POSITIVE SOLUTIONS OF A CLASS OF SEMILINEAR EQUATIONS WITH ABSORPTION AND SCHRÖDINGER EQUATIONS

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ABSTRACT. Several results about positive solutions -in a Lipschitz domain- of a nonlinear elliptic equation in a general form $\Delta u(x) - g(x, u(x)) = 0$ are proved, extending thus some known facts in the case of $g(x, t) = t^q$, $q > 1$, and a smooth domain. Our results include a characterization -in terms of a natural capacity- of a (conditional) removability property, a characterization of moderate solutions and of their boundary trace and a property relating arbitrary positive solutions to moderate solutions. The proofs combine techniques of non-linear p.d.e. with potential theoretic methods with respect to linear Schrödinger equations. A general result describing the measures that are diffuse with respect to certain capacities is also established and used. The appendix by the first author provides classes of functions $g$ such that the nonnegative solutions of $\Delta u - g(., u) = 0$ has some “good” properties which appear in the paper.

1. Introduction

In this paper we study positive solutions of semilinear elliptic equations with absorption of the form

$$\Delta u(x) + g(x, u(x)) = 0$$

in a bounded Lipschitz domain $\Omega$ in $\mathbb{R}^N$, $N \geq 2$. A function $u \in L^1_{\text{loc}}(\Omega)$ is a solution of (1.1) if $g \circ u \in L^1_{\text{loc}}(\Omega)$ and the equation holds in the distribution sense. Here

$$g \circ u(x) := g(x, u(x)) \quad \forall x \in \Omega.$$

1.1. Assumptions. Our basic assumptions on $g$ are

\begin{enumerate}
  \item[(i)] $g \in C(\Omega \times \mathbb{R})$,
  \item[(ii)] $g(x, \cdot)$ is odd and increasing for every $x \in \Omega$,
  \item[(iii)] $g(x, \cdot)$ is convex on $[0, \infty)$ $\forall x \in \Omega$.
\end{enumerate}

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Notice that by (i) and (ii) \( g \) is differentiable with respect to \( t \) at \( t = 0 \). We put

\[
h(x,t) = \begin{cases} 
g(x,t)/t & \text{if } t \neq 0 \\
g'(x,0) & \text{if } t = 0.\end{cases}
\]

and in addition to (1.2) we assume:

\[
\exists \bar{a} > 0 \text{ such that, for every compact set } E \subset \partial \Omega \text{ and every positive solution } u \text{ of (1.1) in } \Omega \text{ which is continuous in } \bar{\Omega} \setminus E,
\]

\[
(1.3) \quad u = 0 \text{ on } \partial \Omega \setminus E \implies h(x,u(x)) \leq \bar{a} \text{dist}(x,E)^{-2}, \forall x \in \Omega.
\]

In particular, taking \( E = \partial \Omega \) this condition says that for every positive solution \( u \) of (1.1) in \( \Omega \),

\[
(1.4) \quad h(x,u(x)) \leq \bar{a} \text{dist}(x,\partial \Omega)^{-2}, \forall x \in \Omega.
\]

As in [27], a basic ingredient in our study is the connection between (1.1) and the Schrödinger equation (1.8)-(1.9) with potential \( V \) given by (1.10). Condition (1.4) plays a crucial role in this context.

In Appendix A it is shown that condition (1.3) follows from (1.4) for some large classes of absorption terms \( g \) (see Theorem A.10). Appendix A also describes concrete classes of absorption terms (not necessarily convex w.r. to \( t \in (0, +\infty) \)) for which (1.4) or (1.3) holds. See Proposition A.1 and Theorem A.7.

Condition (1.4) is closely related to the Keller-Osserman condition [22, 38]. When \( g \) is independent of the space variable the latter condition reads

\[
\psi(a) = \int_a^\infty \frac{dt}{\sqrt{2G(t)}} < \infty, \quad \forall a > 0 \text{ where } G(t) = \int_0^t g(s)ds.
\]

Under condition (1.5) Keller and Osserman obtained a universal estimate for solutions of (1.1) (for details see [22, Theorem 1]) and it was recently observed by one of us that an alternative proof of Proposition A.1 in its full generality can be obtained using Keller’s theorem and its proof. Moreover, in the case when \( g \) is independent of \( x \), it can be shown that assumption (A.2) also implies (1.3) for general domains, not necessarily Lipschitz. Details and related results – including estimates from below for maximal solutions of (1.1) – will be presented in [28].

The basic example of an equation where our conditions are satisfied is

\[
(1.6) \quad -\Delta u + |u|^{q-1}u = 0, \quad q > 1.
\]

It is related to the study of branching processes and superdiffusions (see Dynkin [13, 15] and Le Gall [23, 26]) and has been intensively studied.
We mention two other interesting examples:

\[ g = \text{dist} (\cdot, \partial \Omega)^\alpha g_0, \quad g = \psi_1^\alpha g_0 \quad \alpha \geq 0, \]

and \( g_0 \) satisfies (1.2) and (A.2), e.g. \( g_0(t) = |t|^q \text{sign } t, q \geq 1 \) or \( g_0(t) = (e^{|t|} - 1) \text{sign } t \). Here \( \psi_1 \) denotes the first eigenfunction of \(-\Delta\) in \( \Omega \), \( \psi_1(x_0) = 1 \) at a point \( x_0 \in \Omega \). For a proof and other examples see the Appendix.

### 1.2. Connexion with Schrödinger equations.

Our study of equation (1.1) employs non-linear elliptic p.d.e. techniques combined with results on linear Schrödinger equations of the form

\[ -\Delta u + Vu = 0, \]

where the potential \( V \in L^\infty_{\text{loc}}(\Omega) \) is nonnegative and satisfies the condition

\[ V(x) \leq \bar{a}\delta(x)^{-2}, \quad \forall x \in \partial\Omega, \]

for some real \( \bar{a} > 0 \). This class of potentials is denoted by \( \mathcal{V}(\Omega, \bar{a}) \). The point is that if \( u \) is a positive solution of (1.1) then it satisfies (1.8) with

\[ V(x) = h(x, u(x)) \quad \forall x \in \Omega \]

and (1.4) translates to (1.9). We will write

\[ L^V := -\Delta + V. \]

A solution of equation (1.8) is called an \( L^V \) harmonic function. The term ‘harmonic’ will be reserved for \( L^V = -\Delta \).

Let \( x_0 \in \Omega \) be a fixed reference point. We will denote by \( \mathbb{K} \) (resp. \( \mathbb{K}^V \)) the Martin kernel with respect to the operator \(-\Delta\) (resp. \( L^V = -\Delta + V \)) in \( \Omega \) normalized at \( x_0 \), i.e. \( \mathbb{K}^V : \partial\Omega \ni \zeta \mapsto \mathbb{K}^V_{\zeta} \) where \( \mathbb{K}^V_{\zeta} \) is the unique positive \( L^V \) harmonic function vanishing on \( \partial\Omega \setminus \{ \zeta \} \) such that \( \mathbb{K}^V_{\zeta}(x_0) = 1 \) (see Section 3 for more details). Here we use property (1.9). If \( \mu \in \mathcal{M}(\partial\Omega) \) (the space of finite Borel measures on \( \partial\Omega \)), we denote by \( \mathbb{K}^V[\mu] \) (resp. \( \mathbb{K}[\mu] \)) the corresponding \( L^V \)-harmonic function (resp. \( \Delta \)-harmonic function).

It is known that every positive \( L^V \) harmonic function \( u \) in \( \Omega \) can be uniquely represented in the form

\[ u = \mathbb{K}^V[\mu] := \int_{\partial\Omega} \mathbb{K}^V_{\zeta}(\cdot) d\mu(\zeta) \]

for some \( \mu \in \mathcal{M}_{+}(\partial\Omega) \). The measure \( \mu \) will be called the \textit{L}^V \textit{boundary measure} for \( u \).

Let \( \mathbb{G} \) and \( \mathbb{G}^V \) denote the Green functions in \( \Omega \) of \(-\Delta\) and \( L^V \) respectively. If \( \mu \) is a nonnegative Radon measure in \( \Omega \), we note

\[ \mathbb{G}[\mu] := \int_{\partial\Omega} \mathbb{G}(\cdot, \xi) d\mu(\xi), \quad \mathbb{G}^V[\mu] := \int_{\partial\Omega} \mathbb{G}^V(\cdot, \xi) d\mu(\xi). \]
Note that if $G[\mu]$ is finite at some point in $\Omega$ then it is locally integrable in $\Omega$ and this happens iff $\int_\Omega \psi \, d\mu < \infty$ where $\psi$ is the function $\psi(x) = 1 \land G(x_0, x)$, $x \in \Omega$. So this function plays a special role in a number of conditions below. One may use as well any continuous function $\psi_1$, such that $C^{-1} \psi \leq \psi_1 \leq C \psi$ in $\Omega$ for some real $C \geq 1$. A handy choice for $\psi_1$ is the first eigenfunction of $-\Delta$ with $\psi_1 \geq 0$ and say $\|\psi_1\|_2 = 1$.

1.3. Traces and moderate solutions of (1.1). An exhaustion of $\Omega$ is any increasing sequence of open subsets of $\Omega$ such that $\Omega = \bigcup_{n \geq 1} \Omega_n$. In what follows the reader may as well restrict to exhaustions by smooth domains. The next definition was introduced in [34, Definition 3.6] w.r. to a more restrictive concept of exhaustion.

**Definition 1.1.** If $u$ is a nonnegative Borel function in $\Omega$, we say that $u$ has an $m$-boundary trace $\mu \in M_+^\infty(\partial \Omega)$ if, for any exhaustion $\{\Omega_n\}$ of $\Omega$,

$$\lim_{n \to \infty} \int_{\partial \Omega_n} u \varphi \, d\omega_n = \int_{\partial \Omega} \varphi \, d\mu, \quad \forall \varphi \in C(\Omega).$$

where $\omega_n$ is $\omega_n^{x_0}$, the harmonic measure of $x_0$ in $\Omega_n$. The $m$-boundary trace of $u$ on $\partial \Omega$ is denoted by $\text{tr}_{\partial \Omega} u$. (The subscript will be omitted if it is clear from the context.)

Examples: (i) If $u \in C(\Omega)$ then the $m$-boundary trace of $u|_\Omega$ is the measure $u.\omega_\Omega^{x_0}$ (since $\omega_n^{x_0} \to \omega_\Omega^{x_0}$ in the weak sense). (ii) If $u = K[\nu]$, $\nu \in M_+^\infty(\partial \Omega)$ then $\nu$ is the $m$-boundary trace of $u$ (see [34, Lemma 2.2]). (iii) If $\mu \in M_+^\infty(\Omega)$ (i.e. $\int \psi \, d\mu < \infty$) then $G[\mu]$ has $m$-boundary trace zero. (See [34, Lemma 3.1]).

It follows from the Riesz decomposition theorem that every nonnegative superharmonic function in $\Omega$ admits a $m$-boundary trace which is the $m$-boundary trace of its largest harmonic minorant. In general this statement does not apply to $L^V$ harmonic functions and if $u = K^V[\nu]$, $\nu \in M_+^\infty(\partial \Omega)$ then, in general, $\nu$ is not the $m$-boundary trace of $u$.

We will consider boundary value problems of the form

$$\begin{align*}
-\Delta u + g \circ u &= 0 \quad \text{in } \Omega, \\
\text{tr } u &= \nu \quad \text{on } \partial \Omega,
\end{align*}$$

where $\nu \in M_+^\infty(\partial \Omega)$ and $\text{tr } u$ denotes the $m$-boundary trace of $u$.

The next definition provides a natural class of solutions of (1.1) related to such problems.

**Definition 1.2.** Let $u$ be a nonnegative solution of (1.1). We say that $u$ is a $g$ moderate solution of (1.1) if $g \circ u \in L^1_\psi(\Omega)$ (i.e. $\int_\Omega g(x, u(x)) \psi(x) \, dx < \infty$).

The next proposition recalls among other things other equivalent definitions and the connection with the previous boundary value problems.
Proposition 1.3. If $u$ is a nonnegative solution of (1.1) then $u$ is $g$-moderate iff $u$ admits a harmonic majorant in $\Omega$. This condition is also equivalent to the existence of an $m$-boundary trace for $u$ on $\partial \Omega$. If $u$ is $g$ moderate then $u \in L^1_\psi(\Omega)$ and if $\nu$ is its $m$-boundary trace, $\mathbb{K}[\nu]$ is the least harmonic majorant of $u$ in $\Omega$.

There are also well-known equivalent variational definitions of solutions of (1.1).

Remark 1.4. If $u$ is a positive supersolution of (1.1) such that $g \circ u \in L^1_\psi(\Omega)$ then $u$ has an $m$-boundary trace and the largest solution of (1.1) dominated by $u$ has the same $m$-boundary trace. There is a parallel statement for subsolutions. See [34, Theorem 4.3].

It is known that, for every $\nu \in \mathcal{M}_+(\partial \Omega)$, problem (1.14) has at most one solution. A measure $\nu$ for which a solution exists is called a $g$-good measure and the corresponding solution of (1.14) is denoted by $S^g[\nu]$. Moreover, if $\nu_1, \nu_2$ are $g$-good measures then,

$$\nu_1 \leq \nu_2 \implies S^g[\nu_1] \leq S^g[\nu_2].$$

For a proof see e.g. [34] (where $g$ is assumed to be space independent).

We next recall some basic stability properties of good measures.

Proposition 1.5. (i) If $\mu, \nu \in \mathcal{M}_+(\partial \Omega)$, $\mu$ is a $g$-good measure and $0 \leq \nu \leq \mu$ then $\nu$ is also $g$-good. (ii) If $M \subset \mathcal{M}_+(\partial \Omega)$ is a set of $g$-good measures dominated in $\mathcal{M}_+(\partial \Omega)$ then the least upper bound of $M$ is again $g$-good.

In general the sum of two $g$-good measures is not again $g$-good. There are such examples say for $g(x, t) = \sinh(t)$, but if $g$ satisfies the (uniform) $\Delta_2$ condition –i.e. $g(x, 2t) \leq C g(x, t)$, $\forall(x, t) \in \Omega \times \mathbb{R}_+$ for some $C \geq 0$– then the $g$-good measures form a convex cone of measures.

Denote

$$(1.16) \quad \mathcal{M}^{g, \Delta}_+(\partial \Omega) := \{ \nu \in \mathcal{M}_+(\partial \Omega) : \int_\Omega (g \circ \mathbb{K}[\nu]) \psi \, dx < \infty \}.$$

Notice that this is a convex set of good measures. If $\nu \in \mathcal{M}^{g, \Delta}_+(\partial \Omega)$ we say that $\nu$ is a $(g, \Delta)$ - good measure. When $g$ satisfies the $\Delta_2$-condition the set $\mathcal{M}^{g, \Delta}_+(\partial \Omega)$ is a convex cone of measures.

1.4. Main results. The first set of results concerns $g$-moderate solutions. In all of these we assume that $g$ satisfies our basic conditions, namely, (1.2) and (1.3). For some of the results we assume in addition the $\Delta_2$ condition.

To describe the first main result we still need the following definitions.
Definition 1.6. A compact set $F \subset \partial \Omega$ is conditionally $g$-removable if, for any non-negative $g$-moderate solution $u$ of (1.1),

$$u \in C(\overline{\Omega} \setminus F) \text{ and } u = 0 \text{ on } \partial \Omega \setminus F \implies u = 0.$$ 

When $T \subset M_+(\partial \Omega)$, we also say that a Borel set $A \subset \partial \Omega$ is $T$-null if

$$\tau(A) = 0 \quad \forall \tau \in T.$$ 

Theorem 1.7. A compact set $F \subset \partial \Omega$ is conditionally $g$-removable if and only if $F$ is $M_{g,\Delta}$-null.

