Moduli space of Fedosov structures

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1 Introduction

Fedosov space is a triple: a manifold $M^{2n}$ with a symplectic structure $\omega$, and a compatible symmetric connection $\Gamma$: $(M, \omega, \Gamma)$. Compatibility means that $\omega$ is preserved under geodesic flow of $\Gamma$:

$$\nabla \omega = 0. \quad (1.1)$$

There is a canonical quantization for these manifolds, see [GRS] and references therein.

Here we are interested in local invariants of a Fedosov structure. Namely, we take a space $\mathcal{F}$, of germs of Fedosov structures at a point, and act on them by local coordinate changes, that is the group of all diffeomorphisms

$$G := \text{Diff}(\mathbb{R}^{2n}, 0)$$

fixing the point. A quotient of $\mathcal{F}$ by this action is called the moduli space of Fedosov structures:

$$\mathcal{M} = \mathcal{F}/\text{Diff}(\mathbb{R}^{2n}, 0).$$

This action can be restricted from space of germs $\mathcal{F}$ to space of $k$-jets of Fedosov structures, $\mathcal{F}_k$, with corresponding quotient:

$$\mathcal{M}_k = \mathcal{F}_k/\text{Diff}(\mathbb{R}^{2n}, 0)$$

called the moduli space of $k$-jets. We will only work with generic Fedosov structures. For a generic orbit $\mathcal{O}_k$, its dimension:

$$\dim \mathcal{O}_k = \text{codim } G_{\Phi}$$

is the codimension of the stabilizer $G_{\Phi}$ of a generic Fedosov structure $\Phi$ in $G$. Then we will call

$$\dim \mathcal{M}_k = \dim \mathcal{F}_k - \dim \mathcal{O}_k,$$

and construct the Poincaré series of $\mathcal{M}$:

$$p_{\Phi}(t) = \dim \mathcal{M}_0 + \sum_{k=1}^{\infty} (\dim \mathcal{M}_k - \dim \mathcal{M}_{k-1}) t^k$$
Theorem 1.1 Poincaré series coefficients are polynomial in \( k \), and the series has the form:

\[
p_k(t) = \frac{n[8n(2n^2 - 1)(n + 1) + 11]}{6} + \frac{n(2n + 1)[4n^4 + 2n^3 - 6n^2 - 4n - 3]}{3}t + (t - t^2)\delta_{2n}^2 + 2n \sum_{k=2}^{\infty} \left[ \frac{1}{4} \right] \left( \begin{array}{c} 2n + 2 \\ 2n - 1 \end{array} \right) - \left( \begin{array}{c} 2n + k + 1 \\ 2n - 1 \end{array} \right) \right] t^k.
\]

It represents a rational function.

Remark This confirms the assertion of Tresse, cf. [T], that algebras of "natural" differential-geometric structures are finitely-generated.

Proof Postponed until section 5.

Similar results for other differential-geometric structures were obtained earlier in [Sh] and [D].

To explain significance of Poincaré series represented by a rational function, we make the following:

Remark If a geometric structure is described by a finite number of functional moduli, then its Poincaré series is rational. In particular, if there are \( m \) functional invariants in \( n \) variables, then

\[
p(t) = \frac{m}{(1 - t)^n}
\]

Indeed, dimension of moduli spaces of \( k \)-jets is just the number of monomials up to the order \( k \) in the formal power series of the \( m \) given invariants:

\[
\dim M_k = m \binom{n + k}{n}
\]

For more details and slightly more general formulation see Theorem 2.1 in [Sh2].

2 Action formulas

As usual, two \( C^\infty \)-functions on \( \mathbb{R}^{2n} \) have the same \( k \)-jet at a point if their first \( k \) derivatives are equal in any local coordinates.

We say that two connections \( \nabla \) and \( \tilde{\nabla} \) have the same \( k \)-jet at 0 if for any two \( C^\infty \)-vector fields \( X, Y \) and any \( C^\infty \)-function \( f \), the functions \( \nabla_X Y(f) \) and \( \tilde{\nabla}_X Y(f) \) have the same \( k \)-jet at 0. This is equivalent to connection coefficients of \( \nabla \) and \( \tilde{\nabla} \) having the same \( k \)-jet. We denote by \( j^k\Gamma \) the \( k \)-jet of \( \Gamma \).

There is an action of the group of germs of origin-preserving diffeomorphisms

\[
G = \text{Diff}(\mathbb{R}^{2n}, 0)
\]
on $\mathcal{F}$ and $\mathcal{F}_k$.

For $\varphi \in G$, $(\omega, \nabla)(\text{or } (\omega, \Gamma)) \in \mathcal{F}$ and $j^k \Gamma \in \mathcal{F}_k$:

$$\Gamma \mapsto \varphi^* \Gamma, \quad j^k \Gamma \mapsto j^k (\varphi^* \Gamma),$$

where

$$(\varphi^* \nabla)_X Y = \varphi^{-1}_* (\nabla_{\varphi^*_x \varphi_* X} Y)$$

Let us introduce a filtration of $G$ by normal subgroups:

$$G = G_1 \triangleright G_2 \triangleright G_3 \triangleright \ldots,$$

where

$$G_k = \{ \varphi \in G \mid \varphi(x) = x + (\varphi_1(x), \ldots \varphi_n(x)), \ \varphi_i = O(|x|^k), \ i = 1, \ldots, 2n \}$$

The subgroup $G_k$ acts trivially on $\mathcal{F}_p$ for $k \geq p + 3$.

It means that the action of $G$ coincides with that of $G/G_{p+3}$ on each $\mathcal{F}_p$.

Now $G/G_{p+3}$ is a finite-dimensional Lie group, which we will call $K_p$.

Denote by $\text{Vect}_0(\mathbb{R}^{2n})$ the Lie algebra of $C^\infty$-vector fields, vanishing at the origin. It acts on $\mathcal{F}$ as follows:

**Definition 2.1** For $V \in \text{Vect}_0(\mathbb{R}^{2n})$ generating a local 1-parameter subgroup $g^t$ of $\text{Diff}(\mathbb{R}^n, 0)$, the Lie derivative of a connection $\nabla$ in the direction $V$ is a $(1,2)$-tensor:

$$\mathcal{L}_V \nabla = \left. \frac{d}{dt} \right|_{t=0} g^t \nabla$$

**Lemma 2.2**

$$(\mathcal{L}_V \nabla)(X, Y) = [V, \nabla_X Y] - \nabla_{[V,X]} Y - \nabla_X [V,Y] \quad (2.2)$$

**Proof** Below the composition $\circ$ is understood as that of differential operators acting on functions.

$$\left(\mathcal{L}_V \nabla\right)(X, Y) = \left. \frac{d}{dt} \right|_{t=0} g^t \nabla_{g^t X} Y = \left. \frac{d}{dt} \right|_{t=0} \left[ (g^t)^* \circ \nabla_{g^t X} Y \right] =$$

$$\left. \frac{d}{dt} \right|_{t=0} (g^t)^* \circ \nabla_X Y + \nabla_X Y \circ \left. \frac{d}{dt} \right|_{t=0} (g^t)^* + \nabla_{\left[ \left. \frac{d}{dt} \right|_{t=0} g^t X, Y \right]} + \nabla_X \left. \frac{d}{dt} \right|_{t=0} g^t Y =$$

$$V \circ \nabla_X Y - \nabla_X Y \circ V - \nabla_{\left[ \left. \frac{d}{dt} \right|_{t=0} g^t X, Y \right]} - \nabla_X \left. \frac{d}{dt} \right|_{t=0} g^t Y =$$

$$\mathcal{L}_V (\nabla_X Y) - \nabla_{\mathcal{L}_V X} Y - \nabla_X (\mathcal{L}_V Y)$$

