ON THE FINITE GENERATION OF COORDINATE RINGS OF AFFINE GROUP SCHEMES OVER DISCRETE VALUATION RINGS

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Abstract. We prove a finite generation result for the coordinate ring of certain affine group schemes over a discrete valuation ring. This may be used to simplify the use of results of Prasad and Yu on quasi-reductive groups by Mirkovic and Vilonen in their work on geometric Langlands duality.

In [4], Prasad and Yu defined quasi-reductive group schemes and proved some structure theorems for them. Their work was motivated by a question of Vilonen, and their results are required in the work of Mirkovic and Vilonen [2] on the geometric construction of the Langlands dual group over arbitrary fields. One of their results is that quasi-reductive group schemes are always of finite type. In this note we give a simple proof of a more general finite generation result which is essentially the best possible.

1.

Let $R$ be a DVR with quotient field $K$, residue field $k$, and let $\bar{k}$ be an algebraic closure of $k$.

Theorem 1. Let $\mathcal{G}$ be a flat affine group scheme over $R$ and assume that $R$ is excellent. If the generic fibre $\mathcal{G}_K$ is reduced and of finite type over $K$, the reduced special fibre $(\mathcal{G}_k)_{\text{red}}$ is of finite type over $k$ and $\dim(\mathcal{G}_K) = \dim(\mathcal{G}_k)$, then $\mathcal{G}$ is of finite type over $R$.

The following is an immediate consequence.

Corollary 2. Let $\mathcal{G}$ be a flat affine group scheme over (an arbitrary DVR) $R$. If the generic fibre $\mathcal{G}_K$ is smooth and of finite type over $K$, the reduced geometric special fibre $(\mathcal{G}_k)_{\text{red}}$ is of finite type over $k$ and $\dim(\mathcal{G}_K) = \dim(\mathcal{G}_k)$, then $\mathcal{G}$ is of finite type over $R$.

Proof. By faithfully flat descent we may replace $R$ by a complete discrete valuation ring with algebraically closed residue field. Such a ring is excellent, hence we may apply Theorem 1 to conclude.

Proof of Theorem. Since $R$ is excellent, the generic fibre of $\mathcal{G} \times_R \hat{R}$ is reduced, hence by faithfully flat descent we may assume that $R$ is complete. Let $\mathcal{A} = \Gamma(\mathcal{G}, \mathcal{O}_{\mathcal{G}})$ be the coordinate ring of $\mathcal{G}$. By the discussion in §5.3 of [4] (here we use that $R$ is complete), we may write $\mathcal{G} = \varinjlim_{i \in I} \mathcal{G}_i$, where $I$ is a directed set and $\mathcal{G}_i$ are flat finite type affine group schemes over $R$. Since $\mathcal{G}_K$ is of finite type over $K$ we may assume that the induced morphisms $\mathcal{G}_K \to (\mathcal{G}_i)_K$ are isomorphisms for all $i \in I$, so that we have $\mathcal{A} = \varinjlim_{i \in I} \mathcal{A}_i$, where $\mathcal{A}_i = \Gamma(\mathcal{G}_i, \mathcal{O}_{\mathcal{G}_i})$ and all the induced maps $\mathcal{A}_i \to \mathcal{A}$ are injective. Since tensor products commute with direct limits, we have $\varinjlim_{i \in I} \mathcal{A}_i \otimes_R k = \mathcal{A} \otimes_R k$. Since the $\mathcal{G}_i$ are flat and of finite type, we have $\dim((\mathcal{G}_i)_k) = \dim(\mathcal{G}_k)$ for all $i \in I$.

The group scheme $(\mathcal{G}_k)_{\text{red}}$ is of finite type, so it has only finitely many connected (= irreducible) components. For each $i$, let $\mathcal{G}'_i$ be the union of those components of $(\mathcal{G}_i)_k$ which contain points of $\mathcal{G}_i((\mathcal{G}_k)_k)$, where $\pi_i : \mathcal{G} \to \mathcal{G}_i$ are the maps in the directed system. Since the $\pi_i$ are homomorphisms of group schemes, it follows that $\mathcal{G}'_i$ is an affine open subgroup scheme of $(\mathcal{G}_i)_K$. By [1, 2.2.6],
there exist affine open subschemes $G_i'$ of $G_i$ such that $(G_i')_k = G_i'_{k}$ for all $i \in I$. The $G_i'$ are subgroup schemes of $G_i$ and by construction the maps $\pi_i$ factor through the inclusion $G_i' \subset G_i$. If $A_i'$ denotes the coordinate ring of $G_i'$, then the induced map $\lim_{i \in I} A_i' \to A$ is an isomorphism, hence $G = \lim_{i \in I} G_i'$. So by replacing $G_i$ with $G_i'$, we may assume that the map on component groups induced by the morphisms $G_k \to (G_i)_k$ are all isomorphisms. By Lemma [3] we may now also assume that all the morphisms $(G_k)_{\text{red}} \to ((G_i)_k)_{\text{red}}$ are isomorphisms.

