GRADIENT ESTIMATES FOR THE LAGRANGIAN MEAN CURVATURE EQUATION WITH CRITICAL AND SUPERCRITICAL PHASE

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Abstract. In this paper, we prove interior gradient estimates for the Lagrangian mean curvature equation, if the Lagrangian phase is critical and supercritical and $C^2$. Combined with the a priori interior Hessian estimates proved in [Bha21, Bha22], this solves the Dirichlet boundary value problem for the critical and supercritical Lagrangian mean curvature equation with $C^0$ boundary data. We also provide a uniform gradient estimate for lower regularity phases that satisfy certain additional hypotheses.

1. Introduction

In this paper, we study a priori interior gradient estimates in all dimensions for the Lagrangian mean curvature equation

$$F(D^2u) = \sum_{i=1}^{n} \arctan \lambda_i = \psi(x), \quad x \in B_1(0) \subset \mathbb{R}^n,$$

under the assumption that $|\psi| \geq \frac{(n - 2)}{2}$. Here, $u : B_1 \to \mathbb{R}$ has gradient $Du$ and Hessian matrix $D^2u$, with eigenvalues $\lambda_i$. We will denote $B_r = B_r(0)$ throughout.

When the phase $\psi$ is constant, denoted by $c$, $u$ solves the special Lagrangian equation

$$\sum_{i=1}^{n} \arctan \lambda_i = c,$$

or equivalently,

$$\cos c \sum_{1 \leq 2k+1 \leq n} (-1)^k \sigma_{2k+1} - \sin c \sum_{0 \leq 2k \leq n} (-1)^k \sigma_{2k} = 0.$$
Equation (1.2) originates in the special Lagrangian geometry by Harvey-Lawson [HL82]. The Lagrangian graph \((x, Du(x)) \subset \mathbb{R}^n \times \mathbb{R}^n\) is called special when the argument of the complex number \((1+i\lambda_1) \cdots (1+i\lambda_n)\), or the phase \(\psi\), is constant, and it is special if and only if \((x, Du(x))\) is a (volume minimizing) minimal surface in \((\mathbb{R}^n \times \mathbb{R}^n, dx^2+dy^2)\) [HL82].

More generally, for (1.1), it was shown in [HL82, (2.19)] that the mean curvature vector \(\tilde{H}\) of the Lagrangian graph \((x, Du(x))\) is \(J\nabla_g \psi\), where \(\nabla_g\) is the gradient, and \(J\) is the almost complex structure on \(\mathbb{R}^n \times \mathbb{R}^n\). Note that \(|\tilde{H}|_g\) is bounded for \(C^2\). In the complex setting, a local version of the deformed Hermitian-Yang-Mills equation for a holomorphic line bundle over a compact Kähler manifold is represented by equation (1.1).

The notions of critical and supercritical phases were introduced by Yuan [Yua06]. The Lagrangian angle \(\theta(\lambda) = \sum_i \arctan \lambda_i\) is critical if \(|\theta| = (n-2)\pi/2\) and supercritical if \(|\theta| > (n-2)\pi/2\). We recall that the variable phase \(\psi(x)\) is called critical and supercritical if \(|\psi(x)| \geq (n-2)\pi/2\), and supercritical if \(|\psi(x)| \geq (n-2)\pi/2 + \delta\) for some \(\delta > 0\). It was shown in [Yua06, Lemma 2.1] that the level sets \(\{\lambda : \theta = c\}\) are convex for critical and supercritical phases. In particular, there are Evans [Eva82]-Krylov [Kry83]-Safonov [Saf84, Saf89] \(C^{2,\alpha}\) estimates if \(D^2 u\) is bounded, and \(\psi(x)\) is Hölder continuous.