Theorem 1.8. Let $u$ be a positive $g$-moderate solution of (1.1) in $\Omega$. Then there exists a sequence $\{\mu_n\} \subset M_{g,\Delta} := M_{g,\Delta}^+(\partial \Omega)$ such that

$$u = \lim_{n \to \infty} S_g[\mu_1 + \cdots + \mu_n].$$

If $g$ satisfies the $\Delta_2$ condition, there is an increasing sequence $\{\mu_n\} \subset M_{g,\Delta}$ such that

$$u = \lim_{n \to \infty} S_g[\mu_n].$$

Remark. Note that in the first part of the theorem, $\mu_1 + \cdots + \mu_n$ needs not be $(g, \Delta)$-good. But, as it is dominated by $\text{tr}(u)$, the sum is $g$-good.

Theorem 1.9. (i) If $\mu \in M_+ := M_+(\partial \Omega)$ is a $g$-good measure then $\mu$ vanishes on every $M_{g,\Delta}$-null set.

(ii) If $\mu \in M_+$ vanishes on $M_{g,\Delta}$ null sets then there exists a Borel function $f : \partial \Omega \mapsto (0, 1]$ such that $f \mu$ is a $g$-good measure.

(iii) If, in addition to the basic conditions, $g$ satisfies the $\Delta_2$ condition then, $\mu \in M_+$ is a $g$-good measure if and only if $\mu$ vanishes on every $M_{g,\Delta}$-null set.

Notice that (ii) means that the greatest $g$-good measure $\nu$ smaller than $\mu$ is in the form $\nu = f \mu$ with $f > 0$ $\mu$-a.e.

Theorems 1.7–1.9 are proved in section 5.

Much of the research on the problems discussed above focused on the case of power nonlinearities, i.e. equation (1.6). In this case, for domains of class $C^2$, the results of Theorems 1.7–1.9 have been previously established. In the subcritical case, $1 < q < q_c = (N + 1)/(N - 1)$, every measure is $g$-good (Gmira and Veron [21]). In the supercritical case $q \geq q_c$, $N \geq 3$ the results were obtained by Le Gall [24, 25] in the case $q = 2$, Dynkin and Kuznetsov [16, 18] in the case $q_c \leq q \leq 2$ (employing mainly
probability methods) and Marcus and Veron [31] in the case $q \geq 2$ (using purely analytic methods). A unified approach that applies to all $q \geq q_c$ was provided in [32].

In the case of Lipschitz domains, equation (1.6) has been investigated in [34] under conditions implying that $q$ is subcritical. The supercritical case was treated in [35] in the case that the domain is a polyhedron. For general Lipschitz domains, Theorems 1.7–1.9 are new even in the case of power nonlinearities.

We note here that equation (1.1) with $g(x,t) = \delta(x) |t|^{q-1} t, \alpha > -2$ has been studied in [33], for domains of class $C^2$ and $q$ in the subcritical range, $1 < q < (N+\alpha+1)/(N-1)$. Equation (1.1) with $g(t) = e^t - 1$ was studied in [40] for domains of class $C^2$.

For the statement of our next result on arbitrary positive solutions of (1.1) (not necessarily $g$-moderate) we introduce some additional notation and terminology.

Denote by $C_{g,\Delta}$ the set function on the Borel field $\mathcal{B}(\partial \Omega)$ defined by

$$
C_{g,\Delta}(F) = \sup \{ \tau(F) : \tau \in \mathcal{M}_+ (\partial \Omega), \int_\Omega (g \circ K_\tau) \psi \, dx \leq 1 \}
$$

for every Borel set $F \subset \partial \Omega$. This set function is a capacity (in the sense of [36]).

The family of $\mathcal{M}^g_{\Delta}$ - null sets can also be described as the family of Borel sets $A \subset \partial \Omega$ such that $C_{g,\Delta}(A) = 0$.

A measure $\mu \in \mathcal{M}_+(\partial \Omega)$ is said to be diffuse relative to $C_{g,\Delta}$ if $\mu$ vanishes on $C_{g,\Delta}$ null sets. The measure $\mu$ is concentrated (or singular) w.r. to $C_{g,\Delta}$ if it is concentrated on a $C_{g,\Delta}$ null set. It is known that every measure $\mu \in \mathcal{M}_+(\partial \Omega)$ can be uniquely written in the form $\mu = \mu_d + \mu_c$ where $\mu_d$ is diffuse and $\mu_c$ is singular w.r. to $C_{g,\Delta}$ (see Section 6).

**Theorem 1.10.** Assume that $g$ satisfies (1.2) and (1.3). Let $u$ be a positive solution of (1.1), let $V = h \circ u$ and let $\mu \in \mathcal{M}_+(\partial \Omega)$ be the $L^V$ boundary measure of $u$, i.e. $u = \mathbb{K}^V [\mu]$.

If $\mu$ is not singular relative to $C_{g,\Delta}$ then $u$ dominates a positive $g$ moderate solution $v$ such that $\mu_d$ is absolutely continuous w.r. to $\text{tr } v$.

For the proof of this theorem and some additional results see Section 6.

In connexion with the characterization of positive solutions of (1.6) in terms of their boundary behavior, Dynkin and Kuznetsov [18] (see also [14]) raised a central question: is every positive solution of (1.6) $\sigma$-moderate, i.e. the limit of an increasing sequence of $g$-moderate solutions?

The question was answered affirmatively in the case of $C^2$ domains in a series of works: first, in the subcritical case (Marcus and Veron [29,30]), then in the supercritical case,
by Mselati [37] for $q = 2$ and Dynkin [13] for $q_c \leq q \leq 2$ and finally by Marcus [27] for every $q \geq q_c$. A crucial step in the proof of [27] was to show that if $u$ is a positive solution of (1.6) and $\mu$ is defined as in Theorem 1.10, then $\mu$ is diffuse relative to $C_{g,\Delta}$ and $u$ dominates a positive $g$-moderate solution. Theorem 1.10 constitutes a first step in an investigation of possible extensions of [27] to Lipschitz domains and to more general non-linearities.

Our study relies on results of [27], here adapted to Lipschitz domains, and potential theoretic results about linear Schrödinger equations, in particular a recent result of [4] and a boundary Harnack principle based on [2]. See also Sections 3 and 4 where some auxiliary results, needed in the later sections, are proved.

We also use and prove a general result (see Section 2, Theorem 2.1) that characterizes the positive Borel measures which are diffuse with respect to a capacity defined as the supremum of a family of measures. It extends standard results, in particular results from [19], [11] and [10].

2. A MEASURE THEORETIC APPROXIMATION RESULT.

Let $X$ be a metric space and let $\mathcal{M}_+(X)$ denote the set of all finite nonnegative Borel measures on $X$.

Given $\mathcal{T} \subset \mathcal{M}_+(X)$, we say that a Borel set $A \subset X$ is $\mathcal{T}$-null if

$$\tau(A) = 0 \quad \forall \tau \in \mathcal{T}. \quad (2.1)$$

**Theorem 2.1.** If $X$ is compact and $\mathcal{K} \subset \mathcal{M}_+(X)$ satisfies the conditions

$$\begin{align*}
(a) \quad & \mathcal{K} \text{ is nonempty, convex and weakly closed in } \mathcal{M}_+(X), \\
(b) \quad & \nu \in \mathcal{K}, \ \tau \in \mathcal{M}_+(X), \ \tau \leq \nu \implies \tau \in \mathcal{K},
\end{align*} \quad (2.2)$$

then every $\mu \in \mathcal{M}_+(X)$ that vanishes on $\mathcal{K}$-null sets is in the form $\mu = \sup_{n \geq 1} n \nu_n$ where $\nu_n \in \mathcal{K}$ for all $n \geq 1$ and $\{n \nu_n\}$ is an increasing sequence in $\mathcal{M}_+(X)$.

Note that $\mathcal{E}_+ := \bigcup_{n \geq 1} n \mathcal{K}$ is the convex cone generated by $\mathcal{K}$ and that the conclusion says that $\mu = \lim_{n \to \infty} \lambda_n$ for some increasing sequence $\{\lambda_n\}$ in $\mathcal{E}_+$.

This theorem extends well-known results, in particular results of Feyel–de La Pradelle [19] and Dal Maso [11] in which $\mathcal{K}$ generates the positive cone of the dual of a Dirichlet space (resp. a Sobolev space $W^{-\alpha,q}(\partial\Omega)$). In the proof below we use a result due to G. Mokobodzky [36].
To prove Theorem 2.1 we first notice that upon replacing \( K \) by \( \tilde{K} := \{ \mu \in K; \mu(X) \leq 1 \} \), we may assume that \( K \) is weakly compact in \( \mathcal{M}_+(X) \). Without loss of generality we will also assume that \( \bigcup_{\mu \in K} \text{supp}(\mu) \) is dense in \( X \).

Define then a set function as follows. For any Borel subset \( A \) of \( X \), set
\[
\text{Cap}(A) := \sup \left\{ \int_A d\mu; \mu \in K \right\}.
\]

Cap is a capacity on \( X \) in the sense of [36]. That is Cap is finite nonnegative and

(i) \( \text{Cap}(\emptyset) = 0 \),
(ii) \( \text{Cap}(\bigcup_{n \geq 1} A_n) \leq \sum_{n \geq 1} \text{Cap}(A_n) \) for every sequence \( \{A_n\} \) of Borel subsets of \( X \)
(iii) \( \text{Cap}(\bigcup_{n \geq 1} A_n) = \sup_{n \geq 1} \text{Cap}(A_n) \) for every increasing sequence \( \{A_n\} \) of Borel subsets of \( X \),

as easily verified. So the result in [36] says the following.

**Lemma 2.2.** Let \( \nu \in \mathcal{M}_+(X) \) be such that \( \nu(A) = 0 \) for every Borel \( K \)-null set. Then there exists an increasing sequence \( \{\nu_n\} \) in \( \mathcal{M}_+(X) \) such that \( \nu = \sup_n \nu_n \) and \( \nu_n \leq n \text{Cap} \) for all \( n \geq 1 \).

Thus to prove Theorem 2.1 it suffices to show that every measure \( \nu \in \mathcal{M}_+(X) \) majorized by Cap (i.e. \( \nu(A) \leq \text{Cap}(A) \) for every Borel set \( A \subset X \)) is the upper enveloppe of a set of measures in \( \mathcal{E}_+ = \mathbb{R}_+ \cdot K \).

To begin with, we consider \( E = \mathcal{E}_+ - \mathcal{E}_+ \) the vector subspace of \( \mathcal{M}(X; \mathbb{R}) \) generated by \( K \) and define below a topology on \( E := \mathcal{C}(X; \mathbb{R}) \) that agrees with the duality \( (E; \mathcal{E}) \).

Set for each \( f \in E \):
\[
\|f\| = \sup\{ |\int f \, d\mu|; \mu \in K - K \}.
\]

This defines a norm \( \|\cdot\| \) on \( E \) and this norm is equivalent to the norm
\[
|||f||| = \sup\{ \int |f| \, d\mu; \mu \in K \}, \quad f \in E.
\]

In effect, \( \mu \in K - K \) if and only if \( \mu^+, \mu^- \in K \) (using (b) in [2.2] and \( \mu = \mu^+ - \mu^- \)). Thus for \( f \in E, \mu \in K - K \), we have \( |\int f \, d\mu| \leq \int |f| \, d\mu^+ + \int |f| \, d\mu^- \) and \( \|f\| \leq 2 \|||f||| \).

On the other hand if \( \mu \in K \) and \( f \in E \), one has \( \int |f| \, d\mu = \int f 1_{f \geq 0} \, d\mu - \int f 1_{f < 0} \, d\mu = \int f \, d\mu' = \int f \, d\mu'' \) where \( \mu' = 1_{f \geq 0} \mu - 1_{f < 0} \mu \) and \( \mu'' \in K - K \). Hence \( \|||f||| \leq \|f\| \).

The topology on \( E \) defined by the norm \( \|\cdot\| \) is the topology of uniform convergence on \( K - K \). Since \( K - K \) is (convex, symmetric) vaguely compact (i.e. compact for \( \sigma(\mathcal{E}, E) \))
and generates $E$, Mackey’s Theorem (see [7] or [39]) says that the topological dual of $(E, ||.||)$ is $E$. For sake of completeness we recall the argument in our situation: by the bipolar theorem, the polar of the unit ball $B_E$ in $(E, ||.||)$ with respect to the duality $(E; E^*)$ ($E^*$ is the algebraic dual) is $K - K$ (which -being $\sigma(E^*, E)$ compact- is closed for $\sigma(E^*, E)$ in $E^*$). One uses here that $B_E$ is by definition of the norm $||.||$, the polar of $K - K$ for the duality $(E, E^*)$. But the polar of $B_E$ is also the unit ball of the dual of $(E, ||.||)$. So the set of linear forms on $(E, ||.||)$ whose norm is less than 1 is exactly $K - K$ and the dual of $(E, ||.||)$ is $E$, equipped with the gauge norm of $K - K$.

In what follows $\nu$ denotes a nonnegative Borel measure in $X$ majorized by $\text{Cap}$.

**Proposition 2.3.** The map $f \mapsto \int f^+ d\nu$ is lower semicontinuous in $(E, ||.||)$.

We first notice the following simple Markov-type inequality:

**Lemma 2.4.** If $f \in E$ and $a > 0$ then

$$\text{Cap}(\{|f| \geq a\}) \leq |||f|||/a.$$  

It suffices to note that $\mu\{|f| \geq a\} \leq \frac{1}{a}(\int |f| d\mu) \leq \frac{1}{a}|||f|||$ for $\mu \in \mathcal{K}$. Taking the supremum over all $\mu \in \mathcal{K}$ the result follows. \hfill \Box

**Proof of proposition 2.3** Let $f_n \to f$ in $E$. Since $|f_n^+ - f^+| \leq |f_n - f|$ we have $|||f_n^+ - f^+||| \to 0$, and since $|f_n^+ \wedge f^+ - f^+| \leq |f_n^+ - f^+|$ we also have $|||f_n^+ \wedge f^+ - f^+||| \to 0$.

Now, given $\varepsilon > 0$ we have for every integer $n \geq 1$,

$$\int (f_n^+ \wedge f^+) \, d\nu \geq \int f^+ \, d\nu - \int_{f_n^+ \wedge f^+ \geq f^+ - \varepsilon} (f^+ - f_n^+ \wedge f^+) \, d\nu$$

$$- \int_{f_n^+ \wedge f^+ < f^+ - \varepsilon} (f^+ - f_n^+ \wedge f^+) \, d\nu$$

$$\geq \int f^+ \, d\nu - \varepsilon |||f||| - ||f||_{\infty} \nu(f_n^+ \leq f^+ - \varepsilon)$$

$$\geq \int f^+ \, d\nu - \varepsilon |||f||| - ||f||_{\infty} \text{Cap}(\{f_n^+ \leq f^+ - \varepsilon\})$$

using in the last line the inequality $\nu \leq \text{Cap}$. By the preliminary remark

$$\text{Cap}(\{f_n^+ \leq f^+ - \varepsilon\}) \leq |||f_n^+ \wedge f^+ - f^+|||/\varepsilon \to 0 \text{ as } n \to \infty.$$  

Thus writing $\int f_n^+ d\nu = \int (f_n^+ \wedge f^+) \, d\nu + \int (f_n^+ - f_n^+ \wedge f^+) \, d\nu$ and observing that $\int (f_n^+ - f_n^+ \wedge f^+) \, d\nu \geq 0$ one sees that $\lim inf \int f_n^+ d\nu \geq \int f^+ d\nu - \varepsilon |||f|||$. Letting $\varepsilon \to 0$ the result follows: $\lim inf \int f_n^+ d\nu \geq \int f^+ d\nu$. \hfill \Box
We may now conclude the proof of Theorem 2.1. Recall that $\nu \in M_+(X)$ and $\nu \leq \text{Cap}$.

