Controlling multimode coupling by boundary-wave scattering

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We show that coupling among multiple resonances can be conveniently introduced and controlled by boundary wave scattering. This is demonstrated in optical microcavities of quasicircular shapes, where the couplings of multiple modes are determined by the scattering from harmonic boundary deformations. We analyze these couplings using a perturbation theory, which gives an intuitive understanding of the lowest-order and higher-order scattering processes. Different scattering paths between two boundary waves can either enhance or reduce their coupling strength. The effect of controlled multimode coupling is most pronounced in the direction of output from an open cavity, as the coupling can cause a dramatic change of the external cavity field distribution.

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I. INTRODUCTION

Eigenmodes are fundamental in understanding all quantum and wave phenomena. Take optical microcavities [1,2], for example; the study of their eigenmodes has attracted considerable interest in the pursuit of compact coherent light sources [3–5], understanding quantum chaos in open systems [6,7], and achieving strong light-matter interactions [8,9]. For microdisk lasers in particular, one focus has been optimizing their output directionality while maintaining a high quality (Q) factor. One approach is deforming the circular cavity boundary [6,7,10–12], which alters the dynamics of light trapped inside the cavity. Its effect is depicted by the ray model for cavities much larger than the laser wavelength, which, however, does not apply to the cavities of the wavelength scale [13,14]; in this regime the output directions of high-Q modes in a given cavity are no longer universal or determined by the intracavity ray dynamics. Instead, mode coupling plays an important role, as found in previous studies, and it is important to understand it beyond the usual phenomenological model [15].

Mode coupling in general occurs when the orthogonality or biorthogonality in the system is modified, which can be introduced, for example, by matter-mediated interaction in cavity quantum electrodynamics [16], by nonlinearity in multimode lasers [17], and by linear scattering from a local defect or a gradual boundary deformation in waveguides [18] and microcavities [19,20]. In the latter scenario, several of the authors recently found that a minute boundary deformation can lead to a drastic change of the outcoupling direction in an open system, which was attributed to enhanced two-mode coupling caused by quasidegeneracy in the unperturbed eigenmode spectrum [21].

In this paper we show that multimode coupling can be achieved and controlled by boundary wave scattering. This applies to the general eigenvalue problem

\[ (-\nabla^2 + V(\vec{r}))\psi(\vec{r}) = E\psi(\vec{r}), \]

which can be realized, for example, in a dielectric microcavity [1,2], a vibrating membrane [22], an optical trap for exciton-polariton condensate [23], and a quantum dot [24]. The scalar eigenmodes \( \psi(\vec{r}) \) represent vibrational amplitudes, field components of the electromagnetic waves, or probability wave functions in the corresponding physical systems.

Below we exemplify the properties of boundary wave scattering in open quasicircular cavities, with \( V(\vec{r}) = -(n^2 - 1)E \) inside and \( V(\vec{r}) = 0 \) outside. Equation (1) then becomes the scalar two-dimensional Helmholtz equation, which describes, for example, the propagation of transverse-electric (TE) or transverse-magnetic (TM) waves in a dielectric microdisk cavity of refractive index \( n \). For simplicity, we assume the cavity shape is nearly circular and symmetric along the horizontal axis \( \theta = 0^\circ, 180^\circ \), and we describe the cavity boundary using harmonic series

\[ \rho(\theta) = R \left[ 1 + \sum_v \epsilon_v \cos(v\theta) \right] \quad (v = 2, 3, \ldots), \]

(2)

where \( R \) is the average radius of the cavity. The dipolar term \( (\epsilon_1 \cos \theta) \) is not included because it mostly leads to a lateral shift of the cavity if \( |\epsilon_1| \ll 1 \) and can be eliminated by choosing a proper origin. The harmonic boundary deformations in (2) can be employed as individual turning knobs to introduce and control couplings among multiple modes of different angular momenta, which is a generalization of the procedure introduced in Ref. [21]. Such a scheme can be utilized to alter the outcoupling direction of an eigenmode deterministically, through first-order and higher-order boundary wave scattering.

