Holographic energy density in the Brans-Dicke theory

Hungsoo Kim, H. W. Lee and Y. S. Myung

Relativity Research Center and School of Computer Aided Science, Inje University
Gimhae 621-749, Korea

Abstract

We study cosmological applications of the holographic energy density. Considering the holographic energy density as a dynamical cosmological constant, we need the Brans-Dicke theory as a dynamical framework instead of general relativity. In this case we use the Bianchi identity as a consistency relation to obtain physical solutions. It is shown that the future event horizon as the IR cutoff provides the dark energy in the Brans-Dicke theory. Furthermore the role of the Brans-Dicke scalar is clarified in the dark energy-dominated universe by calculating its equation of state.
1 Introduction

Supernova (SN Ia) observations suggest that our universe is accelerating and the dark energy contributes $\Omega_{DE} \simeq 0.60 - 0.70$ to the critical density of the present universe [1]. Also cosmic microwave background (CMB) observations [2] imply that the standard cosmology is given by the inflation and FRW universe [3]. A typical candidate for the dark energy is the cosmological constant in general relativity. Recently Cohen et al showed that in quantum field theory, the UV cutoff $\Lambda$ is related to the IR cutoff $L_\Lambda$ due to the limit set by forming a black hole [4]. In other words, if $\rho_\Lambda$ is the quantum zero-point energy density caused by the UV cutoff, the total energy of the system with size $L_\Lambda$ should not exceed the mass of the system-size black hole: $L^3_\Lambda \rho_\Lambda \leq L_\Lambda / G$. Here the Newtonian constant $G$ is related to the Planck mass ($G = 1/M_p^2$). The largest $L_\Lambda$ is chosen as the one saturating this inequality and its holographic energy density is then given by $\rho_\Lambda = 3c^2M_p^2/8\pi L^2_\Lambda$ with a factor $3c^2$. Taking $L_\Lambda$ as the size of the present universe (Hubble horizon: $R_{HH}$), the resulting energy is comparable to the present dark energy [5]. Even though this holographic approach leads to the data, this description is incomplete because it fails to explain the equation of state for the dark energy-dominated universe [6]. In order to resolve this situation, one introduces another candidates for the IR cutoff. One is the particle horizon $R_{PH}$. This provides $\rho_\Lambda \sim a^{-2(1+1/c)}$ which gives the equation of state $\omega_\Lambda = 1/3$ for $c = 1$ [7]. It corresponds to a radiation-dominated universe and unfortunately it is a decelerating universe. In order to find an accelerating universe, we need the future event horizon $R_{FH}$. In the case of $L_\Lambda = R_{FH}$ one finds $\rho_\Lambda \sim a^{-2(1-1/c)}$ which describes the dark energy with $\omega_\Lambda = -1$ for $c = 1$. This is close to the data [1]. The related works appeared in ref.[8, 9, 10].

However, the above choices for the IR cutoff have something to be needed to clarify. $\rho_\Lambda = 3M_p^2/8\pi L^2_\Lambda$ with $L_\Lambda = R_{HH}, R_{PH}, R_{FH}$ correspond to dynamical cosmological constants. But authors investigated its cosmological applications in the framework of general relativity. We need the dynamical framework, for example the Brans-Dicke theory, to study the cosmological application of the dynamical cosmological constant.

In this work we ask how the dark energy can be realized from the Brans-Dicke theory with the holographic energy density. In the framework of general relativity, the choice of IR cutoff as the Hubble horizon did not give us a correct equation of state. Furthermore a recent work on this direction did not use the Bianchi identity as a consistency relation to derive the dark energy from the Brans-Dicke theory with holographic energy density [11]. If one does not use the Bianchi identity, it is not easy to discriminate between physical and unphysical solution. We show that the future event horizon as the IR cutoff provides
the dark energy in the Brans-Dicke theory with the holographic energy density. Here we use all of equations including the Bianchi identity as a consistency relation [12]. These are consisted of four equations but it reduces to three equations after eliminating the pressure by making use of the energy-momentum conservation for a matter-fluid. In addition, the role of the Brans-Dicke scalar is clarified by deriving its equation of state.

