Baxter $Q$-operators for integrable DST chain.

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Abstract

Following the procedure, described in the paper [10], for the integrable DST chain we construct Baxter $Q$-operators [1] as the traces of monodromy of some $M$-operators, that act in quantum and auxiliary spaces. Within this procedure we obtain two basic $M$-operators and derive some functional relations between them such as intertwining relations and wronskian-type relations between two basic $Q$-operators.

1 Introduction

Integrable periodic quantum DST (Discrete Self-Trapping) chain is a quantum system described by the following hamiltonian (it corresponds to a certain set of parameters $\omega_0, \gamma, \epsilon$ and $m_{ij}$ in the hamiltonian considered in [2])

$$H = \sum_{k=1}^{N} \left[ \varphi_k^+ \varphi_k + (\varphi_k^+ \varphi_k)^2/2 + \varphi_{k+1}^+ \varphi_k \right], \quad (1)$$

where the canonical variables $\varphi_k^+$ and $\varphi_k$ satisfy commutation relations $[\varphi_i, \varphi_j^+] = \delta_{ij}$ and periodic boundary conditions $\varphi_{k+N} = \varphi_k, \, \varphi_{k+N}^+ = \varphi_k^+$. The system can be considered within the framework of the quantum inverse scattering method or $R$-matrix method. There exists the Lax operator connected with DST chain. It acts in the tensor product of $n$-th quantum space and two-dimensional auxiliary space $\mathbb{C}^2$ (see [3], [5]):

$$L_n(x) = \begin{pmatrix} x - i/2 - i\varphi_n^+ \varphi_n & \varphi_n^+ \\ \varphi_n & i \end{pmatrix}, \quad (2)$$

where $x$ is the spectral parameter. The fundamental relations for the matrix elements of the Lax operator could be written in the following $R$-matrix form:

$$R_{12}(x - y)L_n^1(x)L_n^2(y) = L_n^2(y)L_n^1(x)R_{12}(x - y), \quad (3)$$

where the indicies 1, 2 indicate different auxiliary spaces and $R$–matrix is given by

$$R_{12}(x) = x + iP_{12}, \quad (4)$$

where $P_{12}$ is the permutation operator in the auxiliary spaces 1 and 2. The same intertwining relations are true for the monodromy matrix $T(x) = \prod_{n=1}^{N} L_n(x)$ (here the multipliers
are ordered from right to left), this in turn means that \([t(x), t(y)] = 0\), where we denote \(t(x) = TrT(x)\) (the trace is taken over the auxiliary space). Thus coefficients of the polynomials (over the spectral parameter) \(t(x)\) form the family of commuting operators and the hamiltonian \(H\) and the number of particle operator belong to this family (namely, if we expand \(t(x) = \sum_{k=0}^{N} (x - i/2)^k H^{(k)}\), the number of particle operator \(\hat{n} = \sum_{k=1}^{N} \varphi_k^+ \varphi_k = iH^{(N-1)}\) and hamiltonian \(H = iH^{(N-1)} - (H^{(N-1)})^2/2 + H^{(N-2)}\).

The eigenvectors and the eigenvalues of \(t(x)\) could be constructed in the framework of ABA \([8]\). In this approach one considers the monodromy matrix given by

\[
T(x) = \prod_{n=1}^{N} L_n(x) = \begin{pmatrix} \hat{A}(x) & \hat{B}(x) \\ \hat{C}(x) & \hat{D}(x) \end{pmatrix}.
\]

There exists the so-called Bethe vacuum: \(C(x)\Omega = 0(\Omega = \prod \otimes \omega_k\), where \(\varphi_k \omega_k = 0\). Vectors \(\hat{B}(x_1)\ldots\hat{B}(x_l)\Omega\) will be eigenvectors of \(t(x) = TrT(x) = \hat{A}(x) + \hat{D}(x)\) with eigenvalues

\[
t(x) = (x - i/2)^N \prod_{j=1}^{l} \frac{(x - x_j + i)}{(x - x_j)} + i^N \prod_{j=1}^{l} \frac{(x - x_j - i)}{(x - x_j)}
\]

provided that \(x_i\) obey the Bethe equations

\[
\prod_{j=1}^{l} \frac{(x_i - x_j - i)}{(x_i - x_j + i)} = -\frac{(x_i - i/2)^N}{i^N}
\]

So the polynomial \(q(x) = \prod_{j=1}^{l}(x - x_j)\) satisfies Baxter equation:

\[
t(x)q(x) = (x - i/2)^N q(x - i) + i^N q(x + i) . \tag{5}
\]

According to Baxter \([1]\) let us define the operator \(Q(x)\) such that:

\[
t(x)Q(x) = (x - i/2)^N Q(x - i) + i^N Q(x + i) , \tag{6}
\]

and \([t(x), Q(y)] = 0\), \([Q(x), Q(y)] = 0\).

