TOPOLOGICAL FUNCTORS AS FAMILIARLY-FIBRATIONS

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Abstract. In this paper we develop the theory of topological categories over a base category, that is, a theory of topological functors. Our notion of topological functor is similar to (but not the same) the existing notions in the literature (see [2] 7.3), and it aims at the same examples. In our sense, a (pre) topological functor is a functor that creates cartesian families. A topological functor is, in particular, a fibration, and our emphasis is put in this fact.

Introduction In this paper we develop the theory of topological categories over a base category, that is, a theory of topological functors. Our notion of topological functor is similar to (but not the same) the existing notions in the literature (see [2] 7.3), and it aims at the same examples.

Recall that a (pre) fibration is a functor that creates cartesian arrows. In our sense, a pre-topological functor is a functor that creates cartesian families, and it is topological provided that these families compose. A topological functor is, in particular, a fibration, and our emphasis is put in this fact. We develop an adequate generalization utilizing cartesian families (instead of cartesian arrows) of the basic ideas of Grothendieck’s theory of fibered categories.

In section 1 we set the basic facts of a systematic theory of families of arrows in a category. In section 2 we consider $u$-final and $u$-surjective families in a category $T$ with respect to a functor $T \to S$, and prove a general theorem that characterizes the intrinsic strict epimorphic families in $T$ as the $u$-final and $u$-surjective families. This theorem proves to be very useful in practice, allowing to generalize to a general setting many of the usual arguments and constructions known for the category of topological spaces. The assumption on $u$ in this theorem defines the concept of $E$-functor, which determines the right generality for these constructions. In section 3 we develop the basic yoga of cartesian arrows introduced by Grothendieck in [6], but we do so using families instead of single arrows. In particular, given a fibration, we clarify the relation between cartesian families and initial families, and, related to this, the relation of these families with products in the fibers. In section 4 we define topological functors and prove in our context all the properties corresponding to the usual properties of topological functors, and several new characterizations of these functors. In particular, a characterization of topological functors in terms of the pseudofunctor associated to the fibration.

For all the concepts considered in this paper there is a corresponding dual concept, and all the corresponding dual statements (dual assumption and dual conclusion) hold. We will explicitly dualize concepts and statements only when it is convenient or necessary.
1. Families of arrows

In this section we recall some notions and results that we shall explicitly need in
the following sections, and in this way fix notation and terminology.

Given a category $T$ and an object $X$ in $T$, we shall work with families $(X_\alpha \xrightarrow{g_\alpha} X)_{\alpha \in \Gamma}$ of arrows of $T$ with codomain $X$. Dually, we can also consider families of arrows with domain $X$.

1.1. Notation. Given a family $(X_\alpha \xrightarrow{g_\alpha} X)_{\alpha \in \Gamma}$, we shall simply write $X_\alpha \xrightarrow{g_\alpha} X$, omitting as well a label for the index set (the context will always tell whether we are considering a single $\alpha$ or the whole family).

The diagrammatic notation always denotes a commutative diagram, unless otherwise explicitly indicated.

We denote $T(-, X)$ the family of all arrows $Y \rightarrow X$ of $T$. It is important to point out that we allow the families to be large, that is, not indexed by a set.

Recall that a crible is a sub-family $P \subseteq T(-, X)$ such that any composite $Z \rightarrow Y \rightarrow X$ belongs to $P$ if $Y \rightarrow X$ belongs to $P$.

1.2. Definition. We say that a family $Y_\lambda \rightarrow X$ refines (is a refinement of) a family $X_\alpha \rightarrow X$ if there is a function between the indices $\lambda \mapsto \alpha_\lambda$ together with arrows $Y_\lambda \rightarrow X_{\alpha_\lambda}$ such that

\[
\begin{array}{ccc}
Y_\lambda & \longrightarrow & X_{\alpha_\lambda} \\
\downarrow & & \downarrow \\
X & & \\
\end{array}
\]

Any family $X_\alpha \rightarrow X$ is a refinement (but not a sub-family) of $T(-, X)$ in a canonical way. We denote $P(X_\alpha \rightarrow X)$ the crible of all arrows $Y \rightarrow X$ factorizing by some arrow $X_\alpha \rightarrow X$. The family $X_\alpha \rightarrow X$ refines in a canonical way the family $P(X_\alpha \rightarrow X)$, and in the other direction, $P(X_\alpha \rightarrow X)$ also refines $X_\alpha \rightarrow X$, but there is no canonical refinement.

1.3. Definition. Given an arrow $Y \rightarrow X$, we say that a family $Y_\lambda \rightarrow Y$ is a r-pullback of a family $X_\alpha \rightarrow X$, if $Y_\lambda \rightarrow Y \rightarrow X$ refines $X_\alpha \rightarrow X$. That is, if there is a function between the indices $\lambda \mapsto \alpha_\lambda$ together with arrows $Y_\lambda \rightarrow X_{\alpha_\lambda}$ such that

\[
\begin{array}{ccc}
Y_\lambda & \longrightarrow & X_{\alpha_\lambda} \\
\downarrow & & \downarrow \\
Y & \longrightarrow & X \\
\end{array}
\]

1.4. Remark. Given an arrow $Y \rightarrow X$, among the r-pullbacks there is a largest one, namely, the pullback crible $P \subseteq T(-, Y)$ defined by:

\[
\begin{array}{ccc}
Z & \longrightarrow & X_\alpha \\
\downarrow & & \downarrow \\
Y & \longrightarrow & X \\
\end{array}
\]

All the r-pullbacks are refinements of $P$. 
We consider collections $\mathcal{A}$ of classes of families of arrows with common codomain, one class $\mathcal{A}_X$ (eventually empty) for each object $X$ in $\mathbb{T}$. We say that a family in $\mathcal{A}_X$ is an $\mathcal{A}$-family over $X$.

Properties of families of arrows determine a corresponding collection. For instance, epimorphic families define a collection $\mathcal{E}_{\text{pi}}$ by: $X_{\alpha} \xrightarrow{\beta_{\alpha}} X \in \mathcal{E}_{\text{pi}}$ if given arrows $f, f' : X \to Y$, the condition $(f \circ g_{\alpha} = f' \circ g_{\alpha} \text{ for all } \alpha)$ implies $f = f'$.

We now define some operations on these collections that yield new collections out of given ones:

1.5. Definition (operations on collections).

(1) We denote by $\mathcal{I}_{\text{so}}$ the collection whose only arrows are the isomorphisms.

