On the expected moments between two identical random processes with application to sensor network

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Abstract

We give a closed analytical formula for expected distance to the power $a$ between two identical general random processes, when $a$ is an even positive number. The following identity is valid

$$E[|X_k - Y_k|^a] = a! (\text{Var}[\xi_1])^{\frac{a}{2}} \frac{k^{\frac{a}{2}}}{\lambda^a} + O\left(\frac{k^{-1}}{\lambda^a}\right)$$

(see Theorem 2).

As an application to sensor network we prove that the optimal transportation cost to the power $b > 0$ of the maximal random bicolored matching with edges $\{X_k, Y_k\}$ is

- in $\Theta\left(\frac{n}{\lambda^b}\right)$ when $b \geq 2$,
- in $O\left(\frac{n}{\lambda^b}\right)$ when $0 < b < 2$.

Keywords: Random process, Moment distance, Sensor movement, Matching

1. Motivation

The aim of the note is to study the problem of the expected distance to a power $b > 0$ between two identical general random processes. We define general random process as follows.

Assumption 1 (general random process). Fix $b > 0$. Let $c$ be the smallest even integer greater than or equal to $b$. Consider two identical independent sequences $\{\xi_i\}_{i \geq 1}$, $\{\tau_i\}_{i \geq 1}$ of identically distributed positive, absolutely continuous random variables. Assume that

$$E[\xi_i] = E[\tau_i] = 1,$$

$$E[\xi_i^p] = E[\tau_i^p] \leq C_c, \quad p \in \{2, 3, \ldots, c\}$$

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for some constants $C_c$ independent on $\lambda > 0$, 

$$\forall_{p_i \in \mathbb{N}, 0 \leq p_1 + p_2 + \cdots + p_j \leq c} \left( E \left[ \xi_{i_1}^{p_1} \xi_{i_2}^{p_2} \cdots \xi_{i_j}^{p_j} \right] = E \left[ \xi_{i_1}^{p_1} \right] E \left[ \xi_{i_2}^{p_2} \right] \cdots E \left[ \xi_{i_j}^{p_j} \right], 
E \left[ \tau_{i_1}^{p_1} \tau_{i_2}^{p_2} \cdots \tau_{i_j}^{p_j} \right] = E \left[ \tau_{i_1}^{p_1} \right] E \left[ \tau_{i_2}^{p_2} \right] \cdots E \left[ \tau_{i_j}^{p_j} \right] \right) \quad (3)$$

Let $X_k = \frac{1}{\lambda} \sum_{i=1}^{k} \xi_i$, $Y_k = \frac{1}{\lambda} \sum_{i=1}^{k} \tau_i$.

We are interested in the moments (of each $b > 0$)

$$E \left[ |X_{k+r} - Y_k|^b \right], \quad \text{when} \quad k \geq 1, r \geq 0, \text{(see Assumption 1)}.$$

More importantly, our work is closely related to the paper [5] where the author studied the distance between two i.i.d. Poisson processes with respective arrival times $P_1, P_2, \ldots$ and $Q_1, Q_2, \ldots$ on a line and derived a closed form formula for the distance

$$E \left[ |P_{k+r} - Q_k| \right], \quad \text{for any} \quad k \geq 1, r \geq 0. \quad (4)$$

In addition, he provided an application to sensor networks concerning the optimal expected movement and transportation cost of sensor on the half-infinite interval $[0, \infty)$. The paper [5] treats only the very special case when $P_k, Q_k$ obeys the gamma distribution with parameters $k, \lambda$.

The following open problem was proposed in [5] to study the moments

$$E \left[ |P_{k+r} - Q_k|^b \right],$$

where $b > 0$ is fixed for more general random processes.

We extend the work in [5] by considering the expected distance to all exponents $b > 0$ between two i.i.d. general random processes and thus solve the open problem. The main advantage of our approach is to derive closed form asymptotic formulas for the moments without use of any specific density function (gamma distribution) for a wide class of distributions.

As another point of motivation for studying these distances to exponent arise in sensor networks. We consider two sequences $\{X_i\}_{i=1}^{n}, \{Y_i\}_{i=1}^{n}$ of general i.i.d. random variables (see Assumption 1). The sensors in $X_1, X_2, \ldots, X_n$ are colored black and the sensors in $Y_1, Y_2, \ldots, Y_n$ are colored white. We are interested in expected minimum sum of length to exponent of a maximal bicolored matching (the vertices of each matching edge have different colors).