In this paper, for \(C^2\) critical and supercritical phases, we solve the Dirichlet problem for \(C^0\) boundary data by establishing the missing interior gradient estimates. Interior Hessian estimates for supercritical \(C^{1,1}\) phases were shown by Bhattacharya in [Bha21, Theorem 1.1]; interior Hessian estimates for critical and supercritical phases follow verbatim from the calculations done in [Bha21] (see [Bha22, Remark 2.1]); interior gradient estimates for supercritical \(C^1\) phases were derived in [Bha21, Theorem 1.2].

Our main result is a gradient estimate for arbitrary \(C^2\) critical and supercritical phases.

**Theorem 1.1.** Let \(u\) be a \(C^3(B_1)\) solution of (1.1) on \(B_1(0) \subset \mathbb{R}^n\), where \(\psi \in C^2(B_1)\) satisfies \(\psi \geq (n-2)\pi/2\). Then

\[
|Du(0)| \leq C(n, ||D^2\psi||_{L^\infty(B_1)}) (1 + (osc_B u)^2).
\]

We state the following Hessian estimate combining [Bha21, Theorem 1.1] and [Bha22, Remark 2.1].
Theorem 1.2. Let $u$ be a $C^4$ solution of (1.1) on $B_R(0) \subset \mathbb{R}^n$, where $\psi \in C^2(B_R)$, and $\psi \geq (n-2)\frac{\pi}{2}$. Then we have
\begin{equation}
|D^2u(0)| \leq C \exp[B_{R^2}(0) \max_{B_{R^2}(0)} |Du|^{2n-2}/R^{2n-2}]
\end{equation}
where $C$ is a positive constant depending on $\|\psi\|_{C^2(B_R)}$ and $n$.

As an application, we solve the following Dirichlet boundary value problem with $C^0$ boundary data.

Corollary 1.1. Suppose that $\phi \in C^0(\partial\Omega)$ and $\psi : \overline{\Omega} \to [(n-2)\frac{\pi}{2}, n\frac{\pi}{2})$ is in $C^2(\overline{\Omega})$, where $\Omega$ is a uniformly convex, bounded domain in $\mathbb{R}^n$. Then there exists a unique solution $u \in C^3(\Omega) \cap C^0(\overline{\Omega})$ to the Dirichlet problem
\begin{equation}
\begin{cases}
F(D^2u) = \sum_{i=1}^n \arctan \lambda_i = \psi(x) \text{ in } \Omega \\
u = \phi \text{ on } \partial\Omega
\end{cases}
\end{equation}
The solution $u$ is, in fact, in $C^{3,\alpha}(\Omega)$ for any $\alpha \in (0, 1)$, by classical uniformly elliptic theory.

The Dirichlet problem for a broad class of fully nonlinear, elliptic equations of the form $F(\lambda[D^2u]) = f(x)$ was first studied by Caffarelli-Nirenberg-Spruck in [CNS85], where they proved the existence of classical solutions under various hypotheses on the function $F$ and the domain $\Omega$. In [HL09], Harvey-Lawson studied the Dirichlet problem for fully nonlinear, degenerate elliptic equations of the form $F(D^2u) = 0$ on a smooth bounded domain in $\mathbb{R}^n$. The existence and uniqueness of continuous viscosity solutions to the Dirichlet problem for (1.2) with continuous boundary data was shown in [HL09, Yua08]; see also [Bha20]. In [BW10], Brendle-Warren studied a second boundary value problem for the special Lagrangian equation.