(2) Given two collections $\mathcal{A}, \mathcal{B}$, we define the composite $\mathcal{C} = \mathcal{A} \circ \mathcal{B}$ by means of the following implication:

$$X_{\alpha} \xrightarrow{f_{\alpha}} X \in \mathcal{A}_{X} \text{ and } \forall \alpha X_{\alpha, \beta} \xrightarrow{g_{\alpha, \beta}} X_{\alpha} \in \mathcal{B}_{X_{\alpha}} \implies X_{\alpha, \beta} \xrightarrow{f_{\alpha} \circ g_{\alpha, \beta}} X \in \mathcal{C}_{X}$$

(3) Given $\mathcal{A}$ we define a new collection, denoted $\pi \mathcal{A}$, by:

$$Y_{\alpha} \xrightarrow{f_{\alpha}} Y \in \pi \mathcal{A} \iff \text{there is } X_{\alpha} \xrightarrow{f_{\alpha}} X \in \mathcal{A}_X \text{ and } Y \xrightarrow{g_{\alpha}} X \text{ such that:}$$

$$\text{the squares } \\
\begin{array}{c}
Y_{\alpha} \\
\downarrow
\end{array} \\
\begin{array}{c}
X_{\alpha} \\
\downarrow
\end{array} \quad \begin{array}{c}
Y \\
\downarrow
\end{array} \quad \begin{array}{c}
X
\end{array}$$

are pullbacks for all $\alpha$.

(4) Given $\mathcal{A}$ we define a new collection, denoted $\mathcal{s}\mathcal{A}$, by:

$$X_{\alpha} \xrightarrow{f_{\alpha}} X \in \mathcal{s}\mathcal{A} \iff \text{there is a refinement by a family } Y_{\lambda} \xrightarrow{f_{\lambda}} X \in \mathcal{A}_X.$$
(iii) If the collection \( \mathcal{A} \) satisfies (S), and the category has finite limits, then (U) is equivalent to the condition of stability under pulling-back, that is, \( \pi \mathcal{A} \subseteq \mathcal{A} \) (hence \( \mathcal{A} = \pi \mathcal{A} \)).

In practice some collections are determined by the conjunction of two properties, therefore they are the intersection of two different collections. Concerning this we have:

1.8. Remark. Given two collections \( \mathcal{A}, \mathcal{B} \), consider the collection \( \mathcal{A} \cap \mathcal{B} \). Then:
   
   i) If \( \mathcal{A} \) and \( \mathcal{B} \) both satisfy (I), (resp. (C)), (resp. (S)), then so does \( \mathcal{A} \cap \mathcal{B} \).
   
   This is not the case for conditions (U) and (F). However, it will be so for saturated collections (argue with the pullback crible):
   
   ii) If \( \mathcal{A} \) and \( \mathcal{B} \) both satisfy (S) and (U), (resp. (S) and (F)), then so does \( \mathcal{A} \cap \mathcal{B} \).

An important collection of families are the strict epimorphic families. We recall now this notion from SGA4 [1, I, 10.3, p. 180]:

1.9. Definition. Given two families of arrows \( f_\alpha : X_\alpha \to X \), \( g_\alpha : X_\alpha \to Y \), with the same indexes and domains, we say that \( g_\alpha \) is compatible with \( f_\alpha \) if for any pair of arrows \( (x_\alpha : Z \to X_\alpha, x_\beta : Z \to X_\beta) \) with the same domain the following condition holds: \( f_\alpha \circ x_\alpha = f_\beta \circ x_\beta \) implies \( g_\alpha \circ x_\alpha = g_\beta \circ x_\beta \).

A family \( f_\alpha : X_\alpha \to X \) is strict epimorphic if for any family \( g_\alpha : X_\alpha \to Y \) which is compatible with \( f_\alpha \), there exists a unique \( g : X \to Y \) such that \( g \circ f_\alpha = g_\alpha \) for all \( \alpha \).

The situation is described in the following diagram, where the family \( g_\alpha \) is compatible with the family \( f_\alpha \):

\[
\begin{array}{ccc}
X_\alpha & \xrightarrow{f_\alpha} & X \\
\downarrow{g_\alpha} & & \downarrow{g} \\
Y
\end{array}
\]

In Set the category of sets (as in any topos), every epimorphic family is strict epimorphic and the collection satisfies all five properties in definition 1.6. It is an important exactness property for a category when the collection of strict epimorphic families satisfy all five properties in definition 1.6. This is the case for the regular categories.

It follows from the uniqueness that strict epimorphic families are epimorphic, but the converse is not true in general. Recall that the following holds:

1.10. Remark. The family of all the inclusions of a colimit cone is a strict epimorphic family.

Let \( \mathcal{E}_X \) be the class of all strict epimorphic families with codomain \( X \). The collection \( \mathcal{E}_X \) satisfies (I) and (S), but in general it fails to satisfy (C), (U) and (F) in definition 1.6. Recall the following:

1.11. Proposition. If a functor has a right adjoint, then it preserves \( \mathcal{E}_X \)-families and \( \mathcal{E}_S \)-families (dually, we have that monomorphic, and strict monomorphic families are preserved by functors with a left adjoint).

The following is a technical fact that we shall need later, and it is easy to prove:

1.12. Proposition. If a functor has a left adjoint, then it preserves the relation of compatibility between families.

An strict epimorphisms is the coequalizer of its kernel pair (if the later exists). In this case some authors call them regular epimorphisms. Strict epimorphisms rather than epimorphisms are relevant in the important cases of factorizations.
2. Strict epimorphic and final surjective families

In $\text{Top}$, the category of topological spaces and continuous maps, a family is strict epimorphic if and only if it is surjective and the codomain has the final topology. In this section we develop a similar characterization of strict epimorphic families in a general setting.

We consider a category $\mathcal{T}$ endowed with a functor $u : \mathcal{T} \to \mathcal{S}$ which will satisfy some general conditions, and we consider notions associated to the functor $u$ for families in $\mathcal{T}$.

We start with some definitions:

2.1. Definition. An object $X$ in $\mathcal{T}$ sits over an object $S$ in $\mathcal{S}$ when $u(X) = S$. We say also that $X$ is an object over $S$. An arrow $f : X \to Y$ in $\mathcal{T}$ sits over an arrow $\varphi : S \to T$ in $\mathcal{S}$ when $u(f) = \varphi$, so that $X$ (resp. $Y$) sits over $S$ (resp. $T$). We say also that $\varphi$ lifts to an arrow in $\mathcal{T}$ when there exists $f$ over $\varphi$. A family $f_\alpha : X_\alpha \to Y_\alpha$ in $\mathcal{T}$ sits over a family $\varphi_\alpha : S_\alpha \to T_\alpha$ in $\mathcal{S}$ when $u(f_\alpha) = \varphi_\alpha$, for any $\alpha$. We say also that the family $\varphi_\alpha$ lifts to a family in $\mathcal{T}$ when there exists $f_\alpha$ over $\varphi_\alpha$.

