Symmetric versions of Laman’s Theorem

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Abstract
Recent work has shown that if an isostatic bar and joint framework possesses non-trivial symmetries, then it must satisfy some very simply stated restrictions on the number of joints and bars that are ‘fixed’ by various symmetry operations of the framework.

For the group $C_3$ which describes 3-fold rotational symmetry in the plane, we verify the conjecture proposed in [4] that these restrictions on the number of fixed structural components, together with the Laman conditions, are also sufficient for a framework with $C_3$ symmetry to be isostatic, provided that its joints are positioned as generically as possible subject to the given symmetry constraints.

In addition, we establish symmetric versions of Henneberg’s Theorem and Crapo’s Theorem for $C_3$ which provide alternate characterizations of ‘generically’ isostatic graphs with $C_3$ symmetry.

As shown in [19], our techniques can be extended to establish analogous results for the symmetry groups $C_2$ and $C_s$ which are generated by a half-turn and a reflection in the plane, respectively.

1 Introduction

A bar and joint framework is said to be isostatic if it is minimal infinitesimally rigid, in the sense that it is infinitesimally rigid and the removal of

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any bar results in an infinitesimally flexible framework (see [8, 9, 30, 32], for example).

In 1970, G. Laman provided an elegant characterization of generically 2-isostatic graphs, that is, graphs whose generic 2-dimensional realizations as bar and joint frameworks are isostatic [14]. There are well known difficulties in extending this result to higher dimensions (see [8, 9, 32], for example).

Using techniques from group representation theory, it was recently shown in [4] that if a 2-dimensional isostatic bar and joint framework possesses non-trivial symmetries, then it must not only satisfy the Laman conditions, but also some very simply stated extra conditions concerning the number of joints and bars that are fixed by various symmetry operations of the framework (see also [18, 19]). In particular, these restrictions imply that a 2-dimensional isostatic framework must belong to one of only six possible point groups. In the Schoenflies notation [3], these groups are denoted by $C_{1}, C_{2}, C_{3}, C_{s}, C_{2v}$, and $C_{3v}$.

It was conjectured in [4] that the Laman conditions, together with the corresponding additional conditions concerning the number of fixed structural components, are not only necessary, but also sufficient for a symmetric framework to be isostatic, provided that its joints are positioned as generically as possible subject to the given symmetry constraints.

In this paper, we use the definition of `generic' for symmetry groups established in [17] to verify this conjecture for the symmetry group $C_{3}$ which describes 3-fold rotational symmetry in the plane ($\mathbb{Z}_3$ as an abstract group). The result is striking in its simplicity: to test a `generic’ framework with $C_{3}$ symmetry for isostaticity, we just need to check the number of joints that are ‘fixed’ by the 3-fold rotation, as well as the standard conditions for generic rigidity without symmetry.

By defining appropriate symmetrized inductive construction techniques, as well as appropriate symmetrized tree partitions of graphs, we also establish symmetric versions of Henneberg’s Theorem (see [9, 13]) and Crapo’s Theorem ([3, 9, 24]) for the group $C_{3}$. These results provide us with some alternate techniques to give a ‘certificate’ that a graph is ‘generically’ isostatic modulo $C_{3}$ symmetry. Furthermore, they enable us to generate all such graphs by means of an inductive construction sequence.

With each of the main results presented in this paper, we also lay the foundation to design algorithms that decide whether a given graph is generically isostatic modulo $C_{3}$ symmetry.

It is shown in [19] that our techniques can be extended to establish symmetric versions of Laman’s Theorem, Henneberg’s Theorem, and Crapo’s Theorem for the symmetry groups $C_{2}$ and $C_{s}$ which are generated by a half-turn and a reflection, respectively, as well. However, it turns out that these
proofs, in particular the ones for $C_s$, are considerably more complex than the ones for $C_3$. For simplicity, we therefore restrict our attention to the group $C_3$ in this paper.

The Laman-type conjectures for the dihedral groups $C_{2v}$ and $C_{3v}$ are still open. For a discussion on the difficulties that arise in proving these conjectures (as well as a variety of related conjectures), we refer the interested reader to [19].

2 Rigidity theoretic definitions and preliminaries

2.1 Graph theory terminology

All graphs considered in this paper are finite graphs without loops or multiple edges. The vertex set of a graph $G$ is denoted by $V(G)$ and the edge set of $G$ is denoted by $E(G)$. Two vertices $u \neq v$ of $G$ are said to be adjacent if $\{u, v\} \in E(G)$, and independent otherwise. A set $S$ of vertices of $G$ is independent if every two vertices of $S$ are independent. The neighborhood $N_G(v)$ of a vertex $v \in V(G)$ is the set of all vertices that are adjacent to $v$ and the elements of $N_G(v)$ are called the neighbors of $v$.

A graph $H$ is a subgraph of $G$ if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$, in which case we write $H \subseteq G$. The simplest type of subgraph of $G$ is that obtained by deleting a vertex or an edge from $G$. Let $v$ be a vertex and $e$ be an edge of $G$. Then we write $G - \{v\}$ for the subgraph of $G$ that has $V(G) \setminus \{v\}$ as its vertex set and whose edges are those of $G$ that are not incident with $v$. Similarly, we write $G - \{e\}$ for the subgraph of $G$ that has $V(G)$ as its vertex set and $E(G) \setminus \{e\}$ as its edge set. The deletion of a set of vertices or a set of edges from $G$ is defined and denoted analogously.

If $u$ and $v$ are independent vertices of $G$, then we write $G + \{\{u, v\}\}$ for the graph that has $V(G)$ as its vertex set and $E(G) \cup \{\{u, v\}\}$ as its edge set. The addition of a set of edges is again defined and denoted analogously.

For a nonempty subset $U$ of $V(G)$, the subgraph $\langle U \rangle$ of $G$ induced by $U$ is the graph having vertex set $U$ and whose edges are those of $G$ that are incident with two elements of $U$.

The intersection $G = G_1 \cap G_2$ of two graphs $G_1$ and $G_2$ is the graph with $V(G) = V(G_1) \cap V(G_2)$ and $E(G) = E(G_1) \cap E(G_2)$. Similarly, the union $G = G_1 \cup G_2$ is the graph with $V(G) = V(G_1) \cup V(G_2)$ and $E(G) = E(G_1) \cup E(G_2)$. 

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An automorphism of a graph $G$ is a permutation $\alpha$ of $V(G)$ such that $\{u, v\} \in E(G)$ if and only if $\{\alpha(u), \alpha(v)\} \in E(G)$. The automorphisms of a graph $G$ form a group under composition which is denoted by $\text{Aut}(G)$.

Let $H$ be a subgraph of $G$ and $\alpha \in \text{Aut}(G)$. We define $\alpha(H)$ to be the subgraph of $G$ that has $\alpha(V(H))$ as its vertex set and $\alpha(E(H))$ as its edge set, where $\{u, v\} \in \alpha(E(H))$ if and only if $\alpha^{-1}(\{u, v\}) = \{\alpha^{-1}(u), \alpha^{-1}(v)\} \in E(H)$.

We say that $H$ is invariant under $\alpha$ if $\alpha(V(H)) = V(H)$ and $\alpha(E(H)) = E(H)$, in which case we write $\alpha(H) = H$.

![Figure 1: An invariant (b) and a non-invariant subgraph (c) of the graph $G$ under $\alpha = (v_1 v_2 v_3)(v_4 v_5 v_6) \in \text{Aut}(G)$.](image)

The graph $G$ in Figure 1 (a), for example, has the automorphism $\alpha = (v_1 v_2 v_3)(v_4 v_5 v_6)$. The subgraph $H_1$ of $G$ is invariant under $\alpha$, but the subgraph $H_2$ of $G$ is not, because $\alpha(E(H_2)) \neq E(H_2)$.

### 2.2 Infinitesimal rigidity

**Definition 2.1** [8, 9, 30, 32] A framework in $\mathbb{R}^d$ is a pair $(G, p)$, where $G$ is a graph and $p : V(G) \to \mathbb{R}^d$ is a map with the property that $p(u) \neq p(v)$ for all $\{u, v\} \in E(G)$. We also say that $(G, p)$ is a $d$-dimensional realization of the underlying graph $G$.

An ordered pair $(v, p(v))$, where $v \in V(G)$, is a joint of $(G, p)$, and an unordered pair $\{(u, p(u)), (v, p(v))\}$ of joints, where $\{u, v\} \in E(G)$, is a bar of $(G, p)$.

**Definition 2.2** [9, 30, 32] Let $(G, p)$ be a framework in $\mathbb{R}^d$ with $V(G) = \{v_1, v_2, \ldots, v_n\}$. An infinitesimal motion of $(G, p)$ is a function $u : V(G) \to \mathbb{R}^d$. 
\[ \mathbb{R}^d \text{ such that} \]
\[ (p(v_i) - p(v_j)) \cdot (u(v_i) - u(v_j)) = 0 \text{ for all } \{v_i, v_j\} \in E(G). \]

An infinitesimal motion is a set of displacement vectors, one at each joint of the framework, which preserve the lengths of all bars at first order (see also Figure 2).

**Definition 2.3** [9, 30, 32] An infinitesimal motion \( u \) of a framework \((G, p)\) is an infinitesimal rigid motion if there exists a skew-symmetric matrix \( S \) (a rotation) and a vector \( t \) (a translation) such that \( u(v) = Sp(v) + t \) for all \( v \in V(G) \). Otherwise \( u \) is an infinitesimal flex of \((G, p)\).

**Definition 2.4** [9, 30, 32] A framework \((G, p)\) is infinitesimally rigid if every infinitesimal motion of \((G, p)\) is an infinitesimal rigid motion. Otherwise \((G, p)\) is said to be infinitesimally flexible.

![Figure 2: The arrows indicate the non-zero displacement vectors of an infinitesimal rigid motion (a) and infinitesimal flexes (b, c) of frameworks in \( \mathbb{R}^2 \).](image)

For a framework \((G, p)\) whose underlying graph \( G \) has a vertex set that is indexed from 1 to \( n \), say \( V(G) = \{v_1, v_2, \ldots, v_n\} \), we will frequently denote \( p(v_i) \) by \( p_i \) for \( i = 1, 2, \ldots, n \). The \( k \)th component of a vector \( x \) is denoted by \( (x)_k \).

**Definition 2.5** [8, 9, 30, 32] Let \( G \) be a graph with \( V(G) = \{v_1, v_2, \ldots, v_n\} \) and let \( p : V(G) \to \mathbb{R}^d \). The rigidity matrix of \((G, p)\) is the \(|E(G)| \times dn \)
matrix

\[
\begin{pmatrix}
0 & \ldots & 0 & p_i - p_j & 0 & \ldots & 0 & p_j - p_i & 0 & \ldots & 0 \\
\vdots & & & & & & & & & & \\
\end{pmatrix},
\]

that is, for each edge \(\{v_i, v_j\} \in E(G)\), \(R(G, p)\) has the row with \((p_i - p_j)_1, \ldots, (p_i - p_j)_d\) in the columns \(d(i - 1) + 1, \ldots, di, (p_j - p_i)_1, \ldots, (p_j - p_i)_d\) in the columns \(d(j - 1) + 1, \ldots, dj\), and 0 elsewhere.