For subcritical phases $|c| < (n-2)\pi/2$, interior regularity is not understood. For critical $|c| = (n-2)\pi/2$ and supercritical $|c| > (n-2)\pi/2$ phases, interior gradient estimates were established by Warren-Yuan [WY09a, WY10], and also Yuan’s unpublished notes from 2015. Interior Hessian estimates for dimension $n = 2$ were shown by Heinz, for $|c| = \pi/2$ in dimension $n = 3$ by Warren-Yuan [WY09b], and for general dimension $|c| \geq (n-2)\pi/2$ by Wang-Yuan [WY14]; see also Li [Li19] for a compactness approach and Zhou [Zho22] for estimates requiring Hessian constraints which generalize criticality.
Because the level set of the PDE is convex for critical and supercritical phases, the Evans-Krylov theory yields interior analyticity. The singular $C^{1,\alpha}$ subcritical phase solutions by Nadirashvili-Vlăduţ [NV10] and Wang-Yuan [WY13] show that interior regularity is not possible for subcritical phases, without an additional convexity condition, as in Bao-Chen [BC03], Chen-Warren-Yuan [CWY09], and Chen-Shankar-Yuan [CSY22], and that the Dirichlet problem is not classically solvable for arbitrary smooth boundary data. Interior gradient estimates for continuous boundary data are widely open. Global gradient estimates requiring Lipschitz boundary data were shown by [Lu22]. Homogeneous viscosity solutions of degree less than two were shown to not exist by Nadirashvili-Yuan [NY06]. The non-existence result of Mooney [Moo22] shows that counterexamples for interior $C^{1}$ regularity may be difficult to construct.

If the Lagrangian angle is not necessarily constant, then less is understood. In [HL19], Harvey-Lawson introduced a condition called “tameness” on the operator $F$, which is a little stronger than strict ellipticity and allows one to prove comparison. In [HL21], tamability was established for the supercritical Lagrangian mean curvature equation. In [CP21], Cirant-Payne established comparison principles for the Lagrangian mean curvature equation provided the Lagrangian phase is restricted to the intervals $((n - 2k)rac{\pi}{2}, (n - 2(k - 1))\frac{\pi}{2})$ where $1 \leq k \leq n$, which in turn solves the Dirichlet problem on these intervals as shown in [HL21, Theorem 6.2]. Hessian estimates for convex smooth solutions with $C^{1,1}$ phase $\psi = \psi(x)$ were obtained by Warren [War08, Theorem 8]. For convex viscosity solutions, interior regularity was established for $C^{2}$ phases; see Bhattacharya-Shankar [BS20b, BS20a]. For supercritical phases $|\psi(x)| \geq (n - 2)\pi/2 + \delta$, there is a comparison principle, and the Dirichlet problem was solved in Collins-Picard-Wu [CPW17], Dinew-Do-Tô [DDT18], Bhattacharya [Bha20], and interior gradient estimates were established in [Bha21]. Interior Hessian estimates for supercritical phases were established in [Bha21]. Interior Hessian estimates for critical and supercritical phases $|\psi(x)| \geq (n - 2)\frac{\pi}{2}$, follow verbatim from the calculations done in [Bha21] (see [Bha22, Remark 2.1]): The proof of the Hessian estimate in [Bha21, Theorem 1.1] does not require a negative lower bound on the lowest eigenvalue. For supercritical phases in dimension $n = 2$, a simplified proof [Bha22] was given for interior Hessian estimates using
the super-isoperimetric inequality of Warren-Yuan [WY09a], avoiding the Michael-Simon mean value inequality [MS73]. In the case that \( \phi \) is Lipschitz, Corollary 1.1 can be obtained by proving a global gradient estimate, as in [Lu22]. The existence of interior gradient estimates for the challenging borderline case of critical and supercritical phase has until now remained open. In this paper, we successfully solve this problem for \( C^2 \) phases.

Our approach to prove interior gradient estimate Theorem 1.1 accounts for the smallness of the gradient of the phase near its minimizing, critical values, using a pointwise interpolation inequality [NT70, Equation (3.11), pg. 19], see also [Hor83, Lemma 7.7.2], valid for \( C^2 \) phases. For constant phases, the gradient estimate is established using a maximum principle inequality. The variable phase contribution to the inequality is a “bad term” depending on the phase’s gradient. Although the PDE’s ellipticity degenerates at the critical phase, making the bad term large, the smallness of the gradient at such points provides a balance. Our proof, more generally, shows that an interior gradient estimate holds when \( \psi \) satisfies a certain first order differential inequality; see Remark 2.2. Such an inequality is valid when the phase is any of \( C^2 \), semi-concave, concave, or a supersolution of the infinity-Laplace equation; see Remark 2.3.

