Note on 3-Choosability of Planar Graphs with Maximum Degree 4

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Abstract

Deciding whether a planar graph (even of maximum degree 4) is 3-colorable is NP-complete. Determining subclasses of planar graphs being 3-colorable has a long history, but since Grötzsch’s result that triangle-free planar graphs are such, most of the effort was focused to solving Havel’s and Steinberg’s conjectures. In this paper, we prove that every planar graph of maximum degree 4 obtained as a subgraph of the medial graph of any bipartite plane graphs is 3-choosable. These graphs are allowed to have close triangles (even incident), and have no short cycles forbidden, hence representing an entirely different class than the graphs inferred by the above mentioned conjectures.

Keywords: medial graph, plane graph, 3-colorability, 3-choosability, Alon-Tarsi Theorem.

1 Introduction

In this paper we consider the problem of 3-colorability of a subclass of 4-regular planar graphs. We are motivated by the following problem, proposed by Czap, Jendroľ, and Voigt [9, Problem 3.9].

Problem 1. Is there a bipartite plane graph $G$ such that its medial graph has chromatic number 4?

We prove that such a graph does not exist, even more, the medial graphs of bipartite plane graphs are 3-choosable.

Medial graphs are planar and 4-regular, hence the problem reduces to investigating 3-colorability of a subclass of planar graphs with maximum degree 4. While every graph of maximum degree 3 which is not isomorphic to $K_4$, is 3-colorable by Brooks’ Theorem, already in the class of planar graphs with maximum degree 4 deciding whether a graph...
admits a 3-coloring is NP-complete [17]. Due to this fact, and even more due to the famous Four Color Theorem, the problem of 3-coloring received a lot of attention in the class of planar graphs. For any plane triangulations, Heawood found a necessary and sufficient condition [21] showing that it is 3-colorable if and only if all its vertices have even degrees. Generalizations of this statement have been given in [10, 16] and just recently in [22].

On the other hand, a well-known Grötzsch’s [19] result shows that if there are no cycles of length 3 in a planar graph, then it is 3-colorable. This result was later improved by Grünbaum [18] to planar graphs with at most three triangles. The original proof was faulty, but was later corrected by Aksenov [1]. Finally, a simpler proof of his result was recently given in [6].

Allowing some triangles in a graph, but still retaining 3-colorability yielded two intriguing conjectures. First, Havel [20] conjectured that a 3-colorable planar graph may contain many triangles as long as they are sufficiently far apart. This conjecture was recently proved by Dvořák, Král’, and Thomas [13] (they announced the result already in 2009, but included it in a series of papers on 3-colorability of triangle-free graphs on surfaces; cf. [12]).

The second conjecture is due to Steinberg [23]. It allows arbitrarily many triangles but it forbids short cycles. Namely, Steinberg conjectured that every planar graph without cycles of length 4 and 5 is 3-colorable. The conjecture was disproved by Cohen-Addad et al. [8]; however a number of weaker results have been proved, perhaps the closest being due to Borodin et al. [1], stating that every planar graph without cycles of length 5 and 7, and without adjacent triangles is 3-colorable (see also [5, 6, 7] for other results on this conjecture).

All the problems listed above are even harder in a more general setting of list coloring. As shown by Voigt [26], planar graphs are not 4-choosable; Thomassen [24] found a beautiful proof that they are 5-choosable. It is a rather simple observation that planar bipartite graphs are not 2-choosable, and a bit harder that they are 3-choosable [3]. Therefore it is not surprising that an equivalent of Grötzsch’s result does not hold in this setting; as shown by Voigt [27], there are triangle-free planar graphs which are not 3-choosable. Thomassen [25] however proved that girth 5 is a sufficient condition for their 3-choosability.

An analogue of Havel’s conjecture hence requires cycles of length 3 and also 4 to be sufficiently distant. Dvořák [11] proved that distance 26 between them is sufficient, while the best lower bound requires distance at least 4 [2]. The list-version of Steinberg’s conjecture clearly requires more excluded cycles. So far, there has been a number of partial results towards solving it (cf. [13] and references therein for more details), with currently the best one, due to Dvořák and Postle [15], showing that planar graphs without cycles of lengths from 4 to 8 are 3-choosable. It is still not known if it suffices to forbid only cycles from 4 to 7 or even from 4 to 6.

We contribute to the above described rich field of research with the following theorem, which does assume a special structure of a graph, but does not particularly bound the number of triangles, they can even have common vertices (so the distance between them can be as small as one), and the graph can contain cycles of any length.
Theorem 1. Every loopless planar graph with maximum degree 4 obtained as a subgraph of the medial graph of a bipartite plane graph is 3-choosable.

