The superintegrable chiral Potts quantum chain and
generalized Chebyshev polynomials

G. von Gehlen\textsuperscript{a} and Shi-shyr Roan\textsuperscript{b}

\textsuperscript{a}Physikalisches Institut der Universität Bonn
Nussallee 12, 53115 Bonn, Germany
\textit{e-mail: gehlen@th.physik.uni-bonn.de}

\textsuperscript{b}Institute of Mathematics, Academia Sinica
Taipei, Taiwan
\textit{e-mail: maroan@ccvax.sinica.edu.tw}

Abstract

Finite-dimensional representations of Onsager’s algebra are characterized by the zeros
of truncation polynomials. The $\mathbb{Z}_N$-chiral Potts quantum chain hamiltonians (of which the
Ising chain hamiltonian is the $N = 2$ case) are the main known interesting representations
of Onsager’s algebra and the corresponding polynomials have been found by Baxter and
Albertini, McCoy and Perk in 1987-89 considering the Yang-Baxter-integrable 2-dimensional
chiral Potts model. We study the mathematical nature of these polynomials. We find that
for $N \geq 3$ and fixed charge $Q$ these don’t form classical orthogonal sets because their pure
recursion relations have at least $N + 1$-terms. However, several basic properties are very
similar to those required for orthogonal polynomials. The $N + 1$-term recursions are of the
simplest type: like for the Chebyshev polynomials the coefficients are independent of the
degree. We find a remarkable partial orthogonality, for $N = 3, 5$ with respect to Jacobi-,
and for $N = 4, 6$ with respect to Chebyshev weight functions. The separation properties
of the zeros known from orthogonal polynomials are violated only by the extreme zero at one
end of the interval.

1 Introduction

Onsager’s algebra (OA) has played a crucial role in the early history of exactly solvable statistical
systems: it served as the tool for the first solution of the two-dimensional Ising model \cite{Onsager}. How-
ever, since soon after many other methods for solving the Ising model have been invented, for
many years Onsager’s algebra received little attention. It reappeared when the superintegrable
$\mathbb{Z}_N$-symmetrical chiral Potts quantum chain (SCPC) was introduced as a natural generalization
of the transverse Ising quantum chain \cite{Baxter, Albertini}, and was shown to be integrable \cite{Albertini}
because it forms representations of the OA (or the closely related Dolan-Grady algebra \cite{Dolan, Grady}). The actual solution
for the lowest energy eigenvalues of the SCPC by Baxter \cite{Baxter} and Albertini, McCoy and Perk
\cite{Albertini} did not use the representation theory of Onsager’s algebra but rather inversion relations, or,
more generally, functional relations for the transfer matrix of the 2-dimensional Yang-Baxter
integrable chiral Potts model \cite{Baxter, Albertini}. In 1990 B.Davies \cite{Davies} constructed finite-dimensional repre-
sentations of Onsager’s algebra. In his approach zeros of truncation polynomials play a crucial
role and the representations are determined by a set of zeros and by $sl(2, \mathbb{C})$ representations.
While still it is not known how to obtain the Onsager algebra truncation polynomials directly
from the SCPC hamiltonian, equivalent polynomials have been obtained from the 2-dimensional superintegrable Potts model functional relations in \([3, 8]\).

Given the central role of the truncation polynomials for models solvable due to Onsager’s algebra, it seems desirable to improve our understanding of their mathematical properties. In the Ising case, after a simple mapping, these are Chebyshev polynomials (a fact implicit e.g. in \([14]\)). For \(N \geq 3\) we find many properties reminiscent of, but not in full agreement with those of orthogonal polynomials. Looking into recursion relations, differential equations, zero distributions and eventual weight functions for orthogonality, we find simple structures and in particular, a remarkable partial orthogonality with Jacobi-weight functions.

In the next Section we review Onsager’s algebra, the appearance of the polynomials and Baxter’s results for the \(Z_N\)-polynomials. In Sec.3 we discuss three different versions of the polynomials with different locations of the zeros and check how these look in the Ising case. Sec.4 gives some properties of the \(Z_3\)-polynomials with the zeros on the negative real axis. Sec.5 gives our main results, which are for the polynomials with zeros in the interval \(-1 < c < +1\): recursion relations, differential equations, zeros separation and, finally, partial orthogonality. Sec.6 contains our conclusions.

2 Finite dimensional representations of Onsager’s algebra

2.1 Onsager’s algebra

Onsager’s algebra \(\mathcal{A}\) is formed \([1]\) from elements \(A_m, G_l, m \in \mathbb{Z}, l \in \mathbb{N}, l \geq m,\) satisfying

\[
[A_l, A_m] = 4G_{l-m}; \quad [G_l, A_m] = 2A_{m+l} - 2A_{m-l}; \quad [G_l, G_m] = 0. \tag{1}
\]

Eqs.(1) imply an infinite set of constraints:

\[
[A_{m+1}, A_m] = [A_m, A_{m-1}] \tag{2}
\]

and the existence of the infinite set of commuting operators \(Q_m: \)

\[
Q_m = \frac{1}{2} (A_m + A_{-m} + k(A_{m+1} + A_{m-1})); \quad [Q_l, Q_m] = 0. \tag{3}
\]

where \(k\) is a parameter which usually is taken to be real. \(A_0\) and \(A_1\) generate \(\mathcal{A}\) if they satisfy the Dolan-Grady-relations

\[
[A_0, [A_0, [A_0, A_1]]]] = 16 [A_0, A_1]; \quad [A_1, [A_1, [A_1, A_0]]]]] = 16 [A_1, A_0]. \tag{4}
\]

Finite dimensional representations of \(\mathcal{A}\) are obtained \([13, 14, 15]\) if the \(A_m\) satisfy a pure finite recurrence or difference equation:

\[
\sum_{k=-n}^{n} \alpha_k A_{k-l} = 0 \tag{5}
\]

(implying the same equation for the \(G_l\)). For solving this relation B.Davies \([13]\) introduced the polynomial

\[
\mathcal{F}(z) = \sum_{k=-n}^{n} \alpha_k z^{k+n}. \tag{6}
\]

From \(\mathcal{A}\) it follows that \(\alpha_k\) is either even or odd in \(k\): \(\alpha_k = \pm \alpha_{-k}\), so that

\[
\mathcal{F}(z) = \pm z^{2n+1} \mathcal{F}(1/z) \tag{7}
\]
and the zeros of $F(z)$ come in reciprocal pairs $z_j, z_j^{-1}$ ($j = 1, \ldots, n$). The $A_m$ and $G_m$ can be now be expressed in terms of a set of operators $E_j^\pm, H_j$:

$$A_m = 2\sum_{j=1}^{n} \left(z_j^m E_j^+ + z_j^{-m} E_j^\mp\right); \quad G_m = \sum_{j=1}^{n} \left(z_j^m - z_j^{-m}\right) H_j$$

(8)

which from $A$ obey $sl(2, C)$-commutation rules:

$$[E_j^+, E_k^-] = \delta_{j,k} H_k; \quad [H_j, E_k^\pm] = \pm 2 \delta_{j,k} E_k^\pm.$$  \hspace{1cm} (9)

So $A$ is isomorphic to a subalgebra of the loop algebra of a direct sum of $sl(2, C)$ algebras.

We shall be interested in the eigenvalues of hermitian hamiltonians $H \equiv A_0 + k A_1$ ($H \equiv Q_0$ of eq.(3)), with parameter $k$ and $A_0$ and $A_1$ satisfying (4). Write $E_j^\pm = J_{x,j} \pm i J_{y,j}$, then in a representation $Z(n, s)$ characterized by $z_1, \ldots, z_n$ and by a spin-$s$ representation $\vec{J}_{j}$ of all the $\vec{J}_j$, we have

$$(A_0 + k A_1) Z(n, s) = 2 \sum_{j=1}^{n} \left\{ (2 + k (z_j + z_j^{-1})), J_{x,j}^{(s)} + i(z_j - z_j^{-1}) J_{y,j}^{(s)} \right\}$$

$$= 4 \sum_{j=1}^{n} \sqrt{1 + 2k c_j + k^2} J_{x,j}^{(s)}$$

where $J_{x,j}^{(s)}$ is a $SU(2)$-operator and

$$c_j = \cos \theta_j = \frac{1}{2}(z_j + z_j^{-1})$$  \hspace{1cm} (10)

with $\theta_j$ real for hermitian $H$.

