Duality in Quantum Toda theory and W-algebras

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ABSTRACT

We consider Quantum Toda theory associated to a general Lie algebra. We prove that the conserved quantities in both conformal and affine Toda theories exhibit duality interchanging the Dynkin diagram and its dual, and inverting the coupling constant. As an example we discuss the conformal Toda theories based on $B_2$, $B_3$ and $G_2$ and the related affine theories.
1 Introduction

It was noticed some time ago that there is a relation between the conformal Toda theory based on a simply-laced algebra $g$ with coupling constant $\beta$, and $1/\beta$. This has also been extended to the case of a non-simply-laced Lie algebra $g$; here the conformal Toda theory given by $g$, with coupling constant $\beta$ is related to the Toda theory of $g^\vee$ with coupling constant $1/\beta$. (The Dynkin diagram of $g^\vee$ is obtained by changing the directions of all the arrows which appear in the Dynkin diagram of $g$.)

This relation was first deduced in the Liouville model, which is $sl(2)$ Toda theory. A necessary requirement that the Liouville theory be conformally invariant is that the potential term has conformal weight $(1,1)$. One may obtain the Virasoro algebra from the Lagrangian by adding a conformal improvement to the stress-energy tensor. This then ensures that the potential term has the correct weight. Having obtained the Virasoro algebra for the Liouville theory at coupling constant $\beta$, there is no other value $\beta'$ for which the potential term in the classical Liouville theory with coupling constant $\beta'$ has weight $(1,1)$ with respect to the Virasoro algebra in the Liouville theory with coupling constant $\beta$.

In the quantum theory, however, there are $O(\hbar)$ corrections to the stress-energy tensor which allow the potential term in the quantum theory with coupling constant $1/(\beta\hbar)$ to have weight $(1,1)$ with respect to the Virasoro algebra in the Liouville theory with coupling constant $\beta$.

This was used by Mansfield [1] to explain a problem that was present in the interpretation of the Liouville model at imaginary coupling constant as the Lagrangian theory of the conformal minimal models; this problem being that the quantum solutions to the equations of motion only generated the $(1,q)$ or $(p,1)$ conformal primary fields. With the addition of the potential term of the ‘dual’ theory, all the conformal primary fields $(p,q)$ could be obtained.

In the case of the Liouville theory, the Virasoro algebra is the full chiral algebra. For Toda theories based on more general simply laced algebras, one obtains a W-algebra. The Drinfeld-Sokolov construction for a Hamiltonian reduction based on the maximal regular embedding of $sl(2) \hookrightarrow g$ allows one to find explicit expressions for the chiral algebras of classical conformal Toda theories, and the conserved quantities in classical affine theories, through the Miura transformation. One finds that the conserved quantities are polynomial in the derivatives of the Toda fields $\phi^i$ and $1/\beta$. When people sought free-field representations of W-algebras (the chiral algebras of conformal Toda theories), they were brought to the idea of quantising the Miura transformation. The expressions found for $WA_n$ and $WD_n$ by Fateev and Luk’yanov [2, 3] are exactly the same as the classical expressions, after normal ordering, with the replacement of the coupling constant $1/\beta$ by $1/\beta - (\hbar/\beta)$. These W-algebras have a representation theory quite analogous to that of the Virasoro algebra, and similarly the naive quantisation of the Lagrangian only generates some of the primary fields. However, these quantum W-algebras also admit an extension to the Lagrangian by the potential term at the inverted coupling constant [1], so that the full set of primary fields can be obtained from a Lagrangian theory which contains the...
potential terms for both coupling constants $\beta$ and $1/(\hbar\beta)$.

When considering the quantisation of the W-algebra symmetries of non-simply laced Toda theories there was much less success, and a quantum construction of the chiral algebra in the $B_2$ Toda theory has only been found very recently [4]. Although the fields in this quantum construction are $O(\hbar)$ corrections to the classical expressions, this does not occur simply as a renormalisation of the coupling constant. This may be seen from the fact that the extra potential terms which may be added to the $B_2$ theory are in fact those of the $C_2$ theory at the dual coupling constant; although $B_2$ and $C_2$ are isomorphic as Lie algebras, their root systems are naturally thought of as rotated by $45^\circ$ with respect to each other. Thus, equally well the W-algebra fields are $O(1/\hbar)$ corrections to the $C_2$ theory expressions. We see that the quantum W-algebras may be thought of as a simple renormalisation of the coupling constants in the classical expressions only when the root spaces of $g$ and $g^\vee$ coincide, which means that $g$ must be simply-laced if it is a Lie algebra; if $g$ is a super-algebra, the same condition holds for $WB(0,n)$ too, for which many results analogous to the simply-laced algebras hold [5]. Thus, the chiral algebra of any Toda theory of a non-simply laced algebra will be $O(\hbar)$ corrections to the classical expression for the chiral algebra, if we replace $\beta$ by $1/(\hbar\beta)$, then they are $O(\hbar)$ corrections to the classical expressions for the chiral algebras of the theory based on $g^\vee$. This result has been very recently proven by Frenkel [6] using homological techniques.

When we come to discuss affine Toda theories we may well ask if similar statements hold. It has long been known that the classical affine Toda theories have an infinite set of conserved quantities [7], with spins given by the exponents of the associated affine Lie algebra. Some initial work was done on quantum affine Toda theory by Eguchi and Yang [8], and by Palla [9]. Feigin and Frenkel have recently shown the existence of a full set of quantum conserved currents through the use of homological techniques [10]. Affine Toda theory splits into two regimes; where the coupling constants is real, and where it is imaginary. The real coupling constant regime is more closely associated to perturbed conformal field theories, but it is for the imaginary coupling constant behaviour that most progress has been made in understanding the particle content of the theories [11, 12, 5]. In this letter we give a short proof that the conserved quantities of the affine theories in these regimes obey the duality already found for the conformal theories. We give several examples based on our calculations for conformal Toda theories [4]. We end with a discussion of the implications of this duality on the mass spectrum for these theories, and the recent conjectures of Grisaru et al. [13] for the scattering matrices of some non-simply laced affine theories.

2 Quantum Toda Theory

The quantum Toda field theory associated to a finite dimensional algebra $g$ has the same Lagrangian as the classical theory,

$$\mathcal{L} = \frac{1}{2} \partial_\mu \phi \partial^\mu \phi - \sum_j m_j e^{\beta \alpha_j \cdot \phi}, \quad (2.1)$$
where $\Sigma(g) = \{\alpha_j\}$ denotes the set of simple roots of $g$, $m_j$ are rank $g$ arbitrary non-zero constants and $\beta$ is the coupling constant of the theory. Removing the divergencies by normal ordering introduces a mass scale \cite{14}. However, the algebraic structure does not depend on the mass scale, and one can set the constants $m_j$ to any arbitrary non-zero values by a constant shift in $\phi$.

We work in light-cone coordinates, $z, \bar{z}$ given by
\begin{equation}
    z = e^{i\sqrt{2}x^+}, \quad \bar{z} = e^{-i\sqrt{2}x^-}, \tag{2.2}
\end{equation}
where $0 \leq x \leq \sqrt{2}\pi$, $x^\pm = (t \pm x)/\sqrt{2}$. We shall also denote $\partial/\partial z$ by $\partial$, and define new fields $X, H$ by
\begin{align}
    X(z, \bar{z}) &= -i\sqrt{2}\phi(x^+, x^-), \tag{2.3} \\
    H(z, \bar{z})dz &= i\sqrt{2}\partial_+\phi(x^+, x^-)dx^+, \tag{2.4}
\end{align}
and similarly for $\bar{H}$. Note that $H = i\partial X$. In the quantum theory the conformal symmetry is generated by an improved stress-energy tensor of the form
\begin{equation}
    T_{zz} = \frac{1}{2}H \cdot H + (a\rho - \rho' / a) \cdot \partial H, \tag{2.5}
\end{equation}
where $\rho = \frac{1}{2}\Sigma^+ \alpha$, $\rho^\prime = \frac{1}{2}\Sigma^+ \alpha^\prime$, $\hbar$ has been set to 1, and $a = \beta/\sqrt{2}$ is the coupling constant. $\Sigma^+$ is the set of positive roots of $g$, and the dual roots are given by $\alpha^\prime = 2\alpha/\alpha^2$.

