Scalar runnings and a test of slow roll from CMB distortions

Brian A. Powell
Institute for Defense Analyses, Alexandria, VA 22311, USA
(Dated: September 10, 2012)

A future measurement of cosmic microwave $\mu$-distortions by an experiment with the specifications of PIXIE will provide an equivalent $3\sigma$ detection of the running of the spectral index of scalar perturbations, $\beta = d\alpha/d\ln k$, if $\mu > 7.75 \times 10^{-5}$, covering much of the PIXIE sensitivity range. This corresponds to a resolution limit of $\beta \gtrsim 0.015$ which is relatively large given any presumption of slow roll, a result of the current tight constraints on $\alpha < 0$ on CMB scales. We show that a detection of $\beta$ at this level is in conflict with slow roll conditions if the primordial signal can be distinguished from any post-inflationary contamination.

I. INTRODUCTION

Understanding the character of the primordial density perturbation has been a chief endeavor of modern precision cosmology. Our knowledge of the shape and statistics of the primordial power spectrum has grown substantially over the past two decades, from the Cosmic Background Explorer’s first glimpse of the cosmic microwave background (CMB) anisotropies, to today’s swarm of ground-based and balloon-borne CMB and large scale structure observatories. Our knowledge of the primordial density perturbations derives almost exclusively from these data sources – the temperature and polarization anisotropies of the cosmic microwave background and galaxy correlation measurements – both of which probe large length scales. These observations have revealed a nearly scale invariant spectrum of Gaussian, adiabatic perturbations on comoving scales $k \lesssim 0.1$ Mpc$^{-1}$, in agreement with the simplest models of inflation.

But inflation might not have been so simple. Deviations from Gaussianity and adiabaticity are possible, and the power spectrum might deviate from a power law [1–3]. A spectrum with a nontrivial scale dependence might indicate a complicated inflaton potential exhibiting nonslow roll behavior. A correspondence exists between the scale dependence of the power spectrum, $P(k)$, and the field-dependence of the inflaton potential, $V(\phi)$: specifically, the $n^{th}$-order term of the Taylor expansion of $V(\phi)$ in $\phi$ is linear in the $n^{th}$-order term of the Taylor expansion of $P(k)$ in $\ln k$ [4]. By resolving the scale dependence of $P(k)$, in particular the higher-order runnings of the spectral index: $\alpha = dn_s/d\ln k$, $\beta = d\alpha/d\ln k$, etc., the paradigm of single field, slow roll inflation can be tested.

In order to observe these higher-order runnings, the spectrum must be measured across a wide range of length scales. The Wilkinson Microwave Anisotropy Probe (WMAP) and Sloan’s observations of baryon acoustic oscillations combine to probe scales $10^{-4}$ Mpc$^{-1} \lesssim k \lesssim 0.1$ Mpc$^{-1}$; while these data exhibit a preference for $\alpha < 0$, it is not statistically significant [5]. The Lyman alpha forest, which probes large scale structure on scales $k \approx 1$ Mpc$^{-1}$ [6,8], is promising, but converting flux measurements to a density power spectrum is difficult with our currently incomplete understanding of the properties of the intergalactic medium. Recent ground-based observatories, like the South Pole Telescope [7] and the Atacama Cosmology Telescope [8] have begun to place nearly decisive limits on the running of the spectral index by probing the damping tail of CMB fluctuations: $\alpha = -0.034 \pm 0.018$ [9] and $\alpha = -0.024 \pm 0.0112$ [10], and Planck will do even better. Current data, however, can go no further: in order to resolve higher-order runnings, we need a longer lever arm. By extending the observational window to smaller scales, $P(k)$ can be recovered across a greater range of $\ln k$, and subtle higher-order terms in its Taylor expansion will become relevant. One promising probe of small scale fluctuations is the 21cm transition of neutral hydrogen, which will extend the observational window down to scales $k \approx 100$ Mpc$^{-1}$ [11], and resolve $\alpha$ at the level of $10^{-4}$ [12,13]. We can expect these results in the coming years.

