GLOBAL EXISTENCE AND ASYMPTOTIC BEHAVIOR OF SOLUTIONS TO A CHEMOTAXIS SYSTEM WITH CHEMICALS AND PREY-PREDATOR TERMS

MIHAELA NEGREANU
Depto. de Análisis Matemático y Matemática Aplicada
Universidad Complutense de Madrid, 28040 Madrid, Spain

(Communicated by Christina Surulescu)

Abstract. This paper is concerned with a general asymptotic stabilization of arbitrary global positive bounded solutions for the Lotka Volterra reaction diffusion systems, with an additional chemotactic influence and constant coefficients. We consider the dynamics of a mathematical model involving two biological species, both of which move according to random diffusion and are attracted/repulsed by chemical stimulus produced by the other. The biological species present the ability to orientate their movement towards the concentration of the chemical secreted by the other species. The nonlinear system consists of two parabolic equations with Lotka-Volterra-type kinetic terms coupled with chemotactic cross-diffusion, along with two elliptic equations describing the behavior of the chemicals. We prove that the solution to the corresponding Neumann initial boundary value problem is global and bounded for regular and positive initial data. Moreover, for different ranges of parameters, we show that any positive and bounded solution converges to a spatially constant homogeneous state.

1. Introduction. Chemotaxis is a biological process through which living organisms orientate their movement along a chemical concentration gradient; such a process is present in different types of biological phenomena as bacteria aggregation, immune system response or angiogenesis in the embryo formation and in tumor development. Many mathematicians study a chemotaxis system which describes a part of the life cycle of cellular slime molds with chemotaxis. For a broad survey on the progress of various chemotaxis models and a rich selection of references, we refer the reader to the survey papers [5], [16], [18], [45], [47]. Multi-species chemotaxis systems have been proposed and investigated after the pioneering work of Keller Segel [24], by e.g., [27], [28], [34], [38], [49], [50] and previous works can be found concerning the two biological species with chemotactic abilities problem.

The predator-prey system of Lotka-Volterra with intra-specific concurrence describes the problem of uncontrolled growth of prey in the absence of predators, so it adds a term that limits this growth and it has been studied in the last hundred years by different authors, from the pioneering works of Lotka 1925 and Volterra in 1926.
where the evolution of the species is given in terms of a system of two ODEs. The growth of predators is also limited by attaching to the corresponding equation an additional term. The formulation for the two-dimensional case, one of the important models in biology is as follows (see [20], [46]):

\[
\begin{cases}
\frac{du_1}{dt} = g_1(u_1, u_2), & t > 0, \\
\frac{du_2}{dt} = g_2(u_1, u_2), & t > 0,
\end{cases}
\]

with

\begin{align*}
g_1^{(p_1)}(u_1, u_2) &= u_1(a_{01} - a_{11}u_1^{p_1} - a_{12}u_2), \\
g_2^{(p_2)}(u_1, u_2) &= u_2(-a_{02} - a_{22}u_2^{p_2} + a_{21}u_1),
\end{align*}

where the coefficients \(a_{ij}\) (for \(i = 0, 1, 2\) and \(j = 1, 2\)) are positive given constants, for some positive constants \(p_i\), with \(i = 1, 2\).

If diffusion is considered, the problem becomes a PDEs system of two parabolic equations for the species \(u_1\) and \(u_2\), e.g., the Fisher–KPP equations

\[
\begin{cases}
\frac{\partial u_1}{\partial t} = \Delta u_1 + g_1^{(1)}(u_1, u_2), & x \in \Omega, \ t > 0, \\
\frac{\partial u_2}{\partial t} = \Delta u_2 + g_2^{(1)}(u_1, u_2), & x \in \Omega, \ t > 0.
\end{cases}
\]

To describe the spatial effects in the evolution of ecosystems in ecology, this mathematical model has been used. In (3), the functions \(g_i^{(1)}\), \(i = 1, 2\), describe intra- and inter-specific interactions of two species in an open, bounded and regular domain with smooth boundary \(\Omega \subset \mathbb{R}^n\), \(n \geq 1\). From a mathematical point of view, the system has been already studied for a large range of interactions \(g_i^{(1)}\), see, for example Pao [35], where the stability of (3) is obtained for a competitive case with \(g_i^{(1)} = u_i(a_{0i} - a_{1i}u_i - a_{2j}u_j)\) for constant coefficients \((a_{ij})_{i,j}\) for \(i, j = 1, 2\) and \(i \neq j\).

It is natural to consider the case where the parameters describing the amount of resources of the environment present some kind of periodicity in its asymptotic behavior. In [9] is studied system (3) with periodic in time coefficients \((a_{0i})_{i=1,2}\), satisfying the Gopalsamy condition (see [13]). Ahmad and Lazer in [1] extended these results to the periodic in time dependence of all the coefficients \(a_{ij}\) (for \(i = 0, 1, 2\) and \(j = 1, 2\)). If the coefficients have a periodic behavior in time and space, the existence of periodic solutions was proved in [11]. A generalization of these results to almost periodic functions for one and also to several species is obtained in [15] and [14]. The competitive case is also analyzed in [19].

The previous non exhaustive review of the two biological species with diffusive movement is only a small part of the large existing literature in the problem which shows the interest for the problem.

In nature, there exist common examples, where the biological species movement is oriented by chemicals gradients, “chemotaxis” where predator moves towards the prey. One finds different situations depending of the ability of the predator and the prey to orient their movement towards these chemical gradients: the predator is able to orient its movement towards the higher concentration of the chemical secreted by the prey; the prey can move away from the higher concentration of the predator (chemorepulsion).
In contrast to previous works studying Lotka-Volterra models featuring diffusion and taxis towards/away from chemical signals secreted by two competing populations [10], the present paper considers a predator-prey type of interactions, still in connection with taxis towards/away from chemical signals produced by the prey and the predator, respectively. Concretely, the following initial-boundary value problem is addressed:

\[
\begin{aligned}
\frac{\partial u_1}{\partial t} &= \Delta u_1 - \chi_1 \nabla \cdot (u_1^{m_1} \nabla v_1) + g_1^{(p_1)}(u_1, u_2), \quad x \in \Omega, \quad t > 0, \\
\frac{\partial u_2}{\partial t} &= \Delta u_2 - \chi_2 \nabla \cdot (u_2^{m_2} \nabla v_2) + g_2^{(p_2)}(u_1, u_2), \quad x \in \Omega, \quad t > 0, \\
-\Delta v_1 + \alpha_1 v_1 &= \beta_1 u_2, \quad x \in \Omega, \quad t > 0, \\
-\Delta v_2 + \alpha_2 v_2 &= \beta_2 u_1, \quad x \in \Omega, \quad t > 0, \\
(\nabla u_i - \chi_i u_i^{m_i} \nabla v_i) \cdot \nu &= 0, \quad \nabla v_i \cdot \nu = 0 \quad x \in \partial \Omega, \quad t > 0, \\
u_1(0, x) &= u_1^0(x), \quad u_2(0, x) = u_2^0(x), \quad x \in \Omega,
\end{aligned}
\]

where \(u_1\) and \(u_2\) denote population densities of the prey and the predator, respectively, \(v_1\) and \(v_2\) represent concentrations of the (chemical) signals they respectively produce; \(\alpha_i, \beta_i, m_i, p_i\), for \(i = 1, 2\) are positive constants and \(\Omega\) is a bounded, open regular domain of \(\mathbb{R}^n\), for \(n \geq 1\) with regular boundary; functions \(g_1^{(p_1)}, g_2^{(p_2)}\) are given by (1) and (2), respectively.

Thereby, \(\chi_i\) are the tactic coefficients: \(\chi_1 < 0\) and \(\chi_2 > 0\) model the situation where the prey is avoiding the predator by moving away from its signal gradient, while the predator is following the prey by following \(\nabla v_1\).

In this work we analyze the more general case where \(\chi_i \in \mathbb{R} \ (i = 1, 2)\), although none of the combinations involving \(\chi_1 > 0\) or \(\chi_2 < 0\) have a reasonable biological interpretation in the predator-prey context, but are interesting from the mathematical viewpoint. Under some specific assumptions on the tactic sensitivity coefficients and those involved in the reaction terms, we prove global existence of adequately defined solutions to (4) and their asymptotic behavior, the latter under the further assumption \(p_i = m_i\) made for \(i = 1, 2\).

We assume that the bounded initial data \((u_1^0, u_2^0)\) satisfy

\[
u_1^0(x) \in C^2(\Omega), \quad \frac{\partial u_1^0}{\partial \nu} = 0, \quad 0 < u_2^0 \leq u_1^0(x), \quad x \in \Omega, \quad i = 1, 2,
\]

for some \(\gamma > 0\) and \(u_2^0 > 0\) are positive given constant data. We define \(\Omega_T := \Omega \times (0, T)\), for any \(T < \infty\) and use this notation through the paper.

In the stated problem it can be seen that the mechanism that limits the growth of prey and predators is given by the terms \(-a_{11} u_1^{p_1}\) and \(-a_{22} u_2^{p_2}\), respectively. The term \(-a_{11} u_1^{p_1}\) describes the intra-prey competition for limited external resources while the term \(-a_{22} u_2^{p_2}\) reflects competition among predators for the limited number of prey, and they generalize the most frequent case \(p_i = 1\). The non-linear nature of the chemotaxis term has been studied in the literature by different authors, see [16] and references therein. The exponents \(m_i\) indicate nonlinearities with respect to \(u_i\) in the tactic sensitivity functions; intuitively, there is a reinforcement of movement in direction of \(\nabla v_i\) where the population \(u_i\) is greater than a normalized value and presents a weaker movement where it is less than it. These terms with \(p_i \geq 1\) induce a negative feedback that slows growth as populations approach their maximum size and a stronger intra-specific concurrence, (via exponents of the involved density).
Our intention is to show that given \( a_{ij} > 0, i = 0, 1, 2, j = 1, 2 \), for the real cross-diffusivities \( \chi_1 \) and \( \chi_2 \), all solutions of (4) will stabilize towards an equilibrium. In Negreanu and Tello [34], the predator prey system with diffusion and taxis is also studied for a general case where the coefficients \( a_{ij} = a_{i,j}(x,t) \) (for \( i = 0, 1, 2 \) and \( j = 1, 2 \)) of (1)–(2) are smooth functions in \( \Omega_\infty \) and present a periodic asymptotic behavior. Under suitable assumptions of converge in \( a_{ij} \) to periodic functions, the authors obtained a periodic asymptotic behavior of the solution. In particular if \( g_1^{(p_1)} = g_2^{(p_2)} = 0 \), blow up occurs for a range of initial data as it is shown in [39].

In the last decade many types of systems for two biological species with kinetic interaction have been considered. In [42], the authors proved the stability of homogeneous steady states for one chemical (see also [38], [4], [8]). The evolution of the ecosystem for predators following a chemical secreted by the prey which doesn’t present chemotactic ability is described by predator-prey models of indirect taxis (see examples in [43], [26], [25], and [40], among others).

In [21], [22] the authors establish sufficient conditions for the existence of solutions and its asymptotic dynamics for competitive systems of two biological species and a chemical with non-constant coefficients and in [23] it is presented the problem for one species with time and space dependence coefficients and growth term.

