THE ASYMPTOTIC GEOMETRY OF THE TEICHMÜLLER METRIC

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ABSTRACT. We determine the asymptotic behaviour of extremal length along arbitrary Teichmüller rays. This allows us to calculate the endpoint in the Gardiner–Masur boundary of any Teichmüller ray. We give a proof that this compactification is the same as the horofunction compactification. An important subset of the latter is the set of Busemann points. We show that the Busemann points are exactly the limits of the Teichmüller rays, and we give a necessary and sufficient condition for a sequence of Busemann points to converge to a Busemann point. Finally, we determine the detour metric on the boundary.

1. Introduction

Let $S$ be an oriented surface of genus $g$ with $n$ punctures. We assume that $3g - 3 + n \geq 1$. The Teichmüller space $T(S)$ of $S$ may be defined as the space of marked conformal structures on $S$ up to conformal equivalence. Viewed in this way, the most natural metric of $T(S)$ is the Teichmüller metric. Kerckhoff [10] has shown that a useful tool for studying the geometry of the this metric is the extremal length of a measured foliation. Here we examine the behaviour of the extremal length as one travels along a Teichmüller geodesic ray.

Our main result is as follows. Let $q$ be the initial quadratic differential of a geodesic ray, and let $V(q)$ and $H(q)$ be, respectively, its vertical and horizontal measured foliations. Recall that removing the critical graph of a measured foliation $G$ decomposes the surface into a finite number of connected components, each of which is either a cylinder of closed leaves or a minimal component in which all leaves are dense. Furthermore, the transverse measure on a minimal component $D$ may be written as a sum of a finite number of projectively-distinct ergodic measures: $\nu|D = \sum_j \nu_{D,j}$. We say that a measured foliation $G'$ is an indecomposable component of $G$ if it is either one of the cylindrical components of $G$, or it is topologically equivalent to one of the minimal components $D$ and has as transverse measure one of the $\nu_{D,j}$. A measured foliation is indecomposable if it has only one indecomposable component, namely itself.

**Theorem 1.1.** Let $R(q; \cdot) : \mathbb{R}_+ \to T(S)$ be the Teichmüller ray with initial unit-area quadratic differential $q$, and let $F$ be a measured foliation. Then,

$$\lim_{t \to \infty} e^{-2t} \text{Ext}_{R(q; t)}[F] = \sum_j \frac{i(G_j, F)^2}{i(G_j, H(q))},$$

where the $\{G_j\}$ are the indecomposable components of the vertical foliation $V(q)$.

Date: May 3, 2014.

2000 Mathematics Subject Classification. Primary 32G15; 30F60.
This result was proved by Kerckhoff [10] in the case of Jenkins–Strebel rays, that is, when all the indecomposable components \( \{G_j\} \) are annular. See also [8], [19], and [3] for a more explicit treatment.

In [5], Gardiner and Masur introduced a compactification of Teichmüller space by embedding it into the projective space of \( \mathbb{R}^S \) using the (square root of) the extremal length function, and showing that the image is relatively compact. This is analogous to the Thurston compactification, the only difference being that extremal lengths are used rather than hyperbolic lengths.

Theorem 1.1 is exactly what is needed to show that Teichmüller rays converge in the Gardiner–Masur compactification, and to calculate their limits.

Corollary 1.2. The Teichmüller ray \( R(q; \cdot) \) converges in the Gardiner–Masur compactification to the projective class of

\[
E_q(\cdot) := \left( \sum_j \frac{i(G_j, \cdot)^2}{i(G_j, H(q))} \right)^{1/2}.
\]

The Jenkins–Strebel case of this corollary appears in [18] and [19].

There exists a very general means of compactifying a metric space, namely the horofunction compactification, introduced by Gromov [6]. In [24], it was shown that the horofunction compactification of Teichmüller space with Thurston’s Lipschitz metric [24] is in fact just the usual Thurston compactification. Recall that this metric is the logarithm of the least possible Lipschitz constant over all diffeomorphisms of the surface isotopic to the identity, and that a formula similar to Kerckhoff’s holds, but with extremal length replaced by hyperbolic length.

We show in Section 6 that the horofunction compactification of Teichmüller’s metric is the same as the Gardiner–Masur compactification. This was previously proved by Liu and Su [14].

Theorem 1.3. The Gardiner–Masur compactification and the horofunction compactification of \( T(S) \) are the same.

It seems that it may be more appropriate to consider the Gardiner–Masur compactification when one takes the conformal view of Teichmüller space, and the Thurston compactification when one takes the hyperbolic view. Results about the convergence of Teichmüller geodesics to points in the Thurston boundary, such as in [12] and [17], may be seen as attempts to relate these two geometries of Teichmüller space.

A particularly interesting subset of the horofunction boundary is its set of Busemann points. These are the boundary points that can be reached as a limit along an almost-geodesic, which is a slight weakening of the usual notion of geodesic. Because Teichmüller rays are geodesic in the Teichmüller metric, it is clear that the horofunctions corresponding to the points of the Gardiner–Masur boundary identified in Corollary 1.2 are Busemann. We show that these are the only ones.

Theorem 1.4. A horofunction is a Busemann point if and only if it corresponds to a point of the form \( E_q \) in the Gardiner–Masur boundary, where \( q \) is a quadratic differential.

It is also of interest to know which Teichmüller rays converge to the same boundary point. Kerckhoff answered a related question in the case of Jenkins–Strebel rays using the notion of modular equivalence of Jenkins–Strebel quadratic differentials.
He showed that two Jenkins–Strebel rays are asymptotic, that is, the distance between them converges to zero, if and only if their initial quadratic differentials are modularly equivalent. We generalise this notion to arbitrary quadratic differentials.

**Definition 1.5.** Let \( q \) and \( q' \) be two quadratic differentials whose vertical foliations can be simultaneously be written in the form 
\[
V(q) = \sum_j \alpha_j G_j \quad \text{and} \quad V(q') = \sum_j \alpha'_j G_j,
\]
where \( \{G_j\} \) is a set of mutually non-intersecting indecomposable measured foliations and \( \{\alpha_j\} \) and \( \{\alpha'_j\} \) are sets of positive coefficients. We say that \( q \) and \( q' \) are **modularly equivalent** if
\[
\frac{\alpha_j}{i(G_j, H(q))} = C \frac{\alpha'_j}{i(G_j, H(q'))}, \quad \text{for all } j,
\]
where \( C \) is a positive constant independent of \( j \).

**Theorem 1.6.** Two Busemann points \( E_q \) and \( E_{q'} \) are identical if and only if \( q \) and \( q' \) are modularly equivalent.

The Jenkins–Strebel case of the following result appears in [9] and [23].

**Theorem 1.7.** Every modular equivalence class of quadratic differentials has a representative at each point of Teichmüller space. This representative is unique up to multiplication by a positive constant.

The above theorems have the following geometric interpretation.

**Theorem 1.8.** Let \( p \) be a point of \( \mathcal{T}(S) \) and \( \xi \) be a Busemann point of the horofunction boundary. Then, there exists a unique geodesic ray starting at \( p \) and converging to \( \xi \).

The uniqueness part if this theorem was proved independently by Miyachi [20].

We have seen that the set of Busemann points may be identified with the set of unit-area quadratic differentials at the basepoint. This is also the case for the Teichmüller boundary of Teichmüller space, so it is interesting to compare the two. Let \( x_n \) be a sequence in \( \mathcal{T}(S) \), and write \( x_n = R(q_n; t_n) \), where \( t_n \) is the distance to the basepoint and \( q_n \) is the initial quadratic differential at the basepoint. Recall that \( x_n \) converges to a point in the Teichmüller boundary if and only if \( q_n \) converges to a unit area quadratic differential \( q \), and \( t_n \) converges to infinity. An equivalent condition is that the geodesic segment connecting the basepoint to \( x_n \) converges uniformly on compact sets of \( \mathbb{R}^+ \) to \( R(q; \cdot) \).

Let \( q \) and \( q' \) be Jenkins–Strebel differentials. Kerckhoff [10] showed that if \( V(q) \) and \( V(q') \) have the same single core curve, then \( R(q; \cdot) \) and \( R(q'; \cdot) \) have the same limit in the Teichmüller boundary. He also showed that the same result is true when the vertical foliations have the same \( 3g - 3 \) core curves, which is the maximum number possible. Concerning the general case, he says “it seems likely that some non-convergent rays exist”. We show that, in fact, no such rays exist.

**Theorem 1.9.** Each Teichmüller ray converges in the Teichmüller compactification. Two rays have the same limit if and only if their initial quadratic differentials are modularly equivalent.

We describe the topology that the set of Busemann points inherits from the horofunction boundary in Theorem 7.12. It turns out to be strictly finer than the topology on the Teichmüller boundary. This implies in particular that there exist...
non-Busemann points in the horofunction boundary when $3g - 3 + n \geq 2$, a result that has also been proved by Miyachi [21].

Our final result concerns the detour metric on the set of Busemann points of Teichmüller space. An explicit formula for this may be found in Corollary 8.4 where it is seen that the distance between two Busemann points $E_q$ and $E_{q'}$ is finite if and only if their vertical foliations can be simultaneously be written $V(q) = \sum_j \alpha_j G_j$ and $V(q') = \sum_j \alpha'_j G_j$, where $\{G_j\}_j$ is a set of mutually non-intersecting indecomposable measured foliations and $\{\alpha_j\}_j$ and $\{\alpha'_j\}_j$ are sets of positive coefficients. It is interesting that this is exactly the criterion for when the two Teichmüller rays $R(q; \cdot)$ and $R(q'; \cdot)$ stay a bounded distance apart—the various cases are considered in [14, 16, 8, 13]. Of course, one may easily show in general that when rays stay a bounded distance apart, the detour metric between the corresponding Busemann points is finite. However, the converse is not true for general metric spaces.

The layout of this paper is as follows. In Section 2 we recall some background material on Teichmüller space, including its Gardiner–Masur compactification. In Section 3 we prepare to prove Theorem 1.1 by calculating a lower bound on the extremal length. The upper bound completing the proof is established in Section 4, which is considerably longer. We recall the basics about the horofunction compactification in Section 5 and prove Theorem 1.3 in Section 6. Section 7 is devoted to modular equivalence and the various convergence results detailed above. Finally, in Section 8 we calculate the detour cost on the boundary.

2. Background

Let $S$ be an oriented surface of genus $g$ with $n$ punctures. We assume that $S$ has negative Euler characteristic and is not the 3-punctured sphere, in other words, that $3g - 3 - n \geq 1$. The Teichmüller space $\mathcal{T}(S)$ of $S$ is the space of marked conformal structures $(X, f)$ on $S$ up to conformal equivalence. Here $X$ is a surface and $f : S \to X$ is a quasi-conformal map. Recall that two marked conformal structures $(X_1, f_1)$ and $(X_1, f_1)$ are conformally equivalent if there exists a conformal map $f : X_1 \to X_2$ such that $f \circ f_1$ is homotopic to $f_2$.

Let $x_1 := (X_1, f_1)$ and $x_2 := (X_2, f_2)$ be two marked conformal structures on $S$. The Teichmüller distance between $x_1$ and $x_2$ is defined to be

$$d(x_1, x_2) := \frac{1}{2} \log \inf_f K(f),$$

where the infimum is over all quasi-conformal homeomorphisms $f : X_1 \to X_2$ that are homotopic to $f_2 \circ f_1^{-1}$, and $K(f)$ is the quasi-conformal dilatation of $f$. Obviously, $d(x_1, x_2)$ remains the same if $x_1$ or $x_2$ are replaced by a conformally equivalent structure, and so $d$ defines a metric on $\mathcal{T}(S)$, called the Teichmüller metric. This metric is complete and geodesic [11].

A (holomorphic) quadratic differential on a Riemann surface $X$ is a tensor of the form $q(z) dz^2$, where $q$ is holomorphic. Quadratic differentials are allowed to have 1st order poles at the punctures.

A quadratic differential has a finite number of zeros. In a neighbourhood of any other point, there is a natural parameter $z = x + iy$. Thus, a quadratic differential $q$ gives rise to two measured foliations on $S$: the horizontal foliation $H(q)$ and the vertical foliation $V(q)$. The leaves of $H(q)$ are defined by $y = \text{constant}$, and the transverse measure is $|dz|$. Similarly, the leaves of $V(q)$ are defined by $x = \text{constant}$, and the transverse measure is $|dy|$. The foliations $H(q)$ and $V(q)$ each have a
singularity at every zero of $q$. At a zero of order $k$, the singularity is $(k + 2)$-pronged. See [4] for a detailed account of measured foliations.

We always consider there to be singularities at the punctures. They may be one-pronged, two-pronged, or higher order.

The metric $dx^2 + dy^2$ is the singular flat metric determined by $q$. Its total area is finite. A quadratic differential is said to be of unit area if the area of its associated flat singular metric is 1.

At each point $x = (X, f)$ in Teichmüller space, there is a one-to-one correspondence between the set of geodesic rays starting at $x$ and the set of unit-area quadratic differentials on $X$. Given such a quadratic differential $q$ on $X$ and a scalar $K > 0$, one multiplies the transverse measure of $V(q)$ by $K$ and the transverse measure of $H(q)$ by $1/K$. The resulting pair of measured foliations determines a conformal structure on $S$, and hence a point in Teichmüller space. We denote this point by $R(q; t)$, where $t = \log K$. The function $t \mapsto R(q; t)$ is a unit-speed geodesic ray.

We say that a leaf of a foliation is a saddle connection if it joins two not necessarily distinct singularities. The critical graph is the union of all saddle connections. The complement of the critical graph has a finite number of connected components. Each is either a cylinder swept out by closed leaves or a so-called minimal component, in which all leaves are dense. On each minimal component $D$, there exists a finite set of ergodic transverse measures $\mu_1, \ldots, \mu_n$ such that any transverse measure $\mu$ on $D$ can be written as a sum $\mu = \sum_{i=1}^{n} f_i \mu_i$, with non-negative coefficients $\{f_i\}$. There is an upper bound on the number $n$ of ergodic transverse measures that just depends on the topology of the surface.

A conformal metric on a Riemann surface is a metric that is locally of the form $\rho(z) |dz|$, where $\rho$ is a non-negative measurable real-valued function on the surface. Let $S$ be the set of free homotopy classes of essential, non-peripheral simple closed curves of $S$. We define the $\rho$-length of a curve class $\alpha \in S$ to be the length of the shortest curve in $\alpha$ measured with respect to $\rho$, that is,

$$L_{\rho}(\alpha) := \inf_{\alpha' \in [\alpha]} \int_{\alpha'} \rho \, |dz|,$$

where $|dz|$ denotes the Euclidean length element. The area of $\rho$ is defined to be $A(\rho) := \int_S \rho^2 \, dx \, dy$.

The extremal length of a curve class $\alpha \in S$ on a Riemann surface $X \in T(S)$ is

$$\text{Ext}_X(\alpha) := \sup_{\rho} \frac{L_{\rho}(\alpha)^2}{A(\rho)},$$

where the supremum is over all Borel-measurable conformal metrics of finite area. This is the so-called analytic definition of extremal length. There is also the following geometric definition:

$$\text{Ext}_X(\alpha) := \inf_{C} \frac{1}{\text{mod}(C)},$$

Here the infimum is over all embedded cylinders $C$ in $X$ with core curve isotopic to $\alpha$, and $\text{mod}(C)$ is the modulus of $C$. 

