LINEAR INDEPENDENCE OF TIME-FREQUENCY SHIFTS?

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Abstract. We investigate finite sections of Gabor frames and study the asymptotic behavior of their lower Riesz bound. From a numerical point of view, these sets of time-frequency shifts are linearly dependent, whereas from a rigorous analytic point of view, they are conjectured to be linearly independent.

1. Introduction

A famous conjecture of Heil, Ramanathan, and Topiwala [23], often called the HRT-conjecture, states that finitely many time-frequency shifts of a non-zero $L^2$-function are linearly independent. Denoting a time-frequency shift of $g \in L^2(\mathbb{R}^d)$ along $z = (x, \xi) \in \mathbb{R}^{2d}$ by

$$\pi(z)g(t) = M_\xi T_x g(t) = e^{2\pi i \xi \cdot t} g(t-x), \quad t \in \mathbb{R}^d,$$

the question is whether

$$\sum_{j=1}^n c_j \pi(z_j)g = 0 \implies c_j = 0 \quad \forall j,$$

for arbitrary points $z_1, \ldots, z_n \in \mathbb{R}^{2d}$.

To this day this conjecture is open, it is known to be true only under restrictive conditions on either $g$ or the set $\{z_j\}$.

(a) Linnell’s Theorem [26]: Let $\Lambda \subseteq \mathbb{R}^{2d}$ be a lattice and $g \in L^2(\mathbb{R}^d)$ arbitrary, then for every finite subset $F \subseteq \Lambda$ the set $\{\pi(\lambda)g : \lambda \in F\}$ is linearly independent. This is a deep result obtained with von Neumann algebra techniques; special cases have been reproved with more analytic arguments in [7, 13].

(b) Bownik and Speegle [8] proved the HRT-conjecture for $g$ with one-sided super-exponential decay. This result contains the early results of [23].

In view of these general results, it is rather surprising that it is not known whether four arbitrary time-frequency shifts of $g \in L^2(\mathbb{R}^d)$ are linearly independent. Even for rather special constellations the linear independence of four time-frequency shifts is highly non-trivial [12].

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Further contributions to the HRT-conjecture investigate the kernel of a linear combination of time-frequency shift operators \([1]\) and estimates of the frame bounds of finite sets of time-frequency shifts \([10]\).

For a detailed survey of the linear independence conjecture we refer to Heil’s article \([22]\).

In this note we adopt a different point of view and investigate the numerical linear independence of time-frequency shifts. In other words, can we determine numerically whether a given finite set of time-frequency shifts is linearly independent? We will argue that the answer is negative. To formulate a precise result, we will study the lower Riesz bound of finite sections of a Gabor frame and estimate its asymptotics. By taking larger and larger finite sections, the lower Riesz converges to zero, and in many cases this convergence is super-fast. Thus from a numerical point of view even small sets of time-frequency shifts may look linearly dependent.

The main result will illustrate the spectacular difference between a conjectured mathematical truth and a computationally observable truth.

Let us explain the problem in detail. Let \(\lambda = (\lambda_1, \lambda_2) \in \mathbb{R}^d \times \mathbb{R}^d \simeq \mathbb{R}^{2d}\) be a point in the time-frequency plane (or phase space in the terminology of physics). The time-frequency shift \(\pi(\lambda)\) acts on a function \(g \in L^2(\mathbb{R}^d)\) by

\[
\pi(\lambda)f(t) = e^{2\pi i \lambda_2 \cdot t} g(t - \lambda_1).
\]

For fixed \(g \in L^2(\mathbb{R}^d)\) and a countable subset \(\Lambda \subseteq \mathbb{R}^{2d}\), the set \(G(g, \Lambda) = \{\pi(\lambda)g : \lambda \in \Lambda\}\) is called a Gabor system, and for \(n > 0\) the set

\[
G(g, \Lambda_n) = G(g, \Lambda \cap B_n(0)) = \{\pi(\lambda)g : \lambda \in \Lambda, |\lambda| \leq n\}
\]

is a finite section of \(G(g, \Lambda)\). We are interested in the quantity

\[
A_n = A(g, \Lambda_n) = \min_{c \neq 0} \frac{\|\sum_{|\lambda| \leq n} c_\lambda \pi(\lambda)g\|^2}{\sum_{|\lambda| \leq n} |c_\lambda|^2}.
\]

