RADIATION REACTION
FOR A CHARGED BROWNIAN PARTICLE

Alexander A. Vlasov
High Energy and Quantum Theory
Department of Physics
Moscow State University
Moscow, 119899
Russia

As it is known a model of a charged particle with finite size is a good tool to consider the effects of self-action and backreaction, caused by electromagnetic radiation. In this work the "size" of a charged particle is induced by its stochastic Brownian vibration. Appropriate equation of particle's motion with radiation force is derived. It is shown that the solutions of this equation correctly describe the effects of radiation reaction.

It is known from the classical electrodynamics that a charged particle, moving with acceleration, must radiate electromagnetic waves and thus must feel the backreaction of such radiation. How one can take into account this backreaction in the equation of motion of a particle?

This problem is very old (count from the pioneer works of Abragam and Lorentz [1]), but till now it is under focus of different investigations, handling it as from quantum field theory point of view, so from point of view of the classical physics.

The origin of the problem is the following. If in frames of some theoretical investigation the size of radiating body can be neglected (in comparison with other characteristic lengths), then one can try to use the notion of "point-like" particle for such body and its mathematical representation - the Dirac's delta-function. But the use of the delta-function inevitably leads to divergences in some physical quantities, such as self-electromagnetic energy of point charged particle, its effective mass and so on, and consequently to the necessity of mass renormalization. Dirac was the first [2] to do this renormalization and as the result the famous equation with relativistic radiation
reaction for point-like particle was derived (in the literature - the Abragam-Lorentz-Dirac equation (ALD)).

But immediately scientists found out (and Dirac was among them) that ALD equation leads to many paradoxes. Among them in the literature are usually mentioned [3] the following:

a) the existence of runaway solutions (in the absences of external forces the radiating particle begins to move with growing velocity up to that of light);

b) preacceleration (after supplementary condition, excluding the runaway solutions, the solutions remain, describing particle, "feeling" the external force with some advance in time);

c) the existence of "exotic" ALD solutions for head-on-collisions (i.e. for two opposite charged particles there are solutions, describing their mutual repelling);

and so on.

Thus one can understand the scientists, declaring that "the ALD equation must be modified".

There are two basic ways to do such modification:

a) to consider problem of "point-like" particle not in the frames of classical theory, but within quantum field theory, studying the processes of interaction of charged quantum particles with their quantum electromagnetic fields;

b) to stay in frames of classical theory, but refusing the notion of "point-like" particle and considering small object of finite size like charged particles of dusty plasma, Brownian charged particles and others.

In his works on radiation theme the author follows the second path, considering nonquantum charged particles of small finite size [4].

In particular in this work is investigated the influence of induced by Brownian vibration the effective size of charged particle (in appropriate time scale) on the effect of radiation reaction. The equation of center mass motion of such particle is derived. Some solutions of this equation are investigated.

At first let us remind that the explicit expression of electric field $\vec{E}$ for some moving charged body,

$$\vec{E} = -\nabla \phi - \frac{1}{c} \cdot \frac{\partial \vec{A}}{\partial t},$$

taking into account the effects of retardation:
\[ \phi(t, \vec{r}) = \int dt' d\vec{r}' \delta(t' - t + |\vec{r} - \vec{r}'|/c) \cdot \frac{1}{|\vec{r} - \vec{r}'|} \cdot \rho(t', \vec{r}'); \]

\[ \vec{A}(t, \vec{r}) = \int dt' d\vec{r}' \delta(t' - t + |\vec{r} - \vec{r}'|/c) \cdot \frac{1}{c|\vec{r} - \vec{r}'|} \cdot \vec{j}(t', \vec{r}'); \]

and with the help of the charge conservation law:

\[ \frac{\partial \rho}{\partial t} + \text{div} \vec{j} = 0, \]

can be expanded in series in retardation [5]

\[ \vec{E}(t, \vec{r}) = -\int d\vec{r}' \rho(\vec{r}' - \vec{r})^{-1} + \]

\[ + \sum_{m=1}^{\infty} \frac{(-1)^m}{c^m m!} \cdot \frac{\partial^{m-1}}{\partial t^{m-1}} \int d\vec{r}' (\vec{j}', \nabla) \nabla |\vec{r} - \vec{r}'|^{m-1} - \]

\[ - \sum_{m=0}^{\infty} \frac{(-1)^m}{c^{m+2} m!} \cdot \frac{\partial^{m+1}}{\partial t^{m+1}} \int d\vec{r}' \vec{j}' |\vec{r} - \vec{r}'|^{m-1} \quad (1) \]

here \( \vec{j}' \equiv \vec{j}(t, \vec{r}') \).