The model under consideration occupies an intermediate place between two other integrable models: the XXX spin chain and the Toda chain (Lax operators of these models are intertwined by the rational \(R\)-matrix \([3]\) too). As in the case of XXX spin chain there is the \(Q\)-operator with polynomial eigenvalues in the spectral parameter. It corresponds to the ABA. If we consider the equation \((3)\) as discrete analog of a second order differential equation and immediately there arises the question about the second solution of \((3)\). These second solutions have been intensively discussed in \([1, 8]\). The eigenvalues of the second \(Q\)-operator for DST model are meromorphic functions (in the case of Toda chain there is no ABA but there exist two \(Q\)-operators: one with entire eigenvalues, second with meromorphic eigenvalues \([10]\)). In the next part these two solutions of \((3)\) will be constructed.

The existence of the second \(Q\)-operator, which is linear independent from the first one could be seen from the following simple consideration (similar discussions for the case of XXX-spin chain see in \([3]\)). Let us consider Baxter equation for the eigenvalue of the first
Q-operator - $q(x)$, which is a polynomial of degree $n$ (in the case of DST chain $n$ equals the eigenvalue of number of particle operator $\sum_{i=1}^{N} \varphi_i^+ \varphi_i$), and the eigenvalue of the trace of monodromy matrix $t(x)$, which is a polynomial of degree $N$:

$$t(x)q(x) = (x - i/2)^N q(x - i) + i^N q(x + i)$$

or

$$\frac{t(x)}{q(x + i)q(x - i)} = \frac{(x - i/2)^N}{q(x)q(x + i)} + i^N \frac{q(x)}{q(x)q(x - i)}$$

Multiplying this equation by $\Gamma^N(-i(x - i/2))$ we get:

$$\frac{t(x)\Gamma^N(-i(x - i/2))}{q(x + i)q(x - i)} = \frac{i^N \Gamma^N(-i(x + i/2))}{q(x)q(x + i)} + \frac{i^N \Gamma^N(-i(x - i/2))}{q(x)q(x - i)}$$

Let us denote

$$S(x) = \frac{i^N \Gamma^N(-i(x + i/2))}{q(x)q(x + i)},$$

then

$$\frac{t(x)\Gamma^N(-i(x - i/2))}{q(x + i)q(x - i)} = S(x) - S(x - i).$$

$S(x)$ can be rewritten as follows:

$$S(x) = i^N \Gamma^N(-i(x + i/2)) \left[ \frac{q_1(x)}{q(x + i)} + \frac{q_2(x)}{q(x)} \right],$$

where $q_1(x)$ and $q_2(x)$ are polynomials of degree less than $n$. Substituting this expansion into Baxter equation (8) we get:

$$\frac{t(x)}{q(x + i)q(x - i)} = (x - i/2)^N \left[ \frac{q_1(x)}{q(x + i)} + \frac{q_2(x)}{q(x)} \right] + i^N \left[ \frac{q_1(x - i)}{q(x)} + \frac{q_2(x - i)}{q(x - i)} \right]$$

Since $t(x)$ is a polynomial we see that $(x - i/2)^N q_2(x) + i^N q_1(x - i) = r(x)q(x)$, where $r(x)$ is a polynomial with the degree less than $N$. Expressing then $q_1(x)$ via $q_2(x)$ and $r(x)$, let us substitute it in the expression for $S(x)$:

$$S(x) = i^N \Gamma^N(-i(x + i/2)) r(x + i) + i^N \Gamma^N(-i(x + i/2)) \frac{q_2(x)}{q(x)} \Gamma^N(-i(x + 3i/2)) \frac{q_2(x + i)}{q(x + i)}$$

Now our task is to present $S(x)$ in the following form

$$S(x) = \frac{p(x + i)}{q(x + i)} - \frac{p(x)}{q(x)},$$

and $p(x)$ will be the eigenvalue of the second $Q$-operator. Indeed from (10) we get

$$i^N \Gamma^N(-i(x + i/2)) = p(x + i)q(x) - p(x)q(x + i),$$

and

$$S(x) = \frac{p(x + i)}{q(x + i)} - \frac{p(x)}{q(x)}.$$
and \(i^N \Gamma^N(-i(x + i/2))^N \Gamma^N(-i(x - i/2)) = p(x)q(x - i) - p(x - i)q(x)\). Multiplying the last equation by \((-i(x - i/2))^N\) and subtracting it from the previous one we see, that \(p(x)\) satisfies the same Baxter equation:

\[
t(x)p(x) = (x - i/2)^N p(x - i) + i^N p(x + i). \tag{17}
\]

Thus the next step is to find the function \(g(x)\) such that

\[
g(x + i) - g(x) = i^N \Gamma^N(-i(x + i/2))r(x + i). \tag{18}
\]

