Maximally Chaotic Dynamical System
of
Infinite Dimensionality

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Abstract

We analyse the infinite-dimensional limit of the maximally chaotic dynamical systems that are defined on N-dimensional tori. These hyperbolic systems found successful application in computer algorithms that generate high-quality pseudorandom numbers for advanced Monte Carlo simulations. The chaotic properties of these systems are increasing with N because the corresponding Kolmogorov-Sinai entropy grows linearly with N. We calculated the spectrum and the entropy of the system that appears in the infinite dimensional limit. We demonstrated that the limiting system has exponentially expanding and contracting foliations and therefore belongs to the Anosov C-systems of infinite dimensionality. The liming system defines the hyperbolic evolution of the continuous functions very similar to the evolution of a velocity function describing the hydrodynamic flow of fluids. We compare the chaotic properties of the limiting system with those of the hydrodynamic flow of incompressible ideal fluid on a torus investigated by Arnold. This maximally chaotic system can find application in Monte Carlo method, statistical physics and digital signal processing.

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1 \hspace{1em} \textit{Introduction}

The maximally chaotic dynamical systems continue to attract great attention of the researchers due to their rich physical properties and extended areas of application. In recent years the classical and quantum-mechanical concepts of maximally chaotic systems were developed in series of publications with application to the physics of black holes\cite{1, 2, 3, 4, 5, 6}, of non-Abelian gauge fields \cite{7, 8} and string theory \cite{9}, to the fluid dynamics \cite{13, 14, 15} and astrophysics \cite{16, 17, 18, 19}, foundation of statistical physics \cite{20, 21, 22, 23, 24} and computer science \cite{25, 26, 27}. There is a growing evidence that intrinsic properties of the Hawking black hole radiation can be understood in terms of classical and quantum theory of chaos \cite{28, 29, 30, 31, 32, 33, 34}. It was conjectured that the resolution of the black hole information paradox of black holes radiation, which behaves as a black body radiation with finite temperature and is similar to the thermodynamic system characterised by entropy and other thermodynamic quantities, can be formulated in terms of maximally chaotic dynamical systems \cite{1, 2, 3, 4, 5, 6, 8}. We are interested to analyse deterministic dynamical systems, which show up physical properties intrinsically related to the thermodynamical behaviour \cite{20, 21, 22, 23, 24, 35, 36, 37}.

There is an intuitive understanding about how strong chaotic behaviour of a classical system can be, but it seems natural to define a maximally chaotic system as a system that has a nonzero Kolmogorov-Sinai entropy and therefore belongs to the so called K-systems \cite{38, 39, 40, 41, 8}. The examples of maximally chaotic systems were investigated and constructed in the earlier investigations of Artin, Hadamard, Hedlund, Hopf, Birkhoff, von Neumann and many others researchers working in ergodic theory \cite{42, 43, 44, 45, 46, 47}. A large class of maximally chaotic dynamical systems was constructed by Anosov \cite{41, 48, 49, 51, 52, 53, 54}. These are the systems that fulfil the C-condition that leads to the exponential instability of the trajectories \cite{55}, to the mixing of all orders and to a positive Kolmogorov-Sinai entropy. As a result it is natural to call them C-K systems \cite{8}.

The geodesic flows on closed Riemannian manifolds of variable negative sectional curvatures fulfil the C-condition and therefore represent a rich class of maximally chaotic dynamical systems \cite{31, 35}. A progress in understanding of the chaotic behaviour of the non-Abelian gauge fields, of the N-body systems in Newtonian gravity and of some cosmological models in general relativity were achieved by the application of the methods and results of the ergodic theory and the theory of geodesic C-K flows \cite{7, 8, 56, 57, 58, 59, 60, 61, 16, 17, 18, 19}.
Figure 1: The left figure shows the distribution of the eigenvalues of the operators $A(N)$, where $N = 256$. On the right figure is the distribution of the eigenvalues of the infinite-dimensional operator $A$ in $\lambda$ plane. The unit circles are shown in both figures.

A class of hyperbolic C-K systems defined on a torus found application in computer algorithms that generate high-quality pseudorandom numbers for the advanced Monte Carlo simulations [25, 27]. The chaotic properties of C-K systems on a torus are increasing with the dimensionality $N$ of the systems because the entropy increases linearly with $N$ as $h(A) = \frac{2}{\pi} N$. It seems important to investigate the infinite-dimensional limit of these systems that found successful application in Monte Carlo method [52, 63, 64, 65, 67] and to understand the exceptional properties of the limiting system.

