Space-sequential particle filters for high-dimensional dynamical systems described by stochastic differential equations

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Abstract

We introduce a novel methodology for particle filtering in dynamical systems where the evolution of the signal of interest is described by a SDE and observations are collected instantaneously at prescribed time instants. The new approach includes the discretisation of the SDE and the design of efficient particle filters for the resulting discrete-time state-space model. The discretisation scheme converges with weak order 1 and it is devised to create a sequential dependence structure along the coordinates of the discrete-time state vector. We introduce a class of space-sequential particle filters that exploits this structure to improve performance when the system dimension is large. This is numerically illustrated by a set of computer simulations for a stochastic Lorenz 96 system with additive noise. The new space-sequential particle filters attain approximately constant estimation errors as the dimension of the Lorenz 96 system is increased, with a computational cost that increases polynomially, rather than exponentially, with the system dimension.

Besides the new numerical scheme and particle filters, we provide in this paper a general framework for discrete-time filtering in continuous-time dynamical systems described by a SDE and instantaneous observations. Provided that the SDE is discretised using a weakly-convergent scheme, we prove that the marginal posterior laws of the resulting discrete-time state-space model converge to the posterior marginal posterior laws of the original continuous-time state-space model under a suitably defined metric. This result is general and not restricted to the numerical scheme or particle filters specifically studied in this manuscript.

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1 Introduction

1.1 Motivation and background

Particle filters (PFs) are popular tools for the estimation, tracking and prediction of the time-varying state variables of dynamical systems for which observations are collected and processed sequentially. Many such systems are described by multivariate stochastic differential equations (SDEs) of the form

\[ dX = f(X, t)dt + s(X, t)dW, \]  

where \( X(t) \) is the \( d \)-dimensional state vector, \( t \) is the (continuous) time variable, \( f(X, t) \) is the drift term, \( s(X, t) \) is the diffusion coefficient and \( W(t) \) is a multi-dimensional Wiener process. Noisy (and possibly nonlinear) observations \( Y(t) \) are collected at times \( t_k, k = 1, 2, \ldots \). For most practical applications, this continuous-time system is converted into a discrete-time state-space model (see, e.g., \([9]\) and references therein). A common feature of many applications where PFs are used is that the dimension \( d \) of the state vector \( X(t) \) is very large.

The design of particle filtering algorithms that can perform efficiently in high dimensional state space systems remains a challenge and the effort to tackle this problem over the past decade has been remarkable. It includes significant advances, especially methodological \([12, 23, 11, 18, 5, 2, 20]\), but also "negative" results \([3, 22]\). PFs are built around the importance sampling (IS) method \([19, 6]\), which is aimed at approximating probability distributions using \( N \) Monte Carlo samples with associated weights. These weights, however, degenerate as the target probability distribution and the probability distribution used for sampling become mutually singular. This degeneracy implies that all weights but one become negligible \([22]\), which effectively renders the Monte Carlo approximation useless. This phenomenon occurs even for low or moderate dimensional systems when the likelihood induced by the observed data is narrow with respect to (w.r.t.) the a priori law of the state variables of interest \([15]\). In the context of filtering, weight degeneracy becomes acute as the dimension of the state space increases. Several authors \([3, 22, 18]\) have noted that one general way to prevent weight degeneracy is to increase the number of Monte Carlo samples exponentially with the state dimension \( d \), i.e., choose \( N = \mathcal{O}(c^d) \) for some constant \( c > 1 \). This difficulty in the design of high-dimensional particle filters is usually referred as a curse of dimensionality \([3, 22]\).

While the picture presented in part of the literature is not encouraging, a number of particle filtering schemes have been shown to perform well in high dimensional problems, often taking advantage of specifics of the model at hand, see e.g. \([12, 23, 1, 18, 5, 20]\). The methods in \([12] \) and \([18]\) partition the dynamical state of the system of interest into lower dimensional subsets of variables. Particle filters are then used to approximate the posterior distribution over these subsets of variables. This approach is termed “multiple particle filtering” in \([12]\) and “block particle filtering” in \([18]\). These techniques apply quite naturally to certain applications, including multi-target tracking \([12]\). However, in general they rely on assumptions on the posterior...
distributions of interest which are hard to verify and can be expected to yield biased approximations. Another approach, that tackles directly the issue of weight degeneracy, is the design of PFs that generate nearly-equal importance weights \cite{23} and \cite{1}. These algorithms essentially construct sophisticated proposal distributions for the generation of Monte Carlo samples that match the available observations tightly. They have been applied to moderate-to-high dimensional problems in oceanography. The methodology in \cite{20} relies on a tempering scheme to tackle the curse of dimensionality. The core idea is to construct a sequence of intermediate distributions that bridge the gap between the prior and the posterior distribution. The PF then proceeds by generating approximations of each intermediate distribution starting from the previous one. This approach had been proved useful in designing adaptive importance samplers for non-dynamical inference problems \cite{4}.

In this paper we adopt the approach of \cite{5}, that suggests the design of PFs that evolve sequentially not only over time but also over the coordinates of the state vector. Such methods have been termed space-time particle filters (STPFs): if the state vector is a $d$-dimensional vector $X = (X_1, \ldots, X_d)^\top$ then the subscript $i = 1, \ldots, d$ is interpreted as a discrete-space coordinate. The algorithms in \cite{5} illustrate this strategy by assuming a certain dependence structure in the state space. Such structure, however, cannot be taken for granted in general. In fact, for physical systems described in continuous time by SDEs (or stochastic partial differential equations as well) the structure of the state space is essentially given by the discretisation scheme used to convert the SDE into a discretised difference equation suitable for computer implementation. Classical numerical schemes \cite{14} do not yield the sequential structure across the state space that is needed for STPFs.

Recent surveys on high-dimensional particle filtering can be found in \cite{21} or \cite{24}.

1.2 Contributions

In this paper, we propose, analyse and study numerically several PFs that run sequentially both over time and over the coordinates of the state vector $X(t) = (X_1(t), \ldots, X_d(t))^\top$, in a framework similar to \cite{5}. Our approach relies on the idea that a sequential structure can be imposed on the state variables by using a suitable discretisation of the SDE \cite{1}. The main results to be presented in the rest of this manuscript are outlined below.

- In Section 2 we introduce a numerical discretisation scheme for SDEs which is specifically designed to yield a sequential dependence along the dimensions of the state vector. Let $t_n$, $n = 0, \ldots, N$ be a time grid. The scheme operates in two stages. First, deterministic auxiliary estimates of the state variables at time $t_n$ are computed, from the states at time $t_{n-1}$, using an explicit rule. In the second stage, these deterministic estimates are used to update the (stochastic) state variables sequentially, from $X_1(t_n)$ to $X_d(t_n)$. This scheme creates the sequential structure that is needed to apply the STPFs on \cite{5}. We refer to it as a
semi-implicit scheme because of the way the deterministic auxiliary estimates are used in the second stage.

In Section 2 we also prove that the new scheme converges with weak order 1. We identify the Markov kernel generated by the SDE (in the state-space setup), the approximate Markov kernel generated by the semi-implicit scheme and introduce a distance that can be used to compare them.

- In Section 3 we describe the observation model and then compare the state space systems obtained when using either the exact kernel generated by the SDE or its discrete-time approximation obtained via the semi-implicit scheme with time step $h > 0$. In particular, we prove that, as $h \to 0$, the posterior distributions$^1$ generated by the approximate model converge to the posterior distributions generated by the exact model. In particular, the optimal filter for the discrete-time, approximate model approaches the optimal continuous-time filter as $h \to 0$. Our analysis draws from arguments in [10] and is valid for any weakly convergent numerical discretisation scheme (it is not limited to the semi-implicit scheme of Section 2).

- In Section 4 we introduce particle filtering algorithms that exploit the sequential structure induced by the semi-implicit discretisation scheme. We outline algorithms that are, in general, simpler than the STPFs of [5]. We refer to the proposed algorithms as space-sequential particle filters (SSPFs) as they run sequentially through the “spatial” coordinates $i = 1, \ldots, d$ of the state vector. We also show how these space sequential algorithms can actually be mimicked even when the discretisation scheme is explicit. All the PFs described in this section can be proved to be consistent.

- In Section 5 we illustrate the performance of the proposed SSPFs by applying them on a stochastic Lorenz 96 model. We compare the new PFs with conventional algorithms as the model dimension $d$ increases. For the range of values of $d$ covered by our simulations, the approximation error of the new PFs remains approximately constant when the computational effort increases polynomially with $d$.

1.3 Notation

We complete this introductory section with a summary of notation used throughout the manuscript.

- $a \in \mathbb{R}^d$ denotes a $d$-dimensional column vector, $a = [a_1, \ldots, a_d]^\top$, with real entries, while $A \in \mathbb{R}^{d \times m}$ is a real matrix with $d$ rows and $m$ columns.

- Let $X : \Omega \mapsto \mathbb{R}^d$ be a $d$-dimensional random variable (r.v.) defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Then $\mathbb{E}[X] := \int X \, d\mathbb{P}$ denotes the expected value of $X$. If $X = [X_1, \ldots, X_d]^\top$ then $X_n$.
\[ \|X\|_p = \left( \mathbb{E} \left[ \sum_{i=1}^{d} |X_i|^p \right] \right)^{\frac{1}{p}} \] is the \( L_p \) norm of \( X \).

- For \( \mathcal{R} \subseteq \mathbb{R}^d \) denote by \( \mathcal{B}(\mathcal{R}) \) the Borel sigma algebra of open subsets of \( \mathcal{R} \).
- \( \mathcal{P}(\mathcal{R}) \) is the class of probability measures on the measurable space \((\mathcal{R}, \mathcal{B}(\mathcal{R}))\).
- \( \delta_{x'}(dx) \) is the delta measure that assigns a unit probability mass to the point \( x' \). In particular, for an integrable test function \( \phi \), \( \int_{A} \phi(x) \delta_{x'}(dx) = \begin{cases} \phi(x'), & \text{if } x' \in A, \\ 0, & \text{otherwise}. \end{cases} \)
- Let \( \pi \) denote the probability law of \( X \) and let \( \phi : \mathbb{R}^d \mapsto \mathbb{R} \) be a real test function. If \( \phi \) is integrable with respect to (w.r.t.) \( \pi \), then we denote \( \pi(\phi) := \int \phi(x) \pi(dx) = \mathbb{E}[\phi(X)] \).
- Let \( \alpha = (\alpha_1, \ldots, \alpha_d) \in \mathbb{N} \cup \{0\} \) denote a multi-index and let \( f : \mathbb{R}^d \mapsto \mathbb{R} \) be a real function. If \( x = [x_1, \ldots, x_d]^\top \), then \( f^{(\alpha)}(x) = \frac{\partial^{\alpha_1}}{\partial x_1} \frac{\partial^{\alpha_2}}{\partial x_2} \cdots \frac{\partial^{\alpha_d}}{\partial x_d} f(x) \) denotes a derivative of order \( \|\alpha\| = \sum_{i=1}^{d} \alpha_i \).
- Choose a finite constant \( B > 0 \). \( \mathcal{C}^l_B(\mathcal{R}) \) denotes the set of real continuous functions \( \mathcal{R} \mapsto \mathbb{R} \) with uniformly bounded derivatives up to order \( l \), i.e.,
  \[ \mathcal{C}^l_B(\mathcal{R}) := \left\{ f : \mathcal{R} \mapsto \mathbb{R} \text{ such that } \sup_{x \in \mathcal{R}, \|\alpha\| \leq l} |f^{(\alpha)}| \leq B \right\} \]
- The stochastic process \( X(t) \) is \( d_x \)-dimensional and it can be split into \( q \) \( m_x \)-dimensional processes, denoted \( X_1(t), \ldots, X_q(t) \), where \( m_x = \frac{d_x}{q} \).
- Given a sequence \( x_i, x_{i+1}, \ldots, x_j \) we use \( x_{i:j} \) to denote either the set \( \{x_i, x_{i+1}, \ldots, x_j\} \) or the \((j - i + 1) \times 1\) vector \([x_i, \ldots, x_j]^\top\).

