On a quasi-local mass

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Received 1 September 2009
Published 8 December 2009
Online at stacks.iop.org/CQG/26/245018

Abstract

We modify the previous quasi-local mass definition. The new definition provides expressions of the quasi-local energy, the quasi-local linear momentum and the quasi-local mass. And they are equal to the ADM expressions at spatial infinity. Moreover, the new quasi-local energy has the positivity property.

PACS number: 04.20.Cv

1. Introduction

Recently, there are many attempts to define quasi-local mass [7–11, 16, 19–21] using certain ideas initiated by Brown and York [1, 2]. The definitions are valid for spacelike 2-surfaces. The positivity property requires that the 2-surfaces be spheres which are in certain initial data sets.

It is well known that the Brown–York mass decreases for the round sphere in time slices in the Schwarzschild spacetime. In order to obtain a quasi-local mass of Brown–York’s type with increasing monotonicity property, we proposed a new definition by choosing certain spinor norm as lapse function [23, 24]. But, as pointed out in [23], our quasi-local mass is not equal to the ADM mass at spatial infinity.

In this paper, we modify our definition and provide expressions of the quasi-local energy, the quasi-local linear momentum and the quasi-local mass. In particular, we observe that the norm of the quasi-local linear momentum 3-vector is well defined although this 3-vector cannot be defined in a quasi-local sense. This is unlike the ADM total linear momentum at spatial infinity, where both the norm and the 3-vector are well defined. We also show that these quantities are equal to the corresponding ADM quantities at spatial infinity. Moreover, the new modified quasi-local energy has the positivity property. In particular, the vanishing of the quasi-local mass implies that the spacetime is flat along the 2-spheres, which is in consequence of applying Brown and York’s isometric embedding approach. This later result does not seem to hold true in the other recent definitions of quasi-local mass mentioned above.

The essential idea of our approach in this paper as well as in [23, 24] is to localize Witten’s proof of the positive energy theorem [12, 14, 15, 22] to the (spacelike) bounded domain. The
similar approach was used by Dougan and Mason earlier to define the quasi-local mass [4]. In our previous formulation, the 2-spheres are assumed to be apparent horizon only. In order that the Dirac–Witten equation has a solution, we have to assume that the mean curvature of spacelike hypersurfaces does not change sign. In this paper we include the case that at least one boundary 2-sphere satisfies the strict inequality in apparent horizon conditions. This will ensure the existence of the Dirac–Witten equation on any spacelike hypersurfaces whose mean curvature may change sign, and this existence theorem would be required in Dougan and Mason’s definition [4, 17, 18].

Surprisingly, our quasi-local mass increases for the round sphere in time slices in the Schwarzschild spacetime [23] although we are not able to prove the increasing monotonicity property in general case. This should have a deep physical reason which can be seen naively as follows: in classical mechanics, the total mass is the space integral of the local mass density. But this fact is no longer true in general relativity as the vacuum can have the total energy as follows: in classical mechanics, the total mass is the space integral of the local mass density. However, Witten’s energy formula [12, 22] indicates that the total energy and the total mass are the space integrals of the ‘generalized’ local mass density and the total mass as well. However, Witten’s energy formula [12, 22] indicates that the total energy and the total mass are the space integrals of the ‘generalized’ local mass density involving both the energy–momentum tensors and spinors. From this point of view, we think our approach goes some way towards the full understanding of quasi-local quantities.

2. Dirac–Witten equations

Let $(N, \tilde{g})$ be a four-dimensional spacetime which satisfies the Einstein field equations. Let $(M, g, p)$ be a smooth initial data set. Fix a point $z \in M$ and an orthonormal basis $\{e_a\}$ of $T_z N$ with $e_0$ future-time-directed normal to $M$ and $e_i$ tangent to $M$ $(1 \leq i \leq 3)$.

Denote by $\mathcal{S}$ the (local) spinor bundle of $N$. It exists globally over $M$ and is called the hypersurface spinor bundle of $M$. Let $\tilde{\nabla}$ and $\nabla$ be the Levi-Civita connections of $\tilde{g}$ and $g$ respectively, the same symbols are used to denote their lifts to the hypersurface spinor bundle. There exists a Hermitian inner product $(\cdot, \cdot)$ on $\mathcal{S}$ along $M$ which is compatible with the spin connection $\tilde{\nabla}$. The Clifford multiplication of any vector $\tilde{X}$ of $N$ is symmetric with respect to this inner product. However, this inner product is not positive definite. As $e_0^2 = \text{id}$, $\mathcal{S}$ is decomposed into the direct sums of two subbundles which are the eigenbundles of $e_0$ along $M$. This indicates that there exists a positive definite Hermitian inner product defined by $(\cdot, \cdot) = (e_0 \cdot \cdot)$ on $\mathcal{S}$ along $M$.

