Pulse Propagation in Resonant Tunneling

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We consider the analytically solvable model of a Gaussian pulse tunneling through a transmission resonance with a Breit-Wigner characteristic. The solution allows for the identification of two opposite pulse propagation regimes: if the resonance is broad compared to the energetic width of the incident Gaussian pulse a weakly deformed and slightly delayed transmitted Gaussian pulse is found. In the opposite limit of a narrow resonance the dying out of the transmitted pulse is dominated by the slow exponential decay characteristic of a quasi-bound state with a long life time (decaying state). We discuss the limitation of the achievable pulse transfer rate resulting from the slow decay. Finally, it is demonstrated that for narrow resonances a small second component is superimposed to the exponential decay which leads to characteristic interference oscillations.

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INTRODUCTION

The transmission of pulses in tunneling transport is a basic problem in quantum mechanics\textsuperscript{1,2,3}. Here the case of an incident wave packet transmitted through a resonant quasi-bound level of a quantum system (QS) is a fundamental issue under current study\textsuperscript{4,5,6,7,8,9,10}. In our previous paper\textsuperscript{11} an S-matrix representation of the quasi-bound resonance levels of a given QS has been derived: depending on the complex asymmetry parameter a Breit-Wigner or a general Fano resonance line results. In our study we obtain analytical solutions for the case of an incident Gaussian wave packet interacting with a Breit-Wigner resonance. Here we focus on the simplest case when the central position of the incident wave packet in the momentum space coincides with that of the resonance level. Our analytical model allows for a continuous transition between the limits of a broad and a narrow resonance on scale of the energetic width of the incident packet: in the presence of a broad resonance only a weak deformation and a small delay of the incident Gaussian pulse is found\textsuperscript{12}. In the limit of a narrow resonance the signal is dominated by a decaying state in the QS\textsuperscript{10}. We study the shape of the transmitted pulse in this regime: the dropping flank of the transmitted pulse is not Gaussian any more but dominated by a slow exponential time characteristic of the decaying state. In addition, there is a weak second component in the transmitted signal representing a small deformed Gaussian pulse. Its superposition with the primary exponentially decaying component causes characteristic interference oscillations for which we give an explicit expression.

MODEL

We analyze a set-up as depicted in Fig. 1, an incident standard Gaussian wave packet

$$\psi_0(x,t) = \frac{\psi_0}{\sqrt{1 + \frac{i\hbar}{m}t}} \exp\left[-\frac{(x-v_0 t)^2}{2a^2(1 + \frac{i\hbar}{m}t)}\right] \exp\left[ik_0 x - \frac{i\hbar k_0^2}{2m} t\right].$$

is approaching a tunneling barrier with a group velocity $v_0 = \hbar k_0/m$. At $t = 0$ the incident packet has a width of $a$ in real space. The transmitted pulse is given by the expression

$$\psi(x > d_2, t) = \int \frac{dk}{\sqrt{2\pi}} \psi_0(k) S(k) \exp\left[ik x - \frac{i\hbar}{2m} k^2 t\right].$$

where $\psi_0(k)$ is the Fourier transform of $\psi_0(x, t = 0)$. A short range scattering potential is assumed so that $V(x) = 0$ outside the interval $d_1 \leq x \leq d_2$. Furthermore, it is assumed that there is an isolated resonance at $E_1 \approx \hbar^2 k_0^2/(2m)$, i. e. close to the mean kinetic energy of the incident pulse. Then in the range of finite $|\psi_0(k)|$ the transmission coefficient can be approximated by a Breit-Wigner characteristic

$$S(k) = S_0 \frac{\frac{\hbar^2 k_0^2}{2m}}{k - k_1 + i\frac{\hbar}{2}}.$$
From Eqs. (1) - (3) we find with the help of Ref. [13]

