CHOW-WITT RINGS OF SPLIT QUADRICS

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Abstract. We compute the Chow-Witt rings of split quadrics over a field of characteristic not two. We also determine the $\mathbf{I}^\bullet$-cohomology of these quadrics, in all bidegrees, and determine its ring structure in geometric bidegrees. This is in fact the main step of the computation.

The results on $\mathbf{I}^\bullet$-cohomology corroborate the general philosophy that $\mathbf{I}^\bullet$-cohomology is an algebro-geometric version of singular cohomology of real varieties: our explicit calculations show that the $\mathbf{I}^\bullet$-cohomology ring of a split quadric over the reals is isomorphic to the singular cohomology ring of the space of its real points.

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1. Introduction

Chow-Witt groups were introduced by Barge and Morel [BM00] as a cohomology theory containing an Euler class, detecting if a vector bundle over a smooth affine base scheme splits off a trivial line bundle in the critical range. This generalized a previous result of Murthy [Mu94] for Chow groups and algebraically closed base fields. In recent years, subsequent work of Morel [Mor12], Fasel [Fas08], Fasel-Srinivas [FS09] and Asok-Fasel [AF16] has completed the picture. As Chow-Witt groups are a quadratic refinement of Chow groups, one might hope for generalizations of other results using Chow-Witt groups in the future.

In this paper, we determine the Chow-Witt ring of a split quadric $Q_n$ of dimension $n \geq 3$ over a field $F$ of characteristic not two as follows (see Theorem 8.4):

**Theorem.** The graded $GW(F)$-algebra $\widetilde{CH}^\bullet(Q_n, O \oplus O(1))$ is given by

$$\widetilde{CH}^\bullet(Q_{p,q}, O \oplus O(1)) \cong H^\bullet(Q_{p,q}, \mathbf{I}^\bullet, O \oplus O(1)) \times_{CH^\bullet(Q_{p,q})} (\ker \partial \oplus \ker \partial O(1)).$$

Here the Chow ring $CH^\bullet(Q_{p,q})$ is already known, $\ker \partial \oplus \ker \partial O(1)$ is given in Theorem 8.4 and $H^\bullet(Q_{p,q}, \mathbf{I}^\bullet, O \oplus O(1))$ is given by Theorem 6.6.

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Our computation follows the general strategies laid out by Fasel, Wendt and the first author in their computations for projective spaces, Grassmannians and classifying spaces [Fas13, HW17, W18]. The main step is thus the computation of $\mathbb{P}^*$-cohomology. We do slightly more than strictly necessary here, in that we compute these cohomology groups in all bidegrees, and over a smooth base. That is, we compute the group

$$H^i(Q_n, \mathbb{P}^*, \mathcal{O}(k))$$

for all values of $i$, $j$ and $k$, where $\mathcal{O}(k)$ denotes the $k$th tensor power of an ample line bundle on $Q_n$, and where $Q_n$ is a split quadric over a smooth scheme as in the additive computation of $H^i(\mathbb{P}^p, \mathbb{P})$ of Fasel [Fas13]. Our result is summarized in Theorem 5.4 below.

For the computation of the Chow-Witt ring, we are mainly interested in the $\mathbb{P}^*$-cohomology groups in “geometric” bidegrees. With untwisted coefficients, their direct sum yields the graded ring $H^*(\mathbb{Q}, \mathbb{P}^*) := \bigoplus_{i \geq 0} H^i(Q_n, \mathbb{P}^*)$, which is a graded commutative $W(F)$-algebra [HW17, Proposition 2.5]. Adding twisted coefficients, this extends to a $\mathbb{Z} \oplus \mathbb{Z}/2$-graded $W(F)$-algebra

$$H^*(Q_n, \mathbb{P}^*, \mathcal{O} \oplus \mathcal{O}(1)) := \bigoplus_{i \geq 0} \left( H^i(Q_n, \mathbb{P}^*, \mathcal{O}) \oplus H^i(Q_n, \mathbb{P}^*, \mathcal{O}(1)) \right).$$

The explicit ring structure can often be written down more concisely when twisted coefficients are taken into account. Indeed, this is already true for projective spaces. The following result is Proposition 4.5 in a recent preprint of Matthias Wendt [W18], based on the additive computations of Fasel already mentioned. We include a proof for the reader’s convenience, see Theorem 6.2.

**Theorem** (Fasel/Wendt). Let $S = \text{Spec}(F)$ for a field $F$ of characteristic not two. For any $p \geq 0$, we have a $\mathbb{Z} \oplus \mathbb{Z}/2$ graded ring isomorphism

$$H^*(\mathbb{P}^p, \mathbb{P}^*, \mathcal{O} \oplus \mathcal{O}(1)) = W(F)[\xi, \alpha]/(I(F)\xi, \xi^{p+1}, \xi\alpha, \alpha^2)$$

with $|\xi| = (1, 1), |\alpha| = (p, p + 1)$.

For split quadrics, we obtain the following result (Theorem 6.6):

**Theorem.** For any $n \geq 3$, we have a $\mathbb{Z} \oplus \mathbb{Z}/2$-graded ring isomorphism

$$H^*(Q_n, \mathbb{P}^*, \mathcal{O} \oplus \mathcal{O}(1)) \cong \begin{cases} W(F)[\xi, \alpha, \beta]/(I(F)\xi, \xi^{p+1}, \xi\alpha + \xi\beta, \alpha^2 - u\beta^2, \alpha\beta) & \text{if } n = 2p \text{ and } p \text{ is even} \\
W(F)[\xi, \alpha, \beta]/(I(F)\xi, \xi^{p+1}, \xi\alpha + \xi\beta, \alpha^2, \beta^2) & \text{if } n = 2p \text{ and } p \text{ is odd} \\
W(F)[\xi, \alpha, \beta]/(I(F)\xi, \xi^{p+1}, \xi\alpha, \alpha^2, \beta^2) & \text{if } n = 2p + 1 \end{cases}$$

for some unit $u \in W(F)$. Here, generators of degrees $|\xi| = (1, 1), |\alpha| = (p, p + 1), |\beta| = (q, q + 1)$ where $q = p$ if $n = 2p$ is even and $q = p + 1$ if $n = 2p + 1$ is odd.

For a real variety $X$, $\mathbb{P}^*$-cohomology is closely related to the singular cohomology of the real points $X(\mathbb{R})$ equipped with the analytic topology. The strongest general result in this direction is the following theorem of Jacobson [Jac17]:

**Theorem (Jacobson).** For a real variety $X$, the group $H^i(X, \mathbb{P}^*)$ is isomorphic to the singular cohomology group $H^i(X(\mathbb{R}), \mathbb{Z})$ in all bidegrees with $j > \dim(X)$.

Note that this general result does not apply to the interesting geometric bidegrees $i = j$ corresponding to Chow groups. However, the known computations for projective spaces and Grassmannians indicate that the situation for cellular varieties is even better. In forthcoming work with Wendt, we will show that realization induces a ring isomorphism $H^*(X, \mathbb{P}^*) \cong H^*(X(\mathbb{R}), \mathbb{Z})$ for all smooth cellular real varieties $X$ [HWXZ]. This isomorphism also extends to twisted coefficients. Here, we verify this result for split real quadrics, through explicit computations. Unfortunately, we could not find the integral cohomology ring of a real split quadric in the literature. All we could find was the additive structure of cohomology with $\mathbb{Z}/2$-coefficients, which can be computed using the Gysin sequence, see [D82, § 24.39, problem 10], [WL62] or [CGT]. We therefore include computations of the integral cohomology rings of real quadrics as a preliminary section in
2. Conventions and notation

2.1. Topology. We fix the following notation:

- \( Q_{p,q} \) is the \((p+q)\)-dimensional real quadric defined as a subspace of \( \mathbb{RP}^{p+q+1} \) by the equation
  \[
  x_0^2 + x_1^2 + \cdots + x_p^2 = y_0^2 + y_1^2 + \cdots + y_q^2,
  \]
  where \([x_0 : \ldots : x_p : y_0 : \ldots : y_q] \) are homogeneous coordinates on \( \mathbb{RP}^{p+q+1} \). Note that \( Q_{p,q} \cong Q_{q,p} \). We always assume \( 1 \leq p \leq q \).

Additional notation is introduced at the beginning of Section 3.

2.3. Algebraic geometry. We fix a base scheme \( S \) which is smooth over a field \( F \) of characteristic different from two. More precisely, \( S \) is assumed to be noetherian, separated, smooth and of finite type over \( F \). We may and will moreover assume without loss of generality that \( S \) is connected. These are precisely the assumptions on the base scheme \( S \) used in [Fas13]. The following schemes and morphisms are all defined over \( S \):

- \( Q_n \) denotes the \( n \)-dimensional split quadric, i.e. the closed subvariety of \( \mathbb{P}^{n+1} \) defined by the equation
  \[
  \begin{align*}
  &x_0 y_0 + \cdots + x_p y_p = 0 &\text{when } n = 2p \\
  &x_1 y_1 + \cdots + x_p y_p + z^2 = 0 &\text{when } n = 2p + 1
  \end{align*}
  \]
  When \( n = 2p \), \( Q_n \) is isomorphic to the subvariety \( Q_{p,p} \) defined by equation (2.2). When \( n = 2p + 1 \), it is isomorphic to \( Q_{p,p+1} \).

- \( \mathbb{P}^p_x, \mathbb{P}^p_y, \mathbb{P}^p_z \) and \( \mathbb{P}^p_{xy}, \mathbb{P}^p_{xz}, \mathbb{P}^p_{yz} \) denote the following subvarieties of \( Q_n \):
  - When \( n = 2p \), we define \( \mathbb{P}^p_x, \mathbb{P}^p_y \subset \mathbb{P}^{2p+1} \) by the equations \( y_0 = y_1 = \ldots = y_{p} = 0 \) and \( x_0 = x_1 = \ldots = x_{p} = 0 \) respectively. We define \( \mathbb{P}^p_z, \mathbb{P}^p_{xy}, \mathbb{P}^p_{xz}, \mathbb{P}^p_{yz} \subset \mathbb{P}^{2p+1} \) by the equations \( x_0 = y_1 = \ldots = y_{p} = 0 \) and \( y_0 = x_1 = \ldots = x_{p} = 0 \).
  - When \( n = 2p + 1 \), we define \( \mathbb{P}^p_x, \mathbb{P}^p_y \subset \mathbb{P}^{2p+2} \) by the equations \( y_0 = y_1 = \ldots = y_{p} = z = 0 \) and \( x_0 = x_1 = \ldots = x_{p} = z = 0 \), respectively.

- \( q: Q_n \to S \) and \( p: \mathbb{P}^p \to S \) denote the projection maps

- \( i_y: \mathbb{P}^p_y \to Q_n - \mathbb{P}^p_x, i_z: \mathbb{P}^p_z \to Q_n - \mathbb{P}^p_y, j: Q_n - \mathbb{P}^p_z \to Q \) and \( i: Q_n \to \mathbb{P}^{n+1} \) are the obvious open embeddings; likewise for \( i_z, i_x, i_y, i_{xy}, i_{xz}, i_{yz} \) and \( i_y \).

- \( \rho: Q - \mathbb{P}^p \to \mathbb{P}^p \) is the morphism given by
  \[
  \begin{align*}
  &\rho: [x_0 : \ldots : x_p : y_0 : \ldots : y_p] \mapsto [0 : \ldots : 0 : y_0 : \ldots : y_p] &\text{if } n = 2p \\
  &\rho: [x_0 : \ldots : x_p : y_0 : \ldots : z] \mapsto [0 : \ldots : 0 : y_0 : \ldots : y_p : 0] &\text{if } n = 2p + 1
  \end{align*}
  \]
  (Some of this notation is summarized in Figure 1.)

- \( \mathcal{O}(1) \) is defined as the restriction along \( i: Q_n \to \mathbb{P}^{n+1} \) of the canonical line bundle \( \mathcal{O}(1) \) on \( \mathbb{P}^{n+1} \). When there is no danger of confusion, we simply denote this restriction again by \( \mathcal{O}(1) \).
3. Singular cohomology of real quadrics

In this section, and we study the real quadrics $Q_{p,q} \subset \mathbb{RP}^{p+q+1}$ (with the analytic topology). Note first that $Q_{p,q}$ is a two-fold covering space of $\mathbb{RP}^p \times \mathbb{RP}^q$, via the obvious map that takes $[x:y]$ to $(|[x],[y]|)$. Let $\pi_1$ and $\pi_2$ for the two components of this map. A two-fold cover of the quadric itself is given by $S^p \times S^q$: the real quadric $Q_{p,q}$ is the quotient of $S^p \times S^q$ modulo the involution $\tau := \tau_p \times \tau_q$, where $\tau$ denotes the involution of $S^n$ sending $x$ to $-x$. The composition of the two covering maps is the canonical four-fold cover of $\mathbb{RP}^p \times \mathbb{RP}^q$. The situation is summarized by the central vertical column of Figure 2. Also displayed there are the “diagonal” embedding $\Delta^S: \mathbb{RP}^p \hookrightarrow S^p \times S^q$ that sends $x$ to $(x,(x,0))$ and the two induced maps $\Delta: \mathbb{RP}^p \to Q_{p,q}$ and $\mathbb{RP}^p \to \mathbb{RP}^p \times \mathbb{RP}^q$, respectively. Note that $\Delta$ splits the projection $\pi_1: Q_{p,q} \to \mathbb{RP}^p$.

![Figure 2. The various continuous maps relating spheres, projective spaces and real quadrics](image)

3.1. Twisted coefficients. The two-fold cover $S^p \to \mathbb{RP}^p$ determines a representation $\rho: \pi_1(\mathbb{RP}^p) \to \{\pm 1\} = \text{Aut}(\mathbb{Z})$. We write $\mathbb{Z}(1)$ for the corresponding local coefficient system on $\mathbb{RP}^p$. For $p \geq 2$, the two-fold cover is the universal cover, and $\rho$ is an isomorphism, but the system $\mathbb{Z}(1)$ also exists for $p = 1$. The situation for the real quadrics is similar. The two-fold cover $S^p \times S^q \to Q_{p,q}$ determines a representation $\pi_1(Q_{p,q}) \to \{\pm 1\}$, and we again write $\mathbb{Z}(1)$ for the corresponding local coefficient system. Again, $S^p \times S^q$ is the universal cover for $q \geq p \geq 2$. Note that $\mathbb{Z}(1)$ pulls back to $\mathbb{Z}(1)$ under the maps $\pi_1$ and $\Delta$ in Figure 2, so the notation is consistent.

More generally, given a coefficient ring $R$ and an integer $s$, we define $R(s) := R \otimes_{\mathbb{Z}} \mathbb{Z}(1)^{\otimes s}$. Of course, this really only depends on $R$ and the value of $s$ mod 2. Also note that $\mathbb{Z}/2(s) = \mathbb{Z}/2$ for all $s$. The direct sum $R \oplus R(1)$ is a $\mathbb{Z}/2$-graded ring, and hence cohomology with local coefficients in $R \oplus R(1)$ is a $(\mathbb{Z} \times \mathbb{Z}/2)$-graded ring, which decomposes additively as

$$H^*(Q_{p,q}, R \oplus R(1)) = H^*(Q_{p,q}, R) \oplus H^*(Q_{p,q}, R(1)).$$

Elements $\alpha \in H^i(Q_{p,q}, R)$ have degree $|\alpha| = (i,0)$; elements $\beta \in H^i(Q_{p,q}, R(1))$ have degree $|\beta| = (i,1)$. We sometimes also view $R \oplus R$ as a $\mathbb{Z}/2$-graded coefficient ring for $S^p \times S^q$ and

$$H^*(S^p \times S^q, R \oplus R) = H^*(S^p \times S^q, R) \oplus H^*(S^p \times S^q, R)$$

as a $(\mathbb{Z} \times \mathbb{Z}/2)$-graded ring, in the evident way.
3.2. **Rational cohomology ring.** The description of $Q_{p,q}$ as a quotient of $S^p \times S^q$ implies that $H^\ast(Q_{p,q}, \mathbb{Q})$ is isomorphic to the subring of $H^\ast(S^p \times S^q, \mathbb{Q})$ fixed by $\tau^\ast$ [Ha02, Proposition 3G.1]. Similar arguments show that, more generally, $H^\ast(Q_{p,q}, \mathbb{Q} \oplus \mathbb{Q}(1))$ is isomorphic to the subring of the $\mathbb{Z} \times \{\pm 1\}$-graded ring $H^\ast(S^p \times S^q, \mathbb{Q} \oplus \mathbb{Q}(1))$ fixed by the action of $\tau^\ast$ on homogeneous elements in the $+1$-graded part and the action of $-\tau^\ast$ on the $(-1)$-graded part. As $\tau^\ast: H^\ast(S^n) \to H^\ast(S^n)$ is given by multiplication with $(-1)^{n+1}$, we find:

$$H^\ast(Q_{p,q}, \mathbb{Q} \oplus \mathbb{Q}(1)) \cong \mathbb{Q}[\alpha, \beta]/(\alpha^2, \beta^2) \quad \text{with} \quad \begin{cases} |\alpha| = (p,p+1) \\ |\beta| = (q,q+1) \end{cases} \quad \text{(3.3)}$$

In particular, $Q_{p,q}$ is orientable if and only if $p+q$ is even (use [Ha02, Theorem 3.26]).

