Homological dimensions of analytic Ore extensions

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Abstract

If $A$ is an algebra with finite right global dimension, then for any automorphism $\alpha$ and $\alpha$-derivation $\delta$ the right global dimension of $A[t; \alpha, \delta]$ satisfies

$$\text{rgld } A \leq \text{rgld } A[t; \alpha, \delta] \leq \text{rgld } A + 1.$$ 

We extend this result to the case of holomorphic Ore extensions and smooth crossed products by $\mathbb{Z}$ of $\otimes$-algebras.

1 Introduction

Recall the following theorem:

**Theorem 1.1** ([Wei94], Theorem 4.3.7). If $R$ is a ring then the following estimate takes place for every $n \in \mathbb{N}$:

$$\text{rgld } R[x_1, \ldots, x_n] = n + \text{rgld } R.$$ 

This theorem can be generalized to the Ore extensions $R[t; \alpha, \delta]$, as shown in [MR01]. It turns out that if the global dimension of $R$ is finite, then the global dimension of $R[t; \alpha, \delta]$ either stays the same, or increases by one.

In this paper we adapt the arguments used in [MR01, ch. 7.5] to the topological setting in order to obtain the estimates for the right homological dimensions of analytic Ore extensions (see [Pir08, ch. 4.1]) and smooth crossed products by $\mathbb{Z}$ (see [Sch93] and [PS94]). We want this paper to be self-contained, so we will provide the necessary definitions and constructions.

Below we state the result in the purely algebraic situation, which is provided in [MR01] and then we present its topological version.

**Remark.** There is an ambiguity in defining Ore extensions, which will be demonstrated below, so, to state the result in the algebraic setting, we need to fix an appropriate definition of Ore extensions:

**Definition 1.1.** Let $A$ be an algebra, $\alpha \in \text{End}(A)$ and let $\delta : A \rightarrow A$ be a $\mathbb{C}$-linear map, such that the following relation holds for every $a, b \in A$:

$$\delta(ab) = \delta(a)b + \alpha(a)\delta(b).$$

Then the Ore extension of $A$ w.r.t $\alpha$ and $\delta$ is the vector space

$$A[t; \alpha, \delta] = \left\{ \sum_{i=0}^{n} a_i t^i : a_i \in A \right\}$$

with the multiplication defined uniquely by the following conditions:

1. The relation $ta = \alpha(a)t + \delta(a)$ holds for any $a \in A$.

2. The natural inclusions $A \hookrightarrow A[t; \alpha, \delta]$ and $\mathbb{C}[t] \hookrightarrow A[t; \alpha, \delta]$ are algebra homomorphisms.
Also, if $\delta = 0$ and $\alpha$ is invertible, then one can define the Laurent Ore extension of $A$

$$A[t, t^{-1}; \alpha] = \left\{ \sum_{i=-n}^{n} a_i t^i : a_i \in A \right\}$$

with the multiplication defined the same way.

The thing is, the authors of [MR01] denote slightly different type of algebras by $A[t; \alpha, \delta]$:

**Definition 1.2.** Let $A$ be an algebra, $\tilde{\alpha} \in \text{End}(A)$ and let $\tilde{\delta} : A \to A$ be a $\mathbb{C}$-linear map, such that the following relation holds for every $a, b \in A$:

$$\tilde{\delta}(ab) = \tilde{\delta}(a)\tilde{\alpha}(b) + a\tilde{\delta}(b).$$

Then the Ore extension of $A$ w.r.t $\tilde{\alpha}$ and $\tilde{\delta}$ is the vector space

$$A_{\text{op}}[t; \tilde{\alpha}, \tilde{\delta}] = \left\{ \sum_{i=0}^{n} t^i a_i : a_i \in A \right\}$$

with the multiplication defined uniquely by the following conditions:

1. The relation $at = t\tilde{\alpha}(a) + \tilde{\delta}(a)$ holds for any $a \in A$.
2. The natural inclusions $A \hookrightarrow A_{\text{op}}[t; \tilde{\alpha}, \tilde{\delta}]$ and $\mathbb{C}[t] \hookrightarrow A_{\text{op}}[t; \tilde{\alpha}, \tilde{\delta}]$ are algebra homomorphisms.

Also, if $\tilde{\delta} = 0$ and $\tilde{\alpha}$ is invertible, then one can define the Laurent Ore extension of $A$

$$A_{\text{op}}[t, t^{-1}; \tilde{\alpha}] = \left\{ \sum_{i=-n}^{n} t^i a_i : a_i \in A \right\}$$

with the multiplication defined the same way.

It is easily seen that in the case of invertible $\alpha$, the following algebra isomorphisms take place:

$$A[x; \alpha, \delta] \cong A_{\text{op}}[x, \alpha^{-1}, -\delta \alpha^{-1}], \quad A[x, x^{-1}; \alpha] \cong A_{\text{op}}[x, x^{-1}; \alpha^{-1}].$$  \hspace{1cm} (1)

Throughout the paper, we will work with Ore extensions in the sense of Definition 1.1 (if not stated otherwise).

Now we are ready to state the result in the purely algebraic case, which is, essentially, contained in the [MR01, Theorem 5.7.3]:

**Theorem 1.2 ([MR01], Theorem 5.7.3).** Let $A$ be an algebra, let $\sigma$ be an automorphism and let $\delta$ be a $\sigma$-derivation in the sense of a Definition 1.2. Denote the right global dimension of a ring $R$ by $\text{rgld}(R)$. Then the following estimates hold:

1. $\text{rgld} A \leq \text{rgld} A_{\text{op}}[t; \sigma, \delta] \leq \text{rgld} A + 1$ if $\text{rgld} R < \infty$
2. $\text{rgld} A \leq \text{rgld} A_{\text{op}}[t, t^{-1}; \sigma] \leq \text{rgld} A + 1$
3. $\text{rgld} A_{\text{op}}[t, \sigma] = \text{rgld} A + 1$
4. $\text{rgld} A[t, t^{-1}] = \text{rgld} A + 1$

**Remark.** In fact, the above theorem still holds if we replace $A_{\text{op}}[t; \sigma, \delta]$ with $A[t; \sigma, \delta]$ due to (1).

We noticed that some of the statements in the above theorem can, indeed, be carried over to the case of topological algebras. The following theorem is the main result or our paper:

**Theorem 1.3.** Denote the global dimension of the $\hat{\otimes}$-algebra $A$ (in the fixed category of locally convex spaces) by $\text{dg}(A)$ and its bidimension by $\text{db}(A)$. Let $R$ be a $\hat{\otimes}$-algebra and let $A$ be one of two algebras:
(1) $A = \mathcal{O}(C, R; \alpha, \delta)$, where $\alpha$ is an automorphism, $\delta$ is an $\alpha$-derivation and the pair \{\alpha, \delta\} is localizable.

(2) $A = \mathcal{O}(C^\times, R; \alpha)$, where $\alpha$ is an automorphism and the pair \{\alpha, \alpha^{-1}\} is localizable.

(3) $A = \mathcal{S}(Z, R; \alpha)$, where $R$ is a Fréchet-Arens-Michael algebra, and $\alpha$ defines an $m$-tempered action of $Z$ on $R$.

Then we have
\[ db(A^{op}) \leq db(R^{op}) + 1, \quad dg(A^{op}) \leq dg(R^{op}) + 1. \]

If $R$ is a $\hat{\otimes}$-algebra with $db(R^{op}) < \infty$ and $A$ is one of the two algebras:

(1) $A = \mathcal{O}(C, R; \alpha)$, where $\alpha$ is an automorphism, and the pair \{\alpha, \alpha^{-1}\} is localizable.

(2) $A = \mathcal{O}(C^\times, R; \alpha)$, where $\alpha$ is an automorphism and the pair \{\alpha, \alpha^{-1}\} is localizable.

(3) $A = \mathcal{S}(Z, R; \alpha)$, where $R$ is a Fréchet-Arens-Michael algebra, and $\alpha$ defines an $m$-tempered action of $Z$ on $R$.

Then we have
\[ dg(R^{op}) \leq dg(A^{op}). \]

This paper is organized as follows: in the Section 2 we recall the important notions related to homological properties of topological modules, in particular, we provide definitions of homological dimensions for topological algebras and modules. In the Section 3 we compute the estimates for the homological dimensions of holomorphic Ore extensions; we use the bimodules of relative differentials to construct the required projective resolutions. In the Section 4 we compute the estimates for the smooth crossed products by $Z$. In the Section 5 we list some of the problems related to homological dimensions of Ore extensions which have not been solved in the paper.

In the Appendix A we provide the computations of algebraic and topological bimodules of relative differentials for different types of Ore extensions.