### 2.2 The superintegrable chiral Potts quantum chain

The main example of hamiltonians $H$ satisfying the above conditions (for generalisations see [18, 19]) are the $Z_N$-SCPC-hamiltonians [3, 4] given by

$$H^{(s)} = -\sum_{j=1}^{L} \sum_{l=1}^{N-1} \frac{2}{1 - \omega^{-l}} \left( X_j^l + k Z_j^l Z_{j+1}^{N-l} \right)$$

(11)

where $Z_j$ and $X_j$ are $Z_N$-spin operators acting in the vector spaces $\mathbb{C}^N$ at the sites $j$ ($j = 1, \ldots, L$):

$$Z_i X_j = X_j Z_i \omega^{\delta_{i,j}}; \quad Z_j^N = X_j^N = 1; \quad \omega = e^{2\pi i/N}. $$

The normalization agrees with [4] if we write $H^{(s)} = -\frac{1}{2} N(A_0 + k A_1)$.

In order to find the representations $Z(n, s)$ corresponding to the various sectors of (11), no direct way is known. Albertini et al. [3] have recursively found the low-$k$ finite-size ground state energy using only eq.(13) below and the order $k$-approximation of (11), but this has not lead to a closed formula for the polynomials. However, these polynomials have been derived by solving the two-dimensional chiral Potts model [3, 10], which for the "superintegrable" choice
of parameters contains the hamiltonian as a transfer matrix derivative. First, using an inversion relation, Baxter found one low-lying sector which for \( k \to 0 \) contains the ground state. Then, for the \( \mathbb{Z}_3 \)-case, using functional relations for the transfer matrix, Albertini, McCoy and Perk obtained also the excited level sectors which involve solutions of Bethe-equations. Finally, Baxter, exploiting functional relations derived from the relation of the 2-dimensional integrable chiral Potts model to the six-vertex model, solved the general \( \mathbb{Z}_N \)-superintegrable case. For all sectors of the spin representation turns out to be \( s = \frac{1}{2} \) or the trivial one. Accordingly, all eigenvalues of the hamiltonians have the form

\[
E^{(s)} = -\frac{N}{2} \left\{ a + b k + 4 \sum_{j=1}^{n} m_j \sqrt{1 + 2 k \cos \theta_j + k^2} \right\}; \quad m_j = \pm \frac{1}{2}. \tag{13}
\]

\( a \) and \( b \) are integers originating from the trivial representation.

In this talk, we will consider only the low-lying (no Bethe-excitations) sector polynomials of Baxter. Baxter’s polynomials are written in the variable \( t \) or \( s = t^N \) which is related to \( z = \cos \theta \) by

\[
\frac{z + z^{-1}}{2} = c = \frac{1 + s}{1 - s} \tag{14}
\]

The polynomials are (for details we refer to Baxter’s paper, for simplicity we put there \( P_b = 0 \)):

\[
P^{(L)}_Q(s) = \frac{1}{N} \sum_{j=0}^{N-1} \left( \frac{1 - t^N}{1 - \omega^j t} \right)^L (\omega^j t)^{-P_a}; \quad -P_a = Q + r + L \mod N. \tag{15}
\]

Here \( Q \) denotes the \( \mathbb{Z}_N \)-charge sector \( (Q = 0, 1, \ldots, N - 1) \), and \( r \) \((r = 0, 1, \ldots, N - 1)\) labels the boundary condition to defined by \( Z_{L+1} = \omega^r Z_1 \). Since in \( \mathbb{Z}_N \) \( Q \) and \( r \) appear only in the sum \( Q + r \), we can restrict us to consider the periodic case \( r = 0 \). The degree of the polynomials \( P^{(L)}_Q(s) \) in the variable \( s \) is

\[
b_{L,Q} = \left[ \frac{(N - 1)L - Q}{N} \right] \tag{16}
\]

where \([x]\) denotes the integer part of \( x \). Inversion of the variable \( s \) leads to simple relations:

\[
P^{(L)}_Q(s) = s^{b_{L,Q}} P^{(L)}_{N-(Q+L)}(s^{-1}), \tag{17}
\]

where the charge index \( N - (Q + L) \) is understood \( \mod N \). These relations correspond to the self-reverse property (3) of the interpolation polynomials.

Considering sequences of these polynomials for fixed \( Q \) and \( L \in \mathbb{N} \), from (16) we see that the dimensions \( b_{L,Q} \) do not always increase by one when increasing \( L \) by one: at every \( N \)th step the dimension stays the same: the dimensions of the \( P^{(L)}_Q \) for \( L + Q \mod N = 0 \) and \( L + Q \mod N = 1 \) coincide, see e.g. Table 1.

For \( N = 3 \) the definition, written explicitly, gives

\[
P^{(L)}_{Q+r}(s) = \frac{t^{-P_a}}{3} \left\{ (t^2 + t + 1)^L + \omega^{Q+r}(t^2 + \omega t + \omega)^L + \omega^{-(Q+r)}(t^2 + \omega t + \omega^2)^L \right\}. \tag{15}
\]

\(^1\) Superintegrable\(^2\): if integrable both due to Onsager’s algebra and a Yang-Baxter equation.

\(^2\) For reviews see e.g. [16, 17].
3 \( \mathbb{Z}_2 \): The Ising case

In (14) we have seen three different variables in which to write the polynomials: \( z, c \) and \( s \). Since for (13) we only need the zeros \( z_j \) or \( c_j \) or \( s_j \), the choice which polynomials we should prefer can be decided by convenience. The range in which the zeros appear is dictated by the hermiticity of \( \mathcal{H}^{(s)} \) which requires

\[
-1 < c_j < +1 \quad \text{or} \quad -\infty < s_j < 0 \quad \text{or} \quad |z_j| = 1.
\]  

(18)

Let us warm up by considering the Ising case \( N = 2 \) for which there are two charge sectors \( Q = 0, 1 \) and we have \( s = t^2 \). From (15) Baxter’s polynomials are

\[
P^{(L)}_Q (s = t^2) = \frac{t^{-a_{L,Q}}}{2} \left\{ (t + 1)^L + (-1)^Q(t - 1)^L \right\}; \quad a_{L,Q} = L + Q \mod 2.
\]  

(19)

Their zeros are well known [12]:

\[
s_j = -\tan^2 \left\{ \frac{\pi}{L} \left( j - \frac{1 - Q}{2} \right) \right\}.
\]

One easily derives (see e.g. the next section) that for \( L \) even these polynomials satisfy the differential equation (DE)

\[
4s(s-1) \frac{d^2}{ds^2} P^{(L)}_Q - 2f_Q \frac{d}{ds} P^{(L)}_Q + g_Q P^{(L)}_Q = 0.
\]

with

\[
f_0 = s(2L - 3) + 1; \quad g_0 = L(L - 1);
\]

\[
f_1 = s(2L - 5) + 3; \quad g_1 = (L - 1)(L - 2)
\]

(20)

(for \( L \) odd interchange \( f_0 \leftrightarrow f_1 \) and \( g_0 \leftrightarrow g_1 \)). These are hypergeometric DEs with the polynomial solutions

\[
P^{(L)}_Q(s) \sim _2F_1 \left( \frac{Q+1-L}{2}, \frac{Q-L}{2}; Q + \frac{1}{2}; s \right) \quad \text{for} \ L \ \text{even},
\]

\[
P^{(L)}_Q(s) \sim _2F_1 \left( 1 - \frac{Q+L}{2}, \frac{1}{2} - \frac{Q+L}{2}; \frac{3}{2} - Q; s \right) \quad \text{for} \ L \ \text{odd}.
\]

How does the same information appear in terms of the variable \( c \) (it is the \( c_j \) which are directly required in (13))? We define polynomials \( \Pi^{(L)}_Q(c) \) by

\[
\Pi^{(L)}_Q(c) = (c + 1)^{b_{L,Q}} P^{(L)}_Q \left( s = \frac{c - 1}{c + 1} \right),
\]

