An Analogue of the Kac-Wakimoto Formula and Black Hole Conditional Entropy

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Abstract. A local formula for the dimension of a superselection sector in Quantum Field Theory is obtained as vacuum expectation value of the exponential of the proper Hamiltonian. In the particular case of a chiral conformal theory, this provides a local analogue of a global formula obtained by Kac and Wakimoto within the context of representations of certain affine Lie algebras. Our formula is model independent and its version in general Quantum Field Theory applies to black hole thermodynamics. The relative free energy between two thermal equilibrium states associated with a black hole turns out to be proportional to the variation of the conditional entropy in different superselection sectors, where the conditional entropy is defined as the Connes-Størmer entropy associated with the DHR localized endomorphism representing the sector. The constant of proportionality is half of the Hawking temperature. As a consequence the relative free energy is quantized proportionally to the logarithm of a rational number, in particular it is equal to a linear function the logarithm of an integer once the initial state or the final state is taken fixed.

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orthochronous

**Introduction.**

In the first part of this paper we shall derive a formula for the dimension of a superselection sector in Conformal Quantum Field Theory. However, as we shall see, the role played by conformal invariance is not essential, and indeed we will subsequently deal with general Quantum Field Theory and apply our formula to the computation of the relative free energy between two thermal equilibrium states of the background system for a black hole. The reader mainly interested in the latter topic may at first read the second part of this introduction and then get directly to Section 3.

A local analogue of the Kac-Wakimoto formula. There is a general phenomenon relating the distribution of the Hamiltonian density levels to a dimension, a classical example being given by Weyl’s theorem on the asymptotic distribution of the Laplacian eigenvalues on a compact Riemann manifold.

A similar occurrence appears in the context of lowest weight representations of certain affine Kac-Moody algebras. If $L_{\rho}$ and $L_0$ are the conformal Hamiltonians in the vacuum representation and in the representation $\rho$ of such an infinite dimensional Lie algebra, then there exists the limit

$$\lim_{\beta \to 0^+} \frac{\text{Tr}(e^{-\beta L_{\rho}})}{\text{Tr}(e^{-\beta L_0})} = d(\rho)$$

and the thus defined $d(\rho)$ has the formal properties of a dimension [31].

One expects such a formula to hold in more generality in conformal Quantum Field Theory on $S^1$ with $\rho$ a superselection sector and $d(\rho)$ the statistical dimension of $\rho$ (one has to assume at least that $\text{Tr}(e^{-\beta L_0}) < \infty$ or any structural property to guarantee this), but no result in this direction has been so far obtained (cf. [44]).

However, if we restrict the vacuum state $\omega$ to the local von Neumann algebra $\mathcal{A}(I)$ associated with an interval $I$, then $\omega$ is faithful by the Reeh-Schlieder theorem and hence, by the Tomita-Takesaki theory, $\omega$ is a Gibbs state with respect to its modular group\(^1\). Such a modular group has a geometric meaning and we may interpret it as a local dynamics, in other words the logarithm of the modular operator can be regarded as a local Hamiltonian. One can then argue from [10] that in this local situation the distribution of the energy density levels should be tested in mean, at a specific value of the inverse temperature parameter $\beta$, and hence also a local version of formula (0.1) should not be asymptotic as $\beta \to 0^+$, but evaluated at a specific $\beta$.

Let $\Lambda_I(t)$ be the one-parameter group of special conformal transformations of $S^1$ associated with the interval $I$ of $S^1$ (formula (2.1) below) and $K_\rho$ the generator of the corresponding one-parameter unitary group in the representation $\rho$. We shall obtain the formula

$$\langle e^{-2\pi K_\rho} \xi, \xi \rangle = d(\rho)$$

\(^1\)The reader should be aware of the different meaning of the term modular in refs. [49] and [31].
where $\xi$ is any cyclic vector for $\rho(A(I'))$ such that $(\rho(\cdot)\xi,\xi)$ is the vacuum state on $A(I')$ and $I' = S^1\setminus I$.

In comparison with the formula (0.1), we first note that the right hand side $d(\rho)$ in (0.2) has not only the properties of a dimension, but it is actually identified with the Doplicher-Haag-Roberts [18] statistical dimension of $\rho$.

Moreover the local formula (0.2) holds in full generality, independently of any requirement on the growth of Hamiltonian spectral density. Hopefully it might lead to a model independent proof of the formula (0.1), but it has its own interest. As we shall see, its proof makes use of the knowledge of the modular structure of the local von Neumann algebras [8,22], the description of the conjugate sector in terms of the modular involutions [23,24], and a result on actions of groups on tensor categories that determines $K_\rho$ as a linear function of the logarithm of the relative modular operator (as in formula (3.8) below).

Finally the validity of formula (0.2) goes beyond the context of Conformal Quantum Field Theory. Indeed the same structure is present in the general setting Quantum Field Theory on Minkowski space [48], provided we consider the local von Neumann algebra associated with a wedge region, because in this case the modular structure has the geometric interpretation described by the Bisognano-Wichmann theorem. We then treat this case, where a further physical interpretation can be given. Mutatis mutandis, formulae in Section 2 are also valid Section 3 and vice versa, with the exception of Corollary 3.6. We avoid repetitions and state in each of them the results closer to the spirit of the section.

Quantum numbers for the relative free energy associated with a black hole. As was indicated in [3,4], a black hole looks from the outside with some features of a thermodynamical system in equilibrium. In particular Bekenstein suggested the entropy of a black hole to be equal to $\lambda A$ where $A$ is the surface area of the black hole and $\lambda$ a constant, an hypothesis related the Generalized Second Low of Thermodynamics

$$dS + \lambda dA \geq 0$$

(0.3)

where, in any process, $dA$ is the increment of $A$ and $dS$ is the increment the entropy of the outside region.

Taking into account Quantum effects and General Relativity, Hawking [28] was led to the conclusion that the black hole has a surface temperature

$$T = \frac{\hbar}{k c} \frac{a}{2\pi}$$

where $a$ is the acceleration of a freely falling object at the surface of the black hole$^2$. This computation was made in the context of a free Quantum Field Theory on a curved space-time.

$^2T = \frac{c^3 \hbar}{8\pi k MG}$ in terms of the mass $M$ of the star and the gravitational constant $G$. In the following we shall always use natural units so that the Planck constant $\hbar$, the speed of light $c$ and the Boltzmann constant $k$ are all set equal to 1, thus $T = a/2\pi$. 

Sewell [45] noticed that, at least in the case of a spherically symmetric eternal black hole, a model independent derivation of the Hawking temperature is possible, also in analogy with the Unruh effect [52], see also [17], by means of the Bisognano-Wichmann theorem [6] on the Minkowski space-time. One considers the Rindler space-time $W$ as an approximation of the Schwarzschild space-time and realizes $W$ as a wedge region in the Minkowski space-time, say $W = \{ x, x_1 > |x_0| \}$. The evolution
\[
\begin{aligned}
x_0(t) &= a^{-1} \text{sht} \\
x_1(t) &= a^{-1} \text{cht}
\end{aligned}
\]
corresponds to an observer moving within $W$ with uniform acceleration $a$, and his proper time is equal to $t/a$. $W$ is a natural horizon for this observer, since he cannot send a signal out of $W$ and receive it back. The von Neumann algebra $\mathcal{A}(W)$ of the observables on the Minkowski space localized in $W$ is therefore the proper (global) observable algebra for such a mover. The proper time translations for him are thus given by the one-parameter automorphism group $\alpha_{at}$ of $\mathcal{A}(W)$ corresponding to the rescaled pure Lorentz transformations leaving $W$ invariant. By the Bisognano-Wichmann theorem $\alpha_{at}$ satisfies the KMS condition at inverse temperature $\beta = \frac{2\pi}{a}$ with respect to the (restriction of the Minkowski) vacuum state $\omega$ to $\mathcal{A}(W)$, i.e. the latter is a thermal equilibrium state [26]. By Einstein equivalence principle one can identify $W$ with the outside region of the black hole and interpret the thermal outcome as a gravitational (black hole) effect.

We refer to [25,32,53] for a more complete account of these arguments and further references. We note here that this description has certain restrictions. One is due to the appearance of the Minkowskian vacuum tied up with the Poincarè symmetries that do not exist globally on a general curved space-time. Another point concerns the choice of the Rindler space-time, that only near the horizon is a good approximation of the more appropriate Schwarzschild space-time. We shall briefly discuss these aspects in the final comments. Despite its limitations, this viewpoint is strikingly simple and powerful.

Let now consider a thermodynamical system $\Sigma$ placed in the asymptotically flat outside region of a black hole $B$. The Hawking radiation creates a heat bath for $\Sigma$ and therefore $\Sigma$ is an open system. Taking into account only observable quantities, Sewell [46] inferred that the right thermodynamical potential for $\Sigma$ is the Gibbs free energy, rather than the entropy, and rederived the generalised second low (0.3), where the area term represents now a contribution to the mechanical work done by $\Sigma$ on $B$.

Due to the Hawking effect, we have spontaneous creation of particles, so that the system undergoes a change of quantum numbers. From the Quantum Field Theory point of view, the system goes in a different superselection sector
[54]. Following [27,18] we thus consider a representation \( \rho \) of the quasi-local C*-algebra \( \mathcal{A} = \bigcup \mathcal{A}(\mathcal{O})^- \) that is localizable in any space-like cone and has finite statistics. Under general conditions \( \rho \) is (and we assume to be) Poincaré covariant with positive energy-momentum [23]. We may assume that \( \rho \) is localized within \( W \) and therefore the restriction of \( \rho \) to \( \mathcal{A}(W) \cap \mathcal{A} \) has a normal extension to an endomorphism of its weak closure \( \mathcal{A}(W) \), that will still be denoted by \( \rho_{|\mathcal{A}(W)} \).

The index-statistics theorem [37,40] shows that the map

\[ \rho \rightarrow \rho_{|\mathcal{A}(W)} \]

is a faithful full functor of tensor categories, namely all the information on the superselection structure (charge transfers, statistics, \ldots) is visible within \( \mathcal{A}(W) \), in particular

\[ d(\rho) = \text{Ind}(\rho)^\frac{1}{2} \]

where \( d(\rho) \) is the DHR statistical dimension, i.e. the order of the parastatistics, and \( \text{Ind}(\rho) \) is the Jones index of \( \rho_{|\mathcal{A}(W)} \) (more precisely the minimal index, see [35,40]).

In the sector \( \rho \) the proper time evolution is given by a one-parameter automorphism group \( \alpha_{\rho}^t \) of \( \mathcal{A}(W) \) corresponding to the pure Lorentz transformations leaving \( W \) invariant,

\[ \alpha_{\rho}^t \cdot \rho = \rho \cdot \alpha_t. \]

As we shall see, \( \alpha_{\rho}^t \) admits a unique thermal equilibrium normal state \( \varphi_\rho \) at the same inverse Hawking temperature \( \beta = 2\pi/a \). If \( K_\rho \) is the generator of the unitary implementation of the pure Lorentz in the \( x_1 \)-direction, then

\[ H_\rho = a K_\rho \]

is the proper Hamiltonian for our system \( \Sigma \) in the sector \( \rho \).

A similar structure appears in the analysis of the chemical potential [2]. Adding one particle is not a drastic change as our thermodynamical system has essentially infinitely many particles, so we do not obtain an inequivalent representation, namely \( \rho \) is normal on \( \mathcal{A}(W) \). There is however an important difference. The chemical potential labels different equilibrium states at the same temperature for the same dynamics, while we look at the \( \varphi_\rho \)'s as equilibrium states with respect to their own different dynamics \( \alpha_{\rho}^t \), a fact compatible with the General Relativity context where the dynamics should depend on the state as the matter has influence on the metric tensor.

Motivated by the above discussion, we will consider the relative free energy between two thermal equilibrium states for the system \( \Sigma \) associated with the external region of a black hole. Guided by the thermodynamical expression

\[ dF = dE - TdS \]

we express the relative free energy of the states \( \omega \) and \( \varphi_\rho \) by

\[ F(\omega|\varphi_\rho) = \varphi_\rho(H_\rho) - \beta^{-1} S(\omega|\varphi_\rho) \]
where $S$ is the Araki relative entropy [1] of the two states and $\varphi_\rho(H_\rho)$ is the relative mean (internal) energy.