On the other hand, the gradient vanishes at slower rates for \( C^{1,\alpha} \) phases, and does not appear to balance the degeneration of the ellipticity in our proof. But we note that certain Hölder continuous phases allow for gradient estimates; see Remark 2.4. In such cases, the phase separates from the critical value at such a large speed that the solution is nearly semi-convex, as in the supercritical case.

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2. Proof of the gradient estimate

We modify the pointwise proof of [WY09a] and [Bha21] to bridge the constant critical phase estimate of [WY09a] with the supercritical estimate of [Bha21]. The difference comes from how to treat the bad term involving $D\psi$.

**Notation:** we denote $a \sim b$ if $ca \leq b \leq Ca$, and $a \lesssim b$ if $a \leq Cb$. Here, $c$ and $C$ are positive constants depending on $n$. We will denote $a \lesssim_{\psi} b$ if the constant also depends on $\|D^2\psi\|_{L^\infty(B_1)}$. We denote $a \ll_n 1$ if we are choosing a small fixed constant $a$ depending on $n$. We assume summation under repeated indices unless otherwise indicated.

Let $M = \text{osc}_{B_1} u > 0$; replacing $u$ with $u - \min_{B_1} u + M$, we assume that
\begin{equation}
M \leq u \leq 2M \quad \text{in } B_1.
\end{equation}

Let $w = \eta|Du| + Au^2/2$, where $\eta = 1 - |x|^2$, and $A = 3\sqrt{n}/M$. Let $x_0 \in B_1$ be where $w$ is maximized. After a rotation, we assume that $D^2u$ is diagonal, with $u_{ii} = \lambda_i$. Let us assume that $u_n \geq |Du|/\sqrt{n} > 0$. Then for each $k$, at the max point $x_0$,
\begin{equation}
0 = \partial_k w(x_0) = \eta \frac{u_k \lambda_k}{|Du|} + \eta_k |Du| + Au u_k.
\end{equation}

Since $A = 3\sqrt{n}/M$ is sufficiently large, it follows that
\begin{equation}
\eta \lambda_n \frac{u_n}{|Du|} \in -(c(n), C(n))|Du|.
\end{equation}

It follows that $\lambda_n < 0$ and
\begin{equation}
\eta \sim \frac{|Du|}{|\lambda_n|}.
\end{equation}

Since $|Du| \lesssim \eta |\lambda_n|$, we may assume that $|\lambda_n| > 1$, since otherwise the estimate is done. Moreover, as shown in [WY14, Lemma 2.5], we know that $\lambda_k \geq |\lambda_n|$ for $k < n$ follows from $\psi \geq (n - 2)\pi/2$.

We now proceed to the second derivatives of $w$. Let $g = I + (D^2u)^2$ be the induced metric $dx^2 + dy^2$ on $(x, Du(x))$, with $g^{-1} = (g^{ij})$ its
inverse, and \( g^{ij} = (1 + \lambda_i^2)^{-1} \delta_{ij} \) at \( x_0 \). Then at \( x_0 \),
\begin{equation}
0 \geq g^{ij} \partial_j w(x_0)
\end{equation}
\begin{align}
&= g^{ij} \eta_{ij} |Du| + 2g^{ij} \eta_i |Du|_j + \eta g^{ij} |Du|_{ij} + \text{Aug}^{ij} u_{ij} + Ag^{ij} u_i u_j .
\end{align}

The good term \((G)\) absorbs the constant-phase terms \((C1), (C2),\) and \((C3)\) in the constant phase and supercritical cases, as in \([WY09a]\) and \([Bha21]\). The bad term \((B)\) contains variable phase contributions and will require closer examination.

