Superfluid density of a photo-induced superconducting state

A. Shtyk, G. Goldstein, C. Aron, and C. Chamon

1Department of Physics, Harvard University, Cambridge, MA 02138, USA
2Cavendish Laboratory, University of Cambridge, Cambridge, CB3 0HE, United Kingdom
3Laboratoire de Physique Théorique, École Normale Supérieure, CNRS, PSL University, Sorbonne Université, Paris 75005, France
4Instituut voor Theoretische Fysica, KU Leuven, Belgium
5Department of Physics, Boston University, Boston, MA 02215, USA

Nonequilibrium conditions offer novel routes to superconductivity that are not available at equilibrium. For example, by engineering nonequilibrium electronic populations, pairing may develop between electrons in different energy bands. A concrete proposal has been made to photo-induce superconductivity in a semiconductor, where pairing occurs between electrons in the conduction and valence bands, even for repulsive interactions. Here, we calculate the superfluid density for such a nonequilibrium paired state, and find it to be positive for repulsive interactions and interband pairing. The positivity of the superfluid density implies the stability of the photo-induced superconducting state as well as the existence of the Meissner effect.

PACS numbers:

I. INTRODUCTION

The subject of nonequilibrium superconductivity has recently gained much interest, in part due to experiments on photo-induced transient states in YBa$_2$Cu$_3$O$_{6+\delta}$, and the subsequent experimental and theoretical progress (see Ref. 2 for a review). In fact, ideas to extend the temperature regime where superconductivity exists by optical excitation have a long history. Their root can be traced back to the pioneering theoretical predictions by Eliashberg and Gor’kov, who showed that microwave radiation may assist the formation of the superconducting gap and thus raise the transition temperature $T_c$. These predictions were later confirmed by experiments. The applied electromagnetic radiation shifts the electronic occupation numbers, extending the temperature regime in which the BCS self-consistency equation has a non-zero solution.

These ideas of population control can also be applied to systems that are not superconductors at equilibrium. It was proposed that superconductivity could be induced in narrow, indirect gap semiconductors, with pairing between electrons in the same band (intraband pairing). A non-zero superconducting gap was shown to be possible with attractive and, notably, with repulsive electronic interactions as well. The latter case is particularly important because repulsion is prevalent in electronic systems. However, in the latter case it was also noted that there was no Meissner effect accompanying the formation of a gap, because the sign of the current response was opposite to that in a standard superconductor: the system would respond as a perfect paramagnet instead of a perfect diamagnet. This strange response, corresponding to a negative superfluid density, signals that the state is unstable for repulsive interactions.

Recently, Ref. 10 proposed to use optical pumping to achieve interband, rather than intraband, pairing. In this scheme, electrons sitting in two bands at widely different energies and far away from the chemical potential can form interband Cooper pairs in the $s$-wave channel, even in the case of repulsive interactions. (See Figure 1 for a sketch of the relevant mechanism.)

In this paper, we investigate the stability of the corresponding photo-induced interband superconducting state by computing its superfluid density. We find a positive superfluid density for repulsive interactions for all parameters (e.g., band curvatures, quasi-particle populations) where a non-trivial mean-field solution of the self-consistent BCS equation exist. The positivity of the superfluid density implies the stability of the state as well as the existence of a Meissner effect. The later could be used as a reliable alternative to transport properties to confirm the presence of a nonequilibrium superconducting order in a semiconductor.

II. MODEL

We consider a two-band semiconductor model with electronic dispersions $E_{1p}$ and $E_{2p}$. The chemical potential of the system is set in the middle of the two bands, see Fig. 1. For simplicity, we consider symmetric bands, i.e. $E_{\alpha p} = E_{\alpha -p}$ for $\alpha = 1, 2$. The semiconductor is optically pumped with a broad-band light source, as described in Ref. 10. In this setup, the optical pumping creates a nonequilibrium distribution of the quasiparticles, which is key to achieve the interband pairing.

A. Hamiltonian

Let us assume that, after a transient regime, the interband pairing has already built up. The mean-field description of the system consists in the following BCS
ter introducing the Nambu-Gor’kov spinor notation. Without loss of generality, we assume that $\Delta$ is real. As self-consistently, see below.

The Hamiltonian can be readily diagonalized via the following transformations in Nambu-Gor’kov space

$$\hat{U}_p = \exp \left[ \frac{1}{2} \beta_p \hat{\tau}_y \right], \quad \text{with} \quad \tan \beta_p \equiv \frac{\Delta}{E_p},$$

leading to

$$H = \int_d \Psi_p^\dagger [\xi_p \hat{\tau}_z + \epsilon_p \hat{\tau}_z - \Delta \hat{\tau}_x] \Psi_p,$$

with $\xi_p \equiv \sqrt{E_p^2 + \Delta^2}$. We adopt the underlining to indicate the use of the quasiparticle basis.

### B. Keldysh action

We compute the electrodynamic response of a generic steady state within the Schwinger-Keldysh formalism. The Keldysh action corresponding to the Hamiltonian in Eq. (3) reads

$$S = \int_T dt \int_d \Psi_p^\dagger \left[ (i\partial_t - \epsilon_p) \hat{I} - E_p \hat{\tau}_z - \Delta \hat{\tau}_x \right] \Psi_p,$$

where the time integral over $t$ goes along the standard Keldysh contour $T$ going from $-\infty$ to $+\infty$ and then back to $-\infty$. It is customary to perform a so-called Keldysh rotation of the fields living on those two time branches. In the resulting $2 \times 2$ Keldysh space, the electron Green’s functions are organized as

$$\hat{G} = \begin{pmatrix} \hat{G}^R & \hat{G}^A \\ \hat{G}^A & \hat{G}^K \end{pmatrix},$$

where $\hat{G}^R$, $\hat{G}^A$, and $\hat{G}^K$ are the retarded, advanced and Keldysh Green’s functions, respectively. The retarded Green’s function encodes the spectral properties of the steady-state and reads $\hat{G}^R = \hat{U} \hat{G}^R \hat{U}^\dagger$ with

$$\hat{G}^R (\epsilon, p)^{-1} = (\epsilon - \epsilon_p + i0) \hat{I} - \xi_p \hat{\tau}_z,$$

