On Spectrum of Nonlinear Continuous Operators

Kamal N. Soltanov

Abstract. This article proposed a new approach to the determination of the spectrum for nonlinear continuous operators in the Banach spaces and using it investigated the spectrum of some classes of operators. Here shows that in nonlinear operators case is necessary to seek the spectrum of a given nonlinear operator relatively to another nonlinear operator. Moreover, the order of nonlinearity of examined operator and operator relatively to which seek the spectrum must be identical. Here provided different examples relative to how one can find the eigenvalue and also studied solvability problems.

1. Introduction

Well-known that the spectral theory for linear operators is one of the most important topics of linear functional analysis, as in many cases for the study of the linear operator, it is enough to study its spectrum (e.g. it play an essential role in the theory of linear differential operators). It should be noted the spectral theory of linear operators has essential application in the many topics of the natural sciences (moreover, the spectral theory is at the foundation of quantum mechanics). The essential information on a linear operator contains in its spectrum, consequently, knowledge of the spectrum is knowing of many properties of the operator.

As the basic problems of physics, mechanics etc. are nonlinear, at least, the corresponding differential equations are nonlinear, consequently, these problems generates nonlinear operators in some Banach spaces. It needs to be noted in the literature there exist sufficiently many approaches for the definition of the spectrum for nonlinear continuous operators that beginning at 60 years of the previous century. In the above-mentioned time were introduced the various definitions for the spectra for various classes of nonlinear continuous operators. Unfortunately, many of the introduced definitions in the really were found only infimum of the spectrum of a continuous nonlinear operator which allows these authors to study the nonlinear equations of the form \( f(\cdot) - \lambda_1 g(\cdot) = 0 \) in the Banach spaces (or in the vector topological spaces) where \( f \) is the basic operator but the operator \( g \) is a continuous compact operator, \( \lambda_1 \) is the same infimum (see, e.g. 3, 5, 6, 9, 10, 11, 24, 36).

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etc.). The works \cite{6,7,8,9,10,11,16,23,26} were investigated the Strum-Liouville type problem for the nonlinear perturbed of linear operators and considered the equation of the form \( L = \lambda I + g \) and bifurcation problem.

Well-known that founding of eigenvalues of nonlinear continuous operators allow studying of the bifurcations, which appear under the investigation of the nonlinear equations when it has the form as

\[
f(x) - g(x) = h, \quad x \in X,
\]

where the exponent of the nonlinearity of operator \( g \) is greater than the exponent of the nonlinearity of operator \( f \) although \( f \succ g \) (see, Definition \cite{2} Sec. 2), e.g. if both of operators are positive (see, e.g., \cite{12,16,33}, etc.).

The works \cite{1,2,4,13,14,15,18,19,20,21,22,25,27,28} have been introduced spectra (or first eigenvalue) starting from the equation of the form \( f(\cdot) - \lambda_1 I \cdot = 0 \) as in the theory of the linear operators. And the spectra for the classes that are Frechet differentiable operators, Lipschitz continuous operators, continuous operators, special continuous operators, kept operators, and linearly bounded operators were investigated. This approach supposed that the spectrum of the operator \( f \) acting in the Banach space \( X \) can define such as in the theory of linear operators. In these works is defined a resolvent subset of \( \mathbb{K} \) and it denoted as \( \rho (f) \subset \mathbb{K} \) of the elements \( \lambda \) under which the resolvent operator \( f - \lambda I \) is invertible. Consequently, then a subset \( \sigma(f) = \mathbb{K} - \rho(f) \) is called the spectra of the operator \( f : X \rightarrow X \), where \( I \equiv id \) (identical operator). This approach and obtained results in enough form explained in the book \cite{1} (see also the survey \cite{2}). All of the above works for the study was used degree theory that requires the compactness that later on was generalized to the condition that uses the Kuratowski measure of noncompactness. Unfortunately, these definitions couldn’t fulfill properties from the viewpoint of the above requirements since in this case often the defined \( \lambda \) can dependent on elements of the space \( X \) (see, e.g. examples provided in next).

Therefore, is arise the natural question: Could be to introduce a reasonable definition of a spectrum of a continuous nonlinear operator that satisfies some basic requirements, which were analogous to the existing properties in the theory of the linear operators?

But how one will see later on from the explanation of the characters of the nonlinear operators for the study of the spectrum of the nonlinear operator one needs to approach by another way.

This paper is proposed a new approach for the study of the spectrum of continuous nonlinear operators in the Banach spaces. Really here we find the first eigenvalue of the nonlinear continuous operator in Banach space and this shows how one can seek the other eigenvalues. Moreover, we investigate the solvability of the nonlinear equations in the Banach spaces. Here shows that if use the proposed definition of the spectrum of nonlinear continuous operators in Banach spaces then the spectra will satisfy some properties, that are similar to properties having in the theory of the linear operators.

Later on (in Section 2) will be shown that really the obtained numbers aren’t the spectra of the operator as, in general, these can be to depends on elements of

\textsuperscript{1}see, also Lopez-Gomez, J. Spectral Theory and Nonlinear Functional Analysis (1st ed.). (2001) Chapman and Hall/CRC, T&F Books https://doi.org/10.1201/9781420035506
the domain of the examined operator. The founded numbers $\lambda_1$ allow investigating the solvability of nonlinear equations that have the form $f(\cdot) - \lambda g(\cdot) = y$ containing this continuous nonlinear operator $f$ and another continuous operator $g$ under some complementary conditions such as $|\lambda| \leq \lambda_1$, where $\lambda$ is some number.

In this paper, we will study the spectrum of nonlinear operators acting in Banach spaces, and also the solvability of the depended on parameters nonlinear equations using the solvability theorems and fixed-point theorems of the works \[29, 30, 31, 32\]. Let $X, Y$ be real Banach spaces on the field $\mathbb{R}$ and $X^*, Y^*$ be of their dual spaces, let $Y$ be reflexive space. Let $f : X \to Y$ and $g : X \to Y$ be nonlinear continuous operators such, that $f(0) = 0, g(0) = 0$.

For investigation of the spectrum of continuous nonlinear operators, will consider the following equation

$$ f(x) = \lambda g(x), \quad x \in M \subseteq X, $$

where $f, g$ are continuous operators, in particular, $g$ can be the identical operator, and also to consider the depended on a parameter $\lambda$ equation

$$ f_{\lambda}(x) \equiv f(x) - \lambda g(x) = y, \quad \text{for } y \in Y; $$

where, generally, $\lambda$ is an element of $\mathbb{C}$. Roughly speaking, here is required to study for which $\lambda$ these equations are solvable.

Our goal is the investigation the spectrum of the continuous nonlinear operator in a generalized sense, and also study in the Banach spaces the solvability of the nonlinear operator equations dependent on the parameter.

The paper is organized as follows. Section 2 provided a definition of the spectrum of nonlinear continuous operators in the Banach spaces, some complementary to the definition with explanations and examples, the general theorems on the solvability, and fixed-point theorem, the solvability theorem for the dependent on a parameter nonlinear equation with continuous operators is proved. Here is shown how can be to find the first eigenvalue of nonlinear continuous operators relative to another nonlinear continuous operator. Section 3 provided some examples of nonlinear differential operators, for which founded first eigenvalues relative to another nonlinear differential operators and showed the relations between founding first eigenvalues with first eigenvalues of linear differential operators. Section 4 studied the existence of the first eigenvalues of the fully nonlinear continuous operator relatively to other nonlinear continuous operators and provided examples.

2. Spectral properties of the continuous nonlinear operators

Here a concept for the spectrum of the continuous nonlinear operator with respect to another continuous nonlinear operator is introduced, as above noted we would like to determine such numbers, which are independent of the element of the space. It isn’t difficult to see that if one determines the spectrum of the continuous nonlinear operator in the same way as for the linear continuous operator then the finding number will be the function of elements of the definition domain of the operator.

Let $X$ and $Y$ be the real Banach spaces, $F : D(f) = X \to Y, G : X \subseteq D(G) \to Y$ be the continuous bounded nonlinear operators (for generality) and $\lambda \in \mathbb{R}$ be the number and $F(0) = 0, G(0) = 0$.

So, we will investigate the spectrum of operator $F$ with respect to operator $G$, i.e. we will study the question: For which number $\lambda$ the following equation will be
solvable?
\[ f_\lambda(x) \equiv F(x) - \lambda G(x) = 0, \quad \text{or} \quad F(x) = \lambda G(x), \quad x \in X \]

And also we will study the following equation
\[ f_\lambda(x) \equiv F(x) - \lambda G(x) = y, \quad y \in Y. \]

In the beginning, we will introduce concepts that will be necessary for this paper.

**Definition 1.** The operator \( f : D(f) \subseteq X \rightarrow Y \) is called bounded if there is a continuous function \( \mu : R^1_+ \rightarrow R^1_+ \) such that
\[ \|f(x)\|_Y \leq \mu(\|x\|_X), \quad \forall x \in D(f) \]
and denote this class of operators as \( \mathfrak{B} \) and the bounded continuous class of operators by \( \mathfrak{BC}^0 \).

Now we introduce an order relation in the class of the continuous operators acting in the Banach spaces.

**Definition 2.** Let \( X_0, Y_0 \) be a Banach spaces and \( F : D(f) \subseteq X_0 \rightarrow Y_0 \), \( G : D(G) \subseteq X_0 \rightarrow Y_0 \) be some continuous operators. Denote by \( \mathcal{F}_F(Z), \mathcal{F}_G(Z) \) sets defined in the form
\[ \mathcal{F}_F(Z) \equiv \{x \in X_0 \mid \|F(x)\|_Z < \infty\} \neq \emptyset \quad \land \quad \mathcal{F}_G(Z) \equiv \{x \in X_0 \mid \|G(x)\|_Z < \infty\} \neq \emptyset \]
that are subsets of \( X_0 \) for each Banach space \( Z \subseteq Y_0 \) satisfying conditions \( \text{Im} F \cap Z \neq \emptyset, \text{Im} G \cap Z \neq \emptyset \). If \( \mathcal{F}_F(Z) \subseteq \mathcal{F}_G(Z) \) holds for each of the above mentioned Banach space \( Z \subseteq Y_0 \), then we will say that the operator \( F \) is greater than the operator \( G \) that denote as \( F \succ G \).

