Extremal graphs for odd-ballooning of paths and stars

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Abstract

The odd-ballooning of a graph $G$, denoted by $G_q$, is the graph obtained from replacing each edge in $G$ by a odd cycle of the same size where the new vertices of the odd cycles are all different. In 2002, Erdős et al. determined the extremal graphs of $k$-fan. In 2016, Hou et al. determined extremal graphs of the odd-ballooning of stars for $q \geq 5$. In 2020, Zhu et al. determined extremal graphs of the odd-ballooning of paths for $q \geq 3$. In this article, we use progressive induction lemma of Simonovits to determine the extremal graphs of both odd-ballooning of stars and odd-ballooning of paths for $q \geq 3$.

Keywords: Turán number, odd-ballooning, paths, stars

1 Introduction

In this article, we only consider the undirected graphs without loop and multiple edges. For a graph $G$, denote by $E(G)$ the set of edges and $V(G)$ the set of vertices of $G$. We denote the number of vertices in $G$ by $\nu(G) = |V(G)|$ and the number of edges in $G$ by $e(G) = |E(G)|$, respectively. Denote by $N_G(v) = \{u \in V(G) : uv \in E(G)\}$ the neighborhood of $v$ in $G$, and the closed of neighborhood $N_G[v]$ of $v$ is $N_G(v) \cup \{v\}$. For a vertex $v \in G$, the degree of the vertex $v$, denoted by $d_G(v)$, is the size of $N_G(v)$. We use $\delta(G)$ and $\Delta(G)$ to denote the minimum and maximum degrees of $G$, respectively. A matching $M$ in $G$ is a subgraph of $G$ with $\delta(M) = \Delta(M) = 1$. The matching number of $G$, denoted by $\nu(G)$, is the maximum number of edges in a matching of $G$. For a subset $X \subset V(G)$, let $G[X]$ denote the subgraph of $G$ induced by $X$. Let $G$ and $H$ be two vertex disjoint graphs, we denote the vertex disjoint union of $G$ and $H$ by $G \cup H$ and the vertex disjoint union of $k$ copies of $G$ by $kG$. Denote by $G \vee H$ the graph obtained from $G \cup H$ by adding edges between each vertex of $G$ and each vertex of $H$. For convenience, we indicates the number of vertices of a given graph by subscript, e.g. denote by $P_k$ a path on $k$ vertices, $S_k$ a star on $k$ vertices, $C_k$ a cycle on $k$ vertices, $K_n$ a complete graph on $n$ vertices, $K_{n_1, n_2}$ a complete bipartite graph with parts of cardinality $i$ and $j$, $M_{2l}$ a matching on $2l$ vertices.

Given a graph $H$ and an integer $p \geq 2$, the (edge) blow-up of $H$, denoted by $H^{p+1}$, is the graph obtained from $H$ by replacing each edge of $H$ by a clique of order $p + 1$ where the new vertices of the cliques are all different. Similarly,
Theorem 1.2 \([5], [6]\) For any integer \(n\) is sufficiently large, we have the following results:

Given a graph \(H\) and an odd integer \(q \geq 3\), the odd-ballooning of \(H\), denoted by \(H_q\), the graph obtained from \(H\) by replacing each edge of \(H\) by an odd cycle of length \(q\), say \(q\)-cycle, where the new vertices of the odd cycles are all different. Let \(P_{k,q}\) and \(S_{k,q}\) denote the odd-ballooning of path \(P_k\) and star \(S_k\). When \(q = 3\), the graph \(S_{k,3}\) is also called \(k\)-fan or friendship graph.

The Turán number of a class of graph \(\mathcal{L}\), \(\text{ex}(n, \mathcal{L})\), is the maximum number of edges in \(G\) of order \(n\) that does not contain any \(L \in \mathcal{L}\) as a subgraph. Denote by \(EX\) the class of graphs on \(n\) vertices with \(ex(n, L)\) edges containing no \(L \in \mathcal{L}\) as a subgraph and call a graph from \(EX\) extremal graph for \(L\) or \(\mathcal{L}\)-extremal graph. If \(\mathcal{L}\) contains only one graph, say \(\mathcal{L} = \{L\}\), we can simply use \(ex(n, L)\) and \(EX(n, L)\) to instead. If \(\mathcal{L}\) contains multiple graphs, we can use \(ex(n, H_1, H_2, \cdots, H_k)\) and \(EX(n, H_1, H_2, \cdots, H_k)\) to denote the maximum number of edges and the extremal graph with no subgraph \(H_i\) for \(1 \leq i \leq k\), respectively.

For integer \(1 \leq p \leq n\), the Turán graph, denoted by \(T_{n,p}\), is the complete \(p\)-partite graph on \(n\) vertices in which all parts are balanced. In [1], Turán showed that \(T_{n,p}\) is the unique extremal graph for the complete graph \(K_{p+1}\), and this theorem is often seen as the beginning of extremal graph theory. The number of edges in \(T_{n,p}\) is denoted by \(t_{n,p}\).

For \(s \geq 1, p \geq 2\), let \(H(n, p, s)\) be the graph \(K_{s-1} \vee T_{n-s+1, p}\) and \(H'(n, p, s)\) be any of the graphs obtained from \(H(n, p, s)\) by putting one extra edge in any part of \(T_{n-s+1, p}\) in \(H(n, p, s)\). Given a family of graph \(\mathcal{F}\), the subchromatic number \(p(\mathcal{F})\) of \(\mathcal{F}\) is defined by \(p(\mathcal{F}) = \min_{F \in \mathcal{F}} \chi(F) - 1\), where \(\chi(F)\) is the chromatic number of \(F\). Zhu [2] used a classical decomposition theorem of Simonovits (see for more information in [3], [4]) to determine the extremal number of odd-ballooning of paths and obtained the following result.

