W-GRAPH IDEALS II

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Abstract. In [5], the concept of a W-graph ideal in a Coxeter group was introduced, and it was shown how a W-graph can be constructed from a given W-graph ideal. In this paper, we describe a class of W-graph ideals from which certain Kazhdan-Lusztig left cells arise. The result justifies the algorithm as illustrated in [5] for the construction of W-graphs for Specht modules for the Hecke algebra of type A.

1. Introduction

Let (W, S) be a Coxeter system and \( \mathcal{H}(W) \) its Hecke algebra over \( \mathbb{Z}[q, q^{-1}] \), the ring of Laurent polynomials in the indeterminate \( q \). In [5], we introduced the concept of a W-graph ideal in \( (W, \leq_L) \) with respect to a subset \( J \) of \( S \), where \( \leq_L \) is the left weak Bruhat order on \( W \), and gave a Kazhdan-Lusztig like algorithm to produce, for any such ideal \( \mathcal{I} \), a W-graph with vertices indexed by the elements of \( \mathcal{I} \). In particular, \( W \) itself is a W-graph ideal with respect to \( \emptyset \), and the W-graph obtained is the Kazhdan-Lusztig W-graph for the regular representation of \( \mathcal{H}(W) \) (as defined in [6]). More generally, it was shown that if \( J \) is an arbitrary subset of \( S \) then \( D_J \), the set of distinguished left coset representatives of \( W_J \) in \( W \), is a W-graph ideal with respect to \( J \) and also with respect to \( \emptyset \), and Deodhar’s parabolic analogues of the Kazhdan-Lusztig construction are recovered. In this paper we continue the work in [5], and describe a larger class of W-graph ideals for an arbitrary Coxeter group. Our main aim is to show how to construct W-graphs for a wide class of Kazhdan-Lusztig left cells, without having to first construct the full Kazhdan-Lusztig W-graph corresponding to the regular representation.

To this end we investigate conditions that are sufficient for a sub-ideal of a given W-graph ideal (with respect to the left weak order) to itself be a W-graph ideal. We find that if the sub-ideal is a union of cells then it is a W-graph ideal. It should be noted that, during the course of the proof, we are required to verify a technical result that says that certain structural constants of the associated \( \mathcal{H}(W) \)-module are polynomials that are divisible by \( q \). In particular, for the Kazhdan-Lusztig W-graph for the regular representation, we find that if \( C \) is the left cell that contains \( w_J \), the longest element of the finite standard parabolic subgroup \( W_J \), then \( Cw_J \) is a W-graph ideal with respect to \( J \). Moreover, the W-graph associated with the cell \( C \) is isomorphic to the W-graph constructed from the ideal \( Cw_J \). The result shows that the algorithm in [5] can be applied to construct W-graphs for Kazhdan-Lusztig left cells that contain longest elements of standard parabolic subgroups. In type A, it is known that each such cell is parametrized by the standard tableaux of a fixed shape, and that the cell module is isomorphic to the corresponding Specht module; hence the result justifies the algorithm described in [5] for the construction of W-graphs for Specht modules.
This paper is organized as follows. We start by recalling basic definitions and facts concerning $W$-graphs; in particular, cells and subquotients are discussed. In Section 3, we review the notion of a $W$-graph ideal and the procedure for constructing a $W$-graph from a $W$-graph ideal. Next, in Section 4, we give a description of $W$-graph ideals that arise from sub-ideals of a given $W$-graph ideal, assuming certain conditions. We deduce that if the cells of the associated $W$-graph are ordered in the natural way, based on the preorder by which cells are defined, then there is a unique maximal cell, and this maximal cell is constructed from a $W$-graph ideal. In Section 5 we show that the $W$-graph for the left cell that contains the longest element of a standard parabolic subgroup arises in the manner just described. The result has some significant consequences when it is applied to Coxeter groups of type $A$ and the associated Hecke algebras. Specifically, it justifies the way $W$-graphs for Specht modules, which are exactly $W$-graphs for the corresponding left cells, are calculated in [5]. These topics are included in the discussion in Section 6.

2. W-GRAPHS, CELLS AND SUBQUOTIENTS

Since this is a continuation of the work in [5], we will assume the notation introduced there. In particular, for a Coxeter system $(W, S)$, we let $l$ be its length function and let $\leq$ and $\leq_L$ be its the Bruhat order and the left weak Bruhat order respectively.

Let $A = \mathbb{Z}[q, q^{-1}]$, the ring of Laurent polynomials with integer coefficients in the indeterminate $q$, and let $A^+ = \mathbb{Z}[q]$, and let $\mathcal{H}(W)$ be the Hecke algebra associated with the Coxeter system $(W, S)$. Our convention is that $\mathcal{H}(W)$ is the associative algebra over $A$ generated by $\{T_s \mid s \in S\}$, subject to the following defining relations:

\[
T_s^2 = 1 + (q - q^{-1})T_s \quad \text{for all } s \in S,
\]

\[
T_s T_{s'} T_s \cdots = T_{s'} T_s T_{s'} \cdots \quad \text{for all } s, s' \in S,
\]

where in the second of these there are $m(s, s')$ factors on each side, $m(s, s')$ being the order of $ss'$ in $W$. (We remark that the traditional definition has $T_s^2 = q + (q - 1)T_s$ in place of the first relation above; our version is obtained by replacing $q$ by $q^2$ and multiplying the generators by $q^{-1}$.) It is well known that $\mathcal{H}(W)$ is $A$-free with an $A$-basis $(T_w \mid w \in W)$ and multiplication satisfying

\[
T_s T_w = \begin{cases} 
T_{sw} & \text{if } l(sw) > l(w), \\
T_{sw} + (q - q^{-1})T_w & \text{if } l(sw) < l(w). 
\end{cases}
\]

for all $s \in S$ and $w \in W$.

Let $a \mapsto \overline{a}$ be the involutory automorphism of $A = \mathbb{Z}[q, q^{-1}]$ defined by $\overline{q} = q^{-1}$.

This extends to an involution on $\mathcal{H}(W)$ satisfying

\[
\overline{T_s} = T_s^{-1} = T_s - (q - q^{-1}) \quad \text{for all } s \in S.
\]

A $W$-graph $\Gamma$ is a triple consisting of a set $V$, a function $\mu : V \times V \rightarrow \mathbb{Z}$ and a function $\tau$ from $V$ to the power set of $S$, subject to the requirement that the free $A$-module with basis $V$ admits an $\mathcal{H}$-module structure satisfying

\[
T_s v = \begin{cases} 
-q^{-1} v & \text{if } s \in \tau(v) \\
qv + \sum_{u \in V, s \in \tau(u)} \mu(u, v) u & \text{if } s \notin \tau(v),
\end{cases}
\]

for all $s \in S$ and $v \in V$. 


The set $V$ is called the vertex set of $\Gamma$, and there is a directed edge from a vertex $v$ to a vertex $u$ if and only if $\mu(u,v) \neq 0$. When there is no ambiguity we may use the notation $\Gamma(V)$ for the W-graph with vertex set $V$. We call the integer $\mu(u,v)$ the \textit{weight} of the edge from $v$ to $u$, and we call the set $\tau(v)$ the $\tau$-\textit{invariant} of the vertex $v$. The $\mathcal{H}(W)$-module $\mathcal{A}V$ encoded by $\Gamma$ will be denoted by $M_{\Gamma}$.

Since $M_{\Gamma}$ is $\mathcal{A}$-free it admits a unique $\mathcal{A}$-semilinear involution $\alpha \mapsto \overline{\alpha}$ such that $\tau = \overline{\alpha}$ for all elements $\alpha$ of the basis $V$. It follows from \[\text{[21]}\] that $\overline{h\alpha} = \overline{h}\alpha$ for all $h \in \mathcal{H}$ and $\alpha \in M_{\Gamma}$.

