Article

On Geodesic Behavior of Some Special Curves

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Received: 2 March 2020; Accepted: 17 March 2020; Published: 1 April 2020

Abstract: In this paper, geometric structures on an open subset \( D \subseteq \mathbb{R}^2 \) are investigated such that the graphs associated with the solutions of some special functions become geodesics. More precisely, we determine the Riemannian metric \( g \) such that Bessel (Hermite, harmonic oscillator, Legendre and Chebyshev) ordinary differential equation (ODE) is identified with the geodesic ODEs produced by the Riemannian metric \( g \). The technique is based on the Lagrangian (the energy of the curve) \( L = \frac{1}{2} \| \dot{x}(t) \|^2 \), the associated Euler–Lagrange ODEs and their identification with the considered special ODEs.

Keywords: auto-parallel curve; geodesic; Euler–Lagrange equations; Lagrangian; special functions

MSC: 34A26; 53B15; 53C22

1. Introduction and Preliminaries

The concept of connection plays an important role in geometry and, depending on what sort of data one wants to transport along some trajectories, a variety of kinds of connections have been introduced in modern geometry. Crampin et al. [1], in a certain vector bundle, described the construction of a linear connection associated with a second-order differential equation field and, moreover, the corresponding curvature was computed. Ermakov [2] established that linear second-order equations with variable coefficients can be completely integrated only in very rare cases. Further, some aspects of time-dependent second-order differential equations and Berwald-type connections have been studied, with remarkable results, by Sarlet and Mestdag [3]. Michor and Mumford [4] considered some Riemannian metrics on the space of smooth regular curves in the plane, viewed as the orbit space of maps from \( S^1 \) to the plane modulo the group of diffeomorphisms of \( S^1 \), acting as reparameterizations. For an excellent survey on geometric dynamics, convex functions and optimization methods on Riemannian manifolds, the reader is directed to Udrişte [5,6]. Relatively recently, Udrişte et al. [7], by using an identity theorem for ordinary differential equations (ODEs), investigated some geometrical structures that transform the solutions of a second order ODE into auto-parallel graphs. Later, Treanţă and Udrişte [8], in accordance to Udrişte et al. [7], studied the auto-parallel behavior of some special plane or space curves by using the theory of identifying of two ODEs.

In the present paper, as a natural continuation of some results obtained in Treanţă and Udrişte [8], we are looking for an appropriate geometric structure such that important graphs in applications (like Bessel functions, Hermite functions etc) become geodesics. Specifically, our aim is to determine the Riemannian metric \( g_{ij} \) such that Bessel ordinary differential equation, Hermite ODE, harmonic oscillator ODE, Legendre ODE and Chebyshev ODE, respectively, is identified with the geodesic ODEs produced by \( g_{ij} \). The technique is based on the Lagrangian (the energy of the curve) \( L = \frac{1}{2} \| \dot{x}(t) \|^2 \), the associated Euler–Lagrange ODEs and their identification to the Bessel ODE, Hermite
ODE, harmonic oscillator ODE, Legendre ODE and Chebyshev ODE, respectively. By applying this new technique, we developed an original point of view by introducing some new results regarding the geodesics curves literature. For this aim, in the following, we present some basics to be used in the sequel.

Let \( M \) be a differentiable manifold of dimension \( n \) and denote by \( \mathcal{X}(M) \) the Lie algebra of vector fields on \( M \). A Riemannian metric on \( M \) is a family of (positive definite) inner products \( g_p : T_p M \times T_p M \to \mathbb{R} \), \( p \in M \), such that for any two differentiable vector fields \( X, Y \in \mathcal{X}(M) \), the application \( p \to g_p(X(p), Y(p)) \) defines a smooth function \( M \to \mathbb{R} \).

The local expression of a Riemannian metric \( g \) is

\[
 g = g_{ij} \, dx^i \otimes dx^j, \quad g_{ij} = g \left( \frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j} \right), \quad i, j \in \{1, \ldots, n\},
\]
determined at every point \( p \in M \) by a symmetric positive definite matrix. The inverse of the tensor field \( g \) is

\[
 g^{-1} = g^{ij} \, \frac{\partial}{\partial x^i} \otimes \frac{\partial}{\partial x^j}, \quad g^{ij} = g_{ij}^{-1},
\]
de the entries of the inverse matrix of \( (g_{ij}) \). Endowed with this metric, the differentiable manifold \( (M, g) \) is called a Riemannian manifold. Let \( (M, g) \) be a Riemannian manifold and \( g_{ij} \) the components of \( g \). The Riemannian metric \( g \) determines the symmetric linear connection

\[
 \Gamma^i_{jk} = \frac{1}{2} g^{il} \left( \frac{\partial g_{lj}}{\partial x^k} + \frac{\partial g_{lk}}{\partial x^j} - g_{lj} \Gamma^l_{ik} - g_{lk} \Gamma^l_{ij} \right), \quad i, j, k \in \{1, 2, \ldots, n\} \quad \text{(Christoffel symbols),}
\]
which is called the Riemannian (Levi–Civita) connection of \( M \), whose fundamental property is

\[
 \nabla_k g_{ij} = \frac{\partial g_{ij}}{\partial x^k} - g_{ij} \Gamma^l_{ik} - g_{lk} \Gamma^l_{ij} = 0 \quad \text{(the tensor field \( g \) is parallel with respect to the symmetric linear connection \( \nabla \)).}
\]

Let \( (M, g) \) be a \( C^\infty \)-class \( n \)-dimensional Riemannian manifold and \( \nabla \) a linear connection on \( M \). We say that a vector field \( X \in \mathcal{X}(M) \) is a parallel vector field if \( X \) is parallel with respect to \( \nabla \).

