QUASI-VERTEX-TRANSITIVE MAPS ON THE PLANE

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Abstract. Quasi-vertex-transitive maps are the homogeneous maps on the plane with finitely many vertex orbits under the action of their automorphism groups. We show that there exist quasi-vertex-transitive maps of types \([p^3, 3]\) for \(p \equiv 1 \pmod{6}\), but there doesn’t exist vertex-transitive map of such types. In particular, we determine the surface with the lowest possible genus that admit a polyhedral map of type \([5^3, 3]\).

1. Introduction

The subject of maps on surfaces lies at the interfaces of discrete geometry, graph theory and combinatorial topology. Maps with symmetry (e.g., vertex-transitive maps) are objects of interest to group theorists. The concept of quasi-vertex-transitive map is a natural generalization of the concept of vertex-transitive map on the plane. A map on a surface \(S\) (2-manifold) is a simple, connected, locally finite graph \(G\) embedded on \(S\) satisfying: 1) the closure of each connected component of \(S \setminus G\), called a face, is homeomorphic to a closed 2-disc, 2) each vertex of \(G\) has degree at least 3.

A vertex \(v\) of a map is said to be polyhedral if the intersection of any two distinct faces incident to \(v\) is either \(v\) or an edge incident to \(v\). A map is said to be polyhedral if all its vertices are polyhedral [5].

The vertex-type of a vertex \(v\) of a map is a cyclic sequence, denoted by a cyclic-tuple \([k_1, k_2, \cdots, k_d]\), of the lengths of the faces around \(v\). A vertex-type and its mirror image will be considered to be the same. A map is said to be homogeneous if all its vertices are of the same type [14]. The type of a homogeneous map is the vertex-type of its vertices. A vertex-type is usually written in multiplicative form, e.g., \([5, 5, 3, 3] = [5^2, 3^2]\). Homogeneous maps on the plane are always polyhedral (see Lemma 2.3).

An isomorphism between two maps on a surface \(S\) is an isomorphism between their underlying graphs that extends to a self-homeomorphism of \(S\). We say that a map \(K\) has \(m\) vertex orbits if the action of the group of automorphisms of \(K\), \(Aut(K)\), on the set of vertices of \(K\) has \(m\) orbits. Vertex-transitive maps are the maps with single vertex orbit. They are obviously homogeneous. A map is assumed to be homogeneous, unless otherwise stated.

A homogeneous map \(K\) on the plane is said to be quasi-vertex-transitive if it has finitely many vertex orbits, or equivalently, it is isomorphic to lift of a homogeneous map on some closed surface (see Proposition 2.6). We define the minimal characteristic(respectively minimal polyhedral characteristic) of a quasi-vertex-transitive map \(K\) to be the smallest number \(-\chi\) such that \(K\) is the lift of a map (resp. polyhedral map) on a surface (which exists by Lemma 2.4) with Euler characteristic \(\chi\).

It is known that there doesn’t exist vertex-transitive maps on the plane of types \([p^3, 3]\), \(p \equiv 1 \pmod{6}\) ([13], Proposition 3.1). Here, we apply the Poincaré polygon theorem to prove the following.

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Theorem 1.1. There exist quasi-vertex-transitive maps on the plane of types \([p^3, 3], \) \(p \) odd, \(p \geq 5.\)

To the author’s knowledge, \([p^3, 3] \) with \(p \equiv 1 \) (mod 6) are the first known examples of cyclic-tuples that are the types of quasi-vertex-transitive maps but not the types of vertex-transitive maps on the plane.

The cyclic-tuple \([5^3, 3] \) is of special importance to us and we have the following.

Theorem 1.2. There exists a quasi-vertex-transitive map on the plane of type \([5^3, 3] \) with the minimal characteristic 1 and the minimal polyhedral characteristic 2.

This implies, in particular, that all the hyperbolic surfaces \((\chi < 0)\) admit a map of type \([5^3, 3] \), but none of them are vertex-transitive, the type is unique in that sense.

