Radiation hydrodynamics integrated in the code PLUTO

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ABSTRACT

Aims. The transport of energy through radiation is very important in many astrophysical phenomena. In dynamical problems the time-dependent equations of radiation hydrodynamics have to be solved. We present a newly developed radiation-hydrodynamics module specifically designed for the versatile MHD code PLUTO.

Methods. The solver is based on the flux-limited diffusion approximation in the two-temperature approach. All equations are solved in the co-moving frame in the frequency independent (grey) approximation. The hydrodynamics is solved by the different Godunov schemes implemented in PLUTO and for the radiation transport we use a fully implicit scheme. The resulting system of linear equations is solved either using the successive over-relaxation (SOR) method (for testing purposes), or matrix solvers that are available in the PETSc library. We state in detail the methodology and describe several test cases in order to verify the correctness of our implementation. The solver works in standard coordinate systems, such as Cartesian, cylindrical and spherical, and also for non-equidistant grids.

Results. We have presented a new radiation-hydrodynamics solver coupled to the MHD-code PLUTO that is a modern, versatile and efficient new module for treating complex radiation-hydrodynamical problems in astrophysics. As test cases, either purely radiative situations, or full radiation-hydrodynamical setups (including radiative shocks and convection in accretion discs) have been studied successfully. The new module scales very well on parallel computers using MPI. For problems in star or planet formation, we have added the possibility of irradiation by a central source.

Key words. radiation transport – irradiation – hydrodynamics – accretion disc

1. Introduction

Radiative effects play a very important role in nearly all astrophysical fluid flows, ranging from planet and star formation to the largest structures in the universe. Coupling the equations of radiation transport to those of (magneto-)hydrodynamics (MHD) has been studied for decades, and comprehensive treatments can be found for example in textbooks by Mihalas & Mihalas (1984) or Pomraning (1973). The numerical implementation of two-temperature radiation hydrodynamics (in the diffusion approximation) into multi-dimensional MHD/HD-codes has been done already over twenty years ago in various implementations, for example by Eggum et al. (1988), Kley (1989), in the ZEUS-code (Stone et al. 1992), and later by Turner & Stone (2001).

In order to study, for example, the dynamics and characteristics of stellar atmospheres together with convection, more accurate solvers for the radiation transport based on the method of short characteristics have been developed, see Davis et al. (2012) and Freytag et al. (2012) for the present status. This can then be coupled to the hydrodynamics using the Variable Eddington Tensor method (Jiang et al. 2012). Another approach is the M1 closure model where the radiative moment equations are closed at a higher level (González et al. 2007; Aubert & Teyssier 2008). Despite this progress it is still useful and desirable to have a method at hand which solves the interaction of matter and radiation primarily within the bulk part of the matter which may be optically thick. In such type of applications, the method of flux-limited diffusion (FLD, see Levermore & Pomraning (1981)) has its clear merits and is still implemented into existing MHD-codes, for example in NIRVANA (Kley et al. 2009) to study the planet formation process, in RAMSES (Commerçon et al. 2011) for protostellar collapse simulations, and in combination with a multi-frequency irradiation tool into PLUTO (Kuiper et al. 2010) for massive star formation.

Since the 3D-MHD code PLUTO (Mignone et al. 2007) is becoming increasingly popular within the computational astrophysics community, we added a publicly available radiation module, which is based on the two-temperature FLD-approximation, as described by Commerçon et al. (2011). PLUTO solves the equations of hydrodynamics and magnetohydrodynamics including the non-ideal effects of viscosity, thermal conduction and resistivity by means of shock-capturing Godunov-type methods. Several Riemann solvers, several time stepping methods and interpolation schemes can be chosen. Additionally, we added a ray-tracing routine that allows for additional irradiation by a point source in the center. Treating the irradiation in a ray-tracing approach, guarantees the long-range character of the radiation better than FLD (Kuiper et al. 2012; Kuiper & Klessen 2013).

The paper is organized as follows. In section 2.1, we briefly introduce the equations of hydrodynamics including radiation transport. Additionally we describe the general idea behind the flux limited diffusion approximation. In section 3, we present the discretization of the equations and the solver of the resulting matrix equation, and present our numerical implementation of irradiation. In section 4, we present six different test cases.
to show the correctness of the implemented equations: four test cases with an analytical solution (section 4.1 to 4.4) and two others in which our results are compared with those from other codes (section 4.5 and 4.6). We end with a summary and conclusions.

2. Radiation hydrodynamics

2.1. The equations

Even though the PLUTO-environment includes the full MHD-equations and non-ideal effects such as viscosity, we restrict ourselves here to the Euler equations of ideal hydrodynamics. Radiation effects are included in the two-temperature approximation, which implies an additional equation for the radiation energy. In order to follow the transport of radiation, we apply the flux-limited diffusion approximation and treat the exchange of energy and momentum between the gas and the radiation field with additional terms in the gas momentum and energy equations. The system of equations then read:

\[
\frac{\partial}{\partial t} \rho + \nabla \cdot (\rho v) = 0 \tag{1}
\]

\[
\frac{\partial}{\partial t} \rho v + \nabla \cdot (\rho v \otimes v) + \nabla p = \rho (a_{\text{ext}} + a_{\text{rad}}) \tag{2}
\]

\[
\frac{\partial}{\partial t} \varepsilon + \nabla \cdot [(\varepsilon + p) v] = \rho v \cdot (a_{\text{ext}} + a_{\text{rad}}) - \kappa_p \rho \varepsilon (a_T T^4 - E) \tag{3}
\]

\[
\frac{\partial}{\partial t} E + \nabla \cdot F = \kappa_p \rho c (a_T T^4 - E) \tag{4}
\]

The first three equations (1-3) describe the evolution of the gas motion, where \( \rho \) is the gas density, \( p \) the thermal pressure, \( v \) the velocity, \( \varepsilon = \rho \varepsilon + 1/2 \rho v^2 \) the total energy density (i.e., the sum of internal and kinetic) of the gas without radiation, and \( a_{\text{ext}} \) an acceleration caused by external forces (e.g. gravity), not induced by the radiation field (see below). This system of equations is closed by the ideal gas relation

\[
p = (\gamma - 1) \rho \varepsilon = \rho \frac{k_B T}{\mu m_H}, \tag{5}
\]

where \( \gamma \) is the ratio of specific heats, \( T \) the gas temperature, \( k_B \) the Boltzmann constant, \( \mu \) the mean molecular weight, and \( m_H \) the mass of hydrogen. \( \varepsilon \) is the specific internal energy which can be written as \( \varepsilon = c_T T \), with the specific heat capacity \( c_T \) given by

\[
c_T = \frac{k_B}{(\gamma - 1) \mu m_H}. \tag{6}
\]

Here, we assume constant \( \gamma \) and \( \mu \), which also implies a constant \( c_T \).

The evolution of the radiation energy density \( E \) is given by eq. (4), where \( F \) denotes the radiative flux, \( k_B \) the Planck mean opacity, \( c \) the speed of light and \( a_T \) the radiation constant. The fluid is influenced by the radiation in two different ways. First, the radiation may be absorbed or emitted by the fluid leading to a variation of its energy density. This variation is given by the expression \( k_{p \rho} c (a_T T^4 - E) \), see right hand side of equation (3) and (4). The second effect is that of radiation pressure. We include this term as an additional acceleration to the momentum equation, \( a_{\text{rad}} = \frac{c}{\varepsilon} F \). The present implementation does not include the advective transport terms for the radiation energy and radiative pressure work in eqs. (3) and (4). For the relatively low temperature protoplanetary disk application that we consider here these terms are of minor importance. If required, these terms can be treated in our implementation straightforwardly within PLUTO by adding additional source terms.

