Sufficient conditions for the unique solution of a class of new Sylvester-like absolute value equation

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Abstract

In this paper, a class of new Sylvester-like absolute value equation (AVE) $AXB - |CXD| = E$ with $A, C \in \mathbb{R}^{m \times n}$, $B, D \in \mathbb{R}^{p \times q}$ and $E \in \mathbb{R}^{m \times q}$ is considered, which is quite distinct from the published work by Hashemi [Applied Mathematics Letters, 112 (2021) 106818]. Some sufficient conditions for the unique solution of the Sylvester-like AVE are obtained.

Keywords: New Sylvester-like absolute value equation; unique solution; sufficient condition

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1 Introduction

As is known, the standard absolute value equation (AVE)

\[ Ax + |x| = f \] (1.1)

and its general version

\[ Ax + C|x| = f \] (1.2)

are very strong tools in the field of optimization, including the complementarity problem, linear programming and convex quadratic programming, where $A$ and $C$ may be rectangular matrices of the same order. Based on this, the AVE (1.1)/(1.2) has caused wide public concern over the recent years.

The AVE (1.2) was first introduced in [1] by Rohn. Therewith, its main research contents consist of two aspects: one is to multifarious numerical methods for obtaining its numerical

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solution, see [2–11], and the other is theoretical analysis, including the existence of solvability, bounds for the solutions, various equivalent reformulations, and so on, see [12–18].

Recently, in [19], Hashemi generalized the concept of absolute value equation and considered the following Sylvester-like absolute value equation (AVE)

\[ AXB + C|X|D = F, \]

where \( A, C \in \mathbb{R}^{m \times n}, B, D \in \mathbb{R}^{p \times q}, F \in \mathbb{R}^{m \times q} \) are given. For the Sylvester-like AVE (1.3), Hashemi in [19] established some sufficient conditions for its unique solution. Further, in [20], Wang and Li considered the Sylvester-like AVE (1.3) with square coefficient matrices. Some new sufficient conditions were gained in [20], which are different from the results in [19].

Further, in [21], Wu found a type of new generalized absolute value equation (NGAVE), which is below

\[ Ax - |Bx| = d, \]

with \( A, B \in \mathbb{R}^{n \times n} \) and \( d \in \mathbb{R}^n \). Likewise, Wu in [21] established some necessary and sufficient conditions for the unique solution of the NGAVE (1.4). Clearly, the NGAVE (1.4) is quite different from the AVE (1.2).

Inspired by the work in [21], together with the Sylvester-like AVE (1.3), in this paper, we consider a type of new Sylvester-like absolute value equation (AVE) below

\[ AXB - |CXD| = F, \]

where \( A, C \in \mathbb{R}^{m \times n}, B, D \in \mathbb{R}^{p \times q}, F \in \mathbb{R}^{m \times q} \) are given. Here, the new Sylvester-like AVE (1.5) not only is the generalization form of the GAVE (1.4), but also is from other fields, such as interval matrix equations [22–27], robust control [28], and so on. Similar to the Sylvester-like AVE (1.3), the theory and practice of the new Sylvester-like AVE (1.5) is still interested and challenged because of the nonlinear and nondifferentiable term \(|CXD|\) in (1.5). This is our motivation for this paper. At present, to our knowledge, for the unique solution of the new Sylvester-like AVE (1.5), the necessary and sufficient condition is vacant. Based on this, the goal of the present paper is to fill in this vacant, gain some sufficient conditions for the unique solution of the new Sylvester-like AVE (1.5). What’s more, some useful necessary and sufficient conditions for the unique solution of the new Sylvester-like AVE (1.5) are obtained with square coefficient matrices.

2 Main result

In this section, we will present some conditions for the unique solution of the new Sylvester-like AVE (1.5). To achieve this goal, by using the Kronecker product and the vec operator, the new Sylvester-like AVE (1.5) can be expressed as the NGAVE below

\[ Sx - |Tx| = f \]

with \( S = B^T \otimes A, T = D^T \otimes C, x = \text{vec}(X) \) and \( f = \text{vec}(F) \), where ‘\( \otimes \)’, ‘\( \text{vec} \)’ stand for the Kronecker product and the vec operator, respectively.
To discuss the sufficient condition for the unique solution of the new Sylvester-like AVE (1.5), Lemmas 2.1, 2.2, 2.3 and 2.4 are required.

**Lemma 2.1** [21] Let matrix $A$ in (1.4) be nonsingular. If
\[ \rho((I - 2D)BA^{-1}) < 1 \] (2.2)
for any diagonal matrix $D = \text{diag}(d_i)$ with $d_i \in [0, 1]$, then the NGAVE (1.4) for any $d \in \mathbb{R}^n$ has a unique solution.