A map \(K \) on a surface \(S \) is said to be geometric if \(S \) admits a metric with constant curvature with respect to which the faces of \(K \) are regular polygons in \(S \). One of the interesting facts about homogeneous maps is that they can always be made geometric (see Fact 2.2). So, the results of this paper can also be presented in a geometric way. Tilings produced by geometric vertex-transitive and homogeneous maps on a surface are known as uniform and semi-regular tilings respectively.

It is not very difficult to construct maps of types \([p^3, 3], p \geq 5 \) on the plane with infinitely many vertex orbits. Our work was motivated by the problem of finding a finite aperiodic set of regular polygons (tiles) for the hyperbolic plane, or even stronger, a cyclic-tuple \(\mathfrak{t} \) such that there exist homogenous maps on the plane of the type \(\mathfrak{t} \) but none of them are quasi-vertex-transitive, thereby strengthening the result of [10].

2. Preliminaries

A cyclic-tuple \(\mathfrak{t} = [k_1, k_2, \ldots, k_d] \) with \(d \geq 3 \) is said to be of the spherical, Euclidean, or hyperbolic type if the “angle sum” \(\alpha(\mathfrak{t}) = \sum_{i=1}^{d} \frac{k_i-2}{k_i} \) is \(< 2, = 2, \) or \(> 2 \) respectively.

Fact 2.1 ([3] [7]). For a cyclic-tuple \(\mathfrak{t} = [k_1, k_2, \ldots, k_d] \) with \(d \geq 3 \) and \(\alpha(\mathfrak{t}) < 2, = 2, \) or \(> 2, \) there exist spherical, Euclidean, or hyperbolic regular \(k_i\)-gons (with geodesic sides) for \(i = 1, \ldots, d, \) respectively, which fits around a vertex, or equivalently, their inner angles \(\theta_i \) sum to \(2\pi, \) that is, \(\sum_{i=1}^{d} \theta_i = 2\pi.\)

A map on a surface lifts to a map on one of the universal covers, the sphere, \(S^2\) or the plane. On the other hand, any geometric map on a surface is quotient of a geometric map on one of the universal covers, \(S^2,\) the Euclidean plane, \(\mathbb{E}^2,\) or the hyperbolic plane, \(\mathbb{H}^2,\) under the free action of a discrete group of isometries. Thus, Fact 2.1 implies the following.

Fact 2.2 ([3] [6] [7]). (Geometrization) A homogeneous map on a surface \(S\) is isomorphic to a geometric map on \(S.\) Moreover, each automorphism of a geometric map on \(S\) corresponds to an unique isometry of \(S.\)

If a cyclic-tuple \(\mathfrak{t}\) is the type of a homogeneous map on the sphere, then by the Euler formula \(\alpha(\mathfrak{t}) < 2.\) All such types are known, they correspond to the boundaries of Platonic & Archimedean solids, the prism, the pseudohombicuboctahedron, and additionally two infinite families, \([4^2, r]\) for some \(r \geq 5\) and \([3^3, s]\) for some \(s \geq 4\) (antiprisms) [2] [17].

If a cyclic-tuple \(\mathfrak{t}\) is the type of a homogeneous map on the plane, then it can be deduced from the Euler formula for finite planar graph that \(\alpha(\mathfrak{t}) \geq 2\) [6]. If \(\alpha(\mathfrak{t}) = 2,\) then \(K\) is isomorphic to a geometric map on the Euclidean plane, there are exactly 11 such types [12]. The maps on the torus and the Klein bottles are naturally quotients of maps of those 11 types [8].

The following fact should be well-known though the author was not able to trace a reference.
Lemma 2.3. Homogeneous maps on simply connected surfaces are always polyhedral.