Let $\tilde{G}_i$ (resp. $\tilde{G}$) be the normalisation of $G_i$ (resp. $G$), so that we have a commutative diagram:

$$
\begin{array}{ccc}
\tilde{G} = \lim_{i \in I} \tilde{G}_i & \xrightarrow{\nu} & \tilde{G}_i \\
\downarrow \pi_i & & \downarrow \nu_i \\
G & \xrightarrow{\nu} & G_i \\
\end{array}
$$

Since $R$ is complete, hence excellent, the $\nu_i$ are finite morphisms. The $\pi_{i,j}$ are finite type bijections, hence $\pi_{i,j}$ are birational, quasi-finite and finite type morphisms. Since $\tilde{G}_i$ are also normal, it follows from Zariski’s main theorem that the $\pi_{i,j}$ are open immersions. The morphisms induced by $\pi_{i,j}$ from $(G_i)_k$ to $(G_j)_k$ are finite since the induced morphisms on reduced schemes are isomorphisms. Since the $\nu_i$ are also finite, it follows that the morphisms from $(\tilde{G}_i)_k$ to $(\tilde{G}_j)_k$ induced by $\pi_{i,j}$ are also finite. The images of these morphisms are therefore both open and closed, hence a union of connected components. It then follows that all the $\pi_{i,j}$ are isomorphisms for $i, j$ in a cofinal subset of $I$. By replacing $I$ with such a subset, we may assume that $\tilde{G} = \tilde{G}_i$ for all $i$; in particular, we may assume that $\tilde{G}$ is of finite type over $R$.

Now for any $i$, we have morphisms

$$
\tilde{G} \to G \to G_i
$$

and the morphism $\tilde{G} \to G_i$ is finite. Since $G_i$ is a noetherian scheme, it follows that $G$ is finite over $G_i$, hence of finite type over $R$.

Lemma 3. Let $X = \lim_{i \in I} X_i$ be a directed inverse limit of irreducible affine schemes over a field $k$, with $\dim(X) = \dim(X_i)$ for all $i$ and $X_{\text{red}}, X_i$ of finite type over $k$ for all $i$. Then there exists $J \subset I$, a cofinal subset, such that the induced morphisms $X_{\text{red}} \to (X_j)_{\text{red}}$ are isomorphisms for all $j \in J$.

Proof. Since $\lim_{i \in I} (X_i)_{\text{red}} = (\lim_{i \in I} X_i)_{\text{red}}$ we may assume that all the $X_i$ (hence also $X$) are reduced and $X$ is of finite type.

Let $\pi_j : X \to X_j$ be the natural morphisms, $A := \Gamma(X, \mathcal{O}_X)$, $A_i := \Gamma(X_i, \mathcal{O}_{X_i})$ and $\sigma_j : A_i \to A$ the induced $k$-algebra homomorphisms. Let $f_1, f_2, \ldots, f_r$ be generators of $A$. We may find a cofinal $J \subset I$ such that $f_k \in \sigma_j(A_j)$ for all $k = 1, 2, \ldots, r$ and $j \in J$. This implies that $\pi_j$ are closed embeddings for all $j \in J$. Since the $X_j$’s are irreducible, the dimension condition implies that $\pi_j$ are isomorphisms for all $j \in J$.

Remarks 4. (1) We do not know whether Theorem [1] is true without assuming $G$ affine (the assumption is used to write $G$ as an inverse limit of finite type group schemes).

(2) If $K$ is of characteristic 0 then by Cartier’s theorem the assumption that the generic fibre be reduced in Theorem [1] is redundant. However, if $\text{char}(K) = p > 0$ this is not the case as shown by the following example: Let $G = \lim_{i \in N} \alpha_{p^i}R$ where the transition maps are all equal to the endomorphism of $\alpha_{p^i}k$ induced by multiplication by a uniformizer of $R$. Then $G_K = \alpha_{p^i}k$, $G_k = \text{Spec}(k)$ and $G$ is flat over $R$. However $G$ is not of finite type over $R$.

(3) By faithfully flat descent, Theorem [1] and Corollary [2] have obvious analogues for group
schemes over Dedekind schemes such that the group scheme is of finite type over a Zariski open subset of the base. The group scheme \( \text{Spec}(\mathbb{Z}[x/2, x/3, x/5, \ldots]) \) over \( \mathbb{Z} \) ([3] Remark 4.7)), all of whose fibres are isomorphic to \( \mathbb{G}_a \) over the corresponding residue field, shows that in general the finite type condition over a Zariski open subset of the base cannot be replaced by a more local condition. However, Prasad and Yu have shown [4 Theorem 1.5] that this can be done when all the fibres are reductive

(4) Onoda [3] has proved finite generation results for a large class of affine schemes over more general bases. However, his results do not apply in the situation of Theorem 1 since his assumption on the special fibre is not satisfied.

Acknowledgements. I thank Brian Conrad and Gopal Prasad for their comments; in particular, the proof of Lemma 3, which is simpler than the original, was suggested by Brian Conrad. I also thank S. M. Bhatwadekar for informing me of the article [3] and for related discussions.

The first version of this note was written while I was visiting the Korea Institute of Advanced Study; I thank Andreas Bender and the staff of KIAS for their hospitality.

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