where the operator $\hat{U}$ was defined in Eq. (6). A convenient representation is given by

$$\hat{G}^R (\epsilon, p) = \frac{1}{\epsilon - \epsilon_p - \xi_p + i0} \hat{P}_p + \frac{1}{\epsilon - \epsilon_p + \xi_p + i0} \hat{P}_p^\dagger,$$

with $\hat{P}_p$ being the projectors onto the states of the quasiparticle basis, i.e., $\hat{P}_p = (\hat{I} \pm \hat{\tau}_z)/2$. $\hat{G}^A$ can be determined from $\hat{G}^R$ by a simple time-reversal operation. The Keldysh Green’s function encodes the nonequilibrium state populations,

$$\hat{G}^K (\epsilon, p) = [\hat{G}^R \hat{F} - \hat{F} \hat{G}^A] (\epsilon, p),$$

with the matrix $\hat{F} (\epsilon, p)$ encoding a general nonequilibrium electron distribution function. In general, $\hat{F}$ does...
not commute with the Green’s function \( \tilde{G}(A) \), but in thermal equilibrium the situation is greatly simplified and \( \tilde{F}(\epsilon, p) \) becomes \( \tilde{F}(\epsilon, p) = \tanh \epsilon/2T \cdot \hat{1} \).

The electromagnetic vector potential \( \vec{A} \) generates an electric current \( \vec{J} \) whose spatial components \( J^\mu (\mu = x, y, z) \) read

\[
J^\mu = e \sum \langle \Psi_p^\dagger \left[ \tilde{v}_p^\mu - \epsilon \partial_\mu \tilde{v}_p A^\nu \right] \Psi_p \rangle ,
\]

with \( \epsilon < 0 \) is the charge of the electron and we set the speed of light \( c = 1 \), while the “velocity” and “mass” are

\[
\tilde{v}_p^\mu = V_p^\mu \hat{I} + v_p^\mu \hat{z}, \quad \partial_\mu \tilde{v}_p^\nu = \partial_\mu V_p^{\nu} \hat{z} + \partial_\mu v_p^\nu \hat{I},
\]

with \( \partial_\mu \equiv \partial/\partial p^\mu \). In the quasiparticle basis, they read (omitting the \( p \) indices)

\[
\begin{align*}
\tilde{v}_p^\mu = V_p^\mu \hat{I} + v_p^\mu (\hat{z} \cos \beta - \hat{\tau} \sin \beta), \\
\partial_\mu \tilde{v}_p^\nu = \partial_\mu V_p^{\nu} (\hat{z} \cos \beta - \hat{\tau} \sin \beta) + \partial_\mu v_p^\nu \hat{I}.
\end{align*}
\]

C. Self-consistency equation

In this manuscript, we consider a generic steady state with a diagonal quasiparticle distribution function,

\[
\langle \Psi_p^\dagger \Psi_p^\beta \rangle = \hat{U} \begin{pmatrix} n_{1p} & 0 \\ 0 & 1 - n_{1p} \end{pmatrix} \hat{U}.
\]

The equation (16) implies that the matrix \( \tilde{F} \) encoding the quasiparticle distribution in Eq. (12) is also diagonal in quasiparticle basis and is given by

\[
\tilde{F}(\epsilon, p) = \begin{pmatrix} 1 - 2n_{1p} & 0 \\ 0 & 2n_{1p} - 1 \end{pmatrix}.
\]

Moreover, this means that \( \tilde{F} \) commutes with the Green’s functions \( \tilde{G}(R) \) and \( \tilde{G}(A) \). To derive the self-consistency equation, we go back to the original microscopic electron interaction and track the origin of the superconducting pairing in Eq. (1) as stemming from an electronic interaction

\[
H_{int} = V_{int} \int_{p_1 p_2} \langle c_{p_2}^{\dagger} c_{p_1}^\dagger \rangle \langle c_{-p_2} c_{-p_1} \rangle ,
\]

where \( V_{int} \) is positive for repulsive interactions. Within a mean-field treatment, we have

\[
H_{int} = - \int_p \left( \Delta c_p^{\dagger} c_p^{\dagger} + \Delta^* c_p c_{-p} \right) ,
\]

together with a self-consistency equation reading

\[
\Delta = V_{int} \int_p \langle c_p^{\dagger} c_p \rangle .
\]

Using the explicit form of the distribution function in Eq. (16), the self-consistency equation takes a standard form when written in terms of quasiparticle distribution function:

\[
1 = \frac{V_{int}}{2} \int_p \frac{n_{1p} + n_{2p} - 1}{\xi_p} ,
\]

where \( \xi_p = \sqrt{E_p^2 + \Delta^2} \). The nonequilibrium effects enter through changes of the quasiparticle distribution functions with respect to their equilibrium values,

\[
\frac{n_{eq}(1, 2)p}{2}(\epsilon_p \pm \xi_p, \mu) ,
\]

where \( n_{eq}(\epsilon, \mu) = [1 + \exp(-\epsilon - \mu)/T]^{-1} \) is the Fermi-Dirac distribution at the temperature \( T \) and chemical potential \( \mu \). The latter would reproduce the standard BCS self-consistency equation.

III. SUPERFLUID DENSITY

The focus of this paper is on the Meissner effect i.e. the response of the system to a non-uniform static electromagnetic vector potential. In this static limit, the electromagnetic properties are governed by the superfluid density defined through the following relation

\[
J^\mu = -\rho^\mu v^\nu A^\nu .
\]

We consider the isotropic case, when the superfluid density tensor \( \rho^\mu v^\nu \) is reduced to a scalar quantity \( \rho \),

\[
\rho^\mu v^\nu = (e^2/d) \rho \delta^\mu v^\nu .
\]

The superfluid density \( \rho \) in the present text differs from the standard definition by a factor \( e^2/d \), where \( d \) is the spatial dimension, in order to simplify the expressions below.

