Abstract
In this paper we analyze the linear stability of the Linet–Tian solution with negative cosmological constant. In the limit of vanishing cosmological constant the Linet–Tian metric reduces to a form of the Levi–Civita metric, and, therefore, it can be considered as a generalization of the former to include a cosmological constant. The gravitational instability of the Levi–Civita metric was recently established, and the purpose of this paper is to investigate what changes result from the introduction of a cosmological constant. A fundamental difference brought about by a (negative) cosmological constant is in the structure at infinity. This introduces an added problem in attempting to define an evolution for the perturbations because the constant time hypersurfaces are not Cauchy surfaces. In this paper we show that under a large set of boundary conditions that lead to a unique evolution of the perturbations, we always find unstable modes, that would generically be present in the evolution of arbitrary initial data, leading to the conclusion that the Linet–Tian space times with negative cosmological constant are linearly unstable under gravitational perturbations.

Keywords: axial symmetry, line source, perturbations

1. Introduction

The Linet–Tian metric [1, 2] is a static cylindrically symmetric solution of Einstein’s equations that can be interpreted as describing a line source in a vacuum space time with non vanishing cosmological constant \( \Lambda \), which can be chosen to be positive or negative. In the case of vanishing \( \Lambda \) the metric reduces to a form of the Levi–Civita metric [7].

There has been a continued interest in the Linet–Tian metric and in the construction of models in which the metric is involved. As a few examples we may cite [3] where it was found...
that, for a negative cosmological constant, in the limit where the source vanishes, one obtains a static, but cylindrically symmetric anti-de-Sitter universe. General properties were analyzed, for instance in [4]. The properties of the solutions with positive cosmological constant, and their extension to higher dimensions were considered in [5]. The geodesics kinematics and dynamics in the Linet–Tian metric with negative cosmological constant were analyzed in [6].

On the other hand, the question of the evolution of external fields or of the stability of gravitational perturbations on these space times has not received the same attention. The stability of the related Levi–Civita metric under linear perturbations that break the cylindrical symmetry was analyzed in a recent study [8], finding that the spectrum of allowed frequencies contains one unstable (imaginary frequency) mode for every possible choice of the background metric. The main purpose of the present paper is to analyze to what extent these results are modified by the presence of a cosmological constant, and therefore, extend the results obtained in [8] to the Linet–Tian space times.

In the case of the Linet–Tian metric, as compared with the Levi–Civita metric, the first thing to notice is that the structures of the space times that result for the two possible signs of $\Lambda$ are very different, specially as regards their properties at large distances from the symmetry axis. In the positive cosmological case, there is a null horizon at some finite distance from the symmetry axis, while for negative cosmological constants there is a time like horizon at an infinite proper distance from the axis. This implies that in the analysis of the evolution of either external fields, or small perturbations, one has to consider very different boundary conditions. In this paper we study the linear stability of the Linet–Tian solution with negative cosmological constant. The positive cosmological case will be considered in a separate paper.

In the case of a negative cosmological constant ($\Lambda < 0$), the Linet–Tian metric may be written in the form,

$$ds^2 = Q^{2/3}(-P^\rho d\rho^2 + P^\phi d\phi^2 + P^t d^2 + d\rho^2)$$

(1)

where:

$$Q(\rho) = \frac{1}{\sqrt{3|\Lambda|}} \sinh \left( \sqrt{3|\Lambda|} \rho \right)$$

$$P(\rho) = \frac{2}{\sqrt{3|\Lambda|}} \tanh \left( \frac{\sqrt{3|\Lambda|}}{2} \rho \right)$$

(2)

and the parameters $p_i$ satisfy,

$$p_1 + p_2 + p_3 = 0$$

$$p_1^2 + p_2^2 + p_3^2 = \frac{8}{3}$$

(3)

The ranges of the coordinates are $-\infty < t < \infty$, $0 \leq \rho < \infty$, $-\infty < z < \infty$, and $0 \leq \phi \leq 2\pi$. The $p_i$ may be parameterized as,

$$p_1 = \frac{2(1 - 8\sigma + 4\sigma^2)}{3 - 6\sigma + 12\sigma^2}$$

$$p_2 = \frac{-2(1 + 4\sigma + 8\sigma^2)}{3 - 6\sigma + 12\sigma^2}$$

$$p_3 = \frac{4(1 - 2\sigma - 2\sigma^2)}{3 - 6\sigma + 12\sigma^2}$$

(4)
with \( \sigma \) restricted to \( 0 \leq \sigma \leq 1/2 \). In this paper we will use a parametrization due to Thorne \([9]\), where \( \sigma = \kappa/(2 + 2\kappa) \), and,

\[
\begin{align*}
p_1 &= -\frac{2(1 - 2\kappa - 2\kappa^2)}{3(1 + \kappa + \kappa^2)} \\
p_2 &= -\frac{2(1 + 4\kappa + \kappa^2)}{3(1 + \kappa + \kappa^2)} \\
p_3 &= \frac{2(2 + 2\kappa - \kappa^2)}{3(1 + \kappa + \kappa^2)}
\end{align*}
\]

The range of \( \kappa \) is then \( 0 \leq \kappa < +\infty \). In the limit \(|\Lambda| = 0\) the Linet–Tian solution reduces to a form of the Levi–Civita metric. In what follows it will be more useful to change the coordinate \( \rho \) to a new coordinate \( x \), such that,

\[
\sinh\left(\frac{\sqrt{3}|\Lambda|}{2}\rho\right) = x
\]

so that the range of \( x \) is also \( 0 \leq x < \infty \). We then have,

\[
\begin{align*}
Q(\rho(x)) &= \frac{2x\sqrt{1 + x^2}}{\sqrt{3}|\Lambda|} \\
P(\rho(x)) &= \frac{2x}{\sqrt{3}|\Lambda|}\sqrt{1 + x^2}
\end{align*}
\]

and the Linet–Tian metric takes the form,

\[
ds^2 = -x^2 + \frac{\rho_1}{(1 + x^2)^{\frac{1}{2}}} d\rho^2 + x^2 \left(1 + x^2\right)^{\frac{1}{2}} \left(\frac{p_2}{2} + \frac{2\kappa}{3} \right) dz^2 \\
+ x^2 \left(1 + x^2\right)^{\frac{1}{2}} \frac{\rho_2}{2} \left(\frac{2}{3} \right) d\phi^2 + \frac{4}{3|\Lambda|(1 + x^2)} dx^2
\]

where, for simplicity and without loss of generality for the analysis to be carried out in this paper, we have rescaled the coordinates \( t, z, \phi \). The metric \((8)\), up to rescalings of the coordinates \( t, z, \phi \), is the general solution of the static Einstein equations with a negative cosmological constant, with (full) cylindrical symmetry. As already indicated, the purpose of this paper is to extend the stability analysis of the Levi–Civita metric, given in \([8]\) to the Linet–Tian solution with negative cosmological constant. The problem we are interested in here is that of the (linear) evolution of perturbations of \((8)\) that admit initial data of compact support, and that break the cylindrical symmetry of the background, but preserve the axial symmetry, so that we may write the perturbed metric in the form,

\[
ds^2 = g^{(0)}_{\mu\nu}(1 + \epsilon h_{\mu\nu}) dx^\mu dx^\nu
\]

where \( g^{(0)}_{\mu\nu} \) is the Linet–Tian metric \((8)\), \( x^\mu = \{t, x, z, \phi\} \), \( h_{\mu\nu} = h_{\mu
u}(t, x, z) \), and linearity implies that all geometric quantities are computed up to first order in \( \epsilon \). From the linearity of the problem, and the fact that \( g^{(0)}_{\mu\nu} \) depends only on \( x \), and leaving aside for the moment the question of boundary conditions, we will look first for solutions of the perturbation equations that have the form,

\[
h_{\mu\nu}(t, x, z) = e^{i(\Omega t - kz)} H_{\mu\nu}(\Omega, k, x)
\]
We expect to find a 'complete set' of such solutions, in the sense that the evolution of arbitrary initial data may be (formally) written as a Fourier transform of the form,

\[ h_{\mu\nu}(t, x, z) = \int e^{ik(x-tz)}\mathcal{C}(\Omega, k)H_{\mu\nu}(\Omega, k, x) \, d\Omega \, dk \]  

(11)

where \( \mathcal{C}(\Omega, k) \) is determined by the initial data, and we must keep in mind that in the case of linear instabilities some the \( \Omega \) may be complex numbers, and, also, that part or the whole spectrum of \( \Omega \) may be discrete. On this account, we write the perturbed metric in the form,

\[ ds^2 = g_0^{(0)}(1 + \epsilon e^{i(x-\tau-kz)}F_1)dt^2 + g_0^{(2)}(1 + \epsilon e^{i(x-\tau-kz)}F_2)dx^2 + g_0^{(4)}(1 + \epsilon e^{i(x-\tau-kz)}F_4)dy^2 + 2\epsilon e^{i(x-\tau-kz)}F_1dxdr + 2\epsilon e^{i(x-\tau-kz)}F_2d\phi \]

(12)

where the \( F_i \), and \( H_i \) are functions of \( x \) only.

If we compute Einstein’s equations using the form (12) of the metric, and retain only first order in \( \epsilon \), we find that the \( F_i \) satisfy a set of coupled linear O.D.E., decoupled from the \( H_i \), which themselves satisfy a separate set of coupled linear O.D.E. We may then consider as separate problems the cases where all the \( F_i \) are set equal to zero, (which we will call case I), and the cases where only those terms are non vanishing (case II).

The set of perturbation equations that result from computing Einstein’s equations are only local. In trying to extend their solutions to the whole space time we are confronted with the fact that the Linet–Tian solution is not globally hyperbolic. This non global hyperbolicity stems from two facts. The first is that the space time contains a time like singularity for \( x = 0 \). This singularity is the same as that present in the Levi–Civita metric. The problem of analyzing the evolution of perturbations under this condition was studied in [8]. There it was shown that if one imposes certain physically acceptable restrictions as a boundary condition for \( x = 0 \), it is possible to define a unique evolution for arbitrary perturbations that satisfy the imposed conditions. As a consequence it was possible to prove that the Levi-Civita space time is unstable under perturbations that break the translational symmetry along the symmetry axis. We may apply the same criterion here, but in the Linet–Tian case we have another source of non global hyperbolicity, because the boundary \( x \to \pm \infty \) is time like. This has as a consequence that the constant \( t \) hypersurfaces are not Cauchy surfaces, and that there are null geodesics that remain to the future of any given constant \( t \) hypersurface, and never intersect the hypersurface.

Thus, the future evolution of any perturbation may be arbitrarily modified by information incoming from \( x = \pm \infty \). One way out of this problem is to impose boundary conditions, possible with some physical justification that supplements the model, such that the evolution is solely determined by initial data on some constant \( t \) hypersurface. We notice, however, that this is not a unique prescription, and that the resulting evolution turns to be eventually dependent on the boundary condition. We will therefore rephrase our problem, and try to find, for a certain family of possible boundary conditions, under what conditions, if any, we obtain a well defined, although possibly unstable, evolution of appropriate initial perturbative data. These and other points will be clarified as we derive our may results. We refer to [10] for a more detailed discussion of this problem.