The good term \((G)\):
\begin{equation}
Ag^{ij} u_i u_j \geq A u_i^2 \sim A |Du|^2 \sim \frac{A \eta |Du|}{|\lambda_n|}.
\end{equation}

The first constant phase term \((C1)\):
\begin{equation}
g^{ij} \eta_{ij} |Du| = -2 \sum_i \frac{1}{1 + \lambda_i^2} |Du|_j \geq - \frac{1}{\lambda_n^2} |Du| \sim - \frac{\eta}{|\lambda_n|}.
\end{equation}

The second constant phase term \((C2)\):
\begin{equation}
2g^{ij} \eta_i |Du|_j \geq - \sum_i \frac{1}{1 + \lambda_i^2} u_i \lambda_i \geq - \frac{1}{|\lambda_n|}.
\end{equation}

The third constant phase term \((C3)\):
\begin{equation}
\text{Aug}^{ij} u_{ij} = Au \sum \frac{\lambda_i}{1 + \lambda_i^2} \geq - \frac{1}{|\lambda_n|}.
\end{equation}

The bad term \((B)\), using third derivative calculation \([Bha21, (2.4)]\):
\begin{equation}
\eta g^{ij} |Du|_{ij} = \eta g^{ij} \frac{u_{ijk} u_k |Du|}{|Du|} + \eta \sum_i g^{ii} \frac{(|Du|^2 - u_i^2) \lambda_i^2}{|Du|^3} \geq \eta \psi_k \frac{u_k |Du|}{|Du|} \geq - \eta |D\psi|.
\end{equation}

We thus need to bound this inequality at \( x_0 \):
\begin{equation}
\eta |Du| \leq C(n) M (1 + \eta |D\psi| |\lambda_n|).
\end{equation}

Letting \( \phi = \psi - (n - 2)\pi/2 \geq 0 \), we apply the pointwise interpolation inequality for nonnegative \( C^2 \) functions in \([Hor83, Lemma 7.7.2]\) on
where $B_{\delta}(x_0)$, where $\delta = 1 - |x_0|$;
\begin{equation}
|D\phi(x_0)|^2 \leq \frac{\phi(x_0)^2}{(1 - |x_0|)^2} + 2\|D^2\phi\|_{L^\infty(B_1)}\phi(x_0)
\end{equation}
(2.12)
\begin{equation}
\lesssim_\psi \frac{\phi(x_0)^2}{\eta^2} + \phi(x_0),
\end{equation}
where $\|D^2\phi\|_{L^\infty(B_1)}$ denotes the maximum of the absolute values of the eigenvalues of $D^2\phi$. Let us now recall the following algebraic inequality, valid for $\lambda_n < 0$ and $\lambda_k > 0$ for $k < n$:
\begin{equation}
\psi = (n - 1)\frac{\pi}{2} - \sum_{i<n} \arctan\left(\frac{1}{\lambda_i}\right) - \frac{\pi}{2} + \arctan\left(-\frac{1}{\lambda_n}\right)
\end{equation}
(2.13)
\begin{equation}
\leq (n - 2)\frac{\pi}{2} + \frac{1}{|\lambda_n|}.
\end{equation}
Substituting this information into (2.11), combined with (2.4), yields
\begin{equation}
M^{-1}\eta|Du| \lesssim_\psi 1 + \eta|\lambda_n|^{1/2} \lesssim 1 + (\eta|Du|)^{1/2}.
\end{equation}
(2.14)
It follows that
\begin{equation}
\eta|Du| \lesssim_\psi M + M^2 \lesssim C(n, \|\psi\|_{C^2(B_{1.5})})(1 + (\text{osc}_{B_1}u)^2).
\end{equation}
(2.15)
The $Au^2/2$ term in $w(x_0)$ and the estimate of $w$ on $\partial B_1$ are subordinate to this estimate, so we conclude the proof. \hfill $\square$

