Inertial frame rotation induced by rotating
gravitational waves

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Abstract
We calculate the rotation of the inertial frame within an almost flat cylindrical
region surrounded by a pulse of non-axially-symmetric gravitational waves that
rotate about the axis of our cylindrical polar coordinates. Our spacetime has
only one Killing vector. It is along the $z$-axis and hypersurface orthogonal.
We solve the Einstein equations to first order in the wave amplitude and
superpose such linearized solutions to form a wave pulse. We then solve
the relevant Einstein equation to second order in the amplitude to find the
rotation of inertial frames produced by the pulse. The rotation is without time
delay. The influence of gravitational wave angular momentum on the inertial
frame demonstrates that Mach’s principle cannot be expressed in terms of the
influence of the stress–energy–momentum tensor alone but must also involve
influences of gravitational wave energy and angular momentum.

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1. Introduction

Mach [21] rejected Newton’s [26] absolute space and suggested that some average of the
positions and motions of the masses in the universe determines the inertial frame.

Einstein [10, 11] was inspired by Mach’s principle and his earlier work [9] showed that
the local inertial frame was not absolute but was affected by the rotation and acceleration
of matter nearby. Extrapolating from this he said, in essence, the relativity of inertia is not
fulfilled if inertia is affected by the presence of matter it must be entirely caused by it. He
also said there is no inertia of mass against space but only inertia of mass against mass. Thus
he hoped that the inertial frame might be determined by the distribution of the stress tensor4

4 Indices $\lambda, \mu, \nu, \rho, \ldots = 0, 1, 2, 3$; indices $k, l, m, n, \ldots = 1, 2, 3$ and $a, b, c, \ldots = 0, 1, 2$. The metric $g_{\mu\nu}$ has
signature $+−−−$ and $g$ is its determinant. Covariant derivatives are indicated by a $\nabla$, partial derivatives by a $\partial$ and
covariant derivatives in a three subspace by $\nabla$. The permutation symbol in four dimensions is $\epsilon_{\mu\nu\rho\sigma}$ with
$\epsilon_{0123} = 1$ and in three dimensions $\epsilon_{ijk}$ with $\epsilon_{123} = 1$. Finally $n_{\mu\nu\rho\sigma} = \sqrt{-g} \epsilon_{\mu\nu\rho\sigma}$. 
$T_{\mu\nu}$. His efforts to remove the external influence of the boundary conditions at infinity on
the solutions of his equations led him to consider closed spaces with no spatial boundaries
[10]. Here we show that within general relativity, any general statement of Mach’s principle
that attributes all dragging of inertial frames solely to the distribution of $T_{\mu\nu}$ as the origin
of inertia is false (see [5]). One counter example suffices to demonstrate this. We find that there
is an almost flat cylindrical region near the axis of a revolving gravitational wave pulse (which
inevitably has no $T_{\mu\nu}$) and demonstrate that the inertial frame in the cylindrical interior rotates
relative to the inertial frame at great distances.

The exact cylindrical wave solutions [2, 7, 12, 30] carry no angular momentum about
the axis, because their wave fronts have normals with no component around the axis. Bondi [6]
emphasises this point.

In a sense our work pushes at an open door, since now all accept that gravitational waves
exist and carry both energy and angular momentum. This is not merely known theoretically,
but also observationally [27, 28], cf also [24] and [20].

It would be astonishing if that energy and angular momentum did not produce gravity and
rotation of the inertial frame. Indeed, as we described in more detail in paper [4], the work of
Corvino [8] demonstrates that this must be the case and in [4] we have explicitly calculated
the gravity of a nonlinear pulse of gravitational waves.

Here our aim is to produce a nice clean example of the rotation of the inertial frame in an
almost flat region surrounded by rotating gravitational waves.

An extreme example of inertia due to gravitational waves alone is provided by Gowdy’s
[15] universe, a closed world that contains nothing except gravitational waves. One of
our ultimate aims is to discuss the meaning of Mach’s principle in general relativity and
particularly its application to such systems. However, such applications and the question as to
what bearing Mach’s principle has on the existence of dark energy are beyond the scope of this
paper.

When we first discussed Mach’s principle in FRW universes via perturbation theory [18],
we pointed out that our first-order perturbation theory inevitably missed the second-order
contribution expected from gravitational waves (see also our recent comprehensive paper [3]).
Here we show that there must be such a contribution.

