residue formula for an obstruction to coupled kähler-einstein metrics

akito futaki and yingying zhang

abstract. we obtain a residue formula for an obstruction to the existence of coupled kähler-einstein metrics described in [13]. we apply it to an example studied by the first author [6] and hultgren [15] which is a fano manifold with reductive automorphism, does not admit a kähler-einstein metric but still admits coupled kähler-einstein metrics.

1. introduction

a $k$-tuple of kähler metrics $\omega_1, \cdots, \omega_k$ on a compact kähler manifold $m$ is called coupled kähler metrics if it satisfies

$$\text{Ric}(\omega_1) = \cdots = \text{Ric}(\omega_k) = \lambda \sum_{\alpha=1}^k \omega_\alpha$$

for $\lambda = -1$, 0 or 1 where $\text{Ric}(\omega_\alpha)$ is the ricci form of $\omega_\alpha$ (we do not distinguish kähler metrics $g_\alpha$ and their kähler forms $\omega_\alpha$). such metrics were introduced by hultgren and witt nyström [16]. if $\lambda = 0$ this is just a $k$-tuple of ricci-flat metrics and the existence is well-known for compact kähler manifolds with $c_1(M) = 0$ by the celebrated solution by yau [23] of the calabi conjecture. for $\lambda = -1$ or $\lambda = 1$ the existence problem is an extension for the problem for negative or positive kähler-einstein metrics, and an obvious condition is $c_1(M) < 0$ or $c_1(M) > 0$. hultgren and witt nyström [16] proved the existence of the solution for $\lambda = -1$ under the condition $c_1(M) < 0$ extending [23] and [1], and there are many interesting results for $\lambda = 1$ under the condition $c_1(M) > 0$ including attempts to extend [3] and [22]. further studies of coupled kähler-einstein metrics have been done in [19], [15], [13], [5], [20], [4], [21], [18].

in this paper we derive a residue formula for an obstruction to the existence of positive coupled kähler-einstein metrics described in our previous paper [13] and apply to a computation of an example which appeared in hultgren [15].

the obstruction is described as follows. let $m$ be a fano manifold of complex dimension $m$. assume the anticanonical line bundle has a splitting $k_M^\perp = L_1 \otimes \cdots \otimes L_k$ into the tensor product of ample line bundles $L_\alpha \to m$. then we have $c_1(L_\alpha) = \frac{1}{2\pi}[\omega_\alpha]$ for a kähler form $\omega_\alpha = \sqrt{-1}g_\alpha \partial \bar{\partial}z^i \wedge d\bar{z}^j$, and thus

$$c_1(M) = \frac{1}{2\pi} \sum_{\alpha=1}^k [\omega_\alpha].$$

for each $\omega_\alpha$ we have $f_\alpha \in C^\infty(M)$ such that

$$\text{Ric}(\omega_\alpha) = \sum_{\beta=1}^k \omega_\beta + \sqrt{-1}\partial \bar{\partial} f_\alpha,$$
where \( f_\alpha \) are normalized by
\[
e^{f_1 \omega_1} = \cdots = e^{f_k \omega_k}.
\]
Note that this normalization still leaves an ambiguity up to a constant. But we ignore this ambiguity since it does not cause any problem in later arguments. Of course \( \omega_1, \cdots, \omega_k \) are coupled Kähler-Einstein metrics if and only if \( f_\alpha \) are all constant.

Let \( X \) be a holomorphic vector field. Since a Fano manifold is simply connected there exist complex-valued smooth functions defined up to constant \( u_\alpha \) such that
\[
i_X \omega_\alpha = \overline{\partial}(\sqrt{-1} u_\alpha).
\]
By the abuse of terminology we call \( u_\alpha \) the Hamiltonian function of \( X \) with respect to \( \omega_\alpha \) though \( u_\alpha \) is a Hamiltonian function for the imaginary part of \( X \) in the usual sense of symplectic geometry only when \( u_\alpha \) is real valued. In Theorem 3.3 of [13], it is shown for some choices of \( u_\alpha \) we have
\[
\Delta_\alpha u_\alpha + (\text{grad}_\alpha u_\alpha) f_\alpha = - \sum_{\beta=1}^k u_\beta.
\]
where \( \Delta_\alpha = -\overline{\partial}_\alpha \partial \) is the Laplacian with respect to \( \omega_\alpha \) and \( \text{grad}_\alpha u_\alpha \) is the type \((1,0)\)-part of the gradient of \( u_\alpha \) expressed as \( \text{grad}_\alpha u_\alpha = \frac{\sqrt{g_\alpha}}{\sqrt{-1}} \frac{\partial u_\alpha}{\partial \overline{z}^j} \frac{\partial}{\partial z^j} \) in terms of local holomorphic coordinates \((z^1, \ldots, z^m)\). The case of \( k = 1 \) of this result has been obtained in [8]. If we replace \( u_\alpha \) by \( u_\alpha^c = u_\alpha + c_\alpha \) the equations (4) are satisfied for \( u_\alpha^c \) if and only if
\[
\sum_{\alpha=1}^k c_\alpha = 0.
\]

**Definition 1.1** ([13]). With the choice of \( u_\alpha \) satisfying (4) the Lie algebra character is defined as
\[
\text{Fut} : \mathfrak{h}(M) \to \mathbb{C}
\]
(6)
\[
X \mapsto \text{Fut}(X) = \sum_{\alpha=1}^k \frac{\int_M u_\alpha \omega^m_\alpha}{\int_M \omega^m_\alpha}.
\]
Notice that this definition of Fut is not affected by the ambiguity of the choice of \( u_\alpha \) because of (5). Note also Fut is the coupled infinitesimal form of the group character obtained in [7].