Following \[\text{[6]}\], define a preorder $\leq_{\Gamma}$ on the vertex set of $\Gamma$ as follows: $u \leq_{\Gamma} v$ if there exists a sequence of vertices $u = x_0, x_1, \ldots, x_m = v$ such that $\tau(x_{i-1}) \subseteq \tau(x_i)$ and $\mu(x_{i-1}, x_i) \neq 0$ for all $i \in [1,m]$. In other words, the preorder $\leq_{\Gamma}$ is the transitive closure of the relation $\sim_{\Gamma}$ on $V$ given by $u \sim_{\Gamma} v$ if $\tau(u) \subseteq \tau(v)$ and $\mu(u,v) \neq 0$. Let $\sim_{\Gamma}$ be the equivalence relation corresponding to $\leq_{\Gamma}$; that is, $u \sim_{\Gamma} v$ if and only if $u \leq_{\Gamma} v$ and $v \leq_{\Gamma} u$. The equivalence classes with respect to $\sim_{\Gamma}$ are called the \textit{cells} of $\Gamma$. Each equivalence class, regarded as a full subgraph of $\Gamma$, is itself a W-graph, with the $\mu$ and $\tau$ functions being the restrictions of those for $\Gamma$. Thus, if $C$ is a cell, then $\Gamma(C) = \langle C, \mu, \tau \rangle$ is the W-graph associated with the cell $C$. Observe that the preorder $\leq_{\Gamma}$ on the vertices induces a partial order on the cells, via the rule that if $C, C'$ are cells then $C \leq_{\Gamma} C'$ if and only if $u \leq_{\Gamma} v$ for some (or, equivalently, all) $u \in C$ and $v \in C'$.

Let $U \subseteq V$. If $U$ spans a $\mathcal{H}(W)$-submodule of $M_{\Gamma}(V)$, then $U$ is called a \textit{closed subset} of $V$. We see from Equation \[\text{[21]}\] that this happens if and only if for all vertices $u$ and $v$, if $u \in U$ and $v \sim_{\Gamma} u$ then $v \in U$. (Note that in \[\text{[10]}\] the term \textit{forward-closed} is used for this concept.)

Provided that $U$ is a closed subset of $V$, the subgraphs $\Gamma(U)$ and $\Gamma(V \setminus U)$ induced by $U$ and $V \setminus U$ are themselves W-graphs, with edge weights and $\tau$ invariants inherited from $\Gamma(V)$. Moreover, we have

$$M_{\Gamma(V \setminus U)} \cong M_{\Gamma(V)}/M_{\Gamma(U)}$$ as $\mathcal{H}(W)$-modules.

If $U_2 \subseteq U_1 \subseteq V$ is a nested sequence of closed subsets of $V$ then the W-graph $\Gamma(U_1 \setminus U_2)$ is called a \textit{subquotient} of $\Gamma(V)$, as the $\mathcal{H}$-module $M_{\Gamma(U_1 \setminus U_2)}$ is a quotient of a submodule of $M_{\Gamma(V)}$. It can be seen that if $\Gamma(V)$ has no non-empty proper subquotients then it consists of a single cell.

Let $\Gamma(W) = (W, \mu, \tau)$ be the Kazhdan-Lusztig W-graph, as defined in \[\text{[6]}\]. Thus

$$\mu(y,w) = \begin{cases} 
\mu_{y,w} & \text{if } y < w \\
\mu_{w,y} & \text{if } w < y
\end{cases}$$

where $\mu_{y,w}$ is either zero or the leading coefficient of a certain polynomial $P_{y,w}$, and

$$\tau(w) = \mathcal{L}(w) = \{ s \in S \mid l(sw) < l(w) \}.$$  

In fact Kazhdan and Lusztig show that $W$ can be given the structure of a $W \times W^{\circ}$-graph, where $W^{\circ}$ is the opposite of the group $W$, but in the present paper we are concerned only with the W-graph structure. The equivalence classes determined by the preorder $\leq_{\Gamma(W)}$ (as defined above) are called the \textit{left cells} of $W$.

3. \textit{W-graph} ideals

Let $(W, S)$ be a Coxeter system and $\mathcal{H}$ the associated Hecke algebra. As in \[\text{[5]}\], we find it convenient to define $\text{Pos}(X) = \{ s \in S \mid l(xs) > l(x) \text{ for all } x \in X \}$, so that $\text{Pos}(X)$ is the largest subset $J$ of $S$ such that $X \subseteq D_J$. Let $\mathcal{I}$ be an
ideal in the poset \((W, \leq_L)\); that is, \(\mathcal{I}\) is a subset of \(W\) such that every \(u \in W\) that is a suffix of an element of \(\mathcal{I}\) is itself in \(\mathcal{I}\). This condition implies that \(\text{Pos}(\mathcal{I}) = S \setminus \mathcal{I} = \{ s \in S \mid s \notin \mathcal{I} \}\). Let \(J\) be a subset of \(\text{Pos}(\mathcal{I})\), so that \(\mathcal{I} \subseteq D_J\). For each \(w \in \mathcal{I}\) we define the following subsets of \(S\):

\[
\begin{align*}
\text{SA}(\mathcal{I}, w) &= \{ s \in S \mid sw > w \text{ and } sw \in \mathcal{I} \}, \\
\text{SD}(\mathcal{I}, w) &= \{ s \in S \mid sw < w \}, \\
\text{WA}_J(\mathcal{I}, w) &= \{ s \in S \mid sw > w \text{ and } sw \in D_J \setminus \mathcal{I} \}, \\
\text{WD}_J(\mathcal{I}, w) &= \{ s \in S \mid sw > w \text{ and } sw \notin D_J \}.
\end{align*}
\]

Since \(\mathcal{I} \subseteq D_J\) it is clear that, for each \(w \in \mathcal{I}\), each \(s \in S\) appears in exactly one of the four sets defined above. We call the elements of these sets the strong ascents, strong descents, weak ascents and weak descents of \(w\) relative to \(\mathcal{I}\) and \(J\).

In contexts where the ideal \(\mathcal{I}\) and the set \(J\) are fixed we may omit reference to them and write, for example, \(\text{WA}(w)\) rather than \(\text{WA}_J(\mathcal{I}, w)\). We also define the sets of descents and ascents of \(w\) by \(D_J(\mathcal{I}, w) = \text{SD}(\mathcal{I}, w) \cup \text{WD}_J(\mathcal{I}, w)\) and \(A_J(\mathcal{I}, w) = \text{SA}(\mathcal{I}, w) \cup \text{WA}_J(\mathcal{I}, w)\).

**Remark 1.** It follows from Lemma 2.1. (iii) that

\[
\begin{align*}
\text{WA}(w) &= \{ s \in S \mid sw \notin \mathcal{I} \text{ and } w^{-1}sw \notin J \} \\
\text{WD}(w) &= \{ s \in S \mid sw \notin \mathcal{I} \text{ and } w^{-1}sw \in J \},
\end{align*}
\]

since \(sw \notin \mathcal{I}\) implies that \(sw > w\) (given that \(\mathcal{I}\) is an ideal in \((W, \leq_L)\)). Note also that \(J = \text{WD}(1)\).

**Definition 3.1.** With the above notation, the set \(\mathcal{I}\) is said to be a \(W\)-graph ideal with respect to \(J\) if the following hypotheses are satisfied.

1. There exists an \(A\)-free \(\mathcal{H}\)-module \(\mathcal{I} = \mathcal{I}(\mathcal{I}, J)\) possessing an \(A\)-basis \(B = (b_w \mid w \in \mathcal{I})\) on which the generators \(T_s\) act by

\[
T_s b_w = \begin{cases} 
   b_{sw} & \text{if } s \in \text{SA}(w), \\
   b_{sw} + (q - q^{-1})b_w & \text{if } s \in \text{SD}(w), \\
   -q^{-1}b_w & \text{if } s \in \text{WD}(w), \\
   q b_w - \sum_{y \in \mathcal{I}, y < sw} r_{y,w}^* b_y & \text{if } s \in \text{WA}(w),
\end{cases}
\]

for some polynomials \(r_{y,w}^* \in \mathcal{A}^+\).

2. The module \(\mathcal{I}\) admits an \(A\)-semilinear involution \(\alpha \mapsto \overline{\alpha}\) satisfying \(\overline{b_1} = b_1\) and \(\overline{h} = \overline{\alpha h}\) for all \(h \in \mathcal{H}\) and \(\alpha \in \mathcal{I}\).

An obvious induction on \(l(w)\) shows that \(b_w = T_w b_1\) for all \(w \in \mathcal{I}\).

