Higher Derivative Corrections to Lower Order RG Flow Equations

S. P. de Alwis†
Physics Department, University of Colorado,
Boulder, CO 80309 USA

Abstract

We show that the RG flow equation for the cosmological constant (CC) receives contributions (in addition to those coming from the CC the Einstein-Hilbert term and $R^2$ and $R_{\mu\nu}$ terms) only from terms with just two powers of curvature, but having also powers of the covariant derivative, in the Wilsonian effective action. In pure gravity our argument implies that just considering $f(R)$ theories will miss this effect which arises from terms such as $"R^n\Box^nR"," n = 0, 1, 2, . . .$. We expect similar contributions for the flow equation of the Einstein-Hilbert term as well. Finally we argue that the perturbative ghosts coming from curvature squared terms in the action are in fact spurious since they are at the cutoff scale and can be removed by (cutoff dependent) field redefinitions.

† dealwiss@colorado.edu
1 Introduction

The Wilsonian effective action \[^1\] with a cutoff scale \(\Lambda\) for a QFT (including quantum gravity with or without coupling to matter), evaluated at some momentum or curvature scale \(\partial^2 \phi / \phi, R \ll \Lambda^2\) may be written as an infinite series of local operators

\[
I_\Lambda = \sum \tilde{g}^A(\Lambda) \Phi_A[\phi]. \tag{1}
\]

The conceit of the asymptotic safety (AS) program is that this action is ultra-violet complete in the sense that the limit \(\Lambda \to \infty\) exists, and that only a finite (and hopefully small) number of these operators need to be determined by experiment. In order to validate this conjecture it is necessary to use a convenient form of the renormalization group and a truncation of this infinite set of terms to a manageable set.

While there are alternative versions of the renormalization group equation\[^2\] for the Wilsonian action, a particularly convenient formulation that is useful for studying the question of asymptotic safety\[^3\], close in spirit to Polchinski’s equation\[^4\] was derived in\[^5\]. In this short note we will use this equation to argue that the RG equation for the cosmological constant (and plausibly also the gravitational coupling constant) does not acquire corrections from higher derivative terms such as \(R^N, R^N_{\mu\nu}, N > 2\). In explicit calculations in a different scheme using the Wetterich equation\[^6\] up to \(N \sim 30\) for Ricci scalar terms, these corrections have been found to be suppressed\[^7\]. However we also point out that terms which will contribute are those such as \(R\Box^N R\), which (except for the case \(N = 0\)) have not been computed so far\[^8\]. Finally we discuss the elimination of (perturbative) ghosts typically associated with curvature squared terms\[^9\] by a (scale dependent) redefinition of the metric, which will effectively eliminate additional contributions to the propagator since all \(“R^2”\) terms can be removed from the action. Essentially what this shows is that since the putative ghost is at the cutoff scale it is in fact an artifact of the cutoff. This is of course consistent with what happens in string theory and in fact we borrow an argument due to Deser and Redlich\[^10\] given in that context.

2 Pure gravity case

Let us write the Wilsonian action with a UV cutoff \(\Lambda\) for the pure gravity case as an infinite series

\[
I^\text{grav}_\Lambda = \int d^4 x \sqrt{g}[\Lambda^4 g_0(\Lambda) + \Lambda^2 g_1(\Lambda) R + (g_2(\Lambda) R_{\mu\nu} R^{\mu\nu} + g_2(\Lambda) R^2 + g_3 R \Box R \ldots) \\
+ \Lambda^{-2}[(g_3 a(\Lambda) R R_{\mu\nu} R^{\mu\nu} + \ldots) + (g_3 a(1) R \Box R + \ldots) + O(\Lambda^{-4})] \\
+ I^{(\text{G.F.})}_\Lambda + I^{(\text{ghost})}_\Lambda. \tag{2}
\]

One might ask whether this series is convergent or merely an asymptotic expansion. This question obviously arises even if the UV limit of all the couplings of the theory exist. In the corresponding

\[^1\]For reviews see\[^2\][3][4].

\[^2\]For reviews of the recent literature see\[^7\][8][9].

\[^3\]See for example\[^13\] (and references therein) where it is also speculated that this suppression extends to all higher derivative terms.

\[^4\]Theories with higher derivatives of this form seem to have been first considered in\[^14\] where a calculation of the cosmological beta function can also be found. A suggestion for performing calculations beyond the constant curvature background calculations given in the asymptotic safety literature (as for example in\[^3\]) was made recently in\[^15\].
expansion in string theory it is in fact a convergent expansion with a radius of convergence of the order of the string scale. It is essentially like the expansion of the propagators \(1/(p^2 + M_s^2)\) in powers of \(p^2/M_s^2\) at low energies (with \(M_s\) the mass of a string excitation). For \(p^2 > M_s^2\) of course one needs to replace the field theory by string theory.