## 2 Discretisation of stochastic differential equations

### 2.1 Stochastic differential equation

Let \( W(t) \) denote a \( d_w \)-dimensional Wiener process defined on a probability space \((\Omega, \mathcal{F}, \mathbb{P})\), let \( T < \infty \) be an arbitrary time horizon and choose two functions \( f : \mathbb{R}^{d_x} \times [0, T] \mapsto \mathbb{R}^{d_x} \) and \( s : \mathbb{R}^{d_x} \times [0, T] \mapsto \mathbb{R}^{d_x \times d_w} \). We place our attention on dynamical systems that can be modelled by the solution \( X(t) \), \( 0 \leq t \leq T \), of the Itô SDE

\[ dX = f(X, t)dt + s(X, t)dW. \quad (2) \]

The factors \( f(X, t) \) and \( s(X, t) \) are usually termed the drift and diffusion coefficients of the SDE. Under regularity assumptions \[17\], \( X(t) \) exists and the notation in (2) is just convenient shorthand for the \( d_x \)-dimensional stochastic Itô process

\[ X(t) = X(0) + \int_{0}^{t} f(X(u), u)du + \int_{0}^{t} s(X(u), u)dW(u). \]

We assume that the diffusion coefficient \( s(X, t) \in \mathbb{R}^{d_x \times d_w} \) is a block-diagonal matrix,

\[ s(X, t) = \begin{bmatrix} s_0(X, t) & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & s_{q-1}(X, t) \end{bmatrix}, \]
where $s_i(X, t)$, $i \in \{0, \ldots, q-1\}$, is a $\mathbb{R}^{d_x} \times [0, T] \mapsto \mathbb{R}^{m_x \times m_x}$ function and $q = \frac{d_x}{m_x} \geq 1$ is an integer. Similarly, for the $d_x$-dimensional drift coefficient we denote $f(\cdot, t) = \begin{bmatrix} f_0(\cdot, t) \\ \vdots \\ f_{q-1}(\cdot, t) \end{bmatrix}$, where $f_i : \mathbb{R}^{d_x} \mapsto \mathbb{R}^{m_x}$.

### 2.2 Numerical scheme

We introduce a discretisation scheme for Eq. (2) that can be run sequentially and recursively over time $t$ and the $d_x$ coordinates of $X(t)$. To this end, let us denote

$$
X(t) = \begin{bmatrix} X_0(t) \\ \vdots \\ X_{q-1}(t) \end{bmatrix}, \quad W(t) = \begin{bmatrix} W_0(t) \\ \vdots \\ W_{q-1}(t) \end{bmatrix} \quad \text{and} \quad V_n = \begin{bmatrix} W_0(nh) - W_0((n-1)h) \\ \vdots \\ W_{q-1}(nh) - W_{q-1}((n-1)h) \end{bmatrix},
$$

where $h > 0$ is a time step parameter, and $X_i(t)$ and $W_i(t)$ ($i = 0, \ldots, q-1$) are $m_x$-dimensional vectors (note that $qm_x = d_x$). Vector $V_n$ is Gaussian-distributed, with mean $\mathbb{E}[V_n] = 0$ and covariance matrix $\mathbb{E}[V_n V_n^\top] = hI_{d_x}$, where $I_{d_x}$ the $d_x \times d_x$ identity matrix. This is denoted as $V_n \sim \mathcal{N}(0, hI_{d_x})$. We may also decompose $V_n$ into a collection of $m_x$-dimensional r.v.’s $V_{i,n} = W_i(nh) - W_i((n-1)h) \sim \mathcal{N}(0, hI_{m_x})$, for $i = 0, \ldots, q-1$.

We construct a discrete-time approximation of the process $X(t)$, $0 \leq t \leq T$, over the grid $t_n = nh$, where $h = \frac{T}{N}$ and $n = 0, 1, \ldots, N$, using the modified Euler-Maruyama scheme outlined in Algorithm 1. This scheme yields a sequence $X_n = \begin{bmatrix} X_{0,n} \\ \vdots \\ X_{q-1,n} \end{bmatrix}$, where $X_n$ is an estimate of $X(t_n)$ and, correspondingly, $X_{i,n}$ is an estimate of $X_i(t_n)$. We refer to this procedure as a space-sequential method because it operates sequentially not only over time $n$ but also (and crucially) over the indices $i = 0, \ldots, q-1$, which we interpret as spatial coordinates for convenience. This feature enables the design of efficient particle filtering schemes over space and time, in the vein of 5, which are discussed in Section 3.

The other key element of the proposed scheme is the intermediate computation of the explicit Euler estimate $\hat{X}_n$ and the use of the auxiliary $d_x \times 1$ vectors

$$
\hat{X}_n[i] = \begin{bmatrix} X_{0:i-1,n} \\ \hat{X}_{i,q-1,n} \end{bmatrix}, \quad i = 0, \ldots, d_x - 1,
$$

to generate $X_n$. Because of this feature, we also classify this scheme as semi-implicit. It is straightforward to design similar space-sequential, semi-implicit methods based on more sophisticated schemes (e.g., Runge-Kutta 11 or Taylor 14 schemes) by replacing both 2a) and 2b) by adequate update rules.
Algorithm 1 Space-sequential Euler-Maruyama scheme for multidimensional SDEs.

1. **Initialisation:** let $X_0 = X(0)$ and denote $X_0 = \begin{bmatrix} X_{0,0} \\ \vdots \\ X_{q-1,0} \end{bmatrix}$.

2. **Sequential step:** for $n = 1, \ldots, N$:
   a) Compute an auxiliary estimate $\hat{X}_n \in \mathbb{R}^{d_x}$ via the Euler step
      \[ \hat{X}_n = X_{n-1} + hf_{n-1}(X_{n-1}), \]
      where $f_n(\cdot) := f(\cdot, t_n)$.
   b) For $i = 0, \ldots, q - 1$, compute
      \[ X_n[i] = \begin{bmatrix} X_{0:i-1,n} \\ \hat{X}_{i+1,n} \end{bmatrix}, \]
      \[ X_{i,n} = X_{i,n-1} + hf_{i,n}(X_n[i]) + \sqrt{h}s_{i,n-1}(X_{n-1})V_n, \]
      where $s_{i,n}(\cdot) := s_{i}(\cdot, t_n)$ and $f_{i,n}(\cdot) := f_i(\cdot, t_n)$.

2.3 **Weak convergence**

We are interested in the characterisation of the r.v. $X(t_n)$ in terms of its marginal probability law, which we denote as $\mu_n$, i.e.,
\[ \mu_n(A) := \mathbb{P}(X(t_n) \in A), \]
where $A$ is a Borel subset of $\mathbb{R}^{d_x}$. We expect the space-sequential scheme in Algorithm 1 with time step $h = \frac{T}{N} > 0$, to yield a sequence $X_n$ such that, if we denote
\[ \mu_n^h(A) := \mathbb{P}(X_n \in A), \]
then we can guarantee that $\lim_{h \to 0} \mu_n^h = \mu_n$ in a suitable quantitative sense. As a consequence, we adopt the weak convergence criterion of [14] for our analysis of Algorithm 1 and use stochastic Taylor expansions in order to derive appropriate discrete-time approximations. As with strong approximations, the desired order of convergence also determines the truncation that must be used. However, this will be different from the truncation required for the strong convergence of the same order, in general involving fewer terms [14].

Theorem 1 below guarantees that numerical scheme described by Algorithm 1 converges with weak order 1 as the time step $h \to 0$. For the specification of assumptions (i) and (ii) in the statement of the Theorem 1, we note that, for any $x \in \mathbb{R}^{d_x}$ and $t \in [0, T]$,
- the drift coefficient $f(x, t)$ is a $d_x$-dimensional vector and $f(\cdot, t)$ denotes its $r$-th element,
- and, similarly, the diffusion coefficient $s(x, t)$ is a $d_x \times d_x$ matrix and we write $s^{r,c}(x, t)$ for the entry in the $r$-th row and $c$-th column.
Theorem 1. Choose a constant $B < \infty$ and a test function $\phi \in C^1_B(\mathbb{R}^{d_x})$. If 
(i) $f^i(.,t)$ and $s_{i,j}(.,t)$ belong to $C^1_B(\mathbb{R}^{d_x})$ for every $t \in [0,T]$ and $(i,j) \in \{0,\ldots,d_x - 1\}^2$, and 
(ii) $\mathbb{E}[\phi(X(T))] < \infty$ for every $\phi \in C^1_B(\mathbb{R}^{d_x})$,
then
\[ |\mathbb{E}[\phi(X(T))] - \mathbb{E}[\phi(X_N)]| \leq C h \]
where the constant $C = C(T, d_x) < \infty$ is independent of the time step $h = \frac{T}{N}$ and the initial value $X(0) = X_0$.

Remark 1. At time $t_n = nh$ we have $\mathbb{E}[\phi(X_n)] = \mu_n(\phi)$ and, similarly, $\mathbb{E}[\phi(X(t_n))] = \mu_n(\phi)$. Theorem 1 guarantees that $\sup_{n \leq N} |\mu_n(\phi) - \mu_n(\phi)| = \mathcal{O}(h)$ for any test function $\phi \in C^1_B(\mathbb{R}^{d_x})$.

Before we proceed with the proof of Theorem 1 let us introduce the continuous-time approximation

\[ \bar{X}(t) = \begin{bmatrix} \bar{X}_0(t) \\ \vdots \\ \bar{X}_{q-1}(t) \end{bmatrix} \]

constructed as

\[ \bar{X}_i(t) = X_i(0) + \int_0^t f_i \left( \bar{X}^{i,h}(p), p \right) \, dp + \int_0^t s_i \left( \bar{X}, \tau_p \right) \, dW_i(p). \]

where $\tau_p = \left\lceil \frac{t}{h} \right\rceil h$ and the $d_x$-dimensional vectors $\bar{X}^{i,h}(p) = \begin{bmatrix} \bar{X}^{i,h}_0(p) \\ \vdots \\ \bar{X}^{i,h}_{q-1}(p) \end{bmatrix}$ are defined as

\[ \bar{X}^{i,h}_j(p) = \begin{cases} \bar{X}_j \left( \left\lceil \frac{p}{h} \right\rceil h \right), & \text{if } 0 \leq j < i, \\ \bar{X}_j \left( \left\lceil \frac{p}{h} \right\rceil h \right) + h f_i \left( \bar{X}_j \left( \left\lceil \frac{p}{h} \right\rceil h \right) \right), & \text{if } i \leq j < q, \end{cases} \]

for $i = 0, ..., q - 1$. Note that the signals $\bar{X}(t)$ and $\bar{X}^{i,h}(t)$ are estimates of $X(t)$ and, hence, they are $d_x$-dimensional. Their components $\bar{X}_j(t)$ and $\bar{X}^{i,h}_j(t)$, with $j = 0, \ldots, q - 1$, are $m_x \times 1$ vectors. An induction argument shows that $\bar{X}(nh) = X_n$, for $n = 0, 1, \ldots, N$.

Additionally, let us introduce the family of $\sigma$-algebras \( \{ F_{n,i} : n = 0, \ldots, N; \ i = 0, \ldots, d_x - 1 \} \) such that

- \( F_{n,i} \subseteq F_{n,j} \) whenever \( n \leq m \),
- \( F_{n,i} \subseteq F_{n,j} \) whenever \( i \leq j \), and
- the first $i$ entries of $X(nh)$, denoted $\bar{X}^{0:i-1}(nh)$, are measurable w.r.t. $F_{n,i-1}$.

The $\sigma$-algebra $F_{n,i}$ can be generated by the initial condition $\bar{X}(0)$, the $d_x$-dimensional Wiener process $W(t)$ for $0 \leq t \leq (n-1)h$ and $W_{0:i-1}(t)$ for $(n-1)h < t \leq nh$.