(21)

where \( b_{L,Q} \) was given in (16). After some algebra, the DE (20) can be rewritten in terms of \( \Pi^{(L)}_Q(c) \) and we find

\[
\frac{d}{dc} \left( \left(1 - c^2\right)^Q + \frac{1}{2} \frac{d\Pi^{(L)}_Q}{dc} \right) + \frac{(L^2 - 4Q)(1 - c^2)^{Q-\frac{1}{2}}}{4} \Pi^{(L)}_Q = 0.
\]  

(22)
For higher \( N \) Baxter polynomials \( P \), to derive recursion relations and differential equations it proves convenient to start with the occasionally we will quote the result for general \( N \)

\[
\Pi_0^{(2k)}(c) = 2^k T_k(c); \quad \Pi_1^{(2k)}(c) = 2^k U_{k-1}(c); \quad \Pi_0^{(2k+1)}(c) = 2^k (U_k(c) \pm U_{k-1}(c)),
\]

for \( k = 0, 1, 2, \ldots \) and we use

\[
T_n(x) = \cos(n \arccos x); \quad U_n(x) = \sin((n + 1) \arccos x)/\sqrt{1 - x^2}.
\]

As classical orthogonal polynomials the Chebyshev polynomials satisfy three-term pure recursion relations. In terms of the \( \Pi^{(L)}_Q \) these become:

\[
\Pi^{(L+4)}_Q - 4c \Pi^{(L+2)}_Q + 4 \Pi^{(L)}_Q = 0. \tag{23}
\]

A main concern later in this talk will be to investigate whether for \( N \geq 3 \) the polynomials \( \Pi^{(L)}_Q(c) \) still form orthogonal sequences or at least preserve some features of the Chebyshev polynomials.

We also consider briefly eqs. (19) rewritten in terms of the variable \( z \). The interpolation polynomials \( F^{(L)}_Q(z) \) are related to the \( P^{(L)}_Q(s) \) by

\[
F^{(L)}_Q(z) = (z + 1)^{2b_L} P^{(L)}_Q \left( s = \left( \frac{z - 1}{z + 1} \right)^2 \right) \tag{24}
\]

and for \( N = 2 \) we get

\[
L \text{ even} : \quad 2^{1-L} F^{(L)}_0 = z^L + 1; \quad 2^{1-L} F^{(L)}_1 = \frac{z^L - 1}{z^2 - 1}; \quad L \text{ odd} : \quad 2^{1-L} F^{(L)}_0 = \frac{z^L \mp 1}{z \mp 1}.
\]

For \( L \) prime these are standard cyclotomic polynomials \( Q_k(z) \), in some other cases products of \( Q_k(z) \), e.g. \( F_1^{12}(z) = 2^{11} Q_3 Q_4 Q_6 Q_{12}; \quad F_0^{13}(z) = 2^{12} Q_{13}; \quad F_1^{13}(z) = 2^{12} Q_{26} \). The corresponding \( \mathbb{Z}_3 \) interpolating polynomials \( F^{(L)}_Q(z) \) are more complicated and for them we have not yet discovered particular interesting features.

4 **\( \mathbb{Z}_3 \)-Polynomials** \( P^{(L)}_Q(s) \)

To derive recursion relations and differential equations it proves convenient to start with the Baxter polynomials \( P^{(L)}_Q(s) \). We shall describe the details of the derivations for the case \( \mathbb{Z}_3 \).

For higher \( N \) the same methods can be used, just the formulae become more involved. Only occasionally we will quote the result for general \( N \).

4.1 **Recursion relations**

Our starting point is, that equivalent to Baxter’s definition \( [13] \), we can define the \( P^{(L)}_Q(s) \) via the expansion of the multinomial \( (t^{N-1} + t^{N-2} + \ldots + t + 1)^L \) and collecting the terms with powers of \( t^a \) for each \( a \mod N = 0, 1, \ldots, N - 1 \). So, for \( N = 3 \) we write

\[
(t^2 + t + 1)^L = t^{a_{L,0}} P^{(L)}_0(s) + t^{a_{L,1}} P^{(L)}_1(s) + t^{a_{L,2}} P^{(L)}_2(s), \tag{25}
\]
with \( a_{L,Q} = 2(L + Q) \mod 3 \), and the \( \psi^{(L)}_Q \) are required to depend on \( s = t^3 \) only. From (25) we get immediately

\[
(t^2 + t + 1)^{L+1} = (t^2 + t + 1) \left( t^{a_{L,0}} \psi^{(L)}_0(s) + t^{a_{L,1}} \psi^{(L)}_1(s) + t^{a_{L,2}} \psi^{(L)}_2(s) \right)
\]

\[
= t^{a_{L,0}} \psi^{(L+1)}_0(s) + t^{a_{L,1}} \psi^{(L+1)}_1(s) + t^{a_{L,2}} \psi^{(L+1)}_2(s).
\]  

(26)

Collecting the coefficients of mod 3-powers of \( t \), and choosing \( L \) such that \( L \mod 3 = 0 \), this gives

\[
\begin{pmatrix}
\psi^{(L+1)}_0 \\
\psi^{(L+1)}_1 \\
\psi^{(L+1)}_2
\end{pmatrix} = \begin{pmatrix}
1 & 1 & 1 \\
1 & s & 1 \\
1 & s & s
\end{pmatrix} \begin{pmatrix}
\psi^{(L)}_0 \\
\psi^{(L)}_1 \\
\psi^{(L)}_2
\end{pmatrix}, \quad \begin{pmatrix}
\psi^{(L+2)}_0 \\
\psi^{(L+2)}_1 \\
\psi^{(L+2)}_2
\end{pmatrix} = \begin{pmatrix}
1 & 1 & 1 \\
1 & s & 1 \\
1 & s & s
\end{pmatrix} \begin{pmatrix}
\psi^{(L+1)}_0 \\
\psi^{(L+1)}_1 \\
\psi^{(L+1)}_2
\end{pmatrix}.
\]  

(27)

One rotates the column vectors once more to get the \( \psi^{(L+3)}_Q \) in terms of the \( \psi^{(L+2)}_Q \).

Still taking \( L \mod 3 = 0 \), by repeated application of (27) etc. we obtain 4-term pure recursion relations between polynomials of the same \( Q \), e.g.:

\[
\begin{align*}
\psi^{(L+3)}_0 &= -3s \psi^{(L+2)}_0 + 3s(s-1) \psi^{(L+1)}_0 + (s-1)^2 \psi^{(L)}_0 = 0 \\
\psi^{(L+3)}_1 &= -3s \psi^{(L+2)}_1 + 3s(s-1) \psi^{(L+1)}_1 + (s-1)^2 \psi^{(L)}_1 = 0 \\
\psi^{(L+3)}_2 &= -3s \psi^{(L+2)}_2 + 3s(s-1) \psi^{(L+1)}_2 + (s-1)^2 \psi^{(L)}_2 = 0 \\
\psi^{(L+4)}_Q &= (2s+1) \psi^{(L+3)}_Q + 2(s-1)^2 \psi^{(L+1)}_Q + (s-1)^3 \psi^{(L)}_Q = 0 \quad \text{for } Q = 1, 2 \\
sp^{(L+4)}_Q &= (2s+1) \psi^{(L+3)}_Q + 2s(s-1)^2 \psi^{(L+1)}_Q + (s-1)^3 \psi^{(L)}_Q = 0.
\end{align*}
\]  

(28)

In each except the fourth equations pairs of polynomials appear which have the same degree.