Since \(G(g, \Lambda_n)\) spans a finite-dimensional subspace of \(L^2(\mathbb{R}^d)\), the minimum exists. Moreover, \(A_n = 0\), if and only if \(G(g, \Lambda_n)\) is linearly dependent. Thus we may take \(A_n\) as a quantitative measure for the numerical linear dependence of \(G(g, \Lambda_n)\).

Our main result is an asymptotic estimate for \(A_n\) as \(n \to \infty\). Before formulating this estimate, we need to explain some of the basic concepts of Gabor analysis and time-frequency analysis. We refer to the textbooks \([2,9,17]\) for detailed expositions of time-frequency analysis and frame theory.

A Gabor system \(G(g, \Lambda)\) is a frame, a so-called Gabor frame, if there exist frame bounds \(A, B > 0\), such that

\[
A\|f\|^2 \leq \sum_{\lambda \in \Lambda} |\langle f, \pi(\lambda)g \rangle|^2 \leq B\|f\|^2 \quad \forall f \in L^2(\mathbb{R}^d).
\]

For an equivalent and more suitable condition we define the synthesis operator \(D_{g,\Lambda}\)

\[
D_{g,\Lambda}c = \sum_{\lambda \in \Lambda} c_\lambda \pi(\lambda)g,
\]
which is well-defined on finite sequences \( c \). Then \( \mathcal{G}(g, \Lambda) \) is a frame, if and only if
\[
D_{g, \Lambda} : \ell^2(\Lambda) \to L^2(\mathbb{R}^d) \text{ is bounded and onto } L^2(\mathbb{R}^d).
\]
If, in addition to the frame property, \( \ker D_{g, \Lambda} = \{0\} \), then \( \mathcal{G}(g, \Lambda) \) is a Riesz basis for \( L^2(\mathbb{R}^d) \). In this case there exist \( A', B' > 0 \), such that
\[
A' \|c\|_2^2 \leq \left\| \sum_{\lambda \in \Lambda} c_\lambda \pi(\lambda) g \right\|_2^2 \leq B' \|c\|_2^2 \quad \forall c \in \ell^2(\Lambda).
\]
In other words, a Riesz sequence is \( \ell^2 \)-linearly independent. In particular, every finite subset of a Riesz sequence \( \mathcal{G}(g, \Lambda) \) is linearly independent.

If \( \mathcal{G}(g, \Lambda) \) is a frame, but not a Riesz basis, then by definition \( \ker D_{g, \Lambda} \neq \{0\} \). However, if the linear independence conjecture is true, then certainly \( \ker D_{g, \Lambda_n} = \{0\} \) for all \( n \in \mathbb{N} \). This means that for \( n \to \infty \), the finite sets \( \mathcal{G}(g, \Lambda_n) \) must get “more and more linearly dependent”. Quantitatively, this means that the lower Riesz bound \( A_n \) must tend to 0.

Our main theorem shows that this transition to linear dependence may happen very fast.

**Theorem 1.1.** Let \( v : \mathbb{R}^{2d} \to \mathbb{R}^+ \) be a submultiplicative weight function such that \( \lim_{n \to \infty} v(nz)^{1/n} = 1 \) for all \( z \in \mathbb{R}^{2d} \) (\( v \) satisfies the Gelfand-Raikov-Shilov condition).