Proof of Theorem 1. The argument below is a variation of the procedure in the proof of Theorem 14.1.5 in [14]. Let $L^0$ denote the operator

\[ L^0 = \partial_t + \sum_{i=0}^{d_x-1} f^i \partial_{x_i} + \frac{1}{2} \sum_{i=0}^{d_x-1} \sum_{j=0}^{d_x-1} s_{i,j} \partial_{x_i} \partial_{x_j}, \]
where \( s^{i, j} \) and \( s^{-j} \) are the \( i \)-th row and \( j \)-th column, respectively, of the diffusion coefficient \( s \), and for some map \( v : \mathbb{R}^{d_x} \times [0, T] \to \mathbb{R} \), we denote \( \partial_t v(x, t) = \frac{\partial v(x, t)}{\partial t} \), \( \partial_x v(x, t) = \frac{\partial v(x, t)}{\partial x} \) and \( \partial_{x_i, x_j} v(x, t) = \frac{\partial^2 v(x, t)}{\partial x_i \partial x_j} \). From Theorem 4.8.6 in [14], we know that the functional
\[
v(x, t) := \mathbb{E}[\phi(X(T))|X(t) = x], \quad t \in [0, T], \quad x \in \mathbb{R}^{d_x},
\]
is a solution of the final value problem
\[
L^0 v = 0, \quad \text{with} \quad v(T, x) = \phi(x).
\]
Moreover, \( v(x, t) \) is four times continuously differentiable in the argument \( x = [x_0, \ldots, x_{d_x-1}]^T \) and these partial derivatives are uniformly bounded, hence \( v(\cdot, t) \in C^4_B(\mathbb{R}^{d_x}) \).

From (6) and Itô’s formula we readily obtain that \( \mathbb{E}[v(X(t), t)] = \mathbb{E}[v(X(0), 0)] \) for any \( t \in [0, T] \) and, since \( X(0) = X_0 \), we have the identity
\[
\phi(X(T)) = \mathbb{E}[v(X(T), T)] = \mathbb{E}[v(X_0, 0)].
\]
From Eq. (5), it also follows that \( \phi(X_N) = \mathbb{E}[v(X_N, T)] \). Since \( X_N = \bar{X}(T) \) and \( \bar{X}(0) = X_0 \), we finally obtain the relationship
\[
|\mathbb{E}[\phi(X(T))] - \mathbb{E}[\phi(X_N)]| = |\mathbb{E}[v(X(0), 0)] - \mathbb{E}[v(\bar{X}(T), T)]|
= |\mathbb{E}[v(\bar{X}(0), 0)] - \mathbb{E}[v(\bar{X}(T), T)]|
= |\mathbb{E}[v(\bar{X}(T), T) - v(X_0, 0)]|.
\]
Hence, we have rewritten the error \( |\mathbb{E}[\phi(X(T))] - \mathbb{E}[\phi(X_N)]| \) in terms of the signal \( \bar{X}(t) \) alone, which has been constructed to satisfy \( \bar{X}(nh) = X_n \) for \( n = 0, \ldots, N \), and \( Nh = T \).

Using Itô’s formula for the process \( \bar{X}(t) \), we readily obtain
\[
v(\bar{X}(T), T) - v(X_0, 0) = \int_0^T \left[ \partial_t v(\bar{X}(u), u) + \sum_{i=0}^{d_x-1} f^i(\bar{X}(u), u) \partial_{x_i} v(\bar{X}(u), u) \\
+ \frac{1}{2} \sum_{i=0}^{d_x-1} \sum_{j=0}^{d_x-1} s^{i, j}(\bar{X}(u), u) \partial_{x_i} \partial_{x_j} v(\bar{X}(u), u) \right] du \\
+ \int_0^T \sum_{i=0}^{d_x-1} s^{i, i}(\bar{X}(u), u) \partial_{x_i} v(\bar{X}(u), u) dW(u)
\]
while, using (6),
\[
0 = \int_0^T L^0 v(\bar{X}(u), u) du = \int_0^T \left[ \partial_t v(\bar{X}(u), u) + \sum_{i=0}^{d_x-1} f^i(\bar{X}(u), u) \partial_{x_i} v(\bar{X}(u), u) \\
+ \frac{1}{2} \sum_{i=0}^{d_x-1} \sum_{j=0}^{d_x-1} s^{i, j}(\bar{X}(u), u) \partial_{x_i} \partial_{x_j} v(\bar{X}(u), u) \right] du.
\]
Combining Eqs. (8) and (9) and taking expectations yields

\[
\mathbb{E} \left[ v(X(T), T) - v(X_0, 0) \right] = \\
\mathbb{E} \left\{ \int_0^T \sum_{i=0}^{d_x-1} \left[ f^i \left( \tilde{X}^{[i/m_x], h}, u \right) - f^i \left( \tilde{X}, u \right) \right] \partial_{x_i} v \left( \tilde{X}(u), u \right) \, du \right\} \\
+ \frac{1}{2} \mathbb{E} \left\{ \int_0^T \sum_{i=0}^{d_x-1} \sum_{j=0}^{d_x-1} \left[ s^{i,j} \left( \tilde{X}, \tau_u \right) s^{i,j} \left( \tilde{X}, \tau_u \right) - s^{i,j} \left( X, \tau_u \right) s^{i,j} \left( X, \tau_u \right) \right] \partial_{x_i} \partial_{x_j} v \left( \tilde{X}(u), u \right) \, du \right\},
\]

as \( \mathbb{E} \left[ \int_0^T \sum_{i=0}^{d_x-1} s^{i,i} \left( \tilde{X}, \tau_u \right) \partial_{x_i} v \left( \tilde{X}(u), u \right) \, dW(u) \right] = 0. \)

Let \( f \) denote either \( f = f^i \partial_{x_i} v \) or \( f = \partial_{x_i} v \), for \( i = 0, \ldots, d_x - 1 \). Using the uniform bound \( B < \infty \) on the derivatives of \( f^i \) and \( v \) and the fact that \( \| \tilde{X}(u) - \tilde{X}^{[i/m_x], h}(u) \| = \mathcal{O}(h) \) it is straightforward to prove that

\[
\mathbb{E} \left[ f \left( \tilde{X}^{[i/m_x], h}(u), u \right) - f \left( \tilde{X}(u), u \right) \mid \mathcal{F}_{\tau_u, i} \right] \leq c_{1,f} h,
\]

where \( c_{1,f} < \infty \) is a constant independent of \( h \), \( i \) and \( X_0 \). We denote \( c_f = c_{1,f} \vee c_{2,f} < \infty \).

Combining (10) with the inequalities (11) and (12) we can obtain a suitable upper bound for \( \mathbb{E} \left[ v(X(T), T) - v(X_0, 0) \right] \). To be specific, for the first term on the right-hand side of (10) we obtain

\[
\left\| \mathbb{E} \left\{ \int_0^T \sum_{i=0}^{d_x-1} \left[ f^i \left( \tilde{X}^{[i/m_x], h}, u \right) - f^i \left( \tilde{X}, u \right) \right] \partial_{x_i} v \left( \tilde{X}(u), u \right) \, du \right\} \right\| \\
\leq \mathbb{E} \left\{ \int_0^T \sum_{i=0}^{d_x-1} \left\| f^i \left( \tilde{X}^{[i/m_x], h}, u \right) \partial_{x_i} v \left( \tilde{X}^{[i/m_x], h}(u), u \right) - f^i \left( \tilde{X}, u \right) \partial_{x_i} v \left( \tilde{X}(u), u \right) \right\| \, du \right\} \\
+ \mathbb{E} \left\{ \int_0^T \sum_{i=0}^{d_x-1} \left\| f^i \left( \tilde{X}^{[i/m_x], h}, u \right) \partial_{x_i} v \left( \tilde{X}(u), u \right) - \partial_{x_i} v \left( \tilde{X}^{[i/m_x], h}, u \right) \right\| \, du \right\} \\
\leq T d_x c_f h + \mathbb{E} \left\{ \int_0^T \sum_{i=0}^{d_x-1} \left\| f^i \left( \tilde{X}^{[i/m_x], h}, u \right) \right\| \left\| \partial_{x_i} v \left( \tilde{X}(u), u \right) - \partial_{x_i} v \left( \tilde{X}^{[i/m_x], h}, u \right) \right\| \, du \right\} \\
\leq T d_x c_f h + c_f h \mathbb{E} \left\{ \int_0^T \sum_{i=0}^{d_x-1} \left\| f^i \left( \tilde{X}^{[i/m_x], h}, u \right) \right\| \, du \right\} \\
\leq T d_x c_f \left( 1 + \sup_{0 \leq s \leq T} \mathbb{E} \left[ \left\| f^i \left( \tilde{X}^{[i/m_x], h}, u \right) \right\| \right] \right) h \leq C' h,
\]

where the second inequality follows from (11), the third inequality follows from (12) and \( C' = \mathcal{O}(T d_x) \) is independent of \( h \) and \( X_0 \).
By a similar argument, for the second term on the right-hand side of (10) there exists a constant $C'' = O(Td_h^2) < \infty$, independent of $h$ and $X_0$, such that

\[
\frac{1}{2} \mathbb{E} \left\{ \int_0^T \sum_{i=0}^{d_x-1} \sum_{j=0}^{d_x-1} \left[ s^{i,j}(\bar{X}, \tau_u) s^{i,j}(\bar{X}, \tau_u) - s^{i,j}(\bar{X}, \tau_u) s^{i,j}(\bar{X}, \tau_u) \right] \partial_{x_i,x_j} v(\bar{X}(u),u) \, du \right\} \leq C''h. \tag{13}
\]

If we set $C = C' + C'' < \infty$, then

\[
|\phi(X(T)) - \phi(X_N)| = \mathbb{E} \left[ |v(\bar{X}(T), T) - v(\bar{X}(0), 0)| \right] \leq Ch, \tag{14}
\]

where $C = O(Td_h^2)$ is independent of $h$ and $X_0$. \hfill \square

**Remark 2.** If the drift and diffusion functions, $f(\cdot, t)$ and $s(\cdot, t)$, respectively, are not sufficiently smooth then the scheme can still be shown to converge, albeit with a lesser order.

### 2.4 Continuous- and discrete-time Markov kernels

Both the continuous-time Itô process $X(t)$ and its discrete-time approximation $X_n$ are Markov, i.e., for any Borel set $A \subset \mathbb{R}^{d_x}$, any time $t \in (0, T]$ and any function $x : \mathbb{R} \to \mathbb{R}^{d_x}$, we have

\[
\mathbb{P}(X(t) \in A | X(\tau) = x(\tau), \text{ for } \tau \in [0, t-h]) = \mathbb{P}(X(t) \in A | X(t-h) = x(t-h))
\]

for the process $X(t)$, while

\[
\mathbb{P}(X_n \in A | X_k = x(kh), \text{ for } k = 0, \ldots, n-1) = \mathbb{P}(X_n \in A | X_{n-1} = x((n-1)h))
\]

for the random sequence $X_n$. Therefore, the dynamics of the process $X(t)$ over the time grid $t_n = nh, n = 0, \ldots, N$, can be described by the Markov kernels

\[
K^*_n(x,A) := \mathbb{P}(X(t_n) \in A | X(t_{n-1}) = x) \tag{15}
\]

which we expect to be close to the kernels

\[
K^h_n(x,A) := \mathbb{P}(X_n \in A | X_{n-1} = x) \tag{16}
\]

that generate the sequence $X_n$, at least under the assumptions of Theorem [1].

