Mathematical modeling of multilayer road surfaces

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Abstract. The paper is devoted to the mathematical study of elastic states and deformations of multilayer road surfaces under the force of water. The filtration coefficient in the multilayer strip is piecewise constant. Young's modulus and Poisson's ratio are given in each band. The problem is reduced to a planar boundary value problem for a system of partial differential equations. A method of constructing solutions to this problem in the form of almost-periodic Bohr functions is investigated. A computational experiment was carried out. Graphs of the required mechanical parameters are constructed, their convergence to the boundary values is shown. The Maple software package was used to build the solutions.

The constructed algorithm can be used in solving problems in continuum mechanics.

1. Introduction
When solving plane problems for heterogeneous media in the theory of elasticity, filtration theory, diffusivity theory, thermal conductivity, electrodynamics and magneto dynamics in the case when the region is an l-layer band, boundary value problems for systems of differential equations with boundary conditions given both on the band boundary and on the gluing line are solved [1, 2]. It is especially important to solve such problems in the study of deformations and elastic states of multilayer road surfaces [3]. Due to the porosity of the road surface, liquid can be filtered through it [4]. Under the influence of water, the strength properties of the material can change, as a result of which it can collapse. This is typical of any porous materials. For example, Boskovich D., Clark P. and others showed in [5] that in the development of coal seams for methane production, injection under water pressure with subsequent rapid pumping is used. When water is injected into the well, the strength properties of the coal can change, as a result, the coal can collapse and clog the cleavages, making it difficult to extract gas from the coal seams.

In many textbooks on mathematical physics, analytical solutions to boundary value problems for systems of partial differential equations are written using Fourier integrals, which entails great computational difficulties. The representation of solutions in the form of absolutely convergent Fourier series greatly simplifies the task and makes the further process of computer simulation quite simple. It is worth noting that if a periodic load is applied to a region having a periodic structure, the stress-strain state will also be determined by periodic functions. However, in practice, it is quite difficult to create a periodic load on the body. Then the load will be expressed in the general case by almost-periodic functions (Bohr-Fourier series), the use of which for the study of the stress state of a multilayer elastic band will be devoted to this article.

The apparatus of almost-periodic functions and generalized discrete Fourier transform previously proposed in this paper was used by the author to solve other problems [1, 6], as well as by other authors [7, 8].
2. General statement of the boundary value problem for a system of differential equations

The General mathematical formulation of the problem has the following form. It is required to find functions $U_{km}(x, y)$ such that they satisfy a system of differential equations:

$$
\sum_{k=1}^{N} \left( a_{y}^{(m)} \frac{\partial^2 U_{km}(x, y)}{\partial x^2} + b_{y}^{(m)} \frac{\partial^2 U_{km}(x, y)}{\partial y^2} + c_{y}^{(m)} \frac{\partial U_{km}(x, y)}{\partial x} + d_{y}^{(m)} \frac{\partial U_{km}(x, y)}{\partial y} + e_{y}^{(m)} U_{km}(x, y) \right) = F_{km}(x, y), \quad j = \overline{1, N}, \quad m = \overline{1, l}.
$$

in each of the $m$-th band (figure 1): $-\infty < x < +\infty, \quad a_{m-1} < y < a_{m}, \quad m = \overline{1, l}$.

![Figure 1. Example l-layer strip.](image)

The boundary conditions must be set at the strip boundary, and the gluing conditions must be set at the media interface lines. For example, for a two-layer strip $-\infty < x < +\infty, \quad -a < y < 0, \quad 0 < y < a$, these conditions might look like this:

$$
\begin{align*}
U_{km}(x, y) & \bigg|_{y=a} = \Phi_{km}(x), \\
U_{km}(x, y) & \bigg|_{y=a} = \Psi_{km}(x), \\
U_{x1}(x, y) & \bigg|_{y=0} = U_{x2}(x, y) \bigg|_{y=0}, \\
\frac{\partial U_{x1}(x, y)}{\partial y} & \bigg|_{y=0} = \frac{\partial U_{x2}(x, y)}{\partial y} \bigg|_{y=0}, \\
k & = \overline{1, N}, \quad m = \overline{1, 2}.
\end{align*}
$$

We will construct almost-periodic in the sense of Bohr solutions on each line $y = a_m$, using the generalized discrete Fourier transform, considered in [9].

