Generalized Adaptive Smoothing Using Matrix Completion for Traffic State Estimation

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Abstract. The Adaptive Smoothing Method (ASM) is a data-driven approach for traffic state estimation. It interpolates unobserved traffic quantities by smoothing measurements along spatio-temporal directions defined by characteristic traffic wave speeds. The standard ASM consists of a superposition of two a priori estimates weighted by a heuristic weight factor. In this paper, we propose a systematic procedure to calculate the optimal weight factors. We formulate the a priori weights calculation as a constrained matrix completion problem, and efficiently solve it using the Alternating Direction Method of Multipliers (ADMM) algorithm. Our framework allows one to further improve the conventional ASM, which is limited by utilizing only one pair of congested and free flow wave speeds, by considering multiple wave speeds. Our proposed algorithm does not require any field-dependent traffic parameters, thus bypassing frequent field calibrations as required by the conventional ASM. Experiments using NGSIM data show that the proposed ADMM-based estimation incurs lower error than the ASM estimation.

1 INTRODUCTION

Accurate knowledge of road traffic conditions plays a key role in real-time traffic management systems such as traffic lights, vehicle routing, and road performance evaluations \cite{12, 13, 28}. However, field traffic data obtained from the stationary detectors and floating cars, the two most popular traffic data sources, remain sparse in practice. One resorts to estimation techniques to infer missing traffic data from these sparse measurements.

A popular data-driven method for traffic state estimation is the Adaptive Smoothing Method (ASM), originally developed for stationary detector data \cite{23} and then later applied to floating cars data \cite{24}. ASM is an interpolation method based on the simplified kinematic wave theory of traffic flow \cite{16}. It assumes that perturbations in macroscopic traffic propagates forward (driving direction) in free-flow traffic and backward (in the opposite direction of traffic) in congested traffic. Accordingly, the ASM first builds two traffic estimates for congested and free flow traffic using an anisotropic low-pass filters. The final estimate is taken as a convex combination of these free-flow and congested traffic estimates using weights which depend on field traffic conditions.

Despite the simplicity and wide application of ASM, there are some limitations to be addressed. First, the ASM can only accommodate two traffic waves (one forward and one backward). In real-world traffic, a range of forward and backward traffic waves are observed. Second, the ASM weights used for combining free-flow and congested estimates is based on a heuristic formula that requires field calibration. Further, these weights are sensitive to near-capacity traffic conditions and produce inaccurate estimation results.

To address these shortcomings, we pose the weight calculation in the ASM as a constrained optimization problem, namely a kind of matrix completion problem. The proposed frameworks allows one to accommodate multiple traffic waves during estimation. The optimization problem can be efficiently solved using the alternating direction method of multipliers (ADMM). Our proposed method is calibrated with fewer hyperparameters and gives a better estimation error. Moreover, its performance can be further improved by including more a priori estimates, which were originally limited to a single congested and a single...
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non-linear regression functions such as deep neural net-
build parametric/non-parametric machine learning models
are more accurate than the model-based methods. But,
niques better capture the higher order traffic features and
orks or support vector machines, the data-driven tech-
propsition in ASM is not ad-
[23, 24] in the category of structured-learning methods,
prove the interpretability [5, 19, 21, 22, 27]. The physi-
ations from a physical traffic flow model with field mea-
ations from model-based methods are physically rea-
sonable but are often limited by the capacity of traffic flow
model and the filtering assumptions. Data-driven methods
lack of physical interpretability (black box nature) often
limits these data-driven methods from practical applica-
The third group of estimation methods consist of
structured-learning methods, where the data-driven meth-
ods are built that honor traffic physics constraints such
as conservation laws and kinematic wave theory to im-
prove the interpretability [5, 19, 21, 22, 27]. The physical
straints are honored either during the model fitting
stage [5, 19] or infused in the model architecture [8–10, 22].
These methods have shown robust estimation performance
and requires limited data in the function fitting process.

