Bandlimited Field Sampling Using Mobile Sensors in the Absence of Location Information

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Abstract—Sampling of physical fields with mobile sensor is an emerging area. In this context, this work introduces and proposes solutions to a fundamental question: can a spatial field be estimated from samples taken at unknown sampling locations? Unknown sampling location, sample quantization, unknown bandwidth of the field, and presence of measurement-noise present difficulties in the process of field estimation. In this work, except for quantization, the other three issues will be tackled together in a mobile-sampling framework. Spatially bandlimited fields are considered. It is assumed that measurement-noise affected field samples are collected on spatial locations obtained from an unknown renewal process. That is, the samples are obtained on locations obtained from a renewal process, but the sampling locations and the renewal process distribution are unknown. In this unknown sampling location setup, it is shown that the mean-squared error in field estimation decreases as $O(1/n)$, where $n$ is the average number of samples collected by the mobile sensor. The average number of samples collected is determined by the inter-sample spacing distribution in the renewal process. An algorithm to ascertain spatial field’s bandwidth is detailed, which works with high probability as the average number of samples $n$ increases. This algorithm works in the same setup, i.e., in the presence of measurement-noise and unknown sampling locations.

Index Terms—Additive white noise, nonuniform sampling, signal reconstruction, signal sampling, wireless sensor networks

I. INTRODUCTION

Consider a mobile sensor which has to acquire a spatially smooth field by moving along a path or spatial trajectory [1], [2]. If the sensor is equipped with precise location information, high-precision quantizers, and negligible measurement-noise, then the field reconstruction process reduces to classical (noiseless) sampling and interpolation problems (see [3], [4], [5]). With precise location information, spatial field reconstruction or estimation has been addressed in the presence of quantization as well as measurement-noise (for example, see [6], [7], [8], [9], [10], [11]). In the context of spatial sampling with a mobile-sensor, a more challenging setup is when the location of samples collected is not known. Unknown sampling locations is a fundamentally new topic in spatial field sampling and will be the central theme of this work.

The motivation for mobile-sampling without the knowledge of locations is elucidated first. In practice, a device such as GPS (global positioning system) or other elaborate distributed localization mechanisms can be used to localize a sensor [12]. If a mobile-sensor’s path and its velocity are known, and if the mobile-sensor has an accurate clock, then the sampling locations can be calculated from sample timestamps [11]. However, all these elaborate mechanisms will add to the cost of mobile sensors, and increase its recording overhead. It would be desirable to get rid of the timestamps, the GPS, the distributed mechanisms for localization and the knowledge of velocity, and still reconstruct the spatial field to a desired accuracy. This is the core motivation behind the paper.

Spatially smooth, temporally fixed, and finite-support fields will be considered in this work. The smoothness of spatial field will be modeled by bandlimitedness. It will be assumed that the mobile sensor samples the field at locations obtained by an unknown renewal process. By unknown renewal process, we mean that the probability distribution of the inter-sample locations and even the locations at which field samples are obtained are not known. It will also be assumed that the field samples are affected by additive and independent noise with zero mean and finite variance. Except for independence, zero-mean, and finite variance, it is assumed that the noise distribution is also not known. In such a challenging setup, sampling rate (corresponding to oversampling) will be used in this work to decrease expected mean-squared error in field-reconstruction. In other words, the mobile sensor will collect a large number of readings on locations determined by an unknown renewal process and estimates will be developed in this work to drive down the expected mean-squared error in field-reconstruction with sampling rate.

The field sampling setup with a mobile sensor is illustrated in Fig. 1. To keep the analysis tractable, the spatial field is assumed to be one-dimensional in space and temporally fixed in this first exposition [1]. The mobile sensor collects the spatial field’s values at unknown locations $s_1, s_2, \ldots, s_m$ derived from a renewal process with unknown inter-sample distribution [13].

The spatial field measurements are affected by additive and independent noise process $W(x)$, which has an unknown distribution with zero-mean and finite variance. In this challenging setup, the goal is to estimate the field from the readings collected by the mobile sensor.

Unknown location information on samples, (low-precision) quantization, knowledge of the spatial field’s bandwidth, and presence of additive measurement-noise are four perils in the process of field estimation (or reconstruction). In this work, leaving quantization issues aside, the other three ailments will be addressed together in a mobile-sampling framework. At first, a reconstruction method is discussed where the bandwidth of the signal is known. Next, an algorithm will be outlined in this work which handles an unknown (but finite) bandwidth

1 It is desirable to extend the analysis to multidimensional fields which evolve with time. However, due to unknown sampling locations and measurement-noise, the setup is already very challenging. For this reason, only one-dimensional fields are considered in this work. It is fair to state that the current work will be applicable to slowly changing spatial fields.

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The mobile sampling scenario under study is illustrated. A mobile sensor, where the field is temporally fixed, collects the spatial field’s values at unknown locations $s_1, s_2, \ldots, s_m$. It is assumed that $s_1, s_2, \ldots, s_m$ are realized from an arrival process with an unknown renewal distribution to model the sensor’s nonuniform velocity. It is also assumed that the samples are affected by additive and independent noise process $W(x)$. Our task is to estimate $g(x)$ from the readings $g(s_1) + W(s_1), \ldots, g(s_m) + W(s_m)$.

The main results shown in this work are as follows:

1) For the field sampling setup illustrated in Fig. 1 and with some regularity conditions on the inter-sample location distribution, the expected mean-squared error in field estimation is upper bounded as $O(1/n)$, where $n$ is the sampling density (or $n$ is the expected number of samples realized by the renewal process in the interval of sampling). This result holds when zero-mean finite-variance additive noise is present during sampling, and when the sampling locations are obtained at unknown locations generated from an unknown renewal process.

2) To address unknown bandwidth, an algorithm will be presented to ascertain the correct bandwidth of the spatial field (with high probability) in the mobile-sensor sampling paradigm. This algorithm works in the presence of measurement-noise and when field samples are obtained at unknown locations generated from an unknown renewal process. The proposed algorithm requires the knowledge of mean and variance of squared-noise.

Prior art: The topic of unknown but uniformly distributed sampling locations was recently introduced in the context of spatial sampling in a finite interval. A detailed differentiation with this earlier work is needed, since the setup and results appear to be similar in nature. The estimates used are also the same in the two works (see [7]). If $n$ sensors are uniformly distributed in an interval, as in [14], then their ordered (noise-affected) readings can be used to obtain a bandlimited field estimate with mean-squared error of $O(1/n)$. The derivation of this result utilizes established facts on the mean-squared deviations of ordered uniformly distributed random variables [14], [15]. It is also known that ordered uniformly distributed random variables can be understood as realizations of a Poisson renewal process [15]. In contrast, in this current work the renewal process distribution is assumed to be unknown. This setup is much more challenging than the previous work, since the distribution properties are assumed to be unknown in the current work. As a result, the mean-squared error analysis is derived from first principles. The final result on mean-squared error analysis results in an upper bound of $O(1/n)$, which is the same as in prior work in spite of unknown distribution properties of the renewal process. One important aspect of result in this work is that both the estimate and the mean-squared error result are universal in nature (see [7] and Theorem 3.1). In summary, the sampling location’s distribution is assumed to be unknown, and is the major difference with past work; and, the problem addressed in the current work is more difficult!

Sampling and reconstruction of discrete-time bandlimited signals from samples taken at unknown locations was first studied by Marziliano and Vetterli [16]. This problem is addressed completely in a discrete-time setup and solutions are combinatorial in nature. A recovery algorithm for bandlimited signals from a finite number of ordered nonuniform samples at unknown sampling locations has been proposed by Browning [17]. This algorithm works in a deterministic setup. Estimation of periodic bandlimited signals, where samples are obtained at unknown locations obtained by a random perturbation of equi-spaced deterministic grid, has been studied by Nordio et al. [18]. More generally, the topic of sampling with jitter on the sampling locations [3],[19] Chap. 3.8 is well known in the literature.