We assume that the functions $\Phi_{km}, \Psi_{km}$ and $U_{km}, F_{km}$ are represented as absolutely convergent series

$$
\begin{align*}
\Phi_{km}(x) & = \sum_{i=0}^{\infty} \phi_{km}(\lambda_i) e^{i\lambda_i x}, \\
\Psi_{km}(x) & = \sum_{n=0}^{\infty} \psi_{km}(\lambda_n) e^{i\lambda_n x}.
\end{align*}
$$
\[
F_{jm}(x, y) = \sum_{n=1}^{\infty} f_{jm}(\lambda_n, y)e^{i\lambda_n x},
\]
\[
U_{ln}(x, y) = \sum_{m=1}^{\infty} u_{lm}(\lambda_n, y)e^{i\lambda_n x}.
\]

This means that \( \Phi_{kn}, \Psi_{kn} \in \Pi^1_W \), \( U_{ln}, F_{jm} \in \Pi^2_W \), where \( \Pi^1_W \) - set of almost-periodic functions of one variable of the form \( A(x) = \sum_{n=1}^{\infty} a_n e^{i\lambda_n x}, \sum_{n=1}^{\infty} |a_n| < \infty, \{\lambda_n\} \in R \), \( \Pi^2_W \) - set of almost-periodic functions of one variable of the form \( A(x, y) = \sum_{n=1}^{\infty} a_n(y)e^{i\lambda_n x} \) with \( y \in [a, b] \). A more detailed description of almost-periodic functions and their properties is given in [10,11].

The unknowns in formulas (3) are the functions \( U_{ln}(x, y) \) that and are from boundary conditions and coupling conditions.

Let us act as an operator \( W_0^{-1} \) on the equations of the system (1).

This operator has the following properties. If the function \( A(x) \) is differentiable and belongs to the set \( \Pi^1_W \) together with its derivatives up to order \( k \), then \( W_0^{-1} \frac{d^k A(x)}{dx^k} = (i\lambda_n)^k a(\lambda_n) \). If the coefficients \( a_n \) depend on \( y \), that is \( a_n = a(\lambda_n, y) \), and there is a sequence of numbers \( \{\alpha_n\}, \sum_{n=1}^{\infty} |\alpha_n| < \infty \) such that the condition \( |a_n(\lambda_n, y)| \leq \alpha_n \) for any \( y \) of the segments \([a, b]\) is satisfied, then the function \( A(x, y) \) is almost periodic in the parameter \( x \) \((A(x, y) \in \Pi^1_W (y \in [a, b])\)) and it is true

\[
W_0^{-1} \frac{\partial^\rho A(x, y)}{\partial x^\rho} = (i\lambda_n)^\rho a(\lambda_n, y),
\]

\[
W_0^{-1} \frac{\partial^\rho A(x, y)}{\partial y^\rho} = \frac{d^\rho a(\lambda_n, y)}{dy^\rho}.
\]

Using the properties of this operator we move from the system of partial differential equations to a system of ordinary differential equations for each \( n \) with respect to a function \( u_{ln} = u_{lm}(\lambda_n, y) \), where \( \lambda_n \) is a parameter:

\[
\sum_{k=1}^{N} \left( d_{k}^{(m)} \frac{d^2 u_{lm}}{dy^2} + b_{k}^{(m)} \frac{du_{lm}}{dy} - 2\lambda^2 \frac{\partial_c^{(m)}}{\partial y} u_{lm} + d_{k}^{(m)} \cdot i\lambda_n \cdot u_{lm} + c_{k}^{(m)} u_{lm} \right) = f_{jm}(\lambda_n, y).
\]

The order of the system is not higher \( 2n \). The solution to this system will be

\[
\sum_{q=1}^{2N} P_{qlm}(\lambda_n) x_{qln}(\lambda_n, y) + \tilde{x}_{qln}(\lambda_n, y),
\]

where \( P_{qlm}(\lambda_n) \) is constant at fixed \( q, k, m, \lambda_n \). They are found from the boundary conditions and gluing conditions. To determine these constants, a finite system of linear algebraic equations is obtained.

3. Construction of a mathematical algorithm for studying the elastic-deformed state of a multilayer strip under the force of water

Consider the following problem. Similar problems arise in the study of elastic states and deformations of road surfaces under the force of water.
There is a stationary filtration of some liquid in a homogeneous isotropic porous region representing \( l \) bands: the first \(( m = 1 )\): \(- \infty < x < + \infty , \ a_0 \leq y \leq a_1 \); the second \(( m = 2 )\): \(- \infty < x < + \infty , \ a_1 \leq y \leq a_2 \), etc.

