ON SHRINKING TARGETS AND SELF-RETURNING POINTS

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Abstract. We consider the set $R_{io}$ of points returning infinitely many times to a sequence of shrinking targets around themselves. Under additional assumptions we improve Boshernitzan’s pioneering result on the speed of recurrence. In the case of the doubling map as well as some linear maps on the $d$-dimensional torus, we even obtain a dichotomy condition for $R_{io}$ to have measure zero or one. Moreover, we study the set of points eventually always returning and prove an analogue of Boshernitzan’s result in similar generality.

1. Introduction

Let $(X,B,\mu,T)$ be a measure preserving system equipped with a compatible metric $d$, i.e. a metric such that open subsets of $X$ are measurable. We consider a sequence $\{B(y,r_n)\}_{n=1}^{\infty}$ of balls in $X$ with center $y$ and radius $r_n$. We will refer to the balls as shrinking targets since the interesting questions arise when $r_n \to 0$ although this is not a formal requirement. Classical shrinking target questions focus on the set of $x \in X$, whose $n$'th iterate under $T$ hits $B(y,r_n)$ for infinitely many $n$. That is, the set

$$H_{io} = H_{io}(y,r_n) := \{ x \in X : T^n(x) \in B(y,r_n) \text{ for } \infty \text{ many } n \in \mathbb{N} \}.$$ 

For many dynamical systems, the measure as well as dimension of this set is well understood under certain assumptions on the measure of the shrinking targets (see [1], [8], and references therein for examples). A different and interesting question arises when we do not consider one fixed center for the shrinking targets, but instead consider the points that return infinitely many times to a sequence of shrinking targets around themselves. That is, the set

$$R_{io} = R_{io}(r_n) := \{ x \in X : T^n(x) \in B(x,r_n) \text{ for } \infty \text{ many } n \in \mathbb{N} \}.$$ 

If the invariant measure $\mu$ is nonuniform, then the measure of the targets depends on their location and we also consider the set

$$\hat{R}_{io} = \hat{R}_{io}(M_n) := \{ x : T^n(x) \in B(x,r_n(x)) \text{ for } \infty \text{ many } n \in \mathbb{N} \},$$

where $r_n(x)$ is such that $\mu(B(x,r_n(x))) = M_n$.

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Another interesting set to consider is the eventually-always analogue of \( R_{io} \) which is defined as
\[
R_{ea} = R_{ea}(r_m) := \{ x \in X : \exists m_0 \in \mathbb{N}, \forall m \geq m_0 : \{ T^k(x) \}_{k=1}^m \cap B(x, r_m) \neq \emptyset \},
\]
i.e., the set of points whose sufficiently long orbit always hits a sequence of shrinking targets around themselves. In addition to obtaining a result on \( R_{ea} \) in broad generality, we study the measure of \( R_{io}, \hat{R}_{io}, \) and \( R_{ea} \) for certain classes of dynamical systems on the unit interval as well as some linear maps on the \( d \)-dimensional torus. Note also that the eventually-always analogue of \( H_{io} \), denoted by \( H_{ea} \) or sometimes by \( E_{ah} \), was investigated for similar dynamical systems by the authors in \cite{14} and by Kleinbock, Konstantoulas and Richter in \cite{15}.

1.1. Known results about \( R_{io}, \hat{R}_{io}, \) and \( R_{ea} \). Generally speaking we are interested in the sizes of these sets and how their sizes depend on the measure of the targets. By “size” we mean measure, but if the measure of the set is zero it is interesting to determine the dimension of the set to get a more nuanced picture of how small it is. In the setting of \( \beta \)-transformations, the Hausdorff dimension of \( R_{ea} \) was computed by Zheng and Wu in \cite{22}.

In this paper we focus on the measure of \( R_{io}, \hat{R}_{io}, \) and \( R_{ea} \). The study of self-returning points invariably starts with the Poincaré Recurrence Theorem, which may be stated as follows.

**Theorem 1.1** (Carathéodory, 1919 [6]). Let \((X, d)\) be a separable metric space and let \( \mu \) be a finite \( T \)-invariant Borel measure. For \( \mu \)-almost every \( x \in X \), there exists a subsequence \( n_k \) such that \( T^{n_k}(x) \to x \) as \( k \to \infty \).

The conclusion of the theorem can also be rewritten as
\[
\mu(\{ x \in X : \exists (r_n(x))_{n \in \mathbb{N}} \text{ s.t. } r_n(x) \to 0 \text{ as } n \to \infty \text{ and } \forall n \in \mathbb{N} : T^n(x) \in B(x, r_n(x)) \text{ for } \infty \text{ many } n \in \mathbb{N} \}) = 1.
\]

We see that the sequence \( r_n \) is allowed to depend on the point \( x \) and the rate of \( r_n \to 0 \) may also be arbitrarily slow. It is natural to ask under which circumstances there exists a certain rate on \( r_n \to 0 \) which is uniform across all \( x \in X \) and which maintains full measure as above. In his pioneering paper \cite{3}, Boshernitzan gave the following answer to this question.

**Theorem 1.2** (\cite{3} Theorem 1.2]). Let \((X, T, \mu)\) be a measure preserving system equipped with a metric \( d \). Let \( H_\alpha \) denote the Hausdorff \( \alpha \)-measure for some \( \alpha > 0 \) and assume that \( H_\alpha \) is \( \sigma \)-finite on \( X \). Then for \( \mu \)-almost every \( x \in X \) we have
\[
\liminf_{n \geq 1} \left\{ n^{\frac{1}{\alpha}} d(T^n(x), x) \right\} < \infty.
\]
Furthermore, if \( H_\alpha(X) = 0 \), then for \( \mu \)-almost every \( x \in X \) we have
\[
\liminf_{n \geq 1} \left\{ n^{\frac{1}{\alpha}} d(T^n(x), x) \right\} = 0.
\]
Note that in general, if \( \alpha > \dim_H(X) \) then \( H_\alpha(X) = 0 \) and \( H_\alpha \) is (trivially) \( \sigma \)-finite on \( X \). In many cases, for example when \( X = \mathbb{R}^k \) with the Euclidian metric, we have that \( H_\alpha \) is \( \sigma \)-finite on \( X \) if and only if \( \alpha \geq \dim_H(X) \). In this paper we will mainly focus on interval maps, and hence we take a closer look at Theorem 1.2 when \( X = [0,1] \). Then \( H_\alpha \) is \( \sigma \)-finite on \( X \) for all \( \alpha \geq 1 \) and \( H_\alpha(X) = 0 \) for all \( \alpha > 1 \). Statement (1.1) then corresponds to the case \( \alpha = 1 \) and can be reformulated as: For \( \mu \)-almost every \( x \in X \) there exists a constant \( \kappa(x) > 0 \) such that if \( r_n(x) \geq \frac{\kappa(x)}{n} \), then \( T^n(x) \in B(x, r_n(x)) \) for infinitely many \( n \in \mathbb{N} \), i.e.

\[
\mu(\{ x : T^n(x) \in B(x, r_n(x)) \text{ for } \infty \text{ many } n \in \mathbb{N} \}) = 1.
\]

Statement (1.2) corresponds to the case \( \alpha > 1 \) and enables us to get rid of the \( x \)-dependence of the radii by decreasing the shrinking rate slightly. As a consequence, for any \( \beta < 1 \) and any \( \kappa > 0 \), we have that

\[
r_n \geq \frac{\kappa}{n^\beta} \Rightarrow \mu(R_{io}(r_n)) = 1.
\]

Boshernitzan’s result is surprisingly strong given its level of generality. Since then much work has been done on the topic of self-returning points. However, it appears that even with much stronger assumptions on the system, few improvements of Boshernitzan’s rate of \( n^{\frac{1}{\alpha}} \) have been obtained. As far as we know, the only improvements were obtained by Pawelec [17] Theorem 3.1; Chang, Wu and Wu [7]; Baker and Farmer [2]; and recently by Hussein, Li, Simmons and Wang [13].

Pawelec proved that Boshernitzan’s rate can be improved by a factor \((\log \log n)^{\frac{1}{\alpha}}\) under the assumption of exponential mixing as well as a regularity assumption on the invariant measure which is related to the value of \( \alpha \).

Chang, Wu and Wu as well as Baker and Farmer obtained improvements to Boshernitzan’s result for self-similar sets. Hussein, Li, Simmons and Wang obtained a dichotomy result for some expanding conformal systems, including piecewise expanding maps with an absolutely continuous invariant measure. For such piecewise expanding maps, their result is that \( \mu(R_{io}(r_n)) = 1 \) if and only if \( \sum r_n = \infty \), and otherwise \( \mu(R_{io}(r_n)) = 0 \).

A different perspective on the Boshernitzan result is given through the strong connection between the speed with which a typical point returns close to itself, and the local property of the measure. Let

\[
\tau_r(x) = \inf\{ n \in \mathbb{N} : d(T^n(x), x) < r \}
\]

and

\[
R(x) = \liminf_{r \to 0} \frac{\log \tau_r(x)}{-\log r} \quad \text{and} \quad \overline{R}(x) = \limsup_{r \to 0} \frac{\log \tau_r(x)}{-\log r}.
\]

Barreira and Saussol [4] proved that if \( \mu \) is an invariant probability measure, then for \( \mu \) almost every \( x \) holds

\[
(1.3) \quad R(x) \leq \underline{d}_\mu(x) \quad \text{and} \quad \overline{R}(x) \leq \overline{d}_\mu(x),
\]
where \( d^{-}_{\mu}(x) \) is the lower pointwise dimension of \( \mu \) at \( x \) and \( d^{+}_{\mu}(x) \) is the upper pointwise dimension of \( \mu \) at \( x \), defined by
\[
d^{-}_{\mu}(x) = \liminf_{r \to 0} \frac{\log \mu(B(x, r))}{\log r} \quad \text{and} \quad d^{+}_{\mu}(x) = \limsup_{r \to 0} \frac{\log \mu(B(x, r))}{\log r}.
\]

Suppose for simplicity that \( x \) is a point such that \( d^{-}_{\mu}(x) = d^{+}_{\mu}(x) = s \) and \([1,3]\) holds. Then for any \( \varepsilon > 0 \) we have \( \tau_{\varepsilon}(x) \leq r^{-s+\varepsilon} \) for small \( r \). This tells us that if \( r_{n} = n^{-\alpha} \), then \( d(T^{n}(x), x) < r_{n} \) holds for infinitely many \( n \) if \( \alpha(s+\varepsilon) \leq 1 \). In other words, for any \( \varepsilon > 0 \) we have that \( d(T^{n}(x), x) < n^{-\frac{1}{s}} \) holds for infinitely many \( n \). If \( d^{-}_{\mu}(x) = d^{+}_{\mu}(x) = s \) for \( \mu \)-almost every \( x \), the conclusion obviously holds almost surely. Formulated in another way, if \( d^{-}_{\mu}(x) = d^{+}_{\mu}(x) = s \) for \( \mu \)-almost every \( x \), then \( \mu(\mathcal{R}_{io}(n^{-\alpha})) = 1 \) if \( \alpha < \frac{1}{s} \).