The average of expression (1) over the body’s volume:

\[ \vec{E} \rightarrow < \vec{E} >, \]

\[ < \vec{E} > \equiv \int d\vec{r} \rho(t, \vec{r}) \vec{E}(t, \vec{r}) / \left( \int d\vec{r} \rho(t, \vec{r}) \right) \quad (2) \]

gives us the average self- electric field of a moving body.

For spherically symmetric charge distribution:

\[ \rho = \rho(t, |\vec{r} - \vec{R}(t)|) \]

the space derivatives in (2) and (1) are averaged according to the rule:

\[ < \nabla_\alpha \nabla_\beta > = \delta_\alpha_\beta \nabla^2 / 3, \quad < \nabla > = 0. \]

Then in (2) the first (Coulomb) term of the expression (1) vanishes, and the remaining two sums can be reduced in such a way that

\[ < \vec{E} > = -\frac{2}{3Qc^2} \sum_{n=0}^{\infty} \frac{(-1)^n}{c^n n!} \int d\vec{r} \rho(t, \vec{r}) \cdot \frac{\partial^{n+1}}{\partial t^{n+1}} \int d\vec{r}' \vec{j}' |\vec{r} - \vec{r}'|^{n-1} \]

\]
This expression can be simplified further:

\[
\langle \vec{E} \rangle = -\frac{2}{3Qc^2} \int d\vec{r} \rho(t, \vec{r}) \cdot \frac{\partial}{\partial t} \int \frac{d\vec{r}'}{|\vec{r} - \vec{r}'|} \sum_{n=0}^{\infty} \left( -\frac{|\vec{r} - \vec{r}'|}{c} \cdot \frac{\partial}{\partial t} \right)^n \frac{1}{n!} \vec{j} = 
\]

\[
= -\frac{2}{3Qc^2} \int d\vec{r} \rho(t, \vec{r}) \cdot \frac{\partial}{\partial t} \int \frac{d\vec{r}'}{|\vec{r} - \vec{r}'|} \vec{j}(t - \frac{|\vec{r} - \vec{r}'|}{c}, \vec{r}') 
\]

(3)

It should be mentioned that the expression (3) is a strict one for spherically symmetric charged body.

Let us made new simplifications.

Let the body be rigid in the sense that

\[
\vec{j}(t_{ret}, \vec{r}') = \rho(t_{ret}, \vec{r}') \cdot \vec{v}(t_{ret}, \vec{r}') 
\]

(4)

here \( t_{ret} \equiv t - \frac{|\vec{r} - \vec{r}'|}{c} \) - the retarded time.

Then let us consider the retardation only for the velocity \( v \) in (4) and neglect the retardation for the density of charge:

\[
\rho(t_{ret}, \vec{r}') \approx \rho(t, \vec{r}') 
\]

(5)

This means that Taylor expansion in powers of \( c \to \infty \) (expansion in retardation) for the velocity is more sufficient then for the density of charge, i.e.

\[
\frac{1}{c} \frac{\partial}{\partial t} \rho \ll \frac{\partial}{\partial t} v \approx \rho F_{ext}/m 
\]

(6)

If \( T_{\rho} \) - is the typical time of density variation, and \( T_v \) - is the typical time of velocity variation, then this inequality can be rewritten as:

\[
T_v \ll T_{\rho} 
\]

The inequality (6) has one more interpretation. Due to the law of charge conservation:

\[
\frac{\partial}{\partial t} \rho = -\left( \nabla, \vec{j} \right) \sim \rho v 
\]

inequality (6) leads us to the linearity condition, when one consider only terms linear in velocity and its time derivatives terms.
Thus in (3) following (4, 5, 6) one can apply the time derivative only to the velocity (and not to the density). This leads to the following expression for average electric field:

\[
< \vec{E} > = \frac{2}{3Qc^2} \int d\vec{r}' d\vec{r} \cdot \frac{\rho(t, \vec{r})\rho(t, \vec{r}')}{|\vec{r} - \vec{r}'|} \frac{\partial}{\partial t} \vec{v}(t - \frac{|\vec{r} - \vec{r}'|}{c})
\] (7)

Expression (7), multiplied on the value of the charge \(Q\), is just the Jackson self-electromagnetic force [5] in linear approximation (the magnetic Lorentz self-force is zero in approximation under consideration).