The values of normal and tangential stresses, as well as the values of filtration rate potentials, are known at the external boundaries \( y = a_0 , y = a_1 \) of the region:

\[
\sigma_y^{(1)} (x, a_0) = F_1 (x), \quad \sigma_y^{(1)} (x, a_1) = G_1 (x),
\]

\[
\tau_{xy}^{(1)} (x, a_0) = F_2 (x), \quad \tau_{xy}^{(1)} (x, a_1) = G_2 (x),
\]

\[
\phi^{(1)} (x, a_0) = F_3 (x), \quad \phi^{(1)} (x, a_1) = G_3 (x),
\]

where the functions \( F_j (x), G_j (x) \in \mathbb{P}_w \), \( j = 1, 2, 3 \) and have the structure

\[
F_j (x) = \sum_{n=1}^{\infty} f_j (\lambda_n) e^{j_n x}, \quad G_j (x) = \sum_{n=1}^{\infty} g_j (\lambda_n) e^{j_n x}.
\]

where \( \{ \lambda_n \}_{n=1}^{\infty} \) is a countable set of real numbers that have no limit point at zero.

At the boundaries of the media \( y = a_m , m = 1, \ldots, l - 1 \) there are conditions of rigid coupling:

\[
\sigma_y^{(m)} (x, a_m) = \sigma_y^{(m+1)} (x, a_m),
\]

\[
\tau_{xy}^{(m)} (x, a_m) = \tau_{xy}^{(m+1)} (x, a_m),
\]

\[
w^{(m)} (x, a_m) = w^{(m+1)} (x, a_m),
\]

\[
v^{(m)} (x, a_m) = v^{(m+1)} (x, a_m),
\]

\[
\phi^{(m)} (x, a_m) = \frac{\phi^{(m+1)} (x, a_m)}{k_m},
\]

\[
\frac{\partial \phi^{(m)}}{\partial y} (x, a_m) = \frac{\partial \phi^{(m+1)}}{\partial y} (x, a_m).
\]

where \( k_m \) are the media filtration coefficients. The coefficients of the filter media are permanent.

Displacements \( w^{(m)}, v^{(m)} \) are expressed in a known way in terms of stresses \( \sigma_y^{(m)} (x, y), \quad \sigma_{xy}^{(m)} (x, y) \), \( \tau_{xy}^{(m)} (x, y) \) [12]. For each layer the coefficient of elasticity and Poisson's ratio are given \( E_m, \nu_m \).

Find the functions of the potential of the filtration rate \( \phi^{(m)} (x, y) \) of the liquid acting in each band of the porous region, as well as the voltage \( \sigma_y^{(m)} (x, y), \quad \sigma_{xy}^{(m)} (x, y), \quad \tau_{xy}^{(m)} (x, y), \quad m = 1, l \).

The problem is reduced to solving a system of differential equations.
\[
\frac{\partial^2 \sigma_{x}^{(m)}}{\partial x^2} + \frac{\partial^2 \sigma_{y}^{(m)}}{\partial y^2} + \frac{\partial^2 \sigma_{z}^{(m)}}{\partial x^2} + \frac{\partial^2 \sigma_{z}^{(m)}}{\partial y^2} = 0,
\]
\[
\frac{\partial \tau_{xy}^{(m)}}{\partial x} + \frac{\partial \tau_{xy}^{(m)}}{\partial y} + \frac{\partial \phi^{(m)}}{\partial y} - \lambda^{(m)} = 0,
\]
\[
\frac{\partial \tau_{xy}^{(m)}}{\partial x} + \frac{\partial \sigma_{y}^{(m)}}{\partial y} + \frac{\partial \phi^{(m)}}{\partial y} - \gamma^{(m)} = 0,
\]
\[
\frac{\partial^2 \phi^{(m)}}{\partial x^2} + \frac{\partial^2 \phi^{(m)}}{\partial y^2} = 0,
\]

where \( \gamma^{(m)}, \gamma_0^{(m)} \) - volume weights of a unit of liquid and a unit of a porous body with water enclosed in it. The system of equations (4) for this case will take the form for each \( n \):

\[
\begin{cases}
- \lambda_n^2 A_n^{(m)} + \frac{d^2 A_n^{(m)}}{dy^2} - \lambda_n^2 B_n^{(m)} + \frac{d^2 B_n^{(m)}}{dy^2} = 0, \\
\dot{\lambda}_n A_n^{(m)} + \frac{d C_n^{(m)}}{dy} + \dot{\lambda}_n \cdot q_n^{(m)} D_n^{(m)} = 0, \\
\dot{\lambda}_n C_n^{(m)} + \frac{d B_n^{(m)}}{dy} + q_n^{(m)} \cdot D_n^{(m)} - \gamma_n^{(m)} = 0, \\
- \lambda_n^2 D_n^{(m)} + \frac{d^2 D_n^{(m)}}{dy^2} = 0, \quad m = 1, I,
\end{cases}
\]

\( A_n^{(m)} = A_n^{(m)}(y), \quad B_n^{(m)} = B_n^{(m)}(y), \quad C_n^{(m)} = C_n^{(m)}(y), \quad D_n^{(m)} = D_n^{(m)}(y), \quad q_n^{(m)} = \frac{\gamma_n^{(m)}}{k_n} \).