We shall find the relation

$$F(\omega|\varphi_\rho) = -\frac{1}{2}\beta^{-1}S_c(\rho)$$  (0.4)

where $S_c(\rho)$ is the Connes-Størmer [16] conditional entropy of $\rho(\mathcal{A}(W)) \subset \mathcal{A}(W)$.

Here we use the fact that, by the Pimsner-Popa theorem [42] and the index-statistics theorem [37], $\frac{1}{2}S_c(\rho)$ equals the logarithm of the statistical dimension $d(\rho)$ of $\rho$. As the latter takes only integral values by the DHR theorem [18], we conclude that the possible values for the relative free energy are

$$F(\omega|\varphi_\rho) = -\beta^{-1}\log(n), \quad n = 1, 2, 3, \ldots$$

namely the integer $n = d(\rho)$ here acquaints the different meaning of a quantum number labeling the relative free energy levels.

The vacuum state $\omega$ should play no specific physical role in (0.4) and it is only a convenient reference state in our setting. Indeed we will extend the formula to the case of two arbitrary thermal equilibrium states $\varphi_\sigma$ and $\varphi_\rho$

$$F(\varphi_\sigma|\varphi_\rho) = \frac{1}{2}\beta^{-1}(S_c(\sigma) - S_c(\rho))$$  (0.5)

and therefore $F(\varphi_\sigma|\varphi_\rho) = \beta^{-1}\log(\frac{n}{m})$ is $\beta^{-1}$-times the logarithm of a rational number $\frac{n}{m}$ where $n$ depends only on $\sigma$ and $m$ only on $\rho$.

Finally we observe that formula (0.5) is consistent with the above recalled interpretations of the increment $dA$ of surface area of the black hole [4,46] and, in a sense, it unifies different points of views: the increments of the conditional entropy of $\Sigma$, an information theoretical concept, is indeed proportional to the increment of its free energy, a statistical mechanics concept.

We summarize the conceptual scheme in the proof of our result in the following diagram

Connes-Størmer Entropy $\xrightarrow{\text{Pimsner-Popa Th.}}$ Jones Index

$\text{Th. 3.4} \xrightarrow{\frac{1}{2}\beta^{-1}} \sqrt{\cdot} \xrightarrow{\text{Index-statistics Th.}}$ Relative Free Energy

$\xleftarrow{\beta^{-1}\log} \xrightarrow{\text{Cor. 2.2}} \text{DHR dimension}$

1. Connes cocycles and endomorphisms.

Let $M$ be a von Neumann algebra and $\varphi$, $\omega$ faithful normal positive linear functionals on $M$ and denote by $\sigma^\omega$ and $\sigma^\varphi$ their modular group given by the Tomita-Takesaki theory [49]. The Connes Radon-Nikodym cocycle [13]

$$u(t) = (D\varphi : D\omega)_t$$
is the map \((t \in \mathbb{R} \rightarrow u(t))\) unitary of \(M\) such that
\[
u(t + s) = u(t)\sigma_t^\omega(u(s))\] (1.1)
characterized by: for any given \(x, y \in M\) there exists a complex function \(F\) bounded and continuous in \(\{0 \leq \text{Im} z \leq 1\}\) and analytic in its interior such that
\[
F(t) = \varphi(\sigma_t^\varphi(y)u(t)x), \quad F(t + i) = \omega(xu(t)\sigma_t^\omega(y)).
\] (1.2)
The relevant property is that \(u(t)\) intertwines \(\sigma_t^\omega\) and \(\sigma_t^\varphi\), namely
\[
\sigma_t^\varphi = u(t)\sigma_t^\omega(\cdot)u(t)^*.
\] (1.3)
Conversely a continuous unitary \(\sigma^\omega\)-cocycle \(u\) (i.e. (1.1) is valid for \(u\)) is the Connes cocycle with respect to a unique faithful normal positive linear functional or semifinite weight \(\varphi\) of \(M\). If \(M\) is a factor, a continuous unitary \(\sigma^\omega\)-cocycle \(u\) satisfying (1.3) is uniquely determined up to a one-dimensional character of \(\mathbb{R}\), hence, in order to check whether \(u(t) = (D\varphi : D\omega)_t\), one may test equation (1.2) in the special case \(x = y = 1\): there must exist a function \(F\) bounded and continuous in \(\{0 \leq \text{Im} z \leq 1\}\) and analytic in its interior such that
\[
F(t) = \varphi(u(t)), \quad F(t + i) = \omega(u(t))
\] (1.4)
In particular, given the \(\sigma^\omega\)-cocycle \(u\), the positive functional \(\varphi\) such that \(u(t) = (D\varphi : D\omega)_t\) may be computed by (1.2) as
\[
\varphi(x) = \text{anal. cont.} \omega(xu(t))
\] (1.5)
Let now \(M\) be an infinite factor and denote by \(\text{End}(M)\) the set \(\text{finite-index}\) (or \(\text{finite-dimensional}\)) endomorphisms of \(M\). Namely \(\rho \in \text{End}(M)\) if \(\rho\) is an endomorphism of \(M\) whose intrinsic dimension \(d(\rho)\) is finite ([41], see Appendix A). Equivalently \(\rho(M)\) is subfactor of \(M\) with finite index in the sense of [35,33], indeed, as shown in [38],
\[
d(\rho) = \text{Ind}(\rho)^{\frac{1}{2}},
\]
where \(\text{Ind}(\rho)\) denotes the minimal index of \(\rho(M) \subset M\). We refer to [40] and references therein for the notions of index theory of our use.

We fix a normal faithful state \(\omega\) of \(M\). Given \(\rho \in \text{End}(M)\) we denote by \(\Phi_\rho\) the minimal left inverse of \(\rho\) and set
\[
\Psi_\rho = d(\rho)\Phi_\rho, \quad \psi_\rho = \omega \cdot \Psi_\rho
\]
so that \(\psi_\rho\) is a normal faithful positive functional of \(M\).

We define the cocycle of an endomorphism \(\rho\) by
\[
u(\rho, t) = (D\psi_\rho : D\omega)_t
\] (1.6)
(see also [34]).

As will be apparent, several of the results in this section are valid (essentially with the same proofs) for endomorphisms of factors with a normal faithful conditional expectation onto the range, but for simplicity we just treat the finite-index case.
**Proposition 1.1** Let \( \rho \in \text{End}(M) \) and set \( \rho_t = \sigma_t^\rho \rho \sigma_t^{-1} \). Then \( u(\rho, t) \) satisfies
\[
\text{Ad}u(\rho, t) \cdot \rho_t = \rho
\] (1.7)
In particular, if \( \rho \) is irreducible, then \( u(\rho, t) \) is characterized by (1.7) up to the multiplication by a one-dimensional character of \( \mathbb{R} \).

**Proof.** The minimal expectation \( E_\rho = \rho \Phi_\rho \) onto \( \rho(M) \) leaves \( \psi_\rho \) invariant since
\[
\psi_\rho \rho \Phi_\rho = \omega \Phi_\rho \rho \Phi_\rho = \omega \Phi_\rho = \psi_\rho
\]
thus
\[
\sigma_{\psi_\rho|\rho(M)} = \sigma_{\psi_\rho|\rho(M)} = \sigma_{\rho^{-1}} = \rho \sigma_{\rho^{-1}},
\]
namely
\[
\sigma_{\psi_\rho} \rho(x) = \rho \sigma_{\rho^{-1}}(x), \quad x \in M
\]
and since \( \sigma_{\psi_\rho} = \text{Ad}u(\rho, t) \sigma_{\rho^{-1}} \) we have
\[
\text{Ad}u(\rho, t) \cdot \sigma_{\rho^{-1}} \cdot \rho \cdot \sigma_{\rho^{-1}} = \rho.
\] (1.8)
Now the equation (1.8) determines the restriction of \( \text{Ad}u(\rho, t) \) to \( \rho(M) \) hence it determines \( u(\rho, t) \) up to multiplication by a unitary in \( \rho(M)' \cap M \) and therefore, if \( \rho \) is irreducible, it determines \( u(\rho, t) \) up to a phase. \( \square \)

As recalled in Appendix A, \( \text{End}(M) \) are the objects of a tensor C*-category where the arrows are given by (A.1). The following in this section has a natural interpretation in the setting of tensor C*-categories, although here below we use the explicit formulas of our setting (see [38]).

As we shall use the greek letter \( \sigma \) to denote an element of \( \text{End}(M) \), the modular group will be always denoted with explicit reference to the functional (e.g. \( \sigma_t^\rho \)).

**Proposition 1.2** Let \( \rho \in \text{End}(M) \) be irreducible and contained in \( \sigma \in \text{End}(M) \), namely there exists an isometry \( w \in (\rho, \sigma) \) (i.e. \( w \in M \) and \( w \rho(x) = \sigma(x)w, \forall x \in M \)), then
\[
\Psi_\rho = \Psi_\sigma(w \cdot w^*).
\] (1.9)

If \( \sigma = \bigoplus_{i=1}^N n_i \rho_i \) is an irreducible decomposition of \( \sigma \) and for each \( i \) \{\( w_k^{(i)} \), \( k = 1, \ldots, n_i \)\} is an orthonormal basis of isometries in \( (\rho_i, \sigma) \), then
\[
\Psi_\sigma = \sum_{i=1}^N \sum_{k=1}^{n_i} \Psi_{\rho_i}(w_k^{(i)*} \cdot w_k^{(i)}).
\] (1.10)

**Proof.** Let \( \Psi'_\sigma \) be defined by the right hand side of (1.10). Then
\[
\Psi'_\sigma(\sigma(x)) = \sum_{i=1}^N \sum_{k=1}^{n_i} \Psi_{\rho_i}(w_k^{(i)*} \sigma(x) w_k^{(i)}) = \sum_{i=1}^N \sum_{k=1}^{n_i} \Psi_{\rho_i}(\rho_i(x)) = \sum_{i=1}^N n_i d(\rho_i)x = d(\sigma)x
\]
thus \( d(\sigma)^{-1}\Psi'_\sigma \) is a left inverse of \( \sigma \). As \( \Psi'_\sigma(w_k^{(i)*} w_k^{(i)*}) = \Psi_{\rho_i}(1) = d(\rho_i) \) we have that \( \Psi'_\sigma = \Psi_\sigma \), showing the validity of the equation (1.10). Formula (1.9) is obtained similarly. \( \square \)
Proposition 1.3 If $T$ is an arrow in $(\rho, \sigma)$, then

$$Tu(\rho, t) = u(\sigma, t)\sigma_t^\omega(T).$$ \hfill (1.11)

Proof. Let first $\rho$ be an irreducible component of $\sigma$ and $w$ an isometry in $(\rho, \sigma)$. Then by (1.9) we have

$$u(\rho, t) = (D\psi_\rho : D\omega)_t = (D\psi_\sigma(w \cdot w^*) : D\omega)_t$$

therefore by the Radon-Nikodym chain rule and using the relation $(D\psi_\sigma(w^* \cdot w) : D\psi_\sigma)_t = w^*\sigma_t^\psi_\sigma(w)$ we have

$$u(\rho, t) = (D\psi_\sigma(w^* \cdot w) : D\psi_\sigma)_t(D\psi_\sigma : D\omega)_t = w^*\sigma_t^\psi_\sigma(w)u(\sigma, t) = w^*u(\sigma, t)\sigma_t^\omega(w)$$

and after multiplying on the left by $w$ all members in the above expression, and using the $\sigma_t^\psi$-invariance of $ww^*$, we obtain the special case of (1.11)

$$wu(\rho, t) = u(\sigma, t)\sigma_t^\omega(w).$$ \hfill (1.12)

If now $\sigma = \bigoplus_{i=1}^N n_i\rho_i$ is an irreducible decomposition of $\sigma$ and the $w_k^{(i)}$ are as in Proposition 1.2, we have from (1.12) $u(\sigma, t)w_k^{(i)}w_k^{(i)*} = w_k^{(i)}u(\rho, t)\sigma_t^\omega(w_k^{(i)*})$ and summing up over $i$ and $k$ we have

$$u(\sigma, t) = \sum_{i,k} w_k^{(i)}u(\rho, t)\sigma_t^\omega(w_k^{(i)*}).$$ \hfill (1.13)

If now $T$ is an arrow in $(\rho, \sigma)$, we may decompose both $\sigma$ and $\rho$ into irreducibles so that the ranges of the $w_k^{(i)}$’s of $\rho$ (resp. of $\sigma$) are either orthogonal or contained in the range of $T$ (resp. of $T^*$). Then multiplying (1.13) on the left by $T$ (resp. on the right by $\sigma_t^\omega(T)$) will kill the indeces corresponding to the orthogonal part and the result is obtained by the equation (1.12). \hfill $\square$

Proposition 1.4 $u(\rho\sigma, t) = \rho(u(\sigma, t))u(\rho, t)$.