3.4. **A (minimal) cell structure.** In order to compute the twisted (co)homology of the real quadric $Q_{p,q}$ with integral coefficients, we need to study its two-fold cover $S^p \times S^q$ in more detail.

**Proposition 3.5.** Assume $p \leq q$. There exists a cell structure on $S^p \times S^q$ consisting of $4p + 4$ cells for which the action of $\tau := \tau_p \times \tau_q$ is cellular: we have characteristic maps $j_{0,0}, \ldots, j_{0,p}$ for cells in dimensions $0, \ldots, p$ and characteristic maps $j_{q,0}, \ldots, j_{q,p}$ for cells in dimensions $q, \ldots, p+q$ such that $\tau \circ j_{i,k} = j_{i,k}$. The skeleton of $X = S^p \times S^q$ with respect to this cell structure are given by:

- $X^k = \Delta^S(S^k)$ for $k \in \{0, \ldots, p\}$
- $X^k = \Delta^S(S^p)$ for $k \in \{p, \ldots, q\}$
- $X^{k+q} = \Delta^S(S^p) \cup (S^k \times S^q)$ for $k \in \{0, \ldots, p\}$.

Here, $\Delta^S: S^p \to S^p \times S^q$ denotes the map $x \mapsto (x, (x,0))$, and $S^k$ is viewed as an “equator” of $S^p$ for $k \leq p$, embedded via the first $k+1$ coordinates.

**Proof.** We give an explicit description of this cell structure. The $2p + 2$ cells in dimensions $0, \ldots, p$ can easily be defined as follows. Equip $S^p$ with the usual cell structure that has two cells in each dimension. That is, consider the cell structure with characteristic maps

$$D^k \longrightarrow S^p \\
j_k^+ : (x_0, \ldots, x_{k-1}) \mapsto (x_0, \ldots, x_{k-1}, \sqrt{1-|x|^2}, 0, \ldots, 0) \quad \text{(3.6)}$$

and $j_k^- := \tau_p \circ j_k^+$ for $k \in \{0, \ldots, p\}$. We give $\Delta^S(S^p) \subset S^p \times S^q$ the cell structure induced by this cell structure on $S^p$. That is, we consider the characteristic maps

$$j_{0,k}^\pm := \Delta^S \circ j_k^\pm: D^k \to S^p \times S^q. \quad \text{(3.7)}$$

Note that $\tau \circ j^+_{0,k} = j^-_{0,k}$ as desired.

The second family of $2p + 2$ cells lives in dimensions $q, \ldots, p+q$. In order to define these cells, we need to fix some further notation. Consider the “northern hemisphere” $j^+_q(D^q) \subset S^q$, i.e. the set of all points $x = (x_0, \ldots, x_q) \in S^q$ with $x_q \geq 0$. Let $e_q := (0, \ldots, 0, 1)$ be the “north pole”. Choose and fix a continuous map

$$R: j^+_q(D^q) \to SO(q+1) \\
x \mapsto R_x$$

with the property that $R_x \cdot e_q = x$. Such a map exists by Lemma 3.9 below. Also, fix a map that “wraps the $q$-disk around the $q$-sphere”, i.e. a surjection $w: D^q \to S^q$ that sends $0 \in D^q$ to the north pole $e_q \in S^q$ and every point on the boundary $\partial D^q$ to the south pole $-e_q \in S^q$, and that induces a homeomorphism $D^q/\partial D^q \cong S^q$. For example, we could take $w(x) := (s(x) \cdot x, 1 - |x|)$, where $s(x)$ is a non-negative scalar determined by the requirement that $w(x) \in S^q$. 
Consider $S^p$ as an “equator” of $S^q$ via the embedding $x \mapsto (x,0)$. For every $k \in \{0, \ldots, p\}$, we define a characteristic map for a $(q + k)$-cell as follows:

$$j^+_{q,k}: D^k \times D^q \to S^p \times S^q$$

$$(x, y) \mapsto (j^+_k(x), -R_{\overline{U}_k(x,0)}(wy))$$

(3.8)

Define $j^-_{q,k} := \tau \circ j^+_{q,k}$. When restricted to the inner points of $D^k \times D^q \cong D^{k+q}$, the characteristic maps $j^{\pm}_{q,k}$ induce homeomorphisms:

$$\tilde{D}^k \times \tilde{D}^q \xrightarrow{\sim} (j^+_k(\tilde{D}^k) \times S^q) \setminus \Delta^S(S^p)$$

The images of these homeomorphisms clearly constitute a cover of $(S^p \times S^q) \setminus \Delta^S(S^p)$ by disjoint sets. Thus, altogether the maps $j^{\pm}_{i,k}$ for $i \in \{0, q\}$ and $k \in \{0, \ldots, p\}$ constitute a cell structure on $S^p \times S^q$. □

**Lemma 3.9.** There exists a continuous map $R: j^+_q(D^q) \to \text{SO}(q+1), x \mapsto R_x$ such that $R_x \cdot e_q = x$ for all $x$. Moreover, any two such maps are homotopic through maps $R(t)$ that also satisfy $R_x(t) \cdot e_q = x$.

**Proof.** The map $R$ is a section of the evaluation map $\text{SO}(q+1) \to S^q$ that takes a matrix $R$ to $R \cdot e_q$. This evaluation map constitutes a principal $\text{SO}(q)$-bundle. Over the contractible space $j^+_q(D^q) \cong D^q$, this bundle is necessarily trivial. In particular, sections of this bundle correspond to continuous maps $j^+_q(D^q) \to \text{SO}(q)$. Such maps certainly exist, and as $j^+_q(D^q)$ is contractible, they are all homotopic to constant maps. As $\text{SO}(q)$ is path-connected, it follows that all sections are homotopic (via sections). □

### 3.10. Cellular (co)homology.

Let $\mathbb{Z}[\tau]$ denote the group ring associated with the group with two elements, i.e., $\mathbb{Z}[\tau] := \mathbb{Z} \oplus \mathbb{Z} \tau$ with $\tau^2 = 1$. Let $\overline{C}^{(n)}(0)$ denote the chain complex of free $\mathbb{Z}[\tau]$-modules

$$\overline{C}^{(n)}(0): \mathbb{Z}[\tau] \xleftarrow{1+\tau} \mathbb{Z}[\tau] \xleftarrow{1-\tau} \mathbb{Z}[\tau] \xleftarrow{1+\tau} \mathbb{Z}[\tau] \xleftarrow{1-\tau} \mathbb{Z}[\tau] \xleftarrow{1+\tau} \cdots \leftarrow \mathbb{Z}[\tau]$$

concentrated in degrees 0 to $n$. This is the cellular chain complex of $S^n$ with respect to the cell structure arising from the two-fold covering map $S^n \to \mathbb{RP}^n$ and the standard cell structure on $\mathbb{RP}^n$. Multiplication by $\tau$ on the complex is the chain map induced by the involution $\tau_n$ of $S^n$. The usual homology of $\mathbb{RP}^n$ with coefficients in $\mathbb{Z}$ is the homology of $\overline{C}^{(n)}(0) \otimes_{\mathbb{Z}[\tau]} \mathbb{Z}$, where $\mathbb{Z}$ is viewed as trivial $\mathbb{Z}[\tau]$-module; if instead we tensor with the $\mathbb{Z}[\tau]$-module $\mathbb{Z}(1)$ on which $\tau$ acts as $-1$, we obtain the homology of $\mathbb{RP}^n$ with twisted coefficients.

Let $\overline{C}^{(n)}(1)$ denote the complex with the same groups in each degree, but with the roles of multiplication by $1-\tau$ and multiplication by $1+\tau$ reversed:

$$\overline{C}^{(n)}(1): \mathbb{Z}[\tau] \xleftarrow{1+\tau} \mathbb{Z}[\tau] \xleftarrow{1-\tau} \mathbb{Z}[\tau] \xleftarrow{1+\tau} \mathbb{Z}[\tau] \xleftarrow{1-\tau} \mathbb{Z}[\tau] \xleftarrow{1+\tau} \cdots \leftarrow \mathbb{Z}[\tau]$$

**Proposition 3.11.** For $p \leq q$, the cellular chain complex associated with cell structure on $S^p \times S^q$ described in 3.3 has the form $C(S^p \times S^q) \cong \overline{C}^{(p)}(0) \oplus \overline{C}^{(p)}(q+1)[q]$, where $q+1$ indicates the value of $q+1$ modulo two, and $[q]$ indicates that the second summand is shifted $q$ degrees to the right. Multiplication by $\tau$ on the complex is the chain map induced by the involution $\tau$ on $S^p \times S^q$.

**Proof sketch.** We clearly have a decomposition

$$C(S^p \times S^q) = C(\Delta^S(S^p)) \oplus C'$$

for some complex $C'$ concentrated in degrees $q, \ldots, p+q$, except that, a priori, there might be some differentials from $C'$ to $C(\Delta(S^p))$. However, there are no such differentials. This can easily be seen by considering the map on chain complexes induced by the projection $\pi_1^S: S^p \times S^q \to S^p$: this map is zero on $C'$ and maps $C(\Delta^S(S^p))$ isomorphically to $C(S^p)$. Thus, the above decomposition is an honest decomposition of chain complexes.

We may identify $C(\Delta^S(S^p))$ with $C(S^p)$ with respect to the usual cell structure on $S^p$ consisting of two cells in each dimension. This is precisely the cell structure used in the usual computation of the the homology of $\mathbb{RP}^p$ (cf. [DK01, 5.2.1]). The complex $C'$ can be computed in an analogous
fashion. The non-zero boundary maps $d$ of $C'$ are $\mathbb{Z}[\tau]$-linear maps between free $\mathbb{Z}[\tau]$-modules of rank one, each with a generator $\{\tilde{j}^+_{q,k}\}$ represented by one of the map characteristic maps $j^+_{q,k}$. Write $d: C'_{q+k} \rightarrow C'_{q+k-1}$ as

$$d\{\tilde{j}^+_{q,k}\} = p+\{\tilde{j}^+_{q,k-1}\} + d-\tau\{\tilde{j}^+_{q,k-1}\}. $$

Both coefficients $d_+$ and $d_-$ can easily be seen to be $\pm 1$. The crucial value we need to know for the homology/cohomology calculations is the relative sign $d_+/d_- \in \{\pm 1\}$.

Recall how the coefficients of the boundary map $d$ in the cellular chain complex of a CW complex $X$ are defined. Let $X^i$ denotes the $i$-skeleton. Given a characteristic map $j: D^i \rightarrow X^{i+1}$, write $\pi(j): X^i \rightarrow S^i$ for the map that is constant outside of the cell defined by $j$ and satisfies $\pi(j) \circ j = w$ (the “wrapping map” fixed in the previous section). Given an $(i+1)$-cell with characteristic map $j'$, the coefficient of an $(i)$-cell $[j]$ in $d[j']$ is given by the degree of the composition $\pi(j') \circ j_\partial D^i$.

So let $X = S^p \times S^q$ with our chosen cell structure. The coefficients $d_\pm$ are the degrees of the following two compositions:

$$\partial(D^k \times D^q) \xrightarrow{j^+_{q,k-1}|_{\partial D^{k-1} \times D^q}} X^{k+q-1} \xrightarrow{\pi^+_{q,k-1}} S^{k+q-1}$$

$$\partial(D^k \times D^q) \xrightarrow{j^-_{q,k-1}|_{\partial D^{k-1} \times D^q}} X^{k+q-1} \xrightarrow{\pi^-_{q,k-1}} S^{k+q-1}$$

Consider the two open subsets $(\partial D^k)^+ \times D^q$ and $(\partial D^k)^- \times D^q$ of $\partial(D^k \times D^q)$, defined as follows: $(\partial D^k)^+$ is the open upper hemisphere, i.e. the set of points $(x_0, \ldots, x_{k-1}) \in \partial D^k$ with $x_{k-1} > 0$, and $(\partial D^k)^-$ is the open lower hemisphere, i.e. the set of points with $x_{k-1} < 0$. The restriction of $j^+_{q,k}$ to these two sets is a homeomorphism onto its image. Moreover, explicit computations in coordinates show:

$$j^+_{q,k-1}|_{\partial D^{k-1} \times D^q \cong (\partial D^k)^+ \times D^q} \xrightarrow{\partial(D^k)^+ \times D^q \xrightarrow{j^+_{q,k}} \partial D^k \times D^q} (\partial D^k)^+ \times D^q \xrightarrow{\pi^+_{q,k-1}} S^p \times S^q$$

$$j^+_{q,k-1}|_{\partial D^{k-1} \times D^q \cong (\partial D^k)^- \times D^q} \xrightarrow{\partial(D^k)^- \times D^q \xrightarrow{j^+_{q,k}} \partial D^k \times D^q} (\partial D^k)^- \times D^q \xrightarrow{\pi^+_{q,k-1}} S^p \times S^q$$

We can therefore compute $d_+/d_-$ as the relative degree of the two maps on the right of these equalities. The relative degree of $j^+_{q,k-1}$ and $j^-_{q,k-1}$ is $(-1)^k$, and the degree of $\tau_q$ is $(-1)^{q+1}$. So altogether we find that $d_+/d_- = (-1)^{k+q+1}$. \hfill \Box

Now let $Z(0) = \mathbb{Z}$ and $Z(1)$ denote the trivial $\mathbb{Z}[\tau]$-module and the $\mathbb{Z}[\tau]$-module on which $\tau$ acts as multiplication by $-1$, respectively. Let $C^{(n)}(\tilde{t})$ denote the following chain complexes of abelian groups:

$$C^{(n)}(\tilde{0}): \mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{2} \mathbb{Z} \xrightarrow{2} \mathbb{Z} \xrightarrow{2} \cdots \xleftarrow{\cdots} \mathbb{Z}$$

$$C^{(n)}(\tilde{1}): \mathbb{Z} \xrightarrow{\tilde{2}} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{2} \mathbb{Z} \xrightarrow{2} \cdots \xleftarrow{\cdots} \mathbb{Z}$$

These are related to the chain complexes of $\mathbb{Z}[\tau]$-modules considered above by canonical isomorphisms $C^{(n)}(\tilde{t}) \otimes_{\mathbb{Z}[\tau]} \mathbb{Z}(s) \cong C^{(n)}(s+t)$.

**Corollary 3.12.** Assume $p \leq q$. There exists a cell structure on $Q_{p,q}$ with $2p+2$ cells: one $i$-cell for each $i \in \{0, \ldots, p\}$, and one $i$-cell for each $i \in \{q, \ldots, p+q\}$. The associated chain complexes with coefficients in $\mathbb{Z}(s)$ are given by

$$C(Q_{p,q}, \mathbb{Z}(s)) \cong C^{(p)}(\tilde{s}) \oplus C^{(p)}(s+q+1)[q].$$

This description of $C(Q_{p,q}, \mathbb{Z}(s))$ immediately gives us additive descriptions of the homology and cohomology of $Q_{p,q}$ with arbitrary – untwisted and twisted – coefficients. For example, $H_{*}(Q_{p,q}, \mathbb{Z}/2)$ is just a direct sum of two (shifted) copies of $H_{*}(\mathbb{R}P^p, \mathbb{Z}/2)$. Table 1 displays the cohomology groups of some split real quadrics with untwisted and twisted integral coefficients.
Corollary 3.13. For coefficients $R \in \{\mathbb{Z}, \mathbb{Z}(1), \mathbb{Z}/2\}$, the embedding $\Delta: \mathbb{RP}^p \to Q_{p,q}$ and the projection $\pi_1: \mathbb{RP}^p \to Q_{p,q}$ induces mutually inverse isomorphisms $H^i(Q_{p,q}, R) \cong H^i(\mathbb{RP}^p, R)$ in all degrees $i < p$. In degree $p$, $\Delta^*$ is an epimorphism split by the monomorphism $\pi_1^*$.

