Composition-Diamond Lemma for Non-associative Algebras over a Commutative Algebra

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Abstract: We establish the Composition-Diamond lemma for non-associative algebras over a free commutative algebra. As an application, we prove that every countably generated non-associative algebra over an arbitrary commutative algebra $K$ can be embedded into a two-generated non-associative algebra over $K$.

Key words: Gröbner-Shirshov basis; non-associative algebra; commutative algebra.

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1 Introduction

Gröbner bases and Gröbner-Shirshov bases theories were invented independently by A.I. Shirshov [23] for non-associative algebras and commutative (anti-commutative) non-associative algebras [21], for Lie algebras (explicitly) and associative algebras (implicitly) [22], for infinite series algebras (both formal and convergent) by H. Hironaka [19] and for polynomial algebras by B. Buchberger (first publication in [13]). Gröbner bases and Gröbner-Shirshov bases theories have been proved to be very useful in different branches of mathematics, including commutative algebra and combinatorial algebra, see, for example, the books [11, 12, 14, 15, 17, 18], the papers [2, 3, 4, 5, 16], and the surveys [6, 9, 10, 11].

It is well known that every countably generated non-associative algebra over a field $k$ can be embedded into a two-generated non-associative algebra over $k$. This result follows from Gröbner-Shirshov bases theory for non-associative algebras by A.I. Shirshov [21].

Composition-Diamond lemmas for associative algebras over a polynomial algebra is established by A.A. Mikhalev and A.A. Zolotykh [20], for associative algebras over an associative algebra by L.A. Bokut, Yuqun Chen and Yongshan Chen [17], for Lie algebras over a polynomial algebra by L.A. Bokut, Yuqun Chen and Yongshan Chen [8]. In this paper, we establish the Composition-Diamond lemma for non-associative algebras over

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a polynomial algebra. As an application, we prove that every countably generated non-associative algebra over an arbitrary commutative algebra $K$ can be embedded into a two-generated non-associative algebra over $K$, in particular, this result holds if $K$ is a free commutative algebra.

2 Composition-Diamond lemma for non-associative algebras over a commutative algebra

Let $k$ be a field, $K$ a commutative associative $k$-algebra with unit, $X$ a set and $K(X)$ the free non-associative algebra over $K$ generated by $X$.

Let $K$ denote the free abelian monoid generated by $Y$, $X^*$ the free monoid generated by $X$ and $X^{**}$ the set of all non-associative words in $X$. Denote by

$N = [Y]X^{**} = \{u = u^Y u^X | u^Y \in [Y], u^X \in X^{**}\}.$

Let $kN$ be a $k$-linear space spanned by $N$. For any $u = u^Y u^X, v = v^Y v^X \in N$, we define the multiplication of the words as follows

$uv = u^Y v^Y u^X v^X \in N.$

It is clear that $kN$ is the free non-associative $k[Y]$-algebra generated by $X$. Such an algebra is denoted by $k[Y](X)$, i.e., $kN = k[Y](X)$. Clearly,

$k[Y](X) = k[Y] \otimes k(X).$

Now, we order the set $N = [Y]X^{**}$.

Let $>$ be a total ordering on $X^{**}$. Then $>$ is called monomial if

$(\forall u, v, w \in X^{**}) \ u > v \Rightarrow uw > wv \text{ and } uv > vw.$

For example, the deg-lex ordering on $X^{**}$ is monomial: $uv > u_1 v_1$, if $\text{deg}(uv) > \text{deg}(u_1 v_1)$, otherwise $u > u_1$ or $u = u_1, v > v_1$. Similarly, we define the monomial ordering on $[Y]$.

Suppose that both $>_X$ and $>_Y$ are monomial orderings on $X^{**}$ and $[Y]$, respectively. For any $u = u^Y u^X, v = v^Y v^X \in N$, define

$u > v \iff u^X >_X v^X \text{ or } (u^X = v^X \text{ and } u^Y >_Y v^Y).$

It is obvious that $>$ is a monomial ordering on $N$ in the sense of

$(\forall u, v, w \in [Y]X^{**}) \ u > v \Rightarrow uw > wv, \ uv > vw \text{ and } w^Y u > w^Y v.$

We will use this ordering in this paper.

