Existence of Small Separators Depends on
Geometry for Geometric Inhomogeneous Random
Graphs
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Abstract

We show that Geometric Inhomogeneous Random Graphs (GIRGs) with power law weights may either have or not have separators of linear size, depending on the underlying geometry. While it was known that for Euclidean geometry it is possible to split the giant component into two linear size components by removing at most $n^{1-\varepsilon}$ edges, we show that this is impossible if the geometry is given by the minimum component distance.

1 Introduction

Geometric Inhomogeneous Random Graphs (GIRGs) were introduced by Bringmann, Keusch and Lengler [9,11] as a model for graphs which are scale-free (i.e. the degree sequence follows a power law), but also have a community structure. In the model, each vertex draws independently a weight and a position in some geometric space, and for each pair of vertices an edge is inserted independently with a probability that depends on the weights and distance of the endpoints. The model was inspired by Chung-Lu random graphs [14], and it contains hyperbolic random graphs (HypRGs) [25] as a special case [9]. It falls into a class of Chung-Lu type random graph models [10], which implies that w.h.p. a random graph from the model has a giant component of linear size, all other components are of polylogarithmic size, the diameter of each component is polylogarithmic (small-world), and the average distance in the giant component is $O(\log \log n)$ (ultra-small world). On the other hand, for the case of euclidean GIRGs (EuGIRGs, the underlying geometric space has euclidean geometry) it was shown [9,11] that w.h.p. the graphs have clustering coefficient $\Omega(1)$, and that they have sublinear separators.\footnote{with high probability, i.e., with probability tending to one as the number of vertices tends to $\infty$.}

Other geometries, in particular the non-metric Minimum Component Distance (MCD, [10, Example 7.2]), have been proposed, but not intensively studied. It is clear from [9–11] that the results on component structure, diameter, clustering coefficient, and the existence of separators carry over to non-Euclidean geometries.\footnote{Essentially, the clustering coefficient is the probability that two random neighbours of a random vertex are adjacent, see Definition A.1 on page 21 for a precise definition.}

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average distance, and clustering coefficient carry over.\textsuperscript{d} However, it was unclear whether small separators exist in these graphs. In this paper we give a negative answer: w.h.p. the giant component cannot be split in two linear size parts by removing a sublinear number of edges. Thus in the GIRG model, it depends on the underlying geometry whether small separators exists.

For Erdős-Rényi random graphs with edge probability \( p = 1 + \Omega(1)/n \), there is a classical technique by Luczak and McDiarmid [27] to prove that the giant component is stable, i.e. that there are no sublinear separators. The main idea is to draw the edges in two batches with probabilities \( p_1 \) and \( p_2 \), respectively, where \( p_2 = \varepsilon \) for some small \( \varepsilon > 0 \), and \( p_1 \approx p - p_2 \). After the first batch there is already a giant component, and Luczak and McDiarmid prove that for any connected graph there are relatively few sparse cuts, i.e., cuts with a small number of cross-edges, see Lemma 4.1 below for the exact statement. Since the second batch is drawn independently of the first one, all these sparse cuts will be destroyed by the second batch. Moreover, since \( p_2 \) is small, the giant component is not changed significantly by the second batch. Bollobás, Janson, and Riordan [7] have shown that the proof can be generalised to a much more general class of random graph models.\textsuperscript{e} A crucial ingredient for this approach is that the edges can be inserted in independent batches. (Or alternatively, graph exposition can be linked to a branching process.) This is not the case for GIRGs, since the edges depend on the position of the vertex, and are thus highly dependent of each other. The main technical contribution of this paper is how these dependencies can be overcome. An essential ingredient is a version of Azuma’s inequality with two-sided error that has recently been developed independently (in slightly different forms) by Bringmann, Lengler, and Keusch [10, Theorem 3.3], and by Combes [15]. We refer the reader to Section 4 for a more detailed overview over the proof and its main obstacles.

Background on the euclidean GIRG model. The EuGIRG model, and in particular the HypRG model\textsuperscript{f}, have been intensively studied in the last years in mathematics [3, 10, 12, 13, 21], physics [4–6, 25], and computer science [1, 2, 8, 11, 19, 20, 22–24, 28, 29]. A closely related model of infinite graphs is scale-free percolation [16, 17], see [30] for a comparison. EuGIRGs have been used as a model for social networks [8] and for the internet graph [6, 28], especially for evaluating routing protocols, both experimentally [2, 4–6, 24, 25, 28, 29] and theoretically [8]. Mathematically, EuGIRGs have been studied with respect to their separators and entropy [11], their spectral properties [21], and under bootstrap percolation [12, 23]. Computationally, sampling algorithms [11, 22] and compression algorithms [11] have been studied.

The analysis of GIRGs is often considerably harder than the analysis of non-geometric random graph models like the Erdős-Rényi model or the Chung-Lu model. Most notably, an important and classic tool for the latter are branching

\textsuperscript{d}For the clustering coefficient this was not explicit in [9, 11]. For convenience, we give a proof in the appendix.

\textsuperscript{e}This includes Chung-Lu random graphs, but not GIRGs; the latter is most evident by observing that GIRGs have different properties than the graphs in [7], for example EuGIRGs have sublinear separators.

\textsuperscript{f}In hyperbolic random graphs the positions are drawn uniformly at random in a hyperbolic disc. This is a special case of one-dimensional GIRGs, where the angular coordinate of the hyperbolic plane corresponds to a one-dimensional geometric space, and the radial coordinate corresponds to the weight, cf. [11, Section 7].
processes. For example, exploring an Erdős-Rényi graph with a breadth-first search is intimately linked to a branching process, since for a long time any new edge will lead to an unexplored vertex w.h.p.. This connection fails for GIRGs because they are strongly clustered. Hence the probability that a new edge leads to an already explored vertex is rather large and, crucially, depends strongly on the positions of the uncovered vertices. This broken connection leads to many graph properties which are not observed in non-geometric graphs. For example, a branching process is very unlikely to reach $\omega(\log n)$ leaves and still die out (except in threshold cases), which is why there are no components in the range $\omega(\log n) \cap o(n)$ in non-threshold Erdős-Rényi graphs or Chung-Lu graphs. On the other hand, GIRGs are easily seen to have components of size $(\log n)^{1+\Theta(1)}$ w.h.p., which shows that they behave very different from branching processes.

On the other hand, GIRGs have still a certain degree of independence among the edges: conditioned on the position of the vertices, the edges are generated by independent coin flips. This still allows to use concentration inequalities in some settings. For example, for any fixed vertex $v$, the degree of $v$ is the sum of $n-1$ independent random variables (and thus, approximately Poisson distributed). However, for a pair of vertices $u, v$ the situation is already much more involved: if $u$ and $v$ are adjacent, then their degrees in the remaining graph are still Poisson distributed (in the limit), but they are not distributed like two independent Poisson distributions.

**Background on the Minimum Component Distance (MCD).** In the GIRG model, the geometric position of a vertex is supposed to represent properties and preferences of that vertex, like place of residence, profession, interests, and hobbies for a social network. It has been argued that the euclidean distance does not well reflect the underlying geometry of social networks, since two vertices are only close in the euclidean geometry if they are similar in all their components, while nodes in social networks should be likely to connect if they agree in some aspects. Thus other distance functions have been proposed, in particular the Minimum Component Distance (MCD [10, Example 7.2]), where the distance between two $d$-dimensional vectors is the minimum of the distances in each components. Note that this distance function is not a metric, which is also considered realistic for social networks. The resulting model is still of Chung-Lu type [10, Theorem 7.3], and the results in [10] still apply (component structure, diameter, average distance). Moreover, it is easy to see that the MCD still has a clustering coefficient of $\Omega(1)$ (see appendix), which is considered important for a model of social and technological networks. Non-geometric random graph models like (sparse) Erdős-Rényi graphs or Chung-Lu graphs tend to have very small clustering coefficients [31].

**Structure of the Paper** In Section 2 we will give a formal definition of the GIRG model. In Section 3 we formally define $(\delta, \eta)$-cuts and gives the formal statement of our main theorem that MCD-GIRGs have no small separators. In Section 4 we give a proof outline, and explain why the classical approach

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Footnotes:

6although recently a related approach has been proposed in [30] for scale-free percolation.

7In fact, quite a lot of them. Every box of volume $\frac{1}{n}(\log n)^{1+\varepsilon}$ has probability $n^{o(1)}$ to contain such a component if $\varepsilon > 0$ is a sufficiently small constant, so the expected number of such components is $n^{1-o(1)}$. 
of Luczak and McDiarmid is not directly applicable. Section 5 collects some tools and describes the uncovering procedure that we use in our proof, while 6 contains the proof of the main theorem. Finally, in the appendix we show that w.h.p. MCD-GIRGs have clustering coefficient $\Omega(1)$ as the MCD satisfies a stochastic triangle inequality.

2 Model

We now come to the definition of the GIRG model. The GIRG model was first introduced in [11], but only for euclidean distances\(^1\), and we refer to this case as euclidean GIRG or EuGIRG. We follow the definition in [10, Section 7], which allows a much wider class of distance functions.

The GIRG model is defined on the vertex set $V = \{1, \ldots, n\}$, and comes with the following set of parameters.

- A geometric ground space, which we assume to be the $d$-dimensional torus $T^d = \mathbb{R}^d/\mathbb{Z}^d$ for some constant $d \geq 1$.

- A (non-necessarily metric) distance function on $T^d$, cf. below.

- For every $n$ a sequence $w = (w_1, \ldots, w_n)$ of $n$ positive weights\(^2\), such that a weak power law condition for some constant power law exponent $\beta \in (2, 3)$ is satisfied, see (PL1) and (PL2) below.

- For each $n$ a function $p = p(x_u, x_v, w_u, w_v) \in [0, 1]$ that assigns to each quadruple of two positions and two weights a probability and that satisfies (EP) below for some constant parameter $\alpha > 1$.

Before we elaborate on the precise requirements of the parameters, let us define the model. A random graph in the GIRG model on the vertex set $V = \{1, \ldots, n\}$ is obtained by the following procedure. Each vertex $v$ is equipped with weight $w_v$, and every vertex draws independently a uniformly randomly position $x_v \in T^d$. Afterwards, for each pair $1 \leq u < v \leq n$ of different vertices we independently flip a coin, and insert the edge $\{u, v\}$ with probability $p_{uv} := p(x_u, x_v, w_u, w_v)$.

