Distribution of Gaussian Entanglement in Linear Optical Systems

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Entanglement is an essential ingredient for building a quantum network that can have many applications. Understanding how entanglement is distributed in a network is a crucial step to move forward. Here we study the conservation and distribution of Gaussian entanglement in a linear network using a new quantifier for bipartite entanglement. We show that the entanglement can be distributed through a beam-splitter in the same way as the transmittance and the reflectance. The requirements on the entangled states and the type of networks to satisfy this relation are presented explicitly. Our results provide a new quantification for quantum entanglement and further insights into the structure of entanglement in a network.

I. Introduction

Continuous-variables (CV) quantum systems have attracted tremendous interest due to its intriguing properties as well as the remarkable possibilities it can provide in areas such as quantum communication and quantum computing [1–5]. As the representative of CV systems, Gaussian states have always been the center of attention for the fact that they are easy to produce [6] and convenient to operate experimentally [7].

Many studies have been reported about witnessing and even quantifying Gaussian entanglement [8–15]. Positive partial transpose (PPT), which gives a direct criterion of separability [10–13], is widely used to determine the entanglement of bipartite Gaussian systems. Negativity N and logarithmic negativity [14, 15] are constructed to quantify bipartite entanglement based on PPT criterion. In a multipartite Gaussian system, structures of how quantum correlations are shared among many parties, such as monogamy property [16], have been studied [17–20]. In particular, monogamy inequalities have been proved using entanglement quantifications related to negativity and logarithmic negativity [18, 19]. For example, for a generic tripartite system ρ_{ABC}, the following inequality is satisfied:

\[ \mathcal{E}(A, BC) \geq \mathcal{E}(A, B) + \mathcal{E}(A, C), \tag{1} \]

where \( \mathcal{E}(A, BC) \) represents the one-mode-vs-two-mode entanglement among the bipartition \( A : BC \) and \( \mathcal{E}(A, B) \ (\mathcal{E}(A, C)) \) is the bipartite entanglement of the reduced system after tracing the party \( C \ (B) \). Within the monogamy inequality, the summation of entanglement is proposed. However, it remains an open question which measure can be the 'proper candidate for approaching the task of quantifying entanglement sharing in CV system' [18]. The difficulties lie in the restriction of each measure. For example, Eq. (1) can be violated with the logarithmic negativity measure (examples in Sec.III C), which is used widely as the entanglement measure in CV systems [9, 15].

Gaussian entanglement can be generated in linear-optical networks consisting beam-splitters (BS) by transforming non-classical non-entangled Gaussian states to entangled ones [21–25]. BS linear networks have been studied in demonstrated distributed quantum metrology tasks [21], quantum computing supremacy [26] and quantum communication [27]. Recently, different conservation relations of single-mode non-classicality and two-mode entanglement between the input states and the output states have been reported [28, 29]. However, such a conservation or distribution of entanglement in optical systems has not been studied, for example, in linear-optical systems with two or more BSs.

In this paper, we study the distribution pattern of Gaussian entanglement in a linear network consisting of multiple BSs with single-mode Gaussian states being the inputs. First, we investigate the difference of single-mode nonclassicalities before and after going through a BS, termed as 'residual non-classicality', in a linear network. It has been shown that the residual nonclassicality can quantify two-mode entanglement and form a conservation relation of nonclassicality and entanglement in a single BS system [28, 30]. Thus, it is an interesting question whether the residual nonclassicality could be extended to a linear optical system with multiple modes. However, our results show that it does not reveal how entanglement is distributed when the system becomes more complex. Therefore, we propose a new bipartite entanglement quantifier \( \xi \), which is defined via logarithmic negativity, and study entanglement distribution using this new quantifier. Our results show that bipartite entanglement distributed through a BS can follow the same relation as how light is distributed at the BS. Based on this relation, we obtain a monogamy equality of a tripartite Gaussian state in a network of two BSs, where the tripartite state is generated from a two-mode entangled state mixed with a single-mode quantum state at a BS. We identify the conditions of the input states in order for the equality to hold. The distribution of entanglement is further extended to more complex networks using the new quantifier \( \xi \).

The paper is organized as follows. In Sec.II, we introduce the properties of Gaussian states in a BS optical system as well as three definitions of entanglement quantification. In Sec.III, we derive the conservation relation and distribution pattern of entanglement on the base of the quantifier \( \xi \) and connect

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II. Gaussian state entanglement preliminaries

In this section, we first introduce some basics of single-mode Gaussian states as well as nonclassicality in terms of the characteristic function and the covariance matrix. Then we discuss entanglement quantification of two-mode Gaussian states using logarithmic negativity, residual nonclassicality, and a new entanglement quantifier.

A. Gaussian state and nonclassicality

A quantum state \( \rho \) can be completely described by the characteristic function [31]

\[
\chi(x) = \text{Tr}[\rho D(x)],
\]

where \( D(x) = \exp[-i \sum_{k=1}^{n} (q_k \hat{X}_k + p_k \hat{P}_k)] \), \( x^T = (q_1, p_1, \ldots, q_n, p_n) \) are the space-momentum operators.

A Gaussian state [13, 31] is defined such that its characteristic function is Gaussian (Appendix A). For a single-mode Gaussian state, its characteristic function is given by

\[
\chi(\alpha^*, \alpha) = \exp(-\frac{1}{2} x'^{\dagger} V x'),
\]

with

\[
x' = (\alpha^*, \alpha), \quad V = \begin{bmatrix} a & b \\ b^* & a \end{bmatrix}.
\]

In the above equations, \( V \) is the covariance matrix with \( a^2 - |b|^2 \geq 1/4 \) from the uncertainty principle. Instead of quadrature field variables \( q_i, p_i \), we use bosonic field expressions \((\alpha^*, \alpha)\) here (see Appendix A for transformation relations).

The nonclassicality of a quantum state \( \rho \) can be quantified by the nonclassical depth \( \tau \), which, for a Gaussian state, is related to [32, 33] the minimum eigenvalue \( \lambda = a - |b| \) of its covariance matrix \( V \) as \( \tau = \max \{ 0, 1/2 - \lambda \} \). A quantum state is nonclassical if \( \tau > 0 \) or \( \lambda < 1/2 \). Alternatively, we can consider the quantity

\[
N = -\log_2(2\lambda)
\]

as the nonclassicality of a single-mode Gaussian state [28]. In contrast to \( \lambda \), \( \Lambda = a + |b| \) is the maximum eigenvalue which will be used later.

