On atomic analogue of Landau quantization

B. Basu

Physics and Applied Mathematics Unit
Indian Statistical Institute
Kolkata-700108

S. Dhar

Bankura Sammilani College, Bankura, W.B., India

S. Chatterjee

Bispara High School, Chinsurah, Hooghly W.B., India

Abstract

We have studied the physics of atoms with permanent electric dipole moment and non-vanishing magnetic moment interacting with an electric field and inhomogeneous magnetic field. This system can be demonstrated as the atomic analogue of Landau quantization of charged particles in a uniform magnetic field. This Landau-like atomic problem is also studied with space-space noncommutative coordinates.

PACS numbers: 71.70.Di, 73.43.-f, 42.50.Dv, 32.10.Dk

keywords: Landau quantization; electric dipoles; atomic Hall effect; noncommutative geometry

*Electronic address: banasri@isical.ac.in, Fax:91+(033)2577-3026
†Electronic address: sarmi30@rediffmail.com
I. INTRODUCTION

A charged particle in two dimensions under a transverse uniform magnetic field gives rise to the well known Landau quantization \[1\] where a discrete energy spectrum is observed. This can be shown as the simplest model to explain the integer quantum Hall effect (QHE) \[2\]. Although the Hall effect of neutral atoms was predicted long ago \[3\], the possibility of an analogue of the Hall effect in Bose Einstein condensates was studied recently by Paredes et. al. \[4, 5\]. Inspired by this work \[4, 5\], the Aharonov-Casher interaction was used to generate an analogue of Landau levels in a system of neutral atoms \[6\]. This is a step towards an atomic QHE using electric fields realizable in Bose Einstein condensates. The quantum dynamics of cold atoms in presence of external fields is also of current interest \[7, 8, 9, 10, 11, 12, 13\]. Recently, the idea of Ericsson and Sjöqvist \[6\] has been extended to investigate the quantum dynamics of a neutral particle in the presence of an external electromagnetic field \[14, 15\]. It is shown that in a specific field-dipole configuration, a quantization similar to Landau quantization may be obtained \[14, 15\]. These results motivated us to study the quantum dynamics of an atom in presence of external electric and inhomogeneous magnetic fields.

We have organised the paper in the following way. In sec.II the Landau like quantization of a neutral atom in a specific configuration of external fields is studied. In sec.III, we have studied the system in symmetric gauge and derived the eigenfunction and the energy spectrum. Sec.IV is devoted to investigate the system in space-space noncommutative coordinates. Finally, we have discussed an overview of our work in section V.

II. LANDAU-LIKE QUANTIZATION OF NEUTRAL PARTICLES

In the nonrelativistic limit, the Hamiltonian for a neutral particle (of mass \(m\)) that possesses permanent electric dipole moment \(d_e\) and nonvanishing magnetic moment \(d_m\) in presence of external electric and magnetic fields is given by \[16\]

\[
H = -\frac{1}{2m} \left(\nabla - (d_m \times \mathbf{E}) + (d_e \times \mathbf{B})\right)^2 - \frac{d_m}{2m} \nabla \cdot \mathbf{E} + \frac{d_e}{2m} \nabla \cdot \mathbf{B}
\]  

(1)

\(^{1}\)Electronic address: sankar2050@yahoo.co.in
In this case, the terms of order \((E^2 \text{ and } B^2)\) are neglected, and the electric and magnetic dipoles are aligned in the z-direction. It is also assumed that the fields generated by the source are radially distributed in the space. The momentum operator may be written as,

\[ k_i = mv_i = p_i - (d_m \times E)_i + (d_e \times B)_i \] (2)

We may now consider a different set up. The neutral atoms, with a permanent electric dipole moment \(d_e\) and nonvanishing magnetic moment \(d_m\) are made to interact with an electric field \(E\) and a specific type of magnetic field \(B\). Let an inhomogeneous magnetic field be generated in the neighbourhood of the dipole. In particular, a non zero gradient of the magnetic field component perpendicular to the surface \(S\) (xy plane), varying in the direction of the dipole is subjected. In the nonrelativistic limit, the interaction between the atom and this type of external field may be described by the Hamiltonian as \((c = \hbar = 1)\)

\[ H = \frac{\Pi^2}{2m} + \frac{d_m}{2m} \nabla.E + \frac{d_e}{2m} |(n', \nabla)B| \] (3)