We find that the improved stress energy tensor is conserved \cite{15} and so we can decompose it into modes as
\begin{align}
    T_{zz} &= \sum_n L_n z^{-n-2}, \quad T_{\bar{z} z} = \sum_n \bar{L}_n \bar{z}^{-n-2}, \tag{2.6}
\end{align}
where $\bar{\partial}L_n = 0$ by the Heisenberg equations of motion. We use light-cone commutation relations, which are of the form
\begin{equation}
    [H^j(z), H^k(z')] = \delta^{jk} \delta'(z - z'). \tag{2.7}
\end{equation}
We expand the fields $X(z, \bar{z})$ and $H(z, \bar{z})$ in equal-$x^-$ modes,
\begin{align}
    X^j(z, \bar{z}) &= q^j(\bar{z}) - ip^j(\bar{z})\ell nz + \sum_{n \neq 0} iH^j(\bar{z})_nz^{-n}/n, \tag{2.8} \\
    H^j(z, \bar{z}) &= \sum_n H^j_n(\bar{z})z^{-n-1} \tag{2.9}
\end{align}
for $x^- < x^+ < x^- + \pi/\sqrt{2}$. We have also set $H_0^j \equiv p^j$. Similarly we expand in equal-$x^+$ modes for $x^+ < x^- < x^+ + \pi/\sqrt{2}$. The commutation relations for the modes are
\begin{align}
    [H^j(\bar{z})_m, H^k(\bar{z})_n] = m\delta^{jk}\delta_{m+n}, \quad [q^j(\bar{z}), p^k(\bar{z})] = i\delta^{jk} \tag{2.10}
\end{align}
We shall in future drop reference to the coordinate $\bar{z}$ where the meaning is unambiguous. We have a Hilbert space for each $\bar{z}$ on which these modes act, with a vacuum $|0\rangle$. A field $\psi(z, \bar{z})$ satisfies
\[
\psi(z, \bar{z})|0\rangle = e^{zL-1}|\psi, \bar{z}\rangle.
\] (2.11)

Thus, we may evaluate the equal-$\bar{z}$ commutation relations of fields and states exactly as in standard conformal field theory [16], providing we remember that the $\bar{z}$ dependence of our states and modes is non-trivial.

If we define $E^j(z, \bar{z})dzd\bar{z} = \oint e^{ia\alpha_j \cdot X(z, \bar{z})}e^{2dx^+dx^-}$ then using standard conformal field theory normal ordering we obtain
\[
E^j(z, \bar{z}) = \exp \left[ \sum_{n>0} \frac{a\alpha_j \cdot H(\bar{z})_n z^n}{n} \right] e^{ia\alpha_j \cdot q(\bar{z})} z^{a\alpha_j \cdot p(\bar{z})}.
\] (2.12)

The quantum equations of motion are
\[
\bar{\partial} H(\bar{z})_n = -\frac{1}{2} \sum_j a\alpha_j \cdot E^j(\bar{z})_n,
\] (2.13)

where $E^j_n$ are the modes of the vertex operator,
\[
E^j(z, \bar{z}) = \sum_n E^j(\bar{z})_n z^{-n-1}.
\] (2.14)

By the uniqueness of vertex operators and the conservation of the stress-energy tensor, to evaluate the derivative of a field, we need only evaluate the derivative of the state to which it corresponds,
\[
\bar{\partial} \psi(z, \bar{z})|0\rangle = \bar{\partial} e^{zL-1}|\psi, \bar{z}\rangle
= e^{zL-1}\bar{\partial}|\psi, \bar{z}\rangle.
\] (2.15)

Thus, we may easily evaluate the derivative of any polynomial in the field $H$.
\[
\bar{\partial}|W, \bar{z}\rangle = -\frac{1}{2} \left[ \sum_{n,j} a\alpha_j \cdot E^j(\bar{z})_{-n} \alpha_j \cdot \frac{\delta}{\delta H(\bar{z})_{-n}} \right] |W, \bar{z}\rangle.
\] (2.17)

Conversely, since $E^j(\bar{z})_0|0\rangle = 0$, and $[L_n, E^j(\bar{z})_0|0\rangle = 0$, we may check directly from the commutation relations that
\[
[E^j(\bar{z})_0, W(0, \bar{z})]|0\rangle = E^j(\bar{z})_0|W, \bar{z}\rangle
= -\left[ \sum_n aE^j(\bar{z})_{-n} \alpha_j \cdot \frac{\delta}{\delta H(\bar{z})_{-n}} \right] |W, \bar{z}\rangle.
\] (2.18)
And thus a field $W[H]$ is chiral if and only if it commutes with all the operators $Q^j = E^j(z)_0$. These operators $Q^j$ are commonly called the screening charges. The Virasoro algebra, generated by the modes of the stress energy tensor, commutes with all the screening charges $Q^j$ and is given by the field corresponding to the state

$$|L\rangle = \left[ \frac{1}{2}(H_{-1} \cdot H_{-1}) + (\alpha - \rho^\vee/a) \cdot H_{-2} \right]|0\rangle .$$ (2.20)

### 3 Proof of duality of chiral algebras

There is another set of screening charges

$$Q^\vee_j = \int dz \circ e^{-i/a\alpha^\vee_j \cdot X}$$ (3.1)

which corresponds to the Toda theory of $g^\vee$, at coupling constant $a^\vee = -1/a$. We find that the W-algebra given by the commutant of the screening charges $Q^j$ of $g$ also commutes with the screening charges $Q^\vee_j$ of $g^\vee$. This has been proven in full generality by Frenkel [6]. For our extension to the affine case we need only consider the regime $a^2$ negative. We think it useful to present here a version of the proof of the duality of the chiral algebra in this regime.

We consider each simple root in turn. We are seeking the commutant of $Q^j$ in the space of fields spanned by polynomials in $H^j$ and their derivatives. Since we know the relation between states and fields (2.11) is unique, $[Q^j, \psi(z)] = 0$ is equivalent to $Q^j|\psi\rangle = 0$. Since the modes $H^j_m$ have the commutation relations (2.7), we can decompose the space of modes at each level into the modes $K^i_m = \alpha^i/|\alpha^i| \cdot H_m$ and an $n-1$ dimensional space of modes which commute with the modes $\alpha^i \cdot H_m$. For each $i$ the Fock space of the modes $H^j_m$ splits into the tensor product of the Fock spaces of the modes $K^i_m$ and the orthogonal modes,

$$H = H_K \otimes H^\perp .$$ (3.2)

Since the action of the operator $Q^i$ can be expressed entirely in terms of the modes $K^i_m$, we see that

$$\text{Ker}_H(Q^i) = H^\perp \otimes \text{Ker}_{H_K}(Q^i) .$$ (3.3)

Thus we only need consider this simpler space $\text{Ker}_{H_K}(Q^i)$. We have chosen the modes $K^i_m$ to have free field commutation relations,

$$[K^i_m, K^i_n] = m\delta_{m+n,0}$$ (3.4)

and the operator $Q^i$ is expressed as $\int dz \exp(i a \int K^i(z)dz)$. We know at least one state in $H_K$ which is not in $\text{Ker}_{H_K}(Q^i)$, this being $\tilde{K}^{-1}_i|0\rangle$, since

$$Q^i\tilde{K}^{-1}_i|0\rangle = a|a\alpha^i\rangle .$$ (3.5)

