THE SPACE OF VECTOR FIELDS ON QUANTUM GROUPS

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Abstract

We construct the space of vector fields on quantum groups. Its elements are products of the known left invariant vector fields with the elements of the quantum group itself. We also study the duality between vector fields and 1-forms. The construction is easily generalized to tensor fields. A Lie derivative along any (also non left invariant) vector field is proposed. These results hold for a generic Hopf algebra.

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1 Introduction

Following the program of generalizing the differential geometry structures to the noncommutative case, we construct on a Hopf algebra the analogue of the space of vector fields.

Indeed in the literature the quantum Lie algebra of left invariant vector fields as well as the space of 1-forms has been extensively analyzed [1, 2, 3, 4, 5], while the notion of generic vector field on a Hopf algebra and the duality relation with the space of 1-forms deserves more study [6].

We will see how left invariant vector fields generate the whole space of vector fields. This space can be also characterized as the bicovariant bimodule (vector bundle) dual to that of 1-forms.

Throughout this paper we will deal with a Hopf algebra $A$ over $C$ with coproduct $\Delta : A \rightarrow A \otimes A$, counit $\varepsilon : A \rightarrow C$, and invertible antipode $S : A \rightarrow A$. Particular cases of Hopf algebras are quantum groups, which for us will be Hopf algebras with one (or more) continuous parameter $q$; when $q=1$ the product $\cdot$ in $A$ becomes commutative and we obtain the algebra of functions on a group. When we will speak of commutative case we will refer to the Hopf algebra $C^\infty(G)$ of smooth functions on a (compact) Lie group $G$.

In Section 2 we briefly recall how to associate a differential calculus to a given Hopf algebra and we emphasize the role played by the tangent vectors. This construction will be effected along the lines of Woronowicz’ work [2]. Indeed the results in [2] apply also to a general Hopf algebra with invertible antipode (not necessary a compact matrix pseudogroup). This can be shown by checking that all the formulae used for the construction described in [2] (which are collected in the appendix of [2]) hold also in the case of a Hopf algebra with invertible antipode $S$.

In Section 3 we construct the space of vector fields, while in Section 4 we study the action of the Hopf algebra on the vector fields; i.e., we will study the pushforward of vector fields on Hopf algebras. Then we deal with (covariant and contravariant) tensor fields.

In the last section we propose a Lie derivative and a contraction operator on differential forms along generic vector fields. These two operators are basic tools for the formulation of deformed gravity theories, where the relevant Lie algebra is the $q$-Poincaré Lie algebra.

2 Differential Geometry on Hopf Algebras

In the commutative case, given the differential calculus on a (compact) Lie group $G$, we can consider the subspace in the space of all smooth functions $f : G \rightarrow C$

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1Formula (A.22) in [2] is the most difficult to prove and necessitates the further axiom of the invertibility of the antipode $S$; also the invertibility of the map $s$ in (A.18) relies on the existence of $S^{-1}$. All the other formulae are direct consequences of the Hopf algebra axioms.
defined by:

\[ R \equiv \{ h \in C^\infty(G) \mid h(1_G) = 0 \text{ and } dh(1_G) = 0 \} , \]  

(2.1)

where \( 1_G \) is the unit of the group.

\( R \) is a particular ideal of the Hopf algebra \( C^\infty(G) \). All the information about the differential calculus on \( G \) is contained in \( R \).

Indeed the space of tangent vectors at the origin of the group is given by all the linear functionals which annihilate \( R \) and any constant function. Locally we write a basis as \( \{ \partial_i |_{1_G} \} \). Once we have this basis, using the tangent map (namely \( TL_g \)) induced by the left multiplication of the group on itself:

\[ L_g g' = gg' , \forall g, g' \in G \]

we can construct a basis of left invariant vector fields \( \{ t_i \} \). Then a generic 1-form can be written \( \rho = f_i \omega^i \) \( [f_i \in C^\infty(G)] \) where \( \{ \omega^i \} \) is the dual basis of \( \{ t_i \} \). Finally, the differential on functions is

\[ d = \omega^i t_i \quad \text{that is} \quad df = t_i(f) \omega^i . \]

In \cite{2} the quantum analogue of \( R \subset A \) is studied. Given a Hopf algebra \( A \), it turns out that we can always find an \( R \) and construct a differential calculus (in general not unique).

The space of tangent vectors on \( A \) is then defined as:

\[ T \equiv \{ \chi : A \to C \mid \chi \text{ linear functionals, } \chi(I) = 0 \text{ and } \chi(R) = 0 \} , \]

where \( I \) is the unit of \( A \) (in the commutative case it is the constant function \( I(g) = 1 \forall g \in G \)).

Let \( \{ \chi_i \} \ i = 1, \ldots, n \) be a basis of \( T \). Let \( x^i \in A \) be \( n \) elements such that

\[ \varepsilon(x^i) = 0 , \]  

(2.2)

\[ \chi_i(x^j) = \delta^i_j . \]  

(2.3)

These \( x^i \) can always be found \cite{2}. We can then define the \( n^2 \) linear functionals \( f^i_j : A \to C \)

\[ \forall a \in A \quad f^i_j(a) \equiv \chi_j(x^i a) . \]  

(2.4)

The \( f^i_j \) are well defined [they are independent from the set of \( x^i \) chosen to satisfy (2.2) and (2.3)] and we have

\[ \chi_i(ab) = \chi_j(a)f^j_i(b) + \varepsilon(a)\chi_i(b) . \]  

(2.5)

This is the deformed Leibniz rule for the operators \( \chi_i \). In the \( q = 1 \) case, when \( R \) becomes the set defined in (2.1), we have \( \chi_i = \partial_i|_{1_G} \), \( f^i_j = \delta^i_j \varepsilon \) and we write (2.5) as

\[ \partial_i(fh)|_{1_G} = (\partial_i f|_{1_G})h(1_G) + f(1_G)(\partial_i h|_{1_G}) . \]
For consistency with (2.5) the \( f^i_j \) must satisfy the conditions:

\[
\begin{align*}
  f^i_j(ab) &= f^i_k(a)f^k_j(b) \quad (2.6) \\
  f^i_j(I) &= \delta^i_j. \quad (2.7)
\end{align*}
\]