We group the Adaptive Smoothing Method (ASM)
[23, 24] in the category of structured-learning methods,
since the two dimensional interpolation in ASM is not ad-
hoc but takes into account the wave propagation charac-
teristics in free-flow and congested traffic conditions. Dif-
ferent researchers tried to improve and modify the con-
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the weights used for combining a priori traffic estimates of
free-flow and shockwave wave speeds are unchanged and
are based on a heuristic formula. The method proposed
in our paper overcomes this limitation by systematically
deriving the optimal weights for different wave speeds con-
sidered in ASM. The recent study from [22] uses the ASM
anisotropic kernels in designing efficient convolutional neu-
networks for traffic speed estimation, where the wave
speeds and their respective weights are learned from data.

2 Related Works

Traffic state estimation techniques are broadly grouped
into model-based, data-driven learning and structured
learning methods. Model-based methods combine estima-
tions from a physical traffic flow model with field mea-
urements using an exogenous filter [1, 6–8, 17, 29]. The
estimations from model-based methods are physically rea-
sonable but are often limited by the capacity of traffic flow
model and the filtering assumptions. Data-driven methods
build parametric/non-parametric machine learning models
from historic traffic data [2, 9, 11, 14, 15, 26]. Exploiting

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speeds and their respective weights are learned from data.

3 METHODOLOGY

In this section, we first briefly describe the ASM, then we
specify how to compute the weights using ADMM.

3.1 Notation

We denote matrices using uppercase bold Roman (W, Z)
or Greek (Λ) letters. We specify J as the all-ones matrix,
in which every element is equal to one. The symbols ◦
and ⊘ represent the Hadamard product and Hadamard
division, respectively. ||·||_F is the Frobenius norm of a
matrix. ⟨·, ·⟩_F is Frobenius inner product of two matrices.

3.2 Conventional Adaptive Smoothing Method (ASM)

Denote by Z(x, t) the macroscopic traffic speed field with
space and time indices x and t. Note that Z(x, t) can also
analogously denote the traffic flux or density field. Given
a set of traffic speed measurements \{⟨x_n, t_n, v_n⟩\}_n, ASM
first calculates two (a priori) speed fields,

\[ Z^\text{free}(x, t) = \frac{1}{N(x, t)} \sum_n \phi \left( x - x_n, t - t_n - \frac{x - x_n}{c_{\text{free}}} \right) v_n, \]

\[ Z^\text{cong}(x, t) = \frac{1}{N(x, t)} \sum_n \phi \left( x - x_n, t - t_n - \frac{x - x_n}{c_{\text{cong}}} \right) v_n, \]

corresponding to two wave speeds c_{\text{free}} and c_{\text{cong}}, where
N(x, t) is a normalization constant and \phi(·, ·) is a kernel
function (e.g., bi-variate Gaussian).

The ASM defines Z(x, t) using the following convex com-
bination, written in a compact form as

\[ Z = W^\text{cong} \odot Z^\text{cong} + (I - W^\text{cong}) \odot Z^\text{free} \]

with the weight field W^\text{cong}(x, t) ∈ [0, 1] defined as:

\[ W^\text{cong} = \frac{1}{2} \left[ 1 + \tanh \left( \frac{V_{\text{thr}} - \min\{Z^\text{free}, Z^\text{cong}\}}{\Delta V} \right) \right], \]

where V_{\text{thr}} is the threshold speed and ΔV is the transi-
tion width, which depends on field traffic conditions and
requires independent tuning. Note the operators \min{·, ·}
and \tanh(·) in (3) are applied element-wise.

The ASM weight field (3) is based on the observation
that propagating structures in congested traffic are very
persistent, so that it favours the congested estimation if any of the two a priori estimates indicates congested traffic. It is a heuristic formula that might not adapt well to free-flow and congested traffic speeds, and is sensitive at near-capacity traffic conditions. Also, the dependence on field parameters \( V_{th} \) and \( \Delta V \) requires frequent calibration to handle dynamic traffic conditions. Further, (3) is only defined for two traffic wave speeds, which is inadequate when reproducing heterogeneous traffic dynamics (wave speeds can be any convex combination of these two extreme speeds).

To this end, we formulate the weight calculation (3) as a constrained optimization problem that adapts to traffic conditions in the observed data and eliminates the dependence on field parameters. Our framework can also accommodate multiple wave speeds, which allows one to reproduce richer traffic dynamics. This is shown next.