**Proposition 2.5.** The measure $\nu$ is the upper envelope of the measures $\mu \in \mathcal{E}_+ := \cup_{n \geq 1} nK$ such that $\mu \leq \nu$.

The function $\Phi : f \rightarrow \int f^+ d\mu$ is convex, l.s.c. and homogeneous on $(E, \| \cdot \|)$. The epigraph of $\Phi$ is hence a closed convex cone in $E \times \mathbb{R}$. By the Hahn-Banach separation theorem, $\Phi$ is the upper envelope of the set of all continuous affine functions in $E$ that are less than $\Phi$ in $E$. Now if $F = a + \ell$ is such a function with $\ell \in E$ and $a \in \mathbb{R}$, then $\ell$ must be nondecreasing in $E$ (since $\ell$ is upper bounded in $E_- := -E_+$ and hence $\leq 0$ in $-E_+$). Since for $f \in E_+$, $a + t\ell(f) \leq t\Phi(f)$ for all $t \geq 0$, we have $\ell(f) \leq \Phi(f)$ and

$$\int f \, d\nu = \sup \{ \ell(f) ; \ell \in E' \text{ such that } \ell \leq \Phi \}$$

$$= \sup \{ \int f \, d\mu ; \mu \in E_+ , \mu \leq \Phi \text{ in } E_+ \}.$$ 

Thus $\nu$ is the smallest measure that majorizes $\{ \mu \in \mathcal{E}_+ ; \mu \leq \nu \}$. This set being sup-stable in $M_+(X)$, $\nu$ is the limit in $M_+(X)$ of an increasing sequence in $\mathcal{E}_+$. □

The following special cases is needed in the other parts of this paper. We use here the notations and general assumptions introduced in section 1. Denote for $a \geq 0$,

$$\mathcal{K}_{g,a} := \{ \mu \in M_+(\partial \Omega) ; \int_{\Omega} g(x, \mathbb{K}_\mu(x)) \psi(x) \, dx \leq a \}$$

**Corollary 2.6.** Let $\nu$ be a finite positive Borel measure in $\partial \Omega$ such that $\nu(K) = 0$ whenever $K \subset \partial \Omega$ is compact and $\mathcal{K}_{g,a}$–null (i.e. $\mu(K) = 0$ for every $\mu \in \mathcal{K}_{g,a}$). Then there is an increasing sequence $\{ \nu_n \}$ in $M_+(\partial \Omega)$ such that $\nu_n \in n\mathcal{K}_{g,a}$ for $n \geq 1$ and $\nu = \lim_{n \to \infty} \nu_n$.

It suffices to note that $\mathcal{K}_{g,a}$ is convex and weakly closed in $M_+(\partial \Omega)$. These properties follow from the convexity of $g(x, \cdot)$, the continuity of $g$ and Fatou’s theorem. □

**Corollary 2.7.** Suppose that $g$ satisfies the $\Delta_2$-condition and that $\mu \in M_+^{\alpha, \Delta}(\partial \Omega)$ vanishes on $M_+^{\alpha, \Delta}$–null sets. Then $\nu = \lim_{n \to \infty} \mu_n$ for some increasing sequence $\{ \mu_n \}$ in $M_+^{\alpha, \Delta}(\partial \Omega)$.

It suffices to apply the previous result with $a = 1$ and observe that by the assumptions the cone $M_+^{\alpha, \Delta}(\partial \Omega)$ is generated by $\mathcal{K}_{g,1}$. □
3. Some known results on Schrödinger equations

In this section, we present some notions and results related to the Schrödinger equation \((1.8)\) that are needed to obtain our results. Here \(V\) is a nonnegative continuous function in \(\Omega\) satisfying condition \((1.9)\), namely \(V \in \mathcal{V}(\Omega, \overline{\alpha})\) with \(\overline{\alpha} \geq 0\).

We start with some definitions.

**\(L^V\) harmonicity, \(L^V\) superharmonicity.** By an \(L^V\) harmonic function in \(\Omega\) we mean a continuous function in \(\Omega\) that solves \(L^V(u) = 0\) in the sense of distributions. Then \(u \in W^{2,p}_{\text{loc}}(\Omega)\) for every \(p < \infty\) and \(L^V(u) = 0\) a.e. As well known every \(u \in L^1_{\text{loc}}(\Omega)\) solving \(L^V(u) = 0\) in the distribution sense in \(\Omega\) can be viewed as an \(L^V\) harmonic function.

It is also known that if \(u \in L^1_{\text{loc}}(\Omega)\) is a supersolution of \((1.8)\) (i.e. \(L^V(u) \geq 0\) in the weak sense) then the limit \(\tilde{u}(x) := \lim_{r \to 0} \int_{B(x, r)} u(y) dy / |B(x, r)| \in (-\infty, +\infty]\) exists at every \(x \in \Omega\), \(\tilde{u}\) is l.s.c. and \(L^V\)-superharmonic w.r. to the harmonic sheaf of \(L^V\)-harmonic functions (in the sense of [8]). Conversely every \(L^V\)-superharmonic function in a domain \(\omega \subset \Omega\) which is not the constant \(+\infty\) is a locally integrable supersolution in \(\omega\). In the sequel we will only need to consider continuous \(L^V\) superharmonic functions.

**\(L^V\) potentials.** A non-negative, \(L^V\) superharmonic function in \(\Omega\) is an \(L^V\) potential if it does not have a positive \(L^V\) harmonic minorant in \(\Omega\).

**Integral representations of nonnegative harmonic functions.** By the results in [2] (see also the presentation in [4]) there exists a continuous kernel \(K^V : \partial \Omega \times \Omega \ni (\zeta, x) \mapsto K^V_{\zeta}(x) \in (0, +\infty)\) such that \(K^V_{\zeta}(x) = \lim_{z \to \zeta} G(z, x) / G(z, x_0)\) for \(\zeta \in \partial \Omega, x \in \Omega\) (thus \(K^V_{\zeta}(x_0) = 1\)). Moreover, every positive \(L^V\)-harmonic function \(u\) in \(\Omega\) can be uniquely written in the form \(u = K^V_{\mu} := \int K^V_{\zeta}(\cdot) \, d\mu(\zeta)\) where \(\mu \in \mathcal{M}_+(\partial \Omega)\). The function \(K\) is the \(L^V\)-Martin kernel for \(\Omega\) with normalization at \(x_0\) and if as above \(u = K^V_{\mu}, \mu\) will be called the \(L^V\) boundary measure of \(u\).

**\(L^V\) moderate solutions and supersolutions.** The next definition and statements are parallel to Definition 1.2, Proposition 1.3 and Remark 1.4.

**Definition 3.1.** An \(L^V\) harmonic function \(u\) is \(L^V\)-moderate (or \(V\)-moderate) if \(uV \in L^{1,\psi}_{\text{loc}}(\Omega)\) that is: \(\int_\Omega |u| V \psi \, dx < \infty\). Similarly a positive \(L^V\) superharmonic function \(u\) is \(L^V\)-moderate if \(uV \in L^1_{\psi}(\Omega)\).

**Proposition 3.2.** Let \(u\) be nonnegative and \(L^V\) harmonic in \(\Omega\). Then the following are equivalent: (i) \(u\) is \(V\)-moderate, (ii) \(u\) has a \(m\)-boundary trace in \(\partial \Omega\), (iii) \(u\) admits a harmonic majorant in \(\Omega\).
Proof. (i) implies (ii): see proof of Lemma 3.3.

(ii) implies (iii): the harmonic function with the same m-boundary trace dominates \( u \).

(iii) implies (i): The infimum of all superharmonic majorants, say \( \bar{u} \), is superharmonic. Therefore \( v := \bar{u} - u \) is a non-negative superharmonic function. Thus \( v \) has an m-boundary trace \( \tau \geq 0 \). Since \( v' := \bar{u} - K_\tau \geq u \) and \( v' \) is superharmonic it follows that \( v = v' \) so that \( \tau = 0 \). It follows that \( u \) has an m-boundary trace equal to \( \text{tr} \bar{u} \). \( \square \)

The following lemma is proved in [27] in the case of \( C^2 \) domains.

Lemma 3.3. Let \( w \) be a positive \( L^V \) superharmonic function in \( \Omega \). If \( w \) is \( L^V \) moderate then it possesses an m-boundary trace. The largest \( L^V \) harmonic function dominated by \( w \) is \( L^V \) moderate and its m-boundary trace equals \( \text{tr} w \).

Proof. The function \( \hat{w} := w + G[wV] \) is positive superharmonic in \( \Omega \). Consequently it has an m-boundary trace \( \nu \in \mathcal{M}(\partial \Omega) \) and since \( \text{tr} G[wV] = 0 \) it follows that \( \nu = \text{tr} w \).

Also, because \( \hat{w} := w + G[wV] \) is positive superharmonic, \( -\psi \Delta \hat{w} \in \mathcal{M}_+(\Omega) \), i.e. \( -\psi(\Delta w - V w) = -\psi L^V(w) \in \mathcal{M}_+(\Omega) \). Since \( 0 \leq \mathcal{G}(-\Delta w + V w) \leq \mathcal{G}(-\Delta w + V w) \neq +\infty \) this implies that \( v := \mathcal{G}(-\Delta w + V w) \) has a vanishing boundary trace. Finally \( w - v \) is the largest \( L^V \) harmonic function dominated by \( w \) and \( \text{tr}(w - v) = \text{tr} w \). \( \square \)

A Fatou-Doob-Naïm type result. For easy reference we state some additional potential theoretic results that are used in the sequel and which extend well-known results of Fatou, Doob and Naïm. We refer the reader to [2]. See also [1] or [4]. Recall that the Riesz decomposition theorem for \( L^V \) says that a non-negative \( L^V \) superharmonic function \( u \) in \( \Omega \), admits a greatest \( L^V \)-harmonic function \( h \) dominated by \( u \) and that \( u - h \) is a potential.

Theorem 3.4. a) If \( u, v \) are positive \( L^V \) harmonic functions in \( \Omega \) then,

\[
\lim_{x \to \zeta} \frac{u(x)}{v(x)} = \frac{d\mu_u}{d\mu_v}(\zeta) \quad L^V - \text{finely, } \mu_u - \text{a.e. on } \partial \Omega
\]

where \( \mu_u \) and \( \mu_v \) are the \( L^V \) boundary measures of \( u \) and \( v \) respectively and the term on the right hand side denotes the Radon-Nikodym derivative.

(b) If \( p \) is an \( L^V \) potential and \( v \) is positive \( L^V \) harmonic function:

\[
\lim_{x \to \zeta} \frac{p(x)}{v(x)} = 0 \quad L^V - \text{finely, } \mu_v - \text{a.e. on } \partial \Omega
\]

where \( \mu_v \) is the \( L^V \) boundary measure of \( v \).

*See the comments on \( L^V \)-fine convergence, following the theorem.
For the definition of $L^V$ fine convergence see e.g. [1] (where proofs of the above statements are provided). It is useful and handy to note that if $f$ is a non-negative function in $\Omega$ satisfying the strong Harnack property in $\Omega$ then (see [4] p. 409).

\begin{equation}
\lim_{x \to \zeta} f(x) = a \quad L^V \text{-- finely } \iff f(x) \to a \text{ as } x \to \zeta \text{ non--tangentially.}
\end{equation}

Here we say that a nonnegative function $f$ in $\Omega$ satisfies the strong (uniform) Harnack property in $\Omega$ if for every $\varepsilon > 0$ there is a positive $\eta$ such that

\begin{equation}
|f(x) - f(y)| \leq \varepsilon f(x) \quad \text{whenever } x, y \in \Omega \text{ and } |x - y| \leq \eta d(x; \partial \Omega)
\end{equation}

Nonnegative solutions $u$ of (1.8) or (1.1) have this property. If $f$ and $g$ satisfy the Harnack property and $g$ is $> 0$ in $\Omega$ then the ratio $f/g$ also satisfies the strong Harnack property in $\Omega$. In the present paper (3.4) will be applied only in cases where $p, u, v$ satisfy the Harnack property in $\Omega$. Therefore in our applications we may replace “$L^V$ fine convergence” by “n.t. convergence” (where n.t. is an abbreviation for non-tangential).

$L^V$-regularity of boundary points. For the next definition note that for $\zeta \in \partial \Omega$ the harmonic function $\mathcal{K}_\zeta$ is $L^V$ superharmonic. Therefore, by the Riesz decomposition theorem, $\mathcal{K}_\zeta$ can be uniquely represented in the form

\begin{equation}
\mathcal{K}_\zeta = \mathcal{K}^V_\zeta + p_\zeta
\end{equation}

where $\mathcal{K}^V_\zeta$ is the largest $L^V$ harmonic function dominated by $\mathcal{K}_\zeta$ and $p_\zeta$ is an $L^V$-potential. ($\mathcal{K}^V_\zeta$ may be the zero function.) $\mathcal{K}^V_\zeta$ is the unique $L^V$ harmonic function that vanishes on $\partial \Omega \setminus \{\zeta\}$ and is normalized by $\mathcal{K}^V_\zeta(x_0) = 1$. Therefore,

\begin{equation}
\mathcal{K}^V_\zeta := c_V(\zeta)\mathcal{K}^V_\zeta,
\end{equation}

for some $c_V(\zeta) \geq 0$. Note that $c_V(\zeta) \in [0, 1]$.

**Definition 3.5.** (see [3], [4]) A point $\zeta \in \partial \Omega$ is $L^V$ regular if $\mathcal{K}_\zeta$ is not an $L^V$ potential, i.e. if $c_V(\zeta) > 0$. The point $\zeta$ is $L^V$ singular if it is not regular i.e. if $c_V(\zeta) = 0$.

The set of $L^V$ regular points is denoted by $\mathcal{R}^V$ while the set of $L^V$ singular points is denoted by $\mathcal{S}^V$.

The next facts are taken from [3] (see also [4]). Let $\zeta \in \partial \Omega$. Let $\nu$ be a unit vector in $\mathbb{R}^N$ be such that $\lim_{x \to \zeta, \nu} \nu = 0 \iff \text{a pseudo-inner normal at } \zeta$.

**Proposition 3.6.** (i) $\zeta \in \mathcal{S}^V$ if and only if $\mathcal{K}_\zeta(x)/\mathcal{K}^V_\zeta(x) \to 0$ as $x \to \zeta$ n.t.,

(ii) $\zeta \in \mathcal{R}^V$ if and only if $\liminf_{t \to 0} \mathcal{K}_\zeta(x_0 + tv)/\mathcal{K}^V_\zeta(x_0 + tv) > 0$,

(iii) $\zeta \in \mathcal{S}^V$ if and only if $\mathcal{G}_\zeta(x_0)/\mathcal{G}^0_\zeta(x_0) \to 0$ as $x \to \zeta$ n.t.,

(iv) $\zeta \in \mathcal{R}^V$ if and only if $\liminf_{t \to 0} \mathcal{G}_\zeta(x_0 + tv)/\mathcal{G}^0_\zeta(x_0 + tv) > 0$. 

The next useful fact connecting the $L^V$ boundary measure and the $m$ boundary trace for moderate $\geq 0$ solutions is proved in \cite{27} in the case of $C^2$ domains.

**Proposition 3.7.** If $u$ is a positive $L^V$ moderate harmonic function then its $m$-boundary trace and its $L^V$ boundary measure are mutually absolutely continuous.

**Remark.** Using Lemma 3.3 this is proved in the same way as Proposition 3.8 of \cite{27}. An alternative argument is given below.

**Proof.** Suppose $K^V_\mu$ is moderate ($\mu \in M_+(\partial \Omega)$) and let $m$ be its $m$-trace on $\partial \Omega$. If $\zeta$ is regular then, by (3.3) and (3.4),

$$c_V(\zeta)^{-1}K^V_\zeta = K^V_\zeta + p'_\zeta$$

and $p'_\zeta = G(VK^V_\zeta)$. It follows that $K_{c^{-1}\mu} = K^V_\mu + \text{potential}$ and $c^{-1} \mu = m$.

