Canonical quantization of spontaneously broken topologically massive gauge theory

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Abstract

In this paper we investigate the canonical quantization of a non-Abelian topologically massive Chern-Simons theory in which the gauge fields are minimally coupled to a multiplet of scalar fields in such a way that the gauge symmetry is spontaneously broken. Such a model produces the Chern-Simons-Higgs mechanism in which the gauge excitations acquire mass both from the Chern-Simons term and from the Higgs-Kibble effect. The symmetry breaking is chosen to be only partially broken, in such a way that in the broken vacuum there remains a residual non-Abelian symmetry. We develop the canonical operator structure of this theory in the broken vacuum, with particular emphasis on the particle-content of the fields involved in the Chern-Simons-Higgs mechanism. We construct the Fock space and express the dynamical generators in terms of creation and annihilation operator modes. The canonical apparatus is used to obtain the propagators for this theory, and we use the Poincaré generators to demonstrate the effect of Lorentz boosts on the particle states. 11.10.Ef, 03.70.+k, 11.15.—q
Field theories in \((2 + 1)\)-dimensional space-time exhibit many interesting and important properties related to the masses of the particle excitations of the quantum fields. For example, gauge theories involving a Chern-Simons term support massive gauge field excitations \([1,2]\), which differ from the Higgs-Kibble excitations produced in conventional spontaneous symmetry breaking \([3]\). The combination of both spontaneous symmetry breaking and a Chern-Simons term for the gauge field leads to the Chern-Simons-Higgs (CSH) mechanism, in which the physical fields are transmuted in a process that combines the Chern-Simons and Higgs-Kibble mass-generating effects in a particularly interesting and instructive manner.

An analysis of the covariant gauge field propagator \([4,5]\) indicates the presence of two distinct mass poles, with masses given as complicated functions of the Higgs mass scale (set by the tree-approximation minimum of the symmetry breaking potential) and the Chern-Simons mass scale (coming from the Chern-Simons coupling parameter which has dimensions of mass in three dimensional space-time). The two distinct mass poles may also be seen in a factorization of the Chern-Simons-Proca equations of motion \([6]\). A Schrödinger representation approach \([7]\) provides a simple physical picture based on a quantum mechanical analogue which identifies the two masses precisely with the two characteristic frequencies of the planar quantum mechanical model of charged particles moving in both a uniform magnetic field and a harmonic potential well. In this present paper we investigate field theoretic aspects of the Chern-Simons-Higgs mechanism more deeply, presenting a detailed analysis of the canonical quantization of spontaneously broken Chern-Simons theories. In this work, we pay particular attention to the relation between the quantized fields and their particle excitation modes and to the structure of the Poincaré generators as functionals of these particle excitation operators.

We have chosen to consider a non-Abelian theory in which the non-Abelian gauge symmetry is spontaneously broken in a manner that preserves a residual non-Abelian symmetry in the broken vacuum. This choice is motivated by the question of how a spontaneously
broken Chern-Simons theory ‘knows’ to quantum-mechanically protect the residual non-Abelian gauge symmetry from topologically nontrivial gauge transformations. For non-Abelian Chern-Simons theories, quantum consistency requires that the Chern-Simons coupling parameter takes quantized integer values, in appropriate units. Qualitatively, this consistency condition is reminiscent of Dirac’s quantum mechanical quantization condition for the magnetic monopole, but since the Chern-Simons theory is a field theory further subtleties (such as renormalization) arise. Pisarski and Rao showed that for a Chern-Simons-Yang-Mills theory (with no matter fields or symmetry breaking) a consistent one-loop renormalization involves a finite additive renormalization of the Chern-Simons mass, with the finite shift depending on the gauge group and being such that the integer quantization condition is preserved. Subsequent calculations have confirmed the conjecture that there are no further radiative corrections to this result. Perturbative analyses of Abelian Chern-Simons theories subject to spontaneous symmetry breaking confirm the topological basis of the integer quantization of the renormalization of the Chern-Simons term. This work has shown that, in the Abelian case, in which topological arguments do not apply, the Chern-Simons mass receives a shift, in the broken vacuum, which is not an integer, but a complicated function of the various bare mass scales. In a spontaneously broken non-Abelian Chern-Simons theory, with a completely broken symmetry in which the invariance of the effective theory to gauge transformations is no longer supported, similar behavior was found. More interesting is the situation in which the non-Abelian gauge symmetry is

1 In Ref. it is suggested that this shift should not be interpreted as a finite renormalization of the Chern-Simons mass, but rather as an indication of the appearance of parity-violating terms in the effective action. This reformulation of the result extends the Coleman-Hill theorem, concerning the absence of loop corrections to the Chern-Simons mass, to the case of Abelian spontaneously broken Chern-Simons theories.

2 Note that the explicit formula for the finite shift reported in Ref. is incorrect, although
only partially broken, leaving a residual non-Abelian symmetry in the broken phase. The presence of the non-Abelian residual symmetry suggests that the Chern-Simons coupling parameter should again be quantized, and indeed a direct perturbative computation shows that the Chern-Simons coupling parameter receives a quantized finite shift which preserves the quantum consistency condition in the broken vacuum. This work confirms the validity of the effective theory that describes the quantum fluctuations of the field about the spontaneously broken vacuum; and it motivates an investigation into the origins of the massive propagating particle excitations of this model, and the mechanisms by which they obtain their mass.

In this paper, we consider the canonical quantization of such a non-Abelian model, with a partially broken symmetry leaving a residual non-Abelian symmetry in the broken phase, and develop the underlying dynamical theory. We make explicit the representation of the operator-valued fields in terms of excitations that correspond to observable, propagating particles in the spontaneously broken vacuum. We formulate the model in (2 + 1)-dimensional Minkowski space-time and for definiteness we consider an octet of SU(3) gauge fields interacting with a triplet of scalar fields in the fundamental representation of SU(3). The scalar fields $\Phi$ are coupled gauge-invariantly to the gauge fields, and self-coupled through a quartic potential $V(\Phi^\dagger \Phi) = \mu^2 \Phi^\dagger \Phi - \frac{1}{2} h (\Phi^\dagger \Phi)^2$, where $\mu^2 > 0$ and $h > 0$, so that, in the tree approximation, the scalar fields have nonvanishing vacuum expectation values. The vacuum expectation values of the three constituent fields in $\Phi$ are chosen so that the residual “effective” fields, which represent fluctuations of these scalar fields about their tree approximation vacuum expectation values, still maintain an unbroken SU(2) symmetry in their coupling to the gauge fields. In the canonical quantization of this model we construct

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this does not affect the qualitative conclusions of that paper. The corrected integral appears in Ref. [16].

3This model was first considered in Ref. [17], but the opposite conclusion was reported.
time-dependent fields in an interaction picture that includes, in the “free” Hamiltonian that
drives it, the interaction terms that become bilinear in fields when the charged scalar field
Φ is expanded about its constant vacuum expectation value. We use these time-dependent
interaction-picture fields to evaluate the propagators. And finally, we construct the parti-
cle states that correspond to the two different mass singularities in the propagator for this
model. We express the trilinear and quartic interaction Lagrangian as a functional of these
interaction-picture fields, and obtain a set of vertices that can be used to describe the theory.
In addressing these problems, we make use of technical developments that originated from
separate earlier work by the authors [18,22].

In Section II, we formulate the model and describe the spontaneous symmetry breaking
process. In Section III, we construct the required Fock spaces, express the scalar and gauge
fields as superpositions of particle and ghost excitations, and implement Gauss’s law and
the gauge condition. In Section IV, we construct the interaction-picture scalar and gauge
fields; and we evaluate their time-ordered vacuum expectation values in the spontaneously
broken vacuum state, to obtain the propagators for this theory. In Section V, we construct
the Poincaré generators for this theory, demonstrate the validity of the Poincaré algebra,
and evaluate the effect of Lorentz boosts on each of the massive gluon states. Detailed forms
of the interaction Lagrangian are given in an Appendix.

II. FORMULATION OF THE MODEL

The Lagrangian for this model is given by

\[ \mathcal{L} = -\frac{1}{4} F_{\mu\nu}^a F^{a\mu\nu} + \frac{1}{3} m e^{\mu\nu\rho} \left( F_{\mu\nu}^a A_\rho^a + \frac{2}{3} e f^{abc} A_\mu^a A^b_\nu A^c_\rho \right) + (D^\nu \Phi)^\dagger (D_\mu \Phi) + \mu^2 \Phi^\dagger \Phi - \frac{1}{2} h (\Phi^\dagger \Phi)^2 + \mathcal{L}_\text{fp} \]  

(1)

The implied summations over repeated Latin superscripts, such as a, b, and c, are from 1 to 8
unless otherwise specified.
\[ + \frac{1}{2}(1-\gamma)G^aG^a - \left[ \partial_\mu A^{a\mu} - ie(1-\gamma)(\langle \Phi \rangle_0^\dagger \lambda^a \Phi' - \Phi'^\dagger \lambda^a \langle \Phi \rangle_0) \right] G^a, \]  

(2)

where \( F^a_{\mu\nu} \) designates the \( SU(3) \) gauge field strength

\[ F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu - 2ef^{abc}A^b_\mu A^c_\nu; \]

(3)

we denote by \( F^a_{\mu\nu} \) the “Abelian” part of the field strength,

\[ F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu, \]

(4)

and \( f^{abc} \) represents the \( SU(3) \) structure constants. The covariant derivative of the scalar triplet \( D_\mu \Phi \) is given by

\[ D_\mu \Phi = \partial_\mu \Phi + ie\lambda^a A^a_\mu \Phi, \]

(5)

where \( \lambda^a \) represents the Gell-Mann matrices which satisfy the commutation relations [\( \lambda^a, \lambda^b ] = 2if^{abc}\lambda^c \). The Lagrangian also contains the gauge-fixing term, with gauge-fixing parameter \( \gamma \), for the covariant gauge — in this case, the t’Hooft gauge, which involves both the tree-approximation vacuum expectation value \( \langle \Phi \rangle_0 \) and the fluctuation of the scalar field about that vacuum expectation value \( \Phi' = \Phi - \langle \Phi \rangle_0 \). \( \mathcal{L}_{fp} \) is the part of the Lagrangian that couples the gauge fields to the Faddeev-Popov ghosts, and is given by

\[ \mathcal{L}_{fp} = i\partial_\mu \sigma^a_\mu \partial^\mu \sigma^a_\mu + 2ife f^{abc}A^a_\mu \sigma^b_\mu \partial^\mu \sigma^c_\mu, \]

(6)

where \( \sigma^a_\mu \) and \( \sigma^a_0 \) are the two self-adjoint operator-valued anticommuting scalar Faddeev-Popov fields.