Hence, the result of Barreira and Saussol is similar to the result of Boshernitzan. However, the result of Barreira and Saussol gives information about the return time \( \tau_{\varepsilon} \), which the result of Boshernitzan does not.

1.2. Outline of the paper. In this paper we prove various strengthenings of the known results on the measure of \( \mathcal{R}_{io} \). In Theorem \([A]\) we show that the rate given by Pawelec can be significantly improved for a large class of interval maps, including some quadratic maps. For this result, we need only an assumption on decay of correlations and that the invariant measure is absolutely continuous with respect to Lebesgue measure. A similar result is in Theorem \([B]\) where we are also to obtain sufficient conditions for \( \mu(\hat{\mathcal{R}}_{io}) = 1 \) for systems with an invariant measure that is not absolutely continuous with respect to Lebesgue measure, but satisfies a regularity assumption of the same type as used by Pawelec.

In Theorem \([C]\) we give general sufficient conditions for \( \mathcal{R}_{io} \) and \( \hat{\mathcal{R}}_{io} \) to be of zero measure under mixing assumptions.

We then turn our attention to the case of the doubling map as well as some linear maps on the \( d \)-dimensional torus for which we are able to prove in Theorem \([D]\) an exact dichotomy for when \( \mathcal{R}_{io} \) is of zero and full measure. In addition to rotations \([E]\) exact dichotomy results were previously only known for some self-similar sets equipped with the transformation induced by the left shift on the coding, which have recently been shown under the strong separation condition by Chang, Wu and Wu \([F]\) and under the open set condition by Baker and Farmer \([G]\). And as previously mentioned, Hussein, Li, Simmons and Wang \([I,3]\) have shown exact dichotomy results for some conformal and expanding systems.

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1Note that for a rotation \( R_{\alpha} \), we have \(|R_{\alpha}^{n}(x) - x| = |R_{\alpha}^{n}(0) + x - x| = |R_{\alpha}^{n}(0)| \). Hence \(|R_{\alpha}^{n}(x) - x| < r_{n} \) for infinitely many \( n \) if \( |R_{\alpha}^{n}(0)| < r_{n} \) for infinitely many \( n \). Hence there is a kind of dichotomy which gives either \( \mathcal{R}_{io} = \emptyset \) or \( \mathcal{R}_{io} = S^{1} = X \), depending on a condition on \( \alpha \) and \( r_{n} \). By the Duffin–Schaeffer conjecture (now a theorem of Koukoulopoulos and Maynard \([J]\)), for almost all \( \alpha \), we have \(|R_{\alpha}(0)| < r_{n} \) for infinitely many \( n \) if \( \sum_{n} \varphi(n)r_{n} \) diverges, where \( \varphi \) is Euler’s totient function. Hence for almost all \( \alpha \) we have the dichotomy that \( \mathcal{R}_{io} \) is empty or the entire circle depending on the convergence or divergence of this series. However, for a given rotation number it is not clear whether it belongs to this full measure set and hence if the divergence of the series is the condition which determines the dichotomy.
Finally, we consider the set $R_{ea}$ of eventually always returning points. In Theorem $E$ we prove a result in similar generality as Boshernitzan’s Theorem on $R_{io}$. For the doubling map we give sufficient conditions for $R_{ea}$ to be of zero and full measure in Theorem $F$. As for all known results on $H_{ea}$ there is a range of shrinking rates not allowing any conclusions on the size of $R_{ea}$. It is an open question whether one can prove a dichotomy condition on $H_{ea}$ or $R_{ea}$ for any system (see [15, Question 28]).

In the next section we state the main theorems and provide some intuition to the results and their significance.

2. Main results

On the measure of $R_{io}$ for a class of mixing interval maps. Here we consider the case $X = [0,1]$. We will need the following definition.

Definition 2.1 (Decay of correlations for $L^1$ against $BV$). Let $([0,1], T, \mu)$ denote a measure-preserving system. We say that correlations for the system decay as $p: \mathbb{N} \to \mathbb{R}$ for $L^1$ against $BV$ (bounded variation), if

$$\left| \int f \circ T^n g \, d\mu - \int f \, d\mu \int g \, d\mu \right| \leq \|f\|_1 \|g\|_{BV} p(n)$$

holds for all $n$ and all functions $f$ and $g$ with $\|f\|_1 := \int |f| \, d\mu < \infty$, $\|g\|_{BV} := \text{var} g + \text{sup} |g| < \infty$, where var $g$ denotes the total variation of $g$. If $\sum_n p(n) < \infty$, then we say that the correlations are summable.

Our first main result is the following.

Theorem A. Suppose that the system $([0,1], T, \mu)$ has exponential decay of correlations for $L^1$ against $BV$, and that $\mu$ is absolutely continuous with respect to Lebesgue measure with a density $h$ that is bounded away from zero and which belongs to $L^q$ for some $q > 1$.

Let $r_n$ be a sequence of real numbers such that for any $c > 0$ we have

$$\limsup_{N \to \infty} \sum_{n = c \log N}^{N} r_n = \infty$$

Then $\mu(R_{io}) = 1$.

Remark 2.2. The condition (2.1) is strictly stronger than the condition $\sum_{n=1}^{\infty} r_n = \infty$. However, (2.1) is satisfied for many sequences, for instance if

$$r_n \geq \frac{1}{n} \frac{1}{\prod_{j=1}^{p} \log_j n}$$

holds for some natural number $p$, where $\log_j$ denotes the logarithm iterated $j$ times.

In particular, in the language of Boshernitzan and Pawelec the above result states that for any $p$ and for $\mu$-almost any $x \in [0,1]$

$$\liminf_{n \geq 1} \left\{ n \left( \prod_{j=1}^{p} \log_j n \right) d(T^n(x), x) \right\} = 0.$$
Remark 2.3. Systems which satisfies the assumptions of Theorem A include some piecewise expanding maps [19], and quadratic maps with Benedicks–Carleson parameters as was proved by Young [21]. For piecewise expanding maps, the result of Hussein, Li, Simmons and Wang [13] is stronger than ours, since they only require that \( \sum r_n = \infty \). However, the result for quadratic maps is new.

We remark also that recently Bylund has obtained results about the recurrence of the critical point in the quadratic family \( f_a(x) = 1 - ax^2 \) [5]. He gives a condition on \( r_n \) which implies that the critical point 0 belongs to \( R_{io} \) for a positive measure set of parameters. His condition is satisfied for instance for \( r_n \geq \kappa / (n \log \log n) \), where \( \kappa > 0 \).

Our method to prove Theorem A allows us to also consider more general measures than those that are absolutely continuous with respect to Lebesgue measure. For such systems, it is more natural to consider the set

\[
\hat{R}_{io}(M_n) = \{ x : T^n(x) \in B(x, r_n(x)) \mbox{ for \infty many } n \in \mathbb{N} \},
\]

where \( r_n(x) \) is such that \( \mu(B(x, r_n(x))) = M_n \). We will prove the following theorem.

**Theorem B.** Suppose that the system \( ([0,1],T,\mu) \) has exponential decay of correlations for \( L^1 \) against \( BV \), and that there are constants \( c \) and \( s \) such that

\[
\mu(B(x,r)) \leq cr^s
\]

holds for all balls \( B(x,r) \).

Let \( M_n \) be a sequence of real numbers such that for any \( c > 0 \) we have

\[
(2.2) \quad \limsup_{N \to \infty} \sum_{n = c \log N}^{N} M_n = \infty
\]

Then \( \mu(\hat{R}_{io}(M_n)) = 1 \).

**Remark 2.4.** When the invariant measure \( \mu \) is not absolutely continuous with respect to Lebesgue measure, Theorem B is new also for piecewise expanding systems. For piecewise expanding systems, the result of Hussein, Li, Simmons and Wang [13] is only valid for measures that are absolutely continuous with respect to Lebesgue measure.

Our next theorem concerns a sufficient condition for \( R_{io} \) and \( \hat{R}_{io} \) to be of zero measure.

**Theorem C.** Let \( ([0,1],T,\mu) \) denote a measure-preserving system for which correlations for \( L^1 \) against \( BV \) are summable. Then

\[
\sum_{n=1}^{\infty} \int \mu(B(x,r_n)) \, d\mu(x) < \infty \quad \Rightarrow \quad \mu(R_{io}(r_n)) = 0
\]

and

\[
\sum_{n=1}^{\infty} M_n < \infty \quad \Rightarrow \quad \mu(\hat{R}_{io}(r_n)) = 0.
\]
Dichotomy results on the measure of $R_{io}$ for some linear maps.
For some linear maps we are able to prove an exact dichotomy for when $R_{io}$ is of zero and full measure.

**Theorem D.** Let $X = \mathbb{T}^d = [0,1]^d$, $T(x) = Ax \mod 1$, where $A$ is an integer matrix such that no eigenvalue is a root of unity. Let $\mu$ denote the Lebesgue measure on $X$ and let $r_n$ be a sequence of non-negative numbers. Then

$$\sum_{n=1}^{\infty} r_n^d < \infty \Rightarrow \mu(R_{io}(r_n)) = 0.$$  

Moreover, if all eigenvalues of $A$ are outside the unit circle, then

$$\sum_{n=1}^{\infty} r_n^d = \infty \Rightarrow \mu(R_{io}(r_n)) = 1.$$  

Note that the doubling map is a special case of the setting in Theorem D. In fact, in dimension $d = 1$, Theorem D also follows from the results obtained with different methods in [2], [7] and [13].

**Quantitative uniform recurrence results.** We turn to the set $R_{ea}$ of eventually always returning points. To state our result on speed of uniform recurrence in its full generality we need the subsequent definition.

**Definition 2.5.** Let $(X,d)$ be a metric space of finite diameter. For any $r > 0$, let $N(r)$ denote the minimal number of balls of radius $r$ that are needed to cover the space $X$. The number

$$\dim_B X = \limsup_{r \to 0} \log N(r) / -\log r$$

is called the upper box dimension of $X$.