Note that if we identify an infinitesimal motion of \((G, p)\) with a column vector in \(\mathbb{R}^{dn}\) (by using the order on \(V(G)\)), then the kernel of the rigidity matrix \(R(G, p)\) is the space of all infinitesimal motions of \((G, p)\).

**Theorem 2.1** [2, 7] A framework \((G, p)\) in \(\mathbb{R}^d\) is infinitesimally rigid if and only if either \(\text{rank}(R(G, p)) = d|V(G)| - \binom{d+1}{2}\) or \(G\) is a complete graph \(K_n\) and the points \(p(v), v \in V(G)\), are affinely independent.

**Definition 2.6** [9, 30, 32] A framework \((G, p)\) is independent if the row vectors of the rigidity matrix \(R(G, p)\) are linearly independent. A framework which is both independent and infinitesimally rigid is called \textit{isostatic}.

An isostatic framework is minimal infinitesimally rigid, in the sense that the removal of any bar results in an infinitesimally flexible framework (see also [8, 9, 30, 32]).

**Theorem 2.2** [9, 32] For a \(d\)-dimensional realization \((G, p)\) of a graph \(G\) with \(|V(G)| \geq d\), the following are equivalent:

(i) \((G, p)\) is isostatic;

(ii) \((G, p)\) is infinitesimally rigid and \(|E(G)| = d|V(G)| - \binom{d+1}{2}\);

(iii) \((G, p)\) is independent and \(|E(G)| = d|V(G)| - \binom{d+1}{2}\);

### 2.3 Generic rigidity

Generic rigidity is concerned with the infinitesimal rigidity of ‘almost all’ geometric realizations of a given graph. One of the ‘standard’ definitions of ‘generic’ that is frequently used in rigidity theory is the following.
Definition 2.7 [8, 9] Let $G$ be a graph with $V(G) = \{v_1, \ldots, v_n\}$ and $K_n$ be the complete graph on $V(G)$. A framework $(G, p)$ is generic if the determinant of any submatrix of $R(K_n, p)$ is zero only if it is (identically) zero in the variables $p'_i$.

There are two fundamental facts regarding this definition of generic. First, it follows immediately from Definition 2.7 that the set of all generic realizations of a given graph $G$ in $\mathbb{R}^d$ forms a dense open subset of all possible realizations of $G$ in $\mathbb{R}^d$ [9]. Secondly, the infinitesimal rigidity properties are the same for all generic realizations of $G$, as the next result shows:

Theorem 2.3 [8, 9, 32] For a graph $G$ and a fixed dimension $d$, the following are equivalent:

(i) $(G, p)$ is infinitesimally rigid (independent, isostatic) for some map $p : V(G) \to \mathbb{R}^d$;

(ii) every $d$-dimensional generic realization of $G$ is infinitesimally rigid (independent, isostatic).

It follows that for generic frameworks, infinitesimal rigidity is purely combinatorial, and hence a property of the underlying graph. This gives rise to the following definition of infinitesimal rigidity for graphs:

Definition 2.8 A graph $G$ is generically $d$-rigid ($d$-independent, $d$-isostatic) if $d$-dimensional generic realizations of $G$ are infinitesimally rigid (independent, isostatic).

In 1970, G. Laman proved the following combinatorial characterization of generically 2-isostatic graphs.

Theorem 2.4 (Laman, 1970) [14] A graph $G$ with $|V(G)| \geq 2$ is generically 2-isostatic if and only if

(i) $|E(G)| = 2|V(G)| - 3$;

(ii) $|E(H)| \leq 2|V(H)| - 3$ for all $H \subseteq G$ with $|V(H)| \geq 2$. 

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Various proofs of Laman’s Theorem can be found in [8], [9], [16], [24], and [29], for example.

Throughout this paper, we will refer to the conditions (i) and (ii) in Theorem 2.4 as the Laman conditions.

A combinatorial characterization of generically isostatic graphs in dimension 3 or higher is not yet known. See [8, 9, 25], for example, for a detailed discussion of this problem.

There are some inductive construction techniques that preserve the generic rigidity properties of a graph. These construction techniques can be used to prove theorems such as Laman’s Theorem, to analyze graphs for generic rigidity, and to characterize generically 1-isostatic and 2-isostatic graphs. For all dimensions $d$, they provide a tool to generate classes of generically $d$-isostatic graphs.

**Definition 2.9** [25, 30] Let $G$ be a graph, $U \subseteq V(G)$ with $|U| = d$ and $v \notin V(G)$. Then the graph $\hat{G}$ with $V(\hat{G}) = V(G) \cup \{v\}$ and $E(\hat{G}) = E(G) \cup \{\{v, u\}|u \in U\}$ is called a vertex $d$-addition (by $v$) of $G$.

**Theorem 2.5 (Vertex Addition Theorem)** [8, 9, 25, 30] A vertex $d$-addition of a generically $d$-isostatic graph is generically $d$-isostatic. Conversely, deleting a vertex of valence $d$ from a generically $d$-isostatic graph results in a generically $d$-isostatic graph.

**Definition 2.10** [25, 30] Let $G$ be a graph, $U \subseteq V(G)$ with $|U| = d + 1$ and $\{u_1, u_2\} \in E(G)$ for some $u_1, u_2 \in U$. Further, let $v \notin V(G)$. Then the graph $\hat{G}$ with $V(\hat{G}) = V(G) \cup \{v\}$ and $E(\hat{G}) = (E(G) \setminus \{\{u_1, u_2\}\}) \cup \{\{v, u\}|u \in U\}$ is called an edge $d$-split (on $u_1, u_2; v$) of $G$.

**Theorem 2.6 (Edge Split Theorem)** [8, 9, 25, 30] An edge $d$-split of a generically $d$-isostatic graph is generically $d$-isostatic. Conversely, if one deletes a vertex $v$ of valence $d + 1$ from a generically $d$-isostatic graph, then one may add an edge between one of the pairs of vertices adjacent to $v$ so that the resulting graph is generically $d$-isostatic.

In 1911, L. Henneberg gave the following characterization of generically 2-isostatic graphs.

**Theorem 2.7 (Henneberg, 1911)** [13] A graph is generically 2-isostatic if and only if it may be constructed from a single edge by a sequence of vertex 2-additions and edge 2-splits.
Figure 3: Illustrations of the Vertex Addition Theorem (a) and the Edge Split Theorem (b) in dimension 2.

For a proof of Henneberg’s Theorem, see [9] or [25], for example.

Another way of characterizing generically 2-isostatic graphs is due to H. Crapo and uses partitions of a graph into edge disjoint trees.

Definition 2.11 [5, 15, 24] A $3\text{Tree2}$ partition of a graph $G$ is a partition of $E(G)$ into the edge sets of three edge disjoint trees $T_0, T_1, T_2$ such that each vertex of $G$ belongs to exactly two of the trees.

A $3\text{Tree2}$ partition is called proper if no non-trivial subtrees of distinct trees $T_i$ have the same span (i.e., the same vertex sets).

Figure 4: A proper (a) and a non-proper (b) $3\text{Tree2}$ partition.

Remark 2.1 If a graph $G$ has a $3\text{Tree2}$ partition, then it satisfies $|E(G)| = 2|V(G)| - 3$. This follows from the presence of exactly two trees at each vertex of $G$ and the fact that for every tree $T$ we have $|E(T)| = |V(T)| - 1$. Moreover, note that a $3\text{Tree2}$ partition of a graph $G$ is proper if and only if every non-trivial subgraph $H$ of $G$ satisfies the count $|E(H)| \leq 2|V(H)| - 3$ [15].

Theorem 2.8 (Crapo, 1989) [5] A graph $G$ is generically 2-isostatic if and only if $G$ has a proper $3\text{Tree2}$ partition.
3 Symmetric frameworks

A symmetry operation of a framework \((G, p)\) in \(\mathbb{R}^d\) is an isometry \(x\) of \(\mathbb{R}^d\) such that for some \(\alpha \in \text{Aut}(G)\), we have \(x(p(v)) = p(\alpha(v))\) for all \(v \in V(G)\) \([11, 17, 18, 19]\).

The set of all symmetry operations of a framework \((G, p)\) forms a group under composition, called the point group of \((G, p)\) \([1, 3, 11, 19, 17]\). Since translating a framework does not change its rigidity properties, we may assume wlog that the point group of any framework in this paper is a symmetry group, i.e., a subgroup of the orthogonal group \(O(\mathbb{R}^d)\) \([18, 19, 17]\).

We use the Schoenflies notation for the symmetry operations and symmetry groups considered in this paper, as this is one of the standard notations in the literature about symmetric structures (see \([1, 3, 4, 6, 10, 11, 17, 18, 19]\), for example). The three kinds of possible symmetry operations in dimension 2 are the identity \(I\), rotations \(C_m\) about the origin by an angle of \(\frac{2\pi}{m}\), where \(m \geq 2\), and reflections \(s\) in lines through the origin. In the Schoenflies notation, this gives rise to the following families of possible symmetry groups in dimension 2: \(C_1\), \(C_s\), \(C_m\) and \(C_{mv}\), where \(m \geq 2\). \(C_1\) denotes the trivial group which only contains the identity \(I\). \(C_s\) denotes any symmetry group in dimension 2 that consists of the identity \(I\) and a single reflection \(s\). For \(m \geq 2\), \(C_m\) denotes any cyclic symmetry group of order \(m\) which is generated by a rotation \(C_m\), and \(C_{mv}\) denotes any symmetry group in dimension 2 that is generated by a pair \(\{C_m, s\}\).

Given a symmetry group \(S\) in dimension \(d\) and a graph \(G\), we let \(\mathcal{R}_{(G,S)}\) denote the set of all \(d\)-dimensional realizations of \(G\) whose point group is either equal to \(S\) or contains \(S\) as a subgroup \([17, 18, 19]\). In other words, the set \(\mathcal{R}_{(G,S)}\) consists of all realizations \((G, p)\) of \(G\) for which there exists a map \(\Phi : S \to \text{Aut}(G)\) so that

\[
x(p(v)) = p(\Phi(x)(v)) \text{ for all } v \in V(G) \text{ and all } x \in S. \quad (1)
\]

A framework \((G, p) \in \mathcal{R}_{(G,S)}\) satisfying the equations in \((1)\) for the map \(\Phi : S \to \text{Aut}(G)\) is said to be of type \(\Phi\), and the set of all realizations in \(\mathcal{R}_{(G,S)}\) which are of type \(\Phi\) is denoted by \(\mathcal{R}_{(G,S,\Phi)}\) (see again \([17, 18, 19]\)).