2. Main Results

In this section, we formulate and prove the main results of the paper. First, we study the geodesic behavior of some special curves on an open subset \( D \subseteq \mathbb{R}^2 \) is analyzed. In the second part of this section, the general case is investigated and, in this way, the results obtained in the first part of Section 2 become non-trivial illustrative examples of the developed theory. Finally, Section 3 contains conclusions and other development ideas.

2.1. ODE, harmonic oscillator ODE, Legendre ODE and Chebyshev ODE, respectively. By applying this new technique, we developed an original point of view by introducing some new results regarding the geodesics curves literature. For this aim, in the following, we present some basics to be used in the sequel.

Let \( M \) be a differentiable manifold of dimension \( n \) and denote by \( \mathcal{X}(M) \) the Lie algebra of vector fields on \( M \). A Riemannian metric on \( M \) is a family of (positive definite) inner products \( g_p : T_p M \times T_p M \to \mathbb{R} \), \( p \in M \), such that for any two differentiable vector fields \( X, Y \in \mathcal{X}(M) \), the application \( p \to g_p(X(p), Y(p)) \) defines a smooth function \( M \to \mathbb{R} \).

The local expression of a Riemannian metric \( g \) is

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\]
determined at every point \( p \in M \) by a symmetric positive definite matrix. The inverse of the tensor field \( g \) is

\[
 g^{-1} = g^{ij} \, \frac{\partial}{\partial x^i} \otimes \frac{\partial}{\partial x^j}, \quad g^{ij} = g_{ij}^{-1},
\]
de the entries of the inverse matrix of \( (g_{ij}) \). Endowed with this metric, the differentiable manifold \( (M, g) \) is called a Riemannian manifold. Let \( (M, g) \) be a Riemannian manifold and \( g_{ij} \) the components of \( g \). The Riemannian metric \( g \) determines the symmetric linear connection

\[
 \Gamma^i_{jk} = \frac{1}{2} g^{il} \left( \frac{\partial g_{lj}}{\partial x^k} + \frac{\partial g_{lk}}{\partial x^j} - g_{lj} \Gamma^l_{ik} - g_{lk} \Gamma^l_{ij} \right), \quad i, j, k \in \{1, 2, \ldots, n\} \quad \text{(Christoffel symbols),}
\]
which is called the Riemannian (Levi–Civita) connection of \( M \), whose fundamental property is

\[
 \nabla_k g_{ij} = \frac{\partial g_{ij}}{\partial x^k} - g_{ij} \Gamma^l_{ik} - g_{lk} \Gamma^l_{ij} = 0 \quad \text{(the tensor field \( g \) is parallel with respect to the symmetric linear connection \( \nabla \)).}
\]

Let \( (M, g) \) be a \( C^\infty \)-class \( n \)-dimensional Riemannian manifold and \( \nabla \) a linear connection on \( M \). We say that a vector field \( X \in \mathcal{X}(M) \) is a parallel vector field if \( X \) is parallel with respect to \( \nabla \).

Consider \( c : [a, b] \to M \), \( c(t) = (x^1(t), \ldots, x^n(t)) \), \( t \in [a, b] \subseteq \mathbb{R} \), a regular differentiable curve on \( M \). We say that a vector field \( X \in \mathcal{X}(M) \) is a parallel vector field along the curve \( c \) with respect to \( \nabla \) if \( \nabla_{c(t)} X = 0 \). The curve \( c \) is called auto-parallel if \( \nabla_{c(t)} \dot{c}(t) = 0 \) or, equivalently, it satisfies

\[
 \frac{d^2 x^i}{dt^2} + \Gamma^i_{jk} \frac{dx^j}{dt} \frac{dx^k}{dt} = 0, \quad i = 1, \ldots, n.
\]

We may also write

\[
 \frac{d^2 x^i}{dt^2} + \Gamma^i_{jk} \frac{dx^j}{dt} \frac{dx^k}{dt} = 0, \quad i, j, k \in \{1, 2\},
\]
or equivalently, the function \( y : I \to R \) is a solution of the following differential equations

\[
 \begin{align*}
 \Gamma^1_{12}(t, y)y^2 + 2 \Gamma^1_{12}(t, y)y + \Gamma^1_{11}(t, y) = 0 & \quad (1) \\
 y + \Gamma^2_{12}(t, y)y^2 + 2 \Gamma^2_{12}(t, y)y + \Gamma^2_{11}(t, y) = 0. & \quad (2)
\end{align*}
\]

Let assume that the previous symmetric linear connection \( \nabla \) on \( D \), of components \( \Gamma^i_{jk} \), \( i, j, k \in \{1, 2\} \), is the Levi–Civita connection, that is, its components fulfill the following differential equations

\[
 \begin{align*}
 \Gamma^1_{12}(t, y)y^2 + 2 \Gamma^1_{12}(t, y)y + \Gamma^1_{11}(t, y) = 0 & \quad (1) \\
 y + \Gamma^2_{12}(t, y)y^2 + 2 \Gamma^2_{12}(t, y)y + \Gamma^2_{11}(t, y) = 0. & \quad (2)
\end{align*}
\]
where \( g_{ij} \), \( i, j \in \{1, 2\} \), represent the components of the Riemannian metric \( g = g_{ij}dx^i \otimes dx^j \) on \( D \). Taking into account the complete integrability conditions (closeness conditions) for the components of the Riemannian metric \( g = (g_{ij}) \) on \( D \), and also, \( \Gamma^i_{jk} = 0 \) and \( \Gamma^i_{jk} = \Gamma^j_{ik} \) for \( i, j, k \in \{1, 2\} \), we can rewrite the differential Equation (3) as follows

\[
2g_{21}\Gamma^2_{11} = \frac{\partial g_{11}}{\partial t}, \quad 2g_{22}\Gamma^2_{22} = \frac{\partial g_{22}}{\partial y}, \quad 2g_{21}\Gamma^2_{12} = \frac{\partial g_{12}}{\partial y}.
\]

