Classical field theory
via Cohesive homotopy types

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In the year 1900, at the International Congress of Mathematics in Paris, David Hilbert stated his famous list of 23 central open questions of mathematics [Hi1900]. Among them, the sixth problem (see Cor04 for a review) is arguably the one that Hilbert himself regarded as the most valuable: “From all the problems in the list, the sixth is the only one that continually engaged [Hilbert’s] efforts over a very long period, at least between 1894 and 1932.” [Cor06]. Hilbert stated the problem as follows

Hilbert’s mathematical problem 6. To treat by means of axioms, those physical sciences in which mathematics plays an important part.

Since then, various aspects of physics have been given a mathematical formulation. The following table, necessarily incomplete, gives a broad idea of central concepts in theoretical physics and the mathematics that captures them.

| physics | maths |
|---|---|
| prequantum physics | differential geometry |
| 18xx-19xx | mechanics | symplectic geometry |
| 1910s | gravity | Riemannian geometry |
| 1950s | gauge theory | Chern-Weil theory |
| 2000s | higher gauge theory | differential cohomology |
| quantum physics | noncommutative algebra |
| 1920s | quantum mechanics | operator algebra |
| 1960s | local observables | co-sheaf theory |
| 1990s-2000s | local field theory | (\(\infty, n\))-category theory |

These are traditional solutions to aspects of Hilbert’s sixth problem. Two points are noteworthy: on the one hand the items in the list are crown jewels of mathematics; on the other hand their appearance is somewhat unconnected and remains piecemeal.

Towards the end of the 20th century, William Lawvere, the founder of categorical logic and of categorical algebra, aimed for a more encompassing answer that rests the axiomatization of physics on a decent unified foundation. He suggested to

1. rest the foundations of mathematics itself in topos theory [Law65];
2. build the foundations of physics synthetically inside topos theory by
   a. imposing properties on a topos which ensure that the objects have the structure of differential geometric spaces [Law98];
   b. formalizing classical mechanics on this basis by universal constructions (“Categorical dynamics” [Law67], “Toposes of laws of motion” [Law97]).
While this is a grandiose plan, we have to note that it falls short in two respects:

1. Modern mathematics prefers to refine its foundations from topos theory to
   higher topos theory [L06] viz. homotopy type theory [UFP13].
2. Modern physics needs to refine classical mechanics to quantum mechanics and
   quantum field theory at small length/high energy scales [Fe85, SaSc11].

Concerning the first point, notice that indeed, as conjectured in [Jo11] and proven by
CiSh12:
Homotopy type theory is the internal language of locally Cartesian closed \( \infty \)-categories
\( C \). Moreover [Sh12a]: The univalence axiom encodes the presence of the small object
classifier in locally cartesian closed \( \infty \)-categories which are in fact \( \infty \)-toposes \( H \).

Therefore our task is to: refine Lawvere’s synthetic approach on Hilbert’s sixth prob-
lem from classical physics formalized in synthetic differential geometry axiomatized in
topos theory to high energy physics formalized in higher differential geometry axiomati-
zized in higher topos theory. Specifically, the task is to add to (univalent) homotopy
type theory axioms that make the homotopy types have the interpretation of differential
geometric homotopy types in a way that admits a formalization of high energy physics.

The canonical way to add such modalities on type theories is to add modal operators
which in homotopy type theory are homotopy modalities [Sh12b]. The \( \infty \)-categorical
semantics of a homotopy modality is an idemponent \( \infty \)-(co-)monad as in [L06]. For
these it is clear what an adjoint pair is. We say:

**Definition 1** ([ScSh12]). Cohesive homotopy type theory is univalent homotopy type
theory equipped with an adjoint triple of homotopy (co-)modalities
\[ \int \dashv \flat \dashv \sharp, \]
to be called: shape modality \( \int \) flat co-modality \( \flat \) sharp modality \( \sharp \), such that there is a
canonical equivalence of the \( \flat \)-modal types with the \( \sharp \)-modal types, and such that \( \int \)
preserves finite product types.

This has been formalized in HoTT-Coq by Mike Shulman, see [ScSh12] for details.
With hindsight one finds that this modal type theory is essentially what Lawvere was
envisioning in [Law91], where it is referred to as encoding “being and becoming”, and
later more formally in [Law94, Law07], where it is referred to as encoding “cohesion”.