2.2. The flux-limited diffusion approximation

The system of equations shown cannot be solved without further assumptions for the radiative flux \( F \). Here we use the flux limited diffusion approximation (FLD) where the radiation flux is given by a diffusion approximation

\[
F = -\lambda \frac{c}{\kappa_p \rho} \nabla E, \tag{7}
\]

with the Rosseland mean opacity \( \kappa_r \). The flux-limiter \( \lambda \) describes approximately the transition from very optically thick regions with \( \lambda = 1/3 \) to optically thin regimes, where \( F \rightarrow -cE \frac{\kappa_r}{\kappa_p} \).

This leads to the formal definition of the flux-limiter which is a function of the dimensionless quantity

\[
R = \frac{|\nabla E|}{\kappa_p \rho E}, \tag{8}
\]

with the following behaviour:

\[
\lambda(R) = \begin{cases} \frac{1}{R}, & R \to 0 \\ \frac{1}{3}, & R \to \infty \end{cases} \tag{9}
\]

Physically sensible flux-limiters thus have to fulfill the equation (9) in the given limits and describe the behaviour between the limits approximately. We have implemented three different flux-limiters:

\[
\lambda(R) = \begin{cases} \coth R - \frac{1}{R}, & 0 \leq R \leq \frac{1}{3} \\ \frac{1}{1 + 2(1 + \sqrt{2})R}, & \frac{1}{3} < R \leq \frac{1}{2} \\ \frac{1}{1 + 2(1 + \sqrt{2})R}, & \frac{1}{2} < R \leq \infty \end{cases} \tag{10}
\]

\[
\lambda(R) = \begin{cases} \frac{2}{3 + \sqrt{3}} \frac{1}{1 + R + \sqrt{R+2}}, & 0 \leq R \leq 2 \\ \frac{1}{2}, & 2 < R \leq \infty \end{cases} \tag{11}
\]

from Levermore & Pomraning (1981), Minerbo (1978), and Kley (1989), respectively. A comparison of them is presented in Kley (1989).

In general it is necessary to solve the equations for each frequency which appears in the physical problem. However, here we use the grey approximation in which all radiative quantities including the opacities are integrated over all frequencies. In our treatment scattering is not accounted for directly, but it is included in the effective isotropic absorption and emission coefficients.

3. Solving the radiation part

3.1. Reformulation of the equations

Instead of solving system of equations (1-4) directly as a whole, the problem is split into two steps. In the first step, PLUTO is used to solve the equations of fluid dynamics with the additional force caused by the radiation. This corresponds to the equations (1) to (3) with the additional acceleration, \( a_{\text{rad}} \), but without the interaction term between the matter and radiation (last term in
eq. 3). By using PLUTO for solving the non-radiative part of the equations, we are not limited to the Euler equations, but are able to use the full capabilities of PLUTO for solving the equations of hydrodynamics or magnetohydrodynamics, including the effects of viscosity and magnetic resistivity.

In a second, additional step we solve the radiation energy equation (4) and for the corresponding heating-cooling term in the internal energy of the fluid:

\[
\begin{align*}
\frac{\partial}{\partial t}E - \nabla \cdot \left( \frac{c^4}{k_B \rho} \nabla E \right) &= \kappa_B \rho c (a_R T^4 - E) \\
\frac{\partial}{\partial t} \rho \epsilon &= -\kappa_B \rho c (a_R T^4 - E)
\end{align*}
\]

(13)

In order to obtain the radiation energy density, we solve the system of coupled equations (13). Within one time step PLUTO advances the hydrodynamical quantities, i.e. the density \( \rho \), the velocity \( v \) and a temporary pressure \( p \) from time \( t^\text{old} \) to the time \( t^\text{new} \), where the time step, \( \Delta t = t^\text{new} - t^\text{old} \), is determined by PLUTO using the CFL conditions, presently without including the radiation pressure. These depend on the used time stepping method in PLUTO, for more information see Mignone et al. (2007) and the userguide of PLUTO.

The physical process of radiation transport takes place on time scales much shorter than the one in hydrodynamics. In order to use the same time step for hydrodynamics and the radiation transport, we apply an implicit scheme to handle the radiation diffusion and the coupling between matter and radiation described by equation (13). Because of the coupling of the equations, the method will update \( T \) and \( E \) simultaneously, which leads formally to a nonlinear set of coupled equations. As outlined below, the system is solved for the radiation energy density \( E \). From the new values for \( E \), we compute the new fluid temperature (see eq. 17 below) and update the fluid pressure by using the ideal gas relation from equation (5). This is then used within PLUTO to calculate a new total gas energy \( e \).

### 3.2. Discretization

In order to discretize the equations (13), we apply a finite volume method. For that purpose we integrate over the volume of a grid cell and transform the divergence into a surface integral. Furthermore, we replace the gradient of \( E \) by finite differences, and apply an implicit scheme. The discretization scheme has been implemented in 3D for Cartesian, cylindrical and spherical polar coordinates including all the necessary geometry terms for the divergence and gradient. Since the density has been updated already in the hydrodynamical part of the solver, we can replace \( \frac{\partial \rho}{\partial t} \) with \( \rho \frac{\partial c_V}{\partial t} \), which is valid for a constant heat capacity. Then the resulting discretized equations for the radiative part can be written as

\[
\begin{align*}
E^{n+1}_{i,j,k} - E^n_{i,j,k} \\
&= G_{x}K^n_{x,i,j,k} E^{n+1}_{i+1,j,k} - G_{x}K^n_{x,i,j,k} E^{n+1}_{i-1,j,k} + \frac{\Delta x_{1x}}{2} K^n_{x,i,j,k} \\
&+ G_{y}K^n_{y,i,j,k} E^{n+1}_{i,j+1,k} - G_{y}K^n_{y,i,j,k} E^{n+1}_{i,j-1,k} + \frac{\Delta y_{1y}}{2} K^n_{y,i,j,k} \\
&+ G_{z}K^n_{z,i,j,k} E^{n+1}_{i,j,k+1} - G_{z}K^n_{z,i,j,k} E^{n+1}_{i,j,k-1} + \frac{\Delta z_{1z}}{2} K^n_{z,i,j,k} \\
&+ \kappa_{\rho} \rho_{i,j,k} \epsilon_{i,j,k} c (a_R (T^{n+1}_{i,j,k})^4 - E^{n+1}_{i,j,k}),
\end{align*}
\]

(14)

and for the thermal energy (or temperature, respectively)

\[
\frac{T^{n+1}_{i,j,k} - T^n_{i,j,k}}{\Delta t} = -\kappa_B \rho c \left( a_R (T^{n+1}_{i,j,k})^4 - E^{n+1}_{i,j,k} \right).
\]

(15)