2.2. Definition. We say that two families $f_\alpha : X_\alpha \to Y_\alpha$ which sit over the same family $\varphi_\alpha : S_\alpha \to S$ in $\mathcal{S}$ are $u$-isomorphic if there exists an isomorphism $\theta : X \to Y$ over $\varphi$ such that $\theta \circ f_\alpha = g_\alpha$ for all $\alpha$.

2.3. Remark. All collections considered in this paper are assumed to be closed under $u$-isomorphisms without need to say so explicitly.

2.4. Definition. Let $\mathcal{A}$ be a collection of classes of families in $\mathcal{T}$. We say that $\mathcal{A}$-families are unique up to isomorphisms if given any two $\mathcal{A}$-families $f_\alpha : X_\alpha \to Y_\alpha$ which sit over the same family $\varphi_\alpha : S_\alpha \to S$ in $\mathcal{S}$, they are $u$-isomorphic by a unique isomorphism.

Notice that when $u$ is faithful strict epimorphic families are unique up to isomorphisms.

2.5. Definition. Consider collections $\mathcal{A}$ in $\mathcal{T}$ and $\mathcal{B}$ in $\mathcal{S}$. We say that $u$ creates $\mathcal{A}$-families over $\mathcal{B}$-families if given any $\mathcal{B}$-family $\phi_\alpha : S_\alpha \to S$ and an object $X_\alpha$ over $S$, for every $\alpha$, there exists an $\mathcal{A}$-family $f_\alpha : X_\alpha \to X$ over $\phi_\alpha : S_\alpha \to S$. When the class $\mathcal{B}$ in $\mathcal{S}$ is the “same” class than the class $\mathcal{A}$ in $\mathcal{T}$ (that is, if they are denoted by the same letter), we simply say that $u$ creates $\mathcal{A}$-families.

Complementing remark 1.10 it is immediate to check:

2.6. Remark. A functor which creates and preserves strict epimorphic families (finite strict epimorphic families) create and preserve any colimits (resp. finite colimits) that may exists.

The following lemma is the key to the proof of the characterization theorem 2.20.

2.7. Lemma. Let $\mathcal{A}, \mathcal{C}$ be collections in $\mathcal{T}$, and $\mathcal{B}$ in $\mathcal{S}$ such that:

- $\mathcal{A}$-families are $\mathcal{C}$-families (i.e., $\mathcal{A} \subseteq \mathcal{C}$).
- $u$ sends $\mathcal{C}$-families into $\mathcal{B}$-families (i.e., $u(\mathcal{C}) \subseteq \mathcal{B}$).
- $u$ creates $\mathcal{A}$-families over $\mathcal{B}$-families.
- $\mathcal{C}$-families are unique up to isomorphisms.

Then $\mathcal{C} \subseteq \mathcal{A}$, thus $\mathcal{C} = \mathcal{A}$

Proof. Let $X_\alpha \xrightarrow{f_\alpha} X$ be a $\mathcal{C}$-family over $S_\alpha$ $\xrightarrow{\varphi_\alpha} S$. It is isomorphic with the $\mathcal{A}$-family created over $S_\alpha$ $\xrightarrow{\varphi_\alpha} S$. It follows that it is a $\mathcal{A}$-family. □
2.8. Definition. Given a functor $u : T \to S$, we say that a family $f_\alpha : X_\alpha \to X$ in $T$ is $u$-surjective when the family $u(f_\alpha)$ is strict epimorphic in $S$.

We shall often omit the $u$ when we write “$u$-surjective” and simply write “surjective”. Notice that when $u = \text{id}$, the surjective families are the strict epimorphic families.

Let $S_X$ be the class of all surjective families with codomain $X$. The collection $S$ satisfies conditions (I) and (S), but fails in general to satisfy (C), (U) and (F).

2.9. Definition. Given a functor $u : T \to S$, let $f_\alpha : X_\alpha \to X$ be a family in $T$ over $\phi_\alpha : S_\alpha \to S$ in $S$. The family $f_\alpha$ is $u$-final if for any family $g_\alpha : X_\alpha \to Y$ in $T$ and arrow $\phi : S \to T$ in $S$ such that $g_\alpha$ sits over $\phi \circ \phi_\alpha$, there exits a unique $g : X \to Y$ over $\phi$ such that $g \circ f_\alpha = g_\alpha$.

We shall often omit the $U$ when we write “$u$-final” and simply write “final”. Notice that when $u = \text{id}$ all families are final. The situation for final families is described in the following double diagram, where the top diagram sits over the bottom diagram.

\begin{equation}
\begin{array}{c}
\begin{array}{ccc}
X_\alpha & \xrightarrow{f_\alpha} & X \\
\downarrow & & \downarrow \exists ! g \\
S_\alpha & \xrightarrow{\phi_\alpha} & S \\
\end{array}
\end{array}
\end{equation}

Final families are unique up to isomorphisms in the sense of definition 2.4.

Let $F_X$ be the class of all final families with codomain $X$. The collection $F$ satisfies conditions (I), (C) and (S), but fail in general to satisfy (U) and (F).

By $FS = F \cap S$ we shall denote the collection of all final and surjective families. Notice that $FS$-families are unique up to isomorphisms. Moreover, the collection $FS$ satisfies conditions (I) and (S) by remark 1.8.

Corresponding to proposition 1.11 it is easy to prove the following:

2.11. Proposition. Given a commutative triangle of functors, if $F$ has a right adjoint $R$ such that $u \circ R = u'$, then $F$ preserves final families.

\begin{equation}
\begin{array}{c}
\begin{array}{ccc}
T & \xrightarrow{F} & T' \\
\downarrow u & & \downarrow u' \\
S & \xrightarrow{u} & S \\
\end{array}
\end{array}
\end{equation}

We remark that if the right adjoint $R$ fails to make the triangle commutative, then $F$ will not preserve general final families. However, very often in practice it will preserve those final families which are also surjective.

We study now the implication: \text{Strict epimorphic } \Rightarrow \text{Final surjective}

2.12. Proposition. If $u$ preserves strict epimorphic families then by definition strict epimorphic families are surjective, that is, $sE \subseteq S$. By Proposition 1.11, this is the case when $u$ has a right adjoint.