The space of left invariant vector fields \( \text{inv}\Xi \) is easily constructed from \( T \). Using the coproduct \( \Delta \) we define \( \chi^*a = (id \otimes \chi)\Delta(a) \) and

\[\text{inv}\Xi \equiv \{ t \mid t = \chi^* \text{ where } \chi \in T \} \quad (2.8)\]

There is a one to one correspondence \( \chi_i \leftrightarrow t_i = \chi_i^* \). In order to obtain \( \chi_i \) from \( \chi_i^* \) we simply apply \( \varepsilon \) (here and in the following we use the notation \( \Delta(a) = a_1 \otimes a_2 \)):

\[
(\varepsilon \circ t_i)(a) = \varepsilon(id \otimes \chi_i)\Delta(a) = \varepsilon(a_1\chi_i(a_2)) = \varepsilon(a_1)\chi_i(a_2) = \chi_i(\varepsilon \otimes id)\Delta(a) = \chi_i(a) \quad (2.9)
\]

\( \text{inv}\Xi \) is the vector subspace of all linear maps from \( A \) to \( A \) that is isomorphic to \( T \).

We have chosen this perspective to introduce the space of left invariant vector fields in order to point out that (also in the case of a general Hopf algebra) it has an existence on its own, independent of the space of 1-forms.

The space of 1-forms \( \Gamma \) is formed by all the elements \( \rho \) that are written as formal products and sums of the type

\[
\rho = a_i\omega^i. \quad (2.10)
\]

Here \( a_i \in A \) and \( \omega^i \ i = 1, \ldots, n \) is the basis dual to \( \{t_i\} \). We express this duality with a bracket:

\[
\langle \omega^i, \chi_j \rangle = \delta^i_j. \quad (2.11)
\]

Relation (2.10) tells us that the space of 1-forms is freely generated by the elements \( \omega^i \). By definition any \( \rho \) is decomposed in a unique way as \( \rho = a_i\omega^i \) and \( \Gamma \) is a left \( A \)-module with the trivial product \( b(a_i\omega^i) \equiv (ba_i)\omega^i \). \( \Gamma \) is also a right \( A \)-module with the following right product:

\[
\forall b \in A \quad \omega^ib = (f^i_j * b)\omega^j \equiv (id \otimes f^i_j)\Delta(b)\omega^j. \quad (2.12)
\]

From this relation it follows that (2):

\[
\forall a \in A \quad a\omega^i = \omega^j[(f^i_j \circ S^{-1}) * a] \quad (2.13)
\]

and that any \( \rho \) can be written in a unique way in the form

\[
\rho = \omega^ib_i \quad (2.14)
\]

with \( b_i \in A \).

Finally, the differential operator \( d : A \rightarrow \Gamma \) can be defined through the relation:

\[
\forall a \in A \quad da = (\chi_i * a)\omega^i. \quad (2.15)
\]
As a consequence, the differential calculus obtained has the following properties:

i) The differential operator satisfies the Leibniz rule

\[ d(ab) = (da)b + a(db) \quad \forall a, b \in A. \]  \hspace{1cm} (2.16)

Moreover any \( \rho \in \Gamma \) can be expressed as

\[ \rho = a_\alpha db_\alpha \]  \hspace{1cm} (2.17)

for some \( a_\alpha, b_\alpha \) belonging to \( A \).

The operator \( d \) can be extended in a unique way to an exterior differential \( d \) mapping \( n \)-forms into \((n+1)\)-forms and such that \( d^2 = 0 \).

**Remark.** Once we know the operator \( d \), the space of tangent vectors on \( A \), like in the commutative case, can be defined as:

\[ T = \{ \chi / \chi(a) = 0 \text{ if and only if } Pda = 0 \} \]  \hspace{1cm} (2.18)

where \( Pda \equiv S(a_1)da_2 \) with \( \Delta(a) = a_1 \otimes a_2 \). The linear map \( P \) is a projection operator; to a given form \( \rho = a_\omega \) it associates the form \( P(\rho) = \varepsilon(a_i)\omega^i \). In the commutative case \( \varepsilon(a_i) \) is the value that \( a_i \in A = C^\infty(G) \) takes in the origin \( 1_G \) of the Lie group \( G \). \( P(\rho) \) is then the left invariant 1-form whose value in the origin \( 1_G \) of the Lie group equals the value of the 1-form \( \rho \) in \( 1_G \).

ii) The differential calculus is called bicovariant because using \( d \) and the coproduct \( \Delta \) we can define two linear compatible maps \( \Delta \Gamma \) and \( \Gamma \Delta \)

\[ \Delta \Gamma(ab) = \Delta(a)(id \otimes d)\Delta(b), \quad \Delta : \Gamma \rightarrow A \otimes \Gamma \quad \text{(left covariance)} \]  \hspace{1cm} (2.19)

\[ \Gamma \Delta(ab) = \Delta(a)(d \otimes id)\Delta(b), \quad \Gamma : \Gamma \rightarrow \Gamma \otimes A \quad \text{(right covariance)} \]  \hspace{1cm} (2.20)

which represent the left and right action of the Hopf algebra on \( \Gamma \). In the commutative case they express the pull-back on 1-forms induced by the left or right multiplication of the group on itself \[4\]. \( \Delta \Gamma \) and \( \Gamma \Delta \) are compatible in the sense that \( (id \otimes \Gamma \Delta) \Delta = (\Delta \otimes id) \Gamma \Delta \). In the commutative case this formula tells us that the left and right actions of the group on \( \Gamma \) commute: \( R_{g'}^*L_g^* = L_g^*R_{g'}^* \quad \forall g, g' \in G \).

