ρ-arbitrage and ρ-consistent pricing for star-shaped risk measures*

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Abstract

This paper revisits mean-risk portfolio selection in a one-period financial market, where risk is quantified by a star-shaped risk measure ρ. We make three contributions. First, we introduce the new axiom of sensitivity to large expected losses and show that it is key to ensure the existence of optimal portfolios. Second, we give primal and dual characterisations of (strong) ρ-arbitrage. Finally, we use our conditions for the absence of (strong) ρ-arbitrage to explicitly derive the (strong) ρ-consistent price interval for an external financial contract.

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1 Introduction

The aim of a risk measure is to quantify the risk of a financial position by a single number. This number can be interpreted in different ways in different contexts, as summarised by Wang [44, p. 337]: “In banking, it represents the capital requirement to regulate a risk; in insurance, it calculates the premium for an insurance contract; and in economics it ranks the preference of a risk for a market participant.”

In this paper, we interpret the risk measure ρ as a regulatory constraint imposed by the regulator on a financial agent seeking to optimise a portfolio. Because our focus is on the effectiveness of the risk constraint, we ignore any idiosyncratic risk aversion of the agent. Denoting by $X_\pi$ the excess return of a portfolio $\pi \in \mathbb{R}^d$, we consider the following two problems:

1. Given a minimal desired expected excess return $\nu^* \in [0, \infty)$, minimise the risk $\rho(X_\pi)$ among all portfolios $\pi \in \mathbb{R}^d$ that satisfy $\mathbb{E}[X_\pi] \geq \nu^*$;

2. Given a maximal risk threshold $\rho^* \in [0, \infty)$, maximise the return $\mathbb{E}[X_\pi]$ among all portfolios $\pi \in \mathbb{R}^d$ that satisfy $\rho(X_\pi) \leq \rho^*$.

We refer to either problem as mean-ρ portfolio selection in the sequel. The way to tackle these two problems is to first study problem (1) with an equality constraint:

1’. Given $\nu \geq 0$, minimise the risk $\rho(X_\pi)$ among all portfolios $\pi \in \mathbb{R}^d$ that satisfy $\mathbb{E}[X_\pi] = \nu$.

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Portfolios minimising (1’) are referred to as $\rho$-optimal and the corresponding minimal risk is denoted by $\rho_\nu := \inf\{\rho(X_\pi) : \pi \in \Pi_\nu\}$, where $\Pi_\nu$ denotes all portfolios $\pi \in \mathbb{R}^d$ with $\mathbb{E}[X_\pi] = \nu$. One can then study the $\rho$-optimal boundary $\mathcal{O}_\rho := \{(\rho_\nu, \nu) : \nu \geq 0\}$ and use this to find the solutions to (1) and (2) – in case that they are well posed.

There is a large literature on mean-$\rho$ portfolio selection. Deviation risk measures, which are a generalisation of the standard deviation, have been considered by de Giorgi [19] and Rockafellar et al. [41]. As one would expect, the results in this case share a lot of similarities with the classical mean-variance framework of Markowitz [36]. However, deviation risk measures quantify the degree of uncertainty in a random variable, while regulators are more concerned with the overall seriousness of possible losses. In particular, deviation risk measures are not monotone (or cash-invariant).

Presently, the most popular risk measures are normalised, monotone, cash-invariant and positively homogeneous, i.e., $\rho(\lambda X) = \lambda \rho(X)$ for $\lambda \in (0, \infty)$, with Value at Risk (VaR) and Expected Shortfall (ES) being the most famous examples [1]. However, in this setting, $\rho$-optimal portfolios need not exist. And even if they do exist, mean-$\rho$ portfolio selection may be ill-posed in the sense: there is a sequence of portfolios $(\pi_n)_{n \in \mathbb{N}}$ such that $\mathbb{E}[X_{\pi_n}] \uparrow \infty$ and $\rho(X_{\pi_n}) \leq 0$ for all $n$; or even worse (from the perspective of a regulator), there is a sequence of portfolios $(\pi_n)_{n \in \mathbb{N}}$ such that $\mathbb{E}[X_{\pi_n}] \uparrow \infty$ and $\rho(X_{\pi_n}) \downarrow -\infty$. These phenomena are referred to as $\rho$-arbitrage and strong $\rho$-arbitrage, respectively. They are undesirable from a regulatory point of view since positions that are highly rewarding should come with a risk – and the risk is being nullified here. Both concepts are generalisations of the classical notions of arbitrage of the first and second kind. Indeed if $\rho$ is the worst-case risk measure, then $\rho$-arbitrage is classical arbitrage of the first kind and strong $\rho$-arbitrage is classical arbitrage of the second kind; see [29] Proposition 3.22 for details. For coherent risk measures, the concepts of (strong) $\rho$-arbitrage have been studied from a theoretical perspective in Herdegen and Khan [29]; the practical relevance of $\rho$-arbitrage has been discussed by Armstrong and Brigo in [3] and [4].

Positive homogeneity is a strong property, which has been questioned on economic grounds early on. It triggered the introduction of convex risk measures by Föllmer and Schied [23] and Fritelli and Gianin [24]. It is easy to check that convexity together with normalisation implies that the risk measure is star-shaped, i.e., $\rho(\lambda X) \geq \lambda \rho(X)$ for $\lambda \in [1, \infty)$ [4]. This encapsulates the idea that financial positions become more risky when there is more at stake, for which there is empirical evidence, cf. [11, 12].

The first objective of this paper is to study mean-$\rho$ portfolio selection when $\rho$ is a risk functional, i.e., star-shaped, monotone and normalised. For some of our results, in particular, for our dual characterisations, we assume in addition that $\rho$ is cash-invariant (cash-invariant risk functionals are referred to as risk measures), convex or satisfies the Fatou property. Assuming that $\rho$ lives on some Riesz space $L^\infty \subset L \subset L^1$ and is $(-\infty, \infty]$-valued, we first seek to answer the following three questions for a given market:

(Q1) Existence of optimal portfolios. What conditions guarantee that $\rho$-optimal portfolios for a desired excess return $\nu \geq 0$ exist, i.e., $\text{argmin}_{\pi \in \Pi_\nu} \rho(X_\pi) \neq \emptyset$, where $\Pi_\nu$ denotes all portfolios with $\mathbb{E}[X_\pi] = \nu$?

(Q2) Absence of (strong) $\rho$-arbitrage. What are necessary and sufficient conditions to ensure that the market does not admit (strong) $\rho$-arbitrage?

(Q3) Well-posedness of mean-$\rho$ portfolio selection. When do the mean-$\rho$ problems (1) and (2) admit solutions for all $\nu^*, \rho^* \geq 0$?

(Q1) and (Q3) are important from a practical perspective, whilst (Q2) is crucial for the regulator. To the best of our knowledge, all three questions are open at the level of generality we consider

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1. Among many other papers, mean-VaR portfolio selection has been studied by Alexander and Baptista [11] and Gaivoronski and Pflug [26] and mean-ES portfolio selection has been studied by Rockafellar and Uryasev [40] and Embrechts et al. [21]. For an axiomatic justification of mean-ES portfolio selection, we refer to Han et al. [28].

2. Star-shaped risk measures have recently been studied by Castagnoli et al. [14] in a setting where there is no underlying probability measure.
here. The extant literature on mean-$\rho$ portfolio selection when $\rho$ fails to be positively homogeneous is sparse, which is maybe due to the popularity/practical relevance of VaR/ES/coherent risk measures.\footnote{The literature on mean-$\rho$ portfolio selection for positively homogeneous risk measures has been discussed in detail in \cite{29}, and we refer the interested reader there.} Notwithstanding, the minimisation of convex risk measures has been studied by Ruszczyński and Shapiro \cite{12}, and their results were later extended to \textit{quasiconvex} risk measures by Mastrogiacomo and Gianin \cite{37}. These two papers are related to ours in the sense that they study the following question: Given a vector space $Z$ representing the set of actions and a function $F : Z \to L$ which maps each action to a payoff, when is $\min_{z \in C} \rho (F(z))$ well-posed for some given convex subset $C$ of $Z$. Whilst their setting is more general than ours ($Z$ may be infinite dimensional), their assumptions on $\rho$ are stronger (a “nice” dual representation). As an application, they consider the mean-$\rho$ problem \eqref{eq:1} and provide sufficient conditions that guarantee the existence of a solution to \eqref{eq:1}. In particular, these conditions imply the existence of optimal portfolios (in our sense) and thereby answer \eqref{Q1}, at least partially. Nevertheless, their results do not contribute to answering \eqref{Q2} or \eqref{Q3}. Indeed, even if the mean-$\rho$ problem \eqref{eq:1} has a solution, there might still be $\rho$-arbitrage and the mean-$\rho$ problem \eqref{eq:2} may not have any solutions; cf. \cite{29, Corollary 3.21}.

We first address \eqref{Q1} and show in Theorem \ref{thm:3.8} that the crucial ingredient for the existence of optimal portfolios is that $\rho$ satisfies on the set of returns $X \subset L$ the new axiom \textit{sensitivity to large expected losses}: For any $X \in X \setminus \{0\}$ with $E[X] \leq 0$, there exists $\lambda > 0$ such that $\rho(\lambda X) > 0$. The economic meaning of this axiom is simple and intuitive: Apart from the riskless portfolio, any portfolio with a nonpositive expected excess return has a positive risk if it is scaled by a sufficiently large amount.

We then turn our attention to \eqref{Q2}. A key methodological tool here is to consider $\rho^\infty$, the \textit{smallest positively homogeneous} risk functional that dominates $\rho$. A key result is that under sensitivity to large expected losses and the Fatou property, $\rho$-arbitrage is equivalent to $\rho^\infty$-arbitrage. Therefore, if $\rho$ satisfies this and has a dual representation, then so does $\rho^\infty$, and we can lift the results from \cite{29} on the dual characterisation of $\rho$-arbitrage for coherent risk measures to the case of convex risk measures; see Theorem \ref{thm:1.8}. This link fails in the case of \textit{strong} $\rho$-arbitrage and the results are more involved. However, using methods from convex analysis, we are still able to provide a dual characterisation of \textit{strong} $\rho$-arbitrage for convex risk measures, see Theorem \ref{thm:4.6}.

As a byproduct of our results on \eqref{Q1} and \eqref{Q2}, we are able to answer \eqref{Q3}. We show in Theorem \ref{thm:3.20} that under sensitivity to large expected losses and the Fatou property, the absence of $\rho$-arbitrage is equivalent to the well-posedness of the mean-$\rho$ problems \eqref{eq:1} and \eqref{eq:2}.

The notion of (strong) $\rho$-arbitrage gives rise to the new concept of (strong) $\rho$-\textit{consistent pricing}. The second objective of this paper is to study (strong) $\rho$-consistent pricing by answering the following question for a given market:

\begin{description}
\item \textbf{(Q4)} \textbf{(Strong) $\rho$-consistent pricing}. What are the set of (strong) $\rho$-consistent prices for a financial contract $Y$ that lives \textit{outside} the market, where consistency is determined through the absence of (strong) $\rho$-arbitrage?
\end{description}

Pricing in a “market consistent” way has been an active area of research in mathematical finance for the past few decades. The idea is simple. Once we fix a property $\mathcal{E}$ that translates to forbidding positions in the market that are “too good to be true”, then the set of market consistent (with respect to $\mathcal{E}$) prices for $Y$ is given by

$$I_{\mathcal{E}}(Y) := \{ y \in \mathbb{R} : \text{the augmented market with } Y \text{ priced at } y \text{ satisfies } \mathcal{E} \}.$$ 

When $\mathcal{E}$ stands for no-arbitrage (or no-free lunch with vanishing risk), we are in the classical setting of arbitrage pricing; see e.g. Delbaen and Schachermayer \cite{20} and the references therein. While the absence of arbitrage is universally accepted, its implications for pricing are often rather weak, since for incomplete markets, the pricing intervals can be too large to provide any useful information. Sharper bounds can be obtained by requiring for $\mathcal{E}$ more/something different than just absence of arbitrage.
There have been many proposals in the literature on how to define $\mathcal{E}$. Cochrane and Saa-Requejo [18] considered Sharpe ratios, Bernardo and Ledoit [9] looked at gain-loss ratios, and Černý and Hodges [15] used utility functions to define $\mathcal{E}$. Later contributions in multi-period and continuous-time settings include Klöppel and Schweizer [33] and Araki [3].

Not surprisingly, risk measures have played an important role for defining $\mathcal{E}$, starting with Jaschke and Küchler [32], who studied the absence of good-deals of the second kind in a topological framework. Their results were generalised by Staum [43], and sharpened by Cherny [17]. More recently, Arduca and Munari [4] have studied the absence of good-deals of the first kind and scalable good-deals. We elaborate on these papers and discuss how they compare to our work in Remarks 4.7(b) and 4.9(b).

In this paper, market consistency is defined through the absence of (strong) $\rho$-arbitrage. This terminology is motivated by the observation that markets that admit (strong) $\rho$-arbitrage are not consistent with $\rho$ since the risk constraint becomes void. We answer (Q4) by using our answers to (Q2). In Theorem 3.27 we are able to compute the (strong) $\rho$-consistent price bounds for a new asset in a market with elliptical returns. Theorem 4.10 4.11 gives explicit (strong) $\rho$-consistent price bounds for $Y$ in a general market using (absolutely continuous) equivalent martingale measures.

The remainder of the paper is organised as follows. Section 2 describes our modelling framework. Section 3 is devoted to primal answers to (Q1)-(Q3), and also to (Q4) in the case of elliptical markets. Section 4 provides a dual characterisation of (strong) $\rho$-arbitrage for convex risk measures and dual answers to (Q4). In Section 5 we apply our theoretical results to two classes of examples: risk functionals based on loss functions and $g$-adjusted Expected Shortfall risk measures. Section 6 concludes. Appendix A contains some counterexamples complementing the theory, Appendix B contains key definitions and results on convex analysis (that are used in Section 4), and Appendix C contains some additional technical results and all proofs.

## 2 Modelling framework

### 2.1 Risk framework

We fix a probability space $(\Omega, F, \mathbb{P})$ and work on a Riesz space $L^\infty \subset L \subset L^1$ (for a background on Riesz spaces, see [2, Chapter 8]). Key examples for $L$ include $L^p$-spaces for $p \in [1, \infty]$, or more generally Orlicz hearts and Orlicz spaces. We consider one period of uncertainty, where the elements in $L$ represent (discounted) payoffs at time $t = 1$ of financial positions held at $t = 0$. The reward for any $X \in L$ is quantified by its expectation, $\mathbb{E}[X]$. As for the associated risk, we consider a risk functional $\rho : L \rightarrow (-\infty, \infty]$ satisfying the following axioms:

- **Monotonicity**: For any $X_1, X_2 \in L$ such that $X_1 \leq X_2$ $\mathbb{P}$-a.s., $\rho(X_1) \geq \rho(X_2)$;
- **Normalisation**: $\rho(0) = 0$;
- **Star-shapedness**: For all $X \in L$ and $\lambda \in [1, \infty)$, $\rho(\lambda X) \geq \lambda \rho(X)$.

Here, monotonicity means that higher payoffs have lower risk. Normalisation encodes that no investment means no risk. These two axioms imply that 0 lies in the acceptance set $\mathcal{A}_\rho := \{X \in L : \rho(X) \leq 0\}$ of $\rho$ and $\mathcal{A}_\rho + L_+ \subset \mathcal{A}_\rho$, where

$$L_+ := \{X \in L : X \geq 0 \ \mathbb{P}\text{-a.s.}\}.$$

Finally, star-shapedness captures the idea that a position’s risk should increase at least proportionally when scaled by a factor greater than one. This is economically sounder and strictly weaker than positive homogeneity, where the inequality is replaced by an equality (and $\lambda$ valued in $(0, \infty)$). Examples of risk functionals include the worst-case risk measure, $\text{WC} : L^1 \rightarrow (-\infty, \infty]$ given by

$$\text{WC}(X) := \text{ess sup}(-X),$$

and the Value at Risk (VaR) and Expected Shortfall (ES) defined by

$$\text{VaR}^\alpha(X) := \inf \{m \in \mathbb{R} : \mathbb{P}[m + X < 0] \leq \alpha\} \quad \text{and} \quad \text{ES}^\alpha(X) := \frac{1}{\alpha} \int_0^\alpha \text{VaR}^\alpha(X) \, du,$$
where $\alpha \in (0,1)$ denotes the confidence level and $X \in L^1$. See Section 5 for more examples.

Our definition of a risk functional is very general, but for some of our results, in particular for our dual characterisations, we also need some of the following axioms:

- **Cash-invariance**: For all $X \in L$ and $c \in \mathbb{R}$, $\rho(X + c) = \rho(X) - c$;
- **Convexity**: For any $X, Y \in L$ and $\lambda \in [0,1]$, $\rho(\lambda X + (1-\lambda)Y) \leq \lambda \rho(X) + (1-\lambda)\rho(Y)$;
- **Fatou property** on $\mathcal{Y} \subset L$: If $X_n \to X$ $\mathbb{P}$-a.s. for $X_n, X \in \mathcal{Y}$ and $|X_n| \leq Y$ $\mathbb{P}$-a.s. for some $Y \in L$ then $\rho(X) \leq \lim \inf_{n \to \infty} \rho(X_n)$.

All three axioms are widely used in the literature. Cash-invariance allows us to write $\rho(X) = \inf\{m \in \mathbb{R} : X + m \in \mathcal{A}_p\}$ and interpret the value as the minimal amount of capital required to make the position $X$ acceptable. Such risk functionals are fully characterised by their acceptance set. Convexity represents the idea that diversification should not increase risk and implies $\mathcal{A}_p$ is convex. Note that under normalisation, convexity implies star-shapedness but the converse is false. Finally, the Fatou property ensures that risk is never underestimated by approximations; for our applications, it sometimes suffices to consider this on a subset $\mathcal{Y} \subset L$.

It will be made clear whenever an additional axiom is assumed. In line with the extant literature, we refer to cash-invariant risk functionals as risk measures and positively homogeneous convex risk measures as coherent risk measures; see Föllmer and Schied [24] Chapter 4 for an excellent overview of risk measures – note however, that their definition of risk measure does not include normalisation.

Whilst the key point of this paper is not to insist on positive homogeneity of $\rho$, it turns out that its smallest positively homogeneous majorant $\rho^\infty : L \to (-\infty, \infty]$ plays a key role. This is also a risk functional. It is explicitly given by

$$\rho^\infty(X) := \lim_{t \to \infty} \frac{\rho(tX)}{t}. \tag{2.1}$$

For future reference, note that $\mathcal{A}_{\rho^\infty} = (\mathcal{A}_p)^\infty$, where the latter is used to denote the largest cone contained in $\mathcal{A}_p$, that is,

$$(\mathcal{A}_p)^\infty := \{X \in \mathcal{A}_p : \lambda X \in \mathcal{A}_p \text{ for all } \lambda \in (0, \infty)\}.$$ 

Moreover, if $\rho$ satisfies convexity, cash-invariance or the Fatou property on some $\mathcal{Y} \subset L$, then so does $\rho^\infty$.

### 2.2 Portfolio framework

We consider a one-period $(1+d)$-dimensional market $(S^0_0, \ldots, S^d_0)_{t \in (0,1)}$. We assume that $S^0$ is riskless and satisfies $S^0_0 = 1$ and $S^1_0 = 1 + r$, where $r > -1$ denotes the riskless rate. We further assume that $S^0_0, \ldots, S^d_0$ are risky assets, where $S^0_0, S^d_0 > 0$ and $S^0_1, \ldots, S^d_1 \in L$. We denote the (relative) return of asset $i \in \{0, \ldots, d\}$ by

$$R^i := \frac{S^i_1 - S^0_i}{S^0_0},$$

and its expectation by $\mu^i := \mathbb{E}[R^i]$. For notational convenience, we set $S := (S^0_1, \ldots, S^d_0)$. $R := (R^1, \ldots, R^d)$ and $\mu := (\mu^1, \ldots, \mu^d) \in \mathbb{R}^d$. We may assume without loss of generality that the market is nonredundant, i.e., $\sum_{i=0}^{d} \vartheta^i S^0_i = 0$ $\mathbb{P}$-a.s. implies that $\vartheta^i = 0$ for all $i \in \{0, \ldots, d\}$. We also assume that the risky returns are nondegenerate in the sense that for at least one $i \in \{1, \ldots, d\}$, $\mu^i \neq r$. (If $\mu^i = r$ for all $i \in \{1, \ldots, d\}$, then every portfolio $\pi \in \mathbb{R}^d$ has zero expected excess return. There would be no incentive to invest and mean-risk portfolio optimisation becomes meaningless.) Note that this implies that $\mathbb{P}$ itself is not an equivalent martingale measure for the discounted risky assets $S/S^0$.