Proof. We first note that

$$\rho(u(\rho, t)) = (D\psi_{\rho\sigma} : D\psi_\rho)_t.$$ \hfill (1.14)

Indeed as both functionals $\psi_{\rho\sigma}$ and $\psi_\rho$ leave invariant the conditional expectation $\rho\Phi_\rho$ onto $\rho(M)$, their Radon-Nikodym cocycle coincides with the cocycle of their restriction to $\rho(M)$, thus

$$(D\psi_{\rho\sigma} : D\psi_\rho)_t = (D\psi_{\rho\sigma|\rho(M)} : D\psi_\rho|\rho(M))_t$$

$$= (D\psi_\sigma \cdot \rho^{-1}|_{\rho(M)} : D\omega \cdot \rho^{-1}|_{\rho(M)})_t$$

$$= \rho((D\psi_\sigma : D\omega)_t) = \rho(u(\rho, t)).$$
The Proposition then follows by the Radon-Nikodym chain rule for the Connes cocycles. □

Propositions 1.3 and 1.4 state that \( u(\rho, t) \) is a two-variable-cocycle (Appendix A), with respect to the action of \( \mathbb{R} \) on \( \text{End}(M) \) given by

\[
t \to \rho_t = \sigma^\omega_1 \cdot \rho \cdot \sigma^\omega_{-t}, \quad t \to \sigma^\omega_1(T),
\]

where \( \rho \) is an object and \( T \) is an arrow in \( \text{End}(M) \).

Recall now that each \( \rho \in \text{End}(M) \) has a conjugate object \( \bar{\rho} \), namely there exist \( R_\rho \in (\iota, \bar{\rho}) \) and \( \bar{R}_\rho \in (\iota, \rho \bar{\rho}) \) standard solutions of the equation (A.1), i.e.

\[
\bar{R}_\rho^* \rho(R_\rho) = 1, \quad R_\rho^* \bar{\rho}(\bar{R}_\rho) = 1
\]

and \( \|R_\rho\| = \|\bar{R}_\rho\| = (\sqrt{d(\rho)}) \) is minimal.

As explained in the Appendix A, given an arrow \( T \in (\rho_1, \rho_2) \), the conjugate arrow \( T^* \in (\bar{\rho}_1, \bar{\rho}_2) \) is defined by

\[
T^* = \bar{\rho}_2(\bar{R}_{\rho_1}^*, T^*) \bar{R}_{\rho_2}
\]

where \( R_{\rho_i} \) and \( \bar{R}_{\rho_i} \) give a standard solution for the conjugate equation defining the conjugate \( \bar{\rho}_i \).

Note now that given \( \rho \in \text{End}(M) \), once we choose \( \bar{\rho} \) defined by the \( R_\rho \) and \( \bar{R}_\rho \), than \( (\bar{\rho})_t \) is a conjugate of \( \rho_t \) defined by \( \bar{R}_\rho^t = \sigma^\omega_t(\bar{R}_\rho) \), \( \bar{R}_\rho^t = \sigma^\omega_t(\bar{R}_\rho) \), that we may simply denote by \( \bar{\rho}_t \). In the following \( u(\rho, t)^* \) is defined by this choice of the \( \bar{R} \)-operators.

**Proposition 1.5** \( u(\bar{\rho}, t) = u(\rho, t)^* \).

**Proof.** By definition

\[
u(\rho, t)^* = \bar{\rho}(\bar{R}_\rho^*, u(\rho, t)^*) R_\rho
\]

\[
= \bar{\rho}(\sigma^\omega_1(\bar{R}_\rho^*)u(\rho, t)^*) R_\rho
\]

\[
= \bar{\rho}(\sigma^\omega_1(\bar{R}_\rho^*)\sigma^\omega_1(u(\rho, -t))) R_\rho
\]

\[
= \bar{\rho}(\sigma^\omega_1(\bar{R}_\rho^*)u(\rho, -t))) R_\rho
\]

\[
= u(\bar{\rho}, t)\sigma^\omega_1(\bar{\rho}(\bar{R}_\rho^*u(\rho, -t)/\sigma^\omega_1(u(\rho, -t)))R_\rho
\]

\[
= u(\bar{\rho}, t)\sigma^\omega_1(\bar{\rho}(\bar{R}_\rho^*u(\rho, -t))/\sigma^\omega_1(u(\rho, -t)))R_\rho
\]

Thus we have to show that \( \sigma^\omega_1(\bar{\rho}(\bar{R}_\rho^*u(\rho, -t)))R_\rho = 1 \) or, by applying \( \sigma^\omega_{-t} \), that \( \bar{\rho}(\bar{R}_\rho^*u(\rho, -t))u(\bar{\rho}, -t)R_{\rho_{-t}} = 1 \). Indeed we have

\[
\bar{\rho}(\bar{R}_\rho^*u(\rho, -t))u(\bar{\rho}, -t)R_{\rho_{-t}} = \bar{\rho}(\bar{R}_\rho^*)\bar{\rho}(u(\rho, -t))u(\bar{\rho}, -t)R_{\rho_{-t}}
\]

\[
= \bar{\rho}(\bar{R}_\rho^*)u(\bar{\rho}, -t)R_{\rho_{-t}} = \bar{\rho}(\bar{R}_\rho^*)R_\rho = 1,
\]

where \( u(\bar{\rho}, -t)R_{\rho_{-t}} = R_\rho \) by Proposition 1.3. □
Lemma 1.6 If \( j \) is an \( \omega \)-preserving anti-automorphism of \( M \), then

\[
u(\rho, t) = j(u(j \cdot \rho \cdot j^{-1}, -t)).
\]

Proof. Since \( j \) preserves \( \omega \), by the KMS condition \( j \sigma_t^\omega j^{-1} = \sigma_{-t}^\omega \), thus \( j(u(j \cdot \rho \cdot j^{-1}, -t)) \) is a \( \sigma_t^\omega \)-cocycle and it coincides with \( u(\rho, t) \) because it satisfies the equation (1.5). \[\square\]

Proposition 1.7 Let \( \mathcal{T} \) be a \( C^* \)-tensor subcategory with conjugates of \( \text{End}(M) \) and \( z \) a two-variable cocycle for the action \( \mathbb{R} \to \text{Aut} \mathcal{T} \) given by the modular group \( \sigma^\omega \) (equation (1.15)). Suppose there is an anti-automorphism \( j \) of \( M \) such that \( j \rho j^{-1} \) is a conjugate of \( \rho \) and \( z(j \rho j^{-1}, t) = j(z(\rho, -t)) \) for a given \( \rho \in \mathcal{T} \). Then \( z(\rho, t) \) coincides with the Connes cocycle \( u(\rho, t) \) defined by (1.5). As a consequence

\[
d(\rho) = \lim_{t \to i} \omega(z(\rho, t)). \tag{1.16}
\]

Proof. We have \( z(\rho, t) = \mu(\rho, t)u(\rho, t) \) for some character \( \mu(\rho, \cdot) \) of \( \mathbb{R} \). Since \( \bar{\rho} = j\rho j^{-1} \) is a conjugate of \( \rho \), we have by the Lemma 1.6

\[
z(\bar{\rho}, t) = j(z(\rho, -t)) = j(\mu(\rho, -t)u(\rho, -t)) = \mu(\rho, t)j(u(\rho, -t)) = \mu(\rho, t)u(\bar{\rho}, t)
\]

On the other hand by Lemma A.3 of the Appendix A

\[
z(\bar{\rho}, t) = z(\rho, t) \ast \mu(\rho, t) \ast u(\rho, t) \ast = \mu(\rho, t)u(\bar{\rho}, t)
\]

thus \( \mu(\rho, t) = 1 \). Formula (1.16) is thus a consequence of (1.5) in the case \( x = 1 \). \[\square\]

Before concluding this section we recall the notions of entropy of later use. If \( N \subset M \) is an inclusion of \( II_1 \)-factors, the Connes-Størmer (conditional) entropy \( H(M|N) \) is defined in [16]. By the Pimsner-Popa Theorem [42]

\[
H(M|N) = \log[M : N] \quad \tag{1.17}
\]

where \([M : N]\) is the Jones index of \( N \subset M \) (if \( N \subset M \) is irreducible or extremal). \( H(M|N) \) is generalized to the type \( III \) setting in [15]. If \( \varphi \) is a normal faithful state of \( M \) one defines

\[
H_\varphi(M|N) = \sup_{(\varphi_i)} \left\{ \sum_i S(\varphi|\varphi_i) - S(\varphi|_N|\varphi_i|_N) \right\}
\]

where \((\varphi_i)\) varies among the sets of finitely many normal positive linear functionals \( \varphi_1, \varphi_2, \ldots, \varphi_n \) of \( M \) such that \( \sum_{i=1}^n \varphi_i = \varphi \), and \( S(\cdot|\cdot) \) denotes the Araki relative entropy between states, see [7] and eq. (3.7) below.
If $E$ is a normal conditional expectation of $M$ onto $N$ one sets
\[ HE(M|N) = \sup \{ H_\varphi(M|N), \varphi \cdot E = \varphi \}. \]
If moreover $N$ is a $III_1$-factor, then
\[ HE(M|N) = H_\varphi(M|N) \]
for any normal faithful state $\varphi$ such that $\varphi \cdot E = \varphi$ [29]. $HE(M|N)$ depends on the choice of a normal conditional expectation $E$, but we simply write
\[ H(M|N) = HE_{\text{min}}(M|N) \]
where $E_{\text{min}}$ is the minimal conditional expectation. The Pimsner-Popa equality (1.17) holds true without restrictions, provided $[M : N]$ denotes the minimum index, see [29]. Finally, if $\rho$ is an endomorphism of $M$, we consider the conditional entropy of $\rho$
\[ Sc(\rho) = H(M|\rho(M)). \] (1.18)

2. The formula in conformal field theory.
2.1 Finite index case. We now consider a precosheaf (net) $\mathcal{A}$ of von Neumann algebras associated with a chiral conformal quantum field theory. Namely $\mathcal{A}$ is a map
\[ I \rightarrow \mathcal{A}(I) \]
from the (proper) intervals of $S^1$ to von Neumann algebras $\mathcal{A}(I)$ on a fixed Hilbert space $\mathcal{H}$ that satisfies:
- **Isotony:** if $I \subset \tilde{I}$ then $\mathcal{A}(I) \subset \mathcal{A}(\tilde{I})$,
- **Locality:** $\mathcal{A}(I)$ and $\mathcal{A}(I')$ commute elementwise, where $I' = S^1 \setminus I$,
- **Möbius covariance with positive energy:** there exists a unitary representation $U$ of Möbius group $SL(2, \mathbb{R})/\{1, -1\}$, that for convenience we regard as a representation of its universal covering group $G$, such that $U(g)\mathcal{A}(I)U(g)^* = \mathcal{A}(gI)$ and the generator of the one-parameter rotation subgroup is positive. We set
\[ \alpha_g(X) = U(g)XU(g)^* \]
where $X$ is a local operator, namely $X$ belongs to some $\mathcal{A}(I)$,
- **Existence and uniqueness of the vacuum:** there exists a unique (up to a phase) $U$-invariant unit vector $\Omega \in \mathcal{H}$, and it is cyclic for the algebra generated by of local operators. We denote by $\omega = (\cdot, \Omega, \Omega)$ the vacuum state.
For a discussion of these properties and their consequences, see [24].

Let $\rho$ be a covariant positive energy representation of $\mathcal{A}$ on a separable Hilbert space $\mathcal{H}_\rho$, namely for every proper interval $I$ we have a representation $\rho_I$ of $\mathcal{A}(I)$ on $\mathcal{H}_\rho$ so that
\[ \rho_{I|\mathcal{A}(I)} = \rho_I \quad \text{if } I \subset \tilde{I} \]
and a positive energy representation $U_\rho$ of $G$ on $\mathcal{H}_\rho$ such that

$$\rho_{gI}(\alpha_g(X)) = U_\rho(g)\rho_I(X)U_\rho(g)^*, \quad X \in \mathcal{A}(I)$$

(if $\mathcal{A}$ is strongly additive the covariance property is automatic [23]). We shall always assume the representations to be covariant with positive energy.

