On the waiting time in quantum repeaters with probabilistic entanglement swapping

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The standard approach to realize a quantum repeater relies upon probabilistic but heralded entangled state manipulations and the storage of quantum states while waiting for successful events. In the literature on this class of repeaters, calculating repeater rates has typically depended on approximations assuming sufficiently small probabilities. Here we propose an exact and systematic approach including an algorithm based on Markov chains theory to compute the average waiting time (and hence the transmission rates) of quantum repeaters with arbitrary numbers of links. For up to four repeater segments, we present the exact rate formulae for arbitrary entanglement swapping probabilities. The effect of finite memory times is also considered and the relative influence of the classical communication (of heralded signals) is shown to grow significantly for larger probabilities. Conversely, we demonstrate that for small swapping probabilities the statistical behavior of the waiting time in a quantum repeater cannot be characterized by its average value alone and additional statistical quantifiers are needed.

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Introduction. Quantum repeaters are among the most popular research objects in the field of quantum information. These enable one, in principle, to extend (optical-fiber-based) quantum communication schemes such as quantum cryptography to distances as large as 1000 km and beyond despite channel losses that typically increase exponentially with distance. Thanks to some recent results on the bounds of repeaterless communication [1–4], it is now well-defined when a scheme deserves to be called a quantum repeater and when it does not. These bounds are expressed in terms of a secret key rate per mode (per channel use) and, when ignoring all imperfections besides channel losses, they correspond to an optimal raw qubit transmission rate without quantum repeater [5].

Since the standard approach to quantum repeaters is based on quantum memories and on (at least partially) probabilistic operations on entangled states (distribution, swapping, and purification) [6, 7], which currently may still be the most promising way to exceed the above bounds, some recent proposals consider small-scale versions of quantum repeaters with a minimal number of memory stations (repeater links) [8–11]. While the rate analysis for the smallest repeater with only two segments and one link is fairly straightforward, quantum repeaters with two or more links become increasingly complex to analyze when the entanglement swapping is probabilistic. Indeed there is no explicit expression for the average repeater waiting time in the literature for such advanced cases [12]. However, entanglement swapping based on heralded but non-deterministic Bell measurements is a rather natural situation, especially in one of the most prominent approaches based on atomic ensembles and linear optics [13] where normally even the ideal Bell measurement cannot exceed an efficiency of 1/2, [13], which is further reduced in the presence of memory losses. Thus, so far, typically approximate rate formulae have been applied that depended on the assumption of sufficiently small probabilities [15–17]. Nonetheless, the most efficient memory-based quantum repeater schemes would rely on high swapping probabilities, for instance, based on suitable atom-light interactions [18] or enhanced linear-optics Bell measurements [19–21] [22]. While exact analytical rate formulae are known only for the extreme case of fully deterministic entanglement swapping [23], in the present work, we will address the entire range of arbitrary swapping probabilities. Here we consider repeater schemes like those in Refs. [13, 15] where entanglement purification is built into the distribution and swapping procedures.

We propose an approach based on Markov chains theory which allows one to obtain the waiting time for a quantum repeater with an arbitrary number of segments n. We give explicit expressions for the case of three and four segments and demonstrate how one could obtain these expressions for an arbitrary quantum repeater. These expressions quickly (for n > 5) become impractical. However, we can still get the desired result numerically. This approach works for ~16 segments.

Classical communication and finite memory times can also be included into our formalism. These refinements increase the size of the corresponding Markov chains, but this approach is still feasible for practical applications. We show that for small probabilities the contribution of the classical communication time to the waiting time is negligible by comparison with the time needed for entanglement distribution. We further demonstrate that in addition to the average time the standard deviation should also be taken into account. For example, in the case of finite memory times (as usually adapted to the average waiting time), any deviation to a longer storage time can be problematic.
Probability generating functions. If $X$ is a random variable taking only non-negative integer values, then its probability generating function (PGF) $g(t)$ and cumulative PGF $c(t)$ are defined by $g(t) = \sum_{k=0}^{\infty} P(X = k) t^k$ and $c(t) = \sum_{k=0}^{\infty} P(X \leq k) t^k$. The PGF $g(t)$ contains full statistical information about the corresponding random variable $X$, for example, the average value $\overline{X}$ and the variance $\sigma^2 = \overline{X^2} - \overline{X}^2$ via $\overline{X} = g'(1)$ and $\sigma^2 = g''(1) + g'(1) - g'(1)^2$. The two PGFs are related by $g(t) = (1 - t)c(t)$.

Markov chains. We first consider the special case of deterministic swapping. In this case a quantum repeater with $n$ segments can be modeled by a Markov chain with $N = 2^n$ states, which are labeled by binary strings of length $n$ (1 and 0 mean whether entanglement is distributed or not) or by subsets of the set $\{1, \ldots, n\}$ (subsets of segments with successfully distributed entanglement). The strings $0\ldots0$ and $1\ldots1$ correspond to $\emptyset$ and $[n]$ respectively. The Markov chains corresponding to the quantum repeaters with one and two links are shown in Fig. 1 (with set notation). These chains are examples of so-called absorbing Markov chains where there is a state which, once entered, cannot be escaped from. Such a state is denoted by a double circle in Fig. 1. The number of steps needed to reach the absorbing state is called the absorption time. The real absorption time is $T = k_i \tau$, where $X \equiv k_i$ is the number of steps to reach the absorbing state starting from the $i$-th state and $\tau$ is the time it takes to perform one step. To compute $k_i$ we employ the method of PGFs presented in [24].