This work is different from previous literature in the following non-trivial aspects—(i) the sampling locations are generated according to an unknown renewal process, where even the distribution defining the renewal process is unknown; and, (ii) the field is affected by zero-mean additive independent noise with finite variance, where the distribution of noise is also not known.

**Notation:** Spatial fields which are temporally fixed will be denoted by $g(x)$ and its variants. The $L^\infty$-norm of a field $g(x)$ will be denoted by $\|g\|_\infty$. The spatial derivative of $g(x)$ will be denoted by $g'(x)$. The number of (random) samples will be denoted by $M$, while $n$ will denote the expected value of $M$. Expectation will be denoted by $E$. The set of integers, real numbers, and complex numbers will be denoted by $\mathbb{Z}$, $\mathbb{R}$, and $\mathbb{C}$, respectively. Finally, $j = \sqrt{-1}$.

**Organization:** Spatial field’s model, the reconstruction distortion criterion, the sampling model using a mobile sensor, and the noise model on measurement (or field) is explained in Section II. The estimation of spatial field with known bandwidth and samples taken on an unknown renewal process is addressed in Section III. An algorithm for spatial field estimation with unknown bandwidth and samples taken on an unknown renewal process addressed in Section IV. Measurement-noise is considered both in Section III and Section IV. Simulation results are presented in Section V. Finally, conclusions will be presented in Section VI.
II. FIELD MODEL, DISTORTION, Sampling Process by Mobile Sensor, AND MEASUREMENT-NOISE MODEL

The models used for theoretical analysis presented in Section III and Section IV will be discussed in this section. Field models are discussed first.

A. Field model

It will be assumed that the field of interest is one-dimensional, temporally fixed, and spatially bandlimited. Let \( g(x) \) be the field, where \( x \in \mathbb{R} \) is the spatial dimension. For reasons stated in Sec. I, it will be assumed that the temporal variation of \( g(x) \) is negligible, i.e., \( g(x,t) \equiv g(x) \) during the sampling process. Temporally fixed assumption on the spatial field is suitable when the speed of the mobile sensor is much higher than the temporal rate of change of the field (also see I for discussions). Temporally fixed fields are more tractable to analysis, and this assumption will also help in understanding the effect of location- unawareness on the field sampling process in isolation.

Without loss of generality, it is assumed that \( |g(x)| \leq 1 \), \( g(x) \) is periodic and bandlimited with period equal to 1, and has a bandwidth \( 2\pi b \), where \( b \) is a known positive integer. All these assumptions imply that \( g(x) \) has a Fourier series in the interval \([0,1]\) with bounded Fourier series coefficients

\[
g(x) = \sum_{k=-b}^{b} a[k] \exp(j2\pi kx) \tag{1}
\]

with \( a[k] = \int_{0}^{1} g(x) \exp(-j2\pi kx) dx. \tag{2} \)

Based on Bernstein inequality \( [20] \), it follows that

\[
|g'(x)| \leq 2b\pi ||g||_\infty \leq 2b\pi. \tag{3}
\]

B. Distortion criterion

For a first exposition, a mean-squared error will be used as the distortion metric. If \( \hat{G}(x) \) is any estimate of the field \( g(x) \), then the distortion is defined as

\[
D := \mathbb{E} \left[ \int_{0}^{1} \left( \hat{G}(x) - g(x) \right)^2 dx \right] \nonumber
\]

\[
= \sum_{k=-b}^{b} \mathbb{E} \left[ |A[k] - a[k]|^2 \right] \tag{4}
\]

where \( A[k] \) is the Fourier series representation of \( \hat{G}(x) \).

C. Renewal process based sampling model

It will be assumed that \( X_1, X_2, \ldots \) be a renewal process which are the separations between the sampling locations of a mobile sensor.\(^4\) The mobile sensor traveling in the interval \([0,1]\) obtains samples at the locations \( S_1 := X_1, S_2 := (X_1 + X_2), \ldots, S_M := (X_1 + X_2 + \ldots + X_M) \), where \( M \) is the random number of samples from the renewal process that fall in the interval \([0,1]\). The random number of samples \( M \) that will be obtained in \([0,1]\) satisfies a stopping rule

\[
X_1 + X_2 + \ldots X_M \leq 1 \text{ and } X_1 + X_2 + \ldots X_{M+1} > 1
\]

and, therefore, is a well defined (measurable) random variable \( [21] \). Let \( X_1, X_2, X_3, \ldots \) have the common distribution \( X \). For analysis purposes, it will be assumed that

\[
0 < X \leq \frac{\lambda}{n} \text{ and } \mathbb{E}(X) = \frac{1}{n} \tag{5}
\]

where \( \lambda > 1 \) is a finite constant independent of \( n \). In other words, \( 0 < nX \leq \lambda \), or \( nX \) is a bounded random variable.

III. FIELD ESTIMATION FROM SAMPLES OBTAINED ON AN UNKNOWN RENEWAL PROCESS

In this section, the Fourier series of the periodic bandlimited field \( g(x) \) will be estimated by observing measurement-noise affected samples obtained on an unknown renewal process. As outlined in Section III and Section IV, the term unknown renewal process refers to unknown sampling locations and unknown inter-sample spacing distribution.

Recall that the sampling locations are defined as

\[
S_1 = X_1,
\]

\[
S_2 = X_1 + X_2 = S_1 + X_2,
\]

and \( S_M = X_1 + \ldots + X_M = S_{M-1} + X_M. \tag{6} \)

The sampling locations \( S_1, S_2, \ldots, S_M \) are unknown, as well as with unknown distribution. Noise-aﬀected field values \( g(S_1) + W(S_1), \ldots, g(S_M) + W(S_M) \) are available for the estimation of the field \( g(x) \). The key to our estimation procedure will be the field’s Fourier series coefficient estimate \( \hat{A}_{gen}[k] \) defined by

\[
\hat{A}_{gen}[k] := \frac{1}{M} \sum_{i=1}^{M} \{g(S_i) + W(S_i)\} \exp \left( -j2\pi ki \right) \tag{7}
\]

\(^4\)For a renewal process, each \( X_i > 0 \).
where \(-b \leq k \leq b\). This estimate works without the knowledge of \(S_1, \ldots, S_M\) since the noise-affected field values \(g(S_1) + W(S_1), \ldots, g(S_M) + W(S_M)\) are the samples recorded by the mobile sensor in our model. This formula is a Riemann-sum like approximation to the Fourier series integral formula in (2) with two assumptions: (i) the sample locations given by \(S_i, 1 \leq i \leq M\) are “near” the grid points \(i/M, 1 \leq i \leq M\); and (ii) the measurement-noise part in (2) averages out to “near-zero”. The effect of these assumptions is analyzed; and, in the estimation of Fourier series coefficients from \(\hat{A}_{\text{gen}}[k]\) (in the mean-squared sense) the following theorem is noted.

**Theorem 3.1:** Let \(\hat{A}_{\text{gen}}[k]\) be as defined in (2). Let \(X_i\) be i.i.d. positive (inter-sample spacing) random variables such that \(E(X_i) = 1/n\), and distribution of \(X_i\) has support in \((0, \lambda/n)\). Let \(W(x)\) be a measurement-noise process independent of \(X_i, i \in \mathbb{Z}\) with zero-mean and finite variance. Then

\[
E \left[ |\hat{A}_{\text{gen}}[k] - a[k]|^2 \right] \leq \frac{C}{n} \tag{8}
\]

where \(C\) is a constant that is independent of \(n\), and depends on renewal process parameter \(a\) and the signal bandwidth parameter \(b\). Correspondingly the distortion in (4) is bounded as \(D_{\text{gen}} \leq (2b + 1)C/n \) or \(D_{\text{gen}} = O(1/n)\).