B. Entanglement quantification of two-mode Gaussian states

We consider a lossless BS with two single-mode Gaussian fields being the inputs as shown in Fig. 1. We denote \( \cos^2 \theta \) as the transmittance and \( \varphi \) as the phase shift of the BS. For two separable single-mode Gaussian states, their combined characteristic function is given by

\[
\chi_{in}(\alpha_1^*, \alpha_1, \alpha_2^*, \alpha_2) = \exp(-\frac{1}{2} x'^{\dagger} V_{in} x'),
\]

where \( x' = (\alpha_1^*, \alpha_1, \alpha_2^*, \alpha_2) \), and

\[
V_{in} = \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}, \quad A = \begin{bmatrix} a & b \\ b^* & a \end{bmatrix}, \quad B = \begin{bmatrix} c & d \\ d^* & c \end{bmatrix}.
\]

The covariance matrix of the output field, \( V_{out} \), is derived by a unitary transformation of \( V_{in} \) as

\[
V_{out} = U^{\dagger}(\theta, \varphi)V_{in} U(\theta, \varphi).
\]

Here we give the exact form of \( U(\theta, \varphi) \),

\[
\begin{pmatrix}
\cos \theta & 0 & \sin \theta e^{i\varphi} & 0 \\
0 & \cos \theta & 0 & \sin \theta e^{-i\varphi} \\
-\sin \theta e^{i\varphi} & 0 & \cos \theta & 0 \\
0 & -\sin \theta e^{-i\varphi} & 0 & \cos \theta
\end{pmatrix}.
\]

B1. Logarithmic negativity \( E_N \)

The entanglement of the two-mode system \( \rho_{out} \) after the BS can be evaluated using the logarithmic negativity \( E_N = \log_2 \| \rho_{out}^{T_B} \| \), where \( \| R \| \) denotes the trace norm \( \text{Tr} \sqrt{R R^{\dagger}} \) and \( \rho^{T_B} \) is the partial transpose of a state. The condition for two-mode Gaussian state to be entangled is \( E_N > 0 \).

For Gaussian states, entanglement can be determined totally by its covariance matrix (CM) instead of the density matrix [34]. Logarithmic negativity can be calculated from CM directly, which is much more convenient to manipulate. The
logarithmic negativity for the output Gaussian state $\rho_{A_1 B_1}$ in Fig. 1 is given by [9]

$$E_N = \max \left\{ 0, -\frac{1}{2} \log_2(S - \sqrt{S^2 - 16 \det[V_{\text{out}}]}) \right\}, \quad (8)$$

where $S = 2 \det[A_1] + 2 \det[B_1] - 4 \det[(AB)_1]$, and $A_1, B_1, (AB)_1, (AB)_1^\dagger$ are two-dimensional matrices coming from the output covariance matrix $V_{\text{out}} = \begin{bmatrix} A_1 & (AB)_1 \\ (AB)_1^\dagger & B_1 \end{bmatrix}$ (see details in Appendix C).

**B2. Residual nonclassicality $S_N$**

Alternatively, the degree of entanglement of the two-mode Gaussian state can be quantified via the difference between the nonclassicalities before and after the BS [28]. This quantity is denoted as residual nonclassicality $S_N \equiv N_{\text{in}} - N_{\text{out}}$, where the subscripts in and out denote the total nonclassicality of the input modes and the output modes, respectively. With the definition in Eq. (4), we obtain

$$S_N = N_A + N_B - N_{A_1} - N_{B_1} = \log_2 \frac{\lambda_{A_1} \lambda_{B_1}}{\lambda_A \lambda_B}, \quad (9)$$

where $\lambda_{A_1} = \cos^2 \theta \cdot \lambda_A + \sin^2 \theta \cdot \lambda_B$ and $\lambda_{B_1} = \sin^2 \theta \cdot \lambda_A + \cos^2 \theta \cdot \lambda_B$ (Appendix C). Since BS does not create extra nonclassicality as a linear optical device, $S_N$ can be related to the degree of entanglement. In fact, it is shown that [28]

$$S_N > 0 \iff E_N > 0 \quad (10)$$

for certain input Gaussian states (see Appendix C for detailed proof) with certain constraints.

Those constraints, which are explained in details in Appendix C, include

(a) both of $\rho_A$ or $\rho_B$ are pure states,

(b) $\varphi = \frac{1}{2}[\arg(b) - \arg(d)], \quad (11)$

**B3. Entanglement quantifier $\xi$**

From Eq. (8), it can be derived that $E_N > 0$ is equivalent to

$$\xi \equiv S - \frac{1}{2} - 8 \det[V_{\text{out}}] > 0, \quad (12)$$

where the expression for $S$ can be calculated as

$$S = \frac{1}{2||p_{\text{out}}||_1^2} + 8 \det[V_{\text{out}}] \cdot ||p_{\text{out}}||_1^2. \quad (13)$$

The purity of two-mode Gaussian state follows $||p_{\text{out}}||_1 = \frac{1}{4}||V_{\text{out}}||_2$. Together with Eq. (12,13) we obtain

$$\xi = \frac{1}{2} \left( 1 - \frac{1}{||p_{\text{out}}||_1^2} \right) \left( \frac{||p_{\text{out}}||_1^2}{||p_{\text{out}}||_1^2} - 1 \right). \quad (14)$$

The above expression shows that $\xi > 0$ is equivalent to the PPT criterion in terms of determining whether entanglement exists. This is a sufficient and necessary condition for an arbitrary two-mode Gaussian state to be entangled [5, 10]. In the following, we show that this new quantifier can lead to an entanglement conservation and a distribution relation in a linear network.

**III. Distribution of entanglement in linear optical systems**

**A. Two BS system**

In this section, we investigate how bipartite entanglement is distributed in a linear system with two beam-splitters as shown in Fig. 2. Two single-mode Gaussian states are mixed at the first BS to generate a bipartite entangled state $\rho_{A_1 B_1}$. After that, one of the output mode $\rho_{B_i}$ is mixed with the third single-mode input Gaussian state $\rho_C$ at the second BS. The final output state is a tripartite Gaussian state $\rho_{\text{out}}$ (or $\rho_{A_1 B_1 C_1}$). In general, there are both bipartite entanglement and tripartite entanglement in the system after the two BSs.