Proof. This is clear from Figure 2 and the fact that any split monomorphism or epimorphism $\mathbb{Z} \to \mathbb{Z}$ or $\mathbb{Z}/2 \to \mathbb{Z}/2$ is an isomorphism. □

Corollary 3.14. For $p < q$ and $s \in \mathbb{Z}$, the mod-2-reduction maps $H^i(Q_{p,q}, \mathbb{Z}(s)) \to H^i(Q_{p,q}, \mathbb{Z}/2)$ are surjections in all degrees $i$ in which $H^i(Q_{p,q}, \mathbb{Z}(s)) \cong \mathbb{Z}$, and isomorphisms in all degrees in which $H^i(Q_{p,q}, \mathbb{Z}(s)) \cong \mathbb{Z}/2$. The same is true in the case $p = q$ in all degrees $i \neq p$. The reduction maps $H^p(Q_{p,q}, \mathbb{Z}(s)) \to H^p(Q_{p,q}, \mathbb{Z}/2)$ are epimorphisms or monomorphisms as follows:

$$
\begin{align*}
\mathbb{Z} \oplus \mathbb{Z} & \twoheadrightarrow \mathbb{Z}/2 \oplus \mathbb{Z}/2 \quad \text{when } p + s \text{ is odd;} \\
\mathbb{Z}/2 & \hookrightarrow \mathbb{Z}/2 \oplus \mathbb{Z}/2 \quad \text{when } p + s \text{ is even.}
\end{align*}
$$

Proof. This is immediate from the cellular chain complexes. □

3.15. Alternative additive computations. The additive structure of $\text{H}^*(Q_{p,q}, \mathbb{Z} \oplus \mathbb{Z}(1))$ can also be computed using localization sequences, (twisted) Thom isomorphisms and a topological version of the blow-up setup described in Section 4. That is, there is a topological variant of the algebro-geometric computations that will follow, which we leave as an exercise to the diligent reader. A third approach would be to derive the topological results from the geometric ones, using the results of the forthcoming article [HWXZ], which will establish an isomorphism given by the real realization functor between the $I^*$-cohomology ring of any smooth real cellular variety and the singular cohomology ring of its real points. The topological computations presented here are intentionally independent of these considerations.

3.16. Mod-two cohomology ring.

Proposition 3.17. The cohomology ring of the real quadric $Q_{p,q}$ with coefficients in $\mathbb{Z}/2$ has the form

$$
\text{H}^*(Q_{p,q}, \mathbb{Z}/2) = \begin{cases}
\mathbb{Z}/2[\xi, \zeta]/(\xi^{p+1}, \zeta^2 + \xi^p \zeta) & \text{if } p = q \text{ and } p \text{ is even} \\
\mathbb{Z}/2[\xi, \zeta]/(\xi^{p+1}, \zeta^2) & \text{in all other cases}
\end{cases}
$$

with $|\xi| = 1$ and $|\zeta| = q$. For $p < q$, the generators $\xi$ and $\zeta$ are the unique non-zero elements of the respective degrees. In the case $p = q$, $\xi$ is the unique non-zero element of degree 1, and $\zeta$ is the unique non-zero element in the kernel of $\Delta^*$: $\text{H}^p(Q_{p,q}, \mathbb{Z}/2) \to \text{H}^p(\mathbb{RP}^p, \mathbb{Z}/2)$ (cf. Figure 2).

Proof of 3.17, apart from products of elements of degree $p$ when $p = q$. Write $h^i$ for $H^i(Q_{p,q}, \mathbb{Z}/2)$. Let $\xi$ be the generator of $h^1$. By 3.13 and the known cohomology of $\mathbb{RP}^p$, each of the powers $\xi^i$ generates $h^i$ for $i \in \{0, \ldots, p-1\}$. Moreover, $\xi^p \in h^p$ is non-zero, and $\xi^{p+1} = 0$. Let $\zeta = \zeta_0$ denote the generator of the kernel of $\Delta^*$ on $h^q$. When $p < q$, $\ker(\Delta^*) = h^q$, so $\zeta_0$ is simply a generator of $h^q$. When $p = q$, the elements $\xi^p$ and $\zeta_0$ together form a basis of $h^p$. Pick generators $\zeta_1, \ldots, \zeta_p$ in the remaining degrees $h^{1+q}, \ldots, h^{p+q}$, so that $|\zeta_i| = i + q$. Poincaré duality implies $\xi^{p-i} \zeta_i = \zeta_p$ for all $i \in \{0, \ldots, p\}$, except possibly when $p = q$ and $i = 0$. Thus, in all cases with $p < q$, we deduce that $\xi^i \zeta_0 = \zeta_i$ for $i \in \{0, \ldots, p\}$. It is moreover clear for degree reasons that $\zeta^2 = 0$ in these cases, so for $p < q$ the proof is complete. In the case $p = q$, we deduce that the elements $\zeta_1, \ldots, \zeta_p$ generate degrees $p+1, \ldots, p+q$. It remains to verify that $\xi^p \zeta$ is non-zero (and hence that $\zeta_1 = \zeta_2$), and to compute $\zeta^2$. This will be done after the proof of 3.22 below. □

3.18. Integral cohomology ring. The integral cohomology ring of $\mathbb{RP}^p$ with twisted integral coefficients is as follows:

$$
\text{H}^*(\mathbb{RP}^p, \mathbb{Z} \oplus \mathbb{Z}(1)) = \mathbb{Z}[\xi, \alpha]/(2\xi, \xi^{p+1}, \xi \alpha, \alpha^2) \quad \text{with } \begin{cases}
|\xi| = (1, 1) \\
|\alpha| = (p, p+1)
\end{cases}
$$

Mod-2-reduction to $\text{H}^*(\mathbb{RP}^p, \mathbb{Z}/2) = \mathbb{Z}/2[\xi]/\xi^{p+1}$ is determined by $\xi \mapsto \xi$ and $\alpha \mapsto \xi^p$. For the split quadrics $Q_{p,p}$ with odd $p$, we will need some information about the covering maps in Figure 2.
Lemma 3.19. The maps

\[ H^p(\mathbb{RP}^p \times \mathbb{RP}^p, \mathbb{Z}) \xrightarrow{[\pi_1 \times \pi_2]^*} H^p(Q_{p,p}, \mathbb{Z}) \quad \text{and} \quad H^p(Q_{p,p}, \mathbb{Z}) \xrightarrow{\pi} H^p(S^p \times S^p, \mathbb{Z}) \]  

are either both multiplication by 2 or both multiplication by −2. The composition

\[ H^p(\mathbb{RP}^p, \mathbb{Z}(p+1)) \oplus H^p(\mathbb{RP}^p, \mathbb{Z}(p+1)) \xrightarrow{\pi_1 + \pi_2} H^p(Q_{p,p}, \mathbb{Z}(p+1)) \xrightarrow{\pi} H^p(S^p \times S^p, \mathbb{Z}) \cong H^p(S^p, \mathbb{Z}) \oplus H^p(S^p, \mathbb{Z}) \]  

is of the form \( \begin{pmatrix} \pm 2 & 0 \\ 0 & \pm 2 \end{pmatrix} \).

Proof. The 2-fold cover \( S^p \to \mathbb{RP}^p \) induces multiplication by \( \pm 2 \) on the top cohomology group \( H^p(-, \mathbb{Z}(p+1)) \). This can be seen either explicitly from the cellular computations or from considering the degree of this map. It follows that (3.21) is multiplication by \( \pm 2 \), and that the composition of the two maps in (3.20) is multiplication by 4. Degree considerations show that each factor must be multiplication by \( \pm 2 \).

\[ \square \]

Theorem 3.22. The integral cohomology ring of the real quadric \( Q_{p,q} \) has the following form:

\[ H^*(Q_{p,q}, \mathbb{Z} \oplus \mathbb{Z}(1)) \cong \begin{cases} \mathbb{Z}[\xi, \alpha, \beta]/(2\xi^p + 1, \xi\alpha - \alpha\beta) & \text{if } p = q \text{ and } p \text{ is even} \\ \mathbb{Z}[\xi, \alpha, \beta]/(2\xi^p + 1, \xi\alpha^2 + \beta\alpha) & \text{in all other cases} \end{cases} \]

with generators of degrees \( |\xi| = (1, 1), |\alpha| = (p, p+1), |\beta| = (q, q+1) \). Under mod-2-reduction, \( \xi \mapsto \xi, \alpha \mapsto \alpha \) and \( \beta \mapsto \xi \). See Table 1 for some examples.

Proof of 3.22. Most of this is immediate from 3.14 and 3.13. Indeed, these show how deduce the products of 2-torsion elements from the products of their images under mod-2-reduction, and all of these products except \( \xi^2 \) have already been computed in the proof of Proposition 3.17. To verify the ring structure displayed in 3.22, it only remains to compute the products of the non-torsion classes.
When \( p < q \), non-torsion classes \( \alpha \) and \( \beta \) of the degrees specified in 3.22 are unique up to signs. It moreover follows for degree reasons that \( \alpha^2 = \beta^2 = 0 \) in these cases, and it follows from (the twisted version of) Poincaré duality (see [DK01, Theorem 5.7 and following remarks]) that their product \( \alpha \beta \) is a generator.

We now turn to the case \( p = q \). We need to understand the product

\[
\text{H}^p(Q_{p,p}, \mathbb{Z}(p + 1)) \times \text{H}^p(Q_{p,p}, \mathbb{Z}(p + 1)) \to \text{H}^{2p}(Q_{p,p}, \mathbb{Z}),
\]

where \( \text{H}^p(Q_{p,p}, \mathbb{Z}(p + 1)) = \mathbb{Z} \oplus \mathbb{Z} \). Recall from Figure 2 our notation \( \pi \) for the two-fold cover \( \pi: S^p \times S^p \to Q_{p,p} \), the notation \( \pi_1, \pi_2: Q_{p,p} \to \mathbb{R}P^p \) for the compositions of the two-fold cover \( Q_{p,p} \to \mathbb{R}P^p \times \mathbb{R}P^p \) with the projections onto the two factors, and \( \pi_1^S, \pi_2^S: S^p \times S^p \to S^p \) for the projections onto the two sphere factors. Choose generators \( \alpha \in \text{H}^p(\mathbb{R}P^p, \mathbb{Z}(p + 1)) \) and \( \sigma \in \text{H}^p(S^p, \mathbb{Z}) \) such that \( \alpha \) pulls back to \( 2\sigma \) under the canonical two-fold covering map. Define \( \alpha_i := \pi_i^*(\alpha) \) and \( \sigma_i := (\pi^S_i)^*\sigma \). Then:

\[
\pi^*(\alpha_i) = 2\sigma_i.
\]

Choose a generator \( \beta \) of \( \ker(\Delta_{\ast}) \in \text{H}^p(Q_{p,p}, \mathbb{Z}(p + 1)) \). As \( \pi_1^* \) is split by \( \Delta^* \), it follows that \( \alpha_1, \beta \) form a \( \mathbb{Z} \)-basis of \( \text{H}^p(Q_{p,p}, \mathbb{Z}(p + 1)) \). The pullback of \( \beta \) lives in the kernel of \( (\Delta^S)^\ast \), hence can be written as

\[
\pi^\ast(\beta) = a(\sigma_1 - \sigma_2)
\]

for some \( a \in \mathbb{Z} \). As the generators \( \alpha_i \) are pulled back from \( \mathbb{R}P^p \), we find that \( \alpha_1^2 = \alpha_2^2 = 0 \). In \( \text{H}^p(S^p \times S^p, \mathbb{Z}) \), we know that similarly \( \sigma_1^2 = \sigma_2^2 = 0 \), while \( \sigma_1 \sigma_2 \) is a generator in degree \( 2p \). Moreover, by 3.19, we can choose a generator \( \gamma \in \text{H}^{2p}(Q_{p,p}, \mathbb{Z}) \) such that \( \pi^\ast\gamma = 2\sigma_1 \sigma_2 \), and it then follows that \( \alpha_1 \alpha_2 = 2\gamma \). For the remaining products of \( \alpha_1 \) and \( \beta \), we find:

\[
\begin{align*}
\pi^\ast(\beta^2) &= a^2(\sigma_1 \sigma_2 + (-1)^p \sigma_2 \sigma_1) = a^2 \frac{1}{2}(1 + (-1)^p)\pi^\ast\gamma, \quad \text{so} \quad \beta^2 = a^2 \frac{1}{2}(1 + (-1)^p)\gamma, \\
\pi^\ast(\alpha_1 \beta) &= 2a\sigma_1(\sigma_1 - \sigma_2) = -2a\sigma_1 \sigma_2 = -a\pi^\ast\gamma, \quad \text{so} \quad \alpha_1 \beta = -a\gamma.
\end{align*}
\]

Thus, in the basis of \( \text{H}^p(Q_{p,p}, \mathbb{Z}(p + 1)) \) given by \( \alpha_1, \beta \), the product is described by the following matrix:

\[
\begin{pmatrix}
a^2 \frac{1}{2}(1 + (-1)^p) & a \\
-a & 0
\end{pmatrix}
\]

By Poincaré duality, this matrix must define a perfect pairing on \( \mathbb{Z}^2 \), so \( a = \pm 1 \). This shows that \( \alpha \beta \) is a generator of \( \text{H}^{2p}(Q_{p,p}, \mathbb{Z}) \). For even \( p \), we moreover find that \( \beta^2 = \pm \alpha \beta \), and by changing the sign of \( \beta \) if necessary we may assume \( \beta^2 = \alpha \beta \). For odd \( p \), we find that \( \beta^2 = 0 \). Altogether, this is precisely the result displayed above, with \( \alpha_1 \) written as \( \alpha \). (We also find that \( 2 \beta = \alpha_1 - \alpha_2 \).)

It remains to determine the images of the various generators under reduction mod 2. The elements \( \xi \) and \( \alpha \) are pulled back from cohomology classes of \( \mathbb{R}P^p \) under \( \pi_1 \), so the formulas for these elements follow from the corresponding formulas for \( \mathbb{R}P^p \). The elements \( \beta \in \text{H}^p(Q_{p,p}, \mathbb{Z}(1)) \) and \( \zeta \in \text{H}^p(Q_{p,p}, \mathbb{Z}(2)) \) were both chosen as generators of the kernel of the map induced by \( \Delta: \mathbb{R}P^p \to Q_{p,p} \). As \( \Delta \) is split by \( \pi_1 \), it follows that \( \beta \mapsto \zeta \), as claimed.

\textit{End of proof of Proposition 3.17.} From the given formulas for reduction modulo two, we find that the generator \( \alpha \beta \) reduces to \( \xi^p \zeta \), hence using 3.14 we deduce \( \xi^p \zeta \neq 0 \). The formula for \( \zeta^2 \) displayed in 3.17 is immediate from the formula for \( \beta^2 \) computed in 3.22.

\( \square \)

4. The blow-up setup of Balmer-Calmès

We now turn to algebraic geometry. Recall from Section 2 that \( Q_n \) denotes the split \( n \)-dimensional quadric over a smooth scheme \( S \) over a field \( F \), where \( n \geq 3 \). We begin by summarizing some material from [Nen09].

**Lemma 4.1.** Let \( N_{p^p} \) denote the normal bundle of \( \mathbb{P}^p_x \) in \( Q_n \). Then \( \det(N_{p^p}) \cong O_{p^p}(n - p - 1) \).

**Proof.** See [Nen09, Lemma 6.1 and above Theorem 6.4].

\( \square \)
In the following lemma, an affine space bundle of rank r is understood to be a Zariski locally trivial fibre bundle with fibres isomorphic to $\mathbb{A}^r$. (Such bundles are frequently referred to as “affine bundles” in the literature, but note that this terminology is ambiguous. The bundles we consider here have no fixed linear structure on the fibres, and no associated vector bundle.)

**Lemma 4.2.** The morphism

$$\rho: Q_n - \mathbb{P}_y^r \to \mathbb{P}_y^r : [x_0 : \ldots : x_p : y_0 : \ldots : y_p : z] \mapsto [y_0 : \ldots : y_p]$$

is an affine space bundle with $i_y: \mathbb{P}_y^r \hookrightarrow Q_n - \mathbb{P}_x^r$ as a global section. When n is even, $\rho$ is even a vector bundle, and $i_y$ is its zero section.

**Proof.** This seems to be well-known (see [Nen09, Section 6]). As we are not aware of a suitable reference, we provide here a proof for the case of odd n for the reader’s convenience. The case of even n is similar. Let $U \cong \mathbb{A}^p$ be the open subscheme of $\mathbb{P}_y^r$ defined by $y_i \neq 0$. We will check that $\rho^{-1}(U) \cong U \times \mathbb{A}^{p+1}$. We may assume that $S = \text{Spec}(F)$, as all schemes are already defined over F. Let us moreover assume for ease of notation that $i = 0$. Then $\rho^{-1}(U)$ is an affine quadric in $\mathbb{A}^{2p+2} \cong \mathbb{P}^{2p+1} - V(y_0)$, defined in terms of the coordinates $[x_0 : \ldots : x_p : 1 : y_1 : \ldots : y_p : z]$ by the equation $x_0 + x_1y_1 + \cdots + x_py_p + z^2 = 0$. This affine quadric is isomorphic to $U \times \mathbb{A}^{p+1}$ via an isomorphism of coordinate rings as follows:

$$F[x_0, x_1, \ldots, x_p, y_1, \ldots, y_p, z] \to F[x_1, \ldots, x_p, y_1, \ldots, y_p, z]$$

$$x_0 \mapsto -(x_1y_1 + \cdots + x_py_p + z^2)$$

$$x_i \mapsto x_i \quad \text{for } i \neq 0$$

$$y_i \mapsto y_i \quad \text{for all } i$$

This isomorphism is clearly compatible with the projections to U. \qed

Chow groups satisfy homotopy invariance for affine space bundles over smooth bases. Homotopy invariance also holds for $\mathbf{I}^\ast$-cohomology, and this will be frequently used in the next section:

**Theorem 4.3.** Let $f: X \to Y$ be an affine space bundle over a smooth variety Y over a base field F with $\text{char}(F) \neq 2$. Then $f^\ast$ is an isomorphism in $\mathbf{I}^\ast$-cohomology.

