Cop and Robber with road building ability

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Abstract. We investigate the game of cop and robber, playing on a finite graph between one cop and one robber. The robber can spend one move to build a road to any other vertices on the graph or stay at his position. Several results are provided toward to find characteristics of graphs on which the robber can win.

1. Introduction
The game of Cop and Robber is played on a reflexive graph, that is a graph with loops at every vertex. The cop chooses a vertex to occupy then the Robber chooses one. The two sides then move alternatively, Cop first, where a move is to slide along an edge. The loops are technical devices which allow the cop and the robber to pass. There is perfect information, that is each side knows the location of the other. The Cop wins if he and the Robber are on the same vertex at the same time. Graphs in which the Cop can win are called cop-win graphs and are characterised in [1] and [2]. In [3], a corner ranking procedure was described and it had proved that a graph is cop-win if and only if it has a finite corner rank.

In our research, we consider the situation in which the Robber can build a road to any other vertices or just stay at his position using his move. To avoid confusion, we regard the graph that the winner is the robber as builder-win in such a game. We have made some contribution in characterising builder-win graphs.

We show that a graph is builder-win if for every vertex on the graph, if there exists another vertex that is 3 unit of distance from it. However, the converse is not true. We give an example of a cop-win graph, which has a vertex that its distance from any other vertices on the graph is at most 2, but the graph is also builder-win. In addition, we show that there exist graphs with arbitrarily high corner rank which are not builder-win.

2. Sufficient conditions for builder-win graphs
As the robber has the extra ability to build roads in our game compared to the traditional cop and robber game, any robber-win graph is builder-win. Therefore, we will focus on cop-win graphs in our research. On a cop-win graph, if the robber can find a vertex which is far from the cop, he may be able to have time to build roads before the cop comes and finally win the game. The results are as follows:

2.1. Theorem 2.1. For a graph G, for every vertex v on the graph, if we can find another vertex u with distance at least 3 from v, then the graph is builder-win.

Proof. Consider a graph G satisfying the conditions in the theorem. We assume G is a connected graph as otherwise the graph is clearly builder-win. Let v be the vertex chosen by the cop, the robber can
then choose a vertex \( u \) with distances 3 from the cop. Let \( v_0, u_0 \) be two vertices so that \( v, v_0, u_0, u \) forms a path. We only need to consider the case in which the cop never stays at its own position because the robber can pass as well. Now, the cop moves away from \( v \), the robber can build a road from \( u \) to \( v \). As the distance from \( u \) and \( v \) is 3, \( u \) and \( v \) has no common neighbours and the cop can not catch the robber in the next step.

**Figure 1:** A cycle formed by the four vertices \( u, v, u_0, v_0 \)

Now, we show that the robber can win by staying in the cycle shown on Figure 1. We consider the following strategy played by the robber:

- If the robber’s position is at \( v \) (or \( u \)), stay there unless the cop is at one of the robber’s neighbours. If the cop is next to the robber’s position, but not at \( u \) (or \( v \)), the robber then can move to \( u \) (or \( v \)). If the cop is at \( u \) (or \( v \)), the robber can move to \( v_0 \) (or \( u_0 \)). Since \( u \) and \( v \) has no common neighbours, the cop cannot catch the robber in the next step.
- So, the robber can only be chased away from vertices \( u \) and \( v \) if the cop is on the other vertex. Assume without loss of generality, the cop moves to \( u \), chasing the robber to \( v_0 \). If the cop moves to \( v \), the robber moves to \( u_0 \). Otherwise, if the cop moves to any other vertices, the robber moves to \( v \).
- So, after the movement of robbers, the situations are:
  - The robber is at \( u \) or \( v \), the cop is not at robber’s neighbours.
  - The robber is at \( u_0 \) or \( v_0 \) and the cop is at the diagonally opposite vertex of the robber as shown on Figure 1.

In both cases, the cop can not catch the robber.

One might expect that if, on a cop-win graph, the cop can choose a vertex which has distance at most 2 to any other vertices, then the cop can win. However, we testify that this is not the case through a counter example.

### 2.2. Theorem 2.2

**There exist cop-win graphs, in which the distance between any two vertices is at most 2, are also builder-win.**

**Proof.** We set the following graph with vertices set \((i,j), 1 \leq i \leq 3, 1 \leq j \leq 4\) and there will be an edge between vertices \((i,j)\) and \((k,l)\), if and only if at least one of the following two conditions holds.

- \( i = k = 3 \)
- \( |i - k| \leq 1 \) and \( j \equiv 2 \) (mod 4)

First, we show that this graph is cop-win. We consider the following strategies of cops.

- Place at \((3,1)\) initially.
- If the robber places at \((i_1,j_1)\) as the beginning point, go to \((3,j_1)\).
- Now, the robber goes to \((i_2,j_2)\), if \( i_2 = 1 \), cops either catch the robbers or go to \((2,j_2)\).
- Then, if the robber is not caught yet and goes to \((i_3,j_3)\), the cop can always catch the robber.

