Spectral signatures of recursive magnetic field reconnection

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ABSTRACT

We use 2.5D Magnetohydrodynamic simulations to investigate the spectral signatures of the nonlinear disruption of a tearing unstable current sheet via the generation of multiple secondary current sheets and magnetic islands. During the nonlinear phase of tearing mode evolution, there develops a regime in which the magnetic energy density shows a spectrum with a power-law close to $B(k)^2 \sim k^{-0.8}$. Such an energy spectrum is found in correspondence of the neutral line, within the diffusion region of the primary current sheet, where energy is conveyed towards smaller scales via a “recursive” process of fast tearing-type instabilities. Far from the neutral line we find that magnetic energy spectra evolve towards slopes compatible with the “standard” Kolmogorov spectrum. Starting from a self-similar description of the nonlinear stage at the neutral line, we provide a model that predicts a reconnecting magnetic field energy spectrum scaling as $k^{-4/5}$, in good agreement with numerical results. An extension of the predicted power-law to generic current sheet profiles is also given and possible implications for turbulence phenomenology are discussed. These results provide a step forward to understand the “recursive” generation of magnetic islands (plasmoids), which has been proposed as a possible explanation for the energy release during flares, but which, more in general, can have an impact on the subsequent turbulent evolution of unstable sheets that naturally form in the high-Lundquist number and collisionless plasmas found in most of the astrophysical environments.

Key words: Magnetic reconnection – (Magnetohydrodynamics) MHD – Plasmas – Turbulence

1 INTRODUCTION

Understanding catastrophic processes of energy storage and release via magnetic field reconnection requires, first and foremost, an understanding of under which conditions current sheets become unstable and then evolve nonlinearly. In collisionless or weakly collisional plasmas like those found in many astrophysical environments, thin current sheets spontaneously arise as the result of the plasma dynamics. For example, current sheets are thought to necessarily form in the solar corona as a result of the photospheric displacement of closed coronal loop field lines (Parker 1972; Rappazzo and Parker 2013). Locally, the dissipation or instability of such current sheets is thought to produce heating and particle acceleration, while at larger scales, where topological invariants (helicity conservation) may play a role, current sheet evolution may be driven self-consistently in the eruption process leading to flares (van den Oord 1993). Similarly, in the magnetosphere, a current sheet forms in the tail as a consequence of the external solar wind driving (Otto et al. 2015). Numerical simulations of these processes have only begun to reach the Lundquist and Reynolds numbers sufficient to study the formation of thin sheets and the triggering of their instabilities.

It is now well established that the behavior of current sheets changes dramatically with increasing macroscopic Lundquist number $S = L v_a / \eta$, where $L$ is the (half) length of the sheet, $v_a$ the upstream Alfvén speed and $\eta$ the magnetic diffusivity. For low values of the Lundquist number, $S < S_c$, $S_c$ being a (non-universal) critical threshold around $S_c \approx 10^3$ (Ni et al. 2010; Shi et al. 2018), stable Sweet-Parker-type current sheets can form where laminar, slow reconnection is sustained by plasma flows into and outwards from the sheet (Sweet 1958; Parker 1957). For Lundquist numbers larger than $S_c$, however, Sweet-Parker sheets, whose aspect ratio scales as $L/a \sim S^{1/2}$, are so thin that they become unstable to extremely fast tearing (Biskamp 1986), with normalized growth rate $\gamma \sim S^{1/4}$ (Tajima and Shibata 2002; Loureiro et al. 2007; Bhatcharjee et al. 2009). For example, an active region in the solar corona has a typical spatial scale $L$ of about $L \approx 10^9$ cm, a magnetic field on the order of $B \approx 50$ G, density $\rho \approx 10^9$ cm$^{-3}$ and a temperature $T \approx 10^6$ K that give a macroscopic Lundquist number $S \approx 10^{13}$. At such large values of $S$, a Sweet-Parker sheet would be unstable on a too short of a timescale (less than a second). The opposite happens instead for a current sheet which is macroscopic, which would be unstable on indefinitely long timescale.

