Solution of piezothermoelastic laminated double curvature shell based on 9-node isoparametric element

Ruishan Xing, Lei Kong
Aircraft Engineering Department of Civil Aviation College of Guangzhou,
Guangdong Guangzhou 510403

Abstract. Compared with 8-node and 4-node isoparametric element, the 9-node isoparametric element is more flexible and higher rate of convergence. Based on state-space techniques, the 9-node isoparametric element of piezothermoelastic material under orthogonal curvilinear coordinate system is established. Finally, the example shows the practicability and accuracy of the 9-node isoparametric element of piezothermoelastic material under orthogonal curvilinear coordinate system.

Keywords: Piezothermoelastic material; Orthogonal double curvature shell; 9-node isoparametric element.

1. Introduction
Laminated shell structures of smart composite materials are widely used in the field of Aeronautics and Astronautics. Mechanical-electrical-thermal coupling of piezoelectric thermoelastic materials is one of the hot research issues [1-4], due to theoretical limitations and anisotropic properties of composite materials, it is difficult to obtain the exact solution of its three-dimensional control equation. Mindlin [5] deduced the control equation of piezothermoelastic materials with mechanical-electrical-thermal coupling. Nowacki [6] and Iesan [7] made a general theoretical analysis on piezothermoelastic materials. Based on the classical theory of plates and shells, Tauchert [8] and Jonnalagadda [9] have deduced the approximate solution of laminated plates of piezothermoelastic materials. In recent years, with the rise of the state space method of elasticity, it has become an effective method to solve the structure of decompressed electric plates and shells [10], for example, Tian Xiuyun and Liu Yanhong [11] have obtained the exact solution of the laminated open shell of piezoelectric electrothermal elastomer. In reference [12], the precise solution of simply supported piezoelectric thermoelastic materials on four sides in rectangular coordinate system is given by using precise integration method, combining state space method and dual theory.

Reference [4] shows that the accuracy of 8-node and 9-node isoparametric elements can be significantly improved compared with 4-node isoparametric elements on the premise of avoiding the increase of the stiffness matrix caused by the increase of the number of finite element mesh. Although the accuracy of 8-node isoparametric element is improved, the stiffness of isoparametric element will become hard when the boundary of finite element mesh is obviously irregular [3]. Therefore, for the finite element model with irregular mesh, the accuracy of 8-node isoparametric element will be significantly reduced, 9-node isoparametric element can obviously overcome this disadvantage.
According to the method in [1-4] and based on the state space theory, this paper puts forward the method of 9-node isoparametric element of laminated orthogonal hyperbolic shell in hyperbolic coordinate system. Finally, the feasibility and accuracy of this method are proved by an example.

2. Basic equation

In the orthogonal hyperbolic shell as shown in Figure 1, $l_1, l_2$ are the arc length of the middle surface, and $R_1, R_2$ are the curvature of the middle surface curve respectively, $K_1, K_2$ are the corresponding principal curvatures, and $K_1 = \frac{1}{R_1}, K_2 = \frac{1}{R_2}$, the lame coefficients of $\alpha, \beta$ direction are as follows:

$$H_1 = 1 + K_1 \gamma, H_2 = 1 + K_2 \gamma$$

In the orthogonal hyperbolic coordinate system, without volume force, static, the incremental constitutive equation [13] of anisotropic piezothermoelastic body is as follows:

$$
\begin{pmatrix}
\sigma_\alpha \\
\sigma_\beta \\
\sigma_\gamma \\
\tau_{\alpha\gamma} \\
\tau_{\beta\gamma} \\
\tau_{\alpha\beta} \\
D_\alpha \\
D_\beta \\
D_\gamma \\
p_\alpha \\
p_\beta \\
p_\gamma
\end{pmatrix}
=
\begin{pmatrix}
c_{11} c_{12} c_{13} & 0 & 0 & 0 & 0 & e_{31} & 0 & 0 & 0 & -\lambda_1 \\
c_{12} c_{22} c_{23} & 0 & 0 & 0 & 0 & e_{32} & 0 & 0 & 0 & -\lambda_2 \\
c_{13} c_{23} c_{33} & 0 & 0 & 0 & 0 & e_{33} & 0 & 0 & 0 & -\lambda_3 \\
0 & 0 & 0 & e_{44} & 0 & 0 & e_{24} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & e_{55} & 0 & e_{15} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & e_{66} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & e_{15} & 0 & -\kappa_{11} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & e_{24} & 0 & 0 & -\kappa_{22} & 0 & 0 & 0 & 0 \\
e_{31} e_{32} e_{33} & 0 & 0 & 0 & 0 & -\kappa_{33} & 0 & 0 & \gamma_3 \\
0 & 0 & 0 & 0 & 0 & 0 & -k_{11} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -k_{22} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -k_{33} & 0
\end{pmatrix}
\begin{pmatrix}
e_\alpha \\
e_\beta \\
e_\gamma \\
\gamma_{\alpha\gamma} \\
\gamma_{\beta\gamma} \\
\gamma_{\alpha\beta} \\
E_\alpha \\
E_\beta \\
E_\gamma \\
T_\alpha \\
T_\beta \\
T_\gamma
\end{pmatrix}
$$

