LEAST AREA SPHERICAL CATENOIDS IN HYPERBOLIC THREE-DIMENSIONAL SPACE

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ABSTRACT. In this paper we try to characterize the least area spherical catenoids in hyperbolic 3-space \( \mathbb{H}^3 \).

1. INTRODUCTION

Suppose that \( \Sigma \) is a surface (compact or complete) and that \( M \) is a 3-dimensional Riemannian manifold (compact or complete). An immersion \( f : \Sigma \to M \) is called a minimal surface if its mean curvature is identically equal to zero. An introduction on minimal surfaces can be found in the book [7, Chapter 1].

In this paper, \( M \) is always the three dimensional hyperbolic space \( \mathbb{H}^3 \). For any immersed minimal surface \( \Sigma \), we define the Jacobi operator on \( \Sigma \) by

\[
L = \Delta_{\Sigma} + (|A|^2 - 2),
\]

here \( \Delta_{\Sigma} \) is the Lapalican on \( \Sigma \) and \( |A|^2 \) is the square norm of the second fundamental form on \( \Sigma \).

If \( \Sigma \) is compact with nonempty boundary, the Morse index or index of \( \Sigma \) is the number of negative eigenvalues of the Jacobi operator \( L \) (counting with multiplicity) acting on the space of smooth sections of the normal bundle that vanishes on \( \partial \Sigma \). In this case, we say that \( \Sigma \) is stable if its Morse index is zero, or equivalently if

\[
\int_{\Sigma} |\nabla_{\Sigma} u|^2 > \int_{\Sigma} (|A|^2 - 2)u^2
\]

for all \( u \in C^\infty_0(\Sigma) \). If \( \Sigma \) is a complete (non-compact) minimal surface without boundary, then its Morse index is the supremum of the Morse indices of compact subdomains of \( \Sigma \). In this case, it is globally stable or stable if any compact subdomain of \( \Sigma \) is stable. For a complete minimal surface \( \Sigma \) without boundary, an subdomain \( D \subset \Sigma \) is said to be weakly stable if

\[
\int_{D} |\nabla_{\Sigma} u|^2 \geq \int_{D} (|A|^2 - 2)u^2
\]

for all \( u \in C^\infty_0(D) \). An open subdomain \( D \subset \Sigma \) is maximally weakly stable if it is weakly stable and any other open subdomain that is larger than \( D \) is unstable.

If \( \Sigma \) is a compact minimal surface, then it is least area if its area is smaller than that of any other surface in the same homotopic class; it is area minimizing if its
area is no larger than that of any surface in the same homological class. If $\partial \Sigma \neq \emptyset$, we require that the other surfaces should have the same boundary as $\Sigma$. If $\Sigma$ is a complete minimal surface, then it is least area or area minimizing if any compact subdomain of $\Sigma$ is least area or area minimizing.

In [8], do Carmo and Dajczer studied three types of rotationally symmetric minimal hypersurfaces in $\mathbb{H}^{n+1}$. A rotationally symmetric minimal hypersurface is called a spherical catenoid if it is foliated by spheres, a hyperbolic catenoid if it is foliated by totally geodesic hyperplanes, and a parabolic catenoid if it is foliated by horospheres.

We are interested in two dimensional catenoids in $\mathbb{H}^3$. Do Carmo and Dajczer proved that the hyperbolic and parabolic catenoids are always globally stable (see [8, Theorem 5.5]), then Candel proved that the hyperbolic and parabolic catenoids are also least area minimal surfaces (see [6, p. 3574]).

Compared with the hyperbolic and parabolic catenoids, the spherical catenoids are more complicated. Let $\mathbb{B}^2_+$ be the upper half unit disk on the $xy$-plane with metric given by (2.6), and let $\sigma_\lambda$ be the catenary given by (2.9), which is symmetric about the $y$-axis and passes through the point $(0, \lambda)$, here $\lambda > 0$. Let $\Pi_\lambda$ be the spherical catenoid generated by $\sigma_\lambda$. Mori, Do Carmo and Dajczer, Bérard and Sa Earp, and Seo proved the following result.

**Theorem 1.1** ([17, 8, 4, 18]). There exist two positive numbers $\Lambda_1 \leq \Lambda_2$ such that $\Pi_\lambda$ is unstable if $0 < \lambda < \Lambda_1$, and $\Pi_\lambda$ is globally stable if $\lambda > \Lambda_2$.

**Remark 1.** Do Carmo and Dajczer showed that $\Lambda_1 \approx 0.42315$ in [8]. Seo showed that $\Lambda_1 \approx 0.46288$ in [18]. Mori showed that $\Lambda_2 = \cosh^{-1}(3) \approx 1.7627$ in [17]. Bérard and Sa Earp showed that $\Lambda_2 = \frac{1}{2} \cosh^{-1}\left(\sqrt{\frac{11 + 8\sqrt{2}}{7}}\right) \approx 0.5915$ in [4].