We consider waves in spaces that have a hypersurface orthogonal Killing vector along
$Oz$. If such waves have axial symmetry their wave normals converge on the axis and they
carry no angular momentum. To get truly rotating waves that carry angular momentum around
the axis, we cannot impose axial symmetry. As no exact non-axisymmetric rotating wave
solutions are known, we work via perturbation theory. First we solve the flat space wave
equation for linearized gravitational waves with symmetry along $Oz$, using cylindrical polar
coordinates. Then we superpose these linear solutions to form a wave pulse that rotates
around the axis. This pulse generalizes to non-axial-symmetry the wave pulse found by
Bonnor [7] and by Weber and Wheeler [30]. In the spacetime generated by such a pulse
of linear waves, we then solve one of Einstein’s equations to second order in the wave
amplitude. Following [1] and [4], we see that the wave pulse generates an ‘effective source
term’ of second order in the wave amplitude. This means that the first order changes in
the metric due to the wave can be neglected when calculating the extra perturbation due
to these effective sources. (Keeping the changes could only lead to terms of third order
in the wave amplitude.) Thus we have to calculate in flat space the effects of sources
that are quadratic in the wave amplitude. Such perturbative techniques are well discussed
in [14].
2. Calculations

We consider, as in [4], source free metrics with one hypersurface orthogonal Killing vector of spacelike translations $\zeta$. Coordinates can be chosen so that $z = x^3$ and 
\[ \{x^a, x^3\} \quad \text{with} \quad a = 0, 1, 2 \quad \text{that} \quad \zeta = \{0, 0, 0, 1\}, \] (2.1)
and the metric can be written in the form introduced by Ashtekar et al [1]:
\[ ds^2 = e^{-2\psi} g_{ab} dx^a dx^b - e^{2\psi} dz^2, \] (2.2)
$\psi$ and $g_{ab}$ are functions of the $x^c$ only. The source-free Einstein’s equations $R_{\mu\nu} = 0$ take then the following form in terms of the Ricci tensor $R_{ab}$ of the $(2+1)$-space $d\sigma^2 = g_{ab} dx^a dx^b$:
\[ R_{ab} = 0 \quad \Rightarrow \quad R_{ab} = 2 \partial_a \psi \partial_b \psi, \] (2.3)
\[ R_{33} = 0 \quad \Rightarrow \quad g^{ab} \nabla_a \nabla_b \psi = 0. \] (2.4)

Because $\psi$ satisfies (2.4), the right-hand side of (2.3) automatically satisfies the Bianchi identities in $(2+1)$ spacetime. To zero order $g_{ab}$ reduces to the flat spacetime metric and we choose cylindrical coordinates $x^0 = t, x^1 = \rho, x^2 = \varphi$, so, in the notations of [4] where a dot means a derivative with respect to $t$ and a prime a derivative with respect to $\rho$, our linearized waves obey the wave equation
\[ \ddot{\psi} - \psi'' - \frac{1}{\rho} \psi' - \frac{1}{\rho^2} \partial^2 \varphi \psi = 0. \] (2.5)

The solutions are expressed in Bessel functions $J_m(\omega \rho)$:
\[ \psi = A e^{i m \varphi} \sum_{m=0} J_m(\omega \rho). \] (2.6)

We note that the special class of these waves with $m = 0$ constitute the exact solutions mentioned in the introduction. We follow [7] and [30] in choosing a superposition of such waves but in order to have rotating linearized waves we need $m \neq 0$ and we choose waves of the form $e^{i m \rho - \omega t}$ with $m \omega > 0$ so that the wave fronts rotate positively about the $z$ axis. Our linear wave pulse is given by the superposition with $a > 0$ being the effective duration of the pulse,
\[ \psi = B \int_{0}^{\infty} (a \omega)^m e^{-a \omega} e^{i m \rho - \omega t} J_m(\omega \rho) a d\omega + \text{c.c.}, \] (2.7)
$B$ is a dimensionsless amplitude. It is convenient to define $\alpha = \alpha(t) = a + i t$. Thanks to the Bateman Manuscript Project of Erdélyi et al [13], formula (8.6.5), we find
\[ \psi = B a^{m+1} \rho^{m+1} \frac{2^m \Gamma(m + \frac{1}{2})}{\sqrt{\pi} \rho^m (\alpha^2 + \rho^2)^{-\frac{m-1}{2}}} + \text{c.c.} \] (2.8)