Proof. Let $K$ be a homogeneous map on a simply connected surface of type $\mathfrak{t} = [k_1, k_2, \ldots, k_d]$. By Fact 2.2 above, we may assume that $K$ is a geometric map on $\mathbb{E}^2$, $\mathbb{H}^2$ or $S^2$, depending on $\alpha(\mathfrak{t}) = 2, > 2$ or $< 2$ respectively. A geodesic ray on a surface $S$ with initial point $x \in S$ is a geodesic $\gamma : [0, \infty) \to S$ with $\gamma(0) = x$.

Suppose $K$ is a geometric map on $\mathbb{E}^2$ or $\mathbb{H}^2$. Then for any vertex $v$ of $K$, the geodesic rays with initial point $v$ that extend the $d$ edges incident to $v$, divide $\mathbb{E}^2$ or $\mathbb{H}^2$ respectively into $d$ connected components. So, the interior of each of the faces (regular and therefore convex) surrounding $v$ lies in only one of these components. Note that by definition of a map, $d \geq 3$. It follows that $K$ is polyhedral.

Suppose $K$ is a geometric map on $S^2$. Then for any vertex $v$ of $K$, the shortest geodesics (or the semi-circles) between $v$ and its antipodal point $v_\infty$ that extend the $d$ edges incident to $v$, divide $S^2$ into $d$ connected components. The interior of each of the faces surrounding $v$ lies in only one of these components. Since two distinct great circles meet only at antipodal points and the underlying graph of $K$ is simple, therefore none of the faces (regular) surrounding $v$ have $v_\infty$ as a vertex. It follows that $K$ is polyhedral. \qed

But the same cannot be said about homogeneous maps on non-simply connected surfaces, as we shall see an example in Figure 4.4(a) below. The following folklore, however, asserts that polyhedrality is a weak condition for homogeneous maps on any surface.

Lemma 2.4. Let $K$ be a homogeneous map on a surface $S$. Then, there is a finite covering $\tilde{S}$ of $S$ such that the lift of $K$ to $\tilde{S}$ is a polyhedral map.

Proof. The proof essentially follows from the above lemma and the residual finiteness property of the surface groups. If $S$ is the real projective plane, the sphere, or the plane, then the proof directly follows from the above lemma. In all other cases, i.e., if $S$ is closed and $\chi(S) \leq 0$, then a map $K$ on $S$ lifts to a map $K_U$ on the universal cover of $S$, the plane. Let $F$ be a minimal subset of the set of faces of $K_U$ that generates $K_U$ under the action of the fundamental group $\pi_1(S)$. Here one may assume that the union of the faces in $F$ is a fundamental domain for $\pi_1(S)$. Let $C$ be the union of the faces that intersect any face in $F$. Since $\pi_1(S)$ is residually finite [17], there is a finite covering $f : \tilde{S} \to S$ such that the covering map $\tau_{\tilde{S}} : \mathbb{R}^2 \to \tilde{S}$ is injective on the compact set $C$. This means that all the vertices in $\tau_{\tilde{S}}(F)$ are polyhedral. Now, since every finite index subgroup of a group $\mathcal{G}$ contains a finite index normal subgroup of $\mathcal{G}$, so we can choose $\tilde{S}$ to be a normal covering. Then all the vertices in $\tau_{\tilde{S}}(K_U)$ are polyhedral. But $\tau_{\tilde{S}}(K_U)$ is nothing but the lift of $K$ to $\tilde{S}$, thus the proof follows. \qed

Remark 2.5. The proof of the above lemma also works for homogeneous maps with singular faces, faces that are not contractible but their interiors are.

Now, as mentioned in the introduction, we prove the following.

Proposition 2.6. A homogeneous map on the plane is quasi-vertex-transitive if and only if it is isomorphic to a lift of a map on some closed surface.

