LOCAL INVARIANTS OF NON-COMMUTATIVE TORI

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Abstract. We consider a generic curved non-commutative torus extending the notion of conformally deformed non-commutative torus from [7]. In general, a curved non-commutative torus is no longer represented by a spectral triple, not even by a twisted spectral triple. Therefore, the geometry of this manifold is governed by a positive second order differential operator (Laplace-Beltrami operator) rather than a first order differential operator (Dirac operator). For this manifold, we prove an asymptotic expansion of the heat semi-group generated by Laplace-Beltrami operator and provide an algorithm to compute the local invariants which appear as coefficients in the expansion. This allows to extend the results of [7], [5], [12] (beyond conformal case and/or for multi-dimensional tori).

1. Introduction

We begin by reviewing the classical (commutative) roots of our work, and then move to the non-commutative generalisation prompted by [7]. Finally, we explain our results for the non-commutative torus.

1.1. Minakshisundaram-Plejel theorem and local invariants in the classical geometry.

For a $d$–dimensional Riemannian manifold $(X, g)$, there exists a natural first order differential operator $D_g$ on the space of forms called Hodge-de Rham operator. Its square $D_g^2$ is the Hodge-Laplace operator (denoted further by $\Delta_g$) and its component acting on 0 order forms being the Laplace-Beltrami operator (also denoted by $\Delta_g$) [25]. The heat semi-group is now defined by the formula

$$t \rightarrow e^{-t\Delta_g}, \quad t > 0.$$  

If $X$ is compact, then the resolvent of the Laplace-Beltrami operator $\Delta_g$ is compact. Hence, $e^{-t\Delta_g}$ is compact for $t > 0$. In fact, it happens that $e^{-t\Delta_g}$ belongs to the trace class for $t > 0$.

In his seminal work [29], Weyl proved that, for a compact manifold,

$$\lim_{t \downarrow 0} (4\pi t)^{\frac{d}{2}} \text{Tr}(e^{-t\Delta_g}) = \text{Vol}(X), \quad t \downarrow 0.$$ (1.1)

Following Weyl’s work, it became an established custom to measure various geometric (and often topological) quantities associated with a Riemannian manifold $X$ in terms of its heat semi-group expansion $t \rightarrow e^{-t\Delta_g}, \quad t > 0$. The mere existence of such expansion is a famous theorem of Minakshisundaram and Plejel (among all approaches to that theorem, a particularly detailed account is given in [25]; even though Theorem 3.24 there concerns only a special case $f = 1$, the proof of the formula stated below in the general case is very similar).
Thus, for every \( f \in C^\infty(X) \), the Minakshisundaram-Plejel theorem asserts an existence of an asymptotic expansion

\[
\text{Tr}(Mf e^{-t\Delta}) \approx (4\pi t)^{-\frac{d}{2}} \sum_{k \geq 0 \mod 2} a_k(f) t^\frac{k}{2}, \quad t \downarrow 0.
\]

Here, \( d \) is the dimension of \( X \) and \( Mf : L^2(X) \to L^2(X) \) is the operator of pointwise multiplication by \( f \). Moreover, there exist functions \( A_k \in C^\infty(X) \) such that

\[
a_k(f) = \int_X A_k \cdot f \, d\text{vol}_g, \quad k \geq 0, \quad k = 0 \mod 2,
\]

where \( \text{vol}_g \) is the standard volume element on \( X \) given in local coordinates by the formula

\[
d\text{vol}_g = (\det(g))^{\frac{1}{2}}(x) dx.
\]

Here, the summation goes over even \( k \) only because the manifold is assumed not to have a boundary. For manifolds with boundary, one should also include the terms with odd \( k \).

An easy computation shows that \( A_0 = 1 \), which is consistent with (1.1). Further computations (see e.g. Proposition 3.29 in [25]) show that

\[
A_2 = \frac{1}{6} R,
\]

where \( R \) is the scalar curvature of \( (X, g) \). In particular, \( a_2(1) \) is the Einstein-Hilbert action (see e.g. [11]). Further, the elements \( a_k, k > 2 \) are related to local invariants of higher order [25].

Note that \( a_0 \) extends to a normal state \( h \) on \( L^\infty(X) \) by the obvious formula

\[
h(f) = \int_X f \, d\text{vol}_g, \quad f \in L^\infty(X).
\]

Equation (1.3) can be re-written as

\[
a_k(f) = h(A_k \cdot f), \quad f \in C^\infty(X).
\]

This paper aims to find suitable extensions of the Minakshisundaram-Plejel theorem (and, consequently, of the Weyl theorem — see formula (1.1)) for non-commutative tori with generic, non-flat, metric tensor. In the spectral geometry of Riemannian manifolds, the local invariants (such as Riemannian curvature) can be detected in the asymptotic expansion of the heat semigroup with respect to the Laplace-Beltrami operator. The paradigm of Non-commutative Geometry is to define local invariants via the asymptotic expansion of a heat semi-group associated to the Laplace-Beltrami operator.

1.2. Local invariants in the non-commutative geometry. This grand program began in [7] (published only in 2011, but the main concepts and techniques were developed yet in the 1990’s), where special Riemannian metric (conformal deformations of a flat one) on 2–dimensional non-commutative manifolds was considered. The authors of [7] proved that Euler characteristic of such manifold is 0 by means of Gauss-Bonnet theorem (recall that the classical Gauss-Bonnet theorem asserts that Euler characteristic of the 2–dimensional Riemannian manifold equals to the average of its scalar curvature). Subsequently, the scalar curvature (for the conformal deformation of the 2–dimensional non-commutative torus) was explicitly computed in [5] and [11] and, later, the term \( a_4 \) (the first place where the
Riemann curvature tensor manifests itself beyond the scalar curvature) was further computed in \cite{4} (intermediate computations include about a million terms!).

We now briefly restate the whole program as it can be surmised from \cite{7}. Relevant definitions concerning non-commutative torus $\mathbb{T}^d_\theta$ are given in Subsection \ref{1.3} below.

\textbf{Problem 1.1.} Let $g$ be a Riemannian metric on the non-commutative torus and let $\Delta_g$ be the Laplace-Beltrami operator.

(a) prove, for every $x \in C^\infty(\mathbb{T}^d_\theta)$, the existence of the asymptotic
\[ \text{Tr}(\lambda_t(x)e^{-t\Delta_g}) \sim t^{-\frac{d}{2}} \sum_{k=0}^{\infty} t^k a_k(x) \quad t \downarrow 0. \]

Here, Tr denotes the classical trace on the ideal $\mathcal{L}_1(L_2(\mathbb{T}^d_\theta))$ and (b) provide explicit formulae for the functionals $x \to a_k(x)$, $k \geq 0$.

\subsection*{1.3. Non-commutative Riemannian geometry}

Let $d \geq 2$ and let $\theta \in M_d(\mathbb{R})$ be anti-symmetric. Let $L_\infty(\mathbb{T}^d_\theta)$ be the (von Neumann algebra of a) non-commutative torus defined with the help of the matrix $\theta$. It is represented on the Hilbert space $L_2(\mathbb{T}^d_\theta)$ via left regular representation $\lambda_t$. This algebra can be viewed as the weak closure of the algebra $C^\infty(\mathbb{T}^d_\theta)$ (as introduced in \cite{11}). It is equipped with a faithful tracial state $\tau$, which happens to be normal. All these notions and notations are fully explained in Section \ref{2}.

Ha and Ponge \cite{14} presented a general notion of Riemannian metric $g$ on the non-commutative torus which includes the conformally deformed metric considered in \cite{7} as a special case. Namely, Riemannian metric $g$ on the non-commutative torus is simply a positive element in $\text{GL}_d(C^\infty(\mathbb{T}^d_\theta))$ (the group of invertible $d \times d$ matrices with coefficients in $C^\infty(\mathbb{T}^d_\theta)$) such that the elements $g_{ij}$ and $(g^{-1})_{ij}$ are self-adjoint for all $1 \leq i, j \leq d$.

A von Neumann algebra corresponding to a curved non-commutative torus is the same as for the flat non-commutative torus. It is still represented on the same Hilbert space $L_2(\mathbb{T}^d_\theta)$ via left regular representation. The only difference between flat and non-flat Hilbert spaces is the inner product on $L_2(\mathbb{T}^d_\theta)$ given now by the formula
\[ (u, v)_{\nu} = \tau(u^* \nu v), \quad u, v \in L_2(\mathbb{T}^d_\theta). \]

Here, $\nu \in C^\infty(\mathbb{T}^d_\theta)$ given in formula \cite{31} below should be thought of as a "square root of the determinant" of the metric tensor $g \in \text{GL}_d(C^\infty(\mathbb{T}^d_\theta))$.

On the Hilbert space $L_2(\mathbb{T}^d_\theta)$ (equipped with the inner product $\langle \cdot, \cdot \rangle_\nu$) we define a Laplace-Beltrami operator $\Delta_g$ by setting \cite{14}
\[ \Delta_g = \lambda_t(\nu^{-1}) \sum_{i,j=1}^{d} D_i \lambda_t(\nu^{-1}(g^{-1})_{ij} \nu), \]

Here, $\{D_i\}_{i=1}^d$ are "partial derivations" on $C^\infty(\mathbb{T}^d_\theta)$.

We view this operator as a starting point for Riemannian geometry on the non-commutative torus since it dualises the notion of Riemannian metric in the same spirit as in the commutative case.

It should be noted that the element $e^{-t\Delta_g}$ belongs (see e.g. \cite{21}) to the trace ideal $\mathcal{L}_1(L_2(\mathbb{T}^d_\theta))$ that is to the class of all bounded operators on $L_2(\mathbb{T}^d_\theta)$ whose singular value sequence is summable.
1.4. **Main result.** Our main result stated below provides a resolution to the Problem 1.1 (a), (b) above in the most general situation.

**Theorem 1.2.** Let \( d \geq 2 \) and \( 0 \leq g \in \text{GL}_d(C^\infty(\mathbb{T}^d_\theta)) \) be such that the elements \( g_{ij} \) and \( (g^{-1})_{ij} \) are self-adjoint for all \( 1 \leq i, j \leq d \). For every \( x \in L_\infty(\mathbb{T}^d_\theta) \), there exists an asymptotic expansion

\[
\text{Tr}(\lambda(x)e^{-t\Delta}) \sim t^{-\frac{d}{2}} \sum_{k \geq 0} t^\frac{k}{2} \tau(x \cdot \nu^{-\frac{1}{2}} I_k \nu^\frac{1}{2}), \quad t \downarrow 0.
\]

Here, \( I_k \) is given in Notation 4.7 and the algorithm to compute it is presented in Section 4.

1.5. **Connections to earlier works.** In existing literature such theorems are proved by means of pseudo-differential calculus on the non-commutative tori [3] (developed for toric manifolds in [20]). An alternative approach was introduced in [15, 16] where the case of almost commutative torus was considered. The approach of [15] is based on Duhamel formula. The resulting expression in [15, 16] for the coefficients appears to be the same as the ones in [5, 11, 12].

In our approach, we avoid pseudo-differential calculus or Duhamel formula replacing them with repeated resolvent identity and borrowing methods from non-commutative harmonic analysis.

The outcomes of the presented approach are of potentially wider applicability. Its main advantages are multifold:

- Theorem 1.2 holds for every \( x \in L_\infty(\mathbb{T}^d_\theta) \), not just for a smooth \( x \);
- Theorem 1.2 holds for an arbitrary metric tensor \( g \in \text{GL}_d(C^\infty(\mathbb{T}^d_\theta)) \) and not just for a conformal deformation of a flat noncommutative torus;
- We supply the formulae for all \( I_k \), \( k \geq 0 \), not just for \( k = 0, 2, 4 \);
- Our approach is designed to be applicable to other important examples where pseudo-differential calculus is unavailable e.g. non-commutative spheres;

We caution the reader that Theorem 1.2 is not a generalisation of [5] et al. In fact, in [5] a version of Theorem 1.2 is taken as a starting point and the main focus of [5, 11, 4, 17] is on representing the element \( I_2 \) (or \( I_k \)) in terms of multiple operator integrals.

Computation of \( I_0 \) (note that the algorithm in Section 4 yields \( I_0 = \nu \)) is, in fact, related to Connes Trace Theorem [2] (if we ignore the fact we do not have a bona fide spectral triple). Indeed, the equality

\[
\text{Tr}(\lambda(x)e^{-t\Delta}) = t^{-\frac{d}{2}} \tau(x\nu) + O(t^{-\frac{d}{2}}), \quad t \downarrow 0,
\]

is expected to imply (if \( \Delta \) is replaced by \( D^2 \) for some Dirac-type operator \( D \), then such an implication is known to hold [27]) that

\[
\varphi(\lambda(x)(1 + \Delta)^{-\frac{d}{2}}) = c_d \tau(x\nu)
\]

for every normalised trace on \( L_{1,\infty} \) (the principal ideal generated by the harmonic sequence). However, a Laplace-Beltrami operator \( \Delta_g \) introduced above is not a square of any Dirac-type operator (or, at least, such a Dirac-type operator \( D \) is not yet constructed). Nevertheless, [15, 6] holds in full generality [21].
1.6. Acknowledgements. We thank Professor Connes for supplying us with "little lemma" (see Lemma 5.4 and Theorem 5.5 which radically shortened and streamlined our proof. We thank our colleagues R. Ponge (for explaining to us his approach to the Laplace-Beltrami operator in [14] and for drawing our attention to [24]), B. Iochum (for interest to our work and detailed comparison with [15, 16]), Y. Liu (for explaining to us the interplay between analytical and geometrical ideas), M. Lesch (for discussing [17] with us). We also thank N. Azamov, A. Ber and E. McDonald for verification of our proofs and supplying numerous suggestions which improved the exposition.

