RAD-SUPPLEMENTING MODULES

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Abstract. Let $R$ be a ring, and let $M$ be a left $R$-module. If $M$ is Rad-supplementing, then every direct summand of $M$ is Rad-supplementing, but not each factor module of $M$. Any finite direct sum of Rad-supplementing modules is Rad-supplementing. Every module with composition series is (Rad-)supplementing. $M$ has a Rad-supplement in its injective envelope if and only if $M$ has a Rad-supplement in every essential extension. $R$ is left perfect if and only if $R$ is semilocal, reduced and the free left $R$-module $(R^R)^{(0)}$ is ample Rad-supplementing.

1. Introduction

All rings consider in this paper will be associative with an identity element. Unless otherwise stated, $R$ denotes an arbitrary ring and all modules will be left unitary $R$-modules. For a module $M$, by $X \subseteq M$, we mean $X$ is a submodule of $M$ or $M$ is an extension of $X$. As usual, $\text{Rad} M$ denotes the radical of $M$ and $J$ denotes the Jacobson radical of the ring $R$. $E(M)$ will be the injective envelope of $M$. For an index set $I$, $M^{(I)}$ denotes the direct sum $\oplus_I M$. By $\mathbb{N}$, $\mathbb{Z}$ and $\mathbb{Q}$ we denote as usual the set of natural numbers, the ring of integers and the field of rational numbers, respectively. A submodule $K \subseteq M$ is called small in $M$ (denoted by $K \ll M$) if $M \neq K + T$ for every proper submodule $T$ of $M$. Dually, a submodule $L \subseteq M$ is called essential in $M$ (denoted by $L \unlhd M$) if $L \cap X \neq 0$ for every nonzero submodule $X$ of $M$.

The notion of a supplement submodule was introduced in [12] in order to characterize semiperfect modules, that is projective modules whose factor modules have projective cover. For submodules $U$ and $V$ of a module $M$, $V$ is said to be a supplement of $U$ in $M$ or $U$ is said to have a supplement $V$ in $M$ if $U + V = M$ and $U \cap V \ll V$. The module $M$ is called supplemented if every...
submodule of $M$ has a supplement in $M$. See [19, §41] and [9] for results and the definitions related to supplements and supplemented modules. Recently, several authors have studied different generalizations of supplemented modules. In [1], $\tau$-supplemented modules were defined for an arbitrary preradical $\tau$ for the category of left $R$-modules. For submodules $U$ and $V$ of a module $M$, $V$ is said to be a $\tau$-supplement of $U$ in $M$ or $U$ is said to have a $\tau$-supplement $V$ in $M$ if $U + V = M$ and $U \cap V \subseteq \tau(V)$. $M$ is called a $\tau$-supplemented module if every submodule of $M$ has a $\tau$-supplement in $M$. For the particular case $\tau = \text{Rad}$, Rad-supplemented modules have been studied in [6]; rings over which all modules are Rad-supplemented were characterized. Also, in the recent paper [7], the relation between Rad-supplemented modules and local modules have been investigated. See [18]; these modules are called generalized supplemented modules. Note that Rad-supplements $V$ of a module $M$ are also called coneat submodules which can be characterized by the fact that each module with zero radical is injective with respect to the inclusion $V \subseteq M$; see [1], [9, §10] and [15]. On the other hand, modules that have supplements in every module in which it is contained as a submodule have been studied in [22]; the structure of these modules, which are called modules with the property $(E)$, has been completely determined over Dedekind domains. Such modules are also called Moduln mit Ergänzungseigenschaft in [3] and supplementing modules in [9, p. 255]. We follow the terminology and notation as in [9]. We call a module $M$ supplementing if it has a supplement in each module in which it is contained as a submodule. By considering these modules we define and study (ample) Rad-supplementing modules as a proper generalization of supplementing modules. A module $M$ is called (ample) Rad-supplementing if it has a (an ample) Rad-supplement in each module in which it is contained as a submodule, where a submodule $U \subseteq M$ has ample Rad-supplements in $M$ if for every $L \subseteq M$ with $U + L = M$, there is a Rad-supplement $L'$ of $U$ with $L' \subseteq L$.