To formulate the localization formula let \( Z = \bigcup_{\lambda \in \Lambda} Z_\lambda \) be zero sets of \( X \) where \( Z_\lambda \)'s are connected components. Let \( N_\alpha(Z_\lambda) = (T_M|Z_\lambda)/T_{Z_\lambda} \) be the normal bundle of \( Z_\lambda \) with respect to \( \omega_\lambda \). Then the Levi-Civita connection \( \nabla_\alpha \) of \( \omega_\alpha \) naturally induces an endomorphism \( L^{N_\alpha}(X) \) of \( N_\alpha(Z_\lambda) \) by
\[
L^{N_\alpha}(X)(Y) = (\nabla^\nabla_\alpha Y X)^{-1} \in N_\alpha(Z_\lambda), \quad \text{for any } Y \in N_\alpha(Z_\lambda).
\]
We also assume \( Z \) is nondegenerate in the sense that \( L^{N_\alpha} \) is nondegenerate. Let \( K_\alpha \) be the curvature of \( N_\alpha(Z_\lambda) \). The localization formula of Fut\((X)\) we obtain is the following.

**Theorem 1.2.** Let \( M \) be a Fano manifold with \( K_M^{-1} = L_1 \otimes \cdots \otimes L_k \). Let \( X \) be a holomorphic vector field with nondegenerate zero sets \( Z = \bigcup_{\lambda \in \Lambda} Z_\lambda \), then
\[
\text{Fut}(X) = \frac{1}{m+1} \sum_{\alpha=1}^k \left( \frac{\int_{Z_\lambda} \left( (E_\alpha + c_1(L_\alpha))|Z_\lambda \right)^{m+1}}{\det \left( (2\pi)^{-1}(L^{N_\alpha}(X) + \sqrt{-1}K_\alpha) \right)} \right),
\]
where \( E_\alpha \in \Gamma(\text{End}(L_\alpha)) \) is given by \( E_\alpha s = u_\alpha s \) with \( L^{N_\alpha} \) and \( K_\alpha \) being as above.
Corollary 1.3. If $Z$ contains only discrete points, then

\[
\text{Fut}(X) = \frac{1}{m+1} \sum_{\alpha=1}^{k} \left( \frac{\sum_{p \in Z} (u_\alpha(p))^{m+1} / \det(\nabla X)(p)}{\sum_{p \in Z} (u_\alpha(p))^{m} / \det(\nabla X)(p)} \right)
\]

\[
= \frac{1}{m+1} \left( \sum_{\alpha=1}^{k} \frac{\sum_{p \in Z} (u_\alpha(p))^{m+1}}{\sum_{p \in Z} (u_\alpha(p))^{m}} \right).
\]

We can apply the obtained localization formula for the invariant Fut in the coupled situation to verify the example considered in Hultgren’s paper [15]. This example was first considered by the first author in [6], where he showed that the invariant Fut is non-vanishing, hence there does not exist a Kähler-Einstein metric on this example though the automorphism group is reductive and thus Matsushima’s condition [17] is satisfied. Later, in [11], the localization formula in [12] was used to show a much simpler computation of the invariant Fut can be done. Hultgren [15] considered decompositions of the anticanonical line bundle, and proved in a special case of the decomposition there does exist coupled Kähler-Einstein metric on this manifold.

The rest of the paper proceeds as follows. In section 2 we prove Theorem 1.2. In section 3 we verify the existence result of Hultgren in [15] by checking the vanishing of Fut as an application of Theorem 1.2.

2. Localization Formula

We first consider an ample line bundle $L \to M$ with $c_1(L) = \frac{1}{\pi i} [\omega]$ where $[\omega]$ is a Kähler class of $M$. Let $e_U$ be a non-vanishing local holomorphic section of $L|_U$ where $U$ is an open set of $M$. Then $e_U$ determines a local trivialization of the line bundle $L|_U \cong U \times \mathbb{C}$, given by $ze_U \mapsto (p, z)$, where $z$ is the fiber coordinate. Let $h$ be the Hermitian metric of $L$, and $h_U = h(e_U, e_U)$. The local connection form is given by $\theta_U = \partial \log h_U$. Let

\[
\theta = \theta_U + \frac{dz}{z},
\]

then $\theta$ is a globally defined connection form on the associated principle $\mathbb{C}^*$-bundle. To see this, we first remark that $\frac{dz}{z}$ is the Maurer-Cartan form of $\mathbb{C}^*$. If $U \cap V \neq \emptyset$, and we take another trivialization on $L|_V \cong V \times \mathbb{C}$, given by $we_V \mapsto (p, w)$, where $e_V$ is a non-vanishing local holomorphic section and $w$ is the fiber coordinate. Let $f$ be the non-vanishing holomorphic function such that $e_V = f e_U$, then $h_V = |f|^2 h_U$ and $z = fw$. Then,

\[
\theta_V + \frac{dw}{w} = \partial \log |f|^2 h_U + \frac{f}{z} d\left( \frac{z}{f} \right) = \frac{df}{f} + \partial \log h_U + \frac{dz}{z} - \frac{df}{f} = \theta_U + \frac{dz}{z}.
\]