We prove the theorem in Section 3 while in Section 2 we introduce notation and auxiliary results. We conclude with two open problems in Section 4.

2 Preliminaries

For a plane graph \( G \), with \( V(G), E(G), \) and \( F(G) \) we denote its set of vertices, edges, and faces, respectively. The medial graph \( M(G) \) of \( G \) is the graph with the vertex set \( V(M(G)) = E(G) \), two vertices \( u \) and \( v \) being connected if the edges of \( G \) corresponding to \( u \) and \( v \) are consecutive on some facial trail of \( G \). Thus, every medial graph is 4-regular.

We define the boundary, interior, and exterior of any plane Eulerian graph \( H \) in the following way. First, color the faces of \( H \) with two colors such that adjacent faces receive distinct colors (this is possible as the dual of \( H \) is bipartite). Let the outerface of \( H \) be colored green and its adjacent faces blue. The boundary of \( H \), \( \partial(H) \), is the graph \( H \) itself. The interior \( \text{int}(H) \) is the graph induced by the vertices of \( G \) lying in the blue faces of \( H \) together with the vertices of \( H \) without the edges of \( H \), and the exterior \( \text{ext}(H) \) is the graph induced by the vertices of \( G \) lying in the green faces of \( H \) together with the vertices of \( H \) without the edges of \( H \). Similarly, for a subgraph \( X \) of \( G \), we define

\[
\partial_X(H) = \partial(H) \cap X, \quad \text{int}_X(H) = \text{int}(H) \cap X, \quad \text{ext}_X(H) = \text{ext}(H) \cap X.
\]

For a directed graph (digraph) \( D \), we define the indegree (denoted \( d^-(v) \)) and outdegree (denoted \( d^+(v) \)) of a vertex \( v \) as the number of directed edges having \( v \) as a terminal and an initial vertex, respectively. A subdigraph \( H \) of a digraph \( D \) is called Eulerian if the indegree \( d^+_H(v) \) of every vertex \( v \) of \( H \) is equal to its outdegree \( d^-_H(v) \). The digraph \( H \) is even if it has an even number of directed edges, otherwise, it is odd. Let \( E^e(D) \) and \( E^o(D) \) be the numbers of even and odd spanning Eulerian subgraphs of \( D \), respectively.

By an intersection of graphs \( G_1 \) and \( G_2 \), \( G_1 \cap G_2 \), we mean the graph with the sets of vertices and edges that are contained in both graphs.

The following well-known result due to Alon and Tarsi [3] plays the key role in our proof of Theorem 1.

Theorem 2 (Alon & Tarsi, 1992). Let \( D \) be a directed graph, and let \( L \) be a list-assignment such that \( |L(v)| \geq d^+_D(v)+1 \) for each \( v \in V(D) \). If \( E^e(D) \neq E^o(D) \), then \( D \) is \( L \)-colorable.

We say that \( L \) is a list-assignment for the graph \( G \) if it assigns a list \( L(v) \) of possible colors to each vertex \( v \) of \( G \). If \( G \) admits a proper coloring \( \varphi_l \) such that \( \varphi_l(v) \in L(v) \) for all vertices in \( V(G) \), then we say that \( G \) is \( L \)-colorable or \( \varphi_l \) is an \( L \)-coloring of \( G \). The graph \( G \) is \( k \)-choosable if it is \( L \)-colorable for every assignment \( L \), where \( |L(v)| \geq k \) for every \( v \in V(G) \). The list chromatic number \( \chi_l(G) \) of \( G \) is the smallest \( k \) such that \( G \) is \( k \)-choosable.
3 Proof of Theorem 1

In this section, we present a proof of Theorem 1 using the result of Alon and Tarsi mentioned in the previous section. We first discuss the assumptions of Theorem 1. The medial graph of a graph with a vertex \( v \) of degree 1 contains a loop, since the facial trail traversing \( v \) uses the edge incident to \( v \) twice. As a graph with loops cannot admit a proper vertex coloring, since a vertex incident with a loop is adjacent to itself, we rather consider its subgraph with the loops removed.

