Sigma models with off-shell $N = (4, 4)$ supersymmetry and noncommuting complex structures

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Abstract

We describe the conditions for extra supersymmetry in $N = (2, 2)$ supersymmetric nonlinear sigma models written in terms of semichiral superfields. We find that some of these models have additional off-shell supersymmetry. The $(4, 4)$ supersymmetry introduces geometrical structures on the target-space which are conveniently described in terms of Yano $f$-structures and Magri-Morosi concomitants. On-shell, we relate the new structures to the known bi-hypercomplex structures.
1 Introduction

The target-space geometry of two-dimensional supersymmetric nonlinear sigma models has been extensively discussed in the literature. In [1], and partly in [2], the general case including a $B$-field was described in $(1, 1)$ superspace. For $(2, 2)$ supersymmetry the target-space geometry was shown to be bihermitean, i.e., the metric is hermitean with respect to two complex structures $J_{(\pm)}$. Off-shell, a manifest $(2, 2)$ formulation was only found when the complex structures commute$^1$. Similar results hold for $(4, 4)$ supersymmetry: A manifest $(2, 2)$ formulation was only found when (some of) the complex structures commute.

$^1$Some other models with off-shell $(2,2)$ supersymmetry were found in [3]–[7].
More recently the hermitian geometry of [1] has been described as generalized Kähler geometry [8], a subclass of generalized complex geometry [9]. The intimate relation of this description to sigma models is elucidated in, e.g., [10]–[14]. In particular, as shown in [15], a complete $(2,2)$ superspace description of generalized Kähler geometry, including the case when the complex structures do not commute, requires semichiral fields [3] in addition to the chiral and twisted chiral fields; this had been conjectured but not proven by Sevrin and Troost [4]. The superspace lagrangian $K$ is further shown to be a potential for the metric and $B$-field [16];

The bi-hypercomplex geometry of [1] has likewise been described as generalized hyperkähler geometry in [17] and [18].

In the present paper, we discuss models written in terms of semichiral fields only. We ask under which conditions such a model can carry $(4,4)$ supersymmetry. A limited class of such models was recently discussed in [19]. There the extra supersymmetry transformations were taken to be linear in the derivatives of the fields, and the target-space was restricted to be four-dimensional. It was found that the target-space must have pseudo-hypercomplex geometry to allow $(4,4)$ supersymmetry.

Some models including semichiral but no chiral or twisted chiral fields had been treated previously in [21]; they include additional auxiliary $(4,4)$ fields, and only become purely semichiral models on-shell.

Models with commuting complex structures, described by $n$ chiral and $m$ twisted chiral fields, have off-shell $(4,4)$ supersymmetry when $n = m$ and the Lagrangian $K$ satisfies certain differential constraints [1]. Purely semichiral models have to have an equal number of left and right semichiral fields [3]. Here we find that for some such models whose Lagrangian again satisfies certain differential constraints, there is an off-shell algebra. This algebra has an interpretation in terms of an integrable Yano $f$-structure on $TM \oplus TM$, the sum of two copies of the tangent bundle of the target-space. We already know from [1] that a sigma model with $(4,4)$ supersymmetry has two quaternion-worth of complex structures, $J^A_{(\pm)}$, living on $TM$ and we find that all of these structures fit together nicely. In particular we resolve the interplay between the various integrability conditions involving Nijenhuis tensors and Magri-Morosi concomitants.

The generalized Kähler potential for those semichiral models that are invariant under the off-shell algebra satisfy a constraint. This is analogous to the $(4,4)$ conditions in [1] which are realized for commuting complex structures by the $N = 4$ twisted chiral multiplet. For a subclass of our models, we can give a geometric interpretation of the condition as a kind of hermiticity condition: a certain tensor is preserved by the $f$-structures.
We follow the method used in previous discussions of additional nonmanifest supersymmetries, e.g., in [1] and [20]. To study the additional symmetries, we make the most general ansatz compatible with the properties of the superfields, and then read off the constraints that follow from closure of the supersymmetry algebra and invariance of the action. The constraints from the algebra are discussed in section 3, the invariance of the action is presented in section 5. Often in these investigations field-equations arise and the algebra only closes on-shell. In section 4 we analyze off-shell closure while postponing the on-shell discussion to section 6.

2 Preliminaries

This section contains background material needed for the discussions in later sections.

The (2, 2) supersymmetry algebra for the covariant derivatives is given by

\[ \{ \mathbb{D}_\pm, \mathbb{D}_\pm \} = i \partial_\pm , \]

and the left and right semichiral fields \( X^{a,a'} \), and left and right anti-semichiral fields \( \bar{X}^{\bar{a},\bar{a}'} \) satisfy

\[ \mathbb{D}_+ X^a = 0 \, , \quad \mathbb{D}_- X^{a'} = 0 \, , \quad \mathbb{D}_+ \bar{X}^{\bar{a}} = 0 \, , \quad \mathbb{D}_- \bar{X}^{\bar{a}'} = 0 . \]

A useful collective notation, often used in previous papers, is \( X^L = (X^a, \bar{X}^{\bar{a}}) \) and \( X^R = (X^{a'}, \bar{X}^{\bar{a}'}) \). When we need a notation for all of the fields we write \( X^i \) with \( i = (L, R) \).

We shall consider the generalized Kähler potential \( K \) and the sigma model it defines through the action

\[ S = \int d^2 \xi d^2 \bar{\xi} K(X^i) . \]

The target-space manifold \( \mathcal{M}^{4d} \) coordinatized by the \( d \) left and \( d \) right semichiral fields (and their conjugates) carries bihermitean geometry. This means that there are two complex structures \( J_{(\pm)} \), a metric \( g \) hermitean with respect to both of these and a closed three form \( H \) such that

\[ J_{(\pm)}^2 = -1 \]
\[ \nabla^{(\pm)} J_{(\pm)} = 0 \, , \quad \Gamma^{(\pm)} = \Gamma^0 \pm \frac{1}{2} g^{-1} H \]
\[ J_{(\pm)}^0 g J_{(\pm)} = g \]
\[ d_+^c \omega_+ + d_-^c \omega_- = 0 \, , \quad H = d_+^c \omega_+ = -d_-^c \omega_- , \]

3
where $\Gamma^0$ is the Levi-Civita connection for the metric $g$ and $d^c_{(\pm)} := J_{(\pm)}(d)$. The expression for $d^c$ becomes most simple in complex coordinates: $d^c = i(\partial - \bar{\partial})$.

In later sections we shall also need the explicit form of the complex structures: They are defined in terms of the matrices \[^{15}\]

\[
K_{LR} := \begin{pmatrix} K_{\alpha\alpha'} & K_{\bar{\alpha}\bar{\alpha}'} \\ K_{\bar{\alpha}\alpha'} & K_{\bar{\alpha}\bar{\alpha}'} \end{pmatrix},
\]

and with $C := [j, K]$, they read

\[
J_{(+)} = \begin{pmatrix} j & 0 \\ K_{RL}^{-1}C_{LL} & K_{RL}^{-1}jK_{LR} \end{pmatrix},
\]

\[
J_{(-)} = \begin{pmatrix} K_{LR}^{-1}jK_{RL} & K_{RL}^{-1}C_{RR} \\ 0 & j \end{pmatrix},
\]

(2.6)

with $j$ denoting a canonical $2d \times 2d$ complex structure

\[
j := \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}.
\]

(2.7)

The description (2.4) applies to bihermitean geometry in general, which may be described using chiral, twisted chiral and semichiral fields \[^{15}\]. A special feature of the case we are interested in here is that, although locally we may always write $H = dB$, for the model with only semichiral fields $B$ is globally defined (away from type change loci \[^{8}\]). For more aspects of the global structure of bihermitean geometry, see \[^{22}\].

The data $(g, B, J_{(\pm)})$ in (2.4) may be packaged as structures on $TM \oplus T^*M$ in the form of generalized Kähler geometry \[^{8}\].

3 Nonmanifest supersymmetries

3.1 Ansatz for non-manifest supersymmetries

Requiring that the derivatives are covariant with respect to the additional supersymmetries, e.g., $\mathcal{D}_+(\delta X^a) = \delta(\mathcal{D}_+ X^a) = 0$, leads to the following general ansatz for $N = (4, 4)$
supersymmetry:

\[
\begin{align*}
\delta X^a &= \epsilon^+ \bar{D}+ f^a(X^{L,R}, \bar{X}^{L,R}) + g^a_b(X^c) \epsilon^- \bar{D}^- X^b + h^a_b(X^c) \epsilon^- \bar{D}^- X^b, \\
\delta \bar{X}^\alpha &= -\epsilon^+ D_+ \bar{f}^\alpha(X^{L,R}, \bar{X}^{L,R}) - g^\alpha_b(\bar{X}^c) \epsilon^- D_+ \bar{X}^b - h^\alpha_b(\bar{X}^c) \epsilon^- D_+ \bar{X}^b, \\
\delta X^{a'} &= \epsilon^+ \bar{D}_- \bar{f}^{a'}(X^{L,R}, \bar{X}^{L,R}) + \bar{g}^{a'}_{b'}(\bar{X}^c') \epsilon^+ D_- X^{b'} + \bar{h}^{a'}_{b'}(\bar{X}^c') \epsilon^+ D_- X^{b'}, \\
\delta \bar{X}^{\alpha'} &= -\epsilon^+ D_- \bar{f}^{\alpha'}(X^{L,R}, \bar{X}^{L,R}) - \bar{g}^{\alpha'}_{b'}(\bar{X}^c') \epsilon^+ D_- \bar{X}^{b'} - \bar{h}^{\alpha'}_{b'}(\bar{X}^c') \epsilon^+ D_- \bar{X}^{b'}, 
\end{align*}
\]

(3.1)

where \( \epsilon^\pm \) are the transformation parameters. This ansatz is covariant under left and right holomorphic transformations, i.e., coordinate transformations of the form^2

\[
X^a \rightarrow X'^a(X^b), \quad \bar{X}^\alpha \rightarrow \bar{X}'^\alpha(\bar{X}^\beta), \\
X^{a'} \rightarrow X'^{a'}(\bar{X}^{b'}), \quad \bar{X}^{\alpha'} \rightarrow \bar{X}'^{\alpha'}(\bar{X}^{\beta'}). 
\]

(3.2)

A useful way of rewriting these nonmanifest transformations introduces the matrices \( U^{(\pm)} \) and \( V^{(\pm)} \) defined as

\[
\begin{align*}
\bar{\delta}^{\pm} X &= \bar{\delta}^{\pm} \begin{pmatrix} X^a \\ \bar{X}^\alpha \\ X^{a'} \\ \bar{X}^{\alpha'} \end{pmatrix} = \bar{\delta}^{\pm} \begin{pmatrix} X^L \\ \bar{X}^R \\ X^{L'} \\ \bar{X}^{R'} \end{pmatrix} = U^{(\pm)} \epsilon^{\pm} \bar{D}_\pm X, \\
\delta^{\pm} X &= V^{(\pm)} \epsilon^{\pm} D_\pm X \quad (3.3)
\end{align*}
\]

where^3

\[
U^{(+)} = \begin{pmatrix}
* & f^a_b & f^a_{b'} & f^a_{b''} \\
* & 0 & 0 & 0 \\
* & 0 & \bar{g}^{a'}_{b'} & 0 \\
* & 0 & 0 & -\bar{h}^{a'}_{b''}
\end{pmatrix}, \quad U^{(-)} = \begin{pmatrix}
g^a_b & 0 & * & 0 \\
0 & -h^a_b & * & 0 \\
\bar{f}^{a'}_{b'} & \bar{f}^{a'}_{b''} & * & \bar{f}^{a'}_{b''} \\
0 & 0 & * & 0
\end{pmatrix} \quad (3.4)
\]

and

\[
V^{(\pm)} = -\begin{pmatrix}
\sigma_1 & 0 \\
0 & \sigma_1
\end{pmatrix} \bar{U}^{(\pm)} \begin{pmatrix}
\sigma_1 & 0 \\
0 & \sigma_1
\end{pmatrix}. \quad (3.5)
\]

Here

\[
\sigma_1 = \begin{pmatrix}
0 & 1 \\
1 & 0
\end{pmatrix}. \quad (3.6)
\]

Note that one column in each of the transformation matrices \( U^{(\pm)} \) and \( V^{(\pm)} \) is arbitrary. For the remainder of the paper, we set the arbitrary entries to zero. Doing so provides us

^2Strictly speaking, these are not the most general left and right holomorphic transformations, as they also preserve the choice of polarization, i.e., the separation into left and right coordinates.

^3The fundamental tensorial objects are defined in (3.1). Additional covariant indices denote partial derivatives, e.g., \( f^a_i := \partial_i f^a \), etc.
with full integrability of the transformation matrices and an interpretation of the off-shell algebra in terms of Yano $f$-structures. The consequences of keeping the arbitrariness is discussed briefly in section 7.

For later use, we introduce the projection operators $P^\pm, \hat{P}^\pm$:

\[
P_+ = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad \hat{P}_+ = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},
\]

\[
P_- = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \hat{P}_- = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}.
\]

(3.7)

### 3.2 Magri-Morosi concomitant

To interpret the expressions we find below, we use the Magri-Morosi concomitant \[23, 24\] defined for two endomorphisms $I$ and $J$ of the tangent bundle $TM$ of a manifold $M$ as

\[
\mathcal{M}(I, J)_{jk}^i := -\mathcal{M}(J, I)_{kj}^i = I_j^i J_{k,l}^j - J_j^i I_{k,l}^j - I_j^i J_{k,j}^j + J_j^i I_{k,j}^j.
\]

(3.8)

This concomitant has previously been used when discussing supersymmetry algebra, e.g., in discussing $(1, 0)$ and $(1, 1)$ formulations of certain $(p, q)$ sigma models in \[25\] and discussing generalized complex geometry for $(2, 2)$ models in \[12\].