Here, the superscript \( n \) refers to the values of all variables after the most recent update from the hydrodynamical step. In order to simplify the notation for the separate radiation module, we assume the update takes place from time \( n \) to \( n+1 \). The subscripts \( i, j, k \) refer to the 3 spatial directions of the computational grid, where all variables are located at the cell centers. Half-integer indices refer to cell interfaces. The physical sizes (proper length) of each cell in the 3 spatial directions \( m (m = 1, 2, 3) \) are given by \( \Delta x_m \), where we additionally allow for non-equidistant grids. The effective radiative diffusion coefficient (defined at cell centers) is given by

\[
K^n_{i,j,k} = \frac{c \ell (R_{i,j,k})}{\kappa_B \rho_{i,j,k} \rho_{i,j,k}^n},
\]

where \( R_{i,j,k} \) is calculated from eq. (8) by central differencing. Values at cell interfaces are obtained by linear interpolation. The factors \( G_{x,y,z} \) are geometrical terms defined, respectively, as the left and right surface areas divided by the cell volume in the direction given by \( m = 1, 2, 3 \). In the recent work by Bitsch et al. (2013b) the difference equations have been written out in more detail for Cartesian, equidistant grids. The required opacities are evaluated using the values of \( \rho \) and \( T \) after the hydrodynamical update at time \( t^\text{old} \).

As mentioned before, equations (13) constitute a set of coupled nonlinear equations. The non-linear term \( (T^{n+1}_{i,j,k})^4 \) that appears in equation (15) is linearised using the method outlined in Commerçon et al. (2011)

\[
(T^{n+1}_{i,j,k})^4 = (T^n_{i,j,k})^4 \left( 1 + \frac{T^{n+1}_{i,j,k} - T^n_{i,j,k}}{T^n_{i,j,k}} \right) \approx 4(T^n_{i,j,k})^3(T^{n+1}_{i,j,k} - 3(T^n_{i,j,k})^3).
\]

(16)

Using this approximation, we obtain an equation for computing the new temperature in terms of the new radiation energy density, \( E^{n+1}_{i,j,k} \), and the old temperatures, \( T^n_{i,j,k} \)

\[
T^{n+1}_{i,j,k} = \frac{\kappa_B \rho_{i,j,k}^n}{\kappa_B \rho_{i,j,k}^n} \left( 3(4(T^n_{i,j,k})^3 + E^{n+1}_{i,j,k}) \Delta t + c_V T^n_{i,j,k} \right) + 4\kappa_B \kappa_{\rho} \rho_{i,j,k}^n c \rho_{i,j,k}^n \left( (T^n_{i,j,k})^3 \Delta t \right).
\]

(17)

The expression can be substituted into eq. (14) to obtain a linear system of equations for the new radiation energies \( E^{n+1}_{i,j,k} \), that can be solved using standard matrix solvers, see section 3.4. The new temperature can then be calculated from eq. (17). We implemented several boundary conditions for the radiation energy density including periodic, symmetric and fixed value.

### 3.3. Irradiation

In order to couple possible irradiation to the radiation transport equations, a new source term, \( S \), has to be added to the right hand side of the thermal energy equation in system (13)

\[
\frac{\partial \rho}{\partial t} = -\kappa_B \rho c (a_R T^4 - E) + S.
\]

(18)

This results in an additional term, \( S_{i,j,k} / \rho_{i,j,k} c_V \), in Eq. 15, corresponding in equation (17), and in a modification of the right hand side of the resulting matrix equation for \( E^{n+1}_{i,j,k} \).
For the present implementation, we assume that the irradiating source is located at the centre of a spherical coordinate system. Therefore it is straightforward to compute the optical depth \( \tau_{i,j,k} \) even for simulations using parallel computers. Assuming that a ray of light travels along the radial direction from the origin to the grid cell \( i, j, k \) under consideration, the optical depth from the inner radius \( r_0 \) to the \( i \)th grid cell with radius \( r_i \) can be simply expressed as the integral along the radial coordinate,

\[
\tau_{i,j,k} = \int_{r_0}^{r_i} \kappa_\ast \rho(r) \, dr \approx \sum_{n=0}^{i-1} \kappa_{n,jk} \rho_{n,jk} \Delta r_n
\]  

(19)

where \( \Delta r_n \) is the radial length of the \( n \)th grid cell, and \( \kappa_\ast \) the opacity used for irradiation. For the sake of readability, we write \( \tau_i \) instead of \( \tau_{i,j,k} \) in the following. We use \( \kappa_\ast = \kappa_0 \) in the test case with irradiation presented in section 4.3. Additionally \( \kappa_\ast \) can be defined by the user as well as the other opacities. Re-emission of the photons which were absorbed in the cell volume is handled in our treatment by the heating-cooling term see equation (13).

The luminosity of the source is given by

\[
L_\ast = 4\pi R_\ast^2 \sigma T_\ast^4,
\]

(20)

where \( \sigma \) denotes the Stefan-Boltzmann constant, \( T_\ast \) is the temperature of the star and \( R_\ast \) its radius. In order to compute the amount of irradiated energy which is absorbed by a specific grid cell we have to know the surface area \( A \) of a grid cell oriented perpendicular to the radiation from the star and the flux \( f \) at the radius \( r \). This surface area \( A \) is given by the expression

\[
A_{i,j,k} = \int_{\theta_j}^{\theta_i} \int_{\phi_k}^{\phi_i} dA = r_i^2 (\phi_{k+1} - \phi_k) (\cos \theta_j - \cos \theta_{j+1}),
\]

(21)

where \( \theta \) is the azimuthal and \( \phi \) the polar angle in the spherical coordinate system. Without absorption the flux \( f \) is given by the expression

\[
f = \frac{L_\ast}{4\pi r^2} = \sigma T_\ast^4 \left( \frac{R_\ast}{r} \right)^2.
\]

(22)

The amount of energy per time which arrives at the surface of the grid cell \((i,j,k)\) is

\[
H_{i,j,k} = A_{i,j,k} f = (\phi_{k+1} - \phi_k) (\cos \theta_j - \cos \theta_{j+1}) \sigma T_\ast^4 R_\ast^2,
\]

(23)

again without absorption. If the irradiated energy is partly absorbed, the remaining amount of energy per time is then

\[
S_{i,j,k} = \frac{H_{i,j,k} e^{-\tau_i} - H_{i,j,k} e^{-\tau_{i+1}}}{V_{i,j,k}} = \frac{H_{i,j,k} (e^{-\tau_i} - e^{-\tau_{i+1}})}{V_{i,j,k}}
\]

\[
= \frac{3\sigma T_\ast^4 R_\ast^2 (e^{-\tau_i} - e^{-\tau_{i+1}})}{(r_{i+1}^3 - r_i^3)},
\]

(24)

with the volume of a grid cell

\[
V_{i,j,k} = \int_{r_i}^{r_{i+1}} \int_{\theta_j}^{\theta_i} \int_{\phi_k}^{\phi_i} r^2 \sin \theta \, dr \, d\theta \, d\phi
\]

\[
= \frac{1}{3} (r_{i+1}^3 - r_i^3) (\cos \theta_j - \cos \theta_{j+1}) (\phi_{k+1} - \phi_k).
\]

(25)

The absorbed energy density per time, \( S_{i,j,k} \), is computed for each grid cell before solving the matrix equation. A similar treatment of irradiation has been described recently by Bitsch et al. (2013b), for a multi-frequency implementation see Kuiper et al. (2010).

