Scale Invariance from Modified Dispersion Relations

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In this paper, inspired by the investigations on the theory of cosmological perturbations in Hořava-Lifshitz cosmology, we calculated the spectrum of primordial perturbation leded by a scalar field with modified dispersion relation \( \omega \sim k^p/a^{p-1} \), in which \( z \) is the critical exponent and \( p \) is generally not equal to \( z \). We discussed that for fixed \( z \), if the spectrum is required to be scale invariant, how should \( p \) depend on the background evolution. We concluded that there is always a room of parameters for the generation of scale invariant spectrum.

Keywords: scale invariance; dispersion relation; HL gravity

I. INTRODUCTION

Inflation, e.g. see [1], is the most prevailing scenario for early universe since it first provides an solution for the horizon, flatness and entropy problems of SBB. While the most important success of inflation is it suggests a causal generation mechanism for the superhorizon, adiabatic, scale invariant density perturbation. However, the causal generation of primordial perturbation can also be implemented in the case with the decaying speed of sound [2],[3],[4],[5], and also [6] for contracting phase. During an earlier period of universe, when the sound speed was decaying, the sound horizon contracted, thus the causal perturbation initially deep into the sound horizon can emerge and be stretched to the super sound horizon scale, and become the primordial perturbation responsible for structure formation at late time. In such a case, in principle, inflation may not be required at least for the generation of primordial perturbation. The modification of sound speed actually corresponds to the modification of dispersion relation. The cosmological applications of modified dispersion relation have been studied, which include trans-Planckian physics [7], noncommutative field approach [8], loop quantum gravity [9], Lorentz-violating models [10].

Recently, a candidate for quantum gravity, dubbed Hořava-Lifshitz (HL) gravity [11], has been studied intensively. This theory has an action that is power counting renormalizable with respect to a scaling symmetry which treats space and time differently. In the IR region, this theory naturally flows to general relativity. The black hole solutions in Hořava gravity were studied in [12],[13],[14],[15],[16],[17],[18]. The cosmological solutions was first addressed in [19],[20],[21]. It was found that the early universe in HL cosmology may be able to escape singularity and has a nonsingular bounce. This might give an alternative to inflation, as has been discussed in [22],[23]. There has been many detailed analysis for the theory of cosmological perturbations in HL cosmology [24],[25],[26],[27],[28],[29],[30],[31],[32],[33], and also [34],[35] for gravitational wave. In [24], it was argued in UV regime of HL gravity which has a critical exponent \( z = 3 \), the spectrum of primordial perturbation induced by a scalar field may be scale invariant for any power law expansion with \( a \sim t^n \) and \( n > 1/3 \). This was subsequently confirmed by solving the motion equation of perturbation mode on super sound horizon scale for any background evolution of early universe [25]. In [25], not only how the primordial perturbation generated in UV regime is matched to the observations on large IR scale was illustrated, but also the dependence of spectral index on any \( z \) and \( n \) was showed, in addition, it was also pointed out that the case of \( n < 1/3 \) for the generation of scale invariant spectrum actually corresponds to that in contracting phase. There are also many studies on other aspects of HL cosmology [36],[37],[38],[39],[40],[41],[42], and other relevant issues.

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In this paper, inspired by the investigations on the theory of cosmological perturbations in HL cosmology, we will study the spectrum of primordial perturbation induced by a scalar field with modified dispersion relation $\omega \sim \frac{k^z}{a^{p-1}}$, in which $z$ is the critical exponent. A specific case of our work with $p = z$ have been studied in \cite{25}. We will generalize the calculations of \cite{25} to the case with arbitrary $p$ and then give our discussion on the scale invariance of spectrum. The rest of the paper is organized as follows. In sec.2 we will calculate the spectrum of primordial perturbation leaded by a scalar field with modified dispersion relation $\omega \sim \frac{k^z}{a^{p-1}}$. In sec.3 we will discuss that for $z$ ranged from 1 to 4, if the spectrum is required to be scale invariant, what should the value of $p$ be. We will depict the parameter space that guarantees scale invariant spectrum. The final is our conclusion.