The cost of sensor movement has been studied extensively in the research community (e.g., see [1, 2, 7, 8, 10]). The book [10] addresses the matching theorems for $N$ random variables independently uniformly distributed in the $d-$dimensional unit cube $[0, 1]^d$, where $d \geq 2$. The authors of [6] deal with covering of the unit interval with uniformly and independently at random placed sensors. Further, in [4] the authors investigate the coverage problem in the high dimension when the cost of movement of sensor is proportional to some (fixed) power $b > 0$. In [3] the author considers absolute moments (of each integer power $a$) of the difference between the $k$-th arrival time in two i.i.d. Poisson processes and gives a closed formula involving the gamma function.
2. Main results

Fix \( b > 0 \). In this section the expected distance to the power \( b \) between two i.i.d. general random processes is analyzed.

Firstly, we derive closed form formula for expected distance to the power \( a \) between two identical general random processes, when \( a \) is an even positive integer. We proof Theorem 2.

**Theorem 2.** Let us fix an even positive integer \( a \). Let Assumption 1 hold for \( b := a \) and let \( k > \frac{a}{2} \). Then the following identity is valid

\[
E \left[ |X_k - Y_k|^a \right] = \frac{a! (\text{Var} \left[ \xi_1 \right])^{\frac{a}{2}} k^\frac{a}{2}}{\left( \frac{a}{2} \right)!} \lambda^a + O \left( k^{\frac{a}{2} - 1} \right) \lambda^a.
\]

The general strategy of our combinatorial proof of Theorem 2 is the following. Applying multinomial theorem we write

\[
E \left[ |X_k - Y_k|^a \right] \text{ as the sum (see Equation (5)).}
\]

Next, we make an important observation that

\[
E \left[ (\xi_i - \tau_i)^{2d+1-j} \right] = 0 \quad \text{(see Equation (6)).}
\]

Using this, we rewrite \( E \left[ |X_k - Y_k|^a \right] \) as the sums (13) and (15). Finally, the asymptotic depends on the expression given by the first sum (see Equation (13)), while the second sum (see Equation (15)) is negligible.

**Proof.** Fix an even positive integer \( a \). Assume that \( k > \frac{a}{2} \).

Firstly, combining together multinomial theorem, Equation (3) as well as Assumption 1 we deduce that

\[
E \left[ |X_k - Y_k|^a \right] = E \left[ (X_k - Y_k)^a \right]
\]

\[
= \sum_S \frac{a!}{(l_1)! (l_2)! \cdots (l_k)!} \frac{1}{\lambda^a} \left[ \prod_{i=1}^k (\xi_i - \tau_i)^{l_i} \right]
\]

\[
= \sum_S \frac{a!}{(l_1)! (l_2)! \cdots (l_k)!} \frac{1}{\lambda^a} \prod_{i=1}^k E \left[ (\xi_i - \tau_i)^{l_i} \right], \quad (5)
\]

where

\[
S = \{ (l_1, l_2, \ldots, l_k) \in \mathbb{N}^k : l_1 + l_2 + \cdots + l_k = a \}.
\]

Let \( d \) be natural number. Using Assumption 1 and the basic binomial identity

\[
\binom{2d+1}{j} (-1)^{2d+1-j} = -\binom{2d+1}{2d+1-j} (-1)^j
\]

we have

\[
E \left[ (\xi_i - \tau_i)^{2d+1} \right] = \sum_{j=0}^{2d+1} \binom{2d+1}{j} E \left[ \xi_i^j \right] (-1)^{2d+1-j} E \left[ \tau_i^{2d+1-j} \right]
\]

\[
= \sum_{j=0}^{2d+1} \binom{2d+1}{j} E \left[ \tau_i^j \right] (-1)^{2d+1-j} E \left[ \tau_i^{2d+1-j} \right]
\]

\[
= \sum_{j=0}^d E \left[ \tau_i^j \right] E \left[ \tau_i^{2d+1-j} \right] \left( \binom{2d+1}{j} (-1)^{2d+1-j} + \binom{2d+1}{2d+1-j} (-1)^j \right)
\]

\[
= 0. \quad (6)
\]
Combining together (5) and (6) we deduce that

\[
E [|X_k - Y_k|^a] = \sum_{S_1} \frac{a!}{(l_1)! l_2! \ldots l_k!} \frac{1}{\lambda^a} \prod_{i=1}^{k} E \left[ (\xi_i - \tau_i)^{l_i} \right],
\]

where

\[ S_1 = \{(l_1, l_2, \ldots, l_k) \in \mathbb{N}^k : l_1 + l_2 + \cdots + l_k = a, \ l_i \ \text{are even for} \ i = 1, 2, \ldots, k \}. \]