The following recursion is valid for all \( L \geq 0 \) and \( Q \):

\[
\psi^{(L+9)}_Q - 3((s-1)^2 + 9s) \psi^{(L+6)}_Q + 3(s-1)^4 \psi^{(L+3)}_Q - (s-1)^6 \psi^{(L)}_Q = 0.
\]  

(29)

### 4.2 Differential equations

Differentiating (23), for \( L \mod 3 = 0 \) we get:

\[
(2t + 1)L \left( \psi^{(L)}_0 + t^2 \psi^{(L)}_1 + t \psi^{(L)}_2 \right)
\]

\[
= (t^2 + t + 1) \left\{ 3t^2 \left( \psi^{(L)}_0' + t^2 \psi^{(L)}_1' + t \psi^{(L)}_2' \right) + 2t \psi^{(L)}_1 + \psi^{(L)}_2 \right\}
\]

where \( \psi^{(L)}_Q' \equiv d\psi^{(L)}_Q / ds \), and comparing the coefficients of mod 3-powers of \( t \):

\[
3s(s-1) \begin{pmatrix}
\psi^{(L)}_0' \\
\psi^{(L)}_1' \\
\psi^{(L)}_2'
\end{pmatrix} = \mathcal{B} \begin{pmatrix}
\psi^{(L)}_0 \\
\psi^{(L)}_1 \\
\psi^{(L)}_2
\end{pmatrix}, \quad \mathcal{B} = \begin{pmatrix}
2sL & -sL & -sL \\
-L & 2sL - 2(s-1) & -L \\
-L & -sL & 2sL - (s-1)
\end{pmatrix}.
\]  

(30)

Writing (23) shorthand as \( 3s(s-1)\mathcal{P}' = \mathcal{B} \mathcal{P} \), and differentiating again we have

\[
9s^2(s-1)^2 \mathcal{P}'' = \left\{ \mathcal{B}^2 - 3(2s-1)\mathcal{B} + 3s(s-1)\mathcal{B}' \right\} \mathcal{P}
\]

\[
27s^3(s-1)^3 \mathcal{P}'' = \left\{ \mathcal{B}^3 - 9(2s-1)\mathcal{B}^2 + 18(3s(s-1) + 1)\mathcal{B} + 3s(s-1)(2\mathcal{B}' \mathcal{B} + \mathcal{B} \mathcal{B}') - 18s(s-1)(2s-1)\mathcal{B}' \right\} \mathcal{P}
\]  

(31)
We can use these relations to obtain *decoupled* DEs for the three charge sectors as follows: We form the three expressions ($Q = 0, 1, 2$)

$$27s^2(s - 1)^2P_Q^{(L)'''} - 27s(s - 1)f_Q P_Q^{(L)''} + 3g_Q P_Q^{(L)'} - (L - 1)h_Q P_Q^{(L)} = 0. \quad (32)$$

which contain 9 coefficients $f_Q$, $g_Q$ and $h_Q$. Then using (31) we express the $P_Q^{(L)'''}$, $P_Q^{(L)''}$ and $P_Q^{(L)'}$ in terms of $P_Q^{(L)}$ which gives a matrix equation $\mathcal{MP} = 0$. Requiring $\mathcal{M}$ to be diagonal gives 9 linear equations for the coefficients $f_Q$, $g_Q$, $h_Q$. Solving these, the result for $L \mod 3 = 0$ is

$$f_Q = s(2L - 4 - \tilde{Q}) + 2 + \tilde{Q}; \quad \tilde{Q} = 3 - Q \mod 3;$$
$$g_Q = 3s(4s - 1)L^2 - 3sL \left\{ 4(s - 1)\tilde{Q} + 10s - 7 \right\} + (s - 1) \left\{ 3\tilde{Q}^2(s - 1) + 3\tilde{Q}(5s - 1) + 20s - 2 \right\};$$
$$h_0 = (8s + 1)L - 4(s - 1)L; \quad h_1 = (8s + 1)L - 12(s - 1)L - 2;$$
$$h_2 = (8s + 1)L^2 - (16s - 7)L + 6(s - 1). \quad (33)$$

For the other values of $L \mod 3$ we get the same expressions, just for rotated values of $Q$. For $L \mod 3 = 1$, $Q = 0$ and $L \mod 3 = 2$, $Q = 2$. These DE’s are not anti-selfadjoint as one would like them to be.

For higher $N$ the derivations are completely analogous, just quite lengthy. One obtains $N$th-order DE’s for the $\mathbb{Z}_N$-polynomials. For $L \mod N = 0$ the relation generalizing (30) is

$$Ns(s - 1)\frac{dP}{ds} = \mathcal{B}^{(N)}P$$

$$\mathcal{B}^{(N)} = \begin{pmatrix}
\mathcal{N} & -sL & -sL & -sL & \cdots \\
-L & \mathcal{N} - (N-1)(s-1) & -L & -L & \cdots \\
-L & -sL & \mathcal{N} - (N-2)(s-1) & -L & \cdots \\
-L & -sL & -sL & \mathcal{N} - (N-3)(s-1) & \cdots \\
\cdots & \cdots & \cdots & \cdots & \cdots
\end{pmatrix} \quad (34)$$

using the abbreviation $\mathcal{N} = (N - 1)sL = Ns b_{L,0}$ with $b_{L,Q}$ being defined in (10).

### 4.3 Approximation of the zeros

In [23] it was shown that changing the variable $t$ into $\beta$ according to

$$t = \frac{\sin (\beta - \frac{K\pi}{3})}{\sin (\beta + \frac{K\pi}{3})}, \quad (35)$$

the zeros of the polynomials come approximately equidistant in the interval $-\frac{\pi}{N} < \beta < \frac{\pi}{N}$. Without approximations, for $\mathbb{Z}_3$ the equation $\mathcal{P}_{L,Q}(t) = 0$ becomes

$$(2 \cos \beta)^{-L} = 2 \cos \left( L \left\{ \beta + \frac{\pi}{3} \right\} + \frac{2Q\pi}{3} \right). \quad (36)$$

Neglecting the left-hand side of (36) for $L \gg 1$ (this is exponentially good for $|\beta| \ll \frac{\pi}{3}$ and dangerous only for $2 \cos \beta \approx 1$, i.e. $\beta \approx \pm \frac{\pi}{3}$) we get the approximate solutions

$$\beta_k \approx K - \frac{\pi}{3}; \quad K = \frac{6k + 2Q - 3}{6L} \frac{\pi}{3}; \quad k = 1, 2, \ldots, b_{L,Q}. \quad (37)$$
Table 1: \(\mathbb{Z}_3\)-Polynomials \(\Pi^{(L)}_Q(c)\) for \(L \leq 8\).

| \(L\) | \(\Pi^{(L)}_0\) | \(\Pi^{(L)}_1\) | \(\Pi^{(L)}_2\) |
|---|---|---|---|
| 1 | 1 | 1 | 1 |
| 2 | \(3c + 1\) | \(3c - 1\) | 3 |
| 3 | \(9c^2 - 5\) | \(9c + 3\) | \(9c - 3\) |
| 4 | \(27c^2 + 18c - 5\) | \(27c^2 - 11\) | \(27c^2 - 18c - 5\) |
| 5 | \(81c^3 + 27c^2 - 57c - 11\) | \(81c^3 - 27c^2 - 57c + 11\) | \(81c^2 - 21\) |
| 6 | \(243c^4 - 270c^2 + 43\) | \(243c^3 + 81c^2 - 135c - 21\) | \(243c^3 - 81c^2 - 135c + 21\) |
| 7 | \(3^6c^4 + 486c^3 - 549c^2 - 270c + 43\) | \(3^6c^4 - 702c^3 + 85\) | \(3^6c^4 - 486c^3 - 549c^2 + 270c + 43\) |
| 8 | \(3^7c^5 + 3^6c^4 - 2754c^3 - 702c^2\) | \(3^7c^5 - 3^6c^4 - 2754c^3 + 702c^2\) | \(3^7c^4 - 1782c^3 + 171\) |

\[
+711c + 85
\]

or

\[
c_k = \frac{1 + s_k}{1 - s_k} \approx -\frac{\sin^3(K + \frac{\pi}{3}) - \sin^3K}{\sin^3(K + \frac{\pi}{3}) + \sin^3K}.
\]  

For \(N = 2\) the analogous approach leads to exact results.

Here we shall not further discuss this useful trigonometric mapping of the zeros to the interval \(-\pi < \beta < \pi\). Rather we focus our attention to the rational mapping from \(-\infty < s < 0\) to \(-1 < c < 1\) and study whether traces of the orthogonality present in the \(N = 2\) case survive for \(N \geq 3\).

5 The polynomials \(\Pi^{(L)}_Q(c)\)

The general definition of the \(\Pi^{(L)}_Q(c)\) has already been given in (21). For illustration Table 1 lists the \(\Pi^{(L)}_Q\) for \(L \leq 8\). The inversion relations (17) simply imply that

\[
\Pi^{(L)}_Q(c) = (-1)^{b_{L,Q}} \Pi^{(L)}_{N - Q - L}(-c),
\]

from which it follows that for \(L + 2Q \mod N = 0\) the polynomials \(\Pi^{(L)}_Q\) for \(N\) odd are functions of \(c^2\) only.