We choose a scheme for breaking the \( SU(3) \) symmetry that preserves an \( SU(2) \) symmetry in the effective Lagrangian. In the tree-approximation vacuum state for this effective Lagrangian, the self-interaction \( V(\Phi^\dagger \Phi) \) takes on its classical minimum value for the tree-approximation vacuum expectation value \( \langle \Phi \rangle_0 \). A choice for \( \langle \Phi \rangle_0 \) that satisfies this requirement is
\[ \langle \Phi \rangle_0 = \frac{v}{\sqrt{2}} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \equiv \frac{v}{\sqrt{2}} \langle \phi \rangle_0, \quad (7) \]

where \( v = (2\mu^2/h)^{1/2} \). To analyze this model in the broken vacuum, we expand the scalar field \( \Phi \) in terms of its fluctuations about the v.e.v. \( \langle \Phi \rangle_0 \)

\[ \Phi' = \Phi - \langle \Phi \rangle_0 \quad \text{(8)} \]

and expand the Lagrangian as

\[ \mathcal{L} = \mathcal{L}_0 + \mathcal{L}_1 + \mathcal{L}_2. \quad \text{(9)} \]

Here \( \mathcal{L}_0 \) represents the “free” Lagrangian, in which the interaction have been shut off, and \( \mathcal{L}_1 \) and \( \mathcal{L}_2 \) represent terms that are first and second order in \( e \), respectively. Note that there are several coupling constants and mass scales to consider when making this expansion, and we need to be specific about how coupling constants are “shut off” in taking \( \mathcal{L} \) to its noninteracting limit \( \mathcal{L}_0 \). The Chern-Simons coupling constant \( m \) has dimensions of mass, as do \( e^2 \) (the square of the scalar-gauge coupling), \( v^2 \) (the square of the magnitude of the scalar field v.e.v.), and \( ev \). The noninteracting limit \( \mathcal{L}_0 \) of the full Lagrangian \( \mathcal{L} \) is defined to be the limit \( e \to 0 \) and \( h \to 0 \) with the “Higgs” mass scale \( ev \) kept constant, and the Chern-Simons mass scale unaffected. Then the noninteracting Lagrangian is

\[ \mathcal{L}_0 = -\frac{1}{4} F_{\mu\nu}^a F^{a\mu\nu} + \frac{m}{4} \epsilon^{\mu\nu\rho} F_{\mu\nu}^a A^a_\rho + \frac{e^2 v^2}{4} A^a_\mu A^{b\mu} \langle \phi \rangle_0^\dagger \{ \lambda^a, \lambda^b \} \langle \phi \rangle_0 + |\partial_\mu \Phi'|^2 \\
- \frac{\mu^2}{2} \left( \langle \phi \rangle_0^\dagger \Phi' + \Phi'^\dagger \langle \phi \rangle_0 \right)^2 + i \frac{e v}{\sqrt{2}} A^a_\mu \left[ (\partial^\mu \Phi')^\dagger \lambda^a \langle \phi \rangle_0 - \langle \phi \rangle_0^\dagger \lambda^a \partial^\mu \Phi' \right] \\
- \left[ \partial_\mu A^{a\mu} - i \frac{e v}{\sqrt{2}} (1 - \gamma) \left( \langle \phi \rangle_0^\dagger \lambda^a \Phi' - \Phi'^\dagger \lambda^a \langle \phi \rangle_0 \right) \right] G^a \\
+ \frac{1}{2} (1 - \gamma) G^a G^a + i \partial_\mu \sigma_1^a \partial^\mu \sigma_1^a. \quad \text{(10)} \]

The \( \mathcal{O}(e) \) interaction Lagrangian is

\[ \mathcal{L}_1 = e \left[ f^{abc} F_{\mu\nu}^a A^{b\mu} A^{c\nu} - \frac{m}{3} \epsilon^{\mu\nu\rho} f^{abc} A_\mu^a A_\nu^b A_\rho^c + 2i f^{abc} A_\mu^a \sigma_1^b \partial^\mu \sigma_1^c \right] \]
\[ + \frac{e v}{2\sqrt{2}} A_{\mu} A^{\mu} \left( \Phi \Phi' \{ \lambda^a, \lambda^b \} \langle \phi \rangle_0 + \langle \phi \rangle_0 \{ \lambda^a, \lambda^b \} \Phi' \right) \]
\[- i A_{\mu} \left[ \Phi \Phi' \lambda^a \partial^\mu \Phi' - (\partial^\mu \Phi')^\dagger \lambda^a \Phi' \right] - \sqrt{2} \frac{\mu^2}{e v} |\Phi'|^2 \left( \langle \phi \rangle_0 \Phi' + \Phi' \langle \phi \rangle_0 \right) \]  

(11)

and the \( \mathcal{O}(e^2) \) interaction Lagrangian is
\[ \mathcal{L}_2 = e^2 \left( - f^{abc} f^{ade} A_{\mu} A^{a} A_{\nu} A^{e} - \frac{\mu^2}{e^2 v^2} |\Phi'|^4 + \frac{1}{2} A_{\mu} A^{\nu} A_{\sigma} \Phi' \Phi' \{ \lambda^a, \lambda^b \} \Phi' \right). \]  

(12)

We note that the presence of the Chern-Simons term in the original Lagrangian Eq. (9) introduces a new quadratic piece \( \sim \epsilon FA \) in \( \mathcal{L}_0 \) and a new 3-gluon vertex piece \( \sim \epsilon AAA \) in \( \mathcal{L}_1 \).

To identify the physical and unphysical fields in the broken vacuum, we first express \( \Phi' \) in terms of real scalar fields
\[ \Phi' = \frac{1}{\sqrt{2}} \begin{pmatrix} i \xi^4 + \xi^5 \\ i \xi^6 + \xi^7 \\ -i \xi^8 + \psi \end{pmatrix}. \]  

(13)

Then, using the explicit form given in Eq. (7) of the v.e.v. \( \langle \phi \rangle_0 \), together with the Gell-Mann matrix anticommutation relations
\[ \{ \lambda^a, \lambda^b \} = \frac{1}{3} \delta^{ab} 1 + 2 d^{abc} \lambda^c, \]  

(14)

we can write the free Lagrangian \( \mathcal{L}_0 \) as
\[ \mathcal{L}_0 = - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{m^2}{4} \epsilon^{\mu\nu\rho} F_{\mu\nu} A^a + \frac{1}{2} \sum_{a=4}^{8} M_{(a)}^2 A_{\mu} A^{a\mu} + \sum_{a=4}^{8} M_{(a)} A_{\mu} A^{a\mu} \partial_\mu \xi^a \]
\[- \frac{1}{2} \sum_{a=4}^{8} \partial_\mu \xi^a \partial^\mu \xi^a + \frac{1}{2} \partial_\mu \psi \partial^\mu \psi - \mu^2 \psi^2 + \frac{1}{2} (1 - \gamma) G^a G^a \]
\[- \partial_\mu A^{a\mu} G^a + (1 - \gamma) \sum_{a=4}^{8} M_{(a)} \xi^a G^a + i \partial_\mu \sigma^a_1 \partial^\mu \sigma^a_2. \]  

(15)

Here the symmetry breaking mass scales \( M_{(a)} \) are given by
\[ M_{(a)} = \begin{cases} M_D = e v & a = 4, 5, 6, 7 \\ M_S = \frac{2}{\sqrt{3}} e v & a = 8 \end{cases}. \]  

(16)
From Eq. (15), we recognize $\psi$ as the Higgs scalar field, with mass $\sqrt{2}|\mu|$, and $\xi^a$ ($a = 4, \ldots, 8$) as massless unphysical scalar fields. Furthermore, the gauge fields $A_{\mu}^a$ ($a = 1, 2, 3$) have a quadratic Lagrangian of the Maxwell-Chern-Simons form, while the gauge fields $A_{\mu}^a$ ($a = 4, \ldots, 8$) have an additional Proca-like quadratic term with mass scale parameters $M_{(a)}$ as given in Eq. (16).

The interaction Lagrangians $\mathcal{L}_1$ and $\mathcal{L}_2$ can also be expanded in terms of the real fields in Eq. (13) and the symmetry breaking mass scales in Eq. (16), and the resulting expansions are recorded in Appendix A. It is important to observe that (as expected) the gauge field $A_{\mu}^a$ ($a = 1, 2, 3$) form an $SU(2)$ triplet corresponding to the residual $SU(2)$ symmetry of the broken vacuum. It proves convenient to group the real scalar fields into $SU(2)$ “isospinors”:

$$\Psi_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} i\xi^4 + \xi^5 \\ i\xi^6 + \xi^7 \end{pmatrix},$$  \hspace{1cm} (17)

$$\Psi_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} i\xi^4 + \xi^5 \\ \psi - i\xi^8 \end{pmatrix},$$  \hspace{1cm} (18)

and

$$\Psi_3 = \frac{1}{\sqrt{2}} \begin{pmatrix} i\xi^6 + \xi^7 \\ \psi - i\xi^8 \end{pmatrix}.$$  \hspace{1cm} (19)

With this notation, the fields $A_{\mu}^a$ ($a = 1, 2, 3$) couple to $\Psi_1$ in an $SU(2)$-invariant manner, while the two gauge field doublets ($A_{\mu}^4, A_{\mu}^5$) and ($A_{\mu}^6, A_{\mu}^7$) couple to $\Psi_2$ and $\Psi_3$ so that the part of the isospin invariance that corresponds to rotation in the $i = 1, 2$ plane is preserved; but this latter interaction is not invariant to rotation in the entire isospin space. The remaining gauge field $A_{\mu}^8$ is an $SU(2)$ singlet.

In earlier work on Abelian theories with Chern-Simons interactions [18–20], we implemented Gauss’s law and developed a canonical formulation for the entire Lagrangian, with all interactions included. In a non-Abelian gauge theory, such a program becomes much more problematical. We will therefore implement Gauss’s law only for the partial theory
described by $\mathcal{L}_0$. In this case, however, because of the spontaneously broken symmetry, even the Abelian $\mathcal{L}_0$ contains part of the interaction — not only the part of the $\Phi^4$ self-interaction implicit in the spontaneously broken vacuum state, but also the part of the gauge-invariant coupling of the gauge field to the “charged” scalar $\Phi$ that remains bilinear in operator-valued fields after $\Phi$ has been expressed as $\Phi = \Phi' + \langle \Phi \rangle_0$. Although this part of the interaction term is proportional to $e$, it does not vanish in the “interaction-free” limit, because $e$ combines with $h^{-1/2}$ to become one of the masses that are kept constant in the $\mathcal{L} \to \mathcal{L}_0$ limit. Implementing Gauss’s law and the gauge condition, and developing the canonical formulation of the part of the theory described by $\mathcal{L}_0$, will enable us to construct the Fock space for the particle states observed in the broken vacuum. In the course of this work, we will demonstrate the process by which the masses that stem from the Higgs-Kibble effect combine with the topological mass to form the masses of the propagating modes of the gauge field in this model. $\mathcal{L}_0$, defined as we have specified here, is the Lagrangian that drives the interaction-picture fields when a Higgs-Kibble effect occurs. The corresponding “free” Hamiltonian $H_0$, which is the $e \to 0$ limit of $H$ obtained by this same limiting process, accounts for the particle spectrum of this model. Once $\mathcal{L}_0$ and $H_0$ have been identified, and Gauss’s law and the covariant gauge condition have been imposed, the resulting apparatus can be used to develop a Fock space as well as propagators and vertices for evaluating the $S$-matrix elements and renormalization constants for the full theory, with $\mathcal{L}_1$ and $\mathcal{L}_2$ included.