**Proof.** This may be deduced from [Fas08, Corollaire 11.2.8] by the usual Mayer-Vietoris argument. The required Mayer-Vietoris sequence can be deduced from excision along open embeddings for $\mathbf{I}^\ast$-cohomology. Excision along open embeddings holds as the corresponding Gersten complexes with support are isomorphic. More generally, we have flat excision for $\mathbf{I}^\ast$-cohomology, i.e. excision along flat morphisms, cf. [CF17, Lemma 3.7]. \qed

**Lemma 4.4.** Let $n \geq 3$. Let $i_w: \mathbb{P}_w^p \to Q_n$ denote one of the closed embeddings $i_x$ or $i_y$ (or $i_z$, or $i_y$, if n is even). This morphism induces an isomorphism $i_w^* : \text{Pic}(Q_n) \cong \text{Pic}(\mathbb{P}_w^p)$ under which $i_w^*\mathcal{O}_Q(k) \cong \mathcal{O}_{\mathbb{P}_w^p}(k)$. In particular, each of the mentioned embeddings induces the same isomorphism on Picard groups.

**Proof.** We only deal with $\mathbb{P}_y^p$ here, as the other cases are analogous. Recall that $\text{Pic}(X) \cong \text{CH}^1(X)$ when X is smooth over a field. Consider the pullback map induced by the morphism along the top row of Figure 1:

$$\mathbb{Z} \oplus \text{Pic}(S) \cong \text{CH}^1(\mathbb{P}_y^p) \leftarrow \text{CH}^1(Q_n - \mathbb{P}_x^p) \leftarrow \text{CH}^1(\mathbb{P}_x^p) \leftarrow \text{CH}^1(\mathbb{P}_y^p) \cong \mathbb{Z} \oplus \text{Pic}(S)$$

Using Lemma 4.2 and homotopy invariance for Chow groups, we see that $i_y^*$ are isomorphism. By n $\geq 3$ and the localization sequence of Chow groups, we see $j^\ast$ is an isomorphism. As the composition $i \circ j \circ i_y$ is a linear embedding of $\mathbb{P}_y^p$ into $\mathbb{P}^{n+1}$, the composition of all these pullback maps is likewise an isomorphism, sending $\mathcal{O}_{\mathbb{P}_{n+1}}(1)$ to $\mathcal{O}_{\mathbb{P}_y^p}(1)$. The claim follows. \qed
The closed immersion $ι : \mathbb{P}_y^p \hookrightarrow Q_n$ is a regular immersion of codimension $p$, since $\mathbb{P}_y^p$ and $Q_n$ are both smooth (cf. [EGA4, Ch. 4, 17.12.1]). Let $Bl$ denote the blow-up of $\mathbb{P}_y^p$ along $Q_n$ and let $E$ denote the exceptional fibre. The following proposition shows that this setup satisfies [BC09, Hypothesis 1.2].

**Proposition 4.5.** There exists a morphism $\tilde{ρ} : Bl \to \mathbb{P}_y^p$ making the following diagram commutative:

\[
\begin{array}{c}
\xymatrix{ 
\mathbb{P}_y^p 
\ar@{.>}[dr]^{\tilde{ρ}} & Q_n 
\ar[l]_{i} \ar[dr]^{j} 
\ar[d]_{\pi} & U := Q_n - \mathbb{P}_y^p \cong Bl - E 
\ar[dl]_{\hat{j}} 
\ar[d]_{\rho} & 
\mathbb{P}_y^p
}
\end{array}
\]

Here, $i$ and $\tilde{i}$ are closed immersions, $\pi$ and $\hat{\pi}$ are projections, $j$ and $\hat{j}$ are open immersions, and $\rho$ is the affine space bundle from Lemma 4.2.

**Proof.** We concentrate on the case $n = 2p$; the case of odd $n$ is similar. Then $Bl$ is contained in (in fact equal to) the closed subscheme of

$$Q_n \times \mathbb{P}^p = \operatorname{Proj}(\mathcal{O}[x_0, \ldots, x_p, y_0, \ldots, y_p]/(x_0y_0 + \cdots + x_py_p)) \times \operatorname{Proj}(\mathcal{O}[T_0, \ldots, T_p])$$

defined by the homogeneous polynomials $y_kT_j - y_jT_k$ (for $k, j = 0, 1, \ldots, p$) and $\sum_{i=0}^p x_iT_i = 0$. This follows from the universal property of a blow-up, or from the geometric description of the blow-up along $\mathbb{P}_y^p$ as the closure of the graph of the rational map $Q_n \rightarrow \mathbb{P}_y^p, [x_0 : \ldots : x_p : y_0 : \ldots : y_p] \mapsto [y_0 : \ldots : y_p]$, see [Ha92, paragraph above Exercise 7.19]. Define $\tilde{ρ}$ to be the composition $Bl \hookrightarrow Q_n \times \mathbb{P}^p \rightarrow \mathbb{P}^p_y$, where the first map is the inclusion and the last map is the projection onto the second factor. The relations $y_kT_j = y_jT_k$ guarantee the commutativity of diagram (4.6). \qed

Diagram (4.6) descends to the following diagram of Picard groups:

\[
\begin{array}{c}
\xymatrix{ 
\operatorname{Pic}(Q_n) 
\ar[r]^{j^*} & \operatorname{Pic}(U) 
\ar[d]_{\pi^*} & \operatorname{Pic}(U) 
\ar[d]_{(\rho^*)^{-1}} & 
\operatorname{Pic}(Q_n) \oplus \mathbb{Z}[E] \cong \operatorname{Pic}(Bl) 
\ar[l]_{i_0^*} & 
\xymatrix{ \operatorname{Pic}(\mathbb{P}_y^p) & 
\ar[l]_{\tilde{ρ}^*} & 
\ar[r]^{(1)} & 
\ar[l]_{\tilde{ρ}^*} & 
\ar[r] & 
}\end{array}
\]

where the map $λ : \operatorname{Pic}(\mathbb{P}_y^p) \rightarrow \mathbb{Z}[E]$ does not vanish in general. Hence, the square does not generally commute (cf. [BC09, Remark 2.2]), but both triangles in this diagram commute. For even $n$, we now compute the value of $λ(\mathcal{O}(1))$ as $-1$:

**Proposition 4.8.** Assume $n = 2p$ with $p \geq 2$. The pullback homomorphism $\tilde{ρ} = (1) : \operatorname{Pic}(\mathbb{P}_y^p) \rightarrow \operatorname{Pic}(Q_n) \oplus \mathbb{Z}[E]$ sends $\mathcal{O}_{\mathbb{P}^p}(1)$ to $\mathcal{O}(1), -1$.

**Proof.** All schemes in sight are smooth over a field, so we can identify all Picard groups with codimension-one Chow groups as above. The dimensions of our schemes are $\dim(Bl) = \dim(Q_n) = 2p$ and $\dim(E) = 2p - 1$. The identification of $\operatorname{CH}^1(Bl)$ with $\operatorname{CH}^1(Q_n) \oplus \mathbb{Z}[E]$ is given by $\pi^* : \operatorname{CH}^1(Q_n) \rightarrow \operatorname{CH}^1(Bl)$ and by $ι^* : \operatorname{CH}^0(E) \oplus \mathbb{Z}[E] \rightarrow \operatorname{CH}^1(Bl)$. This identification is obtained by noting that the usual localization sequence of Chow groups is split exact:

\[
0 \rightarrow \operatorname{CH}^0(E) \xrightarrow{ι^*} \operatorname{CH}^1(Bl) \xrightarrow{\tilde{j}^*} \operatorname{CH}^1(U) \rightarrow 0
\]

To see this, recall that for smooth schemes this exact sequence fits into a long exact sequence of motivic cohomology (cf. [D107, §3]):

$$\cdots \rightarrow H^{1,1}(Bl) \xrightarrow{\tilde{j}^*} H^{1,1}(U) \xrightarrow{\rho^*} H^{0,0}(E) \xrightarrow{ι^*} H^{1,1}(Bl) \rightarrow \operatorname{CH}^1(U) \rightarrow 0$$

The sequences breaks down into short split exact sequences because $\tilde{j}^*$ is split surjective via $\tilde{ρ}^* \circ (\rho^*)^{-1}$ for all degrees. The map $j^* : \operatorname{CH}^1(Bl) \rightarrow \operatorname{CH}^1(U)$ in degree one can also be split by $\pi^* \circ (j^*)^{-1}$. Next, we describe the maps in diagram 4.7 explicitly in terms of generators. Let us write $[y_0] \in \operatorname{CH}^1(Q)$ for the cycle on $Q_n$ corresponding to the subscheme defined by $y_0 = 0$, and analogously
for cycles on other schemes. In this notation, each of the groups \( \text{CH}^1(\mathbb{P}^n) \), \( \text{CH}^1(U) \) and \( \text{CH}^1(Q_n) \) is generated by the cycle \([y_0]\) corresponding to the line bundle \( \mathcal{O}(1) \) in each Picard group. The pullback maps are determined by \( p^*[y_0] = [y_0] = j^*[y_0] \), \( \hat{\mu}^*[y_0] = [T_0] \) and \( \pi^*[y_0] = [y_0] \). By the commutativity of both triangles in Diagram (4.7), we see \( j^*[T_0] = [y_0] = j^*[y_0] \) in \( \text{CH}^1(U) \). So the exact sequence 4.9 shows that

\[
[T_0] - [y_0] = \lambda[E]
\]

in \( \text{CH}^1(Bl) \) for some integer \( \lambda \). This is the integer that we need to compute.

To compute \( \lambda \), first note that for the closed subschemes \( V(y_0) \) and \( V(T_0) \) of \( Bl \) we have an equality of sets \( V(y_0) = V(T_0) \cup E \) with neither of \( E \) or \( V(T_0) \) contained in one another. Here, the exceptional divisor \( E \) is the smooth subscheme defined by \( \{[x_0 : \ldots : x_p] \times [T_0 : \ldots : T_p] \in \mathbb{P}^p \times \mathbb{P}^p : \sum x_i T_i = 0\} \). In particular, \( E \) is integral, hence an irreducible component of \( V(y_0) \). Using [Ful78, Section 1.5], we conclude

\[
[y_0] = [T_0] + \ell(E),
\]

where \( \ell \) is the length of \( \mathcal{O}_{V(y_0),E} \) as a module over itself. Now note that \( \mathcal{O}_{V(y_0),E} \) is a field: \( \mathcal{O}_{Bl,E} \) is a discrete valuation ring with maximal ideal given by \((y_0, y_1, \ldots, y_p); \) as \( y_i = y_0 \frac{T_i}{x_i} \), this ideal is principal, generated by \( y_0 \). The relation \( y_0 = 0 \) in \( \mathcal{O}_{V(y_0),E} \) kills this maximal ideal of \( \mathcal{O}_{Bl,E} \), so \( \mathcal{O}_{V(y_0),E} \) is a field as claimed. It follows that \( \ell = 1 \).

By comparing (4.10) and (4.11), we conclude that \( \lambda = -1 \).  

5. \( \mathbf{P}^* \)-cohomology: additive structure

We now embark on our computations of \( \mathbf{P}^* \)-cohomology of split quadrics, keeping the notation established in Section 2 and in the previous section. We refer to [Fas08] for twisted coefficients and how they appear when studying push-forwards, and to Lemmas 4.1 and 4.4 for possible twists that appear for split quadrics.

5.1. Fasel’s computations for projective spaces. Let \( S \) be a smooth scheme over a field \( F \) of characteristic not two. The following isomorphisms of graded abelian groups are proved by Fasel in [Fas13]: see Corollary 5.8, Definition 5.9, and Theorems 9.1, 9.2 and 9.4 of loc. cit. (Note that the sequence in 9.4 of loc. cit. splits as the bundle \( E \) Fasel considers is trivial in our case.)

**Theorem 5.2** (Fasel).

\[
\text{H}^i(\mathbb{P}^p, \mathbf{P}, \mathcal{O}(k)) \cong \begin{cases} 
(2 \leq m \leq p) \bigoplus_{m \text{ even}} \mathbf{H}^{i-m}(S, \mathbf{P}^{i-m}) & \text{if } k \text{ is even and } p \text{ is even} \\
(2 \leq m \leq p) \bigoplus_{m \text{ even}} \mathbf{H}^{i-m}(S, \mathbf{P}^{i-m}) & \text{if } k \text{ is even and } p \text{ is odd} \\
\bigoplus_{m \text{ odd}} \mathbf{H}^{i-m}(S, \mathbf{P}^{i-m}) & \text{if } k \text{ is odd and } p \text{ is even} \\
\bigoplus_{m \text{ odd}} \mathbf{H}^{i-m}(S, \mathbf{P}^{i-m}) & \text{if } k \text{ is odd and } p \text{ is odd}
\end{cases}
\]

Let \( p: \mathbb{P}^p \to S \) be the projection map and \( s: S \to \mathbb{P}^p \) a rational point. The isomorphism above on the component \( \mathbf{H}^{i-m}(S, \mathbf{P}^{i-m}) \to \mathbf{H}^i(\mathbb{P}^p, \mathbf{P}, \mathcal{O}(k)) \) is given by the pullback \( p^* \) if \( k = m = 0 \) and by the pushforward \( s_* \) if \( m = p \) and \( k = p - 1 \). The map

\[
\mu^m_\mathcal{L}: \mathbf{H}^{i-m}(S, \mathbf{P}^{i-m}) \to \mathbf{H}^i(\mathbb{P}^p, \mathbf{P}, \mathcal{L}(-m))
\]

in the isomorphism above is defined in [Fas13, Definition 5.1]; explicitly it is the composition

\[
\mathbf{H}^{i-m}(S, \mathbf{P}^{i-m}) \xrightarrow{p^*} \mathbf{H}^{i-m}(\mathbb{P}^p, \mathbf{P}^{i-m}) \xrightarrow{\partial_{\mathcal{L}(-1)}} \mathbf{H}^{i-m+1}(\mathbb{P}^p, \mathbf{P}^{i-m+1}, \mathcal{L}(-1)) \xrightarrow{c(\mathcal{O}(1))^{m-1}} \mathbf{H}^i(\mathbb{P}^p, \mathbf{P}, \mathcal{L}(-m))
\]

where \( \partial_{\mathcal{L}(-1)} \) is the connecting homomorphism (or Bockstein homomorphism) [Fas13, §2.1] and \( c(\mathcal{O}(1)) \) is the Euler class homomorphism [Fas13, §3].
5.3. \( \Gamma \)-cohomology of completely split quadrics. \( \) The aim of this section is to obtain corresponding results for split quadrics, i.e. to determine the \( \Gamma \)-cohomology of split quadrics in all bidegrees, with all twists. In geometric bidegrees, we will later also compute the ring structure; see Section 6.

We will always assume \( n \geq 3 \). Recall that \( Q_1 \cong \mathbb{P}^1 \) (for which the previous computation applies) and that \( Q_2 \cong \mathbb{P}^1 \times \mathbb{P}^1 \) (using the Segre embedding in \( \mathbb{P}^3 \)).

