Abstract

Tverberg’s theorem is one of the cornerstones of discrete geometry. It states that, given a set \( X \) of at least \( (d + 1)(r - 1) + 1 \) points in \( \mathbb{R}^d \), one can find a partition \( X = X_1 \cup \cdots \cup X_r \) of \( X \), such that the convex hulls of the \( X_i \), \( i = 1, \ldots, r \), all share a common point. In this paper, we prove a strengthening of this theorem that guarantees a partition which, in addition to the above, has the property that the boundaries of full-dimensional convex hulls have pairwise nonempty intersections. Possible generalizations and algorithmic aspects are also discussed.

As a concrete application, we show that any \( n \) points in the plane in general position span \( \lfloor n/3 \rfloor \) vertex-disjoint triangles that are pairwise crossing, meaning that their boundaries have pairwise nonempty intersections; this number is clearly best possible. A previous result of Rebollar et al. guarantees \( \lfloor n/6 \rfloor \) pairwise crossing triangles. Our result generalizes to a result about simplices in \( \mathbb{R}^d, d \geq 2 \).
Acknowledgements Part of the research leading to this paper was done during the 16th Gremo Workshop on Open Problems (GWOP), Waltensburg, Switzerland, June 12-16, 2018. We thank Patrick Schnider for suggesting the problem, and Stefan Felsner, Malte Milatz and Emo Welzl for fruitful discussions during the workshop. We also thank Stefan Felsner and Manfred Scheucher for finding and communicating the example from the last section.

1 Introduction

The following theorem was published by Johann Radon in 1921 [17]: any set of $d + 2$ points in $\mathbb{R}^d$ can be partitioned into two (disjoint) subsets, whose convex hulls intersect. In 1966, Helge Tverberg [22] proved the following important generalization of Radon’s result.

Theorem 1 (Tverberg [22]). Let $X$ be a set of at least $(d + 1)(r - 1) + 1$ points in $d$-space. Then $X$ can be partitioned into $r$ sets whose convex hulls all have a point $o$ in common. (In the literature, the point $o$ is referred to as a Tverberg point and the partition as a Tverberg partition.)

Radon’s theorem covers the case $r = 2$. Figure 1 illustrates Tverberg’s theorem for $d = 2$ and $r = 3$.

Figure 1 Tverberg’s theorem for $d = 2$ and $r = 3$: any set of at least 7 points can be partitioned into three sets whose convex hulls all have a point in common. Our example uses 9 points and shows two Tverberg partitions as well as corresponding Tverberg points. Many other Tverberg partitions exist in this example.

This theorem largely influenced the course of discrete geometry and spurred a lot of research in the area. We do not go into more details in this paper and refer the reader to a recent survey by Bárány and Soberón [5].

Another question, motivating our work, comes from the field of geometric graphs. In [1], Aronov et al. conjectured that there exists an absolute constant $c > 0$, such that, given any set of $n$ points in general position in the plane, one can find at least $cn$ disjoint pairs among them such that their connecting segments pairwise cross. Such a collection of segments is called a crossing family. Despite considerable interest in this problem, the best published bound still comes from the original paper [1], stating that one can always find a crossing family of size at least $c\sqrt{n}$ for some absolute $c > 0$.

In an attempt to approach this problem, Rebollar et al. asked whether one can find at least $cn$ disjoint triples whose connecting triangles cross pairwise [18]. Throughout this paper, we say that two convex bodies in $\mathbb{R}^d$ cross if their boundaries have a non-empty intersection. We remark that if a convex body in $\mathbb{R}^d$ is not full-dimensional then it has no interior and therefore coincides with its boundary. Our main results below would be false in general if relative boundaries were considered in the above definition of crossing of two convex bodies (an easy counterexample in this situation is a set of points lying on a line).
Rebollar et al. showed the following: For every finite point set of size $n$ in the plane in general position, i.e., no three points on a line, there exist $\left\lceil \frac{n}{2} \rightceil$ vertex-disjoint and pairwise crossing triangles with vertices in $P$. As at most $\left\lceil \frac{n}{3} \right\rceil$ disjoint triangles can be found, this leaves a factor-2 gap.

If we only want triangles that have a common point, this gap can be closed using a simple strengthening of Tverberg’s theorem for point sets of size at most $(d + 1)r$ that we present next. However, these triangles might not be pairwise crossing, since triangles can be nested; see Figure 1 (right).

**Theorem 2.** Let $X$ be a set of at least $(d + 1)(r - 1) + 1$ and at most $(d + 1)r$ points in $d$-space. Then $X$ can be partitioned into $r$ disjoint sets $X_1, \ldots, X_r$ of size at most $d + 1$, whose convex hulls all have a point in common.