2.13. Proposition. If $u$ is faithful, strict epimorphic families are final, that is, $sE \subseteq F$.

\textbf{Proof.} Consider the double diagram 2.10, and suppose that $f_\alpha$ is strict epimorphic. Since the family $\phi \circ \phi_\alpha$ is compatible with $\phi_\alpha$, it follows from the faithfulness of $u$ that $g_\alpha$ is compatible with $f_\alpha$. The existence of $g$ follows.
2.14. **Corollary.** If \( u \) is faithful and preserves strict epimorphic families then each strict epimorphic family is final and surjective, that is, \( sE \subseteq FS \).

*Proof.* Apply propositions 2.12 and 2.13. □

From proposition 1.10 it follows:

2.15. **Corollary.** If \( u \) is faithful, the family of all the inclusions of a colimit cone is a final family. If in addition \( u \) preserves strict epimorphic families, it is final surjective. □

We pass now to the implication: \textit{Final surjective} \( \Rightarrow \) \textit{Strict epimorphic}

2.16. **Proposition.** If \( u \) has a left adjoint, then each final surjective family is strict epimorphic, that is \( FS \subseteq sE \).

*Proof.* Let \( X_\alpha \xrightarrow{f_\alpha} X \) be a final surjective family. Given any compatible family \( X_\alpha \xrightarrow{g_\alpha} Y \), by proposition 1.12 \( uX_\alpha \xrightarrow{ug_\alpha} uY \) is compatible with \( uX_\alpha \xrightarrow{uf_\alpha} uX \), which by assumption is strict epimorphic in \( S \). Thus we have \( uX \xrightarrow{uf} uY \). This finishes the proof since we are assuming also that \( X_\alpha \xrightarrow{f_\alpha} X \) is a final family. □

From 1.11, 2.14 and 2.16 it follows:

2.17. **Theorem.** If the functor \( u \) is faithful, and has left and right adjoints, then a family is strict epimorphic if and only if is final and surjective, that is, \( sE = FS \). □

Notice that the hypothesis in this theorem are self dual, so that a dual theorem also holds:

2.18. **Theorem.** If the functor \( u \) is faithful, and has left and right adjoints, then a family is strict monomorphic if and only if is initial and injective. □

We have the same characterization of strict epimorphic families under a different assumption which has also many other important consequences. This assumption defines the right generality for the validity of many constructions and results, and merits to be treated by itself.

2.19. **Definition.** A functor \( \mathcal{T} \xrightarrow{u} \mathcal{S} \) is an \( E \)-functor (resp. \( M \)-functor) if it is faithful and creates and preserves strict epimorphic families (resp. strict monomorphic families).

If we consider only finite families in definition 2.19, we have the notions of \( fE \)-functor and \( fM \)-functor.

2.20. **Theorem.** Given a \( E \)-functor \( \mathcal{T} \xrightarrow{u} \mathcal{S} \), a family in \( \mathcal{T} \) is strict epimorphic if and only if is final and surjective, that is, \( sE = FS \).

*Proof.* By Corollary 2.14, \( sE \subseteq FS \), and we know that \( FS \)-families are unique up to isomorphisms. By Lemma 2.7, it remains to see that \( u \) creates \( sE \)-families over \( u(\mathcal{F}S) \)-families. But in \( \mathcal{S} \) by definition we have \( u(\mathcal{F}S) \subseteq u(S) \subseteq sE \), and by assumption \( u \) creates strict epimorphic families. □

The reader can easily check that all the arguing above still holds if we consider only finite families. We state now explicitly the dual statement of theorem 2.20 in the finite case:

2.21. **Theorem.** Given a \( fM \)-functor \( \mathcal{T} \xrightarrow{u} \mathcal{S} \), a finite family in \( \mathcal{T} \) is strict monomorphic if and only if is initial and injective. □
3. Cartesian families

In this section we develop the basic yoga of cartesian arrows introduced by Grothendieck in [6], but we do so using families instead of single arrows. The consequences of replacing arrows by families are strong and conform a very different theory which furnishes the framework for the theory of topological functors.

Given a functor $u : T \to S$ and an object $S \in S$, we denote $T_S$ the fiber of $u$ over $S$, that is, the subcategory of $T$ formed by all arrows sitting over the identity $S \xrightarrow{id} S$ of the object $S$. Notice that each fiber $T_S$ is a poset if $u$ is faithful.

We explicitate now the definition of initial family by dualizing definition 2.9.

3.1. Definition. Given a functor $u : T \to S$ and a family $f_\alpha : X \to X_\alpha$ in $T$ over $\varphi_\alpha : S \to S_\alpha$ in $S$, we say that the family $f_\alpha$ is $u$-initial if for any family $g_\alpha : Y \to X_\alpha$ in $T$ and arrow $\varphi : R \to S$ in $S$ such that $g_\alpha$ sits over $\varphi_\alpha \circ \varphi$, there exits a unique $g : X \to Y$ over $\varphi$ such that $f_\alpha \circ g = g_\alpha$. We shall often omit the $u$ when we mean “$u$-initial” and simply say “initial”.

The situation for initial families is described in the following double diagram, where the top diagram sits over the bottom diagram.

\[
\begin{array}{ccc}
Y & \xrightarrow{g_\alpha} & X \\
\downarrow \varphi & & \downarrow \varphi_\alpha \\
R & \xrightarrow{f_\alpha} & X_\alpha \\
\end{array}
\]

If we consider, for any object $X$ in $T$, the class $\mathcal{I}_X$ of all initial families with domain $X$, then the collection $\mathcal{I}$ satisfies (I), (C) and (S\textsuperscript{op}), but fail in general to satisfy (U\textsuperscript{op}) and (F\textsuperscript{op}).

Notice that an initial family $f_\alpha$ acts as a monomorphic family on arrows $g, g'$ over the same $\varphi$.

3.3. Definition. Given a functor $u : T \to S$ and a family $f_\alpha : X \to X_\alpha$ in $T$ over $\varphi_\alpha : S \to S_\alpha$ in $S$, we say that the family $f_\alpha$ is $u$-cartesian if for any family $g_\alpha : Y \to X_\alpha$ in $T$ over $\varphi_\alpha : S \to S_\alpha$, there exits a unique $g : Y \to X$ in $T_S$ such that $f_\alpha \circ g = g_\alpha$. We shall often omit the $u$ when we mean “$u$-cartesian” and simply say “cartesian”.

The situation for cartesian families is described in the following double diagram, where the top diagram sits over the bottom diagram.

\[
\begin{array}{ccc}
Y & \xrightarrow{g_\alpha} & X \\
\downarrow g & & \downarrow f_\alpha \\
S & \xrightarrow{\varphi_\alpha} & S_\alpha \\
\end{array}
\]

3.5. Notation. In a double diagram, the top part always sits over the bottom part, and all vertical arrows always sit over the respective identity arrow (that is, vertical arrows are always in the fibers).

