MIRROR PAIRS OF CALABI–YAU THREEFOLDS FROM MIRROR PAIRS OF QUASI-FANO THREEFOLDS

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Abstract. We present a new construction of mirror pairs of Calabi–Yau manifolds by smoothing normal crossing varieties, consisting of two quasi-Fano manifolds. We introduce a notion of mirror pairs of quasi-Fano manifolds with anticanonical Calabi–Yau fibrations using recent conjectures about Landau–Ginzburg models. Utilizing this notion, we give pairs of normal crossing varieties and show that the pairs of smoothed Calabi–Yau manifolds satisfy the Hodge number relations of mirror symmetry. We consider quasi-Fano threefolds that are some blow-ups of Gorenstein toric Fano threefolds and build 6518 mirror pairs of Calabi–Yau threefolds, including 79 self-mirrors.

1. Introduction

A Calabi–Yau manifold is a compact Kähler manifold with trivial canonical class such that the intermediate cohomologies of its structure sheaf are all trivial \( (h^i(M, \mathcal{O}_M) = 0 \text{ for } 0 < i < \dim(M)) \). A K3 surface is a Calabi–Yau twofold in this definition. Calabi–Yau manifolds have special places in the classification of algebraic varieties and they are also among important manifolds that have special holonomy.

To the eye-opening surprise of mathematicians, Calabi–Yau threefolds happen to be compact factors of the spacetime on which physicists are building their physics theory. They have been investigating Calabi–Yau threefolds in their own way. They found that different Calabi–Yau threefolds may give rise to the same physics. Those manifolds are called mirror manifolds and the relationship between them is called mirror symmetry. A mirror pair \((M, M^\circ)\) of Calabi–Yau threefolds is supposed to satisfy

\[
\begin{align*}
    h^{1,1}(M) &= h^{1,2}(M^\circ), \\
    h^{1,2}(M) &= h^{1,1}(M^\circ).
\end{align*}
\]

One can say that the mirror symmetry for Calabi–Yau threefolds from physics has impacts in the geometry of Calabi–Yau threefolds as follows.

- Seemingly different two Calabi–Yau threefolds may be deeply related. An enumerative problem on one can be translated into another enumerative problem on another which sometimes is simpler than the original one ([17, 30]).
The mirror symmetry expects that Calabi-Yau threefolds exist as pairs. Nowadays it is not an unreasonable question to ask what a mirror partner for a certain Calabi-Yau threefold is.

Physicists constructed many Calabi-Yau threefolds as hypersurfaces in weight projective spaces, which generate an almost symmetric plot of $h^{1,1} - h^{1,2}$ vs. $h^{1,1} + h^{1,2}$ ([11, 22]). V. Batyrev generalized the construction and gave completely symmetric mirror construction of Calabi-Yau threefolds as hypersurfaces in Gorenstein toric Fano fourfolds, using the polar duality of reflexive 4-polytopes and proving the Hodge number relation (1.1) ([6]). This construction was generalized further ([7, 10]) and has inspired many researches from both of mathematics and physics.

In this paper, we suggest another systematic construction of mirror pairs of Calabi-Yau threefolds by using the smoothing method. By smoothing, we mean the reverse process of the semistable degeneration of a manifold to a normal crossing variety. If a normal crossing variety is the central fiber of a semistable degeneration of Calabi-Yau manifolds, it can be regarded as a member in a deformation family of those Calabi-Yau manifolds. A remarkable difference between two-dimensional cases of $K3$ surfaces and higher dimensional cases is that there are multiple deformation types for higher dimensional Calabi-Yau manifolds. So building a normal crossing variety smoothable to a Calabi-Yau manifold can be regarded as building a deformation type of Calabi-Yau manifolds. The construction by smoothing is intrinsically up to deformation.

We consider the simplest case of smoothing where the normal crossing variety is composed of two manifolds. Those two component manifolds will be called quasi-Fano manifolds. We further assume that the anticanonical linear systems of those quasi-Fano manifolds induce fibrations whose generic fibers are Calabi-Yau manifolds of codimension one. Even in this simplest case, building the mirror pairs of the smoothing of a normal crossing variety is a challenging problem. A. Tyurin only vaguely suggested that the mirror pair of a smoothing should come from the Landau-Ginzburg models of components of the normal crossing variety in the very last part of his posthumous paper ([33]).

Shortly after mirror symmetry was formulated as a duality between Calabi-Yau manifolds, it was suggested that Fano manifolds also may exhibit mirror symmetry. In this case, the mirror of a Fano manifold is not a compact manifold, but rather a Landau-Ginzburg model, a non-compact manifold equipped with a regular function called superpotential. Recently up comes an interesting conjecture, claiming that the mirror of the Calabi-Yau smoothing of a normal crossing variety may be topologically obtained by gluing Landau-Ginzburg models of components of the normal crossing variety ([14]). Also in [19], a conjectural Hodge number relation between a variety and its Landau-Ginzburg model has been suggested. With these as hints, we try to construct mirror partners of smoothings of normal crossing
varieties. Our key idea is the realization that we may regard a quasi-Fano manifold with anticanonical fibration as a compactification of a Landau–Ginzburg model of another quasi-Fano manifold.

Roughly speaking, we regard a pair of quasi-Fano manifolds with anticanonical Calabi–Yau fibrations as a mirror pair, if each of them is a compactification of the Landau–Ginzburg model of the other. After establishing this notion, we define mirror pairs of normal crossing varieties, smoothable to Calabi–Yau threefolds and show that those Calabi–Yau threefolds satisfy the relation (1.1). It turns out that there is a deep connection between mirror symmetry of quasi-Fano threefolds and mirror symmetry of $K3$ surfaces.

We consider quasi-Fano threefolds that are some blow-ups of Gorenstein toric Fano threefolds and build 6518 mirror pairs of Calabi–Yau threefolds, including 79 self-mirrors by smoothing. One can find tables for them in [29].

The structure of this paper is as follows.

Section 2 is a background section for the smoothing method. We introduce basic definitions of quasi-Fano manifolds and the smoothing theorem of Kawamata–Namikawa, which is the main tool of the construction of Calabi–Yau manifolds in this paper.

We start Section 3 by recalling some basic notions and definitions about reflexive polytopes. We construct a Calabi–Yau threefold from a quasi-Fano threefold that is a blow-up of a Gorenstein toric Fano threefold. This Gorenstein toric Fano threefold comes from a reflexive 3-polytope.

In Section 4, we start our journey to find a mirror partner of the Calabi–Yau threefold constructed in Section 3 under the guidance of the conjectures regarding Landau–Ginzburg models. We make some elementary observations about quasi-Fano manifolds and their Landau–Ginzburg models.

In Section 5, using hints from Section 4, we build a Calabi–Yau threefold by smoothing a normal crossing variety. The components of the normal crossing variety are obtained by sequentially blowing up another Gorenstein toric Fano threefold that comes from the polar dual of the previous reflexive 3-polytope. We prove that these Calabi–Yau threefolds satisfy the relation (1.1). Each of the equivalence classes of reflexive 3-polytopes gives a mirror pair of Calabi–Yau threefolds. Hence we obtain a big list of mirror pairs of Calabi–Yau threefolds, which are summarized in Table 1 of [29].

In Section 6, motivated by our success in the previous sections, we define a notion of mirror pairs of quasi-Fano threefolds. This definition utilizes the conjectural Hodge number relations in [19] and the mirror symmetry of $K3$ surfaces in [13]. Then each of equivalence classes of reflexive 3-polytopes gives a mirror pair of quasi-Fano threefolds, which are also listed in Table 1 of [29]. Those will be building blocks for the construction of more mirror pairs of Calabi–Yau threefolds in Section 7. We also introduce other examples of mirror pairs of quasi-Fano threefolds which come from non-symplectic involutions on $K3$ surfaces.

We start Section 7 by introducing a notion of mirror pairs of normal crossing varieties smoothable to Calabi–Yau threefolds. We prove Theorem
7.2, claiming that the expected relation (1.1) hold for the smoothing of those pairs. Combining mirror pairs of quasi-Fano threefolds we obtained before, we give another large table of mirror pairs of Calabi–Yau threefolds, including 79 self-mirrors. Those are listed in Table 2 of [29].

Section 8 is about whether our pairs of Calabi–Yau threefolds are new. We pick up a particular example of Calabi–Yau threefolds constructed in the previous sections and show that it is not homeomorphic to any of the Calabi–Yau threefolds that are desingularizations of anticanonical sections of Gorenstein toric Fano fourfolds ([6, 26]).

Section 9 is devoted to some discussion on the higher dimensional generalization of notions from the previous sections. We suggest a definition of a mirror pair of higher dimensional quasi-Fano manifolds and prove a topological mirror relation.

In Section 10, we discuss quasi-Fano manifolds with anticanonical fibrations that do not have quasi-Fano manifolds as their mirror partners.

Rigorously speaking, the pairs of Calabi–Yau manifolds constructed in this paper are conjectural because we only check the Hodge number relations of mirror symmetry. However, seeing that those ingredients from Landau–Ginzburg models, mirror symmetry of $K3$ surfaces and mirror symmetry of Calabi–Yau manifolds are merged very naturally to produce expected results, we expect that they are genuine mirror pairs.

2. Preliminaries

By a variety, we mean a reduced complex analytic space. We start with defining basic terminologies.

**Definition 2.1.** A quasi-Fano manifold $X$ is a smooth projective variety whose anticanonical linear system $| - K_X|$ contains a Calabi–Yau manifold and

$$h^i(X, \mathcal{O}_X) = 0$$

for $i > 0$.

We denoted the Calabi–Yau manifold by $D_X$. If a generic element of $| - K_X|$ is smooth, then $D_X$ will be referred to one of those generic ones.

Let

$$\text{Pic}_X(D_X) = i^*(\text{Pic}(X)) \subset \text{Pic}(D_X)$$

and $\alpha_X = \text{rkPic}_X(D_X)$ be the rank of the group $\text{Pic}_X(D_X)$, where $i : D_X \hookrightarrow X$ is the inclusion map. Note that $\text{Pic}_X(D_X)$ is a subgroup of $H^2(D_X, \mathbb{Z})$. If the normal bundle $N_{D_X/X}$ to $D_X$ in $X$ is trivial, then the anticanonical linear system $|D_X|$ induces a fibration (to be called anticanonical fibration)

$$\overline{W}_X : X \to \mathbb{P}^1$$

(2.1)

with $\overline{W}_X^{-1}(\infty) = D_X$ and $X$ is said to have an anticanonical Calabi–Yau fibration.
Let $\mathcal{X} = X_1 \cup X_2$ be a variety whose irreducible components are two smooth varieties $X_1$ and $X_2$. $\mathcal{X}$ is called a normal crossing variety if, near any point $p \in X_1 \cap X_2$, $\mathcal{X}$ is locally isomorphic to
\[ \{(x_0, x_1, \cdots, x_n) \in \mathbb{C}^{n+1} | x_{n-1}x_n = 0\} \]
with $p$ corresponding to the origin and $X_1$, $X_2$ locally corresponding to the hypersurfaces $x_{n-1} = 0, x_n = 0$ respectively in $\mathbb{C}^{n+1}$. Note that the variety $D_{\mathcal{X}} := X_1 \cap X_2$ is smooth. Suppose that there is a proper map $\psi : \mathcal{X} \rightarrow B$ from a Kähler manifold $B$ onto the unit disk $B = \{ t \in \mathbb{C} ||t|| \leq 1 \}$ such that the fiber $\mathcal{X}_t = \psi^{-1}(t)$ is a smooth manifold for every $t \neq 0$ and $\mathcal{X}_0 = \mathcal{X}$. We say that $\mathcal{X}$ is a semistable degeneration of a smooth threefold $M = \mathcal{X}_t$ ($t \neq 0$) and that $M$ is a semistable smoothing (simply smoothing) of $\mathcal{X}$ ([20]).

