Queueing Analysis of a Large-Scale Bike Sharing System through Mean-Field Theory

Quan-Lin Li, Chang Chen, Rui-Na Fan, Liang Xu and Jing-Yu Ma
School of Economics and Management Sciences
Yanshan University, Qinhuangdao 066004, P.R. China
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Abstract

The bike sharing systems are fast increasing as a public transport mode in urban short trips, and have been developed in many major cities around the world. A major challenge in the study of bike sharing systems is that large-scale and complex queueing networks have to be applied through multi-dimensional Markov processes, while their discussion always suffers a common difficulty: State space explosion. For this reason, this paper provides a mean-field computational method to study such a large-scale bike sharing system. Our mean-field computation is established in the following three steps: Firstly, a multi-dimensional Markov process is set up for expressing the states of the bike sharing system, and the empirical process of the multi-dimensional Markov process is given to partly overcome the difficulty of state space explosion. Based on this, the mean-field equations are derived by means of a virtual time-inhomogeneous $M(t)/M(t)/1/K$ queue whose arrival and service rates are determined by the mean-field computation. Secondly, the martingale limit is employed to investigate the limiting behavior of the empirical process, the fixed point is proved to be unique so that it can be computed by means of a nonlinear birth-death process, the asymptotic independence of this system is discussed simply, and specifically, these lead to numerical computation of the steady-state probability of the problematic (empty or full) stations. Finally, some numerical examples are given for valuable observation on how the steady-state probability of the problematic stations depends on some crucial parameters of the bike sharing system.

Keywords: Bike sharing system; queueing network; empirical process; mean-field
equation; nonlinear birth-death process; martingale limit; fixed point; probability of problematic stations.

1 Introduction

The bike sharing systems are fast developing widespread adoption in major cities around the world, and are becoming a public mode of transportation devoted to short trips. Up to now, there have been more than 500 cities having equipped with the bike sharing systems. Also, it is worth noting that the bike sharing systems are being regarded as a promising solution to jointly reduce, such as, traffic congestion, parking difficulty, transportation noise, air pollution and global warming. As a history overview of the bike sharing systems, readers may refer to, such as, DeMaio [18] and Shaheen et al. [64] for more details. Also, DeMaio [16] provided a valuable prospect of the bike sharing systems in the 21st century.

For the status of the bike sharing systems in some countries or cities, important examples include the United States by DeMaio and Gifford [17], France by Faye [21], the European cities with OBIS Project by Janett and Hendrik [36], London by Lathia et al. [40], Montreal by Morency et al. [56], Beijing by Liu et al. [51], several famous cities by Shu et al. [66], and some more general introduction by Shaheen et al. [63] and Meddin and DeMaio [54].

To understand the recent key research directions, here it is necessary to discuss some basic issues in design, operations and optimization of the bike sharing systems. The literature of bike sharing systems may be classified as two classes: The primary issues, and the higher issues. The primary issues are to discuss the number of stations, the station location, the number of bikes, the parking positions, and the types of bikes, all of which may be regarded as the strategic design. The higher issues are to analyze the demand prediction, the path scheduling, the inventory management, the repositioning (or rebalancing) by trucks, the price incentive, and applications of the intelligent information technologies. For analysis of the primary issues, readers may refer to, such as, Dell’Olio et al. [15], Lin and Yang [49], Kumar and Bierlaire [38], Martinez et al. [53] and Nair et al. [58]. While the higher issues were discussed by slightly more literature. Readers may refer to recent publications or technical reports for more details, among which are the repositioning by Forma et al. [22], Vogel and Mattfeld [71], Benchimol et al. [5], Raviv et al. [60], Contardo et al. [13], Caggiani and Ottomanelli [10], Fricker et al. [24], Chemla
et al. [11], Shu et al. [66], Fricker and Gast [23] and Labadi et al. [39]; the inventory management by Lin et al. [50], Raviv and Kolka [60] and Schuijbroek et al. [65]; the price incentives by Waserhole and Jost [74], Waserhole et al. [77] and Fricker and Gast [23]; the fleet management by George and Xia [32, 31], Godfrey and Powell [33], Nair and Miller-Hooks [57] and Guerriero et al. [34]; the simulation models by Barth and Todd [3] and Fricker and Gast [23]; the data analysis by Froehlich and Oliver [26], Vogel et al. [72], Borgnat et al. [8], Côme et al. [12] and Katzev [37].

Based on the above recalling the basic literature, it is necessary to further observe a basic solution to operations of the bike sharing systems. In a bike sharing system, a customer arrives at a station, takes a bike, and uses it for a while; then she returns the bike to a destination station. In this case, the bikes are always distributed unbalance among the stations over time, thus an arriving customer may be confronted with two problematic cases: (1) A station is empty when a customer arrives at the station to rent a bike, and (2) a station is full when a deriving-bike customer arrives at the station to return her bike. Here, the empty or full station is called a problematic station. Since a crucial question for the operational efficiency of the bike sharing system is its ability not only to meet the fluctuating demand for renting bikes at each station but also to provide enough vacant lockers to allow the renters to return the bikes at their destinations, the two types of problematic stations reflect a common challenge faced by the bike sharing systems in practice due to the stochastic and time-inhomogeneous nature of the customer arrivals and bike returns. Based on this, it is a key to be able to measure the steady-state probability of the problematic stations in the study of the bike sharing systems. Furthermore, analysis of the steady-state probability of the problematic stations is very useful in design, operations and optimization of the bike sharing systems, because it can provide necessary help in terms of numerical computation and associated comparison. Up to now, it is still difficult (and even impossible) to provide an explicit expression for the steady-state probability of the problematic stations because the bike sharing system is a more complicated closed queueing network with various geographical interactions, which come both from the bikes parked in multiple stations and from the other bikes ridden on multiple roads. For this, Section 2 explains that the corresponding Markov process of the closed queueing network is of dimension $N^2$.

To compute the steady-state probability of the problematic stations, it is better to develop a stochastically dynamic method through applications of the queueing theory as
well as Markov processes to the study of bike sharing systems. However, the available works on such a research direction are still few up to now. To survey the recent literature, some significant methods and results are listed as follows. Leurent [41] used the $M/M/1/C$ queue to consider a vehicle-sharing system in which each station contains an expanded waiting room only for those customers arriving at either a full station for returning a bike or an empty station for renting a bike, and analyzed performance measures of this vehicle-sharing system in terms of a geometric distribution. Schuijbroek et al. [65] first computed the transient distribution of the $M/M/1/C$ queue, which is used to measure the service level and further to establish a mixed integer programming for the bike sharing system. Then they dealt with the inventory rebalancing and the vehicle routing by means of the optimal solution to the mixed integer programming. Raviv et al. [60] and Raviv and Kolka [59] provided an effective method for computing the transient distribution of a time-inhomogeneous $M(t)/M(t)/1/C$ queue, which is used to evaluate the expected number of bike shortages at any station. Savin et al. [62] used a loss network as well as the admission control to discuss capacity allocation of a rental model with two classes of customers, and studied the revenue management and fleet sizing decision. Adelman [1] applied a closed queueing network to propose an internal pricing mechanism for managing a fleet of service units, and also used a nonlinear flow model to discuss the price-based policy for the vehicle redistribution. George and Xia [32] provided an effective method of closed queueing networks in the study of vehicle rental systems, and determined the optimal number of parking spaces for each rental location. In addition, the mean-field method as well as the queueing theory are recently applied to analyzing the bike sharing systems. Fricker et al. [24] considered a space-inhomogeneous bike sharing system, and expressed the minimal proportion of problematic stations within each cluster. Fricker and Gast [23] provide a detailed analysis for a space-homogeneous bike sharing system in terms of the $M/M/1/K$ queue and some simple mean-field models, and crucially, they derived the closed-form solution to the minimal proportion of problematic stations. Fricker and Tibi [25] first studied the central limit and local limit theorems for the independent (non identically distributed) random variables, which support analysis of a generalized Jackson network with product-form solution; then they used the limit theorems to give a better outline of the stationary asymptotic analysis of the locally space-homogeneous bike sharing systems. On the other hand, the Markov decision processes are also applied to analysis of the bike sharing systems. Examples include Waserhole and Jost [74], Waserhole and Jost
and Waserhole et al. [75], where they used a closed queuing network to propose a Markov decision model for discussing the bike sharing system, and also established a fluid approximation to compute the static optimal policy.

For convenience of readers, it may be useful to recall some basic references in which the mean-field theory is applied to analysis of large-scale stochastic systems. Readers may refer to Spitzer [67], Dawson [14], Sznitman [68], Vvedenskaya et al. [73], Mitzenmacher [55], Turner [70], Graham [29, 30], Benaim and Le Boudec [4], Gast and Gaujal [27, 28], Bordenave et al. [7], Li [43, 44], Li and Lui [48], Li et al. [45, 46], Fricker et al. [24] and Fricker and Tibi [25]. On the other hand, the metastability of Markov processes can play a key role in analysis of the nonlinear Markov processes in the bike sharing systems. Reader may refer to, such as, Bovier [9], Den Hollander [19], Antunes et al. [2], Tibi [69], Li [44] and more references therein.

The main contributions of this paper are twofold. The first contribution is to describe a mean-field queueing model to analyze the large-scale bike sharing systems, where the arrival, walk, deriving-bike (or return) processes among the stations are given some simplified assumptions whose purpose is to guarantee applicability of the mean-field theory. For such a mean-field queueing model, we develop a mean-field queueing method in terms of the mean-field theory, the time-inhomogeneous queue, the martingale limits and the nonlinear birth-death processes. Based on this, we provide a complete picture of applying the mean-field theory to the study of bike sharing systems in practice. Concretely, the mean-field queueing method is established through four basic steps: (1) The system of mean-field equations is set up by means of a virtual time-inhomogeneous $M(t)/M(t)/1/K$ queue whose arrival and service rates are determined by means of the mean-field computation; (2) the asymptotic independence (or propagation of chaos) is proved in terms of the martingale limit and the uniqueness of the fixed point; (3) numerical computation of the fixed point is given by means of a system of nonlinear equations; and (4) performance numerical analysis of the bike sharing system.

The second contribution of this paper is to provide a detailed analysis for the steady-state probability of the problematic stations, which is one of the most key measures in the study of bike sharing systems. It is worth noting that the service level, optimal design and control mechanism of bike sharing systems can be computed by means of the steady-state probability of the problematic stations. Furthermore, this paper develops effective algorithms for computing the steady-state probability of the problematic stations, this gives a
numerically computational framework in the study of bike sharing systems. Furthermore, we use some numerical examples to give valuable observation and understanding on how the performance measures depend on some crucial parameters of the bike sharing system. Notice that Fricker et al. [24], Fricker and Gast [23] and Fricker and Tibi [25] are the only important references to be closely related to this paper by using the mean-field theory, but differently, this paper provides more work focusing on some key theoretical points such as the virtual time-inhomogeneous $M(t)/M(t)/1/K$ queue, the mean-field equations, the martingale limits, the nonlinear birth-death processes, numerical computation of the fixed point, and numerical analysis for the steady-state probability of the problematic stations. With successful exposition of the key theoretical points, such a numerical computation can greatly enable a broad study of bike sharing systems. Therefore, the methodology and results of this paper gain new insights on how to establish the mean-field queueing models for discussing more general bike sharing systems by means of the mean-field theory, the time-inhomogeneous queues and the nonlinear Markov processes.