2 Chaotic Dynamical Systems on $N$-dimensional Tori

The C-K system that realises automorphisms of a torus with coordinates $(u_1, \ldots, u_N)$ is defined as an integer matrix transformation [41, 25, 26, 27, 10, 46, 47, 54]:

$$u_i^{(k+1)} = \sum_{j=1}^{N} A_{ij}(N) u_j^{(k)} \mod 1, \quad k = 0, 1, 2, \ldots \quad (2.1)$$

where the components of the vector $u^{(k)}$ are $u^{(k)} = (u_1^{(k)}, \ldots, u_N^{(k)})$. The phase space of the system is the N-dimensional torus $T^N$ appearing in factorisation of the Euclidean space $E^N$ with coordinates $u = (u_1, \ldots, u_N)$ over an integer lattice $Z^N$ endowed with the invariant Liouville’s measure $d\mu = du_1\ldots du_N$ [11]. The automorphisms (2.1) fulfils the C-condition if and only if the integer matrix $A(N)$ has no eigenvalues on the unit circle and has the determinant equal to one, that is, the eigenvalues $\{\Lambda = \lambda_1, \ldots, \lambda_N\}$ fulfil the following conditions:

1) $\text{Det} A(N) = \lambda_1 \lambda_2 \ldots \lambda_N = 1$,  
2) $|\lambda_i| \neq 1, \quad \forall \ i$. \quad (2.2)
The eigenvalues are divided into sets with modulus smaller and larger than one:

\[ 0 < |\lambda_\alpha| < 1 \text{ for } \alpha = 1 \ldots d, \quad 1 < |\lambda_\beta| < \infty \text{ for } \beta = d+1 \ldots N. \tag{2.3} \]

The C-K system (2.1) has nonzero entropy and its value \( h(A) \) can be calculated in terms of its eigenvalues \[ h(A) = \sum_\beta \ln |\lambda_\beta| = -\sum_\alpha \ln |\lambda_\alpha|. \tag{2.4} \]

The eigenvalues \( \lambda_\beta \) larger than one are nothing else but the Lyapunov exponents and characterise the chaotic properties of the system as it follows from the above definition of entropy (2.4).

We are interested to consider the infinite-dimensional limit \( N \to \infty \) of the system (2.1) when the operator \( A(N) \) is given by the \( N \times N \) matrix with all integer entries \( A_{ij} \in \mathbb{Z} \) and has the following form \[ A(N) = \begin{pmatrix}
1 & 1 & 1 & 1 & \cdots & 1 & 1 \\
1 & 2 & 1 & 1 & \cdots & 1 & 1 \\
1 & 1 & \cdots & \cdots & \cdots & 1 & 1 \\
1 & N & N-1 & N-2 & \cdots & 2 & 1
\end{pmatrix}. \tag{2.5} \]

It has the determinant equal to one and the spectrum of inverse matrix has the form \[ \lambda_j = 1 - 2 \exp(i \pi j/N) + \exp(2i \pi j/N), \quad j = -N, -N+2, \ldots N-2, N \tag{2.6} \]
shown in Fig.1. The kernel \( A(N)u_0 = 0 \) of the operator \( A(N) \) consists of only one vector with zero components \( u_0 = (0, \ldots, 0) \). The eigenvalues fulfil the C-condition (2.2) and the entropy of the system can be calculated for the large values of \( N \) as a sum over eigenvalues:

\[ h(A) = \sum_\beta \ln |\lambda_\beta| = \sum_{-2\pi/3 < \phi_j < 2\pi/3} \ln(4 \cos^2(\phi_j/2)) \sim \frac{2}{\pi} N, \tag{2.7} \]

where the eigenvalues are given in (2.6) and \( \phi_j = \pi j/N \). The entropy increases linearly with the dimension \( N \) of the operator \( A(N) \). We note that the special form of the matrix \( A(N) \) in (2.5) has highly desirable property of having a widely spread, nearly continuum spectrum of eigenvalues (2.6) shown in Fig.1 and indicating that the exponential mixing takes place in many scales (2.5).