2.1 Requirements of the GIRG model.

Distance Function. For simplicity we require the distance function to be translation invariant and symmetric. I.e., we can describe it by a measurable function $\|\cdot\| : T^d \to \mathbb{R}_{\geq 0}$, where $\|x\| = \|y\|$ for all $x \in T^d$, and $\|0\| = 0$. Then we define the distance of $x, y \in T^d$ as $\|x - y\|$, where $x - y \in T^d$ is taken in the additive group $(T^d, +)$. We additionally require that the mapping $V : \mathbb{R}^d_+ \to [0, 1]; V(r) := \text{vol}(\{x \in T^d | \|x\| \leq r\})$ is surjective.

Note crucially that $\|\cdot\|$ does not need to be a norm.

\(^1\)or for any other metric that comes from a norm – they all give equivalent models.

\(^2\)where we are interested in the case $n \to \infty$. All Landau notation $O(\cdot), \Omega(\cdot)$ etc. is w.r.t. this limit, and likewise we mean by “a constant” a quantity that does not depend on $n$. We suppress from the notation that $w$ depends on $n$. 

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Edge probabilities. Let $\alpha > 1$ be a constant. We require that the edge probability function $p(x_u, x_v, w_u, w_v)$ satisfies for all $x_u \neq x_v \in T^d$ and all $w_u, w_v > 0$:

$$p(x_u, x_v, w_u, w_v) = \Theta\left(\min\left\{1, \left(\frac{w_u w_v}{n \cdot V(||x_u - x_v||)}\right)^\alpha\right\}\right), \quad (\text{EP})$$

where as before we denote $V(r) := \text{vol}(\{x \in T^d \mid ||x|| \leq r\})$. Since only $x_u$ and $x_v$ are random, we also write $p_{uv} := p_{uv}(x_u, x_v) := p(x_u, x_v, w_u, w_v)$.

The bounds represented by $\Theta$ are assumed to be global, i.e. there are constants $c_L, c_U > 0$ such that for sufficiently large $n$ the inequalities

$$p_{uv}(x_u, x_v) \geq c_L \min\left\{1, \left(\frac{w_u w_v}{n \cdot V(||x_u - x_v||)}\right)^\alpha\right\} \quad \text{and} \quad (\text{EPL})$$

$$p_{uv}(x_u, x_v) \leq c_U \min\left\{1, \left(\frac{w_u w_v}{n \cdot V(||x_u - x_v||)}\right)^\alpha\right\} \quad \text{and} \quad (\text{EPU})$$

hold for all $x_u \neq x_v \in T^d$ and all $w_u, w_v > 0$.

We remark that (EP) implies that a vertex $v$ with weight $w_v$ has expected degree $\mathbb{E}[\text{deg}(v)] = \Theta(w_v)$, and two vertices $u, v$ with weights $w_u, w_v$, respectively, have probability $\Pr\{u, v \in E\} = \Theta(\min\{1, w_u w_v / n\})$ to be adjacent [10].

Power Law Weights.\(^m\) Let $2 < \beta < 3$ be a constant. We demand that

$$w_{\min} := \min\{w_v \mid v \in V\} = \Omega(1), \quad (\text{PL1})$$

and that there exists $\bar{w} = \bar{w}(n) \geq n^{\omega(1/\log \log n)}$ such that for all constants $\eta > 0$ there are $c_1, c_2 > 0$ such that for all sufficiently large $n$,

$$c_1 \frac{n}{w^{\beta - 1 + \eta}} \leq \#\{v \in V \mid w_v \geq w\} \leq c_2 \frac{n}{w^{\beta - 1 - \eta}}, \quad (\text{PL2})$$

where the second inequality holds for all $w \geq w_{\min}$, and the first one only holds for all $w_{\min} \leq w \leq \bar{w}$. These conditions directly imply that the total weight $W := \sum_{v \in V} w_v$ is in $\Theta(n)$ [10, Lemma 4.2].

2.2 Minimum Component Distance.

Since we are dealing with a torus, we define the absolute difference for $a, b \in [0, 1]$ as

$$|a - b|_T := \min\{|a - b|, 1 - |a - b|\}. \quad (1)$$

For $x, y \in T^d$, the minimum component distance MCD can then be expressed as

$$\|x - y\|_{\min} := \min_{1 \leq i \leq d} |x_i - y_i|_T. \quad (2)$$

\(^1\)This constant essentially enforces convergences. Variations with $\alpha \leq 1$ are possible, but make the requirements on the model more technical, see [10]. Similarly, in [9] also the threshold case $\alpha = \infty$ was considered. We omit both variants for the sake of simplicity.

\(^m\)We remark that all the proofs in the paper go through under much weaker assumptions. More precisely, the power law condition (PL2) could be replaced by the weaker requirement that $\frac{1}{d} \sum w_j = O(1)$ and $\frac{1}{d} \sum w_j^2 \rightarrow \infty$. However, since we want to cite results from [10] (in particular, the existence of a giant component), we stick with the conditions from [10]. Note that some properties that make the model popular would not hold without power law weights, for example the graphs would not be (ultra-)small worlds in general.
Note that the MCD coincides with the euclidean distance on \( T^1 \) and that MCD is not a norm for \( d > 1 \), since the triangle inequality does not hold. While the volume function \( V(r) \) for the euclidean norm grows as \( V_{eu}(r) = \Theta(r^d) \) for \( 0 \leq r \leq 1/2 \), for MCD it is given by \( 1 - (1 - 2r)^d \). As this is somewhat unwieldy, we observe that \( 2r \leq 1 - (1 - 2r)^d \leq 2dr \). Therefore, if we somewhat abusively redefine \( V_{min}(r) := r \), we are off by at most a constant factor; this does not change the model, because such constant factors are swallowed for by the \( \Theta \)-notation in EP.

2.3 Notation

We denote the \( i \)-th component of a vector \( e \) by \( e_i \). In particular, for the position \( x_v \in T^d \) of vertex \( v \) we denote by \( x_{vi} \) the \( i \)-th component, for \( 1 \leq i \leq d \). For \( r \geq 0 \) and \( x \in T^d \), let \( B_r(x) := \{ y \in T^d \mid \|x - y\| \leq r \} \) denote the \( r \)-ball around \( x \) with respect to \( \|\| \| \). and we let \( V(r) := r = \Theta(\text{vol}(B_r(x))) \) whenever we are concerned with MCD. We denote by \( w_{\text{min}} \) the minimum weight, and by \( W := \sum_{v \in V} w_v \) the total weight. For convenience, we remind the reader that \( \alpha > 1 \) is the convergence-enforcing constant of the GIRG model, and that \( \beta \in (2, 3) \) is the power law exponent. The position and weight of a vertex \( v \) are denoted by \( x_v \) and \( w_v \), respectively. The connection probability of two vertices \( u \) and \( v \) is \( p_{uv} := p(x_u, x_v, w_u, w_v) \).

We say that a family of events \( E_n \) holds with high probability (w.h.p.) if \( \Pr[E_n] \to 1 \) as \( n \to \infty \). If \( V_1, V_2 \) are two sets of vertices in a graph \( G \), then we denote by \( E(V_1, V_2) \) the set of edges from \( V_1 \) to \( V_2 \).

By default, in the proofs we use \( n \) for the number of vertices of the GIRG that we study.

3 Our Result

The main result of this paper is that the giant component in the MCD model is stable under the removal of a sublinear number of edges. It is known that w.h.p. euclidean GIRGs have small (edge) separators of size \( n^{1-\epsilon} \) [9]. In contrast, we will prove that for MCD-GIRGs, w.h.p. the removal of any sublinear number of edges only decreases the size of the largest component by a sublinear number of vertices. In particular, this shows that the existence of small separators in the GIRG model depends on the properties of the underlying geometry.

In order to formulate the precise statement, we introduce some terminology.

**Definition 3.1.** For a graph \( G = (V, E) \) and \( \delta, \eta > 0 \), a \((\delta, \eta)\)-cut is a partition of \( V \) into two sets of size at least \( \delta|V| \) such that there are at most \( \eta|V| \) cross-edges (i.e. edges that have one endpoint in each of the sets).

Somewhat loosely speaking, we say that a family of graphs has small separators if for a graph with \( n \) vertices it is possible to split its giant component into parts of linear size by removing only \( o(n) \) edges. Note that we always refer to splitting the giant component and not the whole graph; this is necessary because a GIRG of size \( n \) has \( \Omega(n) \) isolated vertices w.h.p., yielding a trivial cut between isolated and non-isolated vertices. According to Theorem 5.1 below, w.h.p. there is only one component of size \( \Omega(n) \), which justifies the term “giant component”.

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More formally, we will consider \((\delta, \eta)-\)cuts of the largest component of \(G = (V, E)\). We will use the convention that the two sets into which that component is partitioned still need to have size at least \(\delta |V|\), and the cut must have at most \(\eta |V|\) cross-edges (i.e. we do not use the size of the component instead of \(|V|\)).

Our main theorem states that w.h.p. MCD-GIRGs for dimension \(d \geq 2\) have no small separators.

**Theorem 3.2 (Main Theorem).** Let \(G\) be a graph on \(n\) vertices drawn according to the GIRG model with minimum component distance and with dimension \(d \geq 2\). Then, for every \(\delta > 0\) there is an \(\eta > 0\) such that w.h.p. the largest component of \(G\) has no \((\delta, \eta)\)-cut.

Note that for \(d = 1\), the minimum component distance agrees with the euclidean distance. Hence, for \(d = 1\) there exist sublinear separators.

### 4 Proof Outline

In this section we will review the classic approach by Łuczak and McDiarmid [27] for Erdős-Rényi random graphs \(G(n, p)\) with edge probability \(p = c/n\) where \(c > 1\) is a constant (for \(c < 1\) there is no giant component). Then we will explain why the proof cannot be straightforwardly transferred to our setting, and how we can overcome the obstacles. A key ingredient for both proofs is the following lemma, which states that in every connected graph there are only relatively few sparse cuts.