where \(n\) is the unit vector in the direction of \(d_m\) and \(n'\) is the direction of electric dipole moment \(d_e\). The minimal coupling of the particle with the external field is obtained by substituting its momentum as

\[ \Pi = -i\nabla - d_m(n \times E) - d_e(n' \cdot \nabla)A \] (4)

where \(A_{eff} = d_m(n \times E) + d_e(n' \cdot \nabla)A\) is defined as the effective vector potential. A careful analysis of eqn.\(3\) shows that the particles can give rise to different phenomena that generate quantum phases while undergoing a cyclic trajectory. The second term is the origin of Aharanov Casher effect, while the last term is due to differential Aharanov Bohm effect \((\text{DABE})[17,18]\). Actually, this term involves the coupling of the electric dipole moment to a differential form of the vector potential and so may be defined as \(\text{DABE} [17,18]\).

It is noted that the momentum \(\Pi\) follows the standard algebra:

\[ [\Pi_i, \Pi_j] = i\epsilon_{ijk}(B_{eff})_k \] (5)

where the effective field strength \(B_{eff}\) is written as

\[ B_{eff} = \nabla \times A_{eff} \]

\[ = \nabla \times [d_m(n \times E) + d_e(n' \cdot \nabla)A] \] (6)
We can impose the specific field dipole configurations:
(i) electric dipole moment in the x- direction i.e. \( n = (0, 0, 1) \)
(ii) magnetic dipole moment along the z- direction i.e. \( n' = (1, 0, 0) \)
(iii) Electric field on the xy surface: \( E = (E_x(x, y), E_y(x, y), 0) \) with \( \partial_t E = 0 \) and \( \nabla \times E = 0 \) and
(iv) with \( B = (0, 0, B_z(x, y)) \) so that \( \nabla \cdot B = 0 \) and \( \nabla \times B = 0 \).

A simple algebra and with the help of Gauss’ law, eqn.(6) becomes

\[
B_{eff} = d_m [\nabla \times (n \times E)] + d_e [\nabla \times (n' \cdot \nabla)A]
\]

\[
= d_m [n(\nabla \cdot E) + \nabla (n \cdot E)] + d_e [\nabla \times (B \times n') + \nabla \times (n' \cdot \nabla)A]
\]

\[
= d_m [n(\nabla \cdot E)] + d_e [\nabla \times (B \times n')]
\]

\[
= d_m n_{\rho_0} + d_e (n' \cdot \nabla)B
\]

It may be noted that a non zero effective phase

\[
\phi = \int_s [d_m n_{\rho_0} + d_e (n \cdot \nabla)B] dS
\]

is produced if the z-component of magnetic field has a non-vanishing gradient along the direction of \( d_e \), i.e a non-zero effective phase exists when \( \frac{\partial B_z}{\partial x} \neq 0 \)

The kinetic part of the Hamiltonian operator satisfies the relation

\[
[\Pi_x, \Pi_y] = im\omega_{eff} = \frac{i\sigma}{l_{eff}^2}
\]

where the cyclotron frequency is given by

\[
\omega_{eff} = \frac{\sigma |d_m \rho_0 + d_e (\frac{\partial B_z}{\partial x})|}{m}
\]

and the sign \( \sigma = \pm 1 \) denotes the direction of revolution. In this specific problem, the magnetic length is defined as

\[
l_{eff} = \left[ \frac{1}{|d_m \rho_0 + d_e (\frac{\partial B_z}{\partial x})|} \right]^{\frac{1}{2}}
\]

Defining the annihilation and creation operator as

\[
a_{eff} = \frac{1}{\sqrt{2m} |\omega_{eff}|} (\Pi_x + i\sigma \Pi_y) \), \quad a_{eff}^\dagger = \frac{1}{\sqrt{2m} |\omega_{eff}|} (\Pi_x - i\sigma \Pi_y)
\]
with \([a_{eff}, a_{eff}^+] = 1\), we can write the Hamiltonian as

\[
H = (a_{eff}^+ a_{eff} + \frac{1}{2} (1 + \sigma)) |\omega_{eff}| + \frac{p_z^2}{2m} \tag{13}
\]

where we have used the relation \( \frac{d}{dt} \nabla \cdot \mathbf{E} + \frac{d}{dt} (\mathbf{n} \cdot \nabla) \mathbf{B} = \frac{1}{2} \omega_{eff} \)