5.1 Recursion relations

We first derive recursion relations for the case \(\mathbb{Z}_3\). We rewrite the recursive relations obtained for the \(P^{(L)}_Q(s), (27)\) in terms of the polynomials \(\Pi^{(L)}_Q(c)\), with the result (taking \(L \mod 3 = 0\)):

\[
\begin{pmatrix}
\Pi^{(L+i+1)}_0 \\
\Pi^{(L+i+1)}_1 \\
\Pi^{(L+i+1)}_2
\end{pmatrix}
= C(i)
\begin{pmatrix}
\Pi^{(L+i)}_0 \\
\Pi^{(L+i)}_1 \\
\Pi^{(L+i)}_2
\end{pmatrix};
\]
\[
\begin{align*}
C^{(0)} &= \begin{pmatrix} 1 & c_+ & c_+ \\ 1 & c_- & c_+ \\ 1 & c_- & c_- \end{pmatrix}; \quad C^{(1)} = \begin{pmatrix} c_- & c_+ & c_+ \\ c_- & c_- & c_+ \\ 1 & 1 & 1 \end{pmatrix}; \quad C^{(2)} = \begin{pmatrix} c_- & c_+ & c_-c_- \\ 1 & 1 & 1 \end{pmatrix}. \quad (40)
\end{align*}
\]

with \(c_\pm = c \pm 1\). Several interesting relations follow immediately from the particular structure of the matrices \(C^{(i)}\): Still taking \(L\) such that \(L \mod 3 = 0\), we have

\[
\begin{align*}
\Pi_0^{(L+1)} - \Pi_2^{(L+1)} &= 2(\Pi_2^{(L)} + \Pi_2^{(L)}); \quad \Pi_0^{(L+1)} - \Pi_1^{(L+1)} = 2\Pi_1^{(L)} \\
\Pi_0^{(L+2)} - \Pi_1^{(L+2)} &= 2\Pi_1^{(L+1)}; \quad \Pi_1^{(L+3)} - \Pi_2^{(L+3)} = 2\Pi_2^{(L+2)}, \quad (41)
\end{align*}
\]

which shows that the 9 polynomials of the subset with \(L \mod 3 = 0, 1, 2\) and \(Q = 0, 1, 2\) are not independent. There are also derivative relations between these polynomials, see later eqs. (48), (49).

The relations \((\Pi)\) mix the three charge sectors. In order to find recursion relations which do not mix the charge sectors \(Q\), we form the matrices \(C\), \(C^2\), \(C^3\):

\[
C \equiv C^{(2)}C^{(1)}C^{(0)} = \begin{pmatrix} \Pi_0^{(3)} & c_+c_-\Pi_2^{(3)} & c_+c_-\Pi_2^{(3)} \\ \Pi_1^{(3)} & \Pi_0^{(3)} & c_+\Pi_2^{(3)} \\ \Pi_2^{(3)} & c_-\Pi_1^{(3)} & \Pi_0^{(3)} \end{pmatrix}; \quad \Pi_0^{(3)} = 9c^2 - 5; \quad \Pi_1^{(3)} = 3c + 1; \quad \Pi_2^{(3)} = 3c - 1.
\]

\[
C^2 = \begin{pmatrix} \alpha_2 & c_+\beta c_+-\beta_2+ & c_+\beta c_+-\beta_2- \\ \beta_2- & \alpha_2 & c_+\beta c_+-\beta_2+ \\ \beta_2+ & c_-\beta c_+-\beta_2- & \alpha_2 \end{pmatrix}; \quad \alpha_2 = 3^5c^4 - 270c^2 + 43; \quad \beta_2\pm = 3^5c^3 + 3\cdot 4^2 - 135c \pm 21
\]

\[
C^3 = \begin{pmatrix} \alpha_3 & c_+\beta c_+-\beta_3+ & c_+\beta c_+-\beta_3- \\ \beta_3- & \alpha_3 & c_+\beta c_+-\beta_3+ \\ \beta_3+ & c_-\beta c_+-\beta_3- & \alpha_3 \end{pmatrix}; \quad \alpha_3 = 3^8c^6 - 3^7\cdot 5c^4 + 3^4\cdot 59c^2 - 341; \quad \beta_3\pm = 9(3^6c^5 + 3^5c^4 - 3^4\cdot 10c^3 + 3^2\cdot 22c^2 + 177c \mp 19)
\]

The matrix equation \(C^3 + fC^2 + gC + hI = 0\) contains three independent equations and has the solution \(f = -27c^2 + 15 = -3\Pi_0^{(3)}; \quad g = 48; \quad h = -64\), corresponding to the pure recursion relation (valid for all \(L \geq 0\) and all \(Q = 0, 1, 2\))

\[
\Pi_Q^{(L+9)} - (27c^2 - 15)\Pi_Q^{(L+6)} + 48\Pi_Q^{(L+3)} - 64\Pi_Q^{(L)} = 0. \quad (42)
\]

The degrees of the polynomials appearing in this relation increase by two from the right to the left. If we consider \(Q = L\) mod 3, then we have only polynomials in \(c^2\), so that the degrees appearing are consecutive in powers of \(z = 9c^2\) ("simple sets of polynomials") with integer coefficients. These recursion relations are of simplest type (like those for the Chebyshev polynomials), with the coefficients not depending on \(L\). Compare also the same equation \((29)\) for the \(P_Q^{(L)}\).

In order to form simple sets out of the \(\Pi_Q^{(L)} (c)\) with the same \(Q\) having the degree in \(c^2\) increasing in steps of one without repetitions, every third \(L\) must be left out. E.g. for \(Q = 0\), we can form the sequences \(L = 0, 2, 3, 5, 6, 8, 9, \ldots\) or \(L = 1, 2, 4, 5, 7, 8, \ldots\), i.e. leaving out either all \(L\) mod 3 = 1 or all \(L\) mod 3 = 0. Analogous sequences can be formed for \(Q = 1\) and for \(Q = 2\).

\[\text{If we define } \Pi_0^{(c)} = \frac{1}{3}; \quad \Pi_2^{(c)} = 0; \quad \Pi_0^{(-c)} = \frac{1}{2}; \quad \Pi_0^{(-c)} = -\frac{1}{2}, \quad \text{the relations (43) are valid also for } L = -1, -2.\]
Using again (43), and solving similar matrix equations, we find the following 4-term pure recursion relations: For all \( L \geq 0 \) with \( L \mod 3 = 0 \) we get

\[
\Pi_{Q}^{(L+4j)} + (1 - 3c) \Pi_{Q}^{(L+3j)} + 8 \Pi_{Q}^{(L+1+j)} + 8 \Pi_{Q}^{(L+j)} = 0; \\
\text{for } j = 0, 1, 2 \text{ with } Q = 4 - j, 2 - j \mod 3; \tag{43}
\]

\[
\Pi_{Q}^{(L+5j)} + (1 - 9c) \Pi_{Q}^{(L+3j)} + (8 - 12c) \Pi_{Q}^{(L+2+j)} + 8 \Pi_{Q}^{(L+j)} = 0; \\
\text{for } j = 0, 1, 2 \text{ with } Q = 3 - j, 2 - j \mod 3. \tag{44}
\]

Classical orthogonal polynomials must have 3-term recursion relations (see e.g. [20]). The separation of the polynomials \( P_{Q}^{(L)} \) for a fixed \( Q \) into two sets: the ones selected in (43,44) and the ones left out, reminds of Konhauser biorthogonal polynomials [24] (KBOP). KBOPs have 4-term pure recursion relations if the first set consists of the polynomials in \( c \) and second set of polynomials in \( c^2 \). In some cases of (43) this is what we just did, (the \( \Pi_{Q}^{(L+Q)} \) are functions of \( c^2 \)), but not in all cases: for \( j = 2 \) in (43) \( \Pi_{Q}^{(L+1)} \) are left out and these depend on \( c \).