II. PRIMORDIAL PERTURBATION FROM MODIFIED DISPERSION RELATIONS

We will briefly calculate the primordial perturbation induced by a scalar field with modified dispersion relation $\omega \sim \frac{k^z}{a^{p-1}}$. In the momentum space, the motion equation of perturbation of scalar field $\phi$ is given by

$$u''_k + \left( \omega^2 - \frac{a''}{a} \right) u_k = 0,$$

where $u_k$ is related to the perturbation $\delta \phi$ of $\phi$ by $a\delta \phi_k = \frac{1}{\sqrt{2}}[a^-_k u^+_k(\eta) + a^+_k u^-_k(\eta)]$, $a^-_k$ and $a^+_k$ are mode annihilation and generation operators. The prime denotes the derivative with respect to the conformal time $\eta$. We can define $\Omega^2(\eta) = \omega^2 - \frac{a''}{a}$ for following discussions. Here, $\omega$ is taken as

$$\omega = \frac{k^z}{a^{p-1}M^2-1}.\tag{2}$$

When $p = z$, the case is same as that studied in \cite{25}. However, here we will regard $p$ as arbitrary value, which may be equal to $z$ and may be not. Recently, in the studies on the theory of cosmological perturbations in HL cosmology, e.g. \cite{27}, \cite{28}, \cite{29}, it has been found that there are such some terms, which might imply that the factor before $\frac{k^z}{a^{p-1}M^2-1}$ has a dependence on the time, i.e. $\omega \sim f(\eta)\frac{k^z}{a^{p-1}M^2-1}$. This actually can be rewritten as (2), since $a$ is the function of $\eta$. It is this observation that motivates our study.

The calculations of perturbation spectrum is actually a generalization of that in \cite{25}. In term of \cite{25}, the sound speed is given by $c_s = \frac{k^z}{a^{p-1}M^2-1}$ and the sound horizon is $\frac{c_s}{H}$, where $H$ is the Hubble parameter. In this case, how the primordial perturbation generated in UV regime of HL gravity is matched to the observations on large IR scale has been illustrated in e.g. Fig.1 of \cite{25}. We assume that initially all modes of fluctuation starts in their vacuum state. But in order to define vacuum in analogy with that in flat spacetime, adiabatic condition should be admitted. We need $\Omega$ change quite small during one period $\sim \Omega$ of oscillation, i.e. $|\frac{\Omega'}{\Omega^2}| << 1$. In this case, the mode function is approximatelly that in Minkovsky spacetime,

$$u_k \approx \frac{1}{\sqrt{2\Omega(k, \eta)}} \exp \left( -i \int^{\eta} \Omega(k, \eta')d\eta' \right), \text{ when } |\frac{\Omega'}{\Omega^2}| << 1 \tag{3}$$

notice $\Omega \approx \omega$ here, since we need $\frac{a''}{a} \ll \omega^2$ (equivalent to adiabatic condition) to validate the approximation, i.e. the perturbation mode is deep into sound horizon.

However, what we concern is that with $\omega \eta \ll 1$, i.e. on super sound horizon scale, which is directly related with our observation. This can be obtained by solving Eq.(1) with the initial condition \cite{25}, as has been done in the calculations of inflationary perturbation. We take a power law evolving background $a(t) = a_s(t_s)^n$ for study, where $n$ is assumed to be positive real constant, and $t_s$ is regarded as a reference time and the corresponding scale factor is
In conformal time, \( a = a_*(\frac{\eta}{\eta_*})^{(1-n)} \) where \( \eta_* = t_*(1-n) \). Eq. (1) can be reduced to Bessel equation. Thus with the initial condition (3), its solution is
\[
\delta\phi = \frac{|n_0|^2}{2\pi} \text{e}^{-|\eta|^\alpha},
\]
where \( \alpha = v(\frac{np-1}{n-1}) - 0.5(\frac{3n-1}{n-1}) \). Thus dependent on the sign of \( \frac{3n-1}{np-1} \), \( \delta\varphi_k \) evolves in 2 different cases. We can see that \( \alpha = 0 \) when \( \frac{3n-1}{np-1} > 0 \). In this case, \( \delta\varphi_k \) is constant on super horizon scale, which corresponds to the constant mode. While \( \frac{3n-1}{np-1} < 0, \alpha = \frac{3n-1}{n-1} \). In this case, the amplitude will change with time, which can be increasing or decaying, dependent on different background evolution. For example, if \( n > 1, \alpha > 0, \) and \( |\eta| \) increases in contracting phase and decays in expansion phase, thus \( \delta\varphi \) increases in contracting phase and decrease in expansion phase, since \( \eta \) runs initially from \( -\infty \) to \( 0_- \) in the expanding phase with \( n > 1 \), whereas in contracting phase \( \eta \) runs initially from \( 0_+ \) to \( \infty \). The various cases are listed in the table.