Proof. Let \( B \) be a bipartite plane graph. For technical reasons described above, we recursively remove all its vertices of degree at most 1 to obtain the graph \( B' \) with minimum degree 2. Let \( G \) be the medial graph of \( B' \). Below, we describe the procedure to list color \( G \) with at most 3 colors. It is straightforward to observe that any coloring of \( G \) can be extended to a list 3-coloring of \( M(B) \) with loops removed by subsequently coloring the removed vertices in the opposite order, since every vertex to be colored has at most two colored neighbors.

There are two types of faces in \( G \): the ones corresponding to the vertices of \( B \) (we call them black), and the ones corresponding to the faces of \( B \) (we call them white). Notice that all white faces have even length. Moreover, every edge in \( G \) is incident to two faces, one black and one white (see the left graph in Figure 1 for an example).

\[ \text{Figure 1: The left graph is the medial graph } M(Q_3) \text{ of the cube } Q_3; \text{ its vertices and edges are depicted as full circles and solid edges, while the vertices and edges of } Q_3 \text{ are depicted with empty circles and dashed edges. The black faces of } M(Q_3) \text{ are shaded. The right graph is the directed graph } \vec{M}(Q_3) \text{ such that the edges have a black face always on their left hand side.} \]

Let \( \vec{G} \) be a directed graph obtained from \( G \) by directing the edges such that each edge has its black face on the left hand side when going from its initial to its terminal vertex. This in particular means that every vertex has precisely two incoming and two outgoing edges, and therefore \( d^-(v) = d^+(v) = 2 \) for every \( v \in V(\vec{G}) \). Apart from the regularity, we will also use the following fact.
Observation 1. Let $D_1$ and $D_2$ be two directed cycles in $\vec{G}$ intersecting (i.e., having some common vertices) in such a way that $\partial(D_2) \cap \text{int}(D_1) \neq \emptyset$ and $\partial(D_2) \cap \text{ext}(D_1) \neq \emptyset$. Then $E(D_1) \cap E(D_2) \neq \emptyset$.

This is implied by the choice of orientation: two consecutive edges on a directed cycle are always also consecutive on some facial trail.

In what follows, we will show that every odd Eulerian spanning subgraph of $\vec{G}$ can be injectively mapped to an even Eulerian spanning subgraph of $\vec{G}$. We will also show there is an even Eulerian subgraph of $\vec{G}$ to which no odd subgraph is mapped, and thus fulfill the assumptions of Theorem 2. That will imply 3-choosability of $G$.

We distinguish two types of directed cycles in $\vec{G}$: by the definition and Observation 1, all the edges of a given directed cycle $C$ are incident either to black faces or to white faces in the interior of $C$. We refer to the former as black cycles (see Figure 2 for an example) and to the latter as white cycles. Similarly, we say that an Eulerian graph is white if it is comprised of white cycles only.

For a graph $X$, we denote its complement by $\overline{X}$. For a cycle $D$, the $D$-complement of a spanning Eulerian subgraph $X$ of $\vec{G}$ is the spanning Eulerian subgraph $\overline{X}^D$ with the edge set

$$E(\overline{X}^D) = E(\text{ext}_X(D)) \cup E(\text{int}_X(D)) \cup E(\partial_X(D)).$$

The fact that $\overline{X}^D$ is also Eulerian follows from Observation 1.

Claim 1. For an odd black cycle $D$, the $D$-complement of an odd (even) Eulerian spanning subgraph $X$ is an even (odd) Eulerian spanning subgraph $\overline{X}^D$.

Proof. All the edges of $\text{int}(D)$ comprise a disjoint union of cycles around white faces. Since every white face has even length, there is an even number of edges in $\text{int}(D)$. Hence, the parity of the number of edges in $\text{int}_X(D)$ is the same as the parity of the edges in $\text{int}_{\overline{X}}(D)$, and so the parity of the number of edges of $X$ is different from the parity of the edges of
its $D$-complement as one of them contains an odd number of edges of $D$, and the other one an even number of edges of $D$. (Recall that $E(\text{ext}(\overline{X}^D)) = E(\text{ext}(X)).$) ♦

Before we proceed, we prove a claim about cycles in the intersection of an Eulerian graph and its $D$-complement.

**Claim 2.** Let $X$ be an Eulerian spanning subgraph of $\overline{G}$, and let $D$ be a white odd Eulerian subgraph of $X$. Then, there is an odd black cycle in $\text{int}_X(D)$ or $\text{int}_{\overline{X}^D}(D)$.