2 Brans-Dicke theory with a perfect fluid

We start with the action for the Brans-Dicke theory with a perfect fluid [12]

\[ S_{BD} = \int d^4x \sqrt{-g} \left[ \frac{1}{16\pi} \left( \phi R - \omega \frac{\nabla \phi \nabla \phi}{\phi} \right) - \mathcal{L}_m(\psi, g_{\mu\nu}) \right] \] (1)

in the Jordan frame. Here \( \omega \) is a parameter to be determined. Their equations are given by

\[ G_{\mu\nu} = 8\pi T_{\mu\nu}^{BD} + \frac{8\pi}{\phi} T^{m}_{\mu\nu}, \] (2)

\[ \nabla^2 \phi = \frac{8\pi}{2\omega + 3} T^{m}_{\lambda\lambda}, \] (3)

where the energy-momentum tensor \( T^{m}_{\mu\nu} \) for a matter-fluid and \( T^{BD}_{\mu\nu} \) for a Brans-Dicke scalar take the forms

\[ T^{m}_{\mu\nu} = (\rho + p)u_\mu u_\nu + pg_{\mu\nu}, \] (4)

\[ T^{BD}_{\mu\nu} = \frac{1}{8\pi} \left[ \frac{\omega}{\phi^2} \left( \nabla_\mu \phi \nabla_\nu \phi - \frac{1}{2} g_{\mu\nu} (\nabla \phi)^2 \right) + \frac{1}{\phi} \left( \nabla_\mu \nabla_\nu \phi - g_{\mu\nu} \nabla_\alpha \nabla^\alpha \phi \right) \right], \] (5)

\[ T^{BD}_{\mu\nu} = (\rho_{BD} + p_{BD})u_\mu u_\nu + p_{BD}g_{\mu\nu} \equiv T^{BD}_{\mu\nu}/G_0. \] (6)

Here the last expression corresponds to the definitions of energy density and pressure for the Brans-Dicke scalar with a constant \( G_0 \). Let us introduce a (3 + 1)-dimensional Friedman-Robertson-Walker (FRW) metric

\[ ds^2 = -dt^2 + a(t)^2 \left[ \frac{dr^2}{1 - kr^2} + r^2 d\Omega_2^2 \right], \] (7)

where \( a \) is the scale factor of the universe and \( d\Omega_2^2 \) denotes the line element of a two-dimensional unit sphere. Here \( k = -1, 0, 1 \) represent that the universe is open, flat, closed, respectively. Assuming that \( \phi = \phi(t) \), a cosmological evolution is determined by the four equations [11, 12]

\[ H^2 + \frac{\dot{\phi}}{\phi} - \frac{\omega}{6} \left( \frac{\dot{\phi}}{\phi} \right)^2 = \frac{8\pi}{3\phi} \rho - \frac{k}{a^2}, \] (8)
\[
\ddot{\phi} + 3H\dot{\phi} = -\frac{8\pi}{2\omega + 3}(\rho - 3p),
\]
(9)

\[
\dot{\rho} + 3H(\rho + p) = 0,
\]
(10)

\[
\dot{\rho}_{BD} + 3H(\rho_{BD} + p_{BD}) = \frac{\rho}{G_0}\left(\frac{\dot{\phi}}{\phi^2}\right).
\]
(11)

Here \(H = \dot{a}/a\) represents the Hubble parameter and the overdot stands for derivative with respect to the cosmic time \(t\). The first equation corresponds to the Friedmann equation, the second comes from the equation \((3)\) for a Brans-Dicke scalar, and the third is the conservation law \(\nabla_\nu T^{\mu\nu} = 0\) for a matter-fluid. The last comes from the Bianchi identity of \(\nabla_\nu G^{\mu\nu} = 0\) and plays a role of the consistency relation \((12)\). If a solution does not satisfy the four equations Eqs.\((8)-(11)\) simultaneously, it is no longer a physical solution. In order to investigate a role of the Brans-Dicke scalar, one needs its energy density and pressure derived from Eqs.\((5)\) and \((6)\) as

\[
\rho_{BD} = -\frac{1}{16\pi G_0} \left[ \omega \left(\frac{\dot{\phi}}{\phi}\right)^2 - 6H \left(\frac{\dot{\phi}}{\phi}\right) \right],
\]
(12)

\[
p_{BD} = \frac{1}{16\pi G_0} \left[ \omega \left(\frac{\dot{\phi}}{\phi}\right)^2 + 4H \left(\frac{\dot{\phi}}{\phi}\right) + 2\left(\frac{\ddot{\phi}}{\phi}\right) \right].
\]
(13)

In the Brans-Dicke scalar with \(\rho = 0\), from Eq.\((11)\) its equation of state is given by \(p_{BD} = \tilde{\omega}_{BD}\rho_{BD}\) with \(\tilde{\omega}_{BD} = -\frac{1}{3}\). This means that a Brans-Dicke scalar by itself gives zero acceleration. Thus one expects that it plays an intermediate role between matter and cosmological constant.