The plan of the paper is as follows. In the next section we analyze the gauge problem, with the purpose of extracting appropriate gauge invariant quantities, and show that, as already indicated, the evolution equations for a general perturbation separate into two independent sets, that we call Case I and Case II. In section 3 we analyze Case I. We derive a ‘master’ equation and obtain the conditions under which there are unstable modes associated to the evolution of type of
perturbation. Case II is analyzed in section 4. This is the main part of the paper. We show that the perturbations can be reduced to a diagonal form and we obtain a ‘master’ variable, such that solving its equation of motion one obtains a complete solution for the full perturbation. Nevertheless this function is not gauge invariant, but we show how to extract the corresponding gauge invariant part. Next we show that the solutions of the equation satisfied by this gauge invariant part are in one-to-one correspondence with the solutions of the eigenvalue—eigenfunction problem for an operator in a one dimensional, single particle, Schrödinger like system. The self adjointness of this operator requires imposing appropriate boundary conditions that are analyzed in detail. The structure of the resulting spectrum for the operator is studied in section 5, where we find that essentially in all cases it contains negative eigenvalues which correspond to imaginary frequencies, thus establishing our main result that the Linet–Tian space times are gravitationally unstable.

The limit $k = 0$, corresponding to purely radial perturbations is analyzed in section 6. An interesting and somewhat unexpected result of this analysis is that, contrary to what happens for the Levi–Civita case, the Linet–Tian space times with negative cosmological constant are also unstable under purely radial perturbations. In section 7 we consider the limit $\kappa = 0$, corresponding to the Bonnor metric, and find that it is stable under the linear perturbations considered here. We add, mostly for completeness, a brief comment on the ‘hoop conjecture’ in section 8. Some final comments are given in section 9. Mostly for completeness we have included in the appendix details of some complicated expressions which were omitted in the main text for readability.

2. The gauge problem

A central problem in the analysis of the evolution equations (as in [8]) is the elucidation of the gauge invariance of the perturbations. We consider this problem in this section. We first notice that a general coordinate transformation from $(t, x, z, \phi)$ to new coordinates $(T, X, Z, \Phi)$ that preserves the form (12) of the perturbed metric may be written as,

$$
t = T + \epsilon e^{i(\Omega - k^2)}T_0(X)
$$

$$
x = X + \epsilon e^{i(\Omega - k^2)}X_0(X)
$$

$$
z = Z + \epsilon e^{i(\Omega - k^2)}Z_0(X)
$$

$$
\phi = \Phi + \epsilon e^{i(\Omega - k^2)}\Phi_0(X)
$$

(13)

Indicating with a tilde the transformed coefficients, under this transformation we have,

$$\tilde{F}_i(X) = F_i(X) + 2i \Omega T_i(X) + \frac{(3p_i + 4X^2 + 2)}{3X(1 + X^2)}X_i(X)
$$

$$\tilde{F}_2(X) = F_2(X) - \frac{2X}{1 + X^2}X_2(X) + 2 \frac{dX_0(X)}{dX}
$$

$$\tilde{F}_3(X) = F_3(X) - 2i k Z_3(X) + \frac{(3p_3 + 4X^2 + 2)}{3X(1 + X^2)}X_3(X)
$$

$$\tilde{F}_4(X) = F_4(X) + \frac{4X^2 + 2 + 3p_4}{3X(1 + X^2)}X_4(X)
$$

$$\tilde{F}_5(X) = F_5(X) + \frac{4i \Omega}{3[A(1 + X^2)]}X_5(X) - X^{p_5 + 2/3}(1 + X^2)^{1/3 - p_5/2} \frac{dT_i(X)}{dX}
$$

$$\tilde{F}_6(X) = F_6(X) + \frac{ikX^{p_6 + 2/3}}{(1 + X^2)^{p_6/2 - 1/3}}T_i(X) + \frac{i \Omega X^{p_6 + 2/3}}{(1 + X^2)^{p_6/2 - 1/3}}Z_i(X)
$$

$$\tilde{F}_7(X) = F_7(X) - \frac{4ik}{3[A(1 + X^2)]}X_7(X) + \frac{X^{p_7 + 2/3}}{(1 + X^2)^{p_7/2 - 1/3}} \frac{dX_0(X)}{dX}
$$

(14)
and,
\[
\begin{align*}
\tilde{H}_1(X) &= H_1(X) + \frac{i \Omega X^{2/3} + p_3}{(1 + X^2)^{p_2/2 - 1/3}} \Phi_1(X) \\
\tilde{H}_2(X) &= H_2(X) + \frac{X^{2/3} + p_3}{(1 + X^2)^{p_2/2 - 1/3}} d\Phi_1(X) \\
\tilde{H}_3(X) &= H_3(X) - \frac{i k X^{2/3} + p_3}{(1 + X^2)^{p_2/2 - 1/3}} \Phi_1(X)
\end{align*}
\]

so that the \( F_i \) and the \( H_i \) transform separately, which is, of course, consistent with the fact that they satisfy separate systems of coupled O.D.E. We will consider the consequences of these forms of the transformations on the gauge issues and the resulting equations of motion in the following sections.

### 3. Case I

In this section we consider Case I. Here, in accordance with (15) we only have one free function (\( \Phi_1(x) \)) at our disposal, so we may choose a gauge where any one, (or some particular combination) of the functions \( H_i \) is set equal to zero, and this removes any gauge ambiguity. In more detail, it should be clear, from (15), that the expressions,
\[
G_0(x) = k H_1(x) + \Omega H_3(x) \\
G_2(x) = H_2(x) + \frac{i dH_1}{\Omega} - \frac{i (4x^2 + 2 + 3p_3)}{3 \Omega x (1 + x^2)} H_1
\]

are gauge invariant. On the other hand, in accordance with our previous discussion, without loss of generality, we may choose a gauge where \( H_1(x) = 0 \). In this case (i.e. if we set \( H_1 = 0 \)) the functions \( H_2 \) and \( H_3 \) correspond to gauge invariants. We may, therefore consider the perturbation equations that result when we set \( H_1(x) = 0 \) from the start. A simple computation then shows that Einstein’s equations imply that \( H_2 \) and \( H_3 \) must satisfy the coupled set of ODEs,
\[
\frac{dH_2}{dx} = \frac{(3p_1 - 1 - 5x^2) H_2 + 4ikx^{p_2 - 2/3}}{3x(1 + x^2)} H_3 \\
\frac{dH_3}{dx} = \frac{k \left[ \frac{\Omega^2 x^{p_2 - p_1}}{(1 + x^2)^{p_2/2 - p_1/2}} - k^2 \right] H_2 + \frac{(2 + 4x^2 + 3p_3)}{3(1 + x^2)} H_3}
\]

We may use the first of these to solve for \( H_3 \) in terms of \( H_2 \) and \( dH_2/dx \). Replacing in the second we find a second order equation for \( H_2 \) that can be written in the form,
\[
\frac{d^2H_2}{dx^2} = -\frac{(6p_2 + 1 + 11x^2)}{3x(1 + x^2)} dH_2/dx - \frac{4 \Omega^2 (1 + x^2)^{p_2/2 - 4/3}}{3|\Lambda| x^{2/3 + p_1}} H_2 \\
+ H_2 \left[ \frac{4k^2(1 + x^2)^{p_2/2 - 4/3}}{3|\Lambda| x^{2/3 + p_1}} + \frac{5x^4 + (10p_2 + 8p_1 + 4)x^2 + (1 - 3p_1)(p_1 + 2p_2 - 1)}{3x^2(1 + x^2)^2} \right]
\]
We immediately notice in this equation that $|\Lambda|$ appears only combined with $k$ and $\Omega$ in the forms $k^2/|\Lambda|$ and $\Omega^2/|\Lambda|$. We may therefore define two new parameters $\tilde{k} = k/\sqrt{|\Lambda|}$, and $\tilde{\Omega} = \Omega/\sqrt{|\Lambda|}$, and the solutions of (18), for fixed $\kappa$, will be parameterized by $\tilde{k}$, and $\tilde{\Omega}$, which is (formally) equivalent to setting $|\Lambda| = 1$ in (18). We shall adopt this latter choice, remembering that for $|\Lambda| \neq 1$ we must rescale the values of $k$ and $\Omega$.

Suppose now we fix the value of $\kappa$, which fixes the unperturbed space time. In principle, we may specify $k$ freely. Then, if we impose boundary conditions on (17) for $x = 0$, and $x \to \infty$, we turn the problem of solving the equation into a boundary value problem that determines the allowed values of $\Omega^2$. In principle we would have an infinite set of such solutions, that can be considered as functions of $k$, and $\Omega$, and which, through the use (17), provides a complete solution of the perturbation equations for the given parameters. Then, plugging these solutions in (11) we would have an infinite set of solutions of the evolution equations for the perturbations.

We are now confronted with several problems. One is that we do not have closed form solutions of this equation. Another is that it is not clear how to impose acceptable boundary conditions that define and restrict the range of possible values of $\Omega$. And, most importantly, how can we assure that our set of solutions is ‘complete’, in the sense that it can, at least in principle, describe the evolution of acceptable but arbitrary initial data.

These problems may be analyzed by constructing an equivalent self adjoint operator, whose eigenvalues and eigenfunctions are in one to one correspondence with the solutions of (10), and the corresponding values of $\Omega$. This can be achieved by introducing a new function $H_4(y)$, where $y$ is a function of $x$, and an ‘integration’ function $K(x)$ such that,

$$H_4(x) = K(x)H_4(y(x))$$

(19)

where $y(x)$ satisfies the equation,

$$\frac{dy}{dx} = \frac{2}{\sqrt{3} x^{p_1/2 + 1/3}(1 + x^2)^{2/3 - p_1/4}}$$

(20)

and,

$$K(x) = x^{p_1/4 - p_2}(1 + x^2)^{p_1/2 - p_2 - 1/2}$$

(21)

Replacing in (18) we find that $H_4(y)$ satisfies the equation,

$$-\frac{d^2H_4}{dy^2} + V_4(y)H_4 = \Omega^2 H_4$$

(22)

where,

$$V_4(y) = \frac{x^{p_1 - p_2}k^2}{|\Lambda|(1 + x^2)^{p_1/2 - p_2/2}}$$

$$+ \frac{(9p_1 + 12p_2 - 4)(5p_1 + 4p_2 - 4) - 16(7p_1 + 8p_2)x^2}{64x^{4/3 - p_2}(1 + x^2)^{2/3 + p_1/2}}$$

(23)

and $x$ should be given as a function of $y(x)$. We recognize that (22) has the standard form of the Schrödinger equation for a particle in a one dimensional potential, and,
therefore, its spectrum can be analyzed using well-known standard procedures. In particular, its self-adjoint extensions provide a complete, orthonormal, basis of functions in its domain.