In [10], systems of two biological species with chemotactic abilities have been studied, i.e., the competitive system is considered for a general case and the global existence and asymptotic behavior are obtained for positive and bounded initial data under the restrictions \( 2|\chi_1|\beta_i + a_{ij} < a_{jj}, i, j = 1, 2, i \neq j \). Coexistence and extinction are studied for different parameters and initial data for the fully parabolic problem in [7] for constant coefficients and the global existence of solutions is obtained for \( \mu_i > \frac{1}{2} \chi_i^2 \).

The mathematical model analyzed in this article is related to the systems modeling the competitive interaction between the species with constant and positive coefficients \( a_{ij} \) (considered in [10], [49], [48] and [50]), while (4) presents a predator-prey interaction. The system also extends the predator-prey models with indirect taxis (see [43]) to the case where the prey has the ability to orient its movement following a chemical gradient related to the predator. As much as we know, the predator-prey system with two chemicals has not been considered before in the literature from a mathematical point of view.

The basic technical tools in proving the main results here will be the Alikakos method, the fixed point [37] and the Rectangle Method introduced by Pao in [36], which was already exploited in many papers on related models (see [10], [34] for more details). For the case \( a_{11} = a_{22} = 0 \) in (1) and (2), not included in this paper, the system of ordinary differential equations has periodic solutions, not constant and the techniques used here to study the asymptotic behavior of the solutions to the Lotka-Volterra model of a predator-prey interaction with diffusion and chemotactic terms can not possibly be applied directly for technical reasons.

Throughout the present article we work under the following technically motivated assumptions

\begin{align}
2|\chi_1|\beta_1 + a_{12} &< a_{22}, \quad (6) \\
2|\chi_2|\beta_2 + a_{21} &< a_{11} \quad (7)
\end{align}

and

\begin{align}
p_i &\geq m_i, \quad i = 1, 2. \quad (8)
\end{align}
We find some conditions on the parameters which guarantee the global existence and boundedness of classical solutions with nonnegative initial functions. The main theorem of the paper describing the global existence of solutions of (4) is stated as follows:

**Theorem 1.1.** Assume that \( \Omega \in \mathbb{R}^n \), \( n \geq 1 \) is a bounded domain with regular boundary, coefficients \( a_{i,j}, \alpha_j, \beta_j, p_j, m_j \) are positive and \( \chi_j \in \mathbb{R} \) for \( i = 0, 1, 2 \) and \( j = 1, 2 \). Suppose further that these parameters satisfy (6)-(8). Then for any nonnegative initial data \( (u_1^0, u_2^0) \) as in (5), there exists a unique positive solution of (4) globally in time verifying \( u_i, v_i \in C^{2+\gamma,1+\gamma}(\Omega_T) \), for \( i = 1, 2 \) and any \( T < \infty \).

Furthermore, the solutions \( u_i \) and \( v_i \), \( i = 1, 2 \) are uniformly bounded, i.e., there exists a positive constant \( C > 0 \) such that

\[
\sup_{t \geq 0} \left( \|u_i(\cdot,t)\|_{L^\infty(\Omega)} + \|v_i(\cdot,t)\|_{L^\infty(\Omega)} \right) \leq C.
\]

The second main objective is to prove an asymptotic stabilization property for the solution obtained in Theorem 1.1 for the case \( p_i = m_i = 1 \), with \( i = 1, 2 \).

**Theorem 1.2.** Under the assumptions of Theorem 1.1, the global solution of (4), \( (u_1, u_2, v_1, v_2) \), fulfills

\[
\lim_{t \to \infty} \left( \|u_i - u_i^*\|_{L^\infty(\Omega)} + \|v_j - \frac{\beta_j}{\alpha_j} u_i^*\|_{L^\infty(\Omega)} \right) = 0, \quad i, j = 1, 2, \quad i \neq j
\]

where \( u_i^* \) are given by

1. \( u_1^* = \frac{a_{01}}{a_{11}} \quad \text{and} \quad u_2^* = 0 \),

if relation

\[
a_{02}a_{11} - a_{01}a_{21} > 0 \quad (10)
\]

holds, and

2. \( u_1^* = \frac{a_{01}a_{22} + a_{02}a_{12}}{a_{11}a_{22} + a_{12}a_{21}}, \quad \text{and} \quad u_2^* = \frac{-a_{02}a_{11} + a_{01}a_{21}}{a_{11}a_{22} + a_{12}a_{21}} \),

if

\[
a_{02}a_{11} - a_{01}a_{21} < 0. \quad (12)
\]

As in the competitive case [10] with

\[
g_2^{(1)} = u_2(a_{02} - a_{21}u_1 - a_{22}u_2),
\]

in the first case (10), the species \( u_1 \) persists and its density converges to an homogeneous spatial distribution while species \( u_2 \) vanishes as \( t \) goes to infinity. If (12) holds, both species coexist and the densities stabilize in some constant steady state given by (11).

The results obtained in Theorem 1.2 are valid for particular case \( \chi_1 = \chi_2 = 0 \) where the solutions have the same asymptotic behavior as the ODE system (71). In that case, the results are already known, see for instance [19] and reference therein.

An outline of this paper is as follows: In Section 3 we consider a system of ordinary predator-prey differential equations associated to the nonlinear system of PDE's (4) and deduce qualitative solution properties that will be used in the proofs of our main results. In Section 2, we study the global existence of classical solutions...
of (4) with given initial data. In Section 4 we study the asymptotic behavior of positive solutions of (4) with \( p_i = m_i = 1 \) using the relation between the solution \((u_i, \pi_i)\) of the ODE system and the solution \((u_i, v_i)\) of the PDE system (4). Under some order relation between initial condition, we obtain that such order is preserved, i.e., we bound the solution of (4) between \( u_i \) (lower bound) and \( u_i \) (upper bound). The proof follows the rectangle method used in Pao [36] for reaction diffusion systems, (see also Negreanu and Tello [30] and [33] where the method is applied to parabolic-elliptic systems with chemotactic terms). We obtain the asymptotic behavior of the solution to (4), the availability of coexistence states, and an extinction phenomenon in the sense that one of the species dies out asymptotically and the other reaches its carrying capacity as time goes to infinity (Theorem 1.2).

**Remark 1.1.** The predator-prey system with two chemicals has not been considered before in the literature from a mathematical point of view. For the asymptotic behavior, even though the analysis is done in the case \( p_i = m_i = 1 \) therein provided some interest. One issue when attempting to investigate the asymptotic behavior in the general case, is due to the multitude of parameters and the difficulty encountered when applying the comparison method and the known results on the ODE systems. Possibly, for the general case, a Liapunov type energy method could be applied but that is beyond the scope of this article, it is an open problem that we plan to solve.

2. **Global existence of solution.** We start with the following important result on the local existence of classical solutions of system (4) with nonnegative initial functions. The result is enclosed in the following lemma.

**Lemma 2.1.** Under assumptions (6)-(8), there exists a unique solution \((u_1, u_2, v_1, v_2)\) to (4) in \((0, T_{\text{max}})\) satisfying

\[
u_i, v_i \in C^{2+\gamma,1+\gamma}_{x,t}(\Omega_T), \quad \text{for } i = 1, 2 \text{ and any } T < \infty.
\]

Moreover, for \( i = 1, 2 \),

\[
u_i(t,x) \geq 0, \quad v_i(x,t) \geq 0, \quad x \in \Omega, \quad t < \infty.
\]

**Proof.** We take \( T_{\text{max}} \) verifying

\[
\limsup_{t \to T_{\text{max}}} (\|u_i(\cdot, t)\|_{L^\infty(\Omega)} + \|v_i(\cdot, t)\|_{L^\infty(\Omega)} + t) = \infty.
\]

The proof follows standard fixed point theory, see for instance Horstmann [17], Biler [6], Horstmann and Winkler [18], or Negreanu and Tello [33], [34] in order to obtain the local existence of the solutions in \( L^2(0, T; H^1(\Omega)) \cap L^\infty(\Omega_T) \). Uniqueness of solutions is obtained by contradiction, following standard arguments. The non-negativity of \( u_i \) is a consequence of the maximum principle for which also implies the non-negativity of \( v_i \).

**Lemma 2.2.** Suppose \( p_i \geq m_i \) and \( a_{ii} \) large enough for \( i = 1, 2 \). For any \( \gamma > 1 \), there exist \( c_i = c_i(\gamma, \|u_0\|_{L^\gamma(\Omega)}) \) such that

\[
\|u_i\|_{L^\gamma(\Omega)} \leq c_i, \quad \forall t \in (0, T_{\text{max}})
\]

and the following inequalities hold

\[
\int_0^T \int_\Omega u_i^{p_i+\gamma} + \int_0^T \int_\Omega |\nabla u_i|^2 \leq c_i(T + 1) \quad \forall T \in (0, T_{\text{max}}),
\]

with \( i = 1, 2 \).
Proof. We prove the lemma for all \(\chi_i \in \mathbb{R}\). We consider in detail the first case where both \(\chi_1, i = 1, 2\) are positive and the second case, \(\chi_1 < 0\) (for chemorepulsion by its predator) and \(\chi_2 > 0\) (for chemotaxis attraction of predator toward its prey).

For the other possible cases, the proof is similar and we emphasize only the terms where the signs of \(\chi_i\) have an effect on the inequalities.

By multiplying the first and the second equations of (4) by \(u_i^{\gamma-1}\) and \(u_2^{\gamma-1}\) respectively, and integrating by parts over \(\Omega\), we get

\[
\frac{1}{\gamma} \frac{d}{dt} \int_{\Omega} u_i^\gamma + (\gamma - 1) \int_{\Omega} u_i^{\gamma-2}\left|\nabla u_i\right|^2 = (\gamma - 1)\chi_1 \int_{\Omega} u_i^{m_1+\gamma-2}\nabla u_i \nabla v_1 + \\
+ \int_{\Omega} u_i^\gamma (a_{01} - a_{11}u_1^n - a_{12}u_2),
\]

(16)

\[
\frac{1}{\gamma} \frac{d}{dt} \int_{\Omega} u_2^\gamma + (\gamma - 1) \int_{\Omega} u_2^{\gamma-2}\left|\nabla u_2\right|^2 = (\gamma - 1)\chi_2 \int_{\Omega} u_2^{m_2+\gamma-2}\nabla u_2 \nabla v_2 + \\
+ \int_{\Omega} u_2^\gamma (-a_{02} - a_{22}u_2^n + a_{12}u_1).
\]

(17)

After multiplying by \(u_i^{m_i+\gamma-1}\) the third and fourth equations of (4) for \(i = 1, 2\), then by integrating by parts we obtain

\[
(m_1 + \gamma - 1) \int_{\Omega} u_1^{m_1+\gamma-2}\nabla u_1 \nabla v_1 = \beta_1 \int_{\Omega} u_1^{m_1+\gamma-1}u_2 - \alpha_1 \int_{\Omega} u_1^{m_1+\gamma-1}v_1
\]

(18)

and

\[
(m_2 + \gamma - 1) \int_{\Omega} u_2^{m_2+\gamma-2}\nabla u_2 \nabla v_2 = \beta_2 \int_{\Omega} u_1u_2^{m_2+\gamma-1} - \alpha_2 \int_{\Omega} u_2^{m_2+\gamma-1}v_2.
\]

(19)

We have from (18) and (19) that (16) and (17) can be rewritten as follows

\[
\frac{1}{\gamma} \frac{d}{dt} \int_{\Omega} u_1^\gamma + (\gamma - 1) \int_{\Omega} u_1^{\gamma-2}\left|\nabla u_1\right|^2 = -\frac{\alpha_1(\gamma - 1)\chi_1}{m_1 + \gamma - 1} \int_{\Omega} u_1^{m_1+\gamma-1}v_1 + \\
+ \frac{\beta_1(\gamma - 1)\chi_1}{m_1 + \gamma - 1} \int_{\Omega} u_1^{m_1+\gamma-1}u_2 + \int_{\Omega} u_1^\gamma (a_{01} - a_{11}u_1^n - a_{12}u_2)
\]

(20)

and

\[
\frac{1}{\gamma} \frac{d}{dt} \int_{\Omega} u_2^\gamma + (\gamma - 1) \int_{\Omega} u_2^{\gamma-2}\left|\nabla u_2\right|^2 = -\frac{\alpha_2(\gamma - 1)\chi_2}{m_2 + \gamma - 1} \int_{\Omega} u_2^{m_2+\gamma-1}v_2 + \\
+ \frac{\beta_2(\gamma - 1)\chi_2}{m_2 + \gamma - 1} \int_{\Omega} u_2^{m_2+\gamma-1}u_1 + \int_{\Omega} u_2^\gamma (-a_{02} - a_{22}u_2^n - a_{21}u_1)
\]

(21)

In order to study the terms of the above equations we consider two cases (the differences are in the terms in which appear \(\chi_i\)):

- **Case** \(\chi_i > 0\).