We can apply the same technique to find recursion relations also for higher \( Z_N \) polynomials. Generally, we find \( N + 1 \)-term pure recursion relations. For \( Z_4 \) the result analogous to (42) is

\[
\Pi_{Q}^{(L+16)} - 4 \Pi_{Q}^{(4)} \Pi_{Q}^{(L+12)} - 128(14c^2 - 17)\Pi_{Q}^{(L+8)} - 2048c \Pi_{Q}^{(L+4)} + 4096 \Pi_{Q}^{(L)} = 0. \tag{45}
\]

with \( \Pi_{Q}^{(4)} = 64c^3 - 56c \). For \( Z_5 \):\n
\[
\Pi_{Q}^{(L+25)} - 5 \Pi_{Q}^{(5)} \Pi_{Q}^{(L+20)} + 2a_2 \Pi_{Q}^{(L+15)} + a_3 \Pi_{Q}^{(L+10)} + a_4 \Pi_{Q}^{(L+5)} + a_5 \Pi_{Q}^{(L)} = 0. \tag{46}
\]

with \(-5\Pi_{Q}^{(5)} = -3125c^4 + 3750c^2 - 705;\) \(a_2 = -2^6 \cdot 5(375c^2 - 383)\); \(a_3 = -2^{10}(5c^2 - 45 \cdot 13);\) \(a_4 = 2^{16} \cdot 5;\) \(a_5 = -2^{20}\).

There are also \( N + 1 \)-term recursion relations analogous to (43), (44) for sequences formed of \( \Pi_{Q}^{(L)} (c) \) with consecutive degrees in \( c \). Generally, for this, certain every \( N \)th \( L \) must be left out. If these were KBOPs, for having \( N + 1 \)-term pure recursions, there should be subsets of polynomials depending on \( c^N \). However, this is not the case: for any \( N \) there are only subsets depending on \( c^2 \) because of (33).

One might think that orthogonal polynomials which usefully approximate the \( \Pi_{Q}^{(L)} \) could be obtained recursively, neglecting the last term in (42). Using monic polynomials \( \tilde{\Pi}_{Q}^{(L)} \), this looks promising because eq.(42) becomes

\[
\tilde{\Pi}_{Q}^{(L+9)} = \left(c^2 - \frac{5}{9}\right) \tilde{\Pi}_{Q}^{(L+6)} - \frac{16}{243} \tilde{\Pi}_{Q}^{(L+3)} + \frac{64}{19683} \tilde{\Pi}_{Q}^{(L)}.
\]

with the last coefficient looking small. However, new polynomials \( \tilde{\Pi}_0^{(L)} \) defined through (considering \( L \mod 3 = 0, Q = 0 \)):

\[
\tilde{\Pi}_0^{(L+6)} = \left(c^2 - \frac{5}{9}\right) \tilde{\Pi}_0^{(L+3)} - \frac{16}{243} \tilde{\Pi}_0^{(L)}
\]

with \( \tilde{\Pi}_0^{(0)} = 3; \) \( \tilde{\Pi}_0^{(3)} = c^2 - \frac{5}{9} \) don’t seem to be useful, since the \( \tilde{\Pi}_0^{(L)} \) have zeros which for increasing \( L \) spread out of the interval \(-1 < c < +1\), violating the hermiticity of the hamiltonian. The method using eq.(36) does not suffer from this deficiency.
5.2 Differential equations for the $\Pi_{Q}^{(L)}(c)$

The derivative relation (34) can be rewritten for the $\Pi_{Q}^{(L)}$, giving e.g. for $L \mod N = 0$:

$$
\begin{pmatrix}
\Pi_0' \\
c_+ \Pi_1' \\
c_+ \Pi_2' \\
\ldots
\end{pmatrix} = \frac{1}{Nc_+c_-} \begin{pmatrix}
0 & Lc_- & Lc_- & Lc_- & \ldots \\
Lc_+ & -Nc_+ + 2 & Lc_+ & Lc_+ & \ldots \\
Lc_+ & Lc_- & -Nc_+ + 4 & Lc_+ & \ldots \\
\ldots & \ldots & \ldots & \ldots & \ldots
\end{pmatrix}
\begin{pmatrix}
\Pi_0 \\
c_+ \Pi_1 \\
c_+ \Pi_2 \\
\ldots
\end{pmatrix}
$$

(47)

where the primes denote differentiation $d/dc$ and we omitted the superscripts $(L)$ on the $\Pi_{Q}$. The first component of this equation gives the nice simple relation

$$
\Pi_0' = \frac{L}{N} \sum_{Q=1}^{N-1} \Pi_{Q}.
$$

(48)

We now concentrate on the case $N = 3$. From relations analogous to (17) we get the following simple derivative relations:

$$
3c_- \Pi_0' = L(\Pi_1 + \Pi_2) - 2\Pi_0; \quad 3c_+ \Pi_2' = L(\Pi_0 + \Pi_1) - 2\Pi_2
$$

$$
3(c^2 - 1)\Pi_1' = L(c_+ \Pi_2 + c_- \Pi_0) - 2c\Pi_1.
$$

(49)

Using these and similar relations, after considerable algebra, we get the DE

$$
27(c^2 - 1) \left\{ (c^2 - 1) \frac{d^3 \Pi_{Q}^{(L)}}{dc^3} + \alpha(c) \frac{d^2 \Pi_{Q}^{(L)}}{dc^2} \right\} - 3 \left\{ 3(c^2 - 1)L(L + 1) + \beta(c) \right\} \frac{d\Pi_{Q}^{(L)}}{dc}
$$

$$
+ \gamma(L, c) \Pi_{Q}^{(L)} = 0.
$$

(50)

We collect the coefficients in Table 2. Observe that these DE for the $\Pi_{Q}^{(L)}(c)$ are much simpler

| $L \mod 3$ | $Q$ | $\alpha(c)$ | $\beta(c)$ | $\gamma(L, c)$ |
|-----------|-----|-------------|-------------|----------------|
| 0         | 0   | 4c          | -18c^2 + 10 | -2cL^2(L + 3)  |
| 0         | 1/2 | 7c ± 1      | -90c^2 ± 24c + 34 | -2cL^2(L + 3) - 3(3c ± 1)(L(L + 1) - 6) |
| 1         | 1   | 6c          | -60c^2 + 28 | -2c \{L^2(L + 6) + 3L - 10\} |
| 1         | 2/0 | 6c ± 2      | -60c^2 ± 36c + 16 | -2c \{L^2(L + 6) + 3L - 10\} ± 6(L(L + 1) - 2) |
| 2         | 2   | 8c          | -126c^2 + 46 | -2c \{L^2(L + 9) + 6L - 56\} |
| 2         | 0/1 | 5c ± 1      | -36c^2 ± 12c + 16 | -c \{L^2(2L + 9) + 3L - 4\} ± 3L(L + 1) |

than the corresponding eqs.(12) for the $P_{Q}^{(L)}(s)$: there the coefficients $f_{Q}$ of the second derivatives $P_{Q}^{(L)^{\prime\prime}}(33)$ were $L$-dependent, which here is not the case. However, although in (50) the third and second derivative terms can be combined into a total derivative, we don’t see how to put the whole DE into an anti-selfconjugate form.
5.3 Separation property of the zeros.

We want to collect further evidence that the polynomials $\Pi^{(L)}_Q$ violate features which are crucial for classical orthogonal polynomials. Table 3 shows the zeros of the polynomials $\Pi^{(L)}_Q$ for low $L$. Fully satisfied is the confinement of the zeros $c_j$ to the interior of the basic interval $-1 < c_j < +1$ (this must be so because of the hermiticity of the hamiltonian). However, the separation property of the zeros of polynomials sequences successive in $c$ can not be satisfied: If we take out the $L = 3, 6, 9, \ldots$-polynomials as we did in (43) for $j = 1$ with $Q = 4 - j$ then the zeros $.78444; .77622$ for $L = 5, 7$ are in wrong order. Alternatively, taking out $L = 4, 7, 11, \ldots$ as in (44) for $j = 2$ with $Q = 2 - j$ then $95835; .93330$ for $L = 6, 8$ are in wrong order too. Considering the polynomials in $c^2$ only ($Q = 0$ and $L = 3, 6, 9, \ldots$ as in (42)) gives the correct separation of the $c_j^2$.