But it might be possible to go even further: the thermal spectrum of the cosmic microwave background radiation traces fluctuations on scales $50$ Mpc$^{-1} \lesssim k \lesssim 10^4$ Mpc$^{-1}$, offering an unprecedented glimpse into the character of the small-scale power spectrum. This is a fundamentally different kind of measurement: rather than probing the fluctuations themselves, the focus is on the dissipation of these fluctuations into the baryon-photon plasma and the resulting deviation of the thermal spectrum of the radiation field from that of a black body. The proposed PIXIE [14] mission will be sensitive to these spectral distortions at a level useful for power spectrum reconstruction, and there has been recent interest in its prospects for improving constraints on $\alpha$ [15,18]. In this paper we investigate the potential of such a measurement for resolving the running of the running of the spectral index, $\beta = d\alpha/d\ln k$.

1 http://pole.uchicago.edu/
2 http://www.physics.princeton.edu/act/

* Email:brian.powell007@gmail.com
II. CMB DISTORTIONS

Fluctuations in the baryon-photon plasma below the Jeans scale oscillate as acoustic waves during radiation domination. The energy of these fluctuations is dissipated via diffusion damping, resulting in a redistribution of this energy from the fluctuations into the baryon-photon plasma [19,26]. At early times (z ≳ 2 × 10^6), efficient double Compton emission and bremsstrahlung, and the up-scattering of these photons by energetic electrons quickly thermalizes the baryon-photon plasma, preserving its black body spectrum. As the universe cools $-$ between redshifts $5 \times 10^4 \lesssim z \lesssim 2 \times 10^6$ $-$ Compton scattering ensures statistical equilibrium of the radiation field, but the production of photons is insufficient to maintain a black body: a frequency-dependent chemical potential, $\mu(\omega)$, develops and the spectrum shifts to that of a Bose-Einstein distribution. So, while small-scale perturbations are erased before recombination, they leave behind a record of their existence: the energy lost by the perturbations is injected into the CMB distorting the spectrum away from a black body by an amount that depends on the shape and amplitude of the power spectrum on the relevant scales.

The time evolution of the $\mu$-distortion amplitude obeys [24]

$$\frac{d\mu}{dt} \approx -\frac{\mu}{t_{DC}} + 1.4 \frac{Q}{\rho_\gamma},$$  

(1)

where $t_{DC}$ is the double Compton thermalization time, and $Q/\rho_\gamma$ is the fractional rate of energy dissipation, and double Compton emission has been assumed the dominant source of photon production [25,27]. This has the solution

$$\mu \approx 1.4 \int_{z_\mu}^{\infty} \exp \left( -\frac{(z/z_{DC})^{5/2}}{\rho_\gamma} \right) \frac{dQ}{dz} dz,$$  

(2)

where $z_\mu = 5 \times 10^4$ and $z_{DC} = 1.98 \times 10^6$. The energy density of the acoustic waves is $Q = 3\rho_s c_s^2 \langle \delta_s(x)^2 \rangle / 4$, where $c_s^2 \approx 1/3$ is the squared sound speed, and the photon density perturbation, $\delta_\gamma(x)$, has the ensemble average,

$$\langle \delta_\gamma(x)^2 \rangle = \int \frac{d^3k}{(2\pi)^3} P(k),$$  

(3)

where $P(k) = \Delta^2 P^D(k)$, $\Delta^2$ the transfer function, and $P^D(k)$ is the primordial spectrum generated by inflation. Since we are considering acoustic waves below the Jeans scale during radiation domination, the transfer function is appropriate to modes that are well inside the horizon,

$$\Delta \approx 3 \cos(k r_s) e^{-k^2/k_D^2},$$  

(4)

where $r_s$ is the sound horizon and the diffusion scale, $k_D$, varies with redshift during radiation domination as

$$k_D = A_D^{-1/2}(1 + z)^{3/2},$$  

(5)

where $A_D = 8 (135 H_0^2 \sqrt{T_\gamma n_\gamma \sigma T})^{-1} = 5.92 \times 10^{10}$ Mpc$^{-1}$. Here $n_\gamma$ is the free electron number density at $z = 0$ and $\sigma_T$ is the Thompson scattering cross section. The portion of the power spectrum, $P^D(k)$, that is primarily responsible for generating the $\mu$-distortion is set by the diffusion scales at the beginning and end of the epoch, $5 \times 10^4 \lesssim z \lesssim 2 \times 10^6$. Equation (5) gives $k \in [50,10^5]$ Mpc$^{-1}$.