Let us look for \(g(x)\) in the following form:

\[
g(x) = \sum_{k=0}^{\infty} f(-ix - k). \tag{19}
\]

In this case \(g(x + i) - g(x) = -f(-ix)\), and we see that if

\[
f(-ix) = -i^N \Gamma^N(-i(x + i/2))r(x + i),
\]

then

\[
g(x) = -i^N \sum_{k=0}^{\infty} \Gamma^N(-i(x + i/2) - k)r(x + i - ik), \tag{20}
\]

and the desired eigenvalue will be given

\[
p(x) = g(x)q(x) - i^N \Gamma^N(-i(x + i/2))q_2(x) \tag{21}
\]

Apparently (21) is a meromorphic function with respect to the spectral parameter \(x\) which has the poles at the integer values of \(y = -ix + 1/2\) (the convergency of the series for \(g(x)\) at \(-ix + 1/2 \neq \mathbb{Z}\) is provided by the term \(-k\) in the gamma function argument).

As an illustration consider a simple example for the concrete polynomial solution of Baxter equation for the case of two degrees of freedom

\[
q(x) = x^2 - 2ix + 1/4.
\]

This solution corresponds to the Bethe vector \(\frac{1}{\sqrt{2}}(|2, 0 > > 0, 2 >)\) and the eigenvalue of

\[
t(x) = x^2 - 3ix - 9/4.
\]

Here we have introduced following notation for the vectors of quantum space:

\[
|k_1, k_2 >= (\varphi_1^+)^{k_1}(\varphi_2^+)^{k_2}|0, 0 >,
\]

where \(|0, 0 >\) is the Bethe vacuum \((\varphi_1|0, 0 >= \varphi_2|0, 0 >= 0\).

The explicit construction of the polynomials \(q_1, q_2, r\) using the method described above gives:

\[
q_1(x) = -i/2x + 1/4, \quad q_2(x) = i/2x + 3/4, \quad r(x) = i/2x + 1/4.
\]

And for the eigenvalue of the second \(Q\)-operator we obtain

\[
p(x) = (x^2 - 2ix + 1/4) \sum_{k=0}^{\infty} \Gamma^2(-ix + 1/2 - k)(i/2x + k/2 - 1/4) +
\]
\[
+ \Gamma^2(-ix + 1/2)(i/2x + 3/4)
\]
2 Basic $Q$-operators for the DST model

In the present paper we shall construct the basic $Q$-operators using the method described in \cite{10}. In this approach the $Q^{(1,2)}$-operators are the traces of monodromies $\hat{Q}^{(1,2)}$ of appropriate $M^{(1,2)}_n$-operators acting in $n$-th quantum space and in the auxiliary space $\Gamma$, which is the representation space of Heisenberg algebra $[\rho, \rho^+] = 1$. As we shall need to consider the product of $L(x)M^{(1,2)}(x)$, the mutual auxiliary space for this object are $\Gamma \otimes \mathbb{C}^2$, where we can introduce the projectors:

$$
\Pi^+_{ij} = \left( \frac{1}{\rho} \right)_{ij} \frac{1}{(\rho^+\rho + 1)} (1, \rho^+)_j, \quad \Pi^-_{ij} = \left( -\frac{\rho^+}{1} \right)_{ij} \frac{1}{(\rho^+\rho + 2)} (-\rho, 1)
$$

(22)

According to the method of \cite{10} we impose the condition that the products $L(x)M(x)$ and $M(x)L(x)$ have triangle forms in the sense of projectors $\Pi^\pm$:

$$
\left\{ \begin{array}{ll}
\Pi^-_{ik} (L_n(x))_{kl} M^{(1)}_n(x) \Pi^+_{lj} = 0 \\
\Pi^+_{ik} M^{(1)}_n(x) (L_n(x))_{kl} \Pi^-_{lj} = 0
\end{array} \right. ,
$$

(23)

for $M^{(1)}_n(x)$ and

$$
\left\{ \begin{array}{ll}
\Pi^+_{ik} (L_n(x))_{kl} M^{(2)}_n(x) \Pi^-_{lj} = 0 \\
\Pi^-_{ik} M^{(2)}_n(x) (L_n(x))_{kl} \Pi^+_{lj} = 0
\end{array} \right. ,
$$

(24)

for $M^{(2)}_n(x)$. Consider first the system for $M^{(1)}$. It follows from the first equation in (23) that:

$$
\left\{ \begin{array}{l}
M^{(1)}(x) \left( \frac{1}{\rho} \right) = \tilde{L}(x)_{ij} \left( \frac{1}{\rho} \right) A^{(1)}(x) \\
B^{(1)}(x) \left( \frac{1}{\rho} \right) = \tilde{L}(x + i)_{ij} \left( \frac{1}{\rho} \right) M^{(1)}(x)
\end{array} \right. ,
$$