**Definition 3.** Let operators \( F : X \rightarrow Y \) and \( G : X \subseteq D(G) \rightarrow Y \), moreover \( F \succ G \). We say \( \lambda \in K \) belongs to the \( G \)-resolvents subset of operator \( F \) relative to operator \( G \) iff
\[ \lambda \in \rho_G(F) \equiv \left\{ \lambda \in \mathbb{R} \mid f^{-1}_\lambda = (F - \lambda G)^{-1} : F(X) \cap G(X) \subseteq Y \rightarrow X, \quad f^{-1}_\lambda \in \mathfrak{BC}^0 \right\} \]
holds and denote this by \( \rho_G(F) \subseteq K \), where \( f_\lambda(\cdot) \equiv F(\cdot) - \lambda G(\cdot) \). Consequently, we call the element \( \lambda \in K \) the \( G \)-spectrum of the operator \( F \) if \( \lambda \in K - \rho_G(F) \), that we denote by \( \sigma_G(F) \).

**Notation.** Unfortunately, the above definition of a spectrum in such a way does not approach the pair of operators, which are chosen by the independent way, that will be shown next. We will call the number \( \lambda \) is the first eigenvalue of the examined operator relative to another operator as in Definition 3 which is independent of the elements from the domain of the examined operators. This definition allows us to seek also the following eigenvalues of examined operator.

So, for simplicity, we start to consider the case when \( F \succ G \) and consider the case when one of these operators has the inverse operator from the class \( \mathfrak{BC}^0 \). Let operator \( F \) be invertible, i.e. there is \( F^{-1} : F(X) \subseteq Y \rightarrow X \). Then using \( F^{-1} \) we get the equation
\[ y - \lambda G(F^{-1}(y)) = 0, \quad y = F(x), \quad x \in X, \]
that is needed to study on the $F(X) \subseteq Y$. Thus we derive an equation that is equivalent to the equation for investigation of the existence of a spectrum as in the works \cite{1, 2, 3, 4, 6, 9, 13, 18, 19, 20, 22, 23, 25, 28} and their references. Unlike a usual case, here the operator $G \circ F^{-1}$ is defined on subset $F(X)$ and acts as $G \circ F^{-1} : F(X) \rightarrow G(X) \subseteq Y$. If $G$ is invertible then by the same way as above we get to equation

$$F\left(G^{-1}(y)\right) - \lambda y = 0, \quad y = G(x), \quad x \in X,$$

where $G^{-1}$ denotes the inverse operator of $G$. Consequently, this equation needs to investigate on subset $G(X)$ of $Y$.

Thus if $F$ (or $G$) is invertible then we obtain the operator

$$\tilde{F}_\lambda (\cdot) \equiv I - \lambda G\left(F^{-1}(\cdot)\right), \quad \tilde{F}_\lambda : D\left(\tilde{F}_\lambda\right) \subseteq Y \rightarrow Y$$

that depends on the parameter $\lambda$, consequently the equation (2.4) is transformed to the problem on the study of the eigenvalue of operator $G \circ F^{-1}$ (or $F \circ G^{-1}$), in other words, we derived problem about the existence of the fixed-points of operator $G \circ F^{-1}$ (or $F \circ G^{-1}$) in subset $F(X)$ (or $G(X)$) of $Y$.

It is clear that if $F$ is the linear continuous operator with the inverse operator $F^{-1}$ then the problem (2.4) is equivalent to the problem

$$x - \lambda F^{-1} \circ G(x) = 0$$

that becomes the problem about the existence of the fixed-points of the operator $\lambda F^{-1} \circ G$. In many articles, problems of such types were studied \cite{9, 10, 11, 17, 26, 35}, etc.) but we wish to investigate the problem (2.4) in the general case, without such type conditions. Section 3 will be given some explanations relative to the above-provided cases.

Before starting the investigation of the spectrum of the nonlinear operators relative to other nonlinear operators in the general sense necessary to investigate the solvability of nonlinear equations (2.2). To investigate these problems, we will use the general existence and fixed-point theorems of \cite{29, 30}.

Therefore, in the beginning, we will lead these results.

### 2.1. General Solvability Results.

Let $X, Y$ be real Banach spaces such as above, $f : D(f) \subseteq X \rightarrow Y$ be an operator and $B_{r_0}^X(0) \subseteq D(f)$ is the closed ball with a center of 0 $\in X$.

Consider the following conditions.

(i) $f : D(f) \subseteq X \rightarrow Y$ be a bounded continuous operator;

(ii) There is a mapping $g : X \subseteq D(g) \rightarrow Y^*$ such that $g\left(B_{r_0}^X(0)\right) = B_{r_1^*}^{Y^*}(0)$ and

$$\langle f(x), \tilde{g}(x) \rangle \geq \nu(\|x\|_X) = \nu(r), \quad \forall x \in S_r^X(0)$$

holds\textsuperscript{2} where $\tilde{g}(x) \equiv \frac{g(x)}{\|g(x)\|} : R^1_+ \rightarrow R^1$ and $\nu(r_0) \geq \delta_0$ is a continuous function ($\nu \in C^0$), moreover $\nu(\tau)$ is the nondecreasing function for $\tau : r_0 \leq \tau \leq r_0$; $\delta_0 > 0, r_0 \geq 0$ are a constant.

(iii) Almost each $x_0 \in \operatorname{int} B_{r_0}^X(0)$ possess such neighborhood $V_\varepsilon(x_0)$, $\varepsilon \geq \varepsilon_0$ that the following inequality

$$\|f(x_2) - f(x_1)\|_Y \geq \Phi(\|x_2 - x_1\|_X, x_0, \varepsilon)$$

\textsuperscript{2}In particular, the mapping $g$ can be a linear bounded operator as $g \equiv L : X \rightarrow Y^*$ that satisfy the conditions of (ii).
holds for any \( \forall x_1, x_2 \in V_{x}(x_0) \cap B^{X}_{r_0}(0) \), where \( \varepsilon_0 > 0 \) and \( \Phi(\tau, x_0, \varepsilon) \geq 0 \) is the continuous function of \( \tau \) and \( \Phi(\tau, \bar{x}, \varepsilon) = 0 \) if \( \tau = 0 \) (in particular, maybe \( x_0 \equiv 0 \), \( \varepsilon = \varepsilon_0 = r_0 \) and \( V_{x}(x_0) = V_{r_0}(0) \equiv B^{X}_{r_0}(0) \), consequently \( \Phi(\tau, x_0, \varepsilon) \equiv \Phi(\tau, x_0, r_0) \) on \( B^{X}_{r_0}(0) \)).

**Theorem 1.** Let \( X, Y \) be real Banach spaces such as above, \( f : D(f) \subseteq X \rightarrow Y \) be an operator and \( B^{X}_{r_0}(0) \subseteq D(f) \) is the closed ball with a center of \( 0 \in D(f) \). Assume conditions (i) and (ii) are fulfilled. Then the image \( f(B^{X}_{r_0}(0)) \) of the ball \( B^{X}_{r_0}(0) \) contains in an absorbing subset of \( Y \) an everywhere dense subset of \( M \) that is defined as follows

\[
M \equiv \left\{ y \in Y \mid \langle y, \bar{g}(x) \rangle \leq \langle f(x), \bar{g}(x) \rangle, \forall x \in S^{X}_{r_0}(0) \right\}.
\]

Furthermore, if the condition (iii) also is fulfilled then the image \( f(B^{X}_{r_0}(0)) \) of the ball \( B^{X}_{r_0}(0) \) is a bodily subset of \( Y \), moreover \( B^{X}_{r_0}(0) \subseteq M \).

The proof of this theorem, and also its generalization was provided in [29] (see also, [30, 31, 32]).

The condition (iii) can be generalized, for example, as in the following proposition.

**Corollary 1.** Let all conditions of Theorem 1 be fulfilled except for the inequality (2.6) of condition (iii) instead the following inequality

\[
\|f(x_2) - f(x_1)\| \geq \Phi(\|x_2 - x_1\|_X, x_0, \varepsilon) + \psi(\|x_1 - x_2\|_Z, x_0, \varepsilon)
\]

holds, where \( Z \) is Banach space and \( X \subset Z \) is compact, \( \psi(\cdot, x_0, \varepsilon) : R^1_+ \rightarrow R^1 \) is a continuous function relatively \( \tau \in R^1_+ \) and \( \psi(0, x_0, \varepsilon) = 0 \).

Then the statement of Theorem 1 is true.

From Theorem 1 immediately follows

**Theorem 2. (Fixed-Point Theorem).** Let \( X \) be a real reflexive separable Banach space and \( f_1 : D(f_1) \subseteq X \rightarrow X \) be a bounded continuous operator. Moreover, let on closed ball \( B^{X}_{r_0}(0) \subseteq D(f_1) \), with the center of \( 0 \in D(f_1) \), operators \( f_1 \) and \( f \equiv I_1 - f_1 \) satisfy the following conditions:

(I) The following inequations

\[
\|f_1(x)\|_X \leq \mu(\|x\|_X), \quad \forall x \in B^{X}_{r_0}(0),
\]

(2.8) \[
\langle f(x), \bar{g}(x) \rangle \geq \nu(\|x\|_X), \quad \forall x \in B^{X}_{r_0}(0),
\]

hold, where \( f_1(B^{X}_{r_0}(0)) \subseteq B^{X}_{r_0}(0) \), \( g : D(g) \subseteq X \rightarrow X^* \), \( D(f_1) \subseteq D(g) \) and satisfy the condition (ii) (in particular, \( g \equiv J : X \equiv X^* \), i.e. \( g \) is a duality mapping), \( \mu \) and \( \nu \) are such functions as in Theorem 1.