**Definition 3.2.** If \(w \in W\) and \(\mathcal{I} = \{ u \in W \mid u \leq_L w \}\) is a \(W\)-graph ideal with respect to some \(J \subseteq S\) then we call \(w\) a \(W\)-graph determining element.

**Remark 2.** It has been verified in [5, Section 5] that if \(W\) is finite then \(w_S\), the maximal length element of \(W\), is a \(W\)-graph determining element with respect to \(\emptyset\), and \(d_J\), the minimal length element of the left coset \(w_S W_J\), is a \(W\)-graph determining element with respect to \(J\) and also with respect to \(\emptyset\).
Let \( \mathcal{I} \) be a \( W \)-graph ideal with respect to \( J \subseteq S \) and let \( \mathcal{I}(\mathcal{I}, J) \) be the corresponding \( H \)-module (from Definition 3.1). From these data one can construct a \( W \)-graph \( \Gamma \) with \( M_\Gamma = \mathcal{I}(\mathcal{I}, J) \). Specifically, the following results are proved in [5].

**Lemma 3.3.** [5 Lemma 7.2.] The \( H \)-module \( \mathcal{I}(\mathcal{I}, J) \) in Definition 3.1 has a unique \( A \)-basis \( C = \{ c_w \mid w \in \mathcal{I} \} \) such that for all \( w \in \mathcal{I} \) we have \( \tau(w) = c_w \) and

\[
\tag{3.2}
 b_w = c_w + q \sum_{y < w} q_{y,w} c_y
\]

for certain polynomials \( q_{y,w} \in \mathcal{A}^+ \).

Define \( \mu_{y,w} \) to be the constant term of \( q_{y,w} \). The polynomials \( q_{y,w} \), where \( y < w \), can be computed recursively by the following formulae.

**Corollary 3.4.** [5 Corollary 7.4] Suppose that \( w < sw \in \mathcal{I} \) and \( y < sw \). If \( y = w \) then \( q_{y,sw} = 1 \), and if \( y \neq w \) we have the following formulas:

\[
\begin{align*}
& (i) \quad q_{y,sw} = q_{y,w} \text{ if } s \in A(y), \\
& (ii) \quad q_{y,sw} = -q^{-1}(q_{y,w} - \mu_{y,w}) + \sum_x \mu_{y,x} q_{x,w} \text{ if } s \in SD(y), \\
& (iii) \quad q_{y,sw} = -q^{-1}(q_{y,w} - \mu_{y,w}) + \sum_x \mu_{y,x} q_{x,w} \text{ if } s \in WD(y),
\end{align*}
\]

where \( q_{y,w} \) and \( \mu_{y,w} \) are regarded as 0 if \( y \not< w \), and in (ii) and (iii) the sums extend over all \( x \in \mathcal{I} \) such that \( y < x < w \) and \( s \notin D(x) \).

Let \( \mu : C \times C \to \mathbb{Z} \) be given by

\[
\mu(c_y, c_w) = \begin{cases} 
\mu_{y,w} & \text{if } y < w, \\
\mu_{w,y} & \text{if } w < y, \\
0 & \text{otherwise},
\end{cases}
\]

and let \( \tau \) from \( C \) to the power set of \( S \) be given by \( \tau(c_w) = D(w) \) for all \( y \in \mathcal{I} \).

**Theorem 3.5.** [5 Theorem 7.5.] The triple \( (C, \mu, \tau) \) is a \( W \)-graph.

It is immediate from Corollary 3.3 that if \( w < sw \in \mathcal{I} \) then \( \mu_{w,sw} = q_{w,sw} = 1 \), and since also \( D(sw) \notin D(w) \) (since \( s \in D(sw) \setminus D(w) \)) it follows that \( c_{sw} \leq \Gamma(C) c_w \).

A straightforward induction on length now yields the first part of the following result, which in turn immediately yields the second part.

**Corollary 3.6.**

(i) Let \( x \) and \( y \) be in \( \mathcal{I} \). If \( x \leq_L y \) then \( c_y \leq \Gamma(C) c_x \).

(ii) Let \( C \) be a cell of \( \Gamma(C) \) and let \( \mathcal{I}(C) = \{ w \in \mathcal{I} \mid c_w \in C \} \). If \( x, y \in \mathcal{I}(C) \) then the interval \([x, y]\) is contained in \( \mathcal{I}(C) \).

Inverting Equation (3.2), we have

\[
c_w = b_w - \sum_{y < w} q_{y,w} b_y,
\]

where \( p_{y,w} \in \mathcal{A}^+ \) are defined recursively by

\[
p_{y,w} = q_{y,w} - \sum_{y < x < w} q_{y,x} q_{x,w} \quad \text{if } y < w.
\]

Note that \( \mu_{y,w} \) is the constant term of \( p_{y,w} \).
4. Subideals

The main result of this section says (essentially) that a subideal of a W-graph ideal is a W-graph ideal provided that its complement is closed. We assume that $\mathcal{J}$ is an ideal in $(W,\leq_L)$ with $\mathcal{J} \subseteq \mathcal{A}_0$, where $\mathcal{A}_0$ is a W-graph ideal with respect to $J \subseteq \text{Pos}(\mathcal{A}_0)$. We adapt the notation of Section 3 by attaching a subscript or superscript 0 to objects associated with the W-graph ideal $\mathcal{A}_0$. Thus we write $\mathcal{A}_0$ for the $\mathcal{H}$-module associated with $\mathcal{A}_0$ and $(b^0_w \mid w \in \mathcal{A}_0)$ for the basis of $\mathcal{A}_0$ that satisfies the conditions of Definition 3.1. By Lemma 3.3 and Theorem 3.5 we know that $\mathcal{A}_0$ has a W-graph basis $C_0 = \{c^0_w \mid w \in \mathcal{A}_0\}$ such that

\begin{equation}
 b^0_w = c^0_w + \sum_{y \in \mathcal{A}_0} q^0_{y,w} c^0_y
\end{equation}

and

\begin{equation}
 c^0_w = b^0_w - \sum_{y \in \mathcal{A}_0} q^0_{y,w} b^0_y
\end{equation}

where the polynomials $p^0_{y,w}, q^0_{y,w} \in \mathcal{A}^+$ are defined whenever $y < w$ and are related by

\begin{equation}
 p^0_{y,w} = q^0_{y,w} - \sum_{y < x < w} q^0_{y,x} q^0_{x,w}.
\end{equation}

Let $\mu^0_{y,w}$ be the constant term of $q^0_{y,w}$ (or, equivalently, of $p^0_{y,w}$), so that, by Theorem 3.4, the triple $\Gamma(C_0) = (C_0, \mu, \tau)$ is a W-graph, where the functions $\mu$ and $\tau$ are given by

\begin{equation}
 \mu^0(c^0_y, c^0_w) = \begin{cases} 
 \mu^0_{y,w} & \text{if } y < w \\
 \mu^0_{w,y} & \text{if } w < y
 \end{cases}
\end{equation}

and $\tau(c^0_w) = D_J(\mathcal{A}_0, w) = SD(\mathcal{A}_0, w) \cup WD_J(\mathcal{A}_0, w)$, for all $y, w \in \mathcal{A}_0$.

Assuming that $\mathcal{J}$ is a sub-ideal of $(\mathcal{A}_0,\leq_L)$, let $\mathcal{J}' = \mathcal{A}_0 \setminus \mathcal{J}$. Throughout this section, we assume that the set $C' = \{c^0_w \mid w \in \mathcal{J}'\}$ spans an $\mathcal{H}$-submodule of $\mathcal{A}_0$. Recall that this is equivalent to saying that $C'$ is a closed subset of $C_0$. By this assumption, the $\mathcal{H}$-module $\mathcal{J}' = \mathcal{AC}'$ is obtained from a W-graph, namely the subgraph of $\Gamma(C_0)$ with $C'$ as its vertex set and with $\tau$ invariants and edge weights inherited from $\Gamma(C_0)$. Moreover, the subgraph of $\Gamma(C_0)$ with vertex set $\{c^0_w \mid w \in \mathcal{J}'\} = C_0 \setminus C'$ and $\tau$ invariants and edge weights inherited from $\Gamma(C_0)$ is also a W-graph, and the corresponding $\mathcal{H}$-module $\mathcal{J}'$ is isomorphic to $\mathcal{A}_0/\mathcal{J}'$. We define $f: \mathcal{J}_0 \to \mathcal{J}$ to be the homomorphism with kernel $\mathcal{J}'$, and define $c_w = f(c^0_w)$ for all $w \in \mathcal{J}$, so that $C = \{c_w \mid w \in \mathcal{J}\}$ is the W-graph basis of $\mathcal{J}$.