Table 3: Zeros $c_j$ of the $Z_3$-polynoms $\Pi^{(L)}_0(c)$ showing the separation property violation at the upper corner near $c = 1$.

| $L$  | $c_j$ ($j = 1, \ldots, [2L/3]$)         |
|------|----------------------------------------|
| 2    | -.33333                                |
| 3    | -.74536 .74536                         |
| 4    | -.87766 .21100                         |
| 5    | -.93203 -.18575 .78444                 |
| 6    | -.95835 -.43894 .43894 .95835          |
| 7    | -.97264 -.60038 .13013 .77622          |
| 8    | -.98106 -.70632 -.11267 .53341 .93330 |
| 9    | -.98634 -.77818 -.29702 .29702 .77818 | .98634 |
| 10   | -.98983 -.82848 -.53636 .09047 .58910 | .90843 |
| 11   | -.99222 -.86467 -.54241 -.08238 .39948 | .77767 | .97120 |
| 12   | -.99392 -.89136 -.62401 -.22459 .22459 | .62401 | .89136 | .99392 |

5.4 The $\Pi^{(L)}_Q$ expressed in terms of Jacobi polynomials: Partial orthogonality

In order to find out whether the $\Pi^{(L)}_Q(c)$ satisfy some generalized kind of orthogonality, we looked for the eventual weight functions. The definition interval is certainly $-1 < c < +1$ and as a simple guess one may try Jacobi weights

$$w_{\alpha, \beta}(c) = (1 - c)^{\alpha}(1 + c)^{\beta}.$$ 

Since for $Z_3$ the $\Pi^{(L)}_Q$ are Chebyshev polynomials, i.e. there $\alpha = \beta = \pm 1/N$, we tried whether e.g. the $Z_3$ polynomials are related to $\alpha, \beta = \pm 1/3, \pm 2/3$-Jacobi polynomials. Indeed, expanding the $\Pi^{(L)}_Q(c)$ in terms of Jacobi polynomials $P^{(\alpha, \beta)}_n(c)$ and playing around with several choices of $a$ and $b$ in $\alpha = a/N$ and $\beta = b/N$ we find remarkable features which we report now.
by inspection of Table 4 for $N=3$, we conjecture the "partial orthogonality"

$$(\Pi_{Q}^{(3k)}, \Pi_{Q'}^{(3k')})_{L/4} = 0 \quad \text{for} \quad 2k + 1 \leq k'.$$

because the expansion vector of $\Pi_{0}^{3k}$ has $2k + 1$ components and for $k \leq 35$ we have checked that its $k$ lowest components are zero. Interestingly, the first non-zero components are very small.
relations are not at all simple if expressed in terms of the $\Pi$ terms of Chebyshev polynomials $U \mod 4 = 0$) or, replacing $L$ by $T$, an analogously for $T$. Table 5 lists results for $L \leq 13$. Observe the remarkable symmetry between pairs of sectors obtained by replacing $T_k$ by $U_{k-1}$ (e.g. for $\Pi^{(L+1)}_{1+1}$ and $\Pi^{(L+2)}_{0+2}$ taking $L \mod 4 = 0$) or, replacing $T_k$ by $U_k$ for $\Pi^{(L)}_{0+1}$ and $\Pi^{(L+1)}_{1+3}$ (taking still $L \mod 4 = 0$). These relations are not at all simple if expressed in terms of the $\Pi^{(L)}_Q(c)$.

and increase strongly before reaching a plateau level around the $kth$ component: E.g. for $\Pi^0_{105}$ we have $a_0 = \ldots = a_{34} = 0; a_{35} = -0.0007395; a_{38} = 14.6665; a_{52} = -3.186 \cdot 10^3$. It is a simple consequence of the reflection relation (33) that for fixed $L$ always one pair of $Z_3$ charge sectors differs only by the interchange of $\alpha$ and $\beta$.

Several other partial orthogonality relations can be inferred from other vanishing first components in Table 4.

For the $Z_4$-polynomials we find that these show a similar partial orthogonality if expanded in terms of Chebyshev polynomials $U_n(c)$ and $T_n(c)$, i.e. for $\alpha = \beta = \pm \frac{1}{2}$. Here using the normalisation of the Chebyshev polynomials leads to much simpler coefficients than using $P_n^{(\pm \frac{1}{2}, \pm \frac{1}{2})}(c)$, so we define

$$[ a_0, a_1, \ldots, a_n ]_U \equiv \sum_{k=0}^{n} a_k U_k(c), \quad (54)$$

analogously for $T$. Table 5 lists results for $L \leq 13$. Observe the remarkable symmetry between pairs of sectors obtained by replacing $T_k$ by $U_{k-1}$ (e.g. for $\Pi^{(L+1)}_{1+1}$ and $\Pi^{(L+2)}_{0+2}$ taking $L \mod 4 = 0$) or, replacing $T_k$ by $U_k$ for $\Pi^{(L)}_{0+1}$ and $\Pi^{(L+1)}_{1+3}$ (taking still $L \mod 4 = 0$). These relations are not at all simple if expressed in terms of the $\Pi^{(L)}_Q(c)$.

| $L_{Q1+Q2}$ | $2^{-[3L/4]} \frac{1}{2} (\Pi^{(L)}_{Q1}(c) + \Pi^{(L)}_{Q2}(c))$ | $L_Q$ | $2^{-[3L/4]} \Pi^{(L)}_Q(c)$ |
|-------------|---------------------------------|-------|---------------------------------|
| 11+2        | $[1]_U$                         | 21    | $[0, 2]_T$                     |
| 20+2        | $[0, 1]_U$                      | 23    | $[2]_U$                        |
| 30+1        | $[0, 0, 2]_T$                   | 40    | $[0, -1, 0, 2]_T$              |
| 32+3        | $[0, 2]_U$                      | 42    | $[-1, 0, 2]_U$                 |
| 41+3        | $[0, 0, 2]_U$                   | 50+3  | $[0, 1, 0, 4]_U$               |
| 51+2        | $[0, -3, 0, 4]_U$               | 61    | $[0, 0, -2, 0, 8]_T$           |
| 60+2        | $[0, 0, -3, 0, 4]_U$            | 63    | $[0, -2, 0, 8]_U$              |
| 70+1        | $[0, 0, 0, -4, 0, 8]_T$         | 80    | $[0, 0, 1, 0, -8, 0, 8]_T$     |
| 72+3        | $[0, 0, -4, 0, 8]_U$            | 82    | $[0, 1, 0, -8, 0, 8]_U$        |
| 81+3        | $[0, 0, 0, -4, 0, 8]_U$         | 90+3  | $[0, 0, -3, 0, -4, 0, 16]_U$   |
| 91+2        | $[0, 0, 5, 0, -20, 0, 16]_U$    | 101   | $[0, 0, 0, 2, 0, -24, 0, 32]_T$|
| 100+2       | $[0, 0, 0, 5, 0, -20, 0, 16]_U$ | 103   | $[0, 0, 2, 0, -24, 0, 32]_U$   |
| 110+1       | $[0, 0, 0, 0, 6, 0, -32, 0, 32]_T$ | 120   | $[0, 0, 0, -1, 0, 18, 0, -48, 0, 32]_T$ |
| 112+3       | $[0, 0, 0, 6, 0, -32, 0, 32]_U$ | 122   | $[0, 0, -1, 0, 18, 0, -48, 0, 32]_U$ |
| 121+3       | $[0, 0, 0, 6, 0, -32, 0, 32]_U$ | 130+3 | $[0, 0, 0, 5, 0, -8, 0, -48, 0, 64]_U$ |
We conclude these observations giving in Table 6 few results for \( N = 5 \) and \( N = 6 \): For \( N = 5 \) we find partial orthogonality due to vanishing first components with \( \alpha = -\beta = \pm \frac{2}{5} \) and \( \alpha = -\beta = \pm \frac{1}{5} \) according to the sectors \( L \mod 5 \) and \( Q \) considered. For \( N = 6 \) both Chebyshev- and e.g. \( \alpha = -\beta = \pm \frac{1}{6} \)-Jacobi weights are useful. As for \( N = 4 \), various symmetries show up, e.g. for \( N = 6 \), \( L \mod 6 = 0 \) exchanging \( Q = 0 \) with \( Q = 3 \) corresponds to exchanging the Chebyshev polynomials \( T_k \) and \( U_{k-1} \). A convenient method to perform expansions in terms of Jacobi polynomials is given in \[25\], Sec.7.2.