We also know that the Virasoro algebra $L^i_m$ given by

$$L^i(z) = \frac{1}{2} \circ K^i(z)^{2 \circ} + [a|\alpha^i|/2 - 1/(a|\alpha^i|)] \partial K^i(z)$$ (3.6)
commutes with $Q^i$. Thus the space $\text{Ker}_{\mathcal{H}_K}(Q^i)$ is a representation of the Virasoro algebra $L^i$. Similarly, the states produced by the action of $Q^i$ on $\mathcal{H}_K$, which we denote by $\text{Im}_{\mathcal{H}_K}(Q^i)$ form a representation of the Virasoro algebra. Since $Q^i K_{-1}^i |0\rangle \neq 0$, we know that this representation has at least one state with $h = 1$, and so it must be a highest weight representation with $h = 1$. Using the Kac determinant formula [17], we know that if $a^2$ is negative, then the $h = 1$ representation is irreducible. This representation thus has character

$$\chi_{h=1}(q) = q \prod_{n=1}^{\infty} (1 - q^n)^{-1} \quad (3.7)$$

which must be a lower bound on the character of $\text{Im}_{\mathcal{H}_K}(Q^i)$. Consequently there is an upper bound on the character of the space $\text{Ker}_{\mathcal{H}_K}(Q^i)$, namely

$$(1 - q) \prod_{n=1}^{\infty} (1 - q^n)^{-1} \quad (3.8)$$

$\text{Ker}_{\mathcal{H}_K}(Q^i)$ contains one copy of a highest weight representation with $h = 0$. However, we know the character of the $h = 0$ representation, and it is exactly given by (3.8). Since (3.8) was an upper bound on the character of $\text{Ker}_{\mathcal{H}_K}(Q^i)$ we can deduce that $\text{Ker}_{\mathcal{H}_K}(Q^i)$ is an irreducible highest weight representation with $h = 0$. Thus all the states in $\text{Ker}_{\mathcal{H}_K}(Q^i)$ are given by the action of creation modes of the algebra $L^i$ acting on the vacuum.

So, we can decompose the Fock Space $\mathcal{H}$ as

$$\mathcal{H} = \mathcal{H}^{\perp} \otimes (V_0 \oplus V_1) \quad (3.9)$$

where $V_0$ is given by the action of all creation modes of $L^i$ on $|0\rangle$, and $V_1$ by the action of all creation modes of $L^i$ on $K_{-1}^i |0\rangle$.

Thus we can write a generic state $|\psi\rangle$ in $\mathcal{H}$ as

$$|\psi\rangle = \sum_{j \in I} a_j |p_j\rangle \otimes \left(q_j[L^i]|0\rangle + r_j[L^i]K_{-1}^i |0\rangle\right) \quad (3.10)$$

where $|p_j\rangle$ is an arbitrary state in $\mathcal{H}^{\perp}$ and $q_j, r_j$ are polynomials in the lowering modes of $L^i$. If we consider the condition that this state defines a field which commutes with $Q^i$ we obtain

$$Q^i |\psi\rangle = a \sum_{j \in I} a_j |p_j\rangle \otimes r_j[L^i] |a\alpha_i\rangle = 0 \quad (3.11)$$

We can now ask that the same state defines a field which commutes with the dual screening charge $Q^{\dagger\dagger}$. The $Q^{\dagger\dagger}$ have the property that $[L_i, Q^{\dagger\dagger}] = 0$ and so

$$Q^{\dagger\dagger} |\psi\rangle = (-1/a) \sum_{j \in I} a_j |p_j\rangle \otimes r_j[L^i] - 1/a \alpha^{\dagger\dagger}_i \rangle = 0 \quad (3.12)$$
These are the same equations for the state \(|p_j\rangle\) and the polynomials \(r_j\), and so any state which commutes with \(Q^j\) also commutes with \(Q^{\overline{j}}\),

\[
\text{Ker}_{\mathcal{H}_K}(Q^j) = \text{Ker}_{\mathcal{H}_K}(Q^{\overline{j}}).
\] (3.13)

The chiral algebra of the Toda theory for \(a^2\) negative is given as

\[
\mathcal{A} = \bigcap_i \text{Ker}_{\mathcal{H}_K}(Q^i)
\] (3.14)

and so all the fields in \(\mathcal{A}\) also automatically commute with the operators \(Q^{\overline{j}}\). Thus if we denote the chiral algebra of the Toda theory based on \(g\) with coupling constant \(a\) by \(\mathcal{A}(g, a)\), then we have

\[
\mathcal{A}(g, a) = \mathcal{A}(g^{\overline{\gamma}}, -1/a).
\] (3.15)

Note that in (3.15) the simple roots of \(g^{\overline{\gamma}}\) are \(2\alpha/\alpha^2\), where \(\alpha\) are the simple roots of \(g\).

For \(a^2\) positive, rational, there are subtleties in that the Virasoro representation \(\text{Ker}(Q^j)\) is not irreducible and so the decomposition (3.9) is invalid.

### 4 Affine algebras and examples

The classical affine Toda field theories are integrable field theories of scalar fields related to affine Lie algebras. If \(\Sigma = \{\alpha_i\}_{i=0,\ldots,l}\) is the set of simple roots of an affine Lie algebra \(\hat{g}\) of rank \(l\), then one may define the Lagrangian for a theory of \(l\) scalar fields as

\[
\mathcal{L} = \frac{1}{2} \partial_\mu \phi \partial^\mu \phi - \sum_{i=0}^l m_i e^{\beta \alpha_i \cdot \phi}.
\] (4.1)

We shall again consider this theory on the light-cone, and use the conventions of the previous section. When one considers normal ordering the Lagrangian, one must now be more careful. However, one can show that a change in the mass scale can again be compensated for by a shift in \(\phi\), in the sense that requiring \(\langle \phi \rangle = 0\) implies that the \(m_j = M n_j\), where \(\sum n_j \alpha_j = 0\), and \(M\) is a constant which depends on the mass scale. Thus the form of the Lagrangian and in particular the value of \(\beta\) are independent of the regularisation scheme [18]. For the conformal Toda theories, the conserved quantities were local fields, giving a W-algebra as the symmetry algebra. However, for affine theory this is not the case. Instead of all the modes of a local field commuting with the Hamiltonian, only the integrals of fields will commute. The conserved quantities of this integrable system are given by

\[
\mathcal{I}_a = \int W_a dz,
\] (4.2)

where \(\mathcal{I}_a\) and \(W_a\) satisfy

\[
[\mathcal{H}_0, \mathcal{I}_a] = 0, \quad [\mathcal{H}_0, W_a] = \partial V_a.
\] (4.3)

Here \(\mathcal{H}_0\) is the Hamiltonian of the theory and \(V_a\) are arbitrary local fields. A field \(W\) satisfies (4.3) if and only if

\[
\mathcal{H}_0 |W_a\rangle = L_{-1} |V_a\rangle,
\] (4.4)
where $L_{-1} = \int dz z^\circ H \cdot H^\circ$. The quantum Hamiltonian is

$$
H_0 = \int dz \sum_{j=0}^l m_j \cdot \exp(ia\alpha_j \cdot X)^\circ,
$$

and so, by the algebraic independence of the exponential terms, we see that the conserved quantities $\{I_a\}$ must commute with each of the terms in the Hamiltonian individually. The conserved quantity is called trivial if $W_a = \partial Y_a$ where $Y_a$ is a local field. Since $L_{-1}$ commutes with $H_0$ we have for a trivial conserved quantity $|W_a\rangle = L_{-1} |Y_a\rangle$. Let us denote by $\text{Aff}(g, a, j)$ the space of states $|W\rangle$ such that

$$
Q_j |W\rangle = L_{-1} |V\rangle,
$$

where $Q_j = \int dz \exp(ia\alpha_j \cdot X)$.$^\circ$. Then the total space of states satisfying (4.3), which we denote by $\text{Aff}(g, a)$, is given by

$$
\text{Aff}(g, a) = \bigcap_{i=0}^l \text{Aff}(g, a, i) / L_{-1} \mathcal{H}.
$$

Feigin and Frenkel have recently shown that the space of states so defined has the same dimensionality as in the classical case [10]. There is one conserved quantity of spin $e_a$ for each exponent of the affine algebra $\hat{g}$, corresponding to a state $|W_a\rangle$ of conformal weight $e_a + 1$. We shall now show that in the regime $a^2$ negative, the conserved quantities of the affine theory $\hat{g}$ are also conserved quantities of the theory $\hat{g}^\vee$ with inverted coupling constant.

