Locally Differentially Private Minimum Finding

Kazuto Fukuchi\textsuperscript{*1}, Chia-Mu Yu\textsuperscript{†2}, Arashi Haishima\textsuperscript{‡3}, and Jun Sakuma\textsuperscript{§3,1}

\textsuperscript{1}RIKEN AIP, Japan
\textsuperscript{2}National Chung Hsing University, Taiwan
\textsuperscript{3}University of Tsukuba, Japan

May 28, 2019

Abstract

We investigate a problem of finding the minimum, in which each user has a real value and we want to estimate the minimum of these values under the local differential privacy constraint. We reveal that this problem is fundamentally difficult, and we cannot construct a mechanism that is consistent in the worst case. Instead of considering the worst case, we aim to construct a private mechanism whose error rate is adaptive to the easiness of estimation of the minimum. As a measure of easiness, we introduce a parameter $\alpha$ that characterizes the fatness of the minimum-side tail of the user data distribution. As a result, we reveal that the mechanism can achieve $O((\ln^6 N/\epsilon^2 N)^{1/2\alpha})$ error without knowledge of $\alpha$ and the error rate is near-optimal in the sense that any mechanism incurs $\Omega((1/\epsilon^2 N)^{1/2\alpha})$ error. Furthermore, we demonstrate that our mechanism outperforms a naive mechanism by empirical evaluations on synthetic datasets. Also, we conducted experiments on the MovieLens dataset and a purchase history dataset and demonstrate that our algorithm achieves $\tilde{O}((1/N)^{1/2\alpha})$ error adaptively to $\alpha$.

1 Introduction

Statistical analyses with individuals’ data have a significant benefit to our social lives. However, using individuals’ data raises a serious concern about privacy, and privacy preservation is therefore increasingly demanding by social communities. For example, the European Commission (EC) approved a new regulation regarding data protection and privacy, the General Data Protection Regulation (GDPR), which has been in effect since May 2018. With this regulation, any service provider in the world must follow GDPR when providing services to any individuals in the EU.

Motivated by the privacy concern, many researchers developed statistical analysis methods with a guarantee of Differential privacy [Dwork et al., 2006]. The differential privacy prevents privacy leakage in the central model in which a trusted central server\textsuperscript{1} gathers the individuals’ data and then publishes some statistical information about the gathered data to an untrusted analyst. One
limitation of this model is that it requires a trusted central server that processes a differentially private algorithm.

A notion of local differential privacy (LDP) was introduced for preventing privacy leakage to the untrusted central server. Warner [1965] firstly introduced it in the context of surveying individuals. Afterward, Evfimievski et al. [2003] provided a definition of local privacy in a general situation. Many researchers proposed some statistical analysis methods with a guarantee of the local differential privacy. For example, methods for mean and median estimation [Duchi et al., 2013], distribution estimation [Erlingsson et al., 2013, Fanti et al., 2016, Ren et al., 2018], and heavy hitter estimation [Bassily and Smith, 2015] under the LDP guarantee have been investigated so far.

In this paper, we deal with the minimum finding problem under the local differential privacy constraint. In this problem, a number of users hold a real value individually, which can be a sensitive value, and an analyst is interested in the minimum among the values. The minimum finding problem is a primitive but fundamental component for statistical analysis. Even under the privacy constraint, the minimum finding is a necessary first step of statistical analyses.

As we describe later, our mechanism employs binary search to find the interval that contains the minimum. Binary search with local differential privacy has been employed in Gaboardi et al. [2019] for the first time as a component of a mechanism to estimate mean with a confidence interval. Naive application of this mechanism enables minimum finding with local differential privacy, whereas a straightforward application of their utility analysis to minimum finding does not necessarily result in consistent estimation. This is because their utility analysis is specifically derived for their main task, mean estimation with a confidence interval. Further analysis with an additional assumption is needed for deriving a locally private mechanism that can consistently estimate the minimum.

Our contributions are listed as follows:

**Hardness in the worst case** We reveal that the minimum finding problem under the local differential privacy constraint is fundamentally difficult in the worst case. We will prove the fact that there is no locally differentially private mechanism that consistently estimates the minimum under the worst case users’ data distribution.

**LDP mechanism with adaptiveness to α-fatness** Instead of considering the worst case, we construct a locally differentially private mechanism that is adaptive to the easiness of estimation of the minimum, which is determined by the underlying user data distribution. As a measure of easiness, we introduce α-fatness which characterizes the fatness of the minimum-side tail of the user data distribution. Here, a smaller α indicates that the tail is fatter. The minimum finding problem becomes apparently easier when the underlying distribution is fat because we can expect that a greater portion of data is concentrated around the minimum if the distribution is fatter. Hence, we can expect that the decreasing rate of the estimation error becomes smaller as α decreases. The definition of α-fatness is given as follows:

**Definition 1 (α-fatness).** Let $F$ be the cumulative function of the user data distribution supported on $[-1, 1]$. For a positive real α, the distribution of $F$ is α-fat if there exist universal constants $C > 0$ and $\bar{x} \in [-1, 1]$ such that for all $x_{\text{min}} < x < \bar{x}$, $F(x) \geq C(x - x_{\text{min}})^{\alpha}$ where $x_{\text{min}} = \inf\{x \in [-1, 1] : F(x) > 0\}$ is the minimum under $F$.

For example, any truncated distribution, such as the truncated normal distribution, satisfies Definition 1 with $\alpha = 1$. The beta distribution with parameters $\alpha$ and $\beta$ is α-fat. For simplicity, we say $F$ is α-fat if the $F$’s distribution is α-fat.

**Utility analyses** We derive adaptive upper bounds on the mean absolute error of the present mechanism as utility analyses and reveal that these bounds are nearly tight. Under the assumption
that the server knows a lower bound on $\alpha$, we show that the mean absolute error is $O((\ln^2 N/\epsilon^2 N)^{1/2\alpha})$, where $N$ denotes the number of users, and $\epsilon$ is the privacy parameter. If $\alpha$ is unknown to the server, we show that the mean absolute error is $O((\ln^2 N/\epsilon^2 N)^{1/2\alpha})$. Also, we prove that these upper bounds are nearly tight in the sense that any locally differentially private mechanism incurs $\Omega((1/\epsilon^2 N)^{1/2\alpha})$ error under the $\alpha$-fatness assumption. The error rates of our mechanism become slower as $\alpha$ increases; this reflects the intuition about the easiness of estimation mentioned before. Note that this decreasing rate can be achieved even though the algorithm can use only imperfect knowledge on $\alpha$ (e.g., lower bound on $\alpha$) or no information about $\alpha$.

**Empirical evaluation** We conducted some experiments on real and synthetic datasets for evaluating the performance of the proposed mechanism. In the experiment on the synthetic datasets, we firstly confirm the tightness of the theoretical bounds regarding $N$ and $\epsilon$. Furthermore, we demonstrate by the experiment that the present mechanism outperforms a baseline method based on the Laplace mechanism. In the experiment on the real datasets, we evaluate the performance of the proposed mechanism on the MovieLens dataset and a customers’ purchase history dataset. As a result, we present that the proposed mechanism succeeds to achieve $\tilde{O}(1/N^{1/2\alpha})$ rate adaptively to $\alpha$, where the notation $\tilde{O}$ ignores the logarithmic factor.

All the missing proofs can be found in the appendix.

## 2 Preliminaries

We introduce the interactive version of the privacy definition given by Duchi et al. [2013]. Suppose that an individual has a data on a domain $X$. To preserve privacy, the data is converted by a random mechanism $M$ before sending the data to the untrusted server, where the domain of the converted value is denoted as $Z$. In the interactive setting, a mechanism $M$ is a probabilistic mapping from $X \times Z^*$ to $Z$, where $Z^* = \bigcup_{n=1}^{\infty} Z^n$. This means that when the mechanism converts the user’s data, it can utilize the sanitized data that have been already revealed to the server. Privacy of the mechanism $M$ is defined as follows:

**Definition 2** (Local differential privacy [Duchi et al., 2013]). A stochastic mechanism $M$ is $\epsilon$-locally differentially private if for all $x, x' \in X$, all $z \in Z^*$, and all $S \in \sigma(Z)$,

$$\mathbb{P}\{M(x, z) \in S\} \leq e^\epsilon \mathbb{P}\{M(x', z) \in S\},$$

where $\sigma(Z)$ denotes an appropriate $\sigma$-field of $Z$.