With these results, the amount of $\mu$-distortion Eq. (2) can be well-approximated by [18]

$$\mu \approx 2.2 \int_{k_{min}}^{\infty} P^D(k) \left[ \exp \left( -\frac{k}{5400} \right) - \exp \left( -\frac{k^2}{998.6} \right) \right] dk,$$  

(6)

where $\cos^2(k r_s)$ has been replaced by $1/2$ $-$ its average value over one oscillation. In contrast to the anisotropy measurements for which CMB data on scale $\ell$ constrains the spectrum at corresponding wavenumber, $k$, the $\mu$-distortion measurement comes in the form of a weighted integral constraint on $P^D(k)$ over the $\mu$-distortion epoch.

III. OBSERVING $\beta$

The question of whether an observation detects a physical quantity, $\theta$, is a problem in hypothesis testing: are the data sufficient to overturn the null hypothesis, in which $\theta = 0$? In Bayesian statistics, hypotheses are interpreted as models, and hypothesis testing proceeds via a comparison of the Bayesian evidence of the first model, $p(d|M_0)$, with that of the second, $p(d|M_1)$,

$$B_{01} = \frac{p(d|M_0)}{p(d|M_1)},$$  

(7)

where $d$ is the data and $M$ the model. The Bayesian evidence is the average of the likelihood function, $L(\theta|d,M)$, weighted by the prior probability of the parameters, $\theta$,

$$p(d|M) = \int L(\theta|d,M) \pi(\theta|M) d\theta.$$  

(8)

Once the evidence of each model is computed, the Bayes factor Eq. (7) can be evaluated and inferences made based on the Jeffrey's scale [28]: $\ln B_{01} < 1$ means that the data is insufficient to distinguish the models, while $1 < \ln B_{01} < 2.5$, $2.5 < \ln B_{01} < 5$, and $\ln B_{01} \geq 5$ signify positive, moderate, and strong evidence, respectively, in favor of $M_1$. In what follows, we will quote results that satisfy the threshold for strong evidence in favor of $M_1$, $\ln B_{01} \geq 5$, corresponding to a 0.993 posterior probability for the favored model [29].

We are interested in the prospects of detecting and constraining higher-order runnings of the scalar power spectrum, specifically, the running of the running of the spectral index, $\beta = d\alpha/d\ln k$, with future observations with the specifications of the PIXIE experiment [14], a
The Savage-Dickey density ratio is a convenient route to our objective is to determine how large the $\mu$-distortion signal of the quality expected from PIXIE. The former data set includes the latest measurements from the Wilkinson Microwave Anisotropy Probe (WMAP) [3] and the South Pole Telescope (SPT) [10]. These data are well-described by the standard inflationary cosmology with base parameters: the baryon and cold dark matter densities, $\Omega_b h^2$ and $\Omega_c h^2$, the ratio of the sound horizon to the angular diameter distance at decoupling, $\theta_s$, the optical depth to reionization, $\tau$, and a power law spectrum of density perturbations parameterized by amplitude, $A_s$, and spectral tilt, $n_s$. The SPT data requires three additional foreground parameters: the Poisson point source power from randomly distributed galaxies, $D_f^{\text{CL}}$, the clustered point source power, $D_f^{\text{CL}}$, and the secondary emission from the Sunyaev-Zeldovich effect from clusters, $A_{\text{SZ}}$. The model likelihood calculated from these parameters, which we denote $L_{\text{SPT}}$, to indicate that it refers to the WMAP and SPT $C_\ell$ anisotropy spectra, is described in [5] [10]. The second data source, that arising from $\mu$-distortions, is notional. For a given fiducial amount of $\mu$-distortion, $\mu_0$, the likelihood of the model with a theoretical value of $\mu$ given by Eq. [9] is computed assuming a Gaussian posterior for $\mu$, 

$$
L_\mu \propto p(\mu)/\pi(\mu), \quad \propto \exp \left[-\frac{(\mu - \mu_0)^2}{2\sigma_\mu^2}\right], \quad (11)
$$

where the prior, $\pi(\mu)$, is taken to be flat within the range $\mu \in [0, 10^{-4}]$, and $\sigma_\mu = 10^{-5}$ [31]. Then, for a particular set of parameter values, the full likelihood of the model can be determined by simply summing the log-likelihoods of the two data sources,

$$
\ln L_{\text{tot}} = \ln L_{\text{SPT}} + \ln L_\mu. \quad (12)
$$

To summarize, we are considering an overall likelihood that derives from two separate data sources, CMB anisotropies and $\mu$-distortions, that probe different cosmological scales.

