Explicit form of the Yablonskii - Vorob’ev polynomials

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Abstract

Special polynomials associated with rational solutions of the second Painlevé equation and other members of its hierarchy are discussed. New approach, which allows one to construct each polynomial is presented. The structure of the polynomials is established. Formulas of their coefficients are found. Correlations between the roots of every polynomial are obtained.

Keywords: the Yablonskii - Vorob'ev polynomials, the second Painlevé equation, power expansion, power geometry, the second Painlevé hierarchy

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1 Introduction

It is well known that the second Painlevé equation \((P_2)\)

\[ w_{zz} = 2w^3 + zw + \alpha \]  (1.1)

has rational solutions only at integer values of the parameter \(\alpha\) \((\alpha = n \in \mathbb{Z})\). These solutions can be written in terms of the Yablonskii – Vorob’ev polynomials \(Q_n(z)\) [1, 2]

\[ w(z; n) = \frac{d}{dz} \left\{ \ln \left[ \frac{Q_{n-1}(z)}{Q_n(z)} \right] \right\}, \quad n \geq 1, \quad w(z; -n) = -w(z; n). \]  (1.2)
The polynomials $Q_n(z)$ satisfy the differential–difference equation

$$Q_{n+1} Q_{n-1} = zQ_n^2 - 4(Q_n Q'' - (Q'_n)^2),$$

(1.3)

where $Q_0(z) = 1$, $Q_1(z) = z$. It is not clear from the first sight that this relation defines exactly polynomials however it is so. Moreover $Q_n(z)$ are monic polynomials with integer coefficients. These polynomials can be regarded as nonlinear analogues of classical special polynomials. They possess a certain number of interesting properties. In particular, for every integer positive $n$ the polynomial $Q_n(z)$ has simple roots only and besides that, two successive polynomials $Q_n(z)$ and $Q_{n+1}(z)$ do not have a common root. Partial solutions of the Korteweg–de Vries equation, the modified Korteweg–de Vries equation, the nonlinear Schrödinger equation can be expressed via the polynomials $Q_n(z)$ [3].

One of the most important problems concerning the Yablonskii-Vorob’ev polynomials includes constructing explicit formulas of their coefficients [4]. The attempt of solving this problem can be found in recent work [5], where the coefficient of the lowest degree term was discussed.

In this work we present a new method, which allows one to find formulas for the coefficients of every polynomial, to determine the polynomial structure and to obtain correlations between its roots. Our approach can be also applied for constructing other polynomials related to nonlinear differential equations. In particular, we will briefly review the case of some other equations of the $P_2$ hierarchy.

The outline of this paper is as follows. In section 2 the algorithm of our method is presented and main theorems are proved. Correlations between the roots of the Yablonskii-Vorob’ev polynomials are established in section 3. Formulas of coefficients are found in section 4. The polynomials associated with the second and the third equations of the $P_2$ hierarchy are discussed in sections 5 and 6, accordingly.

## 2 Method applied

Being of degree $n(n+1)/2$ the polynomial $Q_n(z)$ can be written as

$$Q_n(z) = \sum_{k=0}^{n(n+1)/2} A_{n,k} z^{n(n+1)/2-k}, \quad A_{n,0} = 1.$$  \hspace{1cm} (2.1)

Let us show that it is possible to obtain each polynomial without leaning on the recurrence formula (1.3). For this aim we will use a power expansion
at infinity for solutions of the equation (1.1). This expansion found in [7] is the following

\[ w(z; \alpha) = \frac{c_{\alpha-1}}{z} + \sum_{l=1}^{\infty} c_{\alpha-3l-1} z^{-3l-1}, \quad z \to \infty. \]  

(2.2)

Here all the coefficients \( c_{\alpha-3l-1} \) can be sequently found. Taking into account four members (2.2) can be written as

\[ w(z; \alpha) = -\frac{\alpha}{z} + \frac{2\alpha}{z^4} \left( \alpha - 1 \right) \left( \alpha + 1 \right) + \frac{4\alpha}{z^7} \left( \alpha - 1 \right) \left( \alpha + 1 \right) \left( 3\alpha^2 - 10 \right) + \frac{8\alpha}{z^{10}} \left( \alpha - 1 \right) \left( \alpha + 1 \right) \left( 12\alpha^4 - 117\alpha^2 + 280 \right) + O\left( \frac{1}{z^{13}} \right). \]  

(2.3)

For convenience of use let us present this series in the form

\[ w(z; \alpha) = \sum_{m=0}^{\infty} c_{\alpha-(m-1)} z^{-m-1}, \]  

(2.4)

where \( c_{\alpha-(m-1)} = 0 \) unless \( m \) is divisible by 3. Suppose \( a_{n,k} \) \((1 \leq k \leq n(n+1)/2)\) are the roots of the polynomial \( Q_n(z) \), then by \( s_{n,k} \) we denote the symmetric functions of the roots

\[ s_{n,m} \overset{def}= \sum_{k=1}^{n(n+1)/2} (a_{n,k})^m, \quad m \geq 1. \]  

(2.5)

Our next step is to express \( s_{n,m} \) through coefficients of the series (2.4).