The Magri-Morosi concomitant relates to the simultaneous integrability of two structures and is a tensor only when $[I, J] = 0$. More precisely, two commuting complex structures are simultaneously integrable if and only if their Magri-Morosi concomitant vanishes. The part antisymmetric in $j, k$ is the Nijenhuis concomitant $\mathcal{N}(I, J)$; when $I = J$ this becomes the Nijenhuis tensor $\mathcal{N}(I)$. If $\mathcal{N}(I) = 0$, then $I$ is integrable.

Assuming that we have one $I$-connection $\nabla^{(I)}$ and one $J$-connection $\nabla^{(J)}$ differing only in the sign of the torsion $\Gamma^{(I/J)} = \Gamma^{(0)} \pm T$, we can rewrite $\mathcal{M}$ as

\[
\mathcal{M}(I, J)_{jk}^i = I_j^i \nabla_l^{(J)} J_k^j - J_k^j \nabla_l^{(I)} I_j^i - I_j^i \nabla_j^{(J)} J_k^j + J_k^j \nabla_j^{(I)} I_j^i - [I, J]_{m}^i \Gamma_{jk}^{(J)} m \\
= : \hat{\mathcal{M}}(I, J)_{jk} - [I, J]_{m}^i \Gamma_{jk}^{(J)} m.
\]

(3.9)

We shall need this version in section 6.4 below.
Finally, we note that in the special case when $I^i_j$ and $J^i_j$ are curl-free in the lower indices, the concomitant simplifies to
\[
\mathcal{M}(I, J)^i_{jk} = (IJ)^i_{jk} - (IJ)^i_{kj}. \tag{3.10}
\]

### 3.3 Constraints from the supersymmetry algebra

Imposing the left-with-right commutator algebra for the ansatz (3.3) relates the Magri-Morosi concomitant of transformation matrices to the commutator of the same matrices as follows
\[
\begin{align*}
[\bar{\delta}^\pm, \delta^\mp]X^i = 0 & \iff \mathcal{M}(U^{(\pm)}, U^{(\mp)})^i_{jk} \bar{\nabla}_\pm X^j \bar{\nabla}_\mp X^k = [U^{(\pm)}, U^{(\mp)}]^i_m \bar{\nabla}_\pm \bar{\nabla}_\mp X^m, \\
[\bar{\delta}^\pm, \delta^\mp]X^i = 0 & \iff \mathcal{M}(U^{(\pm)}, V^{(\mp)})^i_{jk} \bar{\nabla}_\pm X^j \bar{\nabla}_\mp X^k = [U^{(\pm)}, V^{(\mp)}]^i_m \bar{\nabla}_\pm \bar{\nabla}_\mp X^m. \tag{3.11}
\end{align*}
\]

These relations can be rewritten covariantly using $\bar{\mathcal{M}}$ defined in (3.9) as
\[
\begin{align*}
\bar{\mathcal{M}}(U^{(\pm)}, U^{(\mp)})^i_{jk} \bar{\nabla}_\pm X^j \bar{\nabla}_\mp X^k &= [U^{(\pm)}, U^{(\mp)}]^i_m \bar{\nabla}_\pm X^m + \Gamma_{jk}^{(\mp)} \bar{\nabla}_\pm \bar{\nabla}_\mp X^m, \\
\bar{\mathcal{M}}(U^{(\pm)}, V^{(\mp)})^i_{jk} \bar{\nabla}_\pm X^j \bar{\nabla}_\mp X^k &= [U^{(\pm)}, V^{(\mp)}]^i_m \bar{\nabla}_\pm X^m + \Gamma_{jk}^{(\mp)} \bar{\nabla}_\pm \bar{\nabla}_\mp X^m. \tag{3.12}
\end{align*}
\]

In the last equalities we have identified the pullback of the covariant derivative, for use in the on-shell section. Note that constraints on the semichiral fields imply that some of the equations vanish trivially.

The constraints from the left-with-left and right-with-right part of the algebra involve the Nijenhuis tensor:
\[
[\bar{\delta}^\pm, \delta^\pm]X^i = 0 \iff \mathcal{N}(U^{(\pm)})^i_{jk} \bar{\nabla}_\pm X^j \bar{\nabla}_\pm X^k = 0. \tag{3.13}
\]

Finally, using the algebra (2.1), the commutator $[\delta^\pm, \bar{\delta}^\mp]X^i = i\epsilon^\pm \epsilon^\mp \partial^\pm X^i$ yields
\[
\begin{align*}
\mathcal{M}(U^{(\pm)}, V^{(\mp)})^i_{jk} \bar{\nabla}_\pm X^j \bar{\nabla}_\mp X^k &= \left[(UV)^{i(\pm)}_{j(\mp)} + \delta^i_j \bar{\nabla}_\pm X^k + \delta^i_j \bar{\nabla}_\pm X^k \right]. \tag{3.14}
\end{align*}
\]

### 4 Off-shell interpretation of the algebra constraints

In this section we analyze the constraints found in section 3.3, separating the conditions into algebraically independent parts.
4.1 The conditions for off-shell invariance

Off-shell, $\text{DXDX}$ and $\text{DDX}$ are independent structures and hence both sides in equation (3.11) and (3.14) must vanish independently. This gives the conditions

\[
\mathcal{M}(U^{(\pm)}, U^{(\mp)})^i_{jk} = 0, \quad j \neq a, k \neq a' \\
\mathcal{M}(U^{(\pm)}, V^{(\mp)})^i_{jk} = 0, \quad j \neq a, k \neq \bar{a}' \\
\mathcal{M}(U^{(\pm)}, V^{(\pm)})^i_{jk} = 0, \quad j \neq a, k \neq \bar{a}, \tag{4.1}
\]

and

\[
[U^{(\pm)}, U^{(\mp)}]^i_j = 0, \quad j \neq a, a' \\
[U^{(\pm)}, V^{(\mp)}]^i_j = 0, \quad j \neq a, \bar{a}' , \tag{4.2}
\]

and finally

\[
(UV)^{(+i)}_j = -\delta^i_j, \quad j \neq \bar{a} , \quad (VU)^{(+i)}_j = -\delta^i_j, \quad j \neq a , \tag{4.3}
\]

together with their complex conjugate equations. Setting the arbitrary entries in the transformation matrices to zero sets the undetermined columns in (4.3) to zero,

\[
(UV)^{(+i)}_{\bar{a}} = (VU)^{(+i)}_a = (UV)^{(-i)}_{a'} = (VU)^{(-i)}_{\bar{a}'} = 0 . \tag{4.4}
\]

The constraint (3.13) implies that $U^{(\pm)}$ and $V^{(\pm)}$ are integrable on some subspace. When we impose (4.4), the integrability extends to the full space:

\[
\mathcal{N}(U^{(\pm)})^i_{jk} = 0 . \tag{4.5}
\]

The conditions in (4.1) may be written as in (3.10) plus curl terms;

\[
\mathcal{M}(U^{(\pm)}, V^{(\pm)})^i_{kj} = (UV)^{(+i)}_{j,k} - (UV)^{(+i)}_{k,j} + U^{(+i)}_j V^{(\pm)}_{k,j} - V^{(+i)}_k U^{(\pm)}_{j,k} = 0 . \tag{4.6}
\]

The first two terms vanish due to (4.3). The form of the ansatz (3.4) reveals that most of the third and fourth terms also vanish identically. The remaining ones may be shown to be zero due to (4.3) and the integrability (4.5). As an example of the last statement consider

\[
U^{(+i)}_j V^{(+i)}_{k,l} - V^{(+i)}_k U^{(+i)}_{j,l} = 0 . \tag{4.7}
\]

which is nonvanishing for $j, k = b', d'$ when it becomes

\[
\tilde{h}^{a',c'}_{[a',c']} \tilde{g}_{d'}^{c'} - \tilde{g}_{[a',c']}^{a',c'} \tilde{h}_{d'}^{c'} . \tag{4.8}
\]
A short calculation then shows that this combination is zero due to (4.3)
\[ \tilde{h}_c d' = \tilde{g}_a c' \delta'_{d'} , \]  
and (4.5)
\[ \tilde{g}_c [d', c'], c' - \tilde{g}_a [b', d'], d' = 0 . \]  

In summary, off-shell we find the following algebraic constraints in all sectors not projected out by the semi-chiral constraints:

- The transformation matrices $U^{(\pm)}, V^{(\pm)}$ all commute.
- The products $U^{(\pm)}V^{(\pm)}$ and $V^{(\pm)}U^{(\pm)}$ equal minus one.
- The transformation matrices are all separately integrable.
- The Magri-Morosi concomitant vanishes for all two pairs of the transformation matrices. We showed that some of these, namely the last one in (4.1) relating $U^{(\pm)}$ with $V^{(\pm)}$, follow from the above three constraints.

The zeros in the arbitrary columns of the transformation matrices gives full integrability as in (4.5) and the relations (4.4). This makes the products $U^{(\pm)}V^{(\pm)}$ and $V^{(\pm)}U^{(\pm)}$ act as projection operators and we find a nice geometric interpretation in terms of $f$-structures.

### 4.2 A Yano $f$-structure

The fact that the matrices $U^{(\pm)}$ (and $V^{(\pm)}$) are degenerate and satisfy (4.3) and (4.4),
\[ U^{(+)}V^{(+)} = -\text{diag}(1, 0, 1, 1), \quad V^{(+)}U^{(+)} = -\text{diag}(0, 1, 1, 1), \]
\[ U^{(-)}V^{(-)} = -\text{diag}(1, 1, 1, 0), \quad V^{(-)}U^{(-)} = -\text{diag}(1, 1, 0, 1) \]  

prevents a direct interpretation in terms of complex structures on the tangent space $TM$. We are led to consider endomorphisms on $TM \oplus TM$ and the weaker $f$-structures instead. The following $8d \times 8d$ matrices are $f$-structures in the sense of Yano [26]:
\[ F^{(\pm)} := \begin{pmatrix} 0 & U^{(\pm)} \\ V^{(\pm)} & 0 \end{pmatrix} \implies F^{(\pm)}_a + F^{(\pm)} = 0 . \]  

This follows directly from conditions in (4.3). Moreover, $-F^{(\pm)}_a$ and $1 + F^{(\pm)}$ define integrable distributions, as can be shown using (4.3) and (4.5). More explicitly: Using the projectors (3.7), the conditions (4.11) may be written as
\[ \tilde{P}_\pm = 1 + V^{(\pm)}U^{(\pm)}, \quad P_\pm = 1 + U^{(\pm)}V^{(\pm)} . \]
Then we may define
\[
m_{(\pm)} := 1 + F_{(\pm)}^2 = \begin{pmatrix} P_{\pm} & 0 \\ 0 & \bar{P}_{\pm} \end{pmatrix}, \quad l_{(\pm)} := -F_{(\pm)}^2 = \begin{pmatrix} 1 - P_{\pm} & 0 \\ 0 & 1 - \bar{P}_{\pm} \end{pmatrix}
\] (4.14)

These fulfill
\[
l_{(\pm)} + m_{(\pm)} = 1, \quad l_{(\pm)}^2 = l_{(\pm)}, \quad m_{(\pm)}^2 = m_{(\pm)}, \quad l_{(\pm)}m_{(\pm)} = 0
\] (4.15)

and
\[
F_{(\pm)}l_{(\pm)} = l_{(\pm)}F_{(\pm)} = F_{(\pm)}, \quad m_{(\pm)}F_{(\pm)} = F_{(\pm)}m_{(\pm)} = 0.
\] (4.16)

The operators \(l_{(\pm)}\) and \(m_{(\pm)}\) applied to the tangent space at each point of the manifold are complementary projection operators and define complementary distributions in the sense of Yano: \(\Lambda_{\pm}\), the first fundamental distribution, and \(\Sigma_{\pm}\), the second fundamental distribution, corresponding to \(l_{\pm}\) and \(m_{\pm}\), of dimensions \(6d\) and \(2d\), respectively.

Let \(\mathcal{N}_{F_{(\pm)}}\) denote the Nijenhuis tensor for the \(f\)-structures \(F_{(\pm)}\). By a theorem of Ishihara and Yano [27] we have that
\[\]

i. \(\Lambda_{\pm}\) is integrable iff \(m_{(\pm)}^i \mathcal{N}_{F_{(\pm)}}^{ij} = 0\),

ii. \(\Sigma_{\pm}\) is integrable iff \(\mathcal{N}_{F_{(\pm)}}^{ij} m_{(\pm)}^j m_{(\pm)}^k = 0\).

From the definition of the \(f\)-structures in (4.12), one can derive that these two conditions are fulfilled. Hence, the distributions \(\Lambda_{\pm}\) and \(\Sigma_{\pm}\) are integrable.

5 Invariance of the action

The bihermitean geometry of [1] is derived from the \((1,1)\) sigma model via two requirements: closure of the algebra and invariance of the action. More precisely, the supersymmetry algebra implies the existence of the complex structures, whereas invariance the action implies the bihermiticity of the metric and the covariant constancy of the complex structures. Similarly, for \((4,4)\) supersymmetry, the algebra implies that the transformations are given in terms of left and right hypercomplex structures whereas invariance of the action implies the metric is hermitean with respect to all of these structures and the left and right connections preserve the the left and right structures respectively. When, in later sections, we use the knowledge from [1] in understanding our algebra conditions on-shell we can thus use the existence of a hypercomplex structures freely, but only require them to be covariantly constant if we assume that the action is invariant.
At the manifest \((2,2)\) level the discussion of additional supersymmetries in the model with (anti)chiral fields (the hyperkähler case) follows similar lines \([20]\). Extra supersymmetries lead to new complex structures as part of the conditions for closure of the algebra and invariance of the \((2,2)\) action leads to the requirement that they are covariantly constant and that the metric is hermitean with respect to all of them.

