Holographic Superfluid Solitons with Backreaction

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Abstract: The formation of Cooper pairs in the Bardeen-Cooper-Schrieffer (BCS) theory of superfluidity, and the condensation of pre-formed bosonic particles in Bose-Einstein condensation (BEC) are connected to each other by a continuous deformation of the interaction strength. This is referred to as the BCS-BEC crossover, and may be probed by solitonic configurations that exist in both cases. In holographic superfluids that describe strongly interacting phases of matter, the BCS-BEC crossover can be achieved by changing the boundary conditions for the charged scalar field dual to the order parameter condensate. This mechanism has been considered previously in the probe limit, neglecting the backreaction of the order parameter condensate on the dual geometry. In this work, we include the backreaction to investigate the BCS and BEC limits by constructing the corresponding dark soliton configurations in holography. The strength of backreaction is parametrized by the Newton constant in the holographic bulk dual. We study the dependence of the charge depletion and energy density in the core of the soliton on the backreaction strength. We find that the charge depletion at the core of the soliton decreases with increasing backreaction strength. We interpret this qualitatively in terms of the balance between uncondensed and condensed charge in the boundary theory as a homogeneous state is reached at strong backreaction. Finally, the inclusion of backreaction enables us to obtain the effective energy density of the soliton configurations, which together with a calculation of the surface tension enables us to confirm the explanation of the snake instability of dark solitons by a simple physical argument within holography.

Keywords: Soliton, Holography, Backreaction
1 Introduction

Superfluidity is collective quantum phenomena occurring in both bosonic and fermionic systems at low temperatures. In particular, fermionic systems can interpolate in a smooth way between the Bardeen-Cooper-Schrieffer (BCS) condensation of weakly correlated pairs and the Bose-Einstein-Condensation (BEC) of pre-formed bosonic pairs bilinear in the elementary fermions. This is known as the BCS-BEC crossover. This crossover has been realized by cooling fermionic atomic gases to ultra-low temperature and tuning the interactions in the fermionic gas with a controllable external magnetic field in the laboratory (see [1] for a review).

The qualitative essence of the BCS-BEC crossover can be understood from the phase diagram depicted in figure 1: The horizontal axis interpolates between the BCS regime of a weakly attractive interaction, and the BEC limit of very strong attraction between fermions. Above the pairing onset temperature $T^*$, the system is a normal Fermi liquid consisting of the unpaired fermions on the BCS side, and a normal Bose liquid in which the fermions have formed bosonic molecules on the BEC side. As the temperature is lowered on the BCS side, loosely bound Cooper pairs of fermions start to form at $T < T^*$ and condense below a critical temperature $T_c$. On the BEC side, Bose-Einstein condensation of the bosonic molecules occurs for $T < T_c$. Between these two limit and at $T < T_c$, there exists a strongly coupled regime of unconventional superconductivity and superfluidity around the point of infinite scattering length $\frac{1}{k_F a} = 0$, the so-called unitary Fermi gas.

Gauge/gravity duality [2–4] is a powerful tool to describe strongly coupled and correlated systems. Many problems associated with strongly interacting condensed matter physics are tractable in corresponding holographic models [5]. One of these problems is
unconventional superconductivity and superfluidity [6–8], making it relevant for the description of the unitary fermion regime in the BCS-BEC crossover.

Superfluids are quantum fluids which can sustain nonperturbative solitonic excitations. One such kind of solitons, the so-called dark solitons [9], interpolate from the symmetry broken phase far away from the soliton to the symmetry restored phase at the solitons core. Holographic dark solitonic configurations were used in [10] to study the behavior of such holographic superfluids. In [11, 12], dark solitons were studied in the probe limit. As suggested in [11, 12], the two regimes in the BCS-BEC crossover are realized holographically by the two different identifications of source and VEV with the leading and subleading terms in the boundary expansion of the charged scalar dual to the condensate [15–17].

By comparing the particle number density depletion within the holographic solitons core with results from superfluid experiments, the standard (alternative) quantization [15–17] respectively was found to correspond to the BCS (BEC) limit in figure 1. In particular, it was found experimentally that the depletion factor of the particle number density behaves very differently in the two regimes: In the BEC limit the soliton core contains nearly no particles, i.e. the depletion is nearly maximal (100%), while in the BCS limit the soliton core still contains some particles at very low temperatures, i.e. the depletion is smaller (less than 60%). Moreover, the authors of [11, 12] conjectured that one may implement the BCS-
BEC crossover in the holographic superfluid systems via varying the scaling dimension of the condensing operator, which was constructed by means of a double trace deformation in [13], very similar to the double trace deformation in a holographic Kondo model proposed in [14]. Introducing such a deformation [15–17] for the charged scalar operator $O$ describing the condensation can be achieved in the large $N$ boundary theory by imposing a linear combination of Dirichlet and Neumann type boundary conditions for the field dual to $O$. The two limiting cases of pure Dirichlet and pure Neumann type boundary conditions respectively correspond, in the language of the AdS/CFT correspondence, to the standard and alternative quantization.