The parameter $\epsilon$ determines a level of privacy; that is, smaller $\epsilon$ indicates stronger privacy protection. Roughly speaking, the local differential privacy guarantees that the individual’s data cannot be certainly inferred from the privatized data even if an adversary has unbounded computational resources and any prior knowledge.

As a simple implementation of the locally differentially private mechanism, the randomized response proposed by Warner [1965] is known. This is a mechanism for binary data and outputs binary value. Let $X = Z = \{-1, 1\}$, and let $x$ and $z$ be the individual’s data and privatized data by the randomized response, respectively. Then, the randomized response flips the individual’s data $x$ with probability $\frac{1}{1+e^\epsilon}$, and thus we have $z = x$ with probability $e^\epsilon/1+e^\epsilon$ and $z = -x$ with probability $1/1+e^\epsilon$. This mechanism ensures $\epsilon$-local differential privacy.

**Notations.** We denote the indicator function as $\mathbb{1}_x$ for a predicate $x$. Let $\text{sign}(x) = 1$ if $x \geq 0$, and $\text{sign}(x) = -1$ if $x < 0$. For an event $\mathcal{E}$, we denote its complement as $\mathcal{E}^c$. 

3
3 Problem Setup

Suppose there are two stakeholders; users and aggregator. There are $N$ users. They have real-valued data $x_i \in [-1, 1]$ and want $x_i$ to be private. Let $x^{(1)} \leq x^{(2)} \leq \ldots \leq x^{(N)}$ be ordered data. We investigate two similar settings regarding users’ data generation process.

(Fixed data) The users’ data are fixed by some unknown rule.

(i.i.d. data) The users’ data are drawn i.i.d. from some unknown distribution.

The aggregator in the fixed data setting aims to obtain the minimum among the users’ data, whereas he/she in the i.i.d. data setting aims to obtain the minimum within the support of the underlying users’ data distribution.

The unknown rule or distribution is described by a non-decreasing function $F: [0, 1] \rightarrow [-1, 1]$. In the fixed data setting, the function $F$ determines the empirical cumulative distribution of the users’ data. More precisely, the users’ data are determined such that $F(x^{(i)}) = (i - 1)/(N - 1)$ for all $i = 1, \ldots, N$. In the i.i.d. data setting, $F$ is the cumulative distribution function of the unknown user data distribution. In both settings, the minimum of the users’ data is defined as $x_{\text{min}} = \inf\{x : F(x) > 0\}$.

The utility of the estimation is measured by the absolute mean error between the true and estimated values of the minimum. Let $\tilde{x}$ be the estimated value. Then, the absolute mean error is defined as

$$\text{Err} = \mathbb{E}[|x_{\text{min}} - \tilde{x}|],$$

where the expectation is taken over randomness of the sanitization mechanism and data generation (in the i.i.d. data setting). When the users send information regarding $x_i$ to the aggregator, it must be sanitized in the locally differentially private way. Given the privacy parameter $\epsilon > 0$, our goal is to estimate $x_{\text{min}}$ that minimizes the absolute mean error in Eq (1) under the constraint of the $\epsilon$-local differential privacy.

In later discussions, we use $\tilde{F}(x) = \frac{1}{N} \sum_{i=1}^{N} 1_{x_i \leq x}$. We define the quantile function of $F$ and $\tilde{F}$ as $F^*(\gamma) = \inf\{\tau : F(\tau) \geq \gamma\}$ and $\tilde{F}^*(\gamma) = \inf\{\tau : \tilde{F}(\tau) \geq \gamma\}$, respectively.

4 Algorithm

In this section, we derive an algorithm for the locally private finding minimum problem. Algorithm 1 shows the non-private version of the proposed algorithm. It employs the binary search algorithm to find the interval containing the minimum from $2^L$ distinct intervals obtained by evenly dividing the data domain $[-1, 1]$, where $L$ is some positive integer. More precisely, Algorithm 1 iteratively updates the interval $[\ell_t, r_t]$, where the left-endpoint, midpoint, and right-endpoint of the interval are denoted as $\ell_t$, $\tau_t$, and $r_t$, respectively. In Line 1, Algorithm 1 initializes the first interval $[\ell_1, r_1]$ as the data domain. Then, for each round $t$, the algorithm halves the interval into $[\ell_t, \tau_t]$ and $[\tau_t, r_t]$ and then chooses either of them that contains the minimum $x^{(1)}$ (in Lines 3-10). After $L$ iterations, the interval becomes the desired one. The algorithm outputs the middle of the interval as the estimated value (in Line 11). Because the length of the last interval is $2^{-L+1}$ by construction, the error of the estimated value is up to $2^{-L}$.

To identify which $[\ell_t, \tau_t]$ and $[\tau_t, r_t]$ contains the minimum, Algorithm 1 first asks each user whether or not his/her data is smaller than $\tau_t$ (in Line 4). After that, Algorithm 1 calculates the empirical
cumulative distribution at \( \tau_t, \tilde{F}(\tau_t) \), based on their responses. In Algorithm 1, it is denoted as \( \Phi(z) \) in Line 6. Then, \([\ell_t, \tau_t]\) contains the minimum if \( \tilde{F}(\tau_t) > 0 \), and \([\tau_t, \tau_t]\) does otherwise.

Algorithm 2 shows the privatized version of Algorithm 1. Algorithm 1 accesses the users’ data only through a query that asks whether or not their data is smaller than \( \tau_t \). We sanitize the query by using the randomized response described in Section 2 in Line 4 of Algorithm 2. Since the randomized response introduces a noise into the query’s response, we modify Lines 6 and 7 of Algorithm 1. In Line 6, instead of calculating \( \Phi(z) \), Algorithm 2 calculates the unbiased estimated value of \( \tilde{F}(\tau_t) \), which is denoted as \( \Phi'(z') \). Unbiasedness of the estimated value can be confirmed by an elementary calculation. In Line 7, because \( \Phi'(z') \) involves error due to sanitization, we introduce a threshold \( \gamma \) instead of 0.

In Algorithm 2, there are two free parameters; \( L \) and \( \gamma \). We investigate an appropriate choice of \( L \) and \( \gamma \) by analyzing the absolute mean error of this algorithm. The results of the analyses are demonstrated in the next section. We remark that due to the binary search strategy in our proposed method, one can easily see that our proposed method can be easily adapted to maximum finding.

5 Analyses

Hardness in the worst case. We first show that the private finding minimum problem is fundamentally difficult. Indeed, we cannot construct a locally differentially private algorithm that consistently estimates the minimum in the worst case users’ data:

**Theorem 1.** Suppose \( \epsilon \) is fixed. In the both setting, for any \( \epsilon \)-locally differentially private mechanism, there exists \( F \) such that \( \text{Err} = \Omega(1) \) with respect to \( N \).
From the theorem, we can see that the finding minimum problem cannot be solved with a reasonable utility. In Theorem 1, we consider a situation where the minimum point is isolated to all the other points; that is, \( x_1 = -1 \) and \( x_i = 1 \) for \( i = 2, ..., N \). The worst case distribution is not \( \alpha \)-fat for any finite \( \alpha \).

**Adaptive upper bounds and privacy of Algorithm 2.** Next, assuming \( \alpha \)-fatness of the user’s distribution, we reveal the privacy guarantee and the dependency of the error on \( \epsilon \) and \( N \) regarding Algorithm 2.