By removing the non-positive terms on the right side of (20) and (21), we have

\[
\frac{1}{\gamma} \frac{d}{dt} \int_{\Omega} u_1^\gamma + (\gamma - 1) \int_{\Omega} u_1^{\gamma-2}\left|\nabla u_1\right|^2 \leq \frac{\beta_1(\gamma - 1)\chi_1}{m_1 + \gamma - 1} \int_{\Omega} u_1^{m_1+\gamma-1}u_2 \\
+ \int_{\Omega} u_1^\gamma (a_{01} - a_{11}u_1^n - a_{12}u_2)
\]

(22)
and similarly, we obtain
\[
\frac{1}{\gamma} \frac{d}{dt} \int_{\Omega} u_1^\gamma - (\gamma - 1) \int_{\Omega} \frac{\partial u_1}{\partial t}^\gamma \leq \frac{\beta_2(\gamma - 1)\chi_2}{m_2 + \gamma - 1} \int_{\Omega} u_1^{m_2 + \gamma - 1} + \int_{\Omega} u_1^\gamma (a_0 - a_1 a_{22} u_1^{p_1} + a_{21} u_1).
\]
\begin{align}
\int_{\Omega} u_1^{m_1 + \gamma - 1} u_2 & \leq \varepsilon_1 \int_{\Omega} u_1^{\gamma + p_1} + C_1(\varepsilon_1, \gamma, \chi_1, m_1) \int_{\Omega} u_1^{\frac{p_1 + \gamma}{m_1 + \gamma - 1}}, \\
\int_{\Omega} u_2^{m_2 + \gamma - 1} u_1 & \leq \varepsilon_2 \int_{\Omega} u_2^{\gamma + p_2} + C_2(\varepsilon_2, \gamma, \chi_2, m_2) \int_{\Omega} u_2^{\frac{p_2 + \gamma}{m_2 + \gamma - 1}},
\end{align}

for arbitrary positive constants \(\varepsilon_i, i = 1, 2\).

Inserting the above inequalities in (22), (23), for \(\chi > 0\), it yields:
\[
\frac{1}{\gamma} \frac{d}{dt} \int_{\Omega} \frac{\partial u_1}{\partial t}^\gamma \leq \frac{\beta_2(\gamma - 1)\chi_2}{m_2 + \gamma - 1} \left[ \varepsilon_1 \int_{\Omega} u_1^{\gamma + p_1} + C_1(\varepsilon_1, \gamma, \chi_1, m_1) \int_{\Omega} u_1^{\frac{p_1 + \gamma}{m_1 + \gamma - 1}} \right] + a_{01} \int_{\Omega} u_1 - a_{11} \int_{\Omega} u_1^{\gamma + p_1} - a_{12} \int_{\Omega} u_1^\gamma u_2
\]
\begin{align}
\frac{1}{\gamma} \frac{d}{dt} \int_{\Omega} \frac{\partial u_2}{\partial t}^\gamma \leq \frac{\beta_2(\gamma - 1)\chi_2}{m_2 + \gamma - 1} \left[ \varepsilon_2 \int_{\Omega} u_2^{\gamma + p_2} + C_2(\varepsilon_2, \gamma, \chi_2, m_2) \int_{\Omega} u_2^{\frac{p_2 + \gamma}{m_2 + \gamma - 1}} \right] - a_{02} \int_{\Omega} u_2^\gamma - a_{22} \int_{\Omega} u_2^{\gamma + p_2} + a_{21} \int_{\Omega} u_1 u_2^\gamma.
\end{align}

For every \(p_1 \geq 1\), by Young’s inequality we get
\[
\int_{\Omega} u_1^\gamma \leq \varepsilon_3 \int_{\Omega} u_1^{p_1 + \gamma} + C_3(\varepsilon_3, |\Omega|, \gamma, \chi_1),
\]
thus
\[
\frac{1}{\gamma} \frac{d}{dt} \int_{\Omega} u_1^\gamma \leq \frac{4(\gamma - 1)}{\gamma^2} \int_{\Omega} |\nabla u_1|^2 \leq \left[ \frac{\beta_1(\gamma - 1)\chi_1}{m_1 + \gamma - 1} \varepsilon_3 + a_{01} \varepsilon_3 - a_{11} \right] \int_{\Omega} u_1^{\gamma + p_1} + \frac{\beta_1(\gamma - 1)\chi_1}{m_1 + \gamma - 1} C_1(\varepsilon_1) \int_{\Omega} u_1^{\frac{p_1 + \gamma}{m_1 + \gamma - 1}} + a_{01} C_3(\varepsilon_3) - a_{12} \int_{\Omega} u_1^\gamma u_2.
\]
\begin{align}
\frac{1}{\gamma} \frac{d}{dt} \int_{\Omega} u_2^\gamma \leq \frac{4(\gamma - 1)}{\gamma^2} \int_{\Omega} |\nabla u_2|^2 \leq \left[ \frac{\beta_2(\gamma - 1)\chi_2}{m_2 + \gamma - 1} \varepsilon_2 + a_{22} \right] \int_{\Omega} u_2^{\gamma + p_2} - a_{02} \int_{\Omega} u_2^\gamma + \frac{\beta_2(\gamma - 1)\chi_2}{m_2 + \gamma - 1} C_2(\varepsilon_2) \int_{\Omega} u_2^{\frac{p_2 + \gamma}{m_2 + \gamma - 1}} + a_{21} \int_{\Omega} u_1 u_2^\gamma.
\end{align}

We estimate the last integral term in (30), for every \(\gamma > 1\), by Young’s inequality, if \(p_2 > 1\) (for \(p_2 = 1\) we don’t need this estimate)
\[
\int_{\Omega} u_2^{\gamma + 1} \leq \varepsilon_4 \int_{\Omega} u_2^{p_2 + \gamma} + C_4(\varepsilon_4, |\Omega|, \gamma, p_2),
\]
and

\[
\int_{\Omega} u_1 u_2^\gamma \leq \varepsilon_5 \int_{\Omega} u_1^\gamma u_2 + C_5(\varepsilon_5, \gamma) \int_{\Omega} u_2^{\gamma+1}.
\] (32)

Thus, we get

\[
\int_{\Omega} u_1 u_2^\gamma \leq \varepsilon_5 \int_{\Omega} u_1^\gamma u_2 + \varepsilon_4 C_5(\varepsilon_5) \int_{\Omega} u_2^{\gamma+p_2} + C_5(\varepsilon_5) C_4(\varepsilon_4)
\] (33)

and (30) is equivalent to

\[
\frac{1}{\gamma} \frac{d}{dt} \int_{\Omega} u_2^\gamma + \frac{4(\gamma - 1)}{\gamma^2} \int_{\Omega} |\nabla u_2^\gamma|^2
\leq \left[ \frac{\beta_2(\gamma - 1) \chi_2}{m_2 + \gamma - 1} + a_2 \varepsilon_4 C_5(\varepsilon_5) - a_{22} \right] \int_{\Omega} u_2^{\gamma+p_2} + a_2 \varepsilon_3 C_4(\varepsilon_4) - a_{02} \int_{\Omega} u_2^\gamma
\]
\[
+ \beta_2(\gamma - 1) \chi_2 \int_{\Omega} u_2^{\gamma-2+\varepsilon_2} + a_2 \varepsilon_5 \int_{\Omega} u_1^\gamma u_2.
\] (34)

For \(\chi_1 > 0\), since \(p_i + \gamma > m_i + \gamma - 1\), \((\frac{p_i+\gamma}{m_i+\gamma-1}) \leq \gamma + p_j\) and \(i,j = 1,2\), thanks to Young’s inequality, by adding (29) and (34) we claim

\[
\frac{1}{\gamma} \frac{d}{dt} \int_{\Omega} u_1^\gamma + \frac{1}{\gamma} \frac{d}{dt} \int_{\Omega} u_2^\gamma + \frac{4(\gamma - 1)}{\gamma^2} \int_{\Omega} |\nabla u_1^\gamma|^2 + \frac{4(\gamma - 1)}{\gamma^2} \int_{\Omega} |\nabla u_2^\gamma|^2
\leq \left[ \frac{\beta_1(\gamma - 1) \chi_1}{m_1 + \gamma - 1} + a_{01} \varepsilon_3 + \frac{\beta_2(\gamma - 1) \chi_2}{m_2 + \gamma - 1} C_2(\varepsilon_2) \varepsilon_6 - a_{11} \right] \int_{\Omega} u_1^{\gamma+p_1}
\]
\[
+ \left[ \frac{\beta_2(\gamma - 1) \chi_2}{m_2 + \gamma - 1} C_2(\varepsilon_2) \varepsilon_6 + a_2 \varepsilon_4 C_5(\varepsilon_5) + \frac{\beta_1(\gamma - 1) \chi_1}{m_1 + \gamma - 1} C_1(\varepsilon_1) \varepsilon_7 - a_{22} \right] \int_{\Omega} u_2^{\gamma+p_2}
\]
\[
+ C_6(\varepsilon_6, \varepsilon_2, |\Omega|, p_2, \gamma, m_2, \beta_2, \chi_2) + C_7(\varepsilon_7, \varepsilon_1, |\Omega|, p_1, \gamma, m_1, \beta_1, \chi_1)
\]
\[
+ a_{01} C_3(\varepsilon_3) + a_{21} C_4(\varepsilon_4) C_5(\varepsilon_5) + (a_2 \varepsilon_5 - a_{12}) \int_{\Omega} u_1^\gamma u_2,
\] (35)

for arbitrary positive constants \(\varepsilon_6\) and \(\varepsilon_7\).