To prove this, we apply Theorem 1 to $X$. We get sets $X'_1, \ldots, X'_r$ whose convex hulls contain a common point, say, the origin. Using Carathéodory’s theorem, from every $X'_j$ of size larger than $d + 1$ we can select $d + 1$ points $X_i \subseteq X'_i$, whose convex hull still contains the origin. Finally, some of the sets $X'_i$ of size smaller than $d + 1$ are filled up to size $d + 1$ with the points removed from other $X'_i$’s. The origin is still a common point for all conv($X_i$)’s.

The main result of this paper provides a crossing version of Tverberg’s Theorem 2.

**Theorem 3.** Let $X$ be a set of at least $(d + 1)(r - 1) + 1$ and at most $(d + 1)r$ points in $d$-space. Then $X$ can be partitioned into $r$ disjoint sets $X_1, \ldots, X_r$ of size at most $d + 1$, whose convex hulls all have a point in common. Moreover, for any $X_i, X_j$ of size $d + 1$, conv($X_i$) and conv($X_j$) cross, meaning that their boundaries have a non-empty intersection.

We call such a partition a crossing Tverberg partition. An easy calculation shows that the number of sets with exactly $d + 1$ elements is at least $|X| - d \in \{r - d, r - d + 1, \ldots, r\}$. In particular, for sets $X$ of size exactly $(d + 1)r$, we immediately deduce the following simpler-looking corollary.

**Corollary 4.** Let $X$ be a set of $(d + 1)r$ points in $d$-space. Then $X$ can be partitioned into $r$ sets $X_1, \ldots, X_r$ of size $d + 1$, whose convex hulls all have a point in common and such that for any $i, j \in [n]$, conv($X_i$) and conv($X_j$) cross, meaning that their boundaries have a non-empty intersection.

We also obtain an optimal strengthening of the result by Rebollar et al. [13] that moreover generalizes to all dimensions $d \geq 2$.

**Corollary 5.** For every finite point set $X$ of size $n$ in the plane in general position, i.e., no three points on a line, there exist $\left\lceil \frac{n}{2} \right\rceil$ vertex-disjoint and pairwise crossing triangles with vertices in $X$.

More generally, for every finite point set $X$ of size $n$ in $\mathbb{R}^d$ in general position, i.e., no $d$ points lying on a hyperplane, there exist $\left\lceil \frac{n}{d+1} \right\rceil$ vertex-disjoint and pairwise crossing simplices with vertices in $X$.

To derive this from Theorem 3 we remove $n \mod (d + 1)$ points from $X$ and then apply Theorem 3 on the remaining set of $(d + 1)\left\lceil \frac{n}{d+1} \right\rceil$ points.

Finally, we get a crossing version of the actual Tverberg’s theorem 1 for point sets of arbitrarily large size.

**Theorem 6 (Crossing Tverberg theorem).** Let $X$ be a set of at least $(d + 1)(r - 1) + 1$ points in $d$-space. Then $X$ can be partitioned into $r$ sets whose convex hulls all have a point in common. Moreover, for any $X_i, X_j$ of size at least $d + 1$, conv($X_i$) and conv($X_j$) cross, meaning that their boundaries have a non-empty intersection.
This is also easy to prove, using Theorem 3 and Corollary 4. If $n \leq (d+1)r$, we apply Theorem 3. Otherwise, we apply Corollary 4 to an arbitrary subset $Y \subset X$ of size $(d+1)r$, resulting in a crossing Tverberg partition into $r$ sets $Y_1, \ldots, Y_r$ of size $d+1$ each. Now we consecutively add the remaining points to suitable sets in such a way that the crossings between the convex hulls are maintained (the Tverberg point automatically remains valid).

It remains to prove Theorem 3 which we will do in the next section. The main idea is the following. We show that we can assume the points to be in general position. We start with a Tverberg partition according to Theorem 2, i.e. a partition into sets of size at most $d+1$ such that their convex hulls have a point in common. Such a partition might look like in Figure 1 (right). If the full-dimensional convex hulls (which are simplices) cross pairwise, we are done. In general, however, we still have pairs of nested simplices. As long as this is the case, we fix one pair of nested simplices at a time until no pair of nested simplices exists anymore, and our desired partition is obtained.

By fixing, we mean that we repartition the $2(d+1)$ points involved in the two simplices in such a way that the resulting simplices are not nested anymore but still contain the common point. In the example of Figure 1, there is one pair of nested simplices (red edges), and after fixing it (blue edges), we are actually done in this case; see Figure 2.

![Figure 2](image)

**Figure 2** Fixing a pair of nested simplices spanned by the 6 black points: two nested simplices (red) are transformed into two simplices that cross (blue).