From the definitions (2.19) and (2.20) one deduces the following properties [3]:

\[ \Delta \Gamma(ab) = \Delta(a)\Delta \Gamma(\rho)\Delta(b), \quad \Gamma \Delta(ab) = \Delta(a)\Gamma \Delta(\rho)\Delta(b) \]  \hspace{1cm} (2.21)

\[ (\varepsilon \otimes id)\Delta \Gamma(\rho) = \rho, \quad (id \otimes \varepsilon)\Gamma \Delta(\rho) = \rho \]  \hspace{1cm} (2.22)

\[ (\Delta \otimes id)\Delta \Gamma = (id \otimes \Delta \Gamma)\Delta \Gamma, \quad (id \otimes \Delta)\Gamma \Delta = (\Gamma \Delta \otimes id)\Gamma \Delta . \]  \hspace{1cm} (2.23)

An element \( \omega \) of \( \Gamma \) is said to be **left invariant** if

\[ \Delta \Gamma(\omega) = I \otimes \omega \]  \hspace{1cm} (2.24)
and right invariant if
\[ r\Delta(\omega) = \omega \otimes I \]  
(2.25)

We have seen that any \( \rho \) is of the form \( \rho = a_i \omega^i \). We have that the \( \omega^i \) are left invariant and form a basis of \( \text{inv} \Gamma \), the linear subspace of all left invariant elements of \( \Gamma \). Relation (2.11) tells us that \( \text{inv} \Gamma \) and \( \text{inv} \Xi \) are dual vector spaces.

iii) There exists an adjoint representation \( M_j^i \) of the Hopf algebra, defined by the right action on the \( \omega^i \):
\[ r\Delta(\omega^i) = \omega^j \otimes M_j^i ; \quad M_j^i \in A . \]  
(2.26)

The co-structures on the \( M_j^i \) can be deduced [2]:
\[ \Delta(M_j^i) = M_j^l \otimes M_l^i \]  
(2.27)
\[ \varepsilon(M_j^i) = \delta^i_j \]  
(2.28)
\[ S(M_l^i)M_j^j = \delta^j_i = M_l^i S(M_j^j) \]  
(2.29)

The elements \( M_j^i \) can be used to build a right invariant basis of \( \Gamma \). Indeed the \( \eta^i \) defined by
\[ \eta^i \equiv \omega^j S(M_j^i) \]  
(2.30)
are a basis of \( \Gamma \) (every element of \( \Gamma \) can be uniquely written as \( \rho = \eta^i b_i \)) and their right invariance can be checked directly.

Moreover, from (2.29), using (2.30) and (2.13) one can prove the relation
\[ M_j^i (a * f^i)_k = (f^j_i * a) M_k^i \]  
(2.31)
with \( a * f^i_j \equiv (f^i_k \otimes id)\Delta(a) \).

3 Construction of the space of Vector Fields.

In this section we study the space \( \Xi \) of vector fields over Hopf algebras defining a right product between elements of \( \text{inv} \Xi \) and of \( A \).

In the commutative case a generic vector field can be written in the form \( f^i t_i \) where \( \{t_i\} \ i = 1, \ldots , n \) is a basis of left invariant vector fields and \( f^i \) are \( n \) smooth functions on the group manifold.

In the commutative case \( f^i t_i = t_i \circ f^i \) i.e. left and right products (that we have denoted with \( \circ \)) are the same, indeed \( (t_i \circ f^i)(h) \equiv t_i(h) f^i = f^i t_i(h) \). These considerations lead to the following definition.

Let \( t_i = \chi_i \ast \) be a basis in \( \text{inv} \Xi \) and let \( a^i, \ i = 1, \ldots , n \) be generic elements of \( A \):

**Definition**
\[ \Xi \equiv \{ V / \ V : A \longrightarrow A ; \ V = t_i \circ a^i \} , \]  
(3.1)
where the definition of the right product $\circ$ is given below:

**Definition**

$$\forall a, b \in A, \forall t \in \text{inv} \Xi \quad (t \circ a)b \equiv t(b)a = (\chi \ast b)a \ .$$

(3.2)

The product $\circ$ has a natural generalization to the whole $\Xi$:

$$\circ : \Xi \times A \longrightarrow \Xi$$

where $\forall b \in A \quad (V \circ a)(b) \equiv V(b)a \ .$$

(3.3)

It is easy to prove that $(\Xi, \circ)$ is a right $A$-module:

$$V \circ (a + b) = V \circ a + V \circ b \ ; \ V \circ (ab) = (V \circ a) \circ b \ ; \ V \circ (a + b) = V \circ a + V \circ b$$

(3.4)

(we have also $V \circ \lambda a = \lambda V \circ a$ with $\lambda \in C$).

For example $V \circ (ab) = (V \circ a) \circ b$ because

$$\forall c \in A \quad [(V \circ a)b]c = [(V \circ a)(c)b = (V(c)a)b = V(c)ab = [V \circ ab]c.$$

**Note.** To distinguish the elements $V \circ (ab) \in \Xi$ and $V(ab) \in A$ we have not omitted the simbol $\circ$ representing the right product.

$\Xi$ is the analogue of the space of derivations on the ring $C^\infty(G)$ of the smooth functions on the group $G$. Indeed we have:

$$V(a + b) = V(a) + V(b) \ , \ V(\lambda a) = \lambda V(a) \quad \text{Linearity} \quad (3.5)$$

$$V(ab) \equiv (t_i \circ c^j)(ab) = t_j(a)f^j_i \ast b)c^i + aV(b) \quad \text{Leibniz rule} \quad (3.6)$$

in the classical case $t_j(a)(f^j_i \ast b)c^i = V(a)b$ (recall $f^j_i = \delta^j_i \varepsilon \ ; \ \varepsilon \ast b = b$).

We have seen the duality between $\text{inv} \Gamma$ and $\text{inv} \Xi$. We now extend it to $\Gamma$ and $\Xi$, where $\Gamma$ is seen as a left $A$-module (not necessarily a bimodule) and $\Xi$ is our right $A$-module.

**Theorem 1.** There exists a unique map

$$\langle \ , \ \rangle : \Gamma \times \Xi \longrightarrow A$$

such that:

1) $\forall V \in \Xi$; the application

$$\langle \ , V \rangle : \Gamma \longrightarrow A$$

is a left $A$-module morphism, i.e. is linear and $\langle a \rho, V \rangle = a \langle \rho, V \rangle$.