- **Case** \(\chi_1 < 0\) and \(\chi_2 > 0\)

For this case, we will only write the equations that change with respect to the previous case; instead of (20) we get the expression

\[
\frac{1}{\gamma} \frac{d}{dt} \int_{\Omega} u_1^\gamma + \frac{1}{\gamma} \frac{d}{dt} \int_{\Omega} u_2^\gamma + \frac{4(\gamma - 1)}{\gamma^2} \int_{\Omega} |\nabla u_1^\gamma|^2 \leq -\frac{a_1(\gamma - 1) \chi_1}{m_1 + \gamma - 1} \int_{\Omega} u_1^{m_1+\gamma-1} u_1
\]
\[
+ \int_{\Omega} u_1^\gamma (a_{01} - a_{11} u_1^{p_1} - a_{12} u_2).
\] (36)

In order to obtain bounds for \(\int_{\Omega} u_1^{m_1+\gamma-1} u_1\), by applying Young’s inequality and the regularity results for elliptic equations with Neumann boundary conditions, for the third equation of (4), see [2] and [12], we have the additional inequality

\[
\int_{\Omega} u_1^{m_1+\gamma-1} u_1 \leq \varepsilon_1 \int_{\Omega} u_1^{\gamma+p_1} + C_1'(\varepsilon_1, \gamma, p_1, m_1) \int_{\Omega} u_1^{\frac{m_1+\gamma-1}{m_1+p_1}}
\]
\[
\leq \varepsilon_1 \int_{\Omega} u_1^{\gamma+p_1} + \tilde{C}_1(\varepsilon_1, |\Omega|, \gamma, p_1, m_1) \int_{\Omega} u_2^{\frac{m_1+\gamma-1}{m_1+p_1}}.
\] (37)
For $\chi_1 < 0$ and $\chi_2 > 0$, instead of inequality (29) we have

\[
\frac{1}{\gamma} \frac{d}{dt} \int_{\Omega} u_1^\gamma + \frac{4(\gamma - 1)}{\gamma^2} \int_{\Omega} |\nabla u_1^\gamma|^2 \leq \left[ -\frac{\alpha_1(\gamma - 1)\chi_1}{m_1 + \gamma - 1} \varepsilon_1 + a_0 \varepsilon_3 - a_{11} \right] \int_{\Omega} u_1^{\gamma + p_1} - \frac{\alpha_1(\gamma - 1)\chi_1}{m_1 + \gamma - 1} \tilde{C}_1(\varepsilon_1) \int_{\Omega} u_2^{\gamma + p_1} + a_0 C_3(\varepsilon_3) - a_{12} \int_{\Omega} u_1^2 u_2,
\]  
(38)

due to (20). By adding (38) and (34) (and thanks to Young’s inequality as in (35)), it yields

\[
\frac{1}{\gamma} \frac{d}{dt} \int_{\Omega} u_1^\gamma + \frac{1}{\gamma} \frac{d}{dt} \int_{\Omega} u_2^\gamma + \frac{4(\gamma - 1)}{\gamma^2} \int_{\Omega} |\nabla u_1^\gamma|^2 + \frac{4(\gamma - 1)}{\gamma^2} \int_{\Omega} |\nabla u_2^\gamma|^2 \leq \left[ -\frac{\alpha_1(\gamma - 1)\chi_1}{m_1 + \gamma - 1} \varepsilon_1 + a_0 \varepsilon_3 + \frac{\beta_2(\gamma - 1)\chi_2}{m_2 + \gamma - 1} C_2(\varepsilon_2) \varepsilon_6 - a_{11} \right] \int_{\Omega} u_1^{\gamma + p_1} + \frac{\beta_2(\gamma - 1)\chi_2}{m_2 + \gamma - 1} \varepsilon_2 + a_{21} \varepsilon_4 C_5(\varepsilon_5) - a_0 \frac{(\gamma - 1)\chi_1}{m_1 + \gamma - 1} \tilde{C}_1(\varepsilon_1) \varepsilon_7 - a_{22} \int_{\Omega} u_2^{\gamma + p_2} + C_6(\varepsilon_6, \varepsilon_2, |\Omega|, p_2, \gamma, m_2, \beta_2, \chi_2) + C_7(\varepsilon_7, \varepsilon_1, |\Omega|, p_1, \gamma, m_1, \alpha_1, \chi_1) + a_{01} C_3(\varepsilon_3) + a_{21} C_4(\varepsilon_4) C_5(\varepsilon_5) + (a_{21} \varepsilon_5 - a_{12}) \int_{\Omega} u_1^2 u_2
\]  
(39)

for arbitrary positive constants $\varepsilon_6$ and $\varepsilon_7$.

- For every $\chi_1 \in \mathbb{R}$

For $\alpha_i$ large enough and $\varepsilon_5 \leq a_{12}/a_{21}$ (such that the last terms in (35) and (39) are non-positive), then (35) and (39), respectively are as follows

\[
\frac{1}{\gamma} \frac{d}{dt} \int_{\Omega} u_1^\gamma + \frac{1}{\gamma} \frac{d}{dt} \int_{\Omega} u_2^\gamma + \frac{4(\gamma - 1)}{\gamma^2} \int_{\Omega} |\nabla u_1^\gamma|^2 + \frac{4(\gamma - 1)}{\gamma^2} \int_{\Omega} |\nabla u_2^\gamma|^2 \leq -\delta \int_{\Omega} (u_1^{\gamma + p_1} + u_2^{\gamma + p_2}) + C(\delta, |\Omega|, m_i, \gamma, p_i)
\]  
(40)

for $i = 1, 2, \delta > 0$. We apply now the Hölder inequality to the term $\int_{\Omega} u_i^{\gamma + p_i}$

\[
|\Omega|^\frac{p_i}{\gamma} \int_{\Omega} u_i^{\gamma + p_i} \geq \left( \int_{\Omega} u_i^\gamma \right)^\frac{\gamma + p_i}{\gamma},
\]

in order to obtain

\[
\frac{1}{\gamma} \frac{d}{dt} \int_{\Omega} u_1^\gamma + \frac{1}{\gamma} \frac{d}{dt} \int_{\Omega} u_2^\gamma + \frac{4(\gamma - 1)}{\gamma^2} \int_{\Omega} |\nabla u_1^\gamma|^2 + |\nabla u_2^\gamma|^2 \leq -\delta C(|\Omega|, p_i, \gamma) \left[ \left( \int_{\Omega} u_1^\gamma \right)^\frac{\gamma + p_i}{\gamma} + \left( \int_{\Omega} u_2^\gamma \right)^\frac{\gamma + p_2}{\gamma} \right] + C(\delta),
\]  
(41)

with $C(|\Omega|, p_i, \gamma) > 0$.

By denoting by $\int_{\Omega} u_1^\gamma(t) = x(t)$ and $\int_{\Omega} u_2^\gamma(t) = y(t)$, we get the ordinary differential equation

\[
\frac{d}{dt} (x + y) + K_1 [x^{\gamma + p_1} + y^{\gamma + p_2}] \leq K_2.
\]  
(42)
For $1 < a := \min_{i=1,2}\left\{ \frac{p_i+\gamma}{\gamma} \right\}$, thanks to the inequalities $z^{\frac{\gamma}{p_i+\gamma}} + 1 \geq z^a$ and $(x + y)^a \leq 2^{a-1}(x^a + y^a)$ we get

$$\frac{d}{dt}(x + y) \leq -K(x + y)^a + K'$$

(43)
in $(0, T_{max})$, with positive constants $K$ and $K'$ depending on $p_i, \delta, \gamma, |\Omega|$ and then

$$x(t) + y(t) \leq \max\left\{ x(0) + y(0), \left( \frac{K}{K'} \right) \right\}^{\frac{1}{a}}, \forall t \in [0, T_{max}).$$

By integrating over $(0, T_{max})$ in (40) we finish the proof. □

Proof of Theorem 1.1. Solutions $u_1$ and $u_2$ verify

$$\frac{\partial u_1}{\partial t} = \Delta u_1 - m_1 \chi u_1^{m_1-1} \nabla u_1 \nabla v_1 + \chi u_1^{m_1} (\beta_1 u_2 - \alpha_1 v_1) + g_1^{(p_1)}(u_1, u_2),$$

$$\frac{\partial u_2}{\partial t} = \Delta u_2 - m_2 \chi u_2^{m_2-1} \nabla u_2 \nabla v_2 + \chi u_2^{m_2} (\beta_2 u_1 - \alpha_2 v_2) + g_2^{(p_2)}(u_1, u_2).$$

Since $u_i$ are uniformly bounded in $L^k(\Omega)$ for any $k < \infty$ and $m_i, p_i, \gamma$ verify the hypotheses of Theorem 1.1, we get

$$\|v_i\|_{W^{2, k}(\Omega)} \leq C, \quad \text{for} \quad k < \infty$$

and

$$\|v_i\|_{W^{1, \infty}(\Omega)} \leq C \quad \text{uniformly.}$$

Therefore

$$-m_1 \chi u_1^{m_1-1} \nabla u_1 \nabla v_1 + \chi u_1^{m_1} (\beta_1 u_2 - \alpha_1 v_1) + g_1^{(p_1)}(u_1, u_2) \in L^r(\Omega_T)$$

$$-m_2 \chi u_2^{m_2-1} \nabla u_2 \nabla v_2 + \chi u_2^{m_2} (\beta_2 u_1 - \alpha_2 v_2) + g_2^{(p_2)}(u_1, u_2) \in L^r(\Omega_T)$$

for $r < 2$ and $T < T_{max}$. The regularity of $u_i$ (for $i = 1, 2$) is a consequence of the parabolic and elliptic regularity of the equations due to the regularity of the coefficients and the boundedness of $u_i$ and $v_i$. Then, we have $u_i \in C^{2+\gamma, 1+\gamma}_{x,t}(\Omega_T)$ see Remark 48.3 (ii) in Quittner-Souplet [37]. The well known standard regularity result gives the global existence of solutions. □

3. Associated ODE system; super- and sub- solutions; properties. In this section we obtain the asymptotic behavior of the solutions of an auxiliary system of ordinary differential equations related to the original nonlinear system (4) when $p_i = m_i$. To do this, we recall some known lemmas on non-autonomous logistic equation and Lotka-Volterra predator prey systems.

We first rewrite the first two equations of (4) as

$$\begin{cases}
\frac{\partial u_1}{\partial t} = \Delta u_1 - \chi_1 \nabla u_1 \cdot \nabla v_1 + \chi_1 u_1 (\beta_1 u_2 - \alpha_1 v_1) + g_1^{(1)}(u_1, u_2)
\\
\frac{\partial u_2}{\partial t} = \Delta u_2 - \chi_2 \nabla u_2 \cdot \nabla v_2 + \chi_2 u_2 (\beta_2 u_1 - \alpha_2 v_2) + g_2^{(1)}(u_1, u_2).
\end{cases}$$

(44)

We denote by $(\bar{u}_1, \underline{u}_1, \bar{u}_2, \underline{u}_2) = (\bar{u}_1(t), \underline{u}_1(t), \bar{u}_2(t), \underline{u}_2(t))$ the solution of the initial value problem

$$\begin{cases}
\bar{u}'_1 = \chi_1 \beta_1 \bar{u}_2 (\bar{u}_2 - \underline{u}_2) + \bar{u}_1 (a_{01} - a_{11} \bar{u}_1 - a_{12} \bar{u}_2),
\\
\underline{u}'_1 = \chi_1 \beta_1 \underline{u}_1 (\bar{u}_2 - \underline{u}_2) + \underline{u}_1 (a_{01} - a_{11} \underline{u}_1 - a_{12} \underline{u}_2),
\\
\bar{u}'_2 = \chi_2 \beta_2 \bar{u}_2 (\bar{u}_1 - \underline{u}_1) + \bar{u}_2 (-a_{02} - a_{22} \bar{u}_2 + a_{21} \bar{u}_1),
\\
\underline{u}'_2 = \chi_2 \beta_2 \underline{u}_2 (\bar{u}_1 - \underline{u}_1) + \underline{u}_2 (-a_{02} - a_{22} \underline{u}_2 + a_{21} \underline{u}_1),
\end{cases}$$