For any polynomial $f \in k[Y](X)$, $f$ has a unique presentation of the form

$f = \alpha_f \tilde{f} + \sum \alpha_i u_i,$

where $\tilde{f}, u_i \in [Y]X^{**}, \tilde{f} > u_i, \alpha_f, \alpha_i \in k$. $\tilde{f}$ is called the leading term of $f$. $f$ is monic if the coefficient of $\tilde{f}$ is 1.
Let \(* \notin X\). By a \(*\)-word we mean any expression in \([Y](X \cup \{\ast\})^{**}\) with only one occurrence of \(*\). Let \(u\) be a \(*\)-word and \(s \in k[Y](X)\). Then we call \(u|_s = u|_{\ast \rightarrow s}\) an \(s\)-word.

It is clear that for \(s\)-word \(u|_s\), we can express \(u|_s = u^Y(asb)\) for some \(a, b \in X^*\).

Since \(>\) is monomial on \([Y]X^{**}\), we have following lemma.

**Lemma 2.1** Let \(s \in k[Y](X)\) be a non-zero polynomial. Then for any \(s\)-word \(u|_s = u^Y(asb)\), \(u^Y(asb) = u^Y(asb)\).

Now, we give the definition of compositions.

**Definition 2.2** Let \(f\) and \(g\) be monic polynomials of \(k[Y](X)\), \(w = w^Yw^X \in [Y]X^{**}\) and \(a, b, c \in X^*\), where \(w^Y = L(f^Y, g^Y) \triangleq L\) and \(L(f^Y, g^Y)\) is the least common multiple of \(f^Y\) and \(g^Y\) in \(k[Y]\). Then we have the following compositions.

1. **X-inclusion**
   
   If \(w^X = \bar{f}^X = (a(\bar{g}^X)b)\), then
   
   \[
   (f, g)_w = \frac{L}{L_Yf}f - \frac{L}{L_Yg}(a(g)b)
   \]
   
   is called the composition of \(X\)-inclusion.

2. **Y-intersection only**

   If \(|\bar{f}^Y| + |\bar{g}^Y| > |w^Y|\) and \(w^X = (a(\bar{f}^X)b(\bar{g}^X)c)\), then

   \[
   (f, g)_w = \frac{L}{L_Yf}(a(f)b(\bar{g}^X)c) - \frac{L}{L_Yg}(a(\bar{f}^X)b(g)c)
   \]

   is called the composition of \(Y\)-intersection only, where for \(u \in [Y]\), \(|u|\) means the degree of \(u\).

   \(w\) is called the ambiguity of the composition \((f, g)_w\).

**Remark 1.** In the case of \(Y\)-intersection only in Definition 2.2, \(\bar{f}^X\) and \(\bar{g}^X\) are disjoint.

**Remark 2.** By Lemma 2.1, we have \(w > (f, g)_w\).

**Remark 3.** In Definition 2.2, the compositions of \(f, g\) are the same as the ones in \(k(X)\), if \(Y = \emptyset\). If this is the case, we have only composition of \(X\)-inclusion.

**Definition 2.3** Let \(S\) be a monic subset of \(k[Y](X)\) and \(f, g \in S\). A composition \((f, g)_w\) is said to be trivial modulo \((S, w)\), denoted by \((f, g)_w \equiv 0 \mod(S, w)\), if

\[
(f, g)_w = \sum_i \alpha_i u_i|_{s_i},
\]

where each \(s_i \in S\), \(\alpha_i \in k\), \(u_i|_{s_i}\), \(s_i\)-word and \(w > u_i|_{\bar{s}_i}\).

Generally, for any \(p, q \in k[Y](X)\), \(p \equiv q \mod(S, w)\) if and only if \(p - q \equiv 0 \mod(S, w)\).

\(S\) is called a Gröbner-Shirshov basis in \(k[Y](X)\) if all compositions of elements in \(S\) are trivial modulo \(S\).
Lemma 2.4 Let \( S \) be a Gröbner-Shirshov basis in \( k[Y](X) \) and \( s_1, s_2 \in S \). Let \( u_1|s_1, u_2|s_2 \) be \( s_1, s_2 \)-words respectively. If \( w = u_1|s_1 = u_2|s_2 \), then \( u_1|s_1 \equiv u_2|s_2 \mod(S, w) \).

**Proof:** Clearly, \( w^Y = L(\bar{s}_1^Y, \bar{s}_2^Y) \cdot t = L \cdot t \) for some \( t \in [Y] \).

There are three cases to consider.