The following somewhat related fact will be useful later.

**Lemma 3.8.** Suppose $V \in \mathcal{V}(\bar{a}, \Omega)$, $\mu, \nu \in M_+(\partial \Omega)$ and assume that $K^V_\nu \leq K_\mu$ in $\Omega$. Then for every Borel set $A \subset \partial \Omega$,

$$K^V_{1_A \nu} \leq K_{1_A \mu}$$

Moreover if $\mu$ is the boundary trace of $K^V_\nu$, then for every closed set $A \subset \partial \Omega$,

$$\text{tr}(K^V_{1_A \nu}) = 1_A \mu$$

**Proof.** For the first claim, it suffices -by standard measure approximation procedures- to show that $K^V_{1_L \nu} \leq K_{1_U \mu}$ whenever $F$ is compact, $U$ is open in $\partial \Omega$ and $F \subset U$. To prove this observe that $\inf\{K_{1_U \mu} - K^V_{1_F \nu}, 0\}$ is a classical superharmonic function in $\Omega$ which has a nonnegative inferior limit at every point of $\partial \Omega$ (use the fact that $K^V_\lambda$ goes to zero at every $\zeta \in \partial \Omega \setminus \text{supp} \lambda$). Thus it is nonnegative and the claim follows.

Now if $\mu = \text{tr}(K^V_\nu)$, $K^V_\nu \leq K_\mu$ in $\Omega$. Therefore, if $F$ is a compact subset of $\partial \Omega$, $K_{1_F \mu} - K^V_{1_F \nu}$ is a non-negative superharmonic function. Again, by the first part of the lemma (applied to $A = \partial \Omega \setminus F$) we have

$$K_{1_F \mu} - K^V_{1_F \nu} \leq K_\mu - K^V_\nu.$$ 

Since $K_\mu - K^V_\nu$ is a potential in $\Omega$, the nonnegative superharmonic function $K_{1_L \mu} - K^V_{1_F \nu}$ is also a potential and its $m$ boundary trace is zero.

**Two regularity criteria.** The next result was proved in \cite{1} and \cite{27}. (In \cite{27} Proposition 3.9 the result was proved for $C^2$ domains. Using Lemma 3.3 the proof applies equally well to Lipschitz domains.)
Proposition 3.9. For every $\zeta \in \partial \Omega$

\begin{equation} \label{eq:3.5}
\int_{\Omega} \mathbb{K}_V V \psi \, dx < \infty \iff \zeta \in \mathcal{R}^V.
\end{equation}

Furthermore, if $u$ is a positive $L^V$ moderate harmonic function and $\nu = \text{tr}_{\partial \Omega} u$ then,

\begin{equation} \label{eq:3.6}
\nu(S^V) = 0.
\end{equation}

Finally the following characterization of $L^V$ regularity is the main result in [4].

Proposition 3.10. Let $\zeta \in \partial \Omega$. Then

\begin{equation} \label{eq:3.7}
\zeta \in \mathcal{R}^V \iff \int_{\Omega} \mathbb{K}_\zeta V \psi \, dx < \infty.
\end{equation}

4. The Boundary Harnack Principle and related auxiliary results

Let $r, \rho$ be positive numbers and let $f: \mathbb{R}^{N-1} \to \mathbb{R}$ be Lipschitz with a Lipschitz constant $\leq \frac{\rho}{r^2}$ and such that $f(0) = 0$. We set

$$
\omega_f(r, \rho) := \{\xi = (\xi', \xi_N) \in \mathbb{R}^{N-1} \times \mathbb{R}; |\xi'| < r, f(\xi') < \xi_N < \rho\}.
$$

This region will be called the standard Lipschitz domain of height $\rho$, radius $r$ and defining function $f$. We denote $A = A(\rho) := (0, \ldots, 0, \frac{\rho}{2}), A' = A'(\rho) := \frac{3}{4} A$.

We assume at first that $\Omega$ is a region in $\mathbb{R}^N$ whose intersection with the cylinder $T(r, \rho) := \{\xi = (\xi', \xi_N) \in \mathbb{R}^{N-1} \times \mathbb{R}; |\xi'| < r, -\rho < \xi_N < \rho\}$ is $\omega = \omega_f(r, \rho)$.

We fix some $V \in \mathcal{V}(\bar{a}, \Omega), \bar{a} > 0$. As before, $G^V$ denote the Green’s function for $L^V$ in $\Omega$ and $G = G^0$. The following boundary Harnack principle for our operators $L^V$ is proved in [2] (see also a presentation in [4]).

Theorem 4.1. There is a constant $c$ depending only on $N, \bar{a}$ and $\frac{\rho}{r}$ such that whenever $u$ is a positive $L^V$ harmonic function in $\omega$ that vanishes continuously in $\partial \omega \cap T(r, \rho)$ then

\begin{equation} \label{eq:4.1}
c^{-1} r^{N-2} G^V(x, A') \leq \frac{u(x)}{u(A)} \leq c r^{N-2} G^V(x, A'), \quad \forall x \in \Omega \cap \overline{T(\frac{r}{2}, \frac{\rho}{2})}
\end{equation}

In particular, for any pair $u, v$ of positive $L^V$ harmonic functions in $\omega$ that vanish on $\partial \omega \cap T(r, \rho)$:

\begin{equation} \label{eq:4.2}
u(x)/\nu(x) \leq Cu(A)/v(A), \quad \forall x \in \omega_f(r/2, \rho/2)
\end{equation}

where $C = c^2$. 
Proposition 4.3. Suppose that for some constant kernels $K$ statements that the reference point well-known consequence (see e.g. [4, Lemma 3.5]). It will be assumed in the next four to gether with its following this end we use the boundary Harnack principle Theorem 4.1 togethe r with its following

and $\Delta v = 0$. Various results of this type are known, in particular it can be obtained as a consequence of an assumption on the growth of $V$ necessarily stronger than (1.9) (see e.g. [3, 6] and the references therein). Here we will derive estimates of the ratio of $K_\zeta$ and $K^V_\zeta$ in $\Omega$ uniform w.r. to $\zeta$ in certain subsets of $R^V$ (see Proposition 4.6 below). To this end we use the boundary Harnack principle Theorem 4.1 together with its following well-known consequence (see e.g. [4, Lemma 3.5]). It will be assumed in the next four statements that the reference point $x_0$ is in $\Omega \setminus T(r, \rho)$.

Proposition 4.2. There exists a constant $C$ such that for all $x = (0, \ldots, t)$, $|t| \leq \frac{3}{4} \rho$,

\[
C^{-1} t^{2-N} \leq \mathcal{G}_0^V(x) \mathcal{G}^V_{x_0}(x) \leq C t^{2-N}.
\]

and $C$ can be chosen depending only on $\bar{a}$, $\frac{2}{r}$ and $N$.

We now formulate conditions in terms of the Green kernels $G_{x_0}$ and $G^V_{x_0}$ for the Martin kernels $K_\zeta$ and $K^V_\zeta$ to be equivalent.

Proposition 4.3. Suppose that for some constant $C \geq 1$ it holds that

\[
\mathcal{G}(x_0, y) \leq C \mathcal{G}^V(x_0, y) \quad \text{for all } y \in \omega_f(r, \rho).
\]

Then there is a constant $C_1 \geq 1$ depending on $C$, $\frac{2}{r}$, $N$ and $\bar{a}$ such that for all $\zeta \in \partial \Omega \cap T(r/8, \rho/8)$ and $x \in \Omega \cap T(r/4, \rho/4)$

\[
C_1^{-1} K_\zeta(x) \leq K^V_\zeta(x) \leq C_1 K_\zeta(x)
\]

Recall that $\mathcal{G}^V \leq \mathcal{G}$ always holds since $V \geq 0$.

Proof. It suffices to consider the case $\zeta = 0$. Choose $0 < t \leq 1/4$ such that $x \in \partial(T(tr, t\rho))$. It follows from the remark following Theorem 4.1 that

\[
C_1^{-1} \mathcal{G}^V(z, A'(t\rho))) \mathcal{G}^V(A'(t\rho), x) (tr)^{N-2} \leq \mathcal{G}^V(z, x)
\]

\[
\leq C_1 \mathcal{G}^V(z, A'(t\rho))) \mathcal{G}^V(A'(t\rho), x) (tr)^{N-2}
\]

when $z \in \Omega \cap \partial T(2tr, 2t\rho)$. By the maximum principle for $L^V$ these relations extend to all $z \in \Omega \setminus T(2tr, 2t\rho)$ and thus hold at $z = x_0$. Using (4.4) at $z = x_0$, together with their analogues for $V = 0$ and the hypothesis, one obtains that $\mathcal{G}^V(A'(t\rho), x)$ and $\mathcal{G}(A'(t\rho), x)$ are uniformly comparable. By Theorem 4.1 $K^V_\zeta(x)$ is uniformly comparable.
to $K_x^V(A'(t\rho)) \mathcal{G}^V(A'(t\rho); x)(tr)^{N-2}$, hence also to the ratio $\mathcal{G}^V(A'(t\rho); x)/\mathcal{G}^V(x_0, A'(t\rho))$ (using Proposition 4.2). The proposition follows. $\Box$

**Corollary 4.4.** Assume that $\mathcal{G}(x_0, y) \leq C \mathcal{G}^V(x_0, y)$ for $y \in \Omega$ (not only in $\Omega \cap T(r, \rho)$ as above). Then there is a constant $C_2 = C_2(\Omega, r, \rho, x_0, \bar{a}) \geq 1$ such that for $\zeta \in \partial \Omega \cap T(r/8, \rho/8)$ and $x \in \Omega$:

$$C_2^{-1} K_\zeta(x) \leq K_\zeta^V(x) \leq C_2 K_\zeta(x)$$

In fact it follows from Theorem 4.1 that $K_\zeta^V(x)$ is -for $x \in \Omega \setminus T(d/2, d/2)$ and given $\Omega$, $x_0$, $r$, $\rho$, $\bar{a}$-- uniformly equivalent to $\mathcal{G}^V(A'(\frac{d}{2}), x)$ (it suffices to consider the case $x \in \partial T(d/2, d/2)$). Using Harnack inequalities (for $\Omega \ni y \mapsto \mathcal{G}^V(y, x)$) one then sees that $K_\zeta(x)$ is uniformly equivalent to $1 \wedge \mathcal{G}^V(x_0, x)$ -under the same assumptions on $x$ and $\zeta$. Using this also for $V = 0$ together with the hypothesis the result follows for $x \in \Omega \setminus T(d/2, d/2)$. $\Box$

**Lemma 4.5.** Suppose that $0 \leq V \leq \frac{C}{r^2}$ in $\omega = \omega_f(r, \rho)$. Then, with $A = A(\frac{d}{2})$ and for some constant $C_1 = C_1(d, N, C)$

$$\mathcal{G}^V_{x_0} \leq \mathcal{G}_{x_0} \leq C \frac{\mathcal{G}(x_0, A)}{\mathcal{G}^V(x_0, A)} \mathcal{G}^V_{x_0} \text{ in } \omega_f\left(\frac{9r}{10}, \frac{9\rho}{10}\right)$$

We remark here that by homogeneity, one may assume that $r = 1$ and $\rho$ is fixed. Some general similar comparability results (w.r. to two different operators) appear in [6].

**Proof.** Using the Harnack boundary principle Theorem 4.1 it suffices to show that the Green’s functions $g$, $g^V$ for $L^0$ and $L^V$ in $\omega$ are such that $g(A, \cdot) \leq c g^V(A, \cdot)$ in $\omega$ with a constant $c = c(N, \frac{d}{2}, C)$. A proof of the later fact is provided in [5]. In fact it would suffice for the proof to use [5, Proposition 3].

From now on $\Omega$ denotes a general bounded Lipschitz domain in $\mathbb{R}^N$. By definition for each $P \in \partial \Omega$ there is an isometry $I : \mathbb{R}^N \to \mathbb{R}^N$ and a cylinder $T(r, \rho)$ such that $I(P) = 0$ and $I(\Omega) \cap T(r, \rho)$ is a standard Lipschitz domain in $\mathbb{R}^N$ with height $\rho$ and radius $r$. So the previous propositions can be applied to $\Omega_0 = I(\Omega)$ and $T(r, \rho)$.

If $\zeta \in \partial \Omega, x \in \Omega$ and $0 < \theta < \frac{\pi}{2}$, we will say that $[\zeta, x] := \{\zeta + tx; 0 < t \leq 1\}$ is inner $\theta$-non-tangential at $\zeta$ (w.r. to $\Omega$) if the truncated cone $C(\zeta, x, \theta) := \{z; \cos \theta |z - \zeta| |x - \zeta| < |z - \zeta| < |x - \zeta|^2\}$ is contained in $\Omega$.

In the next statement $F$ denotes a compact subset of $\partial \Omega$ such that $V(x) \leq \frac{a}{d(x, F)^2}$ for $x \in \Omega$. As before a reference point $x_0 \in \Omega$ is fixed.
Proposition 4.6. Let $0 < \theta < \frac{\pi}{2}$, $c_1, c_2 > 0$ and assume that for every $\zeta \in F$ there is an inner $\theta$-non-tangential segment $[\zeta, b_\zeta]$ at $\zeta$ (w.r. to $\Omega$) with $|b_\zeta - \zeta| \geq c_2$ and

$$K_\zeta(x) \leq c_1 K_\zeta(x), \quad \forall \zeta \in F, \forall x \in (\zeta, b_\zeta)$$

(thus $F \subset R^V$). Then there exists a constant $\bar{c} \geq 1$ depending only on $\bar{a}$, $\theta$, $c_1$, $c_2$ and $\Omega$ such that, for all $\zeta \in \partial \Omega$ and all $x \in \Omega$

$$\bar{c}^{-1} \leq \frac{K_\zeta(x)}{K_\zeta(x)} \leq \bar{c}$$

Proof. By Proposition 4.2 the assumption means that for some constant $c \geq 1$, $G(x_0, y) \leq c G^V(x_0, y)$ for all $y \in \bigcup(\zeta, b_\zeta]$. By Harnack inequalities and the definition of a Lipschitz domain this means that for every given $\kappa > 0$, it holds that for some constant $c' = c'(c_1, c_2, \bar{a}, \Omega) > 0$,

$$G(x_0, y) \leq c' G^V(x_0, y)$$

for $y \in U(\Omega, \kappa) := \Omega \cap \{ x ; d(x, \partial F) \leq \kappa d(x, \partial \Omega) \}$. Using this for $\kappa > 0$ small enough, Lemma 4.3 shows that $G(x_0, y) \leq c G^V(x_0, y)$ whenever $y \in \Omega \setminus U(\Omega, 2\kappa)$. The proposition follows using Corollary 4.4 above. \(\square\)

We close this section with another auxiliary result. Here we assume that the region $\Omega$ is Lipschitz and bounded, that $g : \Omega \times \mathbb{R}_+ \to \mathbb{R}_+$ is continuous and that $g(x, t)/t$ is non decreasing in $t, t > 0$.

Proposition 4.7. Let $u$ be a nonnegative solution of the equation

$$\Delta u(x) = g(x, u(x))$$

and let $F \subset \partial \Omega$ be closed. Assume further that $g(x, u(x)) \leq \frac{c_0}{|\partial \Omega|^2} u(x)$ in $\Omega$ for some constant $c_0 \geq 1$. Then there is a largest solution $v$ dominated by $u$ in $\Omega$ and vanishing on $\partial \Omega \setminus F$. This solution is also the largest nonnegative (say continuous) subsolution dominated by $u$ that vanishes on $\partial \Omega \setminus F$.

a) If $v$ is a nonnegative solution dominated by $u$ in $\Omega$, then $\Delta v = V v$ for some potential $V = V_\mu \in \mathcal{V}(\Omega, c_0)$ and $v = K^V_{\mu}$ for some $\mu \in \mathcal{M}_+(\partial \Omega)$.