In the AS case one might ask what is this infinite series an expansion of. Indeed it is not clear to the author whether this question is even meaningful. We will nevertheless assume that the series makes sense (converges) for \(\partial^2/\Lambda^2 < 1\) and the UV limit will be taken satisfying this constraint.

The coupling constants satisfy RG equations of the form

\[
\dot{g}^A + (4 - n_A)g^A = \eta^A(\{g\}),
\]

where \(n_A\) is the canonical dimension of the corresponding operator and the LHS is in principle dependent on all the couplings in the effective action. If we truncate the system to the first two operators then the RG equations are (with \(\dot{x} \equiv \Lambda dx/d\Lambda\))

\[
\dot{g}_0 + 4g_0 = \frac{1}{(4\pi)^2}\left[10e^{-g_0/g_1} - 4\right],
\]

\[
\dot{g}_1 + 2g_1 = -\frac{1}{(4\pi)^2}\frac{1}{3}\left[13e^{-g_0/g_1} + 5\right].
\]

Now it has been observed “experimentally” (i.e. by explicit calculation in truncated theories), that these equations are corrected by the addition of \(R \ldots R \ldots R \ldots \ldots R^2\) terms but are remarkably stable under the addition of higher powers of the Ricci scalar, i.e. for a gravitational theory of the form \(f(R)\).

Let us first consider the higher derivative corrections to the CC equation since they can be computed in flat space. We will also ignore gauge fixing and ghost action terms since they are irrelevant for the point we wish to make. As in our previous work [11] we will use the equation

\[
\Lambda \frac{d}{d\Lambda} I_\Lambda[\phi_c] = \frac{1}{2} k \frac{d}{dk} \text{Tr} \ln K_{k,\Lambda}[\phi_c]|_{k=\Lambda} = \text{Tr} \exp\left\{-\frac{1}{\Lambda^2} \frac{\delta}{\delta \phi_c} \otimes \frac{\delta}{\delta \phi_c} I_\Lambda[\phi_c]\right\},
\]

to calculate the contributions of higher derivative terms in (2) to the RHS of (4). These terms can be calculated without invoking the full machinery of the covariant heat kernel expansion (and its generalizations). This is because after performing the differentiation on the RHS of (4), the curvatures can be set to zero, as what we need on the LHS is the coefficient of the unit operator.

Let us begin with the flat space heat kernel,

\[
H_0(x, y) = \langle x|e^{-s_0\hat{p}^2}|y\rangle = \frac{1}{(2\pi)^4} \left(\frac{\pi}{s_0}\right)^2 e^{-((x-y)^2)/s_0}; \quad s_0 \equiv \frac{1}{\Lambda^2}, \quad \hat{p} = -i\partial
\]

The last step was taken by going to momentum space and doing a Gaussian integral. This simple relation is the basis of the heat kernel expansion in curved space but its applicability is clearly restricted to two derivative kinetic terms. However here we need to evaluate the heat kernel when the kinetic term contains all higher powers of \(\hat{p}^2\) (as well as more complicated operators which we ignore here). Consider then an operator of the form

\[
\hat{K} = \hat{p}^2 + s_0h_1(\hat{p}^2)^2 + s_0^2h_2(\hat{p}^2)^3 + \ldots = \hat{p}^2 + \sum_{n=1}^{\infty} s_0^n h_n(\hat{p}^2)^{n+1}.
\]
Now we may write

\[ H(x; x) = < x | e^{-s_0 \hat{K}} | x > = e^{-\sum_{n=1}^{\infty} s_0^{n+1} h_n \frac{\partial^{n+1}}{\partial s^{n+1}}} < x | e^{-\hat{p}^2} | x > |_{s=s_0} \]

\[ = e^{-\sum_{n=1}^{\infty} s_0^{n+1} h_n \frac{\partial^{n+1}}{\partial s^{n+1}}} \frac{1}{(2\pi)^4} \left( \frac{\pi}{s} \right)^2 |_{s=s_0} \equiv \frac{1}{16\pi^2} G(h) \Lambda^4. \]  

(8)

Suppose we are interested in the contributions to the RHS of the RG equation for the CC - namely the equation (4). Then in calculating the relevant terms in \( \hat{K} \) we can set the background fields (in particular the curvatures) to zero after performing the differentiation that define the background field dependent kinetic operator. Schematically this is tantamount to setting \( \delta R \sim \kappa \Box \delta g \) and taking the limit \( R \to 0 \) after differentiation. Clearly the contributions to \( \hat{K} \) will only come from the Einstein-Hilbert term, the “\( R^2 \)” terms and terms of the form \( R \Box^n R \). In particular none of the higher than quadratic powers of a truncation of the form \( f(R) = \sum a_n R^n \), will contribute to the RG of the CC.