(45)
for \( t \in (0, \infty) \), with initial data

\[
\begin{align*}
\pi_1(0) &= \pi_1^0; & u_1(0) &= u_1^0; & \pi_2(0) &= \pi_2^0; & u_2(0) &= u_2^0;
\end{align*}
\]

(46)

verifying

\[
0 < u_1^0 < \pi_1^0 < \infty \quad 0 < u_2^0 < \pi_2^0 < \infty.
\]

(47)

We want to obtain a similar relationship between solutions \((\pi_1, u_1, \pi_2, u_2)\) when the initial data satisfies (47), i.e., we investigate that the initial ordering (47) is inherited by the ODE's solution. Furthermore we prove that \((\pi_1, u_1, \pi_2, u_2)\) are actually global in time and bounded. We prove that the two pairs of solutions of the ODE's system (45), i.e., \((\pi_1, u_1)\) and \((\pi_2, u_2)\) have the same constant limit \(u_1^*\) and \(u_2^*\), respectively and, hence, also any function between them. The non trivial stationary states of (45) satisfying \(u_1 = \pi_1, u_2 = \pi_2\), are

- the semi-trivial steady states

\[
\left( \frac{a_{01}}{a_{11}}, \frac{a_{01}}{a_{11}}, 0, 0 \right)
\]

(48)

- and a fourth steady state given by

\[
\left( \frac{a_{01}a_{22} + a_{02}a_{12} - a_{01}a_{12} + a_{02}a_{11} + a_{01}a_{21} - a_{02}a_{11} + a_{01}a_{21}}{a_{11}a_{22} + a_{12}a_{21} - a_{11}a_{21} + a_{12}a_{12}}, \frac{a_{01}a_{22} + a_{02}a_{12} - a_{01}a_{12} + a_{02}a_{11} + a_{01}a_{21} - a_{02}a_{11} + a_{01}a_{21}}{a_{11}a_{22} + a_{12}a_{21} - a_{11}a_{21} + a_{12}a_{12}} \right).
\]

(49)

**Lemma 3.1.** The solution \((\pi_1, u_1, \pi_2, u_2)\) of system (45), with initial data (46) verifying (47), exists locally and satisfies \([C^1(0, T_{\text{max}})]^4\) for \(T_{\text{max}}\) defined as follows

\[
\limsup_{t \to T_{\text{max}}} (|\pi_1(t)| + |u_1(t)| + |\pi_2(t)| + |u_2(t)| + t) = \infty.
\]

Moreover the solution satisfies the following order relation

\[
0 < u_1 < \pi_1, \quad 0 < u_2 < \pi_2
\]

(50)

for \( t \in (0, T_{\text{max}}) \).

**Proof.** The standard ODE theory gives the local existence of solutions of the system (45)–(46) in \((0, T_{\text{max}})\) since the right hand side terms of (45) is a second order polynomial. The regularity is a consequence of the nature of coefficients which are constants. We check that \(u_i > 0\), for \( i = 1, 2 \). Writing the corresponding equations of \( u_i \) from system (45) as \( u'_i = f_i(\pi_1, u_1, \pi_2, u_2) \), \( f_i \) being smooth functions, one proves easily that \( u_i = 0 \) is a solution of the previous equation. Taking into account that the initial data is positive \( u_i^0 > 0 \), by existence and uniqueness of the solution, we claim that \( u_i(t) > 0 \) for all \( t > 0 \). To obtain \( u_i > \pi_i \) we proceed by contradiction. Suppose (50) is false, i.e., there exists \( t_0 > 0 \) such that

\[
(\pi_1(t) - u_1(t))(\pi_2(t) - u_2(t)) > 0
\]

for all \( t \in (0, t_0) \). Then, one of the following cases occurs

\[
\begin{align*}
\pi_1(t_0) &= \pi_1(t_0) \quad \pi_2(t_0) > u_2(t_0), \quad (51) \\
\pi_1(t_0) > \pi_1(t_0) \quad u_2(t_0) = \pi_2(t_0), \quad (52) \\
\pi_1(t_0) = \pi_1(t_0) \quad u_2(t_0) = \pi_2(t_0). \quad (53)
\end{align*}
\]

The positivity of the solutions gives that in case (51)

\[
\pi'_1 > u'_1, \quad \forall \ t \leq t_0
\]
Lemma 3.2. Under the assumptions (6)–(7), for each 
\[ \leq \]
because of 
\[ c \]
For 
\[ \text{we have} \]
which contradicts 
\[ u_1(t_0) = \overline{u}_1(t_0) \] and (47). In the same way (52) contradicts 
\[ u_2(t_0) = \overline{u}_2(t_0) \] and (47). To prove case (53), we introduce the functions \( \varphi_1 \) and \( \varphi_2 \) defined by 
\[ \varphi_1 = \overline{u}_1 - u_1, \varphi_2 = \overline{u}_2 - u_2. \]
We have the following differential equations for \( \varphi_i \), for \( i = 1, 2 \)
\begin{align*}
\varphi'_1 &= |\chi_1|\beta_1(\overline{u}_2 - u_2)(\overline{u}_1 + u_1) + a_{01}(\overline{u}_1 - u_1) - a_{11}(\overline{u}_1^2 - u_1^2) - a_{12}(\overline{u}_1 u_2 - u_1 \overline{u}_2), \\
\varphi'_2 &= |\chi_2|\beta_2(u_1 - \overline{u}_1)(\overline{u}_2 + u_2) - a_{02}(\overline{u}_2 - u_2) - a_{22}(\overline{u}_2^2 - u_2^2) + a_{21}(\overline{u}_1 u_2 - u_1 \overline{u}_2),
\end{align*}
which can be rewritten as 
\begin{align*}
\varphi'_1 &= \phi_1(\overline{u}_1, u_1, \overline{u}_2, u_2) \varphi_1 + \phi_2(\overline{u}_1, u_1, \overline{u}_2, u_2) \varphi_2, \\
\varphi'_2 &= \phi_3(\overline{u}_1, u_1, \overline{u}_2, u_2) \varphi_2 + \phi_4(\overline{u}_1, u_1, \overline{u}_2, u_2) \varphi_2.
\end{align*}
(54)
We turn our attention to the system (54) with initial data \( \varphi_1(t_0) = 0 \) \( \varphi_2(t_0) = 0 \), where it can be easily checked that \( (\varphi_1, \varphi_2) = (0, 0) \) is a solution of the system and by uniqueness of solution together with the initial data, we conclude that \( \varphi_i = 0 \), for \( i = 1, 2 \) in the interval \( [0, t_0) \), which contradicts the definition of \( t_0 \) and ends the proof.

Certain useful properties of the super-solution \( (\overline{u}_1, \overline{u}_2) \) of (45) are presented in the next lemma

Lemma 3.2. Under the assumptions (6)–(7), for each \( t > 0 \), the super-solution \( (\overline{u}_1, \overline{u}_2) \) of (45) satisfies

\( \bullet \) There exists a positive constant \( K > 0 \), such that 
\[ \overline{u}_1 \leq K, \quad \overline{u}_2 \leq K. \]
\[ (55) \]
Proof. Before checking (55), we show another bound for the product of the super-solution \( (\overline{u}_1, \overline{u}_2) \):

\( \bullet \) There exists a positive constant \( C > 0 \) such that 
\[ \overline{u}_1 \overline{u}_2 \leq C, \quad \text{for} \quad t \in (0, \infty). \]
\[ (56) \]
Dividing the first equation in (45) by \( \overline{u}_1 \) and the third one by \( \overline{u}_2 \), after adding up, we have 
\[ \frac{\overline{u}_1'}{\overline{u}_1} + \frac{\overline{u}_2'}{\overline{u}_2} = \left( |\chi_2|\beta_2 - a_{11} + a_{21} \right) \overline{u}_1 - |\chi_2|\beta_2 u_1 + \left( |\chi_1|\beta_1 - a_{22} \right) \overline{u}_2 - \left( |\chi_1|\beta_1 + a_{12} \right) u_2 + a_{01} - a_{02} \]
\[ \leq a_{01} + \left( |\chi_2|\beta_2 + a_{21} - a_{11} \right) \overline{u}_1 + \left( |\chi_1|\beta_1 - a_{22} \right) \overline{u}_2. \]
\[ (57) \]
For \( c = \min \{ a_{11} - |\chi_2|\beta_2 - a_{21}, a_{22} - |\chi_1|\beta_1 \} > 0 \) due to (6)–(7), we claim 
\[ (\ln(\overline{u}_1 \overline{u}_2))' \leq a_{01} - c(\overline{u}_1 + \overline{u}_2), \]
which is equivalent to 
\[ (\ln(\overline{u}_1 \overline{u}_2))^' \leq a_{01} - c \exp \left( \frac{1}{2} \ln(\overline{u}_1 \overline{u}_2) \right) \]
because of \( \overline{u}_1 + \overline{u}_2 \geq 2 \sqrt{\overline{u}_1 \overline{u}_2} = 2 e^{\frac{1}{2} \ln(\overline{u}_1 \overline{u}_2)}. \)

Thus, \( \ln(\overline{u}_1 \overline{u}_2) \) is a sub-solution of the ordinary differential equation 
\[ z' = a_{01} - c 2 e^{\frac{1}{2} z}, \]
with initial \( z(0) := \ln(\overline{u}_1^0 \overline{u}_2^0) \). A stationary state of the last equation is \( 2 \ln \frac{a_{01}}{2c} \), so 
\[ z \leq \max \{ z(0), 2 \ln \frac{a_{01}}{2c} \}. \]
Because \( \ln(\overline{u}_1(t) \overline{u}_2(t)) \leq z(t) \), for all \( t > 0 \), we find an upper bound for \( \ln(\overline{u}_1 \overline{u}_2) \) and we conclude the proof of (56).

CHEMOTAXIS SYSTEM WITH CHEMICALS AND PREY-PREDATOR TERMS 3347
To obtain inequalities (55), applying (56) to the first equation in (45), we get
\[ \tilde{u}_1' = \tilde{u}_1'|\chi_1|\beta_1(\tilde{u}_2 - u_2) + a_01 - a_11 \tilde{u}_1 - a_12 \tilde{u}_2 \]
\[ = |\chi_1|\beta_1 \tilde{u}_1 \tilde{u}_2 - |\chi_1|\beta_1 \tilde{u}_1 u_2 - a_01 \tilde{u}_1 - a_11 \tilde{u}_1^2 - a_12 \tilde{u}_1 \tilde{u}_2 \]
\[ \leq |\chi_1|\beta_1 C + a_01 \tilde{u}_1 - a_11 \tilde{u}_1^2. \]

By comparison, the previous inequality proves
\[ \tilde{u}_1 \leq \max \left\{ \frac{a_01}{\tilde{u}_1}, \frac{a_01 + \sqrt{a_01^2 + 4|\chi_1|\beta_1 C a_11}}{2a_11} \right\} := K_1. \]

For the existence of an upper bound for \( \tilde{u}_2 \), using the third equation of system (45) and (56)
\[ \tilde{u}_2' = \tilde{u}_2'|\chi_2|\beta_2(\tilde{u}_1 - u_1) - a_02 - a_22 \tilde{u}_2 + a_21 \tilde{u}_1 \]
\[ = |\chi_2|\beta_2 \tilde{u}_1 \tilde{u}_2 - |\chi_2|\beta_2 \tilde{u}_1 u_1 - a_02 \tilde{u}_2 - a_22 \tilde{u}_2^2 + a_21 \tilde{u}_1 \tilde{u}_2 \]
\[ \leq |\chi_2|\beta_2(\tilde{u}_1 \tilde{u}_2 - a_02 \tilde{u}_2 - a_22 \tilde{u}_2^2 + a_21 \tilde{u}_1 \tilde{u}_2 \]
\[ \leq (|\chi_2|\beta_2 + a_21) \tilde{u}_2 - a_02 \tilde{u}_2 - a_22 \tilde{u}_2^2. \]

As in the case of \( \tilde{u}_1 \), we get, by comparison principle,
\[ \tilde{u}_2 \leq \max \left\{ \frac{a_02}{\tilde{u}_2}, \frac{a_02 - \sqrt{a_02^2 + 4a_22(|\chi_2|\beta_2 + a_21)C}}{2a_22} \right\} := K_2. \]

Taking \( K = \max\{K_1, K_2\} \), the proof of (55) is complete and Lemma 3.2 is demonstrated.