Bérard and Sa Earp also provided better estimates on $\Lambda_1$ and $\Lambda_2$ in [3, § 7.3.4]. Here we should remind the readers that the authors of the papers [17, 8, 18] used the hyperboloid model, so we should use a formula given by Bérard and Sa Earp in [3, p. 34] to get the above constants.

Similar to the hyperbolic and parabolic catenoids, we want to know whether the globally stable spherical catenoids are least area minimal surfaces. In this paper, we will proved that there exists a positive number $\Lambda_0$ such that $\Pi_\lambda$ is of least area if $\lambda > \Lambda_0$. More precisely, we will prove the following result.

**Theorem 1.2.** There exists a number $\Lambda_0 \approx 1.10055$ such that for any $\lambda > \Lambda_0$ the catenoid $\Pi_\lambda$ is a least area minimal surface.

2. **Minimal Catenoids in Hyperbolic 3-Space**

In this paper, we will work in the ball model of $\mathbb{B}^3$, i.e.,

$$\mathbb{B}^3 = \{(u,v,w) \in \mathbb{R}^3 \mid u^2 + v^2 + w^2 < 1\},$$
equipped with metric
\[ ds^2 = \frac{4(du^2 + dv^2 + dw^2)}{(1 - r^2)^2}, \]
where \( r = \sqrt{u^2 + v^2 + w^2} \). The hyperbolic space \( \mathbb{B}^3 \) has a natural compactification:
\( \overline{\mathbb{B}}^3 = \mathbb{B}^3 \cup S^2_\infty \), where \( S^2_\infty \cong \mathbb{C} \cup \{ \infty \} \) is called the Riemann sphere. The orientation preserving isometry group of \( \mathbb{B}^3 \) is denoted by \( \text{Möb}(\mathbb{B}^3) \), which consists of Möbius transformations that preserve the unit ball (see [14] Theorem 1.7).

Suppose that \( X \) is a subset of \( \mathbb{B}^3 \), we define the asymptotic boundary of \( X \) by
\[ \partial_\infty X = \overline{X} \cap S^2_\infty, \]
where \( \overline{X} \) is the closure of \( X \) in \( \overline{\mathbb{B}}^3 \).

Using the above notation, we have \( \partial_\infty \mathbb{B}^3 = S^2_\infty \). If \( P \) is a geodesic plane in \( \mathbb{B}^3 \), then \( P \) is perpendicular to \( S^2_\infty \) and \( C \) is an Euclidean circle on \( S^2_\infty \). We also say that \( P \) is asymptotic to \( C \).

Suppose that \( G \) is a subgroup of \( \text{Möb}(\mathbb{B}^3) \) that leaves a geodesic \( \gamma \subset \mathbb{B}^3 \) pointwise fixed. We call \( G \) the spherical group of \( \mathbb{B}^3 \) and \( \gamma \) the rotation axis of \( G \). A surface in \( \mathbb{B}^3 \) invariant under \( G \) is called a spherical surface or a surface of revolution. For two circles \( C_1 \) and \( C_2 \) in \( \mathbb{B}^3 \), if there is a geodesic \( \gamma \) such that each of \( C_1 \) and \( C_2 \) is invariant under the group of rotations that fixes \( \gamma \) pointwise, then \( C_1 \) and \( C_2 \) are said to be coaxial, and \( \gamma \) is called the rotation axis of \( C_1 \) and \( C_2 \).

If \( C_1 \) and \( C_2 \) are two disjoint circles on \( S^2_\infty \), then they are always coaxial. In fact, there always exists a unique geodesic \( \gamma \) such that \( \gamma \) is perpendicular to both \( P_1 \) and \( P_2 \). Then \( C_1 \) and \( C_2 \) are coaxial with respect to \( \gamma \).

In [20], we construct quasi-Fuchsian 3-manifolds which contain arbitrarily many incompressible minimal surface by using spherical minimal catenoids as the barrier surfaces, so we want to know whether there exists a minimal spherical catenoid asymptotic to any given pair of circles on \( S^2_\infty \). In order to answer this question, at first we need define the distance between two circles on \( S^2_\infty \). Let \( C_1 \) and \( C_2 \) are two disjoint circles on \( S^2_\infty \) and let \( P_1 \) and \( P_2 \) be geodesic planes asymptotic to \( C_1 \) and \( C_2 \) respectively. Then the distance between \( C_1 \) and \( C_2 \) is given by
\[ d_L(C_1, C_2) = \text{dist}(P_1, P_2), \]
here \( \text{dist}(\cdot, \cdot) \) is the hyperbolic distance on \( \mathbb{B}^3 \).