(4)

\[
\frac{\partial g_{12}}{\partial y} \Gamma^2_{11} + g_{12} \frac{\partial \Gamma^2_{11}}{\partial y} = \frac{\partial g_{12}}{\partial t} \Gamma^2_{12} + g_{12} \frac{\partial \Gamma^2_{12}}{\partial t}
\]

Moreover, putting the condition that the above mentioned components of the Riemannian metric \( g = g_{ij}dx^i \otimes dx^j \) on \( D \) to satisfy the remaining equations in Equation (4), we find

\[
\varphi(t) = tc, \quad c > 0, \quad k(t) = \frac{2\psi(t)\varphi(t)}{tc} - \frac{\psi^2(t)}{t^2c}.
\]

2.1. Bessel Geodesics

We begin by recalling Bessel ODE,

\[
y'(x) + \frac{1}{x}y(x) + \left(1 - \frac{\alpha^2}{x^2}\right)y(x) = 0.
\]

According to Proposition 3.1 in Treanţă and Udrişte [8], we have

\[
\Gamma^1_{22} = 0, \quad \Gamma^1_{12} = \Gamma^1_{21} = 0, \quad \Gamma^1_{11} = 0
\]

\[
\Gamma^2_{11} = \left(1 - \frac{\alpha^2}{t^2}\right)y, \quad \Gamma^2_{12} = \Gamma^2_{21} = \frac{1}{2t}, \quad \Gamma^2_{22} = 0.
\]

Therefore, using Equation (4) (see the second equation), we obtain \( \frac{\partial g_{22}}{\partial y} = 0 \), that is, \( g_{22} = \varphi(t) \).

Also, using the sixth equation in Equation (4), we get

\[
2\frac{\partial g_{12}}{\partial y} = \varphi(t) \iff \frac{\partial g_{12}}{\partial y} = \frac{1}{2}\varphi(t) \iff g_{12} = g_{21} = \frac{1}{2} \varphi(t)y + \psi(t).
\]

From the third equation, \( 2g_{21}\Gamma^2_{12} = \frac{\partial g_{11}}{\partial y} \), it follows

\[
\frac{\partial g_{11}}{\partial y} = \frac{\varphi(t)y + 2\psi(t)}{2t} \iff g_{11} = \frac{\varphi(t)}{4t}y^2 + \frac{\psi(t)}{t}y + k(t).
\]

Moreover, putting the condition that the above mentioned components of the Riemannian metric \( g = g_{ij}dx^i \otimes dx^j \) on \( D \) to satisfy the remaining equations in Equation (4), we find

\[
\varphi(t) = tc, \quad c > 0; \quad k(t) = \frac{2\psi(t)\varphi(t)}{tc} - \frac{\psi^2(t)}{t^2c}.
\]
\[
c (\alpha^2 - t^2) = 0, \quad c \left(4\alpha^2 - 4t^2 - 1 \right) = 0, \quad \frac{1}{2} \left[ \frac{\psi(t)}{t} \right]' - \left(1 - \frac{\alpha^2}{t^2} \right) \psi(t) = 0.
\]

In summary, all the previous computations allow us to formulate the following result.

**Proposition 1.** Each Bessel graph \( t \rightarrow y(t) \) is not a geodesic with respect to the foregoing metric family constrained by non-singularity and constant signature conditions and by

\[
\varphi(t) = tc, \ c > 0; \quad k(t) = \frac{2\psi(t)\psi(t)}{tc} - \frac{\psi^2(t)}{t^2c}
\]

\[
c \left(\alpha^2 - t^2 \right) = 0, \quad c \left(4\alpha^2 - 4t^2 - 1 \right) = 0
\]

\[
\frac{1}{2} \left[ \frac{\psi(t)}{t} \right]' - \left(1 - \frac{\alpha^2}{t^2} \right) \psi(t) = 0, \quad g_{11}g_{22} - (g_{12})^2 > 0.
\]

**Proof.** Assume that each Bessel graph \( t \rightarrow y(t) \) is a geodesic with respect to the foregoing constrained metric family. The following two relations

\[
c \left(\alpha^2 - t^2 \right) = 0, \quad c \left(4\alpha^2 - 4t^2 - 1 \right) = 0,
\]

where \( c > 0 \), lead us to a contradiction. The proof is complete. \( \square \)

2.2. Hermite Geodesics

We consider Hermite ODE,

\[
y''(x) - 2xy'(x) + 2ny(x) = 0, \quad n \in \mathbb{N}.
\]