**Theorem 2.** For any choice of \( \epsilon, L, \) and \( \gamma \), Algorithm 2 is \( \epsilon \)-locally differentially private. Moreover, for some \( \alpha > 0 \), suppose \( F \) is \( \alpha \)-fat. For a sequence \( h_N \), let \( \gamma = \sqrt{4e^{\gamma/L} (1 + e^{\gamma/L})} h_N / (e^{\gamma/L} - 1)^2 N \). Then, in both of the fixed and i.i.d. data settings, if \( L^2 h_N / \epsilon^2 N = o(1) \), Algorithm 2 incurs an error as

\[
\text{Err} = O \left( \left( \frac{L^2 h_N / \epsilon^2 N}{\epsilon} \right)^{1/2} + e^{-h_N} + 2^{-L} \right).
\]

In Theorem 2, there are two free parameters, \( h_N \) and \( L \), which should be selected by the aggregator. We obtain \( O((L^2 h_N / \epsilon^2 N)^{1/2}) \) error rate by choosing \( h_N \) and \( L \) so that the second and third terms in Theorem 2 are smaller than the first term.

Let us consider the case where the aggregator has a prior knowledge regarding a lower bound on \( \alpha \). In this case, an appropriate choice of \( h_N \) and \( L \) is shown in the following corollary.

**Corollary 1.** For some \( \alpha > 0 \), suppose \( F \) is \( \alpha \)-fat. Let \( h_N = \ln(N)/2\alpha \) and \( L = \Theta(\log_2 N) \) such that \( L \geq \log_2(N)/2\alpha \). Then, Algorithm 2 incurs an error as

\[
\text{Err} = O \left( \left( \frac{\ln^2(N)/\epsilon^2 N}{\epsilon} \right)^{1/2} \right).
\]

The next corollary is useful if the aggregator does not have any prior information about \( \alpha \). In this case, the decreasing rate of the error is slightly worse than Corollary 1.

**Corollary 2.** For some \( \alpha > 0 \), suppose \( F \) is \( \alpha \)-fat. Let \( h_N = \Theta(\log^2(N)) \) and \( L = \Theta(\log^2(N)) \). Then, Algorithm 2 incurs an error as

\[
\text{Err} = O \left( \left( \frac{\ln^2(N)/\epsilon^2 N}{\epsilon} \right)^{1/2} \right).
\]

As well as the intuition, the decreasing rate becomes faster as \( \alpha \) decreases in both settings.

**Lowerbound for the locally private minimum finding.** For confirming tightness of Corollaries 1 and 2, we derive a minimax lower bound for the locally private minimum finding.

**Theorem 3.** Fix \( \epsilon \in [0, 22/35] \), \( \alpha \), and \( C \). In the i.i.d. data setting, for any \( \epsilon \)-locally differentially private mechanism, there exists \( F \) satisfies Definition 1 with \( \alpha \) and \( C \) such that for an increasing sequence of \( N \) and a decreasing sequence of \( \epsilon \),

\[
\text{Err} = \Omega \left( \left( \frac{1}{\epsilon^2 N} \right)^{1/2} \right).
\]

As proved in Theorem 3, any locally private mechanism incurs \( \Omega((1/\epsilon^2 N)^{1/2}) \) error which matches the upper bounds shown in Corollaries 1 and 2 up to log factors. Note that we derive the lower bound in Theorem 3 in a situation where the aggregator knows the fatness parameter \( \alpha \). If the aggregator does not know \( \alpha \), the minimax error might be greater than the one shown in Theorem 3.
6 Experiment

Here, we present experimental results on synthetic, the MovieLens, and a purchase history datasets to show the accuracy advantage of our proposed method and confirm the correctness of the our theoretical analysis.

6.1 Synthetic Data

We investigated the error between the real and estimated minimum with synthetic data. The data were generated from a cumulative distribution $F$ according to either of the fixed or i.i.d. data setting. We used the beta distribution to construct $F$. More precisely, let $[x_{\min}, x_{\min} + \Delta]$ be the support of the data, and let $X$ be a random variable that follows the beta distribution with parameter $\alpha$ and $\beta$. Then, $F$ is the cumulative distribution of $x_{\min} + \Delta X$. $\Delta$, $\alpha$, and $\beta$ are varied as combination of $\Delta \in \{0.3, 0.6, 0.9\}$, $\alpha \in \{0.5, 0.9, 1, 2, 4\}$, and $\beta \in \{1, 2\}$. For stabilizing an error caused by discretization, we report the worst case mean absolute error among $x_{\min} \in \{0 \times (2 - \Delta) - 1, 0.2 \times (2 - \Delta) - 1, \ldots, 1 \times (2 - \Delta) - 1\}$. The mean absolute errors were calculated from average of 1000 runs. We also report the 0.05 and 0.95 quantiles of the errors.

We evaluated two different choices of $L$ and $h_N$ corresponding to Corollaries 1 and 2:
The comparison between our method and the baseline method is shown in Figure 2. We can see from
\[ L = \left\lfloor \log_2(N)/2 \right\rfloor \text{ and } h_N = \ln(N)/2, \]
(\textbf{Unknown } \alpha) \quad \text{and } h_N = \ln^2(N)/2 \ln(1000).

The lower \( \alpha \) case is a suitable parameter choice when the aggregator knows \( \alpha \geq 1 \), whereas the unknown \( \alpha \) case is a suitable parameter choice when the aggregator has no information about \( \alpha \).

Here, we only show partial results in the fixed data setting such that \( \alpha = 1, \beta = 1, \) and \( \Delta = 0.3 \).

Note that the beta distribution with \( \alpha = \beta = 1 \) is in fact the uniform distribution. More experiment results on different configurations can be found in the supplementary material.

\textbf{Error v.s. } \( N \) We first demonstrate that Corollaries 1 and 2 are tight with respect to both \( N \). To this end, we evaluated the error of our mechanism corresponding to \( N \in \{2^{10}, 2^{11}, \ldots, 2^{20}\} \).

The left and middle figures in Figure 1 show the errors of our proposed mechanism with varied \( N \) and \( \epsilon \in \{1, 4\} \). We choose \( L \) and \( h_N \) according to \textbf{Lower } \( \alpha \) in the left and \textbf{Unknown } \( \alpha \) in the middle, respectively. The blue lines denote the theoretical guidelines from Corollaries 1 and 2. We can see from Figure 1 that the slopes of the errors are almost the same as the slope of the theoretical guideline regardless of choice of \( \epsilon \) in both \textbf{Lower } \( \alpha \) and \textbf{Unknown } \( \alpha \). This indicates that the decreasing rates with respect to \( N \) shown in Corollaries 1 and 2 are tight.

\textbf{Error v.s. } \( \epsilon \) Next, we show tightness of Corollaries 1 and 2 regarding \( \epsilon \). To this end, we evaluated the error of our mechanism corresponding to \( \epsilon \in \{2^{-3}, 2^{-2}, \ldots, 2^{0}\} \).

The right figure in Figure 1 shows the errors of our proposed mechanism with varying \( \epsilon \) and \( N = 2^{15}, 2^{20} \). The yellow line represents the theoretical guideline from Corollaries 1 and 2. If \( \epsilon \) is not large, slopes of the error are almost the same as the slope of the theoretical guideline, where the error is saturated up to 2 for small \( \epsilon \) since the data are supported on \([-1, 1]\). We therefore can conclude that the rates in Corollaries 1 and 2 with respect to \( \epsilon \) are tight in the range of small \( \epsilon \). Looseness in large \( \epsilon \) comes from Theorem 2. When deriving Theorem 2, we use a bound \( (e^{\epsilon/2} (1+e^{-\epsilon/2})/(e^{\epsilon/2} - 1)^2)^{\frac{1}{\alpha}} \leq (2L^2/\epsilon)^{\frac{1}{2}} \), which is valid only if \( \epsilon \) is sufficiently small. The experimental results reflect this behavior.

In both experiments of error v.s. \( N \) and \( \epsilon \), the rate looks faster than the theoretical guideline when both \( N \) and \( \epsilon \) is small. This is acceptable because the big-O notation in Theorem 2 indicates that the rate is satisfied only if \( L^2 h_N/\epsilon^2 N \) is sufficiently small.