In this paper, we follow Hsiang’s idea to describe the spherical catenoids in \( \mathbb{B}^3 \) ([11] [1]). Suppose that \( G \) is the spherical group of \( \mathbb{B}^3 \) along the geodesic \( \gamma_0 = \{ (u, 0, 0) \in \mathbb{B}^3 \mid -1 < u < 1 \} \), then \( \mathbb{B}^3/G \cong \mathbb{B}^2_+ \), here
\[ \mathbb{B}^2_+ = \{ (u, v) \in \mathbb{B}^2 \mid v \geq 0 \} \]
For any point \( p = (u, v) \in \mathbb{B}^2_+ \), there is a unique geodesic segment \( \gamma' \) passing through \( p \) that is perpendicular to \( \gamma_0 \) at \( q \). Let \( x = \text{dist}(O, q) \) and \( y = \text{dist}(p, q) = \text{dist}(p, \gamma_0) \), then by [2] Theorem 7.11.2], we have
\[ \tan x = \frac{2u}{1 + (u^2 + v^2)} \quad \text{and} \quad \sinh y = \frac{2v}{1 - (u^2 + v^2)}. \]
Equivalently, we also have

\[
(2.5) \quad u = \frac{\sinh x \cosh y}{1 + \cosh x \cosh y} \quad \text{and} \quad v = \frac{\sinh y}{1 + \cosh x \cosh y}.
\]

Thus \( \mathbb{B}_+^2 \) can be equipped with a metric of warped product in terms of the parameters \( x \) and \( y \)

\[
(2.6) \quad ds^2 = \cosh^2 y \cdot dx^2 + dy^2.
\]

If \( \Pi \) is a minimal surface of revolution in \( \mathbb{B}^3 \) with respect to the axis \( \gamma_0 \) (or the \( u \)-axis), then the curve \( \sigma = \Pi \cap \mathbb{B}_+^2 \) is called the generating curve of \( \Pi \).

Suppose that \( \sigma \) is given by the parametric equations:
\[
(2.7) \quad \frac{2\pi \sinh y \cdot \cosh^2 y}{\sqrt{\cosh^2 y + (y')^2}} = 2\pi \sinh y \cdot \cosh y \cdot \sin \alpha = k \quad \text{(constant)},
\]

where \( y' = dy/dx \) and \( \alpha \) is the angle between the tangent vector of \( \sigma \) and the vector \( e_y = \partial / \partial y \) at the point \( (x(s), y(s)) \).

By the discussion in [11, pp. 486–488], the curve \( \sigma \) satisfies the following equations

\[
(2.8) \quad \sin \alpha = \frac{\sinh(y_0) \cosh(y_0)}{\sinh(y) \cosh(y)} = \frac{\sinh(2y_0)}{\sinh(2y)}.
\]

Now solve \( x \) in terms of \( y \) from (2.7) and take the definite integral from \( y_0 \) to \( y \) for any \( y \geq y_0 \), we have

\[
(2.9) \quad x(y) = \int_{y_0}^{y} \frac{\sinh(2y_0)}{\cosh y} \frac{dy}{\sqrt{\sinh^2(2y) - \sinh^2(2y_0)}}.
\]

Let \( y \to \infty \), we get

\[
(2.10) \quad x(\infty) = \int_{y_0}^{\infty} \frac{\sinh(2y_0)}{\cosh y} \frac{dy}{\sqrt{\sinh^2(2y) - \sinh^2(2y_0)}},
\]

which is exactly equal to \( \frac{1}{4} d_{L}(C_1, C_2) \), here \( C_1 \cup C_2 = \partial_{\infty} \Pi \).

Now replace \( y_0 \) by a parameter \( \lambda \in [0, \infty) \). Let \( \sigma_{\lambda} \) be the catenary given by (2.9) and let \( \Pi_{\lambda} \) be the minimal surface of revolution along the axis \( \gamma_0 \) whose generating curve is the catenary \( \sigma_{\lambda} \). Next we define a definite integral as follows

\[
(2.11) \quad d_0(\lambda) = \int_{\lambda}^{\infty} \frac{\sinh(2\lambda)}{\cosh t} \frac{dt}{\sqrt{\sinh^2(2t) - \sinh^2(2\lambda)}}.
\]

Gomes proved the following theorem (see [10, Proposition 3.2]).
**Theorem 2.1** (Gomes). $d_0(0) = 0$, and as $\lambda$ increases $d_0(\lambda)$ increases monotonically, reaches a maximum, then decreases asymptotically to zero as $\lambda$ goes to infinity.

**Corollary 2.2.** There exists a finite constant $D_0 > 0$ such that for two disjoint circles $C_1, C_2 \subset S^2_\infty$, if $d_L(C_1, C_2) \leq D_0$, then there exist a spherical minimal catenoid $\Pi$ which is asymptotic to $C_1 \cup C_2$.