Similarly to the equilibrium case, there are two contributions to the electric current, paramagnetic and diamagnetic, which stem respectively from the first and second term of the current expression in Eq. (13),

\[
\rho = \rho_{para} + \rho_{dia}. \]

The paramagnetic term is also often called gradient term, for in the case of the parabolic band with electron mass \( m \) we have \( v = p/m = (-ih/m) \nabla \). Below, we compute these two contributions for generic quasiparticle distributions, and later specialize to the case of photo-induced superconductivity.

A. Diamagnetic contribution

We start with the diamagnetic contribution stemming from the second term in Eq. (13),

\[
J_{(dia)}^\mu = -e^2 \sum_{p, \nu} \langle \Psi_p^\dagger (\partial_\nu v^\nu A^\nu) \Psi_p \rangle .
\]
This yields the superfluid density

$$\rho_{\text{dia}} = -\frac{i}{2} \int_{p,\mu} \text{Tr} \left[ \partial_\mu \bar{\psi}_\mu \left( \hat{G}^K - \hat{G}^R + \hat{G}^A \right) (\epsilon, p) \right],$$  \hspace{1cm} (27)

where the trace runs in Nambu-Gor’kov space. The equation (27) has contributions from both quasiparticle bands. The contribution from the upper quasiparticle band can be computed using equations (10), (15), and (16), yielding

$$\int_{p,\mu} \left( \partial_\mu V^\mu \cos \beta + \partial_\mu v^\mu \right) n^1_{1p},$$  \hspace{1cm} (28)

since the actual occupation number\textsuperscript{12} is given by

$$n^1_{1p} = \frac{1}{2} \int \left( \hat{G}^K_{11} - \hat{G}^R_{11} + \hat{G}^A_{11} \right) (\epsilon, p).$$  \hspace{1cm} (29)

The second quasiparticle band contributes with

$$\int_{p,\mu} \left( -\partial_\mu V^\mu \cos \beta_p + \partial_\mu v^\mu \right) (1 - n^2_{2p}),$$  \hspace{1cm} (30)

such that, at last, we obtain

$$\rho_{\text{dia}} = \int_{p,\mu} \left( \partial_\mu V^\mu \cos \beta_p \left( n^1_{1p} + n^2_{2p} - 1 \right) + \partial_\mu v^\mu \right) \left( n^1_{1p} - n^2_{2p} + 1 \right).$$  \hspace{1cm} (31)

### B. Paramagnetic contribution

The paramagnetic contribution stems from the first term of the current expression in Eq. (13), and it reduces to the current–current correlation function,

$$J^\mu_{\text{para}} = -\frac{e^2}{2} \left( \Psi^- \bar{\psi}_n \psi \cdot \bar{\psi} A^\nu \Psi \right).$$  \hspace{1cm} (32)

Therefore, the superfluid density acquires the contribution

$$\rho_{\text{para}} = \frac{1}{2} \int_{p,\mu} \text{Tr} \left[ \hat{G}^K \bar{\psi}_n \hat{G}^R \bar{\psi}_n + \hat{G}^A \bar{\psi}_n \hat{G}^K \bar{\psi}_n \right]$$  \hspace{1cm} (33)

that is the sum of two qualitatively different terms. The details of the derivation are given in App. B. The first term comes from quasiparticle intra-branch processes, and the ordering of limits in the external frequency and in the momenta \( \omega, q \to 0 \) is crucial. We focus on the case relevant for the Meissner effect, i.e., when the static limit (zero frequency) is taken first, yielding

$$\rho_{\text{para}}^{\text{intra}} = \int_{p,\mu} \left( V^\mu + v^\mu \cos \beta_p \right)^2 n^1_{1p} + \left( V^\mu - v^\mu \cos \beta_p \right)^2 n^2_{2p},$$  \hspace{1cm} (34)

where we introduced the quantities

$$n^1_{1p} = \lim_{\epsilon \to 0} \frac{n^1_{1p} + q - n^1_{1p}}{\epsilon_{p+q} + \xi_p} = \frac{\partial_p n^1_{1p} (2p)}{\partial_{\epsilon_{p+q}} \epsilon^p_{1(2)p}},$$  \hspace{1cm} (35)

In the last step above, we assumed isotropic dispersion relations and \( v^\mu_{\text{par}} \) is the quasiparticle velocity \( (\alpha = 1, 2) \).

The second component corresponds to quasiparticle inter-branch processes with pair creation and annihilation. Here, the order of limits \( \omega, q \to 0 \) does not matter. A straightforward calculation gives

$$\rho_{\text{para}}^{\text{inter}} = \frac{1}{2} \int_{p} v^2 \sin^2 \beta_p \frac{n^1_{1p} + n^2_{2p} - 1}{\xi_p}. $$  \hspace{1cm} (36)

### C. Total superfluid density

Integrating by parts the diamagnetic contribution in Eq. (31) and summing with both paramagnetic terms, we obtain the final result for the net superfluid density of the system (see appendix for detailed steps):

$$\rho = -\int_{p} \left( V^2 - v^2 \right) \sin^2 \beta_p \left[ \frac{n^1_{1p} + n^2_{2p} - 1}{\xi_p} - \frac{n^1_{1p} - n^2_{2p}}{\xi_p} \right].$$  \hspace{1cm} (37)

The equation above is the central result of our manuscript. It is applicable to a wide class of electronic dispersions as long as the superconducting state is stable and with a diagonal quasiparticle distribution function. The immediate application of interest is to use this result to study the superfluid density of a photo-induced inter-band superconductor, which is the topic of the next Section.