2) $\forall \rho \in \Gamma$; the application

$$\langle \rho, \ \rangle : \Xi \longrightarrow A$$

is a right $A$-module morphism, i.e. is linear and $\langle \rho, Vb \rangle = \langle \rho, V \rangle b$. 6
3) Given $\rho \in \Gamma$

$$\langle \rho, \; \rangle = 0 \Rightarrow \rho = 0 ,$$  \hspace{1cm} (3.7)

where $\langle \rho, \; \rangle = 0$ means $\langle \rho, V \rangle = 0 \; \forall V \in \Xi$.

4) Given $V \in \Xi$

$$\langle \; , V \rangle = 0 \Rightarrow V = 0 ,$$  \hspace{1cm} (3.8)

where $\langle \; , V \rangle = 0$ means $\langle \rho, V \rangle = 0 \; \forall \rho \in \Gamma$.

5) On $\text{inv} \Gamma \times \text{inv} \Xi$ the bracket $\langle \; , \; \rangle$ acts as the one introduced in the previous section.

Remark. Properties 3) and 4) state that $\Gamma$ and $\Xi$ are dual $A$-moduli, in the sense that they are dual with respect to $A$.

Proof

Properties 1), 2) and 5) uniquely characterize this map. To prove the existence of such a map we show that the following bracket

**Definition**

$$\langle \rho, V \rangle \equiv a_\alpha V(b_\alpha) ,$$  \hspace{1cm} (3.9)

where $a_\alpha, b_\alpha$ are elements of $A$ such that $\rho = a_\alpha db_\alpha$, satisfies 1),2) and 5).

We first verify that the above definition is well given, that is:

Let $\rho = a_\alpha db_\alpha = a'_\beta db'_\beta$ then $a_\alpha V(b_\alpha) = a'_\beta V(b'_\beta)$.

Indeed, since

$$a_\alpha db_\alpha = a'_\beta db'_\beta \iff a_\alpha t_i(b_\alpha)\omega^i = a'_\beta t_i(b'_\beta)\omega^i \iff a_\alpha t_i(b_\alpha) = a'_\beta t_i(b'_\beta)$$

[we used the uniqueness of the decomposition (2.10)] the definition is consistent because

$$a_\alpha V(b_\alpha) = a'_\beta V(b'_\beta) \iff a_\alpha t_i(b_\alpha)c^i = a'_\beta t_i(b'_\beta)c^i$$

where $V = t_i c^i$.

Property 1) is trivial since $a_\rho = a(a_\alpha db_\alpha) = (aa_\alpha)db_\alpha$.

Property 2) holds since

$$\langle \rho, V \omega c \rangle = a_\alpha (V \omega c)(b_\alpha) = a_\alpha V(b_\alpha)c = \langle \rho, V \rangle c .$$

Property 5). Let $\{\omega^i\}$ and $\{t_i\}$ be dual bases in $\text{inv} \Gamma$ and $\text{inv} \Xi$. Since $\omega^i \in \Gamma$ , $\omega^i = a_\alpha db_\alpha$ for some $a_\alpha$ and $b_\alpha$ in $A$. We can also write $\omega^i = a_\alpha db_\alpha = a_\alpha t_k(b_\alpha)\omega^k$ , so that, due to the uniqueness of the decomposition (2.10), we have

$$a_\alpha t_k(b_\alpha) = \delta^i_k I \quad (I \text{ unit of } A);$$
we then obtain

$$\langle \omega^i, t_j \rangle = a_i t_j (b_\alpha) = \delta^i_j I .$$

Property 3). Let $$\rho = a_i \omega^i \in \Gamma .$$
If $$\langle \rho, V \rangle = 0 \ \forall V \in \Xi ,$$ in particular $$\langle \rho, t_j \rangle = 0 \ \forall j = 1, ..., n ;$$ then $$a_i \langle \omega^i, t_j \rangle = 0 \Leftrightarrow a_j = 0 ,$$ and therefore $$\rho = 0 .$$

Property 4). Let $$V = t_i \omega^i \in \Xi .$$
If $$\langle \rho, V \rangle = 0 \ \forall \rho \in \Gamma ,$$ in particular $$\langle \omega^i, V \rangle = 0 \ \forall j = 1, ..., n ;$$ then $$\langle \omega^i, t_j \rangle a^j = 0 \Leftrightarrow a_j = 0 ,$$ and therefore $$\rho = 0 .$$

By construction every $$V$$ is of the form

$$V = t_i \omega^i .$$

We can now show the unicity of such a decomposition.

**Theorem 2.** Any $$V \in \Xi$$ can be uniquely written in the form

$$V = t_i \omega^i$$

**Proof**

Let $$V = t_i \omega^i = t_i \omega^i$$ then

$$\forall i = 1, ..., n \quad a^i = \langle \omega^i, t_j \rangle a^j = \langle \omega^i, V \rangle = \langle \omega^i, t_j \rangle a^j = a^i .$$

Notice that once we know the decomposition of $$\rho$$ and $$V$$ in terms of $$\omega^i$$ and $$t_i ,$$ the evaluation of $$\langle \ , \rangle$$ is trivial:

$$\langle \rho, V \rangle = \langle a_i \omega^i, t_j \omega^j \rangle = a_i \langle \omega^i, t_j \rangle b^j = a_i b^i .$$

Viceversa from the previous theorem $$V = t_i \omega^i (V)$$ and $$\rho = \langle \rho, t_i \rangle \omega^i .$$

We conclude this section by remarking the three different ways of looking at $$\Xi .$$

(I) $$\Xi$$ as the set of all deformed derivations over $$A$$ [see (3.1), (3.5) and (3.6)].

(II) $$\Xi$$ as the right $$A$$-module freely generated by the elements $$t_i , \ i = 1, ..., n .$$

The latter is the set of all the formal products and sums of the type $$t_i a^i ,$$ where $$a^i$$ are generic elements of $$A .$$ Indeed, by virtue of Theorem 2, the map that associates to each $$V = t_i \omega^i$$ in $$\Xi$$ the corresponding element $$t_i a^i$$ is an isomorphism between right $$A$$-moduli.