The second fundamental form of the initial data set is defined as $p_{ij} = \tilde{g}(\tilde{\nabla}_i e_0, e_j)$. Suppose $M$ has boundary $\Sigma$ which has finitely many connected components $\Sigma^1, \ldots, \Sigma^l$, each of which is a topological 2-sphere and endowed with its induced Riemannian and spin structures. Fix a point $z \in \Sigma$ and an orthonormal basis $\{e_i\}$ of $T_z M$ with $e_i$ outward normal to $\Sigma$ and $e_a$ tangent to $\Sigma$ for $2 \leq a \leq 3$. Let $h_{ab} = (\tilde{\nabla}_a e_+, e_b)$ be the second fundamental form of $\Sigma$. Let $H = \text{tr}(h)$ be its mean curvature. $\Sigma$ is a future/past apparent horizon if

$$H \equiv \text{tr}(p|_{\Sigma}) \geq 0$$  \hspace{1cm} (2.1)

holds on $\Sigma$.

Denote by $\nabla$ the lift of the Levi-Civita connection of $\Sigma$ to the spinor bundle $\mathcal{S}|_{\Sigma}$. The Dirac–Witten operator along $M$, the Dirac operator of $M$ acting on $\mathcal{S}$ and the Dirac operator of $\Sigma$ acting on $\mathcal{S}|_{\Sigma}$ are defined as $\tilde{D} = e_i \cdot \tilde{\nabla}_i, \tilde{D} = e_i \cdot \tilde{\nabla}_i, D = e_a \cdot \nabla_a$ respectively. Let $P_\pm = \frac{1}{2}(\text{id} \pm e_0 \cdot e_\pm)$ be the projective operators on $\mathcal{S}|_{\Sigma}$. Then $\mathcal{S}|_{\Sigma} \cong \mathcal{S}|_{\Sigma}^+ \oplus \mathcal{S}|_{\Sigma}^-$ is decomposed into the direct sum of two equivalent complex two-dimensional sub-bundles $\mathcal{S}|_{\Sigma}^\pm = \{ \phi \in \mathcal{S}|_{\Sigma} : P_\pm \phi = \phi \}$. 2
The following integral identity is given by the relation $\nabla_a = \nabla_a + \frac{1}{2} h_{ab} e_r \cdot e_b - \frac{1}{2} P_{aw} e_0 \cdot e_w$, and the Weitzenböck type formula for the Dirac–Witten operator

$$\int_M [\nabla_a \phi]^2 + \phi, T \phi - |\phi|^2 = \int \left( \phi \left( e_r \cdot D - \frac{H}{2} \right) \phi \right) + \frac{1}{2} \int \langle \phi, \text{tr}(p|_\Sigma) e_0 \cdot e_r \cdot \phi \rangle - \frac{1}{2} \int \langle \phi, p_{aw} e_0 \cdot e_w \cdot \phi \rangle,$$

where $T = \frac{1}{2} (T_{00} + T_{0j} e_0 \cdot e_j)$. Let $\phi = \phi^+ \pm \phi^-$. Since $e_0 \cdot e_r \cdot (\phi^\pm) = \mp e_r \cdot D \phi^\pm$, we have $e_0 \cdot e_r \cdot (e_0 \cdot e_a \cdot \phi^\pm) = \mp e_0 \cdot e_a \cdot \phi^\pm$; therefore, the right-hand side of (2.2) is

$$\text{RHS} = \int \langle \phi^+, e_r \cdot D \phi^- \rangle + \langle \phi^-, e_r \cdot D \phi^+ \rangle$$

$$- \frac{1}{2} \int \langle H - \text{tr}(p|_\Sigma) \rangle |\phi^+|^2 - \frac{1}{2} \int \langle H + \text{tr}(p|_\Sigma) \rangle |\phi^-|^2$$

$$- \frac{1}{2} \int \langle \phi^+, p_{aw} e_0 \cdot e_w \cdot \phi^- \rangle - \frac{1}{2} \int \langle \phi^-, p_{aw} e_0 \cdot e_w \cdot \phi^+ \rangle.$$

Parts (a) and (b) of the following existences are proved in [23] which correspond to the case all $\Sigma_i$ are either future or past apparent horizons. Part (c) corresponds to the case that at least one boundary 2-sphere satisfies the strict inequality in apparent horizon conditions.

