Exact solvability of the quantum Rabi model using Bogoliubov operators

Qing-Hu Chen,1,2 Chen Wang,2 Shu He,2,3 Tao Liu,3 and Ke-Lin Wang4
1Center for Statistical and Theoretical Condensed Matter Physics, Zhejiang Normal University, Jinhua 321004, People’s Republic of China
2Department of Physics, Zhejiang Normal University, Hangzhou 310027, People’s Republic of China
3School of Science, Southwest University of Science and Technology, Mianyang 621010, People’s Republic of China
4Department of Modern Physics, University of Science and Technology of China, Hefei 230026, People’s Republic of China

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Within extended coherent states, a recent exact solution to the quantum Rabi model (QRM) [D. Braak, Phys. Rev. Lett. 107, 100401 (2011)] can be recovered in an alternative simpler and more physical way, without use of any extra conditions. In the same framework, the two-photon QRM is solved exactly by treating extended squeezed states on an equal footing. Concise transcendental functions responsible for the exact solutions are derived. The isolated Juddian solutions are also analytically obtained in terms of degeneracy. Both the extended coherent states and squeezed states employed here are essentially Fock states in the space of the corresponding Bogoliubov operators, which result in free-particle number operators. The present approach can be summarized concisely in a unified way and easily extended to various spin-boson systems with multiple levels, even multiple modes.

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I. INTRODUCTION

Matter-light interaction is fundamental and ubiquitous in areas of modern physics ranging from quantum optics and quantum information science to condensed-matter physics. The simplest paradigm is a two-level atom (qubit) coupled to the electromagnetic mode of a cavity (oscillator), which is called the Rabi model [1]. If the coupling strength $g/\omega$ (where $\omega$ is the cavity frequency) between the atom and the cavity mode exceeds the loss rate, the atom and the cavity can repeatedly exchange excitations before coherence is lost. Rabi oscillations can be observed in this atom-cavity system, which is usually known as cavity quantum electrodynamics (QED) [2]. Typically, the coupling strength in cavity QED reaches $g/\omega \sim 10^{-10}$. It can be described by the Jaynes-Cummings (JC) model [3] where the rotating-wave approximation (RWA) is made and analytically closed-form exact solutions are available.

Recently, for superconducting qubits, a one-dimensional (1D) transmission line resonator [4] or an $LC$ circuit [5–8] has been shown to be able to play the role of the cavity; this is known today as circuit QED. More recently, an $LC$ resonator inductively coupled to a superconducting qubit [9–11] has been realized experimentally, where the qubit-resonator coupling can be strengthened by 10%. In this ultrastrong-coupling regime of circuit QED, evidence for the breakdown of the RWA has been provided by the transmission spectra [9]. The remarkable Bloch-Siegert shift associated with the counter-rotating terms also demonstrates the failure of the RWA [10]. So the quantum Rabi model (QRM) has been reconsidered by many authors recently.

On the other hand, the two-photon QRM has also attracted a lot of attention. It is a phenomenological model describing a three-level system interacting with two photons. When the intermediate transition frequencies are strongly detuned from the cavity frequency, after adiabatically eliminating the intermediate levels, one arrives at the effective Hamiltonian. It can describe the two-photon processes occurring in rubidium atoms [12] and quantum dots [13]. The two-photon QRM has also been studied for a long time both with the RWA [14] and beyond the RWA [15–17].

More recently, Braak presented an exact solution [18] to the one-photon QRM, in a representation of bosonic creation and annihilation operators in the Bargmann space [19] of analytical functions in a complex variable. A transcendental function, whose zeros could give exact eigenvalues, was derived. By a proposed criterion for quantum integrability, Braak further shows that the QRM is integrable due to the parity symmetry. However, the derivations are outlined in a mathematical way. It was suggested [20] that an intense dialogue between mathematics and physics is needed. In other words, it is useful to shed some physical insights on Braak’s mainly mathematical solutions.

