ON THE ASYMPTOTIC PROPERTIES FOR STATIONARY SOLUTIONS TO THE NAVIER-STOKES EQUATIONS

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Abstract. In this paper we study solutions of the stationary Navier-Stokes system, and investigate the minimal decay rate for a nontrivial velocity field at infinity in outside of an obstacle. We prove that in an exterior domain if a solution \( v \) and its derivatives decay like \( O(|x|^{-k}) \) for sufficiently large \( k \), depending on the velocity field, as \( |x| \to \infty \), then \( v \) is zero on that exterior domain. Constructive estimate for \( k \) is given. In the case where velocity field is only bounded at infinity, we show that the infimum of \( L^2 \) norm of a velocity field on a unit ball located at distance \( t \) from an origin is bounded from below as \( Ce^{-\beta t^{\frac{3}{4}} \ln(t)} \). The proof of these results are based on the Carleman type estimates, and also the Kelvin transform.

1. Introduction. Let \( \Omega \) be a simply connected open, bounded subset of \( \mathbb{R}^3 \) containing the origin. We consider the stationary Navier-Stokes equations in an exterior domain \( \mathbb{R}^3 \setminus \overline{\Omega} \):

\[
\begin{aligned}
- \Delta v + (v, \nabla) v &= - \nabla p \quad \text{in} \quad \mathbb{R}^3 \setminus \overline{\Omega}, \\
\text{div } v &= 0 \quad \text{in} \quad \mathbb{R}^3 \setminus \overline{\Omega},
\end{aligned}
\]

where \( v = (v_1, v_2, v_3) = v(x) \) and \( p = p(x) \), are the velocity and the scalar pressure of the fluid flows respectively. In the following we consider problem of behaviors at spatial infinity of a nontrivial solutions of (1) in the exterior domain, \( \mathbb{R}^3 \setminus \overline{\Omega} \). The first result in this direction was obtained by Finn [5], who showed that a velocity field satisfying the zero Dirichlet boundary condition on \( \partial \Omega \), which decays at spatial infinity as \( o(|x|^{-1}) \), is trivial. In the case without any boundary condition on \( \partial \Omega \), Dyer and Edmunds [4] proved that any \( C^2 \) bounded velocity field decaying at spatial infinity as \( O(\exp(-\exp(s|x|^3))) \) for any positive \( s \) is trivial.

One can construct nontrivial solution of the (1) in any exterior domain, having a zero of any sufficiently large order at spatial infinity. Indeed, given \( k \geq 1 \) let us set

\[
\phi_k(x) = \partial_{x_1}^k \frac{1}{|x|},
\]

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Then, $\phi_k$ is harmonic on $\mathbb{R}^3 \setminus \{0\}$, and $|\phi_k(x)| = O(|x|^{-k-1})$ as $|x| \to \infty$. For any domain $\Omega$ containing the origin the pair $(v, p)$ defined by

$$v = \nabla \phi_k, \quad p = -\frac{1}{2} |\nabla \phi_k|^2$$

is a smooth solution to (1) on $\mathbb{R}^3 \setminus \overline{\Omega}$, which has zero of order $k + 1$ at spatial infinity. In the theorem below we try to estimate the decay rate on infinity for nontrivial, non-gradient solutions to (1) based on some properties of this solution.

Below we denote

$$B(0, r) = \{ x \in \mathbb{R}^3 \mid |x| < r \}, \quad S(0, r) = \{ x \in \mathbb{R}^3 \mid |x| = r \}.$$

**Definition 1.1.** Let $\Omega$ be a bounded domain and $w \in H^2(\mathbb{R}^3 \setminus \overline{\Omega})$. We say $w$ has a zero of order $s_*$ at spatial infinity if

$$\lim_{r \to +\infty} \int_{S(0, r)} (|\nabla w(x)|^2 |x|^{2s_*} + |w(x)|^2 |x|^{2s_* - 2}) dS < +\infty.$$

Let $\Delta_\omega$ is the Laplace-Beltrami operator on $\mathbb{S}^2$. We set

$$\Lambda = \sqrt{-\Delta_\omega} + \frac{1}{4}.$$

Let $v$ be a nontrivial, non-gradient solution of (1) on $\mathbb{R}^3 \setminus \overline{\Omega}$. We introduce parameter $\gamma(\text{curl} v)$ as

$$\gamma = \gamma(\text{curl} v) = \lim_{r \to +\infty} \| \Lambda(\text{curl} v) \|_{L^2(S(0, t))}/\| \text{curl} v \|_{L^2(S(0, t))}. \quad (3)$$

We have

**Theorem 1.2.** Let $\Omega \subset \mathbb{R}^3$ be a bounded domain containing the origin, and $v$ be a nontrivial, non-gradient solution of (1) on $\mathbb{R}^3 \setminus \overline{\Omega}$ for some positive $\epsilon$ satisfying

$$|v(x)| = O\left( |x|^{-1-\epsilon} \right), \quad |\nabla v(x)| = O\left( |x|^{-2-\epsilon} \right) \quad \text{as} \quad |x| \to \infty. \quad (4)$$

Then $\gamma \in (0, +\infty)$ and solution $v$ can not have zero at spatial infinity of order $s_*$ greater then $\gamma + 1$.

Proof of theorem 1.2 is based on the analysis of the vorticity equation (66), using Carleman estimate with boundary with the logarithmic weight function. Decay assumption (4) on velocity field allow us to treat first order terms in the vorticity equation (66) as a perturbation and use factorization of the Laplace operator into product of two first order operators.

Let us set

$$M(t) = \inf_{|x|=t} \int_{|x-y|<1} |v(y)|^2 dy.$$

This function was introduced in paper [1] to study the localization effect for the Schrödinger equation in $\mathbb{R}^n$. In particular they proved that the minimal decay rate on infinity for nontrivial solution is $M(t) \geq C e^{-\beta t} t^n$ with some positive $\beta$. Several efforts have been done to extend such a result for the stationary Navier-Stokes and the Stokes systems. In [11] the authors proved that the decay rate for $M(t)$ is greater then double exponent, and in [12] they improved the estimate $M(t) \geq C e^{-Ct^2} t^n$ for some positive $C, \hat{C}$. In this paper this result is improved in the following two theorems: In order to formulate the first theorem we introduce parameter $\tilde{\gamma}$:

$$\tilde{\gamma} = \tilde{\gamma}(v) = \lim_{t \to +\infty} \| \Lambda v \|_{L^2(S(0, t))}/\| v \|_{L^2(S(0, t))}. \quad (5)$$

We have
Theorem 1.3. Let $\Omega \subset \mathbb{R}^3$ be a bounded domain containing the origin, and $v$ be a nontrivial solution of (1) on $\mathbb{R}^3 \setminus \overline{\Omega}$ satisfying
\[ |v(x)| = O(|x|^{-3}), \quad |\nabla v(x)| = O(|x|^{-4}) \quad \text{as } |x| \to \infty. \quad (6) \]
Then there exist $k > 0$ depending on $v$ and a constant $t_0$ depending on $v$ such that
\[ \sqrt{M(t)} \geq \frac{1}{t^k} \quad \forall t \geq t_0, \quad (7) \]
If $\text{curl} \, v$ not identically equal to zero, then in inequality (7) one can take
\[ k = 4s + 1, \quad s > \max\{1, \mu\}, \quad (8) \]
where $\mu$ is the maximal root of the polynomial $16\lambda^2 + 8\lambda + 11 - 4\gamma^2$ and parameter $\gamma$ defined by (3).
If $\text{curl} \, v \equiv 0$ in inequality (7) one can take $k$ defined by (8) where $\mu$ is the maximal root of the polynomial $16\lambda^2 + 8\lambda + 11 - 4\tilde{\gamma}^2$ and parameter $\tilde{\gamma}$ defined by (5).

For our next result we make no assumptions on decay of the velocity field on infinity.

Theorem 1.4. Let $\Omega \subset \mathbb{R}^3$ be a bounded domain containing the origin, and $v$ be a nontrivial solution of (1) on $\mathbb{R}^3 \setminus \overline{\Omega}$ satisfying
\[ |v(x)| = O(1) \quad \text{as } |x| \to \infty. \quad (9) \]
Then, there exist constants $C_0 = C_0(v)$, $\beta = \beta(v) > 0$ and $t_0 = t_0(v)$ such that
\[ M(t) \geq C_0 e^{-\beta t} \quad \forall t \geq t_0(v). \quad (10) \]

The estimate (10) proved in theorem 1.4 improves the corresponding estimate from [11]. The machinery used in the proof is the reduction of the stationary Navier-Stokes system to the Shrödinger equation (136). This allows us apply approach developed in [1]. Results of [1] can be applied directly if we consider the Navier-Stokes system in the whole space $\mathbb{R}^3$. In can when $\Omega$ is nonempty bounded set the technique of [1] requires some modification (see Appendix). Assumption on boundedness of domain $\Omega$ is essential. If domain $\Omega$ is unbounded the counterexample to estimate (10) can be easily constructed. The estimate obtained in theorem 1.3 is stronger then estimate (10). The main reason for the improvement is the following. Proof is based on the Carleman estimates with boundary and the decay assumption on the vector field (6) allow us to use $\ln |x|$ as a weight function in the Carleman estimate. The choice of such a function under assumptions of theorem 1.4 is impossible.

The following theorem states the asymptotic behavior of the pressure of the Navier-Stokes equations.

Theorem 1.5. Let a pair $(v, p)$ be a nontrivial solution to (1) on $\mathbb{R}^3$, satisfying
\[ |v(x)| + |p(x) - p_0| \to 0 \quad \text{as } |x| \to +\infty \]
for a constant $p_0$. Let us define
\[ \mathcal{M}(t) = \inf_{|x|=t} \int_{|x-y|<1} \frac{|p(y) - p_0|}{|x-y|} \, dy. \]
Then there exist positive constants $C_1 = C_1(v)$ and $C_2 = C_2(v)$ such that
\[ \mathcal{M}(t) \geq C_1 e^{-C_2 t} \quad \forall t \geq 0. \quad (11) \]
2. The Carleman estimates. In this section we obtain a Carleman type estimate for use in the proof of the main theorems of the previous section. A Carleman type estimates originally were introduced in [3] by T. Carleman in order to prove uniqueness of solutions for Cauchy problems for elliptic equations. Later Carleman estimates we obtained for general scalar partial differential equations under condition of existence of a pseudoconvex functions (see e.g. [8], [9]) and under more restrictive conditions Carleman estimates were obtained for systems of partial differential equations (see e.g. [2], [8]). Recently it was realized that Carleman estimates are very effective tool for proving observability estimates for solutions of evolution equations and consequently for obtaining exact boundary controllability results for evolution equations. This theory requires so-called Carleman estimates with boundary (survey of recent results can be found in [6].) In this section we will prove a Carleman estimate with boundary and with singular weight function. The main Carleman estimate is obtained in lemma 2.2 for the scalar Schrödinger equation. In proposition 1 this estimate is generalized for the elliptic system with the first order terms. Lemma 2.3 and proposition 2 are used in the proof of theorem 1.2 and proposition 5 is used in the proof of theorem 1.3. Proposition 3 and proposition 4 are of independent interest.

Let $\Omega$ be a bounded domain in $\mathbb{R}^3$ with $C^1$ boundary, which contains the origin. We consider the boundary value problem,

$$\Delta w = f \quad \text{in } \Omega, \quad w|_{\partial \Omega} = 0. \quad (12)$$

We define

$$L(s, x, D)u := |x|^{-2s}\Delta(|x|^{2s}u) = \Delta u + \frac{4s}{|x|^2} (x, \nabla u) + \frac{2s(1 + 2s)}{|x|^2} u,$$

and

$$\bar{L}(x, s, D)u := \Delta u - \frac{4s}{|x|^2} (x, \nabla u) + \frac{2s(3 + 2s)}{|x|^2} u.$$

We also set

$$L_+(x, s, D)u := \frac{1}{2} (L(x, s, D) + \bar{L}(x, s, D))u = \Delta u + \frac{4s + 4s^2}{|x|^2} u,$$

and

$$L_-(x, s, D)u := \frac{1}{2} (L(x, s, D) - \bar{L}(x, s, D))u = \frac{4s}{|x|^2} (x, \nabla u) - \frac{2su}{|x|^2}.$$

Definition 2.1. We say that a function $w \in H^2(\Omega)$ has a zero of order $s_0$ at the point $x = 0$ if there exists the following limit,

$$\lim_{r \to +0} \int_{S(0, r)} \left( \frac{|
abla w(x)|^2}{|x|^{2s_0}} + \frac{|w(x)|^2}{|x|^{2s_0 + 2}} \right) dS < +\infty.$$

Given a bounded domain $\Omega$, we define $\rho = \rho(\Omega)$ as follows

$$\rho(\Omega) = \inf\{r > 0 \mid \Omega \subset B(0, r)\}.$$ 

Our main purpose in this section is to prove the following:

Lemma 2.2. Let $\Omega$ be a bounded domain in $\mathbb{R}^3$ with $\partial \Omega$ belonging to $C^1$ class. Suppose $w \in H^2(\Omega)$ is a solution of (12), having zero at origin of the order $s_0$ for
We first have facts during computations. Let us compute the last term in this equality. We will repeatedly use the following

For all sufficiently small positive $s$ for all $u$

Then, we have the following estimate:

for all $s \in [1, \frac{2}{3} - \frac{1}{2})$ and $\epsilon \in (0, 1)$ provided that $|x|^{-2s} f \in L^2(\Omega)$.

**Proof.** For all sufficiently small positive $\delta$ we set $\Omega_\delta = \Omega \setminus B(0, \delta)$. We take the $L^2(\Omega_\delta)$ norm of the left and right hand side of (13):

Let us compute the last term in this equality. We will repeatedly use the following facts during computations.

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Using (20), we write the equality (15) as

\[ I_2(\delta) = 8s \int_{\Omega_\delta} \frac{1}{|x|^2} \Delta u (x, \nabla u) dx \]
\[ = 8s \int_{\partial \Omega \delta} \frac{1}{|x|^2} (x, \nabla u) \frac{\partial u}{\partial \nu} dS - 4s \int_{\Omega_\delta} \frac{1}{|x|^2} (x, \nabla |\nabla u|^2) dx \]
\[ -8s \sum_{i=1}^3 \int_{\Omega_\delta} \partial_{x_i} u (\partial_{x_i} (\frac{x}{|x|^2}), \nabla u) dx = 8s \int_{\partial \Omega \delta} \frac{1}{|x|^2} \left| \frac{\partial u}{\partial \nu} \right|^2 (x, \nu) dS \]
\[ -4s \int_{\partial \Omega \delta} \frac{1}{|x|^2} \left| \frac{\partial u}{\partial \nu} \right|^2 (x, \nu) dS + 4s \int_{\Omega_\delta} |\nabla u|^2 dx \]
\[ - \frac{8s}{\delta^3} \int_{S(0, \delta)} (x, \nabla u)^2 dS + \frac{4s}{\delta} \int_{S(0, \delta)} |\nabla u|^2 dS \]
\[ -8s \int_{\Omega_\delta} \frac{|\nabla u|^2}{|x|^2} dx + 16s \int_{\Omega_\delta} \frac{(x, \nabla u)^2}{|x|^4} dx. \quad (18) \]

\[ I_3(\delta) = -4s \int_{\Omega_\delta} \left( \Delta u + \frac{2s(1 + 2s)}{|x|^2} u \right) \frac{u}{|x|^2} dx \]
\[ = 4s \int_{\Omega_\delta} \left( \frac{|\nabla u|^2}{|x|^2} - \frac{2s(1 + 2s)}{|x|^4} u^2 \right) dx - 2s \int_{\Omega_\delta} u^2 \Delta \frac{1}{|x|^2} dx \]
\[ + \frac{4s}{\delta^3} \int_{S(0, \delta)} (\nabla u, x) u dS - \frac{4s}{\delta^3} \int_{S(0, \delta)} u^2 dS \]
\[ = 4s \int_{\Omega_\delta} \left( \frac{|\nabla u|^2}{|x|^2} - \frac{2s(1 + 2s)}{|x|^4} u^2 \right) dx - 4s \int_{\Omega_\delta} u^2 dx \]
\[ + \frac{4s}{\delta^3} \int_{S(0, \delta)} (\nabla u, x) u dS - \frac{4s}{\delta^3} \int_{S(0, \delta)} u^2 dS. \quad (19) \]

Therefore,

\[ I_1(\delta) + I_2(\delta) + I_3(\delta) + I_4(\delta) = 4s \int_{\partial \Omega \delta} \frac{1}{|x|^2} \left| \frac{\partial u}{\partial \nu} \right|^2 (x, \nu) dS \]
\[ + 16s \int_{\Omega_\delta} \frac{(x, \nabla u)^2}{|x|^4} dx - 4s \int_{\Omega_\delta} \frac{u^2}{|x|^4} dx \]
\[ + \frac{4s}{\delta^3} \int_{S(0, \delta)} (\nabla u, x) u dS - \frac{16s^3 + 8s^2 + 4s}{\delta^3} \int_{S(0, \delta)} u^2 dS \]
\[ - \frac{8s}{\delta^3} \int_{S(0, \delta)} (x, \nabla u)^2 dS + \frac{4s}{\delta} \int_{S(0, \delta)} |\nabla u|^2 dS. \quad (20) \]

Using (20), we write the equality (15) as

\[ |||x|^{-2s} f||^2_{L^2_2(\Omega_\delta)} = ||L_+(x, s, D) u||^2_{L^2_2(\Omega_\delta)} + ||L_-(x, s, D) u||^2_{L^2_2(\Omega_\delta)} \]
\[ -4s \int_{\Omega_\delta} \frac{u^2}{|x|^4} dx + 4s \int_{\partial \Omega \delta} \frac{1}{|x|^2} \left| \frac{\partial u}{\partial \nu} \right|^2 (x, \nu) dS + 16s \int_{\Omega_\delta} \frac{(x, \nabla u)^2}{|x|^4} dx \]
\[ + \frac{4s}{\delta^3} \int_{S(0, \delta)} (\nabla u, x) u dS - \frac{16s^3 + 8s^2 + 4s}{\delta^3} \int_{S(0, \delta)} u^2 dS \]
\[ - \frac{8s}{\delta^3} \int_{S(0, \delta)} (x, \nabla u)^2 dS + \frac{4s}{\delta} \int_{S(0, \delta)} |\nabla u|^2 dS. \quad (21) \]
Simple computations imply
\[
\|L_- (x, s, D)u\|_{L^2(\Omega_s)}^2 = 16s^2 \int_{\Omega_s} \frac{(x, \nabla u)^2}{|x|^4} \, dx + 4s^2 \int_{\Omega_s} \frac{u^2}{|x|^4} \, dx - 16s^2 \int_{\Omega_s} \frac{(x, \nabla u)u}{|x|^4} \, dx
\]
\[
= 16s^2 \int_{\Omega_s} \frac{(x, \nabla u)^2}{|x|^4} \, dx + 4s^2 \int_{\Omega_s} \frac{u^2}{|x|^4} \, dx - 8s^2 \int_{\Omega_s} \frac{(x, \nabla u)^2}{|x|^4} \, dx
\]
\[
= 16s^2 \int_{\Omega_s} \frac{(x, \nabla u)^2}{|x|^4} \, dx + 4s^2 \int_{\Omega_s} \frac{u^2}{|x|^4} \, dx + 8s^2 \int_{\Omega_s} u^2 \text{div} \, \frac{x}{|x|^3} \, dx
\]
\[
+ \frac{8s^2}{\delta^2} \int_{S(0, \delta)} u^2 \, dS
\]
\[
= 16s^2 \int_{\Omega_s} \frac{(x, \nabla u)^2}{|x|^4} \, dx - 4s^2 \int_{\Omega_s} \frac{u^2}{|x|^4} \, dx + \frac{8s^2}{\delta^2} \int_{S(0, \delta)} u^2 \, dS.
\]  
(22)

