ERROR BOUNDS AND STABILITY IN THE $\ell_0$ REGULARIZED FOR CT RECONSTRUCTION FROM SMALL PROJECTIONS

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Abstract. Due to the restriction of the scanning environment and the energy of X-ray, few projections of an object can be obtained in some practical applications of computed tomography (CT). In these situations, the projection data are incomplete and inconsistent, and the conventional analytic algorithm such as filtered backprojection (FBP) algorithm will not work. The streak artifacts can be significantly reduced in few-view reconstruction if the total variation minimization (TVM) based CT reconstruction algorithm is used. However, in the premise of preserving the resolution of image, it will not effectively suppress slope artifacts and metal artifacts when dealing with some few-view of the limited-angle reconstruction problems. To solve this problem, we focus on the image reconstruction algorithm based on $\ell_0$ regularized of wavelet coefficients. In this paper, the error bound between the reference or desire image and the reconstructed result, and the stability of solution were shown in theoretical and experimental, a reconstruction experiment on metal laths from few-view of the limited-angle projections was given. The experimental results indicate that this algorithm outperforms classical CT reconstruction algorithms in preserving the resolution of reconstructed image and suppressing the metal artifacts.

1. Introduction. Computed tomography (CT) has been extensively applied in nondestructive testing (NDT), clinical and archeological, etc.. The fundamental problem of CT is to reconstruct an object from its projections obtained by measuring the attenuation of X-rays passed through the object, which is referred to as an inverse problem. The filtered backprojection (FBP) algorithm (see e.g. Chap.7 in [11] and Chap.3 in [53]) can reconstruct images accurately when the projections obtained are adequate, complete and consistent. However, it is likely that projection data are inconsistent in some practical applications. For example, in the industrial field, the radiation is most absorbed due to the bigger thickness of the materials, and then the reconstructed image will presents some metal artifacts. And again in the medical field, the metal artifacts are also prominent in the reconstructed image due

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to the metals which are introduced into the human body, such as dental fillings or hip prostheses [11]. It is also likely that projection data are incomplete in some practical applications. For instance, in the industrial field, we use the straight-line trajectory CT [27, 23] to detect the defect of long object and use CT to detect the pipeline in service. The projections of the object are obtained from different projection views within a limited scanning range, which is smaller than 180° plus a fan angle, due to the restriction of the scanning environment [40, 45, 3]. Meanwhile in the medical field, the dental CT [13, 38], the C-arm CT [4], and the imaging in the breast and chest [48], in these situations, the projections of object are also obtained from different projection views within a limited scanning range. Dose reduction by reducing the number of views at which data are acquired is considered due to the radiation of X-ray which is extremely harmful to human health [45, 44]. In these situations, image reconstruction by conventional analytic algorithms such as FBP algorithm will lead to artifacts and distortion nearby edges [49]. The artifacts and distortion nearby edges are presented in image will make the defect detection and the disease diagnosis more difficult. Therefore, how to reconstruct images relatively accurately has been a hot research topic when the incomplete and inconsistent projection data is utilized.

One known issue in CT reconstruction is the appearance of metal artifacts which caused by the inconsistent data in Radon or projection space. In [11], inconsistent projection data can also be restored with surrogate data created by interpolation or the inconsistent projection data parts can be ignored.

Iterative algorithms as the optimization approaches have been widely researched in image reconstruction because the superiority in controlling noise and dealing with incomplete data.

1.1. Iterative reconstruction algorithm. The simultaneous algebraic reconstruction technique (SART) were used to research CT image reconstruction by some researchers [2, 37, 34, 15, 35]. The SART algorithm [2] can be formulated as

\[
\begin{align*}
    f_j^{n+1} &= f_j^n + \omega \frac{1}{M} \sum_{i=1}^{M} \sum_{j=1}^{N} a_{i,j} (g_i - A_i f^n),
    \quad j = 1, 2, \ldots, N.
\end{align*}
\]

where \( f_j^n \) denotes the \( j \)th component of the image \( f \) to be reconstructed in \( n \)th step, \( \omega \) is a relaxation factor, \( A_i \) is the \( i \)th row of CT system matrix \( A \), \( a_{i,j} \) is the \( i \)th row and the \( j \)th column element of \( A \), \( g_i \) is the \( i \)th component of the projection data \( g \), \( N \) denotes the total number of image size, and \( M \) denotes the total number of ray. The SART algorithm can suppress the noise if the projection data is complete, and it need not store the system matrix \( A \) which can deal with the large-scale problems. However, this method will leads to obvious artifacts and noise in reconstructed images when projection data is incomplete.

A general procedure to preserve the edge of image and suppress the noise is to additive the priori knowledge about the solution [18, 44, 26, 55, 42]. In 2006, the sparsity of the image gradient magnitudes is considered as a priori knowledge, then, an adapted TV minimization algorithm (TVM) was proposed for few-view and limited-angle CT reconstruction problem. Let \( f(i,j) \) be the image pixel at position \((i,j)\) , then, the \( l_1 \) norm of the magnitude of image gradient is defined as...
The TVM algorithm [45] can be formulated as
\[
\min \| f \|_{TV} \quad \text{such that} \quad Af = g, f \geq 0
\]
The TVM based algorithm can reconstruct an image with better quality from few-view sampling over 360° and suppress streak artifacts. However, the reconstructed image suffer from slope artifacts in the object edges when the scanning projection views is limited and smaller than 180° (such as 100° or 130°) [54]. To further improve the quality of reconstructed images for limited-angle CT reconstruction problem, the classical TV based CT reconstruction algorithm has been improved by some researchers [39, 16], however, the object edge information is distorted partly.

In recent years, the theory of the wavelet frames have been well developed [29, 41, 19, 20]. The wavelet frames have been used in image reconstruction [55, 56, 17, 36, 24, 52, 25] and image restoration [56, 12, 22, 46]. The analysis based method in [56] with ℓ₀ quasi-norm can be formulated as
\[
f^* \in \arg \min_{f \in y} \left\{ \frac{1}{2} \| Af - g \|_2^2 + \sum \lambda_i \| (Wf)_i \|_0 \right\}
\]
where \( y \) is a convex subset of \( \mathbb{R}^n \), \( \lambda_i \) denotes the weight sparsity parameters, \( W \) denotes the multi-level wavelet tight frame transform (i.e., \( W^T W = I \)), \( W^T \) denotes the wavelet inverse transform, \( D \) denotes the weighted matrix, and \( \| (Wf)_i \|_0 \) is the number of nonzero terms \( \sharp \{ i | (Wf)_i \neq 0 \} \). The coefficients of an L-level framelet decomposition of \( f \) are denoted
\[
\alpha = \{ W_{l,k} f : 0 \leq l \leq L - 1, k \in I \}
\]
where \( I \) is the index set of all framelet bands. Analysis based method with ℓ₀ quasi-norm can improve image quality in terms of sharp features as well as smoothness. In [56], the authors only give an experiment about image reconstruction from few-view projection and adopt the penalty decomposition (PD) algorithm to solve (3).