(25)

where we introduce

$$
\tilde{L}(x) = \left( \begin{array}{cc}
i & -\varphi^+ \\
-\varphi & x - 3i/2 - \varphi^+ \varphi
\end{array} \right),
$$

with properties $L(x)\tilde{L}(x) = i(x - i/2) \cdot I$ ($I$ is the identity matrix) and $L(x) + \tilde{L}(x + i) = Tr L(x) \cdot I$ (this identity provides the argument shift in (24) and leads to the finite differences equation). System (23) has the solution of the form $B^{(1)}(x) = cM^{(1)}(x + i)$, $A^{(1)}(x) = c^{-1}M^{(1)}(x)$, where $c$ is a number. Let us choose $c = i$. Along with the analogous consideration of the triangularity condition for right multiplication $\Pi^+_{ik} M^{(1)}_n(x) (L_n(x))_{kl} \Pi^-_{lj} = 0$ it leads to the system:

$$
\left\{ \begin{array}{l}
\tilde{L}(x + i)_{ij} \left( \frac{1}{\rho} \right) M^{(1)}(x) = M^{(1)}(x + i) \left( \frac{1}{\rho} \right) \\
M^{(1)}(x)L(x)_{ij} \left( -\frac{\rho^+}{1} \right) = i \left( -\frac{\rho^+}{1} \right) M^{(1)}(x + i)
\end{array} \right. ,
$$

(26)

For $M^{(2)}$ we get:

$$
\left\{ \begin{array}{l}
\tilde{L}(x + i)_{ij} \left( -\frac{\rho^+}{1} \right) M^{(2)}(x) = M^{(2)}(x + i) \left( -\frac{\rho^+}{1} \right) \\
M^{(2)}(x)L(x)_{ij} \left( \frac{1}{\rho} \right) = i \left( \frac{1}{\rho} \right) M^{(2)}(x + i)
\end{array} \right. ,
$$

(27)
The full multiplication rules have the following form:

\[
(L_n(x))_{ij} M_n^{(1)}(x) = \left( \frac{1}{\rho} \right)_i M_n^{(1)}(x - i) \frac{1}{\rho^+ \rho + 1} (1, \rho^+)_{ij} + \left( -\rho^+ \right)_i \frac{1}{\rho^+ \rho + 2} M_n^{(1)}(x + i)(-\rho, 1)_j + \Pi_{ik}^+ (L_n(x))_{kl} M_n^{(1)}(x) \Pi_{ij}^{+}
\]

\[
(L_n(x))_{ij} M_n^{(2)}(x) = \left( \frac{1}{\rho} \right)_i M_n^{(2)}(x + i)(1, \rho^+)_{ij} + \left( -\rho^+ \right)_i \frac{1}{\rho^+ \rho + 2} M_n^{(2)}(x - i)(-\rho, 1)_j + \Pi_{ik}^+ (L_n(x))_{kl} M_n^{(2)}(x) \Pi_{ij}^{+}
\]

\[
M_n^{(1)}(x) (L_n(x))_{ij} = \left( \frac{1}{\rho} \right)_i M_n^{(1)}(x - i)(1, \rho^+)_{ij} + \left( -\rho^+ \right)_i \frac{1}{\rho^+ \rho + 2} M_n^{(1)}(x + i)(-\rho, 1)_j + \Pi_{ik}^+ (L_n(x))_{kl} M_n^{(1)}(x) \Pi_{ij}^{+}
\]

\[
M_n^{(2)}(x) (L_n(x))_{ij} = \left( \frac{1}{\rho} \right)_i M_n^{(2)}(x + i)(1, \rho^+)_{ij} + \left( -\rho^+ \right)_i \frac{1}{\rho^+ \rho + 2} M_n^{(2)}(x - i)(-\rho, 1)_j + \Pi_{ik}^+ (L_n(x))_{kl} M_n^{(2)}(x) \Pi_{ij}^{+}
\]

We do not consider the irrelevant structures of the last terms in the rhs of these rules. Apparently, the triangle structure \([23,24]\) will be valid also for products of \(L_n\) and \(M_n\), as the quantum operators with different \(n\) commute with each other, therefore these relations guarantee that both the traces of monodromies (if they exist)

\[
Q^{(1,2)}(x) = TrQ^{(1,2)}(x) = Tr \prod_{k=1}^{N} M_k^{(1,2)}(x),
\]

satisfy Baxter equation \([6]\).