Different choices of types \(\Phi : S \to \text{Aut}(G)\) frequently lead to very different geometric types of realizations of \(G\) within \(\mathcal{R}_{(G,S)}\). This is illustrated by the realizations of the complete bipartite graph \(K_{3,3}\) with mirror symmetry depicted in Figure 5. The framework in Figure 5 (a) is a realization in
\( \mathcal{R}(K_{3,3}, C_s) \) of type \( \Phi_a \), where \( \Phi_a : C_s \rightarrow \text{Aut}(K_{3,3}) \) is defined by

\[
\Phi_a(Id) = id \\
\Phi_a(s) = (v_1 v_2)(v_5 v_6)(v_3 v_4),
\]

and the framework in Figure 5 (b) is a realization in \( \mathcal{R}(K_{3,3}, C_s) \) of type \( \Phi_b \), where \( \Phi_b : C_s \rightarrow \text{Aut}(K_{3,3}) \) is defined by

\[
\Phi_b(Id) = id \\
\Phi_b(s) = (v_1 v_4)(v_2 v_5)(v_3 v_6).
\]

Note that ‘almost all’ realizations in \( \mathcal{R}(K_{3,3}, C_s, \Phi_a) \) are isostatic, whereas all realizations in \( \mathcal{R}(K_{3,3}, C_s, \Phi_b) \) are infinitesimally flexible since the joints of any realization in \( \mathcal{R}(K_{3,3}, C_s, \Phi_b) \) are forced to lie on a conic section [17, 28].

![Figure 5: 2-dimensional realizations in \( \mathcal{R}(K_{3,3}, C_s) \) of different types.](image)

**Remark 3.1** A set \( \mathcal{R}(G,S) \) can possibly be empty. For example, there clearly exists no 2-dimensional realization of \( K_2 \) in the set \( \mathcal{R}(K_2, C_2) \).

Given a non-empty set \( \mathcal{R}(G,S) \), it is also possible that \( \mathcal{R}(G,S, \Phi) = \emptyset \) for some map \( \Phi : S \rightarrow \text{Aut}(G) \).

Consider, for example, the non-empty set \( \mathcal{R}(K_2, C_2) \), where \( C_2 = \{Id, C_2\} \) is the half-turn symmetry group in dimension 2, and let \( I : C_2 \rightarrow \text{Aut}(K_2) \) be the map which sends both \( Id \) and \( C_2 \) to the identity automorphism of \( K_2 \). If \((K_2, p) \in \mathcal{R}(K_2, C_2, I)\), then both joints of \((K_2, p)\) must be located at the origin (which is the center of \( C_2 \)). This contradicts Definition 2.1 of a framework, and hence we have \( \mathcal{R}(K_2, C_2, I) = \emptyset \).

The following symmetry-adapted notion of generic for the set \( \mathcal{R}(G,S, \Phi) \) was introduced in [17] (see also [19]).
Definition 3.1 Let $G$ be a graph with $V(G) = \{v_1, \ldots, v_n\}$ and let $K_n$ be the complete graph on $V(G)$. Further, let $S$ be a symmetry group and $\Phi$ be a map from $S$ to $\text{Aut}(G)$. A framework $(G, p) \in \mathcal{R}_{(G,S,\Phi)}$ is $(S, \Phi)$-generic if the determinant of any submatrix of $\mathbf{R}(K_n, p)$ is zero only if it is zero for all $p'$ satisfying the symmetry equations in (1).

Intuitively, an $(S, \Phi)$-generic realization of a graph $G$ is obtained by placing the vertices of a set of representatives for the symmetry orbits $Sv = \{\Phi(x)(v) \mid x \in S\}$ into ‘generic’ positions. The positions for the remaining vertices of $G$ are then uniquely determined by the symmetry constraints imposed by $S$ and $\Phi$ (see [17, 19], for further details).

It is shown in [17] that the set of $(S, \Phi)$-generic realizations of a graph $G$ is an open dense subset of the set $\mathcal{R}_{(G,S,\Phi)}$. Moreover, the infinitesimal rigidity properties are the same for all $(S, \Phi)$-generic realizations of $G$, as the following theorem shows.

Theorem 3.1 [17, 19] Let $G$ be a graph, $S$ be a symmetry group, and $\Phi$ be a map from $S$ to $\text{Aut}(G)$ such that $\mathcal{R}_{(G,S,\Phi)} \neq \emptyset$. The following are equivalent.

(i) There exists a framework $(G, p) \in \mathcal{R}_{(G,S,\Phi)}$ that is infinitesimally rigid (independent, isostatic);

(ii) every $(S, \Phi)$-generic realization of $G$ is infinitesimally rigid (independent, isostatic).

So, being infinitesimally rigid (independent, isostatic) is an $(S, \Phi)$-generic property. This gives rise to

Definition 3.2 Let $G$ be a graph, $S$ be a symmetry group, and $\Phi$ be a map from $S$ to $\text{Aut}(G)$. Then $G$ is said to be $(S, \Phi)$-generically infinitesimally rigid (independent, isostatic) if all realizations of $G$ which are $(S, \Phi)$-generic are infinitesimally rigid (independent, isostatic).

Using techniques from group representation theory, it is shown in [4] that if a symmetric isostatic framework $(G, p)$ belongs to a set $\mathcal{R}_{(G,S,\Phi)}$, where $S$ is a non-trivial symmetry group and $\Phi : S \to \text{Aut}(G)$ is a homomorphism, then $(G, p)$ needs to satisfy certain restrictions on the number of joints and bars that are ‘fixed’ by various symmetry operations of $(G, p)$ (see also [6, 17, 19]). In the following, we summarize the key result for dimension 2.
Definition 3.3 [18, 19] Let $G$ be a graph with $V(G) = \{v_1, \ldots, v_n\}$, $S$ be a symmetry group, $\Phi$ be a map from $S$ to $\text{Aut}(G)$, $(G, p)$ be a framework in $\mathcal{R}(G, S, \Phi)$, and $x \in S$. A joint $(v_i, p_i)$ of $(G, p)$ is said to be fixed by $x$ with respect to $\Phi$ if $\Phi(x)(v_i) = v_i$.

Similarly, a bar $\{(v_i, p_i), (v_j, p_j)\}$ of $(G, p)$ is said to be fixed by $x$ with respect to $\Phi$ if $\Phi(x)\{v_i, v_j\} = \{v_i, v_j\}$.

The number of joints of $(G, p)$ that are fixed by $x$ with respect to $\Phi$ is denoted by $j_{\Phi(x)}$ and the number of bars of $(G, p)$ that are fixed by $x$ with respect to $\Phi$ is denoted by $b_{\Phi(x)}$.

Remark 3.2 If a joint $(v, p(v))$ of a framework $(G, p) \in \mathcal{R}(G, S, \Phi)$ is fixed by $x \in S$ with respect to $\Phi$, then we have $x(p(v)) = p(\Phi(x)(v)) = p(v)$. In particular, if $(G, p)$ is a 2-dimensional framework, then a joint that is fixed by a rotation $C_m \in S$ must lie at the center of $C_m$, and a joint that is fixed by a reflection $s \in S$ must lie on the mirror line corresponding to $s$. Similar geometric restrictions of course also apply for bars of $(G, p)$ that are fixed by various symmetry operations in $S$ (see [4, 17, 19] for details).

Theorem 3.2 [4, 19] Let $G$ be a graph, $S$ be a symmetry group in dimension 2, $\Phi : S \to \text{Aut}(G)$ be a homomorphism, and $(G, p)$ be an isostatic framework in $\mathcal{R}(G, S, \Phi)$ with the property that the points $p(v), v \in V(G)$, span all of $\mathbb{R}^2$. Then

(i) the Laman conditions are satisfied;
(ii) if $S = C_2$, then $j_{\Phi(C_2)} = 0$ and $b_{\Phi(C_2)} = 1$;
(iii) if $S = C_3$, then $j_{\Phi(C_3)} = 0$;
(iv) if $S = C_s$, then $b_{\Phi(s)} = 1$;
(v) if $S = C_{2v}$, then $j_{\Phi(C_2)} = 0$ and $b_{\Phi(C_2)} = b_{\Phi(s)} = 1$ for both reflections $s \in C_{2v}$;
(vi) if $S = C_{3v}$, then $j_{\Phi(C_3)} = 0$ and $b_{\Phi(s)} = 1$ for all reflections $s \in C_{3v}$;
(vii) $S$ is either the trivial group $C_1$ or one of the five non-trivial symmetry groups listed above.

Examples of isostatic frameworks for each of the point groups listed in Theorem 3.2 are given in [4, 19].
It was conjectured in [4] that the conditions identified in Theorem 3.2 (ii) - (vi), together with the Laman conditions, are also sufficient for an \((S, \Phi)\)-generic realization of \(G\) to be isostatic. In the following, we verify this conjecture for the symmetry group \(C_3\). In addition, we provide Henneberg-type and Crapo-type characterizations of \((C_3, \Phi)\)-generically isostatic graphs. The techniques used in these proofs are extended in [19] to prove the corresponding Laman-type conjectures for the groups \(C_2\) and \(C_s\) as well as analogous Henneberg-type and Crapo-type results for these groups. Characterizations of \((C_{2v}, \Phi)\)- or \((C_{3v}, \Phi)\)-generically isostatic graphs, however, have not yet been established (see again [19]).

While, initially, the fact that \(C_3\) allows the easiest and most natural proof for the Laman-type conjecture (as well as for a symmetric version of Crapo’s Theorem) came as somewhat of a surprise, we can now identify some clear indications for this.

For example, Crapo’s Theorem uses partitions of the edges of \(G\) into three edge-disjoint trees, so that it is most natural to extend this result to the cyclic group \(C_3\) of order three. Moreover, the condition \(j_{\Phi(C_3)} = 0\) implies that for any subgraph \(H\) of \(G\) with full \(C_3\) symmetry we must have that both \(|V(H)|\) and \(|E(H)|\) are multiples of three, so that \(H\) cannot satisfy the count \(|E(H)| = 2|V(H)| - 4\) or \(|E(H)| = 2|V(H)| - 5\). As we will see, this turns out to be extremely useful in the proof of the Laman-type result for \(C_3\).

4 Symmetric Henneberg moves and 3Tree2 partitions for \(C_3\)

We need the following inductive construction techniques to obtain a symmetrized Henneberg’s Theorem for \(C_3\).