This defines the action on the germs of connections. Now we can define the action of $\text{Vect}_0(\mathbb{R}^{2n})$ on jets $\mathcal{F}_k$. For $V \in \text{Vect}_0(\mathbb{R}^{2n})$:

$$\mathcal{L}_V (j^k \Gamma) = j^k (\mathcal{L}_V \Gamma),$$

3
where $\Gamma$ on the right is an arbitrary representative of the $j^k\Gamma$ on the left. This is well-defined, since in the coordinate version of (2.2):

$$
(\mathcal{L}_V\Gamma)_{ij} = V^k \frac{\partial \Gamma^l_{ij}}{\partial x^k} - \Gamma^l_{ij} \frac{\partial V^k}{\partial x^l} + \Gamma^l_{kj} \frac{\partial V^k_i}{\partial x^j} + \Gamma^l_{ik} \frac{\partial V^k_j}{\partial x^j} + \frac{\partial^2 V^l_i}{\partial x^j \partial x^j} \tag{2.3}
$$
elements of $k$-th order and less are only coming from $j^k\Gamma$, because $V(0) = 0$. Einstein summation convention in (2.3) above and further on is assumed. Consequently, the action is invariantly defined. This can also be expressed as commutativity of the following diagram:

$$
\begin{array}{c}
j^0\mathcal{F} \leftarrow \ldots \leftarrow j^{k-1}\mathcal{F} \leftarrow \pi_k j^k\mathcal{F} \leftarrow \ldots \leftarrow \mathcal{F} \\
\downarrow \mathcal{L}_V \downarrow \mathcal{L}_V \downarrow \mathcal{L}_V \\
j^0\Pi \leftarrow \ldots \leftarrow j^{k-1}\Pi \leftarrow \pi_k j^k\Pi \leftarrow \ldots \leftarrow \Pi
\end{array}
$$

where $\pi_k$ is projection from $k$-jets onto $(k-1)$-jets, $\mathcal{F}$ and $\Pi$ denote spaces of germs of connections and that of $(1,2)$-tensors respectively, at 0.

3 Stabilizer of a generic $k$-jet

[ The following discussion closely mirrors that of section 3 in [D]. ] Dimensions of stabilizers of generic $k$-jets $G_\Phi$ are required to find orbit dimensions for orbits $O_k$ of generic $k$-jets. The subalgebra generating $G_\Phi$ consists of those $V \in \text{Vect}_0(\mathbb{R}^{2n})$ that

$$
\mathcal{L}_V (j^k\Phi) = 0.
$$

Since $\Phi = (\omega, \Gamma)$, this entails two conditions:

$$
\mathcal{L}_V (j^k\omega) = 0 \quad \mathcal{L}_V (j^k\Gamma) = 0. \tag{3.4}
$$

In the next two sections devoted to finding the stabilizer $G_\Phi$ we assume that $\omega$ is reduced to canonical symplectic form in Darboux coordinates. In these coordinates compatibility (1.1) is written as:

$$
\omega_{\alpha\beta} \Gamma^\beta_{kj} = \omega_{\beta\alpha} \Gamma^\beta_{kj}, \tag{3.5}
$$

where $\omega = J = \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix}$, a standard symplectic matrix, cf. [GRS], p.110. We can introduce grading in homogeneous components on $V$:

$$
V = V_1 + V_2 + \ldots
$$

($V_0 = 0$, so that $V$ preserve the origin), on $\Gamma$:

$$
\Gamma = \Gamma_0 + \Gamma_1 + \ldots,
$$

and on $\omega$:

$$
\omega = \omega_0,
$$

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where $\omega_0 = J$ is a standard symplectic form.

Then (3.3) is rewritten as follows:

$$L_V(j^k \omega) = L_{V_1 + V_2 + \cdots} (\omega_0) = 0$$

$$L_V(j^k \Gamma) = j^k L_V(\Gamma) = j^k L_{V_1 + V_2 + \cdots} (\Gamma_0 + \Gamma_1 + \ldots + \Gamma_k + \ldots) =$$

$$= L_{V_1 \Gamma_0} + \frac{\partial^2 V_2}{\partial x^2} + L_{V_1 \Gamma_1} + \tilde{L}_{V_2 \Gamma_0} + \frac{\partial^2 V_3}{\partial x^2} + \cdots$$

$$\cdots + \tilde{L}_{V_{k+1} \Gamma_0} + \tilde{L}_{V_1 \Gamma_1} + \cdots + L_{V_1 \Gamma_k} + \frac{\partial^2 V_{k+2}}{\partial x^2} + \cdots,$$

where

$$(\frac{\partial^2 V_2}{\partial x^2})_{ij} = \frac{\partial^2 V_{ij}^l}{\partial x^1 \partial x^j}$$

and

$$\tilde{L}_V \Gamma = L_V \Gamma - \frac{\partial^2 V}{\partial x^2}.$$

$\tilde{L}_V \Gamma$ with indexes looks like this:

$$(\tilde{L}_V \Gamma)_{ij}^l = V_k \frac{\partial \Gamma_{ij}^l}{\partial x^k} - \Gamma_{ij}^k \frac{\partial V^k}{\partial x^k} + \Gamma_{ij}^l \frac{\partial V^k}{\partial x^k} + \Gamma_{ik}^l \frac{\partial V^k}{\partial x^j},$$

so $\tilde{L}_V \Gamma$ is just the first 4 terms of $(L_V \Gamma)$, cf. (2.3).

The stabilizer condition therefore results in a system:

$$\begin{cases} L_{V_1} \omega_0 = 0 \\ L_{V_2} \omega_0 = 0 \\ \vdots \\ L_{V_{k+1}} \omega_0 = 0 \\ L_{V_1} \Gamma_0 + \frac{\partial^2 V_2}{\partial x^2} = 0 \\ L_{V_1} \Gamma_1 + \tilde{L}_{V_2} \Gamma_0 + \frac{\partial^2 V_3}{\partial x^2} = 0 \\ \vdots \\ L_{V_1} \Gamma_k + \tilde{L}_{V_2} \Gamma_{k-1} + \ldots + \tilde{L}_{V_{k+1}} \Gamma_0 + \frac{\partial^2 V_{k+2}}{\partial x^2} = 0 \end{cases} \tag{3.6}$$

Our present goal is finding all $(V_1, V_2, \ldots, V_{k+2})$ solving the above system for a generic $\Phi$. Let us start with the $\Gamma$-part. Assuming $V_1$ is arbitrary, we can uniquely find $V_2$ from the first equation, as guaranteed by the following lemma on mixed derivatives:
Lemma 3.1 Given a family \( \{ f_{ij} \}_{1 \leq i, j \leq N} \) of smooth functions, solution \( u \) for the system:
\[
\begin{align*}
  u_{,kl} &= f_{kl} \\
  1 \leq k, l \leq N
\end{align*}
\]
(indexes after a comma henceforth will denote differentiations in corresponding variables) exists if and only if
\[
\begin{align*}
  f_{ij} &= f_{ji} \\
  f_{ij, k} &= f_{kj, i}
\end{align*}
\]
(3.7)

If \( f_{ij} \) are homogeneous polynomials of degree \( s \geq 0 \), then \( u \) can be uniquely chosen as a polynomial of degree \( s + 2 \).