**Theorem 1.1** [2] For any integer \(k \geq 1\) and any odd integer \(q \geq 3\), when \(n\) is sufficiently large, we have the following results:

\[
\text{ex}(n, P_{k+1,q}) = t_{n-p,2} + e(K_{p,n-p}) + e(K_p),
\]

where \(p = \left\lfloor \frac{k-1}{2} \right\rfloor\), and \(H(n, 2, p + 1)\) is the extremal graph of \(P_{k+1,q}\) when \(k\) is odd, \(H'(n, 2, p + 1)\) is the unique extremal graph of \(P_{k+1,q}\) when \(k\) is even.

If a graph \(G\) with vertex number \(n\) has \(n-1\) vertices degree \(d\) and the remaining vertex degree \(d-1\), the graph is called almost \(d\)-regular graph and denoted by \(R(n, d)\). Erdős [5] and Hou [6] determined the extremal number of \(k\)-fan (which also seen as odd-ballooning of stars for \(q = 3\)) and the extremal number of odd-ballooning of stars for \(q \geq 5\), respectively.

**Theorem 1.2** ([5], [6]) For any integer \(k \geq 1\) and any odd integer \(q \geq 3\), when \(n\) is sufficiently large, we have the following results:

1. When \(q = 3\),

\[
\text{ex}(n, S_{k+1,3}) = t_{n,2} + \begin{cases} 
  k^2 - k & \text{if } k \text{ is odd,} \\
  k^2 - \frac{3}{2}k & \text{if } k \text{ is even.}
\end{cases}
\]

For odd \(k\), the extremal graph is constructed by taking a Turán graph \(T_{n,2}\) and embedding two vertex-disjoint copies of \(K_k\) in one partite set. For even \(k\), the extremal graph is constructed by taking a Turán graph \(T_{n,2}\)
and embedding a graph with $2k - 1$ vertices, $k^2 - \frac{3}{2}k$ edges with maximum degree $k - 1$ in one partite set, that is $R(2k - 1, k - 1)$.

2. When $q \geq 5$,

$$ex(n, S_{k+1,q}) = t_{n,2} + (k - 1)^2.$$  

The extremal graph of $S_{k+1,q}$ must be a Turán graph $T_{n,2}$ with a copy of complete bipartite graph $K_{k-1,k-1}$ in one partite set.

The following lemma indicates that the extremal graph without both $P_{k+1,q}$ and $S_{k+1,q}$ exists.

**Lemma 1.3** Let $k \geq 3$ be an integer, $q \geq 3$ be an odd integer, then we have the following:

1. the extremal graphs for $S_{k+1,q}$ contain $P_{k+1,q}$.

2. the extremal graphs for $P_{k+1,q}$ contain $S_{k+1,q}$.

**Proof** (1). Let $G$ be the extremal graph for $S_{k+1,q}$. It is clear that there is a $T_{n,2}$ in $G$. We denote by $A_0$, $A_1$ the two partite sets of $T_{n,2}$ in $G$. Then we will show that $P_{k+1,q} \subseteq G$.

When odd integer $q \geq 5$, without loss of generality, we assume that the copy of $K_{k-1,k-1}$ is embedded in $A_0$. Let $v_1, v_2, \cdots v_{2(k-1)}$ be the vertices of $K_{k-1,k-1}$ in $A_0$. It is easy to find a path $P_{k+1} : v_1v_2\cdots v_{2(k-1)}$ in $G$ where each edge $v_i v_{i+1} (i \in \{1, 2, \cdots, k\})$ satisfies that $v_i, v_{i+1}$ belong to different parts of $K_{k-1,k-1}$. Then the edge $v_i v_{i+1}$ can be extended to a $q$-cycle by using $q - 2$ vertices respectively in $A_1$ and $A_2$ except the vertices in $P_{k+1}$. Hence, we obtain $P_{k+1,q}$ in $G$.

When $q = 3$ and $k$ is odd, without loss of generality, we assume that two vertex-disjoint copies of $K_k$ are embedded in $A_0$. Let $u_1, u_2, \cdots, u_k$ be the vertices of $K_k$ in $A_0$ and $w_1, w_2, \cdots, w_k$ be the vertices of another $K_k$ in $A_0$. We can easily find a path $P_{k+1} : u_1u_2\cdots u_{k-1}u'w_1$ in $G$ where $u'$ is a common neighbor of $u_{k-1}$ and $w_1$ in $A_1$ since $A_0, A_1$ are completely joined. The edge $u_j u_{j+1} (j \in \{1, 2, \cdots, k-2\})$ can be extended to a $q$-cycle by using $q - 2$ vertices respectively in $A_0$ and $A_1$ except the vertices in $P_{k+1}$. The edge $u_{k-1}u'$ together with $u_k$ and other $q - 3$ vertices respectively in $A_0$ and $A_1$ except the vertices in $P_{k+1}$ and the vertices forming the $q$-cycle of each edge $u_j u_j'$ forms a copy of $q$-cycle. The edge $u'w_1$ together with $w_2$ and other $q - 3$ vertices respectively in $A_0$ and $A_1$ except the vertices in $P_{k+1}$, the vertices forming the $q$-cycle of each edge $u_j u_{j+1}$ and the vertices forming the $q$-cycle of edge $u_{k-1}u'$ forms a copy of $q$-cycle. Hence, we obtain $P_{k+1,q}$ in $G$.