![Diagram of Henneberg moves for \(C_3\)](image)

Figure 6: A \((C_3, \Phi)\) vertex addition of a graph \(G\), where \(\Phi(C_3) = \gamma\) and \(\Phi(C^2_3) = \gamma^2\).
Definition 4.1 Let $G$ be a graph, $\mathcal{C}_3 = \{Id, C_3, C_3^2\}$ be a symmetry group in dimension 2, and $\Phi : \mathcal{C}_3 \rightarrow \text{Aut}(G)$ be a homomorphism. Let $v_1, v_2$ be two distinct vertices of $G$ and $v, w, z \notin V(G)$. Then the graph $\hat{G}$ with $V(\hat{G}) = V(G) \cup \{v, w, z\}$ and $E(\hat{G}) = E(G) \cup \{\{v, v_1\}, \{v, v_2\}, \{w, \Phi(C_3)(v_1)\}, \{w, \Phi(C_3)(v_2)\}, \{z, \Phi(C_3^2)(v_1)\}, \{z, \Phi(C_3^2)(v_2)\}\}$ is called a $(\mathcal{C}_3, \Phi)$ vertex addition (by $(v, w, z)$) of $G$.

![Figure 7: A $(\mathcal{C}_3, \Phi)$ edge split of a graph $G$, where $\Phi(C_3) = \gamma$ and $\Phi(C_3^2) = \gamma^2$.](image)

Definition 4.2 Let $G$ be a graph, $\mathcal{C}_3 = \{Id, C_3, C_3^2\}$ be a symmetry group in dimension 2, and $\Phi : \mathcal{C}_3 \rightarrow \text{Aut}(G)$ be a homomorphism. Let $v_1, v_2, v_3$ be three distinct vertices of $G$ such that $\{v_1, v_2\} \in E(G)$ and not both of $v_1$ and $v_2$ are fixed by $\Phi(C_3)$ and let $v, w, z \notin V(G)$. Then the graph $\hat{G}$ with $V(\hat{G}) = V(G) \cup \{v, w, z\}$ and $E(\hat{G}) = (E(G) \setminus \{\{v_1, v_2\}, \{\Phi(C_3)(v_1), \Phi(C_3)(v_2)\}, \{\Phi(C_3^2)(v_1), \Phi(C_3^2)(v_2)\}\}) \cup \{\{v, v_i\} | i = 1, 2, 3\} \cup \{\{w, \Phi(C_3)(v_i)\} | i = 1, 2, 3\} \cup \{\{z, \Phi(C_3^2)(v_i)\} | i = 1, 2, 3\}$ is called a $(\mathcal{C}_3, \Phi)$ edge split (on $(\{v_1, v_2\}, \{\Phi(C_3)(v_1), \Phi(C_3^2)(v_2)\}, \{\Phi(C_3^2)(v_1), \Phi(C_3^2)(v_2)\}); (v, w, z)$) of $G$.

Definition 4.3 Let $G$ be a graph, $\mathcal{C}_3 = \{Id, C_3, C_3^2\}$ be a symmetry group in dimension 2, and $\Phi : \mathcal{C}_3 \rightarrow \text{Aut}(G)$ be a homomorphism. Let $v_0$ be a vertex of $G$ that is not fixed by $\Phi(C_3)$ and let $v, w, z \notin V(G)$. Then the graph $\hat{G}$ with $V(\hat{G}) = V(G) \cup \{v, w, z\}$ and $E(\hat{G}) = E(G) \cup \{\{v, w\}, \{w, z\}, \{z, v\}, \{v, v_0\}, \{w, \Phi(C_3)(v_0)\}, \{z, \Phi(C_3^2)(v_0)\}\}$ is called a $(\mathcal{C}_3, \Phi) \Delta$ extension (by $(v, w, z)$) of $G$.

Remark 4.1 Each of the constructions in Definitions 4.1, 4.2, and 4.3 has the property that if the graph $G$ satisfies the Laman conditions, then so does...
\[ \gamma^2(v_0) \quad \gamma(v_0) \]

\[ v \quad \gamma^2(v_0) \quad \gamma(v_0) \quad w \]

Figure 8: A \((C_3, \Phi) \Delta\) extension of a graph \(G\), where \(\Phi(C_3) = \gamma\) and \(\Phi(C_2^2) = \gamma^2\).

\(\tilde{G}\). This follows from Theorems 2.4 and 2.7 and the fact that we can obtain a \((C_3, \Phi)\) vertex addition of \(G\) by a sequence of three vertex 2-additions, a \((C_3, \Phi)\) edge split of \(G\) by a sequence of three edge 2-splits, and a \((C_3, \Phi)\) \(\Delta\) extension of \(G\) by a vertex 2-addition followed by two edge 2-splits.

In order to extend Crapo’s Theorem to \(C_3\) we need the following symmetrized definition of a 3Tree2 partition.

**Definition 4.4** Let \(G\) be a graph, \(C_3 = \{Id, C_3, C_3^2\}\) be a symmetry group in dimension 2, and \(\Phi : C_3 \rightarrow \text{Aut}(G)\) be a homomorphism. A \((C_3, \Phi)\) 3Tree2 partition of \(G\) is a 3Tree2 partition \(\{E(T_0), E(T_1), E(T_2)\}\) of \(G\) such that \(\Phi(C_3)(T_i) = T_{i+1}\) for \(i = 0, 1, 2\), where the indices are added modulo 3.

\[ T_0 \quad T_1 \quad T_2 \]

\[ v \quad \gamma^2(v) \quad \gamma(v) \quad \gamma^2(v) \quad \gamma(v) \quad w \quad \gamma^2(w) \]

Figure 9: \((C_3, \Phi)\) 3Tree2 partitions of graphs, where \(\Phi(C_3) = \gamma\) and \(\Phi(C_3^2) = \gamma^2\).
5 The main result

Theorem 5.1 Let $G$ be a graph with $|V(G)| \geq 3$, $C_3 = \{\text{Id}, C_3, C_3^2\}$ be a symmetry group in dimension 2, and $\Phi : C_3 \to \text{Aut}(G)$ be a homomorphism. The following are equivalent:

(i) $R(G, C_3, \Phi) \neq \emptyset$ and $G$ is $(C_3, \Phi)$-generically isostatic;

(ii) $|E(G)| = 2|V(G)| - 3$, $|E(H)| \leq 2|V(H)| - 3$ for all $H \subseteq G$ with $|V(H)| \geq 2$ (Laman conditions), and $j_{\Phi(C_3)} = 0$;

(iii) there exists a $(C_3, \Phi)$ construction sequence

\[(K_3, \Phi_0) = (G_0, \Phi_0), (G_1, \Phi_1), \ldots, (G_k, \Phi_k) = (G, \Phi)\]

such that

(a) $G_{i+1}$ is a $(C_3, \Phi_i)$ vertex addition, a $(C_3, \Phi_i)$ edge split, or a $(C_3, \Phi_i)$ $\Delta$ extension of $G_i$ with $V(G_{i+1}) = V(G_i) \cup \{v_{i+1}, w_{i+1}, z_{i+1}\}$ for all $i = 0, 1, \ldots, k - 1$;

(b) $\Phi_0 : C_3 \to \text{Aut}(K_3)$ is a non-trivial homomorphism and for all $i = 0, 1, \ldots, k - 1$, $\Phi_{i+1} : C_3 \to \text{Aut}(G_{i+1})$ is the homomorphism defined by $\Phi_{i+1}(x)|_{V(G_i)} = \Phi_i(x)$ for all $x \in C_3$ and $\Phi_{i+1}(C_3)|_{\{v_{i+1}, w_{i+1}, z_{i+1}\}} = (v_{i+1} w_{i+1} z_{i+1})$;

(iv) $G$ has a proper $(C_3, \Phi)$ 3Tree2 partition.

We break the proof of this result up into four Lemmas.

Lemma 5.2 Let $G$ be a graph with $|V(G)| \geq 3$, $C_3 = \{\text{Id}, C_3, C_3^2\}$ be a symmetry group in dimension 2, and $\Phi : C_3 \to \text{Aut}(G)$ be a homomorphism. If $R(G, C_3, \Phi) \neq \emptyset$ and $G$ is $(C_3, \Phi)$-generically isostatic, then $G$ satisfies the Laman conditions and we have $j_{\Phi(C_3)} = 0$.

Proof. The result follows immediately from Laman’s Theorem (Theorem 2.4) and Theorem 3.2. □

Lemma 5.3 Let $G$ be a graph with $|V(G)| \geq 3$, $C_3 = \{\text{Id}, C_3, C_3^2\}$ be a symmetry group in dimension 2, and $\Phi : C_3 \to \text{Aut}(G)$ be a homomorphism. If $G$ satisfies the Laman conditions and we also have $j_{\Phi(C_3)} = 0$, then there exists a $(C_3, \Phi)$ construction sequence for $G$. 

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**Proof.** We employ induction on $|V(G)|$. Note first that if for a graph $G$, there exists a homomorphism $\Phi : C_3 \to \text{Aut}(G)$ such that $j_{\Phi(C_3)} = 0$, then $|V(G)| \equiv 0 \pmod{3}$. The only graph with three vertices that satisfies the Laman conditions is the graph $K_3$ and if $\Phi : C_3 \to \text{Aut}(K_3)$ is a homomorphism such that $j_{\Phi(C_3)} = 0$, then $\Phi$ is clearly a non-trivial homomorphism. This proves the base case.

So we let $n \geq 3$ and we assume that the result holds for all graphs with $n$ or fewer than $n$ vertices.

Let $G$ be a graph with $|V(G)| = n + 3$ that satisfies the Laman conditions and suppose $j_{\Phi(C_3)} = 0$ for a homomorphism $\Phi : C_3 \to \text{Aut}(G)$. In the following, we denote $\Phi(C_3)$ by $\gamma$ and $\Phi(C_3^2)$ by $\gamma^2$.

Since $G$ satisfies the Laman conditions, it is easy to verify that $G$ has a vertex of valence 2 or 3 (see [8, 9, 19], for example).

We assume first that $G$ has a vertex $v$ of valence 2, say $N_G(v) = \{v_1, v_2\}$. Note that $v, \gamma(v)$ and $\gamma^2(v)$ are three distinct vertices of $G$, because $j_{\gamma} = 0$. Suppose two of these vertices are adjacent, wlog $\{v, \gamma(v)\} \in E(G)$. Then $\{\gamma(v), \gamma^2(v)\}, \{\gamma^2(v), v\} \in E(G)$, because $\gamma \in \text{Aut}(G)$. Let $G' = G - \{v, \gamma(v), \gamma^2(v)\}$. Then

$$|E(G')| = |E(G)| - 3 = 2|V(G)| - 6 = 2|V(G')|.$$ 

Since $|V(G)| \geq 6$, we have $|V(G')| \geq 3$, and hence $G'$ violates the Laman conditions, a contradiction.