Observe that

\[ S_1 = S_2 \cup S_3, \]

\[ S_2 = \{(l_1, l_2, \ldots, l_k) \in \mathbb{N}^k : l_1 + l_2 + \cdots + l_k = a, \ l_i \in \{0, 2\} \ for \ i = 1, 2, \ldots, k \}, \]

\[ S_3 = \{(l_1, l_2, \ldots, l_k) \in \mathbb{N}^k : l_1 + l_2 + \cdots + l_k = a, \ l_i \ \text{are even for} \ i = 1, 2, \ldots, k, \ \exists i (l_i \neq 2) \}, \]

\[ |S_2| = \left(\frac{k}{a} \right), \quad |S_3| = O \left( k^{\frac{p}{p-1}} \right). \]

Let \( f(t) \) be the probability density function of the random variables \( \xi_i, \tau_i \). We use Hölder’s inequalities with \( p \in \{2, 3, \ldots, a\}, \quad q = \frac{p-1}{p} \) and get the following sharp inequalities

\[
\int_0^\infty tf(t)dt < \left( \int_0^\infty t^p f(t)dt \right)^{1/p} \left( \int_0^\infty f(t)dt \right)^{1/q}.
\]

Putting together Inequality (10) and Equality (11) in Assumption (11) we deduce that

\[
E \left[ \xi_i^p \right] = E \left[ \tau_i^p \right] > 1, \quad \text{when} \quad p \in \{2, 3, \ldots, a\}.
\]

Observe that

\[
E \left[ (\xi_i - \tau_i)^2 \right] = 2 \left( E \left[ \xi_i^2 \right] - 1 \right) = 2 \left( \text{Var} \left[ \xi_i \right] \right) > 0.
\]

Together, (9), (12) and (11) imply

\[
\sum_{S_2} \frac{a!}{(l_1)! l_2! \ldots l_k!} \frac{1}{\lambda^a} \prod_{i=1}^{k} E \left[ (\xi_i - \tau_i)^{l_i} \right] = \frac{a! \left( \text{Var} \left[ \xi_i \right] \right)^{\frac{p}{p-1}}}{\lambda^a} |S_2| = \frac{a! \left( \text{Var} \left[ \xi_i \right] \right)^{\frac{p}{p-1}} k^{\frac{p}{p-1}}}{(\frac{q}{p})!} \frac{k^{\frac{p}{p-1}}}{\lambda^a} + O \left( k^{\frac{p}{p-1}} \right).
\]

Using Inequality (2) in Assumption (11) we have

\[
E \left[ (\xi_i - \tau_i)^{l_i} \right] = E \left[ (|\xi_i| + |\tau_i|)^{l_i} \right] \leq \sum_{j=0}^{l_i} \binom{l_i}{j} E \left[ \xi_i^j \right] E \left[ \tau_i^{l_i-j} \right] \leq C_0^n \sum_{j=0}^{l_i} \binom{l_i}{j} = C_0^n 2^{l_i}.
\]
Together, (14), (9) and $C_a > 1$ (see (11)) imply
\[
\sum_{S_3} \frac{a!}{(l_1)!l_2! \ldots (l_k)!} \frac{1}{\lambda^a} \prod_{i=1}^k E \left[ (\xi_i - \tau_i)^{l_i} \right] \leq a! 2^{a} C_a \frac{|S_3|}{\lambda^a} = O \left( \frac{(k^a - 1)}{\lambda^a} \right). \tag{15}
\]

Finally, combining together (7), (8), (13), (15) finishes the proof of Theorem 2.

The next results supports our earlier result whereby the expected distance to the power $a$ between two i.d.d. random processes, remains in $\Theta \left( k^a \lambda \right)$ provided that $r = o \left( \frac{k^a}{\lambda} \right)$.