2. Preliminaries

Everything in this section is folklore. We refer the reader to [23] for deformation quantization (which includes non-commutative torus as a special case), to [26] and [30] for Sobolev spaces on the non-commutative torus and to [14], [22] for various related information.

Let $\theta \in M_d(\mathbb{R})$ be an anti-symmetric matrix. Let $A_\theta$ be a $\ast-$algebra generated by elements $(U_k)_{1 \leq k \leq d}$ of infinite order satisfying the conditions

$$U_k U_l = e^{i\theta_{kl}} U_l U_k, \quad U_k U_k^* = U_k^* U_k = 1.$$ 

Natural Hamel basis in $A_\theta$ is $(e_n)_{n \in \mathbb{Z}^d},$ 

$$e_n = U_1^{m_1} U_2^{m_2} \cdots U_d^{m_d}, \quad n \in \mathbb{Z}^d.$$ 

Note that 

$$e_m e_n = e^{-i \sum_{j<k} \theta_{jk} m_j n_j} e_{m+n}, \quad e_n^* = e^{-i \sum_{j<k} \theta_{jk} n_j m_k} e_{-n}.$$ 

Consider a linear functional $\tau$ on $A_\theta$ defined by the formula

$$\tau(e_n) = \begin{cases} 1, & n = 0 \\
0, & n \neq 0 \end{cases}$$

We have (sums are finite)

$$\tau\left(\sum_{m \in \mathbb{Z}^d} \alpha_m e_m \big| \sum_{n \in \mathbb{Z}^d} \beta_n e_n\right) = \sum_{m, n \in \mathbb{Z}^d} \alpha_m \beta_n \tau(e_m e_n) = \sum_{n \in \mathbb{Z}^d} e^{i \sum_{j<k} \theta_{jk} n_j m_k} \alpha_n \beta_n.$$ 

It is now immediate that 

$$\tau(xy) = \tau(yx), \quad x, y \in A_\theta.$$ 

Let us equip $A_\theta$ with an inner product defined by the formula

$$\langle x, y \rangle = \tau(x^* y).$$ 

This inner product is non-degenerate. Indeed, for $x = \sum_{n \in \mathbb{Z}^d} \alpha_n e_n,$ we have

$$\tau(x^* x) = \tau\left(\sum_{m \in \mathbb{Z}^d} \alpha_m e_m \big| \sum_{n \in \mathbb{Z}^d} \alpha_n e_n\right) = \sum_{m, n \in \mathbb{Z}^d} \overline{\alpha_m} \beta_n \tau(e_m^* e_n) = \sum_{n \in \mathbb{Z}^d} |\alpha_n|^2.$$ 

Hence, $\tau(x^* x) = 0$ implies $x = 0.$

We have that $(A_\theta, \langle \cdot, \cdot \rangle)$ is a pre-Hilbert space. Define a Hilbert space $H$ as the completion of $(A_\theta, \langle \cdot, \cdot \rangle).$
For $x \in A_\theta$, let $\lambda_l(x) : A_\theta \to A_\theta$ be a linear mapping defined by the formula
\[
\lambda_l(x) : y \to xy, \quad y \in A_\theta.
\]
Obviously,
\[
\lambda_l(x) = \sum_{n \in \mathbb{Z}^d} \alpha_n \lambda_l(e_n), \quad x = \sum_{n \in \mathbb{Z}^d} \alpha_n e_n.
\]
Note that
\[
\langle \lambda_l(e_n)y, \lambda_l(e_n)y \rangle = \langle e_n y, e_n y \rangle = \tau(y^* e_n \cdot e_n y) = \tau(y^* y) = \langle y, y \rangle, \quad y \in A_\theta.
\]
In particular, $\lambda_l(e_n)$ is a unitary operator on $H$. Hence, $\lambda_l(x)$ is a bounded operator on $H$ for every $x \in A_\theta$. Now, the mapping
\[
x \to \lambda_l(x), \quad x \in A_\theta
\]
is the left regular representation of the $*$-algebra $A_\theta$. Similarly, we define the right regular representation $\lambda_r$ (even though in the present paper we only use $\lambda_r(e_n)$, $n \in \mathbb{Z}^d$).

We define $L_\infty(T^d_\theta)$ as the weak (or, equivalently, strong) closure of the algebra $\lambda_l(A_\theta)$. It is convenient to denote elements of this algebra by $\lambda_l(x)$.

The state
\[
A \to \langle e_0, Ae_0 \rangle, \quad A \in B(H)
\]
is tracial on $L_\infty(T^d_\theta)$. Indeed,
\[
\langle e_0, \lambda_l(x)\lambda_l(y)e_0 \rangle = \langle e_0, xy e_0 \rangle = \tau(xy) = \tau(yx) = \langle e_0, \lambda_l(y)\lambda_l(x)e_0 \rangle, \quad x, y \in A_\theta.
\]

For $x, y \in L_\infty(T^d_\theta)$, choose $x_n, y_n \in A_\theta$ such that
\[
\lambda_l(x_n) \to \lambda_l(x), \quad \lambda_l(y_n) \to \lambda_l(y)
\]
strongly as $n \to \infty$. Hence,
\[
\lambda_l(x_n)\lambda_l(y_n) \to \lambda_l(x)\lambda_l(y), \quad \lambda_l(y_n)\lambda_l(x_n) \to \lambda_l(y)\lambda_l(x)
\]
strongly as $n \to \infty$. In particular, we have
\[
\langle e_0, \lambda_l(x)\lambda_l(y)e_0 \rangle = \lim_{n \to \infty} \langle e_0, \lambda_l(x_n)\lambda_l(y_n)e_0 \rangle = \lim_{n \to \infty} \langle e_0, \lambda_l(y_n)\lambda_l(x_n)e_0 \rangle = \lim_{n \to \infty} \langle e_0, \lambda_l(y)\lambda_l(x)e_0 \rangle.
\]

Hence, our state is indeed tracial. This trace extends $\tau$ and, for this reason, is also denoted by $\tau$.

Normality of the tracial state $\tau$ follows directly from the definition. We claim that $\tau$ is a faithful trace. Indeed, if $p \in L_\infty(T^d_\theta)$ is a projection with $\tau(p) = 0$, then
\[
\langle p(e_n), p(e_n) \rangle = \langle (p\lambda_l(e_n))(e_0), (p\lambda_l(e_n))(e_0) \rangle = 
\]
\[
= \langle e_0, (p\lambda_l(e_n))^* (p\lambda_l(e_n))(e_0) \rangle = \langle e_0, (\lambda_l(e_n))^* p\lambda_l(e_n)(e_0) \rangle = \tau(\lambda_l(e_n)^* p\lambda_l(e_n)) = \tau(p) = 0.
\]

Hence, $p(e_n) = 0$ for every $n \in \mathbb{Z}^d$. Since $\{e_n\}_{n \in \mathbb{Z}^d}$ is an orthonormal basis in $H$, it follows that $p = 0$. Hence, $\tau$ is faithful.

**Example 2.1.** Take $d' > d$ and consider $d' \times d'$ matrix $\theta'$ whose left upper corner is $\theta$. Suppose that $\theta_{kl}' = 0$ when $k > d$ or when $l > d$. We have $L_\infty(T^d_{\theta'}) = L_\infty(T^d_{\theta'}) \otimes L_\infty(T^{d'-d})$.

**Proof.** Let $\{U_k\}_{1 \leq k \leq d'}$ be the unitaries in the definition of $T^d_{\theta'}$. Note that
\[
(1) \text{ elements } \{U_k\}_{1 \leq k \leq d} \text{ generate the algebra } A_\theta;
Thus, \( \pi \) preserves the trace, it follows that
\[
\langle \pi(x), \pi(y) \rangle = \langle x, y \rangle, \quad x, y \in A_\theta \otimes A_0.
\]

Thus, \( \pi \) extends to a Hilbert space isomorphism \( U : H_\theta \otimes H_0 \to H_{\theta'} \).

It is immediate that
\[
\lambda_i(\pi(x)) = U(\lambda_i \otimes \lambda_i)(x)U^{-1}, \quad x \in A_\theta \otimes A_0.
\]

Hence, the mapping \( z \to UzU^{-1} \) delivers a \(*\)-isomorphism from the algebra \( L_\infty(T^d_\theta) \otimes L_\infty(T^d_{\theta'}) \) to \( L_\infty(T^d_{\theta'}) \).

As usual, \( L_p(T^d_\theta) \) is the \( L_p \)-space associated to the von Neumann algebra \( L_\infty(T^d_\theta) \) and the trace \( \tau \).

The Hilbert space \( H \) is naturally identified with \( L_2(T^d_\theta) \). Every element \( x \in L_2(T^d_\theta) \) admits a unique representation of the form
\[
x = \sum_{n \in \mathbb{Z}^d} \hat{x}(n)e_n, \quad \{\hat{x}(n)\}_{n \in \mathbb{Z}^d} \in l_2(\mathbb{Z}^d).
\]

This Fourier picture allows us to define Sobolev spaces \( W^{k,2}(T^d_\theta) \) by setting
\[
W^{k,2}(T^d_\theta) = \left\{ x \in L_2(T^d_\theta) : \sum_{n \in \mathbb{Z}^d} |n|^{2k} |\hat{x}(n)|^2 < \infty \right\}.
\]

For \( 1 \leq k \leq d \), define self-adjoint operators \( D_k : W^{1,2}(T^d_\theta) \to L_2(T^d_\theta) \) by setting
\[
D_k(x) = \sum_{n \in \mathbb{Z}^d} n_k \hat{x}(n)e_n.
\]

**Fact 2.2.** We have (the second equality holds for all \( x \in L_\infty(T^d_\theta) \))
\[
D_1 \lambda_r(e_n) = \lambda_r(e_n) D_1 + n_i \lambda_r(e_n), \quad \lambda_i(\lambda_r(e_n)) = \lambda_r(e_n) \lambda_i(x).
\]

**Proof.** Second equality is obvious. Let’s check the first equality. Recall that
\[
e_m e_n = c_{m,n} e_{m+n}.
\]

We have
\[
(D_1 \lambda_r(e_n))(e_m) = D_1(e_m) e_n = c_{m,n} D_1(e_{m+n}) = c_{m,n} (m_i + n_i) e_{m+n} =
\]
\[
= (m_i + n_i) e_m e_n = (m_i e_m) \cdot e_n + n_i e_m e_n = (\lambda_r(e_n) D_i + n_i \lambda_r(e_n))(e_m).
\]

Set
\[
D^\alpha(x) = \left( \prod_{k=1}^d D_k^{\alpha_k} \right)(x), \quad \alpha = (\alpha_1, \ldots, \alpha_d) \in \mathbb{Z}_+^d.
\]

Set
\[
C^{\infty}(T^d_\theta) = \left\{ x \in L_\infty(T^d_\theta) : D^\alpha(x) \in L_\infty(T^d_\theta) \text{ for all } \alpha \right\},
\]
\[
W^{k,p}(T^d_\theta) = \left\{ x \in L_p(T^d_\theta) : D^\alpha x \in L_p(T^d_\theta), \quad |\alpha|_1 \leq k \right\}, \quad p > 0, \quad k \in \mathbb{Z}_+.
\]

Here, \( |\alpha|_1 \) is the \( l_1 \)-length of the vector \( \alpha \in \mathbb{Z}^d \).
We equip $W^{k,p}(\mathbb{T}_d^d)$ with its natural norm
\[
\|x\|_{W^{k,p}} = \sum_{|\alpha|_1 \leq k} \|D^\alpha x\|_p.
\]

For $p = 2$, the space $W^{k,p}(\mathbb{T}_d^d)$ coincides with earlier defined $W^{k,2}(\mathbb{T}_d^d)$. Indeed,
\[
D^\alpha \left( \sum_{n \in \mathbb{Z}^d} \hat{x}(n)e_n \right) = \sum_{n \in \mathbb{Z}^d} n^n \hat{x}(n)e_n, \quad n^n = \prod_{k=1}^d n^n_k.
\]
Hence,
\[
\|D^\alpha x\|_2 = \left( \sum_{n \in \mathbb{Z}^d} |n^n|_p \cdot |\hat{x}(n)|^2 \right)^{\frac{1}{2}}.
\]

Therefore,
\[
\sum_{|\alpha|_1 \leq k} \|D^\alpha x\|_2 = \sum_{|\alpha|_1 \leq k} \left( \sum_{n \in \mathbb{Z}^d} |n^n|_p \cdot |\hat{x}(n)|^2 \right)^{\frac{1}{2}}.
\]

Obviously,
\[
|n^n| \leq |n^n_2|, \quad n \in \mathbb{Z}^d, \quad \alpha \in \mathbb{Z}^d_+,
\]
and, therefore
\[
\sum_{|\alpha|_1 \leq k} \|D^\alpha x\|_2 \geq \sum_{|\alpha|_1 \leq k} \left( \sum_{n \in \mathbb{Z}^d} |n^n_2| \cdot |\hat{x}(n)|^2 \right)^{\frac{1}{2}} = \left( \sum_{n \in \mathbb{Z}^d} |\hat{x}(n)|^2 \cdot \sum_{|\alpha|_1 \leq k} \prod_{k=1}^d n^n_k \right)^{\frac{1}{2}}.
\]

On the other hand, we can consider only
\[
\alpha = (k,0,\cdots,0), \quad \alpha = (0,k,0,\cdots,0), \cdots.
\]

We have
\[
\sum_{|\alpha|_1 \leq k} \|D^\alpha x\|_2 \geq \sum_{l=1}^d \left( \sum_{n \in \mathbb{Z}^d} |n^n_{2l}| \cdot |\hat{x}(n)|^2 \right)^{\frac{1}{2}} \geq \left( \sum_{l=1}^d \sum_{n \in \mathbb{Z}^d} |n^n_{2l}| \cdot |\hat{x}(n)|^2 \right)^{\frac{1}{2}} = \left( \sum_{n \in \mathbb{Z}^d} |\hat{x}(n)|^2 \cdot \sum_{l=1}^d |n^n_{2l}| \right)^{\frac{1}{2}}.
\]

On the other hand, we have
\[
\sum_{l=1}^d |n^n_{2l}| \geq d^{1-k} |n^n_{2l}|, \quad n \in \mathbb{Z}^d.
\]
Hence,
\[
\sum_{|\alpha|_1 \leq k} \|D^\alpha x\|_2 \geq d^{1-k} \left( \sum_{n \in \mathbb{Z}^d} |n^n_{2l}| \cdot |\hat{x}(n)|^2 \right)^{\frac{1}{2}}.
\]

Thus,
\[
\sum_{|\alpha|_1 \leq k} \|D^\alpha x\|_2 \approx \left( \sum_{n \in \mathbb{Z}^d} |n^n_{2l}| \cdot |\hat{x}(n)|^2 \right)^{\frac{1}{2}}.
\]

We also set
\[
\Delta = \sum_{k=1}^d D^2_k.
\]

Obviously,
\[
\Delta : W^{2,2}(\mathbb{T}_d^d) \to L_2(\mathbb{T}_d^d)
\]
is self-adjoint (and positive).