Let $\mu$ and $\pi$ be two probability measures on $\mathcal{P}(\mathbb{R}^d)$. If we denote

\[
D_l(\mu, \pi) := \sup_{\phi \in C_B^1(\mathbb{R}^d) : \|\phi\|_{\infty} \leq 1} |\mu(\phi) - \pi(\phi)|, \tag{17}
\]

then it is easy to show that $D_l(\cdot, \cdot)$ is a proper metric on the space of probability measures $\mathcal{P}(\mathbb{R}^d)$. For $l \geq 4$ we can state the lemma below, which is a straightforward consequence of Theorem [1].

**Lemma 1.** Assume that $f^i, s^{i,j} \in C^l_B(\mathbb{R}^d)$ for some $l \geq 4$ and $(i,j) \in \{0, \ldots, d_x-1\}^2$. There is a finite constant $C < \infty$, independent of $h = \frac{1}{n}$ and $x$, such that

\[
\sup_x D_l \left( K^*_n(x, \cdot), K^h_n(x, \cdot) \right) \leq Ch \tag{18}
\]
for every $n = 1, \ldots, N$. In particular, $\lim_{h \to 0} D_t(K_n^h(x, \cdot), K_n^h(x, \cdot)) = 0$ uniformly on $x \in \mathbb{R}^{d_x}$ and $n = 1, \ldots, N$.

**Remark 3.** We write $\lim_{h \to 0} K_h^n(x) = K_n^\ast(x)$ to denote that $\lim_{h \to 0} D_t(K_n(x, \cdot), K_h^n(x, \cdot)) = 0$ for every $n = 1, \ldots, N$. Intuitively, the kernels $K_h^n$ that generate the random sequence $X_n$ converge to the kernels $K_n^\ast$ that generate the true signal process $X(t)$ on the time grid $t_n = nh$. This argument holds for any weakly-convergent numerical scheme, not just the space-sequential scheme outlined in Algorithm 1.

### 3 State-space models

#### 3.1 Observations over a time grid

Assume that the signal $X(t)$ can be partially observed at time instants $t_n = nh$, $n = 1, \ldots, N$, $N = \frac{T}{h}$. To be specific, the observations are $d_y$-dimensional r.v.’s of the form

$$Y_n = b(X(t_n), t_n) + U_n,$$

where $b : \mathbb{R}^{d_x} \times [0, T] \mapsto \mathbb{R}^{d_y}$ is an observation function and $U_n$, $n = 1, \ldots, N$, is a sequence of independent, zero-mean, $d_y$-dimensional real r.v.’s. We assume that $U_n$ has a pdf $g_n : \mathbb{R}^{d_y} \mapsto (0, \infty)$ and we construct the conditional pdf of the observation $Y_n$ conditional on the state $X(t_n) = x$ as

$$\bar{g}_n(y|x) = g_n(y - b_n(x)),$$

where $b_n(x) := b(x, t_n)$. Typically, this pdf cannot be computed exactly, but only up to a proportionality constant. Hence, we introduce the non-normalised density

$$g_n(y|x) := \bar{c}_n \bar{g}_n(y|x), \quad (19)$$

where $0 < \bar{c}_n < \infty$ is an arbitrary constant (possibly unknown) independent of $x$. If we assume that the sequence of observations $\{Y_n = y_n, \ n = 1, \ldots, N\}$ is arbitrary but fixed, then we may interpret $g_n(y_n|\cdot)$ in (19) as a likelihood function $\mathbb{R}^{d_y} \mapsto (0, \infty)$ that takes $x$ as an argument. Hence, we denote

$$g_n(x) := \bar{g}_n(y_n|x) = \bar{c}_n \bar{g}_n(y_n - b_n(x)), \quad n = 1, \ldots, N. \quad (20)$$

**Remark 4.** In typical applications, data are collected sparsely over time. If at time instant $t_n = nh$ there is no observation available, then $b(\cdot, t_n) = 0$ and, as a consequence, $Y_n = U_n$ and $g_n(x) \propto 1$ for every $x \in \mathbb{R}^{d_x}$.

#### 3.2 Exact and approximate state-space Markov models

We refer to the discrete-time, random dynamical system described by

- the initial condition $X_0^\ast = X(0)$, with a priori law $\pi_0$,
the Markov sequence $X^*_n$ generated by the kernels $K^s_n$, $n = 1, \ldots, N$,

- a sequence of likelihood functions $g_n$, $n = 1, \ldots, N$, described by Eq. (20),

as the exact state space model (SSM) for the continuous-time signal $X(t)$. We denote it by the triple $(\pi_0, K^s, g)$, where $K^s := \{K^s_1, \ldots, K^s_N\}$ and $g := \{g_1, \ldots, g_N\}$.

The term ‘exact’ points out that the a priori law of $X^*_n$ is

$$
\eta_n(\text{dx}_n) := \int \cdots \int K^s_n(\text{x}_{n-1}, \text{dx}_n) \prod_{i=1}^{n-1} K^s_i(\text{x}_{i-1}, \text{dx}_i) \pi_0(\text{dx}_0),
$$

which coincides with the a priori law $\mu_n$ of $X(t_n)$ by the precise definition of the kernel $K^s_n$ in Eq. (15). Therefore, $\eta_n = \mu_n$ and, for any integrable test function $\phi : \mathbb{R}^d \to \mathbb{R}$,

$$
\mathbb{E} [\phi(X(t_n))] = \mu_n(\phi) = \eta_n(\phi) = \mathbb{E} [\phi(X_n^*)].
$$

We can construct an approximate SSM for the space-time discretised sequence $X_n$. In particular, we choose

- the same initial condition $X_0 = X(0)$ with law $\pi_0$,
- the Markov sequence $X_n$ generated by the kernels $K^h_n$ (or, equivalently, by Algorithm 1), and
- the same likelihood functions $g_n$, $n = 1, \ldots, N$, described by Eq. (20) as in the exact SSM.

We describe this model by the triple $(\pi_0, K^h, g)$, where $K^h := \{K^h_1, \ldots, K^h_N\}$. By construction, the a priori law of $X_n$ is $\mu^h_n$ and, from Remark 4, $\lim_{h \to 0} \mu^h_n = \mu_n$, i.e., the prior law of $X_n$ converges to the prior law of $X^*_n$ and $X(t_n)$.

### 3.3 Marginal posterior laws

Let $Y_n = y_n$, $n = 1, \ldots, N$, be the observations that yield the likelihoods $g_n$, $n = 1, \ldots, N$. Let us take the SSM $(\pi_0, K^*, g)$ first. Since $X^*_1$ has marginal law $\pi_0$, the one-step-ahead predictive probability law of $X^*_1$ can be easily obtained from $\pi_0$. In particular,

$$
\xi_1(\text{dx}_1) = \mathbb{P}(X^*_1 \in \text{dx}_1) = \int K^*_1(\text{x}_0, \text{dx}_1) \pi_0(\text{dx}_0)
$$

and we denote $\xi_1 = K^*_1 \pi_0$ for conciseness. Then, given an observation $Y_1 = y_1$ and the resulting likelihood function $g_1$, Bayes’ theorem yields the posterior marginal law

$$
\pi_1(\text{dx}_1) := \mathbb{P}(X^*_1 \in \text{dx}_1|Y_1 = y_1) = \frac{g_1(\text{x}_1) \xi_1(\text{dx}_1)}{\xi_1(g_1)}
$$

and we denote $\pi_1 = g_1 \cdot \xi_1 = g_1 \cdot K^*_1 \pi_0$ for conciseness. By induction, one can construct the posterior laws $\xi_n$ and $\pi_n$ recursively, as

$$
\xi_n = K^*_n \pi_{n-1} \quad \text{and} \quad \pi_n = g_n \cdot \xi_n,
$$

for each $n = 1, \ldots, N$. The law $\pi_n$ characterises the so-called filtering probability distribution of $X^*_n$ given the observations $Y_{1:n} = y_{1:n}$.
Remark 5. Since the prior probability laws $\eta_1$ and $\mu_1$ of $X_1^*$ and $X(t_1)$, respectively, coincide (indeed, $\eta_1 = \mu_1 = \xi_1$), then it is apparent that $\pi_1$ is also the posterior law of $X(t_1)$ conditional on the observation $Y_1 = y_1$. A simple induction argument shows that the laws $\xi_n$ and $\pi_n$ also characterise the probability distribution of $X(t_n)$ conditional on $Y_{1:n-1} = y_{1:n-1}$ and $Y_{1:n} = y_{1:n}$, respectively.

By the same argument, one can construct the sequence of posterior laws $\xi_n^h$ and $\pi_n^h$ for the SSM $(\pi_0, \Omega^h, \mathcal{G})$. Indeed, at discrete-time $n = 1$ we obtain $\xi_1^h = K_1^h \pi_0$ and $\pi_1^h = g_1 \cdot \xi_1^h$ while, for $n = 2, \ldots, N$,

$$\xi_n^h = K_n^h \pi_{n-1}^h \quad \text{and} \quad \pi_n^h = g_n \cdot \xi_n^h = g_n \cdot K_n^h \pi_{n-1}^h.$$ 

Following an argument similar to [10, Lemma 2.4] it is possible to prove that $\xi_n^h \xrightarrow{h \to \infty} \xi_n$ and $\pi_n^h \xrightarrow{h \to \infty} \pi_n$ under suitable regularity assumptions. This is made precise by Theorem 2 below.

Theorem 2. Let $Y_{1:N} = y_{1:N}$ be an arbitrary but fixed sequence of observations and assume that, for some $l \geq 4$,

(i) $g_n > 0$, $\|g_n\|_\infty \leq 1$ and $g_n \in C^l_B(\mathbb{R}^d)$ for every $n = 1, \ldots, N$;

(ii) if $\phi \in C^l_B(\mathbb{R}^d)$ then $\phi_n \in C^l_B(\mathbb{R}^d)$, where $\phi_n(x) = \int \phi(x') K_n(x, dx')$.

Then, there are finite constants $\{C_n, C_n' : n = 1, \ldots, N\}$ such that

$$D_l(\xi_n, \xi_n^h) \leq C_n h \quad \text{and} \quad D_l(\pi_n, \pi_n^h) \leq C_n h,$$

for $n = 1, \ldots, N$.

Proof. We follow an induction argument. By construction, the exact and approximate SSMs share the same prior law $\pi_0$, hence $\pi_0 = \pi_0^h$. Let us now assume that

$$D_l(\pi_{n-1}, \pi_{n-1}^h) \leq C_{n-1} h$$

for some $1 \leq n < N$ and let $\phi \in C^l_B(\mathbb{R}^d)$ be a test function such that $\|\phi\|_\infty \leq 1$. We readily see that

$$|\xi_n^h(\phi) - \pi_n(\phi)| = \left| K_n^h \pi_{n-1}^h(\phi) - K_n^h \pi_{n-1}^h(\phi) \right| = \left| \pi_{n-1}^h(\phi_n^h) - \pi_{n-1}(\phi_n) \right|,$$

where

$$\phi_n = \int \phi(x_n) K_n(x_{n-1}, x_n) dx_n \quad \text{and} \quad \phi_n^h = \int \phi(x_n) K_n^h(x_{n-1}, x_n) dx_n,$$

hence a triangle inequality yields

$$|\pi_{n-1}^h(\phi_n^h) - \pi_{n-1}(\phi_n)| \leq |\pi_{n-1}^h(\phi_n^h) - \pi_{n-1}^h(\phi_n)| + |\pi_{n-1}^h(\phi_n) - \pi_{n-1}(\phi_n)|.$$ 

We have $\phi_n \in C^l_B(\mathbb{R}^d)$ because of assumption (ii) in the statement of Theorem 2 and it is apparent that $\|\phi\|_\infty \leq 1$ implies $\|\phi_n\|_\infty \leq 1$. Then, we can apply the induction hypothesis (23) to obtain

$$\sup_{\|\phi\|_\infty \leq 1} \left| \pi_{n-1}^h(\phi_n^h) - \pi_{n-1}(\phi_n) \right| = C_{n-1} h.$$
For the first term on the right-hand side of (25), we note that Lemma 1 implies

\[
\sup_{\|\phi\|_\infty \leq 1, x \in \mathbb{R}^d} |\tilde{\phi}_n^h(x) - \tilde{\phi}_n(x)| \leq Ch
\]

hence

\[
\sup_{\|\phi\|_\infty \leq 1} |\pi_{n-1}^h(\tilde{\phi}_n^h) - \pi_{n-1}(\tilde{\phi}_n)| = \sup_{\|\phi\|_\infty \leq 1} |\pi_{n-1}^h(\tilde{\phi}_n^h) - \tilde{\phi}_n)| \leq Ch. \tag{27}
\]

Taking (27), (26) and (25) together yields

\[
D_l(\xi_n^h, \xi_n) \leq \tilde{C}_n h, \tag{28}
\]

where \( \tilde{C}_n = C_{n-1} + C < \infty \).