(II) Almost each \( x_0 \in \text{int}B^{X}_{r_0}(0) \) possess a neighborhood \( V_{x}(x_0), \varepsilon \geq \varepsilon_0 > 0 \) such that for each \( x_0 \in \text{int} B^{X}_{r_0}(0) \) the following inequality

\[
\|f(x_2) - f(x_1)\|_X \geq \varphi(\|x_2 - x_1\|_X, x_0, \varepsilon),
\]

holds for any \( \forall x_1, x_2 \in V_{x}(x_0) \cap B^{X}_{r_0}(0) \), where the function \( \varphi(\tau, x_0, \varepsilon) \) satisfies the condition such as conditions on functions of right part of (2.7).

Then operator \( f_1 \) possess a fixed-point in the closed ball \( B^{X}_{r_0}(0) \).

\[\text{We note Theorem 2 is the generalization of Theorem of such type from Soltanov K.N., On equations with continuous mappings in Banach spaces. Funct. Anal. Appl. (1999) 33, 1, 76-81.}\]
Now we introduce the following concept.

**Definition 4.** An operator \( f : D(f) \subseteq X \rightarrow Y \) possesses the P-property iff each precompact subset \( M \subseteq \text{Im} f \) of \( Y \) contains such subsequence (maybe generalized) \( M_0 \subseteq M \) that \( f^{-1}(M_0) \subseteq G \) and \( M_0 \subseteq f(G \cap D(f)) \), where \( G \) is a precompact subset of \( X \).

**Notation 1.** It is easy to see that the condition (iii) of Theorem 7 one can replace by the condition: \( f \) possesses P-property.

It should be noted if \( f^{-1} \) is a lower or upper semi-continuous mapping then operator \( f : D(f) \subseteq X \rightarrow Y \) possesses of the P-property.

In the above results, condition (iii) is required for the completeness of the image of the considered operator \( f \). One can describe and other complementary conditions on \( f \) under which \( \text{Im} f \) will be a closed subset (see, e.g. \([30, 31, 32]\)). In particular, the following results are true.

**Lemma 1.** Let \( X, Y \) be Banach spaces such as above, \( f : D(f) \subseteq X \rightarrow Y \) be a bounded continuous operator, and \( D(f) \) is a weakly closed subset of the reflexive space \( X \). Let \( f \) have a weakly closed graph and for each bounded subset \( M \subset Y \) the subset \( f^{-1}(M) \) is the bounded subset of \( X \). Then \( f \) is a weakly closed operator.

We want to note the graph of operator \( f \) is weakly closed iff from \( x_m \stackrel{X}{\rightarrow} x_0 \in D(f) \) and \( f (x_m) \stackrel{Y}{\rightarrow} y_0 \in Y \) follows equation \( f (x_0) \equiv y_0 \) \( \text{Im} f \subset Y \) (for the general case see \([30, 31]\)).

For the proof is enough to note, if \( \{y_m\}_{m=1}^{\infty} \subset \text{Im} f \subset Y \) is the weakly convergent sequence of \( Y \) then \( f^{-1}(\{y_m\}_{m=1}^{\infty}) \) is a bounded subset of \( X \) consequently this has such subsequence \( \{x_m\}_{m=1}^{\infty} \) that \( x_m \in f^{-1}(y_m) \) and \( x_m \stackrel{X}{\rightarrow} x_0 \in D(f) \) for some element \( x_0 \in D(f) \) by virtue of the reflexivity of \( X \).

**Lemma 2.** Let \( X, Y \) be reflexive Banach spaces, and \( f : D(f) \subseteq X \rightarrow Y \) be a bounded continuous mapping that satisfies the condition: if \( G \subseteq D(f) \) is a closed convex subset of \( X \) then \( f(G) \) is the weakly closed subset of \( Y \). Then if \( G \subseteq D(f) \) is a bounded closed convex subset of \( X \) then \( f(G) \) is a closed subset of \( Y \).

For the proof enough to use the reflexivity of the space \( X \) and properties of the bounded closed convex subset of \( X \) (see, e.g. \([29, 30]\)).

**Lemma 3.** Let \( X \) be a Banach space such as above, \( f : X \rightarrow X^* \) be a monotone operator satisfying conditions of Theorem 7 and \( r \geq r_1 \) be some number. Then \( f(G) \) is a bounded closed subset containing a ball \( B_{r_1}^X (f(0)) \) for every such bounded closed convex body \( G \subset X \) that \( B_{r_1}^X (0) \subset G \), where \( r_1 = r_1 (r) \geq \delta_1 > 0 \).

**2.2. Investigation of equations** \((2.1), (2.2)\) and existence of the spectra. We start with the study of the equation \((2.2)\), in order to explain the role of the number \( \lambda \) under the investigation of posed questions. Let \( X, Y \) be real reflexive Banach spaces, \( F : X \rightarrow Y \), \( G : X \subseteq D(G) \rightarrow Y \) are nonlinear operators and \( B_{r_0}^Y (0) \) (\( r_0 > 0 \)) be a closed ball with a center at \( 0 \in X \) that belongs to \( D(F) \). Since in this work, we will consider only operators acting in real spaces, therefore will seek real numbers \( \lambda_0 \), under which the considered equation may be solvable.

Assume on the ball \( B_{r_0}^X (0) \) are fulfilled the following conditions:
1) \( F : B^X_r(0) \to Y, G : B^X_r(0) \to Y \) are a bounded continuous operators, i.e. there exist such continuous functions \( \mu_j : \mathbb{R}_+ \to \mathbb{R}_+ \), \( j = 1, 2 \) that
\[
\| F(x) \|_Y \leq \mu_1 (\| x \|_X) ; \quad \| G(x) \|_Y \leq \mu_2 (\| x \|_X) ,
\]
hold for any \( x \in B^X_r(0) \), in addition \( F \succ G : \)

2) Let \( f_\lambda \equiv F - \lambda G \) be the operator of \( \text{(2.2)} \). Assume there exists such parameter \( \lambda_0 \in \mathbb{R}_+ \) that for each \( (y^*, r, \lambda) \) exists such \( x \in S^X_\kappa(0) \) that the following inequality
\[
\langle f_\lambda (x) , y^* \rangle \geq \nu_\lambda (\| x \|_X) , \quad \exists x \in S^X_\kappa(0) , \quad g(x) = y^*
\]
holds, where \( (y^*, r, |\lambda|) \in S^Y_\kappa(0) \times (0, r_0] \times (0, \lambda_0] \) and \( \nu_\lambda : \mathbb{R}_+ \to \mathbb{R} \) is the continuous function satisfying the condition \( (ii) \) of Theorem \( \text{[1]} \) in this case \( \delta_0 = \delta_{00} \geq 0 \) if \( |\lambda| \not\in |\lambda_0| \).

3) Assume for almost every point \( x_0 \) from \( B^X_r(0) \) there exist numbers \( \varepsilon \geq \varepsilon_0 > 0 \) and such continuous of \( \tau \) functions \( \varphi_\lambda (\tau, x_0, \varepsilon) \geq 0, \psi_\lambda (\tau, x_0, \varepsilon) \) that the following inequality
\[
\| f_\lambda (x_1) - f_\lambda (x_2) \|_Y \geq \varphi_\lambda (\| x_1 - x_2 \|_X, x_0, \varepsilon) + \psi_\lambda (\| x_1 - x_2 \|_Z, x_0, \varepsilon)
\]
holds for any \( x_1, x_2 \in B^X_r(0) \), where \( \varphi (\tau, x_0, \varepsilon) = 0 \Leftrightarrow \tau = 0, \psi (\tau, x_0, \varepsilon) : \mathbb{R}_+ \to \mathbb{R} \), \( \psi_\lambda (0, x_0, \varepsilon) = 0 \) for any \( (x_0, \varepsilon) \) and \( Z \) be a such Banach space that the inclusion \( X \subset Z \) is compact.

**Theorem 3.** Let conditions 1, 2 and 3 are fulfilled on the closed ball \( B^X_r(0) \subset X \). Then equation \( \text{(2.2)} \) is solvable for \( \forall y \in V_\lambda \subset Y \) and each \( \lambda : 0 \leq |\lambda| \leq \lambda_0 \); moreover, the inclusion \( \text{\inclusion} B^Y_{\kappa_0}(0) \subset f_\lambda (B^X_r(0)) \) holds for \( \delta_0 \equiv \delta_0(\lambda) > 0 \) from the condition 2, where \( V_\lambda \) can be defined as follows
\[
V_\lambda \equiv \{ y \in Y | \langle y, g(x) \rangle \leq \langle f_\lambda (x) , g(x) \rangle , \quad \forall x \in S^X_\kappa(0) \} .
\]

For the proof is sufficient to note that all conditions of Theorem \( \text{[1]} \) are fulfilled for each fixed \( \lambda : |\lambda| < \lambda_0 \) due to conditions of Theorem \( \text{[3]} \), therefore applying Theorem \( \text{[1]} \) we get the correctness of Theorem \( \text{[3]} \).

Consequently, the equation \( \text{(2.4)} \) also is solvable in \( B^X_r(0) \) under the conditions on \( f_\lambda \) of the above type that depends at \( \lambda_0 \), e.g.
\[
\| G (F^{-1}(y_1)) - G (F^{-1}(y_2)) \|_Y \leq C (x_0, \varepsilon) \| y_1 - y_2 \|_Y + \psi_\lambda (\| y_1 - y_2 \|_Z, x_0, \varepsilon),
\]
where \( C (x_0, \varepsilon) \lambda_0 < 0 \) and the inclusion \( Y \subset Z \) is compact.