Thus the RG equation for the CC is corrected from (4) to be of the form

\[ \dot{g}_0 + 4g_0 = \frac{1}{16\pi^2} [10e^{-g_0/g_1} - 4 + e^{-\lambda_1} + G(h)]. \]  

(9)

This then explains why in \( f(R) \) theories the RG equation for the CC is not affected beyond the \( R^2 \) terms. We expect a similar result to hold for Newton’s constant \( G_N \) though in that case the flat space argument that we gave above will need to be modified. For instance in analogy with the above argument one expects the class of operators of the form \( k_n R^2 \Box^n R \) to contribute to the beta function equation for \( g_1 \) modifying (5) to

\[ \dot{g}_1 + 2g_1 = -\frac{1}{(4\pi)^2} \frac{1}{3} [13e^{-g_0/g_1} + 5 + H(k)], \]  

(10)

where \( H(k) \) is in principle a computable function of the set \( \{k_n\} \) for which however we are unable at this point to give an explicit expression like (8).

In other words the above calculation suggests that the RG equations (and hence the non-trivial fixed points) for the lowest order operators in Einstein gravity with a cosmological constant, may be seriously affected once one includes higher derivative operators of the above classes. Of course it may turn out that the fixed point values of the couplings \( h_n, k_n \) are indeed small as was the case with couplings of other higher dimensional operators. It is nevertheless important check this directly since \textit{a priori} one would expect them to be \( O(1) \) numbers.

Note that the issue here is not the scaling dimensions of these higher dimensional operators near the UV fixed point. Indeed it may well be that these are essentially given by their canonical values with small corrections so that all these operators would then be irrelevant. Thus their couplings would have to be set equal to their fixed point values so as to be on the critical surface. However if these values are of \( O(1) \) then it would be difficult to establish the existence of an RG trajectory joining the UV fixed point of the theory to the IR fixed point where current experiments are done.

### 3 Scalar field theory

Let us now add a (scalar) matter action of the form

\[ I_{\Lambda}^{\text{matter}} = \int d^4x \sqrt{g} [Z(\phi^2/\Lambda^2) \frac{1}{2} \phi(-\Box)\phi + V(\phi, \Lambda) + \xi(\phi, \Lambda) R + O(\partial^4)], \]  

(11)
\[ V(\phi, \Lambda) = \frac{1}{2} \lambda_1(\Lambda) \Lambda^2 \phi^2 + \frac{1}{4!} \lambda_2(\Lambda) \phi^4 + \frac{1}{6!} \lambda_3(\Lambda) \Lambda^{-2} \phi^6 + \ldots, \] (12)

\[ Z \left( \frac{\phi^2}{\Lambda^2} \right) = Z_0 + \frac{1}{2} Z_1 \frac{\phi^2}{\Lambda^2} + \ldots \] (13)

\[ \xi(\phi, \Lambda) = \frac{1}{2} \xi_1 \phi^2 + \frac{1}{4!} \xi_2 \frac{\phi^4}{\Lambda^2} + \ldots \] (14)

The situation here is very similar to the case of pure gravity. If we ignore operators of the form \( \phi^n \Box^m \phi \) with \( m > 1 \) then the RG equation for the potential is not affected. But inclusion of these terms can seriously affect in principle the evolution of the couplings in the potential. Let us consider this in more detail.

In order to highlight the issue it is enough again to simply consider the flat space case. Consider a set of higher derivative terms of the form \( s_0 \equiv 1/\Lambda^2 \)

\[ \frac{1}{2} \phi \sum_{n=2}^{\infty} z_{n-1} s_0^{n-1} (-\Box)^n \phi. \]

The usual kinetic operator \( \hat{K} = \hat{p}^2 \) then gets replaced by

\[ \hat{K} = \hat{p}^2 + \sum_{n=1}^{\infty} z_n s_0^n (\hat{p}^2)^{n+1}. \]

Now if we ignore the higher derivative terms in \( \hat{K} \) we have the local potential equation for a scalar field theory;

\[ \Lambda \frac{d}{d\Lambda} V_\Lambda(\phi) = \frac{\Lambda^4}{16\pi^2} e^{-Z_0^{-1} V''(\phi)/\Lambda^2}. \] (15)