**Lemma 4.1 (Cut bound [27, Lemma 7]).** For every \(\varepsilon > 0\) there exist \(\eta > 0\) and \(n_0\) such that the following holds. For all \(n \geq n_0\), and for all connected graphs \(G\) with \(n\) vertices, there are at most \((1 + \varepsilon)^n\) bipartitions of \(G\) with at most \(\eta n\) cross-edges.

**Erdős-Rényi Graphs.** With Lemma 4.1, the proof for the Erdős-Rényi case becomes rather straightforward. Our goal is to show that for every \(\delta > 0\), there is \(\eta > 0\) such that w.h.p. the giant of \(G\) has no \((2\delta, \eta)\)-cut (the factor 2 is chosen for convenience of exposition). We fix \(\gamma = \gamma(\eta, c)\) sufficiently small, and uncover the edges in two batches, where in the first batch we insert each edge with probability \(p_1 := (c - \gamma)/n\), thus obtained \(G'\). Afterwards, in the second batch we insert each edge with probability \(p_2 \approx \gamma/n\), where \((1 - p_1)(1 - p_2) = (1 - p)\). Note that the resulting graph is distributed as \(G(n, p)\).

By Lemma 4.1, the number of \((\delta, \eta)\)-cuts of the giant component of \(G'\) is at most \((1 + \varepsilon)^n\). By a standard Chernoff bound, each such cut will receive at least \(\eta n\) cross-edges from the second batch with probability \(1 - e^{-\Omega(n)}\), if \(\eta > 0\) is sufficiently small. Thus we can afford a union bound over all \((1 + \varepsilon)^n\) such cuts in \(G'\) if \(\varepsilon > 0\) is sufficiently small, and w.h.p. all of them will be destroyed in \(G\) (i.e., all of them will have at least \(\eta n\) cross-edges in \(G\)). Finally, let \(K_{\text{max}}\) and \(K'_{\text{max}}\) denote the vertex set of the giant component of \(G\) and \(G'\), respectively. If \(\gamma\) is small enough then w.h.p. \(|K_{\text{max}} \setminus K'_{\text{max}}| < \delta n\). In this case every \((2\delta, \eta)\)-cut

\(^{n}For euclidean GIRGs, the separator is constructed by taking two parallel, axis-parallel half-planes, and removing all edges across these two half-planes. By [9, Theorem 2.5], w.h.p. the number of edges across each axis-parallel half-plane is at most \(n^{1-\varepsilon+o(1)}\) for \(\varepsilon := \min\{\alpha - 1, \beta - 2, 1/d\}\).
of $K_{\text{max}}$ in $G$ would give rise to a $(\delta, \eta)$-cut of $K_{\text{max}}'$ in $G$ by deleting the vertices in $K_{\text{max}} \setminus K_{\text{max}}'$. Since we have shown that w.h.p. the latter does not exist, there is no $(2\delta, \eta)$-cut of $K_{\text{max}}$ in $G$.

**MCD-GIRG graphs** While we will still use Lemma 4.1 for the MCD-GIRG case, the remainder of the proof does not carry over. Firstly, it is not possible to generate the edge set by two independent batches, a large one and a small one. In fact, it is possible to generate the graph in two “almost” independent batches, by first uncovering all edges that “come from” any of the first $d-1$ components of the positions, giving a graph $G_1$, and afterwards uncovering all edges that “come from” the last component of the position. We will make use of this two-stage process in the proof, see (LB1) and (LB2) in Section 5.2. However, unlike in the Erdős-Rényi case, the second batch of edges is not small: it comprises for roughly a $1/d$ fraction of all edges, and the giant component will grow substantially in this last step. In particular, we cannot avoid that many new bipartitions of the giant component are created by the second batch.

The basic idea is, instead of uncovering the edges in batches, we rather want to uncover the vertices in batches. However, we only want to use vertices for the second batch that are contained in the giant component, as otherwise the incident edges would not count towards the cut across the giant component. These objectives seem contradictory, but fortunately it is possible to reconcile them by first uncovering partial geometric information. More precisely, as indicated above we first uncover the first $d-1$ components of each position, and draw edges conservatively according to these coordinates, i.e., we never overestimate the probability of an edge to be in $G$. This gives us a graph $G_1$ which has a giant component w.h.p.. Within the vertices of bounded weight in this giant component we draw randomly a small (but linear size) subset $F$. The yet uncovered incident edges to $F$ will serve as the second batch.

Next we uncover the full geometric information of all vertices in $V \setminus F$, and insert all edges between these vertices. In the proof, this intermediate graph is denoted by $G_3$ (as $G_2$ is used for an earlier phase which we skip over in this outline). Finally, we draw the $d$-th coordinate for vertices in $F$, which allows us to add all edges incident to $F$, completing the graph. We show that these additional edges incident to $F$, which result from vertices being close along the $d$-th coordinate, destroy all $(\delta, \eta)$-cuts in the giant of $G_3$ (Lemma 6.1). This is less trivial than in the Erdős-Rényi case, since the edge probabilities depend on the (already uncovered) $d$-th coordinate of the vertices in $V \setminus F$. Thus the edge probability are strongly correlated. However, in Lemma 6.2 we show that the vertices in $V \setminus F$ are spread uniformly enough over the $d$-th dimension to alleviate this issue.

Finally, we need to show that the edges incident to $F$ added in the last step do not increase the size of the giant by too much (Lemma 6.4). Again, this part is significantly more involved than in the Erdős-Rényi case, since not even the exact size of the giant component of a GIRG is known. We proceed in two steps. Firstly we show that the number of edges going of from $F$ is concentrated (Lemma 6.7); this already requires an Azuma-Hoeffding type bound with two-sided error that has only recently been developed (Theorem 5.5, originally from [10]). Secondly, we show that the number of vertices in large non-giant components in $G_3$ (Lemma 6.5) can be coupled to the result of a random walk.

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if in each step the \(d\)-th coordinates of the uncovered vertices are sufficiently uniformly spread. In this way we derive an upper bound on the number of vertices in such components. Together, this implies that on average each edge incident to \(F\) adds only a constant number of vertices to the giant. This allows us to show that w.h.p. the edges uncovered in the last step increase the giant component only by \(\delta n\) vertices. The theorem then follows analogously to the Erdős-Rényi case.

5 Preliminaries and Tools

5.1 Tools

We start with some basic tools. Concerning GIRGs, we mainly need the existence of a giant component:

**Theorem 5.1** (Component Structure of GIRGs, [10, Theorems 5.9 + 7.3]).

(i) There is a constant \(s_{\text{max}}\) such that with high probability, there is a connected component containing at least \(s_{\text{max}}n = \Omega(n)\) vertices.

(ii) With high probability, all other components have at most \(\log O(1) n\) vertices (polylogarithmic size).

Note that \(s_{\text{max}}\) depends on the parameters of the model, including the factors hidden by \(\Theta\) in the edge probability equation (EP).

Moreover we need that the degree distribution follows a power law. (This is not surprising but recall that the definition only requires the weights to follow a power law.)

**Lemma 5.2** ([10, Theorem 6.3 and Lemma 4.5]). For all \(\eta > 0\) there exist constants \(c_3, c_4 > 0\) such that w.h.p.

\[
c_3 \frac{n}{e^{s-1+\eta}} \leq \# \{v \in V \mid \deg(v) \geq c\} \leq c_4 \frac{n}{e^{s-1-\eta}}.
\] (3)

where the second inequality holds for all \(c \geq 1\) and the first one hold for all \(1 \leq c \leq \bar{w}\), where \(\bar{w}\) is the same as in (PL2). Moreover, there is a constant \(C > 0\) such that \(\mathbb{E} [\deg(v)] \leq C w_v\), and such that with probability \(1 - n^{-\omega(1)}\)

\[
\deg(v) \leq C \cdot (w_v + \log^2 n)
\] (4)

holds for all \(v \in V\).

Note that the exponents in (3) are the same as for the weights (see (PL2)).

Next we give some classical concentration inequality. In the proof we will use the following Chernoff bounds:

**Lemma 5.3** (”Weak” Chernoff bounds [18, Theorem 1.1]). Let \(X := \sum_{i=1}^{n} X_i\) where for all \(1 \leq i \leq n\), the random variables \(X_i\) are independently distributed in \([0, 1]\). Then

(i) \(P[X > (1 + \varepsilon)\mathbb{E}[X]] \leq \exp\left(\frac{-\varepsilon^2}{3}\mathbb{E}[X]\right), \text{ for all } 0 < \varepsilon < 1,\)

(ii) \(P[X < (1 - \varepsilon)\mathbb{E}[X]] \leq \exp\left(\frac{-\varepsilon^2}{3}\mathbb{E}[X]\right), \text{ for all } 0 < \varepsilon < 1,\)
(iii) $P[X > t] \leq 2^{-t}$ for all $t > 2eE[X]$.

In particular, this holds when the $X_i$ are i.i.d. indicator random variables.

In one case, this simple version will not do, and we need to use the stronger form:

**Lemma 5.4** ("Strong" Chernoff bound [14, Theorem 2.15]). Let $X := \sum_{i=1}^{n} X_i$ where for all $1 \leq i \leq n$, the random variables $X_i$ are independently distributed in $\{0, 1\}$ (i.e. they are indicator variables). Then, for any $\varepsilon > 0$ we have

$$P[X \geq (1 + \varepsilon)E[X]] \leq \left(\frac{e^\varepsilon}{(1 + \varepsilon)^{1+\varepsilon}}\right)^{E[X]} \leq \left(\frac{e}{1 + \varepsilon}\right)^{(1+\varepsilon)E[X]}.$$ (5)

The second inequality is a trivial simplification that will come in handy when we apply this lemma.

Finally, we will need a variant of the Azuma-Hoeffding bound with two-sided error event. The crucial difference to other versions of the Azuma-Hoeffding bound is that for both the event $\omega$ and $\omega'$ we only need to consider "good" events.