\textbf{Comparison with Naive Mechanism} We also carried out empirical comparison between our proposed method and a baseline solution. Since there is no existing locally private method for finding the minimum, we consider the straightforward Laplace method as a baseline. In particular, each user with \( x_i \) reports \( \hat{x}_i = x_i + \delta_i \) with \( \delta_i \sim \mathcal{L}(0, 2/\epsilon) \), where \( \mathcal{L}(\mu, b) \) is the Laplace distribution with mean \( \mu \) and scale parameter \( b \). The server simply considers the min, \( \hat{x} \), as the estimated minimum. In this experiment, we use \( N \in \{2^{10}, 2^{11}, \ldots, 2^{20}\} \) and \( \epsilon \in \{1, 4\} \).

The comparison between our method and the baseline method is shown in Figure 2. We can see from Figure 2 that the baseline mechanism suffers from an error larger than 1 for all \( N \). Since the data are supported on \([-1, 1]\), the baseline mechanism fails in reasonable estimation. On the other hand, our proposed mechanism achieves significantly smaller error than the baseline method and successes to decrease its error as \( N \) increases.
6.2 MovieLens Data

We conducted experiments on the MovieLens dataset\(^2\). We used the full dataset consisting of 27,753,444 ratings for 53,889 movies obtained by 283,228 users. We carried out the following tasks. (Task1) the server estimates the minimum and maximum of the users’ average rating. The domain of the rating is \([0, 5]\). (Task2) the server estimates the minimum and maximum numbers of the rated movies per user. We can naturally assume that no user exists that evaluate all the movies. We here assumed that the number of the movies rated by a single user was within \([0, 53,889/2]\). We evaluated the error of our mechanism with varying \(N \in \{2^{14}, ..., 2^{18}\}\) by subsampling the dataset, where we use \(\epsilon \in \{1, 4\}\). Since \(\alpha\), the fatness, of the distributions is unknown, we used the Unknown \(\alpha\) parameter setting shown in Section 6.1. The reported value is an average of 1000 runs. We also report the 0.05 and 0.95 quantiles of the errors.

**Results** The histograms of the dataset for each task are depicted in Figure 3. As shown in Figure 3, the left-side tail of the average review distribution is longer than the right-side tail. Regarding the distribution of the number of reviews per user, the right-side tail is extremely long compared to the left-side tail. We, therefore, can expect that in Task 1, \(\alpha\) of the left-side tail is larger than that of the right-side tail, and in Task 2, \(\alpha\) of the right-side tail is extremely larger than that of the right-side tail.

Figure 4 shows the experimental results. We can see from Figure 4 that the decreasing rates of the estimation error are changed adaptively to \(\alpha\), and the obtained \(\alpha\) shown in the subcaptions corresponds to the fatness of the tail.

6.3 Purchase History Dataset

We also conducted experiments on a purchase history dataset collected in the shopping service provided by Yahoo Japan Corporation. This dataset consists of user attribute information, such as gender and birthday. Also, the dataset contains histories of purchase orders of users in Dec. 2015.

\(^2\)Available at [https://grouplens.org/datasets/movielens/latest/](https://grouplens.org/datasets/movielens/latest/)
Each order consists of a multiset of items purchased. We carried out the following tasks with this dataset:

- **Task1**: The server estimates the minimum age of users whose total amount of purchase on this month was in some range. The ranges are varied as \([¥0, ¥10,000], [¥10,000, ¥20,000], [¥20,000, ¥30,000], [¥30,000, ¥40,000], [¥40,000, ¥50,000], \) and \([¥50,000, ¥60,000]\).

- **Task2**: The server estimates the minimum age of the users who purchased items in a specific product category. Here, the age is rescaled from \([0, 150]\) to \([-1, 1]\). The items are categorized into 23 types of products (e.g., fashion, food, sports), whereas only 19 categories were used so that the number of users who purchased an item in a category is larger than \(2^{15}\). We evaluated the error of our mechanism with varying \(N \in \{2^{11}, ..., 2^{15}\}\) by subsampling the dataset, where we use \(\epsilon \in \{1, 4\}\). Since alpha, the fatness, of the distributions are unknown, we used the **Unknown** \(\alpha\) parameter setting shown in Section 6.1. The reported value is an average of 1000 runs. We also report the 0.05 and 0.95 quantiles of the errors.
10
3 × 10
4 × 10
6 × 10
Err
N
Fitted
Actual = 1
Actual = 4
103 104
N
100
3 × 10 1
4 × 10 1
6 × 10 1
ErrN
Fitted
Actual = 1
Actual = 4
178
(a) Task1: \( \alpha = 3.98 \)
182
(b) Task1: \( \alpha = 4.31 \)
333
(c) Task2: \( \alpha = 3.60 \)
375
(d) Task2: \( \alpha = 4.79 \)

Figure 5: Err v.s. \( N \) on the purchase history dataset. The dotted line represents a function \( N \rightarrow C \log^B N/N^A \) where \( A \) and \( B \) are obtained by the least square method. We show the concrete value of \( \alpha = 1/2A \) in the subcaptions.

Results) Figure 5 shows the experimental results with the real datasets. The figure only consists of the results for Task 1 with the ranges \([¥0, ¥10,000]\) (left) and \([¥40,000, ¥50,000]\) (right), Task 2 with the categories music-software (left) and baby-kids-maternity (right). The results with the other settings can be found in the appendix.

We can see from Figure 5 that for these task, the estimation error of our proposed mechanism decreases as \( N \) increases. Thus, we can expect that our mechanism can be consistently estimate the minimum in the real data. Furthermore, the decreasing rates of the estimation error are changed adaptively to the ranges (in Task 1) and categories (in Task 2).

7 Related Work

LDP gains the first real-world application in Google Chrome’s extension, RAPPOR [Erlingsson et al., 2013] and thereafter also finds applications on the other problems such as distribution estimation [Erlingsson et al., 2013, Fanti et al., 2016, Ren et al., 2018] and heavy hitter estimation [Bassily
and Smith, 2015] for categorical-valued data. Different from existing works, our proposed method addresses finding the minimum over numeric-valued data. Simply bucketizing numeric-valued data as categorical data introduces the estimation error. Thus, to handle numeric-valued data, more elaborate protocol design and analysis are needed. There are also local differential privacy methods for numeric-valued problems. For example, Ding et al. [2017], Duchi et al. [2013], and Nguyen et al. [2016] estimate the mean of numeric-valued data under LDP. Ding et al. [2018] studied hypothesis testing to compare population means while preserving privacy. Kairouz et al. [2016] studied the optimal trade-off between privacy and utility. However, these techniques deal with fundamentally different problems from ours and cannot be extended to the minimum finding problem easily.

Essentially, our proposed method adopts binary search-based strategy, together with randomized response, to find the minimum. Cyphers and Veeramachaneni [2017] developed AnonML to estimate the median over real-valued data under LDP. This method shares the same spirit with ours, i.e., binary search-based strategy with randomized response. However, the estimation error of their mechanism was not analyzed, for which we cannot set the number of rounds for binary search reasonably.

Gaboardi et al. [2019] dealt with a problem of estimating a mean with a confidence bound from a set of i.i.d. Gaussian samples. When calculating the confidence bound, they utilize a locally private quantile estimation mechanism that is almost the same as the one we propose. An utility analysis of the quantile estimation mechanism was also provided by them; however, their analysis does not necessarily guarantee a small estimation error. This is because that their analysis employs two utility criteria and provides the sample complexity to achieve that either of them is small. More precisely, their utility criteria for $p$-quantile estimation are $|\hat{x} - F^*(p)|$ and $|F(\hat{x}) - p|$, where $\hat{x}$ denotes the estimated value. Their utility analysis provides the sample complexity to achieve either of $|\hat{x} - F^*(p)| \leq \tau$ or $|F(\hat{x}) - p| \leq \lambda$. Noting that the former criterion is the absolute error of the $p$-quantile estimation, we can see that their analysis does not guarantee small error of $p$-quantile estimation if $|F(\hat{x}) - p|$ is small.

Our minimum finding mechanism (which can be easily adapted to maximum finding) can be employed as a preprocessing for various types of locally differentially private data analysis. For example, we can use our method for locally differentially private itemset mining [Qin et al., 2016, Wang et al., 2018] over set-valued data. The crucial assumption employed for these methods is that the server knows the maximum number of data items owned by users. The maximum number can be estimated by our mechanism in a local differential privacy manner.