Assume that
\[
\int_{\mathbb{R}^{2d}} |\langle g, \pi(z)g \rangle| v(z) \, dz < \infty.
\]
If \( \mathcal{G}(g, \Lambda) \) is a frame for \( L^2(\mathbb{R}^d) \), but not a Riesz basis, then the lower Riesz bound \( A_n \) of \( \mathcal{G}(g, \Lambda_n) \) decays like
\[
A_n \leq C \sup_{|\lambda| > n} v(\lambda)^{-2}.
\]

For the polynomial weight \( v(z) = (1 + |z|)^s \), the lower bound decays like \( A_n = \mathcal{O}(n^{-2s}) \), and for the sub-exponential weight \( v(z) = e^{a|z|^b} \) for \( a > 0 \) and \( 0 < b < 1 \) we have \( A_n = \mathcal{O}(e^{-an^b}) \). This means that the lower bound \( A_n \) tends to zero almost exponentially. The finite Gabor system \( \mathcal{G}(g, \Lambda_n) \) is extremely badly conditioned, and numerically \( \mathcal{G}(g, \Lambda_n) \) behaves like a linearly dependent set. On the other hand, if \( \Lambda \) is a lattice, then by Linnell’s theorem \( \mathcal{G}(g, \Lambda_n) \) is always linearly independent. Theorem 1.1 states a striking contrast between the numerical linear dependence of finite sets of time-frequency shifts and their conjectured abstract linear independence.

In the remainder of this note we prepare the necessary background on time-frequency analysis and spectral invariance of matrix algebras and then prove Theorem 1.1 and a variation. The proof will be relatively short, but it combines several non-trivial statements from harmonic analysis. In a sense, we extend the quantitative analysis of the finite section method in [21] to elements in the kernel of a matrix.
Operators related to Gabor systems. If $D_{g,A}$ is bounded from $\ell^2(\Lambda)$ to $L^2(\mathbb{R}^d)$, then $G(g, \Lambda)$ is called a Bessel sequence. Its adjoint operator is the analysis operator $D_{g,A}^* \, f = \langle f, \pi(\lambda)g \rangle : \lambda \in \Lambda \rangle \in \ell^2(\Lambda)$ for $f \in L^2(\mathbb{R}^d)$.

We also consider the frame operator of $G(g, \Lambda)$ defined to be

$$S_{g,A} \, f = D_{g,A} \, D_{g,A}^* \, f = \sum_{\lambda \in \Lambda} \langle f, \pi(\lambda)g \rangle \pi(\lambda)g$$

for $f$ in a suitable space of test functions. The Gram matrix is the matrix $G_{g,A} = D_{g,A}^* \, D_{g,A}$ acting on $\ell^2(\Lambda)$ with entries

$$(G_{g,A})_{\lambda,\mu} = \langle \pi(\mu)g, \pi(\lambda)g \rangle \quad \lambda, \mu \in \Lambda.$$ 

The algebraic identity

$$\| \sum_{|\lambda| \leq n} c_\lambda \pi(\lambda)g \|_2^2 = \sum_{|\lambda|,|\mu| \leq n} \langle \pi(\lambda)g, \pi(\mu)g \rangle c_\lambda \overline{c_\mu}$$

shows that the Riesz bounds of $G(g, \Lambda_n)$ are just the extremal eigenvalues of the finite sections of the Gramian matrix of $G(g, \Lambda)$.

Weights and modulation spaces. To measure the time-frequency concentration of a function, we use weighted modulation spaces. In time-frequency analysis one uses the several conditions for weight functions [18]:

(i) a weight $v : \mathbb{R}^{2d} \to \mathbb{R}^+$ is submultiplicative, if $v(z_1 + z_2) \leq v(z_1)v(z_2)$ for all $z_1, z_2 \in \mathbb{R}^{2d}$, and

(ii) $v$ is subconvolutive, if $(v^{-1} * v^{-1})(z) \leq Cv(z)^{-1}$ for all $z \in \mathbb{R}^{2d}$.

(iii) A weight $v$ satisfies the Gelfand-Raikov-Shilov (GRS) condition

$$\lim_{n \to \infty} v(nz)^{1/n} = 1 \quad \text{for all } z \in \mathbb{R}^{2d}.$$ 

The main examples for weights are the polynomial weights $z \mapsto (1 + |z|)^s$ for $s \geq 0$ and the sub-exponential weights $z \mapsto e^{a|z|^b}$ for $a > 0$ and $0 < b < 1$. The exponential weight $z \mapsto e^{a|z|}$ for $a > 0$ does not satisfy the GRS-condition.