Also, we note that every two vertices in the graph have distances at most 2. We now show that the graph can also be builder-win by conceiving the following strategies adopted by the robber.

- Suppose the cop chooses the position \((i_1,j_1)\) initially, the robber then can move to \((1,l_1)\) where \( l_1 = 2 + j_1 \) (mod 4).
- At step \( n \), if the cop’s position is at \((i_n,j_n)\), go to \((1,l_n)\) with \( l_n = 2 + j_n \) (mod 4) whenever possible.
• The robber can always do this unless the cop moves to \((3,j_n)\) at step \(n\) with \(j_n = ln−1\). The robber then builds a road between \((1,ln−1)\) and \((3,j_n−1)\).

Now we show that the robber can win after building the road between \((1,ln−1)\) and \((3,j_n−1)\) while the cop is at \((3,ln−1)\) after both players made their \(n\)’s turn. We can assume without loss of generality that \(ln−1 = 1\). Now, consider the set of vertices \(S=\{(1,1),(2,1),(2,2),(2,4),(3,1),(3,3)\}\). After the robber builds the road between \((1,1)\) and \((3,3)\), we observe that if the robber is at a vertex in \(S\) and the cop moves to a neighbour of the robber, the robber can always move to another vertex in \(S\) which is not the cop’s neighbour. Therefore, the robber can always stay in \(S\) and make sure that the cop cannot catch him.

To conclude, we have constructed a graph which is cop-win, the distance between any two vertices on the graph is at most 2, and the graph is also builder-win.

3. Builder-win graphs and corner ranking

In this section, we will investigate the relationship between builder-win graphs and corner rank developed in [3].

First, we state the relevant definitions about corner rank which are necessary for this paper. For a reflexive graph \(G\), \(V(G)\) refers to the vertices of \(G\) and \(E(G)\) refers to the edges of \(G\). If \(G\) is a graph and \(X\) is a vertex or set of vertices in \(G\), then we let \(G−X\) be the subgraph of \(G\) induced by \(V(G)\setminus X\). We say that a vertex \(u\) dominates a set of vertices \(X\) if \(u\) is adjacent to every vertex in \(X\). Given a vertex \(v\) in a graph, by \(N(v)\), the closed neighbourhood of \(v\), we mean the set of vertices adjacent to \(v\).

As the graph is reflexive, we always have \(v \in N(v)\). For distinct vertices \(v\) and \(w\), if \(N(v) \subseteq N(w)\), then we say that \(v\) is a corner and \(w\) corners \(v\); if \(N(v) \subset N(w)\), we say that \(v\) is a strict corner and \(w\) strictly corners \(v\); if \(N(v) = N(w)\), we call \(v\) and \(w\) twins.

We now define the corner ranking procedure as described in [3].

3.1. Definition 3.1. For any graph \(G\), we define a corresponding corner rank function, \(cr\), which maps each vertex of \(G\) to a positive integer or \(\infty\). We also define a sequence of associated graphs \(G[1],...,G[\alpha]\).

- Initialize \(G[1] = G\), and \(k = 1\).
- If \(G[k]\) is a clique, then:
  - Let \(cr(x) = k\) for all \(x \in G[k]\).
  - Then stop.
- Else if \(G[k]\) is not a clique and has no strict corners, then:
  - Let \(cr(x) = \infty\) for all \(x \in G[k]\).
  - Then stop.
- Else
  - Let \(V\) be the set of strict corners in \(G[k]\).
  - For all \(x \in V\), let \(cr(x) = k\).
  - Let \(G[k+1] = G[k]−V\), increment \(k\) by 1 and return to step 1.

Furthermore, we denote \(cr(G) = \max x \in G cr(x)\).

Lemma 3.2. For a graph \(G\) and integer \(k\), if \(cr(G) > k\), then any vertex \(u \in G\) with \(cr(u) = k\) is strictly cornered by a vertex \(v \in G[k+1]\) in the graph \(G[k]\).

Proof. Let \(S\) be the set of vertices in \(G[k]\) which strictly corners \(u\), as \(cr(u) = k\), \(S\) cannot be empty. For a vertex \(x\), denote \(d(x)\) the number of adjacent vertices of \(x\) in \(G[k]\). Then, there exists a vertex \(v \in S\) such that \(d(v) \geq d(x)\) for all \(x \in S\). We show that this \(v\) is in \(G[k+1]\). Suppose we have a vertex \(w\) which strictly corners \(v\) in \(G[4]\). As \(v\) strictly corners \(u\) in \(G[4]\), \(w\) will also strictly corner \(u\) in \(G[4]\) and
thus $w \in S$. However, we would then have $d(v) \geq d(w)$, which means $w$ cannot strictly corner $v$.