Proof is a straightforward integration of the right-hand sides. \( \square \)

Therefore, if we treat highest-order \( V_k \) in each equation in \( \Gamma \)-part of (3.6) as an unknown, we see that various (combinations of) \( \mathcal{L}_V \Gamma \) must satisfy (3.7). The first condition is satisfied automatically since \( \Gamma \) is symmetric. The second one gives:
\[
(\mathcal{L}_V \Gamma)^l_{ij, p} = (\mathcal{L}_V \Gamma)^l_{pj, i}
\]
This condition for the first equation in \( \Gamma \)-part of (3.6) is satisfied trivially, since \( V_1 \) is of the first degree in \( x \), and \( \Gamma_0 \) is constant. Hence, \( V_2 \) exists and, since it must be of the second degree, is unique. However, if \( k \geq 1 \) (so there is need for more than one equation) there is a non-trivial condition on the second equation:
\[
\mathcal{L}_{V_1} \Gamma_1 + \mathcal{L}_{V_2} \Gamma_0 + \frac{\partial^2 V_3}{\partial x^2} = 0
\]

It follows from the Lemma 3.1 that for the existence of \( V_3 \) it is necessary (and sufficient) to have the following condition:
\[
(\mathcal{L}_V \Gamma_1 + \mathcal{L}_{V_2} \Gamma_0)_{ij, p} = (\mathcal{L}_V \Gamma_1 + \mathcal{L}_{V_2} \Gamma_0)_{pj, i}, \ i < p \quad (3.8)
\]

Outside exceptional dimension two this condition fails for a generic connection unless \( V_1 = 0 \). In other words (3.8), considered as a condition on \( V_1 \) implies \( V_1 = 0 \) (and hence \( V_2 = V_3 = ... = 0 \)). The rest is a proof of this assertion.

Let us consider (3.8) as a linear homogeneous system on the components of \( V_1 \). We will present a Fedosov structure for which (3.8) is non-degenerate. Since nondegeneracy is an open condition on the space \( \mathcal{F}_k \), the same (namely non-degeneracy and resulting trivial solution \( V = 0 \) for the stabilizer) would hold for a generic structure.

Since symplectic part of the structure is fixed by our choice to work in symplectic coordinates, we need only to present the corresponding connection 1-jet. In this 1-jet we choose to have \( \Gamma_0 = 0 \), which simplifies (3.8) to:
\[
(\mathcal{L}_V \Gamma_1)^l_{ij, p} = (\mathcal{L}_V \Gamma_1)^l_{pj, i}
\]
Let us expand it using (2.3):
(Γ^l_{1i,j,k} - Γ^l_{1k,j,i})V^k_{i,p} + (Γ^l_{1k,j,p} - Γ^l_{1p,j,k})V^k_{i,i} +
(Γ^k_{1p,j,i} - Γ^k_{1i,j,p})V^l_{i,k} + (Γ^l_{1i,k,p} - Γ^l_{1p,k,i})V^k_{i,j} = 0, \ i < p \quad (3.9)

Recall that summation over repeated indexes above is assumed.

In local symplectic coordinates:

\[ V^k_1 = \sum_{s=1}^n b^k_s x^s, \quad Γ^l_{1i,j} = \sum_{m=1}^n c_{ij}^l x^m, \quad c_{ij}^l = c_{ji}^l \text{ (connection is symmetric)}, \]

and (3.9) becomes the system on \( b^k_s \):

\[ (c^l_{jk} - c^l_{kj})b^k_p + (c^l_{kp} - c^l_{pk})b^k_i + (c^l_{ki} - c^l_{ik})b^l_k + (c^l_{lp} - c^l_{pl})b^k_j = 0, \ i < p \quad (3.10) \]

The requirement \( 1.1 \) on \( Γ \) to be a symplectic connection is passed through to each of its homogeneous components \( Γ^\alpha_k \) as the following symmetry condition:

\[ ω_i^\alpha Γ^\alpha_{kj} = ω_l^\alpha Γ^\alpha_{jk}, \]

that can be thought of as ‘\( Γ \) with lowered indexes’ is completely symmetric (cf. the discussion on p.110 (especially equation (1.5)) in [GRS]).

Of course \( ω_{iα} \) in our setting is just the standard symplectic matrix. Another way to think about it in terms of e.g. coefficients of \( Γ^1 \) is that they ‘form a symplectic matrix’, namely \( Γ^1 ∈ sp(2n) \) in the left upper and lower indexes:

\[ c_{jk}^{im} = c_{jk}^{mi}, \quad c_{jk}^{jm} = c_{ik}^{jm}, \quad c_{jk}^{jm} = -c_{ik}^{jm} \quad (3.11) \]

∀\( m, k ∈ [1, . . . , 2n] \), \( i, j ∈ [1, . . . , n] \), \( I, J ∈ [n + 1, . . . , 2n] \), \( \tilde{i} = i + n, \tilde{I} = I - n \)

We will also consider only such \( Γ^1 \) that

\[ c^l_{i,j} ≠ 0 \text{ only if } \{i, j, l, p\} = \{α, β, γ\}, \ α ≠ β, β ≠ γ, α ≠ γ \quad (3.12) \]

In other words nonzero coefficients may only occur among those with indexing set consisting of three distinct numbers, and must be zero otherwise.

Let us now turn to \( ω \)-part of (3.10). The fact that \( V_1 \) preserves \( ω \) implies that it is hamiltonian, so its coefficients:

\[ b = J \nabla H \]

for some hamiltonian \( H = \frac{1}{2} \sum_{i,j=1}^n h_{ij} x^i x^j \), homogeneous second degree polynomial, i.e.:

\[ b^k_s = ±h_{k±n,s} \quad (3.13) \]
In the similar manner we obtain the next four equations:

\[ (c_{ijk}^k - c_{ij}^i)h_k + (c_{ijk}^l - c_{ij}^i)h_l + (c_{ijk}^{l'} - c_{ij}^i)h_{l'} + (c_{ijk}^{l''} - c_{ij}^i)h_{l''} = (c_{ij}^i - c_{ij}^j)h_{i} - (c_{ij}^j - c_{ij}^j)h_{j} - (c_{ij}^j - c_{ij}^j)h_{j'} = 0 \]

There are 3 distinct indexes present in each coefficient in the above equation. If it seems that there are only 2, we must use their symmetries \( \{3.11\} \), to explicitly present all three. For example, the first coefficient \( c_{ij}^k = -c_{ij}^k \), and similar for other coefficients. That implies that in each summation the dummy index \( k \) has to turn into one of the fixed ones, e.g. into \( j \), \( i \) or \( j \) in the first term:

\[ (c_{ij}^j - c_{ij}^j)h_j + (c_{ij}^j - c_{ij}^j)h_j + (c_{ij}^j - c_{ij}^j)h_j = (c_{ij}^j - c_{ij}^j)h_{j} - (c_{ij}^j - c_{ij}^j)h_{j} - (c_{ij}^j - c_{ij}^j)h_{j} = 0 \]

Note that even though indexes do repeat in the first term above, the summation convention does not apply, because the designated summation dummy \( k \) is absent! After the similar work is done for the remaining 3 terms, and many cancellations (due to symmetries \( \{3.11\} \)), the original equation simplifies to:

\[ c_{ij}^i h_{ij} + c_{ij}^j h_{ij} + c_{ij}^j h_{ij} - c_{ij}^j h_{ij} = 0 \quad i \neq j . \]