Lemma 2.1 (Gardiner–Masur [5]). For all measured foliations $F$ and $G$, and points $x \in \mathcal{T}(S)$,

$$\text{Ext}_x(G) = \sup_{F \in \mathcal{P}} \frac{i(G, F)^2}{\text{Ext}_x(F)}.$$ 

2.1. The Gardiner–Masur boundary. Define the map $\Phi : \mathcal{T}(S) \to \mathcal{P}_{R^S}$, so that $\Phi(x)$ is the projective class of $(\text{Ext}_x(\alpha)^{1/2})_{\alpha \in S}$. In [5], Gardiner and Masur showed that $(\Phi, \text{cl \, Im \, } \Phi)$ is a compactification of Teichmüller space. Following [19], we call this the Gardiner–Masur compactification, and its topological boundary the Gardiner–Masur boundary.

3. Lower bound

We use $\mathbb{R}_+$ to denote the set of non-negative real numbers. Recall that we have defined, for any quadratic differential $q$, the function $\mathcal{E}_q : \mathcal{MF} \to \mathbb{R}_+$,

$$\mathcal{E}_q(\cdot) := \left( \sum_j \frac{i(G_j, \cdot)^2}{i(G_j, H(q))} \right)^{1/2},$$

where the $\{G_j\}_j$ are the indecomposable components of $V(q) = \sum_j G_j$. 

Lemma 3.1. Let $a$ and $b$ be vectors in $\mathbb{R}_+^n \setminus \{0\}$, $n \geq 1$, and assume that there is no coordinate $j$ for which both $a_j$ and $b_j$ are zero. Then, the function from $\mathbb{R}_+^n \setminus \{0\}$ to $\mathbb{R}$ defined by

$$x \mapsto \frac{(\sum_j a_j x_j)^2}{\sum_j b_j x_j^2}$$

attains its supremum when $x_j = Ca_j/b_j$, where $C > 0$ is a constant independent of $j$. The supremum is $\sum_j a_j^2/b_j$.

Proof. This is elementary. \hfill \Box

Lemma 3.2. Let $q$ be a quadratic differential, and let $R(q; \cdot)$ be the associated geodesic ray. Then,

$$e^{-2t \text{Ext}_{R(q; t)}(F)} \geq \mathcal{E}_q^2(F),$$

for all $t \in \mathbb{R}_+$ and $F \in \mathcal{MF}$.

Proof. Fix $t \in \mathbb{R}_+$ and let $\alpha \in S$. Decompose the vertical foliation of $q$ into its indecomposable components: $V(q) = \sum_{j=0}^J G_j$.

Define a conformal metric $\rho : S \to \mathbb{R}_+$ as follows.

On the annulus associated to each annular indecomposable component $G_j$, let $\rho$ take some positive value $\rho_j$, which we will choose later.

Let $D$ be a minimal domain of $V(q)$, and take a horizontal arc $I$ in the interior of $D$. By considering the point of first return of leaves starting on $I$, we obtain a (non-oriented) interval exchange map, and hence a decomposition of $D$ into a finite number of rectangles $\{R_i\}_i$. There is a one-to-one correspondence between the ergodic measures of the interval exchange map and the indecomposable measured foliations that are supported on $D$. Consider the subset of these indecomposable measured foliations that appear as indecomposable components of $V(q)$. Denote this subset by $\{G_j\}_j : j \in J_D$, where $J_D \subset J$. Write $G_j = (G, \nu_j)$ for all $j \in J_D$, where $G$ is the unmeasured foliation obtained from $V(q)$ by forgetting the measure.
Consider one of the rectangles $R_l$. We can write $R_l = X \times Y$, where $X$ is a horizontal arc and $Y$ is a vertical arc. Since the transverse measures $\{\nu_j\}; j \in J_D$ are mutually singular, there exists a decomposition $X = \bigsqcup_{j \in J_D} X_j$ of $X$ into disjoint Borel subsets $\{X_j\}; j \in J_D$ such that $\nu_j[X_k]$ equals $\nu_j[X]$ when $j = k$, and is zero otherwise. Define $\rho$ to take some positive value $\rho_j$ on $X_j \times Y$, for each $j \in J_D$.

Do this for every rectangle $R_l$ and every minimal domain $D$. Note that points on horizontal edges belong to more than one rectangle, and hence the value of $\rho$ has been defined more than once on these points. This is not a problem however since the set of such points where the definitions differ has $V(q)$-measure zero.

The value of $\rho$ on vertical edges and on the critical graph is not important for the present argument.

For any simple closed curve $\alpha$,

$$\int_{\alpha} \rho |dz| \geq e^t \int_{\alpha} \rho \, dV(q)$$

$$= e^t \sum_{j=0}^J \rho_j \int_{\alpha} dG_j,$$

since $\rho$ is $G_j$-almost everywhere constant along $\alpha$, for all $j$. But $\int_{\alpha} dG_j \geq i(G_j, \alpha)$, and so

$$L_\rho(\alpha) \geq e^t \sum_{j=0}^J \rho_j i(G_j, \alpha).$$

The area of $\rho$ is independent of $t$:

$$A(\rho) = \sum_{j=0}^J \rho_j^2 i(G_j, H(q)).$$

Therefore,

$$e^{-2t} \text{Ext}_{R(q,t)}(\alpha) \geq \frac{\left(\sum_{j=0}^J \rho_j i(G_j, \alpha)\right)^2}{\sum_{j=0}^J \rho_j^2 i(G_j, H(q))}.$$

According to Lemma 3.1, the expression on the right-hand-side attains its maximum when $\rho_j = C i(G_j, \alpha)/i(G_j, H(q))$ for all $j$, where $C$ is any positive constant. Moreover, its maximum is $\sum_{j=0}^J i(G_j, \alpha)^2/i(G_j, H(q))$.

This proves the theorem in the case where $F$ is a curve class. The general case now follows using the continuity and homogeneity of extremal length. \(\square\)

4. Upper bound

As Kerckhoff observed in [10], one often uses the analytic definition to obtain a lower bound on the extremal length, and the geometric definition to obtain an upper bound. However, we will not use this technique in this paper. Instead, we will use the analytic definition a second time to establish another lower bound with a different scaling, and then convert it into an upper bound using Lemma 2.1.

Let $\bar{S}$ denote the completion of $S$. The punctures of $S$ are considered to be distinguished points of $\bar{S}$. 

We define a rectangulation of a quadratic differential $q$ on $S$ to be a map $r$ from a disjoint union of a finite number $n$ of rectangles $\Gamma := \bigsqcup_{k=1}^{n} [0, X_k] \times [0, Y_k]$ to $\bar{S}$ satisfying the following conditions:

1. $r$ is surjective and continuous;
2. $\{x\} \times (0, Y_k)$ is mapped into a leaf of $V(q)$, and $(0, X_k) \times \{y\}$ is mapped into a leaf of $H(q)$, for all $k$, and $x \in (0, X_k)$ and $y \in (0, Y_k)$;
3. $r$ restricted to the union of the interiors of the rectangles is injective, and the image is in $\bar{S}$;
4. $r$ restricted to the interior of any rectangle is an isometry, using the Euclidean metric on $\Gamma$ and the singular flat metric associated to $q$ on $\bar{S}$.

Denote by $\Omega$ the set of unordered pairs $(p, q) \in \Gamma \times \Gamma$ such that $p$ and $q$ lie in the boundary of the same rectangle. We take on this set its natural topology coming from the product topology on $\Gamma$. For $(p, q) \in \Omega$, we denote by $[p, q]$ the closed line segment between $p$ and $q$ in the rectangle in which they both lie. The expressions $[p, q)$, $(p, q)$, and $(p, q]$ will have their obvious meanings.

Let $M$ be the space of Borel measures on $\Omega$. For any measure $\mu$, let $|\mu|$ denote its total mass.

We say a point of $S$ is a corner point if it is the image under $r$ of a corner of a rectangle. A weighting $\rho$ of a rectangulation is an assignment of a positive real number $\rho_k$ to each rectangle.

Define on $\Omega$ the functions

$$v(p, q) := \int_{r[p, q]} dH(q) \quad \text{and} \quad h(p, q) := \int_{r[p, q]} dV(q).$$

The length of $r[p, q]$ in the singular flat metric associated to $q$ is then

$$||(p, q)|| := (v(p, q)^2 + h(p, q)^2)^{1/2}.$$

Let

$$l := \min\{||(p, q)|| \mid (p, q) \in \Omega, r(p) \text{ and } r(q) \text{ are distinct corner points}\}.$$

Let $A$ be the set of elements $\mu$ of $M$ consisting of a finite number of atoms of mass 1 on pairs $(p_n, q_n)$ such that

(i) after reordering if necessary, $r(p_{n+1}) = r(q_n)$ for all $n$, cyclically;
(ii) if, for any $n$, neither $[p_n, q_n]$ nor $[p_{n+1}, q_{n+1}]$ are horizontal, then $r[p_n, q_n]$ concatenated with $r[p_{n+1}, q_{n+1}]$ is an arc transverse to the horizontal foliation $H(q)$;
(iii) if there is an atom on $(p, q)$ with both $p$ and $q$ lying in the same horizontal edge of a rectangle, then both $r(p)$ and $r(q)$ are corner points of $S$;
(iv) if for any $n$, we have $||(p_n, q_n)|| < l$, then either $(p_{n+1}, q_{n+1})$ is horizontal, or $(p_{n-1}, q_{n-1})$ is.

Note that each element $\mu$ of $A$ defines a closed curve $a(\mu)$ on $S$, although this curve is not necessarily simple. When considering an element of $A$, we always reorder the atoms in such a way that (i) holds, and treat the index as being cyclical.

**Lemma 4.1.** Assume a rectangulation $r : \Gamma \to \bar{S}$ and a weighting $\rho$ is given, and let $\epsilon > 0$. Then, for every simple closed curve $\alpha \in S$ there exists $\mu \in A$ such that
Figure 1. figure1

\[ a(\mu) \text{ is homotopic to } \alpha, \text{ and} \]
\[
\int_{a(\mu)} \rho \, dH(q) \leq \int_{\alpha} \rho \, dH(q) + \epsilon, \quad \text{and} \]
\[
\int_{a(\mu)} dV(q) \leq \int_{\alpha} dV(q) + \epsilon. \tag{2b}
\]

Proof. By perturbing \( \alpha \) if necessary, we may assume that it passes only finitely many times through the image under \( r \) of the boundaries of the rectangles.

Construct a measure \( \mu \) on \( \Omega \) as follows. For each piece of \( \alpha \) lying in the image of a rectangle \( R \) and having endpoints \( r(p) \) and \( r(q) \) with \( p \) and \( q \) in \( R \), put an atom of mass one on \((p, q)\). Clearly, \( \mu \) satisfies (i), and \( a(\mu) \) is homotopic to \( \alpha \). Moreover, (2) holds if the perturbation is small enough.

So, order the pairs as in (i). Suppose that, for some \( n \), the arc \( r[p_n, q_n] \) concatenated with \( r[p_{n+1}, q_{n+1}] \) is not transverse to the horizontal foliation, and that neither \([p_n, q_n] \) nor \([p_{n+1}, q_{n+1}] \) is horizontal. So there exists a leaf segment with one end on \( r[p_n, q_n] \) and the other on \( r[p_{n+1}, q_{n+1}] \) that forms a disk when concatenated with a subarc of \( r[p_n, q_n] \cdot r[p_{n+1}, q_{n+1}] \). Choose the leaf segment in such a way as to maximise the size of the disk.

If \((p_n, q_n)\) and \((p_{n+1}, q_{n+1})\) lie in the same rectangle, then simply remove these two atoms from \( \mu \), and replace them with an atom on \((p_n, q_{n+1})\).

Otherwise, there is a point \( x \) lying on the leaf segment, and elements \( q' \) and \( p' \) of \( \Gamma \) lying in vertical edges of the rectangles containing, respectively, \( q_n \) and \( p_{n+1} \), such that \( x = r(q') = r(p') \). Since the leaf segment was chosen to maximize the size of the disk, either \( r(q') = r(p') \) is a singular point, or one or both of \((p_n, q')\) or \((p', q_{n+1})\) is horizontal. So, in \( \mu \), replace the atoms on \((p_n, q_n)\) and \((p_{n+1}, q_{n+1})\) with atoms on \((p_n, q')\) and \((p', q_{n+1})\).

Note that this replacement does not increase the number of atoms in \( \mu \).

Replacing the procedure if necessary, we obtain an element \( \mu \) on \( M \) satisfying (i), (ii), and (2), such that \( a(\mu) \) is homotopic to \( \alpha \).

Now suppose there is an atom \((p_n, q_n)\) in \( \mu \) with both \( p_n \) and \( q_n \) lying in the same horizontal edge of a rectangle and \( r(p_n) \) is not a corner point. Consider the points along the straight line segment from \( p_n \) to \( q_n \) that are mapped by \( r \) to corner points. Let \( p' \) be the closest one to \( p \) if one exists; otherwise, let \( p' := q_n \). Since \( \mu \) satisfies (i), we have \( r(q_{n-1}) = r(p_n) \). None of the points between \( p_n \) and \( p' \) are mapped to corner points, and so there is a point \( q' \) in the same rectangle as \( q_{n-1} \) such that \( r(q') = r(p') \). See Figure 1.

Replace the atoms on \((p_{n-1}, q_{n-1})\) and \((p_n, q_n)\) with atoms on \((p_{n-1}, q')\) and \((p', q)\) if \( p' \neq q_n \), or just with \((p_{n-1}, q')\) if \( p' = q_n \). In the former case, condition (ii) is preserved. In the latter case, this condition may not be preserved, so we must go back to the previous step to re-establish it. Note, however, that in this case the number of atoms in \( \mu \) is decreased. This ensures that our construction will eventually terminate.

By repeating this process as many times as necessary, we ensure that our measure \( \mu \) satisfies (i), (ii), (iii), and (2), and that \( a(\mu) \) is homotopic to \( \alpha \).
Now suppose that there is an atom \((p_n, q_n)\) in \(\mu\) satisfying \(||(p_n, q_n)|| < l\). If \(h(p_n, q_n) > 0\) then either \(p_n\) or \(q_n\) lies in the interior of a horizontal edge and the other point lies on a vertical edge. Without loss of generality, assume the former case. We can move \(p_n\) without increasing \(\int_{a(p)} dV(q)\) until \(p_n\) coincides with a corner of the rectangle in which it lies. If \(p_n\) now equals \(q_n\), we remove this atom and return to re-establish (ii). If they differ, we have now reduced to the case where \(h(p_n, q_n) = 0\).