We introduce the ℓ₀ minimization of wavelet coefficients that proposed for the limited-angle CT reconstruction problem in our previous work, the model can be formulated as
\[
\arg \min_{f \in \Omega} \left\{ \frac{1}{2} \| Af - g \|_2^2 + \lambda \| Wf \|_0 + \frac{\gamma}{2} \| f \|_2^2 \right\}
\]
where \( \Omega \) denotes the convex subspace of ℓ₂, \( \lambda \) is a scalar parameter, \( \gamma \) denotes a smooth parameter, and \( \| Wf \|_0 \) denotes the total number of nonzero terms \( \sharp \{ i | (Wf)_i \neq 0 \} \). The \( \frac{\gamma}{2} \| f \|_2^2 \) term make the object function become a coercive function (i.e., the object function \( \rightarrow \infty \), as \( \| f \| \rightarrow \infty \), for each \( \gamma > 0 \)), then the solution of object function exists by the Section 3.4 (Existence of Optimal Solutions) of [1], the \( \frac{\gamma}{2} \| f \|_2^2 \) term is differentiable that make the reconstructed image to be smooth, and the \( \frac{\gamma}{2} \| f \|_2^2 \) term will make the energy of reconstructed image to be minimized.

In our previous work, a alternating direction method of multipliers (ADMM)-like algorithm was proposed to solve this model, and the SART algorithm is incorporated into the ADMM-like algorithm because it need not store the system matrix \( A \) which can deal with the large-scale problems. We have proved that the sequence, which generated by the ADMM-like, exists a subsequence convergent to a local minimizer of the object function (4). The difference between the ADMM-like algorithm and the traditional ADMM algorithm are as follows: 1) the traditional
ADMM algorithm is utilized to solve the convex and smooth optimization problem, however, the ADMM-like algorithm is utilized to solve the non-convex and non-smooth optimization problem; 2) the traditional ADMM algorithm for convex and smooth optimization problem can obtain the global optimization solution, however, the ADMM-like algorithm can just obtain the local optimization solution; 3) the traditional ADMM algorithm not include the SART algorithm, the iterative hard thresholding (IHT) technology and the projection contraction method (PCM) algorithm, however, the SART algorithm, the IHT technology and the PCM algorithm be incorporated into the ADMM-like algorithm.

1.2. Motivations and contribution. In our previous work, the convergence analysis of the ADMM-like algorithm is our major contributions, however, the error bounds and stability analysis of the ADMM-like algorithm are not given. In [18], the authors analyse the error bound of linear inverse problem by singular value decomposition (SVD). Motived by this idea, we will give the bound of the error between the reference or desire image and the reconstructed result, and the stability analysis of the solution of model under some conditions in this paper.

In some nondestructive testing (NDT) practical applications, the projection data are incomplete and inconsistent. For instance, there are three difficult problems in our metal laths reconstruction experiments. First, some X-rays can not pass through the metal laths due to the energy of X-ray and the thickness of metal laths. Second, the projection data is incomplete due to the scanning angular which is limited. Third, the resolution of reconstructed image will decline if we use fewer projection data. In this paper, we will consider the small projections that means the few-view of the limited-angle projection data is used, we will use the ADMM-like algorithm to deal with the metal laths problems.

In [11], within the missing data approach is applied to eliminate the inconsistence projection data. We will not use the projection data whose X-rays can not pass through the metal laths so as to reduce the metal artifacts.

Contributions. The main contributions of this work as follows:

1. We analyse the error bound between the reference or desire image and the reconstructed image using the ADMM-like algorithm and the SVD method.
2. We analyse the stability of the solution of model in theoretical and experimental under some conditions.
3. To better suppress the slope artifacts and the metal artifacts in the few-view of the limited-angle CT reconstruction problems, and preserve the resolution of reconstructed image, we adopt the wavelet frame based model, which was proposed in our previous work, penalizes the \( \ell_0 \) minimization of the wavelet coefficients. We give a reconstruction experiment on metal laths, and the experimental results indicate that the ADMM-like algorithm outperforms classical reconstruction algorithms.

1.3. Outline. The rest parts of this work are organized in the following. In section 2, we introduce the model and the algorithm to deal with the few-view of the limited-angle CT reconstruction problem that proposed in our previous work. In section 3, we analyse the error bound between the reference or desire image and the reconstructed result, and the stability of the solution of model under some conditions. In section 4, we analyse the stability by simulation experiment and presented the numerical experimental results of the few-view of the limited-angle CT reconstruction problem on metal laths. In section 5, we give the conclusions.
2. Model and algorithm. In this section, we will introduce the model of the limited-angle CT reconstruction and the alternating direction method of multipliers (ADMM)-like algorithm for solving this model that proposed in our previous work. For solving the storage of the system $A$, the SART algorithm is incorporated to the ADMM-like algorithm and the projection contraction method (PCM) algorithm is utilized to solve the constraint subproblem. For the $\ell_0$ quasi-norm, a hard thresholding (HT) technology is utilized. And the ADMM-like algorithm is utilized to solve the few-view of the limited-angle CT reconstruction problem.

2.1. Model. In this paper, we assume that $f$ and $g$ belong to a subspace $\Omega$ of $\mathbb{R}^n$, we choose $\Omega = \{f \mid f \geq 0, f \in \mathbb{R}^n\}$. $A : \Omega \to \Omega$ is a bounded linear operator, and $W : \Omega \to \mathbb{R}^n$ is a multi-level wavelet tight frame transform operator. The CT reconstruction problems can be formulated as

$$ Af + e = g $$

where $e$ denotes some noise and physical factors in the projection data that should be consider in the practical application. The formula (4) can be rewritten as

$$ \arg\min_{f, \alpha} \left\{ \frac{1}{2} \| Af - g \|_2^2 + \lambda \| \alpha \|_0 + \frac{\gamma}{2} \| f \|_2^2 \right\} \text{ s.t. } \alpha = Wf, \ f \geq 0 $$

where $\lambda$ is a scalar parameter, $\gamma \to 0$ is a scalar parameter, and $\| Wf \|_0$ is the total number of nonzero terms $\sharp \{ i | (Wf)_i \neq 0 \}$.

2.2. Algorithm. Let $f_{i,j}$ be the discretized image fixed on a rectangular grid with the size $m \times n$, $i = 1, ..., m$, $j = 1, ..., n$, and $g_{i,j}$ be the discretized projection of projection space, $i = 1, ..., q$, $j = 1, ..., k$, where $q$ denotes the number of detector bin and $k$ denotes the number of projection views. We rearrange the $f_{i,j}$, $g_{i,j}$ into one dimensional vector by appending the rows of the matrix to each other, which is implemented from the topmost. We choose piecewise constant B-spline framelet and the associated filters [56] are

$$ h_0 = \frac{1}{4} \begin{bmatrix} 1, 2, 1 \end{bmatrix}, \quad h_1 = \frac{\sqrt{2}}{4} \begin{bmatrix} 1, 0, -1 \end{bmatrix}, \quad h_2 = \frac{1}{4} \begin{bmatrix} -1, 2, -1 \end{bmatrix} $$

Furthermore, we define the basis functions as the formula (7) of [2] and the element of $A$ is the intersection length of an X-ray with a pixel.