Cartesian families are unique up to isomorphisms in the sense of definition 2.4. Notice then that the isomorphism lives in the fiber $T_S$.

When the family $f_\alpha$ is formed by a unique arrow $X \xrightarrow{f} X'$, we recover Grothendieck’s SGA-notion of cartesian arrow.
We say that functor \( u \) creates \( \text{cartesian} \) (resp. \( \text{initial} \)) arrows if for any arrow \( T \xrightarrow{\varphi} S \) in \( S \) and object \( X \in T \) over \( S \), there exists \( Y \in T \) over \( T \), and a cartesian (resp. initial) arrow \( Y \xrightarrow{f} X \) over \( \varphi \).

Recall [1, t.2, VI, 6.1, p. 102] the following definition:

3.6. Definition. A functor \( u : T \rightarrow S \) is a \( \text{prefibration} \) (equivalently, \( T \) is \( \text{pre-fibered} \)) if \( u \) creates cartesian arrows, and it is a \( \text{fibration} \) (equivalently, \( T \) is \( \text{fibered} \)) if it is a \( \text{prefibration} \) and cartesian arrows compose.

Recall that initial arrows compose and are cartesian, and that when \( T \) is pre-fibered, if cartesian arrows compose then they are initial. Thus:

3.7. Proposition. A functor \( u : T \rightarrow S \) is a fibration if and only if it creates initial arrows.  \( \square \)

We study now this situation for families. Clearly initial families are cartesian, but the converse is not true. Cartesian families do not compose, that is, they do not satisfy property (C).

We establish first a technical lemma relating products in the fibers with cartesian arrows.

3.8. Lemma. Given a functor \( u : T \rightarrow S \), consider the situation described in the following double diagram:

\[
\begin{array}{ccc}
Y & \xrightarrow{f} & X \\
\downarrow{\pi} & & \downarrow{p} \\
Y_\alpha & \xrightarrow{f_\alpha} & X_\alpha \\
\downarrow{\pi_\alpha} & & \downarrow{p_\alpha} \\
S & \xrightarrow{\varphi} & S_\alpha
\end{array}
\]

(i) Assume each \( f_\alpha \) cartesian, then:
\[ \pi_\alpha \text{ is a product in } T_S \Leftrightarrow p_\alpha \text{ is a cartesian family}. \]

(ii) Assume each \( f_\alpha \) initial, then:
\[ \pi_\alpha \text{ is an initial family } \Leftrightarrow p_\alpha \text{ is an initial family}. \]

Proof. (i): We shall argue over the situation described in the following double diagram:

\[
\begin{array}{ccc}
Y & \xrightarrow{g} & X \\
\downarrow{h} & & \downarrow{p} \\
Y_\alpha & \xrightarrow{f_\alpha} & X_\alpha \\
\downarrow{\pi_\alpha} & & \downarrow{p_\alpha} \\
S & \xrightarrow{\varphi} & S_\alpha
\end{array}
\]

\( (\Leftarrow) \): Given a family \( h_\alpha \), consider the composites \( f_\alpha \circ h_\alpha \). Then there is a unique \( g \) such that \( p_\alpha \circ g = f_\alpha \circ h_\alpha \). But \( p_\alpha = f_\alpha \circ \pi_\alpha \), so that \( f_\alpha \circ \pi_\alpha \circ g = f_\alpha \circ h_\alpha \). Since each \( f_\alpha \) is cartesian, it follows that \( \pi_\alpha \circ g = h_\alpha \). Given \( g' \) such that \( \pi_\alpha \circ g' = h_\alpha \), we have \( f_\alpha \circ \pi_\alpha \circ g' = f_\alpha \circ h_\alpha \), so that \( p_\alpha \circ g' = f_\alpha \circ h_\alpha \). This implies \( g' = g \).

\( (\Rightarrow) \): Given a family \( g_\alpha \), since each \( f_\alpha \) is cartesian, there exists a unique \( h_\alpha \) such that \( f_\alpha \circ h_\alpha = g_\alpha \). It follows that there exists in turn a unique \( g \) such that
\[ \pi_\alpha \circ g_\alpha = h_\alpha. \] We have \( f_\alpha \circ \pi_\alpha \circ g = f_\alpha \circ h_\alpha, \) thus \( p_\alpha \circ g = g_\alpha. \) Given \( g' \) such that \( p_\alpha \circ g' = g_\alpha, \) we have \( f_\alpha \circ \pi_\alpha \circ g' = f_\alpha \circ h_\alpha. \) Since each \( f_\alpha \) is cartesian, it follows \( \pi_\alpha \circ g' = h_\alpha. \) This implies \( g' = g. \)

(ii): It follows the same lines than the preceding proof, and it is left to the reader. \( \square \)

Considering the case in which \( f_\alpha = id_{X_\alpha} \) for all indices \( \alpha, \) we have from part i):

3.9. Corollary. A family \( \pi_\alpha : X \to X_\alpha \) in \( T_S \) is a product in \( T_S \) if and only if it is a cartesian family.

Concerning composition of cartesian families, it suffices to consider composition of arrows with families as follows:

3.10. Definition. Given a functor \( u : T \to S \) we say that cartesian arrows compose with cartesian families if for any cartesian arrow \( f : Y \to X \) and cartesian family \( f_\alpha : X \to X_\alpha, \) the family \( f_\alpha \circ f : Y \to X_\alpha \) is cartesian.

3.11. Proposition. Given a prefibration \( u : T \to S, \) the following statements are equivalent:

(i) Cartesian arrows compose with cartesian families.

(ii) Cartesian families are initial families.

(iii) Cartesian families compose.

Proof. Since initial families compose, it remains to prove the implication \((i) \Rightarrow (ii).\)

Let \( X \rightarrow X_\alpha \) be a cartesian family over \( S_\alpha. \) Consider \( g_\alpha \) and \( \varphi \) as in the following double diagram:

\[
\begin{array}{ccc}
Z & \xrightarrow{g_\alpha} & Y \\
\downarrow h & & \downarrow g \\
\varphi & \xrightarrow{f_\alpha} & X_\alpha \\
\end{array}
\]

Take \( g \) cartesian over \( \varphi. \) Since \( f_\alpha \circ g \) is cartesian, we have a unique \( h \) in \( T_H \) such that \( f_\alpha \circ g \circ h = g_\alpha. \) Let \( \ell = g \circ h. \) We have that \( \ell \) sits over \( \varphi \) since \( h \) sits over \( id_H, \) and \( f_\alpha \circ \ell = g_\alpha \) since \( f_\alpha \circ \ell = f_\alpha \circ g \circ h = g_\alpha. \) We leave the reader to check the uniqueness of \( \ell. \) \( \square \)

We conclude then that in a prefibration cartesian families compose if and only if they are initial families.