Proof. The "only if" part of the statement is obvious. For the "if" part, let $K$ be a homogeneous map on the plane of the hyperbolic type (resp. the Euclidean type) with $m$ vertex orbits. By Fact 2.2 we may assume that $K$ is geometric and $Aut(K)$ to be a subgroup of isometries of $\mathbb{H}^2$ (resp. $\mathbb{E}^2$). Due to the fact that $Aut(K)$ is also the automorphism group of the dual of $K$ acting with $m$ face orbits, the fundamental polygon, $F$ of the dual map consists of $m$ faces of the dual. This also means that $F$ is fundamental polygon for a subgroup $\mathcal{H}_1$ of $Aut(K)$. $\mathcal{H}_1$
must be finitely generated because fundamental polygon of infinitely generated subgroup of
isometries of $\mathbb{H}^2$ (resp. $\mathbb{E}^2$) have infinitely many sides.

Now by Selberg’s lemma [11], $\mathcal{H}_1$ has a finite index, torsion free subgroup $\mathcal{H}_2$ of $Aut(K)$. This
means $\mathcal{H}_2$ is a discrete subgroup of isometries of $\mathbb{H}^2$ (resp. $\mathbb{E}^2$) acting freely on $\mathbb{H}^2$ (resp. $\mathbb{E}^2$).
It follows that $K/\mathcal{H}_2$ is a homogeneous map on the closed surface $\mathbb{H}^2/\mathcal{H}_2$ (resp. $\mathbb{E}^2/\mathcal{H}_2$). □

The work of enumeration of vertex-transitive maps on closed surfaces was done in [2] in
terms of their genus. The Euler formula produces a bound on the number of possible types
of homogeneous maps on a surface. For a fixed type, the number of maps on a surface with
$\chi(S) \neq 0$ is finite. Since closed surfaces are countable by their genus and cross-cap numbers,
so the number of quasi-vertex-transitive maps on the plane are countable (up to isomorphism).

Examples 2.7. The examples of homogeneous maps that appear in the literatures are mostly
vertex-transitive or have infinitely many vertex orbits. In [9, 10], the authors showed that
the maps of types $[p^3]$ and $[2p_1, 2p_2, \cdots, 2p_k]$ are vertex-transitive, and also determined their
minimal characteristics. One can construct vertex-transitive maps of various other types by
applying standard operations (e.g. truncation, rectification, cantellation) on these types. Since
most of these operations are reversible, one can also determine the minimal characteristics of
the derived maps.

Examples 2.8. Quasi-vertex-transitive maps of a fixed type are, in general, not unique. In
fact, there are non-isomorphic vertex-transitive maps of the same type, e.g., $[(6n)^3, 3], [4^3, 6]$.

3. QUASI-VERTEX-TRANSITIVE MAPS OF TYPES $[p^3, 3]$

Among all the homogeneous maps on the sphere, only the map of type $[4^3, 3]$ is not vertex-
transitive, and it corresponds to the boundary of pseudorhombicuboctahedron [11]. All the
homogeneous maps on the plane of the Euclidean types are vertex-transitive, but most of the
maps of the hyperbolic types are not so. Here we give a simple proof of the following result
from [13].

Proposition 3.1. There doesn’t exist vertex-transitive map on the plane of type $[p^3, 3]$, for
$p \equiv 1 \mod(6)$.

Proof. We say that a $p$-gon in a map $K$ of type $[p^3, 3]$ is of type-$k$ if the number of triangles
intersecting the $p$-gon is $k$. Suppose there are more than one type of $p$-gons in $K$. Since $p$ is
odd by our hypothesis, each $p$-gon has at least one vertex such that a triangle intersect the
$p$-gon only by that vertex. Consider two such vertices of two different types of $p$-gons in $K$,
then clearly no automorphism of $K$ can map one to the other, so $K$ is not vertex-transitive.
Suppose now that all $p$-gons in $K$ are of the same type, say of the type-$l$. Let us consider a
disc region $V_R$ in the plane of radius $R$ containing a large number of vertices of $K$. Then we
have a count $ct(e)_{ed}$ of edges of $K$ that are common to a $p$-gon and a triangle in the region $V_R$.
We also have a count $ct(\tau)$ of vertices that are the only intersection of a $p$-gon and a triangle in
$V_R$. Since each triangle contribute 3 to both the counts, therefore $ct(e) \sim ct(\tau)$ for sufficiently
large $R$. If each $p$-gon contributes $i$ and $j$ (with $i + j = l$) to $ct(e)$ and $ct(\tau)$ respectively, then
we must have $i = j$. It is easy to see that $i \neq j$ unless $p$ is multiple of 3. Since $p$ is odd,
therefore the proposition follows. □