Solving the system (10), we obtain:

\[
\begin{align*}
A_n^{(m)}(y) &= \sum_{k=1,2} \left( b_{1k}^{(m)} + \frac{(-1)^k \cdot 2}{l_n} \cdot b_{2k}^{(m)} - b_{2k}^{(m)} y - 2 q_n^{(m)} d_k^{(m)} \right) e^{(-1)^k \cdot l_n \cdot y}, \\
B_n^{(m)}(y) &= \sum_{k=1,2} \left( b_{1k}^{(m)} + b_{2k}^{(m)} \right) e^{(-1)^k \cdot l_n \cdot y}, \\
C_n^{(m)}(y) &= - \frac{\gamma_n^{(m)}}{l_n} + i \sum_{k=1,2} \left[ \frac{1}{l_n} \cdot b_{2k}^{(m)} + (-1)^{k+1} \left( b_{1k}^{(m)} + b_{2k}^{(m)} y + q_n^{(m)} d_k^{(m)} \right) \right] e^{(-1)^k \cdot l_n \cdot y}, \\
D_n^{(m)}(y) &= \sum_{k=1,2} d_k^{(m)} \cdot e^{(-1)^k \cdot l_n \cdot y}.
\end{align*}
\]

The coefficients \( b_{1k}^{(m)}, b_{2k}^{(m)}, d_k^{(m)}, \quad k = 1, 2, \) for each value of the variable \( \lambda \neq 0 \) are derived from a system of linear algebraic equations, which is a consequence of the boundary conditions and coupling conditions. The determinant of this system is nonzero at any \( \lambda \neq 0 \). In General, the system is quite cumbersome, so let's consider a special case.
4. Numerical experiment

We solve the above problem for the case when the construction is a two-layer strip: $-\infty < x < +\infty$, $-1 \leq y \leq 0$, $0 \leq y \leq 1$. Dimensionless constants are given for each band $E_1 = 1.2$, $E_2 = 1$, $\nu_1 = 1.2$, $\nu_2 = 1$, $k_1 = 1$, $k_2 = 10$. $\gamma^{(1)} = 1$, $\gamma^{(2)} = 1$, $\gamma_0^{(1)} = 2$, $\gamma_0^{(2)} = 2.5$.

The boundary conditions at the external boundaries of heterogeneous media have the form:

$$\begin{align*}
\sigma_y^{(1)}(x, -1) &= F_1(x) = 4\cos(x) + 2\sin(x), \\
\tau_{xy}^{(1)}(x, -1) &= F_2(x) = 2\cos(x) + 2\sin(x), \\
\varphi^{(1)}(x, -1) &= F_3(x) = -3\cos(x) + 4\sin(x), \\
\sigma_y^{(2)}(x, 1) &= G_1(x) = 2\cos(x) - 4\sin(x), \\
\tau_{xy}^{(2)}(x, 1) &= G_2(x) = 2\cos(x) - 3\sin(x), \\
\varphi^{(2)}(x, 1) &= G_3(x) = \cos(x) + 2\sin(x).
\end{align*}$$

(12)

The software module is developed in the Maple environment. Graphs of tangent and normal stresses, as well as functions of filtration rate potentials are presented in figure 2(a, b), 3(a, b), 4(a, b).

Figure 2a. Stress $\sigma_y^{(1)}$.

Figure 2b. Stress $\sigma_y^{(2)}$. 

Figure 3a. Stress $\tau_{xy}^{(1)}$.

Figure 3b. Stress $\tau_{xy}^{(2)}$.

Figure 4a. Function of potentials $\varphi^{(1)}$. 
Figure 4b. Function of potentials $\varphi^{(2)}$.

Solid lines represent these mechanical parameters on the boundaries and clutch lines: $\sigma_x^{(1)}(x,0)$, $\sigma_y^{(1)}(x,0)$, $\sigma_y^{(2)}(x,1)$, $\tau_x^{(1)}(x,0)$, $\tau_y^{(1)}(x,-1)$, $\tau_x^{(2)}(x,0)$, $\tau_x^{(2)}(x,1)$, $\varphi^{(1)}(x,0)$, $\varphi^{(1)}(x,-1)$, $\varphi^{(2)}(x,0)$, $\varphi^{(2)}(x,1)$, point lines inside areas: the parameter $y$ takes values inside the first layer $-0.2$, $-0.4$, $-0.6$, $-0.8$, inside the second layer $0.2$, $0.4$, $0.6$, $0.8$. It is possible to trace the approximation of the constructed functions to the boundary conditions.

Test cases confirm the effectiveness of this method. All obtained series converge absolutely and uniformly with respect to the variable $x$ if the series $\sum \frac{1}{|\rho_n|}$ converges.

5. References
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