This gives an infinite set of RG equations for the couplings \( \hat{\lambda}_i \equiv \lambda_i/(Z_0)^i \) of (12) of the form

\[ \dot{\hat{\lambda}}_n + (n\gamma + (4 - 2n))\hat{\lambda}_n = (2n)! \frac{\Lambda^4}{16\pi^2} e^{-Z_0^{-1} V''(\phi)/\Lambda^2} \phi^{2n}, \] (16)

(with \( \gamma = \ln Z_0 \)) where the RHS carries an instruction to pick the coefficient of \( \phi^{2n} \) in the expansion of the exponential in powers of \( \phi^2 \). For instance we have for the first few couplings,

\[ \dot{\hat{\lambda}}_1 + (2 + \gamma)\hat{\lambda}_1 = -\frac{e^{-\hat{\lambda}_1}}{16\pi^2} \hat{\lambda}_2 \] (17)

\[ \dot{\hat{\lambda}}_2 + (0 + 2\gamma)\hat{\lambda}_2 = \frac{e^{-\hat{\lambda}_1}}{16\pi^2} (3\hat{\lambda}_2^2 - \hat{\lambda}_3), \] (18)

\[ \dot{\hat{\lambda}}_3 + (-2 + 3\gamma)\hat{\lambda}_3 = \frac{e^{-\hat{\lambda}_1}}{16\pi^2} 6!(\frac{1}{31} \hat{\lambda}_2^3 - \frac{1}{2} \frac{1}{4!} \hat{\lambda}_2 \hat{\lambda}_3 - \frac{1}{6!} \hat{\lambda}_4), \] (19)

etc. Thus it would appear that one is able to iteratively solve for a (non-trivial) fixed point \( (\hat{\lambda}_n = \lambda^*_n; \dot{\hat{\lambda}}_n = 0, \gamma = 0, \forall n \) in terms of \( \lambda_1^* \). For instance given a value \( \lambda_1 = \lambda_1^* \neq 0 \) the first

\[ ^5 \text{In general we should allow for the possibility that the anomalous dimension} \ \eta \equiv \frac{1}{2} \gamma \text{ at a fixed point is non-zero as in the Wilson-Fischer case, although it is zero at the trivial fixed point. We've ignored this possibility for simplicity since one needs to go beyond the local potential equation to discuss this.} \]
The equation above determines \( \dot{\lambda}_2 \), the second equation then determines \( \dot{\lambda}_3 \), the third \( \dot{\lambda}_4 \) and so on for all the couplings in the potential, which is therefore determined at the fixed point as an infinite series. This fixed point however depends on \( \lambda_1 \) which is arbitrary, and hence we appear to have a fixed line\(^6\) parametrized by \( \lambda_1 \).

However one cannot ignore the higher derivative terms - there is no reason \textit{a priori} to assume that dimensionless numbers \( z_n \) are small. The local potential equation gets replaced by (see equations (6)-(8)),

\[
\Lambda \frac{d}{d\Lambda} V_\lambda(\phi) = F(z) \frac{\Lambda^4}{16\pi^2} e^{-z_0^{-1}V''(\phi)/\Lambda^2}, \quad F(z) \frac{\Lambda^4}{16\pi^2} \equiv e^{-\sum_{n=1}^{\infty} z_0^n z_n \frac{g^n}{g_n} + 1} \left( \frac{\pi}{s} \right)^2 |_{s = s_0 = 1/\Lambda^2}.
\]

Thus without any knowledge of the (infinite) number of higher derivative couplings \( z_n \), one cannot really extract any useful information from this (exact!) equation. In particular the fixed line that appears to exist in the iterative solution that ignored the higher derivative couplings, may be destabilized. One cannot really evaluate the potential at the fixed point without knowing the function \( F(z^*) \) at the unknown fixed point values of all these higher derivative couplings - which may or may not have finite values!

### 3.1 Coupling to gravity

Now let us consider the ramifications of the above, firstly to scalar field theory coupled to gravity, and then to the standard model coupled to gravity.