Next, we write the difference \( \pi_n^h(\phi) - \pi_n(\phi) \) in terms of \( \xi_n^h \) and \( \xi_n \) as

\[
\pi_n^h(\phi) - \pi_n(\phi) = \frac{\xi_n^h(g_n(\phi))}{\xi_n(g_n)} - \frac{\xi_n(g_n(\phi))}{\xi_n(g_n)} \pm \frac{\xi_n^h(g_n(\phi))}{\xi_n(g_n)} - \frac{\xi_n^h(g_n(\phi))}{\xi_n(g_n)} + \frac{\xi_n^h(g_n(\phi))}{\xi_n(g_n)} - \frac{\xi_n(g_n(\phi))}{\xi_n(g_n)},
\]

which readily yields the bound

\[
|\pi_n^h(\phi) - \pi_n(\phi)| \leq \frac{|\xi_n^h(g_n(\phi)) - \xi_n(g_n(\phi))|}{\xi_n(g_n)} + \|\phi\|_\infty |\xi_n(g_n) - \xi_n^h(g_n)|. \tag{29}
\]

The difference \( |\xi_n^h(g_n(\phi)) - \xi_n(g_n(\phi))| \) can be upper bounded as

\[
|\xi_n^h(g_n(\phi)) - \xi_n(g_n(\phi))| \leq \|\phi\|_\infty |\xi_n(g_n) - \xi_n^h(g_n)| \tag{30}
\]

and, since \( g_n \in C^1_b(\mathbb{R}^d) \) and \( \|g_n\|_\infty \leq 1 \), the inequality (28) implies that

\[
|\xi_n(g_n) - \xi_n^h(g_n)| \leq \tilde{C}_n h. \tag{31}
\]

Combining (31), (30) and (29) we obtain the bound

\[
|\pi_n^h(\phi) - \pi_n(\phi)| \leq \frac{2\|\phi\|_\infty \tilde{C}_n h}{\xi_n(g_n)} \tag{32}
\]

which holds for all \( \phi \in C^1_b(\mathbb{R}^d) \). Since \( g_n > 0 \), the inequality (32) yields

\[
D_l(\pi_n^h, \pi_n) \leq C_n h,
\]

where \( C_n = \frac{2\|\phi\|_\infty \tilde{C}_n h}{\xi_n(g_n)} < \infty \).

\[\square\]

Remark 6. The argument of Theorem 2 can be applied to any approximate SSM \((\pi_0, K^h, g)\) where the kernels \( K^h = \{K^h_1(x, dx), \ldots, K^h_N(x, dx)\} \) result from a weakly-convergent numerical scheme.
4 Particle filtering

The purpose of the semi-implicit, space-sequential numerical scheme of Algorithm 1 is to enable the design of efficient particle filtering algorithms for the approximation of the marginal posterior laws $\xi^h_n$ and $\pi^h_n$ in applications where the dimension $d_x$ of the state space is large. Since $\xi^h_n \to \xi_n$ and $\pi^h_n \to \pi_n$, as $h \to 0$, such algorithms also guarantee the efficient estimation of the marginal posterior laws generated by the exact SSM, with continuous-time state $X(t)$.

The sequential procedure for the generation of $X_{0,n}, X_{1,n}, \ldots, X_{q-1,n}$ provided by the scheme in Algorithm 1 enables the direct application of the STPFs described in [5]. However, it is also possible to devise simpler particle filtering algorithms with the potential to perform well in high dimensional problems by running sequentially over the ‘spatial’ index $i = 0, 1, \ldots, q - 1$.

In what follows, we describe three algorithms of this type. The first one is a plain space-sequential bootstrap filter implemented on the approximate SSM $(\pi^0, K^h, g)$, while the second one is a space-sequential PF with an a jittering step [8] aimed at improving the diversity of the particle set. These two methods rely directly on the space-sequential discretisation scheme of Algorithm 1. For the third filter, we assume an explicit Euler-Maruyama discretisation of the SDE but we otherwise mimic the steps of the space-sequential PF with jittering. This yields a similar algorithm with reduced computational cost, as we avoid the computation of the auxiliary states $\hat{X}_{i,n}$ in step 2.a) of Algorithm 1. Let us remark that the Euler-Maruyama method has weak order 1, hence all the approximation results in Sections 2.4 and 3 still hold.

We describe the three algorithms, for simplicity, for models where the likelihood function can be factored in agreement with the space-sequential decomposition of the state $X_n$ as $X_{0,n}, \ldots, X_{q-1,n}$, but then we explain how to remove this assumption in Section 4.2. To complete the presentation of the particle filtering algorithms we prove their consistency in Section 4.3. We also establish an explicit relationship between the number of particles of the PF, the time grid $t_n = nh, n = 0, \ldots, N$, for the discretisation of the SDE and the approximation error of the marginal posteriors $\xi_n$ and $\pi_n$ of the exact SSM.

4.1 Conditionally independent observations over space

Recall the partition of the state vector, $X_n = \begin{bmatrix} X_{0,n} \\ \vdots \\ X_{q-1,n} \end{bmatrix}$, where $q = \frac{d_x}{m_x}$ and each vector $X_{i,n}$ is $m_x \times 1$. In the same way, denote

$\hat{X}_n = \begin{bmatrix} \hat{X}_{0,n} \\ \vdots \\ \hat{X}_{q-1,n} \end{bmatrix}, \quad \check{X}_n = \begin{bmatrix} \check{X}_{0,n} \\ \vdots \\ \check{X}_{q-1,n} \end{bmatrix}, \quad \text{etc.}$

We initially assume that that the likelihood function $g_n(x)$ in (20) can be factored as

$$g_n(X_n) = \prod_{i=0}^{q-1} g_{i,n}(X_{i,n})$$

(33)
in order to describe the proposed PFs in a simpler way.

### 4.1.1 Vanilla space-sequential particle filter

We may see the numerical scheme of Algorithm 1 as a description of the state equation in a SSM and then devise a simple bootstrap filter that runs on this model, i.e., using the space-sequential scheme to sample each component $X_{i,n}$ of the state and then assigning weights proportional to the factors $g_{i,n}(X_{i,n})$ in Eq. (33). This procedure is outlined in Algorithm 2.

#### Algorithm 2 Vanilla space-sequential PF with $M$ particles and resampling at each spatial step.

- At time $n = 0$, sample $M$ independent and identically distributed (iid) particles from the prior distribution $\pi_0$, denoted $X_{j,0} \sim \pi_0$, $j = 1, \ldots, M$.
- Recursive step: given \(\{X_{n-1}^j\}_{j=1}^M\), generate \(\{X_{n}^j\}_{j=1}^M\) as described below.
  1. Compute auxiliary particles $\hat{X}_{i,n}^j = X_{i,n-1}^j + hf(X_{i,n-1}^j, X_{0:i-1,n}^j) + \sqrt{h}s_i(X_{n-1}^j)V_{i,n}^j$, where $V_{i,n}^j \sim N(0, I_{m_x})$.
  2. For $i = 0, \ldots, q-1$
     - For $j = 1, \ldots, M$,
       - i. generate new $m_x$-dimensional samples $\tilde{X}_{i,n}^j = X_{i,n-1}^j + hf(X_{0:i-1,n}^j, \hat{X}_{i,n}^j) + \sqrt{h}s_i(X_{n-1}^j)V_{i,n}^j$, where $V_{i,n}^j \sim N(0, I_{m_x})$.
       - ii. and compute importance weights $\tilde{w}_{i,n}^j = g_{i,n}(\tilde{X}_{i,n}^j)$, $j = 1, ..., M$.
  3. Normalise the importance weights, $w_{i,n}^j = \frac{\tilde{w}_{i,n}^j}{\sum_{j=1}^M \tilde{w}_{i,n}^j}$, for $j = 1, \ldots, M$.
  4. Resample the weighted set $\{X_{n-1}^j, \hat{X}_{n}^j, X_{0:i-1,n}^j, \tilde{X}_{i,n}^j, w_{i,n}^j\}_{j=1}^M$ with replacement, $m$ times. Denote the resampled set as $\{X_{n-1}^j, \hat{X}_{n}^j, X_{0:i,n}^j\}_{j=1}^M$ and we carry it forward to the next step.

#### Particle approximations.

Assume the observations $Y_{i,n} = y_{i,n}$ are arbitrary but fixed (deterministic) for $0 < n \leq N$ and $0 \leq i < q$. For $0 \leq i \leq q-1$, the point-mass random measures

\[
\xi_{i,n}^M(d\tilde{x}_{0:i,n}) = \frac{1}{M} \sum_{j=1}^M \delta_{(\tilde{X}_{i,n}^j, X_{0:i-1,n}^j)}(d\tilde{x}_{0:i,n}),
\]

\[
\hat{\pi}_{i,n}^M(d\tilde{x}_{0:i,n}) = \sum_{j=1}^M w_{i,n}^j \delta_{(\tilde{X}_{i,n}^j, X_{0:i-1,n}^j)}(d\tilde{x}_{0:i,n}),
\]

17
are Monte Carlo estimates of the deterministic probability laws

\[ \xi^h_{i,n}(dx_{0,i,n}) = \mathbb{P}(X_{0,i,n} \in dx_{0,i,n}|Y_{1,n-1} = y_{1:n-1}, Y_{0,i-1,n} = y_{0,i-1,n}), \]
\[ \pi^h_{i,n}(dx_{0,i,n}) = \mathbb{P}(X_{0,i,n} \in dx_{0,i,n}|Y_{1,n-1} = y_{1:n-1}, Y_{0,i,n} = y_{0,i,n}), \]

respectively. After resampling, the random measure

\[ \pi^M_{i,n}(dx_{0,i,n}) = \frac{1}{M} \sum_{j=1}^{M} \delta_{X_{0,i,n}^j}(dx_{0,i,n}), \]

is also an approximation of \( \pi^h_{i,n}(dx_{0,i,n}) \). Given a real test function \( \phi : \mathbb{R}^{m_{i-1}} \mapsto \mathbb{R} \) it is straightforward to compute integrals w.r.t. \( \xi^M_{i,n}, \pi^M_{i,n} \) or \( \pi^M_{i,n} \), e.g.,

\[ \hat{\pi}^M_{i,n}(\phi) = \sum_{j=1}^{M} w^i_j \phi(X_{0,i,n}^j, X_{i,n}^j) \quad \text{and} \quad \hat{\pi}^M_{i,n}(\phi) = \frac{1}{M} \sum_{j=1}^{M} \phi(X_{0,i,n}^j). \]

These quantities are Monte Carlo estimators of posterior expectations of \( \phi(X_{0,i,n}) \), namely

\[ \xi^M_{i,n}(\phi) = \mathbb{E}[\phi(X_{0,i,n})|Y_{1,n-1} = y_{1:n-1}, Y_{0,i-1,n} = y_{0,i-1,n}], \]
\[ \pi^M_{i,n}(\phi) = \mathbb{E}[\phi(X_{0,i,n})|Y_{1,n-1} = y_{1:n-1}, Y_{0,i,n} = y_{0,i,n}], \]

For \( i = q - 1 \) we have \( \pi^h_{q-1,n} = \pi^h_n \), hence we naturally define \( \pi^M_n = \pi^M_{q-1,n} \) as an estimator of the filter \( \pi^h_n \). In particular,

\[ \pi^M_n(\phi) = \frac{1}{M} \sum_{j=1}^{M} \phi(X_{0,i,n}^j) \approx \pi^h_n(\phi) = \mathbb{E}[\phi(X_n)|Y_{1:n} = y_{1:n}]. \]

**Path degeneracy.** An obvious difficulty with Algorithm 2 is the well-known phenomenon of path degeneracy: because of the successive resampling steps for \( i = 0, 1, 2, \ldots \) the particles \( X_{0,i-1}^j \) and \( X_{i,n}^j \) collapse into a single (or just a few) different values; see, e.g., Fig. 11 in [7] for an illustration of path degeneracy over time for a conventional PF. For high dimensional systems, where \( q = \frac{d\pi}{dx} \) is large, path degeneracy can be expected to become a significant problem for Algorithm 2.