Consider a normal crossing variety $\mathcal{X} = X_1 \cup X_2$ of quasi-Fano manifolds $X_1, X_2$ such that $D_{\mathcal{X}} = X_1 \cap X_2$ is an anticanonical section of both $X_1, X_2$. If $\mathcal{X}$ is projective and the bundle
\[ N_{D_{\mathcal{X}}/X_1} \otimes N_{D_{\mathcal{X}}/X_2} \]
on $D_{\mathcal{X}}$ is trivial (called $d$-semistability), then $\mathcal{X}$ is smoothable to a Calabi–Yau manifold (will be denoted by $M_{\mathcal{X}}$) (Theorem 4.2, [20]).

For dimension three, the Hodge numbers of $M_{\mathcal{X}}$ are given by (Corollary 8.2, [28]):

\[ h^{1,1,1}(M_{\mathcal{X}}) = h^{1,1}(X_1) + h^{1,1}(X_2) - \alpha_{\mathcal{X}} - 1, \]
\[ h^{1,2}(M_{\mathcal{X}}) = 21 + h^{1,2}(X_1) + h^{1,2}(X_2) - \alpha_{\mathcal{X}}, \]
where
\[ \alpha_{\mathcal{X}} = \text{rk} (\text{Pic}_{X_1}(D_{\mathcal{X}}) + \text{Pic}_{X_2}(D_{\mathcal{X}})). \]

Let $X_i$ be a quasi-Fano manifold with a smooth anticanonical section $D_{X_i}$ for each $i = 1, 2$. If $D_{X_1}$ and $D_{X_2}$ are isomorphic, one can make a $d$-semistable normal crossing variety $X_1 \cup_D X_2$ by gluing along $D_{X_1}$ and $D_{X_2}$, where ‘$\cup_D$’ means gluing $X_1, X_2$ along $D_{X_1}, D_{X_2}$ (see §2 (Corollary 2.4) of [21] for details for the gluing process).

Let $X$ be a quasi-Fano manifold such that the normal bundle $N_{D_{X}/X}$ to $D_{X}$ in $X$ is trivial. Then we have the anticanonical fibration (2.1)
\[ \overline{W}_X : X \rightarrow \mathbb{P}^1 \]
with $\overline{W}_X^{-1}(\infty) = D_{X}$. Let $X_1, X_2$ be copies of $X$. We denote the copy in $X_i$ of $D_{X}$ by $D_{X_i}$. We make a normal crossing variety $\mathcal{X} = X_1 \cup_D X_2$ (also to be denoted by $X \cup_D X$). It is easy to see that $\mathcal{X}$ is projective and $d$-semistable. Hence, by Theorem 4.2 in [20], it is smoothable to a Calabi–Yau manifold $M_{\mathcal{X}}$. Since the manifold $M_{\mathcal{X}}$ is determined by $X$ up to deformation, we will denote it by $\Xi X$. In fact, one can construct $\Xi X$ as a branched double cover over $X$, branched along $S = \overline{W}_X^{-1}([0, \infty))$ but we will keep this point of
view of smoothing for the time being and we will come back to this point later in Section 9.

3. The Calabi–Yau threefold $\Xi_{X_{\Delta}}$

We recall some notations from toric geometry. An integral polytope $\Delta$ in $\mathbb{R}^n$ is a convex hull of finitely many integral points (points with integer coordinates). If, for integral polytopes $\Delta_1, \Delta_2$, there is a $\mathbb{Z}^n$-preserving affine transformation $\sigma$ satisfying $\Delta_2 = \sigma(\Delta_1)$, then $\Delta_1, \Delta_2$ are said to be equivalent. For a set $A \subset \mathbb{R}^n$, its polar dual $A^\circ$ is defined by

$$A^\circ = \{ u \in \mathbb{R}^n | u \cdot v \leq -1 \text{ for any } v \in A \},$$

where $\cdot$ is the standard inner product in $\mathbb{R}^n$. An integral polytope $\Delta \subset \mathbb{R}^n$ is called a reflexive $n$-polytope if $(0, \cdots, 0)$ is in the interior of $\Delta$ and its polar dual $\Delta^\circ$ is also a lattice polytope. For a face $\Gamma$ of $\Delta$, $l(\Gamma)$ and $l^*(\Gamma)$ are the numbers of integral points in $\Gamma$ and in the relative interior of $\Gamma$ respectively. We let $\Delta[k]$ be the set of $k$-dimensional faces of $\Delta$. For each $\Gamma \in \Delta[k]$, let $\Gamma^\circ$ be the dual $(n - k - 1)$-dimensional face of $\Delta^\circ$.

For a fan $\Sigma$ in $\mathbb{R}^n$, we denote by $X(\Sigma)$ the associated toric variety and let $\Sigma[1]$ be the set of primitive ray generators of $\Sigma$. Hence $\Sigma[1]$ is a set of integral points.

For a reflexive polytope $\Delta$, we denote by $\mathbb{P}(\Delta)$ the toric variety that is associated with the fan consisting of cones over all the proper faces over $\Delta$ — this is different from notations in [6]. It is known that $\mathbb{P}(\Delta)$ is a Gorenstein toric Fano variety. Fix a fan $\Sigma_{\Delta}$, consisting of cones over simplices in $\partial \Delta$ in a maximal coherent triangulation of $\Delta$. A maximal coherent triangulation is defined and proved to exist in [16]. Note $\Sigma_{\Delta}[1] = \partial \Delta \cap \mathbb{Z}^n$. The toric variety $X(\Sigma_{\Delta})$, which is projective, is called a maximal partial projective crepant desingularization of $\mathbb{P}(\Delta)$ ([6]).

For a fan $\Sigma$, a reflexive polytope $\Delta$ and a quasi-Fano manifold $X$, we summarize our notations, including ones to be defined:

- $X(\Sigma)$ is the associated toric variety of $\Sigma$.
- $\Sigma[1]$ is the set of primitive ray generators of the fan $\Sigma$.
- $\Delta[k]$ is the set of $k$-dimensional faces of $\Delta$.
- For a face $\Gamma$ of a polytope, $l(\Gamma) = |\Gamma \cap \mathbb{Z}^n|$ is the number of integral points in $\Gamma$.
- $l^*(\Gamma)$ is the number of integral points in the relative interior of $\Gamma$.
- $\mathbb{P}(\Delta)$ is the Gorenstein toric Fano variety that is associated with the fan consisting of cones over all the proper faces over $\Delta$.
- $\Sigma_{\Delta}$ is the fan consisting of cones over a maximal projective triangulation of $\Delta$. $\Sigma_{\Delta}[1] = \partial \Delta \cap \mathbb{Z}^n$.
- $X(\Sigma_{\Delta})$ is the toric variety associated with the fan $\Sigma_{\Delta}$. It is a maximal partial projective crepant desingularization of $\mathbb{P}(\Delta)$.
- $D_X$ is a (generic) smooth anticanonical section of $X$.
- $\text{Pic}_X(D_X) = i^*\left( \text{Pic}(X) \right)$, where $i : D_X \hookrightarrow X$ is the inclusion.
- $\alpha_X = \text{rkPic}_X(D_X)$. 
• $X_\Delta$ is the blow-up of $X(\Sigma_\Delta)$ along a smooth curve $c \in \left| -K_{X(\Sigma_\Delta)}|_{D_X(\Sigma_\Delta)} \right|$.
• $Y_\Delta$ is a sequential blow-up of $X(\Sigma_\Delta)$ along the smooth irreducible curves $c_1, c_2, \ldots, c_k$ (p. 13).
• $X_1 \cup_D X_2$ is the normal crossing variety of quasi-Fano manifolds $X_1, X_2$, made by gluing along their isomorphic smooth anticanonical sections.
• $M_X$ is a smoothing of a normal crossing variety $X = X_1 \cup X_2$.
• $X_1 \cup_D X_2$ is a normal crossing variety, made by gluing two copies of $X$ along the two copies of $D_X$.
• $\Xi_X = M_X$ for $X = X_1 \cup D X_2$.

For dimension three, there are 4319 equivalence classes of reflexive 3-polytopes ([25]) and a maximal partial projective crepant desingularization $X(\Sigma_\Delta)$ of $\mathbb{P}(\Delta)$ is always smooth ([6]). The following lemma will be used several times.

**Lemma 3.1.** For a reflexive 3-polytope $\Delta$,

$$3!\text{vol}(\Delta) = 2l(\Delta) - 6$$

and

$$l(\Delta) + l(\Delta^o) + \sum_{\theta \in \Delta[1]} l^*(\theta)l^*(\theta^o) - \sum_{v \in \Delta[0]} l^*(v^o) - \sum_{\Gamma \in \Delta[2]} l^*(\Gamma) = 28.$$  

**Proof.** The first equality is well-known with the following basic property of reflexive 3-polytope (see, for example, [18]):

$$\sum_{\theta \in \Delta[1]} (l^*(\theta) + 1)(l^*(\theta^o) + 1) = 24.$$  

Combining this with

$$l(\Delta) = \sum_{\Gamma \in \Delta[2]} l^*(\Gamma) + \sum_{\theta \in \Delta[1]} l^*(\theta) + |\Delta[0]| + 1,$$  

$$l(\Delta^o) = \sum_{v \in \Delta[0]} l^*(v^o) + \sum_{\theta \in \Delta[1]} l^*(\theta^o) + |\Delta[2]| + 1$$  

we have the second equality.  

One can choose a smooth $K3$ surface $D_X(\Sigma_\Delta)$ from the linear system $|-K_{X(\Sigma_\Delta)}|$ (Corollary 4.2.3 of [6]). Note that the line bundle $-K_{X(\Sigma_\Delta)}$ is
semi-ample (Lemma 4.1.2 in [12]). Hence the line bundle $-K_{X(\Sigma_\Delta)}|_{D_{X(\Sigma_\Delta)}}$ is nef and 

$$
\left( -K_{X(\Sigma_\Delta)}|_{D_{X(\Sigma_\Delta)}} \right)^2 = (-K_{X(\Sigma_\Delta)})^3 = (-K_{\mathbb{P}(\Delta)})^3 \geq 4.
$$

Accordingly, $-K_{X(\Sigma_\Delta)}|_{D_{X(\Sigma_\Delta)}}$ is very ample (Lemma 2.4 of [23]) and the linear system $|-K_{X(\Sigma_\Delta)}|_{D_{X(\Sigma_\Delta)}}|$ contains a smooth curve $c$. Let $X_\Delta \to X(\Sigma_\Delta)$ be the blow-up along $c$ and $D_{X_\Delta}$ be the proper transform of $D_{X(\Sigma_\Delta)}$. Now $N_{D_{X_\Delta}/X_\Delta}$ to $D_{X_\Delta}$ in $X_\Delta$ is trivial and hence $X_\Delta$ is a quasi-Fano threefold with the anticanonical fibration

$$
\overline{W}_{X_\Delta} : X_\Delta \to \mathbb{P}^1
$$

and $\overline{W}_{X_\Delta}^{-1}(\infty) = D_{X_\Delta}$. We obtain a Calabi–Yau threefold $\Xi_{X_\Delta}$ by smoothing $X_\Delta \cup D_{X_\Delta}$ as in Section 2. From (2.2), we can calculate the Hodge numbers $\Xi_{X_\Delta}$ as follows.