In this paper, without the use of extra conditions, like analyticity of the eigenfunction in the Bargmann representation, we alternatively rederive the same transcendental functions as in Ref. [18] quantum-mechanically within extended coherent states [21,22]. Both zero-bias and biased QRM can be treated simultaneously. Our method is more intuitional and more easily understandable. Similarly, we also study the two-photon QRM [16,17] within extended squeezed states. Compact transcendental functions which are responsible for the exact solution are derived. The Juddian solutions [23] are then easily discussed.

The paper is organized as follows. In Sec. II, we describe the present approach in detail for the one-photon QRM. Braak’s exact solution is recovered straightforwardly. Discussions and comparisons and a brief tutorial for the approach are also given. In Sec. III, the two-photon QRM is solved exactly on an equal footing, concise transcendental functions are derived, and Juddian solutions are discussed. A brief summary is given finally.

II. THE QRM WITHIN BOGOLIUBOV OPERATORS

A. Rederivation of Braak’s solution

The Hamiltonian of a generalized QRM can be describe as follows:

$$H = -\frac{i}{2} (\sigma_z + \Delta \sigma_x) + a^\dagger a + g(a^\dagger + a)\sigma_z,$$  (1)
where $\varepsilon$ and $\Delta$ are the qubit static bias and tunneling matrix element, $a^\dagger$ and $a$ are the photon creation and annihilation operators of the single-mode cavity with frequency $\omega$, $g$ is the qubit-cavity coupling constant, and $\sigma_k$ ($k=x,y,z$) are the Pauli matrices. To facilitate the study, we write the Hamiltonian in the matrix form in units of $\hbar = \omega = 1$.

$$H = \begin{pmatrix} a^\dagger a + g(a^\dagger + a) - \varepsilon & -\Delta_2 \\ -\Delta_2 & a^\dagger a - g(a^\dagger + a) + \varepsilon \end{pmatrix}. \quad (2)$$

To remove the linear terms in the $a^\dagger (a)$ operators, we perform the following two Bogoliubov transformations:

$$A = a + g, \quad B = a - g. \quad (3)$$

In Bogoliubov operators $A (B)$, the matrix element $H_{11} (H_{22})$ can be reduced to the free-particle number operators $A^\dagger A (B^\dagger B)$ plus a constant, which is very helpful for further study.

Unlike the previous ansatz, the Hamiltonian is expressed in the two operators $A$ and $B$ at the same time [21], we here use the single operators after the first. In terms of operator $A$, the Hamiltonian can be written as

$$H = \begin{pmatrix} A^\dagger A - \alpha & -\Delta_2 \\ -\Delta_2 & A^\dagger A - 2g(A^\dagger + A) + \beta \end{pmatrix}. \quad (4)$$

where

$$\alpha = g^2 + \varepsilon, \quad \beta = 3g^2 + \varepsilon. \quad (5)$$

The wave function is then proposed as

$$|\rangle = \begin{pmatrix} \sum_{n=0}^{\infty} \sqrt{n!} e_n |n\rangle_A \\ \sum_{n=0}^{\infty} \sqrt{n!} f_n |n\rangle_A \end{pmatrix}. \quad (6)$$

where $e_n$ and $f_n$ are the expansion coefficients. $|n\rangle_A$ is called an extended coherent state with the following properties:

$$|n\rangle_A = \frac{(A^\dagger)^n}{\sqrt{n!}} |0\rangle_A = \frac{(a^\dagger + g)^n}{\sqrt{n!}} |0\rangle_A, \quad (7)$$

where the vacuum state $|0\rangle_A$ in Bogoliubov operators $A$ is well defined as the eigenstate of the one-photon annihilation operator $a$, and is known as a pure coherent state [24].