**Proof:** To maintain the flow of the results, the key ideas and inequalities in the proof will be proved in this section while the technically detailed (mundane) statistical calculations will be presented in Appendix A and Appendix B.

The estimate in (7) has been designed with the assumption that \(g(S_1)\) has been observed at \(t = 1/m\), \(g(S_2)\) has been observed at \(t = 2/m\), and \(g(S_i)\) has been observed at \(t = i/m\) for various values of \(i\). This is our key statistical approximation. The mean-squared error in making this approximation will be analyzed next. The estimate \(\hat{A}_{\text{gen}}[k]\) consists of two conceptual parts

\[
\hat{A}_{\text{gen}}[k] = \frac{1}{M} \sum_{i=1}^{M} g(S_i) \exp \left( -j2\pi ki \frac{1}{M} \right) + \frac{1}{M} \sum_{i=1}^{M} W(S_i) \exp \left( -j2\pi ki \frac{1}{M} \right) \tag{9}
\]

where the first and second terms correspond to the signal and the measurement-noise part, respectively. These terms will be analyzed separately. Let

\[
A_R[k] := \frac{1}{M} \sum_{i=1}^{M} g(i/M) \exp(-j2\pi ki/M) \quad \tag{10}
\]

be the \(M\)-point Riemann approximation of \(a[k]\) in (3). The approximation \(A_R[k]\) is random due to presence of \(M\). Since \(|a_1 + a_2|^2 \leq 2|a_1|^2 + 2|a_2|^2\), therefore

\[
E \left[ |\hat{A}_{\text{gen}}[k] - a[k]|^2 \right] = 2E \left[ A[k] - a[k] \right]^2 + 2E \left[ |W_{\text{avg}}[k]|^2 \right] \tag{11}
\]

Next, from the triangle inequality (22),

\[
|\hat{A}[k] - a[k]| \leq |\hat{A}[k] - A_R[k]| + |A_R[k] - a[k]| \tag{12}
\]

and \(|a_1 + a_2|^2 \leq 2|a_1|^2 + 2|a_2|^2\), it follows that

\[
E \left[ |\hat{A}[k] - a[k]|^2 \right] \leq 2E \left[ |\hat{A}[k] - A_R[k]|^2 \right] + 2E \left[ |A_R[k] - a[k]|^2 \right] \tag{13}
\]

The term \(|A_R[k] - a[k]|\) will be bounded using the smoothness properties of \(g(x)\) and the mean-value theorem; and, the mean-squared value of \(|\hat{A}[k] - A_R[k]|\) will be upper-bounded using exchangepability of \(X_1, X_2, \ldots, X_M\) conditioned on the stopping time \((M+1)\) (20). The next parts are devoted to these analyses. First, \(|\hat{A}[k] - A_R[k]|\) is considered. Note that

\[
|\hat{A}[k] - A_R[k]|^2 \leq \frac{1}{M^2} \sum_{i=1}^{M} g(S_i) - g \left( \frac{i}{M} \right) \tag{14}
\]

where the last step uses the triangle inequality (22). Since \((a_1 + \ldots + a_m)^2 \leq m(a_1^2 + \ldots + a_m^2)\), for any real numbers \(a_1, \ldots, a_m \in \mathbb{R}\) and any integer \(m\), so

\[
|\hat{A}[k] - A_R[k]|^2 \leq \frac{1}{M^2} \sum_{i=1}^{M} |g(S_i) - g \left( \frac{i}{M} \right)|^2 \tag{15}
\]

where the last step uses \(|g(x_1) - g(x_2)| \leq |g'|_{\infty} |x_1 - x_2|\) for any \(x_1, x_2 \in [0, 1]\). Taking expectations in (17),

\[
E \left[ |\hat{A}[k] - A_R[k]|^2 \right] \leq |g'|_{\infty}^2 E \left[ \frac{1}{M} \sum_{i=1}^{M} |S_i - i/M|^2 \right] \tag{18}
\]

From Appendix A, it is noted that

\[
E \left[ \frac{1}{M} \sum_{i=1}^{M} |S_i - i/M|^2 \right] \leq \frac{C_1}{n} \tag{19}
\]

for any \(F\) and as \(n\) becomes large, and the constant \(C_1 > 0\) depends on \(\lambda\) and is independent of \(n\).

For \(|a[k] - A_R[k]|\), it is shown in Appendix B that

\[
|a[k] - A_R[k]| \leq \frac{C_2}{M} \tag{20}
\]

or \(E \left[ |a[k] - A_R[k]|^2 \right] \leq E \left( \frac{C_2}{M^2} \right)^2 \leq \frac{C_2 \lambda^2}{n^2}. \tag{21}\)

The constant \(C_2 > 0\) depends on the field’s bandwidth parameter \(b\) and is independent of \(n\).
Finally, the mean-squared value of \( W_{\text{avg}}[k] \) has to be characterized. For this part, note that

\[
\mathbb{E}(\|W_{\text{avg}}[k]\|^2) = \mathbb{E} \left( \frac{1}{M} \sum_{i=1}^{M} W(S_i) \exp \left( -\frac{j2\pi ki}{M} \right) \right)^2
\]

\[
= \mathbb{E} \left( \frac{1}{M} \sum_{i=1}^{M} W(S_i) \right)^2 \times \exp \left( -\frac{j2\pi ki}{M} \right) \exp \left( \frac{j2\pi ki}{M} \right)
\]

\[
\overset{(a)}{=} \mathbb{E} \left\{ \frac{1}{M^2} \sum_{i=1}^{M} W(S_i)^2 \right\}
\]

\[
= \mathbb{E} \left[ \frac{\sigma^2}{M} \right] \leq \frac{\sigma^2 \lambda}{n}.
\]

The equality in (a) follows since the noise process \( W(x) \) is independent, \( S_1, S_2, \ldots, S_M \) are distinct, and \( M \) (which depends on sampling process) is independent of the measurement-noise process \( W(x) \).

Putting together results from (18), (19), (21), and (24) in (11).

\[
\mathbb{E} \left[ |A_{\text{gen}}[k] - a[k]|^2 \right] \leq 4\|g'\|^2 \frac{C_1}{n} + 4C_2^2 \lambda^2 \frac{\lambda^2}{n^2} + 2\sigma^2 \lambda
\]

\[
\leq \left( 4(2b + 1)^2 \pi^2 C_1 + \frac{4C_2^2 \lambda^2}{n} + 2\sigma^2 \lambda \right) \frac{1}{n}
\]

\[
\leq \frac{C}{n}
\]

for some \( C > 0 \) which does not depend on \( n \). Observe that the constant \( C \) becomes larger with larger bandwidth \( (b) \), larger noise variance \( (\sigma^2) \), and a larger spread of renewal distribution \( (\lambda) \). A larger value of \( \lambda \), indicates that the (unknown) sampling locations are more spread-out around their mean value of \( 1/n \).

This results in a worse proportionality constant. The main result of the theorem is complete.

Remark 3: From the proof in Appendix A, the \( O(1/n) \) decay in distortion will hold if \( \mathbb{E}(1/M) \) decreases proportionally to \( 1/n \) and \( \mathbb{E}(M + 1) \) is proportional to \( n \). The latter condition can be established easily by Wald’s identity [21]. The former condition will need some sophisticated statistical analysis with stopping-times and has been left as a future work. At a high level, \( \mathbb{E}(1/M) \) can be expected to decrease as \( O(1/n) \) since \( (M/n) = 1 \) almost-surely as the sampling rate \( n \) increases. The assumption that \( nX \leq \lambda \) makes the mean-squared error analysis a little convenient.