3.12. Definition. Given any functor \( u : T \to S, \) we say that products (in the fibers) are stable if given \( \varphi : S \to T \) in \( S \) and a double diagram as follows:

\[
\begin{array}{ccc}
Y & \xrightarrow{g} & X \\
\downarrow \rho_\alpha & & \downarrow \pi_\alpha \\
Y_\alpha & \xrightarrow{g_\alpha} & X_\alpha \\
\end{array}
\]

\[
\begin{array}{ccc}
S & \xrightarrow{\varphi} & T \\
\end{array}
\]

with \( g \) and each arrow \( g_\alpha \) cartesian, then:

\[ \text{if } \pi_\alpha \text{ is a product (in } T_T), \text{ so it is } \rho_\alpha \text{ (in } T_S). \]
3.13. **Proposition.** Given a prefibration $u : T \to S$, the following statements are equivalent:

(i) Cartesian arrows compose with cartesian families.

(ii) Cartesian arrows compose (that is, $u$ is a fibration) and products (in the fibers) are stable.

**Proof.** (ii) $\Rightarrow$ (i). Given a cartesian arrow $Y \xrightarrow{g} X$ and a cartesian family $X \xrightarrow{g} Z$ over $T$, we shall argue over the double diagram which follows:

\[
\begin{array}{c}
Y \xrightarrow{g} X \\
| \quad | \quad | \\
\rho_\alpha \quad \pi_\alpha \quad p_\alpha \\
Y_\alpha \xrightarrow{g_\alpha} X_\alpha \xrightarrow{f_\alpha} Z_\alpha \\
\end{array}
\]

Take $f_\alpha$ and $g_\alpha$ cartesian for each $\alpha$. It follows that there exists the vertical arrows $\pi_\alpha$ and $\rho_\alpha$. By lemma 3.8 we have that $\pi_\alpha$ is a product. Since products are stable, so it is $\rho_\alpha$. But $f_\alpha \circ g_\alpha$ is cartesian, thus again by lemma 3.8 we have that $p_\alpha \circ g$ is a cartesian family.

(i) $\Rightarrow$ (ii). Consider an square as in definition 3.12, with $\pi_\alpha$ a product. By corollary 3.9 we have that $\pi_\alpha$ is a cartesian family, thus so it is $\pi_\alpha \circ g$. Then by lemma 3.8 we have that $\rho_\alpha$ is product. $\blacksquare$

In this paper we shall deal with prefibrations whose fibers are posets. Concerning this we have:

3.14. **Proposition.** Given a prefibration $u : T \to S$, $u$ is faithful if and only if $T_S$ is a poset for any object $S$ in $S$.

**Proof.** Clearly if $u$ is faithful $T_S$ is a poset. Conversely, given $u, v : Y \to X$ over $\varphi : T \to S$, take a cartesian arrow $f : Z \to X$ over $\varphi$; then there exist $u', v' : Y \to Z$ in $T_S$ such that $f \circ u' = u, f \circ v' = v$. But $u' = v'$, thus $u = v$. $\blacksquare$

4. **Topological functors**

A (pre) topological functor behaves as a (pre) fibration, but with respect to families instead of single arrows. In this section we consider four notions on a functor related as follows:

\[
\text{Pretopological} \quad \Rightarrow \quad \text{Prefibration} \\
\uparrow \quad \uparrow \\
\text{Topological} \quad \Rightarrow \quad \text{Fibration}
\]

We say that a functor $u$ creates cartesian (resp. initial) families if for any family $S \xrightarrow{\varphi_S} S_\alpha$ and objects $X_\alpha \in T$ over $S_\alpha$, there exists $Y \in T$ over $S$ and a cartesian (resp. initial) family $Y \xrightarrow{f_\alpha} X_\alpha$ over $S \xrightarrow{\varphi_S} S_\alpha$. Notice that in the terms of the dual case of definition 2.5, this means that $u$ creates cartesian (resp. initial) families over the class $S(S, -)$ of all families in $S$.

4.1. **Definition.** A functor $u : T \to S$ is called pretopological if it creates cartesian families.
From corollary 3.9, it follows that the fibers of a pretopological functor have all products. This property characterize such functors as prefibrations.

4.2. Proposition. A functor $u : T \to S$ is pretopological if and only if it is a prefibration such that the fibers $T_S$ have all products.

Proof. Given $X_\alpha$ over $S_\alpha$ and $S \xrightarrow{\varphi_\alpha} S_\alpha$, we shall argue over the following diagram:

$$
\begin{array}{ccc}
X & \xleftarrow{\pi_\alpha} & S \\
\downarrow{p_\alpha} & & \downarrow{\varphi_\alpha} \\
Y_\alpha & \xrightarrow{f_\alpha} & X_\alpha
\end{array}
$$

Take $f_\alpha$ cartesian for each $\alpha$, let $\pi_\alpha$ be a product in $T_S$, and set $p_\alpha = f_\alpha \circ \pi_\alpha$. By lemma 3.8(i), it follows that $p_\alpha$ is a cartesian family. \hfill \Box

Notice that the proof of Proposition 4.2 also proves that each fiber has small products if and only if the functor creates small cartesian families, a fact that can be of independent interest although we will not have a use for it in this paper.

As it is well known (Freyd [4]) and easy to show, a category with all products collapses into a complete lattice (eventually large), thus we have

4.3. Proposition. A functor $u : T \to S$ is pretopological if and only if it is a prefibration such that the fibers $T_S$ are complete lattices. \hfill \Box

It follows then from proposition 3.14 that pretopological functors are necessarily faithful. We give now a direct proof of this fact.

4.4. Proposition (compare [2], 7.3.4). Any pretopological functor is faithful.

Proof. Let $u : T \to S$ be a pretopological functor. Given $f, g : X \to Y$ over $S \xrightarrow{\varphi} T$, we shall show that $f = g$. Consider all arrows $Y \xrightarrow{\alpha} (-)$ with domain $Y$ in $T_S$, and for each such $\alpha$ an arrow $S \xrightarrow{\varphi_\alpha} T$, $\varphi_\alpha = \varphi$. This determines the following double diagram:

$$
\begin{array}{ccc}
Y & \xrightarrow{h_\alpha} & X \\
\downarrow{h} & & \downarrow{f_\alpha} \\
A & \xrightarrow{f} & T
\end{array}
$$

where $f_\alpha$ is a cartesian family over $\varphi_\alpha$, and $h_\alpha$ is defined as follows:

$$
h_\alpha = \begin{cases} 
  f & \text{if codomain}(\alpha) = A \text{ and } f_\alpha \circ \alpha = g \\
  g & \text{otherwise}
\end{cases}
$$

It follows there exists a unique $h$ such that $f_\alpha \circ h = h_\alpha$ for any $\alpha$. In particular, $f_h \circ h = h_h$. We have then:

$$
h_h = \begin{cases} 
  f & \text{i.e. } h_h = g \\
  g & \text{i.e. } h_h = f
\end{cases}
$$

Since $h_h = f$ or $h_h = g$, it follows $f = g$, both being equal to $h_h$. \hfill \Box

4.5. Definition. A functor $u : T \to S$ is called topological if it creates cartesian families and cartesian families compose.
As an immediate consequence of proposition 3.11 we have the analogous result to proposition 3.7.