**Theorem 2.1.** Let \( c_{i-(m-1)} \) be the coefficient in expansion (2.4) at integer \( \alpha = i \in N \). Then for each \( m \geq 1 \) and \( n \geq 2 \) the following relation holds

\[ s_{n,m} = -\sum_{i=2}^{n} c_{i-(m+1)}. \]  

(2.6)

**Proof.** As far as \( Q_n(z) \) is a monic polynomial with simple roots, then it can be written in the form

\[ Q_n(z) = \prod_{k=1}^{n(n+1)/2} (z - a_{n,k}). \]  

(2.7)
This implies that
\[
\frac{Q'_n(z)}{Q_n(z)} = \sum_{k=1}^{n(n+1)/2} \frac{1}{z - a_{n,k}}.
\] (2.8)

Substituting (2.8) into the expression (1.2) yields
\[
w(z; n) = \frac{Q'_{n-1}(z)}{Q_{n-1}(z)} - \frac{Q'_n(z)}{Q_n(z)} = \sum_{k=1}^{n(n-1)/2} \frac{1}{z - a_{n-1,k}} - \sum_{k=1}^{n(n+1)/2} \frac{1}{z - a_{n,k}}.
\] (2.9)

Expanding this function in a neighborhood of infinity we get
\[
w(z; n) = \sum_{k=1}^{n(n-1)/2} \frac{1}{z(1 - \frac{a_{n-1,k}}{z})} - \sum_{k=1}^{n(n+1)/2} \frac{1}{z(1 - \frac{a_{n,k}}{z})} = \frac{\delta_{0,a_{n-1},1}}{z} - \frac{\delta_{0,a_{n},1}}{z} + \sum_{m=0}^{\infty} \left[ \sum_{k=1+\delta_{0,a_{n-1},1}}^{n(n-1)/2} (a_{n-1,k})^m - \sum_{k=1+\delta_{0,a_{n},1}}^{n(n+1)/2} (a_{n,k})^m \right] z^{-(m+1)}, |z| > \max\{\tilde{a}_{n-1}, \tilde{a}_n\},
\] (2.10)

where the first or the second term is present only if the polynomial $Q_{n-1}(z)$ or $Q_n(z)$ has a zero root, which without loss of generality is the first in the set of roots. In our designations the previous expression can be rewritten as

\[
w(z; n) = -\frac{n}{z} + \sum_{m=1}^{\infty} [s_{n-1,m} - s_{n,m}] z^{-(m+1)}, |z| > \max\{\tilde{a}_{n-1}, \tilde{a}_n\}.
\] (2.11)

The absence of a zero term in sum is essential only at $m = 0$. Comparing expansions (2.11) and (2.4) we obtain the equality

\[s_{n,m} - s_{n-1,m} = -c_{n, -(m+1)}.
\] (2.12)
Decreasing the first index by one in (2.12) and adding the result to the original one yields

\[ s_{n,m} - s_{n-2,m} = -(c_{n, -(m+1)} + c_{n-1, -(m+1)}). \]  \hspace{1cm} (2.13)

Note that \( c_{1, -(m+1)} = 0 \), \( m \geq 1 \) and \( a_{1,1} = 0 \). Then proceeding in such a way we get the required relation \( \text{(2.6)} \).

Remark 1. It was proved many times that \( P_2 \) has a unique rational solution whenever \( \alpha \) is an integer. All these solutions possess convergent series at infinity. Note that every rational solution \( w(z; n) \) has the asymptotic behavior

\[ w(z; n) \sim -\frac{n}{z}, \quad z \to \infty; \]

i.e. the point \( z = \infty \) is a simple root. This fact can be easily seen from \( \text{(2.11)} \). Thus the formal series \( \text{(2.2)} \) at \( \alpha = n \) coincides with the expansion \( \text{(2.10)} \) and is also convergent.

Theorem (2.1) enables us to prove the following theorem.

**Theorem 2.2.** All the coefficients \( A_{n,m} \) of the Yablonskii – Vorob’ev polynomial \( Q_n(z) \) can be obtained with a help of \( n(n+1)/2 + 1 \) first coefficients of the expansion \( \text{(2.4)} \) for the solutions of \( P_2 \).