When the complex structures commute and the sigma model is describable in \((2,2)\) superspace using (an equal number of) chiral and twisted chiral superfields, \((4,4)\) supersymmetry comes at the price of extra conditions on the potential \(K\) \([1]\). This is also true for the linear-transformation model in \([19]\). We expect the same to be true here.

The action \((2.3)\) is invariant under the supersymmetry transformations \((3.3)\) provided that
\[
\left(K_i U^{(+)[j]} u^j\right)_{k l} = 0, \quad j, k \neq a, \quad (5.1)
\]
and analogously for \(U^{(-)}\) and \(V^{(±)}\). We can write this out as \((5.1)\) a system of equations for \(K\):

\[
\begin{align*}
K_{a[a} f_{b]}^a & = 0 \\
k_{c[b} f_{a']^c} + k_{e[b} \tilde{g}_{a']^e} & = 0 \\
k_{a[a} f_{b]}^a + k_{e[a} \tilde{h}_{b]}^e & = 0 \\
k_{c[b} f_{a']^c} + k_{e[b} \tilde{g}_{a']^e} + k_{e[a'} \tilde{h}_{b]}^{e'} & = 0 \\
k_{c[b} f_{a']^c} + k_{e[b} \tilde{g}_{a']^e} + k_{e[a'} \tilde{g}_{b]}^{e'} & = 0 \\
k_{c[b} f_{a']^c} - k_{e[b} \tilde{h}_{a']}^{e} - k_{e[a'} \tilde{h}_{b]}^{e'} & = 0,
\end{align*}
\]

plus analogous relations from \(U^{(-)}\) and \(V^{(±)}\).

The conditions \((5.2)\) (or \((5.1)\)) have to be satisfied for the generalized Kähler potential \(K\) to allow \((4,4)\) supersymmetry in a model with noncommuting complex structures whose commutator has empty kernel. In this sense it plays a similar role to the Monge-Ampère equation for models with vanishing torsion.

In the four-dimensional case with linear transformations, it turned out to be possible to solve \((5.2)\), (see \([19]\)) but this is much harder in general. However, when the curl of \(\tilde{g}\) and \(\tilde{h}\) vanish, the condition has an interpretation on \(TM \oplus TM\) much like a hermiticity condition, which we now turn to.

We combine the Hessian \(K_{ij}\) of the Kähler potential into an antisymmetric tensor on \(\mathcal{B}\) on \(TM \oplus TM\) as
\( \mathfrak{B} = \begin{pmatrix} 0 & K \\ -K^t & 0 \end{pmatrix} \). \hspace{1cm} (5.3)

The relation (5.1) can be used to show that off-shell the \( f \)-structures (4.12) preserve \( \mathfrak{B} \) on a subspace projected out by the second fundamental projection operators \( l_{(\pm)} \) defined in (4.16),

\[ l_{(\pm)} \mathfrak{F}_{(\pm)}^t \mathfrak{B} \mathfrak{F}_{(\pm)} l_{(\pm)} = l_{(\pm)} \mathfrak{B} l_{(\pm)}. \hspace{1cm} (5.4) \]

This may be easily verified using (1.3), which implies \( V^t K^t U = -K \) (except for one column and one row).

6 On-shell interpretation of the algebra constraints

In this section we discuss two main issues: How the conditions derived in section 3.3 have a larger set of solutions on-shell, and the relation to the underlying (hermitean) bi-hypercomplex geometry derived in [1]. In spirit the treatment is similar to both the \((1,1)\) discussion in [1] of extended supersymmetry and to the hyperkähler derivation in [20]. In [1] it was found that the left and right complex structures had to commute to get off-shell closure since the algebra gives a term proportional to this commutator times the field-equations. In [20] it was found that field equations as well as conditions from the invariance of the action were needed for closure of the algebra of non-manifest additional supersymmetries.

Below we separate the conclusions we may draw from closure of the algebra only and those where in addition we need invariance of the action.

6.1 On-shell algebra

In this subsection we use a coordinate transformation to derive an explicit relation between the components of the transformation matrices and the underlying hypercomplex structure. The field equations that follow from the action (2.3) are

\[ \bar{D}_+ K_a = 0, \hspace{0.2cm} D_+ K_{\bar{a}} = 0, \hspace{0.2cm} \bar{D}_- K_{a'} = 0, \hspace{0.2cm} D_- K_{\bar{a}'} = 0. \hspace{1cm} (6.1) \]

These imply that on-shell, \( K_a \) is a semichiral superfield on equal footing with \( X^a \); we may change coordinates to a left-holomorphic or right-holomorphic basis with coordinates \( Z^A = \{ X^a, Y_a := K_a \} \) or \( Z^{A'} = \{ X^{a'}, Y_{a'} := K_{a'} \} \), respectively [7]. In the left basis, the \( \delta^+, \bar{\delta}^+ \) transformations become very simple, whereas in the right basis, the \( \delta^-, \bar{\delta}^- \) transformations
become simple. Since $K(\mathbf{X}^a, \mathbf{X}^{a'})$ is the generating function for the transformation between the bases, on-shell it is sufficient to study the transformations that are simple in one particular basis.

The ansatz for the $\delta^+, \bar{\delta}^+$ transformations is simple in the left basis:

$$\delta^+ Z^A = 0, \quad \delta^+ \bar{Z}^A = -\epsilon^+ \mathbb{D}_+ \bar{\mathbf{f}}^\bar{A}, \quad \bar{\delta}^+ Z^A = \epsilon^+ \mathbb{D}_+ \mathbf{f}^A, \quad \bar{\delta}^+ \bar{Z}^\bar{A} = 0. \quad (6.2)$$

Closure of this part of the algebra is very simple; it implies

$$f^A_B \bar{f}^B_C = \delta^A_C, \quad (6.3)$$

and

$$f^A_C [\bar{f}^B_D] = 0, \quad (6.4)$$

where $f^A_B$ again denotes derivation with respect to $\bar{Z}^B$. These are precisely the conditions found in section 10 of [20], and imply that $J^{(1)}(\pm) = \frac{1}{\mathbb{I}} f^a(\mathbf{X}^i) = f^a(\mathbf{X}^a, \bar{\mathbf{X}}^\bar{a}, K_a(\mathbf{X}^i), K_{\bar{a}}(\mathbf{X}^{\bar{i}})). \quad (6.5)$

We still need to impose the $\delta^+ - \delta^-$ part of the algebra and want to compare to the off-shell transformations (3.1). For both of these tasks, we need to go back to the $\mathbf{X}^a, \mathbf{X}^{a'}$ coordinate basis. For illustrative purposes, we focus on $\delta^+$. Comparing (3.1) and (6.2), we immediately find that on-shell

$$f^a(\mathbf{X}^i) = f^a(\mathbf{X}^a, \bar{\mathbf{X}}^\bar{a}, K_a(\mathbf{X}^i), K_{\bar{a}}(\mathbf{X}^{\bar{i}})). \quad (6.7)$$

Off-shell, $f^a$ may differ from $\mathbf{f}^a$ by a factor $\Delta f^a$, which satisfies $\overline{\mathbb{D}}_+(\Delta f^a(\mathbf{X}^a, K_a(\mathbf{X}^i))) = 0$ on-shell. This gives an off-shell ambiguity in $f^a$. We also have (trivially) that $\bar{\delta}^+ \bar{\mathbf{X}}^\bar{a} = 0$.

Next we have

$$\bar{\delta}^+ Y_{\bar{a}} = K_{a\bar{b}} \bar{\delta}^+ \mathbf{X}^\bar{b} + K_{a\bar{R}} \bar{\delta}^+ \mathbf{X}^\bar{R} = 0 \quad (6.8)$$

and

$$\bar{\delta}^+ Y_a := K_{ab} \bar{\delta}^+ \mathbf{X}^b + K_{aR} \bar{\delta}^+ \mathbf{X}^R = \epsilon^+ \mathbb{D}_+ f_a = \epsilon^+ (f_{ab} \mathbb{D}_+ \mathbf{X}^b + f_{aR} \mathbb{D}_+ \mathbf{X}^R), \quad (6.9)$$

13
where \( f_a(X^i) := f_a(X^a, \overline{X}^\alpha, K_a(X^i), K_a(\overline{X}^i)) \). We can rewrite these equations as

\[
K_{LR} \delta^+ \overline{X}^R = \epsilon^+ \left( f_{ab} \overline{D}_+ \overline{X}^b + f_{ar} \overline{D}_+ \overline{X}^R - K_{ab} \overline{D}_+ f^b \right) \tag{6.10},
\]

where the matrix \( K_{LR} \) is defined as in (2.5). Since \( K_{LR} \) is invertible, we can find the on-shell transformations \( \delta^+ \overline{X}^R \). To find the corresponding functions \( \tilde{g}_{f'}^e \) and \( \tilde{h}_{f'}^e \) in (3.1), since we are on-shell, we need to eliminate one type of term, e.g., \( \overline{D}_+ \overline{X}^b \), using the field equations.\(^4\) Then (6.10) becomes

\[
K_{LR} \delta^+ \overline{X}^R = \epsilon^+ \left( - (f_{ac} - K_{ab} f^b_c) (K^{-1})^{cd} K_{dR} + f_{ar} - K_{ab} f^b_R \right) \overline{D}_+ \overline{X}^R \tag{6.11},
\]

and we find

\[
\tilde{g}_{f'}^e = (K^{-1})^{e'a} [f_{a f'} - K_{ab} f^b] - (f_{ac} - K_{ab} f^b_c) (K^{-1})^{cd} K_{d f'} \\
- (K^{-1})^{e'a} [K_{ab} f^b_{f'} - K_{ab} f^b_{c}] (K^{-1})^{cd} K_{d f'} ,
\]

\[
\tilde{h}_{f'}^e = - (K^{-1})^{e'a} [f_{a f'} - K_{ab} f^b] - (f_{ac} - K_{ab} f^b_c) (K^{-1})^{cd} K_{d f'} \\
+ (K^{-1})^{e'a} [K_{ab} f^b_{f'} - K_{ab} f^b_{c}] (K^{-1})^{cd} K_{d f'} \tag{6.12},
\]

as well as the constraints

\[
0 = (K^{-1})^{e'a} [f_{a f'} - K_{ab} f^b] - (f_{ac} - K_{ab} f^b_c) (K^{-1})^{cd} K_{d f'} \\
- (K^{-1})^{e'a} [K_{ab} f^b_{f'} - K_{ab} f^b_{c}] (K^{-1})^{cd} K_{d f'} ,
\]

\[
0 = (K^{-1})^{e'a} [f_{a f'} - K_{ab} f^b] - (f_{ac} - K_{ab} f^b_c) (K^{-1})^{cd} K_{d f'} \\
- (K^{-1})^{e'a} [K_{ab} f^b_{f'} - K_{ab} f^b_{c}] (K^{-1})^{cd} K_{d f'} \tag{6.13},
\]

In a similar way, we can find \( g_{a}^e, h_{a}^e \) as well as their complex conjugates. The full set of relations will now be discussed in the original coordinates \( X^i \).

### 6.2 Closure modulo field-equations and relations from invariance of the action

Though conceptually simple, the final expressions that we found (6.12)–(6.13) are rather involved and complicate the discussion on the on-shell \( [\delta^+, \delta^-] \) algebra. Here we present

\(^4\)We assume that \( K_{ab} \) is invertible, otherwise, we would need to eliminate another type of term, but the net effect would be the same.
an alternative description that uses only $X^i$ coordinates and relates directly to the bi-
hypercomplex geometry of \cite{1}. We start from the ansatz \cite{3.1} and only use the field
equations to show that the conditions from closure of the algebra have more solutions on-
shell. Whereas in the previous subsection discussing the on-shell algebra, it was convenient
to change coordinates, here it turns out to be convenient to change the basis for the
covariant derivatives.

Recall the field equations \cite{6.1}

$$
K_{a_i} \bar{D}_+ X^i = 0, \quad K_{a_i} \bar{D}_- \check{X}^i = 0,
$$

$$
K_{\bar{a}_i} D_+ X^i = 0, \quad K_{\bar{a}_i} \bar{D}_- \check{X}^i = 0.
$$

(6.14)

These equations are first order in spinorial derivatives. To be able to use them to
understand the conditions \cite{3.12} \cite{3.14}, which contain second order spinorial deivatives,
we must differentiate \cite{6.14}. We are then faced with the task of relating the plus/minus
connections to second and third derivatives of the generalized Kähler potential $K$. Since
the metric is a nonlinear function of the Hessian of $K$, this is not easy. Instead we choose to
express the on-shell condition in terms of the complex structures $J(\pm)$ defined in \cite{2.6} and
use $\nabla \!
\check{J}(\pm)J(\pm) = 0$ to relate them to the connections (assuming invariance of the action).

