Some convolution inversion questions

Michel Valadier
14 mai 2019

Abstract
Blurring of a photographic image by a wrong focus can be modeled by convolution. Is inversion a possible answer? This paper adds complements to a foregoing paper [Va] discussing convolution-inversion of some measures.

MSC2010: 46F10 (Operations with distributions), 65R30 (Improperly posed problems), 94A08 (Image processing).

1 Introduction
The inverse operation of the “blurring” effect induced by convolution (for example wrong focus, cf. [Va, Introduction]) seems not too badly achieved by a high-pass filter i.e. by convolution with a kernel as the mexican hat\(^1\) (or with more oscillations; cf. the cardinal sine function which infinitely oscillates). My naive explanation: using linear mixing\(^2\) of values of the blurred signal seems natural; and this using only values at neighbouring points; and with some negative coefficients, otherwise this would increase the blurring\(^3\). As for values at distant points, only academical examples [Va, equation (10) and Section 5 “Case of almost perfect grey”] could use them.

This is a small side of “Image Processing”. For a large point of view see [J] (book for engineers?), for a quasi-philosophical view see [M2] (maybe [M1] is a shorter version) and there are abundance of papers using wavelets (Ingrid DAUBECHIES being surely the most famous author; her most quoted work: [Da]). In all the literature these problems are said ill-posed. For a now old book cf. [TA, Ch.IV pp.91–115].

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\(^1\) In many documents, too often, this is shown with a \(3 \times 3\) or \(5 \times 5\) matrix of pixels!

\(^2\) With coefficients whose algebraic sum is 1 in order to keep the same mean value.

\(^3\) This is rather qualitative: I will not give values of amplitude and wavelength for any mexican hat.
We don’t bring any general solution but continue the observations of [Va] about prospective inverses. We recall some definitions in Section 2, prove an inversion result in Section 3 where the value 1/2 plays a central role, and give a strange inverse in Section 4 (completing thus [Va, Th.3.6]). Finally we give some naive observations about the gaussian kernel in Section 5.

2 About inverses

Let $T$ be a distribution on $\mathbb{R}^d$ (about distributions an historical reference is [S]). Its convolution product with any other distribution makes sense if $T$ has compact support but also in many different cases. The unit element is $\delta_0$ (the measure with mass 1 at the point 0). The distributions $T$ and $D$ constitute a couple of zero divisors if $T \neq 0$, $D \neq 0$ and

$$\tag{1} T * D = 0.$$ 

And $V$ (this for $T$ necessarily $\neq 0$) is an inverse of $T$ if

$$\tag{2} T * V = \delta_0,$$

in which case also $V \neq 0$. The set possibly empty of all inverses of $T$ could be denoted by $T^{*(-1)}$. With this notation (2) writes

$$V \in T^{*(-1)}.$$ 

If $V_1$ and $V_2$ are inverses of $T$, for any $\lambda \in \mathbb{R}$, $\lambda V_1 + (1 - \lambda)V_2$ is also an inverse of $T$. If (2) and (1) hold, $V + D$ is still an inverse of $T$. If $V_1$ and $V_2$ are two different inverses of $T$, then $T$ and $V_2 - V_1$ constitute a couple of zero divisors. We gave examples in [Va, Section 2].

One may ask if there exist distributions which are not zero divisors. Easy answer: $\delta_x$ for any $x \in \mathbb{R}^d$. When associativity holds any distribution admitting an inverse is not a zero divisor.

One could ask also: does there exist distributions with not any inverse? In my opinion surely: see Section 5.

Surely spaces of distributions (or of general measures) are too large. Images have compact supports! They have density and even more should be considered only as pixels. And the Fourier transform with the “tempérées distributions” (French terminology) is too magical: cf. the Dirac comb. Precise frameworks would be essential.
3 An easy inversion

For \( x \in \mathbb{R}, |x| < 1 \):

\[
(1 + x)^{-1} = 1 - x + x^2 - x^3 + \ldots,
\]

and this series seems the idea of Van Cittert\(^4\).

Let \( d \) be an integer \( \geq 1 \), \( \mathcal{M}^b(\mathbb{R}^d) \) or shortly \( \mathcal{M}^b \) the set of bounded measures on \( \mathbb{R}^d \). The total variation norm of \( \mu \in \mathcal{M}^b \) is

\[
\|\mu\| = \mu^+(\mathbb{R}^d) + \mu^-(\mathbb{R}^d).
\]

Denoting \( C_k(\mathbb{R}^d) \) the space or continuous functions with compact support on \( \mathbb{R}^d \), and \( \|\varphi\| \) the uniform norm of \( \varphi \in C_k(\mathbb{R}^d) \), there holds

\[
\|\mu\| = \sup \left\{ \int \varphi \, d\mu : \varphi \in C_k(\mathbb{R}^d), \|\varphi\| \leq 1 \right\}.
\]

Equipped with the convolution product, \( \mathcal{M}^b \) is a Banach algebra. For \( n \geq 1 \), we note \( \mu^{\ast n} \) the \( n \)-power convolution of \( \mu \) i.e.

\[
\mu^{\ast n} = \mu \ast \cdots \ast \mu.
\]

**Theorem 1** Let \( \mu \in \mathcal{M}^b(\mathbb{R}^d) \) satisfying \( \|\mu\| < 1 \). Then the series

\[
\delta_0 + \sum_{k=1}^{\infty} (-1)^k \mu^{\ast k}
\]

(3)

converges in \( \mathcal{M}^b \) to a measure \( \nu \) which is a convolution inverse of \( \delta_0 + \mu \) in the space of bounded measures on \( \mathbb{R}^d \).