We shall now prove Theorem [11] by applying the Poincaré polygon theorem. The theorem,
in particular, gives sufficient conditions for a convex hyperbolic polygon with finitely many
sides (geodesic segments) to be a fundamental polygon for a discrete subgroup of isometries
of the hyperbolic plane. We refer the readers to [4] for more details about the theorem.
Proof of Theorem [1.7] We will construct a fundamental polygon for the dual of a geometric map of type \([p^3, 3]\) for \(p \geq 5\), \(p\) odd. By Fact [2.1] for the cyclic-tuple \([p^3, 3]\), we have a configuration of three regular hyperbolic \(p\)-gons and one regular hyperbolic triangle around a vertex, see Figure 3.1. Let \(P_4 := \{v_1, v_2, v_3, v\}\) be the hyperbolic quadrilateral obtained by joining the incenters of the faces by geodesic segments, where \(v_2\) is the incenter of the triangle. Then we have \(\angle v_1 = \angle v_3 = \angle v = 2\pi/p, \angle v_2 = 2\pi/3\). So we can form a cyclic sequence of polygons \([V_1, V_2, \cdots, V_p]\) around the vertex \(v\) such that a) \(V_{(p-1)/2} = P_4\), b) \(V_i\) is obtained by reflecting \(P_4\) successively in its dotted sides \(\{v, v_1\}\) and \(\{v, v_3\}\), for \(i = 1, 2, \cdots, \frac{p-1}{2} - 1, \frac{p-1}{2} + 1, \cdots, p-2\), c) \(V_{p-1}\) and \(V_p\) are obtained by rotating \(V_{p-2}\) and \(V_1\) respectively about the midpoint of their dotted sides by an angle \(\pi\). Then the edges of the faces around \(v\) that are not adjacent to \(v\) form a \(2p\)-gon which we may denote by \(\{v_1, v_2, \cdots, v_{2p}\}\), see Figure 3.2 for \(p = 7\). We shall now prove that the hyperbolic \((2p + 2)\)-gon \(F(p)\) obtained from \(2p\)-gon \(\{v_1, v_2, \cdots, v_{2p}\}\) by declaring the midpoints of its two dotted sides \(\{v_p, v_{p+1}\}\) and \(\{v_{p+3}, v_{p+4}\}\) as vertices, denoted by \(v'_p\) and \(v'_{p+3}\) respectively, is our required fundamental polygon. Let \(e_i\) denote the side \(\{v_i, v_{i+1}\}\) for \(i \neq p, p + 3\), and \(e_p, e'_p, e_{p+3}, e'_{p+3}\) denote the sides \(\{v_p, v'_p\}, \{v'_p, v_{p+1}\}, \{v_{p+3}, v'_{p+3}\}, \{v'_{p+3}, v_{p+4}\}\) respectively. The sides are equipped with orientations given by the arrows pointing outwards from the vertices \(v'_p, v'_{p+3}\) and \(v_i\), for \(i = 4, 6, \cdots, p-1, p+5, p+7, \cdots, 2p\), and inward to the vertices \(v_2\) and \(v_{p+2}\). Then we consider the side-pairing isometries \([e_1, e_{p+2}], [e_2, e_{p+1}], [e_p, e'_p], [e_{p+3}, e'_{p+3}], [e_i, e_{i-1}]\) for \(i = 4, 6, \cdots, p-1, p+5, p+7, \cdots, 2p\). So the elliptic cycles are

\[
\{v_2, v_{p+2}\}, \{v_3, v_5, \cdots, v_p, v_{p+1}\}, \{v_{p+3}, v_{p+4}, v_{p+6}, \cdots, v_{2p-1}, v_1\}, \{v'_p, v'_{p+3}\}, \{v_i\} \quad \text{for} \quad i = 4, 6, \cdots, p-1, p+5, p+7, \cdots, 2p.
\]