3.1. Case \( a_{02}a_{11} - a_{01}a_{21} > 0 \). In this subsection we study the case of exclusion of species under assumption (10) for the coefficients \( (a_{i,j})_{i,j} \); we denote by \( u_1^* \) and \( u_2^* \) the semi-trivial steady state given by (9), i.e.,
\[ u_1^* = \frac{a_{01}}{a_{11}}, \quad u_2^* = 0. \]

We state the stabilization property result, the extinction phenomena in the sense that two of the species \( (u_2, \tilde{u}_2) \) die out asymptotically and the others \( (u_1, \tilde{u}_1) \) reach its carrying.

**Theorem 3.1.** Under assumptions (6), (7) and (10), the solution to system (45),
\( (\tilde{u}_1, u_1, \tilde{u}_2, u_2) \), converges to \( (u_1^*, u_1^*, u_2^*, u_2^*) \) given by (48) as time tends to infinity in the sense that
\[ |\tilde{u}_i(t) - u_i^*| \to 0, \quad |u_i(t) - u_i^*| \to 0, \quad \text{as} \quad t \to \infty, \quad i = 1, 2. \]  

To prove Theorem 3.1, we need some qualitative properties of the solutions, that we enclose in the following lemmas. First of all, we investigate the boundedness of \( (\tilde{u}_1, u_1, \tilde{u}_2, u_2) \).

**Lemma 3.3.** Suppose that \( \chi_i, \beta_i \) and \( a_{i,j} \) satisfy (6)–(7) and (10). For the subsolution \( (\tilde{u}_1, u_2) \) of system (45), we have the following bounds
\[ \tilde{u}_1(t) \leq \frac{a_{01}}{a_{11}} \text{ provided } \tilde{u}_1(0) \leq \frac{a_{01}}{a_{11}} \text{ and } \limsup_{t \to \infty} u_2(t) = 0, \quad \text{for } t \in (0, \infty). \]  


Proof. First, recall that $(\eta_1, \pi_1)$ verify (50), thus, dividing the second equation in (45) by $\eta_1$, we get

$$\frac{\eta'_1}{\eta_1} \leq a_{01} - a_{11} \eta_1. \tag{60}$$

We take the ordinary differential equation

$$y'_1 = a_{01} y_1 - a_{11} y_1^2 \tag{61}$$

whose solution is given by

$$y_1 := \frac{1}{c_1 e^{-a_{01}t} + \frac{a_{11}}{a_{01}}}$$

for $c_1 := \eta_1^{-1}(0) - \frac{a_{11}}{a_{01}}$. From comparison principle, function $u_1(t)$ is a sub-solution of (61)

$$u_1 \leq \frac{1}{c_1 e^{-a_{01}t} + \frac{a_{11}}{a_{01}}} \leq \frac{a_{01}}{a_{11}} := u_1^* \tag{62}$$

Dividing the last equation in (45) by $u_2$, taking into account (50) and (62), it follows that

$$\frac{u'_2}{u_2} \leq -a_{02} - a_{22} u_2 + a_{21} \frac{a_{01}}{a_{11}} \left( a_{01} a_{21} - a_{11} a_{02} \right) = -a_{22} u_2. \tag{63}$$

Thus $u_2(t)$ is a sub-solution of the ordinary differential equation

$$y'_2 = \left( a_{01} a_{21} - a_{11} a_{02} \right) \frac{a_{01}}{a_{11}} y_2 - a_{22} y_2^2 \tag{64}$$

which can be solved easily and its solution is given by

$$y_2 := \frac{1}{c_2 e^{-a_{01} a_{21} - a_{11} a_{02} t} \frac{a_{01}}{a_{11}} + \frac{a_{11} a_{22}}{a_{01} a_{21} - a_{11} a_{02}}} \tag{64}$$

Taking into account that we work under assumption $a_{01} a_{21} - a_{11} a_{02} < 0$ for the coefficients $a_{i,j}$ with $i = 0, 1, 2$, $j = 1, 2$, passing to limit for $t \to \infty$ in (64) we obtain

$$\lim_{t \to \infty} \sup \eta_2(t) \leq \lim_{t \to \infty} y_2(t) = 0,$$

and the proof ends.

Now we prove that the super- and sub-solutions $\pi_2$ and $u_2$ are comparable in both directions. The result is enclosed in the following lemma

**Lemma 3.4.** There exists a positive constant $M > 0$ such that, under the assumptions (6)–(7), for each $t > 0$, we have

$$\pi_2 \leq M u_2.$$

**Proof.** The demonstration is similar to the competitive case (see e.g.,[10] and [41] for more details), so we present only the sketch of the proof. Equation (57) can be written as

$$\frac{\pi'_1}{\pi_1} - \frac{u'_1}{u_1} + \frac{\pi'_2}{\pi_2} - \frac{u'_2}{u_2} = \frac{d}{dt} \left( \ln \frac{\pi_1}{\eta_1} + \ln \frac{\pi_2}{u_2} \right) = A_1 (\pi_1 - \eta_1) + A_2 (\pi_2 - u_2), \tag{65}$$

with $A_1 := 2|\chi_2| \beta_2 + a_{21} - a_{11}$ and $A_2 := 2|\chi_1| \beta_1 + a_{12} - a_{22}$. For each $i = 1, 2$, both $A_i$ are negative due to hypothesis (6)-(7). Taking $\epsilon = \min\{-A_1, -A_2\}$ this implies that

$$\frac{d}{dt} \left( \ln \frac{\pi_1}{\eta_1} + \ln \frac{\pi_2}{u_2} \right) \leq -\epsilon[(\pi_1 - \eta_1) + (\pi_2 - u_2)] \leq 0. \tag{66}$$
Thus by integrating over \((0, t)\), by the positivity of \(\ln(\overline{u}_1/\underline{u}_1)\) because \(\overline{u}_1 > \underline{u}_1\), we get
\[
\ln \left(\frac{\overline{u}_2}{\underline{u}_2}\right) \leq c_0 \Rightarrow \overline{u}_2 \leq M\underline{u}_2, \quad (67)
\]
with \(M = e^{c_0}\), for some positive constant \(c_0 > 0\).

A direct consequence of Lemmas 3.3 and 3.4 is that \(\underline{u}_2\) converges to zero as time tends to infinity as follows

Lemma 3.5. Let \(t > 0\) be positive. Under the assumptions of Theorem 3.1, the sub-solution \(\underline{u}_2\) of (45) satisfies
\[
\underline{u}_2 \to 0, \quad \text{as} \quad t \to \infty. \quad (68)
\]

Proof. Taking limit when \(t \to \infty\) in (64) and by the nonnegativity of \(\underline{u}_2\), the proof ends.

Notice that as a consequence of the previous Lemmas, we have that \(\overline{T}_{max} = \infty\) which implies the global existence of the solutions.

For now, our goal is to obtain that \(\overline{u}_1\) and \(\underline{u}_1\) converge to \(\overline{u}_1^* = \frac{a_{01}}{a_{11}}\) as time tends to infinity.

Lemma 3.6. For every \(t > 0\), under assumptions of Theorem 3.1, we have
1.
\[
\overline{u}_1(t) - \underline{u}_1(t) \to 0 \quad \text{as} \quad t \to \infty \quad \text{and} \quad \int_0^\infty (\overline{u}_1(t) - \underline{u}_1(t))dt < C. \quad (69)
\]
2.
\[
\liminf_{t \to \infty} \overline{u}_1(t) \geq u_1^*. \quad \text{Proof.} \quad \text{The demonstration is similar to the competitive case studied in [10] and we suppress the details (see Lemma 2.6 and Lemma 2.10 in [10] for } i = 1). \quad \square
\]

End of the proof of Theorem 3.1. Applying Lemmas 3.4 and (68) we achieve to
\[
\lim_{t \to \infty} \overline{u}_2 = \lim_{t \to \infty} \underline{u}_2 = 0.
\]
By mean of Lemmas 3.3 and 3.6, we get the following bound and convergence for the sub- and super- solutions \(\underline{u}_1\) and \(\overline{u}_1\), respectively
\[
\liminf_{t \to \infty} \underline{u}_1 \leq u_1^* \leq \limsup_{t \to \infty} \overline{u}_1, \quad \underline{u}_1(t) \to u_1^*, \quad \overline{u}_1 \to u_1^*, \quad \text{as} \quad t \to \infty.
\]

The proof of Theorem 3.1 is done. \quad \square

3.2. Case \(a_{01}a_{21} - a_{02}a_{11} > 0\). In this section we prove that under assumption
\[
a_{01}a_{21} - a_{02}a_{11} > 0,
\]
there exists a unique globally stable steady with positive coordinates which corresponds to the coexistence of the prey and predator.

The following theorem gives a precise description of the asymptotic behavior of the solutions of system (45).
Theorem 3.2. Under assumptions (6), (7) and (12), the solution to system (45), 
\((\overline{u}_1, \overline{u}_2, \overline{u}_3, \overline{u}_4)\), converges to \((u^*_1, u^*_2, u^*_3, u^*_4)\) given by (49), i.e., 
\[
\begin{align*}
  u^*_1 &= \frac{a_{01}a_{22} + a_{02}a_{12}}{a_{11}a_{22} + a_{12}a_{21}}, \\
  u^*_2 &= \frac{-a_{02}a_{11} + a_{01}a_{21}}{a_{11}a_{22} + a_{21}a_{12}}
\end{align*}
\]
as time tends to infinity in the sense that 
\[
\overline{u}_i(t) - u^*_i \to 0, \quad |\overline{u}_i(t) - u^*_i| \to 0, \quad \text{as} \quad t \to \infty, \quad i = 1, 2.
\]
The existence and the stability of the solutions of a prey-predator system with constant coefficients are well known and they can be analyzed with techniques that are similar to those employed for the competition system. Let \((a_{ij})_{i,j}\) be as in (1)–(2), for \(i = 0, 1, 2, j = 1, 2\) and let \(\tilde{u}_1(t), \tilde{u}_2(t)\) be the solutions of the system 
\[
\begin{align*}
  \tilde{u}'_1 &= \tilde{u}_1(a_{01} - a_{11}\tilde{u}_1 - a_{12}\tilde{u}_2), \\
  \tilde{u}'_2 &= \tilde{u}_2(-a_{02} - a_{22}\tilde{u}_2 + a_{21}\tilde{u}_1),
\end{align*}
\]
for \(t \in (0, \infty)\), with initial data 
\[
\tilde{u}_1(0) = \tilde{u}^0_1, \quad \tilde{u}_2(0) = \tilde{u}^0_2,
\]
satisfying 
\[
0 < \tilde{u}^0_1 < \tilde{u}^0_2 < \bar{u}_1^0 \quad 0 < \tilde{u}^0_2 < \tilde{u}^0_2 < \bar{u}_2^0.
\]
For system (71) the following result is proved in [3] Proposition 2.1, p. 479 (see the same result in Theorem 3.2 in [44])