Consider a chain with $N$ states $s_i, i = 1, \ldots, N$, and single absorbing state $s_N$. The transition probability matrix $P = (p_{ij})_{i,j=1}^{N}$, where $p_{ij} = P(s_i \rightarrow s_j)$, can be written in the form $P = \left( \begin{array}{c|c} Q & u \\ \hline \otimes & 1 \end{array} \right)$. For any state $s_i$ let $g_i(t)$ denote the PGF of $k_i$. These PGFs are connected by the equations $g_i(t) = \sum_{j=1}^{N} p_{ij} g_j(t)$, where the right-hand side is multiplied by $t$ since every transition increases the absorption time by one unit. It is clear that $g_N(t) = 1$, since it takes zero steps to be absorbed when starting in the absorbing state. We thus get the system of $N - 1$ linear equations for $N - 1$ functions $g_i(t) = (g_1(t), \ldots, g_{N-1}(t))^T$: $g(t) = tQg(t) + tu$. The solution of this system reads as $g(t) = t(E - tQ)^{-1}u$. All rows of $P$ sum up to 1, so we have $u = (E - Q)1$, where $E$ is the identity matrix of size $(N - 1)$ and $1$ is the $(N - 1)$-vector with all components equal to 1. Thus, the solution reads as $g(t) = (E - tQ)^{-1}(E - Q)1$. In a more general situation where some transitions do not contribute to the absorption time (we ignore the time taken by those transitions) we simply do not mark these by $t$. In this case we have a matrix $Q(t)$ and a vector $u(t)$ where some elements are multiplied by $t$ and the solution is $g(t) = (E - Q(t))^{-1}u(t)$. This expression can be applied to an arbitrary absorbing chain with a single absorbing state. If all transitions contribute to the absorption time, then all elements are multiplied by $t$ and $Q(t) = tQ, u(t) = tu$, so the more general expression reduces to the expression given earlier.

We are typically interested in the properties of the absorption time of the system starting in the state $s_1$, so we just need the first component of the vector $g(t)$. In this case we have $g(t) = g_1(t) = \frac{\det(E - Q(t)|u(t))}{\det(E - Q(t))}$, (1)

where $(E - Q(t)|u(t))$ is obtained from $E - Q(t)$ by replacing its first column by $u(t)$. From this PGF we can obtain all statistical quantities, for example, the average value $\overline{k} \equiv k_1$ and the variance $\sigma^2$.

In fact, when all transitions contribute to the absorption time it is possible to express these quantities for all $g_i(t)$ in a compact form as $\overline{k} = R1$, $\sigma^2 = (2R - E)\overline{k} - \overline{k}^2$, where $\overline{k}$ and $\overline{k}^2$ are the vectors of average values and variances of $g(t)$, $z^2$ is the component-wise square of the vector $z$ and $R = (E - Q)^{-1}$ is the fundamental matrix of the chain. These solutions are also given in [25] and obtained there without the explicit use of PGFs. We can compute the vectors of probabilities $\{P(k_i > k)\} = Q^k1$ and $\{P(k_i = k + 1)\} = Q^k u, k \geq 0$.

Non-deterministic swapping. In the Supplemental Material we show that our method correctly reproduces the case of deterministic swapping. Now consider the case of non-deterministic heralded swapping. Let us denote the swapping probability by $a$. We start with a quantum repeater with two segments. Now the language of binary strings is more appropriate, so the states are 00, 01, 10 and 11 (≡ $\emptyset$). The state 11 is the single absorbing state. This Markov chain is shown in Fig. 1a. The matrix $Q$ reads as

$$Q = \begin{pmatrix} (1 - p)^2 + (1 - a)p^2 & p(1 - p) & p(1 - p) \\ (1 - a)p & 1 - p & 0 \\ (1 - a)p & 0 & 1 - p \end{pmatrix}.$$ (2)

From Eq. [A1] we can easily obtain the expression for the generating function $g(t)$. For $a = 1$ this expression coincides with the expression for the deterministic case with $n = 2$. For the average value and the variance we

![FIG. 1: Markov chains with deterministic swapping.](image-url)
obtain
\[ \bar{k} = \frac{3 - 2p}{ap(2 - p)}, \quad \sigma^2 = \frac{k^2 - 4 - p(2 + p(3 - 2p))}{ap^2(2 - p)^2}. \] (3)

The expression for \( \bar{k} \) is a well-known result [23]. However, with our new result for \( \sigma^2 \), we see that \( \sigma/\bar{k} \to 1 \) as \( \alpha \to 0 \), so for small swapping probabilities the variance is comparable to the average value (independently of the value of \( p \)). This result is in strong contrast with the results from other disciplines like the theory of algorithms [20] and may have important consequences depending on the application [27]. Below we present examples of the full probability distribution, which demonstrates that even for moderately small \( p \) and \( \alpha \) the average value alone is an insufficient characteristic of quantum repeaters.

Classical communication. Up to now we ignored the classical communication times, for example, the time it takes to restart the process if the swapping fails. Let us consider a more general problem, where restarting the process takes the time \( \tau' \), which is independent of \( \tau \) (while, for simplicity, here we ignore the classical communication times when the swapping succeeds, see Supplemental Material). We split the state 11 into two states 11 and \( \bar{11} \), where now 11 is the state just ready to try to perform swapping and \( \bar{11} \) is the state after successful entanglement swapping. This Markov chain is shown in Fig. [11b]. The absorption time reads as \( T = k\tau + l\tau' \), where \( l \) is the number of transitions \( 11 \to 00 \) and the transition \( 11 \to \bar{11} \) is ignored.