There are of course some things to prove here. First of all, we need to show that we can actually perform the desired repartitioning in all cases. Then, by fixing one pair, new pairs might arise that need fixing, so it has to be proved that fixing terminates.

The key for repartitioning the points is the following lemma, which may be of independent interest.

**Lemma 7.** Let $T, T'$ be two disjoint $(d+1)$-element sets in $\mathbb{R}^d$ such that $0 \in \text{conv}(T) \cap \text{conv}(T')$. Then there exist two disjoint $(d+1)$-element sets $S, S'$ such that $S \cup S' = T \cup T'$, $0 \in \text{conv}(S) \cap \text{conv}(S')$, and moreover, $\text{conv}(S)$ and $\text{conv}(S')$ cross.

Since applications of Lemma 7 allow to keep the Tverberg point and the sizes of the parts in the (Tverberg) partition, we actually have the following strengthening of Theorem 3.
Theorem 8. Let $X$ be a set of points in $d$-space. Suppose that there is a Tverberg partition of $X$ into $r$ parts of sizes $s_1, \ldots, s_r$, for which $o$ is a Tverberg point, and $s_i \leq d+1$, $i = 1, \ldots, r$. Then there is also a crossing Tverberg partition of $X$ into $r$ parts of sizes $s_1, \ldots, s_r$, for which a Tverberg point is $o$.

In the next section, we prove Theorem 3 and (on the way) Lemma 7. We remark that for each of the other theorems and corollaries in this section, we have already explained how to derive them from Tverberg’s theorem, or from our main Theorem 3. In Section 3, we discuss possible generalizations as well as some algorithmic aspects of the problem.

2 Proof of Theorem 3

First of all, using a standard topological argument, we may assume that the points of $X$ are in general position, meaning that no $d+1$ points lie in a common hyperplane. To justify this assumption, we observe that the set of point sets of some fixed size $n$ that allow a crossing Tverberg partition is closed, since its complement is open. Indeed, if we take a point set that does not allow a crossing Tverberg partition, then all (finitely many) partitions have the property that there is no common point in all the convex hulls, or that some convex hulls are (properly) nested. This property is maintained under any sufficiently small perturbation. Since every point set is the limit of a sequence of point sets in general position, it follows from closedness that if all points sets in general position allow a crossing Tverberg partition, then all point sets do.

Under this general position assumption, we may also assume that the union of the $X_i$’s of size $d+1$ and the common point of all convex hulls are in general position. Henceforth, we assume general position without explicitly mentioning it.

We start with a Tverberg partition of $X$ into sets $X_1, \ldots, X_r$ of size at most $d+1$; such a partition is guaranteed by Tverberg’s Theorem 2. W.l.o.g. we assume that the Tverberg point is the origin.

2.1 Fixing Pairs

The sets $X_1, \ldots, X_r$ may not yet satisfy the requirements of Theorem 3 since there may be pairs $(X_i, X_j)$, $X_i \neq X_j$ such that $\text{conv}(X_i)$ and $\text{conv}(X_j)$ do not cross. As the simplices $\text{conv}(X_i)$ and $\text{conv}(X_j)$ themselves intersect in the origin, the only remaining possibility is that one of them contains the other one, i.e., they are nested; see Figure 1 (right). Lemma 7 whose proof we approach next is a key step, allowing us to “unnest” simplices. In fact, it is enough to show that for any pair $(T, T')$ as in the lemma, there exists a different pair $(S, S')$ such that $S \cup S' = T \cup T'$, $\mathbf{0} \in \text{conv}(S) \cap \text{conv}(S')$. The reason is that at most one of these pairs can be nested: the outer simplex of a nested pair coincides with $\text{conv}(T \cup T')$ and is therefore uniquely determined. Hence, Lemma 7 is implied by the following.

Lemma 9. Let $V \subset \mathbb{R}^d$, $|V| = 2(d+1)$, such that $V \cup \{\mathbf{0}\}$ is in general position. Then

$$\left| \left\{ (F, G) : F, G \in \binom{V}{d+1}, F \cap G = \emptyset, \mathbf{0} \in \text{conv} F \cap \text{conv} G \right\} \right|$$

is even.

In particular, if there is one such pair $(F, G)$, then there is another.

Remark. We note that this statement and its application is quite unusual. Typically, one proves that the number of objects of a certain type is always odd and thus at least one object of the type exists.
2.1.1 Geometric proof of Lemma 9 for $d = 2$

The purely combinatorial proof that we give in the next section for general $d$ is not difficult, but does not provide any geometric intuition. We therefore start with a simple proof in the plane.

Consider a set $V$ of $2(d + 1) = 6$ points in the plane. We remark that the statement is invariant under scaling points, and thus we may assume that all points lie on a circle with the center in the origin. Due to our general position assumption, no two points from $V$ lie on a line passing through the origin; see Figure 3 (left).