**Lemma 2.2** [21] The NGAVE (1.4) for any $d \in \mathbb{R}^n$ has a unique solution if and only if matrix $A + (I - 2D)B$ is nonsingular for any diagonal matrix $D = \text{diag}(d_i)$ with $d_i \in [0, 1]$.

**Lemma 2.3** [29] Let $A, B \in \mathbb{R}^{n \times n}$. Then
\[ \sigma_i(A + B) \geq \sigma_i(A) - \sigma_1(B), i = 1, 2, \ldots, n, \]
where $\sigma_1 \geq \ldots \geq \sigma_n \geq 0$ are the singular values of matrix.

Based on Lemmas 2.2 and 2.3, Lemma 2.4 can be obtained.

**Lemma 2.4** If
\[ \sigma_1(B) < \sigma_n(A) \] (2.3)
where $\sigma_1$ and $\sigma_n$ denote the largest and smallest singular value, respectively, then the NGAVE (1.4) for any $d \in \mathbb{R}^n$ has a unique solution.

**Proof.** Based on Lemma 2.3, the NGAVE (1.4) has a unique solution for any $d \in \mathbb{R}^n$ when the matrix $A + (I - 2D)B$ is nonsingular for any diagonal matrix $D = \text{diag}(d_i)$ with $0 \leq d_i \leq 1$. So, let $\sigma_n(A + (I - 2D)B)$ stand for the minimal singular value of the matrix $A + (I - 2D)B$. Based on Lemma 2.3, we have
\[ \sigma_n(A + (I - 2D)B) \geq \sigma_n(A) - 2\sigma_1((I - 2D)B). \]
Since $\sigma_1((I - 2D)B) \leq \sigma_1((I - 2D))\sigma_1(B) \leq \sigma_1(B)$, the result in Lemma 2.4 holds under the condition (2.3). \qed

Based on the above lemmas, we can present some conditions for the unique solution of the new Sylvester-like AVE (1.5) for any $F$. First, by using Lemma 2.1 indirectly, we can obtain the following result, see Theorem 2.1.

**Theorem 2.1** Let $A, B$ be square nonsingular matrix in (1.5). If
\[ \rho((I - 2\Lambda)((B^{-1}D)^T \otimes CA^{-1})) < 1 \] (2.4)
for any diagonal matrix $\Lambda = \text{diag}(\lambda_i)$ with $\lambda_i \in [0, 1]$, then the new Sylvester-like AVE (1.5) has a unique solution for any $F$. 

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Proof. Since 
\[ S^{-1} = (B^T \otimes A)^{-1} = B^{-T} \otimes A^{-1}, \]
we have 
\[ TS^{-1} = (D^T \otimes C)(B^{-T} \otimes A^{-1}) = D^T B^{-T} \otimes CA^{-1} = (B^{-1} D)^T \otimes CA^{-1}. \]

Based on Lemma 2.1, clearly, the result in Theorem 2.1 is right. \(\square\)

Clearly, we can use \(\rho(((B^{-1} D)^T \otimes CA^{-1})(I - 2\Lambda)) < 1\) instead of the condition (2.4) in Theorem 2.1.

In fact, the condition (2.4) in Theorem 2.1 is not easy to implement in practice. Even if the condition (2.4) in Theorem 2.1 can be executed, the number of arithmetic operations is required to compute the spectral radius of the huge matrix. For instance, for \(A, C \in \mathbb{R}^{n \times n}\) and \(B, D \in \mathbb{R}^{m \times m}\), the number of arithmetic operations is \(O(n^3 m^3)\). Therefore, the sum of the powers here is \(3 + 3 = 6\), i.e., the complexity here is sextic. Faced with this situation, we have to get some conditions that can be detected. By the simple calculation, we have
\[
(I - 2\Lambda)((B^{-1} D)^T \otimes CA^{-1}) \leq |(I - 2\Lambda)((B^{-1} D)^T \otimes CA^{-1})| \\
\leq |I - 2\Lambda||(B^{-1} D)^T \otimes CA^{-1})| \\
\leq |(B^{-1} D)^T \otimes CA^{-1})| \\
= |(B^{-1} D)^T| \otimes |CA^{-1}|.
\]

Note that
\[
\rho((|B^{-1} D|^T \otimes |CA^{-1}|)) = \rho(|(B^{-1} D)^T|)\rho(|CA^{-1}|) = \rho(|B^{-1} D|)\rho(|CA^{-1}|).
\]

So, we have the following result, see Theorem 2.2.

**Theorem 2.2** Let \(A, B\) be square nonsingular matrix in (1.5). If
\[
\rho(|B^{-1} D|)\rho(|CA^{-1}|) < 1,
\]
then the new Sylvester-like AVE (1.5) has a unique solution for any \(F\).