So, consider the case where \(h(p_n, q_n) = 0\). If \((p_{n+1}, q_{n+1})\) is horizontal, then we have established the conclusion of (iii). If not, then \(r[p_n, q_n]\) concatenated with \(r[p_{n+1}, q_{n+1}]\) is transverse to \(H(q)\). Let \(R_j\) and \(R_k\) be the rectangles containing \((p_n, q_n)\) and \((p_{n+1}, q_{n+1})\), respectively. If \(\rho_j < \rho_k\), then we can move \(q_n\) towards \(p_n\) without increasing \(\int_{a(\mu)} \rho dH(q)\), until either \(r(q_n)\) is a corner point or \(q_n\) equals \(p_n\). Similarly, if \(\rho_j > \rho_k\), then we can move \(q_n\) away from \(p_n\) without increasing \(\int_{a(\mu)} \rho dH(q)\), until either \(r(q_n)\) is a corner point or \((p_{n+1}, q_{n+1})\) is horizontal. In the same way, we can move \(p_n\) until either \(r(p_n)\) is a corner point, \((p_{n-1}, q_{n-1})\) is horizontal, or \(p_n\) and \(q_n\) coincide. If \(p_n\) and \(q_n\) now coincide, we may remove this atom from \(\mu\) and then go back to re-establish (ii). If \((p_n)\) and \((q_n)\) are both corner points, then \(||(p_n, q_n)|| \geq l\) and (iii) no longer applies. If \((p_{n-1}, q_{n-1})\) or \((p_{n+1}, q_{n+1})\) is horizontal, then the conclusion of (iii) is true. \(\square\)

Let \(P\) be the subset of \(\Omega\) consisting of points of the form \((p, p)\).

**Lemma 4.2.** Let \(\mu_n\) be a sequence in \(A\), and let \(\lambda_n\) be a sequence of positive real numbers such that \(\lambda_n \mu_n\) converges to \(\mu \in M\). Then, \(\mu[P] \leq 2\mu[H]\).

**Proof.** Observe that the set

\[
P^l := \{(p, q) \in \Omega \mid ||(p, q)|| < l\}
\]

is open, and the set \(H\) is closed. Also, by (iii) and (iv), \(\mu_n[P^l] \leq 2\mu_n[H]\), for all \(n\). The conclusion now follows since \(P \subset P^l\). \(\square\)

For any subset \(\mathcal{R}\) of the set of rectangles, define

\[
\mathcal{X}_\mathcal{R} := \left\{ x \in S \mid r^{-1}(x) \subset \bigcup_{R \in \mathcal{R}} R \right\}.
\]

We call \(\mathcal{R}\) a **patch** if \(\bigcup_{R \in \mathcal{R}} r(R)\) is connected and simply connected, and \(\mathcal{X}_\mathcal{R}\) contains no singularities. We say an arc \(\alpha\) in \(S\) is **short** if \(r^{-1}(\alpha)\) is contained within \(\mathcal{X}_\mathcal{R}\) for some patch \(\mathcal{R}\).

Given a patch \(\mathcal{R}\), we may choose in a consistent way one of the horizontal edges of each rectangle \(R\) in \(\mathcal{R}\) to be the “upper” edge. By consistent, we mean that if a vertical leaf segment is common to the image under \(r\) of two rectangles of \(\mathcal{R}\), then the induced orientations are the same. This lets us define a relation \(<\) on each rectangle \(R\) of \(\mathcal{R}\), where \(p < q\) for \(p, q \in R\) if \(p\) is “lower” than \(q\), that is, further from the “upper” edge of \(R\).

For each \(X \subset \mathcal{X}_\mathcal{R}\), let

\[
U_X := \{(p, q) \in \Omega \mid r(p) \in X \text{ and } p < q, \text{ or } r(q) \in X \text{ and } q < p\}, \quad \text{and}
\]

\[
D_X := \{(p, q) \in \Omega \mid r(p) \in X \text{ and } q < p, \text{ or } r(q) \in X \text{ and } p < q\}.
\]
Define the set of horizontal segments:

\[ \hat{H} := \{(p, q) \in \Omega \mid p < q, q \neq p, \text{ and } p \neq q\}. \]

We will also need the following subset of this set. Let \( H \) be the set of \((p, q)\) in \( \hat{H} \) such that if \( p \) and \( q \) are in the same horizontal edge of a rectangle, then both \( r(p) \) and \( r(q) \) are corner points.

Denote by \( \delta_{(p,q)} \) the Dirac measure on \((p, q) \in \Omega\), that is, the measure consisting of an atom of mass 1 on \((p, q)\).

**Lemma 4.3.** Let \( \mu_n \) be a sequence in \( A \), and let \( \lambda_n \) be a sequence of positive real numbers such that \( \lambda_n \mu_n \) converges to \( \mu \in M \) with \( \mu[\hat{H}] = 0 \). Then, for any patch \( \mathcal{R} \), we have \( \mu[U_X] = \mu[D_X] \), for all Borel subsets \( X \) of \( \mathcal{X}_R \).

**Proof.** Fix \( n \in \mathbb{N} \) and a Borel subset \( X \) of \( S \) satisfying \( \text{cl} X \subset \mathcal{X}_R \). Since \( \mu_n \) is in \( A \) it can be written \( \mu_n = \sum_{j=1}^{\arrowvert \mu_n \arrowvert} \delta_{(p_j,q_j)} \), with the ordered pairs \( \{(p_j,q_j)\}\) satisfying (i)–(iv). Define the sets

\[ U^+ := \{ j \mid r(p_j) \in X \text{ and } p_j < q_j \}, \]
\[ U^- := \{ j \mid r(q_j) \in X \text{ and } q_j < p_j \}, \]
\[ D^+ := \{ j \mid r(p_j) \in X \text{ and } q_j < p_j \}, \]
\[ D^- := \{ j \mid r(q_j) \in X \text{ and } p_j < q_j \}, \]
and
\[ H^\pm := \{ j \mid (p_j,q_j) \in H \}. \]

From (i) and (ii), we see that if \( j \) is in \( U^+ \), then \( j - 1 \) is in either \( D^- \) or \( H^\pm \). Similarly, if \( j \) is in \( U^- \), then \( j + 1 \) is in either \( D^+ \) or \( H^\pm \). So,

\[ \mu_n[U_X] = 2U^+ + 2U^- \leq 2D^- + 2H^\pm + 2D^+ + 2H^\pm = \mu_n[D_X] + 2\mu_n[H]. \]

Here, “\( \arrowvert \)” denotes the number of elements in a set. A similar inequality with \( U_X \) and \( D_X \) interchanged can also be derived in the same way. We conclude that

\[ \arrowvert \mu_n[U_X] - \mu_n[D_X] \arrowvert \leq 2\mu_n[H]. \quad (3) \]

Let \( \partial X := \text{cl} X \backslash \text{int} X \) be the boundary of \( X \). By assumption, \( \partial X \subset \mathcal{X}_R \). We have

\[ \partial U_X \subset \hat{H} \cup P \cup U_{\partial X}, \text{ and} \]
\[ \partial D_X \subset \hat{H} \cup P \cup D_{\partial X}. \]

Let \( Z \) be the set of Borel subsets \( X \) of \( S \) such that \( \text{cl} X \subset \mathcal{X}_R \) and \( \mu[U_{\partial X}] = \mu[D_{\partial X}] = 0 \).

By assumption, \( \mu[\hat{H}] = 0 \), so using Lemma 4.2 we get that \( \mu[P] = 0 \). Therefore, we may apply the Portmanteau theorem to get that \( \mu_n[U_X] \) and \( \mu_n[D_X] \) converge, respectively, to \( \mu[U_X] \) and \( \mu[D_X] \), for all \( X \in Z \). Also, since \( H \) is closed, \( \limsup_n \mu_n[H] \leq \mu[H] = 0 \). We see therefore that \( \mu[U_X] = \mu[D_X] \) for all \( X \in Z \).

Both \( \mu[U_X] \) and \( \mu[D_X] \) are finite measures on \( \mathcal{X}_R \). Since, for any subsets \( X \) and \( Y \) of \( \mathcal{X}_R \), one has \( \partial(X \cap Y) \subset \partial X \cup \partial Y \), we have that \( Z \) is closed under finite intersections.
Take $G$ to be an open Borel subset of the space $X_R$. Choose some metric $d$ on $S$ compatible with the topology, and define, for each $\epsilon \in (0, 1)$,

$$G_\epsilon := \{ x \in G \mid d(x, \partial G) \geq \epsilon \},$$

(4)

where $\partial G$ denotes the boundary of $G$ in $X_R$. Since every point $x$ of $\partial G_\epsilon$ satisfies $d(x, \partial G) = \epsilon$, the sets $\{\partial G_\epsilon\}$, are pairwise disjoint. Therefore, only countably many such sets satisfy $\mu[\cup \partial G_\epsilon] > 0$ and only countably many satisfy $\mu[D_{\partial G_\epsilon}] > 0$. So, $G_\epsilon$ is in $Z$ for $\epsilon$ in some dense subset of $(0, 1)$. Hence, $G$ can be written as a countable union of elements of $Z$.

We have shown that $Z$ is a $\pi$-system that generates the Borel $\sigma$-algebra of $X_R$. So, since the measures $\mu[U]$ and $\mu[D]$ agree on $Z$, they agree on every Borel subset of $X_R$. This concludes the proof. \hfill $\square$

Given a patch $\mathcal{R}$, define $F_Y := \{(p, q) \in \Omega \mid p < q \text{ and } [p, q] \cap Y \neq \emptyset\}$, for any $Y \subset X_R$.

Let $G$ be an unmeasured foliation. A \textit{generalised transverse measure} $\mu$ on $G$ is a map associating a measure to each transverse arc that does not pass through a singular point, with the following condition: if $\alpha : [0, 1] \rightarrow S$ and $\beta : [0, 1] \rightarrow S$ are two such arcs that are isotopic through transverse arcs whose endpoints remain in the same leaf, then $\mu(\alpha) = \mu(\beta)$. We do not require that the measure is regular with respect to the Lebesgue measure.

\textbf{Lemma 4.4.} \textit{Assume that, for every patch $\mathcal{R}$, a measure $\mu \in M$ satisfies $\mu[U_X] = \mu[D_X]$ for all Borel $X \subset X_R$, and that $\int h\,d\mu = 0$. Then, there exists a generalised transverse measure $\tilde{\mu}$ on $G$ such that}

$$\mu[F_{r^{-1}(\alpha)}] = \int_{\alpha} \,d\tilde{\mu},$$

(5)

for every short transverse arc $\alpha$.

\textbf{Proof.} For any transverse arc $\alpha$ avoiding singularities, write $\alpha$ as a concatenation of short transverse arcs $\{\alpha_j\}_j$, and define

$$\int_{\alpha} \,d\tilde{\mu} := \sum_j \mu[F_{r^{-1}(\alpha_j)}],$$

where each $F_{r^{-1}(\alpha_j)}$ is relative to some patch containing $\alpha_j$, which we fix. That the same result is obtained when one takes a different decomposition of $\alpha$ can be seen by considering a common refinement of the two decompositions.

We must show that $\tilde{\mu}$ is a generalised transverse measure. Let $\alpha_0, \alpha_1 : [t_0, t_1] \rightarrow S$ be two transverse arcs isotopic through an isotopy $\mathcal{I} : [t_0, t_1] \times [0, 1] \rightarrow S$, along which each point stays in the same leaf. We write $\alpha_s := \mathcal{I}(\cdot, s)$, for all $s \in [0, 1]$. Since $\mathcal{I}$ is continuous, $\alpha_{s'}$ converges uniformly to $\alpha_s$ as $s'$ tends to $s$; see [25, Lemma 3.1].

Let $s \in [0, 1]$. Write $\alpha_s$ as a concatenation of short transverse arcs $\{\alpha'_j\}_j$, where the domains are pairwise disjoint intervals $\{I_j\}_j$ satisfying $\cup_j I_j = [t_0, t_1]$.

Fix $j$. For $s'$ close enough to $s$, the arcs $\alpha'_j$ and $\alpha_{s'}$ restricted to $I_j$ are in the image under $r$ of the same rectangular patch $\mathcal{R}$.

Fix such an $s'$. Recall that we may define a notion of “upwards” on $X_R$. Decompose $I_j$ into three sets $T^0$, $T^-$, and $T^+$, depending on whether $\alpha_{s'}(t)$ is identical to, below, or above $\alpha_s(t)$, respectively, for $t \in I_j$. 

Similarly, \( U \) ends on the boundary of \( \mathcal{X}_\mathcal{R} \) and does not contain \( \alpha_s(t) \).

Let \( X \) denote the union over \( T^+ \) of the half-open leaf segments \( (\alpha_s(t), \alpha_s'(t)] \), and let \( Y \) denote the union over \( T^+ \) of the open leaf segment that starts on \( \alpha_s'(t) \), ends on the boundary of \( \mathcal{X}_\mathcal{R} \), and does not contain \( \alpha_s(t) \).

Since \( \mu \) is supported on \( V \), we have \( F_{T^+}^\prime \setminus F_{T^+} = U_X \cap D_Y \). So
\[
(U_X \cap D_X) \cup (F_{T^+}^\prime \setminus F_{T^+}) = U_X \cap (D_X \cup D_Y) = U_X.
\]

Similarly,
\[
(U_X \cap D_X) \cup (F_{T^+} \setminus F_{T^+}^\prime) = D_X.
\]

But \( U_X \cap D_X \) is disjoint from both \( F_{T^+} \) and \( F_{T^+}^\prime \), and, by assumption, \( \mu[U_X] = \mu[D_X] \). We deduce that \( \mu[F_{T^+}] = \mu[F_{T^+}^\prime] \).

One may deduce in a similar fashion that \( \mu[F_{T^-}] = \mu[F_{T^-}^\prime] \). So, we have proved that \( \mu[F_{T^j}] = \mu[F_{T^j}^\prime] \).

Since this works for each \( j \), we see that, for all \( s' \) in some neighbourhood of \( s \), the transverse lengths with respect to \( \tilde{\mu} \) of \( \alpha_s \) and \( \alpha_{s'} \) are equal. Using that \( s \) was chosen arbitrarily and that \([0,1]\) is connected, we get that \( \int_{\alpha_s} d\tilde{\mu} \) is independent of \( s \). \( \square \)

For the next two lemmas, we will need the following notation. Given a patch \( \mathcal{R} \), define \( \tilde{F}_Y := \{(p,q) \in \Omega \mid [p,q] \cap Y \neq \emptyset \} \), for all \( Y \subset \mathcal{X}_\mathcal{R} \). For two parameterised closed curves or arcs \( \alpha \) and \( \beta \) on a surface, we define \( \sharp(\alpha, \beta) \) to be the cardinal number of the set \( \{(s,t) \mid \alpha(s) = \beta(t)\} \). By a straight arc on \( S \), we mean one that is straight in the singular flat metric associated to a given quadratic differential.

**Lemma 4.5.** Suppose a rectangulation is given. Let \( \mu \in A \), and let \( \beta \) be a closed curve that can be written as a concatenation of a finite number of straight short arcs \( \{\beta_j\} \). Then, \( i(\alpha(\mu), \beta) \leq \sum_j \mu[\tilde{F}_{r^{-1}(\beta_j)}] \).

**Proof.** If \( x \in S \) is such that no element of \( r^{-1}(x) \) lies on \( [p,q] \) for some \( (p,q) \in \Omega \), then there is some neighbourhood of \( x \) all of whose points have the same property. One may use this to show that any sufficiently small perturbation of the straight short arcs \( \{\beta_j\} \) will not increase \( \sum_j \mu[\tilde{F}_{r^{-1}(\beta_j)}] \).