Proposition 3.1. Suppose that 
\[
-\frac{a_{21}}{a_{11}} < -\frac{a_{02}}{a_{01}} < \frac{a_{22}}{a_{12}} < \frac{a_{11}}{a_{21}} < a_{12}.
\]
Then, system (71) has a unique solution \((u^*_1, u^*_2)\) such that 
\[
(\tilde{u}_1(t) - u^*_1, \tilde{u}_2(t) - u^*_2) \to (0, 0) \quad \text{as} \quad t \to \infty,
\]
for any positive solution \((\tilde{u}_1, \tilde{u}_2)\) of (71) where 
\[
\begin{align*}
  u^*_1 &= \frac{a_{01}a_{22} + a_{02}a_{12}}{a_{11}a_{22} + a_{12}a_{21}}, \\
  u^*_2 &= \frac{-a_{02}a_{11} + a_{01}a_{21}}{a_{11}a_{22} + a_{21}a_{12}}.
\end{align*}
\]
Remark that the restrictions (74) are verified in our case and (75) holds due to (6)–(7). Taking into account Lemma 3.1, we can rewrite system (45) as follows 
\[
\begin{align*}
  \bar{u}'_1 &\geq \bar{u}_1(a_{01} - a_{11}\bar{u}_1 - a_{12}\bar{u}_2), \\
  \bar{u}'_2 &\geq \bar{u}_2(-a_{02} - a_{22}\bar{u}_2 + a_{21}\bar{u}_1),
\end{align*}
\]
and 
\[
\begin{align*}
  \bar{u}'_1 &\leq \bar{u}_1(a_{01} - a_{11}\bar{u}_1 - a_{12}\bar{u}_2), \\
  \bar{u}'_2 &\leq \bar{u}_2(-a_{02} - a_{22}\bar{u}_2 + a_{21}\bar{u}_1),
\end{align*}
\]
with initial data (46) satisfying (47), for \(t \in (0, \infty)\). As in the previous subsection, we prove that the two pairs of solutions of the ODE’s system (45), i.e., \((\overline{u}_1, \overline{u}_1)\) and \((\overline{u}_2, \overline{u}_2)\) have the same constant limits \(u^*_1\) and \(u^*_2\), respectively and, hence, also any function between them.
**Lemma 3.7.** The pairs of solutions \((\pi_1, \pi_2)\) and \((u_1, u_2)\) are super and sub solutions of the prey-predator system \((71)\) if the following relations between the initial data are satisfied
\[
0 < u_1^0 < u_1^0 < \pi_1^0, \quad 0 < u_2^0 < \tilde{u}_2^0 < \pi_2^0.
\]  
Thus, we have the ordering
\[
\begin{align*}
&u_1(t) \leq \tilde{u}_1(t) \leq \pi_1(t), \\
u_2(t) \leq \tilde{u}_2(t) \leq \pi_2(t).
\end{align*}
\]  

**Proof.** To obtain (80) we take into account the local stability of \((45)\) and we apply a contradiction argument: assuming that \(\exists t_0 \in (0, \infty)\) such that
\[
\begin{align*}
&\pi_i(t) > \tilde{u}_i(t), \quad u_i(t) < \tilde{u}_i(t), \text{ for } t < t_0 \text{ and } i = 1, 2.
\end{align*}
\]  
Combining \((77), (78)\) and Mean Value Theorem we reach a contradiction (see [34] for a detailed proof).

Thus, \((\pi_1, \pi_2)\) and \((u_1, u_2)\) are super and sub solutions of the prey-predator system \((71)\) for \(t \in (0, \infty)\). Moreover, the result follows from the fact that \((u_1^*, u_2^*)\) is an uniformly asymptotically stable solution for the system of ODEs \((71)\), i.e.,
\[
\lim \inf_{t \to \infty} \pi_i \leq u_i^* \leq \lim \sup_{t \to \infty} \pi_i.
\]

To demonstrate Theorem 3.2 it is enough to obtain that \(\pi_i(t) - u_i(t) \to 0\) for \(i = 1, 2\), when \(t \to \infty\).

**Lemma 3.8.** Under hypothesis \((6)-(7)\) we have
\[
|\pi_i - u_i| \to 0, \quad \text{as} \quad t \to \infty, \quad i = 1, 2.
\]

**Proof.** Operating with the equations of \((45)\) as in Lemma 3.4, we obtain
\[
\left[ \ln \left( \frac{\pi_1}{u_1} \right) + \ln \left( \frac{\pi_2}{u_2} \right) \right]' \leq -\varepsilon \left[ \ln \left( \frac{\pi_1}{u_1} \right) + \ln \left( \frac{\pi_2}{u_2} \right) \right]
\]
for some \(\varepsilon > 0\). If we integrate now, an ODE comparison beside \((76)\) and Lemma 3.7 show
\[
\left[ \ln \left( \frac{\pi_1}{u_1} \right) + \ln \left( \frac{\pi_2}{u_2} \right) \right] \leq e^{-\varepsilon t} \left[ \ln \left( \frac{\pi_1^0}{u_1^0} \right) + \ln \left( \frac{\pi_2^0}{u_2^0} \right) \right].
\]
According to Lemma 3.1, this entails the inequalities (see [41] and [34] for a more detailed demonstration):
\[
\ln \left( \frac{\pi_i}{u_i} \right) \leq e^{-\varepsilon t} \left[ \ln \left( \frac{\pi_i^0}{u_i^0} \right) + \ln \left( \frac{\pi_i^0}{u_i^0} \right) \right]
\]
for \(i = 1, 2\). Taking limits as \(t \to \infty\), we reach the desired goal (81).

**Lemma 3.9.** Under hypothesis \((6)-(7)\), there exists \(M > 0\) such that the following inequalities holds
\[
\pi_i \leq Mu_i
\]

**Proof.** Applying (82) by Lemma 3.8 and Lemma 3.1, we get (83).
End of the proof of Theorem 3.2. Theorem 3.2 is a direct consequence of the properties of the solutions obtained in the previous subsection. Note that Lemmas 3.2, 3.8 and relation (76) fulfill under restrictions (6)–(7), independently of (10) and (12). So we conclude that (58) holds.

4. Comparison principle and asymptotic behavior of solutions. In this section we investigate the asymptotic behavior of the positive solutions of PDE’s (4) and we relate its solutions with the solutions of the ODE’s system (45). As a preliminary, we state the two-sided pointwise estimates for the solution of (4). The following important theorem provides sufficient conditions for the boundedness of classical solutions of system (4).

Theorem 4.1. Let \((u_1^0, u_2^0) \in (L^\infty(\Omega))^2\). The solution of (4) with initial data verifying (86), (87) is bounded and satisfies

\[
\begin{align*}
  u_1(t) &\leq u_1(t, x) \leq \pi_1(t), & \frac{\beta_2}{\alpha_2} u_1(t) &\leq v_2(t, x) \leq \frac{\beta_2}{\alpha_2} \pi_1(t), \\
  u_2(t) &\leq u_2(t, x) \leq \pi_2(t), & \frac{\beta_1}{\alpha_1} u_2(t) &\leq v_1(t, x) \leq \frac{\beta_1}{\alpha_1} \pi_2(t),
\end{align*}
\]

(84)

(85)

where \((u_i, \pi_i)\) is the solution of the ODE system (45) and \((x, t) \in \Omega_\infty\).

We bound the solution of (4) between a lower bound \(u_i\) and an upper bound \(\pi_i\) in order to obtain the same qualitative behavior than \(u_i\) and \(\pi_i\), for \(i = 1, 2\). The proof is based on the Rectangle Method used in Pao [36], see also Cruz, Negreanu and Tello [10] and Negreanu and Tello [33], [34] where the method is applied to competitive Parabolic-Parabolic-Elliptic-Elliptic, Parabolic-Elliptic systems or Predator Prey Lotka Volterra with periodic coefficients reaction diffusion systems with chemotactic terms.

To this end, we shall derive an appropriate differential inequality for some functional involving the functions \(U_i, \overline{U}_i, \overline{V}_i, \underline{V}_i\), for \(i = 1, 2\). which are defined by setting

\[
\overline{U}_i(x, t) := u_i(t, x) - \pi_i(t), \quad \underline{U}_i(x, t) := u_i(t, x) - \underline{u}_i(t),
\]

and

\[
\overline{V}_i(x, t) := v_i(t, x) - \pi_i(t), \quad \underline{V}_i(x, t) := v_i(t, x) - \underline{v}_i(t), \quad i \neq j,
\]

where \((u_1, u_2, v_1, v_2)\) and \((\underline{u}_1, \pi_1, \underline{u}_2, \pi_2)\) are the solutions of (4) and (45), respectively for \((x, t) \in \Omega_\infty\). We aim to prove that the positive and negative parts \((\overline{U}_i)_+, (\underline{U}_i)_-\) are identically zero and therefore the solutions inherit the order \(\underline{u}_i < u_i < \pi_i\). We use for the positive part of a function \(U\) \((U)_+ = U\) if \(\alpha \geq 0\), and 0 otherwise. The negative part function is defined by \((U)_- = (U)_+\).

Due to (5), for all \(x \in \Omega\), there exist positive numbers \((\underline{u}_1^0, \pi_1^0, \underline{u}_2^0, \pi_2^0)\) such that

\[
0 < \underline{u}_1^0 \leq u_1(x) \leq \pi_1^0,
\]

(86)

\[
0 < \underline{u}_2^0 \leq u_2(x) \leq \pi_2^0.
\]

(87)

Because the functions \((u_1, u_2, v_1, v_2)\) are continuous and differentiable in \(\Omega \times (0, T)\), for an arbitrary \(T > 0\), we can find \(c(T) \geq 0\) such that, for every \(i = 1, 2\),

\[
u_i(x, t) \leq c(T) \quad \text{and} \quad v_i(x, t) \leq c(T).
\]
Proof of Theorem 4.1. We consider $0 < T < \infty$. First we deduce the equation that satisfy $\mathcal{U}_i$ and $\mathcal{U}_i$ subtracting equations in (4) and (45), respectively. We multiply the obtained equations by the test function $(\mathcal{U}_i)_+$ or $(\mathcal{U}_i)_-$, operating and integrating by parts, we reach
\[
\frac{d}{dt} \left( \int_{\Omega} (\mathcal{U}_1^2 + (\mathcal{U}_1)_+^2 + (\mathcal{U}_2)_+^2 + (\mathcal{U}_3)_+^2) \right) \leq k(T) \left( \int_{\Omega} (\mathcal{U}_1^2 + (\mathcal{U}_1)_+^2 + (\mathcal{U}_2)_+^2 + (\mathcal{U}_3)_+^2) \right),
\]
with $k(T) > 0$ (the details of the proof, including the study of every resulting terms can be found in [10]). This together with $(\mathcal{U}_i)_+ = (\mathcal{U}_i)_- = 0$, $i = 1, 2$ and Gronwall’s inequality, recalling hypothesis (86), (87) implies
\[
\int_{\Omega} ((\mathcal{U}_1)_+^2 + (\mathcal{U}_1)_-^2 + (\mathcal{U}_2)_+^2 + (\mathcal{U}_3)_+^2) = 0, \quad \forall t \in (0, T)
\]
so we obtain that $\mathcal{U}_{i,+} = \mathcal{U}_{i,-} = 0$, $i = 1, 2$ (for more details, see for instance, [29], [32] or [34]). Hence we have
\[
\omega_1(t) \leq u_1(t, x) \leq \pi_1(t) \quad \omega_2(t) \leq u_2(t, x) \leq \pi_2(t) \quad (x, t) \in \Omega \times (0, T),
\]
By Lemma 3.2 in [30] and [31], we also get that $V_{i,+} = V_{i,-} = 0$, $i = 1, 2$, equivalent to
\[
\frac{\beta_2}{\alpha_2} \omega_1(t) \leq v_2(t, x) \leq \frac{\beta_2}{\alpha_2} \pi_1(t), \quad \frac{\beta_1}{\alpha_1} \omega_2(t) \leq v_1(t, x) \leq \frac{\beta_1}{\alpha_1} \pi_2(t), \quad (x, t) \in \Omega \times (0, T).
\]
Taking limits as $T \to \infty$ (recall that $T > 0$ is arbitrary), we finish the proof of Theorem 4.1. □

End of the proof of Theorem 1.1. The global existence of solutions is given by Lemmas 2.1 and 3.1, Theorem 4.1 and relation (55) by Lemmas 3.2 and 3.8, respectively.