### 3 Brans-Dicke theory with Holographic energy density

The Brans-Dicke scalar \(\phi\) plays a role the inverse of Newtonian constant \((\phi \sim 1/G)\). In this case we have a relation of \(\phi_0 \sim 1/G_0\). For definiteness we choose the holographic dark energy with \(c = 1\) as\([7, 11]\)

\[
\rho_\Lambda = \frac{3\phi}{8\pi L_\Lambda^2}.
\]
(14)

From now on we explore the role of the Brans-Dicke scalar in the holographic energy-dominated universe with \(k = 0\). First we choose the IR cutoff as the present universe-size (Hubble horizon: HH) such that \(L_\Lambda = 1/H\). Then one finds the power-law solutions from Eqs.\((8)-(11)\) as

\[
a(t) = a_0 t^{\omega/(4\omega + 6)}, \quad \phi(t) = \phi_0 t^{3/(2\omega + 3)}, \quad \rho_\Lambda = \frac{3\omega^3\phi_0}{8\pi(4\omega + 6)^2} \left(\frac{a}{a_0}\right)^{-2(4\omega + 3)/\omega}.
\]
(15)
We note that the Bianchi identity in Eq. (11) is satisfied with any $\omega$. The last relation implies the equation of state defined by $p_\Lambda = \tilde{\omega}_\Lambda^{HH} \rho_\Lambda$,

$$\tilde{\omega}_\Lambda^{HH} = -1 + \frac{2(4\omega + 3)}{3\omega}. \quad (16)$$

On the other hand the Brans-Dicke scalar gives

$$\rho_{BD}^{HH} = 0, \quad p_{BD}^{HH} = \frac{6}{8\pi G_0} \frac{H^2}{\omega} \quad (17)$$

which provides a zero energy density and positive pressure. In the general relativistic limit of $\omega \to \infty$, one finds $\rho_{BD} = p_{BD} = 0$ which means that the Brans-Dicke scalar does not play any role.

On the other hand, one recovers the equation of state in the limit of $\omega \to \infty$ as

$$\tilde{\omega}_\Lambda^{HH} \to \frac{5}{3}. \quad (18)$$

This equation of state is regarded as the correct one for the Hubble horizon. We remark the general relativistic equation of state. The Friedmann equation $H^2 = 8\pi G \rho_\Lambda/3$ with $\rho_\Lambda = 3c^2 H^2/8\pi G$ does not give the equation of state. This gives nothing but $c = 1$. Further, combining the matter density $\rho_m = a_0/a^3$ with holographic energy density $\rho_\Lambda$, its Friedmann equation takes the form of $\rho_m = 3(1 - c^2)H^2/8\pi G$. This implies that $\rho_m$ behaves as $H^2 (\rho_\Lambda)$ [8, 7]. It leads to a dust-like equation of state: $\tilde{\omega}_\Lambda^{HH} = 0$. Especially for $c = 1$, one cannot find any information. This means that the equation of state for the holographic energy density with $L_\Lambda = R_{HH}$ cannot be determined within the framework of general relativity. In this case the Brans-Dicke theory is more favorable to fixing the equation of state. Actually it is given not by $\tilde{\omega}_\Lambda^{HH} = 0$ but an decelerating universe with $\tilde{\omega}_\Lambda^{HH} = 5/3$. However, we want to find the accelerating universe with $\omega < -1/3$ and thus this is not the case.

In order to resolve this situation, one is forced to introduce the particle horizon (PH): $L_\Lambda = R_{PH} = a \int_0^t (dt/a) = a \int_0^t (da/Ha^2)$. We assume the power-law solutions of $a(t) = a_0 t^r$, $\phi = \phi_0 t^s$ with $\rho_\Lambda = \frac{3\phi}{8\pi R_{PH}^2}$. In this case, an accelerating solution is possible for $r > 1$. One obtains three relevant equations from Eqs. (8), (9), and (11) after eliminating the pressure $p$ using Eq. (10) as [11]

$$\begin{align*}
(s + 2)r &= \frac{\omega s^2}{6} + 1, \\
(2\omega + 3)(3r + s - 1)rs &= (r - 1)^2(12r + 3s - 6), \\
rs(\omega + 1) - 2r^2 + 3r - \frac{\omega s}{3} &= 1,
\end{align*} \quad (19)$$

$$\quad (20)$$

$$\quad (21)$$

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where the last relation corresponds to the Bianchi identity $[12]$. From $\rho_\Lambda$ the equation of state is given by

$$\tilde{\omega}_\Lambda^{\text{PH}} = -1 - \frac{s - 2}{3r}$$

(22)