**Proposition 1.** Let $(N, \tilde{g})$ be a spacetime which satisfies the dominant energy condition. Let $(M, g, p)$ be a smooth initial data with the boundary $\Sigma$ which has finitely many multi-components $\Sigma_i$, each of which is a topological 2-sphere.

(a) If $\text{tr}_e(p) \geq 0$ and all $\Sigma_i$ are past apparent horizons, then the following Dirac–Witten equation has a unique smooth solution $\phi \in \Gamma(\tilde{S})$:

$$\begin{cases}
\tilde{D} \phi = 0 & \text{in } M \\
P_\phi = P_\phi_0 & \text{on } \Sigma_i
\end{cases}$$

for given $\phi_0 \in \Gamma((\tilde{S}|_{\Sigma_i})$.

(b) If $\text{tr}_e(p) \leq 0$ and all $\Sigma_i$ are future apparent horizons, then the following Dirac–Witten equation has a unique smooth solution $\phi \in \Gamma(\tilde{S})$:

$$\begin{cases}
\tilde{D} \phi = 0 & \text{in } M \\
P_- \phi = P_- \phi_0 & \text{on } \Sigma_i
\end{cases}$$

for given $\phi_0 \in \Gamma((\tilde{S}|_{\Sigma_i})$.

(c) If $\Sigma_i, 1 \leq i \leq k_0$, are past apparent horizons and $\Sigma_j, k_0 + 1 \leq j \leq l$, are future apparent horizons and there exists at least $l_0$, $1 \leq l_0 \leq l$, such that the strict inequality in (2.1) holds for $l_0$, then the following Dirac–Witten equation has a unique smooth solution $\phi \in \Gamma(\tilde{S})$:

$$\begin{cases}
\tilde{D} \phi = 0 & \text{in } M \\
P_\phi = P_\phi_0 & \text{on } \Sigma_i \\
P_- \phi = P_- \phi_0 & \text{on } \Sigma_j
\end{cases}$$

for given $\phi_0 \in \Gamma((\tilde{S}|_{\Sigma_i})$, $\phi_0 \in \Gamma((\tilde{S}|_{\Sigma_j})$.

**Proof.** To establish the existence of (2.3), (2.4) and (2.5), we only need to show that any solution $\tilde{D} \phi = 0$ is trivial when $\phi_0 = \phi_0 = 0$. We first study case (c); in this case (2.2) gives

$$\int_M [\nabla \phi]^2 + \langle \phi, T \phi \rangle = - \frac{1}{2} \sum_{1 \leq i \leq k_0} \int_{\Sigma_i} [H + \text{tr}(p) ] |\phi|^2$$

$$- \frac{1}{2} \sum_{k_0 + 1 \leq j \leq l} \int_{\Sigma_j} [H - \text{tr}(p) ] |\phi|^2.$$
Since the strict inequality in (2.1) holds for \( l_0 \), we must have \( \phi|_{\Sigma_0} = 0 \) and \( \tilde{\nabla}_i \phi = 0 \) along \( M \). However, in cases (a) and (b), we cannot conclude that \( \phi|_{\Sigma_i} = 0 \) on some \( i \) directly from the apparent horizon condition (2.1). We need to use some other argument. Since \( \tilde{D}(e_0 \cdot \phi) = -\text{tr}_g(p)\phi \), we have

\[
\sum_{1 \leq i \leq l} \int_{\Sigma_i} \langle e_r \cdot \phi, e_0 \cdot \phi \rangle = \int_M \langle \tilde{D}(e_0 \cdot \phi), \tilde{D}(e_0 \cdot \phi) \rangle = \int_M \text{tr}_g(p)|\phi|^2.
\]

If all \( \Sigma_i \) are past apparent horizons and \( \text{tr}_g(p) \geq 0 \), or all \( \Sigma_i \) are future apparent horizons and \( \text{tr}_g(p) \leq 0 \), then the above equality gives that \( \phi|_{\Sigma_i} = 0 \) for all \( i \). We have also \( \tilde{\nabla}_i \phi = 0 \) along \( M \) in these cases. Now \( |d|\phi|^2| \leq 2|\nabla\phi||\phi| \leq |p||\phi|^2 \leq C_\rho|\phi|^2 \), where \( C_\rho = \max|p| \). If there exists a point \( x_0 \in M \) such that \( \phi(x_0) \neq 0 \), then

\[
|\phi(x)| \geq |\phi|(x_0)\ e^{-C_\rho p_0(x)}.
\]

Taking \( x \) to the boundary \( \Sigma_{x_0} \) will give rise to a contradiction. Therefore, \( \phi = 0 \) over \( M \) and the existence of (2.3), (2.4) and (2.5) follow. \( \square \)