**A1. Distribution of entanglement via residual nonclassicality**

An interesting question is to understand how nonclassicality is shared within the system. As described by the caption in Fig. 2, we can apply Eq. (9) at both beam-splitters individually to obtain

$$N_A + N_B = N_{A_1} + N_{B_1} + S_N(A_1, B_1), \quad (15)$$

and

$$N_{B_1} + N_C = N_{B_2} + N_{C_1} + S_N(B_2, C_1). \quad (16)$$

FIG. 2. Based on Fig. 1, the Gaussian state $\rho_{B_1}$ mixes with another Gaussian state $\rho_C$ via $BS_2$, generating $\rho_{B_2 C_1}$. With trace operation $Tr_C$ and $Tr_B$ on $\rho_{B_2 C_1}$, we can obtain $\rho_{B_1}$ and $\rho_{C_1}$, respectively. The whole system of the mode $A$, $B$ and $C$ after the two BSs is represented by $\rho_{\text{out}} = \rho_{A_1 B_1 C_1}$, in which case $\rho_{B_1 C_1} = Tr_A[\rho_{\text{out}}]$. 
The summation of Eq. (15) and Eq. (16) leads to
\[ N_A + N_B + N_C = N_{A_1} + N_{B_2} + N_{C_1} + S_N, \]
with
\[ S_N = S_N(A_1, B_1) + S_N(B_2, C_1), \]
where \( S_N \) quantifies the difference of nonclassicality before and after the two BSs, which is given by
\[
S_N = \log_2 \frac{\lambda_A \lambda_B \lambda_{C_1}}{\lambda_A \lambda_B} + \log_2 \frac{\lambda_B \lambda_{C_1}}{\lambda_B \lambda_C} = \log_2 \frac{\lambda_A \lambda_B \lambda_{C_1}}{\lambda_A \lambda_B \lambda_C},
\]
where the two terms in the first line agree with \( S_N(A_1, B_1) \) and \( S_N(B_2, C_1) \), respectively. \( S_N(B_2, C_1) \) stands for the entanglement of \( \rho_{B_2, C_1} \), while \( S_N(A_1, B_1) \) stands for the entanglement of \( \rho_{A_1, B_1} \) or the entanglement of \( \rho_{A_1, B_2, C_1} \). In order to explore how the bipartite entanglement of \( \rho_{A_1, B_2, C_1} \) is distributed at \( BS_2 \), we need to quantify these two contributions from the entanglement of \( \rho_{A_1, B_2} \), \( \rho_{A_1, C_1} \). However, quantifying the entanglement of \( \rho_{A_1, B_2} \), \( \rho_{A_1, C_1} \) in terms of the residual nonclassicality is not straightforward since both the two bipartite systems are not directly generated from two separable modes. Therefore, we seek for an alternative solution using another quantifier for entanglement, which is \( \xi \) as we introduced in Eq. (14).

**A2. Distribution relation of entanglement via \( \xi \)**

Using the entanglement quantifier \( \xi \), we denote the bipartite entanglement of the states \( \rho_{A_1, B_2} \), \( \rho_{A_1, B_1} \) and \( \rho_{A_1, C_1} \) as \( \xi(A_1, B_1) \), \( \xi(A_1, B_2) \) and \( \xi(A_1, C_1) \), respectively.

We derive a mathematical result based on the expression of \( \xi(A_1, B_1) \), \( \xi(A_1, B_2) \) and \( \xi(A_1, C_1) \). The three quantities satisfy the following relation (see Appendix D for details, the calculation tricks are introduced in Appendix B)
\[
\xi(A_1, B_2) = \cos^2 \theta_2 \cdot \xi(A_1, B_1), \\
\xi(A_1, C_1) = \sin^2 \theta_2 \cdot \xi(A_1, B_1),
\]
where \( \cos^2 \theta_2 \) is the transmittance of \( BS_2 \). With the new entanglement quantifier \( \xi \), it exactly shows that entanglement could be distributed by a BS in the way how transmittance and reflectance are distributed, which are \( \cos^2 \theta \) and \( \sin^2 \theta \), respectively.

We discover the type of input states and the requirements of a linear network in order for the above distribution relation to hold. These constraints are summarized as follows:

(a) \( \rho_{A_1, B}, \rho_{A_1, C} \) should all be pure Gaussian states.

(b) The phase shift of \( BS_2 \), \( \varphi_2 = \frac{1}{2} [\text{arg}(d) - \text{arg}(f)] \),

where \( f \) comes from CM \( V_c \).

**B. Monogamy of quantum entanglement**

Considering the tripartite Gaussian state \( \rho_{A_1, B_2, C_1} \), we reveal further a connection between the distribution relation and the monogamy of entanglement.

For the configuration in Fig. 2, the entanglement of \( \rho_{A_1, B_2, C_1} \), denoted as \( \xi(A_1, B_2, C_1) \), is related to the entanglement of \( \rho_{A_1, B_1} \), denoted as \( \xi(A_1, B_1) \). We prove that (Appendix E)
\[
\| \rho_{A_1, B_2, C_1} \| = \| \rho_{A_1, B_1} \|,
\]
and after the two BSs, which is given by
\[
\| \rho_{A_1, B_2, C_1} \| = \| \rho_{A_1, B_1} \|,
\]
based on which, both the entanglement quantifiers \( \xi \) defined in Eq. (14) and logarithmic negativity \( E_N \) defined in Eq. (8) provide that
\[
\xi(A_1, B_1) = \xi(A_1, B_2) + \xi(A_1, C_1),
\]
which means
\[
\xi(A_1, B_2, C_1) = \xi(A_1, B_2) + \xi(A_1, C_1).
\]
On the other hand, monogamy of quantum entanglement is expressed as [17, 20]
\[
\xi(A, B) > \xi(A, BC) + \xi(A, C).
\]

It implies that the conservation relation in Eq. (24) is actually a special case when monogamy inequality becomes an equality, which means in the most case, there is no such a conservation relation. Thus in Eq. (20), conditions and constraints must be employed to make sure such an equality holds. Besides of the trivial constraint about the phase shift, the pure state condition is the only thing left, which we believe to be of enough generality to be accepted. For the case where one of the constraints is not satisfied, to be specific, \( \rho_C \) is not a pure state, the conservation equality is destroyed and becomes (according to the proof procedures in Appendix D)
\[
\xi(A_1, B_1) > \xi(A_1, B_2) + \xi(A_1, C_1),
\]
which agrees with the monogamy inequality in Eq. (25).

**C. Comparison between \( E_N \) and \( \xi \)**

The most important point of the quantifier \( \xi \) lies in the fact it provides a precise \( \cos^2 \theta \cdot \sin^2 \theta \) distribution pattern. A conservation relation is also naturally satisfied. However, these doesn’t happen when using the measure of logarithmic negativity \( E_N \).
FIG. 3. a. The ratio of $E_N(A_1, B_1)/E_N(A_1, B_2C_1)$ versus $\theta_2$. b. $\xi(A_1, B_2)$ and $\xi(A_1, C_1)$ versus $\theta_2$. $\xi(A_1, B_2)$ acts in a $\cos^2 \theta_2$ relation while $\xi(A_1, C_1)$ presents a $\sin^2 \theta_2$ dependence. c. The ratio of $[\xi(A_1, B_2) + \xi(A_1, C_1)]/\xi(A_1, B_1)$ versus $\theta_2$. d. $E_N(A_1, B_1)$ and the summation of $E_N(A_1, B_2) + E_N(A_1, C_1)$ versus $\theta_2$.