Let us note that in (7) one can consider varying in time densities if the inequality (6) is valid.

Now turn to the motion of charged Brownian particle.

Let us take those time intervals \(T_v\) (time scales), for which the Brownian motion can be described by distribution function:

\[ T_v \gg T_{Br} \]

here \(T_{Br}\) - is the typical time for Maxwell’s velocity distribution to appear.

Let \(n(t, \vec{r})\) be the probability to find Brownian particle at the moment of time \(t\) in the volume \(d\vec{r}\), thus \(n\) - is the distribution function with norm

\[ \int n(t, \vec{r}) d\vec{r} = 1 \]

This function obeys the conservation law

\[ \frac{\partial n}{\partial t} + \text{div} n \vec{V} = 0 \] (8)

Let the motion of the Brownian particle consists from two motions - the first one, regular, with the velocity \(\vec{v} = \vec{v}(t)\), under the influence of some external regular force \(\vec{F}_{ext}\), and the second one, irregular - Brownian diffusion with velocity \(\vec{u}\):

\[ \vec{V} = \vec{v} + \vec{u}, \]

\[ \vec{u} = -\frac{D}{n} \nabla n \] (9)

here \(D\) - the diffusion parameter.
In other words around the regular trajectory of particle’s center of mass there are Brownian vibration in such a way that the average value of the square particle displacement from the regular trajectory is proportional, following Einstein formula, to $Dt$. Thus appears the effective particle’s “size”, proportional to $\sqrt{Dt}$.

Equation (8) with the help of (9) can be put in the form

$$\frac{\partial n}{\partial t} + (\vec{v}, \nabla) n = D\nabla^2 n \quad (10)$$

As the particle’s center of mass moves along its trajectory with velocity $\vec{v} = d\vec{R}(t)/dt$, the distribution function gives the probability of particle’s displacement from this trajectory, so

$$n = n(t, \vec{r} - \vec{R}(t))$$

Inserting this form of $n$ into (10) and taking into account that

$$\frac{\partial n}{\partial t} = -(\frac{d\vec{R}(t)}{dt}, \nabla)n + \left(\frac{\partial n}{\partial t}\right)_{\vec{R}=\text{const}}$$

one gets equation for $n$:

$$\left(\frac{\partial n}{\partial t}\right)_{\vec{R}=\text{const}} = D\nabla^2 n \quad (11)$$

It is the typical Fokker - Plank equation with solution, for the initial condition

$$n(0, \vec{r}) = \delta(\vec{r} - \vec{R}(0)),$$

in the form

$$n(t, \vec{r}) = \frac{1}{(4\pi Dt)^{3/2}} \exp \left(-\frac{|\vec{r} - \vec{R}(t)|^2}{4Dt}\right) \quad (12)$$

(see., for ex., [6]).

To take into consideration the radiation reaction one must the Newtonian equation of the center of mass motion:

$$\vec{a}(t) = \frac{\vec{F}_{ext}(t)}{m}$$
(here $\ddot{a}(t) = \dddot{R}(t)$) supplement with the self-force $\vec{F}_{self} = Q < \vec{E} >$ (see formula (7), where the spherically symmetric density of charge is $\rho = Q \cdot n$).