Taking into account Proposition 3.2 in Treanță and Udriște [8], where we have established the following components

\[
\Gamma^1_{22} = 0, \quad \Gamma^1_{12} = \Gamma^1_{21} = 0, \quad \Gamma^1_{11} = 0
\]

\[
\Gamma^2_{11} = 2ny, \quad \Gamma^2_{12} = \Gamma^2_{21} = -t, \quad \Gamma^2_{22} = 0,
\]

for our symmetric linear connection \( \nabla \), and using the second equation and the sixth equation in Equation (4), we get \( \frac{\partial g_{22}}{\partial y} = 0 \), that is, \( g_{22} = \phi(t) \), and

\[
2\frac{\partial g_{12}}{\partial y} = \phi(t) \iff \frac{\partial g_{12}}{\partial y} = \frac{1}{2} \phi(t) \iff g_{12} = g_{21} = \frac{1}{2} \phi(t)y + \psi(t).
\]

According to \( 2g_{21}\Gamma^2_{12} = \frac{\partial g_{11}}{\partial y} \), it follows

\[
\frac{\partial g_{11}}{\partial y} = -t\phi(t)y - 2t\psi(t) \iff g_{11} = \frac{-t\phi(t)}{2}y^2 - 2t\psi(t)y + k(t).
\]

Moreover, replacing the above components \( g_{ij}, \ i,j \in \{1,2\} \), in the remaining equations in Equation (4), we get

\[
\phi(t) = ce^{-t^2}, \ c > 0; \quad k(t) = \frac{2\phi(t)\phi(t)}{ce^{-t^2}} + \frac{2\psi^2(t)}{ce^{-t^2}}
\]

\[
c \left(2t^2 - 2n - 1 \right) = 0, \quad c \left[-t^3 + (2n + 1)t \right] = 0, \quad [-t\psi(t)]' - 2n\psi(t) = 0.
\]
Proposition 2. Each Hermite graph \( t \rightarrow y(t) \) is not a geodesic with respect to the foregoing metric family constrained by non-singularity and constant signature conditions and by

\[
\varphi(t) = ce^{-t^2}, \quad c > 0; \quad k(t) = \frac{2\varphi(t)\dot{\psi}(t)}{ce^{-t^2}} + 2t\frac{\dot{\psi}^2(t)}{ce^{-t^2}}
\]

\[
c \left(2t^2 - 2n - 1\right) = 0, \quad c \left[-t^3 + (2n + 1)t\right] = 0
\]

\[
\psi \left(\dot{t}^2 - 2n\psi(t)\right) = 0, \quad g_{11}g_{22} - (g_{12})^2 > 0.
\]

Proof. Let suppose that each Hermite graph \( t \rightarrow y(t) \) is a geodesic with respect to the foregoing constrained metric family. Using the relations

\[
c \left(2t^2 - 2n - 1\right) = 0, \quad c \left[-t^3 + (2n + 1)t\right] = 0
\]

we find a contradiction and the proof is complete. \( \square \)

2.3. Harmonic Oscillator Geodesics

Let consider the harmonic oscillator ODE,

\[
\ddot{y}(x) + \omega^2 y(x) = 0,
\]

and analogous foregoing reasoning. Using Proposition 3.3 in Treanță and Udriște [8] and solving the differential equations given in Equation (4), we obtain the metric family

\[
g_{22} = \varphi(t), \quad g_{12} = g_{21} = \frac{1}{2}\ddot{\psi}(t)\varphi(t) + \dot{\psi}(t), \quad g_{11} = k(t).
\]

Also, we have \( \varphi(t) = c, \quad c > 0 \), and \( \ddot{\psi}(t) - c\omega^2 y = 0, \quad k(t) - 2\omega^2\dot{\psi}(t)y = 0 \). The closeness conditions impose \( c\omega^2 = 0, \omega^2\dot{\psi}(t) = 0 \).

Proposition 3. Each harmonic oscillator graph \( t \rightarrow y(t) \) is a geodesic with respect to the foregoing metric family constrained by non-singularity and constant signature conditions and by

\[
\varphi(t) = c, \quad c > 0; \quad \psi(t) - c\omega^2 y = 0, \quad k(t) - 2\omega^2\dot{\psi}(t)y = 0
\]

\[
c\omega^2 = 0, \quad \omega^2\dot{\psi}(t) = 0, \quad g_{11}g_{22} - (g_{12})^2 > 0,
\]

or, equivalently,

\[
\varphi(t) = c, \quad c > 0; \quad k(t) = \frac{2\varphi(t)\dot{\psi}(t)}{c}
\]

\[
c\omega^2 = 0, \quad \omega^2\dot{\psi}(t) = 0, \quad g_{11}g_{22} - (g_{12})^2 > 0.
\]

Proof. According to \( c\omega^2 = 0 \) and taking into account that \( c > 0 \), we get \( \omega = 0 \). Using the relations \( \ddot{\psi}(t) - c\omega^2 y = 0, \quad k(t) - 2\omega^2\dot{\psi}(t)y = 0 \), we obtain \( \ddot{\psi}(t) = c_1, \quad k(t) = c_2 \), with \( c_1, c_2 \in \mathbb{R} \). Thus, for \( \omega = 0 \), the components of the Riemannian metric \( g \) on \( D \) become

\[
g_{22}(t, y) = c, \quad c > 0; \quad g_{12}(t, y) = g_{21}(t, y) = c_1; \quad g_{11}(t, y) = c_2, \quad \forall (t, y) \in D.
\]

The condition \( g_{11}g_{22} - (g_{12})^2 > 0 \) leads to \( c_2 - c_1^2 > 0 \). In conclusion, for \( \omega = 0 \) and for all the points in \( D \) which satisfy \( k(t)\varphi(t) - \dot{\psi}^2(t) > 0 \), each harmonic oscillator graph \( t \rightarrow y(t) \) is a geodesic. \( \square \)
2.4. Legendre Geodesics