*Proof.* For every polynomial there exists a connection between its coefficients and the symmetric functions of its roots \( s_{n,m} \). This connection is the following

\[ mA_{n,m} + s_{n,1}A_{n,m-1} + \ldots + s_{n,m}A_{n,0} = 0, \quad 1 \leq m \leq n(n+1)/2. \]  \hspace{1cm} (2.14)

Taking into account that \( A_{n,0} = 1 \) we get

\[ A_{n,m} = -\frac{s_{n,m} + s_{n,m-1}A_{n,1} + \ldots + s_{n,1}A_{n,m-1}}{m}, \quad 1 \leq m \leq n(n+1)/2. \]  \hspace{1cm} (2.15)

The function \( s_{n,m} \) can be derived using the expression \( \text{(2.6)} \). Hence recalling the fact that \( \text{(2.4)} \) is exactly \( \text{(2.2)} \) we obtain

\[ s_{n,m} = 0, \quad m \in \mathbb{N} \setminus \{3l, \quad l \in \mathbb{N}\}, \]

\[ s_{n,3l} = -\sum_{i=2}^{n} c_{i, -(3l+1)}, \quad l \in \mathbb{N}. \]  \hspace{1cm} (2.16)
Substituting this into (2.15) yields

\[ A_{n,m} = 0, \quad m \in \{1, 2, \ldots, n(n+1)/2\} / \{3l, \ l \in \mathbb{N}\}; \]

\[ A_{n,3l} = -s_{n,3l} + s_{n,3l-3}A_{n,3} + \cdots + s_{n,3}A_{n,3l-3}, \quad l \in \mathbb{N}, \ 3l \leq n(n+1)/2. \]

(2.17)

Thus we see that the coefficients \( A_{n,k} \) of the Yablonskii – Vorob’ev polynomial \( Q_n(z) \) are uniquely defined by coefficients \( c_{n,−(3l+1)} \) of the expansion (2.2). This completes the proof.

**Remark 2.** Expression (2.17) defines the structure of polynomial \( Q_n(z) \). Namely if \( n(n+1)/2 \) is divisible by 3, i.e. \( n \equiv 0 \mod 3 \) or \( n \equiv 2 \mod 3 \), then \( Q_n(z) \) is a polynomial in \( z^3 \). Otherwise if \( n(n+1)/2 \) is not divisible by 3, i.e. \( n \equiv 1 \mod 3 \), then the absolute term of \( Q_n(z) \) is equal to zero and \( Q_n(z)/z \) is a polynomial in \( z^3 \) (as in this case \( n(n+1)/2 − 1 \) is divisible by 3).

### 3 Symmetric functions of the roots

In this section we are discussing properties of the functions \( s_{n,m} \). It is important to mention that they may be regarded as relations between the roots \( a_{n,k} \) of the Yablonskii - Vorob’ev polynomials. In order to establish our main results we need a lemma.

**Lemma 3.1.** *The coefficient \( c_{α,−3l−1} \) in the expansion (2.2) is a polynomial in \( α \) of degree \( 2l+1 \).*

**Proof.** The proof is by induction on \( l \). For \( l = 0 \), there is nothing to prove as \( c_{α,−1} = −α \). Other coefficients can be obtained from the recursion relation

\[ c_{α,−3(l+1)−1} = (3l+2)(3l+1)c_{α,−3l−1} − 2 \sum_{m=0}^{l} \sum_{n=0}^{m} c_{α,−3n−1} \]

(3.1)

\[ c_{α,−3(m−n)−1}c_{α,−3(l−m)−1}, \quad l \geq 0. \]

Suppose that \( c_{α,−3m−1} \) is a polynomial in \( α \) of degree \( 2m+1 \) (\( 0 < m \leq l \)). Then from (3.1) we see that \( c_{α,−3(l+1)−1} \) is also a polynomial in \( α \) and \( \deg(c_{α,−3(l+1)−1}) = 2n + 1 + 2(m − n) + 1 + 2(l − m) + 1 = 2l + 3 = 2(l + 1) + 1. \) Q.E.D.
Theorem 3.1. The following statements are true:

1. at given \( n \geq 2 \) the functions \( s_{n,m} (m > n(n+1)/2) \) do not contain any new information about the roots of \( Q_n(z) \);
2. \( s_{n,3l} \) is a polynomial in \( n \) of degree \( 2(l+1) \).

Proof. The first statement of the theorem immediately follows from the correlation

\[ s_{n,m} + s_{n,m-1}A_{n,1} + \ldots + s_{n,m-n(n+1)/2}A_{n,n(n+1)/2} = 0, \quad m > n(n+1)/2 \quad (3.2) \]

and the expression \( (2.15) \). Now let us prove the second statement. From \( (2.6) \) and Lemma \((3.1)\) we see that in order to find \( s_{n,3l} \) one should calculate finite amount of sums \( \sum_{i=1}^{n} i^m, \ m \in \mathbb{N} \), \( \max m = 2l + 1 \). Such sum is computable. And the result is a polynomial in \( n \) of degree \( m + 1 \). This completes the proof. \( \square \)

