Phantom dark energy with varying-mass dark matter particles: acceleration and cosmic coincidence problem

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We investigate several varying-mass dark-matter particle models in the framework of phantom cosmology. We examine whether there exist late-time cosmological solutions, corresponding to an accelerating universe and possessing dark energy and dark matter densities of the same order. Imposing exponential or power-law potentials and exponential or power-law mass dependence, we conclude that the coincidence problem cannot be solved or even alleviated. Thus, if dark energy is attributed to the phantom paradigm, varying-mass dark matter models cannot fulfill the basic requirement that led to their construction.

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I. INTRODUCTION

Recent cosmological observations support that the universe is experiencing an accelerated expansion, and that the transition to the accelerated phase realized in the recent cosmological past [1]. In order to explain this unexpected behavior, one may modify the theory of gravity [2], or introduce the concept of dark energy which provides the acceleration mechanism. The most explored dynamically dark energy models of the literature consider a canonical scalar field (quintessence) [3], a phantom field, that is a scalar field with a negative sign of the kinetic term [4], or the combination of quintessence and phantom in a unified model named quintom [5].

The dynamical nature of dark energy introduces a new cosmological problem, namely why are the densities of vacuum energy and dark matter nearly equal today although they scale independently during the expansion history. The elaboration of this “coincidence” problem led to the consideration of generalized versions of the aforementioned scenarios with the inclusion of a coupling between dark energy and dark matter. Thus, various forms of “interacting” dark energy models [6, 7, 8, 9] have been constructed in order to fulfill the observational requirements. In the case of interacting quintessence one can find accelerated attractors which moreover give dark matter and dark energy density parameters of the same order, thus solving the coincidence problem [10, 11], but paying the price of introducing new problems such is the justification of a non-trivial, almost tuned, sequence of cosmological epochs [12]. In interacting phantom models [8, 9, 13], the existing literature remains in some special coupling forms which suggest that the coincidence problem might be alleviated [8, 9].

An equivalent approach is to assume that dark energy and dark matter sectors interact in such a way that the dark matter particles acquire a varying mass, dependent on the scalar field which reproduces dark energy [14]. This consideration allows for a better theoretical justification, since a scalar-field-dependent varying-mass can arise from string or scalar-tensor theories [15]. Indeed, in such higher dimensional frameworks one can formulate both the appearance of the scalar field (which is related to the dilaton and moduli fields) and its effect on matter particle masses (determined by string dynamics, supersymmetry breaking, and the compactification mechanism) [16]. In quintessence scenario, such varying-mass dark matter models have been explored in cases of linear [7, 14, 16, 17], power-law [18] or exponential [19, 20, 21] scalar-field dependence. The exponential case is the most interesting since, apart from solving the coincidence problem, it allows for stable scaling behavior, that is for a large class of initial conditions the cosmological evolution converges to a common solution at late times [20, 21].

In the present work we are interested in investigating varying-mass dark matter models in scenarios where dark energy is attributed to a phantom field. Although such a framework could lead to instabilities at the quantum level [22], there have been serious attempts in overcoming these difficulties and construct a phantom theory consistent with the basic requirements of quantum field theory, with the phantom fields arising as an effective description [23]. Performing a complete phase-space analysis using various forms of mass-dependence and scalar-field potentials, we examine whether there exist stable late-time accelerating solutions which moreover solve the coincidence problem. As we will show, the coincidence problem cannot be solved in any of the investigated models.

The plan of the work is as follows: In section II we construct varying-mass dark matter models in the framework of phantom cosmological scenario and we present the formalism for the transformation into an autonomous dynamical system. In section III we perform the phase-space stability analysis for four different models, using various mass-dependence forms and phantom potentials, and in section IV we discuss the corresponding cosmo-
logical implications. Finally, in section [4] we summarize the obtained results.

II. VARYING-MASS DARK MATTER PARTICLES IN THE FRAMEWORK OF PHANTOM COSMOLOGY

Let us construct a cosmological model where dark energy is attributed to a phantom field, in which the dark matter particles have a varying mass depending on this field. Throughout the work we consider a flat Robertson-Walker metric:

\[ ds^2 = dt^2 - a^2(t)dx^2, \]

with \( a \) the scale factor and \( t \) the comoving time.