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## 2 Homological dimensions

### 2.1 Notation

**Remark.** All algebras in this paper are considered to be complex, unital and associative.

Let us introduce some notation (see [Hel86] and [Pir12] for more details). Denote by $\text{LCS}$, $\text{Fr}$ and $\text{Ban}$ the categories of complete locally convex spaces, Fréchet spaces and Banach spaces, respectively. Also we will denote the category of vector spaces by $\text{Lin}$.

For a locally convex Hausdorff space $E$ we will denote its completion by $\hat{E}$. Also for locally convex Hausdorff spaces $E, F$ the notation $E \hat{\otimes} F$ denotes the completed projective tensor product of $E, F$.

**Definition 2.1.** A full additive subcategory $\mathcal{C}$ of $\text{LCS}$ is called **admissible** if the following statements hold:

1. if $E \in \mathcal{C}$ and $F$ is topologically isomorphic to $E$, then $F \in \mathcal{C}$
2. if $E \in \mathcal{C}$ and $E_0 \subset E$ is a complemented subspace of $E$, then $E_0 \in \mathcal{C}$
3. if $E, F \in \mathcal{C}$ then $E \hat{\otimes} F \in \mathcal{C}$.

Suppose that an admissible category $\mathcal{C}$ satisfies the stronger condition
(2') if $E \in \mathcal{C}$ and $E_0 \subset E$ is a closed subspace of $E$, then $E_0$ and $(E/E_0)^\sim$ belong to $\mathcal{C}$.

Then $\mathcal{C}$ is called strictly admissible.

It is easily seen that $\text{LCS}$, $\text{Fr}$ and $\text{Ban}$ are, indeed, strictly admissible.

**Definition 2.2.** A complete locally convex algebra with jointly continuous multiplication is called a $\hat{\otimes}$-algebra. If $A, B$ are $\hat{\otimes}$-algebras and $\eta : A \to B$ is a continuous algebra homomorphism, then $(B, \eta)$ is called $A$-$\hat{\otimes}$-algebra.

**Definition 2.3.** An Arens-Michael algebra is a complete topological algebra whose topology can be defined by a family of submultiplicative seminorms.

For an arbitrary admissible category $\mathcal{C}$ denote by $\text{alg}(\mathcal{C})$ the full subcategory of the category of $\hat{\otimes}$-algebras, whose underlying locally convex spaces belong to $\mathcal{C}$.

**Definition 2.4.** Suppose that $M$ is a topological vector space with a structure of bimodule over a topological algebra $A$. In that case $M$ is a topological $A$-bimodule if the natural maps $A \times M \to M$, $(a, m) \mapsto am$ and $M \times A \to M$, $(m, a) \mapsto ma$ are separately continuous.

**Definition 2.5.** Let $A$ be a $\hat{\otimes}$-algebra and let $M$ be a complete locally convex space with a structure of a topological $A$-module w.r.t. the locally convex topology on $M$. Also suppose that the natural map $A \times M \to M$ is jointly continuous. Then we will call $M$ a left $A$-$\hat{\otimes}$-module. In a similar fashion we define right $A$-$\hat{\otimes}$-modules and $A$-$B$-$\hat{\otimes}$-bimodules.

For a fixed algebra $A \in \text{alg}(\mathcal{C})$ we denote the categories of left $A$-$\hat{\otimes}$-modules, right $A$-$\hat{\otimes}$-modules and $A$-$B$-$\hat{\otimes}$-bimodules, whose underlying locally convex spaces belong to $\mathcal{C}$, by $A$-$\text{mod}(\mathcal{C})$, $\text{mod}$-$A(\mathcal{C})$, $A$-$\text{mod}$-$B(\mathcal{C})$, respectively.

Suppose that $A \in \text{alg}(\mathcal{C})$ and we are given a complex of $A$-$\hat{\otimes}$-modules:

$$\ldots \xrightarrow{d_{n+1}} M_{n+1} \xrightarrow{d_n} M_n \xrightarrow{d_{n-1}} M_{n-1} \xrightarrow{d_{n-2}} \ldots,$$

then we will denote this complex by $\{M, d\}$.

### 2.2 Projectivity and flatness

The following definitions shall be given in the case of left modules; the definitions in the cases of right modules and bimodules are similar, just use the following category isomorphisms:

$$\text{mod}$-$A(\mathcal{C}) \simeq A^{op}$-$\text{mod}(\mathcal{C})$; $A$-$\text{mod}$-$B(\mathcal{C}) \simeq (A \hat{\otimes} B^{op})$-$\text{mod}(\mathcal{C})$.

**Remark.** Throughout this paper, $\mathcal{C}$ denotes an admissible category, if not stated otherwise. Also fix a $\hat{\otimes}$-algebra $A \in \text{alg}(\mathcal{C})$.

**Definition 2.6.** Consider a complex of $A$-$\hat{\otimes}$-modules $\{M, d\}$. Then it is called admissible $\iff$ it splits in the category $\mathcal{C}$. A morphism of $A$-$\hat{\otimes}$-modules $f : X \to Y$ is called admissible if it is one of the morphisms in an admissible complex.

**Definition 2.7.** Suppose that $F : A$-$\text{mod}(\mathcal{C}) \to \text{Lin}$ is an additive functor. Then we will call it exact $\iff$ for every admissible complex $\{M, d\}$ the corresponding complex $\{F(M), F(d)\}$ in $\text{Lin}$ is exact.

**Definition 2.8.**

1. A module $P \in A$-$\text{mod}(\mathcal{C})$ is called projective $\iff$ the functor $\text{Hom}_A(P, -)$ is exact.

2. A module $Y \in A$-$\text{mod}(\mathcal{C})$ is called flat $\iff$ the functor $(-) \hat{\otimes}_A Y : \text{mod}$-$A(\mathcal{C}) \to \text{Lin}$ is exact.

3. A module $X \in A$-$\text{mod}(\mathcal{C})$ is called free $\iff$ $X$ is isomorphic to $A \hat{\otimes} E$ for some $E \in \mathcal{C}$.

**Remark.** Suppose that $M$ is a left $A$-$\hat{\otimes}$-module. Then the following chain of implications holds:

$$M \text{ is free} \Rightarrow M \text{ is projective} \Rightarrow M \text{ is flat.}$$
Definition 2.9. Suppose that $X \in A\text{-mod}(C)$. Suppose that $X$ can be included in a following admissible complex:

$$0 \leftarrow X \leftarrow P_0 \xleftarrow{d_0} P_1 \xleftarrow{d_1} \ldots \xleftarrow{d_{n-1}} P_n \leftarrow 0 \leftarrow 0 \leftarrow \ldots,$$

where every $P_i$ is a projective module. Then we will call the complex $\{P, d\}$, where

$$\{P, d\} = 0 \leftarrow P_0 \xleftarrow{d_0} P_1 \xleftarrow{d_1} \ldots \xleftarrow{d_{n-1}} P_n \leftarrow 0,$$

the projective resolution of $X$ of length $n$. By definition, unbounded resolutions are of infinite length. Flat resolutions are defined similarly.

This allows us to define the notion of a derived functor in the topological case, for example, see [Hel86, ch 3.3]. In particular, $\text{Ext}_A^k(M, N)$ and $\text{Tor}_A^k(M, N)$ are defined similarly to the purely algebraic situation.

2.3 Homological dimensions

Definition 2.10. Consider an arbitrary module $M \in A\text{-mod}(C)$. The following number is well-defined:

$$\text{dh}_A(M) = \min \{ n \in \mathbb{Z}_{\geq 0} : \text{Ext}^{n+1}_A(M, N) = 0 \text{ for every } N \in A\text{-mod}(C) \} =$$

$$= \{ \text{the length of a shortest projective resolution of } M \} \in \{-\infty\} \cup [0, \infty].$$

It is called the left projective dimension of $M$. If $C = \text{Fr}$, then we can define the left weak projective homological dimension of $M$:

$$\text{w.dh}_A(M) = \min \{ n \in \mathbb{Z}_{\geq 0} : \text{Tor}^{n+1}_A(N, M) = 0 \text{ and } \text{Tor}^n_A(N, M) \text{ is Hausdorff for every } N \in \text{mod-}A(\text{Fr}) \} =$$

$$= \{ \text{the length of the shortest flat resolution of } M \} \in \{-\infty\} \cup [0, \infty].$$

It is called the left weak projective homological dimension of $M$.