**Proposition 2.** Let \( (N, \tilde{g}) \) be a spacetime which satisfies the dominant energy condition. Let \( (M, g, p) \) be a smooth initial data with the boundary \( \Sigma \) which has finitely many multi-components \( \Sigma_i \), each of which is a topological 2-sphere. Suppose \( \phi, \psi \) are smooth spinors which satisfy \( \tilde{\nabla}_i \phi = 0, \tilde{\nabla}_i \psi = 0 \). Denote \( \phi_0^\pm = P_\pm \phi|_{\Sigma_i}, \psi_0^\pm = P_\pm \psi|_{\Sigma_i} \). If either \( \phi_0^+, \psi_0^+ \) or \( \phi_0^-, \psi_0^- \) are linearly independent on \( \Sigma_i \), then \( \phi, \psi \) are linearly independent in \( M \).

**Proof.** Let \( C_1 \phi_0(x_0) + C_2 \psi_0(x_0) = 0 \) for complex constants \( C_1, C_2 \) at some point \( x_0 \in M \). We shall show that \( C_1, C_2 \) must be zero. Define \( \Phi(x) = C_1 \phi(x) + C_2 \psi(x) \). Let \( \Omega \) be an open subset of \( M \) where \( \Phi \) is nonzero. Since \( \tilde{\nabla}_i \Phi = 0 \), we know that \( |\Phi(y)| \geq |\Phi(x)|\ e^{-C_\rho p(x,y)} \) for any \( x, y \in \Omega \) where \( C_\rho = \max|p| \). If \( \Omega \) is nonempty, we can take \( y \) to the boundary of \( \Omega \) or to the point \( x_0 \). This will give rise to a contradiction. Therefore, \( \Phi(x) \equiv 0 \). Hence, \( C_1 \phi_0^+ + C_2 \psi_0^+ = 0 \) on \( \Sigma_i \). Since \( \phi_0^+, \psi_0^+ \) or \( \phi_0^-, \psi_0^- \) is linearly independent on \( \Sigma_i \), we have \( C_1 = C_2 = 0 \) and the proposition follows. \( \square \)

3. Quasi-local quantities

Let \( (N, \tilde{g}) \) be a spacetime which satisfies the dominant energy condition. Let \( (M, g, p) \) be a smooth initial data set with boundary \( \Sigma \) which has finitely many connected components \( \Sigma_1, \ldots, \Sigma_l \), each of which is a topological 2-sphere. Suppose that some \( \Sigma_i \) can be smoothly isometrically embedded into a smooth spacelike hypersurface \( \tilde{M} \) in the Minkowski spacetime \( \mathbb{R}^{3,1} \) and denote by \( \mathfrak{S} \) the isometric embedding. Let \( \tilde{\Sigma}_i \) be the image of \( \Sigma_i \) under the map \( \mathfrak{S} \). Let \( \tilde{e}_i \) be the unit vector outward normal to \( \tilde{\Sigma}_i \) and \( \tilde{h}_{ij}, \tilde{H} \) are the second fundamental form and the mean curvature of \( \tilde{\Sigma}_i \) respectively. Denote by \( p_0 = \tilde{p} \circ \mathfrak{S}, H_0 = \tilde{H} \circ \mathfrak{S} \) the pullbacks to \( \Sigma_i \).

The isometric embedding \( \mathfrak{S} \) also induces an isometry between the (intrinsic) spinor bundles of \( \Sigma_i \) and \( \tilde{\Sigma}_i \) together with their Dirac operators which are isomorphic to \( e_r \cdot D \) and \( \tilde{e}_r \cdot \tilde{D} \) respectively. This isometry can be extended to an isometry over the complex two-dimensional sub-bundles of their hypersurface spinor bundles. Denote by \( \mathfrak{S}^\Sigma_i \) (\( \tilde{\Sigma}_i \)) this sub-bundle of \( \mathfrak{S}^\Sigma \) (\( \tilde{\Sigma}_i \)). We choose the same orientations for \( N \) and \( \mathbb{R}^{3,1} \). Since \( e_2 \cdot e_3 = \tilde{e}_2 \cdot \tilde{e}_3 \), \( P_{\pm} \) is preserved by \( \mathfrak{S} \).

Therefore,

\[
\mathfrak{S}|_{\tilde{\Sigma}_i} \cong \mathfrak{S}|_{\Sigma_i} \cong \mathfrak{S}^\Sigma_i.
\]
Let $\tilde{\phi}$ be a constant section of $\hat{\Sigma}$ and denote $\phi_0 = \tilde{\phi} \circ \eta$. Denote by $\hat{\Sigma}$ the set of all these constant spinors $\tilde{\phi}$ with the unit norm. This set is isometric to $S^3$.