### 3.4. The matrix solver

We implemented two different solvers for the matrix equation. The first one uses the method of successive over-relaxation (SOR), and as a faster and more flexible solver we use the PETSc\(^1\) library. From the PETSc library we use the Krylov subspace iterative method and a preconditioner to solve the matrix equation. For all test cases described we used gmres (Generalized Minimal Residual) as iterative method and bjacobi (Block Jacobi) as preconditioner. Beside others the convergence of the SOR algorithm and the PETSc library can be estimated using the following criteria

\[
\|R^k\| < \max(\epsilon_r \cdot \|b\|, \epsilon_n)
\]

(26)

where \( b \) is the right hand side of the matrix equation \( Ax = b \), \( R^k = b - Ax^k \) is the residual vector for the \( k \)-th iteration of the solver and \( x \) is the solution vector (here the radiation energy density). As norm we used here the \( L_2 \) norm. The quantities \( \epsilon_r \) and \( \epsilon_n \) are the relative and absolute tolerance, respectively, and are problem dependent, with a common value of \( 10^{-50} \) for \( \epsilon_r \). For the test cases in section 4 we use relative tolerances \( \epsilon_r \) between \( 10^{-5} \) and \( 10^{-8} \). The criterion (26) is the default one used by the PETSc library. For more information about the convergence test in PETSc the reader should refer to section 4.3.2. of Balay et al. (2012). The solver performance in a parallel environment is described in section 4.6.4.

### 4. Test cases

In order to verify the implemented method, we simulated several test problems and compared the results with either corresponding analytical solutions or calculations done with different numerical codes. Most of the tests correspond to one-dimensional problems. In order to model those, we have used quasi one-dimensional domains, with a very long cuboid that has the height \( h \), width \( w \) and a length \( l \). The length \( l \) is much larger than the width or height, and for simplicity we use \( w = h \). We performed some of the tests in all three implemented coordinate systems (Cartesian, cylindrical and spherical) and in three different alignments of the cuboid along each coordinate direction. This is done to check whether the geometry factors are correct. In the case of a non-Cartesian coordinate system we placed the cuboid at large distances \( r \) from the origin such that the domain approximately describes a Cartesian setup.

We use for all test cases the solver based on the PETSc library with the default iterative solver gmres and the preconditioner bjacobi.

#### 4.1. Linear diffusion test

The following test is adapted from Commerc\’on et al. (2011). The initial profile of the radiation energy density is set to a delta function which is then evolved in time and compared to the analytical one dimensional solution. We perform this test in all implemented coordinate systems (Cartesian, cylindrical and spherical coordinates) as described above, which results in nine different simulations. The used domain is quasi one-dimensional and the equations of hydrodynamics are not solved in this test. Only the radiation diffusion equation

\[
\frac{\partial}{\partial t} E = \nabla \cdot \left( \frac{c_l}{k_r \rho} \nabla E \right)
\]

(27)

\(^1\) For more information visit the website http://www.mcs.anl.gov/petsc or have a look at Balay et al. (2012).
is solved which we obtain from equations (13) by setting \( \kappa_T = 0 \).

An analytical solution to equation (27) can be calculated in the one dimensional case with a constant flux-limiter \( \lambda = \frac{1}{3} \) and a constant product of the Rosseland opacity and density, here we set \( \kappa_T \rho = 1 \) cm\(^{-1}\). The equation to solve is then given by

\[
\frac{\partial}{\partial t} E(x, t) = \frac{c}{3} \frac{\partial^2}{\partial x^2} E(x, t)
\]

(28)

with solution

\[
E(x, t) = \frac{\tilde{E}_0}{\sqrt{\frac{3}{c\pi t}}} e^{-\frac{x^2}{c\pi t}},
\]

(29)

where \( \tilde{E}_0 \) is the integral over the initial profile of the energy density, \( E(x, t = 0) \). Note that in the quasi one-dimensional case (using a stretched 3D domain) \( \tilde{E}_0 \) has the units erg cm\(^{-2}\).

### 4.1. Setup

The domain is a cuboid with a length of 4 cm and a width and height of 0.04 cm. We used here 301 \( \times \) 3 \( \times \) 3 grid cells. The initial profile of the radiation energy density in the quasi one-dimensional case is set by

\[
E_i = \begin{cases} 
\frac{1 \text{ erg cm}^{-3}}{x}, & i = 1, 2, \ldots, N \text{ with } i \neq \frac{N}{2} \\
\tilde{E}_0 \frac{Ax}{x}, & i = \frac{N}{2}
\end{cases}
\]

(30)

where \( Ax \) is the length of a grid cell. For numerical reasons, we have set \( E_i \) for \( i \neq \frac{N}{2} \) to the value 1 erg cm\(^{-3}\) instead of 0 erg cm\(^{-3}\). This choice is not problematic, since \( \tilde{E}_0/Ax \gg 1 \) erg cm\(^{-3}\) for our chosen value of \( \tilde{E}_0 = 10^5 \) erg cm\(^{-2}\). The initial values for pressure and density are \( P = 1 \) g cm\(^{-1}\) s\(^{-2}\) and \( \rho = 1 \) g cm\(^{-3}\). Furthermore we use \( \kappa_T = 1 \) cm\(^2\) g\(^{-1}\) for the Rosseland opacity. All boundary conditions are set to periodic except for the boundary conditions at the beginning and end of the quasi one-dimensional domain, which are set to outflow. For the matrix solver we used a relative tolerance of \( \epsilon_r = 10^{-8} \). The simulation starts at \( t = 0 \) s with an constant time step of \( \Delta t = 1 \cdot 10^{-14} \) s and stops at \( t = 4.2 \cdot 10^{-12} \) s.

### 4.1.2. Results

The numerical solution \( E_n \) and the analytical solution \( E_a \) from equation (29) are plotted in the figures 1 and 2 together with the absolute value of the relative error. In figure 1 the radiation energy density is plotted against the position at the time \( t = 4.2 \cdot 10^{-12} \) s. The relative error in the relevant range from \(-1 \) cm to 1 cm is always below one percent. In figure 2 the time evolution from \( t = 0 \) s to \( 4.2 \cdot 10^{-12} \) s is shown for the positions \( x = \{0, 0.5, 1.0\} \) cm coded in the colors black, blue and red, respectively. The results shown in this figure depend strongly on the position. For the position \( x = 0 \) cm the error is, for all times later than \( t = 4 \cdot 10^{-13} \) s, below one percent and decreases with time. For the other positions, the behaviour is different. The relative error rises and after a while it decreases. This behaviour can be explained by looking at figure 1. The error is higher at the diffusion front. This region moves with time and causes the effect for the other positions. The test shows that the time evolution of the radiation energy density is reproduced correctly. As described, this test was performed in different coordinate systems and orientations, with the same results.

### 4.2. Coupling test

The purpose of this test from Turner & Stone (2001) is to check the coupling between radiation and the fluid. For this purpose we simulate a stationary fluid which is initially out of thermal equilibrium. In this simulation the radiation energy density is the dominant energy which is constant over the whole simulation. The system of equations (13) decouples in this case and, in addition, it is not necessary to solve the matrix equation for \( E \).