Remark 4: Renewal process with small mean (of \( 1/n \)) result in a 'pontogram', which is connected to the Brownian Bridge [23]. In the spatial-sampling context, if \( M(x) \) is the number of samples taken up till location \( x \) (with \( M(1) = M \) in this work’s notation), then \( \sqrt{n}[M(x) - xM(1)] \) will be a generalized pontogram as a function of \( x \in [0, 1] \). Then, it is known that the worst deviation of the Pontogram from a Brownian bridge is negligible (with high probability) as \( n \) increases [23, Theorem 2.1]. This indicates that, in the limit of \( n \) large, the mobile sensor will be sampling the spatial field on a Brownian bridge! The properties of a bandlimited field being observed on a Brownian bridge (at unknown points) is an interesting topic of study for future research.

Like in a Brownian bridge, this result also suggests that the variance of sample-locations in the middle is larger than those at the edges of the sampling interval. In the future, it would be interesting to design estimates of \( a[k] \) which utilize this property.

Remark 5: An inspection of proof of Theorem 3.1 suggests that its result will hold if \( g(x) \) is a field in any set of bounded dynamic-range fields having an orthonormal basis, and having smoothness properties such as finite derivative over the entire class. Bandlimitedness would translate to having finite degrees of freedom (or finite number of non-zero coordinates in the orthonormal basis). Orthonormal basis would imply that the degrees of freedom can be obtained using a suitable inner product, which can be approximated using a Riemann sum (see (2) and (9)). Smoothness properties of the set of fields will enable counterparts of (9) required for approximation analysis (see (17)). This generalization, we believe, is analogous to the Fourier series development followed in this work. This generalization is not established in the current work due to space constraints, and more importantly for simplicity of exposition.

IV. BANDWIDTH DETERMINATION USING FIELD SAMPLES OBTAINED ON AN UNKNOWN RENEWAL PROCESS

In some applications, the bandwidth or the essential bandwidth of the spatial field may not be known [23]. This can be because the essential bandwidth of spatial fields change with time [23], or because the field being observed is not characterized for bandwidth, or because the sampling path is not a straight line[2]. Under some technical assumptions, an algorithm is outlined to find the bandwidth \( b \) of the field in...
Consider a spatial field \( g(x) \) which has a finite but unknown bandwidth parameter \( b \). From (2) and the result in Theorem 3.1, \( \tilde{A}_{\text{gen}}[k] \) for \( |k| > b \) will converge (in the mean-squared sense, and therefore in probability) to zero. This observation can be used to design a reconstruction algorithm for a spatially bandlimited process with unknown bandwidth. The following assumptions are made for this section:

1. The spatial field has a finite but unknown bandwidth parameter \( b \) (see (2)).
2. All the non-zero Fourier series coefficients are larger than \( \Delta \) in magnitude, where \( \Delta > 0 \) is a constant.
3. The measurement-noise is zero-mean, its second moment (variance) is known, and its fourth moment is finite (i.e., \( \text{E}(W^4) < \infty \)).

A non-zero constant \( \Delta \) (in the second assumption above) is needed to ascertain the bandwidth, since any statistical estimate (such as \( \tilde{A}_{\text{gen}}[k] \)) will be negligible outside the bandwidth \( b \) but not exactly zero. A threshold parameter \( \Delta \) ensures that the sampling rate \( n \) can be made large enough to get rid of negligible but otherwise non-zero Fourier series coefficients.

For any \( k \), an estimate for \( a[k] \) can be obtained from \( \tilde{A}_{\text{gen}}[k] \) in (9). The tricky part is determination of \( b \), that is, when to stop the Fourier series coefficient estimation! In other words, a stopping condition is needed. The next paragraph summarizes this stopping condition and then an algorithm for bandwidth determination is presented, with a sketch of its technical correctness.

Consider a simplified problem, where a bandlimited but unknown bandwidth signal \( h(x) \) is available. It is known that the bandwidth of \( h(x) \) is finite, but its value is not known. To reconstruct the field, the bandwidth of \( h(x) \) is required. Let \( c[k] \) be the Fourier series of \( h(x) \). The Fourier series coefficients \( c[0], c[1], c[-1], \ldots \) can be sequentially computed. The main issue is when the Fourier series coefficients computation should be stopped? To this end, note that since \( h(x) \) is available so is its energy \( \int_0^1 |h(x)|^2 dx \). By Parseval’s theorem, it is known that

\[
\int_0^1 |h(x)|^2 dx = \sum_{-\infty}^{\infty} |c[k]|^2. \tag{29}
\]

So, \( c[0], c[1], c[-1], \ldots \) can be computed till the energy in the collected coefficients matches with that of \( h(x) \). Since the energy of \( h(x) \) is finite, and \( b \) is finite by assumption, so this process will end in \((2b+1)\) number of steps. An adaptation of this idea will be used in the stochastic sampling setup with a mobile sensor. Since accurate approximations of the field’s energy and Fourier series coefficients are available only for \( n \to \infty \), so an approximate adaptation of this algorithm is needed to address finite but large values of \( n \).

An estimate of spatial field energy (see (29)) is needed since the field is not available in entirety but only through noise-affected samples at unknown locations. The spatial field’s energy estimate is defined as

\[
E_g := \frac{1}{M} \sum_{i=1}^{M} [g(S_i) + W(S_i)]^2 - \sigma^2, \tag{30}
\]

where \( \sigma^2 \) is the noise variance and is assumed to be known. The intuition in the above estimate is that \( W(x) \) and \( X_i^M, M \) are independent (and hence uncorrelated), and \( W^2(S_i), i = 1, \ldots, M \) will average near \( \sigma^2/M \). The analysis in Appendix C shows that the mean-squared value of \( |E_g - \int_0^1 g^2(x) dx| \) is bounded as

\[
\text{E} \left[ \left| E_g - \int_0^1 g^2(x) dx \right|^2 \right] \leq \frac{C}{n} \tag{31}
\]

where \( C > 0 \) is some constant independent of \( n \) and depends on \( g(x) \) only through \( b \). Therefore \( E_g \) converges to the field energy \( \int_0^1 g^2(x) dx \) in the mean-squared sense (and hence in probability).

As sampling rate \( n \) becomes larger, the empirical energy in (30) converges in mean-squared sense to the true energy of \( g(x) \). An estimate for Fourier series coefficients has been presented in (9), which converges in mean-squared sense as sampling rate \( n \) becomes large. If this estimate is below the threshold \( \Delta \) by some margin, the Fourier series coefficient can be set to zero by a thresholding operation. For asymptotic \( n \), this process will result in (mean-squared) correct Fourier series coefficients. Similarly, as noted earlier, each \( \tilde{A}_{\text{gen}}[k] \) converges in mean-squared sense to the correct \( a[k] \). This motivates the following estimation algorithm, if each Fourier series coefficient is more than \( \Delta \) in magnitude, where \( \Delta > 0 \) is a positive parameter.