The expression (23) for $V_H$ is not the most useful because $p_1$ and $p_2$ are not independent. A more useful one in terms of $\kappa$ is obtained using (5),

$$
\nu_H(y) = \frac{2(2+\kappa)}{x^{1+\kappa+\kappa^2}k^2} \left| \Lambda \right|(1 + x^2)^{3+\kappa+\kappa^2} + \frac{3(4\kappa + 5)(4\kappa + 3)}{16(1 + \kappa + \kappa^2)^2x^{1+\kappa+\kappa^2}(1 + x^2)^{3+\kappa+\kappa^2}} \right)
$$

We have, therefore, that for $x \to 0$ the potential diverges to $+\infty$, while for $x \to \infty$ we have $V_H \to k^2/|\Lambda|$. We further notice that for $2\kappa^2 \leq 5 + 6\kappa$ (i.e. for $\kappa \leq (3 + \sqrt{19})/2 = 3.67...$), the potential $V_H$ is positive definite, and, therefore, we will have $\Omega^2 \geq 0$ for any self-adjoint extension of (22). On the other hand, for $\kappa$ larger than this value, there are negative terms in the potential, and these, in turn, might dominate in some region, making $V_H$ negative there. To see what effect this may have on the allowed values of $\Omega$ we need to establish the existence of self-adjoint extensions for (22), and this requires making the dependence of $\nu_H(y)$ on $y$ more explicit.

In more detail, we may fix an integration constant in (20) and set,

$$
y(x) = \int_0^x \frac{2}{\sqrt{3}x^{p_1/2+1/3}(1 + x_1^{2/3-p_1/4}}} \, dx_1
$$

Since,

$$
p_1 + \frac{1}{3} = \frac{\kappa + \kappa^2}{1 + \kappa + \kappa^2}
$$

the integral in (25) converges for all $x$, and we have $0 \leq y \leq y_0$ for $0 \leq x < \infty$, with,

$$
y_0 = \int_0^\infty \frac{2}{\sqrt{3}x^{p_1/2+1/3}(1 + x_1^{2/3-p_1/4}}} \, dx_1
$$

We also notice that near $x = 0$ we have (to leading orders) the expansions,

$$
y(x) = \left[ \frac{4\sqrt{3}x^{2/3-p_1/2}}{4 + 3p_1} \right] \left[ 1 + \frac{(3p_1 - 8)(4 + 3p_1)x^2}{12(16 - 3p_1)} \right]
$$

while for $x \to \infty$ we have, to leading order,

$$
y(x) = y_0 - \frac{\sqrt{3}}{x^{1/3}}
$$

$x(y) = \frac{\sqrt{3}}{(y_0 - y)^{1/3}}$ (29)
Using these results, we first notice that from (28), near \( y = 0 \), to leading order, we have,

\[ \mathcal{V}_d(y) \approx \frac{(5 + 4\kappa)(3 + 4\kappa)}{4y^2} \]  (30)

and, therefore, near \( y = 0 \), the general solution of (22) behaves as,

\[ H_d(y) \sim C_1 y^{2\kappa + 5/2} + C_2 y^{-2\kappa - 3/2} \]  (31)

where \( C_1 \) and \( C_2 \) are arbitrary constants. This implies that the second term on the RHS of (31) diverges faster than \( 1/y \), and, therefore, we must set \( C_2 = 0 \) to have a boundary condition appropriate for a self adjoint extension.

The other important limit is \( y \to y_0 \) \((x \to \infty)\). In this limit we have

\[ \mathcal{V}_d(y) \approx \frac{k^2}{|\Lambda|} - \frac{(5 + 6\kappa - 2\kappa^2)}{4\sqrt{3}} (y_0 - y) \]  (32)

and, therefore, the general solution of (22), for \( y \to y_0 \) is of the form,

\[ H_d(y) \sim C_3 + C_4 (y_0 - y) \]  (33)

where \( C_3 \) and \( C_4 \) are arbitrary constants. As regards the possibility of a self adjoint extension this corresponds to the circle limit case. In this case we obtain a self adjoint extension by imposing that all allowed solutions must satisfy the boundary condition

\[ C_3 = \alpha C_4 \]  (34)

for some fixed \( \alpha \). This implies that there are infinite possible different self adjoint extensions. In particular for \( \alpha = 0 \) \((C_3 = 0)\) we have the Dirichlet and for \( \alpha = \infty \) \((C_4 = 0)\) the Neumann boundary condition. The general case of arbitrary \( \alpha \) is called the Robin boundary condition. This large ambiguity in the boundary condition for \( y \to y_0 \) can be traced to the fact, already discussed, that for adS the region \( x \to \infty \) corresponds to a time like boundary.

We now go back to the problem of finding the allowed \( \Omega \), assuming we have chosen a particular self adjoint extension. As remarked, for \( 2\kappa^2 < 5 + 6\kappa \), the potential is positive definite, and, therefore, \( \Omega^2 > 0 \) for all self adjoint extensions, corresponding to a stable evolution of arbitrary initial data. On the other hand for sufficiently large \( \kappa \) the potential is negative in some region, as can be explicitly checked by plotting \( \mathcal{V}_d(y) \) as a function of \( y \), using (24) and (25) as parametric representations of these functions. An example is given in figure 1. It can be seen that the region \( \mathcal{V}_d(y) < 0 \) is comparatively small, and shallow. We have analyzed numerically the possible existence of solutions with \( \Omega^2 < 0 \) and found that even for large values of \( \kappa \) these are possible only for \( \alpha > 0 \). For instance, for the example of figure 1, we find that there are solutions with \( \Omega^2 < 0 \) only for \( \alpha > 1.29 \ldots \). The existence of these solutions can be understood if we consider solving numerically (22), by imposing the appropriate boundary condition at \( y = 0 \), for some negative value of \( \Omega^2 \), and integrating towards \( y = y_0 \). Since the potential is smooth for \( y > 0 \), we expect to find a smooth solution for \( y > 0 \), such that for \( y = y_0 \) we have some finite values for the solution and its first derivative, and, therefore, some finite value of \( \alpha \). We shall not elaborate further on these type of solutions and remark only that there appear to be no solutions for \( \Omega^2 < 0 \) if we impose the Dirichlet boundary condition for \( y = y_0 \), and, therefore, the Case I perturbations are stable with respect to that boundary condition. In the next section we consider Case II.
Consider now Case II. In this case, in accordance with (14), given an arbitrary perturbation, since the functions $T_1$, $X_1$, and $Z_1$ are arbitrary, we may choose them such that $\tilde{F}_1 = 0$, $\tilde{F}_3 = 0$, and $\tilde{F}_4 = 0$. It should be clear that with this choice we remove all gauge ambiguity, because any further transformation would make at least one of these functions different from zero. If we make this choice, and assume that only $F_2(x)$, $F_5(x)$, $F_6(x)$, and, $F_7(x)$ are non zero, replacing in the linearized Einstein equations we obtain a coupled set of O.D.E.s for these functions such that by appropriate replacements we may express explicitly $F_5(x)$, $F_6(x)$, and, $F_7(x)$, as linear functions of $F_2$ and $dF_2/dx$, while $F_2$, in turn, satisfies a second order O.D.E. Unfortunately, the explicit forms of the coefficients that result, because of their length and complexity, are very difficult to handle. For this reason, we have chosen to analyze a different choice of gauge, that, although not free of gauge ambiguities, is much easier to handle, and eventually leads to gauge invariant results. In more detail, since the functions $T_1$, $X_1$, and $Z_1$ are arbitrary, it is clear that we can always choose them such that $\tilde{F}_5 = \tilde{F}_6 = \tilde{F}_7 = 0$. Thus, without loss of generality, we may restrict case II to a ‘diagonal form’, namely, we assume that only $F_1$, $F_2$, $F_3$, and $F_4$, are non vanishing. Once this choice is made, there appears to be no freedom left for further simplifications. It turns out, however, that the ‘diagonal form’ is not gauge invariant. This is because the system,

$$
0 = \frac{4i\Omega}{3|\Lambda|(1 + X^2)} X_1(X) - X^{p_1 + 2/3}(1 + X^2)^{1/3 - p_1/2} \frac{dT_1(X)}{dX}
$$

$$
0 = \frac{i k X^{p_1}}{(1 + X^2)^{p_1/2}} T_1(X) + \frac{i \Omega X^{p_2}}{(1 + X^2)^{p_2/2}} Z_1(X)
$$

$$
0 = -\frac{4ik}{3|\Lambda|(1 + X^2)} X_1(X) + \frac{X^{p_2 + 2/3}}{(1 + X^2)^{p_2/2 - 1/3}} \frac{dZ_1(X)}{dX}
$$

(35)

has the solution,

![Figure 1. $V_d(y)$ as a function of $y$ for $\kappa = 10$ and $k = 0.01$. Notice that only the region where $V_d(y) < 0$ is shown. The curve extends to the left, with positive values, up to $y = 0$ where it diverges to $+\infty$.](image-url)
\[ T_i(X) = -2i \Omega Q_i X^{p_i/2-p_{i/2}}(1 + X^{2})^{p_{i/4}-p_{i/4}} \]
\[ X_i(X) = \frac{3}{4} \Lambda (p_1 - p_2) Q_i X^{p_i/2-p_{i/2}}(1 + X^{2})^{p_{i/4}-p_{i/4}} \]
\[ Z_i(X) = 2k Q_i X^{p_i/2-p_{i/2}}(1 + X^{2})^{p_{i/4}-p_{i/4}} \]  
(36)

where \( Q_i \) is an arbitrary function of \( k \) and \( \Omega \). This implies, as can be easily checked, that the system,
\[ F_i(x) = -Q_i \left( \frac{16 \Omega^2 X^{p_i/2-p_{i/2}}}{3|\Lambda|(1+x^2)^{p_{i/4}-p_{i/4}}} + \frac{(p_1 - p_2)(3p_1 + 4x^2 + 2)x^{p_{i/4}+p_{i/4}+2/3}}{3(1+x^2)^{p_{i/4}+p_{i/4}+2/3}} \right) \]
\[ F_2(x) = \frac{Q_i(p_1 - p_2)(2 + 4x^2 + 3p_3)(1+x^2)^{p_{i/4}+2/3}}{x^{p_{i/4}+2/3}} \]
\[ F_3(x) = -Q_i \left( \frac{16k^2 \Omega^2 (1+x^2)^{p_{i/4}-p_{i/4}}}{3(1+x^2)^{p_{i/4}+p_{i/4}+2/3}} + \frac{(p_1 - p_2)(3p_2 + 4x^2 + 2)x^{p_{i/4}+p_{i/4}+2/3}}{3(1+x^2)^{p_{i/4}+p_{i/4}+2/3}} \right) \]
\[ F_4(x) = -Q_i(p_1 - p_2)(2 + 4x^2 + 3p_3)(1+x^2)^{p_{i/4}+2/3} \]
(37)

with \( F_3(x) = F_6(x) = F_7(x) = 0 \), and \( Q_i \) an arbitrary function of \( \Omega, k \), is a pure gauge solution of the perturbed Einstein equations.