The argument of [10–12] leaves an important caveat: While the experimental results are obtained at nearly zero temperature, the holographic probe limit that ignores the backreaction of the matter fields onto the metric is known to break down in the low temperature regime. In particular the condensate in the alternative quantization diverges near zero temperature in the probe limit [7], which is a sign of the backreaction becoming important at low temperatures. In this work we hence study the behavior of dark solitons in holographic superfluid system including the backreaction onto the metric. The dark soliton configurations are constructed by numerically solving Einstein’s equations coupled to the matter fields holographically dual to boundary superfluid system. We in particular employ the DeTurck method, first introduced in [18] and further developed in [19], for finding stationary solutions. This method explicitly breaks the diffeomorphism invariance of gravity, which results in a manifestly elliptic form variant of Einstein’s equation with a well-posed boundary value problem.

Another drawback of the probe limit is that the boundary stress-energy tensor of the condensate cannot be investigated, which conceals the information about important thermodynamic quantities such as the effective energy (mass) and entropy of the soliton. Taking into account the backreaction allows us to extract these quantities. Interestingly, it turns out that our result for the effective energy (mass) of the dark soliton together with the surface tension of our holographic dark solitons is consistent with previous expectations [31, 33] for the physical mechanism of a particular instability of dark solitons, the so-called \textit{snake instability}. The snake instability is an instability of solitons under transverse perturbations, leading to the spontaneous formation of a snake-like bending of the solitons. The snake instability was observed in different physical systems [36–38], and attracted much theoretical attention [33, 39–42]. In holography, the authors of [32] identified the snake instability of holographic superfluids in the probe limit via the appearance of unstable quasi-normal modes in the bulk, and observed the final decay of the ‘snake’ into vortex-anti vortex pairs. The investigation of [32] is systematic but not as intuitive as effective arguments from mechanics or hydrodynamics (see, e.g. [31, 33]). In this work we holographically confirm the explanation of the \textit{snake instability} of dark solitons [31, 33] by calculating the negative effective mass responsible for the self-acceleration effect [31], as well as the positive surface tension responsible for the spontaneous generation of ripples on the soliton [33].

We furthermore calculate the particle number density depletion at the core of the soliton. As shown in [10–12], the density depletion in the probe limit does not reach 100%. One may expect that the reason for this is the neglected backreaction, and that the deple-
tion will increase if the backreaction is included. However, our results show the opposite behavior: As the Newton constant and hence the backreaction is increased, the charge depletion decreases again compared to the probe limit. This behavior is observed in both the BCS and BEC limit of our holographic model. We give a qualitative interpretation in terms of the balance between uncondensed and condensed charge in the boundary theory as the condensed state is reached: First, we show that the homogeneous part of the condensate far away from the core decreases with backreaction in both quantizations, while the condensate at the core is fixed to be zero by the topological structure of the soliton. Second, we recast the total charge conservation equation for fixed backreaction in terms of the ratios of the respective uncondensed and condensed charge normalized over the total charge. While the total charge itself changes with varying backreaction strength, we find that the two ratios follow a simple balance as the backreaction is increased. Taking these two facts together, we deduce that the charge depletion at the soliton core must decrease as the system approaches a homogeneous uncondensed state for increasing backreaction.

This paper is organized as follows: In next section, we briefly introduce our holographic superfluid model and analyze the Ansätze and boundary conditions necessary for solving the equations of motion. In section 3, our numerical scheme and main numerical results are discussed. Section 4 is devoted to the thermodynamics of the holographic dark soliton and to holographically confirm the mechanism of the snake instability. A summary and outlook are included in section 5.

2 Holographic Setup

We work with the simplest holographic superfluid model which requires gravity coupled to a Maxwell field $A_\mu$ and a massive charged scalar field $\Psi$ with charge $e$. The bulk action reads

$$S = \int d^4x \sqrt{-g} \left[ \frac{1}{2\kappa^2_4} (R - 2\Lambda) - \frac{1}{e^2} \left( \frac{1}{4} F_{\mu\nu} F^{\mu\nu} + |D\Psi|^2 + m^2 |\Psi|^2 \right) \right],$$

(2.1)

where we have rescaled the gauge field $A_\mu$ and the scalar $\Psi$ to $\frac{A_\mu}{e}$, $\frac{\Psi}{e}$ compared to the original form [6, 8]. $L$ is the AdS radius related to the cosmological constant as $\Lambda = -\frac{3}{L^2}$, and $m$ is the mass of the charged scalar. The covariant derivative is $D_\mu = \nabla_\mu - iA_\mu$, where $\nabla_\mu$ is the Christoffel covariant derivative w.r.t. the background metric $g_{\mu\nu}$. $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is the field strength. In the rest of the paper we set $L = 1$ by a rescaling of the radial coordinate. The equations of motion derived from the action take the following form:

$$R_{\mu\nu} - \Lambda g_{\mu\nu} - \frac{2\kappa^2_4}{e^2} \left\{ \frac{1}{2} \left[ D_\mu \Psi (D_\nu \Psi)^\dagger + D_\nu \Psi (D_\mu \Psi)^\dagger + g_{\mu\nu} m^2 |\Psi|^2 \right] 
+ \left( \frac{1}{2} F_{\mu\sigma} F^{\sigma}_\nu \right) - \frac{g_{\mu\nu}}{8} F_{\rho\sigma} F^{\rho\sigma} \right\} = 0,$$