Finally let us find several functions \( s_{n,3l} \). They are

\[ s_{n,3} = -\frac{1}{2}n (n^2 - 1) (n + 2), \quad (3.3) \]

\[ s_{n,6} = 2n (n^2 - 1) (n + 2) (n^2 + n - 5), \quad (3.4) \]

\[ s_{n,9} = -4n (n^2 - 1) (n + 2) (n^2 + n - 7) (3n^2 + 3n - 20), \quad (3.5) \]

\[ s_{n,12} = 8n (n^2 - 1) (n + 2) [11n^6 + 33n^5 - 259n^4 - 573n^3 + \]

\[ + 2348n^2 + 2640n - 7700], \quad (3.6) \]

\[ s_{n,15} = -8n (n^2 - 1) (n + 2) [91n^8 + 364n^7 - 3468n^6 - 11678n^5 + \]

\[ + 57138n^4 + 134164n^3 - 454161n^2 - 523250n + 1401400], \quad (3.7) \]

\[ s_{n,18} = 32n (n^2 - 1) (n + 2) [204n^{10} + 1020n^9 - 11584n^8 - \]

\[ - 52456n^7 + 303649n^6 + 1098827n^5 - 4328687n^4 - \]

\[ - 10551991n^3 + 32064418n^2 + 37532600n - 95295200], \quad (3.8) \]
\[ s_{n,21} = -64 n (n^2 - 1) (n + 2) [969 n^{12} + 5814 n^{11} - 77478 n^{10} - 
-440685 n^9 + 2986374 n^8 + 14653560 n^7 - 66988510 n^6 - 
-254167821 n^5 + 882020165 n^4 + 2205662532 n^3 - 
-6249767920 n^2 - 7397539800 n + 18106088000], \]

\[ s_{n,24} = 128 n (n - 1) (n + 2) (n + 1) [4807 n^{14} + 33649 n^{13} - 
-519091 n^{12} - 3551983 n^{11} + 27883155 n^{10} + 172777587 n^9 - 
-911403269 n^8 - 4722725213 n^7 + 18734279962 n^6 + 
+73498559352 n^5 - 234405400524 n^4 - 597184066192 n^3 + 
+1610723930960 n^2 + 1922407748800 n - 4580840264000], \]

\[ s_{n,27} = -128 n (n^2 - 1) (n + 2) [49335 n^{16} + 394680 n^{15} - 
-6966616 n^{14} - 55673212 n^{13} + 501236020 n^{12} + 3749125816 n^{11} - 
-22662344352 n^{10} - 149052619326 n^9 + 674231177321 n^8 + 
+3634840116452 n^7 - 13091078484596 n^6 - 52664915417010 n^5 + 
+158038744882088 n^4 + 408412972732600 n^3 - 
-1061767782349200 n^2 - 1275405591460000 n + 
+2977546171600000]. \]
4 Coefficients of the Yablonskii - Vorob’ev polynomials

Before coming to the direct computation of the coefficients it is important to mention that $A_{n,3l}$ is a polynomial in $n$ of degree $4l$. This fact can be proved by induction. Using the results of the previous section and the expression (2.17) we obtain

$$A_{n,3} = \frac{n}{6} (n^2 - 1) (n + 2), \quad (4.1)$$

$$A_{n,6} = \frac{n}{72} (n^2 - 1) (n^2 - 4) (n - 4) (n + 5) (n + 3), \quad (4.2)$$

$$A_{n,9} = \frac{n}{1296} (n^2 - 1) (n^2 - 4) (n^2 - 9) (n + 4)$$

$$\quad \left( n^4 + 2 n^3 - 57 n^2 - 58 n + 1120 \right), \quad (4.3)$$

$$A_{n,12} = \frac{n}{31104} (n^2 - 1) (n^2 - 4) (n^2 - 9) (n^2 - 16) (n + 5)$$

$$\quad \left( n^6 + 3 n^5 - 109 n^4 - 223 n^3 + 5148 n^2 + 5260 n - 110880 \right), \quad (4.4)$$

$$A_{n,15} = \frac{n}{933120} (n^2 - 1) (n^2 - 4) (n^2 - 9) (n^2 - 16) (n + 5) [n^{10} +$$

$$\quad + 5 n^9 - 200 n^8 - 830 n^7 + 18917 n^6 + 59677 n^5 - 1072550 n^4 -$$

$$\quad - 2245540 n^3 + 35648392 n^2 + 36781248 n - 484323840], \quad (4.5)$$

$$A_{n,18} = \frac{n}{33592320} (n^2 - 1) (n^2 - 4) (n^2 - 9) (n^2 - 16) (n^2 - 25)$$

$$\quad (n + 6) [n^{12} + 6 n^{11} - 287 n^{10} - 1490 n^9 + 40087 n^8 + 169354 n^7 -$$