The Schrödinger equation gives

$$\sum_{n=0}^{\infty} (n - \alpha) \sqrt{n!} e_n |n\rangle_A - \frac{\Delta}{2} \sum_{n=0}^{\infty} \sqrt{n!} f_n |n\rangle_A = E \sum_{n=0}^{\infty} \sqrt{n!} e_n |n\rangle_A$$

$$-\frac{\Delta}{2} \sum_{n=0}^{\infty} \sqrt{n!} e_n |n\rangle_A + \sum_{n=0}^{\infty} (n + \beta) \sqrt{n!} f_n |n\rangle_A$$

$$-2g \sum_{n=0}^{\infty} (\sqrt{n!} f_n \sqrt{n!} |n-1\rangle_A + \sqrt{n+1} \sqrt{n!} f_n |n+1\rangle_A)$$

$$= E \sum_{n=0}^{\infty} \sqrt{n!} f_n |n\rangle_A. \quad (8)$$

Left-multiplying by $A |m\rangle$ gives

$$(m - \alpha - E) e_m = \frac{\Delta}{2} f_m. \quad (9)$$

So the two coefficients $e_m$ and $f_m$ with the same index $m$ are related by

$$e_m = \frac{\Delta}{2 (m - \alpha - E)} f_m. \quad (10)$$

and the coefficient $f_m$ can be defined recursively,

$$m f_m = \Omega (m - 1) f_{m-1} - f_{m-2}, \quad (11)$$

where

$$\Omega (m) = \frac{1}{2g} \left( (m + \beta - E) - \frac{\Delta^2}{4(m - \alpha - E)} \right). \quad (12)$$

Proceeding as before, the two coefficients $f'_m$ and $e'_m$ satisfy

$$e'_m = \frac{\Delta}{m - \alpha' - E} f'_m, \quad (13)$$

and the recursive relation is given by

$$m f'_m = \Omega' (m - 1) f'_{m-1} - f'_{m-2}, \quad (14)$$

where

$$\alpha' = g^2 - \varepsilon, \quad \beta' = 3g^2 - \varepsilon. \quad (15)$$

The wave function can also be written in terms of $B$ as

$$|\rangle = \begin{pmatrix} \sum_{n=0}^{\infty} (-1)^n \sqrt{n!} f'_n |n\rangle_B \\ \sum_{n=0}^{\infty} (-1)^n \sqrt{n!} e'_n |n\rangle_B \end{pmatrix}. \quad (16)$$

Proceeding as before, the two coefficients $f''_n$ and $e''_n$ satisfy

$$e''_m = \frac{-\Delta}{m - \alpha'' - E} f''_m, \quad (17)$$

and the recursive relation is given by

$$m f''_m = \Omega'' (m - 1) f''_{m-1} - f''_{m-2}, \quad (18)$$

with $f''_0 = 1$ and $f''_1 = \Omega'' (0)$. If both wave functions (5) and (14) are true eigenfunctions for a nondegenerate eigenstate with eigenvalue $E$, they should be in principle only different by a complex constant $r$,

$$\sum_{n=0}^{\infty} \sqrt{n!} e_n |n\rangle_A = r \sum_{n=0}^{\infty} (-1)^n \sqrt{n!} f'_n |n\rangle_B, \quad (19)$$

$$\sum_{n=0}^{\infty} \sqrt{n!} f_n |n\rangle_A = r \sum_{n=0}^{\infty} (-1)^n \sqrt{n!} e'_n |n\rangle_B. \quad (20)$$

Left-multiplying the original vacuum state $|0\rangle$ by both side of the above equations yields

$$\sum_{n=0}^{\infty} \sqrt{n!} e_n |0\rangle_A = r \sum_{n=0}^{\infty} (-1)^n \sqrt{n!} f'_n |0\rangle_B, \quad (21)$$

$$\sum_{n=0}^{\infty} \sqrt{n!} f_n |0\rangle_A = r \sum_{n=0}^{\infty} (-1)^n \sqrt{n!} e'_n |0\rangle_B. \quad (22)$$
where
\[ \sqrt{n!(0)n}_x = (-1)^n \sqrt{n!(0)n}_B = e^{-g^2/2} g^n. \] (22)

Eliminating the ratio constant \( r \) gives
\[ \sum_{n=0}^{\infty} e_n g^n \sum_{n=0}^{\infty} e'_n g^n = \sum_{n=0}^{\infty} f_n g^n \sum_{n=0}^{\infty} f'_n g^n. \]