(2.2)

$$D^\mu D_\mu \Psi - m^2 \Psi = 0,$$

(2.3)

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\[ \nabla_\mu F^{\mu\nu} = ig^{\mu\nu} \left[ \Psi^\dagger (D_\mu \Psi) - \Psi (D_\mu \Psi)^\dagger \right]. \]  

In the so-called probe limit \( \frac{2\kappa^2}{\epsilon^2} \ll 1 \), the backreaction of the terms involving the gauge field and the charged scalar on the background geometry in (2.2) can be neglected. In this limit, one can first solve the Einstein equations \( R_{\mu\nu} - \Lambda g_{\mu\nu} = 0 \) for the fixed background metric \( g_{\mu\nu} \), and then solve the matter equations (2.3) and (2.4) on top of that fixed background. Once \( \frac{2\kappa^2}{\epsilon^2} \) is not small this is not possible any longer, and full coupled set of equations (2.2)-(2.4) have to be solved. In this work we are interested in the effect of backreaction. In the following we will set the charge of the scalar \( e = 1 \) and define the backreaction parameter \( \epsilon \equiv 2\kappa^2 \) as a measure of the strength of backreaction.

In the absence of the charged scalar in (2.1), the solution of Einstein equations is the Reissner-Nordstrom-AdS (RN-AdS) black brane

\[ ds^2 = \frac{1}{z^2} \left[ -f (z) \, dt^2 + \frac{dz^2}{f(z)} + dz^2 + dy^2 \right], \]  

\[ f (z) = 1 - \left( 1 + \frac{\epsilon \mu^2 z_+^2}{4} \right) \left( \frac{z}{z_+} \right)^3 + \frac{\epsilon \mu^2 z_+^2}{4} \left( \frac{z}{z_+} \right)^4, \]  

\[ A = \mu \left[ 1 - \left( \frac{z}{z_+} \right) \right] dt, \]  

where \( \mu \) is the chemical potential and \( z_+ \) parametrizes the black brane temperature via

\[ T = \frac{1}{4\pi z_+} \left( 3 - \frac{\epsilon \mu^2 z_+^2}{4} \right). \]  

For numerical convenience [20], we take following radical coordinate transformation with \( z_h \equiv \frac{1}{z_+} \),

\[ z = \frac{1-r^2}{z_h}. \]  

In order to construct the backreacted geometries, we use the following Ansatz compatible with staticity and translation invariance in the second boundary direction \( y \),

\[ ds^2 = \frac{z_h^2}{(1-r^2)^2} \left[ -Q_1 f (r) \, dt^2 + \frac{4r^2 Q_2 dr^2}{z_h^2 f (r)} + Q_4 \left( dx - \frac{2r}{z_h} Q_3 dr \right)^2 + Q_5 dy^2 \right], \]  

\[ \Psi = \left( \frac{1-r^2}{z_h} \right) Q_6, \]  

\[ A = \mu r^2 Q_7 dt. \]  

Here \( \{Q_i | i = 1, 2, \cdots, 7 \} \) are functions of \( r \) and \( x \) to be determined by solving (2.2)-(2.4). In the coordinate (2.9), the conformal boundary is located at \( r = 1 \), while the horizon is at
\( r = 0 \). The Ansatz (2.10) is chosen such that at the horizon the \( Q_i \) are regular. Expanding the equations of motion near the horizon as a power series in \( r \) yields boundary conditions
\[
Q_1|_{r=0} = Q_2|_{r=0}; \quad (\partial_r Q_j)|_{r=0} = 0, \quad j = 2, 3, \ldots, 7. \tag{2.13}
\]
The Dirichlet condition for \( Q_1 \) in (2.13) in particular ensures that the temperature of the black brane is still given by (2.8). At the conformal boundary, we demand that the metric approaches \( AdS_4 \), i.e.
\[
Q_1|_{r=1} = Q_2|_{r=1} = Q_4|_{r=1} = Q_5|_{r=1} = 1; \quad Q_3|_{r=1} = 0. \tag{2.14}
\]
In the asymptotically AdS regime, the scalar field \( \Psi \) behaves in the \( z \) coordinate as
\[
\Psi = \psi_- z^{\Delta_-} + \psi_+ z^{\Delta_+} + \ldots. \tag{2.15}
\]
Here \( \Delta_\pm = 3/2 \pm \sqrt{9/4 + m^2 L^2} \) is fixed in terms of the scalar mass \( m^2 L^2 \) in units of the AdS radius \( L \). In the rest of this paper we choose \( m^2 L^2 = -2 \). For this value, in standard quantization \([21, 22]\) one identifies \( \psi_- \) with the source for the scaling dimension \( \Delta_+ = 2 \) operator \( O_2 \) dual to \( \psi \), while \( \psi_+ \) can be found to contain the vacuum expectation value \( \langle O_2 \rangle \) of the operator \( O_2 \). We choose this value for the scalar mass, since the fermion bilinear condensates in the weakly interacting BCS limit in the 2+1-dimensional field theory dual to (2.1) will have this engineering dimension. With the double trace deformation of \([15–17]\), one can find another fixed point in which the identification of source and VEV is interchanged. In this alternative quantization, \( \psi \) is dual to the scaling dimension \( \Delta_- = 1 \) operator \( O_1 \) with source \( \psi_+ \) and expectation value \( \psi_- \). Since we want the condensate to form spontaneously, without being sourced, we will impose the \( \psi_- = 0 \) UV boundary condition in the standard quantization corresponding to the BCS limit, and the \( \psi_+ = 0 \) boundary condition in the alternative case corresponding to the BEC limit.