By setting \( \sigma_T = \kappa_T \rho \) and \( T = \frac{E}{\rho \mu m} \) from eq. (5) with the assumption that \( \sigma_T \) and \( \rho \) are constant, we can rewrite the thermal...
With the used approximations, the coefficients $C_1$ and $C_2$ are constant. The solution to Eq. (31) can be calculated analytically in terms of an algebraic equation which would have to be solved iteratively. Hence, we integrate Eq. (31) numerically using a Runge Kutta solver of 4-th order scheme with adaptive step size. In the following we refer to this solution as the reference solution.

Information on the expected behaviour of the solution can be obtained directly from the differential equation. It is clear that in the final equilibrium state (with $\frac{d}{dt} = 0$) the gas temperature has to be equal to the radiation temperature $T = \frac{\sqrt{E}}{\gamma m}$, thus the final gas energy density will be

$$e_{\text{final}} = \left( \frac{C_1}{C_2} \right)^\frac{1}{2}.$$  

If the initial gas energy density $e_0$ is much lower than $e_{\text{final}}$, we can neglect the second term in eq. (31) at the beginning, thus $e(t) = C_1 t + e_0$. The corresponding coupling time can be estimated to

$$\tau = \frac{e_{\text{final}} - e_0}{C_1}.$$  

On the other hand, if $e_0 \gg e_{\text{final}}$, we can neglect the first term in eq. (31) and derive

$$e(t) \propto (C_2 t)^{-\frac{1}{2}} \quad \text{and} \quad \tau = \frac{1}{e_{\text{final}}^3 C_2}.$$  

4.2.1. Setup

The computational domain is identical to that of the linear diffusion test in section 4.1. For the grid we use a resolution of $25 \times 3 \times 3$ grid cells. As before we do not solve the equations of hydrodynamics and the boundary conditions are quite simple. All boundaries are set to periodic boundary conditions. The constants we used are set to: radiation energy density $E = 10^{12}$ erg cm$^{-3}$, density $\rho = 10^{-7}$ g cm$^{-3}$, opacity $\sigma_T = 4 \cdot 10^{-6}$ cm$^{-1}$, mean molecular weight $\mu = 0.6$ and the ratio of specific heats $\gamma = 5/3$.

The simulations starts at $t = 0$ s with an initial time step of $\Delta t = 10^{-20}$ s and evolves until $t = 10^{-8}$ s. After each step the time step is increased by $1\%$ in order to speed-up the computation. The simulation is done with three different initial gas energy densities, $e_0 = 10^{10}$ erg cm$^{-3}$, $e_0 = 10^9$ erg cm$^{-3}$ and $e_0 = 10^8$ erg cm$^{-3}$.

4.2.2. Results

Figure 3 shows the numerical gas density and the reference solution plotted against time for the three different initial values of $e$. The agreement of both results is excellent for all initial values. From the figure we see that in the limit of small and large initial $e_0$, we find exactly the behaviour as predicted by the estimates for eq. (31). The analytic estimates for the coupling time $\tau$ from equation (33) agrees very well with our results. The estimate for $e_0 = 10^9$ erg cm$^{-3}$ is $\tau = 5.88 \cdot 10^{-8}$ s and for $e_0 = 10^8$ erg cm$^{-3}$ we calculated $\tau = 5.78 \cdot 10^{-8}$ s. In the case that $e_{\text{final}} < e_0$ the estimate in eq. (34) is approximate $\tau = 5.88 \cdot 10^{-8}$ s. We have to mention here that this test verifies primarily the correctness of equation (17). As in the linear diffusion test, this test was performed in three different coordinate systems in different orientations, with the same results.
4.3. Coupling test with irradiation

This test is in its basic setup the same as that from section 4.2, but with irradiation enabled, i.e. equation (18) is solved instead of the second equation in (13). As described in section 3.3, irradiation is limited to spherical coordinates, which we use for this test. With the same assumptions as in section 4.2, i.e., that $\sigma_p$ and $\rho$ are constant and with the definitions for $\sigma_p$, $e$, $p$ as well as for $T$, it is possible to rewrite $S$ from equation (24) to

$$S(r) = \frac{3\sigma T^4 R^2 e^{-\tau_{\rho}e(r-\delta)}}{(r + \Delta r)^3 - r^3},$$

and obtain for equation (18)

$$\frac{de}{dt} = \frac{S(r) + c \sigma_T E - c \sigma_T \rho R}{C(r)} \left( \frac{\gamma - 1}{\rho} \right) \frac{\mu m_H}{k_B} e^4.$$

The reference solution is computed in the same way as before although it now depends on the distance $r$ from the star. The quasi one-dimensional domain starts at $r = 9000 \cdot 10^3$ cm and ends at $r = 9003 \cdot 10^5$ cm and we use $300 \times 3 \times 3$ grid cells. The domain size in $\theta$ and $\phi$ direction was chosen in a way such that the grid cells are nearly quadratic. For the simulation we use a constant radiation energy density of $E = 10^{-2}$ erg cm$^{-3}$, a density of $\rho = 10^{-5}$ g cm$^{-3}$, a Rosseland opacity of $\kappa_R = 10$ cm$^2$ g$^{-1}$ and a Planck opacity of $\kappa_p = \kappa_R$ which corresponds to $\sigma_T = 10^{-4}$ cm$^{-1}$. The opacity for the irradiation $\kappa_\epsilon$ is set to $\kappa_p$. For the star, the temperature was set to $T_\star = 6000$ K and the radius to $R_\star = 8.1 \cdot 10^8$ cm. Additionally we make the assumption that there is no absorption in the region between the surface of the star and the inner boundary of the computation domain. Figure 4 shows the gas energy density plotted against time with an initial gas energy of $e = 10^9$ erg cm$^{-3}$ at three different positions $d = 0$ cm, $d = 3 \cdot 10^4$ cm and $d = 3 \cdot 10^5$ cm where $d$ is measured relative to the inner boundary of the quasi one-dimensional domain. In figure 5 the radial dependency of the gas energy density is plotted for the same simulation at five different times. As expected, the results show that the gas energy density at a time later than $t = 10^4$ s becomes constant and depends on the distance from the star. The simulated and reference solution show an excellent agreement.

4.4. A steady state test

The original version of this test was published in Flaig (2011). We consider a one-dimensional stationary setup with a given density stratification. In the steady state, the time derivatives in the equations (13) vanish and the system is reduced to the following equation for the radiation energy density

$$0 = \nabla \cdot \left( \frac{c \lambda}{\kappa_R \rho} \nabla E \right).$$

A further reduction is obtained when we rewrite this equation in one dimension along the $z$-axis in Cartesian coordinates. The equation is then much simpler and can be written as

$$\frac{d}{dz} \left( \frac{c \lambda}{\kappa_R \rho} \frac{dE}{dz} \right) = 0.$$

In general the expression $\frac{c \lambda}{\kappa_R \rho}$ is not known analytically for realistic opacities. In order to circumvent this problem, we define the effective optical depth $\tau_{\text{eff}} = \int \frac{dE}{dz} = \int_{z_b}^{z_a} \kappa_{\text{eff}} dz$ where $z_a$ and $z_b$ are the lower and upper boundaries of the quasi one-dimensional domain, respectively, and $\kappa_{\text{eff}}$ is the effective opacity given by $\kappa_{\text{eff}} = \frac{1}{3} \frac{dE}{dz}$. By using $dE_{\text{eff}} = \kappa_{\text{eff}} dz$, equation (38) can be rewritten as:

$$\frac{d}{dz} \left( \frac{c \lambda}{\kappa_R \rho} \frac{dE_{\text{eff}}}{dz} \right) = 0.$$

The solution of this equation is then given by

$$E = (E(\tau_{\text{eff}} = 1) - E(\tau_{\text{eff}} = 0)) \tau_{\text{eff}} + E(\tau_{\text{eff}} = 0).$$

where $E(\tau_{\text{eff}} = 0)$ and $E(\tau_{\text{eff}} = 1)$ are the radiation energy density at the position where the effective optical depth has the values zero or one, respectively. Thus, in the static case the radiation
energy has a linear dependence on the optical depth $\tau_{\text{eff}}$ for all opacity laws.