Imitating the proof of Boshernitzan’s Theorem 1.2 we prove the following result in Section 7.

**Theorem E.** Let $(X, T, \mu)$ be a measurable dynamical system with an invariant probability measure $\mu$ and a compatible metric $d$ such that $(X,d)$ is a metric space of finite diameter and finite upper box dimension $\alpha > 0$. For every $\beta > \alpha$ and for $\mu$ almost every $x$ holds

$$\lim_{m \to \infty} m^{\beta} \inf_{0 \leq k < m} d(x, T^k x) = 0.$$  

By the same consideration as after Theorem 1.2 this statement can be reformulated for interval maps in the following way: For any $\gamma < 1$ and any $\kappa > 0$ we have that

$$r_n \geq \kappa n^\gamma \text{ for all } n \Rightarrow \mu(R_{ea}(r_n)) = 1.$$  

In case of the doubling map we can improve this rate and our results can be summarized as follows.

**Theorem F.** Let $X = [0,1]$, $T(x) = 2x \mod 1$ and let $\mu$ denote the Lebesgue measure.

(1) Assume that $\lim_{m \to \infty} mr_m = 0$. Then $\mu(R_{ea}(r_n)) = 0$.  


Suppose that $h$ is a function such that $h(n) \to \infty$ as $n \to \infty$, and let

$$r_m = \frac{\log(m)h(m)}{m}.$$ 

Then $\mu(R_{ea}(r_n)) = 1.$

**Remark 2.6.** Theorem $F$ holds true as well for transformations $T(x) = \beta x \mod 1$ for any $\beta \in \mathbb{N}$, $\beta \geq 2$. The generalization is straightforward.

We note that, to our knowledge, these are the first known results on the measure of $R_{ea}$.

### 2.1. Intuition and motivation for the main results.

It is instructive to compare the type of statement presented in Theorem $A$ (as well as Theorem 1.2 and [17, Theorem 3.1]) to the ones in Theorem $C$ and $D$.

In the context of $H_{io}$, analogues of Theorem $D$ are known as *Dynamical Borel–Cantelli lemmas* (DBCL’s) and are known to hold for many systems with nice mixing properties. One desirable feature of this kind of result is that it gives an exact dichotomy for when the set in question is of zero or full measure. Another advantage to this type of statement is that it allows a great deal of flexibility on the rate with which the targets are allowed to shrink.

We do a short intermezzo here, clarifying the use of the word *shrinking* when referring to the targets. In DBCL’s the usual assumption is that the sum of the measure of the targets is either finite or infinite. Hence shrinking in this context refers to the *measure* of the targets. (If the targets are nested, then they are necessarily also shrinking in a geometric sense.) This formulation also allows for more general targets than metric balls when $X$ has more complex geometry than in our case.

The direct analogue of dynamical Borel–Cantelli lemmas for $R_{io}$ is to consider a convergence/divergence criteria for the sum of the *average* measure of the targets. Theorem $C$ gives an example of the convergence part of this type of condition. The averaging is clearly necessitated by each sequence of targets being located in a different region of the space $X$. Hence for any non-uniform measure $\mu$, the measure of the targets depend on their location. This also led to the introduction of the set $\hat{R}_{io}$. In principle Theorem $D$ also gives an example of such a condition, however, due to the uniformity of the Lebesgue measure the averaging condition collapses to a condition simply on the sum of the radii of the targets.

In contrast, Theorem 1.2 makes only an assumption on the rate with which the radii $r_n$ go to zero, hence in this context shrinking refers to the *radii* of the balls around $x$. Since Boshernitzan only assumes invariance of the measure, no explicit connection between the radii of the balls and their measure is assumed. However, the assumption in Theorem 1.2 that the space $X$ is $\sigma$-finite with respect to the $\alpha$-dimensional Hausdorff measure, implies that the set of points for which the local dimension of the invariant measure $\mu$ is larger than $\alpha$, must be a small set. Hence, there is implicitely present a weak assumption on the connection between radii of most balls and their measure.

As for Boshernitzan’s theorem, the result of Pawelec [17, Theorem 3.1] and Theorem $A$ are likewise formulated in terms of shrinking of the radii,
however, due to further assumptions on the invariant measure there exists at least a partial connection between the radii and measures of the targets in these cases.

It seems reasonable to expect that the dichotomy in Theorem D holds also under the assumptions of Theorem A, but we are uncertain if this is true. More generally, under sufficiently strong mixing assumptions, one might expect that the divergence of the series

\[ \sum_{n=1}^{\infty} \int \mu(B(x, r_n)) \, d\mu(x) \]

implies that \( \mathcal{R}_{io} \) has full \( \mu \)-measure.

The reason that we need the stronger assumption on \( r_n \) in Theorem A, rather than the divergence of the series above, is that the proof uses estimates on the correlation of the sets \( \{ x : |x - T^n(x)| < r_n \} \). Our estimates on these correlations and the method of proof are not strong enough to obtain \( \mu(\mathcal{R}_{io}) = 1 \) unless we impose extra assumptions on the radii \( r_n \).

2.2. Structure of the paper. We start by collecting several consequences of sufficiently fast decay of correlation in Section 3. These will prove useful in the proofs of the main theorems on \( \mathcal{R}_{io} \) in Sections 4–6. Finally, we consider the set \( \mathcal{R}_{ea} \) of eventually always returning points in Sections 7–8.

3. Consequences of correlation decay

Throughout this section we set \( X = [0, 1] \) and \( T : X \to X \). We will deduce consequences from assumptions on the decay of correlation for \( L^1 \) against \( BV \). We start with the following adaption of Lemma 3 in [18]. The statement is true also for more general piecewise continuous functions \( F \), but to stay simple we formulate it for the kind of functions that we will apply it to.

**Lemma 3.1.** Assume that \( T : X \to X \) has summable decay \( p(n) \) of correlations for \( L^1 \) against \( BV \). Suppose that \( F : [0, 1]^2 \to \mathbb{R} \) is the indicator function of an open or closed convex subset of \( [0, 1]^2 \). Then

\[
\left| \int F(T^n x, x) \, d\mu(x) - \iint F \, d\mu \, d\mu \right| \leq 3p(n)
\]

**Proof.** Let \( Y \subset [0, 1]^2 \) be the convex subset such that \( F(x, y) = \mathbb{1}_Y(x, y) \). Take \( \varepsilon > 0 \) and fix \( n \). Let \( \hat{F} \) be a continuous function such that

\[
\left| \int F(T^n x, x) \, d\mu(x) - \int \hat{F}(T^n x, x) \, d\mu(x) \right| < \varepsilon
\]

and

\[
\left| \iint F \, d\mu \, d\mu - \iint \hat{F} \, d\mu \, d\mu \right| < \varepsilon.
\]

We may choose \( \hat{F} \) such that for all \( x \in [0, 1] \) the function \( f_x : y \mapsto \hat{F}(x, y) \) satisfies \( \text{var} f_x \leq 2 \) and \( \sup |f_x| \leq 1 \). Indeed, put

\[
\hat{F}_k(x, y) = \begin{cases} 
1 & \text{if } (x, y) \in Y, \\
\max\{0, 1 - kd((x, y), Y)\} & \text{if } (x, y) \not\in Y,
\end{cases}
\]
if $Y$ is closed and

$$
\hat{F}_k(x, y) = \begin{cases} 
0 & \text{if } (x, y) \notin Y, \\
\min\{1, kd((x, y), \bar{Y})\} & \text{if } (x, y) \in Y,
\end{cases}
$$

if $Y$ is open, so that $\hat{F}_k$ is a piecewise linear approximation of $F$.

If $Y$ is closed, then $\hat{F}_k \geq F$ and $\hat{F}_k$ converges pointwise to $F$. If $Y$ is open, then $\hat{F}_k \leq F$ and $\hat{F}_k$ converges pointwise to $F$. It follows that the functions $x \mapsto \hat{F}_k(T^nx, x)$ converges pointwise to $x \mapsto F(T^nx, x)$, and the convergence is monotone. Hence, by the monotone convergence theorem, we may take $k$ so large that

$$
\left| \int \hat{F}_k(T^n x, x) \, d\mu(x) - \int F(T^n x, x) \, d\mu(x) \right| < \varepsilon
$$

and

$$
\left| \iint F \, d\mu d\mu - \iint \hat{F}_k \, d\mu d\mu \right| < \varepsilon.
$$

Let $\hat{F} = \hat{F}_k$ for such a $k$.

Let $I_k = [a_k, a_{k+1})$, $k = 0, \ldots, m - 1$ be a partition of $X$. Set

$$
G(x, y) = \sum_{k=0}^{m-1} \hat{F}(a_k, y) 1_{I_k}(x),
$$

where $1_{I_k}$ denotes the characteristic function on $I_k$. Since $\hat{F}$ is continuous we may choose a partition so that

$$
|\hat{F}(x, y) - G(x, y)| < \varepsilon
$$

and hence

$$
\left| \int \hat{F}(T^n x, x) \, d\mu(x) - \int G(T^n x, x) \, d\mu(x) \right| < \varepsilon.
$$

The second integral can be rewritten as

$$
\int G(T^n x, x) \, d\mu(x) = \sum_{k=0}^{m-1} \int \hat{F}(a_k, x) 1_{I_k}(T^n x) \, d\mu(x).
$$

For the integral on the right hand side we may rewrite

$$
\left| \int \hat{F}(a_k, x) 1_{I_k}(T^n x) \, d\mu(x) - \int \hat{F}(a_k, x) \, d\mu(x) \int 1_{I_k}(x) \, d\mu(x) \right|
\leq \|1_{I_k}\|_1 \|\hat{F}(a_k, x)\|_{BV}(n)
\leq \mu(I_k)3p(n).
$$

Now summing over $k$ on both sides and using that $\sum_k \mu(I_k) = 1$ we get

$$
\left| \int G(T^n x, x) \, d\mu(x) - \sum_{k=0}^{m-1} \hat{F}(a_k, x) \mu(I_k) \, d\mu(x) \right| \leq 3p(n)
$$

and

$$
\left| \int \hat{F}(T^n x, x) \, d\mu(x) - \sum_{k=0}^{m-1} \hat{F}(a_k, x) \mu(I_k) \, d\mu(x) \right| \leq \varepsilon + 3p(n).
$$
Hence
\[ \left| \int F(T^n x, x) \, d\mu(x) - \int \sum_{k=0}^{m-1} \hat{F}(a_k, x) \mu(I_k) \, d\mu(x) \right| \leq 2\varepsilon + 3p(n) \]
and
\[ \left| \int F(T^n x, x) \, d\mu(x) - \iint F \, d\mu d\mu \right| \leq \iint F \, d\mu d\mu - \iint \hat{F} \, d\mu d\mu + \iint \hat{F} \, d\mu d\mu - \sum_{k=0}^{m-1} \hat{F}(a_k, x) \mu(I_k) \, d\mu(x) + 2\varepsilon + 3p(n). \]

Finally, by letting \( \varepsilon \to 0 \) (i.e. \( m \to \infty \)), the sum converges to the integral \( \iint \hat{F} \, d\mu d\mu \) and we get
\[ \left| \int F(T^n x, x) \, d\mu(x) - \iint F(y, x) \, d\mu(y) d\mu(x) \right| \leq 3p(n). \]

In order to find correlation estimates in Section 4, we need a lemma similar to Lemma 3.1 but for functions of three variables. This is provided by the following lemma (with \( M_n = \mu(B(x, r_n(x))) \) for every \( n \in \mathbb{N} \) as before).