Finally, the gauge field admits the usual UV expansion
\[
A_t = \mu - \rho z + \ldots. \tag{2.16}
\]
Here \( \mu \) is the chemical potential, and \( \rho \) is the charge (or particle number) density. Since the ground state of the system is conformal, we can scale out one dimensionfull quantity. Throughout this paper, we do so by normalizing all dimensionfull quantities to the chemical potential, which we set to the fixed value \( \mu = 5.6. \)

### 3 Numerical Scheme and Results

We employ the DeTurck method to numerically solve Einsteins equations, for a recent review c.f. \([23]\). This method consists of adding the gauge fixing term \( \frac{1}{2} (\mathcal{L}_\xi g)_{\mu\nu} = \nabla_{(\mu} \xi_{\nu)} \) term to Einsteins equations (2.2), which breaks all diffeomorphisms and yields elliptic Einstein-DeTurck equations,
\[
R_{\mu\nu} - \Lambda g_{\mu\nu} - \epsilon \left\{ \frac{1}{2} \left[ D_\mu \psi (D_\nu \Psi)^\dagger + D_\nu \Psi (D_\mu \Psi)^\dagger + g_{\mu\nu} m^2 |\Psi|^2 \right] + \left( \frac{1}{2} F_{\mu\nu} F^{\sigma}_{\nu} - \frac{g_{\mu\nu}}{8} F^{\rho\sigma} F_{\rho\sigma} \right) \right\} - \nabla_{(\mu} \xi_{\nu)} = 0. \tag{3.1}
\]
\(^1\)This value turned out to be numerically convenient in terms of convergence speed of our code.
Figure 2: Absolute (a) and normalized (b) critical temperature for varying backreaction parameter $\epsilon$. $T_0$ is the critical temperature in the probe limit $\epsilon = 0$. The critical temperature drops with increasing backreaction for both standard (BCS) as well as alternative quantization (BEC), indicating a suppression of the condensation mechanism.

Here the DeTurck vector $\xi^\nu \equiv g^{\rho\sigma} \left[ \Gamma^\nu_{\rho\sigma}(g) - \Gamma^\nu_{\rho\sigma}(\bar{g}) \right]$ is constructed from the difference of the Christoffel symbols of the metric $g_{\mu\nu}$ which we aim to solve for, and a reference metric $\bar{g}_{\mu\nu}$. The reference metric has to have the same asymptotics and symmetries as the metric $g_{\mu\nu}$ we are trying to solve for. In our scheme, we take the standard AdS Reissner-Nordstrom metric (2.5), corresponding to $Q_1 = Q_2 = Q_4 = Q_5 = 1$ and $Q_3 = 0$ in (2.10). We then find solutions to the Einstein-DeTurck equation under the constraint condition $\xi^\mu = 0$, which ensures that our solution coincides with a solution to Einstein’s equations (2.2).

The nonlinear PDEs (3.1), (2.3), (2.4), together with the boundary condition (2.13), (2.14) are then solved via the Newton-Kantorovich method. To be specific, we first linearize the PDEs and then discretize the linear partial differential equations into algebraic equations via the standard pseudospectral procedure, where we represent unknown functions as a linear combination of Chebyshev polynomials in the $z$ coordinate and a Fourier series in $x$ coordinate. Our integration domains lives on a rectangular grid, $(r, x) \in (0, 1) \times (-L_x^2, L_x^2)$. The resulting linear systems is solved by LU decomposition or other iterative techniques.

The condensation instability of the black brane occurring at $T = T_c$ corresponds to a continuous phase transition in boundary systems. To see when our solutions become unstable to forming scalar hair, we need find the critical temperature for given $\epsilon$. We do this by perturbing the RN background (2.5) by the scalar field $\psi = \phi (r) e^{-i\omega t}$. At the onset of the instability, the unstable mode becomes a zero mode $\omega = 0$. The critical temperature $T_c$ itself is therefore found by looking for a static normalizable solution to the scalar equation of motion with $\omega = 0$. Finally, $z_h$ of critical temperature corresponds to maximum of eigenvalue. The results are shown in figure 2. As is expected from previous analysis [24],

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$^2$We have checked $|\xi^2| < 10^{-10}$ when the size of grids reach $30 \times 150$.