**Remark.** If \( \mu \geq 0 \) (and still \( \|\mu\| < 1 \)) and one considers

\[
\frac{1}{1 + \|\mu\|}(\delta_0 + \mu)
\]

we have, as in Duval [Du], a probability measure where \( \delta_0 \) weight is \( > \frac{1}{2} \) (note that Duval, besides this hypothesis, uses wavelets!).

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\(^4\) See in German [VC2]. This Wikipedia page quotes B. Jähne whose book is now in its sixth edition; I don’t see it. See specially, in the fifth edition [J] 17.8.4 pp. 478–480. Reference [VC1] is given by [B] and in latest versions of Wikipedia.
Proof. For two bounded measures $\lambda_1$ and $\lambda_2$, there holds
\[ \|\lambda_1 * \lambda_2\| \leq \|\lambda_1\| \|\lambda_2\| \]
because, for $\varphi \in C_\kappa(\mathbb{R}^d)$ with norm $\leq 1$, one has
\[
\int_{\mathbb{R}^d} \varphi \, d(\lambda_1 * \lambda_2) = \int_{\mathbb{R}^d} \left[ \int_{\mathbb{R}^d} \varphi(x + y) \, d\lambda_1(x) \right] \, d\lambda_2(y) \\
\leq \int_{\mathbb{R}^d} \|\lambda_1\| \, d\lambda_2(y) \leq \|\lambda_1\| \|\lambda_2\| .
\]
Hence $\|\mu^k\| \leq \|\mu\|^k$. Let us set
\[
\nu_n = \delta_0 + \sum_{k=1}^n (-1)^k \mu^k .
\]
This series does converge (it is absolutely convergent) and, as
\[
(\delta_0 + \mu) * \nu_n = \delta_0 + (-1)^n \mu^{(n+1)^*}
\]
there holds
\[
(\delta_0 + \mu) * \lim_n \nu_n = \delta_0 . \qed
\]

In [Va, Theorem 3.6] an inverse of $\frac{1-a}{2} \delta_{-1} + a \delta_0 + \frac{1-a}{2} \delta_1$, with the parameter $a$ belonging to $\frac{1}{2}, 1$, is given. Surely Theorem 1 could apply because
\[
\frac{1-a}{2} \delta_{-1} + a \delta_0 + \frac{1-a}{2} \delta_1 = a \left[ \delta_0 + \frac{1-a}{2a} (\delta_{-1} + \delta_1) \right]
\]
and, thanks to $a \in \frac{1}{2}, 1$, one has $\|\frac{1-a}{2a} (\delta_{-1} + \delta_1)\| < 1$. Thus setting $\mu = \frac{1-a}{2a} (\delta_{-1} + \delta_1)$, hypotheses of Theorem 1 are verified.

4 A strange inverse

The measure
\[
\mu := \frac{1}{4} \delta_{-1} + \frac{1}{2} \delta_0 + \frac{1}{4} \delta_1 = \frac{1}{2} \left[ \delta_0 + \frac{1}{2} (\delta_{-1} + \delta_1) \right]
\]
is the limit in Theorem 3.6 of [Va] when $a$ tends to 1/2: there the coefficients in the expression of the inverse explode to infinity. And we left open the question of existence of an inverse.
But another way is possible. “Lateral” inversion in $\mathcal{D}_+^\prime$ (or in $\mathcal{D}_-^\prime$) is possible. Indeed

$$\frac{1}{4} \delta_{-1} + \frac{1}{2} \delta_0 + \frac{1}{4} \delta_1 = \frac{1}{4} (\delta_{-1} + \delta_0) * (\delta_0 + \delta_1)$$

and (cf. [Va, Lemma 3.1]) $\delta_0 + \delta_1$ admits as inverse

$$\delta_0 - \delta_1 + \delta_2 - \delta_3 + \ldots$$

just as $\delta_{-1} + \delta_0$ admits as inverse

$$\delta_1 - \delta_2 + \delta_3 - \delta_4 + \ldots$$

Multiplying term by term (5) and (6) gives

$$\left[(\delta_{-1} + \delta_0) * (\delta_0 + \delta_1)\right]^{(-1)} \ni \delta_{-1} - 2 \delta_2 + 3 \delta_3 - 4 \delta_4 + \ldots$$