**Lemma 3.2.** Suppose that \( (X, T, \mu) \) has exponential decay of correlations for \( L^1 \) against \( BV \) (with \( p(n) = Ce^{-rn} \) in Definition 2.1) and that there are constants \( c \) and \( s \) such that
\[ \mu(B(x, r)) \leq cr^s \]
holds for any ball \( B(x, r) \).

There is a constant \( D \) such that for all \( m, n \in \mathbb{N} \), and for \( F \) defined by
\[ F(x, y, z) = \begin{cases} 1 & \text{if } x \in B(z, r_{n+m}(z)) \text{ and } y \in B(z, r_n(z)), \\ 0 & \text{otherwise}, \end{cases} \]
we have
\[ \int F(T^{n+m} x, T^n x, x) \, d\mu(x) \leq (1 + 3Ce^{-\frac{r}{2}}) \iint F \, d\mu d\mu \]
\[ + D(M_ne^{-\frac{r}{2}} + M_{m+n}(e^{-\frac{r}{2}} + e^{-rn} + e^{-\frac{r}{2}})). \]

**Proof.** Consider the integral \( \int F(T^{n+m} x, T^n x, x) \, d\mu(x) \). We first approximate \( F \) by partitioning \([0, 1]\) into \( e^{rn/2} \) subintervals of equal length. The indicator function of the \( k \)-th interval is \( G_k \). Let \( z_k \) be the mid point of the \( k \)-th interval.

We write
\[ F(x, y, z) \leq \hat{F}(x, y, z) := \sum_k F_k(x, y)G_k(z), \]
where \( F_k \) is chosen such that the above inequality is true, and making the approximation close to as good as possible. More precisely, we choose \( F_k \) to be the indicator function of a rectangle \( A_{z_k} \), defined as follows.

Consider for fixed \( z \) the sets \( A_z = \{(x, y) : F(x, y, z) = 1\} \), which is a rectangle of size \( 2r_{n+m} \times 2r_n \). By expanding \( A_{z_k} \) to a rectangle of size \( 2(r_{n+m} + e^{-rn/2}) \times 2(r_n + e^{-rn/2}) \) and with the same centre, we obtain the rectangle \( A_{z_k} = I_k \times J_k \). By the construction, we have that \( F \leq \hat{F} \).
Since $F_k$ is the indicator function of a rectangle $\tilde{A}_z = I_k \times J_k$, we get by decay of correlations that
\[
\int F_k(T^m x, x) \, d\mu(x) = \int 1_{I_k}(T^m x) 1_{J_k}(x) \, d\mu(x) 
\leq \mu(I_k)(\mu(J_k) + 3Ce^{-\tau m}).
\] (3.1)
and we get by the assumption $\mu(B(x, r)) \leq cr^n$ that
\[
\mu(I_k) \leq M_{n+m} + 2Ce^{-\frac{\tau}{2}n} \quad \text{and} \quad \mu(J_k) \leq M_n + 2Ce^{-\frac{\tau}{2}n},
\]
which combined with (3.1), implies that
\[
(3.2)
\int F_k(T^m x, x) \, d\mu(x) \leq K_{m,n}
\]
where
\[
K_{m,n} := (M_{n+m} + 2Ce^{-\frac{\tau}{2}n})(M_n + 2Ce^{-\frac{\tau}{2}n} + 3Ce^{-\tau m}).
\]
Using decay of correlations and (3.2), we now get
\[
\int F(T^{n+m} x, T^n x, x) \, d\mu(x)
\leq \sum_k \int F_k(T^{n+m} x, T^n x) G_k(x) \, d\mu(x)
\leq \sum_k \int F_k(T^n x, x) \, d\mu(x) \left( \int G_k d\mu + 3Ce^{-\tau n} \right)
\leq \sum_k K_{m,n} \left( \int G_k d\mu + 3Ce^{-\tau n} \right)
\leq K_{m,n} \left( 1 + \sum_k 3Ce^{-\tau n} \right),
\]
where in the last step, we used that $\sum_k \int G_k d\mu = 1$.Using that the sum over $k$ has $e^{\tau n/2}$ terms, we get that
\[
\int F(T^{n+m} x, T^n x, x) \, d\mu(x)
\leq (M_{n+m} + 2Ce^{-\frac{\tau}{2}n})(M_n + 2Ce^{-\frac{\tau}{2}n} + 3Ce^{-\tau m})(1 + 3Ce^{-\tau n})
\leq M_n M_{n+m}(1 + 3Ce^{-\frac{\tau}{2}n}) + D(M_n e^{-\frac{\tau}{2}n} + M_{m+n}(e^{-\frac{\tau}{2}n} + e^{-\tau m}) + e^{-\frac{\tau}{2}n}).
\]
Since $\int \int \int F \, d\mu \, d\mu \, d\mu = M_n M_{m+n}$, the lemma follows. \hfill \square

4. PROOF OF THEOREMS A AND B

4.1. Correlation estimates. We are going to first prove Theorem B and then conclude Theorem A from Theorem B. To prove Theorem B we suppose that a sequence $(M_n)_{n=1}^{\infty}$ is given, and we define $r_n(x)$ such that $\mu(B(x, r_n(x))) = M_n$.

Let $E_n = \{ x : T^n(x) \in B(x, r_n(x)) \}$. Here, we state and prove some estimates on the measure of $E_n$ that will be needed in the proof of Theorem B.
Lemma 4.1. There is a constant $C > 0$ such that
\[ M_n - Ce^{-\tau n} \leq \mu(\hat{E}_n) \leq M_n + Ce^{-\tau n}. \]

Proof. Put
\[ F_n(x, y) = \begin{cases} 1 & \text{if } x \in B(y, r_n(y)), \\ 0 & \text{otherwise}. \end{cases} \]
Then
\[ \mu(\hat{E}_n) = \int F_n(T^n(x), x) \, d\mu(x). \]
Consequently, we have by Lemma 3.1 that
\[ \int \int F_n \, d\mu \, d\mu - Ce^{-\tau n} \leq \mu(\hat{E}_n) \leq \int \int F_n \, d\mu \, d\mu + Ce^{-\tau n}. \]
By the definition of $F_n$ and by Fubini’s theorem, we have
\[ \int \int F_n \, d\mu \, d\mu = M_n, \]
which proves the lemma.

The following lemma is a direct consequence of Lemma 3.2.

Lemma 4.2. We have the following correlation estimate.
\[ \mu(\hat{E}_n \cap \hat{E}_{n+m}) \leq (1 + 3Ce^{-2\tau n})M_nM_{n+m} 
+ D(M_ne^{-\frac{s}{2}n} + M_{m+n}(e^{-s\tau n} + e^{-s\tau m}) + e^{-s\tau n}). \]

We will use the correlation estimate from Lemma 4.2 to apply the following inequality by Chung and Erdős.

Lemma 4.3 (The Chung–Erdős inequality [9, Lemma]). For measurable sets $A_1, \ldots, A_n$ holds
\[ \mu(A_1 \cup \ldots \cup A_n) \geq \left( \sum_{j=1}^n \mu(A_j) \right)^2 \sum_{j,k=1}^n \mu(A_j \cap A_k). \]

4.2. Proof of Theorems A and B. In this subsection we first give the proof of Theorem B and later we conclude Theorem A. The proof is based on the Chung–Erdős inequality, Lemma 4.3.

4.2.1. Proof of Theorem B. We let
\[ U_N = \bigcup_{j \in I_N} \hat{E}_j \]
where
\[ I_N = \{ j : \frac{2}{\tau s} \log N \leq j \leq N \}. \]
We note that $U_N$ is defined so that $\hat{R}_{io} = \limsup_{N \to \infty} U_N,$ and to prove that $\hat{R}_{io}$ has large measure, we will prove that $U_N$ has large measure. In the union which defines $U_N$ we consider only set $\hat{E}_j$ with $j \in I_N.$ By introducing the set $I_N,$ we get better correlation control. This has the effect that we need to require that
\[ \lim_{N \to \infty} \sum_{n=c\log N}^N M_n = \infty. \]
in order to prove that $U_N$ has large measure, which is the reason that we cannot only assume that $\sum M_n$ is divergent.

Let

$$S_N = \sum_{j \in I_N} \mu(\hat{E}_j) \quad \text{and} \quad \sigma_N = \sum_{j \in I_N} M_j.$$ 

By Lemma 4.1

$$\sum_{j \in I_N} (M_j - Ce^{-\tau j}) \leq S_N \leq \sum_{j \in I_N} (M_j + Ce^{-\tau j}),$$

so that

$$\sigma_N - c_1 \leq S_N \leq \sigma_N + c_1,$$

for some constant $c_1$.

We let

$$C_N = \sum_{j,k \in I_N} \mu(\hat{E}_j \cap \hat{E}_k),$$

and by Lemma 4.2 we have that

$$C_N = S_N + 2 \sum_{j,k \in I_N, j > k} \mu(\hat{E}_j \cap \hat{E}_k)$$

$$\leq S_N + 2 \sum_{j,k \in I_N, j > k} (1 + 3Ce^{-\frac{s}{2}k})M_j M_k + R_N$$

$$\leq S_N + (1 + 3CN^{-1/s})\sigma_N^2 + R_N,$$

where

$$R_N = 2D \sum_{j,k \in I_N, j > k} (M_k e^{-\frac{\tau}{2}k} + M_j (e^{-\frac{\tau}{2}k} + e^{-\tau(j-k)}) + e^{-\frac{\tau}{2}k})).$$

We will prove that $R_N$ is bounded, and split $R_N$ into four sums in a natural way.