On the other hand, a fast tearing instability can grow on an ideal, i.e., $S$-independent, timescale ($\gamma \sim O(1)$) within current sheets whose aspect ratio $L/a \sim S^{1/3}$—much smaller than the Sweet-Parker one (Pucci and Velli 2014). The fundamental impli-
cation of the existence of such an unstable mode, dubbed “ideal” tearing, is that the predicted critical aspect ratio provides an upper limit for current sheets that can naturally form before they disrupt due to the onset of a fast reconnecting mode. In other words, a current sheet, during its dynamical formation, remains quasi-stable as long as its thickness is larger than critical (in which case the growth rate is infinitely small if \( S \gg 1 \)), but it becomes unstable with a finite growth rate while approaching the critical thickness from above. Such a scenario of the disruption of a forming current sheet, as well as the scaling with \( S \) of the critical aspect ratio, has been confirmed by Magnetohydrodynamic (MHD) numerical simulations of forming current sheets (Tenerani et al. 2015b; Huang et al. 2017). Although “ideal” tearing was first discussed in the framework of resistive MHD, it was later shown that the specific scaling of \( L/a \) with plasma parameters may change depending whether viscous or kinetic effects are included, and also depending on the current sheet profile chosen (Tenerani et al. 2015a; Del Sarto et al. 2016; Pucci et al. 2017, 2018).

The idea of a threshold aspect ratio at which magnetic reconnection is triggered on a timescale compatible with the ideal dynamics—i.e., on a timescale generally independent of the non-ideal terms—has been adopted in recent work aimed at incorporating the effect of the tearing instability, within anisotropic eddies, on the turbulent energy cascade (Loureiro and Boldyrev 2017; Mallet et al. 2017). Here we discuss a related problem, namely that of how magnetic reconnection mediates the formation of small scales, by considering the nonlinear evolution of a (two-dimensional) single current sheet. Instead of a Sweet-Parker current sheet, which has often been considered as the archetype of current sheets, we consider a current sheet at the critical thickness defined above, therefore unstable to the “ideal” tearing mode. We investigate via MHD simulations how the nonlinear, recursive onset of tearing-type instabilities within ever smaller current sheets provides a channel to convey energy across scales all the way down to the dissipative ones.

Numerical studies of magnetic reconnection in large aspect ratio sheets, implementing different initial conditions and adopting different plasma descriptions, have shown that X-points collapse and evolve into secondary current sheets during the nonlinear stage of the instability (e.g., Malara et al. (1992); Jemella et al. (2003); Daughton et al. (2006); Lu et al. (2019)). Secondary current sheets may themselves undergo tearing-type instabilities that lead to the formation of chains of secondary magnetic islands (or plasmoids). Such a process repeats itself progressively over smaller and smaller spatial and temporal scales via a recursive mechanism of X-point generation, collapse and disruption (Daughton et al. 2009; Uzdensky et al. 2010), that can be described by a geometrical progression (Tenerani et al. 2015b; Singh et al. 2019). The resulting hierarchy of magnetic islands and current sheets is consistent with a self-similar sequence of fast tearing type instabilities when starting from an “ideally” unstable sheet (Tenerani et al. 2015b), in a way reminiscent of the “fractal” reconnection model first introduced by Shibata and Tanuma (2001) to explain energy release in flares.

Generally, self-similar processes of energy transfer across scales are associated with energy spectra that follow well defined power-laws, and we therefore expect that the “recursive” reconnection described above displays a developed energy spectrum as well. Recent work based on particle-in-cell simulations of a collisionless plasma has reported the development of developed energy spectra within the diffusion region of an unstable current sheet, and it has been shown that the turbulent state generated by secondary magnetic islands can enhance particle energization (Lu et al. 2019).

Modeling and understanding the mechanisms leading to such an energy cascade inside reconnecting sheets is therefore important for understanding not only flare dynamics (for which the concept of fractal reconnection was in fact introduced first), but also and more in general for understanding both particle energization and scattering in reconnection events and for its implications in turbulence, where current sheets are an intrinsic and inexorable part of the turbulent cascade.

Here we focus on the nonlinear stage of the tearing mode, and show that a regime in which magnetic energy density follows a power-law close to \( B(k)^2 \sim k^{-0.8} \) exists within the diffusion region, at the neutral line. We also find that the energy cascade is anisotropic, approximately satisfying \( k_\perp \sim k^{3/5} \), where \( k_\perp \) represents the inverse of the length scale across the sheet. The energy spectrum changes across the current sheet and farther from the neutral line a Kolmogorov-like spectrum is approached, \( B(k)^2 \sim k^{-5/3} \). By analyzing the nonlinear stage of a tearing unstable current sheet and starting from the recursive model discussed in Tenerani et al. (2015b, 2016), we propose a quantitative model that explains the observed magnetic energy spectrum. Our result predicts a spectrum power-law and anisotropy in \( k \)-space which are independent of the current sheet thickness and that weakly depend on the current sheet configuration. We finally discuss possible implications for phenomenological theories of turbulence by considering the effect of “recursive” tearing on the evolution of intermittent coherent structures.