**Remark 2.1.** While the assumption $S^0_0 > 0$ for $i \in \{1, \ldots, d\}$ is necessary in order to define the (relative) return $R^i$ in a meaningful way and to parametrise trading in fractions of wealth, it is without loss of generality. Indeed, if $S^0_0 \leq 0$ for some $i \in \{1, \ldots, d\}$, then one can define $\tilde{S}^i := S^i + (1-S^0_i)S^0$ (so that $\tilde{S}^0_0 = 1$), and the original market is economically equivalent to the market with $S^i$ replaced by $\tilde{S}^i$.
As \( S_0^d, \ldots, S_k^d > 0 \), we can parametrise trading in fractions of wealth, and we assume that trading is frictionless. More precisely, we fix an initial wealth \( x_0 > 0 \) and describe any portfolio (for this initial wealth) by a vector \( \pi = (\pi^1, \ldots, \pi^d) \in \mathbb{R}^d \), where \( \pi^i \) denotes the fraction of wealth invested in asset \( i \in \{1, \ldots, d\} \). The fraction of wealth invested in the riskless asset is in turn given by \( \pi^0 := 1 - \sum_{i=1}^d \pi^i = 1 - \mu \cdot 1 \), where \( 1 := (1, \ldots, 1) \in \mathbb{R}^d \). The return of a portfolio \( \pi \in \mathbb{R}^d \) can be computed by \( R_{\pi} := (1 - \pi \cdot 1) r + \pi \cdot R \), and its excess return over the riskless rate \( r \) is in turn given by
\[
X_\pi := R_{\pi} - r = (1 - \pi \cdot 1) r + \pi \cdot R - r = \pi \cdot (R - r 1).
\]
Thus, \( \mathcal{X} := \{ X_\pi : \pi \in \mathbb{R}^d \} \) is a subspace of \( L \), independent of the initial wealth \( x_0 \). The expected excess return of a portfolio \( \pi \in \mathbb{R}^d \) over the riskless rate \( r \) can be calculated as \( \mathbb{E}[X_\pi] = \pi \cdot (\mu - r 1) \), and the set of all portfolios with expected excess return \( \nu \in \mathbb{R} \) is given by
\[
\Pi_\nu := \{ \pi \in \mathbb{R}^d : \mathbb{E}[X_\pi] = \nu \}.
\]
Then \( \Pi_\nu \) is nonempty, closed and convex for all \( \nu \in \mathbb{R} \). Moreover,
\[
\Pi_{\lambda \nu} = \lambda \Pi_\nu \quad \text{for any} \ \lambda \in \mathbb{R} \setminus \{0\}.
\]
The risk associated to a portfolio \( \pi \) is given by \( \rho(X_\pi) \).

## 3 Mean-\( \rho \) portfolio selection, \( \rho \)-arbitrage and \( \rho \)-consistent pricing

We start our discussion on mean-\( \rho \) portfolio optimisation (concurrently, mean-\( \rho \infty \) portfolio optimisation) by introducing a preference preorder on the set of portfolios. This preorder formalises the idea that return is “desirable” and risk is “undesirable”.

**Definition 3.1.** A portfolio \( \pi \in \mathbb{R}^d \) is strictly \( \rho \)-preferred over another portfolio \( \pi' \in \mathbb{R}^d \) if \( \mathbb{E}[X_\pi] \geq \mathbb{E}[X_{\pi'}] \) and \( \rho(X_\pi) \leq \rho(X_{\pi'}) \), with at least one inequality being strict.

It gives rise to the mean-\( \rho \) problems:

1. Given a minimal desired expected excess return \( \nu^* \in [0, \infty) \), minimise \( \rho(X_\pi) \) among all portfolios \( \pi \in \mathbb{R}^d \) that satisfy \( \mathbb{E}[X_\pi] \geq \nu^* \).
2. Given a maximal risk threshold \( \rho^* \in [0, \infty) \), maximise \( \mathbb{E}[X_\pi] \) among all portfolios \( \pi \in \mathbb{R}^d \) that satisfy \( \rho(X_\pi) \leq \rho^* \).

The way to tackle mean-\( \rho \) portfolio selection is to first find the set of \( \rho \)-optimal portfolios, and to then use the \( \rho \)-optimal boundary to find the solutions to (1) and (2).

**Definition 3.2.** Let \( \nu \geq 0 \). A portfolio \( \pi \in \Pi_\nu \) is called \( \rho \)-optimal for \( \nu \) if \( \rho(X_\pi) < \infty \) and \( \rho(X_\pi) \leq \rho(X_{\pi'}) \) for all \( \pi' \in \Pi_\nu \). We denote the set of all \( \rho \)-optimal portfolios for \( \nu \) by \( \Pi_{\rho \nu} \). Moreover, we set
\[
\rho_\nu := \inf \{ \rho(X_\pi) : \pi \in \Pi_\nu \}, \quad (3.1)
\]
and define the \( \rho \)-optimal boundary by
\[
\mathcal{O}_\rho := \{ (\rho_\nu, \nu) : \nu \geq 0 \}. \quad (3.2)
\]

In classical mean-variance portfolio selection, optimal portfolios exist for all \( \nu \geq 0 \) and they form the solutions to problems (1) and (2). By contrast, this can all break down in the mean-\( \rho \) framework. Optimal portfolios need not exist, cf. [29, Example A.1]. Even if \( \rho \)-optimal portfolios do exist, the mean-\( \rho \) problems may still be ill-posed, for example if the market admits (strong) \( \rho \)-arbitrage (defined below), cf. Example A.1 in Appendix A.
**Definition 3.3.** The market \((S^0, S)\) is said to admit \(\rho\)-arbitrage if there exists a sequence of portfolios \((\pi_n)_{n \geq 1} \subset \mathbb{R}^d\) such that

\[
\mathbb{E}[X_{\pi_n}] \uparrow \infty \quad \text{and} \quad \rho(X_{\pi_n}) \leq 0 \quad \text{for all} \ n.
\]

It is said to admit strong \(\rho\)-arbitrage if there is a sequence of portfolios \((\pi_n)_{n \geq 1} \subset \mathbb{R}^d\) such that

\[
\mathbb{E}[X_{\pi_n}] \uparrow \infty \quad \text{and} \quad \rho(X_{\pi_n}) \downarrow -\infty.
\]

**Remark 3.4.** (a) This definition of strong \(\rho\)-arbitrage is equivalent to [29, Definition 3.17] when \(\rho\) is positively homogeneous; see [29, Remark 3.19]. Aside from the (very unusual) scenario where \(\Pi_\rho^0 = \emptyset\) for all \(\nu \geq 0\) and \(\rho_\nu \geq 0\) for all \(\nu > 0\), this definition of \(\rho\)-arbitrage is equivalent to [29, Definition 3.17] when \(\rho\) is positively homogeneous. Note, however, that the existence of a portfolio \(\pi\) with \(\mathbb{E}[X_\pi] > 0\) and \(\rho(X_\pi) \leq 0\) alone does not necessarily constitute a \(\rho\)-arbitrage since this “riskless profit” (in terms of \(\rho\)) may not be scaled.

(b) If \(\rho\) is a positively homogeneous risk functional that is expectation bounded (\(\rho(X) \geq \mathbb{E}[-X]\) for all \(X \in L\)), the market admits strong \(\rho\)-arbitrage if and only if there exists a portfolio \(\pi \in \mathbb{R}^d\) (in fractions of wealth) such that \(\rho(X_\pi) < 0\). This is equivalent to the existence of a portfolio \((\vartheta^0, \vartheta) \in \mathbb{R}^{1+d}\) (in numbers of shares) such that

\[
\vartheta^0 S_0^0 + \vartheta \cdot S_0 \leq 0 \quad \text{and} \quad \vartheta^0 S_1^0 + \vartheta \cdot S_1 - \varepsilon \in \mathcal{A}_\rho,
\]

which is referred to as a good-deal of the second kind, see e.g. [32]. A good-deal of the first kind is a portfolio \((\vartheta^0, \vartheta) \in \mathbb{R}^{1+d} \setminus \{0\}\) (in numbers of shares) such that

\[
\vartheta^0 S_0^0 + \vartheta \cdot S_0 \leq 0 \quad \text{and} \quad \vartheta^0 S_1^0 + \vartheta \cdot S_1 \in \mathcal{A}_\rho.
\]

In our setting, this corresponds to a portfolio \(\pi \in \mathbb{R}^d \setminus \{0\}\) (in fractions of wealth) with \(X_\pi \in \mathcal{A}_\rho\). Thus, when \(\rho\) is a positively homogeneous risk measure that satisfies the Fatou property and strict expectation boundedness (\(\rho(X) > \mathbb{E}[-X]\) for all non-constant \(X \in L\)), the existence of a good-deal of the first kind is equivalent to the market admitting \(\rho\)-arbitrage by [29, Theorem 3.11 and Theorem 3.20].

(Strong) \(\rho\)-arbitrage is an extension of arbitrage of the first (second) kind. They coincide when \(\rho\) is the worst-case risk measure, cf. [29, Proposition 3.22].

**Definition 3.5.** We say the market \((S^0, S)\) admits arbitrage of the first kind if there exists a trading strategy \((\vartheta^0, \vartheta) \in \mathbb{R}^{1+d}\) (parametrised in numbers of shares) such that

\[
\vartheta^0 S_0^0 + \vartheta \cdot S_0 \leq 0, \quad \vartheta^0 S_1^0 + \vartheta \cdot S_1 \geq 0 \quad \text{P-a.s.} \quad \text{and} \quad \mathbb{P}[\vartheta^0 S_1^0 + \vartheta \cdot S_1 > 0] > 0.
\]

It admits arbitrage of the second kind if there exists a trading strategy \((\vartheta^0, \vartheta) \in \mathbb{R}^{1+d}\) such that

\[
\vartheta^0 S_0^0 + \vartheta \cdot S_0 < 0, \quad \text{and} \quad \vartheta^0 S_1^0 + \vartheta \cdot S_1 \geq 0 \quad \text{P-a.s.}
\]

It is not difficult to show that in general, arbitrage of the first kind implies \(\rho\)-arbitrage, and when \(\rho\) is unbounded from below (e.g. when \(\rho\) is cash-invariant), arbitrage of the second kind implies strong \(\rho\)-arbitrage. For more connections between classical arbitrage and (strong) \(\rho\)-arbitrage, see Corollary 3.18 and Theorem 3.19.

### 3.1 Sensitivity to large expected losses

We first seek to understand under which conditions \(\rho\)-optimal portfolios exist (i.e., address (Q1)) and what properties \(\rho\)-optimal sets have. To that end, we introduce the following axiom.

**Definition 3.6.** The risk functional \(\rho\) is said to satisfy sensitivity to large expected losses on \(Y \subset L\) if for each \(X \in Y \setminus \{0\}\) with \(\mathbb{E}[X] \leq 0\), there exists \(\lambda \in (0, \infty)\) such that \(\lambda X \not\in \mathcal{A}_\rho\).
The financial interpretation of sensitivity to large expected losses is clear: For any nonzero position (in $\mathcal{Y}$) that has a nonpositive expectation, there is eventually a point where the scaled position is considered unacceptable by the person choosing the risk functional, in our case the regulator. It reflects a new notion of loss aversion and is a special case of the more general concept of sensitivity to large losses which is studied in detail in [39][4].

**Remark 3.7.** (a) We will often take $\mathcal{Y} \in \{X, L\}$. Note that $\rho$ satisfies sensitivity to large expected losses on $\mathcal{Y} \subseteq L$ if and only if $\rho^\infty$ does. When $\mathcal{Y} = L$, this is equivalent to $\mathcal{A}_{\rho^\infty} \setminus \{0\} \subseteq \{X \in L : \mathbb{E}[X] > 0\}$, from which it follows that $\mathcal{A}_{\rho^\infty}$ is pointed, i.e., $\mathcal{A}_{\rho^\infty} \cap (-\mathcal{A}_{\rho^\infty}) = \{0\}$. Pointedness of $\mathcal{A}_{\rho^\infty}$ plays an important role in [4] Section 4; cf. also Remark [4.9] b.

(b) It is often the case (see the examples in Section [4]) that $\rho$ is sensitive to large expected losses on the entire space $L$. This is a more general concept than strict expectation boundedness ($\rho(X) > \mathbb{E}[-X]$ for all non-constant $X \in L$), which played an important role in [29]. The two properties are equivalent when $\rho$ is a positively homogeneous risk measure.

By [29] Theorem 3.11, if $\rho$ is a positively homogeneous risk measure, then sensitivity to large expected losses together with the Fatou property implies that $\Pi^\rho_\nu$ is nonempty and compact for all $\nu$ with $\rho_\nu < \infty$. The same result also holds for risk functionals and this answers (Q1).

**Theorem 3.8.** Assume $\rho$ is a risk functional that satisfies the Fatou property on $X$ and sensitivity to large expected losses on $X$. Then for any $\nu \geq 0$ with $\rho_\nu < \infty$, $\Pi^\rho_\nu$ is nonempty and compact.

### 3.2 Optimal boundary

We next answer (Q2). A big step towards this is to understand the map $\nu \mapsto \rho_\nu$ from $\mathbb{R}_+ \to [\infty, \infty]$ whose graph corresponds exactly to the $\rho$-optimal boundary (but with the axes reversed). To this end, it turns out useful to relate the map $\nu \mapsto \rho_\nu$ to the map $\nu \mapsto \rho^\infty_\nu$. We start by stating some basic properties.

**Proposition 3.9.** For a risk functional $\rho$, the map $\nu \mapsto \rho_\nu$ from $\mathbb{R}_+$ to $[\infty, \infty]$ is star-shaped, i.e., for all $\nu \in \mathbb{R}_+$ and $\lambda \in [1, \infty)$, $\rho_{\lambda \nu} \geq \lambda \rho_\nu$. Moreover, $\rho_0 \leq 0$ and the map $\nu \mapsto \rho^\infty_\nu$ from $\mathbb{R}_+$ to $[\infty, \infty]$ is a positively homogeneous majorant.

As a consequence of this result, $\mathcal{O}_\rho$ lies to the left of $\mathcal{O}_{\rho^\infty}$ in the mean-risk plane. Moreover, the function $\nu \mapsto \rho_\nu$ is increasing on the interval $\{\nu \in [\nu^+, \infty) : \rho_\nu < \infty\}$ where $\nu^+ := \inf\{\nu \geq 0 : \rho_\nu > 0\} \in [0, \infty]$.

However, we lack knowledge concerning its behaviour on $(0, \nu^+]$. The next result shows that sensitivity to large expected losses together with the Fatou property yields a stronger connection between $\mathcal{O}_\rho$ and $\mathcal{O}_{\rho^\infty}$ and gives us information concerning

$$\nu_{\min} := \inf\{\nu \geq 0 : \rho_{\nu'} > \rho_\nu \text{ for all } \nu' > \nu\} \in [0, \nu^+] \quad \text{and} \quad \rho_{\min} := \inf\{\rho_\nu : \nu \geq 0\} \in [-\infty, 0].$$

**Proposition 3.10.** Let $\rho$ be a risk functional that satisfies the Fatou property on $X$ and sensitivity to large expected losses on $X$. Then $\nu \mapsto \rho_\nu$ is $(-\infty, \infty]$-valued and lower semi-continuous. Its smallest positively homogeneous majorant is given by $\nu \mapsto \rho^\infty_\nu$, and $\rho^\infty_1 > 0$ if and only if $\nu^+ < \infty$. Moreover, we have the following three cases:

(a) If $\rho^\infty_1 > 0$, then $\nu_{\min} < \infty$ and $\rho_{\min} = \rho_{\nu_{\min}} \in (-\infty, 0]$.

(b) If $\rho^\infty_1 = 0$, then $\nu_{\min} \in [0, \infty]$ and $\rho_{\min} \in [-\infty, 0].$

(c) If $\rho^\infty_1 < 0$, then $\nu_{\min} = \infty$ and $\rho_{\min} = -\infty$.

**Remark 3.11.** One might be inclined to think that $\nu \mapsto \rho^\infty_\nu$ is always the smallest positively homogeneous majorant of $\nu \mapsto \rho_\nu$. However, we need both the Fatou property on $X$ and sensitivity to large expected losses on $X$ in order for this to hold; cf. Example A.2 in Appendix A.

---

4For other existing notions of risk aversion in financial regulation, see [39].
The next result shows that for a convex risk functional \( \rho \) satisfying the Fatou property and sensitivity to large expected losses, \( O_\rho \) is continuous (except where it possibly jumps to \( \infty \)) and convex. It has a strong connection with \( O_{\rho^\infty} \) by Proposition 3.10 and by Theorem 3.8 every point on the \( \rho \)-optimal boundary (with finite risk) corresponds to a \( \rho \)-optimal portfolio. Figure 1 gives a graphical illustration.

**Proposition 3.12.** Let \( \rho \) be a convex risk functional that satisfies the Fatou property on \( X \) and sensitivity to large expected losses on \( X \). Then the map \( \nu \mapsto \rho_\nu \) from \( \mathbb{R}^+ \) to \( (-\infty, \infty] \) is convex, continuous on the closed set \( \{ \nu \in \mathbb{R}^+: \rho_\nu < \infty \} \) and \( \rho_\infty > 0 \iff \nu^+ < \infty \iff \nu_{\min} < \infty \).

Moreover, we have the following three cases:

(a) If \( \rho_1^\infty > 0 \), the map \( \nu \mapsto \rho_\nu \) is nonincreasing on \([0, \nu_{\min}]\), increasing on the closed interval \( \{ \nu \in [\nu_{\min}, \infty) : \rho_\nu < \infty \} \) and bounded below by \( \rho_{\min} = \rho_{\nu_{\min}} \in (-\infty, 0] \).

(b) If \( \rho_1^\infty = 0 \), the map \( \nu \mapsto \rho_\nu \) is nonincreasing on \( \mathbb{R}^+ \) and \( \rho_{\min} \in (-\infty, 0] \).

(c) If \( \rho_1^\infty < 0 \), the map \( \nu \mapsto \rho_\nu \) is decreasing on \( \mathbb{R}^+ \) and \( \rho_{\min} = -\infty \).

Figure 1: The \( \rho \)-optimal boundary (solid) and \( \rho^\infty \)-optimal boundary (dashed) when \( \rho \) satisfies convexity, the Fatou property on \( X \), sensitivity to large expected losses on \( X \), and \( \rho_\nu < \infty \) for all \( \nu \geq 0 \). The black dot in the right lower panel is part of the \( \rho^\infty \)-optimal boundary.

### 3.3 (Strong) \( \rho \)-arbitrage

We can now give primal characterisations of (strong) \( \rho \)-arbitrage in terms of the sign of \( \rho_1^\infty \). In particular, note that \( \rho \)-arbitrage is fully characterised by the sign of \( \rho_1^\infty \) when \( \rho \) satisfies the Fatou property on \( X \) and sensitivity to large expected losses on \( X \).

**Theorem 3.13.** For a risk functional \( \rho \), we have (a) \( \iff \) (b) \( \implies \) (c) for the statements:

(a) \( \rho_1^\infty < 0 \).

(b) The market \((S^0, S)\) admits strong \( \rho^\infty \)-arbitrage.

(c) The market \((S^0, S)\) admits strong \( \rho \)-arbitrage.

**Remark 3.14.** Remark 5.16 shows that the implication “(c) \( \implies \) (b)” does not hold even if \( \rho \) satisfies the Fatou property and sensitivity to large expected losses on \( X \). That being said, Theorem 4.6 provides conditions for when strong \( \rho \)-arbitrage is equivalent to strong \( \rho^\infty \)-arbitrage.

**Theorem 3.15.** Assume \( \rho \) is a risk functional that satisfies the Fatou property on \( X \) and sensitivity to large expected losses on \( X \). Then the following are equivalent:

(a) \( \rho_1^\infty > 0 \).
(b) The market \((S^0, S)\) does not admit \(\rho^\infty\)-arbitrage.

(c) The market \((S^0, S)\) does not admit \(\rho\)-arbitrage.

Remark 3.16. (a) Note that the implication “(c) \(\implies\) (b)” remains true even without the Fatou property on \(X\) or sensitivity to large expected losses on \(X\). However, Example A.2 in Appendix A shows that in order for the reverse implication to hold true we need both these properties simultaneously (see also Remark 3.11).

(b) By Theorem 3.19 and Remark 3.7(a), when \(\rho\) is sensitive to large expected losses and satisfies the Fatou property, then the market admits \(\rho\)-arbitrage if and only if there exists a portfolio \(\pi \in \mathbb{R}^d \setminus \{0\}\) (in fractions of wealth) with \(X_\pi \in A_{\rho^\infty}\). This is equivalent to the existence of a portfolio \((\vartheta^0, \vartheta) \in \mathbb{R}^{1+d} \setminus \{0\}\) (in numbers of shares) such that

\[
\vartheta^0 S^0_0 + \vartheta \cdot S_0 \leq 0 \quad \text{and} \quad \vartheta^0 S^0_1 + \vartheta \cdot S_1 \in A^\infty_{\rho}.
\]

This is referred to as a scalable good-deal in [4].

Remark 3.17. (a) Combining Theorems 3.13 and 3.15 allows us to conclude the following when \(\rho\) satisfies the Fatou property on \(X\) and sensitivity to large expected losses on \(X\): if \(\rho_1^\infty < 0\), then the market admits strong \(\rho\)-arbitrage; if \(\rho_1^\infty = 0\), then the market admits \(\rho\)-arbitrage and it may or may not admit strong \(\rho\)-arbitrage; if \(\rho_1^\infty > 0\), then the market does not admit \(\rho\)-arbitrage.

(b) Fix \(\alpha \in (0, 1)\). Let \(\rho \equiv \text{OCE}^l\) (see Section 5.1) where \(l\) is a loss function satisfying

\[
\lim_{x \to -\infty} \frac{l(x)}{x} = 0 \quad \text{and} \quad \lim_{x \to \infty} \frac{l(x)}{x} = \frac{1}{\alpha}.
\]

One can verify that \(\rho^\infty \equiv \text{ES}^\alpha\) by Proposition 5.6, together with the dual representation of Expected Shortfall and the fact \([0, \frac{1}{\alpha}] \supset \text{dom} \, l^* \supset (0, \frac{1}{\alpha})\). Performing mean-\(\rho\) portfolio selection in the binomial market from Example A.1 in Appendix A with \(\alpha < \alpha^*, \alpha = \alpha^*\) and \(\alpha > \alpha^*\) illustrates simple cases where, respectively, \(\rho_1^\infty > 0\), \(\rho_1^\infty = 0\) and \(\rho_1^\infty < 0\) can occur.

Theorems 3.13 and 3.15 go a long way in answering (Q2) from the introduction. When \(A_{\rho^\infty}\) is equal to the positive cone, we can further establish a useful relationship between \(\rho\)-arbitrage and arbitrage of the first kind. This is a consequence of Theorem 3.15 and Remark 3.7(a).

