Non Standard Extended Noncommutativity of Coordinates

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Abstract

We present in this short note an idea about a possible extension of the standard noncommutative algebra of coordinates to the formal differential operators framework. In this sense, we develop an analysis and derive an extended noncommutative structure given by $[x_a, x_b]_* = i(\theta + \chi)_{ab}$ where $\theta_{ab}$ is the standard noncommutativity parameter and $\chi_{ab}(x) \equiv \chi^\mu_{ab}(x)\partial_\mu = \frac{1}{2}(x_a \theta^\mu b - x_b \theta^\mu a)\partial_\mu$ is an antisymmetric non-constant vector-field shown to play the role of the extended deformation parameter. This idea was motivated by the importance of noncommutative geometry framework, with nonconstant deformation parameter, in the current subject of string theory and D-brane physics.

Keywords: Star product, differential operators, Noncommutative algebra, string theory.

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1 Introduction

Recently there has been a revival interest in the noncommutativity of coordinates in string theory and D-brane physics[1-6]. This interest is known to concern also noncommutative quantum mechanics and noncommutative field theories [7, 8]. Before going into presenting the aim of our work, we will try in what follows to expose some of the results actually known in literature.

The sharing property between all the above interesting areas of research is that the corresponding space exhibits the following structure

\[ [x_i, x_j]_* = i\theta_{ij} \]  \hfill (1)

where \( x_i \) are non-commuting coordinates which can describe also the space-time coordinates operators and \( \theta_{ij} \) is a constant antisymmetric tensor. Quantum field theories living on this space are necessarily noncommutative field theories. Their formulation is simply obtained when the algebra (1) is realized in the space of fields (functions) by means of the Moyal bracket according to which the usual product of functions is replaced by the star-product as follows [9]

\[ (f \ast g)(x) = f(x)e^{\frac{i}{2}\theta^{ab}\delta_a\delta_b} g(x), \]  \hfill (2)

The link with string theory consist on the correspondence between the \( \theta^{ij} \)-constant parameter and the constant antisymmetric two-form potential \( B_{ij} \) on the brane as follows [1].

\[ \theta^{ij} = (\frac{1}{B})^{ij}, \]  \hfill (3)

such that in the presence of this \( B \)-field, the end points of an open string become noncommutative on the D-brane.

In this letter, we try to go beyond the standard noncommutative algebra (1) by presenting some computations leading to consider among other a non-constant antisymmetric \( \hat{\theta} \)-parameter satisfying an extended noncommutative Heisenberg-type algebra given by

\[ [x_i, x_j]_* = i\hat{\theta}_{ij}(x), \]  \hfill (4)

where \( \hat{\theta}_{ij}(x) = (\theta + \chi)_{ij} \) and where \( \chi_{ij}(x) = \chi(x) \partial_\mu = \frac{1}{2}(x^{[a}\theta^{\mu b]} x^{b]} - x^{[a}\theta^{\mu a]} x^{b]}) \partial_\mu \) describes a non-constant vector-field deformation parameter. This is important since the obtained algebra (4) can be reduced to the standard noncommutative algebra (1) once one forget about the operatorial part of \( \hat{\theta} \) namely \( \chi_{ij} = \chi_{ij}^{\mu} \partial_\mu \).

This construction is also interesting as it may help to build a correspondence between the noncommutative geometry framework, based on the algebra (4), and the string theory with a non-constant B-field.
2 Non standard noncommutative algebra.

Consider the noncommutative space defined by the relation (1) originated from the star product definition of two functions \( f \) and \( g \) of an algebra \( \mathcal{A} \) that is given by

\[
(f \ast' g)(x) = f(x)e^{i \partial_a \partial_b ^{\delta a} g(x)},
\]

where \( \partial_a = \frac{\partial}{\partial x^a} \). We denote the star-product in (5) by a prime for some reasons that we will explain later. With this star product, one can define the Moyal bracket as follows

\[
[f(x), g(x)]_\ast' = f(x) \ast' g(x) - g(x) \ast' f(x).
\]

For functions \( f \) and \( g \) coinciding with the coordinates \( x_i \) and \( x_j \), we recover in a simple way (1). Actually, our idea starts from the observation that the derivatives \( \partial_a = \frac{\partial}{\partial x^a} \) in the exponential (2) are differential operators which act in the following way:

\[
\partial_a : \mathcal{A} \to \mathcal{A},
\]

such that the prime derivative is given by

\[
\partial_a f = f'_a \\
\partial_a (fg) = f'_a g + fg'_a \\
\partial_a x_i = \delta_{ai}.
\]

Furthermore, for two given functions \( f \) and \( g \) of the algebra \( \mathcal{A} \), the term \( f \ast' g \) remains an element of \( \mathcal{A} \). So, the prime introduced in the definition of the \( \ast' \)-product (5) is just to express the prime character of the derivative \( \partial_a \) as shown in (7-8).