We mark different transitions by different variables \( t \), as before, marks attempts to distribute entanglement and \( v \) marks unsuccessful attempts to perform swapping, blue line in the figure). The matrix \( Q \) with this information included reads as
\[
Q(t, v) = \begin{pmatrix}
(t(1 - p)^2 & tp(1 - p) & tp(1 - p) & tp^2 \\
0 & t(1 - p) & 0 & tp \\
0 & 0 & t(1 - p) & tp \\
v(1 - a) & 0 & 0 & 0
\end{pmatrix}. \tag{4}
\]

The PGFs of the absorption times are given by \( g(t, v) = (E - Q(t, v))^{-1}u(t, v) \). If we set \( v = 1 \) in this expression, i.e. if we ignore the additional information, we get the less detailed PGF from the previous case. Taking into account that \( \bar{k} = g'_c(1,1) \) and \( l = g'_c(1,1) \), we obtain \( T = \bar{k}\tau + ((1 - a)/a)\tau' \), where \( \bar{k} \) is given by Eq. (3). The variance \( \sigma^2 \) can be computed in the usual way, and the only new part is computing \( \tilde{g}'_c(1,1) \). For small \( p \) and \( a \) the main contribution to \( T \) is given by the first term, \( \bar{k}\tau \), see the Supplemental Material for more details. There we also compare our exact results (without classical communication) with the approximations used e.g. in Refs. [15, 28–35].

Finite memory times. We now show that the more general case of finite memory times also fits into our Markov chains formalism. For two segments with a finite memory of \( m \) time units the corresponding Markov chain is shown in Fig. [3]. In the case of \( m = 0 \) this chain degenerates to Fig. [10a] with success probability \( p^2 \). The case \( m = +\infty \) corresponds to the perfect \( \infty \)-memory case in Fig. [10b]. For the general chain the absorption time is
\[
\mu = (1 + 2q - 2q^{m+1})/(p(1 + q - 2q^{m+1})),
\]
and this expression coincides with the result obtained in [36]. We could further extend this approach by adding a non-deterministic swapping layer, as we have shown above.

Dynamical schemes. In the case of non-deterministic swapping there is no longer a single unique way to perform swapping. To demonstrate different strategies that can be used we start with three segments. There are two ways: (i) fixed – always swap segments 1-2 and, when successful, try to attach segment 3; (ii) dynamic – start with a pair of segments that can be swapped first (1-2 or 2-3). In the case of a tie (all three segments become ready simultaneously), we first try to swap 1-2, or we can arbitrarily choose one pair to break this tie. The difference between the dynamic and fixed ways is illustrated in Fig. [4] (dashed and solid lines, respectively). It can be seen that the dynamic way is only marginally better than the fixed one.

In the case of four segments there are more choices. We can try to swap segments 1 and 2, then swap them with segment 3 and then add the last one. This way is the least efficient. Or we could independently try to swap two pairs 1-2 and 3-4, and then combine these pairs together. This way is the fastest. We could also dynamically try to perform swapping of any ready neighboring segments. The difference between the dynamic, best, and worst schemes is shown in Fig. [4] (dashed, solid, and dotted lines, respectively). This time, contrary to intuition, the dynamic way is not optimal and fits somewhere in be-
tween the other two ways. The reason is that in this case, we have one best way and several equivalent, equally bad ones. When we dynamically try to swap pairs of segments we sometimes follow the best way and sometimes one of the worst. The process is probabilistic, so in one run even the worst way may succeed much sooner than the best one, but on average, this strategy (combining the best and several bad ways) cannot outperform the single best way, which is illustrated by the figure. This situation is different from the previous case, where we had two equivalent ways, and combining those leads to a better one. For a general number of \( n \) segments the number of ways to swap grows exponentially with \( n \). In the case of \( n = 2^d \) there is one optimal way, where we recursively combine the first \( n/2 \) segments with the last \( n/2 \) segments. The waiting times for \( n = 8 \) and \( n = 16 \) are also shown in Fig. 4. Full probability distributions are shown in Fig. 5. We clearly observe that these distributions are highly non-bell-shaped and thus their average values and standard deviations alone are insufficient characteristics of the corresponding random process. More details are given in the Supplemental Material.

**Conclusion.** We have presented a general and systematic approach to compute the average waiting time of quantum repeaters with arbitrary numbers of links. Our approach is based on Markov chains theory and reduces the average waiting time to the absorption time of the corresponding absorbing Markov chain. Our main result is Eq. (A1), which allows one to easily reproduce known results and to get new ones for larger numbers of links. The non-deterministic swapping, finite memory and classical communications, like Lego bricks, can be combined in an arbitrary way. We also demonstrated that for small swapping probabilities the variance of the absorption time is close to the average time itself, so strong deviations from the average will happen quite often in experiments. In this regime the probability distribution of the waiting time is no longer bell-shaped.

**Appendix A: Average time, variance and probability distribution**

Here we show how to get the absorption time and its variance when all transitions contribute to the absorption time. In this case we have \( Q(t) = tQ \) and \( u(t) = tu \). The PGFs \( g(t) \) satisfy the system of linear equations

\[
(E - tQ)g(t) = t(E - Q)1. \quad (A1)
\]

It is obvious that \( g(1) = 1 \), as it must be for all PGFs. Differentiating both sides of Eq. (A1) by \( t \), we obtain

\[
- Qg(t) + (E - tQ)g'(t) = (E - Q)1. \quad (A2)
\]

Substituting \( t = 1 \) and taking into account that \( g(1) = 1 \), we have

\[
- Q1 + (E - Q)\bar{k} = (E - Q)1, \quad (A3)
\]

from which we derive the equality \( \bar{k} = (E - Q)^{-1}1 = R1 \).