**Remark 2.** By our estimate in (4.2), $D_0 < \pi/2$. According to the numerical computation by Bérard and Sa Earp, $d_0$ achieves its maximum $\approx 0.501143$ when $\lambda \approx 0.4955$ (see [3 § 7.3.4]), so $D_0 \approx 1.0022$.

By Theorem 2.1 we may expect that there exists a constant $\Lambda_d > 0$ such that all catenoids $\sigma_\lambda$ intersect each other if $\lambda < \Lambda_d$ and all catenoids $\sigma_\lambda$ locally foliate the semi-disk $B^2_+$ if $\lambda > \Lambda_d$. In fact, with the help of numerical computation, Bérard and Sa Earp could prove the following results (see [4 Theorem 4.7 and Proposition 4.8] and [3 § 7.3.4]).

**Theorem 2.3** (Bérard and Sa Earp). Let $\sigma_\lambda$ be the catenary given by (2.9) and let $\Pi_\lambda$ be the minimal surface of revolution along the axis $\gamma_0$ whose generating curve is the catenary $\sigma_\lambda$. Let $\Lambda_d \approx 0.4955$.

1. All catenoids $\Pi_\lambda$ have index 1 if $\lambda \in (0, \Lambda_d)$, and all catenoids $\Pi_\lambda$ are stable if $\lambda \in [\Lambda_d, \infty)$.

2. When $\Pi_\lambda$ has index 1, there exists $x(\lambda) > 0$ such that the compact subset $\Sigma_x(\lambda)$ of $\Pi_\lambda$ between two planes $P(\pm x(\lambda))$ is a maximal weakly stable region, here $P(\pm x(\lambda))$ are two planes that pass through the points $(\pm \tanh(x(\lambda)/2), 0, 0)$ respectively and that are both perpendicular to the $u$-axis.

3. For $\lambda_1, \lambda_2 \in (0, \Lambda_d)$, the catenaries $\sigma_{\lambda_1}$ and $\sigma_{\lambda_2}$ intersect exactly at two points. Furthermore, the family $\{\Pi_\lambda\}_{0 < \lambda < \Lambda_d}$ has an envelope such that the points at which $\Pi_\lambda$ touches the envelope correspond to the maximal stable domain $\Sigma_x(\lambda)$.

4. All catenaries $\{\sigma_{\lambda}\}_{\lambda > \Lambda_d}$ locally foliate the semi-disk $B^2_+$.

**Corollary 2.4.** For any two disjoint round circles $C_1$ and $C_2$ on $S^2_\infty$, if $d_L(C_1, C_2) < D_0$, there exist two catenoids $\Pi_{\lambda_1}$ and $\Pi_{\lambda_2}$ that have the same asymptotic boundary $C_1 \cup C_2$, here $\lambda_1$ and $\lambda_2$ are positive numbers satisfying $\lambda_1 < \Lambda_d < \lambda_2$ and $d_0(\lambda_1) = d_0(\lambda_2) = \frac{1}{2} d_L(C_1, C_2)$.

Furthermore, for each $\lambda \in (0, \lambda_1)$, the catenaries $\sigma_\lambda$ and $\sigma_{\lambda_2}$ intersect exactly at two points.

On the other hand, we also have the uniqueness of catenoids in the sense of following theorem that was proved by Levitt and Rosenberg (see [12 Theorem 3.2] and [9 Theorem 3]).
Theorem 2.5 (Levitt and Rosenberg). Let $C_1$ and $C_2$ be two disjoint round circles on $S^2_\infty$ and let $\Pi$ be a connected minimal surface immersed in $\mathbb{H}^3$ with $\partial_{\infty}\Pi = C_1 \cup C_2$. Then $\Pi$ is a spherical catenoid.

3. LEAST AREA MINIMAL CATENOIDS

In this section, we will prove Theorem 1.2. At first, we need an estimate which is crucial for proving Theorem 1.2.

Lemma 3.1. For all real numbers $\lambda > 0$, consider the functions

\[
(3.1) \quad f(\lambda) = \int_{\lambda}^{\infty} \sinh t \left( \frac{\sinh(2t)}{\sqrt{\sinh^2(2t) - \sinh^2(2\lambda)}} - 1 \right) dt ,
\]

and $g(\lambda) = \cosh \lambda - 1$, then we have the following results:

1. $f(\lambda)$ is well defined for each fixed $\lambda \in (0, \infty)$.
2. $f(\lambda) < g(\lambda)$ for sufficiently large $\lambda$.