With the help of Eqs. (10) and (15), we arrive at
\[ \sum_{n=0}^{\infty} n - \alpha - E f_n g^n \sum_{n=0}^{\infty} n - \alpha' - E f'_n g^n = \sum_{n=0}^{\infty} f_n g^n \sum_{n=0}^{\infty} f'_n g^n. \] (23)

If we set \( f_n = K_n^- \), \( f'_n = K_n^+ \), and \( E = x - g^2 \), we can recover Braak’s exact solution [18]
\[ G_e(x) = (\Delta^2 / 2)^{1/2} R^+(x)R^-(x) - R^+(x)R^-(x) = 0, \] (24)
where
\[ R^\pm(x) = \sum_{n=0}^{\infty} K_n^\pm(x) g^n, \]
\[ \overline{R}^\pm(x) = \sum_{n=0}^{\infty} K_n^\pm(x) g^n. \]

If \( \varepsilon = 0 \), the above equation can obviously be reduced to the following zero-bias case [18]:
\[ G_n^0(x) = \sum_{n=0}^{\infty} f_n(x) 1 \mp \Delta/2 x - n x, \] (25)

Therefore Braak’s \( G \) functions are completely rederived in a very intuitive and concise way.

The \( G \) functions can be written [25] in terms of so-called Heun functions [26]. The zeros of these Heun functions cannot be given analytically; a numerical technique to locate the zeros is still needed, so a cutoff for the summation is unavoidable in the practical evaluation.

### B. Discussion

In the above derivation, the crucial step is the proportionality of the two wave functions (5) and (14) with the same eigenvalue. Both Hilbert spaces in the two Bogoliubov operators are complete, if truncation is not done, and the proportionality is justified naturally for nondegenerate states.

Next, we link the degenerate states to the Juddian solutions [23]. Koc et al. [27] have obtained isolated exact solutions in the QRM, which are just the Juddian solutions with doubly degenerate eigenvalues. The degenerate eigenstates are excluded in principle in solutions based on proportionality. It naturally follows that the Juddian solutions are exceptional ones as discussed by Braak [18].

It is very interesting to note that, in the whole derivation above, we do not need any extra conditions, such as the analyticity of the eigenfunction in Bargmann representation as in Braak’s work [18]. We believe that the extra condition in the Bargmann representation is covered in the vacuum state in the space of the Bogoliubov operators. These vacuum states are well defined and known as coherent states, so the present derivation is more physical and simpler.

In addition, the validity of the present approach is independent of the parity symmetry. Parity symmetry would be contained self-consistently in the final \( G \) functions if the system really has, e.g., \( \varepsilon = 0 \).

### C. Comparisons and remarks

Based on two Bogoliubov operators \( A \) and \( B \), three of the present authors and one collaborator have used the following wave function to the Hamiltonian(1) to analyze the spectrum in qubit-oscillator systems [cf. Eq. (6) in Ref. [22]]:
\[ | \rangle = \left( \sum_{n=0}^{N} c_n | n \rangle_A \sum_{n=0}^{N} d_n | n \rangle_B \right), \] (26)
where \( N \) is the truncated number. Numerical exact diagonalization (ED) in the space of the two Bogoliubov operators gave the spectrum exactly. The coefficients \( c_n \) and \( d_n \) can be obtained also.

It is interesting to link coefficients in wave function (26) and those in wave functions (5) and (14) as
\[ c_n = \sqrt{n!} e_n, \quad d_n = r (-1)^n \sqrt{n!} e'_n, \]
although the former are obtained from ED and the latter by the zeros of the \( G \) functions. It can also be confirmed numerically. For practical purposes, there are perhaps no essential differences, except that the avenues used to obtain basically the same coefficients are different. The latter is described in a mathematical way and is of more conceptual interest.

In the mathematical sense, we cannot rule out the possibility that ED gives good results for small \( N \) but gets worse for higher \( N \) without a practical evaluation, although we know empirically that it is generally not the case for large \( N \). For low-order perturbation theory, it happens that third-order perturbation theory will give worse results than second-order perturbation theory in some parameter regime, for instance, but this may not be that case in very high-order perturbation theory. In the calculation, we find that the difference between the exact results, which are easily obtained to any desired accuracy, and those for the cutoff \( N \) decrease monotonically with increasing \( N \), and convergence can be arrived at easily. One can determine that the Heun series converges before numerical calculations.