**Theorem 5.5** (Azuma-Hoeffding Bound with Two-Sided Error Events, [10, Theorem 3.3]). Let $Z_1, \ldots, Z_m$ be independent random variables over $\Omega_1, \ldots, \Omega_m$. Let $Z = (Z_1, \ldots, Z_m)$, $\Omega = \prod_{k=1}^{m} \Omega_k$ and let $g : \Omega \to \mathbb{R}$ be measurable with $0 \leq g(\omega) \leq M$ for all $\omega \in \Omega$. Let $B \subseteq \Omega$ (the subset of "bad" events) such that for some $c > 0$ and for all $\omega, \omega' \in \Omega \setminus B$ that differ in at most two components we have

$$|g(\omega) - g(\omega')| \leq c.$$ (6)

Then for all $t > 0$

$$P[|g(Z) - \mathbb{E}[g(Z)]| \geq t] \leq 2e^{-\frac{t^2}{2cm^2}} + (2 \frac{mM}{2} + 1) P[B].$$ (7)

Finally, LeCam’s inequality will imply that the degree distribution is approximately Poisson distributed.

**Theorem 5.6** (LeCam [26]). Suppose $X_1, \ldots, X_n$ are independent Bernoulli random variables s.t. $\Pr[X_i = 1] = p_i$ for $i \in [n]$, $\lambda_n = \sum_{i \in [n]} p_i$ and $S_n = \sum_{i \in [n]} X_i$. Then

$$\sum_{k=0}^{\infty} \Pr[S_n = k] = \frac{\lambda^k e^{-\lambda}}{k!} < 2 \sum_{i=1}^{n} p^i.$$ 

In particular, if $\lambda_n = \Theta(1)$ and $\max_{i \in [n]} p_i = o(1)$, then $\Pr[S_n = k] = \Theta(1)$ for $k = O(1)$.

For convenience, we also restate here Lemma 4.1, which bounds the number of small cuts in connected graphs.

**Lemma 4.1** (Cut bound [27, Lemma 7]). For any $\varepsilon > 0$ there exist $\eta_0(\varepsilon) > 0$ and $n_0$ such that the following holds. For all $n \geq n_0$, and for all connected graphs $G$ with $n$ vertices, there are at most $(1 + \varepsilon)^n$ bipartitions of $G$ with at most $\eta_0 n$ cross-edges.
5.2 Component-Wise Sampling

Naively, we would generate a graph from the GIRG model by drawing all positions (which determines all \( p_{uv} \) for \( u \neq v \in V \)), and then inserting the edge \( \{u, v\} \) with probability \( p_{uv} \). The latter can be done by drawing independent random variables \( Y_{uv} \) uniformly at random from \([0, 1]\), and inserting the edge \( \{u, v\} \) if and only if \( Y_{uv} < p_{uv} \). Since the random \( Y_{uv}, x_u, x_v \) are independent, we may draw them in any order. In particular, we may draw \( Y_{uv} \) before drawing \( x_u \) and \( x_v \). This point of view will be helpful in several places. In particular, it will allow us to insert some edges even if only the first \( d-1 \) component of \( x_u \) and \( x_v \) have been drawn. More precisely, we proceed as follows.

Inequality (8) is a lower bound for \( p_{uv} \); this means that \( Y_{uv} < \frac{M_{\min \{1, (w_u w_v)^{1/\alpha} \}}}{n} \cdot \|x_u - x_v\|_{\min} \). Therefore, it is a sufficient condition for edge insertion, i.e. the edge between \( u \) and \( v \) is inserted with probability one if it is fulfilled. We are particularly interested in applying this to the MCD model, where we have defined \( V_{\min \{r\}} = r \). For the MCD model, the sufficient condition is thus given as

\[
Y_{uv} < c L \min \left\{ 1, \left( \frac{w_u w_v}{n \cdot V_{\|x_u - x_v\|}} \right)^{\alpha} \right\} =: p_{uv}^L(\|x_u - x_v\|_{\min}).
\]  

(9)

Note that if \( \|x_u - x_v\|_{\min} \leq \frac{w_u w_v}{2^\alpha} \), the \( \min\{\ldots\} \) term becomes 1. In that case, having \( Y_{uv} < c L \) guarantees that the edge between \( u \) and \( u \) is inserted, i.e.

\[
P \left[ u \sim v \mid \|x_u - x_v\|_{\min} \leq \frac{w_u w_v}{n} \right] \geq \min\{c L, 1\}.
\]

(10)

Consider the lower bound (9). The condition is met if the following inequality is true:

\[
\min_{1 \leq i \leq d} |x_{ui} - x_{vi}|_T = \|x_u - x_v\|_{\min} < \frac{1^{1/\alpha}}{(Y_{uv})^{1/\alpha}} \cdot \left( \frac{w_u w_v}{n} \right).
\]

(12)

By the definition of MCD, it is satisfied if and only if for at least one coordinate \( 1 \leq i \leq d \), the absolute difference \( |x_{ui} - x_{vi}|_T \) is smaller than the right hand side. The coordinate values are i.i.d. from \([0, 1]\); conditioned on \( Y_{uv} \), we can consider drawing the positions as \( d \) independent “attempts” to fulfill (12). Under that view, a \( d \)-dimensional MCD graph is lower-bounded by a union of \( d \) one-dimensional EuGIRG instances which all use the same \( Y_{uv} \) and are thus not independent. However, we can make them independent by the “splitting” each
Y_{uv} into two independent random variables in the following way. Let Y_{1uv} and Y_{2uv} be i.i.d. from [0, 1], with the following CDF for c ∈ [0, 1]:
\[
P[Y_{1uv} < c] = P[Y_{2uv} < c] = 1 - \sqrt{1 - c}.
\] (13)
A quick calculation shows that the following two statements hold for each c ∈ [0, 1]:
\[
c/2 \leq P[Y_{1uv} < c] = P[Y_{2uv} < c] \leq c \quad \text{and} \quad (14)
\]
\[
P[\min\{Y_{1uv}, Y_{2uv}\} < c] = P[Y_{uv} < c] = c.
\] (15)
Because of the latter, instead of checking Y_{uv} < p_{uv}(x_u, x_v) for including edges, we can equivalently use
\[
\min\{Y_{1uv}, Y_{2uv}\} < p_{uv}(x_u, x_v)
\]
as Edge Insertion Criterion. The lower bound (9) becomes
\[
\min\{Y_{1uv}, Y_{2uv}\} < p_L^L(\|x_u - x_v\|_{\min}).
\] (16)
Since we have
\[
\|x_u - x_v\|_{\min} = \min\{\min_{1 \leq i < d} \{|x_{ui} - x_{vi}|_T\}, |x_{ud} - x_{vd}|_T\},
\]
Inequality (16) is satisfied if
\[
Y_{1uv} < p_L^L(\min_{1 \leq i < d} \{|x_{ui} - x_{vi}|_T\}) \quad \text{or} \quad (LB1)
\]
\[
Y_{2uv} < p_L^L(|x_{ud} - x_{vd}|_T).
\] (LB2)
Note that (LB1) and (LB2) are independent of each other; furthermore, each of the two implies (16) and thus EIC, giving a lower bound. Using (LB1) as the edge insertion criterion satisfies (EP) on \(T^{d-1}\), this enables us to apply the theorems from the previous section on the resulting graphs. For fixed x_u and x_v, the probability of satisfying Y_{uv} < p_{uv}^L is bounded by
\[
p_{uv}^L/2 \leq P[Y_{uv} < p_{uv}^L] \leq p_{uv}^L
\]
according to (14). Thus, given that p_{uv}^L satisfies (EP), so does (LB1).

Note that for d = 1, Inequality (LB1) is not defined. This is what sets the case d = 1 (which does have small separators) apart from d > 1 (which does not). It does not seem to be possible to lower-bound a one-dimensional MCD graph with two completely independent random graphs that fulfil (EP), which is crucial for the proof.

### 5.3 Uncovering Procedure

Let us describe the procedure we use for uncovering the edges of an MCD-GIRG, as outlined in Section 4. A condensed version can be found in Algorithm 1.

**Phase 1** We start by drawing Y_{1uv} and x_{ui} for all u, v ∈ V and all 1 ≤ i ≤ d-1, i.e. leaving out all Y_{2uv} and x_{ud}. This is already sufficient to determine the graph induced by (LB1), which we call G_1. By Theorem 5.1, there is a constant s_{max} such that w.h.p. the largest component (the giant) of G_1 has at least s_{max}n vertices; we will always assume that this holds. We denote the giant of G_1 by K_{max}.
**Phase 2** According to (PL2), there is a constant $B'$ such that w.h.p. at least $s_{\max} n/2$ vertices of $K^1_{\max}$ have weight less than $B'$. This can be shown using the power-law requirements: In (PL2), set $\eta = 1$ and $B' > (2c_2/s_{\max})^{1/(\beta - 2)}$.

Then, at most $s_{\max} n/2$ vertices have weight at least $B'$. Even if all of them are in $K^1_{\max}$, since we assume $|K^1_{\max}| \geq s_{\max} n$, at least $s_{\max} n/2$ vertices of $K^1_{\max}$ must have weight smaller than $B'$. (As mentioned in the previous phase, that assumption holds with high probability.) We now draw a set of vertices $F'$ by including each vertex (not just those in the giant) with weight less than $B'$ independently with probability $4f/s_{\max}$, where $0 < f < \min\{\delta/12, s_{\max}/12\}$ is a constant to be determined later.

**Phase 3** We set $F := F' \cap K^1_{\max}$. The expected size of $F$ is at least $4f/s_{\max} \cdot s_{\max} n/2 = 2fn$, and at most $4f/s_{\max} \cdot s_{\max} n = 4fn$. Since vertices of $K^1_{\max}$ with weight less than $B'$ are included independently in $F'$ (and thus $F$), we can use the Chernoff bounds (Theorem 5.3 with $\varepsilon = 1/2$) to show that w.h.p. we have $6fn \geq |F| \geq fn$; in our further considerations, we will assume this to be true. Note that the order of phases 1 and 2 is interchangeable, as they are independent (the choice of $B'$ depends only on $s_{\max}$ and not on the outcome of $G_1$).