When $q = 3$ and $k$ is even, without loss of generality, we assume that $R(2k - 1, k - 1)$ is embedded in $A_1$. Similarly, one can easily find a path $P_{k+1}$ in $R(2k - 1, k - 1)$ and each edge of $P_{k+1}$ together with $q - 2$ vertices respectively in $A_0$ and $A_1$ except the vertices in $P_{k+1}$ forms $q$-cycle. Then $P_{k+1,3}$ is a subgraph of $G$.

(2). Let $H$ be the extremal graph for $P_{k+1,q}$, $x$ be a vertex of $K_p$ in $H$, where $p = \lceil \frac{k}{k-1} \rceil (\geq 1)$. Then $k$ different $q$-cycles, which intersect in $x$, can be found in $H$. This is possible since $x$ is adjacent to all the other vertices in $H$ and $n$ is sufficiently large. Hence, $S_{k+1,q}$ is a subgraph of $H$. □
Let $G_{n,p,k}$ be a graph of order $n$ which is constructed by taking a Turán graph $T_{n,p}$ and embedding a copy of $K_k$ in one partite set. Consider Theorem 1.1, Theorem 1.2 and Lemma 1.3, we focus on the extremal number of both odd-ballooning of paths and odd-ballooning of stars, that is $ex(n, S_{k+1,q}, P_{k+1,q})$.

**Theorem 1.4** For any integer $k \geq 1$ and any odd integer $q \geq 3$, when $n$ is sufficiently large, we have

$$ex(n, S_{k+1,q}, P_{k+1,q}) = t_{n,2} + \frac{1}{2}(k^2 - k).$$

Furthermore, $G_{n,2,k}$ is the unique extremal graph for both $P_{k+1,q}$ and $S_{k+1,q}$.

## 2 Preliminaries

In 1946, Erdös and Stone [7] stated the following well-known theorem.

**Theorem 2.1** [7] For all integers $p \geq 2$ and $N \geq 1$, and every $\epsilon > 0$, there exists an integer $n_0$ such that every graph with $n \geq n_0$ vertices and at least $t_{n,p} - 1 + \epsilon n^2$ edges contains a $T_{N,p,p}$ as a subgraph.

The following lemma is powerful to estimate the number of edges of a graph with restricted degrees and matching number.

**Lemma 2.2** [8] For any graph $G$ with maximum degree $\Delta \geq 1$ and matching number $\nu \geq 1$, then

$$e(G) \leq f(\nu, \Delta) = \nu \Delta + \lfloor \frac{\nu}{2} \lfloor \frac{\nu}{\Delta/2} \rfloor \rfloor \leq \nu (\Delta + 1).$$

Recall the definition of subchromatic number, the famous Erdös-Stone-Simonovits Theorem gives a rough range of the extremal number of the graph which the subchromatic number of $\mathcal{L}$ is given.

**Lemma 2.3** [9] If $\mathcal{L}$ is a family of graphs with subchromatic number $p > 0$, then

$$ex(n, \mathcal{L}) = \left(1 - \frac{1}{p}\right)\left(\frac{n}{2}\right) + o(n^2).$$

For every family $\mathcal{L}$ of forbidden graphs, Simonovits [10] defined the decomposition family $\mathcal{M}(L)$.

**Definition 2.4** Given a family $\mathcal{L}$, define $p' = p'(\mathcal{L}) = \min_{L \in \mathcal{L}} \chi(L) - 1$. For any integer $p : 2 \leq p \leq p'$, let $\mathcal{M}_p(\mathcal{L})$ be the family of minimal graph $M$ that satisfy the following: there exists an $L \in \mathcal{L}$ and a $t = t(L)$ such that $L \subseteq M' \cup T_{p-1}(pt - t)$, where $M' = M \cup I_t$. We call $\mathcal{M}(\mathcal{L})$ the $p$-decomposition family of $\mathcal{L}$.

Given a graph $H$ and a vertex $v \in V(H)$ with $d_H(v) \geq 2$, a vertex split on the vertex $v$ is defined as follows: replace $v$ by an independent set of $d(v)$ vertices in which each vertex is adjacent to exactly one distinct neighbor in $N_G(v)$. Given
a vertex subset \( U \subseteq V(H) \), a vertex split on \( U \) means applying vertex split on the vertices in \( U \) one by one. Obviously, the order of vertices we apply vertex split does not matter. Denote by \( \mathcal{H}(H) \) the family of all the graphs that can be obtained from \( H \) by applying vertex split on some \( U \subseteq V(H) \). Note that \( U \) could be empty, therefore \( H \in \mathcal{H}(H) \). Let \( \mathcal{H}_p(H) \subseteq \mathcal{H}(H) \) be the family of all the graphs obtained from \( H \) by applying vertex split on any vertex subset \( U \) of \( H \), which satisfies \( \chi(H[U]) \leq p \). Then we have the following lemma to determined the decomposition family of both \( P_{k+1,q} \) and \( S_{k+1,q} \).

**Lemma 2.5** Let \( H \) be any bipartite graph and \( q \geq 5 \) be an odd integer. Then \( \mathcal{M}_2(H_q) = \mathcal{H}_2(H) \).

**Proof** Let \( H \) be any bipartite graph, \( H_q \) be the odd-ballooning of \( H \) with \( q \geq 5 \). Then \( \chi(H_q) = 3 \). Let \( p' = \chi(H_q) - 1 = 2 \), then \( p = 2 \). By definition 2.4, we have \( H_q \subseteq M' \cup K_1(t) \), where \( M' = M \cup I_t \) and \( M \in \mathcal{M}_2(H_q) \). This means each odd cycle of \( H_q \) must contain at least two of the vertices in \( M' \) since \( \chi(H_q) = 3 \).