1. Calculate an estimate \( E_g \) for the spatial field’s energy as in (30).
2. Start with \( B = 0 \).
3. Calculate the Fourier series coefficient \( \tilde{A}_{\text{gen}}[B] \) and \( \tilde{A}_{\text{gen}}[-B] \) as in (9). If the estimates are more than \( \Delta - \frac{1}{\sqrt{n}} \) in magnitude, retain them. Otherwise, set \( \tilde{A}_{\text{gen}}[B] \) and \( \tilde{A}_{\text{gen}}[-B] \) as zero. It will be shown shortly that non-zero coefficients succeed while zero coefficients fail in this test with high probability, as \( n \) increases asymptotically.
4. Increase \( B \) by +1. Repeat the process in previous step till

\[
\frac{\Delta^2}{2} \leq \sum_{k=-B}^{B} \left| \tilde{A}_{\text{gen}}[k] \right|^2 - E_g \leq \frac{\Delta^2}{2}. \tag{32}
\]

It will be shown shortly that this test will be met only by the correct bandwidth \( b \) with high probability, as \( n \) increases asymptotically.

Claim in Item 3) above follows by Chebychev inequality [21]. If \( |a[k]| \geq \Delta \), it is noted that

\[
P \left[ |\tilde{A}_{\text{gen}}[k] - a[k]| < 1/\sqrt{n} \right] \leq \frac{\text{E} \left[ |\tilde{A}_{\text{gen}}[k] - a[k]|^2 \right]}{(1/\sqrt{n})^2} \tag{33}
\]

\[
\leq \frac{C}{n^{1/3}} \tag{34}
\]
which means that \(|\hat{A}_{\text{gen}}[k] - a[k]| < 1/\sqrt{n}\) with high probability. Since \(|a[k]| \geq 2\) by assumption, so

\[
\Delta - \frac{1}{\sqrt{n}} \leq |a[k]| - |\hat{A}_{\text{gen}}[k] - a[k]| \leq |\hat{A}_{\text{gen}}[k]|
\]

(35)

with high probability. This establishes (the obvious) that \(\hat{A}_{\text{gen}}[k]\) will have a magnitude greater than \(\Delta - 1/\sqrt{n}\) with high probability if \(a[k]\) has a magnitude greater than \(\Delta\).

By similar argument, if \(a[k] = 0\), then

\[
\mathbb{P}\left[|\hat{A}_{\text{gen}}[k] - a[k]| < \Delta - \frac{1}{\sqrt{n}}\right] \leq \frac{\mathbb{E}[\hat{A}_{\text{gen}}[k] - a[k]]^2}{(\Delta - n^{1/3})^2} \leq \frac{C}{n(\Delta - n^{1/3})^2}
\]

(36)

which converges to zero with increasing \(n\). So, if \(a[k] = 0\), then \(\hat{A}_{\text{gen}}[k]\) is smaller than \(\Delta - 1/\sqrt{n}\) with high probability.

Claim in Item 4) follows by Chebychev inequality and finiteness of \(B\). From (31) and Chebychev inequality, it follows that

\[
\mathbb{P}\left[E_g - \int_0^1 g^2(x)dx > \frac{1}{\sqrt{n}}\right] \leq \frac{\mathbb{E}[E_g - \int_0^1 g^2(x)dx]^2}{(\Delta - n^{1/3})^2} \leq \frac{C}{n(\Delta - n^{1/3})^2}
\]

(37)

or with high probability or \(E_g\) is close to \(\int_0^1 g^2(x)dx\). The following inequalities are noted, each of which holds with high probability. From (34) and (36),

\[
a[k] \neq 0, |\hat{A}_{\text{gen}}[k]|^2 - |a[k]|^2 \leq O\left(\frac{1}{\sqrt{n}}\right)
\]

(38)

\[
a[k] = 0, |\hat{A}_{\text{gen}}[k]| = 0
\]

(39)

where the second equality is achieved by thresholding the near-zero \(|\hat{A}_{\text{gen}}[k]|\) against \(\Delta - 1/\sqrt{n}\). That is, each estimated \(\hat{A}_{\text{gen}}[k]\) (for \(-B \leq k \leq B\)) is equal to zero \(a[k]\) or at a maximum distance of \(1/\sqrt{n}\) to a non-zero \(a[k]\). So the maximum difference between coefficient energies is

\[
\sum_{k=-B}^{B} |\hat{A}_{\text{gen}}[k]|^2 - \sum_{k=-B}^{B} |a[k]|^2 = O\left(\frac{1}{\sqrt{n}}\right).
\]

(40)

By Parseval’s relation,

\[
\sum_{k=-B}^{B} |a[k]|^2 = \int_0^1 g^2(x)dx
\]

(41)

where the maximum number of nonzero \(a[k]\) is \(\int_0^1 g^2(x)dx / \Delta^2\), since \(|a[k]| \geq \Delta\) by assumption. Finally, the maximum difference between energy estimate of the field and the actual energy is

\[
E_g - \int_0^1 g^2(x)dx \leq \frac{1}{\sqrt{n}}
\]

(42)

From (40), (41), and (42), it follows that

\[
E_g - \sum_{k=-B}^{B} |\hat{A}_{\text{gen}}[k]|^2 = O\left(\frac{1}{\sqrt{n}}\right)
\]

(43)

only for the correct value of bandwidth \(B\). Therefore, the stopping condition in (32) will be met with high probability.

Remark 6: It must be noted that the above algorithm works for asymptotic \(n\) and convergence rate guarantees are not given in this first exposition. It is possible to select some other function of \(n\) instead of \(1/\sqrt{n}\) in Item 3) and some other threshold than \(\Delta^2/2\), and optimize the probability of error for a finite (but large) \(n\). This analysis is left for future work, but simulations based on the above algorithm will be presented.

V. SIMULATION RESULTS

Some simulation results are presented in Fig. 2. In these simulations, additive measurement-noise was generated using Uniform\([-1, 1]\) random variables. The random sampling locations were obtained using a renewal process with uniform inter-sample spacing distribution. The knowledge of this distribution, as explained in [7] and Section III, was not used in the field reconstruction. The field was generated in different ways for the first and second plots, and the third plot. This is explained next.

In the first and second plots, which evaluates the performance of field estimate in Section III for field with known bandwidth, it is assumed that there is a field \(g(x)\) with Fourier series coefficients given by

\[
a[0] = 0.2445, a[1] = -0.0357 + j0.0478
\]

\[
a[2] = 0.0978 + j0.0729, a[3] = -0.1796 - j0.0756
\]

\[
a[1] = \bar{a}[-k], \text{ and } a[k] = 0 \text{ for } |k| > 3.
\]

(44)

These Fourier series coefficients were obtained by independent trials of Uniform\([-1, 1]\) random variables (to obtain the real and imaginary parts). Conjugate symmetry \(a[k] = \bar{a}[-k]\) ensures that the field is real-valued. Finally, the field was scaled to limit its dynamic range within \([-1, 1]\).

For the third plot, which evaluates the bandwidth estimation algorithm of Section IV, it is assumed that there is a field \(g(x)\) with Fourier series coefficients given by

\[
a[0] = 0.1, a[1] = -0.1, a[2] = 0.1,
\]

\[
a[1] = \bar{a}[-k], \text{ and } a[k] = 0 \text{ otherwise.}
\]

(45)

In these simulations \(\Delta\) is set as 0.1. Larger values of \(\Delta\) are more desirable for the algorithm proposed in this section. Coefficient threshold check tests whether the non-zero coefficients in the Fourier series of \(g(x)\) are estimated as more than \(\Delta - 1/\sqrt{n}\) and the zero coefficients are 0 (see Item 3) in the algorithm above). The stopping rule in (32) requires that the estimated field energy is within \(\Delta^2/2 = 0.05\) of the original. This requirement is very stringent and as a result, the stopping rule condition is violated (never or wrongly met) for smaller values of \(n\). For \(\Delta = 0.1\), the threshold \(\Delta - 1/\sqrt{n}\) is positive only when \(n \geq 1000\). For this reason, simulation begins from \(n = 5000\) in the third plot.