In the phantom cosmological paradigm the energy density and pressure of the phantom scalar field \( \phi \) are:

\[ \rho_\phi = -\frac{1}{2} \dot{\phi}^2 + V(\phi) \]
\[ p_\phi = -\frac{1}{2} \dot{\phi}^2 - V(\phi), \]

where \( V(\phi) \) is the phantom potential and the dot denotes differentiation with respect to comoving time. In such a scenario, the dark energy is attributed to the phantom field, and its equation of state is given by

\[ w_{DE} = \frac{p_\phi}{\rho_\phi}. \]

As was mentioned in the introduction, in varying-mass dark matter models the central assumption is that the dark-matter particles have a \( \phi \)-dependent mass \( M_{DM}(\phi) \), while dark matter is considered as dust. Thus, for the dark matter energy density we have the standard definition

\[ \rho_{DM} = M_{DM}(\phi) n_{DM}, \]

where \( n_{DM} \) is the number density of the dark-matter particles. As usual, in the case of FRW geometry, it is determined by the equation

\[ n_{DM}' + 3Hn_{DM} = 0, \]

with \( H \) the Hubble parameter. Therefore, differentiating \( \dot{n}_{DM} \) and using \( \dot{n}_{DM}' = \frac{1}{M_{DM}(\phi)} \frac{dM_{DM}(\phi)}{d\phi} \dot{\phi} \rho_{DM} \), we obtain the evolution equation for \( \rho_{DM} \), namely:

\[ \dot{\rho}_{DM} + 3H \rho_{DM} = \frac{1}{M_{DM}(\phi)} \frac{dM_{DM}(\phi)}{d\phi} \dot{\phi} \rho_{DM}. \]

Obviously, in a case of \( \phi \)-independent dark-matter particle mass, we re-obtain the usual evolution equation \( \dot{\rho}_{DM} + 3H \rho_{DM} = 0 \). Therefore, we observe that the \( \phi \)-dependent mass reveals the interaction between dark matter and dark energy (that is the phantom field) sectors that lies behind it.

Since general covariance leads to total energy conservation, we deduce that the evolution equation for the phantom energy density will be:

\[ \dot{\rho}_\phi + 3H(\rho_\phi + p_\phi) = -\frac{1}{M_{DM}(\phi)} \frac{dM_{DM}(\phi)}{d\phi} \dot{\phi} \rho_{DM}. \]

Thus, \( \frac{dM_{DM}(\phi)}{d\phi} \dot{\phi} < 0 \) corresponds to energy transfer from dark matter to dark energy, while \( \frac{dM_{DM}(\phi)}{d\phi} \dot{\phi} > 0 \) corresponds to dark energy transformation into dark matter.

Equivalently, using the definitions \( \rho_{DM} \) and \( \rho_{\phi} \), the phantom evolution equation can be written in field terms as:

\[ \dot{\phi} + 3H \dot{\phi} - \frac{\partial V(\phi)}{\partial \phi} = \frac{1}{M_{DM}(\phi)} \frac{dM_{DM}(\phi)}{d\phi} \rho_{DM}. \]

Finally, the system of equations closes by considering the Friedmann equations:

\[ H^2 = \frac{\kappa^2}{3}(\rho_\phi + \rho_{DM}), \]
\[ \dot{H} = -\frac{\kappa^2}{2}(\rho_\phi + p_\phi + \rho_{DM}). \]

where we have set \( \kappa^2 \equiv 8\pi G \). Although we could straightforwardly include baryonic matter and radiation in the model, for simplicity reasons we neglect them.

Alternatively, one could construct the equivalent uncoupled model described by:

\[ \dot{\rho}_{DM} + 3H(1 + w_{DM,eff})\rho_{DM} = 0, \]
\[ \dot{\rho}_\phi + 3H(1 + w_{\phi,eff})\rho_\phi = 0, \]

where

\[ w_{DM,eff} = -\frac{1}{M_{DM}(\phi)} \frac{dM_{DM}(\phi)}{d\phi} \frac{\dot{\phi}}{3H} \]
\[ w_{\phi,eff} = w_\phi + \frac{1}{M_{DM}(\phi)} \frac{dM_{DM}(\phi)}{d\phi} \frac{\dot{\phi}}{3H} \frac{\rho_{DM}}{\rho_\phi}. \]

However, it is more convenient to introduce the “total” energy density \( \rho_{tot} \equiv \rho_{DM} + \rho_\phi \), obtaining:

\[ \dot{\rho}_{tot} + 3H(1 + w_{tot})\rho_{tot} = 0, \]

with

\[ w_{tot} = \frac{p_\phi}{\rho_\phi + \rho_{DM}} = w_\phi \Omega_\phi, \]

where \( \Omega_\phi \equiv \frac{\rho_\phi}{\rho_{tot}} \equiv \Omega_{DE} \). Obviously, since \( \rho_{tot} = \frac{3H^2}{\kappa^2} \) (10) leads to a scale factor evolution of the form \( a(t) \propto t^{2/(3(1 + w_{tot}))} \), in the constant \( w_{tot} \) case. However, at the late-time stationary solutions that we are studying in the present work, \( w_{tot} \) has reached to a constant value and thus the above behavior is valid. Therefore, we
conclude that in such stationary solutions the condition for acceleration is just \( w_{\text{tot}} < -1/3 \).