Using (22), we transform (21) into
\[
\| |x|^{-2s}f \|^2_{L^2(\Omega_s)} = \| L_+ (x, s, D)u \|^2_{L^2(\Omega_s)} + \left( 1 + \frac{1}{s} \right) \| L_- (x, s, D)u \|^2_{L^2(\Omega_s)}
\]
\[
+ 4s \int_{\partial \Omega_s} \left| \frac{\partial u}{\partial \nu} \right|^2 (x, \nu) \, dS
\]
\[
+ 4s \int_{S(0, \delta)} (\nabla u, x) \, u \, dS - \frac{16s^3 + 8s^2 + 12s}{\delta^3} \int_{S(0, \delta)} u^2 \, dS
\]
\[
- \frac{8s}{\delta^2} \int_{S(0, \delta)} (x, \nabla u)^2 \, dS + \frac{4s}{\delta} \int_{S(0, \delta)} |\nabla u|^2 \, dS.
\]  
(23)

We have
\[
\| L_- (x, s, D)u \|^2_{L^2(\Omega_s)} = 4s^2 \int_{\Omega_s} \frac{2(x, \nabla u) - u^2}{|x|^4} \, dx
\]
\[
\geq \frac{4s^2}{\rho^2} \left\{ 4 \int_{\Omega_s} \frac{(x, \nabla u)^2}{|x|^2} \, dx + \int_{\Omega_s} \frac{u^2}{|x|^2} \, dx - 2 \int_{\Omega_s} \frac{(x, \nabla u)^2}{|x|^2} \, dx \right\}
\]
\[
\geq \frac{4s^2}{\rho^2} \left\{ 4 \int_{\Omega_s} \frac{(x, \nabla u)^2}{|x|^2} \, dx + \int_{\Omega_s} \frac{u^2}{|x|^2} \, dx + 2 \int_{\Omega_s} \frac{u^2}{|x|^2} \, dx + \frac{2}{\delta} \int_{S(0, \delta)} u^2 \, dS \right\}
\]
\[
\geq \frac{12s^2}{\rho^2} \int_{\Omega_s} \frac{u^2}{|x|^2} \, dx.
\]  
(24)

We also estimate
\[
\int_{\Omega_s} |\nabla u|^2 \, dx = \int_{\Omega_s} uL_+ (x, s, D) \, u \, dx + 4s(s + 1) \int_{\Omega_s} \frac{u^2}{|x|^2} \, dx
\]
\[
- \int_{\partial \Omega_s} \left( \frac{x}{|x|} \right) \nabla u \, u \, dS
\]
\[
\leq \frac{\rho^2}{2} \| L_+ (x, s, D)u \|^2_{L^2(\Omega_s)} + \frac{1}{2\rho^2} \| u \|^2_{L^2(\Omega_s)} + 4s(s + 1) \int_{\Omega_s} \frac{u^2}{|x|^2} \, dx
\]
\[
- \int_{\partial \Omega_s} \left( \frac{x}{|x|} \right) \nabla u \, u \, dS.
\]
After division of this inequality by $\rho^2$ we have

$$\frac{1}{\rho^2} \int_{\Omega_\delta} |\nabla u|^2 dx \leq \frac{1}{2} \|L_+(x, s, D)u\|^2_{L^2(\Omega_\delta)} + \frac{1}{2\rho^2} \|u\|^2_{L^2(\Omega_\delta)} + \frac{4s(s + 1)}{\rho^2} \int_{\Omega_\delta} \frac{u^2}{|x|^2} dx - \frac{1}{\rho^2} \int_{S(0, \delta)} \langle x \rangle |\nabla u| u dS$$

$$\leq \frac{1}{2} \|L_+(x, s, D)u\|^2_{L^2(\Omega_\delta)} + \frac{1}{2\rho^2} \|u\|^2_{L^2(\Omega_\delta)} + \frac{4s(s + 1)}{\rho^2} \int_{\Omega_\delta} \frac{u^2}{|x|^2} dx - \frac{1}{\rho^2} \int_{S(0, \delta)} \langle x \rangle |\nabla u| u dS = \frac{1}{2} \|L_+(x, s, D)u\|^2_{L^2(\Omega_\delta)} + \left(4s^2 + 4s + \frac{1}{2}\right) \frac{1}{\rho^2} \|u\|^2_{L^2(\Omega_\delta)} - \frac{1}{\rho^2} \int_{S(0, \delta)} \langle x \rangle |\nabla u| u dS. \tag{25}$$

From the estimates (23)-(25) we obtain

$$\|\langle x \rangle^{-2s} f\|^2_{L^2(\Omega_\delta)} \geq \frac{1}{2} \|L_+(x, s, D)u\|^2_{L^2(\Omega_\delta)} + \left(1 + \frac{1}{2s}\right) \|L_-(x, s, D)u\|^2_{L^2(\Omega_\delta)} + \frac{4s}{\rho^2} \int_{\partial \Omega_\delta} \frac{u^2}{|x|^2} (x, \partial \nu) dS + \frac{1}{\rho^2} \|\nabla u\|^2_{L^2(\Omega_\delta)} + \frac{(4s^2 + 4s - 1)}{\rho^2 \delta^3} \int_{S(0, \delta)} u^2 dS$$

$$- \frac{16s^3 + 8s^2 + 12s}{\delta^3} \int_{S(0, \delta)} u^2 dS.$$ \tag{26}

Passing to the limit in (26) as $\delta \to 0$ we obtain

$$\|\langle x \rangle^{-2s} f\|^2_{L^2(\Omega_\delta)} \geq \frac{1}{2} \|L_+(x, s, D)u\|^2_{L^2(\Omega)} + \left(1 + \frac{1}{3s}\right) \|L_-(x, s, D)u\|^2_{L^2(\Omega)} + \frac{4s}{\rho^2} \int_{\partial \Omega} \frac{u^2}{|x|^2} (x, \partial \nu) dS + \frac{(4s^2 + 4s - 1)}{2\rho^2} \|u\|^2_{L^2(\Omega)}$$

$$+ \frac{1}{\rho^2} \|\nabla u\|^2_{L^2(\Omega)}. \tag{27}$$

This estimate implies (14) in the case $s \in \left[1, \frac{5}{2} - \frac{1}{4}\right]$ when the last term in the right hand side is absent.

Next we prove estimate for the last term in the right hand side of (14). For any $\epsilon \in (0, 1)$ we have

$$\|L_-(x, s, D)u\|^2_{L^2(\Omega)} = 4s^2 \int_{\Omega} \frac{|2(x, \nabla u) - u|^2}{|x|^4} dx \geq 4s^2 \frac{\rho^{2\epsilon}}{\rho^{2\epsilon}} \int_{\Omega} \frac{|2(x, \nabla u) - u|^2}{|x|^{4-2\epsilon}} dx. \tag{28}$$
Setting \(v_\varepsilon = u/|x|^{2-\varepsilon}\) we write the integral in the right hand side of the above inequality as
\[
\frac{4s^2}{\rho^{2\varepsilon}} \int_\Omega \frac{|2(x, \nabla u) - u|^2}{|x|^{4-2\varepsilon}} dx - 2(\delta \varepsilon)^2 \int_\Omega |\nabla \varepsilon|^2 dx + \frac{4s^2}{\rho^{2\varepsilon}} \int_\Omega |2(x, \nabla v_\varepsilon) + (3-2\varepsilon)v_\varepsilon|^2 dx.
\] (29)

Denote \(g = 2(x, \nabla) v_\varepsilon + (3-2\varepsilon)v_\varepsilon\) and take the scalar product of this function in \(L^2(\Omega)\) with \(v_\varepsilon\). After short integration by parts we have
\[
\int_\Omega g v_\varepsilon dx = \lim_{\delta \to 0} \left( \int_{\Omega_\delta} g v_\varepsilon dx - \int_{\partial \Omega \setminus \partial \Omega_\delta} \langle \nabla v_\varepsilon, g \rangle dS \right) = -2 \lim_{\delta \to 0} \varepsilon \int_{\Omega_\delta} \delta v_\varepsilon^2 dx - 2 \varepsilon \int_\Omega v_\varepsilon^2 dx.
\] (30)

Using Young’s inequality, \(-g v_\varepsilon \leq \varepsilon v_\varepsilon^2 + \frac{1}{4\varepsilon}g^2\), we obtain
\[
\int_\Omega |2(x, \nabla v_\varepsilon) + (3-2\varepsilon)v_\varepsilon|^2 dx = \int_\Omega g^2 dx \geq 4\varepsilon \left\{ - \int_\Omega g v_\varepsilon dx - \varepsilon \int_\Omega v_\varepsilon^2 dx \right\}
= 4\varepsilon \left\{ 2\varepsilon \int_\Omega v_\varepsilon^2 - \varepsilon \int_\Omega v_\varepsilon^2 dx \right\}
= 4\varepsilon^2 \left\| \frac{u}{|x|^{2-\varepsilon}} \right\|_{L^2(\Omega)}^2.
\] (31)

From (28)-(31) we have
\[
\|L_-(x, s, D)u\|_{L^2(\Omega)}^2 \geq \frac{4s^2}{\rho^{2\varepsilon}} \left\| \frac{u}{|x|^{2-\varepsilon}} \right\|_{L^2(\Omega)}^2.
\] (32)

This concludes the proof of the lemma for the term \(\frac{s^2}{\rho^{2\varepsilon}} \left\| \frac{u}{|x|^{2-\varepsilon}} \right\|_{L^2(\Omega)}^2\) in the right hand side. Taking the scalar product of the function \(-u/(\rho^{2\varepsilon}|x|^{2-2\varepsilon})\) with the function \(L_+(x, s, D)u\) in \(L^2(\Omega)\) we have
\[
\int_{\Omega_\delta} \frac{\nabla u^2}{\rho^{2\varepsilon} |x|^{2-2\varepsilon}} \, dx = - \left( L_+(x, s, D)u, \frac{u}{\rho^{2\varepsilon} |x|^{2-2\varepsilon}} \right)_{L^2(\Omega)}
+ \frac{1}{2\rho^{2\varepsilon}} \int_{\Omega_\delta} u^2 \Delta \left( \frac{1}{|x|^{2-2\varepsilon}} \right) \, dx + \int_{\Omega_\delta} \frac{4s(1+s)}{\rho^{2\varepsilon} |x|^{4-2\varepsilon}} u^2 \, dx
+ \frac{1}{\rho^{2\varepsilon}} \int_{S(0, \delta)} (x, \nabla u) u dS
\]
\[
\leq \frac{1}{s} \|L_+(x, s, D)u\|^2_{L^2(\Omega_\delta)} + \frac{s}{\rho^{2\varepsilon}} \left\| \frac{u}{|x|^{2-2\varepsilon}} \right\|_{L^2(\Omega_\delta)}^2
+ \frac{1-\varepsilon}{\rho^{2\varepsilon}} \int_{\Omega_\delta} u^2 \, dx + \frac{1}{\rho^{2\varepsilon}} \int_{S(0, \delta)} (x, \nabla u) u dS
\]
\[
+ \frac{1}{\rho^{2\varepsilon}} \int_{S(0, \delta)} u^2 \, dS \leq \|L_+(x, s, D)u\|^2_{L^2(\Omega_\delta)} + \frac{s}{\rho^{2\varepsilon}} \left\| \frac{u}{|x|^{2-2\varepsilon}} \right\|_{L^2(\Omega_\delta)}^2
+ \frac{1-\varepsilon}{\rho^{2\varepsilon}} \int_{\Omega_\delta} u^2 \, dx + \frac{1}{\rho^{2\varepsilon}} \int_{S(0, \delta)} (x, \nabla u) u dS
+ \frac{1}{\rho^{2\varepsilon}} \int_{S(0, \delta)} u^2 \, dS = \|L_+(x, s, D)u\|^2_{L^2(\Omega_\delta)} + \left\{ \frac{s}{\rho^{2\varepsilon}} + \right\}
\]
\[
\begin{align*}
&\frac{1 - 3\epsilon + 2\epsilon^2 + 4(s + \epsilon^2)}{\rho^{2\epsilon}} \left\| \frac{u}{|x|^{2-\epsilon}} \right\|_{L^2(\Omega_s)}^2 + \frac{1}{\delta^{3-2\epsilon}\rho^{2\epsilon}} \int_{S(0,\delta)} (x, \nabla u) u dS \\
&+ \frac{1 - \epsilon}{\delta^{2-2\epsilon}\rho^{2\epsilon}} \int_{S(0,\delta)} u^2 dS \leq \| L_+(x, s, D) u \|_{L^2(\Omega)}^2 \\
&+ \frac{1}{\delta^{3-2\epsilon}\rho^{2\epsilon}} \int_{S(0,\delta)} (x, \nabla u) u dS + \frac{1 - \epsilon}{\delta^{2-2\epsilon}\rho^{2\epsilon}} \int_{S(0,\delta)} u^2 dS \\
&+ \left\{ \frac{1}{4\epsilon^2 s} + \frac{1}{4\epsilon^2 s^2} + \frac{1}{2s^2} + \frac{1}{\epsilon^2} \left( \frac{1}{s} + 1 \right) \right\} \| L_-(x, s, D) u \|_{L^2(\Omega)}^2
\end{align*}
\]

for \( s \in [1, s_0 - \frac{1}{4}] \) and \( \epsilon \in (0, 1) \), where we used (32). This implies the inclusion of the last term in right hand side of estimate (14).

\[\square\]

**Remark 1.** The Carleman estimate with the weight \( \ln |x| \) was proved in e.g. in [7] and [14] for the zero Cauchy data on the boundary and in [10] with the Dirichlet boundary conditions for the boundaries which does not contain the origin.

Consider the system of linear elliptic equations

\[\Delta w + \sum_{j=1}^{3} B_j(x) \partial_{x_j} w + C(x) w = f \quad \text{in} \quad \Omega, \quad w|_{\partial\Omega} = 0. \quad \text{(33)}\]

Suppose that for some positive \( \epsilon \in (0, 1) \)

\[\sum_{j=1}^{3} \| |x|^{1-\epsilon} B_j \|_{L^\infty(\Omega)} \leq \lambda. \quad \text{(34)}\]

We have

**Proposition 1.** Let \( |x|^{2-\epsilon} C \in L^\infty(\Omega) \) for some positive \( \epsilon \in (0, 1) \) and (34) holds true. Suppose \( w \in H^2(\Omega) \) is a solution of (33), having zero at origin of the order \( 2s_0 \) for some strictly positive \( s_0 > \frac{3}{2} \), and we assume \( f/|x|^{2s_0} \in L^2(\Omega) \). There exist \( \tilde{r} > 0 \) depending on \( \epsilon, \lambda \) and constants \( C_1 > 0, C_2 > 0 \) independent of \( s \) such that if \( \Omega \subset B(0, \tilde{r}) \) the following estimate for \( u := |x|^{-2s} w \) holds true:

\[C_1 \| |x|^{-2s} f \|_{L^2(\Omega)}^2 \geq \frac{1}{4} \| L_+(x, s, D) u \|_{L^2(\Omega)}^2 + \frac{1}{4} \left( 1 + \frac{1}{s} \right) \| L_-(x, s, D) u \|_{L^2(\Omega)}^2 + 4s \int_{\partial\Omega} \left| \frac{1}{|x|^2} \left| \frac{\partial u}{\partial \nu} \right|^2 (x, \nu) dS \right. + \left. \frac{2s^2}{\rho^2} \| u \|_{L^2(\Omega)}^2 + \frac{1}{\rho^2} \| \nabla u \|_{L^2(\Omega)}^2\right)\]

\[+ \frac{Ca}{2} \left( s^2 \left\| \frac{u}{|x|^{2-\epsilon}} \right\|_{L^2(\Omega)}^2 + \frac{1}{\left( 1 + \rho^2 \right)^2} \left\| \nabla u \right\|_{L^2(\Omega)}^2 \right) \quad \text{(35)}\]

for all \( s \in [1, \frac{5}{2} - \frac{1}{4}] \) and \( \epsilon \in (0, 1) \) provided that \( |x|^{-2s} f \in L^2(\Omega) \).

**Proof.** Let \( s \in [1, \frac{5}{2} - \frac{1}{4}] \). Let the right hand side of (14) be \( I(s, w) \). From the Carleman estimate (14) for the Laplace operator with \( f \) replaced by \( f - Cw - \sum_{j=1}^{n} B_j \partial_{x_j} w \) one has

\[I(s, w) \leq \| |x|^{-2s} (f - Cw - \sum_{j=1}^{3} B_j \partial_{x_j} w) \|_{L^2(\Omega)}^2\]
\begin{align}
&\leq 2\|x|^{-2s}f\|_{L^2(\Omega)}^2 + 2\|x|^{-2s}Cu\|_{L^2(\Omega)}^2 + 2\sum_{j=1}^3 \|x|^{-2s}B_j \partial_x w\|_{L^2(\Omega)}^2 \\
&\leq 2\|x|^{-2s}f\|_{L^2(\Omega)}^2 + 2\|x|^{-2s}C\|_{L^2(\Omega)}^2 + 4s^2 \sum_{j=1}^3 \|x|^{-1}B_j u\|_{L^2(\Omega)}^2 \\
&\quad + 2\sum_{j=1}^3 \|B_j \partial_x u\|_{L^2(\Omega)}^2 \\
&\leq 2\|x|^{-2s}f\|_{L^2(\Omega)}^2 + 2\|x|^{-2s}C\|_{L^2(\Omega)}^2 + 4s^2 \sum_{j=1}^3 \|x|^{-1}B_j\|_{L^\infty(\Omega)}^2 \left\| \frac{u}{|x|^{2-\varepsilon}} \right\|_{L^2(\Omega)}^2 \\
&\quad + 2\sum_{j=1}^3 \|x|^{-1}B_j\|_{L^\infty(\Omega)}^2 \left\| \frac{\nabla u}{|x|^{1-\varepsilon}} \right\|_{L^2(\Omega)}^2 . \tag{36}
\end{align}

Let \( \hat{r} > 0 \) is chosen such that
\[ \frac{1}{2} \frac{C_\varepsilon}{\hat{r}^{2\varepsilon}} - 2\|x|^{2-\varepsilon}C\|_{L^\infty(\Omega)}^2 - 4\sum_{j=1}^3 \|x|^{1-\varepsilon}B_j\|_{L^\infty(\Omega)}^2 > 0, \tag{37} \]
and
\[ \frac{C_\varepsilon}{\hat{r}^{2\varepsilon}(1 + \hat{r}^{\varepsilon})^2} - 2\sum_{j=1}^3 \|x|^{1-\varepsilon}B_j\|_{L^\infty(\Omega)}^2 > 0. \tag{38} \]

Form (36)-(38) we obtain (35).