In the first plot, the convergence of random realizations of \(\hat{G}(\theta)\) to \(g(\theta)\) can be observed with increasing \(n\). The graph for \(n = 10000\) is near identical to the true field and cannot be seen in the graph. The mean-squared error, averaged over 10000 random trials, decreases as \(O(1/n)\) as illustrated in the second (log-log) plot. This is in consonance with our results in Theorem II. Finally, in the third plot, it is observed that the stopping rule check and coefficient threshold check are met successfully if \(n\) exceeds 20000. These numerical values will change depending on the value of \(\Delta\).
VI. CONCLUDING REMARKS

This work introduced the estimation of bandlimited spatial fields from noise-affected samples taken at unknown sampling locations, where the locations are generated from a renewal process with unknown distribution. Sampling rate was used to combat against additive measurement-noise as well as unknown sampling locations. A spatial field estimate, which converges to the true spatial field in the mean-squared sense at the rate \(O(1/\text{sampling rate})\), was presented and its analysis was the first main result of the work. A spatial field bandwidth determination algorithm from field samples collected at unknown sampling locations, which works with probability one as sampling rate increases asymptotically, was proposed and its correctness was the second main result of this work. Simulation results, consonant with the theoretical analysis, were also presented.

This work opens a flurry of interesting ideas related to reconstruction of spatial fields from samples collected at unknown sampling locations. How can the field estimates be derived (developed) when the renewal process distribution or the noise distribution is known? Is the distortion result developed in this paper optimal in certain circumstances? How will the distortion change in the presence of sample quantization? How should the field estimation change if a fraction of field samples are taken at known locations or without measurement-noise or both? What field estimation or reconstruction strategy should be used to tackle the sampling of non-bandlimited fields? In all these, and many more, questions we expect sampling rate to play a fundamental role in the obtained answers.

APPENDIX A

MEAN-SQUARED CLOSENESS OF RENEWAL-PROCESS SAMPLING GRID TO A UNIFORM GRID

For analysis purposes, let

\[ R_M = 1 - (X_1 + X_2 + \ldots + X_M) \]  

(46)

be the remaining distance between the last location of sampling and the end of field support (or sampling vehicle’s terminal stop). Observe that \( R_M \) is bounded since

\[ R_M \leq X_{M+1} \leq \frac{\lambda}{n} \]  

(47)

First the average value of \( X_M \) will be determined, conditioned on \( M = m \). Since \( S_M + R_M = 1 \) by definition, so

\[ E(S_M + R_M | M = m) = 1 \]  

(48)

or \( mE(X_1 | M = m) + E(R_M | M = m) = 1 \)  

(49)

i.e., \( E(X_1 | M = m) = \frac{1}{m} - \frac{E(R_M | M = m)}{m} \).  

(50)

Since \( R_M \leq \lambda/n \) so the second term is expected to be negligible with large sampling rate \( n \). That is, conditional average of \( X_i \) is nearly \( 1/m \) conditioned on \( M = m \). In (49), conditioned on \( M = m \), the exchangeable nature of \( X_1, X_2, \ldots, X_m \) is used along with \( S_m = X_1 + \ldots + X_m \).

To determine the expectation of average mean-squared error between \( S_m^M \) and equi-spaced grid, consider the conditional expectation of the following error-term:

\[ E \left[ \sum_{i=1}^{M} \left| S_i - \frac{i}{M} \right|^2 | M = m \right] \]  

(51)

\[ = E \left[ \frac{1}{m} \sum_{i=1}^{m} \left( \sum_{l=1}^{i} \left( X_l - \frac{1}{m} \right) \right)^2 | M = m \right] \]  

\[ = E \left[ \frac{1}{m} \sum_{i=1}^{m} \sum_{l=1}^{i} \sum_{p=1}^{i} (X_l - \frac{1}{m}) (X_p - \frac{1}{m}) | M = m \right] \]  

(52)

\[ = E \left[ \sum_{i=1}^{m} \left( X_i - \frac{1}{m} \right)^2 + i(i-1) \left( X_1 - \frac{1}{m} \right) (X_2 - \frac{1}{m}) | M = m \right] \]  

\[ = \sum_{i=1}^{m} \left( X_i - \frac{1}{m} \right)^2 + \sum_{i=2}^{m} i(i-1) \left( X_1 - \frac{1}{m} \right) (X_2 - \frac{1}{m}) | M = m \]  

(53)

\[ = \frac{(m+1)}{2} \left( X_1 - \frac{1}{m} \right)^2 + \frac{m^2 - 1}{3} \times \]  

\[ \left( X_1 - \frac{1}{m} \right) (X_2 - \frac{1}{m}) | M = m \]  

\[ = \frac{(m+1)}{2} a_m + \frac{m^2 - 1}{3} b_m \]  

(54)
where \((a)\) follows by exchangeability of \(X_1, X_2, \ldots, X_m\) conditioned on \(M = m\), and
\[
a_m := \mathbb{E} \left[ \left( X_1 - \frac{1}{m} \right)^2 \Bigg| M = m \right]
\]
and
\[
b_m := \mathbb{E} \left[ \left( X_1 - \frac{1}{m} \right) \left( X_2 - \frac{1}{m} \right) \Bigg| M = m \right]
\]
By definition \(S_m - 1 + R_m = 0\) conditioned on \(M = m\), so
\[
\mathbb{E} \left[ (S_m - 1)^2 | M = m \right] = \mathbb{E} \left[ (R_m)^2 | M = m \right]
\]
and therefore,
\[
\mathbb{E} \left[ \sum_{i=1}^m \left( X_i - \frac{1}{m} \right)^2 \right] | M = m \bigg| - \mathbb{E} \left[ R_m^2 | M = m \right] = 0
\]
since \(S_m = X_1 + X_2 + \ldots + X_m\). A rearrangement of the above equation results in,
\[
ma_m + m(m-1)b_m = \mathbb{E} \left[ R_m^2 | M = m \right]
\]
or
\[
b_m = \frac{1}{m(m-1)} \left( -ma_m + \mathbb{E} \left[ R_m^2 | M = m \right] \right). \quad (57)
\]
With \(b_m\) from \((57)\), the conditional error term in \((54)\) can be rewritten as
\[
\mathbb{E} \left[ \frac{1}{m} \sum_{i=1}^m \left( X_i - \frac{1}{m} \right)^2 \right] | M = m
\]
\[
= \frac{m+1}{2} a_m + \frac{m^2 - 1}{3m(m-1)} \left( -ma_m + \mathbb{E} \left[ R_m^2 | M = m \right] \right)
\]
\[
= \frac{(m+1)}{6} a_m + \frac{m + 1}{3m} \mathbb{E} \left[ R_m^2 | M = m \right]
\]
\[
\leq \frac{m+1}{6} a_m + \frac{2\lambda^2}{3n^2} \quad (59)
\]
where the last step is obtained since \((m+1)/3m \leq 2/3\) and \(R_m \leq X_{m+1} \leq (\lambda/n)\) by assumption on the inter-sample spacing distribution. So,
\[
\mathbb{E} \left[ \frac{1}{M} \sum_{i=1}^M \left( S_i - \frac{i}{M} \right)^2 \right]
\]
\[
\leq \frac{1}{6} \mathbb{E} \left[ \frac{M+1}{a_m} \right] + \frac{2\lambda^2}{3n^2} \quad (60)
\]
\[
(a) \quad \frac{1}{6} \mathbb{E} \left[ \frac{M+1}{(X_1 - \frac{1}{M})^2} \right] + \frac{2\lambda^2}{3n^2} \quad (61)
\]
\[
(b) \quad \frac{1}{6} \mathbb{E} \left[ \frac{M+1}{2X_1^2 + \frac{2}{M^2}} \right] + \frac{2\lambda^2}{3n^2} \quad (62)
\]
\[
\leq \frac{1}{3} \mathbb{E} \left[ \frac{M+1}{X_1^2} \right] + \frac{1}{3} \mathbb{E} \left[ \frac{M+1}{M^2} \right] + \frac{2\lambda^2}{3n^2} \quad (63)
\]
\[
\leq \frac{1}{3} \mathbb{E} \left[ \frac{M+1}{X_1^2} \right] + \frac{1}{3} \left( \frac{M+1}{M^2} \right) \mathbb{E} \left[ \frac{X_1}{n} + \frac{\lambda^2}{n^2} \right] + \frac{2\lambda^2}{3n^2} \quad (64)
\]
\[
= \frac{1}{3} \mathbb{E} \left[ \frac{M+1}{X_1^2} \right] + \frac{\lambda^2}{3n} + \frac{\lambda^2}{n^2} \leq \frac{C_1}{n} \quad (65)
\]
for some \(C_1 > 0\), which is independent of \(n\) and \(g(x)\). In the above inequalities, \((a)\) follows by the definition of \(a_m\) in \((55)\), \((b)\) follows by \(|x_1 - x_2|^2 \leq 2|x_1|^2 + 2|x_2|^2\) for any real numbers \(x_1\) and \(x_2\), \((c)\) follows because \(0 < X < \lambda/n\) and therefore \((n/\lambda) < M < \infty\), and \((d)\) follows using Wald’s identity on \((M+1)\) and \(\mathbb{E}(X) = 1/n \) \((21)\). This completes the proof.