Let $\Lambda_{I}(t)$ be the special conformal one-parameter group associated with $I$. If $I$ is the upper semi-circle then

$$\Lambda_{I}(t)z = \frac{(z + 1) - e^{-t}(z - 1)}{(z + 1) + e^{-t}(z - 1)} \quad (2.1)$$

while if $I_0$ is any other interval $\Lambda_{I_0}$ is well defined by conjugation of the transformation (2.1) with a conformal transformation $g \in SL(2, \mathbb{R})$ such that $gI = I_0$. We denote by

$$K^I_\rho = -i\frac{d}{dt}U_\rho(\Lambda_{I}(t))|_{t=0}$$

the infinitesimal generator of $U_\rho(\Lambda_{I}(t))$.

**Theorem 2.1** Let $\rho$ be a representation of $\mathcal{A}$, $I \in S^1$ an interval and $\xi \in \mathcal{H}_\rho$ a cyclic vector for $\rho_{I'}(\mathcal{A}(I'))$ such that $\omega(X) = (\rho_{I'}(X)\xi, \xi)$ for all $X \in \mathcal{A}(I')$. Then

$$(e^{-2\pi K^I_\rho}, \xi) = d(\rho).$$

A vector $\xi$ as above always exists.

As we are interested in the representation $\rho$ up to unitary equivalence, we may identify $\mathcal{H}_\rho$ with $\mathcal{H}$ so that, due to Haag duality (see below), $\rho$ becomes an endomorphism of $\mathcal{A}$ localized in a given interval $I$, namely $\rho_{I'}$ acts identically, and $\rho_{I}(\mathcal{A}(\tilde{I})) \subset \mathcal{A}(\tilde{I})$ if $I \subset \tilde{I}$, see [24]. Because of the Reeh-Schlieder theorem (see [21]), the vacuum vector $\Omega$ is cyclic for any local von Neumann algebra, in particular for $\mathcal{A}(I')$, therefore $\Omega$ satisfies in this representation the properties required for the vector $\xi$ in the statement of Theorem 2.1, showing its existence.

As $\rho$ is covariant, there is a unitary $\alpha$-cocycle $z(\rho, g)$ such that

$$\text{Ad}z(\rho, g) \cdot \alpha_g \cdot \rho \cdot \alpha_g^{-1} = \rho, \quad (2.2)$$

where $\alpha_g = \text{Ad}U(g)$. More precisely the equation $z(\rho, g) = U_\rho(g)U(g)^*$ defines $z(\rho, g)$ if $g$ belongs to a neighbouroude of the identity of $G$, and $z(\rho, g)$ is localized in the sense that it belongs to $\mathcal{A}(\tilde{I})$ is $\tilde{I}$ is any interval containing both $I$ and $gI$ [23], (see also [55]). Then $z(\rho, g)$ is defined for arbitrary $g \in G$ as an element of the universal algebra $C^* (\mathcal{A})$ by the cocycle identity $z(\rho, gh) = z(\rho, g)\rho(z(\rho, h))$, but we do not need this fact.

---

3We shall use the following convention: is $A$ is a positive selfadjoint operator and $\eta$ a vector, then $(A\eta, \eta) = ||A^{1/2}\eta||^2$ if $\eta$ belongs to the domain of $A^{1/2}$, and $(A\eta, \eta) = +\infty$ otherwise.
It is important to note that if $\rho$ is irreducible or a finite direct sum of irreducibles, as is the case of a finite-index $\rho$, then $z$ is uniquely determined by the formula (2.2) as a localized cocycle, because $G$ has no non-trivial unitary finite-dimensional representation, see [24].

We consider the tensor $C^*$-category $\mathcal{E}_I$ whose objects are the (covariant, positive energy) endomorphisms of $\mathcal{A}$ with finite index, namely

$$d(\rho) = d(\rho_I) < \infty,$$

localized in an interval whose closure is contained in the interior of a given interval $I$ and the arrows $(\rho, \sigma)$ are the local operators $T$ such that $T\rho I_1(X) = \sigma I_1(X)T$ for all intervals $I_1$ and local operators $X \in \mathcal{A}(I_1)$. By [37] (see also [24]) the restriction map

$$\rho \in \mathcal{E}_I \to \rho_I \in \text{End}\mathcal{A}(I)$$

is a faithful full functor, therefore we may identify $\mathcal{E}_I$ with a tensor $C^*$-subcategory of $\text{End}\mathcal{A}(I)$, so that $d(\rho)$ is identified with the DHR statistical dimension of $\rho$.

Then $z$ is in a natural sense a local two-variable cocycle for the local action of $G$ on $\mathcal{E}_I$ given by $\rho \to \alpha_g \rho \alpha_g^{-1} \ (\rho \in \mathcal{E}_I)$ and $T \to \alpha_g(T)$ ($T$ arrow), namely the properties a) and b) defining a two-variable cocycle after Lemma A.3 of the Appendix A hold for $z(\rho, g)$, but only if $g$ belongs to a neighbourhood of the identity of $G$ (depending on $\rho$). For example if $\rho, \sigma \in \mathcal{E}_I$ then $\rho(z(\rho, g))z(\sigma, g)$ exists if $g$ lies in a neighbourhood of the identity of $G$ and satisfies (2.2) for $\rho\sigma$, hence it agrees with $z(\rho\sigma, g)$ as the $\alpha$-cocycle property and formula (2.2) determines it. We thus have:

**Lemma 2.2** $z(\rho, g)$ is a local two-variable cocycle for the action of $G$ on the tensor $C^*$-category $\mathcal{E}_I$.

**Proof.** This follows by the above discussion and an elementary direct verification of the two-variable cocycle property. □

Recall now that each localized endomorphism $\rho$ has a conjugate localized endomorphism given by the formula [23]

$$\bar{\rho} = j \cdot \rho \cdot j \quad (2.3)$$

where $j = \text{Ad}J$ is the anti-automorphism of $\mathcal{A}$ implemented by the modular conjugation $J$ of $(\mathcal{A}(I_1), \Omega)$ for any choice of the interval $I_1$. To be definite let $I$ be the upper half-circle and $I_1$ the right half-circle. Due to the geometrical meaning of $J$, $j$ implements an anti-automorphism of $\mathcal{A}(I)$. If $g$ is in the Möbius group, we denote by $g^j$ its conjugate by the anti-automorphism given by the reflection on $S^1$ corresponding to $j$.

**Proposition 2.3** Let $\rho$ be finite-index endomorphism of $\mathcal{A}$ localized in the interval $I$. With the above notations, we have

$$z(\bar{\rho}, g) = z(\rho, g)^* = j(z(\rho, g^j))$$
(see Section 1 and Appendix A for the definition of the •-mapping).

Proof. The first equality follows from Lemma 2.1 and Lemma A.3. However one may see directly the validity of both equalities by the uniqueness of $z(\bar{\rho}, g)$ by checking that also $z(\rho, g) \cdot j(z(\rho, g))$ are local $\alpha$-cocycles and both satisfy formula (2.2) for $\bar{\rho} = j\rho j$.

Now the modular structure of $A$ is computed in [30,8], in particular we have

$$\Delta_I^{it} = U(\Lambda_I(-2\pi t))$$

where $\Delta_I$ is the modular operator of $(A(I), \Omega)$. An important consequence is Haag duality

$$A(I') = A(I)'$$

moreover the $A(I)'s$ are type $III_1$ factors.

Next theorem computes the modular structure of $A$ in the representation $\rho$, see also Proposition 3.5. We set $u_I(\rho, t) = u(\rho I, t) = (D\omega \Psi_{\rho I} : D\omega|_{A(I)})_t$ as in (1.6), with $\omega$ the vacuum state.

**Theorem 2.4** Let $\rho$ be finite-index endomorphism of $A$ localized in the interval $I$. Then

$$u_I(\rho, t) = z(\rho, \Lambda_I(-2\pi t)), \quad t \in \mathbb{R}.$$  

Proof. By the formula (2.3) for the conjugate sector and Lemma 2.2, the theorem follows immediately by Proposition 1.7.

**Corollary 2.5** Let $\rho$ be an irreducible finite-index endomorphism of $A$ localized in the interval $I$ of $S^1$. Then

$$d(\rho) = (e^{-2\pi K_I^t} \Omega, \Omega) = \|e^{-\pi K_I^t} \Omega\|^2.$$  

(2.4)

Proof. Since $z_I(\rho, \Lambda_I(t)) = U_\rho(\Lambda_I(t))U(\Lambda_I(t))^*$, Proposition 1.7 and Theorem 2.3 show that the function $t \to \omega(z(\rho, \Lambda_I(-2\pi t)))$ extends to a function bounded and continuous in the strip $\{-1 \leq \text{Im} z \leq 0\}$ and analytic in its interior so that

$$d(\rho) = \text{anal. cont.} \left. \omega(z(\rho, \Lambda_I(-2\pi t))) \right|_{t \to -i} = \text{anal. cont.} \left. (U_\rho(\Lambda_I(-2\pi t))\Omega, \Omega) \right|_{t \to -i}.$$  

Standard functional analysis arguments then show that the right hand side of the above expression is equal to $(e^{-2\pi K_I^t} \Omega, \Omega)$.  

□

Proof of Theorem 2.1 in the finite index case. Let $V : \mathcal{H}_\rho \to \mathcal{H}$ be the unitary given by

$$V_\rho(X)\xi = X\Omega, \quad X \in A(I')$$

so that $V_\rho(X)V^* = X$ if $X \in A(I')$, thus

$$\rho' = V_\rho(\cdot)V^*$$
is an endomorphism of $A$ localized in $I$. By Corollary 2.5 we then have
\[ d(\rho) = (e^{-2\pi K^I_{\lambda} \Omega}, \Omega) = (V e^{-2\pi K^I_{\lambda} V^* \Omega}, \Omega) = (e^{-2\pi K^I_{\lambda} \xi}, \xi), \]
in case $\rho$ has finite index. We have already commented on the existence of $\xi$. The infinite index case is discussed here below. \[ \square \]

2.2 General case: a criterium for finite index. We now show that formula (2.4) gives a criterium for the finiteness of the index of a sector, namely that Theorem 2.1 holds without restrictions. We start with a general fact.

**Proposition 2.6** Let $N \subset M$ be an inclusion of factors and assume that there exist a normal faithful conditional expectation $E$ of $M$ onto $N$ and a normal faithful conditional expectation $E'$ of $N'$ onto $M'$. Then $N' \cap M$ is a discrete type $I$ von Neumann algebra, i.e. a (possibly infinite) direct sum of type $I$ factors. Moreover for each minimal projection $p$ of $N' \cap M$ the inclusion $Np \subset pMp$ has finite index.

**Proof.** Setting $R = N' \cap M$, $E$ restricts to a faithful expectation of $N \vee R$ onto $N$, hence $N \vee R$ is canonically isomorphic to the von Neumann tensor product $N \otimes R$ and we can assume this isomorphism to be spatial (by tensoring $N$ and $M$ by a type $III$ factor, if necessary). On the other hand $E'$ factors through a normal faithful expectation of $N'$ onto $(N \vee R)' = N' \cap R'$ by Takesaki's theorem [50], hence, with the above identification, we have a normal faithful expectation of $N' \otimes B(\mathcal{H})$ onto $N' \otimes R'$, that restricts to a normal expectation of $\mathbb{C} \otimes B(\mathcal{H})$ onto $\mathbb{C} \otimes R'$, that implies $R$ to be be a type $I$ von Neumann algebra.

As $R$ is a direct sum of homogeneous type $I_n$ von Neumann algebras, by considering the reduced inclusion corresponding to an abelian projection of $R$ (fixed by the modular group of the expectation) we are left to prove our statement in the case $R$ be an abelian von Neumann algebra, namely we have to prove that $R$ is totally atomic, for in this case the finiteness of the index of the reduced inclusions corresponding to minimal projections of $R$ would follow by [38, Proposition 4.4].