**Theorem 5.4.** For the split quadric \( Q_n \) of dimension \( n \geq 3 \), we have isomorphisms of groups as follows:

\[
H^i(Q_n, \mathcal{I}) \cong \begin{cases} 
\left( \bigoplus_{m \in T_n} H^{i-m}(S, I^{i-m}) \right) \oplus H^i(S, \mathcal{I}^p) \oplus H^{i-n}(S, \mathcal{I}^{i-n}) & \text{if } n = 2p \text{ and } p \text{ is even} \\
\left( \bigoplus_{m \in T_n} H^{i-m}(S, I^{i-m}) \right) \oplus H^i(S, \mathcal{I}^p) \oplus H^{i-p}(S, \mathcal{I}^{i-p}) \oplus H^{i-n}(S, \mathcal{I}^{i-n}) & \text{if } n = 2p \text{ and } p \text{ is odd} \\
\left( \bigoplus_{m \in T_n} H^{i-m}(S, I^{i-m}) \right) \oplus H^i(S, \mathcal{I}^p) \oplus H^{i-p+1}(S, \mathcal{I}^{i-p-1}) & \text{if } n = 2p+1 \text{ and } p \text{ is even} \\
\left( \bigoplus_{m \in T_n} H^{i-m}(S, I^{i-m}) \right) \oplus H^i(S, \mathcal{I}^p) \oplus H^{i-p+1}(S, \mathcal{I}^{i-p-1}) \oplus H^{i-n}(S, \mathcal{I}^{i-n}) & \text{if } n = 2p+1 \text{ and } p \text{ is odd}
\end{cases}
\]

where \( T_n := \{ 1 \leq m \leq n : m \text{ even if } 1 \leq m \leq \lfloor \frac{n}{2} \rfloor \text{ and } m \text{ odd if } \lceil \frac{n}{2} \rceil + 1 \leq m \leq n \} \).

\[
H^i(Q_n, \mathcal{I}, \mathcal{O}(1)) \cong \begin{cases} 
\left( \bigoplus_{m \in U_n} H^{i-n}(S, I^{i-m}) \right) \oplus H^i(S, \mathcal{I}^p) \oplus H^{i-p}(S, \mathcal{I}^{i-p}) \oplus H^{i-n}(S, \mathcal{I}^{i-n}) & \text{if } n = 2p \text{ and } p \text{ is even} \\
\left( \bigoplus_{m \in U_n} H^{i-n}(S, I^{i-m}) \right) \oplus H^i(S, \mathcal{I}^p) \oplus H^{i-p+1}(S, \mathcal{I}^{i-p-1}) & \text{if } n = 2p+1 \text{ and } p \text{ is even}
\end{cases}
\]

where \( U_n := \{ 1 \leq m \leq n : m \text{ odd if } 1 \leq m \leq \lfloor \frac{n}{2} \rfloor \text{ and } m \text{ even if } \lceil \frac{n}{2} \rceil + 1 \leq m \leq n \} \).

The remainder of this section constitutes a proof of this theorem. The proof relies on homotopy invariance (see Theorem 4.3), localization and dévissage [Fas13, Théorème 9.3.4 and Remarque 9.3.5] for \( \Gamma \)-cohomology. We need to distinguish cases based on the parities of \( n, p \) and the twist, so there are eight different cases to consider.

5.5. Twisting homomorphisms for quadrics.

**Definition 5.6.** Let \( \lambda^m : H^{i-m}(S, I^{i-m}) \rightarrow H^i(Q, \mathcal{I}, \mathcal{O}(-m)) \) denote the following composition:

\[
H^{i-m}(S, I^{i-m}) \xrightarrow{\partial_{L(-1)}} H^{i-m+1}(Q, I^{i-m+1}, L(-1)) \xrightarrow{c(O(1))^{m-1}} H^i(Q, L, \mathcal{L}(-m))
\]

where \( \partial_{L(-1)} \) is the connecting homomorphism (or Bockstein homomorphism) [Fas13, §2.1] and \( c(O(1)) \) is the Euler class homomorphism [Fas13, §3].

**Remark 5.7.** The homomorphism in Definition 5.6 is analogous to the map \( \mu^m \) defined by Fasel [Fas13, Definition 5.1] on projective spaces.

**Lemma 5.8.** The following diagram is commutative.

\[
\begin{array}{ccc}
H^i(Q, \mathcal{I}, \mathcal{L}(-m)) & \xrightarrow{\lambda^m} & H^i(Q - \mathbb{P}^p, \mathcal{I}, \mathcal{L}(-m)) \\
\xrightarrow{j^*} & & \xrightarrow{\rho^*} \\
H^{i-m}(S, I^{i-m}) & \xrightarrow{\mu^m} & H^i(\mathbb{P}^{p-1}, \mathcal{I}, \mathcal{L}(-m))
\end{array}
\]

**Proof.** Similarly to \( \lambda^m \) for \( Q_n \) and \( \mu^m \) for \( \mathbb{P}^p \), we can define twist homomorphisms \( \delta^m : H^{i-m}(S, I^{i-m}) \rightarrow H^i(Q_n - \mathbb{P}^p, \mathcal{I}, \mathcal{O}(-m)) \) for \( Q_n - \mathbb{P}^p \). The result then follows as the Bockstein homomorphisms and the Euler class homomorphisms commute with the pullback homomorphisms \( j^* \) and \( \rho^* \), cf. [Fas13, Proposition 2.1 and §3]. \( \square \)
5.9. Even dimensional quadrics and even twists. In this subsection, we consider first the case \( n = \dim(Q_n) = 2p \) with \( p \) even.

**Proposition 5.10.** Suppose \( n = 2p \) with \( p \) even. Then
\[
H^i(Q_n, \mathcal{V}) \cong \left( \bigoplus_{m \in T_n} H^{i-m}(S, \mathcal{V}^{m}) \right) \oplus H^i(S, \mathcal{V}) \oplus H^{i-n}(S, \mathcal{V}^{-n})
\]
where \( T_n = \{ 1 \leq m \leq 2p : m \) even if \( 1 \leq m \leq p \) and \( m \) odd if \( p + 1 \leq m \leq 2p \} \).

**Proof.** We will show that the following localization sequence splits:
\[
\cdots \longrightarrow H^i_\mathbb{P}^p(Q, \mathcal{V}) \longrightarrow H^i(Q, \mathcal{V}) \longrightarrow H^i(Q - \mathbb{P}^p, \mathcal{V}) \longrightarrow H^i_{\mathbb{P}^p}(Q, \mathcal{V}) \longrightarrow \cdots
\]
By homotopy invariance, the vector bundle projection \( \rho \) of Lemma 4.2 induces an isomorphism
\[
\rho^*: H^i(\mathbb{P}^p_y, \mathcal{V}) \rightarrow H^i(Q - \mathbb{P}^p, \mathcal{V}).
\]
Consider the following diagram:
\[
\begin{array}{ccc}
H^i_\mathbb{P}^p(Q, \mathcal{V}) & \xrightarrow{j^*} & H^i(Q, \mathcal{V}) \longrightarrow H^i(Q - \mathbb{P}^p, \mathcal{V}) \longrightarrow H^i_{\mathbb{P}^p}(Q, \mathcal{V}) \\
\alpha & & \beta & \rho^* \\
\left( \bigoplus_{m \text{ even}} H^{i-m}(S, \mathcal{V}^{m}) \right) \oplus H^i(S, \mathcal{V}) & \longrightarrow & H^i_{\mathbb{P}^p}(Q, \mathcal{V})
\end{array}
\]
Here, \( \beta \) is the isomorphism \( \sum m \mu^m + p^* \) of [Fas13, Theorem 9.1], cf. Definitions 5.1 and 5.9 and Corollary 5.8 of loc. cit. The map \( \alpha \) is defined similarly as \( \beta \), namely as \( \alpha := \sum m \lambda^m + q^* \). The diagram commutes by Lemma 5.8.

Since \( \beta \) and \( \rho^* \) are both isomorphisms, we obtain a splitting \( \alpha \circ \beta^{-1} \circ (\rho^*)^{-1} \) of \( j^* \). It follows that
\[
H^i(Q, \mathcal{V}) \cong H^i_\mathbb{P}^p(Q, \mathcal{V}) \oplus H^i(\mathbb{P}^p_y, \mathcal{V}) \cong H^i_\mathbb{P}^p(Q, \mathcal{V}) \oplus \left( \bigoplus_{m \text{ even}} H^{i-m}(S, \mathcal{V}^{m}) \right) \oplus H^i(S, \mathcal{V})
\]
By the dévissage theorem, Fasel’s computation of \( \mathcal{V} \)-cohomology of \( \mathbb{P}^p \) and Lemma 4.1, we obtain that
\[
H^i_\mathbb{P}^p(Q, \mathcal{V}) \cong H^{1-p}(\mathbb{P}^p, \mathcal{V}^{1-p}, \mathcal{O}(p-1)) \cong \left( \bigoplus_{m \text{ odd}} H^{1-p-m}(S, \mathcal{V}^{1-p-m}) \right) \oplus H^{2p}(S, \mathcal{V}^{2p})
\]
It follows that
\[
H^i(Q, \mathcal{V}) \cong \left( \bigoplus_{m \text{ even}} H^{1-m}(S, \mathcal{V}^{m}) \right) \oplus H^i(S, \mathcal{V}) \oplus \left( \bigoplus_{m \text{ odd}} H^{1-p-m}(S, \mathcal{V}^{1-p-m}) \right) \oplus H^{n}(S, \mathcal{V}^{-n})
\]
Finally, we rewrite this equation to get the result. \( \square \)

Next, we consider the case \( n = \dim(Q_n) = 2p \) with \( p \) odd. Let \( Bl \) be the blow-up of \( Q_n \) along \( \mathbb{P}^p_x \) and let \( E \) be the exceptional fibre. Recall from Proposition 4.5 above that this setup satisfies Hypothesis 1.2 of [BC09].

**Proposition 5.11.** Suppose \( n = 2p \) with \( p \) odd. Then
\[
H^i(Q_n, \mathcal{V}) \cong \left( \bigoplus_{m \in T_n} H^{i-m}(S, \mathcal{V}^{m}) \right) \oplus H^i(S, \mathcal{V}) \oplus H^{i-p}(S, \mathcal{V}^{-p}) \oplus H^{i-n}(S, \mathcal{V}^{-n})
\]
where \( T_n = \{ 2 \leq m \leq 2p - 1 : m \) even if \( 2 \leq m \leq p \) and \( m \) odd if \( p + 1 \leq m \leq 2p \}. \)
Proof. We use the same notation as in Proposition 4.5. As in the previous case, we want to show $j^*$ is split surjective.

$$\cdots \longrightarrow \mathcal{H}^i_{P^n}(Q, V) \longrightarrow \mathcal{H}^i(Q, V) \xrightarrow{j^*} \mathcal{H}^i(Q - \mathbb{P}^p, V) \xrightarrow{\partial} \mathcal{H}^{i+1}_{p^n}(Q, V) \longrightarrow \cdots$$

We draw the following diagram:

$$
\begin{array}{ccc}
\mathcal{H}^i_{P^n}(Q, V) & \longrightarrow & \mathcal{H}^i(Q, V) \\
\xrightarrow{j^*} & \xrightarrow{\partial} & \mathcal{H}^{i+1}_{p^n}(Q, V) \\
\xrightarrow{\pi_*} & & \\
\mathcal{H}^i(BL, V, \omega_\pi) & \cong & \mathcal{H}^i(\mathbb{P}_y, V) \\
\cong & & \mathcal{H}^i(BL, V) \\
\end{array}
$$

Arguing as in the proof of Theorem 2.3 of [BC09], we see that the middle diagram is commutative (as $\lambda(O) = 0$ and as $p$ odd, $\lambda(O) \equiv p-1 \mod 2$, as required). By dévissage, we obtain that $\mathcal{H}^i_{p^n}(Q, V) = \mathcal{H}^{i-p}(\mathbb{P}_x, V^{-p}, O(p-1)) = \mathcal{H}^{i-p}(\mathbb{P}^p_x, V^{-p})$. Now use Fasel’s computation and summarize the result.

5.12. Odd dimensional quadrics and even twists. In this subsection, we first consider the case $n = 2p + 1 = \dim(Q)$ is odd and $p$ is even.

**Proposition 5.13.** Suppose $n = 2p + 1$ with $p$ even. Then

$$\mathcal{H}^i(Q, V) \cong \bigoplus_{m \in T_n} \mathcal{H}^{i-m}(S, \mathcal{V}^{-m}) \oplus \mathcal{H}^i(S, V) \oplus \mathcal{H}^{i-p-1}(S, \mathcal{V}^{p-1})$$

where $T_n = \{1 \leq m \leq 2p : m \text{ even if } 1 \leq m \leq p \text{ and } m \text{ odd if } p+2 \leq m \leq 2p+1\}$.

**Proof.** We want to show the localization sequence splits

$$\cdots \longrightarrow \mathcal{H}^i_{P^n}(Q, V) \longrightarrow \mathcal{H}^i(Q, V) \xrightarrow{j^*} \mathcal{H}^i(Q - \mathbb{P}^p, V) \xrightarrow{\partial} \mathcal{H}^{i+1}_{p^n}(Q, V) \longrightarrow \cdots$$

By homotopy invariance, we have the isomorphism

$$\rho^* : \mathcal{H}^i(Q - \mathbb{P}^p_x, V) \to \mathcal{H}^i(\mathbb{P}_y, V).$$

We can draw the following diagram

$$
\begin{array}{ccc}
\mathcal{H}^i_{P^n}(Q, V) & \longrightarrow & \mathcal{H}^i(Q, V) \\
\xrightarrow{j^*} & \xrightarrow{\partial} & \mathcal{H}^{i+1}_{p^n}(Q, V) \\
\xrightarrow{\alpha} & & \\
\bigoplus_{\text{odd}} & & \bigoplus_{\text{odd}} \\
\mathcal{H}^{i-m}(S, \mathcal{V}^{-m}) & \oplus & \mathcal{H}^i(S, V) \\
\xrightarrow{\beta} & & \mathcal{H}(\mathbb{P}_y, V) \\
\end{array}
$$

where $\alpha$ and $\beta$ are defined similarly as in Proposition 5.10. The same reason shows that $\rho^*$ and $\beta$ are both isomorphisms, and we obtain a splitting $\alpha \circ \beta^{-1} \circ (\rho^*)^{-1}$ of $j^*$. It follows that

$$\mathcal{H}^i(Q, V) \cong \mathcal{H}^i_{P^n}(Q, V) \oplus \mathcal{H}^i(\mathbb{P}_y, V) \cong \mathcal{H}^i_{P^n}(Q, V) \oplus \bigoplus_{m \text{ even}} \mathcal{H}^{i-m}(S, \mathcal{V}^{-m}) \oplus \mathcal{H}^i(S, V)$$

By the dévissage theorem, we obtain that

$$\mathcal{H}^i_{P^n}(Q, V) \cong \mathcal{H}^{i-p-1}(\mathbb{P}^p_x, V^{-p-1}, O(p)) \cong \mathcal{H}^{i-p-1}(\mathbb{P}^p_x, V^{-p-1})$$
and by Fasel’s computation of $I^j$-cohomology of $\mathbb{P}^p$, we get that
$$H^{i-p-1}(\mathbb{P}^p_x, I^{i-p-1}) \cong \bigoplus_{m \text{ even}}^{2 \leq m \leq p} H^{i-p-1-m}(S, I^{i-p-1-m}) \oplus H^{i-p-1}(S, I^{i-p-1})$$

It follows that
$$H^i(Q, I^p) \cong \left( \bigoplus_{m \text{ even}}^{2 \leq m \leq p} H^{i-m}(S, I^{i-m}) \right) \oplus H^i(S, I^p) \oplus \left( \bigoplus_{m \text{ even}}^{2 \leq m \leq p} H^{i-p-1-m}(S, I^{i-p-1-m}) \right) \oplus H^{i-p-1}(S, I^{i-p-1})$$

Finally, we rewrite this equation to get the result. 

Proposition 5.14. Suppose $n = 2p + 1$ with $p$ is odd. Then
$$H^i(Q_n, I^p) \cong \left( \bigoplus_{m \in T_n} H^{i-m}(S, I^{i-m}) \right) \oplus H^i(S, I^p) \oplus H^{i-p}(S, I^{i-p})$$

where $T_n = \{ 1 \leq m \leq 2p + 1 : m \text{ even if } 1 \leq m \leq p \text{ and } m \text{ odd if } p + 2 \leq m \leq 2p + 1 \}$.

Proof. We show that the localization sequence splits
$$\cdots \to H^i_{P_x}(Q, I^p) \to H^i(Q, I^p) \to H^i(Q - P_x, I^p) \to H^{i+1}_{P_x}(Q, I^p) \to \cdots$$

By dévissage, we obtain that $H^i_{P_x}(Q, I^p) = H^{i-1}(\mathbb{P}^p_x, I^{i-1}, \mathcal{O}(p))$. Now, the localization sequence can be rewritten as
$$\cdots \to H^{i-1}(\mathbb{P}^p_x, I^{i-1}, \mathcal{O}(p)) \xrightarrow{\iota_*} H^i(Q, I^p) \xrightarrow{\partial} H^{i+1}_{P_x}(Q, I^p) \to \cdots$$

Now, we show that $\iota_*$ is split injective. Consider the following diagram:
$$\begin{array}{ccc}
H^{i-1}(\mathbb{P}^p_x, I^{i-1}, \mathcal{O}(p)) & \xrightarrow{\iota_*} & H^i(Q, I^p) \\
\pi \downarrow & & \pi \\
H^{i-1}(\mathbb{P}^p_x, I^{i-1}) & \xrightarrow{\iota_*} & H^i(Q, I^p)
\end{array}$$

The diagram is commutative since push-forward commutes with the mod-2 map. The lower horizontal map $\iota_*: H^{i-1}(\mathbb{P}^p_x, I^{i-1}) \to H^i(Q, I^p)$ is split injective because $I^j$-cohomology is oriented and because $\iota_*$ is part of a localization sequence. Let $s_1: H^i(Q, I^p) \to H^{i-1}(\mathbb{P}^p_x, I^{i-1})$ denote a splitting. The map $\pi_\omega$ is also split injective: a splitting is given by the composition
$$H^{i-1}(\mathbb{P}^p_x, I^{i-1}) \xrightarrow{s_1} H^i(Q, I^p) \xrightarrow{\pi} H^i(Q, I^p)$$

Let this splitting be denoted by $s_2$. Now, the composition
$$H^i(Q, I^p) \xrightarrow{s_2} H^i(Q, I^p)$$

provides a splitting for $\iota_*: H^{i-1}(\mathbb{P}^p_x, I^{i-1}, \mathcal{O}(p)) \to H^i(Q, I^p)$. 