(and multiplying this by 4 would give an inverse of $\mu$ in $\mathcal{D}_+^\prime$). But this can also be done on left, in $\mathcal{D}_-^\prime$: an inverse of $\delta_0 + \delta_1$ is

$$\delta_{-1} - \delta_{-2} + \delta_{-3} + \ldots$$

while $\delta_{-1} + \delta_0$ admits as inverse

$$\delta_0 - \delta_{-1} + \delta_{-2} - \delta_{-3} + \ldots$$

There multiplying (8) and (9) gives

$$\left[(\delta_{-1} + \delta_0) * (\delta_0 + \delta_1)\right]^{(-1)} \ni \delta_{-1} - 2 \delta_{-2} + 3 \delta_{-3} - 4 \delta_{-4} + \ldots$$

and taking the half sum of the right members of (7) and (10) one gets the following symmetric inverse of $(\delta_{-1} + \delta_0) * (\delta_0 + \delta_1)$

$$\ldots + \frac{3}{2} \delta_{-3} - \frac{3}{2} \delta_{-2} + \frac{1}{2} \delta_{-1} + \frac{1}{2} \delta_1 - \frac{3}{2} \delta_2 + \frac{1}{2} \delta_3 + \ldots$$

This can be checked directly. Note that the coefficients oscillate, take negative values and tend to infinity. Multiplying by 4 we get the following

**Theorem 2** The measure $\mu$ defined by (4) admits as inverse in $\mathcal{D}'(\mathbb{R})$ (or also in the space of all measures on $\mathbb{Z}$)

$$\nu := \ldots + 6 \delta_{-3} - 4 \delta_{-2} + 2 \delta_{-1} + 2 \delta_1 - 4 \delta_2 + 6 \delta_3 + \ldots$$

where for $n \in \mathbb{Z}$ the coefficient of $\delta_n$ is $2 |n| (-1)^{|n|+1}$. Thus for any image\(^5\) $f \in \mathbb{R}^{(\mathbb{Z})}$ with compact support there holds $(f * \mu) * \nu = f$.

\(^5\) We denote as Bourbaki by $\mathbb{R}^{(\mathbb{Z})}$ the set of all sequences on $\mathbb{Z}$ with compact support.
As pessimistic (with respect to treatment of badly focused images) observations note that in Theorem 3.3 of [Va] we got for $1/2 (\delta_0 + \delta_1)$ the inverse

$$H = \ldots - \delta_{-4} + \delta_{-3} - \delta_{-2} + \delta_{-1} + \delta_0 - \delta_1 + \delta_2 - \delta_3 + \ldots$$

whose coefficients do not tend to 0, and that in Theorem 2 above the coefficients tend to $+\infty$.

### 5 The gaussian kernel case

Let $f$ be a real valued function on $\mathbb{R}^d$ belonging to $L^1$ or even to $L^1 \cap L^2$. We are interested by $f$ (for example the luminous intensity of a monochrom image; this could be written with $d = 2$) but observe $g$ given by

$$g = f \ast h \quad \text{where} \quad h(x) = \frac{1}{(\sqrt{2\pi})^d} e^{-\|x\|^2/2}.$$  

Note that $h$ is the density of the standard gaussian law $N(0, 1_d)$. Hence

$$g(y) = \int_{\mathbb{R}^d} f(x) \frac{1}{(\sqrt{2\pi})^d} e^{-\|y-x\|^2/2} dx.$$  

Let $\mathcal{F}$ denote the Fourier transform used by Probabilists:

$$\mathcal{F}(\varphi)(u) = \int e^{i(u, v)} \varphi(v) dv,$$

for which

$$\mathcal{F}(N(0, 1_d))(u) = e^{-\|u\|^2/2}$$

and

$$[\mathcal{F}^{-1}(\psi)](y) = \frac{1}{(2\pi)^d} \int e^{-i(y, u)} \psi(u) du.$$  

The Fourier transform of a convolution product being the product of the transforms, (11) implies

$$\mathcal{F}(g) = \mathcal{F}(f) e^{-\|\cdot\|^2/2}.$$  

Hence

$$f = \mathcal{F}^{-1}(\mathcal{F}(g) e^{\|\cdot\|^2/2}).$$
Thus theoretically one can, using Fourier, recover the initial signal. But any error in the knowledge of \( g \) will make (12) unusable: the multiplicative factor \( e^{\|t\|^2/2} \) may have considerable effects.

I doubt that \( h \) may have any inverse for the convolution product. The right-hand side of (12) has value at \( x \)

\[
\frac{1}{(2\pi)^d} \int e^{-i(x,t)} e^{\|t\|^2/2} \left[ \int e^{i(t,y)} g(y) dy \right] dt.
\]

I do not see how one could eliminate \( t \) and get an expression such as

\[
\int k(x - y) g(y) dy.
\]

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