$\psi$ may be written in terms of non-dimensional variables as in [4],
\[ \tilde{\rho} = \frac{\rho}{a}, \quad \tilde{t} = \frac{t}{a} \quad \text{and with} \quad 2^m \frac{\Gamma(m + \frac{1}{2})}{\sqrt{\pi} \rho^m (\alpha^2 + \rho^2)^{-\frac{m-1}{2}}} = (2m - 1)!!, \] (2.9)
$\psi$ can also be written in real terms as follows:
\[ \psi = 2 B (2m - 1)!! \tilde{\rho}^m \cos\left[m\varphi - \left(m + \frac{1}{2}\right) \chi\right], \] where $\chi = \arctan \frac{2 \tilde{t}}{1 + \tilde{\rho}^2 - \tilde{t}^2}$.
\[ \] (2.10)
Figure 1. This shows $\psi/B$, the wave profiles for $m=10$ and $\varphi=0$ at five different times at adapted scales. On the left-hand side $\psi/B$ is shown as circulates inwards, from right to left at times $\tilde{t}=-10, -6, -3$. The right-hand side shows it as circulates at closest approach to the axis $\tilde{t}=0$. Note there is very little closer than $\rho \simeq 0.4$ even at $\tilde{t}=0$. Superposed on the right-hand side is the profile as the wave starts to retreat at $\tilde{t}=3$. Thereafter the profiles are the same as on the left-hand side of the figure but in reverse order. At all times the wave profile centers around $\tilde{\rho} = \sqrt{1 + \tilde{t}^2}$.

So the phase of $\psi$ is $m\varphi - (m + \frac{1}{2}) \chi$. The $\tilde{\rho}$ dependence of this phase gives spirality to the wave and changes sign at $t=0$. Figure 1 shows the profile of $\psi$ at different times for $m=10$ and $\varphi=0$.

To calculate the source term on the right-hand side of (2.3.3) we need derivatives of expression (2.8). In particular the $t,\varphi$ component of the source will be given by $2\dot{\psi} \partial_\varphi \psi$. This product contains two terms that are independent of $\varphi$ and two terms that vary like $\sin 2m\varphi$ or $\cos 2m\varphi$. Those terms cannot cause any rotation of the inertial frame on the axis. We therefore concentrate on the terms that are independent of $\varphi$ and denote them by the averaging symbol $\langle \rangle$, implying an average over $\varphi$. Defining $j$, we find from (2.8) and (2.9),

$$j = -2\langle \dot{\psi} \partial_\varphi \psi \rangle = \frac{2B^2}{a} m(2m+1) [(2m-1)!! B^{2m} Q^{-m-3/2} \left( \frac{\partial Q}{\partial u} + 4\tilde{t}^2 \right) ], \quad (2.11)$$

in which

$$u = \tilde{\rho}^2 \quad \text{and} \quad Q = (1 + u - \tilde{t}^2)^2 + 4\tilde{t}^2. \quad (2.12)$$

Before proceeding further we return to equations (2.3), now regarding the right-hand side as a known second-order source of perturbation of our now first-order metric. As the source terms are already of second order, we may neglect the change of the metric from flat space in calculating their effects, as such changes could only give effects of third order. We can write the fully perturbed metric in form (2.2) in which, see (5.4) in [4],

$$g_{ab} dx^a dx^b = e^{2\nu} (dt^2 - d\rho^2) - W^2 (d\varphi - \omega dt)^2. \quad (2.13)$$

The flat space equation for the axially symmetrical part of $\omega$ is then, cf (5.10) in [4],

$$\langle \omega \rangle = \frac{1}{2\rho} (\rho^3 \langle \omega \rangle')' = 2\langle \dot{\psi} \partial_\varphi \psi \rangle = -j. \quad (2.14)$$

We first integrate once with the boundary condition that $\langle \omega \rangle$ is not singular on the axis:

$$\rho^3 \langle \omega \rangle' = -\int_0^{\tilde{\rho}_1} j d\rho_1^2. \quad (2.15)$$

Using the boundary condition that $\langle \omega \rangle \to 0$ at infinity we then find that

$$\langle \omega \rangle = \frac{1}{2} \int_{\tilde{\rho}^2}^{\infty} \frac{1}{\rho_2^3} \left( \int_0^{\rho_2^2} j d\rho_1^2 \right) d\rho_2^2, \quad (2.16)$$
where \( \rho^2_1 \) and \( \rho^2_2 \) are the dummy variables. Doing the outer integration by parts, we deduce

\[
\langle \omega \rangle = \frac{1}{2u} \int_0^u j \, du + \int_u^\infty \frac{1}{2u} j \, du. \tag{2.17}
\]