Thus the ‘diagonal’ perturbations are not gauge invariant, because there are coordinate transformations that preserve the diagonal form. If we indicate with \( F_i(x) \) a particular representation and with \( \tilde{F}_i(x) \) the transformed perturbation, in accordance with (14) and (36) they are related by,
\[ \tilde{F}_i(x) = F_i(x) + \frac{16 \Omega^2 Q_i (1+x^2)^{p_{i/4}-p_{i/4}}}{3|\Lambda|(p_1 - p_2)x^{p_{i/4}-p_{i/4}}} + \frac{Q_i(4x^2 + 3p_2 + 2)x^{p_{i/4}+p_{i/4}+2/3}}{3(1+x^2)^{p_{i/4}+p_{i/4}+2/3}} \]
\[ \tilde{F}_2(x) = F_2(x) - \frac{Q_i(4x^2 + 3p_3)x^{p_{i/4}+p_{i/4}+2/3}}{3(1+x^2)^{p_{i/4}+p_{i/4}+2/3}} \]
\[ \tilde{F}_3(x) = F_3(x) + \frac{16kQ_i (1+x^2)^{p_{i/4}-p_{i/4}}}{3(p_1 - p_2)x^{p_{i/4}-p_{i/4}}} + \frac{Q_i(4x^2 + 3p_2 + 2)x^{p_{i/4}+p_{i/4}+2/3}}{3(1+x^2)^{p_{i/4}+p_{i/4}+2/3}} \]
\[ \tilde{F}_4(x) = F_4(x) + \frac{Q_i(4x^2 + 3p_3)x^{p_{i/4}+p_{i/4}+2/3}}{3(1+x^2)^{p_{i/4}+p_{i/4}+2/3}} \]  
(38)

This immediately implies that, although the functions \( F_i \) are not gauge invariant, we have that
\[ \tilde{F}_i(x) + \tilde{F}_i(x) = F_2(x) + F_4(x) \]  
(39)

and, therefore, \( F_2(x) + F_4(x) \) is a gauge invariant quantity. Similarly,
\[ G_{1d}(x) = F_2(x) - \frac{16(1+x^2)^{p_{i/2}+2/3} \Omega^2 F_2(x)}{|\Lambda|(p_1 - p_2)x^{p_{i/4}-p_{i/4}}(4x^2 + 2 + 3p_3)} - \frac{(4x^2 + 3p_1 + 2)F_2(x)}{(4x^2 + 2 + 3p_3)} \]
\[ G_{3d}(x) = F_2(x) - \frac{16(1+x^2)^{p_{i/2}+2/3}k^2 F_2(x)}{|\Lambda|(p_1 - p_2)x^{p_{i/4}-p_{i/4}}(4x^2 + 2 + 3p_3)} - \frac{(4x^2 + 3p_2 + 2)F_2(x)}{(4x^2 + 2 + 3p_3)} \]  
(40)
are also gauge invariant quantities. 

Replacing the diagonal form of the perturbations in Einstein’s equations, and expanding to first order in \( \epsilon \), we get a set of coupled second order ordinary differential equations for the functions \( F_i(x) \). A closer examination shows that for \( \Omega \neq 0 \), and \( z \neq 0 \), we have,

\[
F_2(x) = -F_0(x) \tag{41}
\]

while for \( F_1 \) and \( F_3 \) we find,

\[
\frac{dF_1}{dx} = -\frac{dF_3}{dx} = \frac{(3p_1 - 3p_2 + 6p_3 + 8x^2 + 4)}{6x(1 + x^2)} F_1 - \frac{(p_1 - p_2)}{2x(1 + x^2)} F_1
\]

\[
\frac{dF_3}{dx} = -\frac{dF_1}{dx} = \frac{(3p_1 - 3p_2 - 6p_3 - 8x^2 - 4)}{6x(1 + x^2)} F_1 + \frac{(p_1 - p_2)}{2x(1 + x^2)} F_1 \tag{42}
\]

We also find,

\[
\frac{dF_2}{dx} = \mathcal{R}_1 F_1 + \mathcal{R}_2 F_2 + \mathcal{R}_3 F_3 \tag{43}
\]

where,

\[
\mathcal{R}_1 = \frac{8x^2 - 2p_1 + 4)(p_1 - p_2)|\Lambda| + 16k^2x^{3/2} - p_1(1 + x^2)^{3/2} + p_1/2}{|\Lambda|x(4x^2 + 3p_3 + 2)(1 + x^2)}
\]

\[
\mathcal{R}_2 = \frac{(8x^2 - 2p_2 + 4)(p_1 - p_2)|\Lambda| + 16 \Omega^2 x^{3/2} - p_1(1 + x^2)^{3/2} + p_1/2}{|\Lambda|x(4x^2 + 3p_3 + 2)(1 + x^2)}
\]

\[
\mathcal{R}_3 = -\frac{32x^4 + (120p_3 + 32)x^2 + 45p_1p_2 + 44 + 60p_3}{12x(1 + x^2)(4x^2 + 3p_3 + 2)} \cdot \frac{4(1 + x^2)p_{12 - 1/2}x^{1/3} - p_3 \Omega^2}{|\Lambda|(4x^2 + 3p_3 + 2)} - \frac{4(1 + x^2)p_{12 - 1/2}x^{1/3} - p_3 \Omega^2}{|\Lambda|(4x^2 + 3p_3 + 2)} \tag{44}
\]

One can now show that any set of functions \((F_1, F_2, F_3)\) that satisfy (41)–(43) is a solution of the perturbation equations, and therefore, to solve those equations, all we need to do is to solve the system (41)–(43). A first step is to introduce two new functions, \( G_1(x) \) and \( G_2(x) \), defined by,

\[
G_1(x) = F_3(x) + F_0(x)
\]

\[
G_2(x) = F_0(x) - F_3(x) \tag{45}
\]

Now, using (42) it follows that,

\[
G_2 = \frac{6x(1 + x^2)}{2 + 4x^2 + 3p_3} \frac{dG_1}{dx} + \frac{2 + 4x^2 - 6p_1}{2 + 4x^2 + 3p_3} G_1 \tag{46}
\]

Then, from the definitions (45) we obtain,

\[
F_3 = \frac{3x(1 + x^2)}{2 + 4x^2 + 3p_3} \frac{dG_1}{dx} + \frac{8x^2 + 4 - 9p_1 - 3p_2}{8x^2 + 4 + 3p_3} G_1 \tag{47}
\]

\[
F_1 = -\frac{3x(1 + x^2)}{4x^2 + 2 + 3p_3} \frac{dG_1}{dx} + \frac{3p_1 - 3p_2}{8x^2 + 4 + 6p_3} G_1
\]
and, therefore, we have explicit expressions for $F_3$, $F_4$, (and $F_2 = -F_1$) in terms of $G_1$ and $dG_1/dx$.

We also find an expression $F_1$ in terms of $G_1$, which has the form,

$$F_1 = \mathcal{R}_4 \frac{d^2 G_1}{dx^2} + \mathcal{R}_5 \frac{d G_1}{dx} + \mathcal{R}_6 G_1 \quad (48)$$

where $\mathcal{R}_4$, $\mathcal{R}_5$, and $\mathcal{R}_6$ are functions of $x$ and the other parameters in the problem. Their explicit expression is given in (A.1)–(A.3) in the appendix. If we replace these expressions for the $F_i$ in the perturbation equations, we find that all the equations are satisfied, provided $G_1(x)$ is a solution of a third order O.D.E. of the form,

$$\frac{d^3 G_1}{dx^3} + \mathcal{R}_4 \frac{d^2 G_1}{dx^2} + \mathcal{R}_5 \frac{d G_1}{dx} + \mathcal{R}_6 G_1 = 0 \quad (49)$$

where the functions $\mathcal{R}_i$ depend on $x$, the metric parameters, $k$ and $\Omega$ in a complicated way. Again we refer to the appendix for their explicit expressions, given there by (A.4)–(A.6). Thus the problem of analyzing the perturbations considered here is reduced to that of finding the solutions of equation (49). But, because of the complicated form of the functions $\mathcal{R}_i$, this turns out to be a rather difficult task. We may, nevertheless, apply the previous results on gauge invariance, to achieve a certain simplification of the problem and take it to a form more suitable for its analysis. We notice that under a coordinate transformation of the form (36) we have,

$$\bar{G}_i(x) = G_i(x) + C W_i(x) \quad (50)$$

where $C$ is a constant, and,

$$W_i(x) = 16k^2(1 + x^2)^{p_1/4 - p_2/4} (p_1 - p_2) x^{p_2/2 - p_1/2} + \frac{|\Lambda|(8x^2 + 4 - 3p_1)x^{p_2/2 + p_1/2 - 4/3}}{(1 + x^2)^{p_1/4 + p_2/4 + 2/3}} \quad (51)$$

We remark that since $G_1(x)$ is already a first order quantity, the form (and the coefficients) of (49) are invariant under the transformation (40), and, therefore, both $G_1(x)$ and $\bar{G}_1(x)$ must be solutions of (49), which implies, as can explicitly be checked, that $W_i(x)$ is also a solution of (49). In fact, it is a pure gauge solution of (49), and, therefore, corresponds to a pure gauge solution of the perturbation equations. But, if we go back to (50), it is clear that,

$$\frac{d}{dx} \left( \frac{\bar{G}_i(x)}{W_i(x)} \right) = \frac{d}{dx} \left( \frac{G_i(x)}{W_i(x)} \right) \quad (52)$$

and, therefore, $d(G_i/W_i)/dx$ is gauge invariant. If we write the general solution of (49) in the form,

$$G_i(x) = W_i(x) W_2(x) + C W_i(x) \quad (53)$$

replacing in (49) we find that $W_2(x)$ satisfies an equation of the form,

$$\frac{d^3 W_2}{dx^3} + \mathcal{R}_{10} \frac{d^2 W_2}{dx^2} + \mathcal{R}_{11} \frac{d W_2}{dx} = 0 \quad (54)$$

where,

$$\mathcal{R}_{10} = \frac{3}{W_i} \frac{d W_i}{dx} + \mathcal{R}_7$$

$$\mathcal{R}_{11} = \frac{3}{W_i} \frac{d^2 W_i}{dx^2} + \frac{2}{W_i} \frac{d W_i}{dx} \mathcal{R}_7 + \mathcal{R}_8 \quad (55)$$
This implies that the function,

\[ W_3(x) = \frac{dW_2}{dx} \tag{56} \]