In Section 2, we investigate some properties of Rad-supplementing modules. It is clear that every supplementing module is Rad-supplementing, but the converse implication fails to be true; Example 2.3. If a module $M$ has a Rad-supplement in its injective envelope, $M$ need not be Rad-supplementing. However, we prove that $M$ has a Rad-supplement in its injective envelope if and only if $M$ has a Rad-supplement in every essential extension; Proposition 2.5. We prove that for modules $A \subseteq B$, if $A$ and $B/A$ are Rad-supplementing, then so is $B$. Using this fact we also prove that every module with composition series is Rad-supplementing; Theorem 2.12. A factor module of a Rad-supplementing module need not be Rad-supplementing; Example 2.15. For modules $A \subseteq B \subseteq C$ with $C/A$ injective, we prove that if $B$ is Rad-supplementing, then so is $B/A$. As one of the main results, we prove that $R$ is left perfect if and only if $R$ is semilocal, $R$ is reduced and $(R_R)^{(N)}$ is Rad-supplementing; Theorem 2.20. Finally, using a result of [22], we show that
over a commutative ring $R$, a semisimple $R$-module $M$ is Rad-supplementing if and only if it is supplementing and that is equivalent the fact that $M$ is pure-injective; Theorem 2.21.

Section 3 contains some properties of ample Rad-supplementing modules. It starts by proving a useful property that a module $M$ is ample Rad-supplementing if and only if every submodule of $M$ is Rad-supplementing; Proposition 3.1. One of the main results of this part is that $R$ is left perfect if and only if $R R$ is reduced and the free left $R$-module $(R R)^{(3)}$ is ample Rad-supplementing; Theorem 3.3. In the proof of this result, Rad-supplemented modules plays an important role as, of course, every ample Rad-supplementing module is Rad-supplemented. Finally, using the characterization of Rad-supplemented modules given in [6], we characterize the rings over which every module is (ample) Rad-supplementing. We prove that every left $R$-module is (ample) Rad-supplementing if and only if every reduced left $R$-module is Rad-supplementing if and only if $R/P(R)$ is left perfect; Theorem 3.4.

2. Rad-supplementing modules

A module $M$ is called radical if $\text{Rad} M = M$, and $M$ is called reduced if it has no nonzero radical submodule. See [21, p. 47] for details for the notion of reduced and radical modules.

Proposition 2.1. Supplementing modules and radical modules are Rad-supplementing.

Proof. Let $M$ be a module and $N$ be any extension of $M$. If $M$ is supplementing, then it has a supplement, and so a Rad-supplement in $N$. Thus $M$ is Rad-supplementing. Now, if $\text{Rad} M = M$, then $N$ is a Rad-supplement of $M$ in $N$. □

By $P(M)$ we denote the sum of all radical submodules of the module $M$, that is,

$$P(M) = \sum \{U \subseteq M \mid \text{Rad} U = U\}.$$ 

Clearly $M$ is reduced if $P(M) = 0$.

Since $P(M)$ is a radical submodule of $M$ we have the following corollary.

Corollary 2.2. For a module $M$, $P(M)$ is Rad-supplementing.

A subset $I$ of a ring $R$ is said to be left $T$-nilpotent in case, for every sequence $\{a_k\}_{k=1}^{\infty}$ in $I$, there is a positive integer $n$ such that $a_1 \cdot \ldots \cdot a_n = 0$.

In general, Rad-supplementing modules need not be supplementing as the following example shows.

Example 2.3. Let $k$ be a field. In the polynomial ring $k[x_1, x_2, \ldots]$ with countably many indeterminates $x_n$, $n \in \mathbb{N}$, consider the ideal $I = (x_1^2, x_2^2 - x_1, x_3^2 - x_2, \ldots)$ generated by $x_1^2$ and $x_{n+1}^2 - x_n$ for each $n \in \mathbb{N}$. Then the quotient ring $R = k[x_1, x_2, \ldots]/I$ is a local ring with the unique maximal ideal
$J = J^2$ (see [6, Example 6.2] for details). Now let $M = J^{(3)}$. Then we have $\text{Rad } M = M$, and so $M$ is Rad-supplementing by Proposition 2.1. However, $M$ does not have a supplement in $R^{(3)}$. Because, otherwise, by [5, Theorem 1], $J$ would be a left $T$-nilpotent as $R$ is semilocal, but this is impossible. Thus $M$ is not supplementing.

For instance, over a left max ring, supplementing modules and Rad-supplementing modules coincide, where $R$ is called a left max ring if every left $R$-module has a maximal submodule or equivalently, $\text{Rad } M \ll M$ for every left $R$-module $M$.