Once the coupling to gravity is included the evolution of scalar field couplings gets modified. The mass term and the quartic coupling equations (17)-(18) for instance, get replaced by

\[
\begin{align*}
\dot{\lambda}_1 + \gamma \dot{\lambda}_1 + 2 \dot{\lambda}_1 &= -e^{-\lambda_1/16\pi^2} F(z) \left[ \frac{\lambda_2}{2} - \frac{1}{8} g_N \dot{\lambda}_1^2 \right] + \frac{5}{(4\pi)^2} g_N e^{2\lambda_{CC}} \gamma \dot{\lambda}_1, \\
\frac{1}{4!} (\dot{\lambda}_2 + \gamma \dot{\lambda}_2) &= -e^{-\lambda_1/16\pi^2} F(z) \left[ \frac{1}{8} \lambda_2 - \frac{1}{4!} \dot{\lambda}_3 + \frac{1}{3} g_1 \dot{\lambda}_2 \dot{\lambda}_1 \right] + \frac{4 g_N e^{2\lambda_{CC}}}{16\pi^2} \frac{1}{4!} \dot{\lambda}_2.
\end{align*}
\]

In the above we have defined \( 16\pi G_N(\Lambda) \Lambda^2 \equiv g_N \equiv -1/g_1 \) the dimensionless Newton constant.

Similarly the gravitational equations acquire some matter contributions so that (1-5) are replaced by \( \lambda_{CC} \equiv g_N g_0 \).

\[
\begin{align*}
\dot{\lambda}_{CC} + 2 \lambda_{CC} &= \frac{g_N}{16\pi^2} \left[ (5 - \frac{13}{3} \lambda_{CC}) e^{2\lambda_{CC}} - (2 + \frac{5}{3} \lambda_{CC}) + e^{-\lambda_1/2} - \frac{1}{6} \lambda_{CC} (1 - 6 \hat{\xi}_1) \right] \\
&+ \frac{3 G(h) - 2 \lambda_{CC} H(k)}{6}, \\
g_N - 2 g_N &= - \frac{g_N^2}{16\pi^2} \frac{1}{3} \left[ 13 e^{2\lambda_{CC}} + 5 + \frac{1}{2} e^{-\lambda_1/2} (1 - 6 \hat{\xi}_1) + H(k) \right].
\end{align*}
\]

\(^6\)The investigation of scalar field theory in this so-called local potential approximation (LPA) is an old subject with conflicting claims. See for example [18][19] for the original dispute, and [20][21] for more recent work with references to the earlier literature. Clearly the infinite series for the potential at a non-trivial fixed point (with \( \lambda_1^* \neq 0 \)) will be convergent only for \( \hat{\phi} \equiv \phi/\Lambda < O(1) \). If on the other hand one requires the potential \( \hat{V} = V(\phi)/\Lambda^4 = V(\hat{\phi}) \) to exist for all \( \hat{\phi} \) then the continuous scaling dimension solutions implied by the above iterative analysis will not give a true solution and the theory will remain trivial [18][21]. The analysis below introduces an additional layer of uncertainty to the claims of [18][22].
If one ignores the higher derivative corrections $F, G, H$ that we discussed above, this system appears to admit a non-trivial fixed point. For instance the last two equations along with the (20) appear to have a fixed point solution in terms of two undetermined parameters $\hat{\lambda}_1, \lambda_{CC}$ which can be treated as free parameters.

However once one includes the above mentioned higher derivative operators these equations are corrected by the function $G(h)$ defined in (8), another function $H(k)$ of the couplings $k$ of the $R^nR^2$ type terms (see equation (10)), and the function $F(z)$ in (21). Unfortunately a priori there does not seem to be any reason why these extra terms should be negligible at a fixed point. In particular the existence of a (physically required) positive fixed point value $g_N^*$ would appear to depend on the sign and magnitude of $H(k^*)$. However it is perhaps reasonable to assume that the infinite series represented by $H(k^*)$ converges to an $O(1)$ number leaving the original conclusion about $g_N^*$ unchanged.

3.2 The standard model

One of the striking successes of the asymptotic safety program is the calculation of the Higgs mass [23]. This was in fact a prediction, since the calculation was done well before the Higgs was discovered at the LHC. So it is important to understand how this came about and what assumptions went into it. In particular one may worry that the series of higher derivative terms discussed above may affect this calculation and destroy its agreement with experiment.

However the prediction of the Higgs mass is actually unaffected by these extra terms provided the assumption of a non-trivial fixed point for the standard model coupled to gravity remains valid (with positive $g_N^*$), in the presence of these extra terms, and the sign of the gravitational correction to the beta function of the Higgs self-coupling remains unchanged. The latter is essentially the last term on the RHS of (21) and appears to be independent of the additional corrections discussed in this note. In particular of course the existence of a trajectory connecting the trivial IR fixed point to the non-trivial UV fixed should persist in the presence of these terms. If that is the case the crucial argument [23] that the Higgs self-coupling is irrelevant at the UV fixed point, so that its value should be set to its fixed point value namely zero, will persist and we would recover the prediction of the Higgs mass. The same appears to be true for the calculation of the top quark mass in [24].