Braak’s \( G \) functions exhibit a very compact form in power series, which motivates us to reshape our previous work. By use of tunable extended bosonic coherent states, the QRM can be mapped to a polynomial equation with a single variable [28]. We can also write this polynomial in power series in the following more concise form for large truncated number \( M \):
\[ F(\alpha) = \sum_{j=0}^{M} \frac{(2\alpha)^j}{j!} c_{M-j} = 0, \] (27)
hybergeometric functions. But the eigenvalues are given by the

defined, although much more complicated than, e.g., the
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of displaced operators with tunable displacements, which are
different from the present Bogoliubov operators with fixed
displacements.

It is very interesting to note that the zeros of both functions
defined through different power series in Eqs. (25) and (27)

initiated from \( c_0 = 1.0 \) and \( c_1 = 0 \), because the coefficients

with the two highest indices, \( M \) and \( M - 1 \), are negligibly

small due to the required convergence and can thus be omitted.
The zeros of the above function \( F(\alpha) \) can also give the exact
eigenvalue through

\[
E^\pm = \alpha g \mp \frac{\Delta}{2},
\]

where \( \pm \) denotes the parity. The results are shown in Fig. 1.
In Eq. (20) at the end of Ref. [28], it was demonstrated that the
wave function is equivalent to expansion in the Fock space
of displaced operators with tunable displacements, which are
different from the present Bogoliubov operators with fixed
displacements.

It is implied in the viewpoint of [20] that the QRM might
have been solved exactly with an analytical closed-form
solution in Ref. [18]. Nevertheless, whether Braak’s exact
solution could be called closed form is subtle and therefore
still controversial in our opinion. The so-called Heun functions

possibly be generally described as follows; the description helpful for

further applications. The main task is to find the corresponding
Bogoliubov operators. Then, one can expand the wave
functions in terms of each Bogoliubov operator separately.

Eliminating the constant ratio of these wave functions will
give transcendental functions, which are defined through
power series in model parameter-dependent quantities with
coefficients related recursively. Finally, the zeros of these
transcendental functions give the eigenvalues exactly, where
numerical solutions to the one-variable (or finite variables in
other multilevel systems for example) nonlinear equation must
be required. Although the power series are defined through an
infinite summation formally, in a practical calculation, they
should be truncated to a finite summation. Fortunately, the
obtained transcendental function \( G(x) \) can be written in terms
of so-called Heun functions, from which we can determine
the convergence before numerical solution. The unavoidable
“cutoff” in the summation of the \( G \) functions in practical
calculations means that some states in the Hilbert space
are not considered, according to the wave functions (5) and (14);
even though their contribution is negligibly small, they still
belong to the Hilbert space of Bogoliubov operators.

following the approach outlined above, we will study the
two-photon QRM in the next section.

III. THE TWO-PHOTON QRM

The Hamiltonian of the two-photon QRM takes the following
matrix form:

\[
H = \begin{pmatrix}
\alpha a + g((a^\dagger)^2 + a^2) & -\frac{\Delta}{2} \\
-\frac{\Delta}{2} & \alpha^* a^\dagger - g((a^\dagger)^2 + a^2)
\end{pmatrix}.
\]

First, we perform a Bogoliubov transformation,

\[
b = ua + va^\dagger, \quad b^\dagger = ua^\dagger + va,
\]

to generate a new bosonic operator. Comparing to the Hamiltonian, if we set

\[
u = \sqrt{\frac{1 + \beta}{2\beta}}, \quad v = \sqrt{\frac{1 - \beta}{2\beta}},
\]

with \( \beta = \sqrt{1 - 4g^2} \), we have a simple quadratic form of one
diagonal Hamiltonian matrix element:

\[
H_{11} = a^\dagger a + g((a^\dagger)^2 + a^2) = \frac{b^\dagger b - v^2}{u^2 + v^2}.
\]
Similarly, we can introduce another operator
\[ c = u a - va, \quad c^1 = u a^1 - va, \] (33)
which yields a simple quadratic form of the other diagonal Hamiltonian matrix element
\[ H_{22} = a^1 a - g[(a^1)^2 + a^2] = \frac{c^1 c - v^2}{u^2 + v^2}. \]