In order to perform the phase-space and stability analysis of the phantom model at hand, we have to transform the aforementioned dynamical system into its autonomous form \[24\,\text{and}\,23\]. This will be achieved by introducing the auxiliary variables:

\[
\begin{align*}
x &= \frac{\kappa \dot{\phi}}{\sqrt{6}H}, \\
y &= \frac{\kappa \sqrt{V(\phi)}}{\sqrt{3}H}, \\
z &= \frac{\sqrt{6}}{\kappa H}
\end{align*}
\]

(18)

together with \( M = \ln a \). Thus, it is easy to see that for every quantity \( F \) we acquire \( \dot{F} = H \frac{dF}{dM} \). Using these variables we obtain:

\[
\Omega_\phi \equiv \frac{\kappa^2 \rho_\phi}{3H^2} = -x^2 + y^2, \quad (19)
\]

\[
w_\phi = \frac{-x^2 - y^2}{-x^2 + y^2}, \quad (20)
\]

\[
w_{\text{tot}} = -x^2 - y^2. \quad (21)
\]

We mention that relations \[20\] and \[21\] are always valid, that is independently of the specific state of the system (they are valid in the whole phase-space and not only at the critical points). Finally, note that in the case of complete dark energy domination, that is \( \rho_{DM} \to 0 \) and \( \Omega_\phi \to 1 \), we acquire \( w_{\text{tot}} \approx w_\phi \leq -1 \), as expected to happen in phantom-dominated cosmology.

The next step is the introduction of a specific ansatz for the phantom potential \( V(\phi) \), and a specific ansatz for the dark-matter particle mass function \( M_{DM}(\phi) \). In this case the equations of motion \[7\], \[9\], \[10\] and \[11\] can be transformed into an autonomous system containing the variables \( x \) and \( y \) and perhaps \( z \) (\( z \) is present only for some such ansatzes) and their derivatives with respect to \( M = \ln a \).

Having transformed the cosmological system into its autonomous form:

\[
X' = f(X), \quad (22)
\]

where \( X \) is the column vector constituted by the auxiliary variables, \( f(X) \) the corresponding column vector of the autonomous equations, and prime denotes derivative with respect to \( M = \ln a \), we extract its critical points \( X_c \) satisfying \( X' = 0 \). Then, in order to determine the stability properties of these critical points, we expand \[22\] around \( X_c \), setting \( X = X_c + U \) with \( U \) the perturbations of the variables considered as a column vector. Thus, for each critical point we expand the equations for the perturbations up to the first order as:

\[
U' = Q \cdot U, \quad (23)
\]

where the matrix \( Q \) contains the coefficients of the perturbation equations. Thus, for each critical point, the eigenvalues of \( Q \) determine its type and stability.

\section{III. Phase-space analysis}

In the previous section we constructed a cosmological scenario where the dark matter particles have a varying mass, depending on the phantom field. Additionally, we presented the formalism for its transformation into an autonomous dynamical system, suitable for a stability analysis. In this section we introduce specific forms for \( V(\phi) \) and \( M_{DM}(\phi) \), and perform a complete phase-space analysis.

For the scalar field potential we consider two well studied cases of the literature, namely the exponential \[20\], \[21\]:

\[
V(\phi) = V_0 e^{-\kappa \lambda_1 \phi} \quad (24)
\]

and the power-law one \[18\], \[26\]:

\[
V(\phi) = V_0 \phi^{-\lambda_2}. \quad (25)
\]

For the dark matter particle mass we consider two possible cases, namely an exponential dependence \[19\], \[23\], \[21\]:

\[
M_{DM}(\phi) = M_0 e^{-\kappa \mu_1 \phi} \quad (26)
\]

and the power-law one \[18\]:

\[
M_{DM}(\phi) = M_0 \phi^{-\mu_2}. \quad (27)
\]

Therefore, in the following we consider four different models, arising from the aforementioned combinations.