4.4.1. Initial setup

The domain was chosen to have an arbitrary length of 300 cm and a width and height of 3 cm and 300 $\times$ 3 $\times$ 3 grid cells were used. This test is performed without solving the hydrodynamical equations, instead we solved equations (13) for a fixed density and opacity law, and evolved the solution, until a stationary state has been reached. For the radiation boundary conditions, we used boundary conditions with fixed values of $E$ at the lower and upper boundary of the domain. At the lower boundary we have chosen $E = a_0 T^4$ with a temperature of $T = 2000$ K.

Because the stratification is optically thin at the upper boundary we want to allow the radiation to escape freely from the domain. For this reason we simply set the temperature to a very small value at the upper boundary, here $T = 10$ K.

All other boundary conditions have been set to periodic. The density stratification is given by

$$\rho(z) = \rho_0 e^{-\left(\frac{z}{z_b}\right)^2}.$$  \hspace{0.5cm} (41)

The initial temperature profile can be chosen randomly in principle, but in order to speed up the computation we used a linear temperature profile starting at $z_b$ with $T = 2000$ K and ending at $z_b$ with $T = 10$ K. From this temperature profile we assigned pressure values using equation (5). The radiation energy density $E$ inside the domain is also set using the gas temperature profile and $E = a_0 T^4$. The ratio of specific heats and the mean molecular weight are set to $\gamma = 1.43$ and $\mu = 0.6$, respectively.

As flux-limiter we have chosen equation (11), for the Rosseland mean opacity $\kappa_R$ we use data from Lin & Papaloizou (1985), and the Planck mean opacity is set to $\kappa_P = \kappa_R$. The initial time step is $\Delta t = 0.3$ s and it is increased slightly with time in order to speed-up the computation and to keep the number of iterations done by the matrix solver nearly constant. This simulation was performed with a relative tolerance of $\epsilon_r = 10^{-6}$ for the matrix solver.

4.4.2. Results

A steady state is reached approximately after $t = 1200$ s. In figure 6, we plot the radiation energy density against the effective optical depth $\tau_{\text{eff}}$ from our numerical solution (red dots) together with the analytical solution from equation (40). The parameters $E(\tau_{\text{eff}} = 1) - E(\tau_{\text{eff}} = 0)$ and $E(\tau_{\text{eff}} = 0)$ have been obtained by fitting equation (40) to the numerical solution. We have to note here that $E(\tau_{\text{eff}} = 0)$ is determined by interpolation between ghost cells and active cells near the upper boundary $z_b$. Hence, the radiation temperature in the active region can be much larger than 10 K, a value was specifically chosen to be very small. We also plot the absolute value of the relative error $|E_n - E_d|/E_d$. The results from the simulation agree very well with the analytical prediction. As we can see from figure 6, the largest deviation from the analytical solution is at small values of $\tau_{\text{eff}}$ with an relative error around one percent. As the linear diffusion and the coupling test, this test was performed in all three coordinate systems and in different orientations, with the same results.

4.5. Radiation shock

In this section we extend the previous tests and solve now the full equations of hydrodynamics and radiation trans-

Fig. 7. Sub- and supercritical shock test. In both cases we plot the radiation temperature (blue line) and the gas temperature (red line) against $s = z - v \cdot t$ where $z$ is the position along the quasi one-dimensional domain and $v$ the piston velocity. The subcritical shock (a) is shown at time $t = 3.8 \cdot 10^4$ s and the supercritical shock (b) at $t = 7.5 \cdot 10^4$ s.

port simultaneously, testing the complete new module within PLUTO environment.

4.5.1. Initial setup

Following a set-up from Ensman (1994), a shock is generated in a quasi one-dimensional domain. This test case is more complex than the ones before, and it is not possible to derive an analytical solution. Instead we compare our results with the simulations of Commerçon et al. (2011). The computational domain is chosen to have a length of $7 \cdot 10^6$ cm and a width and height of 3.418 $\cdot$ 10$^3$ cm with 2048 $\times$ 4 $\times$ 4 grid cells. The initial density and temperature are set to $\rho = 7.78 \cdot 10^{-10}$ g cm$^{-3}$ and $T = 10$ K. The initial radiation energy density is set by the equation $E = a_0 T^4$. For the flux-limiter we employ the Minerbo-
formulation according to eq. (11), and for the opacity we use \( \kappa_r \rho = \kappa_p \rho = 3.1 \cdot 10^{-10} \text{cm}^{-1} \). Furthermore the ratio of specific heats is set to \( \gamma = 7/5 \) and the mean molecular weight to \( \mu = 1 \), in analogy to Commerçon et al. (2011). The time step is computed through the CFL condition of PLUTO for which we assume a value of 0.4. For the solver we took the not so accurate but robust tdiff which uses a simple Lax-Friedrichs scheme. For generating the radiative shock, the following boundary conditions are used: in the direction of the shock propagation, we employ a reflective boundary condition at the lower boundary and a zero-gradient at the upper boundary of the domain. The remaining boundaries are set to periodic. For the relative tolerance used by the matrix solver we have chosen a value of \( \epsilon_r = 10^{-3} \). The shock is generated by applying an initial velocity \( v \) to the gas. The velocity is directed towards the reflecting boundary condition which acts as a wall. The shock propagates then from the wall back into the domain. Depending on the velocity, the shock is sub- or supercritical, i.e., the temperature behind the shock front is larger or equal than the temperature upstream (in front of the shock front), respectively. In this test we simulate both cases: the subcritical shock with a velocity of \( v = 6 \cdot 10^5 \text{cm s}^{-1} \) and the supercritical shock with \( v = 20 \cdot 10^5 \text{cm s}^{-1} \).