In terms of the Bogoliubov operator \( b \), the Hamiltonian can be written as
\[ H = \left( \begin{array}{cc} \eta b - v^2 & -\frac{\Delta}{2} \\ -\frac{\Delta}{2} & H'_{22} \end{array} \right), \] (34)
with
\[ H'_{22} = (u^2 + v^2 + 4 g u v) b^\dagger b - [u v + g(u^2 + v^2)](b^\dagger b + b^2) + 2 g u v + v^2. \]
The wave function is suggested as
\[ | \rangle = \left( \frac{\sum_{n=0}^{\infty} \sqrt{n!} e_n}{\sum_{n=0}^{\infty} \sqrt{n!} f_n} \right)_{b} | n \rangle, \]
\[ | 0 \rangle_{b} = 0. \] (37)

\( | 0 \rangle_{b} \) is the vacuum state of a linear combination of \( a^\dagger \) and \( a \), which is well known as the single-mode squeezed state [30]; \( | n \rangle_{b} \) is thus called an extended squeezed state. Following the procedures in [31], we derive for later use the explicit expression of \( | 0 \rangle_{b} \) in terms of the operator \( a \) involving either even- or odd-number states as follows:
\[ | 0 \rangle_{b}^{(e)} \propto \sum_{k=0}^{\infty} \frac{\sqrt{2k!}}{2k} \left( \frac{-v}{u} \right)^{k} (2k)_{a}, \] (38)
\[ | 0 \rangle_{b}^{(o)} \propto \sum_{k=0}^{\infty} \frac{2^k k!}{\sqrt{(2k + 1)!}} \left( \frac{-v}{u} \right)^{k} (2k + 1)_{a}. \] (39)

The Schrödinger equation gives
\[ \sum_{n=0}^{\infty} \frac{b a^\dagger}{u^2 + v^2} \sqrt{n!} e_n | n \rangle_{b} = \frac{\Delta}{2} \sum_{n=0}^{\infty} \sqrt{n!} f_n | n \rangle_{b}, \]
\[ = E \sum_{n=0}^{\infty} \sqrt{n!} e_n | n \rangle_{b}, \]
\[ (u^2 + v^2 + 4 g u v) b \sum_{n=0}^{\infty} \sqrt{n!} f_n | n \rangle_{b} \]
\[ - [u v + g(u^2 + v^2)](b^\dagger b + b^2) \sum_{n=0}^{\infty} \sqrt{n!} f_n | n \rangle_{b} \]
\[ + (2 g u v + v^2) \sum_{n=0}^{\infty} \sqrt{n!} f_n | n \rangle_{b} - \frac{\Delta}{2} \sum_{n=0}^{\infty} \sqrt{n!} e_n | n \rangle_{b} \]
\[ = E \sum_{n=0}^{\infty} \sqrt{n!} f_n | n \rangle_{b}. \]

Left-multiplying by \( \delta / (m | n \rangle \) gives
\[ \left( \begin{array}{cc} m - v^2 & -\frac{\Delta}{2} \\ \frac{\Delta}{2} & u^2 + v^2 - E \end{array} \right) e_m - \frac{\Delta}{2} f_m = 0, \]
\[ - [u v + g(u^2 + v^2)][f_{m-2} + (m + 2)(m + 1) f_{m+2}] \]
\[ + [(u^2 + v^2) m + v^2 + 2 g u v (2 m + 1) - E] f_m \]
\[ - \frac{\Delta}{2} e_m = 0. \]

Thus we have built a one-to-one relation for coefficients \( e_m \) and \( f_m \):
\[ e_m = \frac{\Delta}{2 \left[ \frac{m - v^2}{u^2 + v^2} - E \right]} f_m, \] (40)

which will considerably simplify the problem. The recursive relation is then obtained as
\[ (m + 2) (m + 1) f_{m+2} = - f_{m-2} + \frac{\Omega (m)}{u v + g(u^2 + v^2)} f_m, \] (41)

where
\[ \Omega (m) = (u^2 + v^2) m + v^2 + 2 g u v (2 m + 1) - E \]
\[ - \frac{\Delta^2}{4 \left( \frac{m - v^2}{u^2 + v^2} - E \right)}. \]