**Proof.** Suppose the contrary and let $D$ be minimal in terms of the number of faces in its interior. On the internal side of the edges of the boundary $\partial(D)$, there are white faces. Take the edges of these faces which are not in $\partial(D)$. There is also an odd number of them, since white faces have even length. At least one such edge $e$ is in $X$, otherwise there is an odd black cycle in $\text{int}_{\overline{X}^D}(D)$. But, $e$ also belongs to some black cycle $C$ in $\text{int}_X(D)$. If $C$ is even, we add it to $D$ (in which case it is white from the point of view of $D$), removing the edges of $\text{int}(C)$ from $D$ and obtaining a smaller graph, which contradicts the minimality of $D$. Otherwise, $C$ is an odd black cycle, and hence the claim is established. ♦

Denote by $\mathcal{E}$ the set of all Eulerian spanning subgraphs of $\overline{G}$. Let $\mathcal{O}$ be a sorted set of all odd black cycles in $\overline{G}$, sorted in ascending order by the number of faces they contain in their interiors. Suppose there are $k$ cycles, $C_1, C_2, \ldots, C_k$, in $\mathcal{O}$. For every $i$, $1 \leq i \leq k$, in consecutive order, we repeatedly remove all $X \in \mathcal{E}$ which either contain all the edges of $C_i$ or none of them. Hence, if in the step $i$ we remove from $\mathcal{E}$ some $X$, then we also remove its $C_i$-complement if it is still in $\mathcal{E}$. In the proof of the claim below, we show that such pairs are always removed at the same step.

**Claim 3.** The number of odd Eulerian spanning subgraphs removed from $\mathcal{E}$ at step $i$ is equal to the number of even such subgraphs.

**Proof.** By Claim 2 we have that an Eulerian spanning subgraph $X$ is of different parity from its $C_i$-complement. Hence, proving that removing $X$ from $\mathcal{E}$ at step $i$ implies removal of its $C_i$-complement at step $i$ establishes the claim.

Suppose the contrary, and let $i$ be minimal such that there is some $X$ in $\mathcal{E}$ whose $C_i$-complement $\overline{X}^{C_i}$ is not in $\mathcal{E}$, i.e. it has been removed in some step $j$, with $j < i$. First, notice that $C_j$ is not completely contained in $\partial(C_i) \cup \text{int}(C_i)$, since, by the definition of $\overline{X}^{C_i}$, $X$ would also contain all the edges of $C_j$ or none of them, meaning that $X$ would also be removed at the step $j$. Hence, $C_j$ has an edge in $\text{ext}(C_i)$. For the same reason, $C_j$ is not completely contained in $\text{ext}(C_i)$, and thus has an edge in $\partial(C_i) \cup \text{int}(C_i)$. In fact, by Observation 1 it follows that there must be some edge of $C_j$ on $\partial(C_i)$. Finally, there is also some edge of $C_j$ in $\text{int}(C_i)$, otherwise $C_j$ is either contained in $\partial(C_j) \cup \text{int}(C_j)$, in which case $i < j$, or $C_j$ is white. In both cases, we obtain a contradiction.

Now, we show that there is some odd black cycle in $\partial(C_i) \cup \text{int}(C_i)$ (distinct from $C_i$) such that $X$ either contains all its edges or none of them. Suppose first that $X$ contains all the edges of $C_i$. Since, by the above argumentation, $C_i$ and $C_j$ have some edges in common, this implies that none of the edges of $C_j$ is in $\overline{X}^{C_i}$, and so all the edges of $C_j$ in $\partial(C_i) \cup \text{int}(C_i)$ are also in $X$. 

6
By Observation 1, \( X_{\text{Int}} = \partial(C_i) \cup \text{int}_X(C_i) \) is Eulerian with the outerface of odd length. By the Handshake Lemma, there is also an innerface \( f \) of odd length (note that \( f \) is a face of \( X_{\text{Int}} \), but not necessarily of \( \partial(C_i) \cup \text{int}(C_i) \)). Moreover, \( f \) is not bounded by \( C_i \), since, by the argumentation above, there is at least one edge in \( \text{int}_X(C_i) \). Let \( C \) be the cycle bounding \( f \) in \( X_{\text{Int}} \). If \( C \) is black, we are done, since there is some \( \ell < i \) such that \( C = C_\ell \), and so \( X \) would be removed at the step \( \ell \).

Otherwise, \( C \) is white and by Claim 2, there is some odd black cycle \( C_\ell \in \mathcal{O} \) in \( \text{int}(C) \), such that \( X^{C_i} \) contains all the edges of \( C_\ell \) or none of them, and \( X \) contains none of the edges of \( C_\ell \) or all of them, respectively, which means that \( X \) would be removed at the step \( \ell \).