In this paper, we introduce the following conditions on $M$:
(i) $\text{tr}_p(p) \geq 0, H|_{\Sigma} + \text{tr}(p|_{\Sigma}) \geq 0$ for all $i$;
(ii) $\text{tr}_p(p) \leq 0, H|_{\Sigma} - \text{tr}(p|_{\Sigma}) \geq 0$ for all $i$;
(iii) $H|_{\Sigma} + \text{tr}(p|_{\Sigma}) \geq 0$ for $1 \leq i \leq k_0$, $H|_{\Sigma} - \text{tr}(p|_{\Sigma}) \geq 0$ for $k_0 + 1 \leq j \leq l$, and there exists at least $l_0, 1 \leq l_0 \leq l$, such that the strict inequality holds for $l_0$.

Now we provide the definition of quasi-local quantities for some $\Sigma_i$ under these conditions. Recall [23, 24].

Case 1. If $\Sigma_i$ has positive Gauss curvature, then it can be embedded smoothly isometrically into $\mathbb{R}^3$ in the Minkowski spacetime $\mathbb{R}^{3,1}$. Denote by $\tilde{\Sigma}_i$ its image. In this case, $\tilde{p} = 0$.

Let $\phi$ be the unique solution of one of (2.3), (2.4) and (2.5) for some $\phi_i \in \hat{\Sigma}$ and $\phi_j = 0$ for $j \neq i$. Denote
\[
m(\Sigma_i, \phi) = \frac{\text{area}(\Sigma_i)}{8\pi} \int_{\Sigma_i} [(H_0 - H)|\phi|^2 + \text{tr}(p|_{\Sigma})|\phi, e_0 \cdot e_r \cdot \phi] - p_{ar} \langle \phi, e_0 \cdot e_a \cdot \phi \rangle.
\]
Define
\[
m_i^b = \min_{\mathbb{R}} m(\Sigma_i, \phi), \quad m_i^c = \max_{\mathbb{R}} m(\Sigma_i, \phi).
\]

Case 2. If $\Sigma_i$ has negative Gauss curvature, $K_{\Sigma_i} \geq -\kappa^2$ ($\kappa > 0$) where $-\kappa^2$ is the minimum of the Gauss curvature. (Here we must choose the minimum of the Gauss curvature instead of arbitrary lower bound, otherwise the quasi-local mass defined in the following way might depend on this arbitrary lower bound.) By [3, 13], $\Sigma_i$ can be smoothly isometrically embedded into the hyperbolic space $\mathbb{H}^3_{-\kappa^2}$ with constant curvature $-\kappa^2$ as a convex surface which bounds a convex domain in $\mathbb{H}^3_{-\kappa^2}$. Let $(t, x_1, x_2, x_3)$ be the spacetime coordinates of $\mathbb{R}^{3,1}$. Then $\mathbb{H}^3_{-\kappa^2}$ is a one-fold of the space-like hypersurfaces $\{r^2 - x_1^2 - x_2^2 - x_3^2 = \frac{1}{\kappa^2} \}$. The induced metric of $\mathbb{H}^3_{-\kappa^2}$ is $\tilde{g}^{\mathbb{H}^3_{-\kappa^2}} = \frac{dx^2}{1 + r^2} + r^2 d\theta^2 + \sin^2 \theta d\psi^2$. It has the second fundamental form $\tilde{p}^{\mathbb{H}^3_{-\kappa^2}} = \kappa \tilde{g}^{\mathbb{H}^3_{-\kappa^2}}$, for the upper-fold $t > 0$ and $\tilde{p}^{\mathbb{H}^3_{-\kappa^2}} = -\kappa \tilde{g}^{\mathbb{H}^3_{-\kappa^2}}$, for the lower-fold $t < 0$ with respect to the future-time-directed normal.

If condition (i) holds, we embed $\Sigma_i$ into upper-fold $\{t > 0\}$. Since $\Sigma_i$ is convex, we have $H + \text{tr}(\tilde{p}|_{\Sigma_i}) > 0$. If condition (ii) holds, we embed $\Sigma_i$ into lower-fold $\{t < 0\}$. We have $H - \text{tr}(\tilde{p}|_{\Sigma_i}) > 0$ in this case. If condition (iii) holds, we embed $\Sigma_i$ into upper-fold if $H + \text{tr}(\tilde{p}|_{\Sigma_i}) \geq 0$ and embed it into lower-fold if $H - \text{tr}(\tilde{p}|_{\Sigma_i}) \geq 0$. Denote also by $\tilde{\Sigma}_i$ its image.