6 Conclusions

We have studied various properties of the polynomials, which via their zeros determine the energy eigenvalues of hamiltonians satisfying Onsager’s algebra. Three different versions have been considered with the zeros in the hermitian case on the negative real axis, the interval \((-1, +1)\), and the unit circle, respectively. The most simple properties emerged in the second case, the polynomials \( \Pi_Q^{(L)}(c) \) which for \( Z_2 \) are Chebyshev polynomials. For the general \( Z_N \) we argue that the polynomials considered have no shorter than \( N + 1 \)-term pure recursion relations. This excludes that the \( \Pi_Q^{(L)} \) are classical orthogonal polynomials. Konhauser biorthogonality (which allows 4-term and higher recursion relations) is not seen to be a property of the \( \Pi_Q^{(L)} \). However, the expansion in terms of Jacobi polynomials reveals a very remarkable partial orthogonality with respect to Jacobi weight functions. The deeper meaning of the latter remains to be studied. There are more general definitions of biorthogonal polynomials in the literature \[20\], e.g. of Iserles and Norsett \[24\] and Van Iseghem \[28\]. For hypergeometric functions \( _2F_2 \) there are four-term recursion relations, see e.g. \[24\] Chap.14. However, these have a much more complicated structure than the recursions which we found for the \( \Pi_Q^{(L)} \). More work is needed to clarify the possible relevance of these latter structures.

Acknowledgements

GvG is grateful to Michael Baake, Harry Braden and Nikita Slavnov for fruitful discussions. He thanks the Institute of Mathematics, Academia Sinica, Taipei for kind hospitality, INTAS-97-1312 and the National Center for Theoretical Sciences of the Tsing Hua University in Hsinchu, Taiwan, for support.

References

[1] Onsager, L. (1944) Phys. Rev. 65, 117.
[2] Dolan, L. and Grady, G. (1982) Phys. Rev. D25, 1587.
[3] Howes, S., Kadanoff, L.P. and den Nijs, M. (1983) Nucl. Phys. B215 [FS7], 169.
[4] von Gehlen, G. and Rittenberg, V. (1985) Nucl. Phys. B257 [FS14], 351
[5] Perk, J.H.H. (1987) in Theta Functions Bowdoin 1987, Am. Math. Soc., Providence, 1989.
[6] Albertini, G., McCoy, B.M., Perk, J.H.H. and Tang, S. (1989) Nucl. Phys. B314 741–763
[7] Baxter, R.J. (1988) Phys. Lett. 133A, 185.
[8] Albertini, G., McCoy, B.M. and Perk, J.H.H. (1989) Adv. Studies in Pure Math. 19, 1.
[9] Au-Yang, H., McCoy, B.M., Perk, J.H.H., Tang, S. and Yan, M.L. (1987) Phys. Lett. 123A, 219.
[10] Baxter, R.J., Perk, J.H.H. and Au-Yang, H. (1988) *Phys. Lett.* **128A**, 138.
[11] Baxter, R.J. (1994) *J. Phys. A: Math. Gen.* **27**, 1837.
[12] Baxter, R.J. (1982) *Exactly solved models in Statistical Mechanics*, Academic Press.
[13] Davies, B. (1990) *J. Phys. A: Math. Gen.* **23**, 2245; (1991) *J. Math. Phys.* **32**, 2945.
[14] Roan, S.-S. (1991) Preprint Max-Planck-Inst. für Mathem., Bonn, MPI/91-70.
[15] Date, E. and Roan, S.-S. (2000) *J. Phys. A: Math. Gen.* **33**, 3275.
[16] McCoy, B.M. (1990) The Chiral Potts Model: from Physics to Mathematics and back, in M. Kashiwara and T. Miwa (eds.) *Special Functions, Proc. ICM-90 Satellite Conf.*, pp. 245-259.
[17] Au-Yang, A. and Perk, J.H.H. (1997) *Int. J. Mod. Phys.* **B11**, 11–26.
[18] Ahn, C. and Shigemoto, K. (1991) *Int. J. Mod. Phys.* **A6**, 3509.
[19] Uglov, D.B. and Ivanov, I.T. (1996) *Journ. Stat. Phys.* **82**, 87.
[20] Chihara, T.S. (1978) *An Introduction to Orthogonal Polynomials*, Gordon and Breach.
[21] Szegö, G. (1967) *Orthogonal Polynomials*, *Amer. Math. Soc. Collog. Publ.* **23**, 3rd ed., New York.
[22] Bazhanov, V.V. and Stroganov, Yu.G. (1990) *J. Stat. Phys.* **59**, 799.
[23] von Gehlen, G. (1999) *Springer Lecture Notes in Physics* **524**, 307, hep-th/9811123.
[24] Konhauser, J.D.E. (1965) *J. Math. Anal. Appl.* **11**, 242–260.
[25] Carlson, B.C. (1977) *Special Functions of Applied Mathematics*, Academic Press
[26] Brezinski, C. (1992) *Biorthogonality and its applications to numerical analysis*, M. Dekker, New York-Basel-Hong Kong.
[27] Iserles, A. and Norsett, S.P. (1987) *J. Compt. Appl. Math.* **19**, 39–45; (1988) *Transac. Amer. Math. Soc.* **306**, 455–474; (1990) *SIAM J. Math. Anal.* **21**, 483–509.
[28] van Iseghem, J. (1987) *J. Comupt. Appl. Math.* **19**, 141–150.
[29] Rainville, E.D. (1960) *Special functions*, Chelsea Publ. Co.
Table 6: Expansion vectors for some $\mathbb{Z}_5^-$ and $\mathbb{Z}_6^-$-polynomials. Since in several cases we don’t show all components but just the first and last ones, in the column $\text{dim}[]$ we give the lengths of the component vectors.

| $L_Q$ | $\text{dim}[]$ | $5^{-[L/5]} \prod_{Q}^{(L)}(c)$ for $\mathbb{Z}_5$ |
|-------|----------------|--------------------------------------------------|
| 42    | 3              | $[0, 0, \frac{250}{3}]$, $-\frac{2}{5}$          |
| 51    | 4              | $[0, -8, 0, 50]$, $-\frac{3}{5}$                 |
| 61    | 5              | $[0, 0, \frac{-500}{7}, 0, \frac{1000}{7}]$, $\frac{2}{5}$, $-\frac{2}{5}$ |
| 60    | 5              | $[0, \pm 16, 0, \pm 150, \frac{1000}{7}]$, $\frac{1}{5}$, $-\frac{1}{5}$ |
| 92    | 7              | $[0, 0, \frac{4400}{21}, 0, \frac{-26000}{27}, 0, \frac{1250000}{231}]$, $\frac{2}{5}$, $-\frac{2}{5}$ |
| 142   | 11             | $[0, 0, 0, 0, \frac{-1709200}{143}, 0, \frac{8\cdot10^7}{561}, 0, \frac{-92\cdot10^8}{24453}, 0, \frac{125\cdot10^8}{46189}]$, $\frac{2}{5}$, $-\frac{2}{5}$ |
| 192   | 15             | $[0, 0, 0, 0, \frac{-2048000}{221}, 0, \frac{128\cdot10^5}{21}, 0, \ldots, 0, \frac{-55\cdot10^{12}}{2028117}, 0, \frac{25\cdot10^{11}}{260953}]$, $\frac{2}{5}$, $-\frac{2}{5}$ |
| 242   | 19             | $[0, 0, 0, 0, 0, \frac{2301952000}{2737}, 0, \ldots, 0, \frac{-688\cdot10^{15}}{220756273}, 0, \frac{5\cdot10^{16}}{9075135}]$, $\frac{2}{5}$, $-\frac{2}{5}$ |

| $L_Q$ | $\text{dim}[]$ | $6^{-[L/6]} \prod_{Q}^{(L)}(c)$ for $\mathbb{Z}_6$ |
|-------|----------------|--------------------------------------------------|
| 34    | 2              | $[0, 36]$, $-\frac{1}{6}$                        |
| 43    | 3              | $[0, 0, 144]$, $-\frac{1}{6}$                    |
| 103   | 8              | $[0, 0, 8736, 0, \frac{-5045760}{91}, 0, \frac{8957952}{143}]$, $\frac{1}{5}$, $-\frac{1}{5}$ |
| 60    | 6              | $[0, \frac{43}{5}, 0, -90, 0, 81]_T$             |
| 63    | 5              | $[\frac{43}{5}, 0, -90, 0, 81]_U$               |