Whence follows using Theorem \( \text{[2]} \) one can obtain the solvability of the equation \( \text{(2.2)} \). Really, let \( Y = X^* \) and closed ball \( B^X_r(0) \) \( (r_0 > 0) \) belongs to \( D(F) \). Let condition 1 of Theorem \( \text{[3]} \) is fulfilled on ball \( B^X_r(0) \). Assume the following conditions are fulfilled.

2'') There exists such parameter \( \lambda_1 \in \mathbb{R} \) that \( \lambda_1 G (F^{-1}(F(B^X_r(0)))) \subset B^X_r(0) \) and for each \( x^* \in S^X_\kappa(0) \) there exists such \( x \in S^X_r(0) \), for each \( r \in (0 < r \leq r_0) \) that the following inequality
\[
\langle \tilde{f}_{\lambda_1} (x) , x^* \rangle \geq \nu_{\lambda_1} (\| x - x_0 \|_X) = \nu_{\lambda_1} (r) , \quad x \in S^X_r(x_0) \subset B^X_r(x_0)
\]
holds, where \( \nu_{\lambda_1} : \mathbb{R}_+ \to \mathbb{R} \) is the continuous function that satisfies the condition \( (ii) \) of Theorem \( \text{[1]} \).

3') For almost every \( \tilde{x} \in B^X_r(x_0) \) there are such number \( \varepsilon \geq \varepsilon_0 > 0 \) and continuous functions \( \Phi_{\lambda_1} (\cdot, \tilde{x}, \varepsilon) : \mathbb{R}_+ \to \mathbb{R}_+ \), \( \varphi_{\lambda_1} (\cdot, \tilde{x}, \varepsilon) : \mathbb{R}_+ \to \mathbb{R} \) for each \( (\tilde{x}, \varepsilon) \),
that the following inequality
\[ \| \tilde{f}_{\lambda_1} (x_1) - \tilde{f}_{\lambda_1} (x_2) \|_X \geq \Phi_{\lambda_1} (\| x_1 - x_2 \|_X, \bar{\epsilon}, \epsilon) + \varphi_{\lambda_1} (\| x_1 - x_2 \|_Z, \bar{\epsilon}, \epsilon) \]
holds for any \( x_1, x_2 \in U_\epsilon (\bar{x}) \cap B^{N_0}_r (x_0) \), where \( \Phi_{\lambda_1} (\tau, \bar{\epsilon}, \epsilon) \geq 0 \) and \( \varphi_{\lambda_1} (\tau, \bar{\epsilon}, \epsilon) = 0 \Leftrightarrow \tau = 0 \), \( \varphi_{\lambda_1} (0, \bar{\epsilon}, \epsilon) = 0 \), and also \( Z \) be a Banach space, in addition \( X \subset Z \) is compact.

Whence implies, that for defined above \( \lambda_1 \) all conditions of Theorem II are fulfilled for the operator \( \tilde{f}_{\lambda_1} \) on the closed ball \( B^{N_0}_r (x_0) \). Consequently, \( \tilde{f}_{\lambda_1} (B^{N_0}_r (x_0)) \) contains a closed absorbing subset of \( X \) (at least, \( 0 \in X \)) by virtue of the Theorem II

In the other words, \( 0 \in \tilde{f}_{\lambda_1} (B^{N_0}_r (x_0)) \) and therefore there exists an element \( \bar{x} \in B^{N_0}_r (x_0) \) for which \( f_{\lambda_1} (\bar{x}) = 0 \) holds, i.e. \( F (\bar{x}) = \lambda_1 G (\bar{x}) \).

The obtained result one can formulate as follows.

**Corollary 2.** Let \( F, G \) be above determined operators, \( F \succcurlyeq G, D (F) \subset D (G) \), and there exists such number \( \lambda_1 \) that conditions 1), 2'), 3') are fulfilled on the closed ball \( B^{N_0}_r (x_0) \subset D (F) \subset X \). Then there exists an element \( \bar{x} \in B^{N_0}_r (x_0) \) such, that \( F (\bar{x}) = \lambda_1 G (\bar{x}) \) or \( \lambda_1 G (F^{-1} (\cdot)) \) has fixed point.

Let \( X, Y \) be Banach spaces and \( B^{N_0}_r (0) \subset D (F) \subset X \), \( r_0 > 0 \), \( F \succcurlyeq G, F (0) = 0, G (0) = 0 \) be a bounded operators and there are the continuous functions \( \nu_F, \nu_G : \mathbb{R}_+ \rightarrow \mathbb{R} \) satisfying the condition (ii) of Theorem I such that for each \( y^* \in S^{N_1}_r (0) \) there exists \( x \in S^N_r (0) \) for which the inequalities
\[ \langle F (x) , y^* \rangle \geq \nu_F (\| x \|_X), \quad \langle G (x) , y^* \rangle \geq \nu_G (\| x \|_{X_1}) \]
hold, where \( X_1 \) is the Banach space that \( X \subset X_1 \) (we denote the relation between \( x \) and \( y^* \in S^{N_1}_r (0) \) by \( g : S^N_r (0) \rightarrow S^{N_1}_r (0), 0 < r \leq r_0 \), so that \( g (x) = y^* \)).

Then according to condition 2') of Corollary II one may expect that spectrum of the operator \( F : D (F) \subset X \rightarrow Y \) relative to operator \( G : D (G) \subset X \rightarrow Y \) can define in the following way
\[ \lambda = \inf \left\{ \frac{\langle F (x) , g (x) \rangle}{\langle G (x) , g (x) \rangle} \mid x \in B^{N_0}_r (0) \setminus \{0\} \right\}, \quad r_0 > 0. \]

Can we call the determined by (2.9) \( \lambda \) spectrum of the operator \( G \circ F^{-1} \) or spectrum of operator \( F \) relative to operator \( G \)? Generally speaking, one cannot name since the composition \( G \circ F^{-1} \) can be nonlinear and \( \lambda_1 \) may be a function as \( \lambda_1 = \lambda_1 (x_1) \), unlike the linear case, where \( x_1 \) is the element on which the relation (2.9) attains infimum. Moreover, if we define the subspace \( \Gamma_{\lambda_1} = \{ \alpha x_1 \mid \alpha \in R \} \subset X \) then for \( \alpha x_1 \in D (F) \), generally, \( \alpha \lambda_1 x_1 \neq G \circ F^{-1} (\alpha x_1) \) since \( G \circ F^{-1} \) is nonlinear operator.

Indeed, if the power of nonlinearity of the operator \( F \) is great than the power of nonlinearity of operator \( G \), or the inverse of its, then obviously, will the case \( \lambda_1 = \lambda_1 (x_1) \). For example, operators \( F \) and \( G \) defines as the following form
\[ F (u) = -\nabla \circ \left( |\nabla u|^{p_0-2} \nabla u \right), \quad G (u) = |u|^{p_1-2} u, \quad Y = W^{-1,q} (\Omega), \]
where \( X = W^{1,p_0} (\Omega) \cap L^{p_1} (\Omega), \Omega \subset \mathbb{R}^n, n \geq 1, \) with sufficiently smooth boundary \( \partial \Omega \) and \( p = \max \{ p_0, p_1 \} \), \( p_0, p_1 > 2, \quad q = p' = \frac{n}{p-1} \). Assume \( p_0 \neq p_1 \) and \( F : D (F) = W^{1,p_0} (\Omega) \rightarrow W^{-1,q_0} (\Omega), \quad G : D (G) = L^{p_1} (\Omega) \rightarrow L^{q_1} (\Omega) \). Then
using (2.9) we get

$$\lambda = \inf \left\{ \frac{(F(u), u)}{(G(u), u)} \mid u \in B_{r_0}^X(0) \setminus \{0\} \right\} =$$

$$\inf \left\{ \frac{\|\nabla u\|_{L^{p_0}}^{p_0}}{\|u\|_{L^{p_1}}} \mid u \in B_{r_0}^{W^{1,p_0} \cap L^{p_1}}(\Omega) \setminus \{0\} \right\}.$$ 

Whence we have if $p_0 > p_1$ then

$$\lambda = \inf \left\{ \left( \frac{\|\nabla u\|_{L^{p_0}}}{\|u\|_{L^{p_1}}} \right)^{p_1} \right\} \left( \|\nabla u\|_{L^{p_0}}^{p_0-p_1} \mid u \in B_{r_0}^{W^{1,p_0} \cap L^{p_1}}(\Omega) \setminus \{0\} \right\}$$

and if $p_0 < p_1$ then

$$\lambda = \inf \left\{ \left( \frac{\|\nabla u\|_{L^{p_0}}}{\|u\|_{L^{p_1}}} \right)^{p_0} \right\} \left( \|u\|_{L^{p_1}}^{p_0-p_1} \mid u \in B_{r_0}^{W^{1,p_0} \cap L^{p_1}}(\Omega) \setminus \{0\} \right\}.$$

Consequently, $\lambda$ will be a function of $u$ as $\lambda_1 = \lambda(u_1)$, here $u_1$ is the function on which is attained the infimum of the above-mentioned expression. (Section 3 has more examples.)

**Remark 1.** Whence implies the main parts of operators $F$ and $G$ must possess a common degree of nonlinearity in order to $\lambda_0$ couldn't be a function of $x$. From Theorem 1 follows that defined in such way number $\lambda_0$ is the number that was assumed exists in the conditions of this theorem. Moreover, the finding number allows us in mentioned theorem to state the existence of solutions for each $\lambda : 0 \leq |\lambda| \leq \lambda_0$ if the element on the right side is from a determined subsets. Thus, we can find the spectrum in the sense Definition 3 using the proof of the above result, if the main parts of operators $F$ and $G$ have a common degree of nonlinearity.

The spectrum of an operator usually must be to characterize the operator, but the determined here number $\lambda$ can’t of this. Therefore we will fit the above question differently unlike the above-mentioned works.

