MAXIMAL ORTHOGONAL SETS OF UNIMODULAR VECTORS OVER FINITE LOCAL RINGS OF ODD CHARACTERISTIC

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ABSTRACT. Let \( R \) be a finite local ring of odd characteristic and \( \beta \) a non-degenerate symmetric bilinear form on \( R^2 \). In this short note, we determine the largest possible cardinality of pairwise orthogonal sets of unimodular vectors in \( R^2 \).

1. INTRODUCTION

Two famous distinct distances and unit distances problems for the plane \( \mathbb{R}^2 \) were posed by Erdős [3]. The first problem asks for the minimum number of distinct distances among \( n \) points in the plane while the latter problem asks for the maximum number of the unit distances that can occur among \( n \) points in the plane. These problems were generalized to the \( n \)-dimensional Euclidean space case [4].

Points in the \( n \)-dimensional vector space over \( \mathbb{F}_q \) were considered for similar questions, see [2, 5, 6] for example. In [5], the authors defined a specific distance between two points in \( \mathbb{F}_q^n \) where \( q \) is an odd prime power and studied the distinct distances problem. It is natural to ask the two mentioned problems more generally by using an arbitrary quadratic form over \( \mathbb{F}_q^n \). Recently, the problems in this direction were studied as follow. Let \( \beta \) be a non-degenerate symmetric bilinear form over \( \mathbb{F}_q^n \). The largest possible cardinality of a subset \( S \subset \mathbb{F}_q^n \) so that \( \beta(\vec{x}, \vec{y}) = 0 \) for every distinct vectors \( \vec{x}, \vec{y} \in S \) was determined in [9] and the case of \( \beta(\vec{x}, \vec{y}) = l \) for all \( l \in \mathbb{F}_q \) was treated in [1].

Let \( R \) be a finite local ring of odd characteristic with identity and a non-degenerate symmetric bilinear form \( \beta \) over \( R^2 \). In this paper, we consider a subset \( S \) of unimodular vectors in \( R^2 \) and determine the largest possible size of \( S \) so that for any two distinct vectors \( \vec{x}, \vec{y} \in S, \beta(\vec{x}, \vec{y}) = 0 \). This is a generalization of the problem over \( \mathbb{F}_q^2 \).

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Throughout the paper, we assume that all rings have the identity. In Section 2, we review some backgrounds about bilinear forms over $R^n$. We then show the main result when $n = 2$ in Section 3. Finally, we conclude the paper with some comments in Section 4.

2. BILINEAR FORM OVER FINITE LOCAL RINGS

Let $R$ be a commutative ring and $n$ a positive integer. A bilinear form $\beta$ on $R^n$ is a map $\beta : R^n \times R^n \to R$ such that

$$\beta(x + y, z) = \beta(x, z) + \beta(y, z) \quad \text{and} \quad \beta(r x, z) = r \beta(x, z)$$

and

$$\beta(x, z + w) = \beta(x, z) + \beta(x, w) \quad \text{and} \quad \beta(s z, \bar{w}) = s \beta(x, \bar{z})$$

for all $x, y, z, w \in V$ and $r, s \in R$. Suppose that $\{e_1, e_2, \ldots, e_n\}$ is a basis of $R^n$. For each bilinear form $\beta$ on $R^n$, we have the $n \times n$ associate matrix $B = (\beta(e_i, e_j))$. A bilinear form $\beta$ is said to be symmetric if its associate matrix $B$ is symmetric, and $\beta$ is said to be non-degenerate if $B$ is invertible. A determinant of a bilinear form $\beta$, denoted by $\det \beta$, is defined to be $\det B$. Two bilinear forms $\beta_1$ and $\beta_2$ over $R^n$ with corresponding matrices $B_1$ and $B_2$ are equivalent if there exists an invertible matrix $P$ over $R$ such that $B_2 = P^T B_1 P$.