Below we set
\[ \mathcal{L}_\pm(x, s, D)u := L_\pm(x, -s, D), \quad \text{and } r_0 := \min_{x \in \partial \Omega} \ln |x|. \]

We prove the following:

**Lemma 2.3.** Let \( \Omega \) be a bounded domain in \( \mathbb{R}^3 \) with \( \partial \Omega \) belonging to \( C^1 \) class and \( \overline{B(0, 1)} \cap \Omega = \emptyset \). Suppose \( u \in H^2(\Omega) \) is a solution of (12), having zero at infinity of the order \( s_0 > 5/2 \) and we assume \( f|s|^{2s_0} \in L^2(\Omega) \). We set \( u := |s|^{2s}w \), and write the problem (12) in terms of \( u \) as
\[ \begin{align*}
L(x, s, D)u &= \mathcal{L}_+(x, s, D)u + \mathcal{L}_-(x, s, D)u = |s|^{2s} f \quad \text{in } \mathbb{R}^3 \setminus \Omega, \\
u &= 0 \quad \text{on } \partial \Omega. \tag{39}
\end{align*} \]

Then, the following hold true.
\[ \| |s|^{2s} f \|_{L^2(\mathbb{R}^3 \setminus \Omega)}^2 = \| \mathcal{L}_+(x, s, D)u \|_{L^2(\mathbb{R}^3 \setminus \Omega)}^2 + \left( 1 - \frac{1}{s} \right) \| \mathcal{L}_-(x, s, D)u \|_{L^2(\mathbb{R}^3 \setminus \Omega)}^2 \]
\[ + 4s \int_{\partial \Omega} \frac{1}{|x|^2} \left\| \frac{\partial u}{\partial \nu} \right\|^2 (x, \nu) dS, \tag{40} \]
\[ 4s^2 \int_{\mathbb{R}^3 \setminus \Omega} \frac{u^2}{|x|^4 \ln^2 |x|} dx \leq \| \mathcal{L}_-(x, s, D)u \|_{L^2(\mathbb{R}^3 \setminus \Omega)}^2, \tag{41} \]
and
\[
\int_{\mathbb{R}^3 \setminus \Omega} \frac{|\nabla u|^2}{|x|^2 \ln |x|} \, dx \leq \mathcal{K}(r_0, s) \|L_-(x, s, D)u\|_{L^2(\mathbb{R}^3 \setminus \Omega)}^2 + \frac{1}{2sr_0} \|L_-(x, s, D)u\|_{L^2(\mathbb{R}^3 \setminus \Omega)} \|L_+(x, s, D)u\|_{L^2(\mathbb{R}^3 \setminus \Omega)} \tag{42}
\]
for all \(s \in [1, \frac{n}{2} - \frac{1}{4})\), where we set
\[
\mathcal{K}(r_0, s) = \frac{1 + r_0}{sr_0} \left( \frac{1}{2sr_0} - \frac{1}{4s} \right) + \frac{1}{s} + \frac{3}{2}.
\]

**Proof.** Let \(\Omega_R = B(0, R) \setminus \Omega\) where \(R\) is a sufficiently large parameter so that \(B(0, R) \supset \Omega\). The equality (23) still holds true in \(\Omega_R\) with negative \(s\). Taking into account that \(L_\pm(x, s, D) = L_\pm(x, -s, D)\), we have
\[
\|x|^{2s} f\|_{L^2(\Omega_R)}^2 = \|L_+(x, s, D)u\|_{L^2(\Omega_R)}^2 + \left(1 - \frac{1}{s}\right) \|L_-(x, s, D)u\|_{L^2(\Omega_R)}^2
\]
\[
+ 4s \int_{\partial \Omega} \frac{1}{|x|^2} \left| \frac{\partial u}{\partial \nu} \right|^2 (x, \nu) \, dS
\]
\[
+ \frac{4s}{R^3} \int_{S(0, R)} (\nabla u, x) u dS - \frac{16s^3 - 8s^2 + 12s}{R^3} \int_{S(0, R)} u^2 dS
\]
\[
- \frac{8s}{R^3} \int_{S(0, R)} (x, \nabla u)^2 dS - \frac{4s}{R} \int_{S(0, R)} |\nabla u|^2 dS. \tag{43}
\]

Passing to the limit as \(R \to +\infty\), we obtain the equality (40). Setting \(v = u/|x|^2\), we obtain
\[
- \frac{1}{2s} L_-(x, s, D) u = \frac{2(x, \nabla u) - u}{|x|^2} = 2(x, \nabla v) + 3v := g. \tag{44}
\]
We take the scalar product of \(g\) in \(L^2(\mathbb{R}^3 \setminus \Omega)\) with \(v/\ln |x|\). After short integration by parts we have
\[
\int_{\mathbb{R}^3 \setminus \Omega} \frac{g v}{\ln |x|} \, dx = \lim_{R \to +\infty} \int_{\Omega_R} \frac{g v}{\ln |x|} \, dx
\]
\[
= \lim_{R \to +\infty} \int_{\Omega_R} \frac{1}{\ln |x|} \left\{ (x, \nabla) v^2 + 3u^2 \right\} \, dx
\]
\[
= \lim_{R \to +\infty} \int_{\Omega_R} \frac{v^2}{\ln |x|} \, dx + \lim_{R \to +\infty} \int_{S(0, R)} \frac{R v^2}{\ln |x|} \, dS
\]
\[
= \int_{\mathbb{R}^3 \setminus \Omega} \frac{v^2}{\ln |x|} \, dx. \tag{45}
\]
Applying the Young’s inequality to (45), we obtain
\[
\int_{\mathbb{R}^3 \setminus \Omega} \frac{v^2}{\ln |x|} \, dx \leq \int_{\mathbb{R}^3 \setminus \Omega} g^2 \, dx,
\]
and therefore, using (44) one has
\[
4s^2 \int_{\mathbb{R}^3 \setminus \Omega} \frac{u^2}{|x|^4 \ln |x|} \, dx \leq \|L_-(x, s, D) u\|_{L^2(\mathbb{R}^3 \setminus \Omega)}^2. \tag{46}
\]
By (46) proof of inequality (41) is complete. On the other hand
\[
\int_{\mathbb{R}^3 \setminus \Omega} \frac{|\nabla u|^2}{|x|^2 \ln^2 |x|} \, dx = - \int_{\mathbb{R}^3 \setminus \Omega} \mathcal{L}_-(x, s, D) uu \left( \frac{1}{|x|^2 \ln^2 |x|} + \frac{1}{|x|^2 \ln^3 |x|} \right) \, dx \\
- \int_{\mathbb{R}^3 \setminus \Omega} \frac{u^2}{|x|^4 \ln^2 |x|} \left( 1 + \frac{1}{\ln |x|} \right) \, dx - \int_{\mathbb{R}^3 \setminus \Omega} \frac{u}{|x|^2 \ln^2 |x|} \mathcal{L}_+(x, s, D) u \, dx \\
+ \int_{\mathbb{R}^3 \setminus \Omega} \frac{u^2 (4s + 6s^2)}{|x|^4 \ln |x|^2} \, dx.
\]
From this inequality we obtain
\[
\int_{\mathbb{R}^3 \setminus \Omega} \frac{|\nabla u|^2}{|x|^2 \ln^2 |x|} \, dx \leq \frac{1 + r_0}{r_0^2} \| \mathcal{L}_-(x, s, D) u \|_{L^2(\mathbb{R}^3 \setminus \Omega)} \| u \|_{L^2(\mathbb{R}^3 \setminus \Omega)} \\
- \left( \frac{1}{r_0} + 1 \right) \int_{\mathbb{R}^3 \setminus \Omega} \frac{u^2}{|x|^4 \ln^2 |x|} \, dx \\
+ \frac{1}{r_0} \left\| \frac{u}{|x|^2 \ln |x|} \right\|_{L^2(\mathbb{R}^3 \setminus \Omega)} \| \mathcal{L}_+(x, s, D) u \|_{L^2(\mathbb{R}^3 \setminus \Omega)} \\
+ (4s + 6s^2) \int_{\mathbb{R}^3 \setminus \Omega} \frac{u^2}{|x|^4 \ln |x|^2} \, dx \\
\leq \frac{1 + r_0}{2sr_0^2} \| \mathcal{L}_-(x, s, D) u \|_{L^2(\mathbb{R}^3 \setminus \Omega)}^2 - \frac{1}{4s^2} \left( \frac{1}{r_0} + 1 \right) \| \mathcal{L}_-(x, s, D) u \|_{L^2(\mathbb{R}^3 \setminus \Omega)}^2 \\
+ \frac{1}{2sr_0} \| \mathcal{L}_-(x, s, D) u \|_{L^2(\mathbb{R}^3 \setminus \Omega)} \| \mathcal{L}_+(x, s, D) u \|_{L^2(\mathbb{R}^3 \setminus \Omega)} \\
+ \left( \frac{1}{s} + \frac{3}{2} \right) \| \mathcal{L}_-(x, s, D) u \|_{L^2(\mathbb{R}^3 \setminus \Omega)}^2.
\]
(47)
The proof of the proposition is complete.

We also need the Carleman estimate similar to one obtained in proposition 1 for
the case when the function $u$ does not satisfy the Dirichlet boundary conditions.

**Proposition 2.** Let $\Omega$ be a bounded domain in $\mathbb{R}^3$ with $\partial \Omega$ belonging to $C^1$ class
and $0 \in \Omega$. Suppose $w \in H^2(\mathbb{R}^3 \setminus \Omega)$ is a solution of (12), having zero at infinity of
the order $s_0 > 5/2$. We set $u := |x|^{2s} w$. Then, the following holds true.
\[
\| |x|^{2s} \Delta w \|_{L^2(\mathbb{R}^3 \setminus \Omega)}^2 = \| \mathcal{L}_+(x, s, D) u \|_{L^2(\mathbb{R}^3 \setminus \Omega)}^2 \\
+ \left( 1 - \frac{1}{s} \right) \| \mathcal{L}_-(x, s, D) u \|_{L^2(\mathbb{R}^3 \setminus \Omega)}^2 + 8s \int_{\partial \Omega} \frac{1}{|x|^2} \left( \frac{\partial u}{\partial \nu} \right)^2 (x, \nu) dS \\
+ 4s \int_{\partial \Omega} u \frac{\partial u}{|x|^2 \partial \nu} dS + (16s^3 + 8s^2 + 12s) \int_{\partial \Omega} (x, \nu) \frac{u^2}{|x|^4} dS \\
- 4s \int_{\partial \Omega} (x, \nu) \frac{|\nabla u|^2}{|x|^2} dS
\]
(48)
for all $s \in [1, \frac{s_0}{2} - \frac{1}{4})$ provided that $|x|^{2s} \Delta w \in L^2(\mathbb{R}^3 \setminus \Omega)$.

**Remark 2.** In lemma 2.3 and proposition 2 the assumption $0 \in \Omega$ is important.
Without such an assumption $0 \in \mathbb{R}^3 \setminus \Omega$ and we have to assume, in order to prove
the equalities (40) and (48), that function $u$ has the zero of an appropriate order at
0. In lemma 2.3 the more stronger assumption, namely \( B(0, 1) \) does not belong to \( \Omega \), allows us to keep the parameter \( r_0 \) positive.

Using the Carleman estimate (14), we first prove the following.

**Proposition 3.** Let \( \Omega \) be a bounded, star-shaped domain respect to the origin in \( \mathbb{R}^3 \) with \( \partial \Omega \) belonging to \( C^1 \) class, \( c \in L^\infty(\Omega) \). Suppose \( w \) is a solution to the boundary value problem
\[
\Delta w + c(x)w = 0 \quad \text{in} \quad \Omega, \quad w|_{\partial \Omega} = 0.
\]
If the function \( w \) at origin has zero of order \( s_0 > \max\{5/2, \rho^2 \|c\|_{L^\infty(\Omega)} / \sqrt{2}\} \), then \( w \equiv 0 \) on \( \Omega \).

**Proof.** We set \( u := |x|^{-2s} w \). Since the domain \( \Omega \) assumed to be star-shaped respect to the origin, we have
\[
(x, \vec{v}) \geq 0 \quad \text{on} \quad \partial \Omega.
\]
This implies
\[
4s \int_{\partial \Omega} \frac{1}{|x|^2} \left| \frac{\partial u}{\partial \nu} \right|^2 (x, \vec{v}) dS \geq 0.
\]
Let \( s_0 \) be such that
\[
\frac{2s_0^2}{\rho^2} - \|c\|_{L^\infty(\Omega)}^2 > 0.
\]
Then, we apply the estimate (14) to equation (49),
\[

\begin{align*}
\| |x|^{-2s_0} wc \|_{L^2(\Omega)}^2 & \geq \frac{1}{2} \| L_+(x, s_0, D) u \|_{L^2(\Omega)}^2 \\
& \quad + \frac{1}{2} \left( 1 + \frac{1}{s_0} \right) \| L_-(x, s_0, D) u \|_{L^2(\Omega)}^2 \\
& \quad + 4s_0 \int_{\partial \Omega} \frac{1}{|x|^2} \left| \frac{\partial u}{\partial \nu} \right|^2 (x, \vec{v}) dS + \frac{1}{\rho^2} \| \nabla u \|_{L^2(\Omega)}^2 + \frac{2s_0^2}{\rho^2} \| u \|_{L^2(\Omega)}^2.
\end{align*}
\]
This estimate and (50) yields
\[
\| u \|_{L^2(\Omega)}^2 \| c \|_{L^\infty(\Omega)}^2 \geq \| wc \|_{L^2(\Omega)}^2 \geq \frac{2s_0^2}{\rho^2} \| u \|_{L^2(\Omega)}^2.
\]
Thanks to (51) this estimates implies that \( u = w \equiv 0 \). Proof of the proposition is complete.

Below we apply the above argument to the decay problem in the exterior domain.

**Proposition 4.** Let \( \Omega \) be a bounded domain in \( \mathbb{R}^3 \), star-shaped respect to the origin, and the function \( c = c(x) \) satisfies \( c(x)|x|^2 \ln |x| \in L^\infty(\mathbb{R}^3 \setminus \Omega) \). Let \( w \) be a solution to the elliptic equation
\[
\Delta w + c(x)w = 0 \quad \text{in} \quad \mathbb{R}^3 \setminus \Omega, \quad w|_{\partial \Omega} = 0.
\]
Suppose \( w \) has zero of order \( s_0 \), at spatial infinity, satisfying
\[
s_0 > \max \left\{ \frac{5}{2}, \frac{1 + \sqrt{1 + \|c\|_{L^\infty(\Omega)}^2 \|x\|_{L^\infty(\mathbb{R}^3 \setminus \Omega)}}}{2} \right\}.
\]
Then, \( w \equiv 0 \) on \( \mathbb{R}^3 \setminus \Omega \).
Proof. Since $\Omega$ is assumed to be star-shaped respect to the origin $(x, \vec{v})|_{\partial \Omega} \geq 0$ and $0 \in \Omega$. Then from (40) and (41), applied to equation (52), for all $s \in \left[ \frac{5}{2}, \frac{9}{2} - \frac{1}{4} \right]$ we have

$$4(s^2 - s) \left\| \frac{w|x|^{2s-2}}{\ln |x|} \right\|_{L^2(\mathbb{R}^3 \setminus \Omega)}^2 \leq \|cu\|_{L^2(\mathbb{R}^3 \setminus \Omega)}^2 \leq \|c|x|^2 \ln |x|\|_{L^2(\mathbb{R}^3 \setminus \Omega)}^2 \left\| \frac{w|x|^{2s-2}}{\ln |x|} \right\|_{L^2(\mathbb{R}^3 \setminus \Omega)}^2.$$  

From this inequality and the choice of $s_0$ the statement of the proposition follows immediately.

We set

$$I(\partial \Omega, s, \tilde{u}) = 8s \int_{\partial \Omega} \frac{1}{|x|^2} \left| \frac{\partial \tilde{u}}{\partial x} \right|^2 (x, \vec{v})dS - 4s \int_{\partial \Omega} \frac{1}{|x|^2} (\tilde{u}, \frac{\partial \tilde{u}}{\partial x})dS$$  

$$+ (16s^3 + 8s^2 + 12s) \int_{\partial \Omega} (x, \vec{v}) \left| \frac{\partial \tilde{u}}{\partial x} \right|^2 dS - 4s \int_{\partial \Omega} (x, \vec{v}) \left| \nabla \tilde{u} \right|^2 dS. \quad (53)$$

The following proposition will be used in the proof of theorem 1.3.

**Proposition 5.** Let $\Omega = B(0, R), R \in (0, 1), \epsilon \in (0, 1), s > 1$, function $w$ belongs to $H^2(B(0, R))$ and $w \equiv 0$ in some neighborhood of $0$. Then

$$\frac{s^2 \epsilon^2}{R^2} \int_{B(0, R)} \frac{u^2}{|x|^{4-2s}} dx + \epsilon^2 \int_{B(0, R)} \frac{\nabla u}{R^2 |x|^{2-2s}} dx + I(\partial B(0, R), s, u)$$  

$$\leq \| |x|^{-2s} \Delta w \|_{L^2(B(0, R))}^2 + \int_{S(0,R)} \frac{8s^2 u^2}{R^3} dS + \frac{\epsilon^2}{R^3} \int_{S(0,R)} (x, \nabla u)udS, \quad (54)$$

where $u = |x|^{-2s}w$.

**Proof.** Repeating the arguments leading to (23) we have

$$\| |x|^{-2s} \Delta w \|_{L^2(B(0, R))}^2 = \|L_+(x, s, D)w\|_{L^2(B(0, R))}^2 + (1 + \frac{1}{8}) \|L_-(x, s, D)u\|_{L^2(B(0, R))}^2$$  

$$- \frac{4s}{R^2} \int_{S(0,R)} \frac{\partial u}{\partial r}u dS + \frac{16s^3 + 8s^2 + 12s}{R^3} \int_{S(0,R)} u^2 dS$$  

$$+ \frac{8s}{R} \int_{S(0,R)} \left| \frac{\partial u}{\partial r} \right|^2 dS - \frac{4s}{R} \int_{S(0,R)} |\nabla u|^2 dS. \quad (55)$$

For any $\epsilon \in (0, 1)$ we have

$$\|L_-(x, s, D)u\|_{L^2(B(0, R))}^2 = 4s^2 \int_{B(0,R)} \frac{|2(x, \nabla u) - u|^2}{|x|^4} dx$$  

$$\geq 4s^2 \frac{\epsilon^2}{R^2} \int_{B(0,R)} \frac{|2(x, \nabla u) - u|^2}{|x|^{4-2s}} dx. \quad (56)$$

Setting $v_\epsilon = u/|x|^{2-\epsilon}$ we write the integral in the right hand side of (56) as

$$\frac{4s^2}{R^2} \int_{B(0,R)} \frac{|2(x, \nabla u) - u|^2}{|x|^{4-2s}} dx = \frac{4s^2}{R^2} \int_{B(0,R)} |2(x, \nabla v_\epsilon) + (3 - 2\epsilon)v_\epsilon|^2 dx. \quad (57)$$
Denote $g = 2(x, \nabla) v_\epsilon + (3 - 2\epsilon) v_\epsilon$ and take the scalar product of this function in $L^2(\mathbb{B}_0, R)$ with $-v_\epsilon$. After short integration by parts we have

$$- \int_{S(0, R)} R\epsilon^2 dS + 2\epsilon \int_{B(0, R)} u^2 d\omega = - \int_{B(0, R)} \omega d\omega \leq \int_{B(0, R)} (\epsilon v_\epsilon^2 + \frac{1}{4\epsilon} g^2) d\omega. $$

Hence the above inequality combined with (56) and (57) imply

$$\frac{1}{4} ||L_-(x, s, D)u||^2_{L^2(B(0, R))} \geq \frac{4\epsilon^2}{R^2} \left( -\epsilon \int_{S(0, R)} u^2 R^{3-2\epsilon} dS + \epsilon^2 \int_{B(0, R)} u^2 R^3 dS \right).$$

(58)

Taking the scalar product of the function $-u/(R^{2\epsilon}|x|^{2-2\epsilon})$ with $L_+(x, s, D)u$ in $L^2(B(0, R))$ and using (58) we have

$$\int_{B(0, R)} \frac{\nabla u^2}{R^{2\epsilon}|x|^{2-2\epsilon}} d\omega = - \left( L_+(x, s, D)u, \frac{u}{R^{2\epsilon}|x|^{2-2\epsilon}} \right)_{L^2(B(0, R))}$$

$$\leq \frac{1}{\epsilon} \int_{S(0, R)} u^2 R^{2\epsilon} dS + \frac{1}{R^3} \int_{S(0, R)} u^2 dS$$

$$\leq \frac{1}{s} ||L_+(x, s, D)u||^2_{L^2(B(0, R))} + \frac{s}{4R^{4\epsilon}} \int_{B(0, R)} \frac{u^2}{|x|^{4-4\epsilon}} d\omega + \frac{1}{R^3} \int_{S(0, R)} (x, \nabla u) u dS$$

$$\leq \frac{1}{s} ||L_+(x, s, D)u||^2_{L^2(B(0, R))} + \frac{s}{4R^{4\epsilon}} \int_{B(0, R)} \frac{u^2}{|x|^{4-4\epsilon}} d\omega + \frac{1}{R^3} \int_{S(0, R)} (x, \nabla u) u dS$$

$$\leq ||L_+(x, s, D)u||^2_{L^2(B(0, R))} + \frac{1}{R^3} \int_{S(0, R)} (x, \nabla u) u dS$$

$$\leq \frac{1}{s} ||L_+(x, s, D)u||^2_{L^2(B(0, R))} + \frac{1}{R^3} \int_{S(0, R)} (x, \nabla u) u dS$$

From (59) and (58) we obtain (54). Proof of the proposition is complete.