In Fig. 3, we show the value of $E_N$ and $\xi$ versus $\theta_2$ ($BS_2$). The results describe the configuration in Fig. 2, where all the parameters are given proper numerical values: For the CM of three pure Gaussian states, $a = \sqrt{0.34}$, $b = 0.3$, $c = \sqrt{0.5}$, $d = 0.5$, $g = \sqrt{0.74}$, $f = 0.7$. For $BS_1$, $\theta_1 = \pi/4$, $\phi_1 = 0$. For $BS_2$, $\phi_2 = 1$, $\theta_2$ varies from 0 to $\pi$. In Fig. 3a, $E_N(A_1, B_2C_1)$ is the one-mode-vs-two-mode entanglement calculated by the minimum symplectic eigenvalue [35] of the three-mode Gaussian state CM, which is the actually the same thing as Eq. (8). The plot shows clearly that $E_N(A_1, B_1) = E_N(A_1, B_2C_2)$, which agrees with Eq. (21). In Fig. 3b, $\xi(A_1, B_2)$ and $\xi(A_1, C_1)$, which are calculated through Eq. (12), vary with $\theta_2$, present a $\cos^2 \theta_2$,$\sin^2 \theta_2$ dependence to $\theta_2$, respectively. This entanglement distribution pattern is exactly what we get in Eq. (19). Fig 3c shows that $\xi(A_1, B_2) + \xi(A_1, C_1) = \xi(A_1, B_1)$, together with Eq. (22), finally proves the monogamy equality $\xi(A_1, B_2C_1) = \xi(A_1, B_2) + \xi(A_1, C_1)$. In Fig 3d, logarithmic negativity is used as the entanglement measure. It is clear that it violates the monogamy inequality in Eq. (25) and fails in revealing the distribution pattern of entanglement.

D. Complex BS networks

The distribution relation of entanglement in the tripartite system can be extended to multi-partite systems in some more complex linear optical networks.

![Complex BS networks diagram](image)

FIG. 4. Based on Fig. 2, another beam-splitter $BS_3$ and another input state $\rho_D$ are added to the network. The CM of $\rho_D$ is given by $V_D = \begin{bmatrix} h & j \\ j & h \end{bmatrix}$.

We consider a four-mode Gaussian state generated by a three-BS network (Fig. 4). As shown by the figure, three...
BSs are linearly arranged. Detailed calculation (Appendix F) indicates that
\[
\xi(A_1, B_1) = \cos^2 \theta_3 \cdot \xi(A_1, B_3 D_1), \quad \xi(A_1, D_1) = \sin^2 \theta_3 \cdot \xi(A_1, B_2 D_1),
\]
where \(\xi(A_1, B_3 D_1) = \xi(A_1, B_2)\). The distribution relation still works here and it is easy to write down the exact value of entanglement between \(A_1\) and any other states. For \(\xi(A_1, D_1)\), combining Eq. (19) and Eq. (27), it is given that
\[
\xi(A_1, D_1) = \sin^2 \theta_2 \cos^2 \theta_2 \cdot \xi(A_1, B_1). \quad (28)
\]
The summation of all the distribution relation equations provides a conservation equality of entanglement in the four-mode states as well, which is given by
\[
\xi(A_1, B_3) + \xi(A_1, D_1) + \xi(A_1, C_1) = \xi(A_1, B_3 D_1 C_1), \quad (29)
\]
where \(\xi(A_1, B_3 C_1 D_1) = \xi(A_1, B_1)\).

Constraints for the distribution relation to hold in Fig. 4 include:

(b) \(\rho_A, \rho_B, \rho_C, \rho_D\) should all be pure Gaussian states.

(b) The phase shift of \(BS_3\), \(\varphi_3 = \frac{1}{2} [\arg(f) - \arg(j)]\).

(c) The covariance matrix of \(\rho_C\) equals to the one of \(\rho_D\). \quad (30)

As demanded by the last constraint, \(V_C = V_D\), both input states \(C\) and \(D\) can be chosen to be vacuum for simplicity during experimental implementation, which means
\[
V_C = V_D = \begin{bmatrix} 0.5 & 0 \\ 0 & 0.5 \end{bmatrix}. \quad (31)
\]

All the inputs state except for \(\rho_{A_{n}}, \rho_{B_{n}}\) are chosen as vacuum or coherent states. In this case, the distribution relation provides
\[
\xi(A_1, B_n) = \prod_{i=2}^{n} \cos^2 \theta_i \cdot \xi(A_1, B_n C_2 \cdots C_{n-1}), \quad (32)
\]
\[
\xi(A_1, C_k) = \sin^2 \theta_{k+1} \prod_{i=2}^{k} \cos^2 \theta_i \cdot \xi(A_1, B_n C_2 \cdots C_{n-1}), \quad (33)
\]

where \(\xi(A_1, B_n C_2 \cdots C_{n-1}) = \xi(A_1, B_1)\) and \(n > 1, \ k > 1\). Applying the same method as we derive Eq. (29), the conservation relation is given by
\[
\xi(A_1, B_n) + \sum_{i=1}^{n-1} \xi(A_1, C_i) = \xi(A_1, B_n C_2 \cdots C_{n-1}). \quad (33)
\]

IV. Discussion and conclusion

In this paper, we discussed the distribution of Gaussian entanglement in linear-optical networks. A new quantifier of bipartite entanglement is introduced, based on which we show that the monogamy inequality and even an monogamy equality are satisfied. For pure input states, entanglement can be a conserved quantity. Moreover, it obeys the distribution pattern in the same way how transmittance and reflectance are distributed when going through a BS. Such property makes it possible to calculate and even control the entanglement of any two subsystems, which provides us a deeper understanding about monogamy of quantum entanglement and how entanglement is distributed in a quantum network.

The new entanglement quantifier \(\xi\), which is closely related to logarithmic negativity and residual nonclassicality, proves better properties than both measure in monogamy inequality verification and entanglement distribution. It can contribute to the research on residual entanglement [18, 35, 36] \(\mathcal{R}\) given by [18]
\[
\mathcal{R}(A : BC) = C(A : BC) - C(A : B) - C(A : C), \quad (34)
\]
where \(C(i : j)\) is the contangle of subsystem \(i\) and \(j\), defined by squared logarithmic negativity. We compare the deference between the contangle and \(\xi\) measure through Fig. 3c and Fig. 6. It turns out the contangle measure satisfies the monogamy inequality but fails to show the conservation of entanglement. Thus it is better to apply \(\xi\) to redefine the residual entanglement as \(\mathcal{R}(A : BC) = \xi(A : BC) - \xi(A : B) - \xi(A : C)\). This development would help to the research on tripartite entanglement, which is our future interests.