(1)

The normal stress on a node $[\sigma_\alpha \sigma_\beta \sigma_\gamma]^T$, shear stress $[\tau_{\alpha\gamma} \tau_{\beta\gamma} \tau_{\alpha\beta}]^T$, electric displacement components $[D_\alpha D_\beta D_\gamma]^T$, heat flux $[p_\alpha p_\beta p_\gamma]^T$ can be obtained by corresponding normal strain $[e_\alpha e_\beta e_\gamma]^T$; shear strain $[\gamma_{\alpha\gamma} \gamma_{\beta\gamma} \gamma_{\alpha\beta}]^T$; electric field intensity component $[E_\alpha E_\beta E_\gamma]^T$ and temperature field parameters $[T_\alpha T_\beta T_\gamma]^T$. $T(i = \alpha, \beta, \gamma)$ is the temperature change in a specific direction, and the total temperature increment is $T$. 

![Fig. 1 Orthogonal hyperbolic coordinate](image-url)
3. Homogeneous equation of state and precise integration method

According to the method in reference [12, 13], based on the increased dimension constitutive relation equation (1), the state space method is used to eliminate the internal stress component of curved surface \( \alpha \beta \sigma \sigma, \alpha \beta \tau \tau, \alpha \beta \phi, \alpha \beta \), the homogeneous equation of state of piezoelectric electrothermal elastic orthogonal hyperbolic shell can be obtained.

\[
\frac{\partial R_{m r}}{\partial r} = [D_A D_b] R_{m r}
\]  

(2)

Where, \( R_{m r} = [P, Q]^T = [\tau_{\alpha\gamma}, \tau_{\beta\gamma}, \sigma_{\gamma}, D_{\gamma}, p_{\gamma} u, v, w, \phi, T]^T \), \( P = [\tau_{\alpha\gamma}, \tau_{\beta\gamma}, \sigma_{\gamma}, D_{\gamma}, p_{\gamma}]^T \), \( Q = [u, v, w, \phi, T]^T \).

As shown in Figure 1, the boundary conditions of simply supported elastic laminated orthogonal hyperbolic shells on four sides can be expressed as follows:

\[
\begin{align*}
0 &= \sigma_{\alpha} = v = w = \phi = T(\alpha = 0, l_1) \\
0 &= \sigma_{\beta} = u = w = \phi = T(\beta = 0, l_2)
\end{align*}
\]  

(3)

If the problem of simply supported four sides is considered here, the series solution of any layer of laminated hyperbolic shell satisfying the boundary condition (3) can be assumed to be:

\[
\begin{align*}
(\tau_{\alpha\gamma}, u) &= \sum_{n} \sum_{m} (\tau_{\alpha\gamma}^{nm}(\gamma), u^{nm}(\gamma)) \cos(\xi \alpha) \sin(\eta \beta) \\
(\tau_{\beta\gamma}, v) &= \sum_{n} \sum_{m} (\tau_{\beta\gamma}^{nm}(\gamma), v^{nm}(\gamma)) \sin(\xi \alpha) \cos(\eta \beta) \\
(\sigma_{\gamma}, D_{\gamma}) &= \sum_{n} \sum_{m} (\sigma_{\gamma}^{nm}(\gamma), D_{\gamma}^{nm}(\gamma)) \sin(\xi \alpha) \cos(\eta \beta) \\
(p_{\gamma}, w) &= \sum_{n} \sum_{m} (p_{\gamma}^{nm}(\gamma), w^{nm}(\gamma)) \sin(\xi \alpha) \cos(\eta \beta) \\
(\phi, T) &= \sum_{n} \sum_{m} (\phi^{nm}(\gamma), T^{nm}(\gamma)) \sin(\xi \alpha) \sin(\eta \beta)
\end{align*}
\]  

(4)

In which: \( \xi = m \pi / l_1, \eta = n \pi / l_2 \).

If precise integration method is used to solve equation (2), the specific solution method is exactly the same as that in reference [12,13].