**Case 1.** \( X \)-inclusion.

We may assume that \( \bar{s}_1^X = (c\bar{s}_2^X)d \) for some \( c, d \in X^* \) and \( w^X = (a\bar{s}_1^X)b = (a(c\bar{s}_2^X)d)b \) for some \( a, b \in X^* \). Thus,

\[
\frac{L \cdot t}{\bar{s}_1^Y}(a(s_1)b) - \frac{L \cdot t}{\bar{s}_2^Y}(a(c(s_2)d)b) = t \cdot (a(\frac{L}{\bar{s}_1^Y}s_1 - \frac{L}{\bar{s}_2^Y}(c(s_2)d))b) = t \cdot (a(s_1, s_2)w, b) \equiv 0 \mod(S, w)
\]

where \( w_1 = L\bar{s}_1^X \).

**Case 2.** \( Y \)-intersection only.

In this case, \( w^X = (a\bar{s}_1^X)b(\bar{s}_2^X)c \), \( a, b, c \in X^* \) and then

\[
\frac{L \cdot t}{\bar{s}_1^Y}(a(s_1)b(\bar{s}_2^X)c) - \frac{L \cdot t}{\bar{s}_2^Y}(a(\bar{s}_1^X)b(s_2)c) = t \cdot (s_1, s_2)w_1 \equiv 0 \mod(S, w)
\]

where \( w_1 = Lw^X \).

**Case 3.** \( Y \)-disjoint and \( X \)-disjoint.

In this case, \( L = \bar{s}_1^Y \bar{s}_2^Y \) and \( w^X = (a\bar{s}_1^X)b(\bar{s}_2^X)c \), \( a, b, c \in X^* \). We have

\[
\frac{L \cdot t}{\bar{s}_1^Y}(a(s_1)b(\bar{s}_2^X)c) - \frac{L \cdot t}{\bar{s}_2^Y}(a(\bar{s}_1^X)b(s_2)c) = t \cdot (\frac{L}{\bar{s}_1^Y}(a(s_1)b(\bar{s}_2^X)c) - \frac{L}{\bar{s}_2^Y}(a(\bar{s}_1^X)b(s_2)c)) = t \cdot ((a(s_1)b(\bar{s}_2)c) - (a(s_1)b(s_2)c) + (a(s_1)b(s_2)c) - (a(\bar{s}_1)b(s_2)c)) = t \cdot ((a(s_1 - \bar{s}_1)b(s_2)c) - (a(s_1)b(s_2 - \bar{s}_2)c)) \equiv 0 \mod(S, w)
\]

since \( w = (a\bar{s}_1)b(s_2)c > (a(s_1 - \bar{s}_1)b(s_2)c) \) and \( w = (a\bar{s}_1)b(s_2)c > (a(s_1)b(s_2 - \bar{s}_2)c) \).

This completes the proof. \( \square \)
Lemma 2.5 Let $S \subseteq k[Y](X)$ with each $s \in S$ monic and $\text{Irr}(S) = \{w \in [Y].X^{**}|w \neq u|_s, u|_s \text{ is an s-word, } s \in S\}$. Then for any $f \in k[Y](X)$,

$$f = \sum_{u_i|_s \leq f} \alpha_i u_i|_{s_i} + \sum v_j \beta_j,$$

where $\alpha_i, \beta_j \in k$, $u_i|_{s_i}$, $s_i$-word, $s_i \in S$ and $v_j \in \text{Irr}(S)$.

Proof. Let $f = \sum \alpha_i u_i \in k[Y](X)$, where $0 \neq \alpha_i \in k$ and $u_1 > u_2 > \cdots$. If $u_1 \in \text{Irr}(S)$, then let $f_1 = f - \alpha_1 u_1$. If $u_1 \notin \text{Irr}(S)$, then there exists an s-word $u|_s$ such that $\bar{f} = u|_s$. Let $f_1 = f - \alpha_1 u|_s$. In both cases, we have $\bar{f} > \bar{f}_1$. Then the result follows from the induction on $\bar{f}$. □

From the above lemmas, we reach the following theorem:

Theorem 2.6 (Composition-Diamond lemma for $k[Y](X)$) Let $S \subseteq k[Y](X)$ with each $s \in S$ monic, $>$ the ordering on $[Y].X^{**}$ defined as before and $\text{Id}(S)$ the ideal of $k[Y](X)$ generated by $S$ as $k[Y]$-algebra. Then the following statements are equivalent:

(i) $S$ is a Gröbner-Shirshov basis in $k[Y](X)$.