The remainder of this paper is organized as follows. In Section 2, we first describe a large-scale bike sharing system with $N$ identical stations, give a $N$-dimensional Markov process for expressing the states of the bike sharing system, and establish an empirical measure process of the $N$-dimensional Markov process in order to partly overcome the difficulty of state space explosion. In Section 3, we set up a system of mean-field equations satisfied by the expected fraction vector through a virtual time-inhomogeneous $M(t)/M(t)/1/K$ queue whose arrival and service rates are determined by means of the mean-field computation. In Section 4, we establish a Lipschitz condition, and prove the existence and uniqueness of solution to the system of mean-field equations. In Section 5, we provide a martingale limit of the sequences of empirical measure Markov processes in the bike sharing system. In Section 6, we analyze the fixed point of the system of mean-field equations, and prove that the the fixed point is unique. Based on this, we simply analyze the asymptotic independence of the bike sharing system, and also discuss the limiting interchangeability with respect to $N \to \infty$ and $t \to +\infty$. In Section 7, we provide some effective computation of the fixed point, and use some numerical examples to investigate how the steady-state probability of the problematic stations depends on some crucial parameters of the bike sharing system. Some concluding remarks are given in Section 8.
2 Model Description

In this section, we first describe a large-scale bike sharing system with \( N \) identical stations, and establish a \( N \)-dimensional Markov process for expressing the states of the bike sharing system. To overcome the difficulty of state space explosion, we provide an empirical measure process of the \( N \)-dimensional Markov process.

We first explain reasons why it is necessary to develop some simplified models in the study of bike sharing systems, and also indicate that the mean-field theory plays a key role in analyzing such simplified models.

In the bike sharing system, a customer arrives at a station, takes a bike, and uses it for a while; then she returns the bike to any station and immediately leaves this system. Based on this, if the bike sharing system has \( N \) stations for \( N \geq 2 \), then it can contain at most \( N (N - 1) \) roads due to the fact that there may be a road between any two stations. In this case, when the bike sharing system is modeled as a closed queueing network with a fixed total number of bikes, it is seen from these bikes distributed either at the stations or on the roads that the corresponding Markov process is of dimension \( N^2 \), and it is described as \( \{ \mathbf{n}(t) : t \geq 0 \} \), where

\[
\mathbf{n}(t) = (n_1(t), n_2(t), \ldots, n_N(t)),
\]

\[
n_k(t) = (n_k(t); n_{1,k}(t), \ldots, n_{k-1,k}(t), n_{k+1,k}(t), \ldots, n_{N,k}(t)),
\]

\[
\sum_{k=1}^{N} n_k(t) + \sum_{i=1}^{N} \sum_{j \neq i} n_{i,j}(t) = \mathcal{R},
\]

\( n_k(t) \) is the number of bikes parked at Station \( k \), \( n_{j,k}(t) \) is the number of bikes to be traveling on Road \( j \to k \) for \( j \neq k \) and \( 1 \leq j, k \leq N \), and \( \mathcal{R} \) is the total number of bikes in the bike sharing system. It is easy to see from the vector \( \mathbf{n}(t) \) that there usually exist some basic difficulties for dealing with the Markov process \( \{ \mathbf{n}(t) : t \geq 0 \} \) of dimension \( N^2 \) due to two main reasons: (1) The state space explosion as \( N \) increases, and (2) a more complicated random structure with respect to the \( N \) stations and associated \( N (N - 1) \) roads. See Li and Fan [47] for more details. For this, it is necessary (or a key) to provide a simplified model with few parameters in order to be able to discuss numerical computation of the steady-state probability of the problematic stations, and further to analyze performance measures of the bike sharing system. To that end, this motivates us in this paper to provide a mean-field queueing method in the study of bike
sharing systems. On the other hand, to apply the mean-field theory, some basic conditions, which can be refined from the modeling assumptions, are needed to guarantee the exchangeability of the $N$-dimensional Markov process \({\{(n_1(t), n_2(t), \ldots, n_N(t)) : t \geq 0\}}\), where the bike information on the roads are omitted. Although the mean-field modeling assumptions are simplified greatly, we can still give some very useful relations among the basic parameters of the bike sharing systems in practice, and crucially, and also provide effective algorithms both for computing the steady-state probability of the problematic stations and for analyzing performance measures of the bike sharing system.

To apply the mean-field theory, here we make some necessarily simplified assumptions for the bike sharing system as follows:

(1) **The $N$ identical stations**: The bike sharing system consists of $N$ identical stations, each of which has a finite bike capacity. At the initial time $t = 0$, each station contains $C$ bikes and $K$ positions to park the bikes, where $1 \leq C < K < \infty$.

(2) **The arrive processes**: The arrivals of outside customers at the bike sharing system are a Poisson process with arrival rate $N\lambda$ for $\lambda > 0$.

(3) **The walk processes**: If an outside or walking customer arrives at an empty station in which no bike may be rented, then she has to walk to another station again in order to hope being able to rent a bike. We assume that the customer may rent a bike from a station within at most $\omega$ consequent walks, otherwise she will directly leave this system (that is, if she has not rented a bike after $\omega$ consequent walks yet). Notice that one walk is viewed as the walking process of the customer from an empty station to another station, and $\omega$ is the maximal number of the customer consequent walks.

We assume that the walk times between any two stations are all exponential with walk rate $\gamma > 0$. Obviously, the expected walk time is $1/\gamma$.

(4) **The driving-bike (or return) processes**: If a driving-bike customer arrives at a full station in which no parking position is available, then she has to derive the bike to another station again. We assume that the returning-bike process is persistent in the sense that the customer must find a station with an empty position to return her bike (that is, she can not leave this system before her bike is returned), because the bike is the public property so that no one can make it her own.

We assume that the deriving-bike times between any two stations are all exponential with deriving-bike rate $\mu$ for $\gamma \leq \mu < +\infty$. Clearly, the expected deriving-bike time is $1/\mu$. 
Figure 1: The physical interpretation of a bike sharing system

(5) The departure discipline: The customer departure has two different cases: (a) The customer directly leaves the bike sharing system if she has not rented a bike yet after $\omega$ consequent walks; or (b) once one customer takes, uses and returns the bike to a station, she completes this trip, thus she can immediately leave the bike sharing system.

We assume that the arrival, walk and driving-bike processes are independent, and all the above random variables are independent of each other. For such a bike sharing system, Figure 1 provides some physical interpretation.

Remark 1 (1) The assumption of the $N$ identical stations is used to guarantee applicability of the mean-field theory (that is, the multi-dimensional Markov process is exchangeable). On the other hand, from a practical point of view, the stations in a major city are also designed as almost the same, for example, Hangzhou has 3000 stations, and each station contains about 30 bikes.

(2) It is necessary to explain the maximal number $\omega$ of the customer consequent walks. If $\omega = 0$, then the arriving customer immediately leaves this system once she arrives at a full station. If $\omega$ is smaller, then the customer would like to find an available bike at a
lucky station through at most \( \omega \) consequent walks, because a bike can help her to fast deal with a number of important things so that she would like to accept the time delay due to the at most \( \omega \) consequent walks.

(3) The assumption with \( 0 < \gamma < \mu < +\infty \) makes sense in practice, because the driving-bike is faster than the walk on an identical road from a station to another station.

In the remainder of this section, we first establish a \( N \)-dimensional Markov process for expressing the states of the bike sharing system. Then we give an empirical measure process of the \( N \)-dimensional Markov process in order to overcome the difficulty of state space explosion.

Let \( X_i^{(N)} (t) \) be the number of bikes parked in Station \( i \) at time \( t \geq 0 \). Then it is easy to see from the above model descriptions that \( \mathcal{X} = \left\{ \left( X_1^{(N)} (t), X_2^{(N)} (t), \ldots, X_N^{(N)} (t) \right) : t \geq 0 \right\} \) is a \( N \)-dimensional Markov process. In general, it is always more difficult to directly study the \( N \)-dimensional Markov process \( \mathcal{X} \) due to the state space explosion. Thus we need to introduce an empirical measure process of the \( N \)-dimensional Markov process \( \mathcal{X} \) as follows. We write

\[
Y_k^{(N)} (t) = \frac{1}{N} \sum_{i=1}^{N} 1_{\{X_i^{(N)} (t) = k\}},
\]

where \( 1_{\{\cdot\}} \) is an indicator function. Obviously, \( Y_k^{(N)} (t) \) is the proportion of the stations with \( k \) bikes at time \( t \), and \( 0 \leq \sum_{i=1}^{N} 1_{\{X_i^{(N)} (t) = k\}} \leq N \). Let

\[
Y^{(N)} (t) = \left( Y_0^{(N)} (t), Y_1^{(N)} (t), \ldots, Y_{K-1}^{(N)} (t), Y_K^{(N)} (t) \right).
\]

Then it is easy to see that the empirical measure process \( \{Y^{(N)} (t) : t \geq 0\} \) is a Markov process on the state space \( \Omega = [0, 1]^{K+1} \).

To study the empirical measure Markov process, we write

\[
y_k^{(N)} (t) = E \left[ Y_k^{(N)} (t) \right]
\]

and

\[
y^{(N)} (t) = \left( y_0^{(N)} (t), y_1^{(N)} (t), \ldots, y_{K-1}^{(N)} (t), y_K^{(N)} (t) \right).
\]

3 The Mean-Field Equations

In this section, we first describe the bike sharing system as a virtual time-inhomogeneous \( M(t)/M(t)/1/K \) queue whose arrival and service rates are determined by means of the
mean-field theory. Then we set up a system of mean-field equations, which is satisfied by the expected fraction vector $y^{(N)}(t)$, in terms of the virtual time-inhomogeneous $M(t)/M(t)/1/K$ queue.

Note that the $N$ stations are identical according to the above model description on both system parameters and operations discipline, thus we can use the mean-field theory to study the bike sharing system, where we only need to observe a tagged station (for example, Station 1) whose number of bikes is regarded as a virtual time-inhomogeneous $M(t)/M(t)/1/K$ queue (see Figure 2); while the other $N - 1$ stations have some impact on the tagged station, and the impact can be analyzed by means of the empirical measure through a mean-field computation for the arrival and service rates in this virtual queue. Based on this, we also explain the reason why the arrival and service processes of this virtual queue are time-inhomogeneous, e.g., see Figure 2 for more details.

It is necessary to explain the difference of the arrival and service processes between the bike sharing system and the virtual time-inhomogeneous $M(t)/M(t)/1/K$ queue. For example, if a real customer arrives and rents a bike at a tagged station, then the number of bikes parked in the tagged station decreases one, thus the real customer arrivals at the tagged station should be a part of the service process of the $M(t)/M(t)/1/K$ queue; while if a real customer returns a bike to a tagged station and leaves this system (i.e., her trip is completed), then the number of bikes parked in the tagged station increases one, thus the real customers’ returning their bikes to the tagged station should be a part of the arrival process of the $M(t)/M(t)/1/K$ queue. Furthermore, the following Theorem 1 provides a more detailed analysis for various parts of the arrival and service processes in the virtual time-inhomogeneous $M(t)/M(t)/1/K$ queue.
For the time-inhomogeneous $M(t)/M(t)/1/K$ queue, now we use the mean-field theory to discuss its Poisson input with arrival rate $\xi_{l}^{(N)}(t)$ for $0 \leq l \leq K - 1$ and its exponential service times with service rate $\eta_{k}^{(N)}(t)$ for $1 \leq k \leq K$.

The following theorem provides expressions for the arrival and service rates: $\xi_{l}^{(N)}(t)$ for $0 \leq l \leq K - 1$ and $\eta_{k}^{(N)}(t)$ for $1 \leq k \leq K$, respectively. Notice that the time-inhomogeneous arrival and service rates will play a key role in our mean-field study later.

**Theorem 1** For $1 \leq k \leq K$ and $\omega = 0, 1, 2, \ldots$, we have

$$\eta_{k}^{(N)}(t) = \eta^{(N)}(t) = \lambda + \gamma y_{0}^{(N)}(t) \frac{1 - \left[y_{0}^{(N)}(t)\right]^{\omega}}{1 - y_{0}^{(N)}(t)}. \quad (1)$$

At the same time, for $0 \leq l \leq K - 1$ we have

$$\xi_{l}^{(N)}(t) = \begin{cases} \frac{\mu}{N} \frac{1}{1 - y_{0}^{(N)}(t)} \left\{ (C - l) + (N - 1) \left[ C - \sum_{k=1}^{K} ky_{k}^{(N)}(t) \right] \right\}, & 0 \leq l \leq C - 1, \\ \frac{\mu}{N} \frac{1}{1 - y_{0}^{(N)}(t)} \left\{ (N - 1) \left[ C - \sum_{k=1}^{K} ky_{k}^{(N)}(t) \right] \right\}, & C \leq l \leq K - 1. \end{cases} \quad (2)$$

**Proof** We first prove Equation (1). In this case, we need to specifically deal with State 0. If one customer arrives at an empty station, then the customer has to walk from the empty station to another station, while the bikes parked at the new station will have two different cases: (a) There is at least one bike with probability $\sum_{i=1}^{K} y_{i}^{(N)}(t)$; and (b) there is no bike with probability $y_{0}^{(N)}(t)$. For Case (a), the customer can rent a bike for her trip; while for Case (b), the customer will have to walk to another station again until she can rent a bike from a next station. Notice that the role played by State 0 is depicted in Figure 3, thus we can easily observe that the state transitions from State 0 are jointly caused by the arrival, walk and return (or deriving-bike) processes.