A local ring is a commutative ring with unique maximal ideal. If $M$ is the unique maximal ideal of a local ring $R$, then the group of units of $R$ is $R \setminus M$. Note that if $u$ is a unit in $R$, then $u + m$ is a unit for all $m \in M$. In case that $R$ is of odd characteristic, it is shown in [8] the classifications of non-degenerate symmetric bilinear forms over $R$.

**Lemma 2.1.** [8] Let $R$ be a finite local ring of odd characteristic with unique maximal ideal $M$. Suppose that $\beta$ is a non-degenerate symmetric form on $R^n$, then one of the following holds:

1. if $n$ is odd, then $\beta$ is equivalent to

$$\beta(x, y) = x_1 y_1 - x_2 y_2 + \cdots + x_{n-2} y_{n-2} - x_{n-1} y_{n-1} + u x_n y_n;$$

2. if $n$ is even, then $\beta$ is equivalent to

$$\beta(x, y) = x_1 y_1 - x_2 y_2 + \cdots + x_{n-3} y_{n-3} - x_{n-2} y_{n-2} + x_{n-1} y_{n-1} - u x_n y_n;$$

where $u = 1$ or $u$ is a non-square unit in $R$, and $\vec{x} = (x_1, x_2, \ldots, x_n), \vec{y} = (y_1, y_2, \ldots, y_n)$ are in $R^n$. 


A vector \( \vec{x} = (x_1, x_2, \ldots, x_n) \) in \( R^n \) is said to be unimodular if the ideal generated by \( x_1, x_2, \ldots, x_n \) is equal to \( R \). In particular, if \( R \) is a field, then every nonzero vector is unimodular. For the case \( R \) is a finite local ring, we have the following lemma on unimodular vectors.

**Lemma 2.2.** [7] Let \( R \) be a finite local ring. Then a vector \( \vec{x} = (x_1, x_2, \ldots, x_n) \in R^n \) is unimodular if and only if \( x_i \) is a unit for some \( i \in \{1, 2, \ldots, n\} \).

### 3. Main result

For a non-degenerate symmetric bilinear form \( \beta \) on \( R^n \), a unimodular orthogonal set is a set \( S \) of unimodular vectors in \( R^n \) in which \( \beta(\vec{x}, \vec{y}) = 0 \) for any two distinct vectors \( \vec{x}, \vec{y} \in S \). We denote by \( S(R, n) \) the largest possible cardinality of a unimodular orthogonal set in \( R^n \). Here, we only consider \( S(R, 2) \) where \( R \) is a finite local ring of odd characteristic.

**Theorem 3.1.** Let \( M \) be the maximal ideal of \( R \). Then

\[
S(R, 2) = \begin{cases} 
|R| - |M|, & \text{det } \beta \text{ is square;} \\
2, & \text{det } \beta \text{ is non square.}
\end{cases}
\]

**Proof.** Assume that \( \text{det } \beta \) is a non square unit. By Theorem 2.1,

\[
\beta(\vec{x}, \vec{y}) = x_1y_1 - x_2y_2
\]

where \( z \) is a fixed non square unit and \( \vec{x} = (x_1, x_2), \vec{y} = (y_1, y_2) \in R^2 \). Let \( S \) be a unimodular orthogonal set in \( R^2 \) and \( (a, b) \in S \). Since \( (a, b) \) is unimodular, \( a \) or \( b \) is a unit (by Lemma 2.2). Suppose that \( a \) is a unit. Then another vectors in \( S \) are of the form \( (a^{-1}zby, y) \) where \( y \) is a unit in \( R \). If \( \vec{y}_1 = (a^{-1}zby_1, y_1), \vec{y}_2 = (a^{-1}zby_2, y_2) \in S \), then \( \beta(\vec{y}_1, \vec{y}_2) = 0 \) implies \( 1 = z(a^{-1}b)^2 \), so \( 1 \in M \) or \( z \) is a square unit which is a contradiction. We have a similar argument for the case \( b \) is a unit. Thus, \( S(R, 2) \leq 2 \). Clearly, \( \{(1,0), (0,1)\} \) is a unimodular orthogonal set. Therefore, \( S(R, 2) = 2 \).