4.5.2. Results

For a better comparison with the results of simulations, where the material is at rest and a moving piston causes the shock, we introduce the quantity \( s \). This quantity is given by the relation \( s = z - vt \) where \( z \) is the position along the quasi-one-dimensional domain. Note that this quantity is called \( z \) in Commerçon et al. (2011). Fig. 7 shows the radiation temperature (blue line) and the gas temperature (red line) against the previously defined quantity \( s \) for both the subcritical (at \( t = 3.8 \cdot 10^5 \text{s} \)) and supercritical case (at \( t = 7.5 \cdot 10^5 \text{s} \)). In the supercritical case the pre- and post-shock gas temperature are equal, as expected. In the subcritical case these temperatures can be estimated analytically (Ensmann 1994; Mihalas & Mihalas 1984; Commerçon et al. 2011). In table 1, the analytical estimates and the numerical values from our simulations and the results from Commerçon et al. (2011) are shown together. Here \( T_p \) is the post-shock temperature, \( T_s \) the pre-shock temperature and \( T_s \) the spike temperature. In the equations, \( R_0 = \frac{2a}{\mu m_r} \) is the perfect gas constant, \( \sigma_{SB} = \frac{c_m}{T} \) the Stefan-Boltzmann constant, and \( \rho \) is the density of the shock relative to the upstream material (or vice versa) in our case \( \rho = 7.19 \cdot 10^5 \text{cm}^{-1} \).

| analytical estimate | numerical solution | Commerçon et al. |
|---------------------|--------------------|------------------|
| \( T_p \) = \frac{2a}{\mu m_r} \gamma \frac{1}{\rho v^2} \) | \( \approx 865 \text{K} \) | \( 816.6 \text{K} \) | \( 825 \text{K} \) |
| \( T_s \approx \frac{5}{9} \frac{2a}{\mu m_r} T_p \gamma \) | \( \approx 315 \text{K} \) | \( 331.9 \text{K} \) | \( 275 \text{K} \) |
| \( T_s \approx T_p + \frac{5}{9} \frac{2a}{\mu m_r} T_p \gamma \) | \( \approx 1075 \text{K} \) | \( 1147.1 \text{K} \) | \( 1038 \text{K} \) |

Table 1. Comparison of the results from the radiation shock test with analytical estimates and the results from Commerçon et al. (2011) for the pre-shock \( T_p \) and post-shock \( T_s \) gas temperature as well as the spike temperature \( T_s \).

The results agree in general with the analytical estimates and the results from Commerçon et al. (2011). The analytical estimate for the post-shock temperature is higher than the numerical results with both codes. We have to note here that the analytical estimate depends on \( \rho \) and differs therefore from the values given in Commerçon et al. (2011). The pre-shock and spike temperatures agree reasonably well with the analytical estimates in our simulations but are higher than the results from Commerçon et al. (2011). The differences of our numerical solution to the analytical estimates might due to the fact that we ignored the advective terms in the radiation energy density in eq. (4) that may play a role in this dynamic situation. Additionally, it is noteworthy that the position of the shock front is very well reproduced. This test was performed in Cartesian coordinates.

4.6. Accretion disc

The goal of this last test is to compare the results of different codes on a more complex two-dimensional physical problem that involves the onset of convective motions. For this purpose we model a section of an internally heated, viscous accretion disc in spherical coordinates \( (r, \theta, \phi) \) where \( r \) is the distance to the centre of the coordinate system, \( \theta \) the polar angle measured from the \( z \)-axis in cylindrical coordinates and \( \phi \) the azimuth angle. The setup follows the standard disc model used in Kley et al. (2009). The tests proceed in two steps. In a first setup we reduce the complexity of the problem and consider a static problem, i.e., without solving the equations of hydrodynamics. This will demonstrate that the equilibrium between viscous heating and radiative cooling is treated correctly in our implementation. In the second setup we consider the full hydrodynamic problem and study the onset of convection in discs.

4.6.1. The initial setup

For both, the static and the dynamical case we use the same initial setup. The radial extent ranges from \( r_{\text{min}} = 0.4 \) to \( r_{\text{max}} = 2.5 \), where all lengths are given in units of the semi-major axis of Jupiter \( a_{\text{Jup}} = 5.2 \text{AU} \). In the vertical direction the domain extends from \( \theta_{\text{min}} = 83^\circ \) to \( \theta_{\text{max}} = 90^\circ \) and in \( \phi \) direction from \( \phi_{\text{min}} = 0^\circ \) to \( \phi_{\text{max}} = 360^\circ \). In the three coordinate directions \( (r, \theta, \phi) \) we use \( 256 \times 32 \times 4 \) grid cells. The disc aspect ratio \( h \) is set to \( h = \frac{L}{\rho} = 0.05 \) where \( L = r \sin \theta \) describes the (radial) distance from the \( z \)-axis in cylindrical coordinates, and \( H \) is the disc’s vertical scale height. The viscosity \( \nu \) is set to a value of \( \nu = 10^{15} \text{cm}^2 \text{s}^{-1} \) and the mean molecular weight to \( \mu = 2.3 \). For the ratio of specific heats we have used different values, as specified below. The density stratification can be obtained from vertical hydrostatic equilibrium, assuming a temperature that is constant on cylinders, \( T = T(s) \). It follows (Masset et al. 2006)

\[
\rho(r, \theta) = \rho_0 \cdot s^{-1.5} \exp \left( \frac{\sin \theta - 1}{h^2} \right)
\]

where the quantity \( \rho_0 \) was chosen such that the total mass of the disc is \( M_{\text{disc}} = 0.01 \cdot M_* \), where \( M_* \) is the mass of the central star of the system which is set to the mass of the sun, \( M_* = M_{\odot} \). The mass within the computational domain is then \( 1/2M_{\text{disc}} \) because we only compute the upper half of the disc. The radial variation leads to a surface density profile of \( \Sigma \propto r^{-3/2} \), which is the equilibrium profile for constant viscosity, and vanishing mass flux through the disc. The pressure \( p \) is set by the isothermal relation \( p = \rho c_s^2 \), with the speed of sound \( c_s = H \Omega_K \) and the Keplerian angular velocity

\[
\Omega_K = \sqrt{\frac{GM_*}{r^3}},
\]
with the gravitational constant $G$. The temperature can be computed through equation (5) and results in $T = \frac{m_{nu} E}{c_\gamma}$. The initial velocities are set to zero except for the angular velocity $v_\phi$ which is set to

$$v_\phi = \sqrt{\frac{(1 - 2\beta^2)}{\gamma}} \frac{GM_s}{r_s}.$$ 

For the Rosseland mean opacity $\kappa_R$ we use data from Lin & Papaloizou (1985), and the Planck mean opacity is set to $\kappa_P = \kappa_R$. The displayed simulations have been performed in the rotating frame in which the coordinate system rotates with the constant angular velocity of $\Omega_R$ at $a_{jup}$, but for non-rotating systems identical results are obtained. As before the radiation energy density is initialised to $E = a_R T^4$.

For density, pressure and radial velocity we apply reflective boundary conditions and the angular velocity is set to the Keplerian values. In the azimuthal direction periodic boundary conditions are used for all variables. In the vertical direction we apply an equatorial symmetry and reflective boundary condition for $\theta_{min}$. The radiation boundary conditions are set to reflective for the $r$ direction (both lower and upper), in $\theta$-direction we use a fixed value of $E = a_R T^4$ with $T = 5$ K at $\theta_{min}$ (which denotes the disc surface), and a symmetric boundary condition holds at the disc’s midplane $\theta_{max}$. For the $\phi$-direction we use periodic boundary conditions.

In both cases we used for the matrix solver a relative tolerance of $\epsilon_r = 10^{-8}$. In the simulation with hydrodynamics we use the Riemann-solver hllc$^2$.