3. Proof of the main results.

Proof of Theorem 1.2. Denote $\omega = \text{curl } v$. First we prove that the limit in the right hand side of (3) exists and

$$\gamma \in (0, +\infty).$$

(60)

In order to prove (60) we show that $\kappa = \lim_{t \to +\infty} ||\omega||_{L^2(S(0, t+2))}/||\omega||_{L^2(S(0, t))}$ satisfies

$$\kappa \in (0, +\infty).$$

(61)
Proposition 6 implies that \( \kappa < \infty \). We claim that

\[ \kappa > 0. \]

Our proof is by contradiction, suppose that \( \kappa = 0 \). Then

\[ \lim_{t \to +\infty} \| \omega \|_{L^2(S(0,t+2))} / \| \omega \|_{L^2(S(0,t))} = 0. \]  

(62)

By (62) for any positive \( N > 1 \) there exists a constant \( t_0(N) \) such that

\[ \| \omega \|_{L^2(S(0,t+2))} / \| \omega \|_{L^2(S(0,t))} \leq \frac{1}{N} \quad \forall t \geq t_0(N). \]  

(63)

So by (63) we obtain

\[ \| \omega \|_{L^2(S(0,t+2n))} \leq \frac{1}{N^n} \| \omega \|_{L^2(S(0,t))} \quad \forall t \geq t_0(N). \]

Then for any \( n \in \mathbb{N}_+ \) and for any \( t \geq t_0(N) \) we have

\[ \sup_{ \hat{t} \in [t+2n,t+2n+\frac{1}{2}]} \| \omega \|_{L^2(S(0,\hat{t}))} \leq \frac{1}{2^{2n}} \max_{\tau \in (0,2)} \| \omega \|_{L^2(S(0,t+\tau))}. \]  

(64)

We fix positive \( s \) and set \( N = 2^{2s} \) and take in (64) \( t = t_0(2^{2s}) \). Then inequality (64) has the form

\[ \sup_{ \hat{t} \in [t+2n,t+2n+\frac{1}{2}]} \| \omega \|_{L^2(S(0,\hat{t}))} \leq \frac{1}{2^{2n}} \max_{\tau \in (0,2)} \| \omega \|_{L^2(S(0,t+\tau))}. \]  

(65)

Function \( \omega = \text{curl} \ v \) satisfies the vorticity equation

\[ P(x,D)\omega = \Delta \omega - (v, \nabla) \omega + (\omega, \nabla) v = 0 \quad \text{in} \quad \mathbb{R}^3 \setminus \Omega. \]  

(66)

The inequality (65), the assumption (4) and the classical a priori estimates for the Laplace operator imply

\[ \| \omega \|_{H^1(B(0,2n+t_0+\frac{1}{4}) \setminus B(0,2n+t_0+\frac{1}{4}))} \leq \frac{C_1}{2^{2n}} \max_{\tau \in (0,2)} \| \omega \|_{L^2(S(0,t+\tau))}. \]  

(67)

Let \( \hat{\mu}(\hat{t}) \) be a smooth function such that

\[ \hat{\mu}(\hat{t}) = 1 \quad \text{for} \quad \hat{t} < 0 \quad \text{and} \quad \hat{\mu}(\hat{t}) = 0 \quad \text{for} \quad \hat{t} \geq \frac{1}{2}. \]

Let \( \omega_n = \mu_n \omega, \mu_n = \hat{\mu}(|x| - 2n - t_0 - \frac{1}{4}) \). By (66) we have

\[ P(x,D)\omega_n = [\mu_n, P(x,D)] \omega \quad \text{in} \quad \mathbb{R}^3 \setminus \Omega. \]  

(68)

Observe that by (67) we obtain

\[ \| |x|^{2s}[\mu_n, P(x,D)] \omega \|_{L^2(\mathbb{R}^3 \setminus \Omega)} \to 0 \quad \text{as} \quad n \to +\infty. \]  

(69)

Indeed

\[ \| |x|^{2s}[\mu_n, P(x,D)] \omega \|_{L^2(\mathbb{R}^3 \setminus \Omega)} = \]

\[ \| |x|^{2s}[\mu_n, P(x,D)] \omega \|_{L^2(B(0,t_0+2n+\frac{1}{4})) \setminus B(0,t_0+2n+\frac{1}{4}))} \]

\[ \leq |t_0 + 2n + \frac{3}{4}|^{2s} \| \mu_n, P(x,D) \|_{L^2(B(0,t_0+2n+\frac{1}{4})) \setminus B(0,t_0+2n+\frac{1}{4}))} \]

\[ \leq |t_0 + 2n + \frac{3}{4}|^{2s} \| \omega \|_{H^1(B(0,t_0+2n+\frac{1}{4})) \setminus B(0,t_0+2n+\frac{1}{4}))} \]

\[ \leq |t_0 + 2n + \frac{3}{4}|^{2s} \max \{ \| \omega \|_{L^2(S(0,t_0))}, \| \omega \|_{L^2(S(0,t_0+2n))} \} \]

\[ \leq C_2, \]  

(70)
where constant $C_2$ is independent of $n$ and $s$. We set

$$f_n = (v, \nabla)\omega_n - (\omega_n, \nabla)v + |\mu_n, P(x, D)|\omega, \quad u_n = |x|^{2s}\omega_n.$$  

Let $t > 1$. By (4) and (70) there exists a constant $C_3$ independent of $s$ and $t$ such that

$$\leq C_3 \left( \left\| \frac{\omega}{|x|^{2+\tau-2s}} \right\|_{L^2(\mathbb{R}^3 \setminus B(0,t))} + \left\| \frac{\nabla \omega}{|x|^{1+\tau-2s}} \right\|_{L^2(\mathbb{R}^3 \setminus B(0,t))} + 1 \right). \quad (71)$$

Since the function $\omega_0$ has a compact support, the Carleman estimate (48) from the proposition 2 applied to vorticity equation (68) holds true for all sufficiently large $s$ in domain $\mathbb{R}^3 \setminus B(0, t)$ with sufficiently large $t$:

$$\| |x|^{2s}f_n \|_{L^2(\mathbb{R}^3 \setminus B(0,t))} = \| \mathcal{L}_+(x, s, D)u_n \|_{L^2(\mathbb{R}^3 \setminus B(0,t))} + \left( 1 - \frac{1}{s} \right) \| \mathcal{L}_-(x, s, D)u_n \|_{L^2(\mathbb{R}^3 \setminus B(0,t))} + \frac{4s}{t^3} \int_{S(t)} ((x, \nabla)u_n, u_n) dS$$

$$+ \frac{8s}{t} \int_{S(t)} \left| \frac{\partial u_n}{\partial \nu} \right|^2 dS + \left( \frac{16s^3 + 8s^2 + 12s}{t^3} \right) \int_{S(t)} |u_n|^2 dS - \frac{4s}{t} \int_{S(t)} |\nabla u_n|^2 dS. \quad (72)$$

Setting $v_n = u_n/|x|^2$, we obtain

$$-\frac{1}{2s} \mathcal{L}_-(x, s, D)u_n = 2(x, \nabla v_n) + 3v_n = g_n \quad \text{in } \mathbb{R}^3 \setminus \Omega.$$  

We take the scalar product of $g_n$ in $L^2(\mathbb{R}^3 \setminus B(0, t))$ with $v_n/\ln |x|$. After short integration by parts we have

$$\int_{\mathbb{R}^3 \setminus B(0,t)} \frac{(g_n, v_n)}{\ln |x|} dx = \int_{\mathbb{R}^3 \setminus B(0,t)} \frac{1}{\ln |x|} \{ \ln |x| \} \{ (x, \nabla) |v_n|^2 + 3|v_n|^2 \} dx$$

$$= \int_{\mathbb{R}^3 \setminus B(0,t)} |v_n|^2 \ln |x| dx - \frac{t}{\ln t} \int_{S(t)} |v_n|^2 dS. \quad (73)$$

Applying the Young’s inequality to (73), we obtain

$$\int_{\mathbb{R}^3 \setminus B(0,t)} \frac{|u_n|^2}{|x|^{4 \ln^2 |x|}} dx \leq \int_{\mathbb{R}^3 \setminus B(0,t)} |g_n|^2 dx + \frac{2}{t \ln t} \int_{S(t)} |u_n|^2 dS,$$

and therefore, using (44) one has

$$4s^2 \int_{\mathbb{R}^3 \setminus B(0,t)} \frac{|u_n|^2}{|x|^{4 \ln^2 |x|}} dx$$

$$\leq \| \mathcal{L}_-(x, s, D)u_n \|_{L^2(\mathbb{R}^3 \setminus B(0,t))}^2 + \frac{8s^2}{t \ln t} \int_{S(t)} |u_n|^2 dS. \quad (74)$$
On the other hand
\[
\int_{R^3 \setminus B(0,t)} \frac{\nabla u_n}{|x|^2 \ln^2 |x|} dx = -\int_{R^3 \setminus B(0,t)} (L_2(x,s,D)u_n, u_n) \frac{1}{|x|^2 \ln^2 |x|} dx
\]
\[+ \frac{1}{t^3 \ln^2 t} \int_{S(0,t)} \sum_{j=1}^3 \frac{x_j \partial x_j u_n, u_n) dS - \frac{1}{t^3} \left( \frac{1}{\ln^2 t} + \frac{1}{\ln^3 t} \right) \int_{S(0,t)} |u_n|^2 dS\]
\[+ \int_{R^3 \setminus B(0,t)} |u_n|^2 \left( 1 + \frac{3}{2 \ln |x|} + \frac{3}{2 \ln^2 |x|} \right) dx \]
\[+ (4s + 6s^2) \int_{R^3 \setminus B(0,t)} \frac{|u_n|^2}{|x|^2 \ln^2 |x|} dx. \tag{75}
\]

By (75) and (74) we obtain from (72) that there exists a constant $C_4$ independent of $s$ such that
\[
\int_{R^3 \setminus B(0,t)} \frac{\nabla u_n}{|x|^2 \ln^2 |x|} dx + \int_{R^3 \setminus B(0,t)} \frac{|u_n|^2}{|x|^2 \ln^2 |x|} dx
\[\leq C_4(||x|^{2s}f ||L_2(\Omega) + \frac{4s}{t^3} \int_{S(0,t)} ((x, \nabla)u_n, u_n) dS)\]
\[+ \frac{8s}{t} \int_{S(0,t)} |\nabla u_n|^2 dS + \frac{16s^3 + 8s^2 + 12s}{t^3} \int_{S(0,t)} |u_n|^2 dS\]
\[+ \frac{4s}{t} \int_{S(0,t)} |\nabla u_n|^2 dS + \frac{8s^2}{t \ln t} \int_{S(0,t)} |u_n|^2 dS. \tag{76}
\]

Using (71) to estimate the first term in the right hand side of (76) we obtain
\[
\int_{R^3 \setminus B(0,t)} \frac{\nabla u_n}{|x|^2 \ln^2 |x|} dx + \int_{R^3 \setminus B(0,t)} \frac{|u_n|^2}{|x|^2 \ln^2 |x|} dx
\[\leq C_5 \left( 1 + \frac{4s}{t^3} \int_{S(0,t)} ((x, \nabla)u_n, u_n) dS\right)\]
\[+ \frac{8s}{t} \int_{S(0,t)} |\nabla u_n|^2 dS + \frac{16s^3 + 8s^2 + 12s}{t^3} \int_{S(0,t)} |u_n|^2 dS\]
\[+ \frac{8s^2}{t \ln t} \int_{S(0,t)} |u_n|^2 dS + \frac{4s}{t} \int_{S(0,t)} |\nabla u_n|^2 dS \leq C_6(t)(1 + s^3)t^{4s}. \tag{77}
\]

Let $\tilde{x}$ be a point in $\mathbb{R}^3$ such that $|\tilde{x}| = 4t + 1. $ Since $\int_{B(\tilde{x},1)} |\omega|^2 dx \neq 0$ inequality (77) implies
\[
(4t)^{2s} \int_{B(\tilde{x},1)} |\omega|^2 dx \leq C_6(t)(1 + s^3)t^{4s} \tag{78}
\]
for all sufficiently large $s.$

Passing in inequality (78) to the limit as $s \to +\infty$ we obtain that $\omega |_{B(\tilde{x},1)} = 0.$ By the uniqueness of the Cauchy problem for the second order elliptic partial differential equation $\omega |_{\mathbb{R}^3 \setminus \Omega} = 0.$ Proof of (61) is complete.

By (61) there exists a constant $k_0$ and a sequence $\{t_k\}_{k=1}^\infty$ such that $t_k \to +\infty$ and
\[
\|\omega\|_{L^2(S(0,t_k+2))} \geq \frac{k}{2} \|\omega\|_{L^2(S(0,t_k))} \\forall k \geq k_0. \tag{79}
\]
By proposition 6 there exist constants $C_7$ and $C_8$ independent of $k$ such that

$$\|\Lambda \omega\|_{L^2(S(0,t_k+1))} \leq C_7 \|\omega\|_{L^2(S(0,t_k))} \quad \forall k \in \mathbb{N}_+$$

(80)

and

$$\|\omega\|_{L^2(S(0,t_k+2))} \leq C_8 \|\omega\|_{L^2(S(0,t_k+1))} \quad \forall k \in \mathbb{N}_+. \tag{81}$$

Combining estimates (80) and (81) we have

$$\|\Lambda \omega\|_{L^2(S(0,t_k+1))} \leq C_9 \|\omega\|_{L^2(S(0,t_k+1))}.$$  

Proof of (60) is complete.

Using the representation of the Laplace operator in the spherical coordinates as

$$\Delta = \frac{1}{r^2} \left\{ (r \partial_r + \frac{1}{2})^2 - \frac{1}{4} + \Delta_w \right\},$$

we write equation (66) as

$$(\partial_r + \frac{1}{2r})W - \frac{1}{r} \Delta W = f \quad \text{in} \quad \mathbb{R}^3 \setminus \Omega,$$  

(82)

where

$$W = (\partial_r + \frac{1}{2r})\omega + \frac{1}{r} \Delta \omega,$$

$$f = (v, \nabla)\omega - (\omega, \nabla)v$$

with $\Lambda = \sqrt{-\Delta_w + \frac{1}{4}}$. We fix $s^* \in (1+\gamma, s_*)$. Setting $W_{s^*}(x) = |x|^{s^*}W(x)$ we obtain

$$\left(\partial_r + \frac{(1-2s^*)}{2r}\right)W_{s^*} - \frac{1}{r} \Delta W_{s^*} = r^{s^*} f \quad \text{in} \quad \mathbb{R}^3 \setminus \Omega. \tag{83}$$

Combining equation (83) with assumption of decay of our solution on infinity implies the existence of a constant $C_{10}$ independent of $t$ such that

$$\|W_{s^*}/|x|^\frac{3}{2}\|_{L^2(\mathbb{R}^3 \setminus B(0,t))} \leq C_{10} \||x|^{s^*+\frac{3}{2}}f\|_{L^2(\mathbb{R}^3 \setminus B(0,t))} \tag{84}$$

for all sufficiently large $t$. Indeed taking the scalar product of equation (83) with function $-W_{s^*}/|x|^3$ on $B(0,N) \setminus B(0,t)$$

$$\int_t^N \int_0^{2\pi} \int_0^\pi (-\frac{1}{2} \partial_r |W_{s^*}|^2 + \frac{(1+2s^*)}{2r^3} |W_{s^*}|^2 + \frac{1}{r^4} (\Lambda W_{s^*}, W_{s^*})) drd\theta d\varphi =$$

$$-\int_t^N \int_0^{2\pi} \int_0^\pi (-\frac{1+2s^*}{2r^2} |W_{s^*}|^2 + \frac{1}{r} (\Lambda W_{s^*}, W_{s^*})) drd\theta d\varphi$$

$$- \frac{1}{2} \int_{S(0,N)} |W_{s^*}(N, \theta, \varphi)|^2 dS + \frac{1}{2} \int_{S(0,t)} |W_{s^*}(t, \theta, \varphi)|^2 dS =$$

$$- \int_t^N \int_0^{2\pi} \int_0^\pi r^{s^*} (f, W_{s^*}) drd\theta d\varphi$$

$$\leq \int_t^N \int_0^{2\pi} \int_0^\pi r^{s^*} (-\frac{4r}{1+2s^*} |f|^2 + \frac{2s^*+1}{4r^3} |W_{s^*}|^2) drd\theta d\varphi. \tag{85}$$

By

$$\lim_{N \to +\infty} \int_{S(0,N)} |W_{s^*}(N, \theta, \varphi)|^2 dS = 0.$$
So passing to the limit in (85) as \( N \to +\infty \) we obtain
\[
\int_t^\infty \int_0^{2\pi} \int_0^\pi \frac{2s^* + 1}{4r^3} |W_{s^*}|^2 dr d\theta d\varphi \\
\leq \frac{t}{2} \int_{S(0,t)} |W_{s^*}(t,\theta)|^2 dS + \int_t^\infty \int_0^{2\pi} \int_0^\pi \frac{4r^{s^* + 1}}{2s^* - 1} |f|^2 dr d\theta d\varphi.
\]
This proves estimate (84).