Since the real CMB anisotropy data are fixed, our task is to vary $\mu_0$ across the prior range until we achieve $|\ln B_{01}| > 5$. This process sets up an interesting tension between the two data sources. On the one hand, we have CMB temperature and polarization anisotropy data that is well-fit by the model $M_0$ with $\beta = 0$. In fact, the simpler model with $\alpha = 0$ is still marginally preferred by current data [32], and is perfectly consistent with an amount of $\mu$-distortion below the PIXIE detection threshold of $\sim 10^{-8}$. However, as the fiducial amount of $\mu$-distortion is increased, even the model with running, $M_0$, struggles to provide an acceptable fit. This is illustrated in Figure 1: as $\mu_0$ is increased, $\alpha$ must get larger to accommodate the increase in small scale power needed to generate the fiducial amount of $\mu$-distortion. But $M_0$ can only provide increased small scale power at the cost of simultaneously increasing large scale power, and in an effort to fit ever larger values of $\mu_0$, the model is pulled out of agreement with large scale CMB data.
FIG. 2. Marginalized constraints for $n_s$, $\alpha$, and $\beta$ for WMAP+SPT (black) and WMAP+SPT+PIXIE (blue) with $\mu_0 = 7.75 \times 10^{-8}$.

This effect signals the need for an additional parameter, $\beta = d\alpha/d\ln k$, that gives a spectrum with increased small scale power – to accommodate the large $\mu$-distortion – without strongly modifying the shape of the large scale power spectrum that is well-constrained by current CMB data.

In order to calculate $B_{01}$ for a particular value of $\mu_0$, the posterior, $p(\beta, d|\mathcal{M}_1)$, and prior, $\pi(\beta, \mathcal{M}_1)$, distributions must be determined. The prior encodes our current knowledge of $\beta$, and is taken equivalent to the posterior distribution of $\beta$ obtained from WMAP+SPT data: it is close to Gaussian with mean $\bar{\beta} = 0.017$ and standard deviation $\sigma_\beta = 0.024$. The posterior distribution is obtained using Markov chain Monte Carlo (MCMC) over the parameter space of model $\mathcal{M}_1$: \{\$O_bh^2, \Omega_c h^2, \theta_s, \tau, A_s, n_s, \alpha, \beta, r, D_{l}^{PS}, D_{l}^{CL}, Asz\}. For each sample, Eq. [6] is numerically integrated with $P_i(k)$ given by Eq. [9] to obtain the $\mu$-distortion and the overall likelihood, $L_{\text{tot}}$, determined from Eq. [12]. This process is iterative: if the $B_{01}$ thus calculated for the chosen fiducial value of $\mu_0$ does not satisfy $|\ln B_{01}| \approx 5$, a new fiducial $\mu_0$ is selected and another MCMC run is begun.

In Table 1 we present the results of our analysis for different values of $\mu_0$ in the range $\mu_0 \in [10^{-8}, 10^{-7}]$. For each value of $\mu_0$, six chains were run using the parameterization of model $\mathcal{M}_1$, and chain convergence was considered achieved when the Gelman-Rubin statistic satisfied $R - 1 < 0.05$. The CosmoMC\(^3\) software package \cite{33} was used for sampling and chain analysis, with modified likelihood routines to calculate $L_{\text{tot}}$. We find that PIXIE

| $|\ln B_{01}$ | $\mu_0 \times 10^{-8}$ |
|-------------|---------------------|
| 1           | 6                   |
| 3.5         | 7.5                 |
| 5           | 7.75                |
| 6.5         | 8                   |

\(^3\) http://cosmologist.info/cosmomc.
will detect $\beta \neq 0$ if $\mu_0 \geq 7.75 \times 10^{-8}$, covering much of the PIXIE sensitivity range ($\mu \gtrsim 10^{-8}$). The corresponding resolution limit of the running of running is relatively large: $\beta \gtrsim 0.015$. This is principally a result of the good constraints on $\alpha < 0$ on CMB scales: $\beta$ is forced to be big to yield a large enough $\mu$-distortion to be detected by an instrument like PIXIE.