Their solutions which satisfy Eqs. (19) and (20) are given by

1. $r = \frac{1}{2}$, $s = 0$, $\omega = \text{arbitrary}$, $\tilde{\omega}_\Lambda^{\text{PH}} = 1/3$

2. $r = -\frac{s}{4} + \frac{1}{2}$, $s = \text{arbitrary}$, $\omega = -\frac{3}{2}$, $\tilde{\omega}_\Lambda^{\text{PH}} = 1/3$

3. $r = \text{arbitrary}$, $s = -3 + \frac{1}{r}$, $\omega = -\frac{6r^3}{(3r - 1)^2}$, $\tilde{\omega}_\Lambda^{\text{PH}} = -1 + \frac{5r - 1}{3r^2}$

4. $r = \text{arbitrary}$, $s = -2r$, $\omega = -\frac{3(2r^2 - 2r + 1)}{2r^2}$

The first three cases satisfy also the Bianchi identity Eq. (21) as a consistency relation. (1) case recovers the general relativistic solution with the equation of state $\tilde{\omega}_{GR} = 1/3$ in the limit of $\omega \to \infty$. (2) corresponds another radiation-dominated solution in the Brans-Dicke theory with the holographic energy density. In the case of (3), one finds the dark energy equation of state $\tilde{\omega}_\Lambda \to -1$ as $r \to \pm \infty$. The case of $r < 0$ is compatible with $\omega \to \infty$ which correspond to the general relativistic case. However, considering $a(t) = a_0 t^r$ with $r < 0$ leads to a contracting universe. Thus this case is not an interesting solution. In the case of $\omega < 0$ ($r > 0$), one finds a negative Brans-Dicke term which does not lead to general relativity in the limit of $r \to \infty$. However, we find an accelerating phase with $\tilde{\omega}_\Lambda^{\text{PH}} < -1/3$ for $r \geq 3$. On the other hand the BD energy density and pressure are given by

$$\rho_{\text{BD}}^{\text{PH}} = -\frac{3(2r - 1)}{8\pi G_0 r^2} H^2, \quad P_{\text{BD}}^{\text{PH}} = \frac{1}{8\pi G_0 r^4} \left( -9r^3 + 14r^2 - 7r + 1 \right) H^2.$$  

(23)

The solution (4) satisfies Eqs. (19) and (20) only but it does not satisfy the Bianchi identity Eq. (20). Hence it is excluded for our purpose.

Finally one introduces the future event horizon (FH): $L_\Lambda = R_{\text{FH}} = a \int_t^\infty (dt/a) = a \int_0^\infty (da/Ha^2)$. This was used in the holographic description of cosmology by Li [7]. Here we assume the power-law solutions of $\phi/\phi_0 = (a/a_0)^\alpha$, $H/H_0 = (a/a_0)^{\beta - 1}$ with $\rho_\Lambda = \frac{3\beta}{8\pi R_{\text{FH}}^3}$. Here we require $\beta \geq 1$ to find a physical solution. In this case, one obtains three relevant equations from Eqs. (8), (9), and (11) after eliminating the pressure $p$ using Eq. (10) [11]

$$\beta^2 = 1 + \alpha - \frac{\omega}{6} \alpha^2,$$  

(24)

$$ (2\omega + 3)(\alpha + \beta + 2)\alpha = 3\beta^2(\alpha + 2\beta + 2),$$  

(25)

$$ \left( \frac{\omega \alpha}{3} - 1 \right) \beta + \frac{\alpha}{3} (2\omega + 3) = \beta^2,$$  

(26)
where the last equation comes from the Bianchi identity\textsuperscript{12}. The equation of state is given by
\[
\tilde{\omega}_A^{\text{FH}} = -1 - \frac{\alpha + 2\beta - 2}{3}.
\] (27)