$$\psi(x > 0, t) = \psi_0 S_0 \rho \sqrt{\frac{\pi}{2}} \exp \left( -\gamma^2 \beta - iq\gamma \right) \text{erfc}(z) \exp \left[ ik_0 x - \frac{i\hbar k_0^2}{2m} t \right],$$  \hspace{1cm} (4)

where \text{erfc} is the complementary error function taken at the argument

$$z = \left( \frac{q}{2\sqrt{\beta}} - i\gamma \sqrt{\beta} \right),$$  \hspace{1cm} (5)

with \( q = (x - v_0 t)/a, \beta = (1 + i\tau)/2, \tau = t/t_0, t_0 = ma^2/\hbar, \) and \( \gamma = a[k_0 - k_1 + i\Gamma_k/2] \equiv \Delta + i\rho. \)

\section*{RESULTS}

The different pulse propagation regimes are illustrated in Fig. 1(b): a sequence of well separated incident Gaussian pulses is created at integer \( t/t_0 = n \) with the center in real space at \( x = 0 \) (source). For the broad enough resonance with \( \rho > 1 \) the signal detected at \( x = L \) is a sequence of well separated weakly deformed Gaussian pulses. These pulses arrive with a small delay with respect to the times \( t/t_0 = n + 1/2 \) at which the signal without scattering potential would arrive. In contrast, if the resonance is narrow, \( \rho \ll 1, \) the transmitted pulses are strongly weakened and deformed so that they are not separable any more. As we will show below this deterioration of the signal is caused by the exponential time characteristic of a decaying state with a life time exceeding the time interval of two

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{fig1.png}
\caption{Part (a): Schematic representation of the incident Gaussian packet (dashed line) approaching the QS. The transmitted signal (dash-dotted line) is detected at \( x = L. \) Part (b): Absolute value of the wave function versus time. In dashed line the sequence of the incident pulses at \( x = 0. \) In dash-dotted line the quasi-stationary limit for the transmitted pulses detected at \( x = L \) for \( \rho = 2 \) (broad resonance) and in solid line for \( \rho = 0.1 \) (narrow resonance). The further parameters are \( L = 5a, k_0a = k_1a = 10, \) and \( |S_0| = 1. \)}
\end{figure}
FIG. 2: Part (c): Absolute value of the transmitted wave function for a broad resonance $\rho = 2.0$ at $\tau = 1.0$ (dotted line), $\tau = 10$ (solid line), $\tau = 50$ (dash-dotted line), and $\tau = 800$ (dotted line). Thick lines represent the exact result in Eq. (4) and, for comparison, thin solid lines the result for $\psi = \psi_0(x,t)$ in Eq. (1). For $\tau = 800$ the corresponding lines are not resolvable. Part (a) absolute value of $z$ (see Eq. (5)) and part (b) $\arg(z)$ with the same line-coding as in part (a). The thin dash-dotted line in (b) represents $\arg(z) = 3\pi/4$.

subsequent pulses. The parameters used for the calculations in Fig. 1(b) have been chosen to demonstrate how a nearly optimal signal transfer can be achieved for a broad resonance: i.) the detector is positioned in a minimum distance of $L \approx 5 \div 10a$ from the source. As illustrated in Fig. 1(a) this choice results from the requirements, first, that when the pulse is prepared or detected it should be well separated from the resonant tunneling structure and, second, that the spatial width of the signal should be comparable to the size of the tunneling structure. ii.) the traveling time of the signal should not exceed the characteristic time $t_0 = ma^2/h$ of the quantum spread of the free wave packet in Eq. (1). For an electron packet with a width of $a = 10$ nm this time is very short, $t_0 \approx 10^{-12}$ s. We can therefore define a minimum speed $v_{min} = L/t_0 = h\tau_0/\min/m$ (we neglect here the spread induced by the Coulomb interaction). iii.) for a maximum signal we require optimal resonance so that $k_0 = k_1 (\Delta = 0)$. Together with the second requirement we then obtain a minimum energy of the resonance level of $E_1 \approx h^2 k_{0,\min}^2/(2m)$ which is for $a = 10nm$ in the range of a few meV. iv.) the imaginary part of the resonance energy is fixed by the conditions of $\rho > 1$ with the constraint that the resonance should still be isolated. v.) for an effectively symmetrical double barrier structure the absolute value of $S_0$ is very close to unity (14).