The final three phases consist of drawing $x_{ud}$ and $Y^2_{uv}$ for all vertices in a particular order that depends on the outcome of phases 1–3. We enumerate the vertices from 1 to $n$; the $k$-th vertex is denoted by $u_k$. The enumeration has to be such that the vertices of $V \setminus K^1_{\max}$ come first, then the vertices of $K^1_{\max} \setminus F$, and finally the vertices of $F$. More precisely, we have $u_i \in V \setminus K^1_{\max}$ for all $1 \leq i \leq |V \setminus K^1_{\max}|$; for all $|V \setminus K^1_{\max}| < i \leq |V \setminus F|$ we have $u_i \in K^1_{\max} \setminus F$, and finally, for all $|V \setminus F| < i \leq n$ we have $u_i \in F$. We draw the $d$-th coordinates step-by-step in the order given by the enumeration. The $d$-th coordinate of $u_i$ is written as $x_{kd}$, since the more consistent $x_{ud}$ might be somewhat hard to read. Together with $x_{bd}$, we draw $Y^2_{kj}$ for all $1 \leq j < k$ (where $Y^2_{kj}$ is short for $Y^2_{u_k u_j}$). Then, for all vertex pairs $(u_i, u_j)$ with $i, j \leq k$, we have complete information; we add all edges fulfilling (EIC) between such pairs. Therefore, the $k$-th step can be described as

- Draw $x_{kd}$ independently and uniformly from $X_d := [0, 1]$.
- For all $1 \leq j < k$, draw $Y^2_{kj}$ independently according to (13).
- For all $1 \leq j < k$, add the edge between $u_k$ and $u_j$ if (EIC) is satisfied.

The definition of the final three phases reflects the way we chose to enumerate the vertices:

**Phase 4** This phase consists of steps 1 to $|V \setminus K^1_{\max}|$, i.e., it encompasses the following actions: Draw $x_{ud}$ and $Y^2_{uv}$ for all $u \neq v \in V \setminus K^1_{\max}$. Then, for all pairs $(u, v) \in (V \setminus K^1_{\max} \times V \setminus K^1_{\max})$, add the edge between $u$ and $v$ if (EIC) is satisfied. This results in a graph we call $G_2$.

**Phase 5** This phase consists of steps $|V \setminus K^1_{\max}| + 1$ to $|V \setminus F|$, i.e., it encompasses the following actions: Draw $x_{ud}$ and $Y^2_{uv}$ for all $u \neq v \in K^1_{\max} \setminus F$. Then, for all pairs $(u, v) \in (V \setminus F \times V \setminus F)$, add the edge between $u$ and $v$ if (EIC) is satisfied. This results in a graph we call $G_3$. 

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Phase 6   This phase consists of steps $|V\setminus F|+1$ to $n$, i.e. it encompasses the following actions: Draw $x_{ud}$ and $Y^2_{uv}$ for all $u \neq v \in K_{n}^{\max}\setminus F$. Then, for all pairs $(u, v) \in (V \times V)$, add the edge between $u$ and $v$ if (EIC) is satisfied. This results in a graph we call $G_{4}$, i.e. the “actual” random graph from the GIRG model.

Our goal is to show that the sixth and last phase destroys all “small” cuts in the giant of $G_3$ (Lemma 6.1) while adding few vertices to it (Lemma 6.4). For $1 \leq i \leq 4$, we let $K_{i}^{\max}$ denote the connected component that contains $K_{i}^{\max}$.

For proving that “small” cuts are destroyed, we will lower-bound the probability that an edge is present using
\[ Y^2_{uv} < p^l_{uv}(|x_{ud} - x_{vd}|T). \] (LB2)

6 Proof of Theorem 3.2

With all these tools laid out, we are ready to tackle the “core” of the proof. We start by stating the following crucial lemma.

Lemma 6.1. There is a constant $\eta > 0$ such that with high probability, $K_{n}^{\max}$ has no $(\delta, \eta)$-cut in $G_{4}$. (More precisely, the subgraph of $G_{4}$ induced by $K_{n}^{\max}$ has no $(\delta, \eta)$-cut.)

In order to prove this, we need the following auxiliary Lemma 6.2.

Lemma 6.2. Consider the set of $d$-th vertex coordinates $\{x_{ud} \mid u \in V\}$. It is a set of $n$ random variables drawn uniformly and independently from $X_{d} = [0, 1]$. We divide $X_{d}$ into $M := \lfloor n/l \rfloor$ subintervals of equal length, where $0 < l < 1$ is a constant. For all $0 \leq j < M$, we define the $j$-th subinterval as $I_{j} := [j/M, (j+1)/M]$. Then, for every $0 < \delta < 1$, and every $l$, there is a constant $r(\delta, l) > 0$ having the following property: With probability $1 - e^{-\Theta(n)}$, there is no set $S$ of $rn$ subintervals such that there are at least $\delta n/2$ vertices $u$ with $x_{ud} \in S$.

Proof. Fix a set $S$ of $rn$ subintervals. The sum of the lengths of all subintervals in $S$ is $rn \cdot \frac{1}{M} = rn/\lfloor n/l \rfloor$. Therefore, for every vertex $u$, the probability of $u$ getting placed in $S$ (i.e. having $x_{ud} \in S$) is at least $rl/2$, at most $2rl$ (by virtue of $rl/2 \leq rn/\lfloor n/l \rfloor \leq 2rl$), and for $\mu$, the expected number of vertices placed in $S$, we have $2rlen \geq \mu \geq rln/2$.

Since the $x_{ud}$ are drawn independently, we can use Chernoff’s inequality to bound the probability that at least $\delta n/2$ vertices are placed in $S$. More precisely, we use the “strong” Chernoff bound (Theorem 5.4). Note that in our case $\delta/2 \cdot n \geq \delta/2 \cdot \frac{4erl}{2\pi^2} = \frac{4erl}{2\pi^2} \mu$, so we set $(1 + \varepsilon) = \frac{4erl}{2\pi^2}$ in Theorem 5.4. We obtain
\[
P\left[ \# \{u \mid x_{ud} \in S\} \geq \frac{\delta}{2}n \right] \leq \left( \frac{4erl}{\delta} \right)^{\frac{4erl}{2\pi^2} \mu / 2} = \left( \frac{4erl}{\delta} \right)^{16/16^{2n/\mu}} \tag{19} \]

If we choose $r$ small enough such that the term in square brackets is smaller than $1/2$, this upper bound is at most $e^{-\log 2} n/\mu$.

As a final step, we take the union bound over all possible choices of $S$. There are at most $2^{M} \leq 2^{2n/\mu}$ ways to choose $S$, so the probability that for
Algorithm 1 Sample a MCD model graph

\textbf{procedure} SAMPLEMCD\( (d, \alpha, \beta, n, w, \delta) \)

\textbackslash/ Phase 1
Draw \( x_{ud}, Y_{uv}^1 \) for all \( u, v \in V \).
Add edges between pairs satisfying \( (LB1) \rightarrow G_1 \)
\( K_{nax}^1 \leftarrow \text{giant of } G_1 \) \(|K_{max}^1| \geq s_{max}n \text{ w.h.p} \)

\textbackslash/ Phase 2
Choose \( f < \min\{\delta/12, s_{max}/12\} \)
\( F' \leftarrow 0 \)
for all \( u \in V \) with \( w_u < B' \) do
   Add \( u \) to \( F' \) with probability \( 4f/s_{max} \)
end for

\textbackslash/ Phase 3
\( F \leftarrow F' \cap K_{max}^1 \)
Enumerate vertices from 1 to \( n \) such that
\( \{u_1, \ldots, u_{|V\setminus K_{max}^1|}\} = V\setminus K_{max}^1, \)
\( \{u_{|V\setminus K_{max}^1|+1}, \ldots, u_{|V\setminus F|}\} = K_{max}^1 \setminus F, \)
\( \{u_{|V\setminus F|+1}, \ldots, u_n\} = F \)
(Shorthand \( x_{kd} \) and \( Y_{kj}^2 \) for \( x_{u_kd} \) and \( Y_{u_kj}^2 \), respectively)

\textbackslash/ Phase 4 (results in \( G_2 \))
for all \( 1 \leq k \leq |V\setminus K_{max}^1| \) do
   Step\( (k) \)
end for

\textbackslash/ Phase 5 (results in \( G_3 \))
for all \( |V\setminus K_{max}^1| < k \leq |V\setminus F| \) do
   Step\( (k) \)
end for

\textbackslash/ Phase 6 (results in \( G_4 \))
for all \( |V\setminus F| < k \leq n \) do
   Step\( (k) \)
end for
\textbf{end procedure}

\textbf{procedure} Step\( (k) \)
Draw \( x_{kd} \) uniformly from \([0,1]\).
for all \( 1 \leq j < k \) do
   Draw \( Y_{kj}^2 \) independently according to the CDF given in \((13)\).
   Add edge between \( u_k \) and \( u_j \) if \( (EIC) \) is satisfied.
end for
\textbf{end procedure}
at least one such $S$ there are at least $\delta n/2$ vertices in $S$ is upper-bounded by $e^{(1-\log 2)2n/t} = e^{-\Theta(n)}$. \hfill \square

Note that restricting the set of vertices considered for Lemma 6.2 just makes the statement weaker; the lemma implies that for any $V' \subseteq V$, w.h.p. there is no set $S$ of $rn$ subintervals such that there are at least $\delta n/2$ vertices $u \in V'$ with $x_{uv} \in S$. We will make use of this for the following corollary that will provide a lower bound for the edge probabilities in the proofs of Lemmas 6.1 and 6.4.

Corollary 6.3. There is a constant $P > 0$ such that with high probability, the following holds for each step $k$ of Algorithm 1, where $\delta n/2 < k \leq n$:

Let $V_k := \{v_i \in V \mid 1 \leq i < k\}$. For each subset $A \subseteq V_k$ of size at least $\delta n/2$, the probability that step $k$ produces an edge from $u_k$ to $A$ due to (LB2) is at least $P$, i.e.

$$P[\exists v \in A \text{ s.t. } u_k \sim v] \geq P. \quad (20)$$

Proof. Choose $l := w_{\min}^2$. With that choice of $l$, any two vertices $u \neq v$ in the same subinterval have distance at most $|x_{uv} - x_{cd}|_T \leq l/n$, so in particular $p^L_{uv}(|x_u - x_v|_{\min}) \geq c_L$. By (14), this means that

$$P \left[ Y^2_{uv} < p^L_{uv} \mid u, v \text{ in the same subinterval} \right] \geq c_L/2 \quad (21)$$

as a probability over $Y^2_{uv}$.