### 4.1.2 Space-sequential jittered particle filter

In order to mitigate the path degeneracy problem we introduce a modified version of Algorithm 2 which includes a jittering step [8] after resampling. To be specific, we construct a jittering kernel of the form

\[ J_{M,\beta}(x', dx) = M^{-\beta} J(x', dx) + (1 - M^{-\beta}) \delta_{x'}(dx), \]

where \( \beta \in [0, 1] \) and \( J(x, dx) \) is a fixed Markov kernel independent of \( M \). Given a probability measure \( \alpha(dx) \), the jittering kernel introduces a bounded perturbation that can be controlled by a suitable choice of \( \beta \). To be specific, it is not hard to see that [8]

\[ \|\alpha(\phi) - J_{M,\beta} \alpha(\phi)\|_p \leq 2\|\phi\|_\infty M^{-\beta} \]

(35)
for any bounded test function $\phi$ and $p \geq 1$. When applied to a point-mass measure $\alpha^M(dx) = \frac{1}{M} \sum_{j=1}^{M} \delta_{x^j}(dx)$, the kernel $J_{M,\beta}$ leaves each particle unchanged with probability $1 - M^{-\beta}$ and it modifies the particle with probability $M^{-\beta}$. Algorithmically, jittering $\alpha^M$ using $J_{M,\beta}$ can be implemented as follows: for $j = 1,\ldots,M$

(i) draw $\tilde{x}^j \sim J(x^j, dx)$, and then
(ii) set $\bar{x}^j = \{ x^j, \text{ with probability } 1 - M^{-\beta}, \tilde{x}^j, \text{ with probability } M^{-\beta} \}$.

The jittered measure is $\bar{\alpha}^M(dx) = \frac{1}{M} \sum_{j=1}^{M} \delta_{\bar{x}^j}(dx)$.

The space-sequential PF with jittering is outlined in Algorithm 3. The use of a specific jittering kernel and its effect on the performance of the space-sequential PF is illustrated numerically in Section 5.

**Algorithm 3** Space-sequential PF with jittering.

- At time $n = 0$, sample $M$ independent and identically distributed (iid) particles from the prior distribution $\pi_0$, denoted $X^j_0 \sim \pi_0, j = 1,\ldots,M$.

- Recursive step: given $\{X^j_{n-1}\}_{j=1}^{M}$, generate $\{X^j_n\}_{j=1}^{M}$ as described below.

  1. Compute auxiliary particles $\hat{X}^j_n = X^j_{n-1} + hf(X^j_{n-1}),$ for $j = 1,\ldots,M$.

  2. For $i = 0,\ldots,q - 1$

     (a) For $j = 1,\ldots,M$,

        i. generate new $m_x$-dimensional samples

        $$\tilde{X}^j_{i,n} = X^j_{i-1,n} + hf(X^j_{i-1,n}, \hat{X}^j_{i-1,n}) + \sqrt{h \delta_i} V^j_{i,n},$$

        where $V^j_{i,n} \sim N(0, I_{m_x})$,

        ii. and compute importance weights

        $$\tilde{w}^j_{i,n} = g_i(n, \tilde{X}^j_{i,n}), \quad j = 1,\ldots,M.$$  

     (b) Normalise the importance weights, $w^j_{i,n} = \frac{\tilde{w}^j_{i,n}}{\sum_{j=1}^{M} \tilde{w}^j_{i,n}},$ for $j = 1,\ldots,M$.

     (c) Resample the weighted set $\{X^j_{n-1}, X^j_{i-1,n}, \hat{X}^j_{i,n}\}$, $w^j_{i,n}$ with replacement, $m$ times. Denote the resampled set as $\{\tilde{X}^j_{n-1}, \tilde{X}^j_{n-1}, \tilde{X}^j_{0,i,n}\}_{j=1}^{M}.$

     (d) Jitter the resampled particles using a kernel $J_{M,\beta}$ as defined in Eq. (34). Denote the jittered particles as $\{X^j_{n-1}, \tilde{X}^j_{n-1}, X^j_{0,i,n}\}_{j=1}^{M}$ and carry them forward to the next step.

4.1.3 Explicit space-sequential jittered particle filter

The space-sequential PF with jittering involves the computation of auxiliary particles $\tilde{X}^j_{0,q-1,n},$ $j = 1,\ldots,M$, which are discarded before proceeding to the next time instant $n + 1$. Besides, these auxiliary values have to be accounted for in the resampling and jittering steps as well.
One way to avoid these additional computations is to replace the semi-implicit, space-sequential discretisation of the SDE in Algorithm 1 by an explicit Euler-Maruyama scheme. This yields a standard state equation,
\[ X_n = h f_{n-1}(X_{n-1}) + \sqrt{h} s_{n-1}(X_{n-1}) V_n \] (36)
for the approximate SSM. However, instead of implementing a conventional PF where particles are generated on all dimensions at once using (36), we still mimic the space-sequential PF with jittering of Algorithm 3. The result is the particle filtering scheme outlined in Algorithm 4. We refer to this method as explicit space-sequential jittered PF.

**Algorithm 4**

Explicit space-sequential PF with jittering.

- At time \( n = 0 \), sample \( M \) independent and identically distributed (iid) particles from the prior distribution \( \pi_0 \), denoted \( X_{j,0} \sim \pi_0 \), \( j = 1, \ldots, M \).
- Recursive step: given \( \{X_{j,n-1}\}_{j=1}^M \), generate \( \{X_{j,n}\}_{j=1}^M \) as described below.
  1. For \( i = 0, \ldots, q - 1 \)
    - (a) For \( j = 1, \ldots, M \),
      i. generate new \( m_x \)-dimensional samples
        \[ \hat{X}_{i,n} = X_{i,n-1} + h f(X_{i,n-1}) + \sqrt{h} s_i(X_{i,n-1}) V_{i,n}^j, \]
        where \( V_{i,n}^j \sim \mathcal{N}(0, I_{m_x}) \),
      ii. and compute importance weights
        \[ \hat{w}_{i,n}^j = g_{i,n}(\hat{X}_{i,n}^j), \quad j = 1, \ldots, M. \]
    - (b) Normalise the importance weights, \( w_{i,n}^j = \frac{\hat{w}_{i,n}^j}{\sum_{l=1}^M \hat{w}_{i,n}^l} \), for \( j = 1, \ldots, M \).
    - (c) Resample the weighted set \( \{X_{n-1}, X_{0:q-1,j,n}: \hat{X}_{i,n}^j, w_{i,n}^j\}_{j=1}^M \) with replacement, \( m \) times.
      Denote the resampled set as \( \{\tilde{X}_{n-1}, \tilde{X}_{0:i,n}: \hat{X}_{i,n}^j\}_{j=1}^M \).
    - (d) Jitter the resampled particles using a kernel \( J_{M,\beta} \) as defined in Eq. (34). Denote the jittered particles as \( \{X_{n-1}, X_{0:i,n}\}_{j=1}^M \) and carry them forward to the next step.

### 4.2 Conditionally dependent observations over space

It can be expected that (33) may not hold for many systems of interest. The proposed methodology can still be applied and any of the space-sequential particle filters can be implemented if we write down the likelihood function as
\[
g_n(X_n) = g_n(X_{0,q-1,n}) = g_n(X_{0,n}, \hat{X}_{1:q-1,n}) \prod_{i=1}^{q-1} g_n(X_{0:i,n}, \hat{X}_{i:q-1,n}),
\]
\[
g_n(X_n) = g_n(X_{0:n}, \hat{X}_{1:q-1,n}) \times g_n(X_{0:1:n}, \hat{X}_{2:q-1,n}) \times \cdots \times g_n(X_{0:q-1,n}) \quad \text{(37)}
\]

where \( \hat{X}_{0:q-1,n} \) is a given sequence (over space \( i \)) of state estimates, the likelihoods \( \bar{g}_n \) have the form

\[
\bar{g}_n(X_{0:i,n}, \hat{X}_{i:q-1,n}) = \frac{g_n(X_{0:i,n}, \hat{X}_{i+1:q-1,n})}{g_n(X_{0:1:n}, \hat{X}_{1:q-1,n})},
\]

and we assume \( X_{q:q-1,n} = \emptyset \), hence \( g_n(X_{0:q-1,n}, X_{q:q-1,n}) = g_n(X_{0:q-1,n}) \).

As a result, the space-sequential PFs filter are the same as in Section 4.1, except for the computation of the non-normalised weights, which becomes

\[
\tilde{w}_i,n^j = \begin{cases} 
g_n(\hat{X}_{0:i,n}^j, \hat{X}_{1:q-1,n}), & \text{if } i = 0 \\
g_n(\hat{X}_{0:i,n}^j, \hat{X}_{1:q-1,n}), & \text{if } i > 0
\end{cases}, \quad \text{for } j = 1, \ldots, M,
\]
in Algorithms 2 and 3. This approach can be used in Algorithm 4 as well, but it demands the computation of the auxiliary particles \( \hat{X}_n^j, \ j = 1, \ldots, M \), which otherwise can be avoided.

### 4.3 Consistency of space-sequential jittered particle filters

It is straightforward to show that the ‘vanilla’ PF in Algorithm 2 is consistent. The space-sequential jittered PFs (Algorithms 3 and 4) are also consistent in the sense that the approximation errors

\[
\| \xi_{i,n}^M (\phi) - \xi_{i,n}^h (\phi) \|_p \quad \text{and} \quad \| \pi_{i,n}^M (\phi) - \pi_{i,n}^h (\phi) \|_p
\]

are \( O(M^{-\beta}) \) for any \( p \geq 1 \) and any bounded real test function \( \phi : \mathbb{R}^{m_x} \rightarrow \mathbb{R} \). We prove this result for Algorithm 4, which allows for a simpler notation, but the argument is exactly the same for Algorithm 3.

**Artificial time representation.** Assume the state equation (39), which corresponds to an Euler-Maruyama discretisation of the SDE (2). The \( m_x \)-dimensional state \( X_{i,n} \) evolving over space \( i = 0, \ldots, q - 1 \) and time \( n \geq 0 \) can be rewritten as a conventional discrete-time random sequence depending on a single index if we define

\[
X_{\ell} := X_{i,n}, \quad \text{where } \ell = q(n - 1) + i,
\]

and the non-Markov transition kernel

\[
Q_{\ell}(x_{\ell-d,\ell-1}, A) := \mathbb{P} \left( X_{i,n} \in A \mid X_{0:q-1,n-1} = x_{q(n-2):q(n-1)-1} \right), \quad \text{(38)}
\]

where \( d = 2q \) and \( A \in \mathcal{B}(\mathbb{R}^{m_x}) \) is a Borel set. Note that these are just the dynamics induced by Eq. (36).