To solve the equations \([26,27]\) we shall use the holomorphic representation for the operators \(\rho, \rho^+\). Let the operator \(\rho^+\) be the operator of multiplication: \((\rho^+ \psi)(\alpha) = \alpha \psi(\alpha)\), while \(\rho\) is the operator of differentiation \((\rho \psi)(\alpha) = \frac{\partial}{\partial \alpha} \psi(\alpha)\). The action of an operator in the holomorphic representation is defined by its kernel:

\[
(\hat{M}\psi)(\alpha) = \int d^2 \mu(\beta) M(\alpha, \bar{\beta}) \psi(\bar{\beta}),
\]

where the measure is defined as follows: \(d^2 \mu(\beta) = e^{-\beta \bar{\beta}} d\beta d\bar{\beta}\).

In this representation the operators, which satisfy the systems \([26,27]\) have the following forms:

\[
M^{(1)}(x, \alpha, \bar{\beta}) = e^{-i\beta \varphi^+} \frac{\Gamma(-i(x - i/2))}{\Gamma(-\varphi^+ \varphi - i(x - i/2))} e^{-i\alpha \varphi}
\]

\[
M^{(2)}(x, \alpha, \bar{\beta}) = e^{i\alpha \varphi} e^{i\varphi^+ \varphi} \Gamma(-\varphi^+ \varphi - i(x - i/2)) e^{i\beta \varphi^+}
\]

In order to find the monodromy \(\hat{Q}(x, \alpha, \bar{\beta})^{(1,2)}\) one has to perform an ordered multiplication of \(M^{(1,2)}\) operators.
\[
\hat{Q}^{(i)}(x, \alpha, \vec{\beta}) = \int \prod_{i=1}^{N-1} d^2 \mu(\gamma_i) M_N^{(i)}(x, \alpha, \vec{\gamma}_{N-1}) M_{N-1}^{(i)}(x, \gamma_{N-1}, \vec{\gamma}_{N-2}) \ldots \times M_2^{(i)}(x, \gamma_2, \vec{\gamma}_1) M_1^{(i)}(x, \gamma_1, \vec{\beta}).
\]

(35)

Taking the trace of \(\hat{Q}^{(1,2)}\) over the auxiliary space we obtain \(Q^{(1,2)}\)-operators. The trace of an operator \(Q\) in the holomorphic representation is given by

\[
TrQ = \int d^2 \mu(\alpha) \hat{Q}(\alpha, \tilde{\alpha}),
\]

(37)

where \(\hat{Q}(\alpha, \tilde{\alpha})\) is the kernel of \(\hat{Q}\).

The eigenvalues \(Q^{(1)}(x)\) are polynomials in the spectral parameter \(x\). It can be seen from the action of \(Q^{(1)}\) onto the basic vectors \(|n_1, n_2, \ldots, n_N\rangle = (\varphi_1^+)^{n_1} (\varphi_2^+)^{n_2} \ldots (\varphi_N^+)^{n_N}|0\rangle\), where \(|0\rangle\) is the Bethe vacuum: \(\varphi_k|0\rangle = 0\), \(k = 1..N\):

\[
Q^{(1)}(x)|n_1, ..., n_N\rangle = \sum_{m_1, ..., m_N=0}^{n_1, ..., n_N} \prod_{k=1}^{N} \frac{(-1)^{m_k}}{m_k!} \frac{\Gamma(-i(x - i/2))}{\Gamma(-i(x - i/2) - n_k + m_k)} \frac{n_k!}{(n_k - m_k)!} |n_k - m_k + m_{k-1}, ... >.
\]

(38)

We see \(Q^{(1)}(x)\) leaves the subspace of vectors with a common particle number \(n = n_1 + n_2 + \ldots + n_N\) invariant, and all matrix elements of \(Q^{(1)}\) are polynomials in \(x\). We shall see below that \([Q^{(1)}(x_1), Q^{(1)}(x_2)] = 0\), so the eigenvalues of \(Q^{(1)}\) are polynomials in \(x\) too. Constructed in \([4]\) \(Q\)-operator corresponds to \(Q^{(1)}\). Its action onto the basic vectors (in the paper \([4]\) the coordinate representation has been chosen for the quantum operators with the basic vectors: \(x_1^{n_1} \ldots x_N^{n_N}\)) is similar to \(Q^{(1)}\) in \((38)\).

For comparison we give also the action of \(Q^{(2)}\) onto the same basic vectors:

\[
Q^{(2)}(x)|n_1, ..., n_N\rangle = e^{ixn} \sum_{m_1, ..., m_N=0}^{\infty} \prod_{k=1}^{N} \Gamma(-i(x - 1/2 - n_k - m_{k-1}) \cdot \frac{(m_{k-1} + n_k)!}{m_k!(n_k + m_{k-1} - m_k)!} |n_k + m_{k-1} - m_k, ... >.
\]

(39)

Here the summation in contrast to \(Q^{(1)}\) is taken over an infinite set of \(m_k\), restricted however by the conditions \(m_k - m_{k-1} \leq n_k\).