Substituting (2.11) into (2.17), we get, after integrating the \( Q^{-m-3/2}(\partial Q/\partial u) \) terms by parts,

\[
\langle \omega \rangle = \frac{2B^2}{a} m [(2m - 1)!!] \left[ \frac{m}{u} I_{m-1} + 2(2m + 1) \frac{\tilde{t}^2}{u} I_m + (m - 1) H_{m-1} + 2(2m + 1) \tilde{t}^2 H_m \right]. \tag{2.18}
\]

in which

\[
I_m(u) = \int_0^u u^m Q^{-m-3/2} \, du, \quad H_m(u) = \int_u^\infty u^{m-1} Q^{-m-3/2} \, du. \tag{2.19}
\]

The evaluation of \( \langle \omega \rangle \) is thus reduced to the calculation of \( I_m \) and \( H_m \) for all \( m \). This is done in the appendix.

3. Rotation of inertial frames at small and great distances

On axis the first two terms in the large brackets of (2.18) are equal to zero. So using (A.15) with \( b = 1 - \tilde{t}^2 \) and \( c = 1 + \tilde{t}^2 \) we get from (2.18)

\[
\langle \omega \rangle_o = \frac{B^2}{a} \frac{(2m)!}{2^{2m-1}} \left[ \frac{1 + m(1 + \tilde{t}^2)}{(1 + \tilde{t}^2)^2} \right]. \tag{3.1}
\]

This is greatest at \( \tilde{t} = 0 \) exactly. Note that there is no time lag between the wave arriving closest to the axis—most of the energy never gets nearer than \( \tilde{\rho} \simeq 0.4 \) (see figure 1 at \( \tilde{t} = 0 \))—and the resulting rotation of inertial frames. Just as in the work of Lindblom and Brill [16] on inertia induction from falling rotating spheres, the non-local effect appears instantaneously when suitable Machian coordinates (gauge) are chosen. This is a consequence of Mach’s principle being embodied in the ‘instantaneous’ constraint equations of general relativity. In electricity, the presence of a charge within a sphere at time \( t \) can be ascertained ‘instantaneously’ by measuring the flux of its field out of a surrounding surface. You do not have to wait for the field to come out from the charge—it is there already.

Returning to our inertial frame on the axis, a rod lying peacefully at the origin, initially oriented at \( \phi = \phi_0 \), will remain in the flat space near there and feel no torque, but, due to the rotation of the inertial frame it will point towards \( \phi = \phi(t) \) where

\[
\phi(t) = \phi_0 + \int_{-\infty}^t \langle \omega \rangle_o \, dt = \phi_0 + B^2 \frac{(2m + 1)!}{2^{2m}} \left[ \arctan \tilde{t} + \frac{\pi}{2} + \frac{\tilde{t}}{(2m + 1)(1 + \tilde{t}^2)} \right]. \tag{3.2}
\]

When the whole pulse has passed it ends up oriented towards

\[
\phi(\infty) = \phi_0 + B^2 \frac{(2m + 1)!}{2^{2m}} \cdot \pi. \tag{3.3}
\]

For symmetry we choose

\[
\phi_0 = -\frac{B^2 (2m + 1)!}{2^{2m}} \cdot \frac{\pi}{2}, \tag{3.4}
\]

so that \( \phi = 0 \) when \( \tilde{t} = 0 \) and

\[
\phi(t) = B^2 \frac{(2m + 1)!}{2^{2m}} \left[ \arctan \tilde{t} + \frac{\tilde{t}}{(2m + 1)(1 + \tilde{t}^2)} \right]. \tag{3.5}
\]

Figure 2 shows \( \phi(t) \) for \( t \gg 0 \). It is antisymmetric about \( t = 0 \).
Far from the axis we have from (2.18) and (A.9)

$$\langle \omega \rangle \simeq \frac{B^2}{a} \cdot \frac{m(m!)(2m-1)!!}{2^{m-1}} \cdot \frac{1}{\tilde{\rho}^2}, \quad \tilde{\rho} \gg 1. \quad (3.6)$$

As expected this is constant in time for \( \tilde{\rho} \) large, but falls off like \( \tilde{\rho}^{-2} \) as befits a cylindrical system with a total angular momentum per unit height, cf [4],

$$J = \frac{\pi \kappa a B^2}{m} \cdot \frac{m(m!)(2m-1)!!}{2^{m-2}}. \quad (3.7)$$