### 3 Infinite-dimensional Limit of \( A(N) \) System

We are interested to define and investigate the system that appears in the infinite-dimensional limit of \( A(N) \) when \( N \to \infty \). The size of state vector \( u = (u_1, \ldots, u_N) \) tends to infinity and it
seems natural to expect that it can be represented by a continuous function $\psi$ defined in an
infinite-dimensional space $\mathcal{H} = \{\psi(u)\}$, possibly a Hilbert space, and the operator $A(N)$ will
reduce to a differential operator $A$ acting in $\mathcal{H}$. The existence of such a limit would mean
that $A$ defines a "measure preserving" transformation of continuous functions $A^n\psi = \psi^{(n)}$ in
$\mathcal{H}$ that will have maximally strong chaotic properties. Our intension is to define this limiting
system, to explore its properties and possible applications in Monte Carlo method. It seems
that the investigation of infinite-dimensional "fully chaotic" transformation of continuous
functions may also help to understand better a chaotic/turbulent motion of fluids. As it was
demonstrated by Arnold [13, 14], the solutions of the partial differential equation describing
the evolution of the hydrodynamic flow of incompressible ideal fluid can be considered as a
continuous measure preserving transformation of fluid velocity and the evolution is partially
chaotic, that is, the flow is exponentially unstable in some directions and is stable in other
directions (the details will be discussed in the last paragraph).

In many areas of mathematics the consideration of infinite-dimensional limits is an am-
biguous procedure, and a priory there is no guarantee that a sensible limit of a given
finite-dimensional structure exists. In our case the dimension of the N-dimensional torus
$S^1 \otimes \ldots \otimes S^1$ tends to infinity and it is unclear what type of phase space should be taken
in the limit. The hint that a sensible limit may exist comes from the fact that as $N \to \infty$
the eigenvalues (2.6) fill out the cardioid curve more and more dense without producing any
deformation of the cardioid curve that can be seen in figure Fig.[1] [27]. The other impor-
tant hint is that the inverse matrix $A(N)^{-1}$ is reminiscent to the matrix that represents the
discrete version of the second-order differential operator [27]. If one supposes that the state
vector becomes a function $\psi(x)$ with its argument on a real line $x \in R^1$, then it seems natural
to look for a differential operator of the second order and the one that will reproduce the
eigenvalue spectrum distributed on cardioid curve. Having in mind the above consideration
let us consider the differential operator

$$A_x = 1 - 2\exp\left(\frac{d}{dx}\right) + \exp\left(2\frac{d}{dx}\right) = \frac{d^2}{dx^2} + \frac{d^3}{dx^3} + \frac{7}{12}\frac{d^4}{dx^4} + ...$$  \hspace{1cm} (3.8)

acting in the Hilbert space of functions $\mathcal{H} = \{\psi(x)\}$ defined on the interval $x \in [-\infty, +\infty]$. The series expansion of the operator has indeed a second-order differential operator and
also infinitely many high derivative terms. Its spectral characteristics are defined by the
eigenvalue equation $A_x\psi(x) = \lambda\psi(x)$. Searching the eigenfunctions in the form of plane
waves

\[ \psi_a(x) = e^{iax} \]  

(3.9)

one can find that the spectrum represents a continuous cardioid curve on the complex plane

\[ \lambda(a) = 1 - 2e^{ia} + e^{2ia} = 4e^{i(a+\pi)} \sin^2 \left( \frac{a}{2} \right). \]  

(3.10)

It is similar to the discrete spectrum \((2.6)\) and has the periodic structure

\[ \lambda(a + 2\pi k) = \lambda(a), \quad k = 0, \pm 1, \pm 2, \ldots \]  

(3.11)

The spectrum is continuous and the eigenvalues are distributed in the complex plane representing the cardioid curve shown in Fig.1. As the real momentum parameter \(a\) varies on the real line interval the eigenvalues run around the cardioid infinitely many times \((3.11)\). The eigenvalues of the operator \(A_x\) can be divided into two sets \(\{\lambda_\alpha\}\) and \(\{\lambda_\beta\}\) with modulus smaller and larger than one: \(0 < |\lambda_\alpha| < 1, \quad 1 < |\lambda_\beta|\). The eigenvalues \(\lambda_\alpha\) and \(\lambda_\beta\) can be found using \((3.10)\):

\[ \lambda_\alpha = 4e^{i(a+\pi)} \sin^2 \left( \frac{a}{2} \right), \quad \text{when} \quad -\frac{\pi}{3} + 2\pi k < a < +\frac{\pi}{3} + 2\pi k, \]

\[ \lambda_\beta = 4e^{i(a+\pi)} \sin^2 \left( \frac{a}{2} \right), \quad \text{when} \quad +\frac{\pi}{3} + 2\pi k < a < +\frac{5\pi}{3} + 2\pi k, \]  