Now we mirror each point at the origin and obtain another 6 points, drawn in white in Figure 3 (right). We observe that in the circular order of points, there must be two consecutive ones of the same color. Indeed, an alternating pattern (black, white, black, white, . . . ) would lead to pairs of mirrored points having the same color. Let $p, p'$ be two consecutive points of the same color; by going to the mirror points if necessary, we may assume that they are black and hence belong to $V$.

We make two observations (actually, just one). (i) $p$ and $p'$ cannot belong to a triple $F \subset V$ such that $0 \in \text{conv}(F)$, as otherwise, the third point would get mirrored to a white point between $p$ and $p'$; see Figure 4 (left). (ii) For any $q \in V \setminus \{p, p'\}$, the two segments $\text{conv}(\{p, q\})$ and $\text{conv}(\{p', q\})$ have the origin on the same side; see Figure 4 (right).

This implies the following: if $\{F, G\}$ is a partition of $V$ that we count in Lemma 9 then (i) $p$ and $p'$ are in different parts, and (ii) swapping $p$ and $p'$ between the parts leads to a different partition $\{F', G'\}$ that we also count. In other words, $p$ and $p'$ are “combinatorially indistinguishable” with respect to the relevant properties, and the operation of swapping them between parts establishes a matching between the partitions that we want to count. Hence, their number is even.
2.1.2 Combinatorial proof of Lemma 9

Recall that for $|V| = 2(d + 1)$, we need to show that there is an even number of partitions $\{F, G\}$ of $V$ into parts of equal size $d + 1$ such that $0 \in \text{conv}(F) \cap \text{conv}(G)$. We show that this follows from the fact that the $(d + 1)$-element subsets $F$ with $0 \in \text{conv}(F)$ form a cocycle, a concept borrowed from topology. The proof itself does not use any topology, though.

**Definition 10.** Let $n \geq k \geq 1$ be integers, and let $V$ be a set with $n$ elements. A family $C \subset \binom{V}{k}$ of $k$-element subsets of $V$ is a cocycle if

$$|\{F \in C : F \subset M\}|$$

is even for every $M \subset V$ with $|M| = k + 1$.

**Example.** Fix a set $D \in \binom{V}{k}$. Then $\delta D := \{F \in \binom{V}{k} : D \subset F\}$ is a cocycle. Indeed, a $(k + 1)$-element subset $M \subset V$ either contains zero sets in $\delta D$ (if $D \not\subset M$) or exactly two sets in $\delta D$ (if $D = M \setminus \{p, q\}$).

**Lemma 11.** Let $V \subset \mathbb{R}^d$ be such that $V \cup \{0\}$ is in general position. Then

$$C(V) := \{F \in \binom{V}{d+1} : 0 \in \text{conv} F\}$$

is a cocycle.

**Proof.** Let $M := \{v_1, v_2, \ldots, v_{d+2}\} \subset V$. Lift the points to dimension $d + 1$ such that the convex hull of the lifted point set $\tilde{M}$ is a full-dimensional simplex $\Delta$; see Figure 5.

![Figure 5 Geometric proof of Lemma 11](image)

Any set $\tilde{F} \subset \tilde{M}$ of size $d + 1$ spans a facet of $\Delta$, and we have $0 \in \text{conv}(F)$ if and only if the vertical line through $0$ intersects that facet. As $V \cup \{0\}$ is in general position, this line intersects zero or two facets of $\Delta$, hence $|\{F \in C(V) : F \subset M\}| \in \{0, 2\}$.

We remark that there is also an elementary linear algebra version of this “proof by picture” (omitted in this extended abstract).

By Lemma 11, Lemma 9 is now simply a special case of the following main result of this section.

**Theorem 12.** Let $k \geq 1$, $|V| = 2k$ and let $C \subset \binom{V}{k}$ be a cocycle. Set

$$P_C := \{\{F, G\} : F, G \in C, F \cap G = \emptyset\}.$$ 

Then $|P_C|$ is even.
The Crossing Tverberg Theorem

Proof. We double count the edges of a suitable bipartite graph. Let \( v \in V \) be an arbitrary but fixed element, and let one part be

\[
\mathcal{D} := \{ D \in \left( \binom{V \setminus \{v\}}{k-1} \right) : D \cup \{v\} \in \mathcal{C} \}.
\]

The other part is

\[
\mathcal{P} := \{ \{F,G\} : F \in \left( \binom{V}{k} \right), G \in \mathcal{C}, F \cap G = \emptyset \}.
\]

In particular, \( \mathcal{P} \supset \mathcal{P}_C \). We connect \( D \in \mathcal{D} \) and \( \{F,G\} \in \mathcal{P} \) by an edge if and only if \( D \subset F \).