\qed
5.15. **Even dimensional quadrics and odd twists.** In this subsection, we compute the case \( n = 2p = \dim(Q_n) \) is even and \( p \) is odd.

**Proposition 5.16.** Suppose \( n = 2p \) with \( p \) odd. Then
\[
H^i(Q_n, \mathcal{V}, \mathcal{O}(1)) \cong \bigoplus_{m \in U_n} H^{i-m}(S, [\mathcal{I}^{-m}])
\]
where \( U_n = \{1 \leq m \leq 2p : m \text{ odd if } 1 \leq m \leq p \text{ and } m \text{ even if } p + 1 \leq m \leq 2p\} \).

**Proof.** Consider the localization sequence
\[
\xymatrix{ H^i_p(Q, \mathcal{V}, \mathcal{O}(1)) \ar[r] & H^i(Q, \mathcal{V}, \mathcal{O}(1)) \ar[r]^{j^*} & H^i(Q - \mathbb{P}^p_x, \mathcal{V}, \mathcal{O}(1)) \ar[r]^\partial & H^i_{p+1}(Q, \mathcal{V}, \mathcal{O}(1)) }
\]
We claim that \( j^* \) is split surjective. Consider the following commutative diagram
\[
\xymatrix{ H^i(Q, \mathcal{V}, \mathcal{O}(1)) \ar[r]^{j^*} & H^i(Q - \mathbb{P}^p_x, \mathcal{V}, \mathcal{O}(1)) \ar[d]^{\alpha} \ar[r]^{\beta} & H^i(\mathbb{P}^p_y, \mathcal{V}, \mathcal{O}(1)) \ar[d] \ar[r] & H^i(\mathbb{P}^p_y, \mathcal{V}, \mathcal{O}(1)) }
\]
where \( \alpha \) and \( \beta \) are defined similarly as in Proposition 5.10. Since \( \beta \) and \( \rho^* \) are both isomorphisms, we obtain a splitting \( \alpha \circ \beta^{-1} \circ (\rho^*)^{-1} \) of \( j^* \). By the dévissage theorem and the 2-periodicity on the twists, we obtain that
\[
H^i_p(Q, \mathcal{V}, \mathcal{O}(1)) \cong H^{i-p}(\mathbb{P}^p_x, \mathcal{V}^{-p}, \mathcal{O}(p)) \cong H^{i-p}(\mathbb{P}^p_x, \mathcal{V}^{-p}, \mathcal{O}(1))
\]
Therefore, \( H^i(Q_n, \mathcal{V}, \mathcal{O}(1)) \cong H^{i-p}(\mathbb{P}^p_x, \mathcal{V}^{-p}, \mathcal{O}(1)) \oplus H^i(\mathbb{P}^p_y, \mathcal{V}, \mathcal{O}(1)) \). By Fasel’s computation again, we summarize the result. \( \square \)

Now, we consider the case \( n = 2p = \dim(Q_n) \) with \( p \) even.

**Proposition 5.17.** Suppose \( n = 2p \) with \( p \) even. Then
\[
H^i(Q_n, \mathcal{V}, \mathcal{O}(1)) \cong \bigoplus_{m \in U_n} H^{i-m}(S, [\mathcal{I}^{-m}]) \oplus H^{i-p}(S, [\mathcal{I}^{-p}]) \oplus 2
\]
where \( U_n = \{1 \leq m \leq 2p : m \text{ odd if } 1 \leq m \leq p \text{ and } m \text{ even if } p + 1 \leq m \leq 2p\} \).

**Proof.** We want to show the localization sequence splits
\[
\xymatrix{ H^i_p(Q, \mathcal{V}, \mathcal{O}(1)) \ar[r] & H^i(Q, \mathcal{V}, \mathcal{O}(1)) \ar[r]^{j^*} & H^i(Q - \mathbb{P}^p_x, \mathcal{V}, \mathcal{O}(1)) \ar[r]^\partial & H^i_{p+1}(Q, \mathcal{V}, \mathcal{O}(1)) }
\]
We provide a splitting for \( j^* \). In light of Proposition 4.5, we draw the following commutative diagram
\[
\xymatrix{ H^i(Bl, \mathcal{V}, \mathcal{O}(1)) \ar[r]^{\pi_*} & H^i(Q - \mathbb{P}^p_x, \mathcal{V}, \mathcal{O}(1)) \ar[r]^{\rho^*} & H^i(\mathbb{P}^p_y, \mathcal{V}, \mathcal{O}(1)) \ar[d]^{\beta^*} \ar[r] & H^i(Bl, \mathcal{V}, \mathcal{O}(1)) }
\]
By Proposition 4.5, we can apply the proof of Theorem 2.3 [BC09] (because \( \lambda(\mathcal{O}(1)) = -1 \) by Proposition 4.8 and \( p \) is even, \( \lambda(\mathcal{O}) \equiv d - 1 \mod 2 \)). By dévissage, we obtain that \( H^i_p(Q, \mathcal{V}, \mathcal{O}(1)) = H^{i-p}(\mathbb{P}^p_x, \mathcal{V}^{-p}, \mathcal{O}(p)) = H^{i-p}(\mathbb{P}^p_y, \mathcal{V}^{-p}) \). By Fasel’s computation, we summarize the result. \( \square \)
5.18. Odd dimensional quadrics and odd twists. In this subsection, we compute the case $n = 2p + 1 = \dim(Q_n)$ is even and $p$ is odd.

**Proposition 5.19.** Suppose $n = 2p + 1$ with $p$ odd. Then

$$H^i(Q_n, \mathcal{P}, \mathcal{O}(1)) \cong \bigoplus_{m \in U_n} H^{i-m}(S, \mathcal{P}^{i-m}) \oplus H^{i-p-1}(S, \mathcal{P}^{i-p-1}) \oplus H^{i-n}(S, \mathcal{P}^{i-n})$$

where $U_n = \{1 \leq m \leq 2p : m \text{ odd if } 1 \leq m \leq p \text{ and } m \text{ even if } p + 2 \leq m \leq 2p\}$.

**Proof.** Consider the localization sequence again

$$H^i_{\mathbb{P}_x}(Q, \mathcal{P}, \mathcal{O}(1)) \longrightarrow H^i(Q, \mathcal{P}, \mathcal{O}(1)) \longrightarrow H^i(Q - \mathbb{P}_x, \mathcal{P}, \mathcal{O}(1)) \longrightarrow H^i_{\mathbb{P}_x}(Q, \mathcal{P}, \mathcal{O}(1))$$

We claim that $j^*$ is split surjective. Consider the following commutative diagram

$$
\begin{array}{ccc}
H^i(Q, \mathcal{P}, \mathcal{O}(1)) & \longrightarrow & H^i(Q - \mathbb{P}_x, \mathcal{P}, \mathcal{O}(1)) \\
\alpha & \cong & \beta \rho^* \\
\bigoplus_{1 \leq m \leq p, m \text{ odd}} H^{i-m}(S, \mathcal{P}^{i-m}) & \longrightarrow & H^{i-p}(\mathbb{P}_y, \mathcal{P}, \mathcal{O}(1))
\end{array}
$$

where $\alpha$ and $\beta$ are defined similarly as in Proposition 5.10. Since $\beta$ and $\rho^*$ are both isomorphisms, we obtain a splitting $\alpha \circ \beta^{-1} \circ (\rho^*)^{-1}$ of $j^*$. By the dévissage theorem and the 2-periodicity on the twists, we obtain that

$$H^i_{\mathbb{P}_x}(Q, \mathcal{P}, \mathcal{O}(1)) \cong H^{i-p-1}(\mathbb{P}_x, \mathcal{P}^{i-p-1}, \mathcal{O}(p + 1)) \cong H^{i-p-1}(\mathbb{P}_x, \mathcal{P}^{i-p-1})$$

Therefore, $H^i(Q_n, \mathcal{P}, \mathcal{O}(1)) \cong H^{i-p-1}(\mathbb{P}_x, \mathcal{P}^{i-p-1}) \oplus H^{i}(\mathbb{P}_y, \mathcal{P}, \mathcal{O}(1))$. By Fasel’s computation again, we summarize the result.

Finally, we compute the case $n = 2p + 1 = \dim(Q_n)$ is even and $p$ is even. The closed immersion $\iota: \mathbb{P}_x \hookrightarrow Q_n$ is a regular immersion of codimension $p + 1$.

**Proposition 5.20.** Suppose $n = 2p + 1$ with $p$ even. Then

$$H^i(Q_n, \mathcal{P}, \mathcal{O}(1)) \cong \bigoplus_{m \in U_n} H^{i-m}(S, \mathcal{P}^{i-m}) \oplus H^{i-p-1}(S, \mathcal{P}^{i-p-1}) \oplus H^{i-n}(S, \mathcal{P}^{i-n})$$

where $U_n = \{1 \leq m \leq 2p + 1 : m \text{ odd if } 1 \leq m \leq p \text{ and } m \text{ even if } p + 2 \leq m \leq 2p + 1\}$.

**Proof.** We show that the localization sequence splits again.

$$H^i_{\mathbb{P}_x}(Q, \mathcal{P}, \mathcal{O}(1)) \longrightarrow H^i(Q, \mathcal{P}, \mathcal{O}(1)) \longrightarrow H^i(Q - \mathbb{P}_x, \mathcal{P}, \mathcal{O}(1)) \longrightarrow H^i_{\mathbb{P}_x}(Q, \mathcal{P}, \mathcal{O}(1))$$

By dévissage, we obtain that $H^i_{\mathbb{P}_x}(Q, \mathcal{P}, \mathcal{O}(1)) = H^{i-p-1}(\mathbb{P}_x, \mathcal{P}^{i-p-1}, \mathcal{O}(p + 1))$. Now, the localization sequence can be rewritten as

$$H^{i-p-1}(\mathbb{P}_x, \mathcal{P}^{i-p-1}, \mathcal{O}(p + 1)) \longrightarrow H^i(Q, \mathcal{P}, \mathcal{O}(1)) \longrightarrow H^i(Q - \mathbb{P}_x, \mathcal{P}, \mathcal{O}(1)) \longrightarrow \cdots$$

Now, we show that $\iota_*$ is split injective. Consider the following commutative diagram

$$
\begin{array}{ccc}
H^{i-p-1}(\mathbb{P}_x, \mathcal{P}^{i-p-1}, \mathcal{O}(p + 1)) & \longrightarrow & H^i(Q, \mathcal{P}, \mathcal{O}(1)) \\
\pi_* & \cong & \iota_* \rho^* \\
H^{i-p-1}(\mathbb{P}_x, \mathcal{P}^{i-p-1}) & \longrightarrow & H^i(Q, \mathcal{P})
\end{array}
$$

The diagram is commutative, since push-forward commutes with the mod-2 map. Recall from Theorem 5.10 that $\pi_*$ and $\iota_*$ are both split injective, and $s_2, s_1$ splits $\pi_*$, $\iota_*$ respectively. Therefore, $s_2 \circ s_1$ provides a splitting for $\iota_*: H^{i-p-1}(\mathbb{P}_x, \mathcal{P}^{i-p-1}, \mathcal{O}(p)) \to H^i(Q, \mathcal{P})$ as in Proposition 5.14. \qed
6. $\Gamma^*$-COHOMOLOGY: MULTIPLICATIVE STRUCTURE

In this subsection, we assume that all split quadrics $Q = Q_n$ are smooth over a base field $F$ of characteristic different from two (that is $S = \text{Spec}F$).

6.1. On the ring $H^*(\mathbb{P}^p, \Gamma^*, \mathcal{O} \oplus \mathcal{O}(1))$.

**Theorem 6.2.** Let $\mathbb{P}^p = \mathbb{P}^p_F$, where $F$ is a field of characteristic $\neq 2$. For any $p \geq 0$, we have a $\mathbb{Z} \oplus \mathbb{Z}/2$ graded $W(F)$-algebra isomorphism

$$H^*(\mathbb{P}^p, \Gamma^*, \mathcal{O} \oplus \mathcal{O}(1)) \cong W(F)[\xi, \alpha]/(1(F)\xi, \xi^{p+1}, \xi\alpha, \alpha^2) \quad \text{with} \quad \begin{cases} |\xi| = (1, 1) \\ |\alpha| = (p, p+1) \end{cases}$$

**Proof.** From the additive result of Fasel quoted as Theorem 5.2 above, we conclude that

$$H^i(\mathbb{P}^p, \Gamma^i, \mathcal{O}(k)) = \begin{cases} W(F)/I(F) = \mathbb{Z}/2 & \text{if } 1 \leq i \leq p \text{ and } k \equiv i \mod 2 \\ W(F) & \text{if } i = 0 \text{ and } k \text{ is even} \\ W(F) & \text{if } i = p \text{ and } k \equiv -p - 1 \mod 2 \\ 0 & \text{otherwise} \end{cases}$$

Let $\xi$ be the generator of $H^1(\mathbb{P}^p, \Gamma^1, \mathcal{O}(-1)) = \mathbb{Z}/2$, and let more generally $\xi_m \in H^m(\mathbb{P}^p, \Gamma^m, \mathcal{O}(-m))$ be the additive generator for $1 \leq m \leq p$. This generator is the image of 1 under $\mu^m$.

We first check that the product $\phi \xi_m = p^r \phi \cup \xi_m$ vanishes for an element $\phi \in W(F) = H^0(S, \Gamma^0)$ if and only if $\phi \in I(F)$. This claim is equivalent to the commutativity of the left square in the following diagram:

$$\begin{array}{cccc}
H^0(S, \Gamma^0) \times H^0(S, \Gamma^0) & \xrightarrow{(p^r, \mu^m)} & H^0(\mathbb{P}^p, \Gamma^0) \times H^m(\mathbb{P}^p, \Gamma^m, \mathcal{O}(-m)) & \xrightarrow{\cup} & H^0(\mathbb{P}^p, \Gamma^0) \times H^m(\mathbb{P}^p, \Gamma^m) \\
\downarrow & & \downarrow & & \downarrow \\
H^0(S, \Gamma^0) & \xrightarrow{\mu^m} & H^m(\mathbb{P}^p, \Gamma^m, \mathcal{O}(-m)) & \xrightarrow{\cup} & H^m(\mathbb{P}^p, \Gamma^m) \\
\end{array}$$

The composition of the lower maps is identified in Lemma 5.2 of [Fas13] with the map called “$\xi_m$” in loc. cit. (Note that said lemma simplifies under our assumption that the base $F$ is a field.) The commutativity of the outer diagram therefore follows from the ring structure on $H^*(\mathbb{P}^p, \Gamma^*)$. As both factors in the lower horizontal composition are isomorphisms, the commutativity of the left square follows.

A similar argument shows the commutativity of the following diagram:

$$\begin{array}{cccc}
H^0(S, \Gamma^0) \times H^0(S, \Gamma^0) & \xrightarrow{\cup} & H^1(\mathbb{P}^p, \Gamma^1, \mathcal{O}(-l)) \times H^m(\mathbb{P}^p, \Gamma^m, \mathcal{O}(-m)) & \xrightarrow{\cup} & H^0(\mathbb{P}^p, \Gamma^0) \times H^l+m(\mathbb{P}^p, \Gamma^{l+m}, \mathcal{O}(-l - m)) \\
\downarrow & & \downarrow & & \\
H^0(S, \Gamma^0) & \xrightarrow{\mu^{l+m}} & H^l+m(\mathbb{P}^p, \Gamma^{l+m}, \mathcal{O}(-l - m)) & & \\
\end{array}$$

This shows that $\xi_l \xi_m = \xi_{l+m}$ in the appropriate degrees, and hence that $\xi^m = \xi^m_1$ is an additive generator for $1 \leq m \leq p$. Note that $\xi^{p+1} = 0$ for degree reasons.

Finally, let $s: S \to \mathbb{P}^p$ be a rational point, and let $\alpha$ be the image of $1 \in W(F) = H^0(S, \Gamma^0)$ under the isomorphism $s_*: H^0(S, \Gamma^0) \to H^0(\mathbb{P}^p, \Gamma^0, \mathcal{O}(-p - 1))$. Again for degree reasons, $\xi \alpha = 0$ and $\alpha^2 = 0$. It remains to check that $\alpha \phi = \alpha \cup p^r \phi$ vanishes for an element $\phi \in W(F)$ if and only if $\phi = 0$. This is immediate from the identity $\alpha \cup p^r \phi = s_*(\phi)$, which in turn follows from the projection formula of [CF17]: for any $\psi \in W(F)$, this formula gives $s_* \psi \cup p^r \phi = s_*(\psi \cup s^*p^r \phi)$, and we conclude by taking $\psi = 1$ and noting that $p \circ s = \text{id}$.