Looking for a possible generalization of the above analysis to the formal differential operators framework, we shall now introduce another kind of star-product, denoted by \( \ast \) and associated to an operatorial action of the derivative \( \partial_a \). Before going into describing how does it works, let us first introduce the set \( \Sigma^{(p,q)} \), \( p \geq 0 \) [10]. This is the algebra of local differential operators of arbitrary spins and positive degrees. The upper indices \( (p,q) \) carried by \( \Sigma \) are the lowest and the highest degrees. A particular example is given by \( \Sigma^{(0,0)} \) which is nothing but the algebra \( \mathcal{A} \), the structure usually used in the standard \( \ast \)-product computations. Furthermore, in terms of the spin quantum number \( \Delta = s \), the space \( \Sigma^{(p,q)} \) is given by

\[
\Sigma^{(p,q)} = \bigoplus_{s \in \mathbb{N}} \Sigma_s^{(p,q)}.
\]

Typical elements of (9) are given for \( (p,q) = (0,k) \) by

\[
\Sigma_s^{(0,k)} = \sum_{m=0}^{k} \chi_{s-m}(x) D^m = \sum_{m=0}^{k} \chi_{s-m}^{\mu_1...\mu_m}(x) \partial_{\mu_1}...\partial_{\mu_m},
\]
Next, we assume that the derivative \( \partial_a \) acts on the function \( f \) as an operator in the following way
\[
\partial_a f(x) = f'_a(x) + f(x) \partial_a,
\]
a fact which means that our derivative should be defined as
\[
\partial_a : \Sigma^{(0,i)} \to \Sigma^{(0,i+1)}.
\]
This way to define the derivative is induced from the extended \( \ast \)-product operation defined as follows
\[
(f \ast g)(x) = f(x)g(x) + \frac{i}{2} \theta^{ij}(f'_i + f \partial_i)(g'_j + g \partial_j) + \ldots
\]
In section 3 we summarize some non trivial relation satisfied by the product \( \ast \). The major difference between the two star products \( \ast \) and \( \ast \) is that, for a given function \( f \) of the algebra \( A \), the term \( \partial_i f \) belongs on the first case to the algebra \( A \) while on the second case it is an element of the space \( \Sigma^{(0,1)} \). In general \( (\partial_1 \ldots \partial_n f) \) is an element of \( \Sigma^{(0,n)} \) which is a particular set of the space of local differential operators denoted by \( O(A) \) and which we can realize as
\[
O(A) = \bigoplus_{0 \leq p \leq q} \Sigma^{(p,q)}(A),
\]
Now, consider the \( \ast \)-product definition for the coordinates \( x_a \) and \( x_b \), we obtain by using the above analysis
\[
x_a \ast x_b = \sum_{\alpha=0}^{\infty} (x_a \ast x_b)_\alpha
= x_a x_b + \sum_{\alpha=1}^{\infty} \frac{1}{\alpha!(\frac{i}{2})^\alpha} \prod_{\alpha} \theta^{\mu_i \nu_i} \prod_{\alpha} \partial_{\mu_i} x_a (\prod_{k=1}^{\alpha} \partial_{\nu_k} x_b)
\]
and explicitly we have,
\[
(x_a \ast x_b)_\alpha = \frac{1}{\alpha!(\frac{i}{2})^\alpha} \left\{ x_a x_b \prod_{\mu} \left( \prod_{\alpha} (\theta^{\mu_i \nu_i} \partial_{\mu_i} \partial_{\nu_i}) \right) + \sum_{\alpha} \left\{ \theta_{ab} \prod_{k \neq i} (\theta^{\mu_k \nu_k} \partial_{\mu_k} \partial_{\nu_k}) + x_a \theta^{\mu_i \nu_i} \prod_{j \neq i} (\theta^{\mu_j \nu_j} \partial_{\mu_j}) \prod_{k} \partial_{\nu_k} + (x_a \theta^{\nu_i \mu_i} + x_b \theta^{\nu_i a}) \prod_{j \neq i} (\theta^{\mu_j \nu_j} \partial_{\mu_j}) \prod_{k} \partial_{\nu_k} + \sum_{j \neq i} \theta^{\nu_i \mu_i} \theta^{\mu_j \nu_j} \prod_{(i \neq j) k} \partial_{\nu_k} (\prod_{m \neq i, k \neq j} \theta^{\mu_k \nu_k} (\prod_{m} \partial_{\mu_m} \prod_{(i \neq j)}) \right\} \right\}. \tag{16}
\]
This result is obtained by using the derived recurrence formula (30).
Furthermore, using the antisymmetry property of \( \theta \), we can easily check that (16) can be more simplified. Concrete examples are given by the first term \( x_a x_b \prod_{i=1}^{\alpha} (\theta^{\mu_i \nu_i} \partial_{\mu_i} \partial_{\nu_i}) \) which vanishes. Also \( \theta_{ab} \prod_{k \neq i} (\theta^{\mu_k \nu_k} \partial_{\mu_k} \partial_{\nu_k}) \), \( x_a \theta^{\mu_i \nu_i} \prod_{j \neq i} (\theta^{\mu_j \nu_j} \partial_{\mu_j}) \prod_{k} \partial_{\nu_k} \).
as well as \((x_a \theta^{\mu_a} b + x_b \theta^{\alpha_a}) \prod_{j \neq i} (\theta^{\mu_j} \nu_j \partial \nu_j) \prod_{k} \partial \nu_k\) are terms which contribute only for the value \(\alpha = 1\).