### IV. SUPERFLUID DENSITY IN A STEADY STATE

In the optical-pumping setup presented in Ref. 10, a nonequilibrium population of the two electron bands is created by shining a laser on the system. The laser is responsible for the emergence of a resonance surface in momentum space, \( S \), where the sum of the two bands \( 2E_k = E_{1k} + E_{2k} \equiv 0 \). The resonant surface \( S \) has essentially the same role as a Fermi surface in a conventional BCS superconductor in thermal equilibrium. Superconducting pairing takes place around the resonant surface \( S \), where \( E_k = 0 \). For simplicity, we assume that \( S \) is rotationally invariant. The electronic dispersions can be expanded in the vicinity of this surface as

$$E_k \equiv \frac{E_{1k} + E_{2k}}{2} \approx V(k - k_S) + \kappa_+ (k - k_S)^2, $$  \hspace{1cm} (38)

$$\varepsilon_k \equiv \frac{E_{1k} - E_{2k}}{2} \approx \varepsilon_0 + V(k - k_S) + \kappa_- (k - k_S)^2. $$  \hspace{1cm} (39)

A situation which is especially favorable for the formation of the superconducting order corresponds to whenever the velocity matching condition \( V = 0 \) is satisfied, see Ref. 10 for details. Below, we concentrate on this case, such that \( E_k \approx \kappa_+ (k - k_S)^2 \).
where $\Gamma_{12}$, which is diagonal in the quasiparticle basis and can be parametrized as

$$\rho = \frac{s_{11} + s_{22}}{2},$$

in our specific case of photo-induced superconductivity there is a relation between the components of the above matrix,$^{10}$

$$(2E + i\Gamma_{12})s_{21} = -\Delta^*(n_{11}^k + n_{22}^k - 1),$$

where $\Gamma_{12}$ is an interband relaxation rate. At energies $E \gg \Gamma_{12}$, this relation implies a distribution function which is diagonal in the quasiparticle basis and

$$\langle \Psi_k^\dagger \Psi_{-k} \rangle = \hat{U}^\dagger \left( \begin{array}{cc} n_{1k} & 0 \\ 0 & 1 - n_{2k} \end{array} \right) \hat{U}.$$  

This is of course reasonable since quasiparticles are true excitations of the emergent superconducting state.

As it follows from Ref.$^{10}$, the distribution functions in Eq. (42) are relatively smooth around the resonant surface and depend only on the energy $E_k$. Together with the assumed velocity matching condition, $V \approx 0$, this implies that the second term in the superfluid density in Eq. (37) is negligible as compared to the first one, since the quasiparticle velocities $v_{12}^k = v_p \pm V_p \cos \beta_p \approx v$ and

$$\rho = \int_V^\dagger \frac{\Delta^2}{\xi_p} (\bar{n}_{1p} + \bar{n}_{2p} - 1),$$

where, we recall, $\xi_p = \sqrt{E_p^2 + \Delta^2}$.

Now, we make use of the explicit form of distribution functions obtained in Ref. 10 (see also the appendix where we reproduce the derivation of the relevant results for the case of quasiparticles in the steady state with a mean-field pairing field and an external optical pump). In particular, a relevant quantity of interest is

$$n_{1k} + n_{2k} - 1 = \frac{4E_k \sqrt{E_k^2 + \Delta^2}}{4E_k^2 + \gamma_s \Delta^2} N_S,$$

where we introduced the combination

$$N_S = n_F(E_{1k}, \mu_1) + n_F(E_{2k}, \mu_2) - 1,$$

with $n_F(E_{\alpha k}, \mu_\alpha)$ the Fermi-Dirac distributions corresponding to the quasi-thermal equilibrium that sets up in each band. The effective chemical potentials $\mu_\alpha$ can be seen as Lagrange multipliers enforcing the average number of particles in each band, and depend on the balance between the optical drive and the interband relaxation mechanisms (see Ref. 10 for details). Note that the finiteness of the above quantity, $N_S \neq 0$, is crucial to the formation of an interband Cooper pairing. For convenience, we introduced $\gamma_s \equiv \Gamma_{12}(\Gamma_{-1}^{-1} + \Gamma_{1}^{-1})$ with $\Gamma_{12}$ being intraband relaxation rates. We obtain the superfluid density

$$\rho = N_S \frac{\Delta^2}{\xi_p} \int_k \frac{4E_k}{(E_k^2 + \Delta^2)(4E_k^2 + \gamma_s \Delta^2)},$$

and the self-consistency equation for photo-induced superconductivity

$$1 = 2N_S V_{\text{int}} \int_k \frac{E_k}{4E_k^2 + \gamma_s \Delta^2}.$$

The most pressing questions are now

- Is superconductivity possible? (This question is the focus of Ref. 10.)
- What is the sign of the superfluid density? (This is our main focus.)

Answers to these question depend solely on the signs of three parameters, namely: electron-electron interaction $V_{\text{int}}$, curvature of the electron dispersion $\kappa_+$, and $N_S$. Indeed, a rapid inspection of the right-hand side of Eq. (47) implies that the sign of the superfluid density is governed by

$$\text{sgn}(\rho) = \text{sgn}(\kappa_+) \times \text{sgn}(N_S).$$

Similarly, the inspection of the right-hand side of Eq. (48) implies that a solution with a finite superconducting order parameter exists whenever

$$\text{sgn}(V_{\text{int}}) = \text{sgn}(\kappa_+) \times \text{sgn}(N_S).$$

The corresponding outcomes for different cases are summarized in the Table I. We now compute the expression of the superfluid density in two limiting cases: $\Delta \gg \Gamma$ and $\Delta \ll \Gamma$.  

| Parameter | $\checkmark$ | $\checkmark$ | $\checkmark$ | $\checkmark$ |
|-----------|--------------|--------------|--------------|--------------|
| $V_{\text{int}}$ | - | - | + | + |
| $N_S$ | - | + | - | + |
| $\kappa_+ = \frac{s_{11} + s_{22}}{2}$ | + | - | - | + |
| $\rho$ | - | - | + | + |