Consider the Legendre ODE,

\[ \dot{y}(x) - \frac{2x}{1-x^2} y(x) + \frac{n(n+1)}{1-x^2} y(x) = 0, \quad n \in \mathbb{N}. \]

We apply the same reasoning as in the previous cases. Therefore, we find the metric family

\[ g_{22} = \varphi(t), \quad g_{12} = g_{21} = \frac{1}{2} \psi(t) y + \psi(t), \quad g_{11} = -\frac{t \varphi(t)}{2(1-t^2)} y^2 - \frac{2t \varphi(t)}{1-t^2} y + k(t) \]

subject to

\[ \varphi(t) = c(1-t^2), \quad c > 0; \quad k(t) = \frac{2 \psi(t) \varphi(t)}{c(1-t^2)} + \frac{2t \psi^2(t)}{c(1-t^2)^2} \]

\[ c \left( n^2 + n + 1 \right) = 0, \quad c \left[ \left( 4n^2 + 1 \right) t^3 + \left( 4n^2 + 2 \right) t \right] = 0 \]

\[ \left[ \frac{t \varphi(t)}{t^2 - 1} \right] + n(n+1) \frac{\psi(t)}{t^2 - 1} = 0. \]

**Proposition 4.** Each Legendre graph \( t \to y(t) \) is not a geodesic with respect to the foregoing metric family constrained by non-singularity and constant signature conditions and by

\[ \varphi(t) = c(1-t^2), \quad c > 0; \quad k(t) = \frac{2 \psi(t) \varphi(t)}{c(1-t^2)} + \frac{2t \psi^2(t)}{c(1-t^2)^2} \]

\[ c \left( n^2 + n + 1 \right) = 0, \quad c \left[ \left( 4n^2 + 1 \right) t^3 + \left( 4n^2 + 2 \right) t \right] = 0 \]

\[ \left[ \frac{t \varphi(t)}{t^2 - 1} \right] + n(n+1) \frac{\psi(t)}{t^2 - 1} = 0, \quad g_{11} g_{22} - (g_{12})^2 > 0. \]

**Proof.** The equation \( c \left( n^2 + n + 1 \right) = 0 \), where \( c > 0 \), leads to \( n^2 + n + 1 = 0 \). Since it does not admit natural solutions, we conclude that each Legendre graph \( t \to y(t) \) is not a geodesic with respect to the foregoing constrained metric family. \( \square \)

2.5. Chebyshev Geodesics

We consider Chebyshev ODE,

\[ \dot{y}(x) - \frac{x}{1-x^2} \ddot{y}(x) + \frac{n^2}{1-x^2} y(x) = 0, \quad n \in \mathbb{N}. \]

Solving the differential equations given in Equation (4) and taking into account Proposition 3.5 in Treanță and Udriște [8], we get the metric family

\[ g_{22} = \varphi(t), \quad g_{12} = g_{21} = \frac{1}{2} \varphi(t) y + \psi(t), \quad g_{11} = -\frac{t \varphi(t)}{4(1-t^2)} y^2 - \frac{t \psi(t)}{1-t^2} y + k(t) \]

subject to

\[ \varphi(t) = c\sqrt{1-t^2}, \quad c > 0; \quad k(t) = \frac{2 \psi(t) \varphi(t)}{c\sqrt{1-t^2}} + \frac{t \psi^2(t)}{c(1-t^2)\sqrt{1-t^2}} \]

\[ c \left[ -2n^2 t^4 + (4n^2 + 1) t^2 - 2n^2 - 1 \right] = 0, \quad c \left[ (1-4n^2) t^3 + (4n^2 + 2) t \right] = 0 \]

\[ \frac{1}{2} \left[ \frac{t \varphi(t)}{t^2 - 1} \right] + n^2 \frac{\psi(t)}{t^2 - 1} = 0. \]
Proposition 5. Each Chebyshev graph \( t \rightarrow y(t) \) is not a geodesic with respect to the foregoing metric family constrained by non-singularity and constant signature conditions and by

\[
\varphi(t) = c \sqrt{1 - t^2}, \quad c > 0; \quad k(t) = \frac{2\varphi(t)\dot{\varphi}(t)}{c\sqrt{1 - t^2}} + \frac{t\dot{\varphi}^2(t)}{c(1 - t^2)\sqrt{1 - t^2}}
\]

\[
c \left[ -2n^2 t^4 + (4n^2 + 1)t^2 - 2n^2 - 1 \right] = 0, \quad c \left[ (1 - 4n^2)t^3 + (4n^2 + 2)t \right] = 0
\]

\[
\frac{1}{2} \left[ \frac{t\varphi(t)}{1 - t^2} \right]' + n^2 \frac{\varphi(t)}{1 - t^2} = 0, \quad \delta_{11}\delta_{22} - (\delta_{12})^2 > 0.
\]

Proof. Assuming that each Chebyshev graph \( t \rightarrow y(t) \) is a geodesic with respect to the previous constrained metric family and taking into account the following two relations

\[
\varphi(t) = c \sqrt{1 - t^2}, \quad c > 0; \quad k(t) = \frac{2\varphi(t)\dot{\varphi}(t)}{c\sqrt{1 - t^2}} + \frac{t\dot{\varphi}^2(t)}{c(1 - t^2)\sqrt{1 - t^2}}
\]

\[
c \left[ -2n^2 t^4 + (4n^2 + 1)t^2 - 2n^2 - 1 \right] = 0, \quad c \left[ (1 - 4n^2)t^3 + (4n^2 + 2)t \right] = 0
\]

we find a contradiction. The proof is complete. \( \square \)

Further, in order to give a generalization of all the previously mentioned results, we establish the following main result.