By decomposing $R$ into its diffuse and atomic part, we may then assume $R$ to be diffuse abelian and find an absurd. To this end, for notational convenience, we may identify $N$ with $M$ (see [39]), i.e. we set $N = \sigma(M)$ for some endomorphism $\sigma$ of $M$. We may decompose $\sigma = \int \oplus \sigma_\lambda d\mu(\lambda)$ into irreducibles and as $R$ is abelian $\sigma_\lambda$ is disjoint to $\sigma_{\lambda'}$ for almost all pairs $(\lambda, \lambda')$, hence $\sigma_\lambda \sigma_{\lambda'}$ does not contain the identity, except for $(\lambda, \lambda')$ in a set of product measure 0 and we conclude that $\sigma \sigma$ does not contain the identity too. By [37] this shows that there exists no normal faithful expectation onto $N$ contradicting our hypothesis. \[ \square \]

**Lemma 2.7** Let $N \subset M$ be an inclusion of von Neumann algebras on a Hilbert space $\mathcal{H}$, $\Omega$ a cyclic separating vector for $M$ and $\Delta_M$, $\Delta_N$ the corresponding modular operators on $\mathcal{H}$ and on $\overline{N\Omega}$. Then $\|\Delta^\frac{1}{2}_M \xi\| = \|\Delta^\frac{1}{2}_N \xi\|$ for all $\xi$ in the domain of $\Delta^\frac{1}{2}_N$.

**Proof.** If $x \in N$ we have with usual notations
\[ \|\Delta^\frac{1}{2}_N x \Omega\| = \|J_N \Delta^\frac{1}{2}_N x \Omega\| = \|x^* \Omega\| = \|J_M \Delta^\frac{1}{2}_M x \Omega\| = \|\Delta^\frac{1}{2}_M x \Omega\| \]


and as \( N \Omega \) is a core for \( \Delta_{N}^{\frac{1}{2}} \), the equality holds for all the vectors in the domain of \( \Delta_{N}^{\frac{1}{2}} \). 

**Corollary 2.8** Let \( T(t) \) and \( V(s) \) be two unitary one-parameter groups on a Hilbert space \( \mathcal{H} \) such that

\[
V(s)T(t)V(-s) = T(e^{-2\pi s}t), \quad t, s \in \mathbb{R}
\]

(2.5)

and assume \(-i \frac{d}{dt} T(t)|_{t=0}\) to be positive. Then

\[
\|e^{-\pi D} \xi\| = \|e^{-\pi D} T(t) \xi\|
\]

(2.6)

for all \( \xi \) in the domain of \( e^{-\pi D} \) and all \( t \geq 0 \), where \( D \) is the generator of \( V \).

**Proof.** The projection \( P \) onto the \( T \)-fixed vectors commutes both with \( T \) and \( V \) and on such vectors the equation (2.6) trivially holds, hence we may assume that \( P = 0 \). With this assumption all non-zero representations of the commutation relation (2.5) are quasi-equivalent by von Neumann uniqueness theorem \((D \) and the logarithm of the generator of \( T(t) \) satisfy the Heisenberg commutation relations), hence we may verify the equation (2.6) in any given representation.

Let \( \mathcal{B} \) be the conformal net on \( \mathbb{R} = S^1 \setminus \{-1\} \) given by the current algebra (see e.g. [20]), \( T \) and \( V \) the translation and dilation unitary groups. Then \( e^{-2\pi D} = \Delta_M \), where \( M = \mathcal{B}(0, +\infty) \) and \( T(t)e^{-2\pi D} T(-t) = \Delta_N \), where \( N = \mathcal{B}(t, +\infty) \). Lemma 2.7 then applies to the modular operators \( \Delta_M \) and \( \Delta_N \) with respect to the vacuum and gives equation (2.6). \( \square \)

We shall now denote by \( T_I(t) \) the one-parameter unitary group of translations associated with \( I \), namely cutting \( S^1 \) and identifying it with \( \mathbb{R} \) so that \( I \) is identified with \( \mathbb{R}^+ \), then \( T_I(t) \) correspond to the translations on \( \mathbb{R} \). If \( \rho \) is an endomorphism of \( \mathcal{A} \) localized in \( I \), we shall then denote by \( T^I_{\rho} \) the corresponding one-parameter unitary group in the representation \( \rho \).

**Corollary 2.9** Let \( \rho \) be an endomorphism of \( \mathcal{A} \) localized in the interval \( I \). Then

\[
\|e^{-\pi K^I_{\rho}} \xi\| = \|e^{-\pi K^I_{\rho}} T^I_{\rho}(t) \xi\|
\]

for all \( \xi \) in the domain of \( e^{-\pi K^I_{\rho}} \) and \( t \geq 0 \).

**Proof.** Immediate by Corollary 2.8. \( \square \)

**Proposition 2.10** Let \( \rho \) be an endomorphism of \( \mathcal{A} \) localized in the interval \( I \). If \( (e^{-2\pi K^I_{\rho} \Omega}, \Omega) < \infty \), then the formula

\[
\psi_{\rho}(XY^*) = (e^{-\pi K^I_{\rho}} X\Omega, e^{-\pi K^I_{\rho}} Y\Omega)
\]
determines a positive normal linear functional \( \psi_\rho \) on \( A(I) \) such that 
\[
(D\psi_\rho : D\omega_{|A(I)})_t = z(\rho, \Lambda_I(-2\pi t)).
\]

**Proof.** By Connes’ theorem [13] there exists a unique normal faithful semifinite weight \( \psi_\rho \) on \( A(I) \) such that 
\[
(\Delta(\omega_I'|\psi_\rho)_{tt}) = z(\rho, \Lambda_I(-2\pi t)).
\]
where \( \Delta(\cdot | \cdot) \) denotes the Connes spatial derivative [14] (between a weight on a von Neumann algebra and a weight on its commutant), and since by (2.2)
\[
\Delta_{it}^{it} = e^{-2\pi i t K^I}
\]
we have
\[
\psi_\rho(XY^*) = \text{anal. cont.} \omega(Y^* u(\rho, t) \sigma_t^\rho(X)) = \text{anal. cont.} \omega(Y^* z(\rho, \Lambda_I(-2\pi t)) \sigma_t^\rho(X)) = \text{anal. cont.} (e^{-i2\pi t K^I} X \Omega, Y \Omega).
\]

Alternatively one could use directly the expression given by Proposition 3.5. \( \square \)

**Proposition 2.11** Let \( \rho \) and \( \psi_\rho \) be as in Proposition 2.10 and identify \( I \) with \( \mathbb{R}^+ \) as above. Then
\[
\psi_\rho(U_\rho(g)XU_\rho(g)^*) = \psi_\rho(X), \quad X \in A(I), \tag{2.8}
\]
provided \( g \in G \) is a dilation or a positive translation associated with \( I \).

As a consequence there exists a positive linear functional \( \tilde{\psi}_\rho \) on \( A_\ell = \cup_\ell A(\ell, +\infty)^- \), normal on any \( A(\ell, +\infty) \), translation and dilation invariant in the representation \( \rho \), defined by
\[
\tilde{\psi}_\rho(X) = \psi_\rho(T^I_\rho(t)XT^I_\rho(t)^*)
\]
where \( X \in A(\ell, +\infty) \) and \( t + \ell > 0 \).

**Proof.** The second assertion clearly follows from the first one. Formula (2.8) holds if \( g \) is a dilation, as the dilations correspond to the modular automorphisms of \( A(I) \) with respect to \( \psi_\rho \), due to the construction of \( \psi_\rho \). By Proposition 2.10 we thus have to show that
\[
\|e^{-\pi K^I_\rho} T^I_\rho(t)XT^I_\rho(-t)\Omega\| = \|e^{-\pi K^I_\rho} X \Omega\|, \tag{2.9}
\]
where $X \in \mathcal{A}(I)$ and $t \geq 0$. Indeed by Corollary 2.9 for all $X \in \mathcal{A}(I)$ and $t \geq 0$ we have

$$
\|e^{-\pi K_{\rho}^I} T_{\rho}^I(t) X T_{\rho}^I(-t) \Omega\| = \|e^{-\pi K_{\rho}^I} X T_{\rho}^I(-t) \Omega\| = \|e^{-\pi K_{\rho}^I} X \xi\|. \tag{2.10}
$$

where $\xi = T_{\rho}^I(-t) \Omega = T_{\rho}^I(-t) T^I(t) \Omega$. On the other hand $z(\rho, t) = T_{\rho}^I(t) T^I(-t)$ belongs to $\mathcal{A}(I)$ if $t \geq 0$ as $\rho(T^I(t) \cdot T^I(-t))$ is also localized in $I$. Therefore $(X' \xi, \xi) = (X' \Omega, \Omega)$ if $X' \in \mathcal{A}(I')$.

Now $\xi$ is cyclic for $\mathcal{A}(I')$ if $t \geq 0$, because $\xi = u \Omega$ with $u = z(\rho, t)$ a unitary in $\mathcal{A}(I)$. By Proposition 2.10, the equation (2.10) gives

$$
\|e^{-\pi K_{\rho}^I} T_{\rho}^I(t) X T_{\rho}^I(-t) \Omega\| = \|e^{-\pi K_{\rho}^I} X \xi\| = \|e^{-\pi K_{\rho}^I} X u \Omega\| = \psi_{\rho}(X u u^* X^*) = \psi_{\rho}(XX^*)
$$

showing (2.9) as desired.

**Proposition 2.12** Let $\rho = \oplus_k \rho_k$ be a direct sum of endomorphisms of $\mathcal{A}$ all localized in a given interval $I$. Then

$$(e^{-2\pi K_{\rho}^I} \Omega, \Omega) = \sum_k (e^{-2\pi K_{\rho_k}^I} \Omega, \Omega).$$

**Proof.** Let $v_k$ a family of isometries of $\mathcal{A}(I)$ such that $\rho = \sum_k v_k \rho_k(\cdot) v_k^*$. Then

$$e^{-2\pi K_{\rho}^I} = \sum_k v_k e^{-2\pi K_{\rho_k}^I} v_k^*$$

hence by Proposition 2.10 we have

$$(e^{-2\pi K_{\rho}^I} \Omega, \Omega) = \sum_k (e^{-2\pi K_{\rho_k}^I} v_k^* \Omega, v_k^* \Omega) = \sum_k \psi_{\rho_k}(v_k^* v_k) = \sum_k \psi_{\rho_k}(1) = \sum_k d(\rho_k)$$

□

**Proposition 2.13** Let $\rho$ be as in Proposition 2.10. Then $\rho_1(\mathcal{A}(I))' \cap \mathcal{A}(I)$ is equal to the commutant $\{\cup_{I_0} \rho_{I_0}(\mathcal{A}(I_0))\}'$ of the representation $\rho$.

**Proof.** The proof is based on the arguments given in the proof of [24, Theorem 2.3] that concerned the case $\rho$ had a priori finite index. In that context the proof relied on the construction in [24, Corollary 2.5] of a locally normal faithful positive linear functional invariant under dilations and translations in the representation $\rho$. Proposion 2.11 provides us with such a functional $\tilde{\psi}_\rho$ in our setting, therefore, with obvious modifications, the rest of the proof of [24, Theorem 2.3] is valid here.

□
Lemma 2.14 Let $\rho$ be an endomorphism of $A$ localized in the interval $I$. If there exists a normal faithful conditional $E$ expectation of $A(I)$ onto $\rho(A(I))$, then $\rho I$ is a (possibly infinite) direct sum of irreducible endomorphisms of $A(I)$ with finite index.

Proof. By conformal invariance we may assume that $I$ is the upper semi-circle. We now use the formula $ar{\rho} = j \cdot \rho \cdot j$ for the conjugate sector $\bar{\rho}$, where $j = \text{Ad}J$ with $J$ the modular conjugation associated with the right semi-circle. Due to its geometrical meaning, $j$ is an anti-automorphism of $A(I)$, so that $j \cdot E \cdot j$ is a normal faithful expectation onto $\bar{\rho}(A(I))$. Now the inclusion $\bar{\rho}(A(I)) \subset A(I)$ is dual to the inclusion $\rho(A(I)) \subset A(I)$, hence the Lemma follows by Proposition 2.6. □

Theorem 2.15 Let $\rho$ be an endomorphism of $A$ localized in the interval $I$. Then $(e^{-2\pi K^I_\rho} \Omega, \Omega) < +\infty$ if and only if $\rho$ has finite index. Therefore the equality $d(\rho) = (e^{-2\pi K^I_\rho} \Omega, \Omega)$ holds regardless $d(\rho)$ be finite or infinite.