To compute the service rate $\eta_{k}^{(N)}(t)$ for $1 \leq k \leq K$, it is seen from Figure 3 that State 0 (that is, the tagged station is empty) is a key, and it leads to the rate $\eta^{(N)}(t)$ $\frac{1}{1 - y_{0}^{(N)}(t)}$ with respect to $n$ consequent walks, where the $n$ consequent walks correspond to $n$ empty stations with probability $\left[y_{0}^{(N)}(t)\right]^{n}$ for $1 \leq n \leq \omega$. In the final walk with $n = \omega$, either the customer rent a bike at a nonempty station, or she directly leaves the bike sharing system if no bike is rented after $\omega$ consequent walks. Thus the number of the consequent walks to find an available station may be 1 with probability $y_{0}^{(N)}(t)$, 2 with $\left[y_{0}^{(N)}(t)\right]^{2}$, and generally, $n$ with $\left[y_{0}^{(N)}(t)\right]^{n}$ for $1 \leq n \leq \omega$. Thus for the virtual time-inhomogeneous
Figure 3: The state transitions for computing $\eta_k^{(N)}(t)$

For $l = 0$ (i.e., States 0), all the original $C$ bikes in the tagged station are rented to travel on the roads. For the other $N - 1$ stations, our computation for the number of bikes rented to travel on the roads is based on the mean-field theory (i.e., under an average setting), thus the expected number of bikes rented to travel on the roads is given by

$$(N - 1) \cdot \left[ C - \sum_{k=1}^{K} ky_k^{(N)}(t) \right],$$
where $\sum_{k=1}^{K} k y_k^{(N)}(t)$ is the expected number of bikes parked in any station, while $C - \sum_{k=1}^{K} k y_k^{(N)}(t)$ is the expected number of bikes rented to travel on the roads from any station. Thus, for the $N$ stations, the total expected number of bikes rented to travel on the roads is given by

$$C + (N - 1) \cdot \left[ C - \sum_{k=1}^{K} k y_k^{(N)}(t) \right].$$

Notice that the returning-bike process of each bike is persistent in the sense that the customer keeps finding an empty position in one station, it is easy to check that the return rate of each driving bike to the tagged station is given by

$$\mu + \mu y_k^{(N)}(t) + \mu \left[ y_k^{(N)}(t) \right]^2 + \mu \left[ y_k^{(N)}(t) \right]^3 + \cdots = \frac{1}{1 - y_k^{(N)}(t)},$$

where $\left[ y_k^{(N)}(t) \right]^n$ is the probability that a customer $n$ times continuously returns her bike to $n$ full stations. Thus we use the mean-field computation to obtain that for State 0 (for $l = 0$),

$$\xi_0^{(N)}(t) = \frac{1}{N} \left\{ C + (N - 1) \cdot \left[ C - \sum_{k=1}^{K} k y_k^{(N)}(t) \right] \cdot \frac{1}{1 - y_k^{(N)}(t)} \right\} \cdot \mu \frac{1}{1 - y_k^{(N)}(t)}.$$

Similarly, for States $l$ with $1 \leq l \leq C - 1$, we have

$$\xi_l^{(N)}(t) = \frac{\mu}{N} \cdot \frac{1}{1 - y_k^{(N)}(t)} \left\{ (C - l) + (N - 1) \cdot \left[ C - \sum_{k=1}^{K} k y_k^{(N)}(t) \right] \right\}.$$

Finally, for States $l$ with $C \leq l \leq K$, since all the original $C$ bikes are parked in the tagged station, we obtain

$$\xi_l^{(N)}(t) = \frac{\mu}{N} \cdot \frac{1}{1 - y_k^{(N)}(t)} \left\{ (N - 1) \cdot \left[ C - \sum_{k=1}^{K} k y_k^{(N)}(t) \right] \right\},$$

which is independent of the number $l = C, C + 1, \ldots, K$. This completes this proof. ■

**Remark 2** (1) From Equation (1), for the number of consequent walks, it may be useful to observe two special cases: (a) If $\omega = 0$, $y_k^{(N)}(t) = \lambda$. (b) If $\omega \to \infty$, then $y_k^{(N)}(t) = \lambda + \left[ y_k^{(N)}(t) \right] / \left[ 1 - y_k^{(N)}(t) \right].$

(2) The time-inhomogeneous $M(t)/M(t)/1/K$ queue is a fictitious queueing system corresponding to the number of bikes parked in the tagged station, while its virtual arrival
and virtual service rates are determined by means of the empirical measure process through some mean-field computation.

(3) It is seen from the proof of Theorem 1 that the different ages of "finding-bike attempts" and "returning-bike attempts" has not any influence on the mean-field computation due to the memoryless property of the exponential distributions and of the Poisson processes. Thus, the mean-field method can be successfully applied to our current analysis of the bike sharing system. However, it will be very difficult (or an open problem) to apply the mean-field method if there exist general distributions or general renewal processes in the bike sharing system.

In the remainder of this section, we set up a system of mean-field equations by means of the time-inhomogeneous $M(t)/M(t)/1/K$ queue whose state transition relation is depicted in Figure 2 with the arrival rate $\xi^{(N)}(t)$ for $0 \leq l \leq K - 1$, and with service rate $\eta^{(N)}(t) = \eta^{(N)}(t)$ for $1 \leq k \leq K$. For how to establish such mean-field equations, readers may refer to, such as, Li and Lui [48], Li et al. [45, 46] and Fricker and Gast [23] for more details.

To apply the mean-field theory, the number of bikes parked in the tagged station is described as the virtual time-inhomogeneous $M(t)/M(t)/1/K$ queue, thus we can set up a system of mean-field equations in terms of the (nonlinear) birth-death process corresponding to the $M(t)/M(t)/1/K$ queue. To this end, we denote by $Q(t)$ the queue length of the $M(t)/M(t)/1/K$ queue at time $t \geq 0$. Then it is seen from Figure 2 that $\{Q(t) : t \geq 0\}$ is a time-inhomogeneous continuous-time birth-death process whose infinite generator is given by

$$
V_{y^{(N)}(t)} = \begin{pmatrix}
B_1(t) & B_0(t) \\
B_2(t) & -\Theta^{(N)}_C(t) & \xi^{(N)}_C(t) \\
& \eta^{(N)}(t) & -\Theta^{(N)}_C(t) & \xi^{(N)}_C(t) \\
& & \ddots & \ddots & \ddots \\
& & & \eta^{(N)}(t) & -\Theta^{(N)}_C(t) & \xi^{(N)}_C(t) \\
& & & & \eta^{(N)}(t) & -\eta^{(N)}(t)
\end{pmatrix}, \quad (3)
$$

where

$$
\eta^{(N)}(t) = \lambda + \gamma y^{(N)}_0(t) \frac{1 - \left[ y^{(N)}_0(t) \right]^\omega}{1 - y^{(N)}_0(t)}.
$$
for $0 \leq l \leq C$

$$\xi_l^{(N)} (t) = \frac{\mu}{N} \frac{1}{1 - y_K^{(N)} (t)} \left\{ (C - l) + (N - 1) \left[ C - \sum_{k=1}^{K} k y_k^{(N)} (t) \right] \right\}$$

and

$$\Theta_l^{(N)} (t) = \xi_l^{(N)} (t) + \eta^{(N)} (t);$$

$$B_1 (t) = \begin{bmatrix}
    -\xi_0^{(N)} (t) & \xi_0^{(N)} (t) \\
    \eta^{(N)} (t) & -\Theta_1^{(N)} (t) & \xi_1^{(N)} (t) \\
    \vdots & \ddots & \ddots & \ddots \\
    \eta^{(N)} (t) & -\Theta_{C-2}^{(N)} (t) & \xi_{C-2}^{(N)} (t) \\
    \eta^{(N)} (t) & -\Theta_{C-1}^{(N)} (t)
\end{bmatrix}_{C \times C},$$

$$B_0 (t) = \begin{pmatrix} 0, 0, \ldots, 0, \xi_{C-1}^{(N)} (t) \end{pmatrix}^T$$

and

$$B_2 (t) = \begin{pmatrix} 0, 0, \ldots, 0, \eta^{(N)} (t) \end{pmatrix},$$

$A^T$ denotes the transpose of the vector (or matrix) $A$.

Using the birth-death process described in Figure 2, we obtain a system of mean-field (or ordinary differential) equations as follows:

$$\frac{d}{dt} y_0^{(N)} (t) = -\xi_0^{(N)} (t) y_0^{(N)} (t) + \eta^{(N)} (t) y_1^{(N)} (t),$$

for $1 \leq k \leq K - 1$

$$\frac{d}{dt} y_k^{(N)} (t) = \xi_{k-1}^{(N)} (t) y_{k-1}^{(N)} (t) - \left[ \xi_k^{(N)} (t) + \eta^{(N)} (t) \right] y_k^{(N)} (t) + \eta^{(N)} (t) y_{k+1}^{(N)} (t),$$

$$\frac{d}{dt} y_K^{(N)} (t) = \xi_{K-1}^{(N)} (t) y_{K-1}^{(N)} (t) - \eta^{(N)} (t) y_K^{(N)} (t).$$

Now, we write the above system of mean-field equations in a vector form as

$$\frac{d}{dt} y^{(N)} (t) = y^{(N)} (t) V y^{(N)} (t), \quad \tag{4}$$

with the boundary condition

$$y^{(N)} (t) e = 1, \quad \tag{5}$$

and with the initial condition

$$y^{(N)} (0) = g$$

where $g = (g_0, g_1, \ldots, g_K)$ with $g_i \geq 0$ for $0 \leq i \leq K$ and $\sum_{i=0}^{K} g_i = 1$, and $e$ is a column vector of ones with a suitable dimension in the context.
Remark 3 To deal with the time-inhomogeneous continuous-time birth-death process, readers may refer to Chapter 8 in Li [42] for more details, where the detailed literatures are surveyed both for the time-inhomogeneous queues and for the time-inhomogeneous Markov processes.

4 A Lipschitz Condition

In this section, we first establish a Lipschitz condition. Then we prove the existence and uniqueness of solution to the system of ordinary differential equations by means of the Lipschitz condition.

We write
\[ \frac{d}{dt}y(t) = y(t) V_y(t), \]  
with
\[ y(t) = 1, \quad y(0) = g. \]

where
\[ V_{y(t)} = \begin{pmatrix} -a(t) & a(t) \\ b(t) & -c(t) & a(t) \\ & \ddots & \ddots & \ddots \\ b(t) & -c(t) & a(t) \\ b(t) & -b(t) \end{pmatrix}, \]  
\[ b(t) = \lambda + \gamma y_0(t) \frac{1 - [y_0(t)]^\omega}{1 - y_0(t)}, \]
\[ a(t) = \mu \frac{1}{1 - y_K(t)} \left[ C - \sum_{k=1}^{K} ky_k(t) \right] \]
and
\[ c(t) = a(t) + b(t). \]

Obviously, that Equations (6) and (7) are a system of first-order ordinary differential equations.