Proof. (1) Using the substitution $t \to t + \lambda$, we have

\[
(3.2) \quad f(\lambda) = \int_{0}^{\infty} \sinh(t + \lambda) \left( \frac{\sinh(2(t + \lambda))}{\sqrt{\sinh^2(2(t + \lambda)) - \sinh^2(2\lambda)}} - 1 \right) dt .
\]

We will prove that $f(\lambda) < K \cosh \lambda$, where

\[
(3.3) \quad K = \int_{0}^{1} \frac{1}{x^2} \left( \frac{1}{\sqrt{1 - x^4}} - 1 \right) dx
\]

is a constant between 0 and 1.

Let $\Theta(t, \lambda) = \frac{\sinh(2t + 2\lambda)}{\sqrt{\sinh^2(2t + 2\lambda) - \sinh^2(2\lambda)}}$, then for any fixed $t \in [0, \infty)$, it’s easy to verify that $\Theta(t, \lambda)$ is increasing on $[0, \infty)$ with respect to $\lambda$. So we have the estimate

\[
\Theta(t, \lambda) \leq \lim_{\lambda \to \infty} \frac{\sinh(2t + 2\lambda)}{\sqrt{\sinh^2(2t + 2\lambda) - \sinh^2(2\lambda)}} = \frac{\sinh(2t) + \cosh(2t)}{\sqrt{(\sinh(2t) + \cosh(2t))^2 - 1}} = \frac{e^{2t}}{\sqrt{e^{4t} - 1}} = \frac{1}{\sqrt{1 - e^{-4t}}} .
\]

Besides, $\sinh(t + \lambda) < (\sinh t + \cosh t) \cosh \lambda = e^t \cosh \lambda$, therefore we have the following estimate

\[
f(\lambda) < \cosh \lambda \int_{0}^{\infty} e^t \left( \frac{e^{2t}}{\sqrt{e^{4t} - 1}} - 1 \right) dt
\]

\[
= \cosh \lambda \int_{0}^{\infty} e^t \left( \frac{1}{\sqrt{1 - e^{-4t}}} - 1 \right) dt
\]

\[
= \cosh \lambda \int_{0}^{1} \frac{1}{x^2} \left( \frac{1}{\sqrt{1 - x^4}} - 1 \right) dx \quad (t \mapsto x = e^{-t})
\]
Since \(x^2 + 1 \geq 1\), we have
\[
(3.4) \quad K = \int_0^1 \frac{1}{x^2} \left( \frac{1}{\sqrt{1-x^2}} - 1 \right) \, dx < \int_0^1 \frac{1}{x^2} \left( \frac{1}{\sqrt{1-x^2}} - 1 \right) \, dx = 1,
\]
here we use the substitution \(x \rightarrow \sin x\) to evaluate the second integral in (3.4).

(2) By the above estimate, we have \(f(\lambda) < K \cosh \lambda\) for any \(\lambda \in [0, \infty)\). Let
\[
(3.5) \quad \Lambda_0 = \cosh^{-1} \left( \frac{1}{1-K} \right),
\]
then \(f(\lambda) < g(\lambda)\) if \(\lambda \geq \Lambda_0\).

**Remark 3.** The function \(f(\lambda)\) in (3.1) has its geometric meaning: \(2\pi f(\lambda)\) is the difference of the infinite area of one half of the catenoid \(\Pi_\lambda\) and that of the annulus
\[
\mathcal{A} = \{(0, v, w) \in \mathbb{H}^3 \mid \tanh(\lambda/2) \leq \sqrt{v^2 + w^2} < 1\}.
\]

**Remark 4.** The first definite integral in (3.4) is an elliptic integral. By the numerical computation, \(K \approx 0.40093\), and \(\Lambda_0 \approx 1.10055\).

We need the coarea formula that will be used in the proof of Theorem 1.2. The proof of (3.6) in Lemma 3.2 is very easy, which can be found in [19].

**Lemma 3.2** (Calegari and Gabai [5] § 1). Suppose \(\Sigma\) is a surface in the hyperbolic 3-space \(\mathbb{H}^3\). Let \(\gamma \subset \mathbb{H}^3\) be a geodesic, for any point \(q \in \Sigma\), define \(\theta(q)\) to be the angle between the tangent space to \(\Sigma\) at \(q\), and the radial geodesic that is through \(q\) (emanating from \(\gamma\)) and is perpendicular to \(\gamma\). Then
\[
(3.6) \quad \text{Area}(\Sigma \cap \mathcal{N}_s(\gamma)) = \int_0^s \int_{\Sigma \cap \partial \mathcal{N}_s(\gamma)} \frac{1}{\cos \theta} \, dldt,
\]
here \(\mathcal{N}_s(\gamma)\) is the hyperbolic \(s\)-neighborhood of the geodesic \(\gamma\).