**Theorem 3.** Let us fix an even positive integer $a$. Let Assumption 1 hold for $b := a$. If $r = o \left( \frac{k^a}{\lambda} \right)$ then
\[
E \left[ |X_{k+r} - Y_k|^a \right] = \frac{\Theta \left( k^a \right)}{\lambda^a}.
\]

**Proof.** Firstly, we apply multinomial theorem, Equation (3) and get
\[
E \left[ |X_{k+r} - X_k|^a \right] = E \left[ \left( \sum_{i=1}^r X_{k+i} \right)^a \right] = E \left[ \left( \sum_{i=1}^r X_{k+i} \right)^a \right]
\]
\[
= \sum_{l_1 + l_2 + \ldots + l_r = a} \frac{a!}{l_1!l_2! \ldots (l_r)!} \frac{1}{\lambda^a} \prod_{i=1}^r E \left[ \xi_{k+i}^{l_i} \right].
\]
Using Inequality (2) in Assumption 1 and (11) we have
\[
\prod_{i=1}^r E \left[ \xi_{k+i}^{l_i} \right] \leq C_a^a.
\]
Hence
\[
E \left[ |X_{k+r} - X_k|^a \right] \leq \frac{C_a^a}{\lambda^a} \sum_{l_1 + l_2 + \ldots + l_r = a} \frac{a!}{l_1!l_2! \ldots (l_r)!} = \frac{C_a^a}{\lambda^a} r^a.
\]
Since $r = o \left( \frac{k^a}{\lambda} \right)$, we have
\[
E \left[ |X_{k+r} - X_k|^a \right] = o \left( \frac{k^a}{\lambda} \right). \tag{16}
\]
Combining together assumption $r = o \left( \frac{k^a}{\lambda} \right)$ and the result of Theorem 2 for $k := k+r$ we easily deduce that
\[
E \left[ |X_{k+r} - Y_{k+r}|^a \right] = \frac{\Theta \left( k^a \right)}{\lambda^a}. \tag{17}
\]
Notice that
\[
|x + y|^a \leq (|x| + |y|)^a \leq 2^{a-1} \left( |x|^a + |y|^a \right) \quad \text{when} \quad a \geq 1, \ x, y \in \mathbb{R}. \tag{18}
\]
This inequality follows from the fact that \( f(x) = x^a \) is convex over \( \mathbb{R}_+ \) for \( a \geq 1 \). Applying (18) for \( x := X_k - Y_k \), \( y := X_{k+r} - X_k \) we get
\[
E \left[ |X_{k+r} - Y_k|^a \right] \leq 2^{a-1} \left( E \left[ |X_k - Y_k|^a \right] + E \left[ |X_{k+r} - X_k|^a \right] \right). \tag{19}
\]
Combining together (19), the result of Theorem 2 and Equation (16) we have the desired upper bound
\[
E \left[ |X_{k+r} - Y_k|^a \right] = O \left( k^{\frac{2}{a}} \lambda \right) \text{ if } r = o \left( k^{\frac{1}{2}} \right).
\]

Next, applying (18) for \( x := X_{k+r} - Y_k \), \( y := X_k - X_{k+r} \) we have
\[
E \left[ |X_k - Y_k|^a \right] \leq 2^{a-1} \left( E \left[ |X_{k+r} - Y_k|^a \right] + E \left[ |X_k - X_{k+r}|^a \right] \right) \tag{20}
\]
Together (20), the result of Theorem 2 and Equation (16) imply the lower bound
\[
E \left[ |X_{k+r} - Y_k|^a \right] = \Theta \left( k^{\frac{2}{a}} \right) \text{ if } r = o \left( k^{\frac{1}{2}} \right).
\]
This is sufficient to complete the proof of Theorem 3.

The next theorem extends our Theorem 3 to real-valued exponents. In the proof of Theorem 4 we combine together Jensen’s inequality and the results of Theorem 3.

Let us recall Jensen’s inequality for expectations. If \( f \) is a convex function, then
\[
f \left( E[X] \right) \leq E \left[ f(X) \right] \tag{21}
\]
provided the expectations exists (see [9, Proposition 3.1.2]).