$\square$
6.3. On the ring $H^\ast(Q_n, \mathbf{1}^\ast)$.

**Theorem 6.4.** We have an isomorphism of graded commutative rings

$$H^\ast(Q_n, \mathbf{1}^\ast) \cong \left\{ \begin{array}{ll}
\mathbb{Z}/2[\xi, \bar{\beta}]/(\xi^p + 1, \bar{\beta}^2 - \xi^p \bar{\beta}) & \text{if } n = 2p \text{ with } p \text{ is even} \\
\mathbb{Z}/2[\xi, \bar{\beta}]/(\xi^p + 1, \bar{\beta}^2) & \text{if } n = 2p + 1 \text{ or } n = 2p \text{ and } p \text{ is odd}
\end{array} \right.$$ 

Here, $|\xi| = 1$ and $|\bar{\beta}| = q$ where $q = p$ if $n = 2p$ is even and $q = p + 1$ if $n = 2p + 1$ is odd.

**Proof.** Assume $n = 2p + 1$ is odd. The additive structure can be obtained by the following short split exact sequence (use [BC09, Theorem 1.3])

$$0 \to H^{i-p-1}(\mathbb{P}_x, \mathbf{I}^{i-p-1}) \to H^i(Q_n, \mathbf{1}) \to H^i(\mathbb{P}_y, \mathbf{1}) \to 0$$

If $0 \leq i \leq p$, $H^i(Q_n, \mathbf{1}) \cong H^i(\mathbb{P}_x, \mathbf{I}) \cong \mathbb{Z}/2$ is an isomorphism since $H^{i-p-1}(\mathbb{P}_x, \mathbf{I}^{i-p-1}) = 0$. If $p + 1 \leq i \leq 2p + 1$, then $H^{i-p}(\mathbb{P}_x, \mathbf{I}^{i-p}) \cong H^i(Q_n, \mathbf{1}) \cong \mathbb{Z}/2$ is an isomorphism since $H^0(\mathbb{P}_y, \mathbf{1}) = 0$. In this case, we let $\xi$ be the generator corresponding to $H^1(Q_n, \mathbf{1}) \cong H^1(\mathbb{P}_x, \mathbf{I}) \cong \mathbb{Z}/2$ and $\bar{\beta}$ be generator corresponding to $H^{i-p+1}(\mathbb{Q}_n, \mathbf{I}^{i-p+1}) \cong H^0(\mathbb{P}_y, \mathbf{1}) \cong \mathbb{Z}/2$. By [DI07, Appendix A] and Milnor's conjecture, we obtain that $H^\ast(Q_n, \mathbf{1}^\ast) \cong \mathbb{Z}/2[\xi, \bar{\beta}]/(\xi^p + 1, \bar{\beta}^2)$. Assume $n = 2p$ is even. We have the following split exact sequence (use [BC09, Theorem 1.3])

$$0 \to H^{i-p}(\mathbb{P}_x, \mathbf{I}^{i-p}) \to H^i(Q_n, \mathbf{1}) \to H^i(\mathbb{P}_y, \mathbf{1}) \to 0$$

If $0 \leq i \leq p - 1$, $H^i(Q_n, \mathbf{1}) \to H^i(\mathbb{P}_y, \mathbf{1}) \cong \mathbb{Z}/2$ is an isomorphism since $H^{i-p}(\mathbb{P}_x, \mathbf{I}^{i-p}) = 0$. If $p + 1 \leq i \leq 2p$, $H^i(Q_n, \mathbf{1}) \cong H^0(\mathbb{P}_x, \mathbf{I}) \cong \mathbb{Z}/2$ is an isomorphism, since $H^0(\mathbb{P}_y, \mathbf{1}) = 0$. If $i = p$, $H^p(Q_n, \mathbf{1}) \cong H^0(\mathbb{P}_x, \mathbf{I}) \cong \mathbb{Z}/2$ and $\bar{\beta}$ be generator corresponding to $H^0(\mathbb{P}_y, \mathbf{1}) \to H^p(Q_n, \mathbf{1})$. By [DI07, Appendix A] and Milnor’s conjecture, we obtain that $H^\ast(Q_n, \mathbf{1}^\ast) \cong \mathbb{Z}/2[\xi, \bar{\beta}]/(\xi^p + 1, \bar{\beta}^2)$ if $p$ is odd and $H^\ast(Q_n, \mathbf{1}^\ast) \cong \mathbb{Z}/2[\xi, \bar{\beta}]/(\bar{\xi}^p - \bar{\xi})$ if $p$ is even. \hfill \square

6.5. On the ring $H^\ast(Q_n, \mathbf{1}^\ast, \mathcal{O} \oplus \mathcal{O}(1))$. To prove the following result, we use the projection formula [CF17, Corollary 3.2, Remark 3.6] and the base change theorem [CF17, Proposition 3.2], [AF16, Theorem 2.4.1] several times without explicitly mentioning it. This requires even more care than in the additive case when dealing with twists by line bundles, so we will be even more explicit about these whenever necessary. Recall also that $H^\ast(-, \mathbf{1}^\ast)$ is graded commutative, whereas Chow-Witt groups are $(-1)$-graded commutative, see e. g. [HW17, 2.4, 2.5].

**Theorem 6.6.** As before, let $S = \text{Spec}(F)$ and $F$ be a field of characteristic $\neq 2$. For any $n \geq 3$, we have a $\mathbb{Z} \oplus \mathbb{Z}/2$-graded $W(F)$-algebra isomorphism

$$H^\ast(Q_n, \mathbf{1}^\ast, \mathcal{O} \oplus \mathcal{O}(1)) \cong \left\{ \begin{array}{ll}
W(F)[\xi, \alpha, \beta]/(I(F)\xi, \xi^{p+1}, \xi\alpha + \xi\beta, \alpha^2 - u\beta^2, \alpha\beta) & \text{if } n = 2p \text{ and } p \text{ is even} \\
W(F)[\xi, \alpha, \beta]/(I(F)\xi, \xi^{p+1}, \xi\alpha + \xi\beta, \alpha^2, \beta^2) & \text{if } n = 2p \text{ and } p \text{ is odd} \\
W(F)[\xi, \alpha, \beta]/(I(F)\xi, \xi^{p+1}, \xi\alpha, \alpha^2, \beta^2) & \text{if } n = 2p + 1
\end{array} \right.$$ 

for some unit $u \in W(F)$. Here, generators of degrees $|\xi| = (1, \overline{1})$, $|\alpha| = (p, p + 1)$, and $|\beta| = (q, q + 1)$, where $q = p$ if $n$ is even and $q = p + 1$ if $n$ is odd. The reduction map

$$H^\ast(Q_n, \mathbf{1}^\ast, \mathcal{O} \oplus \mathcal{O}(1)) \to H^\ast(Q_n, \mathbf{1}^\ast)$$

is given by $\xi \mapsto \xi$, $\beta \mapsto \beta$ and $u \mapsto 1$.

$$\alpha \mapsto \left\{ \begin{array}{ll}
\xi^p & \text{if } n \text{ is even} \\
\bar{\xi}^p & \text{if } n \text{ is odd}
\end{array} \right.$$
Combining Theorem 3.22 and one of the main results of [HWXZ], we know that \( u = -1 \) if \( F = \mathbb{R} \), and more generally that for a subfield \( F \subset \mathbb{R} \), \( u \) maps to \(-1\) under the base change map \( W(F) \to W(\mathbb{R}) \cong \mathbb{Z} \). See Section 7 for more details.

**Proof.** Assume \( n = 2p \). By the short split exact sequence in the additive computation above,

\[
0 \longrightarrow H^{i-p}(\mathbb{P}^p_x, \mathbb{I}^{i-p}, \omega_x \otimes O(j)) \overset{(i_x)_*}{\longrightarrow} H^i(Q, \mathbb{I}, O(j)) \overset{(i_y)_*}{\longrightarrow} H^i(\mathbb{P}^p_y, \mathbb{I}, O(j)) \longrightarrow 0
\]

we obtain

\[
H^i(Q_{2p}, \mathbb{I}, O(j)) = \begin{cases} 
W(F)/I(I) = \mathbb{Z}/2 & \text{if } 1 \leq i \leq p \text{ and } i = j \\
W(F)/I(I) = \mathbb{Z}/2 & \text{if } p + 1 \leq i \leq 2p - 1 \text{ and } i = j + 1 \\
W(F) \oplus W(F) & \text{if } i = p \text{ and } j = p - 1 \\
W(F) & \text{if } i = 0 \text{ or } i = n \text{ and } j \text{ is even} \\
0 & \text{if otherwise}
\end{cases}
\]

where \( \omega_x := \omega_{\mathbb{P}^p_x} \otimes i_x^*(\omega_Q)^Y \cong \det(N_{\mathbb{P}^p_y/Q})^Y \cong O_{\mathbb{P}^p_y}(1 - p) \) (see Lemma 4.1).

Consider the following two cartesian diagrams:

\[
\begin{array}{ccc}
S & \overset{s_y}{\longrightarrow} & \mathbb{P}^p_x \\
\downarrow{s_{x'}} & & \downarrow{\iota_y} \\
\mathbb{P}^p_{x'} & \overset{\iota_{x'}}{\longrightarrow} & Q_n
\end{array}
\quad \quad
\begin{array}{ccc}
S & \overset{s_x}{\longrightarrow} & \mathbb{P}^p_x \\
\downarrow{s_{x'}} & & \downarrow{\iota_x} \\
\mathbb{P}^p_{x'} & \overset{\iota_{x'}}{\longrightarrow} & Q_n
\end{array}
\]

(6.7)

Let \( s := s_ys_{x'} = s_{x'}s_x \). Let \( s := s_{y}s_{y'} = s_{x}s_{x'} \). Let \( \xi \) be the generator corresponding to \( H^1(Q_n, \mathbb{I}, O(-1)) = \mathbb{Z}/2 \). Let \( \beta \) be the image of \( 1 = (1) \in W(F) \) under the following map

\[
W(F) \cong H^0(S, I^0) \overset{P^*_y}{\longrightarrow} H^0(\mathbb{P}^p_x, I^0) \cong H^0(\mathbb{P}^p_x, \mathbb{I}, \omega_x \otimes O(p - 1)) \overset{(i_x)_*}{\longrightarrow} H^p(Q_n, \mathbb{I}, O(p - 1))
\]

and similarly define \( \alpha = (i_{x'})_*(1) = (i_{x'})_*p^*_x(1) \in H^p(Q_n, \mathbb{I}, O(p - 1)) \). If \( i = p \) and \( j = p - 1 \), we have two splittings by the above two cartesian diagrams

\[
0 \longrightarrow H^0(\mathbb{P}^p_x, I^0) \overset{(i_y)_*}{\longrightarrow} H^0(Q, \mathbb{I}, O(p - 1)) \overset{(i_y)_*}{\longrightarrow} H^0(\mathbb{P}^p_y, \mathbb{I}, O(p - 1)) \longrightarrow 0
\]

\[\overset{(s_x)_*}{\cong} \]

\[
H^0(\mathbb{P}^p_{x'}, \mathbb{I}^0) \overset{(s_{x'})_*}{\cong} H^0(S, I^0)
\]

\[
0 \longrightarrow H^0(\mathbb{P}^p_{y'}, I^0) \overset{(i_y)_*}{\longrightarrow} H^0(Q, \mathbb{I}, O(p - 1)) \overset{(i_y)_*}{\longrightarrow} H^0(\mathbb{P}^p_{y'}, \mathbb{I}, O(p - 1)) \longrightarrow 0
\]

\[\overset{(s_{y'})_*}{\cong} \]

\[
H^0(\mathbb{P}^p_{y'}, \mathbb{I}^0) \overset{(s_{y'})_*}{\cong} H^0(S, I^0)
\]

It follows that \( \alpha \) and \( \beta \) form a base for the degree \( p \) part \( H^p(Q_n, \mathbb{I}, O(p - 1)) \) considered as a rank two free \( W(F) \)-module.

Note that the reduction map \( \pi \): \( H^i(Q, \mathbb{I}, O(j)) \to H^i(Q, \mathbb{I}) \cong \mathbb{Z}/2 \) is an isomorphism for \( 1 \leq i \leq p \) and \( i = j \), and for \( p + 1 \leq i \leq 2p - 1 \) and \( i = j + 1 \). To prove this claim, If \( 1 \leq i \leq p \), the map \( H^i(Q, \mathbb{I}, O(j)) \overset{(i_x)_*}{\longrightarrow} H^i(\mathbb{P}^p_y, \mathbb{I}, O(j)) \) is an isomorphism by the degree reason. If \( p + 1 \leq i \leq 2p - 1 \), the map \( H^{i-p}(\mathbb{P}^p_x, \mathbb{I}^{i-p}, \omega_x \otimes O(j)) \overset{(i_x)_*}{\longrightarrow} H^i(Q, \mathbb{I}) \) is an isomorphism by degree reason again. Consider the
By Fasel’s computation on projective spaces (see Section 5.1 and [Fas13, Lemma 5.2]), the lower arrow of the left square is an isomorphism for $1 \leq i \leq p$ and $i = j$, and the upper arrow of the right square is an isomorphism for $p + 1 \leq i \leq 2p - 1$ and $i = j + 1$. The claim follows.

As in the case of projective spaces, we obtain the relations $I(F)\xi = \xi^{p+1}$ and $\xi(\alpha + \beta)$ in the ring structure by comparing to the ring $H^*(Q_n, \mathbb{P}^r)$. However, other relations can not be obtained by this way.

We form the following diagrams. Mostly, we use $L_x = L_y = L_{x'} = L_{y'} = O(p - 1)$.

\[
\begin{align*}
H^0(S, \mathbb{P}^p, s_y^*(\omega_x \otimes t_y^*\mathbb{L}_y)) \times H^0(S, \mathbb{P}^p, s_y^*(\omega_y \otimes t_y^*\mathbb{L}_y)) & \xrightarrow{(s_y^*, s_y^*)} H^0(S, \mathbb{P}^p, s_y^*(\omega_x \otimes t_y^*\mathbb{L}_y)) \times H^0(S, \mathbb{P}^p, s_y^*(\omega_y \otimes t_y^*\mathbb{L}_y)) \\
\cup & \\
H^0(S, \mathbb{P}^p, s_y^*(\omega_x \otimes t_y^*\mathbb{L}_y)) & \xrightarrow{(t_y^*)_*} H^0(S, \mathbb{P}^p, s_y^*(\omega_y \otimes t_y^*\mathbb{L}_y))
\end{align*}
\]

\[
\begin{align*}
H^p(Q, \mathbb{P}^p, \mathbb{L}_{x'}) \times H^p(Q, \mathbb{P}^p, \mathbb{L}_{y'}) & \xrightarrow{(t_y^*)_*} H^p(Q, \mathbb{P}^p, \mathbb{L}_{x'}) \\
\cup & \\
H^p(Q, \mathbb{P}^p, \mathbb{L}_{x'}) & \xrightarrow{(t_y^*)_*} H^p(Q, \mathbb{P}^p, \mathbb{L}_{y'})
\end{align*}
\]

\[
\begin{align*}
H^0(S, \mathbb{P}^p, s_y^*(\omega_y \otimes t_y^*\mathbb{L}_y)) \times H^0(S, \mathbb{P}^p, s_y^*(\omega_x \otimes t_y^*\mathbb{L}_x)) & \xrightarrow{(s_y^*, s_y^*)} H^0(S, \mathbb{P}^p, s_y^*(\omega_y \otimes t_y^*\mathbb{L}_y)) \times H^0(S, \mathbb{P}^p, s_y^*(\omega_x \otimes t_y^*\mathbb{L}_x)) \\
\cup & \\
H^0(S, \mathbb{P}^p, s_y^*(\omega_y \otimes t_y^*\mathbb{L}_y)) & \xrightarrow{(t_y^*)_*} H^0(S, \mathbb{P}^p, s_y^*(\omega_x \otimes t_y^*\mathbb{L}_x))
\end{align*}
\]

\[
\begin{align*}
H^p(Q, \mathbb{P}^p, \mathbb{L}_{y'}) \times H^p(Q, \mathbb{P}^p, \mathbb{L}_{x'}) & \xrightarrow{(t_y^*)_*} H^p(Q, \mathbb{P}^p, \mathbb{L}_{y'}) \\
\cup & \\
H^p(Q, \mathbb{P}^p, \mathbb{L}_{y'}) & \xrightarrow{(t_y^*)_*} H^p(Q, \mathbb{P}^p, \mathbb{L}_{x'})
\end{align*}
\]

Take $(a, b) \in H^0(\mathbb{P}^p, \mathbb{P}^p, \mathbb{L}_{y'} \otimes t_y^*\mathbb{L}_y) \times H^0(\mathbb{P}^p, \mathbb{P}^p, \mathbb{L}_{x'} \otimes t_y^*\mathbb{L}_x)$. By the left cartesian diagram (6.8), the diagram (6.8) is commutative by the following identities

\[
(t_x^*)_a \cup (t_y)_b = (t_x^*)_a \cup (t_y)_b
\]

\[
= (t_x^*)_a \cup (t_y)_b
\]

where the projection formula in the last equality does produce the trivial sign $(-1)^{0p} = 1$. On the other hand,

\[
(t_x^*)_a \cup (t_y)_b = (-1)^{p^2} (t_y)_b \cup (t_x^*)_a
\]

\[
= (-1)^{p^2} (t_y)_b \cup (t_x^*)_a
\]

\[
= (-1)^{p^2} (t_y)_b \cup (t_x^*)_a
\]

This shows that $(t_x^*)_a \cup (t_y)_b$ differs from $(t_y)_a \cup (t_y)_b$ by a sign $(-1)^{p^2}$.
If \( n = 2p \) with odd \( p \), the map \((\iota_x)_* : \text{H}^0(\mathbb{P}^p, \mathbb{P}^0) \to \text{H}^0(Q, \mathbb{P}, \mathcal{O}(p-1))\) coincides with the map \((\iota_y)_* : \text{H}^0(\mathbb{P}^p, \mathbb{P}^1) \to \text{H}^0(Q, \mathbb{P}, \mathcal{O}(p-1))\) by the following \( A^1 \)-homotopy

\[
\mathbb{P}^p \times A^1 \to Q_{2p}
\]

by

\[
[a_0, a_1, \ldots, a_p] \mapsto [(1-t)a_0, ta_1, (1-t)a_1, -ta_0, \ldots, (1-t)a_{p-1}, ta_p, (1-t)a_p, -ta_{p-1}]
\]

from [DI07, Lemma A.7]. A similar reasoning shows that \((\iota_x')_* = (\iota_y')_*\).