(III) $$\Xi$$ as $$\Xi' = \{ U : \Gamma \rightarrow A , \ U$$ linear and $$U(\alpha \rho) = a U(\rho) \ \forall a \in A \},$$ i.e. $$\Xi$$ as the dual (with respect to $$A$$) of the space of 1-forms $$\Gamma .$$ The space $$\Xi'$$ has a trivial right $$A$$-module structure: $$(Ua)(\rho) \equiv U(\rho)a . \ \Xi$$ and $$\Xi'$$ are isomorphic right $$A$$-moduli because of property $$(3.8)$$ which states that to each $$\langle \ , V \rangle : \Gamma \rightarrow A$$ there corresponds one and only one $$V . [\langle \ , V \rangle = \langle \ , V' \rangle \Rightarrow V = V'] .$$ Every $$U \in \Xi'$$ is of the form $$U = \langle \ , V \rangle ;$$ more precisely, if $$a^i$$ is such that $$U(\omega^i) = a^i$$ then $$U = \langle \ , t_i \omega^i \rangle .$$
4 Bicovariant Bimodule Structure.

In \[2\] the space of 1-forms is extensively studied. A right and a left product are introduced between elements of $\Gamma$ and of $A$, and it is known how to obtain a left product from a right one [e.g. $\omega^i a = (f^i_j \ast a) \omega^j$], i.e. $\Gamma$ is a bimodule over $A$.

Since the actions $\Delta_\Gamma$ and $\Gamma \Delta$ are compatible in the sense that:

\[(id \otimes \Gamma \Delta) \Delta_\Gamma = (\Delta_\Gamma \otimes id) \Gamma \Delta \quad (4.1)\]

the bimodule $\Gamma$ is called a bicovariant bimodule. In \[2\] it is shown that relations (2.6),(2.7),(2.13),(2.27),(2.28) and (2.26) completely characterize the bicovariant bimodule $\Gamma$.

In Section 3 we have studied the right product $\ast$ and we have seen that $\Xi$ is a right bimodule over $A$ [see (3.4)]. In this section we introduce a left product and a left and right action of the Hopf algebra $A$ on $\Xi$. The left and right actions $\Delta_\Xi$ and $\Xi \Delta$ are the $q$-analogue of the push-forward of tensor fields on a group manifold. Similarly to $\Gamma$ also $\Xi$ is a bicovariant bimodule.

The construction of the left product on $\Xi$, of the right action $\Xi \Delta$ and of the left action $\Delta_\Xi$ will be effected along the lines of Woronowicz’ Theorem 2.5 in \[2\], whose statement can be explained in the following steps:

**Theorem 3.** Consider the symbols $t_i$ ($i = 1, \ldots, n$) and let $\Xi$ be the right $A$-module freely generated by them:

$$\Xi \equiv \{t_i a^i \mid a^i \in A\}$$

Consider functionals $O^i_j : A \rightarrow \mathbb{C}$ satisfying [see (2.6) and (2.7)]

$$O^i_j(ab) = O^k_j(a)O^i_k(b) \quad (4.2)$$

$$O^i_j(I) = \delta^i_j \quad (4.3)$$

Introduce a left product via the definition [see (2.13)]

**Definition**

$$b(t_i a^i) \equiv t_j [(O^j_i \circ S^{-1}) \ast b] a^i \quad (4.4)$$

It is easy to prove that

i) $\Xi$ is a bimodule over $A$. (A proof of this first statement as well as of the following ones is contained in \[2\]).

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i) $\Xi$ is a bimodule over $A$. (A proof of this first statement as well as of the following ones is contained in \[2\]).

Introduce an action (push-forward) of the Hopf algebra $A$ on $\Xi$

**Definition**

$$\Delta_\Xi(t_i a^i) \equiv (I \otimes t_i) \Delta(a^i) \quad (4.5)$$
It follows that ii) \((\Xi, \Delta \Xi)\) is a left covariant bimodule over \(A\), that is
\[
\Delta_\Xi(aVb) = \Delta(a)\Delta_\Xi(V)\Delta(b); \quad (\varepsilon \otimes id)\Delta_\Xi(V) = V; \quad (\Delta \otimes id)\Delta_\Xi = (id \otimes \Delta)\Delta_\Xi.
\]

\[\Box\]

Introduce \(N^i_j\) satisfying [see (2.27) and (2.28)]
\[
N^i_j(a \ast O^k_i) = (O^i_j \ast a)N^k_i
\]
\[
\Delta(N^i_j) = N^j_l \otimes N^i_l
\]
\[
\varepsilon(N^i_j) = \delta^i_j,
\]
and introduce \(\varepsilon \Delta\) such that [see (2.26)]

**Definition**
\[
\varepsilon \Delta(a^i t_i) \equiv \Delta(a^i) t_j \otimes N^j_i.
\]

Then it can be proven that iii) The elements [see (2.30)]
\[
h_i \equiv t_j S(N^j_i)
\]
are right invariant: \(\varepsilon \Delta(h_i) = h_i \otimes I\). Moreover any \(V \in \Xi\) can be expressed in a unique way respectively as \(V = h_i a^i\) and as \(V = b^i h_i\), where \(a^i, b^i \in A\).

\[\Box\]

iv) \((\Xi, \Xi \Delta)\) is a right covariant bimodule over \(A\), that is
\[
\varepsilon \Delta(aVb) = \Delta(a)\varepsilon \Delta(V)\Delta(b); \quad (id \otimes \varepsilon)\varepsilon \Delta(V) = V; \quad (id \otimes \Delta)\varepsilon \Delta = (\varepsilon \Delta \otimes id)\varepsilon \Delta.
\]

\[\Box\]

v) The left and right covariant bimodule \((\Xi, \Delta \Xi, \varepsilon \Delta)\) is a bicovariant bimodule, that is left and right actions are compatible:
\[
(id \otimes \varepsilon \Delta)\Delta_\Xi = (\Delta_\Xi \otimes id)\varepsilon \Delta.
\]

\[\Box\Box\Box\]

In the previous section we have seen [remark (II)] that the space of vector fields \(\Xi\) is the free right \(A\)-module generated by the symbols \(t_i\), so that the above theorem applies to our case.

There are many bimodule structures (i.e. choices of \(O^i_j\)) \(\Xi\) can be endowed with. Using the fact that \(\Xi\) is dual to \(\Gamma\) we request compatibility with the \(\Gamma\) bimodule.

In the commutative case \(\langle f \omega^i, t_j \rangle = \langle \omega^i f, t_j \rangle = \langle \omega^i, ft_j \rangle = \langle \omega^i, t_j f \rangle\).