For the first sum, we have

$$\sum_{j,k \in I_N, j > k} M_k e^{-\frac{\tau}{2}k} \leq \sum_{j = \frac{1}{2} \log N}^{N} \sum_{k = \frac{1}{2} \log N}^{j-1} e^{-\frac{\tau}{2}k} \leq \sum_{j = \frac{1}{2} \log N}^{N} c_2 e^{-\log N} \leq c_2.$$ 

The same kind of estimate works for the second sum as well and yields

$$\sum_{j,k \in I_N, j > k} M_j e^{-\frac{\tau}{2}k} \leq \sum_{j = \frac{1}{2} \log N}^{N} \sum_{k = \frac{1}{2} \log N}^{j-1} e^{-\frac{\tau}{2}k} \leq \sum_{j = \frac{1}{2} \log N}^{N} c_2 e^{-\log N} \leq c_2.$$ 

For the third sum, we have

$$\sum_{j,k \in I_N, j > k} M_j e^{-\tau(j-k)} = \sum_{j = \frac{1}{2} \log N}^{N} M_j \sum_{k = \frac{1}{2} \log N}^{j-1} e^{-\tau(j-k)}$$

$$\leq \sum_{j = \frac{1}{2} \log N}^{N} c_3 M_j = c_3 \sigma_N.$$
Finally, the fourth sum is estimated as the first two by
\[ \sum_{j,k \in I \atop j > k} e^{-\frac{s}{2} k} \leq c_2. \]

Hence we have \( R_N \leq 2D(3c_2 + 3c\sigma_N) \) and
\[
C_N \leq S_N + (1 + 3CN^{-1/s})\sigma_N^2 + 2D(3c_2 + 3c\sigma_N)
\leq \sigma_N + c_1 + (1 + 3CN^{-1/s})\sigma_N^2 + 2D(3c_2 + 3c\sigma_N)
= (1 + 3CN^{-1/s})\sigma_N^2 + c_4(\sigma_N + 1).
\]

We now use the Chung–Erdös inequality and conclude that
\[ \mu(U_N) \geq \frac{S_N^2}{C_N} \geq \frac{(\sigma_N - c_1)^2}{(1 + 3CN^{-1/s})\sigma_N^2 + c_4(\sigma_N + 1)}. \]

Since \( \limsup_{N \to \infty} \sigma_N = \infty \) we conclude that
\[ \limsup_{N \to \infty} \mu(U_N) \geq 1, \]
and hence that \( \limsup_{N \to \infty} \mu(U_N) = 1 \). It follows that \( \mu(\limsup U_N) = 1 \) and since \( \limsup U_N = \limsup \hat{E}_j \) we have proved that \( \mu(\hat{R}_{io}) = 1 \). This proves Theorem \( B \).

4.2.2. Proof of Theorem \( A \). We now conclude Theorem \( A \) from Theorem \( B \).
Since \( \mu \) has density in \( L^q \) for \( q > 1 \), it follows by Hölder’s inequality that we may take \( s = 1 - \frac{1}{q} \). Hence the assumptions of Theorem \( B \) are satisfied and we may conclude that for any sequence \( M_n \) which satisfies (2.2) and for almost every \( x \), we have \( T^n(x) \in B(x,r_n(x)) \) for infinitely many \( n \), where \( r_n \) is such that \( \mu(B(x,r_n)) = M_n \). Since \( \mu \) is absolutely continuous with respect to Lebesgue measure with a density which is bounded away from zero, this immediately implies that for almost all \( x \) we have \( d(x,T^n(x)) < r_n \) for infinitely many \( n \), provided the sequence \( r_n \) satisfies (2.1). This proves Theorem \( A \).

5. Proof of Theorem \( C \)

Given a sequence \( (r_n)_{n=1}^\infty \) of non-negative numbers, we will use the notation
\[ E_n := \{ x \in X : T^n(x) \in B(x,r_n) \} \]
so that
\[ \mathcal{R}_{io} = \bigcap_{k=1}^\infty \bigcup_{n=k}^\infty E_n. \]

(5.1)

Note that from (5.1) we get
\[ \mu(\mathcal{R}_{io}) \leq \lim_{k \to \infty} \sum_{n=k}^\infty \mu(E_n). \]
Hence, if for fixed $k$ the sum converges we get $\mu(R_{\alpha}) = 0$, so we are interested in estimating the measure $\mu(E_n)$. For that purpose, we will apply Lemma 3.1 on the function $F_n$ defined by

$$F_n(x, y) = \begin{cases} 1 & \text{if } x \in B(y, r_n), \\ 0 & \text{otherwise.} \end{cases}$$

This allows us to write

$$\mu(E_n) = \int 1_{E_n} \, d\mu = \int F_n(T^n x, x) \, d\mu(x). \tag{5.2}$$

By Lemma 3.1 we then have

$$\left| \mu(E_n) - \int \int F_n(y, x) \, d\mu(y) \, d\mu(x) \right| \leq 3p(n) \tag{5.3}$$

which turns into

$$\left| \mu(E_n) - \int \mu(B(x, r_n)) \, d\mu(x) \right| \leq 3p(n) \tag{5.4}$$

and finally

$$\mu(E_n) \leq \int \mu(B(x, r_n)) \, d\mu(x) + 3p(n) \tag{5.5}$$

The term $p(n)$ is summable by assumption, hence we see that if

$$\sum_{n=1}^{\infty} \int \mu(B(x, r_n)) \, d\mu(x) < \infty$$

then $\sum \mu(E_n) < \infty$ and consequently $\mu(R_{\nu}) = 0$. This proves Theorem C.

6. PROOF OF THEOREM D

6.1. The one dimensional case. In this section we prove Theorem D in the one dimensional case when $T : [0, 1] \to [0, 1]$ and $Tx = ax \mod 1$, where $a$ is an integer with $|a| > 1$. We do this since the proof in this case is simpler. In Section 6.2 we give the proof of the higher dimensional case. The higher dimensional case is similar to the one dimensional case, but has some extra complications that are not present in the one dimensional case.

We let $\mu$ denote the Lebesgue measure on $X = [0, 1]$, which is a $T$ invariant measure. In this case $\mu(B(x, r_n)) = 2r_n$. Note that, in contrast to the general case of Section 5, the right hand side is independent of $x$.

The proof of the theorem will rely on an application of the following lemma with $H = 1$. (See for instance Harman [11, Lemma 2.3], or conclude it yourself from the Chung–Erdős inequality.) The special case with $H = 1$ is the Erdős–Renyi formulation of the Borel–Cantelli lemma [10].

**Lemma 6.1.** Let $H > 0$. If $A_j$ are sets such that

$$\sum_{n=1}^{\infty} \mu(A_n) = \infty \tag{6.1}$$
Since all quantities in these two equations are integers, we can conclude that
\[ \mu(A_i \cap A_j) - H \mu(A_i) \mu(A_j) \]
and
\[ \liminf_{N \to \infty} \sum_{1 \leq i < j \leq N} \left( \sum_{i=1}^{N} \mu(A_i) \right) \leq 0, \]
then \( \mu(\limsup A_j) \geq \frac{1}{H} \).

The strategy is to rewrite the quantity in the numerator of (6.2) using Fourier series. The following two lemmas will be helpful.

**Lemma 6.2.** Let \( f \) be a function of bounded variation on \([0, 1]\). Let \( f \sim \sum_{n \in \mathbb{Z}} c_n e^{2\pi i n x} \) be the Fourier series of the 1-periodic extension of \( f \) to \( \mathbb{R} \). Then \( |c_n| \leq \frac{\text{var} f}{2\pi|n|} \) for any \( n \neq 0 \).

**Proof.** Using Stieltjes integration, and integration by parts, we may write
\[ c_n = \int_0^1 e^{-i2\pi n x} f(x) \, dx = \int_0^1 \frac{1}{i2\pi n} f(x) \, d(e^{-i2\pi n x}) \]
\[ = \int_0^1 \frac{1}{i2\pi n} e^{-i2\pi n x} \, df(x). \]
Hence \( |c_n| \leq \frac{1}{2\pi|n|} \text{var} f \).

For an elementary proof not using Stieltjes integrals, see Taibleson [20].

**Lemma 6.3.** Let \( a, m, n \in \mathbb{N} \). Then \( \gcd(a^m - 1, a^n - 1) = a^{\gcd(m,n)} - 1 \).

**Proof.** Set \( d = \gcd(a^m - 1, a^n - 1) \) and \( k = \gcd(m,n) \). Hence the claim is that \( d = a^k - 1 \). We will prove this by first showing that \( a^k - 1 \mid d \) and afterwards that \( d \mid a^k - 1 \).

Since \( k = \gcd(m,n) \) we have \( k \mid m \) and \( k \mid n \). Say \( m = ks, n = kl \) for some \( s,l \in \mathbb{Z} \). This means that we may write
\[ a^m - 1 = a^{ks} - 1 = (a^k)^s - (1)^s \]
\[ a^n - 1 = a^{kl} - 1 = (a^k)^l - (1)^l. \]
We recall the general identity for \( p, q, r \in \mathbb{N} \),
\[ (p^r - q^r) = (p - q)(p^{r-1} + p^{r-2}q + p^{r-3}q^2 + \ldots + pq^{r-2} + q^{r-1}) \]
which can be verified simply by multiplying the brackets. Applying this identity we get that
\[ (a^k)^s - (1)^s = (a^k - 1)(a^{k(s-1)} + a^{k(s-2)} + \ldots + a^k + 1) \]
\[ (a^k)^l - (1)^l = (a^k - 1)(a^{k(l-1)} + a^{k(l-2)} + \ldots + a^k + 1). \]
Since all quantities in these two equations are integers, we can conclude that \( a^k - 1 \mid a^n - 1 \) and \( a^k - 1 \mid a^m - 1 \). Hence, \( a^k - 1 \mid \gcd(a^m - 1, a^n - 1) = d \).

To show \( d \mid a^k - 1 \) we apply Bézout’s Lemma on \( k = \gcd(m,n) \) which gives us \( u, v \in \mathbb{Z} \) such that \( um + vn = k \). On the one hand, we note that \( u \) and \( v \) cannot be both positive because then \( k \) would be larger than \( m \) and \( n \). On the other hand, \( u \) and \( v \) cannot be both negative because then
would be negative as well. Without loss of generality, we let \( u > 0 \) and \( v \leq 0 \). Notice that if \( v = 0 \), then \( um = k \) which implies \( u = 1 \) and \( k = m \). Clearly, \( d = \gcd(a^m - 1, a^n - 1) \) divides \( a^k - 1 \) in this case. So we examine the remaining situation \( u > 0 \) and \( v < 0 \). Then we use the identity (6.3) again to see that \( d = \gcd(a^m - 1, a^n - 1) \) divides \( a^{um} - 1 \) as well as \( a^{vn} - 1 \). Hence, \( d \) divides \( a^{um} - 1 - a^k(a^{vn} - 1) = a^k - 1 \).