The hard thresholding (HT) [18] operator with the threshold $\lambda \in \mathbb{R}^+$ is defined as

$$ H_\lambda(x) = \begin{cases} 0 & \text{if } |x| < \lambda \\ \{0, x\} & \text{if } |x| = \lambda \\ x & \text{if } |x| > \lambda \end{cases} $$

The key to the ADMM-like algorithm is de-coupling the $\ell_2$ and $\ell_0$ parts of the function (6). A splitting technique [12], which can de-couple the $\ell_2$ and $\ell_0$ parts of the object function, is utilized to separates (6) into two subproblems. The three steps should be performed are

$$ \text{step 1} : \arg\min_{f \in \Omega} \left\{ \frac{1}{2} \| Af - g \|_2^2 + \frac{t}{2} \| Wf - \alpha + v \|_2^2 + \frac{\gamma}{2} \| f \|_2^2 \right\} $$

$$ \text{step 2} : \arg\min_\alpha \{ \lambda \| \alpha \|_0 + \frac{t}{2} \| Wf - \alpha + v \|_2^2 \} $$

$$ \text{step 3} : \quad v^{n+1} = v^n - (\alpha^{n+1} - Wf^{n+1}) $$
To solve the step 1, an algorithm with system matrix $A$ that needs to be stored in memory will not work because the system $A$ is too large to store in memory, although (8) has a closed form solution. To deal with the system matrix $A$, the $\theta(f) := \frac{1}{2}\|Af - g\|_2^2$ proximal linearized at a given point $f^n$ is $\theta(f^n) + (f - f^n, \theta'(f^n)) + \frac{\tau}{2}\|f - f^n\|_2^2,$ where $\tau$ is a parameter, then, the step 1 can be written

$$f^{n+1} \in \arg \min_{f \geq 0} \{ \langle f - f^n, \theta'(f^n) \rangle + \frac{\tau}{2}\|f - f^n\|_2^2 + \frac{t}{2}\|Wf - \alpha + v\|_2^2 + \frac{\gamma}{2}\|f\|_2^2 \}.$$  \hspace{1cm} (11)

The (11) can be written

$$f^{n+1} \in \arg \min_{f \geq 0} \{ \frac{\tau}{2}\|f - f^n\|_2^2 + \frac{1}{\tau}\theta'(f^n)\|f\|_2^2 + \frac{t}{2}\|Wf - \alpha + v\|_2^2 + \frac{\gamma}{2}\|f\|_2^2 \}. \hspace{1cm} (12)$$

We divide the $\tau$ in the objective function

$$f^{n+1} \in \arg \min_{f \geq 0} \{ \frac{1}{2}\|f - f^n\|_2^2 + \frac{1}{\tau}\theta'(f^n)\|f\|_2^2 + \frac{\tilde{t}}{2}\|Wf - \alpha + v\|_2^2 + \frac{\tilde{\gamma}}{2}\|f\|_2^2 \}, \hspace{1cm} (13)$$

where $\tilde{t} = \frac{\tau}{\omega}$ and $\tilde{\gamma} = \frac{\gamma}{\omega}$. Thus, in this step, we only tune two parameters.

Compare with the SART algorithm, we denote $\tilde{f}^n = f^n - \frac{1}{\omega}A'(f^n)$ and find that the SART algorithm is similar with this formula if we let $\tau$ be the normalization of $\frac{1}{\omega}A'A$. Thus, the SART algorithm in (11) or (13) can be used in this step. Another benefit is that the system matrix $A$ need not be stored in memory, and only have to calculate it when required. Then, we solve (13) using the projection contraction method (PCM) algorithm [30, 31] because this algorithm is a fast and convergent algorithm for convex optimization with constraint. We summarize the step 1 as follows

1. Obtain a proximal point using the SART algorithm

$$\tilde{f}_j^n = f_j^n - \omega \frac{1}{M} \sum_{i=1}^{M} \sum_{j=1}^{N} a_{i,j} (A_i f^n - g_i), j = 1, 2, ..., N. \hspace{1cm} (14)$$

2. Optimize the solution of (8) using the projection contraction method (PCM) [30, 31].

$$f^{n+1} = PCM(\tilde{f}^n) \hspace{1cm} (15)$$

To solve step 2, we use the iterative hard thresholding (IHT) [56] to obtain the closed form solution. We denote $\lambda = \sqrt{\frac{2\tau}{\lambda}}$. The minimizer of (9) is

$$\alpha^{n+1} = H_{\lambda}(Wf^{n+1} + v^n) \hspace{1cm} (16)$$

Now, the process of the ADMM-like algorithm are summarized in the form of a pseudo-code. $N_{ite}$ denotes the max iterative times, $\epsilon_1$ is the relative error criterion of image $f^{n+1}$ and $f^n$, $\gamma_{n+1} = 0.9 \times \gamma_n$ will make $\gamma \to 0$ as $n \to +\infty$. Implementation steps of the ADMM-like algorithm are shown as follows:

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Implementation steps of the ADMM-like algorithm

**Step 1. initialization:**
Given $\varepsilon_1$, $\lambda > 0$, $t > 0$, $\gamma = 1$, $f^1 = 0$, $v^1 = \alpha^1 = Wf^1$, $\epsilon = 1$, $\omega = 1$, $n = 1$.
while $(\epsilon > \varepsilon_1$ and $n < N_{\text{ite}})$

**Step 2. SART-PCM updating:**
(1) SART updating as (14);
(2) PCM updating as (15):

**Step 3. IHT updating as (16):**

**Step 4. Lagrangian Multiplier updating as (10):**

\[ \epsilon = \frac{\|f^{n+1} - f^n\|_2}{\|f^{n+1}\|_2}, \gamma_{n+1} = 0.9 * \gamma_n \]
end while

output $f^{n+1}$

We denote $L(f) = \frac{1}{2} \| f - f^n + \frac{1}{t} \theta'(f^n) \|_2^2 + \frac{1}{2} \| Wf - \alpha + v \|_2^2 + \frac{1}{2} \| f \|_W^2$, the PCM [30, 31] updating step of the ADMM-like algorithm are shown as follows

Implementation steps of PCM algorithm

**Step 1.** Set $\beta_0 = 1$, $\nu \in (0, 1)$, $f^0 \in \Omega$ and $k = 0$
**Step 2.** $\tilde{f}^k = P_{\Omega}[f^k - \beta_k \nabla L(f^k)]$
\[ r_k = \frac{\beta_k \| \nabla L(f^k) - \nabla L(\tilde{f}^k) \|_2}{\| f^k - \tilde{f}^k \|_2}, \]
while $r_k > \nu$, $\beta_k := \frac{2}{3} \beta_k \cdot \min \{1, \frac{1}{r_k} \}$
\[ f^k := P_{\Omega}[f^{k-1} - \beta_k \nabla L(f^k)], \]
end(while)
\[ d(f^k, \tilde{f}^k) = (f^k - \tilde{f}^k) - \beta_k \nabla L(f^k) - \nabla L(\tilde{f}^k), \]
\[ \alpha_k = \frac{\| d(f^k, \tilde{f}^k) \|_2^2}{\| d(f^k, \tilde{f}^k) \|_2}, \]
\[ f^{k+1} = f^k - \gamma \alpha_k d(f^k, \tilde{f}^k) \]
If $r_k < 0.4$ then $\beta_k := 1.5 \beta_k$, end(if)
**Step 3.** $\beta_{k+1} = \beta_k$ and $k = k + 1$, go to Step 2.