To compute the variance, we need to compute the second derivatives. Differentiating Eq. (A2), we obtain

\[
-2Qg'(t) + (E - tQ)g''(t) = 0, \quad (A4)
\]

and thus \( g''(1) = 2RQ\bar{k} \). We then have

\[
g''(1) + g'(1) = (2RQ + E)\bar{k}, \quad (A5)
\]

and it is easy to verify that \( 2RQ + E = 2R - E \), so we arrive at the expressions given in the main text. Let us denote \( p_\succ(k) = P(X > k) \) and the corresponding generating functions

\[
f(t) = \sum_{k=0}^{+\infty} p_\succ(k)t^k. \quad (A6)
\]

It is easy to see that the standard PGF can be expressed in terms of this GF as \( g(t) = 1 - (1 - t)f(t) \). For the random variables under discussion we explicitly get this GF from Eq. (A1)

\[
f(t) = (E - tQ)^{-1}1. \quad (A7)
\]

For the \( k \)-th derivative we have

\[
f^{(k)}(t) = k!(E - tQ)^{-(k+1)}Q^k1, \quad (A8)
\]
from which we immediately obtain the probabilities $$p>(k) = \{P(k_i > k)\},$$

$$p>(k) = \frac{f(k)(0)}{k!} = Q^k1.$$  \hspace{1cm} (A9)

The probabilities $$p(k+1) = \{P(k_i = k+1)\}$$ can be computed as

$$p(k+1) = p>(k) - p>(k+1) = Q^k(E - Q)1 = Q^k u,$$

which proves the statement in the main text. One can easily check that these probabilities lead to the same expression for $$\bar{k}$$ and $$\sigma$$ that have been obtained before.

### Appendix B: Deterministic swapping

In this case it is possible to obtain the solution with a simple combinatorial reasoning. The probability that a single segment will finish in $$k$$ or less steps is $$1 - q^k$$, thus the probability that $$n$$ segments will finish in no more than $$k$$ steps is $$(1 - q^k)^n$$. For the cumulative PGF of the absorption time we have

$$c(t) = \sum_{k=0}^{+\infty} (1 - q^k)^n t^k = \sum_{j=0}^{n} (-1)^j \binom{n}{j} \frac{1}{1 - qt},$$  \hspace{1cm} (B1)

from which we easily obtain the ordinary PGF and then

$$\bar{k} = \sum_{j=1}^{n} (-1)^{j+1} \binom{n}{j} \frac{1}{1 - q^j}. \hspace{1cm} (B2)$$

This expression has already been obtained in [23].

We show that this solution agrees with our general method based on Markov chains theory. Here we use subset notation. The transition probability $$p_{IJ}$$ from the state labeled by the subset $$I$$ to the state $$J$$ is given by

$$p_{IJ} = \begin{cases} 
0 & I \not\subseteq J \\
p_j |J| |q^n - |J| & I \subseteq J 
\end{cases}, \hspace{1cm} (B3)$$

where $$q = 1 - p$$. We first show that all probabilities for a fixed subset $$I$$ sum up to one and that the PGFs satisfy Eq. (A1). In fact, we have

$$\sum_{J \subseteq [n]} P_{IJ} = \sum_{J \supseteq [I]} P_{IJ} = \sum_{j=|I|}^{n} \binom{n-|I|}{j-|I|} p_j q^{n-j}$$

$$= \sum_{j=0}^{n-|I|} \binom{n-|I|}{j} p_j q^{n-|I|-j} = (p + q)^{n-|I|} = 1.$$  \hspace{1cm} (B4)

The PGF $$g_I(t)$$ is equal to PGF $$g_{\emptyset}(t)$$ where $$n$$ is replaced by $$n - |I|$$, because the PGF of the $$n$$-system where $$|I|$$ segments are ready is the same as the PGF of the system

| $$n$$ | $$\bar{k}$$, double precision | $$\bar{k}$$, true |
|-------|----------------------------|
| 58    | -2.0                       | 2.50             |
| 59    | 4.0                        | 2.51             |
| 60    | -14.0                      | 2.52             |

### Table I: The quantity $$\bar{k}$$, given by Eq. (B2), for some values of $$n$$. The success probability is $$p = 0.9$$ ($$q = 0.1$$).

We show that these PGFs satisfy Eq. (A1), where the elements $$p_{IJ}$$ of $$Q$$ are given by Eq. (B3), for all $$I, J \subseteq [n]$$. In other words, the matrix $$Q$$ is obtained from the full matrix $$P$$ by removing its last row and last column. We have to prove that

$$g_I(t) - t \sum_{[n] \supseteq J \supseteq I} p_{IJ} q^{n-|J|} g_J(t)$$

$$= t(1 - \sum_{[n] \supseteq J \supseteq I} p_{IJ} q^{n-|J|}) = tp^{n-|I|}. \hspace{1cm} (B6)$$

We have

$$\sum_{[n] \supseteq J \supseteq I} p_{IJ} q^{n-|J|} (1 - q^k)^{n-|I|}$$

$$= \sum_{j=|I|}^{n-1} \binom{n-|I|}{j-|I|} p_j q^{n-j} (1 - q^{k+1})^{n-|I|} \hspace{1cm} (B7)$$

$$= (1 - q^{k+1})^{n-|I|} - p^{n-|I|}.$$  \hspace{1cm}

Multiplying by $$t \cdot t^k = t^{k+1}$$, summing up, and multiplying this by $$(1 - t)$$, we get

$$t \sum_{[n] \supseteq J \supseteq I} p_{IJ} q^{n-|J|} g_J(t) = g_I(t) - tp^{n-|I|}, \hspace{1cm} (B8)$$

from which we obtain Eq. (B6). So, we have just proved that the PGFs satisfy Eq. (A1). However, note that our general approach is exactly the opposite — starting with Eq. (A1) we derive expressions for PGFs.