Setting \( w(x) = |x|^{s^* - \frac{3}{2}} \omega(x) \) we have
\[
\partial_r w + \frac{1 - s^*}{r} w + \frac{1}{r} \Lambda w = W_{s^*} / |x| \quad \text{in} \quad \mathbb{R}^3 \setminus \bar{\Omega}.
\]

We divide both sides of equation (86) by \( |x| \) and take the \( L^2 \) norm of the both sides of equation (86), we obtain the equality
\[
\left\| r^{-1} \partial_r w \right\|^2_{L^2(\mathbb{R}^3 \setminus B(0,t))} + \left\| \frac{1 - s^*}{r} w + \frac{1}{r} \Lambda w \right\|^2_{L^2(\mathbb{R}^3 \setminus B(0,t))} \\
+ \frac{1}{t^2} \int_{S(0,t)} (w, (s^* - 1) w - \Lambda w) dS = \|W_{s^*}/|x|\|^2_{L^2(\mathbb{R}^3 \setminus B(0,t))}.
\]
Using (84), we estimate the right hand side of (87) to obtain
\[
\left\| r^{-1} \partial_r w \right\|^2_{L^2(\mathbb{R}^3 \setminus B(0,t))} + \left\| \frac{1 - s^*}{r} w + \frac{1}{r} \Lambda w \right\|^2_{L^2(\mathbb{R}^3 \setminus B(0,t))} \\
+ \frac{1}{t^2} \int_{S(0,t)} (w, (s^* - 1) w - \Lambda w) dS \leq C_{11} \|f|^{s^* + \frac{3}{2}}\|^2_{L^2(\mathbb{R}^3 \setminus B(0,t))}.
\]
By (4) there exist constants \( C_{12} \) and \( \tilde{t}_0 \) independent of \( t \) such that for all \( t \geq \tilde{t}_0 \)
\[
\left\| f|x|^{s^* + \frac{3}{2}} \right\|^2_{L^2(\mathbb{R}^3 \setminus B(0,t))} \leq \frac{C_{12}}{t^2} \left\| \nabla w / |x| + |w| / |x| \right\|^2_{L^2(\mathbb{R}^3 \setminus B(0,t))}.
\]
So there exist a positive constant \( C_{13} \) and \( \tilde{t}_1 \) such that for all \( t \geq \tilde{t}_1 \)
\[
C_{13} \left( \left\| r^{-1} \partial_r w \right\|^2_{L^2(\mathbb{R}^3 \setminus B(0,t))} + \left\| \frac{1 - s^*}{r} w + \frac{1}{r} \Lambda w \right\|^2_{L^2(\mathbb{R}^3 \setminus B(0,t))} \right) \\
+ \frac{1}{t^2} \int_{S(0,t)} (w, (s^* - 1) w - \Lambda w) dS \leq 0.
\]

Let us take the sequence of \( \{t_k\}_{k=1}^{+\infty}, \ t_k \to +\infty \) such that \( \gamma = \lim_{t_k \to +\infty} \|\Lambda w\|_{L^2(S(0,t_k))}/\|v\|_{L^2(S(0,t_k))} \). Then for all sufficiently large \( k \)
\[
\int_{S(0,t_k)} (w, (s^* - 1) w - \Lambda w) dS = t_k^{2s^* + 2} \int_{S(0,t_k)} (\omega, (s^* - 1) \omega - \Lambda \omega) dS \\
\geq t_k^{2s^* + 2} \left( \|\omega\|^2_{L^2(S(0,t_k))} - \|\omega\|_{L^2(S(0,t_k))} \|\Lambda \omega\|_{L^2(S(0,t_k))} \right) > 0.
\]
From (89) and this inequality the statement of the theorem follows immediately. \( \square \)

**Proof of Theorem 1.3.** First we consider the case when vorticity \( \omega = \text{curl} \ v \) is not identically equal zero. Let us introduce the function
\[
M_1(t) = \inf_{|x|=1} \|\omega\|^2_{W^*_{s^*}(B(x,\frac{3}{2}))}.
\]

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We observe that there exists \( \hat{t}(v) \) and a positive constant \( C_0 \) independent of \( t \) such that

\[
M_1(t) \leq C_0 M(t) \quad \forall t \geq \hat{t}(v).
\]

In order to prove (91) we introduce the function \( \tilde{\rho} \in C_0^\infty(B(0, \frac{19}{20})) \), \( \tilde{\rho}|_{B(0, \frac{19}{20})} = 1 \) and set \( \rho(x, y) = \tilde{\rho}(x - y) \) for \( j \in \{0, 1\} \). Obviously there exists a constant \( C_1 \) independent of \( x \) such that

\[
\|\rho(x, \cdot)\|_{H^{-1}(B(x, \frac{19}{20}))} \leq C_1 \|v\|_{L^2(B(x, \frac{19}{20}))}, \quad \forall |x| = t.
\]

Let \( W \) be solution to the following boundary value problem

\[
\Delta W = \omega \quad \text{in } B(x, \frac{19}{20}), \quad W|_{S(x, \frac{19}{20})} = 0.
\]

By (92) and classical estimate for the Laplace operator

\[
\|W\|_{H^1(B(x, \frac{19}{20}))} \leq C_2 \|\omega\|_{H^{-1}(B(x, \frac{19}{20}))} \leq C_3 \|v\|_{L^2(B(x, \frac{19}{20}))}.
\]

Taking the scalar product of the equation (66) and the function \( \rho^{10}(x, \cdot)W \) in space \( L^2(B(x, \frac{19}{20})) \) after integration by parts we have

\[
- \int_{B(x, \frac{19}{20})} \rho^{10}(x, \cdot) |\omega|^2 dy = \int_{B(x, \frac{19}{20})} (2(\nabla W, \nabla \rho^{10}(x, \cdot)), \omega) \\
+ (\omega, W) \Delta \rho^{10}(x, \cdot) + ((v, \nabla)(\rho^{10}(x, \cdot)W), \omega) - ((\omega, \nabla)v, \rho^{10}(x, \cdot)W) dy.
\]

Using the Cauchy inequality, we estimate the right hand side of (95) as

\[
\int_{B(x, \frac{19}{20})} \rho^{10}(x, \cdot) |\omega|^2 dy \leq C_4 (1 + \|v\|_{W^{1,\infty}(B(0, 1))} \|W\|_{H^1(B(x, \frac{19}{20}))}) \|\rho^{5}(x, \cdot)\|_{L^2(B(x, \frac{19}{20}))}.
\]

From (96) and (94) we obtain the estimate

\[
\|\rho^{5}(x, \cdot)\|_{L^2(B(x, \frac{19}{20}))} \leq C_5 \|v\|_{L^2(B(x, \frac{19}{20}))}.
\]

Taking the scalar product of the equation (66) and the function \( \rho^{20}(x, \cdot)\omega \) we obtain that there exists a constant \( C_6 \) independent of \( x \) such that

\[
\|\rho^{10}(x, \cdot)\nabla \omega\|_{L^2(B(x, \frac{19}{20}))} \leq C_6 \|\rho^{5}(x, \cdot)\omega\|_{L^2(B(x, \frac{19}{20}))} \leq C_7 \|v\|_{L^2(B(x, \frac{19}{20}))}.
\]

Combining (98), (97) we obtain from vorticity equation (66):

\[
\|\rho^{10}(x, \cdot)\nabla \omega\|_{L^6(B(x, \frac{19}{20}))} + \|\rho^{10}(x, \cdot)\omega\|_{H^2(B(x, \frac{19}{20}))} \\
\leq C_8 (\|\rho^{10}(x, \cdot)\nabla \omega\|_{L^2(B(x, \frac{19}{20}))} + \|\rho^{5}(x, \cdot)\omega\|_{L^2(B(x, \frac{19}{20}))}) \\
\leq C_9 \|v\|_{L^2(B(x, \frac{19}{20}))}.
\]

Standard \( L^p \) estimates for elliptic equations, estimate (99) and vorticity equation (66) imply:

\[
\|\rho^{20}(x, \cdot)\nabla \omega\|_{W^{2,2}(B(x, \frac{19}{20}))} + \|\rho^{10}(x, \cdot)\omega\|_{H^2(B(x, \frac{19}{20}))} \\
\leq C_{10} (\|\rho^{10}(x, \cdot)\nabla \omega\|_{L^6(B(x, \frac{19}{20}))} + \|\rho^{5}(x, \cdot)\omega\|_{L^6(B(x, \frac{19}{20}))}) \\
\leq C_{11} \|v\|_{L^2(B(x, \frac{19}{20}))}.
\]

From (100) and Sobolev embedding theorem we obtain (91).
We remind that $\Omega \subset B(0,1)$. Let $R$ be a large positive parameter. Hence one can consider the inversion transform from $\mathbb{R}^3 \setminus B(0,R)$ into $B(0,\frac{1}{R})$ defined by $x \to \frac{x}{|x|^2} := y$. For such change of variables we define functions $w, u$ by

$$v(x) = \frac{1}{|x|} u \left( \frac{x}{|x|^2} \right), \quad \omega(x) = \frac{1}{|x|} w \left( \frac{x}{|x|^2} \right).$$

\[ (101) \]

By the assumption (6) function $w$ is bounded in some neighborhood of origin. Then equation (66) is transformed into

$$\Delta w = - \frac{1}{|y|^3} (u, y) w + \frac{1}{|y|^3} (w, y) u + \frac{1}{|y|^3} (w, \nabla u) - \frac{(w, \nabla) u}{|y|} - \frac{2(y, u)}{|y|^3} (y, \nabla) w + \frac{2(y, w)}{|y|^3} (y, \nabla) u.$$  \[ (102) \]

Let $x_*$ be an arbitrary point on sphere $S(0,t)$. An image of the ball $B(x_*, \frac{1}{2})$ under the inversion transform is the ball $B(y_*, \sqrt{2(t^2 - \frac{1}{2})})$ centered at $y_* = x_*/(t^2 - \frac{1}{2})$. Obviously $y_* \to 0$ in $\mathbb{R}^3$ as $t \to +\infty$.

One can write equation (102) as

$$L(y, D)w = \Delta w + \sum_{j=1}^{3} B_j(y) \partial_{y_j} w + C(y) w = 0 \quad \text{on } B(0, \frac{1}{R}).$$

Denote $w(y_*, y) = w(y + y_*)$, $\tilde{B}_j(y_*, y) = B_j(y + y_*)$, $\tilde{C}(y_*, y) = C(y + y_*)$. Let $r_*(t, R) = \text{sup} \{ r | B(y_*, r) \subset B(0, \frac{1}{R}) \}$. Then on the ball $B(0, r_*(t, R))$ we consider the equation

$$L(y_*, y, D)w = \Delta w(y_*, \cdot) + \sum_{j=1}^{3} \tilde{B}_j(y_*, \cdot) \partial_{y_j} w(y_*, \cdot) + \tilde{C}(y_*, \cdot) w(y_*, \cdot) = 0.$$  \[ (103) \]

Observe that by (6) and (102) we have

$$|\tilde{C}(y_*, y)| \leq 2 \frac{|u(y + y_*)|}{|y + y_*|^3} + 3 \frac{\nabla u(y + y_*)}{|y + y_*|^2} \leq 2 \frac{|v(y + y_*)|}{|y + y_*|^3} + 3 \frac{|v(y + y_*)|}{|y + y_*|^3} + 3 \frac{||\nabla v(y + y_*)||}{|y + y_*|^2} \leq 5 ||y||^3 ||w||_{C^0(\mathbb{R}^3 \setminus B(0,1))} + 3 ||y||^4 ||\nabla w||_{C^0(\mathbb{R}^3 \setminus B(0,1))}.$$  \[ (104) \]

Let $\chi_1$ be a $C^\infty(\mathbb{R}^3)$ nonnegative function such that $\chi_1|_{B(0,1)} \equiv 0$ and $\chi_1(y) = 1$ for $|y| \geq 2$. Let $\tilde{w} = w(y_*, \cdot) \chi(t, \cdot)$ where $\chi(t, \cdot) = \chi(2\sqrt{2}(t^2 - \frac{1}{2})y)$. Then function $\tilde{w}$ satisfies the equation

$$L(y_*, y, D)\tilde{w} = -[\chi(t, \cdot), L(y_*, y, D)]w(y_*, \cdot) \quad \text{on } B(0, r_*(t, R)), \quad \tilde{w}|_{B(0, \frac{1}{R(t^2 - \frac{1}{2})})} = 0.$$  \[ (105) \]

Observe that

$$|\sum_{j=1}^{3} \tilde{B}_j(y_*, y) \partial_{y_j} \tilde{w}(y)| \leq 3 \frac{|w(y + y_*)|}{|y + y_*|} |\nabla \tilde{w}(y)| \leq 3 ||w||_{C^0(\mathbb{R}^3 \setminus B(0,1))} ||\nabla \tilde{w}(y)||.$$  \[ (106) \]
Moreover by (3) and (105), for any $\epsilon \in (0, 1)$ and any positive constant $C_{12}$ one can take parameter $R$ are sufficiently large such that

\[
\frac{\epsilon^2}{2(r_*(t, R))^{2\gamma}} \| \hat{w} \|^2_{L^2(B(0, r_*(t, R)))} + \frac{\epsilon^2}{2(r_*(t, R))^{2\gamma}} \| \nabla \hat{w} \|^2_{L^2(B(0, r_*(t, R)))} \\
- C_{12} \sum_{j=1}^3 \hat{B}_j(y_*, \cdot) \partial_{y_j} \hat{w} + \hat{C}(y_*, \cdot) \hat{w} \|_{L^2(B(0, r_*(t, R)))}^2 \geq 0.
\]

Applying to equation (104) the Carleman estimate (54) and using (106) we obtain that for any $s > 1$ and any $\epsilon \in (0, 1)$ there exist a constant $C_{13}(\epsilon)$ such that

\[
\frac{\epsilon^2}{2(r_*(t, R))^{2\gamma}} \| \nabla (\hat{w} / |y|^{2\gamma}) \| |y|^{-1+\epsilon} \|_{L^2(B(0, r_*(t, R)))}^2 \\
+ \frac{\epsilon^2 s^2}{2(r_*(t, R))^{2\gamma}} \| \hat{w} |y|^{-2s-2+\epsilon} \|_{L^2(B(0, r_*(t, R)))}^2 + \mathcal{I}(\partial B(0, r_*(t, R)), s, \frac{w}{|y|^{2\gamma}}) \\
\leq \int_{S(0, r_*(t, R))} \frac{8s^2 |w|^2}{r_*(t, R)^{4\gamma+3}} dS + \frac{\epsilon^2}{r_*(t, R)^{2\gamma+3}} \int_{S(0, r_*(t, R))} ((y, \nabla)(w / |y|^{-2\gamma}), w) dS \\
+ C_{13}[\chi(t, \cdot), L(y_*, y, D)] |w(y_*, \cdot)| |y|^{-2s} \|_{L^2(B(0, r_*(t, R)))}^2,
\]

where $\mathcal{I}(\partial B(0, r_*(t, R)), s, w)$ is defined by (53).

Since $y_* \to 0$ as $t \to +\infty$ then $r_*(t, R) \to \frac{1}{R}$ as $t \to +\infty$. Therefore we observe that for any positive $s$ and all $R > 1$ we have

\[
\mathcal{I}(\partial B(0, r_*(t, R)), s, \frac{w}{|y|^{2\gamma}}) \to \mathcal{I}(\partial B(0, \frac{1}{R}), s, \frac{w}{|y|^{2\gamma}}) \text{ as } t \to +\infty.
\]

Therefore

\[
\int_{S(0, r_*(t, R))} \frac{8s^2 |w|^2}{r_*(t, R)^{4\gamma+3}} dS + \frac{\epsilon^2}{r_*(t, R)^{2\gamma+3}} \int_{S(0, r_*(t, R))} ((y, \nabla)(w / |y|^{2\gamma}), w) dS \\
\to \int_{S(0, \frac{1}{R})} \frac{8s^2 |w|^2}{r_*(t, R)^{4\gamma+3}} dS + \epsilon^2 R^{2\gamma+3} \int_{S(0, \frac{1}{R})} ((y, \nabla) w / |y|^{2\gamma}, w) dS \text{ as } t \to +\infty.
\]

Let $s > 1$ be such that

\[
16s^3 + 8s^2 + (11 - 4\gamma^2)s > 0.
\]

We estimate the term $\mathcal{I}(\partial B(0, \frac{1}{R}), s, \frac{w}{|y|^{2\gamma}})$ from below.

For any $\delta \in (0, \frac{1}{4})$ we have the following inequality

\[
4s^2 R^{2\gamma} \int_{S(0, \frac{1}{R})} \left( w |y|^{-2s}, \frac{\partial (w |y|^{-2s})}{\partial r} \right) dS \\
\leq (4 - 2\delta)s \int_{S(0, \frac{1}{R})} R \left| \frac{\partial (w |y|^{-2s})}{\partial r} \right|^2 dS + \frac{4s^2 R^{8+4s}}{4 - 2\delta} \int_{S(0, \frac{1}{R})} |w|^2 dS
\]

and short computations imply:

\[
\mathcal{I}(\partial B(0, \frac{1}{R}), s, \frac{w}{|y|^{2\gamma}}) \geq (4 + 2\delta)s \int_{S(0, \frac{1}{R})} R \left| \frac{\partial (w |y|^{-2s})}{\partial r} \right|^2 dS \\
+(16s^3 + 8s^2 + 11s - \frac{2\delta s}{4 - 2\delta}) \int_{S(0, \frac{1}{R})} R^{4\gamma+3} |w|^2 dS - 4s \int_{S(0, \frac{1}{R})} R |\nabla (w |y|^{-2s})|^2 dS =
\]
\begin{align*}
2s\delta \int_{S(0, \frac{1}{4}\rho)} R |\frac{\partial (w|y|^{-2s})}{\partial r}|^2 dS + (16s^3 + 8s^2 + 11s - \frac{2\delta s}{4 - 2\delta}) \int_{S(0, \frac{1}{4}\rho)} R^{4s+3} |w|^2 dS \\
-4s \int_{S(0, \frac{1}{4}\rho)} R^{4s+3} \left( \frac{1}{\sin^2 \phi} |\partial_\phi w|^2 + |\partial_\rho w|^2 \right) dS = \\
2s\delta \int_{S(0, \frac{1}{4}\rho)} R |\frac{\partial (w|y|^{-2s})}{\partial r}|^2 dS + \\
R^{4s-1} (16s^3 + 8s^2 + 11s - \frac{2\delta s}{4 - 2\delta}) \int_{S(0, R)} |\omega|^2 dS \\
-4sR^{4s-1} \int_{S(0, R)} \left( \frac{1}{\sin^2 \phi} |\partial_\phi \omega|^2 + |\partial_\rho \omega|^2 \right) dS = \\
2s\delta \int_{S(0, \frac{1}{4}\rho)} R |\frac{\partial (w|y|^{-2s})}{\partial r}|^2 dS + \\
R^{4s-1} (16s^3 + 8s^2 + 11s - \frac{2\delta s}{4 - 2\delta}) \|\omega\|^2_{L^2(S(0, R))} - 4sR^{4s-1} \|\Lambda \omega\|^2_{L^2(S(0, R))}.
\end{align*}

Let \( \{R_j\}_{j=0}^\infty \) be a sequence such that \( R_j \to +\infty \) as \( j \to +\infty \) and 
\[ \gamma = \lim_{j \to +\infty} \|\Lambda \omega\|^2_{L^2(S(0, R_j))} / \|v\|^2_{L^2(S(0, R_j))}. \]

By definition of \( \gamma \) any positive \( \tilde{\epsilon} \) there exists \( j_0 \) such that \( \|\Lambda \omega\|^2_{L^2(S(0, R_j))} \leq (\gamma^2 + \tilde{\epsilon}) \|\omega\|^2_{L^2(S(0, R_j))} \) for all \( j \geq j_0 \). Hence by (111) we have
\begin{align*}
\mathcal{I}(\partial B(0, \frac{1}{R}) , s, \frac{w}{|y|^{2s}}) &\geq s\delta \int_{S(0, \frac{1}{4}\rho)} R |\frac{\partial (w|y|^{-2s})}{\partial r}|^2 dS \\
+R^{4s-1} (16s^3 + 8s^2 + (11 - 4\gamma^2 - \tilde{\epsilon})s) \|\omega\|^2_{L^2(S(0, R))}
\end{align*}
for any \( R \in \{R_j\}_{j=0}^\infty \).