If we consider the condition of vanishing torque on the dipole, then using the constraint \( <\Pi_z >= <p_z >= 0\), we obtain the energy eigenvalues

\[
E^{(\sigma)}_\nu = (\nu + \frac{1}{2} (1 + \sigma)) |\omega_{eff}| \tag{14}
\]

where \(\nu = 0, 1, 2, 3, \ldots\)

The expression is similar to that of the standard Landau quantization, where charge \(q\) in a uniform magnetic field perpendicular to its motion has energy eigenvalues

\[
E_\nu = (\nu + \frac{1}{2}) \hbar |\omega| + \frac{\hbar^2 k_z^2}{2m} \tag{15}
\]

where \(\nu = 0, 1, 2, 3, \ldots\) and \(k_z\) are real valued.

In the standard Landau problem, the motion in the x-y plane is transformed into one-dimensional harmonic oscillator accompanying free motion in the z-direction and energies are independent of direction of revolution. It is noted that in the atomic analogue case the energy eigenvalues depends on \(\sigma\), the direction of revolution, and the condition of vanishing torque on the dipole puts an additional constraint on the stationary states.

We can also make a comparison on the separation between two energy levels in both the cases. The separation between two Landau levels is given by

\[
\Delta E = \hbar |\omega| = \hbar \frac{|qB|}{m} \tag{16}
\]

with cyclotron frequency \(|\omega| = \frac{|qB|}{m}|.

In our case, from eqn.\,(11) and (10) we find that the separation between two energy levels is

\[
\Delta E = \hbar |\omega_{eff}| = \hbar \left| \frac{d_m \frac{e\omega}{\epsilon_0} + d_e (\frac{\partial B_z}{\partial x})}{m} \right| \tag{17}
\]

With the help of the relation \( B = \frac{\Phi}{S}, (S \text{ being the area and } \Phi \text{ the enclosed flux})\) and comparison of eqn.\,(17) and eqn.\,(16), we can find the analogy

\[
q\Phi \longleftrightarrow \left| \frac{d_m \lambda}{\epsilon_0} + d_e S (\frac{\partial B_z}{\partial x}) \right| \tag{18}
\]
where \( \lambda = \rho_0 S \), is uniform linear charge density. This relation shows the duality between the standard Landau problem and a system of atoms with permanent electric dipole moment and non-vanishing magnetic moment interacting with electric field and inhomogeneous magnetic field.

### III. MOTION IN SYMMETRIC GAUGE

We may now proceed to solve the problem in a specific gauge, for example: symmetric gauge, and derive the energy spectrum.

The Hamiltonian (3) may be rewritten as

\[
H = \left( \frac{p - A_{\text{eff}}}{2m} \right)^2 + \frac{d_m}{2m} \nabla \cdot E + \frac{d_e}{2m} \left| (n' \cdot \nabla) B \right|
\]

Substituting the symmetric gauge \( A_{\text{eff}} = (-\frac{B_{\text{eff}}}{2} y, \frac{B_{\text{eff}}}{2} x) \) and considering the charge density producing the electric field to be constant and specifying a constant gradient of the magnetic field component (i.e. \( \frac{\partial B_z}{\partial x} \) = constant) the Hamiltonian in planer motion become

\[
H = \frac{1}{2m} \left( (p_x - A_{\text{eff}}^x)^2 + (p_y - A_{\text{eff}}^y)^2 \right)
\]

\[= \frac{1}{2m} \left( (p_x + \frac{B_{\text{eff}}}{2} y)^2 + (p_y - \frac{B_{\text{eff}}}{2} x)^2 \right)\]

\[= \frac{1}{2} (p_x^2 + p_y^2) + \frac{1}{2} m \left( \frac{B_{\text{eff}}}{2m} \right)^2 (x^2 + y^2) - \frac{B_{\text{eff}}}{2m} L\]

where \( L = xp_y - yp_x \). To solve \( H \), we change the variables to

\[
z = x + iy, \quad p_z = \frac{1}{2} (p_x - ip_y)\]

and introduce two sets of annihilation and creation operators

\[
a = \frac{1}{2} \sqrt{\frac{B_{\text{eff}}}{2}} \bar{z} + i \sqrt{\frac{B_{\text{eff}}}{2}} p_z \quad a^\dagger = \frac{1}{2} \sqrt{\frac{B_{\text{eff}}}{2}} \bar{z} - i \sqrt{\frac{B_{\text{eff}}}{2}} p_z\]