The following identities show the relation \( \alpha^2 = 0 \). Similarly, we obtain the relation \( \beta^2 = 0 \).

\[
(\iota_x)_*1 \cup (\iota_x)_*1 = (\iota_x)_*1 \cup (\iota_y)_*1 \quad \text{by } A^1\text{-homotopy}
\]

\[
= (\iota_x)_*(1 \cup (\iota_y)_*1) \quad \text{by projection formula}
\]

\[
= 0 \quad \text{by the localization sequence}
\]

(6.10)

The reduction map sends \( \xi \) to \( \bar{\xi} \), \( \beta \) to \( \bar{\beta} \) and \( \alpha \) to \( \bar{\xi}p - \bar{\beta} \). This can be seen by comparing the corresponding split exact localization sequences for \( \mathbb{I}^1 \)-cohomology and for \( \mathbb{I}^j \)-cohomology. (The third setting \( \alpha \) to \( \xi^p - \beta \) may be obtained as in [DI07, Proof of Theorem A.10 line 4 (or Lemma A.9)].)

If \( n = 2p \) with even \( p \), the maps \((\iota_x)_* \) and \((\iota_y)_* : \text{H}^0(\mathbb{P}^p, \mathbb{P}^0) \to \text{H}^0(Q, \mathbb{P}, \mathcal{O}(p-1))\) coincide by the \( A^1 \)-homotopy

\[
\mathbb{P}^p \times A^1 \to Q_{2p}
\]

\[
[a_0, a_1, \ldots, a_p] \mapsto [a_0, 0, (1-t)a_2, ta_1, (1-t)a_1, -ta_2, \ldots, (1-t)a_{p-1}, ta_p, (1-t)a_p, -ta_{p-1}]
\]

from [DI07, Lemma A.7]. A similar reasoning shows that \((\iota_x)_* = (\iota_y)_*\).

By the argument (6.10) and the above \( A^1 \)-homotopy, we deduce the relation \( \alpha \beta = 0 \). By the above \( A^1 \)-homotopy, and by diagrams (6.8), we obtain \( \alpha^2 \) corresponds to a generator in \( W(F) \). (Here we use that the lower horizontal maps in the two diagrams are isomorphism, which follows from the previous additive computations.) The argument for \( \beta^2 \) is similar, using (6.9). This implies that \( \alpha^2 = u\beta^2 \) for some unit \( u \in W(F) \). As in the case of odd \( p \), the reduction map sends \( \xi \) to \( \bar{\xi} \), \( \beta \) to \( \bar{\beta} \) and \( \alpha \) to \( \bar{\xi}^p - \bar{\beta} \).

When \( n = 2p + 1 \), the additive computation shows that

\[
\text{H}^i(Q_{2p+1}, \mathbb{P}, \mathcal{O}(j)) = \begin{cases} 
W(F)/I(F) = \mathbb{Z}/2 & \text{if } 1 \leq i \leq p \text{ and } i = j \\
W(F)/I(F) = \mathbb{Z}/2 & \text{if } p + 2 \leq i \leq 2p + 1 \text{ and } i = j + 1 \\
W(F) & \text{if } i = p \text{ and } j = p + 1 \\
W(F) & \text{if } i = p + 1 \text{ and } j = p \\
W(F) & \text{if } i = j = 0 \text{ or if } i = 2p + 1, j = 1 \\
0 & \text{if otherwise}
\end{cases}
\]

Let \( \xi \) be the generator corresponding to \( \text{H}^1(Q_n, \mathbb{I}^1, \mathcal{O}(1)) = \mathbb{Z}/2 \). Let \( \alpha \) be the generator corresponding to \( \text{H}^p(Q_n, \mathbb{P}, \mathcal{O}(p+1)) = W(F) \), and let \( \beta \) be the generator corresponding to \( \text{H}^{p+1}(Q_n, \mathbb{P}^{p+1}, \mathcal{O}(p)) = W(F) \). The relations \( I(F)\xi, \xi^{p+1} \) and \( \alpha \) follow similarly from the even case by using the reduction map. The relations \( \alpha^2 = \beta^2 = 0 \) follow from comparing the degrees. The reduction map sends \( \xi \) to \( \bar{\xi} \), \( \alpha \) to \( \bar{\xi}^p \) and \( \beta \) to \( \bar{\beta} \).

### 7. Real realization

We now compare Theorem 6.6 with Theorem 3.22. They both apply to the split quadrics \( Q_n \) with \( n = (p, p) \) or \( n = (p+1, p) \) and \( n \geq 3 \), while the latter theorem also holds more generally. Now if \( W(F) \cong \mathbb{Z} \), e.g. if \( F = \mathbb{R} \), then there is an explicit ring isomorphism from the graded commutative ring \( \text{H}^*(Q_n, \mathbb{P}, \mathcal{O} \oplus \mathcal{O}(1)) \) in algebraic geometry to the graded commutative ring \( \text{H}^*(X(\mathbb{R})^{an}, \mathbb{Z}) \) topology, given by mapping \( \alpha \) to \( -\beta \) if \( n = 2p \), and the identity on all other generators. Moreover, there is another obvious isomorphism between the corresponding rings with \( \mathbb{Z}/2 \)-coefficients, and both isomorphisms are compatible via the reduction maps. Nothing of this a coincidence, of course. We will put this isomorphism in perspective, continuing the discussion started in section 3.15.
As observed in that section, the strategy of the previous proof concerning $I^*$-cohomology of the algebro-geometric $Q_n$ also leads to a computation of integral singular cohomology $H^*(Q_n, Z \oplus Z(1))$ for the topological real quadric $Q_n$. Even more is true. We recall from [Jac17] that there is a real realization functor

$$H^*(X, I^*) \to H^*(X(\mathbb{R})^{an}, Z)$$

for $X$ a smooth variety over $\mathbb{R}$, which is known to be an isomorphism in some degrees in special cases, see loc. cit. and previous work by Fasel and others.

By recent work of the authors and Matthias Wendt [HWXZ], we have a much stronger result for smooth cellular varieties.

**Theorem 7.1.** For any smooth variety $X$ over $\mathbb{R}$, the real realization morphism

$$H^*(X, I^*) \to H^*(X(\mathbb{R})^{an}, Z)$$

is a ring homomorphism. If $X$ is cellular, then this is an isomorphism.

*Proof.* See [HWXZ]. □

This applies in particular to the split quadrics $Q_n$. The preprint also studies twisted coefficients and establishes a generalization of Theorem 7.1, replacing $I^*$ and $Z$ by $(I^*, \mathcal{O} \oplus \mathcal{O}(1))$ and $Z \oplus Z(1)$, respectively. If more generally $F$ is a subfield of $\mathbb{R}$, then the base change composed with real realization is still a ring homomorphism. Finally, we deduce that the description of Theorem 6.6 extends to all quadrics covered by Theorem 3.22, in particular $\mathbb{P}^1 \times \mathbb{P}^1$. However, the Picard group is larger in this case, so we haven’t computed cohomology with respect to all possible twists for $\mathbb{P}^1 \times \mathbb{P}^1$.

8. **Chow-Witt rings**

From [KM91], [DI07], we have the following computation:

**Theorem 8.1.** (Karpenko-Merkurjev, Dugger-Isaksen) The Chow rings of the split quadrics $Q_n$ defined above are as follows:

$$\text{CH}^*(Q_n) \cong \begin{cases} 
\mathbb{Z}[x, y]/(x^{p+1} - 2xy, y^2 - x^p y) & \text{if } n = 2p \text{ with } p \text{ even} \\
\mathbb{Z}[x, y]/(x^{p+1} - 2xy, y^2) & \text{if } n = 2p \text{ with } p \text{ odd} \\
\mathbb{Z}[x, y]/(x^{p+1} - 2y, y^2) & \text{if } n = 2p + 1
\end{cases}$$

where $|x| = 1$ and $|y| = p$ if $n$ is even, $|y| = p + 1$ if $n$ is odd.

*Proof.* See [DI07, Theorem A.4, Theorem A.10]. □

In particular, we see that $\text{CH}^*(Q_{p,p})$ and $\text{CH}^*(Q_{p,p+1})$ have no 2-torsion. Hence to compute $\text{CH}^*(Q_{p,p})$ and $\text{CH}^*(Q_{p,p+1})$, we may apply part (1) of Proposition 2.11 of [HW17] which we briefly recall.

**Proposition 8.2.** Let $F$ be a perfect field of characteristic unequal to 2, and let $X$ be a smooth scheme over $F$. The canonical ring homomorphism

$$\text{CH}^*(X) \to H^*(X, I^*) \times_{\text{Ch}^*(X)} \ker \partial$$

induced from the cartesian square defining Milnor-Witt $K$-theory is always surjective, with $\partial: H^n(X) \to H^{n+1}(X, I^{n+1})$ as in loc. cit. The canonical ring homomorphism is injective if $\text{CH}^*(X)$ has no non-trivial 2-torsion. These claims also apply to twisted coefficients.

The statement relies on the isomorphism $H^n(X, I^n) \cong \text{Ch}^n(X)$.

**Lemma 8.3.** In terms of our chosen generators, the isomorphism of commutative $\mathbb{Z}/2$-algebras $\text{Ch}^n(X) \cong H^n(Q_n, I^n)$ has the form

$$\begin{align*}
\bar{x} & \mapsto \bar{\xi} \\
\bar{y} & \mapsto \begin{cases}
\beta + \delta \xi & \text{if } n \text{ is even, where } \delta \in \{0, 1\} \\
\beta & \text{if } n \text{ is odd}
\end{cases}
\end{align*}$$
Proof. This is obvious, given the degree constraints and that we are working over \( \mathbb{Z}/2 \).

Putting everything together, we obtain our main result:

**Theorem 8.4.** Let \( Q_n \) be the split quadric over a field \( F \) with \( \text{char}(F) \neq 2 \) as above, and \( \mathcal{L} \) a line bundle on \( Q_n \). Then the graded \( GW(F) \)-module \( \widehat{CH}^\bullet(Q_n, \mathcal{L}) \) is given by

\[
\widehat{CH}^\bullet(Q_n, \mathcal{L}) \cong H^\bullet(Q_n, \mathcal{I}^\bullet(\mathcal{L})) \times_{\text{CH}^\bullet(Q_n)} \ker \partial \mathcal{L}
\]

with \( \ker \partial \mathcal{L} \) given as follows:

for \( \mathcal{L} = \mathcal{O} \):

\[
\begin{align*}
\mathbb{Z}(2x, 2y, xy, x^2, y^2) & \quad \text{when } n = 2p, \text{ and } p \text{ is even} \\
\mathbb{Z}(2x, x^2, x^p, y) & \quad \text{when } n = 2p, \text{ and } p \text{ is odd} \\
\mathbb{Z}(2x, x^2, y) & \quad \text{when } n = 2p + 1, \text{ and } p \text{ is even} \\
\mathbb{Z}(2x, 2y, xy, x^2, x^p) & \quad \text{when } n = 2p + 1, \text{ and } p \text{ is odd}
\end{align*}
\]

where \( \mathbb{Z}(\text{elements}) \) denotes the subring of \( CH^\bullet(Q_n) \) generated by the specified elements (see Theorem 8.1). The following four twisted cases are described as submodules of \( CH^n(Q_n) \) over the respective four rings above.

For \( \mathcal{L} = \mathcal{O}(1) \):

\[
\begin{align*}
\{ x \cdot \mathbb{Z}(x^2, xy) + y \cdot \mathbb{Z} + \ker(\text{mod } 2) \} & \quad \text{when } n = 2p, \text{ and } p \text{ is even} \\
\{ x \cdot \mathbb{Z}(x^2, x^p, y) + \ker(\text{mod } 2) \} & \quad \text{when } n = 2p, \text{ and } p \text{ is odd} \\
\{ x \cdot \mathbb{Z}(x^2, y, x^{p-1}y) + \ker(\text{mod } 2) \} & \quad \text{when } n = 2p + 1, \text{ and } p \text{ is even} \\
\{ x \cdot \mathbb{Z}(x^2, x^p, xy, x^{p-1}y) + y \cdot \mathbb{Z} + \ker(\text{mod } 2) \} & \quad \text{when } n = 2p + 1, \text{ and } p \text{ is odd}
\end{align*}
\]

Proof. Everything except the computation of \( \ker \partial \mathcal{L} \) has been established already. For the latter, one uses that in the notations of [HW17] we have \( \partial \mathcal{L} = \beta \circ \text{mod } 2 \) and \( \ker \beta = \text{Imp}_\mathcal{L} \). Our computation of the \( W(F) \)-algebra \( H^\bullet(Q_{p,q}, \mathcal{I}^\bullet) \) and the reduction map \( \rho \) (using Lemma 8.3) allows us to completely compute the image of the latter. Note that for \( n = 2p \) and \( p \) even this is independent of the value of \( \delta \), using the relation \( \xi^{p+1} = 0 \) both in the untwisted and twisted case and moreover the relation \( y^2 = x^p y \) in the twisted case. From this we easily deduce the kernels of \( \partial \mathcal{L} \) and \( \beta \) both for \( \mathcal{L} = \mathcal{O} \) and \( \mathcal{L} = \mathcal{O}(1) \).

For a more concise description of the \( (\mathbb{Z} \times \mathbb{Z}/2) \)-graded \( GW(F) \)-algebra \( \widehat{CH}^\bullet(Q_n, \mathcal{O} \oplus \mathcal{O}(1)) \), we introduce an artificial \( \mathbb{Z}/2 \)-grading on \( CH^\bullet(Q_n) \) and \( CH^\bullet(Q_n) \) as follows:

\[
\text{"} CH^\bullet(Q_n, \mathcal{O} \oplus \mathcal{O}(1)) \text{"} := CH^\bullet(Q_n)[\tau]/(\tau^2 - 1)
\]

\[
\text{"} CH^\bullet(Q_n, \mathcal{O} \oplus \mathcal{O}(1)) \text{"} := CH^\bullet(Q_n)[\tau]/(\tau^2 - 1)
\]

In this notation, the same arguments as in the proof above yield the following description of the Chow-Witt ring:

\[
\widehat{CH}^\bullet(Q_n, \mathcal{O} \oplus \mathcal{O}(1)) \cong H^\bullet(X, \mathcal{I}^\bullet, \mathcal{O} \oplus \mathcal{O}(1)) \times_{\text{CH}^\bullet(X, \mathcal{O} \oplus \mathcal{O}(1))} \mathbb{Z}(2\tau, 2x, 2y, x\tau, x^p\tau^{p+1}, y\tau^{q+1}),
\]

where \( \mathbb{Z}(\text{elements}) \) denotes the subring of \( CH^\bullet(Q_n, \mathcal{O} \oplus \mathcal{O}(1)) \) generated by the specified elements.

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