The Euler-Lagrange equations determined by $\mathcal{L}_0$ are

$$\partial_\mu F^{a\mu} - \frac{1}{2} m e^{\mu\rho\nu} F^a_{\mu\rho} - \partial^\nu G^a = M_\alpha^2 A^{\alpha\mu} + \partial^\nu \alpha^a, \quad (20)$$

$$\partial_\mu A^{a\mu} - (1 - \gamma) \alpha^a = (1 - \gamma) G^a, \quad (21)$$

$$\partial_\mu \partial^\mu \psi + 2 \mu^2 \psi = 0, \quad (22)$$

and
\[ \partial_\mu \partial^\alpha \xi^a + M(a) \partial_\mu A^{a\mu} = (1 - \gamma) M(a) G^a, \]  

(23)

where \( \alpha^a = M(a) \xi^a \), and

\[ \partial_\mu \partial^\mu \sigma^a_i = \partial_\mu \partial^\mu \sigma^a_p = 0. \]  

(24)

From these equations, we get

\[ \partial_\mu \partial^\mu G^a = -(1 - \gamma) M(a)^2 G^a. \]  

(25)

Equation (20) represents the Maxwell-Ampere law (for \( \nu = 1, 2 \)) as well as Gauss’s law (for \( \nu = 0 \)); however, as is to be expected in covariant gauges, this equation differs from the classical form of Maxwell-Ampere and Gauss’s laws by the gauge-fixing term — in this case, \( \partial^\nu G^a + \partial^\nu \alpha^a \). Implementation of the correct form of these laws will have the effect of defining a subspace for the dynamical time-evolution of state vectors in which the gauge-fixing term will have vanishing matrix elements. Equation (21) will be used to impose the covariant gauge condition: \( \gamma = 0 \) corresponds to the Feynman, and \( \gamma = 1 \) to the Landau version of the covariant (t’Hooft) gauge.

To quantize this theory, we need to express the Hamiltonian in terms of the canonical momenta given by \( \Pi^{a\mu} = \partial \mathcal{L}_0 / \partial (\partial_\mu A^a_\mu) \). These canonical momenta are:

\[ \Pi^{a\mu} = F^{a\mu 0} + \frac{1}{2} m e^{0\mu} A^a_\nu - g^{0\mu} G^a, \]  

(26)

\[ \Pi_\psi = \partial_0 \psi, \]  

(27)

\[ \Pi^{a}_\xi = \partial_0 \xi^a + M(a) A^a_0, \]  

(28)

\[ \Pi^{a}_f = i \partial_0 \sigma^a_p, \]  

(29)

and

\[ \Pi^{a}_p = -i \partial_0 \sigma^a_f. \]  

(30)
\( \Pi_f \) and \( \Pi_p^a \) are the conjugate momenta to the fields \( \sigma_f \) and \( \sigma_p^a \), respectively.

The only equation that does not contain any time-derivatives of fields (and therefore is a constraint) is \( \Pi^{a0} = -G^a \). This constraint is manifestly consistent with canonical (Poisson) equal-time commutation rules, which we impose. The equal-time commutation rules (ETCR) are:

\[
[A^a(x), \Pi^b(y)] = i\delta_{ln} \delta^{ab} \delta(x - y), \tag{31}
\]

\[
[A^a_0(x), G^b(y)] = -i\delta^{ab} \delta(x - y), \tag{32}
\]

\[
[\xi^a(x), \Pi^b_p(y)] = i\delta^{ab} \delta(x - y), \tag{33}
\]

\[
[\psi(x), \Pi\psi(y)] = i\delta(x - y), \tag{34}
\]

and all other commutators among these fields are zero. The anticommutation rules for the Faddeev-Popov ghost fields are

\[
\{\sigma^a_f(x), \Pi^b_f(y)\} = i\delta^{ab} \delta(x - y), \tag{35}
\]

\[
\{\sigma^a_p(x), \Pi^b_p(y)\} = i\delta^{ab} \delta(x - y), \tag{36}
\]

and all other combinations anticommute.

The Hamiltonian density \( \mathcal{H}_0 \), determined by \( \mathcal{L}_0 \) and by the canonical momenta, will be expressed as

\[
\mathcal{H}_0 = \sum_{a=1}^{8} \mathcal{H}^a + \mathcal{H}_\psi + \mathcal{H}_{fp}; \tag{37}
\]

for \( a = 1, 2, 3 \):

\[
\mathcal{H}^a = \frac{1}{2} \Pi^a_l \Pi^a_l + \frac{1}{4} F^a_{ln} F^a_{ln} + \frac{1}{8} m^2 A^a_l A^a_l + \frac{1}{2} \epsilon_{ln} A_l^a \Pi^a_n \\
+ A^a_0 (\partial_l \Pi^a_l - \frac{1}{4} \epsilon_{ln} F^a_{ln}) + G^a \partial_l A_l^a - \frac{1}{2} (1 - \gamma) G^a G^a; \tag{38}
\]

for \( a = 4, 5, 6, 7, 8 \):
\[ H^a = \frac{1}{2} \Pi^a \Pi^i + \frac{1}{4} F^a_i F^a_i + \frac{1}{2} \left[ \frac{1}{4} m^2 + M^2 \right] A^a_n A^a_n + \frac{1}{2} m \epsilon_{ln} A^a_l \Pi^a_i \\
+ A^a_0 [\partial_l \Pi^a_i - \frac{1}{2} m \epsilon_{ln} F^a_i - M_0 \Pi^a] + G^a [\partial_l A^a_i - (1 - \gamma) M_0 \xi^a_l] \\
- M_0 A^a_l \partial_l \xi^a - \frac{1}{2} (1 - \gamma) G^a G^a + \frac{1}{4} \Pi^a_\xi \Pi^a_\xi + \frac{1}{2} \partial_l \xi^a \partial_l \xi^a; \quad (39) \]

The other parts of \( H_0 \) are

\[ H_\psi = \frac{1}{2} \Pi_\psi \Pi_\psi + \frac{1}{2} \partial_l \psi \partial_l \psi + \mu^2 \psi^2; \quad (40) \]

and

\[ H_{\text{fp}} = i \Pi^a_\text{p} \Pi^i + i \partial_j \sigma^a_i \partial_j \sigma^a_\text{p}. \quad (41) \]

The Hamiltonian, \( H_0 = \int d\mathbf{x} \ H_0(\mathbf{x}) \), is the “free” kinetic energy limit of the entire Hamiltonian, with the proviso that in this model the free kinetic energy limit includes the part of the interaction term in which the constant tree-approximation vacuum expectation value of \( \Phi \) combines with the charge \( e \) to form a new constant, dimensionally a mass, whose operator-valued coefficient is bilinear in fields. This part of the interaction is not shut off in the \( H \rightarrow H_0 \) limit, and is absorbed into a generalized, more encompassing kinetic energy operator \( H_0 \).

**III. PARTICLE STATES AND GAUSS’S LAW**

Equation (20), when \( \nu = 0 \) and the canonical momenta replace the time derivatives of fields, has the form\(^5\)

\[ \partial_l \Pi^a_l + \frac{1}{2} m \epsilon_{ln} \partial_l A^a_n - M_0 \Pi^a_\xi = \partial^l G^a. \quad (42) \]

\(^5\)For notational simplicity, we will, from here on, generally use a noncovariant notation in which the subscript \( l \) denotes a covariant component of a covariant quantity (like \( \partial_l \)), a contravariant component of a contravariant quantity (like \( A_l \)), or the contravariant component of the second rank tensor \( \Pi_l \).
The right-hand side of Eq. (42) would have to vanish to express Gauss’s law. But since \( \partial^0G^a = 0 \) is not one of the Euler-Lagrange equations, we therefore have to take some further measures to implement Gauss’s law. For later reference, we will define the “Gauss’s law operator” \( \mathcal{G}^a \) as

\[
\mathcal{G}^a = \partial_l \Pi^a_l + \frac{1}{2} m \epsilon_{ln} \partial_l A^n_a - M_{(a)} \Pi^a_\xi.
\] (43)

In order to describe the particle states of this theory, we must construct a “suitable” representation for the operator-valued fields in terms of creation and annihilation operators for the observable propagating particles described by this model. We expect these observable particle modes to consist of Higgs scalars as well as gauge field excitations—massive gluons with both topological mass and mass from the spontaneously broken symmetry. The eight gauge fields in this model fall into three classes, and should give rise to particle excitations with different mass: one \( SU(2) \) triplet of gauge fields with \( M_{(a)} = 0 \) and excitation modes that have only topological mass; two doublets of gauge fields with \( M_{(a)} = M_D \) excitations whose mass depends on both the topological mass and \( M_D \); and a singlet similar to the doublet, but with \( M_D \) replaced by \( M_S \). The pole structure of the propagator \([4,5]\) and earlier work on related systems \([6,7]\) suggest that the gauge fields in the doublet and singlet sectors each have two different massive gluon states. The gauge fields in the unbroken \( SU(2) \) triplet have just a single gluon excitation mode. We will make an initial ansatz that incorporates this set of particle states into the representation of the gauge fields. If more particle states are needed than the ones included in our ansatz, or if an entirely different set is required, it will be impossible to construct a suitable representation using these excitation modes. If fewer particle modes are sufficient, then it will be become evident that a mode is redundant. Mistakes in the tentative choices of particle modes will therefore be self-correcting. Conversely, a consistent and suitable representation of the gauge fields will confirm that the identification of the particle excitations is correct.

The first requirement for a suitable representation is that it must be consistent with the equal-time commutation rules given in Eqs. \([33]\)–\([34]\). But it is apparent that the observable,
propagating gluon modes listed above will not suffice to represent all the commutation rules included in Eqs. (31) and (32). Further modes, in the form of ghost excitations, are required. These ghost modes are identical to the ones that appear in Abelian Maxwell-Chern-Simons theory [18–20], and that are also required in (3+1)-dimensional QED \((\text{QED}_4)\) in covariant and axial (except for the spatial axial) gauges [24]. The excitation operators for the massive gluons are the annihilation operator \(a^c(k)\) and its adjoint creation operator \(a^c_d(k)\), which obey the commutation rule \([a^c(k), a^d_d(q)] = \delta^{cd} \delta_{kq}\). For the gauge fields in the doublet and singlet sectors, the second observable, propagating mode will be designated by the annihilation operator \(b^c(k)\) and its adjoint creation operator \(b^c_d(k)\), which obey the commutation rule \([b^c(k), b^d_d(q)] = \delta^{cd} \delta_{kq}\).