(3.12)

where \(k = 0, \pm 1, \pm 2, \ldots\). This structure of the spectrum repeats itself with the period \(2\pi\). There are eigenvalues \(|\lambda| = 1\) corresponding to \(a = \pm \frac{\pi}{3} + 2\pi k\) where the cardioid intersects a unit circle shown in Fig.1.

4  The Kolmogorov-Sinai Entropy of Limiting System

In order to establish the fact that the operator \(A_x\) is defining a measure-preserving transformation one should calculate the determinant of the operator \(A_x\). The measure-preserving transformations of the phase spaces is a characteristic property of Hamiltonian systems that is expressed in terms of the Liouville’s theorem and represent a large class of dynamical systems that are considered in ergodic theory \([20, 21, 22, 23, 24]\). Using the fact that \(\ln \det A_x = Tr \ln A_x\) we will have

\[ \ln \det A_x = \sum_{k=-\infty}^{\infty} \int_{-\pi+2\pi k}^{\pi+2\pi k} \ln[e^{i(a+\pi)}] \sin^2 \left( \frac{a}{2} \right) \frac{da}{2\pi} = 0, \]  

(4.13)

that is, the determinant is equal to one \(\det A_x = 1\) and the operator \(A_x\) is defining a measure-preserving transformation (the kernel subspace \(K\) of the operator \(A_x\) will be defined in \((4.23)\)).
One can define now the homomorphism of the Hilbert space $\mathcal{H} = \{\psi(x)\}$ in terms of the operator $A_x$ as
\begin{equation}
\psi^{(1)}(x) = A_x \psi(x)
\end{equation}
and the dynamical system on the infinite dimensional phase space $\mathcal{H}$ as:
\begin{equation}
\psi^{(n)}(x) = A_x^n \psi(x) \mod 1, \quad n = 0, 1, 2, \ldots
\end{equation}
The homomorphism (4.15) is defined by mod 1 operation meaning that the functions $\psi^{(n)}(x)$ are wrapping around an infinitely long cylinder $R^1 \otimes S^1$ ($\psi : R^1 \to S^1$).