Therefore, $\{v, \gamma(v), \gamma^2(v)\}$ is an independent subset of $V(G)$, which says that the six edges $\{v, v_i\}, \{\gamma(v), \gamma(v_i)\}, \{\gamma^2(v), \gamma^2(v_i)\}, i = 1, 2$, are all pairwise distinct. Thus,

$$|E(G')| = |E(G)| - 6 = 2|V(G)| - 9 = 2|V(G')| - 3.$$ 

Also, for $H \subseteq G'$ with $|V(H)| \geq 2$, we have $H \subseteq G$, and hence

$$|E(H)| \leq 2|V(H)| - 3.$$ 

Therefore, $G'$ satisfies the Laman conditions.

Let $\Phi' : C_3 \to \text{Aut}(G')$ be the homomorphism with $\Phi'(x) = \Phi(x)|_{V(G')}$ for all $x \in C_3$. Then we have $j_{\Phi'(C_3)} = 0$, and hence, by the induction hypothesis, there exists a sequence

$$(K_3, \Phi_0) = (G_0, \Phi_0), (G_1, \Phi_1), \ldots, (G_k, \Phi_k) = (G', \Phi')$$

satisfying the conditions in Theorem 5.1 (iii). Since $G$ is a $(C_3, \Phi')$ vertex addition of $G'$ with $V(G) = V(G') \cup \{v, \gamma(v), \gamma^2(v)\}$,

$$(K_3, \Phi_0) = (G_0, \Phi_0), (G_1, \Phi_1), \ldots, (G', \Phi'), (G, \Phi)$$

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is a sequence with the desired properties.

Suppose now that $G$ has a vertex $v$ of valence 3, say $N_G(v) = \{v_1, v_2, v_3\}$, and no vertex of valence 2. Note that $v, \gamma(v)$ and $\gamma^2(v)$ are again three distinct vertices of $G$, as are $v_i, \gamma(v_i)$ and $\gamma^2(v_i)$ for each $i = 1, 2, 3$, because $j_\gamma = 0$. We need to consider the following three cases (see also Figure 10):

**Case 1:** $\{v, \gamma(v), \gamma^2(v)\}$ is an independent subset of $V(G)$ and all three of these vertices share a common neighbor, say wlog $v_1$. Since $\gamma \in \text{Aut}(G)$, this says that each of $v, \gamma(v)$ and $\gamma^2(v)$ has the same neighbors, namely $v_1, \gamma(v_1)$ and $\gamma^2(v_1)$.

**Case 2:** $\{v, \gamma(v), \gamma^2(v)\}$ is an independent subset of $V(G)$ and there is no vertex in $G$ that is adjacent to all three of these vertices.

**Case 3:** $\{v, \gamma(v), \gamma^2(v)\}$ is not independent in $G$ and hence $\{v, \gamma(v)\}, \{\gamma(v), \gamma^2(v)\}, \{\gamma^2(v), v\} \in E(G)$.

![Figure 10](image-url)

Figure 10: If a graph $G$ satisfies the conditions in Theorem 5.1 (ii) and has a vertex $v$ of valence 3, then $G$ is a graph of one of the types depicted above.

**Case 1:** By Theorems 2.4 and 2.6 there exists a pair $\{a, b\}$ of vertices in $\{v_1, \gamma(v_1), \gamma^2(v_1)\}$ such that $G - \{v\} + \{a, b\}$ satisfies the Laman conditions. By the same argument, applied two more times, it follows that the graph $\Gamma = G - \{\{v, \gamma(v), \gamma^2(v)\}\} + \{\{v_1, \gamma(v_1)\}, \{\gamma(v_1), \gamma^2(v_1)\}, \{\gamma^2(v_1), v_1\}\}$ also satisfies the Laman conditions.

Further, if we define $\tilde{\Phi}$ by $\tilde{\Phi}(x) = \Phi(x)|_{V(\Gamma)}$ for all $x \in C_3$, then $\tilde{\Phi}(x) \in \text{Aut}(\Gamma)$ for all $x \in C_3$ and $\tilde{\Phi} : C_3 \to \text{Aut}(\Gamma)$ is a homomorphism. Since we clearly also have $j_{\Phi(C_3)} = 0$, it follows from the induction hypothesis that there exists a sequence

$$(K_3, \Phi_0) = (G_0, \Phi_0), (G_1, \Phi_1), \ldots, (G_k, \Phi_k) = (\Gamma, \tilde{\Phi})$$
satisfying the conditions in Theorem 5.1 (iii). Since $G$ is a $(C_3, \Phi)$ edge split of $\tilde{G}$ with $V(G) = V(\tilde{G}) \cup \{v, \gamma(v), \gamma^2(v)\}$,

$$(K_3, \Phi_0) = (G_0, \Phi_0), (G_1, \Phi_1), \ldots, (\tilde{G}, \tilde{\Phi}), (G, \Phi)$$

is a sequence with the desired properties.

**Case 2:** By Theorems 2.24 and 2.26 there exists $\{i_2, j_2\} \subseteq \{1, 2, 3\}$ such that $G_2 = G - \{\gamma^2(v)\} + \{(\gamma^2(v_{i_2}), \gamma^2(v_{j_2}))\}$ satisfies the Laman conditions. By the same argument, there exist $\{i_1, j_1\} \subseteq \{1, 2, 3\}$ and $\{i_0, j_0\} \subseteq \{1, 2, 3\}$ such that both $G_1 = G_2 - \{\gamma(v)\} + \{(\gamma(v_{i_1}), \gamma(v_{j_1}))\}$ and $G_0 = G_1 - \{v\} + \{(v_{i_0}, v_{j_0})\}$ also satisfy the Laman conditions. We assume wlog that $\{i_0, j_0\} = \{1, 2\}$. Then for every subgraph $H$ of $G_0 - \{v_1, v_2\}$ with $v_1, v_2 \in V(H)$ we have $|E(H)| \leq 2|V(H)| - 4$. Moreover, for every subgraph $H$ of $G' = G - \{v, \gamma(v), \gamma^2(v)\}$ with $v_1, v_2 \in V(H)$ we also have $|E(H)| \leq 2|V(H)| - 4$, because $G'$ is obtained from $G_0 - \{v_1, v_2\}$ by deleting the edges $\{\gamma(v_{i_1}), \gamma(v_{j_1})\}$ and $\{\gamma^2(v_{i_2}), \gamma^2(v_{j_2})\}$ (see also Figure 11).

![Figure 11: The graphs $G$, $G_0$ and $G'$ in Case 2 of the proof of Lemma 5.3.](image-url)

Since $G'$ is invariant under $\gamma$, every subgraph $H$ of $G'$ with $\gamma(v_1), \gamma(v_2) \in V(H)$ or $\gamma^2(v_1), \gamma^2(v_2) \in V(H)$ also satisfies $|E(H)| \leq 2|V(H)| - 4$. Note that $\{v_1, v_2\}, \{\gamma(v_1), \gamma(v_2)\}$ and $\{\gamma^2(v_1), \gamma^2(v_2)\}$ are three distinct pairs of vertices (though not edges, by the above counts) of $G$, as the following argument shows.

Suppose $\{v_1, v_2\} = \{\gamma(v_1), \gamma(v_2)\}$. Then $v_1 = \gamma(v_2)$ and $v_2 = \gamma(v_1)$, because $G$ satisfies $j_\gamma = 0$. Therefore, $\gamma(v_1) = \gamma^2(v_2)$, and hence $v_2 = \gamma^2(v_2)$, contradicting $j_\gamma = 0$. Similarly, $\{\gamma(v_1), \gamma(v_2)\} \neq \{\gamma^2(v_1), \gamma^2(v_2)\}$ and $\{v_1, v_2\} \neq \{\gamma^2(v_1), \gamma^2(v_2)\}$.

We claim that $\tilde{G} = G' + \{\gamma(v_1), \gamma(v_2), \gamma^2(v_1), \gamma^2(v_2)\}$ satisfies the Laman conditions. We clearly have

$$|E(\tilde{G})| = |E(G')| + 3 = |E(G)| - 6 = 2|V(G)| - 9 = 2|V(\tilde{G})| - 3.$$

Suppose there exists a subgraph $H$ of $G'$ with $v_1, v_2, \gamma(v_1), \gamma(v_2) \in V(H)$ and $|E(H)| = 2|V(H)| - 4$. Then there also exists $\gamma(H) \subseteq G'$ with $\gamma(v_1), \gamma(v_2), \gamma^2(v_1), \gamma^2(v_2) \in V(\gamma(H))$ and $|E(\gamma(H))| = 2|V(\gamma(H))| - 4$, as
well as $\gamma^2(H) \subseteq G'$ with $\gamma^2(v_1), \gamma^2(v_2), v_1, v_2 \in V(\gamma^2(H))$ and $|E(\gamma^2(H))| = 2|V(\gamma^2(H))| - 4$, because $G'$ is invariant under $\gamma$. Let $H' = H \cup \gamma(H)$. Then

$$|E(H')| = |E(H)| + |E(\gamma(H))| - |E(H \cap \gamma(H))|$$

$$\geq 2|V(H)| - 4 + 2|V(\gamma(H))| - 4 - (2|V(H \cap \gamma(H))| - 4)$$

$$= 2|V(H')| - 4,$$

because $H \cap \gamma(H)$ is a subgraph of $G'$ with $\gamma(v_1), \gamma(v_2) \in V(H \cap \gamma(H))$. Since $H'$ is also a subgraph of $G'$ with $\gamma(v_1), \gamma(v_2) \in V(H')$, it follows that

$$|E(H')| = 2|V(H')| - 4.$$

Similarly, it can be shown that $H'' = H' \cup \gamma^2(H)$ satisfies

$$|E(H'')| = 2|V(H'')| - 4,$$

because $H' \cap \gamma^2(H)$ is a subgraph of $G'$ with $v_1, v_2 \in V(H' \cap \gamma^2(H))$. However, $H''$ is invariant under $\gamma$ and satisfies $j_{\gamma''(v_1)} = 0$ (mod 3) and $|E(H'')| \equiv 0$ (mod 3), contradicting the count $|E(H'')| = 2|V(H'')| - 4$.

Therefore, every subgraph $H$ of $G'$ with $v_1, v_2, \gamma(v_1), \gamma(v_2) \in V(H), \gamma(v_1), \gamma(v_2), \gamma^2(v_1), \gamma^2(v_2) \in V(H)$, or $\gamma^2(v_1), \gamma^2(v_2), v_1, v_2 \in V(H)$ satisfies $|E(H)| \leq 2|V(H)| - 5$.