Then the equation of motion of the center of mass in nonrelativistic approximation will be

$$\ddot{a}(t) = \frac{\vec{F}_{ext}(t)}{m} - \frac{2Q^2}{3mc^2} \int \int d\vec{r}d\vec{r}' \frac{n(t, \vec{r}) n(t, \vec{r}')}{|\vec{r} - \vec{r}'|} \ddot{a}(t - \frac{|\vec{r} - \vec{r}'|}{c})$$  \hspace{1cm} (13)

Using the result (12) for distribution function, introducing new dimensionless variables

$$\vec{r} - \vec{R}(t) \equiv \vec{\mu}; \quad |\vec{\mu}| \equiv \sqrt{4Dt} x;$$
$$\vec{r}' - \vec{R}(t) \equiv \vec{\nu}; \quad |\vec{\nu}| \equiv \sqrt{4Dt} y;$$

and taking into consideration the spherical symmetry, equation (13) finally reduces to this form

$$\ddot{a}(t) = \frac{\vec{F}_{ext}(t)}{m} - \frac{4Q^2}{3\pi mc} \cdot \frac{1}{Dt} \int_0^\infty \int_0^\infty dx dy xy e^{(-x^2-y^2)} \left[ \vec{\nu}(t - \frac{\sqrt{4Dt}}{c}|x-y|) - \vec{\nu}(t - \frac{\sqrt{4Dt}}{c}|x+y|) \right]$$  \hspace{1cm} (14)

Thus equation (13) (or (14)) take into account the existence of the finite size of the charged particle, induced by Brownian vibration (in time scale $T_{Br} \ll T_v \ll T_{\rho}$). Due to the finiteness of particle’s size, the self-force, which is the radiation reaction force in our approach, has finite value and there is no need to do mass renormalization.

Equation (13) differs from ALD equation, but has much in common with the Sommerfeld finite-size models of a charged particle.

Let us mention the following features of equation (13) ((14)):

1. If the external force is constant: $\vec{F}_{ext} = \vec{F}_0 = const$ then eq. (13) has the solution

$$\ddot{a} = \frac{\vec{F}_0}{m} \left( 1 + \frac{2Q^2}{3mc^2} \int \int d\vec{r}d\vec{r}' \frac{n(t, \vec{r}) n(t, \vec{r}')}{|\vec{r} - \vec{r}'|} \right)^{-1} \equiv \frac{\vec{F}_0}{m + m^*_{em}}$$  \hspace{1cm} (15)

here $m^*_{em}$ - is the effective mass of the self-electromagnetic field:

$$m^*_{em} = \frac{2Q^2}{3c^2} \int \int d\vec{r}d\vec{r}' \frac{n(t, \vec{r}) n(t, \vec{r}')}{|\vec{r} - \vec{r}'|}$$
If the effective mass is approximated as \( \frac{Q^2}{c^2 L_{Br}} \), where \( L_{Br} \) - the typical "size" of Brownian spread, equal according to Einstein formula, to \( \sqrt{4D t} \), then, following (15), the acceleration of Brownian particle slightly increases in time up to its maximum value \( \frac{F_0}{m} \), achieved at \( t \to \infty \). In other words, in this process the self-force (force of radiation reaction) is not equal to zero and together with the effective self- electromagnetic mass vanishes at \( t \to \infty \).

2). If the external force is absent, one can find by ordinary substitution that eq. (14) has no "free" harmonic solutions like

\[
\vec{v} = \vec{A} \cos(\omega t)
\]

That is, contrary to Sommerfeld models (see, for ex., [4] ) there are no oscillations, free of radiation damping.

3). In details, "free" solutions of eq. (14) are exponentially damped. Indeed, after substitution in (14) the velocity in the form

\[
\vec{v} \sim e^{\left(p t\right)}, \quad p = p' + ip''
\]

where the real part of parameter \( p \) is small enough: \( p' \to 0 \left( p'T_v \ll 1 \right) \), one gets the following algebraic equation:

\[
p = -\frac{4Q^2}{3\pi mc} \cdot \frac{1}{Dt} \int_0^\infty \int_0^\infty dx dy \ xy \ e^{(-x^2-y^2)} \left[ e^{(-\delta p|x-y|)} - e^{(-\delta p|x+y|)} \right]
\]

here \( \delta \equiv \sqrt{4Dt} \).