### 3.3 Optimal ASM Weight Calculation

We formulate the weight calculation as the following matrix completion problem:

\[
\begin{aligned}
\text{Minimize} & \quad \| P_\Omega \left( \sum_i W^i \odot Z^i \right) - Z \|_F \\
\text{s.t.} & \quad \sum_i W^i = J,
\end{aligned}
\]

(4)

where \( Z^i \) is the \( i \)th a priori speed field estimate corresponding to a wave speed \( c_i \), determined using (1). The associated weights \( W^i \) are the decision variables of (4). The binary mask operator \( P_\Omega \) evaluates the objective function only at the observed indices \( \Omega \).

For a given set of wave speeds \( c_i \) and priori estimate \( Z^i \), the problem (4) determines the optimal convex coefficients \( W^i \) that results in least norm error. For \( m = 2 \), (4) solves for weights of two wave speeds, say \( c_{\text{free}} \) and \( c_{\text{cong}} \), similar to the ASM setting. Naturally, (4) can easily consider multiple wave speeds \( m > 2 \) in the final speed field estimation instead of just two wave speeds as in the ASM.

### 3.4 Solving (4) using Alternating Direction Method of Multipliers

We apply Alternating Direction Method of Multipliers (ADMM) [3] to solve (4). Using an auxiliary variable \( \hat{Z} \), we first reformulate (4) as:

\[
\begin{aligned}
\text{Minimize} & \quad (W^i)_{m=1}^i, \hat{Z} \quad \| P_\Omega (\hat{Z} - Z) \|_F \\
\text{s.t.} & \quad \hat{Z} = \sum_i W^i \odot Z^i \\
& \quad \sum_i W^i = J
\end{aligned}
\]

(5)

With the introduction of \( \hat{Z} \), we can simplify the objective function that was hard to directly minimize. Based on this reformulation, we write the augmented Lagrangian function of (5) as follows:

\[
L_\beta (\hat{Z}, W^1, ..., W^m, \Lambda_1, \Lambda_2) = \frac{1}{2} \| M \odot (Z - \hat{Z}) \|_F^2 + \beta \frac{1}{2} \| \hat{Z} - \sum_i W^i \odot Z^i \|_F^2 + \langle \Lambda_1, \hat{Z} - \sum_i W^i \odot Z^i \rangle_F + \frac{\beta}{2} \| \hat{Z} - \sum_i W^i \odot Z^i \|_F^2
\]

(6)

where \( \hat{Z}, W^1, ..., W^m \) are the primal variables and \( \Lambda_1, \Lambda_2 \) are the dual variables for (5).

The ADMM then proceeds iteratively by alternatively optimizing the primal variables and dual variables. This is summarized in Algorithm 1.

The primal variable updates in Algorithm 1 can be solved in closed form:

\[
\hat{Z} = (M \odot Z - \Lambda_1 + \beta (\sum_i W^i \odot Z^i)) \odot (M + \beta J)
\]

\[
W^i = (\beta (\hat{Z} \odot Z^i + J - \sum_{r \neq i} W^r \odot (Z^r \odot Z^i + J)) + \Lambda_1 \odot Z^i - \Lambda_1) \odot (\beta (Z^i \odot Z^i + J))
\]

The dual variables are updated using simple gradient ascent. Algorithm 1 terminates when the primal and dual residuals reach their pre-defined feasibility tolerances [3].

### 4 EXPERIMENTAL SETUP

We test the estimation performance of our proposed algorithm using the NGSIM US 101 highway data [25]. The data consists of vehicle trajectories from a road section 670 m in length over a 2700 s time period. This data is converted to a ground-truth macroscopic speed field \( Z \) of dimension \( 67 \times 2700 \). Heatmaps of these data are shown in Fig. 1a and Fig. 2a for 100 – 700 s and and 1400 – 2200 s as two cases. The 100 – 700 s case is a mixture of free flow and congestion while 1400 – 2200 s consists mostly of congested traffic (backward shockwaves). We assume
Algorithm 1: ADMM for solving problem (5)