While def. [1] may look simple, its consequences are rich. In [Sc13a] we show how
cohesive homotopy type theory synthetically captures not just differential geometry, but
the theory of (generalized) differential cohomology (e.g. [Bun12]). This is the coho-
mology theory in which physical gauge fields (such as the field of electromagnetism) are
 cocycles. We show in [Sc13a] that cohesion implies the existence of geometric homotopy
types Phases such that

1. the dependent homotopy types over Phases are prequantized covariant phase
   spaces of physical field theories;
2. correspondences between these dependent types are spaces of trajectories equipped
   with local action functionals;
3. group actions on such dependent types encode the Hamilton-de Donder-Weyl
   equations of motion of local covariant field theory;
4. the “motivic” linearization of these relations over suitable stable homotopy types
   yields the corresponding quantum field theories.

An exposition of what all this means is in section 1.2 of [Sc13a]. See [Nu13] for details
on the last point. See [Sc13b] for a general overview.
Specifically, cohesive homotopy type theory has semantics in the ∞-topos $H$ of ∞-stacks over the site of smooth manifolds (section 4.4 of [Sc13a]). This contains a canonical line object $A^1 = \mathbb{R}$, the \textit{continuum}, abstractly characterized by the fact that the shape modality exhibits (in the sense of [Sh12b]) the corresponding $A^1$-homotopy localization. Forming the quotient type by the type of integers yields the \textit{smooth circle group} $U(1) \simeq \Omega^n \mathbb{B}^n U(1)$ is the $n$-fold loop type. Write then

$$\theta_{\mathbb{B}^n U(1)} := \text{fib}(\text{fib}(\epsilon)) : \mathbb{B}^n U(1) \to \flat B^{n+1} U(1)$$

for the second homotopy fiber of the co-unit $\epsilon : \flat B^{n+1} U(1) \to B^{n+1} U(1)$ of the flat co-modality. Cohesion implies that we may think of this as the \textit{universal Chern-character} for ordinary smooth cohomology (section 3.9.5 in [Sc13a]). Hence we write $\text{Phases} := B^n U(1)_{\text{conn}}$ for the dependent sum of “all” homotopy fibers of $\theta_{\mathbb{B}^n U(1)}$ (for some choice of “all”, see section 4.4.16 of [Sc13a]). Then a dependent type $\nabla$ over $B U(1)_{\text{conn}}$ is a \textit{prequantized phase space} (see section 3.9.13 of [Sc13a]) in classical mechanics [Ar89].

An equivalence of dependent types over $B U(1)_{\text{conn}}$ is a \textit{Hamiltonian symplectomorphism} and a (concrete) function term

$$H : B \mathbb{R} \to \prod_{B U(1)_{\text{conn}}} \text{BEquiv}(\nabla, \nabla)$$

of the function type from the delooping of $\mathbb{R}$ to the delooping of the dependent product of the type of auto-equivalences of $\nabla$ is equivalently a choice of \textit{Hamiltonian}. It sends the (“time”) parameter $t : \mathbb{R}$ to the Hamiltonian evolution $\exp(t\{H, -\})$ with Hamilton-Jacobi action functional $\exp(\frac{i}{\hbar} \int L dt)$. In the ∞-categorical semantics this is given by a diagram in $H$ of the following form\cite{FRS13a}:

\begin{center}
\begin{tikzcd}
\text{graph } (\exp(t\{H, -\})) \arrow[swap]{dr}{\nabla} \arrow{d}{\text{exp}(\frac{i}{\hbar} \int L dt)} & X \arrow{dl}{\nabla} \\
X & X
\end{tikzcd}
\end{center}

Here $X := \sum_{B U(1)_{\text{conn}}} \nabla$ is the phase space itself and $\nabla$ is its \textit{pre-quantum bundle} [FRS13a].

This statement concisely captures and unifies a great deal of classical Hamilton-Lagrange-Jacobi mechanics, as in [Ar89]. Moreover, when replacing $B U(1)_{\text{conn}}$ here with $B^n U(1)_{\text{conn}}$ for general $n \in \mathbb{N}$, then the analogous statement similarly captures $n$-dimensional classical field theory in its “covariant” Hamilton-de Donder-Weyl formulation on dual jet spaces of the field bundle\cite{Rom05} (see e.g. [Rom05]). This is shown in section 1.2.11 of [Sc13a].

\footnote{1This is a pre-quantization of the \textit{Lagrangian correspondences} of [We83].
\footnote{2 I am grateful to Igor Khavkine for discussion of this point.}
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