Suppose that $v = 0$ on $T(r, \rho) \cap \partial \Omega$ for some cylinder $T(r, \rho)$ such that $\omega(r, \rho) = \Omega \cap T(r, \rho)$ is a standard Lipschitz domain of height $\rho$ and radius $r$. Then by the boundary Harnack principle applied to the $L^V$-harmonic function $v$,

$$v(x) \leq C v(A') r^{N-2} G^V_{A'}(x) \leq C u(A') r^{N-2} G_{A'}(x), \quad x \in \omega(r/2, \rho/2)$$

with $C$ depending only on $\frac{\rho}{r}$, $N$ and $c_0$. Here $A' = (0, \frac{3}{4}\rho)$. 
This shows that the family of all non-negative solutions of $\Delta u - g(x, u) = 0$ dominated by $u$ and vanishing on $\partial \Omega \setminus F$ must uniformly vanish at each point $\zeta \in \partial \Omega \setminus F$

Remark. Recall that since $s = 1$ is $L^V_e$ superharmonic, the Fatou-Doob property available here implies that $\mu(\partial \omega \cap T(r, \rho)) = 0$ (see Remark 2.7 in [14]).

b) Let again $T(r, \rho)$ be a cylinder such that $\omega(2r, 2\rho) = \Omega \cap T(r, \rho)$ is a standard Lipschitz domain of height $\rho$ and radius $r$ and let now $v$ be finite non-negative lsc function in $\overline{\omega}(r, \rho)$ which is $\leq u$ in $\Omega \cap \overline{\omega}(r, \rho)$, vanishes continuously on $\partial \Omega \cap \overline{T}(r, \rho)$ and which is a subsolution in $\omega$. Since e.g. $\Delta v \geq 0$ we may assume after a modification on a negligible set that $v$ is subharmonic.

Setting again $V = g(x, v(x))/v(x)$ ($= 0$ if $v(x) = 0$), $v$ is clearly $L^V_e$ subharmonic in $\omega$ and $V_v \in \mathcal{V}(\Omega, c_0)$.

Fix $\varepsilon > 0$ small. The function $(v - \varepsilon)_+$ is again a subsolution on $\omega' := T'(3r/4, \rho) \cap (\Omega + \varepsilon' e_N)$ that vanishes on $T(3r/4, \rho) \cap (\partial \Omega + \varepsilon' e_N)$ provided $\varepsilon'$ is sufficiently small. Since the equation $\Delta u - g(x, u) = 0$ has its absorption term continuous in $\overline{\omega'} \times \mathbb{R}$ a standard result asserts that there is a minimum solution $w_{\varepsilon, \varepsilon'}$, with $(v - \varepsilon)_+ \leq w_{\varepsilon, \varepsilon'} \leq u$ and that $w_{\varepsilon, \varepsilon'} = 0$ on $T(P, r) \cap (\partial \Omega + \varepsilon' e_N)$.

Now by part, a) (and Harnack) we have on $T(r/2, \rho/2) \cap (\Omega + \varepsilon' e_N)$

$$(v(x) - \varepsilon)_+ \leq w_{\varepsilon, \varepsilon'}(x) \leq cu(A')r^{N-2}G_{A'}(x) \leq cu(A')r^{N-2}G_{A'}(x)$$

Notice that $c$ here is independent of $\varepsilon$ and $\varepsilon'$. So $(v(x) - \varepsilon)_+ \leq cu(A')r^{N-2}G_{A'}(x)$ and letting $\varepsilon$ goes to zero we get that the set of all the considered functions $v$ uniformly goes to zero on $\partial \Omega \cap T(r/2, \rho/2)$.

c) Finally the supremum of all the nonnegative continuous subsolutions $v$ vanishing on $\partial \Omega \setminus F$ and dominated by $u$ vanishes on $\partial \Omega \setminus F$. It is also well known to be a (continuous) solution and the proposition is proved. \( \square \)

The next statement complements Proposition 4.7

**Proposition 4.8.** In addition to the basic conditions, suppose that $g$ satisfies

$$g(x_1, t)/t \to \infty \quad \text{as} \quad t \to \infty,$$

for some $x_1 \in \Omega$. Then there is a largest solution $U_F$ of $(1.1)$ in $\Omega$ such that $U_F \in C(\bar{\Omega} \setminus F)$ and $U_F = 0$ on $F' := \partial \Omega \setminus F$.

**Proof.** Let $\mathcal{U}$ denote the set of all solutions of $(1.1)$ in $\Omega$. Then $\{h \circ v : v \in \mathcal{U}\}$ is locally uniformly bounded in $\Omega$ by $(1.3)$, so that by $(1.5)$, $\sup\{v(x_1) : v \in \mathcal{U}\} < \infty$. Since the potentials $V_v := h \circ v$ are locally uniformly bounded, the solutions $v$ satisfy Harnack inequalities uniformly w.r. to $v \in \mathcal{U}$ in $\Omega$. Therefore $U := \sup \mathcal{U}$ is locally
bounded and in fact a solution of (1.1). This proves the proposition for \( F = \partial \Omega \). The general case follows using Proposition 4.7 with \( u = U_{\partial \Omega} \).

5. \((g, \Delta)\)-Moderate solutions and \(g\)-good measures

In this section we assume that \( g \) satisfies conditions (1.2) and (1.3). Additional assumptions will be mentioned if needed.

**Theorem 5.1.** Let \( u \) be a positive \( g \)-moderate solution of (1.1) in the bounded Lipschitz domain \( \Omega \). If \( \nu := \text{tr}_{\partial \Omega} u \) then there exists a positive measure \( \tau \in \mathcal{M}_+(\partial \Omega) \) such that \( \tau \leq \nu \), \( \tau \neq 0 \) and \( \tau \in \mathcal{M}^{g, \Delta}_+ \), i.e.,

\[
\int_\Omega g \circ K[\tau] \psi \, dx < \infty.
\]

**Proof.** Let \( V(x) := h(x, u(x)) \), \( x \in \Omega \). By assumption

\[
\int_\Omega (g \circ u) \psi \, dx = \int_\Omega u V \psi \, dx < \infty.
\]

and \( u \) is a moderate \( L^V \) harmonic function in \( \Omega \). Denote \( \nu' := \mathbb{K} V_{\nu'} \) and recall that by Proposition A.1 the measures \( \nu' \) and \( \nu \) are mutually absolutely continuous. By Proposition 3.9 and Fubini’s theorem, \( \nu' \)-almost every point \( \zeta \in \partial \Omega \) is \( L^V \) regular. Thus using Proposition 3.6 and fixing some \( \theta \in (0, \frac{\pi}{2}) \) sufficiently small (depending on \( \Omega \)) we may find a compact subset \( F \) of \( \partial \Omega \) with \( \nu(F) > 0 \) and positive constants \( c_1 \) and \( r \) such that whenever \( \zeta \in F \) there is a point \( x = x(\zeta) \in \Omega \) such that \( |\zeta - x| = r \), \((\zeta, x] \) is \( \theta \) non-tangential at \( \zeta \) for \( \Omega \) and

\[
G(x_0, z) \leq c_1 G^V(z, x_0), \quad \forall z \in (\zeta, x).
\]

Set \( \nu_F = 1_F \nu' \). Then \( v := \mathbb{K}^V_{\nu_F} \) is a positive subsolution of (1.1) and the maximal solution \( w \) dominated by \( u \) and vanishing on \( \partial \Omega \setminus F \) is moderate, its \( m \)-boundary trace being \( \nu_F := 1_F \nu \). This follows from Lemma 3.8 which implies that \( \mathbb{K}^V_{\nu_F} \leq w \leq \mathbb{K}_{\nu_F} \) in \( \Omega \) and then that \( \text{tr}_{\partial \Omega}(w) = \nu_F \).

In particular if \( W(x) := h(x, w(x)) \), \( x \in \Omega \), we have \( w = \mathbb{K}^W_\lambda \) for a positive measure \( \lambda \in \mathcal{M}_+(\partial \Omega) \) which is equivalent to \( \nu'' \) (and hence to \( 1_F \nu \)).

Also, since \( 0 \leq W \leq V \), we again have

\[
G(x_0, z) \leq c_1 G^W(x_0, z)
\]

whenever \( z \in (\zeta, x] \) with \( \zeta \in F \) and \( x = x(\zeta) \) as above. By Proposition 4.2 this means that for some other constant \( c_2 \geq 1 \)

\[
(c_2)^{-1} \mathbb{K}^W_\zeta(z) \leq \mathbb{K}^W_\zeta(z) \leq c_2 \mathbb{K}^W_\zeta(z), \quad \forall z \in (\zeta, x(\zeta)], \zeta \in F.
\]
and by Proposition 4.6 above – notice that \( W \leq c \) \[ d(x, F) \] \( -2 \) – we conclude that for some other constant \( c_3 > 1 \) we even have

\[
(c_3)^{-1} K_{\zeta}^W(z) \leq K_{\zeta}(z) \leq c_3 K_{\zeta}^W(z), \quad \text{for all } \zeta \in F \text{ and all } z \in \Omega
\]

So that finally, taking \( \tau \not\equiv 0 \) with \( c_3 \tau \leq \nu'' \wedge \lambda \)

\[
\int (g \circ K_\tau) \psi \, dx \leq \int (g \circ K_{\Delta}^W) \psi \, dx \leq \int (g \circ u) \psi \, dx < \infty. \quad \Box
\]

Based on Theorem 5.1 above, the next result provides a complete characterization of conditionally \( g \)-removable sets. In addition, if \( g \) satisfies the \( \Delta_2 \) condition, we obtain a full characterization of \( g \)-good measures and show that every \( g \) moderate solution is the limit of an increasing sequence of \( (g, \Delta) \) moderate solutions. In the absence of the \( \Delta_2 \) condition we establish a modified version of the last two results

At this point it is convenient to introduce the following notation:

\[
\mathcal{T}^{g, \Delta} := \cup_1^n M_{\Delta}^{g, \Delta}(\partial \Omega).
\]

Thus \( \mathcal{T}^{g, \Delta} \) is the convex cone generated by \( M_{\Delta}^{g, \Delta}(\partial \Omega) \). If \( \mu_1, \ldots, \mu_n \in M_{\Delta}^{g, \Delta} \) then

\[
\frac{1}{n} \sum_{i=1}^n \mu_i \in M_{\Delta}^{g, \Delta}
\]

and \( \mathcal{T}^{g, \Delta} \) may be described as the family of all finite sums of measures from \( M_{\Delta}^{g, \Delta} \). Note that if \( \nu_1, \nu_2 \in \mathcal{T}^{g, \Delta} \) and \( \nu_1 \leq \nu_2 \) then \( \nu_2 - \nu_1 \in \mathcal{T}^{g, \Delta} \).

Clearly, if \( g \) satisfies the \( \Delta_2 \) condition then \( \mathcal{T}^{g, \Delta} = \mathcal{M}_{\Delta}^{g, \Delta} \).

**Theorem 5.2.** A compact set \( F \subset \partial \Omega \) is conditionally \( g \)-removable if and only if \( F \) is \( M_{\Delta}^{g, \Delta} \) null.

**Proof.** If \( F \) is not \( M_{\Delta}^{g, \Delta} \) null then, by definition, there exists a positive measure \( \tau \in M_{\Delta}^{g, \Delta} \) which is concentrated on \( F \). Thus \( \tau \) is \( g \)-good and \( S^g_{\nu} \) is a positive moderate solution of (1.1) that vanishes on \( \partial \Omega \setminus F \).

Suppose now that \( F \) is \( M_{\Delta}^{g, \Delta} \) null. Let \( u \) be a non-negative \( g \)-moderate solution that vanishes on \( \partial \Omega \setminus F \) and let \( \mu := tr u \). Clearly \( \mu(\partial \Omega \setminus F) = 0 \). If \( u \) is positive then, by Theorem 5.1 there exists a positive measure \( \tau \in M_{\Delta}^{g, \Delta} \) such that \( \tau \leq \mu \). But this is impossible because \( F \) is \( M_{\Delta}^{g, \Delta} \) null. Thus \( u = 0 \). \( \Box \)

**Theorem 5.3.** If \( u \) is a positive \( g \)-moderate solution of (1.1) in \( \Omega \) then \( u \) is the limit of an increasing sequence of solutions of (1.1) whose boundary trace belongs to \( \mathcal{T}^{g, \Delta} \).
Consequently, there exists a sequence \( \{ \mu_n \} \subset \mathcal{M}^g_{+\Delta} \) such that

\[
\text{tr}(u) = \sum_1^{\infty} \mu_n \tag{5.4}
\]

If \( g \) satisfies the \( \Delta_2 \) condition then every \( g \)-moderate solution is the limit of an increasing sequence of \( (g, \Delta) \) moderate solutions.

Proof. Let \( u \) be a positive \( g \)-moderate solution with m-boundary trace \( \mu \). Let \( u^* \) be the supremum of all \( g \)-moderate solutions \( v \) such that \( \text{tr}(v) \in T^{g,\Delta} \) and \( v \leq u \). Clearly \( u^* \) is \( g \)-moderate (see Proposition 1.5), \( \mu^* := \text{tr}(u^*) \leq \mu \) and \( \mu^* \) is the upper envelope of all \( v \in T^{g,\Delta} \) which are majorized by \( \mu \).

Suppose that \( u^* \not\equiv u \) so that \( \mu^* := \text{tr} u^* \not\equiv \mu \). Then \( \mu - \mu^* \) is a nonzero \( g \)-good measure and by Theorem 5.1 there exists a nonzero measure \( \tau' \leq \mu - \mu^* \) in \( \mathcal{M}^g_{+\Delta} \). If \( \nu \in T^{g,\Delta} \) then \( \nu + \tau' \leq \mu \). Therefore \( \mu^* + \tau' \leq \mu^* \), which contradicts the fact that \( \tau' \) is positive. Hence, \( u^* = u \). Finally, every positive measure \( \mu \) that is the limit of an increasing sequence of measures in \( T^{g,\Delta} \) can be represented as in (5.4).

Now assume that \( g \) satisfies the \( \Delta_2 \) condition. In this case \( \mathcal{M}^g_{+\Delta} = T^{g,\Delta} \). Therefore the last assertion of the theorem is a consequence of the first. \( \square \)

**Theorem 5.4.**

(i) If \( \mu \in \mathcal{M}_+ \) is a \( g \)-good measure then \( \mu(A) = 0 \) for every \( \mathcal{M}^g_{+\Delta} \) null set \( A \).

(ii) If \( \mu \in \mathcal{M}_+ \) vanishes on \( \mathcal{M}^g_{+\Delta} \) null sets then there exists a Borel function \( f : \partial \Omega \rightarrow (0,1] \) such that \( f\mu \) is a \( g \)-good measure.

(iii) If \( g \) satisfies the \( \Delta_2 \) condition then, \( \mu \in \mathcal{M}_+ \) is a \( g \)-good measure if and only if \( \mu(A) = 0 \) for every \( \mathcal{M}^g_{+\Delta} \) null set \( A \).

**Proof.**

(i) Let \( A \) be a compact \( \mathcal{M}^g_{+\Delta} \) null set. If \( \mu \) is \( g \)-good and \( \mu(A) > 0 \) then \( \tau := \mu 1_A \) is a \( g \)-good positive measure. Since \( A \) is compact, \( S^g[\tau] \) is a positive \( g \)-moderate solution that vanishes on \( \partial \Omega \setminus A \). By Theorem 5.2 \( u = 0 \), which is absurd. This implies (i).