### 4.6.2. The static case

In this test case only the radiative equations are solved without the hydrodynamics. In order to account for the viscous heating in this case, we add an additional dissipation contribution, $D$, to the right hand side of the internal energy equation in (13). We consider standard viscous heating, and include only the main contribution due to the approximately Keplerian shear flow. At the individual grid points the dissipation is then given by

$$D_{i,j,k} = \nu^2 \rho_{i,j,k} \left( \frac{\partial \Omega_{i,j,k}}{\partial r_i} \right)^2,$$  \hspace{1em} (43)

where $\nu$ is the constant viscosity and $\Omega_{i,j,k}$ the angular velocity at the individual grid points. In summary we solve the same equations as in the case with irradiation, when we substitute $S_{i,j,k}$ with $D_{i,j,k}$.

In the steady state, the time derivatives in the equations (13) vanish and the system is reduced to the following equation for the radiation energy density

$$\nabla \cdot \left( \frac{c_\lambda}{\kappa_R \rho} \nabla E \right) = D.$$ \hspace{1em} (44)

In optically thick regions, $E = a_R T^4$ and eq. (44) determines the temperature stratification within the disc.

The simulation starts at $t = 0$ orbits and is evolved until $t = 100$ orbits are reached, where one orbit corresponds to the Keplerian orbital period at the distance of $a_{jup}$ which is given here by $3.732 \cdot 10^{5}$ s. The initial and overall time step was chosen as $\Delta t = 10^{-3}$ orbits $= 3.732 \cdot 10^{5}$ s. The results for the static case are shown in figure 8 using here a value of $\gamma = 7/5$ for the adiabatic index. The plots show the radial temperature profile of the accretion disc in the mid-plane for the simulations after 10 orbits (top panel) and after 100 orbits (bottom panel). We display results of two different simulations, one done with the code PLUTO (red dots) using the described methods, and the second (black lines) run with the code RH2D (Kley 1989). The result from both codes are nearly identical. Even after 100 orbits the absolute value of the relative error is always less than 2%. The test shows that the time-scale of the radiative evolution, as well as the equilibrium state is captured correctly. We note that the code RH2D uses the one-temperature approach of radiation transport in this case.

### 4.6.3. The dynamical case

The final equilibrium of the described static case does not depend on the magnitude of $\gamma$, because the viscous heating is independent of it, see eq. (44). The situation is different, however,
for the dynamical cases, where the hydrodynamical evolution of the flow is taken into account. Since the time scale of the radiative transport depends on γ (through eq. 6), one might expect the possibility of convective instability, see for example the recent work by Bitsch et al. (2013a). This is indeed the case for small enough values of γ. In order to demonstrate the correctness of our implementation also for the full dynamical problem, we modelled two discs, one with γ = 5/3 which clearly shows no convection, and the other with γ = 1.1 which shows strong convection. The initial setup was identical to that described before, but now we solve the equations of viscous hydrodynamics with radiation transport, but without irradiation and explicit dissipation. Please note that for viscous flows the energy generation due to viscous dissipation is automatically included in the total energy equation. The equations (1) to (3) are solved by PLUTO, and the system of equations (13) are solved as described in section 3. Since this setup is very dynamical and requires a more complex interplay of hydrodynamics and radiative transport, we use an additional third code, NIRVANA, for comparison. The NIRVANA code has been used in Kley et al. (2009) and Bitsch et al. (2013a) on very similar setups. The results of the two cases are shown in Fig. 9. In the top panel (a) we display the result for the γ = 5/3 case which is not convective. Here, the agreement between the codes is excellent with the maximum deviation in the percentage range. In the lower panel (b) we display the results for the γ = 1.1 case. Here the radiative transport time-scale is enhanced which leads to a strongly convective situation, which can be seen in the raggedness of the curves. In this simulation we doubled the spatial resolution, compared with the γ = 5/3 case, such that the convection cells are reasonably well resolved, see figure 10. The agreement between the three different codes is very good, despite of the very different solution methods for the hydrodynamics equations: PLUTO uses the total energy equation with a Riemann-solver while RH2D and NIRVANA use a second-order upwind scheme and the thermal energy equation. Additionally, the latter two codes use the full dissipation function and the one-temperature approach.

4.6.4. Parallel scaling

In order to test the parallel scaling of our new implementation, we used the same setup as in section 4.6.3 and increased the number of grid cells to 1024 × 64 × 256. The computations were only run until \( t = 5 \) orbits, and we used the solver PETSc. So we were able to run the test on 64 up to 1024 processor cores within a reasonable time. The simulations were run on clusters of the BWGrid which are equipped with Intel Xeon E5440 cpus and have a low latency InfiniBand network. In figure 11 we show the results of the simulations performed with full hydrodynamics and radiation transport. The run-time increases nearly by a factor of two when doubling the number of cores. With this setup, solving the hydrodynamics equations needs between 40% and 50% of the computation time and the radiation transport the remaining 60% to 50%, however, these numbers are strongly problem-dependent. Therefore even up to 1024 cores, we see good agreement with ideal scaling. According to Amdahl’s law the full code, including the original part of PLUTO and our implementation of the radiation transport, is well parallelised.

5. Summary and conclusions

We described the implementation of a new radiation module to the PLUTO code. The module solves for the flux-limited diffusion approximation in the two-temperature approach. For discretisation the finite volume method is used, and the resulting difference equations couple the updates of the temperature and radiation energy density. Due to possibly severe time step limitations, the set of equations is solved implicitly. For treating the non-linearity of the temperature in the matter-radiation coupling term, we utilize the method of Commerçon et al. (2011).

The accuracy of the implementation has been verified using different physical and numerical setups. The first set of tests deals with purely radiative problems that include the purely diffusive evolution towards an equilibrium, and special setups to test the coupling terms between radiative and thermal energy. A newly developed setup checks for the correct inclusion of the irradiation from a central source in a spherical coordinate system.

In the second test suite we study the full simultaneous evolution of hydrodynamics and radiation. First, sub- and super-
critical radiative shock simulations are performed and their outcomes agree very well with published results of identical setups. Finally, we study the onset of convection in internally heated viscous discs, and find very good agreement between 3 different, independent hydrodynamical codes. This last test also allowed us to test the correct implementation in a spherical coordinate system and a non-equidistant logarithmic grid. Our numerical performance tests indicate excellent parallel scaling, up to at least 1024 processors.

The current version of the radiation module comes with routines for the Rosseland mean opacity from Lin & Papaloizou (1985) and Bell & Lin (1994). Additionally it is possible to use the Rosseland and Planck mean opacities from Semenov et al. (2003).

Fig. 9. Radial mid-plane temperature profile in the simulation with PLUTO (red line), RH2D (black line) and NIRVANA (blue line) in the quasi-equilibrium state after 100 orbits in the case with $\gamma = 5/3$ without convection (a) and in the strongly convective case with $\gamma = 1.1$ (b). Additionally we added the results of a simulation performed with PLUTO where we use a logarithmic grid in $r$-direction (green line).

Fig. 11. Parallel scaling benchmark results for the static accretion disc test case. We plot here the number of processor cores against $t_N$ where $t_N$ is the runtime used on $N$ processors accordingly for $t_{64}$. The used run-times with full hydrodynamics and radiation transport for 64, 128, 256, 512 and 1024 cpu cores (red crosses) are shown together with the ideal case (black dashed line).

The described radiation module can be easily used within the PLUTO-environment. It can be found on the webpage\footnote{http://www.tat.physik.uni-tuebingen.de/~pluto/pluto_radiation/} as a patch for the version 4.0 of PLUTO.

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