In the similar manner we obtain the next four equations:

ii) \( i = P, \; j = L \neq i \)

\[ 2(c_{ij}^j - c_{ij}^j)h_{jj} + (c_{ij}^j - 2c_{ij}^j)h_{ij} + (c_{ij}^j - c_{ij}^j)h_{ij} + (2c_{ij}^j - c_{ij}^j)h_{ij} - c_{ij}^j h_{ij} = 0 \quad i \neq j . \]

iii) \( I = j (= i), \; l = P, \; I < P \Rightarrow j (= i) < l \)

\[ (c_{ii}^l - c_{ii}^l)h_{ll} - c_{ll}^l h_{ll} + (2c_{ii}^l - c_{ii}^l)h_{ll} + (2c_{ii}^l - c_{ii}^l)h_{ll} + (2c_{ii}^l - c_{ii}^l)h_{ll} = 0 \quad i < l . \]

\[ I = l, \; j = P, \; I < P \Rightarrow l < j \]

\[ (c_{jj}^j - c_{jj}^j)h_{jj} - c_{jj}^j h_{jj} + (2c_{jj}^j - c_{jj}^j)h_{jj} + (c_{jj}^j - c_{jj}^j)h_{jj} + (2c_{jj}^j - c_{jj}^j)h_{jj} = 0 \quad j > l . \]
Since these two equations are the same modulo changing \( i \) into \( j \), we can keep just the last equation, but for \( j \neq l \). And finally we rewrite it in \( i \) and \( j \) in conformity with others:

\[
(c_{ij}^*-c_{ji}^*)h_{ii}-c_{ij}^*h_{ij}+(2c_{ij}^*-c_{ji}^*)h_{ij}^*+(c_{ij}^*-c_{ji}^*)h_{ii}^*+(2c_{ij}^*-c_{ji}^*)h_{jj} = 0 \quad i \neq j .
\]

iv) \( i = I, J(=:\vec{j}) = P \neq \vec{i} \)

\[
(c_{ij}^*-c_{ji}^*)h_{ii}+(c_{ij}^*-c_{ji}^*)h_{ii}+(c_{ij}^*-c_{ji}^*)h_{jj} + (2c_{ij}^*-c_{ji}^*)h_{ij} + (2c_{ij}^*-c_{ji}^*)h_{ij} + c_{ij}^*h_{ij} = 0 \quad i \neq j .
\]

v) \( i = \vec{I}, J(=:\vec{j}) = P \neq \vec{i} \)

\[-c_{ij}^*h_{ij}+2(c_{ij}^*-c_{ji}^*)h_{ii}+(c_{ij}^*-c_{ji}^*)h_{jj} + (2c_{ij}^*-c_{ji}^*)h_{ij} + (2c_{ij}^*-c_{ji}^*)h_{ij} + c_{ij}^*h_{ij} = 0 \quad i \neq j .
\]

In each of the equations i)-v) above we are free to interchange \( i \) with \( j \) to obtain another five: i'), ii'), iii'), iv') and v'). Thus we obtain the system of ten equations for ten variables: \( h_{ij}, h_{ij}, h_{ij}, h_{ij}, h_{ii}, h_{jj}, h_{ij}, h_{ii}, \) and \( h_{ij} \). However, the last two variables are only found in the equation v), which allows us to consider first four equations and their 'primed' i), i'),... iv') as an 8 X 8 system for the first eight variables:

\[
\begin{pmatrix}
-h_{ij} & h_{ij} & h_{ij} & h_{ij} & h_{ij} & h_{ij} & h_{ij} & h_{ij}
\end{pmatrix}
\]

\[
\begin{pmatrix}
-c_{ij}^i & -c_{ij}^* & -c_{ij}^* & c_{ij}^i & -c_{ij}^* & -c_{ij}^* & c_{ij}^i & -c_{ij}^*
\end{pmatrix}
\]

\[
\begin{pmatrix}
(\vec{c}_{ij}^* - 2c_{ij}^i) & \vec{c}_{ij}^* & (2c_{ij}^i - c_{ij}^i) & -c_{ij}^i & 2(c_{ij}^i - c_{ij}^i)
\end{pmatrix}
\]

\[
\begin{pmatrix}
(\vec{c}_{ij}^* - 2c_{ij}^i) & (2c_{ij}^i - c_{ij}^i) & \vec{c}_{ij}^* & -c_{ij}^i & 2(c_{ij}^i - c_{ij}^i)
\end{pmatrix}
\]

\[
\begin{pmatrix}
-c_{ij}^i & (2c_{ij}^i - c_{ij}^i) & (c_{ij}^i - c_{ij}^i) & (c_{ij}^i - c_{ij}^i) & (2c_{ij}^i - c_{ij}^i)
\end{pmatrix}
\]

We set:

\[
2c_{ij}^i - c_{ij}^i(= 2c_{ij}^i - c_{ij}^i) = 0, \quad c_{ij}^i(= c_{ij}^i) = 0, \quad c_{ij}^* = 0, \quad * \in \{i, \vec{i}\}, \quad \alpha \notin \{i, \vec{i}\}.
\]
This doesn’t completely separate the system, but it does annihilate the lower left block. Consider the lower right block:

\[
\begin{pmatrix}
    h_{ii} & h_{\overline{ii}} & h_{jj} & h_{\overline{j}j} \\
    \begin{pmatrix}
        c_{\overline{j}j}^{\overline{i}} - c_{\overline{i}j}^{\overline{j}} \\
        0 \\
        (c_{\overline{i}j}^{\overline{j}} - c_{\overline{j}j}^{\overline{i}}) \\
        0
    \end{pmatrix} & \begin{pmatrix}
        (c_{\overline{i}j}^{\overline{i}} + c_{\overline{j}j}^{\overline{j}}) \\
        0 \\
        (c_{\overline{j}j}^{\overline{i}} - c_{\overline{i}j}^{\overline{j}}) \\
        0
    \end{pmatrix} & 0 & \begin{pmatrix}
        (2c_{\overline{i}j}^{\overline{i}} - c_{\overline{i}j}^{\overline{j}} - c_{\overline{j}j}^{\overline{i}}) \\
        (2c_{\overline{i}j}^{\overline{j}} - c_{\overline{j}j}^{\overline{i}} - c_{\overline{j}j}^{\overline{j}}) \\
        (2c_{\overline{j}j}^{\overline{i}} + c_{\overline{j}j}^{\overline{j}}) \\
        (2c_{\overline{j}j}^{\overline{j}} + c_{\overline{j}j}^{\overline{i}})
    \end{pmatrix}
\end{pmatrix}
\]

It has ten independent coefficients, which we can set as follows:

\[
c_{\overline{j}j}^{\overline{i}} = \begin{cases} 
+1 & i > j \\
-1 & i < j 
\end{cases}, \quad c_{\overline{i}j}^{\overline{i}} = c_{\overline{i}j}^{\overline{j}} = 2, \text{ and the rest all equal to 1.}
\]

It is clearly non-degenerate, hence we just need to show that the upper left block also can be chosen non-degenerate:

\[
\begin{pmatrix}
    h_{ij} & h_{\overline{i}j} & h_{ij} & h_{\overline{j}j} \\
    0 & -c_{ij}^{\overline{i}} & -c_{ij}^{\overline{j}} & c_{ij}^{\overline{j}} \\
    0 & -c_{ij}^{\overline{j}} & -c_{ij}^{\overline{i}} & c_{ij}^{\overline{i}} \\
    -3c_{ij}^{\overline{j}} & 0 & (2c_{ij}^{\overline{i}} - c_{ij}^{\overline{j}}) & 0 \\
    -3c_{ij}^{\overline{i}} & (2c_{ij}^{\overline{j}} - c_{ij}^{\overline{i}}) & 0 & 0
\end{pmatrix}
= \begin{pmatrix}
    0 & -c_{ij}^{\overline{i}} & \frac{1}{2}c_{ij}^{\overline{j}} & c_{ij}^{\overline{j}} \\
    0 & \frac{1}{2}c_{ij}^{\overline{i}} & -c_{ij}^{\overline{j}} & c_{ij}^{\overline{i}} \\
    -\frac{3}{2}c_{ij}^{\overline{i}} & 0 & (2c_{ij}^{\overline{j}} - c_{ij}^{\overline{i}}) & 0 \\
    -\frac{3}{2}c_{ij}^{\overline{j}} & (2c_{ij}^{\overline{i}} - c_{ij}^{\overline{j}}) & 0 & 0
\end{pmatrix}
\]