**APPENDIX B**

**APPROXIMATION ERROR IN INTEGRALS BY RIEMANN SUM**

It is noted that
\[
|a[k] - A_R[k]| = \left| \int_0^1 g(x) \exp(-j2\pi kx)dx - \frac{1}{M} \sum_{i=1}^M g \left( \frac{i}{M} \right) \exp \left( -j2\pi ki \right) M \right| \quad (66)
\]
\[
= \frac{1}{M} \sum_{i=1}^M \int_{\frac{i}{M}}^{\frac{i+1}{M}} g(x) \exp(-j2\pi kx)dx - \frac{1}{M} \sum_{i=1}^M g \left( \frac{i}{M} \right) \exp \left( -j2\pi ki \right) \quad (67)
\]
\[
= \frac{1}{M} \sum_{i=1}^M \frac{1}{M} g(Z_{i,M}) \exp(-j2\pi kZ_{i,M}) - \frac{1}{M} \sum_{i=1}^M g \left( \frac{i}{M} \right) \exp \left( -j2\pi ki \right) \quad (68)
\]
where \(Z_{i,M} \in (l/M, (l+1)/M)\) is some constant that depends on \(g(x) \exp(-j2\pi kx)\) by the Lagrange mean-value theorem \((22)\). So,
\[
|a[k] - A_R[k]| = \left| \frac{1}{M} \sum_{i=1}^M g(Z_{i,M}) \exp(-j2\pi kZ_{i,M}) - \frac{1}{M} \sum_{i=1}^M g \left( \frac{i}{M} \right) \exp \left( -j2\pi ki \right) \right| \quad (69)
\]
\[
\leq \frac{1}{M} \sum_{i=1}^M \left| g(Z_{i,M}) \exp(-j2\pi kZ_{i,M}) - g \left( \frac{i}{M} \right) \exp \left( -j2\pi ki \right) \right| \quad (70)
\]
\[
\leq \frac{1}{M} \sum_{i=1}^M \left| g \left( \frac{i}{M} \right) \exp \left( -j2\pi ki \right) \right| \quad (71)
\]
\[
\leq \frac{1}{M} \sum_{i=1}^M \left| g(x) \exp(-j2\pi kx) \right| \quad (72)
\]
\[
\leq \frac{1}{M} \left| \frac{d}{dx} g(x) \exp(-j2\pi kx) \right|_\infty \quad (73)
\]
\[
\leq \frac{1}{M} \left| \frac{d}{dx} g(x) \exp(-j2\pi kx) \right|_\infty \quad (74)
\]
As a result
\[
\mathbb{E}[|a[k] - A_R[k]|^2] \leq \mathbb{E} \left( \frac{C_2^2}{M^2} \right) \quad (75)
\]
\[
\leq \frac{C_2^2 \lambda^2}{n^2}. \quad (76)
\]
APPENDIX C
MEAN-SQUARED ERROR IN THE ENERGY ESTIMATION

Observe that,

\[ |E_g - \int_0^1 g^2(x)dx| \leq \left| \frac{1}{M} \sum_{i=1}^{M} g^2(S_i) - \int_0^1 g^2(x)dx \right| \]
\[ \leq \left| \frac{1}{M} \sum_{i=1}^{M} g^2(S_i) - \int_0^1 g^2(x)dx \right| + \left| \frac{2}{M} \sum_{i=1}^{M} g(S_i)W(S_i) \right| \]
\[ \leq \frac{2}{M} \left( \sum_{i=1}^{M} g(S_i) - g \left( \frac{i}{M} \right) \right) + \frac{4\pi b}{M} \]
\[ \leq 2 \left( \sum_{i=1}^{M} g(S_i) - g \left( \frac{i}{M} \right) \right) + \frac{4\pi b}{M} \]

The three terms in the above expression will be analyzed one by one and it will be argued that their mean-squared values do not exceed \(O(1/n)\). The first term can be bounded as

\[ \left| \frac{1}{M} \sum_{i=1}^{M} g^2(S_i) - \int_0^1 g^2(x)dx \right| \]
\[ \leq \left| \frac{1}{M} \sum_{i=1}^{M} g^2(S_i) - \frac{1}{M} \sum_{i=1}^{M} g^2 \left( \frac{i}{M} \right) \right| + \left| \frac{1}{M} \sum_{i=1}^{M} g^2 \left( \frac{i}{M} \right) - \int_0^1 g^2(x)dx \right| \]
\[ \leq \frac{1}{M} \sum_{i=1}^{M} \left| g^2(S_i) - g^2 \left( \frac{i}{M} \right) \right| + \left| \frac{1}{M} \sum_{i=1}^{M} g^2 \left( \frac{i}{M} \right) - \int_0^1 g^2(x)dx \right| \]

By Lagrange mean-value theorem and the continuity of \(g(x)\), it follows that

\[ \int_{\frac{i}{M}}^{\frac{i+1}{M}} g^2(x)dx = g^2 \left( Z_{i,M} \right) \frac{1}{M} \]

where \(\frac{i}{M} \leq Z_{i,M} \leq \frac{i+1}{M}\) or \(Z_{i,M} - \frac{i}{M} \leq \frac{1}{M}\). Further, note that

\[ |g^2(x) - g^2(y)| \leq 2 \|g\|_{\infty} |g(x) - g(y)| \]
\[ \leq 2 \|g\|_{\infty} \|g'\|_{\infty} |x - y| \]

By the sequential use of (83), (84), and (85) in (82), the following inequalities are obtained:

\[ \left| \frac{1}{M} \sum_{i=1}^{M} g^2(S_i) - \int_0^1 g^2(x)dx \right| \]
\[ \leq \frac{1}{M} \sum_{i=1}^{M} \left| g^2(S_i) - g^2 \left( \frac{i}{M} \right) \right| + \frac{1}{M} \sum_{i=1}^{M} \left| g^2 \left( \frac{i}{M} \right) - g^2 \left( Z_{i,M} \right) \right| \]
\[ \leq 2 \|g\|_{\infty} \sum_{i=1}^{M} \left| g(S_i) - g \left( \frac{i}{M} \right) \right| + \frac{4\pi b}{M} \]

since \(\|g\|_{\infty} \leq 1\) by assumption and \(\|g'\|_{\infty} \leq 2\pi b\) from (3).