We conclude that \( a^k - 1 = d \) and the lemma is proved. \( \square \)

We are now ready to prove Theorem D in the one dimensional case.

**Proof of Theorem D when \( d = 1 \).** In the case of \( \sum_{n=1}^{\infty} r_n < \infty \) the result will follow from the easy part of the Borel–Cantelli lemma. In the case \( \sum_{n=1}^{\infty} r_n = \infty \) the statement will follow from the special case of Lemma 6.1 with \( H = 1 \).

In our use of Lemma 6.1, we let \( A_n = E_n = \{ x \in X : T^n(x) \in B(x, r_n) \} \) recalling that \( R_{\text{io}} = \bigcap_{k=1}^{\infty} \bigcup_{n=k}^{\infty} E_n \).

To analyse \( \mu(E_n) \) and \( \mu(E_n \cap E_m) \), we define the function

\[
G_n(x) = \begin{cases} 
1 & \text{if } |x| < r_n \\
0 & \text{otherwise.}
\end{cases}
\]

Since \( G_n(x) \) is a function on \( \mathbb{R}/\mathbb{Z} \), we may periodically extend it to all of \( \mathbb{R} \) and write it via its Fourier series, i.e.

\[
G_n(x) = \sum_{l \in \mathbb{Z}} c_{n,l} e^{2\pi i l x}.
\]

The function \( G_n(T^n x - x) \) is the indicator function of \( E_n \). Hence, we have

\[
\mu(E_n) = \int G_n(T^n x - x) \, dx = \sum_{l \in \mathbb{Z}} c_{n,l} \int e^{-2\pi i l (a^n - 1)x} \, dx.
\]

In the sum above, all integrals are zero, except for \( l = 0 \). Hence we have

\[
\mu(E_n) = c_{n,0} = \int G_n(x) \, dx = 2r_n.
\]

It now follows that if \( \sum r_n < \infty \) then \( \sum \mu(E_n) < \infty \) and the easy part of the Borel–Cantelli lemma implies that \( \mu(R_{\text{io}}) = 0 \). Of course, for \( d = 1 \) this also follows directly from Theorem C.

We assume from now on that \( \sum r_n = \infty \).

Using the Fourier series for \( G_n \) we rewrite the quantity \( \mu(E_m \cap E_n) \). We have

\[
\mu(E_m \cap E_n) = \int 1_{E_m \cap E_n} \, d\mu = \int 1_{E_m} 1_{E_n} \, d\mu = \int G_m(T^m x - x) G_n(T^n x - x) \, d\mu.
\]

Hence we may write

\[
\int G_m(T^m x - x) G_n(T^n x - x) \, d\mu
\]
Recall that $\mu$ now estimate the quantity $(6.6)$ solving \[(a_m, n, p) - (a_n, p)(-j), j \in \mathbb{Z}. \] This means that the sum in (6.5) may be rewritten as
\[
c_m, 0 \cdot c_n, 0 + \sum_{j \in \mathbb{Z} \setminus \{0\}} c_m, a_{(m, p)} \cdot c_n, a_{(n, p)}(-j).
\]
Recall that $c_{m, 0} = 2r_m = \mu(E_m)$. Since var $G_n \leq 2$, we have by Lemma 6.2 that $|c_{n, k}| \leq \frac{1}{\pi |k|}$. Using these estimates on the Fourier coefficients, we can now estimate the quantity $\mu(E_m \cap E_n) - \mu(E_m)\mu(E_n)$, namely,
\[
\mu(E_m \cap E_n) - \mu(E_m)\mu(E_n) = \sum_{j \in \mathbb{Z} \setminus \{0\}} c_{m, a_{(m, p)} \cdot c_n, a_{(n, p)}(-j)} \leq \sum_{j \in \mathbb{Z} \setminus \{0\}} \frac{1}{|a_{(m, p)}|} \left| c_{n, a_{(n, p)}(-j)} \right| \leq \frac{1}{\pi^2} \sum_{j \in \mathbb{Z} \setminus \{0\}} \frac{1}{|a_{(m, p)}|} \left| a_{(n, p)}(-j) \right| \leq \frac{1}{\pi^2} \sum_{j \in \mathbb{Z} \setminus \{0\}} \frac{1}{\left| \frac{a_m - 1}{a^n - 1} \right|} \left| \frac{a^n - 1}{a^n - 1} \right| \cdot \left| \frac{a_m - 1}{a^n - 1} \right|.
\[
\begin{align*}
&\frac{1}{\pi^2} \sum_{j \in \mathbb{Z} \setminus \{0\}} \frac{1}{|j|^2 (a^m - 1)(a^n - 1)} \\
&< \frac{4}{\pi^2} \sum_{j \in \mathbb{Z} \setminus \{0\}} \frac{1}{|j|^2 a^{m+n}} \\
&\leq 2a^{2p-(m+n)}.
\end{align*}
\]

Inserting this in condition (6.2) of Lemma 6.1 with \(H = 1\) we get
\[
\sum_{1 \leq n < m \leq k} 2a^{2p-(m+n)} = \frac{2 \sum_{m=1}^{k} \sum_{n=1}^{m-1} a^{2p-(m+n)} \left(\sum_{n=1}^{\lfloor \frac{m}{2} \rfloor} a^{2n-(m+n)} + \sum_{n=\lceil \frac{m}{2} \rceil}^{m-1} a^{2(m-n)-(m+n)}\right)}{\left(\sum_{n=1}^{k} \mu(E_n)\right)^2}.
\]

By assumption we know that the denominator goes to infinity and we will show that the numerator converges for \(k \to \infty\). We will do this by splitting the sum in the numerator as follows
\[
\sum_{1 \leq n < m \leq k} 2a^{2p-(m+n)} \leq \sum_{m=1}^{k} \left(\sum_{n=1}^{\lfloor \frac{m}{2} \rfloor} a^{2n-m} + \sum_{n=\lceil \frac{m}{2} \rceil}^{m-1} a^{m-3n}\right)
\]
\[
= \sum_{m=1}^{k} \left(\frac{m}{2} a^{\frac{m}{2}} - m a^{-\frac{m}{2}} + \frac{m}{2} a^{-\frac{3m}{2}}\right)
\]
\[
= \sum_{m=1}^{k} \left(\frac{m}{2} a^{-\frac{m}{2}} + \frac{m}{2} a^{-\frac{3m}{2}}\right)
\]
\[
\leq \sum_{m=1}^{\infty} ma^{-\frac{m}{2}}.
\]

This series converges and hence condition (6.2) is satisfied. The result then follows from Lemma 6.1. \(\square\)

6.2. **The case of general dimension.** The proof follows the strategy employed for the one dimensional case but with certain adaptations. The notation \(E_n\) and \(G_n\) remains unchanged.

We first take care of the measure zero case which is just a simple adaptation. Written as Fourier series \(G_n\) becomes
\[
G_n(x) = \sum_{l \in \mathbb{Z}^d} c_{nl} e^{2\pi i \langle l, x \rangle}.
\]
In the same way as in the one dimensional case, we have
\[ \mu(E_n) = \int G_n(T^n x - x) \, dx = \sum_{l \in \mathbb{Z}^d} c_{n,l} \int e^{i2\pi \langle l, (A^n - I)x \rangle} \, dx = \sum_{l \in \mathbb{Z}^d} c_{n,l} \int e^{i2\pi \langle (A^n - I)^T l, x \rangle} \, dx. \]

Since \( A \) has no eigenvalues that are roots of unity, the matrix \((A^n - I)^T\) is an invertible integer matrix and \((A^n - I)^T l = \vec{0}\) only if \( l = \vec{0} \), where \( \vec{0} \) denotes the zero-vector in \( d \) dimensions. Hence, all integrals in the sum above are zero, unless \( l = \vec{0} \). It follows that
\[ \mu(E_n) = c_{n,\vec{0}} = \int G_n \, dx = c_d r_n^d, \]
where \( c_d \) is the volume of the \( d \) dimensional unit ball. Now, if \( \sum r_n^d < \infty \) then \( \sum \mu(E_n) < \infty \) and the easy part of the Borel–Cantelli lemma implies that \( \mu(\mathcal{R}_{lo}) = 0 \).

We assume from now on that \( \sum r_n^d = \infty \) and that all eigenvalues of \( A \) lie outside the unit circle. There is a number \( \lambda > 1 \) such that all eigenvalues of \( A \) have modulus strictly larger than \( \lambda \). In this case we approximate the functions \( G_n \) by \( C^r \)-functions. As a parameter in this approximation, we choose \( \varepsilon > 0 \). Let \( f \in C^r([0,1]) \) be such that \( f \) is monotone, \( f(0) = 1 \), \( f(1) = 0 \) and \( f' \) has compact support in \((0,1)\). Put \( f_n(t) = f\left(\frac{t - r_n}{\varepsilon r_n}\right) \).

We approximate \( G_n \) by
\[ \tilde{G}_n(x) = \begin{cases} 1 & \text{if } |x| \leq r_n \\ f_n(|x|) & \text{if } r_n < |x| \leq (1 + \varepsilon)r_n \\ 0 & \text{if } (1 + \varepsilon)r_n < |x| \end{cases} \]
where \( |x| \) denotes the length of the vector \( |x| \). Note that \( G_n \leq \tilde{G}_n \). Written as Fourier series \( \tilde{G}_n \) becomes
\[ \tilde{G}_n(x) = \sum_{l \in \mathbb{Z}^d} \tilde{c}_{n,l} e^{2\pi i \langle l, x \rangle}. \]

Analogous to above, we get
\[ (1 + \varepsilon)^d \mu(E_n) \geq \int \tilde{G}_n \, dx = \tilde{c}_{n,\vec{0}}. \]

Since \( \tilde{G}_n \) is a \( C^r \)-function, a scaling argument gives the estimate
\[ |\tilde{c}_{n,l}| \leq \frac{C r_n^{-r}}{|l|^r}, \]
where \( C \) is a uniform constant.