In addition to detecting $\beta \neq 0$, PIXIE will place relatively good constraints on its value: $\beta = 0.015^{+0.02}_{-0.01}$ at $2\sigma$, for $\mu_0 = 7.75 \times 10^{-8}$. Current constraints on the spectral parameters for WMAP+SPT are compared with projected constraints when PIXIE is included at this level in Figure 2. It is interesting to note that while $\mu$-distortion data greatly improves constraints on $\beta$, there is little change in the posterior distributions of either $n_s$ or $\alpha$ compared to current results. This is because both $n_s$ and $\alpha$ are already well-constrained by WMAP+SPT: the existence of sufficient small scale power to generate a detectable level of $\mu$-distortion establishes the need for $\beta$ and constrains it without affecting the tight limits on $n_s$ and $\alpha$ on CMB scales.

IV. IMPLICATIONS FOR SLOW ROLL INFLATION

The minimum amount of running of running that can be detected with statistical significance (strong evidence, or 0.993 posterior odds) is relatively large: $\beta \gtrsim 0.015$, possibly in conflict with slow roll inflation, provided that the observed $\mu$-distortion is primordial. The slow roll approximation holds whenever the acceleration of the scalar field, $\ddot{\phi}$, can be neglected relative to the cosmological drag, $\dot{\phi} \ll 3H\dot{\phi}$, with the result that

$$3H\dot{\phi} \approx -V',$$  

(13)

where dots denote time derivatives and primes denote derivatives with respect to $\phi$. This is the case as long as the scalar field is potential-dominated: $\phi^2/2 \ll V(\phi)$. Slow roll can be maintained as long as the potential is sufficiently flat; this condition is often stated in terms of the smallness of the parameters,

$$\epsilon \equiv 2M_P^2 \left( \frac{H'}{H} \right)^2 \approx \frac{M_P^2}{2} \left( \frac{V'}{V} \right)^2 \ll 1$$

$$\eta \equiv 2M_P^2 \frac{H''}{H} \approx \frac{M_P^2}{2} \frac{V''}{V} \ll \mathcal{O}(1)$$

where the dots denote terms involving higher derivatives of $H(\phi)$. But how many terms must we consider? How small do they need to be? The slow roll approximation can be formally derived by considering the Taylor expansion of the Hubble parameter [34],

$$\frac{H(\phi)}{H_0} = 1 + \sum_{\ell=0} \frac{\ell\lambda_H}{2(\ell+1)!B_1^{\ell+1}} \left( \frac{\phi}{M_{Pl}} \right)^{\ell+1},$$  

(14)

where $B_1 = \sqrt{\epsilon_0/2}$ and the subscript “0” indicates that the coefficients should be evaluated when the pivot scale, $k_0$, exits the horizon, and we have chosen $\phi_0 = 0$. The coefficients,

$$\ell\lambda_H \equiv (2M_{Pl})^\ell \frac{(H')^{\ell-1} d^{(\ell+1)} H}{H^\ell d\phi^{(\ell+1)}}, \; \ell \geq 0$$  

(15)

generate the hierarchy of slow roll parameters derived from $H(\phi)$: $\lambda_1^2 H = \eta$, $\lambda_2^2 H = \xi^2$, etc. In this representation, a cosmological solution given by Eq. (14) truncated at some order $M$ is interpreted as a perturbation expansion about de Sitter space.

The parameters $\epsilon$ and the $\lambda_H$ determine approximate expressions for the spectral parameters: $n_s$, $\alpha$, $\beta$, $\cdots$, that are themselves coefficients of the Taylor expansion of $\ln P(k)$ in $\ln k$. The lowest order spectral parameter, $n_s$, is lowest order in slow roll, $n_s(\epsilon, \eta, \cdots)$, where ... includes all higher-order terms $\epsilon^2, \epsilon\eta, \eta^2$, and so on. The second order spectral parameter, $\alpha$, is likewise second order in slow roll, $\alpha(\xi^2, \epsilon^2, \epsilon\eta, \cdots)$. The slow roll approximation can be considered valid if the $n^\text{th}$-order spectral parameter is dominated by the $n^{\text{th}}$-order slow roll parameters, with higher-order terms comprising ever smaller corrections. It is therefore a prediction of slow roll inflation that higher-order spectral parameters should be vanishingly small, and that a general solution should be described by only the few lowest-order terms.