As the observations \( Y_{i,n} \) depend on space \( i \) and time \( n \) coordinates as well, we apply the same change of index \( \ell = q(n - 1) + i \) and let \( y_{\ell} := Y_{i,n} \). The likelihood of \( X_{\ell} \) given the observation \( Y_{\ell} \) is \( g_{\ell}(x_{\ell}) := g_{i,n}(X_{\ell}) \), where \( \ell = q(n - 1) + i \). We assume the observations \( Y_{\ell} = y_{\ell} \) are arbitrary but fixed, hence we omit them in the notation.
Algorithm 5 PF on artificial time

1. At time $\ell = 0$, draw $X^j_{\ell-1} \sim \mu_0$, $j = 1, \ldots, M$, where $\mu_0$ is the prior distribution.
2. Recursive step: given $\{X^j_{\ell-1}\}_{j=1}^N$, generate $\{X^j_{\ell}\}_{j=1}^M$ as described below.
   (a) Draw $Y^j_{\ell} \sim Q_{\ell}(X^j_{\ell-1}, dx_{\ell})$, $j = 1, \ldots, M$.
   (b) Compute weights $w^j_{\ell} \propto g_{\ell}(Y^j_{\ell})$, $j = 1, \ldots, M$.
   (c) Resample $M$ times, with replacement, the set $\{X^j_{\ell-1} \cup Y^j_{\ell}, w^j_{\ell}\}_{j=1}^M$. The resampled set is denoted $\{\bar{Y}^j_{\ell}\}_{j=1}^M$.
   (d) Jitter the resampled particles using a kernel $J_{\ell,\beta}$ as defined in Eq. (34). Denote the jittered particles as $\{\bar{X}^j_{\ell}\}_{j=1}^M$.

The random measures

$$\tilde{\mu}^M_{\ell}(dx_{\ell-1}) = \frac{1}{M} \sum_{j=1}^M \delta_{\bar{X}^j_{\ell}}(dx_{\ell-1})$$

$$\hat{\mu}^M_{\ell}(dx_{\ell-1}) = \frac{1}{M} \sum_{j=1}^M \delta_{\bar{X}^j_{\ell}}(dx_{\ell-1})$$

are approximations of the posterior law $\mu_{\ell}$ obtained after the resampling step ($\tilde{\mu}^M_{\ell}$) and after the jittering step ($\hat{\mu}^M_{\ell}$). The theorem below guarantees the consistency of these two estimators.

**Theorem 3.** Assume that the observations $Y_\ell = y_\ell$ are fixed and the likelihood functions $g_\ell$ are bounded and strictly positive for $\ell = 1, \ldots, Nq - 1$, where $N = \frac{q}{R}$ is a finite integer. Let the jittering kernel be defined as in Eq. (34) with $0 < \beta \leq \frac{1}{2}$. For every bounded test function $\phi : \mathbb{R}^{2NqR} \mapsto \mathbb{R}$ there exist constants $C_{\phi,\ell} < \infty$ independent of $M$ and $\beta$ such that

$$\|\mu_{\ell}(\phi) - \mu^M_{\ell}(\phi)\|_p \leq C_{\phi,\ell} M^{-\beta} \quad \text{for } p \geq 1 \text{ and } \ell = 0, \ldots, Nq - 1.$$  \hfill (39)
Proof. We apply the same induction argument as in the proof of Lemma 1 in [16]. For \( \ell = 0 \) the particles \( X_{-d+1,0}^j \) are iid samples from the prior \( \mu_0 \), hence

\[
\| \mu_0^M(\phi) - \mu_0(\phi) \|_p \leq C_{\phi,0}M^{-2}\beta \leq C_{\phi,0}M^{-\beta}. \tag{40}
\]

for the induction step, assume that \( \| \mu_{\ell-1}^M(\phi) - \mu_{\ell-1}(\phi) \|_p \leq C_{\phi,\ell-1}M^{-\beta} \). If we follow step by step the argument in the proof of Lemma 1 in [16] we can proceed through steps 2(a), 2(b) and 2(c) of Algorithm 5 and arrive at

\[
\| \tilde{\mu}_\ell^M(\phi) - \mu_\ell(\phi) \|_p \leq \tilde{C}_{\phi,\ell}M^{-\beta}. \tag{41}
\]

The measure \( \mu_\ell^M \) is obtained by applying the jittering kernel, i.e., \( \mu_\ell^M = J_{\mu,\beta}\tilde{\mu}_\ell^M \). However, from [35] we have

\[
\| J_{\mu,\beta}\tilde{\mu}_\ell^M(\phi) - \tilde{\mu}_\ell^M(\phi) \|_p \leq 2\| \phi \|_\infty M^{-\beta}. \tag{42}
\]

Taking (41) and (42) together yields, by way of an obvious triangle inequality, the relationship in the statement of Theorem 3 with the constant \( C_{\phi,\ell} = \tilde{C}_{\phi,\ell} + 2\| \phi \|_\infty < \infty \). \( \square \)

Remark 8. The posterior law \( \pi_{i,n}^h \) is a marginal of \( \mu_\ell \) when \( \ell = q(n-1) + i \) and, similarly, the particle approximation \( \pi_{i,n}^h \) from Algorithm 2 is just a marginal of \( \mu_\ell^M \) obtained with Algorithm 5.

As a consequence, for a bounded test function \( \phi : \mathbb{R}^m \rightarrow \mathbb{R} \), we have \( \| \pi_{i,n}^M(\phi) - \pi_{i,n}^h(\phi) \| = O(M^{-\beta}) \) and, for \( i = q - 1 \) in particular, we obtain \( \| \pi_{n}^M(\phi) - \pi_{n}^h(\phi) \| = O(M^{-\beta}) \).

Moreover, recall \( \pi_n(dx) = P(X(t_n) \in dx|Y_1;n) \) is the filter for the exact SSM. If we choose a test function \( \phi \in C_B^{\ell}(\mathbb{R}^d) \) such that \( \| \phi \|_\infty \leq 1 \) we readily obtain

\[
\mathbb{E} \left[ |\pi^M_n(\phi) - \pi_n(\phi)| \right] \leq \mathbb{E} \left[ |\pi^h_n(\phi) - \pi^M_n(\phi)| \right] + |\pi^h_n(\phi) - \pi_n(\phi)|
\leq C_{\phi,nq-1}M^{-\beta} + |\pi^h_n(\phi) - \pi_n(\phi)|
\leq C_{\phi,nq-1}M^{-\beta} + C_nh
\]

where we have successively applied a triangle inequality, Theorem 2 and Theorem 3. Now, if we choose \( N = O(M^{-\beta}) \), in such a way that the step size for the time grid is \( h = \frac{1}{N} = O(M^{-\beta}) \) we arrive at

\[
\mathbb{E} \left[ |\pi^M_n(\phi) - \pi_n(\phi)| \right] = O(M^{-\beta}).
\]

5 Example: stochastic Lorenz 96 model

5.1 State space model

We illustrate the application of the proposed methodology to implement a particle filter for a stochastic Lorenz 96 model with additive noise, see e.g. [13]. In particular, we consider the \( d_x- \)

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2These steps of the algorithm are standard. The sampling step 2(a) is handled using the Marcinkiewicz-Zygmund inequality, for the update step 2(b) we resort to Bayes’ theorem and an inequality analogous to (29), and the resampling step 2(c) is dealt with using Marcinkiewicz-Zygmund again.
dimensional SDE

\[ dX_i = -X_{i-1}X_{i-2} - X_{i+1} - X_i + F + \sigma dW_i, \quad i = 0, \ldots, d_x - 1, \]  

(43)

where \( F \) is a forcing constant (we set \( F = 8 \) unless otherwise explicitly stated), \( W_i(t), i = 1, \ldots, d_x - 1, \) are standard Wiener processes and \( \sigma \) is a constant diffusion coefficient. The operations on the index \( i \) are performed modulo \( d_x \) (hence, \( X_{d_x} = X_0 \)).

The SDE can be discretised using the semi-implicit space-sequential scheme of Algorithm 1. At each time \( n \), we first obtain the auxiliary discrete-time variables

\[ \hat{X}_{i,n} = X_{i,n-1} + h ( -X_{i-1,n-1}X_{i-2,n-1} - X_{i+1,n-1} - X_{i,n-1} + F ), \quad i = 0, \ldots, d_x - 1, \]

and then proceed to generate the state variables sequentially for \( i = 0, \ldots, d_x \), i.e.,

\[ X_{i,n} = X_{i,n-1} + h \left( -\hat{X}_{i-1,n}\hat{X}_{i-2,n} - \hat{X}_{i+1,n-1} - \hat{X}_{i,n-1} + F \right) + \sqrt{h} \sigma Z_{i,n}, \quad i = 0, \ldots, d_x - 1, \]

where \( \{Z_{i,n} : i = 1, \ldots, d_x - 1\} \) is a set of iid standard Gaussian r.v.'s and

\[ \hat{X}_{i-1,n} = \begin{cases} \hat{X}_{d_x-1,n}, & \text{if } i = 0 \\ X_{i-1,n}, & \text{otherwise} \end{cases}, \]

\[ \hat{X}_{i-2,n} = \begin{cases} \hat{X}_{d_x-2,n}, & \text{if } i = 0 \\ \hat{X}_{d_x-1,n}, & \text{if } i = 1 \\ X_{i-2,n}, & \text{otherwise} \end{cases}, \]

and

\[ \hat{X}_{i+1,n} = \begin{cases} X_{0,n}, & \text{if } d_x - 1 = 0 \\ X_{i+1,n}, & \text{otherwise} \end{cases}. \]

For this example, the partition of the state vector \( X_n \) can be carried out in 1-dimensional elements, i.e., \( m_x = 1 \) and \( q = d_x \) when comparing with the general case of Section 2.

We assume that the state variables can be observed independently. In particular, the observations have the form

\[ Y_n = A_n X_n + \sigma_y U_n, \]  

(44)

where \( Y_n \) is a \( d_y \times 1 \) vector of observations \( (d_y \leq d_x) \), \( A_n \) is a known \( d_y \times d_x \) observation matrix of the form

\[ A_n = a_n \begin{bmatrix} u_{n}^{k_1} & u_{n}^{k_2} & \cdots & u_{n}^{k_{d_y}} \end{bmatrix}, \]

(45)

where \( a_n \in \{0, 1\}, \) \( u_{n}^{k_j} \) is a vector of 0’s with a single value of 1 in the \( j \)-th entry, and \( k_1 < k_2 < \cdots < k_{d_y}, \sigma_y \) is a known scale parameter and \( U_n \) is a sequence of iid \( \mathcal{N}(0, I_{d_y}) \) Gaussian r.v.'s. Intuitively, this observation model implies that

- at each time \( n \) there are \( d_y \) state variables which may be observed in Gaussian noise, with \( d_y \leq d_x \);
- the observed variables may vary from time \( n \) to time \( n + 1 \) (the matrix \( A_n \) is time-varying) but the set of observable variables is always known; and
- actual observations are only collected for time instants where \( a_n = 1 \), for which \( Y_n \sim \mathcal{N}(A_n X_n, \sigma_y^2 I_{d_y}) \). When \( a_n = 0, Y_n \sim \mathcal{N}(0, \sigma_y^2 I_{d_y}) \) and, therefore, \( g_n(x) \propto 1 \) for all \( x \in \mathbb{R}^{d_x} \).
5.2 Numerical results

We have carried out computer simulations to assess the performance of the three space-sequential PFs in tracking the stochastic Lorenz 96 model as the model dimension $d_x$ is increased and, specifically, how this performance compares to a standard PF built upon a standard Euler-Maruyama discretisation of (43). Hereafter, we refer to

- the standard PF based on the Euler-Maruyama discretisation of (43) as ‘PF’;
- the space-sequential PF of Algorithm 2 as ‘SSPF’;
- the space-sequential jittered PF of Algorithm 3 as ‘SSJPF’; and
- the space-sequential jittered PF of Algorithm 4 as ‘ESSJPF’.