Without loss of generality we may assume that the sequence \( r_n \) satisfies
\[ r_n \leq \frac{1}{n^2} \quad \Rightarrow \quad r_n = 0. \]
We will prove that the set \( \mathcal{R}_{lo} \) has full measure under this assumption. If this assumption is not satisfied, then we may simply replace each \( r_n \) which satisfies \( r_n \leq 1/n^2 \) by \( r_n = 0 \). This does not change \( \sum r_n^d = \infty \) and the
Lemma 6.4. Let $U$ be a square integer matrix such that no eigenvalue is a root of unity. Let $p = \gcd(m, n)$. Then

$$(B^m - I)k = (B^n - I)l, \quad l, k \in \mathbb{Z}^d$$

if and only if

$$\begin{align*}
k &= (I + B^v + \ldots + B^{n-v})j \\
l &= (I + B^v + \ldots + B^{m-v})j
\end{align*}$$

for some $j \in \mathbb{Z}^d$.

Proof. Replacing $B^p$ by $B$, we may assume that $\gcd(m, n) = 1$. We are then to prove that $(B^m - I)k = (B^n - I)l$ holds if and only if

$$\begin{align*}
k &= (I + B + \ldots + B^{n-1})j \\
l &= (I + B + \ldots + B^{m-1})j
\end{align*}$$

for some $j \in \mathbb{Z}^d$.

Since $(B^m - I)(I + B + \ldots + B^{n-1}) = (B^n - I)(I + B + \ldots + B^{m-1})$ it is clear that (6.10) are solutions to the equation $(B^m - I)k = (B^n - I)l$. It remains to prove that these are the only solutions.

We first prove that there are integer polynomials $u$ and $v$ such that

$$(6.11) \quad u(x)(1 + x + \ldots + x^{m-1}) + v(x)(1 + x + \ldots + x^{n-1}) = 1.$$

Let $Z$ be the set of pairs $(m, n)$ of natural numbers for which (6.11) holds for some integer polynomials $u$ and $v$. Clearly, $(1, 1) \in Z$.

Suppose that $m > n$. Then $(m - n, n) \in Z$ implies that $(m, n) \in Z$. Similarly, if $n > m$, then $(m, n - m) \in Z$ implies that $(m, n) \in Z$.

Since $\gcd(m, n) = 1$, we can repeatedly reduce the pair $(m, n)$ by replacing it with $(m - n, n)$ or $(m, n - m)$, and as in the Euclidean algorithm, this procedure will eventually end up in the pair $(1, 1) \in Z$. Hence $(m, n) \in Z$. This proves that there are integer polynomial $u$ and $v$ such that (6.11) holds.
From (6.11), we get
\[ u(B)(I + B + \ldots + B^{m-1}) + v(B)(I + B + \ldots + B^{n-1}) = I, \]
and in particular
\[ (6.12) \quad k = u(B)(I + B + \ldots + B^{m-1})k + v(B)(I + B + \ldots + B^{n-1})k \]
for any vector \( k \).

Let \( P \) be such that \( P = (B^m - I)k = (B^n - I)l \) for some \( k, l \in \mathbb{Z}^d \). Then
\[
\begin{align*}
P &= (B - I)(I + B + \ldots + B^{m-1})k \\
&= (B - I)(I + B + \ldots + B^{n-1})l \\
&= (B^m - I)k = (B^n - I)l,
\end{align*}
\]
and hence
\[
(I + B + \ldots + B^{m-1})k = (I + B + \ldots + B^{n-1})l.
\]
We have
\[ u(B)(I + B + \ldots + B^{m-1})k = u(B)(I + B + \ldots + B^{n-1})l. \]
Using (6.12) and the fact that \( u(B), v(B) \) are polynomials in \( B \), we get that
\[
\begin{align*}
k &= u(B)(I + B + \ldots + B^{m-1})k + v(B)(I + B + \ldots + B^{n-1})k \\
&= u(B)(I + B + \ldots + B^{m-1})l + v(B)(I + B + \ldots + B^{n-1})k \\
&= (I + B + \ldots + B^{m-1})(u(B)l + v(B)k) \\
&= (I + B + \ldots + B^{m-1})j
\end{align*}
\]
where \( j = u(B)l + v(B)k \) is an integer vector. Similarly, we get
\[ l = (I + B + \ldots + B^{m-1})j \]
with the same \( j \).

In the following, set
\[
\begin{align*}
A_m &:= I + (A^T)^p + \ldots + (A^T)^{m-p}, \\
A_n &:= I + (A^T)^p + \ldots + (A^T)^{n-p},
\end{align*}
\]
where \( p \) always denotes \( p = \gcd(m, n) \).

Lemma 6.4 tells us that the sum in (6.9) may be rewritten as
\[ \tilde{c}_{m,0} \tilde{c}_{n,0} + \sum_{j \in \mathbb{Z}^d \setminus \{0\}} \tilde{c}_{m,A_n,j} \tilde{c}_{n,-A_m,j} \]
and as was noted above
\[ \tilde{c}_{n,0} = \int \tilde{G}_n(x) \, d\mu \leq (1 + \varepsilon)^d \mu(E_n). \]
Hence, if we let \( H = (1 + \varepsilon)^{2d} \), then
\[
\sum_{m,n=1}^{N} \left( \mu(E_n \cap E_m) - H \mu(E_n) \mu(E_m) \right) \leq \sum_{m,n=1}^{N} \sum_{j \in \mathbb{Z}^d \setminus \{0\}} \tilde{c}_{m,A_n,j} \tilde{c}_{n,-A_m,j}.
\]
Using the estimate (6.7) and the assumption (6.8) we have if \( r_n, r_m \neq 0 \) that
\[
|\tilde{c}_{m,A_n} \tilde{c}_{n,A_m}| \leq C(r_n r_m)^{-r} \frac{1}{|A_{n,j}|^r |A_{m,j}|^r} \leq C' n^{2r} m^{2r} \left( \frac{1}{|A_{n,j}|^r |A_{m,j}|^r} \right).
\]

We may estimate that
\[
|A_{n,j}| \geq c \lambda^{n-p} |j| \quad \text{and} \quad |A_{m,j}| \geq c \lambda^{m-p} |j|,
\]
for some uniform constant \( c \). Hence
\[
|\tilde{c}_{m,A_n} \tilde{c}_{n,A_m}| \leq C' n^{2r} m^{2r} \lambda^{r(2p-m-n)} |j|^{-2r},
\]
an estimate which is certainly true also when either \( r_n = 0 \) or \( r_m = 0 \), since then all the corresponding Fourier coefficients are zero and \( |\tilde{c}_{m,A_n} \tilde{c}_{n,A_m}| = 0 \).

We conclude that if \( r \) is sufficiently large, then
\[
\sum_{m,n=1}^{N} \left( \mu(E_n \cap E_m) - H \mu(E_n) \mu(E_m) \right) \leq C' \sum_{m,n=1}^{N} j \in \mathbb{Z} \setminus \{0\} C^2 n^{2r} m^{2r} \lambda^{r(2p-m-n)} |j|^{-2r} = C' \sum_{m,n=1}^{N} n^{2r} m^{2r} \lambda^{r(2p-m-n)}.
\]

Just as in the one dimensional case, this is bounded as \( N \to \infty \). Thus, Lemma 6.1 implies that \( \mu(R_{io}) \geq \frac{1}{H} = (1 + \varepsilon)^{-2d} \). As \( \varepsilon \) can be taken as small as we wish, we can make \( H \) arbitrarily close to \( 1 \) and we conclude that \( \mu(R_{io}) = 1 \).

7. Proof of Theorem \([\text{E}]\)

In this section we prove our very general result on quantitative uniform recurrence.

Proof of Theorem \([\text{E}]\) If \( V \) is a measurable set and \( t \in \mathbb{N} \), then we let
\[
V(t) = \{ x \in V : T^i(x) \notin V \text{ for all } 1 \leq i \leq t \}.
\]
We have \( \mu(V(t)) \leq 1/t \) because \( T^{-i}(V(t)) \), \( 1 \leq i \leq t \), are disjoint sets of equal measure \( \mu(V(t)) \).

Fix \( \beta > \alpha \) and choose \( \beta' \) with \( \beta > \beta' > \alpha \). The proof contains two parameters \( \gamma > 0 \) and \( \theta > 1 \) that will be chosen later. For each \( m \in \mathbb{N} \), we let \( R_m = m^{-\gamma} \), and we take \( p_m \) so that
\[
m = \frac{4p_m^2}{R_m^{\beta'}} \iff p_m = \frac{1}{2} m^{1 - \frac{\beta'}{2}}.
\]
We require that \( \gamma \in (1/\beta, 1/\beta') \) so that \( p_m \to \infty \) with \( m \).
When $m$ is large enough, since $\dim B X < \beta' < \beta$, there is a cover $\{B_{m,i} \mid i \in I_{m,0}\}$ of $X$ by balls of diameter $|B_{m,i}| = R_m$ such that the number of balls is at most $R_m^\beta$. We then have

$$(7.1) \quad \sum_{i} |B_{m,i}|^{\beta'} \leq R_m^{\beta - \beta} < 1$$

if $m$ is large enough.

By Vitali’s covering lemma which holds true in every metric space [12, Theorem 1.2], there is $I_{m} \subset I_{m,0}$ such that the balls $B_{m,i}$, $i \in I_{m}$ are pairwise disjoint and such that $\{5B_{m,i} \mid i \in I_{m}\}$ covers $X$. (If $B$ is a ball, then $5B$ denotes the ball with same centre as $B$ and with radius 5 times as large.)

By replacing the balls in the cover $\{5B_{m,i} \mid i \in I_{m}\}$ by subsets, we can get a cover $\{V_{m,i} \mid i \in I_{m}\}$ of $X$, such that the sets $V_{m,i}$ are pairwise disjoint and such that

$$1 \leq |V_{m,i}| \leq 5.$$ 

Let $J_{m} = \{ i \in I_{m} : 2p_m \mu(V_{m,i}) \leq |B_{m,i}|^{\beta'} \}$. Then

$$(7.2) \quad \mu \left( \bigcup_{i \in J_{m}} V_{m,i} \right) = \sum_{i \in J_{m}} \mu(V_{m,i}) \leq \frac{1}{2p_m} \sum_{i} |B_{m,i}|^{\beta'} < \frac{1}{2p_m},$$

by (7.1).