For general \( \tilde{t} \) and \( \tilde{\rho} \), \( \langle \omega \rangle \) is given in (2.18) with \( I_m \) and \( H_m \) from (A.6) and (A.12). But for \( \tilde{\rho}^2 \ll 1 + \tilde{t}^2 \) we need to show that our space is almost flat so that \( \langle \omega \rangle \) can be properly interpreted as a rotation of the inertial frame. We therefore evaluate \( \langle \omega \rangle \) not just at \( \tilde{\rho} = 0 \), but in this larger region, and show that at each time \( \langle \omega \rangle \) varies weakly with \( \tilde{\rho} \) there. Indeed from (A.17) and (A.18)

$$\langle \omega \rangle \simeq \langle \omega \rangle_0 \left[ 1 - \frac{(2m-1)!!}{(m+1)!} \cdot \frac{1}{(1 + \tilde{t}^2)^{m+1}} \cdot \frac{(2m-5) + (2m+7)\tilde{t}^2}{(1 + \tilde{t}^2)} \cdot \left( \frac{\tilde{\rho}^2}{1 + \tilde{t}^2} \right)^m \right]. \quad (3.8)$$

Figures 3 and 4 show \( j, \kappa \) times the angular momentum density, and the resulting \( \langle \omega(t, \rho) \rangle \).

The fact that the source terms in (2.3) all vary as \( \tilde{\rho}^{2m} \) shows that the other terms in the metric will vary similarly, so wherever \( [\tilde{\rho}^2/(1 + \tilde{t}^2)]^m \) is negligible our second-order metric will take the form

$$ds^2 = e^{2\psi} \left[ dt^2 - d\rho^2 - \rho^2 (d\varphi - \langle \omega \rangle_0 (dr)^2) \right] - e^{-2\psi} \, dz^2, \quad \tilde{\rho}^2 \ll (1 + \tilde{t}^2) \quad (3.9)$$

with \( \psi \) given by (2.10). If we now go to rotating axes by writing \( \tilde{\varphi} = \varphi - \varphi(t) \), then our metric becomes

$$ds^2 = e^{2\tilde{\varphi}} \left[ d\tilde{t}^2 - d\rho^2 - \rho^2 d\tilde{\varphi}^2 \right] - e^{-2\tilde{\varphi}} \, dz^2, \quad \tilde{\rho}^2 \ll (1 + \tilde{t}^2) \quad (3.10)$$

which is flat in the reduced space. Furthermore the wave itself never penetrates significantly within \( \tilde{\rho}/\sqrt{1 + \tilde{t}^2} \simeq 0.4 \).
Figure 3. The profile of the angular momentum density at $t \approx 0, 1$ and 2. Again the drawing is for $m = 10$.

Figure 4. The profile $\langle \omega \rangle$ as a function of $\tilde{\rho}$ at selected times $\tilde{t} = 0, 0.5, 1$ and 2. Note the still flat uniformly rotating central region when $\tilde{t} = 0$.

4. Angular momentum transport by gravitational torques

Those who have studied angular momentum transport in spiral galaxies, know that the classical gravitational stress tensor is [17, 25]

$$\sigma_{kl} = \frac{1}{\kappa} \left( 2 \partial_k \psi \partial_l \psi - \delta_{kl} \sum_m |\partial_m \psi|^2 \right), \quad (4.1)$$

where $\psi$ is the classical gravitational potential. The gravitational couple transferring angular momentum outwards through a cylinder of radius $\rho$ is

$$C_g = \int \epsilon_{3kl} X^k (\sigma^{lm} dS_m), \quad (4.2)$$

where $d\tilde{S}$ is the outward pointing surface element and $\tilde{X}$ the radius vector from the axis. Evidently this reduces to