8 Conclusion

In this study, we propose a method for finding the minimum of individuals’ data values under local differential privacy. We reveal that the absolute error of the proposed mechanism is $O((\ln N/\epsilon N)^{1/2\alpha})$ under the $\alpha$-fatness assumption and it is optimal up to log factors. The mechanism is adaptive to $\alpha$; that is, it can obtain that rate without knowing $\alpha$.

Acknowledgments

This work was partly supported by KAKENHI (Grants-in-Aid for scientific research) Grant Numbers JP19H04164 and JP18H04099. We would like to express our gratitude to Yahoo Japan Corporation.
for providing the purchase history dataset.

References

Raef Bassily and Adam D. Smith. Local, Private, Efficient Protocols for Succinct Histograms. In Proceedings of the Forty-Seventh Annual ACM on Symposium on Theory of Computing (STOC), pages 127–137. ACM, 2015. doi: 10.1145/2746539.2746632.

Stéphane Boucheron, Gábor Lugosi, and Pascal Massart. Concentration inequalities using the entropy method. The Annals of Probability, 31(3):1583–1614, 7 2003. doi: 10.1214/aop/1055425791.

Bennett Cyphers and Kalyan Veeramachaneni. AnonML: Locally private machine learning over a network of peers. In IEEE International Conference on Data Science and Advanced Analytics (DSAA), pages 549–560. IEEE, 2017. doi: 10.1109/DSAA.2017.80.

Bolin Ding, Janardhan Kulkarni, and Sergey Yekhanin. Collecting telemetry data privately. In Advances in Neural Information Processing Systems 30: Annual Conference on Neural Information Processing Systems (NIPS), pages 3574–3583, 2017.

Bolin Ding, Harsha Nori, Paul Li, and Allen Joshua. Comparing Population Means under Local Differential Privacy: with Significance and Power. In Proceedings of the Thirty-Second AAAI Conference on Artificial Intelligence, pages 26–33. AAAI Press, 2018.

John Duchi, Martin Wainwright, and Michael Jordan. Minimax Optimal Procedures for Locally Private Estimation. ArXiv e-prints, 2016. URL http://arxiv.org/abs/1604.02390.

John C. Duchi, Michael I. Jordan, and Martin J. Wainwright. Local privacy and statistical minimax rates. In 54th Annual IEEE Symposium on Foundations of Computer Science (FOCS), pages 429–438, 2013. doi: 10.1109/FOCS.2013.53.

Cynthia Dwork, Frank McSherry, Kobbi Nissim, and Adam Smith. Calibrating noise to sensitivity in private data analysis. In Shai Halevi and Tal Rabin, editors, Theory of Cryptography, pages 265–284, Berlin, Heidelberg, 2006. Springer Berlin Heidelberg. ISBN 978-3-540-32732-5.

Úlfar Erlingsson, Vasyl Pihur, and Aleksandra Korolova. RAPPOR: Randomized Aggregatable Privacy-Preserving Ordinal Response. In Proceedings of the 2014 ACM SIGSAC Conference on Computer and Communications Security (CCS), pages 1054–1067. ACM, 2013. doi: 10.1145/2660267.2660348.

Alexandre Evfimievski, Johannes Gehrke, and Ramakrishnan Srikant. Limiting privacy breaches in privacy preserving data mining. In Proceedings of the twenty-second ACM SIGMOD-SIGACT-SIGART symposium on Principles of database systems, pages 211–222. ACM, 2003.

Giulia C. Fanti, Vasyl Pihur, and Úlfar Erlingsson. Building a RAPPOR with the Unknown: Privacy-Preserving Learning of Associations and Data Dictionaries. Proceedings on Privacy Enhancing Technologies (PoPETs), 2016(3):41–61, 2016.

Marco Gaboardi, Ryan Rogers, and Or Sheffet. Locally private mean estimation: z-test and tight confidence intervals. In Kamalika Chaudhuri and Masashi Sugiyama, editors, Proceedings of Machine Learning Research, volume 89 of Proceedings of Machine Learning Research, pages 2545–2554. PMLR, 2019.
A Examples of $\alpha$-Fat Distributions

For giving a better understanding of $\alpha$-fatness, we introduce some concrete values of $\alpha$, $C$, and $\bar{x}$ for some $F$.

**Example 1** (Beta distribution). Let $X$ be a random variable following the beta distribution with parameters $\alpha$ and $\beta$. Suppose $F$ is the cumulative distribution of $x_{\text{min}} + (x_{\text{max}} - x_{\text{min}})X$. Then, Definition 1 is satisfied with the same $\alpha$, and with $C = \max\{1, (\alpha B(\alpha, \beta))^{-1}\}/(x_{\text{max}} - x_{\text{min}})^{\alpha}$ and $\bar{x} = x_{\text{max}}$, where $B(\alpha, \beta)$ denotes the beta function.

**Example 2** (Truncated (normal) distributions). Suppose $F$ is the cumulative distribution of the truncated normal distribution supported on $[x_{\text{min}}, x_{\text{max}}]$ with parameters $\mu \in [-1, 1]$ and $\sigma^2 > 0$. Then, Definition 1 is satisfied with $\alpha = 1$ and $C = \min\{\phi((x_{\text{min}} - \mu)/\sigma), \phi((x_{\text{max}} - \mu)/\sigma)\}/\sigma \Phi((x_{\text{max}} - \mu)/\sigma) - \Phi((x_{\text{min}} - \mu)/\sigma)$, where $\phi(x)$ and $\Phi(x)$ denote a density function and cumulative function of the standard normal distribution, respectively. More generally, any truncated distribution satisfies Definition 1 with $\alpha = 1$. 

Figure 6: The density function of the beta distribution with $\alpha \in \{0.5, 1, 1.5, 2, 4, 6, 10, 14\}$ and $\beta = 3$. 

Peter Kairouz, Sewoong Oh, and Pramod Viswanath. Extremal Mechanisms for Local Differential Privacy. *Journal of Machine Learning Research (JMLR)*, 17:17:1–17:51, 2016.

Thong T. Nguyen, Xiaokui Xiao, Ying Yang, Siu Cheung Hui, Hyejin Shin, and Junbum Shin. Collecting and analyzing data from smart device users with local differential privacy. *ArXiv e-prints*, 2016. URL https://arxiv.org/abs/1606.05053.

Zhan Qin, Yin Yang, Ting Yu, Issa Khalil, Xiaokui Xiao, and Kui Ren. Heavy hitter estimation over set-valued data with local differential privacy. *ACM Conference on Computer and Communications Security (CCS)*, 2016.

Xuebin Ren, Chia-Mu Yu, Weiren Yu, Shusen Yang, Xinyu Yang, Julie A. McCann, and Philip S. Yu. LoPub: High-Dimensional Crowdsourced Data Publication With Local Differential Privacy. *IEEE Transactions on Information Forensics and Security*, 13(9):2151–2166, 2018. doi: 10.1109/TIFS.2018.2812146.

Tianhao Wang, Ninghui Li, and Somesh Jha. Locally Differentially Private Frequent Itemset Mining. In *IEEE Symposium on Security and Privacy (S&P)*, pages 127–143. IEEE, 2018. doi: 10.1109/SP.2018.00035.

Stanley L. Warner. Randomized Response: A Survey Technique for Eliminating Evasive Answer Bias. *Journal of the American Statistical Association*, 1965. doi: 10.1080/01621459.1965.10480775.
Figure 6 shows the probability density function of the beta distribution with different parameter settings. In Figure 6, we set $\beta = 3$, and $\alpha$ are varied as shown in the legend. We can see from Figure 6 that density around the minimum becomes larger as $\alpha$ decreases.

## B Analyses of Algorithm 2

### B.1 Utility Analysis

Here, we will prove the following two theorems corresponding to the fixed and i.i.d. data settings.