For $i \notin J_{m}$ we have

$$(7.3) \quad \mu(V_{m,i}(m)) \leq \frac{1}{m} \leq \frac{R_m^{\beta'}}{4p_m^2} \leq \frac{2p_m \mu(V_{m,i})}{4p_m^2} = \frac{1}{2p_m} \mu(V_{m,i}).$$

Let

$$G_{m} = \bigcup_{i \in J_{m}} V_{m,i} \cup \bigcup_{i \notin J_{m}} V_{m,i}(m).$$

By (7.2) and (7.3) we then have

$$\mu(G_{m}) \leq \frac{1}{2p_m} + \frac{1}{2p_m} = \frac{1}{p_m}.$$

Notice that if $x \in \overline{G}_{m}$, then there is an $i$ and a $k$ with $1 \leq k \leq m$ such that $x, T^k x \in V_{m,i}$. Hence, $d(x, T^k x) \leq 5R_m$.

Put $m_j = j^\theta$. Then

$$\mu(G_{m_j}) \leq \sum_{j=1}^{\infty} p_{m_j}^{-1} = \sum_{j=1}^{\infty} 2j^{-\theta(1-\frac{\beta'}{2})},$$

which is convergent provided $\theta(1-\frac{\beta'}{2}) > 1$. Since $1 - \beta' \gamma > 0$, we can choose $\theta > 1$ sufficiently large so that the above series is convergent.

It then follows by the Borel–Cantelli lemma that for such a choice of $\theta$, we have

$$\mu(\limsup_{j \to \infty} G_{m_j}) = 0.$$

Therefore, we have

$$\mu(\liminf_{j \to \infty} \overline{G}_{m_j}) = 1.$$
Let \( F = \liminf_{j \to \infty} G_{m_j} \). Whenever \( x \in F \), there is a \( j_0 \) which depends on \( x \) such that for any \( j > j_0 \), there is a \( k \leq m_j = j^\theta \) with 
\[
d(x, T^k x) \leq 5 R_{m_j} = 5j^{-\theta \gamma}.
\]
Let \( x \in F \) and suppose that \( m > j_0^\theta \). There is then a \( j \) such that \( j^\theta < m \leq (j + 1)^\theta \). There is therefore a \( k \leq j^\theta < m \) such that 
\[
d(x, T^k x) \leq 5 R_{m_j} = 5\left(\frac{j + 1}{j}\right)^{\theta \gamma} (j + 1)^{-\theta \gamma} \leq 5 \cdot 2^{\theta \gamma} m^{-\gamma}.
\]
Consequently, for any large enough \( m \), there is a \( k < m \) with 
\[
d(x, T^k x) \leq 5 \cdot 2^{\theta \gamma} m^{-\gamma}.
\]
Since \( \gamma > 1/\beta \) the theorem follows from this statement. \( \Box \)

8. **Eventually always returning points for the doubling map**

We consider the set of \emph{eventually always returning points} defined by
\[
R_{ea} := \{ x \in X : \exists n \in \mathbb{N} \forall m \geq n : \{ T^k x \}_{k=1}^m \cap B(x, r_m) \neq \emptyset \}
\]
for the doubling map \( T(x) = 2x \mod 1 \) on \( X = [0, 1] \). We write 
\[
E_{k,m} := \{ x \in X : T^k x \in B(x, r_m) \},
\]
\[
C_m := \bigcup_{k=1}^m E_{k,m},
\]
\[
A_n := \bigcap_{m=n}^{\infty} C_m.
\]
Since \( A_n \subset A_{n+1} \) we have \( \mu(R_{ea}) = \lim_{n \to \infty} \mu(A_n) \).

8.1. **Sufficient condition for measure one.**

**Proposition 8.1.** Let \( X = [0, 1], T(x) = 2x \mod 1 \) and let \( \mu \) denote the Lebesgue measure. We consider \( r_m = \frac{\Delta_m h(\Delta_m)}{m} \) with \( h(n) \to \infty \) as \( n \to \infty \), \( \Delta_m \to \infty \) as well as \( \frac{m}{\Delta_m} \to \infty \) and \( \frac{m^{2+\sigma}}{\Delta_m} 2^{-\Delta_m} \leq 1 \) for some \( \sigma > 0 \) for \( m \) sufficiently large. Then \( \mu(R_{ea}) = 1 \).

**Proof.** Let \( \varepsilon_m = \frac{1}{m^{1+\sigma}} \). We define the function 
\[
F_m(t) = \begin{cases} 
0 & \text{if } |t| < r_m, \\
1 & \text{otherwise.}
\end{cases}
\]
In order to describe the return of the point \( x \) under \( T^m \) we can use the map \( F_m \) in the following way:
\[
G_{k,m}(x) := F_m(T^k x - x) = \begin{cases} 
0 & \text{if } |T^k x - x| < r_m, \\
1 & \text{otherwise.}
\end{cases}
\]
Then \( G_{k,m} \) is the characteristic function of \( \bar{E}_{k,m} \) using the notation from above. In that notation we also have

\[
\mu(\bar{C}_m) = \mu\left( \bigcap_{k=1}^{m} E_{k,m} \right) = \int \prod_{k=1}^{m} G_{k,m}(x) \, dx \leq \int \prod_{k=1}^{m/\Delta_m} G_{p_k,m}(x) \, dx
\]

with \( p_k = k \cdot \Delta_m \). Thinking of \([0,1]\) as the circle and identifying the endpoints of \([0,1]\), we note that \( G_{p_1,m}(x) = F_m((2^{p_1} - 1)x) \) attains the value 1 on \( 2^{p_1} - 1 \) many intervals of length \( \frac{1-2r_m}{2^{p_1} - 1} \), see Figure 1. On each of these intervals, the function \( G_{p_2,m} \) takes the value 0 on at least

\[
\left(1 - 2(1 + \varepsilon m) r_m \right) \cdot \frac{2^{p_2} - 1}{2^{p_1} - 1}
\]

many intervals of length \( \frac{2r_m}{2^{p_2} - 1} \).

For the rest of the proof, all inequalities and estimates should be considered true for \( m \) sufficiently large. Since \( \frac{m^{2+\sigma}}{\Delta_m} 2^{-\Delta_m} \leq 1 \) and \( h(\Delta_m) > 1 \), we have

\[
2\varepsilon m r_m = 2 \frac{h(\Delta_m) \Delta_m}{m^{2+\sigma}} \geq 2 \frac{2^{p_1} - 1}{2^{p_2} - 1}
\]

We can therefore estimate that the number of intervals in (8.1) is bounded from below by

\[
(1 - 2(1 + \varepsilon m) r_m) \cdot \frac{2^{p_2} - 1}{2^{p_1} - 1}
\]

Hence, the number of intervals where the function \( G_{p_1,m} \cdot G_{p_2,m} \) is zero, but \( G_{p_1,m} \) is not, is at least

\[
(2^{p_1} - 1) \left( (1 - 2(1 + \varepsilon m) r_m) \cdot \frac{2^{p_2} - 1}{2^{p_1} - 1} \right) = (1 - 2(1 + \varepsilon m) r_m) \cdot (2^{p_2} - 1).
\]

The total length of these intervals is therefore at least

\[
\frac{2r_m}{2^{p_1} - 1} (1 - 2(1 + \varepsilon m) r_m) \cdot (2^{p_2} - 1) \geq 2r_m (1 - 2(1 + \varepsilon m) r_m).
\]

\footnote{For the sake of convenience we treat the numbers \( p_k \) and \( \frac{m}{\Delta_m} \) as integers avoiding the use of floor functions.}
Continuing like this, we observe that the product \( \prod_{k=1}^{m} G_{k,m}(x) \) is 0 on a length of at least
\[
2r_m \cdot \sum_{k=1}^{m/\Delta_m} \left( 1 - 2(1 + \varepsilon_m) r_m \right)^{k-1} = 2r_m \cdot \frac{1 - (1 - 2(1 + \varepsilon_m) r_m \frac{m}{\Delta_m})^{m}}{2(1 + \varepsilon_m) r_m}
\]
\[
= \frac{1}{1 + \varepsilon_m} \cdot \left( 1 - (1 - 2(1 + \varepsilon_m) r_m \frac{m}{\Delta_m}) \right)
\]
\[
\geq \frac{1}{1 + \varepsilon_m} \cdot \left( 1 - (1 - 2r_m \frac{m}{\Delta_m}) \right)
\]
\[
\geq 1 - \varepsilon_m,
\]
provided that
\[
\lim_{m \to \infty} (1 - 2r_m \frac{m}{\Delta_m}) = 0.
\]
Under condition (8.2) we therefore obtain
\[
\mu(\mathbb{C}_m) \leq \int \prod_{k=1}^{m/\Delta_m} G_{k,m}(x) \, dx < \varepsilon_m.
\]
Since \( \sum m \varepsilon_m < \infty \), we have \( \mu(A_n) \to 1 \) as \( n \to \infty \). Hence, \( \mu(R_{\text{ea}}) = 1 \).

To conclude we note that condition (8.2) is satisfied for the choice \( r_m = \Delta_m / m \), since
\[
\lim_{m \to \infty} (1 - 2r_m \frac{m}{\Delta_m}) = \exp(-2) < 1.
\]

**Proof of part (2) in Theorem F.** The choice \( \Delta_m = (2 + \sigma) \log_2(m) \) satisfies the assumptions of Proposition 8.1 since \( m \frac{2+\sigma}{(2+\sigma) \log_2(m)} \to \infty \) and
\[
\frac{m^{2+\sigma}}{\Delta_m} \to 1 \frac{1}{(2 + \sigma) \log_2(m)} < 1.
\]

### 8.2. Sufficient condition for measure zero

In the converse direction, part (1) in Theorem F follows from the next proposition.

**Proposition 8.2.** Let \( X = [0, 1] \), \( T(x) = 2x \mod 1 \) and let \( \mu \) denote the Lebesgue measure. Suppose that
\[
\lim_{m \to \infty} mr_m = 0.
\]
Then \( \mu(R_{\text{ea}}) = 0 \).

**Proof.** Identifying \([0, 1]\) with the circle, the set \( E_{k,m} \) consists of \( 2^k - 1 \) intervals (sectors) of length \( 2r_m / (2^k - 1) \). Hence the measure of \( E_{k,m} \) is \( 2r_m \).

It follows immediately that \( \mu(C_m) \leq 2mr_m \). Hence, the condition
\[
\lim_{m \to \infty} mr_m = 0
\]
implies that \( \mu(R_{\text{ea}}) = 0 \).
Remark 8.3. From the equation $\mu(\overline{C_m}) \geq 1 - 2r_m m$ we can also deduce the necessary condition for $\mu(\mathcal{R}_{ea}) = 1$ that

$$\lim_{m \to \infty} \mu(B_m)m = \lim_{m \to \infty} 2r_m m \geq 1$$

In particular, $\mathcal{R}_{ea}$ cannot have full measure for $\mu(B_m) = \frac{c}{m}$ with any $c < 1$.

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