Corollary 3.18. Assume \(\rho\) is a risk functional that satisfies the Fatou property on \(X\) and \(A_{\rho^\infty} = L_+\). Then the following are equivalent:

(a) The market \((S^0, S)\) does not admit \(\rho\)-arbitrage.

(b) The market \((S^0, S)\) does not admit arbitrage of the first kind.

The following result shows that when \(A_{\rho^\infty} \supseteq L_+\), this equivalence breaks down. It can be viewed as a generalisation of [29 Theorem 3.23].

Theorem 3.19. Assume \(\rho\) is a (cash-invariant) risk functional such that \(A_{\rho^\infty} \supseteq L_+\). Then there exists a market with returns in \(L\) that does not admit arbitrage of the first kind, but admits (strong) \(\rho\)-arbitrage.

The primal characterisations of (strong) \(\rho\)-arbitrage in Theorems 3.13 and 3.15 are also useful when returns are elliptically distributed. We will apply them to that setting in Section 3.4 but before this we turn to (Q3).

Theorem 3.20. Assume \(\rho\) is a risk functional that satisfies the Fatou property on \(X\) and sensitivity to large expected losses on \(X\). The following are equivalent:

(a) The market \((S^0, S)\) does not admit \(\rho\)-arbitrage.

(b) For any \(\nu^*, \rho^* \geq 0\), the mean-\(\rho\) problems (1) and (2) admit at least one solution.
3.4 (Strong) ρ-consistent pricing for elliptical returns

The criteria in Theorems 3.13 and 3.15 are rather indirect as they rely on computing the sign of \( \rho \). This is nontrivial in general. However, one can easily extend [29, Section 3.4] to give a characterisation of (strong) ρ-arbitrage in terms of the maximal Sharpe ratio when returns are elliptical. As a consequence, in this setting, we can answer (Q4) from the introduction. To this end, we introduce the concept of (strong) ρ-consistent pricing.

**Definition 3.21.** Let \( X \in L \). We say \( x \in \mathbb{R} \) is a (strong) ρ-consistent price for \( X \) if the augmented market \((S^0, S, S^{d+1})\) with \( S^{d+1}_0 = x \) and \( S^{d+1}_t = X \) does not admit (strong) ρ-arbitrage. We denote the set of (strong) ρ-consistent prices for \( X \) by \( I_\rho(X) \) (\( I^\rho_\rho(X) \)).

The idea behind the above definition is as follows: If the original market \((S^0, S)\) is free of (strong) ρ-arbitrage, but the augmented market \((S^0, S, S^{d+1})\) is not, then the new asset is priced “incorrectly” because a situation arises that is “too good to be true”, in the sense that the augmented market allows for arbitrary high returns with non-positive risk (with risk tending to \(-\infty\)).

**Remark 3.22.** (a) Since (strong) WC-arbitrage coincides with arbitrage of the first (second) kind, (cf. [29, Proposition 3.22]) we have that \( I_{WC}(X) \) coincides with the set of arbitrage-of-the-first-kind-free prices (\( I^a_{WC}(X) \) coincides with the set of arbitrage-of-the-second-kind-free prices).

(b) Due to the fact that arbitrage of the first kind implies ρ-arbitrage, we have \( I_\rho(X) \subset I_{WC}(X) \). Similarly, \( I^a_\rho(X) \subset I^a_{WC}(X) \) when \( \rho \) is unbounded from below. Thus, ρ-consistent pricing can be seen as a generalisation of classical arbitrage pricing that may result in sharper and more acceptable bounds, cf. Theorem 4.11 and Example 3.29.

(c) Since (strong) ρ-arbitrage is closely linked to good deals of the first (second) kind, cf. Remark 3.4 (strong) ρ-consistent pricing is closely linked to good-deal pricing, see e.g. [15, 32]. Indeed, the main goal is to price in a way that ensures the absence of “attractive” investment opportunities.

We begin by outlining some basic properties of (strong) ρ-consistent pricing.

**Proposition 3.23.** Assume \( \rho \) is a risk functional, let \( X \in L \) and \((S^0, S)\) the original market.

(a) We have \( I_\rho(X) \subset I^a_\rho(X) \subset I^{\infty}_\rho(X) \). If \( \rho \) satisfies the Fatou property and sensitivity to large expected losses, then \( I_\rho(X) = I^{\infty}_\rho(X) \). In this case, if \( A_{\rho^\infty} = L_+ \), then the set of ρ-consistent prices coincides with the set of arbitrage-of-the-first-kind-free prices.

(b) If the market \((S^0, S)\) admits (strong) ρ-arbitrage, then \( I_\rho(X) = \emptyset \) (\( I^a_\rho(X) = \emptyset \)).

(c) If the market \((S^0, S)\) does not admit (strong) ρ-arbitrage and \( X = a \cdot (S^0_0, S^1_0, \ldots, S^d_0) \) for some \( a \in \mathbb{R}^{d+1} \), then \( I_\rho(X) = \left\{ a \cdot (S^0_0, S^1_0, \ldots, S^d_0) \right\} \) (\( I^a_\rho(X) = \left\{ a \cdot (S^0_0, S^1_0, \ldots, S^d_0) \right\} \)), if \( \rho \) is unbounded from below, and \( I^a_\rho(X) = \mathbb{R} \) otherwise.

Using only the primal characterisations of Theorems 3.13 and 3.15 not much can be said about the set of (strong) ρ-consistent prices for an external contract beyond Proposition 3.23 in general. Notwithstanding, if asset prices are elliptically distributed and \( \rho \) is law-invariant we can give explicit pricing intervals.

**Definition 3.24.** An \( \mathbb{R}^d \)-valued random vector \( X = (X_1, \ldots, X_d) \) has an elliptical distribution if there exists a location vector \( a \in \mathbb{R}^d \), a \( d \times d \) nonnegative definite dispersion matrix \( B \in \mathbb{R}^{d \times d} \), and a characteristic generator \( \psi : [0, \infty) \to \mathbb{R} \) such that the characteristic function of \( X \), \( \phi_X \) can be expressed as

\[
\phi_X(t) = e^{it^Ta\psi(t^TB)t} \quad \text{for all } t \in \mathbb{R}^d.
\]

In this case we write \( X \sim E_d(a, B, \psi) \).

**Remark 3.25.** If \( X \) has an elliptical distribution with finite second moments, \( X \) is also characterised by its mean vector \( \mu \in \mathbb{R}^d \), covariance matrix \( \Sigma \in \mathbb{R}^{d \times d} \) and characteristic generator \( \psi \). Therefore, we may write \( X \sim E_d(\mu, \Sigma, \psi) \); see [38, Remark 3.27] for details.
Definition 3.26. A risk functional \( \rho : L \rightarrow (-\infty, \infty] \) is called law-invariant if \( \rho(X_1) = \rho(X_2) \) whenever \( X_1, X_2 \in L \) have the same law.

The next result gives the precise set of (strong) \( \rho \)-consistent prices for an external contract in an elliptical market setting. To this end, in order to easily deal with the case that \( x \leq 0 \), we replace the asset \( S^{d+1} \) by \( \tilde{S}^{d+1} := S^{d+1} + (1 - x)S^0 \) as described in Remark 2.1 and similarly set \( \tilde{S}^i := S^i + (1 - S^0_i)S^0 \), and then check if the market \( (S^0, \tilde{S}, \tilde{S}^{d+1}) \) admits (strong) \( \rho \)-arbitrage.

Theorem 3.27. Let \( (S^0, S) \) be a market, \( X \in L \) and assume \( (S_1, X) \) has an elliptical distribution with mean vector \( \mu \), positive definite covariance matrix \( \Sigma \) and characteristic generator \( \psi \). Assume \( \{Y \sim E_t(\mu, \sigma_1, \psi) : \mu_Y \in \mathbb{R}, \sigma_1^T \geq 0\} \subset L \) and let \( Z \sim E_t(0, 1, \psi) \). For \( x \in \mathbb{R} \), define \( \tilde{\mu}(x) \in \mathbb{R}^{d+1} \) and \( \text{SR}_{\max}(x) \) by

\[
\tilde{\mu}(x) := \begin{cases} 
\mu_i - S_0^i(1 + r), & i \in \{1, \ldots, d\}, \\
\mu_{d+1} - x(1 + r), & i = d + 1,
\end{cases}
\quad \text{and} \quad \text{SR}_{\max}(x) := \sqrt{\tilde{\mu}(x)^T \Sigma^{-1} \tilde{\mu}(x)}.
\]

(a) If \( \rho \) is a law-invariant risk measure and \( \rho \)-arbitrage is equivalent to \( \rho^\infty \)-arbitrage, then

\[
I_\rho(X) = I_{\rho^\infty}(X) = \{x \in \mathbb{R} : \text{SR}_{\max}(x) < \rho^\infty(Z)\}.
\]

(b) If \( \rho \) is a law-invariant risk measure and strong \( \rho \)-arbitrage is equivalent to strong \( \rho^\infty \)-arbitrage, then

\[
I_{\rho^s}(X) = I_{\rho^s}(X) = \{x \in \mathbb{R} : \text{SR}_{\max}(x) \leq \rho^\infty(Z)\}.
\]

Remark 3.28. (Strong) \( \rho \)-arbitrage and (strong) \( \rho^\infty \)-arbitrage are in particular equivalent when \( \rho \equiv \rho^\infty \), i.e., when \( \rho \) is positively homogeneous. More generally, Theorem 3.15 (Theorem 4.6) give conditions when (strong) \( \rho \)-arbitrage is equivalent to (strong) \( \rho^\infty \)-arbitrage.

If \( \rho^\infty(Z) < \infty \) (which is generically the case if \( \rho^\infty \neq \text{WC} \)), Theorem 3.27 implies that the set of (strong) \( \rho \)-consistent prices is a bounded interval and hence meaningful – unlike the set of no-arbitrage prices (which is \( (-\infty, \infty) \) in this situation). We illustrate this with an example, where \( \rho^\infty = \rho \equiv \text{ES}^\alpha \), the Expected Shortfall at level \( \alpha \in (0, 1) \).

Example 3.29. Assume \( (S^1, X) \) has an elliptical distribution with characteristic generator \( \psi \), mean vector \( \mu = (\mu_1, \mu_2) \) and covariance matrix

\[
\Sigma = \begin{pmatrix} \sigma_1^2 & \gamma \sigma_1 \sigma_2 \\
\gamma \sigma_1 \sigma_2 & \sigma_2^2 \end{pmatrix},
\]

where \( \gamma \in (-1, 1) \) denotes the correlation between \( S^1 \) and \( X \). Theorem 3.27 then gives

\[
\text{SR}_{\max}(x) := \sqrt{\tilde{\mu}(x)^T \Sigma^{-1} \tilde{\mu}(x)} = \sqrt{\frac{1}{1 - \gamma} (\text{SR}_1^2(x) - 2\gamma \text{SR}_1 \text{SR}_2(x) + \text{SR}_2^2)}
\]

where \( \text{SR}_1 := (\mu_1 - S_0^1(1 + r))/\sigma_1 \) and \( \text{SR}_2(x) := (\mu_2 - x(1 + r))/\sigma_2 \) denote the Sharpe ratios of \( S^1 := S^1 + (1 - S^0)S^0 \) and \( \tilde{S}^2 = X + (1 - x)S^1 \), respectively.

Fix \( \alpha \in (0, 1) \) and let \( Z \sim E_t(0, 1, \psi) \). Note that by strict expectation boundedness \( \text{ES}^\alpha(Z) \in (0, \infty) \). By Theorem 3.27, the original market \( (S^0, S^1) \) is free of \( \text{ES}^\alpha \)-arbitrage (and hence free of strong \( \text{ES}^\alpha \)-arbitrage) if and only if \( \text{SR}_1 \in (-\text{ES}^\alpha(Z), \text{ES}^\alpha(Z)) \), which is equivalent to

\[
S^0_1 \in \left( \frac{\mu_1 - \sigma_1 \text{ES}^\alpha(Z)}{1 + r}, \frac{\mu_1 + \sigma_1 \text{ES}^\alpha(Z)}{1 + r} \right).
\]

(3.3)

So assume (3.3) and set

\[
(S^0_1)^* := \frac{\mu_2 - \gamma \text{SR}_1 \sigma_2}{1 + r} \quad \text{and} \quad \kappa := \frac{\sigma_2}{1 + r} \sqrt{(1 - \gamma^2)(\text{ES}^\alpha(Z))^2 - \text{SR}_1^2}.
\]

\[
\text{If } Z \sim \mathcal{N}(0, 1), \text{ then } \text{ES}^\alpha(Z) = \phi(\Phi^{-1}(\alpha))/\alpha \text{ where } \phi \text{ and } \Phi \text{ denote the pdf and cdf of a standard normal distribution, respectively.}
\]
Then Theorem 3.27 gives, after some algebra,
\[
I_{\text{ES}}(X) = ((S_0^2)^* - \kappa, (S_0^2)^* + \kappa) \quad \text{and} \quad I_{\text{ES}}^* (X) = [(S_0^2)^* - \kappa, (S_0^2)^* + \kappa].
\]

In particular, the length of the (strong) \(\rho\)-consistent price interval for \(X\) is given by
\[
2\kappa.
\]

If we allow \(\gamma\) to vary (cf. Figure 2, left panel), then the length of the interval is largest when \(\gamma = 0\) (perfectly uncorrelated), and decreases to 0 as \(\gamma \uparrow 1\) or \(\gamma \downarrow -1\) (perfectly correlated).

If we allow \(\alpha\) to vary (cf. Figure 2, right panel), then the length of the interval diverges to \(\infty\) and the interval itself to \((-\infty, \infty)\) as \(\alpha \downarrow 0\). Moreover, the length of the interval decreases to zero and the endpoints of the intervals converge to the midpoint \((S_0^2)^*\) as \(\alpha \uparrow \alpha^*\), where \(\alpha^*\) is the critical threshold for the original market to be free of ES\(\alpha\)-arbitrage.

![Figure 2: Interval length of \(I_{\text{ES}}(X)\). The left panel shows the dependence on the correlation \(\gamma\) between \(S_1^1\) and \(X\); the right panel shows the dependence on the ES sensitivity parameter \(\alpha\).](image)

### 4 Dual characterisation of \(\rho\)-arbitrage and \(\rho\)-consistent pricing

In this section, we consider the case that \(\rho\) is a convex risk measure on \(L\) that admits a dual representation. There are many relevant examples that fall into this category, cf. Section [3].

Let \(\mathcal{D} := \{Z \in L^1 : Z \geq 0 \ \text{P-a.s. and} \ E[Z] = 1\}\) be the set of all Radon-Nikodym derivatives of probability measures that are absolutely continuous with respect to \(\mathbb{P}\). Throughout this section, we assume that \(\rho : L \to (-\infty, \infty)\) admits a dual representation
\[
\rho(X) = \sup_{Z \in \mathcal{D}} \{E[-ZX] - \alpha(Z)\}
\]  
for some penalty function \(\alpha : \mathcal{D} \to [0, \infty]\) with effective domain \(\mathcal{Q}^\alpha := \text{dom} \alpha = \{Z \in \mathcal{D} : \alpha(Z) < \infty\}\) \(\neq \emptyset\). The penalty function determines how seriously we treat probabilistic models in \(\mathcal{D}\). Since \(\rho\) is normalised, \(\inf \alpha = 0\). Moreover, replacing \(\alpha\) if necessary by its quasi-convex hull, we may assume without loss of generality that \(\mathcal{Q}^\alpha\) is convex; see Remark 4.1 for details.

**Remark 4.1.** (a) The supremum in (4.1) is over the effective domain of \(\alpha\). This captures the idea that only the measures "contained" in \(\mathcal{Q}^\alpha\) are seen as plausible. Since \(-ZX\) may not be integrable, we define \(E[-ZX] := E[ZX^-] - E[ZX^+]\), with the conservative convention \(E[-ZX] := \infty\) if \(E[ZX^-] = \infty\).

(b) The class of risk measures satisfying (4.1) is very large. In particular, we do not impose lower semi-continuity on \(\alpha\), or \(L^1\)-closedness or uniform integrability on \(\mathcal{Q}^\alpha\).

(c) The representation in (4.1) is not unique. However, it is not difficult to check that the minimal penalty function for which (4.1) is satisfied is given by
\[
\alpha^\rho(Z) := \sup \{E[-ZX] - \rho(X) : X \in L \text{ and } \rho(X) < \infty\}.
\]

Note that \(\alpha^\rho\) is automatically convex. Moreover, its effective domain \(\mathcal{Q}^\rho := \{Z \in \mathcal{D} : \alpha^\rho(Z) < \infty\}\) is also convex and the maximal dual set. Notwithstanding, it is sometimes useful not to consider \(\alpha^\rho\) or \(\mathcal{Q}^\rho\); cf. [29, Remark 4.1(c)].
Proposition 4.2. Suppose that Condition I is satisfied. Then for any portfolio \( \pi \in \mathbb{R}^d \),
\[
\rho(X_\pi) = \sup_{c \in C_{\mathcal{Q}^\alpha}} (\pi \cdot c - f_\alpha(c)),
\]
where \( C_{\mathcal{Q}^\alpha} := \{E[-Z(R - r1)] : Z \in \mathcal{Q}^\alpha \} \subset \mathbb{R}^d \) is convex and \( f_\alpha : \mathbb{R}^d \to [0, \infty) \), defined by
\[
f_\alpha(c) = \inf\{\alpha(Z) : Z \in \mathcal{Q}^\alpha \text{ and } E[-Z(R - r1)] = c\}
\]
satisfies \( \text{dom} \ f_\alpha = C_{\mathcal{Q}^\alpha} \). Moreover, \( \rho \) satisfies the Fatou property on \( \mathcal{X} \).

Condition UI is a uniform version of Condition I. For \( X \in \mathcal{X} \), it allows us to replace \( \alpha \) in (4.1) by its \( L^1 \)-lower semi-continuous convex hull \( \overline{\overline{\alpha}} \), and the supremum by a maximum.

Proposition 4.3. Suppose that Condition UI is satisfied. Denote by \( \overline{\overline{\alpha}} : \mathcal{D} \to [0, \infty] \) the \( L^1 \)-lower semi-continuous convex hull of \( \alpha \). Then for \( X \in \mathcal{X} \),
\[
\rho(X) = \max_{Z \in \mathcal{Q}^{\overline{\overline{\alpha}}}} \{E[-ZX] - \overline{\overline{\alpha}}(Z)\}.
\]

As a consequence of Proposition 4.3 we obtain the following result which is crucial in establishing the dual characterisation of strong \( \rho \)-arbitrage.

Proposition 4.4. Suppose that Condition UI is satisfied. Then for any portfolio \( \pi \in \mathbb{R}^d \),
\[
\rho(X_\pi) = \max_{c \in C_{\mathcal{Q}^{\overline{\overline{\alpha}}}}} (\pi \cdot c - f_{\overline{\overline{\alpha}}}(c)),
\]
where \( C_{\mathcal{Q}^{\overline{\overline{\alpha}}}} := \{E[-Z(R - r1)] : Z \in \mathcal{Q}^{\overline{\overline{\alpha}}} \} \subset \mathbb{R}^d \) is convex and bounded, and \( f_{\overline{\overline{\alpha}}} : \mathbb{R}^d \to [0, \infty) \), defined by
\[
f_{\overline{\overline{\alpha}}}(c) = \inf\{\overline{\overline{\alpha}}(Z) : Z \in \mathcal{Q}^{\overline{\overline{\alpha}}} \text{ and } E[-Z(R - r1)] = c\}
\]
is the lower semi-continuous convex hull of \( f_\alpha \) defined in Proposition 4.2 and satisfies \( \text{dom} \ f_{\overline{\overline{\alpha}}} = C_{\mathcal{Q}^{\overline{\overline{\alpha}}}} \). Moreover, the infimum in (4.6) is a minimum if \( c \in C_{\mathcal{Q}^{\overline{\overline{\alpha}}}} \).

Remark 4.5. (a) By \( \overline{\overline{\alpha}} \leq \alpha \) and (B.2) it follows that \( \mathcal{Q}^\alpha \subset \mathcal{Q}^{\overline{\overline{\alpha}}} \subset \mathcal{Q}^{\alpha} \), where \( \mathcal{Q}^{\alpha} \) is the \( L^1 \)-closure of \( \mathcal{Q}^\alpha \). Moreover, if \( \alpha \) is bounded from above on its effective domain, then \( \mathcal{Q}^{\overline{\overline{\alpha}}} \) is uniform.

(b) Since the \( L^1 \)-lower semi-continuous convex hull of \( \alpha \) coincides with its \( \sigma(L^1, L^\infty) \)-lower semi-continuous convex hull by [45] Theorem 2.2.1, \( \overline{\overline{\alpha}} \) coincides with \( \alpha^{**} \), the biconjugate of \( \alpha \) under the pairing \( \langle \cdot, \cdot \rangle : L^1 \times L^\infty \to \mathbb{R}, (Z, X) \mapsto E[-ZX] \), by the Fenchel-Moreau theorem (and the fact that \( \alpha \) is nonnegative); see Appendix B for details.
The final object that we need to recall is the “interior” of $Q^\alpha$, which will be crucial in the dual characterisation of $\rho$-arbitrage. This is done in an abstract way. More precisely, we look for (nonempty) subsets $\tilde{Q}^\alpha \subset Q^\alpha$ satisfying:

**Condition POS.** $\tilde{Z} > 0$ $\mathbb{P}$-a.s. for all $\tilde{Z} \in \tilde{Q}^\alpha$.

**Condition MIX.** $\lambda Z + (1 - \lambda) \tilde{Z} \in \tilde{Q}^\alpha$ for all $Z \in Q^\alpha$, $\tilde{Z} \in \tilde{Q}^\alpha$, and $\lambda \in (0, 1)$.

**Condition INT.** For all $\tilde{Z} \in \tilde{Q}^\alpha$, there is an $L^\infty$-dense subset $E$ of $D \cap L^\infty$ such that for all $Z \in E$, $\tilde{Z} \in \tilde{Q}^\alpha$, and $\lambda \in (0, 1)$ such that $\lambda Z + (1 - \lambda) \tilde{Z} \in Q^\alpha$.