In the beginning, we will study the case when the defined in (2.9) number $\lambda$ is independent of $x$. As was explained above from the existence of the fixed point $x_1$ of the operator $G \circ F^{-1}$ does not follow that $x_1$ is the eigenvector and the number $\lambda_1$ is the eigenvalue for this operator. Consequently, the existence of the inverse operator of $F$ or $G$ also does not resolve this question. All of these shows that for the study of the posed question is necessary some relations between the operators $F$ and $G$.

So, we assume one of the following conditions are fulfilled: (1) $F$ and $G$ are homogeneous with common exponent $p > 0$ or a common function $\phi(\cdot)$, i.e. $F(\mu x) = \mu^p F(x)$, $G(\mu x) = \mu^p G(x)$ or $F(\mu x) = \phi(\mu) F(x)$, $G(\mu x) = \phi(\mu) G(x)$ for any $\mu > 0$; (2) Investigate the problem locally, i.e. study the problem on the closed ball $B_{r_0}^X(0) \subseteq D(f_\lambda)$ for selected $r > 0$ and to seek of $\lambda$ in the form $\lambda \equiv \lambda(r)$.

We start to study case (1), i.e. when $F$ and $G$ are homogeneous with exponent $p > 0$. Whence imply that (2.9) defines a number $\lambda$ independent from $x$ hence if denote this minimum by $\lambda_1$ and the element at which minimum is attained by $x_1$ then (2.9) will fulfill for $\forall x \in \Gamma_{\lambda_1} \cap D(F)$. Consequently, in this case, one can define $x_1$ as the first eigenvector and $\lambda_1$ as the first eigenvalue of operator $F$ relatively to operator $G$ (as in the linear case), or one can define as the fixed point $y_1 = F(x_1)$ of operator $\lambda G \circ F^{-1}$.
Now let be the case (2). Then if $F$ and $G$ have of the different orders of the homogeneous that are given by different functions, e.g. by polynomial functions with exponents $p_F \neq p_G$ then possible 2 variants: (a) $p_F > p_G$ and (b) $p_F < p_G$.

If the case (a) occur then $F(x) = r^{p_F} F(\bar{x})$ and $G(x) = r^{p_G} G(\bar{x})$ since for any $x \in X_0$ one can write $x \equiv r\bar{x}$ where $\|x\|_{X_0} \equiv r$ and $\bar{x} = \frac{x}{r} \in S_1 X_0 (0) \subset X_0$. Hence due to Theorem 3 we get that $G$ only can be the perturbation of the operator $F$, therefore this case is not essential. Let be the case (b). In this case, if there exists $\lambda_0$ and $x_0$ that $F(x_0) = \lambda_0 G(x_0)$ then

$$F(x_0) = r_0^{p_F} F(\bar{x}_0), \quad G(x) = r_0^{p_G} G(\bar{x}_0) \implies r_0^{p_F} F(\bar{x}_0) = \lambda_0 r_0^{p_F} G(\bar{x}_0)$$

$$\implies F(\bar{x}_0) = \lambda(\lambda_0, r_0) G(\bar{x}_0) \implies \lambda(\lambda_0, r_0) = \lambda_0 r_0^{p_G - p_F}$$

holds. Hence follows, that if we change $x_0 \equiv r_0 \bar{x}_0$ to $x_1 \equiv r_1 \bar{x}_0$ then $\lambda$ will change to $\lambda = \lambda_0 r_1^{p_G - p_F}$. In other words from here implies if $p_F \neq p_G$ then any existing number $\lambda$ will depend on element $x \in X$, i.e. $\lambda = \lambda(r)$ on the line $\{x \in X \mid x = r \bar{x}, r \in R\}$. The previous discussion shows there are two variants either $p_F = p_G$ or $\lambda_0 = \lambda_0(x_0)$ and will be sufficient to investigate these cases. If to assume the operator $f$ is linear and $g$ is an identical operator then firstly one needs to investigate the number $\lambda$ belonging to the spectrum of the linear operator $f$.

Well-known in the case when $\lambda$ less than the first eigenvalue (or less than of the $\inf \{ |\lambda| \mid \lambda \in \sigma(f) \}$) then one can use the resolvent of the operator $f$ for the study of the nonhomogeneous equation.

So, here we will study the posed question mainly in the case when condition $p_F = p_G$ holds.

Consequently, the concept defined in the articles [1] [2] [3] [11] [13] [18] [19] [22] [25] [27] [28] etc of the semilinear spectral set is special case of the Definition 3 by virtue of (2.3) and (2.4).

### 3. Some Application of General Results

Consider the following problems

$$(3.1) \quad -\nabla \circ (|\nabla u|^{p-2} \nabla u) - \lambda |u|^{p_0-2} u |\nabla u|^{p_1} = 0, \quad u |_{\partial \Omega} = 0, \quad \lambda \in \mathbb{C},$$

$$(3.2) \quad -\nabla \circ (|u|^{p-2} \nabla u) - \lambda |u|^{p_0-2} u |\nabla u|^{p_1} = 0, \quad u |_{\partial \Omega} = 0, \quad \lambda \in \mathbb{C},$$

where $\Omega \subset \mathbb{R}^n$ is an open bounded domain with sufficiently smooth boundary $\partial \Omega$, $n \geq 1$, $p_0 + p_1 = p$ and $\nabla \equiv (D_1, ..., D_n)$. We denote this operator by $f_0$, which acts from $W_0^{1,p}(\Omega)$ to $W^{-1,q}(\Omega)$. It is easy to see that $f_0 : W_0^{1,p}(\Omega) \rightarrow W^{-1,q}(\Omega)$ is a continuous operator and for any $u \in W_0^{1,p}(\Omega)$

$$0 = \langle f_0(u), u \rangle \equiv -\nabla \circ (|\nabla u|^{p-2} \nabla u) - \lambda |u|^{p_0-2} u |\nabla u|^{p_1}, u \rangle =$$

$$\|\nabla u\|_p^p - \int_{\Omega} \lambda |u|^{p_0} |\nabla u|^{p_1} dx \quad \implies \|\nabla u\|_p^p = \lambda \int_{\Omega} |u|^{p_0} |\nabla u|^{p_1} dx$$

holds. Whence follows $\lambda \geq 0$ since both of these expressions are positive.

(1) We will investigate of problem 3.1 by using of the Theorem [1] or Theorem 3 but we interest to study the question on the spectrum therefore here we will
use Corollary 2. According to the previous section, we can introduce the following denotations
\[
F(u) = -\nabla \left( |\nabla u|^{p-2} \nabla u \right), \quad F: W_0^{1,p}(\Omega) \rightarrow W^{-1,q}(\Omega),
\]
\[
G(u) = |u|^{p_0-2} u |\nabla u|^{p_1}, \quad G: W_0^{1,p}(\Omega) \rightarrow W^{-1,q}(\Omega).
\]
So, it needs to seek the minimal value of \( \lambda \) and function \( u_\lambda(x) \) (if it exists) for which the equality
\[
\| \nabla u \|_p^p = \lambda \int_\Omega |u|^{p_0} |\nabla u|^{p_1} \, dx, \quad u \in B_1^{W^{1,p}(\Omega)}(0)
\]
or the equality
\[
\lambda = \frac{\| \nabla u \|_p^p}{\int |u|^{p_0} |\nabla u|^{p_1} \, dx} = \int_\Omega \left( \frac{\| \nabla u \|_p}{|u|} \right)^{p_0} \left( \frac{\| \nabla u \|_p}{|\nabla u|} \right)^{p_1} \, dx
\]
holds. Consequently, we need to find the following number
\[
(3.3) \quad \lambda_1 = \inf \left\{ \frac{\| \nabla u \|_p^p}{\int |u|^{p_0} |\nabla u|^{p_1} \, dx} \mid u \in B_1^{0 \cdot W^{1,p}(\Omega)}(0) \right\}.
\]
It is clear that \( \lambda_1 \) exists and \( \lambda_1 > 0 \).
Whence follows
\[
(3.4) \quad \lambda_1 \geq \left( \frac{\| \nabla u \|_p}{|u|} \right)^{p_0} \quad \Rightarrow \quad \lambda_1 \geq \frac{\| \nabla u \|_p}{|u|}
\]
for any \( u \in W_0^{1,p}(\Omega), \ u(x) \neq 0 \).

We denote by \( \lambda_{p_0,p_1} \) the first spectrum of posed problem that one can define as
\[
(3.5) \quad \lambda_{p_0,p_1} = \inf \left\{ \| \nabla u \|_p \left[ \int_\Omega |u|^{p_0} |\nabla u|^{p_1} \, dx \right]^{-\frac{1}{p}} \mid u \in \mathcal{S}_1^{W^{1,p}(\Omega)}(0) \right\}.
\]
From (3.3) we obtain
\[
\lambda_{p_0,p_1} \geq \lambda_1^{\frac{1}{p_0}} = \inf \left\{ \frac{\| \nabla u \|_p}{|u|} \mid u \in W_0^{1,p}(\Omega) \right\}
\]
that is well-known \( \lambda_1^{\frac{1}{p_0}} = \lambda_1(L_p) \) was defined as the first spectrum of \( p-\)Laplacian (see, e.g. \cite{23}) that is
\[
\lambda_1(-\Delta_p) = \inf \left\{ \frac{\| \nabla u \|_p}{|u|} \mid u \in W_0^{1,p}(\Omega) \right\},
\]
consequently, this inequality shows that \( \lambda_{p_0,p_1} \) is comparable with the spectrum \( \lambda_1(-\Delta_p) \) of the \( p-\)Laplacian, i.e. \( \lambda_{p_0,p_1} \) satisfy the inequality \( \lambda_{p_0,p_1} \geq \lambda_1(-\Delta_p) \).