Suppose that \( \sharp(r[p,q], \beta_j) \) is infinite for some atom \( (p,q) \) of \( \mu \) and some \( j \). Then, we may perturb an endpoint of \( \beta_j \) so that \( \sharp(r[p,q], \beta_j) \) becomes either zero or one and \( \sum_j \mu[\tilde{F}_{r^{-1}(\beta_j)}] \) is not increased. So we may assume, without loss of generality, that \( \sharp(r[p,q], \beta_j) \) is either zero or one for all atoms \( (p,q) \) of \( \mu \) and all \( j \). Note that in this case \( \delta_{(p,q)}[\tilde{F}_{r^{-1}(\beta_j)}] = \sharp(r[p,q], \beta_j) \). Write \( \mu = \sum_k \delta_{(p_k,q_k)} \). So,
\[
\begin{align*}
i(\alpha(\mu), \beta) & \leq \sharp(\alpha(\mu), \beta) \\
& \leq \sum_j \sum_k \sharp(r[p_k,q_k], \beta_j) \\
& = \sum_j \mu[\tilde{F}_{r^{-1}(\beta_j)}].
\end{align*}
\]

\( \square \)

**Lemma 4.6.** Suppose that a rectangulation is given. Let \( \mu_n \) be a sequence in \( A \), and \( \lambda_n \) be a sequence of positive real numbers. Assume that \( \lambda_n \alpha(\mu_n) \) converges
to $F \in \mathcal{MF}$, and that $\lambda_n \mu_n$ converges to $\mu \in M$ satisfying $\int h \, d\mu = 0$. Then, $i(F, \beta) \leq \int_{\beta} d\tilde{\mu}$, for all closed curves $\beta$ avoiding singularities.

**Proof.** Since $\lambda_n[a(\mu_n)]$ converges to $F$, we have that $\lambda_n i(a(\mu_n), \beta)$ converges to $i(F, \beta)$. Perturb $\beta$ so that it is a concatenation of closed straight transverse short arcs $\beta_j$, and so that $\mu[D_{\{x\}}] = 0$ for all points $x \in \cup_j \beta_j$. We may do this in such a way that $\int_{\beta_j} d\tilde{\mu}$ is not increased by more than an arbitrarily small $\epsilon > 0$. By Lemma 4.5, $\lambda_n i(a(\mu_n), \beta) \leq \sum_j \mu_n[\bar{F}_{r^{-1}(\beta_j)]}$, for all $n$.

Each set $\bar{F}_{r^{-1}(\beta_j)}$ is closed, and so

$$\limsup_n \lambda_n \mu_n[\bar{F}_{r^{-1}(\beta_j)]} \leq \mu[\bar{F}_{r^{-1}(\beta_j)]}, \quad \text{for each } j.$$ 

We also have $\mu[\bar{F}_{r^{-1}(\beta_j)]} = \mu[F_{r^{-1}(\beta_j)]$, for all $j$. Applying Lemmas 4.3 and 4.4, we see that $\mu[F_{r^{-1}(\beta_j)] = \int_{\beta_j} d\tilde{\mu}$, for all $j$. Putting all of this together, and using the fact that $\epsilon$ is arbitrary, we deduce the result. \hfill $\square$

### 4.1. Generalised transverse measures with no atoms

Suppose we are given an unmeasured foliation $G$. Consider the set of generalised transverse measures on $G$ that have no atoms on saddle connections. We regard two of them as being equivalent if they agree on each minimal component of $G$ and give the same height to each annular component. Let $\mathcal{G}(G)$ be the space of equivalence classes under this equivalence relation. We see that $\mathcal{G}(G)$ is a closed finite-dimensional cone. There is one extremal ray for each annular component of $G$, and one for each projective class of ergodic transverse measure on a minimal component. Let a set $J$ index these extremal rays, and, for each $j \in J$, choose an element $\nu_j \in \mathcal{G}(G)$ of the $j$th extremal ray. Every $\nu \in \mathcal{G}(G)$ can be written $\nu = \sum_{j \in J} f_j \nu_j$, for some collection of non-negative coefficients $\{f_j\}_{j \in J}$.

Any element of $\mathcal{G}(G)$ gives rise to an element of $\mathcal{MF}$. We define the intersection number of a generalised transverse measure $\tilde{\nu} \in \mathcal{G}(G)$ and a curve class $[\beta]$ in $\mathcal{S}$ to be $i(\tilde{\nu}, \beta) := \inf_{\beta} \int_{\beta} d\tilde{\nu}$, where the infimum is taken over all curves in the curve class. Clearly, $i(\cdot, \beta)$ is linear for fixed $\beta$.

The following lemma is [25] Lemma 6.3, restated in terms of measured foliations rather than measured laminations.

**Lemma 4.7.** Let $\{F_j\}_{j \in J} \subset \{0, \ldots, n\}$ be a set of projectively-distinct indecomposable elements of $\mathcal{MF}$ such that $i(F_j, F_k) = 0$ for all $j$ and $k$, and let $\epsilon > 0$. Then, there exists a curve class $[\beta]$ in $\mathcal{S}$ such that $i(F_j, \beta) < i(F_0, \beta) \epsilon$, for all $j \neq 0$.

**Lemma 4.8.** Let $\tilde{\nu} \in \mathcal{G}(G)$ be written $\tilde{\nu} = \sum_{j} f_j \nu_j$. Let $F \in \mathcal{MF}$ be such that $i(F, \beta) \leq i(\tilde{\nu}, \beta)$, for all $\beta \in \mathcal{S}$. Then $F$ has a representation of the form $(G, \tilde{\nu}')$, where $\tilde{\nu}' = \sum_j f_j' \nu_j$ with non-negative coefficients $\{f_j'\}$. Moreover, $f_j' \leq f_j$, for all $j$.

**Proof.** Since intersection number is continuous, we have $i(F, H) \leq i(\tilde{\nu}, H)$, for all $H \in \mathcal{MF}$. In particular, taking $H := (G, \tilde{\nu})$, we get $i(F, \tilde{\nu}) = 0$. If $F$ had an indecomposable component $F'$ that is not a multiple of $(G, \nu_j)$ for any $j$, then we could use Lemma 4.7 to find a curve $\beta \in \mathcal{S}$ such that $i(F', \beta)$ is much greater than $i(f_j \nu_j, \beta)$, for all $j$. However, this is impossible by assumption. Hence, $F$ can be written $F = (G, \tilde{\nu}')$, where $\tilde{\nu}' = \sum_j f_j' \nu_j$ with non-negative coefficients $\{f_j'\}$.
Choose \( \epsilon > 0 \), and let \( \tilde{\mu}^c := \sum_{k \neq j} f_k \nu_k \). By Lemma 4.7 there is a curve class \( \beta \in S \) such that \( i(\tilde{\mu}^c, \beta) < i(f_j \nu_j, \beta) \epsilon \). Therefore,

\[
(1 + \epsilon) f_j i(\nu_j, \beta) > i(\tilde{\mu}, \beta) \geq i(\tilde{\mu}^c, \beta) \geq f_j' i(\nu_j, \beta).
\]

The result follows since \( \epsilon \) is arbitrary. \( \square \)

### 4.2. Construction of a weighted rectangulation.

In this subsection, we construct a particular weighted rectangulation.

We will need the following measure theoretic lemma.

**Lemma 4.9.** Let \( \{\nu_j\}; j \in \{0, \ldots, n\} \) be mutually-singular non-atomic finite measures on an interval \( I \) of the real line. Then, for any \( \delta > 0 \), there exists a decomposition of \( I \) into disjoint subsets \( P_j; j \in \{0, \ldots, n\} \) such that each \( P_j \) is composed of a finite number of intervals, and \( \nu_j[I] - \nu_j[P_j] < \delta \), for all \( j \).

**Proof.** The measures \( \{\nu_j\} \) are mutually singular, so there exists a decomposition \( I = X_0 \cup \ldots \cup X_n \) of \( I \) into disjoint Borel sets such that \( \nu_j[X_k] \) equals \( \nu_j[I] \) when \( j = k \), and is zero otherwise.

Fix \( j \in \{0, \ldots, n\} \), and let \( \delta > 0 \) be given. We may approximate \( X_j \) from above by open sets of \( I \):

\[
0 = \sum_{k \neq j} \nu_k[X_j] = \inf \left\{ \sum_{k \neq j} \nu_k[U_j] \mid X_j \subset U_j \subset I, \text{ and } U_j \text{ is open} \right\}.
\]

So, there is an open set \( U_j \) of \( I \) such that \( X_j \subset U_j \subset I \) and \( \sum_{k \neq j} \nu_k[U_j] < \delta \). In particular, \( \nu_k[U_j] < \delta \), for all \( k \neq j \).

Since \( U_j \) is open, it is the disjoint union of a countable collection of open intervals. We can choose a finite number of these with union \( V_j \) such that

\[
\nu_j[V_j] + \delta > \nu_j[U_j] \geq \nu_j[X_j] = \nu_j[I].
\]

Do this for all \( j \).

For all \( j \in \{1, \ldots, n\} \), let

\[
P_j := V_j \setminus \bigcup_{k \neq j} V_k,
\]

and let

\[
P_0 := I \setminus \bigcup_{k \neq 0} P_k \supset V_0 \setminus \bigcup_{k \neq 0} V_k.
\]

Each \( P_j \) is a finite disjoint union of intervals (not necessarily open). The \( \{P_j\} \) are pairwise disjoint. Clearly,

\[
\nu_j[P_j] \geq \nu_j[V_j] - \sum_{k \neq j} \nu_j[V_k], \quad \text{for all } j.
\]

However,

\[
\sum_{k \neq j} \nu_j[V_k] \leq \sum_{k \neq j} \nu_j[U_k] \leq n \delta, \quad \text{for all } j.
\]

The conclusion follows. \( \square \)

**Construction 4.10.** Let \( G \) be the unmeasured foliation obtained from \( V(q) \) by forgetting the transverse measure. As before, let \( J \) be a set indexing the extremal rays of \( \mathcal{G}(G) \). Suppose we are given, positive real numbers \( \delta, \epsilon \), and \( \{\theta_j\}; j \in J \).
Let $J_S \subset J$ be the subset of indices associated to annuli. For each $j \in J_S$, let $A_j$ be the annulus. By cutting $A_j$ along a horizontal leaf, we obtain a rectangle, to which we give weight $\theta_j$.

Let $D$ be a minimal domain of $G$, and take a horizontal arc $I$ in the interior of $D$. By considering the point of first return of leaves starting on $I$, we obtain a (non-oriented) interval exchange map, and hence a rectangular decomposition $\{R_j\}$ of $D$. There is a one-to-one correspondence between the ergodic measures of the interval exchange map and the indecomposable projective measured foliations that are supported on $D$. Let $J_D \subset J$ index this set, and let $\{\nu_j\}; j \in J_D$ be the ergodic measures of the interval exchange map. Consider one of the rectangles $I$. We get a decomposition of $I$ into $\sharp J$ disjoint sets $\{P_j\}$, each the disjoint union of a finite number of intervals, such that $\nu_j[I] - \nu_j[P_j] < \delta$, for all $j \in J_D$. By sweeping the edge $I$ along the rectangle $R_i$, this gives us a decomposition of $R_i$ into sub-rectangles. Associate the weight $\theta_j$ to each rectangle swept out by an interval contained in $P_j$. We repeat this construction for all rectangles in $\{R_i\}$ and for all minimal domains of $G$. Since the annuli and the minimal components of $G$ make up the whole surface, the construction so far gives us a weighted rectangulation $\Gamma$.

However, we wish to give special treatment to the saddle connections. To do this, we will define another rectangulation. Let $C_\epsilon$ be the closure of the set of points of $S$ that are connected to a non-singular point of the critical graph by a horizontal arc of length less than $\epsilon^2$ in the singular flat metric coming from $q$. We assume that $\epsilon$ is small enough that this is a union of rectangles, one for each saddle connection.

Give weight $1/\epsilon$ to each of these rectangles. The remaining part of the surface $S \setminus C_\epsilon$ may be decomposed into rectangles, to which we give weight zero. The conformal metric on $S$ induced by this rectangulation we denote by $\rho_\epsilon$.

We combine the two rectangulations we have constructed as follows. We say that one rectangulation $r_1 : \Gamma_1 \to \tilde{S}$ is finer than another $r_2 : \Gamma_2 \to \tilde{S}$ if for every rectangle $R_1$ in $\Gamma_1$ there exists a rectangle $R_2$ in $\Gamma_2$ such that $r_1(R_1) \subset r_2(R_2)$. Given two rectangulations, one may find a third rectangulation that is finer than both. On this rectangulation we may choose a weighting in such a way that the conformal metric induced on $S$ is the sum of the conformal metrics induced by the original rectangulations. So we obtain a conformal metric $\rho = \rho_\theta + \rho_\epsilon$.

Since $V(q) \in \mathcal{G}(G)$, we may write it as $V(q) = \sum_{j \in J} g_j \nu_j$, where the $\{g_j\}$ are non-negative coefficients. Note that some of the $\{g_j\}$ may be zero. We let $a_j := i(g_j \nu_j, H(q))$, for all $j \in J$, and consider it to be the area of the $j$th component of $V(q)$. Observe that $\sum_{j \in J} a_j = i(V(q), H(q))$ is the area of the singular flat metric associated to the quadratic differential $q$.

We use the notation $O(\epsilon)$ to stand for any function that is less than some linear function of $\epsilon$, for $\epsilon$ small enough.

**Lemma 4.11.** In construction 4.10 fix the quadratic differential $q$, the parameters $\theta_j$, and the choice of horizontal arc in each minimal domain. Then, the area of the conformal metric $\rho$ obtained from the constructed weighted rectangulation satisfies $A(\rho) \leq \sum_j \theta_j^2 a_j + O(\epsilon) + O(\delta)$.

**Proof.** We have $A(\rho) = A(\rho_\theta) + A(\rho_\epsilon)$. 

First observe that the area of \( \rho_e \) is \( \epsilon^2 L/\epsilon \), where \( L \) is the total length of the critical graph with respect to \( H(q) \). This is \( O(\epsilon) \).

Let \( j \in J_S \). The corresponding annular component \( A_j \) of \( G \) contributes \( \theta_j^2 a_j \) to the area of \( \rho_e \).

Now consider a minimal domain \( D \) of \( G \), and let \( J_D \in J \) be the set of indices of the ergodic measured foliations supported on it. In the construction, \( D \) was decomposed into a finite number of rectangles \( \{ R_l \} \), each having a horizontal edge \( I_l \) that is further subdivided into \( 2J_D \) disjoint sets \( \{ P_{ij} \} \), each composed of a finite number of intervals.

Observe that \( a_j = \sum_i h_i \nu_j[I_i] \), for all \( j \in J_D \), where \( h_i \) is the height of \( R_l \) with respect to \( H(q) \).