Let us fix some \( R \) from \( \{R_j\}_{j=0}^\infty \). Then by (108) - (112) for all sufficiently large \( t \) one can take \( \epsilon \in (0, 1) \) such that
\begin{align*}
\mathcal{I}(\partial B(0, r_*(t, R)) , s, \frac{w}{|y|^{2s}}) - \int_{S(0, r_*(t, R))} \frac{8s^2|w|^2}{r_*(t, R)^{4s+3}} dS \\
-\frac{\epsilon^2}{r_*(t, R)^{2s+3}} \int_{S(0, r_*(t, R))} (|y| \nabla)(\frac{w}{|y|^{2s}}) dS > 0.
\end{align*}
So we can write the inequality (107) as
\begin{align*}
\|\nabla \tilde{w} \|^2_{L^2(B(0, r_*(t, R)))} + s\|\tilde{w} \|^2_{L^2(B(0, r_*(t, R)))} \\
\leq C_{14} \|\tilde{w} \|^2_{H^1(B(0, \frac{1}{R}^{1/2} - \frac{1}{2}))} \|\tilde{w} \|^2_{L^2(B(0, \frac{1}{R^{1/2} - \frac{1}{2}}))} \\
\leq C_{15} \|\tilde{w} \|^2_{H^1(B(0, \frac{1}{R^{1/2} - \frac{1}{2}}))} < 4s+1. \quad (113)
\end{align*}
Observe that the left hand side of this equality is always greater or equal strictly positive constant $C_{17} = C_{17}(\omega, s)$ such that for all sufficiently large $t$ we have
\[ \|\nabla \tilde{w}(y)\|_{L^2(B(0, r_*(t, R)))}^{2s - 1 + \varepsilon} + s \|\nabla \tilde{w}(y)\|_{L^2(B(0, r_*(t, R)))}^{2s - 2 + \varepsilon} \geq C_{17} > 0. \] (114)
Combining two inequalities (114) and (113), we obtain
\[ \|\omega\|_{W^{1, \infty}(B(x, \frac{1}{2}))} \geq \frac{C_{17}}{C_{16} t^{4s+1}} \] as $t \to +\infty$.

Taking the infimum of the right hand side of the above inequality over the set \{ $x \in \mathbb{R}^3 | |x| = t$ \} for all $t$ sufficiently large we obtain
\[ \sqrt{M_1(t)} \geq \frac{C_{17}}{C_{16} t^{4s+1}} \] as $t \to +\infty$.
This inequality and estimate (91) imply (7).

If the vorticity $\omega$ is equal to zero then $v$ is the gradient solution, namely locally there exist a function $\phi$ such that $v = \nabla \phi$, $p = -\frac{1}{2} |\nabla \phi|^2$. Since $\text{div} v = 0$ function $\phi$ is the harmonic function on a domain of definition. Then all components of the velocity vector $v$ are harmonic functions in $\mathbb{R}^3 \setminus \Omega$. The Kelvin transform $u$ of the function $v$ defined by (101) is harmonic in some neighborhood of zero possibly excluding the origin. On the other hand, thanks to our assumption (6) on the decay rate of the velocity field on infinity, the Kelvin transform of the function $v$ is bounded in some neighborhood of origin and therefore is harmonic at this neighborhood. The function $u$ can have at origin the zero of finite order. The inequality (7) with (8) can be obtained repeating the above arguments where instead of equation (102) one use equation $\Delta u = 0$. Proof of theorem is complete.  

**Proof of Theorem 1.4.** First we show that under the assumption of our theorem there exists a constant $C_1$ such that
\[ \|v\|_{H^{2}(\mathbb{R}^3 \setminus B(0, R))} \leq C_1, \] (115)
where $B(0, R)$ is some ball centered at zero of sufficiently large radius such that $\Omega \subset \subset B(0, R - 2)$.

Let $W$ be solution to the following boundary value problem
\[ \Delta W = \omega \quad \text{in} \ B(x, 2), \quad W|_{S(x, 2)} = 0, \] (116)
where $x \in \mathbb{R}^3$ be a point such that $B(x, 2) \subset \mathbb{R}^3 \setminus B(0, R)$ and $\omega = \text{curl} v$. By classical estimate for the Laplace operator there exists a constant $C_2$ independent of $x$ such that
\[ \|W\|_{H^1(B(x, 2))} \leq C_2 \|\omega\|_{H^{-1}(B(x, 2))} \leq C_3 \|v\|_{L^2(B(x, 2))} \leq C_4. \] (117)
Here we used (9) to obtain the last inequality.

Let $\rho$ be a smooth function such that
\[ \rho \in C^\infty\left(\overline{B(0, 2)}\right), \quad \rho|_{S(0, 2)} = 0, \]
\[ \rho(y) > 0 \quad \forall y \in B(0, 2); \quad \rho(x, \cdot) = \rho(x - \cdot). \] (118)
Taking the scalar product of the equation (66) and the function \( \rho^{100}(x, \cdot)W \) in \( L^2(B(x, 2)) \) we have

\[
\int_{B(x, 2)} \rho^{100}(x, \cdot)|\omega|^2 dy = \int_{B(x, 2)} \left\{-2 \sum_{k=1}^{3} \partial_{x_k} \rho^{100}(x, \cdot)(\partial_{x_k} W, \omega) - (\omega, W)\Delta \rho^{100}(x, \cdot) - ((v, \nabla)(\rho^{100}(x, \cdot)W), \omega) + ((\omega, \nabla)\rho^{100}(x, \cdot)V, v)\right\} dy.
\]

(119)

We estimate each term in the right hand side of inequality (119) separately. Applying the inequality \(|ab| \leq \frac{1}{2}(a^2 + b^2)\) we obtain that for any positive \( \varepsilon \)

\[
|\int_{B(x, 2)} (\omega, W)\Delta \rho^{100}(x, \cdot)dy| \leq C_5 \int_{B(x, 2)} \rho^{98}(x, \cdot)|\omega| dy \\
\leq \frac{1}{8} \int_{B(x, 2)} \rho^{100}(x, \cdot)|\omega|^2 dy + C_6 \|W\|^2_{L^2(B(x, 2))} \\
\leq \frac{1}{8} \int_{B(x, 2)} \rho^{100}(x, \cdot)|\omega|^2 dy + C_7.
\]

(120)

Here in order to obtain the last inequality in (120) we used (117). Using (117) and the inequality \(|ab| \leq \frac{1}{2}(a^2 + b^2)\) we have

\[
\int_{B(x, 2)} 2\sum_{k=1}^{3} \partial_{x_k} \rho^{100}(x, \cdot)(\partial_{x_k} W, \omega) dy \leq C_9 \int_{B(x, 2)} 2\rho^{99}(x, \cdot)|\nabla W|\omega dy \\
\leq \frac{1}{8} \int_{B(x, 2)} \rho^{100}(x, \cdot)|\omega|^2 dy + C_9 \|W\|^2_{H^1(B(x, 2))} \\
\leq \frac{1}{8} \int_{B(x, 2)} \rho^{100}(x, \cdot)|\omega|^2 dy + C_{10}.
\]

(121)

Using the fact that velocity field \( v \) belongs to the space \( L^\infty(\mathbb{R}^2 \setminus \Omega) \) we obtain

\[
\int_{B(x, 2)} |(v, \nabla)(\rho^{100}(x, \cdot)W), \omega)| dy \\
\leq C_{11} \|v\|_{L^\infty(\mathbb{R}^2 \setminus \Omega)} \int_{B(x, 2)} \rho^{99}(x, \cdot)(|\nabla W| + |W|)|\omega| dy \\
\leq \frac{1}{8} \int_{B(x, 2)} \rho^{100}(x, \cdot)|\omega|^2 dy + C_{12} \|W\|^2_{H^1(B(x, 2))} \\
\leq \frac{1}{8} \int_{B(x, 2)} \rho^{100}(x, \cdot)|\omega|^2 dy + C_{13}.
\]

(122)

In the similar way we estimate the following integral

\[
\int_{B(x, 2)} |(\omega, \nabla)\rho^{100}(x, \cdot)W, v)| dy \\
\leq C_{14} \|v\|_{L^\infty(\mathbb{R}^2 \setminus \Omega)} \int_{B(x, 2)} \rho^{99}(x, \cdot)(|\nabla W| + |W|)|\omega| dy \\
\leq \frac{1}{8} \int_{B(x, 2)} \rho^{100}(x, \cdot)|\omega|^2 dy + C_{15} \|W\|^2_{H^1(B(x, 2))} \\
\leq \frac{1}{8} \int_{B(x, 2)} \rho^{100}(x, \cdot)|\omega|^2 dy + C_{16}.
\]

(123)
Estimating the right hand side of (119) using (120)-(123) we obtain
\[ \sup_{x \in \mathbb{R}^3 \setminus B(0, R)} \int_{B(x, 2)} \rho^{100}(x, \cdot) |\omega|^2 dy \leq C_{17}, \]
where constant \( C_{17} \) is independent of \( x \). This inequality, classical estimate for elliptic equation (66) imply
\[ \sup_{x \in \mathbb{R}^3 \setminus B(0, R)} \int_{B(x, 2)} \left( \rho^{100}(x, \cdot) |\omega|^2 + \rho^{110}(x, \cdot) |\nabla \omega|^2 \right) dy \leq C_{18}, \]
where constant \( C_{18} \) is independent of \( x \).

Since \( \Delta v = \text{curl} \omega \) in \( \mathbb{R}^3 \setminus \Omega \), (124) by (9) and classical estimates for the Laplace operator we have
\[ \sup_{x \in \mathbb{R}^3 \setminus B(0, R)} \int_{B(x, 2)} \left( \rho^{100}(x, \cdot) |\omega|^2 + \rho^{110}(x, \cdot) |\nabla \omega|^2 + \rho^{120}(x, \cdot) |\nabla v|^2 \right) dy \leq C_{19}. \]
This in turn implies
\[ \sup_{x \in \mathbb{R}^3 \setminus B(0, R)} \int_{B(x, 2)} \left( \rho^{100}(x, \cdot) |\omega|^2 + \rho^{110}(x, \cdot) |\nabla \omega|^2 + \rho^{120}(x, \cdot) |\nabla v|^2 + \sum_{|\alpha| = 2} |\partial^\alpha \omega|^2 \right) dy \leq C_{20}. \]
From (124) for any multi-index \( \alpha \) we have
\[ \Delta \partial^\alpha v = \partial^\alpha \text{curl} \omega \quad \text{in} \quad \mathbb{R}^3 \setminus \Omega. \]
This equality and elliptic estimate for the Laplace operator imply that
\[ \|v\|_{H^{k+1}(B(x, r_a))} \leq C_{21}(k) (\|\omega\|_{H^{k}(B(x, r_b))} + \|v\|_{L^\infty(\mathbb{R}^3 \setminus \Omega)}) \forall B(x, r_b) \subset \mathbb{R}^3 \setminus \bar{\Omega}, \ 0 < r_a < r_b, \]
for all \( k = 1, 2, \cdots \), where the constant \( C_{21} \) depends only on \( r_a, r_b \). Inequalities (126) with \( k = 2 \) and (125) imply
\[ \|v\|_{H^3(B(x, \frac{r_b}{4}))} \leq C_{22}. \]
The inequality (127) combined with (125) implies
\[ \sup_{x \in \mathbb{R}^3 \setminus B(0, R)} \left( \|\omega\|_{L^\infty(B(x, \frac{r_b}{4}))} + \|\nabla v\|_{L^\infty(B(x, \frac{r_b}{4}))} + \|v\|_{H^3(B(x, \frac{r_b}{4}))} \right) < \infty. \]
On the other hand, writing (66) in the form,
\[ \Delta \omega = \text{div} (v \otimes \omega) - \text{div} (\omega \otimes v) \quad \text{in} \quad \mathbb{R}^3 \setminus \Omega, \]
and applying the operator \( \partial^\alpha \), \(|\alpha| = 2\) on the both sides of equation (129), one has by the elliptic estimate
\[
\sum_{|\beta|=3} \| \partial^\beta \omega \|_{L^2(B(x, \frac{\gamma}{2}))} \leq C_{23} \sum_{|\alpha|=2} \left( \| \partial^\alpha (v \otimes \omega) \|_{L^2(B(x, \frac{\gamma}{4}))} + \| \partial^\alpha (\omega \otimes v) \|_{L^2(B(x, \frac{\gamma}{4}))} \right)
\]
\[
\leq C_{24} \left( \sum_{|\alpha|=2} \| \partial^\alpha \omega \|_{L^2(B(x, \frac{\gamma}{4}))} \right) \| v \|_{L^\infty(B(x, \frac{3\gamma}{4}))}
\]
\[
+ \| \nabla \omega \|_{L^2(B(x, \frac{1\gamma}{4}))} \| \nabla v \|_{L^\infty(B(x, \frac{1\gamma}{4}))}
\]
\[
+ \sum_{|\alpha|=2} \| \partial^\alpha v \|_{L^2(B(x, \frac{\gamma}{4}))} \| \omega \|_{L^\infty(B(x, \frac{3\gamma}{4}))} \leq C_{25}.
\]
Hence, using (128) and (127) we obtain from (130):
\[
\sup_{x \in \mathbb{R}^3 \setminus B(0,R)} \left( \| \nabla \omega \|_{L^\infty(B(x, \frac{10\gamma}{4}))} + \sum_{|\beta|=2} \| \partial^\beta v \|_{L^\infty(B(x, \frac{10\gamma}{4}))} \right) < \infty. \tag{131}
\]
Applying the operator \( \partial^\alpha \), \(|\alpha| = 3\) on the both sides of (129), and using the elliptic estimate again, one obtains
\[
\sum_{|\beta|=4} \| \partial^\beta \omega \|_{L^2(B(x, 1))} \leq C_{26} \left( \sum_{|\alpha|=3} \| \partial^\alpha (v \otimes \omega) \|_{L^2(B(x, \frac{\gamma}{4}))} \right)
\]
\[
+ \| \partial^\alpha (\omega \otimes v) \|_{L^2(B(x, \frac{\gamma}{4}))} \right) \leq C_{27} \left( \sum_{|\alpha|=3} \| \partial^\alpha \omega \|_{L^2(B(x, \frac{\gamma}{4}))} \right) \| v \|_{L^\infty(B(x, \frac{3\gamma}{4}))}
\]
\[
+ \sum_{|\alpha|=2} \| \partial^\alpha v \|_{L^2(B(x, \frac{\gamma}{4}))} \| \nabla \omega \|_{L^\infty(B(x, \frac{1\gamma}{4}))}
\]
\[
+ \sum_{|\alpha|=2} \| \partial^\alpha v \|_{L^2(B(x, \frac{\gamma}{4}))} \| \nabla v \|_{L^\infty(B(x, \frac{1\gamma}{4}))}
\]
\[
+ \sum_{|\alpha|=3} \| \partial^\alpha v \|_{L^2(B(x, \frac{\gamma}{4}))} \| \omega \|_{L^\infty(B(x, \frac{3\gamma}{4}))} \leq C_{28}, \tag{132}
\]
where the last bound by constant follows by applying the previous estimates (125), (128), (130) and (131). Therefore, by (132) and (126), one has
\[
\sup_{x \in \mathbb{R}^3 \setminus B(0,R)} \left( \sum_{|\beta|=2} \| \partial^\beta \omega \|_{L^\infty(B(x, 1))} + \sum_{|\beta|=3} \| \partial^\beta v \|_{L^\infty(B(x, 1))} \right) < \infty. \tag{133}
\]
The estimate (133) implies (115).

From the Navier-Stokes equations, one has
\[
\partial_t \mathbf{u} = \Delta \mathbf{u} - \nabla \pi - \nabla \times \mathbf{u} \times \mathbf{u} - \frac{1}{3} \nabla \left( \sum_{k=1}^{3} \frac{1}{k} \partial_k \partial \mathbf{u} \cdot \partial \mathbf{u} \right) \quad \text{in} \ \mathbb{R}^3 \setminus \overline{\Omega}, \tag{134}
\]
for \( i, j \in \{1, 2, 3\} \). Therefore, estimating the right hand side of (134) by (128)-(133) one has
\[
\sup_{x \in \mathbb{R}^3 \setminus B(0,R)} \sum_{|\beta|=2} \| \partial^\beta \mathbf{u} \|_{L^\infty(B(x, 2))} < \infty. \tag{135}
\]
We set \( \Phi = p + \frac{1}{2} |v|^2 \). Consider the vector function
\[
W = (v, \omega, \nabla \Phi) : \mathbb{R}^3 \setminus \overline{\Omega} \to \mathbb{R}^9.
\]
Observe that this function satisfies to the Schrödinger equation
\[ \Delta W + VW = 0 \quad \text{in} \quad \mathbb{R}^3 \setminus \overline{\Omega}, \]
where potential \( V \) is given by formula
\[ V(x) = \begin{pmatrix} A & 0 & -I \\ B & C & 0 \\ D & E & F \end{pmatrix} \]
with matrices \( A, B, C, D, E, F \) defined by
\[
A_{ij} = -\partial x_i v_i + \partial x_j v_j, \quad B_{ij} = -\partial x_i \omega_i, \quad C_{ij} = \partial x_j v_i, \\
D_{ij} = -\partial^2 x_{i,j} \Phi, \quad E_{ij} = -2\partial x_i \omega_j, \quad F_{ij} = -\partial x_i v_j
\]
for \( i, j \in \{1, 2, 3\} \). We note that in order to obtain the last three equations in the system (136) we used the fact that
\[ \Delta \Phi - (v, \nabla \Phi) - |\omega|^2 = 0 \quad \text{in} \quad \mathbb{R}^3 \setminus \overline{\Omega}. \]
We claim that under the hypothesis (9), we have for sufficiently large \( R \)
\[ |V| + |W| \in L^\infty(\mathbb{R}^3 \setminus B(0, R)). \] (137)
The boundedness in \( \mathbb{R}^3 \setminus B(0, R) \) follows for the components \( A_{ij}, B_{ij}, C_{ij}, D_{ij}, E_{ij}, F_{ij} \) of the function \( V \). In order to show the boundedness for \( D_{ij} \) using the equation (134) we estimate
\[ \|\partial^2 x_{i,j} \Phi\|_{L^\infty(\mathbb{R}^3 \setminus B(0, R))} \leq \|\partial^2 x_{i,j} \Phi\|_{L^\infty(\mathbb{R}^3 \setminus B(0, R))} + \sum_{|\beta|=2} \|\partial^3 v\|_{L^\infty(\mathbb{R}^3 \setminus B(0, R))}. \]
The first norm in the right hand side of this inequality is finite by (135). The second and third norms are finite thanks to (133) and (115). The proof of (137) is complete.

Let function \( M_0(t) \) be introduced by formula
\[ M_0(t) = \inf_{|x|=t} \int_{|x-y|<1/2} |W(y)|^2 dy. \]
We claim that there exists a positive constant \( C_{29} = C_{29}(V, W) \) and positive constant \( \beta = \beta(v) \) such that
\[ M_0(t) \geq C_{29} e^{-\beta t^2 \ln(t)} \quad \forall t \geq t_0(v). \] (138)
The proof of this fact is given in appendix. (We note that for the case \( \Omega = 0 \) this inequality first established in [1].) On the other hand there exists a constant \( C_{30} \) such that
\[ M_0(t) \leq C_{30} M(t). \] (139)
Indeed, since \( -\nabla (p + \frac{1}{2} |v|^2) = \text{curl} \omega + (v, \nabla)v - \frac{1}{2} \nabla |v|^2 \) we have
\[
\int_{|x-y|<1/2} |W(y)|^2 dy \leq 2M(t) + 2 \int_{|x-y|<1/2} \left| \text{curl} \omega + (v, \nabla)v - \frac{1}{2} \nabla |v|^2 \right| dy \\
+ 2M_1(t) \leq C_{31}(M(t) + M_1(t) + M(t)\|v\|_{W^1_\infty(\mathbb{R}^3 \setminus B(0, R))}) \leq C_{32} M(t).
\]
Here $M_1(t)$ is given by (90) and in order to get the last equality we used (91). From (139) and (138) we obtain (10).