Theorem 1. Consider \( \mu, \nu : I \rightarrow R \) two \( C^2 \)-class functions which determine the following second order linear homogeneous differential equation

\[
y + \mu(t)y + \nu(t)y = 0, \quad (5)
\]

where \( I \subseteq R \) is an open real interval. Also, let \( \nabla \) be a symmetric linear connection on \( D = I \times R \). Then, the following assertions are equivalent:

(i) the \( C^2 \)-class curve \( x : J \subseteq I \rightarrow D, \ x(t) = (t, y(t)), \ \forall t \in J \), is geodesic with respect to the symmetric linear connection \( \nabla \), for any solution \( y : J \subseteq I \rightarrow R \) of the ODE Equation \( 5 \);

(ii) for any solution \( y : J \subseteq I \rightarrow R \) of the ODE Equation \( 5 \), each graph \( t \rightarrow y(t) \) is geodesic with respect to the symmetric linear connection \( \nabla \);

(iii) the symmetric linear connection \( \nabla \) has the components

\[
\Gamma_{11}^1(t, y) = 0, \quad \Gamma_{12}^1(t, y) = \Gamma_{21}^1(t, y) = 0, \quad \Gamma_{22}^1(t, y) = 0
\]

\[
\Gamma_{11}^2(t, y) = \nu(t)y, \quad \Gamma_{12}^2(t, y) = \Gamma_{21}^2(t, y) = \frac{1}{2} \mu(t), \quad \Gamma_{22}^2(t, y) = 0,
\]

for any \( t \in J, \ y \in R \), and the components of the Riemannian metric \( g = g_{ij} dx^i \otimes dx^j \) on \( D \) fulfill

\[
g_{22}(t, y) = \varphi(t), \quad g_{12}(t, y) = g_{21}(t, y) = \frac{1}{2} \varphi(t)y + \psi(t)
\]

\[
g_{11}(t, y) = \frac{1}{4} \varphi(t)\mu(t)y^2 + \psi(t)\mu(t)y + k(t)
\]

subject to

\[
\varphi(t) = ce^{\int \mu(t)dt}, \quad c > 0; \quad k(t) = \frac{2\psi(t)\dot{\varphi}(t)}{\varphi(t)} - \mu(t)\frac{\dot{\varphi}^2(t)}{\varphi(t)}
\]

\[
\frac{1}{2} [\varphi(t)\mu(t)]' - \varphi(t)v(t) = 0, \quad \frac{1}{4} [\varphi(t)\mu(t)]' - \varphi(t)v(t) = 0
\]

\[
\frac{1}{2} [\varphi(t)\mu(t)]' - \psi(t)v(t) = 0, \quad \delta_{11}(t, y)g_{22}(t, y) - (\delta_{12}(t, y))^2 > 0.
\]
Proof. The equivalence between (i) and (ii) is obvious. Further, we consider the equivalence (ii) \iff (iii). Let \( y: J \subseteq I \to R \) be a solution of the ODE Equation (5) and consider the differential Equations (1) and (2),

\[
\Gamma_{12}(t,y)\dot{y}^2 + 2\Gamma_{12}(t,y)\dot{y} + \Gamma_{11}(t,y) = 0 \\
\dot{y} + \Gamma_{22}(t,y)\dot{y}^2 + 2\Gamma_{12}(t,y)\dot{y} + \Gamma_{11}(t,y) = 0.
\]

Each graph \( t \to y(t) \) is geodesic with respect to the symmetric linear connection \( \nabla \) on \( D \), of components \( \Gamma^i_{jk} \), \( i,j,k \in \{1,2\} \), if and only if any solution \( y: J \subseteq I \to R \) of the ODE Equation (5) is also a solution of the foregoing differential equations, that is,

\[
\Gamma^1_{11}(t,y) = 0, \quad \Gamma^1_{12}(t,y) = \Gamma^1_{21}(t,y) = 0, \quad \Gamma^1_{22}(t,y) = 0 \\
\Gamma^2_{11}(t,y) = v(t)y, \quad \Gamma^2_{12}(t,y) = \Gamma^2_{21}(t,y) = \frac{1}{2} \mu(t), \quad \Gamma^2_{22}(t,y) = 0,
\]

for any \( t \in J \), \( y \in R \) (see Theorems 1.3 and 1.4 in Udriște et al. [7], and the components of the Riemannian metric \( g = g_{ij} dx^i \otimes dx^j \) on \( D \) satisfy the differential equations in Equation (4), that is, by a direct computation and considering that \( g \) determines on \( D \) a symmetric positive definite matrix, the following relations hold:

\[
g_{22}(t,y) = \varphi(t), \quad g_{12}(t,y) = g_{21}(t,y) = \frac{1}{2} \psi(t) \mu(t) \\
g_{11}(t,y) = \frac{1}{4} \varphi(t) \mu(t) + \psi(t) \mu(t) + k(t)
\]

subject to

\[
\varphi(t) = c e^{\int \mu(t) dt}, \quad c > 0; \quad k(t) = \frac{2 \psi(t) \mu(t)}{\varphi(t)} - \mu(t) \frac{\psi^2(t)}{\varphi(t)}
\]

\[
\frac{1}{2} [\varphi(t) \mu(t)]' - \varphi(t) v(t) = 0, \quad \frac{1}{4} [\psi(t) \mu(t)]' - \psi(t) v(t) = 0
\]

\[
\frac{1}{2} [\varphi(t) \mu(t)]' - \psi(t) v(t) = 0, \quad g_{11}(t,y) g_{22}(t,y) - (g_{12}(t,y))^2 > 0.
\]