which, on account of (52) is gauge invariant, and, therefore, contains the non trivial part of the perturbation, satisfies a second order O.D.E. The main difficulty in the analysis of this equation is due to the presence in the coefficients of (54) of powers of \( x \) (and of \( 1 + x^2 \)) that are rational functions of \( \kappa \) that do not combine in a simple way. Nevertheless, we may show (details are given in the appendix) that in general the equation for \( W_3(x) \) may be written in the form,

\[ -\frac{|\Lambda|^\frac{n}{3} + 2/3(1 + x^2)^{4/3 - \rho/2}}{4} \frac{d^2W_3}{dx^2} + \mathcal{R}_{12}\frac{dW_3}{dx} + \mathcal{R}_{13}W_3 = \Omega^2 W_3 \tag{57} \]

where the functions \( \mathcal{R}_{12} \) and \( \mathcal{R}_{13} \) depend on \( x, \kappa, |\Lambda|, \) and \( k \), but are independent of \( \Omega \). But, since we expect the solutions to satisfy some restrictions in order to be acceptable as perturbations, for instance in their behaviour for both \( x \to 0 \) and \( x \to \infty \), it should be clear that once these restrictions are imposed, (57) may be taken as a boundary value—eigenvalue problem that determines the acceptable values of \( \Omega \). We may then try applying known techniques, such those used in [8], to find the allowed spectrum for \( \Omega \), and look for possible instabilities, signaled by a negative value of \( \Omega^2 \). The first step would be to find appropriate boundary conditions, such that the resulting functions are acceptable as perturbations. However, because of the complicated form of the equation satisfied by \( W_3 \), it turns out that it is somewhat simpler to proceed directly to an analysis of its solutions and then relate those solutions to the general problem. Just as in Case I, this can be made to look like a quantum mechanical problem for a particle in a one dimensional potential by introducing a new variable \( y \), and a new function, \( W_4(y) \), such that,

\[ W_3(x) = \mathcal{K}(x)W_4(y(x)) \tag{58} \]

where \( \mathcal{K}(x) \) is an ‘integration factor’, determined as follows. We first write (57) in the form,

\[ \frac{d^2W_3}{dx^2} + q_3(x)\frac{dW_3}{dx} + q_4(x)W_3 + q_5(x) \Omega^2 W_3 = 0 \tag{59} \]

where \( q_3 = -q_1\mathcal{R}_{12}, q_4 = -q_1\mathcal{R}_{13}, \) and,

\[ q_5(x) = -\frac{4}{3|\Lambda|^\frac{n}{3} + 2/3(1 + x^2)^{4/3 - \rho/2}} \tag{60} \]

Replacing (58) in (59) we get,

\[ \frac{-d^2W_4}{dy^2} - \left( \frac{dy}{dx} \right)^{-1}\frac{dK}{dx} \left( \frac{dy}{dx} \right)^{-1} + \left( \frac{dy}{dx} \right)^{-1}\frac{d^2y}{dx^2} + q_5 \right) \frac{dW_4}{dy} = \Omega^2 q_1 \left( \frac{dy}{dx} \right)^{-1} \frac{dW_4}{dy} \]

\[ = \Omega^2 q_1 \left( \frac{dy}{dx} \right)^{-1} \frac{dW_4}{dy} \tag{61} \]

If we choose now \( y(x) \) as a solution of,
\[
\frac{dy}{dx} = \sqrt{\frac{1}{\Lambda} |q(y)}
\]
\[
= \frac{2}{\sqrt{3} x^{1/2} x^{1/3} (1 + x^2)^{2/3} - p_1/p_2^1},
\]
(62)

(which is identical to (20)), and impose the condition that \( K \) is a solution of the equation,
\[
2 \frac{dK}{K} \frac{dx}{dx} = - \left( \frac{dy}{dx} \right)^{-1} \frac{d^2 y}{dx^2} - q_3
\]
(63)

so that the coefficient of \( dW_2/\dy \) in (61) vanishes, replacing in (61) we finally find that \( W_4(\gamma) \) satisfies an equation of the form,
\[
\mathcal{H} W_2 = \frac{\Omega^2}{\Lambda} W_4
\]
(64)

where,
\[
\mathcal{H} = - \frac{d^2}{dy^2} + V_1(y)
\]
(65)

and
\[
V_1(y) = - \frac{1}{\kappa} \left( \frac{dy}{dx} \right)^{-2} \left( \frac{d^2 y}{dx^2} + q_3 \frac{dy}{dx} + q_4 K \right)
\]
(66)

Actually, as will be clear in what follows, \( V_1 \) is readily obtained only as a function of \( x \), but it is implicitly dependent on \( y \) through the function \( x(y) \) obtained by inverting \( y(x) \). The main difficulty here is that, unfortunately, for general values of \( \kappa \) the function \( q_3(x) \) is rather complicated and we could not find explicit expressions for \( \kappa(x) \). Nevertheless one can check that with the choice (62) of \( y(x) \) the right hand side of (63) is regular in \( 0 < x < \infty \), and therefore, the solution \( \kappa(x) \) is also regular in \( 0 < x < \infty \). Moreover, if we assume that \( \kappa(x) \) satisfies (63), replacing in (66) we obtain,
\[
V_1(y) = \frac{1}{4} \left( \frac{dy}{dx} \right)^{-4} \left( 2 \frac{dy}{dx} \frac{d^2 y}{dx^2} - 3 \left( \frac{d^2 y}{dx^2} \right)^2 \right) + \frac{1}{4} \left( \frac{dy}{dx} \right)^{-2} \left( q_3^2 + 2 \frac{dq_3}{dx} - 4 q_4 \right)
\]
(67)

Notice that an important consequence of this result is that, besides \( q_3, q_4, \) and \( dy/dx \) (given by (62)), no explicit knowledge of \( \kappa, \) or of \( y(x) \), is required to construct an explicit form of \( V_1 \) as a function of \( x \). It is, again, understood that \( x \) should be given as function \( y \) by inverting (62), but, as we will see in what follows, the \( x \)-dependent form (67) will be all we need to obtain the general properties of \( V_1(y) \), and, therefore, of the solutions of (64). One further important property that results from the explicit \( x \)-dependent form of \( V_1 \) is that, besides \( \kappa \), it depends only on \( k/\sqrt{\Lambda} \). Therefore, if we introduce,
\[
\tilde{\Omega} = \Omega \frac{1}{\sqrt{\Lambda}},
\]
\[
\tilde{k} = k/\sqrt{\Lambda},
\]
(68)

then (65) takes the form
\[
-d^2W_4 \frac{dy^2}{\Omega^2} + V_{\Omega}(\kappa, \Omega, y)W_4 = \Omega^2 W_4
\] (69)

which contains no explicit reference to $|\Lambda|$. Thus, for fixed $\kappa$, the effect of changing $|\Lambda|$ in the unperturbed metric is just a rescaling of the parameters $k$, and $\Omega$ in the solutions of the perturbation equations.

Going back to (64), we notice that it has the familiar form of a (one dimensional) Schrödinger equation, where $V_{\Omega}(y)$ is the potential. Therefore, the central problem of the present analysis is then to find, if they exist, appropriate self adjoint extensions of $\mathcal{H}$, where by appropriate we mean that the resulting solutions are acceptable as perturbations, and that any acceptable perturbation can, in principle, be expanded using the solutions, which, in turn, provides a solution of the evolution equations. We will elaborate further on this point in what follows.

We recall at this point that on account of (4), $p_1$ has the range $-p_1 \leq y_0 \leq 0\,,$ and, therefore, considering (62), we see that $y$ is a monotonic function of $x$, such that with an appropriate choice of an integration constant, corresponding to the range $0 < x < \infty$, we have $0 < y < y_0$, where $y_0$ is given by

\[
y_0 = \int^{\infty}_0 \frac{2}{\sqrt{3} x^{p_1/2} + 1/3 (1 + x^{1/2} y_0^{1/2} y_0^{1/2})} dx
\] (70)

and, therefore, is finite.

4.1. Some properties of $V_{\Omega}(y)$

As already remarked we could not find closed form expressions for $K(x)$ for general values of $\kappa$. We notice, however, that to analyze the possible behaviour of the solutions of (64) at the boundaries ($y = 0$, and $y = y_0$), all that is required are adequate expansions of $\gamma(x)$, and of $K(x)$. Let us consider first the boundary $y = 0$. In accordance with (62), (and a similar analysis carried out for Case I), near $x = 0$ we have,

\[
x \approx \left( \frac{\sqrt{3} (4 - 3p_1) y_0}{12} \right)^{6/(4 - 3p_1)}
\] (71)

A simple computation shows that near $x = 0$, to leading order we have,

\[
V_{\Omega} \approx -\frac{(4 - 3p_1)^2}{192} x^{p_1 - 4/3}
\] (72)

and, therefore, near $y = 0$, again to leading order, we have,

\[
V_{\Omega}(y) \approx -\frac{1}{4y^2}
\] (73)

Thus, near $y = 0$, the behaviour of the potential $V_{\Omega}(y)$ is independent of $\kappa$ and $k$. The general solution $W_4(y)$ of (64) for a potential of the form (73) takes the form,

\[
W_4(y) \approx C_1 y + C_2 \ln(y) y
\] (74)

where $C_1$ and $C_2$ are arbitrary constants. It is well known that all that is required in this case to obtain a self adjoint extension (at least as regards the boundary $y = 0$), is to keep to ratio $C_1/C_2$ at some fixed value for all the solutions of (64). The situation is similar to that found and discussed in our previous analysis of the perturbations of the Levi–Civita space times [8]. For similar reasons as those considered there we shall impose $C_2 = 0$, as our boundary
condition. We refer the reader to [8] for further details and references. We only remark that setting $C_4 = 0$ is the most restrictive condition as regards possible eigenvalues $\Omega^2 < 0$, as can be seen by considering the situation for a pure $-1/(4y^2)$ potential.