We verify that

\[
\angle v_2 + \angle v_{p+2} = 2\pi, \quad \angle v_3 + \angle v_5 + \cdots + \angle v_p + \angle v_{p+1} = 2\pi,
\]

\[
\angle v_{p+3} + \angle v_{p+4} + \angle v_{p+6} + \cdots + \angle v_{2p-1} + \angle v_1 = 2\pi,
\]

\[
\angle v'_p = \angle v'_{p+3} = \pi, \quad \angle v_i = 2\pi/p \quad \text{for} \quad i = 4, 6, \cdots, p-1, p+5, p+7, \cdots, 2p. \quad (3.1)
\]
So, all the elliptic cycles are proper elliptic cycles. Therefore, by the Poincaré polygon theorem, $F(p)$ is the fundamental polygon for the group of isometries of the hyperbolic plane generated by the side pairings. Thus we have a geometric map $K$ on $\mathbb{H}^2$ whose faces are copies of $F(p)$. It is easy to see that the map obtained by subdividing each face of $K$ into quadrilaterals $P_4$s is the dual of a map $\tilde{K}$ of type $[p^3, 3]$. Clearly, the map $\tilde{K}$ has at most $p$ vertex orbits. This completes the proof of the theorem. 

4. Polyhedral maps of type $[5^3, 3]$ 

The vertex-transitive polyhedral maps on orientable surfaces of genus 2, 3 and 4 have been studied in [15]. The question of the existence of polyhedral maps of all the possible types on the surface with $\chi = -1$ has been settled except for the type $[5^3, 3]$ (in a private communication with Dipendu Maity, will appear elsewhere). Here we take up the task of verifying the remaining case.

**Proof of Theorem** 4.2. Let us assume that $K$ be a polyhedral map of type $[5^3, 3]$ on the surface with $\chi = -1$. Then the number of vertices, edges and faces are $-15$, $-30$ and $-14$ respectively. The number of triangles and pentagons in $K$ are 5 and 9 respectively, and the triangles are mutually disjoint. We call a pentagon is of type-$n$ if the number of triangles adjacent to it is $n$. Let $p_n$ be the number of type-$n$ pentagons in $K$ for $n = 3, 4, 5$. Then we have

$$p_3 + p_4 + p_5 = 9.$$  \hspace{1cm} (4.1) 

Let us count the pairs $(P, T)$ such that $P$ is a pentagon adjacent to the triangle $T$. Since the total number of pentagons adjacent to each triangle is 6, so the number of distinct pairs $(P, T)$ counted in two ways yield the equation

$$3p_3 + 4p_4 + 5p_5 = 6 \times 5 = 30.$$  \hspace{1cm} (4.2)

Solving the equations 4.1 and 4.2 we obtain $(p_3, p_4, p_5) = (6, 3, 0)$ or $(7, 1, 1)$.

**Case 1:** We consider the possibility that there are three mutually disjoint type-4 pentagons $P_1, P_2, P_3$ in $K$. Then, the union of their vertices comprises all 15 vertices of $K$. Consequently, every vertex has one and only one type-4 polygon adjacent to it. Starting with one such type-4 polygon, say $P_1$, a part of $K$ can be represented by Figure 4.1 with $P_2$ and $P_3$ being other two type-4 pentagons.

It follows that the triangles $T_1, T_2, T_3$ are mutually disjoint. Since $K$ is a polyhedral map therefore the type-4 polygon $P \neq P_1, P_3$. But $P = P_2 \implies T_1T_2 \implies T_1 = T_3$, a contradiction, therefore $P \neq P_2$ either. This is again impossible since $p_4 = 3$.