To discuss the existence and uniqueness of solution to the system of ordinary differential equations (6) and (7), in what follows we need to establish a Lipschitz condition by means of a computational method given in Section 4.1 of Li et al. [45].
For simplicity of description, we need to suppress time $t$ from the vector $y(t)$ and its entries $y_k(t)$ for $0 \leq k \leq K$, and then rewrite Equations (6) and (7) in a simple form as

$$\frac{d}{dt}y = F(y), \quad ye = 1, y(0) = g,$$

where

$$F(y) = yV_y = (y_0, y_1, \ldots, y_K) \begin{pmatrix} -a & a \\ b & -c & a \\ \vdots & \ddots & \ddots \\ b & -c & a \\ b & -b \end{pmatrix},$$

$$b = \lambda + \frac{\gamma y_0 (1 - y_0^m)}{1 - y_0}, \quad a = \frac{\mu}{1 - y_K} \left( C - \sum_{k=1}^{K} k y_k \right),$$

$$c = \frac{\mu}{1 - y_K} \left( C - \sum_{k=1}^{K} k y_k \right) + \left( \lambda + \frac{\gamma y_0 (1 - y_0^m)}{1 - y_0} \right).$$

Let

$$F(y) = (F_0(y), F_1(y), \ldots, F_{K-1}(y), F_K(y)).$$

Then for $k = 0$

$$F_0(y) = -y_0 \frac{\mu}{1 - y_K} \left( C - \sum_{k=1}^{K} k y_k \right) + y_1 \left[ \lambda + \frac{\gamma y_0 (1 - y_0^m)}{1 - y_0} \right],$$

for $1 \leq i \leq K - 1$

$$F_i(y) = (y_{i-1} - y_i) \frac{\mu}{1 - y_K} \left( C - \sum_{k=1}^{K} k y_k \right) + (y_i - y_{i+1}) \left[ \lambda + \frac{\gamma y_0 (1 - y_0^m)}{1 - y_0} \right]$$

and for $k = K$

$$F_K(y) = y_{K-1} \frac{\mu}{1 - y_K} \left( C - \sum_{k=1}^{K} k y_k \right) - y_K \left[ \lambda + \frac{\gamma y_0 (1 - y_0^m)}{1 - y_0} \right].$$

Now, we define the norms of a vector $x = (x_0, x_1, \ldots, x_K)$ and a matrix $A = (a_{i,j})_{0 \leq i, j \leq K}$ as follows:

$$\|x\| = \max_{0 \leq i \leq K} \{|x_i|\}$$

and

$$\|A\| = \max_{0 \leq j \leq K} \left\{ \sum_{i=0}^{K} |a_{i,j}| \right\}.$$
It is easy to check that
\[ \|x A\| \leq \|x\| \|A\|. \]

From (41) of Li et al. [45], the matrix of partial derivatives of the vector function \( F(y) \) of dimension \( K + 1 \) is given by
\[
DF(y) = \begin{pmatrix}
\frac{\partial F_0(y)}{\partial y_0} & \frac{\partial F_1(y)}{\partial y_0} & \cdots & \frac{\partial F_K(y)}{\partial y_0} \\
\frac{\partial F_0(y)}{\partial y_1} & \frac{\partial F_1(y)}{\partial y_1} & \cdots & \frac{\partial F_K(y)}{\partial y_1} \\
\vdots & \vdots & \ddots & \vdots \\
\frac{\partial F_0(y)}{\partial y_K} & \frac{\partial F_1(y)}{\partial y_K} & \cdots & \frac{\partial F_K(y)}{\partial y_K}
\end{pmatrix}.
\]  

(10)

To establish the Lipschitz condition of the vector function \( F(y) \) of dimension \( K + 1 \), it is seen from Lemma 5 of Li et al. [45] that we need to provide an upper bound of the norm \( \|DF(y)\| \). To this end, it is necessary to first give an assumption with respect to the two key numbers \( y_0 \) and \( y_K \) as follows:

**Assumption of Problematic Stations:** Let \( \delta \) be a sufficiently small positive number. We assume that
\[ 0 \leq y_0, y_K \leq 1 - \delta. \]

Now, we provide some interpretation for practical rationality of the assumption of problematic stations. Firstly, the probability \( y_0(t) + y_K(t) \) of problematic stations is always smaller through management mechanism or control methods (for example, repositioning by trucks, price incentives, and applications of information technologies), thus it is natural to take the condition: \( 0 \leq y_0, y_K \leq 1 - \delta \) in practice. Secondly, Theorem 5 in Section 6 will further demonstrate from the steady-state viewpoint that \( \lim_{t \to +\infty} y_0(t) = p_0 \leq 1/2 \) and \( \lim_{t \to +\infty} y_K(t) = p_K \leq 1 - \delta \). Finally, if \( y_0(t) = 1 \), then \( y_k(t) = 0 \) for \( 1 \leq k \leq K \); and if \( y_K(t) = 1 \), then \( y_k(t) = 0 \) for \( 0 \leq k \leq K - 1 \). Therefore, such a case with either \( y_0(t) = 1 \) or \( y_K(t) = 1 \) will lead to the unavailability of the bike sharing system.

**Theorem 2**

(1) Under the assumption of problematic stations, \( \|DF(y)\| \leq M \), where
\[
M = 2\lambda + \gamma \frac{(\omega + 5)}{2} + \frac{\mu}{\delta} \left[ 1 + \frac{1}{\delta} \right] C + \frac{K(K + 1)}{2}.
\]

(2) The vector function \( F(y) \) of dimension \( K + 1 \) is continuous and also satisfies the Lipschitz condition for \( (t, y) \in [0, +\infty) \times \left\{ [0, 1 - \delta] \times [0, 1]^{K-1} \times [0, 1 - \delta] \right\} \).

(3) There exists a unique solution to the system of ordinary differential equations \( \frac{dy}{dt} = F(y), y_e = 1 \) and \( y(0) = g \) for \( (t, y) \in [0, +\infty) \times \left\{ [0, 1 - \delta] \times [0, 1]^{K-1} \times [0, 1 - \delta] \right\} \).
Proof: (1) It follows from (10) that
\[ \|DF(y)\| = \max_{0 \leq j \leq K} \left\{ \sum_{i=0}^{K} \left| \frac{\partial F_j(y)}{\partial y_i} \right| \right\}. \]

It is easy to check that
\[ \frac{\partial F_0(y)}{\partial y_0} = -\frac{\mu}{1 - y_K} \left( C - \sum_{k=1}^{K} ky_k \right) + \gamma y_1 \sum_{k=1}^{\omega-1} k y_0, \]
\[ \frac{\partial F_0(y)}{\partial y_1} = y_0 \frac{\mu}{1 - y_K} + \lambda + \gamma y_0 \sum_{k=0}^{\omega-1} y_k, \]
and for \(2 \leq i \leq K\)
\[ \frac{\partial F_0(y)}{\partial y_i} = y_0 \frac{i \mu}{1 - y_K}. \]

By using
\[ |y_k| \leq 1, 0 \leq k \leq K; \quad \frac{1}{1 - y_K} \leq \frac{1}{\delta}; \quad 0 \leq C - \sum_{k=1}^{K} ky_k \leq C, \]
we obtain
\[ \sum_{i=0}^{K} \left| \frac{\partial F_0(y)}{\partial y_i} \right| \leq \lambda + \frac{\mu}{\delta} \left( C + \frac{K(K+1)}{2} \right) + \gamma \frac{\omega(\omega + 3)}{2}. \]

Similarly, we obtain that for \(1 \leq j \leq K - 1\)
\[ \sum_{i=0}^{K} \left| \frac{\partial F_j(y)}{\partial y_i} \right| \leq 2\lambda + \frac{\mu}{\delta} \left( 2C + \frac{K(K+1)}{2} \right) + \gamma \frac{\omega(\omega + 5)}{2} \]
and
\[ \sum_{i=0}^{K} \left| \frac{\partial F_K(y)}{\partial y_i} \right| \leq \lambda + \frac{\mu}{\delta} \left[ \left( 1 + \frac{1}{\delta} \right) C + \frac{K(K+1)}{2} \right] + \gamma \frac{\omega(\omega + 3)}{2}. \]
Let
\[ \mathbf{M} = 2\lambda + \gamma \frac{\omega(\omega + 5)}{2} + \frac{\mu}{\delta} \left[ \left( 1 + \frac{1}{\delta} \right) C + \frac{K(K+1)}{2} \right]. \]

Then
\[ \|DF(y)\| = \max_{0 \leq j \leq K} \left\{ \sum_{i=0}^{K} \left| \frac{\partial F_j(y)}{\partial y_i} \right| \right\} \leq \mathbf{M}. \]

(2) By means of Lemma 5 in Li et al. [45], we obtain that for any two vectors \(x, y \in [0, 1 - \delta] \times [0, 1]^{K-1} \times [0, 1 - \delta],\)
\[ \|F(x) - F(y)\| \leq \sup_{0 \leq t \leq 1} \|DF(x + t(y - x))\| \|y - x\| \leq \mathbf{M} \|y - x\|. \]
This shows that $F(y)$ is continuous and also satisfies the Lipschitz condition for $(t, y) \in [0, +\infty) \times \left\{ [0, 1 - \delta] \times [0, 1]^{K-1} \times [0, 1 - \delta] \right\}$.

(3) Notice that $F(y)$ is continuous and also satisfies the Lipschitz condition for $(t, y) \in [0, +\infty) \times \left\{ [0, 1 - \delta] \times [0, 1]^{K-1} \times [0, 1 - \delta] \right\}$, it follows from Chapter 1 of Hale [35] that there exists a unique solution to the system of ordinary differential equations $\frac{dy}{dt} = F(y)$, $ye = 1$ and $y(0) = g$ for $(t, y) \in [0, +\infty) \times \left\{ [0, 1 - \delta] \times [0, 1]^{K-1} \times [0, 1 - \delta] \right\}$. This completes the proof. □

In the remainder of this section, we set up a simple relation between the two systems of ordinary differential equations (4) and (5); and (6) and (7) through a limiting assumption $\lim_{N \to \infty} y^{(N)}(t) = y(t)$, the correctness of which will be proved in the next section. To this end, from Equation (4) we set

$$G\left(y^{(N)}(t)\right) = y^{(N)}(t) V_{y^{(N)}(t)}$$

or a simple form by suppressing $t$

$$G\left(y^{(N)}\right) = y^{(N)} V_{y^{(N)}}.$$ 

It follows from (4) and (5) that

$$\frac{d}{dt} y^{(N)} = G\left(y^{(N)}\right), \quad ye = 1.$$ 

By using $\lim_{N \to \infty} y^{(N)}(t) = y(t)$, we obtain that for $0 \leq k \leq K - 1$

$$\lim_{N \to \infty} \xi^{(N)}_k (t) = a(t),$$

and

$$\lim_{N \to \infty} \eta^{(N)} (t) = b(t).$$

Thus comparing the matrix $G\left(y^{(N)}\right)$ with the matrix $F(y)$, we obtain

$$\lim_{N \to \infty} G\left(y^{(N)}\right) = F(y).$$

Since

$$\frac{d}{dt}\left(\lim_{N \to \infty} y^{(N)}(t)\right) = \frac{d}{dt} y = F(y)$$

and

$$\lim_{N \to \infty} \left(\frac{d}{dt} y^{(N)}(t)\right) = \lim_{N \to \infty} G\left(y^{(N)}\right) = F(y),$$

we obtain

$$\frac{d}{dt}\left(\lim_{N \to \infty} y^{(N)}(t)\right) = \lim_{N \to \infty} \left(\frac{d}{dt} y^{(N)}(t)\right).$$
5 The Martingale Limit

In this section, we provide a martingale limit (i.e., the weak convergence in the Skorohod space) for the sequences of empirical measure Markov processes in the bike sharing system.

We define a \((K+1)\)-dimensional simplex
\[
F = \left\{ f = (f_0, f_1, \ldots, f_{K-1}, f_K) : f_k \geq 0 \text{ and } \sum_{k=0}^{K} f_k = 1 \right\},
\]
and endow \(F\) with the metric
\[
d(x, y) = \sup_{0 \leq k \leq K} \frac{|x_k - y_k|}{k + 1}, \quad x, y \in F.
\]
Obviously, \(d(x, y) \leq 1\) for \(x, y \in F\). Under the metric, the space \(F\) is compact, complete and separable. Let \(D_F[0, +\infty)\) be the space of right-continuous paths with left limits in \(F\) endowed with the Skorohod metric. For the Skorohod space and the weak convergence, readers may refer to Billingsley [6] and Chapter 3 of Ethier and Kurtz [20] for more details.