\subsection{A. Model 1: Exponential potential and exponentially-dependent dark-matter particle mass}

Inserting the auxiliary variables \[18\] into the equations of motion \[7\], \[9\], \[10\] and \[11\], we result in the following autonomous system:

\[
\begin{align*}
x' &= -3x + \frac{3}{2}x(1 - x^2 - y^2) - \sqrt{\frac{3}{2}} \lambda_1 y^2 - \sqrt{\frac{3}{2}} \mu_1 (1 + x^2 - y^2) \\
y' &= \frac{3}{2}y(1 - x^2 - y^2) - \sqrt{\frac{3}{2}} \lambda_1 xy.
\end{align*}
\]

(28)

Note that in this case, the auxiliary variable \( z \) is not needed.
The critical points \((x_c, y_c)\) of the autonomous system [25] are obtained by setting the left hand sides of the equations to zero. The real and physically meaningful (that is corresponding to \(y > 0\) and \(0 \leq \Omega_\phi \leq 1\)) of them are:

\[
\begin{pmatrix}
  x_c &= -\frac{\lambda_1}{\sqrt{6}}, \\
  y_c &= \sqrt{1 + \frac{\lambda_1^2}{6}}
\end{pmatrix},
\]

\[
Q = \begin{bmatrix}
\frac{1}{2} \left( -9x_c^2 - 2\sqrt{6}\mu_1 x_c - 3 \left( y_c^2 + 1 \right) \right) & \frac{1}{2} \left( -9y_c^2 - x_c \left( 3x_c + \sqrt{6}\lambda_1 \right) + 3 \right) \\
-\frac{1}{2} y_c \left( 6x_c + \sqrt{6}\lambda_1 \right) & \mu_1 - 1
\end{bmatrix}.
\]

Therefore, for each critical point of table I we examine the signs of the real parts of the eigenvalues of \(Q\), which determine the type and stability of this specific critical point. In table I we present the results of the stability analysis. In addition, for each critical point we calculate the values of \(w_{\text{tot}}\) (given by relation (21)), and of \(\Omega_\phi\) (given by (19)). Thus, knowing \(w_{\text{tot}}\) we can express the acceleration condition \(w_{\text{tot}} < -1/3\) in terms of the model parameters.

The critical point A exists always and it is either a saddle point (the \(Q\)-eigenvalues have real parts of different sign) or an attractor (the \(Q\)-eigenvalues have negative real parts). The critical point B, if it exists, it is always a saddle point. The cosmological model at hand admits another critical point, namely C, which is unphysical since it leads to \(\Omega_\phi < 0\). This point has coordinates \((x_c = -\sqrt{\frac{2}{3}}\mu_1, y_c = 0)\) and it is either a saddle point or an attractor. If \(\mu_1 (\mu_1 - 1) > 3/2\) it is an attractor and in this case, although unphysical, it can attract an open set of orbits from the interior of the physical region of the phase space.

In order to present this behavior more transparently, we evolve the autonomous system numerically for \(\lambda_1 = 0.4\) and \(\mu_1 = 2\), and the results are shown in figure 1. Depending on which region of the phase-space does the system initiates, it lies in the basin of attraction of either A or C, and thus it is attracted by one or the other point. In particular, the orbits initially below the stable manifold of B-points converge towards C, while the orbits initially above this curve converge to A. Interestingly, A is not the global attractor for points at the physical region (region corresponding to \(0 \leq \Omega_\phi \leq 1\), bounded by the dashed (red) curves) in two regions. The orbits initially below this curve converge towards C. The orbits initially above this curve converge to A.

with \(\phi\)-independent dark matter particle mass.

\[
\begin{pmatrix}
  x_c &= \sqrt{\frac{2}{\lambda_1 - \mu_1}}, \\
  y_c &= \sqrt{\frac{2}{\lambda_1 - \mu_1}} \left( \mu_1 - 1 - \frac{2}{\lambda_1 - \mu_1} \right)
\end{pmatrix}.
\]

and in table I we present the necessary conditions for their existence. The \(2 \times 2\) matrix \(Q\) of the linearized perturbation equations writes:

\[
\begin{pmatrix}
  x_c &= \sqrt{\frac{2}{\lambda_1 - \mu_1}}, \\
  y_c &= \sqrt{\frac{2}{\lambda_1 - \mu_1}} \left( \mu_1 - 1 - \frac{2}{\lambda_1 - \mu_1} \right)
\end{pmatrix}.
\]