(ii) If \( \mu \in \mathcal{M}_+ \) vanishes on \( \mathcal{M}^g_{+\Delta} \) null sets then, by Theorem 2.1 it is the limit of an increasing sequence of measures belonging to \( T^{g,\Delta}(\partial \Omega) \) (see Corollary 2.2). Therefore it can be represented by a series \( \sum_1^{\infty} \mu_n \) where \( \mu_n \in \mathcal{M}^g_{+\Delta} \). Every measure in \( \mathcal{M}^g_{+\Delta} \) is \( g \)-good and the maximum of two \( g \)-good measures is \( g \) good (see Proposition 1.5).

Therefore \( \nu_n := \max(\mu_1, \cdots, \mu_n) \) is \( g \)-good. Consequently \( \nu := \lim_{n \rightarrow \infty} \max(\mu_1, \cdots, \mu_n) \) is \( g \) good, \( \nu \leq \mu \) and \( \nu \) is absolutely continuous w.r. to \( \mu \).
(iii) Now assume that \( g \) satisfies the \( \Delta_2 \) condition. Recall that in this case \( \mathcal{M}_+^{g,\Delta} = \mathcal{T}^{g,\Delta} \). If \( \mu \in \mathcal{M}_+ (\partial \Omega) \) is a positive measure that vanishes on \( \mathcal{M}_+^{g,\Delta} \)-null sets then, by Corollary 2.7, it is the limit of an increasing sequence of measures belonging to \( \mathcal{M}_+^{g,\Delta} (\partial \Omega) \). Since every measure in \( \mathcal{M}_+^{g,\Delta} (\partial \Omega) \) is \( g \)-good it follows that \( \mu \) is \( g \)-good.

\[ \square \]

6. Positive solutions and \( g \)-moderate solutions

As in the previous section it will always be assumed that \( g \) satisfies conditions (1.2) and (1.3).

Let \( u \) be a positive solution of (1.1) in \( \Omega \). Let \( V \) be defined as in (1.10) and let \( \mu \) be the \( L^V \) boundary measure for \( u \), i.e., \( u = \mathcal{K}^V_\mu \).

Set

\[ u^* = \sup \{ v ; v \text{ is a } g \text{-moderate solution in } \Omega \text{ and } 0 \leq v \leq u \}. \]

The function \( u^* \) is the largest \( \sigma \)-moderate solution of (1.1) dominated by \( u \).

Before stating our final result we recall a classical lemma on decomposition of measures that is used in our proof.

**Lemma 6.1.** (\cite{20}, see also \cite{9, Appendix 4A}) Let \( \mu \) be a finite measure on a measurable space \((Y, \mathcal{F})\) and let \( \mathcal{G} \) be a subset of \( \mathcal{F} \) such that:

1. \( \emptyset \in \mathcal{G} \) and \( \mathcal{G} \) is closed with respect to finite or countable unions,
2. \( A \in \mathcal{G}, A' \in \mathcal{F} \) and \( A' \subset A \implies A' \in \mathcal{G} \).

Then \( \mu \) can be uniquely written in the form \( \mu = \mu_0 + \mu_1 \), where \( \mu_0 \) and \( \mu_1 \) are finite measures on \((Y, \mathcal{F})\), \( \mu_1 (A) = 0 \) for all \( A \in \mathcal{G} \) and \( \mu_0 \) is concentrated on a set \( A_0 \in \mathcal{G} \).

Combining this lemma and Theorem 2.1 we obtain,

**Proposition 6.2.** Let \( \tau \in \mathcal{M}_+ (\partial \Omega) \) be a positive finite Borel measure. Then:

(i) There is a unique decomposition \( \tau = \tau_d + \tau_c \) with \( \tau_d, \tau_c \in \mathcal{M}_+ (\partial \Omega) \), \( \tau_d \) diffuse and \( \tau_c \) singular with respect to \( C_{g,\Delta} \).

(ii) \( \tau_d = \sum_{1}^{\infty} \tau_n \) for some increasing sequence \( \{ \tau_n \} \) in \( \mathcal{M}_+^{g,\Delta} (\partial \Omega) \)

**Proof.** The first statement is a special case of Lemma 6.1. The second statement follows from Theorem 2.1 (see Corollary 2.6) by the same argument as in the proof of Theorem 5.3. \( \square \)
If $F$ is a closed subset of $\partial \Omega$, we denote (recall $u$ is a nonnegative solution of (1.1) in $\Omega$),

$$[u]_F := \sup\{v \in U_F : v \leq u\}. \tag{6.2}$$

where $U_F$ is the set of all non-negative solutions of (1.1) in $\Omega$ vanishing on $\partial \Omega \setminus F$. By Proposition 4.7, $[u]_F$ is the largest solution dominated by $u$ and vanishing on $\partial \Omega \setminus F$.

We now state this section’s main result. Recall that $u$ is a positive solution of (1.1) and that $u^*$ is defined by (6.1).

**Theorem 6.3.** If $\mu$ is not singular relative to $C_{g, \Delta}$ then $u$ dominates a positive moderate solution. Furthermore, there exists a Borel function $f : \partial \Omega \to (0, 1]$ such that $f \mu$ is $g$-good and

$$\mathbb{S}_{f \mu}^{g} \leq u. \tag{6.3}$$

Let $u^*$ be as in (6.1). For every compact subset $F$ of $\mathcal{R}(u) := \mathcal{R}^V$,

$$[u]_F = [u^*]_F. \tag{6.4}$$

With the next lemma we begin the proof of Theorem 6.3. This lemma is an adaptation of [27, Lemma 5.5]. For the convenience of the reader we reproduce the proof here.

**Lemma 6.4.** Let $\nu \in \mathcal{M}^g(\partial \Omega)$ be a positive measure. Suppose that there exists no positive solution of (1.1) dominated by the supersolution $v = \inf(u, K_\nu)$. Then $\mu \perp \nu$.

**Proof.** Set $V' = h(v)$. Clearly $v$ is $L^{V'}$ superharmonic since $v$ is a supersolution of (1.1).

Moreover $v$ is an $L^{V'}$ potential. For if $w$ is a nonnegative $L^{V'}$ harmonic function dominated by $v$, then $w$ is a subsolution of (1.1). Hence, since $v$ is a supersolution, there exists a solution $w'$ of (1.1) with $w \leq w' \leq v$. By the assumption $w' = 0$ and hence $w = 0$. This shows that $v$ is an $L^{V'}$ potential.

Evidently, the harmonic function $K_\nu$ is $L^{V'}$ superharmonic. Since

$$\int_{\Omega} K_\nu V' \psi \, dx \leq \int_{\Omega} g(K_\nu) \psi \, dx < \infty,$$

$K_\nu$ is moreover an $L^{V'}$-moderate nonnegative superharmonic function. By Lemma 3.3, the largest $L^{V'}$ harmonic dominated by $K_\nu$, say $w$, is $L^{V'}$ moderate and has the $m$-boundary trace $\nu$. Of course

$$p := K_\nu - w$$

is an $L^{V'}$-potential and by the results in [2] (see Section 3 above)

$$w = K_\nu^{V'}.$$
where \( \nu' \) is a positive finite measure on \( \partial \Omega \). By Lemma A.1, \( \nu, \nu' \) are mutually a.c.

By the relative Fatou theorem (for \( L^{V'} \)), since \( v, p \) are \( L^{V'} \) potentials and \( w \) is \( L^{V'} \) harmonic,

\[
(6.5) \quad \frac{v}{w} \to 0, \quad \frac{K_{\nu}/w}{K_{\nu'}} \to 1 \quad L^{V'}\text{-finely } \nu'\text{-a.e.}
\]

Because \( v = \inf(u, K_{\nu}) \), (6.5) implies that

\[
(6.6) \quad \frac{u}{w} \to 0 \quad L^{V'}\text{-finely } \nu'\text{-a.e.}
\]

Further, by (6.5) and (6.6)

\[
(6.7) \quad \frac{u}{K_{\nu}} \to 0 \quad L^{V'}\text{-finely } \nu'\text{-a.e.}
\]

Thus, since the function \( u/K_{\nu} \) satisfies the strong Harnack condition in \( \Omega \),

\[
(6.8) \quad \frac{u}{K_{\nu}} \to 0 \quad \text{n.t. } \nu\text{-a.e.}
\]

where we have also used the fact that \( \nu \) and \( \nu' \) are equivalent.

So \( K_{\nu}/u \to +\infty \) n.t. at \( \nu\)-a.e. point in \( \partial \Omega \). But on the other hand, viewing \( K_{\nu} \) as a non negative \( L^{V} \)-superharmonic function, the \( L^{V} \) relative Fatou-Doob theorem says that \( K_{\nu}/u \) has a finite n.t. limit \( \mu\)-a.e. in \( \partial \Omega \). These two asymptotic behaviors for \( K_{\nu}/u \) imply that \( \mu \perp \nu \). \( \square \)

Recall that by definition \( u = K_{\mu}^{V} \) but that \( \mu \) is not necessarily \( g \)-good.

**Corollary 6.5.** Suppose that the measure \( \nu \in \mathcal{M}_{+}^{g,\Delta}(\partial \Omega) \) is absolutely continuous with respect to \( \mu \). Then there exists a Borel function \( f \) on \( \partial \Omega \) such that \( 0 < f \) \( \nu\)-a.e. and

\[
(6.9) \quad v := S_{f,\nu}^{g} \leq u.
\]

Moreover when \( u \) admits a trace \( \mu' \) we can choose \( f \) such that \( f\nu = \nu \wedge \mu' \).

**Proof.** By the previous Lemma, for every positive measure \( \nu' \leq \nu \), there is a positive solution \( w \) of (1.1) in \( \Omega \) such that \( 0 < w \leq K_{\nu'} \wedge u \). It follows that the least upper bound of the set of good measures \( \lambda \leq \nu \) and such that \( S_{\lambda}^{g} \leq u \) is a \( g \)-good measure in the form \( f\nu \) with \( 0 < f \leq 1 \) \( \nu\)-a.e. \( \square \)

**Lemma 6.6.** If \( \mu \)-almost every \( \zeta \in \partial \Omega \) is \( L^{V} \)-regular, then \( u \) is \( \sigma \)-moderate.

**Proof.** We exhaust \( \mu \) with an increasing sequence \( \{F_{n}\} \) of compact subsets of \( \partial \Omega \) chosen as follows: \( F_{n} \) is the set of all \( \zeta \in \partial \Omega \) such that \( K_{\zeta}^{V} \leq nK_{\zeta} \) in \( \Omega \) and \( \int_{\Omega} V K_{\zeta} \psi \, dx \leq n \). The fact that \( \mu(\partial \Omega \setminus F_{n}) \to 0 \) follows from the assumptions together with Propositions 3.10 and Definition 3.5.
Consider $u_n = \hat{K}_{\mu_n}^V$ the smallest solution of (1.1) larger than $K_{\mu_n}^V$ (the later is a subsolution smaller than $u$). Clearly $u_n \uparrow u$. Moreover by its very construction $K_{\mu_n}^V \leq nK_{\mu}$ and so $u_n \leq nK_{\mu}$. This proves that $u_n$ is $g$-moderate.

**Corollary 6.7.** If $F \subset \partial \Omega$ is compact and $F \subset R(u)$ then $[u]_F$ is $\sigma$-moderate.

*Proof. We may assume that $v = [u]_F > 0$. Consider $V_F = h \circ v$. It follows from $V_F \leq V$–which implies $G^V \leq G^{V_F}$–and Proposition 3.6 (iii) that every $L^V$ regular boundary point is also $L^{V_F}$ regular. Thus $L^{V_F}v = 0$, $v$ and $V_F$ vanish on $\partial \Omega \setminus F$ and every $\zeta \in \partial \Omega$ is $L^{V_F}$ regular. And the result follows from the previous Lemma.*

*Proof of Theorem 6.3.* By Theorem 5.4 (ii) there exists a Borel function $f_0 : \partial \Omega \mapsto (0, 1]$ such that $f_0 \mu_d$ is a $g$-good measure. By Theorem 5.3 there exists a sequence $\{\tau_n\} \subset M^g_\Delta$ such that

\[(6.10) \quad f_0 \mu_d = \sum_{1}^{\infty} \tau_n.\]

By Corollary 6.5 for every $n \in \mathbb{N}$ there exists a function $h_n : \partial \Omega \mapsto (0, 1]$ such that $h_n : \partial \Omega \mapsto (0, 1]$ and

\[v_n := S_{h_n \tau_n}^g \leq u.\]

Further, $\nu_n := \max(h_1 \tau_1, \ldots, h_n \tau_n)$ is $g$-good, $\nu_n \leq \mu_d$ and

\[w_n = S_{\nu_n}^g \leq u.\]

Therefore $\nu$ is $g$-good and

\[w := \lim w_n = S_{\nu}^g \leq u.\]

Finally (6.10) implies that $\mu_d$ is absolutely continuous w.r. to $\nu$. This proves (6.3).

By Corollary 6.7 if $F$ is a compact subset of $R(u)$ then $[u]_F$ is $\sigma$-moderate. Therefore $[u]_F \leq [u^*]_F$. Since $u^* \leq u$ we obtain (6.4).
Appendix A. Some classes of “good” absorption terms
by Alano Ancona

Summary. We first describe in section A.1 a class of absorption terms $g(x,t)$ such that the condition (1.4) from section 1 holds (i.e. every positive solution $u_0$ of (A.1) below is a solution of a Schrödinger equation that behaves well with respect to dilations). Classes of functions $g$ such that the stronger condition (1.3) holds is later described in section A.2. In section A.4 Theorem A.10 gives (for a somewhat more restricted class of $g$) a simpler way to condition (1.3) by deriving it from condition (1.4).

Recall that condition (1.4) implies both (uniform inner) Harnack inequalities for $u_0$ as well as a useful uniform boundary Harnack principle. See remark A.5.

The exposition is made essentially independent of the previous sections.

A.1. We consider in a domain $\Omega$ of $\mathbb{R}^N$, $N \geq 1$, a semi-linear equation

(A.1) $$-\Delta u(x) + g(x, u(x)) = 0$$

where the function $g : \Omega \times \mathbb{R}_+ \to \mathbb{R}_+$ is continuous.

A nonnegative function $u$ in $\Omega$ is a solution of (A.1) if $u$ and $g(., u(.))$ are in $L^1_{loc}(\Omega)$ and $\Delta u = g(., u(.))$ -in the distribution sense-. In particular $u$ is subharmonic -thus locally bounded- and $\Delta u \in L^\infty_{loc}(\Omega)$. So $u$ is continuous and in $W^{2,p}_{loc}(\Omega)$ for all $p<\infty$.

It is once for all assumed that $g$ satisfies the following three assumptions.

(ND) for each $x \in \Omega$, $t \mapsto g(x,t)$ is $\geq 0$ nondecreasing on $[0, +\infty)$,

(H) there is a real $C_1 \geq 1$ such that $g(x', a) \leq C_1 g(x, a)$ whenever $|x' - x| \leq \frac{1}{2}\delta(x)$.

Here $\delta(x) := \text{dist}(x, \mathbb{R}^N \setminus \Omega)$, and (H) is reminiscent of Harnack inequalities. It is also assumed that there are positive constants $C_2$, $c_0$ such that the following holds:

(KOT) If $x \in \Omega$, $a \in \mathbb{R}_+$ are such that $\frac{g(x,a)}{a} > c_0 \delta(x)^{-2}$, then

(A.2) $$\int_2^\infty \frac{dt}{\sqrt{G(x,t)}} \leq C_2 \sqrt{\frac{a}{g(x,a)}},$$

where $G(x,t) = \int_0^t g(x,s) \, ds$. When $g$ is $x$-independent the finiteness -for some $a > 0$- of the l. h. s. of (A.2) is the well-known Keller-Osserman condition (see the comments in the beginning of section 1).
Note that \((KOT)\) forces \(g(x,0) = \lim_{t \to 0^+} g(x,t) = 0\).