Hence $\theta = \theta_U + \frac{dz}{z}$ is independent of the trivialization. Obviously $\sqrt{-1} \partial \bar{\partial} \theta = \omega$. Let $u$ be a complex-valued smooth function such that

\[
i_X \omega = \bar{\partial}(\sqrt{-1} u).
\]

It is well-known (c.f. [10] for example) that a Hamiltonian vector field $X$ written in this way lifts $L$ uniquely up to $cz \frac{\partial}{\partial z}$ for a constant $c$. Let $\tilde{X}$ be a lift of $X$ to $L$. Then obviously $u_X := -\theta(\tilde{X})$ is a Hamiltonian function for $X$ and $-\theta(\tilde{X} - cz \frac{\partial}{\partial z}) = u_X + c$. Thus, the ambiguity of $c_\alpha$ for $L_\alpha$ above appears in this way. The connection form $\theta$ determines a horizontal lift $X^h$ of $X$, given by

\[
X^h = \tilde{X} - \theta(\tilde{X}) z \frac{\partial}{\partial z}.
\]

Apparently, this expression is independent of the lift $\tilde{X}$ and $\theta(X^h) = 0$. 

\[
= \frac{1}{m+1} \left( \sum_{\alpha=1}^{k} \frac{\sum_{p \in Z} (u_\alpha(p))^{m+1}}{\sum_{p \in Z} (u_\alpha(p))^{m}} \right).
\]
Now, for each ample line bundle $L_\alpha \to M$, $\alpha = 1, \ldots, k$, choose Hermitian metric $h_\alpha$, let $\theta_\alpha$ be corresponding connection form on the associated principal $\mathcal{C}^\ast$-bundle, and $\Theta_\alpha$ is the curvature form such that $\Theta_\alpha = \partial\bar{\partial}\log h_\alpha = -\sqrt{-1}\omega_\alpha$.

Hence, with a choice of a Hamiltonian function $u_\alpha$, the lifted holomorphic vector field $X_\alpha$ (omitting the tilde) of $X$ on $L_\alpha$ is

$$X_\alpha = X^h_\alpha - u_\alpha z \frac{\partial}{\partial z}$$

where $X^h_\alpha$ is the horizontal lift of $X$. Then of course

$$u_\alpha = -\theta_\alpha(X_\alpha).$$

The infinitesimal action on the space $\Gamma(L_\alpha)$ of holomorphic sections of $L_\alpha$ is given by

$$\Lambda_\alpha : \Gamma(L_\alpha) \to \Gamma(L_\alpha)
\begin{equation}
  s \mapsto \Lambda_\alpha(s) = \nabla^\alpha_X s + u_\alpha s
\end{equation}$$

where $\nabla^\alpha$ is the covariant derivative determined by $\theta_\alpha$. Then we can check that for $f \in C^\infty(M)$, $s \in \Gamma(L_\alpha)$,

1. $\Lambda_\alpha$ satisfies the Leibniz Rule.

$$\Lambda_\alpha(fs) = \nabla^\alpha_X(fs) + u_\alpha fs
= X(f)s + f\nabla^\alpha_Xs + f u_\alpha s
= X(f)s + f \Lambda_\alpha s.$$  

2. $\overline{\partial} \Lambda_\alpha = \Lambda_\alpha \overline{\partial}$. This follows from

$$\overline{\partial}\Lambda_\alpha s = \overline{\partial}(i_X \nabla^\alpha s + u_\alpha s) = -i_X \overline{\partial}\nabla^\alpha s + \overline{\partial} u_\alpha s
= ( - i_X \Theta_\alpha + \overline{\partial} u_\alpha ) s = \sqrt{-1}(i_X \omega_\alpha - \overline{\partial}(\sqrt{-1} u_\alpha)) s = 0.$$  

3. It is obvious that $\Lambda_\alpha|_{\text{Zero}(X)} = u_\alpha|_{\text{Zero}(X)}$ is a linear map on $\Gamma(L_\alpha|_{\text{Zero}(X)})$. This implies $\Lambda_\alpha|_{\text{Zero}(X)} \in \text{End}(L_\alpha|_{\text{Zero}(X)})$. This endomorphism along the zero set of $X$ can be extended a global endomorphism of $L_\alpha$ by letting for $s \in \Gamma(L_\alpha)$

$$E_\alpha s = \Lambda_\alpha s - \nabla^\alpha_X s = u_\alpha s = -\theta_\alpha(X_\alpha)s.$$  

Then $E_\alpha \in \text{End}(L_\alpha)$ and

$$\overline{\partial} E_\alpha = \overline{\partial} u_\alpha = i_X(-\sqrt{-1}\omega_\alpha) = i_X \Theta_\alpha.$$  