| \( n \) | expansion | contraction |
|---|---|---|
| \( n > 1 \) | decay | increase |
| \( 1/3 < n < 1 \) | decay | increase |
| \( n < 1/3 \) | increase | decay |

In next section, it will be showed that \( np > 1 \) in expanding phase and \( np < 1 \) in contracting phase are required for the emergence of primordial perturbation, i.e. initially the perturbation mode is deep into the sound horizon, and then is stretched to the super sound horizon scale. This condition together with \( \frac{3n-1}{np-1} < 0 \) or \( \frac{3n-1}{np-1} > 0 \) will prohibit some possibilities. For example, since \( \frac{3n-1}{np-1} < 0, \) if \( n > 1/3, \) then \( p < 1/n, \) in this case only the contracting phase is possible, while the decaying solution in the expanding phase is prohibited since the perturbation can not emerge in the expanding phase with \( np < 1 \).

The perturbation spectrum is given by
\[
\mathcal{P}_\varphi = \beta H_*^2 (\eta H_*)^{2a} \left( \frac{M}{H_*} \right)^{(2\pi - 2)v} \left( \frac{k}{H_*} \right)^{n_s - 1}.
\]
where the constant \( \beta \) is
\[
\beta = \frac{(\Gamma (v))^2}{4\pi^3} \left[ 2 \left( \frac{np - 1}{1 - n} \right) \right]^{2v-1} \left( \frac{n}{1 - n} \right)^{2n - 2npv + 2nv}.
\]
since \( \alpha = 0 \) and \( n_s = 1 \), which is consistent with well known result. For \( p = z = 3 \), we have \( \alpha = 0, v = 0.5 \) and \( n_s = 1 \), thus \( \mathcal{P}_\phi \sim M^2 \), which is consistent with that obtained in [24], [27]. The spectral index \( n_s \) is

\[
 n_s - 1 = 3 - z \left| \frac{3n - 1}{np - 1} \right|
\]  

(10)

When \( p = z = 1 \), [11] will reproduce the result obtained with normal dispersion relation, i.e. the spectrum is scale invariant only when \( n \gg 1 \) (inflation) or \( n = \frac{3}{2} \) (contraction with \( w \simeq 0 \)). [53], [54], [55]. While when \( p = z \), the result in [27] is recover, in which the parameter space of scale invariance of spectrum is studied in details. However, here since \( p \) can be generally not equal to \( z \), it can be expected that the parameter space for scale invariance could be enlarge, i.e. there can be more possibilities to obtain the scale invariant spectrum, which will be investigated fully in the following section.

### III. PARAMETER SPACE FOR SCALE INVARIANCE

In general, the fluctuations of field can only seed primordial perturbations inside Hubble radius, otherwise no stable vacuum of \( \delta \phi_k \) can be defined. The generation of an adiabatic, Gaussian and scale invariant spectrum of superhubble perturbations can not happen in any period of standard big bang. It is generally argued that only during a period of accelerated expansion, in which \( a/k \) grows faster then \( \frac{1}{H} \), can subhubble modes being stretched to become superhubble modes. In this case fluctuations can be generated when they were subhubble and become superhubble during inflation.