We introduce a real basis for the spinor derivatives:

$$
\mathbb{D}_\pm := \frac{1}{2}(D_\pm - iQ_\pm), \tag{6.15}
$$

then \cite{6.14} becomes\footnote{Note however that we use full (2,2) superfield expressions in, e.g., \cite{6.16}; we can reduce to (1,1)
superspace by restricting to superfields to depend only on half the spinor coordinates.}

$$
Q_+ \check{X}^R = J_{k}(\pm) D_+ X^k,
$$

$$
Q_- X^L = J_{(-)k} D_- X^k
$$

(6.16)

where we have introduced (components of) the complex structures $J(\pm)$ as defined in \section{2}

The semichiral conditions rewritten in terms of the real operators \cite{6.15} and \cite{2.7} read

$$
Q_+ X^L = j D_+ X^L,
$$

$$
Q_- X^R = j D_- X^R.
$$

(6.17)

Combining this with \cite{6.16} and \cite{2.6} we find that on-shell

$$
Q_\pm \check{X} := Q_+ \left( \begin{array}{c} X^L \\ X^R \end{array} \right) = J(\pm) D_\pm \left( \begin{array}{c} X^L \\ X^R \end{array} \right) = J(\pm) D_\pm \check{X},
$$

(6.18)
which using (6.15) implies
\[\mathbb{D}_\pm X^i = \pi_k^{(\pm)i} D_\pm X^k\]
\[\mathbb{D}_\pm X^i = \pi_k^{(\pm)i} D_\pm X^k\]  \hspace{1cm} (6.19)

where we have introduced the projection operator
\[\pi := \frac{1}{2} (1 + iJ)\]  \hspace{1cm} (6.20)
and its complex conjugate.

### 6.3 Relations to bi-hypercomplex geometry

In subsection 6.1 we constructed the bi-hypercomplex structures directly in terms of the transformations of the left and right holomorphic coordinates, and related bi-hypercomplex structures to the f-structures implicitly by constructing the tensors in the ansatz (3.1) in terms of the same transformations. In this subsection we analyze the relation using the real basis; this makes some aspects clearer while complicating others.

From the \(N = (1,1)\) analysis of \(\mathbb{I}\) we know that when the model has \((4,4)\) supersymmetry there exists an \(SU(2)\) worth \(6\) of left and right complex structures \((J^{(1)}_{(\pm)}, J^{(2)}_{(\pm)}, J^{(3)}_{(\pm)})\) on the \(4d\) dimensional space, satisfying the bi-hypercomplex algebra (6.6). We now relate the f-structures to \(J^{(A)}_{(\pm)}\).

The complex structures \(J_{(\pm)}\) are part of the \(SU(2)\) worth of complex structures, and we set \(J^{(3)}_{(\pm)} := J_{(\pm)}\). In the real basis (6.15), the additional supersymmetries take the form
\[
\delta_s X := \delta^\pm X + \tilde{\delta}^\pm X = \frac{1}{2} \left[ (J^{(1)}_{(\pm)} + iJ^{(2)}_{(\pm)}) \epsilon^\pm D_\pm X + (J^{(1)}_{(\pm)} - iJ^{(2)}_{(\pm)}) \bar{\epsilon}^\pm D_\pm X \right],
\]  \hspace{1cm} (6.21)

Identifying (6.21) with (6.3) we deduce that
\[
\frac{1}{2} \left( J^{(1)}_{(\pm)} - iJ^{(2)}_{(\pm)} \right) = U^{(\pm)} \pi^{(\pm)}\]
\[
\frac{1}{2} \left( J^{(1)}_{(\pm)} + iJ^{(2)}_{(\pm)} \right) = V^{(\pm)} \bar{\pi}^{(\pm)}.
\]  \hspace{1cm} (6.22)

This relation implies
\[
(UV)^{(\pm)} \pi^{(\pm)} = -\pi^{(\pm)}
\]
\[
(VU)^{(\pm)} \pi^{(\pm)} = -\bar{\pi}^{(\pm)}.
\]  \hspace{1cm} (6.23)

---

For positive definite metric.
A further consequence of the algebra (6.6) is, e.g., that

\[ U^{(\pm)}_{\pi^{(\pm)}} = \pi^{(\pm)} U_{\pi^{(\pm)}}^{(\pm)} , \quad V^{(\pm)}_{\bar{\pi}^{(\pm)}} = \pi^{(\pm)} V_{\bar{\pi}^{(\pm)}}^{(\pm)} . \]  

(6.24)

On \( TM \oplus TM \) we have that

\[ \frac{1}{2} \begin{pmatrix} 0 & J_{(\pm)}^{(1)} - i J_{(\pm)}^{(2)} \\ J_{(\pm)}^{(1)} + i J_{(\pm)}^{(2)} & 0 \end{pmatrix} = F^{(\pm)} \begin{pmatrix} \pi^{(\pm)} & 0 \\ 0 & \bar{\pi}^{(\pm)} \end{pmatrix} =: F^{(\pm)} \Pi^{(\pm)} , \]  

(6.25)

and the relations (6.24) can be used to show that both sides square to \(-\Pi^{(\pm)}\).

Finally, assuming that the action is invariant we have \( \nabla^{(\pm)} J^{(A)}_{(\pm)} = 0 \), (see (2.4)) which implies that

\[ \nabla^{(\pm)} U^{(\pm)}_{\pi^{(\pm)}} = 0 \]
\[ \nabla^{(\pm)} V^{(\pm)}_{\bar{\pi}^{(\pm)}} = 0 , \]  

(6.26)

The equations (6.22)–(6.25) expresses the relation between the bi-hypercomplex geometry and the extra supersymmetries (3.1). The relation does not seem to be one-to-one since only, e.g., \( U^{(\pm)}_{\pi^{(\pm)}} \) enters. However, the particular form (3.4) of \( U^{(\pm)} \) may be used in combination with the explicit expressions (2.6) of \( J^{(\pm)} \) to show that all of \( U^{(\pm)} \) is in fact determined by \( J^{(A)}_{(\pm)} \). This is evident from the explicit expressions for the components of \( U^{(\pm)} \) in section 6.1.

### 6.4 On-shell interpretation of the constraints.

On-shell, there are more cases when the algebra of the extra supersymmetries close, in analogy to, e.g., models written in terms of (anti)chiral fields. To illustrate the line of argument we first discuss (3.14).

Modulo the curl-part,

\[ U^{(\pm)}_{j} V^{(\pm)}_{i} X_{[k,l]} - V^{(\pm)}_{j} U^{(\pm)}_{i} X_{[k,l]} \]  

(6.27)

we may use (4.6) to rewrite (3.14) as

\[ - \left[ (UV)^{(\pm)}_{j,k} - (UV)^{(\pm)}_{k,j} \right] \tilde{D}_{\pm} X^{j} \tilde{D}_{\pm} X^{k} \]  

(6.28)

\[ + \left[ (UV)^{(\pm)}_{j} + \delta^{j} \right] \tilde{D}_{\pm} D_{\pm} X^{j} + \left[ (UV)^{(\pm)}_{j} + \delta^{j} \right] D_{\pm} \tilde{D}_{\pm} X^{j} = 0 . \]  

(6.29)

Since the LHS is

\[ \tilde{D}_{\pm} \left[ (UV)^{(\pm)}_{j} D_{\pm} X^{j} \right] + D_{\pm} \left[ (UV)^{(\pm)}_{j} \tilde{D}_{\pm} X^{j} \right] + \{D_{\pm}, \tilde{D}_{\pm}\} X^{i} \]  

(6.30)
and we know from (6.19) and (6.23) that on-shell the square brackets become $-D^\pm X^i$ and $-\bar{D}^\pm X^i$ respectively, we see that the LHS vanishes on-shell. It remains to consider the terms in (6.27).

Writing the term out in full, including the derivatives, we have

$$
(U_{j,l}^{\pm}) V_{i}^{\pm} - V_{k,l}^{\pm} U_{j,l}^{\pm} \bar{D}^\pm X^i \bar{D}^\pm X^k
$$

Using the relations (6.22) and (6.24) it is possible to show that one can replace all the $U$’s and $V$’s by, e.g., combinations of $\pi^{(\pm)}$’s and $J^{(1)}_{(\pm)}$ yielding the following expression for the curl-terms:

$$
\left( J_{k}^{(1)} \mathcal{N}(\pi)^{r}_{p} J_{j}^{(1)} \pi^{j}_{p} - J_{k}^{(1)} \mathcal{N}(\pi)^{r}_{p} J_{j}^{(1)} \pi^{j}_{q} + \mathcal{N}(J^{(1)})^{i}_{jk} \pi^{j}_{p} \pi^{k}_{q} \right) D X^p D X^q,
$$

where the ($\pm$)-indices were omitted for clarity. The integrability of the $J^{(A)}_{(\pm)}$’s means that all the Nijenhuis-tensors and thus all of terms in (6.32) vanish. We thus see that on-shell (3.14) implies no new constraints.

Next we consider (3.12). Off-shell we had to set the terms with independent structures separately to zero (4.2). On-shell we find no conditions on the tensors if we also assume invariance of the action.

The RHS of (3.12) is

$$
[U^{(+)} , U^{(-)}]^{i} \nabla^{(-)}_{j} \bar{D}^\pm X^j,
$$

where $\nabla^{(-)}_{j}$ is the pull-back of the minus-covariant derivative $\nabla^{(-)}_{i}$ in the $\bar{D}^\pm$ basis. We want to avoid the off-shell conclusion that the commutator vanishes and observe that the commutator multiplies something that looks like a field equation. However, we have to use (6.18) to see if it actually vanishes on-shell.

In the remainder of this section, we use the conditions that follow from invariance of the action [1], which imply that the metric is hermitean with respect to all the complex structures and the connections $\Gamma^{(\pm)}$ preserve the hypercomplex structures $J_{(\pm)}$: $\nabla^{(\pm)} J_{(\pm)} = 0$. A straightforward calculation shows that

$$
\nabla^{(-)}_{j} \bar{D}^\pm X^i = -\frac{1}{2} \{ \pi^{(-)} , \pi^{(+)} \}^i_{k} \nabla^{(-)}_{j} D^\pm X^k.
$$

---

7 This is equivalent to restricting the holonomy of the connections $\Gamma^{(\pm)}$ to a symplectic group.

8 Here the operator $\nabla^{(-)}_{\pm}$ is the pullback in the $D^\pm$-basis.
To lowest order, the RHS is proportional to the $(1,1)$ field equation. Since it is written in manifest $(2,2)$ form, one may expect that it also vanishes to all orders. In fact, the $(2,2)$ relation
\[
\{Q_+, Q_-\} X^i = 0 ,
\] 
has the on-shell content
\[
[J_{(-)}, J_{(+)}]^i_j \nabla^{(-)}_j D_- X^j = 0 ,
\] 
where again covariant constancy of the complex structures is used. Since the commutator is invertible in a model with only semichiral fields,
\[
\nabla^{(-)}_+ D_- X^j = 0 ,
\] 
and that the RHS of (6.34) vanishes.

Using the connections with skew torsion $T = \pm \frac{1}{2} dB$ we have from the definition (3.9) that the LHS of (3.12) is
\[
\hat{\mathcal{M}}(U^{(+)}, U^{(-)})^{i} j k D_+ X^j D_- X^k = 
\left( U^{(+)}_j \nabla^{(-)}_l U^{(-)}_k - U^{(-)}_j \nabla^{(+)}_l U^{(+)i} - U^{(+)}_i \nabla^{(-)}_j U^{(-)}_k + U^{(-)}_i \nabla^{(+)}_j U^{(+)l} \right) D_+ X^j D_- X^k .
\] 
Given the results for the RHS, the appropriate projections of $\hat{\mathcal{M}}(U^{(+)}, U^{(-)})$ thus have to vanish. However, we know from (6.19) that on-shell
\[
\hat{D}_+ X^j \hat{D}_- X^k = \pi^{(+)}_l D_+ X^l \pi^{(-)}_m D_- X^m ,
\] 
and invoking invariance of the action, we may use (6.26) to conclude that then indeed $\hat{\mathcal{M}}(U^{(+)}, U^{(-)}) = 0$.

In summary our result is very similar to the hyperkähler discussion in [20], we need to invoke invariance of the action to show that there are more solutions on-shell to the conditions from the algebra.

The only constraints we get on the transformation matrices on-shell for invariant actions are the integrability condition
\[
\mathcal{N}(U^{(+)})^{i} j k \pi^{(+)i}_l \pi^{(+)j}_m D_\pm X^l D_\pm X^m = 0 .
\] 
(together with the identification (6.22).

\[\footnote{In hyperkähler case the the algebra only closes on-shell.}\]
7 Discussion

Throughout this paper, the arbitrary entries in the transformation matrices $U^{(\pm)}$ (and $V^{(\pm)}$) were set to zero. Off-shell, this has the advantages of yielding geometric structures on the full target-space. Keeping the arbitrariness would restrict the features (e.g., integrability) of these structures to certain subspaces.

We have identified new geometric structures on the target-space of sigma models written in terms of semichiral fields. These structures arise when we study additional off-shell supersymmetries. We have discussed the $f$-structures as living on the sum of two copies of the tangent bundle $TM \oplus TM$. Clearly one would like to identify the relation to generalized complex geometry on $TM \oplus T^*M$. Formally, this may be achieved using the existence of a metric [15]

$$g = \Omega[J_{(+)}, J_{(-)}],$$

(7.1)

where

$$\Omega := \begin{pmatrix}
0 & 2iK_{LR} \\
-2iK_{RL} & 0
\end{pmatrix}.$$  

(7.2)

We use $g$ to relate $TM$ and $T^*M$ to write $\mathcal{F}$ as an $f$-structure on $TM \oplus T^*M$:

$$\tilde{\mathcal{F}} := \begin{pmatrix}
0 & U^{-1}g^{-1} \\
gV & 0
\end{pmatrix}.$$  

(7.3)

We plan to return to the geometry of $f$-structures in the context of generalized complex geometry in a later publication.

A related question concerns the condition for invariance of the action. As we have shown for a subclass of our transformations, this amounts to the conservation of an antisymmetric tensor $\mathcal{B}$ on certain subspaces of $TM \oplus TM$ by the $f$-structures. Again, the corresponding object on $TM \oplus T^*M$ can be found using the metric $g$:

$$\tilde{\mathcal{B}} = \begin{pmatrix}
0 & Kg^{-1} \\
-gK^t & 0
\end{pmatrix}.$$  

(7.4)

It remains to clarify where this object fits into the generalized complex picture. This also ties in with the question of how the conditions for invariance that we have described relate to those found in [21], where (4, 4) models with auxiliary fields are discussed.