2 NUMERICAL SET UP AND INITIAL CONDITIONS

We employ a 2.5D (two spatial coordinates and three dimensional fields) compressible MHD code, in which we take an adiabatic closure and a viscous stress tensor. The code is periodic in the \( x \) direction whereas non reflecting boundary conditions are imposed in the inhomogeneous \( y \) direction with the method of projected characteristics (Thompson 1987; Landi et al. 2005). Fast Fourier Transform is used along the \( x \) direction and a sixth order compact finite difference scheme is employed along \( y \). A sixth order compact filter is adopted in both \( x \) and \( y \) coordinate to damp energy accu-
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Figure 2. Contour plots of the out-of-plane current density $J_z$ (color coded) and of the flux function $\psi$ (black lines) from the early nonlinear stage to $t = 25$. Snapshots are shown, form left to right, top to bottom, at time $t = 10, 12, 14, 16, 18, 20, 22, 25$.

mulation at the grid scale. An explicit fourth order Runge-Kutta method is used for time integration.

As initial condition we choose a plasma equilibrium with homogeneous mass density $\rho_0$ and pressure $p_0$. The magnetic field is force-free and it is given by a (force-free) Harris sheet profile with fixed (half) thickness $a$ plus an out-of-plane magnetic field that ensures magnetic pressure balance:

$$ B_0 = B_0 \tanh \left( \frac{y}{a} \right) \hat{x} + B_0 \text{sech} \left( \frac{y}{a} \right) \hat{z}. $$

(1)

We normalize the magnetic field and velocity to $B_0$ and to the Alfvén speed $v_A = B_0 / \sqrt{\pi \rho_0}$, respectively, lengths to the half-length $L$ of the sheet, time to the Alfvén time $\tau_A = L / v_A$, density
and pressure to the background density \( \rho_0 \) and to magnetic pressure \( B_0^2/4\pi \), respectively. As usual, it is useful to introduce also the flux function \( \psi \) such that the in-plane magnetic field is \( \mathbf{B}_\perp = \nabla \times \psi \mathbf{\hat{z}} \). In this simulation the magnetic diffusivity is set to \( \eta = 10^{-6} \), so that the macroscopic Lundquist number defined with the half-length of the current sheet is \( S = L\eta/a \equiv 10^6 \), the Prandtl number \( P \) is set to one, the equilibrium pressure \( p_0 = 0.8 \), and we choose a current sheet aspect ratio \( L/a = 51/3 = 100 \). The simulation box has sides \( L_x \times L_y = (2 \times 0.4) L \) with 4096\(^2 \) mesh points. We seed the instability with a random noise of magnetic fluctuations with rms amplitude \( \delta b \approx 0.01 \) and wave numbers \( m \) in the periodic \( x \)-direction in the interval \( 1 \leq m \leq 1024 \), and localized along \( y \) within the magnetic field shear region.

3 NUMERICAL RESULTS

3.1 Overview of the linear and nonlinear stages

In Fig. 1 we show for reference the temporal evolution of some unstable Fourier modes of \( \psi \) taken along the neutral line \( y = 0 \). During the initial phase of the simulation, from \( t = 0 \) until about \( t \approx 9 \), we see the exponential growth of several modes close to the most unstable one \( m = 3 \), corresponding to wave vector \( k = m2\pi/L_x \approx 10 \). The inferred growth rate is about \( \gamma = 0.45 \) (dashed line in Fig. 1), slightly smaller than the one expected theoretically with the present parameters (\( \gamma = 0.47 \)) as compressibility tends to decrease reconnection rates. At the end of the linear stage, towards \( t = 9 \), a superposition of modes from \( m = 4 \) to \( m = 6 \) has grown. At that time the nonlinear stage begins, corresponding to a faster growth of low wave number modes (essentially \( m = 1 \)) and to the emergence of two intense currents. In general the nonlinear stage is characterized by a competition and interplay between coalescence of magnetic islands (inverse cascade) and X-point collapse and disruption into smaller scale secondary sheets in correspondence of the most intense currents (direct cascade). We give an overview of the overall evolution of the current sheet from the early nonlinear stage until saturation of the inverse cascade in Fig. 2.

