An Intermittency Model For Passive-Scalar Turbulence

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Abstract

A phenomenological model for the inertial range scaling of passive-scalar turbulence is developed based on a bivariate log-Poisson model. An analytical formula of the scaling exponent for three-dimensional passive-scalar turbulence is deduced. The predicted scaling exponents are compared with experimental measurements, showing good agreement.

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The inertial range dynamics of a passive scalar advected by fluid turbulence is of great theoretical interest [1]. In particular, recent success [2] in the deduction of the anomalous scaling exponent for the passive scalar field advected by a white and Gaussian velocity has inspired renewed enthusiasm for the problem [3].

For the three dimensional turbulence diffusion, the advective velocity field is governed by the Navier-Stokes equation whose statistics is far from white and Gaussian. The extension of Kraichnan’s theory [2] for this problem is intriguing and non-trivial. In fact it has even been a difficult task to experimentally measure the scaling exponents of the passive scalar. The existing experimental data do not yield convergent results [1 4]. Direct numerical simulation (DNS) has been a useful alternative for studying scaling behaviors in fluid turbulence [7]. In a recent note, using the DNS data, we have tried to explain the discrepancies among experimental results [8].

The existing phenomenological theories, such as the β model [9] and a bivariate log-Normal model [10], are based on fractal structures of turbulence [5], yielding analytical predictions of scaling exponents. Nevertheless, it has been noted that log-normal distribution leads to negative values of scaling exponents for high order passive scalar structure functions, contradicting existing numerical and experimental measurements.

In this letter, we develop a phenomenological theory based on a bivariate log-Poisson model. The analytical formula for the scaling exponents is obtained. The scaling exponents from the theory are compared with those from experimental measurements, showing good agreement.

According to Kolmogorov’s refined similarity hypothesis (RSH), or K62 theory [11] for the velocity field and similar RSH for the passive scalar field proposed by Obukhov [12],

\begin{align*}
\Delta u_\ell &= v_u (\ell \epsilon_\ell)^{1/3}, \\
\Delta T_\ell &= v_T \ell^{1/3} \epsilon_\ell^{-1/6} N_\ell^{1/2},
\end{align*}

(1)

where $\Delta u_\ell = u(x + \ell) - u(x)$ and $\Delta T_\ell = T(x + \ell) - T(x)$ are the longitudinal velocity increment and the scalar increment respectively, $l$ is in the inertial range, $v_u$ and $v_T$ are
random variables whose statistics only depend on the Reynolds number. \( \epsilon \) and \( N \) are the locally averaged velocity dissipation and scalar dissipation, respectively [4]. Equation (1) has been verified by experiments and is in general believed to be correct [13]. Suppose that

\[ S_p(\ell) = \langle (\Delta u_\ell)^p \rangle \sim \ell^{\zeta_p}, \quad S_T^p(\ell) = \langle (\Delta T_\ell)^p \rangle \sim \ell^{z_p}, \]

then to deduce scaling exponents from (1), a probability density function (PDF) of velocity field and a joint PDF for velocity and passive scalar must be modeled. Here \( \langle \cdot \rangle \) denotes an ensemble and \( \zeta_p \) and \( z_p \) are the \( p \)-th order scaling exponents for the velocity and scalar structure functions, respectively. The log-normal assumption in K62 theory [11] has been challenged recently by She and Leveque [14] who argued that a log-Poisson PDF for \( \epsilon \) leads to a better description of the statistics of \( \epsilon \). A hierarchy model based on the log-Poisson PDF was derived for the inertial range scaling exponents, showing good agreement with numerical simulations [7] and experiments [15].

The distribution function of a random variable \( x \) satisfying Poisson distribution can be written as [16]:

\[ p_i = e^{-a} a^i / i!, \]

where \( a \) is the mean for the variable. The generating function of \( p_i \) has the form:

\[ P(s) = \sum_i p_i s^i = e^{-a + as}. \]

The generating function for a bivariate Poisson distribution \( p_{i,j} = p(x = i, y = j) \) can be defined as

\[ P(s_1, s_2) = \sum_{i,j} p_{i,j} s_1^i s_2^j = e^{-a_1 - a_2 - b + a_1 s_1 + a_2 s_2 + bs_1 s_2}, \]

where \( a_1, a_2 \) are variances for \( x \) and \( y \), respectively, and \( b \) is a constant representing the correlation between \( x \) and \( y \). Let us assume that the velocity dissipation and passive scalar dissipation have the hierarchy structure relation:

\[ \epsilon_{\ell_1} = W_{\ell_1 \ell_2} \epsilon_{\ell_2} \quad \text{and} \quad N_{\ell_1} = V_{\ell_1 \ell_2} N_{\ell_2}, \]