### Appendix C: Numerical computing

One must be careful when numerically computing the average absorption time, Eq. (B2), in the case of deterministic swapping for large values of $$n$$ (typically for $$n > 50$$). Computing this expression with standard double precision for $$p = 0.9$$ gives the results presented in the second column of Table I. The negative values are obviously wrong, but the positive value is also incorrect. The
problem is in loss of precision, and to get correct results one has to use multiple precision arithmetic. The correct values are shown in the third column (these values are rounded to two digits). Note that this erroneous behaviour does not depend on the programming language or computer algebra system, it is a feature of limited precision of floating point arithmetic and is reproducible on any hardware that conforms to the IEEE 754 standard.

Appendix D: Quantum repeaters with three segments

Here we consider perfect, $\infty$-time memories and ignore the classical communication times. As we said in the main text, for three segments there are two ways to perform swapping. In the first way we always follow a fixed path: either we first swap segments 1 and 2, and when we are successful we try to attach segment 3 to them, or we start with segments 2 and 3 and then attach segment 1. These two approaches are equivalent (the probabilistic properties are the same), so for definiteness we start with segments 1-2. We refer to this way as 2|1. We have eleven states: 000, 001, 010, 011, 100, 101, 110, 111, 11\overline{1}, \overline{1}0, \overline{1}1, \overline{1}1. The last state is absorbing. The bar on top of the ones means that the corresponding segments have been successfully swapped. For example, in the state 110 the entanglement in the first two segments has been distributed and the states are ready to attempt swapping. Depending on the result of this attempt this state can either go to the state 11\overline{1} with probability $a$ or to the state 001 (with probability $1-a$). Since here we ignore the time it takes to perform the swapping, such transitions do not contribute to the absorption time. Note that in the case of a tie, 111, we choose to try to swap segments 1-2, so this state either goes to 11\overline{1} (with probability $a$) or to 001 (with probability $1-a$). There is no state 11\overline{1}, since in the state 011 we do not swap segments 2-3; we have to wait until segment 1 is ready and in such a case we have the tie that has already been discussed. The matrix $Q_{2|1}$ (corresponding to transition probabilities between non-absorbing states) reads as (where the order of the states is the same as listed above),

$$Q_{2|1} = \begin{pmatrix} (1-p)^3 & p(1-p)^2 & p(1-p)^2 & p^2(1-p) & p(1-p)^2 & p^2(1-p) & p^2(1-p) & p^3 & 0 & 0 \\ 0 & (1-p)^2 & 0 & p(1-p) & 0 & p(1-p) & 0 & 0 & 0 & 0 \\ 0 & 0 & (1-p)^2 & p(1-p) & 0 & 0 & p(1-p) & 0 & 0 & 0 \\ 0 & 0 & 0 & 1-p & 0 & 0 & 0 & 0 & a & 0 \\ 1-a & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1-a & 0 & 0 & 0 & 0 & 0 & 0 & 1-p & p \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1-p & p \\ 1-a & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} . \quad (D1)$$

The matrix $Q_{2|1}(t)$ is given by

$$Q_{2|1}(t) = \begin{pmatrix} (1-p)^3 t & p(1-p)^2 t & p(1-p)^2 t & p^2(1-p) t & p(1-p)^2 t & p^2(1-p) t & p^2(1-p) t & p^3 t & 0 & 0 \\ 0 & (1-p)^2 t & 0 & p(1-p) t & 0 & p(1-p) t & 0 & 0 & 0 & 0 \\ 0 & 0 & (1-p)^2 t & p(1-p) t & 0 & 0 & p(1-p) t & 0 & 0 & 0 \\ 0 & 0 & 0 & (1-p) t & 0 & 0 & 0 & 0 & pt & 0 \\ 1-a & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & a \\ 0 & 1-a & 0 & 0 & 0 & 0 & 0 & 0 & 1-p & pt \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1-p & pt \\ 1-a & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} . \quad (D2)$$

The PGF can be obtained from our general expression \[A1\] and the absorption time in this case becomes

$$\overline{k} = \frac{a^2 (p^4 - 5p^3 + 10p^2 - 10p + 4) + a (2p^4 - 9p^3 + 17p^2 - 16p + 6) - 4p^3 + 16p^2 - 23p + 12}{a^2(p-2)p (a(p^3 - 3p^2 + 4p - 2) - 2p^2 + 5p - 4)} . \quad (D3)$$
One can confirm that for \( a = 1 \) this expression coincides with Eq. (B2) for \( n = 3 \).