\[
b = \frac{1}{2} \sqrt{\frac{B_{\text{eff}}}{2}} \bar{z} + i \sqrt{\frac{B_{\text{eff}}}{2}} p_z \quad b^\dagger = \frac{1}{2} \sqrt{\frac{B_{\text{eff}}}{2}} \bar{z} - i \sqrt{\frac{B_{\text{eff}}}{2}} p_z\]

satisfying the following commutation relations

\[
[a, a^\dagger] = [b, b^\dagger] = 1
\]
We can write the $L$ and $H$ in terms of these operators

$$L = (a^\dagger a - b^\dagger b)$$

$$H = \frac{B_{\text{eff}}}{2m}(a^\dagger a + b^\dagger b + 1) - \frac{B_{\text{eff}}}{2m}(a^\dagger a - b^\dagger b)$$

$$= \frac{B_{\text{eff}}}{m}(b^\dagger b + \frac{1}{2})$$

Thus the motion is similar to that of a one dimensional harmonic oscillator. So the eigenfunction and energy spectrum of our system is given by

$$\psi_\nu = \frac{1}{\sqrt{(2m\omega_{\text{eff}})^\nu \nu!}}(b^\dagger)^\nu|0>$$

$$E = \frac{B_{\text{eff}}}{m}(\nu + 1/2) = \frac{\omega_{\text{eff}}}{2}(2\nu + 1), \quad \nu = 0, 1, 2, ...$$

respectively.

**IV. MOTION IN NONCOMMUTATING PLANE**

We may now study the specific system in noncommutating (NC) plane. The space coordinate is noncommutating following the algebra

$$[\tilde{x}, \tilde{y}] = i\theta$$

where $\theta$ is a constant noncommutative parameter.

Noncommutating coordinates can be written in terms of commutating $(x, y)$ as

$$\tilde{x} \equiv x - \frac{\theta}{2}p_y \quad \tilde{y} \equiv y + \frac{\theta}{2}p_x$$

So NC analogue of Hamiltonian (20) will be reduced to

$$H_{nc} = \frac{1}{2m}\left( [1 + k]p_x + \frac{B_{\text{eff}}}{2}y \right)^2 + \frac{1}{2m}\left( [1 + k]p_y - \frac{B_{\text{eff}}}{2}x \right)^2$$

$$= \frac{1}{2m}\left( \tilde{p}_x + \frac{B_{\text{eff}}}{2}y \right)^2 + (\tilde{p}_y - \frac{B_{\text{eff}}}{2}x)^2$$

where

$$k = \frac{B_{\text{eff}}}{2}\theta, \quad [1 + k]p_i = \tilde{p}_i, \quad \tilde{\omega} = [1 + k]\omega \quad \text{and} \quad \tilde{L} = x\tilde{p}_y - y\tilde{p}_x$$
To solve $H_{nc}$, we change the variables to

$$z = x + iy, \quad \tilde{p}_z = \frac{1}{2}(\tilde{p}_x - i\tilde{p}_y)$$

and introduce two sets of annihilation and creation operators

$$\tilde{a} = \frac{1}{2}\sqrt{\frac{B_{eff}}{2}} z + i\sqrt{\frac{B_{eff}}{2}} \tilde{p}_z \quad \tilde{a}^\dagger = \frac{1}{2}\sqrt{\frac{B_{eff}}{2}} z - i\sqrt{\frac{B_{eff}}{2}} \tilde{p}_z$$

$$\tilde{b} = \frac{1}{2}\sqrt{\frac{B_{eff}}{2}} z + i\sqrt{\frac{B_{eff}}{2}} \tilde{p}_z \quad \tilde{b}^\dagger = \frac{1}{2}\sqrt{\frac{B_{eff}}{2}} z - i\sqrt{\frac{B_{eff}}{2}} \tilde{p}_z$$

satisfying the following commutation relations

$$[\tilde{a}, \tilde{a}^\dagger] = [\tilde{b}, \tilde{b}^\dagger] = 1$$

We can write the $\tilde{L}$ and $H_{nc}$ in terms of these operators

$$\tilde{L} = (\tilde{a}^\dagger \tilde{a} - \tilde{b}^\dagger \tilde{b})$$

$$H_{nc} = \frac{B_{eff}}{2m}(\tilde{a}^\dagger \tilde{a} + \tilde{b}^\dagger \tilde{b} + 1) - \frac{B_{eff}}{2m}(\tilde{a}^\dagger \tilde{a} - \tilde{b}^\dagger \tilde{b})$$