Table I: Sign of the superfluid density $\rho$ for different parameters: electronic interaction $V_{\text{int}}$ (repetitive when positive), nonequilibrium population imbalance $N_S$ [see Eq. (46)] and average band curvature $\kappa_+$. Only the cases allowing for an inter-band superconducting state are displayed.
In the regime $\Delta \gg \Gamma$, we start by expanding the energy $E_k$ around the resonance surface (using the velocity matching condition, $V \approx 0$), $E \approx \kappa_+ (k - k_S)^2$, so that the superfluid density becomes
\[
\rho = \frac{(\text{sgn} \kappa_+ N_S)}{\sqrt{\Delta |\kappa_+|}} A_S v^2 B_\rho(\gamma_+), \quad (51)
\]
where $A_S = 4\pi k_S^2$ is the area of the resonant surface ($2\pi k_S$ for a two-dimensional system) and $B_\rho(x)$ is a positive function
\[
B_\rho(x) = \int_{-\infty}^{\infty} dt \frac{4t^2}{(t^4 + 1)(4t^4 + x)} = \begin{cases} \frac{\pi x^{-1/4}}{2\sqrt{2}\pi x^{-1}} & x << 1 \\ \frac{\pi}{2\sqrt{2}\pi} & x \gg 1 \end{cases}
\]
Equation (51) displays an anomalous scaling of the superfluid density $\rho$ with the order parameter $\Delta$,
\[
\rho \propto \frac{1}{\sqrt{\Delta}} \text{ for } \Delta \gg \Gamma. \quad (53)
\]
Such a divergent scaling survives only while $\Delta \gg \Gamma$. In the regime where $\Delta \ll \Gamma$, after re-including properly the factors of $\Gamma$ that have been neglected so far (see the appendix for a detailed derivation), one finds
\[
\rho \propto \Delta^2 \text{ for } \Delta \ll \Gamma, \quad (54)
\]
which is similar to the conventional BCS scenario in equilibrium. The two scalings in Eqs. (53) and (54) signal that the superfluid density reaches a maximum in the crossover regime.

V. CONCLUSION

We have computed the superfluid density of the superconducting state that can be induced by optically pumping valence band electrons to the conduction band. We found a positive superfluid density in the presence of repulsive electronic interactions, and this constitutes an important check of the stability of the superconducting order announced in Ref. 10. The next check is to make sure that the heating caused by the optical pumping is slow enough to allow for the superconducting order to develop (on the order of hundreds on 1/$\Gamma$’s) and, perhaps more importantly, for the transport measurements to be performed. The power dissipated can estimated to be $P \sim \Gamma_{\text{interband}} N \times h\omega_{\text{gap}}$ with an interband recombination rate $\Gamma_{\text{interband}} \sim 10^{-8}$ eV, a density of states $N \sim 10^{20}/\text{cm}^3$, and $h\omega_{\text{gap}} \sim 0.3$ eV, amounting to $P \sim 10^{6-7} \text{ J/s,cm}^3$, and leading to a generous window of $10^{-4}$ s to perform the experiments before the sample temperature increases by approximately 10 K (in the absence of external cooling).

Acknowledgments

This work has been supported by the DOE Grant DE-FG02-06ER46316 (C.C.), and by the Engineering and Physical Sciences Research Council (EPSRC) and No. EP/M007065/1 (G.G.), and by the EPSRC Network Plus on “Emergence and Physics far from Equilibrium”. Statement of compliance with the EPSRC policy framework on research data: this publication reports theoretical work that does not require supporting research data.

Appendix A: Equations of motion for the quasiparticle distribution functions

1. Equations of motion

The equations of motion for the optical pumping setup employed in the present paper are derived in Ref. 10. They read as:
\[
\frac{d}{dt} n_{1k}^{11} = i\Delta s_k^{12} - i\Delta^* s_k^{12*} - 2 \Gamma_1 n_{1k}^{11}, \quad (A1)
\]
\[
\frac{d}{dt} n_{2k}^{22} = i\Delta s_k^{12} - i\Delta^* s_k^{12*} - 2 \Gamma_2 n_{2k}^{22},
\]
\[
\frac{d}{dt} s_k^{12} = -(2E_k - i\Gamma_2) s_k^{12} + i\Delta^* (n_{1k}^{11} + n_{2k}^{22} - 1),
\]
where $n_{k}^{\alpha\alpha} = \langle c_k^{\alpha\dagger} c_k^{\alpha} \rangle$, and $s_k^{12} = \langle c_k^{1} c_k^{2\dagger} \rangle$. The tilded quantities $\tilde{n}_{k}^{\alpha\alpha} = n_{k}^{\alpha\alpha} - n_F(E_{\alpha k}, \mu_\alpha)$, where $n_F(E_{\alpha k}, \mu_\alpha)$ are distribution functions in the external thermal bath (see Ref. 10 for details).

In the steady state all time derivatives are zero. The last equation gives a useful relation
\[
s_k^{12} = \frac{\Delta^*}{2E_k - i\Gamma_2} (n_{1k}^{11} + n_{2k}^{22} - 1). \quad (A2)
\]

Proceeding with solving equations of motion, we get
\[
n_{1k}^{11} + n_{2k}^{22} - 1 = \frac{4E^2 + \Gamma_1^2}{4E^2 + \Gamma_2^2 + \gamma_+ \Delta^2} N_S, \quad (A3)
\]
\[
s_k^{12} = \frac{\Delta^*(2E + i\Gamma_2)}{4E^2 + \Gamma_2^2 + \gamma_+ \Delta^2} N_S \quad (A4)
\]
where $N_S = (n_F(E_{1k}, \mu_1) + n_F(E_{2k}, \mu_2) - 1)$ and $\gamma_+ = \Gamma_2(N_0^{-1} + \Gamma_2^{-1})$. Only these two quantities, (A3) and (A4) are of interest, as we will see below.

2. Quasiparticle distribution functions

As we have stated in the main text, for energies larger than the decay rates $\Gamma$ we deal with well-defined quasiparticle distribution functions. Using equation (16), we get for the matrix quasiparticle distribution function
\[
\begin{pmatrix}
    n_{1k}^{11} & 0 \\
    0 & 1 - n_{2k}^{22}
\end{pmatrix}
= \hat{U} \begin{pmatrix}
    n_{1k}^{11} & s_k^{12} \ \\
    s_k^{12} & 1 - n_{2k}^{22}
\end{pmatrix} \hat{U}^+. \quad (A5)
\]
\[ n_{1k} = n_{k}^{11} \cos^{2} \frac{\beta}{2} + (1 - n_{k}^{22}) \sin^{2} \frac{\beta}{2} + (\text{Re} s_{k}^{12}) \sin \beta, \]
\[ 1 - n_{2k} = n_{k}^{11} \sin^{2} \frac{\beta}{2} + (1 - n_{k}^{22}) \cos^{2} \frac{\beta}{2} - (\text{Re} s_{k}^{12}) \sin \beta. \]
\[ (A6) \]