$^3$Since a single soliton has no periodicity in the $x$ direction, for simplicity in spatial boundary conditions and efficiency in numerics, we follow [25] and instead construct the double soliton (kink-anti-kink) configuration, and then analyze a single soliton.
Figure 3: The profile of (a) the function $Q_1$ setting the $tt$ metric component, (b) the function $Q_3$ setting, together with $Q_4$, the $rr$ metric component, (c) the function $Q_6$ setting the charged scalar profile, and (d) the function $Q_7$ setting the $t$ component of the gauge field, at $\epsilon = 0.25, T/T_c = 0.5$ in the BCS case. Panel (c) shows the profile of the soliton condensate close to the asymptotic boundary $r = 1$. Panel (d) shows the depletion of the charge density close to the core of the soliton at $x = 0$. Panels (a) and (b) show that the perturbation of the background geometry is localized around the soliton core.

the critical temperature drops with increasing strength of backreaction in the standard BCS case, i.e. the backreaction hinders the generation of a condensate. However, in the alternative BEC case, find the critical temperature to be $T_c/\mu \approx 0.2$ almost independently of the backreaction parameter.

Having found the critical temperature, we then construct the solution with backreaction for $T < T_c$. These are hairy charged black holes dual to the superfluid phase. The seed configurations of matter field can be chosen as $\psi_\pm \sim \tanh(x - L_x/4) \tanh(-x - L_x/4)$. As a result, the part components of metric and the configuration of the matter fields are shown in figure 3 and figure 4. We can see that larger deformations of the spacetime metric appear only near the core of soliton.

From the asymptotic form of the matter fields, one then reads off the expectation value
Figure 4: The profile of (a) the function $Q_1$ setting the $tt$ metric component, (b) the function $Q_3$ setting, together with $Q_4$, the $rr$ metric component, (c) the function $Q_6$ setting the charged scalar profile, and (d) the function $Q_7$ setting the $t$ component of the gauge field, at $\epsilon = 0.25, T/T_c = 0.5$ in the BEC case. Panel (c) shows that the charged scalar field has a considerably flatter profile in $r$ direction for the alternative quantization employed here. Panel (d) shows a larger depletion of the charge density close to the core of the soliton compared to the BCS case in figure 3c. Also, the width of the soliton is found to be smaller compared to the BCS case from panels (a) and (b).

of the charged condensate and and the particle number density in the dual field theory. These are shown in figure 5. As found in [11, 12] in the probe limit, the soliton shows a larger charge depletion in the BEC phase.

Figure 6 shows that the depletion decreases as the backreaction increases. On the other hand, as can be seen from figure 7, the condensate away from the soliton core also decreases strongly with increasing backreaction. From figure 7 it seems that the system is returning to a homogeneous uncondensed state in the limit of large backreaction. This is in particular obvious for the BEC case (figure 7b), but the trend is also obvious for the BCS case (figure 7a). This interpretation of the data is furthermore consistent with the general expectation that backreaction inhibits the formation of the condensate in a holographic superconductor [24]. Since the condensate is charged, the suppression of the
Figure 5: The condensate (a) and the particle number density (b) as a function of $x$ at $\epsilon = 0.25, T/T_c = 0.5$. $\varrho_+ (\varrho_-)$ are particle number densities normalized to their equilibrium values at $x \to \pm \infty$. Red solid lines correspond to the BEC (alternatively quantized) case, while blue dashed lines to the BCS (standard quantized) case. While the condensate is similar in the BEC and BCS case, the charge depletion is considerably larger in the BEC case compared to the BCS one.

Figure 6: The depletion of particle number density as a function of $\epsilon$ at $T/T_c = 0.5$. The inset subfigure shows the change of particle number density far away from the core of soliton and at the core. In both cases, increasing backreaction reduces the charge depletion.

condensate towards a homogeneous uncondensed state implies that the ratio of condensed to total charge monotonically decreases with increasing backreaction. This can be seen from figure 8. Since the ratios of condensed and uncondensed charge over total charge is bound to add up to one by charge conservation (c.f. appendix B for a proof of this fact in our holographic system), the ratio of uncondensed to total charge hence must increase with increasing backreaction. This can also be verified from figure 8. However, since the system seems to be forced back into a homogeneous uncondensed state with increasing backreaction, and since the cause for the charge depletion at the soliton core was the steep profile in the condensate at the soliton core, the charge density distribution must also return to a homogeneous state with increasing backreaction. This qualitative interpretation
consistently explains our numerical results for both the BEC as well as the BCS choice of boundary conditions.

Finally, the dependency of the depletion factor on temperature for different $\epsilon$ is plotted in figure 9. Different from the expectation in [11], for the BEC soliton, the depletion in the core is considerably smaller than 100%, and even lower than in the probe limit at low temperature.\footnote{One possible explanation for this finding is that the temperature here is not low enough. In fact, the solution with backreaction at lower temperature is still very hard and unreliable to obtain in numerics.} This is consistent with our finding that backreaction itself results in the reduction of the charge depletion discussed above. We think that the underlying reason for this behavior is the nature of the condensed zero temperature IR fixed point, which may be uncharged. We plan to analyze this fixed point using analytic methods along the lines of [35], and also construct other fixed points which show increasing depletion with lowering temperature, in the near future.
Figure 9: The depletion of particle number density as a function of $T/T_c$ for different $\epsilon$ in the (a) BCS and (b) BEC case. In both cases, the depletion factor decreases with increasing backreaction, i.e. more charge is present at the soliton core.