Fig. 2 displays a set of contour plots of the out-of-plane current density \( J_z \) (color coded) and of \( \psi \) (black lines). During the early nonlinear stage (between \( t \approx 9 \) and \( t \approx 14 \)), merging of magnetic islands and X-point collapse lead to the formation of two islands separated by two secondary current sheets near \( x = 1 \) and \( x = 1.5 \), which both become unstable to tearing at about \( t = 14 \) (see Fig. 2, first and second rows). Subsequently, magnetic island merging leads to roughly one elongated magnetic island with a single, highly inhomogeneous current sheet (thicker on the left end side than on the right end side), within which a sequence of tearing type instabilities that generate smaller and smaller structures (Fig. 2, second last and last columns) is triggered in a recursive fashion as reported in previous work (e.g., Daughton et al. (2009); Tenerani et al. (2015b)). As can be seen, the overall nonlinear dynamics leads to a final state characterized by one big magnetic island, when the mode \( m = 1 \) has reached a steady amplitude level (cfr. also Fig. 1), and an elongated current sheet with embedded small scale secondary islands and sheets.

3.2 Current density evolution

The different stages in the overall evolution of the current sheet described in the previous section can be recognized also in Fig. 3, left panel, where we show \( < J_z^2 > \) as a function of time, the brackets denoting spatial average: the blue color corresponds to the average over the simulation box (and multiplied by a factor of 10 to enhance visibility) and along the neutral line (red color). Bottom: blow out of the unstable secondary current sheet at \( t = 17.6 \). Upper panel: contour plot of \( J_z \) (color coded) and of \( \psi \) (black lines). Middle panel: cut of \( J_z \) as a function of \( x \) at the neutral line (\( y = 0 \)). Bottom panel: cut of \( J_z \) across the current sheet at \( x = 1.0 \).

Figure 3. Top: time evolution of the normalized average current density \( < J_z^2 > \) over the whole simulation box (blue line, multiplied by a factor of 10 to enhance visibility) and along the neutral line (red color). Bottom: blow out of the unstable secondary current sheet at \( t = 17.6 \). Upper panel: contour plot of \( J_z \) (color coded) and of \( \psi \) (black lines). Middle panel: cut of \( J_z \) as a function of \( x \) at the neutral line (\( y = 0 \)). Bottom panel: cut of \( J_z \) across the current sheet at \( x = 1.0 \).
of new, smaller and more intense current sheets (increase phase of the spike), and their relaxation via tearing instability (decreasing phase of the spike). The profile of $< J_x^2 >$ at $y = 0$ has a maximum at $t = 17.6$. For reference, we show in Fig. 3, bottom panel, a blow out of the secondary current sheet at $t = 17.6$ that displays the contours of $J_x$ and $\psi$ (upper plot), a cut of the current density along the sheet, $J_x(x, y = 0)$ (middle plot) and a cut across the sheet at $x = 1$, $J_x(x = 1, y)$ (lower plot). We remark that the small scale, large amplitude oscillations seen in the middle right panel of Fig. 3 are well resolved and do not correspond to the well known Gibbs numerical effect arising when structures are poorly resolved.

As can be seen from Fig. 3, bottom panels, and in general by inspection of Fig. 2, the most intense currents are localized at the neutral line where $J_x(x, 0)$ evolves into intense filaments with local values reaching up to 50 times the initial current density. The profile of $< J_x^2 >$ averaged over the entire box (blue color) displays the same spikes although the average current has a longer rising phase, as it increases until about $t = 21$. This is due to the fact that current density spreads farther from the neutral line due to the growing size of magnetic islands during merging, and in fact $< J_x^2 >$ begins to decrease when the mode $m = 1$ has reached a steady value and the inverse cascade has saturated. At the very end of our simulation the current starts to increase again, and we ascribe this effect not to reconnection but to the development of sharp gradients at the edges of the main magnetic island, clearly visible in the vicinity of $x = 1.75$ in the very last panel of Fig. 2.