4.6. **Proposition.** A functor \( u : T \to S \) is topological if and only if it creates initial families. □

By definition an injective family in \( T \) is a family over an strict monomorphic family in \( S \). Thus, creation of initial injective families means creation of initial families over strict monomorphic families. A topological functor, in particular, creates initial arrows and initial injective families. We see next that these two particular instances of creation imply the creation of arbitrary initial families.

4.7. **Proposition.** A functor \( u : T \to S \) is topological if and only if it is a fibration (that is, it creates initial arrows) and it creates initial injective families.

**Proof.** Given \( X_\alpha \) over \( S_\alpha \) and \( S \xrightarrow{S_{\varphi}} S_{\alpha} \), we shall argue over the following diagram:

\[
\begin{array}{c}
X \\
\downarrow \pi_\alpha \\
Y_\alpha \\
\downarrow f_\alpha \\
X_\alpha
\end{array}
\]

\[
S \xrightarrow{\varphi} S_{\alpha}
\]

Take an initial arrow \( f_\alpha \) for each \( \alpha \). Consider the family of identity arrows \( \text{id}_{S_\alpha} \), one for each \( \alpha \) (obviously a strict monomorphic family in \( S \)), let \( \pi_\alpha \) be an initial family, and set \( p_\alpha = f_\alpha \circ \pi_\alpha \). By lemma 3.8(ii), it follows that \( p_\alpha \) is an initial family. □

Clearly, a topological functor is a fibration, and as an immediate corollary of propositions 3.13 and 4.3 we have the following characterization of topological functors as fibrations:

4.8. **Proposition.** A functor \( u : T \to S \) is topological if and only if it is a fibration such that the fibers \( T_S \) are complete lattices and infima are stable. □

We state now a characterization of topological functors \( u : T \to S \) in terms of the pseudofunctor \( P : S^{\text{op}} \to \text{Cat} \) associated to the fibration. Recall that the data which defines a fibration \( u \) is equivalent to the data that defines its associated pseudofunctor \( P \) (Grothendieck’s construction, [6]). Recall that standard notation for the action of \( P \) on arrows is an upper star. In this way, given \( S \xrightarrow{\varphi} T \) in \( S \), \( P(S \xrightarrow{\varphi} T) = T_T \xrightarrow{\phi^*} T_S \). Recall also that given \( X \) in \( T_T \), \( \varphi^* X \) in \( T_S \) is characterized by a cartesian arrow \( \varphi^* X \to X \) in \( T \) over \( S \xrightarrow{\varphi} T \).

An immediate consequence of proposition 4.8 is the following:

4.9. **Theorem.** Given a pseudofunctor \( P : S^{\text{op}} \to \text{Cat} \), the associated fibration \( u : T \to S \) is a topological functor if and only if:

(i) For all objects \( S \) in \( S \), \( P(S) \) is a complete lattice.

(ii) For all arrows \( \varphi \) in \( S \), \( \varphi^* \) preserves infima, or, equivalently, it has a left adjoint \( \varphi_! \vdash \varphi^* \).

We introduce now some background and notations.

4.10. **Notation.** Given a functor \( u : T \to S \) and an object \( S \in S \),

1) We denote by \( S_\top \in T \) the initial family over the empty family with domain \( S \). If it exists, \( S_\top \) is a terminal object of the fiber \( T_S \), and the assignment \( S \mapsto S_\top \) furnishes a right adjoint \( (-)_\top \vdash u \).

2) We denote by \( S_\perp \in T \) the final family over the empty family with codomain \( S \). If it exists, \( S_\perp \) is an initial object of the fiber \( T_S \), and the assignment \( S \mapsto S_\perp \) furnishes a left adjoint \( u \vdash (-)_\perp \).
Notice that a right adjoint \( r : S \to T \) of \( u \) not necessarily furnishes an initial family over the empty family. That is, \( r(S) \) will not be \( S_\top \). We have \( r(S) = S_\top \) when the counit of the adjunction is the identity arrow, that is, essentially, when \( r \) is full and faithful. Same considerations apply to a left adjoint for \( u \) and \( S_\bot \).

If \( 1 \) is a terminal object of \( S \), then \( 1_\top \) is a terminal object of \( T \), and if \( 0 \) is an initial object of \( S \), then \( 0_\bot \) is an initial object of \( T \).

We warn the reader that, contrary than in the usual examples, the objects \( 1_\top \) and \( 1_\bot \) may be different. That is, the fiber over \( 1 \), even in the case in which the category \( S \) is the category of sets, will not in general be the singleton category (there may be many different "structures" on the singleton set).

4.11. Example. It is clear that the forgetful functor \( u : \text{Top} \to \text{Set} \) is topological, and \( 1 \) has a unique topology (discrete = indiscrete). If we consider each set endowed with a filter of subsets instead of a topology, and the same definition of morphisms, then we obtain a category \( \text{Filt} \) and a forgetful functor \( u : \text{Filt} \to \text{Set} \) which is topological (initial families like in \( \text{Top} \)) but now \( 1 \) has two different structures (discrete \( \neq \) indiscrete). This type of structure appears when considering convenient categories for proper homotopy theory (categories of exterior spaces, see [5]).

Since a topological functor creates, in particular, the empty initial family, there exists \((−)_\top \). But also there exists \((−)_\bot \):

4.12. Proposition. Given a faithful functor \( u : \mathcal{T} \to \mathcal{S} \), let \( X \to Z \) be an initial family over the family of all arrows with domain \( S \), \( S \to u(Z) \), \( Z \in \mathcal{T} \). Then \( X \) is the final family over the empty family with codomain \( S \), \( X = S_\bot \) (and \( \pi_\alpha \) is the arrow that corresponds to \( \alpha \) by adjointness).