4 Elimination of curvature squared terms and removing a ghost

Finally we would like to make some remarks about the curvature squared terms that we have considered in this paper and the issue of the spin two ghost that appears to be present in such theories [16].

It is well known that curvature squared terms $R^2, R_{\mu\nu}^2, R_{\mu\nu\lambda\sigma}^2$ terms can be replaced by the Euler density - which gives no contribution to the scattering amplitudes. This is accomplished by a (scale dependent in our case) (metric) field redefinition of the form

$$g_{\mu\nu} \rightarrow g_{\mu\nu} + b_0 R_{\mu\nu} + b_1 R g_{\mu\nu}$$

7In [23] it has been argued that higher dimension operators may not affect the AS results for standard model couplings.
which (with an appropriate choice of the coefficients $b_i$) will remove the Ricci scalar and tensor squared terms in equation (2) and replace the Riemann squared term by the Euler density. As is well known such a redefinition is expected to leave the physical results of the theory (the S-matrix) unchanged. Of course this does not mean that the RG equations for the coefficients $g_{2a}, g_{2b}, g_{2c}$ are eliminated since in deriving these equations we worked in a field basis which is cutoff independent - the cutoff dependence coming entirely from the coupling constants. However in order to accomplish the replacement of the quadratic terms by the Euler density, the field redefinition above clearly must be cutoff dependent - thus we can no longer take the redefined fields to be cutoff independent, hence the flow of these terms will contribute in a different way. Also this field redefinition will change all higher order couplings in a $b_i$ dependent way, though since the original action by definition must contain all terms which are consistent with the symmetries of the theory this will simply redefine the coefficients of these terms.

What is however somewhat less well known is the fact that all terms that are quadratic in curvature with additional derivatives can be eliminated by a (scale dependent) field redefinition. This appears to have been first observed by Deser and Redlich [17]. Essentially the argument is to observe first that by making repeated use of Bianchi identities any term in the action involving just two Riemann tensors, but also (necessarily even say $= 2n \neq 0$) powers of the covariant derivative, can always be written in the form

$$4 R_{\mu\nu} \Box^n R^{\mu\nu} - R \Box^n R + O(R^3) + \text{total derivatives}. \tag{24}$$

Thus the entire collection of terms in the action that are just quadratic in the curvatures (but with arbitrary numbers of derivatives), up to the Euler density term, total derivatives and higher powers of the curvature can be written in the form

$$\mathcal{L} \sim \sqrt{g}[a_1 R_{\mu\nu} R^{\mu\nu} + a_2 R^2 + \sum_{n=1} (a_1^{(n)} R_{\mu\nu} \Box^n R^{\mu\nu} + a_2^{(n)} R \Box^n R)]. \tag{24}$$

However all such terms can be removed by a generalization of the field redefinition (23) of the form,

$$g_{\mu\nu} \rightarrow g_{\mu\nu} + b_0 R_{\mu\nu} + b_1 R g_{\mu\nu} + \sum_{n=1} (b_1^{(n)} \Box^n R_{\mu\nu} + b_2^{(n)} \Box^n R). \tag{25}$$

Thus one can eliminate all curvature squared terms (apart from the Euler density which does not give any contribution to amplitudes in perturbation theory) from the Lagrangian. This implies that the graviton propagator in this transformed basis is the same as in the Einstein theory. Hence it appears that one could eliminate the perturbative spin two ghost found in $R^2$ theories by Stelle [16]. Similar arguments could be used to eliminate all higher derivative terms which are quadratic in the scalar field $\phi$ such as $\phi \Box^n \phi$.

Let us now discuss the relevance of this observation to the discussion of higher derivative contributions to the beta functions in this paper, and more generally to the AS program as a whole.

The coefficients $a_i^{(n)}$, $n \geq 0$ in equation (24) are cutoff dependent. Therefore the coefficients $b_i^{(n)}$ are necessarily cutoff dependent. However as mentioned earlier, when we computed the beta functions we worked in a basis where the fields (including the metric), were taken to be independent of the cutoff with the only dependence on $\Lambda$ coming from the coupling constants. If we assumed that for the old basis of fields, this is no longer true for the new basis so one would have to rederive the beta function equations with the metric now being dependent on the cutoff. Alternatively one may start with the basis in which the $R^2$ terms are absent - and take the fields to be independent
of the cutoff as before. However the RG evolution will generate these terms again. Clearly this is no different from saying that even if one took as an initial condition at some arbitrary scale the coefficients \( a_i^{(n)} \), \( n \geq 0 \) to be zero - the RG flow will generate them. Thus it is clear that the additional terms in the beta function equation for the cosmological constant (9)(22) will need to be taken into account. It is also clear from the above that similar arguments can be made for fields (such as the scalar field treated above) to remove the apparent ghosts that result from higher derivative terms that are quadratic in the fields and which are generated by RG flows even if set to zero at any given scale.