**Proposition 2.4.** Every direct summand of a Rad-supplementing module is Rad-supplementing.

**Proof.** Let $U$ be a direct summand of a Rad-supplementing module $M$, and let $N$ be any extension of $U$. Then $M = A \oplus U$ for some submodule $A \subseteq M$. By hypothesis $M$ has a Rad-supplement in the module $A \oplus N$ containing $M$, that is, there exists a submodule $V$ of $A \oplus N$ such that $$(A \oplus U) + V = A \oplus N \quad \text{and} \quad (A \oplus U) \cap V \subseteq \text{Rad } V.$$ Now, let $g : A \oplus N \to N$ be the projection onto $N$. Then

$$U + g(V) = g(A \oplus U) + g(V) = g((A \oplus U) + V) = g(A \oplus N) = N; \quad \text{and}$$

$$U \cap g(V) = g((A \oplus U) \cap V) \subseteq g(\text{Rad } V) \subseteq \text{Rad}(g(V)).$$

Hence $g(V)$ is a Rad-supplement of $U$ in $N$. □

If a module $M$ has a Rad-supplement in its injective envelope $E(M)$, $M$ need not be Rad-supplementing. For example, for $R = \mathbb{Z}$, the $R$-module $M = 2\mathbb{Z}$ has a Rad-supplement in $E(M) = \mathbb{Q}$ since $\text{Rad } \mathbb{Q} = \mathbb{Q}$ (and so $\mathbb{Q}$ is Rad-supplemented). But, $M$ does not have a Rad-supplement in $\mathbb{Z}$, and thus $M$ is not Rad-supplementing. However, we have the following result.

**Proposition 2.5.** Let $M$ be a module. Then the following are equivalent.

(i) $M$ has a Rad-supplement in every essential extension;

(ii) $M$ has a Rad-supplement in its injective envelope $E(M)$.

**Proof.** (i)⇒(ii) is clear.

(ii)⇒(i) Let $M \subseteq N$ with $M \cong N$, and let $f : M \to N$ and $g : M \to E(M)$ be inclusion maps. Then we have the following commutative diagram with $h$ necessarily monic:

\[
\begin{array}{ccc}
M & \xrightarrow{f} & N \\
\downarrow{g} & & \downarrow{h} \\
E(M) & & \\
\end{array}
\]

By hypothesis, $M$ has a Rad-supplement in $E(M)$, say $K$. That is, $M + K = E(M)$ and $M \cap K \subseteq \text{Rad } K$. Since $M \subseteq h(N)$, we obtain that $h(N) = \ldots$
\[ h(N) \cap E(M) = h(N) \cap (M + K) = M + h(N) \cap K. \] Now, taking any \( n \in N \), we have \( h(n) = m + h(n_1) = h(m + n_1) \) where \( m \in M \) and \( h(n_1) \in h(N) \cap K \). So, \( n = m + n_1 \in M + h^{-1}(K) \) since \( h \) is monic, and so \( M + h^{-1}(K) = N \). Moreover, \( M \cap h^{-1}(K) = h^{-1}(M) \cap K \subseteq h^{-1}(\text{Rad} K) \subseteq h^{-1}(h^{-1}(K)) \) since \( h^{-1}(M) = M \) as \( h \) is monic. Hence \( h^{-1}(K) \) is a Rad-supplement of \( M \) in \( N \).

**Proposition 2.6.** Let \( B \) be a module, and let \( A \) be a submodule of \( B \). If \( A \) and \( B/A \) are Rad-supplementing, then so is \( B \).

**Proof.** Let \( B \subseteq N \) be any extension of \( B \). By hypothesis, there is a Rad-supplement \( V/A \) of \( B/A \) in \( N/A \) and a Rad-supplement \( W \) of \( A \) in \( V \). We claim that \( W \) is a Rad-supplement of \( B \) in \( N \). We have epimorphisms \( f : W \rightarrow V/A \) and \( g : V/A \rightarrow N/B \) such that \( \ker f = W \cap A \subseteq \text{Rad } W \) and \( \ker g = V/A \cap B/A \subseteq \text{Rad } (V/A) \). Then \( g \circ f : W \rightarrow N/B \) is an epimorphism such that \( W \cap B = \ker (g \circ f) \subseteq \text{Rad } W \) by [20, Lemma 1.1]. Finally, \( N = V + B = (W + A) + B = W + B \).

**Remark 2.7.** The previous result holds for supplementing modules; see [22, Lemma 1.3-(c)].

**Corollary 2.8.** If \( M_1 \) and \( M_2 \) are Rad-supplementing modules, then so is \( M_1 \oplus M_2 \).

**Proof.** Consider the short exact sequence

\[ 0 \rightarrow M_1 \rightarrow M_1 \oplus M_2 \rightarrow M_2 \rightarrow 0. \]

Thus the result follows by Proposition 2.6.

\( R \) is said to be a left hereditary ring if every left ideal of \( R \) is projective.

**Corollary 2.9.** If \( M/P(M) \) is Rad-supplementing, then \( M \) is Rad-supplementing. For left hereditary rings, the converse is also true.