Proof. We only have to show that if $(e^{-2\pi K^I_\rho} \Omega, \Omega) < +\infty$ then $\rho$ has finite index. Now in this case Proposition 2.10 gives us a faithful positive normal linear functional on $A(I)$ whose modular group leaves $\rho(A(I))$ globally invariant by Proposition 1.1. By Takesaki theorem [50] we have a normal faithful expectation onto $\rho(A(I))$, whence by Proposition 2.12 and Lemma 2.14 $\rho$ is a direct sum of irreducible finite index sectors $\rho_k$. As for each $\rho_k$ the formula $(e^{-2\pi K^I_k} \Omega, \Omega) = d(\rho_k)$ holds true by Corollary 2.5, the results follows by the additivity expressed in Proposition 2.12. □

Before concluding the section, we mention further applications of the above methods to the analysis of superselection sectors with infinite index [23, Section 11], in particular regarding the positivity of the energy in these representations. This matter will be discussed in [5].

3. Hawking temperature in a charged state and conditional entropy.

3.1 General setting and a first expression. Following the discussion made in the introduction, we consider a Quantum Field Theory on the Minkowski space $\mathbb{R}^4$, identify the Rindler space-time $W$ with a wedge region in $\mathbb{R}^4$, and look at $W$ in analogy with the Schwarzschild space-time.

For convenience we fix the Lorentz frame so that

$$W = \{ x \in \mathbb{R}^4, x_1 > |x_0| \}$$

and denote by $\Lambda_W(t)$ the corresponding one-parameter group of pure Lorentz transformations in the $x_1$-direction:

$$\Lambda_W(t) = \begin{pmatrix} \text{ch}(t) & \text{sh}(t) & 0 & 0 \\ \text{sh}(t) & \text{ch}(t) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$
Let $\mathcal{A}(\mathcal{O})$ be the von Neumann algebra on the Hilbert space $\mathcal{H}$ of the observables localized within the region $\mathcal{O}$ of the Minkowski space. Let $U$ denote the unitary covariant, positive energy, representation of the Poincaré group $\mathcal{P}_+^+$ on $\mathcal{H}$ and $\Omega$ the vacuum vector.

We assume the local algebras to be generated by a Wightman field [48], in order to have the Bisognano-Wichmann theorem that identifies the Tomita-Takesaki modular operator $\Delta$ and the modular conjugation $J$ associated with $(\mathcal{A}(W), \Omega)$:

$$\Delta_W^{it} = U(\Lambda_W(-2\pi t)), \ t \in \mathbb{R},$$

and $J$ is the PCT anti-unitary composed with the unitary implementation of the change of sign of the $x_2, x_3$-coordinates. Therefore $U(\Lambda_W(t))$ implements a one-parameter automorphism group $\alpha_t$ of $\mathcal{A}(W)$ that satisfies the Kubo-Martin-Schwinger equilibrium condition at inverse temperature $\beta = 2\pi$ with respect to the restriction of the vacuum state $\omega = (\Omega, \Omega)$ to $\mathcal{A}(W)^4$; in other words, by restriction to $\mathcal{A}(W)$, the pure ground state $\omega$ becomes faithful (by the Reeh-Schlieder theorem) and thermal for the geodesic evolution on the Rindler space provided boost transformations.

As already explained, there is a relation of this setting with the Hawking and the Unruh effects, first noted by Sewell [45]. The space-time $W$ can be identified with the outside region of a black hole. Then the observable algebra for the background system of the black hole is $\mathcal{A}(W)$, the corresponding proper Hamiltonian is

$$H = aK = -i \frac{d}{dt} U(\Lambda_W(at))|_{t=0},$$

where $a$ is the surface gravity of the black hole, and the dynamics in the Heisenberg picture is given by

$$\alpha_{at}(X) = e^{iHt} X e^{-iHt}, \ X \in \mathcal{A}(W).$$

Accordingly $\omega|_{\mathcal{A}(W)}$ is a KMS state (i.e. Gibbs state at infinite volume [26]) at inverse Hawking temperature $\beta = \frac{a}{2\pi}$. We refer to [25,47] for more details and further literature.

We shall consider the black hole as a heat reservoir for its background system and treat the latter as an open system.

Because of the particle production due to Hawking effect, the background system undergoes a change in its quantum numbers, namely the system goes in different superselection sectors, and we shall consider the thermal equilibrium charged state corresponding to a given sector.

We thus consider a superselection sector, namely the unitary equivalence class of a representation $\rho$ of the quasi-local $C^*$-algebra $\mathcal{A}$, the norm closure $\bigcup \mathcal{A}(\mathcal{O})^-$ of the union of all local algebras associated to bounded regions $\mathcal{O}$. The representation

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4One may start with a modular covariance condition and encode the space-time symmetries intrinsically into the net structure [9].
\(\rho\) is assumed to be localizable in each space-like cone \(S^5\), namely \(\rho\) and the identity (vacuum) representation have unitarily equivalent restrictions to the \(C^*\)-algebra \(\cup\{A(\mathcal{O}), \mathcal{O} \subset S', \mathcal{O} \text{ bounded}\}\) generated by the local observables in \(S'\).

We may then assume \(S \subset W\) and, by identifying the representation Hilbert spaces, \(\rho\) to act as the identity on \(A(O)\) if \(O \subset S'\), namely \(\rho\) is a DHR localized endomorphism \([18]\). By wedge duality (consequence of the Bisognano-Wichmann theorem)

\[
\mathcal{A}(W') = \mathcal{A}(W)',
\]

therefore \(\rho\) restricts to a normal endomorphism of \(\mathcal{A}(W)\), still denoted by \(\rho\) (more precisely \(\rho\) restrict to \(\mathcal{A}(W) \cap \mathcal{A}\) and has a normal extension to \(\mathcal{A}(W)\)).

We assume that \(\rho\) is irreducible and Poincaré covariant with positive energy-momentum, namely there exists a unitary representation \(U_{\rho}\) of universal covering group \(\tilde{\mathcal{P}}^+_+\) of the the Poincaré group such that

\[
U_{\rho}(g)\rho(X)U_{\rho}(g)^* = \rho(U(g)XU(g)^*), \quad X \in \mathcal{A}, \; g \in \tilde{\mathcal{P}}^+_+.
\] (3.1)

The covariance is automatic under general conditions \([23]\).

As shown in \([18]\), the notion statistics is intrinsically associated with \(\rho\), in particular the statistical dimension \(d(\rho)\) is defined and turns out to be a positive integer or \(+\infty\). We shall assume \(d(\rho) < \infty\). By the index-statistics theorem \([37]\)

\[
\text{Ind}(\rho) = d(\rho)^2,
\]

where \(\text{Ind}(\rho)\) is the minimal index of \(\rho|_{\mathcal{A}(W)}\), so we may equivalently assume that the restriction of \(\rho\) to \(\mathcal{A}(W)\) has finite index.

The representation \(U_{\rho}\) giving the covariance is uniquely defined by formula (3.1): since \(\rho\) is irreducible any other representation would differ from \(U_{\rho}\) by a one-dimensional character of \(\tilde{\mathcal{P}}^+_+\), that has to be trivial because \(\tilde{\mathcal{P}}^+_+\) has no non-trivial finite-dimensional unitary representation.

Now

\[
z(\rho, g) = U_{\rho}(g)U(g)^*
\]

is a \(\text{Ad}U(g)\)-cocycle and is localized, in particular \(z(\rho, g)\) belongs to \(\mathcal{A}(W)\) if also \(\rho \cdot \text{Ad}U(g)\) is localized in \(W\). In particular \(z(\rho, \Lambda_W(t))\) is a \(\alpha_t\)-cocycle localized in \(W\). As a consequence we have:

**Lemma 3.1** \(\alpha_t^\rho(X) = U_{\rho}(\Lambda_W(t))XU_{\rho}(\Lambda_W(-t)), \; X \in \mathcal{A}(W)\) defines a one-parameter automorphism group \(\alpha_t^\rho\) of \(\mathcal{A}(W)\).

**Proof.** We have

\[
\alpha_t^\rho(\mathcal{A}(W)) = z(\rho, \Lambda_W(t))\alpha_t(\mathcal{A}(W))z(\rho, \Lambda_W(t))^* = z(\rho, \Lambda_W(t))\mathcal{A}(W)z(\rho, \Lambda_W(t))^* = \mathcal{A}(W)
\]

\(^5\)This class exhausts all the translation covariant, positive energy representations with an isolated mass shell \([12]\), but possibly not charges with long range interactions.
because \( z(\rho, \Lambda_W(t)) \) belongs to \( \mathcal{A}(W) \).

The one-parameter automorphism group \( \alpha^\rho_{at} \) is the dynamics of our system in the sector \( \rho \) and the corresponding proper Hamiltonian is given by

\[
H_\rho = aK_\rho = -i \frac{d}{dt} U_\rho(\Lambda_W(at))|_{t=0}
\]

Theorem 2.1, or equivalently formula (2.4), has its version here, by an analogous proof,

\[
d(\rho) = (e^{-\beta H_\rho \Omega, \Omega})|_{\beta=\frac{2\pi}{a}} = (e^{-2\pi K_\rho \Omega, \Omega})
\]

where \( \Omega \) is the vacuum vector or any other cyclic vector for \( \mathcal{A}(W) \) such that \( (\cdot, \Omega) \) coincides with the vacuum state \( \omega \) on \( \mathcal{A}(W') \).

As in Proposition 2.8 we have a normal faithful state of \( \mathcal{A}(W) \) given by

\[
\varphi_\rho(XX^*) = d(\rho)^{-1} \| e^{-\pi K_\rho \Omega} X \|^2, \; X \in \mathcal{A}(W).
\]

**Lemma 3.2** If \( \rho \) is irreducible then the one-parameter automorphism group \( \alpha^\rho_{at} \) is ergodic on \( \mathcal{A}(W) \), namely its fixed points are the scalars.

**Proof.** The proof is similar to the one given in [24] in the case of a conformal theory. Details will be given somewhere else.

Next we show that the system in the sector \( \rho \) admits a thermal equilibrium state at the same Hawking inverse temperature \( \beta = 2\pi/a \).

**Theorem 3.3** \( \alpha^\rho_{at} \) admits a unique normal KMS state \( \varphi_\rho \) at inverse temperature \( \beta = \frac{2\pi}{a} \). The state \( \varphi_\rho \) is given by the equation (3.3) or equivalently by

\[
\varphi_\rho = \omega \Phi_\rho
\]

where \( \Phi_\rho \) is the minimal left inverse of \( \rho \) on \( \mathcal{A}(W) \) and, if \( \rho \) is irreducible, \( \varphi_\rho \) is the unique normal \( \alpha^\rho \)-invariant state of \( \mathcal{A}(W) \). If \( \beta \neq \frac{2\pi}{a} \), no \( \alpha^\rho_{at} \)-KMS normal state exists.

**Proof.** The situation is similar to the one discussed in the previous section. Again, relying on on formula (3.2) and Proposition 1.7, we see that the Connes cocycle \( (D\varphi_\rho : D\omega_{|\mathcal{A}(W)})_t \), where \( \varphi_\rho \) is defined by eq. (3.4), is equal to \( d(\rho)^{-it} z(\rho, \Lambda_W(-2\pi t)) \), thus the modular group of \( \varphi_\rho \) is given by

\[
\sigma_t^{\varphi_\rho} = \alpha^\rho_{-2\pi t}
\]

i.e. \( \varphi_\rho \) is \( \alpha^\rho_{at} \)-KMS at \( \beta = 2\pi/a \). The non-existence of normal states at different temperatures is an immediate consequence of the outerness of \( \sigma^{\varphi_\rho} \) (cf. [13]), since \( \mathcal{A}(W) \) is a type III_1 factor (see [36]).

The uniqueness of \( \varphi_\rho \) as a normal \( \alpha^\rho \)-invariant state is equivalent to the ergodicity of \( \alpha^\rho \), hence a consequence of Lemma 3.2.
Now a finite volume consideration (see Appendix B) suggests that to regard $(e^{-\beta H_\rho}, \Omega)$ as the ratio of the (here undefined) partition functions $Z_0(\beta)$ of the state $\omega$ and $Z_\rho(\beta)$ of the state $\varphi_\rho$, namely
\[
\log(e^{-\beta H_\rho}, \Omega, \Omega) = \log Z_0(\beta) - \log Z_\rho(\beta),
\]
whence we expect the quantity
\[
F(\omega|\varphi_\rho) = -\beta^{-1} \log(e^{-\beta H_\rho}, \Omega, \Omega)
\]
to represent the increment of the free energy between $\omega$ and $\varphi_\rho$. We shall see the above formula to hold true in a precise sense.