(ii) If $0 \neq f \in \text{Id}(S)$, then $\bar{f} = u|_s$ for some s-word $u|_s$, $s \in S$.

(iii) $\text{Irr}(S) = \{w \in [Y].X^{**}|w \neq u|_s, u|_s \text{ is an s-word, } s \in S\}$ is a k-linear basis for the factor algebra $k[Y](X)/\text{Id}(S)$.

Proof: (i) $\Rightarrow$ (ii). Suppose $0 \neq f \in \text{Id}(S)$. Then $f = \sum \alpha_i u_i|_{s_i}$ for some $\alpha_i \in k$, $s_i$-word $u_i|_{s_i}$, $s_i \in S$. Let $w_i = u_i|_s$ and $w_1 = w_2 = \cdots = w_l > w_{l+1} \geq \cdots$. We will prove the result by using induction on $l$ and $w_1$.

If $l = 1$, then the result is clear. If $l > 1$, then $w_1 = u_1|_s = u_2|_s$. Now, by (i) and Lemma 2.4, $u_1|_{s_1} \equiv u_2|_{s_2} \mod(S, w_1)$. Thus,

$$\alpha_1 u_1|_{s_1} + \alpha_2 u_2|_{s_2} = (\alpha_1 + \alpha_2) u_1|_{s_1} + \alpha_2 (u_2|_{s_2} - u_1|_{s_1}) \equiv (\alpha_1 + \alpha_2) u_1|_{s_1} \mod(S, w_1).$$

Therefore, if $\alpha_1 + \alpha_2 \neq 0$ or $l > 2$, then the result follows from the induction on $l$. For the case $\alpha_1 + \alpha_2 = 0$ and $l = 2$, we use the induction on $w_1$. Now the result follows.

(ii) $\Rightarrow$ (iii). By Lemma 2.5, $\text{Irr}(S)$ generates the factor algebra. Moreover, if $0 \neq h = \sum \beta_j u_j \in \text{Id}(S)$, $u_j \in \text{Irr}(S)$, $u_1 > u_2 > \cdots$ and $\beta_1 \neq 0$, then $u_1 = \bar{h} = u|_s$, a contradiction. This shows that $\text{Irr}(S)$ is a k-linear basis of the factor algebra.

(iii) $\Rightarrow$ (i). For any $f$, $g \in S$, since $k[Y]|S \subseteq \text{Id}(S)$, we have $h = (f, g)|_w \in \text{Id}(S)$. The result is trivial if $(f, g)|_w = 0$. Assume that $(f, g)|_w \neq 0$. Then, by Lemma 2.5, (iii) and by noting that $w > (f, g)|_w = \bar{h}$, we have $(f, g)|_w \equiv 0 \mod(S, w)$.

This shows (i). □

Remark: Theorem 2.6 is the Composition-Diamond lemma for non-associative algebras when $Y = 0$.  

5
3 Applications

Let \( A \) be an arbitrary \( K \)-algebra and \( A \) be presented by generators \( X \) and defining relations \( S \)

\[
A = K(X|S).
\]

Let \( K \) have a presentation by generators \( Y \) and defining relations \( R \)

\[
K = k[Y|R]
\]
as a quotient algebra of the polynomial algebra \( k \).

Then with a natural way, as \( k[Y]-\)algebras, we have an isomorphism

\[
k[Y|R](X|S) \rightarrow k[Y](X|S', Rx, x \in X), \sum (f_i + Id(R))u_i + Id(S) \mapsto \sum f_iu_i + Id(S'),
\]

where \( f_i \in k[Y], u_i \in X^*, S' = S' \cup \{gx|g \in R, x \in X\}, S^l = \{\sum f_iu_i \in k[Y](X)|\sum(f_i + Id(R))u_i \in S\}. \) Then \( A \) has an expression

\[
A = k[Y|R](X|S) = k[Y](X|S', gx, g \in R, x \in X).
\]

**Theorem 3.1** Each countably generated non-associative algebra over an arbitrary commutative algebra \( K \) can be embedded into a two-generated non-associative algebra over \( K \).