Let us define the global dimensions:

Definition 2.11. Suppose that $C$ is an admissible category and let $A \in \text{alg}(C)$. Then we can define the following invariants of $A$:

$$\text{dg}(A) = \sup \{ \text{dh}_A(M) : M \in A\text{-mod}(C) \} - \text{the (left) global dimension of } A.$$

$$\text{db}(A) = \text{dh}_{A \otimes A^{op}}(A) - \text{the bidimension of } A.$$

If $C = \text{Fr}$, then we can define the weak global dimensions:

$$\text{w.dg}(A) = \sup \{ \text{w.dh}_A(M) : M \in A\text{-mod}(\text{Fr}) \} - \text{the weak global dimension of } A.$$

$$\text{w.db}(A) = \text{w.dh}_{A \otimes A^{op}}(A) - \text{the weak bidimension of } A.$$

3 Estimates for the bidimension and projective global dimensions of holomorphic Ore extensions

3.1 Bimodules of relative differentials

Firstly, let us give several necessary algebraic definitions:

Definition 3.1. Let $S$ be an algebra, $A$ be an $S$-algebra and $M$ be an $A$-bimodule. Then an $S$-linear map $\delta : A \rightarrow M$ is called an $S$-derivation if the following relation holds for every $a, b \in A$:

$$\delta(ab) = \delta(a)b + a\delta(b).$$
Definition 3.2. Let $A$ be an algebra and let $M$ be a right $A$-module (or a $A$-bimodule, resp.). For any endomorphism $\alpha : A \to A$ denote by $M_\alpha$ a right $A$-module (or an $A$-bimodule, resp.), which coincides with $M$ as an abelian group (left $A$-module, resp.), and whose structure of right $A$-module is defined by $m \circ a = m\alpha(a)$.

In a similar fashion one defines $aM$ for left modules.

The following definition is due to J. Cuntz and D. Quillen, see [CQ95]:

Definition 3.3. Suppose that $S$ is an algebra and $(A, \eta)$ is an $S$-algebra, where $\eta : S \to A$ is an algebra homomorphism. Denote by $\overline{A} = A/\text{Im}(\eta(S))$ the $S$-bimodule quotient. Then we can define the bimodule of relative differential 1-forms $\Omega^1_S(A) = A \otimes_S \overline{A}$. The elementary tensors in $\Omega^1_S(A)$ are usually denoted by $a_0 \otimes \overline{a_1} = a_0a_1$. The $A$-bimodule structure on $\Omega^1_S(A)$ is uniquely defined by the following relations:

$$b \circ (a_0a_1) = ba_0a_1, \quad (a_0a_1) \circ b = a_0d(a_1b) - a_0a_1db.$$

The bimodule of relative differential 1-forms together with the canonical $S$-derivation $d_A : A \to \Omega^1_S(A)$, $d_A(a) = 1 \otimes \overline{a} = da$

has the following universal property:

Proposition 3.1 ([CQ95], Proposition 2.4). For every $A$-bimodule $M$ and an $S$-derivation $D : A \to M$ there is a unique $A$-bimodule morphism $\varphi : \Omega^1_S(A) \to M$ such that the following diagram is commutative:

$$\begin{array}{ccc}
\Omega^1_S(A) & \xrightarrow{\varphi} & M \\
\downarrow{d_A} & & \downarrow{D} \\
A & & \\
\end{array}$$

(2)

Suppose that $A$ is an algebra and $M$ is an arbitrary $A$-bimodule. Then the bimodule $\Omega^1_A(T_A(M))$, where $T_A(M)$ denotes the tensor algebra of $M$, admits a relatively simple description.

Proposition 3.2 ([CQ95], Proposition 2.6). Suppose that $A$ is an algebra and $M$ is an arbitrary $A$-bimodule. Then there is a canonical $A$-bimodule isomorphism $T_A(M) \otimes_A M \otimes_A T_A(M) \simeq \Omega^1_A(T_A(M))$.

The proof of this proposition relies on utilizing the universal properties of $\Omega^1_A(M)$ and $T_A(M)$. In the appendix of this paper we compute the bimodule of relative differential 1-forms of Ore extensions, see Proposition A.1.

It turns out that $\Omega^1_S(A)$ can be included in the following short exact sequence:

Proposition 3.3 ([CQ95], Proposition 2.45). The following sequence of $A$-bimodules is exact:

$$0 \longrightarrow \Omega^1_S A \xrightarrow{j} A \otimes_S A \xrightarrow{m} A \longrightarrow 0,$$

(3)

where $j(a_0 \otimes \overline{a_1}) = j(a_0a_1) = a_0a_1 \otimes 1 - a_0 \otimes a_1$ and $m$ denotes the multiplication.

3.2 A topological version of the bimodule of relative differentials and holomorphic Ore extensions

Remark. Here $\mathcal{C}$ is an admissible category which contains $\mathcal{O}(\mathbb{C})$ and $\mathcal{O}(\mathbb{C}^\times)$ as objects. For example, let $\mathcal{C} = \text{LCS}$ or $\text{Fr}$.

Now we are ready to introduce a topological analogue of $\Omega^1_S(A)$, (see [Pir08]).

Definition 3.4. Suppose that $R$ is a $\hat{\otimes}$-algebra and $(A, \eta)$ is a $R$-$\hat{\otimes}$-algebra. Denote by $A/\text{Im}(\eta(R)) = \overline{A}$ the $R$-$\hat{\otimes}$-bimodule quotient. Then we can define the (topological) bimodule of relative differential 1-forms $\Omega^1_R(A) := A \otimes_R \overline{A}$. The elementary tensors are usually denoted by $a_0 \otimes \overline{a_1} = a_0a_1$.

The structure of $A$-$\hat{\otimes}$-bimodule on $\Omega^1_R(A)$ is uniquely defined by the following relations:

$$b \circ (a_0a_1) = ba_0a_1, \quad (a_0a_1) \circ b = a_0d(a_1b) - a_0a_1db.$$
As the reader can easily see, the construction in Definition 3.1 closely resembles the construction in Definition 3.3. It turns out that the analytic versions of Propositions 3.1-3.3 and Corollary 3.1 also hold for \( \Omega^1_R(A) \), see \cite[ch. 7]{Pir08}. Moreover, the topological version of the Proposition 3.3 yields an admissible short sequence, as the reader will see later.

To give the definition of the holomorphic Ore extension of a \( \hat{\otimes} \)-algebra, associated with an endomorphism and derivation, we need to recall the definition of localizable and \( m \)-localizable morphisms.

**Definition 3.5.** Suppose that \( E \) is a locally convex space. A linear operator \( T \in L(E) \) is \( m \)-localizable \( \iff \) there exists a generating family of seminorms \( \{ || \cdot ||_\nu : \nu \in \Lambda \} \) in \( A \) such that for every \( \nu \in \Lambda \) there exists \( C > 0 \) such that \( ||T(x)||_\nu \leq C||x||_\nu \) for every \( n > 0 \) and \( x \in A \).

A family of continuous linear maps \( T \subset L(E) \) is called localizable \( \iff \) there exists a generating family of submultiplicative seminorms \( \{ || \cdot ||_\nu : \nu \in \Lambda \} \) in \( A \) such that for every \( T \in T \) and \( \nu \in \Lambda \) there exists \( C > 0 \) such that \( ||T(x)||_\nu \leq C||x||_\nu \) for every \( x \in A \).

**Definition 3.6.** Let \( A \) be an Arens-Michael algebra. \( T \) is \( m \)-localizable \( \iff \) there exists a generating family of submultiplicative seminorms \( \{ || \cdot ||_\nu : \nu \in \Lambda \} \) in \( A \) such that for every \( \nu \in \Lambda \) there exists \( C > 0 \) such that \( ||T(x)||_\nu \leq C||x||_\nu \) for every \( n > 0 \) and \( x \in A \).

A family of continuous linear maps \( T \subset L(E) \) is called \( m \)-localizable \( \iff \) there exists a generating family of submultiplicative seminorms \( \{ || \cdot ||_\nu : \nu \in \Lambda \} \) in \( A \) such that for every \( T \in T \) and \( \nu \in \Lambda \) there exists \( C > 0 \) such that \( ||T(x)||_\nu \leq C||x||_\nu \) for every \( x \in A \).

Let \( A \) be a \( \hat{\otimes} \)-algebra and suppose that \( \alpha : A \rightarrow A \) is a localizable endomorphism of \( A \), \( \delta : A \rightarrow A \) is a localizable \( \alpha \)-derivation of \( A \). In the paper \cite{Pir08} A. Pirkovskii proves that there is a unique multiplication on the projective tensor product \( \hat{\otimes} \mathcal{O}(\mathbb{C}) \) such that the following statements hold:

1. \( A \hat{\otimes} \mathcal{O}(\mathbb{C}) \) is a \( \hat{\otimes} \)-algebra.

2. The natural inclusion \( A[t; \alpha, \delta] \hookrightarrow A \hat{\otimes} \mathcal{O}(\mathbb{C}) \) is a continuous \( A \)-algebra homomorphism.