(2) Now we will consider of the problem \( 3.2 \) then it is get
\[
\int_\Omega |u|^{p-2} |\nabla u|^2 \, dx = \lambda \int_\Omega |u|^{p_0} \, dx
\]
or
\[ \frac{4}{p^2} \left\| \nabla \left( |u|^{\frac{p-2}{2}} u \right) \right\|_2^2 = \lambda \left\| |u|^{\frac{p-2}{2}} u \right\|_2^2, \]
here if assume \( p_0 = p \) and \( |u|^{\frac{p-2}{2}} u \equiv v \) then we get
\[ \frac{4}{p^2} \left\| \nabla v \right\|_2^2 = \frac{4}{p^2} \left\| \nabla \left( |u|^{\frac{p-2}{2}} u \right) \right\|_2^2 = \lambda \left\| |u|^{\frac{p-2}{2}} u \right\|_2^2 = \lambda \|v\|_2^2. \]

Whence follows, that the first eigenvalue \( \lambda_1 (p) \) of the operator \(-\nabla \cdot (|u|^{p-2} \nabla u)\) relative to the operator \( |u|^{p-2} u \) can be defined using the first eigenvalue \( \lambda_1 (-\Delta) \) of Laplacian that is defined by expression
\[ \lambda_1 (-\Delta) = \inf \left\{ \frac{\| \nabla v \|_2}{\|v\|_2} \mid v \in W^{1,2}_0(\Omega) \right\}. \]

Consequently, the first eigenvalue \( \lambda_1 (p) \) of the operator \(-\nabla \cdot (|u|^{p-2} \nabla u)\) relative to the operator \( |u|^{p-2} u \) (in appropriate to problem (3.2)) one can define by the equality \( \lambda_1 (p) = \left( \frac{2}{p} \lambda_1 (-\Delta) \right)^{\frac{1}{2}} \).

Thus we get

**Proposition 1.** (1) Let \( \Delta_p \) is the \( p \)-Laplacian operator with homogeneous boundary conditions on the bounded domain of \( R^n \) with smooth boundary \( \partial \Omega \) and \( p_0 + p_1 = p \). Then the first eigenvalue \( \lambda_{p_0,p_1} \) for the operator \(-\Delta_p \) relative to the operator \( G : G (u) \equiv |u|^{p_0-2} u \nabla u |^{p_1} \) exists and be defined with equality (3.6).

(2) If the \( F \) and \( G \) are operators generated by the problem (3.2) and \( p_0 = p \) then the first eigenvalue of the operator \( F \) relative to the operator \( G \) is determined as \( \lambda_1 (p) = \left( \frac{2}{p} \lambda_1 (-\Delta) \right)^2 \), where \( \lambda_1 (-\Delta) \) is the first eigenvalue of Laplacian.

**Remark 2.** It should be noted by the same way one can define the spectrum of operator \( \Delta_p \) (and also of the operator \( F : F (u) \equiv \sum_{i=1}^{n} D_i \left( |D_i u|^{p-2} D_i u \right) \)) relative to operator \( G_0 : G_0 (u) \equiv \sum_{i=1}^{n} |u|^{p_0-2} u |D_i u|^{p_1} \).

Now, using the previous results we will investigate the solvability of the problem with the following equation in the case when the homogeneous boundary condition
\[ \begin{align*}
\lambda_1 (u) & \equiv -\nabla \left( |\nabla u|^{p-2} \nabla u \right) - \lambda |u|^{p_0-2} u |\nabla u|^{p_1} = h (x)
\end{align*} \]
on the open bounded domain \( \Omega \subset R^n \) with smooth boundary \( \partial \Omega \). For this problem the following result holds.

**Theorem 4.** Let numbers \( p, p_0, p_1 \geq 0 \) are such that \( p_0 + p_1 = p \leq 2 \), \( \lambda_{p_0,p_1} \) is the number defined in (3.6). Then if \( \lambda < \lambda_{p_0,p_1} \) then the posed problem is solvable in \( W^{1,p}_0(\Omega) \) for each \( h \in W^{-1,q} (\Omega) \).

**Proof.** Let
\[ \begin{align*}
f_\lambda (u) & \equiv -\nabla \left( |\nabla u|^{p-2} \nabla u \right) - \lambda |u|^{p_0-2} u |\nabla u|^{p_1} \end{align*} \]
is the operator generated by the posed problem for (3.6) acts as \( f_\lambda : X \rightarrow Y \), where \( X \equiv W^{1,p}_0(\Omega) \), \( Y \equiv W^{-1,q} (\Omega) \) according to the above conditions. For the proof the solvability of this problem we will use of Corollary 1.
Whence follows that the inequality
\[
\langle f \lambda (u), u \rangle \equiv \| \nabla u \|^p_p - \lambda \int_{\Omega} |u|^{p_0} |\nabla u|^{p_1} \ dx \geq 0
\]
holds for any \( u \in W_0^{1,p} (\Omega) \) under conditions of Theorem 4.

Consequently, if \( \lambda < \lambda_{p_0,p_1} \) then \( f \lambda \) satisfies the condition (ii) of the Theorem 4 moreover it is fulfilled for \( x_0 = 0 \) and \( g \equiv Id \). The realization of the condition (i) of Theorem 4 for \( f \lambda \) is obvious.

We provide some inequalities, which show the fulfillment of the condition (iii) of the Theorem 4 (Corollary 1) for this problem. It isn’t difficult to see
\[
\langle f (u) - f (v), u - v \rangle \equiv \left( \| \nabla u \|^{p-2} \nabla u - |\nabla u|^{p-2} \nabla v \right), \left( \nabla (u - v) \right) - \\
\lambda \left( \| u |^{p_0} u - |v|^{p_0} |\nabla v|^{p_1} v \right), u - v \rangle \geq c_0 \| \nabla (u - v) \|^{p_0} - \\
\lambda \left( |u|^{p_0} u (|\nabla u|^{p_1} - |\nabla v|^{p_1}) , u - v \rangle - \lambda \left( |\nabla v|^{p_1} \left( |u|^{p_0} u - |v|^{p_0} v \right) , u - v \rangle \right.
\]
hold for any \( u, v \in W_0^{1,p} (\Omega) \). Here the second term of the right side one can estimate as
\[
\left| \left( |u|^{p_0} u (|\nabla u|^{p_1} - |\nabla v|^{p_1}) , u - v \rangle \right| \leq c_1 \left( |u|^{p_0} |\nabla u|^{p_1} |\nabla v|^{p_1} - |u - v| , |u - v| \right) \leq \\
\varepsilon \| \nabla (u - v) \|^{p} + C (\varepsilon) \| u |^{(p_0 - 1)p'} \| \nabla u |^{(p_1 - 1)p'} \| u - v |^{p'} , p' = \frac{p}{p-1}.
\]

Thus we get that all of the conditions of Theorem 4 (the case of Corollary 1) are fulfilled for the problem (3.6). Consequently, applying Theorem 4 we get the correctness of Theorem 4. \( \square \)

**4. Fully Nonlinear Operator**

Now we will study the question on the existence of the spectrum of the fully nonlinear operator. Let \( X, Y, Z \) are the real Banach spaces, the inclusion \( Y \subset Z \) is continuous and dense, where \( Z^{\ast} \) is the dual space of \( Z \). Let \( L : D (L) \subseteq X \rightarrow Y \) is linear operator, \( D (L) \) is dense in \( X \); \( f : D (f) \subseteq Y \rightarrow Z \) and \( g : X \subseteq D (f) \rightarrow Z \) are nonlinear operators. Assume that \( L (D (L)) \subseteq D (f) \).

We wish define the spectrum of the operator \( f \circ L : D (L) \subseteq X \rightarrow Z \) with respect to the operator \( g \), where \( D (L) \subseteq D (g) \). (We should to noted the nonlinearity of operator \( g \) depends on the natures of nonlinearity of the operator \( f \).)

We will consider problem

\[
(4.1) \quad f (Lx) - \lambda g (x) = h, \quad h \in Z,
\]
where \( \lambda \in \mathbb{R} \) be a parameter and \( h \) be an element of \( Z \).

We will investigate two questions:

(a) Do can define the spectrum of operator \( f \circ L \) with respect to operator \( g \)?

(b) For which \( \lambda \in \mathbb{R} \) and \( h \in Z \) equation (4.1) is solvable?
We should be noted our goal is to study the spectrum of the nonlinear operator in the sense of Definition 3, but as above noted the subset determined according to this definition contains numbers that depend on the elements of the domain of the examined operator. Therefore, we will call a number is the spectrum of the nonlinear operator if it characterizes the examined operator as the theory of the linear operators. Whence follows according to the above explanation that necessary to assume the identical homogeneity of the nonlinearities of the operators \( f \) and \( g \).