**Proof.** Let the notation be as before. Let \( A \) be the non-associative algebra over \( K = k[Y|R] \) generated by \( X = \{x_i|i = 1,2,\ldots\} \). We may assume that \( A = k[Y|R](X|S) \) is defined as above. Then \( A \) can be presented as \( A = k[Y](X|S', gx, g \in R, i = 1,2,\ldots) \). By Shirshov algorithm, we can assume that, with the deg-lex ordering \( >_Y \) on \( [Y], R \) is a Gröbner-Shirshov basis in the free commutative algebra \( k[Y] \). Let \( >_X \) be the deg-lex ordering on \( X^* \), where \( x_1 > x_2 > \ldots \). We can also assume, by Shirshov algorithm, that with the ordering on \( [Y]X^* \) defined as before, \( S' = S^l \cup \{gx|g \in R, x \in X\} \) is a Gröbner-Shirshov basis in \( k[Y](X) \).

Let \( B = k[Y](X,a,b|S_1) \) where \( S_1 \) consists of

\[
\begin{align*}
    f_1 &= S^l, \\
    f_2 &= \{gx|g \in R, x \in X\}, \\
    f_3 &= \{a(b^i) - x_i|i = 1,2,\ldots\}, \\
    f_4 &= \{ga|g \in R\}, \\
    f_5 &= \{gb|g \in R\}.
\end{align*}
\]

Clearly, \( B \) is a \( K \)-algebra generated by \( a,b \). Thus, to prove the theorem, by using our Theorem 2.6, it suffices to show that with the ordering on \( [Y](X \cup \{a,b\})^* \) as before, where \( a > b > x_i, i = 1,2,\ldots, S_1 \) is a Gröbner-Shirshov basis in \( k[Y](X,a,b) \).

Denote by \( (i \wedge j)_{wij} \) the composition of the type \( f_i \) and type \( f_j \) with respect to the ambiguity \( w_{ij} \). Since \( S' \) is a Gröbner-Shirshov basis in \( k[Y](X) \), we need only to check all compositions related to the following ambiguities \( w_{ij} \):

\[
1 \wedge 4, \quad w_{14} = L(\tilde{f}Y, \tilde{g})(z_1(\tilde{f}X)z_2az_3);
\]
1 ∧ 5, \ w_{15} = L(\bar{f}'Y, \bar{g})(z_1(\bar{f}'X)z_2b_3);
2 ∧ 4, \ w_{24} = L(g', \bar{g})(z_1x_2a_3);
2 ∧ 5, \ w_{25} = L(g', \bar{g})(z_1x_2a_2b_3);
3 ∧ 4, \ w_{34} = g\alpha(b);
3 ∧ 5, \ w_{35} = g\alpha(b);
4 ∧ 1, \ w_{41} = L(g, \bar{f}'Y)(z_1a_2z_2(\bar{f}'X)z_3);
4 ∧ 2, \ w_{42} = L(g, \bar{g}')(z_1a_2x_3);
4 ∧ 4, \ w_{44} = L(\bar{g}_1, \bar{g}_2)a;
4 ∧ 5, \ w_{45} = L(g, \bar{g}')(z_1a_2b_2);
5 ∧ 1, \ w_{51} = L(\bar{g}, \bar{f}'Y)(z_1b_2z_2(\bar{f}'X)z_3);
5 ∧ 2, \ w_{52} = L(\bar{g}, \bar{g}')(z_1b_2x_3);
5 ∧ 4, \ w_{54} = L(\bar{g}, \bar{g}')(z_1b_2a_3);
5 ∧ 5, \ w_{55} = L(\bar{g}_1, \bar{g}_2)b;

where \( g, g', g_1, g_2 \in R, f \in S^t, z_1, z_2, z_3 \in (X \cup \{a, b\})^* \) and \( (z_1v_1z_2v_2z_3) \) is some bracketing.
Now, we prove that all the compositions are trivial.
1 ∧ 4, \ w_{14} = L(\bar{f}'Y, \bar{g})(z_1(\bar{f}'X)z_2a_3), \ \text{where} \ f \in S^t, \ g \in R.
We can write \( \bar{f}'X = (uvx) \), where \( u, v \in X^* \). Since \( S' = \{ S^t, Rx, x \in X \} \) is a Gröbner-Shirshov basis in \( k[Y](X) \), we have \( (f, gx)_w = \sum \alpha_i u_i | s_i \), where \( w = L(\bar{f}'Y, \bar{g})\bar{f}'X \), each \( \alpha_i \in k \), \( s_i \in S' \), \( u_i \in [Y]X^* \) and \( w > u_i | s_i \). Then

\[
(1, 4)_{w_{14}} = \frac{L}{f'}(z_1f z_2a_3) - \frac{L}{g}(z_1(\bar{f}'X)z_2g a_3)
= \frac{L}{f'}(z_1f z_2a_3) - \frac{L}{g}(z_1(ug xv)z_2a_3) + \frac{L}{g}(z_1(ug xv)z_2a_3) - \frac{L}{g}(z_1(\bar{f}'X)z_2g a_3)
= (z_1(\frac{L}{f'}f - \frac{L}{g}(ug xv))z_2a_3) + \frac{L}{g}((z_1(ug xv)z_2a_3) - (z_1(\bar{f}'X)z_2a_3))
= \sum \alpha_i (z_1u_i | s_i z_2a_3)
\equiv 0 \mod(S_1, w_{14}).
\]