In the second way of performing swapping we start with the pair of segments that is ready, and in the case of a tie (when all three segments are ready) we choose to try to swap segments 1 and 2 first. We show that it does not matter how we break the tie. We refer to this way of performing swapping simply as 3. In this case we have thirteen states: 000, 001, 010, 011, 100, 101, 110, 111, 001, 110, 111, 011, 111. Note that here we have two additional states 011 and 111 in comparison with the previous case. These states exist since here we do not wait for the first two segments and can swap the last two immediately when they happen to be ready. The matrix \( Q_3 \) reads as

\[
Q_3 = \begin{pmatrix}
(1-p)^3 & p(1-p)^2 & p(1-p)^2 & p(1-p)^2 & p^2(1-p) & p^2(1-p) & p^2(1-p) & p^2(1-p) & p^3 & 0 & 0 & 0 & 0 \\
0 & (1-p)^2 & 0 & p(1-p) & 0 & p(1-p) & 0 & p(1-p) & 0 & p^2 & 0 & 0 & 0 \\
1-a & 0 & 0 & 0 & 0 & 0 & 0 & 0 & a & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & (1-p)^2 & p(1-p) & p(1-p) & p^2 & 0 & 0 & 0 & 0 & a \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1-a & 0 & 0 & 0 & 0 & 0 & 0 & 0 & a & 0 & 0 & 0 & 0 \\
0 & 1-a & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1-a & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1-a & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{pmatrix}.
\]

The matrix \( Q_3(t) \) can be obtained from \( Q_3 \) by marking all transitions whose probabilities depend on \( p \) and multiplying the corresponding entries by \( t \). For the absorption time we then have

\[
\bar{\kappa} = \frac{a^2 \left(p^3 - 4p^3 + 6p^2 - 5p + 2\right) + a \left(2p^3 - 10p^3 + 21p^2 - 22p + 9\right) - 4p^3 + 16p^2 - 22p + 11}{a^2(p-2)p(a(p^3 - 2p^2 + 2p - 1) - 3p^2 + 7p - 5)}.
\]

Appendix E: Quantum repeaters with four segments

Here we again consider perfect, \( \infty \)-time memories and ignore the classical communication times. In the case of four segments there are three different ways to perform swapping. The first way is to always wait for the first two segments, 1 and 2, try to perform swapping on these, in the case of success try to attach segment 3, and if this attempt is successful try to attach segment 4. Note that we could also start with segment 3 and 4, and try to attach segment 2 and 1 (in this order), or we could start with segments 2 and 3, and then try to attach segments 1 and 4 — all these ways are equivalent from the probabilistic point of view.

A more efficient way is to independently swap the pairs of segments 1-2 and 3-4 and, when both swapping have been successful, try to swap these pairs. The corresponding matrices are rather large, so we present only the absorption times. For these two schemes we have, respectively
where these matrices it can be shown that

\[
\kappa = \frac{a^2 (p^5 - 4p^4 + 9p^3 - 12p^2 + 9p - 3) + a(p-1) (p^2 - 2p + 2)^2 + 2p^5 - 13p^4 + 37p^3 - 57p^2 + 48p - 18}{a^3(p-2)p(p^2 - 3p + 3)(p^2 - 2p + 2)}, \tag{E1}
\]

Though these expressions are not extremely large, it may be less error-prone to solve the corresponding systems of linear equations numerically than to use the expressions directly. For the dynamic scheme (when we immediately try to swap the segments that are ready) the explicit expression is a few times larger than the above two, so we do not present it here (nonetheless we used that large expression obtained with Mathematica to plot the curve shown in the main text).

### Appendix F: Quantum repeaters with finite memory

In the case of finite memories (provided that \(m > 0\)) the matrix \(E - Q\) and vector \(u\) can be written as

\[
E - Q = \begin{pmatrix}
1 - q^2 & -pq & 0 & \ldots & 0 & -pq & 0 & \ldots & 0 \\
0 & 1 & -q & \ldots & 0 & 0 & 0 & \ldots & 0 \\
\vdots & \vdots & \ddots & \ddots & \vdots & \ddots & \ddots & \ddots & \vdots \\
0 & 0 & 0 & \ldots & -q & 0 & 0 & \ldots & 0 \\
-q & 0 & 0 & \ldots & 1 & 0 & -q & \ldots & 0 \\
0 & 0 & 0 & \ldots & 0 & 1 & -q & \ldots & 0 \\
\vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\
0 & 0 & 0 & \ldots & 0 & 0 & 0 & \ldots & -q \\
-q & 0 & 0 & \ldots & 0 & 0 & 0 & \ldots & 1 \\
\end{pmatrix}, \quad u = (E - Q)1 = \begin{pmatrix}
p^2 \\
p \\
\vdots \\
p \\
p \\
\vdots \\
p \\
p \\
\end{pmatrix} \tag{F1}
\]

To compute the determinant of \(E - Q\), we expand it with respect to its first row. We have

\[
\det(E - Q) = 1 - q^2 + pqD_1 + (-1)^m pqD_2, \tag{F2}
\]

where \(D_1\) and \(D_2\) are the determinants of the matrices obtained from \(E - Q\) by removing the first row and column 2 and \(m + 2\) respectively. By simple manipulations with these matrices it can be shown that

\[
D_1 = -q^m, \quad D_2 = (-1)^{m+1} q^m. \tag{F3}
\]

Combining these results, we obtain

\[
\det(E - Q) = 1 - q^2 - 2pq^{m+1} = p(1 + q - 2q^{m+1}). \tag{F4}
\]

This gives us the denominator. The numerator can be computed in a similar way, and leading to the expression as presented in the main text.

### Appendix G: Comparison with known results

As we have already demonstrated in the main text, our approach correctly reproduces all known exact results. In this section we compare the exact results with widely used approximations and demonstrate that those approximations can be used only in a narrow region of small \(p\) and \(a\) probabilities and that this region shrinks as the number of segments increases.