We have \( \nu_k[P_{ij}] \leq \nu_k[I_i] - \nu_k[P_{ik}] < \delta \), for all distinct \( j \) and \( k \) in \( J_D \). Therefore,

\[
\nu[P_{ij}] = \nu_j[P_{ij}] + \sum_{k \in J_D, k \neq j} \nu_k[P_{ij}] \\
\leq \nu_j[I_i] + (\# J_D - 1)\delta, \quad \text{for all } j \in J_D.
\]

So, the contribution of \( D \) to the area \( \rho_e \) is

\[
\sum_l h_l \sum_{j \in J_D} \theta_j^2 \nu[P_{ij}] \leq \sum_{j \in J_D} \theta_j^2 a_j + O(\delta). \tag*{□}
\]

Define the \( \rho \)-length of a generalised transverse measure \( \tilde{\mu} \) to be

\[
\rho \text{-length}(\tilde{\mu}) := \int_S \rho(d\tilde{\mu} \times dH(q)).
\]

**Lemma 4.12.** In construction \([4.10]\), fix the quadratic differential \( q \), the parameters \( \theta_j \), and the choice of horizontal arc in each minimal domain. Let \( \tilde{\mu} \in \mathcal{G}(G) \) be a generalised transverse measure on \( G \) with no atoms. Write \( \tilde{\mu} = \sum_{j \in J} f_j \nu_j \). Then the \( \rho \)-length of \( \tilde{\mu} \) satisfies \( \rho \text{-length}(\tilde{\mu}) \geq \sum_j \theta_j f_j i(\nu_j, H(q)) - O(\epsilon) - O(\delta) \).

**Proof.** The proof is similar to the proof of Lemma \([4.11]\). \( \square \)

**Definition 4.13.** Let

\[
\mathcal{E}_q^*(\{F\}) := \begin{cases} 
\sum_j f_j^2 a_j, & \text{if } F = \sum_j f_j G_j \text{ with } V(q) = \sum G_j, \\
+\infty, & \text{otherwise},
\end{cases}
\]

\( a_j := i(G_j, H(q)) \) is the area of the indecomposable component \( j \) of \( V(q) \) relative to \( q \).

Now we are ready to prove the key lemma of this section.

**Lemma 4.14.** Let \( R(q; \cdot) \) be ray in Teichmüller space with initial quadratic differential \( q \). Let \( F_n \) be a sequence in \( \mathcal{MF} \) converging to an element \( F \) of \( \mathcal{MF} \), and let \( t_n \) be a sequence of times diverging to infinity. Then,

\[
\liminf_{n \to \infty} e^{2t_n} \text{Ext}_{R(q;t_n)}[F_n] \geq \mathcal{E}_q^*(F).
\]

**Proof.** Since \( S \) is dense in \( \mathcal{PMF} \) and \( \text{Ext}_{R(q;t_n)}[\cdot] \) is continuous for fixed \( t \), there exists a sequence \( ([\alpha_n])_n \) of curve classes, and a sequence of positive real numbers \( \lambda_n \) such that \( \lambda_n [\alpha_n] \) converges to \( F \), and

\[
\left| e^{2t_n} \text{Ext}_{R(q;\alpha_n)}[\lambda_n \alpha_n] - e^{2t_n} \text{Ext}_{R(q;t_n)}[F_n] \right| \to 0, \quad \text{as } n \to \infty.
\]
So, to establish the lemma, it suffices to show that
\[ L := \liminf_{n \to \infty} e^{2t_n} \text{Ext}_{R(q; t_n)} [\lambda_n \alpha_n] \geq e^L[F]. \]

By taking a subsequence if necessary, we may assume that \( e^{2t_n} \text{Ext}_{R(q; t_n)} [\lambda_n \alpha_n] \) converges to \( L \).

Let \( G \) be the unmeasured foliation obtained from \( V(q) \) by forgetting the measure, and let \( J \) be a set indexing the extremal rays of \( G(G) \). For each \( j \in J \), choose an representative \( \nu_j \in G(G) \) of the \( j \)th extremal ray. Choose positive real numbers \( \delta \) and \( \epsilon \), and \( \{ \theta_j \}; j \in J \). Using these parameters, define the weighted rectangulation \( (\{R_i\}, r, \rho) \) according to construction 4.10. This gives us a conformal metric \( \rho \) on the Riemann surface \( R(q; 0) \).

Recall that, for each \( t \in \mathbb{R}_+ \), one goes from \( R(q; 0) \) to \( R(q; t) \) by stretching the vertical foliation and shrinking the horizontal foliation by a factor \( e^t \). Let \( \rho_t \) be the conformal metric on \( R(q; t) \) obtained from \( \rho \) by stretching the surface in this way. The area of \( \rho_t \) is identical to that of \( \rho \), for all \( t \in \mathbb{R}_+ \), because the stretching in the vertical direction is compensated by the shrinking in the horizontal direction.

From the analytic definition of extremal length,
\[ \text{Ext}_{R(q; t_n)} [\alpha_n] \geq \inf_{\alpha \in [\alpha_n]} \frac{L_{\rho_{t_n}}(\alpha)^2}{A(\rho_{t_n})}, \quad \text{for all} \quad n \in \mathbb{N}. \]

For each \( n \in \mathbb{N} \), choose a representative \( \alpha_n \) of \([\alpha_n]\) in such a way that
\[ \lambda_n^2 e^{2t_n} \left| L_{\rho_{t_n}}(\alpha_n)^2 - \inf_{\alpha \in [\alpha_n]} L_{\rho_{t_n}}(\alpha)^2 \right| \to 0, \quad \text{as} \quad n \to \infty. \]

Choose a sequence \( \epsilon'_n \) of positive real numbers converging to zero. We apply Lemma 4.11 to get a sequence \( \mu_n \) in \( A \) such that \( a(\mu_n) \) is homotopic to \( \alpha_n \), and
\[ \int \rho v \, d\mu_n \leq \int_{\alpha_n} \rho \, dH(q) + \epsilon'_n, \quad \text{and} \]
\[ \int h \, d\mu_n \leq \int_{\alpha_n} dV(q) + \epsilon'_n. \]

By taking a subsequence if necessary, we may assume that the sequence of probability measures \( \mu_n/|\mu_n| \) converges weakly to a probability measure \( \mu \).

If \( L \) is infinite, then there is nothing to prove. So, assume that \( L \) is finite.

We claim that \( \lambda_n |\mu_n| \) can not converge to infinity. If it has some subsequence that converges to zero, then obviously the claim is true. So, consider the case where it is bounded away from zero. We have
\[ \lambda_n e^{t_n} L_{\rho_{t_n}}(\alpha_n) \geq \lambda_n e^{2t_n} \int_{\alpha_n} \rho \, dV(q) \]
\[ \geq \lambda_n |\mu_n| e^{2t_n} (\min \rho) \left( \int h \, d\mu_n - \epsilon'_n \right) / |\mu_n|, \]
for \( n \) large enough. Using that \( L \) is finite and that \( e^{2t_n} \) grows without bound, we see that
\[ \int h \, d\mu = \lim_{n \to \infty} \int h \, d\frac{\mu_n}{|\mu_n|} = 0. \]
In particular, $\mu[\hat{H}] = 0$, and so by Lemma 1.2 we get $\mu[P] = 0$. Since $\mu$ is a probability measure, we deduce that $\int v \, d\mu > 0$. However,

$$\lambda_n e^{\alpha_n} L_{p_{\alpha_n}}(\alpha_n) \geq \lambda_n \int_{\alpha_n} \rho \, dH(q)$$

$$\geq \lambda_n |\mu_n| (\min \rho) \left( \int v \, d\mu_n - \epsilon_n' \right) / |\mu_n|.$$

So, again using that $L$ is finite, we see that $\lambda_n |\mu_n|$ can not converge to infinity.

Therefore, by restricting to a subsequence if necessary, we may assume that $\lambda_n \mu_n$ converges weakly to a finite measure $\mu'$, which of course will be a multiple of $\mu$. A similar argument to that just given shows that $\int h \, d\mu' = 0$ and that $L \geq (\int \rho v \, d\mu')^2 / A(\rho)$.

Applying Lemmas 4.3, 4.4, and 4.6, we obtain a generalised transverse measure $\tilde{\mu}^\epsilon$ on $G$ such that

$$i(F, \beta) \leq \int \tilde{d}\tilde{\mu}^\epsilon,$$

for every closed curve $\beta$ avoiding singularities. Moreover,

$$L \geq \frac{(\rho\text{-length}(\tilde{\mu}^\epsilon))^2}{A(\rho)}.$$

We have made the dependence on $\epsilon$ explicit because we will now let $\epsilon$ approach zero. Since $\int \rho v \, d\mu' \, \epsilon$ is bounded above uniformly in $\epsilon$, so also is $\int _\alpha \tilde{d}\tilde{\mu}^\epsilon$ for all transverse arcs $\alpha$ avoiding singularities. We conclude that there is a sequence $\epsilon_n$ converging to zero and a generalised transverse measure $\tilde{\mu}$ such that $\tilde{\mu}^\epsilon$ converges to $\tilde{\mu}$ as $n$ tends to infinity, in the sense of weak convergence.

The contribution of a rectangle $R$ containing a saddle connection to the $\rho^\epsilon$-length of $\tilde{\mu}^\epsilon$ is $\epsilon^{-1} h_R \int _{\beta} \tilde{d}\tilde{\mu}^\epsilon$, where $\beta^\epsilon$ is an open transverse arc crossing the rectangle, and $h_R$ is the height of $R$ with respect to $H(q)$. Thus, $\int _{\beta} \tilde{d}\tilde{\mu}^\epsilon$, the mass of $\tilde{\mu}^\epsilon$ crossing this rectangle, converges to zero as $\epsilon$ tends to zero. From this and the properties of $\tilde{\mu}^\epsilon$, we deduce that $\tilde{\mu}$ has no atoms. This implies that the $\rho^\epsilon$-length of $\tilde{\mu}^\epsilon$ converges to the $\rho^0$-length of $\tilde{\mu}$.

From (6), we get that $i(F, \beta) \leq \int _{\beta} \tilde{d}\tilde{\mu}$, for every closed curve $\beta$. Hence, $i(F, \beta) \leq i(\tilde{\mu}, \beta)$, for all $\beta \in S$. We apply Lemma 4.8 to get that we may write $F = (G, \tilde{\mu}^\epsilon)$, where $\tilde{\mu}^\epsilon = \sum _j f_j^\epsilon \nu_j$ with non-negative coefficients $\{f_j^\epsilon\}$ satisfying $f_j^\epsilon \leq f_j$, for all $j \in J$. By Lemma 4.12

$$\rho^0\text{-length}(\tilde{\mu}) \geq \sum _j \theta_j f_j^\epsilon i(\nu_j, H(q)) - O(\delta).$$

By Lemma 4.11

$$A(\rho) \leq \sum _j \theta_j^2 g_j^\epsilon i(\nu_j, H(q)) + O(\delta).$$

Therefore,

$$L \geq \frac{(\sum _j \theta_j f_j^\epsilon i(\nu_j, H(q)))^2}{\sum _j \theta_j^2 g_j^\epsilon i(\nu_j, H(q))},$$

where we have used the fact that $\delta$ is arbitrary. Using the fact that the $\theta_j$ are also arbitrary, and applying Lemma 3.4, we get that $L \geq \sum _j f_j^2 a_j / g_j^2 = E_g^\epsilon [F]$. □
Lemma 4.15. For all measured foliations $F$ and quadratic differentials $q$,

$$E_q^2[F] = \sup_{F' \in \mathcal{MF}\setminus\{0\}} \frac{i(F, F')^2}{E_q^*[F]}.$$  \hfill (7)

Proof. Let $V(q) = \sum_j G_j$ be the decomposition of $V(q)$, the vertical foliation of $q$, into indecomposable components. Looking at the definition of $E_q^*[F']$, we see that the right-hand-side of (7) equals

$$\sup_{f \in \mathbb{R}_+ \setminus \{0\}} \frac{\sum_j i(F, G_j) f_j}{\sum_j f_j^2 a_j},$$

where $a_j := i(G_j, H(q))$ for each $j$. By Lemma 3.1, this supremum is equal to $\sum_j i(F, G_j)^2 / a_j$, as required. $\Box$

The following is the main result of this section.

Lemma 4.16. Let $R(q; \cdot)$ be the Teichmüller ray with initial quadratic differential $q$. Then,

$$\limsup_{t \to \infty} e^{-2t} \operatorname{Ext}_{R(q; t)}[F] \leq E_q^2([F]), \quad \text{for all } F \in \mathcal{MF}.$$  

Proof. Take a sequence of times $t_n$ such that

$$\limsup_{t \to \infty} e^{-2t} \operatorname{Ext}_{R(q; t)}[F] = \lim_{n \to \infty} e^{-2t_n} \operatorname{Ext}_{R(q; t_n)}[F]$$

By Lemma 2.1, for each $t \in \mathbb{R}_+$, there exists $[F'_t] \in \mathcal{PMF}$ such that

$$\operatorname{Ext}_{R(q; t)}[F] \operatorname{Ext}_{R(q; t)}[F'_t] = i(F, F'_t)^2.$$  

Let $[F'] \in \mathcal{PMF}$ be a limit point of $[F'_{t_n}]$, and choose representatives such that $F'_{t_n}$ converges to $F'$. Using Lemma 4.14 and the continuity of $i(F, \cdot)^2$, we get

$$\limsup_{n \to \infty} \frac{i(F, F'_{t_n})^2}{e^{2t_n} \operatorname{Ext}_{R(q; t_n)}[F'_{t_n}]} \leq i(F, F')^2 \frac{E_q^*[F']}{E_q^*[F]}.$$  

So,

$$\limsup_{t \to \infty} e^{-2t} \operatorname{Ext}_{R(q; t)}[F] = \lim_{n \to \infty} \frac{i(F, F'_{t_n})^2}{e^{2t_n} \operatorname{Ext}_{R(q; t_n)}[F'_{t_n}]} \leq i(F, F')^2 \frac{E_q^*[F']}{E_q^*[F]}.$$  

The result now follows on applying Lemma 4.15. $\Box$

We may now prove Theorem 1.1

Proof of Theorem 1.1. The result follows on combining Lemmas 3.2 and 4.16. $\Box$
5. The horofunction boundary

We recall the definition of the horofunction boundary of a metric space, which first appeared in [6]. See also [2] for more information.

Let \((X, d)\) a metric space. Choose a basepoint \(b \in X\), and to each point \(z \in X\) associate the function \(\phi_z : X \to \mathbb{R}\), with
\[
\phi_z(x) = d(x, z) - d(b, z)
\]
for \(x \in X\).

Assume that \((X, d)\) is proper, meaning that closed balls are compact, and geodesic, meaning that every pair of points is connected by a geodesic segment. Under these assumptions, the map \(\Phi : X \to C(X)\) given by \(\Phi(z) = \phi_z\) embeds \(X\) into the space of continuous functions on \(X\), which is endowed with the topology of uniform convergence on bounded subsets of \(X\). We identify \(X\) with its image under this embedding. The horofunction boundary of \(X\) is defined to be
\[
X(\infty) = \text{cl} \Phi(X) \setminus \Phi(X),
\]
and its members are called horofunctions. Under our assumptions on \((X, d)\), the space \(X \cup X(\infty)\) is a compactification of \(X\).

It is easy to verify that choosing a different base-point \(b\) just has the effect of altering each horofunction by an additive constant, and that the horofunction boundaries coming from different basepoints are homeomorphic.

A path \(\gamma : \mathbb{R}_+ \to X\) is called an almost-geodesic if, for each \(\epsilon > 0\),
\[
|d(\gamma(0), \gamma(s)) + d(\gamma(s), \gamma(t)) - t| < \epsilon, \quad \text{for } s \text{ and } t \text{ large enough, with } s \leq t.
\]
Rieffel [22] proved that every almost-geodesic converges to a limit in \(X(\infty)\). A horofunction is called a Busemann point if there exists an almost-geodesic converging to it. We denote by \(X_B(\infty)\) the set of all Busemann points in \(X(\infty)\).