In addition, based on Theorem 5.6.10 in [30], we also have
\[
\rho((I - 2\Lambda)((B^{-1} D)^T \otimes CA^{-1})) \leq \sigma_1((I - 2\Lambda)((B^{-1} D)^T \otimes CA^{-1})) \\
\leq \sigma_1(I - 2\Lambda)\sigma_1((B^{-1} D)^T \otimes CA^{-1}) \\
\leq \sigma_1((B^{-1} D)^T \otimes CA^{-1}) \\
= \sigma_1((B^{-1} D)^T)\sigma_1(CA^{-1}) \\
= \sigma_1(B^{-1} D)\sigma_1(CA^{-1}).
\]

Based on this, Theorem 2.3 can be obtained.
Theorem 2.3 Let $A, B$ be square nonsingular matrix in $\mathbb{R}^{n \times n}$. If
\[ \sigma_1(B^{-1}D)\sigma_1(CA^{-1}) < 1, \] 
then the new Sylvester-like AVE (1.1) has a unique solution for any $F$.

Compared with Theorem 2.1, indeed, the conditions of Theorems 2.2 and 2.3 can be easy to execute. Here, it is noted that the conditions of Theorems 2.2 and 2.3 only work if $A$ and $B$ are nonsingular. In addition, for a general square matrix $H$, there is no relations between $\sigma_1(H)$ and $\rho(|H|)$ unless one adds yet more additional requirements. Based on this fact, Theorem 2.3 sometimes performs better than Theorem 2.2, vice versa.

To obtain the sufficient condition that is more general, based on Lemma 2.4, Theorem 2.4 can be obtained and its proof is omitted.

Theorem 2.4 If
\[ \sigma_1(C)\sigma_1(D) < \sigma_n(A)\sigma_n(B) \] 
then the new Sylvester-like AVE (1.1) has a unique solution for any $F$.

Compared with Theorems 2.2 and 2.3, Theorem 2.4 not only is fit for the square matrix, but also is fit for the rectangular matrix. This implies that the condition (2.7) in Theorem 2.4 is indeed more general.

It is not difficult to find that all the conditions of Theorems 2.2, 2.3 and 2.4 can be checked. Not only that, the computational complexity of all the conditions in Theorems 2.2, 2.3 and 2.4 is cubic.

By the way, for $m = n = p = q$ in (1.3), combining Theorems 3.1 and 3.2 in [21], with Lemma 2.4, we can obtain the following necessary and sufficient conditions for the unique solution of the new Sylvester-like AVE (1.3), see Theorem 2.5.

Theorem 2.5 Let $S = B^T \otimes A$, $T = D^T \otimes C$. Then the following statements are equivalent:
1. the new Sylvester-like AVE (1.1) has a unique solution for any $F \in \mathbb{R}^{n \times n}$;
2. $\{S + T, S - T\}$ has the row $\mathcal{W}$-property;
3. \( \text{det}(F_1(S+T)+F_2(S-T)) \neq 0 \) for arbitrary nonnegative diagonal matrices $F_1, F_2 \in \mathbb{R}^{n \times n}$ with $\text{diag}(F_1 + F_2) > 0$;
4. matrix $(S - T)(S + T)^{-1}$ is a $P$-matrix (all its principal minors are positive), where matrix $S + T$ is invertible;
5. matrix $S + (I - 2\Lambda)T$ is nonsingular for any diagonal matrix $\Lambda = \text{diag}(\lambda_i)$ with $\lambda_i \in [0, 1]$.

Since the order of the matrices $S$ and $T$ in Theorem 2.5 is $n^2 \times n^2$, the number of arithmetic operations required to check both parts 2, 3, 4 and 5 of Theorem 2.5 is at least sextic, i.e., $\mathcal{O}(n^6)$. Therefore, the computational complexity of all the conditions in Theorem 2.5 is at least $\mathcal{O}(n^6)$. 

5
3 Conclusions

In this paper, the unique solution of a type of new Sylvester-like absolute value equation (AVE) $AXB - |CXD| = E$ with $A, C \in \mathbb{R}^{m \times n}$, $B, D \in \mathbb{R}^{p \times q}$ and $E \in \mathbb{R}^{m \times q}$ has been discussed. Some useful sufficient conditions for the unique solution of the new Sylvester-like AVE are obtained. Particularly, all the sufficient conditions of Theorems 2.2, 2.3 and 2.4 can be checked with a cubic complexity in the light of the order of the input matrices.