**Remark 2.1.** It is straightforward to refine the $|\lambda_n| < 1$ case and thereby improve the estimate (1.3) to the following
\begin{equation}
|Du(0)| \leq C(n, \|D^2\psi\|_{L^\infty(B_1)})(\text{osc}_{B_1}u + (\text{osc}_{B_1}u)^2).
\end{equation}
(2.16)
**Remark 2.2.** More generally, let $\psi \in C^1(B_1)$ be critical and super-critical, or $\psi - (n - 2)\pi/2 =: \phi \geq 0$, and also satisfy the following first order differential inequality on $B_1$:
\begin{equation}
|D\phi| \leq \eta f(\eta^{-2}\phi),
\end{equation}
(2.17)
where $f(t) \searrow 0$ as $t \searrow 0$, and $\eta = 1 - |x|^2$. Then a $C^1$ estimate is valid for $C^2(B_1)$ solutions of (1.1):
\begin{equation}
|Du(0)| \leq C(n, f, \text{osc}_{B_1}u).
\end{equation}
(2.18)
To prove the estimate, we insert (2.17) in the determining inequality (2.11) and use (2.13) and (2.4). We obtain at $x_0$, using that $f$ is
increasing,

$$M^{-1} |Du| \lesssim 1 + \eta^2 |\lambda_n| f \left( \frac{1}{\eta^2 |\lambda_n|} \right)$$
(2.19)

$$\lesssim 1 + \eta |Du| f \left( \frac{C(n)}{\eta |Du|} \right).$$

If $\eta |Du| \geq C(n, f, M) =: H$ for large enough $H$ such that $f(C(n)/H) \ll_{n} M^{-1}$, then $M^{-1} \eta |Du| \lesssim 1$, and the estimate follows. In the alternative case that $\eta |Du| \leq C(n, f, M)$, the estimate (2.18) is already done.

**Remark 2.3.** Let us list some examples of phases which satisfy a first order inequality of the form (2.17).

1. For $f(t)^2 = t^2 + Ct$, we recover the $C^2$ interpolation inequality (2.12). Note that general $C^{1,\alpha}$ phases fail to satisfy the inequality (2.17).

2. Interpolation inequality (2.12) can be generalized to phases $\psi \in C^1(B_1)$ which are semi-concave, with $D^2 \psi \leq K I$ for some $K > 0$. In this case, the dependence on $\|D^2 \psi\|_{L^\infty(B_1)}$ is replaced with $K$. Indeed, by semi-concavity, there holds for $x_0 \in B_1(0)$ and $x \in B_\delta(x_0)$:

$$0 \leq \phi(x_0) + (x - x_0) \cdot D\phi(x_0) + K|x - x_0|^2/2.$$  
(2.20)

The proof in [Hor83, Lemma 7.7.2] can then be repeated verbatim. This generalizes Theorem 1.1 to semi-concave phases.

3. The choice $f(t) = 2t$ corresponds to $\psi \in C^1(B_1)$ concave. Choosing $x - x_0 = -(1 - |x_0|)D\phi(x_0)/|D\phi(x_0)|$ with $K = 0$ in (2.20) gives

$$|D\phi(x_0)| \leq \frac{\phi(x_0)}{1 - |x_0|} \leq \frac{2\phi(x_0)}{\eta}.$$  
(2.21)

This is the first term in (2.12), so as in (2.14), we obtain $\eta |Du| \leq C(n)M$. We thus obtain the linear estimate

$$|Du(0)| \leq C(n)(1 + \text{osc}_{B_1} u).$$  
(2.22)

This can be improved to $|Du(0)| \leq C(n)\text{osc}_{B_1} u$, as in Remark 2.1. One novelty here is the independence of $\psi$. For example, if

$$\psi(x) = (n - 2)\frac{\pi}{2} + \epsilon(1 - |x|^1)$$  
(2.23)

for some $\epsilon, \alpha \in (0, 1)$, then (2.22) is independent of $\epsilon$. The interior gradient estimate for $C^1$ supercritical phases in [Bha21] would degenerate as $\epsilon \to 0$. 