We shall again consider each simple root in turn and introduce the operators $K_m$ associated to the particular simple root under consideration. If we consider the decomposition (3.9) of the space $\mathcal{H}$, then we may again take an arbitrary state in $\mathcal{H}$ to be

$$
|\psi\rangle = \sum_{j \in I} a_j |p_j\rangle \otimes (q_j [L]^j |0\rangle + r_j [L]^j K_{-1}^i |0\rangle),
$$

where $|p_j\rangle$ is an arbitrary state in $\mathcal{H}^{i\perp}$ and $q_j, r_j$ are polynomials in the creation modes of $L^i$. If we consider the condition that this state defines a field whose integral commutes with $Q^j$ we obtain

$$
Q^j |\psi\rangle = a \sum_{j \in I} a_j |p_j\rangle \otimes r_j [L]^j |a\alpha_i\rangle
$$

$$
= L_{-1} \sum_{j \in I} a_j |p_j\rangle \otimes (q_j [L]^j |0\rangle + r_j [L]^j K_{-1}^i |0\rangle).
$$

If we ask that the same state defines a field which commutes with the dual screening charge $Q^\vee$, we obtain

$$
Q^\vee |\psi\rangle = (-1/a) \sum_{j \in I} a_j |p_j\rangle \otimes r_j [L]^j |1/a\alpha^\vee\rangle
$$

$$
= L_{-1} \sum_{j \in I} a_j |p_j\rangle \otimes (q_j [L]^j |0\rangle + r_j [L]^j K_{-1}^i |0\rangle).
$$
These are the same equations for the states $|p_j\rangle$ and the polynomials $r_j, q_j$ and so the integral of any field which commutes with $Q^i$ also commutes with $Q^i\lor$, and so

$$\text{Aff}(g, a, i) = \text{Aff}(g^\lor, -1/a, i). \quad (4.11)$$

Since $L_{-1}$ is independent of $g$ and $a$ we have

$$\text{Aff}(g, a) = \text{Aff}(g^\lor, -1/a) \quad (4.12)$$

and the two affine Toda theories have the same conserved quantities.

## 5 Examples

The classification of affine Dynkin diagrams may be found in [19]; we shall adopt this notation. Each affine diagram with $(l + 1)$ vertices defines a theory of $l$ scalar fields. We have investigated the theories $C_2^{(1)}, D_3^{(2)}, G_2^{(1)}, D_4^{(3)}, B_3^{(1)}, A_5^{(2)}, C_3^{(1)}, D_5^{(2)}, A_4^{(2)}, A_6^{(2)}$. With the exception of all but the last two, these can be split into dual pairs, for which the Dynkin diagrams are as follows,

$$\begin{align*}
\hat{g} & \quad \hat{g}^\lor \\
C_2^{(1)} & \quad D_3^{(2)} \\
G_2^{(1)} & \quad D_4^{(3)} \\
B_3^{(1)} & \quad A_5^{(2)} \\
C_3^{(1)} & \quad D_5^{(2)}
\end{align*} \quad (5.1)$$

For the case $A_4^{(2)}$ and $A_6^{(2)}$, these are self dual with Dynkin diagrams
We have searched for conserved quantities of the affine Toda theories based on these Dynkin diagrams up to spin 5 for the rank three cases, and to spin 7 for the rank two cases. This was done using the algebraic manipulation language REDUCE to evaluate the action of $Q^i$ and $Q^{iv}$. If we consider fields in the chiral algebra $\mathcal{A}(g,a)$, and equation (4.6) for the screening charge corresponding to the additional root, we only need consider states in $\mathcal{H}$ which are formed from the action of lexicographically ordered modes of the fields in the chiral algebras. We can then compare this with an arbitrary states of the form $L_{-1}|V\rangle$.

5.1 $B_2$

Here explicit expressions for the chiral algebra are known [4]. One finds unique states $|W\rangle$ such that $\mathcal{H}_0|W\rangle = L_{-1}|V\rangle$ for the spin of $W$ equal to 2, 4 and 6.

We take the simple roots of $B_2$ to be

$$(0,1), \quad (1,-1) \quad (5.3)$$

If we consider the conformal theory Toda theory based on $B_2$ then we can find fields which commute with the screening charges for the simple roots of the algebra. Starting with an arbitrary state at each level we implemented these constraints, and up to conformal weight 6, we found two independent fields which satisfied them. One was the Virasoro algebra generator, which corresponded to the state (2.20). The other was a spin 4 conformal primary field corresponding to the state $W_{-4}|0\rangle$, which we have described in our paper [20]. The field has an extremely lengthy expression in terms of free fields which is an $O(1/a)$ perturbation of the classical expression, and we choose to normalise the state $W_{-4}|0\rangle$ by

$$\langle 0|W_{-4}W_{-4}|0\rangle = -72(3a - 2/a)(a - 3/a)(5a - 6/a)(3a - 5/a)(5a - 8/a)$$

$$\times (4a - 5/a)(7a - 6/a)(3a - 7/a)/(75a^2 - 226 + 150/a^2) \quad (5.4)$$

We also found of course the Virasoro descendants of $L_{-2}|0\rangle$ and of $W_{-4}|0\rangle$, and we give the whole list in Table 3.

Having found these fields, we may search for conserved quantities for the quantum affine theories related to $B_2$. These affine theories are $C_2^{(1)}$ and $D_3^{(2)}$, which form a pair of dual
algebras. We can also consider $A^{(2)}_4$ to be an extension of $B_2$ in two different ways; this is a self-dual algebra. The additional roots are as follows:

| Table 1: Affine extensions of $B_2$ |
|-----------------------------------|
| algebra             | $C^{(1)}_2$ | $D^{(2)}_3$ | $A^{(2)}_4$ (i) | $A^{(2)}_4$ (ii) |
| root                | ($-1, -1$)  | ($-1, 0$)   | ($-1/2, -1/2$) | ($-2, 0$)       |

The conserved quantities for these affine theories are of the form

$$I = \int \Phi(z) dz$$  \hspace{0.5cm} (5.5)

where $\Phi(z)$ is some field which is polynomial in the derivatives of the Toda fields. This charge $I$ must certainly commute with the Hamiltonian of the conformal theory, which comprises the screening charges for the conformal theory, and so we can take $\Phi(z)$ to be some field in the W-algebra of the conformal Toda theory. With $\Phi(z)$ of this form, we now need only check that the integral $I$ commutes with the screening charges corresponding to the extra term in the potential of the affine theory.

We systematically searched for fields $\Phi(z)$ of this form up to spin 8. We found only three such fields, although there are many more in the W-algebra up to conformal spin 8 which we list in Table 3 below. To find conserved quantities, we only need consider fields $\Phi(z)$ which cannot be expressed as the $\partial/\partial z$ derivative of another local field, or for which the state $L_1|\Phi\rangle = 0$. Thus, when we look for conserved quantities, we need only look at fields $\Phi$ of conformal spins 2, 4 and 6. We find, as expected, that the momentum

$$\int L(z) dz$$  \hspace{0.5cm} (5.6)

is a conserved quantity. There are no conserved quantities at spin 2, or suitable $\Phi$ at spin 3. At $\Phi$ of spin 4, for a conserved quantity of spin 3, we found a conserved charge for each of the affine theories. If we write the field $\Phi$ as

$$|\Phi\rangle = W_{-4}|0\rangle + xL_{-2}L_{-2}|0\rangle$$  \hspace{0.5cm} (5.7)

then we obtain the conserved quantities for the choices of $x$ below.

| Table 2: Spin 3 charges |
|-------------------------|
| $x$ | $C^{(1)}_2$ | $D^{(2)}_3$ | $A^{(2)}_4$ (i) | $A^{(2)}_4$ (ii) |
|-----|-------------|-------------|-----------------|-----------------|
| $x$ | $-18(3a^2-2)2(3a^2-7)/(3a^2-4)(75a^2-226a^2+150)$ | $-18(a^2-3)^2(7a^2-6)/(2a^2-3)(75a^2-226a^2+150)$ | $18(4a^2-5)(5a^2-22)/5(75a^2-226a^2+150)$ | $-18(5a^2-8)(11a^2-5)/5(75a^2-226a^2+150)$ |