**Proposition 3.2.**

$$
h^{1,1}(\Xi_{X_\Delta}) = l(\Delta) + \sum_{\Gamma \in \Delta[2]} l^*(\Gamma) - 3
$$

and

$$
h^{1,2}(\Xi_{X_\Delta}) = 3! \text{vol}(\Delta^\circ) - l(\Delta) + \sum_{\Gamma \in \Delta[2]} l^*(\Gamma) + 27.
$$

**Proof.** By (2.2), we have

$$
h^{1,1}(\Xi_{X_\Delta}) = 2h^2(X_\Delta) - 1 - \alpha_{X_\Delta}
$$

and

$$
h^{1,2}(\Xi_{X_\Delta}) = 21 + h^3(X_\Delta) - \alpha_{X_\Delta}.
$$

Firstly,

$$
h^2(X_\Delta) = h^2(X(\Sigma_\Delta)) + 1 = l(\Delta) - 3.
$$

Recall $\Sigma_\Delta[1] = \partial \Delta \cap \mathbb{Z}^3$. So an integral point $v$ of $\partial \Delta$ corresponds to a torus invariant divisor $D_v$ of $X(\Sigma_\Delta)$. If $v$ lies in a relative interior of a face (codimension one) of $\Delta$, $D_v$ does not meet with $D_{X_\Delta}$. Let $G$ be a subgroup of $\text{Pic}(X(\Sigma_\Delta))$ that is generated by $D_v$’s, where $v$ does not lie in a relative interior of a face of $\Delta$. One can show that the map $G \to \text{Pic}(D_{X(\Sigma_\Delta)})$ is injective ($\S$2, [4]). Hence

$$
\alpha_{X_\Delta} = \alpha_{X(\Sigma_\Delta)} = \text{rk}G = l(\Delta) - \sum_{\Gamma \in \Delta[2]} l^*(\Gamma) - 4.
$$

On the other hand, we have

$$
h^3(X_\Delta) = h^3(X(\Sigma_\Delta)) + h^1(c) = 0 + (-K^3_{X(\Sigma_\Delta)}) + 2 = (-K^3_{\mathbb{P}(\Delta)}) + 2 = 3! \text{vol}(\Delta^\circ) + 2
$$

because $X(\Sigma_\Delta) \to \mathbb{P}(\Delta)$ is a crepant resolution.
Therefore
\[ h^{1,1}(\Xi_{X_{\Delta}}) = 2h^2(X_{\Delta}) - 1 - \alpha_{X_{\Delta}} \]
\[ = 2(l(\Delta) - 3) - 1 - \left( l(\Delta) - \sum_{\Gamma \in \Delta^{[2]}} l^* (\Gamma) - 4 \right) \]
\[ = l(\Delta) + \sum_{\Gamma \in \Delta^{[2]}} l^* (\Gamma) - 3 \]
and
\[ h^{1,2}(\Xi_{X_{\Delta}}) = 21 + h^3(X_{\Delta}) - \alpha_{X_{\Delta}} \]
\[ = 21 + 3! \text{vol}(\Delta^\circ) + 2 - \left( l(\Delta) - \sum_{\Gamma \in \Delta^{[2]}} l^* (\Gamma) - 4 \right) \]
\[ = 3! \text{vol}(\Delta^\circ) - l(\Delta) + \sum_{\Gamma \in \Delta^{[2]}} l^* (\Gamma) + 27. \]

Our next goal is to find a mirror partner of the Calabi–Yau threefold \( \Xi_{X_{\Delta}} \). One may apply the procedures in the previous section to the polar dual \( \Delta^\circ \) of the reflexive polytope \( \Delta \) and construct a Calabi–Yau threefold \( \Xi_{X_{\Delta^\circ}} \). But one can check immediately that the relation (1.1) does not hold for \( \Xi_{X_{\Delta}} \), \( \Xi_{X_{\Delta^\circ}} \). Hence this naive try does not work. Despite this, one may still suspect that a mirror partner of \( \Xi_{X_{\Delta}} \) is somehow related with the variety \( X(\Sigma_{\Delta^\circ}) \). Our plan is to modify \( X(\Sigma_{\Delta^\circ}) \) to some other quasi Fano threefold \( Y \) so that the Calabi–Yau threefolds \( \Xi_{X_{\Delta}}, \Xi_{Y} \) satisfy the relation (1.1). But then the question would be what kind of modification to \( X(\Sigma_{\Delta^\circ}) \) is needed. In the next section, we will have a discussion on this question.

4. Conjectures and speculations

For a quasi-Fano threefold \( X \) with anticanonical fibration, \( W_X : X \to \mathbb{P}^1 \). We have built a Calabi–Yau threefold \( \Xi_X \). Let us assume that there is a quasi-Fano threefold \( Y \) with anticanonical fibration such that \( \Xi_Y \) is a mirror partner of \( \Xi_X \). In this section, we discuss how those two quasi-Fano threefolds \( X,Y \) should be related. This section is speculative and is intended for explaining how the author came up with the relations (4.5), (4.6) and (4.7) that serve as hints in constructing the threefold \( Y_{\Delta^\circ} \) in Section 5 so that the Calabi–Yau threefolds \( \Xi_{X_{\Delta}}, \Xi_{Y_{\Delta^\circ}} \) satisfy the relation (1.1).

We start with recent conjectures about Landau–Ginzburg models. A Landau–Ginzburg model is a pair \((Z,W)\), where \( Z \) is a quasi-projective manifold and \( W : Z \to \mathbb{C} \) is a fibration (called superpotential) whose generic fiber is a Calabi–Yau manifold of codimension one.

In [14] (and also in [5] for simpler case), an interesting conjecture has been made:
Let $M$ be a Calabi–Yau manifold and suppose that $M$ admits a degeneration to a union $X_1 \cup X_2$ of two quasi-Fano varieties glued along an anticanonical hypersurface. Its mirror partner $M^\circ$ can be constructed topologically by gluing together the Landau–Ginzburg models $(Z_1, W_1)$ and $(Z_2, W_2)$ of $X_1$ and $X_2$ respectively.

Note that $\Xi_X$ is a smoothing of a normal crossing variety $X_1 \cup_D X_2$ of $X_1$, $X_2$ which are copies of $X$. Note $D_{X_i} = X_1 \cap X_2$ for each $i$. We have an anticanonical fibration $\overline{W}_{X_i} : X_i \to \mathbb{P}^1$ with $\overline{W}_{X_i}^{-1} = D_{X_i}$. Let $X_i^* = X_i - D_{X_i}$ and consider the map $W_{X_i} : X_i^* \to \mathbb{C}$, where $W_{X_i} = \overline{W}_{X_i}|_{X_i^*}$.

Regarding the above conjecture, we observe the following:

1. Topologically $\Xi_X$ can be made by gluing the open ends of $X_1^*, X_2^*$ ([33]).
2. The fibration $W_i : X_i^* \to \mathbb{C}$ can be regarded as a superpotential of a Landau–Ginzburg model.

Following the line of thoughts in the conjecture, we boldly conjecture that the mirror partner $(\Xi_X)^\circ$ of $\Xi_X$ is a semistable smoothing of a normal crossing variety $Y_1 \cup Y_2$ of quasi-Fano threefolds $Y_1, Y_2$ and that $(X_1^*, W_{X_1})$ and $(X_2^*, W_{X_2})$ are Landau–Ginzburg models of $Y_1, Y_2$ respectively.

Since $X_1, X_2$ are the same copies of $X$, we expect that $Y_1, Y_2$ are also copies of a single quasi-Fano manifold $Y$ with a smooth anticanonical section $D_Y$. For the generality of discussion, let us not restrict the dimension $n = \dim Y$ to be three. The fact that $Y_1 \cup_D Y_2$ is $d$-semistable implies that the normal bundle $N_{D_Y/Y}$ on $D_Y$ is trivial. So the anticanonical linear system of $Y$ induces a fibration $\overline{W}_Y : Y \to \mathbb{P}^1$ with $\overline{W}_Y^{-1}(\infty) = D_Y$. We note that the map

$$W_Y : Y^* \to \mathbb{C}$$

also can be viewed as a superpotential of a Landau–Ginzburg model, where $Y^* = Y - D_Y$ and $W_Y = \overline{W}_Y|_{Y^*}$. Noting that $(\Xi_X)^\circ = \Xi_Y$ can be topologically made by gluing the open ends of $Y_1^*, Y_2^*$, it is reasonable to conclude that $(Y_1^*, W_{Y_1})$ and $(Y_2^*, W_{Y_2})$ are Landau–Ginzburg models of $X_1, X_2$ respectively. In sum, we speculate:

If $(X^*, W_X)$ is a Landau–Ginzburg model of $Y$ and $(Y^*, W_Y)$ is a Landau–Ginzburg model of $X$, then $(\Xi_X, \Xi_Y)$ is a mirror pair of Calabi–Yau manifolds.

In order to find the quasi-Fano manifold $Y$, we need more information about it. A conjectural Hodge number relation, suggested in [19], is relevant to this task. The authors in [19] conjecture that if $(X^*, W_X)$ is a Landau–Ginzburg model of $Y$, then the following holds ((3.1.3), [19]):

$$h^{a+n}(X^*, W_X^{-1}(t)) = \sum_{p-q=a} h^{p,q}(Y),$$

(4.1)
where \( t \) is a generic point in the image of \( W_X \) and \( n = \dim X \). If \( (Y^*, W_Y) \) is a Landau–Ginzburg model of \( X \), it becomes

\[
(4.2) \quad h^{a+n}(Y^*, W_Y^{-1}(t)) = \sum_{p-q=a} h^{p,q}(X).
\]

These two equations imply relations in topological Euler characteristics

\[
\chi(X^*, W_X^{-1}(t)) = (-1)^n \chi(Y), \quad \chi(Y^*, W_Y^{-1}(t)) = (-1)^n \chi(X).
\]

From the pair \( (Y^*, W_Y^{-1}(t)) \), we have an exact sequence

\[
\cdots \to H^i(Y^*, W_Y^{-1}(t))) \to H^i(Y^*) \to H^i(W_Y^{-1}(t))) \to \cdots,
\]

which gives a relation of topological Euler characteristic:

\[
\chi(Y^*) = \chi(Y^*, W_Y^{-1}(t))) + \chi(W_Y^{-1}(t))).
\]

Similarly we have

\[
\chi(X^*) = \chi(X^*, W_X^{-1}(t))) + \chi(W_X^{-1}(t))).
\]

Note also \( \chi(Y) = \chi(Y^*) + \chi(D_Y) \), \( \chi(X) = \chi(X^*) + \chi(D_X) \). Combining these equations, we have

\[
(4.3) \quad \begin{cases} 
\frac{1 - (-1)^n}{2} \cdot (\chi(X) + \chi(Y)) = \chi(D_X) + \chi(D_Y), \\
\frac{1 + (-1)^n}{2} \cdot (\chi(X) - \chi(Y)) = \chi(D_X) - \chi(D_Y),
\end{cases}
\]

which implies

\[
(4.4) \quad \chi(D_X) = (-1)^{n-1} \chi(D_Y).
\]

Note \( \dim D_X = \dim D_Y = n - 1 \). So this suggests that \( (D_X, D_Y) \) should be a mirror pair of Calabi–Yau manifolds.