The well-known approximate expression for the waiting time of a repeater with \(n = 2^d\) segments (for the most efficient scheme) reads as

\[
\kappa \approx \left( \frac{3}{2a} \right)^d \frac{1}{p} \equiv \kappa_{\infty}. \tag{G1}
\]

Fig. 5 shows the quality of this approximation for different values of \(n\). This approximation is used and discussed in Refs. [15, 28, 32] without an explicit comparison with exact expressions (while the regime of interest in those references was that of very small probabilities and it is pointed out there that the approximation only holds in that regime). Here we can eventually perform such a comparison based on our exact results. It would be a mistake to use this approximation in the deterministic case \((a = 1)\). Fig. 7a illustrates that in this case the expression (G1) is not even asymptotically exact, except in the simplest case of \(n = 2\). It is asymptotically exact only when both \(p\) and \(a\) are small, as illustrated in Fig. 7b for the case of \(p = a\).

The expression for the waiting time of a quantum repeater with an arbitrary number of \(n\) segments (not necessarily power of 2) was obtained in Ref. [23] and is given by [192]. In the more recent Refs. [34, 35], the following
expression was used:

$$\bar{k} = \sum_{j=1}^{n} \frac{1}{1 - q^j}. \quad \text{(G2)}$$

This expression is indistinguishable from Eq. (B2) and works very well for sufficiently small $p$, i.e., near-unity $q$ (which is the regime of interest in Refs. [34, 35]). However, for larger $p$, the expression ceases to work and no longer coincides with the exact expression in Eq. (B2) that works for arbitrary $q$ values. In fact, in the extreme case of $q = 0$ ($p = 1$), one can easily see that the expression in Eq. (G2) is wrong, because then we should have $\bar{k} = 1$, but this expression gives $\bar{k} = n$. The exact expression for deterministic swapping in Eq. (B2) can be obtained as a special case from our general rate analysis (see main text and below).

In the main text we mentioned that attaching segments (i.e. performing swapping) one by one is the least efficient way to distribute entanglement among those schemes that we consider. However, there is an even less efficient scheme. That scheme is equivalent to first distributing entanglement in all the segments and then trying to perform all the swappings simultaneously, which means exploiting quantum memories only for the initial distribution and completely wasting it for the swappings. If at least one swapping fails then the whole process (including entanglement distribution) starts from scratch. If the swapping probability is $a$ then, having successfully distributed entanglement, this process succeeds with probability $a^{n-1}$, so the entanglement distribution should be tried $1/a^{n-1}$ times on average. We conclude that the waiting time in this scheme is $K = \frac{\bar{k}}{a^{n-1}}$, which increases exponentially with distance, whereas in the fastest case (entanglement doubling) the waiting time grows polynomially (since $n \sim L$ and $d \sim \ln n \sim \ln L$). Let us compare $K$ with that of two other schemes: the fastest scheme and the one-by-one attaching schemes.

It can be shown that the PGF of a nondeterministic one-by-one attaching quantum repeater with two segments reads as

$$g_2(t) = \frac{ap^2t(1+qt)}{1-(2-3p+(2-a)p^2)t+q(1-2p+ap^2)t^2}, \quad \text{(G3)}$$

and for $n > 2$ segments it is given by the following recur-
The number of possibilities, which corresponds to the $n$-th coefficient of $V(z)$, grows exponentially

$$ [z^n V(z)] \sim \lambda^n \beta^n n^{-3/2}, \quad (H2) $$

where $\lambda$ and $\beta$ are some constants. The performance corresponding to these trees depends on their depth. For the case of $n = 2^d$ there is a unique perfectly balanced tree of the smallest depth and a correspondingly optimal way to perform swapping.

FIG. 9: Different fixed ways to perform swapping in quantum repeaters with small number of segments.

The size of the Markov chain (and thus the size of the corresponding system of linear equations) describing a quantum repeater with $n$ segments depends on the scheme. For the most efficient scheme (if $n = 2^d$ is a power of two) this size grows as $2^n$, i.e. exponentially with $n$. However, the Markov chain given in the main text is not the most efficient one. Using a technique known as “lumping”, we can construct a smaller chain which still correctly describes the random process. This technique can only be applied to Markov chains with some degree of symmetry. For the perfectly balanced case of $n = 2^d$ the corresponding Markov chains exhibit a high degree of symmetry, so they can be represented much more compactly. The idea is to combine several states into one state, and then recompute the transition probabilities appropriately. The details when this can be done and how to recompute probabilities are given in Ref. [35].

Appendix H: Arbitrary number of segments and full distributions

In general there are many possibilities to combine neighboring segments in a fixed way. The number of these ways for $n$ segments is equal to the number of non-plane binary trees of size $n$. For some small $n$ these trees are shown in Fig. 9. The generating function of these numbers $V(z)$ satisfies the equation

$$ V(z) = z + \frac{1}{2} V^2(z) + \frac{1}{2} V(z^2). \quad (H1) $$

\[ \text{FIG. 8: The absorption time for three different schemes for } n = 4 \text{ segments, } p = 0.1. \]
We illustrate this with a simple example. For \( n = 2 \) segments the original Markov chain has \( 2^2 = 4 \) states: 00, 01, 10 and 11. The states 01 and 10 are symmetric and can be considered identical, so they can be lumped into one state (one can check that the corresponding lumpability conditions are satisfied) and we get three states \( s_1 = 00, s_2 = 01|10 \) and \( s_3 = 11 \). When we go to the next step, we would have \( 2^3 = 9 \) states \( s_is_j, i, j = 1, 2, 3 \), but the states \( s_is_j \) and \( s_js_i \) for \( i \neq j \) can be combined into one state, so we get only 3 \( \cdot \) \( (3 + 1)/2 = 6 \) states. This process can be repeated on any level. For \( n = 2^d \) we had \( N_1 = 2^2 \) and \( N_{d+1} = N_d^2 \) and thus \( N_d = 2^n = 2^{2^d} \). For \( n = 16 = 2^d \) we had \( N_4 = 2^{16} = 65536. \) Now we have \( N_1 = 3 \) and \( N_{d+1} = N_d(N_d + 1)/2 \). There is no explicit expression for this sequence, but it grows more slowly than the previous one. For \( n = 16 \) we have \( N_4 = 231 \). Thus, instead of solving a system with \( > 65000 \) equations, we need to solve a system of only 230 equations, which is a rather trivial task. The curves presented here for \( n = 4, 8, 16 \) have been obtained with this Markov chain compression trick.