Now, we make the following two claims about this bipartite graph.

(i) Every set \( D \in \mathcal{D} \) has even degree.

(ii) A pair \( \{F,G\} \in \mathcal{P} \) has odd degree if and only if \( \{F,G\} \in \mathcal{P}_C \).

The statement of the lemma immediately follows from (i) and (ii) due to the fact that the number of odd-degree vertices in every graph is even. It remains to prove the two claims.

To see (i), fix \( D \in \mathcal{D} \). The edge condition \( D \subset F \) is equivalent to \( G \subset V \setminus D \). As \( \mathcal{C} \) is a cocycle and the size of \( M := V \setminus D \) is \( k+1 \), the number of such \( G \in \mathcal{C} \) is even, and each one determines a unique pair \( \{V \setminus G, G\} \in \mathcal{P} \) that is connected to \( D \).

For (ii), fix a pair \( \{F,G\} \in \mathcal{P} \). If \( v \in F \), there is only one candidate for a neighbor, namely \( D = F \setminus \{v\} \). By definition of \( D \), this candidate is actually a neighbor if and only if \( F = D \cup \{v\} \in \mathcal{C} \). Hence, \( \{F,G\} \) has degree 1 if \( F \in \mathcal{C} \) and degree 0 otherwise. If \( v \notin F \), there are \( |F| \) candidates for neighbors, namely the sets \( D_w = F \setminus \{w\}, w \in F \). As before, \( D_w \) is actually a neighbor if and only if \( D_w \cup \{v\} = F \cup \{v\} \setminus \{w\} \in \mathcal{C} \). Consider the set \( M := F \cup \{v\}, |M| = k+1 \). Its \( k \)-elements subsets are \( F \) as well as all the \( D_w \cup \{v\} \). As \( \mathcal{C} \) is a cocycle, an even number of them is contained in \( \mathcal{C} \). Hence, if \( F \in \mathcal{C} \), then \( \{F,G\} \) has an odd number of neighbors \( D_w \), otherwise an even number. \( \Box \)

2.2 Fixing terminates

Now we are prepared to finish the proof of Theorem 3 according to the outline already given in the introduction. We start with an arbitrary Tverberg partition \( \mathcal{X} = \{X_1, \ldots, X_n\} \).

As long as there exists a pair of nested simplices \( \text{conv}(X_i) \subset \text{conv}(X_j) \), we apply the fixing operation (Lemma 7) to replace \( T = X_i \) and \( T' = X_j \) with \( S, S' \) such that \( \text{conv}(S) \) and \( \text{conv}(S') \) cross. We need to show that after finitely many fixes, there are no nested pairs anymore, in which case we have a crossing Tverberg partition.

To see termination, we observe that in any fixing operation that replaces \( X_i \) and \( X_j \), such that \( \text{conv}(X_i) \subset \text{conv}(X_j) \), the simplex \( \text{conv}(X_j) \) is volume-wise the unique largest \( d \)-dimensional simplex that can be formed from the \( 2(d+1) \) points \( X_i \cup X_j \) involved in the operation. Hence, the two simplices \( \text{conv}(S) \) and \( \text{conv}(S') \) replacing \( \text{conv}(X_i) \) and \( \text{conv}(X_j) \) are volume-wise both strictly smaller than \( \text{conv}(X_i) \). Therefore, if we order all full-dimensional simplices by decreasing volume, the sequence of these volumes goes down lexicographically in every fixing operation.

Formally, let \( \mathcal{V} = (V_1, V_2, \ldots, V_s), s \leq r \) be the sequence of volumes in decreasing order before the fix, and \( \mathcal{V}' = (V'_1, V'_2, \ldots, V'_s) \) the decreasing order after the fix. Moreover, suppose that \( k \) is the largest index at which the volume of \( \text{conv}(X_j) \) appears in \( \mathcal{V} \) (by even more general position, we could assume that there is a unique such index, but this does not really help here). The volume of \( \text{conv}(X_j) \) appears at a position \( \ell > k \). As the fixing operation removes a volume equal to \( V_k \) and inserts two volumes smaller than \( V_k \), we have...
that $V$ and $V'$ agree in the first $k-1$ positions, but $V'_k < V_k$. This exactly defines the relation "$V' < V$" in decreasing lexicographical order, and as this is a total order, fixing must eventually terminate.

Remark. Instead of volume, we may use the number of points from $X$ inside $\text{conv}(X_j)$ as a measure.