Fix a $\delta n/2 < k \leq n$. Apply Lemma 6.2 to $V_k$; this yields that with probability $1 - e^{\Theta(n)}$, there is no set $S$ of $rn$ subintervals such that there are at least $\delta n/2$ vertices $v \in V_k$ with $x_{uv} \in S$. In particular, this implies that for every subset $A \subseteq V_k$ of size at least $\delta n/2$, there are more than $rn$ subintervals $I_j$ that contain at least one vertex of $A$ (i.e. there is at least one $v \in A$ such that $x_{uv} \in I_j$). Thus, with probability at least $rn/M = rn/[n/l]$, the position $x_{uv}$ is in a subinterval that contains at least one vertex of $A$. If $x_{uv}$ is in the same subinterval as $x_{cd}$ for some $v \in A$, we have $P[u \sim v] \geq c_L/2$ (see (21)), and thus

$$P[\exists v \in A \text{ s.t. } u_k \sim v] \geq rn/[n/l] \cdot c_L/2 \geq P, \quad (22)$$

for some constant $P > 0$. Note that this statement holds independent of the $Y^2_{uv}$ and $x_{uv}$ for $0 \leq i < d$.

The probability that Lemma 6.2 fails for a particular $k$ is at most $e^{-\Theta(n)}$.

The claim thus follows by a union bound over all $k$.

We are now equipped to complete the proof of Lemma 6.1:

Proof of Lemma 6.1. At the end of Phase 5, consider bipartitions of $K^3_{\max}$ into two sets $C_1$ and $C_2$, both containing at least $\delta n$ vertices. Let $\mu := P \cdot fn/2$, where $P$ is the same constant as in Corollary 6.3. We apply Lemma 4.1 to $K^3_{\max}$ with an $\varepsilon > 0$ such that $\log(1+\varepsilon) < P fn/16 = \mu/(8n)$. The lemma provides an $\eta' > 0$ such that there are at most $(1+\varepsilon)^n$ bipartitions of $K^3_{\max}$ into two subsets $C_1$ and $C_2$ with at most $\eta'n$ cross-edges at the end of Phase 5. In particular, this is an upper bound for the number of bipartitions into two sets which both have size at least $\delta n$, with at most $\eta'n$ cross-edges.

Furthermore, a partition of $K^3_{\max}$ also induces a bipartition of $F$. Since $F$ has at least $fn$ vertices w.h.p., we have for at least one $i \in \{1, 2\}$ that $|F \cap C_i| \geq fn/2$; we may assume that this holds for $i = 1$. As we require
Theorem 5.3 with \( \varepsilon = 1/2 \), with probability at least \( 1 - e^{-\frac{\mu}{2}} \) there are at least \( \mu/2 \) such edges.

Finally, let \( \eta := \min \{ \eta', \mu/(2n) \} \). According to the first part, there are at most \((1 + \varepsilon)^n\) bipartitions of \( K_{\max}^3 \) into two sets such that both have size at least \( \delta n \), and have at most \( \eta n \) cross-edges. According to the second part, any such bipartition has at least \( \mu/2 \geq \eta n \) cross-edges after Phase 6 with probability not less than \( 1 - e^{-\frac{\mu}{e}} \). Therefore, by a union bound over all such bipartitions, the probability that there is a \((\delta, \eta)\)-cut (i.e., a bipartition with less than \( \eta n \) cross-edges) in the subgraph of \( G_4 \) induced by \( K_{\max}^3 \) is at most \((1 + \varepsilon)^n e^{-\frac{\mu}{2}} = e^{-\Theta(n)} \).

What remains is to show that going through Phase 6 does not add too many vertices to the giant.

**Lemma 6.4.** There is a constant \( f(\delta) > 0 \) such that with high probability

\[
|K_{\max}^3| \leq |K_{\max}^4| + 3\delta n. \tag{23}
\]

Note that trivially \( K_{\max}^3 \subseteq K_{\max}^4 \).

We will prove Lemma 6.4 by showing that for an appropriate \( f \), after Phase 5 there are few vertices outside of \( K_{\max}^3 \) that are in large components, or in components which have at least one “heavy” vertex. So these components cannot increase \( K_{\max}^3 \) by much. Afterwards, we will show that \( F \) has few edges to vertices which fall in neither of these categories.

Let us begin with showing that few vertices are in large components which are not the giant:

**Lemma 6.5.** There is a constant \( s_t \) such that with high probability, the graph \( G_t \) has less than \( \delta n \) vertices which are in a component that \( a \) is not the giant, and \( b \) has size at least \( s_t \).

**Proof.** Choose \( s_t \) to be larger than \( 4/(s_{\max} P) \), where \( P \) is the same as in Corollary 6.3. Consider Phase 5 of Algorithm 1, which is used to generate \( G_3 \). It consists of steps from \( |V\setminus K_{\max}^1| + 1 \) to \( |V\setminus F| \). Recall that

\[
\left\{ u_k \in V \mid |V\setminus K_{\max}^1| < k \leq |V\setminus F| \right\} = K_{\max}^1 \setminus F. \tag{24}
\]

Let \( A_k \) be the set of vertices in \( G_2 \) which have the following property: At the end of step \( k - 1 \), they are in a component of size at least \( s_t \), but not in the giant. For each step \( k \) of Phase 5, we define the indicator random variable \( Z_k \) which is 1 if and only if \( A_{k+1} \neq A_k \), i.e. if and only if step \( k \) adds an edge from \( u_k \) to at least one vertex in \( A_k \) (thus attaching it to the giant). We apply Corollary
yielding that with high probability, at each step \( k \) with \( |A_k| \geq \delta n \), vertex \( u_k \) receives an edge to \( A_k \) with probability at least \( P \), i.e.

\[
\mathbb{P}[Z_k \mid |A_k| \geq \delta n] \geq P;
\]

we call this a “hit”. Additionally, we define indicator random variables \( Z'_k \) as follows:

\[
Z'_k := \begin{cases} 
Z_k, & \text{if } |A_k| \geq \delta n \\
B_P, & \text{otherwise.}
\end{cases}
\]

Here, \( B_P \) is an indicator random variable with success probability \( P \). Note that while the \( Z'_k \) are not independently, we have \( \mathbb{P}[Z'_k = 1 \mid A_k] \geq P \) for all \( k \) and all \( A_k \subseteq V_k \), i.e. not depending on the outcome of the previous steps. Therefore, we may lower-bound the sum over the \( Z'_k \) by a sum over \( |K^1_{\text{max}} \setminus F| \) independent instances of \( B_P \). We get

\[
\mathbb{E} \left[ \sum_{k = |V \setminus F|} |V \setminus F| \right] \geq \mathbb{E} \left[ \sum_{k = |V \setminus K^1_{\text{max}}|+1} |V \setminus F| \times B_P \right] > P \cdot \frac{s_{\text{max}}}{2} =: \mu.
\]

For the last step, we use linearity of expectation and the fact that we require \( f < s_{\text{max}}/12 \) (and \( F \) has at most \( 6fn \) vertices w.h.p.), which implies \( |K^1_{\text{max}} \setminus F| > s_{\text{max}}/2 \). Applying the Chernoff bound (Theorem 5.3) to \( \sum_k B_P \) yields that w.h.p. \( \sum_k B_P \geq \mu/2 \) and therefore also that w.h.p. \( \sum_k Z'_k \geq \mu/2 \). Now, under the assumption of \( \sum_k Z'_k \geq \mu/2 \), there are two possibilities: If \( A_{|V \setminus F|+1} \) is smaller than \( \delta n \), we are done (as this is exactly the set of vertices in components of size at least \( s_t \) but not the giant in \( G_3 \)). Otherwise, each case of \( Z'_k = 1 \) actually resulted from a “hit”, so at least \( s_t \cdot \mu/2 \) vertices were eliminated from \( A \). However, by our choice of \( s_t > 4/(s_{\text{max}}P) = 2n/\mu \), this leads to a contradiction. Therefore, under the assumption of \( \sum_k Z'_k \geq \mu/2 \) (and thus with high probability), the set \( A_{|V \setminus F|+1} \) has less than \( \delta n \) vertices as desired.

Furthermore, most vertices in small components also have small weight:

**Lemma 6.6.** There is a constant \( B > 0 \) such that there are at most \( \delta n/s_t \) vertices with weight at least \( B \) which are in a component of size at most \( s_t \) in \( G_3 \). In particular, there are at most \( \delta n \) vertices in \( G_3 \) which are in a component of size at most \( s_t \) and share that component with a vertex with weight at least \( B \). For convenience, we choose \( B \geq B' \) (so that the weight of vertices in \( F \) is guaranteed to be at most \( B \)).

**Proof.** We use (PL2) to bound the total number of vertices with weight at least \( B \). In particular, we set \( \eta = 1 \) in (PL2) and \( w \) to \( \max\{B', (c_2s_t/\delta)^{1/(3-2)}\} =: B \). Inequality (PL2) then yields that there are at most \( \delta n/s_t \) vertices with weight at least \( B \). Furthermore, there are clearly no more than \( \delta n \) vertices which are in a component of size at most \( s_t \) and share that component with a vertex with weight at least \( B \).

It only remains to bound the number of edges that go from \( F \) to low-weight vertices.

**Lemma 6.7.** There is an \( f > 0 \) such that with high probability, there are at most \( \delta n/s_t \) edges from \( F \) to vertices of weight at most \( B \).
Proof. Recall how $F$ was defined: We include each vertex with weight less than $B'$ independently with probability $4f/s_{\text{max}}$ in a set $F'$. (The weight bound $B' \leq B$ was chosen such that there are at least $s_{\text{max}}n/2$ vertices with weight less than $B'$ in $K_1^\text{max}$.) Then, the set $F$ is given as $F' \cap K_1^\text{max}$. Note that the generation of $F'$ is completed independent of sampling the actual graph, so we might as well draw $F'$ before generating even $G_1$. Using a Chernoff bound (Theorem 5.3i with $\varepsilon = 1/2$), we can show that w.h.p we have $|F'| \leq (6f n / s_{\text{max}}) =: f'n$. Let $S$ be the set of all vertices which have weight at most $B$; note that $F' \subseteq S$. We will show that for small enough $f$ (and thus $f'$), there are at most $\delta n / s_t$ edges from $F'$ to $S$ in $G_4$; this completes the argument, as $F = F' \cap K_1^\text{max} \subseteq F'$.