3. Error bound and stability analysis. In this section, we discuss the error bound between the reference or desire image and the reconstructed result, and the stability of the solution of model under some conditions. Let $\{(f^n, \alpha^n, v^n)\}$ be the sequence generated by the (15), (16) and (10), and $(f^*, \alpha^*, v^*)$ is the stop iterate point and $f^*$ is a local minimizer of (4) that have been proved in our previous work. We need to recall a lemma.

**Lemma 3.1.** (The corollary of [32]) Suppose $\Omega \subset \mathbb{R}^n$ and $\text{int} \Omega \neq \emptyset$, $G : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is differentiable in $f^* \in \text{int} \Omega$. If $f^*$ is a local minimizer of $G$, we have $\nabla G(f^*) = 0$. Furthermore, if $G$ is a convex function and $\Omega$ is a convex set, $\nabla G(f^*) = 0$ if and only if $f^*$ is a global minimizer of $G$. 
Theorem 3.2. Let \( \hat{f} \) be a reference image, the sequence \( \{(f^n, \alpha^n, v^n)\} \) generated by the (15), (16) and (10), \((f^*, \alpha^*, v^*)\) is the stop iterate point using the ADMM-like algorithm, and \( \epsilon \) denotes some noise and physical factors. We can estimate the error bound between the reference image and the reconstructed image as follows:

\[
T(t, \gamma) - R(t, \gamma) - \frac{\|\epsilon\|^2}{2\sqrt{t}} \leq \|\hat{f} - f^*\|_2 \leq T(t, \gamma) + R(t, \gamma) + \frac{\|\epsilon\|^2}{2\sqrt{t}},
\]

where \( T(t, \gamma) = \|\sum_{k=1}^{K} t(\hat{f} - W^T(\alpha^* - v^*), p_k) p_k \|_2 \), \( R(t, \gamma) = \|\sum_{k=1}^{K} \gamma(\hat{f}, p_k)p_k \|_2 \), \( K \) is the number of non-zero singular values and \( \delta_k^2 \) is the non-zero eigenvalues of \( AA^T \) and \( A^T A \).

Proof. We know that the sequence \( \{(f^n, \alpha^n, v^n)\} \) generated by the (15), (16) and (10), then we obtain the explicit solution of \( f^* \) by the optimization condition of (8). By Lemma 3.1, we can obtain that

\[
f^* = (A^T A + (t + \gamma)I)^{-1}(A^T g + tW^T(\alpha^* - v^*)).
\]

By singular value decomposition (SVD), we have that

\[
Af = \sum_{k=1}^{K} \delta_k (f, p_k) u_k
\]

where \( K \) is the number of non-zero singular values, \( \{u_k\} \), \( \{p_k\} \) are the orthonormal bases of eigenvectors of \( AA^T \) and \( A^T A \) with corresponding non-zero eigenvalues \( \delta_k^2 \), then (17) can be rewritten as

\[
f^* = \sum_{k=1}^{K} \frac{1}{\delta_k^2 + t + \gamma}(\delta_k (g, u_k)p_k + t(W^T(\alpha^* - v^*), p_k)p_k)
\]

Next, we analyse the discrepancy between the reconstructed result \( f^* \) and the reference image \( \hat{f} \),

\[
\|\hat{f} - f^*\|_2 = \left\| \sum_{k=1}^{K} \langle \hat{f}, p_k \rangle p_k - \sum_{k=1}^{K} \delta_k (g, u_k)p_k + t(W^T(\alpha^* - v^*), p_k)p_k \right\|_2
\]

First, we prove the right side of the inequality of Theorem 3.2. Plugging (5) into the above equation (20), we have that

\[
\|\hat{f} - f^*\|_2 \leq \left\| \sum_{k=1}^{K} \langle \hat{f}, p_k \rangle p_k - \sum_{k=1}^{K} \delta_k (A\hat{f} + e, u_k)p_k + t(W^T(\alpha^* - v^*), p_k)p_k \right\|_2 + \sum_{k=1}^{K} \frac{\delta_k (e, u_k)p_k}{\delta_k^2 + t + \gamma} \|_2
\]

Plugging (18) into the above inequality, we have that

\[
\|\hat{f} - f^*\|_2 \leq \left\| \sum_{k=1}^{K} \frac{(t + \gamma)(\hat{f}, p_k)p_k - t(W^T(\alpha^* - v^*), p_k)p_k}{\delta_k^2 + t + \gamma} \right\|_2 + \sum_{k=1}^{K} \frac{\delta_k (e, u_k)p_k}{\delta_k^2 + t + \gamma} \|_2
\]

\[
\leq \left\| \sum_{k=1}^{K} \frac{t\hat{f} - W^T(\alpha^* - v^*), p_k)p_k}{\delta_k^2 + t + \gamma} \right\|_2 + \sum_{k=1}^{K} \frac{\delta_k (e, u_k)p_k}{\delta_k^2 + t + \gamma} \|_2 + \sum_{k=1}^{K} \frac{\gamma\hat{f}, p_k)p_k}{\delta_k^2 + t + \gamma} \|_2 + \sum_{k=1}^{K} \frac{\delta_k (e, u_k)p_k}{\delta_k^2 + t + \gamma} \|_2
\]
Next, we prove the left side of the inequality of Theorem 3.2. Plugging (5) into the above equation (20), we have that

\[
\|\hat{f} - f^*\|_2 = \|\sum_{k=1}^K (\hat{f} - W^T(\alpha^* - v^*), p_k) p_k - \sum_{k=1}^K \frac{\delta_k(A\hat{f} + e, u_k)p_k + t(W^T(\alpha^* - v^*), p_k)p_k}{\delta_k^2 + t + \gamma} - \frac{\delta_k(e, u_k)p_k}{\delta_k^2 + t + \gamma} \|_2
\]

Plugging (18) into the above inequality, we have that

\[
\|\hat{f} - f^*\|_2 = \|\sum_{k=1}^K (t + \gamma)(\hat{f}, p_k)p_k - \sum_{k=1}^K \frac{\delta_k(A\hat{f} + e, u_k)p_k + t(W^T(\alpha^* - v^*), p_k)p_k}{\delta_k^2 + t + \gamma} - \frac{\delta_k(e, u_k)p_k}{\delta_k^2 + t + \gamma} \|_2
\]

Plugging (18) into the above inequality, we have that

\[
\|\hat{f} - f^*\|_2 \leq \|\sum_{k=1}^K t(W^T(\alpha^* - v^*), p_k)p_k\|_2 + \|\sum_{k=1}^K \frac{\delta_k(e, u_k)p_k}{\delta_k^2 + t + \gamma} \|_2
\]

So, we finish this proof.

\[\Box\]

**Corollary 1.** Let \(\gamma \to 0\) in Theorem 3.2, then we can obtain a better error bound between the reference image and the reconstructed image as follows:

\[
T(t) - \frac{\|e\|_2}{2\sqrt{t}} \leq \|\hat{f} - f^*\|_2 \leq T(t) + \frac{\|e\|_2}{2\sqrt{t}}.
\]

Where \(T(t) = \|\sum_{k=1}^K \frac{t(W^T(\alpha^* - v^*), p_k)p_k}{\delta_k^2 + t + \gamma} \|_2\).