However, in the general cases with a modified dispersion relation, the argument should be changed as follows. The adiabatic condition for Eq.(11) is \( \left| \frac{\omega}{a} \right| \ll 1 \), which is satisfied when \( \omega^2 \gg a''/a \), since in this case \( \left| \frac{\omega}{a} \right| = \left| \frac{\omega}{\omega_k} \right| \) and \( \left| \frac{\omega}{a} \right| \ll 1 \) is equivalent to \( \omega^2 \gg a''/a \). Because \( a''/a \sim 1/\eta^2 \), we conclude that \( |\omega\eta| \gg 1 \) is needed for the existence of adiabatic vacuum. This result can be viewed in 2 perspectives. The first one is to define an effective wavelength as \( a/\omega \). The initial condition \( |\omega\eta| \gg 1 \) corresponds the effective wavelength deep inside the Hubble horizon at the beginning \( a/\omega \ll 1/h \). The second one is to define a sound horizon as \( c_s/\eta \), in which \( c_s = \omega/k \). \( |\omega\eta| \gg 1 \) means the physical wavelength \( a/k \) is much smaller than the sound horizon \( c_s/\eta \), thus a casual relationship can be established on super horizon scale. The details of this argument can be found in [25]. The next step is to analysis the subsequent evolution of \( |\omega\eta| \) once initial requirement is satisfied. In power law expansion or contracting \( a \sim |t|^p \) universe, \( t = 0 \) is defined to be big bang singularity. We have \( |\omega\eta| \sim |t|^{1-np} \), since \( \omega \sim a^{-p} \sim |t|^n(1-p) \) and \( |\eta| \sim |t|^{1-n} \). When our universe starts with big bang and expands with a power law expansion \( a \sim t^n \), \( np > 1 \) is required to guarantee \( |\omega\eta| \gg 1 \). What follows is \( |\omega\eta| \) decreases monopoly to much smaller than 1, and this means corresponding mode leaves the sound horizon. Otherwise, the requirement that physical wavelength should be deep inside the sound horizon in the early universe can not be assured. But what if we have another explanation for 'early universe'? Even if \( |\omega\eta| \ll 1 \) in the vicinity of big bang, there are still possibility that during a power law contracting phase, our universe starts with \( t < 0 \) and \( |t| \) quite big, thus \( |\omega\eta| \sim |t|^{1-np} \gg 1 \) if \( np < 1 \). This scenario can also make sure adiabatic approximation satisfied and \( |\omega\eta| \) decreases in the contracting phase, the physical wavelength stretched to super Hubble scale, seed the energy density perturbation once it reenters the Hubble radius. Thus the emergence of primordial perturbation can be reduced to the criteria

\[
 np > 1 \text{ in an expanding phase} \\
 np < 1 \text{ in an contracting phase} .
\]

(11)

In term of Eq.(10), the possibilities of the scale invariant spectrum for fixed \( z \) are listed in Table.(I). We set the shaded region as the Case A, which satisfies \( \frac{3n-1}{pn-1} > 0 \), while the white region as Case B, which satisfies \( \frac{3n-1}{pn-1} < 0 \). We
take $z = 1$ as a simple example for analysis. In order to get scale invariant primordial perturbation in an expanding phase, $p > 1/n$ is required at first. When $p = 1$, then in Case A, the result with normal dispersion relation is recovered, i.e. scale invariance only for $n \gg 1$. We can conclude that accelerated expansion is need if $z = 1$ and $p = 1$, this is just inflation. In Case B, the scale invariant spectrum can also be obtained when $n = \frac{2}{3}$, which corresponds to a period dominated by matter, while $np < 1$ means this period must be a contracting phase, according to (11). There is another interesting example, i.e. the case with $z = 3$ and $p = 3$. In this case, $n_s - 1 \approx 0$ for any value of $n$. This means no matter how background evolves, scale invariant spectrum can be generated. We can check Case B, and find that the solution $n = \frac{1}{3}$ and $p = 3$ lies in curve $pn = 1$ which is prohibited, since the points in this curve correspond to condition that $\omega \eta$ remains constant, and thus fails to cross the sound horizon.