B. Model 2: Power-law potential and power-law-dependent dark-matter particle mass

Inserting the auxiliary variables [18] into the equations of motion (7), (9), (10) and (11), we result in the follow-
In this case, the critical points are non-hyperbolic, that is there exists always at least a zero eigenvalue. We mention that for non-hyperbolic critical points the result of linearization cannot be applied in order to investigate the local stability of the system (the system can be unstable to small perturbations on the initial condition or to small perturbations on the parameters) \[^2, 28, 29\]. However, it is possible to get information about the existence and the dimensionality of the stable manifold by applying the center manifold theorem \[^28\]. Doing so we deduce that the dimensionality of the local stable manifold is 1 and 2 for D and E respectively. In particular, the stable manifold of D is tangent, at the critical point, to the x-axis, while the stable manifold of E is tangent, at the critical point, to the xy-plane. The existence of a 1D stable manifold for D, implies that the orbits asymptotic to D as \(t \to -\infty\) are contained in either an unstable or center manifold (each one of dimensionality 1, that is a curve). There are some exceptional orbits converging to D as \(t \to +\infty\), but these have a zero measure. On the other hand, the fact that E has a 2D stable manifold implies that there exists a non-zero-measure set of orbits that converges to E as \(t \to +\infty\). Finally, there are some exceptional orbits contained in its center manifold that cannot be classified by linearization. In summary, using more sophisticated tools such as the Normal Forms theorem \[^28\], we indeed find that the center manifold of E attracts an open set of orbits provided \(\lambda_2 \leq 0\). On the
other hand, if $\lambda_2 > 0$ the orbits located near the center manifold of E blow up in a finite time. Since this point does not allow for a solution of the coincidence problem (it always possesses $\Omega_\phi = 1$) we do not present the aforementioned procedure in detail.

Numerical investigation reveals the above features. In fig. 3 we depict orbits projected in the xy-plane, as they arise from numerical evolution in the case of $\lambda_2 = -0.5$ and $\mu_2 = 0.5$.

![Image](image.png)

**FIG. 3:** (Color Online) xy-projection of the phase-space of Model 2, for the parameter values $\lambda_2 = -0.5$ and $\mu_2 = 0.5$. The critical point E (representing de Sitter solutions) is the attractor of the system. The dashed (red) curves bound the physical part of the phase space, that is corresponding to $0 \leq \Omega_\phi \leq 1$.

### C. Model 3: Power-law potential and exponentially-dependent dark-matter particle mass

In this case the autonomous system reads:

\[
x' = -3x + \frac{3}{2}x(1 - x^2 - y^2) - \frac{\lambda_2 y^2 z}{2} - \sqrt{\frac{3}{2}} \mu_1 (1 + x^2 - y^2)
\]
\[
y' = \frac{3}{2}y(1 - x^2 - y^2) - \frac{\lambda_2 x y z}{2}
\]
\[
z' = -x z^2.
\]

The real and physically meaningful critical points are

\[
(x_{c6} = 0, \ y_{c6} = 1, \ z_{c6} = 0), \\
(x_{c7} = -\frac{\sqrt{3}}{\mu_1}, \ y_{c7} = \sqrt{1 - \frac{3}{2\mu_1^2}}, \ z_{c7} = 0)
\]

and the necessary conditions for their existence are shown in table III. The $3 \times 3$ matrix $Q$ of the linearized perturbation equations writes:

\[
Q = \begin{bmatrix}
\frac{1}{2} (-9x_c^2 - 2\sqrt{6}\mu_1 x_c - 3(y_c^2 + 1)) \\
-\frac{1}{2} x_c (6x_c + z_c \lambda_2) \\
-\frac{1}{2} y_c (3x_c + z_c \lambda_2 + 3) \\
\frac{1}{2} (-9y_c^2 - 2\sqrt{6}\mu_1 y_c) \\
\frac{1}{2} x_c (3y_c + z_c \lambda_2) \\
-2x_c z_c
\end{bmatrix}
\]

In the model at hand, all critical points are non-hyperbolic and the dimensionality of their stable manifold is presented in table III. Additionally, we mention that there exists also an unphysical critical point H, with coordinates $(x_{c8} = -\sqrt{\frac{3}{2}} \mu_1, \ y_{c8} = 0, \ z_{c8} = 0)$.

### TABLE II: The real and physically meaningful critical points of Model 2 and their behavior.

| Cr. P. | $x_c$ | $y_c$ | $z_c$ | Existence | Stable manifold | $\Omega_\phi$ | $w_{tot}$ | Acceleration |
|--------|------|------|------|-----------|----------------|-----------|-----------|-------------|
| D      | $x_{c4}$ | $y_{c4}$ | $z_{c4}$ | Always    | 1-Dimensional | $0$       | $0$       | Never       |
| E      | $x_{c5}$ | $y_{c5}$ | $z_{c5}$ | Always    | 2-Dimensional | $1$       | $-1$      | Always      |

In the model at hand, all critical points are non-hyperbolic and the dimensionality of their stable manifold is presented in table III. Additionally, we mention that there exists also an unphysical critical point H, with coordinates $(x_{c8} = -\sqrt{\frac{3}{2}} \mu_1, \ y_{c8} = 0, \ z_{c8} = 0)$.