The similar statement can be formulated for \( A \hat{\otimes} \mathcal{O}(\mathbb{C}^\times) \) and the natural inclusion

\[ A[t, t^{-1}; \alpha] \hookrightarrow A \hat{\otimes} \mathcal{O}(\mathbb{C}^\times). \]

The resulting \( \hat{\otimes} \)-algebras are denoted by \( \mathcal{O}(\mathcal{C}, A; \alpha, \delta) \) and \( \mathcal{O}(\mathcal{C}^\times, A; \alpha) \), respectively, and are called the holomorphic Ore extensions of \( A \) with respect to \( \alpha \) and \( \delta \). If \( A \) is an Arens-Michael algebra and the respective pairs of morphisms are \( m \)-localizable, then these algebras are Arens-Michael, as well.

In this paper we compute the topological bimodules of relative differential 1-forms of holomorphic Ore extensions and smooth crossed products by \( \mathbb{Z} \), see Propositions \ref{A2} and \ref{A3}.

### 3.3 Upper estimates for the bidimension

**Lemma 3.1.** Let \( R \in \text{alg}(\mathcal{C}) \) and let \( A \) be a \( R \)-\( \hat{\otimes} \)-algebra.

1. For every projective right module \( P \in \text{mod}-R \) the right \( A \)-module \( P \hat{\otimes}_RA \) is projective.

2. For every projective bimodule \( P \in R-\text{mod}-R \) and \( \alpha \in \text{Aut}(A) \) the bimodule \( A_\alpha \hat{\otimes}_RP \hat{\otimes}_RA \in A-\text{mod}-A \) is projective.

**Proof.**

1. \( P \in \text{mod}-R \) is projective iff \( P \) is a retract of a right free \( R \)-module \cite[Theorem 3.1.27]{Hel86}, in other words, there are \( E \in \mathcal{C} \) and a retraction \( \sigma : E \hat{\otimes} R \rightarrow P \). But then the map

\[ \sigma \otimes \text{Id}_A : E \hat{\otimes} A \cong E \hat{\otimes} R \hat{\otimes} R A \rightarrow P \hat{\otimes} R A \]

is a retraction, as well.
2. Any projective bimodule is a retract of a free bimodule, in other words, there exist $E \in C$ and a retraction $\sigma : R \hat{\otimes} E \hat{\otimes} R \to P$. Notice that the map

$$\text{Id}_{A_\alpha} \otimes \sigma \otimes \text{Id}_A : A_\alpha \hat{\otimes} E \hat{\otimes} A \simeq A_\alpha \hat{\otimes} R \hat{\otimes} E \hat{\otimes} R \hat{\otimes} R A \to A_\alpha \hat{\otimes} R P \hat{\otimes} R A$$

is a retraction, $A_\alpha$ is, obviously, a projective left $A$-$\hat{\otimes}$-module and $E \hat{\otimes} A$ is a projective right $A$-$\hat{\otimes}$-module. Therefore, [Hel86, Proposition 4.1.4] implies that $A_\alpha \hat{\otimes} E \hat{\otimes} A$ is projective as an $A$-$\hat{\otimes}$-bimodule, and we know that a retract of a projective bimodule is projective.

\[\square\]

**Theorem 3.1.** Suppose that $R \in \text{alg}(C)$, and $A$ is one of the two $\hat{\otimes}$-algebras:

1. $A = \mathcal{O}(C, R; \alpha, \delta)$, where the pair $\{\alpha, \delta\}$ is localizable.
2. $A = \mathcal{O}(C^\times, R; \alpha)$, where the pair $\{\alpha, \alpha^{-1}\}$ is localizable.

Then we have

$$\text{db}(A^{\text{op}}) \leq \text{db}(R^{\text{op}}) + 1.$$ 

**Proof.** Due to [Pir08, Proposition 7.2] and Proposition A.2, we have the following sequence of $A$-$\hat{\otimes}$-bimodules, which splits in the categories $R$-$\text{mod}$-$A$ and $A$-$\text{mod}$-$R$:

$$0 \longrightarrow A_\alpha \hat{\otimes} R A \longrightarrow \overset{j}{A \hat{\otimes} R A} \longrightarrow \overset{m}{A} \longrightarrow 0,$$

where $m$ is the multiplication operator. Let

$$0 \leftarrow R \leftarrow P_0 \leftarrow \cdots \leftarrow P_n \leftarrow 0$$

be a projective resolution of $R$ in $R$-$\text{mod}$-$A$. Notice that (5) splits in $R$-$\text{mod}$ and $\text{mod}$-$R$, because all objects in the resolution are projective as left and right $R$-$\hat{\otimes}$-modules. Therefore, we can apply the functors $A_\alpha \hat{\otimes} R(-)$ and $A_\alpha \hat{\otimes} R(-)$ to (5) and the resulting complexes of $A$-$\hat{\otimes}$-bimodules are still admissible:

$$0 \leftarrow A \leftarrow A \hat{\otimes} R P_0 \leftarrow \cdots \leftarrow A \hat{\otimes} R P_n \leftarrow 0$$

(6)

$$0 \leftarrow A_\alpha \leftarrow A_\alpha \hat{\otimes} R P_0 \leftarrow \cdots \leftarrow A_\alpha \hat{\otimes} R P_n \leftarrow 0.$$ 

(7)

Recall that $A$ is a free left $R$-$\hat{\otimes}$-module, so the functor $(-) \hat{\otimes} R A$ preserves admissibility, therefore the following complexes of $A$-$\hat{\otimes}$-bimodules are admissible:

$$0 \leftarrow A \hat{\otimes} R A \leftarrow A \hat{\otimes} R P_0 \hat{\otimes} R A \leftarrow \cdots \leftarrow A \hat{\otimes} R P_n \hat{\otimes} R A \leftarrow 0$$

(8)

$$0 \leftarrow A_\alpha \hat{\otimes} R A \leftarrow A_\alpha \hat{\otimes} R P_0 \hat{\otimes} R A \leftarrow \cdots \leftarrow A_\alpha \hat{\otimes} R P_n \hat{\otimes} R A \leftarrow 0$$

(9)

Lemma 3.1 implies that (8) and (9) are projective resolutions for $A \hat{\otimes} R A$ and $A_\alpha \hat{\otimes} R A$. Now we can apply [Hel86, Proposition 3.5.5] to (4), so we get

$$\text{db}(A) = \text{dh}_{A^*}(A) \leq \max\{\text{dh}_{A^*}(A \hat{\otimes} R A), \text{dh}_{A^*}(A_\alpha \hat{\otimes} R A) + 1\} \leq n + 1.$$ 

In other words, we have obtained the desired estimate

$$\text{db}(A) \leq \text{db}(R) + 1.$$ 

\[\square\]
3.4 Upper estimates for the global dimensions

**Proposition 3.4.** Let \( R \in \mathfrak{alg}(\mathcal{C}) \), let \( A \) be a \( R\hat{\otimes} \)-algebra and let \( M \) be a right \( A\hat{\otimes} \)-module.

1. For every projective resolution of \( M \) in \( \text{mod}-R \)

\[
0 \leftarrow M \leftarrow P_0 \leftarrow P_1 \leftarrow \ldots
\]

the complex

\[
0 \leftarrow M\hat{\otimes}_RA \leftarrow P_0\hat{\otimes}_RA \leftarrow P_1\hat{\otimes}_RA \leftarrow P_2\hat{\otimes}_RA \leftarrow \ldots
\]  

(10)

is a projective resolution of \( M\hat{\otimes}_RA \) in the category of right \( A\hat{\otimes} \)-modules. In particular,

\[
\text{dh}_{A^{op}}(M\hat{\otimes}_RA) \leq \text{dh}_{R^{op}}(M).
\]

2. Moreover, if \( \mathcal{C} \subset \text{Fr} \), then for every flat resolution of \( M \) in \( \text{mod}-R \)

\[
0 \leftarrow M \leftarrow F_0 \leftarrow F_1 \leftarrow \ldots
\]

the complex

\[
0 \leftarrow M\hat{\otimes}_RA \leftarrow F_0\hat{\otimes}_RA \leftarrow F_1\hat{\otimes}_RA \leftarrow F_2\hat{\otimes}_RA \leftarrow \ldots
\]  

(11)

is a flat resolution of \( M\hat{\otimes}_RA \) in the category of right \( A\hat{\otimes} \)-modules. In particular,

\[
w.dh_{A^{op}}(M\hat{\otimes}_RA) \leq w.dh_{R^{op}}(M).
\]

To prove this proposition we need the following simple lemma.

**Lemma 3.2.** Let \( F \in \text{mod}-R \) be a flat module. Then for any \( R\hat{\otimes} \)-algebra \( A \), the module \( F\hat{\otimes}_RA \in \text{mod}-A \) is flat.

**Proof.** Just use the fact that \( F\hat{\otimes}_RA \hat{\otimes}_AX \) is isomorphic to \( F\hat{\otimes}_RX \). \( \square \)

**Proof of Proposition 3.4.**

1. **Lemma 3.1** implies that \( P_i\hat{\otimes}_RA \) is a projective right \( A\hat{\otimes} \)-module for all \( i \). The complex \( \text{[1]} \) is admissible, because the functor \( (-)\hat{\otimes}_RA \) for a free \( A \) preserves admissibility, so it defines a projective resolution of \( M\hat{\otimes}_RA \) in the category \( \text{mod}-A \).