**Theorem 4.** Suppose $F$ is $\alpha$-fat, and $\gamma$ satisfies $2\gamma < C(\bar{x} - x_{\min})^{\alpha}$.

In the fixed data setting, for any $N$, we have

$$\text{Err} \leq 2\left(\frac{2\gamma}{C}\right)^{\frac{1}{\alpha}} + \exp\left(-\frac{(e^{\epsilon/L} - 1)^{2}\gamma^2N}{4(e^{\epsilon/L} + 1)e^{\epsilon/L}}\right) + 2^{-L}.$$  

**Theorem 5.** Suppose $F$ is $\alpha$-fat, and $\gamma$ satisfies $2\gamma < C(\bar{x} - x_{\min})^{\alpha}$.

In the i.i.d. data setting, for any $N$, we have

$$\text{Err} \leq 2\left(\frac{1}{C}\right)^{\frac{1}{\alpha}}\left(\frac{[2\gamma N]^{\frac{1}{\alpha}}}{(N + 1)^{\frac{1}{\alpha}}}\right) + \exp\left(-\frac{(e^{\epsilon/L} - 1)^{2}\gamma^2N}{4(e^{\epsilon/L} + 1)e^{\epsilon/L}}\right) + 2^{-L},$$

where $(x)^{\alpha}$ denotes the rising factorial. That is, letting $\Gamma$ be the gamma function, $(x)^{\alpha} = \Gamma(x + \alpha)/\Gamma(x)$.

We obtain Theorem 2 by substituting the specified $\gamma$. Note that the first term in Theorem 5 matches the first term in Theorem 2 because if $\gamma = \omega(1/N)$, we have

$$\lim_{N \to \infty} \left(\frac{[2\gamma N]^{\frac{1}{\alpha}}}{(N + 1)^{\frac{1}{\alpha}}}\right) \left(\frac{N + 1}{2\gamma N}\right)^{\frac{1}{\alpha}} = 1.$$  

Here, we give the proof sketch of Theorems 4 and 5. Algorithm 2 can be seen as an algorithm that estimates $\gamma$-quantile of the users’ data because the algorithm finds $x \in [-1, 1]$ such that $\tilde{F}(x) = \gamma$. Hence, the mean absolute error of Algorithm 2 can be decomposed as

$$\text{Err} \leq E\left|\tilde{F}^*(\gamma) - x_{\min}\right| + E\left|\bar{x} - \tilde{F}^*(\gamma)\right|. \quad (2)$$

The first term in Eq (2) denotes the error between the minimum and $\gamma$-quantile, and the second term denotes the estimation error of the $\gamma$-quantile.

To analyze the second term in Eq (2), we define events of mistake. For each round $t$, define an event

$$M_t = \left\{\tau_t < \tilde{F}^*(\gamma) \implies \Phi(z) \geq \gamma, \tau_t > \tilde{F}^*(\gamma) \implies \Phi(z) < \gamma\right\}.$$  

Then, $M_t$ represents an event that, at round $t$, the algorithm chooses an interval that is far from the $\gamma$-quantile, and hence we say the algorithm mistakes at round $t$ if $M_t$ occurs. Then, we obtain the following lemma regarding the estimation error of the $\gamma$-quantile:
Lemma 1. Let $\tau$ be determined by Algorithm 2. Then, for any random variable $\delta > 0$ that can depend on $x_1, \ldots, x_N$, we have

$$E \left[ \left| \tilde{x} - \tilde{F}^*(\gamma) \right| \right] = \delta + E \left[ \max_{t=1,\ldots,L} \mathbb{P} \{|M_t| \leq |\tilde{F}^*(\gamma) - \tau_t| \geq \delta\} \right] + 2^{-L}.$$ 

The concentration inequality gives a bound on the second term in Lemma 1.

Lemma 2. Let $z = (z_1, \ldots, z_N)$ be the sanitized version of $(\text{sign}(\tau - x_1), \ldots, \text{sign}(\tau - x_N))$ using the randomized response with the privacy parameter $\epsilon$. If $\gamma > \tilde{F}(\tau)$,

$$\mathbb{P} \{ \Phi(z) > \gamma \} \leq \exp \left( -\frac{(e^\epsilon - 1)^2(\tilde{F}(\tau) - \gamma)^2N}{4(e^\epsilon + 1)e^\epsilon} \right).$$

Moreover, if $\gamma < \tilde{F}(\tau)$,

$$\mathbb{P} \{ \Phi(z) < \gamma \} \leq \exp \left( -\frac{(e^\epsilon - 1)^2(\tilde{F}(\tau) - \gamma)^2N}{4(e^\epsilon + 1)e^\epsilon} \right).$$

Choose $\delta$ such that $\tilde{F}^*(2\gamma) - \tilde{F}^*(\gamma) \geq \delta$ or $\tilde{F}^*(\gamma) - \tilde{F}^*(0) \geq \delta$. Then, for any $t$, $|\tilde{F}(\tau_t) - \gamma| \geq \gamma$. Thus, we obtain a bound on the second term in Lemma 1 from Lemma 2. We can prove Theorems 4 and 5 by deriving bounds on the first term in Eq (2) and the first term in Lemma 1, where bounds on these terms can be obtained from Definition 1.

B.2 Privacy Analysis

Privacy of Algorithm 2 can be proved easily with application of the sequential composition theorem. We confirm that Algorithm 2 ensures $\epsilon$-local differential privacy.

Theorem 6. Algorithm 2 is $\epsilon$-locally differentially private.

Proof. Algorithm 2 uses the randomized response $L$ times with privacy parameter $\epsilon/L$. By the sequential composition theorem of the local differential privacy, the total privacy loss is at most $\epsilon$. 

Note that Algorithm 2 is $\epsilon$-locally differentially private for any choice of $L$ and $\gamma$.

C Proofs

C.1 Proof for Hardness

We introduce the definition of the differential privacy for proving Theorem 1. The differential privacy is weaker than the local differential privacy in the sense that any analysis using data satisfying the local differential privacy ensures the differential privacy. Thus, if the minimum finding problem is difficult under the differential privacy, the problem is also difficult under the local differential privacy. The formal definition of the differential privacy is given as follows:
Definition 3 (Differential privacy [Dwork et al., 2006]). A stochastic mechanism $\mathcal{M}$ mapping from $X^N$ to $Z$ is $\epsilon$-differentially private if for all $X, X' \in X^N$ differing at most one record, and all $S \in \sigma(Z)$,

$$\mathbb{P}\{\mathcal{M}(X) \in S\} \leq e^\epsilon \mathbb{P}\{\mathcal{M}(X') \in S\},$$

where $\sigma(Z)$ denotes an appropriate $\sigma$-field of $Z$.

Then, we prove Theorem 1.

Proof of Theorem 1. Fixed data case. Let $F_0$ be a cumulative distribution such that $F_0(x) = 0$ for $x \in [-1, 1)$. Let $F_1$ be another cumulative distribution such that $F_1(x) > 0$ for any $x \in (-1, 1]$. Let $X_0$ and $X_1$ be the users’ data generated from $F_0$ and $F_1$, respectively. Then, $X_0$ and $X_1$ have different minimum, whereas the other records are equivalent.