### TABLE III: The real and physically meaningful critical points of Model 3 and their behavior.

| Cr. P. | $x_c$ | $y_c$ | $z_c$ | Existence | Stable manifold | $\Omega_\phi$ | $w_{tot}$ | Acceleration |
|--------|------|------|------|-----------|----------------|-----------|-----------|-------------|
| D      | $x_{c9}$ | $y_{c9}$ | $z_{c9}$ | Always    | 1-Dimensional | $0$       | $0$       | Never       |
| E      | $x_{c10}$ | $y_{c10}$ | $z_{c10}$ | Always    | 2-Dimensional | $1$       | $-1$      | Always      |
F. This behavior is depicted in fig. 4 which has arisen from numerical evolution using $\lambda_2 = 1$ and $\mu_1 = 1.8$. If we restrict ourselves in the region $|\mu_1| < \sqrt{\frac{3}{2}}$, then the critical point G does not exists and thus there are not scaling solutions. In this case F is indeed the attractor for a positive-measure set of initial conditions. Moreover, there exist exceptional orbits contained on a 1D center manifold of F whose dynamical behavior cannot be anticipated from the linear analysis. However, since this scenario does not lead to a solution of the coincidence problem ($\Omega_\phi = 1$ always) we do not present an advanced stability analysis for F.

D. Model 4: Exponential potential and power-law-dependent dark-matter particle mass

In this case the autonomous system writes:

$$
egin{align*}
    x' &= -3x + \frac{3}{2}x(1 - x^2 - y^2) - \frac{\sqrt{3}}{2} \lambda_1 y^2 - \frac{\mu_2}{2} z (1 + x^2 - y^2) \\
y' &= \frac{3}{2} y (1 - x^2 - y^2) - \frac{\sqrt{3}}{2} \lambda_1 xy \\
z' &= -x z^2.
\end{align*}
$$

(34)

The real and physically meaningful critical points are

$$
(x_{c9} = 0, \ y_{c9} = 0, \ z_{c9} = 0), \quad (x_{c10} = -\frac{\lambda_1}{\sqrt{6}}, \ y_{c10} = \sqrt{1 + \frac{\lambda_1^2}{6}}, \ z_{c10} = 0),
$$

(35)

and in table IV we present the necessary conditions for their existence. The $3 \times 3$ matrix $Q$ of the linearized perturbation equations reads:

$$
Q = \begin{bmatrix}
\frac{1}{2} (-9x_c^2 - 2z_c \mu_2 x_c - 3y_c^2 - 3) & y_c (-3x_c - \sqrt{6} \lambda_1 + z_c \mu_2) & -\frac{1}{2} (x_c^2 + y_c^2 + 1) \mu_2 \\
-\frac{1}{2} y_c (6x_c + \sqrt{6} \lambda_1) & \frac{1}{2} (-9y_c^2 - x_c (-3x_c + \sqrt{6} \lambda_1) + 3) & 0 \\
-2x_c z_c & 0 & 0
\end{bmatrix}.
$$

The aforementioned critical points are non-hyperbolic since at least one eigenvalue of $Q$ is always zero. Linear analysis in not conclusive in these cases, but information about the dimensionality of the stable manifold can be obtained by applying the center manifold theorem \cite{28}. The corresponding results are shown in table IV. Since both I and J cannot solve the coincidence problem ($\Omega_\phi = 1$), we do not present the aforementioned analysis in detail. Finally, in order to acquire a more transparent picture of the phase-space behavior, we evolve the system numerically for $\lambda_1 = 1$ and $\mu_2 = 1.8$ and we depict the results in fig. 5.

| Cr. P. | $x_c$ | $y_c$ | $z_c$ | Existence | Stable manifold | $\Omega_\phi$ | $\omega_{tot}$ | Acceleration |
|--------|-------|-------|-------|-----------|----------------|-------------|--------------|-------------|
| F      | $x_{c0}$ | $y_{c0}$ | $z_{c0}$ | Always   | 2-Dimensional | 1           | -1           | Always       |
| G      | $x_{c7}$ | $y_{c7}$ | $z_{c7}$ | $|\mu_1| > \sqrt{3}$ | 1-Dimensional | $1 - \frac{1}{\sqrt{2}}$ | -1 | Always       |