**Case 2:** If $K$ does not have three mutually disjoint type-4 pentagons, then for both the solutions of $(p_3, p_4, p_5) = (6, 3, 0)$ or $(7, 1, 1)$ above, there is a vertex, say $v_1 \in V(K)$ which is not a vertex of any of the type-4 pentagons. In such a situation, without loss of generality, a portion of $K$ can be represented by Figure 4.2. The possible values of $a, b = v_2, v_4, v_5, v_{14}, v_{15}$, and $c, d = v_3, v_{10}, v_{11}, v_{12}, v_{13}$. We now check their viability.

1) $c = v_3$ implies either $v = 2$ or $d = 2$, which is not possible.

2) Similarly, $b = v_2$ is not possible by the symmetry of Figure 4.2 about the vertex $v_1$.

3) $c = v_{11}, d = v_3 \implies u = v_{13}$ ($u \neq v_7, v_8$ for degree reason) $\implies b = v_{14} \implies \deg(v_{14}) > 4$, which is not possible.
4) \( c = v_{12}, d = v_{13} \implies \text{no choice left for } b \) (as \( v_{13}, v_{10} \in \{v_9, v_{10}, v_b, v_a, v_{13}\} \)).

5) \( c = v_{13}, d = v_{12} \implies \text{no choice left for } u.

1), 2) and 3) together implies that \( (c, d) \) must be an edge of the triangle \( \{v_{10}, v_{11}, v_{12}\} \), and again by the symmetry of Figure 4.2, \( (a, b) \) of the triangle \( \{v_4, v_5, v_{15}\} \).

6) \( (c, d) \neq (v_{10}, v_{11}), (v_{11}, v_{10}) \) as the pentagon \( \{v_{10}, v_{11}, v_1, v_3, v_9\} \neq \{v_{10}, v_{11}, v_5, v_6, v_{14}\} \).

7) \( \{c, d\} = \{v_{11}, v_{12}\} \implies u = v_{13} \implies a, b = v_{15}, v_5 \implies \deg(v_5) > 4. \)

8) \( \{c, d\} = \{v_{12}, v_{11}\} \implies b = v_4 \implies \text{no choice left for } u.

9) \( \{c, d\} = \{v_{12}, v_{10}\} \implies (a, b) \) is not an edge of the triangle \( \{v_4, v_5, v_{15}\} \).

10) \( \{c, d\} = \{v_{10}, v_{12}\} \) leads to Figure 4.3.

![Fig. 4.2. Part of K of Case 2](image1)

![Fig. 4.3. Cyclic sequence for \( \{c, d\} = \{v_{10}, v_{12}\} \)](image2)

We observe now that, because of cyclic repetition of vertices on the horizontal line \( L_1 \), the triangles on the horizontal line \( L_2 \) must be placed either in the shown order or all translated by one edge. In both the cases, if we form a graph with vertices representing the triangles of the map and edges representing type-3 pentagons having common edges with two triangles, then we would have a graph with 5 vertices each having degree 3 and 6 edges, which is absurd. Thus, we have exhausted all the possibilities in Case 2 as well. This leaves us with no choice but concluding that \( K \) is not polyhedral. Without the polyhedral condition, however, the example in Figure 4.1(a) is a map of type \( [5^3, 3] \) on the surface with \( \chi = -1 \), \( \mathbb{R}P^2 \# \mathbb{R}P^2 \# \mathbb{R}P^2 \). This completes the proof of the first part of the theorem.

In view of Lemma 2.4, the example in Figure 4.1(a) suggests that for a polyhedral map we could look on surfaces with \( \chi = -2 \), i.e., on double torus \( T^2 \) (orientable) and \( \mathbb{R}P^2 \# \mathbb{R}P^2 \# \mathbb{R}P^2 \# \mathbb{R}P^2 \) (non-orientable). Indeed, we found the following examples shown in Figures 4.4(b) and 4.4(c).

![Fig. 4.4. Maps of type \( [5^3, 3] \) on surface with \( \chi = -1 \) and \( \chi = -2 \)](image3)

This completes the proof of Theorem 1.2. \( \square \)
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