**Proof of Theorem 1.5.** Let us set $\Phi = p - p_0 + \frac{1}{2}|v|^2 := \bar{p} + \frac{1}{2}|v|^2$. Then, the steady Navier-Stokes system implies that

$$L_v(x, D)\Phi := \Delta \Phi - (v, \nabla \Phi) = |\omega|^2 \quad \text{in} \quad \mathbb{R}^3.$$  

For each $a \in \mathbb{R}^3$ and $t > 0$ we define,

$$\Phi_{t,a}(x) := \Phi(tx - a), \quad v_{t,a}(x) := v(tx - a), \quad \omega_{t,a}(x) = \omega(tx - a).$$

Then,

$$\Delta \Phi_{t,a} - t(v_{t,a}, \nabla \Phi_{t,a}) - t^2|\omega_{t,a}|^2 = 0 \quad \text{in} \quad \mathbb{R}^3. \quad (140)$$

By the maximum principle, we have $\Phi_{t,a} \leq 0$. Multiplying equation (140) by $e^{-s|x|}$ with some positive $s$ and integrating over $\mathbb{R}^3$, we obtain

$$\int_{\mathbb{R}^3} \left\{ s^2 - \frac{2s}{|x|} \right\} |\Phi_{t,a}| e^{-s|x|} dx + t^2 \int_{\mathbb{R}^3} |\omega_{t,a}|^2 e^{-s|x|} dx = st \int_{\mathbb{R}^3} |\Phi_{t,a}| e^{-s|x|} \frac{(v_{t,a}, x)}{|x|} dx.$$  

Let us choose parameter $s$ so that

$$s \geq \max\{2, 4t\|v\|_{L^\infty(\mathbb{R}^3)}\}. \quad (141)$$

Then, we have

$$\frac{s^2}{2} \int_{\mathbb{R}^3 \setminus B(0,2)} |\Phi_{t,a}| e^{-s|x|} dx + t^2 \int_{\mathbb{R}^3} |\omega_{t,a}|^2 e^{-s|x|} dx \leq st \int_{\mathbb{R}^3} |\Phi_{t,a}| e^{-s|x|} \frac{(v_{t,a}, x)}{|x|} dx - \int_{B(0,2)} (s^2 - \frac{2s}{|x|}) |\Phi_{t,a}| e^{-s|x|} dx. \quad (142)$$

Taking into account the estimate

$$st \int_{\mathbb{R}^3} |\Phi_{t,a}| e^{-s|x|} \frac{(v_{t,a}, x)}{|x|} dx \leq st \int_{\mathbb{R}^3} |\Phi_{t,a}| e^{-s|x|} |v_{t,a}| dx \leq \frac{s^2}{4} \int_{\mathbb{R}^3} |\Phi_{t,a}| e^{-s|x|} dx,$$

which is obvious from (141) for all $s$ greater than one, we obtain from (142) and (143) that

$$\frac{s^2}{4} \int_{\mathbb{R}^3 \setminus B(0,2)} |\Phi_{t,a}| e^{-s|x|} dx + t^2 \int_{\mathbb{R}^3} |\omega_{t,a}|^2 e^{-s|x|} dx \leq - \int_{B(0,2)} \frac{3s^2}{4} - \frac{2s}{|x|} |\Phi_{t,a}| e^{-s|x|} dx \leq 2s \int_{B(0,2)} \frac{|\Phi_{t,a}|}{|x|} e^{-s|x|} dx. \quad (144)$$
Since \( \Phi_{t,a} \leq 0 \) on \( \mathbb{R}^3 \) again, we have
\[
|\Phi_{t,a}(x)| = -\Phi_{t,a}(x) = -\bar{p}(tx - a) - \frac{|v(tx - a)|^2}{2} \leq \bar{p}(tx - a) \leq |\bar{p}(tx - a)| \text{ on } \mathbb{R}^3.
\] (145)

We take \( \tau > 6 \). Let us consider the sequence \( \{x_{\tau}\} \) such that
\[
|x_{\tau}| = \tau, \quad \int_{|x_{\tau} - y| < 1} \frac{|\bar{p}(y)|}{|x_{\tau} - y|} dy \leq 2M(\tau).
\] (146)

We change variables \( x = y - \frac{x_{\tau}}{2} \). Then, we observe
\[
\{x \in \mathbb{R}^3 \mid |x| < 1\} \leftrightarrow \{y \in \mathbb{R}^3 \mid |y - \frac{x_{\tau}}{2}| < 1\},
\]
and
\[
|y| \leq |y - \frac{x_{\tau}}{2}| + \frac{|x_{\tau}}{2} < 1 + \frac{\tau}{2} \leq 2\tau.
\]

Using (144), (145) and (146), observing \( B(\frac{x_{\tau}}{2}, 1) \subset \mathbb{R}^3 \setminus B(0, 2) \) for \( \tau > 6 \), we have
\[
\frac{s^2}{4} \int_{B(0,1)} |\Phi_{2,0}|e^{-2\tau s} dx \leq \frac{s^2}{4} \int_{B(\frac{x_{\tau}}{2}, 1)} |\Phi_{2,x_{\tau}}|e^{-s|x|} dx
\]
\[
\leq \frac{s^2}{4} \int_{\mathbb{R}^3 \setminus B(0,2)} |\Phi_{2,x_{\tau}}|e^{-s|x|} dx \leq 2s \int_{B(0,2)} \frac{|\Phi_{2,x_{\tau}}|}{|x|} e^{-s|x|} dx
\]
\[
\leq 2s \int_{B(0,2)} \frac{1}{|x|} |\bar{p}(2x - x_{\tau})| dx
\]
\[
\leq CsM(\tau),
\]
where constant \( C \) is independent of \( s \) and \( \tau \). From this inequality we have
\[
\frac{s^2}{4} \int_{B(0,1)} |\Phi_{2,0}| dx \leq CsM(\tau)e^{2\tau s}.
\]

We consider two sub-cases. If
\[
\int_{B(0,1)} |\Phi_{2,0}| dx \neq 0,
\]
then
\[
\lim_{\tau \to +\infty} M(\tau)e^{2\tau s} > 0.
\]

If \( \Phi_{2,0} = 0 \) on \( B(0,1) \), then this implies \( \Phi = 0 \) on \( B(0,2) \). Since \( \Phi(x) \leq 0 \) on \( \mathbb{R}^3 \), and \( \Phi(x) \to 0 \) as \( |x| \to \infty \), by the maximum principle applied to \( L_v(x, D)\Phi = |\omega|^2 \) we obtain \( \Phi = 0 \) on \( \mathbb{R}^3 \). Hence, \( \omega = 0 \) on \( \mathbb{R}^3 \). Then \( v = \nabla g \), where \( g \) is a harmonic function in \( \mathbb{R}^3 \). But since \( |v(x)| \to 0 \) as \( |x| \to +\infty \) we have that \( g = \text{constant} \), and therefore \( v = 0 \). Proof of theorem is complete. \( \square \)

**Appendix.** We give the proof of inequality (138).

**Proof.** Without a loss of generality we may assume that \( 0 \in \Omega \) and \( \Omega \subset B(0, \frac{1}{20}) \). Let us choose constant \( \tilde{\beta} \) such that
\[
\frac{\tilde{\beta}}{3^2} > \max \left\{ \frac{64}{35} \|V\|^2_{L^\infty(\mathbb{R}^3 \setminus \Omega)}, 1 \right\}.
\] (147)

Our proof of (138) is by contradiction. Suppose that (138) is false. Then for any \( \tilde{\beta} \) satisfying (147) there exist a constant \( C \) and a sequence \( t_j \to +\infty \) in general
depending on \( \beta \) such that \( \sqrt{M_0(t_j)} \leq Ce^{-\beta t_j^{4/3} \ln(t_j)} \). Consider functions \( \tilde{W}(x) = W(x/3), \tilde{V}(x) = \frac{1}{3} V(x/3) \). These functions satisfy the equation

\[
\Delta \tilde{W} + \tilde{V} \tilde{W} = 0 \quad \text{on} \quad \mathbb{R}^3 \setminus B(0,1).
\]

The short computations imply

\[
\inf_{|x|=t_j} \int_{|y| \leq 3/2} |W(y/3)|^2 dy = \sqrt{M_0(t_j)} \int_{|y| \leq 3/2} |W(y)|^2 dy = C e^{-\beta t_j^{4/3} \ln(t_j)}.
\]

From (148) we have

\[
\|\tilde{W}\|_{H^1(B(z_j,1))} \leq C_1 e^{-\frac{\beta}{3/4} R_j \frac{4}{3} \ln(R_j/3)}.
\]

We set \( \chi_0(x) \) be a cut-off function such that

\[
\chi_0(x) = 1 \quad \text{on} \quad B(0,\frac{1}{2}), \quad 0 \leq \chi_0(x) \leq 1 \quad \text{on} \quad B(0,1),
\]

\[
\chi_0(x) = 0 \quad \text{on} \quad \mathbb{R}^3 \setminus B(0,1).
\]

For any \( R > 2 \) let \( \tilde{\chi}_R(x) \in C_0^\infty(B(0,1)) \) be cut-off function such that

\[
\tilde{\chi}_R|_{B(0,1\frac{1}{4})} = 1, \quad |\tilde{\chi}_R(x)| \leq 1 \quad \text{on} \quad B(0,1),
\]

\[
|\partial^\alpha \tilde{\chi}_R(x)| \leq C_R |\alpha| \quad \forall |\alpha| \geq 0.
\]

Consider the functions \( W_{R_j}(x) = \tilde{W}((R_j - 1)x + z_j), V_{R_j}(x) = \tilde{V}((R_j - 1)x + z_j) \). The function \( W_{R_j} \) satisfies

\[
\Delta W_{R_j} + (R_j - 1)^2 V_{R_j} W_{R_j} = 0 \quad \text{on} \quad B(0,1).
\]

From (148) we have

\[
\|W_{R_j}\|_{H^1(B(0,\frac{1}{3}))} = \sqrt{\int_{B(0,\frac{1}{3})} ((R_j - 1)^2 |\nabla \tilde{W}|^2 + \tilde{W}^2) \circ ((R_j - 1)x + z_j) dx =
\]

\[
(R_j - 1)^{-\frac{1}{2}} \sqrt{\int_{B(z_j,1)} ((R_j - 1)^2 |\nabla \tilde{W}|^2 + \tilde{W}^2) dx}
\]

\[
\leq (R_j - 1)^{-\frac{1}{2}} \sqrt{\int_{B(z_j,1)} (|\nabla \tilde{W}|^2 + \tilde{W}^2) dx \leq C_1 (R_j - 1)^{-\frac{1}{2}} e^{-\frac{\beta}{3} R_j \frac{4}{3} \ln(R_j/3)}.
\]

Next we establish the Carleman estimate for the Laplace operator with the weight function \( \varphi(|x|) = -\ln(|x| - \frac{3}{2}|x|^2) \). Let us introduce the operators

\[
L(x, D, s) = e^{s\varphi} \Delta e^{-s\varphi} = \Delta - 2s(\nabla \varphi, \nabla \cdot) + s^2 |\nabla \varphi|^2 - s \Delta \varphi,
\]

\[
L_+(x, D, s) = \Delta + s^2 |\nabla \varphi|^2, \quad L_-(x, D, s) = -2s(\nabla \varphi, \nabla \cdot) + s \Delta \varphi.
\]
Integration by parts implies simple integration by parts provide the formulae

$$u$$

Moreover, since by (150) the function

$$\phi$$

Observe that

$$\nabla$$

Setting

$$\Delta u = L_+(x, D, s)u + L_-(x, D, s)u = f_s$$ on \(B(0,1)\).

Taking the \(L^2\)- norm of left hand side and right hand side of this equation we obtain the identity

$$\|f_s\|_{L^2(B(0,1))}^2 = \|L_+(x, D, s)u\|_{L^2(B(0,1))}^2 + \|L_-(x, D, s)u\|_{L^2(B(0,1))}^2 + 2\langle L_+(x, D, s)u, L_-(x, D, s)u \rangle_{L^2(B(0,1))}.$$  \hspace{1cm} (152)

Observe that \(\nabla \phi = \frac{r}{|x|^3} \varphi_r = -\frac{2}{3} \frac{(8-2|x|)}{8|x|^3}, \varphi_r = -\frac{1}{8|r|^2}, \varphi_{rr} = \frac{1}{r} + \frac{1}{8|x|^2} \Delta \varphi = \frac{6}{r} + \frac{1}{8|x|^2}\), \(\nabla \varphi = \frac{(1-4|x|^2)}{|x|^3}, \varphi_{rr} = \frac{1}{r} + \frac{1}{4|x|^2}\), and \(\Delta \varphi = \frac{6}{r} + \frac{1}{8|x|^2}\). Moreover, since by (150) the function \(u\) has a compact support in \(B(0,1 - \frac{1}{r^4})\), the simple integration by parts provide the formulae

$$s^3 (-2(\nabla \varphi, \nabla u) - \varphi_{rr} u, |\nabla \varphi|^2 u)_{L^2(B(0,1))} =$$

$$s^3 \int_{B(0,1)} (\nabla \varphi |\nabla \varphi|^2 |u|^2 - \varphi_{rr} |\nabla \varphi|^2 |u|)^2 \, dx =$$

$$2s^3 \int_{B(0,1)} \left( \frac{\varphi^2_3}{|x|} + \frac{\varphi^2 r}{r^2} \varphi_{rr} \right) u^2 \, dx =$$

$$16s^3 \int_{B(0,1)} \frac{\varphi^2 r u^2}{|x|(8 - |x|)^2} \, dx$$  \hspace{1cm} (153)

and

$$\langle \Delta u, -2s(\nabla \varphi, \nabla u) - s \varphi_{rr} u \rangle_{L^2(B(0,1))} =$$

$$s \int_{B(0,1)} (2(\varphi'' \nabla u, \nabla u) - 2\Delta \varphi |\nabla u|^2 - \frac{1}{2} \Delta \varphi_{rr} u^2) \, dx =$$

$$s \int_{B(0,1)} (2(-\frac{(x, \nabla u)^2}{|x|^3} \varphi_r + \frac{(x, \nabla u)^2}{|x|^2} \varphi_{rr}) - \frac{1}{2} \Delta \varphi_{rr} |u|^2) \, dx.$$  \hspace{1cm} (154)

Using (153) and (154) and the fact that functions \(-\varphi_r, \varphi_{rr}, \Delta \varphi\) are nonnegative on \(B(0,1)\) from (152) we have

$$\|f_s\|_{L^2(B(0,1))} \geq \|L_+(x, D, s)u\|_{L^2(B(0,1))}^2 - s \int_{B(0,1)} \Delta \varphi_{rr} |u|^2 \, dx$$

$$+ 12s^3 \int_{B(0,1)} \frac{(4 - |x|)^2}{|x|^3 (8 - |x|)^4} u^2 \, dx + \|L_-(x, D, s)u\|_{L^2(B(0,1))}^2.$$  \hspace{1cm} (155)

Integration by parts implies

$$\frac{1}{s} \|L_-(x, D, s)u\|_{L^2(B(0,1))}^2 = 4s(\nabla \varphi, \nabla u)_{L^2(B(0,1))}^2$$

$$- 2s \int_{L^2(B(0,1))} \text{div} (\varphi_{rr} \nabla \varphi) |u|^2 \, dx + s \|\varphi_{rr} u\|_{L^2(B(0,1))}^2.$$  \hspace{1cm} (156)

Since there exists a function \(G \in C^0(\overline{B(0,1)})\) such that

$$\text{div} (\varphi_{rr} \nabla \varphi) = \frac{1}{r^4} + \frac{1}{r^3} G(x)$$
there exists a constant $C_2$ independent of $s$ such that
\[
\int_B \frac{-4(x, \nabla u)^2}{|x|^3} \varphi_r dx - s \int_B \Delta \varphi_r |u|^2 dx \geq -C_2 \int_B \frac{|u|^2}{|x|^3} dx.
\] (156)

By (155) and (156) we have
\[
\|f_s\|_{L^2(B(0,1))}^2 \geq \|L_+(x, D, s)u\|_{L^2(B(0,1))}^2
\]
\[
+ (128s^3 - s^2) \int_B \frac{(4 - |x|)^2}{|x|^4(8 - |x|)^4} |u|^2 dx + \frac{1}{2} \|L_-(x, D, s)u\|_{L^2(B(0,1))}^2
\] (157)

for all $s \geq \tilde{s}_0$.

We set in the Carleman estimate (157) $s = s_j = \beta R_j^4$ where parameter $\beta$ such that
\[
\beta \in \left( \max \left\{ \frac{64}{3}\|V\|_{L^\infty(\mathbb{R}^3 \setminus \Omega)}, 1 \right\}, \frac{\tilde{\beta}}{3} \right).
\] (158)

By (147) such $\beta$ exists and since $\beta > 1$ for all sufficiently large $j$ we have $s_j > s_0$.

Then by (147) and (158) for all $R_j \geq 10$ and for all $x \in B(0,1)$
\[
\frac{128\beta^3 R_j^4(4 - |x|)^2}{(8 - |x|)^4|x|^3} - \frac{2}{34}(R_j - 1)^4 \|V\|_{L^\infty(\mathbb{R}^3 \setminus \Omega)}^2 \geq
\]
\[
\frac{84}{128\beta^3 R_j^4|s|^2} - \frac{2}{34}(R_j - 1)^4 \|V\|_{L^\infty(\mathbb{R}^3 \setminus \Omega)}^2 \geq
\]
\[
\frac{2\beta(R_j - 1)^4 |s|^2}{64} - \frac{2}{34}(R_j - 1)^4 \|V\|_{L^\infty(\mathbb{R}^3 \setminus \Omega)}^2 > 0.
\] (159)

Using definition of the function $f_s$ we have
\[
\|f_s\|_{L^2(B(0,1))} \leq 2 \|-(R_j - 1)^2 V_{R_j} u\|_{L^2(B(0,1))}^2
\]
\[
+ 2\|2s \Delta \varphi u + ([\chi_{R_j}, \Delta] W_{R_j} e^{s \varphi})\|_{L^2(B(0,1))}^2
\]
\[
\leq 2(R_j - 1)^4 \|V\|_{L^\infty(\mathbb{R}^3 \setminus \Omega)}^2 \|u\|_{L^2(B(0,1))}^2
\]
\[
+ 2\|2s \Delta \varphi u + ([\chi_{R_j}, \Delta] W_{R_j} e^{s \varphi})\|_{L^2(B(0,1))}^2.
\]

By (157), (159) and the above inequality we obtain for all sufficiently large $R_j$ there exist a positive constant $C_3$ independent of $j$ such that
\[
\|([\chi_{R_j}, \Delta] W_{R_j} e^{s \varphi})\|_{L^2(B(0,1))} \geq C_3 \int_{B(0,1)} s_j^2 |\chi_{R_j} W_{R_j}|^2 e^{2s_j \varphi} dx.
\] (160)

Hence by (149), (150) and (160) for all $R_j \geq R_0 > 0$ and some positive constant $C_4$ independent of $R_j$ such that
\[
\sqrt{\int_{B(0,1)} |\chi_{R_j} W_{R_j}|^2 e^{2s_j \varphi} dx} \leq \|[\chi_{R_j}, \Delta] W_{R_j} e^{s \varphi}\|_{L^2(B(0,1))}
\]
\[
\leq C_4 \left( \sqrt{\int_{B(0,1)} (\nabla W_{R_j})^2 dx + \int_{B(0,1)} (R_j^4 |\nabla W_{R_j}|^2) dx e^{s_j \varphi(\frac{7}{4 R_j^4})}}
\]
\[
+ \sqrt{\int_{B(0,1) \setminus B(0,1 - \frac{1}{R_j})} (R_j^4 |\nabla W_{R_j}|^2 + R_j^8 |W_{R_j}|^2) dx e^{s_j \varphi(1 - \frac{1}{R_j})}} \right).
\] (161)
Moreover observe that \( B(\frac{z_j}{R_j} (\frac{3}{R_j} - 1), \frac{1}{R_j}) \subset B(0, 1 - \frac{1}{R_j}) \). This follows from the inequality
\[
|x| \leq |x - \frac{z_j}{R_j} (\frac{3}{R_j} - 1)| + \left| \frac{z_j}{R_j} (\frac{3}{R_j} - 1) \right| \leq \frac{1}{R_j} + (1 - \frac{3}{R_j}) = 1 - \frac{2}{R_j}. \tag{162}
\]
Therefore by (149) and (150) we note that
\[
(x_{R_j} W_{R_j})(x) = W_{R_j}(x) = \tilde{W}((R_j - 1)x + z_j) \quad \text{on} \quad B(\frac{z_j}{R_j} (\frac{3}{R_j} - 1), \frac{1}{R_j})
\]
for all sufficiently large \( R_j \).