The proof is complete. \( \Box \)

Remark 1. Let us consider the Lagrangian (the energy of curve)

\[
L = \frac{1}{2} \| \dot{x}(t) \|^2 = \frac{1}{2} g_{ij}(x(t)) \dot{x}^i(t) \dot{x}^j(t), \quad i, j \in \{1,2\},
\]

where \( x(t) = \left( x^1(t), x^2(t) \right) \) is a \( C^2 \)-class curve on \( D \subseteq \mathbb{R}^2 \). We write the Euler–Lagrange equations,

\[
\frac{\partial L}{\partial x^i} - \frac{d}{dt} \frac{\partial L}{\partial \dot{x}^i} = 0, \quad i \in \{1,2\},
\]

associated to the previous Lagrangian, and taking into account that \( x(t) = \left( x^1(t), x^2(t) \right) := (t,y(t)) \), we obtain the following differential equations

\[
y + \frac{\partial g_{12}}{\partial y} \frac{1}{2} \frac{d}{dt} y^2 + \frac{\partial g_{11}}{\partial y} \frac{1}{2} \frac{d}{dt} g_{11} = 0 \quad (6)
\]

\[
y + \frac{1}{2} \frac{\partial g_{22}}{\partial y} y^2 + \frac{\partial g_{12}}{\partial y} \frac{1}{2} \frac{d}{dt} g_{12} - \frac{1}{2} \frac{d}{dt} \frac{\partial g_{11}}{\partial y} = 0.
\]
\textbf{Theorem 2.} Let \( \mu, \nu : I \to R \) be two \( C^\infty \)-class functions which determine the second order linear homogeneous differential equation given in Equation (5), where \( I \subseteq R \) is an open real interval. Also, let \( \nabla \) be a symmetric linear connection on \( D = I \times R \) having the components defined as in Theorem 1. The following assertions are equivalent:

(i) the \( C^2 \)-class curve \( x : J \subseteq I \to D, x(t) = (t, \psi(t)), \forall t \in J \), for any solution \( y : J \subseteq I \to R \) of the ODE Equation (5), is geodesic with respect to the symmetric linear connection \( \nabla \);

(ii) the components of the Riemannian metric \( g = g_{ij} dx^i \otimes dx^j \) on \( D \) satisfy the following relations

\[
\frac{\partial g_{12}}{\partial y} = \frac{1}{2} \frac{\partial^2 g_{22}}{\partial y \partial t} - \frac{1}{2} \frac{\partial g_{22}}{\partial t} = \Gamma_{12}^2, \quad \frac{\partial g_{11}}{\partial y} = \frac{\partial g_{22}}{\partial t} = 2\Gamma_{12}^2
\]

\[
\frac{1}{2} \frac{\partial g_{11}}{\partial y} = \frac{\partial g_{12}}{\partial t} - \frac{1}{2} \frac{\partial g_{11}}{\partial y} = \Gamma_{11}^2
\]

\[
\frac{\partial g_{12}}{\partial y} \Gamma_{11}^2 + g_{12} \frac{\partial \Gamma_{11}^2}{\partial y} = \frac{\partial g_{12}}{\partial t} \Gamma_{11}^2 + g_{12} \frac{\partial \Gamma_{11}^2}{\partial t}
\]

\[
\frac{\partial g_{22}}{\partial y} \Gamma_{11}^2 + g_{22} \frac{\partial \Gamma_{11}^2}{\partial y} + \frac{\partial g_{22}}{\partial t} \Gamma_{11}^2 + g_{22} \frac{\partial \Gamma_{11}^2}{\partial t}
\]

\[
g_{12}(t, y)g_{22}(t, y) - (g_{12}(t, y))^2 > 0.
\]

(iii) the components of the Riemannian metric \( g = g_{ij} dx^i \otimes dx^j \) on \( D \) fulfill

\[
g_{22}(t, y) = \varphi(t), \quad g_{12}(t, y) = g_{21}(t, y) = \frac{1}{2} \psi(t) y + \psi(t)
\]

\[
g_{11}(t, y) = \frac{1}{4} \varphi(t) \mu(t) y^2 + \psi(t) \mu(t) y + k(t)
\]

subject to

\[
\varphi(t) = ce^{\int \mu(t) dt}, \quad c > 0; \quad k(t) = \frac{2\psi(t) \varphi(t)}{\varphi(t)} - \mu(t) \frac{\varphi^2(t)}{\varphi(t)}
\]

\[
\frac{1}{2} [\varphi(t) \mu(t)]' - \varphi(t) \nu(t) = 0, \quad \frac{1}{4} [\varphi(t) \mu(t)]' - \psi(t) \nu(t) = 0
\]

\[
\frac{1}{2} [\psi(t) \mu(t)]' - \psi(t) \nu(t) = 0, \quad g_{11}(t, y)g_{22}(t, y) - (g_{12}(t, y))^2 > 0.
\]