where \( \ell_1 \) and \( \ell_2 \) are two length scales; \( W_{\ell_1 \ell_2} \) and \( V_{\ell_1 \ell_2} \) are multiplicative factors depending on \( \ell_1 \) and \( \ell_2 \). For \( \ell_1 \) and \( \ell_2 \) within the inertial range, the multiplicative factors can be written in the forms of

\[ W_{\ell_1 \ell_2} = (\ell_1 / \ell_2)^{h X} \quad \text{and} \quad V_{\ell_1 \ell_2} = (\ell_1 / \ell_2)^{h T} \beta_T, \]

where \( \beta, \beta_T, h \) and \( h_T \) are constants to be determined later; \( (X, Y) \) are stochastic variables. In the following, we assume that \( (X, Y) \) follow the bivariate Poisson distribution \( p_{i,j} \).
If there is a scaling, then for $\ell$ in the inertial range, $\langle \epsilon^p_{\ell} N^q_{\ell} \rangle \sim \ell^{\tau(p,q)}$, where $\tau_{p,q}$ is the scaling exponent of power order $p$ and $q$. Using the bivariate Poisson distribution assumption, we have:

$$\langle W^p_{\ell_1 \ell_2} V^q_{\ell_1 \ell_2} \rangle = (\ell_1/\ell_2)^{ph+qh} \sum_{i,j} \beta^p_i \beta^q_j p_{ij}.$$  

It is easy to recognize that the summation in the right hand side of the above equation is nothing but the generating function for the bivariate Poisson distribution. It is then straightforward to write a formula for $\tau(p,q)$:

$$\tau(p,q) = -hp - h_T q + (a + a_T + b - a\beta^p - a_T \beta^q_T - b\beta^p_T \beta^q_T) / \ln(\ell_1/\ell_2),$$  

where constants $a$, $a_T$ and $b$ are functions of $\ell_1$ and $\ell_2$. The obvious physical constrains $\tau(1,0) = \tau(0,1) = 0$ result the following relations: $a = [h/(1 - \beta)] \ln(\ell_1/\ell_2 - b)$ and $a_T = [h_T/(1 - \beta_T)] \ln(\ell_1/\ell_2 - b)$. If we further assume the correlation between $\epsilon_{\ell}$ and $N_{\ell}$ satisfying power law scaling in the inertial range, i.e. $b = \gamma_b \ln(\ell_1/\ell_2)$ and $\gamma_b$ is a constant, we obtain the scaling exponent relation:

$$\tau(p,q) = \tau_p + \tau_q + \tau_b(p,q),$$  

where

$$\tau_p = -hp + [h/(1 - \beta)](1 - \beta^p),$$

and

$$\tau_q = -h_T q + [h_T/(1 - \beta_T)](1 - \beta^q_T),$$

are scaling exponents for moments $\langle \epsilon^p_{\ell} \rangle$ and $\langle N^q_{\ell} \rangle$ respectively, and

$$\tau_b(p,q) = \gamma_b (1 - \beta^p - \beta^q_T + \beta^p \beta^q_T)$$

is the contribution from the cross correlation. There have been several discussions about how to determine the model constants in $\tau_{p,q}$. The parameter $\beta$ depends on the
constant $h$ and the intermittency parameter $\mu = \tau_2$ \cite{17}, i.e. $\beta = 1 - \mu/h$. It has been argued that the co-dimension, $C$, for the most singular structure of the velocity dissipation \cite{14} is related to $\beta$ and $h$:

\begin{equation}
C = h/(1 - \beta) = h^2/\mu. \tag{4}
\end{equation}

FIG. 1. Theoretical scaling exponents $\zeta_p$ ($p \leq 0$) for the velocity structure function field as a function of $p$ ($\times$), compared with numerical measurements ($\square$) \cite{18}.

The original hierarchy structure model by She-Leveque assumes that the high intensity dissipation region tends to form tube-like structures which implies: $C = 2$. Using $\mu = 2/9$ \cite{4}, we obtain $h = \pm 2/3$. The sign associated with $h$ is due to the square operation in (4). If we further assume $z_p$ grows slower than exponential as $p$ tends to $\pm \infty$, the positive (or negative) $h$ has to be chosen for the positive (or negative) power $p$ in (4). The original SL model with $h = 2/3$ corresponds to the range of positive powers, yielding $\beta = h = 2/3$. As shown in \cite{18} for $S_p^T$ with $p \geq 0$, only the negative power $p$ for $\tau_p$ is involved. Hence, for $p \leq 0$, we choose $h = -2/3$, yielding $\beta = 4/3$ and:

\begin{equation}
\tau_p = \frac{2}{3}p + 2[1 - (4/3)^p]. \tag{5}
\end{equation}

Using RSH relation in (4), we have

\begin{equation}
\zeta_p = \frac{5}{9}p + 2[1 - (4/3)^{p/3}]. \tag{6}
\end{equation}