Let us notice that in some realizations of quantum and auxiliary operators there may appears the factorization of \(Q\)-operators first considered by Bazhanov and Stroganov \([4]\) (see also \([4]\)), if, for example, we choose the coordinate representation for quantum and auxiliary operators, we will get the following factorized form for \(Q\)-operator \([4,8]\):

\[
Q(x_1, ..., x_N, x'_1, ..., x'_N) = \prod_{k=1}^{k=N} q_k(x_k, x'_{k+1}, x'_k)
\]

(40)
In the paper [5] one of the \(Q\)-operators in the form (40) was constructed. It is also possible to construct the second \(Q\)-operator (it was also notice in [5]) in the same factorized form without use of an axilliary space. However from the point of view of approach [10] the origin of such kind of factorization is not clear. and trace of factorized defined by in complex canonical onto also above \([Q(x), t(y)] = 0\), these

In the simplest case of one quantum degree of freedom \(N = 1\) we obtain \((n = \varphi^+\varphi)\)

\[
Q^{(1)}(x) = \sum_{k=0}^{n} \frac{n!}{k!(n-k)!} \frac{\Gamma(-ix + 1/2)}{\Gamma(-ix + 1/2 - n - k)}
\]

\[
Q^{(2)}(x) = e^{i\pi n} \sum_{m=0}^{\infty} \frac{(n+m)!}{m!n!} \frac{\Gamma(-ix - 1/2 - n - m)}{\Gamma(-ix - 1/2 - n - m)}.
\]

As we have expected the eigenvalues of \(Q^{(1)}\) are polynomial of degree \(n\) and the eigenvalues of \(Q^{(2)}\) are meromorphic functions over the spectral parameter.

It is possible to find the solutions of (26,27) in the form:

\[
M^{(1)}(x, \rho) = P^{\rho} \varphi^{i\pi n} \exp \left[ i\pi/2(\varphi^+ - \varphi^-) - i(x-i/2) \right] e^{-x\pi/2}
\]

\[
M^{(2)}(x, \rho) = \Pi^{\rho} \varphi^{i\pi n} \exp \left[ i\pi/2(\rho^+ - \varphi^-) - i(x-i/2) \right]
\]

Where \(P^{\rho}\) is an operator equal:

\[
P^{\rho} = \exp \left[ \pi/2(\varphi^+ - \varphi^-) \right] \exp \left[ i\pi/2(\rho^+ - \varphi^-) \right].
\]

It has properties of the permutation operator:

\[
P^{\rho} \varphi = i\rho P^{\rho}, \quad P^{\rho} \varphi^+ = -i\rho^+ P^{\rho},
\]

\[
\varphi P^{\rho} = -iP^{\rho} \rho, \quad \varphi P^{\rho}^+ = iP^{\rho} \rho^+,
\]

\(\Pi^{\rho}\) is an operator with the following properties:

\[
\varphi \Pi^{\rho} = -i\rho \Pi^{\rho}, \quad \Pi^{\rho} \varphi^+ = -i\Pi^{\rho} \rho^+,
\]

however

\[
\varphi^+ \Pi^{\rho} = i[\Pi^{\rho}, \rho^+], \quad \Pi^{\rho} \varphi = -i[\Pi^{\rho}, \rho].
\]

The explicit expression for \(\Pi^{\rho}\) is

\[
\Pi^{\rho} = \left[ 1 + \sum_{k=1}^{\infty} (i\varphi^+ \rho)^k \frac{\Gamma(\rho^+ + k + 1)}{\Gamma(\rho^+ + k + 1)} \right] \cdot \frac{\Gamma(\rho^+ + \varphi^+ + 1)}{\Gamma(\varphi^+ + 1)} e^{i\pi \varphi^+ \varphi}.
\]

The operators \(M_n^{(1,2)}\) and \(L_n(x)\) satisfy certain intertwining relations which will imply the mutual commutativity of \(Q\)-operators and the transfer matrix. Here we will present the intertwining relations without its derivation because the method used and the intertwining \(R\)-matrix are the same as in the case of the Toda chain [10].
They are: 1) for the Lax and $M$-operators:

$$R_{kl}^{(i)}(x - y) (L_n(x))_{tm} M_n^{(i)}(y) = M_n^{(i)}(y) (L_n(x))_{kl} R_{tm}^{(i)}(x - y)$$  \hspace{1cm} (46)

The corresponding $R$-matrices are:

$$R_{kl}^{(1)}(x - y) = \begin{pmatrix} x - y + i\rho^+ \rho & -i\rho^+ \\ -i\rho & i \end{pmatrix}$$  \hspace{1cm} (47)

$$R_{kl}^{(2)}(x - y) = \begin{pmatrix} i & i\rho^+ \\ i\rho & x - y + i + i\rho^+ \rho \end{pmatrix}$$  \hspace{1cm} (48)

Note that in the paper [5] the equation (46) with $R^{(1)}$ was considered as the defining equation for local $M$-operators and the trace of their monodromy is the $Q$-operator.