There are four independent coefficients in this system:

\[c_{ij}^{\overline{i}}, \ c_{ij}^{\overline{j}}, \ c_{ij}^{\overline{i}}, \text{ and } c_{ij}^{\overline{j}}.\]

Setting them all equal to 1 achieves non-degeneracy for this block, and for the 8 X 8 system. This leaves us equations v) and v'), which reduce to this 2 X 2 system for the remaining two unknowns \(h_{\overline{i}j}\) and \(h_{\overline{j}j}\):

\[
\begin{pmatrix}
    2(c_{\overline{i}j}^{\overline{i}} + c_{\overline{i}j}^{\overline{j}}) & -(c_{\overline{j}j}^{\overline{i}} + c_{\overline{j}j}^{\overline{j}}) \\
    -(c_{\overline{j}j}^{\overline{i}} + c_{\overline{j}j}^{\overline{j}}) & 2(c_{\overline{i}j}^{\overline{i}} + c_{\overline{i}j}^{\overline{j}})
\end{pmatrix} = \begin{pmatrix}
    4 & -2 \\
    -2 & 4
\end{pmatrix}
\]

Since we required 3 distinct indexes for our non-zero coefficients, this method is only good for dimensions 4 or larger.
Proposition 3.2 The stabilizer of a k-jet of a generic connection for \( n \geq 4 \) is: \( G_1/G_2 \) for \( k = 0 \), and \( 0 \) for \( k \geq 1 \).

The lowest dimension 2 has to be treated separately.

4 Exceptional dimension 2

In this case the stabilizer of the 1-jet is non-trivial (it has dimension one), stabilizers of the higher jets are all trivial.

Since the general method of previous section fails here, we must reconsider (3.10) with \( i = 1 \) and \( p = 2 \):

\[
(c_{ij}^k - c_{kj}^l)b_k^k + (c_{kj}^l - c_{jk}^l)b_j^k + (c_{jk}^l - c_{kj}^l)b_k^l + (c_{lk}^2 - c_{kl}^2)b_j^k = 0
\]

Summing over two indexes, we obtain:

\[
(c_{ij}^{12} - c_{ij}^{21})b_i^l + (c_{ij}^{21} - c_{ij}^{22})b_j^l + (c_{ij}^{21} - c_{ij}^{11})h_{2j} - (c_{ij}^{22} - c_{ij}^{21})h_{1j} = 0
\]

Varying pair \((ij)\) we obtain next 4 equations on 3 variables \( h_{11}, h_{12} \) and \( h_{22} \):

\[
(c_{21}^{11} - c_{22}^{11})h_{21} + (c_{21}^{21} - c_{22}^{22})h_{22} + (c_{22}^{11} - c_{21}^{11})h_{21} - (c_{22}^{22} - c_{21}^{22})h_{21} = 0
\]

\[
-(c_{21}^{11} - c_{22}^{12})h_{11} - (c_{21}^{21} - c_{22}^{21})h_{12} + (c_{22}^{22} - c_{21}^{21})h_{21} - (c_{22}^{22} - c_{21}^{21})h_{11} = 0
\]

\[
(c_{22}^{11} - c_{21}^{11})h_{21} + (c_{21}^{22} - c_{22}^{22})h_{22} + (c_{22}^{11} - c_{21}^{11})h_{22} - (c_{22}^{22} - c_{21}^{22})h_{12} = 0
\]

\[
-(c_{22}^{11} - c_{22}^{22})h_{11} - (c_{21}^{22} - c_{22}^{22})h_{12} + (c_{22}^{22} - c_{21}^{22})h_{22} - (c_{22}^{22} - c_{21}^{22})h_{12} = 0
\]

The coefficient matrix of the system is this:

\[
\begin{pmatrix}
-c_{12}^{11} & (c_{21}^{21} - c_{22}^{22}) \\
-(c_{21}^{11} - c_{22}^{12}) & 2(c_{22}^{11} - c_{22}^{12}) \\
-(c_{22}^{11} - c_{22}^{12}) & (c_{22}^{22} - c_{21}^{22})
\end{pmatrix}
\]

Setting

\[
a := c_{11}^{21} - c_{21}^{11}, \quad b := c_{12}^{21} - c_{22}^{12}, \quad \text{and} \quad c := c_{21}^{21} - c_{22}^{22},
\]

we transform it into:

\[
\begin{pmatrix}
h_{12} & h_{11} & h_{22} \\
0 & -b & c \\
0 & b & -c \\
-2c & 2a & 0 \\
-2b & 0 & 2a
\end{pmatrix}
\]
It is clearly degenerate, and has rank 2 in general position.

This means we need to consider (3.8) in full generality: for arbitrary \( \Gamma_0 \) and \( \Gamma_1 \). (4.10) is (3.8) under assumption that \( \Gamma_0 = 0 \), which now has to be lifted.) (3.8) involves \( \mathcal{L}_{V_2} \Gamma_0 \), so we need to express \( V_2 \) from the first equation of \( \Gamma \)-part of (4.10):

\[
\mathcal{L}_{V_1} \Gamma_0 + \frac{\partial^2 V_2}{\partial x^2} = 0.
\]

Setting \( \Gamma^k_{\theta i j} =: \gamma^k_{ij} \), it can be rewritten in index form as:

\[
V_{2,ij} = \gamma^k_{ij} b^l_k - \gamma^l_{kj} b^k_i - \gamma^l_{ik} b^k_j =: v^l_{ij}
\]

Actually, second derivatives of \( V_2 \) is all we need in (3.8), where they appear in (4.15) allows us to rewrite

\[
(\mathcal{L}_{V_2} \Gamma_0)^l_{ij,p} = -\gamma^k_{ij} V^l_{2,kp} + \gamma^l_{kj} V^k_{2,ip} + \gamma^l_{ik} V^k_{2,jp},
\]

which we can now rewrite as:

\[
(\mathcal{L}_{V_2} \Gamma_0)^l_{ij,p} = -\gamma^k_{ij} (\gamma^s_{kp} b^l_s - \gamma^l_{ps} b^s_k - \gamma^l_{ks} b^s_p) + \gamma^l_{kj} (\gamma^s_{sp} b^k_s - \gamma^k_{ps} b^s_p - \gamma^s_{ksp})
\]

\[
+ \gamma^l_{ik} (\gamma^s_{js} b^k_s - \gamma^k_{js} b^s_k - \gamma^s_{jsk})
\]

(4.14)

One note about coefficients \( \gamma \). Compatibility conditions (1.1) in dimension \( n = 2 \) become:

\[
\gamma^1_{ij} = -\gamma^2_{ij}.
\]

That leaves 4 independent coefficients: \( \gamma^2_{11}, \gamma^1_{11}, \gamma^1_{12} \) and \( \gamma^1_{22} \).