**Proof of Theorem 1.2** At first, we will prove that some special compact domains of \(\Pi_\lambda\) are least area if \(\lambda\) is sufficiently large. Suppose that \(\partial_{\infty} \Pi_\lambda = C_1 \cup C_2\), and let \(P_i\) be the geodesic plane asymptotic to \(C_i\) \((i = 1, 2)\). Let \(\sigma_i\) be the generating curve of the catenoid \(\Pi_\lambda\), then \(\sigma_i\) is perpendicular to the asymptotic boundary of \(\mathbb{B}^2_+\).

For \(x \in (-d_0(\lambda), d_0(\lambda))\), let \(P(x)\) be the geodesic plane perpendicular to the \(u\)-axis such that \(\text{dist}(O, P(x)) = |x|\). Now let
\[
(3.7) \quad \Sigma = \bigcup_{|x| \leq x_1} (\Pi_\lambda \cap P(x)),
\]
for some \(0 < x_1 < d_0(\lambda)\). Let \(\partial \Sigma = C_+ \cup C_-\). Note that \(C_+\) and \(C_-\) are coaxial with respect to the \(u\)-axis or \(\gamma_0\).

**Claim 1.** \(\text{Area}(\Sigma) < \text{Area}(P_+) + \text{Area}(P_-)\), here \(P_\pm\) are the compact subdomains of \(P(\pm x_1)\) that are bounded by \(C_\pm\) respectively.

**Proof.** Recall that \(P_\pm\) are two (totally) geodesic disks with hyperbolic radius \(y_1\), so the area of \(P_\pm\) is given by
\[
(3.8) \quad \text{Area}(P_+) = \text{Area}(P_-) = 4\pi \sinh^2 \left( \frac{y_1}{2} \right) = 2\pi (\cosh y_1 - 1),
\]
here \((x_1, y_1) \in \Pi_\lambda\) satisfies the equation (2.9).

Recall that \(\text{Area}(\Sigma) = \text{Area}(\Sigma \cap N(y_0))\), by the co-area formula we have

\[
\text{Area}(\Sigma) = \int_{\lambda}^{\lambda'} \left( \text{Length}(\Sigma \cap \partial N(y_0)) \cdot \frac{1}{\cos \alpha} \right) dt
= \int_{\lambda}^{\lambda'} \left( 4\pi \sinh t \cdot \frac{\sinh(2t)}{\sqrt{\sinh^2(2t) - \sinh^2(2\lambda)}} \right) dt,
\]

here the angle \(\alpha\) is given by (2.8). Consider the functions

\[
\ell(\lambda) = \int_{\lambda}^{\lambda'} \sinh t \left( \frac{\sinh(2t)}{\sqrt{\sinh^2(2t) - \sinh^2(2\lambda)}} - 1 \right) dt
\]

and \(g(\lambda) = \cosh \lambda - 1\). By Lemma [3.1] \(\ell(\lambda) < g(\lambda)\) for all \(\lambda > \Lambda_0\). So for any \(y_1 \in (\lambda, \infty)\) we have

\[
4\pi \int_{\lambda}^{\lambda'} \sinh t \frac{\sinh(2t)}{\sqrt{\sinh^2(2t) - \sinh^2(2\lambda)}} dt < 4\pi(\cosh \lambda - 1),
\]

and then \(\text{Area}(\Sigma) < \text{Area}(P_+) + \text{Area}(P_-)\).

\(\square\)

**Claim 2.** There is no minimal annulus with the same boundary as that of \(\Sigma\) which has smaller area than that of \(\Sigma\).

**Proof.** Let \(\Omega\) be the subregion of \(\mathbb{B}^3\) bounded by \(P(-x_0)\) and \(P(x_0)\), and let \(T_\lambda\) be the simply connected subregion of \(\mathbb{B}^3\) bounded by \(\Pi_\lambda\).

Assume that \(\Sigma'\) is a least area annulus with the same boundary as that of \(\Sigma\), and \(\text{Area}(\Sigma') < \text{Area}(\Sigma)\). Since \(\Sigma'\) is a least area annulus, it must be a minimal surface. By [15 Theorem 5] and [16 Theorem 1], \(\Sigma'\) must be contained in \(\Omega\), otherwise we can use cutting and pasting technique to get a minimal surface contained in \(\Omega\) that has smaller area. Furthermore, recall that \(\{\Pi_\lambda\}_{\lambda > \Lambda_0}\) locally foliate \(\Omega \subset \mathbb{B}^3\), here \(\Lambda_0 \approx 0.4955\) is given by Theorem [2.3] therefore \(\Sigma'\) must be contained in \(T_\lambda\) by the Maximum Principle. It’s easy to verify that the boundary of \(T_\lambda \cap \Omega\) is given by \(\partial(T_\lambda \cap \Omega) = \Sigma \cup P_+ \cup P_-\).