Similarly, \( (1, 5)_{w_{15}} \equiv 0 \), \( (4, 1)_{w_{41}} \equiv 0 \), \( (5, 1)_{w_{51}} \equiv 0 \).

2 ∧ 4, \ w_{24} = L(g', \bar{g})(z_1x_2a_3), \ \text{where} \ g, g' \in R.
If \( |g'| + |\bar{g}| > |L| \), then since \( R \) is a Gröbner-Shirshov basis in \( k[Y] \), \( (g', g)_w = (\frac{L}{g'}g' - \frac{L}{g}g) = \sum \alpha_i u_i h_i \), where \( w = L(\bar{g}', \bar{g}), \) each \( \alpha_i \in k, u_i \in [Y], \ h_i \in R \) and \( w > u_i | h_i \). Thus

\[
(2, 4)_{w_{24}} = \frac{L}{g'}(z_1g'x_2a_3) - \frac{L}{g}(z_1x_2g a_3)
= (\frac{L}{g'}g' - \frac{L}{g}g)(z_1x_2a_3)
= \sum \alpha_i u_i h_i (z_1x_2a_3)
= \sum \alpha_i u_i (z_1x_2h_i a_3)
\equiv 0 \mod(S_1, w_{24}).
\]

Similarly, \((2, 5)_{w_{25}} \equiv 0\), \((4, 2)_{w_{42}} \equiv 0\), \((4, 5)_{w_{45}} \equiv 0\), \((5, 2)_{w_{52}} \equiv 0\) and \((5, 4)_{w_{54}} \equiv 0\).

\[3 \land 4, \quad w_{34} = \bar{g}a(b^i), \text{ where } g \in R.\]

Let \(g = \bar{g} + r \in R\). Then

\[(3, 4)_{w_{34}} = -\bar{g}x_i - ra(b^i) \equiv -\bar{g}x_i - rx_i \equiv gx_i \equiv 0 \mod(S_1, w_{34}).\]

Similarly, \((3, 5)_{w_{35}} \equiv 0\).

\[4 \land 4, \quad w_{44} = L(\bar{g}_1, \bar{g}_2)a, \text{ where } g_1, g_2 \in R.\]

If \(|\bar{g}_1| + |\bar{g}_2| > |L|\), then since \(R\) is a Gröbner-Shirshov basis in \(k[Y]\), \((g_1, g_2)_w = (\frac{L}{g_1}g_1 - \frac{L}{g_2}g_2) = \sum \alpha_i u_i h_i\), where \(w = L(\bar{g}_1, \bar{g}_2)\), each \(\alpha_i \in k, u_i \in [Y], h_i \in R\) and \(w > u_i h_i\). Thus

\[(4, 4)_{w_{44}} = \frac{L}{g_1}(g_1a) - \frac{L}{g_2}(g_2a) = \frac{L}{g_1}g_1 - \frac{L}{g_2}g_2 = \sum \alpha_i u_i h_i a \equiv 0 \mod(S_1, w_{44}).\]

If \(|\bar{g}_1| + |\bar{g}_2| = |L|\), then

\[(4, 4)_{w_{44}} = \frac{L}{g_1}(g_1a) - \frac{L}{g_2}(g_2a) = (\bar{g}_2g_1 - \bar{g}_1g_2)a \equiv ((g_1 - \bar{g}_1)g_2 - (g_2 - \bar{g}_2)g_1)a \equiv 0 \mod(S_1, w_{44}).\]

Similarly, \((5, 5)_{w_{55}} \equiv 0\).

Now we have proved that \(S_1\) is a Gröbner-Shirshov basis in \(k[Y](X, a, b)\).

The proof is complete. □

A special case of Theorem 3.1 is the following corollary.

**Corollary 3.2** Every countably generated non-associative algebra over a free commutative algebra can be embedded into a two-generated non-associative algebra over a free commutative algebra.

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