For comparison we exemplify the cases of a broad and a narrow resonance in Fig. 2 and Fig. 3 respectively. The basic differences can be understood in a proper expansion of the complementary error function in Eq. (4).

For a broad resonance $\rho$ is large and we find for the relevant values of $q$ (where $|\psi|$ deviates significantly from zero) that the argument of $z$ is less than $3\pi/4$ (see Fig. 2(b)). Under this condition we can introduce an asymptotic expansion of the error function as given in (15) yielding

$$
\psi(x,t) = S_0 \frac{\sqrt{3}}{z} \left[ 1 + \sum_{m=1}^{\infty} \frac{a_m}{z^2} \right] \psi_0(x,t) = \psi_{DG}(x,t),
$$

with $a_m = (-1)^m(1 \times 3 \times 5 \times \cdots \times (2m - 1))$. The expression in Eq. (6) describes a weakly distorted Gaussian (DG) pulse. The relatively weak distortion of the incident Gaussian pulse $\psi_0(x,t)$ follows, first, from the fact that at large $\rho$ the parameter $1/(2|z|^2)$ is small compared to unity. Therefore, the second term in the square bracket in Eq. (6) gives only a small correction. In fact, only the first two terms in the sum over $m$ in Eq. (6) are necessary to approximate the analytical result from Eq. (4) within plot resolution. Second, for large $\rho$ the argument $z$ depends only very weakly on
This explains that the factor $1/z$ in front of the square bracket in Eq. (6) merely leads to a minor deformation of the packet as well. Note that in the illustrated example $\rho = 2$ takes a moderate value and that the approximation in Eq. (6) works even better for larger $\rho$ for a limited amount of terms in the $m$ summation.

An inspection of Fig. 3(a) shows immediately that for a narrow resonance the unperturbed incident Gaussian pulse is no good approximation for the transmitted pulse. The difference is most pronounced for short times. For example, at $\tau = 1$ a narrow Gaussian transmitted peak results in the case of a broad resonance (Fig. 2(c)) while for the narrow resonance (Fig. 3(a)) a much weaker, broader, and strongly asymmetric resonance results which is nearly completely restricted to negative $q$. To explain the difference we observe that for the broad resonance the argument of $z$ is smaller than $3\pi/4$ in the $q$-region of the transmitted peak ($-5 \leq q \leq 5$ at $\tau = 1$, see Figs. 2(b) and (c)). In contrast, for the narrow resonance the argument of $z$ is larger than $3\pi/4$ in the $q$-region of the transmitted peak ($q \leq 0$ see Figs. 3(a) and (c)). To obtain for small $\rho$ an approximate expression for $q \leq 0$ one expands $	ext{erfc}(-z)$ according to [15] instead of $	ext{erfc}(z)$ as has been done for the broad resonance. Then, a superposition of two factors

$$\psi(x, t) = \psi_{DG}(x, t) + \psi_{DS}(x, t),$$

is obtained with

$$\psi_{DS}(x, t > 0) = \psi_0 S_0 \rho \sqrt{2\pi} \exp(\beta \rho^2) \exp\left(\frac{\Gamma_k}{2} (x - v_0 t)\right) \exp\left(ik_0 x - \frac{\hbar k_0^2}{2m} t\right).$$

For the broad resonance the factor $\psi_{DS}(x, t > 0)$ is the dominant contribution to the signal and $\psi_{DG}(x, t)$ is small. The dominant factor $\psi_{DS}$ results from a decaying state (DS). This can be seen, first, from the exponential decay $\propto \exp(-\Gamma t/2)$ with $\Gamma = v_0 \Gamma_k$ at a fixed space coordinate $x$. Second, at a fixed time the wave function grows $\propto \exp(\Gamma_k x/2)$ which is typical for a DS [16]. This growth continues up to $q \approx 0$ where the wave front of the decaying state signal is located. As can be gathered from Fig. 3(b) this wave front can well be described by replacing in Eq. 4 the complementary error function by its small $z$-expansion $1 - \frac{2}{\sqrt{\pi}} z + ...$.