By [29, Proposition 4.9], the maximal “interior” of $Q^\alpha$ can be described explicitly by $\tilde{Q}^\alpha_{\max} := \{0 < \tilde{Z} \in Q^\alpha : \text{there is an } L^\infty\text{-dense subset } E \text{ of } D \cap L^\infty \text{ such that for all } Z \in E, \text{ there is } \lambda \in (0, 1) \text{ such that } \lambda Z + (1 - \lambda) \tilde{Z} \in Q^\alpha\}$.

## 4.2 Dual characterisation of (strong) $\rho$-arbitrage

We are now in a position to state and prove the dual characterisation of strong $\rho$-arbitrage in terms of absolutely continuous martingale measures (ACMMs) for the discounted risky assets, $M := \{Z \in D : \mathbb{E}[Z(R^i - r)] = 0 \text{ for all } i = 1, \ldots, d\}$; and the dual characterisation of $\rho$-arbitrage in terms of equivalent martingale measures (EMMs) for the discounted risky assets, $P := \{Z \in M : Z > 0 \text{ } \mathbb{P}\text{-a.s.}\}$.

We first consider the dual characterisation of strong $\rho$-arbitrage. Since strong $\rho$-arbitrage is in general not equivalent to strong $\rho^\infty$-arbitrage (cf. Remark 5.16), the following result and its proof are rather delicate and require some more advanced notions of convex analysis; see Appendix B for a brief review on some key notions of details.

**Theorem 4.6.** Assume Condition UI is satisfied and $1 \in Q^\alpha$. Denote by $\overline{Q}^\alpha : D \rightarrow [0, \infty]$ the $L^1$-lower semi-continuous convex hull of $\alpha$. Then the following are equivalent:

(a) The market $(S^0, S)$ does not admit strong $\rho$-arbitrage.

(b) $Q^{\overline{Q}^\alpha} \cap M \neq \emptyset$.

When $\alpha$ is bounded on its effective domain, then $Q^{\overline{Q}^\alpha} = \tilde{Q}^\alpha$ and strong $\rho$-arbitrage is equivalent to strong $\rho^\infty$-arbitrage.

**Remark 4.7.** (a) Note that in order to apply Theorem 4.6, we do not necessarily need to find $\overline{Q}^\alpha$ but rather only its effective domain $Q^{\overline{Q}^\alpha}$.

(b) When $\rho$ is coherent, then $Q^{\overline{Q}^\alpha} = \tilde{Q}^\alpha$ by Remark 4.5(a), and Theorem 4.6 reduces to [29, Theorem 4.15]. By Remark 3.4(b), this is comparable to Cherny’s result [17, Theorem 3.1] concerning the absence of good-deals of the second kind for coherent risk measures. Good-deals of the second kind is a concept that was first explored by Jaschke and Küchler [32] in a topological framework. They proved that its absence is equivalent to the existence of a consistent price system, and their results were generalised by Staum [43]. [32] and [43] are theoretical, and unlike [17], are difficult to apply. On the other hand, Theorem 4.6 (which leads to Theorem 4.10) can be used in practice, and also holds for convex risk measures.

We next consider the dual characterisation of $\rho$-arbitrage.

**Theorem 4.8.** Suppose that Condition I is satisfied, $\rho$ satisfies sensitivity to large expected losses on $L$ and $\tilde{Q}^\alpha_{\max} \neq \emptyset$. Then the following are equivalent:

(a) The market $(S^0, S)$ does not admit $\rho$-arbitrage.
Theorem 4.11) tells us

As a consequence of Theorem 4.6/Theorem 4.8, one can obtain price bounds for a financial contract

4.3 Dual characterisation of (strong)

In order to apply Theorem 4.8, we are aware of that does this. They relate the absence of good-deals with the existence of a consistent price deflator

expected losses on the entire space

it follows that

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we are aware of that does this. They relate the absence of good-deals with the existence of a consistent price deflator

1

In this section, we apply our theory to various examples. Our main focus is on convex risk measures that are not coherent since the latter have been discussed in [29, Section 5]. We do not make any assumptions on the returns, other than our standing assumptions that they are contained in a Riesz space and that the market is nonredundant and nondegenerate.

4.3 Dual characterisation of (strong) \(\rho\)-consistent pricing

As a consequence of Theorem 4.6/Theorem 4.8 one can obtain price bounds for a financial contract \(X\) that lies outside a given market based on the absence of strong \(\rho\)-arbitrage/\(\rho\)-arbitrage. This involves running through the ACMMs that lie in \(Q^{\rho}\)/the EMMs that lie in an “interior” \(\tilde{Q}\) of \(Q\) (note that the choice of “interior” does not matter) and taking the corresponding discounted expectation. This is what the next two results show.

Theorem 4.10. Suppose \(Q^\alpha\) is uniformly integrable and 1 \(\in Q^\alpha\). Let

\[
\tilde{L} := \{Y \in L^1 : \lim_{a \to \infty} \sup_{Z \in \tilde{Q}} E[Z\{Y|Y|>a\}] = 0\}
\]

and \((S^0, S)\) be a \((1 + d)\)-dimensional market with returns in \(\tilde{L}\). Then, for any financial contract \(X \in \tilde{L}\), we have

\[
I_\rho(X) = \{E[ZX/(1 + r)] : Z \in Q^{\rho\alpha} \cap M\}
\]

(4.7)

where \(M\) is the set of ACMMs for the original market.

Theorem 4.11. Suppose \(\rho\) satisfies sensitivity to large expected losses and admits a dual representation (4.7), where \(\emptyset \neq Q^\alpha \subset Q^\rho\) satisfies Conditions POS, MIX and INT. Let

\[
\tilde{L} := \{Y \in L^1 : ZY \in L^1 \text{ for all } Z \in Q^\alpha\}
\]

and \((S^0, S)\) be a \((1 + d)\)-dimensional market with returns in \(\tilde{L}\). Then, for any financial contract \(X \in \tilde{L}\), we have

\[
I_\rho(X) = \{E[ZX/(1 + r)] : Z \in \tilde{Q}^\alpha \cap \mathcal{P}\}
\]

(4.8)

where \(\mathcal{P}\) is the set of EMMs for the original market.

Remark 4.12. While there is a strong link between good-deals of the first (second) kind and (strong) \(\rho\)-arbitrage, cf. Remark 3.22(c), let us stress that – unlike the extant literature on good-deal pricing, which often yields only abstract price intervals – (strong) \(\rho\)-consistent price bounds are explicitly computable once \((\mathcal{M})\) \(\mathcal{P}\) is known since \((Q^{\rho\alpha})\) \(\tilde{Q}\) is not difficult to find in general, cf. Section 5.

5 Examples

In this section, we apply our theory to various examples. Our main focus is on convex risk measures that are not coherent since the latter have been discussed in [29, Section 5]. We do not make any assumptions on the returns, other than our standing assumptions that they are contained in a Riesz space and that the market is nonredundant and nondegenerate.
5.1 Risk functionals based on loss functions

The examples in this section are based around the theme of loss functions.

**Definition 5.1.** A function \( l : \mathbb{R} \to \mathbb{R} \) is called a loss function if it is nondecreasing, convex, \( l(0) = 0 \) and \( l(x) \geq x \) for all \( x \in \mathbb{R} \).

A loss function \( l \) reflects how risk averse an agent is, and so it is natural to assume that it is nondecreasing and \( l(0) = 0 \). The assumption \( l(x) \geq x \) means that compared to the risk neutral evaluation, there is more weight on losses and less on gains. Finally, convexity of \( l \) encodes that diversified positions are less risky than concentrated ones.

The growth rate of \( l \) will play an important role in the sequel. To that end, we let

\[
a_l := \lim_{x \to -\infty} \frac{l(x)}{x} \quad \text{and} \quad b_l := \lim_{x \to \infty} \frac{l(x)}{x}.
\]

Note that \( 0 \leq a_l \leq b_l \leq \infty \), where \( a_l < b_l \) unless \( a_l = b_l = 1 \), in which case \( l \) is the identity function. We will also repeatedly make use of the fact \((a_l,b_l) \in \text{dom } l^* \subset [a_l,b_l] \), where \( l^* \) is the convex conjugate of \( l \). In particular, this means \( l^* \) is bounded on any compact subset of \((a_l,b_l)\).

**5.1.1 Expected weighted loss**

The expected weighted loss of \( X \in H^{\Phi_l} \) with respect to a loss function \( l \) is given by

\[
\text{EW}^l(X) := \mathbb{E}[l(-X)]
\]

where \( H^{\Phi_l} \) is the Orlicz heart corresponding to the Young function \( \Phi_l := l|_{\mathbb{R}_+} \). By the properties of \( l \) and the definition of \( H^{\Phi_l} \), \( \text{EW}^l \) is a real-valued convex risk functional (but never cash-invariant unless \( l(x) = x \)). It is also not difficult to check that it satisfies the Fatou property on \( H^{\Phi_l} \). Therefore, by Corollary 3.18, Theorem 3.19 and Proposition C.4, we have the following.

**Corollary 5.2.** Let \( l \) be a loss function for which \( a_l = 0 \) or \( b_l = \infty \). Assume the market \((S^0,S)\) has returns in \( H^{\Phi_l} \). The following are equivalent:

(a) The market \((S^0,S)\) does not admit \( \text{EW}^l \)-arbitrage.

(b) The market \((S^0,S)\) does not admit arbitrage of the first kind.

Moreover, if \( a_l > 0 \) and \( b_l < \infty \), then there exists a market with returns in \( H^{\Phi_l} = L^1 \) that admits \( \text{EW}^l \)-arbitrage, but does not admit arbitrage of the first kind.

**Remark 5.3.** One can check that this result (including Proposition C.4) extends to functions \( l : \mathbb{R} \to \mathbb{R} \) that are nondecreasing, convex, and satisfy \( l(0) = 0 \) as well as \( \lim_{x \to \infty} l(x) = \infty \) (which is weaker than \( l(x) \geq x \) for all \( x \in \mathbb{R} \)).

**5.1.2 Shortfall risk measures**

Shortfall risk measures were first introduced as a case study on \( L^\infty \) by Föllmer and Schied in [23, Section 3]. Here, we work on Orlicz hearts. To that end, let \( l \) be a loss function and define the acceptance set

\[
\mathcal{A}_l := \{ X \in H^{\Phi_l} : \text{EW}^l(X) \leq 0 \},
\]

where \( H^{\Phi_l} \) is the Orlicz heart corresponding to the Young function \( \Phi_l := l|_{\mathbb{R}_+} \). Then the shortfall risk measure associated with \( l \) is given by \( \text{SR}^l : H^{\Phi_l} \to (-\infty,\infty] \) where

\[
\text{SR}^l(X) := \inf\{m \in \mathbb{R} : X + m \in \mathcal{A}_l\} = \inf\{m \in \mathbb{R} : \text{EW}^l(X + m) \leq 0\}.
\]

This is a convex risk measure that satisfies the Fatou property. It can be interpreted as the cash-invariant analogue of \( \text{EW}^l \) in the sense that it is cash-invariant and \( \text{SR}^l(X) \leq 0 \) if and only if \( \text{EW}^l(X) \leq 0 \). It admits a dual representation, which we now recall.
Proposition 5.4. Let \( l \) be a loss function and \( l^* \) its convex conjugate. Then for \( X \in H^{\Phi_l} \)
\[
\text{SR}^l(X) = \sup_{Z \in \mathcal{Q}^l} \left\{ \mathbb{E}[-Z X] - \alpha^l(Z) \right\}, \quad \text{where} \quad \alpha^l(Z) := \inf_{\lambda > 0} \frac{1}{\lambda} \mathbb{E}[l^*(\lambda Z)]
\]
and \( \mathcal{Q}^l = \{ Z \in \mathcal{D} : \text{there exists } \lambda > 0 \text{ such that } \mathbb{E}[l^*(\lambda Z)] < \infty \} \).

The characterisation of \( \text{SR}^l \)-arbitrage depends on the constants \( a_l \) and \( b_l \). There are two cases. Firstly, when \( a_l = 0 \) or \( b_l = \infty \), then since \( \text{dom} \ l^* \supset (a_l, b_l) \), Proposition 5.4 implies
\[
\mathcal{Q}^l \supset \{ Z \in \mathcal{D} \cap L^\infty : \text{there exists } \varepsilon > 0 \text{ such that } Z \geq \varepsilon \mathbb{P}\text{-a.s.} \}.
\]
As a consequence of this, (4.2) and [29 Proposition C.6], we have \( (\text{SR}^l)^\infty \equiv \text{WC} \) in this case. Whence, Corollary 3.18 can be applied.

Corollary 5.5. Let \( l \) be a loss function where \( a_l = 0 \) or \( b_l = \infty \). Assume the market \((S^0, S)\) has returns in \( H^{\Phi_l} \). The following are equivalent:

(a) The market \((S^0, S)\) does not admit \( \text{SR}^l \)-arbitrage.

(b) The market \((S^0, S)\) does not admit arbitrage of the first kind.

When \( a_l > 0 \) and \( b_l < \infty \), as \( (a_l, b_l) \subset \text{dom} \ l^* \subset [a_l, b_l] \), we have
\[
\{ Z \in \mathcal{D} : \text{there exists } \lambda > 0 \text{ such that } a_l < \text{ess inf } \lambda Z \text{ and } \text{ess sup } \lambda Z < b_l \} \subset \mathcal{Q}^l \quad \text{and} \quad \{ Z \in \mathcal{D} : \text{there exists } \lambda > 0 \text{ such that } a_l \leq \lambda Z \leq b_l \mathbb{P}\text{-a.s.} \} \subset \mathcal{Q}^l.
\]

Conditions I and UI are satisfied since \( ||Z||_{\infty} \leq b_l/a_l \) for \( Z \in \mathcal{Q}^l \). And the dual characterisation of (strong) \( \text{SR}^l \)-arbitrage follows from Theorem 4.6, Proposition C.5 and Proposition C.6.

Corollary 5.6. Let \( l \) be a loss function, and assume that \( a_l > 0 \) and \( b_l < \infty \). Assume the market \((S^0, S)\) has returns in \( H^{\Phi_l} \).

(a) The market \((S^0, S)\) does not admit \( \text{SR}^l \)-arbitrage if and only if there exists \( Z \in \mathcal{P} \) such that \( a_l + \varepsilon < \lambda Z < b_l - \varepsilon \mathbb{P}\text{-a.s.} \) for some \( \lambda, \varepsilon > 0 \).

(b) The market \((S^0, S)\) does not admit strong \( \text{SR}^l \)-arbitrage if and only if there exists \( Z \in \mathcal{M} \) such that \( \mathbb{E}[l^*(\lambda Z)] < \infty \) for some \( \lambda > 0 \).

Remark 5.7. (a) All of the above results for shortfall risk measures hold for functions \( l : \mathbb{R} \to \mathbb{R} \) that are nondecreasing, convex and satisfy \( l(0) = 0 \) and \( l(x) > 0 \) for all \( x > 0 \).

(b) Using numerical examples, it was shown in [27] Section 5] that shortfall risk measures corresponding to functions of the form \( l(x) = cx^\alpha 1_{\{x>0\}} \) where \( \alpha > 1 \) and \( c > 0 \) “adequately account for event risk”. In light of part (a) above, Corollary 5.5 reinforces this.

5.1.3 OCE risk measures

Optimised Certainty Equivalents (OCEs) were first introduced by Ben-Tal and Teboulle [7] and later linked to risk measures on \( L^\infty \) by the same authors in [8]. We follow [10] Section 5.4 and define the OCE risk measure associated with a loss function \( l \) as the map \( \text{OCE}^l : H^{\Phi_l} \to \mathbb{R} \),
\[
\text{OCE}^l(X) := \inf_{\eta \in \mathbb{R}} \{ \mathbb{E}[l(\eta - X)] - \eta \},
\]
where \( H^{\Phi_l} \) is the Orlicz heart corresponding to the Young function \( \Phi_l := l|_{\mathbb{R}_+} \). By [10] Section 5.1 (with \( V \equiv \text{EW}^l \)), \( \text{OCE}^l \) is the largest real-valued convex risk measure on \( H^{\Phi_l} \) that is dominated by \( \text{EW}^l \) By [10] Theorem 4.3, it also satisfies the Fatou property on \( H^{\Phi_l} \). Like shortfall risk measures, OCE risk measures admit a dual representation.

6More generally, cash-invariant hulls of convex functionals have been studied by [22, 34].
Proposition 5.8. Let \( l \) be a loss function and \( l^* \) its convex conjugate. Then for \( X \in H^{q_l} \)

\[
\text{OCE}^l(X) = \sup_{Z \in \mathcal{Q}^l} \{ \mathbb{E}[-ZX] - \alpha'(Z) \}, \quad \text{where} \quad \alpha'(Z) := \mathbb{E}[l^*(Z)]
\]

and \( \mathcal{Q}^l = \{ Z \in \mathcal{D} : \mathbb{E}[l^*(Z)] < \infty \} \).

Remark 5.9. Normalisation of \( \text{OCE}^l \) is equivalent to \( l(x) \geq x \) for all \( x \in \mathbb{R} \). If \( l(x) > x \) for all \( x \) with \(|x|\) sufficiently large, then \( \lim_{|x| \to \infty} (l(x) - x) = \infty \) by convexity of \( x \), and the infimum in (5.2) is attained; cf. [16, Lemma 5.2]. However, if \( l(x) = x \) for either \( x \geq 0 \) or \( x \leq 0 \), then the infimum is not necessarily attained, and it is easy to check \( \text{OCE}^l(X) = \mathbb{E}[-X] \) for \( X \in H^{q_l} \).

Just like for shortfall risk measures, the characterisation of \( \text{OCE}^l \)-arbitrage depends on the constants \( a_l \geq 0 \) and \( b_l \leq \infty \). There are two cases. Firstly, when \( a_l = 0 \) and \( b_l = \infty \), then dom \( l^* \supset (0, \infty) \) and it follows that

\[
\mathcal{Q}^l = \{ Z \in \mathcal{D} : \mathbb{E}[l^*(Z)] < \infty \}.
\]

As a consequence of this, (4.2) and [29, Proposition C.6], it follows that \( \text{OCE}^l(X) = \mathbb{E}[-X] \) for \( X \in H^{q_l} \).

Corollary 5.10. Let \( l \) be a loss function where \( a_l = 0 \) and \( b_l = \infty \). Assume the market \((S^0, S)\) has returns in \( H^{q_l} \). The following are equivalent:

(a) The market \((S^0, S)\) does not admit \( \text{OCE}^l \)-arbitrage.

(b) The market \((S^0, S)\) does not admit arbitrage of the first kind.

When \( a_l > 0 \) or \( b_l < \infty \), we can derive a dual characterisation of (strong) \( \text{OCE}^l \)-arbitrage. By Remark 5.9 it suffices to consider the case \( a_l < 1 < b_l \).

Corollary 5.11. Let \( l \) be a loss function and assume that either \( a_l > 0 \) or \( b_l < \infty \), and \( a_l < 1 < b_l \). Assume the market \((S^0, S)\) has returns in \( H^{q_l} \). Then,

(a) The market \((S^0, S)\) does not admit \( \text{OCE}^l \)-arbitrage if and only if there exists \( Z \in \mathcal{P} \) such that \( \mathbb{E}[l^*(Z)] < \infty \) and \( a_l + \varepsilon < Z < b_l - \varepsilon \) \( \mathbb{P} \)-a.s. for some \( \varepsilon > 0 \).

(b) If in addition \( b_l < \infty \), the market \((S^0, S)\) does not admit strong \( \text{OCE}^l \)-arbitrage if and only if there exists \( Z \in \mathcal{M} \) such that \( \mathbb{E}[l^*(Z)] < \infty \).

5.2 Adjusted risk functionals

Suppose we have a family of risk functionals \( (\rho_\alpha)_{\alpha \in \mathcal{I}} \) on a Riesz space \( L^\infty \subset L \subset L^1 \) indexed by a set \( \mathcal{I} \). Let \( g : \mathcal{I} \to [0, \infty] \) be a function such that \( \inf g = 0 \). Then the functional defined by

\[
\rho^g(X) := \sup_{\alpha \in \mathcal{I}} \{ \rho_\alpha(X) - g(\alpha) \}, \quad X \in L,
\]

is also a risk functional\(^7\). It is cash-invariant/convex/satisfies the Fatou property whenever \( \rho_\alpha \) is cash-invariant/convex/satisfies the Fatou property for \( \alpha \in \mathcal{I} \). The way to interpret this \( g \)-adjusted risk functional, is to look at its acceptance set. Indeed, \( X \in \mathcal{A}_{\rho^g} \) if and only if \( \rho_\alpha(X) \leq g(\alpha) \) for all \( \alpha \in \mathcal{I} \). Thus, whilst the risk of a random variable \( X \) is ultimately represented by a single number \( \rho^g(X) \), whether or not \( X \) is acceptable depends on the entire continuum of values \( (\rho_\alpha(X))_{\alpha \in \mathcal{I}} \). In this way, \( g \) can be considered as a target risk profile.

When \( \mathcal{I} \) is equipped with a partial order, \( \rho_\alpha \geq \rho_{\alpha'} \) for \( \alpha \leq \alpha' \) and \( \rho_\alpha \) is cash-invariant for all \( \alpha \in \mathcal{I} \), then we may assume without loss of generality that \( g \) is nonincreasing. Indeed, otherwise, we may replace \( g \) by \( \tilde{g} : \mathcal{I} \to [0, \infty] \) where

\[
\tilde{g}(\alpha) := \inf \{ g(\alpha') : \alpha' \leq \alpha \},
\]

and note that the acceptance set of \( \rho^g \) coincides with the acceptance set of \( \rho^{\tilde{g}} \), which implies that \( \rho^g = \rho^{\tilde{g}} \) since they are cash-invariant. We will consider the cases where \( \rho_\alpha \) is given by either Value at Risk or Expected Shortfall.