**Proof.** This result can be obtained by Theorem 3.2 directly. \[\Box\]

If some conditions that \(W\) is a linear transformation and \(W^TW = I\) are assumed, the following theorem will be established.

**Theorem 3.3.** Let \(\hat{f}\) be a reference image, the sequence \(\{(f^n, \alpha^n, v^n)\}\) generated by the (15), (16) and (10), \((f^*, \alpha^*, v^*)\) is the stop iterate point using the ADMM-like algorithm. \(W\) is a linear transformation and \(W^TW = I\), and \(e\) denotes some noise and physical factors, then, we obtain

\[
\|\hat{f} - f^*\|_2 \leq T(t, \gamma) + R(t, \gamma) + \frac{t + \gamma + \delta_{\min}^2}{\gamma + \delta_{\min}^2} \cdot \frac{\|e\|_2}{2\sqrt{t}}.
\]
First, we prove the (21), according to the Theorem 3.2, we have that

$$\delta$$

Next, we prove the (22), according to the proof of Theorem 3.2, we have that

$$R'(t, \gamma) = \frac{\gamma \|f\|_2}{2t + \gamma + \delta_{max}^2}$$

and

$$\|f - f^*\|_2 \geq T'(t, \gamma) - R'(t, \gamma) = \frac{t + \gamma + \delta_{max}^2}{2t + \gamma + \delta_{max}^2} \cdot \|e\|_2 \sqrt{t},$$

where $$T(t, \gamma) = \frac{t \|W^T v\|_2}{\gamma + \delta_{min}^2}$$, $$R(t, \gamma) = \frac{\gamma \|f\|_2}{\gamma + \delta_{min}^2}$$, $$T'(t, \gamma) = \frac{t \|W^T v\|_2}{2t + \gamma + \delta_{max}^2}$$ and $$R'(t, \gamma) = \frac{\gamma \|f\|_2}{2t + \gamma + \delta_{max}^2}$$. $$\delta_{min}^2$$ and $$\delta_{max}^2$$ are the minimum and maximum non-zero eigenvalue of $$A^T A$$, respectively.

Proof. First, we prove the (21), according to the Theorem 3.2, we have that

$$\|f - f^*\|_2 \leq \frac{\sum_{k=1}^{K} t (f - W^T (\alpha^* - v^*), p_k) p_k}{\delta_k^2 + t + \gamma} + \frac{\|W^T v\|_2}{\gamma + \delta_{min}^2} \cdot \|e\|_2 \sqrt{t}.$$}

$$\|f - f^*\|_2 \leq \frac{t \|f - f^*\|_2}{t + \gamma + \delta_{min}^2} + \frac{t \|W^T v\|_2}{t + \gamma + \delta_{min}^2} + \frac{\gamma \|f\|_2}{t + \gamma + \delta_{min}^2} \cdot \|e\|_2 \sqrt{t}$$

According to (10) and $$W^T W = I$$, we have $$W f^* = \alpha^*$$ and $$f^* = W^T \alpha^*$$.

Let $$\delta_{min}^2$$ be the minimum non-zero eigenvalue of $$A^T A$$, and $$W$$ is a linear transformation, we obtain

$$\|f - f^*\|_2 \leq \frac{t \|f - f^*\|_2}{t + \gamma + \delta_{min}^2} + \frac{t \|W^T v\|_2}{t + \gamma + \delta_{min}^2} + \frac{\gamma \|f\|_2}{t + \gamma + \delta_{min}^2} \cdot \|e\|_2 \sqrt{t}$$

Next, we prove the (22), according to the proof of Theorem 3.2, we have that

$$\|f - f^*\|_2 = \frac{\sum_{k=1}^{K} t (f - W^T (\alpha^* - v^*), p_k) p_k}{\delta_k^2 + t + \gamma} + \frac{\gamma (f, p_k) p_k}{\delta_k^2 + t + \gamma} - \frac{\delta_k (e, u_k) p_k}{\delta_k^2 + t + \gamma} \|e\|_2 \sqrt{t}.$$}

$$\|f - f^*\|_2 \geq \frac{\sum_{k=1}^{K} t (f - W^T (\alpha^* - v^*), p_k) p_k}{\delta_k^2 + t + \gamma} + \frac{\gamma (f, p_k) p_k}{\delta_k^2 + t + \gamma} - \frac{\delta_k (e, u_k) p_k}{\delta_k^2 + t + \gamma} \|e\|_2 \sqrt{t}.$$}

Let $$\delta_{max}^2$$ be the maximum non-zero eigenvalue of $$A^T A$$, and $$W$$ is a linear transformation, we obtain

$$\|f - f^*\|_2 \geq \frac{\sum_{k=1}^{K} t (f - W^T (\alpha^* - v^*), p_k) p_k}{\delta_k^2 + t + \gamma} + \frac{\gamma (f, p_k) p_k}{\delta_k^2 + t + \gamma} - \frac{\delta_k (e, u_k) p_k}{\delta_k^2 + t + \gamma} \|e\|_2 \sqrt{t}.$$}

$$\|f - f^*\|_2 \geq \frac{t \|W^T v\|_2}{t + \gamma + \delta_{max}^2} - \frac{t \|f - f^*\|_2}{t + \gamma + \delta_{max}^2} + \frac{\gamma \|f\|_2}{t + \gamma + \delta_{max}^2} \cdot \|e\|_2 \sqrt{t}$$

$$\geq \frac{t \|W^T v\|_2}{2t + \gamma + \delta_{max}^2} - \frac{2t + \gamma + \delta_{max}^2}{2t + \gamma + \delta_{max}^2} \cdot \|f\|_2 - \frac{t + \gamma + \delta_{max}^2}{2t + \gamma + \delta_{max}^2} \cdot \|e\|_2 \sqrt{t}$$

So, we finish this proof. □

**Corollary 2.** Let $$\gamma \to 0$$ in Theorem 3.3, $$W$$ is a linear transformation and $$W^T W = I$$, and $$e$$ denotes some noise and physical factors. We have
Theorem 3.4. \[ \| \hat{f} - f^* \|_2 \leq \frac{t}{\delta^2_{\min}} \cdot \| W^T v^* \|_2 + \frac{t + \delta^2_{\min}}{2\sqrt{t}} \cdot \| e \|_2 \] and
\[ \| \hat{f} - f^* \|_2 \geq \frac{t}{2t + \delta^2_{\max}} \cdot \| W^T v^* \|_2 - \frac{t + \delta^2_{\max}}{2t + \delta^2_{\max}} \cdot \| e \|_2 \]

Proof. This result can be obtained by Theorem 3.3 directly. \qed

Remark 1. The result of the Theorem 3.2 and the Corollary 1 are implicit, on the contrary, the Theorem 3.3 and the Corollary 2 are explicit.