4. Suppose that $\phi(x) \geq 0$ is a $C^1(B_1)$ viscosity supersolution of the infinity-Laplace equation/Aronsson’s equation:

$$D^2\phi(D\phi, D\phi) = \phi_{ij}\phi_i\phi_j \leq 0.$$  

Then using comparison with cones, there is a pointwise estimate [CEG01, Lemma 2.5] for the gradient:

$$|D\phi(x)| \leq \frac{\phi(x)}{1 - |x|}.$$  

In fact, this is concavity inequality (2.21), and this corresponds to $f(t) = 2t$ in (2.17). We conclude that a linear gradient estimate (2.22) is valid.

**Remark 2.4.** If $u$ is a viscosity solution of (1.1) for Hölder phase

$$\psi = (n - 2)\frac{\pi}{2} + |x|^\alpha,$$

where $0 < \alpha < 1$, the function

$$u(x) + C|x|^{2-\alpha}$$

is convex, if $C(\alpha)$ is large enough. This follows from the algebraic relation (2.13), which gives $|\lambda_{\text{min}}| < |x|^{-\alpha}$. It follows that $u(x)$ is locally Lipschitz continuous.

**References**

[BC03] Jiguang Bao and Jingyi Chen, *Optimal regularity for convex strong solutions of special Lagrangian equations in dimension 3*, Indiana University mathematics journal (2003), 1231–1249.

[Bha20] Arunima Bhattacharya, *The Dirichlet problem for Lagrangian mean curvature equation*, arXiv:2005.14420 (2020).

[Bha21] ———, *Hessian estimates for Lagrangian mean curvature equation*, Calculus of Variations and Partial Differential Equations 60 (2021), no. 6, 1–23.

[Bha22] ———, *A note on the two dimensional Lagrangian mean curvature equation*, arXiv preprint arXiv:2110.01728, to appear in Pacific Journal of Mathematics (2022).

[BS20a] Arunima Bhattacharya and Ravi Shankar, *Optimal regularity for Lagrangian mean curvature type equations*, Preprint arXiv:2009.04613 (2020).

[BS20b] ———, *Regularity for convex solutions of Lagrangian mean curvature equation*, arXiv:2006.02030 (2020).

[BW10] Simon Brendle and Micah Warren, *A boundary value problem for minimal Lagrangian graphs*, J. Differential Geom. 84 (2010), no. 2, 267–287.
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[CEG01] Michael G Crandall, Lawrence C Evans, and Ronald F Gariepy, Optimal Lipschitz extensions and the infinity Laplacian, Calculus of Variations and Partial Differential Equations 13 (2001), no. 2, 123–139.

[CNS85] L. Caffarelli, L. Nirenberg, and J. Spruck, The Dirichlet problem for nonlinear second order elliptic equations, III: Functions of the eigenvalues of the Hessian, Acta Math. 155 (1985), 261–301.

[CP21] Marco Cirant and Kevin R Payne, Comparison principles for viscosity solutions of elliptic branches of fully nonlinear equations independent of the gradient, Mathematics in Engineering, 3(4): 1-45 (2021).

[CPW17] Tristan C Collins, Sebastien Picard, and Xuan Wu, Concavity of the Lagrangian phase operator and applications, Calculus of Variations and Partial Differential Equations 56 (2017), no. 4, 89.

[CSY22] Jingyi Chen, Ravi Shankar, and Yu Yuan, Regularity for convex viscosity solutions of special Lagrangian equation, arXiv:1911.05452, to appear in Communications on Pure and Applied Mathematics (2022).

[CWY09] Jingyi Chen, Micah Warren, and Yu Yuan, A priori estimate for convex solutions to special Lagrangian equations and its application, Communications on Pure and Applied Mathematics 62 (2009), no. 4, 583–595.

[DDT18] Slawomir Dinew, Hoang-Son Do, and Tat Dat Tô, A viscosity approach to the Dirichlet problem for degenerate complex Hessian-type equations, Analysis & PDE 12 (2018), no. 2, 505–535.

[Eva82] Lawrence C Evans, Classical solutions of fully nonlinear, convex, second-order elliptic equations, Communications on Pure and Applied Mathematics 35 (1982), no. 3, 333–363.