\(^7\)Here, we tacitly assume that the supremum is only taken over those \( \alpha \in \mathcal{I} \) for which \( g(\alpha) < \infty \).
5.2.1 Adjusted Value at Risk

For a nonincreasing function \( g : (0, 1) \to [0, \infty] \) with \( \inf g = 0 \), we define the \( g \)-adjusted Value at Risk as the map \( \text{VaR}^g : L^1 \to (-\infty, \infty] \) given by

\[
\text{VaR}^g(X) := \sup_{\alpha \in (0, 1)} \{ \text{VaR}^\alpha(X) - g(\alpha) \}.
\]

This is a family of risk measures that are neither convex nor positively homogeneous (unless the function \( g \) is constant on its effective domain). It was introduced by Bignozzi, Burzoni and Munari \[10]\footnote{Our definition is based on \[10\] Proposition 4. There, they consider nondecreasing functions that are left-continuous (since these are the properties of the generalised inverse of a benchmark loss distribution. However, in line with the way we defined VaR, the functions \( g \) must be nonincreasing for us. They also do not necessarily have to be left-continuous.} By Corollary 3.18, Theorem 3.19 and Proposition \[C.8\] we have the following result concerning VaR\(^g\)-arbitrage.

**Proposition 5.12.** Assume the market \((S^0, S)\) has returns in \(L^1\). Let \( g : (0, 1) \to [0, \infty] \) be a nonincreasing function with \( \inf g = 0 \). If \( g \) is real-valued, then the following are equivalent:

(a) The market \((S^0, S)\) does not admit VaR\(^g\)-arbitrage.

(b) The market \((S^0, S)\) does not admit arbitrage of the first kind.

If \( g \) is not real-valued and the probability space is atomless, then there exists a market with returns in \(L^1\) that admits strong VaR\(^g\)-arbitrage, but does not admit arbitrage of the first kind.

5.2.2 Adjusted Expected Shortfall

Let \( G \) be the set of nonincreasing functions \( g : (0, 1] \to [0, \infty] \) with \( g(1) = 0 \) and \( \{1\} \subseteq \text{dom} \ g \). For \( g \in G \), we define the \( g \)-adjusted Expected Shortfall as the map \( \text{ES}^g : L^1 \to (-\infty, \infty] \), given by

\[
\text{ES}^g(X) := \sup_{\alpha \in [0,1]} \{ \text{ES}^\alpha(X) - g(\alpha) \}, \quad \text{where} \quad \text{ES}^1(X) := \lim_{a \to 1} \text{ES}^\alpha(X) = \mathbb{E}[X].
\]

This is a family of convex risk measures introduced by Burzoni, Munari and Wang \[13]\footnote{Our definition is based on \[13\] Proposition 2.2. There, they consider nondecreasing functions. However, in line with the way we defined ES, the functions \( g \) must be nonincreasing for us. The case \( \text{dom} \ g = \{1\} \) corresponds to the expected-loss risk measure and is not interesting.}. We proceed to state the dual representation of \( g \)-adjusted ES. To this end, for \( \beta \in (0,1) \) set \( G_\beta := \{ g \in G : \text{inf} \ \text{dom} \ g = \beta \} \).

**Proposition 5.13.** Let \( g \in G \). Then \( \text{ES}^g : L^1 \to (-\infty, \infty] \) satisfies the dual representation

\[
\text{ES}^g(X) = \sup_{Z \in \mathcal{Q}^g} \{ \mathbb{E}[-ZX] - g(\|Z\|^{-1}_\infty) \}
\]

where the penalty function \( \alpha^g : \mathcal{D} \to [0, \infty] \) is given by \( \alpha^g(Z) = g(\|Z\|^{-1}_\infty) \) if \( Z \in \mathcal{D} \cap L^\infty \) and \( \alpha^g(Z) = \infty \) otherwise \[10\]. Moreover, \( \mathcal{Q}^g = \{ Z \in \mathcal{D} \cap L^\infty : g(\|Z\|^{-1}_\infty) < \infty \} \) is convex and satisfies

\[
\mathcal{Q}^g = \begin{cases} 
\mathcal{D} \cap L^\infty, & \text{if } g \in \mathcal{G}_0, \\
\{ Z \in \mathcal{D} : \|Z\|_\infty \leq \frac{1}{\beta} \}, & \text{if } g \in \mathcal{G}_\beta, \beta \in (0,1) \text{ and } g(\beta) < \infty, \\
\{ Z \in \mathcal{D} : \|Z\|_\infty < \frac{1}{\beta} \}, & \text{if } g \in \mathcal{G}_\beta, \beta \in (0,1) \text{ and } g(\beta) = \infty.
\end{cases}
\]

When \( g \in \mathcal{G}_0 \), we have that \( (\text{ES}^g)^\infty \equiv \text{WC} \) by \[4.2\], \[5.4\] and \[29\] Proposition C.6. Therefore, we may apply Corollary 3.18.

**Corollary 5.14.** Let \( g \in \mathcal{G}_0 \) and assume the market \((S^0, S)\) has returns in \(L^1\). The following are equivalent:

\[
\text{Ra}_{\text{C}2}
\]
(a) The market \((S^0, S)\) does not admit \(ES^q\)-arbitrage.

(b) The market \((S^0, S)\) does not admit arbitrage of the first kind.

We can further provide a dual characterisation of (strong) \(ES^q\)-arbitrage when \(g \in G_\beta\) and \(\beta \in (0, 1)\). In this case, since \(Q^{\alpha^g}\) is \(L^\infty\)-bounded, Conditions I and UI are both satisfied if the returns lie in \(L^1\). Moreover, it is not difficult to check that

\[
\tilde{Q}^{\alpha^g} = \{0 < Z \in D : \|Z\|_\infty < 1/\beta\}
\]

is a subset of \(Q^{\alpha^g}\) that satisfies Conditions POS, MIX and INT and contains 1; see [29, Proposition B.6] for details. Finally, Proposition C.10 shows that

\[
\bar{Q}^{\alpha^g} = \begin{cases} 
Z \in D : \|Z\|_\infty \leq \frac{1}{\beta}, & \text{if } g \in G^\infty_{\beta}, \\
Z \in D : \|Z\|_\infty < \frac{1}{\beta}, & \text{if } g \in G^\infty_{\beta} \setminus G^\infty_{\beta'},
\end{cases}
\]

where \(G^\infty_{\beta'} := \{g \in G_\beta : g \text{ is bounded on its effective domain}\}\). Thus, Theorems 4.6 and 4.8 yield the following result.

**Corollary 5.15.** Let \(g \in G_\beta\) where \(\beta \in (0, 1)\) and assume the market \((S^0, S)\) has returns in \(L^1\).

(a) \((S^0, S)\) does not admit \(ES^q\)-arbitrage if and only if there exists \(Z \in \mathcal{P}\) such that \(\|Z\|_\infty < \frac{1}{\beta}\).

(b) When \(g \in G^\infty_{\beta'} (g \in G_\beta \setminus G^\infty_{\beta'}), (S^0, S)\) does not admit strong \(ES^q\)-arbitrage if and only if there exists \(Z \in \mathcal{M}\) with \(\|Z\|_\infty \leq (<) \frac{1}{\beta}\).

**Remark 5.16.** This result shows that the implication “(c) \(\Rightarrow\) (b)” in Theorem 3.13 does not hold. Indeed, if \(g \in G_{\beta'} \setminus G^\infty_{\beta'}\) for \(\beta \in (0, 1)\) and there exists no \(Z \in \mathcal{M}\) with \(\|Z\|_\infty < \frac{1}{\beta}\) but a \(Z \in \mathcal{M}\) with \(\|Z\|_\infty = \frac{1}{\beta}\), then the market admits strong \(\rho\)-arbitrage for \(\rho = ES^q\). However, since the \(L^1\)-closure of \(Q^{\alpha^g}\) is \(\{Z \in D : \|Z\|_\infty \leq \frac{1}{\beta}\}\), it follows from [4, 2] and [29, Theorem 4.15] that the market does not admit strong \(\rho^\infty\)-arbitrage.

### 6 Conclusion and outlook

The goal of this paper has been to answer the four questions posed in the introduction. We have seen that essentially (Q1) has a positive answer if \(\rho\) satisfies sensitivity to large expected losses on the set of excess returns. However, this axiom is not enough to avoid (strong) \(\rho\)-arbitrage, which is the main concern of the regulator.

In order to characterise the absence of \(\rho\)-arbitrage, we discovered the key relationship between mean-\(\rho\) portfolio selection and mean-\(\rho^\infty\) portfolio selection, where \(\rho^\infty\) is the smallest positively homogeneous risk function that dominates \(\rho\). This relationship is crucial for the dual characterisation of \(\rho\)-arbitrage when \(\rho\) is a convex risk measure since it allows to lift the results on mean-\(\rho\) portfolio selection from the coherent case to the convex case. This link between \(\rho\) and \(\rho^\infty\) breaks in the case of strong \(\rho\)-arbitrage. Nevertheless, we were still able to derive a dual characterisation of strong \(\rho\)-arbitrage by using tools from convex analysis.

We answered (Q3) by showing that the well-posedness of the mean-\(\rho\) problems (1) and (2) is equivalent to the absence of \(\rho\)-arbitrage, when \(\rho\) is sensitive to large expected losses and satisfies the Fatou property.

Finally, as a byproduct of (Q2), we were able to answer (Q4). In the case of elliptical markets, this led to the elegant result in the form of Theorem 3.27. More generally, when \(\rho\) is a convex risk measure, this boils down to taking discounted expectations with respect to certain (but not necessarily all) absolutely continuous/equivalent martingale measures for the discounted risky assets, cf. Theorems 4.10 and 4.11. In particular, we stress that unlike the majority of literature on pricing, (strong) \(\rho\)-consistent intervals can be expressed in a precise way.
A Counterexamples

In this appendix we give some counterexamples to complement the results in Section 3.

Example A.1. Suppose that risk is quantified by the Expected Shortfall at level \( \alpha \in (0,1) \), \( \ES^\alpha \). Consider the binomial model with one riskless asset and one risky asset with returns \( R^0 \) and \( R^1 \), respectively, satisfying

\[
R^0 = r \text{ P-a.s., } \mathbb{P}[R^1 = u] = p \quad \text{and} \quad \mathbb{P}[R^1 = d] = 1 - p,
\]

where we assume that \( p \in (0,1) \) and \(-1 < d < r < u\). We also assume that \( \mathbb{E}[R^1] > r \), i.e., \( up + d(1-p) > r \). For given desired excess return \( \nu > 0 \), we have to invest \( \pi^1_\nu = \frac{nu + d(1-p) - r}{up + d(1-p) - r} \) into the risky asset to obtain an expected excess return of \( \nu \). Denote the corresponding ES at level \( \alpha \) by \( \ES^\alpha_\nu = \nu \ES^\alpha_1 \) where

\[
\ES^\alpha_1 = \begin{cases} \pi^1_1(r - d), & \text{if } \alpha \leq 1 - p, \\ \pi^1_1 \frac{u-r}{\alpha} \left( \frac{1-p}{1-q} - \alpha \right), & \text{if } \alpha > 1 - p. \end{cases}
\]

Setting \( \alpha^* := \frac{1-p}{1-q} \), we obtain

\[
\ES^\alpha_1 = \begin{cases} > 0, & \text{if } \alpha < \alpha^*, \\ = 0, & \text{if } \alpha = \alpha^*, \\ < 0, & \text{if } \alpha > \alpha^*. \end{cases}
\]

Thus, for \( \alpha \geq \alpha^* \), \( \ES^\alpha_1 \leq 0 \) for all \( \nu \geq 0 \) (the market admits \( \ES^\alpha \)-arbitrage). And for \( \alpha > \alpha^* \), we have that \( \ES^\alpha_1 \downarrow -\infty \) as \( \nu \uparrow \infty \) (the market admits strong \( \ES^\alpha \)-arbitrage).

Example A.2. Consider a three-dimensional market where \( r = 0, R^1 \sim N(0,1) \) and \( R^2 \sim N(1,1) \) and assume that \( R^1 \) and \( R^2 \) are not perfectly correlated. Then for \( \nu \in \mathbb{R} \), \( \Pi_\nu = \{ (\pi^1, \nu) : \pi^1 \in \mathbb{R} \} \). Let \( \mathcal{X} := \{ \pi^1 R^1 + \nu R^2 : (\pi^1, \nu) \in \mathbb{R}^2 \} \).

(a) Define \( \eta : \mathcal{X} \to (-\infty, \infty] \) by

\[
\eta(\pi^1 R^1 + \nu R^2) = \begin{cases} \max\{\nu^2 e^{-\pi^1/\nu} - 1, 0\}, & \text{if } \nu > 0 \text{ and } \pi^1 \in \mathbb{R}, \\ 0, & \text{if } \nu = 0 \text{ and } \pi^1 \in \mathbb{R}, \\ \infty, & \text{otherwise}. \end{cases}
\]

Then \( \eta \) is normalised and star-shaped but not sensitive to large expected losses on \( \mathcal{X} \). By Proposition C.3 we can extend \( \eta \) to a risk measure \( \rho : L^1 \to (-\infty, \infty] \) that is normalised, star-shaped, monotone and satisfies \( \rho|_{\mathcal{X}} = \eta \). It also satisfies the Fatou property on \( \mathcal{X} \) but not sensitivity to large expected losses on \( \mathcal{X} \). For \( \nu \geq 0 \), \( \rho_\nu \) is attained and equal to 0. Thus, the market admits \( \rho \)-arbitrage. Now it is not difficult to check that

\[
\rho^\infty(\pi^1 R^1 + \nu R^2) = \begin{cases} 0, & \text{if } \nu = 0 \text{ and } \pi^1 \in \mathbb{R}, \\ \infty, & \text{if } \nu \neq 0 \text{ and } \pi^1 \in \mathbb{R}. \end{cases}
\]

Whence, \( \rho^\infty_1 = \infty \). Therefore, in the absence of sensitivity to large expected losses (even if the Fatou property is satisfied), \( \rho \)-arbitrage does not imply \( \rho^\infty \)-arbitrage.

(b) Alter the above risk functional by defining \( \eta(\pi^1 R^1 + \nu R^2) = \infty \) for \( \nu = 0 \) and \( \pi^1 \neq 0 \). Then \( \rho \) satisfies sensitivity to large expected losses on \( \mathcal{X} \), but no longer satisfies the Fatou property on \( \mathcal{X} \). By arguing as above, it is not difficult to check that the market admits \( \rho \)-arbitrage but not \( \rho^\infty \)-arbitrage. Whence, if the Fatou property is not satisfied (even if sensitivity to large expected losses is satisfied), \( \rho \)-arbitrage does not imply \( \rho^\infty \)-arbitrage.
B Key definitions and results on convex analysis

In this appendix, we recall some key definitions and results regarding convex functions and convex conjugates.

Let $X$ be a topological vector space and $f : X \to [-\infty, \infty]$ a function.

- The epigraph of $f$ is given by
  \[ \text{epi } f := \{(x, t) \in X \times \mathbb{R} : f(x) \leq t\} . \]

Note that $f$ can be recovered from its epigraph, $f(x) = \inf \{t \in \mathbb{R} : (x, t) \in \text{epi } f\}$. Also, a function $g : X \to [-\infty, \infty]$ is dominated by $f$ if and only if $\text{epi } f \subset \text{epi } g$.

- The effective domain of $f$ is given by
  \[ \text{dom } f := \{x \in X : f(x) < \infty\} . \]

We say $f$ is proper if $\text{dom } f \neq \emptyset$ and $f(x) > -\infty$ for all $x \in X$.

- We say $f$ is convex if $\text{epi } f$ is a convex subset of $X \times \mathbb{R}$. Note that if $f$ is convex, $\text{dom } f$ is a convex subset of $X$.

- We say $f$ is quasi-convex if $\{x \in X : f(x) \leq t\}$ is a convex subset of $X$ for all $t \in \mathbb{R}$. Every convex function is quasi-convex, but the converse is not true. However, if $f$ is quasi-convex, $\text{dom } f$ is a convex subset of $X$.

- We say $f$ is lower semi-continuous if $\text{epi } f$ is a closed subset of $X \times \mathbb{R}$.

- The convex hull of $f$, $\text{co } f : X \to [-\infty, \infty]$, is the largest convex function majorised by $f$,
  \[ \text{co } f(x) := \sup\{g(x) : |g : X \to [-\infty, \infty] \text{ is convex and } g \leq f\} . \]

By [39] Equation (3.5), $\text{epi } \text{co } f = \{(x, t) \in X \times \mathbb{R} : (x, s) \in \text{co } \text{epi } f \text{ for all } s > t\}$, where $\text{co } \text{epi } f = \bigcap\{C \subset X \times \mathbb{R} : \text{epi } f \subset C \text{ and } C \text{ is convex}\}$. Moreover, it is not difficult to check that $\text{dom } \text{co } f = \text{co } \text{dom } f$, where $\text{co } \text{dom } f = \bigcap\{C \subset X : \text{dom } f \subset C \text{ and } C \text{ is convex}\}$.

- The quasi-convex hull of $f$, $\text{qco } f : X \to [-\infty, \infty]$, is the largest quasi-convex function majorised by $f$,
  \[ \text{qco } f(x) := \sup\{g(x) : |g : X \to [-\infty, \infty] \text{ is quasi-convex and } g \leq f\} . \]

Since every convex function is quasi-convex, it follows that $\text{co } f \leq \text{qco } f \leq f$. Moreover, it is not difficult to check that $\text{dom } \text{qco } f = \text{dom } \text{co } f = \text{co } \text{dom } f$.

- The lower semi-continuous hull of $f$, $\text{lsc } f : X \to [-\infty, \infty]$ is the largest lower semi-continuous function majorised by $f$,
  \[ \text{lsc } f(x) := \sup\{h(x) : |h : X \to [-\infty, \infty] \text{ is lower semi-continuous and } h \leq f\} . \]

By [39] Equation (3.6), $\text{epi } \text{lsc } f = \text{cl } \text{epi } f$, or equivalently we have [39] Equation (3.7),

\[ \text{lsc } f(x) = \inf\{\liminf_{i \in I} f(x_i) : \lim x_i = x\} . \tag{B.1} \]

In particular, this implies that $\text{dom } \text{lsc } f \subset \text{cl } \text{dom } f$.

- The lower semi-continuous convex hull of $f$, $\overline{\text{co } f} : X \to [-\infty, \infty]$ is given by $\overline{\text{co } f} := \text{lsc } \text{co } f$ (which may not be the same as $\text{co } \text{lsc } f$). Since the closure of a convex set is again convex and $\text{epi } \overline{\text{co } f} = \text{cl } \text{co } \text{epi } f$, it follows that $\overline{\text{co } f}$ is the largest lower semi-continuous convex function majorised by $f$. Moreover,
  \[ \text{dom } \overline{\text{co } f} \subset \text{cl } \text{co } \text{dom } f . \tag{B.2} \]
• If $Y$ is a nonempty subset of $X$ and $f : Y \to [-\infty, \infty]$ a function, we can extend $f$ to $X$ by considering the function $\bar{f} : X \to [-\infty, \infty]$ defined by

$$\bar{f}(x) = \begin{cases} f(x), & \text{if } x \in Y, \\ \infty, & \text{if } x \in X \setminus Y. \end{cases}$$

This extension is natural in that epi $\bar{f} \subset Y \times \mathbb{R}$, dom $\bar{f} \subset Y$, dom $\text{co} \bar{f}$, dom $\text{qco} \bar{f} \subset Y$ if $Y$ is convex, dom $\text{lsc} \bar{f} \subset Y$ if $Y$ is closed and dom $\text{co} \bar{f} \subset Y$ if $Y$ is convex and closed. For this reason, if $Y$ is convex, we may define the functions $\text{co} f, \text{qco} f : Y \to [-\infty, \infty]$ by $\text{co} f(x) := \text{co} \bar{f}(x)$, $\text{qco} f(x) := \text{qco} \bar{f}(x)$ and call this the convex hull and quasi-convex hull of $f$, respectively. Similarly, if $Y$ is closed (and convex), we may define the functions $\text{lsc} f : Y \to [-\infty, \infty]$ and $\text{co} f : Y \to [-\infty, \infty]$ by $\text{lsc} f(x) := \text{lsc} \bar{f}(x)$ and $\text{co} f(x) := \text{co} \bar{f}(x)$ and call this the lower semi-continuous (convex) hull of $f$.

In order to discuss convex conjugates, we assume that $(X, X')$ is a dual pair under the duality $\langle \cdot, \cdot \rangle : X \times X' \to \mathbb{R}$, i.e., $X$ and $X'$ are vector spaces together with a bilinear functional $\langle x, x' \rangle \mapsto \langle x, x' \rangle$ such that

• If $\langle x, x' \rangle = 0$ for each $x' \in X'$, then $x = 0$;

• If $\langle x, x' \rangle = 0$ for each $x \in X$, then $x' = 0$.

We endow $X$ with the weak topology, $\sigma(X, X')$,

$$x_\alpha \rightharpoonup x \text{ in } X \text{ if and only if } \langle x_\alpha, x' \rangle \to \langle x, x' \rangle \text{ in } \mathbb{R} \text{ for each } x' \in X',$$

and $X'$ with the weak* topology, $\sigma(X', X)$,

$$x'_\alpha \rightharpoonup^* x' \text{ in } X' \text{ if and only if } \langle x, x'_\alpha \rangle \to \langle x, x' \rangle \text{ in } \mathbb{R} \text{ for each } x \in X.$$

These topologies are locally convex and Hausdorff; the topological dual of $(X, \sigma(X, X'))$ is $X'$; and the topological dual of $(X', \sigma(X', X))$ is $X$; see [2, Section 5.14] for details.