For any two horofunctions \(\xi\) and \(\eta\), we define the detour cost by
\[
H(\xi, \eta) = \sup_{W \ni \xi} \inf_{x \in W} \left( d(b, x) + \eta(x) \right),
\]
where the supremum is taken over all neighbourhoods \(W\) of \(\xi\) in the compactification \(X \cup X(\infty)\). This concept originated in [1]. An equivalent definition is
\[
H(\xi, \eta) = \inf_{\gamma} \lim_{t \to \infty} \inf_{x \in \gamma} d(b, \gamma(t)) + \eta(\gamma(t)), \quad (8)
\]
where the infimum is taken over all paths \(\gamma : \mathbb{R}_+ \to X\) converging to \(\xi\).

One can show that a horofunction \(\xi\) is a Busemann point if and only if \(H(\xi, \xi) = 0\). The following result is useful for calculating the detour cost; see [26, Lemma 3.3] and [25, Lemma 5.2].

**Proposition 5.1.** Let \(x \in X\), and let \(\gamma\) be an almost-geodesic converging to a Busemann point \(\xi\). Then, for any horofunction \(\eta\),
\[
\lim_{t \to \infty} d(x, \gamma(t)) + \eta(\gamma(t)) = \xi(x) + H(\xi, \eta).
\]

By symmetrising the detour cost, the set of Busemann points can be equipped with a metric. For \(\xi\) and \(\eta\) in \(X_B(\infty)\), we define
\[
\delta(\xi, \eta) = H(\xi, \eta) + H(\eta, \xi) \quad (9)
\]
and call \(\delta\) the detour metric. This construction appears in [1, Remark 5.2]. The function \(\delta : X_B(\infty) \times X_B(\infty) \to [0, \infty]\) is a metric, which might take the value \(+\infty\). Note that we can partition \(X_B(\infty)\) into disjoint subsets such that \(\delta(\xi, \eta)\) is finite for
each pair of horofunctions $\xi$ and $\eta$ lying in the same subset. We call these subsets the parts of the horofunction boundary of $(X, d)$, and $\delta$ is a finite-valued metric on each one.

The detour metric $\delta$ is independent of the base-point. Isometries of $(X, d)$ extend to homeomorphisms on the horofunction compactification, and preserve the detour metric.

6. The horofunction boundary is the Gardiner–Masur boundary

We show in this section that the horofunction compactification of Teichmüller space with the Teichmüller metric is just the Gardiner–Masur compactification. This result has also appeared in the work of Liu and Su [14]. Our proof uses the bound from Section 3 but does not use any of the material from Section 4.

A compactification of a topological space $X$ is a pair $(f, \bar{X})$, where $\bar{X}$ is a compact topological space and $f : X \to \bar{X}$ is a homeomorphism onto its image, with $f(X)$ open and dense in $\bar{X}$. Let $Y$ be a Hausdorff space. If $g$ is a continuous function from $X$ to $Y$, then we say that a function $g$ from $\bar{X}$ to $Y$ is a continuous extension of $g$ to $\bar{X}$ if $g \circ f = g$. A compactification $(f_1, X_1)$ of $X$ is said to be finer than another one $(f_2, X_2)$ if there exists a continuous extension of $f_2$ to $X_1$. The two compactifications are said to be isomorphic if each is finer than the other.

Lemma 6.1. Let $(f_1, X_1)$ and $(f_2, X_2)$ be two compactifications of $X$ such that $f_2$ extends continuously to an injective map $g : X_1 \to X_2$. Then, the two compactifications are isomorphic.

Proof. Clearly, $X_1$ is finer than $X_2$.

We have

$$f_2(X) = g \circ f_1(X) \subset g(X_1).$$

(10)

The denseness of $f_2(X)$ in $X_2$ gives that $\text{cl } f_2(X) = X_2$. Also, since $X_1$ is compact and $g$ is continuous, $g(X_1)$ is compact, and hence closed in $X_2$. Therefore, taking the closure of (10), we get $X_2 \subset g(X_1)$. So, $g$ is surjective. As a continuous bijection from a compact space to a Hausdorff one, $g$ is a homeomorphism. Its inverse satisfies $g^{-1} \circ f_2 = f_1$, and so is a continuous extension of $f_1$ from $X_2$ to $X_1$. □

We will show that the Gardiner–Masur compactification and the horofunction compactification are isomorphic by showing that each is isomorphic to a third compactification.

For each $x$ in $T(S)$, define

$$K_x := \sup_{F \in \mathcal{P}} \frac{\text{Ext}_x(F)}{\text{Ext}_d(F)}.
$$

and $\mathcal{E}_x : \mathcal{MF} \to \mathbb{R}_+$:

$$\mathcal{E}_x(F) := \left(\frac{\text{Ext}_x(F)}{K_x}\right)^{1/2}.
$$

Define $E := \{\mathcal{E}_x | x \in T(S)\}$. Let $\bar{E} := \text{cl } E$ be its closure in the space of continuous functions on $\mathcal{MF}$ with the topology of uniform convergence on compact sets.

Consider the compactification $(\mathcal{E}, \bar{E})$, where $\mathcal{E} : x \mapsto \mathcal{E}_x$. 
Lemma 6.2. The Gardiner–Masur compactification is isomorphic to the compactification \((\mathcal{E}, \bar{E})\).

Proof. Define the map \(\Psi : \bar{E} \to \mathcal{P}(\mathbb{R}^S), f \mapsto [f|_S]\). Here, \(f|_S\) is the restriction of \(f\) to the set \(S\), and \([\cdot]\) denotes projective equivalence class. The map \(\Psi\) is clearly continuous when we take on \(\bar{E}\) the topology of uniform convergence on compact sets, and on \(\mathcal{P}(\mathbb{R}^S)\) the quotient topology of the product topology.

For any \(x \in \mathcal{T}(S)\), we have \(\Psi \circ \mathcal{E}(x) = [\mathcal{E}_x|_S] = [\text{Ext}_x(\cdot)]\), which is the vector in the Gardiner–Masur compactification associated to the point \(x\).

We conclude that \(\Psi\) is a continuous extension of the map \(x \mapsto [\text{Ext}_x(\cdot)]\).

Suppose that \(\Psi f = \Psi g\) for some \(f\) and \(g\) in \(\bar{E}\). This means that \(f|_S = \lambda g|_S\) for some \(\lambda > 0\). By continuity, \(f = \lambda g\) on all of \(\mathcal{MF}\). Taking the supremum over \(\mathcal{MF}\), we see that \(\lambda = 1\), and so \(f = g\). This proves that \(\Psi\) is injective.

We now apply Lemma 6.1. \(\square\)

For each \(f \in \bar{E}\), let \(\Psi f\) be the function from \(\mathcal{T}(S)\) to \(\mathbb{R}_+\) defined by

\[
\Psi f(x) := \frac{\log \sup_{F \in \mathcal{PF}} \frac{f(F)}{\text{Ext}_x(F)}}{f(F)},
\]

for all \(x \in \mathcal{T}(S)\).

Lemma 6.3. Let \(q\) be a quadratic differential with uniquely-ergodic vertical foliation \(V(q)\), and let \(f : \mathcal{PF} \to \mathbb{R}_+\) be a bounded function such that \(f(V(q)) > 0\). For each \(t \in \mathbb{R}_+\), let \(\frac{f(\cdot)}{\text{Ext}_{R(q;t)}(\cdot)}\) attain its maximum over \(\mathcal{PF}\) at \(F_t\). Then, \(F_t\) converges to \(V(q)\) as \(t\) tends to infinity.

Proof. Fix \(t \in \mathbb{R}_+\). We have \(e^{2t} \text{Ext}_{R(q;t)}(V(q)) = 1\). Also, by Lemma 3.2, \(\text{Ext}_{R(q;t)}(F_t) \geq e^{-2ti} i(F_t, V(q))^2\). Combining these, and using the maximising property we have assumed for \(F_t\), we get

\[
i(F_t, V(q))^2 \leq \frac{f(F_t)}{e^{4t} f(V(q))}.
\]

So, as \(t\) tends to infinity, \(i(F_t, V(q))\) converges to zero. Since \(V(q)\) is uniquely-ergodic and all the \(F_t\) are in \(\mathcal{PF}\), this implies that \(F_t\) converges to \(V(q)\). \(\square\)

Lemma 6.4. The horofunction compactification is isomorphic to the compactification \((\mathcal{E}, \bar{E})\).

Proof. The continuity of \(\Psi\) follows immediately from the compactness of \(\mathcal{PF}\) and the topology we are using on \(\bar{E}\).

For \(y \in \mathcal{T}(S)\),

\[
\Psi \mathcal{E}_y(x) = \log \sup_{F \in \mathcal{PF}} \frac{\text{Ext}_y(F)}{\text{Ext}_x(F)} - \log K_y = d(\cdot, y) - d(b, y).
\]

So, \(\Psi\) is a continuous extension to \(E\) of the map \(y \mapsto d(\cdot, y) - d(b, y)\).

It remains to show that \(\Psi\) is injective. Let \(f\) and \(g\) be distinct elements of \(\bar{E}\). Exchanging \(f\) and \(g\) if necessary, we have \(f(G) < g(G)\) for some uniquely ergodic \(G \in \mathcal{PF}\), since such foliations are dense in \(\mathcal{PF}\). Since \(f\) and \(g\) are continuous, we may choose a neighbourhood \(N\) of \(G\) in \(\mathcal{P}\) small enough that there are real numbers \(u\) and \(v\) such that

\[
f(F) \leq u < v \leq g(F), \quad \text{for all } F \in N.
\]
By Lemma 6.3 we can find a point \( p \) in \( T(S) \) such that the supremum over \( PMF \) of \( f(\cdot)/Ext_p(\cdot) \) is attained in the set \( N \). Putting all this together, we have

\[
\sup_{PMF} \frac{f(\cdot)}{Ext_p(\cdot)} = \sup_{N} \frac{f(\cdot)}{Ext_p(\cdot)} \leq \sup_{N} \frac{u}{Ext_p(\cdot)} < \sup_{N} \frac{v}{Ext_p(\cdot)} \leq \sup_{PMF} \frac{g(\cdot)}{Ext_p(\cdot)}.
\]

Thus, \( \Psi f(p) < \Psi g(p) \), which implies that \( \Psi f \) and \( \Psi g \) differ. We have proved that \( \Psi \) is injective.

The result now follows on applying Lemma 6.4. \( \square \)

**Theorem 1.3.** The Gardiner–Masur compactification and the horofunction compactification of \( T(S) \) are isomorphic.

**Proof.** This follows from Lemmas 6.2 and 6.4. \( \square \)

7. The set of Busemann points

For distinct points \( x \) and \( y \) in \( T(S) \), let \( Q(x, y) \) be the unit-area quadratic differential at \( x \) such that \( R(q; \cdot) \) passes through \( y \).

**Theorem 7.1.** Let \( q \) be a quadratic differential. Then, the Teichmüller geodesic ray \( R(q; \cdot) \) converges in the Gardiner–Masur compactification to the point

\[
[\mathcal{E}_q(\cdot)] := \left[ \sum_{j} \frac{i(G_j, \cdot)^2}{i(G_j, H(q))} \right],
\]

where the \( \{G_j\} \) are the indecomposable components of \( V(q) \).

**Proof.** This is essentially a restatement of Theorem 1.1. \( \square \)

**Corollary 7.2.** Two geodesics with initial unit-area quadratic differentials \( q \) and \( q' \) converge to the same point of the Gardiner–Masur boundary if \( q \) and \( q' \) are modularly equivalent.

**Proof.** Since \( q \) and \( q' \) are modularly equivalent, \( V(q) = \sum_{j} \alpha_j G_j \) and \( V(q') = \sum_{j} \alpha'_j G_j \) for some set of mutually non-intersecting indecomposable measured foliations \( \{G_j\}_j \) and positive coefficients \( \{\alpha_j\}_j \) and \( \{\alpha'_j\}_j \) satisfying (1). So \( \mathcal{E}_q = \mathcal{E}_{q'} \). We now apply the theorem. \( \square \)

**Remark.** We will prove the converse to this corollary in Theorem 1.6.

**Definition 7.3.** Let \( T \) be a sub-interval of \( \mathbb{R}_+ \). A path \( \gamma : T \to X \) in a metric space \( (X, d) \) is an optimal path for a function \( f : X \to \mathbb{R} \) if it is geodesic and if \( f(\gamma(s)) = d(\gamma(s), \gamma(t)) + f(\gamma(t)) \) for all \( s, t \in T \) with \( s < t \).

**Lemma 7.4.** Let \( (X, d) \) be a metric space, and let \( \gamma : \mathbb{R} \to X \) be a geodesic line converging in the forward direction to \( \xi \) in the horofunction boundary of \( X \). Then, \( \gamma \) is an optimal path for \( \xi \).

**Proof.** For all \( s, t \in \mathbb{R} \) with \( s < t \),

\[
\xi(\gamma(s)) - \xi(\gamma(t)) = \lim_{u \to \infty} \left( d(\gamma(s), \gamma(u)) - d(\gamma(t), \gamma(u)) \right) = d(\gamma(s), \gamma(t)).
\]
Recall that a function $f : X \to \mathbb{R}$ is 1-Lipschitz if $f(x) - f(y) \leq d(x, y)$ for all points $x$ and $y$ in $X$. The following lemma shows that optimal paths for 1-Lipschitz functions may be “spliced” together.

**Lemma 7.5.** Let $T_1$ and $T_2$ be two sub-intervals of $\mathbb{R}$ with non-empty intersection. Let $\gamma_1 : T \to X$ and $\gamma_2 : T \to X$ be optimal paths for a 1-Lipschitz function $f : X \to \mathbb{R}$, such that $\gamma_1$ and $\gamma_2$ agree on $T_1 \cap T_2$. Then, the path defined, for $t \in T_1 \cup T_2$, by

$$
\gamma(t) := \begin{cases} 
\gamma_1(t), & \text{if } t \in T_1, \\
\gamma_2(t), & \text{if } t \in T_2,
\end{cases}
$$

is an optimal path for $f$.

**Proof.** Swap the indices if necessary so that $T_1 \setminus T_2 \subset (-\infty, t)$ and $T_2 \setminus T_1 \subset (t, \infty)$, for some $t \in T_1 \cap T_2$. Let $t_1, t_2 \in T_1 \cup T_2$ be such that $t_1 \leq t \leq t_2$. Since $\gamma_1$ and $\gamma_2$ are optimal paths for $f$,

$$
d(\gamma(t_1), \gamma(t)) = t - t_1 = f(\gamma(t_1)) - f(\gamma(t)) \quad \text{and} \quad d(\gamma(t), \gamma(t_2)) = t_2 - t = f(\gamma(t)) - f(\gamma(t_2)).
$$

Adding these equations and using the 1-Lipschitzness of $f$ gives

$$
d(\gamma(t_1), \gamma(t)) + d(\gamma(t), \gamma(t_2)) = t_2 - t_1 = f(\gamma(t_1)) - f(\gamma(t_2)) \leq d(\gamma(t_1), \gamma(t_2)).
$$

Applying the triangle inequality, we get that these inequalities are actually equalities. The same equalities hold trivially when $t_1$ and $t_2$ are both less than or both greater than $t$, since in this case they are both in $T_1$ or both in $T_2$, respectively. □

For any $x \in \mathcal{T}(S)$, and $F \in \mathcal{MF}\{0\}$, define $\tau_x(F)$ to be the unique $G \in \mathcal{MF}\{0\}$ such that $F$ and $G$ are the vertical and horizontal foliations of a quadratic differential based at $x$. In other words, $F$ and $\tau_x(F)$ together define a singular flat metric on $S$ that is in the conformal class of metrics $x$. By the Hubbard–Masur theorem, $\tau_x(F)$ is jointly continuous in $x$ and $F$.