At $\Phi$ of spin 6, for a conserved quantity of spin 5, we found a unique conserved charge for the affine theories $C^{(1)}_2, D^{(2)}_3$. If we write

$$|\Phi\rangle = (L_{-2}W_{-4} + xL_{-2}L_{-2}L_{-2} + yL_{-4}L_{-2})|0\rangle$$  \hspace{0.5cm} (5.8)
Table 3: W-algebra states for $B_2$

| spin | state |
|------|-------|
| 2    | $L_{-2}|0\rangle$ |
| 3    | $L_{-3}|0\rangle$ |
| 4    | $L_{-4}|0\rangle$, $W_{-4}|0\rangle$, $L_{-2}L_{-2}|0\rangle$ |
| 5    | $L_{-5}|0\rangle$, $W_{-5}|0\rangle$, $L_{-3}L_{-2}|0\rangle$ |
| 6    | $L_{-6}|0\rangle$, $W_{-6}|0\rangle$, $L_{-3}L_{-3}|0\rangle$, $L_{-4}L_{-2}|0\rangle$, $L_{-2}W_{-4}|0\rangle$, $L_{-2}L_{-2}L_{-2}|0\rangle$ |
| 7    | $L_{-7}|0\rangle$, $W_{-7}|0\rangle$, $L_{-4}L_{-3}|0\rangle$, $L_{-5}L_{-2}|0\rangle$, $L_{-2}W_{-5}|0\rangle$, $L_{-3}W_{-4}|0\rangle$, $L_{-3}L_{-2}L_{-2}|0\rangle$ |
| 8    | $L_{-8}|0\rangle$, $W_{-8}|0\rangle$, $L_{-4}L_{-4}|0\rangle$, $L_{-5}L_{-3}|0\rangle$, $L_{-6}L_{-2}|0\rangle$, $L_{-2}W_{-6}|0\rangle$, $L_{-3}W_{-5}|0\rangle$, $L_{-4}W_{-4}|0\rangle$, $W_{-4}W_{-4}|0\rangle$, $L_{-2}L_{-2}W_{-4}|0\rangle$, $L_{-3}L_{-3}L_{-2}|0\rangle$, $L_{-4}L_{-2}L_{-2}|0\rangle$, $L_{-2}L_{-2}L_{-2}L_{-2}|0\rangle$ |

then we get:

Table 4: Spin 5 charges

|       | $C_2^{(1)}$ | $D_3^{(2)}$ |
|-------|-------------|-------------|
| $x$   | $\frac{2(a^2-3)(93a^4-160a^2+84)}{(3a^2-4)(75a^4-226a^2+150)}$ | $\frac{-2(3a^2-2)(21a^4-80a^2+93)}{(2a^2-3)(75a^4-226a^2+150)}$ |
| $y$   | $\frac{-6(a^2-3)(120a^8-493a^6+720a^4-464a^2+120)}{a^2(3a^2-4)(75a^4-226a^2+150)}$ | $\frac{6(3a^2-2)(15a^8-116a^6+360a^4-493a^2+240)}{a^2(2a^2-3)(75a^4-226a^2+150)}$ |

We also found a conserved quantity of spin 7 for each of the affine theories. This charge
corresponds to a field $\Phi$ of spin 8, which can be written in terms of the W-algebra fields as

$$
|\Phi\rangle = W_{-4}W_{-4}|0\rangle + xL_{-2}L_{-2}W_{-4}|0\rangle + yL_{-4}W_{-4}|0\rangle + zL_{-2}L_{-2}L_{-2}W_{-4}|0\rangle
+ uL_{-1}L_{-2}W_{-2}|0\rangle + vL_{-6}W_{-2}|0\rangle
$$

(5.9)

The conserved quantities are then obtained for the choices:

| Table 5: Spin 7 charges |
|-------------------------|
| $C_2^{(1)}$             |
| $x$                     |
| $\frac{84 (3 a^6 - 107 a^4 + 84 a^2 - 12)}{(3 a^2 - 4)(75 a^4 - 226 a^2 + 150)}$ |
| $y$                     |
| $\frac{28 (180 a^{10} - 1299 a^8 + 38675 a^6 - 3444 a^4 + 1564 a^2 - 240)}{a^2 (3 a^2 - 4)(75 a^4 - 226 a^2 + 150)}$ |
| $z$                     |
| $\frac{12 (47583 a^{12} - 427614 a^{10} + 1478251 a^8 - 2557008 a^6 + 2366936 a^4 - 1108752 a^2 + 201168)}{(3 a^2 - 4)^2(75 a^4 - 226 a^2 + 150)^2}$ |
| $u$                     |
| $72 (40770 a^{16} - 471519 a^{14} + 2246016 a^{12} - 5799543 a^{10} + 8966752 a^8 - 8586320 a^6 + 5032712 a^4 - 1644112 a^2 + 227040)$ |
| $v$                     |
| $\frac{1480281944 a^8 - 944031488 a^6 + 385622640 a^4 - 91324800 a^2 + 9504000}{(5 a^4 (3 a^2 - 4)^2(75 a^4 - 226 a^2 + 150)^2}$ |

| $D_3^{(2)}$             |
| $x$                     |
| $\frac{84 (3 a^6 - 42 a^4 + 107 a^2 - 66)}{(2 a^2 - 3)(75 a^4 - 226 a^2 + 150)}$ |
| $y$                     |
| $\frac{28 (30 a^{10} - 391 a^8 + 1722 a^6 - 3257 a^4 + 2598 a^2 - 720)}{a^2 (2 a^2 - 3)(75 a^4 - 226 a^2 + 150)}$ |
| $z$                     |
| $\frac{12 (12573 a^{12} - 138594 a^{10} + 591734 a^8 - 1278504 a^6 + 1478251 a^4 - 855228 a^2 + 190332)}{(2 a^2 - 3)^2(75 a^4 - 226 a^2 + 150)^2}$ |
| $u$                     |
| $72 (7095 a^{16} - 102757 a^{14} + 629089 a^{12} - 2149580 a^{10} + 4483376 a^8 - 5799543 a^6 + 4492032 a^4 - 1886076 a^2 + 326160)$ |
| $v$                     |
| $\frac{1081746337 a^8 - 99661582 a^6 + 575709060 a^4 - 187642800 a^2 + 6136000}{(5 a^4 (2 a^2 - 3)^2(75 a^4 - 226 a^2 + 150)^2}$ |
| \( A_{4}^{(2)} \) (i) |
|---|
| \( x \) | \( \frac{84 (30 a^4 - 113 a^2 + 90)}{5 (75 a^4 - 226 a^2 + 150)} \) |
| \( y \) | \( \frac{42 (75 a^8 - 700 a^6 + 2332 a^4 - 2960 a^2 + 1200)}{5 a^2 (75 a^4 - 226 a^2 + 150)} \) |
| \( z \) | \( \frac{108 (4 a^2 - 5)(5 a^2 - 22)(360 a^4 - 1243 a^2 + 930)}{25 (75 a^4 - 226 a^2 + 150)^2} \) |
| \( u \) | \( \frac{36 (82125 a^{12} - 1019625 a^{10} + 5013330 a^8 - 12708296 a^6 + 17503220 a^4 - 12371400 a^2 + 3492000)}{25 a^2 (75 a^4 - 226 a^2 + 150)^2} \) |
| \( v \) | \( \frac{72 (928125 a^{16} - 16155000 a^{14} + 119232825 a^{12} - 485268080 a^{10} + 1183204611 a^8 - 1764835120 a^6 + 1572362700 a^4 - 766170000 a^2 + 156600000)}{125 a^4 (75 a^4 - 226 a^2 + 150)^2} \) |

| \( A_{4}^{(2)} \) (ii) |
|---|
| \( x \) | \( \frac{-84 (45 a^4 - 113 a^2 + 60)}{5 (75 a^4 - 226 a^2 + 150)} \) |
| \( y \) | \( \frac{168 (75 a^8 - 370 a^6 + 583 a^4 - 350 a^2 + 75)}{5 a^2 (75 a^4 - 226 a^2 + 150)} \) |
| \( z \) | \( \frac{108 (5 a^2 - 8)(11 a^2 - 5)(465 a^4 - 1243 a^2 + 720)}{25 (75 a^4 - 226 a^2 + 150)^2} \) |
| \( u \) | \( \frac{-72 (218250 a^{12} - 1546425 a^{10} + 4375805 a^8 - 6354148 a^6 + 5013330 a^4 - 2039250 a^2 + 328500)}{25 a^2 (75 a^4 - 226 a^2 + 150)^2} \) |
| \( v \) | \( \frac{72 (978750 a^{16} - 95771250 a^{14} + 393090675 a^{12} - 882417560 a^{10} + 1183204611 a^8 - 970536160 a^6 + 476913300 a^4 - 129240000 a^2 + 14850000)}{125 a^4 (75 a^4 - 226 a^2 + 150)^2} \) |