TABLE III: The real and physically meaningful critical points of Model 3 and their behavior.
TABLE IV: The real and physically meaningful critical points of Model 4 and their behavior.

| Cr. P. | $x_c$ | $y_c$ | $z_c$ | Existence | Stable manifold | $\Omega_\phi$ | $w_{\text{tot}}$ | Acceleration |
|--------|-------|-------|-------|------------|----------------|---------------|----------------|--------------|
| 1      | $x_{c1}$ | $y_{c1}$ | $z_{c1}$ | Always     | 1-Dimensional  | 0             | 0              | Never        |
| J      | $x_{c10}$ | $y_{c10}$ | $z_{c10}$ | Always     | 2-Dimensional  | $1 - \frac{1}{2}(3 + \lambda_j^2)$ | Always        |

FIG. 5: (Color Online) $xy$-projection of the phase-space of Model 4 for the parameter values $\lambda_1 = 1$ and $\mu_2 = 1.8$. The critical point J (corresponding to a super-accelerating universe) attracts all the orbits in this invariant set. The dashed (red) curves bound the physical part of the phase space, that is corresponding to $0 \leq \Omega_\phi \leq 1$.

IV. COSMOLOGICAL IMPLICATIONS AND DISCUSSION

Having performed a complete phase-space analysis of several varying dark-matter-mass models, we can discuss the corresponding cosmological behavior. A general remark is that this behavior is radically different from the corresponding quintessence scenarios with the same potentials and mass-functions. Additionally, a common feature of almost all the phantom models previously studied is the existence of attractors with $w_\phi \leq -1$ in the whole phase-space, and thus, independently of the specific scenario and of the imposed initial conditions, the universe always lies below the phantom divide, as it is expected for phantom cosmology. This global behavior is not always realized in the case of exponentially dependent dark-matter mass, and additional constraints must be imposed.

Apart from acquiring acceleration, in this work we examine whether the above constructed varying dark-matter-mass models can solve or alleviate the coincidence problem. Thus, assuming as usual that the present universe is already at a late-time attractor, we calculate $\Omega_\phi$ in all stable fixed points, and if $0 < \Omega_\phi < 1$ then the coincidence problem is solved since $\Omega_\phi$ and $\Omega_{DM}$ will be of the same order of magnitude as suggested by observations. On the contrary, $\Omega_\phi = 1$ corresponds to a universe completely dominated by dark energy, while $\Omega_\phi = 0$ (that is $\Omega_{DM} = 1$) to one completely dominated by dark matter, both in contrast with observations.

Finally, we mention that as long as the interaction responsible for the varying dark-matter particle mass is not too strong, the standard cosmology can be always recovered. On the other hand, since we assume that the universe is currently at an attractor, its state is independent of the initial conditions. Thus, we can switch on the interaction and consider as initial conditions the end of the known epochs of standard Big Bang cosmology, in order to avoid disastrous interference.

A. Model 1

In this model the critical point B is unstable, and therefore it cannot be a late-time cosmological solution. The only relevant critical point is A, which is a stable fixed point for $\lambda_1 (\mu_1 - \lambda_1) < 3$. As can be seen from table IV, it corresponds to an accelerating universe with $\Omega_\phi = 1$, that is to complete dark-energy domination. Thus, this specific cosmological solution cannot solve the coincidence problem. Furthermore, the fact that $w_{\text{tot}}$ is not only less than $-1/3$, as required by the acceleration condition, but it is always less than $-1$, leads to $\dot{H} > 0$ at all times. Therefore, this solution corresponds to a super-accelerating universe, that is with a permanently increasing $H$, resulting to a Big Rip. This behavior is common in phantom cosmology.
previous studies of phantom cosmology \cite{8,9,13}, which further weakens the applicability of the model.

In summary, Model 1, that is an exponential potential and an exponentially-dependent dark-matter particle mass, cannot act as a candidate for solving the coincidence problem.

\section*{B. Model 2}

In this case, both real and physically meaningful critical points, namely D and E, have a stable manifold of smaller dimensionality than that of the phase-space, and as was mentioned in subsection III B they have very small probability to be the late-time attractors of the system. However, even if the cosmological evolution is managed to be attracted by these solutions, the coincidence problem will not be solved, since D represents a flat, non-accelerating universe dominated by dark matter, and E correspond to de Sitter universe completely dominated by dark energy. These critical points are located in the region where the scalar field and the Hubble parameter diverge. Divergencies in a cosmological scenario are represented as asymptotic states, in particular associated with the past and future asymptotic dynamics \cite{11,32}. In the present Model 2, due to the non-compactness of the phase-space, such a behavior can lead either to an asymptotic state acquired at infinite time, or to a singularity reached at a finite time. If $H \rightarrow \infty$ or $\rho_\phi \rightarrow \infty$ at $t \rightarrow \infty$ then we acquire an eternally expanding universe, while if $H \rightarrow \infty$ at $t \rightarrow t_{\text{BR}} < \infty$ then the universe results to a Big Rip \cite{33}.