On the other hand the Deser-Redlich argument implies that at any arbitrary scale at which we do perturbation theory, one can always find a basis in which these ghosts [16] are absent. Hence in a theory which incorporates all higher derivative terms allowed by symmetries such ghosts are spurious. This means that this particular ghost problem is not an issue for the AS program. Similarly it is clear that the scalar ghost(s) coming from higher derivative terms that are quadratic in \( \phi \) are also spurious. Clearly this is a consequence of the fact that these ghosts appear at the cutoff scale, which if the AS conjecture is valid can be pushed to arbitrarily high scales.

It is important to point out that the problem identified in [16] relates to a theory with just the Einstein term and the curvature squared terms with no higher derivative terms. The reason the latter can be set to zero is that this theory is renormalizable. On the other hand if we make the field redefinition (23) to get rid of these terms to eliminate the additional (quartic) terms in the inverse propagator, we necessarily generate higher powers of curvature. This is because the coefficients of additional terms are not infinitesimal, so we are really doing a finite shift of the field variable resulting in an infinite functional Taylor series. Thus the price of removing the ghost is the loss of renormalizability and hence predictivity at high (close to Planck) scales - unless of course the theory is asymptotically safe!

A different way of making the same point is that in "\( R^2 \)" gravity theories one has fixed finite scale - the Planck scale. The aim is then to calculate scattering amplitudes not just at low energies but also at arbitrary energies \( E > M_P \). Since the theory is renormalizable this is possible but given that there is a spin two ghost at \( E = M_P \) the theory cannot be unitary. In the AS case the situation is radically different. Here the calculations that can be performed are always at energies \( E < \Lambda \) which is an arbitrary cutoff and if AS is valid can be taken to be arbitrarily large. The ghost in this formulation is always at the scale \( \Lambda \) and hence is beyond the regime of validity of the calculation - hence it is spurious.

When one has an infinite number of higher derivative terms, as will be the case when these ghosts are eliminated, and in any case is a given in the AS program, one is immediately confronted with questions of convergence and non-locality. In fact the original motivation of the Deser-Redlich paper was to understand how the low energy expansion of string theory avoids having perturbative ghosts as it must, since string theory (at least around asymptotically flat backgrounds) is free of ghosts. In string theory this infinite series is expected to sum up to a non-local string field theory. Unfortunately (in the closed string - i.e. gravity case) there is no closed form action (unlike for the open string). What we do have are expressions for the scattering amplitudes (involving gravitons) in a flat background giving well defined unitary analytic scattering amplitudes (in principle) to all orders in perturbation theory. It is unclear what the meaning of the corresponding infinite series in the AS program is, if the AS program is an alternative to, rather than a complementary way of

---

8The fact that the simplest background solution for string theory is flat 10 (or 26) dimensional space is being ignored in this discussion. Effectively what is being assumed is that the space has been compactified in to a 4d space with some six dimensional space with fixed moduli and or some abstract CFT with the appropriate central charge.
looking at, string theory.

On the other hand one may take a more pragmatic view of the whole program. In fact this appears to be the point of view taken by Weinberg (see for example [26]). In this approach suppose one wishes to calculate (say) the amplitude for a scattering process involving gravitons at some energy scale $E$. Then the cutoff $\Lambda$ should be chosen such that $\Lambda > E$ but not too much larger, so that the calculation to some low order in perturbation theory will suffice, assuming of course that the AS program is valid, and that the (finite number of) relevant couplings at this scale have been determined by the RG and the (infinite number of) irrelevant couplings are all determined by their fixed point values. Now the above discussion of ghost elimination means that we still need to find the field redefinition (at this value of the cutoff) that eliminates all the higher derivative propagator terms. In any practical calculation presumably one needs only to determine a finite number of such terms since (how many will depend on the desired accuracy of the calculation), all but a finite set can be ignored. Of course given that the S-matrix should be independent of field redefinitions all that the above argument establishes is that an apparent (perturbative) ghost at the cutoff scale is really a spurious state. Thus as long as one is working at an energy scale which is below this cutoff the scattering amplitudes should not be affected.