**Proof.** Since \( P(M) \) is Rad-supplementing by Corollary 2.2, the result follows by Proposition 2.6. Over left hereditary rings, any factor module of a Rad-supplementing module is Rad-supplementing (see Corollary 2.18).

We give the proof of the following known fact for completeness.

**Lemma 2.10.** Every simple submodule \( S \) of a module \( M \) is either a direct summand of \( M \) or small in \( M \).

**Proof.** Suppose that \( S \) is not small in \( M \), then there exists a proper submodule \( K \) of \( M \) such that \( S + K = M \). Since \( S \) is simple and \( K \neq M \), \( S \cap K = 0 \). Thus \( M = S \oplus K \).

**Proposition 2.11.** Every simple module is \((\text{Rad})\)-supplementing.
Proof. Let $S$ be a simple module and $N$ any extension of $S$. Then by Lemma 2.10, $S \ll N$ or $S \oplus S' = N$ for a submodule $S' \subseteq N$. In the first case, $N$ is a (Rad-)supplement of $S$ in $N$, and in the second case, $S'$ is a (Rad-)supplement of $S$ in $N$. So, in each case $S$ has a (Rad-)supplement in $N$, that is, $S$ is (Rad-)supplementing. □

**Theorem 2.12.** Every module with composition series is (Rad-)supplementing.

**Proof.** Let $0 = M_0 \subseteq M_1 \subseteq M_2 \subseteq \cdots \subseteq M_n = M$ be a composition series of a module $M$. The proof is by induction on $n \in \mathbb{N}$. If $n = 1$, then $M = M_1$ is simple, and so $M$ is (Rad-)supplementing by Proposition 2.11. Suppose that this is true for each $k \leq n - 1$. Then $M_{n-1}$ is (Rad-)supplementing. Since $M_n/M_{n-1}$ is also (Rad-)supplementing as a simple module, we obtain by Proposition 2.6 that $M = M_n$ is (Rad-)supplementing. □

**Corollary 2.13.** A finitely generated semisimple module is (Rad-)supplementing.

In general, a factor module of a Rad-supplementing module need not be Rad-supplementing. To give such a counterexample we need the following result.

$R$ is called Von Neumann regular if every element $a \in R$ can be written in the form $axa$, for some $x \in R$.

**Proposition 2.14.** Let $R$ be a commutative Von Neumann regular ring. Then an $R$-module $M$ is Rad-supplementing if and only if $M$ is injective.

**Proof.** Suppose that $M$ is a Rad-supplementing module. Let $M \subseteq N$ be any extension of $M$. Then there is a Rad-supplement $V$ of $M$ in $N$, that is, $V + M = N$ and $V \cap M \subseteq \text{Rad} V$. Since all $R$-modules have zero radical by [13, 3.73 and 3.75], we have $\text{Rad} V = 0$, and so $N = V \oplus M$. Conversely, if $M$ is injective and $M \subseteq N$ is any extension of $M$, then $N = M \oplus K$ for some submodule $K \subseteq N$. Thus $K$ is a Rad-supplement of $M$ in $N$. □

It is known that a ring $R$ is lefty hereditary if and only if every quotient of an injective $R$-module is injective (see [8, Ch.I, Theorem 5.4]).

**Example 2.15.** Let $R = \prod_{i \in I} F_i$ be a ring, where each $F_i$ is a field for an infinite index set $I$. Then $R$ is a commutative Von Neumann regular ring. Indeed, let $a = (a_i)_{i \in I} \in R$ where $a_i \in F_i$ for all $i \in I$. Taking $b = (b_i)_{i \in I} \in R$ where $b_i \in F_i$ such that

$$b_i = \begin{cases} a_i^{-1} & \text{if } a_i \neq 0, \\ 0 & \text{if } a_i = 0. \end{cases}$$

Then we obtain that

$$ab = (a_i)b_i(a_i) = (a_ib_i)_{i \in I} = (a_i)_{i \in I} = a.$$  

Now, by Proposition 2.14, $R$ is a Rad-supplementing module over itself since it is injective (see [13, Corollary 3.11B]). Since $R$ is not noetherian, it cannot be
semisimple (by [14, Corollary 2.6]). Thus $R$ is not hereditary by [16, Corollary]. Hence, there is a factor module of $R$ which is not injective.

The following technical lemma will be useful to show that Rad-supplementing modules are closed under factor modules, under a special condition.