Ghost excitation operators exist in pairs. In this work, we will use the ghost annihilation operators \(a^c_Q(k)\) and \(a^c_R(k)\) and their respective adjoint creation operators \(a^d_d(k)\) and \(a^d_d(k)\) in the representations of the gauge field. Ghost states have zero norm, but the single-particle ghost states \(a^c_Q(k)|0\rangle\) and \(a^c_R(k)|0\rangle\) have a nonvanishing inner product; similar nonvanishing inner products also arise for \(n\)-particle states with equal numbers of \(Q\) and \(R\) ghosts. These properties of the ghost states are implemented by the commutator algebra

\[
[a^c_Q(k), a^d_d(q)] = [a^c_R(k), a^d_d(q)] = \delta^{cd} \delta_{kq}
\]

(44)

and

\[
[a^c_Q(k), a^d_d(q)] = [a^c_R(k), a^d_d(q)] = 0,
\]

(45)

which, in turn, imply that the unit operator in the one-particle ghost (OPG) sector is

\[
1_{\text{OPG}} = \sum_k \left[a^d_d(k)|0\rangle\langle 0|a^c_R(k) + a^c_R(k)|0\rangle\langle 0|a^d_d(k)\right];
\]

(46)

the obvious generalization of Eq. (44) applies in the \(n\)-particle sectors. The ghost excitations enable us to satisfy the equal-time commutation relations, Eqs. (31) and (32).

Another requirement we will impose on a “suitable” representation is that the Gauss’s law operator \(G^c(x)\) be restricted to a linear combination of ghost operators for a single
kind of ghost. We will stipulate the specific requirement that $\mathcal{G}(x)$ be a superposition of $a_Q^c(k)$ and $a_Q^{c*}(k)$ operators. There is yet another criterion that a representation must satisfy in order to be suitable: The gluon modes (propagating and ghost) must appear in the Hamiltonian $H_0$ in such a manner that dynamical time-evolution—i.e. translation by the time-displacement operator $\exp\left(-iH_0t\right)$—never propagates state vectors into the “dangerous” part of Hilbert space in which inner products between the two different types of ghost states drain probability from observable particle states.

We have found the required suitable representation of the fields by a combination of unitary transformations similar to the ones used in previous work [18–20] and of “trial fields” with arbitrary parameters which we then adjusted to arrive at “suitable” field representations. For example, we used the trial field

$$A_i^c(x) = \sum_k [\alpha_1(k)\epsilon_{in}k_n + \alpha_2(k)k_i] \left[ a^c(k)e^{ik\cdot x} + a^{c\dagger}(k)e^{-ik\cdot x}\right]$$

$$+ \sum_k i[\alpha_3(k)\epsilon_{in}k_n + \alpha_4(k)k_i] \left[ a^c(k)e^{ik\cdot x} - a^{c\dagger}(k)e^{-ik\cdot x}\right]$$

$$+ \sum_k [\alpha_5(k)\epsilon_{in}k_n + \alpha_6(k)k_i] \left[ b^c(k)e^{ik\cdot x} + b^{c\dagger}(k)e^{-ik\cdot x}\right]$$

$$+ \sum_k i[\alpha_7(k)\epsilon_{in}k_n + \alpha_8(k)k_i] \left[ b^c(k)e^{ik\cdot x} - b^{c\dagger}(k)e^{-ik\cdot x}\right]$$

$$+ \sum_k [\alpha_9(k)\epsilon_{in}k_n + \alpha_{10}(k)k_i] \left[ a_Q^c(k)e^{ik\cdot x} + a_Q^{c*}(k)e^{-ik\cdot x}\right]$$

$$+ \sum_k i[\alpha_{11}(k)\epsilon_{in}k_n + \alpha_{12}(k)k_i] \left[ a_Q^c(k)e^{ik\cdot x} - a_Q^{c*}(k)e^{-ik\cdot x}\right]$$

$$+ \sum_k [\alpha_{13}(k)\epsilon_{in}k_n + \alpha_{14}(k)k_i] \left[ a_R^c(k)e^{ik\cdot x} + a_R^{c*}(k)e^{-ik\cdot x}\right]$$

$$+ \sum_k i[\alpha_{15}(k)\epsilon_{in}k_n + \alpha_{16}(k)k_i] \left[ a_R^c(k)e^{ik\cdot x} - a_R^{c*}(k)e^{-ik\cdot x}\right], \quad (47)$$

where $\alpha_1(k), \ldots, \alpha_{16}(k)$ are arbitrary real parameters. Similar substitutions were made for the other fields in the model. The requirements of “suitability” were then translated into a set of equations which was solved using a customized operator algebra manipulation package in MATHEMATICA [23]. The resulting gauge field representations for the $SU(2)$-symmetric triplet ($c = 1, 2, 3$) that has topological mass only are

$$A_i^c(x) = \sum_k \frac{8ik\epsilon_{in}k_n}{m^{5/2}} \left[ a_Q^c(k)e^{ik\cdot x} - a_Q^{c*}(k)e^{-ik\cdot x}\right]$$
\[ + (1 - \gamma) \sum \frac{2k_l}{m^{3/2}} \left[ a^c_\mathcal{Q}(k)e^{ik\cdot x} + a^c_\mathcal{Q}^*(k)e^{-ik\cdot x} \right] \]
\[ - \sum \frac{4k^2k_l}{m^{7/2}} \left[ a^c_\mathcal{Q}(k)e^{ik\cdot x} + a^c_\mathcal{Q}^*(k)e^{-ik\cdot x} \right] \]
\[ + \sum \frac{m^{3/2}k_l}{16k^3} \left[ a^c_R(k)e^{ik\cdot x} + a^c_R^*(k)e^{-ik\cdot x} \right] \]
\[ - \sum \sqrt{\omega(k)k_l} \left[ a^c(k)e^{ik\cdot x} + a^c(k)^+e^{-ik\cdot x} \right] \]
\[ + \sum \frac{i\epsilon_{ln}k_n}{k\sqrt{2\omega(k)}} \left[ a^c(k)e^{ik\cdot x} - a^c(k)^+e^{-ik\cdot x} \right], \quad (48) \]
\[ \Pi^c_i(x) = -\sum \frac{4ikk_l}{m^{3/2}} \left[ a^c_\mathcal{Q}(k)e^{ik\cdot x} - a^c_\mathcal{Q}(k)e^{-ik\cdot x} \right] \]
\[ + (1 - \gamma) \sum \frac{\epsilon_{ln}k_n}{\sqrt{m}} \left[ a^c_\mathcal{Q}(k)e^{ik\cdot x} + a^c_\mathcal{Q}(k)^+e^{-ik\cdot x} \right] \]
\[ + \sum \frac{6k^2\epsilon_{ln}k_n}{m^{5/2}} \left[ a^c_\mathcal{Q}(k)e^{ik\cdot x} + a^c_\mathcal{Q}(k)^+e^{-ik\cdot x} \right] \]
\[ + \sum \frac{m^{5/2}\epsilon_{ln}k_n}{32k^3} \left[ a^c_R(k)e^{ik\cdot x} + a^c_R(k)^+e^{-ik\cdot x} \right] \]
\[ + \sum \frac{i\epsilon_{ln}k_n}{23/2k\sqrt{\omega(k)}} \left[ a^c(k)e^{ik\cdot x} - a^c(k)^+e^{-ik\cdot x} \right] \]
\[ + \sum \frac{\sqrt{\omega(k)\epsilon_{ln}k_n}}{23/2k} \left[ a^c(k)e^{ik\cdot x} + a^c(k)^+e^{-ik\cdot x} \right], \quad (49) \]
\[ A^c_i(x) = -\sum \frac{4k^3}{m^{7/2}} \left[ a^c_\mathcal{Q}(k)e^{ik\cdot x} + a^c_\mathcal{Q}(k)^+e^{-ik\cdot x} \right] \]
\[ - (1 - \gamma) \sum \frac{2k}{m^{3/2}} \left[ a^c_\mathcal{Q}(k)e^{ik\cdot x} + a^c_\mathcal{Q}(k)^+e^{-ik\cdot x} \right] \]
\[ + \sum \frac{m^{3/2}}{16k^2} \left[ a^c_R(k)e^{ik\cdot x} + a^c_R(k)^+e^{-ik\cdot x} \right] \]
\[ - \sum \frac{k}{m\sqrt{2\omega(k)}} \left[ a^c(k)e^{ik\cdot x} + a^c(k)^+e^{-ik\cdot x} \right], \quad (50) \]
and
\[ G^c(x) = \sum \frac{8ik^2}{m^{3/2}} \left[ a^c_\mathcal{Q}(k)e^{ik\cdot x} - a^c_\mathcal{Q}(k)^+e^{-ik\cdot x} \right], \quad (51) \]
where \( \omega(k) = \sqrt{m^2 + k^2} \); and for the doublet and singlet sectors with combined topological and “Higgs-Kibble” mass \( (c = 4, \ldots, 8) \), the fields are represented by
$$A^c_i(x) = -\sum_k \frac{\omega_c(k)}{2m_c(m_c + \bar{m}_c)} \frac{k_{lt}}{k} [a^c(k)e^{ik\cdot x} + a^{\dagger c}(k)e^{-ik\cdot x}]$$

$$+ \sum_k \sqrt{\frac{m_c}{2\omega_c(k)(m_c + \bar{m}_c)}} \frac{i\epsilon_{ln}k_{lt}}{k} [b^c(k)e^{ik\cdot x} - b^{\dagger c}(k)e^{-ik\cdot x}]$$

$$+ \sum_k \sqrt{\frac{\bar{\omega}_c(k)}{2m_c(m_c + \bar{m}_c)}} \frac{ik_{lt}}{k} [b^c(k)e^{ik\cdot x} - b^{\dagger c}(k)e^{-ik\cdot x}]$$

$$- \sum_k \sqrt{\frac{m_c}{2\bar{\omega}_c(k)(m_c + \bar{m}_c)}} \frac{\epsilon_{ln}k_{lt}}{k} [b^c(k)e^{ik\cdot x} + b^{\dagger c}(k)e^{-ik\cdot x}]$$

$$- \sum_k \kappa_c(\gamma)m_c\bar{m}_c(m_c - \bar{m}_c)^{3/2} \left[a^c_Q(k)e^{ik\cdot x} + a^{\dagger c}_Q(k)e^{-ik\cdot x}\right]$$

$$+ \sum_k \frac{(m_c - \bar{m}_c)^{3/2}k_{lt}}{16k^3} \left[a^c_R(k)e^{ik\cdot x} + a^{\dagger c}_R(k)e^{-ik\cdot x}\right], \quad (52)$$

$$\Pi^c_i(x) = \sum_k \sqrt{\frac{m_c(m_c + \bar{m}_c)}{8\omega_c(k)}} \frac{i\epsilon_{ln}k_{lt}}{k} [a^c(k)e^{ik\cdot x} - a^{\dagger c}(k)e^{-ik\cdot x}]$$

$$+ \sum_k \sqrt{\frac{\omega_c(k)(m_c + \bar{m}_c)}{8m_c}} \frac{\epsilon_{ln}k_{lt}}{k} [a^c(k)e^{ik\cdot x} + a^{\dagger c}(k)e^{-ik\cdot x}]$$

$$+ \sum_k \sqrt{\frac{\bar{m}_c(m_c + \bar{m}_c)}{8\bar{\omega}_c(k)}} \frac{k_{lt}}{k} [b^c(k)e^{ik\cdot x} + b^{\dagger c}(k)e^{-ik\cdot x}]$$