It is now only left to show that for every subgraph $H$ of $G'$ with $v_1, v_2, \gamma(v_1), \gamma(v_2), \gamma^2(v_1), \gamma^2(v_2) \in V(H)$, we have $|E(H)| \leq 2|V(H)| - 6$. Suppose to the contrary that there exists a subgraph $H$ of $G'$ with $v_1, v_2, \gamma(v_1), \gamma(v_2), \gamma^2(v_1), \gamma^2(v_2) \in V(H)$ and $|E(H)| = 2|V(H)| - 5$. Then there also exist $\gamma(H) \subseteq G'$ and $\gamma^2(H) \subseteq G'$ with the same properties, because $G'$ is invariant under $\gamma$. Let $H' = H \cup \gamma(H)$. Then

$$|E(H')| = |E(H)| + |E(\gamma(H))| - |E(H \cap \gamma(H))|$$

$$\geq 2|V(H)| - 5 + 2|V(\gamma(H))| - 5 - (2|V(H \cap \gamma(H))| - 5)$$

$$= 2|V(H')| - 5,$$

because $H \cap \gamma(H)$ is a subgraph of $G'$ with $v_1, v_2, \gamma(v_1), \gamma(v_2) \in V(H \cap \gamma(H))$. Since $H'$ is also a subgraph of $G'$ with $v_1, v_2, \gamma(v_1), \gamma(v_2) \in V(H')$, it follows that

$$|E(H')| = 2|V(H')| - 5.$$

Similarly, it can be shown that $H'' = H' \cup \gamma^2(H)$ satisfies

$$|E(H'')| = 2|V(H'')| - 5,$$

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because $H' \cap \gamma^2(H)$ is a subgraph of $G'$ with $v_1, v_2, \gamma(v_1), \gamma(v_2) \in V(H' \cap \gamma^2(H))$. However, $H''$ is invariant under $\gamma$ and we have $j_{\gamma|V(H'')} = 0$, so that $|V(H'')| \equiv 0 \pmod{3}$ and $|E(H'')| \equiv 0 \pmod{3}$, contradicting the count $|E(H'')| = 2|V(H'')| - 5$.

Thus, $\tilde{G} = G' + \{v_1, v_2\}, \{\gamma(v_1), \gamma(v_2)\}, \{\gamma^2(v_1), \gamma^2(v_2)\}$ indeed satisfies the Laman conditions.

Further, if we define $\tilde{\Phi}$ by $\tilde{\Phi}(x) = \Phi(x)|_{V(\tilde{G})}$ for all $x \in C_3$, then $\tilde{\Phi}(x) \in \text{Aut}(\tilde{G})$ for all $x \in C_3$ and $\tilde{\Phi} : C_3 \to \text{Aut}(\tilde{G})$ is a homomorphism. Since we also have $j_{\Phi(C_3)} = 0$, it follows from the induction hypothesis that there exists a sequence

$$(K_3, \Phi_0) = (G_0, \Phi_0), (G_1, \Phi_1), \ldots, (G_k, \Phi_k) = (\tilde{G}, \tilde{\Phi})$$

satisfying the conditions in Theorem 5.1 (iii). Since $G$ is a $(C_3, \Phi)$ edge split of $\tilde{G}$ with $V(G) = V(\tilde{G}) \cup \{v, \gamma(v), \gamma^2(v)\}$,

$$(K_3, \Phi_0) = (G_0, \Phi_0), (G_1, \Phi_1), \ldots, (\tilde{G}, \tilde{\Phi}), (G, \Phi)$$

is a sequence with the desired properties.

Case 3: Note that $G' = G - \{v, \gamma(v), \gamma^2(v)\}$ satisfies

$$|E(G')| = |E(G)| - 6 = 2|V(G)| - 9 = 2|V(G')| - 3.$$

Also, for $H \subseteq G'$ with $|V(H)| \geq 2$, we have $H \subseteq G$, and hence

$$|E(H)| \leq 2|V(H)| - 3,$$

so that $G'$ satisfies the Laman conditions.

If we define $\Phi'$ by $\Phi'(x) = \Phi(x)|_{V(G')}$ for all $x \in C_3$, then $\Phi'(x) \in \text{Aut}(G')$ for all $x \in C_3$ and $\Phi' : C_3 \to \text{Aut}(G')$ is a homomorphism. Since we also have $j_{\Phi'(C_3)} = 0$, it follows from the induction hypothesis that there exists a sequence

$$(K_3, \Phi_0) = (G_0, \Phi_0), (G_1, \Phi_1), \ldots, (G_k, \Phi_k) = (G', \Phi')$$

satisfying the conditions in Theorem 5.1 (iii). Since $G$ is a $(C_3, \Phi') \Delta$ extension of $G'$ with $V(G) = V(G') \cup \{v, \gamma(v), \gamma^2(v)\}$,

$$(K_3, \Phi_0) = (G_0, \Phi_0), (G_1, \Phi_1), \ldots, (G', \Phi'), (G, \Phi)$$

is a sequence with the desired properties. □
Lemma 5.4 Let $G$ be a graph with $|V(G)| \geq 3$, $C_3 = \{Id, C_3, C_3^2\}$ be a symmetry group in dimension 2, and $\Phi : C_3 \to Aut(G)$ be a homomorphism. If there exists a $(C_3, \Phi)$ construction sequence for $G$, then $G$ has a proper $(C_3, \Phi)$ 3Tree2 partition.

Proof. We proceed by induction on $|V(G)|$. Let $V(K_3) = \{v_1, v_2, v_3\}$ and wlog let $\Phi : C_3 \to Aut(K_3)$ be the homomorphism defined by $\Phi(C_3) = (v_1 v_2 v_3)$. Then $K_3$ has the proper $(C_3, \Phi)$ 3Tree2 partition $\{E(T_0), E(T_1), E(T_2)\}$, where $T_0 = \{\{v_1, v_2\}\}$, $T_1 = \{\{v_2, v_3\}\}$ and $T_2 = \{\{v_3, v_1\}\}$.

Assume, then, that the result holds for all graphs with $n$ or fewer than $n$ vertices, where $n \geq 3$.

Let $G$ be a graph with $|V(G)| = n + 3$ and let $\Phi : C_3 \to Aut(G)$ be a homomorphism such that there exists a $(C_3, \Phi)$ construction sequence

$$(K_3, \Phi_0) = (G_0, \Phi_0), (G_1, \Phi_1), \ldots, (G_k, \Phi_k) = (G, \Phi)$$

satisfying the conditions in Theorem 5.1 (iii). By Remark 4.1, $G$ satisfies the Laman conditions, and hence, by Remark 2.1 any 3Tree2 partition of $G$ must be proper. Therefore, it suffices to show that $G$ has some $(C_3, \Phi)$ 3Tree2 partition. We let $\Phi(C_3) = \gamma$ and $\Phi(C_3^2) = \gamma^2$.

By the induction hypothesis, $G_{k-1}$ has a $(C_3, \Phi_{k-1})$ 3Tree2 partition $\{E(T_{0}^{(k-1)}), E(T_{1}^{(k-1)}), E(T_{2}^{(k-1)})\}$. In the following, we compute the indices $i$ of the trees $T_{i}^{(k-1)}$ modulo 3.

Suppose first that $G$ is a $(C_3, \Phi_{k-1})$ vertex addition by $(v w z)$ of $G_{k-1}$ with $N_G(v) = \{v_1, v_2\}$, where $w = \gamma(v)$ and $z = \gamma^2(v)$. Since $\Phi_{k-1}(C_3) = \gamma|V(G_{k-1})$ we have $N_G(w) = \{\gamma(v_1), \gamma(v_2)\}$ and $N_G(z) = \{\gamma^2(v_1), \gamma^2(v_2)\}$. Note that both $v_1$ and $v_2$ belong to exactly two of the trees $T_i^{(k-1)}$. Therefore, there exists $l \in \{0, 1, 2\}$ such that $v_1 \in V(T_{l}^{(k-1)})$ and $v_2 \in V(T_{l+1}^{(k-1)})$. It follows that $\gamma(v_1) \in V(T_{l+1}^{(k-1)})$, $\gamma^2(v_1), \gamma(v_2) \in V(T_{l+2}^{(k-1)})$ and $\gamma^2(v_2) \in V(T_{l}^{(k-1)})$.

So, if we define $T_{l}^{(k)}$ to be the tree with

$V(T_{l}^{(k)}) = V(T_{l}^{(k-1)}) \cup \{v, z\}$

$E(T_{l}^{(k)}) = E(T_{l}^{(k-1)}) \cup \{\{v, v_1\}, \{z, \gamma^2(v_2)\}\}$,

$T_{l+1}^{(k)}$ to be the tree with

$V(T_{l+1}^{(k)}) = V(T_{l+1}^{(k-1)}) \cup \{v, w\}$

$E(T_{l+1}^{(k)}) = E(T_{l+1}^{(k-1)}) \cup \{\{v, v_2\}, \{w, \gamma(v_1)\}\}$,
and $T_{l+2}$ to be the tree with
\[
V(T_{l+2}) = V(T_{l+2}^{(k-1)}) \cup \{w, z\}
\]
\[
E(T_{l+2}^{(k)}) = E(T_{l+2}^{(k-1)}) \cup \{w, \gamma(v_2), \{z, \gamma^2(v_1)\}\},
\]
then \( E(T_0^{(k)}), E(T_1^{(k)}), E(T_2^{(k)}) \) is a \((C_3, \Phi) 3\text{Tree2 partition of } G. \)

![Diagram](image_url)

**Figure 12:** Construction of a \((C_3, \Phi) 3\text{Tree2 partition of } G \) in the case where \( G \) is a \((C_3, \Phi_{k-1}) \) vertex addition of \( G_{k-1} \).

Suppose next that \( G \) is a \((C_3, \Phi_{k-1}) \) edge split on \( \{v_1, v_2\}, \{\gamma(v_1), \gamma(v_2)\}, \{\gamma^2(v_1), \gamma^2(v_2)\} \); \( (v, w, z) \) of \( G_{k-1} \) with \( E(G_k) = (E(G_{k-1}) \setminus \{v_1, v_2\}, \{\gamma(v_1), \gamma(v_2)\}, \{\gamma^2(v_1), \gamma^2(v_2)\}) \) \( \cup \{v, v_i\} \mid i = 1, 2, 3 \} \cup \{z, \gamma^2(v_i) \mid i = 1, 2, 3\} \), where \( w = \gamma(v) \) and \( z = \gamma^2(v) \). Wlog we may assume that \( \{v_1, v_2\} \in E(T_0^{(k-1)}) \). Then \( \{\gamma(v_1), \gamma(v_2)\} \in E(T_1^{(k-1)}) \) and \( \{\gamma^2(v_1), \gamma^2(v_2)\} \in E(T_2^{(k-1)}) \). Note that \( v_3 \) belongs to a tree \( T_1^{(k-1)} \), where \( l \neq 0 \). Suppose \( v_3 \in T_1^{(k-1)} \). Then \( \gamma(v_3) \in T_2^{(k-1)} \) and \( \gamma^2(v_3) \in T_0^{(k-1)} \). So, if we define \( T_0^{(k)} \) to be the tree with
\[
V(T_0^{(k)}) = V(T_0^{(k-1)}) \cup \{v, z\}
\]
\[
E(T_0^{(k)}) = (E(T_0^{(k-1)}) \setminus \{v_1, v_2\}) \cup \{\{v, v_1\}, \{v, v_2\}, \{z, \gamma^2(v_3)\}\},
\]

\( T_1^{(k)} \) to be the tree with
\[
V(T_1^{(k)}) = V(T_1^{(k-1)}) \cup \{w, v\}
\]
\[
E(T_1^{(k)}) = (E(T_1^{(k-1)}) \setminus \{\gamma(v_1), \gamma(v_2)\})
\]
\[
\cup \{w, \gamma(v_1), \{w, \gamma(v_2), \{v, v_3\}\},
\]

and \( T_2^{(k)} \) to be the tree with
\[
V(T_2^{(k)}) = V(T_2^{(k-1)}) \cup \{z, w\}
\]
\[
E(T_2^{(k)}) = (E(T_2^{(k-1)}) \setminus \{\gamma^2(v_1), \gamma^2(v_2)\})
\]
\[
\cup \{z, \gamma^2(v_1), \{z, \gamma^2(v_2), \{w, \gamma(v_3)\}\},
\]

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then \( \{ E(T_0^{(k)}), E(T_1^{(k)}), E(T_2^{(k)}) \} \) is a \((C_3, \Phi)\) 3Tree2 partition of \( G \). If \( v_3 \in T_2^{(k-1)} \), then we obtain a \((C_3, \Phi)\) 3Tree2 partition of \( G \) in an analogous manner.