Observe that $J \subseteq \text{Pos}(\mathcal{A}_0) \subseteq \text{Pos}(\mathcal{J})$, and so it makes sense to ask whether $\mathcal{J}$ is a W-graph ideal with respect to $J$. Note that the definitions immediately imply that $\text{SD}(\mathcal{A}_0, w) = \text{SD}(\mathcal{J}, w)$ and $\text{WD}_J(\mathcal{A}_0, w) = \text{WD}_J(\mathcal{J}, w)$, and that $\text{WA}_J(\mathcal{A}_0, w) \subseteq \text{WA}_J(\mathcal{J}, w)$, for all $w \in \mathcal{J}$. Since there may exist an $s \in S$ with $sw \in \mathcal{J}_0 \setminus \mathcal{J}$, it is not necessarily the case that $\text{WA}_J(\mathcal{J}, w) = \text{WA}_J(\mathcal{A}_0, w)$.

For each $w \in \mathcal{J}$ we define $b_w = T_w c_1$, and we put $B = \{b_w \mid w \in \mathcal{J}\}$.

**Lemma 4.1.** The $\mathcal{H}$-module $\mathcal{J}$ is $\mathcal{A}$-free with $\mathcal{A}$-basis $B$.

**Proof.** Let us compute $f(b^0_w)$ for each $w \in \mathcal{J}$. We have

\begin{equation}
 f(b^0_w) = f(T_w b^1_w) = T_w f(b^0_1) = T_w f(c^0_1) = T_w c_1 = T_w b_1 = b_w
\end{equation}
for each \( w \in \mathcal{J} \). It now follows from Equation \( \text{(4.1)} \) that for all \( w \in \mathcal{J} \)

\[
b_w = f(b_w^0)
\]

\[
= f(c_w^0 + \sum_{y \in \mathcal{J}_0 \setminus \mathcal{J}, y < w} q_{y,w}^0 c_y^0)
\]

\[
= f(c_w^0) + \sum_{y \in \mathcal{J}_0 \setminus \mathcal{J}, y < w} q_{y,w}^0 f(c_y^0)
\]

\[
= c_w + \sum_{y \in \mathcal{J}, y < w} q_{y,w}^0 c_y
\]

since \( f(c_y^0) = 0 \) for all \( y \in \mathcal{J}_0 \setminus \mathcal{J} \). Thus, for each \( w \in \mathcal{J} \)

\[
b_w = c_w + \sum_{y \in \mathcal{J}, y < w} q_{y,w}^0 c_y
\]

where we have defined \( q_{y,w} = q_{y,w}^0 \) whenever \( y, w \in \mathcal{J} \) and \( y < w \). Now since \( C \) is an \( \mathcal{A} \)-basis for \( \mathcal{J} \), we see that \( B \) is also an \( \mathcal{A} \)-basis for \( \mathcal{J} \), as claimed. \( \square \)

**Lemma 4.2.** For each \( w \in \mathcal{J}' \),

\[
f(b_w^0) = \sum_{y \in \mathcal{J}, y < w} r_{y,w} b_y \quad \text{for some } r_{y,w} \in q\mathcal{A}^+.
\]

**Proof.** We have, for each \( w \in \mathcal{J}' \),

\[
0 = f(c_w^0) = f(b_w^0 - \sum_{y \in \mathcal{J}_0, y < w} q_{y,w}^0 b_y)
\]

\[
= f(b_w^0) - \sum_{y \in \mathcal{J}_0, y < w} q_{y,w}^0 f(b_y^0)
\]

\[
= f(b_w^0) - \sum_{y \in \mathcal{J}', y < w} q_{y,w}^0 f(b_y^0) - \sum_{y \in \mathcal{J}, y < w} q_{y,w}^0 b_y.
\]

Now by rearranging terms, Equation \( \text{(4.6)} \) becomes

\[
f(b_w^0) = \sum_{y \in \mathcal{J}', y < w} q_{y,w}^0 f(b_y^0) + \sum_{y \in \mathcal{J}, y < w} q_{y,w}^0 b_y
\]

for each \( w \in \mathcal{J}' \). We now use induction on \( l(w) \) to show that Equation \( \text{(4.5)} \) holds for all \( w \in \mathcal{J}' \).

Suppose first that \( w \) is of minimal length subject to \( w \in \mathcal{J}' \). Equation \( \text{(4.7)} \) gives

\[
f(b_w^0) = \sum_{y \in \mathcal{J}, y < w} q_{y,w}^0 b_y
\]

since minimality of \( w \) implies that the set \( \{ y \in \mathcal{J}' \mid y < w \} \) is empty. Thus

\[
f(b_w^0) = \sum_{y \in \mathcal{J}} r_{y,w} b_y \quad \text{where } r_{y,w} = q_{y,w}^0 \in q\mathcal{A}^+,
\]

as required.
Now let \( w \in \mathcal{J}' \) be arbitrary and assume that the result holds for all \( y \in \mathcal{J}' \) such that \( l(y) < l(w) \); that is, assume that
\[
(4.8) \quad f(b^0_y) = \sum_{x \in \mathcal{J}, x < y} r_{x,y} b_x \quad \text{for some } r_{x,y} \in qA^+.
\]

Equation (4.7) and Equation (4.8) give
\[
f(b^0_w) = \sum_{y \in \mathcal{J}', y < w} \left( \sum_{x \in \mathcal{J}, x < y} q^0_{y,w} \left( \sum_{x \in \mathcal{J}, x < y} r_{x,y} b_x \right) + \sum_{y \in \mathcal{J}', y < w} q^0_{y,w} b_y \right) \\
= \sum_{x \in \mathcal{J}, x < w} \left( \sum_{y \in \mathcal{J}', y < w} q^0_{y,w} r_{x,y} \right) b_x + \sum_{x \in \mathcal{J}, x < w} q^0_{x,w} b_x \\
= \sum_{x \in \mathcal{J}, x < w} \left( q^0_{x,w} + \sum_{y \in \mathcal{J}', y < w} q^0_{y,w} r_{x,y} \right) b_x.
\]

It follows that
\[
f(b^0_w) = \sum_{y \in \mathcal{J}, y < w} r_{y,w} b_y,
\]
where \( r_{y,w} = q^0_{y,w} + \sum_{x \in \mathcal{J}, y < x < w} q^0_{x,w} r_{x,y} \in qA^+ \), and we are done. \( \Box \)

**Remark 3.** In the expression for \( r_{y,w} \) in the proof above, we see that if \( y = sw < w \) then \( r_{y,w} = q \), since \( \{ x \mid y < x < w \} = \emptyset \) and \( p^0_{y,w} = q^0_{y,w} = 1 \).

For future reference, we state the formula for the coefficients \( r_{y,w} \) in the proof above in the following corollary (minding Remark 3).

**Corollary 4.3.** For each \( w \in \mathcal{J}' \) and \( y \in \mathcal{J} \), the coefficients appearing in Equation (4.8) are given by
\[
(4.9) \quad r_{y,w} = q^0_{y,w} + \sum_{x \in \mathcal{J}, y < x < w} q^0_{x,w} r_{x,y} \in qA^+.
\]

In particular, \( r_{y,w} = q \) if \( y = sw < w \).

We now prove the first main result of the paper.