Finally let us stress that the elimination of the higher derivative quadratic in curvatures/fields terms which pertain to putative ghosts depends on an iterative procedure that is valid to arbitrarily high values of $\Lambda$ the cutoff, only under the assumption that the the large cutoff limit exists - i.e. there exists an UV fixed point. This of course is the fundamental assumption behind the AS program and while there is considerable evidence for this, it is far from being proven yet.

5 Acknowledgements

I wish to thank Astrid Eichhorn, Jan Pawlowski and Roberto Percacci for discussions and comments on the manuscript and Stanley Deser for comments on the last section. I also wish to thank Tim Morris for an e-mail on the controversy referred to in footnote (6). Finally I would like to acknowledge the hospitality of the Abdus Salam ICTP, Trieste, Italy, where some of this work was done and to thank the Dean of the College of Arts and Sciences at the University of Colorado for partial support of this research.

References

[1] K. G. Wilson and J. B. Kogut, Phys. Rept. 12, 75 (1974).
[2] T. R. Morris, Prog. Theor. Phys. Suppl. 131, 395 (1998), hep-th/9802039
[3] C. Bagnuls and C. Bervillier, Phys. Rept. 348, 91 (2001), hep-th/0002034
[4] O. J. Rosten, Phys. Rept. 511, 177 (2012), 1003.1366
[5] S. Weinberg, in "Critical Phenomena for Field Theorists" 14th International School of Sub-nuclear Physics: Understanding the Fundamental Constitutents of Matter Erice, Italy, July 23-August 8, 1976 (1976), p. 1.
[6] S. Weinberg, General Relativity, an Einstein Centenary Survey, S.W. Hawking and W. Israel (Eds) Cambridge Univ. Press (1979).
[7] A. Codello, R. Percacci, and C. Rahmede, Annals Phys. **324**, 414 (2009), [0805.2909](https://arxiv.org/abs/0805.2909).

[8] M. Reuter and F. Saueressig, New J. Phys. **14**, 055022 (2012), [1202.2274](https://arxiv.org/abs/1202.2274).

[9] R. Percacci, *Introduction to covariant quantum gravity and asymptotic safety* (World Scientific, 2017), ISBN 9813207175.

[10] J. Polchinski, Nucl. Phys. **B231**, 269 (1984).

[11] S. P. de Alwis, JHEP **03**, 118 (2018), [1707.09298](https://arxiv.org/abs/1707.09298).

[12] C. Wetterich, Phys. Lett. **B301**, 90 (1993).

[13] K. Falls, D. F. Litim, K. Nikolakopoulos, and C. Rahmede (2013), [1301.4191](https://arxiv.org/abs/1301.4191).

[14] M. Asorey, J. L. Lopez, and I. L. Shapiro, Int. J. Mod. Phys. **A12**, 5711 (1997), [hep-th/9610006](https://arxiv.org/abs/hep-th/9610006).

[15] M. S. Ruf and C. F. Steinwachs, Phys. Rev. **D97**, 044049 (2018), [1711.04785](https://arxiv.org/abs/1711.04785).

[16] K. S. Stelle, Phys. Rev. **D16**, 953 (1977).

[17] S. Deser and A. N. Redlich, Phys. Lett. **B176**, 350 (1986), [Erratum: Phys. Lett.B186,461(1987)].

[18] K. Halpern and K. Huang, Phys. Rev. **D53**, 3252 (1996), [hep-th/9510240](https://arxiv.org/abs/hep-th/9510240).

[19] T. R. Morris, Phys. Rev. Lett. **77**, 1658 (1996), [hep-th/9601128](https://arxiv.org/abs/hep-th/9601128).

[20] H. Gies, Phys. Rev. **D63**, 065011 (2001), [hep-th/0009041](https://arxiv.org/abs/hep-th/0009041).

[21] I. Hamzaan Bridle and T. R. Morris, Phys. Rev. **D94**, 065040 (2016), [1605.06075](https://arxiv.org/abs/1605.06075).

[22] K. Halpern and K. Huang, Phys. Rev. Lett. **77**, 1659 (1996).

[23] M. Shaposhnikov and C. Wetterich, Phys. Lett. **B683**, 196 (2010), [0912.0208](https://arxiv.org/abs/0912.0208).

[24] A. Eichhorn and A. Held, Phys. Lett. **B777**, 217 (2018), [1707.01107](https://arxiv.org/abs/1707.01107).

[25] A. Eichhorn and A. Held, Phys. Rev. **D96**, 086025 (2017), [1705.02342](https://arxiv.org/abs/1705.02342).

[26] S. Weinberg, Phys. Rev. **D81**, 083535 (2010), [0911.3165](https://arxiv.org/abs/0911.3165).