[HL82] Reese Harvey and H. Blaine Lawson, Calibrated geometries, Acta Math. 148 (1982), 47–157.

[HL09] ______, Dirichlet duality and the nonlinear Dirichlet problem, Communications on Pure and Applied Mathematics 62 (2009), no. 3, 396–443.

[HL19] ______, The inhomogeneous Dirichlet problem for natural operators on manifolds, Annales de l’Institut Fourier 69 (2019), no. 7, 3017–3064.

[HL21] F Reese Harvey and H Blaine Lawson, Pseudoconvexity for the special Lagrangian potential equation, Calculus of Variations and Partial Differential Equations 60 (2021), no. 1, 1–37.

[Hor83] Lars Hormander, The analysis of linear partial differential operators, Springer-Verlag, Berlin, 1983.

[Kry83] N. V. Krylov, Boundedly inhomogeneous elliptic and parabolic equations in a domain, Izv. Akad. Nauk SSSR Ser. Mat. 47 (1983), no. 1, 75–108. MR 688919

[Li19] Caiyan Li, A compactness approach to Hessian estimates for special Lagrangian equations with supercritical phase, Nonlinear Analysis 187 (2019), 434–437.

[Lu22] Siyuan Lu, On the Dirichlet problem for Lagrangian phase equation with critical and supercritical phase, arXiv preprint arXiv:2204.05420 (2022).
Connor Mooney, *Homogeneous functions with nowhere vanishing hessian determinant*, arXiv preprint arXiv:2201.01260 (2022).

James H Michael and Leon M Simon, *Sobolev and mean-value inequalities on generalized submanifolds of $\mathbb{R}^n$*, Communications on Pure and Applied Mathematics 26 (1973), no. 3, 361–379.

Louis Nirenberg and Francois Treves, *On local solvability of linear partial differential equations part I: Necessary conditions*, Communications on Pure and Applied Mathematics 23 (1970), no. 1, 1–38.

Nikolai Nadirashvili and Serge Vlăduţ, *Singular solution to Special Lagrangian Equations*, Annales de l'I.H.P. Analyse non linéaire 27 (2010), no. 5, 1179–1188 (en).

Nikolai Nadirashvili and Yu Yuan, *Homogeneous solutions to fully nonlinear elliptic equations*, Proceedings of the American Mathematical Society (2006), 1647–1649.

MV Safonov, *On the classical solution of Bellman’s elliptic equation*, Soviet Math. Dokl, vol. 30, 1984, pp. 482–485.

———, *On the classical solution of nonlinear elliptic equations of second order*, Mathematics of the USSR-Izvestiya 33 (1989), no. 3, 597.

Micah Warren, *Special Lagrangian Equations*, University of Washington, 2008.

Micah Warren and Yu Yuan, *Explicit gradient estimates for minimal Lagrangian surfaces of dimension two*, Math. Z. 262 (2009), no. 4, 867–879. MR 2511754

———, *Hessian estimates for the sigma-2 equation in dimension 3*, Comm. Pure Appl. Math. 62 (2009), no. 3, 305–321. MR 2487850

———, *Hessian and gradient estimates for three dimensional special Lagrangian equations with large phase*, American Journal of Mathematics 132 (2010), no. 3, 751–770.

Dake Wang and Yu Yuan, *Singular solutions to special Lagrangian equations with subcritical phases and minimal surface systems*, American Journal of Mathematics 135 (2013), no. 5, 1157–1177.

———, *Hessian estimates for special Lagrangian equations with critical and supercritical phases in general dimensions*, American Journal of Mathematics 136 (2014), no. 2, 481–499.

Yu Yuan, *Global solutions to special Lagrangian equations*, Proceedings of the American Mathematical Society (2006), 1355–1358.

Yu Yuan, *Lecture notes for the summer school at ITCP, Trieste, Italy*, June 2008.

Xingchen Zhou, *Hessian estimates to special Lagrangian equation on general phases with constraints*, Calculus of Variations and Partial Differential Equations 61 (2022), no. 1, 1–14.
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