Let $\phi(t) = e^{-\pi t^2}$ be the Gaussian and $v$ a weight function on $\mathbb{R}^{2d}$. A function $g$ belongs to the modulation space $M^1_v(\mathbb{R}^d)$, if

$$\|g\|_{M^1_v} := \int_{\mathbb{R}^{2d}} |\langle g, \pi(z)\phi \rangle| \, v(z) \, dz < \infty.$$ 

Likewise $g \in M^\infty_v(\mathbb{R}^d)$, if

$$\|g\|_{M^\infty_v} := \sup_{z \in \mathbb{R}^{2d}} |\langle g, \pi(z)\phi \rangle| \, v(z) < \infty.$$ 

From the theory of modulation spaces we need the following facts about the modulation spaces $M^1_v$ and $M^\infty_v$. See [17] and [14] for a historical survey about modulation spaces.

Lemma 1.2. (A) Assume that $v$ is a submultiplicative weight on $\mathbb{R}^{2d}$. Then the following conditions are equivalent:

(i) $g \in M^1_v(\mathbb{R}^d)$
(ii) $\int_{\mathbb{R}^{2d}} |\langle g, \pi(z)g \rangle| \, v(z) \, dz < \infty$. 
(iii) The function \( z \mapsto \langle g, \pi(z)g \rangle \) belongs to the amalgam space \( W(C, \ell^1_v) \), i.e., it is continuous and

\[
\sum_{k \in \mathbb{Z}^{2d}} \sup_{z \in [0,1]^{2d}} |\langle g, \pi(k+z)g \rangle | v(k) < \infty.
\]

(B) Assume that \( v \) is submultiplicative and subconvolutive. Then \( g \in M_v^\infty(\mathbb{R}^d) \) if and only if \( \sup_{z \in \mathbb{R}^{2d}} |\langle g, \pi(z)g \rangle | v(z) < \infty \).

For a proof see [17], Propositions 12.1.2, 12.1.11 and Theorem 13.5.3.

Note that condition (2) in Theorem 1.1 amounts to saying that \( g \in M_v^\infty(\mathbb{R}^d) \).

Spectral invariance of matrices with off-diagonal decay. Let \( \Lambda \) be a countable set in \( \mathbb{R}^{2d} \) satisfying the condition

\[
\max_{z \in \mathbb{R}^{2d}} \#\{\lambda \in \Lambda : |\lambda - z| \leq 1\} < \infty.
\]

\( \Lambda \) is said to be relatively separated. Let \( v \) be a submultiplicative weight on \( \mathbb{R}^{2d} \).

We will use the following classes of infinite matrices over the index set \( \Lambda \).

(i) The class \( C_v^\infty(\Lambda) \) consists of matrices \( A = (a_{\lambda,\mu})_{\lambda,\mu \in \Lambda} \) with off-diagonal decay \( v^{-1} \) and is equipped with the norm

\[
\|A\|_{C_v^\infty} = \sup_{\lambda,\mu \in \Lambda} |a_{\lambda,\mu}| v(\lambda - \mu).
\]

For polynomials weights \( v(z) = (1 + |z|)^s, C_v^\infty \) is often called the Jaffard class.

(ii) A matrix \( A \) belongs to the class \( C_v = C_v(\Lambda) \) of convolution-dominated matrices, if there exists an envelope function \( \Theta \in W(C, \ell^1_v) \), such that

\[
|a_{\lambda,\mu}| \leq \Theta(\lambda - \mu) \quad \forall \lambda, \mu \in \Lambda.
\]

The norm on \( C_v \) is \( \|A\|_{C_v} = \inf \{ ||\Theta||_{W(C, \ell^1_v)} : \Theta \text{ is an envelope} \} \).