**Lemma 2.16.** Let $A \subseteq B \subseteq C$ be modules with $C/A$ injective. Let $N$ be a module containing $B/A$. Then there exists a commutative diagram with exact rows:

$$
\begin{array}{ccccccccc}
0 & \rightarrow & A & \rightarrow & B & \rightarrow & B/A & \rightarrow & 0 \\
\downarrow{id} & & \downarrow{\varphi} & & \downarrow{\beta} & & \downarrow{\sigma} & & 0 \\
0 & \rightarrow & A & \rightarrow & P & \rightarrow & N & \rightarrow & 0
\end{array}
$$

**Proof.** By pushout we have the following commutative diagram, where $\varphi$ exists since $C/A$ is injective:

$$
\begin{array}{ccccccccc}
0 & \rightarrow & B/A & \rightarrow & N & \rightarrow & N/(B/A) & \rightarrow & 0 \\
\downarrow{\varphi} & & \downarrow{\alpha} & & \downarrow{id} & & \downarrow{g} & & 0 \\
0 & \rightarrow & C/A & \rightarrow & N' & \rightarrow & N/(B/A) & \rightarrow & 0
\end{array}
$$

In the diagram, since the triangle-(1) is commutative, there exists a homomorphism $\alpha : N/(B/A) \rightarrow N'$ making the triangle-(2) is commutative by [11, Lemma I.8.4]. So, the second row splits. Then we can take $N' = (C/A) \oplus (N/(B/A))$, and so we may assume that $\beta : C/A \rightarrow N'$ is an inclusion. Therefore, we have the following commutative diagram since $B/A = \beta(B/A) = g(B/A) \subseteq N'$:

$$
\begin{array}{ccccccccc}
0 & \rightarrow & A & \rightarrow & B & \rightarrow & B/A & \rightarrow & 0 \\
\downarrow{id} & & \downarrow{\phi} & & \downarrow{\gamma} & & \downarrow{\sigma} & & 0 \\
0 & \rightarrow & A & \rightarrow & C \oplus (N/(B/A)) & \rightarrow & N' & \rightarrow & 0
\end{array}
$$

where $\gamma(a) = (a, 0)$ for every $a \in A$, $\phi(b) = (b, 0)$ for every $b \in B$, and $\sigma(c, \overline{\tau}) = (c + A, \overline{\tau})$ for every $c \in C$ and $\overline{\tau} \in N/(B/A)$. Finally, taking $P = \sigma^{-1}(g(N))$ and defining a homomorphism $\tilde{\sigma} : P \rightarrow g(N)$ by $\tilde{\sigma}(x) = \sigma(x)$ for every $x \in P$ (in fact, $\tilde{\sigma}$ is an epimorphism as so is $\sigma$), we obtain the following desired commutative diagram:

$$
\begin{array}{ccccccccc}
0 & \rightarrow & A & \rightarrow & B & \rightarrow & B/A & \rightarrow & 0 \\
\downarrow{id} & & \downarrow{\varphi} & & \downarrow{\beta} & & \downarrow{\tilde{\sigma}} & & 0 \\
0 & \rightarrow & A & \rightarrow & P & \rightarrow & g(N) \cong N & \rightarrow & 0
\end{array}
$$

**Proposition 2.17.** Let $A \subseteq B \subseteq C$ with $C/A$ injective. If $B$ is Rad-supplementing, then so is $B/A$. 

\[\square\]
Proof. Let \( B/A \subseteq N \) be any extension of \( B/A \). By Lemma 2.16, we have the following commutative diagram with exact rows since \( C/A \) is injective:

\[
\begin{array}{c}
0 \longrightarrow A \longrightarrow B \xrightarrow{\sigma} B/A \longrightarrow 0 \\
\downarrow{id} & \downarrow{h} & \downarrow{f} \\
0 \longrightarrow A \longrightarrow P \xrightarrow{g} N \longrightarrow 0
\end{array}
\]

Since \( h \) is monic and \( B \) is Rad-supplementing, \( B \cong \text{Im} \ h \) has a Rad-supplement in \( P \), say \( V \). That is, \( \text{Im} \ h + V = P \) and \( \text{Im} \ h \cap V \subseteq \text{Rad} \ V \). We claim that \( g(V) \) is a Rad-supplement of \( B/A \) in \( N \).

\[
N = g(P) = g(h(B)) + g(V) = (f \sigma)(B) + g(V) = (B/A) + g(V), \quad \text{and}
\]

\[
(B/A) \cap g(V) = f(\sigma(B)) \cap g(V) = g[h(B) \cap V] \subseteq g(\text{Rad} \ V) \subseteq \text{Rad} \ g(V). \quad \square
\]

Corollary 2.18. If \( R \) is a left hereditary ring, then every factor module of Rad-supplementing module is Rad-supplementing.