Therefore, power-law potentials with power-law-dependent dark-matter particle masses, cannot solve the coincidence problem.

\section*{C. Model 3}

In this model we see that the critical point F exists always, while G exists only for $|\mu_1| > \sqrt{3}$. However, in both cases the stable manifold is of smaller dimensionality than that of the phase-space. Furthermore, in order to avoid the treatment of unphysical attracting states we have to impose the additional constraint $|\mu_1| < \sqrt{2}/3$. For this choice of parameters, G does not exists and thus there are not scaling solutions, while F is the attractor for a positive-measure set of initial conditions. Point F corresponds to a dark-energy dominated de Sitter universe, while G to a flat accelerating universe with $\Omega_\phi = 1 - \frac{3}{\mu_1^2}$, that is with $0 < \Omega_\phi < 1$ in the region that it exists. In both points the phantom field diverges. However, even if G possesses $0 < \Omega_\phi < 1$, it can not solve the coincidence problem since it is not a relevant late-time attractor.

In summary, power-law potentials with exponentially-dependent dark-matter particle masses cannot solve or even alleviate the coincidence problem.

\section*{D. Model 4}

In this case, the critical points I and J exist always. The point I corresponds to a flat, non-accelerating, matter-dominated universe. J corresponds to a dark-energy dominated universe, that super-accelerates \cite{30}. However, similarly to the previous cases, the stable manifolds of I and J are 1D or 2D respectively, and thus almost all orbits of the cosmological system cannot be attracted by them at late times. In addition, they cannot lead to $0 < \Omega_\phi < 1$. Therefore, an exponential potential and a power-law-dependent dark-matter particle mass, cannot solve the coincidence problem.

\section*{V. CONCLUSIONS}

In the present work we investigated the phantom cosmological scenario, with varying-mass dark-matter particles due to the interaction between dark-matter and dark-energy sectors. In particular, we performed a detailed phase-space analysis of various models, with either exponentially or power-law dependent dark-matter particle mass, in exponential or power-law scalar field potentials. These functions cover a wide range of the possible forms, and they correspond to the cases that can accept a reasonable theoretical justification \cite{13,18,19,20,21}. In each case we extracted the critical points, we determined their stability, and we calculated the basic cosmological observables, namely the total equation-of-state parameter $w_{\text{tot}}$ and $\Omega_{\text{DE}}$ (attributed to the phantom field). Our basic goal was to examine whether there exist late-time attractors, corresponding to accelerating universe and possessing $\Omega_{\text{DE}}/\Omega_{\text{DM}} \approx O(1)$, thus satisfying the basic observational requirements.

Apart from the case of an exponential potential with an exponentially-dependent dark-matter particle mass, which possesses a relevant late-time (phantom) attractor, in all the other models we found that physical, well-motivated solutions have a very small chance to attract the universe at late times. In addition, in all the examined cases, solutions having $\Omega_{\text{DE}}/\Omega_{\text{DM}} \approx O(1)$ are not relevant attractors at late times. Therefore, summarizing, the coincidence problem cannot be solved or even alleviated in varying-mass dark matter particles models in the framework of phantom cosmology, in a radical contrast with the corresponding quintessence case \cite{18,20,21}. This conclusion agrees with that of \cite{9}, that interacting phantom cosmology cannot solve the coincidence problem. It seems that interacting phantom cosmology, either directly or through the dependence of the dark-matter particle mass, cannot fulfill the basic requirements that led to its construction, that is to provide stable accelerating late-time solutions which can solve the coincidence problem. An alternative direction could be to consider a specially constructed potential or dark-matter particle mass in order to solve the coincidence problem, but this would imply significant loss of simplicity, gener-
alinity, and theoretical justification of the model.

The aforementioned conclusion has been extracted by the negative-kinetic-energy realization of phantom, which does not cover the whole class of phantom models. However, since it is a qualitative statement it should intuitively be robust for general phantom scenarios, too. Therefore, phantom cosmology with varying-mass dark matter particles cannot easily act as a successful candidate to describe dark energy.

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