3. Definition of a Curved non-commutative torus

3.1. Curved non-commutative torus. Here we define curved non-commutative torus and Laplace-Beltrami operator on it.

For a positive invertible element \( \nu \in L_\infty(T^d_\theta) \), consider
\[
\phi_\nu : x \rightarrow \tau(x\nu), \quad x \in L_\infty(T^d_\theta).
\]

Define a new inner product on \( L_2(T^d_\theta) \) by setting
\[
\langle u, v \rangle_\nu = \phi_\nu(u^*v) = \tau(u^*rv).
\]

Let \( \text{GL}_d(L_\infty(T^d_\theta)) \) be the set of invertible matrices with matrix elements from \( L_\infty(T^d_\theta) \). Let \( \text{GL}_d(C^\infty(T^d_\theta)) \) be the set of invertible matrices with matrix elements from \( C^\infty(T^d_\theta) \).

Riemannian metric on \( T^d_\theta \) (see [24] or [14]) is a positive element of \( \text{GL}_d(C^\infty(T^d_\theta)) \) such that the elements \( g_{ij} \) and \( (g^{-1})_{ij} \) are self-adjoint for all \( 1 \leq i, j \leq d \) (see [14]).

3.2. Laplace-Beltrami operator. Let \( g \in \text{GL}_d(C^\infty(T^d_\theta)) \) be a Riemannian metric. In the classical differential geometry, Laplace-Beltrami operator involves the square root of the determinant of \( g \). In the non-commutative case, there is no notion of a determinant for a matrix with non-commuting elements. We propose the following substitution for a "square root of the determinant" of \( g \). Set
\[
\nu = \pi^{-\frac{d}{2}} \int_{\mathbb{R}^d} e^{-\sum_{i,j=1}^d t_i t_j (g^{-1})_{ij}} dt.
\]

Note that \( (g^{-1})_{ij} \in C^\infty(T^d_\theta) \) (see [14]). Hence, \( \sum_{i,j=1}^d t_i t_j (g^{-1})_{ij} \in C^\infty(T^d_\theta) \) for all \( t \in \mathbb{R}^d \). It follows that \( e^{-\sum_{i,j=1}^d t_i t_j (g^{-1})_{ij}} \in C^\infty(T^d_\theta) \) for all \( t \in \mathbb{R}^d \). Moreover, the integrand in (3.1) is Bochner integrable in every \( C^m(T^d_\theta) \), \( m \geq 0 \). Thus, \( \nu \in C^\infty(T^d_\theta) \) and \( \nu^{\frac{1}{2}} \in C^\infty(T^d_\theta) \).

This choice of \( \nu \) may seem unexpected, however it appears to be very natural. In fact, this is the only choice of \( \nu \) which makes the Laplace-Beltrami operator defined below compatible with Connes Integration Formula (see [24]).

Laplace-Beltrami operator \( \Delta_g \) is defined on the Hilbert space \( L_2(T^d_\theta) \) equipped with the inner product \( \langle \cdot, \cdot \rangle_\nu \) by the formula (see [14] or [24])
\[
\Delta_g = \lambda_l(\nu^{-1}) \sum_{i,j=1}^d D_i \lambda_l(\nu^{\frac{1}{2}}(g^{-1})_{ij} \nu^{\frac{1}{2}}) D_j.
\]

Laplace-Beltrami operator is self-adjoint and positive on the domain \( W^{2,2}(T^d_\theta) \) (see Proposition 9.12 in [14]).

3.3. Statement of the task. The task is to find the asymptotic for the function
\[
t \rightarrow \text{Tr}(\lambda_l(x)e^{-t\Delta_g}), \quad t \downarrow 0.
\]
Here, \( x \in L_\infty(T^d_\theta) \) and \( g \in \text{GL}_d(C^\infty(T^d_\theta)) \) is a Riemannian metric.

First, note that the mapping \( U = \lambda_l(\nu^{-\frac{1}{2}}) \) is a unitary operator from \( (L_2(T^d_\theta), \langle \cdot, \cdot \rangle) \) to \( (L_2(T^d_\theta), \langle \cdot, \cdot \rangle_\nu) \) (this follows directly from the definition of these inner products). Define a self-adjoint (and positive) operator \( A_g \) on the Hilbert space \( (L_2(T^d_\theta), \langle \cdot, \cdot \rangle_\nu) \) with the domain \( W^{2,2}(T^d_\theta) \) by setting
\[
A_g = U^{-1} \Delta_g U.
\]
Equivalently,

\[(3.2) \quad A_g = \lambda_i(\nu^{-\frac{1}{2}}) \sum_{i,j=1}^{d} D_i \lambda_i(\nu^{-\frac{1}{2}} (g^{-1})_{ij} \nu^{\frac{1}{2}}) D_j \lambda_i(\nu^{-\frac{1}{2}}).\]

**Example 3.1 (Conformal deformation of a flat torus).** For example, if \( d = 2 \) and \( g = (h \delta_{ij}) \), then

\[A_g = \lambda_i(h^{-\frac{1}{2}}) \Delta \lambda_i(h^{-\frac{1}{2}})\]

exactly as it should be.

**Proof.** Obviously, \((g^{-1})_{ij} = h^{-1} \delta_{ij}\). Hence,

\[\nu = \frac{1}{\pi} \int_{\mathbb{R}^2} e^{-|t|^2/h} dt = h.\]

Hence,

\[\nu^{\frac{1}{2}} (g^{-1})_{ij} \nu^{\frac{1}{2}} = \delta_{ij}.\]

This completes the proof. \(\square\)

The task can be now equivalently restated as follows: to find an asymptotic for the function

\[t \to \mathrm{Tr}(\lambda_i(\nu^{\frac{1}{2}} x \nu^{-\frac{1}{2}}) e^{-t A_g}), \quad t \downarrow 0.\]

Here, \(x \in L^\infty(T^d)\), \(g \in \text{GL}_d(C^\infty(T^d_\theta))\) is a Riemannian metric and \(\nu\) is defined by \((3.1)\).

Indeed, we have

\[
\mathrm{Tr}(\lambda_i(x)e^{-t\Delta_g}) = \mathrm{Tr}(U^{-1}\lambda_i(x)e^{-t\Delta_g}U) = \\
= \mathrm{Tr}(U^{-1}\lambda_i(x)U \cdot U^{-1}e^{-t\Delta_g}U) = \mathrm{Tr}(\lambda_i(\nu^{\frac{1}{2}} x \nu^{-\frac{1}{2}}) e^{-t A_g}).
\]

**4. Definitions and Notations**

In this short section, we introduce the notations used in the statement and proof of Theorem 1.2, particularly, the functions \(\text{good}_k\) and \(\text{bad}_n\).

**Notation 4.1.** For \( s \in \mathbb{R}^d \), set

\[V(s) = \sum_{i=1}^{d} s_i A_i,
\]

where

\[A_i = \sum_{j=1}^{d} \lambda_i((g^{-1})_{ij} \nu^{\frac{1}{2}}) D_j \lambda_i(\nu^{-\frac{1}{2}}) + \sum_{j=1}^{d} \lambda_i(\nu^{-\frac{1}{2}}) D_j \lambda_i(\nu^{\frac{1}{2}} (g^{-1})_{ji}), \quad 1 \leq i \leq d.
\]

**Notation 4.2.** Let \( g = (g_{ij}) \in \text{GL}_d(C^\infty(T^d_\theta)) \) be a Riemannian metric. For every \( s \in \mathbb{R}^d \), set

\[x(s) = \sum_{i,j=1}^{d} (g^{-1})_{ij} s_i s_j.
\]

**Notation 4.3.** For every \( z \in \mathbb{C}\backslash \mathbb{R}_- \) and for every \( s \in \mathbb{R}^d \), set \(x_0(s, z) = 1\) and

\[x_m(s, z) = (V(s) + A_g) ((x(s) + z)^{-1} x_{m-1}(s, z)), \quad m \in \mathbb{N}.
\]

Here, \(A_g\) is defined in \((3.2)\).
Notation 4.4. Let $\mathcal{A} \subset \mathbb{N}$. For every $z \in \mathbb{C}\setminus\mathbb{R}_-$ and for every $s \in \mathbb{R}^d$, set $x_0^\mathcal{A}(s, z) = 1$ and
\[
x_m^\mathcal{A}(s, z) = (V(s))((x(s) + z)^{-1}x_{m-1}^\mathcal{A}(s, z)), \quad 1 \leq m \in \mathcal{A},
\]
x_m^\mathcal{A}(s, z) = A_g((x(s) + z)^{-1}x_{m-1}^\mathcal{A}(s, z)), \quad 1 \leq m \notin \mathcal{A}.
\]
Observe that, for $\mathcal{A} \subset \{1, \ldots, m\}$, we have
\[
x_m^\mathcal{A}(rs, r^2 z) = r^{1\mathcal{A}|^{-2m}}x_m^\mathcal{A}(s, z).
\]

Notation 4.5. For every $z \in \mathbb{C}\setminus\mathbb{R}_-$ and for every $s \in \mathbb{R}^d$, set
\[
good_k(s, z) = (x(s) + z)^{-1} \sum_{\frac{k}{2} \leq m \leq \min(k, d)} (-1)^m \sum_{\mathcal{A} \subset \{1, \ldots, m\}} x_m^\mathcal{A}(s, z),
\]
\[
corr_k(s, z) = (x(s) + z)^{-1} \sum_{\frac{k}{2} \leq m \leq k} (-1)^m \sum_{\mathcal{A} \subset \{1, \ldots, m\}} x_m^\mathcal{A}(s, z),
\]
\[
\bad_n(z) = \left(\lambda_r(e_n)^* \frac{1}{A_g + z} \lambda_r(e_n)\right)(x_{d+1}(n, z)).
\]

Obviously, $\good_k = \corr_k$ for $k \leq d$.

Key feature of the term $\good_k$ is its homogeneity
\[
good_k(rs, r^2 z) = r^{-k-2}\good_k(s, z),
\]
which follows immediately from (4.1).

Notation 4.6. For every $0 \neq s \in \mathbb{R}^d$ set
\[
\Good_k(s) = \frac{1}{2\pi} \int_\mathbb{R} \good_k(s, i\lambda)e^{i\lambda}d\lambda.
\]
\[
\Corr_k(s) = \frac{1}{2\pi} \int_\mathbb{R} \corr_k(s, i\lambda)e^{i\lambda}d\lambda.
\]

For $k > 0$, integrals are well defined; for $k = 0$, integrals should be understood in the sense of principal value.

Obviously, $\Good_k = \Corr_k$ for $k \leq d$.

Notation 4.7. For every $k \in \mathbb{Z}_+$, we set
\[
I_k = \int_\mathbb{R}^d \Corr_k(s)ds.
\]

5. Strategy

In the subsequent lemma, weak convergence is asserted, not assumed.

Lemma 5.1. Let $g \in \text{GL}_d(C^\infty(\mathbb{T}_d^d))$ be a Riemannian metric and let $A_g$ be the operator defined by (3.2). For every $x \in L_2(\mathbb{T}_d^d)$, we have
\[
\text{Tr}(\lambda(x)e^{-tA_g}) = \tau(x \cdot F(t)).
\]
Here $F(t) \in L_2(\mathbb{T}_d^d)$ is given by the series (converging weakly in $L_2(\mathbb{T}_d^d)$)
\[
F(t) = \sum_{n \in \mathbb{Z}^d} (\lambda_r(e_n)^* e^{-tA_g} \lambda_r(e_n))(1).
\]
Proof. It follows from [21] that
\[ \lambda_l(x)e^{-tA_s} \in L_1 \]
for every \( x \in L_2(\mathbb{T}^d) \).