$$+ \sum_k \sqrt{\frac{\bar{\omega}_c(k)(m_c + \bar{m}_c)}{8\bar{m}_c}} \frac{i\epsilon_{ln}k_{lt}}{k} [b^c(k)e^{ik\cdot x} - b^{\dagger c}(k)e^{-ik\cdot x}]$$

$$- \sum_k \kappa_c(\gamma)m_c\bar{m}_c(m_c - \bar{m}_c)^{3/2} \left[a^c_Q(k)e^{ik\cdot x} + a^{\dagger c}_Q(k)e^{-ik\cdot x}\right]$$

$$+ \sum_k \frac{(m_c - \bar{m}_c)^{5/2}\epsilon_{ln}k_{lt}}{32k^3} \left[a^c_R(k)e^{ik\cdot x} + a^{\dagger c}_R(k)e^{-ik\cdot x}\right], \quad (53)$$

$$A^0_i(x) = -\sum_k \frac{\epsilon_{ln}k_{lt}}{\sqrt{2\omega_c(k)m_c(m_c + \bar{m}_c)}} [a^c(k)e^{ik\cdot x} + a^{\dagger c}(k)e^{-ik\cdot x}]$$

$$+ \sum_k \frac{ik}{\sqrt{2\omega_c(k)m_c(m_c + \bar{m}_c)}} [b^c(k)e^{ik\cdot x} - b^{\dagger c}(k)e^{-ik\cdot x}]$$

$$- \sum_k \frac{4k^3\epsilon_{ln}k_{lt}}{m_c\bar{m}_c(m_c - \bar{m}_c)^{3/2}} [a^c_Q(k)e^{ik\cdot x} + a^{\dagger c}_Q(k)e^{-ik\cdot x}]$$

$$+ \sum_k \frac{\kappa_c(\gamma)(m_c - \bar{m}_c)^{3/2}}{16k^3} \left[a^c_R(k)e^{ik\cdot x} + a^{\dagger c}_R(k)e^{-ik\cdot x}\right], \quad (54)$$

and
The unphysical scalar fields for $c = 4, \ldots, 8$ are

\[
\xi^c(x) = -\sum_k \frac{4ik^3}{\kappa_c(\gamma)(m_c - \bar{m}_c)^{3/2}} \left[ a_Q^c(k)e^{ik\cdot x} - a_Q^c(k)e^{-ik\cdot x} \right] - \sum_k \frac{i(m_c\bar{m}_c)^{1/2}(m_c - \bar{m}_c)^{3/2}}{16k^3} \left[ a_R^c(k)e^{ik\cdot x} - a_R^c(k)e^{-ik\cdot x} \right] \tag{56}
\]

and their canonically conjugate momenta

\[
\Pi^c_\xi(x) = -\sum_k k \sqrt{\frac{m_c}{2\omega_c(k)(m_c + \bar{m}_c)}} \left[ a^c(k)e^{ik\cdot x} + a^\dagger(k)e^{-ik\cdot x} \right] + \sum_k \frac{8k^3}{(m_c\bar{m}_c)^{1/2}(m_c - \bar{m}_c)^{3/2}} \left[ a^c_Q(k)e^{ik\cdot x} + a^c_Q(k)e^{-ik\cdot x} \right] \tag{57}
\]

where $\omega_c(k) = \sqrt{m_c^2 + k^2}$, $\bar{\omega}_c(k) = \sqrt{\bar{m}_c^2 + k^2}$, and

\[
\kappa_c(\gamma) = \sqrt{k^2 + (1 - \gamma)m_c\bar{m}_c}. \tag{58}
\]

$m_c$ and $\bar{m}_c$ are the masses of $a^c(k)$ and $b^c(k)$ modes, respectively. They are combinations of the Chern-Simons topological mass $m$ and of the Higgs-Kibble mass $m_c$; their values are

\[
m_c = \frac{\sqrt{4M^2_c + m^2} + m}{2} \tag{59}
\]

and

\[
\bar{m}_c = \frac{\sqrt{4M^2_c + m^2} - m}{2}. \tag{60}
\]

The masses $M_c$ are given by Eq. (16). The Higgs field $\psi$ and its canonical momentum $\Pi_\psi$ are represented as

\[
\psi(x) = \sum_k \frac{1}{\sqrt{2\Omega(k)}} \left[ \alpha(k)e^{ik\cdot x} + \alpha(k)e^{-ik\cdot x} \right] \tag{61}
\]

and
\[ \Pi_\psi(x) = -\sum_k i \sqrt{\frac{\Omega(k)}{2}} \left[ \alpha(k) e^{i k \cdot x} - \alpha^\dagger(k) e^{-i k \cdot x} \right], \] (62)

where \( \Omega(k) \) is given by

\[ \Omega(k) = \sqrt{2 \mu^2 + k^2}. \] (63)

The Faddeev-Popov ghost fields are represented as [25]

\[ \sigma_c^f(x) = \sum_k \frac{1}{\sqrt{2k}} \left[ g_c^e(k) e^{i k \cdot x} + g_c^\ast e(k) e^{-i k \cdot x} \right], \] (64)

\[ \sigma_c^p(x) = -\sum_k \frac{i}{\sqrt{2k}} \left[ g_p^e(k) e^{i k \cdot x} - g_p^\ast e(k) e^{-i k \cdot x} \right], \] (65)

\[ \Pi_c^f(x) = \sum_k i \sqrt{\frac{k}{2}} \left[ g_c^e(k) e^{i k \cdot x} + g_c^\ast e(k) e^{-i k \cdot x} \right], \] (66)

and

\[ \Pi_c^p(x) = \sum_k \sqrt{\frac{k}{2}} \left[ g_c^e(k) e^{i k \cdot x} - g_c^\ast e(k) e^{-i k \cdot x} \right], \] (67)

where \( g_c^e(k), g_p^e(k), g_c^\ast e(k), \) and \( g_p^\ast e(k) \) obey the anticommutation rules

\[ \{ g_c^a(k), g_p^{b \ast}(q) \} = \{ g_p^a(k), g_c^{b \ast}(q) \} = \delta^{ab} \delta_{kq} \] (68)

and

\[ \{ g_c^a(k), g_l^{b \ast}(q) \} = \{ g_p^a(k), g_l^{b \ast}(q) \} = 0. \] (69)

When Eqs. (52)–(67) are substituted into the Hamiltonian \( H_0 \) given in Eq. (37), we obtain the expression

\[ H_0 = \sum_{c=1}^{8} H_c + H_\psi + H_{fp}, \] (70)

where \( H_c \) is given by

\[ H_c = \sum_k \omega(k) a_c^\dagger(k) a_c^e(k) + \sum_k k \left[ a_{R_1}^\ast(k) a_{Q_1}^c(k) + a_{R_2}^\ast(k) a_{Q_2}^c(k) \right] \]

\[ - (1 - \gamma) \sum_k \frac{64k^4}{m^3} a_{Q}^\ast(k) a_{Q}^c(k) \] (71)
for the $c = 1, 2, 3$ sector of unbroken $SU(2)$ gluon triplet, and

\[
H_c = \sum_k \left[ \omega_c(k) a^c(k) a^c(k) + \bar{\omega}_c(k) b^c(k) b^c(k) \right] \\
+ \sum_k \kappa_c(\gamma) \left[ a_R^c(k) a_Q^c(k) + a_Q^c(k) a_R^c(k) \right],
\]

(72)

for $c = 4, \ldots, 8$. For the doublet ($c = 4, 5, 6, 7$) and singlet ($c = 8$) sectors, $m_c$ and $\bar{m}_c$ are given by Eqs. (59) and (60) respectively; for $c = 1, 2, 3$ there is only a single gluon mode and the mass $m$ is the topological mass. The Higgs Hamiltonian $H_\psi$ is given by

\[
H_\psi = \sum_k \Omega(k) a_\uparrow(k) a(k);
\]

(73)

and the Faddeev-Popov ghost part of the Hamiltonian $H_{fp}$, by

\[
H_{fp} = \sum_k k \left[ g_r^c(k) g_p^c(k) + g_p^c(k) g_r^c(k) \right].
\]

(74)

Inspection confirms that $H_0$ is diagonal in the particle number for the observable, propagating particle modes (the massive gluons and the Higgs excitations) of this model and that to this extent the representations of the gauge fields have turned out to be “suitable.” Explicit construction of a Fock space for this model will demonstrate that the ghost components of the Hamiltonian also satisfy the suitability requirement. We can construct a Fock space $\{ |h\rangle \}$ for this model, on the foundation of the perturbative vacuum, $|0\rangle$, which is annihilated by all the annihilation operators: $a^c(k), b^c(k), a_Q^c(k)$ and $a_Q^c(k)$, as well as $\alpha(k)$ and the Faddeev-Popov ghosts, $g_r^c(k)$ and $g_p^c(k)$. In this construction, we make use of techniques developed in earlier work \[18,20,24,26\]. This perturbative Fock space includes all multiparticle states, $|N\rangle$, consisting of observable, propagating particles (Higgs particles and massive gluons) that are created when $\alpha^\dagger(k), a^c(k)$ and $b^c(k)$ respectively act on $|0\rangle$. All such states $|N\rangle$ are eigenstates of $H_0$. States, such as $a_Q^c(k)|N\rangle$ or $a_Q^c(k) a_Q^c(q)|N\rangle$, in which a single variety of ghost creation operator acts on one of these multiparticle states $|N\rangle$ have zero norm; they have no probability of being observed, and have vanishing expectation values of energy, momentum, as well as all other observables. We will designate as $\{ |n\rangle \}$ that subspace of $\{ |h\rangle \}$ which consists of all states $|N\rangle$ and of all states in which a chain of

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\(a^c_R(k)\) operators — but no \(a^c_Q(k)\) operators — act on \(|N\rangle\). States in which both varieties of ghosts appear simultaneously, such as \(a^c_Q(k)a^d_R(q)|N\rangle\), are in the Fock space \(\{|h\rangle\}\), but not in \(\{|n\rangle\}\); because these states have a nonvanishing norm and contain ghosts, they are not probabilistically interpretable. Their appearance in the course of time evolution signals a defect in the theory. Since the states \(|N\rangle\) constitute the set of states in \(\{|n\rangle\}\) from which all zero norm states (the ones with ghost constituents) have been excised, we will sometimes speak of the set of \(|N\rangle\) as a quotient space of observable propagating states. The time-evolution operator \(\exp(-iH_0t)\) has the important property that, if it acts on a state vector \(|n_i\rangle\) in \(\{|n\rangle\}\), it can only propagate it within \(\{|n\rangle\}\). We observe that the only parts of \(H_0\) that could possibly cause a state vector to leave the subspace \(\{|n\rangle\}\), are those that contain either \(a^c_R(k)\) or \(a^c_Q(k)\) operators. The only part of \(H_0\) that has that feature contains the combination of operators \(\Gamma^c = a^c_R(k)a^c_Q(k) + a^d_Q(k)a^d_R(k)\). When \(a^c_R(k)\) acts on a state vector \(|n_i\rangle\), it either annihilates the vacuum or annihilates one of the \(a^c_Q(k)\) operators in \(\{|n\rangle\}\). In the latter case, \(\Gamma^c\) replaces the annihilated \(a^c_Q(k)\) operator with an identical one. When \(a^c_Q(k)\) acts on a state vector \(|n_i\rangle\), it always annihilates it. It is therefore impossible for \(\Gamma^c\) to transform a state vector in \(\{|n\rangle\}\) to one external to \(\{|n\rangle\}\) in which an \(a^c_R(k)\) operator acts on \(|n_i\rangle\). The only effect of \(\Gamma^c\) is to translate \(|n_i\rangle\) states within \(\{|n\rangle\}\). These features of the Hamiltonian \(H_0\) confirm that Eqs. (52)–(57) are suitable representations of the gauge fields. \(H_0\) counts the number of massive gluons of momentum \(k\) belonging to the unbroken \(SU(2)\) sector of the original \(SU(3)\) system, and assigns an energy \(\omega(k)\) to each of them. It similarly counts the two varieties of massive gluons in the doublet and singlet sectors, and assigns the energy \(\omega_c\) and \(\bar{\omega}_c(k)\) to the \(a^c(k)\) and \(b^c(k)\) varieties respectively. And lastly, \(H_0\) counts the number of Higgs particles of mass \(\sqrt{2}\mu\) and assigns the energy \(\sqrt{2\mu^2 + k^2}\) to each. Beyond that, the form of \(H_0\) guarantees that any state vector initially in \(\{|n\rangle\}\) is propagated by \(\exp(-iH_0t)\) entirely within \(\{|n\rangle\}\).