4. Hyperbolic 9-node isoparametric element for piezoelectric thermoelastic materials

According to the variational principle in reference [12,13], considering the homogeneous form of formula (1), the variational principle formula of piezoelectric electrothermal elastic material is:

\[
\delta \Pi = \delta \int_{\gamma} (P^T Q_{\gamma} - H) dV + \delta \int_{S_A} \lambda_{\gamma}^T B_{\gamma\tau} - \lambda_{\gamma}^T B_{\gamma\phi} dS
\]  

(5)

Where, \( H \) is a generalized Hamiltonian function, and the symbol \( \lambda_{\gamma}, B_{\gamma\phi}, \lambda_{\gamma}, B_{\gamma\phi} \) is a mark about boundary condition processing. Except for the inconsistency of variable number, its meaning is the same as that of reference [4].

If the shape function of nine node isoparametric element is the same as that in reference [4], then for the shape function \((i = 1, 3, 5, 7)\) on all corner nodes, when it is written as the same expression, it is:

\[
\begin{align*}
N_i(\xi, \eta) &= \frac{1}{4} (1 + \xi)(1 + \eta)(\xi \xi + \eta \eta)(i = 1, \cdots, 8) \\
N_9(\xi, \eta) &= 1 - \xi^2(1 - \eta^2)
\end{align*}
\]  

(6)
Therefore, the physical components can be marked as follows:

\[
\begin{align*}
\sigma_a & = \sigma_b = \sigma_c \\
\tau_{ay} & = \tau_{by} = \tau_{cy} \\
(D_a & = D_b = D_c) = N'(D_a^e, D_b^e, D_c^e) \\
(p_a & = p_b = p_c) = N'(p_a^e, p_b^e, p_c^e)
\end{align*}
\]

(7)

In formula (7), \( N' = [N_1, N_2, N_3, N_4, N_5, N_6, N_7, N_8, N_9] \).

By variation (and integration by parts) of equation (5), the following nine node isoparametric element formulation can be obtained,

\[
\int_0^{[M' = \begin{bmatrix} 0 & M' \end{bmatrix}] J d\xi d\eta \begin{bmatrix} P'(z) \end{bmatrix} = \int_0^{[K' = \begin{bmatrix} K_{11}^e & K_{12}^e \\ K_{21}^e & K_{22}^e \end{bmatrix}] J d\xi d\eta \begin{bmatrix} P'(0) \end{bmatrix}}
\]

(8)

Where, \( P' = [\tau_{ay}^e, \tau_{by}^e, \tau_{cy}^e, D_y^e, p_y^e]^T \), \( Q = [u^e, v^e, w^e, \phi^e, T^e]^T \); in which \( J \) is Jacobian matrix, \( K_y \), \( i,j = 1,2 \) is the equivalent stiffness matrix related to material parameters, \( M' \) is a diagonal matrix; the dimension of the nine node isoparametric element is \( 90 \times 90 \).

5. Numerical example

The solution of PZT-5A five layer orthogonal hyperbolic shell \([90 \times 0 \times 90 \times 0\] \), verify the correctness of Laplace solution. material stiffness coefficient (GPa): \( c_{11} = c_{22} = 99.201, c_{13} = c_{33} = 54.016, c_{33} = 86.856, c_{12} = 54.016, c_{44} = c_{55} = 21.1, c_{66} = 22.6 \). dielectric constant \( (10^8 \text{C}^2/\text{Nm}^2) \): \( \kappa_1 = \kappa_2 = 1.53, \kappa_{33} = 1.5 \), piezoelectric constant \( (\text{C/m}^2) \): \( e_{31} = e_{32} = -7.209, e_{33} = e_{15} = 12.322, e_{35} = 15.1184 \), stress temperature constant \( (10^4 \text{NK}^{-1}/\text{m}^2) \): \( \lambda_1 = \lambda_2 = 3.314, \lambda_3 = 3.26 \), thermoelectric constant \( (10^{-6}\text{C/N})\): \( y_3 = 0.0007 \), thermal conductivity \( (\text{W/mK}) \): \( k_1 = k_{22} = k_{33} = 1.8, H = 0.01m \), the thickness of each layer is \( 0.2H \), \( l_1 = l_2 = 100H \), \( R_l = R_2 = 100H \).

Considering working conditions: \( \sigma_y = T = \sin(\xi \alpha) \sin(\eta \beta) \), Loading on the outer surface of hyperbolic shell. In which \( m = n = 1 \). Considering the problem of simply supported four sides, the treatment method of boundary conditions is the same as that of reference [4].