\textbf{Proof.} Considering the components of the symmetric linear connection \( \nabla \) on \( D \), given in Theorem 1 (see (iii)), the equivalence (ii) \( \iff \) (iii) follows by a direct computation. Let remark that the relations that appear at (ii) in our theorem represent only a rearrangement of the terms in Equation (4). Consequently, the proof of the equivalence (i) \( \iff \) (ii) follows in the same manner as in Theorem 1 (see the second part of the proof for the equivalence (ii) \( \iff \) (iii)). The proof is complete. \( \square \)

\textbf{Theorem 3.} The \( C^2 \)-class curve \( x : J \subseteq I \to D, x(t) = (t, \psi(t)), \forall t \in J \), for any solution \( y : J \subseteq I \to R \) of the ODE Equation (5), is geodesic with respect to the symmetric linear connection \( \nabla \) of components

\[
\Gamma_{11}^1(t, y) = 0, \quad \Gamma_{12}^1(t, y) = \Gamma_{21}^1(t, y) = 0, \quad \Gamma_{22}^1(t, y) = 0
\]
Theorem 4. The C^2-class curve x: \[ J \subseteq I \rightarrow D, \] of components \( \Gamma^i_{jk} \), \( i, j, k \in \{1, 2\} \), if and only if the differential Equations (2), (6) and (7) have the same set of solutions and the following complete integrability and positive defining conditions hold

\[
\frac{\partial g_{12}}{\partial y} \Gamma^1_{11} + g_{12} \frac{\partial \Gamma^1_{11}}{\partial y} = \frac{\partial g_{12}}{\partial t} \Gamma^1_{12} + g_{12} \frac{\partial \Gamma^2_{12}}{\partial t} 
\]

\[
\frac{\partial g_{22}}{\partial y} \Gamma^2_{12} + g_{22} \frac{\partial \Gamma^2_{12}}{\partial y} = \frac{\partial g_{22}}{\partial t} \Gamma^2_{22} + g_{22} \frac{\partial \Gamma^3_{22}}{\partial t} 
\]

\[
\frac{\partial g_{22}}{\partial y} \Gamma^2_{11} + g_{22} \frac{\partial \Gamma^2_{11}}{\partial y} + \frac{\partial g_{12}}{\partial y} \Gamma^1_{12} + g_{12} \frac{\partial \Gamma^2_{12}}{\partial y} = \frac{\partial g_{12}}{\partial t} \Gamma^2_{12} + g_{12} \frac{\partial \Gamma^3_{12}}{\partial t} + \frac{\partial g_{22}}{\partial t} \Gamma^2_{12} + g_{22} \frac{\partial \Gamma^3_{12}}{\partial t} 
\]

\[
g_{11}(t, y)g_{22}(t, y) - (g_{12}(t, y))^2 > 0.
\]

**Proof.** Using the previous two theorems (see Theorem 1 and Theorem 2) and Remark 1, the proof is immediately clear. \( \square \)

**Remark 2.** Let us notice that the components of the symmetric linear connection \( \nabla \) mentioned in the previous result (see Theorem 3) can be obtained from the condition that the differential Equations (1), (2) and the ODE (3) to have the same set of solutions. Consequently, we get the next result.

**Theorem 4.** The C^2-class curve x: \( J \subseteq I \rightarrow D, \) of components \( \Gamma^i_{jk} \), \( i, j, k \in \{1, 2\} \), if and only if the differential Equations (1), (2), (6), (7) and the ODE Equation (5) have the same set of solutions and

\[
\frac{\partial g_{12}}{\partial y} \Gamma^1_{11} + g_{12} \frac{\partial \Gamma^1_{11}}{\partial y} = \frac{\partial g_{12}}{\partial t} \Gamma^1_{12} + g_{12} \frac{\partial \Gamma^2_{12}}{\partial t} 
\]

\[
\frac{\partial g_{22}}{\partial y} \Gamma^2_{12} + g_{22} \frac{\partial \Gamma^2_{12}}{\partial y} = \frac{\partial g_{22}}{\partial t} \Gamma^2_{22} + g_{22} \frac{\partial \Gamma^3_{22}}{\partial t} 
\]

\[
\frac{\partial g_{22}}{\partial y} \Gamma^2_{11} + g_{22} \frac{\partial \Gamma^2_{11}}{\partial y} + \frac{\partial g_{12}}{\partial y} \Gamma^1_{12} + g_{12} \frac{\partial \Gamma^2_{12}}{\partial y} = \frac{\partial g_{12}}{\partial t} \Gamma^2_{12} + g_{12} \frac{\partial \Gamma^3_{12}}{\partial t} + \frac{\partial g_{22}}{\partial t} \Gamma^2_{12} + g_{22} \frac{\partial \Gamma^3_{12}}{\partial t} 
\]

\[
g_{11}(t, y)g_{22}(t, y) - (g_{12}(t, y))^2 > 0.
\]

**Proof.** Taking into account the foregoing remark (see Remark 2) and Theorem 3, the proof is obvious. \( \square \)

### 3. Conclusions and Further Developments

In this paper, we have studied the geodesic behavior of some special plane curves. We have developed an original point of view by introducing some new results regarding the geodesics and auto-parallel curves literature. Concretely, we have studied the geometric structures on an open subset \( D \subseteq \mathbb{R}^2 \) such that the graphs associated with the solutions of some special functions to become geodesics. A potential application of the obtained results is the inverse problem of the calculus of variations, which seeks the conditions for a system of second-order differential equations to be equivalent to the Euler–Lagrange equations of some Lagrangian function.

**Funding:** The APC was funded by University Politehnica of Bucharest, “PubArt” program.

**Conflicts of Interest:** The author declares no conflict of interest.
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