Moreover, with Eq. (14), it implies \(\xi\) can be used beyond CV systems, for example, in determining the entanglement of discontinuous q-bits. \(\xi\) has the potential to be a general entanglement measure which may deserve further studies on its properties, such as monotonicity under local operations and classical communications.
The characteristic function is rewritten by
\[
\chi(\alpha^*, \alpha) = \chi(q, p) = \exp(-\frac{1}{2}x^T \gamma x)
\]
\[
= \exp(-\frac{1}{2} |\alpha^*, \alpha|^2 \begin{bmatrix} 1 & i \\ -i & -1 \end{bmatrix} \gamma \begin{bmatrix} 1 & i \\ -i & -1 \end{bmatrix} \begin{bmatrix} \alpha^* \\ \alpha \end{bmatrix})
\]
\[
= \exp(-\frac{1}{2} x'^T V x').
\]
Therefore, instead of \( x \), under Bosonic field variables vector \( x' = (\alpha^*, \alpha) \), the covariance matrix \( V \) is related to \( \gamma \) by
\[
V = \frac{1}{2} \begin{bmatrix} 1 & i \\ 1 & -i \end{bmatrix} \gamma \begin{bmatrix} 1 & i \\ -i & -1 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} \gamma_{11} + \gamma_{22} & \gamma_{12} & \gamma_{12} & \gamma_{12} \\ \gamma_{12} & \gamma_{11} - \gamma_{22} + 2i\gamma_{12} & \gamma_{12} & \gamma_{12} \\ \gamma_{12} & \gamma_{11} - \gamma_{22} - 2i\gamma_{12} & \gamma_{11} + \gamma_{22} & \gamma_{12} \\ \gamma_{12} & \gamma_{12} & \gamma_{12} & \gamma_{11} + \gamma_{22} \end{bmatrix}
\]
where \( \gamma_{ij} \) has been presented in Eq. (A2).

**Appendix B: Important theorems**

In this paper, all the two-dimensional matrices during the calculation are symmetric matrix, and their diagonal values are equal to each other. Some tricks when manipulating these matrices are frequently used throughout the following text.

We consider two matrices \( A \) and \( B \) given by
\[
A = \begin{bmatrix} a & b \\ b & a \end{bmatrix}, \quad B = \begin{bmatrix} c & d \\ d & c \end{bmatrix}.
\]

(a) Theorem 1. It can be proved that
\[
AB = BA. \tag{B1}
\]

(b) Theorem 2. Given that
\[
|A + B| = (a + c)^2 - (b + d)^2
\]
\[
= a^2 - b^2 + c^2 - d^2 + 2(ac - bd)
\]
where \(|\cdot|\) stands for the determinant of a matrix. Since \( |B|B^{-1} = \begin{bmatrix} c & -d \\ -d & c \end{bmatrix} \), it follows
\[
|A + B| = |A| + |B| + \text{Tr}(AB^{-1})|B|
\]
\[
= |A| + |B| + \text{Tr}(BA^{-1})|A|. \tag{B2}
\]

(c) Theorem 3. It can be proved that
\[
|A + B|(A + B)^{-1} = \begin{bmatrix} a + c & -b - d \\ -b - d & a + c \end{bmatrix}
\]
\[
= \begin{bmatrix} a & -b \\ -b & a \end{bmatrix} + \begin{bmatrix} c & -d \\ -d & c \end{bmatrix} \tag{B3}
\]
\[
= |A|A^{-1} + |B|B^{-1}.
\]
(d) Theorem 4. A, B, C are two-dimensional matrices and A is reversible. Then
\[
\det \begin{bmatrix} A & B \\ B & C \end{bmatrix} = |A| \cdot |C - BA^{-1}B| = |AC - AB^{-1}B| = |AC - BB|. \tag{B4}
\]
where the last step is based on Eq. (B1).

(e) Theorem 5. Define \( A_1 = \cos^2 \theta \cdot A + \sin^2 \theta \cdot B, \)
\( B_1 = \cos^2 \theta \cdot B + \sin^2 \theta \cdot A, \)
and \((AB)_1 = (A - B) \sin \theta \cos \theta,\)
then
\[
A_1B_1 - (AB)_1(AB)_1 = AB. \tag{B5}
\]
Note that \((AB)_1\) is a matrix related to \(A\) and \(B\). As shown later in Appendix C, it denotes the off-diagonal matrix of \(V_{\text{out}}\), which is different from \(AB\), the matrix product of \(A\) and \(B\).