**Proposition A.1.** There is a positive constant \(\bar{c}\) such that for every nonnegative solution \(u\) in \(\Omega\) of equation \((A.1)\)

\[
\tag{A.3}
g(x,u(x)) \leq \bar{c} u(x) \delta(x)^{-2}
\]

for all \(x \in \Omega\). The constant \(\bar{c}\) may be chosen depending only on \(N, c_0, C_1\) and \(C_2\).

We first note the following elementary fact.

**Lemma A.2.** Set \(C_3 = 2 + C_2\). Under the above assumptions, we have for \(x \in \Omega\) and \(a \in \mathbb{R}_+^*\) such that \(\frac{g(x,a)}{a} > c_0 \delta(x)^{-2}\),

\[
\tag{A.4}
\int_a^\infty \frac{dt}{\sqrt{G(x,t) - G(x,a)}} \leq C_3 \sqrt{\frac{a}{g(x,a)}}.
\]

**Proof.** By the convexity of \(t \mapsto G(x,t)\) in \(\mathbb{R}_+^*\), \(G(x,t) - G(x,a) \geq g(x,a)(t-a)\). Thus

\[
\int_a^{2a} \frac{dt}{\sqrt{G(x,t) - G(x,a)}} \leq 2 \sqrt{\frac{a}{g(x,a)}}.
\]

On the other hand for \(t \geq 2a\), \(G(x,t) \geq 2G(x,a)\) since \(G(x,t)/t\) is nondecreasing in \(t\). Whence \(G(x,t) - G(x,a) \geq G(x,a)\) and

\[
\int_{2a}^\infty \frac{dt}{\sqrt{G(x,t) - G(x,a)}} \leq \int_{2a}^\infty \frac{dt}{\sqrt{G(x,t)}}
\]

On using \((A.2)\) the result follows. \(\square\)

The next lemma appears in [12] -under some slightly stronger assumptions -. Consider an ordinary differential equation

\[
\tag{*}
y''(r) + \frac{N-1}{r} y'(r) = f(y(r))
\]

with \(f \geq 0\) and nondecreasing in \(\mathbb{R}_+^*\). Set \(F(t) = \int_0^t f(s) ds\).

Recall that a nonnegative subsolution of \((*)\) in \((a,b), 0 \leq a < b < +\infty\), is a nonnegative locally bounded function \(y\) in \((a,b)\) such that the distribution \([-r^{N-1} y'(r)]'\) is a positive measure in \((a,b)\) larger than \(r^{N-1} f(y(r)) dr\). So \(y\) may be viewed as locally Lipschitz function in \((a,b)\).

**Lemma A.3.** (See [12] Lemma 2.7) Suppose that \(y\) is a nondecreasing and nonnegative subsolution of \((*)\) in a neighborhood of \([r_0,r_1]\), \(0 < r_0 < r_1 < \infty\). Suppose also that
\[ F(y(r_0)) > 0. \] Then,
\[ \int_{y(r_0)}^{y(r_1)} \frac{du}{\sqrt{2(F(u) - F(y(r_0)))}} \geq \frac{r_0}{N - 2} \left[ 1 - \frac{r_0^{N-2}}{r_1^{N-2}} \right] \quad \text{(or } \geq r_0 \log \frac{r_1}{r_0} \text{, if } N = 2). \]

To prove the lemma one may easily adapt the argument given in [12] for smooth \( y \). We skip the details.

**Corollary A.4.** If there exists a nonnegative and nondecreasing subsolution \( y \) in \((0, T)\) with \( F(y(0+)) > 0 \), then
\[ \int_{y(0)}^{\infty} \frac{du}{\sqrt{2(F(u) - F(y(0)))}} \geq r_0 \log \frac{r_1}{r_0}, \quad \text{if } N = 2. \]

The corollary follows from the lemma by taking \( r_0 = T/2, r_0 < r_1 < T \), and letting \( r_1 \to T \). (Notice that \( \int_{a}^{+\infty} \frac{du}{\sqrt{F(t) - F(a)}} \) is a decreasing function of \( a \), since \( F \) is convex.)

**Proof of Proposition A.1.** Consider a point \( x_0 \in \Omega \) and the ball \( B = B(x_0, \delta(x_0)/2) \). Then, by \((H)\), \( u \) is a nonnegative subsolution of the equation
\[ \Delta u(x) - \frac{1}{C_1} g(x_0, u(x)) = 0 \]
in \( B \). Letting \( g_0(t) = \frac{1}{C_1} g(x_0, t) \) we may rewrite this equation: \( \Delta u - g_0(u) = 0 \).

For every isometry \( R \) of \( \mathbb{R}^N \) that fixes \( x_0 \), \( u \circ R \) is again a subsolution of \((A.7)\) in \( B \) and the function \( w := \sup \{ u \circ R : R \text{ isometry fixing } x_0 \} \) is also a (continuous) subsolution of \((A.7)\) in \( B \) in the form \( w(x) = \tilde{u}(d(x, x_0)) \) for some continuous \( \tilde{u} : [0, \delta(x_0)/2] \to \mathbb{R}_+ \).

The function \( w \) is a classical subharmonic function in \( B \) and is invariant under every rotation that fixes \( x_0 \). Thus \( \tilde{u} \) is nondecreasing and evidently \( \tilde{u} \) is a subsolution of the ODE : \( y''(r) + \frac{N-1}{r} y'(r) - g_0(y(r)) = 0 \) in \((0, \delta(x_0)/2)\).

Applying Corollary A.4 we see that if \( g_0(u(x_0)) > 0 \) -hence \( u(x_0) > 0 \)- and if say \( N \geq 3, \)
\[ \frac{\delta(x_0)}{N-2} \frac{2^{N-2} - 1}{2^{N-1}} \leq \sqrt{2} \int_{u(x_0)}^{\infty} \frac{du}{\sqrt{G_0(u) - G_0(u(x_0))}} \]
\[ = \sqrt{2C_1} \int_{u(x_0)}^{\infty} \frac{du}{\sqrt{G(x_0, u) - G(x_0, u(x_0))}}. \]
So if moreover \( \frac{g(x_0,u(x_0))}{u(x_0)} > c_0 \delta(x_0)^{-2} \) it follows from (A.4) that
\[
c_1 \delta(x_0) \leq C_3 \sqrt{2C_1 \sqrt{\frac{u(x_0)}{g(x_0,u(x_0))}}} \]
with \( c_1 = \frac{1}{N-2} \left( \frac{2N-1}{2N-2} \right) \). This implies that \( g(x_0,u(x_0)) \leq c \delta(x_0)^{-2} u(x_0) \), \( c = \frac{2C_1(C_3)^2}{c_1^2} \).

The only other possibility is that \( g(x_0,u(x_0)) \leq c_0 u(x_0) \delta(x_0)^{-2} \). Proposition A.1 is proved.

**Remark A.5.** Proposition A.1 says that \( u \) is a solution of a Schrödinger equation
\[
L^V(u) := \Delta u - Vu = 0 \text{ for a potential } V \in L^\infty_{loc}(\Omega), \quad V(x) \leq \bar{c} \delta(x)^{-2}, \quad \bar{c} \text{ depending only on the constants } C_1, C_2, c_0 \text{ and } N. \]
In particular there is a constant \( c \) depending only on \( \bar{c} \) such that \( u(x) \leq c u(x') \) whenever \( |x' - x| \leq \frac{1}{2} \delta(x) \) (Harnack inequalities). Moreover, the Harnack boundary inequalities of [2] (see [4]) are available for \( L^V \).

A.2. We next show that if \( \Omega \) is Lipschitz and if \( g \) fulfills a supplementary assumption then the conclusion in Proposition A.1 can be strengthened: property (1.3) holds, i.e. there is a positive real \( \bar{c} > 0 \) such that for every closed subset \( F \) of \( \partial \Omega \) and every nonnegative solution \( u \) of (A.1) in \( \Omega \) that vanishes on \( \partial \Omega \setminus F \),

\[
(A.9) \quad g(x,u(x)) \leq \frac{\bar{c}}{d(x,F)^2} u(x), \quad x \in \Omega. \]

From now on, \( \Omega \) is assumed to be bounded and Lipschitz. For \( \varepsilon > 0 \), define \( \Omega(\varepsilon) := \{x \in \Omega; \delta(x) < \varepsilon \} \).

As before it is assumed that \( g \) is continuous in \( \Omega \times \mathbb{R}_+ \), \( g(x,0) = 0 \) and that \( g(x,.) \) is nondecreasing for all \( x \in \Omega \); moreover the Harnack type condition \( (H) \) holds. In other words, there is a real \( C_1 \geq 1 \) such that \( g(x',a) \leq C_1 g(x,a) \) whenever \( |x' - x| \leq \frac{1}{2} \delta(x) \).

**Definition A.6.** We say that \( g \) is quasi inwardly increasing (near the boundary) in \( \Omega \) if there are constants \( \varepsilon_0 > 0 \) and \( C \geq 1 \) such that:

\( (QI) \) whenever \( x \in \Omega(\varepsilon_0) \), \( 0 < r < \varepsilon_0 \) and \( a > 0 \), we have \( g(x,a) \leq C g(x',Ca) \) for at least one point \( x' \in \{z \in \mathbb{R}^N; |z-x| = r, \delta(z) \geq \varepsilon_0 r \} \).

When \( (QI) \) holds we also say that \( g \) is \( (C,\varepsilon_0) \) quasi inwardly increasing (near the boundary) in \( \Omega \).

Using property \( (H) \) it is easy to see that -after a change of constant \( C \)- we may choose the point \( x' \) independently of \( a > 0 \).

It should also be noted that property \( (QI) \) is (up to change of constants) a property of \( g \) with respect to the pseudo-hyperbolic geometry in \( \Omega \). To be more specific recall that
for $c > 0$, an arc $γ : [0, ℓ] → Ω$ such that $δ(γ(t)) ≥ cd(γ(t), γ(0))$ is called a $c$-John arc in $Ω$. It can be shown that the validity of (QI) (for some $C$ and $ε_0$) means (up to change of constants) that from every point $x_0 ∈ Ω(ε_0/4)$ starts a $c$-John arc $γ : [0, ℓ] → Ω$ in $Ω$, such that $δ(γ(ℓ)) = ε_0$ and $g(x, a) ≤ Cg(γ(s), Ca)$ for all $a > 0$ and $s ∈ (0, ℓ]$.

**Theorem A.7.** Suppose that for some constants $C_2 ≥ 1$ and $ε_0 > 0$:

\[
(KOT')' \int_{2a}^{∞} \frac{dt}{\sqrt{G(x,t)}} ≤ C_2 \sqrt{\frac{a}{g(x,a)}} \quad \text{whenever} \ a > 0, \ x ∈ Ω \ \text{and} \ g(x,a) ≥ C_2 a,
\]

(QI) $g$ is $(C_2, ε_0)$-quasi inwardly increasing (near the boundary) in $Ω$, where $G$ is as in (A.2). Then property (A.9) holds for some real $τ ≥ 1$ depending only on $Ω$ and $g$.

**Remark.** For a more restricted class of $g$, a different and simpler approach to this result is given in (A.4) For the proof we require the following lemmas.

**Lemma A.8.** Let $y_0 ∈ Ω$ and let $u$ be a positive solution of (A.7) in $Ω$ that vanishes on $∂Ω ∩ B(y_0, 2s)$, $0 < s ≤ s_0$. Then, for $z ∈ ∂B(y_0, s) ∩ Ω$:

\[
u(z) ≥ \frac{1}{4} \sup \{ u(y) ; y ∈ Ω, |y − y_0| = s \} \quad ⇒ \quad d(z, ∂Ω) ≥ η s.
\]

Here $s_0$ and $η$ are (small) positive constants depending only on $Ω$ and $g$. Moreover for all $z ∈ ∂B(y_0, s)$ such that $δ(z) ≥ η s$, we have, for a constant $c' = c'(Ω, g) ≥ 1$

\[
u(z) ≥ \frac{1}{c'} \sup \{ u(y) ; y ∈ Ω, |y − y_0| = s \}.
\]

**Proof.** The point here is that by Proposition (A.1) and Remark (A.5) some uniform Harnack (or boundary Harnack) inequalities are available for $u$.

In particular, by the (inner uniform) Harnack inequalities satisfied by $u$, we may for the proof restrict to the case where $B(y_0, 2s) ∩ ∂Ω ≠ ∅$, i.e. $δ(y_0) < 2s$.

Then, provided $s_0$ is chosen sufficiently small, and because $Ω$ is bounded and Lipschitz, there is a cylinder $T(r, ρ) ∈ \mathbb{R}^N$ -with $0 < r = r(Ω) < ρ = ρ(Ω)$, $ρ/r = κ(Ω)$- and an isometry $\mathcal{J}$ of $\mathbb{R}^N$ such that $ω(r, ρ) := \mathcal{J}(Ω) ∩ T(r, ρ)$ is a standard Lipschitz domain (see Section 4) of height $ρ$ and radius $r$ and $\mathcal{J}(y_0) = (0, τ) ∈ \mathbb{R}^{N−1} × \mathbb{R} ∼ \mathbb{R}^N$ where $0 < τ ≤ 4κs$, $20s < r$. Denote $A = \mathcal{J}^{-1}(0, 10κs)$, $B = \mathcal{J}^{-1}(0, τ + s)$. By the Harnack boundary principle

\[
u(x) ≤ C u(A) s^{N−2} G_A^V(x) ≤ C u(A) s^{N−2} G_A(x) \quad \text{if} \ \mathcal{J}(x) ∈ ω(5s, 5κs)
\]
where $\mathcal{G}^V_A$ is $L^V$ Green’s function for $\mathfrak{I}^{-1}(\omega(10s,10\kappa s))$, $\mathcal{G}_A := \mathcal{G}^0_A$ and $C \geq 1$ is a constant depending only on $\Omega$ and $\bar{c}$. On the other hand by Harnack inequalities, $u(B) \geq cu(A)$ ($c \geq 0$ depending on $C_1$ and $\Omega$). Thus, when $z \in \partial B(y_0,s) \cap \Omega$

$$u(z) \leq C c^{-1}u(B) s^{-2} \mathcal{G}_A(z)$$

But it is well-known that $s^{-2} \mathcal{G}_A(z) \leq c'(d(z,\partial \Omega)/s)^\alpha$ in $\mathfrak{I}^{-1}(\omega(5s,5\kappa s))$ for some positive reals $c$ and $\alpha$ depending only on $N$ and $\rho/r$. The first part of the lemma follows.

The second part is then a simple consequence of the (uniform inner) Harnack inequalities that are satisfied by $L^V$ solutions. □

**Lemma A.9.** Suppose $F \subset \subset \partial \Omega$ is closed and $u = 0$ in $\Omega \setminus F$. Given $\varepsilon_1 > 0$ there is a (large) $t = t(\varepsilon_1, \Omega, \delta) \geq 2$ and a real $\varepsilon_2 > 0$ such that $\varepsilon_2 t \leq \frac{1}{10}$ and

if $y_0 \in \Omega$ and $\delta(y_0) \leq \varepsilon_2 d(y_0, F)$ then $u(y_0) \leq \varepsilon_1 \sup \{u(z); |z - y_0| = t\delta(y_0)\}$.

**Proof.** If $M = \sup \{u(z); |z - y_0| = t\delta(y_0)\}$ the proof above shows, taking now $A = \mathfrak{I}^{-1}(0, \tau + 2t\delta(y_0))$, that

$$u(y) \leq C Mt^{-\alpha}$$

provided that $t\delta(y_0) \leq \frac{1}{10} d(y_0, F)$. The lemma follows. □

**Proof of Theorem A.7.** Let $u$ be a positive solution of (A.1) and let $F$ be a compact subset of $\partial \Omega \setminus F$. To prove the desired estimate we exploit again the ideas of the proof of Proposition A.1. By (A.3) we may assume $F \neq \emptyset$ and $d(x, F)$ small.