In a recent paper \cite{18}, we have compared DNS data with the theoretical prediction using $h = 2/3$ for $\zeta_p$ when $-1 \leq p \leq 0$. In Fig. 1, we show again this comparison with (4) for $h = -2/3$. 

\begin{figure}

\centering
\includegraphics[width=0.5\textwidth]{fig1.png}
\caption{Theoretical scaling exponents $\zeta_p$ ($p \leq 0$) for the velocity structure function field as a function of $p$ ($\times$), compared with numerical measurements ($\square$) \cite{18}.}
\end{figure}
The similar procedure can be carried out for the passive scalar dissipation. For \( q \geq 0 \), only the positive \( q \) in \( \tau_q \) has contribution. Consequently, a positive \( h_T \) has to be chosen. Using the similar arguments as for the velocity field, we finally obtain \( C_T = h_T/(1 - \beta_T) = h_T^2/\mu_T \), where \( \mu_T \) is the intermittency parameter for the passive scalar dissipation \( N_{\ell} \). It is noted that the high intensity region for the passive scalar dissipation \( N \) tends to form sheet-like structures [8], implying the co-dimension \( C_T = 1 \). We then obtain \( h_T = \sqrt{\mu_T} \) and \( \beta_T = 1 - \sqrt{\mu_T} \). From numerical [8] and experimental [6] data, \( \mu_T \approx 1/4 \), we have:

\[
\tau_q = -\frac{1}{2}q + [1 - (1/2)^q],
\]

(7)

From (5) and (7), \( \tau(p, q) \) can be written as follows:

\[
\tau(p, q) = 3 + \frac{2}{3}p - 2(\frac{4}{3})^p - \frac{1}{2}q - (1/2)^q + \gamma_b[1 - (\frac{4}{3})^p - (1/2)^q + (\frac{4}{3})^p(1/2)^q].
\]

(8)

Using (8) and (9), we have the following analytical formula for \( z_p \):

\[
\begin{align*}
z_p &= \frac{1}{3}p + \tau(-\frac{p}{6}, \frac{p}{2}) \\
&= 3 - \frac{1}{36}p - 2(3/4)^{p/6} - (1/2)^{p/2} \\
&\quad + \gamma_b[1 - (3/4)^{p/6} - (1/2)^{p/2} + (3/4)^{p/6}(1/2)^{p/2}].
\end{align*}
\]

(9)

In the above equation, \( \gamma_b \) is not determined. At this moment, there is no direct measurement of \( \gamma_b \) and we treat it as a free parameter. In Fig. 2a we plot \( z_p \) as functions of \( p \) for \( \gamma_b = 0, 0.1 \) and 0.2. The scaling exponents from the experimental measurement carried out by Antonia et al. [6] have also been included. A good agreement between theory and measurement for \( \gamma_b \approx 0 \) is seen, indicating that the correlation between \( \epsilon_{\ell} \) and \( N_{\ell} \) is weak. In fact, the variation of the scaling exponents is not drastic for \( \gamma_b \) varying from 0 to 0.2. In Fig. 2b, we demonstrate the model dependence on co-dimension \( C_T \). We notice that when \( C_T \) is between 0.75 and 1, the model results agree with the experiment. As \( C_T \) increases further, the model prediction deviates from the experimental data. It has been found that the co-dimension of passive scalar field is around 0.6 [19]. It would be useful to directly measure \( C_T \), the co-dimension of passive scalar dissipation using experimental or numerical simulation data.
The evidence of scaling exponents for $\gamma_b = 0$ matches well with those from experimental measurements might indicate real physics. As a matter of fact, from (1) we notice that for positive and large $p$ values, the major contribution of $\varepsilon^{-p/6}$ to $S_p^T$ comes from small amplitude events whose structures are blob-like, whereas the major contribution of $N_p^{p/2}$ to $S_p^T$ comes from large amplitude events whose structures are sheet-like. The former is often associated with weak stretching regions while the latter is associated with strong stretching regions. We suspect that the weak spatial correlation of these regions might be the direct reason why the nearly-independent log-Poisson with $\gamma_b \approx 0$ gives a good prediction of the scaling exponents. We should mention that (1) fails to be valid if power $p$ is too large. In fact, when $p$ is larger than 36 and $\gamma_b$ equals to zero, $z_p$ starts be negative. Up to now, there is very little understanding about the statistics of very high moments, due to the difficulties in both experiments or numerical simulations. Consequently, there is no solid justification for using the RSH and Hierarchy model in such high power range.

![Graph](image)

**FIG. 2.** Theoretical scaling exponents $z_p$ for the passive scalar field as functions of $p$ compared with experimental measurements [6], showing the model dependence on: (a) $\gamma_b$ while the $C_T$ equals 1; (b) $C_T$ while $\gamma_b$ equals 0.
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