If \( v \) is submultiplicative, then \( C_v \) is a Banach algebra. If \( v^{-1} \in \ell^1(\Lambda) \) and \( v \) is subconvolutive, then \( C_v^\infty \) is a Banach algebra. Both algebras can be embedded into the \( C^* \)-algebra of bounded operators \( \mathcal{B}(\ell^2(\Lambda)) \).

The most important result about these matrix algebras is their spectral invariance asserting that the off-diagonal decay is preserved under inversion.

**Theorem 1.3.** Assume that \( \Lambda \) is relatively separated and the \( v \) is a submultiplicative weight satisfying the GRS-condition.

(i) If \( A \in C_v \) and \( A \) is invertible on \( \ell^2(\Lambda) \), then \( A^{-1} \in C_v \).

(ii) Assume in addition that \( v \) is subconvolutive. If \( A \in C_v^\infty \) and \( A \) is invertible on \( \ell^2(\Lambda) \), then \( A^{-1} \in C_v^\infty \).

We say that both \( C_v \) and \( C_v^\infty \) are inverse-closed in \( \mathcal{B}(\ell^2(\Lambda)) \). Theorem 1.3 has been proved several times and on several levels of generality. We refer to the original work of Baskakov [5], Kurbatov [25], Gohberg-Kaeshoek-Woerdemann [16], and Sjöstrand [27] for (i), and to Baskakov [5], Jaffard [24], and Baskakov [20] for (ii). The attributions for the algebra \( C_v \) are a bit subtle, because the cited references deal only with the case when \( \Lambda \) is a lattice. The case of a relatively separated index set \( \Lambda \) follows by a simple reduction described in [3]: Since \( \max \#(\Lambda \cap ([k \pm [0,1]^{2d}])) = N < \infty \), one can define an explicit map \( a : \Lambda \mapsto \mathbb{Z}^{2d} \) that preserves the off-diagonal
decay properties after re-indexing a given matrix $A$. For the spectral invariance
one may assume therefore without loss of generality that $\Lambda$ is a lattice. Also,
Sjöstrand’s argument [27] works for relatively separated index sets and weights
without any change of the proof. An extended survey about spectral invariance
including matrix algebras can be found in [19].

These matrix classes arise naturally in the analysis of Gabor frames, as is shown
by the following lemma.

Lemma 1.4. Assume that $\Lambda \subseteq \mathbb{R}^{2d}$ is relatively separated and that $\nu$ is a submul-
tiplicative weight on $\mathbb{R}^{2d}$.

(i) If $g \in M^1_v(\mathbb{R}^d)$, then the Gramian $G_{g,\Lambda}$ of $G(g, \Lambda)$ is in $C_v(\Lambda)$.

(ii) If, in addition, $\nu$ is subconvolutive and if $g \in M^\infty_v(\mathbb{R}^d)$, then $G_{g,\Lambda} \in C^\infty_v(\Lambda)$.

Proof. Since $|G_{g,\Lambda}(\mu,\lambda)| = |\langle \pi(\mu)g, \pi(\lambda)g \rangle| = |\langle g, \pi(\lambda - \mu)g \rangle|$
we may take $\Theta(z) = |\langle g, \pi(z)g \rangle|$ as an envelope function. If $g \in M^1_v(\mathbb{R}^d)$,
then $\Theta \in W(\mathcal{C}, \ell^1_v)$ by Lemma 1.2. (ii) is clear from the definitions.

Proof of Theorem 1.1. Theorem 1.1 follows from the combination of several
observations. First an easy lemma.