End of the proof of Theorem 1.2. The asymptotic behavior of the solutions is a consequence of Theorem 4.1 and Theorem 3.1 or Theorem 3.2 for the case $a_{02}a_{11} - a_{01}a_{21} < 0$ or the case $a_{02}a_{11} - a_{01}a_{21} > 0$, respectively. □

Acknowledgments. The author is partially supported by Ministerio de Economía y Competitividad under grant MTM2017-83391-P (Spain). The author is very grateful to the referee for his/her detailed comments and valuable suggestions, which improved the manuscript.

REFERENCES

[1] S. Ahmad and A. Lazer, Asymptotic behavior of solutions of periodic competition diffusion systems, Nonlinear Anal., 13 (1989), 263–284.
[2] S. Agmon, A. Douglas and L. Nirenberg, Estimates near the boundary for solutions of elliptic partial differential equations satisfying general boundary conditions. I, Comm. Pure Appl. Math., 12 (1959), 623–727.
[3] Z. Amine and R. Ortega, A periodic prey-predator system, J. Math. Anal. Appl., 185 (1994), 477–489.
[4] X. Bai and M. Winkler, Equilibration in a fully parabolic two-species chemotaxis system with competitive kinetics, Indiana Univ. Math. J., 65 (2016), 553–583.
[5] N. Bellomo, A. Belloquid, Y. Tao and M. Winkler, Toward a mathematical theory of Keller-Segel models of pattern formation in biological tissues, Math. Models Methods Appl. Sci., 25 (2015), 1663–1763.
[6] P. Biler, Local and global solvability of some parabolic systems modelling chemotaxis, Adv. Math. Sci. Appl., 8 (1998), 715–743.
[7] T. Black, Global existence and asymptotic stability in a competitive two-species chemotaxis system with two signals, *Discrete Contin. Dyn. Syst. Ser. B*, 22 (2017), 1253–1272.

[8] T. Black, J. Lankeit and M. Mizukami, On the weakly competitive case in a two-species chemotaxis model, *IMA J. Appl. Math.*, 81 (2016), 860–876.

[9] C. Cosner and A. C. Lazer, Stable coexistence states in the Volterra-Lotka competition model with diffusion, *SIAM J. Appl. Math.*, 44 (1984), 1112–1132.

[10] E. Cruz, M. Negreanu and J. I. Tello, Asymptotic behavior and global existence of solutions to a two-species chemotaxis system with two chemicals, *Z. Angew. Math. Phys.*, 69 (2018), 20pp.

[11] S. M. Fu and M. Ruyun, Existence of a global coexistence state for periodic competition model systems, *Nonlinear Anal.*, 28 (1997), 1265–1271.

[12] D. Gilbarg and N. S. Trudinger, *Elliptic Partial Differential Equations of Second Order*, Grundlehren der Mathematischen Wissenschaften, 224, Springer-Verlag, Berlin-New York, 1977.

[13] K. Gopalsamy, Exchange of equilibria in two species Lotka-Volterra competition models, *J. Austral. Math. Soc. Ser. B*, 24 (1982), 160–170.

[14] G. Hetzer and W. Shen, Convergence in almost periodic competition diffusion systems, *J. Math. Anal. Appl.*, 262 (2001), 307–338.

[15] G. Hetzer and W. Shen, Uniform persistence, coexistence, and extinction in almost periodic/nonautonomous competition diffusion systems, *SIAM J. Math. Anal.*, 34 (2002), 204–227.

[16] D. Horstmann, From 1970 until present: The Keller-Segel model in chemotaxis and its consequences. I, *Jahresber. Deutsch. Math.-Verein.*, 105 (2003), 103–165.

[17] D. Horstmann, Generalizing the Keller–Segel Model: Lyapunov functionals, steady state analysis, and blow-up results for multi-species chemotaxis models in the presence of attraction and repulsion between competitive interacting species, *J. Nonlinear Sci.*, 21 (2011), 231–270.

[18] D. Horstmann and M. Winkler, Boundedness vs. blow-up in a chemotaxis system, *J. Differential Equations*, 215 (2005), 52–107.

[19] V. Hutson, K. Mischaikow and P. Poláčik, The evolution of dispersal rates in a heterogeneous time-periodic environment, *J. Math. Biol.*, 43 (2001), 501–533.

[20] V. Hutson, K. Mischaikow and P. Poláčik, *Elements of Physical Biology*, Williams and Wilkins Co., Baltimore, MD.

[21] T. B. Issa and W. Shen, Uniqueness and stability of coexistence states in two species models with/without chemotaxis on bounded heterogeneous environments, *J. Dyn. Differential Equations*, 31 (2019), 2305-2338.

[22] T. B. Issa and W. Shen, Persistence, coexistence and extinction in two species chemotaxis models on bounded heterogeneous environments, *J. Dyn. Differential Equations*, 31 (2019), 1839-1871.

[23] T. B. Issa and W. Shen, Dynamics in chemotaxis models of parabolic-elliptic type on bounded domain with time and space dependent logistic sources, *SIAM J. Appl. Dyn. Syst.*, 16 (2017), 926–973.

[24] E. F. Keller and L. A. Segel, Initiation of slime mold aggregation viewed as an instability, *J. Theoret. Biol.*, 26 (1970), 399–415.

[25] F. Kentarou and T. Senba, Application of an Adams type inequality to a two-chemical substances chemotaxis system, *J. Differential Equations*, 263 (2017), 88–148.

[26] H. Li and Y. Tao, Boundedness in a chemotaxis system with indirect signal production and generalized logistic source, *Appl. Math. Lett.*, 77 (2018), 108–113.

[27] M. Mizukami, Boundedness and asymptotic stability in a two-species chemotaxis-competition model with signal-dependent sensitivity, *Discrete Contin. Dyn. Syst. Ser. B*, 22 (2017), 2301–2319.

[28] M. Mizukami and T. Yokota, Global existence and asymptotic stability of solutions to a two-species chemotaxis system with any chemical diffusion, *J. Differential Equations*, 261 (2016), 2650–2669.

[29] M. Negreanu and J. I. Tello, On a two species chemotaxis model with slow chemical diffusion, *SIAM J. Math. Anal.*, 46 (2014), 3761–3781.

[30] M. Negreanu and J. I. Tello, Asymptotic stability of a two species chemotaxis system with non-diffusive chemoattractant, *J. Differential Equations*, 258 (2015), 1592–1617.

[31] M. Negreanu and J. I. Tello, On a parabolic-elliptic chemotactic system with non-constant chemotactic sensitivity, *Nonlinear Anal.*, 80 (2013), 1–13.
[32] M. Negreanu and J. I. Tello, On a competitive system under chemotactic effects with non-local terms, *Nonlinearity*, 26 (2013), 1083–1103.
[33] M. Negreanu and J. I. Tello, On a comparison method to reaction-diffusion systems and its applications to chemotaxis, *Discrete Contin. Dyn. Syst. Ser. B*, 18 (2013), 2669–2688.
[34] M. Negreanu and J. I. Tello, Global existence and asymptotic behavior of solutions to a predator-prey chemotaxis system with two chemicals, *J. Math. Anal. Appl.*, 474 (2019), 1116–1131.
[35] C. V. Pao, Coexistence and stability of a competition-diffusion system in population dynamics, *J. Math. Anal. Appl.*, 83 (1981), 54–76.
[36] C. V. Pao, Comparison methods and stability analysis of reaction-diffusion systems, in *Comparison Methods and Stability Theory*, Lecture Notes in Pure and Appl. Math., 162, Dekker, New York, 1994, 277–292.
[37] P. Quittner and P. Souplet, *Superlinear Parabolic Problems. Blow-Up, Global Existence and Steady States*, Birkhäuser Advanced Texts, Birkhäuser Verlag, Basel, 2007.
[38] C. Stinner, J. I. Tello and M. Winkler, Competitive exclusion in a two-species chemotaxis model, *J. Math. Biol.*, 68 (2014), 1607–1626.
[39] Y. Tao and M. Winkler, Boundedness vs. blow-up in a two-species chemotaxis system with two chemicals, *Discrete Contin. Dyn. Syst. Ser. B*, 20 (2015), 3165–3183.
[40] Y. Tao and M. Winkler, Critical mass for infinite-time aggregation in a chemotaxis model with indirect signal production, *J. Eur. Math. Soc. (JEMS)*, 19 (2017), 3641–3678.
[41] J. I. Tello and M. Winkler, A chemotaxis system with logistic source, *Comm. Partial Differential Equations*, 32 (2007), 849–877.
[42] J. I. Tello and M. Winkler, Stabilization in a two-species chemotaxis system with a logistic source, *Nonlinearity*, 25 (2012), 1413–1425.
[43] J. I. Tello and D. Wrzosek, Predator prey model with diffusion and indirect prey-taxis, *Math. Models Methods Appl. Sci.*, 26 (2016), 2129–2162.
[44] A. Tineo, On the asymptotic behavior of some population models, *J. Math. Anal. Appl.*, 167 (1992), 516–529.
[45] A. M. Turing, The chemical basis of morphogenesis, *Philos. Trans. Roy. Soc. London Ser. B*, 237 (1952), 37–72.
[46] V. Volterra, Variazioni e fluttuazioni del numero d individui in specie animali conviventi, *Mem. R. Accad. Naz. Dei Lincei*, (1926).
[47] M. Winkler, Finite time blow-up in th higher-dimensional parabolic-parabolic Keller-Segel system, *J. Math. Pures Appl. (9)*, 100 (2013), 748–767.
[48] Q. Zhang, Competitive exclusion for a two-species chemotaxis system with two chemicals. *Appl. Math. Lett.*, 83 (2018), 27–32.
[49] Q. Zhang, X. Liu and X. Yang, Global existence and asymptotic behavior of solutions to a two-species chemotaxis system with two chemicals, *J. Math. Phys.*, 58 (2017), 9pp.
[50] P. Zheng, C. Mu and X. Hu, Persistence property in a two-species chemotaxis system with two signals, *J. Math. Phys.*, 58 (2017), 17pp.

Received February 2019; 1st revision July 2019; final revision October 2019.

E-mail address: negreanu@mat.ucm.es