The other important limit is for $y \to y_0$, $x \to \infty$. Using again the explicit form of $V_1$ as a function of $x$, we find, to leading order for $x \to \infty$,

$$V_1 \approx \frac{3(1 + \kappa + \kappa^2)k^6 + 2\kappa^2(\kappa + 2)|\Lambda|^\frac{3}{2}}{3(1 + \kappa + \kappa^2)k^4|\Lambda|} + O(x^{-2/3}) \tag{75}$$

or, in terms of $y$,

$$V_1(y) \approx \frac{3(1 + \kappa + \kappa^2)k^6 + 2\kappa^2(\kappa + 2)|\Lambda|^\frac{3}{2}}{3(1 + \kappa + \kappa^2)k^4|\Lambda|} + O(y_0 - y) \tag{76}$$

Therefore, the potential $V_1(y)$ is finite and regular at $y = y_0$. The general solution $W_4(y)$ is then, to leading order, of the form,

$$W_4(y) \approx C_1 + C_2(y_0 - y) \tag{77}$$

where $C_1$ and $C_2$ are arbitrary constants. This corresponds also, as for $y = 0$, to the circle limit case for self-adjointness. However, in this case we do not have a regularity criterion that we can apply, and, in principle, all fixed ratios $C_2/C_1$ are acceptable, introducing an essential ambiguity in the evolution of the perturbations. As already discussed, this can be traced to the fact that the boundary $x \to \infty$ is time like, as discussed in [10]. Nevertheless, if we assume that a criterion for choosing a particular value of the ratio $C_2/C_1$ can be established, and that that choice leads to a well defined evolution of the perturbations, the question still remains as to which, if any, of the possible choices leads to a stable evolution. For this reason, in what follows we will leave the choice of the value of $C_2/C_1$ open, and proceed to the general analysis of the solutions of the perturbation equations. We notice, on this regards, that (62) and (67) provide a parametric (in terms of $x$) expression for $V_1(y)$. We may use this fact to gain some insight on the nature of the solutions of (64), by plotting $V_1(y(x))$ as a function of $y(x)$, evaluated numerically, using,

$$y(x) = \int_0^x \frac{2}{\sqrt{3} x_1^{p_1/2 + 1/3} (1 + x_1^{2/3 - p_1/4})^{1/4}} \, dx_1 \tag{78}$$

Some examples are given in figure 2 for several values of $\kappa$ and $k = 1$. Recall that the range of $y$ depends on $\kappa$, and that in all cases we have $V_1(y(x)) \sim -1/(4y^2)$ sufficiently close to $y = 0$. The presence of a ‘dip’ region, away from $y = 0$, where the potential has a negative minimum is noticeable in all cases. We must remark here that, just as in Case I, it should be clear that, for a given boundary condition for $y = 0$, the most restrictive condition as regards the possibility of eigenvalues $\Omega^2 < 0$, is obtained setting $C_1 = 0$, that is, imposing that the solutions vanish for $y = y_0$.

5. The spectrum of $\mathcal{H}$

Once we make appropriate choices of the boundary conditions at $y = 0$ and $y = y_0$, the operator $\mathcal{H}$ in (64) becomes self-adjoint and we are assured of the existence of a complete spectrum of eigenvalues and eigenfunctions that can be used to generate the evolution of arbitrary initial data for the perturbations. The details are entirely similar to those analyzed in [8], and will not be repeated here. The important point is that the resulting evolution will be stable only if
there are no eigenvalues with $\Omega^2 < 0$, and, therefore, we will concentrate on finding, if they exist, those eigenvalues and eigenfunctions. If we try to find those solutions, either exact or numerically, directly from (64), we are faced with the difficulty that $V_1$ is known only implicitly as function of $y$. For this reason, and considering that there is a one-to-one correspondence between the solutions of (64), and those of (59), we shall analyze the solutions of the latter instead of those of (64). As a first step, we need to translate the boundary conditions for (64) into those corresponding to (59). To achieve this we would need to solve (63) to obtain $K$, but, fortunately, we only need its behaviour near the boundaries.

Using (63), and expanding $y(x)$ and $q_3$ to leading order near $x = 0$, after some computations we find,

\[ K(x) \simeq x^{2(1+\kappa + \kappa^2)} \]

(79)

We also have $y \propto x^{1+\kappa + \kappa^2}$, so that we finally find that near $x = 0$ we should have,

\[ W_3(x) \simeq x^{1+\kappa + \kappa^2}(C_1 + C_2 \ln(x)) \]

(80)

where $C_1$ and $C_2$ are arbitrary constants. In accordance with our previous discussion, we must impose the condition $C_2 = 0$.

For the boundary at $x \to \infty$ we have,

\[ K(x) \simeq x^{-5/3} + \frac{(p_2 - p_1)}{2k^2}x^{-7/3} \]

(81)

Since $x \to \infty$ we have $y \simeq y_0 - \sqrt{3}x^{-2/3}$, we find that in that limit we should have,

\[ W_3(x) \simeq C_1x^{-5/3} + C_2x^{-7/3} \]

(82)

where $C_1$ and $C_2$ are arbitrary constants. Notice that the case $C_1 = 0$, $C_2 = 0$ corresponds to $W_3(y)$ approaching a non zero constant, with $dW_3/dy = 0$, while for $C_1 = 0$, $C_2 \neq 0$, we have $W_3(y) = 0$, with $dW_3/dy \neq 0$ and finite.

---

**Figure 2.** $\mathcal{V}_I(y(x))$ as a function of $y$ for several values of $\kappa$ and $k = 1$. The dashed curve (leftmost) corresponds to $\kappa = 0.5$. The dotted curve (center) to $\kappa = 1$, and the solid curve (rightmost) to $\kappa = 2$. Notice the presence of a 'dip' region in all cases, where the potential has a minimum away from $y = 0$. All the curves extend to $y = 0$, and diverge as $-1/(dy^2)$ sufficiently close to $y = 0$, although this is not shown in the plots for simplicity.
There are different procedures that one can use to analyze (59) numerically. We notice that, for a general procedure, near $x = 0$ the structure of the coefficients as functions of $x$ is rather complicated because of the presence of exponents that are rational functions of $\kappa$. On other hand, if we consider, for instance, the coefficients of the unperturbed metric, for $x \to \infty$, they all admit expansions of the form,

$$x^{n+2/3}(1 + x^2)^{1/3 - \rho/2} \sim x^{d/3} \sum_{j=0} q_j x^{-2j}$$

so, that, eventually, all relevant quantities in the problem can be expanded in (integer) powers $x^{1/3}$. In particular, it is not very difficult, although too lengthy to be displayed here, to show that all the coefficients in (59) admit expansions similar to (83) for sufficiently large $x$. On this account, one can show that, for sufficiently large $x$, the general solution of (59) admits an expansion of the form,

$$W_3(x) \simeq C_1 \left( x^{-5/3} + \sum_{j=4} q_j x^{-(2j+1)/3} \right) + C_2 \left( x^{-7/3} + \sum_{j=4} b_j x^{-(2j+1)/3} \right)$$

(84)

where $C_1$ and $C_2$ are arbitrary constants, and the coefficients $q_j$ and $b_j$ are functions of $\kappa$, $\Omega$, $|\Lambda|$, and $k$ that can be readily computed by replacing in (59) and expanding in powers of $x$ for $x \to \infty$. Notice that regarding the boundary conditions for $y = y_0$, if we define $\alpha$ by,

$$\alpha = -\left( W_3 \left( \frac{dW_3}{dy} \right) \right)^{-1} \bigg|_{y=y_0}$$

(85)

then, we have,

$$\alpha = -\frac{\sqrt{3} C_1}{C_2}$$

(86)

and, therefore, the ratio of the constants $C_1/C_2$ controls the boundary value imposed at $y = y_0$.

Considering the general form of the potential, and the fact that the range of $y$ is finite, we can see that the most restrictive boundary condition for the existence of negative eigenvalues is $W_3(y_0) = 0$, (with $dW_3/dy|_{y=y_0} = 0$). This condition corresponds to $C_1 = 0$, $C_2 \neq 0$.

We remark once again that we are only interested in finding whether, after imposing boundary condition that imply that $\mathcal{H}$ in (64) is self adjoint, its spectrum contains negative eigenvalues. This amounts to finding solutions of (59) for negative $\Omega^2$, after imposing the corresponding boundary conditions. Notice that, since the range of $y$ is finite, the spectrum is fully discrete, but, because of the complicated form of the coefficients of (59) this search can only be carried out numerically. There are different possibilities here. We have chosen a ‘shooting’ method, where, using the expansion (84), we first obtain an approximation to $W_3(x)$ and $dW_3/dx$ for a sufficiently large value of $x$, and then integrate numerically towards $x = 0$. In more detail, after fixing the values of $\kappa$, $k$, $C_1$, and $C_2$, (recall that we can take $|\Lambda| = 1$, without loss of generality), we give a tentative value for $\Omega^2$, and integrate numerically towards $x = 0$. Since for arbitrary values of $\Omega^2$ the corresponding solution will behave near $x = 0$ as in (80), a sufficiently accurate integration should lead to a solution that approaches zero in all cases. For this reason we look for solutions such that,
where $C$ is some constant, since, as should be clear from (80), this corresponds to solutions where the terms in $\ln(x)$ vanish. Some particular examples are shown figures 3–6. In all cases we have set $C_1 = 0$, as the most restrictive condition. In figure 3 we show a plot of $W_3(x)x^{-\kappa+1/2+\nu}$ for $\kappa = 1$, and $k = 1$, for approximate solutions for the two lowest eigenvalues. Numerically we find that increasing or decreasing $\Omega^2$ slightly from the indicated values causes the curve to diverge either towards $+\infty$ or $-\infty$ as we approach $x = 0$, which we interpret, as indicated above, as indicative of the existence of an eigenvalue and corresponding eigenfunction for the type of boundary conditions imposed on the problem. Mostly for completeness we also show a plot of the corresponding solutions $W_3(x)$ in figure 4. We have explored numerically a range of values of the parameter $\kappa$ an in all cases we have found that the lowest eigenvalue is negative, although we have no proof that this is the case in general for extremely large or small values of $\kappa$. Examples of these lowest eigenvalues for different $\kappa$ are given in figures 5 and 6. We also remark in all cases analyzed the lowest eigenvalue is negative, but the next to lowest is positive, a situation that resembles that found in [8].

6. The limit $k = 0$ (radial perturbations)

In this section we consider the limit $k = 0$. This corresponds to purely radial perturbations that are independent of $z$, and, therefore, preserve the cylindrical symmetry of the unperturbed metric. One can check that the linearized Einstein equations imply that $H_1 = H_5 = H_6 = 0$. If we consider now the linearized equations for the $F_i$, we find that in general we must have,

$$F_7(x) = -\frac{i}{\Omega} \frac{dF_6(x)}{dx} + \frac{i(3p_2 + 2 + 4x^2)}{3\Omega x(1 + x^2)} F_6(x)$$

Without loss of generality we may write,

$$F_6(x) = \frac{i\Omega x^{3/2+p_1}}{(1 + x^2)p_1/2 - 1/3} Z_5(x)$$

and this implies,

$$F_7(x) = \frac{x^{3/2+p_1}}{(1 + x^2)p_1/2 - 1/3} \frac{dZ_5(x)}{dx}$$

Then, in accordance with (14), for $k = 0$, with an appropriate choice of $Z_5(x)$, we may set $F_6(x) = F_7(x) = 0$. This still leaves free the choice of $T_1(x)$ and $X_1(x)$. Considering again (14), for $k = 0$, we conclude that we may always choose a gauge where we also set $F_6(x) = F_5(x) = 0$. Notice that $F_1(x)$ is still defined up to an additive constant. This is, of course, consistent with the resulting equations of motion. In detail we find, as independent equations,

$$\frac{dF_i}{dx} = \frac{(3p_2 - 4)(3p_1 - 4) - 32x^4 - (24p_1 + 24p_2 + 32)x^2}{6x(1 + x^2)(3p_1 - 4 - 8x^2)} F_2 + \frac{16x^{4/3-p_1}(1 + x^2)p_1/2 + 2/3 \Omega^2 + |\Lambda| (8x^2 + 4 - 3p_2)(p_1 - p_2)}{x(1 + x^2)(3p_1 - 4 - 8x^2) |\Lambda|} F_9$$

(91)
and, \( F_3 \) are rather complicated functions of \( x \), that depend on \( \kappa \), but are independent of \( \Omega \).