In the precursor to this article [19] where the nonmanifest transformations were linear, the four-dimensional target-space was seen to carry indefinite signature metric and
vanishing three form $H$, the geometry being pseudo-hyperkähler. In the preceding discussion this case does not show up. This can be traced back to the identification of the extra transformations with those generated by the $J(A)$’s that form an $SU(2)$ algebra. To relate to the results in [19] we would have to consider an $SL(2,\mathbb{R})$ algebra instead (and restrict to four dimensions). This changes the on-shell discussion in sections 6.3 and 6.4 drastically. For linear transformations and four-dimensional target-space this is, however, the only option.

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Sigma models with off-shell $N = (4, 4)$ supersymmetry and noncommuting complex structures

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Abstract

We describe the conditions for extra supersymmetry in $N = (2, 2)$ supersymmetric nonlinear sigma models written in terms of semichiral superfields. We find that some of these models have additional off-shell supersymmetry. The $(4, 4)$ supersymmetry introduces geometrical structures on the target-space which are conveniently described in terms of Yano $f$-structures and Magri-Morosi concomitants. On-shell, we relate the new structures to the known bi-hypercomplex structures.
1 Introduction

The target-space geometry of two-dimensional supersymmetric nonlinear sigma models has been extensively discussed in the literature. In [1], and partly in [2], the general case including a $B$-field was described in (1, 1) superspace. For (2, 2) supersymmetry the target-space geometry was shown to be bihermitean, i.e., the metric is hermitean with respect to two complex structures $J_{(\pm)}$. Off-shell, a manifest (2, 2) formulation was only found when the complex structures commute.\footnote{Some other models with off-shell (2,2) supersymmetry were found in [3–7].} Similar results hold for (4, 4) supersymmetry: A manifest (2, 2) formulation was only found when (some of) the complex structures commute.
More recently the bihermitean geometry of [1] has been described as generalized Kähler geometry [8], a subclass of generalized complex geometry [9]. The intimate relation of this description to sigma models is elucidated in, e.g., [10]–[14]. In particular, as shown in [15], a complete $(2, 2)$ superspace description of generalized Kähler geometry, including the case when the complex structures do not commute, requires semichiral fields [3] in addition to the chiral and twisted chiral fields; this had been conjectured but not proven by Sevrin and Troost [4]. The superspace lagrangian $K$ is further shown to be a potential for the metric and $B$-field [16];

The bi-hypercomplex geometry of [1] has likewise been described as generalized hyperkähler geometry in [17] and [18].

In the present paper, we discuss models written in terms of semichiral fields only. We ask under which conditions such a model can carry $(4, 4)$ supersymmetry. A limited class of such models was recently discussed in [19]. There the extra transformations were taken to be linear in the derivatives of the fields, and the target-space was restricted to be four-dimensional. It was found that no interesting solution for $N = (4, 4)$ supersymmetry exists, but instead one can find an interesting solution for $N = (4, 4)$ twisted supersymmetry. This implied that the target-space must have pseudo-hypercomplex geometry.

Some models including semichiral but no chiral or twisted chiral fields had been treated previously in [21]; they include additional auxiliary $(4, 4)$ fields, and only become purely semichiral models on-shell.

Models with commuting complex structures, described by $n$ chiral and $m$ twisted chiral fields, have off-shell $(4, 4)$ supersymmetry when $n = m$ and the Lagrangian $K$ satisfies certain differential constraints [1]. Purely semichiral models have to have an equal number of left and right semichiral fields [3]. Here we find that for some such models whose Lagrangian again satisfies certain differential constraints, there is an off-shell algebra. This algebra has an interpretation in terms of an integrable Yano $f$-structure on $TM \oplus TM$, the sum of two copies of the tangent bundle of the target-space. We already know from [1] that a sigma model with $(4, 4)$ supersymmetry has two quaternion-worth of complex structures, $J^A(\pm)$, living on $TM$ and we find that all of these structures fit together nicely. In particular we resolve the interplay between the various integrability conditions involving Nijenhuis tensors and Magri-Morosi concomitants.

The generalized Kähler potential for those semichiral models that are invariant under the off-shell algebra satisfy a constraint. This is analogous to the $(4, 4)$ conditions in [1] which are realized for commuting complex structures by the $N = 4$ twisted chiral multiplet. For a subclass of our models, we can give a geometric interpretation of the condition as a
kind of hermiticity condition: a certain tensor is preserved by the $f$-structures.

We follow the method used in previous discussions of additional nonmanifest super-symmetries, e.g., in [1] and [20]. To study the additional symmetries, we make the most general ansatz compatible with the properties of the superfields, and then read off the constraints that follow from closure of the supersymmetry algebra and invariance of the action. The constraints from the algebra are discussed in section 3, the invariance of the action is presented in section 5. Often in these investigations field-equations arise and the algebra only closes on-shell. In section 4 we analyze off-shell closure while postponing the on-shell discussion to section 6.

2 Preliminaries

This section contains background material needed for the discussions in later sections.

The $(2,2)$ supersymmetry algebra for the covariant derivatives is given by

$$\{\mathcal{D}_\pm, \bar{\mathcal{D}}_\pm\} = i\partial_{\mp}, \quad (2.1)$$

and the left and right semichiral fields $X^{a,a'}$, and left and right anti-semichiral fields $\bar{X}^{\bar{a},\bar{a}'}$ satisfy

$$\bar{\mathcal{D}}_+ X^a = 0, \quad \mathcal{D}_- X^{a'} = 0, \quad \mathcal{D}_+ \bar{X}^{\bar{a}} = 0, \quad \bar{\mathcal{D}}_- \bar{X}^{\bar{a}'} = 0. \quad (2.2)$$

A useful collective notation, often used in previous papers, is $X^L = (X^a, \bar{X}^{\bar{a}})$ and $X^R = (X^{a'}, \bar{X}^{\bar{a}'})$. When we need a notation for all of the fields we write $X^i$ with $i = (L, R)$.

We shall consider the generalized Kähler potential $K$ and the sigma model it defines through the action

$$S = \int d^2\xi \mathcal{D}^2 \bar{\mathcal{D}}^2 K(X^i). \quad (2.3)$$

The target-space manifold $\mathcal{M}^{4d}$ coordinatized by the $d$ left and $d$ right semichiral fields (and their conjugates) carries bihermitean geometry. This means that there are two complex structures $J_{(\pm)}$, a metric $g$ hermitean with respect to both of these and a closed three form $H$ such that [1]

$$J^2_{(\pm)} = -\mathbb{1},$$

$$\nabla^{(\pm)} J_{(\pm)} = 0, \quad \Gamma^{(\pm)} = \Gamma^0 \pm \frac{1}{2} g^{-1} H,$nabla^{(\pm)} J_{(\pm)} = 0, \quad \Gamma^{(\pm)} = \Gamma^0 \pm \frac{1}{2} g^{-1} H,$$

$$J_{(\pm)}^* g J_{(\pm)} = g,$$

$$d^c_+ \omega_+ + d^c_- \omega_- = 0, \quad H = d^c_+ \omega_+ = -d^c_- \omega_-, \quad (2.4)$$
where $\Gamma^0$ is the Levi-Civita connection for the metric $g$ and $d^c (\pm) := J_{(\pm)}(d)$. The expression for $d^c$ becomes most simple in complex coordinates: $d^c = i(\bar{\partial} - \partial)$.

In later sections we shall also need the explicit form of the complex structures: They are defined in terms of the matrices $[15]$

$$K_{LR} := \begin{pmatrix} K_{aa'} & K_{a\bar{a}} \\ K_{\bar{a}a'} & K_{\bar{a}\bar{a}} \end{pmatrix},$$

and with $C := [j, K]$, they read

$$J_{(+)} = \begin{pmatrix} j & 0 \\ K_{RL}^{-1}C_{LL} & K_{RL}^{-1}jK_{LR} \end{pmatrix}$$

$$J_{(-)} = \begin{pmatrix} K_{LR}^{-1}jK_{RL} & K_{RL}^{-1}C_{RR} \\ 0 & j \end{pmatrix}$$

with $j$ denoting a canonical $2d \times 2d$ complex structure

$$j := \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}.$$  \hspace{1cm} (2.7)

The description (2.4) applies to bihermitean geometry in general, which may be described using chiral, twisted chiral and semichiral fields $[15]$. A special feature of the case we are interested in here is that, although locally we may always write $H = dB$, for the model with only semichiral fields $B$ is globally defined (away from type change loci $[8]$). For more aspects of the global structure of bihermitean geometry, see $[22]$.

The data $(g, B, J_{(\pm)})$ in (2.4) may be packaged as structures on $TM \oplus T^*M$ in the form of generalized Kähler geometry $[8]$.

### 3 Nonmanifest supersymmetries

#### 3.1 Ansatz for non-manifest supersymmetries

Requiring that the derivatives are covariant with respect to the additional supersymmetries, e.g., $\mathcal{D}_+(\delta X^a) = \delta(\mathcal{D}_+ X^a) = 0$, leads to the following general ansatz for $N = (4,4)$
Note that one column in each of the transformation matrices \( U \) for the remainder of the paper, we set the arbitrary entries to zero. Doing so provides us with a useful way of rewriting these nonmanifest transformations introduces the matrices \( U^{(\pm)} \) and \( V^{(\pm)} \) defined as

\[
\tilde{\delta}^{\pm} X := \tilde{\delta}^{\pm} \begin{pmatrix} X^a \\ \bar{X}^\ad \\ X'^a \\ \bar{X}'^\ad \end{pmatrix} = \tilde{\delta}^{\pm} \begin{pmatrix} X^L \\ \bar{X}^\Ld \\ X'^L \\ \bar{X}'^\Ld \end{pmatrix} = U^{(\pm)} \epsilon^{\pm} \bar{D}_\pm X, \quad \delta^{\pm} X = V^{(\pm)} \epsilon^{\pm} D_\pm X
\]  

(3.3)

where

\[
U^{(+)} = \begin{pmatrix} * & f^a_b & f^a_\bar{b} & f^a_\bar{\bar{b}} \\ * & 0 & 0 & 0 \\ * & 0 & \bar{g}^{\ad'}_b & 0 \\ * & 0 & 0 & \bar{h}^{\ad'}_\bar{b} \end{pmatrix}, \quad U^{(-)} = \begin{pmatrix} g^a_b & 0 & * & 0 \\ 0 & h^a_\bar{b} & * & 0 \\ \bar{f}^{\ad'}_b & \bar{f}^{\ad'}_\bar{b} & * & \bar{f}^{\ad'}_{\bar{\bar{b}}} \\ 0 & 0 & * & 0 \end{pmatrix}
\]  

(3.4)

and

\[
V^{(\pm)} = \begin{pmatrix} \sigma_1 & 0 \\ 0 & \sigma_1 \end{pmatrix} \tilde{U}^{(\pm)} \begin{pmatrix} \sigma_1 & 0 \\ 0 & \sigma_1 \end{pmatrix}. \quad (3.5)
\]

Here

\[
\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \text{.} \quad (3.6)
\]

Note that one column in each of the transformation matrices \( U^{(\pm)} \) and \( V^{(\pm)} \) is arbitrary. For the remainder of the paper, we set the arbitrary entries to zero. Doing so provides us with

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2Strictly speaking, these are not the most general left and right holomorphic transformations, as they also preserve the choice of polarization, i.e., the separation into left and right coordinates.

3The fundamental tensorial objects are defined in (3.1). Additional covariant indices denote partial derivatives, e.g., \( f^a_i := \partial_i f^a \), etc.
with full integrability of the transformation matrices and an interpretation of the off-shell algebra in terms of Yano $f$-structures. The consequences of keeping the arbitrariness is discussed briefly in section 7.

For later use, we introduce the projection operators $P^\pm, \hat{P}^\pm$:

$$
P_+ = \begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}, \quad \hat{P}_+ = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix},
$$

$$
P_- = \begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}, \quad \hat{P}_- = \begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}.
$$

(3.7)

3.2 Magri-Morosi concomitant

To interpret the expressions we find below, we use the Magri-Morosi concomitant \cite{23, 24} defined for two endomorphisms $I$ and $J$ of the tangent bundle $TM$ of a manifold $M$ as

$$
\mathcal{M}(I, J)^i_{jk} := -\mathcal{M}(J, I)^i_{kj} = I^l_j J^i_k + J^i_k I^l_j - I^l_i J^j_k + J^j_i I^l_k.
$$

(3.8)

This concomitant has previously been used when discussing supersymmetry algebra, e.g., in discussing $(1, 0)$ and $(1, 1)$ formulations of certain $(p, q)$ sigma models in \cite{25} and discussing generalized complex geometry for $(2, 2)$ models in \cite{12}.

The Magri-Morosi concomitant relates to the simultaneous integrability of two structures and is a tensor only when $[I, J] = 0$. More precisely, two commuting complex structures are simultaneously integrable if and only if their Magri-Morosi concomitant vanishes. The part antisymmetric in $j, k$ is the Nijenhuis concomitant $\mathcal{N}(I, J)$; when $I = J$ this becomes the Nijenhuis tensor $\mathcal{N}(I)$. If $\mathcal{N}(I) = 0$, then $I$ is integrable.