The above discussion enables us to write the Lie algebra character (6) as

$$\text{Fut}(X) = \sum_{\alpha=1}^{k} \frac{\int_M u_\alpha \omega^m_\alpha}{\int_M \omega^m_\alpha}$$

$$= \frac{1}{m+1} \sum_{\alpha=1}^{k} \frac{\int_M (u_\alpha + \omega_\alpha)^{m+1}}{\int_M (u_\alpha + \omega_\alpha)^m}$$

$$= \frac{1}{m+1} \sum_{\alpha=1}^{k} \frac{\int_M (-\theta_\alpha(X_\alpha) + \sqrt{-1}\Theta_\alpha)^{m+1}}{\int_M (-\theta_\alpha(X_\alpha) + \sqrt{-1}\Theta_\alpha)^m}$$

$$= \frac{1}{m+1} \sum_{\alpha=1}^{k} \frac{\int_M (E_\alpha + \sqrt{-1}\Theta_\alpha)^{m+1}}{\int_M (E_\alpha + \sqrt{-1}\Theta_\alpha)^m}.$$  

Here we remark that the both expressions $\int_M (-\theta_\alpha(X_\alpha) + \sqrt{-1}\Theta_\alpha)^{m+1}$ and $\int_M (-\theta_\alpha(X_\alpha) + \sqrt{-1}\Theta_\alpha)^m$ are independent of the choice of Hermitian metric $h_\alpha$. This could either follow from...
We choose a family of Hermitian metrics $h_\alpha(t)$, let $h_\alpha(t) = e^{t\varphi_\alpha}h_\alpha$, for $\varphi_\alpha \in C^\infty(M)$. Then

$$\theta_\alpha(t) = \partial \log h_\alpha(t) + \frac{dz}{\bar{z}} = \theta_\alpha - t\varphi_\alpha$$

is the corresponding family of connections on associated principle $C^*$-bundle, and the curvature forms are

$$\Theta_\alpha(t) = \Theta_\alpha + t\partial\bar{\partial}\varphi_\alpha,$$

and we compute that

$$i_X\Theta_\alpha(t) = i_X\Theta_\alpha + i_X(t\partial\bar{\partial}\varphi_\alpha) = \partial(u_\alpha + tX(\varphi_\alpha)),$$

we let $u_\alpha(t) = u_\alpha + tX(\varphi_\alpha)$. This $u_\alpha(t)$ is a Hamiltonian function of $X$ for the Kähler form $\omega_\alpha(t)$ corresponding to $h_\alpha(t)$. As we saw above the lifted vector field on $L_\alpha$ is given by

$$X_\alpha(t) = X^h(\alpha(t) - u_\alpha(t)z\frac{\partial}{\partial z}. $$

Then

$$-\theta_\alpha(t)(X_\alpha(t)) = u_\alpha(t) = -\theta_\alpha(X_\alpha(t)) + tX(\varphi_\alpha).$$

We will check the metric independence of $\int_M(-\theta_\alpha(X_\alpha) + \sqrt{-1}\Theta_\alpha)m$ and similar argument works for $\int_M(-\theta_\alpha(X_\alpha) + \sqrt{-1}\Theta_\alpha)m$. We compute that

$$\frac{d}{dt}\int_M\left(-\theta_\alpha(t)(X_\alpha(t)) + \sqrt{-1}\Theta_\alpha(t)\right)^{m+1}
= (m+1)\int_M\left(-\theta_\alpha(t)(X_\alpha(t)) + \sqrt{-1}\Theta_\alpha(t)\right)^m \wedge (X(\varphi_\alpha) + \sqrt{-1}\partial\bar{\partial}\varphi_\alpha)
= (m+1)\left(\int_M X(\varphi_\alpha)(\sqrt{-1}\Theta_\alpha(t))^m - m\theta_\alpha(t)(X_\alpha(t))(\sqrt{-1}\Theta_\alpha(t))^{m-1} \wedge \sqrt{-1}\partial\bar{\partial}\varphi_\alpha\right)
= (m+1)\left(\int_M X(\varphi_\alpha)(\sqrt{-1}\Theta_\alpha(t))^m - m\int_M \bar{\partial}(\theta_\alpha(t)(X_\alpha(t)))(\sqrt{-1}\Theta_\alpha(t))^{m-1} \wedge \sqrt{-1}\partial\varphi_\alpha\right)
= (m+1)\left(\int_M X(\varphi_\alpha)(\sqrt{-1}\Theta_\alpha(t))^m + m\int_M i_X\Theta_\alpha(t) \wedge (\sqrt{-1}\Theta_\alpha(t))^{m-1} \wedge \sqrt{-1}\partial\varphi_\alpha\right)
= 0.$$

**Proof of Theorem 5.2** Now, we follow an argument in the book [9] (see Theorem 5.2.8), originally due to Bott [2] to give the localization formula.