For every \( T \in L_1 \), we have
\[ \text{Tr}(T) = \sum_{n \in \mathbb{Z}^d} \langle e_n, Te_n \rangle. \]

Therefore,
\[ \text{Tr}(\lambda_l(x)e^{-tA_s}) = \sum_{n \in \mathbb{Z}^d} \langle e_n, (\lambda_l(x)e^{-tA_s})(e_n) \rangle. \]

Since \( e_n = \lambda_r(e_n)1 \) and since \( \lambda_r(e_n) \) commutes with \( \lambda_l(x) \), it follows that
\[ \langle e_n, (\lambda_l(x)e^{-tA_s})(e_n) \rangle = \langle (\lambda_r(e_n))(1), (\lambda_l(x)e^{-tA_s}\lambda_r(e_n))(1) \rangle = \]
\[ = (1, (\lambda_r(e_n))^{*}\lambda_l(x)e^{-tA_s}\lambda_r(e_n))(1) = (1, (\lambda_l(x))\lambda_r(e_n)^{*}e^{-tA_s}\lambda_r(e_n))(1). \]

Combining these equalities, we obtain
\[ \text{Tr}(\lambda_l(x)e^{-tA_s}) = \sum_{n \in \mathbb{Z}^d} \langle 1, (\lambda_l(x)\lambda_r(e_n)^{*}e^{-tA_s}\lambda_r(e_n))(1) \rangle = \]
\[ = \left( 1, \lambda_l(x) \left( \sum_{n \in \mathbb{Z}^d} \lambda_r(e_n)^{*}e^{-tA_s}\lambda_r(e_n)1 \right) \right) = (1, \lambda_l(x)(F(t))) = \tau(x \cdot F(t)). \]
\[ \square \]

In Section 6, we prove the following result.

**Theorem 5.2.** For every \( 0 \neq n \in \mathbb{Z}^d \) and for every \( 0 \neq z \in \mathbb{C} \setminus \mathbb{R}_- \), we have
\[ (\lambda_r(e_n)^{*}\frac{1}{A_g + z}\lambda_r(e_n))(1) = \sum_{k=0}^{2d} \text{good}_k(n, z) + (-1)^{d+1}\text{bad}_n(z), \]
where
(i) functions \( \text{good}_k \) and \( \text{bad}_n \) are explicitly defined in Notation 4.3.
(ii) functions \( \text{good}_k(s, \cdot), \ k \geq 0, \) are analytic on \( \mathbb{C} \setminus \mathbb{R}_- \).
(iii) for every \( z \) with \( \Re(z) \leq 0 \), we have
\[ \|\text{bad}_n(z)\|_2 = O\left(|z|^{-1}(|n|^2 + |z|)^{-\frac{d+1}{2}}\right), \quad n \in \mathbb{Z}^d. \]

Functions \( \text{good}_k \) are called good because they have a very concrete representation. Bad terms \( \text{bad}_n \) are, in a certain sense, negligible (see the explanation after Theorem 5.3).

In Section 7, we prove the following result.

**Theorem 5.3.** We have
\[ (\lambda_r(e_n)^{*}e^{-tA_s}\lambda_r(e_n))(1) = \sum_{k=0}^{2d} (\frac{t}{2})^{k}\text{Good}_k(\sqrt{n}t) + (-1)^{d+1}\text{Bad}_n(t), \]
where
(i) functions \( \text{Good}_k \) are explicitly defined in Notation 4.0.
(ii) we have
\[ \|\text{Bad}_n(t)\|_2 = O\left(\frac{\log(|n|)}{|n|^{d+1}}\right) \]
uniformly in \( t \).

Terms \( \text{Good}_k \) are called good because they have very concrete representation. Bad terms \( \text{Bad}_n(t) \) are negligible in the following sense:
\[ \| \sum_{n \in \mathbb{Z}^d} \text{Bad}_n(t) \|_2 = O(1) \]
uniformly in \( t \).

Alain Connes suggested to us the method based on the Poisson summation formula which allows to replace sums with integrals (see Proposition 2.27 in [9]).

Lemma 5.4 (Connes "little lemma"). If \( f \) is a Schwartz function, then
\[ \sum_{n \in \mathbb{Z}^d} f(nt) = t^{-d} \int_{\mathbb{R}^d} f(s)ds + O(t^\infty), \quad t > 0. \]

Proof. By Poisson summation formula, we have
\[ \sum_{n \in \mathbb{Z}^d} f(nt) = t^{-d} \sum_{n \in \mathbb{Z}^d} (\mathcal{F}f)(nt^{-1}). \]

Here,
\[ (\mathcal{F}f)(s) = \int_{\mathbb{R}^d} f(u)e^{-2\pi i \langle u, s \rangle} du. \]

Note that \( f \) is a Schwartz function and so is \( \mathcal{F}f \). For every \( m \), we have \( (\mathcal{F}f)(s) = O(|s|^{-m}) \). Thus,
\[ \sum_{n \neq 0 \in \mathbb{Z}^d} (\mathcal{F}f)(nt^{-1}) = \sum_{n \neq 0 \in \mathbb{Z}^d} O\left(\frac{t^m}{|n|^m}\right) = O(t^m), \quad m > d. \]

Hence,
\[ \sum_{n \in \mathbb{Z}^d} f(nt) = t^{-d} \int_{\mathbb{R}^d} f(s)ds + O(t^{m-d}), \quad t > 0. \]

Since \( m \) is arbitrarily large, the assertion follows. \( \square \)

For vector-valued functions, the notion analogous to that of Schwartz function does not exist (see though a substitute in Appendix C in [14]). However, the following adjustment of Connes "little lemma" is possible (and proved in Section 8).

Theorem 5.5. For every \( f \in W^{p,1}(\mathbb{R}^d, X) \), \( p > d \), we have
\[ \left\| \sum_{n \in \mathbb{Z}^d} f(tn) - t^{-d} \int_{\mathbb{R}^d} f(u)du \right\|_X = O(t^{p-d}), \quad t > 0. \]

In Section 8 we verify the conditions of Theorem 5.5 for \( f = \text{Good}_k \) and infer the following intermediate result.

Theorem 5.6. Let \( d \geq 2 \). For every \( x \in L_2(\mathbb{T}^d) \), we have
\[ \left| \text{Tr}(\lambda(x)e^{-tA}) - \sum_{0 \leq k < d \atop k \equiv 0 \mod 2} i^{\frac{k-d}{2}} \cdot \tau(xI_k) \right| = O(1). \]
Remark 5.7. It is tempting to say that, taking more terms in Theorem 5.2, we should obtain an asymptotic for $\text{Tr}(\lambda_l(x)e^{-tA_n})$ modulo $O(t^N)$ for large $N$ (rather than asymptotic modulo $O(t^n)$). However, proving this does not seem straightforward. In our proof of Theorem 1.2, we use tensoring trick instead.

6. Splitting theorem for resolvent in arbitrary dimension

Lemma 6.1. For every $n \in \mathbb{Z}^d$, we have

$$\lambda_r(e_n)^* A_g \lambda_r(e_n) = \lambda_l(x(n)) + A_g + V(n).$$

Proof. It follows from Fact 2.2 that

$$\lambda_r(\nu^{-\frac{1}{2}}) D_1 \lambda_l(\nu^{\frac{1}{2}}(g^{-1})_{ij}) D_2 \lambda_r(\nu^{-\frac{1}{2}}) \cdot \lambda_r(e_n) =$$

$$= \lambda_r(e_n) \cdot \lambda_l(\nu^{-\frac{1}{2}})(D_i + n_i) \lambda_l(\nu^{\frac{1}{2}}(g^{-1})_{ij})(D_j + n_j) \lambda_l(\nu^{-\frac{1}{2}}).$$

Hence,

$$\lambda_r(e_n)^* A_g \lambda_r(e_n) = A_g + \lambda_l\left(\sum_{i,j=1}^d n_i n_j (g^{-1})_{ij}\right) +$$

$$+ \sum_{i,j=1}^d \lambda_l(\nu^{-\frac{1}{2}}) n_i \lambda_l(\nu^{\frac{1}{2}}(g^{-1})_{ij}) D_j \lambda_l(\nu^{-\frac{1}{2}}) +$$

$$+ \sum_{i,j=1}^d \lambda_l(\nu^{-\frac{1}{2}}) D_i \lambda_l(\nu^{\frac{1}{2}}(g^{-1})_{ij}) n_j \lambda_l(\nu^{-\frac{1}{2}}).$$

Lemma 6.2. For every $z \in \mathbb{C} \setminus \mathbb{R}^-$, we have

$$\left(\lambda_r(e_n)^* \frac{1}{A_g + z} \lambda_r(e_n)\right)(1) = \sum_{k=0}^{2d} \text{good}_k(n, z) + (-1)^{d+1} \text{bad}_n(z).$$

Proof. Iterating the resolvent identity, we obtain

$$\frac{1}{A + z} = \sum_{m=0}^d (-1)^m \frac{1}{B + z} \cdot \left((A - B) \frac{1}{B + z}\right)^m + \frac{(-1)^{d+1}}{A + z} \cdot \left((A - B) \frac{1}{B + z}\right)^{d+1}.$$

Now, we set $A = \lambda_r(e_n)^* A_g \lambda_r(e_n)$ and $B = \lambda_l(x(n))$ and apply both sides to the vector 1. By Lemma [1.1] we have

$$A - B = V(n) + A_g.$$

Using the equality

$$\left(\frac{1}{B + z}\right)^m (1) = \left((V(n) + A_g) \lambda_l((x(n) + z)^{-1})\right)^m (1) = x_m(n, z),$$

we obtain

$$\frac{1}{A + z}(1) = \sum_{m=0}^d (-1)^m (x(n) + z)^{-1} x_m(n, z) + (-1)^{d+1} \text{bad}_n(z).$$

Finally, we have

$$x_m(n, z) = \sum_{\alpha \in \{1, \ldots, m\}} x^\alpha_m(n, z).$$
Thus,
\[
\sum_{m=0}^{d} (-1)^m (x(n) + z)^{-1} x_m(n, z) = \sum_{m=0}^{d} (-1)^m (x(n) + z)^{-1} \sum_{\mathcal{A} \subset \{1, \ldots, m\}} x_{\mathcal{A}'}(n, z).
\]

Obviously,
\[
\sum_{\mathcal{A} \subset \{1, \ldots, m\}} x_{\mathcal{A}'}(n, z) = \sum_{k=m}^{2m} \sum_{\mathcal{A} \subset \{1, \ldots, m\}} x_{\mathcal{A}'}(n, z) = \sum_{k=0}^{2d} \sum_{\frac{k}{2} \leq m \leq \min(k, d)} (-1)^m (x(n) + z)^{-1} \sum_{\mathcal{A} \subset \{1, \ldots, m\}} x_{\mathcal{A}'}(n, z) = \sum_{k=0}^{2d} \text{good}_k(n, z).
\]

Combining the last equality with (6.1), we complete the proof. \qed

In the following lemmas, \(c_k(g)\) are some constants (they may differ in different lemmas) which depend only on \(k\) and the metric \(g\). Their precise values are irrelevant.

**Lemma 6.3.** For every \(k \geq 0\), we have
\[
\|(x(s) + z)^{-1}\|_{W^{k, \infty}} \leq \frac{c_k(g)}{|s|^2 + |z|}, \quad \Re(z) \leq 0, \quad s \in \mathbb{R}^d.
\]

**Proof.** We prove the assertion by induction on \(k\). For \(k = 0\), we have \(W^{0, \infty} = L_\infty\) and the assertion is obvious. Suppose, it is true for \(k\) and let us prove it for \(k + 1\).

By definition (2.1), we have
\[
\|x\|_{W^{k+1, p}} \leq \|x\|_{W^{k, p}} + \sum_{j=1}^{d} \|D_j x\|_{W^{k, p}}.
\]
Thus,
\[
\|(x(s) + z)^{-1}\|_{W^{k+1, \infty}} \leq \|(x(s) + z)^{-1}\|_{W^{k, \infty}} + \sum_{j=1}^{d} \|D_j((x(s) + z)^{-1})\|_{W^{k, \infty}}.
\]

Clearly,
\[
D_j((x(s) + z)^{-1}) = -(x(s) + z)^{-1} D_j(x(s)) (x(s) + z)^{-1}.
\]

Using the inequality
\[
\|xy\|_{W^{k, \infty}} \leq 2^k \|x\|_{W^{k, \infty}} \|y\|_{W^{k, \infty}},
\]
we arrive at
\[
\|D_j((x(s) + z)^{-1})\|_{W^{k, \infty}} \leq 2^{2k} \|(x(s) + z)^{-1}\|_{W^{k, \infty}}^2 \|D_j(x(s))\|_{W^{k, \infty}}.
\]
Obviously,
\[ \left\| D_j(x(s)) \right\|_{W^{k,\infty}} \leq \left\| x(s) \right\|_{W^{k+1,\infty}} \leq |s|^2 \sum_{i,j=1}^d \left\| (g^{-1})_{ij} \right\|_{W^{k+1,\infty}}. \]
Using the inductive assumption, we obtain
\[ \left\| D_j((x(s) + z)^{-1}) \right\|_{W^{k,\infty}} \leq \frac{2^{k} c_k^2(g)|s|^2}{(|s|^2 + |z|)^2}, \sum_{i,j=1}^d \left\| (g^{-1})_{ij} \right\|_{W^{k+1,\infty}} = \frac{c_k(g)|s|^2}{(|s|^2 + |z|)^2}. \]
Hence,
\[ \left\| (x(s) + z)^{-1} \right\|_{W^{k+1,\infty}} \leq \frac{c_k(g)}{|s|^2 + |z|} + \frac{d c_k(g)|s|^2}{(|s|^2 + |z|)^2} \leq \frac{c_k(g) + d c_k(g)}{|s|^2 + |z|}. \]

\[ \square \]