We have checked explicitly that these conserved quantities obey the required duality properties, that is they commute with the terms appearing in the affine Toda Lagrangian for the dual algebra with \( a \to -1/a \). \( C_{2}^{(1)} \) and \( D_{3}^{(2)} \) are dual algebras, as are \( A_{4}^{(2)} \) (i) and \( A_{4}^{(2)} \) (ii). However, the normalisations of the roots differ from that in (3.15), so that e.g. \( x \) of \( C_{2}^{(1)} \) transforms into \( x \) of \( D_{3}^{(2)} \) under \( a \to -\sqrt{2}/a \) up to a change of sign in \( W \).

The spins of the conserved quantities we have found agree with the spins of the classical conserved quantities, namely that the spins are equal to the exponents of the affine Lie algebra (see e.g. Table E, page 216 [19]).
Table 6:

| algebra | exponents |
|---------|-----------|
| $C_2^{(1)}$ | 1 mod 2 |
| $D_3^{(1)}$ | 1 mod 2 |
| $A_4^{(2)}$ | 1, 3, 7, 9 mod 10 |

5.2 $B_3$

We take the simple roots of $B_3$ to be

$$
(0, 0, 1), \quad (-1, 1, 0), \quad (1, 0, -1). \tag{5.10}
$$

If we consider the conformal theory Toda based on $B_3$ then we can find fields which commute with the screening charges for the simple roots of the algebra. Starting with an arbitrary state at each level we implemented these constraints, and up to conformal weight 6, we found three independent fields which satisfied them. One was the Virasoro algebra generator, which corresponded to the state (2.20), and the other two were a spin 4 and a spin 6 field, corresponding to states

$$
W_{-4}|0\rangle, \quad V_{-6}|0\rangle, \tag{5.11}
$$

respectively. These are extremely lengthy expressions which are $O(1/a)$ perturbations of the classical expression. We have used the normalisation

$$
\langle 0|W_4W_{-4}|0\rangle = -12(3a - 5/a)(4a - 3/a)(5a - 7/a)(5a - 9/a)(6a - 7/a)(7a - 8/a) \times (7a - 10/a)(9a - 8/a)/(525a^2 - 1357 + 840/a^2) \tag{5.12}
$$

$$
\langle 0|V_6V_{-6}|0\rangle = -112(a - 3/a)(3a - 4/a)(3a - 5/a)(4a - 3/a)(5a - 3/a) \times (5a - 6/a)(5a - 7/a)(5a - 9/a)(5a - 11/a)(6a - 7/a) \times (7a - 8/a)(7a - 10/a)(7a - 12/a)(9a - 8/a)(11a - 8/a) \times (9(35a - 48/a)(35a^2 - 97 + 56/a^2)(735a^2 - 1937 + 1176/a^2)) \tag{5.13}
$$

There were also the Virasoro descendants of $L_{-2}|0\rangle$ and of $W_{-4}|0\rangle$.

Having found these fields, we may search for conserved quantities for the quantum affine theories related to $B_3$. These affine theories are $B_3^{(1)}$, $D_5^{(2)}$ and $A_6^{(4)}$. The duals to these algebras are $A_5^{(2)}$, $C_3^{(1)}$ and $A_6^{(4)}$ respectively.
Table 7: Affine extensions of $B_3$

| algebra | $B_3^{(1)}$ | $D_5^{(2)}$ | $A_6^{(4)}$ |
|---------|-------------|-------------|-------------|
| root    | $(-1,-1,0)$ | $(-1,0,0)$  | $(-2,0,0)$  |

The conserved quantities for these affine theories are of the form (5.5) where $\Phi(z)$ is some field which is polynomial in the derivatives of the Toda fields. As above, this charge $I$ must certainly commute with the conformal Hamiltonian, and so we need only check that the integral $I$ obtained from some W-algebra field commutes with the screening charges corresponding to the extra term in the potential of the affine theory.

We systematically searched for such fields $\Phi(z)$ up to spin 6. Up to that spin the fields in the W-algebra are:

Table 8: W-algebra states for $B_3$

| spin | state |
|------|-------|
| 2    | $L_{-2}|0\rangle$ |
| 3    | $L_{-3}|0\rangle$ |
| 4    | $L_{-4}|0\rangle$, $W_{-4}|0\rangle$, $L_{-2}L_{-2}|0\rangle$ |
| 5    | $L_{-5}|0\rangle$, $W_{-5}|0\rangle$, $L_{-3}L_{-2}|0\rangle$ |
| 6    | $L_{-6}|0\rangle$, $W_{-6}|0\rangle$, $V_{-6}|0\rangle$ $L_{-3}L_{-3}|0\rangle$, $L_{-4}L_{-2}|0\rangle$ $L_{-2}W_{-4}|0\rangle$, $L_{-2}L_{-2}L_{-2}|0\rangle$ |

We find, as expected, that the there is a conserved quantity of spin one, the momentum

$$\int L(z) dz.$$  \hspace{1cm} (5.14)

There are no conserved quantities of spin 2, but a conserved quantity of spin 3, corresponding to a field $\Phi$ of conformal weight 4, for each of the affine theories. If we write the field $\Phi$ as

$$\Phi_{-4}|0\rangle = W_{-4}|0\rangle + xL_{-2}L_{-2}|0\rangle$$  \hspace{1cm} (5.15)

then we obtain the conserved quantities for the choices of $x$ below.
If we write the corresponding field $\Phi$ of spin 6 as

$$|\Phi\rangle = V_{-6}|0\rangle + xL_{-2}W_{-4}|0\rangle + yL_{-2}L_{-2}|0\rangle + zL_{-4}L_{-2}|0\rangle$$

(5.16)

then we obtain the conserved quantities for the choices below.

For the affine theories $B_{3}^{(1)}$ and $D_{5}^{(2)}$ we found further a conserved quantity of spin 5. If we write the corresponding field $\Phi$ of spin 6 as

$$|\Phi\rangle = V_{-6}|0\rangle + xL_{-2}W_{-4}|0\rangle + yL_{-2}L_{-2}|0\rangle + zL_{-4}L_{-2}|0\rangle$$

(5.16)

then we obtain the conserved quantities for the choices below.

| $B_{3}^{(1)}$ | $D_{5}^{(2)}$ | $A_{6}^{(4)}$ |
|---|---|---|
| $x$ | $\frac{a^2(4a^2-3)(5a^2-9)}{(5a^2-6)(525a^4-1357a^2+840)}$ | $\frac{-3(3a^2-5)(3a^2-7)(9a^2-8)}{(3a^2-4)(525a^4-1357a^2+840)}$ | $\frac{-12(7a^2-10)(13a^2-7)}{3075a^4-9499a^2+5880}$ |
| $y$ | $\frac{10a^2(5a^2-3)(5a^2-11)}{9(5a^2-6)(35a^4-97a^2+56)}$ | $\frac{-5(a^2-3)(3a^2-7)(11a^2-8)}{9(3a^2+4)(35a^4-97a^2+56)}$ |
| $z$ | $\frac{-5(a^2-3)(3a^2-7)(11a^2-8)}{9(3a^2+4)(35a^4-97a^2+56)}$ | $\frac{-(a^2-3)(11a^2-8)(641025a^{10}+4699785a^8+13747319a^6+19963511a^4+14313192a^2-4026240)}{3(3a^2-4)^2(35a^2-48)(525a^4-1357a^2+840)(735a^4-1937a^2+1176)}$ |

The conserved quantities for $A_{5}^{(2)}, C_{3}^{(1)}$ and $A_{4}^{(2)}$ can be obtained by substituting $a \rightarrow -\sqrt{2}/a$ in the above expressions for $B_{3}^{(1)}, D_{5}^{(2)}$ and $A_{4}^{(2)}$.