As we see in the main text, both the self consistency equation and the final approximation for the superfluid density depend only on the combination
\[ n_{1k} + n_{2k} - 1 = (n_{k}^{11} + n_{k}^{22} - 1) \cos \beta + 2(\text{Re} s_{k}^{12}) \sin \beta. \]

Finally, throughout the text we assumed that we deal with a pure quasiparticle state. The general form of the distribution function in the quasiparticle basis,
\[ (\frac{n_{1k} \text{ OD}}{\text{OD}^{+} 1 - n_{2k}}) = \hat{U}(\frac{n_{k}^{11}}{s_{k}^{12} 1 - n_{k}^{22}}) \hat{U}^{\dagger}, \]
\[ (A9) \]

3. Offdiagonal element of the distribution function

Finally, throughout the text we assumed that we deal with a pure quasiparticle state. The general form of the distribution function in the quasiparticle basis,
\[ \rho_{\text{param}}(\omega, q) = \rho_{\text{para}}^{(\text{intra})}(\omega, q) + \rho_{\text{para}}^{(\text{inter})}(\omega, q). \]
\[ (B2) \]

1. Intraband contribution

The intraband contribution can in turn be broken down into contributions of the two quasiparticle bands,
\[ \rho_{\text{para}}^{(\text{intra})}(\omega, q) = \rho_{\text{para}}^{(\text{intra}-1)}(\omega, q) + \rho_{\text{para}}^{(\text{intra}-2)}(\omega, q), \]
\[ (B3) \]

where the contribution of the first quasiparticle band is
\[ \rho_{\text{para}}^{(\text{intra}-1)}(\omega, q) = \frac{-1}{2} \iint_{\epsilon, p, \mu} (\tilde{v}_{11})^{2} (G_{11}^{K R} + G_{11}^{A} + G_{11}^{A} + G_{11}^{K R}), \]
\[ (B4) \]

The first term of (B4) is
\[ \iint_{\epsilon, p, \mu} (\tilde{v}_{11})^{2} G_{11}^{K} G_{11}^{R} = \]
\[ (B5) \]

\[ = \iint_{\epsilon, p, \mu} (\tilde{v}_{11})^{2} F_{11}^{R} (G_{11}^{R} - G_{11}^{A}) G_{11}^{R} = \]
\[ = -2i \pi \iint_{\epsilon, p, \mu} (\tilde{v}_{11})^{2} F_{11}^{R} \delta_{\epsilon_{e} - (\epsilon + \xi)} \frac{1}{\epsilon_{e} - (\epsilon + \xi_{e}) + i0} \]
\[ = -i \iint_{p, \mu} (\xi_{11})^{2} F_{11}^{R} \frac{1}{\epsilon_{e} - (\epsilon + \xi_{e}) - \omega + i0} \]
\[ (B6) \]

where we took into account the fact that the Green’s function combination \((G^{R} - G^{A})\) is related to density of states and
\[ G_{11}^{R} - G_{11}^{A} = \frac{1}{\epsilon_{e} - (\epsilon + \xi_{e}) + i0} \]
\[ = -2i \pi \delta_{\epsilon_{e} - (\epsilon + \xi_{e})}, \]

with \(\delta\) being Dirac delta function.

Similarly, the second term turns out to be
\[ \iint_{\epsilon, p, \mu} (\tilde{v}_{11})^{2} G_{11}^{A} G_{11}^{K} = \]
\[ = -i \iint_{p, \mu} (\xi_{11})^{2} \frac{F_{11}^{R}}{(\epsilon + \xi)_{e} - (\epsilon + \xi)_{e} - \omega + i0} \]
\[ (B7) \]
so that the total contribution of the first band into intraband part is

\[ \rho_{(\text{para})}^{(\text{intra-1})}(\omega, q) = -\frac{1}{2} \oint_{p,\mu} \left( \tilde{v}^\mu_{11} \tilde{v}^\mu_{11} \right) \frac{F_{11+} - F_{11-}}{(\varepsilon + \xi)_+ - (\varepsilon + \xi)_- - \omega + i0}. \]  

(B8)

To proceed we note that the matrix element \( F_{11} \) gives the quasiparticle distribution function

\[ F_{11}(\varepsilon, p) = 1 - 2\eta_{11}p, \]  

(B9)

the matrix element of the velocity operator is

\[ \tilde{v}^\mu_{11} = V^\mu + v^\mu \cos \beta, \]  

(B10)

and the limit yields the derivative (35) introduced in the main text.

\[ \lim_{q \to 0, \omega \to 0} \frac{F_{11+} - F_{11-}}{(\varepsilon + \xi)_+ - (\varepsilon + \xi)_- - \omega + i0} = -2\eta_{11}p. \]  

(B11)

Summing up, the contribution of the first quasiparticle band is

\[ \rho_{(\text{para})}^{(\text{intra-1})} = \oint_{p,\mu} (V^\mu + v^\mu \cos \beta_1p)^2 \eta_{11}p. \]  

(B12)

Similarly, the contribution of the second band is

\[ \rho_{(\text{para})}^{(\text{intra-2})} = \oint_{p,\mu} (V^\mu - v^\mu \cos \beta_2p)^2 \eta_{22}p. \]  

(B13)

Summation of the two reproduces the result (34) from the main text.

The order of limits in \( \omega, q \to 0 \) was crucial throughout the calculation. With the opposite order of limits we would have a zero intraband contribution

\[ \lim_{q \to 0, \omega \to 0} \frac{\eta_{11}(p+q/2) - \eta_{11}(p-q/2)}{(\varepsilon + \xi)_{p+q/2} - (\varepsilon + \xi)_{p-q/2} - \omega + i0} = \lim_{\omega \to 0 - \omega + i0} = 0. \]  

(B14)

Finally, we note that strictly speaking the vertex coupling electrons to the vector potential is

\[ \frac{1}{2} \left( \tilde{v}^\mu_{p+q/2} + \tilde{v}^\mu_{p-q/2} \right) = \tilde{v}^\mu_p + O(q^2), \]  

(B15)

but the corrections coming from finite external vector potential momentum \( q \) are irrelevant for the present calculation.