Assuming that we have one $I$-connection $\nabla^{(I)}$ and one $J$-connection $\nabla^{(J)}$ differing only in the sign of the torsion $\Gamma^{(I/J)} = \Gamma^{(0)} \pm T$, we can rewrite $\mathcal{M}$ as

$$
\mathcal{M}(I, J)^i_{jk} =\Gamma^{(I/J)} - \mathcal{M}(I, J)^i_{jk} = - \mathcal{M}(J, I)^i_{kj} = I^l_j J^i_k - J^i_k I^l_j - I^l_i J^j_k + J^j_i I^l_k - [I, J]^i_{jm} \Gamma^{(J)}_{jk} m.
$$

(3.9)

We shall need this version in section 6.4 below.
These relations can be rewritten covariantly using \( \hat{\mathcal{M}} \) into algebraically independent parts.

In this section we analyze the constraints found in section 3.3, separating the conditions

\[ \delta \text{indices, the concomitant simplifies to} \]

\[ \mathcal{M}(I, J)_{jk} = (JI)^i_{jk} - (IJ)^i_{k,j} . \]  

(3.10)

### 3.3 Constraints from the supersymmetry algebra

Imposing the left-with-right commutator algebra for the ansatz (3.3) relates the Magri-Morosi concomitant of transformation matrices to the commutator of the same matrices as follows

\[ [\tilde{\delta}^\pm, \delta^\mp] X^i = 0 \iff \mathcal{M}(U^{(\pm)}, U^{(\mp)})_{jk} \bar{D}_\pm X^j \bar{D}_\mp X^k = [U^{(\pm)}, U^{(\mp)}]^m_{\ i} \bar{D}_\pm \bar{D}_\mp X^m , \]

\[ [\bar{\delta}^\pm, \delta^\mp] X^i = 0 \iff \mathcal{M}(U^{(\pm)}, V^{(\mp)})_{jk} \bar{D}_\pm X^j \bar{D}_\mp X^k = [U^{(\pm)}, V^{(\mp)}]^m_{\ i} \bar{D}_\pm \bar{D}_\mp X^m . \]  

(3.11)

These relations can be rewritten covariantly using \( \hat{\mathcal{M}} \) defined in (3.9) as

\[ \hat{\mathcal{M}}(U^{(\pm)}, U^{(\mp)})_{jk} \bar{D}_\pm X^j \bar{D}_\mp X^k = [U^{(\pm)}, U^{(\mp)}]^m_{\ i} \left( \bar{D}_\pm \bar{D}_\mp X^m + \Gamma_{jk}^{(\pm)} m \bar{D}_\pm X^j \bar{D}_\mp X^k \right) \]

\[ \hat{\mathcal{M}}(U^{(\pm)}, V^{(\mp)})_{jk} \bar{D}_\pm X^j \bar{D}_\mp X^k = [U^{(\pm)}, V^{(\mp)}]^m_{\ i} \left( \bar{D}_\pm \bar{D}_\mp X^m + \Gamma_{jk}^{(\mp)} m \bar{D}_\pm X^j \bar{D}_\mp X^k \right) \]

\[ \hat{\mathcal{M}} = [U^{(\pm)}, \bar{D}_\pm X^m \bar{D}_\mp X^m] . \]  

(3.12)

In the last equalities we have identified the pullback of the covariant derivative, for use in the on-shell section. Note that constraints on the semichiral fields imply that some of the equations vanish trivially.

The constraints from the left-with-left and right-with-right part of the algebra involve the Nijenhuis tensor:

\[ [\bar{\delta}^\pm, \delta^\mp] X^i = 0 \iff \mathcal{N}(U^{(\pm)})_{jk} \bar{D}_\pm X^j \bar{D}_\mp X^k = 0 . \]  

(3.13)

Finally, using the algebra (2.1), the commutator \( [\delta^\pm, \bar{\delta}^\mp] X^i = i \epsilon^\pm \epsilon^\mp \partial_{\mp} X^i \) yields

\[ \mathcal{M}(U^{(\pm)}, V^{(\pm)})_{jk} \bar{D}_\pm X^j \bar{D}_\mp X^k = \left[ (UV)^{(\pm)} i_j \bar{D}_\pm \bar{D}_\mp X^j \right] + \left[ (VU)^{(\pm)} i_j \bar{D}_\pm \bar{D}_\mp X^j \right] . \]  

(3.14)

### 4 Off-shell interpretation of the algebra constraints

In this section we analyze the constraints found in section 3.3, separating the conditions into algebraically independent parts.
4.1 The conditions for off-shell invariance

Off-shell, $DXDX$ and $DDX$ are independent structures and hence both sides in equation (3.11) and (3.14) must vanish independently. This gives the conditions

$$\mathcal{M}(U^{(+)}, U^{(-)})^i_{jk} = 0, \quad j \neq a, \; k \neq a'$$

$$\mathcal{M}(U^{(+)}, V^{(-)})^i_{jk} = 0, \quad j \neq a, \; k \neq a'$$

$$\mathcal{M}(U^{(+)}, V^{(+)})^i_{jk} = 0, \quad j \neq a, \; k \neq \bar{a}, \quad (4.1)$$

and

$$[U^{(+)}, U^{(-)}]^i_j = 0, \quad j \neq a, a'$$

$$[U^{(+)}, V^{(-)}]^i_j = 0, \quad j \neq a, \bar{a}', \quad (4.2)$$

and finally

$$(UV)^{(+)i} = \delta^i_j \quad j \neq \bar{a}, \quad (VU)^{(+)i} = -\delta^i_j, \quad j \neq a, \quad (4.3)$$

together with their complex conjugate equations. Setting the arbitrary entries in the transformation matrices to zero sets the undetermined columns in (4.3) to zero,

$$(UV)^{(+)i}_{\bar{a}} = (VU)^{(+)i}_a = (UV)^{(-)i}_{a'} = (VU)^{(-)i}_{\bar{a}'} = 0 \quad (4.4)$$

The constraint (3.13) implies that $U^{(\pm)}$ and $V^{(\pm)}$ are integrable on some subspace. When we impose (4.4), the integrability extends to the full space:

$$\mathcal{N}(U^{(\pm)})^i_{jk} = 0 \quad (4.5)$$

The conditions in (4.1) may be written as in (3.10) plus curl terms;

$$\mathcal{M}(U^{(\pm)}, V^{(\pm)})^i_{kj} = (UV)^{(+)i}_{j,k} - (UV)^{(+)i}_{k,j} + U^{(\pm)l}_{j} V^{(\pm)j}_{[k,l]} - V^{(\pm)l}_{j} U^{(\pm)j}_{[k,l]} = 0 \quad (4.6)$$

The first two terms vanish due to (4.3). The form of the ansatz (3.4) reveals that most of the third and fourth terms also vanish identically. The remaining ones may be shown to be zero due to (4.3) and the integrability (4.5). As an example of the last statement consider

$$U^{(+)l}_{j} V^{(+)a'}_{[k,l]} - V^{(+)l}_{k} U^{(+)a'}_{[j,l]} = 0 \quad (4.7)$$

which is nonvanishing for $j, k = b', d'$ when it becomes

$$\tilde{h}^{a'}_{[b', c']} \tilde{g}^{c'}_{d'} - \tilde{g}^{a'}_{[d', c']} \tilde{h}^{c'}_{b'} \quad (4.8)$$
A short calculation then shows that this combination is zero due to (4.3)
\[ \tilde{h}_{\tilde{c}'} \tilde{g}_{\tilde{c}'} = \tilde{g}_{\tilde{c}'} \tilde{h}_{\tilde{c}'} = -\delta_{\tilde{d}'}, \quad (4.9) \]
and (4.5)
\[ \tilde{g}_{\tilde{d}'} \tilde{g}_{\tilde{c}'} - \tilde{g}_{\tilde{d}'} \tilde{g}_{\tilde{c}'} = 0. \quad (4.10) \]

In summary, off-shell we find the following algebraic constraints in all sectors not projected out by the semi-chiral constraints:

- The transformation matrices $U^{(\pm)}, V^{(\pm)}$ all commute.
- The products $U^{(\pm)} V^{(\pm)}$ and $V^{(\pm)} U^{(\pm)}$ equal minus one.
- The transformation matrices are all separately integrable.
- The Magri-Morosi concomitant vanishes for all two pairs of the transformation matrices. We showed that some of these, namely the last one in (4.1) relating $U^{(\pm)}$ with $V^{(\pm)}$, follow from the above three constraints.

The zeros in the arbitrary columns of the transformation matrices gives full integrability as in (4.5) and the relations (4.4). This makes the products $U^{(\pm)} V^{(\pm)}$ and $V^{(\pm)} U^{(\pm)}$ act as projection operators and we find a nice geometric interpretation in terms of $f$-structures.

### 4.2 A Yano $f$-structure

The fact that the matrices $U^{(\pm)}$ (and $V^{(\pm)}$) are degenerate and satisfy (4.3) and (4.4),
\[ U^{(+)} V^{(+)} = -\text{diag}(1,0,1,1), \quad V^{(+)} U^{(+)} = -\text{diag}(0,1,1,1), \]
\[ U^{(-)} V^{(-)} = -\text{diag}(1,1,1,0), \quad V^{(-)} U^{(-)} = -\text{diag}(1,1,0,1) \quad (4.11) \]
prevents a direct interpretation in terms of complex structures on the tangent space $TM$. We are led to consider endomorphisms on $TM \oplus TM$ and the weaker $f$-structures instead.

The following $8d \times 8d$ matrices are $f$-structures in the sense of Yano [26]:
\[ F^{(\pm)} := \begin{pmatrix} 0 & U^{(\pm)} \\ V^{(\pm)} & 0 \end{pmatrix} \quad \Rightarrow \quad F_{(\pm)}^2 + F_{(\pm)} = 0. \quad (4.12) \]

This follows directly from conditions in (4.3). Moreover, $-F_{(\pm)}^2$ and $1 + F_{(\pm)}^2$ define integrable distributions, as can be shown using (4.3) and (4.5). More explicitly: Using the projectors (3.7), the conditions (4.11) may be written as
\[ \hat{P}_{\pm} = 1 + V^{(\pm)} U^{(\pm)} , \quad P_{\pm} = 1 + U^{(\pm)} V^{(\pm)}. \quad (4.13) \]
Then we may define

\[ m(\pm) := 1 + \mathcal{F}^2(\pm) = \begin{pmatrix} P_\pm & 0 \\ 0 & \bar{P}_\pm \end{pmatrix}, \quad l(\pm) := -\mathcal{F}^2(\pm) = \begin{pmatrix} 1 - P_\pm & 0 \\ 0 & 1 - \bar{P}_\pm \end{pmatrix}. \] (4.14)

These fulfill

\[ l(\pm) + m(\pm) = 1, \quad l^2(\pm) = l(\pm), \quad m^2(\pm) = m(\pm), \quad l(\pm)m(\pm) = 0 \] (4.15)

and

\[ \mathcal{F}(\pm)l(\pm) = l(\pm)\mathcal{F}(\pm) = \mathcal{F}(\pm), \quad m(\pm)\mathcal{F}(\pm) = \mathcal{F}(\pm)m(\pm) = 0. \] (4.16)

The operators \( l(\pm) \) and \( m(\pm) \) applied to the tangent space at each point of the manifold are complementary projection operators and define complementary distributions in the sense of Yano: \( \Lambda_\pm \), the first fundamental distribution, and \( \Sigma_\pm \), the second fundamental distribution, corresponding to \( l_\pm \) and \( m_\pm \), of dimensions \( 6d \) and \( 2d \), respectively.

Let \( \mathcal{N}_{\mathcal{F}(\pm)} \) denote the Nijenhuis tensor for the \( f \)-structures \( \mathcal{F}(\pm) \). By a theorem of Ishihara and Yano [27] we have that

i. \( \Lambda_\pm \) is integrable iff \( m^i_{(\pm)j} \mathcal{N}^j_{\mathcal{F}(\pm)jk} = 0 \),

ii. \( \Sigma_\pm \) is integrable iff \( \mathcal{N}^i_{\mathcal{F}(\pm)jk} m^j_{(\pm)i} m^k_{(\pm)m} = 0 \).

From the definition of the \( f \)-structures in (4.12), one can derive that these two conditions are fulfilled. Hence, the distributions \( \Lambda_\pm \) and \( \Sigma_\pm \) are integrable.

### 4.3 Additional twisted supersymmetry

In a previous paper [19] we investigated the special case of four-dimensional target space and required the transformations (3.1) to be linear. There, it was found that no solution with interesting geometry exists which possesses additional supersymmetry. On the other hand, one could impose additional twisted linear supersymmetry \( [\delta, \delta] = -\partial \) for a solution with interesting geometrical properties.

In the general case treated in this paper, we have found that additional supersymmetry can indeed be imposed. But also additional twisted supersymmetry could be considered. The difference would be that contraint (3.14) would receive a minus sign,

\[
\mathcal{M}(U^{(\pm)}, V^{(\pm)})_{jk} \bar{D}_{\pm}X^j \bar{D}_{\pm}X^k = \left[ (UV)^{(\pm)i}_j - \delta^i_j \right] \bar{D}_{\pm}\bar{D}_{\pm}X^j + \left[ (VU)^{(\pm)i}_j - \delta^i_j \right] D_{\pm}\bar{D}_{\pm}X^j. \] (4.17)
with the effect that the structure defined in (4.12) would be a $f$-structure of hyperbolic type,
\[ \mathcal{F}(\pm)(\mathcal{F}_{(\pm)}^2 - 1) = 0, \] (4.18)
that is, generalizations of product structures instead of complex structures.