The jittering kernel for these computer experiment is implemented as follows. Before the loop over the coordinate indices $i = 0, \ldots, q - 1$ in Algorithms 3 and 4, set $Z_{n-1}^j := X_{n-1}^j$, $j = 1, \ldots, M$, i.e., we keep the particle set obtained at discrete time $n - 1$. For the SSJPF algorithm, let $\tilde{X}_n^j = \left( \tilde{X}_{n-1}^j, \tilde{X}_n^j, \tilde{X}_{0:i,n}^j \right)$. The kernel $J(\tilde{X}_n^j, \tilde{d}X_n^j)$ of Eq. (34) is constructed

$$J(\tilde{X}_n^j, \tilde{d}X_n^j) = N(\tilde{Z}_{n-1}^j, \frac{1}{10} I_{d_x}) \times \delta_{\tilde{X}_n^j}^{\tilde{d}X_n^j} \times \delta_{\tilde{X}_{0:i,n}^j}^{\tilde{d}X_{0:i,n}^j},$$

where $N(\tilde{Z}_{n-1}^j, \frac{1}{10} I_{d_x})$ is a Gaussian measure. With this scheme, only the current ancestor $\tilde{X}_{n-1}^j$ is actually jittered by shifting it to a perturbed copy of the original ancestor $Z_{n-1}^j$ (note that $\tilde{X}_{n-1}^j$ and $Z_{n-1}^j$ are different, in general, because of the resampling steps). We have found that this is enough to create diversity in this example. The other jittering parameter is set as $\beta = 0.1$. The same procedure is applied with the ESSJPF algorithm.

For this comparison, we have carried out multiple independent simulation trials. Other than the dimension $d_x$, the model parameters are common to all computer experiments. In particular:

- The forcing parameter is $F = 8$ and the step-size for time discretisation is $h = 5 \times 10^{-4}$ continuous-time units.
- The diffusion coefficient is $\sigma = 2$, fixed for all $i = 0, \ldots, d_x$ and all simulation runs.
- Observations are collected every 100 discrete-time steps. This can be represented using model (44) by choosing

$$a_n = \begin{cases} 
1, & \text{if } n = 100k \text{ for some } k \in \mathbb{N}, \\
0, & \text{otherwise}
\end{cases}.$$ 

This implies that $g_n(x) \propto 1$ for all $x$ when $n \neq 100k$.
- At every observation step, the matrix $A_n$ has dimension $d_y \times d_x$, where $d_y = \lfloor \frac{3}{5} d_y \rfloor$. For the purpose of the simulation, $A_n$ is generated randomly, with the structure in (45), but it is assumed known for the filtering algorithms. In summary, we observe 60% of the coordinates of the $d_y$-dimensional signal every 100 discrete-time steps (or 0.05 continuous-time units).
- The observation noise variance is $\sigma_y^2 = 1$. 

25
• The simulation covers the continuous-time interval $t \in (0, T)$, with a final time $T = 10$. Hence, the number of discrete time steps is $N = h^{-1}T = 2 \times 10^4$.

With this setup, we run 50 independent simulation trials for each value of the signal dimension $d_x \in \{25, 50, 100, 200, 400\}$. In each trial, we

1. generate a ground-truth signal $X^*_n$, for $n = 0, \ldots, N$, using a standard Euler-Maruyama scheme with $h = 5 \times 10^{-4}$;
2. generate observations $Y_n$, according to Eq. (44), using the signal $X^*_n$;
3. run a standard PF based on the standard Euler-Maruyama scheme.
4. run the SSPF, SSJPF and ESSJPF algorithms.

Rather than resampling after every update step, we adopt (for all algorithms) the strategy of resampling only when the normalised effective sample size falls below a threshold of 0.5. This approach reduces the computational cost and, for this example at least, improves the performance of the filters.

The number of particles for each PF is selected as a function of the dimension $d_x$. In particular, the number of particles for the standard PF is $M = \lfloor 3d_x^{1.3} \rfloor$, while we set $M' = \lfloor d_x^{1.3} \rfloor$ for the SSPF, SSJPF and ESSJPF algorithms. The reason to choose different numbers is to balance the run-times of the two algorithms. Note that the space-sequential PFs are computationally more demanding than the standard PF and (any) one of the space-sequential algorithms, respectively, in the $k$-th simulation run for dimension $d_x$. For each $k = 1, \ldots, 50$ and each $d_x \in \{25, 50, 100, 200, 400\}$ we can compute the mean square errors (MSEs)

$$MSE_{PF}(k, d_x) = \frac{1}{N} \sum_{n=1}^{N} \left\| X^*_n - X^M_n(k, d_x) \right\|^2, \quad MSE_{SSP}(k, d_x) = \frac{1}{N} \sum_{n=1}^{N} \left\| X^*_n - X^{M'}_n(k, d_x) \right\|^2,$$

where $MSE_{SSP}$ stands for $MSE_{SSPF}$, $MSE_{SSJPF}$ or $MSE_{ESSJPF}$. We obtain a population of 50 realisations of the MSE for each algorithm and each $d_x$.

Figure displays a summary of the MSEs normalised by the dimension $d_x$, i.e., $\frac{1}{d_x} MSE_{PF}(k, d_x)$ on the top-left panel, $\frac{1}{d_x} MSE_{SSPF}(k, d_x)$ on the top-right panel, $\frac{1}{d_x} MSE_{SSJPF}(k, d_x)$ on the bottom-left panel and $\frac{1}{d_x} MSE_{ESSJPF}(k, d_x)$ on the bottom-right panel. For each value of $d_x$, the box plots display: the median of the population (with a red horizontal line), a box delimited by the 25% and 75% percentiles of the empirical distribution, upper and lower whiskers extending to ±2.7

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For a particle approximation $\pi_n^M(dx) = \frac{1}{N} \sum_{j=1}^{M} \delta_{X^j_{n-1,n}}(dx)$ we obtain the posterior mean estimate

$$\int X_{0:d_x-1,n} \pi_n^M(dx) = \frac{1}{N} \sum_{j=1}^{M} X^j_{0:d_x-1,n} \approx \mathbb{E}[X_{0:d_x-1,n}|Y_{1:n}].$$

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26
standard deviations from the mean, and outliers marked as red crosses. We observe how the MSEs per dimension of the standard PF and SSPF estimates increase quickly with $d_x$. For the SSJPF and ESSJPF algorithms, on the other hand, the populations of MSEs display similar medians as $d_x$ increases, while the variance of the errors decreases significantly (recall that the number of particles

\[ M' = \lfloor d_x^{1.3} \rfloor \]

Figure 1: Performance of the standard PF, SSPF, SSJPF and ESSJPF algorithms when tracking a stochastic Lorenz 96 system. The mean square error (MSE) is computed w.r.t. the true signal, i.e., if $X^*_n$ is the $d_x$-dimensional actual state and $\hat{X}_n$ is its Monte Carlo estimate, then for each simulation run $MSE = \frac{1}{N} \sum_{n=1}^{N} \| X^*_n - \hat{X}_n \|^2$ (note that $N$ is the total number of discrete-time steps). The MSE is normalised by the model dimension $d_x$.

Figure 2 shows similar numerical results for the various MSEs, but for this figure the errors are normalised by the signal power $P_x = \frac{1}{N} \sum_{n=1}^{N} \| X^*_n \|^2$. We can see the same difference in performance: the MSEs for the standard PF and SSPF grow quickly with $d_x$ and eventually (for $d_x \geq 200$) the median error becomes greater than 1, i.e., the second order moment of the error is greater than the second order moment of the signal, $P_x$. For the vanilla SSPF, performance degrades even more quickly. On the other hand, for the SSJPF and ESSJPF algorithms the error remains low and approximately constant over all values of $d_x$, with a contraction in the variance as both $d_x$ and $M' = \lfloor d_x^{1.3} \rfloor$ increase.

Figure 3 accounts for the computational cost of the algorithms. The panel on the left displays the increase in the number of particles, $M = \lfloor 3d_x^{1.3} \rfloor$ and $M' = \lfloor d_x^{1.3} \rfloor$, versus the model dimension $d_x$. The panel on the right shows the average run time per simulation for each value of $d_x$ for the standard PF and the SSJPF algorithm. The values of $M$ and $M'$ have been selected in such a way...
that the SSJPF and ESSJPF schemes always take smaller run times compared to the standard PF.

In order to provide further illustration of the performance of the SSJPF method, we have conducted an additional computer experiment for a stochastic Lorenz 96 model with $d_x = 1,000$ state variables running from (continuous) time 0 to time $T = 40$. As for the rest of the parameters, $F = 8$ and $h = 5 \times 10^{-4}$. Measurements are generated according to model (44) and we assume that 60% of the state variables (randomly chosen) can be observed every 100 discrete time steps. The SSJPF algorithms is implemented with $M' = [d_x^{1.3}] = 7,943$ particles and resampling is performed when the normalised effective sample size falls below 0.5. Note that updates are carried out as we run sequentially through $i = 0, \ldots, d_x - 1$. Since we observe 60% of the state variables, this implies 600 updates at every observation time (and there are $40 \times 100 \approx 800$ observation times).

Figure 4 shows the square errors versus time, averaged per dimension, for this simulation. Specifically, if $X_{n}^{M'}$ is the estimate of the $d_x$-dimensional vector $X^*_n$, then we plot $\varepsilon_n^{M'} = \frac{1}{d_x} \| X^*_n - X_{n}^{M'} \|_2^2$ versus time $n$. (46)

We observe that the error per dimension remains bounded and non-increasing over the whole simulation.

For the same experiment, Figure 5 shows the actual states $X_{1,n}^*, X_{3,n}^*, \ldots, X_{11,n}^*$ together with their estimates versus the continuous-time grid $t_n = nh$ (for $20 \leq t_n \leq 40$). The signals
are estimated and tracked closely over time, with only minor (relative to the signal amplitude) mismatches.

6 Conclusions

We have introduced a novel methodology for particle filtering in dynamical systems where the evolution of the signal of interest is described by a SDE and observations are collected instantaneously at prescribed time instants. The new approach includes the discretisation of the SDE and the design of efficient particle filters for the resulting discrete-time state-space model.

The numerical scheme for the discretisation of the SDE converges with weak order 1 and it is devised to create a sequential dependence structure along the coordinates of the discrete-time state vector. As a result, it is possible to apply directly any of the space-time particle filters in [5], as well as the class of space-sequential particle filters introduced in this paper.

We have conducted computer simulations for a stochastic Lorenz 96 system with additive noise
Figure 5: State signals $X^*_{1,n}, X^*_{3,n}, \ldots, X^*_{11,n}$ and their estimates computed with a space-time bootstrap filter for a simulation of the stochastic Lorenz 96 model with $d_x = 1,000$. The particle filter propagates $M' = 7,943$ particles. 60% of the variables are observed every 100 discrete time steps. The horizontal axis represents the continuous time grid $t_n = nh$ for $20 \leq t_n \leq 40$.

that show how the new space-sequential particle filters attain a stable performance (approximately constant estimation errors) as the dimension of the Lorenz 96 system is increased. Most importantly, this performance is attained with a computational effort that increases polynomially, rather than exponentially, with the system dimension (the particle filter runs with $M = d_x^{1.3}$ particles, where $d_x$ is the system dimension).

Besides the new numerical scheme and particle filters, we have provided in this paper a general framework for discrete-time filtering in continuous-time dynamical systems described by a SDE and instantaneous observations. Provided that the SDE is discretised using a weakly-convergent scheme, we have proved that the marginal posterior laws of the resulting discrete-time state-space model converge (as $h \to 0$, where $h$ is the step-size of the time grid) to the posterior marginal posterior laws of the original continuous-time state-space model under a suitably defined metric. This guarantees
that any consistent particle filter running on the discrete-time state-space model also converges to the underlying continuous-time optimal filter under the same metric. We emphasise that these results are general and not restricted to the semi-implicit numerical scheme or the space-sequential particle filters specifically studied in this manuscript.

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