For the the empirical measure \(Y^{(N)}(t)\), we write
\[
W\left( Y^{(N)}(t) \right) = \begin{pmatrix}
A_1 (t) & A_0 (t) \\
A_2 (t) & -\Gamma_{C}^{(N)} (t) & \alpha_{C}^{(N)} (t) \\
\beta^{(N)} (t) & -\Gamma_{C}^{(N)} (t) & \alpha_{C}^{(N)} (t) \\
& \ddots & \ddots & \ddots \\
& & \beta^{(N)} (t) & -\Gamma_{C}^{(N)} (t) & \alpha_{C}^{(N)} (t) \\
& & & \beta^{(N)} (t) & -\beta^{(N)} (t)
\end{pmatrix},
\]
where
\[
\beta^{(N)} (t) = \lambda + \gamma Y_0^{(N)} (t) \frac{1 - \left[ Y_0^{(N)} (t) \right]^\omega}{1 - Y_0^{(N)} (t)},
\]
for \(0 \leq l \leq C\)
\[
\alpha_l^{(N)} (t) = \frac{\mu}{N} \frac{1}{1 - Y_K^{(N)} (t)} \left\{ (C - l) + (N - 1) \left[ C - \sum_{k=1}^{K} k Y_k^{(N)} (t) \right] \right\}
\]
and
\[
\Gamma_l^{(N)} (t) = \alpha_l^{(N)} (t) + \beta^{(N)} (t);
\]
\[
A_1 (t) = \begin{pmatrix}
-\alpha_0^{(N)} (t) & \alpha_0^{(N)} (t) \\
\beta^{(N)} (t) & -\Gamma_1^{(N)} (t) & \alpha_1^{(N)} (t) \\
& \ddots & \ddots & \ddots \\
& & \beta^{(N)} (t) & -\Gamma_{C-2}^{(N)} (t) & \alpha_{C-2}^{(N)} (t) \\
& & & \beta^{(N)} (t) & -\Gamma_{C-1}^{(N)} (t)
\end{pmatrix}_{C \times C},
\]

and

\[
A_0 (t) = \left( 0, 0, \ldots, 0, \alpha_{C-1}^{(N)} (t) \right)^T
\]

For the sequence \( \{ Y^{(N)} (t), t \geq 0 \} \) of the empirical measure Markov processes, by means of a similar computation for setting up the system of mean-field equations (4) and (5), we can obtain a system of stochastic differential equations as follows:

\[
\frac{d}{dt} Y^{(N)} (t) = Y^{(N)} (t) W \left( Y^{(N)} (t) \right),
\]

with the boundary condition

\[
Y^{(N)} (t) e = 1.
\]

For the random vector \( Y (t) = (Y_0 (t), Y_1 (t), \ldots, Y_K (t)) \), we write

\[
W (Y (t)) = \begin{pmatrix}
-\alpha (t) & \alpha (t) \\
\beta (t) & -\tau (t) & \alpha (t) \\
& \ddots & \ddots & \ddots \\
& & \beta (t) & -\tau (t) & \alpha (t) \\
& & & \beta (t) & -\beta (t)
\end{pmatrix},
\]

\[
\beta (t) = \lambda + \gamma Y_0 (t) \frac{1 - [Y_0 (t)]^\omega}{1 - Y_0 (t)},
\]

\[
\alpha (t) = \mu \frac{1}{1 - Y_K (t)} \left[ C - \sum_{k=1}^{K} kY_k (t) \right]
\]

and

\[
\tau (t) = \alpha (t) + \beta (t).
\]

Based on this, we write

\[
\frac{d}{dt} Y (t) = Y (t) W (Y (t)),
\]
with the boundary condition
\[ Y(t) e = 1 \]  \hspace{1cm} (15)
and the initial condition
\[ Y(0) = g. \]  \hspace{1cm} (16)

Using a similar analysis to that in Theorem 2, we can show that there exists a unique solution to the system of stochastic differential equations (14) to (16), where the assumption of problematic stations is also necessary.

The following lemma is useful for discussing the mean-field limit \( Y(t) = \lim_{N \to \infty} Y^{(N)}(t) \) for \( t \geq 0 \).

**Lemma 1** For the sequence \( \{Y^{(N)}(t), t \geq 0\} \) of Markov processes,
\[ M^{(N)}(t) = Y^{(N)}(t) - Y^{(N)}(0) - \int_0^t \left\{ Y^{(N)}(x) W\left(Y^{(N)}(x)\right) \right\} dx \]  \hspace{1cm} (17)
is a martingale with respect to \( N \geq 1 \).

**Proof:** Note that the generator of the Markov process \( \{Y^{(N)}(t), t \geq 0\} \) is given by the matrix \( W\left(Y^{(N)}(t)\right) \), thus using Dynkin’s formula, e.g., see Equation (III.10.13) in Rogers and Williams [61] or Page 162 in Ethier and Kurtz [20], and it is easy to check that \( M^{(N)}(t) \) is a martingale with respect to \( N \geq 1 \). This completes the proof. \( \blacksquare \)

The following theorem gives the mean-field limit of the sequence \( \{Y^{(N)}(t), t \geq 0\} \) of Markov processes. Notice that this mean-field limit is a key for proving the asymptotic independence of the queueing processes in the bike sharing system.

**Theorem 3** If \( Y^{(N)}(0) \) converges weakly to \( Y(0) \in \mathcal{F} \) as \( N \to \infty \), then \( \{Y^{(N)}(t), N \geq 1\} \) converges weakly in \( D_F[0, +\infty) \) endowed with the Skorohod topology to the solution \( Y(t) \) to the system of stochastic differential equations (14) to (16).

**Proof:** The proof can be completed by the following three steps.

**Step One:** The relative compactness of \( Y^{(N)}(t) \) in \( D_F[0, +\infty) \)

Note that the space \( \mathcal{F} \) is of dimension \( K + 1 \), we use Paragraphs 8.6 to 8.9 of Chapter 3 of Ethier and Kurtz [20] (see Pages 137 to 139) to prove the relative compactness of \( Y^{(N)}(t) \) in \( D_F[0, +\infty) \). To that end, we only need to indicate three conditions given in Chapter 3 of Ethier and Kurtz [20] as follows:
(a) **EK7.7** For every $\varepsilon > 0$ and rational $r \geq 0$, there exists a compact set $\Gamma_{\varepsilon,r} \in \mathcal{F}$ such that

$$\lim_{N \to \infty} \inf_{y \in \Gamma_{\varepsilon,r}} P \{ d \left( \Y^{(N)}(t) , y \right) < \varepsilon \} \geq 1 - \varepsilon.$$  

(b) **EK8.37** For all $T > 0$, there exists $\chi > 0$, $D > 0$ and $\tau > 1$ such that for all $N \geq 1$ and all $0 \leq h \leq t \leq T + 1$

$$E \left[ d^2 \left( \Y^{(N)}(t+h) , \Y^{(N)}(t) \right) d^2 \left( \Y^{(N)}(t) , \Y^{(N)}(t-h) \right) \right] \leq Dh^\tau.$$  

(c) **EK8.30** For the above value $\chi > 0$

$$\lim_{\delta \to 0} \lim_{N \to \infty} \sup E \left[ d^\chi \left( \Y^{(N)}(\delta) , \Y^{(N)}(0) \right) \right] = 0.$$  

In what follows we prove each of the three conditions.

Firstly, we prove (a) EK7.7. Taking $\Gamma_{\varepsilon,r} = \mathcal{F}$, and noting that the space $\mathcal{F}$ is compact, this directly gives the proof of (a) EK7.7 through a similar analysis to that in Theorem 7.2 of Chapter 3 of Ethier and Kurtz [20] (see Pages 128 to 129).

Secondly, we prove (b) EK8.37. Let $\chi = 2$. Then by using Remark 8.9 of Chapter 3 of Ethier and Kurtz [20] (see Page 139), we obtain

$$E \left[ d^2 \left( \Y^{(N)}(t+h) , \Y^{(N)}(t) \right) d^2 \left( \Y^{(N)}(t) , \Y^{(N)}(t-h) \right) \right] = E \left[ d \left( \Y^{(N)}(t+h) , \Y^{(N)}(t) \right) \right] \cdot E \left[ d \left( \Y^{(N)}(t) , \Y^{(N)}(t-h) \right) \right] \leq \left( \lambda + \mu + \gamma \right) h^2,$$

(18)

this indicates that (b) EK8.37 holds for the parameters: $T$, $t$, $h$, $D = (\lambda + \mu + \gamma)^2$ and $\tau = 2$.

Finally, we prove (c) EK8.30. It follows from (18) that

$$E \left[ d^\chi \left( \Y^{(N)}(\delta) , \Y^{(N)}(0) \right) \right] \leq \left( \lambda + \mu + \gamma \right) \delta^\chi,$$

this gives

$$\lim_{\delta \to 0} \lim_{N \to \infty} \sup E \left[ d^\chi \left( \Y^{(N)}(\delta) , \Y^{(N)}(0) \right) \right] = 0.$$

**Step Two:** The weakly convergent limit of $\{ \Y^{(N)}(t) \}$ has almost surely continuous sample paths for $t \geq 0$

For $Y \in D_{\mathcal{F}[0, +\infty)}$, we define

$$J(Y, u) = \sup_{0 \leq t \leq u} \left\{ d \left( Y(t) , Y(t^-) \right) \right\}$$
\[ J(Y) = \int_0^{+\infty} e^{-u} J(Y, u) \, du. \]

Using Theorem 10.2 (a) of Chapter 3 of Ethier and Kurtz [20] (see Page 148), it is easy to check that for all \( N \geq 1 \) and \( u \geq 0 \), \( J(Y^{(N)}(u)) \leq 1/N \) almost surely, which leads to \( J(Y^{(N)}) \leq 1/N \) almost surely. Thus, as \( N \to \infty \), if \( Y^{(N)}(t) \Rightarrow Y(t) \), then \( Y(t) \) is almost surely continuous if and only if \( J(Y^{(N)}) \Rightarrow 0 \), where “\( \Rightarrow \)” denotes the weak convergence.

**Step Three:** The martingale limit

Given the continuity of any limit point, using the continuous mapping theorem (e.g., see Whitt [78]), we prove that Equations (14) and (15) are satisfied by any limit point: \( Y(t) = \lim_{N \to \infty} Y^{(N)}(t) \) for \( t \geq 0 \) as follows:

Using the martingale central limit theorem (e.g., see Theorem 1.4 of Chapter 7 of Ethier and Kurtz [20] in Page 339), it follows from Lemma 1 that as \( N \to \infty \), \( \langle M_k^{(N)}(t) \rangle \to 0 \) for \( t \geq 0 \), where \( \langle \cdot \rangle \) denotes the quadratic variation. Note that \( \langle M_k^{(N)}(t) \rangle \) only changes at time \( t \) when \( M_k^{(N)}(t) \) jumps, and it increases by the square of the jump sizes, while the jump sizes are of order \( 1/N \) and the jump rates are of order \( N \). Using a similar analysis to that in Theorem 2 of Section 4, we can prove that there exists a unique solution to the system of stochastic differential equations (14) and (15) for any initial value. Noting the relative compactness of \( Y^{(N)}(t) \) in \( D_F[0, +\infty) \) and using Chapter 3 of Ethier and Kurtz [20], this proves that the sequence \( \{ Y^{(N)}(t), N \geq 1 \} \) of Markov processes converges in the space \( D_F[0, +\infty) \) to the Markov process \( \{ Y(t), N \geq 1 \} \). This completes the proof. ■

Finally, it is necessary to provide some interpretation on Theorem 3. If \( \lim_{N \to \infty} Y^{(N)}(0) = y(0) = g \in \Omega \) in probability, then Theorem 3 shows that \( Y(t) = \lim_{N \to \infty} Y^{(N)}(t) \) is concentrated on the trajectory \( \mathcal{S}_g = \{ y(t, g) : t \geq 0 \} \), where \( y(t, g) = E[ Y(t) | Y(0) = g] \), and \( y(0, g) = g \). This indicates the functional strong law of large numbers for the time evolution of the fraction of each state of this bike sharing model, thus the sequence \( \{ Y^{(N)}(t), t \geq 0 \} \) of Markov processes converges weakly to the expected fraction vector \( y(t, g) \) as \( N \to \infty \), that is, for any \( T > 0 \)

\[
\lim_{N \to \infty} \sup_{0 \leq s \leq T} \left\| Y^{(N)}(s) - y(s, g) \right\| = 0 \quad \text{in probability.} \tag{19}
\]

**Remark 4** To study the weak convergence in the Skorohod space for the sequence \( \{ Y^{(N)}(t), t \geq 0 \} \) of Markov processes, there are three frequently used methods: (1) Operator semigroups, e.g., Vvedenskaya et al. [73], Li and Lui [38], and Li et al. [45, 46]:
(2) martingale limits, for example, Turner [70], and Graham [29, 30]; and (3) density-dependent jump Markov processes, for instance, Chapter 11 of Ethier and Kurtz [20], and Mitzenmacher [55]. Here, this paper takes the method of martingale limits to establish an outline of such a proof.