Next we claim that
\[
\inf_{x \in B(\frac{z_j}{R_j} (\frac{3}{R_j} - 1), \frac{1}{R_j})} \varphi(x) = \varphi(1 - \frac{2}{R_j}). \tag{163}
\]
Function \( \varphi \) depends only on \( |x| \), and it is strictly monotone decreasing function of \( |x| \) on interval \((0, 1)\). Therefore function \( \varphi \) will reach minimum on ball \( B(\frac{z_j}{R_j} (\frac{3}{R_j} - 1), \frac{1}{R_j}) \) if norm of \( x \) from \( B(\frac{z_j}{R_j} (\frac{3}{R_j} - 1), \frac{1}{R_j}) \) will reach maximum. If \( x \in B(\frac{z_j}{R_j} (\frac{3}{R_j} - 1), \frac{1}{R_j}) \) then by (162)
\[
|x| \leq 1 - \frac{2}{R_j}.
\]
On the other hand point \( \tilde{x} = \frac{z_j}{R_j} (\frac{3}{R_j} - 1) \) belongs to \( B(\frac{z_j}{R_j} (\frac{3}{R_j} - 1), \frac{1}{R_j}) \) and \( |\tilde{x}| = 1 - \frac{2}{R_j} \). Hence (163) is proved.

We set \( y_j = \left( \frac{z_j(R_j - 1)}{R_j} (\frac{3}{R_j} - 1) + z_j \right) \). Using this equality in (161) for all sufficiently large \( R_j \) by (163) we have
\[
(R_j - 1)^{-\frac{2}{3}} \| \tilde{W} \|_{L^2(B(y_j, \frac{R_j - 1}{R_j}))} = \\
\| \tilde{W}((R_j - 1)x + z_j) \|_{L^2(B(\frac{z_j}{R_j} (\frac{3}{R_j} - 1), \frac{1}{R_j}))} \\
\leq e^{-s_j \varphi(1 - \frac{2}{R_j})} \| W_{R_j} e^{s_j \varphi} \|_{L^2(B(\frac{z_j}{R_j} (\frac{3}{R_j} - 1), \frac{1}{R_j}))}. \tag{164}
\]
Estimating norm of the function \( W_{R_j} e^{s_j \varphi} \) in the right hand side of (164) by the right hand side of inequality (161) we obtain:
\[
(R_j - 1)^{-\frac{2}{3}} \| \tilde{W} \|_{L^2(B(y_j, \frac{R_j - 1}{R_j}))} \\
\leq C_5(R_j e^{s_j \varphi(1 - \frac{1}{R_j}) - s_j \varphi(1 - \frac{1}{R_j})} + \| W_{R_j} \|_{H^1(B(0, \frac{1}{R_j}))}) e^{s_j \varphi(\frac{1}{R_j} - s_j \varphi(1 - \frac{1}{R_j}))}. \tag{165}
\]
By the Taylor’s formula for all sufficiently large \( R_j \) we have
\[
s_j \left( \varphi(1 - \frac{1}{R_j}) - \varphi(1 - \frac{2}{R_j}) \right) = s_j \left( \frac{\varphi(1)}{R_j} + O(\frac{1}{R_j^2}) \right) = -\frac{\beta R_j^3}{2} + O \left( \frac{1}{R_j^3} \right).
\]
Using this equality in (165) and the fact that for all sufficiently large \( R_j \)
\[
-s_j \varphi(1 - \frac{2}{R_j}) \leq 0
\]
we obtain

\[ (R_j - 1)^{-\frac{5}{2}} \| \tilde{W} \|_{L^2(B(y_j, \frac{R_j - 1}{R_j }))} \leq C_0 (e^{-\beta R_j^4/4} + e^{\eta R_j^4/4} \| W_{R_j} \|_{H^1(B(0, \frac{1}{R_j }))}) \]  

(166)

By (151) and (158) there exist a positive \( \beta_0 \) such that for all sufficiently large \( R_j \) we have

\[ e^{\eta R_j^4/4} \| W_{R_j} \|_{H^1(B(0, \frac{1}{R_j }))} \leq C_0 e^{-\beta R_j^4/4} \]

\[ \leq C_0 (R_j - 1)^{-\frac{1}{2}} e^{-\beta R_j^4/4} \]

\[ \leq C_0 (R_j - 1)^{-\frac{1}{2}} e^{-\beta R_j^4/4} \ln(R_j). \]

Applying this inequality to estimate the last term in the right hand side of (166) for all sufficiently large \( R_j \) we obtain that

\[ \| \tilde{W} \|_{L^2(B(y_j, \frac{R_j - 1}{R_j }))} \leq C e^{-\frac{\beta R_j^4}{2}}. \]  

(167)

From the sequence \( y_j \) we take a subsequence \( \{ y_{j_k} \}_{k=1}^{\infty} \) which is convergent to some point \( \hat{y} \) such that \( |\hat{y}| = 4 \). Then by (167) function \( \tilde{W} \) is identically equal to zero on the ball \( B(\hat{y}, 1) \). Hence, uniqueness of solution of the Cauchy problem for the second order elliptic equation implies \( W \equiv 0 \). We arrived to the contradiction since velocity field \( v \) is not identically equal zero. Proof of inequality (138) is complete. \( \Box \)

Let \( \eta \in C_0^\infty[\frac{1}{2}, 1], \eta|_{\frac{1}{4}, \frac{3}{4}} = 1 \). We set \( \mu(t, x) = \eta(|x| - t - 1) \).

**Proposition 6.** Under conditions of the Theorem 1.2 there exist constant \( C_1 \) independent of \( t \) and \( t_1 \) such that

\[ \| \Lambda \omega \|_{L^2(S(0, t+1))} \leq C_1 \int_{B(0,t+2) \setminus B(0,t+\frac{1}{2})} (|\nabla (\mu \omega)|^2 + |(\mu \omega)^2|) dx \]  

(168)

for all \( t \geq t_1 \).

**Proof.** In order to prove (168) we construct the solution to the following boundary value problem

\[ P^*(t, x, D)w = q \quad \text{in} \quad \mathbb{R}^3 \setminus B(0, 1), \quad w|_{S(0,1)} = 0. \]  

(169)

Here \( P^*(t, x, D) \) is the operator formally adjoint to \( P(t, x, D) \) and the operator \( P(t, x, D) \) is obtained from the operator \( P(x, D) \) determined by formula (66) by change of variables \( x \to tx \). Setting \( \omega_t(x) = \omega(tx), v_t(x) = v(tx) \) from (66) we have

\[ P(t, x, D)\omega_t = \Delta \omega_t - t(v_t, \nabla)\omega_t + t(\omega_t, \nabla)v_t = 0 \quad \text{in} \quad \mathbb{R}^3 \setminus B(0, 1). \]  

(170)

Instead of the problem (169) consider the following boundary value problem

\[ R(t, x, D)u = q_* \quad \text{in} \quad B(0, 1), \quad u|_{S(0,1)} = 0. \]  

(171)

The operator \( R(t, x, D) \) has the form

\[ R(t, x, D) = \Delta + \sum_{j=1}^{3} B_j(t, x) \partial x_j + C(t, x). \]

It obtained from the operator \( P(t, x, D) \) in the following way. First we take the inversion change of variables \( x \to \frac{x}{|x|^2} \) in the operator \( P(t, x, D) \). We denote the
corresponding operator as $Q(t, x, D)$. Then $R(t, x, D) = \frac{1}{|x|^\epsilon} Q(t, x, D) \circ |x|$. By our assumptions (4) on decay of the velocity field $v$ there exist constant $C_0(\epsilon)$ independent of $t, x$ and constant $t_1$ such that for any $j \in \{1, 2, 3\}$ we have

$$|C(t, x)| \leq \frac{C_0(\epsilon)}{|x|^{2-\epsilon} t^\epsilon}, \quad |B_j(t, x)| \leq \frac{C_0(\epsilon)}{|x|^{1-\epsilon} t^\epsilon} \quad \forall x \in B(0, 1) \text{ and } t \geq t_1. \quad (172)$$

The first and zero order terms of the operator $R(t, x, D)$ we denote as $R_1(t, x, D)$. We claim that for any positive $\delta$ there exist $\epsilon_0 > 0$ and $t_2$ such that

$$\|R_1(t, x, D)\|_{W^{2, \frac{3}{2} + \epsilon_0}(B(0, 1))} \leq \delta \quad \forall t \geq t_2. \quad (173)$$

In order to prove (173) we estimate the contribution of first order terms and zero order terms to the norm the operator $R_1(t, x, D)$ separately. Let $\epsilon_0 = \epsilon_0(\delta) \in (0, \frac{3\epsilon}{2(2+\epsilon)})$. Using (172) we have

$$\|Cw\|_{L^{3, \frac{3}{2} + \epsilon_0}(B(0, 1))} \leq \|C\|_{L^{3, \frac{3}{2} + \epsilon_0}(B(0, 1))} \|w\|_{L^\infty(B(0, 1))} \leq C_2\|C\|_{L^{3, \frac{3}{2} + \epsilon_0}(B(0, 1))} \|w\|_{W^{2, \frac{3}{2} + \epsilon_0}(B(0, 1))} \leq \frac{C_3}{t^\epsilon} \left( \int_{B(0, 1)} \frac{1}{|x|^{2-\epsilon}(\frac{3}{2} + \epsilon_0)} \right)^{\frac{1}{2} + \epsilon_0} \|w\|_{W^{2, \frac{3}{2} + \epsilon_0}(B(0, 1))} \leq \frac{C_4}{t^\epsilon} \left( \int_0^1 \frac{1}{r^{1+2\epsilon_0-(\frac{3}{2} + \epsilon_0)}} \right)^{\frac{1}{2} + \epsilon_0} \|w\|_{W^{2, \frac{3}{2} + \epsilon_0}(B(0, 1))} \leq \frac{C_5}{t^\epsilon} \|w\|_{W^{2, \frac{3}{2} + \epsilon_0}(B(0, 1))}. \quad (174)$$

Here we used the fact that the integral respect to $\epsilon$ in (174) is convergent since our assumption on $\epsilon_0$ implies $2\epsilon_0 - \epsilon(\frac{3}{2} + \epsilon_0) < 0$.

Applying the Cauchy inequality and using (172) for any $j \in \{1, 2, 3\}$ we have

$$\|B_j \nabla w\|_{L^{3, \frac{3}{2} + \epsilon_0}(B(0, 1))} \leq \frac{C_6}{t^\epsilon} \|w\|_{L^{3, \frac{3}{2} + \epsilon_0}(B(0, 1))} \leq \frac{C_7}{t^\epsilon} \left( \int_{B(0, 1)} \frac{1}{|x|^{3+2\epsilon_0}(1-\epsilon)} \right)^{\frac{3+2\epsilon_0}{2+\epsilon_0}} \|\nabla w\|_{L^{3+2\epsilon_0}(B(0, 1))}. \quad (175)$$

By our assumption on $\epsilon_0$ the number $(3 + 2\epsilon_0)(1 - \epsilon) - 3$ is negative. Then the integral

$$\int_{B(0, 1)} \frac{1}{|x|^{(3+2\epsilon_0)(1-\epsilon)}} dx$$

is convergent. By the Sobolev embedding theorem $W^{1, \frac{3}{2} + \epsilon_0}_0(B(0, 1)) \subset L^q(B(0, 1))$ with $\frac{1}{q} = \frac{1}{\frac{3}{2} + \epsilon_0} - \frac{1}{4}$. Since $q = 3\frac{3+2\epsilon_0}{2+\epsilon_0} > 3 + 2\epsilon_0$ from (175) for all sufficiently large $t$ we have

$$\|B_j \nabla w\|_{L^{3, \frac{3}{2} + \epsilon_0}(B(0, 1))} \leq \frac{C_8}{t^\epsilon} \|w\|_{W^{2, \frac{3}{2} + \epsilon_0}(B(0, 1))} \quad \forall j \in \{1, 2, 3\}. \quad (176)$$

Hence by (174) and (176) the estimate (173) is proved. Therefore, there exists $t_3$ such that for all $t \geq t_3$ the boundary value problem (171) has a unique solution for any function $q_\epsilon \in L^{3, \frac{3}{2} + \epsilon_0}(B(0, 1))$ and this solution satisfies the estimate

$$\|w\|_{W^{2, \frac{3}{2} + \epsilon_0}(B(0, 1))} \leq C_9 \|q_\epsilon\|_{L^{3, \frac{3}{2} + \epsilon_0}(B(0, 1))}. \quad (177)$$
where constant $C_9$ is independent of $t$ and $q_*$. Next we take the Kelvin transform of problem (171) to obtain a solution to problem (169).

Take $q = \omega_t$ and denote the corresponding solution to problem (170) as $w_t$. By (177) the following estimate is true

$$\| \partial_t w_t \|_{L^2(S(0,1))} \leq C_{10} \| \omega \chi_{B(0,3) \setminus B(0,1)} \|_{L^2(B(0,3) \setminus B(0,1))}$$

where constant $C_{10}$ is independent of $t$ provided that $t \geq t_3$.

Taking the scalar product in $L^2(\mathbb{R}^3 \setminus B(0,1))$ of equation (169) with function $w_t$ and integrating by parts and using decay properties of the velocity field we obtain

$$\int_{S(0,1)} (\omega_t, \partial_t w_t) \, dS = \| \omega_t \|_{L^2(B(0,3) \setminus B(0,1))}^2. \quad (179)$$

Combining (178) and (179) and applying the Cauchy inequality we obtain

$$\| \omega_t \|_{L^2(B(0,3) \setminus B(0,1))} \leq C_{11} \| \omega_t \|_{L^2(S(0,1))}. \quad (180)$$

Let $\tilde{\omega} = \mu(t, \cdot) \omega$. The function $\tilde{\omega}$ solves the boundary value problem

$$P(x, D)\tilde{\omega} = [\mu(t, \cdot), P(x, D)]\omega \text{ in } B(0, t + 2) \setminus B(0, t + \frac{1}{2}),$$

$$\tilde{\omega}|_{S(0,t+2)} = \tilde{\omega}|_{S(0,t+\frac{1}{2})} = 0. \quad (181)$$

By the definition of the function $\mu$ and (180) there exist a constant $C_{12}$ independent of $t$ such that

$$\| [\mu(t, \cdot), P(x, D)] \omega \|_{W^{-1}_2(B(0,t+2) \setminus B(0,t+\frac{1}{2}))} \leq C_{10} \| \omega \|_{L^2(B(0,t+2) \setminus B(0,t+\frac{1}{2}))} \leq C_{12} \| \omega \|_{L^2(S(0,t))} \quad \forall t \geq t_4. \quad (182)$$

Taking the scalar product of equation (181) with function $\tilde{\omega}$ in $L^2(B(0, t + 2) \setminus B(0, t + \frac{1}{2}))$ we obtain

$$\int_{B(0,t+2) \setminus B(0,t+\frac{1}{2})} (|\nabla \tilde{\omega}|^2 - (\tilde{\omega}, \nabla v, \tilde{\omega})) \, dx =$$

$$- (\mu(t, \cdot), P(x, D) \omega, \tilde{\omega})_{L^2(B(0,t+2) \setminus B(0,t+\frac{1}{2}))}. \quad (183)$$

By (182) we obtain from (183):

$$\int_{B(0,t+2) \setminus B(0,t+\frac{1}{2})} (|\nabla \tilde{\omega}|^2 - (\tilde{\omega}, \nabla v, \tilde{\omega})) \, dx \leq C_{13} \| \omega \|_{L^2(S(0,t))} \| \tilde{\omega} \|_{W^1_2(B(0,t+2) \setminus B(0,t+\frac{1}{2}))}. \quad (184)$$

By our assumption on decay of the velocity field on infinity from (184) we have that there exists a constant $C_{14}$ independent of $t$ such that

$$\int_{B(0,t+2) \setminus B(0,t+\frac{1}{2})} |\nabla \tilde{\omega}|^2 \, dx \leq C_{14} \| \omega \|_{L^2(S(0,t))} \| \tilde{\omega} \|_{W^1_2(B(0,t+2) \setminus B(0,t+\frac{1}{2}))} \leq C_{15} \| \tilde{\omega} \|_{L^2(B(0,t+2) \setminus B(0,t+\frac{1}{2}))}.$$  \quad (185)

Thanks to the identity

$$\frac{t^2}{r^2} \int_{S(0,\tau)} |\tilde{\omega}|^2 \, dS = \int_{B(0,\tau) \setminus B(0,t)} \frac{t^2}{r^2} (\partial_r \tilde{\omega}, \tilde{\omega}) \, dx \quad \forall \tau > t$$
there exists a constant $C_{16}$ independent of $t$ such that
\[
\int_{B(0,t+2) \setminus B(0,t+\frac{1}{2})} |\hat{\omega}|^2 \, dx \leq C_{16} \int_{B(0,t+2) \setminus B(0,t+\frac{1}{2})} |\nabla \hat{\omega}|^2 \, dx.
\] (186)
So, from (185) and (186), for all sufficiently large $t$ we have
\[
\int_{B(0,t+2) \setminus B(0,t+\frac{1}{2})} (|\nabla \hat{\omega}|^2 + |\hat{\omega}|^2) \, dx \leq C_{17} \|\omega\|_{L^2(S(0,t))}^2.
\] (187)
The vorticity equation (66) and standard estimates for elliptic equation imply the existence of a constant $C_{18}$ independent of $t$ such that
\[
\|\Lambda \omega\|_{L^2(S(0,t+1))}^2 \leq C_{18} \int_{B(0,t+2) \setminus B(0,t+\frac{1}{2})} (|\nabla \hat{\omega}|^2 + |\hat{\omega}|^2) \, dx.
\] (188)
Combining (187) and (188) we obtain (168). \qed

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