$$\quad - 3558197 n^6 - 11273654 n^5 + 213009052 n^4 + 445008120 n^3 -$$

$$\quad - 7958672096 n^2 - 8183083776 n + 131736084480], \quad (4.6)$$

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\[ A_{n,21} = \frac{n}{1410877440} (n^2 - 1) (n^2 - 4) (n^2 - 9) (n^2 - 16) (n^2 - 25) \\
(n + 6) [n^{16} + 8 n^{15} - 420 n^{14} - 3080 n^{13} + 87206 n^{12} + 563640 n^{11} - \\
-11872404 n^{10} - 64603080 n^9 + 1168725105 n^8 + 5068846496 n^7 - \\
-84738361232 n^6 - 272233713360 n^5 + 4305162496688 n^4 + \\
9070093877056 n^3 - 132946555052544 n^2 - 137527838945280 n + \\
+1802149635686400], \\
(4.7) \\
\] \\
\[ A_{n,24} = \frac{n}{67722117120} (n^2 - 1)(n^2 - 4)(n^2 - 9)(n^2 - 16)(n^2 - 25) \\
(n^2 - 36) (n^2 - 49) (n + 8) [n^{16} + 8 n^{15} - 492 n^{14} - 3584 n^{13} + \\
+121478 n^{12} + 775824 n^{11} - 20020068 n^{10} - 107298432 n^9 + \\
+2443900401 n^8 + 10428074408 n^7 - 227804919608 n^6 - \\
-720372362304 n^5 + 15749707896080 n^4 + 32712421425280 n^3 - \\
-723511683854592 n^2 - 739989727488000 n + 16283709208166400], \\
(4.8) \\
\]
\[ A_{n,27} = \frac{n}{3656994324480} \left( n^2 - 1 \right) \left( n^2 - 4 \right) \left( n^2 - 9 \right) \left( n^2 - 16 \right) \]

\[ \left( n^2 - 25 \right) \left( n^2 - 36 \right) (n + 7) \left[ n^{22} + 11 n^{21} - 715 n^{20} - 7535 n^{19} + + 255960 n^{18} + 2520582 n^{17} - 61352518 n^{16} - 548941070 n^{15} + + 110581177409 n^{14} + 87218618983 n^{13} - 1583236002911 n^{12} - 10658996892035 n^{11} + 183545605960118 n^{10} + 1017581723269944 n^9 - - 17057801205684864 n^8 - 74457629190856880 n^7 + + 1221271897326432992 n^6 + 3928812356965402880 n^5 - - 62376250932962964992 n^4 - 131389485437862420480 n^3 + + 1976136057743843819520 n^2 + 2042498326581057945600 n - - 28450896728508334080000 \right] . (4.9) \]

Now let us write out the general form of the Yablonskii - Vorob'ev polynomial \( Q_n(z) \). It is the following

\[ Q_n(z) = z^{n(n+1)/2} + A_{n,3} z^{n(n+1)/2-3} + A_{n,6} z^{n(n+1)/2-6} + + A_{n,9} z^{n(n+1)/2-9} + A_{n,12} z^{n(n+1)/2-12} + A_{n,15} z^{n(n+1)/2-15} + + A_{n,18} z^{n(n+1)/2-18} + A_{n,21} z^{n(n+1)/2-21} + A_{n,24} z^{n(n+1)/2-24} + + A_{n,27} z^{n(n+1)/2-27} + \ldots \] (4.10)

where the first nine coefficients \( A_{n,3k} \), \( (k = 1, \ldots, 9) \) we have just found (see (4.1) - (4.9)). Substituting \( n = 0, 1, 2, 3, 4, 5, 6, 7 \) into expression (4.10)
one can obtain the Yablonskii - Vorob’ev polynomials $Q_0(z)$, $Q_1(z)$, $Q_2(z)$, $Q_3(z)$, $Q_4(z)$, $Q_5(z)$, $Q_6(z)$, and $Q_7(z)$. Rational solutions of the second Painlevé equation \(1.1\) can be found using the correlation \(1.2\).

5 Special polynomials associated with the second equation of the $P_2$ hierarchy

In this section our interest is in the polynomials $Q_n^{[2]}(z)$ associated with the forth-order analogue to the second Painlevé equation

\[ w_{zzzz} - 10 w^2 w_z - 10 w w_z^2 + 6 w^5 - z w - \alpha = 0. \] \(5.1\)

Power expansion of its solutions in a neighborhood of infinity was found in \cite{10} and is the following

\[ w(z; \alpha) = \frac{c_{\alpha,-1}}{z} + \sum_{l=1}^{\infty} c_{\alpha,-5l-1} z^{-5l-1}, \quad z \to \infty. \] \(5.2\)

Again for convenience of use let us rewrite this series in the form

\[ w(z; \alpha) = \sum_{m=0}^{\infty} c_{\alpha,-(m-1)} z^{-m-1}, \] \(5.3\)

where $c_{\alpha,-(m-1)} = 0$ unless $m$ is divisible by 5. The polynomial $Q_n^{[2]}(z)$ is a monic polynomial of degree $n(n+1)/2$

\[ Q_n^{[2]}(z) = \sum_{k=0}^{n(n+1)/2} A_{n,k} z^{n(n+1)/2-k}, \quad A_{n,0} = 1. \] \(5.4\)