• The convex conjugate of $f$, $f^* : X' \to [-\infty, \infty]$, and the biconjugate of $f$, $f^{**} : X \to [-\infty, \infty]$, are defined as

$$f^*(x') := \sup \{ \langle x, x' \rangle - f(x) : x \in X \} \quad \text{and} \quad f^{**}(x) := \sup \{ \langle x, x' \rangle - f^*(x') : x' \in X' \}.$$

• It follows from [39, Theorem 5] that epi $f^{**}$ is the intersection of all the “non-vertical” closed half spaces in $X \times \mathbb{R}$ that contain epi $f$, i.e.,

$$f^{**}(x) = \sup \{ a(x) : a \text{ is affine and continuous and } a \leq f \}, \quad \text{(B.3)}$$

where a function $a : X \to \mathbb{R}$ is affine and continuous if it is of the form $a(x) = \langle x, x' \rangle + c$ for some $x' \in X'$ and $c \in \mathbb{R}$.

• If $\text{co} f(x) > -\infty$ for all $x \in X$, then $f^{**} = \text{co} f$ by [39, Theorems 4 and 5]. In particular if $f$ is convex, lower semi-continuous and proper, then $f = f^{**}$, which is the famous Fenchel-Moreau theorem.

C Additional results and proofs

**Proposition C.1.** Let $\rho$ be a risk functional that satisfies the Fatou property on $X$ and let $c \geq 0$. Assume there exists an unbounded sequence of portfolios $(\pi_n)_{n \geq 1} \subset \mathbb{R}^d$ with $\rho(X_{\pi_n}) \leq c$ for all $n \in \mathbb{N}$. Then there exists a portfolio $\pi \in \mathbb{R}^d \setminus \{0\}$ with $\rho(\lambda X_\pi) \leq c$ for all $\lambda > 0$. Moreover, if $\mathbb{E}[X_{\pi_n}] = 0$ for all $n$, we may further assume $\mathbb{E}[X_\pi] = 0$.

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Proof. By passing to a subsequence and relabelling the assets, we may assume without loss of generality that \(|\pi^1_n| \geq |\pi^1_i|\) for all \(n \in \mathbb{N}\) and \(i \in \{1, \ldots, d\}\). As \(\|\pi_n\| \to \infty\) we must have that \(|\pi^1_n| \to \infty\), and by shifting the sequence we may assume \(|\pi^1_n| > 0\) for all \(n \in \mathbb{N}\). Then for all \(i \in \{1, \ldots, d\}\) we have \(\pi^1_n/|\pi^1_n| \to \pi^1 \in [-1, 1]\) and by compactness we can pass to a further subsequence and assume that \(\pi^1_n/|\pi^1_n| \to \pi^1 \in [-1, 1]\), where \(\pi^1 \in \{-1, 1\}\). It follows that

\[
X_{\pi_n}/|\pi^1_n| \to X_{\pi^1} \text{ P-a.s.},
\]

where \(\pi \neq 0\) since \(\pi^1 \in \{-1, 1\}\). Since \(|\pi^1_n| \to \infty\), for any \(\lambda > 0\), there exists \(N\) such that \(\lambda/|\pi^1_n| \in (0, 1)\) for all \(n \geq N\). Now star-shapedness of \(\rho\) gives

\[
\rho(\lambda X_{\pi_n}/|\pi^1_n|) \leq \lambda \rho(X_{\pi_n})/|\pi^1_n| \leq c, \quad n \geq N.
\]

By the Fatou property \((L \supseteq \mathcal{X}\) being a Riesz space), \(\rho(\lambda X_{\pi^1}) \leq \liminf_{n \to \infty} \rho(\lambda X_{\pi_n}/|\pi^1_n|) \leq c\). Hence

\[
\rho(\lambda X_{\pi^1}) \leq c \text{ for all } \lambda > 0.
\]

If in addition \(\mathbb{E}[X_{\pi_n}] = 0\) for all \(n\), then linearity of the expectation and the dominated convergence theorem gives \(\mathbb{E}[X_{\pi^1}] = 0\). Indeed, since \(\pi^1_n/|\pi^1_n| \in [-1, 1]\) we have

\[
|X_{\pi_n}/|\pi^1_n|| = |X^1 + \frac{\pi^2_n}{|\pi^1_n|} X^2 + \cdots + \frac{\pi^d_n}{|\pi^1_n|} X^d| \leq |X^1| + |X^2| + \cdots |X^d|
\]

where \(X^i := R^1 - \tau_i \in L^1\) for \(i \in \{1, \ldots, d\}\). This, together with \((C.1)\) and the dominated convergence theorem gives \(\mathbb{E}[X_{\pi^1}] = \lim_{n \to \infty} \mathbb{E}[X_{\pi_n}/|\pi^1_n|] = 0\).

Proof of Theorem 3.8. Define the function \(f_\rho : \mathbb{R}^d \to [0, \infty]\) by \(f_\rho(\pi) = \max\{\rho(X_{\pi}), 0\} + |\mathbb{E}[X_{\pi}]|\). Then \(f_\rho\) is lower semi-continuous by the Fatou property of \(\rho\) on \(\mathcal{X}\) (and the fact that \(L \supseteq \mathcal{X}\) is a Riesz space) and linearity of the expectation. Moreover, it is star-shaped, i.e., \(f_\rho(\lambda \pi) \geq \lambda f_\rho(\pi)\) for all \(\lambda \geq 1\) and \(\pi \in \mathbb{R}^d\), by the star-shapedness of \(\rho\) and linearity of the expectation.

For \(\delta \geq 0\), set \(A_\delta := \{\pi \in \mathbb{R}^d : f_\rho(\pi) \leq \delta\}\). Then each \(A_\delta\) is closed by lower semi-continuity of \(f_\rho\). We proceed to show that each \(A_\delta\) is also bounded and hence compact.

For \(\delta = 0\), using \(f_\rho(\pi) \geq |\mathbb{E}[X_{\pi}]| > 0\) for any \(\pi \in \mathbb{R}^d \setminus \Pi_0\), it follows that \(A_0 \subseteq \Pi_0\). Also note that for each \(\pi \in A_0\), \(X_{\pi} \in A_\rho\). If \(A_0\) were unbounded, then Proposition \(C.1\) would imply the existence of a portfolio \(\pi \in \Pi_0 \setminus \{0\}\) with \(\rho(\lambda X_{\pi}) \leq 0\) for all \(\lambda > 0\). But this would contradict \(\rho\) being sensitive to large expected losses on \(\mathcal{X}\). Therefore, \(A_0\) must be bounded.

For \(\delta > 0\), we argue as follows: Since \(A_0\) is bounded, there exists \(d > 0\) such that \(f_\rho(\pi) > 0\) for any portfolio \(\pi\) belonging to the set \(D := \{x \in \mathbb{R}^d : \|x\|_2 = d\}\). Compactness of \(D\) and lower semi-continuity of \(f_\rho\) give \(m := \min\{f_\rho(x) : x \in D\} \in (0, \infty]\). Star-shapedness of \(f_\rho\) in turn implies that \(f_\rho(\pi) \geq m \|\pi\|_2/d\) for all \(\pi \in \mathbb{R}^d\) with \(\|\pi\|_2 \geq d\), which in turn implies that each \(A_\delta\) is bounded.

We finish by a standard argument. Fix \(\nu \geq 0\) and assume \(\rho_\nu < \infty\). By definition, there exists a sequence of portfolios \((\pi_n)_{n \geq 1} \subseteq \Pi_{\nu}\) such that \(\rho(\lambda X_{\pi_n}) \searrow \rho_\nu\) and \(\rho_\nu + 1 \geq \rho(\lambda X_{\pi_n})\) for all \(n\). Setting \(\delta^* := \max\{\rho_\nu + 1, 0\} + \nu\), it follows that \((\pi_n)_{n \geq 1} \subseteq A_{\delta^*}\). Compactness of \(A_{\delta^*}\), closedness of \(\Pi_{\nu}\) and the Fatou property of \(\rho\) imply the existence of a portfolio \(\pi \in \Pi_{\nu}\) with \(\rho(\lambda X_{\pi}) \leq \rho_\nu\), i.e., \(\Pi_{\nu}^\text{c}\) is nonempty. Furthermore, \(\Pi_{\nu}^\text{c}\) is bounded since it is a subset of \(A_{\delta^*}\), and closed since \(\rho\) satisfies the Fatou property.

\[\text{Proposition C.2. Suppose } X \text{ is a topological space and } K \subseteq X \text{ is compact. Then for any nondecreasing sequence of lower semi-continuous functions } f_t : K \to [-\infty, \infty] \text{ with } f(x) := \lim_{t \to \infty} f_t(x) \text{ for all } x \in K, \text{ we have}\]

\[
\min_{x \in K} f(x) = \lim_{t \to \infty} \min_{x \in K} f_t(x).
\]

Furthermore, if \((x_t)_{t \geq 1}\) is a sequence where \(\min_{x \in K} f_t(x) = f_t(x_t)\), then any limit point is a minimiser for \(f\).  \[\text{Proposition C.2 is [11] Lemma 2.7(c)}.\] The application there was to relate finite horizon discrete time Markov decision processes with infinite horizon ones.
Proof. First note that $f$ is lower semi-continuous because it is the supremum of lower semi-continuous functions. By the compactness of $K$ and lower semi-continuity, $f$ and $f_t$ attain their minimum values. Now since $f_t$ is a nondecreasing sequence, it is easy to see that
\[
\min_{x \in K} f(x) \geq \lim_{t \to \infty} \min_{x \in K} f_t(x) =: m.
\]
For the reverse inequality, consider the sets $A_t := \{x \in K : f_t(x) \leq m\}$. These are nonempty (because $\emptyset \neq \arg\min f_t \subset A_t$), closed (by the lower semi-continuity of $f_t$) and compact (since $K$ is compact and $A_t$ is closed). Moreover, they are nested in the sense that $A_t \supset A_{t+1}$. It follows by Cantor’s intersection theorem that
\[
A := \bigcap_{t=1}^{\infty} A_t \neq \emptyset,
\]
i.e., there exists $x^* \in K$ such that $f_t(x^*) \leq m$ for all $t$. Taking the limit as $t \to \infty$ yields
\[
\min_{x \in K} f(x) \leq f(x^*) \leq m = \lim_{t \to \infty} \min_{x \in K} f_t(x).
\]
To prove the final claim, note that $\arg\min f = A$ because $f(x) \leq m$ if and only if $f_t(x) \leq m$ for all $t$. Whence, any limit point of a sequence of minimisers $(x_t)_{t \in \mathbb{N}}$ — that is where $x_t \in \arg\min f_t$ for all $t \geq 1$ — is contained in $A$, and hence, is a minimiser for $f$.

Proof of Proposition 3.11. First note that by Theorem 3.8, the map $\nu \mapsto \rho_\nu$ is $(-\infty, \infty]$-valued.

Next we establish lower semi-continuity. Fix $y \in \mathbb{R}$ and let $B_y := \{\nu \in \mathbb{R}_+ : \nu \cdot y \leq y\}$. We must show that this set is closed. So let $(\nu_n)_{n \geq 1} \subset B_y$ and assume $\nu_n \to \nu$. By Theorem 3.8 for each $n$ there exists a portfolio $\pi_n$ such that $\rho(X_{\pi_n}) = \rho_{\nu_n} \leq y$ and $\mathbb{E}[X_{\pi_n}] = \nu_n$. We proceed to show that the sequence $(\pi_n)_{n \geq 1}$ belongs to a compact set. To this end, let $c \in \mathbb{R}$ be such that $|\nu_n| \leq c$ for all $n$. Setting $\delta := \max\{y, 0\} + c$ it follows that each $\pi_n$ lies in $A_\delta := \{\pi \in \mathbb{R}^d : \max\{\rho(X_{\pi}), 0\} + \mathbb{E}[X_\pi] \leq \delta\}$, which is compact by the proof of Theorem 3.8. Passing to a subsequence, we may assume that $(\pi_n)_{n \geq 1}$ converges to some $\pi \in \mathbb{R}^d$, and by dominated convergence and the Fatou property, it follows that $\mathbb{E}[X_\pi] = \nu$ and $\rho(\pi) \leq \nu$. Whence, $\rho_\nu \leq y$ and so $\nu \in B_y$.

We now show that $\nu \mapsto \rho_\nu^{-1}$ is a nondecreasing sequence and compact (since $\rho^{-\infty}$ is sensitive to large expected losses, $\rho_0^{-\infty} = 0 \geq \rho_0$). Thus, it suffices to show that
\[
\rho_\nu^{-\infty} = \lim_{t \to \infty} \rho_t^{-1}/t.
\]
The key idea is to consider the risk functionals $\rho^t : L \to (-\infty, \infty]$ defined by $\rho^t(X) = \rho(tX)/t$ for $t \geq 1$. They satisfy the Fatou property on $\mathcal{X}$, sensitivity to large expected losses on $\mathcal{X}$ and $\rho_t^{-1} = \rho_1^{-1}$. By star-shapedness of $\rho$ and definition of $\rho^{-\infty}$ in (2.1), we have $\rho^{t+1}(X) \geq \rho^t(X)$ and $\lim_{t \to \infty} \rho^t(X) = \rho^{-\infty}(X)$ for all $X \in L$. This implies $(\rho_t^{-1})_{t \geq 1}$ is a nondecreasing sequence and
\[
\rho_t^{-\infty} \geq m := \lim_{t \to \infty} \rho_t^{-1}.
\]
If $m = \infty$, the reverse inequality is clear, so assume $m < \infty$. Then as $\rho^t \geq \rho$ and $m \geq \rho_1^{-1}$ for each $t \geq 1$, it follows that
\[
\Pi_t^\rho \subset \{\pi \in \Pi_1 : \rho(X_\pi) \leq m\} \subset \{\pi \in \Pi_1 : \max\{\rho(X_\pi), 0\} + |\mathbb{E}[X_\pi]| \leq \max\{m, 0\} + 1\} : = K.
\]

Since $K$ is compact by the proof of Theorem 3.8, (C.2) follows by applying Proposition C.2 to the sequence of functions $f_t : K \to (-\infty, \infty]$ given by $f_t(\pi) := \rho^t(X_{\pi})$.

The statements in (b) and (c) as well as the equivalence between $\rho_t^{-\infty} > 0$ and $\nu^+ < \infty$ follow directly from the fact $\nu \mapsto \rho_\nu^{-\infty}$ is the smallest positively homogeneous majorant of $\nu \mapsto \rho_\nu$.

Finally, we establish (a). If $\rho_t^{-\infty} > 0$, then $\nu^+ \leq \infty$ by the above and hence $\rho_\nu > 0$ for all $\nu > \nu^+$. By lower semi-continuity of $\nu \mapsto \rho_\nu$ and compactness of $[0, \nu^+]$, there exists a global minimum $m \leq \rho_0 \leq 0$ that is attained at $\nu^* := \sup\{\nu \in [0, \nu^+] : \rho_\nu = m\}$. By construction, $\rho_\nu > m$ for all $\nu > \nu^*$. Whence, by definition $\nu_{\min} = \nu^* < \infty$ and $\rho_{\min} = \rho_{\nu_{\min}} \in (-\infty, 0]$. 

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Proof of Proposition 3.12. First, we establish convexity of $\nu \mapsto \rho_\nu$. Let $\nu, \nu' \in \mathbb{R}_+$, $\lambda \in [0, 1]$ and

$A := \Pi_\nu \times \Pi_{\nu'}$. Using convexity of $\rho$ and the fact $(\pi, \pi') \in A$ implies $\lambda \pi + (1 - \lambda) \pi' \in \Pi_{\lambda \nu + (1 - \lambda) \nu'}$, we obtain

$$
\rho_{\lambda \nu + (1 - \lambda) \nu'} \leq \inf_{(\pi, \pi') \in A} \{\rho(X_{\lambda \pi + (1 - \lambda) \pi'})\} \leq \inf_{(\pi, \pi') \in A} \{\lambda \rho(X_\pi) + (1 - \lambda) \rho(X_{\pi'})\} \leq \lambda \rho_\nu + (1 - \lambda) \rho_{\nu'}.
$$

Thus, $\nu \mapsto \rho_\nu$ is convex on $\mathbb{R}_+$.

Next, since $\nu \mapsto \rho_\nu$ is convex, it is continuous in the interior of its effective domain $\{\nu \in \mathbb{R}_+ : \rho_\nu < \infty\}$, which is an interval. This together with lower semi-continuity shown in Proposition 3.10 implies that $\nu \mapsto \rho_\nu$ is finite and continuous on the closure of $\{\nu \in \mathbb{R}_+ : \rho_\nu < \infty\}$, which a fortiori implies that $\{\nu \in \mathbb{R}_+ : \rho_\nu < \infty\}$ is closed. The other claims follow directly from Propositions 3.9 and 3.10 together with standard properties of convex functions.

Proof of Theorem 3.13. “(a) $\iff$ (b)” This is [29, Theorem 3.18]. “(b) $\implies$ (c)” This follows from the definition of strong $\rho$-arbitrage and the fact $\rho^{\infty}$ dominates $\rho$.

Proof of Theorem 3.13. “(c) $\implies$ (b)” This follows from the definition of $\rho$-arbitrage and the fact $\rho^{\infty}$ dominates $\rho$.

“(b) $\implies$ (a)” We prove by contraposition, so assume $\rho_\nu^{\infty} \leq 0$. Since $\rho$ satisfies the Fatou property on $X$ and sensitivity to large expected losses on $X$, so too does $\rho^{\infty}$. Hence, by Theorem 3.8, $\rho_\nu^{\infty} = \rho^{\infty}(X_{\pi_n})$ for some $\pi_n \in \Pi_1$. Letting $\pi_n = n \pi$ yields a sequence of portfolios for which $E[X_{\pi_n}] \uparrow \infty$ and $\rho^{\infty}(X_{\pi_n}) = n \rho_\nu^{\infty} \leq 0$ for all $n$. Thus, the market admits $\rho^{\infty}$-arbitrage.

“(a) $\implies$ (c)” Assume $\rho_\nu^{\infty} > 0$. This implies $\nu^+ < \infty$ by Proposition 3.10. By definition of $\nu^+$, it follows for any sequence of portfolios with $E[X_{\pi_n}] \uparrow \infty$, $\rho(X_{\pi_n}) > 0$ eventually. Therefore, the market does not admit $\rho$-arbitrage.

Proposition C.3. Let $X$ be a subspace of $L^1$ such that $\{(X, Y) \in \mathcal{X}^2 : Y \geq X $ $\mathbb{P}$-a.s. and $\mathbb{P}[Y > X] > 0\} = \emptyset$. Suppose $\eta : \mathcal{X} \rightarrow (-\infty, \infty]$ is normalised, star-shaped and $\eta(X) \leq WC(X)$ for all $X \in \mathcal{X}$. Define $\mathcal{Y} := \{Y \in L^1 : Y \geq X $ $\mathbb{P}$-a.s. for some $X \in \mathcal{X}\}$ and $\rho : L^1 \rightarrow (-\infty, \infty]$ by

$$
\rho(X) = \begin{cases} 
\eta(X), & \text{if } X \in \mathcal{X}, \\
E[-X], & \text{if } X \in \mathcal{Y}, \\
WC(X), & \text{otherwise}.
\end{cases}
$$

Then $\rho$ is a risk functional such that $\rho|_{\mathcal{X}} \equiv \eta$.

Proof. Normalisation of $\rho$ is clear, as is the fact $\rho|_{\mathcal{X}} \equiv \eta$.

To show monotonicity, let $X, Y \in L^1$ and assume $Y \geq X$ $\mathbb{P}$-a.s. and $\mathbb{P}[Y > X] > 0$. If $X \in \mathcal{X}$ or $X \in \mathcal{Y}$, then $Y \in \mathcal{Y}$ and $\rho(X) \geq E[-X] \geq E[-Y] = \rho(Y)$. If $X \in L^1 \setminus (\mathcal{X} \cup \mathcal{Y})$, then $\rho(X) = WC(X) \geq WC(Y) \geq \rho(Y)$. Whence, $\rho$ is monotone.

Finally, to show $\rho$ is star-shaped, fix $\lambda \geq 1$ and $X \in L^1$. If $X \in S$ where $S = \{\mathcal{X}, \mathcal{Y}, L^1 \setminus (\mathcal{X} \cup \mathcal{Y})\}$, then also $\lambda X \in S$ and so it follows that $\rho(\lambda X) \geq \lambda \rho(X)$.

Proof of Theorem 3.13. Assume $A_{\rho^{\infty}} \supseteq L_+$ and let $X \in A_{\rho^{\infty}} \setminus L_+$. Since $\rho$ (and hence $\rho^{\infty}$) are monotone, we may assume without loss of generality that $\mathbb{P}[X > 0] > 0$. Fix $n \in \mathbb{N}$ so that $E[X^n] \geq \frac{1}{n} E[X^n] > 0$ and let $R := X^+ - \frac{1}{2} X^-$. Consider the market $(S^0, S)$ defined by $S^0 \equiv 1$ and $S := S^1$ where $S^1_1 = 1$ and $S^1_1 = 1 + R$. This market does not admit arbitrage of the first kind since $\mathbb{P}[R < 0] > 0$ and $\mathbb{P}[R > 0] > 0$. Moreover, $E[R] > 0$ and $R \in A_{\rho^{\infty}}$ since $R \geq X$. Thus, $\rho_\nu^{\infty} \equiv 0$ and the market admits $\rho^{\infty}$-arbitrage. Whence, the market admits $\rho$-arbitrage by Remark 3.16(a).

When $\rho$ is cash-invariant, $\rho^{\infty}$ is also cash-invariant. By adding $\varepsilon > 0$ sufficiently small to $R$ above, we can find $\tilde{R}$ such that $\mathbb{P}[\tilde{R} < 0] > 0$, $E[\tilde{R}] > 0$ and $\rho^{\infty}(\tilde{R}) < 0$. Replacing $R$ with $\tilde{R}$ in the market above produces a market where $\rho_\nu^{\infty} < 0$. This market does not admit arbitrage of the first kind but admits strong $\rho$-arbitrage by Theorem 3.13.