For each vertex \( v \in M' \), \( N[v] \) in \( H_q \) can span into a \( S_{d_H(v),q} \). If \( v \in M \), then \( M \) must contain another vertices for each odd cycle. This implies that \( v \) is not a split vertex in \( M \). Due to the minimality of \( M \), \( d_M(v) = d_H(v) \). Note that \( I_t \) and \( K_1(t) \) can form a complete bipartite graph \( K_{t,t} \). If \( v \notin M \), then \( K_{t,t} \) does not contain odd cycles, the odd-ballooning of each edge in \( K_{t,t} \) must contain exactly two vertices in \( M \) due to the minimality of the number of edges in \( M \) and \( q \geq 5 \). This means that there must be \( d_H(v) \) independent edges in \( M \). Using the minimality of \( M \) again, then \( e(M) = e(H) \) and \( M \in \mathcal{H}(H) \). Let \( U \) be the set of the vertices of \( H \) which are split. Since \( H[U] \subseteq K_{t,t} \), then \( \chi(H[U]) \leq 2 \) and \( M \in \mathcal{H}_2(H) \). Since \( M \) is arbitrary, hence \( \mathcal{M}_2(H_q) \subseteq \mathcal{H}_2(H) \). For each \( H' \in \mathcal{H}_2(H) \), \( H_q \subseteq (H' \cup I_t) \cup K_1(t) \), by definition 2.4, \( H' \in \mathcal{M}_2(H_q) \). Hence, \( \mathcal{H}_2(H) \subseteq \mathcal{M}_2(H_q) \). \( \Box \)

Thus, using the lemma above, we have the following decomposition family of both \( P_{k+1,q} \) and \( S_{k+1,q} \) where \( q \geq 5 \):

\[
\mathcal{M}(S_{k+1,q}) = \mathcal{H}_2(S_{k+1}) = \{ S_{k+1}, M_{2k} \},
\]

\[
\mathcal{M}(P_{k+1,q}) = \mathcal{H}_2(P_{k+1}) = \{ \text{all linear forests with } k \text{ edges} \}.
\]

For a family of forbidden graph \( \mathcal{L} \) with decomposition family \( \mathcal{M}(\mathcal{L}) \), we have the following proposition:

**Proposition 2.6**

\[
e(T_p(n)) + \text{ex} \left( \left\lceil \frac{n}{p} \right\rceil, \mathcal{M}(\mathcal{L}) \right) \leq \text{ex} \left( n, \mathcal{L} \right) \leq e(T_p(n)) + (1 + o(1)) p \cdot \text{ex} \left( \left\lceil \frac{n}{p} \right\rceil, \mathcal{M}(\mathcal{L}) \right) + cn,
\]

where \( c = c(\mathcal{L},p) \) depends only on \( \mathcal{L} \) and \( p \).

This proposition give a bound of the extremal number of given graph, and for the lower bound, the graph is obtained by embedding a copy of a graph from \( EX \left( \left\lceil \frac{n}{p} \right\rceil, \mathcal{L} \right) \) into the largest part of \( T_{n,p} \). For the upper bound, which was discussed in [11].

Recall the definition of \( G_{n,p,k} \) in Section 1, we have
Lemma 2.7 Let $k \geq 1$, $n \geq 2k$, then
\[ e(n, M(P_{k+1,q}), M(S_{k+1,q})) \geq \frac{1}{2}k(k-1), \]
where the equality holds for the graph in $G_{n,1,k}$.

Proof Let $G$ be any graph in $EX(n, M(P_{k+1,q}), M(S_{k+1,q}))$. It is clear that the graph in $G_{n,1,k}$ do not contain any graph in both $M(P_{k+1,q})$ and $M(S_{k+1,q})$ as a subgraph. It means that
\[ e(G) = ex(n, M(P_{k+1,q}), M(S_{k+1,q})) \geq \frac{1}{2}k(k-1). \]
This implies the desired result. \hfill \Box

By proposition 2.6 and lemma 2.7, it is clear that:

Lemma 2.8 For every integer $k \geq 3$ and odd integer $q \geq 3$, let $G$ be the extremal graph for both $S_{k+1,q}$ and $P_{k+1,q}$, then we have $e(G) \geq e(G_{n,2,k})$.

The following stability result was proved by Erdős [5] and Simonovits [12], which was given a rough structure of the extremal graphs for a graph $H$ with $\chi(H) = r \geq 3$ and $H \neq K_r$.

Lemma 2.9 ([5],[12]) Let $H$ be a graph with $\chi(H) = r \geq 3$ and $H \neq K_r$. Then for every $\gamma > 0$, there exists $\delta > 0$ and $n_0 = n_0(H, \gamma) \in \mathbb{N}$ such that the following holds: If $G$ is an $H$-free graph on $n \geq n_0$ vertices with $e(G) \geq ex(n, H) - \delta n^2$, then there exists a partition of $V(G) = V_1 \cup \cdots \cup V_{r-1}$ such that $\sum_{i=1}^{r-1} e(V_i) < \gamma n^2$.

Let $V_0 \cup V_1$ be a partition of $V(G)$ such that $e(V_0, V_1)$ is maximized, lemma 2.9 implies that $m = e(V_0) + e(V_1) < \gamma n^2$ where $\gamma$ is any positive integer. The following claim asserts that the partition is close to being balanced.

Corollary 2.10 Let $G$ be an $H$-free $(H \neq K_r, r \geq 3)$ graph on $n$ vertices with classes $V_0$ and $V_1$, $\gamma$ be any positive number. Then, we have
\[ \frac{n}{2} - \sqrt{\gamma n} < |V_i| < \frac{n}{2} + \sqrt{\gamma n}, \text{ for } i = 0, 1. \]
Furthermore, $e(V_0) + e(V_1) \geq \frac{1}{2}k(k-1)$ and if the equality holds then $G$ contains a complete graph $K_k$ with classes $V_0$ and $V_1$.