This paper is organized as follows. In Sec. II we review the perturbation theory for the scalar Helmholtz equation in a quasicircular cavity. We relate each perturbation contribution in the presence of the harmonic deformations in (2) to scattering strengths of different orders. In Sec. III we demonstrate multimode coupling via a single harmonic boundary deformation using TM modes. In Sec. IV we examine how multiple harmonic boundary deformations can be introduced to control couplings among TM modes. In Sec. V we apply this
technique to control the emission directionality of microdisk lasers. The summary is given in Sec. VI.

II. PERTURBATION THEORY

We begin by reviewing the perturbation theory for TM [25,26] and TE modes [21] of the scalar Helmholtz equation in a quasicircular system. In the absence of deformation, each eigenmode \( \psi \) of the circular system is characterized by its angular momentum \( m \) and radial quantum number \( n \). The latter indicates the number of intensity peaks in the radial direction inside the cavity, and we will refer to the modes with \( n \ll m \) as the boundary waves since they are confined closely to the inside of the cavity boundary. For convenience, we represent their angular dependence by sine and cosine functions, i.e.,

\[
\psi_{m,n}(\vec{r}) \propto \left\{ \begin{array}{ll}
J_m(nK_{m,n}r) \cos(m \theta), & \text{for TM modes} \\
\frac{J_m(nK_{m,n}r)}{J_0(nK_{m,n}r)} \sin(m \theta), & \text{for TE modes}
\end{array} \right.
\]

inside the cavity, where \( K_{m,n} \) is the complex resonant frequency, corresponding to the square root of \( E \) in Eq. (1). Outside the cavity \( \psi_{m,n}(\vec{r}) \) are similarly defined, with the Bessel functions \( J_m(nK_{m,n}r) \) replaced by the Hankel functions of the first kind \( H_m(nK_{m,n}r) \) and properly normalized to guarantee the continuity of \( \psi_{m,n}(\vec{r}) \) at the cavity boundary. The value of \( K_{m,n} \) is determined by the boundary condition

\[
S_m(Z) = n \frac{J_m'(nZ)}{J_m(nZ)} - \frac{H_m'(Z)}{H_m(Z)} = 0
\]

for TM modes and

\[
T_m(Z) = n \frac{1}{J_m(nZ)} \frac{J_m'(nZ)}{H_m'(Z)} - \frac{H_m(Z)}{H_m(Z)} = 0
\]

for TE modes, where \( Z \equiv K R \).

The eigenmodes inside the cavity are slightly perturbed in the presence of a minute boundary deformation. The perturbed modes each have a dominant angular momentum \( m \) and recognizable radial quantum number \( n \), and they are still parity eigenstates about the horizontal symmetry axis if the boundary takes the form of Eq. (2). We denote them as \( \psi_{m,n} \) and their resonant frequencies as \( K_{m,n} \) to distinguish them from the unperturbed modes \( \psi_{m,0} \) and their frequencies \( K_{m,0} \). Below we focus on the even-parity modes, and the analysis for the odd-parity modes is similar. We drop the indices of \( \psi, \varphi, k, \) and \( K \) unless they are important. Using the ansatz

\[
\psi(\vec{r}) = \sum_{p \geq 0} a_p J_{p\eta}(\eta \rho R) \cos(p \theta), \quad r < \rho(\theta),
\]

and \( a_m \equiv 1 \) for the dominant angular momentum, Dubertrand et al. found that in a cavity with the boundary given by

\[
\rho(\theta) = R[1 + \varepsilon f(\theta)], \quad f(\theta) = f(-\theta),
\]

the perturbed quantities are

\[
k R = Z \left[ 1 - \varepsilon A_{mm} - \varepsilon^2 \left( \frac{Z(n^2 - 1)}{S_p} \sum_{q \neq m} A_{mq} A_{qm} \frac{1}{S_q} \right) \right.
\]