Proposition 2.19. If \( M \) is a reduced, projective and Rad-supplementing module, then \( \text{Rad} M \ll M \).

Proof. Suppose \( X + \text{Rad} M = M \) for a submodule \( X \) of \( M \). Then since \( M \) is projective, there exists \( f \in \text{End} (M) \) such that \( \text{Im} \ f \subseteq X \) and \( \text{Im} (1 - f) \subseteq \text{Rad} M = JM \) where \( J \) is a Jacobson radical of \( R \). Therefore \( f \) is a monomorphism by [4, Theorem 3]. Since \( M \) is Rad-supplementing and \( \text{Im} f \cong M \), \( \text{Im} f \) has a Rad-supplement \( V \) in \( M \), that is, \( \text{Im} f + V = M \) and \( \text{Im} f \cap V \subseteq \text{Rad} V \). Now we have an epimorphism \( g : V \to M/\text{Im} f \) such that \( \text{Ker} g = V \cap \text{Im} f \subseteq \text{Rad} V \). Moreover, since \( M = \text{Im} f + \text{Im} (1 - f) = \text{Im} f + \text{Rad} M \) we have \( \text{Rad}(M/\text{Im} f) = M/\text{Im} f \). Thus \( \text{Rad} V = V \), and so \( V = 0 \) since \( M \) is reduced. Hence \( M = \text{Im} f \ll X \) implies that \( X = M \) as required. \( \square \)

\( R \) is said to be a semilocal ring if \( R/J \) is a semisimple ring, that is a left (and right) semisimple \( R \)-module (see [14, §20]).

Theorem 2.20. A ring \( R \) is left perfect if and only if \( R \) is semilocal, \( rR \) is reduced and the free left \( R \)-module \( F = (rR)^{\infty} \) is Rad-supplementing.

Proof. If \( R \) is left perfect, then \( R \) is semilocal by [2, 28.4], and clearly \( rR \) is reduced. Since all left \( R \)-modules are supplemented and so Rad-supplemented, \( F \) is Rad-supplementing. Conversely, since \( P(rR) = 0 \) we have \( P(F) = (P(rR))^{(\infty)} = 0 \), that is, \( F \) is reduced. Thus by Proposition 2.19, \( JF = \text{Rad} F \ll F \), that is, \( J \) is left \( T \)-nilpotent by, for example, [2, 28.3]. Hence \( R \) is left perfect by [2, 28.4] since it is moreover semilocal. \( \square \)

Supplementing modules over commutative noetherian rings have been studied in [3]; the author showed that if a module \( M \) is supplementing, then it is cotorsion, that is, \( \text{Ext}^1_{\text{fd}}(F, M) = 0 \) for every flat module \( F \) (see [10] for cotor-sion modules). So the question was raised When Rad-supplementing modules
are cotorsion? Since any pure-injective module is cotorsion, the following result gives an answer of the question for a semisimple module over a commutative ring. The relation between (Rad-)supplementing modules and cotorsion modules needs to be further investigated.

The part (iii)⇒(i) of the proof of the following theorem follows from [22, Theorem 1.6-(ii)⇒(i)], but we give it by explanation for completeness.

**Theorem 2.21.** Let $R$ be a commutative ring. Then the following are equivalent for a semisimple $R$-module $M$.

(i) $M$ is supplementing;

(ii) $M$ is Rad-supplementing;

(iii) $M$ is pure-injective.

**Proof.** (i)⇒(ii) is clear.

(ii)⇒(iii) Let $M \subseteq N$ be a pure extension of $M$. By hypothesis $M$ has a Rad-supplement $V$ in $N$, that is, $M + V = N$ and $M \cap V \subseteq \text{Rad} V$. Since $M$ is pure in $N$, we have $\text{Rad} M = M \cap \text{Rad} N$ (as $R$ is commutative). Thus $M \cap V \subseteq M \cap \text{Rad} N = \text{Rad} M = 0$ as $M$ is semisimple. Hence $N = M \oplus V$ as required.