Appendix C: Entanglement and residual nonclassicality of two-mode Gaussian state

Substituting Eq. (7) into Eq. (6),
\[
V_{\text{out}} = \begin{bmatrix} A_1 & (AB)_1 \\ (AB)_1 & B_1 \end{bmatrix}, \tag{C1}
\]
where the matrix \(A_1, B_1, (AB)_1\) are given by
\[
A_1 = \begin{bmatrix} a \cos^2 \theta + c \sin^2 \theta & b \cos^2 \theta + d \sin^2 \theta e^{2i\varphi} \\ b^* \cos^2 \theta + d^* \sin^2 \theta e^{-2i\varphi} & a \cos^2 \theta + c \sin^2 \theta \end{bmatrix},
\]
\[
B_1 = \begin{bmatrix} a \sin^2 \theta + c \cos^2 \theta & b \sin^2 \theta e^{-2i\varphi} + d \cos^2 \theta \\ b^* \sin^2 \theta e^{2i\varphi} + d^* \cos^2 \theta & a \sin^2 \theta + c \cos^2 \theta \end{bmatrix},
\]
\[
(AB)_1 = \begin{bmatrix} (a - c)e^{i\varphi} & be^{-i\varphi} - de^{i\varphi} \\ be^{i\varphi} - de^{-i\varphi} & (a - c)e^{-i\varphi} \end{bmatrix} \sin \theta \cos \theta. \tag{C2}
\]
The constraint in Eq. (11b), which is \( \varphi = \frac{1}{2}[\arg(b) - \arg(d)] \), make it possible to simplify \(b \cos^2 \theta + d \sin^2 \theta e^{2i\varphi} \). It leads us to redefine \(b = |b|, d = |d|\), and \(\varphi = 0\). We have verified that without loss of generality, all the results will remain unchanged in the following derivation. It then follows
\[
A_1 = A \cos^2 \theta + B \sin^2 \theta,
\]
\[
B_1 = A \sin^2 \theta + B \cos^2 \theta,
\]
\[
(AB)_1 = (A - B) \sin \theta \cos \theta. \tag{C3}
\]
As we mentioned in Sec.II A, \(\lambda\) denotes the minimum eigenvalue of a matrix. With the above expression of \(A_1, B_1\), we obtain the transformation relation of \(\lambda_{A_1}, \lambda_{B_1}\) and \(\lambda_A, \lambda_B\) as
\[
\lambda_{A_1} = \cos^2 \theta \cdot \lambda_A + \sin^2 \theta \cdot \lambda_B,
\]
\[
\lambda_{B_1} = \sin^2 \theta \cdot \lambda_A + \cos^2 \theta \cdot \lambda_B. \tag{C4}
\]
After several transformation, Eq. (C4) leads to
\[
\frac{\lambda_{A_1} \lambda_{B_1}}{\lambda_A \lambda_B} - 1 = \frac{(\lambda_B - \lambda_A)^2}{4 \lambda_A \lambda_B} \sin^2(2\theta). \tag{C5}
\]
Considering \(S_N = \log_2(\lambda_{A_1} \lambda_{B_1})/(\lambda_A \lambda_B), \)
\[
S_N > 0 \quad \text{is equivalent to} \quad \frac{\lambda_{A_1} \lambda_{B_1}}{\lambda_A \lambda_B} - 1 > 0. \tag{C6}
\]
On the other hand, as mentioned in Eq. (12), \(E_N > 0\) is equivalent to \(\xi > 0\). The new quantifier \(\xi = S - 1/2 - 8 \det[V_{\text{out}}]\) can be described in terms of \(\lambda_A, \lambda_B\) as well.
\[
S = 2 \det[A_1] + 2 \det[B_1] - 4 \det[(AB)_1] = (2\lambda_A \Lambda_A + \lambda_B \lambda_B) - [(a - c)^2 - (b - d)^2] \sin^2(2\theta)
\]
\[
= 2(\lambda_A \Lambda_A + \lambda_B \lambda_B)(1 - 2 \sin^2 \theta \cos^2 \theta) + 4(\lambda_A \Lambda_A + \lambda_B \lambda_B) \sin^2 \theta \cos^2 \theta - (\lambda_A - \lambda_B)(\lambda_A - \lambda_B) \sin^2(2\theta)
\]
\[
= 2(\lambda_A \Lambda_A + \lambda_B \lambda_B) - 2(\lambda_A - \lambda_B)(\Lambda_A - \Lambda_B) \sin^2(2\theta), \tag{C7}
\]
where \(\Lambda\) is the maximum eigenvalue of a CM in contrast to the minimum eigenvalue \(\lambda\). Recall that \(\det[V_{\text{out}}] = |A| \cdot |B| = \lambda_A \Lambda_A \lambda_B \lambda_B\) and both \(A, B\) are pure states \((\lambda_A \Lambda_A = \lambda_B \lambda_B = 1/4),\) it then follows
\[
S - 1/2 - 8 \det[V_{\text{out}}] = S - 1/2 - 8 \times \frac{1}{4} \times \frac{1}{4} = 2(\lambda_B - \lambda_A)(\Lambda_A - \Lambda_B) \sin^2(2\theta)
\]
\[
= C \cdot (\frac{\lambda_{A_1} \lambda_{B_1}}{\lambda_A \lambda_B} - 1). \tag{C8}
\]
Equation (C5) is applied in the last step where
\[
C = 8 \lambda_A \lambda_B \frac{\Lambda_A - \Lambda_B}{\lambda_B - \lambda_A}. \tag{C9}
\]
Recall that the expression for \(\lambda_A, \lambda_A, \lambda_B, \lambda_B\) are given by
\[
\lambda_A = a - |b| = \sqrt{1/4 + |b|^2} - |b|,
\]
\[
\Lambda_A = a + |b| = \sqrt{1/4 + |b|^2} + |b|,
\]
\[
\lambda_B = c - |d| = \sqrt{1/4 + |d|^2} - |d|,
\]
\[
\Lambda_B = c + |d| = \sqrt{1/4 + |d|^2} + |d|. \tag{C10}
\]
Note that \(\sqrt{1/4 + x^2} - x\) is a monotonically decreasing function while \(\sqrt{1/4 + x^2} + x\) is a monotonically increasing one, which leads to \(\lambda_A < \lambda_B < \Lambda_B < \Lambda_A\) or \(\lambda_B < \lambda_A < \Lambda_A < \Lambda_B\). In both cases, \(C > 0\).

As can be seen from the above expression, \(S_N > 0\) is a necessary and sufficient condition for \(S - 1/2 - 8 \det[V_{\text{out}}] > 0,\) which means the two-mode Gaussian entanglement exists.

Appendix D: Distribution relation of entanglement in a two-BS system

As shown by Fig. 2, three Gaussian states are mixed by a two-BS network. Their output CM is given by
\[
V_{\text{out}} = U^T(\theta_2, \varphi_2)U^T(\theta_1, \varphi_1)V_{\text{in}}U(\theta_1, \varphi_1)U(\theta_2, \varphi_2). \tag{D1}
\]
For $\xi(A_1, B_2)$, we trace out mode $C$ on the output state in order to obtain the covariance matrix $V_{A_1B_2}$ of the state $\rho_{A_1B_1}$. It is given that

$$V_{A_1B_2} = \begin{bmatrix} A_1 & (AB_1) \cos \theta_2 \\ (AB_1) \cos \theta_2 & B_1 \cos^2 \theta + C \sin^2 \theta_2 \end{bmatrix}, \quad C = \begin{bmatrix} g & f \\ f & g \end{bmatrix}, \quad (D2)$$

where $A_1, (AB_1), B_1$ are given by Eq. (C3). Following the idea introduced in Appendix C, we redefine $\varphi_2 = 0$ and $f = |f|$ when calculating the determination of matrix.

For $\xi(A_1, B_2) = S_{A_1B_2} - 1/2 - 8|V_{A_1B_1}|$,

$$S_{A_1B_1} = 2|A_1| + 2|B_1| \cos \theta_2 + C \sin^2 \theta_2| - 4|AB_1| \cos \theta_2, \quad (D3)$$

where the second term $|B_1| \cos \theta_2 + C \sin^2 \theta_2$ is calculated based on Theorem 4

$$|B_1| \cos \theta_2 + C \sin^2 \theta_2 = |B_1| \cos \theta_2 + |C| \sin^2 \theta_2 + \sin^2 \theta_2 \text{Tr}(B_1C^{-1})|C|. \quad (D4)$$

For $|V_{A_1B_1}|$, according to Theorem 4 or Eq. (B4),

$$|V_{A_1B_1}| = |A_1| (A_1 \cos \theta + C \sin^2 \theta_2) - |AB_1| (AB_1) \cos \theta_2 \\
= |\cos \theta_2| |V_{A_1B_1}| + |\sin^2 \theta_2| |A_1||C| + \cos \theta_2 \sin^2 \theta_2 \text{Tr}(A_1B_1 - (AB_1)B_1)C^{-1}A_1|A_1||C| \\
= |\cos \theta_2| |V_{A_1B_1}| + |\sin^2 \theta_2| |A_1| + \cos \theta_2 \sin^2 \theta_2 \text{Tr}(AB(A_1)A_1^{-1}C^{-1})|C|. \quad (D5)$$

Theorems 2, 5 are applied in the third and last step, respectively. According to Theorem 3,

$$AB|A|^i_1 = AB|A^{-1}| \cos \theta_1 + |B| \sin^2 \theta_1 = |A|B \cos^2 \theta_1 + |B| \sin^2 \theta_1 = \frac{1}{4} B. \quad (D6)$$