**Remark 5** Under the weak convergence in the Skorohod space for the sequence \( \{ Y^{(N)}(t), \ t \geq 0 \} \) of Markov processes, Theorem 3 demonstrates the correctness of the system of mean-field equations (6) and (7), i.e., as \( N \to \infty \), Equations (6) and (7) are the limits of Equations (4) and (5), respectively.

6 The Fixed Point and Nonlinear Analysis

In this section, we analyze the fixed point of the limiting system of mean-field equations. We first prove that the fixed point is unique in terms of the Birkhoff center. Then we simply analyze the asymptotic independence of the bike sharing system, and also discuss the limiting interchangeability with respect to \( N \to \infty \) and \( t \to +\infty \). Notice that the uniqueness of the fixed point is a key in numerical computation of the fixed point in terms of a system of nonlinear equations.

Let the vector \( p \) be the fixed point of the limiting expected fraction vector \( y(t) \). Then

\[
    p = \lim_{t \to +\infty} y(t),
\]

where \( p = (p_0, p_1, \ldots, p_{K-1}, p_K) \) and

\[
    p_k = \lim_{t \to +\infty} y_k(t), \quad 0 \leq k \leq K.
\]

This gives

\[
    p = \lim_{t \to +\infty} \lim_{N \to \infty} y^{(N)}(t).
\]

We write

\[
    b(p) = \lim_{t \to +\infty} b(t) = \lambda + \gamma p_0 \frac{1 - p_0^\gamma}{1 - p_0},
\]

\[
    a(p) = \lim_{t \to +\infty} a(t) = \mu \frac{1}{1 - p_K} \left( C - \sum_{k=1}^{K} kp_k \right)
\]

and

\[
    c(p) = a(p) + b(p).
\]
Thus it follows from (8) that

\[ V_p = \lim_{t \to +\infty} V_y(t) = \begin{pmatrix}
-a(p) & a(p) \\
b(p) & -c(p) & a(p) \\
\ddots & \ddots & \ddots \\
& b(p) & -c(p) & a(p) \\
& b(p) & -b(p)
\end{pmatrix}, \tag{20} \]

which is the infinitesimal generator of an irreducible, aperiodic and positive-recurrent birth-death process due to the fact that \( a(p) > 0, b(p) > 0 \), and the size of the matrix \( V_p \) is finite.

On the other hand, it is easy to see that the matrix \( V_y(t) \) given in (13) is also the infinitesimal generator of a continuous-time birth-death process with state space \( \{0, 1, \ldots, K\} \). Since \( a(t) > 0, b(t) > 0 \) and \( V_y(t)e = 0 \), the birth-death process \( V_y(t) \) is irreducible, aperiodic and positive-recurrent. In this case, it is seen from Vvedenskaya et al. [73] or Mitzenmacher [55] that

\[
\lim_{t \to +\infty} \frac{d}{dt} y(t) = 0
\]
or

\[
\lim_{t \to +\infty} y(t) V_y(t) = 0.
\]

Thus it follows from (6) and (7) that

\[
\begin{cases}
p V_p = 0, \\
p e = 1.
\end{cases} \tag{21}
\]

### 6.1 Expressions for the fixed point

Notice that the matrix \( V_p \) may be viewed as the infinitesimal generator of an irreducible, aperiodic and positive-recurrent birth-death process who corresponds to the \( M/M/1/K \) queue with arrival rate \( a(p) \) and service rate \( b(p) \). Let \( \rho(p) = a(p)/b(p) \). It is easy to check that (a) if \( \rho(p) = 1 \), then

\[
p_k = \frac{1}{K+1}, \quad 0 \leq k \leq K;
\]

and (b) if \( \rho(p) \neq 1 \), then

\[
p_k = \rho^k(p) \frac{1 - \rho(p)}{1 - \rho^{K+1}(p)}, \quad 0 \leq k \leq K. \tag{23}
\]
This demonstrates that if $\rho(p) \neq 1$, then the probability vector $p$ is the fixed point of the following nonlinear vector equation

$$p = \left( \frac{1 - \rho(p)}{1 - \rho^{K+1}(p)}, \rho(p), \frac{1 - \rho(p)}{1 - \rho^{K+1}(p)}, \ldots, \rho^K(p), \frac{1 - \rho(p)}{1 - \rho^{K+1}(p)} \right). \quad (24)$$

Notice that Li [44] gave some iterative algorithms for computing the fixed point $p$ by means by the system of nonlinear equations (21) or (24).

In the following, we set up another nonlinear vector equation satisfied by the fixed point $p$. Differently from Equation (24), the new nonlinear vector equation can be employed to study a more general block-structure bike sharing system with either a Markovian arrival process (MAP) or a phase-type (PH) service time, e.g., see Li [42] and Li and Lui [48] for more details.

To solve the system of equations (21) from a more general setting, let $r_{\min}(p)$ and $g_{\min}(p)$ be the minimal nonnegative solutions to the following two nonlinear equations

$$a(p) - [a(p) + b(p)] r(p) + b(p) r^2(p) = 0$$

and

$$a(p) g^2(p) - [a(p) + b(p)] g(p) + b(p) = 0,$$

respectively. Then

$$r_{\min}(p) = \frac{a(p) + b(p) - |a(p) - b(p)|}{2b(p)}$$

and

$$g_{\min}(p) = \frac{a(p) + b(p) - |a(p) - b(p)|}{2a(p)}.$$

Clearly, we have

$$r_{\min}(p) b(p) = g_{\min}(p) a(p) = \frac{a(p) + b(p) - |a(p) - b(p)|}{2}.$$

Let

$$\Omega_p = \left\{ \left( r_{\min}(p), \frac{1}{r_{\min}(p)} \right) : a(p) > b(p) \right\} \cup \left\{ \left( \frac{1}{g_{\min}(p)}, g_{\min}(p) \right) : a(p) < b(p) \right\} \cup \{(1, 1) : a(p) = b(p)\}.$$ 

Then for a pair $(r(p), g(p)) \in \Omega_p$, we have

$$r(p) g(p) = 1.$$
The following theorem illustrates that each element of the fixed point $p$ is a combinatorial sum of two geometric solutions if $a(p) \neq b(p)$.

**Theorem 4** If $a(p) \neq b(p)$ and $(r(p), g(p)) \in \Omega_p$, then for $0 \leq k \leq K$,

$$p_k = c_1r^k(p) + c_2g^{K-k}(p),$$

where the two constants $c_1$ and $c_2$ are determined by

$$c_1 = \frac{\frac{g^{K-1}(p)b(p) - g^K(p)a(p)}{a(p) - r(p)b(p)}}{\frac{g^{K-1}(p)b(p) - g^K(p)a(p)}{a(p) - r(p)b(p)}}$$

and

$$c_2 = \frac{\frac{g^{K-1}(p)b(p) - g^K(p)a(p)}{a(p) - r(p)b(p)}}{\frac{g^{K-1}(p)b(p) - g^K(p)a(p)}{a(p) - r(p)b(p)}}$$

**Proof:** If $a(p) \neq b(p)$, then the proof contains three steps. Firstly, it is easy to check that for $1 \leq k \leq K - 1$, $p_k = c_1r^k(p) + c_2g^{K-k}(p)$ with $(r(p), g(p)) \in \Omega_p$ can satisfy the equation

$$p_{k-1}a(p) - p_k[a(p) + b(p)] + p_{k+1}b(p) = 0.$$  

Secondly, for $k = 0, K$ we obtain

$$- [c_1 + c_2g^K(p)] a(p) + [c_1r(p) + c_2g^{K-1}(p)] b(p) = 0$$

and

$$[c_1r^{K-1}(p) + c_2g(p)] a(p) - [c_1r^K(p) + c_2] b(p) = 0.$$  

It follows from (27) and (28) that

$$c_1 = \frac{g^{K-1}(p)b(p) - g^K(p)a(p)}{a(p) - r(p)b(p)}c_2$$

and

$$c_1 = \frac{b(p) - g(p)a(p)}{r^{K-1}(p)a(p) - r^K(p)b(p)}c_2$$

respectively. Notice that $r(p)g(p) = 1$ for $(r(p), g(p)) \in \Omega_p$, we have

$$\frac{b(p) - g(p)a(p)}{r^{K-1}(p)a(p) - r^K(p)b(p)} = \frac{1}{r^{K-1}(p)}\left[\frac{b(p) - g(p)a(p)}{a(p) - r(p)b(p)}\right] = \frac{g^{K-1}(p)b(p) - g^K(p)a(p)}{a(p) - r(p)b(p)}.$$

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this demonstrates that (29) is the same as (30). Finally, using (25) and $\sum_{k=0}^{K} p_k = 1$ we obtain
\[ c_1 \frac{1 - r^{K+1} (p)}{1 - r (p)} + c_2 \frac{1 - g^{K+1} (p)}{1 - g (p)} = 1, \]
which, together with (29), follows (26) in order to express the constants $c_1$ and $c_2$. This completes the proof. □

Using Theorem 4, the probability vector $p$ is the fixed point of the following nonlinear vector equation
\[ p = (c_1 + c_2 g^K (p), c_1 r (p) + c_2 g^{K-1} (p), \ldots, c_1 r^{K-1} (p) + c_2 g (p), c_1 r^K (p) + c_2). \] (31)

We write
\[ \mathbb{S}_p = \{ p : pV_p = 0, pe = 1 \}. \]
Then it is clear that
\[ \mathbb{S}_p = \left\{ p : p_k = \rho^k (p) \frac{1 - \rho (p)}{1 - \rho^{K+1} (p)}, 0 \leq k \leq K \right\} = \left\{ p : p_k = c_1 r^k (p) + c_2 g^{K-k} (p), 0 \leq k \leq K \right\}. \]

Since the equation $pV_p = 0$ (or $p_k = \rho^k (p) [1 - \rho (p)] / [1 - \rho^{K+1} (p)]$, or $p_k = c_1 r^k (p) + c_2 g^{K-k} (p), 0 \leq k \leq K$) is nonlinear, it is possible for a more complicated bike sharing system that there are multiple elements (solutions) in the set $\mathbb{S}_p$. In fact, an argument by analytic function indicates that the elements of the set $\mathbb{S}_p$ are isolated.

To describe the isolated element structure of the set $\mathbb{S}_p$, we often need to use the Birkhoff center of the mean-field dynamic system, which leads to check whether the fixed point is unique or nor.