**Theorem 4.4.** Let \( \mathcal{I}_0 \) be a W-graph ideal with respect to \( J \subseteq \operatorname{Pos}(\mathcal{I}_0) \) and let \( C_0 = \{ c^0_w \mid w \in \mathcal{I}_0 \} \) be the W-graph basis of the module \( \mathcal{I}_0 = \mathcal{I}(\mathcal{I}_0, J) \). Suppose that \( \mathcal{I} \) is a sub-ideal of \( \mathcal{I}_0 \) such that \( \{ c^0_w \mid w \in \mathcal{I}_0 \setminus \mathcal{I} \} \) is a closed subset of \( C_0 \). Then \( \mathcal{I} \) is a W-graph ideal with respect to \( J \). Moreover, the corresponding W-graph is isomorphic to the full subgraph of \( \Gamma(\mathcal{I}_0) \) on the vertex set \( \{ c^0_w \mid w \in \mathcal{I} \} \subseteq C_0 \), with \( \tau \) and \( \mu \) functions inherited from \( \Gamma(\mathcal{I}_0) \).

**Proof.** We need to verify that the ideal \( \mathcal{I} \) satisfies the hypotheses required in Definition 3.1. All we need to show is that the \( H \)-module \( \mathcal{I} \) as constructed above satisfies the required conditions. By Lemma 4.1, \( \mathcal{I} \) is \( \mathcal{A} \)-free with a free \( \mathcal{A} \)-basis given by \( B = \{ b_w \mid w \in \mathcal{I} \} \), where \( b_w = f(b^0_w) \) for each \( w \in \mathcal{I} \). (Recall that \( f \) is natural homomorphism from \( \mathcal{I}_0 \) onto \( \mathcal{I} \)). To complete the verification of Condition (i) in Definition 3.1, we proceed to work out how the generators \( T_s \) act on the basis elements \( b_w \).
Let \( w \in \mathcal{I} \) and let \( s \in S \). If \( s \in \text{SA}(\mathcal{I}, w) \) then \( w < sw \in \mathcal{I} \subseteq \mathcal{I}_0 \), whence \( s \in \text{SA}(\mathcal{I}_0, w) \), and Equation (3.1) for \( \mathcal{I}_0 \) gives \( T_s b^0_w = b^0_{sw} \). So
\[
T_s b_w = T_s f(b^0_w) = f(T_s b^0_w) = f(b^0_{sw}) = b_{sw}
\]
in accordance with the requirements of Definition 3.1 (i). If \( s \in \text{SD}(\mathcal{I}, w) \) then \( w > sw \), whence \( s \in \text{SD}(\mathcal{I}_0, w) \), and \( T_s b^0_w = b^0_{sw} + (q - q^{-1})b_w \) by Equation (3.1) for \( \mathcal{I}_0 \). So
\[
T_s b_w = T_s f(b^0_w) = f(T_s b^0_w) = f(b^0_{sw} + (q - q^{-1})b^0_w) = b_{sw} + (q - q^{-1})b_w,
\]
again in accordance with the requirements of Definition 3.1 (i). If \( s \in \text{WD}_J(\mathcal{I}, w) \) then \( sw = wt \) for some \( t \in J \), whence \( s \in \text{WD}_J(\mathcal{I}_0, w) \), and Equation (3.1) for \( \mathcal{I}_0 \) gives \( T_s b^0_w = -q^{-1}b_{sw} \). So
\[
T_s b_w = T_s f(b^0_w) = f(T_s b^0_w) = f(b^0_{sw} - q^{-1}b^0_w) = -q^{-1}b_w,
\]
and again the requirements of Definition 3.1 (i) are satisfied.

Finally, suppose that \( s \in \text{WA}_J(\mathcal{I}, w) \), so that \( w < sw \notin \mathcal{I} \). It follows that either \( s \in \text{SA}_J(\mathcal{I}_0, w) \) or \( s \in \text{WA}_J(\mathcal{I}_0, w) \), depending on whether \( sw \in \mathcal{I}_0 \setminus \mathcal{I} \) or \( sw \notin \mathcal{I}_0 \). In the former case Equation (3.1) for \( \mathcal{I}_0 \) gives \( T_s b^0_w = b^0_{sw} \), and so
\[
(4.10) \quad T_s b_w = T_s f(b^0_w) = f(T_s b^0_w) = f(b^0_{sw}) = q b_w - \sum_{y \in J \atop y < sw} r^s_{y,w} b_y,
\]
where \( r^s_{y,w} = -r_{y,sw} \in qA^+ \), by Lemma 4.2 and Corollary 4.3. On the other hand, if \( s \in \text{WA}_J(\mathcal{I}_0, w) \), then, by Equation (3.1) for \( \mathcal{I}_0 \) and Lemma 4.2
\[
T_s b_w = T_s f(b^0_w) = f(T_s b^0_w) = f(qb^0_w - \sum_{y \in J \atop y < sw} r^0_{y,w} b_y)
\]
\[
= qb_w - \sum_{y \in J \atop y < sw} r^0_{y,w} b_y - \sum_{y \in J \atop y < sw} r^0_{y,w} f(b_y)
\]
\[
= qb_w - \sum_{y \in J \atop y < sw} r^0_{y,w} b_y - \sum_{y \in J \atop y < sw} r^0_{y,w} \left( \sum_{x \in J \atop x < y} r_{x,y} b_x \right)
\]
\[
= qb_w - \sum_{y \in J \atop y < sw} r^0_{y,w} b_y - \sum_{y \in J \atop y < sw} \left( \sum_{x \in J \atop x < y} r^0_{y,w} r_{x,y} \right) b_x
\]
\[
= qb_w - \sum_{y \in J \atop y < sw} \left( r^0_{y,w} + \sum_{x \in J \atop x < y} r^0_{x,w} r_{y,x} \right) b_y.
\]
Thus we have shown that
\[
(4.11) \quad T_s b_w = qb_w - \sum_{y \in J \atop y < sw} r^s_{y,w} b_y
\]
where \( r^s_{y,w} = r^0_{y,w} + \sum r^0_{x,w} r_{y,x} \), and \( r^s_{y,w} \in qA^+ \) by Definition 3.1 and Corollary 4.3. Hence in either case the requirements of Definition 3.1 (i) are satisfied.

The second assertion of the theorem is obviously satisfied, by the way we defined the \( \mathcal{H} \)-module \( \mathcal{I} \). This also ensures that Condition (ii) of Definition 3.1 holds, since, as we observed in Section 2, every module arising from a \( W \)-graph admits
a semilinear involution $\alpha \mapsto \overline{\alpha}$ that fixes the elements of the $W$-graph basis and satisfies $h\overline{\alpha} = \overline{h\alpha}$ for all $h \in \mathcal{H}$.

Let $\mathcal{H}$ be a $W$-graph ideal with respect to $J \subseteq \text{Pos}(\mathcal{H})$ and let $C = \{ c_w \mid w \in \mathcal{H} \}$ be the corresponding $W$-graph basis of the module $\mathcal{I}(\mathcal{H}, J)$. Let $\mathcal{C}$ be the set of cells of $\Gamma = \Gamma(C)$. We have the following result.

**Lemma 4.5.** Let $C_1 \in \mathcal{C}$ be the cell that contains $c_1$. Then $C_1$ is the unique maximal element of $(\mathcal{C}, \leq_{\Gamma})$.

*Proof.* The result follows readily from Corollary 3.9 (i) and the fact that $1 \leq_L w$ for all $w \in \mathcal{H}$. □

For every $D \subseteq C$ define $\mathcal{I}_D = \{ w \in \mathcal{H} \mid c_w \in D \}$, and for each cell $\mathcal{C} \in \mathcal{C}$ define $\mathcal{C}$ to be the union of all $D \in \mathcal{C}$ such that $\mathcal{C} \leq_{\Gamma} D$. Note that Lemma 4.5 tells us that $C_1 \subseteq \mathcal{C}$ and hence that $1 \in \mathcal{I}_C$.

**Lemma 4.6.** For each $\mathcal{C} \in \mathcal{C}$ the sets $\mathcal{I}_C$ and $\mathcal{I}_{C \setminus \mathcal{C}}$ are ideals of $(W, \leq_L)$.

*Proof.* To show that $\mathcal{I}_C$ is an ideal it suffices to show that if $w \in \mathcal{I}_C$ and $s \in S$ with $sw <_L w$, then $sw \in \mathcal{I}_C$. Let $D$ and $D'$ be the cells that contain $c_w$ and $c_{sw}$ respectively. Since $w \leq_L w$ it follows from Corollary 3.9 (i) that $D' \geq_{\Gamma} D$. But $D \geq_{\Gamma} \mathcal{C}$ since $w \in \mathcal{I}_C$, so $D' \geq_{\Gamma} \mathcal{C}$. Hence $c_{sw} \in \mathcal{C}$, so that $sw \in \mathcal{I}_C$, as desired.