$$C_g = \frac{2}{\kappa} \int \partial_\rho \psi \partial_\psi \rho \, d\psi \, dz. \quad (4.3)$$
For these gravitational stresses to carry angular momentum outwards there must be a positive correlation between $\partial_\rho \psi$ and $\partial_\phi \psi$ averaged over such a cylinder. Such a correlation gives a trailing sense of spirality to contours of constant $\psi$ in the sense that the outer parts of a spiral galaxy trail that of the inner parts in the sense defined by the rotation. It is no different here; for angular momentum transport outward, the spiral formed by contours of $\psi$ must trail, but contrariwise when they form a leading spiral with the outside further advanced than the inside, the angular momentum is transported inward. Thus our knowledge of spiral structure in galaxies enabled us to predict that there would be leading spirals when the wave was going in and therefore transporting its angular momentum inward, but trailing spirals when our gravitational wave was receding from the axis. At $t = 0$ there should be no angular momentum transport so the contours of $\psi$ should form a rotating cartwheel structure with no spirality. These expectations were beautifully confirmed when we plotted the contours of $\psi$ using Mathematica. The waves always rotate counterclockwise in figure 5. They start as quite tightly wrapped leading spirals; as they move inwards they keep rotating but the wrapping becomes less tight. At $t = 0$ they unwrap to form a rotating cartwheel structure and thereafter they move outward as trailing spirals carrying angular momentum with them. It is when they are closest in that their rotation causes the greatest rotation of the inertial frame in the central region, though the waves themselves never reach much further in than $\tilde{\rho} = 1$. Nevertheless the second-order effect of the angular momentum causes the rotation of the inertial frame within. This is illustrated by the direction of the central arrow that shows the orientation of a rod initially at rest at the origin. Because this effect is second order, the rotation seen in the figure is dependent on the amplitude of the wave, whereas the spiral shapes are not.

5. Energy and angular momentum

In section 3, we used the asymptotic form of $\langle \omega \rangle$ to deduce the total angular momentum per unit height but we could also calculate it via the angular momentum density $j/\kappa$

$$J = \frac{1}{\kappa} \int_0^\infty \frac{\pi}{2} \rho \, d\rho = \frac{4\pi}{\kappa} aB^2 m[(2m - 1)!!]^2[mI_{m-1} + (2m + 1)\tilde{r}^2I_m]$$

$$= \frac{\pi}{\kappa} aB^2 \frac{m(m!)^2(2m - 1)!!}{2^{m-1}}. \quad (5.1)$$

Any quantity $\psi$ that obeys $\Box \psi = 0$ automatically conserves both $J$ and the quantity

$$\frac{1}{\kappa} \int (\psi^2 + |\vec{\nabla}\psi|^2) \, dV. \quad (5.2)$$

The identity

$$\frac{1}{\kappa} \partial_\rho (\psi^2 + |\vec{\nabla}\psi|^2) = \frac{1}{\kappa} \vec{\nabla} \cdot (2\psi \vec{\nabla}\psi) \quad (5.3)$$

follows directly from the wave equation and can be interpreted as a conservation law with $\frac{1}{\kappa} \psi \vec{\nabla}\psi \cdot d\vec{S}$ being the flux through an element of the closed surface surrounding a volume $V$. We note that the flux vector now directed through a surface $\varphi = \text{const}$ will on multiplication by $\rho$ give the angular momentum density $j/\kappa$. Integrating $\frac{1}{\kappa} (\psi^2 + |\vec{\nabla}\psi|^2)$ over a volume of unit height we find the grand total and using the formulae of the appendix,

$$\int_0^\infty \int_0^{2\pi} \frac{1}{\kappa} (\psi^2 + |\vec{\nabla}\psi|^2) \, d\rho \, d\varphi = \frac{\pi}{\kappa} B^2 \frac{m!(2m + 1)!!}{2^{m-1}}, \quad (5.4)$$

which is independent of time as expected. We may identify this conserved quantity with energy per unit length via the following argument. The conservation laws $\nabla_\mu T^\mu_\nu = 0$, see [4].

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Figure 5. This shows the $m = 10$ wave which always rotates anticlockwise. As it comes inwards (a) at $\tilde{t} = -2$ it is in the form of a leading spiral with the outer parts of the arms ahead of the central parts. By $\tilde{t} = -1$ (b) the spiral has started to open. By $\tilde{t} = 0$ (c) the central parts have caught up and the spiral has changed to a cartwheel structure but rotation keeps it beyond $\tilde{\rho} \simeq 0.4$. By $\tilde{t} = 1$ (d) the spiral has become trailing as befits a wave that now feeds angular momentum outwards. By $\tilde{t} = 2$ (e) the spiral becomes tighter and the flat central cylinder becomes larger. We show $\tilde{t} = 10$ (f) at a small scale but note the beautiful tight wrapping of the narrow arms. Also note the opposite spirality of the conjugate pairs $\tilde{t} = \pm 2$ and $\tilde{t} = \pm 1$. Figures encompass a radius $\tilde{\rho} \simeq 7$ ($\tilde{\rho} \simeq 17$ for $\tilde{t} = 10$). The height of $\psi/B$ reduced by a factor of $10^{-4}$ is between 0 and 1. Lighting falls at 45° from the left. The view is along the $z$ axis from above at a distance of $10^{-4} \psi/\tilde{B} = 40$. 
in the reduced $(2+1)$ space follow from its contracted Bianchi identities of which the $x^0 = t$

one reads as

$$\frac{1}{\rho} \left[(\rho T^0_0) + (\rho T^1_0)^\prime + \partial_q (\rho T^2_0)\right] = 0. \quad (5.5)$$