### 3.3 Energy spectra

We consider $t = 16$ the time at which “recursive” reconnection begins to be fully at play. In Fig. 4 we show the amplitude of $\psi$ in Fourier space at three different stages of the evolution. The spectrum is taken along the $x$-direction at the center of the sheet ($y = 0$) where magnetic islands are generated, and we have averaged $|\psi(k)|$ over the time interval indicated in the plot legend. The blue and grey curves of $|\psi(k)|$ in Fig. 4 show that a steady-state spectrum is formed during the “recursive” reconnection: the slope of the spectrum appears to remain only slightly flatter than a simple power law $\psi(k) \sim k^{-7/5}$ (black dashed line) which is approached at wave vectors $k \geq 15$ (close to the dominant mode at the end of the linear stage) until about $k \approx 300$, after which the spectrum steepens. Since the slope is sensitive to the lower and upper bound of the wave vector range considered, we performed several numerical fits around the range $\Delta k = (15, 300) - (10, 400)$, and inferred a slope $\alpha_k = -1.39 \pm 0.03$.

In Fig. 5 and Fig. 6 we show the magnetic and kinetic energy density spectra along the neutral line, at $y = 0$. Energy spectra are displayed in solid black and colored lines and, for reference, we also show some spectral slopes in dashed and dot-dashed lines, to be discussed later in section 4.1. As shown in Fig. 5, we find that $|B_x(k)|^2 \sim k^{-1.24 \pm 0.06}$ (estimated according to the same criterium adopted for $\psi(k)$), a bit flatter than the shown scaling $k^{-1.6}$ (upper panel) while the reconnecting component of the magnetic field

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**Figure 4.** Amplitude of $|\psi(k)|$ at $y = 0$ during three different stages, averaged over the time intervals indicated in the legend. The orange color corresponds to the linear stage, the magenta color to the early nonlinear stage, the blue and grey color correspond to the time interval with “recursive” reconnection and multiple tearing type instabilities.

**Figure 5.** Magnetic energy density spectra at the neutral line $y = 0$, obtained after averaging spectra at times $t = 16 - 22$. Upper panel: $|B_x(k)|^2$. Middle panel: $|B_y(k)|^2$ (the reconnecting magnetic field). Bottom panel: total $|B(k)|^2$ (blue) and in-plane $|B(k)_y|^2$ (magenta) magnetic energy density spectra.
4 DISCUSSION

4.1 Transition to turbulence in a single current sheet

We adopt here a heuristic method to derive a model that predicts specific scaling laws of magnetic energy spectra, following the idea that magnetic reconnection provides a channel for reaching small scales via a scale-invariant (recursive) hierarchy of tearing modes.

To this aim, we consider a simplified model by assuming an inviscid, incompressible plasma and by neglecting the effect of flows on the stability of current sheets.

On the basis of similarity and rescaling of quantities, it is convenient now to label with an $n$ the typical length scales that describe the current sheet generated at step $n$ of the hierarchy, with $n \geq 1$: the half length $L_n$ and thickness $a_n$, the inner singular layer half thickness $\delta_n$, and the wave vector of the most unstable mode $k_n$. The local Lundquist number is defined accordingly as $S_n = L_n a_n / \eta = (L_n / L) S$, where $L$ and $S$ refer to the primary current sheet. We therefore have assumed that as long as the recursive process of reconnection takes place, a given current sheet at step $n$ has a sufficiently large aspect ratio so as to allow for the fastest mode to develop. Also, for the sake of simplicity, we assume that the upstream Alfvén speed does not change significantly. In this regard we have verified that the upstream magnetic field of the secondary sheets is about 80% of $B_0$ and we therefore neglect embedding...
effects (Cassak and Drake 2009; Del Sarto and Ottaviani 2017) by normalizing speeds to \( v_a \).

We recall briefly the properties of the most unstable mode in thin current sheets, and in particular the scalings with normalizing speeds to effects (e.g., Biskamp 2000; Del Sarto and Ottaviani 2009).

\[
\frac{\psi_n}{B_0L_n} \sim 0.25 S_n^{1/2} \left( \frac{a_n}{L_n} \right)^{1/2}.
\]

Equation (4) can be used to find the relation between \( \psi_n \) and the length scale \( k_n \) at which magnetic energy is conveyed. We now consider the more general case in which we assume that a tearing instability at step \( n \) is triggered when a generic aspect ratio \( a_n/L_n \sim S_n^{-\alpha} \) is reached (for us it should be, in principle, \( \alpha = 1/3 \)). With this assumption we obtain

\[
\psi_n \sim \left( \frac{L_n}{a_n} \right)^{1/2} S^{-1/2} \left( \frac{a_n}{L_n} \right)^{-1/2}.
\]