Lemma 1.5. Assume that $G(g, \Lambda)$ is a Bessel sequence with bound $B$ and that
ker $D_{g,\Lambda} \neq \{0\}$. If $c \in \ker D_{g,\Lambda}$, $\|c\|_2 = 1$, then for sufficiently large $n$ we have

$$A_n \leq 2B \sum_{\lambda \in \Lambda: |\lambda| > n} |c_\lambda|^2.$$  

Proof. We split the sum $\sum_{\lambda \in \Lambda} c_\lambda \pi(\lambda)g = 0$ into two parts and then take norms.
We obtain

$$\left\| \sum_{|\lambda| \leq n} c_\lambda \pi(\lambda)g \right\|_2^2 = \sum_{|\lambda| > n} |c_\lambda|^2 \leq B \sum_{|\lambda| > n} |c_\lambda|^2.$$

For $n$ large enough we have $\sum_{|\lambda| \leq n} |c_\lambda|^2 \geq \frac{1}{2}$, whence the lower Riesz bound $A_n$ of
$G(g, \Lambda_n)$ obeys the following estimate:

$$A_n = \inf_{c \neq 0} \frac{\left\| \sum_{|\lambda| \leq n} c_\lambda \pi(\lambda)g \right\|_2^2}{\sum_{|\lambda| \leq n} |c_\lambda|^2} \leq 2B \sum_{|\lambda| > n} |c_\lambda|^2.$$

This estimate holds for every normalized $c \in \ker D_{g,\Lambda}$.

Lemma 1.5 states the obvious fact that the finite sets $G(g, \Lambda_n)$ become “more and
more linearly dependent” in the sense that $A_n \rightarrow 0$. To estimate the asymptotic
behavior of $A_n$ more precisely, we need to construct a “bad” sequence $c$ with fast
decay in $\ker D_{g,\Lambda}$. The possible decay depends on the time-frequency concentration
of the window $g$, as we will prove now.

Proposition 1.6. If $g \in M^1_v(\mathbb{R}^d)$ and $G(g, \Lambda)$ is a frame, but not a Riesz basis for
$L^2(\mathbb{R}^d)$, then $\ker D_{g,\Lambda} \cap \ell^1_v(\Lambda) \neq \{0\}$. 

Proof. 1. Recall that \( G_{g,\Lambda} = D_{g,\Lambda}^* D_{g,\Lambda} \) is the Gramian operator associated to \( \mathcal{G}(g, \Lambda) \). Consequently, \( c \in \ker D_{g,\Lambda} \) if and only if \( \|D_{g,\Lambda}c\|_2^2 = \langle G_{g,\Lambda}c, c \rangle = 0 \) if and only if \( c \in \ker G_{g,\Lambda} \).

2. To relate the spectrum of the frame operator \( S_{g,\Lambda} \) on \( L^2(\mathbb{R}^d) \) and of \( G_{g,\Lambda} \) on \( \ell^2(\Lambda) \), we use the identity

\[
\sigma(S_{g,\Lambda}) \cup \{0\} = \sigma(D_{g,\Lambda} D_{g,\Lambda}^*) \cup \{0\} = \sigma(D_{g,\Lambda}^* D_{g,\Lambda}) \cup \{0\} = \sigma(G_{g,\Lambda}) \cup \{0\},
\]

which follows from a purely algebraic manipulation \cite[p. 199]{[link]}. From this identity we draw the following conclusions: Since \( \mathcal{G}(g, \Lambda) \) is a frame, we have \( \sigma(S_{g,\Lambda}) \subseteq [A, B] \) for \( A, B > 0 \). Since \( \mathcal{G}(g, \Lambda) \) is not a Riesz basis, \( \ker G_{g,\Lambda} \neq \{0\} \) and thus \( 0 \in \sigma(G_{g,\Lambda}) \). Consequently,

\[
(9) \quad \sigma(G_{g,\Lambda}) \subseteq \{0\} \cup [A, B].
\]

The main point is the spectral gap between 0 and \( A \).

3. We now apply an argument developed by Baskakov \cite{[link]} to show that the orthogonal projection onto the kernel of \( G_{g,\Lambda} \) is a matrix with off-diagonal decay. Let \( P \) be the orthogonal projection from \( \ell^2(\Lambda) \) onto \( \ker G_{g,\Lambda} \). With the Riesz functional calculus \cite{[link]}, this projection can be written as

\[
(10) \quad P = \frac{1}{2\pi i} \int_{\gamma} (zI - G_{g,\Lambda})^{-1} \, dz,
\]

where \( \gamma \) is a closed curve in \( \mathbb{C} \) around 0 disjoint from the interval \( [A, B] \), for instance \( \gamma(t) = \frac{A}{2} e^{2\pi it}, t \in [0, 1] \).