5 Invariance of the action

The bihermitean geometry of [1] is derived from the $(1,1)$ sigma model via two requirements: closure of the algebra and invariance of the action. More precisely, the supersymmetry algebra implies the existence of the complex structures, whereas invariance the action implies the bihermiticity of the metric and the covariant constancy of the complex structures. Similarly, for $(4,4)$ supersymmetry, the algebra implies that the transformations are given in terms of left and right hypercomplex structures whereas invariance of the action implies the metric is hermitean with respect to all of these structures and the left and right connections preserve the the left and right structures respectively. When, in later sections, we use the knowledge from [1] in understanding our algebra conditions on-shell we can thus use the existence of a hypercomplex structures freely, but only require them to be covariantly constant if we assume that the action is invariant.

At the manifest $(2,2)$ level the discussion of additional supersymmetries in the model with (anti)chiral fields (the hyperkähler case) follows similar lines [20]. Extra supersymmetries lead to new complex structures as part of the conditions for closure of the algebra and invariance of the $(2,2)$ action leads to to the requirement that they are covariantly constant and that the metric is bihermitean.

When the complex structures commute and the sigma model is describable in $(2,2)$ superspace using (an equal number of) chiral and twisted chiral superfields, $(4,4)$ supersymmetry comes at the price of extra conditions on the potential $K$ [1]. This is also true for the linear-transformation model in [19]. We expect the same to be true here.

The action (2.3) is invariant under the supersymmetry transformations (3.3) provided that
\[ (K_i U^{(+)}_{ji})_{kj} = 0, \quad j, k \neq a, \] (5.1)
and analogously for $U^{(-)}$ and $V^{(\pm)}$. We can write this out as (3.1) a system of equations for $K$:
\[ (K_a f_{ij}^i + K_{a'} g_{ij}^{a'} + K_{a'} \tilde{g}_{ij}^{a'})_{kj} = 0 \] (5.2)
plus analogous relations from $U^{(-)}$ and $V^{(\pm)}$. 

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The conditions (5.2) (or (5.1)) have to be satisfied for the generalized Kähler potential $K$ to allow $(4,4)$ supersymmetry in a model with noncommuting complex structures whose commutator has empty kernel. In this sense it plays a similar role to the Monge-Ampère equation for models with vanishing torsion.

In the four-dimensional case with linear twisted supersymmetry transformations, it turned out to be possible to solve (5.2), (see [19]) but this is much harder in general. However, when the curl of $\tilde{g}$ and $\tilde{h}$ vanish, the condition has an interpretation on $TM \oplus TM$ much like a hermiticity condition, which we now turn to.

We combine the Hessian $K_{ij}$ of the Kähler potential into an antisymmetric tensor on $\mathcal{B}$ on $TM \oplus TM$ as

$$\mathcal{B} = \begin{pmatrix} 0 & K \\ -K^t & 0 \end{pmatrix}. \quad (5.3)$$

The relation (5.1) can be used to show that off-shell the $f$-structures (4.12) preserve $\mathcal{B}$ on a subspace projected out by the second fundamental projection operators $l_{(\pm)}$ defined in (4.16),

$$l_{(\pm)} F^t_{(\pm)} \mathcal{B} F_{(\pm)} l_{(\pm)} = l_{(\pm)} \mathcal{B} l_{(\pm)}. \quad (5.4)$$

This may be easily verified using (4.3), which implies $V^t K^t U = -K$ (except for one column and one row).

6 On-shell interpretation of the algebra constraints

In this section we discuss two main issues: How the conditions derived in section 3.3 have a larger set of solutions on-shell, and the relation to the underlying (hermitean) bi-hypercomplex geometry derived in [1]. In spirit the treatment is similar to both the (1,1) discussion in [1] of extended supersymmetry and to the hyperkähler derivation in [20]. In [1] it was found that the left and right complex structures had to commute to get off-shell closure since the algebra gives a term proportional to this commutator times the field-equations. In [20] it was found that field equations as well as conditions from the invariance of the action were needed for closure of the algebra of non-manifest additional supersymmetries.

Below we separate the conclusions we may draw from closure of the algebra only and those where in addition we need invariance of the action.
6.1 On-shell algebra

In this subsection we use a coordinate transformation to derive an explicit relation between the components of the transformation matrices and the underlying hypercomplex structure. The field equations that follow from the action (2.3) are

\[ \bar{D}_+ K_a = 0, \quad D_+ \bar{K}_a = 0, \quad \bar{D}_- K_{a'} = 0, \quad D_- \bar{K}_{a'} = 0. \]  (6.1)

These imply that on-shell, \( K_a \) is a semichiral superfield on equal footing with \( X^a \); we may change coordinates to a left-holomorphic or right-holomorphic basis with coordinates \( Z_A = \{ X^a, Y^a := K_a \} \) or \( Z'_A = \{ X'^a, Y'^a := K'_{a'} \} \), respectively [7]. In the left basis, the \( \delta^+, \bar{\delta}^+ \) transformations become very simple, whereas in the right basis, the \( \delta^-, \bar{\delta}^- \) transformations become simple. Since \( K(X^a, X'^a) \) is the generating function for the transformation between the bases, on-shell it is sufficient to study the transformations that are simple in one particular basis.

The ansatz for the \( \delta^+, \bar{\delta}^+ \) transformations is simple in the left basis:

\[ \delta^+ Z^A = 0, \quad \delta^+ \bar{Z}^A = \epsilon^+ \bar{D}_+ \bar{f}^A \bar{B}, \quad \bar{\delta}^+ Z^A = \epsilon^+ \bar{D}_+ f^A, \quad \bar{\delta}^+ \bar{Z}^A = 0. \]  (6.2)

Closure of this part of the algebra is very simple; it implies

\[ f^A_B \bar{f}^B_C = -\delta^A_C, \]  (6.3)

and

\[ f^A_C[B f^C_D] = 0, \]  (6.4)

where \( f^A_B \) again denotes derivation with respect to \( \bar{Z}^B \). These are precisely the conditions found in section 10 of [20], and imply that

\[ J^{(1)}_+ = \begin{pmatrix} 0 & f^A_B \\ f^A_B & 0 \end{pmatrix}, \quad J^{(2)}_+ = \begin{pmatrix} 0 & i f^A_B \\ -i f^A_B & 0 \end{pmatrix}, \quad J^{(3)}_+ = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \]  (6.5)

generate an integrable hypercomplex structure. Similarly, in the right basis, the \( \delta^- \) transformations generate a second integrable hypercomplex structure so that in total we get a bi-hypercomplex structure,

\[ J^{(A)}_+ J^{(B)}_+ = -\delta^{AB} + \epsilon^{ABC} J^{(C)}_+ \]  (6.6)

We still need to impose the \([\delta^+, \delta^-]\) part of the algebra and want to compare to the off-shell transformations (3.1). For both of these tasks, we need to go back to the \( X^a, X'^a \)
coordinate basis. For illustrative purposes, we focus on $\delta^+$. Comparing (3.1) and (6.2), we immediately find that on-shell

$$f^a(X^i) = f^a(X^a, \bar{X}^a, K_a(X^i), K_{\bar{a}}(X^i)) .$$

(6.7)

Off-shell, $f^a$ may differ from $f^a$ by a factor $\Delta f^a$, which satisfies $\bar{D}_+(\Delta f^a(X^a, K_a(X^i))) = 0$ on-shell. This gives an off-shell ambiguity in $f^a$. We also have (trivially) that $\delta^+ \bar{X}^a = 0$. Next we have

$$\delta^+ Y_a = K_{ab} \delta^+ \bar{X}^b + K_{aR} \delta^+ \bar{X}^R = 0$$

and

$$\delta^+ Y_a := K_{ab} \delta^+ \bar{X}^b + K_{aR} \delta^+ \bar{X}^R = \bar{e}^+ \bar{D}_+ f_a = \bar{e}^+ (f_{ab} \bar{D}_+ \bar{X}^b + f_{aR} \bar{D}_+ \bar{X}^R) ,$$

(6.9)

where $f_a(X^i) := f_a(X^a, \bar{X}^a, K_a(X^i), K_{\bar{a}}(X^i))$. We can rewrite these equations as

$$K_{LR} \delta^+ \bar{X}^R = \bar{e}^+ \left( f_{ab} \bar{D}_+ \bar{X}^b + f_{aR} \bar{D}_+ \bar{X}^R - K_{ab} \bar{D}_+ f^b + K_{aR} \bar{D}_+ f^b \right) ,$$

(6.10)

where the matrix $K_{LR}$ is defined as in (2.5). Since $K_{LR}$ is invertible, we can find the on-shell transformations $\delta^+ \bar{X}^R$. To find the corresponding functions $\tilde{g}_b^c$ and $\tilde{h}_b^c$ in (3.1), since we are on-shell, we need to eliminate one type of term, e.g., $\bar{D}_+ \bar{X}^b$, using the field equations. Then (6.10) becomes

$$K_{LR} \delta^+ \bar{X}^R = \bar{e}^+ \left( - (f_{ac} - K_{ab} f^b_c) (K^{-1})^{cd} K_{dR} + f_{aR} - K_{ab} f^b_R \right) \bar{D}_+ \bar{X}^R .$$

(6.11)

and we find

$$\tilde{g}_b^c = (K^{-1})^{e^a} \left[ f_{aR} - K_{ab} f^b_{j^c} - (f_{ac} - K_{ab} f^b_c) (K^{-1})^{cd} K_{dR} \right]$$

$$- (K^{-1})^{e^a} \left[ K_{ab} f^b_{j^c} - K_{ab} f^b_{j^c} (K^{-1})^{cd} K_{dR} \right] ,$$

$$\tilde{h}_b^c = (K^{-1})^{e^a} \left[ f_{aR} - K_{ab} f^b_{j^c} - (f_{ac} - K_{ab} f^b_c) (K^{-1})^{cd} K_{dR} \right]$$

$$- (K^{-1})^{e^a} \left[ K_{ab} f^b_{j^c} - K_{ab} f^b_{j^c} (K^{-1})^{cd} K_{dR} \right] ,$$

(6.12)

as well as the constraints

$$0 = (K^{-1})^{e^a} \left[ f_{aR} - K_{ab} f^b_{j^c} - (f_{ac} - K_{ab} f^b_c) (K^{-1})^{cd} K_{dR} \right]$$

$$- (K^{-1})^{e^a} \left[ K_{ab} f^b_{j^c} - K_{ab} f^b_{j^c} (K^{-1})^{cd} K_{dR} \right] ,$$

$$0 = (K^{-1})^{e^a} \left[ f_{aR} - K_{ab} f^b_{j^c} - (f_{ac} - K_{ab} f^b_c) (K^{-1})^{cd} K_{dR} \right]$$

$$- (K^{-1})^{e^a} \left[ K_{ab} f^b_{j^c} - K_{ab} f^b_{j^c} (K^{-1})^{cd} K_{dR} \right] .$$

(6.13)
In a similar way, we can find $g^a_i$, $h^a_i$ as well as their complex conjugates. The full set of relations will now be discussed in the original coordinates $X^i$.

6.2 Closure modulo field-equations and relations from invariance of the action

Though conceptually simple, the final expressions that we found (6.12)–(6.13) are rather involved and complicate the discussion on the on-shell $[\delta^+\, \delta^-]$ algebra. Here we present an alternative description that uses only $X^i$ coordinates and relates directly to the bi-hypercomplex geometry of [1]. We start from the ansatz (3.1) and only use the field equations to show that the conditions from closure of the algebra have more solutions on-shell. Whereas in the previous subsection discussing the on-shell algebra, it was convenient to change coordinates, here it turns out to be convenient to change the basis for the covariant derivatives.

Recall the field equations (6.1)

\[
\begin{align*}
K_{ai} \bar{D} X^i &= 0, \\
K_{ai} D X^i &= 0,
\end{align*}
\]

These equations are first order in spinorial derivatives. To be able to use them to understand the conditions (3.12)–(3.14), which contain second order spinorial derivatives, we must differentiate (6.14). We are then faced with the task of relating the plus/minus connections to second and third derivatives of the generalized Kähler potential $K$. Since the metric is a nonlinear function of the Hessian of $K$, this is not easy. Instead we choose to express the on-shell condition in terms of the complex structures $J^{(\pm)}$ defined in (2.6) and use $\nabla^{(\pm)} J^{(\pm)} = 0$ to relate them to the connections (assuming invariance of the action).

We introduce a real basis for the spinor derivatives:

\[D_\pm := \frac{1}{2} (D \pm i Q)\]

then (6.14) becomes\(^5\)

\[
\begin{align*}
Q_+ X^R &= J^{R}_{(\pm)k} D_+ X^k \\
Q_- X^L &= J^{L}_{(\pm)k} D_- X^k
\end{align*}
\]

\(^5\)Note however that we use full (2,2) superfield expressions in, e.g., (6.16); we can reduce to (1,1) superspace by restricting to superfields to depend only on half the spinor coordinates.
where we have introduced (components of) the complex structures $J_{(\pm)}$ as defined in section 2.

The semichiral conditions rewritten in terms of the real operators (6.15) and (2.7) read

$$
Q_+ X^L = jD_+ X^L \\
Q_- X^R = jD_- X^R .
$$

(6.17)

Combining this with (6.16) and (2.6) we find that on-shell

$$
Q_{\pm} X := Q_{\pm} \left( \begin{array}{c}
X^L \\
X^R 
\end{array} \right) = J_{(\pm)} D_{\pm} \left( \begin{array}{c}
X^L \\
X^R 
\end{array} \right) = J_{(\pm)} D_{\pm} X ,
$$

(6.18)

which using (6.15) implies

$$
\mathcal{D}_{\pm} X^i = \pi^{(\pm)i} D_{\pm} X^k \\
\bar{\mathcal{D}}_{\pm} X^i = \bar{\pi}^{(\pm)i} D_{\pm} X^k
$$

(6.19)

where we have introduced the projection operator

$$
\pi := \frac{1}{2} (1 + iJ) ,
$$

(6.20)

and its complex conjugate.