In the quantum case we know that \(\langle a \omega^i, t_j \rangle = \langle \omega^i, t_j a \rangle\) and we require
\[
\langle \omega^i a, t_j \rangle = \langle \omega^i, at_j \rangle
\]
(4.11)
this condition uniquely determines the bimodule structure of Ξ. Indeed we have
\[ \langle \omega^j, at_j \rangle = \langle \omega^j a, t_j \rangle = \langle (f^i_k * a) \omega^k, t_j \rangle = f^i_k * a \langle \omega^k, t_j \rangle = f^i_k * a \delta^k_j = \delta^i_j f^l_j * a \]
so that
\[ at_i = t_j \circ (f^j_i * a). \] (4.13)

We then define
\[ O^j_i \equiv f^j_i \circ S \] (4.14)
so that also \[ O^j_i \circ S^{-1} = f^j_i \] and (4.13) can be rewritten [see (2.13) and (4.4)]
\[ at_j = t_j \circ [(O^j_i \circ S^{-1}) * a] . \] (4.15)

**Theorem 4.** The functionals \( O^j_i \) satisfy conditions (4.2) and (4.3).

**Proof**
The first condition \( O^j_i(I) = \delta^j_i \) holds trivially.
The second one is also easily checked:
\[ O^j_i(ab) = (f^j_i \circ S)(ab) = f^j_i[S(b)S(a)] = f^j_i[S(b)]f^k_i[S(a)] = f^k_i[S(a)]f^j_k[(S(b)] = O^k_i(a)O^j_i(b) \]

So far Ξ has a bimodule structure. Ξ becomes a left covariant bimodule if we define \( \Delta_\Xi \) as in (4.5) so that \( t_i \) are left invariant vector fields.

As noticed in [2] \( M^k_i \) and \( f^i_j \) are dual in the sense that \( f^i_j(M^k_j) = \Lambda^i_k \delta^j_i \) when \( q = 1 \). This suggests the following definition of the \( n^2 \) elements \( N^i_k \in A \)

**Definition**
\[ N^i_k = S^{-1}(M^i_k) , \] (4.16)
so that also \( N^i_k \) and \( O^j_i \) are dual:
\[ O^j_i(N^i_k) = f^i_j(M^i_k) = \Lambda^i_k \delta^j_i . \]

**Theorem 5.** The \( N^i_k \) elements defined above satisfy relations (4.8), (4.7) and (4.6):

1) \( \varepsilon(N^j_i) = \delta^j_i \)  
2) \( \Delta(N^j_i) = N^j_i \otimes N^i_j \)  
3) \( N^i_k(a * O^j_i) = (O^j_i * a) N^j_i \)

**Proof**

1) is trivial.
2) \( \Delta(N^i_j) = \Delta[S^{-1}(M^i_j)] = \sigma_A(S^{-1} \otimes S^{-1}) \Delta(M^i_j) = \)
\[ \sigma_A[(S^{-1}(M_j^k) \otimes S^{-1}(M_k^i))] = \sigma_A(N_i^k \otimes N_j^i) = N_i^k \otimes N_j^i \]

where \( \sigma_A \) is the flip in \( A \otimes A \), \( \sigma_A(a \otimes b) = b \otimes a \) for any \( a, b \in A \).

3) We know that [see (2.31)]

\[ \forall a \in A \quad M_{i}^{j}(a * f_{i}^{j}) = (f_{i}^{j} * a)M_{i}^{j} \]

or equivalently,

\[ M_{i}^{j}[S(a) * f_{i}^{j}] = [f_{i}^{j} * S(a)]M_{i}^{j} . \]

Now

\[ [S(a) * f_{i}^{j}] = (f_{i}^{j} \otimes \text{id})\Delta[S(a)] = (\text{id} \otimes f_{i}^{j})(S \otimes S)\Delta(a) = S(O_{k}^{i} * a) . \]

Similarly,

\[ [f_{i}^{j} * S(a)] = S(a * O_{i}^{j}) . \]

So we can write

\[ S(a * O_{i}^{j})M_{i}^{j} = M_{i}^{j}S(O_{k}^{i} * a) \]

for all \( a \in A \). Applying \( S^{-1} \) to both members of this last expression we obtain relation 3).

\[ \blacksquare \]

Now that we have all the ingredients, the construction of the bicovariant bimodule \( \Xi \) is easy and straightforward. For example \( \Xi \Delta \) is given in formula (4.9).

We can then conclude that \((\Xi, \Delta_{\Xi}, \Xi \Delta)\) is a bicovariant bimodule.

Notice that, since Theorem 3 completely characterizes a bicovariant bimodule all the formulas containing the symbols \( f_{i}^{j}, M_{k}^{i} \) or elements of \( \Gamma \) are still valid under the substitutions \( f_{i}^{j} \rightarrow O_{i}^{j}, \quad M_{k}^{i} \rightarrow N_{k}^{j}, \quad \Gamma \rightarrow \Xi \).

5 Tensor fields

The construction completed for vector fields is readily generalized to \( p \)-times contravariant tensor fields.

We define \( \Xi \otimes \Xi \) to be the space of all elements that can be written as finite sums of the kind \( \sum V_i \otimes V_i' \) with \( V_i, V_i' \in \Xi \). The tensor product (in the algebra \( A \)) between \( V_i \) and \( V_i' \) has the following properties:

\[ V \otimes a \otimes V' = V \otimes aV', \quad a(V \otimes V') = (aV) \otimes V' \text{ and } (V \otimes V') \circ a = V \otimes (V' \circ a) \]

so that \( \Xi \otimes \Xi \) is naturally a bimodule over \( A \).