So, we will study the following particular case that in some sense maybe to explain the general case. To investigate the posed problems we will use the general results of the articles [29, 30]. Let \( B^X_\nu(0) \subset D(L), \ r_0 > 0 \) and consider the following conditions:

1) There are such constants \( c_1, c_2 > 0 \) that \( c_1 \| x \|_Y \geq \| Lx \|_Y \geq c_2 \| x \|_X \) for any \( x \in D(L) \subseteq X \), moreover, \( Y \) is reflexive space and the inverse to \( L \) is a compact operator;

2) The operator \( f \circ L \) is greater than the operator \( g, \) i.e. \( f \circ L > g \); \( f \) and \( g \) as the functions are continuous and satisfy the following conditions: \( f(t) \cdot t > 0 \) for \( \forall t \in R \setminus \{0\} \); \( f(0) = 0, \ g(0) = 0 \);

3) There is such number \( \lambda_0 > 0 \) that for each \( z^* \in S^Z_1(0) \) there exist \( x(z^*) \in S^X_r(0), 0 \leq r \leq r_0 \) such that the following inequality

\[
\langle f(Lx) - \lambda g(x) \rangle, z^* \rangle \geq \nu(\| Lx \|_Y, \lambda)
\]

holds for \( \forall \lambda : |\lambda| < \lambda_0, \) where \( \nu : R_+ \rightarrow R \) is a continuous function and there exists \( \delta_0(\lambda) > 0 \) such that \( \nu(t, \lambda) \geq \delta_0(\lambda) \) for \( \forall t = \| Lx \|_Y \) when the variable \( x \) run over the sphere \( S^X_{r_0}(0) \);

4) There is such \( \varepsilon_0 > 0 \) that for \( \varepsilon \geq \varepsilon_0 > 0 \) and a.e. \( x \in B^X_\nu(0) \subseteq X \) there exists a neighborhood \( U_\varepsilon(x) \) such that for \( \forall x_1, x_2 \in U_\varepsilon(x) \)

\[
\| f(Lx_1) - f(Lx_2), Lx_1 - Lx_2 \| \geq l(x_1, x_2) \| Lx_1 - Lx_2 \|_Y^2
\]

and

\[
\| g(x_1) - g(x_2) \|_Z \leq l_1(x_1, x_2) \| x_1 - x_2 \|_X
\]

hold, where \( l(x_1, x_2) > 0, l_1(x_1, x_2) > 0 \) are a bounded functionals in the sense of similar to Definition 4.

Theorem 5. Let conditions 1-4 be fulfilled and \( D(L) = X \). Then for each \( h \in M \subset Z \) and \( \forall \lambda : |\lambda| \leq \lambda_0 \) equation (4.7) solvable, where a subset \( M \) determined by the following inequality

\[
M(\lambda) \leq \left\{ z \in Z | \langle z, z^* \rangle \leq \nu(\| Lx \|_Y, \lambda), \ \forall z^* \in S^Z_1(0) \right\},
\]

for each \( x \in S^X_{r_0}(0) \).

Proof. For the proof, it is sufficient to show, that the examined operator satisfies all conditions of the general result of Subsection 2.1. It is clear the operator \( F_\lambda(x) \equiv f(Lx) - \lambda g(x) \) satisfies conditions (i),(ii) of the general results with the \( x_0 = 0 \) according to conditions 1-4 (since \( F_\lambda(0) = 0 \)). It remains to show fulfill of the condition (iii) and for this sufficiently to investigate the following expression

\[
\| F_\lambda(x_1) - F_\lambda(x_2) \|_Z = \| (f(Lx_1) - \lambda g(x_1)) - (f(Lx_2) - \lambda g(x_2)) \|_Z.
\]

Now we will prove this expression satisfies the following inequality

\[
\| F_\lambda(x_1) - F_\lambda(x_2) \|_Z \geq c(l(x_1, x_2) \| x_1 - x_2 \|_X, \lambda) - c_1(l_1(x_1, x_2) \| x_1 - x_2 \|_{X_0}, \lambda).
\]
We set the following expression:
\[
(F_\lambda (x_1) - F_\lambda (x_2), Lx_1 - Lx_2) = \langle (f (Lx_1) - f (Lx_2), Lx_1 - Lx_2) - \\
-\lambda (g (x_1) - g (x_2), Lx_1 - Lx_2)
\]
that is defined correctly, since \( F : D (L) \subseteq X \rightarrow Z \) and \( L : D (L) \subseteq X \rightarrow Y \subset Z^* \), whence by carry out certain necessary operations and considering the conditions of this section we get
\[
\langle F_\lambda (x_1) - F_\lambda (x_2), Lx_1 - Lx_2 \rangle = \langle f (Lx_1) - f (Lx_2), Lx_1 - Lx_2 \rangle - \\
-\lambda \langle g (x_1) - g (x_2), Lx_1 - Lx_2 \rangle \geq l (x_1, x_2) \| Lx_1 - Lx_2 \|_Y^2 - \\
-|\lambda| \| g (x_1) - g (x_2) \|_Z \| L (x_1 - x_2) \|_Y \geq l (x_1, x_2) \| L (x_1 - x_2) \|_Y^2 - \\
-|\lambda| \| g (x_1) - g (x_2) \|_Z \| L (x_1 - x_2) \|_Y
\]
(4.2)
according to condition 1). Now again of taking into account condition 1) we arrive to
\[
|\langle F_\lambda (x_1) - F_\lambda (x_2), Lx_1 - Lx_2 \rangle| \leq \| F_\lambda (x_1) - F_\lambda (x_2) \|_Z \cdot \| Lx_1 - Lx_2 \|_Y.
\]
Thus using inequalities (4.2) and (4.3) and condition 4 we get the following estimate
\[
\| F_\lambda (x_1) - F_\lambda (x_2) \|_Z \geq l (x_1, x_2) \| L (x_1 - x_2) \|_Y - |\lambda| \| g (x_1) - g (x_2) \|_Z \geq \\
l (x_1, x_2) \| L (x_1 - x_2) \|_Y - |\lambda| l (x_1, x_2) \| x_1 - x_2 \|_X.
\]
(4.4)
So, under conditions of this theorem conditions (i)-(iii) of the general theorem are fulfilled, and from the above inequality follows the fulfillment of condition (iv) ensure that the image of the \( F (B^X_{r_0} (0)) \) is closed. Consequently, Theorem 5 is proved.

Remark 3. It needs to note the subset \( M (\lambda) \) defined in Theorem 5 decreases by increases of the number \(|\lambda| \searrow \lambda_0\). The above-mentioned articles actually seek the number \( \lambda_0 \) that the existence was assumed in the previous theorem.

Here we will study how one can determine such numbers that one can use as the number \( \lambda_0 \), moreover, which are independent of the elements from the domain of the examined operators. Really by such way, the founded number can be called the first eigenvalue of the examined operator relative to the different operator as in Definition 7.

In what follow we will use some results from the article Berger [6] (see, also [34]). Therefore, we provide here results that are necessary for us from article [6].

Definition 5. (6) Let \( A : X \rightarrow X^* \) be a variational operator. Then \( A \) is of class \( J \) if:
(i) \( A \) is bounded, i.e. \( \| A (x) \| \leq \mu (\| x \|) \);
(ii) \( A \) is continuous from the strong topology of \( X \) to the weak topology of \( X^* \);
(III) Oddness, i.e. \( A (-x) = -A (x) \);
(iv) Coerciveness, i.e. \( \int_0^1 \langle A (sx), x \rangle \, ds \searrow \infty \) when \( \| x \|_X \searrow \infty \);
(V) Monotonicity: \( \langle A (x_1) - A (x_2), x_1 - x_2 \rangle > 0 \), for any \( x_1, x_2 \in X \).
Lemma 4. Let $A$ be as variational operator of class I, then a $\partial A_R$

$$\partial A_R = \left\{ x \in X \left| \int_0^1 \langle A (sx) , x \rangle \, ds = R \right. \right\}$$

is a closed, bounded set in $X$. Furthermore $\|x\|_X \geq k(R) > 0$ and $\partial A_R$ is a weakly closed, bounded convex set, where $k(R)$ is a constant independent of $x \in \partial A_R$.

Theorem 6. Let $A : X \to X^*$ be an operator of class I or II where $X$ is a reflexive Banach space over the reals. Let $B : X \to X^*$ an operator of class III. Then the eigenvalue problem $A(x) = \lambda B(x)$ has at least one non-trivial solution. This solution is normalized by the requirement that $x \in \partial A_R$ and characterized as a solution of the variational problem for

$$\sup \left\{ \frac{1}{0} \int \langle B(sx), x \rangle \, ds \mid x \in \partial A_R \right\}$$

sufficiently large $R$. Furthermore

$$\lambda = \frac{\langle A(x), x \rangle}{\langle B(x), x \rangle}$$

So, consider the homogeneous equation in order to investigate the existence of the number $\lambda_0$.

Proposition 2. Let $X \subset Y$ and dense in $Y$, $Z = Y^*$. Let conditions 1, 2, 4 of the above Theorem 7 are fulfilled for this case and $f, g$ as the functions are monotone odd functions then there exist such $\lambda_0 > 0$ and $x \lambda_0 \in \partial E^{B_R0} \subset X$ that $F_\lambda_0 (x, \lambda_0) \equiv f (Lx, \lambda_0) - \lambda_0 g (x, \lambda_0) = 0$, for some number $R_0 \gg 1$, where $\partial E^{B_R0}$ be defined as follows

$$\partial E^{B_R0} = \left\{ x \in B^{B_R0} (0) \subset X \left| \int_0^1 \langle f(sLx), Lx \rangle \, ds = R_0 \right. \right\}$$

Moreover, the condition similar to condition 3 of the above theorem satisfies.

Proof. It is clear that $L (B^{B_R0} (0))$ is convex due to the linearity of the operator $L$. From above Lemma follows that $E^{B_R0}$ is a weakly closed, bounded convex set and $\|x\|_X \geq k(r_0) > 0$, where $k(r_0)$ is a constant independent of $x \in E^{B_R0}$, as the operator $f \circ L$ satisfies all conditions of this lemma.

Consequently, the result of Berger follows that $E^{B_R0}$ and $\partial E^{B_R0}$ are weakly closed, bounded convex sets.

Now consider the expression $(g(x), Lx)$ and note that there exists such constant $M$ that

$$0 < \sup \left\{ \|g(x)\|_{Y'} \mid x \in \partial E^{B_R0} \right\} = M < \infty$$

according to the conditions 1, 2 and boundedness of the norm $\|Lx\|_Y$ (as $0 < \|Lx\|_Y < M_1 < \infty$).

Consequently, there is constant $\lambda_0 = \lambda_0 (M)$ such that $0 < \lambda_0 < \infty$. \[\square\]
So, we provide the result on the spectrum of the operator \( f \circ L \) respect to the operator \( g \).