The Hamiltonian can also be expressed in terms of the other Bogoliubov operator \( c \):
\[ H = \left( \begin{array}{cc} \frac{\Delta}{2} & -\frac{\Delta}{2} \\ -\frac{\Delta}{2} & H'_{11} \end{array} \right), \] (42)

with
\[ H'_{11} = (v^2 + u^2 + 4 g u v) c^1 c + [u v + g(v^2 + u^2)](c^1)^2 + c^2 + 2 g u v + v^2. \]

The wave function then can be expanded in the Fock space of the \( c \) operator in the form
\[ | \rangle = \left( \frac{\sum_{n=0}^{\infty} i^l \sqrt{n!} f_n^* | n \rangle_{c}}{\sum_{n=0}^{\infty} i^l \sqrt{n!} f_n | n \rangle_{c}} \right), \] (43)

where \( l = n \) for \( n = 2 k \) and \( l = n + 1 \) for \( n = 2 k + 1 \). Therefore only two values of \( l \) are possible.

Similarly, by the Schrödinger equation, we can obtain the following relations for the coefficients:
\[ e'_m = \frac{\Delta}{2 \left[ \frac{m - v^2}{u^2 + v^2} - E \right]} f'_m, \] (44)

and the recursive relation
\[ (m + 2)(m + 1) f'_{m+2} = - f'_{m-2} + \frac{\Omega (m)}{u v + g(u^2 + v^2)} f'_m, \] (45)

with
\[ \Omega (m) = v^2 + (u^2 + v^2) m + 2 g u v (2 m + 1) - E \]
\[ - \frac{\Delta^2}{4 \left( \frac{m - v^2}{u^2 + v^2} - E \right)}. \]
Note that the two sets of coefficients in the two wave functions have the same form.

Similarly, the two wave functions with the same eigenvalue should be in principle proportional to each other for the nondegenerate state,

$$\left( \sum_{n=0} \sqrt{n!} e_n |n\rangle_b \right) = r \left( \sum_{n=0} \sqrt{n!} f_n^e |n\rangle_c \right) \quad (46)$$

Left-multiplying $|0\rangle$ on both equations gives

$$\sum_{n=0} \sqrt{n!} e_n (0) |n\rangle_b = r \sum_{n=0} i^n \sqrt{n!} f_n^e (0) |n\rangle_c,$$

$$\sum_{n=0} \sqrt{n!} f_n (0) |n\rangle_b = r \sum_{n=0} i^n \sqrt{n!} e_n^e (0) |n\rangle_c.$$

By using Eqs. (38) and (39), we always have

$$i^n \sqrt{n!} (0) |n\rangle_c = \sqrt{n!} (0) |n\rangle_b = L_n^e,$$

where

$$L_{n=2k}^e = \frac{(2k)!}{2^k} \sum_{j=0}^k \frac{(-\frac{1}{2})^j}{j!(k-j)!},$$

$$L_{n=2k+1}^e = \frac{(2k+1)!}{2^k} \sum_{j=0}^k \frac{2^j j!(-\frac{1}{2})^j}{(2j+1)!((k-j)!},$$

for even and odd particle numbers in the Bogoliubov operators $b$ and $c$, respectively. Then we have

$$\sum_n e_n L_n^e = r \sum_n f_n^e L_n^e, \quad \sum_n f_n L_n^c = r \sum_n e_n^c L_n^c.$$