Furthermore, we discuss the stability of the solution of the ADMM-like algorithm in terms of a small perturbation \( e \) on the simulation projection data with free noise. According to (5), we have that
\[ A f = g + e \] with \( \| e \|_2 \rightarrow 0 \).

Theorem 3.4. Under the conditions of Corollary 1, we assume a priori that \( \| e \|_2 < \epsilon \rightarrow 0 \). If we choose \( t \rightarrow 0 \) appropriately that makes \( \| e \|_2 \rightarrow 0 \) or a constant \( t \) appropriately and some parameters make \( \| \hat{f} - W^T (a^* - v^*) \|_2 \rightarrow 0 \), the error in estimation \( \| \hat{f} - f^* \|_2 \) will converge to 0, then, the solution of the ADMM-like algorithm is stable.

Proof. According to the Corollary 1. First, since \( \| e \|_2 < \epsilon \rightarrow 0 \), if we choose \( t \rightarrow 0 \) according to the \( e \) such that \( \| e \|_2 \rightarrow 0 \), and \( T(t) \) will converge to 0, we have \( \| \hat{f} - f^* \|_2 \) will converge to 0. Second, if we choose the constant \( t \) appropriately and some parameters make \( \| \hat{f} - W^T (a^* - v^*) \|_2 \rightarrow 0 \), \( T(t) \) will converge to 0, and because \( \| e \|_2 < \epsilon \rightarrow 0 \), \( \| e \|_2 \rightarrow 0 \), we have \( \| \hat{f} - f^* \|_2 \) will converge to 0. So, the solution of the ADMM-like algorithm is stable if the conditions above are satisfied. \qed

Theorem 3.5. Under the conditions of Corollary 2, we assume a priori that \( \| e \|_2 < \epsilon \rightarrow 0 \). If we choose \( t \rightarrow 0 \) appropriately and \( \frac{t + \delta^2_{\min}}{2\sqrt{t}} \cdot \| e \|_2 \rightarrow 0 \) or a constant \( t \) appropriately and some parameters make \( \| W^T v^* \|_2 \rightarrow 0 \), the error in estimation \( \| \hat{f} - f^* \|_2 \) will converge to 0, then, the solution of ADMM-like algorithm is stable.

Proof. According to the Corollary 2. First, if we let \( t \rightarrow 0 \), \( \frac{t}{\delta^2_{\min}} \cdot \| W^T v^* \|_2 \) will converge to 0. Since \( \| e \|_2 < \epsilon \rightarrow 0 \), we choose \( t \rightarrow 0 \) appropriately according to the \( e \) such that \( \frac{t + \delta^2_{\min}}{2\sqrt{t}} \cdot \| e \|_2 \rightarrow 0 \), we have \( \| \hat{f} - f^* \|_2 \) will converge to 0. Second, if we choose the constant \( t \) and some parameters make \( \| W^T v^* \|_2 \rightarrow 0 \), \( \frac{t}{\delta^2_{\min}} \cdot \| W^T v^* \|_2 \) will converge to 0, and because \( \| e \|_2 < \epsilon \rightarrow 0 \), \( \| e \|_2 \rightarrow 0 \), we have \( \| \hat{f} - f^* \|_2 \) will converge to 0. So, the solution of the ADMM-like algorithm is stable if the conditions above are satisfied. \qed
Remark 2. According to (8), (9) and (10), we should note that the $\| \hat{f} - W^T(a^* - v^*) \|_2$ or $\| W^T v^* \|_2$ is close to 0 or not which is also decided by the parameter $t$, the regularization parameter $\lambda$ (which controls the sparsity of image under a wavelet tight framlets), the scanning range of limited-angle CT (i.e. system matrix $A$) and the projection data with noise. The $\| \hat{f} - W^T(a^* - v^*) \|_2 \to 0$ or $\| W^T v^* \|_2 \to 0$ is not satisfied generally by experience if the scanning angular range of limited-angle is seriously insufficient.

Remark 3. If the parameters of ADMM-like algorithm do not meet $\| \hat{f} - W^T(a^* - v^*) \|_2 \to 0$ or $\| W^T v^* \|_2 \to 0$ when the constant $t$ not close to 0, then, the solution of the ADMM-like algorithm is not stable, but the error bound can be estimated using the Corollary 1 or Corollary 2. Theorem 3.4 or 3.5 is just a sufficient condition for $\| \hat{f} - f^* \|_2$ that converges to 0. In the experiments, we find the ADMM-like algorithm has a slower convergent rate if we choose smaller $t$. Therefore, we will choose the parameters appropriately for better quality of reconstructed image.

4. Numerical results. In this section, we will test the stability by simulation projection with different level of noise and give a reconstruction experiment on metal laths from few-view of the limited-angle real projections. We compare the results of our algorithm with classical iterative reconstruction algorithms which include the SART and the TVM algorithms.

In the experiments, we choose the parameters of all algorithm for best image quality by trial and error. Our experiments are implemented on 3.40 GHz intel(R) Core(TM) i3-4130 CPU processor with 8G memory and coded in C++ language and matlab language.

4.1. Stability by simulation projection. We test the stability by Shepp-Logan phantom for few-view of the limited-angle CT reconstruction. The projection data are generated by the simulation Shepp-Logan phantom and the different level of Gaussian noise are added to projection data. The maximum value of the projection data is $\text{maxg} = 30164$ in our experiment, the mean value of the Gaussian noise is set to zero, and the standard deviation $\delta$ is set to 0.001% (0.01%, 0.05%, 0.1%) of $\text{maxg}$. Table 1 shows the geometrical scanning parameters of the simulated CT imaging system. The scanning angular ranges $[0, 140^\circ]$ are investigated, and the projection views is set to 70. The maximum iteration number of the iterative algorithms is $N_{\text{ite}} = 800$ for different level of Gaussian noise.

| The distance between source and detector | 1200mm |
| The distance between source and rotation center | 981mm |
| The angle interval of two adjacent projection views | $2^\circ$ |
| The angle interval of two adjacent rays | 0.00329$^\circ$ |
| The numbers of detector bin | 256 |
| The diameter of field of view | 143.6222mm |
| Pixel size | $0.5632 \times 0.5632$mm$^2$ |
| The size of image | $256 \times 256$ |
In the PCM algorithm, $\gamma = 1.9$, the maximum iteration number is 2000 and the stopping criterion is $\varepsilon = 1 \times 10^{-5}$, and adding a stopping criterion is $\varepsilon_1 = 1 \times 10^{-6}$ to the ADMM-like algorithm. In the experiment, the reconstruction parameters of our algorithm for different level of Gaussian noise are used as follows: 1) $t = 0.4$, $\lambda = 2.4$ for the standard deviation 0.001% and 0.01% of Gaussian noise; 2) $t = 0.4$, $\lambda = 4$ for the standard deviation 0.05% and 0.1% of Gaussian noise.

![Figure 1. The reconstructed results of Shepp-Logan phantom.](image)

The left of image is the original phantom. The subsequent columns show the reconstructed results for Gaussian noise with standard deviation 0.001心智级, 0.01心智级, 0.05心智级 and 0.1心智级.