**Theorem 7.** Let the functions \( f \) and \( g \) are homogeneous moreover their order of the homogeneity equally and is the function \( \varphi \), i.e. for each \( \tau \in \mathbb{R} \) hold the equality \( f (\tau \cdot y) = \varphi (\tau) \cdot f (y) \), \( g (\tau \cdot y) = \varphi (\tau) \cdot g (y) \). Let all conditions of the above proposition fulfills. Then the operator \( f \circ L \) has the spectrum respect to the operator \( g \). Moreover, this spectrum is the function of the spectrum of the operator \( L \).

**Proof.** From the previous theorem follows the existence of such \( \lambda_0 \in \mathbb{R} \) that the equation \( F_x (\lambda_0) \equiv f (Lx) - \lambda_0 g (x) = 0 \) solvable. Then using of the well-known approach it is necessary to seek an element \( x \in X \) that satisfy the following equality

\[
\lambda = \inf \left\{ \frac{\langle f (Lx), Lx \rangle}{\langle g (x), Lx \rangle} \mid x \in X \right\}.
\]

Due to the conditions of this theorem, it is enough to study the above question only for \( x \in S_{1}^{X} (0) \). We can take into account that \( f \) is an \( N \)–function and the expression \( \langle f (Lx), Lx \rangle \) generates a functional \( \Phi (Lx) \) according to the condition on \( f \).

Whence follows that it is enough to seek the number \( \lambda \) the following way

\[
\lambda = \inf \left\{ \frac{\| f (Lx) \|_Z}{\| g (x) \|_Z} \mid x \in S_{1}^{X} (0) \right\}.
\]

From above theorem follows the existence of the number \( \lambda > 0 \). Moreover, it isn’t difficult to see that all conditions of the Theorem are fulfilled under the conditions of this theorem since due to condition 3 always one can find such a number \( \lambda_0 > 0 \) that condition \((iii)\) will fulfill. Thus follows the existence an element \( x_0 \in S_{1}^{X} (0) \) and a number \( \lambda_0 \) that is the desired infimum.

Let \( x_1 \in S_{1}^{X} (0) \) is the first eigenfunction and \( \lambda_1 \) first eigenvalue of the operator \( L \) then we have

\[
\lambda_0 \leq \frac{\varphi (\lambda_1) \| f (x_1) \|_Z}{\| g (x_0) \|_Z}.
\]

\[\square\]

Now we provide some examples of operators related to the above theorems.

1. Let \( L : W^{m,p} (\Omega) \rightarrow L_p (\Omega) \) be a linear differential operator with the spectrum \( P (L) \subset R_+ \), the operator \( f \) is the function \( f (\tau) = |\tau|^{p-2} \tau \) and \( g \equiv f \).

So, it needs to define the first eigenfunction and eigenvalue of the operator \( f (L \circ) \) relative to operator \( g (\circ) \). Then using the expression \((4.5)\) we get

\[
\lambda_f = \inf \left\{ \frac{\langle f (Lu), Lu \rangle}{\langle g (u), Lu \rangle} \mid u \in W^{m,p} (\Omega) \right\} =
\]

**Remark 4.** In the case when \( L \) is the differential operator \( \langle f (Lx), Lx \rangle \) is a function of the norm \( \| Lx \|_{L_p} \) of some Lebesgue or Orlicz space, where \( \Phi \) be an \( N \)–function.
following equality

then we get

\[ \lambda_{p_2} \]  

i.e., we will seek of the spectrum of the operator \( \nabla u \) (4.3)

\[ \lambda \]

\[ \mu \]

Whence we arrive that \( \lambda_{f_1} = \lambda_{p_1}^{p_2} \), where \( \lambda_{L_1} \) is the first eigenvalue and the function \( u_1 \in S_{1}^{w_{k}, p, \Omega} (0) \) is the first eigenfunction of the operator \( L \).

2. We will study the spectral property of the fully nonlinear operator in the following two special cases

\[ \Delta |u|^p - 2 \Delta u = \lambda |\nabla u|^\mu - 2 u, \quad x \in \Omega, \quad u \mid \partial \Omega = 0, \]

\[ \Delta |u|^p - 2 \Delta u = \lambda |u|^\mu u, \quad x \in \Omega, \quad u \mid \partial \Omega = 0, \]

i.e., we will seek of the spectrum of the operator \( \Delta |u|^p - 2 \Delta u \) relatively of operators \( |\nabla u|^\mu - 2 u \) and \( |u|^\mu u \), separately.

2 (a). Consider problem (4.2). For study of the posed question we will use the following equality

\[ \lambda \]

then we get

\[ \lambda = (\mu - 1)^{-1} |\nabla u|^\mu \]

Hence follows

\[ \lambda_{1} (p, \mu) = (\mu - 1) \inf \left\{ \frac{\| \Delta u \|^p_{L^p}}{\| \nabla u \|^\mu_{L^\mu}} \mid u \in W^{2, p} \cap W^{1, p} \right\} \]

\[ \mu \]

again for to find \( \lambda \) satisfying of the assumed condition we must select such exponent \( \mu \) that the given case requires.

Consequently, we need assume \( \mu = p \), then we get

\[ \lambda_{1} (p, p) = (p - 1) \inf \left\{ \frac{\| \Delta u \|^p_{L^p}}{\| \nabla u \|^p_{L^p}} \mid u \in W^{2, p} \cap W^{1, p} \right\} \]

As is well-known \( \| \nabla u \|^p \leq c (p, \Omega) \| \Delta u \|^p \) under the condition \( u \mid \partial \Omega = 0 \), consequently, \( \lambda_{1} (p, p) \leq (p - 1) c (p, \Omega) \).

Proposition 3. Let \( f_0 : W^{2, p} (\Omega) \cap W^{1, p} (\Omega) \rightarrow L^q (\Omega) \) that has the presentation \( f_0 (u) = - |\Delta u|^p - 2 \Delta u \) and \( f_1 : W^{1, p} (\Omega) \rightarrow L^q (\Omega) \) that has the presentation \( f_1 (u) = |\nabla u|^p - 2 u \). Then \( f_0 \) has minimal spectrum with respect to \( f_1 \) which defined by (4.4).
2 \(b\). Consider problem (4.3) for \(\nu = p - 2\) then we have
\[
\left\langle -|\Delta u|^{p-2} \Delta u, -\Delta u \right\rangle = \left\langle \lambda |u|^{p-2} u, -\Delta u \right\rangle \implies
\]
\[
\|\Delta u\|_p^p = \lambda \frac{4(p-1)}{p^2} \left\| \nabla \left( |u|^\frac{p-2}{2} u \right) \right\|_2^2
\]
or
\[
\|\Delta u\|_p^p = \lambda (p-1) \left\| \left( |u|^{p-2} |\nabla u| \right)^2 \right\|_1.
\]
Thus we get
\[
\tilde{\lambda}_1(p) = \frac{1}{p-1} \inf \left\{ \frac{\|\Delta u\|_p^p}{\left\| |u|^{\frac{p-2}{2}} |\nabla u| \right\|_2^2} \left| u \in W^{2,p} \cap W_0^{1,p} \right\} = \frac{p}{2(p-1)} \inf \left\{ \frac{\left\| |u|^{\frac{p-2}{2}} \right\|_2^2}{\left\| \nabla \left( |u|^{\frac{p-2}{2}} u \right) \right\|_2^2} \left| u \in W^{2,p} \cap W_0^{1,p} \right\}
\]
(4.7)
Hence we obtain
\[
\tilde{\lambda}_1(p) \geq \frac{1}{p-1} \frac{\|\Delta u\|_p^p}{\| |u|^{p-2} \|_p^2 \|\nabla u\|_p^2}
\]
according of the following inequality
\[
\left\| \left( |u|^{p-2} |\nabla u| \right)^2 \right\|_1 \leq \|u\|_p^{p-2} \|\nabla u\|_p^2.
\]
So, we arrive to result

**Proposition 4.** Let \(f_0 : W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega) \to L^q(\Omega)\) that has the presentation \(f_0(u) = -|\Delta u|^{p-2} \Delta u\) and \(f_1 : L^p(\Omega) \to L^q(\Omega)\) that has the presentation \(f_1(u) = |u|^{p-2} u\). Then \(f_0\) has minimal spectrum with respect to \(f_1\) which defined by (4.5) and satisfies the inequation (4.6).

One can approach at the mentioned above question on the relation using also of the following equality
\[
- \int |\Delta u|^{p-2} \Delta u = \lambda \int |u|^{p-2} u \implies
\]
as the operator \(-\Delta\) is positive under the conditions of this section, then we get
\[
\|\Delta u\|_{p-1} = \lambda \|u\|_{p-1} \implies
\]
\[
\lambda = \inf \left\{ \frac{\|\Delta u\|_{p-1}^{p-1}}{\|u\|_{p-1}} \left| u \in W^{2,p} \cap W_0^{1,p} \right\}.
\]
4.0.1. Some remarks on the eigenvalues and bifurcation. From above-mentioned results follows:

1. To seek the eigenvalues of the nonlinear continuous operator in the Banach space is necessary to choose the other operator in such a way that the order of nonlinearity will be identical with the order of nonlinearity of the examined operator. If one uses the proposed approach then it is possible to find and the other eigenvalues of this operator.

2. To study the bifurcation of solutions, in the beginning, is necessary to find eigenvalues of this operator relatively to the nonlinear operator that has identical order of the nonlinearity with the examined operator from the main part, moreover, the choosing another operator must take account of the properties of the second part of the examined equation, e.g. if consider the equation (2.2) usually assumed $f \succ g$, where $f$ is the main part and $g$ is the second part of this equation, but in this case, the order of nonlinearity $g$ must be greater than the order of nonlinearity $f$ (see, e.g. Section 3), as in articles [12, 16, 33], etc.

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National Academy of Sciences of Azerbaijan, Baku, AZERBAIJAN
Email address: sultan_kamal@hotmail.com
URL: https://www.researchgate.net/profile/Kamal-Soltanov/research