Assume \( n = 3 \) for the rest of this section. In the natural map \( H_2(D_Y) \to H_2(Y) \), we have

\[
\dim H_2(D_Y) = \dim \ker(H_2(D_Y) \to H_2(Y)) + \dim \im(H_2(D_Y) \to H_2(Y)).
\]

and, by Poincaré duality,

\[
\dim \im(H_2(D_Y) \to H_2(Y)) = \dim \im(H^2(Y) \to H^2(D_Y)) = \alpha_Y,
\]

which implies

\[
\dim \ker(H_2(D) \to H_2(Y)) = h^2(D_Y) - \alpha_Y.
\]

From the pair \( (Y, D_Y) \), we have an exact sequence

\[
0 = H_3(D_Y) \to H_3(Y) \to H_3(Y, D_Y) \to H_2(D_Y) \to H_2(Y) \to \cdots,
\]

which implies

\[
h_3(Y, D_Y) = h_3(Y) + \dim \ker(H_2(D) \to H_2(Y)) = h^3(Y) + h^2(D_Y) - \alpha_Y.
\]
From the pair \((Y^*, W^{-1}_Y(t))\), we have an exact sequence
\[
0 = H^1(W^{-1}_Y(t)) \rightarrow H^2(Y^*, W^{-1}_Y(t)) \rightarrow H^2(Y^*) \rightarrow H^2(W^{-1}_Y(t)) \\
\rightarrow H^3(Y^*, W^{-1}_Y(t)) \rightarrow H^3(Y^*) \rightarrow H^3(W^{-1}_Y(t)) = 0,
\]
which implies
\[
h^2(Y^*, W^{-1}_Y(t)) + h^2(W^{-1}_Y(t)) + h^3(Y^*) = h^2(Y^*) + h^3(Y^*, W^{-1}_Y(t)).
\]
By Lefschetz duality, \(h_3(Y, D_Y) = h^3(Y^*)\) and Equation (4.2) implies
\[
h^2(Y^*, W^{-1}_Y(t)) = h^{2,1}(X), h^3(Y^*, W^{-1}_Y(t)) = 2 + 2h^{1,1}(X).
\]
Combining these, finally we have
\[
h^3(Y) + h^2(D) - \alpha_Y = 2h^{1,2}(Y) - h^2(Y) + h^{1,2}(X) + 23.
\]
i.e.
\[
(4.5) \quad \quad \alpha_Y = h^2(Y) - h^{1,2}(X) - 1.
\]
Using the assumption that \((Y^*, W_Y)\) is also a Landau–Ginzburg model of \(X\), we have
\[
(4.6) \quad \quad \alpha_X = h^2(X) - h^{1,2}(Y) - 1.
\]
Finally, together with (4.3), we have
\[
(4.7) \quad \quad \alpha_X + \alpha_Y = 20.
\]
In sum, we conjecture
\[
"The relations (4.5), (4.6) and (4.3) hold if \(\Xi_X, \Xi_Y\) are mirror pairs of Calabi–Yau threefolds"
\]
This conjecture will be more concretized to form Definition 6.1 and Definition 9.1.

5. Mirror partner of \(\Xi_{X_\Delta}\)

Now let us come back to the problem of finding a mirror partner of \(\Xi_{X_\Delta}\). For our previous \(X_{\Delta^0}\), we have
\[
\alpha_{X_\Delta} + \alpha_{X_{\Delta^0}} \leq 20,
\]
which may not comply with (4.7) in general and (4.5), (4.6) do not hold. Instead one can show
\[
\alpha_{X_\Delta} + \text{rkPic}(D_{X_{\Delta^0}}) = 20,
\]
using Pic\((D_{X_{\Delta^0}}) \simeq \text{Pic}(D_X(\Sigma_{\Delta^0}))\) and results in [32].

Note that Pic\(_{X_{\Delta^0}}(D_{X_{\Delta^0}})\) is a sublattice of Pic\(_{X_{\Delta^0}}(D_{X_{\Delta^0}})\). Let us investigate which classes of Pic\(_{X_{\Delta^0}}(D_{X_{\Delta^0}})\) are missing in Pic\(_{X_{\Delta^0}}(D_{X_{\Delta^0}})\). For \(v \in \partial \Delta^0 \cap \mathbb{Z}^3\), let \(D_v\) be the corresponding torus invariant divisor of \(X(\Sigma_{\Delta^0})\). Then the intersection \(D_v \cap D_{X(\Sigma_{\Delta^0})}\) may not be irreducible if \(v\) is a point of the relative interior of an edge \(\theta\) of \(\Delta^0\). So some classes in Pic\(_{X_{\Delta^0}}(D_{X_{\Delta^0}})\) that come from components of \(D_v \cap D_{X(\Sigma_{\Delta^0})}\) may not come from classes in Pic\(_{X_{\Delta^0}}(D_{X_{\Delta^0}})\) — these are the missing classes we are looking for. With this observation, we
will find some modification to \( X(\Sigma_{\Delta^o}) \) that leads to a quasi-Fano threefold \( Y_{\Delta^o} \) such that

1. \( \text{Pic}_{Y_{\Delta^o}}(D_{Y_{\Delta^o}}) \simeq \text{Pic}(D_X(\Sigma_{\Delta^o})) \ (\simeq \text{Pic}(D_{X_{\Delta^o}})) \),
2. The normal bundle \( N_{DY_{\Delta^o}/Y_{\Delta^o}} \) is trivial,
3. The relations (4.5), (4.6) and (4.7) hold for \( X_{\Delta}, Y_{\Delta^o} \).

If one blows up sequentially \( X(\Sigma_{\Delta^o}) \) along all the irreducible curves \( c_1, c_2, \ldots, c_k \) such that

\[
(5.1) \quad \sum_{v \in \partial \Delta^o \cap \mathbb{Z}^3} D_v \cap D_X(\Sigma_{\Delta^o}) = c_1 + c_2 + \cdots + c_k,
\]

then he gets a threefold that turns out to satisfy (1), (2) and (3) in the above properties. The sequential blow-up is done as follows. Let \( Y^{(1)} \to X(\Sigma_{\Delta^o}) \) be the blow-up along \( c_1 \) and \( D^{(1)} \) be the proper transform of \( D_X(\Sigma_{\Delta^o}) \). Since the blow-up center \( c_1 \) lies on \( D_X(\Sigma_{\Delta^o}) \), \( D^{(1)} \) is isomorphic to \( D_X(\Sigma_{\Delta^o}) \). So \( D^{(1)} \) contains copies of \( c_1, c_2, \ldots, c_k \). We denote them by \( c_1^{(1)}, c_2^{(1)}, \ldots, c_k^{(1)} \).

Note that \( \sum_{v \in \partial \Delta^o \cap \mathbb{Z}^3} D_v \) is an anticanonical divisor of \( X(\Sigma_{\Delta^o}) \). Hence the divisor \( D^{(1)}|_{D^{(1)}} \) is linearly equivalent to

\[
c_1^{(1)} + c_2^{(1)} + \cdots + c_k^{(1)} - c_1^{(1)} = c_2^{(1)} + c_3^{(1)} + \cdots + c_k^{(1)}.
\]

We construct \( Y^{(2)}, Y^{(3)}, \ldots, Y^{(k)} \) inductively as follows. Let \( Y^{(l+1)} \to Y^{(l)} \) be the blow-up along \( c_{l+1}^{(l)} \) and \( D^{(l+1)} \) be the proper transform of \( D^{(l)} \). Since the blow-up center \( c_{l+1}^{(l+1)} \) lies on \( D^{(l)} \), \( D^{(l+1)} \) is isomorphic to \( D^{(l)} \). So \( D^{(l+1)} \) contains copies of \( c_1^{(l)}, c_2^{(l)}, \ldots, c_k^{(l)} \). We denote them by \( c_1^{(l+1)}, c_2^{(l+1)}, \ldots, c_k^{(l+1)} \).

Let \( Y_{\Delta^o} = Y^{(k)} \). Note that the divisor \( D^{(l)}|_{D^{(l)}} \) is linearly equivalent to

\[
c_{l+1}^{(l)} + c_2^{(l)} + \cdots + c_k^{(l)}.
\]

Hence the normal bundle \( N_{DY_{\Delta^o}/Y_{\Delta^o}} \) is trivial. Since the curves \( c_1, c_2, \ldots, c_k \) all together generate the lattice \( \text{Pic}(D_X(\Sigma_{\Delta^o})) \) ([32]), we have

\[
\text{Pic}_{Y_{\Delta^o}}(D_{Y_{\Delta^o}}) \simeq \text{Pic}(D_X(\Sigma_{\Delta^o})).
\]

We denote the composite of the above blow-ups by \( Y_{\Delta^o} \to X(\Sigma_{\Delta^o}) \).

Note

\[
\alpha_{Y_{\Delta^o}} = \alpha_{X(\Sigma_{\Delta^o})} + \sum_{\theta \in \Delta^o[1]} \left((l(\theta) - 2)(l(\theta^o) - 1) - 1 \right)
\]

\[
= l(\Delta^o) - 4 - \sum_{\Gamma \in \Delta^o[2]} l^{*}(\Gamma) + \sum_{\theta \in \Delta^o[1]} \left((l(\theta) - 2)(l(\theta^o) - 2) \right)
\]

\[
= l(\Delta^o) - \sum_{v \in \Delta^o[0]} l^{*}(v^o) + \sum_{\theta \in \Delta[1]} l^{*}(\theta^o)l^{*}(\theta) - 4.
\]
Hence we have

$$\alpha_X + \alpha_{Y^{\circ}} = \left( l(\Delta) - \sum_{\Gamma \in \Delta^{[2]}} l^*(\Gamma) - 4 \right) + \left( l(\Delta^o) - \sum_{v \in \Delta^{[0]}} l^*(v^o) + \sum_{\theta \in \Delta^{[1]}} l^*(\theta^o)l^*(\theta) - 4 \right)$$

$$= 20,$$

where Lemma 3.1 was used. So we explicitly checked that (4.7) is satisfied.

We have

$$k = |\Delta^o[0]| + \sum_{\theta \in \Delta^o[1]} (l(\theta) - 2)(l(\theta^o) - 1)$$

$$= |\Delta[2]| + \sum_{\theta \in \Delta[1]} l(\theta) = |\Delta[2]| + \sum_{\theta \in \Delta[1]} l^*(\theta^o)l^*(\theta) + \sum_{\theta \in \Delta[1]} l^*(\theta^o).$$

Note

$$h^2(Y^{\circ}) = h^2(X(\Sigma^{\circ})) + k$$

$$= l(\Delta^o) + |\Delta[2]| + \sum_{\theta \in \Delta[1]} l^*(\theta^o)l^*(\theta) + \sum_{\theta \in \Delta[1]} l^*(\theta^o) - 4.$$

Let $g(c_i)$ be the genus of $c_i$, then

$$h^3(Y^{\circ}) = h^1(X(\Sigma^{\circ})){\sum_i h^1(c_i) = 2 \sum_i g(c_i) = 2 \sum_{v \in \Delta^{[0]}} l^*(v^o) = 2 \sum_{\Gamma \in \Delta^{[2]}} l^*(\Gamma).}$$

Now one can check that (4.5), (4.6) hold for $X_{\Delta}$, $Y^{\circ}$.