We consider (3.8) as \( S(V_1) = 0 \) - linear operator acting on \( V_1 \), and split the operator into two parts: \( S = S(\Gamma_0) + S(\Gamma_1) \). Matrix of \( S(\Gamma_1) \) is calculated at the top of this section.

4.13 allows us to rewrite \( S(\Gamma_0) V_1 = (\mathcal{L}_{V_2} \Gamma_0)^l_{ij,p} - (\mathcal{L}_{V_2} \Gamma_0)^l_{p,i} \) as:

\[
(\gamma^k_{pj} \gamma^s_{ki} - \gamma^l_{ij} \gamma^s_{kp}) b^l_s + (\gamma^l_{pk} \gamma^s_{js} - \gamma^k_{pj} \gamma^s_{ks}) b^l_s
\]

\[
+ (\gamma^l_{pk} \gamma^s_{js} - \gamma^l_{ik} \gamma^s_{js} - \gamma^l_{jsk}) b^s_j + (\gamma^l_{ij} \gamma^s_{js} - \gamma^l_{ik} \gamma^s_{js}) b^s_p
\]

(4.15)

Recall that dimension \( n = 2 \), and indexes \( 1 = i < p = 2 \) must therefore stay fixed at \( i = 1, p = 2 \), while the remaining pair of indexes take any values. That turns (4.15) into a system of 4 expressions indexed with \( (j,l) \):

(11) \[
(\gamma^1_{22} \gamma^2_{11} - \gamma^1_{12} \gamma^2_{12}) b^1_1 + (\gamma^1_{11} \gamma^2_{12} - \gamma^1_{12} \gamma^2_{12}) b^2_1 + (\gamma^1_{11} \gamma^2_{12} - \gamma^1_{12} \gamma^2_{12}) b^2_2 +
\]

\[
+ (\gamma^1_{2k} \gamma^2_{1l} - \gamma^1_{1k} \gamma^2_{1l}) b^1_2 + (\gamma^1_{2k} \gamma^2_{1l} - \gamma^1_{1k} \gamma^2_{1l}) b^2_1
\]

(22) \[
(\gamma^1_{21} \gamma^2_{11} - \gamma^1_{12} \gamma^2_{12}) b^1_1 + (\gamma^1_{12} \gamma^2_{12} - \gamma^1_{12} \gamma^2_{12}) b^2_1 + (\gamma^1_{12} \gamma^2_{12} - \gamma^1_{12} \gamma^2_{12}) b^2_2 +
\]

\[
+ (\gamma^2_{2k} \gamma^2_{1l} - \gamma^2_{1k} \gamma^2_{1l}) b^1_2 + (\gamma^2_{2k} \gamma^2_{1l} - \gamma^2_{1k} \gamma^2_{1l}) b^2_1
\]
We use (3.13, $V_1$-hamiltonian) to go from $b$-coefficients for $V_1$ to $h$-coefficients. Then the fact that $V_2$ too is hamiltonian (second equation in $\omega$-part of (3.6)) follows automatically, as a short calculation would show. Considered by itself, this system is degenerate. Indeed, setting

\[(\gamma_{22}^1 \gamma_{11}^2 - \gamma_{12}^1 \gamma_{12}^2) =: A, (\gamma_{2k}^1 \gamma_{k1}^2 - \gamma_{1k}^1 \gamma_{k2}^2) =: B, (\gamma_{21}^1 \gamma_{11}^2 - \gamma_{12}^1 \gamma_{12}^2) =: C,\]

and using $h$-coefficients, the system’s matrix becomes:

\[
\begin{pmatrix}
0 & -B & C \\
0 & B & -C \\
-2C & 2A & 0 \\
-2B & 0 & 2A
\end{pmatrix}
\]

The determinant of this is identically zero.

Notice that the two matrices for $S(\Gamma_0)$ and $S(\Gamma_1)$ obtained so far look exactly the same, up to capitalization of the entries’ names. The matrix for the operator $S = S(\Gamma_0) + S(\Gamma_1)$ is a sum of the two. Since it will have the same structure as either of its degenerate summands, it is also degenerate. It has rank 2 however, since it’s lower right 2 X 2 block is:

\[
\begin{pmatrix}
2(a + A) & 0 \\
0 & 2(a + A)
\end{pmatrix}
\]

This is non-degenerate in general position, since:

\[a + A = \gamma_{22}^1 \gamma_{11}^2 + \gamma_{12}^1 \gamma_{11}^2 - (c_{21}^{22} + c_{12}^{11}) \neq 0,
\]

resulting in a 1-dimensional stabilizer at 1-jet.

Let us now consider the next, second jet of our connection. To calculate its stabilizer, we need to solve the following equation from (3.6) for $V_4$:

\[\mathcal{L}_{V_1} \Gamma_2 + \tilde{\mathcal{L}}_{V_2} \Gamma_1 + \tilde{\mathcal{L}}_{V_3} \Gamma_0 + \frac{\partial^2 V_4}{\partial x^2} = 0\]

Its compatibility conditions are:

\[(\mathcal{L}_{V_1} \Gamma_2 + \tilde{\mathcal{L}}_{V_2} \Gamma_1 + \tilde{\mathcal{L}}_{V_3} \Gamma_0)_{ij,p} = (\mathcal{L}_{V_1} \Gamma_2 + \tilde{\mathcal{L}}_{V_2} \Gamma_1 + \tilde{\mathcal{L}}_{V_3} \Gamma_0)_{pj,i} \]

We will use the same strategy as in the previous section to prove that in this case stabilizer is trivial. Namely we will obtain a connection 2-jet, for which the above equation will be a non-degenerate homogeneous linear system. We
set $\Gamma_0 = \Gamma_1 = 0$. This implies $V_2 = V_3 = 0$, hence hamiltonian, so that $\omega$-part of (4.4) is true for any hamiltonian $V_1$. That simplifies (4.16) to:

$$(\mathcal{L}_V \Gamma_2)_{ij,p} = (\mathcal{L}_V \Gamma_2)_{jpt}$$

(4.17)

We introduce notation for coefficients of $\Gamma_2$:

$$\Gamma_{2ij}^l = \sum_{s,t=1}^2 d_{ijst}^l x^s x^t, \quad d_{ijst}^l = d_{jits}^l$$

Compatibility with $\omega$ [101] impose these restrictions on $d$ in dimension 2:

$$d_{2aji}^l = -d_{1aji}^l$$

There are thus 4 families of independent coefficients: $d_{11ij}^1, d_{11ij}^2, d_{12ij}^1$ and $d_{12ij}^2$. With these,

$$(\mathcal{L}_V \Gamma_2)_{ij,p} = 2d_{ijkt}^l b_k^i x^t + 2d_{ijkp}^l b_k^j x^t - 2d_{ijpt}^l b_k^i x^t + 2d_{ijpt}^l b_k^j x^t + 2d_{ijkt}^l b_k^j x^t$$

