Speedup for Natural Problems and 
\[ NP = \? coNP \]

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1 Introduction

Informally, a language \( L \) has speedup if, for any Turing machine (TM) for \( L \), there exists one that is better. Blum [2] showed that there are computable languages that have almost-everywhere speedup. These languages were unnatural in that they were constructed for the sole purpose of having such speedup. We identify a condition apparently only slightly stronger than \( P \neq NP \) which implies that accepting any \( coNP \)-complete language has an infinitely-often (i.o.) superpolynomial speedup and \( NP \neq coNP \). We also exhibit a natural problem which unconditionally has a weaker type of i.o. speedup based upon whether the full input is read. Neither speedup pertains to the worst case.

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1For a review of related literature, see Monroe [9].
2 Conditional Speedup for $coNP$-Complete Languages

**Def 2.1** Define $BHP = \{\langle N, x, 1^t \rangle |$ there is at least one accepting path of nondeterministic TM $N$ on input $x$ with $t$ or fewer steps}, $DBHP$ is the same but with $N$ deterministic, and $HP = \{\langle N, x \rangle |$ there is at least one accepting path of NTM $N$ on input $x$ (with no bound on the number of steps)$\}$. If $M$ is a deterministic TM then $T_M$ is the function that maps a string $x$ to how many steps $M(x)$ takes.

Note that $BHP$ is NP-complete with the accepting path as a certificate, that $coBHP$ is $coNP$-complete, and $DBHP \in P$.

Suppose $P \neq NP$ and therefore $coBHP \notin P$. The following condition rules out the absurd possibility that some $M$ can nevertheless accept the subset of inputs beginning with any particular machine-input pair within a polynomial bound (for that subset):

\[(*) \text{ Let } M \text{ be a deterministic TM accepting } coBHP. \text{ Then there exists } \langle N', x' \rangle \in coHP \text{ such that the function } f(t) = T_M(N', x', 1^t) \text{ is not bounded by any polynomial.}\]

An intuition for why this condition might hold could be a belief that there is at least one $N', x'$ for which $M$ must infinitely often use brute force to rule out all possible accepting paths of $N'$ on $x'$ with at most $t$ steps.

**Def 2.2** For $M$ and $M'$ accepting a language $L$, write $M \leq_p M'$ if there exists a polynomial $p$ such that for all inputs $x \in L$:

\[T_M(x) \leq p(|x|, T_{M'}(x)).\quad (1)\]

If $L$ has a least element $M$ under $\leq_p$, say that $M$ is $p$-optimal\(^3\) and otherwise say that $L$ has i.o. superpolynomial speedup.

**Theorem 2.3** If $L$ is NP-complete, $L$ does not have superpolynomial speedup.

\(^2\)The function $f$ may depend on $M$, $N'$, and $x'$. For inputs not in $coBHP$, $M$ does not accept, but otherwise its behavior is not constrained.

\(^3\)See Krajíček and Pudlák [6].
Proof: For any \( L \in NP \), there is a \( p \)-optimal TM for finding witnesses for \( L \), by Levin \[7\]. Levin’s universal witness search algorithm works for any \( NP \) language by dovetailing every possible TM, running any output produced through a predetermined witness verifier, and then printing out the first witness that is verified. If \( L \) is \( NP \)-complete, then there is a \( p \)-optimal algorithm accepting \( L \) using the self-reducibility of \( NP \)-complete languages, by Schnorr \[11\].

Theorem 2.4 If (*) holds, then \( coBH P \) has superpolynomial speedup, and \( NP \neq coNP \).

Proof: Given \( M \) accepting \( coBH P \), choose \( N', x' \) for \( M \) in (*), so \( f(t) = T_M(\langle N', x', 1^t \rangle) \) is not polynomially bounded. We create \( M' \) as follows:

1. Input \( \langle N, x, 1^t \rangle \).
2. If \( N, x \neq N', x' \) then run \( M(N, x, 1^t) \).
3. If \( N, x = N', x' \) then reject immediately.

Then \( M' \prec_p M \), and \( coBH P \) therefore has superpolynomial speedup. Since \( coBH P \) is \( coNP \)-complete, and no \( NP \)-complete language has superpolynomial speedup, then \( NP \neq coNP \).

Theorem 2.4 is a striking result: a condition only slightly stronger than \( P \neq NP \), which states that at least one instance of \( coBH P \) is hard, implies \( NP \neq coNP \).

Theorem 2.5 If one \( coNP \)-complete language has superpolynomial speedup, then all of them do.

Proof: For \( coNP \)-complete languages \( L_1 \) and \( L_2 \), suppose \( L_1 \) has superpolynomial speedup and \( L_2 \) does not. Let \( f, g \) be polynomial time reductions from \( L_1 \) to \( L_2 \) and vice versa, i.e., \( x \in L_1 \) if and only if \( f(x) \in L_2 \), and \( x \in L_2 \) if and only if \( g(x) \in L_1 \). Suppose \( M_2 \) is \( p \)-optimal for \( L_2 \). Then \( M'_2 = M_2 \circ f \circ g(x) \) is also \( p \)-optimal for \( L_2 \). Let \( M_1 = M_2 \circ f \). Because \( L_1 \)

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4See Gurevich \[5\], Goldreich \[4\], Ben-Amram \[1\], Messner \[8\], and Sadowski \[10\].