Next, assume that \( \text{det } \beta \) is a square unit. By Theorem 2.1,

\[
\beta(\vec{x}, \vec{y}) = x_1y_1 - x_2y_2
\]

for \( \vec{x} = (x_1, x_2), \vec{y} = (y_1, y_2) \in R^2 \). Clearly,

\[
S = \{(x, x) \mid x \in R \setminus M\}
\]

is a unimodular orthogonal set. Then \( S(R, 2) \geq |R| - |M| \). We show that the converse of the inequality holds. Since \( \text{char}(R) \) is odd, \( |R| - |M| \geq 2 \). If \( |R| - |M| =
2, then \(|R| = 3\) and \(|M| = 1\), i.e., \(R = \mathbb{F}_3\). By Theorem 4 of [1], \(S(R, 2) = 2 = |R| - |M|\). Assume that \(|R| - |M| \geq 3\). Let \(S\) be a maximal unimodular orthogonal set in \(R^2\) and \((a, b) \in S\). Since \((a, b)\) is unimodular, \(a\) or \(b\) is a unit (by Lemma 2.2).

Suppose that \(a\) is a unit. Then another vectors in \(S\) are of the form \((a, y)\), where \(y\) is a unit in \(R\), so \(|S| \leq |R| - |M| + 1\). If \(\mathbf{y}_1 = (a^{-1}by, y_1)\) and \(\mathbf{y}_2 = (a^{-1}by_2, y_2)\) are two distinct vectors in \(S\). then \(\beta(\mathbf{y}_1, \mathbf{y}_2) = 0\) implies \(1 = (a^{-1}b)^2\), and so \((a, b) = (a^{-1}b^2, b)\) is also in that form. It can be argued similarly for the case that \(b\) is a unit. Thus, we have \(S(R, 2) \leq |R| - |M|\). Therefore, the equality holds. \(\square\)

**Remark.** From the proof of Theorem 3.1, we have the following.

1. If \(\det \beta\) is square, then all maximal unimodular orthogonal sets of \(R^2\) are
   - \(\{(a, b), (a^{-1}bzy, y)\}\) where \(a \in R^\times\), \(b \in R\) and \(y \in R^\times\), and
   - \(\{(a, b), (x, (bz)^{-1}ax)\}\) where \(a \in M\) and \(b, x \in R^\times\).

2. If \(\det \beta\) is non-square, then all maximal unimodular orthogonal sets of \(R^2\) are
   \[\{(ux, x) \mid x \in R^\times\}\text{ or }\{(x, ux) \mid x \in R^\times\}\]
   where \(u \in R\) with \(u^2 = 1\). In particular, if \(R = \mathbb{Z}_{p^s}\), then all maximal unimodular orthogonal sets of \(R^2\) are
   \[\{(x, x) \mid x \in R^\times\}, \{(-x, x) \mid x \in R^\times\}\text{ and }\{(x, -x) \mid x \in R^\times\}.\]

4. **Concluding remarks**

The problem of finding the largest cardinality of pairwise orthogonal subset in \(\mathbb{F}_q^n\) has been solved in [1, 9]. In Theorem 3.1, we solve the similar problem for unimodular vectors in \(R^2\) where \(R\) is a finite local ring of odd characteristic by using an elementary counting method from the properties of \(R\). The problem when \(n \geq 3\) could also be considered but it seems to be difficult if we use the method in the proof of Theorem 3.1 because all equations will be more complicated. Unlike the finite fields case [1], the problem when \(n\) is odd and \(R\) is a general finite local ring is not easy to manage even finding a lower bound for \(S(n, R)\) since \(R\) can have zero divisors. We plan to discuss some of these extension works using a new technique in another paper.

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