### 6.2 The Birkhoff center and uniqueness

For the Birkhoff center, our discussion includes the following two cases:

**Case one**: $N \to \infty$. In this case, we denote a solution to the system of differential equations (6) and (7) by $\Phi (t)$. Thus, the Birkhoff center of the solution $\Phi (t)$ is defined as
\[ \Theta = \left\{ \overline{p} \in F : \overline{p} = \lim_{k \to \infty} \Phi (t_k) \text{ for any scale sequence } \{ t_k \} \text{ with } t_l \geq 0 \text{ for } l \geq 1 \text{ and } \lim_{k \to \infty} t_k = +\infty \right\}. \]
Notice that perhaps $\Theta$ contains the limit cycles or the stationary points (i.e., the local extremum points or the saddle points), it is clear that $S_p \subset \Theta$. Obviously, the limiting empirical measure Markov process $\{Y(t) : t \geq 0\}$ spends most of its time in the Birkhoff center $\Theta$.

**Case two:** $t \to +\infty$. In this case, we write

$$\pi^{(N)} = \lim_{t \to +\infty} Y^{(N)}(t),$$

since for each $N = 1, 2, 3, \ldots$, the bike sharing system with $N$ identical stations is stable.

Let

$$\Xi = \left\{ \pi \in \mathcal{F} : \pi = \lim_{k \to \infty} \pi^{(N_k)} \text{ for any positive integer sequence } \{N_k\} \text{ with } 1 \leq N_1 \leq N_2 \leq N_3 \leq \cdots \text{ and } \lim_{k \to \infty} N_k = \infty \right\}. $$

It is easy to see that

$$S_p \subset \Xi \subset \Theta.$$ 

Therefore, the set $\Theta - S_p$ contains the limit cycles or the saddle points.

Notice that

$$\begin{align*}
\begin{cases}
p V_p = 0, \\
p e = 1,
\end{cases}
\end{align*}$$

this gives that for $k = 0$

$$-\mu p_0 (1-p_0) \left( C - \sum_{k=1}^{K} kp_k \right) + p_1 \left[ (1-p_0) + \gamma p_0 (1-p_0^\omega) \right] (1-p_K) = 0, \quad (32)$$

for $1 \leq k \leq K-1$

$$-\mu (1-p_0) \left( C - \sum_{k=1}^{K} kp_k \right) (p_{k-1} - p_k) + \left[ (1-p_0) + \gamma p_0 (1-p_0^\omega) \right] (1-p_K) (p_k - p_{k+1}) = 0, \quad (33)$$

and for $k = K$

$$-\mu p_{K-1} (1-p_0) \left( C - \sum_{k=1}^{K} kp_k \right) + p_K \left[ (1-p_0) + \gamma p_0 (1-p_0^\omega) \right] (1-p_K) = 0, \quad (34)$$
with the boundary condition

\[ p_0 + p_1 + p_2 + \cdots + p_K = 1. \tag{35} \]

Notice that under the condition \( p_0, p_K < 1 \) (i.e., the assumption of problematic stations), the system of nonlinear equations (21) is the same as the system of nonlinear equations (32) to (35).

The following theorem gives an important result: The fixed point is unique. Notice that the uniqueness of the fixed point plays a key role in performance numerical analysis of the bike sharing system. On the other hand, this proof uses the system of nonlinear equations (32) to (35), but the two special solutions \((1, 0, \ldots, 0, 0)\) and \((0, 0, \ldots, 0, 1)\) are excluded from the set \( S_p \).

**Theorem 5** Let \( |S_p| \) denote the number of elements of the set \( S_p \). Then \( |S_p| = 1 \). This shows that the fixed point is unique.

**Proof:** This proof has two parts: (1) The existence of the fixed point \( p \), which is easily dealt with by the fact that \( p \) is the stationary probability vector of the ergodic birth-death process \( V_p \); and (2) the uniqueness of the fixed point \( p \), which can be proved by means of the unique point of intersection either between the quadratic function \( f_0 (p_0) \) and the polynomial function \( h_0 (p_0) \), or between the quadratic function \( f_n (p_n) \) and the linear function \( h_n (p_n) \) for \( 1 \leq n \leq K \) as follows.

Based on the system of nonlinear equations (32) to (35), the uniqueness of the fixed point \( p \) is proved through the following three steps:

**Step one:** Analyzing \( p_0 \). In this case, we write

\[ f_0 (p_0) = \mu p_0 (1 - p_0) \left( C - \sum_{k=1}^{K} kp_k \right) \]

and

\[ h_0 (p_0) = p_1 \left[ \lambda (1 - p_0) + \gamma p_0 (1 - p_0) \right] (1 - p_K). \]

It is easy to check that

\[ f_0 (0) = 0, \ f_0 (1) = 0, \ f_0 \left( \frac{1}{2} \right) = \frac{1}{4} \mu \left( C - \sum_{k=1}^{K} kp_k \right) > 0, \]
and for \( p_0 \in (0, 1) \)

\[
\frac{d}{dp_0} f_0 (p_0) = (1 - 2p_0) \mu \left( C - \sum_{k=1}^{K} kp_k \right)
\]

\[
= \begin{cases} 
> 0, & 0 < p_0 < \frac{1}{2}, \\
= 0, & p_0 = \frac{1}{2}, \\
< 0, & \frac{1}{2} < p_0 < 1,
\end{cases}
\]

and

\[
\frac{d^2}{d (p_0)^2} f_0 (p_0) = -2 \mu \left( C - \sum_{k=1}^{K} kp_k \right) < 0,
\]

this demonstrates that \( f_0 (p_0) \) is a concave function with the maximal value \( f_0 \left( \frac{1}{2} \right) > 0 \) at \( p_0 = 1/2 \).

Now, we analyze the polynomial function \( h_0 (p_0) \) for \( p_0 \in (0, 1) \). It is easy to see that

\[
h_0 (0) = \lambda p_1 (1 - p_K) > 0, \quad h_0 (1) = 0.
\]

For \( p_0 \in (0, 1) \)

\[
\frac{d}{dp_0} h_0 (p_0) = \left[ \gamma - \lambda - \gamma (1 + \omega) p_0^\omega \right] p_1 (1 - p_K)
\]

\[
= \begin{cases} 
> 0, & p_0 > \sqrt[\omega]{\frac{\gamma - \lambda}{\gamma (1 + \omega)}}, \\
= 0, & p_0 = \sqrt[\omega]{\frac{\gamma - \lambda}{\gamma (1 + \omega)}}, \\
< 0, & p_0 < \sqrt[\omega]{\frac{\gamma - \lambda}{\gamma (1 + \omega)}},
\end{cases}
\]

Since \( h_0 (0) > 0 \) and \( h_0 (1) = 0 \), it is seen from (36) that only one case: \( p_0 < \sqrt[\omega]{\frac{\gamma - \lambda}{\gamma (1 + \omega)}} \) can hold; while the other two cases are incorrect because the derivative \( \frac{d}{dp_0} h_0 (p_0) \geq 0 \) for \( p_0 \in (0, 1) \) can not result in such two values: \( h_0 (0) > 0 \) and \( h_0 (1) = 0 \). Thus we obtain

\[
p_0 < \sqrt[\omega]{\frac{\gamma - \lambda}{\gamma (1 + \omega)}} < \sqrt[\omega]{\frac{1}{(1 + \omega)}} \leq 1.
\]

Note that for \( p_0 \in (0, 1) \)

\[
\frac{d^2}{d (p_0)^2} h_0 (p_0) = -\gamma \omega (1 + \omega) p_0^{\omega-1} p_1 (1 - p_K) < 0,
\]

thus \( h_0 (p_0) \) is a decreasing and concave function from Point \((0, h_0 (0))\) to \((1, 0)\) without any extreme value.
Based on the above analysis, it is seen from Figure 4 (a) that there exists a unique solution to the nonlinear equation \( f_0(p_0) = h_0(p_0) \).

**Step two:** Analyzing \( p_k \) for \( 1 \leq k \leq K - 1 \). In this case, we write

\[
f_k(p_k) = \mu(1 - p_0) \left( C - \sum_{k=1}^{K} kp_k \right) (p_{k-1} - p_k)
\]

and

\[
h_k(p_k) = [\lambda (1 - p_0) + \gamma p_0 (1 - p_0')] (1 - p_K) (p_k - p_{k+1}).
\]

Notice that

\[
f_k(0) = \mu(1 - p_0) \left( C - \sum_{i \neq k} ip_i \right) p_{k-1} > 0,
\]

\[
f_k(p_{k-1}) = 0;
\]

and for \( 0 < p_k < p_{k-1} \)

\[
\frac{d}{dp_k} f_k(p_k) = \mu(1 - p_0) \left[ -k(p_{k-1} - p_k) - \left( C - \sum_{k=1}^{K} kp_k \right) \right] < 0,
\]

\[
\frac{d^2}{dp_k^2} f_k(p_k) = 2k\mu(1 - p_0) > 0,
\]

thus the quadratic function \( f_k(p_k) \) is a strictly decreasing convex function for \( 0 < p_k < p_{k-1} \).
Now, we consider the linear function \( h_k(p_k) \). We obtain
\[
h_k(0) = -[\lambda (1 - p_0) + \gamma p_0 (1 - p_0^\omega)] (1 - p_K) p_{k+1} < 0,
\]
and if \( p_k = 1 \), then \( p_i = 0 \) for \( i \neq k \) with \( 1 \leq i \leq K \), and it is clear that
\[
h_k(1) = \lambda > 0.
\]
Since
\[
\frac{d}{dp_k} h_k(p_k) = [\lambda (1 - p_0) + \gamma p_0 (1 - p_0^\omega)] (1 - p_K) > 0,
\]
the linear function \( h_k(p_k) \) is strictly increasing for \( p_k \in (0,1) \). Therefore, it is seen from Figure 4 (b) that there exists a unique solution \( p_k \) to the equation \( f_k(p_k) = h_k(p_k) \).

**Step three:** Analyzing \( p_K \). Since \( p_k \) is the unique solution to the equation \( f_k(p_k) = h_k(p_k) \) for \( 0 \leq k \leq K - 1 \), it is clear that \( p_K \) can uniquely determined by means of the relation that \( p_K = 1 - \sum_{k=0}^{K-1} p_k \). This completes the proof. 

Now, we provide a simple discussion for the limiting interchangeability of the vector \( y^{(N)}(t) \) as \( N \to \infty \) and \( t \to +\infty \). Notice that the limiting interchangeability is always necessary and useful in many practical applications when using the stationary probabilities of the limiting process \( \{Y(t) : t \geq 0\} \) to give an effective approximation for performance analysis of the bike sharing system.

From \( |S_p| = 1 \) by Theorem 5, it is seen from the Birkhoff center that
\[
\lim_{t \to +\infty} \lim_{N \to \infty} y^{(N)}(t) = \lim_{t \to +\infty} y(t) = P
\]
and
\[
\lim_{N \to \infty} \lim_{t \to +\infty} y^{(N)}(t) = \lim_{N \to \infty} P^{(N)} = P.
\]
This gives
\[
\lim_{t \to +\infty} \lim_{N \to \infty} y^{(N)}(t) = \lim_{N \to \infty} \lim_{t \to +\infty} y^{(N)}(t) = p.
\]
Therefore, we have
\[
\lim_{N \to \infty} y^{(N)}(t) = p.
\]

Finally, we provide a simple discussion on the asymptotic independence of this bike sharing system. To this end, the uniqueness of the fixed point given by \( |S_p| = 1 \) of Theorem
5] plays a key role. Using Corollaries 3 and 4 of Benaim and Le Boudec [4], we obtain the asymptotic independence of the queueing processes of the bike sharing system as follows:

\[
\lim_{t \to +\infty} \lim_{N \to \infty} P \left\{ X_1^{(N)}(t) = i_1, \ldots, X_k^{(N)}(t) = i_k \right\} = \lim_{N \to \infty} \lim_{t \to +\infty} P \left\{ X_1^{(N)}(t) = i_1, \ldots, X_k^{(N)}(t) = i_k \right\} = p_{i_1} p_{i_2} \cdots p_{i_k}
\]

and

\[
\lim_{N \to \infty} \lim_{t \to +\infty} \frac{1}{t} \int_0^t 1 \left\{ X_1^{(N)}(t) = i_1, X_2^{(N)}(t) = i_2, \ldots, X_k^{(N)}(t) = i_k \right\} \, dt = \lim_{N \to \infty} \lim_{t \to +\infty} \frac{1}{t} \int_0^t 1 \left\{ X_1^{(N)}(t) = i_1, X_2^{(N)}(t) = i_2, \ldots, X_k^{(N)}(t) = i_k \right\} \, dt = p_{i_1} p_{i_2} \cdots p_{i_k} \text{ a.s.}
\]

**Remark 6** For a more complicated bike sharing system, it is possible to have \(|S_p| \geq 2\). For this case with \(|S_p| \geq 2\), the metastability of the bike sharing system is a key, and it can be roughly described as an interesting phenomenon which occurs when the bike sharing system stays a very long time in some abnormal state before reaching its normal state. To study the metastability, a useful method is to determine a Lyapunov function \(g(y)\) for the system of differential equations (such as, (6) and (7)). Therefore, we need to find a continuously differentiable, bounded from below, function \(g(y)\) defined on \([0, 1]^{K+1}\) such that

\[
y V_y \nabla g(y) \leq 0.
\]

Notice that \(y V_y \nabla g(y) = 0\) if \(y V_y = 0\), which is satisfied by \(y = p\). On the other hand, some properties of the function \(g(y)\) allow one to discriminate the stable points (the local minima of \(g(y)\)) from the unstable points (the local maxima or saddle points of \(g(y)\)) in the study of metastability.