Proof. We have to check that \( X \) is the value on \( S \) of a left adjoint to \( u \). This is immediate, given \( S \to u(Z) \), just take \( X \to Z \). Uniqueness follows from the faithfulness of \( u \).\n
4.13. Corollary (compare [2] 7.3.7). Any topological functor \( u : \mathcal{T} \to \mathcal{S} \) has full and faithful left and right adjoints \((−)_\top \vdash u \vdash (−)_\bot \), \( u(-)_\top = u(-)_\bot = \text{id} \).

From proposition 4.4 and theorems 2.17, 2.18, we have:

4.14. Corollary. Given any topological functor \( u : \mathcal{T} \to \mathcal{S} \), a family in \( \mathcal{T} \) is strict epimorphic if and only if it is final and surjective, and it is strict monomorphic if and only if it is initial and injective.\n
The lack of size limitations on the families not only has as a consequence the faithfulness of pretopological functors, but also implies that the notion of pretopological and topological functor is selfdual. Proposition 4.12 generalizes, that is, if a functor creates initial families, then it creates final families (obviously an implication equivalent to its reciprocal)

4.15. Proposition (compare [2] 7.3.6). A functor \( u : \mathcal{T} \to \mathcal{S} \) is topological if and only if considered as a functor \( u : \mathcal{T}^{\text{op}} \to \mathcal{S}^{\text{op}} \) is topological.

Proof. It is enough to prove that if \( u \) creates initial families then it creates final families. Given a family \( S_\alpha \to S \) in \( \mathcal{S} \) and a family \( X_\alpha \) of objects of \( \mathcal{T} \) over \( S_\alpha \), we must create a final family \( X_\alpha \to X \) over \( \varphi_\alpha \). We shall argue over the following
Consider the family of all arrows with domain $S$, $S \xrightarrow{\beta} u(Z)$, $Z \in T$, such that for all $\alpha$ there exists $X_\alpha \xrightarrow{g_{\alpha\beta}} Z$ in $T$ over $\beta \circ \varphi_\alpha$. Take a initial family $X \xrightarrow{h_\beta} Z$ over the family $\beta$. For each $\alpha$, the family $g_{\alpha\beta}$ indexed by $\beta$ implies the existence of an arrow $X_\alpha \xrightarrow{f_\alpha} X$ over $\varphi_\alpha$ such that $h_\beta \circ f_\alpha = g_{\alpha\beta}$. It remains to prove that $f_\alpha$ is a final family. Suppose we are given $Z_0 \in T$, $S \xrightarrow{\beta_0} u(Z_0)$ in $S$, and for all $\alpha$ an arrow $X_\alpha \xrightarrow{u_\alpha} Z_0$ over $\beta_0 \circ \varphi_\alpha$. Let $h = h_{\beta_0}$. Since $u$ is faithful we have $u_\alpha = g_{\alpha\beta_0}$ so that $h \circ f_\alpha = u_\alpha$. Furthermore, again by the faithfulness of $u$, such an $h$ is unique since it must sit over $\beta_0$. □

From the proof of this theorem we derive a further characterization of topological functors:

4.16. **Proposition.** A functor $u : T \to S$ is topological if and only if

i) It is faithful and creates and preserves strict monomorphic families.

ii) It has a full and faithful right adjoint $u \dashv (-)_T$, $u(-)_T = \text{id}$.

**Proof.** If $u$ is a topological functor, i) follows from proposition 4.4 and corollary 4.14, and ii) from corollary 4.13. For the other direction, we shall show that $u$ creates final families with the same proof that of proposition 4.15. We show that the family in $S$ over which an initial family is created is actually a strict monomorphic family. Consider the double diagram:

\[
\begin{array}{ccc}
X_\alpha & \xrightarrow{f_\alpha} & X \\
\text{id} & \downarrow & \downarrow h_\beta \\
S & \xrightarrow{\varphi_\alpha} & S
\end{array}
\]

where the arrows $g_\alpha$ correspond by adjointness to $\varphi_\alpha$. This shows that $S \xrightarrow{\text{id}} S$ is in the family $S \xrightarrow{\beta} u(Z)$ and thus this family is strict monomorphic. By the dual of theorem 2.20, the strict monomorphic family created over it is an initial family. The same proof continuous then. Notice that we are assuming $u$ to be faithful. □

By proposition 4.15 we have that topological functors are the same thing that functors satisfying the dual of definition 4.5, that is, functors that create final families (or equivalently, cocartesian families that compose). As a consequence, all the dual statements from 4.5 to 4.16 hold for topological functors. We explicitate the dual of this last characterization:

4.17. **Proposition.** A functor $u : T \to S$ is topological if and only if:

i) It is faithful and preserves and creates strict epimorphic families.

ii) It has a full and faithful left adjoint $(-) \dashv u$, $u(-) = \text{id}$.

4.18. **Remark.** Notice that condition i) in the previous propositions mean that a topological functor is a $\mathcal{E}$-functor and a $\mathcal{M}$-functor.
It is immediate to check that from the creation of strict epimorphic (respectively, strict monomorphic) families it follows the creation of all colimits (respectively, limits) that may exists in \( S \) (see remark 2.6). We finish this section writing down a list of properties that topological functors have.

4.19. **Theorem.** If \( u : T \to S \) is a topological functor (definition 4.5), then:

1) It is faithful.
2) It creates final families and initial families.
3) It is a fibration and a cofibration, the fibers are complete (hence also cocomplete) lattices, and for any arrow \( S \xrightarrow{\varphi} T \) in \( S \), there is a pair of adjoint functors \( \varphi ! \dashv \varphi^* \) between \( T_S \) and \( T_T \), \( (\cdot)^* \) is the action of the fibration, and \( ! (\cdot) \) is the action of the cofibration.
4) There are adjunctions \( (\cdot) \top \vdash u \vdash (\cdot) \bot \), with \( u (\cdot) \top = u (\cdot) \bot = id \).
5) In \( T \) strict epimorphic families are the same than final surjective families, and strict monomorphic families are the same than initial injective families.
6) It creates and preserve any limit and colimit that may exists in \( S \).
7) If in \( S \) all epis are strict (for example, when it is a topos), then in \( T \) epimorphic families are the same than surjective families. If in \( S \) all monos are strict (for example, when it is a topos), then in \( T \) monomorphic families are the same than injective families.
8) When \( S = \textbf{Set} \) is the category of sets, then \( u \) is representable by the object \( 1 \bot \).

The forgetful functor from the category of topological spaces and of the several quasitopoi associated with this category are examples of topological functors (see [3] and the references therein). However, forgetful functors from concrete quasitopoi in a more general sense are not always topological functors. They lead to a notion of quasitopological functor, which we shall develop elsewhere.

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