\[
- \frac{3A_{mm}^2 - B_{mm}}{2} - Z \left( A_{mm}^2 - B_{mm} \frac{H_m'}{H_m} \right) \right].
\]

a_p = \varepsilon Z(n^2 - 1) \frac{1}{S_p} \left( A_{pm} + \varepsilon \left( \frac{Z}{S_p} \right) A_{mm} A_{qm} \frac{1}{S_q} \right)
\]

\[+ \frac{B_{pm}}{2} \left[ 1 + Z \left( \frac{H_m'}{H_m} + \frac{H_p'}{H_p} \right) \right]
\]

\[+ \frac{Z(n - 2)}{2} \sum_{q \neq m} A_{pq} A_{qm} \frac{1}{S_q} \right),
\]

up to order \( O(\varepsilon^2) \) for TM modes [25]. We have dropped the argument \( Z \) in \( S, H, \) and their derivatives \( S', H' \) with respect to \( Z \). The coefficients \( A_{pm}, B_{pm} \) are given by

\[
A_{pm} = \frac{c_p}{\pi} \int_0^\pi \varepsilon f(\theta) \cos(p \theta) \cos(m \theta) d\theta,
\]

\[
B_{pm} = \frac{c_p}{\pi} \int_0^\pi \varepsilon f'(\theta) \cos(p \theta) \cos(m \theta) d\theta,
\]

with \( c_p = 2(p > 0), 1(p = 0) \).

The perturbation theory was extended to TE modes in Ref. [21], which is more complicated due to the discontinuity of the radial derivative of \( \psi \) at the cavity boundary. The results up to the order \( O(\varepsilon) \) are given by

\[
k R = Z[1 - \varepsilon A_{mm}],
\]

\[
a_p = \varepsilon Z \sum_{q \neq m} \alpha_{pq} \alpha_{qm} + (\cdots) B_{pm} + (\cdots) A_{pm} A_{mm},
\]

where

\[
\alpha_{pm} = \varepsilon Z(n^2 - 1) \frac{1}{S_p} A_{pm}
\]

can be considered the scattering strength for the first-order process \( m \to p \) by the \( \cos(m \pm p) \theta \) deformation in (2). In our notation the angular momenta \( m, p \) of the boundary waves are non-negative, representing the clockwise (CW) wave with a positive sign and counterclockwise (CCW) wave with a negative sign. For example, the \( \cos(m \pm p) \theta \) deformation can scatter the CW (CCW) wave of angular momentum \( m \) into the CW (CCW) wave of angular momentum \( p \) and \( (m \pm p) \theta \) deformation can scatter the CW (CCW) wave of angular momentum \( m \) into the CCW (CW) wave of angular momentum \( p \).

We emphasize that the scattering strength \( a_{pm} \) is mode dependent; it is proportional to \( S_p^{-1}(K_{m,n}R) \), which we will refer as the spectral function. If there is another resonance

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In Eq. (19) since it does not depend on the spectral function of the intermediate state, which needs to be large. Thus we will refer to them as "virtual" processes. The last $O(\varepsilon^2)$ term in Eq. (16) has a more complicated dependence on the spectral function of the final state. Its scattering strength is proportional to $A_{nm}$, which represents the scattering of the CW and CCW waves of angular momentum $\pm m$ into each other by the $\cos(2m\phi)$ deformation.

III. MULTIMODE COUPLING VIA A SINGLE HARMONIC BOUNDARY DEFORMATION

In this section and the next section we exemplify multimode coupling using the TM polarized modes. Let us first consider a single harmonic perturbation $\cos(\theta)$ with an amplitude $|\varepsilon| \ll 1$. As we have discussed in the previous section, this boundary deformation scatters the boundary wave of angular momentum $m$ into two sidebands $m \pm \nu$ to the leading order, whose amplitudes are given by

$$a_{m \pm \nu, m} \equiv \frac{\varepsilon_m (n^2 - 1) Z}{2S_{m \pm \nu}} \quad (18)$$

from Eq. (17). This boundary wave scattering introduces coupling between $\varphi_{m,n}$ and two other modes, $\varphi_{m\pm\nu,n}$ and $\varphi_{m-n,\nu}$, whose dominant angular momenta are $m + \nu$ and $m - \nu$, respectively.