Finally, suppose that \( G \) is a \((C_3, \Phi_{k-1})\) \( \Delta \) extension by \((v \ w \ z)\) of \( G_{k-1} \) with 

\[
E(G) = E(G_{k-1}) \cup \{ \{v, w\}, \{w, z\}, \{z, v\}, \{v, v_0\}, \{w, \gamma(v_0)\}, \{z, \gamma_2(v_0)\} \},
\]

where \( w = \gamma(v) \) and \( z = \gamma_2(v) \). Wlog we may assume that \( v_0 \in V(T_0^{(k-1)}) \).

Then \( \gamma(v_0) \in V(T_1^{(k-1)}) \) and \( \gamma_2(v_0) \in V(T_2^{(k-1)}) \). So, if we define \( T_0^{(k)} \) to be the tree with 

\[
V(T_0^{(k)}) = V(T_0^{(k-1)}) \cup \{v, w\}, 
E(T_0^{(k)}) = E(T_0^{(k-1)}) \cup \{ \{v, v_0\}, \{v, w\} \},
\]
of a graph $G$ with $V(G) = \{v_1, v_2, \ldots, v_n\}$. A frame in $\mathbb{R}^2$ is a triple $(G, p, q)$, where $p : V(G) \to \mathbb{R}^2$ and $q : E(G) \to \mathbb{R}^2 \setminus \{0\}$ are maps with the property that for all $\{v_i, v_j\} \in E(G)$ there exists a scalar $\lambda_{ij} \in \mathbb{R}$ (which is possibly zero) such that $p(v_i) - p(v_j) = \lambda_{ij}q(\{v_i, v_j\})$.

**Definition 5.2** The generalized rigidity matrix of a frame $(G, p, q)$ in $\mathbb{R}^2$ is the matrix

$$
R(G, p, q) = \begin{pmatrix}
0 & \ldots & 0 & q(\{v_i, v_j\}) & 0 & \ldots & 0 & -q(\{v_i, v_j\}) & 0 & \ldots & 0 \\
\vdots & & & & & & & & & & & \\
0 & \ldots & 0 & q(\{v_i, v_j\}) & 0 & \ldots & 0 & -q(\{v_i, v_j\}) & 0 & \ldots & 0
\end{pmatrix},
$$

i.e., for each edge $\{v_i, v_j\} \in E(G)$, $R(G, p, q)$ has the row with $(q(\{v_i, v_j\}))_1$ and $(q(\{v_i, v_j\}))_2$ in the columns $2i - 1$ and $2i$, $-(q(\{v_i, v_j\}))_1$ and $-(q(\{v_i, v_j\}))_2$ in the columns $2(j - 1)$ and $2j$, and 0 elsewhere.

We say that $(G, p, q)$ is independent if $R(G, p, q)$ has linearly independent rows.

**Remark 5.1** If $(G, p, q)$ is a frame with the property that $p(v_i) \neq p(v_j)$ whenever $\{v_i, v_j\} \in E(G)$, then we obtain the rigidity matrix of the framework $(G, p)$ by multiplying each row of $R(G, p, q)$ by its corresponding scalar $\lambda_{ij}$. Therefore, if $(G, p, q)$ is independent, so is $(G, p)$.
Lemma 5.5 Let $G$ be a graph with $|V(G)| \geq 3$, $C_3 = \{Id, C_3, C_3^2\}$ be a symmetry group in dimension 2, and $\Phi : C_3 \rightarrow \text{Aut}(G)$ be a homomorphism. If $G$ has a proper $(C_3, \Phi)$ 3Tree2 partition, then $\mathcal{R}_{(G, C_3, \Phi)} \neq \emptyset$ and $G$ is $(C_3, \Phi)$-generically isostatic.

Proof. Suppose $G$ has a proper $(C_3, \Phi)$ 3Tree2 partition $\{E(T_0), E(T_1), E(T_2)\}$. By Theorem 3.1 it suffices to find some framework $(G, p) \in \mathcal{R}_{(G, C_3, \Phi)}$ that is isostatic. Since $G$ has a 3Tree2 partition, $G$ satisfies the count $|E(G)| = 2|V(G)| - 3$ (see Remark 2.1), and hence, by Theorem 2.2, it suffices to find a map $p : V(G) \rightarrow \mathbb{R}^2$ such that $(G, p) \in \mathcal{R}_{(G, C_3, \Phi)}$ is independent. In the following, we again denote $\Phi(C_3)$ by $\gamma$ and $\Phi(C_3^2)$ by $\gamma^2$.

Let $e_0 = (0, 0), e_1 = (1, 0)$, and $e_2 = \left(\frac{1}{2}, \frac{\sqrt{3}}{2} \right)$. Also, for $i = 0, 1, 2$, let $V_i$ be the set of vertices of $G$ that are not in $V(T_i)$, and let $(G, p, q)$ be the frame with $p : V(G) \rightarrow \mathbb{R}^2$ and $q : E(G) \rightarrow \mathbb{R}^2$ defined by

$$
p(v) = e_i \quad \text{if } v \in V_i
$$

$$
q(b) = \begin{cases} 
  e_2 - e_1 = \left(-\frac{1}{2}, \frac{\sqrt{3}}{2} \right) & \text{if } b \in E(T_0) \\
  e_0 - e_1 = \left(-\frac{1}{2}, -\frac{\sqrt{3}}{2} \right) & \text{if } b \in E(T_1) \\
  e_1 - e_0 = (1, 0) & \text{if } b \in E(T_2)
\end{cases}
$$

![Figure 15: The frame $(G, p, q)$.](image)

We claim that the generalized rigidity matrix $R(G, p, q)$ has linearly independent rows. To see this, we first rearrange the columns of $R(G, p, q)$ in such a way that we obtain the matrix $R'(G, p, q)$ which has the $(2i - 1)^{st}$ column of $R(G, p, q)$ in its $i^{th}$ column and the $(2i)^{th}$ column of $R(G, p, q)$ in its $(|V(G)| + i)^{th}$ column for $i = 1, 2, \ldots, |V(G)|$. Let $F_0$ denote the row vector of $R'(G, p, q)$ that corresponds to the edge $b \in E(G)$. We then rearrange the rows of $R'(G, p, q)$ in such a way that we obtain the matrix $R''(G, p, q)$ which has the vectors $F_b$ with $b \in E(T_0)$ in the rows $1, 2, \ldots, |E(T_0)|$, the vectors $F_b$ with $b \in E(T_1)$ in the following $|E(T_1)|$ rows, and the vectors $F_b$ with $b \in E(T_2)$ in the last $|E(T_2)|$ rows. So $R''(G, p, q)$ is a matrix of the
form

\[
\begin{pmatrix}
  -\frac{1}{2} & \frac{1}{2} & \sqrt{3} & -\frac{\sqrt{3}}{2} \\
  \vdots & \vdots & \vdots & \vdots \\
  -\frac{1}{2} & \frac{1}{2} & -\frac{\sqrt{3}}{2} & \frac{\sqrt{3}}{2} \\
  \vdots & \vdots & \vdots & \vdots \\
  1 & -1 & 0 & 0
\end{pmatrix}
\]

Clearly, \( R(G, p, q) \) has a row dependency if and only if \( R''(G, p, q) \) does. Suppose \( R''(G, p, q) \) has a row dependency of the form

\[
\sum_{b \in E(T_2)} \alpha_b F_b = 0,
\]

where \( \alpha_b \neq 0 \) for some \( b \in E(T_2) \). Then, since \( T_2 \) is a tree, we have

\[
\sum_{b \in E(T_2)} \alpha_b F_b \neq 0.
\]

Thus, there exists a vertex \( v_s \in V(T_2), s \in \{1, 2, \ldots, |V(G)|\} \), such that

\[
\sum_{b \in E(T_2)} \alpha_b (F_b)_s = C \neq 0.
\]

Since \( v_s \in V(T_2), v_s \) belongs to either \( T_0 \) or \( T_1 \), say wlog \( v_s \in V(T_1) \) and \( v_s \notin V(T_0) \). Therefore, \( (F_b)_s = 0 \) and \( (F_b)_{|V(G)|+s} = 0 \) for all \( b \in E(T_0) \) and

\[
\sum_{b \in E(T_1)} \alpha_b (F_b)_s = -C.
\]

This says that

\[
\sum_{b \in E(T_1)} \alpha_b (F_b)_{|V(G)|+s} = \sum_{b \in E(G)} \alpha_b (F_b)_{|V(G)|+s} = -\sqrt{3}C \neq 0,
\]

a contradiction. Therefore, if \( \sum_{b \in E(G)} \alpha_b F_b = 0 \) is a row dependency of \( R''(G, p, q) \), then \( \alpha_b = 0 \) for all \( b \in E(T_2) \).
So, it is now only left to show that the matrix $\tilde{R}(G, p, q)$ which is obtained from $R''(G, p, q)$ by deleting those rows of $R''(G, p, q)$ that correspond to the edges of $T_2$ has linearly independent rows. This can be done by multiplying $\tilde{R}(G, p, q)$ by appropriate matrices of basis transformations and then using arguments analogous to those above. So, as claimed, the frame $(G, p, q)$ is independent.