It is now straightforward to solve (92) for \( F_2 \), replace in (93), and obtain a second order equation for \( F_3 \) of the form,
Figure 5. $W_\nu(x)/x^{1+\kappa}$ as a function of $x$ for the lowest eigenvalue for $k = 1$ and two different values of $\kappa$. The thick line curve corresponds to $\kappa = 0.5$ and $\Omega^2 = -1.2177...$, while the thin line curve corresponds to $\kappa = 3$, and $\Omega^2 = -0.9097...$. The functions are not normalized.

Figure 6. $W_3(x)$ as a function of $x$ for the lowest eigenvalue for $k = 1$ and two different values of $\kappa$. The thick line curve corresponds to $\kappa = 0.5$ and $\Omega^2 = -1.2177...$, while the thin line curve corresponds to $\kappa = 3$, and $\Omega^2 = -0.9097...$. The functions are not normalized.
\[ \frac{d^2 F_3}{dx^2} + Q_4(x) \frac{dF_3}{dx} + \Omega^2 Q_5(x) F_3 + Q_6(x) F_3 = 0 \]  
\tag{94}

where,

\[ Q_4(x) = [24(1 + \kappa + \kappa^2)x^6 + 2(1 + \kappa + \kappa^2)(4\kappa^2 - 2\kappa + 19)x^4 \\
+ (21 - 15\kappa - 42\kappa^2 - 36\kappa^3)x^2 + 9 + 9\kappa] \\
\times [x(1 + \kappa^2)(2x^2(1 + \kappa + \kappa^2) + 3 + 2\kappa)(4x^2(1 + \kappa + \kappa^2) + 3)]^{-1} \]

\[ Q_5 = \frac{4x}{3|\Lambda|} \left[ \frac{2(1 + \kappa^2)}{1 + x^2} \right] \]

\[ Q_6 = \frac{24(\kappa + 2)(2\kappa + 1)|\kappa|}{(1 + x^2)(2(1 + \kappa + \kappa^2)x^2 + 3 + 3\kappa)(4(1 + \kappa + \kappa^2)x^2 + 3)} \]  
\tag{95}

We can use now the same techniques as in the previous sections, and define,

\[ F_5(x) = K(x) W(y(x)) \]  
\tag{96}

where \( y(x) \) satisfies (62), and

\[ K(x) = \frac{(4(1 + \kappa + \kappa^2)x^2 + 3)x}{2(1 + \kappa + \kappa^2) + 3 + 3\kappa} \]  
\tag{97}

and, one can check that this implies that \( W(y) \) must satisfy the equation,

\[ -\frac{d^2 W}{dx^2} + \mathcal{V}_5(y) W(y) = \frac{\Omega^2}{|\Lambda|} W(y) \]  
\tag{98}

where,

\[ \mathcal{V}_5(y) = [512\mu^2x^8 + 384(2\mu + 1)\mu^3x^6 - 144(16\mu - 15)\mu^2x^4 \\
- 432\mu(8\mu^2 - 7\mu + 1)x^2 - 81] \left[ \frac{x^2(1 + x^2)}{48\mu^2(4\mu^2x^2 + 3)^2} \right] \]  
\tag{99}

with \( \mu = 1 + \kappa + \kappa^2 \).

Again, as in previous sections, we only have a rather complicated expression for \( \mathcal{V}_5(y) \) as a function of \( x \), and only implicitly, as a function of \( y \). This still allows for a parametric representation that we can use to obtain plots of \( \mathcal{V}_5(y) \) as a function of \( y \), as indicated in figure 7. Several important points are clearly visualized in that plot. One is the divergence to \(-\infty\) for \( y \to 0 \). Another is the divergence to \(+\infty\) for \( y \to y_0 \) and, equally important, the ‘deep’ region away from \( y = 0 \). We may, following similar steps as above, determine quantitatively the leading behaviour of \( \mathcal{V}_5(y) \) both for \( y \to 0 \), (that is, for \( x \to 0 \)), and for \( y \to y_0 \) (that is, for \( x \to \infty \)). Leaving aside details similar to those already considered, we find,

\[ \mathcal{V}_5(y) \approx -\frac{1}{4y^2} \quad \text{for } y \to 0 \]  
\tag{100}
which is similar to all the previous cases. However, for \( y \to y_0 \) (\( x \to \infty \)), we find,

\[
V_3'(y) \approx \frac{2}{(y_0 - y)^2} \quad ; \quad \text{for } y \to y_0
\]  

(101)

This immediately implies that the solutions of (98) behave as,

\[
W(y) \approx \frac{C_1}{y_0 - y} + C_2(y_0 - y) \quad ; \quad \text{for } y \to y_0
\]  

(102)

where \( C_1 \) and \( C_2 \) are constants, and, therefore, there is a *unique* self adjoint extension, as far as the boundary \( y = y_0 \) is concerned, corresponding in this case to imposing the condition \( C_1 = 0 \) on all solutions. We complete the self adjoint specification by imposing, as before, that \( W(y) \sim \sqrt{y} \) (no \( \ln(y) \) terms). We can again check that these conditions translate as the following boundary conditions on \( F_3(x) \):

\[
F_3(x) \sim C_3 \left[ 1 - \frac{\Omega^2 (1 + \kappa + \kappa^2)^2}{3|\Lambda|} \frac{2}{x^{1+\kappa+\kappa}} \right] \quad ; \quad \text{for } x \to 0
\]  

\[
F_3(x) \sim C_4 x^{\kappa-2} \quad ; \quad \text{for } x \to +\infty
\]  

(103)

where \( C_3 \) and \( C_4 \) are constants. It is again easy to obtain an asymptotic expansion for \( F_3(x) \) for \( x \to +\infty \), satisfying (103). The first terms are,

\[
F_3(x) \approx C_4 \left[ \frac{1}{x^{2}} - \frac{3 \Omega^2}{10|\Lambda| x^{10/3}} - \frac{(2\kappa^2 + 8\kappa + 5)}{4(1 + \kappa + \kappa^2)\Lambda^4} + \ldots \right]
\]  

(104)

We have used this expansion, together with a ‘shooting’ method to obtain numerical values for \( \Omega \) and solutions for \( F_3(x) \), corresponding to the lowest eigenvalues and eigenfunctions in the self adjoint extension of \( W(y) \). In all the cases analyzed, that correspond to a range of values of \( \kappa \), we find that lowest eigenvalue corresponds to \( \Omega^2 < 0 \), indicating the unexpected result that the radial perturbations become *unstable* in the presence of a negative cosmological constant. An example is given in figure 8.
7. A comment on the limit $\kappa = 0$ (the Bonnor metric)

An interesting limit of the Linet–Tian metric was analyzed by Bonnor, [3] where it was found that, for a negative cosmological constant, in the limit where the source vanishes, ($\kappa = 0$), one obtains a static, but cylindrically symmetric anti-de-Sitter universe. We may analyze linear perturbations of that metric along the lines used in this paper. Using the gauge properties already considered we may restrict to ‘diagonal’ perturbations that do not break the axial symmetry of the unperturbed metric. It is convenient in this case to write the metric (plus perturbations) in the form,

$$\begin{align*}
    \text{d}s^2 = (1 + |\Lambda|^2)^{2/3} \left( 1 + e^{i(|\Lambda| - k)z} F_0(x) \right) \text{d}t^2 + \frac{4(1 + e^{i(|\Lambda| - k)z} F_0(x))}{3(1 + |\Lambda|^2)} \text{d}x^2 \\
    + (1 + |\Lambda|^2)^{2/3} \left( 1 + e^{i(|\Lambda| - k)z} F_0(x) \right) \text{d}z^2 + \frac{x^2(1 + e^{i(|\Lambda| - k)z} F_0(x))}{(1 + |\Lambda|^2)^{1/3}} \text{d}\phi^2
\end{align*}$$

(105)

The linearized Einstein equations for this metric may be written as,

$$\begin{align*}
    F_0(x) &= -F_0(x) \\
    F_0(x) &= F_0(x) + Q
\end{align*}$$

(106)

$$\begin{align*}
    F_3(x) &= -\frac{k^2}{k^2 - \Omega^2} Q + \frac{(3 + 2|\Lambda|^2)(1 + |\Lambda|^2)^{2/3}}{2x(\Omega^2 - k^2)} \frac{\text{d}F_0}{\text{d}x} \\
    &- \frac{3x^2(k^2 - \Omega^2)(1 + |\Lambda|^2)^{1/3} + 2|\Lambda|^2 x^4 + 12|\Lambda|^2 x^2 + 9}{3x^2(k^2 - \Omega^2)(1 + |\Lambda|^2)^{1/3}} F_3(x)
\end{align*}$$

(107)

where $Q$ is an arbitrary constant, and,
\[
\frac{d^2 F_4}{dx^2} = \frac{(1 + 3|\Lambda|x^3) dF_4}{x(1 + |\Lambda|x^2)}\frac{4(8\Lambda^2x^2 - \Omega^2)(1 + |\Lambda|x^2)^{1/3} + 9 - 2|\Lambda|^2x^4 + 6|\Lambda|x^2)}{9x^2(1 + |\Lambda|x^2)^2} F_4(x) \tag{108}
\]

Therefore, every solution of (108) for \(F_4(x)\) provides a solution for the perturbation problem. To analyze the properties of these solutions we may introduce a new function and variable defined by,

\[F_4(x) = K(x)W_4(y(x))\tag{109}\]

where,

\[\frac{dy}{dx} = \frac{2}{\sqrt{3}(1 + |\Lambda|x^2)^{3/6}}\]

\[K(x) = \frac{1}{\sqrt{3}(1 + |\Lambda|x^2)^{1/12}}\tag{110}\]

and we find for \(W_4\) the equation,

\[-\frac{d^2 W_4}{dy^2} + \frac{16x^2(1 + |\Lambda|x^2)^{1/3}k^2 + 40|\Lambda|x^2 + 45}{16x^2(1 + |\Lambda|x^2)^{1/3}} W_4(y) = \Omega^2 W_4(y) \tag{111}\]

But we immediately see that the ‘potential’ in (111) is positive definite, and therefore, the metric is stable under this type of perturbations.