On the other hand, performing straightforward but lengthy computations, we find the following noncommutative extended \(\star\)-algebra

\[
[x_a, x_b]_\star = i \hat{\theta}_{ab}(x),
\]

where the only non-vanishing term among a long mathematical series is given by

\[
\hat{\theta}_{ab} = \theta_{ab} + \frac{1}{2} (x_a \theta^{\mu} b - x_b \theta^{\mu} a) \partial_\mu.
\]

Later on, we will denote the vector-field appearing on the rhs of (18) simply by

\[
\chi_{ab} \equiv \chi_{ab}(x) \partial_\mu = \frac{1}{2} (x_a \theta^{\mu} b - x_b \theta^{\mu} a) \partial_\mu.
\]

In this way, \(\chi\) is interpreted as a deformation parameter term such that the algebra (17) becomes

\[
[x_a, x_b]_\star = i (\theta + \chi)_{ab},
\]

Note by the way that the long series we obtained for the \(\hat{\theta}\) -parameter, before simplifying to (18), is given by

\[
\hat{\theta}_{ab}(x) = \sum_{\alpha=1}^{\infty} \hat{\theta}_{ab}^{\alpha},
\]

with

\[
\hat{\theta}_{ab}^{\alpha} = \frac{1}{\alpha!} \frac{i^{\alpha-1}}{2^{\alpha}} \sum_{i=1}^{\alpha} \{2 \theta_{ab} \prod_{k \neq i} (\theta^{\mu_k} \nu_k \partial \nu_k) + (x_a \theta^{\mu_i} b - x_b \theta^{\mu_i} a) \prod_{j \neq i} (\theta^{\mu_j} \nu_j \partial \nu_j) \prod_{k=1}^{\alpha} \partial \nu_k \}.
\]

3 Some Useful Formulas

1 let \(c\) and \(c'\) be constant numbers, we have

\[
c \star c' = c.c'
\]

2 For each function \(f(x)\) on the algebra \(\Sigma^{(0,0)} \equiv \mathcal{A}\), we can show by using explicit computations that

\[
f(x) \star c = f.c + \sum_{\alpha=1}^{\infty} \frac{\xi(\frac{1}{2})^\alpha}{\alpha!} \prod_{i=1}^{\alpha} \theta^{\mu_i} \nu_i f_{\mu_1 \ldots \mu_{\alpha}}(x) \prod_{j=1}^{\alpha} \partial \nu_j
\]

where for example \(f_{a_1}^{(1)}\) is the prime derivative with respect to \(\partial_{a_1}\)

3 We have also

\[
c \star f(x) = c.f(x)
\]
4 Combining (24-25) we find for the particular case $f = x$

$$[x_\mu, c]_* = \frac{i}{2} c \theta_\mu \partial_\nu$$

(26)

5 Obviously

$$\partial_a \ast \partial_b = \partial_a \partial_b$$

(27)

6 Also we have

$$\partial_a \ast f(x) = \partial_a f(x) = f'_a(x) + f \partial_a$$

(28)