The proof of the other part is similar. □

**Lemma 4.7.** Let $\mathcal{C} \in \mathcal{C}$. Then $\overline{\mathcal{C}} = C \setminus \mathcal{C}$ and $\overline{\mathcal{C}} \cup \mathcal{C}$ are closed subsets of $C$.

*Proof.* To show that $\overline{\mathcal{C}}$ is closed it is sufficient to show that whenever $c_w \in \overline{\mathcal{C}}$ and $c_y \in \mathcal{C}$ is such that $c_y \leq_{\Gamma} c_w$, then $c_y \in \overline{\mathcal{C}}$. Given such elements $c_w$ and $c_y$, let $\mathcal{Y}$ and $\mathcal{W}$ be the cells that contain $c_y$ and $c_w$. Then $\mathcal{W} \not\geq_{\Gamma} \mathcal{C}$ since $c_w \notin \mathcal{C}$, and since $\mathcal{W} \geq_{\Gamma} \mathcal{Y}$ it follows that $\mathcal{Y} \not\geq_{\Gamma} \mathcal{C}$, whence $c_y \notin \overline{\mathcal{C}}$, as required. A similar argument proves that $\overline{\mathcal{C}} \cup \mathcal{C}$ is also closed. □

**Corollary 4.8.** For each $\mathcal{C} \in \mathcal{C}$ the sets $\mathcal{I}_C$ and $\mathcal{I}_{C \setminus \mathcal{C}}$ are $W$-graph ideals with respect to $J$. The associated $W$-graphs are the corresponding full subgraphs of $\Gamma$, with $\tau$ and $\mu$ inherited from $\Gamma$.

*Proof.* Since Lemma 4.6 shows that $\mathcal{I}_C$ and $\mathcal{I}_{C \setminus \mathcal{C}}$ are subideals of the $W$-graph ideal $\mathcal{H}$, and Lemma 4.7 shows that the complements of $\overline{\mathcal{C}}$ and $\overline{\mathcal{C}} \setminus \mathcal{C}$ are closed subsets of $C$, the result follows immediately from Theorem 4.4. □

**Remark 4.** In the above situation, the closed subsets $\overline{\mathcal{C}}$ and $\overline{\mathcal{C}} \setminus \mathcal{C}$ of $C$ span $\mathcal{H}$-submodules $M_{\Gamma(\mathcal{C})}$ and $M_{\Gamma(\overline{\mathcal{C}} \setminus \mathcal{C})}$ of $M_{\Gamma} = \mathcal{I}(\mathcal{H}, J)$. Furthermore, the factor module $M_{\Gamma(\overline{\mathcal{C}} \setminus \mathcal{C})}/M_{\Gamma(\mathcal{C})}$ is isomorphic to the $\mathcal{H}$-module determined by the cell $\mathcal{C}$, which in turn is isomorphic to the kernel of the natural homomorphism $f : \mathcal{I}(\mathcal{H}, J) \rightarrow \mathcal{I}(\mathcal{I}_{\overline{\mathcal{C}} \setminus \mathcal{C}}, J)$.

For later reference, we record the following special case of Corollary 4.8 obtained by setting $\mathcal{C} = \mathcal{C}_1$.

**Lemma 4.9.** The set $\mathcal{F}_1 = \mathcal{F}_{\mathcal{C}_1}$ is a $W$-graph ideal, and the corresponding $W$-graph is exactly that of the maximal cell of $\Gamma$. 

5. Cells in the ideal of minimal coset representatives

Let \((W, S)\) be a Coxeter system, and let \(H = H(W)\) be the associated Hecke algebra. Let \(J\) be an arbitrary subset of \(S\) and \(D_J\) the set of distinguished left coset representatives for \(W_J\). It is easily shown that if \(u \in W\) is a suffix of some element of \(D_J\) then \(u\) is also in \(D_J\); so \(\mathcal{I} = D_J\) is an ideal of \((W, \leq_L)\). Clearly \(\text{Pos}(\mathcal{I}) = J\). In \([5]\) Section 8], it is shown that \(\mathcal{I}\) is a W-graph ideal with respect to \(\emptyset\) and also with respect to \(J\). Here we consider the latter case only. We briefly review the main facts, referring the reader to \([5]\) for the full details.

Let \(H_J\) be the Hecke algebra associated with the Coxeter system \((W_J, J)\). Let \(A_\phi\) be \(A\) made into an \(H_J\)-module via the homomorphism \(\phi: H_J \rightarrow A\) defined by \(\phi(T_u) = (-q)^{-l(u)}\) for all \(u \in W_J\), and let \(A_\phi = H \otimes_{H_J} A_\phi\), the \(H\)-module induced from \(A_\phi\) (so that \(A_\phi\) is essentially the module \(M^J\) of \([2]\) in the case \(u = -1\)). Then \(A_\phi\) is \(A\)-free with \(A\)-basis \(B_J = \{ b_{vw}^J \mid w \in D_J \}\), where \(b_{vw}^J = T_w \otimes 1\) for each \(w \in D_J\).

All the conditions in Definition 3.1 are satisfied, and so \(A_\phi\) has a W-graph basis \(C_J = \{ c_{vw}^J \mid w \in D_J \}\) such that \(c_{vw}^J = b_{vw}^J - \sum_{y < w} q p_{y, w}^J b_{vy}^J\) for all \(w \in D_J\), where the polynomials \(p_{y, w}^J\) are given by the formulas in Section 4.4 above. Note that in the special case \(J = \emptyset\) the module \(A_\phi\) is isomorphic to the left regular module \(H\), and the W-graph basis is \(C_\emptyset = \{ c_w \mid w \in W\}\), the Kazhdan-Lusztig basis of \(H\). In this case the W-graph \(\Gamma(C_J)\) becomes the regular Kazhdan-Lusztig W-graph \(\Gamma(W)\), and \(c_w = T_w - \sum_{y < w} q p_{y, w} T_y\) for all \(w \in W\); see \([5]\) Proposition 8.2).

We shall show that if \(W_J\) is finite then the W-graph \(\Gamma(C_J)\) is isomorphic to the W-graph of a certain union of left cells in \(\Gamma(W)\).

**Proposition 5.1.** If \(J \subseteq S\) and \(W_J\) is finite then the polynomials \(p_{y, w}^J\) and \(p_{y, w}\) defined above are related via the formula \(p_{y, w}^J = p_{yw, wsw}^J\), where \(w_J\) is the longest element in \(W_J\).

**Proof.** In view of the relationship between our polynomials \(p_{y, w}^J\) and Deodhar’s polynomials \(P_{y, w}^J\) (see \([5]\) Proposition 8.4]), and the relationship between our polynomials \(p_{y, w}\) and the Kazhdan-Lusztig polynomials \(P_{y, w}\) (see \([5]\) Proposition 8.2]), this result is immediate from \([2]\) Proposition 3.4].

For each \(w \in W\), define \(L(w) = \{ s \in S \mid sw < w \}\). Note that this is the \(\tau\)-invariant of the vertex \(c_w\) of \(\Gamma(W)\).

**Lemma 5.2.** If \(W_J\) is finite and \(w \in D_J\), then \(L(wsw_J) = D_J(\mathcal{I}, w)\), where \(\mathcal{I} = D_J\) (as above).

**Proof.** Suppose first that \(s \in L(wsw_J)\), so that \(sww_J < wsw\). If \(sw < w\) then \(s \in SD_J(w) \subseteq D_J(w)\). On the other hand, if \(sw > w\) then \(sw \notin D_J = \mathcal{I}\), since otherwise we would have \(l(sw) = l(sw) + l(w_J) > l(w) + l(w_J) = l(ww_J)\). So in this case \(s \in WD_J(w) \subseteq D_J(w)\), and we conclude that \(L(wsw_J) \subseteq SD_J(w)\).