Data: $Z$
Result: $W^1, ..., W^m$
Initialize: $\hat{Z}, W^1, ..., W^m, \Lambda_1, \Lambda_2$

while stopping criterion not met do

\[
\hat{Z} \leftarrow \arg\min_{\hat{Z}} \left( \frac{1}{2} \|M \odot (Z - \hat{Z})\|_F^2 + \langle \Lambda_1, \hat{Z} - \sum_i W^i \odot Z^i \rangle_F + \frac{\beta}{2} \|Z - \sum_i W^i \odot Z^i\|_F^2 \right)
\]

\[
W^i \leftarrow \arg\min_{W^i} \left( \langle \Lambda_1, \hat{Z} - \sum_i W^i \odot Z^i \rangle_F + \frac{\beta}{2} \|Z - \sum_i W^i \odot Z^i\|_F^2 + \langle \Lambda_2, \sum_i W^i - J \rangle_F + \frac{\beta}{2} \|\sum_i W^i - J\|_F^2 \right)
\]

$\Lambda_1 \leftarrow \Lambda_1 + \beta (\hat{Z} - \sum_i W^i \odot Z^i)$

$\Lambda_2 \leftarrow \Lambda_2 + \beta (\sum_i W^i - J)$

end

that the input measurements come from stationary detector data, though the methodology can also be applied to floating car data. Thus, the set of input indices $\Omega$ consists of the number of rows of $Z$ observed.

We conduct three experiments in this study. In the first experiment, we determine optimal weights for two wave speeds (i.e., $m = 2$), and compare the estimation error of the proposed ADMM-based algorithm and the conventional ASM. We use the wave speeds $c_1 = 80$ km/hr and $c_2 = -15$ km/hr, corresponding to the optimal setting for ASM. The estimation methods are compared for two different time periods to evaluate their performance in free-flowing and congested traffic conditions. The true speed fields and input measurements for these two cases are shown in Fig. 1b and Fig. 2b.

In the second experiment, we evaluate the benefits of the proposed ADMM-based estimation algorithm to incorporate multiple wave speeds (i.e., $m > 2$). We consider the following set of wave speeds: \{-20, -17.5, -15, -12.5, +60, +70, +80, +90\} km/hr. These wave speeds are chosen so as to incorporate a range of traffic waves emanating in free-flowing and congested traffic conditions. In the third experiment, we investigate our method’s performance on different level of input sparsity by changing the detector coverage rate.

The parameters of ASM and ADMM used in our experiments are summarized in Table 1. The spatial and temporal smoothing widths, $\sigma$ and $\tau$, are the parameters of the kernel function $\phi(\cdot, \cdot)$. We note that the choice of $\sigma$ and $\tau$ are sensitive to estimation errors, especially for lower reconstruction window. We chose them as half the average inter-detector spacing and sampling time, respectively, as recommended in [24]. $\beta$ is an ADMM hyperparameter that controls the convergence rate; smaller values of $\beta$ means larger step sizes and faster convergence, but this can cause instability [20]. We use $\beta = 1$ in our experiments.

The quality of the estimated speed field $\hat{Z}$ is measured using the relative error in all the experiments,

$$m_r = \frac{\|Z - \hat{Z}\|_F}{\|Z\|_F}$$
5 Result Analysis and Discussion

The experiment results are discussed below:

Expt 1: Comparison of ASM and ADMM-based
The estimation errors \( m_r \) for different cases shown in Fig. 1 and Fig. 2 are summarized in Table 2. The least error is highlighted in bold. We see that the ADMM estimation gives slightly better performance in both cases, which speaks to the utility of the weights estimation using ADMM (the main difference between ADMM and ASM in this case).

We also show the visualization for both method’s estimation result in Fig. 3 and Fig. 4. Both algorithms successfully captured the ground truth wave propagation dynamics. It is also notable that ASM tends to show a thinner and sharper pattern than ADMM, which is not an accurate representation for ground truth. This is more obvious in the estimation shown in Fig. 4. In free flow region, we see that ADMM’s estimation consist of several minor waves, which is not observed in the ASM’s estimation. Nevertheless, ADMM captured the area size and shape more accurately than ASM.