5Hartmanis asked whether is there an optimal search algorithm similar to Levin’s that also rejects when there is no witness (Trakhtenbrot \[12\]); in this case, there is not for \( NP \)-complete languages.
has superpolynomial speedup by assumption, there exists $M'_1 <_p M_1$. That implies $M'_1 \circ g <_p M'_2$ on inputs $x \in L_2$ so in fact $M_2$ was not $p$-optimal, a contradiction.

### 3 Unconditional Speedup for $coBHP$

This section proves unconditionally that $coBHP$ has a different form of speedup which hinges upon whether the full input is read. The intuition is that it is useful for $M$ accepting $coBHP$ to be able to recognize that its input begins with a non-halting $N', x'$, but no $M$ can recognize all non-halting $N', x'$, since $coHP$ is not computably enumerable (c.e.).

**Def 3.1** For $M$ and $M'$ accepting a language $L$, write $M' <_b M$ if (1) there exists an infinite subset of inputs $S \subset L$ on which the runtime of $M$ is not bounded above by a constant but the runtime of $M'$ is bounded above by a constant, and (2) there exists a constant $c_S$ such that the runtime disadvantage of $M'$ on inputs in $L - S$ is less than an additive factor $c_S$. If for any $M$ there exists $M'$ such that $M' <_b M$, say that $L$ has i.o. $b$-speedup. The speedup is effective if $M'$ is computable from $M$. Otherwise, say that $M$ is $b$-optimal.

**Lemma 3.2** For any $M$ accepting $coBHP$, there is some $N', x' \in coHP$ computable from $M$ for which $T_M(N', x', 1^t) \geq t$.

**Proof:** Assume, by way of contradiction, that for some $M$ and for all $N', x' \in coHP$ there exists a $t_0$ such that $T_M(N', x', 1^{t_0}) < t_0$. This computation must have determined that $\langle N', x', 1^{t_0} \rangle \in coBHP$ without reading the entire input. In particular, it only read part of the $1^{t_0}$. Hence for all $t > t_0$, $T_M(N', x', 1^t) < t_0$. Therefore

$$\langle N, x \rangle \in coHP \implies (\exists t_0)[M(N, x, 1^{t_0}) \text{ accepts and } T_M(N, x, 1^{t_0}) < t_0].$$

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6 This consideration is excluded in inequality (1) by the $|x|$ term.

7 The proof below can be seen as a bounded version of the statement that every non-c.e. language has speedup if $M'$ is “better” than $M$ at accepting a language $L$ if $M'$ correctly accepts a strictly larger subset of $L$ than $M$. If $L$ is productive, then the speedup is effective.

8 The trivial linear speedup is not $b$-speedup. Geffert describes nontrivial linear speedups for nondeterministic machines.
Therefore \( coHP \) is c.e., a contradiction. Because \( coHP \) is productive, \( N', x' \) for which no such \( t_0 \) exists is computable from \( M \). \( \square \)

**Theorem 3.3** \( coBP \) and \( coDBHP \) each have \( b \)-speedup, and the speedup is effective\(^9\)

**Proof:** Suppose \( M \) accepts \( coBP \). Compute \( N', x' \in coHP \) for \( M \) by Lemma 3.2. We create \( M' \) as follows:

1. Input \( \langle N', x', 1^t \rangle \) but without yet reading any of \( 1^t \).
2. If \( N, x \neq N', x' \) then run \( M(N, x, 1^t) \).
3. If \( N, x = N', x' \) then reject immediately.

Note that there is a constant \( C \) such that, for all \( t \), \( T_M(N', x', 1^t) \geq t \) and \( T_M'(N', x', 1^t) \leq C \). Hence, \( coBP \) has \( b \)-speedup, with \( S = \{ \langle N', x', 1^t \rangle \} \).

The same proof applies to \( coDBHP \). \( \square \)

4 Conclusion

We conjecture that any \( M \) which might serve as a counterexample to widely believed complexity hypotheses could, as in Lemma 3.2, be modified to perform tasks known to be noncomputable. In particular:

**Conjecture 4.1** If there exists \( M \in P \) accepting a \( coNP \)-complete language (for instance \( coBP \)), then \( M \) can be modified to accept a language that is not c.e. (for instance \( coHP \)).

Similarly, some suspect that integer multiplication has speedup, and it is generally believed that integer multiplication is a one-way function. These conjectured properties could be related to a known property of integer multiplication that apparently has never been used to prove anything about the complexity of multiplication itself: the Presburger arithmetic without multiplication is a decidable while arithmetic with multiplication is undecidable.

**Conjecture 4.2** Suppose \( M \) can factor integers in polynomial time. Then \( M \) can be modified to accept true arithmetic statements.

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\(^9\)There are \( coNP \)-complete languages which do not have \( b \)-speedup. For instance, a \( b \)-optimal \( M \) for \( TAUT \) reads clause \( i + 1 \) only if the first \( i \) clauses are a tautology.
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