We define the relative free energy $F(\omega|\varphi_\rho)$ between the states $\omega$ and $\varphi_\rho$ by
\[
F(\omega|\varphi_\rho) = \varphi_\rho(H_\rho) - \beta^{-1} S(\omega|\varphi_\rho)
\]
where $S(\omega|\varphi_\rho) = S(\omega|\mathcal{A}(W)|\varphi_\rho)$ is the Araki relative entropy of the two states on $\mathcal{A}(W)$,
\[
S(\omega|\varphi_\rho) = -(\log \Delta_{\Omega,\xi_\rho,\xi_\rho});
\]
here $\Delta_{\Omega,\xi_\rho}$ is Araki’s relative modular operator of $\mathcal{A}(W)$ associated with the two cyclic separating vectors $\Omega$ and $\xi_\rho$, where $\xi_\rho$ is any cyclic vector such that $\varphi_\rho = (\cdot, \xi_\rho,\xi_\rho)$ on $\mathcal{A}(W)$. In particular we may assume $\xi_\rho$ to belong to the natural cone associated with $(\mathcal{A}(W), \Omega)$.

The quantity $\varphi_\rho(H_\rho) = (H_\rho \xi_\rho,\xi_\rho)$ in (3.6) represents the relative mean energy in between $\omega$ and $\varphi_\rho$, indeed according to formula (3.5) this has to be given, anticipating Proposition 3.5, by
\[
\frac{d}{d\beta} \log(e^{-\beta H_\rho}, \Omega, \Omega) = \frac{d(\rho)(H_\rho \xi_\rho,\xi_\rho)}{e^{-\beta H_\rho}, \Omega, \Omega} = \frac{(H_\rho \xi_\rho,\xi_\rho)}{\Delta_{\Omega,\xi_\rho}} = \varphi_\rho(H_\rho);
\]
where $J$ is the modular conjugation of both $\Omega$ and $\xi_\rho$ and we have set $\beta = 2\pi/a$ and applied formula (3.2).

More directly one may define the relative mean energy by the formal expression
\[
\tilde{\varphi}_\rho(H_\rho) = \varphi_\rho(H_\rho - H)
\]
where $H_\rho - H$ is the relative Hamiltonian, $\tilde{\varphi}_\rho = (\cdot, \xi_\rho,\xi_\rho)$ and one sets $\tilde{\varphi}_\rho(H) = 0$ motivated by the fact that $(e^{-iHt}\xi_\rho,\xi_\rho)$ is a real even function (because $\Delta_{\Omega}^{it}$ preserves the natural cone) picking a maximum at $t = 0$.

The relative entropy $S(\omega|\varphi_\rho)$ is always non-negative, but it may be equal to $+\infty$, as no volume renormalization has been made (cf. Appendix B); also the relative mean energy $\varphi_\rho(H_\rho)$ may be infinite, but we shall show that the relative free energy between $\varphi_\rho$ and $\omega$ is finite, so in particular $\varphi_\rho(H_\rho)$ shall be bounded below.

Formula (3.6) will have the obvious rigorous meaning as
\[
F(\omega|\varphi_\rho) = (H_\rho + \beta^{-1} \log \Delta_{\Omega,\xi_\rho,\xi_\rho}).
\]
**Theorem 3.4** The relative free energy between the thermal equilibrium states $\omega$ and $\varphi_\rho$ is proportional to the Connes-Størmer entropy of the sector $\rho$:

$$ F(\omega|\varphi_\rho) = -\frac{1}{2} \beta^{-1} S_c(\rho). $$

Here $S_c(\rho)$ denotes the conditional entropy $H(\mathcal{A}(W)||\rho(\mathcal{A}(W))$ (see (1.17), (1.18))

$$ S_c(\rho) = \sum \{ \sum_{i=1}^n S(\varphi_\rho|\varphi_i) - S(\varphi_\rho|\rho) \} $$

where $(\varphi_i)$ varies among the sets of finitely many normal positive linear functionals $\varphi_1, \varphi_2, \ldots, \varphi_n$ of $\mathcal{A}(W)$ such that $\sum_{i=1}^n \varphi_i = \varphi_\rho$.

We note the extensive property of $S_c$: $S_c(\rho_1 \rho_2) = S_c(\rho_1) + S_c(\rho_2)$ [38].

**Proposition 3.5** We have

$$ \beta H_\rho = 2\pi K_\rho = -\log \Delta_{\Omega, \xi_\rho} - \frac{1}{2} S_c(\rho). $$

**Proof.** By Araki’s formula

$$ (D\varphi_\rho : D\omega|_{\mathcal{A}(W)})_t = \Delta_{\Omega, \xi_\rho}^{it} \Delta_{\Omega}^{-it} $$

therefore

$$ u_W(\rho, t) = (Dd(\rho) \varphi_\rho : D\omega|_{\mathcal{A}(W)})_t = d(\rho)^{it} \Delta_{\Omega, \xi_\rho}^{it} \Delta_{\Omega}^{-it}. $$

On the other hand by our formula

$$ u_W(\rho, t) = z(\rho, \Lambda_W(-2\pi t)) = e^{i2\pi K_\rho t} e^{-i2\pi Kt} $$

therefore

$$ e^{-i2\pi K_\rho t} e^{i2\pi Kt} = d(\rho)^{it} \Delta_{\Omega, \xi_\rho}^{it} \Delta_{\Omega}^{-it} $$

and as $\Delta_{\Omega}^{it} = U(\Lambda_W(-2\pi t)) = e^{-iKt}$ we see that

$$ e^{-i2\pi t K_\rho} = d(\rho)^{it} \Delta_{\Omega, \xi_\rho}^{it} $$

so the proposition is obtained by differentiating this expression at $t = 0$.

An alternative argument will appear in Lemma 3.10.

**Proof of Theorem 3.4.** By evaluating on $\varphi_\rho$ both sides of formula (3.7) we have

$$ \beta(H_\rho \xi_\rho, \xi_\rho) = -(\log \Delta_{\Omega, \xi_\rho} \xi_\rho, \xi_\rho) - \log d(\rho). $$

On the other hand $d(\rho)$ is the square root of the minimal index of $\rho|_{\mathcal{A}(W)}$ [37], thus by the Pimsner-Popa equality (1.18) it follows that $\log d(\rho) = \frac{1}{2} S_c(\rho)$ hence proving the theorem. □
Corollary 3.6  The possible values of the relative free energy with initial state $\omega$ are

$$F(\omega|\varphi_\rho) = -\frac{1}{2}\beta^{-1}\log(n), \quad n = 1, 2, 3, \ldots$$

Proof. Immediate by the DHR theorem [18] to the effect that the statistical dimension is a positive integer or $+\infty$. □

Therefore the integer $n$, expressing the order of the parastatistics in [18], here appears as a quantum number labeling the relative free energy levels.

In low space-time dimensions the quantization of the conditional entropy is less restrictive. By Jones theorem [35] and the results in [38,43] we have however restrictions for the possible values of the relative free energy associated with a planar black hole:

Corollary 3.7  In low dimensions the possible values of $e^{-\beta F(\omega|\varphi_\rho)}$ are restricted to $4\cos^2(\frac{\pi}{n})$ in the interval $(0, 4)$. No value in $(4, 5)$ is possible. In the interval $(5, 6)$ only 3 values are possible.

Proof. The first assertion follows from [35], because $e^{-\beta F(\omega|\varphi_\rho)}$ is an index. The rest of the statement is a consequence of the further restrictions on the index values due to the occurrence of the braid group symmetry [38,43]. □

3.2 The increment of the free energy between arbitrary thermal equilibrium states.

Let $\sigma$ be another endomorphism localized in $W$ and $\xi_{\sigma}$ the cyclic separating vector for $A(W)$ such that

$$(X\xi_{\sigma}, \xi_{\sigma}) = \varphi_\sigma(X), \quad X \in A(W)$$

lying in the natural cone associated with $(A(W), \Omega)$.

To extend the definition (3.6) for the free relative energy to the case the initial state is an arbitrary thermal state $\varphi_\sigma$, we note first that the formal relative Hamiltonian between $\varphi_\sigma$ and $\varphi_\rho$ is $H_\rho - H_\sigma$ and hence the relative mean internal energy should be formally given by

$$\tilde{\varphi}_\rho(H_\rho - H_\sigma) = \tilde{\varphi}_\rho(H_\rho - H_\sigma - H) = \tilde{\varphi}_\rho(H_\rho + H_\sigma - H).$$

Here the conjugate charge given by $\tilde{\sigma} = j \cdot \sigma \cdot j$ (see [23]) is localized in $W'$ and $H_{\tilde{\sigma}} = JH_\sigma J = -H_\sigma$.

These premises and the following Lemma will motivate the definition (3.9) below.

Lemma 3.8  We have

$$e^{itH_{\rho\sigma}} = e^{itH_\rho}e^{-itH}e^{itH_{\tilde{\sigma}}}. $$

Proof. By the cocycle property

$$z(\rho\tilde{\sigma}, \Lambda_W(t)) = \rho(z(\tilde{\sigma}, \Lambda_W(t)))z(\rho, \Lambda_W(t)) = z(\tilde{\sigma}, \Lambda_W(t))z(\rho, \Lambda_W(t))$$
because $z(\bar{\sigma},\Lambda W(t))$ is localized in $W'$ and $\rho$ acts identically on $A(W')$ so the Lemma is obtained by multiplying on the right by $e^{-itK}$ the above expression. \(\square\)

We thus define the relative free energy between $\varphi_\sigma$ and $\varphi_\rho$ by

$$F(\varphi_\sigma|\varphi_\rho) = \tilde{\varphi}_\rho(H_{\rho\sigma}) - \beta^{-1}S(\varphi_\sigma|\varphi_\rho)$$

and we give to this expression a rigorous meaning as done for the expression (3.6).

Alternatively, extending the considerations in the previous subsection, we could interpret directly $\log(e^{-\beta H_{\rho\sigma}}\xi_\sigma,\xi_\sigma)$ as the increment of the logarithm of the partition function between the states $\varphi_\sigma$ and $\varphi_\rho$, leading to the expression

$$F(\varphi_\sigma|\varphi_\rho) = -\beta^{-1}\log(e^{-\beta H_{\rho\sigma}}\xi_\sigma,\xi_\sigma).$$

**Theorem 3.9** The relative free energy is given by

$$F(\varphi_\sigma|\varphi_\rho) = \frac{1}{2}\beta^{-1}(S_c(\sigma) - S_c(\rho)).$$

As a consequence $e^{-\beta F(\varphi_\sigma|\varphi_\rho)}$ is equal to the rational number $\frac{d(\rho)}{d(\sigma)}$.

**Lemma 3.10** Let $\rho$ be as above, $\rho'$ an endomorphism localized in $W'$ and $\varphi_{\rho'} = \omega \cdot \Phi_{\rho'}|_{A(W')}$, where $\Phi_{\rho'}$ is minimal left inverse of $\rho'$. Then

$$\Delta(\varphi_{\rho'}|\varphi_\rho) = \frac{d(\rho)}{d(\rho')} e^{-2\pi K_{\rho\rho'}}$$

where $\Delta(\cdot|\cdot)$ denotes the Connes spatial derivative.