Suppose conversely that \(s \in D_J(w)\). If \(s \in SD_J(w)\) then \(sw < w\), and it follows that \(l(sw) = l(sw) + l(w_J) < l(w) + l(w_J) = l(ww_J)\), so that \(s \in L(wsw_J)\). If \(s \in WD_J(w)\) then \(sw \notin D_J = \mathcal{I}\); so \(s \in D_J\), but since \(w \in D_J\) it follows from Lemma 2.1 (iii) of \([2]\) that \(sw = sw'\) for some \(sw'\) in \(J\). Thus \(l(sw) = l(swsw) = l(w) + l(sw)\), since \(w \in D_J\) and \(s'w_J \in W_J\). But \(l(sw) < l(w_J)\), since \(w_J\) is the longest element in \(W_J\), and so \(l(w) + l(s'w_J) < l(w) + l(w_J) = l(ww_J)\). Hence \(s \in L(wsw_J)\), and we conclude that \(D_J(w) \subseteq L(wsw_J)\), as required.

The following result is immediate from \([4]\) Lemma 2.8].
Lemma 5.3. The set $D_Jw_J$ is a union of Kazhdan-Lusztig left cells: we have

$$D_Jw_J = \{ w \in W \mid w \leq_J w_J \}.$$ 

Furthermore, $\mathcal{A}C_J = M_{\Gamma(C_J)} \cong \mathcal{H}c_{w_J} \cong \mathcal{J}_0$ as left $\mathcal{H}$-modules.

More explicitly, the $\mathcal{H}$-module isomorphism $M_{\Gamma(C_J)} \cong \mathcal{H}c_{w_J}$ derives from an isomorphism of $W$-graphs. Lemma 5.2 and Proposition 5.1 show that the mapping $c_w \mapsto c_{w_J}$ from $C_J$ to $C_0$ induces an isomorphism of the $W$-graph $\Gamma(C_J)$ (obtained from $D_J$ considered as a $W$-graph ideal with respect to $J$) with the full subgraph of $\Gamma(W)$ corresponding to the set $D_Jw_J$. The mapping preserves edge-weights and $\tau$-invariants, and hence preserves cells and the partial order on cells.

As an immediate consequence of Lemma 4.9 and the above remarks, we obtain the second main result of this paper.

Theorem 5.4. Let $J \subseteq S$ be such that $W_J$ is finite, and let $\mathcal{C}$ be the Kazhdan-Lusztig left cell that contains $w_J$. Then $\mathcal{C} = \mathcal{J}_1w_J$, where $\mathcal{J}_1 \subseteq D_J$ is a $W$-graph ideal with respect to $J$, and $\mathcal{C}_1 = \{ c_{w} \mid w \in \mathcal{J}_1 \}$ is the maximal cell of $\Gamma(C_J)$.

The cell $\mathcal{C}$ in Theorem 5.4 is the maximal cell in $D_Jw_J$, and the theorem tells us that $\mathcal{H}c_{w_J}/\mathcal{A}(D_Jw_J \setminus \mathcal{C}) \cong \mathcal{A}C_J/\mathcal{A}(C_J \setminus \mathcal{C}_1) \cong \mathcal{J}(\mathcal{J}_1, J)$ as $\mathcal{H}$-modules.

6. $W$-GRAPHS FOR LEFT CELLS IN TYPE $A$

Let $W_n$ be the Coxeter group of type $A_{n-1}$, which we identify with the symmetric group on $[1, n]$, the set of integers from 1 to $n$, by identifying the simple reflections $s_1, s_2, \ldots, s_{n-1}$ in $W_n$ with the transpositions $(1, 2), (3, 4), \ldots, (n-1, n)$ (respectively). We use a left-operator convention for permutations, writing $w$ for the action of $w \in W_n$ on $i \in [1, n]$.

Since the principal objective of this section is to prove Proposition 6.3 of [5], we start by reviewing the conventions and terminology of that paper.

A sequence of nonnegative integers $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$ is called a partition of $n$ if $\lambda_1 + \lambda_2 + \cdots + \lambda_k = n$ and $\lambda_1 \geq \cdots \geq \lambda_k$. We define $P(n)$ to be the set of all partitions of $n$. For each $\lambda = (\lambda_1, \ldots, \lambda_k) \in P(n)$ we define

$$[\lambda] = \{ (i, j) \mid 1 \leq j \leq \lambda_i \text{ and } 1 \leq i \leq k \},$$

and refer to this as the Young diagram of $\lambda$. Pictorially $[\lambda]$ is represented by a left-justified array of boxes with $\lambda_i$ boxes in the $i$-th row; the pair $(i, j) \in [\lambda]$ corresponds to the $j$-th box in the $i$-th row.

For $\lambda \in P(n)$, define $\lambda'$ by $\lambda'_i = | \{ \lambda_j \mid \lambda_j \geq i \} |$, and call $\lambda'$ the partition conjugate to $\lambda$. The Young diagram of $\lambda'$ is the transpose of the Young diagram of $\lambda$: the number of boxes in the $i$-th row of $[\lambda']$ equals the number of rows in the $i$-th row of $[\lambda']$.

If $\lambda$ is a partition of $n$ then a $\lambda$-tableau is a bijection $t: [\lambda] \rightarrow [1, n]$. The partition $\lambda$ is called the shape of the tableau $t$, and we write $\lambda = \text{Shape}(t)$. For each $i \in [1, n]$ we define $\text{row}_i(t)$ and $\text{col}_i(t)$ to be the row index and column index of $i$ in $t$ (so that $t^{-1}(i) = (\text{row}_i(t), \text{col}_i(t))$). We define $\text{Tab}(\lambda)$ to be the set of all $\lambda$-tableaux, and we let $t^\lambda$ be the specific $\lambda$-tableau given by

$$t^\lambda(i, j) = j + \sum_{h=1}^{i-1} \lambda_h$$
for all \((i, j) \in [\lambda]\). That is, the numbers 1, 2, \ldots, \lambda_1 fill the first row of \([\lambda]\) in order from left to right, then the numbers \(\lambda_1 + 1, \lambda_1 + 2, \ldots, \lambda_1 + \lambda_2\) similarly fill the second row, and so on. We also define \(t_\lambda\) to be the \(\lambda\)-tableau that is the transpose of the \(\lambda'\)-tableau \(t^{\lambda'}\), where \(\lambda'\) is the conjugate of \(\lambda\). Thus in \(t_\lambda\) the numbers 1, 2, \ldots, \lambda_1 fill the first column in order from top to bottom, then the numbers \(\lambda_1 + 1, \lambda_1 + 2, \ldots, \lambda_1' + \lambda_2'\) fill the next column from top to bottom, and so on.

It is clear that for any fixed \(\lambda \in P(n)\) the group \(W_n\) acts on the set of all \(\lambda\)-tableaux, via \((wt)(i,j) = w(t(i,j))\) for all \((i,j) \in [\lambda]\), for all \(\lambda\)-tableaux \(t\) and all \(w \in W_n\). Moreover, the map from \(W_n\) to \(\text{Tab}(\lambda)\) defined by \(w \mapsto wt_\lambda\) for all \(w \in W_n\) is bijective. We use this bijection to transfer the left weak Bruhat and the Bruhat partial orders from \(W_n\) to \(\text{Tab}(\lambda)\). Thus if \(t_1, t_2\) are arbitrary \(\lambda\)-tableaux and we write \(t_1 = w_1 t_\lambda\) and \(t_2 = w_2 t_\lambda\) with \(w_1, w_2 \in W_n\), then by definition \(t_1 \leq t_2\) if and only if \(w_1 \leq w_2\), and \(t_1 \leq_L t_2\) if and only if \(w_1 \leq_L w_2\). Similarly, if \(t = wt_\lambda\) is an arbitrary \(\lambda\)-tableau, where \(w \in W_n\), then we define \(l(t) = l(w)\).

A \(\lambda\)-tableau \(t\), where \(\lambda \in P(n)\), is said to be column standard if its entries increase down the columns, that is, if \((i, j) < t(i+1, j)\) whenever \((i, j) \in [\lambda]\) and \((i+1, j) \in [\lambda]\). Similarly, \(t\) is said to be row standard if its entries increase along the rows, that is, if \((i, j) < t(i, j+1)\) whenever \((i, j) \in [\lambda]\) and \((i, j+1) \in [\lambda]\). A standard tableau is a tableau that is both column standard and row standard. We write \(\text{STD}(\lambda)\) for the set of all standard tableaux for \(\lambda\).