Proof Let $|V_0| = \frac{n}{2} + a$, then $|V_1| = \frac{n}{2} - a$. Let $k$ be any positive integer, $m = e(V_0) + e(V_1)$. Since
\[ \frac{n^2}{4} + \frac{1}{2}k(k-1) = t_{n,2} + \frac{1}{2}k(k-1) \leq e(G) \leq |V_0||V_1| + m = \frac{n^2}{4} - a^2 + m, \]
we have $m \geq \frac{1}{2}k(k-1)$ and $m \geq a^2$. By lemma 2.9, $m < \gamma n^2$, $a^2 < \gamma n^2$. Hence, $|a| < \sqrt{\gamma n}$. If $m = \frac{1}{2}k(k-1)$, then
\[ t_{n,2} + \frac{1}{2}k(k-1) \leq e(G) \leq |V_0||V_1| + \frac{1}{2}k(k-1). \]
Hence, $|V_0||V_1| = t_{n,2}$, which means that $V_0, V_1$ are balanced and $G$ contains a complete graph $K_k$ with classes $V_0$ and $V_1$. \hfill \Box
If there exists a Turán graph \( T_{2n,2} \), where \( 2a < n \), in \( G \) with classes \( V'_0 \) and \( V'_1 \) such that \( V'_0 \subseteq V_0 \) and \( V'_1 \subseteq V_0 \), then we denote by \( W \) the set of vertices in \( G - T_{2n,2} \) that are joined to all vertices in \( T_{2n,2} \). Let \( G_i = G[V_i] \), \( \Delta_i = \Delta(G_i) \) and \( \nu_i = \nu(G_i) \), \( i = 0, 1 \). For a vertex \( x \in V_i \), let \( E_{1-i}(x) = \{ e \in E(G_{1-i}) \mid V(e) \cap N_G(x) \neq \emptyset \} \).

**Lemma 2.11** Let odd integer \( q \geq 5 \), \( a \left( < 1/2n \right) \) be a sufficiently large number and \( G \) be a graph with a partition vertices into two parts,

\[
V(G) = V_0 \cup V_1.
\]

Moreover, let \( V'_1 \subseteq V_i, V''_1 \subseteq V_i \setminus V'_1, |V''_1| = a \) and \( G[V'_0 \cup V'_1] = T_{2n,2} \), the vertex in \( V''_1 \) is joined to each vertex of \( V'_1 \) for \( i = 0, 1 \). Then we have:

1. If \( G \) is \( P_{k+1,q} \)-free, then \( G[V''_0 \cup V''_1] \) does not contain linear forest with more than \( k - 1 \) edges. Moreover, the number of matches in \( G \) does not more than \( k \).

2. If \( G \) is \( S_{k+1,q} \)-free, then we have the following two result:
   a. \( |W| = 0 \),
   b. and for any vertex \( x \in V_i \),

\[
\deg_{G_i} (x) + \nu(G_i - N_{G_i} | x) + \nu(G[E_{1-i}(x)]) \leq k - 1.
\]

**Proof** (1). If \( G \) is \( P_{k+1,q} \)-free, we will prove \( G[V''_0 \cup V''_1] \) does not contain a linear forest with \( k \) edges. Suppose to the contrary that there is a linear forest with \( k \) edges in \( G[V''_0 \cup V''_1] \). Define

\[
\xi(j) = \begin{cases} 
1 & \text{if } j \text{ is odd}, \\
0 & \text{if } j \text{ is even}.
\end{cases}
\]

Then we will find a copy of \( P_{k+1,q} \) in \( G \) as follows.

Case 1. The linear forest with \( k \) edges is a path \( P_{k+1} \). Let \( P_{k+1} : u_1 u_2 \cdots u_{k+1} \). Without loss of generality, suppose that there is a path \( P_{k+1} \) in \( G[V''_0] \), then for each \( i \in [1, k] \), we find a sequence of vertices \( w^1_{i1}, w^2_{i1}, \ldots, w^q_{i1} \) with \( w^j_{i1} \in V''_1 \) for \( 1 \leq j \leq q - 2 \), such that \( u_iu_{i+1}w^2_{i1}w^3_{i1}w^q_{i1}w^q_{i1}u_{i+1}u_i \) is a \( q \)-cycle \( (q \geq 5) \). Furthermore, we require that \( w^j_{i1} \) (\( l \in [1, k], j \in [1, q - 2] \)) are pairwise different. Hence, the \( P_{k+1} \) can be extended to a copy of \( P_{k+1,q} \), a contradiction.

Case 2. The linear forest with \( k \) edges contains both \( P_2 \) and the paths with more than \( 2 \) vertices.

Without loss of generality, suppose that there are \( tP_2 \) in the linear forest, denoted as \( M_t \), and the rest paths in the linear forest are

\[
P_{t_1+1}, P_{t_2+1}, \ldots, P_{t_m+1},
\]

where \( t_i \geq 2, i = 1, 2, \ldots, m \). Then \( t + t_1 + t_2 + \cdots + t_m = k \). First, we will show that there exists a path \( P_{t+1} \) which can be extended to a copy of \( P_{t+1,q} \).

If the edges of \( M_t \) are in the same part, say \( G[V''_0] \), then we can easily find a \( P_{t+1} : u_1 u_2 \cdots u_{t+1} \) in \( G[V''_0 \cup V''_1] \), since \( G[V''_0 \cup V''_1] = T_{2n,2} \) and \( a \) is sufficiently large. For each \( i \in [1, t] \), we can find vertices \( w^2_{i1}, w^3_{i1}, \ldots, w^q_{i1} \) one by one,
then a common neighbor of \( w_0^{q-3} \) and \( u_t \), say \( w_1^{l_1} \), in \( V_1 \) and a common neighbor of \( w_0^{q-3} \) and \( u_{t+1} \), say \( w_1^{l_{t+1}} \), in \( V_1 \), such that
\[
 u_t w_1^{l_1} w_0^{q-3} w_1^{l_{t+1}} u_{t+1}
\]
is a \( q \)-cycle. Hence, the path \( P_{t+1} \) can be extended to \( P_{t+1,q} \).