Suppose now that \( X \) contains none of the edges of \( C_i \), and therefore \( X^{C_i} \) contains all of them. In this case, we use analogous argumentation as in the previous paragraph that there is an odd black cycle such that \( \partial(C_i) \cup \text{int}_X(C_i) \) contains either all or none of its edges, meaning that \( X \) contains none or all of its edges, respectively, which again implies that \( X \) would have been removed in some of the previous steps, a contradiction.

Hence, after all cycles from \( \mathcal{O} \) are removed, by Claim 2, there is no odd Eulerian spanning subgraph left in \( \mathcal{E} \). However, there is at least one even Eulerian spanning subgraph, which contains at least one edge of every odd black cycle in \( \vec{G} \), but not all edges of any. We guarantee its existence by the following claim.

Claim 4. White faces of \( G \) can be colored with two colors, red and blue, such that every odd black cycle shares an edge with the boundary of at least one red and at least one blue face.

Proof. Let \( H \) be the graph whose vertex set is formed by the white faces of \( G \) and two vertices of \( H \) are connected if the corresponding white faces share a vertex in \( G \). The graph \( H \) is planar, and hence we can color its vertices with four colors, say 1, 2, 3, and 4, by the Four Color Theorem. Color the white faces of \( G \) whose corresponding vertices in \( H \) are colored with 1 or 2, with red, and the other white faces with blue.

Let \( C \) be an odd black cycle. By the orientation of the graph, every vertex of \( C \) has its two incident edges that are not in \( C \) either both in \( \text{int}(C) \) or both in \( \text{ext}(C) \). Let \( V_i \) be the set of the vertices of \( C \) that have two incident edges in \( \text{int}(C) \) and let \( V_e \) be the set of vertices of \( C \) that have two incident edges in \( \text{ext}(C) \). As every vertex has degree 4, the number of edges in \( \text{int}(C) \) is equal to twice the number of vertices that have all of their edges in \( \text{int}(C) \), plus \( |V_i| \). As the edges in \( \text{int}(C) \) are the edges of the disjoint union of the boundaries of white faces, and as white faces are even, there are an even number of edges in \( \text{int}(C) \), thus \( |V_i| \) is even. The number of white faces that share an edge with \( C \) is equal to \( |V_e| \), and as \( C \) is odd, \( |V_e| \) is odd. Therefore there is an odd number of white faces that share edges with \( C \), and these form an odd cycle in \( H \). As there are three colors needed for coloring an odd cycle, the claim is established.

By taking the edges of the union of the boundaries of the red faces, we obtain an even Eulerian subgraph that contains at least one edge of every odd black cycle in \( \vec{G} \), but not all edges of any. This even eulerian subgraph is still in \( \mathcal{E} \).
Hence, we have proved that there are more even Eulerian spanning subgraphs in $\vec{G}$ as odd Eulerian spanning subgraphs and thus fulfill the assumptions of Theorem 2. This means that $G$ is 3-choosable.

4 Conclusion

In this paper, we answered the question of Czap, Jendroľ, and Voigt [9, Problem 3.9] about chromatic number of medial graphs of bipartite plane graphs. We used an application of the Theorem of Alon and Tarsi and to satisfy the main assumption of it, we strongly used the fact our graphs have maximum degree 4 and that its faces can be properly colored in two colors, where one color class contains only even faces. It is not clear if the former condition is really needed. In fact, we believe that the following conjecture can be answered in affirmative.

**Conjecture 1.** Every simple plane graph whose faces can be properly colored with two colors such that one color class contains only even faces is 3-colorable.

In Conjecture 1 we require a simple graph, since any plane graph with every edge replaced by two parallel edges satisfies the assumption of being 2-face-colorable, with the faces created by the parallel edges being even, and so any plane graph with chromatic number 4 would be a counterexample. Our result, however, allows parallel edges since the maximum degree is limited to 4.

Conjecture 1 can be strengthened to the choosability version, but we are less certain about the answer.

**Question 1.** Is every simple plane graph whose faces can be properly colored with two colors such that one color class contains only even faces also 3-choosable?

One might ask, why limit only to plane graphs. Theorem 1 does not hold in general for graphs that embed to other surfaces. In Figure 3 we present a graph which needs 4 colors for a vertex-coloring (as it is straightforward to check that 3 colors do not suffice to color it, we leave it to the reader).

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Figure 3: A medial graph (depicted in black) of a bipartite graph embedded on torus (depicted in grey) having chromatic number 4.

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