Consider an invariant polynomial $P$ of degree $(m + l)$ for $l = 0, 1$, let

$$P_\alpha(E_\alpha + \sqrt{-1}\Theta_\alpha) = \sum_{r=0}^{m+l} P_{\alpha,r}(E_\alpha, \sqrt{-1}\Theta_\alpha),$$

where

$$P_{\alpha,r}(E_\alpha, \sqrt{-1}\Theta_\alpha) = \binom{m+l}{r}P(E_\alpha, \ldots, E_\alpha; \sqrt{-1}\Theta_\alpha, \ldots, \sqrt{-1}\Theta_\alpha).$$

Since $\bar{\partial}E_\alpha = i_X\Theta_\alpha$, we have

$$\sqrt{-1}\bar{\partial}P_\alpha = i_XP_\alpha.$$

Define a $(1, 0)$ form $\pi_\alpha$ as follows: for a holomorphic vector field $Y$,

$$i_Y\pi_\alpha = \frac{\omega_\alpha(Y, X)}{\omega_\alpha(X, X)},$$

then

$$i_X\pi_\alpha = 1, \quad \text{and} \quad i_X\bar{\partial}\pi_\alpha = 0.$$
We further define
\[ \eta_\alpha = \frac{\pi_\alpha}{1 - \sqrt{-1}\partial \pi_\alpha} \wedge P_\alpha(E_\alpha + \sqrt{-1}\Theta_\alpha), \]
then \( \eta_\alpha \) is defined outside zero sets of \( X \). The computation shows
\[ P_\alpha(E_\alpha + \sqrt{-1}\Theta_\alpha) = -\sqrt{-1}\partial \eta_\alpha + i_X \eta_\alpha. \]
Let \( B_\epsilon(Z) \) be an \( \epsilon \)-neighbourhood of \( Z \). Then
\[
\int_M P_\alpha(E_\alpha + \sqrt{-1}\Theta_\alpha) = \lim_{\epsilon \to 0} \int_{M-B_\epsilon(Z)} P_\alpha(E_\alpha + \sqrt{-1}\Theta_\alpha)
\]
\[
= \sqrt{-1} \lim_{\epsilon \to 0} \int_{M-B_\epsilon(Z)} -\partial \eta_\alpha^{2m-1} = \sqrt{-1} \lim_{\epsilon \to 0} \sum_{\lambda \in \Lambda} \int_{\partial B_\epsilon(z)} \eta_\alpha^{2m-1}
\]
\[
= \sqrt{-1} \sum_{\lambda \in \Lambda} \lim_{\epsilon \to 0} \int_{\partial B_\epsilon(z)} \pi_\alpha \wedge (1 + (\sqrt{-1}\partial \pi_\alpha)^2 + \cdots + (\sqrt{-1}\partial \pi_\alpha)^{m-1})
\]
\[
\wedge \sum_{r=0}^{m-1} P_{\alpha,r}(E_\alpha, \sqrt{-1}\Theta_\alpha).
\]
As computed in Theorem 5.2.8 in [9] or [2],
\[
(2\pi)^{-m} \int_M P_\alpha(E_\alpha + \sqrt{-1}\Theta_\alpha) = \sum_{\lambda \in \Lambda} \frac{P_\alpha(E_\alpha + \sqrt{-1}\Theta_\alpha)|_{Z_\lambda}}{\det((2\pi)^{-1}(L_{\alpha}(X) + \sqrt{-1}K_\alpha))},
\]
where \( K_\alpha \) is the curvature of the normal bundle \( N_\alpha \) with respect to the induced metric. Taking \( P = tr^{m+1} \) and \( P = tr^m \), and apply above to \([11]\), we obtain the localization formula of \( \text{Fut}(X) \) in the coupled case \([7]\). \( \square \)

3. Application of Localization Formula

Before computing the example, we remark that by Theorem 3.2 in [13], \([4]\) is equivalent to
\[
\int_M (u_1 + \cdots + u_k)dV = 0
\]
where \( dV = e^{\omega_\alpha} \omega_\alpha^m \) which is independent of \( \alpha \) by the normalization \([2]\). By Theorem 5.2 in [13] this condition is equivalent to
\[
\sum_{\alpha=1}^k P_\alpha = P_{-K_M}
\]
where \( P_\alpha \) is the moment map image of \( \omega_\alpha \).

We consider the tautological line bundles \( \mathcal{O}_{\mathbb{CP}^1}(-1) \to \mathbb{CP}^1 \) and \( \mathcal{O}_{\mathbb{CP}^2}(-1) \to \mathbb{CP}^2 \), and the bundle \( E = \mathcal{O}_{\mathbb{CP}^1}(-1) \oplus \mathcal{O}_{\mathbb{CP}^2}(-1) \) over \( \mathbb{CP}^1 \times \mathbb{CP}^2 \). Let \( M \) be the total space of the projective line bundle \( \mathbb{P}(E) \) over \( \mathbb{CP}^1 \times \mathbb{CP}^2 \). In local coordinates, we let
\[
\mathbb{CP}^1 = \{(b_0 : b_1)\}, \quad \mathbb{CP}^2 = \{(a_0 : a_1 : a_2)\},
\]
\[
\mathcal{O}_{\mathbb{CP}^1}(-1) = \{[(w_0, w_1), (b_0 : b_1)] | (w_0, w_1) = \lambda(b_0, b_1) \text{ for some } \lambda \in \mathbb{C}\},
\]
\[
\mathcal{O}_{\mathbb{CP}^2}(-1) = \{[(z_0, z_1, z_2), (a_0 : a_1 : a_2)] | (z_0, z_1, z_2) = \mu(a_0, a_1, a_2) \text{ for some } \mu \in \mathbb{C}\},
\]
\[
M = \{[(w_0, w_1), (a_0 : a_1 : a_2), (b_0 : b_1)] | (w_0, w_1) = \lambda(b_0, b_1), (z_0, z_1, z_2) = \mu(a_0, a_1, a_2) \text{ for some } (\lambda, \mu) \neq (0, 0) \text{ in } \mathbb{C} \times \mathbb{C}\}.\]
The \((\mathbb{C}^*)^4\)-action on \(M\) is defined by extending the \(\mathbb{C}^*\)-action on \(\mathbb{CP}^1\) and \((\mathbb{C}^*)^2\)-action on \(\mathbb{CP}^2\). We let \((t_1, t_2, t_3, t_4) \in (\mathbb{C}^*)^4\), then