8. A comment on the hoop conjecture

In an interesting question regarding systems with full cylindrical symmetry, as in our case, is how the hoop conjecture manifest itself as a possible instability of the system. Here we need to remark that we are already dealing with a system that contains a naked singularity along the symmetry axis, and, therefore, it is not clear how one would assign either transverse or longitudinal radiuses to verify the conjecture. In fact, any finite section of the axis, if it could be isolated, would be in a critical situation, as its transverse radius might be thought as being zero. An important point regarding the results of our analysis is that the instabilities that we find are purely gravitational, independent of a matter contents, because they result from the presence of the ‘deep’ region of the effective potential, away from the symmetry axis. Since this happens in the vacuum region, we might expect that they would remain even after the singularity is smoothed out, either by the presence of a regular cylinder of matter, or by some other way of regularizing the metric close to the symmetry axis. In this regards it is interesting to consider the behaviour of ‘deep’ region as a function of \(k\), since \(k\) is inversely proportional to the wavelength of the perturbations in the \(z\)-direction. It turns out that as we increase the value of \(k\) (shorter wavelengths), (an example is shown in figure 9), the ‘deep’ becomes deeper and closer to the axis, so that it would be these shorter wavelengths that would be eliminated by a regularization, while the longer ones would still introduce instabilities, as, at least intuitively, expected from the hoop conjecture. In any case it would be interesting to construct explicit examples, but that is totally outside the context of the present research.

9. Final comments

In this paper we have analyzed the axial gravitational perturbations of the gravitational field of an infinite line source, in the presence of a negative cosmological constant. The analysis generalizes a previous one where similar perturbations were studied for the Levi–Civita
metric, that corresponds to the limit of vanishing cosmological constant. We derived the linearized equations of motion for the perturbations and showed that they can be separated into two sets, which we identified as Case I and Case II. Both sets contain, in principle, a certain degree of gauge ambiguity, which we analyzed in detail allowing us to find the corresponding gauge invariant parts. In both cases we obtained a ‘master variable’ and a corresponding ‘master equation’, in the form of a Schrödinger like equation. The main purpose and usefulness of these equations is that under appropriate boundary conditions they define a self adjoint operator, whose eigenvalues determine the frequencies of the perturbative modes, and whose eigenfunctions provide a complete set for the expansion and subsequent evolution of arbitrary perturbative initial data. But, because of the fact that for the unperturbed metric the constant $t$ hypersurfaces are not Cauchy surfaces, one needs to introduce, somewhat arbitrarily, appropriate boundary conditions at time like infinity in order to obtain well defined evolution equations for the perturbations. Thus, a central problem here was establishing those boundary conditions that fix the behaviour of the solutions at the axis of symmetry and at infinite distance from that axis. We found that in both cases, for the symmetry axis there is either a unique or a physically motivated choice. On the other hand, there is, within the context of the unperturbed metric, no unique natural or physical choice from an infinite set of possibilities. We have considered a set of choices, leading to a Robin type condition, that in some way reflects that idea that ‘nothing’ is added to the initial data in its subsequent evolution. A general feature that results is that for the assumed types of boundary conditions, equivalent to a Robin condition on the wave equation, the resulting evolution contains generically an unstable component. The main result of our analysis is then that the evolution will contain generically unstable components, and, therefore, that the space times considered here are gravitationally unstable. Since these space times contain a naked singularity, one would be tempted to ascribe the instability to the presence of that singularity. We remark, however, that here, as in the cases considered in [8, 11], and [12], the unstable mode is related to the form of a ‘potential’ away from the singularity, indicating the possibility that the instability might remain even after smoothing the (curvature) singularity by considering a radially extended source. In the present case, as in [8], one would have to consider a cylindrical regular source, such as an infinite cylinder (of finite radius) of some kind of matter. The problem in this construction is that the resulting system is considerably more complex than the vacuum case considered here, since,
asides from conditions that must be imposed at the matter—vacuum boundary, we must incorporate an equation of state for the matter, and the resulting appropriate boundary conditions. The question, nevertheless, is interesting, but outside the scope of the present research. We finally include a brief summary of the main results obtained in the paper,

- Case I (non diagonal perturbations): stability for Dirichlet boundary condition for \( x \to \infty \). Unstable solutions for Robin boundary conditions for \( \alpha > 1.29 \ldots \) and large \( \kappa \).
- Case II (diagonal perturbations): unstable solutions for all Robin boundary conditions, including Dirichlet and Neumann.
- Purely radial perturbations: unstable solutions for all Robin boundary conditions, including Dirichlet and Neumann.
- The Bonnor metric (\( \kappa \to 0 \) limit): stable under linear perturbations.

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Appendix

In this appendix we provide more detailed expressions for the functions \( \mathcal{R}_i(x) \) appearing in section 4. First we have,

\[
\mathcal{R}_4 = \frac{12(1+x^2)^2x^2|\Lambda|}{(4-3p_1+8x^2)(p_1-p_2)|\Lambda| + 16(1+x^2)^{2/3+p_1/2}x^{4/3+p_2}k^2} \quad (A.1)
\]

\[
\mathcal{R}_5 = \frac{x(1+x^2)}{(4x^2 + 2 + 3p_3)((4-3p_1+8x^2)(p_1-p_2)|\Lambda| + 16(1+x^2)^{2/3+p_1/2}x^{4/3+p_2}k^2)}
\times \left[ (80x^4 + (228p_3 + 56)x^2 + 27p_1^2 + 56 + 96p_3 + 45p_1p_2)|\Lambda| + 48(1+x^2)^{2/3+p_1/2}x^{4/3+p_2}k^2 \right] \quad (A.2)
\]

\[
\mathcal{R}_6 = \left[ (p_1 - p_2)(-128x^4 + (-48p_1 - 72p_2 - 128)x^2 + (3p_1 - 4)(3p_1 - 4 + 9p_2))|\Lambda| - 32(1+x^2)^{2/3+p_1/2}(4x^2 + 2 + 3p_3)x^{4/3+p_1}\Omega^2 + 48(1+x^2)^{2/3+p_1/2}(-p_2 + p_1)x^{4/3+p_1}\Omega^2 \right]
\times \left[ 2(4x^2 + 2 + 3p_3)((4-3p_1 + 8x^2)(p_2 - p_1)|\Lambda| + 16(1+x^2)^{2/3+p_1/2}x^{4/3+p_2}k^2)^{-1} \right] \quad (A.3)
\]

A straightforward computation then shows that,

\[
\mathcal{R}_7 = \left[ 3(p_1 + 2p_2)(p_1 - p_2)|\Lambda| + 16(1+x^2)^{2/3+p_1/2}x^{4/3+p_2}k^2 \right] \mathcal{R}_4
\]

\[
-4x|\Lambda|(1+x^2)(4x^2 + 2 + 3p_3)\mathcal{R}_5 \right]
\times \left[ 4(1+x^2)(4x^2 + 2 + 3p_3)x|\Lambda|\mathcal{R}_4 \right]^{-1}, \quad (A.4)
\]
\[ \mathcal{R}_8 = -\frac{1}{\mathcal{R}_4} \frac{d\mathcal{R}_5}{dx} \left[ \frac{3|\Lambda|p_1p_2 + 3|\Lambda|p_1^2 - 6|\Lambda|p_2^2 + 16(1 + x^2)^{2/3 + p_2/2}k x^{4/3 - p_2}k^2}{(4x^2 + 2 + 3p_3)4\mathcal{R}_4|\Lambda|(1 + x^2)} \right] \times \left[ |\Lambda| \left( 32x^4 - 24x^2p_3 + 32x^2 + 12p_1 + 9p_1^2 - 16 + 27p_2p_1 + 12p_2 + 36p_2^2 \right) - 48(1 + x^2)^{2/3 + p_2/2}k x^{4/3 - p_2}k^2 \right] \left[ 4\mathcal{R}_4|\Lambda|(1 + x^2)^{2/3 + p_2/2}k x^{4/3 - p_2}k^2 \right]^{-1} - \frac{\mathcal{R}_6}{\mathcal{R}_4}, \quad (A.5) \]

and,

\[ \mathcal{R}_9 = \left[ 32(1 + x^2)^{2/3 + p_2/2}(4x^2 + 2 + 3p_3)k x^{4/3 - p_2} \Omega^2 \right. \\
+ 8x|\Lambda|(1 + x^2)^2(4x^2 + 2 + 3p_3) \frac{d\mathcal{R}_6}{dx} \\
- 2(4x^2 + 2 + 3p_3) \left( 3(p_1 + 2p_2)(p_1 - p_2)|\Lambda| + 16(1 + x^2)^{2/3 + p_2/2}k x^{4/3 - p_2}k^2 \right) \mathcal{R}_6 \\
- 3(p_1 - p_2) \left( (8p_1 - 32 + 16p_2)x^2 - 32x^4 \right) \\
- 15p_1p_2 - 18p_2^2 - 3p_1^2 + 4p_1 + 8p_2 \right] |\Lambda| + 16(1 + x^2)^{2/3 + p_2/2}k x^{4/3 - p_2}k^2 \left. \right] \left[ 4\mathcal{R}_4|\Lambda|(1 + x^2)^{2/3 + p_2/2}k x^{4/3 - p_2}k^2 \right]^{-1} \quad (A.6) \]

To get (57) from (54) we first notice that \( \mathcal{R}_9 \) does not appear in either \( \mathcal{R}_{10} \) or \( \mathcal{R}_{11} \) in (55), and then, that in accordance with (A.1)–(A.5), \( \Omega \) appears only in \( \mathcal{R}_8 \), in the term \( -\mathcal{R}_6\mathcal{R}_4 \). This term can be written in the form,

\[ \frac{\mathcal{R}_6}{\mathcal{R}_4} = \left[ (128x^4 + (48p_1 + 72p_2 + 128)x^2 - (3p_1 - 4)(3p_1 - 4 + 9p_2))|\Lambda| \right. \\
- 48(1 + x^2)^{2/3 + p_2/2}k x^{4/3 - p_2}k^2 \left( \frac{p_2 - p_1}{24(4x^2 + 2 + 3p_3)(1 + x^2)^2 |\Lambda|} \right) \\
- \frac{4(1 + x^2)^{2/3 + p_2/2}k x^{4/3 - p_2}k^2}{3|\Lambda|} \Omega^2 \quad (A.7) \]

It is now straightforward to compute the remaining terms to get (57). We finally have,

\[ \mathcal{R}_{12} = -\frac{1}{q_1} \left[ \frac{3}{W_{i}} \frac{dW_{i}}{dx} + \mathcal{R}_7 \right] \]

\[ \mathcal{R}_{13} = -\frac{1}{q_1} \left[ \frac{3}{W_{i}} \frac{d^2W_{i}}{dx^2} + \frac{2\mathcal{R}_2}{W_{i}} \frac{dW_{i}}{dx} + \mathcal{R}_8 - \Omega^2 q_1 \right] \quad (A.8) \]

where \( q_1(x) \) is given by (60). It is clear from (A.5) and (A.7), that \( \mathcal{R}_{13} \) does not depend on \( \Omega \).

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