Note that we apply the condition that $|A| = |B| = 1/2$ in the last step. Substituting it into the expression of $|V_{A_1B_1}|$,

$$|V_{A_1B_2}| = |\cos \theta_2| |V_{A_1B_1}| + \frac{1}{4} |\sin^2 \theta_2| |A_1| + \frac{1}{4} \cos \theta_2 \sin^2 \theta_2 \text{Tr}(BC^{-1})|C|. \quad (D7)$$

As can be seen, $S_{A_1B_1}$ and $|V_{A_1B_2}|$ are split into many terms which are related to $S_{A_1B_1}$ and $|V_{A_1B_1}|$. Substituting Eq. (D3, D4, D7) to the expression of $\xi(A_1, B_2)$,

$$\xi(A_1, B_2) = S_{A_1B_2} - 1/2 - 8|V_{A_1B_1}| \\
= \cos \theta_2 \cdot (S_{A_1B_1} - 1/2 - 8|V_{A_1B_1}|) + \sin^2 \theta_2 \cos^2 \theta_2 \cdot (4|A_1| - 1 - 4|AB_1|). \quad (D8)$$

Since $|A| = |B| = 1/4$, we calculate $|A_1|$ as

$$|A_1| = |A \cos^2 \theta_1 + B \sin^2 \theta_1| \\
= |A| \cos^2 \theta_1 + |B| \sin^2 \theta_1 = |A| \sin^2 \theta_1 + |B| \cos^2 \theta_1 = |A| \sin^2 \theta_1 + B \cos^2 \theta_1 |B_1|. \quad (D9)$$

Recall that $|V_{A_1B_1}| = |\rho_{AB}| = |B| = \frac{1}{4} \cdot \frac{1}{2}$, it follows

$$4|A_1| - 1 - 4|AB_1| = 2|A_1| + 2|B_1| - 4|AB_1| \cdot 1/2 - 8|V_{A_1B_1}|. \quad (D10)$$

Substituting Eq. (D10) into Eq. (D8),

$$S_{A_1B_1} = 2|A_1| + 2|B_1| - 1/2 - 8|V_{A_1B_1}| \\
= (\cos \theta_2 + \sin \theta_2 \cos \theta_2) \cdot (S_{A_1B_1} - 1/2 - 8|V_{A_1B_1}|) \\
= \cos \theta_2 \cdot (S_{A_1B_1} - 1/2 - 8|V_{A_1B_1}|) \\
= \cos \theta_2 \cdot \xi(A_1, B_1). \quad (D11)$$

With the above calculation, we obtain

$$\xi(A_1, B_2) = \cos \theta_2 \cdot \xi(A_1, B_1). \quad (D12)$$

Using the same steps to calculate $\xi(A_1, C_1)$, it can be obtained that

$$\xi(A_1, C_1) = \sin \theta_2 \cdot \xi(A_1, B_1), \quad (D13)$$

which means entanglement could be distributed as the same way of distributing reflectance and transmittance.

**Appendix E: Trace norm equality relations**

$\rho_{A_1B_1}^{T_A}$ can be written as $\rho_{A_1B_1}^{T_A} = \sum_i \lambda_i \rho_i$ [9], where $\rho_i = |i\rangle \langle i|, (\rho_{AB})$, stands for a series of pure states. $\rho_i = \delta_i \rho_i$.

$$\sqrt{\rho_{A_1B_1}^{T_A}} = \sum_{i,j} \lambda_i \lambda_j \rho_i \rho_j = \sum_i |\lambda_i|^2 \rho_i = \sum_i \rho_i \rho_i \rho_i. \quad (E1)$$

It leads to $\|\rho_{A_1B_1}^{T_A}\|_1 = \text{Tr} \left[ \sqrt{\rho_{A_1B_1}^{T_A}} \left( \sqrt{\rho_{A_1B_1}^{T_A}} \right) \right] = \sum_i |\lambda_i|^2$. According to Fig. 2, $\rho_{A_1B_1C}$ is obtained by a unitary transformation on $\rho_{A_1B_1} \otimes \rho_C$, recall that $\rho_C$ is a pure state,

$$\rho_{A_1B_1C} = \rho_{BC}^{T_A} \rho_{A_1B_1} \otimes \rho_C u_{BC}. \quad (E2)$$

Thus,

$$\rho_{A_1B_1C}^{T_A} = \rho_{BC}^{T_A} \rho_{A_1B_1} \otimes \rho_C u_{BC} = \rho_{BC}^{T_A} \sum_i \lambda_i \rho_i \otimes \rho_C u_{BC}. \quad (E3)$$
It follows that

\[
\sqrt{\langle \rho_{A;B,C}^{T_A} | \rho_{A;B,C}^{T_A} \rangle} = \sqrt{\| u_{BC}^{\dagger} (\sum_i A_i \rho_{i} \otimes \rho_{BC}) u_{BC} \|} = \sqrt{\| u_{BC}^{\dagger} (\sum_i |A_i\| \cdot \rho_i \otimes \rho_{BC}) u_{BC} \|} = u_{BC}^{\dagger} (\sum_i |A_i| \cdot \rho_i \otimes \rho_{BC}) u_{BC}.
\]

The trace norm of \( \rho_{A;B,C}^{T_A} \) is calculated by

\[
\| \rho_{A;B,C}^{T_A} \|_1 = \text{Tr} \left[ u_{BC}^{\dagger} (\sum_i |A_i| \cdot \rho_i \otimes \rho_{BC}) u_{BC} \right] = \sum_i |A_i|, \quad (E5)
\]

which means \( \| \rho_{A;B,C}^{T_A} \|_1 = \| \rho_{A;B,C}^{T_A} \|_1 \).

The second equation, \( \| \rho_{A;B,C}^{T_A} \|_1 = \| \rho_{A;B,C}^{T_A} \|_1 \), can be obtained in the similar method.

**Appendix F: Distribution relation of entanglement in a linearly arranged BS system**

A general proof of the distribution relation in a linearly arranged \( n \)-BS system is presented in this section (Fig. 5).

Define

\[
V_1 = \begin{bmatrix} A_1 & (AB) \end{bmatrix}_{B_1}, \quad \xi_1 = S_{A_1B_1} - 1/2 - 8|V_1|, \quad (F1)
\]

where \( V_1 \) is the covariance matrix of \( \rho_{A_1B_1} \). The one for \( \rho_{A_2B_2} \), \( V_2 \), is related to \( V_1 \) as

\[
\begin{align*}
V_2 &= \begin{bmatrix} A_1 & (AB) \end{bmatrix}_{B_2}, \\
(AB) &= (AB)_1 \cos \theta_2, \\
B_2 &= B_1 \cos^2 \theta_2 + \frac{1}{2} \sin^2 \theta_2, \\
\end{align*} \quad (F2)
\]

where \( \frac{1}{2} \) (to be specific, \( \frac{1}{2} I_2 \)) stands for a new input state, which is vacuum or coherent states as we mentioned before.