**Tab. 1 Calculation results in outer/inner surface of hyperbolic shell**

| Outer surface of hyperbolic shell | variable | \( u (10^{-3}) \) | \( v (10^{-3}) \) | \( w (10^{-3}) \) | \( \bar{D}_y (10^{-10}) \) | \( \bar{P}_y (10^{-3}) \) |
|-----------------------------------|----------|------------------|------------------|------------------|------------------|------------------|
| 9 node element \((4 \times 4)\)   | 3.44246  | 3.44191          | 1.50562          | -4.34375         | 4.62903          |
| 9 node element \((6 \times 6)\)   | 3.58154  | 3.58144          | 1.56207          | -4.51970         | 4.81619          |
| 9 node element \((8 \times 8)\)   | 3.63699  | 3.63868          | 1.59101          | -4.58889         | 4.89583          |
| 9 node element \((10 \times 10)\)| 3.68944  | 3.68909          | 1.61382          | -4.65671         | 4.96253          |
| Precise integration              | 3.74677  | 3.74675          | 1.63913          | -4.74265         | 5.05308          |

| Inner surface of hyperbolic shell | variable | \( u (10^{-3}) \) | \( v (10^{-3}) \) | \( w (10^{-3}) \) | \( \bar{D}_y (10^{-10}) \) | \( \bar{P}_y (10^{-3}) \) |
|-----------------------------------|----------|------------------|------------------|------------------|------------------|------------------|
| 9 node element \((4 \times 4)\)   | 3.00521  | 3.00481          | 1.50020          | -4.25853         | 4.53392          |
| 9 node element \((6 \times 6)\)   | 3.12693  | 3.12657          | 1.55627          | -4.43148         | 4.71771          |
| 9 node element \((8 \times 8)\)   | 3.17501  | 3.17527          | 1.58526          | -4.49931         | 4.79521          |
| 9 node element \((10 \times 10)\)| 3.22080  | 3.22053          | 1.60799          | -4.56532         | 4.86053          |
| Precise integration              | 3.27051  | 3.27054          | 1.63287          | -4.64910         | 4.94872          |
Table 2 Generalized plane external stresses and displacements of neutral surface

| variable | $\tau_{x}(10^{-12})$ | $\tau_{y}(10^{-12})$ | $\sigma_{z}(10^{-12})$ | $\phi(10^{-11})$ | $T(10^{-5})$ |
|----------|------------------------|------------------------|------------------------|-----------------|--------------|
| Finite element | 1.13815 | 1.13827 | -6.22784 | 1.53869 | 2.44612 |
| Precise integration | 1.15095 | 1.15092 | -6.29647 | 1.55814 | 2.48962 |
| variable | $u(10^{-8})$ | $v(10^{-8})$ | $w(10^{-7})$ | $D_{x}(10^{-10})$ | $P_{z}(10^{-5})$ |
| 9 node element (10×10) | 3.47115 | 3.47126 | 1.61731 | -4.59706 | 4.90202 |
| Precise integration | 3.50958 | 3.50958 | 1.63614 | -4.69572 | 4.99956 |
| variable | $\sigma_{x}(10^{-10})$ | $\sigma_{y}(10^{-10})$ | $\tau_{xy}(10^{-10})$ | $D_{y}(10^{-13})$ | $D_{z}(10^{-12})$ |
| 9 node element (10×10) | 5.61126 | 5.62004 | 5.63261 | 8.66500 | -2.29964 |
| Precise integration | 5.69527 | 5.69527 | 5.70923 | 8.86908 | -2.34779 |
| variable | $P_{x}(10^{-7})$ | $P_{y}(10^{-7})$ |
| 9 node element (10×10) | -7.60217 | -7.58623 |
| Precise integration | -7.77413 | -7.77413 |

Table 1 shows the nine node finite element solution and the 10 dimensional precise integration solution. Among them, the maximum error of nine node 4 × 4 finite element mesh generation is 8.4%; the maximum error of nine node 6 × 6 finite element mesh generation is 4.7%; the maximum error of nine node 8 × 8 finite element mesh generation is 3.2%; the maximum error of nine node 10 × 10 finite element mesh generation is 1.8%. It can be concluded that the nine node isoparametric finite element method is convergent. Table 2 shows the solutions of the related variables of the hyperbolic shell's neutral surface. The maximum error of the nine node 10 × 10 finite element mesh is 2.4%. These maximum errors occur in the variables $D_{x}$ and $P_{x}$.

Through the comparison of the data above, the accuracy of nine node isoparametric finite element solution is verified.

6. Conclusion

The nine node isoparametric element of the orthogonal hyperbolic shell made of piezothermoelastic material is proposed. The feasibility, fast convergence and accuracy of the nine node isoparametric material are proved by an example. A new approximate method is provided for the solution of piezoelectric thermoelastic laminated shell in orthogonal hyperbolic coordinate system.

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