2. The proof is essentially the same, we just need to keep in mind the **Lemma 3.2**. \( \square \)

**Lemma 3.3.** Let \( R \in \mathfrak{alg}(\mathcal{C}) \), \( \alpha : R \rightarrow R \) is an automorphism and \( M \) is a right \( R \)-module, then \( \text{dh}_{R^{op}}(M_\alpha) = \text{dh}_{R^{op}}(M) \) and if \( \mathcal{C} \subset \text{Fr} \) then \( w.dh_{R^{op}}(M_\alpha) = w.dh_{R^{op}}(M) \).

**Proof.** The proof relies on the fact that \( \alpha : \text{mod}-R \rightarrow \text{mod}-R \) and \( \alpha(-) : \text{R-mod} \rightarrow \text{R-mod} \) can be viewed as automorphisms of \( \text{mod}-R \) and \( \text{R-mod} \), which preserve admissibility of morphisms, projectivity and flatness of modules.

Indeed, if \( f : M \rightarrow N \) is an admissible module homomorphism, then \( f_\alpha : M_\alpha \rightarrow N_\alpha \) is admissible, because \( \Box f_\alpha = \Box f \), where \( \Box : \text{mod}-R \rightarrow \mathcal{C} \) is the forgetful functor. The same goes for \( \alpha(-) \).

Let \( P \in \text{mod}-R \) be projective. Then for any admissible epimorphism \( \varphi : X \rightarrow Y \) we have the following chain of canonical isomorphisms:

\[
\text{Hom}(P_\alpha, Y) \simeq \text{Hom}(P, Y_{\alpha^{-1}}) \simeq \text{Hom}(P, X_{\alpha^{-1}}) \simeq \text{Hom}(P_\alpha, X).
\]

Let \( F \in \text{mod}-R \) be flat. Then \( F_\alpha\hat{\otimes}_RX \simeq F\hat{\otimes}_R(\alpha^{-1}X) \) and we already know that \( \alpha^{-1}(-) \) preserves admissibility. \( \square \)

Now we are prepared to state the theorem:
Theorem 3.2. Let $R \in \text{alg}(\mathcal{C})$ be a $\hat{\otimes}$-algebra. Suppose that $A$ is one of the two $\hat{\otimes}$-algebras:

1. $A = \mathcal{O}(\mathcal{C}, R; \alpha, \delta)$, where $\alpha$ is invertible, and the pair $\{\alpha, \delta\}$ is localizable.
2. $A = \mathcal{O}(\mathcal{C}^\times, R; \alpha)$, where the pair $\{\alpha, \alpha^{-1}\}$ is localizable.

Then the right global dimension of $A$ can be estimated as follows:

$$\text{dg}(A^{\text{op}}) \leq \text{dg}(R^{\text{op}}) + 1.$$ 

Proof. Suppose that $M$ is a right $A$-$\hat{\otimes}$-module. Then we can apply the functor $M\hat{\otimes}A(-)$ to the sequence (4). Notice that the resulting sequence of right $A$-$\hat{\otimes}$-modules

$$0 \longrightarrow M\hat{\otimes}A_\alpha \hat{\otimes}_R A \overset{\text{Id}_M \otimes i}{\longrightarrow} M\hat{\otimes}A \hat{\otimes}_R A \overset{\text{Id}_M \otimes m}{\longrightarrow} M\hat{\otimes}A \longrightarrow 0$$

is isomorphic to the sequence

$$0 \longrightarrow M_\alpha \hat{\otimes}_R A \overset{j'}{\longrightarrow} M \hat{\otimes}_R A \overset{m}{\longrightarrow} M \longrightarrow 0. \quad (12)$$

Since (4) splits in $A\text{-mod-}R$, (12) splits in $\text{mod-}R$, in particular, this is an admissible short exact sequence.

Now notice that we can apply [Hel86, Proposition 3.5.5] to (12), so we get

$$\text{dh}_{A^{\text{op}}}(M) \leq \max\{\text{dh}_{A^{\text{op}}}(M \hat{\otimes}_R A), \text{dh}_{A^{\text{op}}}(M_\alpha \hat{\otimes}_R A) + 1\} \leq \text{dh}_{R^{\text{op}}}(M) + 1$$

due to Proposition 3.4 and Lemma 3.3. Hence, the following estimate holds:

$$\text{dg}(A^{\text{op}}) \leq \text{dg}(R^{\text{op}}) + 1. \quad \square$$

3.5 Lower estimates

In order to obtain lower estimates, we need to formulate the following lemma:

Proposition 3.5. Suppose that $R, A \in \text{alg}(\mathcal{C})$ and $A$ is a free left $R$-$\hat{\otimes}$-module. Also assume that there exists an $R$-$\hat{\otimes}$-module isomorphism $\varphi : A \rightarrow R \hat{\otimes} E$ such that $\varphi(1) = 1 \otimes x$ for some $x \in E$. Then $i : M \rightarrow M \hat{\otimes}_R A, i(m) = m \otimes 1$ is an admissible monomorphism for every $M \in \text{mod-}R$.

Proof. Look at the following diagram:

$$M \overset{i}{\longrightarrow} M \hat{\otimes}_R A \overset{\text{Id}_M \otimes \varphi}{\longrightarrow} M \hat{\otimes}_R R \hat{\otimes} E \overset{\pi \otimes \text{Id}_E}{\longrightarrow} M \hat{\otimes} E,$$

where $\pi : M \hat{\otimes}_R R \rightarrow M, \quad \pi(m \otimes r) = mr$.

Due to the Hahn-Banach theorem there exists a functional $f \in E^\times$ such that $f(x) = 1$, so the map $m \rightarrow m \otimes x$ admits a right inverse, which is uniquely defined by $n \otimes y \rightarrow f(y)n$, therefore $i$ as a mapping of lcs admits a right inverse too, because $\text{Id}_M \otimes \varphi$ and $\pi \otimes \text{Id}_E$ are invertible. \quad \square

Proposition 3.6. Suppose that $\mathcal{C}$ is a strictly admissible category. Let $R, A \in \text{alg}(\mathcal{C})$ and assume that $\text{dg}(R^{\text{op}}) < \infty$. Suppose that the following conditions hold:

1. $A$ is a free left $R$-$\hat{\otimes}$-module.
2. Moreover, we can choose a $R$-$\hat{\otimes}$-module isomorphism $\varphi : A \rightarrow R \hat{\otimes} E$ in such a way that $\varphi(1) = 1 \otimes x$ for some $x \in E$.
3. $A$ is projective as a right $R$-$\hat{\otimes}$-module.
Then \( \text{dg}(R^{\text{op}}) \leq \text{dg}(A^{\text{op}}) \).

**Proof.** Fix a module \( M \in \text{mod-} R \) such that \( \text{dh}_{R^{\text{op}}}(M) = \text{dg}(R^{\text{op}}) \). The Proposition 3.5 states that the map \( M \to M \otimes R A \), \( m \mapsto m \otimes 1 \) is an admissible monomorphism; \( C \) is strictly admissible, so there exists a short admissible sequence

\[
0 \to M \overset{i}{\to} M \otimes R A \to N \to 0
\]

for some \( N \in \text{mod-} R \).

Notice that \( \text{dh}_{R^{\text{op}}}(M) = \text{dg}(R^{\text{op}}) \) and \( \text{dh}_{R^{\text{op}}}(N) \leq \text{dg}(R^{\text{op}}) \), therefore \( \text{dh}_{R^{\text{op}}}(M \otimes R A) = \text{dg}(R^{\text{op}}) \).

This equality can be proven by looking at the long exact sequence of \( \text{Ext}_R(\cdot, X) \). But \( A \) is projective as a right \( R \)-\( \hat{\otimes} \)-module, therefore \( \text{dh}_{R^{\text{op}}}(M \otimes R A) \leq \text{dh}_{A^{\text{op}}}(M \otimes R A) \), so \( \text{dg}(R^{\text{op}}) \leq \text{dg}(A^{\text{op}}) \).

**Theorem 3.3.** Let \( C \) be a strictly admissible category.

Suppose that \( R \in \text{alg}(C) \) with \( \text{dg}(R^{\text{op}}) < \infty \) and \( A \) is one of the two \( \hat{\otimes} \)-algebras:

1. \( A = \mathcal{O}(C, R; \alpha) \), where the pair \((\alpha, \alpha^{-1})\) is localizable.
2. \( A = \mathcal{O}(C^\times, R; \alpha) \), where the pair \((\alpha, \alpha^{-1})\) is localizable.