**Theorem 1.7.** Every modular equivalence class of quadratic differentials has a representative at each point of Teichmüller space. This representative is unique up to multiplication by a positive constant.

**Proof.** First we prove uniqueness. Let $q$ and $q'$ be two unit-area quadratic differentials at the same point $x$ of $\mathcal{T}(S)$, and suppose that $q$ and $q'$ are modularly equivalent. Consider the geodesics $\gamma : \mathbb{R} \to \mathcal{T}(S)$ and $\gamma' : \mathbb{R} \to \mathcal{T}(S)$ passing through $x$ at time zero and having directions $q$ and $q'$, respectively. By Corollary [7.2] these geodesics both converge in the forward direction to the same Busemann point $\xi$.

By Lemma 7.4 both $\gamma$ and $\gamma'$ are optimal paths for the horofunction $\xi$. So, by Lemma 7.5 the path

$$
\overline{\gamma}(t) := \begin{cases} 
\gamma'(t), & t < 0, \\
\gamma(t), & t \geq 0,
\end{cases}
$$

is also optimal for $\xi$. In particular, $\overline{\gamma}$ is a geodesic. However, Teichmüller geodesics are uniquely extendable [11]. We conclude that $\gamma$ and $\gamma'$ are identical, from which it follows that $V(q)$ and $V(q')$ are identical. This further implies that $q = q'$.

The following proof of existence uses the uniqueness and is similar to the proof in the special case of Jenkins–Strebel foliations; see for example [7] Theorem 3].
We use induction on the number $J$ of indecomposable components comprising each member of the given modular equivalence class.

When $J = 1$, there exists by the Hubbard–Masur theorem a quadratic differential at $x$ whose vertical foliation is proportional to the single component of the modular equivalence class. In this case (11) is trivially satisfied.

Assume the result is true when the number of indecomposable components is less than $J$. Suppose we are given a modular equivalence class whose members have $J$ indecomposable components proportional to $\{G_j\}_{1 \leq j \leq J}$. For each $(\lambda_j)_j$ in $(0, \infty)^J$, define the measured foliation class $V_\lambda := \sum_{j=1}^J \lambda_j G_j$. Consider the map $M$ from $(0, \infty)^J$ to itself given by
\[
(\lambda_j)_j \mapsto \left(\frac{\lambda_j}{i(G_j, \tau_x(V_\lambda))}\right)_j.
\]

By the theorem of Hubbard–Masur, $\tau_x$ is a continuous function. Also, for any $j$, since $i(G_j, V_\lambda) = 0$, we have $i(G_j, \tau_x(V_\lambda)) > 0$. It follows that $M$ can be extended continuously to $\mathbb{R}_+^J \setminus \{0\}$.

Observe that $M$ satisfies $M(\alpha \lambda) = M(\lambda)$, for all $\alpha > 0$ and vectors $\lambda = (\lambda_j)_j$. So $M$ induces a continuous self map $\tilde{M}$ of the projective space $P(\mathbb{R}_+^J \setminus \{0\})$. The uniqueness proved above is precisely that this map is injective. The space $P(\mathbb{R}_+^J \setminus \{0\})$ has the structure of a closed simplex, and $\tilde{M}$ leaves each open face invariant. By the induction hypothesis, $\tilde{M}$ is a surjection on each open face. We conclude that $\tilde{M}$ is a homeomorphism on the boundary of $P(\mathbb{R}_+^J \setminus \{0\})$. But $P(\mathbb{R}_+^J \setminus \{0\})$ has the topology of a closed disk, and every injective map of a closed disk that is a homeomorphism on the boundary is a homeomorphism. Therefore, $\tilde{M}$ is surjective.

For each $x \in T(S)$ and $G \in \mathcal{MF} \setminus \{0\}$, define $q(x, G)$ to be the quadratic differential at $x$ with vertical foliation $G$.

**Lemma 7.6.** Let $\{G_j\}_j$ be a set of mutually non-intersecting indecomposable measured foliations, and define the set of measured foliations
\[
\Delta := \left\{ \sum_{j} \lambda_j G_j \mid \lambda_j \geq 0 \text{ for all } j \right\} \setminus \{0\}.
\]

Then, the set $\{[E_{q(x, G)}] \mid x \in T(S) \text{ and } G \in \Delta\}$ is a closed subset of the Gardiner–Masur boundary.

**Proof.** Combining Theorem 1.7 and Corollary 7.2, we see that
\[
\{[E_{q(x, G)}] \mid x \in T(S) \text{ and } G \in \Delta\} = \{[E_{q(b, G)}] \mid G \in \Delta\} := D.
\]

It follows easily from the arguments in the second part of the proof of Lemma 1.7 that the map from $P(\mathbb{R}_+^J \setminus \{0\})$ to the Gardiner–Masur boundary given by
\[
(\lambda_j)_j \mapsto E_{q(b, V_\lambda)}(\cdot) = \left(\sum_{j} \frac{\lambda_j i(G_j, \cdot)^2}{i(G_j, \tau_b(V_\lambda))}\right)^{1/2}
\]
is continuous. Since the domain is compact, the image is compact.

A min-plus measure is a lower semicontinuous function from some set to $\mathbb{R} \cup \{\infty\}$.

**Theorem 7.7.** A horofunction is a Busemann point if and only if it can be expressed $\Psi f$ for some function $f$ in the set $\{E_q \mid q \text{ is a quadratic differential}\}$.
Theorem 1.2. There is a min-plus measure that is greater than or equal to both \( \nu \) and \( \rho \). By Theorem 7.7, this modular equivalence class has a representative is in the set \( \mathcal{B} \) to \( \xi \). By Theorem 7.1, the geodesic ray \( \gamma \) converges to the Busemann point \( \xi \) (13)

\[
\eta(x) = \xi(x) + H(\xi, \eta),
\]

since \( \xi \) is the Busemann point to which \( \xi \) converges. This is true true for all \( x \in \mathcal{T}(S) \).

We now allow \( x \) to vary. By [25, Lemma 5.1], \( \eta(\cdot) \leq \xi(\cdot) + H(\xi, \eta) \) for each horofunction \( \xi \). So,

\[
\eta(\cdot) := \inf_{x \in \mathcal{T}(S)} \left( \xi(\cdot) + H(\xi, \eta) \right).
\]

It follows from [14, Proposition 5.1] that \( i(V(q^x), V(q^y)) = 0 \) for all \( x \) and \( y \) in \( \mathcal{T}(S) \). Therefore, there exists a finite set \( \{ G_j \} \) of mutually non-intersecting indecomposable measured foliations such that, for all \( x \in \mathcal{T}(S) \), the foliation \( V(q^x) \) is in the set

\[
\Delta := \left\{ \sum_j \lambda_j G_j \mid \lambda_j \geq 0 \text{ for all } j \right\}.
\]

By Lemma 2.6, the set \( D := \{ \xi_{q(x,G)} \mid x \in \mathcal{T}(S) \text{ and } G \in \Delta \} \) is a closed subset of the horofunction boundary. Obviously, \( \xi \) is in \( D \), for each \( x \in \mathcal{T}(S) \).

From [14], we may write

\[
\eta(\cdot) := \inf_{\xi \in B} \left( \xi(\cdot) + \nu(\xi) \right),
\]

where \( B \) is the set of Busemann points and \( \nu \) is a min-plus measure on \( B \) taking the value \( \infty \) outside \( D \). Since \( \eta \) is a Busemann point it may be written \( \eta = \inf_{\xi \in B} (\xi + \nu'(\xi)) \) where \( \nu' \) takes the value \( 0 \) at \( \eta \), and the value \( \infty \) everywhere else. By [26, Theorem 1.2], there is a min-plus measure \( \rho \) on \( B \) satisfying \( \eta = \inf_{\xi \in B} (\xi + \rho(\xi)) \) that is greater than or equal to both \( \nu \) and \( \nu' \). Since \( \eta \) is not identically \( \infty \), neither is \( \rho \), and therefore \( \eta \) must be in \( D \). We have thus proved that \( \eta \) is of the required form.

Theorem 1.8. Let \( p \) be a point of \( \mathcal{T}(S) \) and \( \xi \) be a Busemann point of the horoboundary. Then, there exists a unique geodesic ray starting at \( p \) and converging to \( \xi \).

Proof. By Theorem 1.4, \( \xi = \mathcal{E}_q \) for some modular equivalence class \([q]\) of quadratic differentials. By Theorem 1.4, this modular equivalence class has a representative
q at p. By Theorem 7.1, the geodesic \( R(q; \cdot) \) converges to \( \xi \). This geodesic starts at \( p \).

Suppose that \( \gamma \) and \( \gamma' \) are two geodesics starting at \( p \) and converging to \( \xi \). Using the same reasoning as in the uniqueness part of the proof of Theorem 1.7, one can show that \( \gamma \) and \( \gamma' \) are identical. □

**Theorem 1.6.** Two Busemann points \( E_q \) and \( E_{q'} \) are identical if and only if \( q \) and \( q' \) are modularly equivalent.

**Proof.** It was proved in Corollary 7.2 that \( E_q \) and \( E_{q'} \) are identical when \( q \) and \( q' \) are modularly equivalent. Let \( q \) and \( q' \) be quadratic differentials based at points \( x \) and \( y \), respectively, that are not modularly equivalent. By Theorem 1.7, we can find a quadratic differential \( \tilde{q} \) at \( x \) that is modularly equivalent to \( q' \), and hence different from \( q \). So, \( q \) and \( q' \) define different geodesics emanating from \( p \), and, by Theorem 1.8, the two geodesics have different limits. We conclude that \( E_q \neq E_{\tilde{q}} = E_{q'} \). □

**Lemma 7.8.** Let \( q \) be a quadratic differential. If \( V(q) = \sum_j G_j \) is written as a sum of indecomposable measured foliations, possibly scalar multiples of one another, then

\[
E_2^q(F) = \sum_j \frac{i(G_j, F)^2}{i(G_j, H(q))}, \quad \text{for all } F \in MF. 
\]

**Proof.** Let \( F' \) be some indecomposable component of \( F \), and let \( J' \) be the set of indices \( j \) for which \( G_j = \lambda_j F' \) for some \( \lambda_j > 0 \). Clearly, \( \sum_{j \in J'} \lambda_j = 1 \). So,

\[
\sum_{j \in J'} \frac{i(G_j, F)^2}{i(G_j, H(q))} = \sum_{j \in J'} \lambda_j \frac{i(F', F)^2}{i(F', H(q))} = \frac{i(F', F)^2}{i(F', H(q))}.
\]

Since this is true for every indecomposable component \( F' \) of \( F \), the result follows. □

**Lemma 7.9.** Let \( G = \sum_j G_j \) be written as a sum of measured foliations, possibly scalar multiples of one another, and let \( H \in MF \) be such that \( i(H, G_j) > 0 \) for all \( j \). Then

\[
\frac{i(G, F)^2}{i(G, H)} \leq \sum_j \frac{i(G_j, F)^2}{i(G_j, H)}, \quad \text{for all } F \in MF.
\]

If the \( G_j \) are not all scalar multiples of the same measured foliation, then the inequality is strict for some \( F \in MF \).

**Proof.** Observe first that, for all \( g_1, g_2 \in [0, \infty) \) and \( h_1, h_2 \in (0, \infty) \),

\[
\frac{(g_1 + g_2)^2}{h_1 + h_2} \leq \frac{g_1^2}{h_1} + \frac{g_2^2}{h_2},
\]

and that equality occurs precisely when \( g_1/h_1 = g_2/h_2 \).

We use induction on \( J \). The lemma is trivially true when \( J = 1 \).

Assume that it is true when there are \( J - 1 \) terms in the sum. Write \( G = G' + G_J \), where \( G' := \sum_{j=1}^{J-1} G_j \) is the sum of the first \( J - 1 \) terms. Using the inequality above
and then the induction hypothesis, we get, for all $F \in \mathcal{MF}$,
\[
\frac{i(G', F)^2}{i(G', H)} \leq \frac{i(G_j', F)^2}{i(G_j', H)} + \sum_{j=1}^{J-1} \frac{i(G_j, F)^2}{i(G_j, H)} + \frac{i(G_j, F)^2}{i(G_j, H)}.
\]

Thus the inequality holds when there are $J$ terms.

Equality for $F \in \mathcal{MF}$ is equivalent to
\[
\frac{i(G', F)}{i(G', H)} = \frac{i(G_j, F)}{i(G_j, H)}.
\]

If this is true for all $F$, then $G_j$ is projectively equivalent to $G'$, and hence to $G$. Since the ordering of the sum is arbitrary, the same applies to each term. □

**Lemma 7.10.** Let $G_n$ be a sequence in $\mathcal{MF}$ converging to a non-zero element $G$ of $\mathcal{MF}$, and let $H_n$ be a sequence in $\mathcal{MF}$ such that $H_n$ is proportional to an indecomposable component of $G_n$ for all $n$, and $H_n$ converges to $0$ as $n$ tends to infinity. Then, for all $x \in \mathcal{T}(S)$ and $F \in \mathcal{MF}$,
\[
\lim_{n \to \infty} \frac{i(H_n, F)^2}{i(H_n, \tau_x(G_n))} = 0.
\]

**Proof.** Let $\lambda_n$ be a sequence of positive real numbers such that the quadratic differential $q(x, \lambda_n H_n)$ has unit area, for all $n$. Since the set of unit-area quadratic differentials at $x$ is compact, by taking a subsequence if necessary, we may assume that $\lambda_n H_n$ converges to an element $H$ of $\mathcal{MF}\{0\}$. For any $F \in \mathcal{MF}$, we have that $i(H_n, F)$ converges to $0$, and $i(\lambda_n H_n, F)$ converges to $i(H, F)$. Also, $i(\lambda_n H_n, \tau_x(G_n))$ converges to $i(H, \tau_x(G))$. Since $i(\lambda_n H_n, G_n) = 0$ for all $n$, we have $i(H, G) = 0$, which implies that $i(H, \tau_x(G)) > 0$. Therefore,
\[
\lim_{n \to \infty} \frac{i(H_n, F)^2}{i(H_n, \tau_x(G_n))} = \lim_{n \to \infty} \frac{i(H_n, F)i(\lambda_n H_n, F)}{i(\lambda_n H_n, \tau_x(G_n))} = 0.
\]

**Lemma 7.11.** Let $\gamma : \mathbb{R}_+ \to \mathcal{T}(S)$ be a geodesic ray starting from a point $\gamma(0) = p$ and converging to a Busemann point $\xi$. For any $r \geq 0$, the point $\gamma(r)$ is the unique point $x$ satisfying $d(p, x) = \xi(p) - \xi(x) = r$.