**Theorem 4.** Fix \( b > 0 \). Let Assumption 1 hold. If \( r = o \left( k^{\frac{1}{2}} \right) \), then
\[
E \left[ |X_{k+r} - Y_k|^b \right] = \begin{cases} 
\Theta \left( k^{\frac{b}{2}} \lambda \right) & \text{ if } b \geq 2, \\
O \left( k^{\frac{b}{2}} \right) & \text{ if } 0 < b < 2.
\end{cases}
\]

**Proof.** First we prove the upper bound. Assume that \( b > 0 \). We use Jensen’s inequality (see 21) for \( X = |X_{k+r} - Y_k|^b \) and \( f(x) = x^{\frac{1}{b}} \) and get
\[
\left( E \left[ |X_{k+r} - Y_k|^b \right] \right)^{\frac{1}{b}} \leq E \left[ |X_{k+r} - Y_k|^{2[b]} \right]. \tag{22}
\]
Putting together Theorem 3 for \( a := 2[b] \) and Inequality 22 we deduce that
\[
E \left[ |X_{k+r} - Y_k|^b \right] \leq \left( \Theta \left( k^{[b]} \right) \right)^{\frac{1}{b}} \frac{\Theta \left( k^{\frac{b}{2}} \right)}{\lambda \lambda^b} = \frac{\Theta \left( k^{\frac{b}{2}} \right) \lambda}{\lambda^b}.
\]
This proves the upper bound.
Next we prove the lower bound. Assume that $b \geq 2$. We apply Jensen’s inequality (see (21)) for $X = |X_{k+r} - Y_k|^2$ and $f(x) = x^{\frac{b}{2}}$ and have

$$
(\mathbb{E} [ |X_{k+r} - Y_k|^2 ])^{\frac{b}{2}} \leq \mathbb{E} [ |X_{k+r} - Y_k|^b ].
$$

(23)

Combining together Theorem 3 for $a := 2$ and Inequality (23) we get

$$
\mathbb{E} [ |X_{k+r} - Y_k|^b ] \geq \left( \frac{\Theta (k)}{\lambda^2} \right)^{\frac{b}{2}} = \frac{\Theta \left( \frac{k^{\frac{b}{2}}}{\lambda} \right)}{\lambda^b}.
$$

This is enough to prove the lower bound and completes the proof of Theorem 4.

3. Application to sensor networks

In this section, we consider the optimal transportation cost to the power $b$ of the maximal random bicolored matching, when $b > 0$.

Assume that sensors are initially placed according to two general random processes. Let Assumption 1 hold. The sensors in $X_1, X_2, \ldots, X_n$ are colored black and the sensors in $Y_1, Y_2, \ldots, Y_n$ are colored white.

We would like to find the maximal bicolored matching $M$ so as to:

1. for every pair of sensors $\{X_k, Y_l\} \in M$, the sensors $X_k, Y_k$ have different colors,
2. the expected transportation cost to the power $b > 0$ defined as

$$
T_b(M) := \sum_{\{X_k, Y_l\} \in M} \mathbb{E} [ |X_k - Y_l|^b ]
$$

is minimized.

Firstly, we observe that the minimal transportation cost to the power $b$ is attained by the maximal matching with edges $\{X_k, Y_k\}$ for $k = 1, 2, \ldots, n$.

**Lemma 5.** Fix $b \geq 0$. Let $M_{opt}$ be the maximal matching with edges $\{X_k, Y_k\}$, for $k = 1, 2, \ldots, n$. Then for all matchings $M$ we have

$$
T_b(M) \geq T_b (M_{opt}).
$$

**Proof.** The proof is essentially the same as the proof of [5, Lemma 5].

We are now ready to analyze the maximal matching with edges $\{X_k, Y_k\}$ for $k = 1, 2, \ldots, n$. Applying Theorem 4 from the previous section we can prove the following theorem.

**Theorem 6.** Fix $b > 0$. If $M_{opt}$ denotes the maximal matching with edges $\{X_k, Y_k\}$ for $k = 1, 2, \ldots, n$, then

$$
T_b(M_{opt}) = \begin{cases} 
\Theta \left( \frac{n^{\frac{b+1}{2}}}{\lambda^b} \right) & \text{when } b \geq 2 \\
O \left( \frac{n^{\frac{b+1}{2}}}{\lambda^b} \right) & \text{when } 0 < b < 2.
\end{cases}
$$
Proof. First of all, observe that

\[ T_b(M_{opt}) = \sum_{k=1}^{n} E \left[ |X_k - Y_k|^b \right]. \]

After that, the result of Theorem 6 follows immediately from well known identity

\[ \sum_{k=1}^{n} k^b = \Theta \left( n^{\frac{b}{2}+1} \right) \quad \text{when } b > 0 \]

and Theorem 4 for \( r := 0 \).

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