Then the conditions 1–3 of the Proposition 3.6 are satisfied. As a corollary, the following estimate takes place:

\[
\text{dg}(R^{\text{op}}) \leq \text{dg}(A^{\text{op}})
\]

**Proof.** The proofs for these two cases are pretty much the same, so here we will only provide the argument for the first case.

The first two conditions follow directly from the definition of \( \mathcal{O}(C, R; \alpha) \). To check the third condition, we can prove that \( \mathcal{O}(C, R; \alpha) \) is, in fact, a free right \( R \)-\( \hat{\otimes} \)-module.

Fix a generating family of seminorms \( \{\|\cdot\|_{\lambda} : \lambda \in \Lambda\} \) on \( R \) such that \( \|\alpha^k(r)\|_{\lambda} \leq C|k| \|r\|_{\lambda} \) for all \( k \in \mathbb{Z} \) and \( r \in R \).

Consider the following mapping:

\[
\varphi : \mathcal{O}(C, R; \alpha) \to \mathcal{O}(C) \hat{\otimes} R, \quad \varphi(rz^n) = z^n \otimes \alpha^{-n}(r).
\]

Then it is easily seen that \( \varphi \) is well-defined, because for every \( f = \sum_{i=0}^{m} r_n z^n \in R[t; \alpha], \rho > 0 \) and \( \lambda \in \Lambda \) we have

\[
\|\varphi(f)\|_{\rho, \lambda} = \left\| \sum_{i=0}^{m} z^n \otimes \lambda^{-n}(r_n) \right\|_{\rho, \lambda} \leq \sum_{i=0}^{m} \|z^n \otimes \lambda^{-n}(r_n)\|_{\rho, \lambda} = \sum_{i=0}^{m} \|r_n\|_{\lambda} C^n \|f\|_{\lambda, C^\rho}.
\]

It is also a morphism of right modules, which, obviously, admits a well-defined inverse homomorphism

\[
\varphi^{-1} : \mathcal{O}(C) \hat{\otimes} R \to \mathcal{O}(C, R; \alpha), \quad \varphi^{-1}(z^n \otimes r) = \alpha^n(r)z^n,
\]

because for every \( \lambda \in \Lambda, \rho > 0, f = \sum f_k z^k \in \mathcal{O}(C) \) and \( r \in R \) we have

\[
\|\varphi^{-1}(f \otimes r)\|_{\lambda, \rho} = \sum_{0}^{\infty} \|f_k \alpha^k(r)\|_{\lambda} \rho^k \leq \sum_{0}^{\infty} \|r\|_{\lambda} \|f_k\|_{\lambda} (C^\rho)^k = \|r\|_{\lambda} \|f\|_{C^\rho}.
\]

**Remark.** The author of this paper is not aware whether \( \mathcal{O}(C, R; \alpha, \delta) \) is free (or even projective) as a right \( R \)-\( \hat{\otimes} \)-module when \( \delta \neq 0 \).
4 Homological dimensions of smooth crossed products by $\mathbb{Z}$

The following definitions and theorems are due to L. Schweitzer, see [Sch93] or [PS94] for more detail.

**Definition 4.1.** Suppose that $R$ is a Fréchet algebra. Let $G$ be one of the groups $\mathbb{R}, \mathbb{T}$ or $\mathbb{Z}$ and suppose that $\alpha : G \to \text{Aut}(R)$ is an action of $G$ on $R$. Then $\alpha$ is called an $m$-tempered action if the topology on $R$ can be defined by a family of submultiplicative seminorms $\{\|\cdot\|_\lambda : \lambda \in \Lambda\}$ such that there exists a polynomial $p$ satisfying

$$\|\alpha_x(r)\|_\lambda \leq |p(x)| \|r\|_\lambda$$

for any $x \in G$ and $\lambda \in \Lambda$.

**Theorem 4.1 ([Sch93], Theorem 3.1.7).** Let $R$ be a Fréchet-Arens-Michael algebra with an $m$-tempered action of one of the groups $\mathbb{R}, \mathbb{T}$ or $\mathbb{Z}$. Then the projective tensor product $\mathcal{S}(G)\hat{\otimes} R$ endowed with the following multiplication:

$$(f \ast g)(x) = \int_G f(y)\alpha_y(g(y - x))dy$$

is a Fréchet-Arens-Michael algebra. This algebra is denoted by $\mathcal{S}(G, R; \alpha)$.

For example, let us consider a Fréchet-Arens-Michael algebra $R$ with a $m$-tempered action of $\mathbb{Z}$ and fix a generating family of submultiplicative seminorms $\{\|\cdot\|_\lambda : \lambda \in \Lambda\}$ on $R$, such that

$$\|\alpha_n^\Lambda(r)\|_\lambda \leq p(n) \|r\|_\lambda, \quad (r \in R, n \in \mathbb{Z}),$$

where $p$ is a polynomial. Recall the definition of the space of rapidly decreasing sequences:

**Definition 4.2.**

$$s \cong \left\{(a_n) \in \mathbb{C}^{\mathbb{Z}} : \|a\|_k = \sup_{n \in \mathbb{Z}} |a_n|(|n| + 1)^k \leq \infty \ \forall k \in \mathbb{N}\right\} = \left\{(a_n) \in \mathbb{C}^{\mathbb{Z}} : \|a\|^2_k = \sum_{n \in \mathbb{Z}} |a_n|^2(|n| + 1)^{2k} \leq \infty \ \forall k \in \mathbb{N}\right\} = \left\{(a_n) \in \mathbb{C}^{\mathbb{Z}} : \|a\|_k = \sum_{n \in \mathbb{Z}} |a_n|(|n| + 1)^k \leq \infty \ \forall k \in \mathbb{N}\right\}.$$ 

This is the Example 29.4 in [MDR97]. Then, in our case, $\mathcal{S}(\mathbb{Z}) = s$. Let us formulate the following proposition.

**Proposition 4.1.** The following $R\hat{\otimes}$-algebra isomorphism takes place:

$$\mathcal{S}(\mathbb{Z}, R; \alpha) \cong \left\{f = (f^{(k)})_{k \in \mathbb{Z}} : f^{(k)} \in R : \|f\|_{\lambda,k} = \sum_{n \in \mathbb{Z}} \|f^{(n)}\|_\lambda (|n| + 1)^k < \infty \ \text{for all} \ \lambda \in \Lambda, k \in \mathbb{N}\right\},$$

with multiplication uniquely defined by

$$f^{(k)}e_k \ast g^{(l)}e_l = f^{(k)}\alpha_k(g^{(l)})e_{k+l},$$

where $(e_k)_l = \delta_{kl}$. In other words, the invertible Ore extension $R[t, t^{-1}; \alpha]$ can be embedded into $\mathcal{S}(\mathbb{Z}, R; \alpha)$ as a dense subalgebra:

$$i : R[t, t^{-1}; \alpha] \hookrightarrow \mathcal{S}(\mathbb{Z}, R; \alpha), \quad i(t^n) = e_n.$$
Proof. Denote the RHS of the above isomorphism by $A$. Let us construct the isomorphism explicitly, define

$$
\psi : \mathcal{S}(\mathbb{Z}, R; \alpha) = s \hat{\otimes} R \to A,
\psi(x_n e_n \otimes r) = x_n \alpha_n(r)e_n.
$$

For now it is an algebra homomorphism defined only on a dense subset of $\mathcal{S}(\mathbb{Z}, R; \alpha)$, we will now prove that it is a well-defined continuous map, because for every $x = (x_i) \in c_{00}, r \in R, \lambda \in \Lambda$ and $k \in \mathbb{N}$ we have

$$
\|\psi(x \otimes r)\|_{\lambda, k} = \sum_{n \in \mathbb{Z}} \|x_n \alpha_n(r)\|_{\lambda} |n| + 1)^k \leq \|r\|_{\lambda} \sum_{n=0}^{\infty} |x_n| p(n)(|n| + 1)^k.
$$

Notice that there exists $d \in \mathbb{N}$ and $C > 0$ such that $|p(m)| \leq C(|m| + 1)^d$ for every $m \in \mathbb{Z}$, therefore we have

$$
\|\psi(x \otimes r)\|_{\lambda, k} \leq C \sum_{n \in \mathbb{Z}} |x_n| \|r\|_{\lambda} (|n| + 1)^{d+k} \leq C \|r\|_{\lambda} \|x\|_{d+k}.
$$

Similarly, we define

$$
\psi' : A \to \mathcal{S}(\mathbb{Z}, R; \alpha),
\psi'(r e_n) = e_n \otimes \alpha_n(r)
$$

and prove that it is a well-defined continuous algebra homomorphism. Obviously, $\psi' \psi = \psi' \psi = \text{Id}$. 