4. Spectral invariance: By Lemma \cite{[link]} \( G_{g,\Lambda} \) and \( zI - G_{g,\Lambda} \) are matrices in \( \mathcal{C}_v \). Since \( zI - G_{g,\Lambda} \) is invertible for \( z \in \gamma \), Theorem \cite{[link]} implies that \( (zI - G_{g,\Lambda})^{-1} \) is also in \( \mathcal{C}_v \). From the continuity of the resolvent function \( z \mapsto (zI - G_{g,\Lambda})^{-1} \) we conclude that \( \sup_{z \in \gamma} \|(zI - G_{g,\Lambda})^{-1}\|_{\mathcal{C}_v} < \infty \). Consequently, the integral defining the orthogonal projection onto the kernel of \( G_{g,\Lambda} \) is in the algebra of convolution-dominated matrices \( \mathcal{C}_v \):

\[
P \in \mathcal{C}_v.
\]

This means that there exists an envelope \( \Theta \in W(C, \ell^1_v) \), such that \( |P_{\lambda,\mu}| \leq \Theta(\lambda - \mu) \). If \( \{e_\lambda : \lambda \in \Lambda\} \) with \( e_\lambda(\mu) = \delta_{\lambda,\mu} \) denotes the standard orthonormal basis of \( \ell^2(\Lambda) \), then

\[
|\langle e_\lambda, P e_\mu \rangle| = |P_{\lambda,\mu}| \leq \Theta(\lambda - \mu),
\]

or, equivalently, \( P e_\mu \in \ell^1_v(\Lambda) \) for all \( \mu \in \Lambda \). As the projection \( P \) is non-zero by assumption, \( P e_\mu \neq 0 \) for some \( \mu \), and thus we have found a non-trivial vector in \( \ker G_{g,\Lambda} \cap \ell^1_v = \ker D_{g,\Lambda} \cap \ell^1_v \), and we are done.

Combining Lemma \cite{[link]} and Proposition \cite{[link]}, we now can conclude the proof of Theorem \cite{[link]}. Choose an \( \ell^2 \)-normalized \( c \in \ker D_{g,\Lambda} \cap \ell^1_v(\Lambda) \). Then by (8) we obtain...
that
\[ A_n \leq 2B \sum_{|\lambda|>n} |c_\lambda|^2 \]
\[ \leq 2B \sup_{|\lambda|>n} v(\lambda)^{-2} \sum_{|\lambda|>n} |c_\lambda|^2 v(\lambda)^2 \]
\[ \leq 2B \sup_{|\lambda|>n} v(\lambda)^{-2} \sum_{|\lambda|>n} |c_\lambda| v(\lambda) = C \sup_{|\lambda|>n} v(\lambda)^{-2}. \]
(11)

Theorem 1.1 is proved completely. \( \square \)

The same proof yields the following variation of Theorem 1.1.

**Theorem 1.7.** Let \( v \) be a submultiplicative and subconvolutive weight function satisfying the Gelfand-Raikov-Shilov condition.

Assume that \( g \in M_v^\infty(\mathbb{R}^d) \) and that \( \mathcal{G}(g, \Lambda) \) is a frame for \( L^2(\mathbb{R}^d) \), but not a Riesz basis. Then the lower Riesz bound \( A_n \) of \( \mathcal{G}(g, \Lambda_n) \) decays like
\[ A_n \leq C \sum_{|\lambda|>n} v(\lambda)^{-2}. \]
(12)