**Lemma 6.4.** For every \((m, k) \geq 0\), we have
\[ \left\| x_m'(s,z) \right\|_{W^{k,2}} \leq \frac{c_k(g)}{|s|^2 + |z|} \left\| x_m'(s,z) \right\|_{W^{k+1,2}}, \quad m \in \mathcal{A}, \]
\[ \left\| x_m'(s,z) \right\|_{W^{k,2}} \leq \frac{c_k(g)}{|s|^2 + |z|} \left\| x_m'(s,z) \right\|_{W^{k+2,2}}, \quad m \notin \mathcal{A}. \]

**Proof.** Consider the case \(m \in \mathcal{A}\). By definition, we have
\[ \left\| x_m'(s,z) \right\|_{W^{k,2}} = \left\| (V(s))(x(s) + z)^{-1} x_m'(s,z) \right\|_{W^{k,2}}. \]
By triangle inequality, we have
\[ \left\| x_m'(s,z) \right\|_{W^{k,2}} \leq \sum_{i=1}^d |s_i| \cdot \left\| A_i((x(s) + z)^{-1} x_m'(s,z)) \right\|_{W^{k,2}}. \]
Using obvious inequality
\[ \left\| A_i x \right\|_{W^{k,2}} \leq c_k'(g) \cdot \left\| x \right\|_{W^{k+1,2}}, \quad 1 \leq i \leq d, \]
we obtain
\[ \left\| x_m'(s,z) \right\|_{W^{k,2}} \leq d c_k'(g)|s| \cdot \left\| (x(s) + z)^{-1} x_m'(s,z) \right\|_{W^{k+1,2}}. \]
Using the inequality
\[ \left\| x y \right\|_{W^{k+1,2}} \leq 2^{k+1} \left\| x \right\|_{W^{k+1,2}} \left\| y \right\|_{W^{k+1,2}}, \]
we arrive at
\[ \left\| x_m'(s,z) \right\|_{W^{k,2}} \leq 2^{k+1} d c_k'(g)|s| \cdot \left\| (x(s) + z)^{-1} x_m'(s,z) \right\|_{W^{k+1,2}}. \]
The assertion for the case \(m \notin \mathcal{A}\) follows now from Lemma 6.3.

Consider the case \(m \in \mathcal{A}\). By definition, we have
\[ \left\| x_m'(s,z) \right\|_{W^{k,2}} = \left\| A_g((x(s) + z)^{-1} x_m'(s,z)) \right\|_{W^{k,2}}. \]
Using obvious inequality
\[ \left\| A_g x \right\|_{W^{k,2}} \leq c_k'(g) \left\| x \right\|_{W^{k+2,2}}, \]
we obtain
\[ \left\| x_m'(s,z) \right\|_{W^{k,2}} \leq c_k'(g) \left\| (x(s) + z)^{-1} x_m'(s,z) \right\|_{W^{k+2,2}}. \]
Using the inequality
\[ \|xy\|_{W^{k+2,2}} \leq 2^{k+2}\|x\|_{W^{k+2,\infty}}\|y\|_{W^{k+2,2}}, \]
we arrive at
\[ \|x^{\mathcal{A}}_m(s, z)\|_{W^{k,2}} \leq 2^{k+2}c_k(g)\|x(s) + z\|_{W^{k+2,\infty}}^{-1}\|x^{\mathcal{A}}_{m-1}(s, z)\|_{W^{k+2,2}}. \]
The assertion for the case \( m \in \mathcal{A} \) follows now from Lemma 6.3.

**Lemma 6.5.** For every \((m, k) \geq 0\) and for every \( \mathcal{A} \subset \{1, \ldots, m\} \), we have
\[ \|x^{\mathcal{A}}_m(s, z)\|_{W^{k,2}} \leq \frac{c_{m-k}(g)}{\left(\|s\|^2 + |z|\right)^m}. \]

**Proof.** The assertion follows by induction on \( m \). For \( m = 0 \), we have that \( \mathcal{A} = \emptyset \) and, hence, \( |\mathcal{A}| = 0 \). It is immediate that
\[ \|x^{\emptyset}_0(s, z)\|_{W^{k,2}} = \|1\|_{W^{k,2}} = 1. \]
This establishes base of induction.

We now establish the step of induction. Suppose the assertion is true for \( m - 1 \), for every subset of \( \{1, \ldots, m - 1\} \) and for every \( k \). Let \( \mathcal{B} = \mathcal{A}\\backslash\{m\} \subset \{1, \ldots, m - 1\} \).

If \( m \in \mathcal{A} \), then Lemma 6.3 asserts that
\[ \|x^{\mathcal{A}}_m(s, z)\|_{W^{k,2}} \leq \frac{c_k(g)}{\left(\|s\|^2 + |z|\right)^m} \|x^{\mathcal{A}}_{m-1}(s, z)\|_{W^{k+1,2}}. \]
Applying inductive assumption for the set \( \mathcal{B} \), we obtain
\[ \|x^{\mathcal{A}}_m(s, z)\|_{W^{k,2}} \leq \frac{c_k(g)}{\left(\|s\|^2 + |z|\right)^m} \frac{c_{m-1-k+1}(g)}{\left(\|s\|^2 + |z|\right)^{(m-1)-\frac{1}{2}|\mathcal{B}|}} = \frac{c_k(g)c_{m-1-k+1}(g)}{\left(\|s\|^2 + |z|\right)^{m-\frac{1}{2}|\mathcal{B}|}}. \]

If \( m \notin \mathcal{A} \), then Lemma 6.3 asserts that
\[ \|x^{\mathcal{A}}_m(s, z)\|_{W^{k,2}} \leq \frac{c_k(g)}{\|s\|^2 + |z|} \|x^{\mathcal{A}}_{m-1}(s, z)\|_{W^{k+2,2}} = \frac{c_k(g)}{\|s\|^2 + |z|} \|x^{\mathcal{A}}_{m-1}(s, z)\|_{W^{k+2,2}}. \]
Applying inductive assumption for the set \( \mathcal{B} \), we obtain
\[ \|x^{\mathcal{A}}_m(s, z)\|_{W^{k,2}} \leq \frac{c_k(g)}{\|s\|^2 + |z|} \frac{c_{m-1-k+2}(g)}{\left(\|s\|^2 + |z|\right)^{(m-1)-\frac{1}{2}|\mathcal{B}|}} = \frac{c_k(g)c_{m-1-k+2}(g)}{\left(\|s\|^2 + |z|\right)^{m-\frac{1}{2}|\mathcal{B}|}}. \]
This establishes step of induction.

**Proof of Theorem 5.2.** Firstly, the equality (5.1) is established in Lemma 6.2. The assertion of Theorem 5.2 (ii) does not require any proof. The assertion of Theorem 5.2 (iii) is immediate from the definition of the term good \( g \) (see Notation 4.3).

It remains to show the assertion of Theorem 5.2 (iii). By definition (see Notation 4.3), we have
\[ \text{bad}_n(z) = \left( \lambda_r(e_n)^* \frac{1}{A_g + z} \lambda_r(e_n) \right)(x_{d+1}(n, z)). \]

Therefore,
\[ (6.2) \quad \|\text{bad}_n(z)\|_2 \leq \left\| \lambda_r(e_n)^* \frac{1}{A_g + z} \lambda_r(e_n) \right\|_{L^2 \to L^2} \cdot \|x_{d+1}(n, z)\|_2. \]
Since \( A_g \geq 0 \), it follows that
\[ (6.3) \quad \left\| \lambda_r(e_n)^* \frac{1}{A_g + z} \lambda_r(e_n) \right\|_{L^2 \to L^2} = \left\| \frac{1}{A_g + z} \right\|_{L^2 \to L^2} \leq |z|^{-1}, \quad \Re(z) \geq 0. \]
On the other hand, it follows from Lemma 6.3 (with \( k = 0 \)) that
\[
\|x_{d+1}(n, z)\|_2 \leq \sum_{\|u\| \leq (1, \ldots, d+1)} \|x_{d+1}(n, z)\|_2 \leq c(g) \sum_{\|u\| \leq (|n|^2 + |z|)^{1/2} - 1}.
\]
Since \( |u| \leq d + 1 \) and \( n \neq 0 \), it follows that
\[
(|n|^2 + |z|)^{1/2} - 1 \leq \frac{1}{(|n|^2 + |z|)^{d/2}}.
\]
Thus,
\[
(6.4) \quad \|x_{d+1}(n, z)\|_2 \leq \frac{2^{d+1}c(g)}{|n|^2 + |z|)^{d/2}}.
\]
Combining (6.2), (6.3) and (6.4), we arrive at
\[
\|bad_n(z)\|_2 \leq \frac{2^{d+1}c(g)}{|z|(|n|^2 + |z|)^{d/2}}.
\]
This completes the proof of Theorem 5.2 (iii). \( \square \)

7. Splitting Theorem for Exponential in Arbitrary Dimension

In this section, \( \Gamma \) denotes the contour passing from \(-i\infty \) to \( i\infty \) as follows: along the line \( \{\Re z = 0\} \) from \(-i\infty \) to \(-i \), then along the circle \( \{|z| = 1\} \) in the counter-clockwise direction from \(-i \) to \( i \), then along the line \( \{\Re z = 0\} \) from \( i \) to \( i\infty \). This contour is introduced with a single purpose: to avoid the origin in the integration. However, in the statement of Theorem 6.3 we use Notation 4.6 where the integration is taken over the line \( \{\Re z = 0\} \). This allows us to employ homogeneity of the function \( \text{good}_k \) (as in (6.2)) and to write the respective integral as \( \text{Good}_k \).

Lemma 7.1. For \( 0 \neq s \in \mathbb{R}^d \) and for every \( t > 0 \), we have
\[
t^{\frac{d}{2}} \text{Good}_k(st^{\frac{1}{2}}) = \frac{1}{2\pi i} \int_{\Gamma} \text{good}_k(s, z) e^{iz} dz.
\]
Proof. By definition of \( \text{Good}_k \) (see Notation 4.6), we have
\[
t^{\frac{d}{2}} \text{Good}_k(st^{\frac{1}{2}}) = \frac{t^{\frac{d}{2}}}{2\pi} \int_{\mathbb{R}} \text{good}_k(t^{\frac{1}{2}} s, i\lambda) e^{i\lambda} d\lambda.
\]
By the homogeneity of \( \text{good}_k \), we have
\[
t^{\frac{d}{2}} \text{Good}_k(st^{\frac{1}{2}}) = \frac{t^{-1}}{2\pi} \int_{\mathbb{R}} \text{good}_k(s, t^{-1} \lambda) e^{i\lambda} d\lambda.
\]
Changing the variable \( \lambda = t\mu \), we obtain
\[
t^{\frac{d}{2}} \text{Good}_k(st^{\frac{1}{2}}) = \frac{1}{2\pi} \int_{\mathbb{R}} \text{good}_k(s, i\mu) e^{i\mu} d\mu = \frac{1}{2\pi i} \int_{\Gamma} \text{good}_k(s, z) e^{iz} dz.
\]
Thus,
\[
t^{\frac{d}{2}} \text{Good}_k(st^{\frac{1}{2}}) - \frac{1}{2\pi i} \int_{\Gamma} \text{good}_k(s, z) e^{iz} dz = \frac{1}{2\pi i} \int_{\Gamma} \text{good}_k(s, z) e^{iz} dz,
\]
where the closed contour \( \Gamma \) goes from \( z = -i \) to \( z = i \) along the line \( \{\Re z = 0\} \), then from \( z = i \) to \( z = -i \) along the circle \( \{|z| = 1\} \) clockwise.
Note that \( x(s) \geq c(g)|s|^2 \) for every \( s \in \mathbb{R}^d \). Hence, \( \text{good}_k(s, \cdot) \) is analytic in \( \mathbb{C} \setminus (-\infty, -c(g)|s|^2] \). Since \( \Gamma_1 \) is a closed contour lying inside \( \mathbb{C} \setminus (-\infty, -c(g)|s|^2] \), it follows from Cauchy theorem that

\[
\int_{\Gamma_1} \text{good}_k(s, z) e^{t z} dz = 0.
\]

This completes the proof. \( \square \)

**Proof of Theorem 5.3.** For every \( y > 0 \) and \( t \geq 0 \), we have

\[
e^{-ty} = \frac{1}{2\pi i} \text{p.v.} \int_{\text{IR}} e^{t z} dz.
\]

Here, principal value is needed because the integral is not absolutely convergent at infinity. Hence, for every \( y, t \geq 0 \) we have

\[
e^{-ty} = \frac{1}{2\pi i} \text{p.v.} \int_{\Gamma} e^{t z} dz
\]

By the functional calculus, we have

\[
e^{-tA} = \frac{1}{2\pi i} \text{p.v.} \int_{\Gamma} e^{t z} dz.
\]

Therefore,

\[
(\lambda_r(e_n)^* e^{-tA} \lambda_r(e_n))(1) = \frac{1}{2\pi i} \text{p.v.} \int_{\Gamma} (\lambda_r(e_n)^* \frac{1}{A_g + z} \lambda_r(e_n))(1) e^{t z} dz.
\]

By Lemma 7.1 we have

\[
t^{\frac{s}{2}} \text{Good}_k(st^{\frac{1}{2}}) = \frac{1}{2\pi i} \int_{\Gamma} \text{good}_k(s, z) e^{t z} dz.
\]