The lumping process greatly reduces the number of states, but it can be applied only if the Markov chain has some symmetry that can be exploited. For the case of \( n = 2^d \) this symmetry is maximal, so in this case we can dramatically reduce the size of the chain. Even though this size (our new \( N_d \)) still grows exponentially as \( \approx 2 \cdot 1.346^n \) [38], this trick can substantially simplify the task and make an intractable problem (on a common hardware) quite trivial.

We can also easily compute the full probability distribution, not only the average value. Two examples are shown in Fig. 10. We mentioned in the main text that for small success probabilities \( p \) and \( a \) the variance becomes comparable with the average value itself and thus the average is not a good characteristic of the process. We illustrate this issue for the case of \( n = 16 \). The probabilities \( p_{>}(k) = Q^k1 \) can be easily computed numerically, since the size of \( Q \) is just 230 (we need only the first element of this vector, which we denote \( p_{>}(k) \)). For example, for \( p = a = 0.5 \) we have \( \overline{k} \approx 122.62 \) and \( p_{>}(250) \approx 0.11 \). This means that every tenth attempt will take at least twice as long as the average time. For \( p = a = 0.1 \) this probability is \( \approx 0.12 \), and for \( p = a = 0.01 \) it is \( \approx 0.13 \). Every eighth attempt will take more than twice as long as the average. By the way, in the last case we have \( \overline{k} \approx 5 \cdot 10^{10}, \) so to compute \( p_{>}(2\overline{k}) \) one has to compute \( Q^{10^{11}} \). For a matrix of size 230 this computation (using NumPy) happens instantaneously, which demonstrates the power of the compression trick applied to the Markov chain. It clearly shows that for quantum repeaters of a certain size it is no longer necessary to rely on the ap-
proximation (II1), since our method allows one to easily compute the full probability distribution of a quantum repeater with up to 16 segments. For 32 and more segments that approximation (which works only for powers-of-two numbers) is precise only for small probabilities \( p \) and \( a \), for which the waiting time is impractically large. In a more application-oriented and practical regime of the repeater probabilities and for large numbers of segments the approximation is inapplicable.

Whether the above statistical behavior is acceptable or not depends on the specific application. Our analysis clearly demonstrates that large deviations cannot be simply ignored and the average value becomes an insufficient characteristic of the process, as Figs. II1c II1d illustrate. For truly reliable systems even the knowledge of the variance will be not enough and more detailed information might be needed. Our approach has unique power to obtain not only some integral properties like average time and its variance, but also the full probability distribution and thus it gives insight into deeper properties of the process.

Appendix I: Classical communication in quantum repeaters

Here we show that the relative effect of the classical communication time is more pronounced for large probabilities. Like in the main text, for simplicity, first we ignore the times needed for successful swapping attempts and count only the time needed to restart the process in the case of failure. We assume that for two segments it takes one unit of time and for four segments it takes two units of time (in general, this time doubles when we double the number of segments). We cannot use the relation \( \overline{K} = R\mathbf{1} \), since it was obtained under the assumption that all transitions contribute to the absorption time, and this assumption no longer holds here. In the present simple case the vector \( \mathbf{u} \) does not depend on \( t \), because we ignore the time for successful swapping. Modifying the proof given in the first section of this supplemental material, one can obtain the relation for \( \overline{K} \) in such a case:

\[
\overline{K} = RQ_0 \mathbf{1}, \tag{I1}
\]

where \( Q_0 \) is obtained from \( Q \) by zeroing the elements not marked by \( t \). It can be shown that the relation

\[
T = K\tau + \frac{1 - a}{a}\tau', \tag{I2}
\]

obtained in the main text for two segments, can be extended for the general case of combining two identical systems. In general, it reads as

\[
T = T_0 + \frac{1 - a}{a}\tau', \tag{I3}
\]

where \( T_0 \) is the average waiting time without classical communication, and \( \tau' \) is the time to restart the process in the case of failure on the outermost level. Combining all these results, we can compute the waiting time with classical communication. For two and four segments it is shown in Fig. II1. For two segments the expression for the waiting time with classical communication becomes

\[
T = \left( \frac{3 - 2p}{ap(2 - p)} + \frac{1 - a}{a} \right) \tau. \tag{I4}
\]

For four segments the expression is large, so we do not present it here. Except for two segments, it is almost always more efficient to solve the corresponding system of linear equations than trying to obtain an explicit expression. One can see that for small probabilities the entanglement distribution time gives the main contribution to the total waiting time. Conversely, for larger probabilities, the classical communication times matter much more (see Fig. II1).

We note that usually in a symmetric quantum repeater based on distance doubling, the times needed to classically communicate results of an entanglement swapping attempt can be identical for success and failure events. In this case Eq. (I3) should be modified as

\[
T = T_0 + \frac{1}{a}\tau', \tag{I5}
\]

where \( \tau' = 2^{d-1}\tau \), and \( n = 2^d \) is the number of segments. In general, there are many possibilities, so we cannot give concrete results for all of them. We show that our approach is powerful enough to easily treat most, if not all, practical situations by combining simpler schemes into more complicated ones and further refining them by adding classical communication and finite memory.

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