Now the summation $\sum_n$ is taken for the series with either even or odd number $n$. Eliminating the constant $r$ yields

$$\sum_n \frac{\Delta}{2(n^2+\nu^2)-E} f_n L_n^e = \sum_n \frac{\Delta}{2(n^2+\nu^2)-E} f_n^e L_n^e,$$

$$= \sum_n f_n^e L_n^e,$$  \hspace{1cm} (50)

with the use of Eqs. (40) and (44). Setting $f_n = f_n^e$ and $-x = -\nu^2 - E(\mu^2 + \nu^2)$, we finally have

$$G_{e,o}^\pm = \sum_n f_n \left[ 1 \pm \frac{\Delta(\mu^2 + \nu^2)}{2(n-x)} \right] L_n^e = 0,$$  \hspace{1cm} (51)

where the coefficient $f_n$ is initiated from $f_0 = 1$ ($f_1 = 1$) for the case of the even (odd) $n$ in the recurrence scheme Eq. (41), and $\pm$ denotes the parity. Thus, $G$ functions for the two-photon QRM have been obtained. The zeros of the $G$ functions give the exact eigenvalues, as shown in Fig. 2. It should be straightforward to extend to the biased two-photon QRM, but it is not shown here.

Travênc [17] has extended Braak’s approach to solve this two-photon model, but a concise $G$ function as in Eq. (51) was not obtained. The coefficients are entangled in the two coupled equations, which may prevent such a simple description for the $G$-functions. In the present Eqs. (40) and (44), the two coefficients are related one-to-one with the same index $n$, which facilitates the derivations. This is also the advantage of Bogoliubov operators, which result in free-particle number operators.

The Juddian solution to the two-photon QRM has been studied by Emary and Bishop [32]. With these $G$ functions at hand, we can also discuss the Juddian solution readily in an alternative way, similar to the one-photon model [18]. The $G$ function is also not analytic in $x$ but has simple poles at $x = 0, 1, 2, \ldots$. For special values of the model parameter $g$, there are eigenvalues which do not correspond to zeros of Eq. (51); these are the exceptional solutions. All exceptional eigenvalues are given by the positions of the poles:

$$E = (n + \frac{1}{2}) \beta - \frac{1}{2},$$  \hspace{1cm} (52)

which is exactly the isolated solution obtained in Ref. [32]. The necessary and sufficient condition for the occurrence of the eigenvalue is $f_{\nu}(x = n) = 0$, which provides a condition on the model parameters $g$ and $\Delta$. They occur when the pole of $G_{e,o}^\pm(x)$ at $x = n$ is lifted because its numerator in Eq. (51) vanishes. The condition can be obtained readily by Eq. (41) as follows for $n = 2, 3, \text{and } 4$ respectively:

$$2 - 6\beta^2 + \left( \frac{\Delta}{2} \right)^2 = 0, \quad 6 - 10\beta^2 + \left( \frac{\Delta}{2} \right)^2 = 0,$$

$$8(3 - 30\beta^2 + 35\beta^4) + 2(7 - 17\beta^4) \left( \frac{\Delta}{2} \right)^2 + \left( \frac{\Delta}{2} \right)^4 = 0,$$

which are exactly the same as those in Ref. [32]. These constraints on the model parameters for the Juddian solutions have not been derived in the direct extension of the Braak’s approach to the two-photon model [17]. From Eq. (50), we know that the proportionality is justified only for even or odd photonic numbers, respectively. The Juddian solutions correspond to those states which are degenerate, and therefore are excluded within this proportionality, so the level crossing points of lines from $G_{e,o}^+$ and $G_{e,o}^-$ and those from $G_{o,o}^+$ and $G_{o,o}^-$ correspond to Juddian solutions.

**IV. SUMMARY**

In this paper, by using extended coherent states, Braak’s exact solution in the QRM is recovered explicitly in an
We have expanded the wave function in $N$ and obtained numerically exact solutions previously [21].

For a multilevel spin-boson model, such as the finite-sized Dicke model [33], quantum chaos has been discussed [34]. We have expanded the wave function in $N + 1$ Bogoliubov operators for the Dicke model with finite-$N$ two-level atoms, and obtained numerically exact solutions previously [21].

According to the above discussion and the link with Braak’s solutions, exact solvability is ensured without doubt in this system. The quasi-integrability and quantum chaos in this system should be very interesting. On the other hand, the multimode QRM has also been realized experimentally in circuit QED systems [9]. Extensions to these systems are in progress.

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