$$= \frac{B_{eff}}{m}(\tilde{b}^\dagger \tilde{b} + \frac{1}{2})$$

The Hamiltonian is now reduced to the form of a one dimensional harmonic oscillator with frequency of precession $\tilde{\omega}_{eff}$. So the eigenfunction is given by

$$\psi_\nu = \frac{1}{\sqrt{(2m\tilde{\omega}_{eff})^\nu \nu!}}(\tilde{b}^\dagger)^\nu |0>$$

and the energy spectrum will be denoted as

$$E_{nc} = \frac{\tilde{\omega}_{eff}}{2}(2\nu + 1), \quad \nu = 0, 1, 2, ...$$

It is easily checked that for $\theta = 0$, i.e. for vanishing non-commutative parameter, the equations (40) and (41) are mapped to the equations of the eigenfunctions and eigenenergies of the commutative plane.

V. DISCUSSIONS

We have studied the quantum dynamics of a neutral particle in two dimensions in external fields and found the analogy with the standard Landau quantization. The neutral
atom with permanent electric dipole moment and nonvanishing magnetic moment is considered in external electric and space dependent magnetic fields. The resulting effect is analogous to that of a charged particle moving in presence of a uniform transverse magnetic field. Our analysis is a more generalized version of the Landau-Aharonov-Casher quantization \[6\] where the atomic motion was considered in presence of electric field only. We have taken nonvanishing gradient of magnetic field by considering a varying magnitude of the z-component of the field along x-direction. The same effect may also be obtained by varying the direction of the dipole \[18\]. The eigenfunction and energy spectrum of the system is derived using symmetric gauge and extended our analysis to study the system in noncommutating plane. It is found that the result reduces to that of the commutating plane when the noncommutative parameter is set equal to zero.

Acknowledgement: Thanks to the referee for his fruitful suggestions.

[1] L. D. Landau, Z. Phys. 64, 629 (1930)
[2] A. H. MacDonald, cond-mat/9410047
[3] A. P. Kasantsev, A.M. Dykhne and V.L. Pokrovsky, published in Pis'ma v ZhETF (atom-ph/9604002)
[4] B. Paredes, P. Fedichev, J. I. Cirac and P. Zoller, Phys. Rev. Lett. 87 010402 (2001)
[5] B. Paredes, P. Zoller, J. I. Cirac, Solid State Commun. 127 (2003) 155.
[6] M. Ericsson and E. Sjöqvist, Phys. Rev. A 65 013607 (2001)
[7] J. K. Pachos and E. Rico, Phys. Rev. A 70 (2004) 053620.
[8] L.-M. Duan, E. Demler, M. D. Lukin, Phys. Rev. Lett. 91 (2003) 090402.
[9] R B. Diener, Qi Zhou, Hui Zhai and Tin-Lun Ho, Phys. Rev. Lett. 98 (2007) 180404
[10] W. Yi, G.-D. Lin, L.-M. Duan, Phys. Rev. A 76 (2007) 031602.
[11] S. Murakami, N. Nagaosa, S. C Zhang, Science 301 (2003) 1348
[12] Shi-Liang Zhu, Hao Fu, C.-J. Wu, S. -C. Zhang, L. -M. Duan, Phys. Rev. Lett. 97 (2006) 240401.
[13] Xiong-Jun Liu, Xin Liu, L. C. Kwek and C. H. Oh, Phys. Rev. Lett. 98 (2007) 026602.
[14] C. Furtado, J. R. Nascimento and L. R. Ribeiro, Phys.Lett. A 358 (2006) 336-338
[15] L. R. Ribeiro, Claudio Furtado and J. R. Nascimento, Phys. Lett. A 348 (2006) 135.

[16] E. Passos, L. R. Ribeiro, C. Furtado and J. R. Nascimento; arXiv:hep-th/0610222 (to appear in Phys. Rev A).

[17] Y. Aharonov and D. Bhom, Phys. Rev. 115 (1959) 485

[18] Jiannis K. Pachos, Phy. Lett. A 344 (2005) 441

[19] B. Basu and Subir Ghosh, Phys. Lett. A 346 (2005) 133.

[20] B. Basu, S. Dhar and Subir Ghosh, Europhys. Lett. 76 (2006) 395