The spins of the conserved quantities we have found are again equal to the exponents of the affine Lie algebra, as in the classical case.
Table 11:

| algebra | exponents               |
|---------|-------------------------|
| $B_3^{(1)}$ | $i \equiv 1 \mod 2$     |
| $C_3^{(1)}$ | $i \equiv 1 \mod 2$     |
| $A_5^{(1)}$ | $i \equiv 1 \mod 2$     |
| $D_5^{(2)}$ | $i \equiv 1 \mod 2$     |
| $A_6^{(2)}$ | $i \equiv 1, 3, 5, 9, 11, 13 \mod 14$ |

5.3 $G_2$

We take the simple roots of $G_2$ to be

$$(0, 1), \quad \left(\sqrt{3}/2, -3/2\right). \quad (5.17)$$

If we consider the conformal theory Toda based on $G_2$ then we can find fields which commute with the screening charges for the simple roots of the algebra. Starting with an arbitrary state at each level we implemented these constraints, and up to conformal weight 6, we found two independent fields which satisfied them. One was the Virasoro algebra generator, which corresponded to a state (2.20), and the other was a spin 6 conformal primary field corresponding to the state $W_{-6}|0\rangle$.

$$W_{-6}|0\rangle. \quad (5.18)$$

This state is given by an extremely lengthy expression which is an $O(1/a)$ perturbation of the classical expression. We normalised the state as

$$
\langle 0|W_6W_{-6}|0\rangle = -576(a - 1/a)(3a - 4/a)(a - 3/a)(9a - 4/a) \\
\times (4a - 5/a)(15a - 16/a)(6a - 7/a)(7a - 8/a)(6a - 11/a) \\
\times (11a - 8/a)(7a - 10/a)(15a - 14/a)(7a - 12/a)(9a - 7/a) \\
/ \left(168a^2 - 387 + 224/a^2\right)(294a^2 - 713 + 392/a^2) \quad (5.19)
$$

Having found these fields, we may search for conserved quantities for the quantum affine theories related to $G_2$. These affine theories are $G_2^{(1)}$ and $D_4^{(3)}$, corresponding to the extra roots as below,

Table 12: Affine extensions of $G_2$

| algebra | $G_2^{(1)}$ | $D_4^{(3)}$ |
|---------|-------------|-------------|
| root    | $(-\sqrt{3}, 0)$ | $(-\sqrt{3}/2, -1/2)$ |
The conserved quantities $\mathcal{I}$ for these affine theories are again of the form (5.5). We systematically searched for fields $\Phi(z)$ of this form up to spin 8. We again considered arbitrary linear combinations of the W-algebra fields up to level 8. These are listed below.

### Table 13: W-algebra states for $G_2$

| spin | state                                      |
|------|--------------------------------------------|
| 2    | $L_{-2}|0\rangle$                           |
| 3    | $L_{-3}|0\rangle$                           |
| 4    | $L_{-4}|0\rangle, L_{-2}L_{-2}|0\rangle$    |
| 5    | $L_{-5}|0\rangle, L_{-3}L_{-2}|0\rangle$    |
| 6    | $L_{-6}|0\rangle, W_{-6}|0\rangle, L_{-3}L_{-3}|0\rangle, L_{-4}L_{-2}|0\rangle, L_{-2}L_{-2}L_{-2}|0\rangle$ |
| 7    | $L_{-7}|0\rangle, W_{-7}|0\rangle, L_{-4}L_{-3}|0\rangle, L_{-5}L_{-2}|0\rangle, L_{-3}L_{-2}L_{-2}|0\rangle$ |
| 8    | $L_{-8}|0\rangle, W_{-8}|0\rangle, L_{-4}L_{-4}|0\rangle, L_{-5}L_{-3}|0\rangle, L_{-6}L_{-2}|0\rangle, L_{-2}W_{-6}|0\rangle, L_{-3}L_{-3}L_{-2}|0\rangle, L_{-4}L_{-2}L_{-2}|0\rangle, L_{-2}L_{-2}L_{-2}L_{-2}|0\rangle$ |

We found only three such combinations which yielded conserved quantities. We find, as expected, that the momentum

$$\int L(z)dz$$  \hspace{1cm} (5.20)

is a conserved quantity. We found no conserved quantities at spin less than 5 other than the momentum. At spin 5 we found conserved quantities of the form (5.5), with

$$|\Phi\rangle = (W_{-6} + xL_{-2}L_{-2}L_{-2} + yL_{-4}L_{-2})|0\rangle$$  \hspace{1cm} (5.21)

for both the $G_2^{(1)}$ and $D_4^{(3)}$ theories. The variables $x, y$ take their values as follows.
We have checked explicitly that these conserved quantities obey the required duality properties, that is that they commute with the terms appearing in the affine Toda Lagrangian.

**Table 14:** Spin 5 charges

|       | $G_2^{(1)}$ |
|-------|-------------|
| $x$   | $(6a^2-11)(9a^2-4)(3048a^6-9569a^4+9876a^2-3328)$ |
|       | $(a^2-1)(168a^4-387a^2+224)(294a^4-713a^2+392)$ |
| $y$   | $(6a^2-11)(9a^2-4)(6888a^8-22842a^6+29755a^4-18864a^2+4928)(3a^2-4)$ |
|       | $2a^2(a^2-1)(168a^4-387a^2+224)(294a^4-713a^2+392)$ |

**Table 15:** Spin 7 charges

|       | $G_2^{(1)}$ |
|-------|-------------|
| $x$   | $12(a^2-3)(11a^2-8)(1872a^6-7407a^4+9569a^2-4064)$ |
|       | $(3a^2-4)(168a^4-387a^2+224)(294a^4-713a^2+392)$ |
| $y$   | $6(a^2-1)(11a^2-8)(8316a^8-42444a^6+89265a^4-91368a^2+36736)(a^2-3)$ |
|       | $a^6(3a^2-4)(168a^4-387a^2+224)(294a^4-713a^2+392)$ |

We found a further conserved quantity at spin 7 of the form (5.5), with

$$|\Phi\rangle = L_{-2}W_{-6}|0\rangle + xL_{-2}L_{-2}L_{-2}|0\rangle + yL_{-4}L_{-2}L_{-2}|0\rangle + zL_{-6}L_{-2}|0\rangle$$

(5.22)

for both the $G_2^{(1)}$ and $D_4^{(3)}$ theories. The variables $x, y$ and $z$ take their values as follows.

|       | $D_4^{(3)}$ |
|-------|-------------|
| $x$   | $6(13a^2-8)(168a^4-387a^2+224)(294a^4-713a^2+392)$ |
| $y$   | $3(13a^2-8)(43092a^{12}-374544a^{10}+1361691a^8-2658600a^6+2895033a^4-1664696a^2+393344)$ |
|       | $a^2(3a^2-4)(168a^4-387a^2+224)(294a^4-713a^2+392)$ |
| $z$   | $12(13a^2-8)(317520a^{16}-3663576a^{14}+18539748a^{12}-53719692a^{10}$ |
|       | $+9748252a^8-113310839a^6+82144696a^4-33811904a^2+6021120)$ |
|       | $/(5a^4(3a^2-4)(168a^4-387a^2+224)(294a^4-713a^2+392)$ |
for the dual algebra with $a \to -1/a$. Again, due to the choice of scale the results for $G_2^{(1)}$ and $D_4^{(3)}$ transform into each other under $a \to -2/(\sqrt{3}a)$.