Consider a point $x_0 \in \Omega$ and the ball $B = B(x_0, R)$, $R := d(x_0, F) < \varepsilon_0/10$. Extending $u$ by zero outside $\Omega \setminus F$ we may view $u$ as a subharmonic function in $B$ and consider the function

$$w := \sup \{u \circ \mathcal{R}; \mathcal{R} \in \mathfrak{I}_{x_0}\}$$

in $B$, where $\mathfrak{I}_{x_0}$ is the group of isometries of $R^N$ fixing $x_0$. Clearly $w$ is again a continuous subharmonic function in $B$ in the form $w(x) = \tilde{u}(d(x, x_0))$ for some continuous nondecreasing function $\tilde{u} : [0, R) \to \mathbb{R}_+$.

Denote, for $x \in B$, $\mathcal{U}(x) := \{\mathcal{R} \in \mathfrak{I}_{x_0}; u(\mathcal{R}(x)) > \frac{1}{2} w(x)\}$. Note that $w(x) > 0$.

Given $x_1 \in B$, there is an open neighborhood $\omega$ of $x_1$ such that, in $\omega$, $u \circ \mathcal{R} < w$ for $\mathcal{R} \in \mathfrak{I}_{x_0} \setminus \mathcal{U}(x_1)$, and $u \circ \mathcal{R} \geq \frac{1}{4} w$ if $\mathcal{R} \in \mathcal{U}(x_1)$. So $w = \sup \{u \circ \mathcal{R}; \mathcal{R} \in \mathcal{U}(x_1)\}$ in $\omega$.

It follows that in $\omega$, the positive measure $\Delta w$ satisfies in the distribution sense

$$\Delta w(x) \geq \inf \{g(\mathcal{R}(x), u(\mathcal{R}(x)); \mathcal{R} \in \mathcal{U}(x_1)\}$$
\[ \geq \frac{1}{c_1} \inf \{ g(x_0, c_1^{-1}u(\mathcal{R}(x))) ; \mathcal{R} \in \mathcal{U}(x_1) \} \]
\[ \geq \frac{1}{c_1} g(x_0, \frac{1}{4c_1} w(x)) \]

where after the first inequality sign all terms are functions and where \( c_1 \) is a new constant \( \geq 1 \). For the second inequality, observe that for \( \mathcal{R} \in \mathcal{U}_{x_1} \) and \( x \in \omega \), \( g(x_0, C^{-1}u(\mathcal{R}(x))) \leq C g(x', u(\mathcal{R}(x))) \) for some \( x' \in \partial B(x_0, |x - x_0|) \cap \Omega \) such that \( \delta(x') \geq \varepsilon_0|x-x_0| \). So by Lemma A.8 and (H), \( g(x_0, C^{-1}u(\mathcal{R}(x))) \leq c C g(\mathcal{R}(x), u(\mathcal{R}(x))) \) where \( c = c(g, \Omega) \geq 1 \).

Thus, \( \tilde{u} \) is a subsolution of the ODE: \( y''(r) + \frac{N-1}{r} y'(r) - g_0(y(r)) = 0 \) in \((0, R)\) where \( g_0(t) := \frac{1}{4c_1} g(x_0, \frac{1}{4c_1} t) \).

Using Corollary A.4 we get, if \( g_0(u(x_0)) > 0 \) (and say \( N \geq 3 \)), and if \( G_0(t) := \int_0^t g_0(s) \, ds \),

\[ \frac{R}{N-2} \frac{2^{N-2} - 1}{2^{N-1}} \leq \sqrt{2} \int_{u(x_0)}^{\infty} \frac{du}{\sqrt{G_0(u) - G_0(u(x_0))}} \]
\[ = \sqrt{8c_1} \int_{u(x_0)}^{\infty} \frac{du}{\sqrt{G(x_0, u/4c_1) - G(x_0, u(x_0)/4c_1)}} \]
\[ = 4\sqrt{8} (c_1)^{3/2} \int_{u(x_0)/4c_1}^{\infty} \frac{du}{\sqrt{G(x_0, u) - G(x_0, u(x_0)/4c_1)}}. \]

So if \( g(x_0, u(x_0)/4c_1) > 0 \) it follows from (A.4), \((KOT)'\) and the obvious adaptation of Lemma A.2 that

\[ c_4 R \leq \sqrt{\frac{u(x_0)}{g(x_0, u(x_0)/4c_1)}} \]

for some constant \( c_4 > 0 \) depending only on \( \Omega \) and \( g \).

At this point we have established that for every \( x_0 \in \Omega \) and some constant \( \bar{c} = \bar{c}(\Omega, g) \)

\[ g(x_0, u(x_0)/4c_1) \leq \bar{c} u(x_0) d(x, F)^{-2}. \]

To get rid of the constant \( 4c_1 \) in the r.h.s we use Corollary A.4 with \( \varepsilon_1 = (4cCc_1)^{-1} \) where \( c \geq 1 \) is chosen as follows. There is a (fixed) large \( t \) and a small \( \varepsilon_2 > 0 \) (independent of \( x_0 \)) with \( \varepsilon_2 t \leq \frac{1}{10} \) and whenever \( \delta(x_0) \leq \varepsilon_2 d(x_0, F) \) then \( u(x_0) \leq \varepsilon_1 \sup \{ u(x); |x - x_0| = t \delta(x_0) \} \); so \( u(x_0) \leq c'\varepsilon_1 \sup \{ u(x); |x - x_0| = t \delta(x_0), \delta(x) \geq \varepsilon_0 t \delta(x) \} \) (using property (H)). Here \( c' = c'(\Omega, g, \varepsilon_0) \geq 1 \). We choose \( c := c' \).
By the assumption (QT) there exists \(x_1 \in \partial B(x_0, t\delta(x_0))\) with \(\delta(x_1) \geq \varepsilon_0 t \delta(x_0)\) and \(g(x_0, u(x_0)) \leq C g(x_1, Cu(x_0))\). By the above, \(u(x_0) \leq (4C_1)^{-1} u(x_1)\). Hence

\[
g(x_0, u(x_0)) \leq C g(x_1, Cu(x_0)) \leq C g(x_1, \frac{1}{4c_1} u(x_1)) \leq c c' \frac{u(x_1)}{d(x_1, F)^2} \leq c c' \frac{u(x_0)}{d(x_0, F)^2}
\]

where \(c'\) is again a Harnack constant depending only on the chosen \(t, \Omega, \varepsilon_0\) and \(g\).

This proves (A.9) for \(x \in \Omega\) such that \(\delta(x) \leq \varepsilon d(x, F)\) if \(\varepsilon = \varepsilon(\Omega, g) > 0\) is small enough. By (A.1) the result then extends (with another constant) to all \(x \in \Omega\). \(\square\)

A.3. Few Examples. We first describe some examples of functions \(g\) in the form \(g(x, t) = b(x) \bar{g}(t)\) that satisfy the basic assumptions in section 1 (we then say that \(g\) is “good”), in particular \(\bar{g}\) by the results above. We owe to Moshe Marcus the idea of considering examples of this form. See also the reference \[33\].

a) Recall that \(\Omega\) always denote a bounded Lipschitz domain in \(\mathbb{R}^N\). Suppose that \(\bar{g}\) is convex and nondecreasing in \([0, \infty)\) with \(g(0) = 0\) and that \(b \in C_+ (\Omega)\). Suppose furthermore that for some constants \(\varepsilon > 0\) and \(c \geq 1\),

(1) if \(x, x' \in \partial \Omega, |x - x'| \leq \frac{1}{2} d(x, \partial \Omega)\) then \(b(x') \leq c b(x)\),

(2) whenever \(\zeta \in \partial \Omega, 0 < 2r < r' \leq \varepsilon, x \in \partial B(\zeta, r) \cap \Omega\), there exists a point \(x' \in \partial B(\zeta, r')\) such that \(b(x) \leq c b(x')\) and \(d(x', \partial \Omega) \geq \varepsilon r'\).

(3) \(\int_a^\infty \frac{dt}{\sqrt{G(t)}} \leq c \sqrt{\frac{a}{G(a)}}\) for \(a\) large -where \(G(t) := \int_0^t \bar{g}(t)\). Then \(g(x, t)\) is “good” and satisfies the assumptions of Theorem [A.7] Note that by the main results in [2] any positive \(b \in C_+ (\Omega)\) which solves \(\Delta u - V u = 0\) near \(\partial \Omega\), with \(\sup_0 \delta^2 |V| < \infty, V_0 = o(\delta^2)\) as \(\delta \to 0\), and \(b = 0\) on \(\partial \Omega\) is an admissible function \(b\) as well as any power \(\beta^m\) of \(\beta, m > 0\). Clearly more general second order elliptic operators can be considered.

To obtain more examples notice that \(b(x) = \delta(x)\) is an admissible function \(b\), that powers, products, sums of admissible functions \(b\) are still admissible.

Example 1. \(g(x, t) = c \varphi(x)^\alpha \delta(x)^{\beta t}\) where \(\varphi\) is the ground state of the Dirichlet Laplacian in \(\Omega\) and \(\alpha, \beta \geq 0\).

Example 2. It is easily checked that any increasing function \(\bar{g} : \mathbb{R}_+ \to \mathbb{R}_+\) such that \(g(t) \geq c \left(\frac{t}{s}\right)^\alpha g(s)\) for some \(\alpha > 1\), a constant \(c > 0\) and all \(t > s > 1\) is an admissible function \(\bar{g}\).

b) Other examples can be obtained using the following remark: let \(g_1, \ldots, g_n\) be admissible functions \(g\) in \(\Omega \times \mathbb{R}_+\), in the sense that each satisfies the assumptions of
Proposition A.1 (resp. Theorem A.7) Then if $\beta_1, \ldots, \beta_n$ are positive reals, the function $\sum_{j \leq N} \beta_j g_j$ is again an admissible function $g$. The proof is left as an exercise.

c) We finally mention other classes of examples. If $g(x,t) := g \left( t b_1(x) \right) b_2(x)$ where $g : \mathbb{R}_+ \to \mathbb{R}_+$, $b_1, b_2 : \Omega \to \mathbb{R}_+$ are admissible in the sense of a) above, then, provided $g$ satisfies the $\Delta_2$ condition, $g$ satisfies the assumptions of Theorem A.7.

A.4. A variant of Theorem A.7

In this part, we describe another way to obtain estimates in the form (A.9). We show that under condition (QI) and another mild condition on $g$, property (A.3) implies property (A.9) for solutions.

Theorem A.10. Consider in a bounded Lipschitz domain $\Omega$ of $\mathbb{R}^N$ the equation

$$(A.10) \quad \Delta u(x) = g(x, u(x))$$

where $g : \Omega \times \mathbb{R}_+ \to \mathbb{R}_+$ is continuous and such that $t \mapsto h(x, t) = t^{-1} g(x, t)$ is nondecreasing in $\mathbb{R}_+$. Suppose that for some constants $C \geq 1$ and $\varepsilon_0 > 0$

(i) (QI) $g$ is $(C, \varepsilon_0)$-quasi inwardly increasing (near the boundary) in $\Omega$.

(ii) every positive solution $u$ of (A.10) is such that $g(x, u(x)) \leq C (\delta(x))^{-2} u(x)$ in $\Omega$.

Then property (A.9) holds for $\Omega$ and the equation (A.10).

Proof. a) Let $u$ be a positive solution of (A.10) in $\Omega$ and let $F$ be the smallest closed subset of $\partial \Omega$ such that $u = 0$ in $\partial \Omega \setminus F$. By the assumption we have that $u$ is a solution of an equation $\Delta u - Vu = 0$ in $\Omega$ where the potential $V \in L^\infty_{loc}(\Omega)$ is nonnegative and such that

$$V(x) \leq \frac{C}{\delta(x)^2} \text{ for } x \in \Omega.$$

b) Suppose that for some cylinder $T(r, \rho) = \{(x', x_N) \in \mathbb{R}^{N-1} \times \mathbb{R} \cong \mathbb{R}^N; |x'| < r, |x_N| < \rho\}$ with $0 < r < \rho$, the region $\omega(r, \rho) := \Omega \cap T(r, \rho)$ is a standard Lipschitz domain of radius $r$ and height $\rho$. Suppose also that $u = 0$ on $\partial \Omega \cap T(r, \rho)$ (i.e. $F \cap T(r, \rho) = \emptyset$) and denote $A' = (0, \frac{3}{4} \rho)$.

Then by the Harnack boundary property,

$$u(x) \leq cr^{N-2} u(A') \mathbb{G}^V_{A'}(x) \leq cr^{N-2} u(A') \mathbb{G}_{A'}(x)$$

for $x \in \Omega \cap T(r/2, \rho/2)$ and some $c = c(C, \frac{\varepsilon}{\rho}, N) \geq 1$. Here $\mathbb{G}^V$ is the $L^V$ Green’s function in $\omega := T(r, \rho) \cap \Omega$, $\mathbb{G} = \mathbb{G}^0$ and $A' = (0, \frac{3}{4} \rho)$.
Thus on using the Harnack inequalities for $L^V$ we get for $x' \in \partial B(x, r/2)$, $\delta(x') \geq \varepsilon_0 r/2$,

$$u(x) \leq c' r^{-N-2} u(x') \mathcal{G}_{A'}(x)$$

with a constant $c'$ depending only on $C$, $N$ and $\rho/r$.

Since $r^{-N-2} \mathcal{G}_{A'}(x) \leq c_1 (d(x, \partial \Omega)/r)^{\alpha}$ when $x \in \Omega \cap T(r/2, \rho/2)$ for some constants $c_1 \geq 1$ and $\alpha \in (0, 1)$, it follows that

$$u(x) \leq \frac{1}{C} u(x')$$

if in addition to the previous conditions $x \in T(\varepsilon_1 r, \varepsilon_1 \rho) \cap \Omega$ where the constant $\varepsilon_1 > 0$ is chosen small enough depending only on $c$, $c'$, $N$ and $\frac{\rho}{r}$.

c) The domain $\Omega$ being bounded and Lipschitz, there is a small $\kappa > 0$ such that for every point $x \in \Omega$ for which $\delta(x) \leq \kappa d(x, F)$ the following holds: $d(x, \partial \Omega) \leq \frac{1}{8} \varepsilon_1 \varepsilon_0$ and there is an affine isometry $\mathcal{R}$ of $\mathbb{R}^N$ and numbers $r, \rho > 0$ such that $\mathcal{R}^{-1}(\Omega) \cap T(r, \rho)$ is a standard Lipschitz domain of radius $r$ and height $\rho$, $F \cap \mathcal{R}(T(r, \rho)) = \emptyset$, the point $x$ being inside $\mathcal{R}(T(\varepsilon_1 r, \varepsilon_1 \rho))$. Moreover the ratio $\frac{\rho}{r}$ depends only on $\Omega$ and one may further choose $r \geq c_2 d(x, F)$ for some $c_2 > 0$.

Thus if $x' \in \Omega \cap \partial B(x, r/2)$ and $\delta(x') \geq \varepsilon_0 r/2$, we have by the assumptions $g(x, a) \leq Cg(x', Ca)$ for all $a > 0$ -or $h(x, a) \leq C^2 h(x', Ca)$- and

$$V(x) = h(x, u(x)) \leq h(x, \frac{1}{C} u(x')) \leq C^2 h(x', u(x')) \leq \frac{2 C^3}{r^2} \leq \frac{2 C^3 (c_2)^{-2}}{d(x, F)^2}.$$  

Here we have used the fact that $h(x', u(x')) \leq C/(\delta(x'))^2 \leq 2 C/r^2$.

On the other hand if $x \in \Omega$ is such that $\delta(x) \geq \kappa d(x, F)$ then

$$V(x) \leq \frac{C}{\delta(x)^2} \leq \frac{C \kappa^{-2}}{d(x, F)^2}$$

and the proposition follows. □
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