Since the fastest growing mode within \( L_n \) is \( k_nL_n \sim S^{-1/4} (a_n/L_n)^{-5/4} \), we can write

\[
\frac{L_n}{L} \sim (k_nL)^{\frac{1}{2}} (a_n/L)^{\frac{1}{2}} S^{\frac{1}{2}} \left( \frac{a_n}{L_n} \right)^{-1/2}.
\]

At this point we can use eq. (6) into eq. (5) to express the nonlinear amplitude of \( \psi_n \) in terms of \( k_n \). This last step eliminates the parametric dependence on \( \alpha \). We finally go to the continuum limit, as we are interested in the spectral density, obtaining in this way

\[
\psi(k) \sim S^{-3/5} (kL)^{-7/5}.
\]

It is interesting to note that the reasoning followed to find a scaling for \( \psi(k) \) with \( k \) leads to a universal value of the spectral index, independent of the thickness of the current sheets, i.e., our result does not depend on \( \alpha \). However, simulations with different initial thicknesses, and at larger values of \( S \), are necessary to verify whether and in which way such spectral signatures vary. The slightly flatter slope observed in Fig. 4 might be due to effects that have been neglected in our simple model, including the fact that “recursive” reconnection does not take place homogeneously in space at the neutral line and coalescence of the smaller magnetic islands with the larger ones, which could contribute to a redistribution of energy even at \( k > 15 \) (approximately the dominant mode at the end of the linear stage). However, the fact that the spectrum is very close to a power law with exponent \(-7/5\) suggests that the recursive process...
provides a good description of the nonlinear interactions leading to the direct energy cascade. The recursive reconnection is expected to develop until the local Lundquist number reaches the critical value \( S_c \), and current sheets are stabilized.

The spectrum of the reconnecting magnetic field component can be derived in a straightforward way, because \( B_0(k) = k \phi(k) \), which leads directly to \( B_0(k)^2 \sim k^{-4/5} \), in good agreement with our numerical results. In order to estimate the spectrum of \( B_0(k) \) instead, we have to find a relation between spatial scales along \( (k^{-1}) \) and across \( (\delta) \) the sheet. To this aim we combine eq. (2)–(3) with eq. (6), obtaining \( k_n a_n \sim S^{-2/5}(k_n L)^{2/5} \), that yields \( \delta/L \sim S^{-2/5}(k L)^{-3/5} \) or also \( k L \sim S^{2/5}(k L)^{3/5} \), not far from what is shown in Fig. 9. We finally employ the condition \( \nabla \cdot \mathbf{B} = 0 \), \( k B_0(k) \sim B_0(k)/\delta \), to get \( B_0(k)^2 \sim k^{-8/5} \), a bit steeper than what is observed in the simulation.

We conclude this section by noticing that there might be circumstances in which current sheets may not be well described by a Harris-type profile, such as current sheets generated via a turbulent cascade. It is therefore of interest to see how the power-laws change by assuming a generic current density profile. Dependence on the type of profile enters via the \( \Delta^\prime \) parameter (Farth et al. 1963), which can be taken as scaling with a generic negative power \( p \) of the wave vector, \( \Delta^\prime \sim (k a)^{-p} \), where \( p = 1 \) corresponds to the Harris sheet while for instance \( p = 2 \) corresponds to a magnetic field that goes to zero at the boundaries (Del Sarto et al. 2016; Pucci et al. 2018).

For a generic current density profile we can therefore generalize our scaling laws as follows (see also Singh et al. (2019)):

\[
k_n a_n \sim S_n \frac{1}{(\ell n)^{\frac{1}{1+p}}}
\]

(8)

\[
\frac{\delta_n}{a_n} \sim S_n \frac{1}{(\ell n)^{\frac{1}{1+p}}}
\]

(9)

By repeating the same exercise as for the Harris sheet, but with these new scalings, we obtain a \( \psi(k) \) and anisotropy in \( k \)-space which are again independent of the thickness, but that are now parameterized in \( p \),

\[
\psi(k) \sim S_n \frac{(1+2p)}{\ell n} (k L)^{\frac{1}{1+p}} k_L \sim S_n \frac{(1+2p)}{\ell n} (k L)^{\frac{1}{1+p}}.
\]

(10)

In summary, if \( p \) could take any value in the interval \( 1 \leq p < \infty \), we would find that the reconnecting magnetic field and the anisotropy follow power-law scalings bounded between two rather close values,

\[
B_0(k)^2 \sim (k L)^{-4/5}, \quad k L \sim (k L)^{3}, \quad p = 1,
\]