4 A Simple Mechanism for the Snake Instability

In this section we turn to the discussion of the thermodynamics of our holographic dark soliton solution, as well as its instabilities. Since the boundary chemical potential and temperature are fixed, our system is in the grand canonical ensemble characterized by the grand potential

$$\Omega = E - TS - \mu N.$$  \hfill (4.1)

Here $N$ is the total particle number obtained by integrating the charge density $\rho$ over space. The internal energy is found to be

$$E = \int_{\Sigma_t} d^2x \sqrt{\eta} \left[ T_{\mu\nu} \left( \partial_t \right)^\mu \right] t^\nu.$$  \hfill (4.2)

Here $\eta_{\mu\nu}$ is the induced metric on the surface $\Sigma_t$ at $z = 0$ and $t = \text{const}$, with unit normal vector $t^\nu$, and $T_{\mu\nu}$ is the holographic stress-energy tensor, see appendix A for its detailed calculation. The entropy $S$ is the usual black hole entropy, given by

$$S = \frac{A_h}{4G} = \frac{4\pi \epsilon^2}{\epsilon} \int \sqrt{Q_4 (0, x) Q_5 (0, x)} d^2x.$$  \hfill (4.3)

Since the soliton extends in a noncompact spatial direction, in what follows we will consider densities of the above thermodynamic quantities. In particular, the grand potential $\Omega$ and the energy $E$ are replaced by their respective densities $\omega$ and $\varepsilon$. Far away from the soliton center, these local quantities will approach their homogeneous equilibrium values. Therefore, the soliton core is characterized by the difference of these local densities to their equilibrium values.

The energy density difference is displayed in figure 10b, where we see an obvious energy depletion around the soliton core. Upon integration along $x$, this depletion yields a negative

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5 In this section, we only show the explicit results for the soliton in the BCS superfluid, while the results for the BEC case is similar.
Figure 10: Total (a) and subtracted (b) energy density at $\epsilon = 0.25$, $T = 0.5T_c$. The subtracted energy density, which can be interpreted as the effective mass density of the soliton in the condensed phase, is found to be negative.

Figure 11: Total (a) and subtracted (b) grand potential density at $\epsilon = 0.25$, $T = 0.5T_c$. After integration over the width of the soliton, the latter becomes the surface tension of the soliton, which is found to be positive.

effective energy difference $\Delta E = -7.220$ (in units of the chemical potential $\mu$). This energy difference $\Delta E$ can be seen to set the effective mass of the soliton, which is negative as expected for a dark soliton [31].

The grand potential density difference is plotted in figure 11b, from which we see that there is a grand potential cost for the soliton with respect to the homogeneous background. Upon integration along $x$, the grand potential cost of the soliton yields the surface tension coefficient of the soliton. The surface tension coefficient $\sigma$ of a domain wall such as the soliton is defined as the external work $W$ necessary to enlarge the surface by a unit area while keeping temperature and chemical potential fixed. Under these conditions, the external work is just the increase of the grand potential due to the enlargement of the domain wall surface, $W = \Omega - \Omega_0$, with $\Omega_0$ being the grand potential of the corresponding homogeneous system without the domain wall. For the case displayed in figure 11b, we numerically determined the surface tension in units of the chemical potential to be $\sigma = 6.615$. As a
consistency check for our numerics, we also plot the pressure anisotropy $B \equiv p_x - p$ in figure 12a and check the thermodynamic relation $\omega = -p$ with $p$ the average pressure, which should hold far away from the soliton center.

With the results for these thermodynamic variables, we can confirm the following explanation for the so-called snake instability of the dark soliton [33]: The soliton moves through the condensate as a heavy, i.e. nonrelativistic, particle [31]. Since it has a negative effective mass $M_{eff} = \Delta E < 0$, its energy $E_s = \frac{M_{eff}}{2} \dot{q}^2$ decreases with increasing velocity $\dot{q}$. As shown in [31], for a homogeneous solitonic configuration, the velocity grows exponentially. This is the so-called self-acceleration instability [31] of the dark soliton. The self-acceleration originates from the dissipative interaction of the soliton with the surrounding condensate. As discussed in [31], for a homogeneous soliton configuration the self-acceleration terminates once the soliton velocity reaches the speed of sound, at which point the soliton decays into sound waves that dissipate away in the condensate.

On the other hand, the soliton has a positive surface tension coefficient $\sigma$. As shown in the hydrodynamic approximation in [33], the combined effect of the negative effective mass together with the positive surface tension leads to a growing transverse bending mode with a finite wave vector. The same instability was also found in the holographic quasinormal mode spectrum in [32]. Once this transverse bending mode starts to grow, the self-acceleration instability enhances the local bending, leading to the formation of a snake-like structure.