If the edges of \( M_t \) are in the different parts, we assume that there \( a \) edges in \( G[V_0''] \) and \( b \) edges in \( G[V_1''] \), where \( a + b = t \), \( a, b \geq 1 \). We can find a \( P_{a+1} \) in \( G[V_0' \cup V_1''] \) and at least one endpoint of this path is in \( G[V_i'''] \) (\( i \in \{0, 1\} \)). Let
\[
P_{a+1} : u_1 w_1^{l_1} \cdots w_{a+1}^{l_{a+1}}, \quad u_1 \in G[V_0''].
\]
For each \( l \in [1, a] \), we can find vertices \( w_0^{l_1}, w_1^{l_{a+1}} \) one by one, then a common neighbor of \( w_0^{l_1} \) and \( u_t \), say \( w_1^{l_{t+1}} \), in \( V_1 \) and a common neighbor of \( w_0^{l_{a+1}} \) and \( u_{t+1} \), say \( w_1^{l_{t+2}} \), in \( V_1 \), such that
\[
 u_t w_1^{l_1} w_0^{l_{a+1}} w_1^{l_{t+1}} u_{t+1} w_1^{l_{t+2}}
\]
is a \( q \)-cycle. Then we can find a \( P_{b+1} \) in \( G[V_0' \cup V_1'] \) and one endpoint of this path is \( u_1 \). Similarly, each edge of \( P_{b+1} \) can be extended to \( q \)-cycle by the method above. It is not hard to find that \( P_{a+1} \) together with \( P_{b+1} \) forms a \( P_{a+b+1} = P_{t+1} \). Hence, the path \( P_{t+1} \) can be extended to \( P_{t+1,q} \).

Assume \( P_{t+1} : u_1 w_1^{l_1} \cdots u_{t+1} \) is the path obtained above, and \( P_{t+1} \) is a path \( v_1 v_2 \cdots v_{t+1} \) in \( G[V_i'''] \). \( P_{t+1} \) is a path \( v_1 w_2 \cdots w_{t+1} \) in \( G[V_i'''] \), where \( i \in \{0, 1\} \). Without loss of generality, we can assume that \( u_1 \in V_1'' \). Then we can find a path
\[
P_{t+1,t+1} : u_{t+1} w_1^{l_1} \cdots u_t v_2 \cdots v_t w_2 \cdots w_{t+1},
\]
where \( w \) is a common neighbor of \( v_1 \) and \( w_2 \) in \( G[V_0' \cup V_1'] \). Each edge of the path \( P_{t+1,t+1} \) can be extended into \( q \)-cycle by the method in Case 1. The edge \( u_1 v_2 \) together with the edge \( v_1 v_2 \) and other \( q - 3 \) vertices in \( T_{2a,2} \) except the vertices which form \( q \)-cycle in \( P_{t+1} \) and \( w \) forms a \( q \)-cycle. The edge \( v_1 w_2 \) together with the edge \( v_1 v_2 \) and other \( q - 3 \) vertices in \( T_{2a,2} \) except the vertices which form \( q \)-cycle in \( P_{t+1} \) and \( w \) forms a \( q \)-cycle. The edge \( w_2 w_2 \) together with the edge \( w_1 w_2 \) and other \( q - 3 \) vertices in \( T_{2a,2} \) except the vertices which form \( q \)-cycle in \( P_{t+1} \) and \( w \) forms a \( q \)-cycle. Hence, together with the \( P_{t+1,q} \) obtained above, the path \( P_{t+1,t+1} \) can be extended to a copy of \( P_{t+1,t+1,q} \). Using the same procedure repeatedly, we can find a copy of \( P_{k+1,q} \), a contradiction.

Case 3. The linear forest with \( k \) edges only contains paths with more than \( 2 \) vertices.

Let the paths with more than \( 2 \) edges in the linear forest be:
\[
P_{t+1}, P_{t+1}, \ldots, P_{t_m+1},
\]
where \( 2 \leq t_i \leq k - 2 \), \( i = 1, 2, \ldots, m \). Next, we will find a path \( P_{k+1} \) which can be extended to \( P_{k+1,q} \).

If \( P_{t+1} \) and \( P_{t+1} \), denoted as \( v_1 v_2 \cdots v_{t+1} \) and \( w_1 w_2 \cdots w_{t+1} \) respectively, are in the same part, say \( G[V_i'''] \), then we can find a copy of path
\[
P_{t+1,t+1} : v_1 v_2 \cdots v_{t+1} w w_2 \cdots w_{t+1},
\]
where \( w \) is a common neighbor of \( v_1 \) and \( w_2 \) in \( V_i''' \). That is possible (by Lemma 2.9 and Corollary 2.10) since \( G \) always contains triangle. Then each edge of the
path \( P_{t_1+1} \) and \( P_{t_2+1} \) can be extended into \( q \)-cycle by the method in Case 1. The edge \( v_1w \) together with the edge \( v_1v_{t_1+1} \) and other \( q-3 \) vertices in \( T_{2a,2} \) except the vertices which form \( q \)-cycle in \( P_{t_1+1} \) and \( P_{t_2+1} \) forms a \( q \)-cycle. The edge \( w_2w_2 \) together with the edge \( w_1w_2 \) and other \( q-3 \) vertices in \( T_{2a,2} \) except the vertices which form \( q \)-cycle in \( P_{t_1+1}, P_{t_2+1} \) and \( v_1w \) forms a \( q \)-cycle. Hence, the path \( P_{t_1+t_2+1} \) can be extended to a copy of \( P_{t_1+t_2+1,q} \). Using the same procedure repeatedly, we can find a copy of \( P_{k+1,q} \), a contradiction.