Now, if $(G, p)$ is not a framework, then we need to symmetrically pull apart those joints of $(G, p, q)$ that have the same location $e_i$ in $\mathbb{R}^2$ and whose vertices are adjacent. So, wlog suppose $|V_0| \geq 2$. Then we also have $|V_0| = |V_1| = |V_2| \geq 2$, because $\{E(T_0), E(T_1), E(T_2)\}$ is a $(\mathcal{C}_3, \Phi)$ 3Tree2 partition of $G$. Since $\{E(T_0), E(T_1), E(T_2)\}$ is proper, one of $\langle V_0 \rangle \cap T_i$, $i = 1, 2$, say wlog $\langle V_0 \rangle \cap T_2$, is not connected, and hence $\langle V_1 \rangle \cap T_0$ and $\langle V_2 \rangle \cap T_1$ are also not connected. Let $A$ be the set of vertices in one of the components of $\langle V_0 \rangle \cap T_2$ and $\gamma(A)$ and $\gamma^2(A)$ be the vertex sets of the corresponding components of $\langle V_1 \rangle \cap T_0$ and $\langle V_2 \rangle \cap T_1$, respectively. For $t \in \mathbb{R}$, we define $p_t : V(G) \to \mathbb{R}^2$ and $q_t : E(G) \to \mathbb{R}^2$ by

$$p_t(v) = \begin{cases} (-\frac{1}{2}t, -\frac{2\sqrt{3}}{3}t) & \text{if } v \in A \\ (1 + t, 0) & \text{if } v \in \gamma(A) \\ \left(\frac{1}{2}(1 - t), \frac{2\sqrt{3}}{3}(1 + t)\right) & \text{if } v \in \gamma^2(A) \\ p(v) & \text{otherwise} \end{cases}$$

$$q_t(b) = \begin{cases} (1 + \frac{1}{2}t, \frac{2\sqrt{3}}{3}t) & \text{if } b \in E_{A, V_1 \setminus \gamma(A)} \\ (1 + \frac{2}{3}t, \frac{2\sqrt{3}}{3}t) & \text{if } b \in E_{A, \gamma(A)} \\ (-\frac{1}{2} - t, \frac{2\sqrt{3}}{3}) & \text{if } b \in E_{A, \gamma^2(A)} \\ (-\frac{1}{2} - \frac{3}{2}t, \frac{2\sqrt{3}}{3}(1 + t)) & \text{if } b \in E_{\gamma(A), \gamma^2(A)} \\ (-\frac{1}{2} - \frac{3}{2}t, -\frac{2\sqrt{3}}{3}(1 + t)) & \text{if } b \in E_{\gamma^2(A), V_0 \setminus A} \\ (-\frac{1}{2} - \frac{2\sqrt{3}}{3} - \sqrt{3}t) & \text{if } b \in E_{\gamma^2(A), A} \\ q(b) & \text{otherwise} \end{cases}$$

where for disjoint sets $X, Y \in V(G)$, $E_{X, Y}$ denotes the set of edges of $G$ incident with a vertex in $X$ and a vertex in $Y$. Then $(G, p_t, q_t) = (G, p, q)$ if $t = 0$. Now, if we let $t'$ be an indeterminate, then the rows of $(G, p_{t'}, q_{t'})$ are linearly dependent (over the quotient field of $\mathbb{R}[t']$) if and only if the determinants of all $\left|E(G)\right| \times \left|E(G)\right|$ submatrices of $(G, p_{t'}, q_{t'})$ are identically zero. These determinants are polynomials in $t'$. Thus, the set of all $t \in \mathbb{R}$ with the property that $R(G, p_t, q_t)$ has a non-trivial row dependency is a variety $F$ whose complement, if non-empty, is a dense open set. Since $t = 0$ is in the complement of $F$ we can conclude that for almost all $t$, $(G, p_t, q_t)$ is independent. Therefore, there exists a $t_0 \in \mathbb{R}$, $t_0 \neq 0$, such that the frame $(G, p_{t_0}, q_{t_0})$ is independent. This process can be continued until we obtain an
independent frame \((G, \hat{p}, \hat{q})\) with \(\hat{p}(u) \neq \hat{p}(v)\) for all \(\{u, v\} \in E(G)\). Then, by Remark 5.1, \((G, \hat{p})\) is an independent framework and the right translation of \((G, \hat{p})\) yields an independent framework in the set \(R(G, C_3, \Phi)\). □

Lemmas 5.2, 5.3, 5.4, and 5.5 provide a complete proof for Theorem 5.1.

As shown in [19], there also exists a direct geometric proof for the fact that condition \((iii)\) implies condition \((i)\) in Theorem 5.1 i.e., that the existence of a \((C_3, \Phi)\) construction sequence for \(G\) implies that \(R(G, C_3, \Phi) \neq \emptyset\) and that \(G\) is \((C_3, \Phi)\)-generically isostatic. By generalizing the basic geometric techniques used in this proof, we can construct classes of \((S, \Phi)\)-generically isostatic graphs for a variety of symmetry groups \(S\). These techniques also allow us to prove (or at least conjecture) characterizations of \((S, \Phi)\)-generically isostatic graphs in situations where symmetric tree partitions are too complex. Moreover, they provide significant results for \((S, \Phi)\)-generically independent graphs.

An immediate consequence of the symmetrized Laman’s Theorems for \(C_3\), \(C_2\), and \(C_s\) (and the analogous conjectures for \(C_{2v}\) and \(C_{3v}\)) is that there is (would be) a polynomial time algorithm to determine whether a given graph \(G\) is \((S, \Phi)\)-generically isostatic. In fact, although the Laman conditions involve an exponential number of subgraphs of \(G\), there are several algorithms that determine whether they hold in \(c|V(G)||E(G)|\) steps, where \(c\) is a constant. The pebble game ([12]) is an example for such an algorithm. The additional symmetry conditions for the number of fixed structural components can trivially be checked in constant time, from the graph automorphisms.
6  Further work

6.1  Pinned frameworks

In mechanical and structural engineering, one is often interested in the rigidity and flexibility properties of pinned frameworks, i.e., frameworks that have some of their joints firmly anchored (‘pinned’) to the ground (see, for example, [6, 21, 22]). Using the techniques presented in [4, 6, 19], it is straightforward to show that an isostatic symmetric pinned framework \((G, p)\) must again satisfy some very simply stated restrictions on the number of (unpinned) joints and bars of \((G, p)\) that are fixed by various symmetry operations of \((G, p)\). While there are only six possible point groups that allow isostatic frameworks in the plane, it turns out that an isostatic pinned framework can be constructed for any point group in dimension 2.

We conjecture that the standard Laman-type conditions for a pinned graph \(G\) (see [21], for example), together with the additional necessary conditions concerning the number of fixed joints and bars, are also sufficient for pinned 2-dimensional realizations of \(G\) which are as generic as possible subject to the given symmetry constraints to be isostatic.

In particular, for the symmetry groups \(C_2, C_3,\) and \(C_s\) in dimension 2, we claim that the techniques of this paper extend directly to proofs of the corresponding symmetric versions of Laman’s Theorem for pinned frameworks.

6.2  Frameworks in dimension \(d > 2\)

A combinatorial characterization of generically \(d\)-isostatic graphs in dimension \(d > 2\) has not yet been found [8, 9, 32]. Recall from Section 2.3 however, that there are a number of inductive construction techniques which are known to preserve the generic rigidity properties of a graph (see also [15, 30]).

It is shown in [19] that, unlike in dimension 2, symmetry in dimension \(d > 2\) induces extra conditions for a graph \(G\) to be \((S, \Phi)\)-generically isostatic beyond those of

(a) \(G\) being generically \(d\)-isostatic and

(b) the symmetry conditions derived in [4] concerning the number of fixed structural components of \(G\) (and of all symmetric subgraphs \(H\) of \(G\) with the full count \(|E(H)| = d|V(H)| - \binom{d+1}{2}\)).
We conjecture that ‘flatness’ caused by symmetry is the only additional concern, and that it can be made into a finite set of added combinatorial conditions, for each symmetry group. See [4, 19, 20] for further details.

### 6.3 Body-bar and body-hinge structures

Faced with the difficulties of characterizing generically rigid graphs in dimension $d > 2$, in contrast with the well developed theory in the plane, there has recently been a careful study of a special class of frameworks, the class of body-bar frameworks. These structures have a basically complete combinatorial theory which exhibits all the key theorems and algorithms of the well understood plane frameworks (see, for example, [23, 27, 30]).

For a body-bar framework $(G, p)$ that possesses non-trivial symmetries, joint work with S. Guest and W. Whiteley shows that in addition to the conditions in Tay’s Theorem (see [23]), there exist further necessary conditions for $(G, p)$ to be isostatic [10]. These can be formulated as restrictions on the number of bars and bodies that are fixed by various symmetry operations of $(G, p)$. While these extra conditions are analogous to the ones derived for bar and joint frameworks, the modified context holds the promise of converting them into necessary and sufficient conditions for an arbitrary-dimensional body-bar realization of $G$ to be isostatic, provided that this realization is as generic as possible subject to the given symmetry constraints. These conjectures, as well as various additional conjectures concerning combinatorial characterizations of $d$-dimensional symmetric body-bar frameworks, are stated in [10].

For the groups $C_2$, $C_3$, and $C_s$ in dimension 2, these conjectures can readily be proven by modeling a symmetric body-bar framework as a framework (in the sense of Definition 2.1) with isostatic bar and joint bodies of required symmetry, and then applying the results of this paper.

An interesting special class of body-bar frameworks with some important applications in rigidity theory is the class of body-hinge frameworks [26, 30, 31, 32]. It is shown in [26] that body-hinge realizations of a multi-graph $G$ with generic hinge assignments are infinitesimally rigid if and only if body-bar realizations of $G$ with generic positions for the end-points of the bars are infinitesimally rigid. So, body-hinge frameworks have the same efficient algorithms for testing generic rigidity as body-bar frameworks [31]. Moreover, the Molecular Conjecture posed by T.-S. Tay and W. Whiteley in 1984 proposes that the even more special class of body-bar frameworks that arise in the models of molecular kinematics (i.e., the class of body-hinge frameworks that have all hinges of each body concurrent in a point) also
have the same good combinatorial theory as general body-bar frameworks \cite{26, 31}, so that, under generic conditions, the efficient counting algorithms for body-bar frameworks also apply to molecular body-hinge frameworks.

Given certain symmetry constraints, we conjecture that, analogously to the non-symmetric situation, the results and conjectures in \cite{10} concerning symmetric-generic body-bar frameworks also translate directly to symmetric-generic body-hinge frameworks. We further conjecture that a symmetric version of the Molecular Conjecture holds, i.e., that under symmetric-generic conditions, body-bar frameworks and molecular frameworks also possess the same rigidity properties.

We note that a number of biomolecules possess rotational symmetry, including a number of virus shells which exhibit the symmetry of the rotational icosahedral group. The potential for such applications, as well as for understanding human-built structures which are designed to have symmetry, is a further motivation for giving explicit results for symmetric body-bar, body-hinge, and molecular structures.

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