(iii)⇒(i) Let $M \subseteq N$ be any extension of $M$. Then the factor module $X = (M + \text{Rad} N)/\text{Rad} N$ of $M$ is again semisimple and pure-injective. Since semisimple submodules are pure in every module with zero radical and $\text{Rad}(N/\text{Rad} N) = 0$, it follows that $X$ is a direct summand of $N/\text{Rad} N$. Now let $X = (V/\text{Rad} N) \oplus X = N/\text{Rad} N$

for a submodule $V \subseteq N$ such that $\text{Rad} N \subseteq V$. So we have $V + M = N$ with $V$ minimal, and thus $V$ is a supplement of $M$ in $N$. This is because, if $T + M = N$ for a submodule $T$ of $N$ with $T \subseteq V$, then from $\text{Rad}(N/T) = \text{Rad}((M + T)/T) = \text{Rad}(M/M \cap T) = 0$ as $M/M \cap T$ is semisimple, we obtain that $\text{Rad} N \subseteq T$. Moreover, since $\text{Rad} N = V \cap (M + \text{Rad} N) = V \cap M + \text{Rad} N$, we have $V \cap M \subseteq \text{Rad} N$ and $V = T + V \cap M \subseteq T + \text{Rad} N = T$, thus $T = V$. □

### 3. Ample Rad-supplementing modules

The following useful result gives a relation between Rad-supplementing modules and ample Rad-supplementing modules.

**Proposition 3.1.** A module $M$ is ample Rad-supplementing if and only if every submodule of $M$ is Rad-supplementing.

**Proof.** ($\Leftarrow$) Let $M$ be a module and $N$ be any extension of $M$. Suppose that for a submodule $X \subseteq N$, $X + M = N$. By hypothesis the submodule $X \cap M$ of $M$ has a Rad-supplement $V$ in $X$ containing $X \cap M$, that is, $(X \cap M) + V = X$ and...
(X ∩ M) ∩ V ⊆ \text{Rad} V. Then N = M + X = M + (X ∩ M) + V = M + V and, 
M ∩ V = M ∩ (V ∩ X) = (X ∩ M) ∩ V ⊆ \text{Rad} V. Hence V is a Rad-supplement 
of M in N such that V ⊆ X.

(⇒) Let U be a submodule of M and N be any module containing U. Thus 
we can draw the pushout for the inclusion homomorphisms \( i_1 : U \hookrightarrow N \) and 
\( i_2 : U \hookrightarrow M \):

\[
\begin{array}{c}
\xymatrix{
M \ar[r]^\alpha & F \\
U \ar[u]_{i_2} \ar[r]_\beta & N \ar[u]_{i_1}
}
\end{array}
\]

In the diagram, \( \alpha \) and \( \beta \) are also monomorphisms by the properties of pushout 
(see, for example, [17, Exercise 5.10]). Let \( M' = \text{Im} \alpha \) and \( N' = \text{Im} \beta \). Then 
\( F = M' + N' \) by the properties of pushout. So by hypothesis, \( M' \cong M \) 
has a Rad-supplement \( V \) in \( F \) such that \( V \subseteq N' \), that is, \( M' + V = F \) and 
\( M' \cap V \subseteq \text{Rad} V \). Therefore \( V \) is a Rad-supplement of \( M' \cap N' \) in \( N' \), because 
\( N' = N' \cap F = N' \cap (M' + V) = (M' \cap N') + V \) and \( (M' \cap N') \cap V = \)
\( M' \cap V \subseteq \text{Rad} V \). Now, we claim that \( \beta^{-1}(V) \) is a Rad-supplement of \( U \) in \( N \).

Since \( \beta : N \rightarrow F \) is a monomorphism with \( N' = \text{Im} \beta \), we have an isomorphism 
\( \tilde{\beta} : N \rightarrow N' \) defined as \( \tilde{\beta}(x) = \beta(x) \) for all \( x \in N \). By this isomorphism, since \( V \)
is a Rad-supplement of \( M' \cap N' \) in \( N' \), we obtain \( \tilde{\beta}^{-1}(V) \) is a Rad-supplement of 
\( \tilde{\beta}^{-1}(M' \cap N') \) in \( \tilde{\beta}^{-1}(N') \). Since it can be easily shown that 
\( \tilde{\beta}^{-1}(V) = \beta^{-1}(V) \), 
\( \tilde{\beta}^{-1}(N') = N \), and \( \tilde{\beta}^{-1}(M' \cap N') = U \) the result follows. \( \square \)

**Corollary 3.2.** Every ample Rad-supplementing module is both Rad-supplementing 
and Rad-supplemented.

**Theorem 3.3.** A ring \( R \) is left perfect if and only if \( _RR \) is reduced and the free left \( R \)-module \( F = (\_RR)^{[R]} \) is ample Rad-supplementing.

**Proof.** If \( R \) is left perfect, then \( _RR \) is reduced and all left \( R \)-modules are 
supplemented, and so Rad-supplemented. Thus every submodule of \( F \) is Rad-supplementing.
Hence \( F \) is ample Rad-supplementing by Proposition 3.1. Conversely, if \( F \) is ample Rad-supplementing, then it is Rad-supplemented by 
Corollary 3.2, and so \( R \) is left perfect by [6, Theorem 5.3]. \( \square \)

Finally, we give the characterization of the rings over which every module is 
(ample) Rad-supplementing.