Ordinary derivatives in flat space have replaced covariant ones because $T^b_a$ is already of second

order in the wave amplitude:

$$\kappa T_{ab} = R_{ab} - \frac{1}{2} g_{ab} R = 2 \partial_a \psi \partial_b \psi - g_{ab} (\psi^2 - |\tilde{\nabla} \psi|^2). \quad (5.6)$$

where $\tilde{\nabla} \psi$ is the normal flat space gradient in $\rho$ and $\varphi$.

$$|\tilde{\nabla} \psi|^2 = \psi^2 + \frac{1}{\rho^2} (\partial_\varphi \psi)^2. \quad (5.7)$$

From (5.6) we see that

$$\kappa T^0_0 = \psi^2 + |\tilde{\nabla} \psi|^2 \quad (5.8)$$

which is the density of our conserved quantity. The total energy per unit length as measured from the conicity at infinity, $\gamma_{\infty}$, was shown to be $2 \pi \gamma_{\infty}/\kappa$ by [1].

In [4], equation (5.31), we show that $2 \pi \gamma_{\infty}/\kappa$ is given by the left-hand side of (5.4). This completes the identification of our conserved quantity with the energy per unit height. It also shows that the wave produces a long-range effect on $\gamma$ far from the region $\rho \simeq t$ where its main amplitude lies. It is one of the strange properties of cylindrical general relativity that this conicity is independent of $\rho$ outside the wave. This has the effect that while particle trajectories are globally bent by the defect angle nevertheless there are no orbits along finite sections of circles (about the axis) caused by the long distance gravity of the cylindrical wave. Rather the geodesics are like the geodesics on a cone all of which go to infinity. It would be interesting to calculate the gravity due to a more realistic three-dimensional wave pulse for which the odd behaviour at infinity found in $(2+1)$ dimensions would not obscure the interpretation.

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Appendix. Evaluation of the integrals $I_m$ and $H_m$

$$I_m = \int_0^u u^m Q^{-m-3/2} \, du, \quad H_m = \int_u^\infty u^{m-1} Q^{-m-3/2} \, du. \quad (A.1)$$

We write

$$Q = u^2 + 2bu + c^2, \quad \text{where} \quad b = 1 - \tilde{t}^2 \quad \text{and} \quad c = 1 + \tilde{t}^2, \quad (A.2)$$

so $c \geq b$. Then

$$\int_0^u Q^{-1/2} \, du = \arcsin \left( \frac{u + b}{\sqrt{c^2 - b^2}} \right) - \arcsin \left( \frac{-b}{\sqrt{c^2 - b^2}} \right). \quad (A.3)$$
and
\[ \int_0^u Q^{-3/2} \, du = -\frac{2}{\partial c^2} \left( \int_0^u Q^{-1/2} \, du \right) = \frac{u+b}{c^2-b^2} Q^{-1/2} - \frac{b/c}{c^2-b^2} \equiv X(b, u). \] (A.4)

Now differentiate \( m \) times with respect to \( b \):
\[ (-1)^m (2m+1)!! \int_0^u u^m Q^{-m-3/2} \, du = \frac{\partial^m X(b, u)}{\partial b^m}. \] (A.5)

So
\[ I_m(u) = \frac{(-1)^m}{(2m+1)!!} \frac{\partial^m X(b, u)}{\partial b^m}, \] (A.6)

where in this expression \( c = 1 + \tilde{r}^2 \) and, after differentiation \( b = 1 - \tilde{r}^2 \). Now from (A.4)
\[ X(b, \infty) = \frac{1 - b/c}{c^2-b^2} = \frac{1}{c(c+b)}, \] (A.7)

so
\[ I_m(\infty) = \frac{m!}{(2m+1)!!} \cdot \frac{1}{c(c+b)^{m+1}}. \] (A.8)