Higher-order scattering processes create weaker sidebands at $m \pm 2\nu, m \pm 3\nu, \ldots$, coupling more modes with decreasing strength in general. The strength $\beta_{m \pm 2\nu, m}$ of the second-order scattering can be found in Eq. (16):

$$\beta_{m \pm 2\nu, m} \approx a_{m \pm 2\nu, m},a_{m \pm \nu, m}. \quad (19)$$

We have assumed $\nu < m$, with which $A_{nm}$ and the last term in Eq. (16) vanish. We have also neglected the "virtual" process given by $B_{pm}$ in Eq. (16); it is weak compared with the right-hand side of Eq. (19) since it does not depend on the spectral function $S_{m\pm\nu}$, of the intermediate state, which needs to be large for the second-order scattering strength to be non-negligible.

In Fig. 1 we show one example with a $\cos(3\theta)$ boundary deformation in a circular cavity of index $n = 3.13$. Near $KR = 4.6$ there are three eigenmodes of angular momenta $m = 11$, $m' = 8$, $m'' = 5$ and radial quantum numbers $\eta = 1, 2, 3$. They are calculated using a scattering-matrix method similar to that described in Refs. [10,27]. We refer to them as modes 1, 1', and 1'', and they have increasing cavity decay rates, defined by $\kappa = -2\Im[KR] > 0$. We first focus on mode 1 ($m = 11$) and point out that the two first-order sidebands of mode 1 at $p = 8, 14$ do not have equal strength; the presence of mode 1'' ($m'' = 8$) leads to a spectral function $S_8^{-1} > S_5^{-1}$, and the sideband at $p = 8$ is about four times stronger than that at $p = 14$. Similarly, the proximity of mode 1'' ($m'' = 5$) enhances the scattering into the second-order sideband of mode 1 at $p = 5$, which is even stronger than the first-order one at $p = 14$.

Figures 1(b) and 1(c) for modes 1' and 1'' further display the mutual coupling between them and mode 1. The couplings, however, affect these eigenmodes differently. For example, their decay rates all vary on the scale of $10^{-4}$ when $\varepsilon_3$, the amplitude of the $\cos(3\theta)$ boundary deformation, changes from...
FIG. 2. (Color online) Effect of the cos(3θ) deformation of the cavity boundary on the complex frequencies of the three modes in Fig. 1. Solid lines and black dashed lines show the numerical data and the second-order perturbation result from Eq. (20), respectively. The relative change of the decay rate of mode 1, which has the highest $Q$ factor, is much larger than for modes 1′ and 1″.

From Eqs. (9) and (18) we know that $\alpha_{pm}$ and $\alpha_{pq}$ are then in phase ($\pi$ out of phase) if the amplitudes $\varepsilon_{m-p}$ and $\varepsilon_{q-p}$ of the harmonic modulations $\cos(m-p)\theta$ and $\cos(q-p)\theta$ have the same sign (opposite signs). Therefore the requirement for the aforementioned cancellation at some value of $\varepsilon_{m-p}$ is to have a real $\alpha_{pm}$ and $\alpha_{pq}$. Indeed, we find $\text{Arg}[\alpha_{8,11}] = 0.008$ in the example given above, where $m = 11$, $q = 8$, and $p = 5$, and a negative $\varepsilon_{6} = -0.0023$ is needed to cancel the $p = 5$ sideband at a positive $\varepsilon_{3} = 0.01$. We also note that the phase of $\alpha_{p}$ changes by $\pi$ across $\varepsilon_{6} = -0.0023$ as a result.