(11)

\[
B_0(k)^2 \sim (k L)^{-2/3}, \quad k L \sim (k L)^{5/2}, \quad p \rightarrow \infty.
\]

(12)

### 4.2 Implications for turbulence phenomenology

Because the recursive collapse leads to reconnection occurring on time-scales that do not depend on Lundquist number, it is natural to imagine that the “recursive” reconnection discussed in section 4.1 should impact the energy cascade in evolving MHD turbulence. A number of recent papers have developed phenomenologies addressing this aspect (Lourey and Boldyrev 2017; Boldyrev and Lourey 2017; Mallet et al. 2017). Although these works ultimately lead to different predictions for the turbulent cascade, all of them start from an anisotropic MHD turbulence framework and essentially look for the scale at which the elongated eddies, that they associate with current sheets, become unstable to tearing with growth rates equal to those of the nonlinear cascade rate at the same scale. The energy cascade is then modified and sub-inertial spectral power laws has been derived. Here we consider instead how the onset of recursive tearing within current sheets that form between the eddies may modify the turbulent cascade.

It has been observed in some MHD and kinetic simulations that a power law for the spectral energy density forms when magnetic islands are seen to grow in correspondence of thin current sheets (e.g., Dong et al. (2018); Cerri et al. (2017); Franci et al. (2017)). Such findings support the general idea that the onset of reconnection in an evolving turbulent cascade has an impact on the cross-scale energy transfer. In the context of kinetic simulations, Landi et al. (2019) point out that traditional wave and intermittency based theories fail to explain the observed (steadier) spectra at sub-ion scales, and suggest that the onset of reconnection might be the cause.

The problem of how the evolution of intermittent structures influences the energy cascade is a complex one and far from been understood completely. For example, current sheets might be thought of either as elongated turbulent eddies or as intermittent, coherent structures with depleted nonlinearities forming at boundaries between such eddies. In that case, the phenomenology would require an explicit consideration of eddy filling factors. Here we suggest a possible phenomenological model that describes the feedback of reconnecting intermittent structures on the MHD turbulent cascade, inspired by the intuitive \( \beta \)-model put forward by Frisch et al. (1978).

The idea underlying the \( \beta \)-model is that while the time scale for nonlinear interactions remains the eddy turnover time (e.g., \( \tau_{nl} \sim \ell/B_0 \)), small eddies become more and more sparse ("less filling" Frisch et al. (1978)) and will occupy smaller and smaller volumes. A way to include this effect statistically is to assume that such structures occupy a fraction \( \beta_\ell \) the filling factor of the overall volume, so that the energy density at scale \( \ell \) will be given by \( E_{\ell} \sim (B_0^2)^2 \beta_\ell \), where \( \beta_\ell \leq 1 \). Since the average energy density transfer rate \( \varepsilon \) has to be maintained steady and constant, this typically leads to a steeper spectrum than what predicted traditionally. The usual procedure is to find the scaling of \( (B_0^2) \) via the condition \( \varepsilon \sim (\tau_{nl}\ell) \sim (B_0^2)^2 \beta_\ell/\tau_{nl}\ell \rightarrow (B_0^2)^2 \sim (\ell^{2/3}B_0)^{-2/3} \), and then go to the continuum to get the spectral density, the whole point being to estimate the filling factor at scale \( \ell \).

Consider now a turbulent state that generates elongated 2D current sheets in a plane perpendicular to the local mean magnetic field. Assume that turbulence is magnetically dominated, i.e. neglect the effect of flows on the tearing onset (in reality current sheets are often associated with vorticity sheets so that the flows structure may play a role). Beyond a certain scale \( \ell_c \) (that we leave undetermined at this level of discussion) the turbulent cascade proceeds inside the localized sheets. In this scenario, the filling volume correction applies when "recursive" reconnection develops inside such sheets, and we can estimate \( \beta_\ell \) with the volume of magnetic islands times the current sheet number \( N \). Thus \( \beta_\ell \sim (\ell \delta L) N \), where \( \ell \) and \( \delta L \) are the magnetic island length scales along and across the local 2D sheet, and they correspond to the inverse of our \( k \) and \( \delta \). Now, the scalings given in their general form in eq. (10) are defined with \( L \) being the length scale of the unstable sheet (or \( \ell \) with the present notation), which in this case is not necessarily the macroscopic length, that we label instead \( L_0 \). We therefore need to renormalize scales and Lundquist number to their macroscopic values, \( L_0 \) and
S₀, respectively. This simple procedure leads to
\[ \beta_T \sim NS_0^{-\left(1 + p\right) \left(1 + 2p\right) \over L_0^{\left(2 + 2p\right) \left(1 + 2p\right)}}. \]  