In Fig. 3(a) the curve for $\tau = 10$ shows characteristic oscillations superimposed to the decaying state signal. These oscillations are presented in more detail in Fig. 4. For $q \leq -8$ the exact result agrees within plot resolution with the approximation of Eqs. (7), Eq. (8), and Eq. (6) (in the latter equation the sum over $m$ can be neglected). The

![FIG. 3: (a) Absolute value of the transmitted wave function, (b) absolute value of $z$, and (c) the argument of $z$ at $\rho = .1$, $l_1 = l_0$, $\tau = 1.0$ (dotted line), and $\tau = 10$. (solid line), $\tau = 100$ (dashed line), and $\tau = 300$ (dash-dotted line). The thin dash-dot-dotted line in c represents $\arg(z) = 3\pi/4$.](image-url)
FIG. 4: Parameters like for the solid line in Fig. 3(a) with $\tau = 10$. In solid line detail of the corresponding curve in Fig. 3(a), dashed line $|\psi_{DS}/(\psi_0 S_0)|$ and in circles the result of the approximation $|\psi| = |\psi_{DS}| + \psi_{OSZ}$ with $\psi_{OSZ}$ as given in Eq. (10).

Characteristic oscillations stem from the interference between the dominant component $\psi_{DS}$ (dashed line in Fig. 4) and the small component $\psi_DG$. They are caused by the oscillating phase in the Gaussian factor $\exp[-q^2/(4\beta)]$ in $\psi_DG$ at complex $\beta$. To show this we introduce further approximations in Eq. (9): first, for moderate times $\tau \sim 10$ one may set $\beta \sim i\tau/2$ outside the Gaussian factor $\exp[-q^2/(4\beta)]$ and, second, for small $\rho$ the contribution $\rho\sqrt{\beta}$ in $z$ can be omitted. It results that

$$\psi_DG = S_0\psi_0\sqrt{\tau}/q \exp\left(-\frac{q^2}{4\beta} + i\frac{\pi}{4}\right) \exp\left[ik_0x - i\frac{h\kappa_0^2}{2m}t\right].$$

(9)

With this approximation for $\psi_DG$ and with Eq. (8) a representation $|\psi(x, t)| = |\psi_{DS}(x, t)| + \psi_{OSZ}(x, t)$ follows where

$$\psi_{OSZ} = \frac{\text{Re}(\psi_DG\psi_D^*)}{|\psi_{DS}|} = -|\psi_0 S_0|\rho\sqrt{\tau}/q \exp\left(-\frac{q^2}{2(1 + \tau^2)}\right) \sin\left(\frac{q^2}{2\tau} - \frac{\tau^2}{2} - \frac{\pi}{4}\right).$$

(10)

As shown in Fig. 4 this expression describes the characteristic oscillations very accurately. Apart from the constant phase $-\pi/4$ the argument in the sine-factor in Eq. (10) stems from $\exp[-q^2/(4\beta)]$ in Eq. (9). This argument determines solely the phase and period of the characteristic oscillations.

CONCLUSIONS

We have analyzed an analytical model for the resonant transmission of a wave packet. In the limit of a broad resonance a weakly distorted transmitted pulse results. In the opposite limit of a narrow resonance the transmitted wave is dominated by a decaying state and characteristic oscillations result. Our considerations can easily be extended to the case of a general Fano resonance. The position of the transmitted peaks at different time moments are determined by specific relations between $k_0, k_1$ and $\Gamma_k$. An understanding of these relations opens the possibility to solve an inverse problem in which the resonance characteristics are expressed in terms of the properties of the transmitted pulse.

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