7 The general formula

$$\prod^n_{j=1} \partial_{\nu_j} \ast f(x) = \prod^n_{j=1} \partial_{\nu_j} f(x)$$

(29)

8 Applying to the coordinates $x_a$

$$\partial_{\mu_1} \ldots \partial_{\mu_n} x_a = x_a \partial_{\mu_1} \ldots \partial_{\mu_n} + \sum_{i=1}^n \delta_{a\mu_i} \partial_{\mu_1} \ldots \partial_{\mu_i} \ldots \partial_{\mu_n},$$

(30)

or equivalently

$$\left( \prod^n_{i=1} \partial_{\mu_i} \right) x_a = x_a \left( \prod^n_{i=1} \partial_{\mu_i} \right) + \sum_{i=1}^n \delta_{a\mu_i} \left( \prod^n_{j \neq i} \partial_{\mu_j} \right).$$

(31)

4 Concluding Remarks

Following this construction, some remarks are in order:

{1} A first important remark concerning the obtained algebra (17), is that it does not closes as a standard algebra. This property is easily observed since the extended Moyal bracket of $x_a$ and $x_b$; which are coordinates elements of $\Sigma_{-1}^{(0,0)} = A_{-1}^{(0,0)}$, gives $\hat{\theta}_{ab}(x) = \theta_{ab} + \frac{1}{2}(x_a \theta_{\mu b} - x_b \theta_{\mu a}) \partial_\mu$ which is an element of $\Sigma_{-2}^{(0,1)}$. However, if we forget about the vector field term $\chi_{ab}^{\mu} \partial_\mu$ in $\hat{\theta}_{ab}(x)$, we recover the standard noncommutative structure (1) which is a closed algebra. We can conclude for this point that the fact to transit from prime star product $\ast$ to the operatorial one $\ast$, is equivalent to introduce local vector fields contributions at the level of the deformation parameter $\hat{\theta}_{ab}$ which therefore becomes coordinates dependent.

{2} Related to {1}, we can also check that the associativity with respect to the operatorial $\ast$-product operation is not satisfied. As an example consider

$$(f \ast 1) \ast 1 = D(x) \ast 1,$$

$$f \ast (1 \ast 1) = D(x)$$

(32)

where the differential operator $D(x)$ is just the result of $f \ast 1$. Then we can easily check that $D(x) \neq D(x) \ast 1$ as shown in the formulas (24-25). This property of non associativity of the operatorial star product exhibits a particular interest. In
fact it makes us recall the non associative algebra based on results about open
string correlation functions proposed in [4a] and which deal with D-branes in a
background with non-vanishing H.

\{3\} Concerning the mentioned properties \{1-2\}, the problem of closure of the
derived algebra (17) can be approached by using the analogy with the non-linear
Zamolodchikov $W_3$-algebra which exhibits a similar property. Namely the non-
closure of the algebra due to the presence of the spin-4 term in the commutation
relations of $W_3$ currents. For a review see [11].

\{4\} The noncommutative extended parameter $\hat{\theta}_{ab} = (\chi + \theta)_{ab}$ is not a constant
object contrary to $\theta_{ab}$ and thus the associated algebra (17) is not a trivial structure
as it corresponds to a noncommutative deformation of the standard algebra (1)
by the vector fields $\chi_{ab}$.

\{5\} Using the derived relation (26), we can easily show that the non constant
deformation parameter $\chi_{ab}$ is given by $\chi_{ab} \equiv i\{x_b.(x_a \star 1) - x_a.(x_b \star 1)\}$.

\{6\} $\theta_{ab}$ as well as the antisymmetric tensor $\chi_{ab}(x)$ are objects of conformal weights
$\Delta = -2$, since $\Delta(\partial_{\mu}) = -\Delta(x) = 1$.

\{7\} From the mathematical point of view, $\hat{\theta}_{ab}^{(\alpha)}$ given in (21-22) are general objects
which belong to the subspaces

$$\Sigma^{(2\alpha-2,2\alpha-1)}_{-2},$$

and $\hat{\theta}_{ab}$ given in (18) is nothing but the first contribution for $\alpha = 1$ and conse-
quently is an object of $\Sigma^{(0,1)}_{-2}$.

\{8\} We easily obtain the standard noncommutative algebra (1) from (17) just
by considering the following quotient space

$$\Sigma^{(0,1)}_{-2} / \Sigma^{(1,1)}_{-2}$$

which consist simply on forgetting about vector fields $\chi^\mu \partial_\mu$. 
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