Figure 1 presents the reconstructed results from simulated data by adding different levels of Gaussian noise. The left of Figure 1 is the original phantom of this experiment. The subsequent columns show the reconstructed results for Gaussian noise with standard deviation 0.001心智级, 0.01心智级, 0.05心智级 and 0.1心智级. From Figure 1, the quality of reconstructed results changes to be better gradually as the decrease of standard deviation.

To further verify this observation, we compute the RMSE and the PSNR of Shepp-Logan phantom with the reconstructed results, shown in Figure 2 and Figure 3. The RMSE and PSNR are defined as [56]:

$$\text{RMSE} = \sqrt{\frac{1}{M} \sum_{i=1}^{M} (f(i) - f_{true}(i))^2}$$

$$\text{PSNR} = -20 \log_{10} \sqrt{\frac{\sum_{i=1}^{M} (f(i) - f_{true}(i))^2}{M}}$$

where $f$ denotes the reconstructed image, $f_{true}$ denotes the original image, and $M$ denotes the number of image pixels.

From Figures 2 and 3, with the decrease of the standard deviation, the RMSE of Shepp-Logan phantom with reconstructed results is decreasing and the PSNR of Shepp-Logan phantom with reconstructed results is increasing. And we find that the bigger standard deviation will makes the sequence $f_n$ converge to a cluster point early, but the reconstructed result is worse. It implies that the smaller standard deviation, the better image quality. The Figure 2 shows the solution of the ADMM-like algorithm is stable on small Gaussian noise.

Next, a abdomen phantom [57] is used to test the stability of the ADMM-like algorithm. Table 2 shows the simulated geometrical scanning parameters of
Figure 2. The RMSE of Shepp-Logan phantom with reconstructed images by adding Gaussian noise with standard deviation 0.001%maxg, 0.01%maxg, 0.05%maxg and 0.1%maxg.

Figure 3. The PSNR of Shepp-Logan phantom with reconstructed images by adding Gaussian noise with standard deviation 0.001%maxg, 0.01%maxg, 0.05%maxg and 0.1%maxg.

the few-view of limited-angle CT. The maximum value of the projection data is \( \text{maxg} = 36765 \) in our experiment, the mean value of the Gaussian noise is zero, and the standard deviation \( \delta \) is 0.001%\text{maxg}, 0.01%\text{maxg}, 0.05%\text{maxg} and 0.1%\text{maxg} are added to the projection data. In the experiments, we investigate the scanning ranges \( 0^\circ \sim 160^\circ \), and the number of the projection views is 114, and the maximum iteration number \( N_{\text{ite}} = 800 \) for different level of Gaussian noise.

In this experiment, the reconstruction parameters of our algorithm for different level of Gaussian noise are used as follows: 1) \( t = 0.36 \) and \( \lambda = 2.4 \) for the standard

| Table 2. Geometrical scanning parameters for simulated CT imaging system |
|---------------------------------------------------------------|
| The distance between source and detector | 1200mm |
| The distance between source and rotation center | 981mm |
| The angle interval of two adjacent projection views | 1.41\(^{\circ}\) |
| The angle interval of two adjacent rays | 0.0005\(^{\circ}\) |
| The numbers of detector bin | 560 |
| The diameter of field of view | 279.5mm |
| Pixel size | \( 0.5 \times 0.5\text{mm}^{2} \) |
| The size of image | \( 512 \times 512 \) |
deviation 0.1% and 0.05% of Gaussian noise; 2) \( t = 0.36 \) and \( \lambda = 3 \) for the standard deviation 0.01% and 0.001% of Gaussian noise.

Figure 4 shows the reconstructed results from simulated data by adding different levels of Gaussian noise. The left of Figure 4 is the original phantom of this experiment. The subsequent columns show the reconstructed results for Gaussian noise with standard deviation 0.001\%\( \text{max}_g \), 0.01\%\( \text{max}_g \), 0.05\%\( \text{max}_g \) and 0.1\%\( \text{max}_g \). From Figure 4, the quality of reconstructed results changes to be better gradually as the decrease of standard deviation.

Figure 4. The reconstructed results of abdomen phantom. The left of image is the original phantom. The subsequent columns show the reconstructed results for Gaussian noise with standard deviation 0.001\%\( \text{max}_g \), 0.01\%\( \text{max}_g \), 0.05\%\( \text{max}_g \) and 0.1\%\( \text{max}_g \).

To further verify this observation, we compute the RMSE and the PSNR of abdomen phantom with the reconstructed images, shown in Figures 5 and 6. Figure 5 shows the solution of the ADMM-like algorithm is not stable under our chosen parameters, however, the quality of reconstructed result is acceptable in terms of visual inspection.

Figure 5. The RMSE of abdomen phantom with reconstructed images by adding Gaussian noise with standard deviation 0.001\%\( \text{max}_g \), 0.01\%\( \text{max}_g \), 0.05\%\( \text{max}_g \) and 0.1\%\( \text{max}_g \).

Furthermore, a NCAT phantom [47] is used to test the stability of the ADMM-like algorithm. The simulated geometrical scanning parameters for the few-view of limited-angle CT are shown in Table 1. The mean value of the Gaussian noise is zero, and the standard deviation \( \delta \) is 0.001\% (0.01\%, 0.05\%, 0.1\%) of \( \text{max}_g \) are added to the projection data, where \( \text{max}_g \) denotes the maximum value of the
projection data and is 32584 in these experiments. In the experiments, we consider the scanning ranges $0^\circ \sim 140^\circ$, the number of the projection views is 70, and the maximum iteration number $N_{ite} = 1800$ for different level of Gaussian noise.

In these experiments, the reconstruction parameters of our algorithm for different level of Gaussian noise are used as follows: 1) $t = 0.36$ and $\lambda = 3$ for the standard deviation 0.1% and 0.05% of Gaussian noise; 3) $t = 0.36$ and $\lambda = 0.45$ for the standard deviation 0.01% and 0.001% of Gaussian noise.

Figure 7 shows the reconstructed results from simulated data by adding different levels of Gaussian noise. The left of Figure 7 is the original phantom of this experiment. The subsequent columns show the reconstructed results for Gaussian noise with standard deviation 0.001% maxg, 0.01% maxg, 0.05% maxg and 0.1% maxg.

From Figure 7, the quality of the reconstructed results changes to be better gradually as the decrease of the standard deviation.

To further confirm this observation, we compute the RMSE and the PSNR of NCAT phantom with the reconstructed images, shown in Figures 8 and 9. Figure 8 shows the solution of the ADMM-like algorithm is stable on small Gaussian noise.

4.2. Reconstruction from real data. To further verify the effectiveness and stability of the ADMM-like algorithm, we use the real projection data of metal laths which is obtained from our real CT system. There are three difficult problems in metal laths reconstruction experiment. First, some X-rays can not pass through the metal laths due to the energy of X-ray and the thickness of metal laths. Second, the projection data is incomplete due to the scanning angular which is limited. Third, the resolution of reconstructed image will decline if we use fewer projection data. In these situations, the reconstructed image will present some serious metal artifacts and slope artifacts using the SART and TVM algorithms and the resolution of reconstructed image is damaged (see Figures 12, 13, 14 and 15).