In the appendix we prove the Proposition A.3 which states that the structure of $\hat{\Omega}^1_{R}(\mathcal{S}(\mathbb{Z}, R; \alpha))$ is similar to the algebraic and holomorphic cases. This gives us an opportunity to formulate the following theorem:

**Theorem 4.2.** Let $R$ be a Fréchet-Arens-Michael algebra. If we denote $A = \mathcal{S}(\mathbb{Z}, R; \alpha)$, then we have

$$
db(A^{\text{op}}) \leq \db(R^{\text{op}}) + 1, \quad dg(A^{\text{op}}) \leq dg(R^{\text{op}}) + 1.
$$

The proof of the theorem is verbatim copy of the proofs of Theorems 3.1 and 3.2.

As a simple corollary from the Proposition 4.1 and Proposition 3.6 we get the lower estimates:

**Theorem 4.3.** Let $R$ be Fréchet-Arens-Michael algebra with $dg(R^{\text{op}}) < \infty$ and denote $A = \mathcal{S}(\mathbb{Z}, R; \alpha)$. Then the conditions 1–3 of the Proposition 3.6 are satisfied. In particular, we have

$$
dg(R^{\text{op}}) \leq dg(A^{\text{op}}).
$$

**5 Conclusion**

Here we will formulate some problems, which have not yet been solved in this paper.

1. We still do not know whether

$$
db(R^{\text{op}}) \leq dB(A^{\text{op}})
$$

for any of the three cases.

2. Can we adapt the argument of this paper to the weak dimensions of our algebras? It seems that the answer is “yes”, at least for the upper estimates for w.dg$(A)$.

3. In the paper we haven’t proved the lower estimate for the right projective dimensions of $\mathcal{O}(\mathbb{C}, R; \alpha, \delta)$, when $\delta \neq 0$. In particular, is this algebra free or projective as a right $R$-module for $\delta \neq 0$? It is true for algebraic Ore extensions, because we don’t have to deal with infinite series, for example.
A Relative bimodules of differential 1-forms of Ore extensions

Proposition A.1. Let $R$ be a $\mathbb{C}$-algebra. Suppose that

1. $A = R[t; \alpha, \delta]$, where $\alpha : R \to R$ is an endomorphism and $\delta : R \to R$ is an $\alpha$-derivation.
2. $A = R[t, t^{-1}; \alpha]$, where $\alpha : R \to R$ is an automorphism.

Then $\Omega^1_R(A)$ is canonically isomorphic as an $A$-bimodule to $A_\alpha \otimes_R A$.

Proof. The first part of the proof works for the both cases. Define the map $\varphi : A_\alpha \times A \to \Omega^1_R(A)$ as follows:

$$\varphi(f, g) = f d(t g) - f d(t) g = (f, g \in A).$$

This map is balanced, because

$$\varphi(f, g) + \varphi(f, g') = \varphi(f, g + g'), \quad \varphi(f, g) + \varphi(f', g) = \varphi(f + f', g), \quad (f, f', g, g' \in A)$$

and

$$\varphi(f, rg) = f (d(t) r) g = f (d(t r)) g = f (r) (d(t) g) = \varphi(f \circ r, g).$$

Also we have

$$h \varphi(f, g) = \varphi(h f, g), \quad \varphi(f, g) h = \varphi(f, g h).$$

Therefore, $\varphi$ induces a well-defined homomorphism of $A$-bimodules $\varphi : A_\alpha \otimes_R A \to \Omega^1_R(A)$.

We will use the universal property of $\Omega^1_R(A)$ to construct the inverse morphism.

1. Suppose that $A = R[t; \alpha, \delta]$. Consider the following linear mapping:

$$D : A \to A_\alpha \otimes A, \quad D(rt^n) = \sum_{k=0}^{n-1} r^k t^{n-k-1}.$$ 

Now we want to show that $D$ is an $R$-derivation. First of all, notice that for any $f = \sum_{k=0}^{m} r_k t^k \in R[t; \alpha, \delta]$ and $n \geq 0$ we have

$$D(f t^n) = \sum_{k=0}^{m} r_k D(t^{k+n}) = \sum_{k=0}^{m} r_k (D(t^k) t^n + t^k D(t^n)) = D(f) t^n + f D(t^n).$$

It suffices to show that $D(t^n r) = D(t^n) r$, let us prove it by induction w.r.t. $n$:

$$D(t^{n+1} r) = D(t^n \delta(r) + t^n \alpha(r) t) = D(t^n) \delta(r) + D(t^n) \alpha(r) t + t^n \alpha(r) \otimes 1 = D(t^n) t r + t^n \alpha(r) \otimes 1 = D(t^n) t r + t^n \otimes r = (D(t^n) + t^n D(t)) r = D(t^{n+1}) r.$$

2. Suppose that $A = R[t, t^{-1}; \alpha]$. Consider the following linear mapping:

$$D : A \to A_\alpha \otimes A, \quad D(rt^n) = \begin{cases} \sum_{k=0}^{n-1} r t^k \otimes t^{n-k-1}, & \text{if } n \geq 0, \\ - \sum_{k=1}^{n} r t^{-k} \otimes t^{n+k-1}, & \text{if } n < 0. \end{cases}$$

As in the first case, this map turns out to be an $R$-derivation. We leave the verifications of the following equalities to the reader:

(a) $D(t^n r) = D(t^n) r$ for $n \in \mathbb{Z}, r \in R$.
(b) $D(t^m t^n) = D(t^m) t^n + t^m D(t^n)$ for $n, m \in \mathbb{Z}$. 

14
The following argument works in the both cases. Notice that \( \varphi \circ D = d_A \). Denote the extension of \( D \) by \( \tilde{D} \), so \( D = \tilde{D} \circ d_A \). Therefore, we can derive from the universal property of \( \Omega\beta(A) \) that \( \varphi \circ \tilde{D} = \text{Id}_{\Omega\beta(A)} \). And

\[
\tilde{D} \circ \varphi(a \otimes b) = a(\tilde{D} \circ \varphi(1 \otimes 1))b = \tilde{D}(dt) = a \otimes b \Rightarrow \tilde{D} \circ \varphi = \text{Id}_{A_n \otimes A}.
\]

The following proposition was already proven by A. Yu. Pirkovskii, see [Pir08, Proposition 7.8], but we present another proof, which is similar to the proof of the Proposition A.1, even in the case of \textit{localizable} morphisms. Moreover, the proof can be carried over to the case of smooth crossed products by \( \mathbb{Z} \), as we will see later.

**Proposition A.2.** Let \( R \) be an Arens-Michael algebra. Suppose that \( A \) is one of the following \( \hat{\otimes} \)-algebras:

1. \( A = \mathcal{O}(\mathbb{C}, R; \alpha, \delta) \), where \( \alpha : R \to R \) is an endomorphism and \( \delta : R \to R \) is an \( \alpha \)-derivation, such that the pair \((\alpha, \delta)\) is a localizable pair of morphisms.

2. \( A = \mathcal{O}(\mathbb{C}^\times, R; \alpha) \), where \( \alpha : R \to R \) is an automorphism, such that the pair \((\alpha, \alpha^{-1})\) is a localizable pair of morphisms.

Then \( \Omega\beta_R(A) \) is canonically isomorphic to \( A_\alpha \hat{\otimes} R A \).

**Proof.** Fix a generating family of seminorms \( \{\|\cdot\|_\lambda : \lambda \in \Lambda\} \) on \( R \) such that \( \|\alpha(x)\|_\lambda \leq C \|x\|_\lambda \) and \( \|\delta(x)\|_\lambda \leq C \|x\|_\lambda \). Define the map \( A_\alpha \times A \to \Omega\beta_R(A) \) as in the proof of the Proposition 0.1:

\[
\varphi(f, g) = f d(z g) - f z d g = f(d(t)) g, \quad (f, g \in A).
\]

This is a \( R \)-balanced map (the proof is literally the same as in the Proposition A.1), also it is easily seen from the continuity of the multiplication on \( A \) that this map is continuous. Therefore, this map induces a continuous \( A \hat{\otimes} \)-bimodule homomorphism \( A_\alpha \hat{\otimes} R A \to \Omega\beta_R(A) \).