From (15), (18), and (19), it can be deduced that

\[ \mathbb{E} \left[ \left( \frac{2}{M} \sum_{i=1}^{M} g(S_i) - g \left( \frac{i}{M} \right) \right)^2 \right] \leq \frac{4\|g'\|_{\infty}^2 C_1}{n} \]

Since \(M > (n/\lambda)\), so

\[ \mathbb{E} \left( \frac{4\pi b^2}{M} \right) \leq \frac{16\pi^2 b^2 \lambda^2}{n^2} \]

The second term in (79) has a mean-squared error bounded as follows:

\[ \mathbb{E} \left[ \frac{4}{M^2} \sum_{i=1}^{M} g(S_i)W(S_i) \right] \]
\[ = \mathbb{E} \left[ \frac{4}{M^2} \sum_{i=1}^{M} g(S_i)g(S_i)W(S_i)W(S_i) \right] \]
\[ \leq \mathbb{E} \left[ \frac{4}{M^2} \sum_{i=1}^{M} g^2(S_i) \sigma^2 \right] \]
\[ \leq \mathbb{E} \left[ \frac{4M^2 \sigma^2}{\lambda} \right] \leq \frac{4\lambda^2}{\lambda} \]

where (a) follows since \(W(x)\) process is independent of \(M\) and \(W(S_i), W(S_j)\) are independent if \(S_i \neq S_j\) by assumption, and (b) follows since \(\|g\|_{\infty} \leq 1\) and \(M > n/\lambda\).

The third term in (79) has a mean-squared error bounded as follows:

\[ \mathbb{E} \left[ \left( \frac{1}{M} \sum_{i=1}^{M} (W^2(S_i) - \sigma^2) \right)^2 \right] \]
\[ \leq \mathbb{E} \left[ \frac{\sigma^2(W^2)}{M} \right] \]
\[ \leq \frac{\sigma^2(W^2)}{n} \]

where (a) follows since \(W(x)\) process is independent of \(M\) and \(W(S_i), W(S_j)\) are independent if \(S_i \neq S_j\) by assumption and \(W^2(x)\) has mean \(\sigma^2\) and variance \(\sigma^2(W^2)\) by assumption, and (b) follows since \(M > (n/\lambda)\).
From (79), and (90), (91), (94) and (96), it follows that
\[
\mathbb{E} \left[ \left| g_0(x) - \int_0^1 g(x) dx \right|^2 \right] 
\leq \frac{C}{n}
\leq \frac{C}{n}
\]  \hspace{2cm} (97)
where \( \hat{C} > 0 \) is some constant independent of \( n \) and depends on \( g(x) \) only through \( \lambda \).

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**REFERENCES**

[1] J. Unnikrishnan and M. Vetterli, “Sampling and reconstructing spatial fields using mobile sensors,” in *Proceedings of the IEEE ICASSP*. NY, USA: IEEE, 2012.

[2] ———, “Sampling and reconstruction of spatial fields using mobile sensors,” *IEEE Trans. Signal Proc.*, 2013.

[3] A. Papoulis, “Error analysis in sampling theory,” *Proceedings of the IEEE*, vol. 54, no. 7, pp. 947–955, July 1966.

[4] A. J. Jerri, “The Shannon Sampling Theorem – its Various Extensions and Applications: a Tutorial Preview,” *Proceedings of the IEEE*, vol. 65, pp. 1565–1594, Nov. 1977.

[5] Farokh Marvasti (ed.), *Nonuniform Sampling*. New York, USA: Kluwer Academic Publishers, 2001.

[6] M. S. Pinsker, “Optimal filtration of square-integrable signals in Gaussian noise,” *Problemy Peredachi Informatsii*, vol. 16, no. 2, pp. 52–68, Apr. 1980.

[7] E. Masry, “The reconstruction of analog signals from the sign of their noisy samples,” *IEEE Transactions on Information Theory*, vol. 27, no. 6, pp. 735–745, Nov. 1981.

[8] O. Dabeer and A. Karnik, “Signal parameter estimation using 1-bit dithered quantization,” *IEEE Transactions on Information Theory*, vol. 52, no. 12, pp. 5389–5405, Dec. 2006.

[9] E. Masry and P. Ishwar, “Field estimation from randomly located binary noisy sensors,” *IEEE Transactions on Information Theory*, vol. 55, no. 11, pp. 5197–5210, Nov. 2009.

[10] Y. Wang and P. Ishwar, “Distributed field estimation with randomly deployed, noisy, binary sensors,” *IEEE Transactions on Signal Processing*, vol. 57, no. 3, pp. 1177–1189, Mar. 2009.

[11] A. Kumar and V. M. Prabhakaran, “Estimation of bandlimited signals from the signs of noisy samples,” in Proc. of the 2013 IEEE International Conference on Acoustic Speech and Signal Processing (ICASSP), May 2013, pp. 5815–5819.

[12] N. Patwari, J. N. Ash, S. Kyperountas, A. O. Hero III, R. L. Moses, and N. S. Correal, “Locating the nodes: Cooperative localization in wireless sensor networks,” *IEEE Signal Processing Magazine*, vol. 22, no. 4, pp. 54–69, Jul. 2005.

[13] R. G. Gallager, *Stochastic Processes: Theory for Applications*, 1st ed. Cambridge University Press, 2014.

[14] A. Kumar, “On bandlimited signal reconstruction from the distribution of unknown sampling locations,” *IEEE Trans. Signal Proc.*, vol. 63, no. 5, pp. 1259–1267, Mar. 2015.

[15] H. A. David and H. N. Nagaraja, *Order Statistics*, 3rd ed. New York, NY: John Wiley & Sons, 2003.

[16] P. Marziliano and M. Vetterli, “Reconstruction of irregularly sampled discrete-time bandlimited signals with unknown sampling locations,” *IEEE Transactions on Signal Processing*, vol. 48, no. 12, pp. 3462–3471, Dec. 2000.

[17] J. Browning, “Approximating signals from nonuniform continuous time samples at unknown locations,” *IEEE Transactions in Signal Processing*, vol. 55, no. 4, pp. 1549–1554, Apr. 2007.

[18] A. Nordio, C.-F. Chiasserini, and E. Viterbo, “Performance of linear field reconstruction techniques with noise and uncertain sensor locations,” *IEEE Transactions on Signal Processing*, vol. 56, no. 8, pp. 3535–3547, Aug. 2008.

[19] A. I. Zayed, *Advances in Shannon’s Sampling Theory*. CRC press, 1993.

[20] G. H. Hardy, J. E. Littlewood, and G. Polya, *Inequalities*. London, UK: Cambridge University Press, 1959.

[21] R. Durrett, *Probability: Theory and Examples*, 2nd ed. Belmont, CA: Duxbury Press, 1996.

[22] W. Rudin, *Principles of Mathematical Analysis*. USA: McGraw-Hill Companies, 1976.

[23] J. Steinbach and H. Zhang, “On a weighted embedding for p–norms,” *Stochastic Processes and their Applications*, vol. 47, no. 2, pp. 183–195, 1993.

[24] D. Slepian, “On bandwidth,” *Proceedings of the IEEE*, vol. 64, no. 3, pp. 292–300, Mar. 1976.

[25] J. Ranieri and M. Vetterli, “Sampling and reconstructing diffusion fields in presence of aliasing,” in Proc. of the ICASSP 2013, 2013.

[26] R. J. Marks II, *Introduction to Shannon Sampling and Interpolation Theory*. New York, USA: Springer-Verlag, 1990.