In general, it is not easy to give an analytic solution to the system of nonlinear equations (21), but its numerical solution is always simple and available. In the rest of this paper, we shall develop such a numerical solution, and give performance numerical analysis of this bike sharing system including the steady-state probability of the problematic stations, and the stationary expected number of bikes at any station.
7 Numerical Analysis

In this section, we use some numerical examples to investigate the steady-state probability of the problematic stations. To that end, performance analysis of the bike sharing system will focus on five points: (1) $p_0$; (2) $p_K$; (3) $p_0 + p_K$; (4) $E[Q] = \sum_{k=1}^{K} kp_k$; and (5) the profit $R$.

Notice that

\[
\begin{align*}
V & = 0, \\
\pi & = 1,
\end{align*}
\]

this gives the system of nonlinear equations (32) to (35) whose solution is unique by means of $||S_p|| = 1$ by Theorem 5.

Based on the system of nonlinear equations (32) to (35), we can numerically compute the unique solution, i.e., the fixed point $\mathbf{p}$. Furthermore, the fixed point $\mathbf{p}$ is employed in performance numerical computation of the bike sharing system. Based on this, we use some numerical examples to give valuable observation and understanding with respect to design, operations and optimization of the bike sharing systems. Therefore, such a numerical analysis will become more and more useful in the study of bike sharing systems in practice.

7.1 Analysis of $p_0$

Note that $p_0$ is a probability that there is no bike in a tagged station, thus it is also the probability that the arriving customer can not rent a bike in the tagged station. To design a better bike sharing system, we hope that the value of $p_0$ is as small as possible, and this can be realized through taking a suitable parameters: $C, K, \lambda, \gamma$ and $\omega$, where $C, K$ and $\mu$ are controlled by the station; while $\lambda, \gamma$ and $\omega$ are given by the customers.

In this bike sharing system, we take that $C = 30$, $K = 50$, $\omega = 1$ and $\gamma = 0.25$. The left one of Figure 5 shows how the probability $p_0$ depends on $\lambda \in (10, 30)$ when $\mu = 0.3, 1$ and 8, respectively. It is seen that $p_0$ increases either as $\lambda$ increases or as $\mu$ decreases. Note that the numerical results are intuitively reasonable because what $\lambda$ increases quickens up the rental rate of bikes at the tagged station, while what $\mu$ decreases reduces the return rate of bikes at the tagged station. Hence the probability $p_0$ increases as the number of bikes parked at the tagged station decreases in the two cases.

For the bike sharing system, we take that $C = 30$, $K = 50$, $\omega = 1$ and $\mu = 4$. The right
one of Figure 5 indicates how the probability $p_0$ depends on $\lambda \in (5, 15)$ when $\gamma = 0.05, 0.5$ and 1, respectively. It is seen that $p_0$ increases as $\lambda$ increases or as $\gamma$ decreases.

7.2 Analysis of $p_K$

Differently from $p_0$ given in Subsection 7.1, $p_K$ is a probability that the bikes are full in a tagged station, thus $p_K$ is also the probability that the deriving-bike customer can not return her bike at the tagged station. To design a better bike sharing system, we hope that the value of $p_K$ is as small as possible through taking a suitable parameters: $C, K, \lambda, \mu, \gamma$ and $\omega$.

In this bike sharing system, we take that $C = 30$, $K = 50$, $\omega = 1$ and $\gamma = 0.25$. The left one of Figure 6 shows how the probability $p_K$ depends on $\lambda \in (10, 30)$ when $\mu = 4, 8$ and 12, respectively. It is seen that $p_K$ decreases either as $\lambda$ increases or as $\mu$ decreases. Note that what $\lambda$ increases speeds up the rental rate of bikes at the tagged station, while what $\mu$ decreases reduces the return rate of bikes at the tagged station.

For the bike sharing system, we take that $C = 30$, $K = 50$, $\omega = 1$ and $\mu = 7$. The right one of Figure 6 indicates how the probability $p_K$ depends on $\lambda \in (10, 30)$ when $\gamma = 0.05, 0.5$ and 3, respectively. It is seen that $p_K$ decreases as $\lambda$ increases or as $\gamma$ increases.
7.3 Analysis of $p_0 + p_K$

Based on the above two analysis for $p_0$ and $p_K$, we further hope that the value of $p_0 + p_K$ can be as small as possible through taking a suitable parameters: $C, K, \lambda, \mu, \gamma$ and $\omega$.

In this bike sharing system, we take that $C = 30$, $K = 50$, $\omega = 1$ and $\gamma = 0.25$. The left one of Figure 7 shows how the probability $p_0 + p_K$ depends on $\lambda \in (10, 30)$ when $\mu = 6, 8$ and 10, respectively. It is seen that $p_0 + p_K$ decreases either as $\lambda$ increases or as $\mu$ decreases. Comparing Figure 7 with Figures 5 and 6, it is seen that $p_K$ has a bigger influence on the probability $p_0 + p_K$ than $p_0$.

For the bike sharing system, we take that $C = 30$, $K = 50$, $\omega = 1$ and $\mu = 12$. The
right one of Figure 7 indicates how the probability \( p_0 + p_K \) depends on \( \lambda \in (15, 30) \) when \( \gamma = 0.05, 0.5 \) and 1, respectively. It is seen that \( p_0 + p_K \) decreases as \( \lambda \) increases or as \( \gamma \) increases.

### 7.4 Analysis of \( E[Q] \)

From \( E[Q] = \sum_{k=1}^{K} kp_k \), it is seen that \( E[Q] \) is the stationary expected number of bikes parked at any station. Obviously, a customer who is renting a bike likes a bigger \( E[Q] \), while a customer who is returning a bike likes a smaller \( E[Q] \). In addition, \( E[Q] \) can also be used to express the profit of any station as follows:

\[
R = -cE[Q] + \psi \{ C - E[Q] \},
\]

where \( c \) is the cost price per bike and per time unit when a bike is parked in the tagged station, and \( \psi \) is the benefit price per bike and per time unit when a bike is rented from the tagged station.

In this bike sharing system, we take that \( C = 30, K = 50, \omega = 1 \) and \( \gamma = 0.25 \). The left of Figure 8 shows how the stationary mean \( E[Q] \) depends on \( \lambda \in (10, 30) \) when \( \mu = 2, 5 \) and 8, respectively. It is seen that \( E[Q] \) decreases either as \( \lambda \) increases or as \( \mu \) decreases.

For the bike sharing system, we take that \( C = 20, K = 50, \omega = 1 \) and \( \mu = 7 \). The right of Figure 8 indicates how the stationary mean \( E[Q] \) depends on \( \lambda \in (10, 30) \) when \( \gamma = 0.05, 0.1 \) and 6, respectively. It is seen that \( E[Q] \) decreases as \( \lambda \) increases or as \( \gamma \) increases.
7.5 Parameter optimization

We provide a simple discussion for how to optimize some key parameters of the bike sharing system through numerical experiments. Note that $\lambda$, $\gamma$ and $\omega$ are the arrival and walk information of any customer respectively, thus our parameter optimization will not consider them. Thus our decision variables in the following optimal problems will mainly focus on the three parameters: $C$, $K$ and $\mu$.

(a) Optimization based on the probabilities $p_0$ and $p_K$

Since our purpose is to minimize either $p_0$, $p_K$ or $p_0 + p_K$, we may choose a weighted method in which $\beta_1$, $\beta_2$ and $\beta_3$ are the weighted coefficients with $\beta_1, \beta_2, \beta_3 \geq 0$ and $\beta_1 + \beta_2 + \beta_3 = 1$. In this case, our optimal problem is given by

$$\min \{ \beta_1 p_0 + \beta_2 p_K + \beta_3 (p_0 + p_K) \}$$

s.t. $0 < \gamma < \mu,$

$$1 \leq C \leq K.$$  

For example, when $\beta_2 = 0$ and $\beta_3 = 0$, $\min \{ \beta_1 p_0 + \beta_2 p_K + \beta_3 (p_0 + p_K) \} = \min \{ p_0 \}$; when $\beta_1 = 0$ and $\beta_3 = 0$, $\min \{ \beta_1 p_0 + \beta_2 p_K + \beta_3 (p_0 + p_K) \} = \min \{ p_K \}$; when $\beta_1 = 0$ and $\beta_2 = 0$, $\min \{ \beta_1 p_0 + \beta_2 p_K + \beta_3 (p_0 + p_K) \} = \min \{ p_0 + p_K \}$. Therefore, our above optimal problem is a more general tradeoff among three key factors: $p_0$, $p_K$ and $p_0 + p_K$.

(b) Optimization based on the profit $R$

Now, our optimal purpose is to maximize the profit of any station as follows:

$$\max \{-cE[Q] + \psi \{ C - E[Q] \} \}$$

s.t. $0 < \gamma < \mu,$

$$1 \leq C \leq K.$$  

8 Concluding Remarks

In this paper, we apply the mean-field theory to studying a large-scale bike sharing system, where the mean-field computation can partly overcome the difficulty of state space explosion in more complicated bike sharing systems. We first use an $N$-dimensional Markov process to express the states of the bike sharing system, and construct an empirical measure Markov process of the $N$-dimensional Markov process. Then we set up the system of mean-field equations by means of a virtual time-inhomogeneous $M(t)/M(t)/1/K$ queue
whose arrival and service rates are determined through some mean-field computation. Furthermore, we employ the martingale limit to investigate the limiting behavior of the empirical measure process, and prove that the fixed point is unique. This illustrates the asymptotic independence of the queueing processes in the bike sharing system. Based on this, we can compute the fixed point through a nonlinear birth-death process, and provide some effective algorithms for computing the steady-state probability of the problematic stations. Finally, we use some numerical examples to give valuable observation on how the steady-state probability of the problematic stations depends on some crucial parameters of the bike sharing system.

This paper provides a complete picture on how to use the mean-field theory, the time-inhomogeneous queues and the nonlinear Markov processes to analyze performance measures of the bike sharing systems. This picture is described as the following four key steps: (1) Setting up system of mean-field equations, (2) proofs of the mean-field limit, (3) uniqueness and computation of the fixed point, and (4) performance computation of the bike sharing system. Therefore, the methodology and results of this paper give new highlight on understanding influence of system key parameters on performance measures of the bike sharing systems. Along such a line, there are a number of interesting directions for potential future research, for example:

- Analyzing impact of the intelligent information technologies on operations management of the bike sharing systems;
- discussing the bike sharing systems with non-exponential distributions and non-Poisson point processes, and develop some more general mean-field models;
- studying the periodical or time-inhomogeneous bike sharing systems; and
- modeling a bike sharing system with multiple clusters, where the unbalanced bikes can be redistributed among the stations or clusters by means of optimal scheduling of trucks.

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