5 Conclusion and Discussion

We investigated the implications of including the gravitational backreaction onto solitons in holographic superfluid systems. We numerically solved Einstein’s equations coupled with the relevant matter fields. As compared to the probe limit, and contrary to our original expectations, increasing the backreaction decreased the depletion of the particle number density in the soliton core. We gave a qualitative interpretation of this in terms of the
balance of the ratios of condensed and uncondensed over total charge in the dual field theory as a homogeneous state is reached at strong backreaction. Finally, we computed the holographic stress energy tensor of the system and confirmed a simple holographic explanation for the snake instability of the dark soliton.

In this work, we restricted ourselves to the asymptotic regimes of BEC and BCS superfluidity. In particular we did not investigate the Robin boundary conditions necessary to model the actual crossover regime, in which the strongly coupled unitary fermion system is expected to live. We plan to investigate the behavior of the soliton in the crossover regime in future work. Such an investigation may in particular provide a better description of the intermediate unitary fermion regime, for which the Bogolyubov-de Gennes theory provides only a broad approximation [33]. Furthermore, our qualitative interpretation of the reduction in depletion factor implicitly relied on the assumption of an uncondensed homogenous infrared fixed point at strong backreaction or low temperatures. In our holographic superconductor strong backreaction implies a smaller critical temperature, and hence the limit of strong backreaction is equivalent to the limit of low temperatures. Only if the homogeneous infrared fixed point is uncondensed, i.e. if the condensing operator is irrelevant at the infrared fixed point, can the system return to the uncondensed state in the low temperature limit. We will investigate the possible infrared fixed points in our system along the lines of [35] to support our qualitative interpretation in future work. In the thin domain wall limit of the soliton, an analytic treatment of the domain wall in terms of a brane with junction conditions is also conceivable.

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A Holographic Stress-Energy Tensor

In order to compute the holographic stress energy tensor, following the process of [29, 30], we need to find the asymptotic expansion of the metric at the conformal boundary. The expansion is obtained by solving (3.1) with boundary conditions order by order in $\sqrt{1 - r}$
and in addition imposing \( \xi^\mu = 0 \),

\[
Q_1 (r, x) = 1 - \frac{\epsilon Q_6^{(0)} (x)}{z_h^2} (1 - r)^2 + q_1 (x) (1 - r)^3 + O \left[ (1 - r)^4 \right], \quad i = 1, 4, 5, \tag{A.1a}
\]

\[
Q_2 (r, x) = 1 + \frac{8 \epsilon Q_6^{(0)} (x) Q_6^{(1)} (x)}{3 z_h^2} (1 - r)^3 + O \left[ (1 - r)^4 \right], \tag{A.1b}
\]

\[
Q_3 (r, x) = \frac{2 \epsilon Q_6^{(0)} (x) \partial_x Q_6^{(0)} (x)}{z_h^3} (1 - r)^3 + O \left[ (1 - r)^4 \right], \tag{A.1c}
\]

\[
Q_6 (r, x) = Q_6^{(0)} (x) + Q_6^{(1)} (x) (1 - r) + \cdots, \tag{A.1d}
\]

\[
Q_7 (r, x) = Q_7^{(0)} (x) + Q_7^{(1)} (x) (1 - r) + \cdots. \tag{A.1e}
\]

Here \( q_1 (x), q_4 (x), q_5 (x) \) satisfy (A.3) related with the tracelessness and conservation of boundary stress energy tensor,

\[
T_i^i = 0, \quad \partial_i T^{ij} = 0. \tag{A.2}
\]

Using these relations, one can explicitly show the following conditions:

\[
q_1 (x) + q_4 (x) + q_5 (x) = - \frac{\epsilon Q_6^{(0)} (x) \left(-3 Q_6^{(0)} (x) + 8 Q_6^{(1)} (x)\right)}{z_h^2}, \tag{A.3a}
\]

\[
\partial_x q_4 (r, x) = - \frac{2 \epsilon \left(-3 Q_6^{(0)} (x) + 8 Q_6^{(1)} (x)\right) \partial_x Q_6^{(0)} (x) + 2 Q_6^{(0)} (x) \partial_x Q_6^{(1)} (x)}{3 z_h^2}. \tag{A.3b}
\]

Having obtained the asymptotic behavior of metric functions, one then changes to Fefferman-Graham coordinates \((z, v)\) by an expansion of series (A.4a) and demands \( g_{zz} = \frac{1}{z^2} \) and \( g_{zv} = 0 \) to determine the two functions \( \{a_k (v), b_k (v)\} \) order by order in \( z \). Here we provide the first few terms necessary for the computation of holographic stress energy tensor:

\[
\begin{cases}
  r = 1 - \frac{z_h}{2} z + \sum_{k=2}^{\infty} a_k (v) z^k, \\
  x = v + \sum_{k=1}^{\infty} b_k (v) z^k,
\end{cases} \tag{A.4a}
\]

\[
a_2 (v) = -\frac{z_h^2}{8}, \quad a_3 (v) = -\frac{z_h^3}{16}, \tag{A.4b}
\]

\[
a_4 (v) = \frac{z_h^2 \left(24 \epsilon \mu^2 + 51 z_h^2 + 32 \epsilon Q_6^{(0)} (v) Q_6^{(1)} (v)\right)}{1152}, \tag{A.4c}
\]

\[
b_1 (v) = b_2 (v) = b_3 (v) = 0, \tag{A.4d}
\]

\[
b_4 (v) = -\frac{\epsilon Q_6^{(0)} (v) \partial_v Q_6^{(0)} (v)}{16}. \tag{A.4e}
\]