Setting

\[
\text{Bad}_n(t) = \frac{1}{2\pi i} \int_{\Gamma} \text{bad}_n(z) e^{t z} dz,
\]

we infer from Theorem 5.2 that

\[
(\lambda_r(e_n)^* e^{-tA} \lambda_r(e_n))(1) = \sum_{k=0}^{2d} t^{\frac{k}{2}} \text{Good}_k(nt^{\frac{1}{2}}) + (-1)^{d+1} \text{Bad}_n(t).
\]

Obviously,

\[
\|\text{Bad}_n(t)\|_2 \leq \frac{1}{2\pi} \int_{\Gamma} \|\text{bad}_n(z)\|_2 \cdot |e^{t z}| \cdot |dz|.
\]

By Theorem 5.2 we have

\[
\|\text{bad}_n(z)\|_2 \leq \frac{c(g)}{|z| \cdot (|n|^2 + |z|)^{\frac{d+1}{2}}}
\]

for some constant \( c(g) \), which only depends on \( g \) and not on \( n \). On \( \Gamma \), we have \( |e^{t z}| \leq e^t \leq e \) as \( t \in (0, 1) \). Therefore,

\[
\|\text{Bad}_n(t)\|_2 \leq c(g) \cdot \int_{\Gamma} \frac{|dz|}{|z| \cdot (|n|^2 + |z|)^{\frac{d+1}{2}}}.
\]

Clearly,

\[
\int_{\Gamma} \frac{|dz|}{|z| \cdot (|n|^2 + |z|)^{\frac{d+1}{2}}} = \frac{\pi}{(|n|^2 + 1)^{\frac{d+1}{2}}} + 2 \int_{1}^{\infty} \frac{d\lambda}{\lambda \cdot (|n|^2 + \lambda)^{\frac{d+1}{2}}}.
\]
Furthermore,
\[
\int_1^{[n]^2} \frac{d\lambda}{\lambda \cdot (|n|^2 + \lambda)^{d/2}} \leq \int_1^{[n]^2} \frac{d\lambda}{|n|^d} \cdot \frac{1}{|n|^{-d+1}} = \frac{2 \log(|n|)}{|n|^{d+1}}
\]
and
\[
\int_{|n|^2}^{\infty} \frac{d\lambda}{\lambda \cdot (|n|^2 + \lambda)^{d/2+1}} \leq \int_{|n|^2}^{\infty} \frac{d\lambda}{|n|^d} = \frac{2}{d+1} |n|^{-d-1}.
\]
Combining these inequalities, we obtain
\[
\|\text{Bad}_n(t)\|_2 = O\left(\frac{\log(|n|)}{|n|^{d+1}}\right),
\]
as desired. \qed

8. Poisson summation formula for vector-valued functions

Theorem 8.1. Let \(X\) be a Banach space. Poisson summation formula holds for every \(f \in W^{p,1}(\mathbb{R}^d, X)\), \(p > d\), i.e.
\[
\sum_{n \in \mathbb{Z}^d} f(n) = \sum_{n \in \mathbb{Z}^d} (\mathcal{F} f)(n).
\]
In what follows,
\[
(\mathcal{F} f)(s) = \int_{\mathbb{R}^d} f(u) e^{-2\pi i (u,s)} \, du, \quad s \in \mathbb{R}^d.
\]
\[
(T_s f)(u) = f(u - s), \quad u, s \in \mathbb{R}^d.
\]
In what follows, \(BW C(\mathbb{R}^d, X)\) denotes the space of bounded weak* continuous \(X\)-valued functions on \(\mathbb{R}^d\).

Lemma 8.2. We have \(W^{d,1}(\mathbb{R}^d, X) \subset BW C(\mathbb{R}^d, X)\). Moreover, we have
\[
\|f\|_{L_\infty(\mathbb{R}^d, X)} \leq \|f\|_{W^{d,1}(\mathbb{R}^d, X)}, \quad f \in W^{d,1}(\mathbb{R}^d).
\]
Proof. Step 1: Let us prove the inequality \(\text{(8.1)}\) for \(X = \mathbb{C}\) and for every Schwartz function \(f\).

For \(s \in \mathbb{R}^d\), let
\[
K_s = \{u \in \mathbb{R}^d : u \leq s\}.
\]
We have
\[
f(s) = \int_{K_s} (\partial_0 \cdots \partial_{d-1} f)(u) \, du.
\]
Thus,
\[
|f(s)| \leq \|\partial_0 \cdots \partial_{d-1} f\|_1 \leq \|f\|_{W^{d,1}(\mathbb{R}^d)}.
\]
Taking supremum over \(s \in \mathbb{R}^d\), we complete the proof of Step 1.

Step 2: Let us prove the assertion for \(X = \mathbb{C}\).

Now, recall that Schwartz functions are dense in \(W^{d,1}(\mathbb{R}^d)\). For a given \(f \in W^{d,1}(\mathbb{R}^d)\), choose a sequence \(\{f_n\}_{n \geq 0}\) of Schwartz functions such that \(f_n \rightarrow f\) in \(W^{d,1}(\mathbb{R}^d)\) (and, therefore, in distributional sense). We have
\[
\|f_n - f_m\|_\infty \leq \|f_n - f_m\|_{W^{d,1}} \rightarrow 0, \quad n, m \rightarrow \infty.
\]
Thus, \(\{f_n\}_{n \geq 0}\) is a Cauchy sequence in \(L_\infty(\mathbb{R}^d)\). Therefore, \(f_n \rightarrow h\) in \(L_\infty(\mathbb{R}^d)\) (and, therefore, in distributional sense). By uniqueness of the limit, \(h = f\). Hence, \(f_n \rightarrow f\) in \(L_\infty(\mathbb{R}^d)\). Since each \(f_n\) is continuous, then so is \(f\).
Step 3: To see the assertion in general case, take \( g \in X^* \). The function \( l_g : s \to \langle g, f(s) \rangle \) belongs to \( W^{d,1}(\mathbb{R}^d) \). By Step 2, \( l_g \) is continuous for every \( g \in X^* \) and, therefore, \( f \) is weak* continuous. Clearly,
\[
\|l\|_{W^{d,1}} \leq \|g\|_{X^*} \|f\|_{W^{d,1}(\mathbb{R}^d, X)}.
\]
By Step 2, we have
\[
\|l\|_{\infty} \leq \|g\|_{X^*} \|f\|_{W^{d,1}(\mathbb{R}^d, X)}.
\]
Taking supremum over the unit ball in \( X^* \), we obtain
\[
\|f\|_{L^\infty(\mathbb{R}^d, X)} \leq \|f\|_{W^{d,1}(\mathbb{R}^d, X)}.
\]

Lemma 8.3. We have \( \mathcal{F}((L_1 \cap L_\infty)(\mathbb{R}^d, X)) \subset BW\mathcal{C}(\mathbb{R}^d, X) \).

Proof. Let \( f \in L_1(\mathbb{R}^d, X) \). It is obvious that
\[
\|\mathcal{F}f\|_{L^\infty(\mathbb{R}^d, X)} \leq \|f\|_{L_1(\mathbb{R}^d, X)}.
\]
Let \( \mathbb{B}^d \) be the unit ball in \( \mathbb{R}^d \) centered at \( 0 \).
Fix \( \epsilon > 0 \) and choose \( n \in \mathbb{N} \) such that
\[
\int_{\mathbb{R}^d \setminus \mathbb{B}^d} \|f(u)\|_X du < \epsilon.
\]
We have
\[
\left\| (\mathcal{F}f)(s_1) - (\mathcal{F}f)(s_2) \right\|_X \leq \left\| \int_{\mathbb{R}^d \setminus \mathbb{B}^d} f(u)(e^{-2\pi i(u,s_1)} - e^{-2\pi i(u,s_1)}) du \right\|_X + \left\| \int_{\mathbb{B}^d} f(u)(e^{-2\pi i(u,s_1)} - e^{-2\pi i(u,s_1)}) du \right\|_X \leq 2\epsilon + \|f\|_{L^\infty(\mathbb{R}^d, X)} \int_{\mathbb{B}^d} |e^{-2\pi i(u,s_1)} - e^{-2\pi i(u,s_1)}| du \leq 2\epsilon + 2\pi \|f\|_{L^\infty(\mathbb{R}^d, X)} \cdot \int_{\mathbb{B}^d} \|u\|_2 \|s_1 - s_2\|_2 du \leq 2\epsilon + 2\pi \|f\|_{L^\infty(\mathbb{R}^d, X)} \cdot n^{d+1} \|s_1 - s_2\|_2 \cdot \text{vol}(\mathbb{B}^d).
\]
If
\[
\|s_1 - s_2\|_2 \leq n^{-d-1} \|f\|_{L^\infty(\mathbb{R}^d, X)} \epsilon,
\]
then
\[
\left\| (\mathcal{F}f)(s_1) - (\mathcal{F}f)(s_2) \right\|_X \leq c_d \epsilon.
\]
Since \( \epsilon \) is arbitrarily small, the assertion follows. \( \square \)

Lemma 8.4. We have \( W^{d,1}(\mathbb{R}^d, X) \subset (l_1(L_\infty))(\mathbb{R}^d, X) \). Moreover, we have
\[
\|f\|_{l_1(L_\infty)(\mathbb{R}^d, X)} \leq c_d \|f\|_{W^{d,1}(\mathbb{R}^d, X)}, \quad f \in W^{d,1}(\mathbb{R}^d, X).
\]

Proof. Let \( \phi \) be a smooth function supported on \([{-1,1}]^d\) such that \( \phi = 1 \) on \([{-\frac{1}{2}, \frac{1}{2}}]^d\). We have
\[
\|f\|_{l_1(L_\infty)(\mathbb{R}^d, X)} \leq \sum_{n \in \mathbb{Z}^d} \|f \cdot T_n \phi\|_{L_\infty(\mathbb{R}^d, X)}.
\]
Using Lemma 8.2 we have
\[
\|f\|_{l_1(L_\infty)(\mathbb{R}^d, X)} \leq \sum_{n \in \mathbb{Z}^d} \|f \cdot T_n \phi\|_{W^{d,1}(\mathbb{R}^d, X)} \leq \]
\[ \leq \|\phi\|_{C^d([-1,1]^d)} \cdot \sum_{n \in \mathbb{Z}^d} \|f\|_{W^{d,1}(n+[-1,1]^d, X)} = 2^d \|\phi\|_{C^d([-1,1]^d)} \|f\|_{W^{d,1}(\mathbb{R}^d, X)}. \]

\[ \square \]

**Lemma 8.5.** For \( p > d \), we have \( F(W^{p,1}(\mathbb{R}^d, X)) \subset (l_1(L_\infty))(\mathbb{R}^d, X) \). Moreover, we have
\[ \|Ff\|_{l_1(L_\infty)(\mathbb{R}^d, X)} \leq c_{p,d}\|f\|_{W^{p,1}(\mathbb{R}^d, X)}, \quad f \in W^{p,1}(\mathbb{R}^d, X). \]

**Proof.** We have
\[ |s|^p \cdot \|(Ff)(s)\|_X = \|(\Delta^{\frac{d}{2}} f)(s)\|_X \leq \|(\Delta^{\frac{d}{2}} f)\|_{L_\infty(\mathbb{R}^d, X)} \leq \|\Delta^{\frac{d}{2}} f\|_{L_1(\mathbb{R}^d, X)} \leq \|f\|_{W^{p,1}(\mathbb{R}^d, X)}, \]
\[ \|(Ff)(s)\|_X \leq \|F(f)\|_{L_\infty(\mathbb{R}^d, X)} \leq \|f\|_{L_1(\mathbb{R}^d, X)} \leq \|f\|_{W^{p,1}(\mathbb{R}^d, X)}. \]
That is,
\[ \|(Ff)(s)\|_X \leq \min\{|s|^{-p}, 1\} \cdot \|f\|_{W^{p,1}(\mathbb{R}^d, X)}. \]
Since the mapping
\[ s \to \min\{|s|^{-p}, 1\} \]
belongs to \( (l_1(L_\infty))(\mathbb{R}^d) \), the assertion follows. \[ \square \]

**Proof of Theorem 8.7.** By Lemma 8.2, \( f(s) \) makes sense for every \( f \in W^{d,1}(\mathbb{R}^d, X) \) and for every \( s \in \mathbb{R}^d \). By Lemma 8.3 we have
\[ \sum_{n \in \mathbb{Z}^d} \|f(n)\|_X \leq \sum_{n \in \mathbb{Z}^d} \sup_{s \in n+[-\frac{1}{2},\frac{1}{2}]^d} \|f(s)\|_X = \|f\|_{l_1(L_\infty)(\mathbb{R}^d, X)} \leq c_d\|f\|_{W^{d,1}(\mathbb{R}^d, X)}. \]
In particular, the series in the left hand side converges in \( X \) and
\[ \| \sum_{n \in \mathbb{Z}^d} f(n) \|_X \leq c_d\|f\|_{W^{d,1}(\mathbb{R}^d, X)}. \]
That is, left hand side defines a bounded mapping \( T: W^{d,1}(\mathbb{R}^d) \to X. \)

By Lemma 8.3 \( (Ff)(s) \) makes sense for every \( f \in W^{d,1}(\mathbb{R}^d, X) \) and for every \( s \in \mathbb{R}^d \). By Lemma 8.3 we have
\[ \sum_{n \in \mathbb{Z}^d} \|(Ff)(n)\|_X \leq \sum_{n \in \mathbb{Z}^d} \sup_{s \in n+[-\frac{1}{2},\frac{1}{2}]^d} \|(Ff)(s)\|_X = \|Ff\|_{l_1(L_\infty)(\mathbb{R}^d, X)} \leq c_{p,d}\|f\|_{W^{p,1}(\mathbb{R}^d, X)}. \]
In particular, the series in the right hand side converges in \( X \) and
\[ \| \sum_{n \in \mathbb{Z}^d} (Ff)(n) \|_X \leq c_{p,d}\|f\|_{W^{p,1}(\mathbb{R}^d, X)}. \]
That is, right hand side defines a bounded mapping \( S: W^{p,1}(\mathbb{R}^d) \to X. \)

If \( f \) is vector valued Schwartz function and if \( g \in X^* \), then \( l_g: s \to \langle g, f(s) \rangle \) is a Schwartz function. We have
\[ \langle g, Tf \rangle = \sum_{n \in \mathbb{Z}^d} \langle g, f(n) \rangle = \sum_{n \in \mathbb{Z}^d} l_g(n). \]
We also have
\[ \langle g, Sf \rangle = \sum_{n \in \mathbb{Z}^d} \langle g, (Ff)(n) \rangle = \sum_{n \in \mathbb{Z}^d} (Fl_g)(n). \]
We take Poisson formula for scalar valued Schwartz functions for granted — it follows from the distributional equality
\[ \sum_{n \in \mathbb{Z}^d} \delta(x - n) = \sum_{n \in \mathbb{Z}^d} e^{2\pi i (n, x)}, \quad x \in \mathbb{R}^d. \]

That is, we have
\[ \sum_{n \in \mathbb{Z}^d} l_{n}(n) = \sum_{n \in \mathbb{Z}^d} (\mathcal{F}l_{n})(n). \]

Combining these 3 equalities, we infer
\[ \langle g, T f \rangle = \langle g, S f \rangle, \quad g \in X^*. \]

In other words, \( T f = S f \) for every vector valued Schwartz function.