### 6.3 Relations to bi-hypercomplex geometry

In subsection 6.1 we constructed the bi-hypercomplex structures directly in terms of the transformations of the left and right holomorphic coordinates, and related bi-hypercomplex structures to the $f$-structures implicitly by constructing the tensors in the ansatz (3.1) in terms of the same transformations. In this subsection we analyze the relation using the real basis; this makes some aspects clearer while complicating others.

From the $N = (1,1)$ analysis of [1] we know that when the model has $(4,4)$ supersymmetry there exists an $SU(2)$ worth of left and right complex structures $(J^{(1)}_{(\pm)}, J^{(2)}_{(\pm)}, J^{(3)}_{(\pm)})$ on the 4d dimensional space, satisfying the bi-hypercomplex algebra (6.6). We now relate the $f$-structures to $J^{(A)}_{(\pm)}$.

The complex structures $J_{(\pm)}$ are part of the $SU(2)$ worth of complex structures, and we set $J^{(3)}_{(\pm)} := J_{(\pm)}$. In the real basis (6.15), the additional supersymmetries take the form

$$
\delta_s X := \delta^+ X + \bar{\delta}^\mp X = \frac{1}{2} \left[ (J^{(1)}_{(\pm)} + iJ^{(2)}_{(\pm)}) \epsilon^\pm D_{\pm} X + (J^{(1)}_{(\pm)} - iJ^{(2)}_{(\pm)}) \bar{\epsilon}^\pm D_{\pm} X \right] ,
$$

(6.21)
Identifying (6.21) with (3.3) we deduce that

\[
\frac{1}{2} \left( J_{(\pm)}^{(1)} - i J_{(\pm)}^{(2)} \right) = U^{(\pm)} \pi^{(\pm)},
\]

\[
\frac{1}{2} \left( J_{(\pm)}^{(1)} + i J_{(\pm)}^{(2)} \right) = V^{(\pm)} \bar{\pi}^{(\pm)}.
\] (6.22)

This relation implies

\[
(UV)^{(\pm)} \bar{\pi}^{(\pm)} = -\bar{\pi}^{(\pm)}, \quad (VU)^{(\pm)} \pi^{(\pm)} = -\pi^{(\pm)}.
\] (6.23)

A further consequence of the algebra (6.6) is, e.g., that

\[
U^{(\pm)} \pi^{(\pm)} = \bar{\pi}^{(\pm)} U^{(\pm)} \pi^{(\pm)}, \quad V^{(\pm)} \bar{\pi}^{(\pm)} = \pi^{(\pm)} V^{(\pm)} \bar{\pi}^{(\pm)}.
\] (6.24)

On $TM \oplus TM$ we have that

\[
\frac{1}{2} \begin{pmatrix} 0 & J_{(\pm)}^{(1)} - i J_{(\pm)}^{(2)} \\ J_{(\pm)}^{(1)} + i J_{(\pm)}^{(2)} & 0 \end{pmatrix} = \mathcal{F}(\pm) \begin{pmatrix} \bar{\pi}^{(\pm)} & 0 \\ 0 & \pi^{(\pm)} \end{pmatrix} =: \mathcal{F}(\pm) \Pi(\pm),
\] (6.25)

and the relations (6.24) can be used to show that both sides square to $-\Pi(\pm)$.

Finally, assuming that the action is invariant we have $\nabla^{(\pm)} J_{(\pm)}^{(A)} = 0$, (see (2.4)) which implies that

\[
\nabla^{(\pm)} U^{(\pm)} \pi^{(\pm)} = 0,
\]

\[
\nabla^{(\pm)} V^{(\pm)} \bar{\pi}^{(\pm)} = 0,
\] (6.26)

The equations (6.22)–(6.25) expresses the relation between the bi-hypercomplex geometry and the extra supersymmetries (3.1). The relation does not seem to be one-to-one since only, e.g., $U^{(+) \pi^{(+)}}$ enters. However, the particular form (3.4) of $U^{(\pm)}$ may be used in combination with the explicit expressions (2.6) of $J_{(\pm)}^{(A)}$ to show that all of $U^{(\pm)}$ is in fact determined by $J_{(\pm)}^{(A)}$. This is evident from the explicit expressions for the components of $U^{(\pm)}$ in section 6.1.

### 6.4 On-shell interpretation of the constraints.

On-shell, there are more cases when the algebra of the extra supersymmetries close, in analogy to, e.g., models written in terms of (anti)chiral fields. To illustrate the line of argument we first discuss (3.14).
Modulo the curl-part,
\[ U_j^{(\pm)i} V^{(\pm)i}_{[k,l]} - V_k^{(\pm)i} U_j^{(\pm)i}_{[k,l]} \] (6.27)
we may use (4.6) to rewrite (3.14) as
\[ - \left[ (UV)^{(\pm)i}_{j,k} - (UV)^{(\pm)i}_{k,j} \right] \bar{D}_\pm X^j D_\pm X^k \] (6.28)
\[ + \left[ (UV)^{(\pm)i}_{j} + \delta_j^i \right] \bar{D}_+ D_\pm X^j + \left[ (UV)^{(\pm)i}_{j} + \delta_j^i \right] D_\pm \bar{D}_+ X^j = 0. \] (6.29)
Since the LHS is
\[ \bar{D}_\pm \left[ (UV)^{(\pm)i}_{j} D_\pm X^j \right] + D_\pm \left[ (UV)^{(\pm)i}_{j} \bar{D}_\pm X^j \right] + \{ \bar{D}_+, D_\pm \} X^i \] (6.30)
and we know from (6.19) and (6.23) that on-shell the square brackets become \(-\bar{D}_\pm X^i\) and \(-\bar{D}_\pm X^i\) respectively, we see that the LHS vanishes on-shell. It remains to consider the terms in (6.27).

Writing the term out in full, including the derivatives, we have
\[ (U_j^{(\pm)i} V^{(\pm)i}_{[k,l]} - V_k^{(\pm)i} U_j^{(\pm)i}_{[k,l]} ) \bar{D}_\pm X^j D_\pm X^k \]
\[ = (U_j^{(\pm)i} V^{(\pm)i}_{[k,l]} - V_k^{(\pm)i} U_j^{(\pm)i}_{[k,l]} ) \pi_r^{(\pm)j} \pi_p^{(\pm)k} D_\pm X^p D_\pm X^q \] (6.31)
Using the relations (6.22) and (6.24) it is possible to show that one can replace all the \(U\)'s and \(V\)'s by, e.g., combinations of \(\pi^{(\pm)}\)'s and \(J^{(1)}_{(\pm)}\) yielding the following expression for the curl-terms:
\[ \left( J^{(1)}_k \right) \bar{N}(\pi)_r^k J^{(1)}_j \pi_p^j \pi_p^r - J^{(1)}_k \bar{N}(\pi)_r^k J^{(1)}_j \pi_p^j \pi_p^r \] (6.32)
where the \((\pm)\)-indices were omitted for clarity. The integrability of the \(J^{(A)}_{(\pm)}\)'s means that all the Nijenhuis-tensors and thus all of terms in (6.32) vanish. We thus see that on-shell (3.14) implies no new constraints.

Next we consider (3.12). Off-shell we had to set the terms with independent structures separately to zero (4.12). On-shell we find no conditions on the tensors if we also assume invariance of the action.

The RHS of (3.12) is
\[ [U^{(+)}_j, U^{(-)}_j] \vec{\nabla}^{(-)}_j \bar{D}_- X^j, \] (6.33)
where \(\vec{\nabla}^{(-)}_j\) is the pull-back of the minus-covariant derivative \(\nabla^{(-)}_i\) in the \(D_\pm\) basis. We want to avoid the off-shell conclusion that the commutator vanishes and observe that the
commutator multiplies something that looks like a field equation. However, we have to use (6.18) to see if it actually vanishes on-shell.

In the remainder of this section, we use the conditions that follow from invariance of the action \[1\], which imply that the metric is hermitean with respect to all the complex structures and the connections \[\Gamma(\pm)\] preserve the hypercomplex structures \[J(\pm)\]:

\[
\nabla(\pm) \mathcal{J}(\pm) = 0.
\]

A straightforward calculation shows that

\[
\bar{\nabla} \nabla(\pm) \mathcal{J} = -\frac{1}{2} \{\pi(\pm), \pi(\mp)\}^i_j \nabla(\pm) \mathcal{D} \mathcal{J}^k.
\]

(6.34)

To lowest order, the RHS is proportional to the \((1,1)\) field equation. Since it is written in manifest \((2,2)\) form, one may expect that it also vanishes to all orders. In fact, the \((2,2)\) relation

\[
\{Q_+, Q_-\} \mathcal{J}^i = 0,
\]

(6.35)

has the on-shell content

\[
\left[ J_-(\pm), J_+(\pm) \right] \mathcal{D} \mathcal{J} = 0,
\]

(6.36)

where again covariant constancy of the complex structures is used. Since the commutator is invertible in a model with only semichiral fields,

\[
\nabla(\pm) \mathcal{D} \mathcal{J} = 0,
\]

(6.37)

and that the RHS of (6.34) vanishes.

Using the connections with skew torsion \[T = \pm \frac{1}{2} dB\] we have from the definition (3.9) that the LHS of (3.12) is

\[
\mathcal{M}(U(\pm), U(\pm))^i_j \bar{\mathcal{D}} \mathcal{J} = \left( U_j^i \nabla(\pm) U_k(\pm)^i - U_k^i \nabla(\pm) U_j(\pm)^i - U_j^i \nabla(\pm) U_k(\pm)^i + U_k^i \nabla(\pm) U_j(\pm)^i \right) \bar{\mathcal{D}} \mathcal{J}.
\]

(6.38)

Given the results for the RHS, the appropriate projections of \(\mathcal{M}(U(\pm), U(\pm))\) thus have to vanish. However, we know from (6.19) that on-shell

\[
\bar{\mathcal{D}} \mathcal{J} = \pi(\pm)^i_j \mathcal{D} \mathcal{J}^i \pi(\pm)^k \mathcal{D} \mathcal{J}^k,
\]

(6.39)

and invoking invariance of the action, we may use (6.26) to conclude that then indeed \(\mathcal{M}(U(\pm), U(\pm)) = 0\).

---

6This is equivalent to restricting the holonomy of the connections \(\Gamma(\pm)\) to a symplectic group.

7Here the operator \(\nabla(\pm)\) is the pullback in the \(D(\pm)\)-basis.
In summary our result is very similar to the hyperkähler discussion in [20], we need to invoke invariance of the action to show that there are more solutions on-shell to the conditions from the algebra\footnote{In hyperkähler case the the algebra only closes on-shell.}.

The only constraints we get on the transformation matrices on-shell for invariant actions are the integrability condition

\[ \mathcal{N}(U^{(\pm)})^i_{jk} \pi^{j(\pm)k}_i \pi^{(\pm)k}_m D_\pm \mathcal{X}_i D_\pm \mathcal{X}_m = 0. \]

(6.40)

together with the identification (6.22).

7 Discussion

Throughout this paper, the arbitrary entries in the transformation matrices $U^{(\pm)}$ (and $V^{(\pm)}$) were set to zero. Off-shell, this has the advantages of yielding geometric structures on the full target-space. Keeping the arbitrariness would restrict the features (e.g., integrability) of these structures to certain subspaces.

We have identified new geometric structures on the target-space of sigma models written in terms of semichiral fields. These structures arise when we study additional off-shell supersymmetries. We have discussed the $f$-structures as living on the sum of two copies of the tangent bundle $TM \oplus TM$. Clearly one would like to identify the relation to generalized complex geometry on $TM \oplus T^\ast M$. Formally, this may be achieved using the existence of a metric [15]

\[ g = \Omega[J_{(+)}, J_{(-)}], \]

(7.1)

where

\[ \Omega := \begin{pmatrix} 0 & 2iK_{LR} \\ -2iK_{RL} & 0 \end{pmatrix}. \]

(7.2)

We use $g$ to relate $TM$ and $T^\ast M$ to write $\mathcal{F}$ as an $f$-structure on $TM \oplus T^\ast M$:

\[ \tilde{\mathcal{F}} := \begin{pmatrix} 0 & U^{-1}g \\ gV & 0 \end{pmatrix}. \]

(7.3)

We plan to return to the geometry of $f$-structures in the context of generalized complex geometry in a later publication.

A related question concerns the condition for invariance of the action. As we have shown for a subclass of our transformations, this amounts to the conservation of an antisymmetric tensor $\mathfrak{B}$ on certain subspaces of $TM \oplus TM$ by the $f$-structures. Again, the
corresponding object on $TM \oplus T^*M$ can be found using the metric $g$:

$$\tilde{\mathcal{B}} = \begin{pmatrix}
0 & Kg^{-1} \\
-gK^t & 0
\end{pmatrix}. \tag{7.4}$$

It remains to clarify where this object fits into the generalized complex picture. This also ties in with the question of how the conditions for invariance that we have described relate to those found in [21], where $(4,4)$ models with auxiliary fields are discussed.

In the precursor to this article [19] where the nonmanifest transformations were linear and the target space was four-dimensional, there was no interesting solution with additional supersymmetry. Additional twisted supersymmetry could be imposed, however. The target-space was then seen to carry indefinite signature metric and vanishing three form $H$, the geometry being pseudo-hyperkähler. In the present paper, where the target space is $4d$-dimensional, the transformations close to an ordinary supersymmetry algebra if $d > 1$, i.e. the dimension of the target space is larger than four. This stems from the fact that a complex number $a$ can never fulfill $a\bar{a} = -1$, whereas for a matrix $A$ with complex conjugated components $\bar{A}$, this could indeed be fulfilled. We could also have considered a twisted supersymmetry in the general case. The result would have been hyperbolic $f$-structures, a generalization of the result in [19].

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