3 Discussion

3.1 A Topological version of Theorem 3

If $r$ is a prime power, then Tverberg’s theorem admits a topological generalization, known as the Topological Tverberg Theorem:

> Theorem 13. If $d \in \mathbb{N}$ and if $r$ is a prime power, then for every continuous map from the $(d+1)(r-1)$-dimensional simplex $\Delta_{(d+1)(r-1)}$ to $\mathbb{R}^d$, there exist $r$ pairwise disjoint faces of $\Delta_{(d+1)(r-1)}$ whose images intersect in a common point.

This result was first proved in the case when $r$ is a prime by Bárány, Shlosman and Szücs [4] and later extended to prime powers by Özaydin [14]. On the other hand, Theorem 13 is false if $r$ is not a prime power: By work of Mabillard and Wagner [11] and a result of Özaydin [14], a closely related result (the generalized Van Kampen–Flores Theorem) is false whenever $r$ is not a prime power and, as observed by Frick [9], the failure of Theorem 13 for $r$ not a prime power follows from this by a reduction due to Gromov [10] and to Blagojević, Frick, and Ziegler [6]. The lowest dimension in which counterexamples are known to exist is $d = 2r$ [2, 11]. We refer to the recent surveys [5, 7, 20, 23] for more background on the Topological Tverberg Theorem and its history.

In the same vein, it is natural to wonder if our Theorem 3 also extends to the topological setting. The straightforward approach to generalize our result would be to use the Topological Tverberg Theorem instead of Tverberg’s theorem and then keep fixing the pairs of simplices whose boundaries do not mutually intersect. To this end we need an adaptation of Lemma 11 and the fixing procedure to the topological setting. The rest of our argument is free of any geometry except for the use of Carathéodory’s theorem which is not really crucial. While extending Lemma 11 is easy, showing the termination of the fixing procedure appears to be quite difficult except under the scenario which we discuss below. First, we discuss planar extensions of Theorem 3. An arrangement of pseudolines $\mathcal{P}$ is a finite set of not self-intersecting open arcs, called pseudolines, in $\mathbb{R}^2$ such that (i) For every pair $P_1, P_2 \in \mathcal{P}$ of two distinct pseudolines, $P_1$ and $P_2$ intersect transversely in a single point, and (ii) $\mathbb{R}^2 \setminus \mathcal{P}$ is not connected for every $P \in \mathcal{P}$. A drawing of a complete graph on $n$ vertices $K_n$ in the plane is pseudolinear if the edges can be extended to an arrangement of pseudolines.

In the plane, it is not hard to see that Theorem 3 and its proof almost extends to the setting of pseudolinear drawings of complete graphs. Since the Topological Tverberg Theorem is only valid for prime powers $r$, the number of pairwise crossing triangles is slightly smaller then the number of vertices divided by 3.

> Theorem 14. In a pseudolinear drawing of a complete graph $K_{3m}$ we can find $n = (1 - o(1))m$ vertex-disjoint and pairwise crossing triangles. Moreover, the topological discs bounded by these triangles intersect in a common point.

Proof. Let $\mathcal{D}$ be a pseudolinear drawing of $K_{3m}$. We put $n$ to be the largest prime power not larger than $m + 1$. By the asymptotic law of distribution of prime numbers $n = (1 - o(1))m$. 
The Crossing Tverberg Theorem

Next, apply the Topological Tverberg theorem with \( r = n \) and \( d = 2 \) to a map \( \mu : \Delta_{3n-3} \rightarrow \mathbb{R}^2 \) which extends \( D \) as follows. We define \( \mu \) on the 1-dimensional skeleton of \( \Delta_{3n-3} \) as a restriction of \( D \) to some \( K_{3n-2} \). Note that every triangle of \( K_{3n-2} \) is drawn by \( D \) as a closed arc without self intersections. The map \( \mu \) extends to the 2-dimensional skeleton of \( \Delta_{3n-3} \) so that every 2-dimensional face is mapped homeomorphically in \( \mathbb{R}^2 \). We define the map \( \mu \) on the rest of \( \Delta_{3n-3} \) arbitrarily while maintaining continuity.

Analogously to the proof of Theorem 3 an application of the Topological Tverberg Theorem gives us \( n - 1 \) disjoint 2-dimensional faces \( F_1, \ldots, F_{n-1} \) of \( \Delta_{3n-3} \) whose images under \( \mu \) intersect in a common point. To this end we apply Carathéodory’s theorem for drawings of complete graphs [8] Lemma 4.7] instead of the original version of Carathéodory’s theorem. Let \( T_i \) denote the boundary of \( F_i \), for \( i = 1, \ldots, n - 1 \). Note that each \( T_i \) is a triangle in \( K_{3n-2} \). If all pairs \( T_i \) and \( T_j \) are crossing then we are done. Otherwise, we perform the fixing operations, which can be done since Lemma [11] easily extends to the setting in which we replace simplices by images of 2-dimensional faces of \( \Delta_{3n-3} \) under \( \mu \).