Therefore, we have $v \in G^{[b+1]}$ as desired.

Now, we show some results on the relationship between builder-win graphs and corner rank.

**Theorem 3.3.** Consider a graph $G$, if there exists a vertex $v \in G[2]$ which is adjacent to all other vertices in $G[2]$, then $G$ cannot be a builder-win graph.

**Proof.** The cop can win with the following strategy:

- The cop chooses to be at $v$ at the start.
- If the robber chooses to be in $G[2]$, the cop can catch the robber immediately.
- If the robber chooses to be in a vertex $w$ which is not in $G[2]$, then there must be a vertex $u$ in $G[2]$ which strictly corners $w$. The cop can then move to $u$.
- As $u$ strictly corners $w$, the cop can catch the robber in the next step no matter where the robber goes to.

**Theorem 3.4.** For any positive integer $n$, there exists a graph $G$ with $cr(G) = n$ such that $G$ is not a builder-win graph.

**Proof.** We consider the graph $G$ with vertices $(i,j)$, $1 \leq i \leq 2$, $1 \leq j \leq n$. Vertices $(i,j)$ and $(k,l)$ are adjacent if any of the following conditions hold:

- $i = k$, $|j - l| = 1$,
- $l = j = n$.

We now show that $G$ has corner rank $n$ by showing that $cr(i,j) = j$. We prove this statement using induction. We note that $(1,1)$ is strictly cornered by $(1,3)$ and $(2,1)$ is strictly cornered by $(2,3)$ in $G$. None of the other vertices in $G$ is strictly cornered as they all have the same degree. So, we have $cr((1,1)) = cr((2,1)) = 1$ and $(i,j) \in G[2]$ if and only if $j \geq 2$.

Now, assume for an integer $1 < k < n$, $cr((i,j)) = j$ for all $1 \leq i \leq 2$ and $j < k$. So, $(i,j) \in G[k]$ if and only if $j \geq k$. Now, in $G[k]$, if $k \neq n - 1$, $(1,k)$ will be strictly cornered by $(1,k + 2)$ and $(2,k)$ will be strictly cornered by $(2,k + 2)$, all other vertices will not be strictly cornered as they all have the same degree in $G[k]$. So, we have $cr((1,k)) = cr((2,k)) = k$ and $(i,j) \in G[k+1]$ if and
**Figure 2:** An illustration of the graph when \( n = 5 \). Vertices are adjacent if they are connected by a line segment or if they are on the same bold line. Only if \( j \geq k + 1 \) as desired. If \( k = n - 1 \), then \((1,k)\) and \((2,k)\) will both be strictly cornered by \((1,n)\) and \((2,n)\) and \(G[n]\) will just contain \((1,n)\) and \((2,n)\). This concludes the inductive proof.

Finally, we prove that the cop can win in such graph by adopting the following strategies:

- Choose \((1,n)\) at the start.
- If the robber chooses \((1,l_1)\), the cop can catch him immediately. So, suppose the robber chooses \((2,l_1)\). Choose \((2,j_2)\) with \(j_2 = n\) if \(n \equiv l_1(\mod 2)\) and \(j_2 = n - 1\) otherwise.
- After round \(m\), suppose the robber is at \((k_m,l_m)\) and the cop is at \((i_m,j_m)\), if \(i_m = k_m\), the cop can catch the robber directly, otherwise the cop goes to \((i_m+1,j_m+1)\) with \(i_m+1 = k_m\) and \(j_m+1 = j_m - 1\).

We show that the cop can indeed win with this strategy. To avoid being caught, the robber will have to choose \(k_2 = 1\), \(k_3 = 2\), \(k_4 = 1\) and so on. Therefore, we only need to consider the case when \(k_m = i_{m+1} = 1\) if \(m \equiv 0(\mod 2)\) and \(k_m = i_{m+1} = 2\) otherwise. Moreover, as long as \(j_{m+1} > l_m\), \(l_{m+1} = l_m + 1\) or \(l_m - 1\). As \(j_{m+2} = j_{m+1} - 1\), we would have \(j_{m+2} - l_{m+1} = j_{m+1} - l_m\) or \(j_{m+2} - l_{m+1} = (j_{m+1} - l_m) - 2\). As \(j_2 \equiv l_1(\mod 2)\) and \(j_2 \geq l_1\), eventually, for some \(m\) we will have \(j_{m+1} = l_m\) and thus the robber will be caught. □

4. References

[1] R. Nowakowski, P. Winkler Vertex to vertex pursuit in a graph, Discrete Mathematics 43 (1983), 235-239.
[2] A. Quillot, Thèse d’Etat, Université de Paris VI (1983), 235-239.
[3] D. Offner, K. Ojakian, Cop-Win Graphs: Optimal Strategies and Corner Rank submitted