We begin by writing the spectral observables as follows

$$n_s = 1 - 4\epsilon + 2\eta - 2\xi^2 - 8(C + 1)e^2 + (6 + 10C)e\eta$$

$$\alpha = -2\xi^2 - 8e^2 + 10e\eta - (8 + 14C)e\xi^2$$

$$+ 2C\eta\xi^2 + 2C^3\lambda_H$$

$$\beta = -14\epsilon\xi^2 + 2\eta\xi^2 - 2C\xi^4 + 2\xi^2\lambda_H - 2C^4\lambda_H$$

$$- 32e^3 + 62e\xi^2 - 20e\eta^2 - (46 + 56C)e\xi^2$$

$$+ (26 + 48C)e\eta\xi^2 - 2C\eta^2\xi^2 + (10 + 20C)e^3\lambda_H$$

$$- 6C\eta^3\lambda_H,$$  

(16)

where $C \simeq -0.73$. The subscript “0” has been omitted for brevity; it should be understood that the above expressions are to be evaluated at the pivot scale, $k_0$, and associated field value, $\phi_0$. We have written the observables to next-to-leading order in the slow roll parameters because we would like to study the contributions of these higher-order terms, especially for $\beta$. We first set $\lambda_H = 0$, truncating the hierarchy at order $M = 3$. We then seek a cosmological solution, parameterized by $\{\epsilon(\phi), \eta(\phi), \xi(\phi), \lambda_H(\phi)\}$ that yields observables in agreement with current data in combination with a discovery of $\mu$-distortion by PIXIE with $\mu_0 = 7.75 \times 10^{-8}$ (cf. Figure 2.)

In addition to satisfying parameter constraints, the solution must yield sufficient inflation to solve the horizon and flatness problems: $N \gtrsim 30$ e-folds if inflation is to occur above the electroweak scale, and $N \approx 55 (60)$ for GUT (SUSY) scale inflation. The number of e-folds of
expansion is given by
\[
\frac{dN}{d\phi} = \frac{1}{2M_{Pl}^2} \frac{H}{H'},
\]
where \(H(\phi)\) is determined by the slow roll parameters up to order \(M = 3\) via Eq. 14. For a given set of parameters \(\{\epsilon_0, \eta_0, \xi_0, 3\lambda_{H0}\}\) Eq. 17 can be integrated numerically to obtain the number of e-folds of expansion that occur after the scale \(k_0\) leaves the horizon. We stochastically sampled the \(\epsilon - \eta - \xi^2 - 3\lambda_H\) space, and integrated Eq. 17 for each set of initial conditions. We collected solutions with \(r < 0.3\), \(n_s \in [0.94, 1.01]\), and \(\alpha \in [-0.08, 0]\), ranges that approximate the 95% confidence limits obtained on the parameters in the previous section. No constraints were placed on \(N\) or \(\beta\) so that the full space could be explored. As we show in Figure 3 (a), not only do the models fail to provide enough inflation they all give \(\beta \lesssim 0.005\) – just outside the 2\(\sigma\) limit. This result shows that if \(\beta\) is controlled only by leading-order terms in the slow roll expansion, it is not possible to find a working single field inflation model that fits observations.

What about the next-to-leading order term, \(4\lambda_H\)? If we truncate the slow roll hierarchy at order \(M = 4\), our analysis shows that not only is sufficient inflation obtained, but \(\beta\) can now be large (Figure 3 (b)). One such model is given by the parameter values: \(\{\epsilon_0, \eta_0, \xi_0, 3\lambda_H, 4\lambda_H\} = \{0.015, -0.01, 0.035, 0.018, 0.0025\}\). Not only are the first four parameters of equal magnitude, but the fifth parameter, \(4\lambda_H\), while only amounting to a small correction according to the slow roll approximation, strongly affects the inflationary dynamics and allows for \(3\lambda_H\) to become large. Our analysis therefore shows that for \(\beta \gtrsim 0.005\), the slow roll expansion must be taken to at least fourth order to obtain viable solutions, many of which are described by slow roll parameters of roughly equal magnitude and relevance. Given this observation, there is little reason to expect that higher-order terms can be safely neglected. We conclude that a statistically significant detection of running of running by PIXIE, \(\beta \gtrsim 0.015\), is in conflict with single field, slow roll inflation.