Let us build a Calabi–Yau threefold $\Xi_{Y^{\circ}}$ from $Y^{\circ}$ and calculate the Hodge numbers of $\Xi_{Y^{\circ}}$. Firstly,

$$h^{1,1}(\Xi_{Y^{\circ}}) = 2h^2(Y^{\circ}) - 1 - \alpha_{Y^{\circ}}$$

$$= 2 \left( l(\Delta^o) + |\Delta[2]| + \sum_{\theta \in \Delta[1]} l^*(\theta^o)l^*(\theta) + \sum_{\theta \in \Delta[1]} l^*(\theta^o) - 4 \right) - 1$$

$$- \left( l(\Delta^o) - \sum_{v \in \Delta^{[0]}} l^*(v^o) + \sum_{\theta \in \Delta[1]} l^*(\theta^o)l^*(\theta) - 4 \right)$$

$$= l(\Delta^o) + 2|\Delta[2]| + \sum_{\theta \in \Delta[1]} l^*(\theta^o)l^*(\theta) + 2 \sum_{\theta \in \Delta[1]} l^*(\theta^o) + \sum_{v \in \Delta^{[0]}} l^*(v^o) - 5$$

and
\( h^{1,2}(\Xi_{Y_{\Delta^o}}) = 21 + h^3(Y_{\Delta^o}) - \alpha_{Y_{\Delta^o}} \)

\[
= 21 + 2 \sum_{\Gamma \in \Delta[2]} l^*(\Gamma) - \left( l(\Delta^o) - \sum_{v \in \Delta[0]} l^*(v^o) + \sum_{\theta \in \Delta[1]} l^*(\theta^o)l^*(\theta) - 4 \right)
\]

\[
= 25 + 2 \sum_{\Gamma \in \Delta[2]} l^*(\Gamma) - l(\Delta^o) + \sum_{v \in \Delta[0]} l^*(v^o) - \sum_{\theta \in \Delta[1]} l^*(\theta^o)l^*(\theta).
\]

Now we confirm the Hodge number mirror relation (1.1) for \( \Xi_{X_{\Delta^o}}, \Xi_{Y_{\Delta^o}}. \)

**Theorem 5.1.**

\[
h^{1,1}(\Xi_{X_{\Delta^o}}) = h^{1,2}(\Xi_{Y_{\Delta^o}}) \quad \text{and} \quad h^{1,2}(\Xi_{X_{\Delta^o}}) = h^{1,1}(\Xi_{Y_{\Delta^o}}).
\]

**Proof.** Firstly,

\[
h^{1,1}(\Xi_{X_{\Delta^o}}) - h^{1,2}(\Xi_{Y_{\Delta^o}}) = \left( l(\Delta) + \sum_{\Gamma \in \Delta[2]} l^*(\Gamma) - 3 \right)
\]

\[
- \left( 25 + 2 \sum_{\Gamma \in \Delta[2]} l^*(\Gamma) - l(\Delta^o) + \sum_{v \in \Delta[0]} l^*(v^o) - \sum_{\theta \in \Delta[1]} l^*(\theta^o)l^*(\theta) \right)
\]

\[
= l(\Delta) - \sum_{\Gamma \in \Delta[2]} l^*(\Gamma) + l(\Delta^o) - \sum_{v \in \Delta[0]} l^*(v^o) + \sum_{\theta \in \Delta[1]} l^*(\theta^o)l^*(\theta) - 28
\]

\[
= 0,
\]

where Lemma 3.1 was used.
Secondly
\[ h^{1,2}(\Xi_{X_{\Delta}}) - h^{1,1}(\Xi_{Y_{\Delta^o}}) = 3!\text{vol}(\Delta^o) - l(\Delta) + \sum_{\Gamma \in \Delta[2]} l^*(\Gamma) + 27 \]
\[ - \left( l(\Delta^o) + 2|\Delta[2]| + \sum_{\theta \in \Delta[1]} l^*(\theta^o)l^*(\theta) + 2 \sum_{\theta \in \Delta[1]} l^* (\theta^o) + \sum_{v \in \Delta[0]} l^*(v^o) - 5 \right) \]
\[ = l(\Delta^o) - l(\Delta) + \sum_{\Gamma \in \Delta[2]} l^*(\Gamma) + 26 - 2|\Delta[2]| - \sum_{\theta \in \Delta[1]} l^* (\theta^o)l^*(\theta) \]
\[ - 2 \sum_{\theta \in \Delta[1]} l^*(\theta^o) - \sum_{v \in \Delta[0]} l^*(v^o) \]
\[ = - l(\Delta^o) - l(\Delta) + \sum_{\Gamma \in \Delta[2]} l^*(\Gamma) + 28 \]
\[ - \sum_{\theta \in \Delta[1]} l^*(\theta^o)l^*(\theta) + \sum_{v \in \Delta[0]} l^*(v^o) \quad (\because (3.1)) \]
\[ = 0, \]
where Lemma 3.1 was used again.

Let us take an example.

**Example 5.2.** Consider a reflexive 3-polytope \( \Delta \) whose vertices are
\((1, 0, 0), (0, 1, 0), (0, 0, 1), (-4, -4, -3).\)

This reflexive polytope \( \Delta \) gives the quasi-Fano threefold \( X_{\Delta} \) with
\[ h^2(X_{\Delta}) = 6, h^3(X_{\Delta}) = 38, \alpha_{X_{\Delta}} = 4 \]
and its polar dual gives another quasi-Fano threefold \( Y_{\Delta^o} \) with
\[ h^2(Y_{\Delta^o}) = 36, h^3(Y_{\Delta^o}) = 2, \alpha_{Y_{\Delta^o}} = 16. \]

The Hodge numbers of the pair \( (\Xi_{X_{\Delta}}, \Xi_{Y_{\Delta^o}}) \) of resulted Calabi–Yau threefolds are
\[ h^{1,1}(\Xi_{X_{\Delta}}) = 7, h^{1,2}(\Xi_{X_{\Delta}}) = 55, h^{1,1}(\Xi_{Y_{\Delta^o}}) = 55, h^{1,2}(\Xi_{Y_{\Delta^o}}) = 7. \]

For 4319 equivalence classes of reflexive 3-polytopes, we give all the Hodge numbers of the pairs of \( (\Xi_{X_{\Delta}}, \Xi_{Y_{\Delta^o}}) \)’s in Table 1 of [29].

6. **Mirror pairs of quasi-Fano threefolds**

Now we have many examples of \( X_{\Delta}, Y_{\Delta^o} \) which give pairs of Calabi–Yau threefolds \( (\Xi_{X_{\Delta}}, \Xi_{Y_{\Delta^o}}) \)’s satisfying the relation (1.1). All the pairs \( (X_{\Delta}, Y_{\Delta^o}) \) satisfy
\[ \alpha_{X_{\Delta}} + \alpha_{Y_{\Delta^o}} = 20. \]
It seems that more delicate structure is involved — mirror symmetry for lattice polarized $K3$ surfaces.

Let $L$ be a primitive sublattice of the $K3$ lattice with signature $(1, t - 1)$ and $1 \leq t \leq 19$. An $L$-polarized $K3$ surface is a pair $(S, j)$ where $S$ is a $K3$ surface and $j : L \to \text{Pic}(S)$ is a primitive lattice embedding. One can construct a moduli space $K_L$, parametrizing $L$-polarized $K3$ surfaces, which has dimension $20 - t$. Suppose that there is another lattice $L^\circ$ such that the orthogonal complement $L^\perp$ in the $K3$ lattice has a decomposition $L^\perp \cong U \oplus L^\circ$, where $U$ is the hyperbolic lattice. The moduli space $K_L^\circ$ is defined as a mirror of $K_L$ in [13] and the pair $(K_L, K_L^\circ)$ of moduli spaces was shown to have properties analogous to those of mirror symmetry of Calabi–Yau threefolds. The lattice pair $(L, L^\circ)$ is called a $K3$-mirror pair of lattices and we have $\text{rk}L + \text{rk}L^\circ = 20$.

In [32], it is noticed that $\left(\text{Pic}_{X(\Sigma_\Delta)}(D_X(\Sigma_\Delta)), \text{Pic}_{X(\Sigma_\Delta)}(D_X(\Sigma_\Delta))\right)$ is a $K3$-mirror pair of lattices. Note

$$\text{Pic}_{X(\Sigma_\Delta)}(D_X(\Sigma_\Delta)) \cong \text{Pic}_{X(\Sigma_\Delta)}(D_X(\Sigma_\Delta)), \text{Pic}_{X(\Sigma_\Delta)}(D_X(\Sigma_\Delta)) \cong \text{Pic}(D_X(\Sigma_\Delta)).$$

Hence, for all the pairs of $(X_\Delta, Y_\Delta^\circ)$,

the pair $\left(\text{Pic}_{X(\Sigma_\Delta)}(D_X(\Sigma_\Delta)), \text{Pic}_{X(\Sigma_\Delta)}(D_X(\Sigma_\Delta))\right)$ of lattices is a $K3$-mirror pair.

This indicates that there are some connections between the mirror symmetries for $K3$ surfaces and pairs of quasi-Fano threefolds that lead to pairs of Calabi–Yau threefolds, satisfying the Hodge number relation $(1, 1)$. For a quasi-Fano threefold $X$ whose anticanonical map is a $K3$-fibration, not all $\text{Pic}_{D_X(\Sigma_\Delta)}(D_X(\Sigma_\Delta))$-polarized $K3$ surfaces appear as an anticanonical section of $X$ — the anticanonical linear system $|-K_X|$ is just a pencil and so too small to contain all such $K3$ surfaces. Instead it is expected that a generic $\text{Pic}_{D_X(\Sigma_\Delta)}(D_X(\Sigma_\Delta))$-polarized $K3$ surface may appear as an anticanonical section of some deformation of $X$.

With all these properties, it seems reasonable to call $(X_\Delta, Y_\Delta^\circ)$ a mirror pair of quasi-Fano threefolds. We give a definition for three-dimensional case, collecting properties the pairs $(X_\Delta, Y_\Delta^\circ)$ satisfy.

**Definition 6.1.** A pair $(X, Y)$ of quasi-Fano threefolds whose anticanonical maps are $K3$-fibrations is called a mirror pair if

$$\left(\text{Pic}_{X(\Sigma_\Delta)}(D_X(\Sigma_\Delta)), \text{Pic}_{X(\Sigma_\Delta)}(D_X(\Sigma_\Delta))\right)$$

is a $K3$-mirror pair of lattices and

$$\alpha_X = h^2(X) - h^{1,2}(X) - 1, \alpha_Y = h^2(Y) - h^{1,2}(X) - 1,$$

where $D_X, D_Y$ are generic smooth anticanonical sections of $X, Y$ respectively.
We show that a mirror pair of quasi-Fano threefolds give rise a pair of Calabi–Yau threefolds that satisfy the relation (1.1).

**Proposition 6.2.** Let \((X,Y)\) be a mirror pair of quasi-Fano threefolds and consider Calabi–Yau threefolds \(\Xi_{X}, \Xi_{Y}\) from them. Then

\[
h^{1,1}(\Xi_{X}) = h^{1,2}(\Xi_{Y}), h^{1,2}(\Xi_{X}) = h^{1,1}(\Xi_{Y}).
\]

**Proof.**

\[
h^{1,1}(\Xi_{X}) = 2h^{2}(X) - 1 - \alpha_{X} = 2(\alpha_{X} + h^{1,2}(Y) + 1) - 1 - \alpha_{X} = \alpha_{X} + 2h^{1,2}(Y) + 1 = 20 - \alpha_{Y} + 2h^{1,2}(Y) + 1 = h^{1,2}(\Xi_{Y})
\]

and similarly

\[
h^{1,2}(\Xi_{X}) = 21 + h^{3}(X) - \alpha_{X} = -\alpha_{Y} + 2h^{2}(Y) - 1 = h^{1,1}(\Xi_{Y}).
\]

Let us give more examples of mirror pairs of quasi-Fano threefolds other than those from toric varieties. They come from non-symplectic involutions on K3 surfaces.