The same relationship works for \( V_n \) and \( V_{n-1} \) as

\[
V_n = \begin{bmatrix} A_1 & (AB) \end{bmatrix}_{B_n}, \quad (F3)
\]

\[
(AB)_n = (AB)_{n-1} \cos \theta_n, \\
B_n = B_{n-1} \cos^2 \theta_n + \frac{1}{2} \sin^2 \theta_n.
\]

Following the same way of Eq. (D3-D11),

\[
S_{A_1B_1} = 2|A_1| - 4|(AB)_{n-1}| \cos^2 \theta_n + 2|B_{n-1}| \cos^4 \theta_n + 2 \times \frac{1}{4} \sin^4 \theta_n + 2 \times \frac{1}{2} \sin^2 \theta_n \cos^2 \theta_n \text{Tr}(B_{n-1}). \quad (F4)
\]

On the other hand,

\[
|V_{A_1B_1}| = \cos^4 \theta_n |V_{A_1B_1}| + \frac{1}{4} \sin^4 \theta_n |A_1| + \frac{1}{2} \sin^2 \theta_n \cos^2 \theta_n \text{Tr}(A_1A_1^{-1}[A_1B_{n-1} - (AB)_{n-1}(AB)_{n-1}]). \quad (F5)
\]

Substituting Eq. (F4, F5) into the expression of \( \xi_n \),

\[
\xi_n = S_{A_1B_1} - 1/2 - 8|V_{A_1B_1}| = \cos^4 \theta_n \xi_{n-1} + \sin^4 \theta_n \cos^2 \theta_n (4|A_1| - 1 - 4|(AB)_{n-1}|) + \sin^2 \theta_n \cos^2 \theta_n \text{Tr}(B_{n-1} - 4|A_1|A_1^{-1}[A_1B_{n-1} - (AB)_{n-1}(AB)_{n-1}]). \quad (F6)
\]

The complex term in the trace operator bracket, \( B_{n-1} - 4|A_1|A_1^{-1}[A_1B_{n-1} - (AB)_{n-1}^2] \), can be derived mathematically by the idea of iteration

\[
n = 2, \quad B_2 = 4|A_1|A_1^{-1}[A_1B_1 - (AB)_1], \quad (F7)
\]

\[
= B_1 - 4|A_1|A_1^{-1}[A_1B_1 - (AB)_1] = B_1 - 4 \times \frac{1}{4} B_1 = 0,
\]

\[
n = 3, \quad B_3 - 4|A_1|A_1^{-1}[A_1B_2 - (AB)_2] = \cos^4 \theta_2 \cdot 0 + \sin^4 \theta_2 (\frac{1}{2} - 2|A_1|) \cdot I_2 \quad (F8)
\]

\[
= (1 - \cos^2 \theta_2) (\frac{1}{2} - 2|A_1|) \cdot I_2.
\]

\[
n = 4, \quad B_4 - 4|A_1|A_1^{-1}[A_1B_3 - (AB)_3] = \cos^4 \theta_3 \cdot (1 - \cos^2 \theta_3) (\frac{1}{2} - 2|A_1|) \cdot I_2 + \sin^2 \theta_3 (\frac{1}{2} - 2|A_1|) \cdot I_2 \quad (F9)
\]

\[
= (1 - \cos^2 \theta_2 \cos^2 \theta_3) \cdot (\frac{1}{2} - 2|A_1|) \cdot I_2.
\]

It follows that

\[
\text{Tr} \left[ B_{n-1} - 4|A_1|A_1^{-1}[A_1B_{n-1} - (AB)_{n-1}] \right] = \text{Tr} \left[ (1 - \sum_{i=2}^{n-1} \cos^2 \theta_i) \cdot (\frac{1}{2} - 2|A_1|) \cdot I_2 \right] = (1 - \sum_{i=2}^{n-1} \cos^2 \theta_i) \cdot (1 - 4|A_1|).
\]

Substituting it into Eq. (F6),

\[
\xi_n = \cos^4 \theta_n \cdot \xi_{n-1} + \sin^4 \theta_n \cos^2 \theta_n \left( 4|A_1| - 1 - 4|(AB)_{n-1}| + (1 - \sum_{i=2}^{n-1} \cos^2 \theta_i)(1 - 4|A_1|) \right) = \cos^4 \theta_n \cdot \xi_{n-1} + \sin^2 \theta_n \cos^2 \theta_n \left( \left( \prod_{i=2}^{n-1} \cos^2 \theta_i \right)(4|A_1| - 1) - 4|(AB)_{n-1}| \right) = (\prod_{i=2}^{n-1} \cos \theta_i \cdot (AB)_1, \quad (F9)
\]

Recall that \((AB)_1 = \cos \theta_1 \cdot (AB)_{n-2} = \prod_{i=2}^{n-1} \cos \theta_i \cdot (AB)_1\), it follows that
\[ \xi_n = \cos^4 \theta_n \cdot \xi_n + \sin^2 \theta_n \cos^2 \theta_n \left( \prod_{i=2}^{n-1} \cos^2 \theta_i \right) (4|A_1| - 1) - 4 \left( \prod_{i=2}^{n-1} \cos^2 \theta_i \right) |\langle AB \rangle_1| \]

\[ = \cos^4 \theta_n \cdot \xi_{n-1} + \sin^2 \theta_n \left( \prod_{i=2}^{n} \cos^2 \theta_i \right) (4|A_1| - 1 - 4 |\langle AB \rangle_1|) \]

\[ = \cos^4 \theta_n \cdot \xi_{n-1} + \sin^2 \theta_n \left( \prod_{i=2}^{n} \cos^2 \theta_i \right) \cdot \xi_1. \]

(F10)

Applying the idea of iteration again,

\[ n = 2, \quad \xi_2 = \cos^4 \theta_2 \cdot \xi_1 + \sin^2 \theta_2 \cos^2 \theta_2 \cdot \xi_1 = \cos^2 \theta_2 \cdot \xi_1, \]

\[ n = 3, \quad \xi_3 = \cos^4 \theta_3 \cdot \xi_2 + \sin^2 \theta_3 \cos^2 \theta_3 \cdot \cos^2 \theta_2 \cdot \xi_1 \]

\[ = \cos^4 \theta_3 \cdot \xi_2 + \sin^2 \theta_3 \cos^2 \theta_3 \cdot \xi_2 = \cos^2 \theta_3 \cdot \xi_2. \]

(F11)

It follows that

\[ \xi_n = \cos^2 \theta_n \cdot \xi_{n-1} = \prod_{i=2}^{n} \cos^2 \theta_i \cdot \xi_1. \quad \text{(F12)} \]

Thus, the distribution relation of entanglement in a linearly arranged \( n \)-BS system is proved.

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