The determinant of the operator $A$ is equal to one and the eigenvalues are distributed inside and outside of the unit circle, and we have an example of infinite-dimensional hyperbolic C-K system of the type (2.2). To get convinced that the operator $A_x$ represents a hyperbolic C-K system one should establish the existence of exponentially expanding and contracting foliations [41]. Let us consider the evolution of the infinitesimal perturbation $\psi \to \psi + \delta\psi$ under the action of $A$ operator in analogy with the geodesic deviation equation:
\begin{equation}
\delta\psi^{(n)}(x) = A_x^n \delta\psi(x).
\end{equation}
The deviation $\delta L$ can be evaluated by using the standard inner product in Hilbert space and the mean value theorem [69]:
\begin{align}
\delta L_n &= \langle \delta\psi^+ | A_x^n \delta\psi \rangle = \int_{-\infty}^{+\infty} \delta\psi^+(x) A_x^n \delta\psi(x) dx \\
&= \int_{-\infty}^{+\infty} \int_{\Delta a} \frac{da}{2\pi} \int_{\Delta a'} \frac{da'}{2\pi} e^{-ia' x} \delta\phi^+(a') \left[ 4 \sin^2 \left( \frac{a}{2} \right) \right]^n e^{i\phi(a)} e^{i2\pi a} \delta\phi(a) dx \\
&= \frac{1}{2\pi} \int_{\Delta a} da \ e^{i\phi(a) + \pi} \left[ 4 \sin^2 \left( \frac{a}{2} \right) \right]^n |\delta\phi(a)|^2 = e^{i\phi(a) + \pi} e^{i\arcsin^2 \left( \frac{\pi}{2} \right) |\delta\phi|^2}, \quad (4.17)
\end{align}
where $\bar{a}$ is a number in the interval $\Delta a \subset (\pi/3, 5\pi/3)$ and $|\delta\phi|^2 = \frac{1}{2\pi} \int_{\Delta a} |\delta\phi|^2 da$. The absolute value of the deviation is growing exponentially with the iteration time $n$ as
\begin{equation}
|\delta L_n| \sim |\delta\phi|^2 e^{n\pi \arcsin^2 \left( \frac{\pi}{2} \right)}.
\end{equation}
In the above perturbation the Fourier spectrum of the function $\delta\psi(x)$ is localised in the region of the spectrum $\Delta a$, where the eigenvalues $\lambda_\beta$ are larger than one, $\Delta a \subset (\pi/3, 5\pi/3)$, and are defined in (3.12). The integration over $a$ and $a'$ was specified to be in that region $\Delta a$ and the perturbation function had the following form:
\begin{equation}
\delta\psi(x) = \int_{\Delta a} \delta\phi(a) e^{i\phi} \frac{da}{2\pi}.
\end{equation}
In a similar way one can get convinced that the exponential contraction takes place when
the Fourier spectrum of the perturbation function is localised in the part of the spectrum
$\lambda_\alpha$, where the eigenvalues are less than one $\Delta a \subset (-\pi/3, \pi/3)$ in (3.12). The existence of
exponentially expanding and contracting foliations is a sufficient condition for a dynamical
system to expose strong statistical/chaotic properties and, in particular, to have nonzero
Kolmogorov-Sinai entropy and to be classified as a hyperbolic C-K system. In physical terms
this means that the Fourier amplitudes $\phi(a)$ of the initial state vector $\psi(x)$ are stretched and
compressed depending on whether the value of $a$ is in the interval $a \in (\pi/3 + 2\pi k, 5\pi/3 + 2\pi k)$
or in the interval $a \in (-\pi/3 + 2\pi k, \pi/3 + 2\pi k)$, where $k = 0, \pm 1, \pm 2, \ldots$.

The above consideration allows to calculate the Kolmogorov entropy of the system per
unit of the iteration time $n$ as it is was defined by Kolmogorov [39]. The new aspect that
appears in this infinite-dimensional system case is that the spectrum (3.10) is continuous
and repeats itself infinitely many times. For that reason the standard Kolmogorov definition
[39] of the entropy per unit iteration time is equal to infinity. That can be observed also
from the equation (2.7) when $N \to \infty$. In this circumstances one can propose to calculate
the entropy per unit period (3.11) of the spectrum (3.10):

$$h(A_x) = \sum \alpha \ln \frac{1}{|\lambda_\alpha|} = -\int_{-\pi/3}^{\pi/3} \ln|4 \sin^2(\frac{a}{2})| \frac{da}{2\pi} = 2i[Li_2(e^{i5\pi/3}) - Li_2(e^{i\pi/3})] \sim \frac{2}{\pi}, \quad (4.20)$$

where we used the fact that $\prod_\alpha \lambda_\alpha \prod_\beta \lambda_\beta = 1$ and $Li_n(z)$ is the polylogarithm function.
This result is understandable in the sense that the finite-dimensional matrix system (2.5)
considered above had the entropy $\sim \frac{2}{\pi}N$, where $N$ is the dimension of the matrix operator.
As far as the operator $A$ can be considered as the infinite dimensional limit $N \to \infty$ of (2.5),
the standard Kolmogorov entropy of the system (4.15) tends to infinity but its entropy "per
spectral period" is finite. The Fig.(2) demonstrates an example of the iteration (4.15) of a
smooth function.