We will use the Azuma-Hoeffding bound with two-sided error event, Theorem 5.5: in the following, we describe the choice of parameters for the theorem. The induced subgraph $G_4[S]$ of $G_4$ is determined by the following $2|S| - 1$ independent random variables. We may assume that $S = \{1, \ldots, |S|\}$, so that we have an ordering of the vertices. (It does not need to be the same ordering as in Algorithm 1). Firstly, for every $v \in S$ we include its random position $x_v$ as a random variable. Secondly, for each $v \in S \setminus \{1\}$ we include the random variable $Y_v := (Y_{v,1}, \ldots, Y_{v,v-1})$, which is uniformly at random in $[0,1]^{v-1}$. The random variables $Y_{uv}$ are the same as in Section 5.2, so we include the edge $(u, v)$ if and only if $Y_{uv} \leq p_{uv}$. As in Theorem 5.5, we denote the product space corresponding to the $2|S| - 1$ random variables by $\Omega$. (By abuse of notation, we also denote the sample space by $\Omega$.) For $\omega \in \Omega$, the function $g(\omega)$ is simply defined as the number of edges from $F'$ to $V$ in $\omega$.

Note that $0 \leq g(\omega) \leq |F'| \cdot |S| \leq f'n^2$. This corresponds to $M$ in the theorem. We define the “bad” subset $B$ as follows:

$$B := \{\omega \in \Omega \mid \exists u \in S : \deg(u) \geq 2C \log^2 n\}$$

(28)

where $C > 0$ is the constant from Lemma 5.2. Note that by Lemma 5.2 (and using the fact that the weights of vertices in $S$ are bounded by $B$), we have $P[B] = n^{-\omega(1)}$. Moreover, observe that for any $\omega, \omega' \in \Omega \setminus B$ which differ in at most two components, we have

$$|g(\omega) - g(\omega')| \leq 4C \log^2 n =: c$$

(29)

since the outcome of every $x_u$ and $Y_u$ affects at most $2C \log^2 n$ edges (because $\omega$ and $\omega'$ are both in $\Omega \setminus B$ and all edges a single coordinate affects have a vertex in common). This corresponds to $c$ in the theorem.

Furthermore, Lemma 5.2 also implies that the expected degree of each vertex in $S$, and thus $F'$, is at most $B \cdot C$. Since $g$ is upper-bounded by the sum of degrees of vertices in $F'$, we get that

$$E[g(Z)] \leq E \left[ \sum_{u \in F'} \deg(u) \right] \leq BC f'n.$$  

(30)

This allows us to set $t := 2BCf'n - E[g(Z)] \geq BC f'n$ for Theorem 5.5.

Putting all of this together, Theorem 5.5 yields

$$P[g(Z) - E[g(Z)] \geq t] \leq 2e^{-\frac{(BC f'n)^2}{24(22(2 \log^2 n)^2} + (2 \cdot 2n \cdot f'n^2 \log^2 n + 1)n^{-\omega(1)} = n^{-\omega(1)}.$$  

(31)
If we choose $f$ small enough such that $2BCf' < \delta/s_t$, we get the desired statement, since for such a choice of $f'$, the theorem implies that w.h.p. $g(Z) \leq E[g(Z)] + t = 2BCf'n < \delta n/s_t$.

We now have all we need to complete the proof of Lemma 6.4

Proof of Lemma 6.4. Each vertex in $G_3$ is an element of exactly one of the following sets:

1. $K_{\text{max}}^3$.
2. The set of vertices in a component of size at least $s_t$, but not in $K_{\text{max}}^3$.
3. The set of vertices in a component of size less than $s_t$ which contains at least one vertex with weight at least $B$.
4. The set of vertices in a component of size less than $s_t$ in which all vertices have weight less than $B$. We call this set $T$.

Going through Phase 6 will attach some vertices from sets 2–4 to $K_{\text{max}}^3$; our goal is to show that w.h.p at most $3\delta n$ vertices are attached this way. Lemma 6.5 tells us that w.h.p. the second set contains at most $\delta n$ vertices. Lemma 6.6 gives the same result for the third set. Therefore these two sets can contribute at most $\delta n$ vertices each, and the remaining $\delta n$ must come from $T$ (set 4). However, according to Lemma 6.7, there are only at most $\delta n/s_t$ edges from $F$ to $T$ w.h.p. (since vertices in $T$ all have weight less than $B$). Each such edge can attach at most $s_t$ vertices from $T$ to $K_{\text{max}}^3$, so w.h.p at most $\delta n$ edges from $T$ are attached, completing the argument.

Theorem 6.8. With high probability, $K_{\text{max}}^4$ has no $(4\delta, \eta)$-cut, where $\eta$ is the same as in Lemma 6.4.

Proof. Assume there is a $(4\delta, \eta)$-cut of $K_{\text{max}}^4$; such a cut induces a bipartition of vertices into two set $C_1$ and $C_2$ which are both of size at least $4\delta n$. Consider the restriction of $C_1$ and $C_2$ to $K_{\text{max}}^3$. Lemma 6.4 implies that for $i \in \{1, 2\}$ we have $|K_{\text{max}}^3 \cap C_i| \geq |C_i| - 3\delta n \geq \delta n$. Therefore, any $(4\delta, \eta)$-cut of $K_{\text{max}}^4$ induces a $(\delta, \eta)$-cut of $K_{\text{max}}^3$, but according to Lemma 6.1, such a cut does not exist w.h.p.

Since Theorem 6.8 is just a slight reformulation of our main Theorem 3.2, this concludes the proof.

7 Conclusion

We studied the separator properties of the MCD model for dimensions greater than one, and proved that this class of random graphs does not have small separators. This is a substantial difference to the one-dimensional case and to EuGIRGs. Our result shows that the underlying geometry decides on whether there are small separators. Future research could include investigating distance functions that interpolate between MCD and euclidean distance. Under the MCD model, two vertices are likely to connect if their positions are close along the first axis, OR the second axis, . . ., OR the $d$-th axis, whereas in Euclidean GIRGs, they have to be close along the first axis, AND the second axis, . . .,
AND the \(d\)-th axis. Being able to easily mix and match these two would allow for formulating intuitive models, with connection criteria resembling boolean formulae (such as “Two people are likely to be friends if they both like football AND live close to each other, OR if both like the same particular online game”). Hopefully, the tools laid out in this paper for analyzing the MCD model will prove useful for these questions.

A Appendix: Clustering Coefficient

In this appendix we show that w.h.p. the clustering coefficient of an MCD-GIRG is \(\Omega(1)\). In fact, our result holds for any GIRG model for which the distance function satisfies a stochastic triangle inequality, i.e. two uniformly random points in the \(\epsilon\)-neighbourhood of some position \(x \in \mathbb{T}^d\) have probability \(\Omega(1)\) to have distance at most \(O(\epsilon)\). The proof follows closely the proof for EuGIRGs from [9], except that we replace the deterministic triangle inequality by its stochastic version. We use the notation \(u \sim v\) to indicate that \(u\) and \(v\) are adjacent.

For the definition of clustering coefficient, there are several slightly different definitions. We follow the convention in [32].

**Definition A.1.** For a graph \(G = (V, E)\) the clustering coefficient of a vertex \(v \in V\) is defined as

\[
cc(v) := \begin{cases} 
\frac{1}{\binom{\deg(v)}{2}} \cdot |\text{triangles in } G \text{ containing } v|, & \text{if } \deg(v) \geq 2, \\
0, & \text{otherwise},
\end{cases}
\]

and the (mean) clustering coefficient of \(G\) is \(cc(G) := \frac{1}{|V|} \sum_{v \in V} cc(v)\).

In case of GIRGs, we can use that the edges are sampled independently after the positions are fixed, to get the following expression for \(E[cc(G)]\).

**Lemma A.2.** Let \(G\) be a GIRG, and let \(v \in V\) be a vertex. Then

\[
E[cc(v)] = \Pr[\deg(v) \geq 2] \cdot \frac{1}{\binom{n-1}{2}} \sum_{u, u' \in V \setminus \{v\}, u \neq u'} \Pr[u \sim u' \mid u \sim v, u' \sim v].
\]

**Proof.** Fix the vertex \(v\), and let \(u, \tilde{u} \in V \setminus \{v\}\) uniformly at random subject to \(u \neq \tilde{u}\). Moreover, let \(E_N\) be the event that \(u\) and \(\tilde{u}\) are both adjacent to \(v\), and let \(E_\Delta\) be the event that \(v, u, \tilde{u}\) form a triangle. Then we may write \(E[cc(v)] = \binom{n-1}{2}/\binom{\deg(v)}{2} \cdot \Pr[E_\Delta]\), and for any \(s \geq 2\) we have \(\Pr[E_N | \deg(v) = s] = \binom{s}{2}/\binom{n-1}{2}\). Hence,

\[
E[cc(v)] = \sum_{s=2}^{n-1} \Pr[\deg(v) = s] \cdot \Pr[G, u, \tilde{u}: E_\Delta | E_N, \deg(v) = s].
\]

Now recall that in a GIRG all edges are independent after fixing the geometric positions. Hence, we have \(\Pr[E_\Delta | E_N, x_0 = x, \deg(v) = s] = \Pr[E_\Delta | E_N, x_0 = x]\) for every position \(x \in \mathbb{T}^d\). Consequently, we also get the same for the marginal probabilities, \(\Pr[E_\Delta | E_N, \deg(v) = s] = \Pr[E_\Delta | E_N]\). The claim follows.
We will show that the following property is enough to ensure strong clustering.

**Definition A.3.** A distance function $\|\cdot\| : \mathbb{T}^d \to \mathbb{R}_{\geq 0}$ in the sense of Section 2 satisfies a stochastic triangle inequality if there is a constant $C > 0$ such that the following two conditions hold.