**Theorem 3.4.** For a ring \( R \), the following are equivalent:

(i) Every left \( R \)-module is Rad-supplementing;
(ii) Every reduced left \( R \)-module is Rad-supplementing;
(iii) Every left \( R \)-module is ample Rad-supplementing;
(iv) Every left \( R \)-module is Rad-supplemented;
(v) \( R/P(R) \) is left perfect.
Proof. Let $M$ be a module. (i)⇒(ii) is clear.
(ii)⇒(i) Since $M/P(M)$ is reduced, it is Rad-supplementing by hypothesis. So $M$ is Rad-supplementing by Corollary 2.9.
(i)⇒(iii) Since every submodule of $M$ is Rad-supplementing, $M$ is ample Rad-supplementing by Proposition 3.1.
(iii)⇒(iv) by Corollary 3.2.
(iv)⇒(i) Let $M \subseteq N$ be any extension of $M$. By hypothesis, $N$ is Rad-supplemented, and so $M$ has a Rad-supplement in $N$.
(iv)⇔(v) by [6, Theorem 6.1].

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References

[1] K. Al-Takhman, C. Lomp, and R. Wisbauer, τ-complemented and τ-supplemented modules, Algebra Discrete Math. (2006), no. 3, 1–16.
[2] F. W. Anderson and K. R. Fuller, Rings and Categories of Modules, New-York, Springer, 1992.
[3] J. Averdunk, Module mit Ergänzungseigenschaft, Dissertation, Ludwig-Maximilians-Universität München, Fakultät für Mathematik, 1996.
[4] I. Beck, Projective and free modules, Math. Z. 129 (1972), 231–234.
[5] E. Büyükaşık and C. Lomp, Rings whose modules are weakly supplemented are perfect: Applications to certain ring extensions, Math. Scand. 105 (2009), no. 1, 25–30.
[6] E. Büyükaşık, E. Mermut, and S. Özdemir, Rad-supplemented modules, Rend. Semin. Mat. Univ. Padova 124 (2010), 157–177.
[7] E. Büyükaşık, R. Tribak, On w-local modules and Rad-supplemented modules, J. Korean Math. Soc. 51 (2014), no. 5, 971–985.
[8] H. Cartan and S. Eilenberg, Homological Algebra, Princeton Landmarks in Mathematics and Physics series, New Jersey: Princeton Univesity, 1956.
[9] J. Clark, C. Lomp, N. Vanaja, and R. Wisbauer, Lifting modules, Frontiers in Mathematics, Basel: Birkhäuser Verlag, Supplements and projectivity in module theory, 2006.
[10] E. E. Enochs and O. M. G. Jenda, Relative homological algebra, vol. 30 of de Gruyter Expositions in Mathematics, Berlin: Walter de Gruyter & Co., 2000.
[11] L. Fuchs and L. Salce, Modules over non-Noetherian domains, vol. 84 of Mathematical Surveys and Monographs, Providence, RI: American Mathematical Society, 2001.
[12] F. Kasch and E. A. Mares, Eine Kennzeichnung semi-perfekter Module, Nagoya Math. J. 27 (1966), 525–529.
[13] T. Y. Lam, Lectures on modules and rings, vol. 189 of Graduate Texts in Mathematics, New York: Springer-Verlag, 1999.
[14] , A first course in noncommutative rings, vol. 131 of Graduate Texts in Mathematics, New York: Springer-Verlag, 2001.
[15] E. Mermut, Homological Approach to Complements and Supplements, Ph.D. thesis, Dokuz Eylül University, The Graduate School of Natural and Applied Sciences, İzmir-Turkey, 2004.
[16] B. L. Osofsky, Rings all of whose finitely generated modules are injective, Pacific J. Math. 14 (1964), 645–650.
[17] J. J. Rotman, An Introduction to Homological Algebra, Universitext, New York: Springer, 2009.
[18] Y. Wang and N. Ding, Generalized supplemented modules, Taiwanese J. Math. 10 (2006), no. 6, 1589–1601.
[19] R. Wisbauer, Foundations of Module and Ring Theory, Reading: Gordon and Breach, 1991.
[20] W. Xue, Characterization of semiperfect and perfect rings, Publ. Mat. 40 (1996), no. 1, 115–125.
[21] H. Zöschinger, Komplementierte Moduln über Dedekindringen, J. Algebra 29 (1974), 42–56.
[22] , Moduln, die in jeder Erweiterung ein Komplement haben, Math. Scand. 35 (1974), 267–287.

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