The inverse operator $G$ is defined by the equation

$$A_x G(x - y) = \sum_{k=-\infty}^{\infty} \delta(x - y - k) \quad (4.21)$$

and has the following solution:

$$G(x - y) = \frac{1}{2} (x - y)^2 \sum_{k=-\infty}^{\infty} e^{2\pi ik(x-y)}. \quad (4.22)$$

The general solution of (4.21) can be expressed as a sum the fixed solution (4.22) and an
arbitrary element of the kernel $K$. The kernel subspace $K$ of the operator $A_x$ is defined by
Figure 2: The figure demonstrates the result of a triple iteration \(4.15\) \(\psi^{(3)}(x) = A_x^2 \psi(x)\) of the smooth function \(\psi(x) = \sin(x) + \sin(2x) - \cos(3x) + \sin(4x)\).

the equation \(A \psi_0(x) = 0\) and has the following solution:

\[
\psi_0(x) = (c_1 x + c_2) \sum_{k=\infty}^{-\infty} a_k e^{2\pi ikx},
\]

(4.23)

where \(c_1, c_2, a_k\) are arbitrary constants. The spectrum belongs to the discrete values of spectral parameter \(a = 2\pi k\) in (3.9) and have a zero measure in the continuous spectrum of the limiting system (3.8). The inverse transformation \(\psi^{(-n)} = G^n \psi, n = 0, 1, 2, \ldots\) is therefore defined modulo kernel (4.23). We will defined it in its most simple form (4.22):

\[
\psi^{(-1)} = \int G(x - y) \psi(y) dy = \sum_{k=-\infty}^{\infty} \frac{1}{2} k^2 \psi(x - k) \mod 1.
\]

(4.24)

Thus the evolution of the system is given in both "time directions" by (4.24) and (4.14).

Because the determinant of the operator \(A_x\) is equal to one on a quotient space \(\mathcal{H}/\mathcal{K}\) of functions on a cylinder \(R^1 \otimes S^1\), where \(\mathcal{K}\) is a kernel (4.23), it is natural to think that the operator \(A_x\) defines a measure preserving transformation. In this circumstances one can try to define a measure that is invariant with respect to the transformations generated by \(A_x\). The volume element \(\mathcal{V}\) in the quotient space \(\mathcal{H}/\mathcal{K}\) can be defined by using the Wiener-Feynman functional integral \(\mathcal{V}_C = \int_C F[\psi] D\psi(x)\), where \(C\) is a subset in \(\mathcal{H}/\mathcal{K}\) and the functional \(F[A_x \psi] = F[\psi]\) should be invariant under the action of the transformations generated by the operator \(A_x\). The measure \(D\psi(x)\) is invariant because the determinant of the corresponding Jacobian operator is equal to one (4.13). The invariant functional \(F[\psi]\) can be constructed by projecting the state vectors on the eigenfunctions \(\phi(\pm \frac{\pi}{3} + 2\pi k) = \int_{-\infty}^{\infty} e^{i(\pm \frac{\pi}{3} + 2\pi k)x} \psi(x) dx\) (3.9) that correspond to the eigenvalues of modulus one \(|\lambda(\pm \frac{\pi}{3} + 2\pi k)| = 1, k = 0, \pm 1, \pm 2, \ldots\). The invariant functional \(F[\psi]\) can be defined as:

\[
F[\psi] = \exp \left[ - \sum_{k=\pm 1, \pm 2, \ldots} \frac{|\phi(\frac{\pi}{3} + 2\pi k)|^2 + |\phi(-\frac{\pi}{3} + 2\pi k)|^2}{2} \right].
\]

(4.25)
5 Flow of Incompressible Ideal Fluid

Were similar systems investigated and successfully used in the past? The solutions of the partial differential equation describing the evolution $t \to g_t(x)$ of the hydrodynamic flow of incompressible ideal fluid filled in a two dimensional torus $x \in T^2$ can be considered as a continuous area preserving diffeomorphisms $SDiff(T^2)$ of a torus $T^2 \to T^2$. In Arnold approach [13] the ideal fluid flow is described by the geodesics $g_t(x) \in G$ on the diffeomorphism group $G = SDiff(T^2)$ with the velocity $v_t(x) = \dot{g}_t(x)g_t^{-1}(x)$ belonging to the corresponding algebra $\mathfrak{g}$ = sdiff$(T^2)$ of divergence free vector fields. The Riemannian metric on the group $G$ is induced from the metric on a torus [13] and the stability of the geodesic flows on the group $G$ can be analysed by investigating the behaviour of the corresponding sectional curvatures $K(v, \delta v)$ [13, 41, 71]. It was found that the flows that are defined by a parallel velocity field on $T^2$ are unstable because the sectional curvatures are negative and the flow is exponentially unstable. In other directions the sectional curvatures are positive and the flows are stable [13]. A similar stability analysis was performed for the hydrodynamic flow on two-dimensional sphere $S^2$ in [14, 73] and on high-dimensional torus $T^N$ in [72, 73]. In all these cases the flow is exponentially unstable in some directions and is stable in some other directions, resulting in the limitation of predictability of the hydrodynamic flow and leading to the principal difficulties of a long-term ”weather forecasting” [13]. Comparing these systems with the system considered above one can observe that here we have discrete in time transformations of the phase space and, secondly, the system (3.8), (4.14), (4.24) shows up exponential instability of its geodesics in the full quotient phase space $H/K$. This full phase space chaotic behaviour can find application in Monte Carlo method, statistical physics and in digital signal processing in communication systems, the subjects of future investigation.