Let $M$ be a $\epsilon$-differentially private mechanism. Then, its mean absolute errors for $X_0$ and $X_1$ are obtained as

$$\mathbb{E}[|M(X_0) - x_{\min}|] = \mathbb{E}[|M(X_0) - 1|]$$
$$\mathbb{E}[|M(X_1) - x_{\min}|] = \mathbb{E}[|M(X_1) + 1|].$$

Assume

$$\mathbb{E}[|M(X_0) - 1|] = o(1).$$

Then, by the Markov inequality, we have

$$\mathbb{E}[|M(X_0) - 1|] \geq \mathbb{P}\{|M(X_0) - 1| > 1\} = o(1).$$

Because of the differential privacy assumption, we have

$$\mathbb{P}\{|M(X_0) - 1| > 1\} \geq e^{-\epsilon}\mathbb{P}\{|M(X_1) - 1| > 1\} = o(1).$$

We obtain a lower bound on the error for $X_1$ as

$$\mathbb{E}[|M(X_1) + 1|] \geq \mathbb{P}\{|M(X_1) + 1| > 1\} = 1 - \mathbb{P}\{|M(X_1) - 1| > 1\} = 1 - o(1).$$

This discussion is true even if we exchange $X_0$ and $X_1$. Thus, we obtain the claim.

i.i.d. data case. Let $F_0$ be the same cumulative distribution above. Let $F_1$ be a cumulative distribution such that $F_1(x) = \delta$ for any $x \in (-1, 1)$. Note that the distributions of $F_0$ and $F_1$ are supported only on $\{-1, 1\}$ such that $\mathbb{P}\{X = -1\} = 0$ under $F_0$ and $\mathbb{P}\{X = -1\} = \delta$ under $F_1$. In the similar manner in the fixed data case, assume

$$o(1) = \mathbb{E}[|M(X_1) + 1|] \geq \mathbb{P}\{|M(X_1) + 1| > 1\},$$

where the inequality is obtained by the Markov inequality. Since under $F_0$, all the users’ data are 1, the number of the different records between $X_0$ and $X_1$ follows the binomial distribution with
a parameter $N$ and $\delta$. Let $H(X_0, X_1)$ be the number of the different records between $X_0$ and $X_1$. Then, from the differential privacy assumption, we have

$$
P\{|M(X_1) + 1| > 1\} \geq \mathbb{E}\left[e^{-\epsilon H(X_0, X_1)} P\{|M(X_0) + 1| > 1\}\right]
$$

$$
= \mathbb{E}\left[e^{-\epsilon H(X_0, X_1)}\right] P\{|M(X_0) + 1| > 1\}
$$

$$
\geq (1 - \delta)^N P\{|M(X_0) + 1| > 1\}.
$$

For $\delta = o(1/N)$, we have

$$
P\{|M(X_1) + 1| > 1\} \geq (1 - o(1)) P\{|M(X_0) + 1| > 1\}.
$$

In the same manner of the fixed data case, we get the claim. \qed

### C.2 Proof for Upper Bounds

**Proof of Lemma 1.** Let $t_1 < t_2 < ... < t_M$ be the rounds that the algorithm mistakes. By the definition of $M_t$, we have

$$
\tilde{F}^*(\gamma) \leq \tau_{t_1} \leq \tau_{t_2} \leq ... \leq \tau_{t_M},
$$

or

$$
\tilde{F}^*(\gamma) \geq \tau_{t_1} \geq \tau_{t_2} \geq ... \geq \tau_{t_M},
$$

Since the algorithm does not mistake after $t_M$ round, we have $|t_M - \bar{x}| \leq 2^{-L}$. Let $t_\delta$ be the maximum round $t$ such that $|\tilde{F}^*(\gamma) - \tau_t| \leq \delta$. We remark that $t_\delta$ is the random variable over $[L]$. Then, we have

$$
\mathbb{E}\left[|\bar{x} - \tilde{F}^*(\gamma)|\right] = \mathbb{E}\left[|\tilde{F}^*(\gamma) - \tau_{t_\delta}|\right] + \mathbb{E}[|\tau_{t_\delta} - \tau_{t_M}|] + 2^{-L}.
$$

Since the difference between $\tau_t$ and $\tau_{t+1}$ is $2^{-t}$, we have

$$
|\tau_{t_\delta} - \tau_{t_M}| \leq \sum_{t=t_\delta+1}^{L} \mathbb{I}_{M_t} 2^{-t} \leq \sum_{t=1}^{L} \mathbb{I}_{M_t} |\tilde{F}^*(\gamma) - \tau_t| > \delta 2^{-L}.
$$

Hence,

$$
\mathbb{E}[|\tau_{t_M} - \tau_{t_M}|] \leq \mathbb{E}\left[\sum_{t=1}^{L} \mathbb{P}\{M_t\} \mathbb{I}_{|\tilde{F}^*(\gamma) - \tau_t| > \delta 2^{-t}\}\right]
$$

$$
\leq \mathbb{E}\left[\max_{t=1, ..., L} \mathbb{P}\{M_t\} \mathbb{I}_{|\tilde{F}^*(\gamma) - \tau_t| > \delta \sum_{t=1}^{L} 2^{-t}}\right]
$$

$$
\leq \mathbb{E}\left[\max_{t=1, ..., L} \mathbb{P}\{M_t\} \mathbb{I}_{|\tilde{F}^*(\gamma) - \tau_t| > \delta}\right].
$$

\qed
Proof of Theorem 4. From Eq (2) and Lemmas 1 and 2, with an appropriate \( \delta \), we have

\[
\text{Err} \leq \mathbb{E} \left[ \left\| \tilde{F}^*(\gamma) - \tilde{F}^*(0) \right\|_1 \right] + \mathbb{E} \left[ \left\| \tilde{F}^*(\gamma) - \tilde{F}^*(2\gamma) \right\|_1 \right] + \exp \left( -\frac{(e^\epsilon/L - 1)^2 2^2 N}{4(e^\epsilon/L + 1)e^\epsilon/L} \right) + 2^{-L},
\]

(3)
where we use the fact that \( \tilde{F}^* \) is non-decreasing. The sum of the first two terms is bounded above by

\[
2\mathbb{E} \left[ \left\| \tilde{F}^*(2\gamma) - x_{(1)} \right\|_1 \right].
\]

(4)

If \( 2\gamma < C_1(C_2 - x_{(1)})^\alpha \), we have \( \tilde{F}^*(2\gamma) \in (x_{(1)}, C_2) \). Hence, under \( \alpha \)-fatness, we have \( \left| \tilde{F}^*(2\gamma) - x_{(1)} \right| \leq \left( \frac{2\gamma}{C_2} \right)^{1/\alpha}. \) Substituting this into Eq (3) yields the desired result.

Proof of Theorem 5. The proof follows the same manner of that of Theorem 4 except a bound on Eq (4). Let \( U_{(1)}, \ldots, U_{(N)} \) be the order statistics of the uniform distribution on \([0, 1]\). Then, we have

\[
x_{(k)} = F^*(U_{(k)}).
\]

Hence,

\[
\mathbb{E} \left[ \left\| \tilde{F}^*(2\gamma) - x_{\text{min}} \right\|_1 \right] \\
= \mathbb{E} \left[ \left\| F^*(U_{(2\gamma N)}) - F^*(0) \right\|_1 \right] \\
\leq \frac{1}{C_1^{1/\alpha}} \mathbb{E} \left[ U_{(2\gamma N)}^{1/\alpha} \right].
\]

Since \( U_{(k)} \) follows the beta distribution with parameters \( k \) and \( N - k + 1 \), we have

\[
\mathbb{E} \left[ U_{(k)}^{1/\alpha} \right] = \frac{B(k + \frac{1}{\alpha}, N - k + 1)}{B(k, N - k + 1)} \\
= \frac{\Gamma(k + \frac{1}{\alpha})\Gamma(N + 1)}{\Gamma(N + 1 + \frac{1}{\alpha})\Gamma(k)} = \left( \frac{k}{N + 1} \right)^{1/\alpha}.
\]

Proof of Lemma 2. We use the concentration inequality from [Boucheron et al., 2003]. Let \( Z = f(z_1, \ldots, z_N) = \sum_{i=1}^{N} z_i \). Let \( Z^{(i)} = f(z_1, \ldots, z_{i-1}, z_i', z_{i+1}, \ldots, z_N) \), where \( z_i' \) be the independent copy of \( z_i \). Define

\[
V_+ = \mathbb{E} \left[ \sum_{i=1}^{N} (Z - Z^{(i)})^2 \mathbb{1}_{Z > Z^{(i)}} \right] \\
= \frac{4(e^\epsilon + 1)^2}{(e^\epsilon - 1)^2} \sum_{i=1}^{N} \mathbb{P} \{ z_i' = -1 \} \mathbb{1}_{z_{i} = 1}.
\]
Moreover, we have

\[ V_- = E \left[ \sum_{i=1}^{N} (Z - Z^{(i)})^2 \mathbb{1}_{Z < Z^{(i)}} | z_1, ..., z_N \right] \]

\[ = 4 \left( e^\epsilon + 1 \right)^2 \sum_{i=1}^{N} \mathbb{P}\{z_i' = 1\} \mathbb{1}_{z_i = -1}. \]