A drawing of a graph in the plane is simple if every pair of edges intersect at most once either at a common end point or in a proper crossing. Clearly, all pseudolinear drawings of complete graphs are also simple, but not vice-versa. Hence, it might be worthwhile to extend Theorem [14] to simple drawings of complete graphs.

If we want the interiors of the triangles to be pairwise intersecting, we only known that we can take \( m = O(\log n^{1/6}) \) which is easily derived from the following result of Pach, Solymosi and Tóth [13]. Every simple drawing of \( K_n \) contains a drawing of \( K_m \) that is weakly isomorphic to a so-called convex complete graph or a twisted complete graph, see Figure [3] for which Theorem [14] holds. We omit the proof of the latter which is rather straightforward. For example, if in a twisted drawing of \( K_{3n} \) the vertices are labeled as indicated in the figure, \( \{(0 + i, n + i, 2n + i) \mid i = 0, \ldots, n - 1\} \) is a crossing Tverberg partition.

If we do not insist on the interiors of the triangles to be pairwise intersecting, we know that \( m \) can be taken to be at least \( O(n^\varepsilon) \) for some small \( \varepsilon > 0 \) by the following result of Pach and Tóth [8]. Every simple drawing of \( K_n \) contains \( O(n^\varepsilon) \) pairwise crossing edges for some \( \varepsilon > 0 \).
Theorem 14 could be generalized to hold for an appropriate high dimensional analog of pseudolinear drawings. Since this would require introducing many technical terms and would not offer substantially interesting content we refrain from doing so.

### 3.2 Stronger conditions on crossings

A pair of vertex disjoint $([d/2] - 1)$-dimensional simplices in general position in $d$-space does not intersect. Hence, the pairwise intersection of the boundaries of $\text{conv}(X_i)$'s in the conclusion of Theorem 8 cannot be strengthened to the pairwise intersection of lower than $[d/2]$-dimensional skeleta of the boundaries.

Nevertheless, for $d = 3$ one may ask if in the setting of Lemma 7 we get a stronger property along the following lines. Can we guarantee the existence of a pair of vertex-disjoint tetrahedra $\{S, S'\}$ that both contain the origin and such that the boundary of a 2-dimensional face of $S$ is linked with the boundary of a 2-dimensional face of $S'$? Again, the answer to this question is negative. Stefan Felsner and Manfred Scheucher found the following set of 8 points:

$$(3, -2, 2), (2, -5, 3), (-3, 0, -4), (-1, 2, 0),$$

$$(1, -5, -4), (4, 1, -2), (-2, -5, -4), (-3, 1, 3).$$

This set contains a pair of disjoint tetrahedra both containing the origin $(0, 0, 0)$, but no two disjoint linked tetrahedra both containing the origin.

The example was found using a SAT solver who found an abstract order type with the require property. A realization of the order type with actual points was obtained with a randomized procedure.

### 3.3 Computational complexity of finding a crossing Tverberg partition

A natural question is whether we can find the partition of the point set given by Theorem 8 efficiently, i.e., in polynomial time in the size of $X$. A straightforward way to construct an algorithm is to make the proof of Theorem 8 algorithmic. To this end we first need an algorithm for finding a Tverberg partition and also a Tverberg point. Unfortunately, several results suggest that an efficient algorithm for this problem is rather unlikely to exist. Since a Tverberg partition is guaranteed to exist, NP-complexity theory does not apply to the
algorithm problem of finding it, and for example, PPAD completeness theory \[16\] appears to be more suitable for this problem. Nevertheless, we are not aware of any result in this direction. The only closely related hardness result we are aware of is the one by Teng \[21\] Theorem 8.14], who proved that checking whether a given point is a Tverberg point of a given point set is NP-complete.

A line of research on finding an approximate Tverberg partition efficiently was initiated by Miller and Sheehy \[12\] and further developed in \[13, 19\]. In particular, Mulzer and Werner \[13\] showed that it is possible to find in time \[d^{O(\log d)|X|}\] an approximate Tverberg partition of size \[\left\lceil \frac{|X|}{d+1} \right\rceil\], whereas Theorem \[1\] guarantees the partition of size \[\left\lceil \frac{|X|}{d} \right\rceil\].

If we aim only at an approximate algorithmic version of our Theorem \[3\] along the lines of the result of Mulzer and Werner, we face the problem of efficiently fixing an (approximate) Tverberg partition to make it crossing. Due to the fact that our termination argument for the iterated Fixing Pairs procedure relies on progress in the lexicographical ordering of the simplex volumes, we may potentially need exponentially many (in the size of \(X\)) iterations before we arrive at a crossing partition. We leave it as an interesting open problem to prove or disprove that there is always a way to invoke the Fixing Pairs operation only polynomially (or least subexponentially) many times in order to arrive at a partition required by Theorem \[3\].