The effect of controlled multimode coupling is most pronounced in the outcoupling of the high-$Q$ modes. Recent studies [13,19] show that the output direction of a high-$Q$ mode can be completely overwhelmed by that of a lower-$Q$ mode to which it couples. The situation becomes more interesting if the high-$Q$ mode couples to more than one lower-$Q$ mode, such as the case in Fig. 1. Unlike the intracavity intensity distribution, which is largely determined by the dominant angular momentum and the strong first-order sidebands [see Eq. (21) and its discussion], the weaker sidebands of lower angular momenta can also have a strong influence on the outcoupling due to their stronger leakiness. More specifically,
the wave function (6) in the far field becomes
\[
\psi(r \to \infty) \propto \sum_p (a_p + b_p) e^{-i\pi p/2} H_p(kR) \cos(p\theta)
\]
using the large-argument asymptotic form of the Hankel function of the first kind
\[
H_p(z \to \infty) = \sqrt{\frac{\pi z}{2}} e^{i\pi p/2 + i\pi z/4}.
\]
\[
W_p = \sum_p W_p \cos(p\theta), \quad p = 0, 1, 2, \ldots
\]
We note that the amplitude of \(H_p(kR)\) in the denominator of Eq. (23), evaluated at the average radius of the cavity, reduces dramatically for a smaller angular momentum \(p\), which represents the stronger leakiness mentioned above.

Now let us examine how the outcoupling direction of mode 1 changes with \(\epsilon_6\) in the example shown in Fig. 3. At \(\epsilon_6 = -0.0023\) the \(p = 5\) sideband outside the cavity is very small, similar to what happens inside the cavity. The outcoupling of mode 1 is then dominated by the \(p = 8\) sideband, which leads to an approximate angular dependence of \(\cos(16\theta)\) for the far-field intensity [Fig. 4(a)]. As \(\epsilon_6\) changes from \(-0.0023\), the cancellation of the two scattering paths is removed, and the coupling between modes 1 and 1' increases rapidly; the

\[|W_5 - W_8| = \sqrt{2|W_5| \cdot |W_8|} \approx |W_5|,\]

\[\epsilon_6 \approx -0.0023\] |W_5| becomes comparable to the |W_8|, and the outcoupling is enhanced in the \(\epsilon_6 \approx -0.0023\) case [see Fig. 5(b)]. This is because \(W_5\) is approximately proportional to \(|\alpha_6|\) since \(|\alpha_6| \gg |\beta_5|\) in expression (23) for the far field, and we know from our discussion of Eq. (22) that the phase of \(\alpha_6\) jumps by about \(\pi\) across \(\epsilon_6 = -0.0023\). Meanwhile, \(W_8\) varies little for such a small change of the minute \(\epsilon_6\) since it depends on \(\epsilon_6\) only through a second-order scattering path 11 → 5 → 8. As a result, \(W_5\) and \(W_6\) are now approximately in phase, and the outcoupling is enhanced in the \(\theta \approx 60^\circ, 180^\circ, 300^\circ\) directions [Fig. 4(b)]. At \(\epsilon_6 \approx -0.001\), however, the phase of \(W_5\) changes roughly by \(\pi\) [see Fig. 5(b)]. This is because \(W_5\) is approximately proportional to \(|\alpha_6|\) since \(|\alpha_6| \gg |\beta_5|\) in expression (23) for the far field, and we know from our discussion of Eq. (22) that the phase of \(\alpha_6\) jumps by about \(\pi\) across \(\epsilon_6 = -0.0023\). Meanwhile, \(W_8\) varies little for such a small change of the minute \(\epsilon_6\) since it depends on \(\epsilon_6\) only through a second-order scattering path 11 → 5 → 8. As a result, \(W_5\) and \(W_6\) are now approximately in phase, and the outcoupling is enhanced in the \(\theta \approx 0^\circ, 120^\circ, 240^\circ\) directions instead [Fig. 4(c)], which is flipped vertically from that at \(\epsilon_6 \approx -0.004\). In this process the intracavity intensity distribution of mode 1 barely changes from the \(\triangleright\) pattern shown in Fig. 1(a) since the modified \(p = 5\) sideband inside the cavity is very weak [see Fig. 3(b)]. Thus the flipping of the outcoupling direction with \(\epsilon_6\) is different from that reported in Ref. [21], which involves the flipping of the intracavity field pattern as well. As \(\epsilon_6\) moves farther away from \(-0.0023\), the \(p = 5\) sideband in mode 1 gradually becomes the dominant angular momentum outside the cavity [Fig. 5(a)], and the angular dependence of the outcoupling approaches \(\cos(10\theta)\) (not shown).