The spins of the conserved quantities we have found agree with the spins of the classical conserved quantities, namely that the spins are equal to the exponents of the affine Lie algebra.

| algebra | exponents |
|---------|-----------|
| $G_2^{(1)}$ | $i \equiv \pm 1 \mod 6$ |
| $D_4^{(3)}$ | $i \equiv \pm 1 \mod 6$ |

### 6 Conclusions

As expected, we found the same number of conserved quantities in these non-simply laced affine Toda theories as in the classical theories, and the expressions for these are $O(\bar{\hbar})$ deformations of the classical expressions. We also found the purely quantum property of duality, that

$$\text{Aff}(g, a) = \text{Aff}(g', -1/a).$$ (6.1)

We now have the problem of interpretation of this result. There are two fundamentally different domains of affine Toda fields theory, namely a purely imaginary and a real. In the first instance the theory is a massive field theory and it is conjectured that there is purely elastic scattering of the particles for $g$ simply-laced. S-matrices have been conjectured for the scatterings [11, 21, 22]. In the second regime it is believed that the theory corresponds to a perturbation of conformal field theory as first considered by A. B. Zamolodchikov [23]. However here there are problems in that the Hamiltonian does not obviously describe a unitary time evolution, and the related problem of null states in the W-algebra Verma modules which are the particle sectors (although see [24] for a possible route through this problem). Here the W-algebras are well understood for $g$ simply-laced, and it seems very possible that the Hamiltonian is indeed unitary when acting on the physical subspace of the free-field Fock space.

An important requirement of the bootstrap approach to the factorised S-matrix was that the ratios of the particle masses did not change under change of $\beta$; this holds to one loop for the simply-laced theories and for $A_2^{(2)}$, but not for the non-simply-laced theories or other twisted theories.

For the non-simply laced affine theories the masses flow as the coupling constant changes. However, these affine theories are still thought to be quantum integrable theories since the particles do not appear to decay to one loop in standard perturbation theory [25]. Can we say anything interesting from our calculations here? We have found
that there is a relation between the conserved quantities of the affine theories for \((\hat{g}, a)\) and \((\hat{g}^\vee, -1/a)\). One way this should be reflected in the scattering theory is in the flow of the particle masses. Since the theories are “governed” by their conserved quantities, then the large \(a\) limit of the mass-ratios of the Toda theory based on \(\hat{g}\) should be the small \(a\) limit of the theory based on \(\hat{g}^\vee\).

The most interesting recent work on the non-simply laced Toda theories has been the conjecture of the S-matrices for the \(A^{(2)}_{2n-1}, B^{(1)}_n, C^{(1)}_n\) and \(D^{(2)}_{n+1}\) theories by Delius et al. [13, 26]. They find that the mass ratios do renormalise, and on closer inspection this is from those of the \(A^{(2)}_{2n-1}\) theory to those of the \(B^{(1)}_n\) theory and for \(C^{(1)}_n\) to those of \(D^{(2)}_{n+1}\). This indicates that our conjecture on the flow of the masses is correct, although their conjectured S-matrices do not respect this duality for \(G^{(1)}_2 \leftrightarrow D^{(3)}_4\). Further, Monte-Carlo lattice simulations seem to bear out a flow of masses connecting a theory with its dual, for at least some of the other affine theories [27].

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References

[1] P. Mansfield, *Conformally Extended Toda Theories*, Phys. Lett. B242 (1990) 387.

[2] V. A. Fateev and S. L. Luk’yanov, *The models of Two-Dimensional Conformal Quantum Field Theory with $Z_n$ Symmetry*, Int. J. Mod. Phys. A3 (1988) 507.

[3] V. A. Fateev and S. L. Luk’yanov, *Additional Symmetries and Exactly-Solvable Models in Two-Dimensional Conformal Field Theory*, Sov. Sci. Rev. A15 (1990) 1.

[4] H. G. Kausch and G. M. T. Watts, *Quantum Toda theory and the Casimir algebra of $B_2$ and $C_2$*, Durham University Preprint DTP-91-35 (1991), to appear in Int. J. Mod. Phys. A.

[5] C. Destri, H. J. de Vega and V. A. Fateev, *The Exact S-Matrices associated to non-simply laced affine Toda Field Theories: The $B_n^{(1)}$ and $C_n^{(1)}$ cases*, Phys. Lett. b256 (1990) 173.

[6] E. V. Frenkel, *Affine Kac-Moody Algebras at the Critical Level and Quantum Drinfeld-Sokolov Reduction*, PhD thesis, Harvard University, 1991.

[7] V. G. Drinfel’d and V. V. Sokolov, *Lie Algebras and Equations of Korteweg-de Vries Type*, J. Sov. Math. 30 (1985) 1975.

[8] T. Eguchi and S.-K. Yang, *Deformations of Conformal Field Theories and Soliton Equations*, Research Institute for Fundamental Physics, Kyoto University RIFP-797 (1987).

[9] L. Palla, *Perturbed $W$ algebras and affine Toda theories*, Nucl. Phys. B341 (1990) 714.

[10] B. L. Feigin and E. V. Frenkel, *Free Field resolutions and affine Toda theory*, Harvard University Preprint (1991).

[11] H. W. Braden, E. Corrigan, P. E. Dorey and R. Sasaki, *Extended Toda Field Theory and Exact S-Matrices*, Phys. Lett. B227 (1989) 411.

[12] H. W. Braden, E. Corrigan, P. E. Dorey and R. Sasaki, *Aspects of Perturbed Conformal Field Theory, Affine Toda Field Theory and Exact S-Matrices*, University of Durham Preprint UDCPT-89-35 (1989).

[13] G. W. Delius, M. T. Grisaru and D. Zanon, *Exact S-Matrices for the non-simply-laced affine Toda theories $a_{2n-1}^{(2)}$*, Cern preprint CERN-TH.6333/91 (1991).

[14] S. Coleman, *Quantum Sine Gordon Theory as the massive Thirring Model*, Phys. Rev. D11 (1975) 2088.
[15] P. Mansfield, *Solution of Toda systems*, Nucl. Phys. B208 (1982) 277; *Light-cone quantisation of the Liouville and Toda field theories*, B222 (1983) 419.

[16] P. Goddard, *Meromorphic Conformal Field Theory*, *in*: Infinite Dimensional Lie Algebras and Lie Groups, ed. V. G. Kac, World Scientific, 1989, CIRM-Luminy July conference on Infinite dimensional Lie Algebras and Lie Groups, Marseille 1988.

[17] P. Goddard and D. Olive, *Kac-Moody and Virasoro Algebras in Relation to Physics*, Int. J. Mod. Phys. A1 (1986) 303.

[18] C. Destri and H. J. de Vega, *New exact results in affine Toda field theories: Free energy and wave function renormalizations*, Nucl. Phys. B358 (1991) 251.

[19] V. Kac, *Infinite dimensional Lie algebras*, Cambridge University Press, 1985.

[20] H. G. Kausch and G. M. T. Watts, *A Study of W-algebras using Jacobi identities*, Nucl. Phys. (1991) 740.

[21] H. W. Braden, E. Corrigan, P. E. Dorey and R. Sasaki, *Affine Toda Field Theory and Exact S Matrices*, Nucl. Phys. B338 (1990) 689.

[22] P. Christe and G. Mussardo, *Integrable Systems away from criticality: The Toda Field Theory and S Matrix of the Tricritical Ising model*, Nucl. Phys. B330 (1990) 465.

[23] A. B. Zamolodchikov, *Integrable Field Theory from Conformal Field Theory*, *in*: Advances in Pure Mathematics 19, Integrable Systems in Quantum Field Theory and Statistical Mechanics, eds. M. Jimbo, T. Miwa and A. Tsuchiya, pages 641–674, Kinokuniya Company, Tokyo, 1989.

[24] T. J. Hollowood, *Quantum solitons in affine Toda field theories*, Princeton preprint PUPT-1286 (1991).

[25] G. Mussardo, private communication.

[26] G. W. Delius, M. T. Grisaru and D. Zanon, *Exact S-Matrices for non-simply-laced affine Toda theories*, Cern preprint CERN-TH.6337/91 (1991).

[27] R. A. Weston and G. M. T. Watts, work in progress.