That is, we have 2 bounded linear maps from \( W^{p,1}(\mathbb{R}^d, X) \) to \( X \). These maps coincide on the subspace of vector valued Schwartz functions. Since vector valued Schwartz functions are dense in \( W^{p,1}(\mathbb{R}^d, X) \), it follows immediately that these maps coincide on \( W^{p,1}(\mathbb{R}^d, X) \). This completes the proof.

**Proof of Theorem 5.2** By Theorem 8.1 we have
\[ \sum_{n \in \mathbb{Z}^d} f(tn) = \sum_{n \in \mathbb{Z}^d} (\sigma_{\frac{1}{t}} f(n)) = \sum_{n \in \mathbb{Z}^d} (\mathcal{F}(\sigma_{\frac{1}{t}} f))(n) = t^{-d} \sum_{n \in \mathbb{Z}^d} (\mathcal{F}f)(t^{-1}n). \]

In other words, we have
\[ \sum_{n \in \mathbb{Z}^d} f(tn) - t^{-d} \int_{\mathbb{R}^d} f(u) du = t^{-d} \sum_{0 \neq n \in \mathbb{Z}^d} (\mathcal{F}f)(t^{-1}n). \]

Recall that
\[ |s|^p \cdot \| (\mathcal{F}f)(s) \|_X = \| (\mathcal{F}(\Delta_{\frac{1}{t}}^p f))(s) \|_X \leq \| \mathcal{F}(\Delta_{\frac{1}{t}}^p f) \|_{L^\infty(\mathbb{R}^d, X)} \leq \| \Delta_{\frac{1}{t}}^p f \|_{L^1(\mathbb{R}^d, X)} \leq \| f \|_{W^{p,1}(\mathbb{R}^d, X)}. \]

Hence, for \( n \neq 0 \),
\[ \| (\mathcal{F}f)(t^{-1}n) \|_X \leq \frac{t^p}{|n|^p} \| f \|_{W^{p,1}(\mathbb{R}^d, X)}. \]

We now infer that
\[ \left\| \sum_{0 \neq n \in \mathbb{Z}^d} (\mathcal{F}f)(t^{-1}n) \right\|_X \leq \sum_{0 \neq n \in \mathbb{Z}^d} \frac{t^p}{|n|^p} \| f \|_{W^{p,1}(\mathbb{R}^d, X)} = c_p t^p \| f \|_{W^{p,1}(\mathbb{R}^d, X)}. \]

This completes the proof. \( \square \)

**Lemma 9.1.** We have
\[ \text{Good}_k(s) = \frac{1}{2\pi i} \int_{1+i\mathbb{R}} \text{good}_k(s, z) e^z dz. \]

**Proof.** The crucial fact is that, for \( s \neq 0 \), the mapping \( z \to \text{good}_k(s, z) \) is holomorphic in the half-plane \( \{ \Re(z) > -\epsilon \} \), where \( \epsilon \) depends on \( s \). We have
\[ (9.1) \quad \frac{1}{2\pi} \int_{\mathbb{R}} \text{good}_k(s, i\lambda) e^{i\lambda} d\lambda = \frac{1}{2\pi i} \int_{i\mathbb{R}} \text{good}_k(s, z) e^z dz. \]
We claim that
\[ \int_{1+i\mathbb{R}} \text{good}_k(s, z) e^z dz = \int_{i\mathbb{R}} \text{good}_k(s, z) e^z dz. \]
Indeed, we have
\[ \int_{1+i\mathbb{R}} \text{good}_k(s, z) e^z dz = \lim_{N \to \infty} \int_{1-iN}^{1+iN} \text{good}_k(s, z) e^z dz. \]
Using analyticity and Cauchy theorem, we write
\[ \int_{1-iN}^{1+iN} \text{good}_k(s, z) e^z dz + \int_{-iN}^{iN} \text{good}_k(s, z) e^z dz. \]
Obviously,
\[ \left\| \int_{1-iN}^{1+iN} \text{good}_k(s, z) e^z dz \right\|_\infty \leq e \sup_{t \in (0,1)} \| \text{good}_k(s, t-iN) \|_\infty. \]
Thus,
\[ \int_{1-iN}^{1+iN} \text{good}_k(s, z) e^z dz = o(1), \quad N \to \infty. \]
Similarly,
\[ \int_{1+iN}^{1+iN} \text{good}_k(s, z) e^z dz = o(1), \quad N \to \infty. \]
This proves the claim and, hence, the assertion of the lemma.

**Lemma 9.2.** Let \( \alpha, \beta \in \mathbb{Z}_+^d \) and let
\[ g(s, z) = (x(s) + z)^{-N} s^\beta. \]
For \( \Re z > 0 \) and for every \( s \in \mathbb{R}^d \), we have
\[ \left\| \partial^\alpha \frac{d^N}{dNz} g(s, z) \right\|_\infty = O((|s|^2 + |z|)^{\frac{1}{2} |\beta|_1 + \frac{1}{2} |\alpha|_1 - N - 1}). \]

**Proof.** Set \( h(s, z) = (x(s) + z)^{-N} \). We claim that
\[ \left\| \partial^\alpha h(s, z) \right\|_\infty = O((|s|^2 + |z|)^{\frac{1}{2} |\alpha|_1 - N}). \]
We prove the assertion by induction on \( N \). For \( N = 1 \), it is obvious. Let us prove it for \( N + 1 \).

Let \( h = h_1 h_2 \), where \( h_1(s, z) = (x(s) + z)^{-N} \) and \( h_2(s, z) = (x(s) + z)^{-1} \). By Leibniz rule, we have
\[ \partial^\alpha h = \sum_{\alpha_1 + \alpha_2 = \alpha} c^\alpha_{\alpha_1} \partial^{\alpha_1} h_1 \cdot \partial^{\alpha_2} h_2. \]
By triangle inequality, we have
\[ \left\| \partial^\alpha h(s, z) \right\|_\infty \leq \sum_{\alpha_1 + \alpha_2 = \alpha} c^\alpha_{\alpha_1} \left\| \partial^{\alpha_1} h_1(s, z) \right\|_\infty \cdot \left\| \partial^{\alpha_2} h_2(s, z) \right\|_\infty. \]
By inductive assumption, we have
\[ \left\| \partial^{\alpha_1} h_1(s, z) \right\|_\infty = O((|s|^2 + |z|)^{\frac{1}{2} |\alpha_1|_1 - N}). \]
Obviously,
\[ \left\| \partial^\alpha \partial^\beta g(s, z) \right\|_\infty = O((|s|^2 + |z|)^{-\frac{1}{2}|\alpha_2| - 1}). \]
Combining these 3 estimates we establish the claim.

The assertion for \( \beta = 0 \) follows immediately from the claim above.

Consider now the general case. Let \( g = h_1 h_2 \), where \( h_1(s, z) = s^\beta \) and \( h_2(s, z) = (x(s) + z)^{-1} \). By Leibniz rule, we have
\[ \partial^\alpha \frac{d^N}{d^N z} g(s, z) = \sum_{\alpha_1 + \alpha_2 = \alpha} c_{\alpha_1}^{\alpha} \partial^\alpha h_1 \cdot \partial^\alpha \frac{d^N}{d^N z} h_2. \]
By triangle inequality, we have
\[ \left\| \partial^\alpha \frac{d^N}{d^N z} g(s, z) \right\|_\infty \leq \sum_{\alpha_1 + \alpha_2 = \alpha} c_{\alpha_1}^{\alpha} \left\| \partial^\alpha h_1(s, z) \right\|_\infty \cdot \left\| \partial^\alpha \frac{d^N}{d^N z} h_2(s, z) \right\|_\infty. \]
By the special case proved above, we have
\[ \left\| \partial^\alpha \frac{d^N}{d^N z} h_2(s, z) \right\|_\infty = O((|s|^2 + |z|)^{-\frac{1}{2}|\alpha_2| - N - 1}). \]
Evidently,
\[ \left\| \partial^\alpha h_1(s, z) \right\|_\infty = \begin{cases} O(|s|^{|\beta| - |\alpha|}), & \alpha \leq \beta \\ 0, & \alpha \not\leq \beta \end{cases}. \]
A combination of these 3 estimates yields the assertion. \( \square \)

**Lemma 9.3.** Let \( a_l \in L_\infty(\mathbb{T}^d_\theta), 1 \leq l \leq \bar{L} \), and let
\[ g(s, z) = \prod_{l=1}^{\bar{L}} a_l(x(s) + z)^{-1}. \]
For \( \Re z > 0 \) and for every \( s \in \mathbb{R}^d \), we have
\[ \left\| \partial^\alpha \frac{d^N}{d^N z} g(s, z) \right\|_\infty = O((|s|^2 + |z|)^{-\frac{1}{2}|\alpha| - L - N}). \]

**Proof:** We prove the assertion by induction on \( \bar{L} \). For \( \bar{L} = 1 \), the assertion follows from Lemma 4.2 (applied with \( \beta = 0 \)). Suppose the assertion holds for \( \bar{L} \). Let us prove it for \( \bar{L} + 1 \).

Let \( g = h_1 h_2 \), where
\[ h_1(s, z) = \prod_{l=1}^{\bar{L}} a_l(x(s) + z)^{-1}, \quad h_2(s, z) = a_{\bar{L} + 1}(x(s) + z)^{-1}. \]
By Leibniz rule, we have
\[ \partial^\alpha \frac{d^N}{d^N z} g(s, z) = \sum_{\alpha_1 + \alpha_2 = \alpha, N_1 + N_2 = N} c_{\alpha_1, \alpha_2}^{\alpha} \partial^\alpha h_1 \cdot \partial^\alpha \frac{d^N}{d^N z} h_2. \]
By triangle inequality, we have
\[ \left\| \partial^\alpha \frac{d^N}{d^N z} g(s, z) \right\|_\infty \leq \sum_{\alpha_1 + \alpha_2 = \alpha, N_1 + N_2 = N} c_{\alpha_1, \alpha_2}^{\alpha} \left\| \partial^\alpha h_1(s, z) \right\|_\infty \cdot \left\| \partial^\alpha \frac{d^N}{d^N z} h_2(s, z) \right\|_\infty. \]
By inductive assumption, we have
\[
\left\| \partial^{\alpha_1} \frac{d^{N_1}}{d^{N_1}z} h_1(s, z) \right\|_\infty = O((|s|^2 + |z|)^{-\frac{1}{2}|\alpha_1|-L-N_1}).
\]
By Lemma 9.2, we have
\[
\left\| \partial^{\alpha_2} \frac{d^{N_2}}{d^{N_2}z} h_2(s, z) \right\|_\infty = O((|s|^2 + |z|)^{-\frac{1}{2}|\alpha_2|-1-N_2}).
\]
A combination of these 3 estimates yields the assertion. □

**Lemma 9.4.** Let \( a_l \in L_\infty(\mathbb{T}_d^d), 1 \leq l \leq L, \) and let
\[
g(s, z) = (x(s) + z)^{-1}s^\beta \prod_{l=1}^L a_l(x(s) + z)^{-1}.
\]
For \( \Re z > 0 \) and for every \( s \in \mathbb{R}^d \), we have
\[
\left\| \partial^\alpha \frac{d^N}{d^N z} g(s, z) \right\|_\infty = O((|s|^2 + |z|)^{\frac{1}{2}|\beta|_1} - \frac{1}{2}|\alpha_1|-L-N-1)).
\]