V. CONTROL OF MICROCAVITY EMISSION PATTERN VIA MULTIMODE COUPLING

To demonstrate potential applications of the presented theory, we will show in this section that the boundary wave scattering can be used to control the directions of microcavity emission. Previously, we fabricated semiconductor microcavities of various shapes and obtained directional emissions [13,14]. Since the laser emission is predominantly
TE polarized, we consider here the TE modes and introduce multiple harmonic terms in the boundary deformation to control the output directions.

We start with $\varepsilon_2 = -0.07$ and $\varepsilon_3 = 0.008$, and three nearby modes with $m = 11, m' = 8, m'' = 5$ can be found around $kR = 4.9$ (see Fig. 6); they are the TE correspondence of the TM modes we have discussed in Figs. 1–4, albeit the deformations are different. We find that mode 1 couples strongly to mode $1'$, with similar $W_p$'s outside the cavity. Consequently, the output directionality of mode 1 is almost identical to that of mode $1'$ [Figs. 6(a) and 6(b)].

To enhance the coupling between modes 1 and $1''$, we introduce a cos(6$\theta$) perturbation, similar to what is done in Figs. 3 and 4. We again find that the output directionality of mode 1, here indicated by

$$U \equiv \frac{\int_0^{2\pi} I(\theta)\cos\theta d\theta}{\int_0^{2\pi} I(\theta)d\theta}$$

(25)

to measure its “skewness” along the horizontal direction, changes dramatically from left pointing ($U < 0$) to right pointing ($U > 0$), while the output directionality of modes $1'$ and $1''$ barely changes [Figs. 7(b) and 7(c)]. We also perform a classical ray-tracing calculation of $U$ for various cavity deformations; Fig. 7(c) [6,11,28] shows clearly that it does not capture the correct deformation dependence of the output directionality of mode 1, which is a wave-interference effect not taken into account in the classical ray dynamics. As shown in Fig. 7(a), the weak intensity of mode 1 near the cavity center at $\varepsilon_6 = -0.008$ is similar to that of mode $1''$ in Fig. 6(c), which is already a hint that the aforementioned change of mode 1 is indeed caused by the newly introduced first-order coupling to
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Fig. 8. (Color online) Analysis of the far-field change of mode 1 shown in Fig. 7. (a) Amplitude change of the angular components \( W_p \) in mode 1 outside the cavity as a function of \( \epsilon_6 \). The black solid line and red dashed line represent \( W_5 \) and \( W_3 \), and low-lying dotted lines show the rest of \( W_p \) up to \( p = 12 \). \( W_6 \) is normalized to be 1. \( |W_5(\epsilon_6) - W_5(0)| \) indicates the strength of the first-order scattering \( 11 \rightarrow 5 \), which varies significantly with \( \epsilon_6 \). (b) Phase change of \( W_5 \) in mode 1 as a function of \( \epsilon_6 \). It is nearly constant on both sides of \( \epsilon_6 = 0 \) and jumps by \( \pi \) across \( \epsilon_6 = 0 \). The two dashed lines are separated by \( \pi \) and are used as references. These results show that the switching of the output direction of mode 1 in Fig. 7 is due to the \( \epsilon_6 \)-dependent coupling to mode 1”. mode 1”. To further confirm this relation, we note that the amplitudes of \( W_5, W_1 \) in mode 1 vary most noticeably and linearly with \( \epsilon_6 \) [see Fig. 8(a)], which are exactly the two most significant components in mode 1” outside the cavity. In addition, the change of \( W_5 \) from its value at \( \epsilon_6 = 0 \) has almost fixed phases for \( \epsilon_6 < 0 \) and \( \epsilon_6 > 0 \), which differ by \( \pi \) [Fig. 8(b)]. These observations indicate that the change of \( W_5 \) is of first order in \( \epsilon_6 \), which is what we expect from the first-order scattering amplitude \( a_5 \) due to the \( \cos(6\theta) \) deformation, given by the TE perturbation result from Eq. (14).