Now we claim that \(\Sigma'\) is symmetric about any geodesic plane that passes through the \(u\)-axis, i.e., \(\Sigma'\) is a surface of revolution. Otherwise, using the reflection along the geodesic planes that pass through the \(u\)-axis, we can find another annulus \(\Sigma''\) with \(\partial \Sigma'' = \partial \Sigma'\) such that either \(\text{Area}(\Sigma'') < \text{Area}(\Sigma')\) or \(\Sigma''\) contains folding curves (see [15 pp. 418–419]) so that we can find smaller area annulus by the discussion in [15 pp. 418–419]. Similarly, \(\Sigma'\) is symmetric about the \(vw\)-plane.

Now let \(\sigma' = \Sigma' \cap \mathbb{B}_+^2\), then \(\sigma'\) satisfies the equations (2.7), which may imply that \(\Sigma'\) is a compact subdomain of some catenoid \(\Pi_\lambda\). Obviously \(\Pi_{\lambda'} \cap \Pi_\lambda = C_+ \cup C_-\).

Since \(\Sigma' \subset T_{\lambda'} \cap \Omega\), we have \(\lambda' < \lambda\). By Theorem [2.3] and the numerical computation \(d_0(\Lambda_0) < d_0(\Lambda_1)\), here \(\Lambda_1 \approx 0.46288\) is the constant given by Seo in [18], so
we have $\lambda' < \Lambda_1$, which implies that $\Pi_{\lambda'}$ is unstable. Besides, according to Theorem 2.3, $\Sigma'$ is also unstable, so it couldn’t be a least area minimal surface unless $\Sigma' \equiv \Sigma$. Therefore any compact annulus of the form (3.7) is a least area minimal surface.

Now let $S$ be any compact domain of $\Pi_{\lambda}$, then we always can find a compact annulus $\Sigma$ of the form (3.7) such that $S \subset \Sigma$. If $S$ is not a least area minimal surface, then we can use the cutting and pasting technique to show that $\Sigma$ is not a least area minimal surface. This is contradicted to the above discussion.

Therefore, $\Pi_{\lambda}$ is least area if $\lambda$ is sufficiently large.

Remark 5. In the proof of Claim 2 in Theorem 1.2, if $\Sigma'$ is an annulus type minimal surface but it is not a least area minimal surface, then it might not be a surface of revolution (see [13], p. 234).

4. REMARKS ON THEOREM 2.1

In [10], Gomes didn’t prove Theorem 2.1 in detail. It is worth giving a proof here (with the help of numerical computation).

Lemma 4.1. Let $\phi(t, \lambda) = \sqrt{5} \cosh(t + \lambda) - \cosh(t + 3\lambda)$ be a function defined on $[0, \infty) \times [0, \infty)$, then there exists $0 < \Lambda_3 < \Lambda_4$ such that $\phi(\cdot, \lambda) > 0$ if $\lambda \leq \Lambda_3$ and $\phi(\cdot, \lambda) \leq 0$ if $\lambda \geq \Lambda_4$.

Proof. It’s easy to verify the function

$$h(\lambda) = \frac{\sqrt{5} \cosh(\lambda) - \cosh(3\lambda)}{\sinh(3\lambda) - \sqrt{5} \sinh(\lambda)}$$

is a decreasing function on $[0, \infty)$.

At first, $\phi(\cdot, \lambda) > 0$ is equivalent to $h(\lambda) > \tanh t$. Since $\tanh t < 1$ and $\tanh t \to 1$ as $t \to \infty$, we need to solve $h(\lambda) \geq 1$. Let $\Lambda_3$ be the solution of the equation $h(\lambda) = 1$, i.e. $\Lambda_3 \approx 0.402359$, then $\phi(\cdot, \lambda) > 0$ if $\lambda \leq \Lambda_3$.

Secondly, $\phi(\cdot, \lambda) \leq 0$ is equivalent to $h(\lambda) \leq \tanh t$. Since $\tanh(0) = 0$, we need solve $h(\lambda) \leq 0$. Let $\Lambda_4$ be the solution of $\sqrt{5} \cosh(\lambda) - \cosh(3\lambda) = 0$, i.e. $\Lambda_4 \approx 0.53068$, then $\phi(\cdot, \lambda) \leq 0$ if $\lambda \geq \Lambda_4$.