V. CONCLUSIONS

The discovery of higher-order runnings of the scalar power spectrum has important implications for the nature of the inflationary era. Large higher-order runnings threaten the validity of the slow roll approximation, and disfavor an explanation in terms of single field, slow roll inflation. Current data from WMAP+SPT have almost conclusively detected \(\alpha \neq 0\), and Planck will furnish a detection if the value of \(\alpha\) is close to current best-fit estimate, \(\alpha \approx -0.024\). Meanwhile, the running of running, \(\beta = d\ln \alpha / d\ln k\), is not well constrained because today’s observatories do not probe the primordial spectrum, \(P(k)\), across a sufficiently large range of length scales to resolve it. In this work, we investigated whether

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4 This is equivalent to solving the system of flow equations truncated at order \(M = 3\) [38, 39].

5 A similar investigation was carried out in [40] regarding the implications for slow roll inflation of \(\alpha\) near the WMAP3 centroid, \(\alpha \approx -0.05\).
a future measurement of CMB $\mu$-distortions by an experiment like PIXIE, taken in combination with CMB anisotropy data, would provide a sufficiently long lever arm to resolve these higher-order runnings.

We framed the detectability of $\beta$ as a problem in Bayesian model comparison between two power spectra: one with $\beta$ and one without. We computed the Bayes factor using MCMC for different fiducial values of $\mu_0$ in order to determine how large the notional $\mu$-distortion signal needs to be to favor the running of running model. We find that if $\mu_0 > 7.75 \times 10^{-5}$, the inclusion of $\beta$ in the model is required by the data with strong significance (equivalent to 0.993 posterior odds.), covering much of the PIXIE sensitivity range. The best-fit value associated with detection is $\beta \approx 0.015$, which is relatively large given any presumption of slow roll. This is a result of the tight constraints on $\alpha$ on CMB scales: with $\alpha < 0$, $\beta$ must be moderately large and positive in order for the spectrum to produce enough $\mu$-distortion on small scales to be detectable by PIXIE. Constraints on $\beta$ are also improved by around a factor of 5 over current estimates.

We next studied the implications that large $\beta$ has for single field slow roll inflation under the assumption that the primordial distortion is distinct from post-inflationary sources. We performed a flow analysis, focusing particularly on the next-to-leading order terms in the slow roll expansion and their role in the inflationary dynamics and observables. It was found that models taken to leading order ($M = 3$) failed to yield sufficient inflation or satisfy constraints on $\beta$, while a truncation at next-to-leading order ($M = 4$) gave models with very different inflationary trajectories that succeeded in meeting these criteria. The inclusion of $4\lambda_H$, while amounting to only a small correction according to the slow roll approximation, strongly alters the inflationary dynamics. Successful $M = 4$ models had slow roll parameters of roughly equal magnitude and importance, challenging the slow roll prescription that higher-order terms in the expansion should indicate ever smaller corrections to the inflationary observables and trajectories. We conclude that a detection of running of running by PIXIE will be in conflict with single field, slow roll inflation.

The search for CMB $\mu$-distortions is a promising avenue for challenging the prevailing conception of inflation. It is important that work continues on understanding the possible foregrounds that might contaminate any $\mu$-distortion signal, as well as identifying possible post-inflationary sources of $\mu$-distortions that might mimic a primordial signal. If a measurement of primordial $\mu$-distortions by PIXIE results in a detection of $\beta$, then this will be a strong indication that inflation either exhibits non-slow roll dynamics or is driven by multiple fields.

ACKNOWLEDGMENTS

The author would like to thank Will Kinney, Jens Chluba, and Enrico Pajer for comments, and Simon DeDeo and Damien Easson for helpful discussions.

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