From Theorem 2 in [Boucheron et al., 2003], for \( \theta > 0 \) and \( \lambda \in (0, 1/\theta) \), we have

\[ \ln E\left[ e^{\lambda (Z - E[Z])} \right] \leq \frac{\lambda \theta}{1 - \lambda \theta} \ln E\left[ e^{\lambda V} \right], \]

and

\[ \ln E\left[ e^{-\lambda (Z - E[Z])} \right] \leq \frac{\lambda \theta}{1 - \lambda \theta} \ln E\left[ e^{-\lambda V} \right]. \]

By definition, we have

\[ \frac{\lambda \theta}{1 - \lambda \theta} \ln E\left[ e^{\lambda V} \right] \]

\[ = \frac{\lambda \theta}{1 - \lambda \theta} \sum_{i=1}^{N} \ln \left( \mathbb{P}\{z_i = 1\} e^{\epsilon} \mathbb{P}\{z_i = -1\} \right) \]

\[ = 4 \lambda^2 \frac{(e^\epsilon + 1)^2}{(e^\epsilon - 1)^2} \sum_{i=1}^{N} P\{z_i = -1\} \]

\[ + \frac{\lambda \theta}{1 - \lambda \theta} \sum_{i=1}^{N} \ln \mathbb{P}\{z_i = 1\}. \]

As \( \theta \to 0 \), we obtain

\[ \lim_{\theta \to 0} \frac{\lambda \theta}{1 - \lambda \theta} \ln E\left[ e^{\lambda V} \right] \]

\[ = 4 \lambda^2 \frac{(e^\epsilon + 1)^2}{(e^\epsilon - 1)^2} \sum_{i=1}^{N} P\{z_i = -1\} \]

\[ \leq 4 \lambda^2 \frac{(e^\epsilon + 1)e^\epsilon N}{(e^\epsilon - 1)^2}. \]

In the similar way, we obtain

\[ \lim_{\theta \to 0} \frac{\lambda \theta}{1 - \lambda \theta} \ln E\left[ e^{-\lambda V} \right] \]

\[ = 4 \lambda^2 \frac{(e^\epsilon + 1)^2}{(e^\epsilon - 1)^2} \sum_{i=1}^{N} P\{z_i = 1\} \]

\[ \leq 4 \lambda^2 \frac{(e^\epsilon + 1)e^\epsilon N}{(e^\epsilon - 1)^2}. \]
From the Markov inequality, we have
\[ P\{Z > \mathbb{E}[Z] + t\} \leq \frac{e^{\lambda(Z - \mathbb{E}[Z])}}{e^{\lambda t}}, \]
and
\[ P\{Z < \mathbb{E}[Z] - t\} \leq \frac{e^{-\lambda(Z - \mathbb{E}[Z])}}{e^{\lambda t}}. \]
Optimizing \( \lambda \) gives that
\[ P\{Z > \mathbb{E}[Z] + t\} \leq \exp\left(-\frac{(e^\epsilon - 1)^2t^2}{16(e^\epsilon + 1)e^\epsilon N}\right), \]
and
\[ P\{Z < \mathbb{E}[Z] - t\} \leq \exp\left(-\frac{(e^\epsilon - 1)^2t^2}{16(e^\epsilon + 1)e^\epsilon N}\right). \]
Noting that
\[ P\{Z > \mathbb{E}[Z] + t\} = P\left\{ \frac{\epsilon^\epsilon + 1}{\epsilon^\epsilon - 1} \sum_{i=1}^{N} z_i > 2N\tilde{F}(\tau) + t \right\}. \]
Thus, setting \( t = 2N(\gamma - \tilde{F}(\tau)) \) yields the desired claim.

C.3 Proof for Lower Bound

Proof of Theorem 3. i.i.d. data case. We use the lower bound from Duchi et al. [2016] for the i.i.d. case.

Theorem 7 ([Duchi et al., 2016]). Given \( \delta > 0 \), let \( F \) and \( F' \) be the cumulative functions such that these minimums, denoted as \( x_{\text{min}} \) and \( x'_{\text{min}} \), respectively, differs at least \( 2\delta \), i.e., \( |x_{\text{min}} - x'_{\text{min}}| \geq 2\delta \). For \( \epsilon \in [0, 22/35] \), for any \( \epsilon \)-locally differentially private mechanism, there exists a cumulative function \( F_0 \) such that the error under \( F_0 \) is lower bounded as
\[ \text{Err} \geq |\delta\left(\frac{1}{2} - \sqrt{N\epsilon^2 \text{TV}(F, F')^2}\right), \]
where \( \text{TV} \) denotes the total variation distance.

From Theorem 7, we can obtain a lower bound by designing \( F \) and \( F' \) so that \( \text{TV}(F, F') \) is minimized while satisfying \( |x_{\text{min}} - x'_{\text{min}}| \geq 2\delta \) simultaneously. We select different choices of \( F \) and \( F' \) for \( \alpha \in (0, 1) \) and \( \alpha \geq 1 \).

Case \( \alpha \in (0, 1) \). Set
\[
F(x) = \begin{cases} 
(x + 1)^\alpha & \text{if } x \in [-1, 0] \\
1 & \text{otherwise}
\end{cases},
\]
\[
F'(x) = \begin{cases} 
0 & \text{if } x \in [-1, -1 + 2\delta) \\
(x + 1 - 2\delta)^\alpha & \text{if } x \in [-1 + 2\delta, 2\delta] \\
1 & \text{otherwise}
\end{cases}.
\]
Then, the total variation distance between $F$ and $F'$ is obtained as
\[
TV(F, F') = 1 - \alpha \int_{-1+2\delta}^{0} \min\{(x+1)^{\alpha-1}, (x+1-2\delta)^{\alpha-1}\} dx
\]
\[
= 1 - \alpha \int_{2\delta}^{1} x^{\alpha-1} dx
\]
\[
= 1 - (1 - (2\delta)^\alpha) = (2\delta)^\alpha.
\]
Hence, setting $\delta = (16\epsilon^2 N)^{-1/2\alpha}/2$ yields that
\[
\text{Err} \geq \frac{1}{8} \left( \frac{1}{16\epsilon^2 N} \right)^{1/2\alpha}.
\]

**Case $\alpha \geq 1$.** Set
\[
F(x) = \begin{cases} 
(x+1)^\alpha & \text{if } x \in [-1, 0] \\
1 & \text{otherwise ,}
\end{cases}
\]
\[
F'(x) = \begin{cases} 
0 & \text{if } x \in [-1, -1+2\delta] \\
(x+1)^\alpha - (2\delta)^\alpha & \text{if } x \in [-1+2\delta, 0] \\
1 & \text{otherwise .}
\end{cases}
\]
Then, the total variation distance between $F$ and $F'$ is obtained as
\[
TV(F, F') = \int_{0}^{2\delta} \alpha x^{\alpha-1} dx
\]
\[
= (2\delta)^\alpha.
\]
Hence, with the same setting of $\delta$ for $\alpha \in (0, 1)$ case yields the same lower bound. \qed

**D More Experiments: Synthetic Datasets**

The full experimental results on the synthetic dataset can be found in expr.pdf.

**E More Experiments: Purchase History Data**

The rest of the experimental results on the purchase history dataset is shown in here.
Figure 7: Experimental results for Task1.
Figure 8: Experimental results for Task2, Part I.
Figure 9: Experimental results for Task 2, Part II.
(a) instrument-hobby-learning, $\alpha = 3.94$

(b) kitchen-and-daily-goods, $\alpha = 4.06$

(c) leisure-and-outdoor, $\alpha = 4.19$

(d) music-software, $\alpha = 3.60$

(e) personal-computer, $\alpha = 4.15$

(f) pet-supplies, $\alpha = 4.64$

Figure 10: Experimental results for Task2, Part III.
(a) sports, $\alpha = 4.21$

Figure 11: Experimental results for Task2, Part IV.