6 Conclusion

We were able to define the infinite-dimensional limit of maximally chaotic dynamical systems on N-dimensional tori when the dimension $N$ tends to infinity. The limiting system $A_x$ is represented by a nonlocal differential operator acting on the infinite-dimensional Hilbert space of continuous functions. We calculated the eigenvalue spectrum of the limiting operator, as well as its determinant and the corresponding Kolmogorov-Sinai entropy. We investigated the exponentially expanding and contracting foliations and demonstrated that the limiting
system belongs to the class of hyperbolic Anosov C-systems. The limiting system represents a unique example of a C-system of infinite dimensionality, which has a quite simple form and is chaotic in its full phase space.

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8 Appendix

The motion of an abstract “rigid body” rotating in high-dimensional Euclidean space \( \vec{r} \in E \) that is invariant under the isometry group \( g(\xi) \in G \) can be described in terms of geodesic flow on a corresponding group manifold \([13]\). The stationary frame coordinates of the “rigid body” are defined as \( \vec{r} = g_t \vec{Q} \), where \( g_t = g(\xi(t)) \) is a time-dependent element of the matrix group \( G \) and \( \vec{Q} \) are the frame coordinates rigidly fixed to the rotating “body” \( \dot{\vec{Q}} = 0 \). Thus

\[
\dot{r} = g_t \dot{\vec{Q}} = \dot{g}_t g_t^{-1} \quad \text{where} 
\]

The matrix of angular velocity in stationary frame

\[
\hat{\omega}_s = \dot{g}_t g_t^{-1} \quad \text{(8.26)}
\]

is a right-invariant one-form ( \( d(gg_0)(gg_0)^{-1} = dgg^{-1} \), where \( g_0 \) is a fixed element of the group \( G \)). The matrix of angular velocity in rotating frame

\[
\hat{\Omega}_c = g_t^{-1} \dot{g}_t \quad \text{(8.28)}
\]

is left invariant because \( (g_0g)^{-1}d(g_0g) = g^{-1}dg \). It follows that

\[
\dot{\omega}_s = \dot{g}_t g_t^{-1} g_t g_t^{-1} = g_t \hat{\Omega}_c g_t^{-1}. \quad \text{(8.29)}
\]

The kinetic energy is defined as a sum of the kinetic energies of all “parts” of the rotating “body” through the velocities (8.26):

\[
T = \frac{1}{2} \sum_a m_a \dot{r}_a \dot{r}_a = \frac{1}{2} \text{Tr}(\hat{I}_s \dot{\omega}_s \dot{\omega}_s^+) = -\frac{1}{2} \text{Tr}(\hat{I}_s \dot{\omega}_s \dot{\omega}_s^-), \quad \text{(8.30)}
\]

*The general relation between operators in stationary and rotating frames is \( A_s = g_t A_c g_t^{-1} \).
where one should use the relation $\vec{r} = g_t \vec{Q}$, and therefore

\[ \hat{I}_s = \sum_a m_a r_a r_a^+ = g_t \hat{I}_g^{-1}, \quad \hat{I} = \sum_a m_a Q_a Q_a^+. \tag{8.31} \]