For general \( b \) and \( u \)
\[ \int_u^\infty Q^{-3/2} \, du = \frac{1}{c^2-b^2} [1 - (u+b) Q^{-1/2}]. \] (A.9)

Differentiating with respect to \( c^2 \) and multiplying by \(-2/3\):
\[ \int_u^\infty Q^{-5/2} \, du = \frac{1}{3(c^2-b^2)^2} \left[ 2 - \frac{(u+b)(2Q+c^2-b^2)}{Q^{3/2}} \right] \equiv Y(b, u). \] (A.10)

Differentiating \( m - 1 \) times with respect to \( b \) we find
\[ (-1)^{m-1} (2m+1)!! \int_u^\infty u^{m-1} Q^{-m-3/2} \, du = \frac{\partial^{m-1} Y(b, u)}{\partial b^{m-1}}. \] (A.11)

Hence,
\[ H_m = \frac{3(-1)^{m-1}}{(2m+1)!!} \frac{\partial^{m-1} Y(b, u)}{\partial b^{m-1}}, \] (A.12)

again \( b \) must be set equal to \( 1 - \tilde{r}^2 \) after the differentiation. From (A.10) we see that
\[ Y(b, 0) = \frac{1}{3c^2} \left[ \frac{1}{c(c+b)} + \frac{1}{(c+b)^2} \right]. \] (A.13)

So
\[ (-1)^{m-1} \frac{\partial^{m-1} Y(b, 0)}{\partial b^{m-1}} = \frac{1}{3c^2} \left[ (m-1)! \frac{(m-1)!}{c(c+b)^m} + \frac{m!}{(c+b)^{m+1}} \right]. \] (A.14)

and
\[ H_m(0) = \frac{(m-1)!}{(2m+1)!!} \cdot \frac{b + (m+1)c}{c^3(c+b)^{m+1}}. \] (A.15)

Whereas (A.6) and (A.12) give \( I_m \) and \( H_m \) for all values of \( u \), (A.8) and (A.15) give their values at \( 0 \) and \( \infty \), nevertheless it is useful to view their behaviours at small and at large \( u \) explicitly. In particular for \( \tilde{r}^2 \ll \frac{1}{2}(1 + \tilde{r}^2) \), the region encompassed by the wave but not ‘feeling’ it, we may expand \( Q^{-m-3/2} \) in the form
\[ Q^{-m-3/2} \simeq \frac{1}{c^{2m+3}} \left[ 1 - (2m-3) \frac{bu}{c^2} \right], \quad u \ll c. \] (A.16)
Then
\[ I_m(u) \simeq \frac{u^{m+1}}{c^{2m+3}} \left[ \frac{1}{m+1} - \frac{2m-3}{m+2} \cdot \frac{bu}{c^2} \right], \quad u \ll c, \quad (A.17) \]
and likewise
\[ H_m(u) \simeq H_m(0) - \frac{u^m}{c^{2m+3}} \left[ \frac{1}{m+1} - \frac{2m-3}{m+2} \cdot \frac{bu}{c^2} \right], \quad u \ll c. \quad (A.18) \]
Likewise for \( \tilde{\rho}^2 \gg \frac{1}{2}(1 + \bar{\ell}^2) \) which is \( u \gg c \)
\[ Q^{-m-3/2} \simeq \frac{1}{u^{2m+3}} \left[ 1 - (2m-3) \frac{b}{u} \right], \quad u \gg c, \quad (A.19) \]
so
\[ I_m(u) \simeq I_m(\infty) - \frac{1}{u^{m+2}} \left[ \frac{1}{m+2} - \frac{2m-3}{m+3} \cdot \frac{b}{u} \right], \quad u \gg c, \quad (A.20) \]
and
\[ H_m(u) \simeq \frac{1}{u^{m+3}} \left[ \frac{1}{m+3} - \frac{2m-3}{m+4} \cdot \frac{b}{u} \right], \quad u \gg c. \quad (A.21) \]
Finally when \( b = c, Q \) becomes a perfect square so the integrals are easier; since both \( b \) and \( c \) are then 1 we express the answer that way:
\[ I_m = \sum_{s=0}^{m} (-1)^{m-s} \binom{m}{s} \left( 1 + u \right)^{s-2m-2} - 1 \left( s - 2m - 2 \right) \]
\[ H_m = \sum_{s=0}^{m-1} (-1)^{m-s} \binom{m}{s} \left( 1 + u \right)^{s-2m-2} \left( s - 2m - 2 \right). \quad (A.22) \]

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