1. For every $\varepsilon > 0$ let $x_1 = x_1(\varepsilon), x_2 = x_2(\varepsilon)$ be uniformly at random in the $\varepsilon$-ball $\{x \in \mathbb{T}^d \mid \|x\| \leq \varepsilon\}$. Then
   \[
   \liminf_{\varepsilon \to 0} \Pr[\|x_1 - x_2\| \leq C\varepsilon] > 0. \tag{32}
   \]

2. Moreover,
   \[
   \liminf_{\varepsilon \to 0} \frac{\text{vol}\{x \in \mathbb{T}^d \mid \|x\| \leq \varepsilon\}}{\text{vol}\{x \in \mathbb{T}^d \mid \|x\| \leq C\varepsilon\}} > 0. \tag{33}
   \]

We note that every metric satisfies a stochastic triangle inequality for $C = 2$, since the probability in (32) is one by the triangle inequality, and (33) is satisfied with limit $C^{-d}$. The minimum component distance satisfies a stochastic triangle inequality for $C = 2$, since a random point $x_1$ in the $\varepsilon$-ball must have at least one “small” component of absolute value at most $\varepsilon$, and the probability that both $x_1$ and $x_2$ have the same small component is at least $1/d$. Thus the limit in (32) is at least $1/d$. Moreover, (33) is satisfied with limit $1/C = 1/2$.

**Theorem A.4.** Consider the GIRG model with a distance function that satisfies the stochastic triangle inequality, and let $G$ be a random instance. Then $\text{cc}(G) = \Omega(1)$ with high probability.

*Proof.* Let $V' := \{v \in V \mid w_v \leq n^{1/8}\}$ and $G' = G[V']$. Considering $G'$ will be helpful later on to prove concentration, but we first show that $\mathbb{E}[\text{cc}(G')] = \Omega(1)$. Since in the definition of distance function we required the volume $V(r) = \text{vol}\{x \in \mathbb{T}^d \mid \|x\| \leq r\}$ to be surjective on $[0, 1]$, there exists $r > 0$ such that $V(r) = 1/n$. For any vertex $v$, we define $U(v) := \{x \in \mathbb{T}^d \mid \|x - x_v\| \leq r\}$ to be the $r$-neighborhood of $x_v$. Note that by (EPL) and (PL1), any two different vertices $u, v$ with $u \in U(v)$ have edge probability $p_{uv} = \Omega(1)$.

Now we restrict the vertex set even further. By (PL2) there exists $w_0 = O(1)$ such that there are $\Omega(n)$ vertices with weight at most $w_0$. Since $\text{cc}(G') = \frac{1}{|V'|} \sum_{v \in V'} \text{cc}_{G'}(v) = \Omega(\frac{1}{n} \sum_{v \in V} \text{cc}_{G'}(v))$, it suffices to show that a vertex $v$ of weight at most $w_0$ has expected clustering coefficient $\mathbb{E}\text{cc}_{G'}(v) = \Omega(1)$. So we fix a vertex $v \in V_{\leq w_0} := \{v \in V \mid w_v \leq w_0\}$ and its position $x_v$.

Then the expected number of vertices in $V_{\leq w_0}$ with position in $U(v)$ is $\Theta(1)$. Consider the event $\mathcal{E} = \mathcal{E}(v)$ that the following three properties hold.

(i) $v$ has exactly two neighbors in $V_{\leq w_0}$ with positions in $U(v)$.

(ii) $v$ does not have neighbors in $V_{\leq w_0}$ with positions in $\mathbb{T}^d \setminus U(v)$.

(iii) $v$ does not have neighbors in $V \setminus V_{\leq w_0}$.

We claim that $\Pr[\mathcal{E}] = \Theta(1)$. For (i), note that the expected number of vertices in $V_{\leq w_0}$ with position in $U(x)$ is $\Theta(1)$. Since the position of every vertex is independent, by LeCam’s theorem (Theorem 5.6) the probability that there are
exactly two vertices in \( V_{\leq w_0} \) with position in \( U(x) \) is \( \Theta(1) \). Moreover, for each such vertex the probability to connect to \( v \) is \( \Theta(1) \), so (i) holds with probability \( \Theta(1) \). Conditioned on these events (i.e., all other vertices are in \( T^d \setminus U(v) \)), we bound (ii). Recall that for any vertex \( u \neq v \) we have \( \Pr[u \in T^d \setminus U(v)] = 1 - 1/n \). Therefore, for any vertex \( u \neq v \), we can bound
\[
\Pr[v \sim u \mid x_v \in T^d \setminus U(v)] \leq \frac{\Pr[v \sim u]}{\Pr[x_v \in T^d \setminus U(v)]} \leq \Pr[v \sim u] + \frac{1}{n - 1} = O(w_u/n),
\]
where the last step follows since \( \Pr[v \sim u] = \Theta(\min\{1, w_u w_v/n\}) \) [10, Lemma 4.3] and \( w_v = \Theta(1) \). Since \( W = \sum_{u \in V} w_u = O(n) \) and \( \max_{v \in V} w_v = o(n) \) by (PL2), we may apply LeCam’s theorem, and (ii) holds with probability \( \Theta(1) \).

Finally, for every fixed position \( x \), (iii) holds independently of (i) or (ii) with probability \( \Theta(1) \), again by LeCam’s theorem. This proves the claim that \( \Pr[\mathcal{E}] = \Theta(1) \).

Conditioned on \( \mathcal{E} \), let \( u \) be a neighbor of \( v \). Then \( x_u \in U(v) \), and \( w_u \leq w_0 \). Moreover, since \( p(w_v, w_u, x_v, x_u) = \Theta(1) \) for all weights \( w_u \leq w_0 \) and all positions \( x_u \in U(v) \), we have for all measurable \( A \subseteq U(v) \),
\[
\Pr[x_u \in A \mid \mathcal{E}, v_1 \sim v] = \int_A \Pr[x_1 = u, v \sim v_1] du \int_{U(v)} \Pr[x_1 = u, v \sim v_1] du = \Theta\left( \frac{\text{VOL}(A)}{\text{VOL}(U(v))} \right).
\]
In other words, conditioned on \( \mathcal{E} \), the position \( x_u \) is uniformly distributed in \( U(v) \) up to constant factors. The same holds for the second neighbor \( u' \) of \( v \). Therefore, by (32), with probability \( \Omega(1) \) we have \( \|u - u'\| \leq C r \). By (33) we also have \( V(C r) = O(V(r)) = O(1/n) \), and thus \( p_{uu'} = \Omega(1) \). So we have shown that \( \mathbb{E}[\text{cc}(G') \mid \mathcal{E}(v)] = \Omega(1) \) for all \( v \in V_{\leq w_0} \). Since \( \Pr[\mathcal{E}(v) = \Theta(1)] \), this proves \( \mathbb{E}[\text{cc}(G') \mid v] = \Omega(1) \) for all \( v \in V_{\leq w_0} \), which implies \( \mathbb{E}[\text{cc}(G')] = \Omega(1) \).

It remains to show concentration. We want to apply the Azuma-Hoeffding bound with two-sided error event (Theorem 5.5) to \( f(G') := \text{cc}(G') \), where we define the error event \( \mathcal{B} \) to be the event that there is a vertex in \( V' \) of degree larger than \( Cn^{1/8} \) for a suitable constant \( C > 0 \). By Lemma 5.2 we have \( \Pr[\mathcal{B}] = n^{-\omega(1)} \) if \( C \) is large enough. Similar as in the proof of Lemma 6.7, we apply Theorem 5.5 to the following \( 2|V'| - 1 \) independent random variables, where we may assume that \( V' = \{1, \ldots, |V'|\} \). Firstly, for every \( v \in V' \) we include its random position \( x_v \) as a random variable. Secondly, for each \( v \in V \setminus \{1\} \) we include a random variable \( Y_v := (Y_{v1}, \ldots, Y_{v|V'|-1}) \), which is uniformly at random in \( [0, 1]^{|V'|-1} \). The random variables \( Y_{uv} \) play the same role as in Section 5.2, so we include the edge \( \{uv\} \) if and only if \( Y_{uv} \leq p_{uv} \). As in Theorem 5.5, we denote the product space corresponding to the \( 2|V'| - 1 \) random variables by \( \Omega \).

Note that if \( \omega, \omega' \) are two events in \( \Omega \setminus \mathcal{B} \) then every random variables \( x_v \) or \( Y_v \) that differs in \( \omega \) and \( \omega' \) can affect at most the \( \text{cc}'s \) of those vertices which are adjacent to \( v \) in either \( \omega \) or \( \omega' \). Hence, the effect of each coordinate on \( \text{cc}(G') \) is at most \( \frac{1}{|V'|} 2 C n^{1/8} \leq 4 C n^{-7/8} \), and thus \( \|f(\omega) - f(\omega')\| \leq 8 C n^{-7/8} \) if \( \omega \) and \( \omega' \) differ in at most two components. Note that for this type of argument it is crucial that both of \( \omega \) and \( \omega' \) are “good” events. Now we may apply Theorem 5.5 with \( c = 8 C n^{-7/8} \), \( M = 1 \), \( m = 2|V'| - 1 \leq 2n \), and \( t := n^{-1/4} \), and obtain
\[
\Pr\left[ |\text{cc}(G') - \mathbb{E}[\text{cc}(G')]| \geq n^{-1/4} \right] \leq 2e^{-\Omega(n^{1/4})} + 8 C n^{1/8} \Pr[\mathcal{B}] = o(1).
\]
Since $\mathbb{E}[\text{cc}(G')] = \Theta(1)$, this shows that w.h.p. $\text{cc}(G') = \Omega(1)$. To conclude the same for $G$, we observe that $|V| \cdot \text{cc}(G) \geq |V'| \cdot \text{cc}(G') - \sum_{v \in V \setminus V'} \deg(v)$. However, an easy calculation shows that w.h.p.

$$\sum_{v \in V \setminus V'} \deg(v) \Rightarrow (1 + o(1)) \sum_{v \in V \setminus V'} w_v \Rightarrow o(n).$$

Since w.h.p. $|V'| = \Theta(n)$, this concludes the argument and proves $\text{cc}(G) = \Omega(1)$ with high probability.

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