Given \(\lambda \in P(n)\) we define \(J_\lambda\) to be the subset of \(S\) consisting of those simple reflections \(s_i = (i, i+1)\) such that \(i\) and \(i+1\) lie in the same column of \(t_\lambda\), and we define \(W_\lambda\) to be the standard parabolic subgroup of \(W_n\) generated by \(J_\lambda\). Thus \(W_\lambda\) is the column stabilizer of \(t_\lambda\). Moreover, the set of minimal left coset representatives for \(W_\lambda\) in \(W_n\) is the set

\[D_\lambda = \{ d \in W_n \mid di < d(i+1) \text{ whenever } s_i \in J_\lambda \}\]

since the condition \(di < d(i+1)\) is equivalent to \(l(ds_i) > l(d)\). It follows that \(\{ dt_\lambda \mid d \in D_\lambda \}\) is precisely the set of column standard \(\lambda\)-tableaux.

We have the following result (see for example [3 Lemma 1.5], [5 Lemma 6.2]).

**Lemma 6.1.** Let \(\lambda \in P(n)\) and define \(v_\lambda \in W_n\) by the requirement that \(t^{\lambda'} = v_\lambda t_\lambda\). Then \(\text{STD}(\lambda) = \{ wt_\lambda \mid w \leq_L v_\lambda \} = \{ t \in \text{Tab}(\lambda) \mid t \leq t^{\lambda'} \}\).

The Robinson-Schensted algorithm associates each \(w \in W_n\) with an ordered pair of standard tableaux of the same shape \(\lambda\) for some \(\lambda \in P(n)\). Moreover, this gives a bijection from \(W_n\) to the set of all such pairs.

**Theorem 6.2.** The Robinson-Schensted map \(w \mapsto RS(w) = (P(w), Q(w))\) is a bijection from \(W_n\) to \(\{(t, u) \in P(n)^2 \mid \text{Shape}(t) = \text{Shape}(u)\}\).

See, for example, [9 Theorem 3.1.1]. Details of the algorithm can also be found (for example) in [9 Section 3.1].

The following lemma, the proof of which relies on the details of the Robinson-Schensted algorithm, is of crucial importance to us.

**Lemma 6.3.** Let \(\lambda \in P(n)\) and let \(w \in W_n\). Then \(RS(w) = (t, t_\lambda)\) for some \(t \in \text{STD}(\lambda)\) if and only if \(w = vh_{t_\lambda}\) for some \(v \in W\) such that \(vt_\lambda \in \text{STD}(\lambda)\). When these conditions hold, \(t = vt_\lambda\).
Theorem 6.6. Let \( v \in W_n \) be the element of \( W_n \) such that \( v_{\lambda} t_\lambda = t^\lambda \) and let \( \mathcal{I}_\lambda = \{ v \in W \mid v \leq_L v_\lambda \} \). Then the Kazhdan-Lusztig left cell that contains \( w_J \) is \( \mathcal{I}_\lambda w_J \).

Remark 5. Lemma 6.4 and Theorem 6.6 justify the procedure used in [5] to compute a \( W \)-graph with vertex set indexed by \( \text{STD}(\lambda) \).

Remark 6. By Theorem 6.6, the \( H \)-module \( \mathcal{I}(\mathcal{I}_\lambda, J) \) derived from the \( W \)-graph ideal \( \mathcal{I}_\lambda \) with respect to \( J \) is isomorphic to the cell module associated with the...
cell $C_\lambda = \mathcal{S}_\lambda w_J$. But it is known—see, for example, [5, Lemma 3.4]—that this cell module is isomorphic to the Specht module $S^\lambda$ associated with the partition $\lambda$. So we conclude that $\mathcal{S}(\mathcal{S}_\lambda, J) \cong S^\lambda$.

Let $\lambda \in \mathcal{P}(n)$ and let $J = J_\lambda$ as above. Recall that $\mathcal{S} = D_J$ is a W-graph ideal with respect to $J$, and the associated $\mathcal{H}$-module $\mathcal{S}(\mathcal{S}, J)$ is the induced module $\mathcal{S}_\theta$, defined in Section 5 above. Since the maximal cell $C_\lambda^\lambda$ of the W-graph $\Gamma(C_J)$ derived from $\mathcal{S}$ gives rise to the W-graph ideal $\mathcal{S}_\lambda = \{ v \in W \mid v \leq_L v_\lambda \}$, it follows that the $\mathcal{H}$-module $\mathcal{S}(\mathcal{S}_\lambda, J)$ is isomorphic to $\mathcal{S}_\theta/\mathcal{A}(C_J \setminus C_\lambda^\lambda)$. Writing $f$ for the homomorphism $\mathcal{S}_\theta \to \mathcal{S}(\mathcal{S}_\lambda, J)$ with kernel $\mathcal{A}(C_J \setminus C_\lambda^\lambda)$, it follows from Lemma 4.1 that $(f(b_w^I) \mid w \in \mathcal{S}_\lambda)$ is an $\mathcal{A}$-basis of $\mathcal{S}(\mathcal{S}_\lambda, J)$, where the elements $b_w^I = T_w \otimes 1$ for $w \in D_J$ make up the basis $B_J$ of $\mathcal{S}_\theta$. Now if $s \in S$ and $w \in D_J$ then

$$T_sb_w^I = \begin{cases} b_w^I & \text{if } sw > w \text{ and } sw \in D_J \\ -q^{-1}b_w^I & \text{if } sw > w \text{ and } sw \notin D_J \\ b_w^I + (q-q^{-1})b_w^s & \text{if } sw < w, \end{cases}$$

and if we assume that $w \in \mathcal{S}_\lambda$ and apply $f$ to these formulas we find that

$$T_sf(b_w^I) = \begin{cases} f(b_w^I) & \text{if } s \in SA(w) \\ -q^{-1}f(b_w^I) & \text{if } s \in WD(w) \\ f(b_w^I) + (q-q^{-1})f(b_w^I) & \text{if } s \in SD(w). \end{cases}$$

If $s \in WA(w)$, so that $sw \in D_J \setminus \mathcal{S}_\lambda$, then by Lemma 4.2

$$T_sf(b_w^I) = f(b_w^I) = \sum_{y \in \mathcal{S}_\lambda \setminus \mathcal{S}_y \setminus sw} r_y^s f(b_y^I)$$

for some polynomials $r_y^s \in q\mathcal{A}^+$. Moreover, since $\mathcal{S}_\lambda$ is generated by the single element $v_\lambda$, it follows from [5, Proposition 7.9] that all the elements $y$ appearing in the sum in Equation (6.1) satisfy $y \leq w$, and we also know by Corollary 4.3 (see Equation (4.11)) that $r_y^s w \equiv q$. So if we now choose an isomorphism $\theta : \mathcal{S}(\mathcal{S}_\lambda, J) \to S^\lambda$ and for each $t \in STD(\lambda)$ define $b_t = \theta(f(b_w^I))$, where $w$ is the unique element of $\mathcal{S}_\lambda$ such that $t = w\lambda$, then we conclude that $(b_t \mid t \in STD(\lambda))$ is a basis of $S^\lambda$, and for all $s \in S$ and $t \in STD(\lambda)$,

$$T_sb_t = \begin{cases} b_t & \text{if } s \in SA(t), \\ b_t + (q-q^{-1})b_t & \text{if } s \in SD(t), \\ -q^{-1}b_t & \text{if } s \in WD(t), \\ qa_t - \sum_{u < t} r_u^{(s)} b_u & \text{if } s \in WA(t), \end{cases}$$

for some $r_u^{(s)} \in q\mathcal{A}^+$. We remark that it is not hard to deduce that these basis elements are uniquely determined, to within a scalar multiple, by the conditions that $T_gb_t = -q^{-1}b_t$ for all $s \in \mathcal{S}_\lambda$, and $b_w t = T_w b_t$ for all $w \in \mathcal{S}_\lambda$.

We have thus proved Proposition 6.3 of [5], the assertion of which was that the polynomials $r_u^{(s)}$ are all divisible by $q$.

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