\[
(t_1, t_2, t_3, t_4) \cdot [(z_0 : z_1 : z_2 : w_0 : w_1), (a_0 : a_1 : a_2), (b_0 : b_1)]
= [(z_0 : t_1 z_1 : t_2 z_2 : t_4 w_0 : t_4 t_3 w_1), (a_0 : t_1 a_1 : t_2 a_2), (b_0 : t_3 b_1)].
\]

There are totally seven \((\mathbb{C}^*)^4\)-invariant divisors;

\[D_1 = \{ z_0 = a_0 = 0 \}, \quad D_2 = \{ z_1 = a_1 = 0 \}, \quad D_3 = \{ z_2 = a_2 = 0 \},\]

which are identified with \(\mathbb{CP}^1\)-bundle over \(\mathbb{CP}^1 \times \mathbb{CP}^1\);

\[D_4 = \{ b_0 = w_0 = 0 \}, \quad D_5 = \{ b_1 = w_1 = 0 \},\]

which are identified with \(\mathbb{CP}^1\)-bundle over \(\mathbb{CP}^2\);

\[D_6 = \{ z_0 = z_1 = z_2 = 0 \}, \quad D_7 = \{ w_0 = w_1 = 0 \},\]

which are identified with \(\mathbb{CP}^1 \times \mathbb{CP}^2\). It is known that

\[K_M^{-1} = \sum_{i=1}^{7} D_i.\]

As in [15], we consider the following decomposition for \(c \in (1/4, 3/4)\) which is ampleness condition for the line bundles associated with \(D(c)\) and \(D(1-c)\) below. Define

\[D(c) = \frac{1}{2} K_M^{-1} + (c - \frac{1}{2})(D_4 + D_5)\]
\[D(1-c) = \frac{1}{2} K_M^{-1} + (\frac{1}{2} - c)(D_4 + D_5),\]

then

\[(13) \quad K_M^{-1} = D(c) + D(1-c).\]

We remark that the torus action preserves above decomposition.

Note also that the invariant Fut is invariant under any automorphism of \(M\) preserving the decomposition [(13)]. Using the automorphism \((b_0, b_1) \mapsto (b_1, b_0)\) one can see \(\text{Fut}(X_3) = \text{Fut}(-X_3)\) and thus \(\text{Fut}(X_3) = 0\) for the infinitesimal generator \(X_3\) for the \(t_3\)-action, and similarly \(\text{Fut}(X_1) = \text{Fut}(X_2) = 0\) for the infinitesimal generators \(X_1\) and \(X_2\) of \(t_1\) and \(t_2\)-actions using the automorphisms induced by the odd permutations of the coordinates \((a_0 : a_1 : a_2)\). Hence, to compute the coupled Fut invariant, it is sufficient to consider the action of one parameter subgroup \((1, 1, 1, t_4)\) on \(M\) by

\[(1, 1, 1, t_4) \cdot [(z_0 : z_1 : z_2 : w_0 : w_1), (a_0 : a_1 : a_2), (b_0 : b_1)]
= [(z_0 : z_1 : z_2 : t_4 w_0 : t_4 w_1), (a_0 : a_1 : a_2), (b_0 : b_1)].\]

For this action, let \(\xi = \lambda/\mu, \eta = 1/\xi\), then the associated holomorphic vector field is

\[X = \xi \frac{\partial}{\partial \xi} = -\eta \frac{\partial}{\partial \eta}.\]

Zero sets are

\[Z_\infty = \{ \mu = 0 \} = D_6, \quad \text{and} \quad Z_0 = \{ \lambda = 0 \} = D_7.\]