Symmetric functions of its roots can be defined as it was done in the case of the Yablonskii-Vorob’ev polynomials

\[ s_{n,m} \overset{\text{def}}{=} \sum_{k=1}^{n(n+1)/2} (a_{n,k})^m, \quad m \geq 1, \] \(5.5\)

where $a_{n,k} (1 \leq k \leq n(n+1)/2)$ are the roots of $Q_n^{[2]}(z)$. The following theorems are true:

**Theorem 5.1.** Let $c_{i,-m-1}$ be the coefficient in expansion \(5.3\) at integer $\alpha = i \in \mathbb{N}$. Then for each $m \geq 1$ and $n \geq 2$ the following relation holds

\[ s_{n,m} = - \sum_{i=2}^{n} c_{i,-(m+1)}. \] \(5.6\)
Theorem 5.2. All the coefficients $A_{n,m}$ of the polynomial $Q_n^2(z)$ can be obtained with a help of $n(n+1)/2 + 1$ first coefficients of the expansion (5.3) for the solutions of (5.1).

These theorems can be proved in the same way as in section 2. Using the expression (5.6) we get

$$s_{n,5} = n \left( n^2 - 1 \right) (n^2 - 4, ) (n + 3)$$

(5.7)

$$s_{n,10} = 6 n \left( n^2 - 1 \right) (n^2 - 4) (n + 3) \left[ 3 n^4 + 6 n^3 - 73 n^2 - 76 n + 504 \right],$$

(5.8)

$$s_{n,15} = 36 n \left( n^2 - 1 \right) (n^2 - 4) (n + 3) \left[ 15 n^8 + 60 n^7 - 1010 n^6 - 3240 n^5 + 28759 n^4 + 62988 n^3 - 388124 n^2 - 420168 n + 2018016 \right],$$

(5.9)

$$s_{n,20} = 72 n \left( n^2 - 1 \right) (n^2 - 4) (n + 3) \left[ 285 n^{12} + 1710 n^{11} - 37965 n^{10} - 205500 n^9 + 2387695 n^8 + 10802590 n^7 - 85963355 n^6 - 296581040 n^5 + 1799452252 n^4 + 4106229664 n^3 - 20241225792 n^2 - 22345634304 n + 93861960192 \right],$$

(5.10)

$$s_{n,25} = 864 n \left( n^2 - 1 \right) (n^2 - 4) (n + 3) \left[ 1035 n^{16} + 8280 n^{15} - 233460 n^{14} - 1779120 n^{13} + 26279210 n^{12} + 181180560 n^{11} - 1828510100 n^{10} + 10846761360 n^9 + 22823297235 n^8 + 398441209080 n^7 - 2435998476560 n^6 - 8750167253280 n^5 + 4458897338072 n^4 + 104249136461184 n^3 - 457807824496512 n^2 + 511460571815424 n + 1994754378000384 \right],$$

(5.11)
s_{n,30} = 864 n \left(n^2 - 1\right) \left(n^2 - 4\right) \left(n + 3\right) [49329 \,n^{20} + 493290 \,n^{19} - \\
-17146575 \,n^{18} - 168377940 \,n^{17} + 3079121634 \,n^{16} + 28513272780 \,n^{15} - \\
-357124460950 \,n^{14} - 3012530148880 \,n^{13} + 28532122080349 \,n^{12} + \\
+211703836490170 \,n^{11} - 1597880669280075 \,n^{10} - \\
-10005358208913420 \,n^9 + 62322893943391984 \,n^8 + \\
+311781921068301760 \,n^7 - 1647676490226842800 \,n^6 - \\
-6078753435828084160 \,n^5 + 27931267677418875264 \,n^4 + \\
+66378780435551109120 \,n^3 - 271399919715872962560 \,n^2 - \\
+305657587619581317120 \,n + 1137057869565290889216] \\
(5.12)

First several coefficients of the polynomials \( \hat{Q}_n^{[2]}(z) \) are the following

\[ A_{n,5} = -\frac{n}{5} \left(n^2 - 1\right) \left(n^2 - 4\right) \left(n + 3\right), \]  
(5.13)

\[ A_{n,10} = \frac{n}{50} \left(n^2 - 1\right) \left(n^2 - 4\right) \left(n^2 - 9\right) \left(n + 4\right) \left(n^4 + 2 \,n^3 - 85 \,n^2 - 86 \,n + 1260\right), \]  
(5.14)

\[ A_{n,15} = -\frac{n}{750} \left(n^2 - 1\right) \left(n^2 - 4\right) \left(n^2 - 9\right) \left(n^2 - 16\right) \left(n + 5\right) \left[n^8 + 4 \,n^7 - 248 \,n^6 - 758 \,n^5 + 26959 \,n^4 + 55186 \,n^3 - 1107792 \,n^2 - \\
-1135512 \,n + 15135120\right], \]  
(5.15)
\[ A_{n,20} = \frac{1}{15000} n \left( n^2 - 1 \right) \left( n^2 - 4 \right) \left( n^2 - 9 \right) \left( n^2 - 16 \right) \left( n^2 - 25 \right) \]