Let $\phi$ be the unique solution of one of (2.3), (2.4) and (2.5) for some $\phi_i \in \hat{\Sigma}$ and $\phi_j = 0$ for $j \neq i$. Denote
\[
m_{\pm}(\Sigma_i, \phi) = \frac{\text{area}(\Sigma_i)}{8\pi} \int_{\Sigma_i} [(H_0 - H)|\phi|^2 + \text{tr}(p|_{\Sigma})|\phi, e_0 \cdot e_r \cdot \phi] - p_{ar} \langle \phi, e_0 \cdot e_a \cdot \phi \rangle
\]
\[
\frac{\text{area}(\Sigma_i)}{4\pi} \int_{\Sigma_i} \langle \phi, e_0 \cdot e_r \cdot \phi \rangle.
\]
Define
\[
m_i^b = \begin{cases} 
\min_{\mathbb{R}} m_{\pm}(\Sigma_i, \phi) & \text{if } \Sigma_i \text{ is embedded into upper-fold,} \\
\min_{\mathbb{R}} m_{-}(\Sigma_i, \phi) & \text{if } \Sigma_i \text{ is embedded into lower-fold.}
\end{cases}
\]
Case 3. If $\Sigma_i$ has nonnegative Gauss curvature which vanishes at some point, then it can be only $C^{1,1}$ embedded isometrically into $\mathbb{R}^3$ [5, 6]. To avoid this regularity problem, we first embed it smoothly isometrically into hyperbolic space $\mathbb{H}^3_{-\kappa}$, and then take the limit $\kappa \to 0$. Define

$$m^b_i = \begin{cases} \max_{\Sigma_i} m_+ (\Sigma_i, \tilde{\phi}) & \text{if } \Sigma_i \text{ is embedded into upper-fold}, \\ \max_{\Sigma_i} m_- (\Sigma_i, \tilde{\phi}) & \text{if } \Sigma_i \text{ is embedded into lower-fold}. \end{cases}$$

Remark 1. The quasi-local quantities do not depend on the choice of spinors after taking maximum and minimum on certain constant spinor space. This procedure was not taken to rule out spinors in Dougan and Mason’s definition [4].

Remark 2. The solutions of (2.3), (2.4) and (2.5) for some $\tilde{\phi}_i \in \tilde{\mathbb{S}}$ and $\tilde{\phi}_j = 0$ for $j \neq i$ are parameterized by the positive or negative part of the constant Dirac spinors in $\mathbb{R}^{3,1}$. So the set of these unit norm constant spinors is $S^3$.

Remark 3. The normalized factor is omitted in the definitions of $m(\Sigma_i, \tilde{\phi})$ and $m_i^\pm (\Sigma_i, \tilde{\phi})$ in [23, 24].

Definition 1. Let $(N, \tilde{g})$ be a spacetime which satisfies the dominant energy condition. Let $(M, g, p)$ be a smooth initial data set with the boundary $\Sigma$ which has finitely many multi-components $\Sigma_i$, each of which is a topological 2-sphere. If one of conditions (i), (ii), (iii) holds, then the quasi-local energy $E_i^{loc}$, the quasi-local linear momentum $|P_i^{loc}|$ and the quasi-local mass $m_i^{loc}$ of $\Sigma_i$ are defined respectively as follows:

$$E_i^{loc} = \frac{m_i^+ + m_i^-}{2}, \quad |P_i^{loc}| = \frac{m_i^+ - m_i^-}{2}, \quad m_i^{loc} = \sqrt{m_i^+ m_i^-}.$$  

From the following theorem, we know that $m_i^+ \geq m_i^- \geq 0$. Therefore, these quasi-local quantities are well defined.

It is straightforward that these quasi-local quantities all vanish if $(N, \tilde{g})$ is the Minkowski spacetime. On the other hand, by the uniqueness, any solution of (2.3) or (2.4) with boundary value in $\tilde{\mathbb{S}}$ approaches a constant spinor in asymptotically flat initial data sets. Therefore, the limits at spatial infinity are

$$\lim_{r \to \infty} m_i^+ = E + |P|, \quad \lim_{r \to \infty} m_i^- = E - |P|,$$

where $E$ and $P_i$ are the ADM total energy and the ADM total linear momentum, respectively, and $|P| = \sqrt{P_1^2 + P_2^2 + P_3^2}$. Denote by $m = \sqrt{E^2 - P_1^2 - P_2^2 - P_3^2}$ the ADM mass. We have

$$\lim_{r \to \infty} E_i^{loc} = E, \quad \lim_{r \to \infty} |P_i^{loc}| = |P|, \quad \lim_{r \to \infty} m_i^{loc} = m.$$  