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Proof of Theorem 3.20. Assume the market admits $\rho$-arbitrage. Then by definition, it follows that the mean-$\rho$ problem (2) does not have any solutions for any $\rho^* \geq 0$.

Assume the market does not admit $\rho$-arbitrage. Then by Theorem 3.15 $\rho^* > 0$. We first show that the set
\[ K_c := \{ \pi \in \mathbb{R}^d : \rho(X_\pi) \leq c \} \]
is compact for all $c < \infty$. Closedness follows from the Fatou property. We show boundedness via contradiction. If $(\pi_n)_{n \geq 1} \subset K_c$ is unbounded, then by Proposition C.1 there exists a nonzero portfolio $\pi \in \mathbb{R}^d$ with $\rho(\lambda X_\pi) \leq c$ for all $\lambda > 0$. Thus $\rho^*(X_\pi) \leq 0$, and by sensitivity to large expected losses, $\mathbb{E}[X_\pi] > 0$. But this would contradict the fact $\rho^*_1 > 0$.

To show that the mean-$\rho$ problem (1) admits a solution, let $\nu^* \geq 0$. If $\rho_{\nu^*} = \infty$, then $\rho_{\nu^*} = \infty$ for all $\nu \geq \nu^*$ and so any portfolio in $\Pi_\nu$ for $\nu \geq \nu^*$ is a solution to (1). If $\rho_{\nu^*} < \infty$, then as $K_{\rho_{\nu^*}}$ is compact, and $\rho_{\nu^*}$ is attained by Theorem 3.8 this means that the set
\[ K_{\rho_{\nu^*}}^*: = \{ \pi \in \mathbb{R}^d : \rho(X_\pi) \leq \rho_{\nu^*} \text{ and } \mathbb{E}[X_\pi] \geq \nu^* \} \]
is nonempty and compact. It follows that there exists a convergent sequence $(\pi_n)_{n \geq 1} \subset K_{\rho_{\nu^*}}^*$ with limit $\pi^* \in K_{\rho_{\nu^*}}^*$ such that
\[ \lim_{n \to \infty} \rho(X_{\pi_n}) = \inf \{ \rho(X_{\pi}) : \pi \in K_{\rho_{\nu^*}}^* \}. \]
By the Fatou property, $\pi^*$ must be a solution to (1).

Finally, to show the mean-$\rho$ problem (2) admits a solution, let $\rho^* \in [0, \infty)$. Then as $K_{\rho^*}$ is nonempty (since $0 \in K_{\rho^*}$) and compact, there exists a convergent sequence $(\pi_n)_{n \geq 1} \subset K_{\rho^*}$ with limit $\pi^* \in K_{\rho^*}$ such that
\[ \lim_{n \to \infty} \mathbb{E}(X_{\pi_n}) = \sup \{ \mathbb{E}(X_{\pi}) : \pi \in K_{\rho^*} \}. \]
As $\mathbb{E}[X_{\pi_n}] \to \mathbb{E}[X_{\pi^*}]$, it follows that $\pi^*$ is a solution to (2). \hfill \Box

Proof of Proposition 3.2. The first part of the statement in (a) is because strong $\rho^*$-arbitrage implies strong $\rho$-arbitrage, which in turn implies $\rho$-arbitrage. The second part of (a) is a consequence of Theorem 3.15 and Corollary 3.18 Parts (b) and (c) are clear. \hfill \Box

Proof of Theorem 3.24. Consider the market $(S^0, S, S^{d+1})$ with $S^{d+1} = x$ and $S^{d+1} = X$. We will work with the economically equivalent market $(S^0, \hat{S}, \hat{S}^{d+1})$ where $\hat{S}^i := S^i + (1 - S^0)S^0$. Then the mean vector of the excess returns is given by $\hat{\mu}(x)$ and its covariance matrix is given by $\Sigma$. The maximal Sharpe ratio in the augmented market is then given by
\[ \text{SR}_{max}(x) := \max_{\pi \in \mathbb{R}^d \setminus \{0\}} \frac{\mathbb{E}[X_{\pi}]}{\sqrt{\text{Var}(X_{\pi})}} = \sqrt{\hat{\mu}(x)^T \Sigma^{-1} \hat{\mu}(x)}. \]
If $\rho$ is a law-invariant risk measure for which (strong) $\rho$-arbitrage is equivalent to (strong) $\rho^*$-arbitrage, then by Corollary 3.28, the market $(S^0, \hat{S}, \hat{S}^{d+1})$ admits (strong) $\rho$-arbitrage if and only if $\text{SR}_{max}(x) < (\leq) \rho^*(Z)$. Whence,
\[ I_\rho(X) = I_{\rho^*}(X) = \{ x \in \mathbb{R} : \text{SR}_{max}(x) < \rho^*(Z) \} \quad \text{and} \]
\[ I^*_\rho(X) = I^*_{\rho^*}(X) = \{ x \in \mathbb{R} : \text{SR}_{max}(x) \leq \rho^*(Z) \}. \]
\hfill \Box

Proof of Proposition 4.3. Let $X \in \mathcal{X}$, i.e., there is $\pi \in \mathbb{R}^d$ such that $X = X_\pi = \pi \cdot (R - r1)$. By Condition UI, this implies that $Q^\alpha$ and $XQ^\alpha$ are UI.

First, co $\alpha$ (whose effective domain is $Q^\alpha$) represents $\rho$ by Remark 4.1(d) since $\alpha^\rho \leq \alpha \leq \alpha$ by Remark 4.1(c) and the definition of the convex hull. This together with co $\alpha \leq \alpha$ and Remark 4.5(a) implies that $Q^{\text{co} \alpha} \subset Q^\alpha$ and
\[ \rho(X) = \sup_{Z \in Q^\alpha} \{ \mathbb{E}[ZX] - \text{co} \alpha(Z) \} \leq \sup_{Z \in Q^\alpha} \{ \mathbb{E}[ZX] - \text{co} \alpha(Z) \}. \quad (C.3) \]
If we can show that the supremum on the right side of \((C.3)\) is attained and the inequality is an equality, then \((4.4)\) follows.

To see that the supremum on the right side of \((C.3)\) is attained, let \((Z_n)_{n \in \mathbb{N}}\) be a maximising sequence in \(\overline{Q}^\alpha\). As \(Q^\alpha\) is uniformly integrable and convex, \(\overline{Q}^\alpha\) is convex and \(\sigma(L^1, L^\infty)\)-sequentially compact by the Dunford-Pettis and the Eberlein-Šmulian theorems. After passing to a subsequence, we may assume that \(Z_n\) converges weakly to some \(Z^* \in \overline{Q}^\alpha\). Then because the map \(\tilde{Z} \mapsto \mathbb{E}[-\tilde{Z}X]\) is weakly continuous on \(\overline{Q}^\alpha\) (by \([29\) Proposition C.2]) and \(\overline{c} \alpha\) is also \(\sigma(L^1, L^\infty)\)-lower semi-continuous by \([15\) Theorem 2.2.1], \(Z^*\) is a maximiser.

Finally, we show that the inequality in \((C.3)\) is an equality. We may assume without loss of generality that the right hand side of \((C.3)\) is larger than \(-\infty\). Hence, \(\overline{c} \alpha\) \((Z^*)\) is finite. Let \(\varepsilon > 0\). Since \(\overline{c} \alpha\) is the \(L^1\)-lower semi-continuous hull of \(c \alpha\) and \(c \alpha(Z) = \infty\) for \(Z \notin Q^\alpha\) and \(\overline{c} \alpha(Z^*) < \infty\), by \([11,\) there exists a sequence \((Z_n)_{n \in \mathbb{N}} \subset Q^\alpha\) that converges in \(L^1\) to the maximiser \(Z^*\) and for which \(\lim_{n \to \infty} c \alpha(Z_n) \leq \overline{c} \alpha(Z^*) + \varepsilon\). Using again that the map \(\tilde{Z} \mapsto \mathbb{E}[-\tilde{Z}X]\) is weakly continuous and hence strongly continuous yields

\[
\rho(X) \geq \lim_{n \to \infty} \{\mathbb{E}[-Z_nX] - c \alpha(Z_n)\} \geq \mathbb{E}[-Z^*X] - \overline{c} \alpha(Z^*) - \varepsilon.
\]

Now the claim follows by letting \(\varepsilon \to 0\).

**Proof of Proposition \([4.4]\)** It is clear by the definition of \(C_{Q^{\overline{c} \alpha}}\) that \(\text{dom } f_{\overline{c} \alpha} = C_{Q^{\overline{c} \alpha}}\). By Remark \([4.5\) (a)], \(Q^{\overline{c} \alpha} \subset Q^\alpha\) and as \(c \alpha(Z) = \infty\) for \(Z \in Q^\alpha \setminus Q^{\overline{c} \alpha}\) it follows that

\[
f_{\overline{c} \alpha}(c) = \inf\{\overline{c} \alpha(Z) : Z \in Q^\alpha \text{ and } \mathbb{E}[-Z(R-r1)] = c\}.
\]

Since \(Q^\alpha\) is \(\sigma(L^1, L^\infty)\)-sequentially compact by Dunford-Pettis and the Eberlein-Šmulian theorems and \(\overline{c} \alpha\) is \(\sigma(L^1, L^\infty)\)-lower semi-continuous by \([15\) Theorem 2.2.1], it follows that the infimum is attained and (finite) if \(c \in C_{Q^{\overline{c} \alpha}}\). Moreover, since \(Q^{\overline{c} \alpha} \subset Q^\alpha\) is convex, it follows that \(C_{Q^{\overline{c} \alpha}} \subset C_{Q^\alpha} = \{\mathbb{E}[-Z(R-r1)] : Z \in Q^\alpha\}\) is convex and bounded since \(C_{Q^\alpha} = \text{cl}(C_{Q^\alpha})\) is a (convex) compact subset of \(\mathbb{R}^d\) by \([29\) Proposition 4.5].

Next, we show that \(f_{\overline{c} \alpha}\) is convex and lower semi-continuous. Convexity follows easily from convexity of \(\overline{c} \alpha\). To argue lower semi-continuity let \((c_n)_{n \in \mathbb{N}}\) be a sequence in \(\mathbb{R}^d\) that converges to \(c \in \mathbb{R}^d\). Without loss of generality, we may assume that each \(c_n\) and \(c\) lies in \(C_{Q^{\overline{c} \alpha}}\). Let \((Z_n)_{n \in \mathbb{N}} \in \mathbb{Q}^{\overline{c} \alpha}\) be a corresponding sequence of minimisers. Since \(Q^\alpha\) is \(\sigma(L^1, L^\infty)\)-sequentially compact by the Dunford-Pettis and the Eberlein-Šmulian theorems, after passing to a subsequence, we may assume that \((Z_n)_{n \in \mathbb{N}}\) converges weakly to some \(Z \in Q^{\overline{c} \alpha}\). As the map \(\tilde{Z} \mapsto \mathbb{E}[-\tilde{Z}X]\) is \(\sigma(L^1, L^\infty)\)-continuous on \(Q^\alpha\) by \([29\) Proposition C.2], it follows that \(\mathbb{E}[-Z(R-r1)] = c\). By \(\sigma(L^1, L^\infty)\)-lower semi-continuity of \(\overline{c} \alpha\) this implies that \(f_{\overline{c} \alpha}(c) \leq \overline{c} \alpha(Z) \leq \liminf_{n \to \infty} \overline{c} \alpha(Z_n) = \liminf_{n \to \infty} f_{\overline{c} \alpha}(c_n)\).

We proceed to show that \(f_{\overline{c} \alpha}\) is the lower semi-continuous convex hull of \(f_\alpha\). To this end, for a function \(g : \mathbb{R}^d \to [0, \infty]\), define the map \(\alpha^g : D \to [0, \infty]\) by

\[
\alpha^g(Z) = \begin{cases} g(\mathbb{E}[-Z(R-r1)]), & Z \in Q^{\overline{c} \alpha} \\ \infty, & \text{otherwise} \end{cases}
\]

If \(g\) is convex and lower semi-continuous, then \(\alpha^g\) is convex and \(\sigma(L^1, L^\infty)\)-lower semi-continuous because the map \(\tilde{Z} \mapsto \mathbb{E}[-\tilde{Z}(R-r1)]\) is linear and \(\sigma(L^1, L^\infty)\)-continuous on \(Q^\alpha \supset Q^{\overline{c} \alpha}\) by \([29\) Proposition \(C.2]\).

Seeking a contradiction, suppose now that there exists a convex lower semi-continuous function \(g : \mathbb{R}^d \to [0, \infty]\) such that \(g \leq f_\alpha\) and \(f_{\overline{c} \alpha}(c*) < g(c*)\) for some \(c* \in C_{Q^{\overline{c} \alpha}}\). Then

\[
\alpha^g(Z) \leq \alpha^g(Z) \leq \alpha(Z), \quad Z \in Q^{\overline{c} \alpha},
\]

and hence \(\alpha^g \leq \overline{c} \alpha\). Let \(Z^* \in Q^{\overline{c} \alpha}\) be such that \(\mathbb{E}[-Z^*(R-r1)] = c^*\) and \(\overline{c} \alpha(Z^*) = f_{\overline{c} \alpha}(c*)\). Then

\[
\overline{c} \alpha(Z^*) = f_{\overline{c} \alpha}(c*) < g(c*) = \alpha^g(Z^*)
\]

and we arrive at a contradiction.

Finally, \([4.5\) follows from Proposition \(4.3\).
Proof of Theorem 4.6. First we show that the market admits strong ρ-arbitrage if and only if \( \inf_{\pi \in \mathbb{R}^d} \rho(X_\pi) = -\infty \). For the nontrivial direction, let \( (\pi_n)_{n \geq 1} \subset \mathbb{R}^d \) be a sequence of portfolios such that \( \rho(X_{\pi_n}) \land -\infty \). By the dual representation of \( \rho \), this implies that \( \mathbb{E}[-X_{\pi_n}] - \alpha(1) \land -\infty \), and since \( \alpha(1) < \infty \), this gives \( \mathbb{E}[X_{\pi_n}] \nearrow \infty \).

Now let \( f_{\pi \alpha} \) be as in Proposition 4.4. Since \( \text{dom } f_{\pi \alpha} = C_{\pi \alpha} \), the convex conjugate of \( f_{\pi \alpha} \) is given by

\[
f_{\pi \alpha}^*(\pi) = \sup_{c \in \mathbb{R}^d} (\pi \cdot c - f_{\pi \alpha}(c)) = \sup_{c \in C_{\pi \alpha}} (\pi \cdot c - f_{\pi \alpha}(c)), \quad \pi \in \mathbb{R}^d.
\]

By (4.5), this implies

\[
f_{\pi \alpha}^*(\pi) = \rho(X_\pi), \quad \pi \in \mathbb{R}^d.
\]

Since \( f_{\pi \alpha} \) is a nonnegative lower semi-continuous convex function, the Fenchel-Moreau theorem (cf. Appendix B) and (C.4) give

\[
-f_{\pi \alpha}(0) = -f_{\pi \alpha}^*(0) = - \sup_{\pi \in \mathbb{R}^d} (-f_{\pi \alpha}^*(\pi)) = \inf_{\pi \in \mathbb{R}^d} \rho(X_\pi).
\]

The result follows since \( Q_{\pi \alpha} \cap M = \emptyset \) if and only if \( f_{\pi \alpha}(0) = \infty \), and the market admits strong ρ-arbitrage if and only if \( \inf_{\pi \in \mathbb{R}^d} \rho(X_\pi) = -\infty \). The final claim is a consequence of Remark 4.5(a).

Proof of Theorem 4.8. The result follows from Theorem 3.15 and [29, Theorem 4.20], noting that by Remark 3.7(b), \( \rho \) satisfying sensitivity to large expected losses on \( L \) implies that \( \rho^\infty \) is strictly expectation bounded.

Proof of Theorem 4.10. If the original market \( (S^0, S) \) admits strong ρ-arbitrage, then \( I_\rho(X) = \emptyset \) by Proposition 3.23(b) and \( Q_{\pi \alpha} \cap M = \emptyset \) by Theorem 4.6. Whence, (4.7) holds.

So assume the original market does not admit strong ρ-arbitrage. Consider the market \( (S^0, S, S^{d+1}) \) where \( S^{d+1}_0 = 1 \) and \( S^{d+1}_i = X + (1 - x)S^0_i \). Then, by Theorem 4.6, \( x \) is a strong \( \rho \) consistent price for \( X \) if and only if there is an ACMM \( Z \subset Q_{\pi \alpha} \) for the extended market, i.e.,

\[
S_0^i = \mathbb{E}[Z S^0_i/(1 + r)], \quad \text{for } i = 1, \ldots, d + 1.
\]

In particular, \( Z \) is necessarily contained in \( Q_{\pi \alpha} \cap M \), and we obtain the inclusion \( \subset \) in (4.7). Conversely, if \( S_0^{d+1} = \mathbb{E}[Z S^{d+1}_i/(1 + r)] \) for some \( Z \in Q_{\pi \alpha} \cap M \), then this \( Z \) is also an ACMM for the extended market model, and so the two sets in (4.7) are equal.

Proof of Theorem 4.11. If the original market \( (S^0, S) \) admits ρ-arbitrage, then \( I_\rho(X) = \emptyset \) by Proposition 3.23(b) and \( Q_{\pi \alpha} \cap \mathcal{P} = \emptyset \) by Theorem 4.8. Whence, (4.8) holds.

So assume the original market does not admit ρ-arbitrage. Consider the market \( (S^0, S, S^{d+1}) \) where \( S^{d+1}_0 = 1 \) and \( S^{d+1}_i = X + (1 - x)S^0_i \). Then, by Theorem 4.8, \( x \in \mathbb{R} \) is a \( \rho \) consistent price for \( X \) if and only if there exists an EMM \( Z \subset Q_{\pi \alpha} \) for the extended market, i.e.,

\[
S_0^i = \mathbb{E}[Z S^0_i/(1 + r)], \quad \text{for } i = 1, \ldots, d + 1.
\]

In particular, \( Z \) is necessarily contained in \( Q_{\pi \alpha} \cap \mathcal{P} \), and we obtain the inclusion \( \subset \) in (4.8). Conversely, if \( S_0^{d+1} = \mathbb{E}[Z S^{d+1}_i/(1 + r)] \) for some \( Z \in Q_{\pi \alpha} \cap \mathcal{P} \), then this \( Z \) is also an EMM for the extended market model, and so the two sets in (4.8) are equal.

Proposition C.4. \( (A_{EW})^\infty = H^b_+ \) if and only if \( a_l = 0 \) or \( b_l = \infty \).

Proof. Assume first that \( b_l = \infty \) and suppose \( X \in H^b_+ \) and \( \mathbb{P}[X < 0] > 0 \). Then since there exists \( a \geq 0 \) and \( b \leq 0 \) such that \( l(x) \geq ax + b \) for all \( x \leq 0 \), for any \( \lambda > 0 \) we have

\[
EW^i(\lambda X) = \mathbb{E}[l(-\lambda X)] \geq \mathbb{E}[-a\lambda X + b I_{\{X \geq 0\}}] + \mathbb{E}[l(-\lambda X) I_{\{X < 0\}}] \geq \lambda k_1 + c + p(\lambda k_2)
\]

where \( k_1 := \mathbb{E}[-aX I_{\{X \geq 0\}}] \leq 0, c := b\mathbb{P}[X \geq 0] \leq 0, k_2 := \min\{1, -\text{ess inf}(X)/2\} > 0 \) and \( p := \mathbb{P}[X \leq -k_2] > 0 \). Now as \( \lambda \to \infty \), \( (\lambda k_1 + c + p(\lambda k_2))/\lambda \to \infty \) since \( b_l = \infty \). Therefore, there exists \( \lambda \geq 1 \) such that \( EW^i(\lambda X) > 0 \) and so \( (A_{EW})^\infty = H^b_+ \).
Now assume that $a_l = 0$ and suppose $X \in H^{\Phi_l}$ and $\mathbb{P}[X < 0] > 0$. If $X$ is constant, then of course there exists $\lambda \geq 1$ such that $\text{EW}^l(\lambda X) > 0$, so assume $X$ is not constant. Then there exists $Y \in H^{\Phi_l}$ such that $Y \geq X$ $\mathbb{P}$-a.s., $\text{ess sup}(Y) > 0$ and $\mathbb{P}[Y < 0] > 0$. By monotonicity, $\text{EW}^l(\lambda X) \geq \text{EW}^l(\lambda Y)$, and since there exists $a > 0$ and $b \leq 0$ such that $l(x) \geq ax + b$ for all $x \geq 0$, for any $\lambda > 0$ we have

$$\text{EW}^l(\lambda Y) = \mathbb{E}[l(-\lambda Y)] \geq \mathbb{E}[(-a\lambda Y + b)\mathbf{1}_{\{Y < 0\}}] + \mathbb{E}[l(-\lambda Y)\mathbf{1}_{\{Y \geq 0\}}] \geq \lambda j_1 + d + q\lambda j_2$$

where $j_1 := \mathbb{E}[-a\lambda Y\mathbf{1}_{\{Y < 0\}}] > 0$, $d := b\mathbb{P}[Y < 0] \leq 0$, $j_2 := \max\{-1, -\text{ess sup}(Y)/\lambda\} < 0$ and $q := \mathbb{P}[Y \geq -c_2] > 0$. As $\lambda \to \infty$, $(\lambda j_1 + d + q\lambda j_2)/\lambda \to j_1 > 0$ since $a_l = 0$. Therefore, there exists $\lambda \geq 1$ such that $\text{EW}^l(\lambda X) > 0$ and so $(\mathbb{A}_{\text{EW}^l})^\infty = H^+_{\Phi_l}$.