Left and right actions on \( \Xi \otimes \Xi \) are defined by:

\[ \Delta_{\Xi}(V \otimes V') \equiv V_1 V'_1 \otimes V_2 \otimes V'_2, \quad \Delta_{\Xi} : \Xi \otimes \Xi \rightarrow A \otimes \Xi \otimes \Xi \quad (5.1) \]
where as usual $V_1$, $V_2$ etc. are defined by
\[
\Delta_\Xi (V) = V_1 \otimes V_2, \quad V_1 \in A, \ V_2 \in \Xi
\]
(5.3)
\[
\varepsilon \Delta (V) = V_1 \otimes V_2, \quad V_1 \in \Xi, \ V_2 \in A.
\]
(5.4)
More generally, we can introduce the action of $\Delta$ on $\Xi^\otimes_p \equiv \Xi \otimes \Xi \otimes \cdots \otimes \Xi$ as
\[
\Delta_\Xi (V \otimes V' \otimes \cdots \otimes V'') \equiv V_1 V'_1 \cdots V''_1 \otimes V_2 \otimes V'_2 \cdots \otimes V''_2
\]
(5.5)
\[
\varepsilon \Delta : \Xi^\otimes_p \longrightarrow A \otimes \Xi^\otimes_p;
\]
(5.6)
Left invariance on $\Xi \otimes \Xi$ is naturally defined as $\Delta_\Xi (V \otimes V') = I \otimes V \otimes V'$ (similar definition for right invariance), so that for example $t_i \otimes t_j$ is left invariant, and is in fact a left invariant basis for $\Xi \otimes \Xi$: each element can be written as $t_i \otimes t_j \equiv t^{ij}$ in a unique way.

It is not difficult to show that $\Xi \otimes \Xi$ is a bicovariant bimodule. In the same way also $(\Xi^\otimes_p, \Delta_\Xi, \varepsilon \Delta)$ is a bicovariant bimodule. An analogue procedure, using $\Delta_\Gamma$ and $\Gamma \Delta$ instead of $\Delta_\Xi$ and $\varepsilon \Delta$, applies also to $\Gamma^\otimes_p$ the $p$-times tensor product of 1-forms.

Any element $v \in \Xi^\otimes_p$ can be written as $v = t_{i_1} \otimes \cdots t_{i_p} \omega^{i_1 \cdots i_p}$ in a unique way, similarly any element $\tau \in \Gamma^\otimes_p$ can be written as $\tau = a_{i_1 \cdots i_p} \omega^{i_1} \otimes \cdots \omega^{i_p}$ in a unique way.

It is now possible to generalize the previous bracket $(\ , \ ) : \Gamma \times \Xi \rightarrow A$ to $\Gamma^\otimes_p$ and $\Xi^\otimes_p$:

\[
(\ , \ ) : \Gamma^\otimes_p \times \Xi^\otimes_p \longrightarrow A
\]
(5.7)
\[
(\tau, v) \longmapsto \langle \tau, v \rangle = a_{i_1 \cdots i_p} \langle \omega^{i_1} \otimes \cdots \omega^{i_p}, t_{j_1} \otimes \cdots t_{j_p}\rangle \delta^{i_1 \cdots i_p}_{j_1 \cdots j_p}
\]
where we have defined
\[
\langle \omega^{i_1} \otimes \cdots \omega^{i_p}, t_{j_1} \otimes \cdots t_{j_p}\rangle \equiv \langle \omega^{i_1}, t_{j_1}\rangle \cdots \langle \omega^{i_p}, t_{j_p}\rangle = \delta^{i_1}_{j_1} \cdots \delta^{i_p}_{j_p}.
\]
Using definition (5.7) it is easy to prove that
\[
\langle \tau a, v \rangle = \langle \tau, av \rangle,
\]
(5.8)
for example
\[
\langle \omega^{i_1} \otimes \cdots \omega^{i_p} a, t_{j_1} \otimes \cdots t_{j_p}\rangle = (f^{i_1}_1 \ast \cdots \ast f^{i_p}_p \ast a) \langle \omega^{k_1} \otimes \cdots \omega^{k_p}, t_{j_1} \otimes \cdots t_{j_p}\rangle
\]
\[ \langle \omega^i \otimes \ldots \omega^p, at_{j_1} \otimes \ldots t_{j_p} \rangle = \langle \omega^i \otimes \ldots \omega^p, t_{l_1} \otimes \ldots t_{l_p} \rangle (f^{l_p}_{j_p} \ast \ldots f^{l_1}_{j_1} \ast a) \]

and these last two expressions are equal if and only if (5.7) holds.

Therefore we have also shown that definition (5.7) is the only one compatible with property (5.8), i.e. property (5.8) uniquely determines the coupling between \( \Xi^\otimes \) and \( \Gamma^\otimes \).

It is easy to prove that the bracket \( \langle \ , \ \rangle \) extends to \( \Gamma^\otimes p \) and \( \Xi^\otimes p \) the duality between \( \Gamma \) and \( \Xi \).

More generally we can define \( \Xi^\otimes \equiv A \oplus \Xi \oplus \Xi^\otimes 2 \oplus \Xi^\otimes 3 \ldots \) to be the algebra of contravariant tensor fields (and \( \Gamma^\otimes \) that of covariant tensor fields).

The actions \( \Delta_\Xi \) and \( \Xi \Delta \) have a natural generalization to \( \Xi^\otimes \) so that we can conclude that \( (\Xi^\otimes, \Delta_\Xi, \Xi \Delta) \) is a bicovariant graded algebra, the graded algebra of tensor fields over the ring “of functions on the group” \( A \), with the left and right “push-forward” \( \Delta_\Xi \) and \( \Xi \Delta \).

\section{Lie derivative}

In this last section we propose a definition of Lie derivative along a generic vector field. We start with the introduction of the contraction operator \( i_V \) with \( V \in \Xi \).