( $b_k^i$ are still coefficients of $V_1$, as in section [5], and (4.17) (with $i = 1, p = 2$) is:

$$(d_{jkt}^l - d_{2jkt}^l) b_k^i + (d_{ijkt}^l - d_{kijt}^l) b_k^j + (d_{1jk2}^l - d_{2jk1}^l) b_k^k +$$

$$(d_{2j1t}^l - d_{1j2t}^l) b_k^k + (d_{1k2t}^l - d_{2k1t}^l) b_k^l = 0$$

With the triple of indexes $(j, l, t)$ arbitrary, we have system of 8 equations in 4 variables: the coefficients of $V_1$. This is the system, equations are labelled by this index triple:

$$(111) \quad 2(d_{1112}^1 - d_{1211}^1) b_1^1 + (d_{1112}^1 - d_{1211}^1) b_2^2 + \left( d_{1112}^1 - d_{1211}^1 \right) b_1^2 + \left( d_{1211}^1 - d_{1112}^1 \right) b_2^2 = 0$$

$$(221) \quad 2(d_{1212}^2 - d_{2211}^2) b_2^1 + (d_{1212}^2 - d_{2211}^2) b_2^2 +$$

$$\left( d_{1222}^2 - d_{2221}^2 \right) b_2^1 + \left( d_{2211}^2 - d_{1212}^2 \right) b_2^2 = 0$$

$$(121) \quad 3(d_{1112}^2 - d_{1211}^2) b_1^1 +$$

$$\left( d_{1122}^2 - d_{2211}^2 + d_{2111}^1 - d_{1112}^1 \right) b_2^2 = 0$$

$$(211) \quad (d_{1122} - d_{2211}) b_2^1 +$$

$$\left( d_{1122} - d_{2211} \right) b_2^2 +$$

$$\left( d_{1122} - d_{2221} \right) b_2^1 + \left( d_{1112} - d_{2111} + d_{2211} - d_{1212} \right) b_2^2 = 0$$

$$(112) \quad (d_{1112}^1 - d_{1212}^1) b_1^1 +$$

$$\left( d_{1122}^1 - d_{2212} \right) b_2^2 +$$

$$\left( d_{1122} - d_{2222} \right) b_2^1 + \left( d_{1112}^1 - d_{2111} + d_{2212} - d_{1212} \right) b_2^2 = 0$$

$$(222) \quad (d_{2222} - d_{2221}) b_1^1 +$$

$$\left( d_{2222} - d_{1222} \right) b_2^2 +$$

$$\left( d_{2221} - d_{1122} \right) b_2^2 + \left( d_{1112}^2 - d_{2211} \right) b_2^2 = 0$$
we see the system take form:

\hspace{1cm}(212) \quad 2(d_{112}^2 - d_{1212}^2)b_1^2 + (d_{112}^2 - d_{1212}^2)b_2^2 + (d_{2221}^2 - d_{1212}^2 + d_{1212}^1 - d_{112}^1)b_1^3 + (d_{112}^2 - d_{2111}^2)b_2^2 = 0

\hspace{1cm}3(d_{1222}^1 - d_{1222}^3)b_2^2 + (d_{2221}^2 - d_{1122}^1 - d_{2221}^3)b_2^2 = 0

Setting:

\begin{align*}
a &= (d_{1112}^1 - d_{1121}^1), \
b &= (d_{1211}^2 - d_{1122}^2), \
c &= (d_{1212}^1 - d_{2221}^1), \
d &= (d_{2222}^1 - d_{2222}^1), \
e &= (d_{2221}^2 - d_{2221}^2), \
f &= (d_{2112}^2 - d_{2112}^2)
\end{align*}

we see the system take form:

\[
\begin{pmatrix}
2a & a & a & b \\
2c & c & d + e & -b \\
-3b & 0 & f + c - a & 0 \\
-e & -2e & g & a - c \\
h & 2h & g & a - f \\
2f & f & d - h & -b \\
0 & 3g & 0 & h - e - d
\end{pmatrix}
\hspace{1cm}=\hspace{1cm}
\begin{pmatrix}
h_2 & h_1 & h \h_1 & h_1 & h_2
\end{pmatrix}
\]

This is non-degenerate for a generic connection. For example, if \(d_{1112}^2 = d_{1122}^2 = 1\), the rest is null, then \(f = 1, b = -1,\) all others zero, and the system is:

\[
\begin{pmatrix}
-1 \\
\end{pmatrix}
\]

Now we can summarize what we know about exceptional stabilizers:

**Proposition 4.1** The stabilizer of a \(k\)-jet of a generic connections for \(n = 2\) is equal to \(G_1/G_2\) for \(k = 0\), is \(1\)-dimensional for \(k = 1\), and is trivial for \(k \geq 2\).

## 5 Poincaré series

Here we will calculate the Poincaré series of \(\mathcal{M}\), the moduli space of Fedosov structures:

\[
p_\Phi(t) = \dim \mathcal{M}_0 + \sum_{k=1}^{\infty} (\dim \mathcal{M}_k - \dim \mathcal{M}_{k-1})t^k
\]
To obtain $\dim M_k$, we need to discuss $F_k$ first. In particular, we need to know how many different local symplectic structures are there. More precisely, we want to find the dimension of the space of $k$-jets of non-degenerate closed 2-forms at a point. Non-degeneracy is an open condition and does not affect dimension. Closedness locally is equivalent to exactness. For a symplectic form $\omega$:

$$\omega = d\alpha,$$

for some 1-form $\alpha$ defined up to $\nabla f$, a gradient of a function, that function in its turn is defined up to a constant. We have the following exact sequence:

$$0 \to \mathbb{R} \to C^\infty(\mathbb{R}^{2n}) \xrightarrow{d^0} \Lambda^1(\mathbb{R}^{2n}) \xrightarrow{d^1} d\Lambda^1(\mathbb{R}^{2n}) \to 0,$$

which descends to jets:

$$0 \to \mathbb{R} \to j^l \Lambda^1(\mathbb{R}^{2n}) \xrightarrow{d^0} j^{l+1} \Lambda^1(\mathbb{R}^{2n}) \xrightarrow{d^1} j^l (d\Lambda^1(\mathbb{R}^{2n})) \xrightarrow{d^2} 0$$

It follows that:

$$\dim[j^l d\Lambda^1(\mathbb{R}^{2n})] = \dim[j^{l+1} \Lambda^1(\mathbb{R}^{2n})] - \dim[j^{l+2}(C^\infty(\mathbb{R}^{2n}))] + \dim \mathbb{R}$$

We are interested in 0-jets since higher jets of $\omega$ are determined by the connection part $\Gamma$ of a given Fedosov structure $\Phi$ through compatibility condition (1.1), see Theorem 4.5 (2) p.124 in [GRS].

$$\dim[j^0 \Lambda^1(\mathbb{R}^{2n})] = \dim[j^1 \Lambda^1(\mathbb{R}^{2n})] - \dim[j^2(C^\infty(\mathbb{R}^{2n}))] + 1 = 2n \left( \begin{array}{c} 2n+1 \\ 2n \end{array} \right) - \left( \begin{array}{c} 2n+2 \\ 2n \end{array} \right) + 1 = \frac{2n(2n-1)}{2}$$

Each $\omega$ is compatible with (or preserved by) all $\Gamma$, such that $\omega_{ia} \Gamma^a_{jk}$ is completely symmetric in $i, j, k$, cf. the last paragraph on p.110 in [GRS]. At 0-jet of Fedosov structure $\Phi_0 = (\omega_0, \Gamma_0)$ there are $\left( \frac{2n+3-1}{2n-1} \right)$ of those, hence:

$$\dim F_0 = \dim \{ \text{all } \omega_0 \} \dim \{ \text{all compatible } \Gamma_0 \} = \frac{2n(2n-1)}{2} (\frac{2n+2}{2n-1})$$

For other $F_k$'s we must remember that each $\Gamma^i_{jk}$ is a homogeneous polynomial of degree $k$ in $2n$ variables:

$$\dim F_k = \frac{2n(2n-1)}{2} \left( \frac{2n+2}{2n-1} \right) \sum_{m=0}^{k} \left( \frac{2n+m-1}{2n-1} \right) = \frac{2n(2n-1)}{2} \left( \frac{2n+2}{2n-1} \right) \left( \frac{2n+k}{2n} \right)$$