References

1. B. Aronov, P. Erdős, W. Goddard, D. J. Kleitman, M. Klugerman, J. Pach, and L. J. Schulman. Crossing families. Combinatorica, 14(2):127–134, 1994.
2. Sergey Avvakumov, Isaac Mabillard, Arkadiy Skopenkov, and Uli Wagner. Eliminating higher-multiplicity intersections, III. Codimension 2. Preprint, arXiv:1511.03501, 2015.
3. Martin Balko, Radoslav Fulek, and Jan Kynčl. Crossing numbers and combinatorial characterization of monotone drawings of \(K_n\). Discrete & Computational Geometry, 53(1):107–143, 2015.
4. Imre Bárány, Senya B Shlosman, and András Szücs. On a topological generalization of a theorem of Tverberg. Journal of the London Mathematical Society, 2(1):158–164, 1981.
5. Imre Bárány and Pablo Soberón. Tverberg’s theorem is 50 years old: A survey. Bulletin of the American Mathematical Society, 55(4):459–492, jun 2018. doi:10.1090/bull/1634.
6. Pavle V. M. Blagojević, Florian Frick, and Günter M. Ziegler. Tverberg plus constraints. Bull. Lond. Math. Soc., 46(5):953–967, 2014.
7. Pavle V. M. Blagojević and Günter M. Ziegler. Beyond the Borsuk–Ulam theorem: The Topological Tverberg story. In: A Journey Through Discrete Mathematics: A Tribute to Jiří Matoušek (M. Loebl, J. Nešetřil, and R. Thomas, eds.), Springer, pp. 273–341, 2017.
8. Jacob Fox and János Pach. Coloring kk-free intersection graphs of geometric objects in the plane. European Journal of Combinatorics, 33(5):853–866, 2012.
9. Florian Frick. Counterexamples to the topological Tverberg conjecture. Oberwolfach Reports, 12(1):318–321, 2015.
10. Mikhail Gromov. Singularities, expanders and topology of maps. Part 2: From combinatorics to topology via algebraic isoperimetry. Geom. Funct. Anal., 20(2):416–526, 2010.
11. Isaac Mabillard and Uli Wagner. Eliminating higher-multiplicity intersections, I. A Whitney trick for Tverberg-type problems. Preprint, arXiv:1508.02349, 2015. An extended abstract appeared (under the title Eliminating Tverberg points, I. An analogue of the Whitney trick) in Proc. 30th Ann. Symp. Comput. Geom., 2014, pp. 171–180.
12. Gary L Miller and Donald R Sheehy. Approximate centerpoints with proofs. Computational Geometry, 43(8):647–654, 2010.
13. Wolfgang Mulzer and Daniel Werner. Approximating tverberg points in linear time for any fixed dimension. Discrete & Computational Geometry, 50(2):520–535, 2013.
Murad Özaydin. Equivariant maps for the symmetric group. Preprint, 1987. Available at http://minds.wisconsin.edu/handle/1793/63829.

János Pach, József Solymosi, and Géza Tóth. Unavoidable configurations in complete topological graphs. *Discrete & Computational Geometry*, 30(2):311–320, 2003.

Christos H Papadimitriou. On the complexity of the parity argument and other inefficient proofs of existence. *Journal of Computer and system Sciences*, 48(3):498–532, 1994.

Johann Radon. Mengen konvexer körper, die einen gemeinsamen punkt enthalten. *Mathematische Annalen*, 83(1-2):113–115, 1921.

José Luis Álvarez Rebollar, Jorge Cravioto Lagos, and Jorge Urrutia. Crossing families and self crossing hamiltonian cycles. *XVI Encuentros de Geometría Computacional*, page 13, 2015.

David Rolnick and Pablo Soberón. Algorithms for Tverberg’s theorem via centerpoint theorems. Preprint, arXiv:1601.03083, 2016.

Arkadiy B. Skopenkov. A user’s guide to the topological Tverberg conjecture. *Russian Mathematical Surveys*, 73(2):323, 2018.

Shang-Hua Teng. *Points, spheres, and separators: a unified geometric approach to graph partitioning*. PhD thesis, Carnegie Mellon University, 1992.

Helge Tverberg. A generalization of Radon’s theorem. *Journal of the London Mathematical Society*, 1(1):123–128, 1966.

Rade T. Živaljević. Topological methods in discrete geometry. In Jacob E. Goodman, Joseph O’Rourke, and Csaba D. Tóth, editors, *Handbook of discrete and computational geometry*, CRC Press Ser. Discrete Math. Appl., chapter 21. CRC, Boca Raton, FL, 2018.