(13)

The number of current sheets can be estimated by considering that the dissipation rate must not depend on \( S₀ \), i.e., \( N/\partial \sim \left( S₀ \right)^0 \). Since we expect that current sheets going unstable have an aspect ratio that scales with \( E \) bounded between \( S₀ \) and \( \left( S₀ \right)^0 \). We obtain \( J^2 \sim \left( S₀ \right)^{(1+p)/(1+2p)} \) and as a consequence \( N \sim \left( S₀ \right)^p/(1+2p) \). If we start from a Kolmogorov-type spectrum, this model predicts a modified energy cascade with a steeper spectral index,

\[ E(k) \sim C S₀^p (kL₀)^{-\left(1 + 2p+ \eta \right)/\left(n + 2p \right)}, \]  

(14)

where \( C \) is a constant that can be determined by matching the cascade in the inertial range with the sub-inertial range, and \( -0.02 \leq a < 0.05 \) for \( 1 \leq p \leq \infty \), and \( a = 0 \) for \( p = 1, 6 \).

To summarize, eq. (14) yields an energy spectrum which is bounded between \( E(k) \sim k^{-11/5} \) for \( p = 1 \), and \( E(k) \sim k^{-20/9} \) for \( p \to \infty \). The present model therefore predicts a magnetic energy spectral index of essentially \(-2.2\). Inclusion of the Alfvén effect as in the Iroshnikov-Kraichnan model (Iroshnikov 1964; Kraichnan 1965) is straightforward and would lead to a similar result, with a spectral power law of \(-2.3\). These results are in good agreement with numerical simulations at high Reynolds and Lundquist number by Dong et al. (2018), who performed simulations of 2D MHD turbulence; it is also in good agreement with 3D numerical results by Huang and Bhattacharjee (2016) of an initially unstable Sweet-Parker sheet that develops into a fully turbulent state with results by MHD turbulence; it is also in good agreement with 3D numerical simulations at high Reynolds and Lundquist number by Dong et al. (2018), who performed simulations of 2D MHD turbulence; it is also in good agreement with 3D numerical results by Huang and Bhattacharjee (2016) of an initially unstable Sweet-Parker sheet that develops into a fully turbulent state with results by MHD turbulence; it is also in good agreement with 3D numerical results by Huang and Bhattacharjee (2016) of an initially unstable Sweet-Parker sheet that develops into a fully turbulent state with results by MHD turbulence; it is also in good agreement with 3D numerical results by Huang and Bhattacharjee (2016) of an initially unstable Sweet-Parker sheet that develops into a fully turbulent state with results by MHD turbulence; it is also in good agreement with 3D numerical results by Huang and Bhattacharjee (2016) of an initially unstable Sweet-Parker sheet that develops into a fully turbulent state with results by MHD turbulence; it is also in good agreement with 3D numerical results by Huang and Bhattacharjee (2016) of an initially unstable Sweet-Parker sheet that develops into a fully turbulent state with results by MHD turbulence; it is also in good agreement with 3D numerical results by Huang and Bhattacharjee (2016) of an initially unstable Sweet-Parker sheet that develops into a fully turbulent state with results by MHD turbulence; it is also in good agreement with 3D numerical results by Huang and Bhattacharjee (2016) of an initially unstable Sweet-Parker sheet that develops into a fully turbulent state with results by MHD turbulence; it is also in good agreement with 3D numerical results by Huang and Bhattacharjee (2016) of an initially unstable Sweet-Parker sheet that develops into a fully turbulent state with results by MHD turbulence; it is also in good agreement with 3D numerical results by Huang and Bhattacharjee (2016) of an initially unstable Sweet-Parker sheet that develops into a fully turbulent state with results by MHD turbulence; it is also in good agreement with 3D numerical results by Huang and Bhattacharjee (2016) of an initially unstable Sweet-Parker sheet that develops into a fully turbulent state with results by MHD turbulence; it is also in good agreement with 3D numerical results by Huang and Bhatch...