Theorem 1. Let $(N, \tilde{g})$ be a spacetime which satisfies the dominant energy condition. Let $(M, g, p)$ be a smooth initial data set with the boundary $\Sigma$ which has finitely many multi-components $\Sigma_i$, each of which is a topological 2-sphere. If one of conditions (i), (ii), (iii) holds, then

(a) $m_i^+ \geq 0 \Leftrightarrow E_i^{loc} \geq |P_i^{loc}|$ for each $i$;

(b) If $E_{i_0}^{loc} = 0$ for some $i_0$, then $N$ is flat along $\Sigma_{i_0}$.
Proof. Part (a) is a direct consequence of proposition 1 and the following inequality proved in [23, 24]:
\[
\int_{\Sigma_{\alpha}} \langle \phi, e_{r} \cdot D\phi \rangle \leq \frac{1}{2} \int_{\Sigma_{\alpha}} \langle \phi, (H_0 - p_{\text{bar}} e_0 \cdot e_{r} + p_{\text{bar}} e_0 \cdot e_{a}) \phi \rangle.
\] (3.2)

For the completeness, we summarize the proof of (3.2) here. Since \( \partial_{t} \tilde{\phi} = 0 \), restricted on \( \Sigma_{i R} \), we have \( \nabla_{a} \tilde{\phi} + \frac{1}{2} \tilde{p}_{ab} e_{r} \cdot \tilde{e}_{b} \cdot \tilde{\phi} - \frac{1}{2} \tilde{p}_{a j} \tilde{e}_{j} \cdot \tilde{\phi} = 0 \). Pullback to \( \Sigma_{i R} \), we obtain
\[
e_{r} \cdot D\phi_{0}^{*} = \frac{H_0}{2} \phi_{0}^{*} + \frac{1}{2} p_{\text{bar}} \phi_{0}^{*} + \frac{1}{2} p_{\text{bar}} e_0 \cdot e_{a} \cdot \phi_{0}^{*}.
\]
Using \( \int_{\Sigma_{\alpha}} \langle \phi_{0}^{*}, e_{r} \cdot D\phi_{0}^{*} \rangle = \int_{\Sigma_{\alpha}} \langle e_{r} \cdot D\phi_{0}^{*}, \phi_{0}^{*} \rangle \), we get
\[
\int_{\Sigma_{\alpha}} (H_0 - p_{\text{bar}}) \langle \phi_{0}^{*}, \phi_{0}^{*} \rangle = \int_{\Sigma_{\alpha}} (H_0 + p_{\text{bar}}) \langle \phi_{0}^{*}, \phi_{0}^{*} \rangle.
\]

The proof of (3.2) as well as the theorem under condition (i) or condition (iii) with \( \Sigma_{i S} \) the past apparent horizon is the same as that under condition (ii) or condition (iii) with \( \Sigma_{i S} \) the future apparent horizon. So we only prove the first case. Let \( \phi \) be the smooth solution of the Dirac–Witten equation with \( \phi^{*} = \phi_{0}^{*} \). We have
\[
\int_{\Sigma_{\alpha}} \langle \phi, e_{r} \cdot D\phi \rangle = 2 \Re \int_{\Sigma_{\alpha}} \langle \phi^{*}, e_{r} \cdot D\phi_{0}^{*} \rangle
\]
\[
\leq 2 \Re \int_{\Sigma_{\alpha}} \langle \phi^{*}, (H_0 + p_{\text{bar}}) \phi_{0}^{*} \rangle
\]
\[
\leq \frac{1}{2} \int_{\Sigma_{\alpha}} (H_0 + p_{\text{bar}}) (|\phi^{*}|^2 + |\phi_{0}^{*}|^2)
\]
\[
+ \Re \int_{\Sigma_{\alpha}} \langle \phi^{*}, p_{\text{bar}} e_0 \cdot e_{a} \cdot \phi_{0}^{*} \rangle
\]
\[
= \frac{1}{2} \int_{\Sigma_{\alpha}} (H_0 + p_{\text{bar}}) (|\phi^{*}|^2 + |\phi_{0}^{*}|^2)
\]
\[
+ \Re \int_{\Sigma_{\alpha}} \langle \phi^{*}, p_{\text{bar}} e_0 \cdot e_{a} \cdot \phi_{0}^{*} \rangle
\]
\[
= \frac{1}{2} \int_{\Sigma_{\alpha}} (H_0 + p_{\text{bar}}) (|\phi^{*}|^2 + |\phi_{0}^{*}|^2)
\]
\[
+ \Re \int_{\Sigma_{\alpha}} \langle \phi^{*}, p_{\text{bar}} e_0 \cdot e_{a} \cdot \phi_{0}^{*} \rangle
\]
\[
= \frac{1}{2} \int_{\Sigma_{\alpha}} \langle \phi, (H_0 - p_{\text{bar}} e_0 \cdot e_{r} + p_{\text{bar}} e_0 \cdot e_{a}) \phi \rangle.
\]