2. Interband contribution

In contrast, the order of limits is irrelevant for the interband contribution due to the presence of the superconducting gap. In the interband contribution we compare energies of the two quasiparticles from the same band and the quantity \((\varepsilon + \xi)_+ - (\varepsilon + \xi)_- \to 0 \) in the denominator results in the importance of the order of limits. Meanwhile, in the interband contribution below we will be comparing two quasiparticles from different bands encountering a well-defined denominator \((\varepsilon + \xi)_+ - (\varepsilon - \xi)_- \to 2\xi \geq 2\Delta \). Thus we omit external frequency \( \omega \) and momentum \( q \) right away.

Similarly to the intraband case, we can divide the interband contribution into processes where the quasiparticle transitions from the first band into the second \((1 \to 2)\), and in the opposite direction \((2 \to 1)\).

\[ \rho_{(\text{para})}^{(\text{inter})} = \rho_{(\text{para})}^{(\text{inter}(1 \to 2))} + \rho_{(\text{para})}^{(\text{inter}(2 \to 1))}. \]  

(B16)

The contribution of \((1 \to 2)\) interband processes is

\[ \rho_{(\text{para})}^{(\text{inter})} = \oint_{p,\mu} (\tilde{v}^\mu_{12} \tilde{v}^\mu_{21}) \left( G^K_{11} G^R_{22} + G^K_{11} G^A_{22} + G^K_{22} G^K_{11} + G^K_{22} G^A_{11} \right). \]  

(B17)

This expression can be conveniently represented as

\[ \rho_{(\text{para})}^{(\text{inter})} = \text{Im} \oint_{p,\mu} (\tilde{v}^\mu_{12} \tilde{v}^\mu_{21}) \left( G^K_{11} G^{R+}_{22} + G^K_{22} G^K_{11} \right). \]  

(B18)

The first term in the equation above gives

\[ \oint_{p,\mu} (\tilde{v}^\mu_{12} \tilde{v}^\mu_{21}) G^K_{11} G^{R+}_{22} = \]  

(B19)

\[ = \oint_{p,\mu} \left( \tilde{v}^\mu_{12} \tilde{v}^\mu_{21} \right) F_{11} \left( \tilde{G}^{R-}_{11} - \tilde{G}^{A+}_{11} \right) G^{R+}_{22} \]  

\[ = -2i\pi \oint_{p,\mu} \left( \tilde{v}^\mu_{12} \tilde{v}^\mu_{21} \right) F_{11} \delta(\varepsilon - (\varepsilon + \xi)) \frac{1}{\varepsilon - (\varepsilon - \xi) + i0} \]  

\[ = -i \oint_{p,\mu} \left( \tilde{v}^\mu_{12} \tilde{v}^\mu_{21} \right) \frac{F_{11}}{2\xi}. \]  

Similarly, the second term is

\[ \oint_{p,\mu} (\tilde{v}^\mu_{12} \tilde{v}^\mu_{21}) G^K_{22} G^{K-}_{11} = -i \oint_{p,\mu} \left( \tilde{v}^\mu_{12} \tilde{v}^\mu_{21} \right) \frac{F_{11}}{2\xi}. \]  

(B20)

Taking into account that the diagonal elements of the quasiparticle distribution function are

\[ F_{11}(\varepsilon, p) = 1 - 2\eta_{11}p, \]  

(B21)

\[ F_{22}(\varepsilon, p) = 2\eta_{22}p - 1, \]  

and matrix elements of the velocity operator are

\[ \tilde{v}^\mu_{21} = -v^\mu \sin \beta, \]  

(B22)

we get

\[ \rho_{(\text{para})}^{(\text{inter})} = -\oint_{p,\mu} \left( v^\mu \sin \beta \right)^2 \frac{F_{11} - F_{22}}{2\xi} \]  

\[ = \oint_{p,\mu} \left( v^\mu \sin \beta \right)^2 \frac{\eta_{11}p + \eta_{22}p - 1}{\xi p}. \]  

(B23)

reproducing the result (36) from the main text.
3. Total superfluid density

Finally, while the summation of the dia- and paramagnetic contributions is a mathematical exercise, it is not entirely straightforward and we illustrate it here for the convenience of the reader. To reproduce the compact expression from the main text, we have to integrate by parts the diamagnetic contribution

\[ \rho_{(\text{dia})} = \iint_{\mathbb{R}^3} \left\{ \partial_\mu V^\mu \cos \beta_p (n_{1p} + n_{2p}) - 1 + (\partial_\mu v^\mu)(n_{1p} - n_{2p}) \right\} \cdot \left( \frac{1}{n_{1p}} + \frac{1}{n_{2p}} \right). \]  

Integrating by parts we have

\[ \rho_{(\text{dia})} = \iint_{\mathbb{R}^3} \left\{ V^\mu \partial_\mu (\cos \beta_p) (n_{1p} + n_{2p}) - 1 + V^\mu \cos \beta_p \partial_\mu (n_{1p} + n_{2p}) + (\partial_\mu v^\mu)(n_{1p} - n_{2p}) \right\}. \]  

Let us label the contribution in each line above as \( \rho^{(\alpha)}_{(\text{dia})}, \alpha = 1, 2, 3 \).

First, since \( \cos \beta_p = E/\sqrt{E_p^2 + \Delta^2} \), we have

\[ \partial_\mu (\cos \beta_p) = \frac{\Delta^2}{(E_p^2 + \Delta^2)^{3/2}} V^\mu = \frac{V^\mu}{\xi_p} \sin^2 \beta_p \]  

and the first contribution becomes

\[ \rho^{(1)}_{(\text{dia})} = -\iint_{\mathbb{R}^3} V^\mu \sin^2 \beta_p \frac{n_{1p} + n_{2p} - 1}{\xi_p}. \]  

Second, we go back to the definition of the derivative of the distribution function over the quasiparticle energy (35) and observe that since

\[ \frac{n_{1(2)p} - \Delta_1(2)p}{\xi_p} \approx \frac{n_{1(2)p} - \Delta_1(2)p}{\xi_p}, \]  

then

\[ \partial_\mu n_{1(2)p} = \frac{\partial \Delta_1(2)p}{\partial \xi_p} \frac{\partial n_{1(2)p}}{\partial \xi_p}. \]  