An involution \(\rho\) of a K3 surface \(S\) is called non-symplectic if \(\rho^{*}(\omega) = -\omega\) for each \(w \in H^{2,0}(S)\). Let \((H^{2}(S, \mathbb{Z}))^{\rho}\) be the invariant sublattice of \(H^{2}(S, \mathbb{Z})\) by \(\rho^{*}\). Non-symplectic involutions can be classified by their invariant lattices and there are 75 isomorphic classes of such invariant lattices ([31]), which will be called non-symplectic involution lattices. For a non-symplectic involution lattice \(L\), a generic element of \(K_{L}\) has a non-symplectic involution whose invariant lattice is \(L\).

Fix a non-symplectic involution lattice \(L\) whose orthogonal complement \(L^{\perp}\) in the K3 lattice has a decomposition

\[
L^{\perp} \cong U \oplus L^{\circ},
\]

i.e. \((L, L^{\circ})\) is a K3 mirror pair of lattices. Choose a generic K3 surface \(S\) from \(K_{L}\) that has a non-symplectic involution \(\rho\) whose invariant lattice is \(L\). The fixed locus of \(\rho\) is a disjoint union of smooth curves. Let \(\iota : \mathbb{P}^{1} \to \mathbb{P}^{1}\) be involution fixing two distinct points. Let \(V_{\rho}\) be the blow-up of the quotient \((S \times \mathbb{P}^{1})/(\rho, \iota)\) along its singular locus. One can check that \(V_{\rho}\) is smooth and it is a quasi-Fano threefold with anticanonical K3 fibration with generic fiber \(D_{V_{\rho}}\) isomorphic to \(S\) (see §4 in [24] for the details of the construction).

It is also easy to see

\[
\text{Pic}_{V_{\rho}}(D_{V_{\rho}}) = L.
\]

It is known that \(L^{\circ}\) is also a non-symplectic involution lattice. For a generic K3 surface \(S^{\circ} \in K_{L^{\circ}}\) with a non-symplectic involution \(\rho^{\circ}\) whose invariant lattice is \(L^{\circ}\), construct a quasi-Fano threefold \(V_{\rho^{\circ}}\). One can easily
check that \((V_\rho, V_{\rho^\circ})\) is a mirror pair of quasi-Fano threefolds (See §4, [24] for the Hodge numbers of them). It turns out that the corresponding pair of Calabi–Yau threefolds

\[ (\Xi_{V_\rho}, \Xi_{V_{\rho^\circ}}) \]

is the famous Borcea–Voisin mirror pair of Calabi–Yau threefolds ([9, 34]). This fact can be explained as follows.

Choose a point \(p \in \mathbb{P}^1\) such that \(\iota(p) \neq p\). Consider a degeneration of an elliptic curve \(E\) to a normal crossing of two projective lines

\[ Z_0 := \mathbb{P}^1 \cup \iota(p) \mathbb{P}^1, \]

made by attaching at two points \(p, \iota(p)\) and denote the degeneration by \(3 \to B\). Consider an involution \(\tilde{\iota}\) acting on \(3\) fiberwise which induces the standard involution \(\xi\), acting on an elliptic fiber as the multiplication by \(-1\) and whose restriction to each component \(\mathbb{P}^1\) is the involution \(\iota\). Let \(X\) be the blow-up of \((S \times \gamma)/\rho, \tilde{\iota}\) along the singular locus. Then \(X\) is smooth and the induced map \(X \to B\) is a degeneration of Calabi–Yau threefold \(U_\rho\) to a \(V_\rho \cup D\), where \(U_\rho\) is the blow-up of \((S \times E)/\rho, \xi\) along the singular locus. Hence we conclude that \(U_\rho\) and \(\Xi_{V_\rho}\) are of the same deformation type. Similarly \(U_{\rho^\circ}\) and \(\Xi_{V_{\rho^\circ}}\) are of the same deformation type. We note that \((U_\rho, U_{\rho^\circ})\) is the mirror pairs of Calabi–Yau threefolds, constructed in [9, 34].

**Remark 6.3.** In Definition 6.1, we imposed the K3-mirror lattice condition, which is much stronger than (4.7). On the other hand, in the proofs of Proposition 6.2 and upcoming theorems, only Equation (4.7) is used instead of fully utilizing the K3-mirror lattice condition. However we note that every example of mirror pairs of quasi-Fano threefolds, including ones from non-symplectic involutions on K3 surfaces, satisfies the K3-mirror lattice condition. We expect that the K3-mirror lattice condition will play an important roll in showing more delicate mirror relations than Hodge numbers relation (1.1) between mirror pairs of Calabi–Yau threefolds, constructed by smoothing normal crossings of quasi-Fano threefolds. Definition 6.1 needs to be refined, and will be completed when the Landau-Ginzburg models of quasi-Fano threefolds are totally understood.

7. Mirror pairs of \(d\)-semistable Calabi–Yau threefolds of type II

In previous sections, we considered smoothing of normal crossing varieties \(X \cup_D X\) whose components are isomorphic. Now we generalize the construction for the case that components are not isomorphic.

Since a normal crossing variety, smoothable to Calabi–Yau manifolds, can be regarded as a member in a deformation family of those Calabi–Yau manifolds, we call the normal crossing variety

\[ X_1 \cup X_2 \cup \cdots \cup X_r \]
of dimension $n$ as a $d$-semistable Calabi–Yau $n$-fold of type II if it has no triple locus, i.e. $\bigcup_{1<j<k} X_i \cap X_j \cap X_k = \emptyset$. This is a generalization of a notion for K3 surfaces ([15, 27]). The normal crossing varieties we are considering are the simplest examples of $d$-semistable Calabi–Yau manifolds of type II which are composed of two quasi-Fano manifolds and they have been actively investigated in [28, 33]. Now we want to discuss mirror pairs of such Calabi–Yau threefolds.

Consider normal crossing varieties $X = X_1 \cup X_2$ and $Y = Y_1 \cup Y_2$ of quasi-Fano threefolds, smoothable to Calabi–Yau threefolds $M_X$ and $M_Y$ respectively. Suppose that $(X_i, Y_i)$ is a mirror pair of quasi-Fano threefolds for each $i = 1, 2$ and $M_X$ and $M_Y$ satisfy the mirror relation (1.1). Then one can show:

$$(\alpha_{X_1} + \alpha_{X_2} - \alpha_X) + \alpha_Y = 20$$

and

$$(\alpha_{Y_1} + \alpha_{Y_2} - \alpha_Y) + \alpha_X = 20,$$

where

$$\alpha_X = \text{rk} (\text{Pic}_{X_1}(D_X) + \text{Pic}_{X_2}(D_X)), \quad \alpha_Y = \text{rk} (\text{Pic}_{Y_1}(D_Y) + \text{Pic}_{Y_2}(D_Y))$$

with $D_X = X_1 \cap X_2$, $D_Y = Y_1 \cap Y_2$. Noting

$$\alpha_{X_1} + \alpha_{X_2} - \alpha_X = \text{rk} (\text{Pic}_{X_1}(D_X) \cap \text{Pic}_{X_2}(D_X))$$

and

$$\alpha_{Y_1} + \alpha_{Y_2} - \alpha_Y = \text{rk} (\text{Pic}_{Y_1}(D_Y) \cap \text{Pic}_{Y_2}(D_Y)),$$

we give the following definition.

**Definition 7.1.** Suppose that $d$-semistable Calabi–Yau threefolds $X = X_1 \cup X_2$, $Y = Y_1 \cup Y_2$ of type II satisfy

1. $(X_i, Y_i)$ is a mirror pair of quasi-Fano threefolds such that $D_X = X_1 \cap X_2$, $D_Y = Y_1 \cap Y_2$ are anticanonical sections of $X_i$, $Y_i$ respectively for each $i = 1, 2$.
2. The pairs of lattices

$$(\text{Pic}_{X_1}(D_X) + \text{Pic}_{X_2}(D_X), \text{Pic}_{Y_1}(D_Y) \cap \text{Pic}_{Y_2}(D_Y)),$$

$$(\text{Pic}_{X_1}(D_X) \cap \text{Pic}_{X_2}(D_X), \text{Pic}_{Y_1}(D_Y) + \text{Pic}_{Y_2}(D_Y))$$

are K3-mirror pairs.

Then the pair $(X, Y)$ is called a mirror pair of $d$-semistable Calabi–Yau threefolds of type II.

Note that the pair $(X_{\Delta} \cup_D X_{\Delta}, Y_{\Delta} \cup_D Y_{\Delta})$, constructed in Sections 3 and 5, is a mirror pair of $d$-semistable Calabi–Yau threefolds of type II in this definition.

**Theorem 7.2.** For normal crossing varieties $X = X_1 \cup X_2$, $Y = Y_1 \cup Y_2$, if $(X, Y)$ is a mirror pair of $d$-semistable Calabi–Yau threefolds of type II, then the Calabi–Yau threefolds $M_X, M_Y$ satisfy

$$h^{1,1}(M_X) = h^{1,2}(M_Y), h^{1,2}(M_X) = h^{1,1}(M_Y).$$
Proof.
\[ h^{1,1}(M_X) = h^{1,1}(X_1) + h^{1,1}(X_2) - \alpha_X - 1 \]
\[ = h^{1,1}(X_1) + h^{1,1}(X_2) - \alpha_{X_1} - \alpha_{X_2} - \alpha_Y + 20 - 1 \]
\[ = -h^{1,2}(Y_1) - h^{1,2}(Y_2) - \alpha_Y + 21 \]
\[ = h^{1,2}(M_Y). \]

Similarly we can get the second equation. \( \square \)

The following immediate corollary of this theorem is very useful for generating many examples of mirror pairs of \( d \)-semistable Calabi–Yau threefolds of type II.

**Corollary 7.3.** Let \( X = X_1 \cup X_2, Y = Y_1 \cup Y_2 \) be \( d \)-semistable Calabi–Yau threefolds of type II such that

1. \((X_i, Y_i)\) is a mirror pair of quasi-Fano threefolds for each \( i = 1, 2 \),
2. \( \text{Pic} X_1(D_X), \text{Pic} Y_2(D_Y) \) are sublattices of \( \text{Pic} X_2(D_X), \text{Pic} Y_1(D_Y) \) respectively.

Then the pair \((X, Y)\) is a mirror pair of \( d \)-semistable Calabi–Yau threefolds of type II.

For a fixed reflexive 3-polytope \( \Delta \), consider \( X_\Delta, Y_\Delta \). From their construction, we can choose \( D_{X_\Delta}, D_{Y_\Delta} \) so that \( D_{X_\Delta} \simeq D_{Y_\Delta} \) (to be denoted by \( D_\Delta \)). Hence we can make a \( d \)-semistable normal crossing variety

\[ Z_\Delta = X_\Delta \cup D Y_\Delta \]

by gluing along \( D_\Delta \).