Since

\[\mathbb{P}(O_{\mathbb{CP}^1}(-1) \oplus O_{\mathbb{CP}^2}(-1)) = \mathbb{P}\left((O_{\mathbb{CP}^1}(-1) \oplus O_{\mathbb{CP}^2}(-1)) \otimes O_{\mathbb{CP}^2}(1)\right)
= \mathbb{P}\left((O_{\mathbb{CP}^1}(-1) \otimes O_{\mathbb{CP}^2}(1)) \oplus O_{\mathbb{CP}^2}\right),\]
the normal bundle of $Z_\infty$ is
\[ \nu(Z_\infty) = \mathcal{O}_{\mathbb{CP}^1}(-1) \otimes \mathcal{O}_{\mathbb{CP}^2}(1), \]
similarly, the normal bundle of $Z_0$ is
\[ \nu(Z_0) = \mathcal{O}_{\mathbb{CP}^1}(1) \otimes \mathcal{O}_{\mathbb{CP}^2}(-1) = \nu(Z_\infty)^{-1}. \]
Let $a, b$ be the positive generators of $H^2(\mathbb{CP}^1, \mathbb{Z})$ and $H^2(\mathbb{CP}^2, \mathbb{Z})$. Then
\[ c_1(\mathbb{CP}^1) = 2a, \quad c_1(\mathbb{CP}^2) = 3b, \]
and
\[ c_1(K_M^{-1})|_{Z_\infty} = c_1(Z_\infty) + c_1(\nu(Z_\infty)) = 2a + 3b - a = a + 4b. \]
Similarly we have
\[ c_1(K_M^{-1})|_{Z_0} = 3a + 2b. \]
Since the line bundle $[D_4]$ restricted to $Z_\infty = D_6$ is isomorphic to the line bundle corresponding to the divisor \{0\} in $\mathbb{CP}^1 \times \mathbb{CP}^2$ we have $c_1([D_4])|_{Z_\infty} = a$. Similarly we have
\[ c_1([D_4])|_{Z_0} = c_1([D_5])|_{Z_\infty} = c_1([D_5])|_{Z_0} = a. \]
Then
\[ c_1(D(c))|_{Z_\infty} = \frac{1}{2}(a + 4b) + (c - \frac{1}{2})2a = (2c - \frac{1}{2})a + 2b, \]
\[ c_1(D(c))|_{Z_0} = \frac{1}{2}(3a + 2b) + (c - \frac{1}{2})2a = (2c + \frac{1}{2})a + b. \]
To see the value of $u$ along the zero sets of $X$ we may use the description of the moment polytope $P(c)$ in [15]
\[ P(c) = \{ y \in \mathbb{R}^4 : \langle y, d_i \rangle \leq \frac{1}{2}, i \neq 4, 5, \langle y, d_i \rangle \leq c, i = 4, 5 \} \]
where $d_i$ are as described in [15]. Since $P(c) + P(1 - c) = P_{-K_M}$ the moment polytopes are those obtained by the Hamiltonian functions satisfying (4) as follows from the arguments of the beginning of this section. From this description for $d_6 = (0, 0, 0, -1)$ and $d_7 = (0, 0, 0, 1)$ we see
\[ u|_{Z_\infty} = -\frac{1}{2}, \quad u|_{Z_0} = \frac{1}{2}. \]
By using the fact
\[ a^2 = b^3 = 0, \]
we first compute
\[ \text{Vol}(D(c)) = \left[ \frac{(u|_{Z_\infty} + c_1(D(c))|_{Z_\infty})^4}{u|_{Z_\infty} + c_1(\nu(Z_\infty))} + \frac{(u|_{Z_0} + c_1(D(c))|_{Z_0})^4}{u|_{Z_0} + c_1(\nu(Z_0))} \right][\mathbb{CP}^1 \times \mathbb{CP}^2] \]
(14)
\[ = \left[ \frac{-1/2 + (2c - 1/2)a + 2b}{-1/2 - a + b} \right]^4 + \left[ \frac{1/2 + (2c + 1/2)a + b}{1/2 + a - b} \right]^4 \right][\mathbb{CP}^1 \times \mathbb{CP}^2] \]
\[ = 112c - 6, \]
replacing $c$ by $1 - c$, we get
(15) \[ \text{Vol}(D(1 - c)) = 106 - 112c. \]
We also need to compute the numerators in the localization formula. For the divisor $D(c)$,

$$\left( \frac{(u|Z_{\infty} + c_1(D(c))|Z_{\infty})^5}{u|Z_{\infty} + c_1(\nu(Z_{\infty}))} + \frac{(u|Z_0 + c_1(D(c))|Z_0)^5}{u|Z_0 + c_1(\nu(Z_0))} \right)[\mathbb{CP}^1 \times \mathbb{CP}^2]$$

(16)

$$= \left( -\frac{1}{2} + \frac{(2c - 1/2)a + 2b}{-1/2 - a + b} \right)^5 + \left( \frac{1/2 + (2c + 1/2)a + b}{1/2 + a - b} \right)^5 \right)[\mathbb{CP}^1 \times \mathbb{CP}^2]$$

$$= -30c - 12,$$

replacing $c$ by $1 - c$, we get for divisor $D(1 - c)$,

$$\left( \frac{(u|Z_{\infty} + c_1(D(1-c))|Z_{\infty})^5}{u|Z_{\infty} + c_1(\nu(Z_{\infty}))} + \frac{(u|Z_0 + c_1(D(1-c))|Z_0)^5}{u|Z_0 + c_1(\nu(Z_0))} \right)[\mathbb{CP}^1 \times \mathbb{CP}^2]$$