If \( P_{t_1+1} \) and \( P_{t_2+1} \) are in the different parts, without loss of generality, we assume that the path \( P_{t_1+1} \) in \( G[V_0^q] \), the path \( P_{t_2+1} \) in \( G[V_1^q] \). Then we can find a path

\[
P_{t_1+t_2+1} : v_1v_2 \cdots v_{t_1+1}w_3 \cdots w_{t_2+1}.
\]

Then each edge of the paths \( P_{t_1+1} \) and \( P_{t_2+1} \) can be extended into \( q \)-cycle by the method in Case 1. The edge \( v_{t_1+1}w_2 \) together with the edge \( w_1w_2 \) and other \( q-3 \) vertices in \( T_{2a,2} \) except the vertices which form \( q \)-cycle in \( P_{t_1+1} \) and \( P_{t_2+1} \) forms a \( q \)-cycle. Hence, the path \( P_{t_1+t_2+1} \) can be extended to a copy of \( P_{t_1+t_2+1,q} \). Using the same procedure repeatedly, we can find a copy of \( P_{k+1,q} \), a contradiction.

Case 4. The linear forest with \( k \) edges only contains paths with 2 edges.

In this case, there are exactly \( kP_2s \) in the linear forest, denoted as \( M_k \). Then we can find a path \( P_{t_1+1} \) in \( G[V_0^q \cup V_1^q] \) by using the method in Case 2 and it can be extended to a copy of \( P_{t_1+1,q} \), a contradiction.

Furthermore, if the number of matches in \( G \) more than \( k \), we can move the vertices, which are the endpoint of the paths, in \( T_{2a,2} \) to \( G \setminus T_{2a,2} \). Let \( T_{2a',2} \) be the new graph from \( T_{2a,2} \) by moving vertices. Hence, we have to move at least \( k \) vertices from \( T_{2a,2} \) to \( G \setminus T_{2a,2} \) such that there is no edges between \( T_{2a',2} \) and \( G \setminus T_{2a',2} \). Then there exists a linear forest with more than \( k \) edges in \( G \setminus T_{2a',2} \) which is contradiction.

(2). If \( G \) is \( S_{k+1,q} \)-free, then we will prove that \( |W| = 0 \). Suppose to the contrary that there exists a vertices \( w \) in \( W \). It is clear that \( k \) paths, which length is \( q-2 \), can be find in \( T_{2a,2} \) where the vertices of the paths are all different and each endpoint of the paths is adjacent to \( w \). Then \( w \) together with these \( k \) \( (q-2) \)-paths form a copy of \( S_{k+1,q} \), a contradiction. \( \square \)

**Remark 1** The second result (b) was proved by Hou in the Claim 10 in [6]. It means that if \( G \) is \( S_{k+1,q} \)-free then there are at most \( k-1 \) edges in \( G \).

In this article, our main method is the so-called progression induction which was introduced by Simonovits in 1968. This method is similar to the mathematical induction and Euclidean algorithm and combine from them in a certain sense.

**Lemma 2.12** [12] Let \( \mathcal{U} = \bigcup_{i=1}^{\infty} \mathcal{U}_n \) be a set of given elements, such that \( \mathcal{U}_n \) are disjoint subsets of \( \mathcal{U} \). Let \( \mathcal{B} \) be a condition or property defined on \( \mathcal{U} \) (i.e. the elements of \( \mathcal{U}_n \) may satisfy or not satisfy \( \mathcal{B} \)). Let \( \Theta \) be a function defined also on \( \mathcal{U} \) such that \( \Theta \) is a nonnegative integer and

1. if \( a \) satisfies \( \mathcal{B} \), then \( \Theta (a) \) vanishes,

2. there is an \( M_0 \) such that if \( n > M_0 \) and \( a \in \mathcal{U}_n \) then either a satisfies \( \mathcal{B} \) or there exist an \( n' \) and an \( a' \) such that

\[
\frac{n}{2} < n' < n, a' \in \mathcal{U}_{n'} \text{ and } \Theta (a) < \Theta (a') .
\]
Then there exists an $n_0$ such that if $n > n_0$, from $a \in \mathcal{U}_n$ follows that $a$ satisfies $\mathfrak{B}$.

3 Proof of the main theorem

In this section, we will prove that $ex(n,S_{k+1,q},P_{k+1,q}) = t_{n,2} + \frac{1}{2}(k^2 - k)$ for any integer $k \geq 1$ and any odd integer $q \geq 3$, when $n$ is sufficiently large. For $q = 3$, the result was already proved by Zhu in [13]. So in the later article, we only consider the extremal graph for both odd-balooning of paths and odd-balooning of stars with odd integer $q \geq 5$.

Let $F_n$ be an extremal graph for both $S_{k+1,q}$ and $P_{k+1,q}$ on $n$ vertices, $G_n$ be any graph in $G_{n,2,k}$. By lemma 2.8, $e(F_n) \geq t_{n,2} + \frac{1}{2}k(k - 1)$. We will show that $e(F_n) = t_{n,2} + \frac{1}{2}k(k - 1)$ and the extremal graph is the Turán graph $T_{n,2}$ embedding a complete graph $K_k$ into one partite set.