To ensure the smoothability of \( Z_\Delta \) to a Calabi–Yau threefold, we need to show that it is projective, i.e. we need to find some ample divisors \( H_{X_\Delta}, H_{Y_\Delta} \) of \( X_\Delta, Y_\Delta \) respectively such that \( H_{X_\Delta}|_{D_\Delta}, H_{Y_\Delta}|_{D_\Delta} \) are linearly equivalent. Note that \( \pi_X : X_\Delta \to X(\Sigma_\Delta) \) is a blow-up along a smooth curve \( c \in |-K_{X(\Sigma_\Delta)}| \) and \( \pi_Y : Y_\Delta \to X(\Sigma_\Delta) \) is a sequential blow-up along a smooth curve \( c_1, c_2, \ldots, c_k \) such that

\[ \sum_{v \in \partial \Delta \cap \mathbb{Z}^3} D_v \cap D_{X(\Sigma_\Delta)} = c_1 + c_2 + \cdots + c_k. \]

Let \( \Delta[0] = \{v_1, v_2, \cdots, v_l\} \) and \( \gamma_i = D_{X(\Sigma_\Delta)} \cap D_{v_i} \). Let \( \{v'_1, \cdots, v'_m\} \) be the set of all the integral points that lie on the relative interiors of some edges of \( \Delta \), then

\[ D_{X(\Sigma_\Delta)} \cap D_{v'_i} = \epsilon_{i1} + \cdots + \epsilon_{ia_i}, \]

where \( \epsilon_{i1}, \cdots, \epsilon_{ia_i} \) are disjoint smooth rational curves and \( a_i = l^*(\theta^0) \) for \( v'_i \in \theta \).

Then

\[ \{c_1, c_2, \cdots, c_k\} = \{\gamma_1, \gamma_2, \cdots, \gamma_l\} \cup \bigcup_{1 \leq i \leq m} \{\epsilon_{i1}, \cdots, \epsilon_{ia_i}\}. \]
Let $E_i, F_{ij}$ be the exceptional divisors over $\gamma_i, \epsilon_{ij}$ in the sequential blow-up \( \pi_Y : Y_{\Delta} \rightarrow X(\Sigma_{\Delta}) \) respectively. For an ample divisor $H$ of $X(\Sigma_{\Delta})$, there are some positive integers $b_i$’s, $d_{ij}$’s such that the divisor
\[
H_1 := N\pi_{\bar{X}}^*(H) - \sum_i b_i E_i - \sum_{i,j} d_{ij} F_{ij}
\]
is ample on $Y_{\Delta}$ for sufficiently large $N$. The point here is that we can assume that $d_1 = \cdots = d_{ia_i} (= d_i)$ since the curve $\epsilon_1, \cdots, \epsilon_{ia_i}$ are all disjoint. Let $E$ be the exceptional divisor of the blow-up $\pi_X : X_{\Delta} \rightarrow X(\Sigma_{\Delta})$. Then the divisor
\[
H_2 := N\pi_{X}^*(H) - \sum_i (b_i - 1)\pi_{X}^*(D_{v_i}) - \sum_i (d_i - 1)\pi_{X}^*(D_{v'}) - E
\]
is ample on $X_{\Delta}$ for sufficiently large $N$. It is trivial to check that $H_{X_{\Delta}}|_{D_{\Delta}}, H_{Y_{\Delta}}|_{D_{\Delta}}$ are linearly equivalent. So we proved that $Z_{\Delta} = X_{\Delta} \cup_D Y_{\Delta}$ is projective. Similarly $Z_{\Delta^o} = Y_{\Delta^o} \cup_D X_{\Delta^o}$ is also $d$-semistable and projective.

Note that $(X_{\Delta}, Y_{\Delta^o}), (Y_{\Delta}, X_{\Delta^o})$ are mirror pairs of quasi-Fano threefolds and
\[
\text{Pic}_{X_{\Delta}}(D_{X_{\Delta}}) \subset \text{Pic}_{Y_{\Delta}}(D_{Y_{\Delta}}), \quad \text{Pic}_{X_{\Delta^o}}(D_{X_{\Delta^o}}) \subset \text{Pic}_{Y_{\Delta^o}}(D_{Y_{\Delta^o}}).
\]
Therefore $(Z_{\Delta}, Z_{\Delta^o})$ is a mirror pair of $d$-semistable Calabi–Yau threefolds of type II. If $\Delta$ is self-dual, i.e. the polytopes $\Delta, \Delta^o$ are equivalent, then $Z_{\Delta} = Z_{\Delta^o}$. So $Z_{\Delta}$ (also $M_{Z_{\Delta}}$) is a self-mirror. There are 79 self-dual 3-polytopes among 4319 equivalence classes of reflexive 3-polytopes. Hence this construction gives us 2199 mirror pairs of Calabi–Yau threefolds, including 79 self-mirrors. Those are listed in Table 2 of [29] and the following are their Hodge numbers.
\[
h^{1,1}(M_{Z_{\Delta}}) = h^{1,2}(M_{Z_{\Delta^o}}) = l(\Delta) + |\Delta[0]| + \sum_{\theta \in \Delta[1]} l^*(\theta) + \sum_{\Gamma \in \Delta[2]} l^*(\Gamma) - 4,
\]
\[
h^{1,2}(M_{Z_{\Delta}}) = h^{1,1}(M_{Z_{\Delta^o}}) = 23 - l(\Delta) + l(\Delta^o) + \sum_{v \in \Delta[0]} l^*(v^o) + \sum_{\Gamma \in \Delta[2]} l^*(\Gamma) - \sum_{\theta \in \Delta[1]} l^*(\theta^o) l^*(\theta).
\]

The polytope $\Delta$ in Example 5.2 gives rise to the pair $(M_{Z_{\Delta}}, M_{Z_{\Delta^o}})$ of Calabi–Yau threefolds with
\[
h^{1,1}(M_{Z_{\Delta}}) = h^{1,2}(M_{Z_{\Delta^o}}) = 13, h^{1,2}(M_{Z_{\Delta}}) = h^{1,1}(M_{Z_{\Delta^o}}) = 37.
\]
Note that there are some multiplicities of mirror pairs in the Batyrev mirror construction in [6], due to the fact that desingularizations of Calabi–Yau hypersurfaces in Gorenstein toric Fano fourfolds may not be unique. There also have been found multiplicities to some of Borcea–Voisin mirror pairs ([3]). There are similar multiplicities in the mirror construction of this paper.
In the construction of quasi-Fano threefolds $X_\Delta, Y_\Delta$ which are blow-ups of $X(\Sigma_\Delta)$, there may be more than one choice of maximal projective triangulations of $\partial \Delta$. This leads to a multiplicity of mirror pairs of quasi-Fano threefolds. Furthermore recall that we built $Y_\Delta$ by sequentially blowing up $X(\Sigma_\Delta)$ along curves $c_1, c_2, \cdots, c_k$. The sequential blow-up depends on the order of blow-ups in general (see comments after Remark 8.3 in [28]) and so this also gives another multiplicity of mirror pairs. These cause the multiplicities in the construction of mirror pairs, $\big(\Xi_{X_\Delta}, \Xi_{Y_{\Delta^c}}\big)$, $\big(M_{Z_\Delta}, M_{\bar{Z}_{\Delta^c}}\big)$.

8. Are they new?

Our examples in Table 1, 2 of [29] are constructed by smoothing method while Calabi–Yau threefolds in [2, 26], which are the largest source of known examples of Calabi–Yau threefolds, come as desingularizations of anticanonical sections of Gorenstein toric Fano fourfolds. A simple way of distinguishing our examples from them is to compare the Hodge numbers but most of their Hodge numbers overlap. That’s probably because there are so many such examples from toric Fano fourfolds. However one still could suspect that the mirror pairs in this paper may overlap with those from toric fourfolds. In this section, we pick up a particular Calabi–Yau threefold from our list and explicitly show that this Calabi–Yau threefold is not homeomorphic to any of those from toric fourfolds. The Calabi–Yau threefold which we pick up is $\Xi_{X_\Delta}$, where $X(\Sigma_\Delta) = \mathbb{P}^3$. Its Hodge numbers are

$$h^{1,1} = 2, h^{1,2} = 86.$$ 

There are exactly ten different Calabi–Yau threefolds with these Hodge numbers that are desingularizations of anticanonical sections of Gorenstein toric Fano fourfolds ([6, 26]). Those are constructed from nine reflexive 4-polytopes — one of the polytopes gives rise to two non-homeomorphic Calabi–Yau threefolds.

For a compact threefold $M$ with $h^2(M) = 2$ and the second Chern class $c_2(M)$ that is not zero in $H^4(M, \mathbb{Z})_f$, where $A_f = A/A_t$ for an Abelian group $A$ with its torsion part $A_t$, we will define a topological invariant $\lambda(M)$ as follows. Note that the subgroup

$$\{ l \in H^2(M, \mathbb{Z})_f | c_2(M) \cdot l = 0 \}$$

of $H^2(M, \mathbb{Z})_f$ is generated by a single element $m$. Then the number

$$\lambda(M) := |m^3|$$

is a topological invariant of $M$. Firstly we calculate $\lambda(\Xi_{X_\Delta})$. $\Xi_{X_\Delta}$ is a smoothing of normal crossing variety $\mathcal{X} = X_1 \cup X_2$, where

$$\pi_i : X_i \to X(\Sigma_\Delta)$$

is a blow-up of $X(\Sigma_\Delta) = \mathbb{P}^3$ along a smooth curve $c \in | - K_{\mathbb{P}^3}|$. $S$ and $D = X_1 \cap X_2$ is the proper transform in $X_i$ of $S$ for $i = 1, 2$, where $S$ is a
smooth quartic surface of $\mathbb{P}^3$. Let
\[ G^k(\mathcal{X}, \mathbb{Z}) = \ker(H^k(X_1, \mathbb{Z}) \oplus H^k(X_2, \mathbb{Z}) \to H^k(D, \mathbb{Z})), \]
where the map $H^k(X_1, \mathbb{Z}) \oplus H^k(X_2, \mathbb{Z}) \to H^k(D, \mathbb{Z})$ is given by
\[ (l_1, l_2) \mapsto l_1|_D - l_2|_D. \]
Note that $G^k(\mathcal{X}, \mathbb{Z})$ inherits the cup product from those of $H^k(X_1, \mathbb{Z})$, $H^k(X_2, \mathbb{Z})$ with the mixed term set to be zero (see §4, [28]). Let
\[ h_1 = (\pi_1^*(H), \pi_2^*(H)), h_2 = (4\pi_1^*(H) - E_1, 0), \]
where $H$ is a hyperplane section of $\mathbb{P}^3$ and $E_i$ be the exceptional divisor of the blow-up $\pi_i$. Then it is easy to check that $h_1$, $h_2$ belong to $G^2(\mathcal{X}, \mathbb{Z})$. Note
\[ \{\pi_1^*(H), 4\pi_1^*(H) - E_1\} \]
is a basis for the lattice $H^2(X_1, \mathbb{Z})$. By Poincaré duality, there are classes $l_1, l_2 \in H^1(X_1, \mathbb{Z})$ such that the cup product matrix of
\[ \{\pi_1^*(H), 4\pi_1^*(H) - E_1\} \quad \text{and} \quad \{l_1, l_2\} \]
is the $2 \times 2$ identity matrix. Let
\[ h_1' = (l_1, 0), h_2' = (l_2, f), \]
where $f$ is a fiber over a point on the blow-up center under $\pi_2$. It is not hard to see that $h_1'$, $h_2'$ belong to $G^4(\mathcal{X}, \mathbb{Z})$. Now the cup product matrix of $\{h_1, h_2\}$ and $\{h_1', h_2'\}$ is the $2 \times 2$ identity matrix, which is unimodular. According to [28], this property guarantees that there is an isomorphism $\phi$ from the sublattice $\langle h_1, h_2 \rangle$ of $G^2(\mathcal{X}, \mathbb{Z})$ to $H^2(M_X, \mathbb{Z})$ with the cup product preserved. Let $c_2 = (c_2(X_1), c_2(X_2))$, then $c_2$ belongs to $G^4(\mathcal{X}, \mathbb{Z})$ (§7, [28]) and
\[ c_2 \cdot h = c_2(M_X) \cdot \phi(h) \]
for any $h \in \langle h_1, h_2 \rangle$. Now we can calculate $\lambda(M_X)$. Note
\[ c_2 \cdot h_1 = 44, c_2 \cdot h_2 = 24, \]
So the group
\[ \{h \in \langle h_1, h_2 \rangle | c_2 \cdot h = 0\} \]
is generated by $6h_1 - 11h_2$. Hence
\[ \lambda(\Xi_{X_{\Delta}}) = \lambda(M_X) = |(6h_1 - 11h_2)^3| = 4320. \]
Next we need to find out the $\lambda$-invariants of those ten Calabi–Yau threefolds. Since those threefolds are hypersurfaces in toric varieties, it is a routine job to calculate the cubic forms on the second integral cohomology groups and the product with the second Chern class. Those calculations are provided in a data base ([1, 2]). Using this data base, we calculate $\lambda$-invariants of those ten Calabi–Yau threefolds in Table 1, where ‘ID #’ is the polytope number in [2]. One can find out the vertex coordinates of the corresponding 4-polytopes in [2] with those polytope ID #’s.
Table 1. $\lambda$-invariants of Calabi–Yau threefolds with $h^{1,1} = 2, h^{1,2} = 86$