The matrix $\hat{I}$ is a symmetric positive definite constant matrix that determines the "moment of inertia" in the frame rigidly fixed to the rotating "body". The matrix of angular momentum in stationary frame is

\[ \hat{m}_s = \hat{I}_s \hat{\omega}_s \tag{8.32} \]

and the corresponding angular momentum in rotating frame can be defined by projection of $\hat{m}_s$ into the rotating frame:

\[ \hat{m}_s = g_t \hat{M}_c g_t^{-1}, \tag{8.33} \]

thus

\[ \hat{M}_c = \hat{I} \hat{\Omega}_c, \tag{8.34} \]

where one should use the relations (8.32), (8.31) and (8.27), (8.28). In terms of rotating frame coordinates the kinetic energy (8.30) will take the form

\[ T = -\frac{1}{2} Tr(\hat{I}_s \hat{\omega}_s \hat{\omega}_s) = -\frac{1}{2} Tr(\hat{m}_s \hat{\omega}_s) = -\frac{1}{2} Tr(\hat{M}_c \hat{\Omega}_c) = -\frac{1}{2} Tr(\hat{\Omega}_c \hat{\Omega}_c), \tag{8.35} \]

where we used the relations (8.33) and (8.29). As it follows from (8.35) and (8.40), in mathematical terms the matrix $\hat{I}$ defines the alternative Euclidean structure on the group algebra $\langle a, b \rangle_I = Tr(T_a \hat{I} T_b)$, where $T^a$ are the generators of the algebra $\mathfrak{g}$ and $\hat{\Omega}_c = \sum_a T^a \Omega^a$.

Using the relation (8.33) and the conservation of the angular momentum in stationary frame $\hat{m}_s = 0$ one can get the generalised Euler equation

\[ \dot{g}_t \hat{M}_c g_t^{-1} + g_t \dot{\hat{M}}_c g_t^{-1} - g_t \hat{M}_c g_t^{-1} \dot{g}_t g_t^{-1} = 0, \]

which can be represented in the following standard form:

\[ \dot{\hat{M}}_c + \hat{\Omega}_c \hat{M}_c - \dot{\hat{M}}_c \hat{\Omega}_c = 0, \]

or equivalently as

\[ \frac{d\hat{M}_c}{dt} = [\hat{M}_c; \hat{\Omega}_c]. \tag{8.36} \]

In terms of angular velocity it takes the following form:

\[ \hat{I} \frac{d\hat{\Omega}_c}{dt} = [\hat{I} \hat{\Omega}_c; \hat{\Omega}_c] , \quad \frac{d\hat{\Omega}_c}{dt} = \hat{I}^{-1} [\hat{I} \hat{\Omega}_c; \hat{\Omega}_c] = \Gamma(\hat{\Omega}_c, \hat{\Omega}_c). \tag{8.37} \]

\[ ^1 \text{The square of angular momentum is conserved: } Tr(\hat{m}_s^2) = Tr[(g_t \hat{M}_c g_t^{-1})^2] = Tr(\hat{M}_c^2). \]
The last Euler equation can be represented in the form of geodesic equation

$$\frac{d\Omega^a}{dt} + \Gamma^a_{bd} \Omega^b \Omega^d = 0, \quad (8.38)$$

where $\Gamma^a_{bd}$ are the Christopher symbols of the metric (8.40). The kinetic energy defines the left invariant metric on the group:

$$ds^2 = Tr(\hat{I} \hat{\Omega}_c \hat{\Omega}_c) dt^2 = Tr(\hat{I} g^{-1} dg \ g^{-1} dg) = Tr(\hat{I} \ g^{-1} \frac{\partial g}{\partial \xi^a} \ g^{-1} \frac{\partial g}{\partial \xi^b}) \ d\xi^a d\xi^b, \quad (8.39)$$

where the $\xi^a$ are parameters of the Lie group $G$ and

$$ds^2 = g_{ab} \ d\xi^a d\xi^b, \quad g_{ab} = Tr(g^{-1} \frac{\partial g}{\partial \xi^b} \hat{I} \ g^{-1} \frac{\partial g}{\partial \xi^a}). \quad (8.40)$$

The calculation of the components of the Riemann tensor and of the sectional curvatures

$$K(\xi, \eta) = \frac{R_{abcd} \xi^a \eta^b \xi^c \eta^d}{|\xi \wedge \eta|^2} \quad (8.41)$$

allows to investigate the stability of the geodesic flows even in the cases when $G$ is the infinite-dimensional group of diffeomorphims $SDiff(M)$ that describes the flow of incompressible ideal fluid filled in a manifold $M$ [13, 14, 72, 73].

9 Data Availability Statement.

The data that support the findings of this study are available within the article and also in the references [25, 26, 27, 62, 63, 64, 65, 67].

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