Let \( t \in inv \Xi \), the operator \( i_t \) on forms is characterized by:

- \( \alpha \) \quad \( i_t (a) = 0 \quad a \in A \)
- \( \beta \) \quad \( i_t (\omega^i) = \delta^j_i I \)
- \( \gamma \) \quad \( i_t (\omega^{i_1} \wedge \ldots \omega^{i_n}) = i_t (\omega^{i_1}) f^{j_1}_{i_1} \ast (\omega^{i_2} \wedge \ldots \omega^{i_n}) - \omega^{i_1} \wedge i_t (\omega^{i_2} \wedge \ldots \omega^{i_n}) \)
- \( \delta \) \quad \( i_t (a \vartheta + \vartheta') = ai_t (\vartheta) + i_t (\vartheta') \quad \vartheta, \vartheta' \text{ generic forms} \)
- \( \varepsilon \) \quad \( i_{\lambda t} = \lambda^i i_{t_i} \quad \lambda^i \in C \)

These relations uniquely define \( i_t \) along \( t \in inv \Xi \); its existence is ensured by the uniqueness of the expansion of a generic n-form on a basis of left invariant 1-forms: \( \vartheta = a_{i_1 i_2 \ldots i_n} \omega^{i_1} \wedge \ldots \omega^{i_n} \). It can be shown that property \( \gamma \) holds in the more general case:

\[ i_t (a_{i_1 \ldots i_n} \omega^{i_1} \wedge \ldots \omega^{i_n}) = i_t (a_{i_1 \ldots i_n} \omega^{i_1} \wedge \ldots \omega^{i_s}) \wedge f^{j_1}_{i_1} \ast (\omega^{i_{s+1}} \wedge \ldots \omega^{i_n}) + \]
\[ + (-1)^s a_{i_1 \ldots i_n} \omega^{i_1} \wedge \ldots \omega^{i_s} \wedge i_t (\omega^{i_{s+1}} \wedge \ldots \omega^{i_n}) \]

with \( a_{i_1 \ldots i_n} \in A \).

For a generic vector field \( V = t_j \otimes b^j \) we define

**Definition**

- \( \zeta \) \quad \( i_V (\vartheta) = i_{t_j \otimes b^j} (\vartheta) \equiv i_{t_j} (\vartheta) b^j \quad \vartheta \text{ generic form.} \)
The following properties are easily proven:

\[ \alpha' \quad i_V(a) = 0 \quad a \in A \]
\[ \beta' \quad i_V(\omega^j) = b^j \quad \text{where } V = t_i \cdot b^i \]
\[ \gamma' \quad i_V(a_{i_1 \ldots i_n} \omega^{i_1} \wedge \ldots \wedge \omega^{i_n}) = i_{t_j}(a_{i_1 \ldots i_n} \omega^{i_1} \wedge \ldots \wedge \omega^{i_s} \wedge \omega^{i_{s+1}} \wedge \ldots \wedge \omega^{i_n}) + (-1)^s a_{i_1 \ldots i_n} \omega^{i_1} \wedge \ldots \wedge \omega^{i_s} \wedge i_V(\omega^{i_{s+1}} \wedge \ldots \wedge \omega^{i_n}) \]

with \( a_{i_1 \ldots i_n} \in A \).

\[ \delta' \quad i_V(a \vartheta + \vartheta') = a i_V(\vartheta) + i_V(\vartheta') \quad \vartheta, \vartheta' \text{ generic forms} \]

\[ \epsilon' \quad i_{\lambda V} = \lambda i_V \quad \lambda \in \mathbb{C} \]

**Remark.** Definition \( \zeta' \) and property \( \delta' \) reduce in the commutative case to the familiar formulae:

\[ i_{f V} \vartheta = f i_V \vartheta \quad \text{and} \quad i_V(h \vartheta) = h i_V \vartheta . \]

The Lie derivative along left invariant vector fields is given by:

\[ \ell_t(\tau) \equiv (id \otimes \chi)r\Delta(\tau) = \chi * \tau \quad \ell_t : \Gamma^{\otimes n} \longrightarrow \Gamma^{\otimes n} \]  

(6.1)

where \( \chi \in inv \Xi \) is such that \( \chi * = t \). It can be proved that \[ ]:

\[ \ell_t = i_t d + di_t . \]  

(6.2)

It is then natural to introduce the Lie derivative along a generic vector field \( V \) through the following

**Definition**

\[ \ell_V = i_V d + div . \]  

(6.3)

**Theorem 6.** The Lie derivative satisfies the following properties:

1) \( \ell_V a = V(a) \)

2) \( \ell_V d \vartheta = d \ell_V \vartheta \)

3) \( \ell_V(\lambda \vartheta + \vartheta') = \lambda \ell_V(\vartheta) + \ell_V(\vartheta') \)

4) \( \ell_{V \circ b}(\vartheta) = (\ell_V b) \vartheta - (-1)^p i_V(\vartheta) \wedge db \)

where \( \vartheta \) is a generic \( p \)-form

5) \( \ell_V(\rho \wedge \Omega) = \rho \wedge \ell_V(\Omega) + \ell_{t_j}(\rho) \wedge (f^k_j * \Omega)b^j + (-1)^p i_{t_k}(\rho)(f^k_j * \Omega) \wedge db^j \)

where \( \rho \in \Gamma, \Omega \) is a left invariant \( p \)-form and \( V = t_j v b^j \).

**Proof**

Properties 1), 2), 3) and 4) follow directly from the definition (6.3).

Property 5) is also a consequence of definition (6.3); the proof is computational and makes use of the identities

\[ i_{t_j}[(d \rho) \wedge \Omega] = i_{t_k}(d \rho) \wedge f^k_j * \Omega + d \rho \wedge i_{t_j} \Omega \; ; \quad d(f^k_j * \Omega) = f^k_j * d \Omega . \]
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References

[1] L.D. Faddeev, N.Yu. Reshetikhin, L.A. Takhtajan, Algebra i Anal. 1 1, (1989) 178.

[2] S.L. Woronowicz, Commun. Math. Phys. 122 (1989) 125.

[3] S. Majid, Int. J. Mod. Phys. A5 (1990) 1.

[4] B. Jurčo, Lett. Math. Phys. 22 (1991) 177.

[5] U. Carow-Watamura, M. Schlieker, S. Watamura, W. Weich, Commun. Math. Phys. 142(1991) 605.

[6] P. Schupp, P. Watts, B. Zumino, Bicovariant Quantum Algebras and Quantum Lie Algebras, LBL-32315 and UCB-PTH-92/14.

[7] P. Aschieri and L. Castellani, Int. J. Mod. Phys. A8 (1993) 1667.

[8] P. Schupp, P. Watts, B. Zumino, Cartan Calculus for Hopf Algebras and Quantum Groups, NSF-ITP-93-75, LBL-34215 and UCB-PTH-93/20.

[9] E. Abe, Hopf Algebras, (Cambridge University Press, 1977).