For part (b), if \( E_{i c}^{\text{loc}} = 0 \) for some \( i_0 \), then \( m_{i_0}^{a} = m_{i_0}^{b} = 0 \). Therefore, for any \( \phi \in \mathcal{Z}, m(\Sigma_{i R}, \phi) = 0 \). This implies \( \nabla_{a} \phi = 0 \). Let \( \phi_1, \phi_2 \in \mathcal{Z} \) be two linearly independent constant spinors. Let \( \phi_1, \phi_2 \) be the smooth solutions of one of (2.3), (2.4) and (2.5) corresponding to the boundary values \( \phi, \phi_{0} \). Then \( \nabla_{a} \phi_1 = \nabla_{a} \phi_2 = 0 \). By proposition 2, \( \phi_1 \) and \( \phi_2 \) are linearly independent. Choose the frame \( e_\alpha \) such that \( \nabla_{a} e_{\beta} = 0 \). Then
\[
-\frac{1}{2} \tilde{R}_{j a b} e_{a} e_{\beta} \phi_{\alpha} = [\tilde{\nabla}_{a}, \tilde{\nabla}_{j}] \phi_{\alpha} = 0
\] (3.3)
for \( a = 1, 2 \). Note that we are not able to obtain the vanishing of \( \tilde{R}_{j a b} \) along \( M \) as \( \phi_{a} \) are complex four-dimensional which might neither belong to the positive eigenbundle of \( e_{0} \) nor the negative eigenbundle of \( e_{0} \). However, in the proof of (3.2), the inequality
\[
2 \Re \langle \phi^{*}, \phi_{0}^{*} \rangle \leq |\phi^{*}|^2 + |\phi_{0}^{*}|^2
\] is used on \( \Sigma_{\alpha} \), and it is forced to be the equality when \( E_{i c}^{\text{loc}} = 0 \). So \( \phi^{*} = \phi_{0}^{*} \) on \( \Sigma_{\alpha} \). Since the boundary value condition gives \( \phi^{*} = \phi_{0}^{*} \), we obtain \( \phi = \phi_{0} \) on \( \Sigma_{\alpha} \). As \( \phi_{0} \) are pullbacks of \( \phi_{a} \), \( \phi_{0} \) belong to the same eigenbundle of \( e_{0} \). This indicates
that the situation can be reduced to complex two-dimensional spinor bundles on $\Sigma_{i0}$ and \((3.3)\) holds true for two linearly independent constant spinors $\phi_{a0}$ on $\Sigma_{i0}$. Therefore, $\tilde{R}_{i0a0} = 0$ on $\Sigma_{i0}$. So $T_{00} = 0$ on $\Sigma_{i0}$ and the dominant energy condition gives $T_{a0} = 0$ which implies $\tilde{R}_{0a0} = 0$ on $\Sigma_{i0}$. Therefore, the spacetime is flat along $\Sigma_{i0}$. □

4. Discussions

In this paper, we define the quasi-local quantities for a 2-sphere in the spacelike hypersurface. For a spacelike 2-sphere $\Sigma$ in spacetimes, we can take infimum of above-defined quasi-local quantities over the set of spacelike hypersurfaces enclosed by $\Sigma$, if the set is nonempty. Because of theorem 1, the infimum exists and is nonnegative. We can define this infimum as the quasi-local quantities of $\Sigma$. If we think the choice of the spacelike hypersurface for $\Sigma$ is gauge, we can call the infimum gauge in this situation. Alternatively, if $\Sigma$ can enclose a unique maximal hypersurface which satisfies $\text{tr}(\rho) = 0$ in spacetime (it is indeed the case in certain situation), then the quasi-local quantities of $\Sigma$ can be defined in terms of the geometry of this 2-sphere and the maximal hypersurface. And we can call it the maximal gauge. The quasi-local quantities of $\Sigma$ in these gauges can be understood not to depend on the spacelike hypersurface.

Acknowledgments

The author is indebted to N Ó Murchadha, Roh-Suan Tung and Naqing Xie for valuable conversations. This work is partially supported by NSF of China (10421001, 10725105, 10731080), NKBRPC(2006CB805905) and the Innovation Project of Chinese Academy of Sciences.

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