\[ (n - 7) (n + 8) (n + 6) \left[ n^{10} + 5 n^9 - 436 n^8 - 1774 n^7 + 94877 n^6 + 290861 n^5 - 11996834 n^4 - 24480516 n^3 + 661271112 n^2 + 673560144 n - 12570798240 \right], \] (5.16)

\[ A_{n,25} = -\frac{n}{375000} \left( n^2 - 1 \right) \left( n^2 - 4 \right) \left( n^2 - 9 \right) \left( n^2 - 16 \right) \left( n^2 - 25 \right) \]

\[ (n^2 - 36) (n + 7) \left[ n^{16} + 8 n^{15} - 800 n^{14} - 5740 n^{13} + 329098 n^{12} + 2049572 n^{11} - 89974936 n^{10} - 468800180 n^9 + 16974375821 n^8 + 70733085892 n^7 - 2040797018848 n^6 - 6371948611280 n^5 + 144781627711680 n^4 + 300266640707328 n^3 - 5485849799351616 n^2 - 5637057042355200 n + 85489473342873600 \right], \] (5.17)
\[ A_{n,30} = \frac{n}{11250000} (n^2 - 1) (n^2 - 4) (n^2 - 9) (n^2 - 16) (n^2 - 25) \]

\[(n^2 - 36) (n^2 - 49) (n + 8) [n^{20} + 10 n^{19} - 1185 n^{18} - 10950 n^{17} +
+ 732102 n^{16} + 6106308 n^{15} - 311393810 n^{14} - 2287492220 n^{13} +
+ 99293702253 n^{12} + 625780701186 n^{11} - 23733970000125 n^{10} -
- 12461428270910 n^9 + 4030862821261084 n^8 +
+ 16877198228842864 n^7 - 456671374284826080 n^6 -
- 1429614037781766720 n^5 + 323001111441025610560 n^4 +
+ 67002855424545877632 n^3 - 128213099493410484800 n^2 -
- 1315873542386590387200 n + 21754933728927759360000] \]

Rational solutions of (5.1) can be expressed via the logarithmic derivative of the polynomials \( Q_n^{(2)}(z) \)

\[ w(z; n) = \frac{d}{dz} \left\{ \ln \left[ \frac{Q_n^{(2)}(z)}{Q_n^{(2)}(z)} \right] \right\}, \quad n \geq 1, \quad w(z; -n) = -w(z; n). \quad (5.19) \]

6 Special polynomials associated with the third equation of the \( P_2 \) hierarchy

In this section we will deal with the polynomials \( Q_n^{(3)}(z) \) associated with the sixth-order analogue to the second Painlevé equation

\[ w_{zzzzzz} - 14 w^2 w_{zzzz} - 56 w w_z w_{zzz} - 42 w w_{zz}^2 - 70 w_z^2 w_{zz} +
+ 70 w^4 w_{zz} + 140 w^3 w_z^2 - 20 w^7 - z w - \alpha = 0. \quad (6.1) \]
Power expansion of its solutions in a neighborhood of infinity can be presented in the form

$$w(z; \alpha) = \frac{c_{\alpha,-1}}{z} + \sum_{l=1}^{\infty} c_{\alpha,-l-1} z^{-l-1}, \quad z \to \infty. \quad (6.2)$$

Again for convenience of use let us rewrite this series as

$$w(z; \alpha) = \sum_{m=0}^{\infty} c_{\alpha,-(m-1)} z^{-m-1}, \quad (6.3)$$

where $c_{\alpha,-(m-1)} = 0$ unless $m$ is divisible by 7. Since the polynomial $Q_n^{[3]}(z)$ is a monic polynomial of degree $n(n + 1)/2$, then it can be written as

$$Q_n^{[2]}(z) = \sum_{k=0}^{n(n+1)/2} A_{n,k} z^{n(n+1)/2-k}, \quad A_{n,0} = 1. \quad (6.4)$$

Symmetric functions of its roots can be defined as it was done in the case of the Yablonskii-Vorob’ev polynomials $Q_n(z)$ and the polynomials $Q_n^{[2]}(z)$

$$s_{n,m} \overset{\text{def}}{=} \sum_{k=1}^{n(n+1)/2} (a_{n,k})^m, \quad m \geq 1, \quad (6.5)$$

where $a_{n,k} (1 \leq k \leq n(n + 1)/2)$ are the roots of $Q_n^{[3]}(z)$. It can be proved the following theorems:

**Theorem 6.1.** Let $c_{i,-m-1}$ be the coefficient in expansion (6.3) at integer $\alpha = i \in N$. Then for each $m \geq 1$ and $n \geq 2$ the following relation holds

$$s_{n,m} = - \sum_{i=2}^{n} c_{i,-(m+1)}. \quad (6.6)$$

**Theorem 6.2.** All the coefficients $A_{n,m}$ of the polynomial $Q_n^{[3]}(z)$ can be obtained with a help of $n(n + 1)/2 + 1$ first coefficients of the expansion (6.3) for the solutions of (6.1).