The size of reconstructed image is $1024 \times 1024$. The scanning angular $[0, 160^\circ]$ are investigated, and the projection views are set to 228 (114, 76). The scanning range $[0, 140^\circ]$ are investigated, and the numbers of the projection views is 199 (100, 66). In this experiments, the weight coefficient $\omega = 1$ for the SART algorithm (14) and the initial image $f_1 = 0$ for the iterative algorithms. The reconstruction parameters of the TVM based reconstruction algorithm are $N_{TV} = 20$ and $\alpha = 0.2$. The maximum iteration number of all the iterative algorithms is $N_{ite} = 100$.

![Figure 6](image-url)

Figure 6. The PSNR of abdomen phantom with reconstructed images by adding Gaussian noise with standard deviation 0.001% maxg, 0.01% maxg, 0.05% maxg and 0.1% maxg.
Figure 7. The reconstructed results of NCAT phantom. The left of image is the original phantom. The subsequent columns show the reconstructed results for Gaussian noise with standard deviation 0.001\%\text{max}_g, 0.01\%\text{max}_g, 0.05\%\text{max}_g and 0.1\%\text{max}_g.

In this algorithm, the parameters of the PCM algorithm is the same as the simulation experiment. In real data experiment, reconstruction parameters of the ADMM-like algorithm are given in the following: 1) for scanning range $[0, 160^\circ]$, $t = 0.65, \lambda = 0.0000000085$; 2) for scanning range $[0, 140^\circ]$, $t = 0.6, \lambda = 0.0000000089$.

Figures 10 and 11 show the reconstructed results from real data using different algorithms for three different projection views in limited-angle $160^\circ$ and $140^\circ$, respectively. The left of Figures 10 and 11 are the reconstructed results using the SART algorithm. The subsequent columns show the reconstructed results using the TVM based algorithm and the ADMM-like algorithm. In these figures, from top to down in each row show the reconstructed results from three kinds of projection views. Figures 12 and 13 show the zoom-in views of the reconstructed results in ROI1 of Figures 10 and 11. And Figures 14 and 15 show the zoom-in views of the reconstructed results in ROI2 of Figures 10 and 11.

Referring to Figures 12 and 14, for scanning ranges $[0, 160^\circ]$, the results using the SART algorithm contain some noise and metal artifacts; the results using the

Figure 8. The RMSE of NCAT phantom with reconstructed images by adding Gaussian noise with standard deviation 0.001\%\text{max}_g, 0.01\%\text{max}_g, 0.05\%\text{max}_g and 0.1\%\text{max}_g.
TVM do not present the same noise, but appear some metal artifacts and lose the resolution of reconstructed image (see Figure 14) where the number of the second metal laths can not be counted, the reconstructed image is over-smooth; the results using the ADMM-like algorithm suppresses the noise and the slope artifacts, and preserves the resolution of reconstructed image although it has some metal artifacts.

Referring to Figures 13 and 15, for scanning ranges $[0, 140^\circ]$, the slope artifacts

![Figure 9. The PSNR of NCAT phantom with reconstructed images by adding Gaussian noise with standard deviation 0.001$\% maxg$, 0.01$\% maxg$, 0.05$\% maxg$ and 0.1$\% maxg$.](image)

![Figure 10. The reconstructed results from real data using different algorithms for different projection views in the limited-angle $160^\circ$.](image)
Figure 11. The reconstructed results from real data using different algorithms for different projection views in the limited-angle $140^0$.

Figure 12. The ROI1 of Figure 10 for different projection views.
Figure 13. The ROI1 of Figure 11 for different projection views.

Figure 14. The ROI2 of Figure 10 for different projection views.
and the metal artifacts are presented using the SART algorithm; the results using the TVM algorithm show that the edge of image was blurry, slope artifacts and metal artifacts are presented, and the resolution of reconstructed image was damaged; The results using the ADMM-like algorithm show that the slope artifacts had been better suppressed than that using the TVM algorithm and the SART algorithm, and the resolution of reconstructed image can be better preserved. More edge structure information can be recovered using the ADMM-like algorithm under no damage the resolution of reconstructed image.

From the above experimental results, we can see that the ADMM-like algorithm has more advantages in reducing the slope artifacts under preserving the resolution of reconstructed image for few-view of limited-angle CT reconstruction. But the metal artifacts are still not eliminated in reconstructed image using the ADMM-like algorithm.

Next we will focus on reducing the metal artifacts using the ADMM-like algorithm. In [11], the author eliminate the inconsistence projection data by missing data. Motivated by this idea, we will miss some projection data that the X-ray can not pass through the metal laths due to the energy of X-ray and the thickness of metal laths. We find the slope artifact will presented in the reconstructed image when we missing some angle included in [0, 160°]. In this work, we rotate the metal laths in some angle so as to avoiding the X-ray can not pass through the metal laths and modify the constrain condition. In the following experiment, the scanning angles are limited in [0, 160°], the number of views is 455, and the constraint condition $f \geq 0$ modified by $f \geq 0.003$ because the projection data includes some inconsistence data. The parameters of the SART, the TVM and the ADMM-like algorithm are same as above. The maximum iteration number of all the iterative algorithms is $N_{ite} = 15$.

Figure 16 shows the reconstructed results using three different algorithms, from left to right show the reconstructed results using the SART, the TVM and the ADMM-like algorithm. Figures 17 and 18 show the ROIs of Figure 10. Referring to Figures 16, 17 and 18, if the constrain condition $f \geq 0$ modified by $f \geq 0.003$, the
ADMM-like algorithm outperforms the SART or the TVM algorithm in suppressing the noise and the metal artifacts, and preserving the resolution of reconstructed image for few-view of limited-angle CT reconstruction.

**Figure 16.** The reconstructed result by different algorithms, from left to right present the results of the SART, the TVM and the ADMM-like algorithm.

**Figure 17.** The ROI1 of Figure 16, from left to right present the results of the SART, the TVM and the ADMM-like algorithm.

5. **Conclusions and perspectives.** To reduce the slope artifacts and metal artifacts of CT reconstruction from the few-view of limited-angle scanning range under the preserve the resolution of reconstructed image, we use an effective image reconstruction algorithm base on $\ell_0$ minimization of wavelet coefficients that have been proposed in our previous work. First, we analyse the bound of the error between the
reference image and reconstructed image by SVD, and give the explicit and implicit error bounded. Second, we analyse the stability of the solution of model in theoretical and experimental, and give a sufficient condition for the stability. And we give a experiment on metal laths from few-view of limited-angle projection. The experimental results indicate that the ADMM-like algorithm causes lesser artifacts in the reconstructed results and can reconstruct the edge structure information more valid than the classical CT reconstruction algorithms. The fan-beam CT system is investigated in this work. We will investigate the $\ell_0$ regularized based CT reconstruction algorithm for Micro-CT system, and investigate the $\ell_p$, $(0 < p < 1)$ regularized based CT reconstruction algorithm for limited-angle scanning CT system in the future.

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![Figure 18](image.png)

**Figure 18.** The ROI of Figure 16, from left to right present the results of the SART, the TVM and the ADMM-like algorithm.
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