Next, we need to know orbit dimensions. $G_1/G_3$ acts on $\Phi_0$ non-trivially, i.e. both first and second component of generating vector field $V_1$ and $V_2$ are acting. The stabilizer $G_{\Phi_0}$ is determined by an arbitrary hamiltonian $V_1$:

$$\dim O_0 = \dim \{ (V_1, V_2) \} - \dim \mathfrak{sp}(2n)$$

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We have:

\[ \dim \mathcal{M}_0 = \frac{2n(2n-1)}{2} \left( \frac{2n+2}{2n-1} \right) - n((2n+1)^2-2) = \frac{n[8n(2n^2-1)(n+1)+11]}{6} \]

\[ \dim \mathcal{M}_k = \dim \mathcal{F}_k - \dim \mathcal{O}_k \]

\[ = \frac{2n(2n-1)}{2} \left( \frac{2n+2}{2n-1} \right) \sum_{m=0}^k \left( \frac{2n+m-1}{2n-1} \right) - 2n \sum_{m=1}^{k+2} \left( \frac{2n+m-1}{2n-1} \right) + \delta^2_{2n} \delta^1_k \]

\[ = \frac{2n(2n-1)}{2} \left( \frac{2n+2}{2n-1} \right) \left( \frac{2n+k}{2n} \right) - 2n \left[ \left( \frac{2n+k+2}{2n} \right) - 1 \right] + \delta^2_{2n} \delta^1_k, \quad k \geq 1 \]

This gives us dimension of moduli space of \( k \)-jets:

\[ \dim \mathcal{M}_0 = \frac{n(2n+1)}{3} [4n^4 + 2n^3 - 6n^2 - 4n - 3] + \delta^2_{2n} \]

The common term will have this coefficient:

\[ \dim \mathcal{M}_k - \dim \mathcal{M}_{k-1} = \frac{2n(2n-1)}{2} \left( \frac{2n+2}{2n-1} \right) \left( \frac{2n+k-1}{2n-1} \right) - 2n \left( \frac{2n+(k+2)-1}{2n-1} \right) \]

\[ = 4n \left( \frac{2n+2}{2n-2} \right) \left( \frac{2n+k-1}{2n-1} \right) - 2n \left( \frac{2n+k+1}{2n-1} \right) - \delta^2_{2n} \delta^2_k, \quad k \geq 2 \]

We have:

\[ p_\Phi(t) = \dim \mathcal{M}_0 + \sum_{k=1}^{\infty} (\dim \mathcal{M}_k - \dim \mathcal{M}_{k-1}) t^k \]

\[ = \frac{n[8n(2n^2-1)(n+1)+11]}{6} + \frac{n(2n+1)}{3} [4n^4 + 2n^3 - 6n^2 - 4n - 3] t + (t-t^2) \delta^2_{2n} + \]

\[ + 2n \sum_{k=2}^{\infty} \left[ \frac{2}{4} \left( \frac{2n+2}{2n} \right) \left( \frac{2n+k-1}{2n-1} \right) - \left( \frac{2n+k+1}{2n-1} \right) \right] t^k \]
Proposition 5.1 The Poincaré series $p_\Phi(t)$ is a rational function. Namely,

$$p_\Phi(t) = \frac{n(20n^2 + 8n + 11)}{6} - \frac{n(2n + 1)}{3} \left[ 4n^4 + 2n^3 + 2n^2 - 4n + 3 \right] t$$

$$+ (t - t^2) \delta_{2n}^2 + 2n D_\Phi \left( \frac{1}{1 - t} \right)$$

where $D_\Phi$ is a differential operator of order $2n - 1$:

$$D_\Phi = 2 \left( \begin{array}{c} 2n+2 \\ 4 \end{array} \right) \left( \begin{array}{c} 2n + t \frac{d}{dt} - 1 \\ 2n-1 \end{array} \right) - \left( \begin{array}{c} 2n + t \frac{d}{dt} + 1 \\ 2n-1 \end{array} \right)$$

Proof Indeed, denote

$$\varphi_m(t) = \sum_{k=0}^{\infty} k^m t^k, \quad m \in \mathbb{Z}_+,$$

then

$$\varphi_m(t) = \sum_{k=0}^{\infty} k^{m-1} k t^{k-1} = t \left( \sum_{k=0}^{\infty} k^{m-1} t^k \right) = \left( \frac{t}{1 - t} \right) \varphi_{m-1}(t) \quad \text{for} \ m \in \mathbb{N}.$$  

Thus

$$\varphi_m(t) = \left( \frac{t}{1 - t} \right)^m \varphi_0(t) = \left( \frac{t}{1 - t} \right)^m \left( \frac{1}{1 - t} \right).$$

Hence,

$$\sum_{k=0}^{\infty} \left[ 2 \left( \begin{array}{c} 2n+2 \\ 4 \end{array} \right) \left( \begin{array}{c} 2n + k - 1 \\ 2n-1 \end{array} \right) - \left( \begin{array}{c} 2n + k + 1 \\ 2n-1 \end{array} \right) \right] t^k$$

$$= \left[ 2 \left( \begin{array}{c} 2n+2 \\ 4 \end{array} \right) \left( \begin{array}{c} 2n + t \frac{d}{dt} - 1 \\ 2n-1 \end{array} \right) - \left( \begin{array}{c} 2n + t \frac{d}{dt} + 1 \\ 2n-1 \end{array} \right) \right] \left( \frac{1}{1 - t} \right).$$

We have:

$$\sum_{k=0}^{\infty} \left[ 2 \left( \begin{array}{c} 2n+2 \\ 4 \end{array} \right) \left( \begin{array}{c} 2n + k - 1 \\ 2n-1 \end{array} \right) - \left( \begin{array}{c} 2n + k + 1 \\ 2n-1 \end{array} \right) \right] t^k$$

$$= \sum_{k=0}^{\infty} \left[ 2 \left( \begin{array}{c} 2n+2 \\ 4 \end{array} \right) \left( \begin{array}{c} 2n + k - 1 \\ 2n-1 \end{array} \right) - \left( \begin{array}{c} 2n + k + 1 \\ 2n-1 \end{array} \right) \right] t^k$$

$$- \left( \begin{array}{c} 2n+2 \\ 3 \end{array} \right) (2n^2 - n - 1)t - \left( \begin{array}{c} 2n+1 \\ 2 \end{array} \right) \frac{2n^2 + n - 4}{3}.$$
So
\[ p_\Phi(t) = \frac{n[8n(2n^2 - 1)(n + 1) + 11]}{6} + \frac{n(2n + 1)}{3} \left[ 4n^4 + 2n^3 - 6n^2 - 4n - 3 \right] t \]
\[ + (t - t^2)\delta_{2n}^2 + 2n \left\{ \sum_{k=0}^{\infty} \left[ \binom{2n + 2}{4} \binom{2n + k - 1}{2n - 1} - \binom{2n + k + 1}{2n - 1} \right] t^k \right\} \]
\[ - \left( \frac{2n + 2}{3} \right)(2n^2 - n - 1) t - \binom{2n + 1}{2} \left( \frac{2n^2 + n - 4}{3} \right) \]
\[ = \frac{n(20n^2 + 8n + 11)}{6} - \frac{n(2n + 1)}{3} \left[ 4n^4 + 2n^3 + 2n^2 - 4n + 3 \right] t \]
\[ + (t - t^2)\delta_{2n}^2 + 2n D_\Phi \left( \frac{1}{1 - t} \right) \]
\[ \square \]

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