Proof of Theorem 2.1 It’s easy to show $d_0(\lambda) \to 0$ as $\lambda \to \infty$. In fact, using the substitution $t \to t + \lambda$, we have

$$d_0(\lambda) = \int_0^\infty \frac{\sinh(2\lambda)}{\cosh(t + \lambda)} \frac{dt}{\sqrt{\sinh^2(2t + 2\lambda) - \sinh^2(2\lambda)}}$$

$$= \int_0^\infty \frac{1}{\cosh(t + \lambda)} \frac{dt}{\sqrt{\left(\frac{\sinh(2t + 2\lambda)}{\sinh(2\lambda)}\right)^2 - 1}}$$

$$< \int_0^\infty \frac{1}{\cosh \lambda} \frac{dt}{\sqrt{\sinh^2(2t) + \cosh^2(2t))}}.$$
Since \( \sinh(2t) + \cosh(2t) = e^{2t} \), we have

\[
(4.2) \quad d_0(\lambda) < \frac{1}{\cosh \lambda} \int_0^\infty \frac{dt}{\sqrt{e^{2t} - 1}} = \frac{1}{\cosh \lambda} \int_0^\infty \frac{e^{-2t}}{\sqrt{1 - e^{-4t}}} dt = \frac{1}{\cosh \lambda} \cdot \frac{\pi}{4}.
\]

Since \( d_0(0) = 0 \) and \( d_0(\lambda) \to 0 \) as \( \lambda \to \infty \), it must have a maximum value.

Next we will prove that \( d'_0 \) has a unique zero \( \Lambda_d \in (0, \infty) \), and so \( d_0 \) has a unique maximum in \((0, \infty)\). It’s easy to verify that

\[
(4.3) \quad d'_0(\lambda) = \frac{\int_0^\infty \sinh(t + \lambda)(5 \cosh^2(t + \lambda) - \cosh^2(t + 3\lambda)) \, dt}{\cosh^2(t + \lambda) \sqrt{\sinh(2t)} \sqrt{\sinh^3(2t + 4\lambda)}}
\]

By Lemma 4.1, \( d'_0(\lambda) > 0 \) on \((0, \Lambda_3)\) and \( d'_0(\lambda) < 0 \) on \((\Lambda_4, \infty)\), therefore \( d_0(\lambda) \) is increasing on \((0, \Lambda_3)\) and decreasing on \((\Lambda_4, \infty)\).

Now we need to prove that \( d'_0 \) has a unique zero \( \Lambda_d \in (\Lambda_3, \Lambda_4) \). Let

\[
(4.4) \quad h(t, \lambda) = \frac{\sinh(t + \lambda)(5 \cosh^2(t + \lambda) - \cosh^2(t + 3\lambda))}{\cosh^2(t + \lambda) \sqrt{\sinh(2t)} \sqrt{\sinh^3(2t + 4\lambda)}}
\]

then

\[
(4.5) \quad \frac{\partial h}{\partial \lambda}(t, \lambda) = \frac{r(t, \lambda)}{16 \cosh^3(t + \lambda) \sqrt{\sinh(2t)} \sqrt{\sinh^5(2t + 4\lambda)}},
\]

where

\[
(4.6) \quad r(t, \lambda) = 76 \sinh(2\lambda) - 22 \sinh(2t) + 29 \sinh(2t + 4\lambda) + \sin(2t + 8\lambda) - 26 \sinh(4t + 6\lambda) - 6 \sin(4t + 10\lambda) - 25 \sinh(6t + 8\lambda) + \sin(6t + 12\lambda).
\]

Let

\[
(4.7) \quad w_1(\lambda) = r(0, \lambda) = 8 \sinh(2\lambda) \cosh^2(\lambda) \cdot (15 - 8 \cosh(2\lambda) - 8 \cosh(4\lambda) - 8 \cosh(6\lambda) + \cosh(8\lambda)),
\]

and

\[
(4.8) \quad w_2(\lambda) = \lim_{t \to \infty} \frac{r(t, \lambda)}{\sinh(6t)} = \exp(12\lambda) - 25 \exp(8\lambda).
\]

Let \( \Lambda_5 = \frac{1}{2} \log 5 \approx 0.804719 \) be the zero of \( w_2(\lambda) \), then both \( w_1 \) and \( w_2 \) are negative if \( 0 \leq \lambda < \Lambda_5 \), which can imply \( r(t, \lambda) < 0 \) for all \( (t, \lambda) \in [0, \infty) \times [0, \Lambda_5) \). Therefore \( d''_0(\lambda) < 0 \) for all \( \lambda \in (0, \Lambda_5) \).

Recall \( \Lambda_3 < \Lambda_4 < \Lambda_5 \), so \( d'_0(\lambda) \) is decreasing on \([\Lambda_3, \Lambda_4]\). Since \( d'_0(\Lambda_3) > 0 \) and \( d'_0(\Lambda_4) < 0 \), \( d'_0(\lambda) \) has exactly one zero \( \Lambda_d \in (\Lambda_3, \Lambda_4) \subset [0, \infty) \), here \( \Lambda_d \approx 0.4955 \) is the constant given by Bérard and Sa Earp. \( \square \)
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