1. Suppose that \( A = \mathcal{O}(\mathbb{C}, R; \alpha, \delta) \). Consider the following linear map:

\[
D : R[z; \alpha, \delta] \to A_\alpha \hat{\otimes} R A, \quad D(rz^n) = \sum_{k=0}^{n-1} rz^k \otimes z^{n-k-1}.
\]

For now it is defined on the dense subset of \( A \); let us prove that this map is continuous. Fix \( \lambda_1, \lambda_2 \in \Lambda \) and \( \rho_1, \rho_2 \in \mathbb{R}_{\geq 0} \). Denote the projective tensor product of \( \|\cdot\|_{\lambda_1, \rho_1} \) and \( \|\cdot\|_{\lambda_2, \rho_2} \) by \( \gamma \). Then for every \( f = \sum_{k=0}^{m} f_k z^k \in R[z; \alpha, \delta] \subset A \) we have

\[
\gamma(D(f)) \leq \sum_{k=1}^{m} \gamma(D(f_k z^k)) = \sum_{k=1}^{m} \gamma \left( \sum_{l=0}^{k-1} f_k z^l \otimes z^{k-l-1} \right) \leq \sum_{k=1}^{m} \|f_k\|_{\lambda_1} \sum_{l=0}^{k-1} \|\rho_1\rho_2\| l^{-l-1} \leq \sum_{k=1}^{m} \|f_k\|_{\lambda_1} (2 \max\{\rho_1, \rho_2, 1\})^k = \|f\|_{\lambda_1, 2 \max\{\rho_1, \rho_2, 1\}}.
\]

2. Suppose that \( A = \mathcal{O}(\mathbb{C}^\times, R; \alpha) \). Consider the following linear map:

\[
D : R[t, t^{-1}; \alpha] \to A_\alpha \otimes A, \quad D(rz^n) = \begin{cases} 
\sum_{k=0}^{n-1} rz^k \otimes z^{n-k-1}, & \text{if } n \geq 0, \\
- \sum_{k=1}^{n} rz^{-k} \otimes z^{n+k-1}, & \text{if } n < 0.
\end{cases}
\]

For now it is defined on the dense subset of \( A \); let us prove that this map is continuous. Fix \( \lambda_1, \lambda_2 \in \Lambda \) and \( \rho_1, \rho_2 \in \mathbb{R}_{\geq 0} \). Denote the projective tensor product of \( \|\cdot\|_{\lambda_1, \rho_1} \) and \( \|\cdot\|_{\lambda_2, \rho_2} \) by \( \gamma \). Suppose that \( n \geq 0 \). Then we have

\[
\gamma(D(rz^n)) = \gamma \left( \sum_{l=0}^{n-1} rz^l \otimes z^{n-l-1} \right) \leq \|r\|_{\lambda_1} \sum_{l=0}^{n-1} \|\rho_1\rho_2\| l^{-l-1} \leq \|rz^n\|_{\lambda_1, 2 \max\{\rho_1, \rho_2, 1\}}.
\]
If \( n < 0 \), then
\[
\gamma(D(rz^n)) = \gamma \left( \sum_{l=1}^{[n]} rz^{-l} \otimes z^{n+l-1} \right) \leq \|r\|_{\lambda_1} \sum_{l=1}^{[n]} \lambda_1^{-l} \rho_2^{-n+l-1} \leq \|rz^n\|_{\lambda_1, 2 \min\{\rho_1, \rho_2\}, 1}.
\]

Therefore, for every \( f = \sum_{k=-m}^{m} f_k z^k \in R[z, z^{-1}; \alpha] \subset A \) we have
\[
\gamma(D(f)) = \sum_{k=-m}^{m} \gamma(D(f_k z^k)) \leq \sum_{k=-m}^{m} \|f_k z^k\|_{\lambda_1, 2(\rho_1 + \rho_2 + 1)} = \|f\|_{\lambda_1, 2(\rho_1 + \rho_2 + 1)}.
\]

Therefore, this map can be uniquely extended to the whole algebra \( A \); we will denote the extension by \( D \), as well. Notice \( D \) is also an \( R \)-derivation: the equality \( D(ab) - D(a)b - aD(b) = 0 \) holds for \( R[z; \alpha, \delta] \times R[z; \alpha, \delta] \subset A \times A \), which is a dense subset of \( A \times A \). Therefore, \( D(ab) = D(a)b + aD(b) \) for every \( a, b \in A \).

Notice that \( \varphi \circ D = d_A \). Denote the extension of \( D : A \to A_\alpha \otimes_R A \) by \( \tilde{D} : \tilde{\Omega}_{k_2}(A) \to A_\alpha \otimes_R A \), so \( D = \tilde{D} \circ d_A \). Therefore we can derive from the universal property of \( \tilde{\Omega}_{k_2}(A) \) that \( \varphi \circ \tilde{D} = \text{Id} \). And
\[
\tilde{D} \circ \varphi(a \otimes b) = a(\tilde{D} \circ \varphi(1 \otimes 1))b = \tilde{D}(dt) = a \otimes b.
\]
Therefore, the equality \( \tilde{D} \circ \varphi = \text{Id} \) holds on a dense subset of \( A_\alpha \otimes_R A \), but \( \tilde{D} \circ \varphi \) is continuous, therefore, \( \tilde{D} \circ \varphi = \text{Id} \) holds everywhere on \( A_\alpha \otimes_R A \).

\[\square\]

**Proposition A.3.** Let \( R \) be a Fréchet-Arens-Michael algebra and consider an \( m \)-tempered action \( \alpha \) of \( \mathbb{Z} \) on \( R \). If we denote the algebra \( \mathcal{S}(\mathbb{Z}, R; \alpha) \) by \( A \), then \( \tilde{\Omega}_{k_2}(A) \) is canonically isomorphic to \( A_\alpha \otimes_R A \).

**Proof.** Fix a generating system of submultiplicative seminorms \( \{\|\cdot\|_\lambda : \lambda \in \Lambda\} \) on \( R \), such that
\[
\|\alpha_n(x)\|_\lambda = \|\alpha_1^n(x)\|_\lambda \leq p(n) \|x\|_\lambda \quad (x \in R, \lambda \in \Lambda).
\]

Define the map \( \varphi : A_{\alpha_1} \times A \to \tilde{\Omega}_{k_2}(A) \) as follows:
\[
\varphi(f, g) = f d(e_1 * g) - (f * e_1) dg = f(de_1)g.
\]
It is a continuous \( R \)-balanced linear map (the proof is literally the same as in the previous propositions).

Now consider the following linear map:
\[
D : A \to A_\alpha \otimes_R A, \quad D(re_n) = \begin{cases} \sum_{k=0}^{n-1} re_k \otimes e_{n-k-1} & \text{if } n \geq 0 \\ - \sum_{k=1}^{[n]} re_{-k} \otimes e_{n+k-1} & \text{if } n < 0 \end{cases}.
\]

Let us prove that it is a well-defined and continuous map from \( A \) to \( A_\alpha \otimes_R A \). Fix \( \lambda_1, \lambda_2 \in \Lambda \) and \( k_1, k_2 \in \mathbb{Z}_{\geq 0} \). Denote the projective tensor product of \( \|\cdot\|_{\lambda_1, k_1} \) and \( \|\cdot\|_{\lambda_2, k_2} \) by \( \gamma \). Let \( n \geq 1 \), then we have
\[
\gamma(D(re_n)) = \gamma \left( \sum_{k=0}^{n-1} re_k \otimes e_{n-k-1} \right) \leq \|r\|_{\lambda_1} \left( \frac{n^{k_2} + 2^{k_1}(n - 1)^{k_2} + \ldots + n^{k_1}}{2} \right)
\]
\[
\leq \|r\|_{\lambda_1} \left( \frac{n^{k_2} + 2^{k_1}(n - 1)^{k_2} + \ldots + n^{k_1}}{2} \right) \leq \|r\|_{\lambda_1} n^{2 \max\{k_1, k_2\} + 1} = \|re_n\|_{\lambda_1, 2 \max\{k_1, k_2\} + 1}.
\]
For \( n < 0 \) the argument is pretty much the same:
\[
\gamma(D(re_n)) = \gamma \left( \sum_{k=1}^{[n]} re_{-k} \otimes e_{n+k-1} \right) \leq \|r\|_{\lambda_1} \left( \frac{2^{k_1}(|n| + 1)^{k_2} + 3^{k_1}|n|^{k_2} + \ldots + (|n| + 1)^{k_1}2^{k_2}}{2} \right)
\]
\[
\leq \|r\|_{\lambda_1} \left( \frac{2^{k_1}(|n| + 1)^{k_2} + 3^{k_1}|n|^{k_2} + \ldots + (|n| + 1)^{k_1}2^{k_2}}{2} \right) \leq \|r\|_{\lambda_1} |n|^{4 \max\{k_1, k_2\}} = 2\|re_n\|_{\lambda_1, 4 \max\{k_1, k_2\} + 1}.
\]
It is easily seen that for every \( f \in R \hat{\otimes} c_0 \) we have

\[
\gamma(Df) \leq \sum_{m \in \mathbb{Z}} \gamma(D(f^{(m)}e_m)) \leq 2 \sum_{m \in \mathbb{Z}} \left\| f^{(m)}e_m \right\|_{\lambda_1,4\max\{k_1,k_2\}+1} = 2 \left\| f \right\|_{\lambda_1,4\max\{k_1,k_2\}+1}.
\]

Then \( D \) is a \( R \)-derivation which can be uniquely extended to the whole algebra \( A \), the extension \( \tilde{D} \) is the inverse of \( \varphi \), the proof is the same as in the Proposition A.2. \( \square \)

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