In general, if $p = z \neq 3$, it is impossible to have scale invariant spectrum for arbitrary $n$, as has been studied in previous works. However, from Table (I), we can see that when $p \neq z$, it is possible to have scale invariant spectrum with any value of $n$, as long as $p$ and $n$ satisfy certain constraint. For example, for $z = 2$, it can be found that when $p = \frac{1+6n}{3n}$, which occurs for the expansion or contraction with Case A, or $p = \frac{5-6n}{3n}$, which occurs for the expansion or contraction with Case B, the spectrum is scale invariant. This means that if $n = \frac{2}{3}$, i.e. the phase dominated by matter, the scale invariance will be obtained in the expansion with $p = 2.5$ or the contraction with $p = 0.5$. Thus if $\omega = f(\eta) \frac{k^2}{a^3}$, we can have scale invariant spectrum for the expansion with $f(\eta) \sim \frac{1}{a^2}$ or the contraction with $f(\eta) \sim \sqrt{a}$. This result might be interesting for the studies on the theory of cosmological perturbations in HL cosmology. In general, it can be showed that if the detailed balance condition is required, the term with $k^6$ in the equation of perturbation mode, like Eq. (1), exactly cancels, while the term with $k^4$ actually dominates, e.g. [29]. However, it seems there is a time dependence in the $k^4$ term. We show that whether the spectrum is actually scale invariant is determined by this time dependence, and there can always be room for the generation of scale invariant spectrum.

The parameter space for scale invariance can also be represented in Fig.1 and 2. The corresponding curves of ‘valid parameter’ is plotted in Fig.1 for Case A and in Fig.2 for Case B. The black line $np = 1$ divide the parameter space into 2 regions. The pink region denotes those for contracting phases, while the uncolored region denotes those for expanding phases.

| $z$ | $n > 1/3$ | $n < 1/3$ | $n > 1/3$ | $n < 1/3$ | $n > 1/3$ | $n < 1/3$ |
|-----|-----------|-----------|-----------|-----------|-----------|-----------|
| 1   | $p > 1/n$, expansion | $p < 1/n$, contraction | $p > 1/n$, expansion | $p < 1/n$, contraction | $p > 1/n$, expansion | $p < 1/n$, contraction |
| 2   | $n > 1/3$ | $n < 1/3$ | $n > 1/3$ | $n < 1/3$ | $n > 1/3$ | $n < 1/3$ |
| 3   | $n > 1/3$ | $n < 1/3$ | $n > 1/3$ | $n < 1/3$ | $n > 1/3$ | $n < 1/3$ |
| 4   | $n > 1/3$ | $n < 1/3$ | $n > 1/3$ | $n < 1/3$ | $n > 1/3$ | $n < 1/3$ |

TABLE I: The possibilities of the scale invariant spectrum for fixed $z = 1, 2, 3, 4$ are listed.
FIG. 1: The parameter space for scale invariant spectrum in Case A. For $z=1,2,3,4$, the constraint for $p$ and $n$ is plotted in the figure as curves. Curves lie in the pink region means corresponding parameters can only lead to scale invariance in contracting phase, while any parameters setup represented by point in the uncolored region can only give scale invariant spectrum in expanding phase. For an example, we can see that for $z = 2$, if $n = \frac{2}{3}$, the spectrum is scale invariant only when $p = 2.5$, which is consistent with the equation in shaded region in the row with $z = 2$ and $n > \frac{1}{3}$ in Table.(I).

FIG. 2: The parameter space for scale invariant spectrum in Case B. For $z = 1,2,3,4$, curves lie in the pink region means corresponding parameters can only lead to scale invariance in contracting phase, while any parameter setup represented by point in the uncolored region can give scale invariant spectrum only in expanding phase. While the yellow region is prohibited. For an example, we can see for $z = 2$, if $n = \frac{2}{3}$, the spectrum is scale invariant only when $p = 0.5$, which is consistent with the equation in white region in the row with $z = 2$ and $n > \frac{1}{3}$ in Table.(I).
IV. CONCLUSION

In this paper, the spectrum of primordial perturbation induced by a scalar field with modified dispersion relation \( \omega \sim \frac{k^{p-1}}{a^2} \) is studied, in which \( z \) is the critical exponent. We generalize the calculations of [25], in which \( p = z \), to the case with arbitrary \( p \) and show the general result for scale invariant spectrum. We find that when \( p \neq z \), there is always a room of parameters for the generation of scale invariant spectrum whenever \( n \) and \( z \) are. It should be pointed out that here what we concern is only how the primordial perturbation can emerge and be scale invariant, and whether other problems of standard cosmology can be solve simultaneously still needs to be analysed. However, this work can be interesting for further studies on HL cosmology.

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