**Proof.** Setting $\omega' = \omega|_{A(W')}$ and using [14] one has

$$\Delta(\varphi_{\rho'}|\varphi_\rho) = (D\varphi_{\rho'} : D\omega')_t \Delta(\omega'|\varphi_\rho) = d(\rho)^t(D\varphi_{\rho'} : D\omega')_te^{-i2\pi t K_{\rho'}}$$

$$= d(\rho)^t d(\rho')^{-it} e^{-i2\pi t K_{\rho'}} e^{i2\pi t K_{\rho}} e^{-i2\pi t K_{\rho'}}$$

$$= d(\rho)^t d(\rho')^{-it} z(\rho',\Lambda W(-2\pi t))z(\rho,\Lambda W(-2\pi t)) e^{-i2\pi t K}$$

$$= d(\rho)^t d(\rho')^{-it} z(\rho',\Lambda W(-2\pi t))z(\rho,\Lambda W(-2\pi t)) e^{-i2\pi t K}$$

$$= d(\rho)^t d(\rho')^{-it} e^{-i2\pi t K_{\rho\rho'}}.$$
Corollary 3.11 We have
\[ \Delta_{\xi_{\sigma},\xi_{\rho}} = \frac{d(\rho)}{d(\sigma)} e^{-2\pi K_{\rho\bar{\sigma}}}. \]

Proof. Immediate by the above discussion and the relation \( \Delta_{\xi_{\sigma},\xi_{\rho}} = \Delta(\varphi_{\sigma} \cdot \text{Ad}J|\varphi_{\rho}) \), where \( \varphi_{\sigma} \cdot \text{Ad}J \) is the vector state \( (\cdot_{\sigma},\xi_{\sigma}) \) on \( \mathcal{A}(W') \).

\[ \square \]

Proof of Theorem 3.9. By the above Corollary we have
\[ \beta H_{\rho\bar{\sigma}} = \beta a K_{\rho\bar{\sigma}} = -\log \Delta_{\xi_{\sigma},\xi_{\rho}} + \frac{1}{2}(S_{\mathcal{C}}(\sigma) - S_{\mathcal{C}}(\rho)) \]
and this clearly implies the desired relation. \( \square \)

Appendix A. Tensor categories and cocycles.
The purpose of this Appendix is to shed light on part of the mathematical structure underlying our results. Indeed a good part of our results depends only on the tensor categorical structure provided by the superselection sectors and are therefore visible without a more detailed description of the theory.

Let \( \mathcal{T} \) be a strict \( C^* \)-tensor category. We assume \( (\iota,\iota) = \mathbb{C} \), where \( \iota \) is the identity object and \( (\cdot,\cdot) \) denotes the intertwiner space. We refer to [41] for the basic notions used here.

A basic and originating example for \( \mathcal{T} \), appearing in [18,19], is obtained by taking \( \text{End}(M) \), \( M \) a unital \( C^* \)-algebra with trivial centre, to be the set of objects, and arrows between objects \( \rho \) and \( \rho' \) given by
\[ (\rho,\rho') = \{ T \in M, T \rho(x) = \rho'(x) T, \forall x \in M \} \]
while the tensor product is given by the composition of maps
\[ \rho \otimes \rho' = \rho \cdot \rho' \]
\[ T \otimes S = \rho_2'(T)S = S \rho_1'(T), \quad T \in (\rho_1,\rho_2), \quad S \in (\rho_1',\rho_2'). \]
The reader unfamiliar with abstract tensor categories might at first focuses on this particular case.

Given an object \( \rho \) of \( \mathcal{T} \), an object \( \bar{\rho} \) of \( \mathcal{T} \) is said to be a conjugate of \( \rho \) if there exist \( R_{\rho} \in (\iota,\bar{\rho} \rho) \) and \( \bar{R}_{\rho} \in (\iota,\rho \bar{\rho}) \) such that
\[ \bar{R}_{\rho}^* \otimes 1_{\rho} \circ 1_{\rho} \otimes R_{\rho} = 1_{\rho}; \quad R_{\rho}^* \otimes 1_{\bar{\rho}} \circ 1_{\bar{\rho}} \otimes \bar{R}_{\rho} = 1_{\bar{\rho}}. \] (A.1)

We shall assume that each object \( \rho \) has a conjugate \( \bar{\rho} \) (this is automatic in \( \text{End}(M) \) [38] if \( M \) is an infinite factor and \( \rho \) has finite index) and shall refer to (A.1) as to the conjugate equation for \( \rho \) and \( \bar{\rho} \). The equation (A.1) has then a standard solution, namely one can choose multiples of isometries \( R_{\rho} \) and \( \bar{R}_{\rho} \) in (A.1) so that
∥R_ρ∥ = ∥R_ρ∥ = \sqrt{d(\rho)} is minimal. This formula defines d(\rho), the dimension of \rho [41].

Now recall that given an arrow T ∈ (\rho, \rho'), the conjugate arrow T^* ∈ (\bar{\rho}, \bar{\rho}') is defined by

\[ T^* = 1_{\bar{\rho}} \otimes \bar{R}_{\rho}^* \circ 1_{\bar{\rho}} \otimes T^* \circ 1_{\bar{\rho}} \otimes R_{\rho'} \otimes 1_{\bar{\rho}} \in (\bar{\rho}, \bar{\rho}'), \]

where \bar{R}_{\rho} and R_{\rho'} are multiples of isometries in the standard solution for the conjugate equations defining the conjugates \bar{\rho} and \bar{\rho}'. The mapping T ↦ T^* is antilinear and enjoys in particular the following properties

a) \(1^*_\rho = 1_{\bar{\rho}},\)
b) \(S^* \circ T^* = (S \circ T)^*\)
c) \(T^{**} = T^*\).

We shall say that \(\alpha\) is an (anti-)automorphism of \(\mathcal{T}\) if \(\alpha\) is an invertible functor of \(\mathcal{T}\) with itself, (anti-)linear on the arrows, commuting with the \(^*\)-operation and preserving tensor products. The action of \(\alpha\) on the object \(\rho\) and on the arrow \(T\) will be denoted by \(\rho^\alpha\) and \(T^\alpha\).

Given an automorphism \(\alpha\) of \(\mathcal{T}\) a cocycle \(u\) with respect to \(\alpha\) is a map

\[ \rho \in \text{Obj}(\mathcal{T}) \to u(\rho) \text{ unitaries in } (\rho, \rho^\alpha) \]

such that

a) \(u(\rho \otimes \rho') = u(\rho) \otimes u(\rho')\)
b) If \(T \in (\rho, \rho')\) then the following diagram commutes

\[ \begin{array}{ccc}
\rho & \xrightarrow{T} & \rho' \\
\downarrow{u(\rho)} & & \downarrow{u(\rho')} \\
\rho^\alpha & \xrightarrow{T^\alpha} & \rho'^\alpha
\end{array} \]

(A.2)

**Proposition A.1** Let \(u\) be a cocycle with respect to \(\alpha\) as above. Then \(u(\bar{\rho}) = u(\rho)^*\).

**Proof.** The proof is obtained similarly as in Proposition 1.5. \(\square\)

**Lemma A.2** Let \(u\) be a cocycle with respect to \(\alpha\) as above and \(j\) be an anti-automorphism of \(\mathcal{T}\). Then \(\rho \to u(\bar{\rho})^j\) is a cocycle with respect to \(j^\alpha j^{-1}\).

**Proof.** The statement is checked by a direct verification. \(\square\)

We now give two uniqueness results that are at the basis of the identifications of the covariance cocycle and the Connes cocycle in this paper.
Lemma A.3 With the notations in Lemma A.2, assume that \( j \alpha j^{-1} = \alpha^{-1} \) and \( \rho^j \) is a conjugate of \( \rho \) for all objects \( \rho \). Then the cocycle \( u \) with respect to \( \alpha \) is unique.

Proof. If \( u' \) is a cocycle with respect to \( \alpha \) and \( \rho \) is irreducible, then \( u'(\rho) = \mu(\rho)u(\rho) \) for some phase \( \mu(\rho) \). By Proposition A.1 \( \mu(\rho) = \bar{\mu}(\rho) \), while by Lemma A.2 \( \mu(\rho) = \mu(\rho) \), hence \( \mu = 1 \) on the irreducibles, thus always because a cocycle is determined by its value on the irreducible objects. \( \square \)

Let now \( G \) be a group and \( \alpha \) an action of \( G \) on \( T \), namely a homomorphism \( g \to \alpha_g \) of \( G \) into the automorphism group of \( T \) (for simplicity we omit topological assumptions).

For any \( \rho \in T \) and \( g \in G \), let \( u(\rho, g) \) be a unitary in \( (\rho^g, \rho) \) (where \( \rho^g \equiv \rho^{\alpha_g} \)). We shall say that \( u \) is a two-variable cocycle if:

a) For any fixed \( g \in G \), \( u(\cdot, g)^* \) is a cocycle with respect to the automorphism \( \alpha_g \).

b) For any fixed \( \rho \in T \), \( u(\rho, \cdot) \) is a \( \alpha \)-cocycle, namely \( u(\rho, gh) = u(\rho, g)u(\rho, h)^{\alpha_g} \).

Proposition A.4 Let \( u \) be a two-variable cocycle as above. If \( G \) is perfect (i.e. \( G \) has no non-trivial one-dimensional unitary representation), then \( u \) is unique.

Proof. As in the proof of Lemma A.3, if \( \rho \) is an irreducible object, a second two-variable cocycle would give rise to a phase \( \mu(\rho, g) \) that, for a fixed \( \rho \), would be a one-dimensional character of \( G \), and thus had to be trivial. \( \square \)

Appendix B. The relative free energy at finite volume.

A finite volume computation with canonical distribution may clarify the notion of relative free energy \( F \) in (3.6). Let the Hamiltonians of the evolutions \( \alpha^{(0)} \) and \( \alpha^{(1)} \) be given by positive selfadjoint operators \( H_0 \) and \( H_1 \), so that \( \alpha^{(k)} \) is implemented by \( e^{itH_k} \). The Gibbs state \( \omega_{\beta}^{(k)} \) for \( \alpha^{(k)} \) is given by

\[
\omega_{\beta}^{(k)} = \text{Tr}(\rho_{k} \cdot)
\]

with density matrix

\[
\rho_{k} = Z_{k}(\beta)^{-1} e^{-\beta H_k}
\]

where \( Z_{k}(\beta) = \text{Tr}(e^{-\beta H_k}) \) is the partition function.

Then

\[
F_{k} = \omega_{k}(H_{k}) - \beta^{-1} S(\rho_{k}) = -\beta^{-1} \log Z_{k}(\beta)
\]

is the Helmholtz free energy in the state \( \omega^{(k)} \), where the entropy in state \( \omega^{(k)} \) is given by

\[
S(\rho_{k}) = -\text{Tr}(\rho_{k} \log \rho_{k})
\]

Then the relative entropy is given by (see [51])

\[
S(\omega_{\beta}^{(0)}|\omega_{\beta}^{(1)}) = -\text{Tr}(\rho_{0} \log \rho_{0} - \rho_{0} \log \rho_{1}) = -\omega_{\beta}^{(0)}(\log \rho_{0} - \log \rho_{1})
\]

\[
= \beta \omega_{\beta}^{(0)}(H_{\text{rel}}) + \log Z_{0}(\beta) - \log Z_{1}(\beta)
\]
where $H_{\text{rel}} = H_0 - H_1$ is the relative Hamiltonian.

The relative free energy is thus given by

$$F(\omega^{(0)}_\beta|\omega^{(1)}_\beta) = \omega^{(0)}_\beta (H_{\text{rel}}) - \beta^{-1}S(\omega^{(0)}_\beta|\omega^{(1)}_\beta)$$

$$= \beta^{-1} \log Z_0(\beta) - \beta^{-1} \log Z_1(\beta) = F_1 - F_0.$$ 

Had we considered a gran canonical distribution on the Fock space, the Hamiltonian for $\alpha_t^{(k)}$ would have been implemented by $e^{it(H_k - \mu_k N_k)}$, with $\mu_k$ the chemical potential and $N_k$ the number operator, and the above expression for $F_k$ would have had accordingly modified.

We note explicitly that

$$e^{-\beta F(\omega^{(0)}_\beta|\omega^{(1)}_\beta)} = \frac{\text{Tr}(e^{-\beta H_1})}{\text{Tr}(e^{-\beta H_0})}$$

providing evidence to the analogy between formulae (0.1) and (0.2).

**Final comments.**

As mentioned, the Rindler space-time is a good approximation of the Schwarzschild space-time only near the horizon. However a version of our results within the context of the Kruskal extension of the Schwarzschild space-time should be possible, as a version of the Bisognano-Wichmann theorem and a model independent derivation of the Hawking temperature has been given in this setting [46].

Another point to comment on is related to the use of the Minkowski vacuum associated with Poincaré symmetries. As is known there exists no vacuum state for a quantum field theory on a general curved space-time. In such a general theory the relative free energy could however be defined by consistency with the fusion rules of the supeselection structure and it seems that our results may be achieved in wider contexts.

Concerning the expression (3.6) for the relative free energy, it would be physically meaningful to derive it by a finite volume approximation, where its expression is given in Appendix B. To this end one should use the split property and the Noether currents, see [11], and this approach might also be useful for the here above discussed extension to more general curved spacetimes. Moreover the development of such techniques could bring up to a model independent derivation of the formula (0.1).

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