Expt 2: Multiple a priori estimates Next, we evaluate the ADMM estimation error when considering multiple wave speeds (a priori estimates). We consider the wave speeds: \([-20, 90]; [-17.5, 80]; [-15, 70]; [-12.5, 65]\). To measure the improvement, we run different sub-experiments, where in each experiment, we add one pair of congested and free-flow prior estimates and calculate the our algorithm’s estimation error. For example, we start with two a priori estimates with \( c_{\text{cong}} = -10 \) and \( c_{\text{free}} = 90 \), then add next pair of wave speeds \( c_{\text{cong}} = -17.5 \) and \( c_{\text{free}} = 80 \), and so on. The results are summarized in Fig. 5. The green star marker represents the estimation error of conventional ASM with two a priori estimates for reference. We observe that increasing the number of wave speeds could further reduce reconstruction error as seen in Fig. 5. The choice of wave speeds considered in this experiment is rather heuristic. A more careful choice of wave speeds could lead to even greater improvement in the performance.

Expt 3: Different input coverage rates We test our proposed algorithm’s performance under different detector coverage rates. As shown in Fig. 6, we applied the ADMM-based (with \( m = 2 \)) and ASM algorithms on the case 2 data for different number of detectors (i.e., input

| Parameter  | Value | Description                  |
|------------|-------|------------------------------|
| \( v_{\text{thr}} \) | 60 km/h | Critical traffic speed       |
| \( \Delta v \)  | 20 km/h | Transition width             |
| \( \sigma \)    | \( \Delta x/2 \) | Space coordinate smoothing width|
| \( \tau \)     | \( \Delta t/2 \) | Time coordinate smoothing width|
| \( \beta \)    | 1     | Step-size in ADMM            |

TABLE 1: Parameter Setting

| Parameter | Case 1 | Case 2 |
|-----------|--------|--------|
| ASM       | 0.12417| 0.19843|
| ADMM      | **0.12054** | **0.19637** |

TABLE 2: Estimation errors for Expt 1

FIGURE 3: Comparison of the ADMM-based and ASM estimation results for the Case 1 data (Refer Fig. 1).

FIGURE 4: Comparison of the ADMM-based and ASM estimation results for the Case 2 data (Refer Fig. 2.).
penetration rates). The curves in Fig. 6 show the error rate trend for increasing the input coverage rates from 1\% to 10\% (i.e., number of detectors from 1 to 7). We observe that the errors from both the algorithms decrease with the increase of input information, with ADMM performing better than ASM. But, the difference in error rates are higher when the input information is very limited, i.e., 1\% − 5\%. This indicates that our ADMM-based algorithm extracts more information from the sparse input measurements in comparison to the conventional ASM.

6 CONCLUSIONS

This study improves the conventional Adaptive Smoothing Method (ASM) for traffic state estimation by proposing a systematic procedure to calculate the weights of congested and free-flow traffic speed fields (a priori estimates), which is otherwise done heuristically. We formulate the a priori estimates’ weight calculation as a matrix completion problem, and efficiently solve it using the Alternating Direction Method of Multipliers algorithm. Our algorithm doesn’t depend on field parameters, and can, thus, reduce the effort involved in field calibrations as required in conventional ASM. Our framework allows one to consider a priori speed field estimates corresponding to multiple wave speeds rather than just two wave speeds as done in the conventional ASM. This is advantageous for reproducing richer traffic dynamics, e.g., vehicular traffic flows with wide range of desired speed distributions. Experiments using real traffic data show that our proposed algorithm reduces the estimation error and achieves better performance than the conventional ASM, particularly when using multiple a priori input speed field estimates. The experiments also show reduction in estimation error when the input measurements are sparse.

We observe in our multiple a priori estimates experiments that proper choice of wave speed $c_i$ can enhance the performance of our algorithm’s performance to a great extent. However, the choice of these values can be rather heuristic. We believe that the present approach can be extended to incorporate tuning these parameters in a more effective manner and improve the estimation accuracy further. We also observe that our algorithm tends to estimate high velocity regions more accurately, which is a natural outcome due to the objective function based on Frobenius error minimization, which tends to focus on high magnitude numbers. In the context of traffic estimation where the congested region is of greater interest, one can utilize a different objective function, which emphasizes errors in low-speed traffic or by changing the estimated traffic state variable from velocity to density. We leave this to future research.

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