Using this identity, we have

\[ \rho^{(2)}_{(\text{dia})} = -\iint_{\mathbb{R}^3} V^\mu \cos \beta_p \left( n_{1p} + n_{2p} \right) + V^\mu \cos \beta_p \right\} \cdot \left( \frac{1}{n_{1p}} + \frac{1}{n_{2p}} \right). \]  

and

\[ \rho^{(3)}_{(\text{dia})} = -\iint_{\mathbb{R}^3} V^\mu \left\{ (n_{1p} + n_{2p}) - (n_{1p} - n_{2p}) \right\} \cdot \left( \frac{1}{n_{1p}} + \frac{1}{n_{2p}} \right). \]  

Adding together \( \rho_{(\text{dia}-2)}, \rho_{(\text{dia}-3)} \) and \( \rho_{(\text{para})} \) (given by Eq.(34)), we have

\[ \rho^{(2)}_{(\text{dia})} + \rho^{(3)}_{(\text{dia})} + \rho_{(\text{para})} = \]  

\[ = -\iint_{\mathbb{R}^3} \left\{ V^\mu + V^\mu \cos \beta_p \right\} \cdot \left( \frac{1}{n_{1p}} + \frac{1}{n_{2p}} \right). \]  

Finally, while the summation of the dia- and paramagnetic contributions is a mathematical exercise, it is not entirely straightforward and we illustrate it here for the convenience of the reader. To reproduce the compact expression from the main text, we have to integrate by parts the diamagnetic contribution

\[ \rho_{(\text{dia})} = \iint_{\mathbb{R}^3} \left\{ \partial_\mu V^\mu \cos \beta_p (n_{1p} + n_{2p} - 1) + (\partial_\mu v^\mu)(n_{1p} - n_{2p} + 1) \right\}. \]  

Integrating by parts we have

\[ \rho_{(\text{dia})} = \iint_{\mathbb{R}^3} \left\{ V^\mu \partial_\mu (\cos \beta_p) (n_{1p} + n_{2p} - 1) + V^\mu \cos \beta_p \partial_\mu (n_{1p} + n_{2p}) + (\partial_\mu v^\mu)(n_{1p} - n_{2p}) \right\}. \]  

Appendix C: Finite \( \Gamma \) effects on superfluid Density

We would like to analyze more closely the superfluid density in the vicinity of the superconducting transition. In order to explore the region \( \Delta \ll \Gamma \) we would like to phenomenologically include the effects of finite dissipation \( \Gamma \). Therefore we write:

\[ \rho \equiv N_S v^2 \int_k \frac{4 \Delta^2 E_k}{(E_k^2 + \Gamma^2/4 + \Delta^2)(4E_k^2 + \Gamma^2 + \gamma_s \Delta^2)} \]  

Furthermore the self consistency equation becomes

\[ 1 = 2N_S V_{\text{in}} \int_k \frac{E_k}{(4E_k^2 + \Gamma^2 + \gamma_s \Delta^2)}, \]  

which after the integration gives

\[ 1 = \frac{\pi V_{\text{in}} N_S}{2\sqrt{2\kappa} \pi} \left( \frac{(\Gamma^2 + \gamma_s \Delta^2)}{4\kappa^2} \right)^{-1/4}, \]  

From this expression we obtain that for superconductivity to exist we must have that

\[ V_{\text{in}} > \frac{2\sqrt{2\kappa} \Gamma}{\pi N_S} \equiv V_{\text{min}}. \]
We then get for $V_{\text{int}} \geq V_{\text{min}}$ and $|V_{\text{int}} - V_{\text{min}}| \ll V_{\text{min}}$

$$\rho \cong N_S v^2 \int \frac{4 \Delta^2 E_k}{(E_k^2 + \Gamma^2/4)(4E_k^2 + \Gamma^2)}$$

$$= N_S v^2 \Delta^2 \frac{1}{\kappa_+^3} \cdot \frac{\pi}{4\sqrt{2}} \left( \frac{\Gamma^2}{4\kappa_+^2} \right)^{-5/4}$$

$$= v^2 N_S \Delta^2 \frac{1}{\Gamma^2 \sqrt{\kappa_+ \Gamma}} \cdot \frac{\pi}{4\sqrt{2}}$$

$$\cong \frac{\Delta^2}{2\sqrt{2}\Gamma^2 V_{\text{min}}},$$

the result presented in the main text.

---

1. S. Kaiser, C. R. Hunt, D. Nicoletti, W. Hu, I. Gierz, H. Y. Liu, M. Le Tacon, T. Loew, D. Haug, B. Keimer, and A. Cavalleri, Phys. Rev. B 89, 184516 (2014).
2. R. Mankowsky, M. Frst, and A. Cavalleri, Reports on Progress in Physics 79, 064503 (2016).
3. L. P. Gor’kov and G. M. Eliashberg, JETP Lett 8, 202 (1968).
4. G. M. Eliashberg, JETP Lett. 11, 114 (1970).
5. V. M. Dmitriev, E. V. Khristenko, A. V. Trubitsyn, and F. F. Mende, Ukr. Fiz. Zh. 15, 1614 (1970).
6. V. Galitskii, V. Elesin, and Y. V. Kopaev, JETP Lett 18, 27 (1973).
7. D. Kirzhnits and Y. V. Kopaev, JETP Lett 17, 270 (1973).
8. V. Elesin, Y. V. Kopaev, and R. K. Timerov, JETP Lett 38, 1170 (1974).
9. V. M. Galitskii, V. F. Elesin, and Y. V. Kopaev, in Nonequilibrium Superconductivity, edited by D. N. Langenberg and A. I. Larkin (Elsevier Science Publishers, 1986) pp. 377–451.
10. G. Goldstein, C. Aron, and C. Chamon, Phys. Rev. B 91, 054517 (2015).
11. L. V. Keldysh, Sov. Phys. JETP 20, 1018 (1965).
12. J. Rammer and H. Smith, Rev. Mod. Phys. 58, 323 (1986).
13. A. Kamenev and A. Levchenko, Advances in Physics 58, 197 (2009).