Finally, the holographic stress energy tensor is computed by using (A.5a) for standard case and (A.5b) for alternative case [26, 27] in
Figure 13: Change of the total charge density (a), condensate charge density (b), uncondensed charge density (c) and flux contribution (d) with backreaction at $T = 0.5T_c$ in the BCS case.

\[ T_{\mu\nu} = \frac{1}{\kappa^2} \lim_{z \to 0} \frac{1}{z} \left( K_{\mu\nu} - \gamma_{\mu\nu} K - 2\gamma_{\mu\nu} - \frac{\epsilon}{2} |\Psi|^2 \gamma_{\mu\nu} \right), \]  

(A.5a)

\[ T_{\mu\nu} = \frac{1}{\kappa^2} \lim_{z \to 0} \frac{1}{z} \left[ K_{\mu\nu} - \gamma_{\mu\nu} K - 2\gamma_{\mu\nu} + \frac{\epsilon}{2} \left( -\Psi^\dagger n^\sigma D_\sigma \Psi - C.C. + |\Psi|^2 \right) \gamma_{\mu\nu} \right]. \]  

(A.5b)

Here $K_{\mu\nu}$ is the extrinsic curvature associated with an inward pointing unit normal vector $n^\sigma$ on the constant $z = \epsilon$ surface near the boundary. $\gamma_{\mu\nu}$ is the induced metric on the cut-off surface. The last term in (A.5) cancels the divergences due to the presence of scalar field [28].

B Particle Number (Charge) Change with Backreaction

For our static but inhomogeneous charged scalar and gauge field configuration, integrating the $t$ component of Maxwell equations (2.4) over the holographic $r$ coordinate, one obtains the total particle number density in terms of three contributions,

\[ \sqrt{-g} F^{tr}_{r=1} = \sqrt{-g} F^{tr}_{r=0} + \int_0^1 (-\sqrt{-g} J^t) dr + \int_0^1 \partial_x \sqrt{-g} F^{xt} dr, \]  

(B.1)
Condensate particle number density.

\[ \rho_{\psi} \]

The spatial domain of all integrations above is \((-\frac{L_y}{2}, \frac{L_y}{2}) \times (-\frac{L_y}{2}, \frac{L_y}{2})\). Due to the translation symmetry along y direction, we normalized these charges with regard to \(L_y\).

Figure 14: Change of the total charge density (a), condensate charge density (b), uncondensed charge density (c) and flux contribution (d) with backreaction at \(T = 0.5T_c\) in the BEC case.

\[ J^\nu = ig^{\mu \nu} \left[ \Psi^\dagger (D_\mu \Psi) - \Psi (D_\mu \Psi)^\dagger \right] . \quad \text{(B.2)} \]

On the right of (B.1), the first term yields the uncondensate particle number density, given by the electric flux evaluated at the horizon. The second term is the condensate particle number density. The third term is a contribution from to the electric flux in \(x\) direction which arises due to the inhomogeneity of our setup.

\[ \int_{N_{tot}} d^2 x \sqrt{-g} F^{tr} |_{r=1} = \int_{N_b} d^2 x \sqrt{-g} F^{tr} |_{r=0} + \int_{N_\psi} d^2 x \int_0^1 (-\sqrt{-g} J^t) dr \quad \text{(B.3)} \]

As can be seen from figure 13 and figure 14, the backreaction suppresses the condensate particle density and promotes the uncondensate particle number density. The flux contribution figure 13d and figure 14d is an even function around \(x = 0\). It is also suppressed with increasing backreaction. A relation between the total charge \(N_{tot}\), the uncondensed charge \(N_b\) and the charge in the condensate \(N_\psi\) can be obtained by integrating (B.1) over boundary spatial part.\(^6\) After integration, the flux contribution in (B.1) becomes a total...
derivative and hence vanishes if we integrate over a symmetric interval around $x = 0$. Note that since the system returns to homogeneous state in $x$ direction away from the soliton, the final ratios in (B.4) will not depend on the size of the integration region. The remaining terms in (B.1) yield a relation (B.3) in which we denote the three terms in order as $N_{\text{tot}}$, $N_b$, and $N_{\psi}$. The ratios of uncondensed and condensed charge over total charge then have to add up to one,

$$\frac{N_b}{N_{\text{tot}}} + \frac{N_{\psi}}{N_{\text{tot}}} = 1. \quad (B.4)$$

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