**Proof.** Let
\[
h_1(s, z) = (x(s) + z)^{-1}s^\beta, \quad h_2(s, z) = \prod_{l=1}^L a_l(x(s) + z)^{-1}.
\]
By Leibniz rule, we have
\[
\frac{d^N}{d^N z} = \sum_{\alpha_1 + \alpha_2 = \alpha} c_{N, \alpha} \frac{d^{N_1}}{d^{N_1}z} h_1 \cdot \frac{d^{N_2}}{d^{N_2}z} h_2.
\]
By triangle inequality, we have
\[
\left\| \frac{d^N}{d^N z} g(s, z) \right\|_\infty \leq \sum_{\alpha_1 + \alpha_2 = \alpha, N_1 + N_2 = N} c_{N, \alpha} \left\| \frac{d^{N_1}}{d^{N_1}z} h_1(s, z) \right\|_\infty \cdot \left\| \frac{d^{N_2}}{d^{N_2}z} h_2(s, z) \right\|_\infty.
\]
By Lemma 9.2, we have
\[
\left\| \partial^{\alpha_1} \frac{d^{N_1}}{d^{N_1}z} h_1(s, z) \right\|_\infty = O((|s|^2 + |z|)^{\frac{1}{2}|\beta|_1} - \frac{1}{2}|\alpha_1|-L-N_1-1)).
\]
By Lemma 9.3, we have
\[
\left\| \partial^{\alpha_2} \frac{d^{N_2}}{d^{N_2}z} h_2(s, z) \right\|_\infty = O((|s|^2 + |z|)^{-\frac{1}{2}|\alpha_2|-L-N_2}).
\]
A combination of these 3 estimates yields the assertion. □

**Lemma 9.5.** For \( \Re z > 0 \) and for every \( s \in \mathbb{R}^d \), we have
\[
\left\| \partial^\alpha \frac{d^N}{d^N z} \text{good}_k(s, z) \right\|_2 = O((|s|^2 + |z|)^{\frac{1}{2}-\frac{1}{2}|\alpha|_1-N}).
\]

**Proof.** By induction, \( \text{good}_k(s, z) \) is a sum of finitely many terms of the shape
\[
g(s, z) = (x(s) + z)^{-1}s^\beta \cdot \prod_{l=1}^L a_l(x(s) + z)^{-1},
\]
where \( |\beta|_1 = 2L - k \) and \( a_l \in C^\infty(\mathbb{T}_d^d), 1 \leq l \leq L. \) The assertion follows from Lemma 9.4 □
Lemma 9.6. For \( k \geq 0 \) and \( p > 0 \), we have \( \text{Good}_k \in W^{p,1}(\mathbb{R}^d, L_2(T^d_\theta)) \).

Proof. Using Lemma 9.1 and integration by parts, we obtain

\[
\text{Good}_k(s) = \frac{(-1)^N}{2\pi i} \int_{1+i\mathbb{R}} \frac{d^N}{dN z} \text{good}_k(s, z) e^z dz.
\]

Heuristically, we have

\[
\partial^\alpha \text{Good}_k(s) = \frac{(-1)^N}{2\pi i} \int_{1+i\mathbb{R}} \partial^\alpha \frac{d^N}{dN z} \text{good}_k(s, z) e^z dz.
\]

This formula is indeed true because the integral in the right hand side converges absolutely by Lemma 9.5. Moreover, we have

\[
\|\partial^\alpha \text{Good}_k(s)\|_2 \leq c N, \alpha, g \int_{\mathbb{R}} \left( |s|^2 + 1 + \lambda \right)^{-\frac{d}{2} - 1 - \frac{1}{2} |\alpha| - N - k - 1} d\lambda \leq c' N, \alpha, g \left( |s|^2 + 1 \right)^{-\frac{d}{2} - 1 - |\alpha| - N}.
\]

In particular, \( \partial^\alpha \text{Good}_k \in L_1(\mathbb{R}^d, L_2(T^d_\theta)) \) for every \( \alpha \in \mathbb{Z}^d_+ \).

\[ \Box \]

Corollary 9.7. For every \( k \geq 0 \), the series

\[
\sum_{n \in \mathbb{Z}^d} \text{Good}_k(n t^{1/2})
\]

converges in \( L_2(T^d_\theta) \). We have

\[
\sum_{n \in \mathbb{Z}^d} \text{Good}_k(n t^{1/2}) = t^{-\frac{d}{2}} \cdot \int_{\mathbb{R}^d} \text{Good}_k(s) ds + O(t^{\infty}).
\]

Proof. The assertion follows immediately from Lemma 9.6 and Theorem 5.5.

\[ \Box \]

Proof of Theorem 5.6. Let

\[
F(t) = \sum_{n \in \mathbb{Z}^d} \left( \lambda_r(e_n)^* e^{-tA_s \lambda_r(e_n)}(1), \right)
\]

where the series converges weakly in \( L_2(T^d_\theta) \) by Lemma 5.1. By Theorem 5.3 we have

\[
\left\| F(t) - \sum_{n \in \mathbb{Z}^d} \left( \sum_{k=0}^{2d} t^{\frac{k}{2}} \text{Good}_k(n t^{1/2}) \right) \right\|_2 = O(1)
\]

By Corollary 9.7, we have

\[
\left\| F(t) - \sum_{k=0}^{2d} t^{\frac{k-d}{2}} \int_{\mathbb{R}^d} \text{Good}_k(s) ds \right\|_2 = O(1).
\]

Obviously, the terms with \( k \geq d \) are bounded. Since \( \text{Good}_k \) is an odd function when \( k \) is odd, it follows that respective summand is 0. Recall that

\[
I_k = \int_{\mathbb{R}^d} \text{Good}_k(s) ds, \quad 0 \leq k \leq d.
\]

We now have

\[
\left\| F(t) - \sum_{0 \leq k < d} t^{\frac{k-d}{2}} I_k \right\|_2 = O(1).
\]
Let Lemma 10.1.

Let $\theta$. For simplicity, it makes sense to set $\theta_{kl} = 0$ when $k > d$ or when $l > d$. We have $L^\infty(T^d_{g'}) = L^\infty(T^d_{g}) \otimes L^\infty(T^{d'}_{d'})$ (see Example 2.1).

Define a metric $g'$ (size of $g'$ is $d'$) whose left upper corner is $g$. We ask that $g_{kl} = \delta_{k,l}$ when either $k > d$ or $l > d$.

**Lemma 10.1.** Let $\nu'$ be a version of $\nu$ constructed from the metric tensor $g'$. We have $\nu' = \nu \otimes 1$.

**Proof.** We have

$$\nu' = \pi^{-\frac{d}{2}} \int_{\mathbb{R}^d} e^{-\sum_{i,j=1}^{d'} t_i t_j (g')_{ij}} dt \otimes \pi^{-\frac{d' - d}{2}} \int_{\mathbb{R}^{d'}} e^{-\sum_{i,j=1}^{d'} t_i t_j (g')_{ij}} dt = \nu \otimes 1.$$

**Lemma 10.2.** Let $\text{Corr}'_k$ be a version of $\text{Corr}_k$ constructed from the metric tensor $g'$. We have

$$\text{Corr}'_k(s) = \text{Corr}_k(u) \otimes e^{-|v|^2}, \quad u = (s_1, \ldots, s_d), \quad v = (s_{d+1}, \ldots, s_{d'}).$$

**Proof.** Let $\text{corr}'_k$ be a version of $\text{corr}_k$ constructed from the metric tensor $g'$.

We have

$$\text{corr}'_k(s, i\lambda) = \text{corr}_k(u, |v|^2 + i\lambda) \otimes 1, \quad u = (s_1, \ldots, s_d), \quad v = (s_{d+1}, \ldots, s_{d'}).$$

The crucial fact is that, for $u \neq 0$, the mapping $z \rightarrow \text{corr}_k(u, z)$ is holomorphic in the half-plane $\{ \Re(z) > -\epsilon \}$, where $\epsilon$ depends on $u$. Therefore, we have

$$\frac{1}{2\pi} \int_{\mathbb{R}} \text{corr}'_k(s, i\lambda) e^{i\lambda} d\lambda = e^{-|v|^2} \frac{1}{2\pi} \int_{|v|^2 + i\mathbb{R}} \text{corr}_k(u, z) e^z dz.$$ (10.1)

We claim that

$$\int_{|v|^2 + i\mathbb{R}} \text{corr}_k(u, z) e^z dz = \int_{i\mathbb{R}} \text{corr}_k(u, z) e^z dz.$$ 

Indeed, we have

$$\int_{|v|^2 + i\mathbb{R}} \text{corr}_k(u, z) e^z dz = \lim_{N \to \infty} \int_{|v|^2 + iN} \text{corr}_k(u, z) e^z dz.$$ 

We now write

$$\int_{|v|^2 - iN} \text{corr}_k(u, z) e^z dz = \int_{|v|^2 - iN} \text{corr}_k(u, z) e^z dz + \int_{iN}^{iN} \text{corr}_k(u, z) e^z dz + \int_{iN}^{-iN} \text{corr}_k(u, z) e^z dz.$$ 

Obviously,

$$\left\| \int_{|v|^2 - iN} \text{corr}_k(u, z) e^z dz \right\|_{\infty} \leq |v|^2 e^{|v|^2} \sup_{t \in (0, |v|^2)} \| \text{corr}_k(u, t - iN) \|_{\infty}.$$
Thus,
\[
\int_{|v|^2-iN}^{-iN} \text{corr}_k(u, z)e^{\tau}dz = o(1), \quad N \to \infty.
\]
Similarly,
\[
\int_{iN}^{[|v|^2+iN} \text{corr}_k(u, z)e^{\tau}dz = o(1), \quad N \to \infty.
\]
This proves the claim.
The assertion follows from the above claim and (10.1). □

Lemma 10.3. Let \( I_k' \) be a version of \( I_k \) constructed from the metric tensor \( g' \). We have
\[
I_k' = \pi^{\frac{d'-d}{2}} I_k \otimes 1.
\]
Proof. Obviously,
\[
\int_{\mathbb{R}^{d'-d}} e^{-|v|^2}dv = \pi^{\frac{d'-d}{2}}.
\]
The assertion follows now from Lemma 10.2. □

Proof of Theorem 1.2. By Lemma 10.1, \( \lambda_l((\nu')^{-\frac{1}{2}}) \sum_{i,j=1}^{d} D_i \lambda_l((\nu')^{\frac{1}{2}}(g^{-1})_{ij}(\nu')^{\frac{1}{2}})D_j \lambda_l((\nu')^{-\frac{1}{2}}) + \sum_{i=d+1}^{d'} D_i^2 \).

By Lemma 10.1, the first summand is exactly \( A_g \otimes 1 \). The second summand is, clearly, \( 1 \otimes \Delta \). Consequently, we have
\[
A_g' = A_g \otimes 1 + 1 \otimes \Delta.
\]
This implies
\[
e^{-tA_g'} = e^{-tA_g} \otimes 1 - t\Delta = e^{-tA_g} \otimes e^{-t\Delta}.
\]
Also, if \( x \in L_\infty(T^d) \), then \( x \otimes 1 \in L_\infty(T^d) \). We have
\[
\lambda_l(x) e^{-tA_g'} = \lambda_l(x) e^{-tA_g} \otimes e^{-t\Delta}.
\]
Therefore,
\[
\text{Tr}(\lambda_l(x) e^{-tA_g'}) = \text{Tr}(\lambda_l(x) e^{-tA_g}) \cdot \text{Tr}(e^{-t\Delta}).
\]
It follows from the Poisson summation formula that
\[
\text{Tr}(e^{-t\Delta}) = \left( \frac{\pi}{t} \right)^{\frac{d'-d}{2}} \cdot \left( 1 + O(t^\infty) \right).
\]
By Theorem 5.6 we have
\[
\text{Tr}(\lambda_l(w) e^{-tA_g'}) = t^{-\frac{d'}{2}} \sum_{0 \leq k < d' \atop k = 0 \mod 2} t^k \tau(wI_k') + O(1), \quad t \downarrow 0,
\]
for every \( w \in L_\infty(T^d) \). Setting \( w = x \otimes 1 \), we infer from Lemma 10.3 that
\[
\text{Tr}(\lambda_l(x) e^{-tA_g}) \cdot \left( \frac{\pi}{t} \right)^{\frac{d'-d}{2}} \cdot \left( 1 + O(t^\infty) \right) = \pi^{\frac{d'-d}{2}} t^{\frac{d'}{2}} \sum_{0 \leq k < d' \atop k = 0 \mod 2} t^k \tau(xI_k') + O(1).
\]
It follows immediately that
\[ \text{Tr}(\lambda_l(x) e^{-tA_g}) = t^{-\frac{d}{2}} \sum_{0 \leq k < d' \atop k = 0 \mod 2} t^{\frac{k}{2}} \tau(xI_k) + O(t^{\frac{d' - d}{2}}). \]

Taking as large $d'$ as needed, we obtain an asymptotic expansion.

Finally, we have
\[ e^{-t\Delta_g} = \lambda_l(\nu^{-\frac{1}{2}}) e^{-tA_g} \lambda_l(\nu^{\frac{1}{2}}). \]

Thus,
\[ \text{Tr}(\lambda_l(x) e^{-t\Delta_g}) = \text{Tr}(\lambda_l(x) \cdot \lambda_l(\nu^{-\frac{1}{2}}) e^{-tA_g} \lambda_l(\nu^{\frac{1}{2}})) = \text{Tr}(\lambda_l(\nu^{\frac{1}{2}} x \nu^{-\frac{1}{2}}) e^{-tA_g}). \]

By the already proved asymptotic expansion, we have
\[ \text{Tr}(\lambda_l(x) e^{-t\Delta_g}) \sim t^{-\frac{d}{2}} \sum_{k = 0 \mod 2} t^{\frac{k}{2}} \tau(\nu^{\frac{1}{2}} x \nu^{-\frac{1}{2}} \cdot I_k) = t^{-\frac{d}{2}} \sum_{k = 0 \mod 2} t^{\frac{k}{2}} \tau(x \cdot \nu^{-\frac{1}{2}} I_k \nu^{\frac{1}{2}}). \]

□

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