(17)

$$= 30c - 18.$$

Plugging above (14), (15), (16), (17) into the localization formula (Theorem 1.2), we obtain

$$\text{Fut}(X) = \frac{[\mathbb{CP}^1 \times \mathbb{CP}^2]}{\text{Vol}(D(c))}$$

$$\left( \frac{(u|Z_{\infty} + c_1(D(c))|Z_{\infty})^5}{u|Z_{\infty} + c_1(\nu(Z_{\infty}))} + \frac{(u|Z_0 + c_1(D(c))|Z_0)^5}{u|Z_0 + c_1(\nu(Z_0))} \right)[\mathbb{CP}^1 \times \mathbb{CP}^2]$$

(18)

$$= \frac{-30c + 12}{112c - 6} + \frac{30c - 18}{106 - 112c}$$

$$= \frac{-15(112c^2 - 112c + 23)}{(56c - 3)(56c - 53)},$$

therefore, the invariant Fut character vanishes when

$$c = \frac{1}{2} \pm \frac{1}{4} \sqrt{5}.$$

This is the same as in [15].

REFERENCES

[1] T. Aubin: Equations du type de Monge-Ampère sur les variétés kählériennes compactes, C. R. Acad. Sci. Paris, 283, 119–121 (1976).
[2] R. Bott: A residue formula for holomorphic vector-fields. J. Diff. Geom. 1(1967), 311-330.
[3] X. X. Chen, S. K. Donaldson, S. Sun: Kähler–Einstein metric on Fano manifolds. III: limits with cone angle approaches $2\pi$ and completion of the main proof, J. Amer. Math. Soc. 28, 235–278 (2015).
[4] V.V. Datar and V.P. Pingali: On coupled constant scalar curvature Kähler metrics. arXiv preprint arXiv:1901.10454 (2019).
[5] T. Delcroix and J. Hultgren: Coupled complex Monge-Ampère equations on Fano horosymmetric manifolds. arXiv preprint arXiv:1812.07218.
[6] A. Futaki: An obstruction to the existence of Einstein Kähler metrics, Invent. Math. 73, 437-443 (1983).
[7] A. Futaki: On a character of the automorphism group of a compact complex manifold, Invent. Math., 87(1987), 655-660.
[8] A. Futaki: The Ricci curvature of symplectic quotients of Fano manifolds. Tohoku Math. J. (2) 39 (1987), no. 3, 329–339.
[9] A. Futaki, Kähler-Einstein Metrics and Integral Invariants. Lecture Notes in Mathematics, 1314. Springer-Verlag, Berlin, 1988. iv+140 pp.
[10] A. Futaki and T. Mabuchi: Moment maps and multilinear bilinear forms associated with symplectic classes, Asian J. Math., 6(2002), 349-371.
[11] A. Futaki, T. Mabuchi and Y. Sakane: Einstein-Kähler metrics with positive Ricci curvature. Kähler metric and moduli spaces, 11-83, Adv. Stud. Pure Math., 18-II, Academic Press, Boston, MA, 1990.
[12] A. Futaki and S. Morita : Invariant polynomials of the automorphism group of a compact complex manifold, J. Diff. Geom., 21, 135–142 (1985).
[13] A. Futaki. and Y. Zhang. Sci. China Math. (2019). https://doi.org/10.1007/s11425-018-9499-y
[14] V. Guillemin : Moment maps and combinatorial invariants of Hamiltonian $T^n$-spaces, Birkhäuser Boston 1994.
[15] J. Hultgren : Coupled Kähler-Ricci solitons on toric manifolds. arXiv preprint [arXiv:1711.09881] (2017).
[16] J. Hultgren and D. Witt Nyström : Coupled Kähler-Einstein metrics. Int. Math. Res. Not. published online in 2018. https://doi.org/10.1093/imrn/rnx298
[17] Y. Matsushima : Sur la structure du groupe d’homéomorphismes analytiques d’une certaine variété kählérienne, Nagoya Math. J., 11, 145-150 (1957).
[18] S. Nakamura : Deformation for coupled Kähler-Einstein metrics, in preparation.
[19] V.P. Pingali : Existence of coupled Kähler-Einstein metrics using the continuity method, Internat. J. Math. 29(2018), 1850041, 8 pp.
[20] R. Takahashi : Ricci iteration, twisted and coupled Kähler-Einstein metrics. arXiv preprint [arXiv:1901.09754] (2019).
[21] R. Takahashi : Geometric quantization of coupled Kähler-Einstein metrics. arXiv preprint [arXiv:1904.12812] (2019).
[22] G. Tian: K-stability and Kähler-Einstein metrics. Comm. Pure Appl. Math. 68, no. 7: 1085–1156 (2015).
[23] S.-T. Yau : On the Ricci curvature of a compact Kähler manifold and the complex Monge-Ampère equation I, Comm. Pure Appl. Math. 31(1978), 339-441.

YAU MATHEMATICAL SCIENCES CENTER, TSINGHUA UNIVERSITY, HAIDIAN DISTRICT, BEIJING 100084, CHINA

E-mail address: futaki@tsinghua.edu.cn

YAU MATHEMATICAL SCIENCES CENTER, TSINGHUA UNIVERSITY, HAIDIAN DISTRICT, BEIJING 100084, CHINA

E-mail address: yingyzhang@tsinghua.edu.cn