We next turn to the implementation of Gauss’s law and the gauge condition. We have previously noted that Gauss’s law, \(G^a(x) = 0\), is not a consequence of the Euler-Lagrange equations, and that further analysis is required to demonstrate that it is properly imple-
mented. We further observe that, when Eqs. (52)–(67) are substituted into Eq. (43), \( G^a \) turns out to be a linear combination of only those ghost excitations that can live in the subspace \( \{|n\rangle\} \) — \( a_Q^a(k) \) and \( a_Q^a(k) \). All other excitation operators — \( a_R^a(k) \) and \( a_R^a(k) \), and the annihilation and creation operators for both varieties of propagating particles which appear in the gauge fields, their canonical momenta, and in \( \Pi^a_\xi \) — cancel in \( G^a \). The explicit expression for \( G^a \) obtained from this substitution is

\[
G^a(x) = \sum_k \frac{8k^3}{(m - \bar{m})^{3/2}} \left[ a_Q^a(k)e^{ik\cdot x} + a_Q^a(k)e^{-ik\cdot x} \right].
\] (75)

The implementation of Gauss’s law is an immediate consequence of this expression for \( G^a \). A state vector that describes an observable state is one of the \( |N\rangle \) states in the quotient space discussed earlier. The time evolution generated by \( \exp(-iH_0t) \) has previously been shown to keep any state vector that initially was an \( |N\rangle \) state contained in the subspace \( \{|n\rangle\} \).

And the Gauss’s law operator \( G^a \), as well as any other operator that is a linear combination of \( a_Q^a(k) \) and \( a_Q^a(k) \) ghost excitation operators only, must vanish in \( \{|n\rangle\} \). These facts provide for the permanent validity of Gauss’s law as long as the state vector representing the system is initially one of the \( |N\rangle \) state — or at least a state in \( \{|n\rangle\} \) — and provided that \( \exp(-iH_0t) \) is the time-evolution operator for the system. Similarly, \( G^c \) is also represented as a superposition of \( a_Q^c(k) \) and \( a_Q^c(k) \) ghost excitation operators only, so that \( \langle n_b|G^c|n_a\rangle = 0 \) for the same reason that \( \langle n_b|G^a|n_a\rangle = 0 \). Equation (21) therefore shows that in the subspace \( \{|n\rangle\} \), the t’Hooft gauge condition, \( \partial_\mu A^a_\mu - (1 - \gamma)\alpha^a = 0 \), holds. We thus have shown not only that the time-displacement operator \( \exp(-iH_0t) \) keeps state vectors permanently within the subspace \( \{|n\rangle\} \), but that it is also precisely in this subspace that Gauss’s law and the gauge condition are permanently implemented.

It is apparent that the explicit representations of the fields we have given in Eqs. (52)–(57) are instrumental in obtaining the results we have demonstrated above. But the confirmation of the particle mode content of these fields that the self-consistency of this formulation provides is not weakened by its dependence on an explicit representation of the fields. A representation in terms of creation and annihilation operators, and the choice of a Hilbert
space in which to embed the formalism — in this case the Fock spaces \(|n\rangle\) and \(|h\rangle\) — are inevitably important parts of the axiomatic structure of the theory. And it is a significant fact that a representation of the operator-valued fields and a Fock space have been found that permit a consistent interpretation of \(H_0\) as a kinetic energy operator for a system of noninteracting particles in a new vacuum state, even though part of the interaction described by \(\mathcal{L}\) is included in \(H_0\). Moreover, a Fock space has been constructed within which \(H_0\) time displaces state vectors so that unitarity, Gauss’s law, and the gauge condition are all permanently guaranteed. It should be noted that when all interactions are included in a complete Hamiltonian \(H\), these conditions no longer apply. Under the influence of the time-evolution operator \(\exp(-iHt)\), state vectors “leak out” of \(|n\rangle\), and probabilistically uninterpretable state vectors that contain combination of ghosts, for example \(\partial_Q^c(k)\partial_R^d(q)|N\rangle\), develop. Combinations of Faddeev-Popov ghosts are then necessary to compensate for such combinations of \(Q\) and \(R\) ghost \([25]\), and loops of Faddeev-Popov ghost play an important role in maintaining the unitarity of the theory. One reason for the interpretability of this model is that the “interaction-free” limit we have described — the limit as \(e \to 0\) and \(h \to 0\) while \(e^2/h\) remains constant — leads to an essentially Abelian theory. The fact that \([G^a(x), G^b(y)] = 0\) confirms that observation. In a non-Abelian theory this commutator would not vanish, but would regenerate the Gauss’s law operator \(G^c(x)\) in a pattern determined by the structure constants of the corresponding Lie group. Because of the Abelian nature of this limiting form of the theory, the Faddeev-Popov ghost are not required in this stage of the work, and have not been included in the Fock space \(|h\rangle\) or \(|n\rangle\).

**IV. THE PERTURBATIVE THEORY**

The propagator for the gauge field is given by

\[
D_{\mu\nu}(x_1, x_2) = \langle 0|T[A_{\mu}(x_1), A_{\nu}(x_2)]|0\rangle, \tag{76}
\]

where \(T\) designates time-ordering, \(A_{\mu}(x)\) is the interaction-picture field
\[ A_\mu(x) = e^{iH_0t}A_\mu(x)e^{-iH_0t}, \quad (77) \]

\( A_\mu(x) \) is the Schrödinger picture field, and \( |0\rangle \) is the vacuum state of the \( \{|n\}\) space. Similarly, the propagator for an unphysical scalar \( \xi(x) \) is

\[ \Delta_\xi(x_1, x_2) = \langle 0 | T[\xi(x_1), \xi(x_2)] | 0 \rangle, \quad (78) \]

and, for the Higgs field,

\[ \Delta_\psi(x_1, x_2) = \langle 0 | T[\psi(x_1), \psi(x_2)] | 0 \rangle. \quad (79) \]

There are other propagators in this theory, but they vanish for \( \gamma = 1 \) (Landau gauge) which we use in our work, and therefore are not of primary interest to us. We find that the relevant interaction picture fields for \( c = 1, 2, 3 \) are

\[ A_c^c(x) = \sum_k \frac{8ik\epsilon_{\mu\nu}k_\mu}{m^{5/2}} \left[ a_Q^c(k)e^{ik\cdot x -ikt} - a_Q^c(k)e^{-ik\cdot x +ikt} \right] \]

\[ + (1 - \gamma) \sum_k \frac{2k_l}{m^{3/2}} \left[ a_Q^c(k)e^{ik\cdot x -ikt} + a_Q^c(k)e^{-ik\cdot x +ikt} \right] \]

\[ - \sum_k \frac{4k^2k_l}{m^{5/2}} \left[ a_Q^c(k)e^{ik\cdot x -ikt} + a_Q^c(k)e^{-ik\cdot x +ikt} \right] \]

\[ + \sum_k \frac{m^{3/2}k_l}{16k^3} \left[ a_R^c(k)e^{ik\cdot x -ikt} + a_R^c(k)e^{-ik\cdot x +ikt} \right] \]

\[ - \sum_k \frac{\sqrt{\omega(k)}k_l}{\sqrt{2mk}} \left[ a_c^c(k)e^{ik\cdot x -i\omega(k)t} + a_c^c(k)e^{-ik\cdot x +i\omega(k)t} \right] \]

\[ + \sum_k \frac{i\epsilon_{\mu\nu}k_\mu}{k\sqrt{2\omega(k)}} \left[ a_c^c(k)e^{ik\cdot x -i\omega(k)t} - a_c^c(k)e^{-ik\cdot x +i\omega(k)t} \right] \quad (80) \]

and

\[ A_0^c(x) = - \sum_k \frac{4k^3}{m^{7/2}} \left[ a_Q^c(k)e^{ik\cdot x -ikt} + a_Q^c(k)e^{-ik\cdot x +ikt} \right] \]

\[ - (1 - \gamma) \sum_k \frac{2k}{m^{3/2}} \left[ a_Q^c(k)e^{ik\cdot x -ikt} + a_Q^c(k)e^{-ik\cdot x +ikt} \right] \]

\[ + \sum_k \frac{m^{3/2}}{16k^2} \left[ a_R^c(k)e^{ik\cdot x -ikt} + a_R^c(k)e^{-ik\cdot x +ikt} \right] \]

\[ - \sum_k \frac{k}{m\sqrt{2\omega(k)}} \left[ a_c^c(k)e^{ik\cdot x -i\omega(k)t} + a_c^c(k)e^{-ik\cdot x +i\omega(k)t} \right]. \quad (81) \]
for \( c = 4, \ldots, 8 \), they are

\[
A^c_t(x) = - \sum_k \sqrt{\frac{\omega_c(k)}{2m_c(m_c + m_c)}} \frac{k_l}{k} \left[ a^c(k)e^{ikx - i\omega_c(k)t} + a^c(k)e^{i\omega_c(k)t} \right] \\
+ \sum_k \sqrt{\frac{m_c}{2\omega_c(k)(m_c + m_c)}} \frac{i\epsilon_{ln}k_n}{k} \left[ a^c(k)e^{ikx - i\omega_c(k)t} - a^c(k)e^{-i\omega_c(k)t} \right] \\
+ \sum_k \sqrt{\frac{\omega_c(k)}{2m_c(m_c + m_c)}} \frac{ik_l}{k} \left[ b^c(k)e^{ikx - i\omega_c(k)t} - b^c(k)e^{-i\omega_c(k)t} \right] \\
- \sum_k \sqrt{\frac{m_c}{2\omega_c(k)(m_c + m_c)}} \frac{\epsilon_{ln}k_n}{k} \left[ b^c(k)e^{ikx - i\omega_c(k)t} + b^c(k)e^{-i\omega_c(k)t} \right] \\
- \sum_k \frac{4k^3}{\kappa_c(\gamma)m_cm_c(m_c - m_c)^{3/2}} \left[ a^c_Q(k)e^{ikx - i\omega_c(k)t} + a^c_Q^*(k)e^{-i\omega_c(k)t} \right] \\
+ \frac{\kappa_c(\gamma)(m_c - \bar{m}_c)^{3/2}}{16k^3} \left[ a^c_R(k)e^{ikx - i\omega_c(k)t} + a^c_R^*(k)e^{-i\omega_c(k)t} \right], \quad (82)
\]