On the other hand, if $b_l \neq \infty$ and $a_l \neq 0$, then define the loss function $\tilde{l} : \mathbb{R} \to \mathbb{R}$ by

$$\tilde{l}(x) = \begin{cases} b_l x, & \text{if } x \geq 0, \\ a_l x, & \text{if } x < 0. \end{cases}$$

Then $\tilde{l} \geq l$, so $\text{EW}^{\tilde{l}} \geq \text{EW}^l$, and to complete the proof, it suffices to find $X \in H^{\Phi_l} = H^{\Phi_{\tilde{l}}} = L^1$ such that $\mathbb{P}[X < 0] > 0$ and $\text{EW}^l(\lambda X) \leq 0$ for all $\lambda > 0$. To that end, let $A \in \mathcal{F}$ be a nontrivial event, $p := \mathbb{P}[A] \in (0, 1)$ and consider the random variable $X = \alpha\mathbf{1}_A - \beta\mathbf{1}_{A^c}$ where $\alpha, \beta > 0$ satisfy $-p\alpha + (1 - p)b\beta < 0$. Then $X \in L^1$, $\mathbb{P}[X < 0] > 0$ and for any $\lambda > 0$,

$$\text{EW}^l(\lambda X) = \mathbb{E}[\tilde{l}(\lambda X)] = \lambda[-pa_l + (1 - p)b_l\beta] \leq 0.$$ 

Proof of Proposition 5.4. If $l|_{\mathbb{R}_-} = 0$, then $\text{SR}^l \equiv \mathbb{W}C$. In this case, (5.1) holds since $l^*|_{[0, 1]} = 0$ and $\alpha^l(Z) = 0$ for $Z \in D \cap L^\infty_{\Phi_l}$. Otherwise, if $l|_{\mathbb{R}_-} \neq 0$, then $\text{SR}^l$ is a real-valued convex risk measure on $H^{\Phi_l}$ and (5.1) follows from [16, Theorem 4.3], and the proof of [23, Theorem 10].

Proposition C.5. Let $l$ be a loss function and assume that $0 < a_l < b_l < \infty$. Consider the penalty function $\alpha^l(Z) := \inf_{\lambda > 0} \frac{1}{\lambda} \mathbb{E}[l^* (\lambda Z)]$ for $Z \in D$. Then

$$\tilde{Q}^{\alpha^l} = \{Z \in D : \text{there exists } k > 0 \text{ and } \varepsilon > 0 \text{ such that } a_l + \varepsilon < kZ < b_l - \varepsilon \text{ } \mathbb{P}\text{-a.s.}\}$$

is a nonempty subset of $\tilde{Q}^{\alpha^l}$ satisfying Conditions POS, MIX and INT.

Proof. First, since $(a_l, b_l) \subset \text{dom } l^* \subset [a_l, b_l]$ and $l^*$ is bounded on any compact subset of $(a_l, b_l)$, $\tilde{Q}^{\alpha^l} \subset Q^{\alpha^l}$. Moreover, it is clear that $1 \in \tilde{Q}^{\alpha^l}$, and by definition $\tilde{Q}^{\alpha^l}$ satisfies Condition POS.

We next show Condition MIX. To that end, let $Z \in \tilde{Q}^{\alpha^l}$, $\tilde{Z} \in \tilde{Q}^{\alpha^l}$ and $\lambda \in (0, 1)$. Then there exists $k, k, \tilde{k} > 0$ such that $\mathbb{E}[l^* (kZ)] < \infty$ and $a_l + \varepsilon < kZ < b_l - \varepsilon$ $\mathbb{P}$-a.s. In particular, since $a_l < kZ < b_l$ $\mathbb{P}$-a.s., it follows that

$$a_l + \varepsilon < k^*(\lambda Z + (1 - \lambda)\tilde{Z}) < b_l - \varepsilon^* \text{ } \mathbb{P}\text{-a.s.},$$

where $k^* := \tilde{k}/(\lambda k + (1 - \lambda)k) > 0$ and $\varepsilon^* := (1 - \lambda)k^*\tilde{\varepsilon} > 0$. Therefore, $\lambda Z + (1 - \lambda)\tilde{Z} \in \tilde{Q}^{\alpha^l}$ and $\tilde{Q}^{\alpha^l}$ satisfies Condition MIX.

We now show Condition INT is satisfied. Let $\tilde{Z} \in \tilde{Q}^{\alpha^l}$, set $\mathcal{E} := D \cap L^\infty_{\Phi_l}$ and let $Z \in \mathcal{E}$. Then there exists $k, \varepsilon > 0$ such that $\text{ess inf } kZ > a_l + \varepsilon$ and $\text{ess sup } kZ < b_l - \varepsilon$. Let

$$\lambda_1 := \begin{cases} \frac{b_l - \varepsilon - k\|Z\|_\infty}{k(1\|\tilde{Z}\|_\infty - \|Z\|_\infty)}, & \text{if } \|Z\|_\infty > \|\tilde{Z}\|_\infty, \\ \frac{1}{2}, & \text{otherwise}, \end{cases}$$

and $\lambda_2 := 1 - (a_l + \varepsilon)/\text{ess inf } kZ$. Then setting $\lambda := \min\{\lambda_1, \lambda_2\}$ yields $\text{ess inf } k(\lambda Z + (1 - \lambda)\tilde{Z}) \geq a_l + \varepsilon$ and $\text{ess sup } k(\lambda Z + (1 - \lambda)\tilde{Z}) \leq b_l - \varepsilon$. Therefore, $\lambda Z + (1 - \lambda)\tilde{Z} \in \tilde{Q}^{\alpha^l}$ and $\tilde{Q}^{\alpha^l}$ satisfies Condition INT.

Proposition C.6. Let $l$ be a loss function such that $a_l > 0$ and $b_l < \infty$. Then the penalty function $\alpha^l(Z) := \inf_{\lambda > 0} \frac{1}{\lambda} \mathbb{E}[l^* (\lambda Z)]$ is convex and $L^1$-lower semi-continuous. Thus, $\tilde{Q}^{\alpha^l} = Q^{\alpha^l}$.
Proof. First note that $\alpha^l$ is the minimal penalty function for $S R^l$ by the proof of [23, Theorem 10], and $S R^l$ is a real-valued convex risk measure on $H^{Q^l}$. Therefore $\alpha^l$ is convex by [16, Theorem 4.3]. To prove $L^1$-lower semi-continuity, consider an arbitrary sequence $(Z_n)_{n \geq 1} \subset Q^{l^*}$ that converges in $L^1$ to $Z$ and assume $\liminf_{n \to \infty} \alpha^l(Z_n) \leq y$. By restricting to a subsequence and relabelling, we may assume without loss of generality $Z_n \to Z$ $P$-a.s. Thus, $Z \in D$ and $\alpha^l(Z)$ is well-defined. To complete the proof, we must show $\alpha^l(Z) \leq y$. Now for any $n \in \mathbb{N}$: since $Z_n \in Q^{l^*}$, $l^*$ is nonnegative and $\text{dom} \ l^* \subset [a_l, b_l]$, that means $\alpha^l(Z_n) = \frac{1}{\lambda^*_n} E[l^*(\lambda^*_n Z_n)]$ for some $\lambda^*_n \in [a_l, b_l]$ (i.e., the infimum is attained – this can be shown by taking a minimising sequence $(\lambda_k)_{k \geq 1} \subset [a_l, b_l]$, restricting to a convergent subsequence and using Fatou’s lemma). By restricting to a further subsequence and relabelling, we may assume that $\lambda^*_n \to \lambda^*$. Then $\frac{1}{\lambda^*_n} l^*(\lambda^*_n Z_n) \to \frac{1}{\lambda^*} l^*(\lambda^* Z)$ $P$-a.s. (since $l^*$ is continuous on its effective domain and $\lambda_n Z_n \in \text{dom} \ l^*$ $P$-a.s.), and so by Fatou’s lemma, which we can apply since $l^*$ is nonnegative:

$$y \geq \liminf_{n \to \infty} \alpha^l(Z_n) = \liminf_{n \to \infty} \frac{1}{\lambda^*_n} E[l^*(\lambda^*_n Z_n)] \geq \frac{1}{\lambda^*} E[l^*(\lambda^* Z)] \geq \alpha^l(Z).$$

\[
\]

Proof of Proposition 5.8. This follows from: [16, Equation (5.23)] and Remark 5.9 in the case that $l(x) > x$ for all $x$ with $|x|$ sufficiently large; and from $\alpha^l(1) = 0$ and $\alpha^l(Z) = \infty$ for all $Z \in D \setminus \{1\}$ in the case that $l$ is equal to the identity either on $R_+$ or $R_-$. 

Proposition C.7. Let $l$ be a loss function and assume that either $a_l > 0$, or $b_l < \infty$ and $a_l < 1 < b_l$. Define $\alpha^l(Z) := E[l^*(Z)]$. Then

$$\hat{Q}^{\alpha^l} = \{Z \in Q^{l^*} : a_l + \varepsilon < Z < b_l - \varepsilon \text{ $P$-a.s. for some } \varepsilon > 0\},$$

is a nonempty subset of $Q^{l^*}$ satisfying Conditions POS, MIX and INT.

Proof. It is clear that $1 \in \hat{Q}^{\alpha^l} \subset Q^{l^*}$, and by definition $\hat{Q}^{\alpha^l}$ satisfies Condition POS. For the rest of the proof, it is useful to recall that $(a_l, b_l) \subset \text{dom} \ l^* \subset [a_l, b_l]$ and $l^*$ is bounded on any compact subset of $(a_l, b_l)$.

To show Condition MIX, let $Z \in Q^{l^*}$, $\tilde{Z} \in \hat{Q}^{\alpha^l}$ and $\lambda \in (0, 1)$. Since $l^*$ is convex, $E[l^*(\lambda Z + (1 - \lambda)\tilde{Z})] < \infty$ so $\lambda Z + (1 - \lambda)\tilde{Z} \in Q^{l^*}$. Furthermore, since $a_l \leq Z \leq b_l$ $P$-a.s. and $a_l + \varepsilon < \tilde{Z} < b_l - \varepsilon$ $P$-a.s. for some $\varepsilon > 0$, it follows that

$$a_l + (1 - \lambda)\varepsilon < \lambda Z + (1 - \lambda)\tilde{Z} < b_l - (1 - \lambda)\varepsilon \text{ $P$-a.s.}$$

Therefore, $\lambda Z + (1 - \lambda)\tilde{Z} \in \hat{Q}^{\alpha^l}$ and $\hat{Q}^{\alpha^l}$ satisfies Condition MIX.

Finally we show Condition INT is satisfied. Assume first that $b_l = \infty$ and let $\tilde{Z} \in \hat{Q}^{\alpha^l}$. Set $E := D \cap L^\infty$ and $Z \in E$. Then there exists $\varepsilon > 0$ such that $\tilde{Z} > a_l + \varepsilon$ $P$-a.s. By choosing $\lambda \in (0, \frac{1}{2}\varepsilon/(a_l + \varepsilon)]$, it follows that

$$\lambda Z + (1 - \lambda)\tilde{Z} \geq (1 - \lambda)(a_l + \varepsilon) \geq a_l + \frac{\varepsilon}{2} \text{ $P$-a.s.}$$

Now since $l^*$ is convex, real-valued on $(a_l, \infty)$ and its minimum is $l^*(1) = 0$, it follows that $l^*$ is nonincreasing on $(a_l, 1)$ and nondecreasing on $(1, \infty)$. Whence, setting $A := \{\lambda Z + (1 - \lambda)\tilde{Z} \leq 1\}$ and $B := \{Z \leq \tilde{Z}\}$, we have

$$E[l^*(\lambda Z + (1 - \lambda)\tilde{Z})] = E[l^*(\lambda Z + (1 - \lambda)\tilde{Z})1_A] + E[l^*(\lambda Z + (1 - \lambda)\tilde{Z})1_{A^c}]$$

$$\leq l^*(a_l + \frac{\varepsilon}{2}) + E[l^*(\lambda Z + (1 - \lambda)\tilde{Z})1_{A^c}] + E[l^*(\lambda Z + (1 - \lambda)\tilde{Z})1_{A \cap B}]$$

$$\leq l^*(a_l + \frac{\varepsilon}{2}) + E[l^*(\tilde{Z})] + l^*(\|Z\|_\infty) < \infty,$$

Therefore, $\lambda Z + (1 - \lambda)\tilde{Z} \in Q^{l^*}$ and $\hat{Q}^{\alpha^l}$ satisfies Condition INT when $b_l = \infty$. Now assume $1 < b_l < \infty$ and let $\tilde{Z} \in \hat{Q}^{\alpha^l}$. Set $E := D \cap L^\infty$ and $Z \in E$. Then there exists $\varepsilon > 0$ such that $a_l + \varepsilon < \tilde{Z} < b_l - \varepsilon$. By choosing $\lambda \in (0, \frac{1}{2}\varepsilon/(a_l + \varepsilon)]$ if $\|Z\|_\infty < b_l$ and choosing $\lambda \in (0, \min\{\frac{1}{2}\varepsilon/(a_l + \varepsilon), \varepsilon/(\frac{1}{2}\varepsilon + \|Z\|_\infty - b_l)\})$ otherwise; it follows that there exists $\varepsilon' > 0$ such that $a_l + \varepsilon' \leq \lambda Z + (1 - \lambda)\tilde{Z} \leq b_l - \varepsilon'$ $P$-a.s. Therefore, $\lambda Z + (1 - \lambda)\tilde{Z} \in Q^{l^*}$ and $\hat{Q}^{\alpha^l}$ satisfies Condition INT when $1 < b_l < \infty$.  \(\square\)
Proof of Corollary 5.11. (a) This follows from Proposition C.7 and Theorem 4.8, noting that Condition I follows from the generalised Hölder inequality; see e.g. [29, Equation (B.1)].

(b) First, note that since \((a_1, b_1) \subset \text{dom} \ l^* \subset [a_i, b_i]\) we have

\[
\{ Z \in \mathcal{D} : a_1 < \text{ess inf} \ Z \text{ and } \text{ess sup} \ Z < b_1 \} \subset Q^\ell \subset \{ Z \in \mathcal{D} : a_1 \leq Z \leq b_1 \} \ P\text{-a.s.}.
\]

Thus, Condition UI holds if \(b_1 < \infty\). Using this, the result follows from Theorem 4.6 if we can show the penalty function \(\mathcal{D} \ni Z \mapsto \alpha^\ell(Z) := E[l^*(Z)]\) is convex and \(L^1\)-lower semi-continuous. Since \(l^*\) is convex, \(\alpha^\ell\) is convex. To show \(L^1\)-lower semi-continuity, it suffices to consider a sequence \((Z_n)_{n \geq 1} \subset Q^\ell\) that converges in \(L^1\) to \(Z\) and show \(\alpha^\ell(Z) \leq \liminf_{n \to \infty} \alpha^\ell(Z_n)\). By restricting to a subsequence and relabelling, we may assume \(Z_n\) converges to \(Z\) \(P\)-a.s., and hence \(Z \in \mathcal{D}\). Moreover, \(l^*(Z_n)\) is a nonnegative sequence that converges to \(l^*(Z)\) \(P\)-a.s., since \(l^*\) is nonnegative, continuous on its effective domain and \(Z_n \in \text{dom} \ l^* \ P\text{-a.s.}\). Applying Fatou’s lemma yields:

\[
\alpha^\ell(Z) = E[l^*(Z)] \leq \liminf_{n \to \infty} E[l^*(Z_n)] = \liminf_{n \to \infty} \alpha^\ell(Z_n). \quad \square
\]

Proposition C.8. Assume \(g : (0, 1) \to [0, \infty]\) is a nonincreasing function with \(\inf g = 0\). If \(g\) is real-valued, then \((A_{\text{VaR}})_{\infty} = L^1_+\). If \(g\) is not real-valued and the probability space is atomless, then \((A_{\text{VaR}})_{\infty} \supseteq L^1_+\).

Proof. Let \(X \in L^1\). By definition, \(\text{VaR}^\beta(\alpha X) \leq 0\) for all \(\alpha \in (0, \infty)\) if and only if \(\text{VaR}^\alpha(\alpha X) \leq g(\alpha)\) for all \(\alpha \in (0, \infty)\) and \(\alpha \in (0, 1)\). By the positive homogeneity of \(\text{VaR}\), this is equivalent to \(\text{VaR}^\alpha(X) \leq 0\) for all \(\alpha \in \text{dom} g\).

If \(g\) is real-valued, this implies that \(0 \geq \lim_{\alpha \to 0} \text{VaR}^\alpha(X) = \text{WC}(X)\) for all \(X \in (A_{\text{VaR}})_{\infty}\). We may conclude that \((A_{\text{VaR}})_{\infty} = L^1_+\).

If \(g\) is not real-valued, then there exists \(\beta \in (0, 1)\) such that \(\inf \text{dom} g = \beta\). It follows that \((A_{\text{VaR}})_{\infty} \supseteq A_{\text{VaR}}^g\) because \(0 \geq \text{VaR}^\beta(X) \geq \text{VaR}^g(X)\) for any \(X \in A_{\text{VaR}}^g\) and \(\alpha \in \text{dom} g\). Since the probability space is atomless, it is straightforward to check that \((A_{\text{VaR}})_{\infty} \supseteq L^1_+\). \quad \square

Proof of Proposition 5.13. The dual representation has been shown in [13, Proposition 3.7]. (5.4) follows directly from the definition of \(\alpha^g\), which also gives convexity of \(Q^\alpha\). \quad \square

Proposition C.9. Let \(\beta \in (0, 1)\) and \(g \in G_\beta \setminus G_\beta^\infty\). Define \(\hat{g} : (0, 1] \to [0, \infty]\) by \(\hat{g}(x) = \text{co } \tilde{g}(1/x)\) where \(\tilde{g} : [1, \infty) \to [0, \infty]\) is given by \(\tilde{g}(x) = g(1/x)\). Then \(\hat{g} \in G_\beta \setminus G_\beta^\infty\) and \(\hat{g}(\beta) = \infty\).

Proof. Since \(g \in G_\beta\), it follows that \(\hat{g}\) is nondecreasing, real-valued on \([1, 1/\beta)\), \(\infty\) on \((1/\beta, \infty)\) and \(\hat{g}(1) = 0\). By the definition of the lower semi-continuous convex hull, it is not difficult to check that \(\text{co } \hat{g}\) has the same properties and so \(\hat{g} \in G_\beta\). It remains to show that \(\text{co } \hat{g}(1/\beta) = \infty\).

Seeking a contradiction, suppose that \(\text{co } \hat{g}(1/\beta) =: k < \infty\). As \(\text{co } \hat{g}\) is a proper lower semi-continuous convex function, the Fenchel-Moreau theorem gives \(\text{co } \hat{g} = \hat{g}^{**}\) where \(\hat{g}^{**}\) is the biconjugate of \(\hat{g}\). Since \(\hat{g}\) is nondecreasing and \(\lim_{x \downarrow 1/\beta} \hat{g}(x) = \infty\), there exists \(c \in [1, 1/\beta)\) such that \(\hat{g}(x) > k + 1\) for all \(x \in (c, 1/\beta)\). Thus, the affine (and continuous) function \(a : [1, \infty) \to \mathbb{R}\) with \(a(c) = 0\) and \(a(1/\beta) = k + 1\) satisfies \(a \leq \hat{g}\) and \(a(1/\beta) > k = g^{**}(1/\beta)\). This is in contradiction to the fact that by (B.3), \(g^{**}\) dominates any affine (and continuous) function dominated by \(\hat{g}\). \quad \square

Proposition C.10. Let \(\beta \in (0, 1)\) and \(g \in G_\beta\). Let \(\text{co } \alpha^g\) be the \(L^1\)-lower semi-continuous convex hull of \(\alpha^g\). Then its effective domain is given by

\[
Q^\alpha = \begin{cases} \{ Z \in \mathcal{D} : ||Z||_{\infty} \leq \frac{1}{2}\}; & \text{if } g \in G_\beta^\infty, \\ \{ Z \in \mathcal{D} : ||Z||_{\infty} < \frac{1}{2}\}; & \text{if } g \in G_\beta \setminus G_\beta^\infty. \end{cases}
\]

Proof. If \(g \in G_\beta^\infty\), then the result follows from Remark 4.5(a). So assume \(g \in G_\beta \setminus G_\beta^\infty\). Define the function \(\hat{g} : [0, 1] \to [0, \infty]\) by \(\hat{g}(x) = \text{co } \tilde{g}(1/x)\) where \(\tilde{g} : [1, \infty) \to [0, \infty]\) is given by \(\tilde{g}(x) = g(1/x)\). By Proposition C.9, \(\hat{g} \in G_\beta \setminus G_\beta^\infty\) and \(\hat{g}(\beta) = \infty\). Moreover, \(\hat{g}(x) \leq \hat{g}(1/x) = g(x)\) for \(x \in (0, 1]\).
Moreover, by the fact that $\varphi \hat{g}$ is convex and lower semi-continuous, nondecreasing, real-valued on $[1, 1/\beta)$ and $\infty$ on $[1/\beta, \infty)$, it follows that $\alpha^\delta : D \to [0, \infty]$, given by

$$
\alpha^\delta(Z) = \begin{cases} 
\varphi \hat{g}(\|Z\|_\infty), & \text{if } Z \in Q^\delta = \{Z \in D : \|Z\|_\infty < 1/\beta\}, \\
\infty, & \text{otherwise},
\end{cases}
$$

is convex and $L^1$-lower semi-continuous. Thus, $\alpha^\delta \geq \varphi \alpha^\delta \geq \varphi \alpha^\delta = \alpha^\delta$, which implies $Q^{\alpha^\delta} \subset Q^{\varphi \alpha^\delta} \subset Q^{\alpha^\delta}$. Since $Q^{\alpha^\delta} = Q^{\alpha^\delta} = \{Z \in D : \|Z\|_\infty < 1/\beta\}$, the result follows. 

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