**Proof.** By Lemma 7.4, $\gamma(r)$ satisfies this condition.

Let $x$ be any point of $\mathcal{T}(S)$ satisfying the condition. By Lemma 7.8 there exists a geodesic ray $\gamma' : \mathbb{R}_+ \to \mathcal{T}(S)$ starting at $x$ and converging to $\xi$, and, by Lemma 7.3, this ray is an optimal path for $\xi$.

Let $\gamma'' : [0, r] \to \mathcal{T}(S)$ be the geodesic segment connecting $p$ and $x$. Since $\xi$ is $1$-Lipschitz,
\[
\xi(\gamma''(t)) - \xi(x) \leq r - t \quad \text{and} \quad \xi(p) - \xi(\gamma''(t)) \leq t,
\]
for all $t \in [0, r]$. Combining this with the assumption on $x$, we get $\xi(p) - \xi(\gamma''(t)) = t$, for all $t \in [0, r]$. It follows that $\gamma''$ is an optimal path for $\xi$. Applying Lemma 7.3, we see that the path
\[
\gamma'''(t) := \begin{cases} 
\gamma''(t), & \text{if } t \in [0, r], \\
\gamma'(t), & \text{if } t \geq r,
\end{cases}
\]
is an optimal path for $\xi$, and hence a geodesic. But, by Lemma 1.8 there is only one geodesic starting at $p$ and converging to $\xi$. Therefore, $\gamma''$ is identical to $\gamma$, and $\gamma(r) = x$.

**Theorem 7.12.** Let $q_n$ be a sequence of unit-area quadratic differentials based at $b \in \mathcal{T}(S)$. Then, $\mathcal{E}_{q_n}$ converges to a Busemann point $\mathcal{E}_q$ if and only if both the following hold:

(i) $q_n$ converges to $q$;

(ii) for every sequence $(G^n)_n$ of indecomposable elements of $\mathcal{MF}$ such that, for each $n \in \mathbb{N}$, $G^n$ is a component of $V(q_n)$, we have that every limit point of $G^n$ is indecomposable.

**Proof.** Assume conditions (i) and (ii) hold. We wish to show that $\mathcal{E}_{q_n}$ converges to $\mathcal{E}_q$ in the Gardiner–Masur compactification. So, consider any limit point of this sequence. By taking a subsequence if necessary, we can assume that $\mathcal{E}_{q_n}$ actually converges to this point.

For each $n \in \mathbb{N}$, we can write $V(q_n) = \sum_{j=1}^{J} G^n_j$, with an upper bound on $J$ depending on the topology of the surface. By taking a subsequence if necessary, we can ensure that $G^n_j$ converges to some $G_j$ in $\mathcal{MF}$ for each $j$. By hypothesis, $V(q) = \sum_j G_j$, and each $G_j$ is indecomposable. Note that this is not necessarily a decomposition of $V(q)$ into indecomposable components since some of the $G_j$ may be scalar multiples of each other.

The convergence of $q_n$ implies that $H(q_n)$ converges to $H(q)$. We deduce that $a^n_j := i(G^n_j, H(q_n))$ converges to $a_j := i(G_j, H(q))$, for each $j$.

Let $F \in \mathcal{MF}$. For each $j$ such that $a_j$ is zero, we have that $G^n_j$ converges to zero, and hence, by Lemma 7.10 that $i(G^n_j, F)^2/a^n_j$ converges to zero. For all other $j$, we have that $i(G^n_j, F)^2/a^n_j$ converges to $i(G_j, F)^2/a_j$. It follows that $\mathcal{E}_{q_n}(F)$ converges to $\mathcal{E}_q(F)$, by Lemma 7.3.

Now assume that $\mathcal{E}_{q_n}$ converges to $\mathcal{E}_q$. So, the associated horofunctions $\xi_n := \Psi E_{q_n}$ converge uniformly on compact sets to $\xi := \Psi E_q$. For each $n \in \mathbb{N}$, let $z_n := R(q_n; 1)$. Observe that $d(b, z_n) = 1$ and $\xi_n(z_n) = -1$ for all $n$. So, for any limit point $z$ of the sequence $(z_n)_n$, we have $d(b, z) = 1$ and $\xi(z) = -1$. But, by Lemma 7.11 $R(q; 1)$ is the only point of Teichmüller space with these properties. We conclude that $z_n$ converges to $R(q; 1)$. It follows that $q_n$ converges to $q$, and hence that (i) holds.

Let $G^n$ be a sequence as in (ii). We may, for each $n \in \mathbb{N}$, write $V(q_n) = \sum_{j=0}^{J} G^n_j$, where $J$ is independent of $n$, each $G^n_j$ is either zero or an indecomposable component of $V(q)$, and $G^n = G^n_0$.

We wish to show every limit point of $(G^n_j)_n$ is indecomposable. By taking a subsequence if necessary, we may assume that, for each $j$, the subsequence $(G^n_j)_n$ converges to some element $G_j$ of $\mathcal{MF}$. Since $q_n$ converges to $q$, we have $V(q) = \sum_{j=0}^{J} G_j$. Write $a_j := i(G_j, H(q))$, for each $j$. As before, for any $F \in \mathcal{MF}$,

$$\lim_{n \to \infty} \mathcal{E}_{q_n}^2(F) = \sum_j \frac{i(G_j, F)^2}{a_j},$$

where the sum is over all $j \in \{0, \ldots, J\}$ such that $G_j$ is not zero.

For each $j$, we can write $G_j = \sum_{l=0}^{L_j} G^l_j$ as a sum of projectively-distinct indecomposable measured foliations, where $L_j$ is bounded depending on the topology.
of the surface. Even though the \( \{G^i_j\}_{j,t} \) are not necessarily projectively distinct, we have, by Lemma 7.8 that

\[
\mathcal{E}_q^2(F) = \sum_{j=0}^J \sum_{t=0}^{L_j} \frac{i(G^i_j, F)^2}{i(G^i_j, H(q))}.
\]

By Lemma 7.9 for each \( j \),

\[
\frac{i(G^i_j, F)^2}{i(G^i_j, H(q))} \leq \sum_{t=0}^{L_j} \frac{i(G^i_j, F)^2}{i(G^i_j, H(q))}.
\]

Since \( \mathcal{E}_q \) converges to \( \mathcal{E}_q \), equality holds in (14) for all \( F \in \mathcal{M}F \), and for all \( j \).

Therefore, according to Lemma 7.10, for each \( j \), the \( \{G^i_j\}_{j,t} \) are all projectively equivalent to \( G_j \), that is, \( G_j \) is indecomposable.

**Theorem 1.9.** Each Teichmüller ray \( R(q; \cdot) \) is convergent to the ray \( R(q'; \cdot) \), where \( q' \) is the unique unit-area quadratic differential at the basepoint that is modularly equivalent to \( q \).

**Proof.** The existence and uniqueness of \( q' \) was proved in Theorem 1.7. By Theorem 1.6 \( \mathcal{E}_q = \mathcal{E}_{q'} \).

By Theorem 7.1 \( R(q; \cdot) \) converges in the Gardiner–Masur compactification to \( \mathcal{E}_q \). But this compactification is the same as the horocompactification by Theorem 1.3 and so \( \Psi_\mathcal{E}_R(q; \cdot) = d(b, R(q; t)) \) converges uniformly on compact sets to \( \Psi_\mathcal{E}_q = \Psi_\mathcal{E}_{q'} \), as \( t \) tends to infinity. Choose \( s \in \mathbb{R}_+ \). For each \( t \), let \( z(t) := R(q(t); s) \), where \( q(t) := Q(b, R(q; t)) \) is the initial quadratic differential of the Teichmüller geodesic segment from \( b \) to \( R(q; t) \). We have \( d(b, z(t)) = s \) and \( \Psi_\mathcal{E}_q(z) = s \), and so, by Lemma 7.11 \( z = R(q'; s) \). We deduce that \( z(t) \) converges to \( R(q'; s) \) as \( t \) tends to infinity. The conclusion now follows, since \( s \) was chosen arbitrarily.

8. The detour metric on the boundary

In this section we calculate the detour cost and detour metric of the Teichmüller metric. The technique will be similar to that used in [25] to calculate the same quantities for Thurston’s Lipschitz metric.

Let \( G' \in \mathcal{M}F \) be expressed as \( G' = \sum_j G_j \) in terms of its indecomposable elements. For \( G \in \mathcal{M}F \), we write \( G \ll G' \) if \( G \) can be expressed as \( G = \sum_j \lambda_j G_j \), where each coefficient \( \lambda_j \) is a non-negative number.

**Lemma 8.1.** Let \( F_j; j \in \{0, \ldots, J\} \) be a finite set of mutually non-intersecting indecomposable non-zero measured foliations such that no two are projectively equivalent, and let \( C > 0 \). Then, there exists a curve class \( \alpha \in \mathcal{S} \) such that \( i(F_0, \alpha) > Ci(F_j, \alpha) \) for all \( j \in J \setminus \{0\} \).

**Proof.** This is a restatement of [25] Lemma 6.3.

**Lemma 8.2.** Let \( q \) and \( q' \) be quadratic differentials at \( b \). If \( V(q) \ll V(q') \), then

\[
\sup \left\{ \frac{\mathcal{E}_q^2(F)}{\mathcal{E}_q^2(F)} \mid F \in \mathcal{P}\mathcal{M}F \right\} = \max_j \frac{\lambda_j i(G_j, H(q'))}{i(G_j, H(q))},
\]

where \( V(q) \) is expressed as \( V(q) = \sum_j \lambda_j G_j \) in terms of the indecomposable components \( G_j \) of \( V(q') \). If \( V(q) \not\ll V(q') \), then the supremum is \( +\infty \).
Theorem 8.3. and hence \\
\{V_j\} negative coefficients such that, for all \((V_j)\), the set where the ratio is well defined, that is, excluding values of \(\gamma\). Here, and in similar situations, we interpret the supremum to be over \(32\) CORMAC WALSH \(\sup\). Therefore, by Proposition 5.1, relabel the indices so that the \(j\) for which \(g_j'\) is the largest is \(j = 0\). So, \(g_jg_0/t_j/0 \leq g_0g_j'/\omega_j',\) for all \(j\). Therefore, for all \(F \in \mathcal{M}F\),  \\
\[\frac{g_0'}{t_0} \sum_j g_j(F, G_j)^2/t_j \leq \frac{g_0}{t_0} \sum_j g_j'(F, G_j)^2/t_j',\]
and hence  \\
\[E(F) := \frac{\mathcal{E}_q^2(F)}{\mathcal{E}_q^2(F)} = \frac{\sum_j g_j(F, G_j)^2/t_j}{\sum_j g_j'(F, G_j)^2/t_j'} \leq \frac{g_0'}{t_0} g_0,\]
For any \(C > 0\), we may apply Lemma 8.1 to get a measured foliation \(F_0 \in \mathcal{M}F\) such that \(\mathcal{E}_q(F_0) > C\mathcal{E}_q(F, G_j)\) for all \(j \in \{1, \ldots, J\}\). By choosing \(C\) large enough, we can make \(E(F_0)\) as close as we like to \(g_0g_0'/\omega_0'/0\).

We conclude that \(\sup_F E(F) = g_0g_0'/\omega_0'/0\). The result follows. \(\square\)

Theorem 8.3. Let \(q\) and \(q'\) be unit area quadratic differentials at \(b\). If \(V(q) \ll V(q')\), then  \\
\[H(\mathcal{E}_q', \mathcal{E}_q) = \frac{1}{2} \log \sup_{F \in \mathcal{PM}F} \mathcal{E}_q^2(F) + \frac{1}{2} \log \max_j \left(\frac{\lambda_j^2 i(G_j, H(q'))}{i(G_j, H(q'))}\right) - \frac{1}{2} \log \sup_{F \in \mathcal{PM}F} \frac{\mathcal{E}_q^2(F)}{\mathcal{E}_q^2(F)} = \frac{1}{2} \log \mathcal{E}_q^2(F) + \frac{1}{2} \log \mathcal{E}_q^2(F),\]
where \(V(q)\) is expressed as \(V(q) = \sum_j \lambda_j G_j\) in terms of the indecomposable components \(G_j\) of \(V(q')\). If \(V(q) \not\ll V(q')\), then \(H(\mathcal{E}_q', \mathcal{E}_q) = +\infty\).

Proof. Let \(\gamma := R(q', \cdot)\) be the geodesic starting at \(b \in T(S)\) and having initial quadratic differential \(q'). By Theorem 7.1, \(\gamma\) converges to the Busemann point \(\mathcal{E}_q'.\)

Therefore, by Proposition 6.1,  \\
\[H(\mathcal{E}_q', \mathcal{E}_q) = \lim_{t \to \infty} \left(\frac{d(b, \gamma(t)) + \Psi \mathcal{E}_q'(\gamma(t))}{2}\right) + \frac{1}{2} \lim_{t \to \infty} \left(\log \sup_{F \in \mathcal{PM}F} \mathcal{E}_q^2(F) + \log \sup_{F \in \mathcal{PM}F} \mathcal{E}_q^2(F) - \frac{1}{2} \log \sup_{F \in \mathcal{PM}F} \mathcal{E}_q^2(F) + \frac{1}{2} \log \mathcal{E}_q^2(F)\right)\]
Combining Lemma 6.2 and Theorem 1.1, we get that \(e^{-2t} \mathcal{E}_q'(\gamma)\) converges uniformly on compact sets to \(\mathcal{E}_q'(\cdot)\). Therefore,  \\
\[\lim_{t \to \infty} \sup_{F \in \mathcal{PM}F} \mathcal{E}_q^2(F) = \sup_{F \in \mathcal{PM}F} \mathcal{E}_q^2(F)\]
From Lemma 3.2, we get  \\
\[\sup_{F \in \mathcal{PM}F} \mathcal{E}_q^2(F) \leq \sup_{F \in \mathcal{PM}F} \mathcal{E}_q^2(F)\] for all \(t\).
But the limit of a supremum is trivially greater than or equal to the supremum of the limits. We conclude that
\[ \lim_{\gamma \to \infty} \sup_{F \in \mathcal{PMF}} e^{2t} \mathcal{E}_q^2(F) = \sup_{F \in \mathcal{PMF}} \mathcal{E}_q^2(F). \]

The result now follows on applying Lemma 8.2. \( \square \)

**Corollary 8.4.** Let \( q \) and \( q' \) be unit area quadratic differentials at \( b \). If \( V(q) = \sum_j g_j G_j \) and \( V(q') = \sum_j g'_j G_j \), where \( \{G_j\} \) is a finite set of mutually non-intersecting indecomposable measured foliations, and the \( g_j \) and \( g'_j \) are positive coefficients, then the detour metric between \( E_q \) and \( E_{q'} \) is
\[
\delta(E_q, E_{q'}) = \frac{1}{2} \log \max_j \frac{g_j(G_j, H(q'))}{g'_j(G_j, H(q))} + \frac{1}{2} \log \max_j \frac{g'_j(G_j, H(q'))}{g_j(G_j, H(q))}.
\]

If \( V(q') \) and \( V(q) \) cannot be simultaneously written in this form, then \( \delta(E_q, E_{q'}) = +\infty \).

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