\[
A^0_\bar{c}(x) = - \sum_k \sqrt{\frac{k}{2\omega_c(k)m_cm_c(m_c + m_c)}} \left[ a^c(k)e^{ikx - i\omega_c(k)t} + a^c(k)e^{-i\omega_c(k)t} \right] \\
+ \sum_k \sqrt{\frac{i\kappa}{2\omega_c(k)m_cm_c(m_c + m_c)}} \left[ b^c(k)e^{ikx - i\omega_c(k)t} - b^c(k)e^{-i\omega_c(k)t} \right] \\
- \sum_k \frac{4k^3}{m_cm_c(m_c - m_c)^{3/2}} \left[ a^c_Q(k)e^{ikx - i\omega_c(k)t} + a^c_Q^*(k)e^{-i\omega_c(k)t} \right] \\
+ \frac{\kappa_c(\gamma)(m_c - \bar{m}_c)^{3/2}}{16k^3} \left[ a^c_R(k)e^{ikx - i\omega_c(k)t} + a^c_R^*(k)e^{-i\omega_c(k)t} \right], \quad (83)
\]

\[
\xi^c(x) = - \sum_k \frac{4ik^3}{\kappa_c(\gamma)(m_c\bar{m}_c)^{1/2}(m_c - m_c)^{3/2}} \left[ a^c_Q(k)e^{ikx - i\omega_c(k)t} - a^c_Q^*(k)e^{-i\omega_c(k)t} \right] \\
- \frac{i(m_c\bar{m}_c)^{1/2}(m_c - m_c)^{3/2}}{16k^3} \left[ a^c_R(k)e^{ikx - i\omega_c(k)t} - a^c_R^*(k)e^{-i\omega_c(k)t} \right], \quad (84)
\]

and

\[
\psi(x) = \sum_k \frac{1}{\sqrt{2\Omega(k)}} \left[ \alpha(k)e^{ikx - i\Omega(k)t} + \alpha^\dagger(k)e^{-i\Omega(k)t} \right], \quad (85)
\]

The Faddeev-Popov ghost fields are

\[
\sigma^c_t(x) = \sum_k \frac{1}{\sqrt{2k}} \left[ g^c_t(k)e^{ikx - ikt} - g^c_t(k)e^{-i\omega_c}\right], \quad (86)
\]

and

26
\[ \sigma^c_p(x) = \sum_k \frac{i}{\sqrt{2}k} \left[ g^c_p(k) e^{ik \cdot x - ikt} - g^c_p^\ast(k) e^{-ik \cdot x + ikt} \right]. \] (87)

The propagators for the gauge fields can be expressed as

\[ D^{ab}_{\mu\nu}(x_1, x_2) = -i\delta^{ab} \int \frac{d^3k}{(2\pi)^3} P^{(a)}_{\mu\nu}(k) e^{-ik \cdot (x_1 - x_2)_\alpha}; \] (88)

for \( a = 1, 2, 3 \):

\[
D^{(a)}_{\mu\nu}(k) = (1 - \gamma) \frac{k_\mu k_\nu}{(k^\alpha k_\alpha + i\epsilon)^2} - \frac{k_\mu k_\nu}{(k^\alpha k_\alpha + i\epsilon)(k^\alpha k_\alpha - m^2 + i\epsilon)} \\
+ \frac{g_{\mu\nu}}{k^\alpha k_\alpha - m^2 + i\epsilon} + \frac{i\epsilon_{\mu\nu\lambda} k^\lambda}{(k^\alpha k_\alpha + i\epsilon)(k^\alpha k_\alpha - m^2 + i\epsilon)}; \] (89)

and for \( a = 4, \ldots, 8 \):

\[
D^{(a)}_{\mu\nu}(k) = \frac{(k^\alpha k_\alpha - m_a m_a) g_{\mu\nu}}{(k^\alpha k_\alpha - m_a^2 + i\epsilon)(k^\alpha k_\alpha - m_a^2 + i\epsilon)} + \frac{i(m_a - \bar{m}_a) \epsilon_{\mu\nu\rho} k^\rho}{(k^\alpha k_\alpha - m_a^2 + i\epsilon)(k^\alpha k_\alpha - \bar{m}_a^2 + i\epsilon)} \\
- \frac{k_\mu k_\nu}{k^\alpha k_\alpha - m_a^2 + i\epsilon} + \frac{\gamma(k^\alpha k_\alpha - m_a m_a) k_\mu k_\nu}{(k^\alpha k_\alpha - m_a^2 + i\epsilon)(k^\alpha k_\alpha - \bar{m}_a^2 + i\epsilon)} - \frac{(1 - \gamma)(m_a - \bar{m}_a)^2 k_\mu k_\nu}{(k^\alpha k_\alpha - m_a^2 + i\epsilon)(k^\alpha k_\alpha - \bar{m}_a^2 + i\epsilon)}. \] (90)

These expressions agree with the gauge field propagators reported in Ref. [9] for \( a = 1, 2, 3 \) and with Refs. [4, 5, 17] for \( a = 4, \ldots, 8 \). These propagators were obtained by inverting the quadratic part of the gauge-fixed Lagrangian. The other propagators are given in terms of the Fourier integral

\[ \Delta(x_1, x_2) = -i \int \frac{d^3k}{(2\pi)^3} \Delta(k^2) e^{-ik^\alpha (x_1 - x_2)_\alpha}; \] (91)

where

\[ \Delta_\psi(k^2) = \frac{-1}{k^\alpha k_\alpha - 2\mu^2 + i\epsilon}; \] (92)

\[ \Delta^{(a)}(k^2) = \frac{-\delta^{ab}}{k^\alpha k_\alpha - (1 - \gamma)m_a m_a + i\epsilon}; \] (93)

and

\[ \Delta_{fp}(k^2) = \frac{-1}{k^\alpha k_\alpha + i\epsilon}. \] (94)
In a canonical theory, the vertices are dictated by the interaction Hamiltonian $H_{\text{int}}$. Since, in this model, time derivatives of operator-valued fields appear in the interaction Lagrangian as well as in $L_0$, $H_{\text{int}}$ will differ from $-\int d\mathbf{x} \ (L_1 + L_2)$. The resulting vertices will be determined by $H_{\text{int}}$, and the propagators will consist of vacuum expectation values of the time-ordered fields that appear in $H_{\text{int}}$. In expanding the $S$-matrix for scattering from an initial state $|i\rangle$ to a final state $|f\rangle$,

$$S_{fi} = \langle f | \mathcal{T} \exp \left( -i \int dt \ e^{iH_0t} H_{\text{int}} e^{-iH_0t} \right) |i\rangle,$$

by using the Wick theorem \[27\], we will sometimes encounter time-ordered products of fields and, at other times, time-ordered products of space-time derivatives of fields. When time derivatives of fields appear as arguments of a time-ordering operation, we will replace the time-ordering operator $\mathcal{T}$ with the “$\mathcal{T}$-star ordering” operator $\mathcal{T}^*$ which is defined so that any derivatives acting on time-ordered fields are to be taken only after time ordering has been carried out. In transforming $\mathcal{T}$-ordered to $\mathcal{T}^*$-ordered fields, additional terms are generated, which contain the $\delta(x_0 - y_0)$ that is produced when time derivatives are extracted from $\mathcal{T}$-ordered products of time-differentiated fields. As was pointed out by Matthews, these extra terms in which $\delta(x_0 - y_0)$-functions appear just cancel the difference between $H_{\text{int}}$ and $-\int d\mathbf{x} \ (L_1 + L_2)$, so that the perturbative theory requires only the propagators given in Eqs. (88)–(93) and the vertices dictated by the interaction Lagrangian \[28\]. Application of the Matthews rule to a model with a spontaneously broken gauge symmetry that produces massive gauge excitations also applies to this case \[21\].

V. POINCARÉ STRUCTURE AND LORENTZ TRANSFORMATIONS OF MASSIVE GAUGE BOSONS

In this section we will construct the six canonical Poincaré generators in $2+1$ dimensions: the time-evolution operator, $P_0 = H_0$; the two-component space-displacement operator $P_l$; the (scalar) rotation operator $J$; and the two-component Lorentz boost $K_l$. We will also use
the Lorentz boost generators to transform the single-particle massive gauge boson states, to display their properties under Lorentz transformations as well as to obtain further confirmation of the consistency of our canonical formulation of this model.

The canonical Poincaré generators for this model are: $P_0 = \int d\mathbf{x} \mathcal{P}_0(\mathbf{x})$, where $\mathcal{P}_0 = \mathcal{H}_0$ with $\mathcal{H}_0$ given by Eq. (37):

$$P_l = \int d\mathbf{x} \mathcal{P}_l(\mathbf{x}) \tag{96}$$

where

$$\mathcal{P}_l = -\Pi_\xi \partial_l \xi - \Pi_\eta \partial_l A_\eta + G \partial_l A_0 - \Pi_\psi \partial_l \psi - \Pi_\tau \partial_l \sigma_\tau - \Pi_\rho \partial_l \sigma_\rho; \tag{97}$$

$$J = \int d\mathbf{x} \epsilon_{ln} x_l \mathcal{P}_n(\mathbf{x}) + \int d\mathbf{x} \kappa_{\text{rotation}}(\mathbf{x}) \tag{98}$$

and

$$K_l = x_0 P_l - \int d\mathbf{x} x_l \mathcal{P}_0(\mathbf{x}) + \int d\mathbf{x} \kappa_l^{\text{boost}}(\mathbf{x}) \tag{99}$$

where

$$\kappa_{\text{rotation}} = \epsilon_{ln} A_l \Pi_n \tag{100}$$

and

$$\kappa_l^{\text{boost}} = -A_l G + A_0 \Pi_l. \tag{101}$$

The term $\kappa_{\text{rotation}}$ implements the mixing of the space components of the fields during a rotation. It arises from the fact that, under an infinitesimal rotation $\delta \theta$ about an axis perpendicular to the 2-D plane, the components of $A^\mu$ transform as follows:

$$\delta A_l(\mathbf{x}) = -[\epsilon_{ij} x_i \partial_j A_l(\mathbf{x}) + \epsilon_{ln} A_n(\mathbf{x})] \delta \theta \tag{102}$$

and

$$\delta A_0(\mathbf{x}) = -\epsilon_{ij} x_i \partial_j A_0(\mathbf{x}) \delta \theta. \tag{103}$$
Under an infinitesimal boost $\delta \beta_l$ along the $l$-direction, the components of $A^\mu$ transform as follows

\[ \delta A_0(x) = -[x_0 \partial_t A_0(x) + x_l \partial_0 A_0(x) - A_l(x)] \delta \beta_l \]  

(104)

and

\[ \delta A_i(x) = -[x_0 \partial_t A_i(x) + x_l \partial_0 A_i(x) - \delta_{il} A_0(x)] \delta \beta_l. \]  

(105)

Use of the canonical commutation rules leads to the following commutation rules for the Poincaré generators:

\[ [P_l, P_n] = 0, \]  

(106)

\[ [H, P_l] = [H, J] = 0, \]  

(107)

\[ [H, K_l] = i P_l, \]  

(108)

\[ [P_l, K_n] = i \delta_{ln} H, \]  

(109)

\[ [P_l, J] = -i \epsilon_{ln} P_n, \]  

(110)

\[ [J, K_l] = i \epsilon_{ln} K_n, \]  

(111)

and

\[ [K_l, K_n] = -i \epsilon_{ln} J. \]  

(112)

We observe that these commutation rules form a closed Lie algebra, and that they are consistent with the transformations given in Eqs. (102)–(105).