The real part of it provides us with this equation

\[
p' = -\frac{4Q^2}{3\pi mc} \cdot \frac{1}{Dt} \int_0^\infty \int_0^\infty dx dy \ xy \ e^{(-x^2-y^2)} \cdot \left[ e^{(-\delta p'|x-y|)} \cos(\delta p''|x-y|) - e^{(-\delta p'|x+y|)} \cos(\delta p''|x+y|) \right] \approx
\]

\[
\approx -\frac{4Q^2}{3\pi mc} \cdot \frac{1}{Dt} \int_0^\infty \int_0^\infty dx dy \ xy \ e^{(-x^2-y^2)} \left[ \cos(\delta p''|x-y|) - \cos(\delta p''|x+y|) \right] =
\]
\[-\frac{4Q^2}{3\pi mc} \cdot \frac{1}{Dt} \left( \int_0^\infty dx \, x \, e^{(-x^2)} \sin(\delta p'' x) \right)^2 < 0\]

Consequently

\[p' < 0.\]

This means that the solutions of (14) for zero external force are exponentially damped.

Such damping is obvious from the physical point of view - it is caused by the radiation energy losses.

4). Let us expand the integrand in (14) in series in ”retardation”:

\[t \gg \frac{\sqrt{4Dt}}{c} |x \pm y|\]

Then, taking into account

\[\vec{v}(t - \delta|x - y|) - \vec{v}(t - \delta|x + y|) \approx \vec{v}(t)(x + y - |x - y|)\delta - 2xy\delta^2 \ddot{\vec{v}}(t) + ...\]

after integration one gets

\[\vec{F}_{\text{self}}/m \approx -\ddot{\vec{v}}(t) \frac{m_{em}}{m} + \frac{2Q^2}{3mc^3} \dddot{\vec{v}}(t) + ... \quad (16)\]

here the second term is the classical Abragam - Lorentz expression for radiation force and the effective electromagnetic mass \(m_{em}\) of the first term equals to

\[\frac{Q^2}{c^2} \cdot \frac{1}{\sqrt{4Dt}} \cdot \frac{\sqrt{2}}{3\sqrt{\pi}} \quad (17)\]

and tends to zero for \(t \to \infty\).

Thus we have found out that the consideration of the Brownian spread of particle’s size (in appropriate time scale) leads to new equation of particle’s motion with solutions having explicit physical sense and avoids the troubles connected with the notion ”point-like” particle.

It should be noted that the idea to consider the existence of particle’s size, induced by Brownian vibration, is not the original one (see, for ex., the discussion and references in the chapter 22 of the book [8]). Nevertheless author does not know the works where this idea was realized in concrete mathematical equations. In our work - this is the equation (13). It is integro
- differential - difference equation with retardation. That is why it may have solutions, besides mentioned above, typical for the models of particles of a finite size - the so called tunneling solutions [4]. It would be interesting to investigate this problem more closely in the further works.
REFERENCES

1. H. Lorentz, "The Theory of Electron", Leipzig, Teubner, 2nd edition, 1916. M.Abragam, "Electromagnetische Theorie der Strahlung", Leipzig, Teubner, 1905.

2. P.Dirac, Proc. Roy. Soc., A167, 148, 1938.

3. N.P.Klepikov, Usp. Fiz. Nauk, 146, 317 (1985). S.Parrott, Relativistic Electrodynamics and Differential Geometry, Springer-Verlag, NY, 1987. T.Erber, Fortschr.Phys., 9, 342 (1961). P.Pearle, in Electromagnetism, ed. D.Tepliz, Plenum, NY, 1982, p.211. A.Yaghjian, Relativistic Dynamics of a Charged Sphere, Lecture Notes in Physics, 11, Springer, Berlin, 1992.

4. Al..Vlasov, Vestnik Mosk.St.Univ., Fizika, N 5, 17 (1998); N 6, 15 (2001). Alexander A.Vlasov, in "Photon: old problems in light of new ideas", p. 126, ed. Valeri V. Dvoeglazov, Nova Sci. Publ., NY, 2000. E-print Archive: physics/9911059, physics/9912054, physics/0004026, physics/0103065, physics/0110003, physics/0205012.

5. J.D. Jackson Classical Electrodynamics, Wiley, NY, 1999.

6. I.A.Kvasnikov Thermodynamics and Statistical Physics. Part 2., Moscow, Mosk.St.Univ., 1988.

7. A.Sommerfeld, Gottingen Nachrichten, 29 (1904), 363 (1904), 201 (1905).

8. ..Sokolov, Yu.M.Loskutov, I.M.Ternov Quantum Mechanics., Moscow, Uchpedgiz, 1962.