in Gorenstein toric Fano fourfolds

| ID # | $\lambda (M)$ |
|------|--------------|
| 12   | 1404         |
| 13   | 108          |
| 13   | 1564         |
| 14   | 3456         |
| 15   | 17280        |
| 16   | 17946        |
| 17   | 137214       |
| 18   | 67230        |
| 19   | 258198       |
| 20   | 457050       |

Since all the $\lambda$-invariants are different from $\lambda (\Xi_\Delta) = 4320$, the Calabi–Yau threefold $\Xi_\Delta$ is not homeomorphic to a desingularization of an anticanonical section of any Gorenstein toric Fano fourfold.

9. Higher dimensional cases

For higher dimensional cases, we give a definition, taking the mirror relation (4.4) for anticanonical sections into account.

Definition 9.1. A pair $(X, Y)$ of quasi-Fano manifolds of dimension higher than three that have anticanonical Calabi–Yau fibrations

$$\overline{\mathcal{W}}_X : X \to \mathbb{P}^1, \overline{\mathcal{W}}_Y : Y \to \mathbb{P}^1,$$

is called a mirror pair if the pairs

$$(X^*, Y), (Y^*, X)$$

satisfy (4.1), (4.2) respectively and $(D_X, D_Y)$ is a mirror pair of Calabi–Yau manifolds, where $D_X, D_Y$ are generic smooth anticanonical sections of $X, Y$ respectively.

We also generalize Definition 7.1 for higher dimensions.

Definition 9.2. Let $\mathcal{X} = X_1 \cup X_2, \mathcal{Y} = Y_1 \cup Y_2$ be $d$-semistable Calabi–Yau manifolds of type II with dimension higher than three. If $(X_i, Y_i)$ is a mirror pair of quasi-Fano manifolds such that $D_{\mathcal{X}} = X_1 \cup X_2, D_{\mathcal{Y}} = Y_1 \cup Y_2$ are anticanonical sections of $X_i, Y_i$ respectively for each $i = 1, 2$. Then the pair $(\mathcal{X}, \mathcal{Y})$ is called a mirror pair of $d$-semistable Calabi–Yau manifolds of type II.

We have a higher dimensional apology of Theorem 7.2 (see also Theorem 2.3, [14]).

Proposition 9.3. For normal crossing varieties $\mathcal{X} = X_1 \cup X_2, \mathcal{Y} = Y_1 \cup Y_2$ of dimension $n$, if $(\mathcal{X}, \mathcal{Y})$ is a mirror pair of $d$-semistable Calabi–Yau manifolds of type II, then

$$\chi (M_\mathcal{X}) = (-1)^n \chi (M_\mathcal{Y}).$$
Proof. Firstly
\[
\chi(X_i) = \chi(X_i^\ast) + \chi(D_{X_i}) \\
= \chi(X_i^\ast, W_{X_i}^{-1}(t)) + \chi(W_{X_i}^{-1}(t)) + \chi(D_{X_i}) \\
= (-1)^n \chi(Y_i) + 2\chi(D_{X_i})
\]
and similarly we have
\[
\chi(Y_i) = (-1)^n \chi(X_i) + 2\chi(D_{Y_i}).
\]
By (4.4),
\[
\chi(D_{X_i}) = \chi(D_{X}) = (-1)^{n-1}\chi(D_{Y}) = (-1)^{n-1}\chi(D_{Y_i}).
\]
Hence we have
\[
\chi(M_{X}) = \chi(X_1) + \chi(X_2) - 2\chi(D_{X}) \\
= (-1)^n \chi(Y_1) + (-1)^n \chi(Y_2) + 2\chi(D_{X}) \\
= (-1)^n \chi(Y_1) + (-1)^n \chi(Y_2) + 2(-1)^{n-1}\chi(D_{Y}) \quad (\because (4.4)) \\
= (-1)^n (\chi(Y_1) + \chi(Y_2) - 2\chi(D_{Y})) \\
= (-1)^n \chi(M_{Y}).
\]

If we try to construct $\Xi_{X_\Delta}, \Xi_{Y_\Delta}$ and $M_{Z_\Delta}$ for higher dimensional reflexive polytope $\Delta$, we come up with some difficulties due to singularities as follows.

1. There may be no smooth maximal partial projective crepant desingularization $X(\Sigma_{\Delta})$ of $\mathbb{P}(\Delta)$ for $\dim \geq 4$.
2. There may be no smooth anticanonical section of $X(\Sigma_{\Delta})$ for $\dim \geq 5$.

Hence we may not apply the smoothing theorem in [20] for these cases. However one can build $\Xi_X$ as a double cover of $X$, branched along $D_X \cup D_X'$, where $D_X'$ is another anticanonical section of $X$, disjoint from $D_X$. So it is still possible to construct $\Xi_{X_\Delta}, \Xi_{Y_\Delta}$ and these will be some singular Calabi–Yau varieties, which are expected to satisfy the properties similar to those in Theorem 5.1. In the case of $Z_\Delta$, which may not be a normal crossing variety anymore, one needs to generalize the smoothing theorem so that some mild singularities may be allowed. It seems natural to allow some mild singularities when one considers higher dimensional quasi-Fano manifolds.

10. ‘Rigid’ quasi-Fano manifolds

In this paper, we have considered a special kind of varieties — quasi-Fano manifolds with anticanonical fibrations and defined notions of mirror pairs of them. Noting that they are of negative Kodaira dimension and have additional fibration structure, their classifications seem reachable at least for the three-dimensional case with very low or very high $\alpha_X$. 
There are Calabi–Yau threefolds that do not have Calabi–Yau threefolds as their mirror partners such as rigid ones \((h^{1,2} = 0)\). Hence it would be worthwhile to ask if quasi-Fano manifolds with anticanonical fibrations always come as mirror pairs. For three-dimensional case, if \((X, Y)\) is a mirror pair of quasi-Fano threefolds with anticanonical fibrations, then we should have \(\alpha_X + \alpha_Y = 20\). So if \(\alpha_X\) has its maximal value, 20, then \(\alpha_Y\) needs to be zero, which is impossible because \(Y\) has an ample divisor. Hence if \(\alpha_X = 20\), \(X\) does not have a quasi-Fano threefold with anticanonical fibration as its mirror partner — in this case, \(X\) could be called ‘rigid’ quasi-Fano threefold. Such examples can be made easily. For example, take an exceptional ( \(\text{rkPic} = 20\)) quartic \(K3\) surface \(D\) on \(\mathbb{P}^3\) that has smooth curves \(c_1, c_2, \ldots, c_k\) such that

- \(c_1 + c_2 + \cdots + c_k\) belongs to the linear system \(|-K_{\mathbb{P}^3}|D|\),
- \(c_1, c_2, \ldots, c_k\) generate \(H^2(D, \mathbb{Q})\).

An example of such \(D\) is the Fermat quartic. Blow up sequentially \(\mathbb{P}^3\) along \(c_1, c_2, \ldots, c_k\) to get a quasi-Fano threefold \(X\). Then \(\alpha_X = 20\) and so \(X\) is a ‘rigid’ quasi-Fano threefold which does not have a quasi-Fano threefold with anticanonical fibration as its mirror partner.

In the case of dimension \(n > 3\), there are also ‘rigid’ quasi-Fano manifolds whose generic anticanonical sections are rigid Calabi–Yau manifolds. Take a rigid Calabi–Yau manifold \(D\) of dimension \(n - 1\) with non-Gorenstein involution \(\rho\) on it, where we call an involution \(\rho\) non-Gorenstein if \(\rho^*(\omega) = -\omega\) for each \(\omega \in H^{n-1,0}(D)\). We assume further that the fixed locus of \(\rho\) is a manifold of dimension \(n - 2\). Let \(X\) be the blow-up of the quotient \((D \times \mathbb{P}^1)/\langle \rho, \tau \rangle\) along the singular locus, where \(\tau\) is an involution of \(\mathbb{P}^1\), fixing two distinct points. Then \(X\) is a quasi-Fano manifold whose anticanonical section \(D_X\) is isomorphic to \(D\), which is a rigid Calabi–Yau manifold. Hence in the view of Definition 9.1, \(X\) does not have a quasi-Fano manifold with anticanonical fibration as its mirror partner. For \(n = 4\), an easy example of such a Calabi–Yau threefold \(D\) is the one that was introduced by Beauville in [8].

Besides those ‘rigid’ quasi-Fano manifolds, there are ‘non-rigid’ quasi-Fano threefolds that do not have quasi-Fano threefolds as its mirrors. Consider a quasi-Fano threefold \(X\) such that the lattice

\[
L^\perp \cap H^2(D_X, \mathbb{Z})
\]

(10.1)
does not contain a hyperbolic lattice, where \(L = \text{Pic}_X(D_X)\). Then \(L\) does not have a \(K3\)-mirror lattice. So \(X\) cannot have a quasi-Fano threefold as its mirror. Some concrete examples are obtained from non-symplectic involutions on \(K3\) surfaces. Choose a non-symplectic involution \(\rho\) on a \(K3\) surface whose invariant lattice does not a \(K3\)-mirror lattice, then the quasi-Fano threefold \(V_\rho\) introduced in Section 6 has no quasi-Fano threefolds as its mirror. According to the classification in [31], there are 11 families of...
such involutions and the resulted quasi-Fano threefold $V_\rho$ satisfies

$$11 \leq \alpha_{V_\rho} \leq 19.$$  

In sum, there are ‘non-rigid’ quasi-Fano threefolds that do not have quasi-Fano threefolds as its mirrors.

If the lattice in (10.1) contains a hyperbolic lattice, then $\text{Pic}_X(D_X)$ has a $K3$-mirror lattice. Hence, an interesting question would be:

**Question 10.1.** For a quasi-Fano threefold $X$ with anticanonical fibration such that $\text{Pic}_X(D_X)$ has a $K3$-mirror lattice, is there a quasi-Fano threefold that has $X$ as its mirror partner?

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