Their solutions which satisfy Eqs. (24) and (25) are given by

1. \(\alpha = -2(\beta + 1), \beta = \text{arbitrary}, \omega = -3/2, \tilde{\omega}_A^{\text{FH}} = 1/3\)
2. \(\alpha = 0, \beta = -1, \omega = \text{arbitrary}, \tilde{\omega}_A^{\text{FH}} = 1/3\)
3. \(\alpha = (\beta + 2)(\beta - 1), \beta = \text{arbitrary}, \omega = \frac{6}{(\beta + 2)(\beta - 1)}\),
\[
\tilde{\omega}_A^{\text{FH}} = -1 - \frac{(\beta - 1)(\beta + 4)}{3},
\]
4. \(\alpha = -2, \beta = \text{arbitrary}, \omega = -\frac{3}{2}(\beta^2 + 1)\).

The first three cases satisfy also the Bianchi identity Eq. (26) as a consistency relation. The solution (1) is a radiation-dominated phase for the Brans-Dicke theory with the holographic energy density. (2) corresponds another radiation-dominated solution in the Brans-Dicke theory with the holographic energy density. In the case of (3), one finds the dark energy equation of state \(\tilde{\omega}_A^{\text{FH}} \rightarrow -1\) when \(\omega \rightarrow \infty (\beta \rightarrow 1)\). Actually this corresponds to the dark energy solution of the general relativistic case with the future event horizon. On the other hand, the Brans-Dicke energy density and pressure are given by
\[
\rho_{BD}^{\text{FH}} = -\frac{3(\beta^2 - 1)}{8\pi G_0}H^2 \leq 0, \quad P_{BD}^{\text{FH}} = \frac{(\beta - 1)}{8\pi G_0}\left(\beta^3 + 4\beta^2 + 3\beta + 1\right)H^2 \geq 0,
\] (28)
where the equalities hold when \(\omega \rightarrow \infty (\beta \rightarrow 1)\). We emphasize that the Bianchi identity Eq. (11) as a consistency relation plays an important role in the Brans-Dicke theory combined with the holographic energy density. However, (4) case does not satisfy the Bianchi identity and thus it is not the case.

## 4 Discussions

We discuss the role of the holographic energy density with the IR cutoff in the Brans-Dicke theory. It is very interesting to investigate the role of dynamical cosmological constant (holographic energy density) in the dynamical framework (Brans-Dicke theory). This may be considered as an analogy that the role of cosmological constant is usually investigated in the framework of general relativity. In this case, one has to use three equations from Eqs. (8), (9), and (11) after eliminating the pressure \(p\) using Eq. (10). Here Eq. (11) is the Bianchi identity as a consistency relation.

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In the case of the Hubble horizon as the IR cutoff, its equation of state is not fixed in general relativity because of handicap of the framework\cite{6,7}. However, one finds that $\tilde{\omega}_\Lambda^{HH} \to 5/3$ in the $\omega \to \infty$ limit of the Brans-Dicke theory. It represents the correct equation of state for the holographic energy density with the Hubble horizon as the IR cutoff. However, it derives a decelerating universe.

We find that the particle horizon as the IR cutoff could provide two radiation-dominated phases as well as an accelerating phase in the Brans-Dicke theory. But we cannot recover the dark energy with $\tilde{\omega}_\Lambda^{PH} \to -1$ in the general relativistic limit. Hence the particle horizon is not suitable for a candidate of the IR cutoff to obtain the dark energy.

In the case of the future event horizon as the IR cutoff we find a phantom-like equation of state $\tilde{\omega}_\Lambda^{FH} \leq -1$ from the Brans-Dicke theory with holographic energy density. In the limit of $\omega \to \infty$, we immediately find the equation of state $\tilde{\omega}_\Lambda^{FH} \to -1$ for a general relativistic case.

Finally we comment on the role of the Brans-Dicke scalar in view of the holographic energy density. In the case of the future event horizon, we have $\rho_\Lambda > 0$, $p_\Lambda < 0$ while $\rho_{BD} \leq 0$, $p_{BD} \geq 0$. This means that the Brans-Dicke scalar plays a different role when comparing to that of the holographic energy density. Actually the Brans-Dicke scalar slows down the expansion-rate of the holographic energy density as the dynamical cosmological constant \cite{12}. This is because $p_{\text{tot}} = p_\Lambda + p_{BD}$ becomes less negative. As the time goes on, the role of the Brans-Dicke scalar is negligible, while the holographic energy density derive an accelerating universe solely.

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