2) for $M^{(1)}$- and $M^{(2)}$-operators acting in different auxiliary spaces

$$M_n^{(1)}(x, \rho) M_n^{(2)}(y, \tau) R^{12}(x - y) = R^{12}(x - y) M_n^{(2)}(y, \tau) M_n^{(1)}(x, \rho),$$  \hspace{1cm} (49)

where the intertwining matrix $R^{12}$ is defined by the kernel in the holomorphic representation

$$R^{12}(x, \alpha, \bar{\beta}; \gamma, \bar{\delta}) = \sum_{n=0} \frac{((\alpha - \gamma)(\bar{\beta} - \bar{\delta}))^n}{n!\Gamma(-ix + n + 1)},$$  \hspace{1cm} (50)

here $\alpha, \bar{\beta}$ and $\gamma, \bar{\delta}$ are the holomorphic variables for the representation space for pairs of operators $\rho, \rho^+$ and $\tau, \tau^+$.

3) for $M^{(1)}$-operators acting in different auxiliary spaces:

$$R^{(11)}(x - y) M^{(1)}(x, \rho) M^{(1)}(y, \tau) = M^{(1)}(y, \tau) M^{(1)}(x, \rho) R^{(11)}(x - y),$$  \hspace{1cm} (51)

where

$$R^{(11)}(x) = P_{\rho\tau} (1 + \rho^+ \tau)^{-ix},$$  \hspace{1cm} (52)

here $P_{\rho\tau}$ is the operator of permutation of the auxiliary spaces.

4) for $M^{(2)}$-operators acting in different auxiliary spaces:

$$R^{(22)}(x - y) M^{(2)}(x, \rho) M^{(2)}(y, \tau) = M^{(2)}(y, \tau) M^{(2)}(x, \rho) R^{(22)}(x - y),$$  \hspace{1cm} (53)

where

$$R^{(22)}(x) = P_{\rho\tau} (1 + \tau^+ \rho)^{-ix}.$$  \hspace{1cm} (54)

These intertwining relations lead to the mutual commutativity of $Q$-operators and $t$-matrix.

$$[t(x), Q^{(i)}(y)] = 0, \quad [Q^{(i)}(x), Q^{(j)}(y)] = 0, \quad i, j = 1, 2.$$  \hspace{1cm} (55)

So far we have constructed two solutions of the operator Baxter equation. Now we are going to establish linear independence of these solutions defining the one-parametric family of a finite difference analogues of the Wronsky determinant

$$W_m = Q_1(x - im)Q_2(x + i) - Q_1(x + i)Q_2(x - im)$$  \hspace{1cm} (56)
where \( m \) is a non-negative integer. Consider properties of these objects following directly from Baxter equation:

\[
\begin{align*}
  t(x)Q_1(x) &= (x - i/2)^N Q_1(x - i) + i^N Q_1(x + i) \\
  t(x)Q_2(x) &= (x - i/2)^N Q_2(x - i) + i^N Q_2(x + i).
\end{align*}
\]  

(57)

Multiplying first equation by \( Q_2(x) \), second by \( Q_1(x) \), and substracting one from another we see that

\[
(x - i/2)^N W_0(x - i) = -i^N W_0(x),
\]

so \( W_0 \) necessarily has the factor \( \Gamma^N(-i(x + i/2)) \). Multiplying then the first equation by \( Q_2(x - i) \) and the second by \( Q_1(x - i) \) we get

\[
W_1(x) = (-i)^N t(x) W_0(x - i).
\]

In the general case of non-negative integer \( m \) multiplying the first equation by \( Q_2(x - im) \) and the second equation by \( Q_1(x - im) \), we obtain:

\[
(x - i/2)^N W_{m-2}(x - 2i) + i^N W_m(x) = t(x) W_{m-1}(x - i).
\]

(58)

These identities completely define all \( W_m \) provided \( W_0 \) is known. These argumentations make sense in the presence of two solutions of Baxter equation with \( W_0 \neq 0 \) identically. The eigenvalues of the transfer matricies \( t_l(x) \)-traces of monodromy matrixes in auxiliary spaces of spin \( l = m/2 \) satisfies the recurrent relations similar to (58). The family of \( t_l \) could be obtained with the help of the expression for the Lax operator in the auxiliary space of spin \( l \):

\[
L_l(x) = i^{2l} e^{-it + \varphi} \frac{\Gamma(l^2 - i x)}{\Gamma(-ix - l)} e^{-it - \varphi}.
\]

(59)