**Proof.** The proof is similar, we just use the versions of Lemma 1.4 and Theorem 1.3 that are valid for \( M_v^\infty(\mathbb{R}^d) \). Instead of Proposition 1.6 we use the following statement: If \( g \in M_v^\infty(\mathbb{R}^d) \) and \( \mathcal{G}(g, \Lambda) \) is a frame, but not a Riesz basis for \( L^2(\mathbb{R}^d) \), then \( \ker D_{g,\Lambda} \cap \ell^\infty_v(\Lambda) \neq \{0\} \). Equation (11) is replaced by
\[ A_n \leq 2B \sum_{|\lambda|>n} |c_\lambda|^2 \]
\[ \leq 2B \sup_{|\lambda|>n} |c_\lambda|^2 v(\lambda)^2 \sum_{|\lambda|>n} v(\lambda)^{-2}. \]

**REMARKS:**
1. Note the importance of assumptions: \( \mathcal{G}(g, \Lambda) \) must be a frame so that there exists a spectral gap for the Gramian. Theorem 1.1 fails, when \( \mathcal{G}(g, \Lambda) \) is not a frame and the spectral gap is missing. This may be the case for Gabor systems at the critical density, for instance, with \( \phi(t) = e^{-\pi t^2} \) the Gabor system \( \mathcal{G}(\phi, \mathbb{Z}^2) \) is neither a frame nor a Riesz basis (but still complete in \( L^2(\mathbb{R}) \)). In this case, the asymptotic decay of the lower Riesz bound \( A_n \) can be investigated with different methods, see [4].

2. Theorem 1.1 quantifies the degree of linear dependence of the finite sets \( \mathcal{G}(g, \Lambda_n) \). Note that good time-frequency localization of \( g \) (corresponding to fast growth of \( v \)) yields a faster decay of the constants \( A_n \). This is somewhat counterintuitive, because the fast decay of \( z \mapsto \langle g, \pi(z)\phi \rangle \) implies that the function \( z \mapsto |\langle g, \pi(z)\phi \rangle|^2 \) is sharply peaked in \( \mathbb{R}^{2d} \), and shifts of sharply peaked bumps (corresponding to the time-frequency shifts of \( \pi(\lambda)g \)) tend to be linearly independent with good constants. According to Theorem 1.1 this is not the case here.
This phenomenon indicates the existence of subtle cancellations in linear combinations of time-frequency shifts and seems to be yet another manifestation of the uncertainty principle.

3. To obtain an upper estimate for $A_n$, we needed to find only a single sequence $c \in \ell^2(\Lambda)$ such that $\| \sum_{|\lambda| \leq n} c_\lambda \pi(\lambda) g \|_2^2 \approx A_n \| c \|_2^2$. In the course of the proof we have constructed such a sequence by using the spectral invariance and the properties of the basis function $g$.

It is natural to ask whether the decay rate of $A_n$ in Theorem 1.1 is best possible. This question, however, is much more difficult, because it amounts to showing that $\| \sum_{|\lambda| \leq n} c_\lambda \pi(\lambda) g \|_2^2 \geq \text{const} \ A_n \| c \|_2^2$ for all $c$. Since every finite set of time-frequency shifts can be extended to a Gabor frame, this statement seems equivalent to the original linear independence conjecture.

4. If $v$ is an exponential weight, $v(z) = e^{a|z|}$ for some $a > 0$, then the matrix algebras $C_v$ and $C_v^\infty$ are no longer inverse-closed in $B(\ell^2(\Lambda))$. The statement of Theorem 1.3 is false and has to be replaced by a weaker version. Nevertheless one can show [4] that for $g \in M^1_v$ with exponential weight $v(z) = e^{a|z|}$ the lower Riesz bound decays exponentially $A_n \lesssim e^{-\epsilon n}$ for some $\epsilon > 0$.

5. In our analysis we have only used that $G(g,\Lambda)$ is a frame with ker $D_{g,\Lambda} \neq \{0\}$ and the decay properties of the Gramian $G_{g,\Lambda}$. The statement about the asymptotic behavior of the lower Riesz bound $A_n$ carries over without change to general localized frames [15] indexed by a relatively separated subset of $\mathbb{R}^{2d}$.

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