Thus we can calculate several functions $s_{n,m}$.

$$s_{n,7} = -\frac{5n}{2} \left( n^2 - 1 \right) \left( n^2 - 4 \right) \left( n^2 - 9 \right) (n + 4) , \quad (6.7)$$
\[ s_{n,14} = 40 \, n \, (n^2 - 1) \, (n^2 - 4) \, (n^2 - 9) \, (n + 4) \, [5 \, n^6 + 15 \, n^5 - 340 \, n^4 - 705 \, n^3 + 8651 \, n^2 + 9006 \, n - 77220], \] 
\[ s_{n,21} = -800 \, n \, (n^2 - 1) \, (n^2 - 4) \, (n^2 - 9) \, (n + 4) \, [35 \, n^{12} + 210 \, n^{11} - 6860 \, n^{10} - 36225 \, n^9 + 629265 \, n^8 + 2736720 \, n^7 - 32792630 \, n^6 - 108110865 \, n^5 + 989372966 \, n^4 + 2162197152 \, n^3 - 16042160664 \, n^2 - 1714174880 \, n + 107749699200], \] 
\[ s_{n,28} = 144000 \, n \, (n^2 - 1) \, (n^2 - 4) \, (n^2 - 9) \, (n + 4) \, [35 \, n^{18} + 315 \, n^{17} - 13930 \, n^{16} - 118580 \, n^{15} + 2790550 \, n^{14} + 21633990 \, n^{13} - 351126160 \, n^{12} - 2393487040 \, n^{11} + 29535328963 \, n^{10} + 170146938815 \, n^9 - 1682680983550 \, n^8 + 7778843727060 \, n^7 + 63914199838220 \, n^6 + 219710453628328 \, n^5 - 1543528005776784 \, n^4 - 3462669359314848 \, n^3 + 21327037897395456 \, n^2 + 23096340571898880 \, n - 127579953840768000] \]  
\[ (6.10) \]

Now let us find the first few coefficients of \( Q_n^{[3]}(z) \).

\[ A_{n,7} = \frac{5}{14} \, n \, (n^2 - 1) \, (n^2 - 4) \, (n^2 - 9) \, (n + 4), \] 
\[ (6.11) \]

\[ A_{n,14} = \frac{5}{392} \, n \, (n^2 - 1) \, (n^2 - 4) \, (n^2 - 9) \, (n^2 - 16) \, (n + 5) \, [5 \, n^6 + 15 \, n^5 - 1105 \, n^4 - 2235 \, n^3 + 56540 \, n^2 + 57660 \, n - 864864], \] 
\[ (6.12) \]
\begin{align*}
A_{n,21} &= \frac{25}{16464} n (n^2 - 1) (n^2 - 4) (n^2 - 9) (n^2 - 16) (n^2 - 25) \\
&\quad (n + 6) [5n^{12} + 30n^{11} - 3235n^{10} - 16450n^9 + 985395n^8 + 4040610n^7 - \\
&\quad -137483057n^6 - 426660726n^5 + 9606229564n^4 + 19928307448n^3 - \\
&\quad -331282163616 n^2 - 341318105856 n + 4505374089216],
\end{align*}
\text{(6.13)}

\begin{align*}
A_{n,28} &= \frac{25}{921984} n (n^2 - 1) (n^2 - 4) (n^2 - 9) (n^2 - 16) (n^2 - 25) \\
&\quad (n^2 - 36) (n + 7) [25 n^{18} + 225 n^{17} - 31900 n^{16} - 260300 n^{15} + \\
&\quad +21092150 n^{14} + 152218150 n^{13} - 8908097220 n^{12} - 55439329220 n^{11} + \\
&\quad +2295894377065 n^{10} + 11991324803825 n^9 - 362820266918120 n^8 - \\
&\quad -1523845124481400 n^7 + 35464390728403088 n^6 + \\
&\quad +111777610107420944 n^5 - 2096257669340843520 n^4 - \\
&\quad -4380613465733418240 n^3 + 68742943019313426432 n^2 + \\
&\quad +70952136972154454016 n - 960217764587156275200]
\end{align*}
\text{(6.14)}

Again rational solutions of \textbf{(6.1)} can be written in terms of the logarithmic derivative of the polynomials $Q_n^{[3]}(z)$. 

19
7 Conclusion

An alternative method for constructing the Yablonskii - Vorob’ev polynomials has been presented. The basic idea of the method is to use power expansions of solutions for the second Painlevé equation. These power expansions can be found with a help of algorithms of power geometry [6,7]. Using our approach we have derived formulas for the coefficients of the Yablonskii - Vorob’ev polynomials and also we have obtained the correlations between the roots of each polynomial. Our method can be also applied for constructing polynomials associated with other nonlinear differential equations [11–15].

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