Here operators \( l^k \) \((k = \pm, 3)\) are the operators of spin \( l \) and the factor \( i^{2l} \Gamma^{-1}(-i(x + i/2)) \) is introduced in order that in the cases of \( l = 0 \) and \( l = 1/2 \) we will obtain correspondingly \( L_0 = 1 \) and \( L_{1/2}(x) = L(x) \)-the Lax operator (2). For the operators considered in the Introduction the relation (10) gives

\[
W_0(x) = i^N \Gamma^N(-i(x + i/2)).
\]

An explicit calculation of \( W_0 \) for the solutions constructed in Section 2 using the method, described in the paper [10] gives

\[
W_0(x) = e^{i \hat{n} \hat{\varphi} \Gamma^N(-i(x + i/2))},
\]

(60)

where \( \hat{n} \) is the number of particles operator. And for this pair it follows that:

\[
W_m(x) = e^{i \hat{n} \Gamma^N(-ix - 1/2)t_l(x)}
\]

(61)

Finally we arrive at the following general Wronskian-type relations:

\[
Q_1(x - im)Q_2(x + i) - Q_1(x + i)Q_2(x - im) = e^{i \hat{n} \Gamma^N(-ix - 1/2)t_l(x)}.
\]

(62)
3 Conclusion

In the present paper we have constructed the basic $Q$-operators in the form of traces of monodromies of basic local $M$-operators for the case of DST integrable model. The obtained $Q$-operators are presented in the form of formal series over the canonical operators $\varphi_k, \varphi_k^+$ and have well defined action onto vectors of quantum space. The intertwining relations indicating the mutual commutativity of $t(x)$ and $Q$-operators are derived. Obtained are functional relations of wronskian-type showing the linear independence of $Q$-operators and connection between the $Q$-operators and the transfer matrices in the auxiliary spaces of higher spins.

Let us notice some unsolved problems. It would be interesting to find $Q$-operators for small numbers of freedom degrees as functions of the family of commuting operators, connected with $t(x)$. The origin of the factorizations a l’a Pasquer-Gaudin, which appear in some representations [3, 4] is not clear.

The described in the present method makes it possible to find $M$-operators in the most interesting case of the XXX-spin chain (they coincide with the Lax operators $L(x)$ and $\tilde{L}(x)$ for DST model with interchaged quantum and auxiliary spaces), but the traces of their monodromy diverge. However, there exists the procedure of $Q$-operator construction, analogous to the described in [3] one, for XXX $\text{SL}(2, \mathbb{C})$ spin chain [11]. So the case of XXX-spin chain deserves further investigation.

This work was supported in part by grants of RFBR 00-15-96645, 01-02-16585, CRDF MO-011-0, of the Russian Minestry on the education E00-3.3-62 and INTAS 00-00561.

References

[1] Baxter R.J. Stud.Appl.Math, L51-69, 1971; Ann.Phys. N.Y., v.70, 193-228, 1972; Ann.Phys. N.Y., v.76, 1-71, 1973.

[2] Scott A. C. and Eilbeck J. C. 1986,
The quantized discrete self-trapping equation. Phys. Lett. A, 119 60-64

[3] Enolskii V.Z., Salerno M., Kostov N.A., Scott A.C.,
Alternate quantization of the discrete self-trapping dimer, 1991 Phisica Scripta 43 229-235.

[4] Bazhanov V.V., Lukyanov S.L., Zamolodchikov A.B.,
Commun. Math. Phys. v.190, 247-78, 1997; v.200, 297-324, 1998.

[5] Kuznetsov V.B., Salerno M., Sklyanin E.K.,
Quantum Backlund transformation for the integrable DST model. 2000 J. Phys. A: Math. Gen. 33 171-189.

[6] Pasquer V. and Gaudin M.,
The periodic Toda chain and a matrix generalization of the Bessel function recursion relations. 1992 J. Phys. A: Math. Gen. 25 5243-52.
[7] Bazhanov V.V., Stroganov Yu.G., 
Chiral Potts model as a descendant of the six vertex model. 
J. Statist. Phys. 59: 799-817, 1990.

[8] Faddeev L. D., 
How Algebraic Bethe Ansatz works for integrable model. 
1995 UMANA 40 214, 
e-Print Archive: hep-th/9605187.

[9] Pronko G.P., Stroganov Yu.G., 
Bethe Equations ”on the Wrong Side of Equator”, 
J. Phys. A: Math. Gen. 32 2333-40, 1999. 
e-Print Archive: hep-th/9808153.

[10] Pronko G.P., 
On Baxter Q-operators for Toda Chain, 
J. Phys. A33: Math. Gen. 33 8251-8266, 2000, 
e-Print Archive: nlin.SI/0003002.

[11] Derkachov S. E., 
Baxter’s Q-operator for the homogeneous XXX spin chain, 
J. Phys. A: Math. Gen. 32 1999, 5299-5316. 
e-Print Archive: solv-int/9902013.