Our main theorem will be proved by progressive induction, where $\mathcal{U}_n$ is the set of extremal graph for both $S_{k+1,q}$ and $P_{k+1,q}$ on $n$ vertices. Property $\mathfrak{B}$ states that $e(F_n) \leq e(G_n)$ and the equality holds if and only if $F_n \in G_{n,2,k}$.

Define $\Theta(F_n) = e(F_n) - e(G_n)$ (\geq 0). By lemma 2.12, it is enough to show that if $e(F_n') \leq e(F_n)$, then either $F_n' \in G_{n,2,k}$ or there exists an $n' \in \left(\frac{2}{3}, n\right)$ such that $\Theta(F_n') > \Theta(F_n)$ when $n$ is sufficiently large.

Now, we will find a subgraph of $F_n$ satisfying the lemma 2.11. By lemma 2.1, for sufficiently large $n$, $F_n$ must contain a Turán graph $T_{2r,2}$, where $r$ is sufficiently large. We divide $T_{2r,2}$ into two partite set $B_0, B_1$ in $F_n$. Denote $\widehat{F}_{n-2r}$ the rest of the graph $F_n - T_{2r,2}$, $e_u$ the number of edges joining $\widehat{F}_{n-2r}$ and $T_{2r,2}$. Then we have

$$e(F_n) = e(\widehat{F}_{n-2r}) + e_u + t_{2r,2}.$$ 

Next, we partition the vertices of $\widehat{F}_{n-2r}$ into several vertex sets. Let $c$ be a sufficiently small constant. Denote by $W$ the vertices joining to at least $cr$ vertices of each class of $T_{2r,2}$. Let $i \in \{0, 1\}$. If $x$ is adjacent to less than $c^2r$ vertices of $B_i$ and is adjacent to at least $(1 - c)cr$ vertices of $B_{1-i}$, then let $x \in C_i$. If $x$ is adjacent to less than $c^2r$ vertices of $B_i$ and is adjacent to less than $(1 - c)cr$ vertices of $B_{1-i}$, then let $x \in D$. We has already known that there is no vertex in $W$ by lemma 2.11.

Obviously, $C_0 \cup C_1 \cup B_0 \cup B_1 \cup D$ is the partition of $V(F_n)$. Since both $\mathcal{M}(P_{k+1,q})$ and $\mathcal{M}(S_{k+1,q})$ contain a matching with size $k$ and each vertex of $C_i$ is adjacent to less than $c^2r$ vertices of $B_i$, is adjacent to at least $(1 - c)cr$ vertices of $B_{1-i}$. Hence, there are at most $k$ independent edges in $B_i \cup C_i$. Consider the edges joining $B_i$ and $C_i$ and select a maximal set of independent edges, say $x_1y_1, \cdots, x_ty_t$ with $x_i \in B_i, y_j \in C_i$ and $i \in \{0, 1\}, j \in \{1, 2, \cdots, t\}$. Since the number of vertices of $B_i$ joining to at least one of $y_1, \cdots, y_t$ is less than $c^2rk$ and the remaining vertices of $B_i$ are not adjacent to any vertices of $C_i$. Hence, we can move $c^2rk$ vertices of $B_i$ to $C_i$, obtain $B'_i$ and $C'_i$ such that there is no edge between $B'_i$ and $C_i$. Let $T_{2r',2}$ be the new graph obtained from above with class $B'_0$ and $B'_1$ and $\widehat{F}$ be the remain graph, that is $\widehat{F} = F_n - T_{2r',2}$, with class $C'_0, C'_1$ and $D$. We conclude that the remain graph $\widehat{F}$, as the induced subgraph of $F_n$, satisfying that the vertices in $C'_i$ is adjacent to no vertex in $B'_i$.  

10
is adjacent to at least \((1 - c - ck) cr\) vertices of \(B_{i-1}^r\) and the vertices in \(D\) is adjacent to less than \(c^2r\) vertices of \(B_i^r\) and less than \((1 - c) cr\) vertices of \(B_{i-1}^r\).

Denote by \(e_u'\) the number of edges joining \(V(\hat{F})\) and \(V(T_{2r',2})\). Then we have

\[
\Theta(F_n) = e(F_n) - e(G_n) = e(T_{2r',2}) - e(T_{2r',2}) + (e_{u'} - e_{v'}) + e(\hat{F}) - e(G_{n-2r'})
\]

\[
\leq (e_{u'} - e_{v'}) + e(F_{n-2r'}) - e(G_{n-2r'})
\]

\[
= (e_{u'} - e_{v'}) + \Theta(F_{n-2r'}). 
\]

If \(e_{u'} - e_{v'} < 0\), then \(\Theta(F_n) < \Theta(F_{n-2r'})\). Since \(n - 2r' \in \left(\frac{n}{2}, n\right)\), we are done. Thus, we may assume that \(e_{u'} - e_{v'} \geq 0\), then

\[
e_{u'} - e_{v'} \leq (n - 2r' - |D|) r' + |D| \left(c^2r + (1 - c) cr\right) - (n - 2r') r'
\]

\[
= |D| \left(c^2r - (1 - c) cr - r'\right)
\]

\[
= (2c^2r - cr - r')|D|
\]

\[
\leq 0,
\]

where the equality holds if and only if \(|D| = 0\). Hence, we have \(e_{u'} = e_{v'}\) and each vertex of \(C_i'\) is adjacent to each vertex of \(B_i'^r\). Since \(F_n\) does not contain both \(P_{k+1,q}\) and \(S_{k+1,q}\) as a subgraph, the vertices in \(C_i'\) satisfy the condition in lemma 2.11 and, by corollary 2.10, the subgraph \(F_n \setminus T_{2r',2}\) is balance and there is a complete graph \(K_k\) in it.

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