References

[1] J. Rohn, Systems of linear interval equations, Linear Algebra Appl., 126 (1989) 39-78.
[2] L. Caccetta, B. Qu, G.-L. Zhou, A globally and quadratically convergent method for absolute value equations, Comput. Optim. Appl., 48 (2011) 45-58.
[3] O.L. Mangasarian, A generalized Newton method for absolute value equations, Optim. Lett., 3 (2009) 101-108.
[4] J. Rohn, An algorithm for solving the absolute value equations, Electron. J. Linear Algebra., 18 (2009) 589-599.
[5] D.K. Salkuyeh, The Picard-HSS iteration method for absolute value equations, Optim. Lett., 8 (2014) 2191-2202.
[6] S.-L. Wu, C.-X. Li, A special Shift-splitting iterative method for the absolute value equations, AIMS Math., 5 (2020) 5171-5183.
[7] C.-X. Li, S.-L. Wu, Modified SOR-like iteration method for absolute value equations, Math. Probl. Eng., 2020 (2020) 9231639.
[8] P. Guo, S.-L. Wu, C.-X. Li, On SOR-like iteration method for solving absolute value equations, Appl. Math. Lett., 97 (2019) 107-113.
[9] Y.-Y. Lian, C.-X. Li, S.-L. Wu, Weaker convergent results of the generalized Newton method for the generalized absolute value equations, J. Comput. Appl. Math., 338 (2018) 221-226.
[10] C.-X. Li, A preconditioned AOR iterative method for the absolute value equations, Inter. J. Comput. Meth., 14 (2017) 1750016.
[11] C.-X. Li, A modified generalized Newton method for the absolute value equations, J. Optim. Theory Appl., 170 (2016) 1055-1059.
[12] J. Rohn, A theorem of the alternatives for the equation $Ax + B|x| = b$, Linear Multilinear A., 52 (2004) 421-426.
[13] S.-L. Wu, C.-X. Li, A note on unique solvability of the absolute value equation, Optim. Lett., 14 (2020) 1957-1960.
[14] O.L. Mangasarian, R.R. Meyer, Absolute value equations, Linear Algebra Appl., 419 (2006) 359-367.
[15] S.-L. Wu, C.-X. Li, The unique solution of the absolute value equations, Appl. Math. Lett., 76 (2018) 195-200.
[16] O. Prokopyev, On equivalent reformulations for absolute value equations, Comput. Optim. Appl., 44 (2009) 363-372.
[17] M. Hladik, Bounds for the solutions of absolute value equations, Comput. Optim. Appl., 69 (2018) 243-266.
[18] S.-L. Wu, S.-Q. Shen, On the unique solution of the generalized absolute value equation, Optim. Lett., 2020, https://doi.org/10.1007/s11590-020-01672-2.

[19] B. Hashemi, Sufficient conditions for the solvability of a Sylvester-like absolute value matrix equation, Appl. Math. Lett., 112 (2021) 106818.

[20] L.-M. Wang, C.-X. Li, New sufficient conditions for the unique solution of a square Sylvester-like absolute value equation, Appl. Math. Lett., 116 (2021) 106966.

[21] S.-L. Wu, The unique solution of a class of the new generalized absolute value equation, Appl. Math. Lett., 116 (2021) 107029.

[22] A. Neumaier, Interval Methods for Systems of Equations, Cambridge University Press, Cambridge, 1990.

[23] N.P. Seif, S.A. Hussein, A.S. Deif, The interval Sylvester equation, Computing., 52 (1994) 233-244.

[24] B. Hashemi, M. Dehghan, Results concerning interval linear systems with multiple right-hand sides and the interval matrix equation $AX = B$, J. Comput. Appl. Math., 235 (2011) 2969-2978.

[25] B. Hashemi, M. Dehghan, The interval Lyapunov matrix equation: analytical results and an efficient numerical technique for outer estimation of the united solution set, Math. Comput. Model., 55 (2012) 622-633.

[26] M. Dehghani-Madiseh, M. Hladík, Efficient approaches for enclosing the united solution set of the interval generalized Sylvester matrix equations, Appl. Numer. Math., 126 (2018) 18-33.

[27] M. Dehghani-Madiseh, M. Dehghan, Generalized solution sets of the interval generalized Sylvester matrix equation $\sum_{i=1}^{p} A_i X_i + \sum_{j=1}^{q} Y_j B_j = C$ and some approaches for inner and outer estimations, Comput. Math. Appl., 68 (2014) 1758-1774.

[28] V.N. Shashikhin, Robust assignment of poles in large-scale interval systems, Autom. Rem. Contr., 63 (2002) 200-208.

[29] F.-Z. Zhang, Matrix Theory: Basic results and techniques (Second edition). Springer, New York, 2011.

[30] R.A. Horn, C.R. Johnson, Matrix Analysis. Cambridge University Press, Cambridge, 1986.