To facilitate this investigation of the Lorentz transformation of states that are eigenstates to $H_0$, we shift to a description of excitation operators that have an invariant norm under
Lorentz transformations. We observe, for example, that the norm of the one-particle state 
\[ a^\dagger(k)|0\rangle, \]
\[ \left| a^\dagger(k)|0\rangle \right|^2 = \sum_q \langle 0|[a^\dagger(q), a^\dagger(k)]|0\rangle = \int dq \ \delta(k - q), \]  
(113)
is not a Lorentz scalar because \( dk \) is not the Lorentz invariant measure for the phase space.

The invariant measure can be established by noting that the invariant delta function
\[ \delta(k - q)\delta(k_0 - q_0)\delta(q_\mu q^\mu - m_c^2)\Theta(q_0) = \frac{\delta(k - q)\delta(k_0 - \omega_c(k))}{2\omega_c(k)}, \]
(114)
so that the states \( A^\dagger(k)|0\rangle \), created by operators that obey
\[ [A^\dagger(k), A^\dagger(q)] = 2\omega_c(k)(2\pi)^2\delta^{cd}\delta(k - q), \]
(115)
have unit norms in every Lorentz frame. Similarly, the normalized operators for the other
modes of the gauge field obey
\[ [B^\dagger(k), B^\dagger(q)] = 2\bar{\omega}_c(k)(2\pi)^2\delta^{cd}\delta(k - q); \]
(116)
and the equivalently normalized ghost operators satisfy
\[ [A^\dagger_Q(k), A^\dagger_R(q)] = [A^\dagger_R(k), A^\dagger_Q(q)] = 2\kappa_c(\gamma)(2\pi)^2\delta^{cd}\delta(k - q). \]
(117)
The normalized operators corresponding to the mode \( \alpha(k) \) of the Higgs field and the two
Faddeev-Papov ghosts \( g^a_t(k) \) and \( g^a_p(k) \) are given by \( \hat{\alpha}(k) \), \( \hat{\gamma}_t^a(k) \) and \( \hat{\gamma}_p^a(k) \), respectively. These normalized operators satisfy the following commutation and anticommutation relations:
\[ [\hat{\alpha}(k), \hat{\alpha}^\dagger(q)] = 2\Omega_k(2\pi)^2\delta(k - q) \]
(118)
and
\[ \{\hat{\gamma}_t^a(k), \hat{\gamma}_p^b^\dagger(q)\} = \{\hat{\gamma}_p^a(k), \hat{\gamma}_t^b^\dagger(q)\} = 2k(2\pi)^2\delta^{ab}\delta(k - q). \]
(119)
Hence, the boost operator \( K_l \) is written as
Using the commutations rules given by Eqs. (115) and (116), we find that

\[
A^c(k) \to \tilde{A}^c(k)
\]

transforms only into itself at a new space-time point. The physically observable consequence of Eqs. (121) and (122) is that, under a Lorentz transformation, the topologically massive gauge excitations behave like the massive excitations of a scalar field—each topologically massive gauge excitation transforms only into itself at a new space-time point.
VI. CONCLUSION

In this paper we have presented a detailed analysis of the canonical quantization of spontaneously broken topologically massive gauge theory. In 2+1 dimensions the possibility of including a Chern-Simons term in the gauge field Lagrangian leads to new forms of mass-generating effects for gauge fields. The resulting Chern-Simons-Higgs mechanism differs in interesting ways from the conventional Higgs-Kibble mechanism, and in this paper we have explored the Chern-Simons-Higgs mechanism by concentrating on the relation between the quantized fields and their particle excitation modes. We have found, by a series of unitary transformations, a consistent particle-mode representation of the operator-valued fields and we have constructed the corresponding Fock space which permits a consistent interpretation of the diagonalized noninteracting Hamiltonian $H_0$ as an energy operator for a system of noninteracting particles in a new vacuum state. Within this Fock space, $H_0$ acts unitarily as a time translation generator, in such a way that Gauss’s law and the gauge condition are manifestly preserved. We have computed the gauge field propagators as vacuum expectation values of time-ordered products of the gauge field operators, and formulated the corresponding perturbative expansion of the interacting theory. We have chosen to present our analysis for a non-Abelian Chern-Simons theory in which the original non-Abelian symmetry is spontaneously broken, but with a residual non-Abelian symmetry in the broken vacuum. Such a non-Abelian model clearly illustrates the interplay of the space-time and algebraic features of the Chern-Simons-Higgs mechanism. This particular model is also motivated by the question of its quantum consistency. Indeed, the result reported in [16], that the bare quantum consistency condition of Deser-Jackiw-Templeton [2] is maintained at one-loop in such a broken vacuum, was in fact first obtained by us using the techniques and formalism described in this paper. An interesting further application would be to the analysis of the non-Abelian versions of the self-dual Chern-Simons-Higgs systems considered in [29].
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APPENDIX A: INTERACTION LAGRANGIAN

In this Appendix, we record the explicit expansions of the interaction Lagrangians $\mathcal{L}_1$ and $\mathcal{L}_2$ in terms of real fields. These interaction Lagrangians define the vertices required for perturbative computations. When the Lagrangians given in Eqs. (11) and (12) are expanded in terms of the real fields in Eq. (13) and the symmetry breaking mass scales in Eq. (16), we obtain the following: the $\mathcal{O}(e)$ interaction Lagrangian becomes

$$
\mathcal{L}_1 = e f^{abc} F^a_{\mu\nu} A^b_{\mu} A^c_{\nu} - \frac{1}{3} e m e^{\mu\rho} f^{abc} A^a_{\mu} A^b_{\nu} A^c_{\rho} + \sum_{a=4}^7 e M_D A^a_{\mu} A^a_{\mu} \psi
+ \frac{2e}{\sqrt{3}} M_S A^8_{\mu} A^8_{\mu} \psi + e M_D d^{ab4} A^a_{\mu} A^b_{\nu} \xi^5 - e M_D d^{ab5} A^a_{\mu} A^b_{\mu} \xi^4
+ e M_D d^{ab6} A^a_{\mu} A^b_{\mu} \xi^7 - e M_D d^{ab7} A^a_{\mu} A^b_{\mu} \xi^6
- i e \left[ \bar{\Psi}^1_{\mu} A^\mu \tau_1 \Psi_1 - (\partial_\mu \Psi_1)^\dagger A^\mu \tau_1 \right]
- i e \left[ \bar{\Psi}^2_{\mu} (A^4_{\mu} \tau^1 + A^5_{\mu} \tau^2) \partial_\mu \Psi_2 - (\partial_\mu \Psi_2)^\dagger (A^4_{\mu} \tau^1 + A^5_{\mu} \tau^2) \Psi_2 \right]
- i e \left[ \bar{\Psi}^3_{\mu} (A^6_{\mu} \tau^1 + A^7_{\mu} \tau^2) \partial_\mu \Psi_3 - (\partial_\mu \Psi_3)^\dagger (A^6_{\mu} \tau^1 + A^7_{\mu} \tau^2) \Psi_3 \right]
- i e (\Phi^8 \lambda^8 \partial_\mu \Phi^8 - \partial_\mu \Phi^8)^\dagger A^8_{\mu} \lambda^8 \Phi^8)
- \frac{e \mu^2}{M_D} \psi \left[ (\xi^4)^2 + (\xi^5)^2 + (\xi^6)^2 + (\xi^7)^2 + (\xi^8)^2 + \psi^2 \right]
+ 2 i e f^{abc} A^a_{\mu} \sigma_1^\mu \sigma_3^\mu \psi
$$

and the $\mathcal{O}(e^2)$ interaction Lagrangian becomes

$$
\mathcal{L}_2 = -e^2 f^{abc} f^{ade} A^a_{\mu} A^d_{\mu} A^e_{\nu} A^{\nu}
+ \frac{1}{3} e^2 A^a_{\mu} A^a_{\nu} \left[ (\xi^4)^2 + (\xi^5)^2 + (\xi^6)^2 + (\xi^7)^2 + (\xi^8)^2 + \psi^2 \right]
+ e^2 d^{ab1} A^a_{\mu} A^b_{\mu} (\xi^5 \xi^7 + \xi^4 \xi^6) + e^2 d^{ab2} A^a_{\mu} A^b_{\mu} (\xi^5 \xi^6 - \xi^4 \xi^7)
+ \frac{1}{2} e^2 d^{ab3} A^a_{\mu} A^b_{\mu} \left[ (\xi^4)^2 + (\xi^5)^2 - (\xi^6)^2 - (\xi^7)^2 \right]
$$
\[ + e^2 d^{ab4} A^a_\mu A^b_\mu (\xi^5 \psi - \xi^4 \xi^8) - e^2 d^{ab5} A^a_\mu A^b_\mu (\xi^5 \xi^8 + \xi^4 \psi) \\
+ e^2 d^{ab6} A^a_\mu A^b_\mu (\xi^7 \psi - \xi^6 \xi^8) - e^2 d^{ab7} A^a_\mu A^b_\mu (\xi^7 \xi^8 + \xi^6 \psi) \\
+ \frac{1}{2\sqrt{3}} e^2 d^{ab8} A^a_\mu A^b_\mu \left[ (\xi^4)^2 + (\xi^5)^2 + (\xi^6)^2 + (\xi^7)^2 - 2(\xi^8)^2 - 2\psi^2 \right] \\
- \frac{e^2 \mu^2}{4 M_D^2} \left[ (\xi^4)^2 + (\xi^5)^2 + (\xi^6)^2 + (\xi^7)^2 + (\xi^8)^2 + \psi^2 \right]^2. \] (A2)

In these expressions, \( \tau \) designates the Pauli spin matrices, and \( A^\mu \) denotes the gauge field triplet \( A^a_\mu \) (\( a = 1, 2, 3 \)) in the unbroken \( SU(2) \) “isospin” subgroup. The isospinors \( \Psi_a \) (\( a = 1, 2, 3 \)) are the combinations of the Higgs field \( \psi \) and the \( \xi^a \) fields given by Eqs. (17)–(19).
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