The coupling between modes 1 and 1” can also be enhanced via a second-order scattering process. By introducing a \( \cos(4\theta) \) boundary deformation and utilizing the large \( \cos(2\theta) \) deformation, the scattering from \( m = 11 \) to \( p = 5 \) is efficiently enhanced from the two paths \( 11 \rightarrow 9 \rightarrow 5 \) and \( 11 \rightarrow 7 \rightarrow 5 \), while the coupling between modes 1 and 1” is still barely affected. As we show in Fig. 9, a similar flipping of the outcoupling direction of mode 1 is observed when \( \epsilon_5 \) varies from \(-0.01 \) to \( 0.01 \), while those of modes 1’ and 1” stay roughly the same.

VI. DISCUSSION AND CONCLUSION

Finally, we compare our study to previous works on surface scattering. Light scattering from a rough surface or inside a corrugated waveguide has been studied extensively [29–36]. For example, Ref. [33] studied the higher-order effect of longitudinal surface roughness in the direction of wave propagation in quasi-one-dimensional waveguides. Although the examples in our paper are for planar systems, the governing equation (1) is equivalent to steady-state wave propagation in quasi-two-dimensional waveguides. In this waveguide analog the surface roughness we consider is transverse instead, with no variation in the longitudinal direction of the waveguide. We also note that Ref. [33] considered a closed (i.e., Dirichlet) boundary condition in the cross section of the waveguide, while we consider an open (i.e., outgoing) boundary condition. Despite the different scattering geometries, there are some similar features in both systems. Reference [33] found the crossover of the first- and second-order terms in the longitudinal scattering mean free path, while we discuss how the first- and second-order scatterings can have comparable amplitudes and interfere destructively or constructively. A key difference lies in the physical factors that influence the relative amplitudes of these terms. One dominant factor shown in our work is the spectral overlap of adjacent resonances or quantized modes of the cross section, which is absent in Ref. [33] given that the one-dimensional cross section considered leads to an equally spaced and nonoverlapping transverse mode spectrum.

In summary, we have shown a convenient approach to achieve and control multimode coupling using boundary wave scattering. The examples given are for solutions of the scalar Helmholtz equation with two types of open boundary conditions in quasicircular systems. Fine-tuning of the harmonic boundary deformation has been demonstrated, for example, in a liquid-jet column [37], and the general principle should also apply for a wide variety of Hamiltonians in other geometries, unless the scattering is prohibited by a topological property of the material [38–40]. The boundary wave scattering presented is a linear and elastic analog of Brillouin scattering [41] in a circular geometry [42], where the angular momentum plays the roles of frequency. The boundary wave scattering can also couple modes within the cavity plane to propagating modes in the free space [43]. The cancellation of the scattering from \( m = 11 \) to \( p = 5 \) by the destructive interference of two scattering paths shown in Fig. 3 closely resembles the vanishing of absorption in electromagnetically induced transparency [44].

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**APPENDIX: QUASIDEGENERACY AND THE SPECTRAL FUNCTION**

In the appendix we discuss how the boundary wave scattering of high-Q modes depends on the frequency regime. We are most interested in the $\eta = 1$ modes, which have the smallest cavity decay rates and thus the lowest thresholds once optical gain is introduced to the cavity. In Ref. [21] it has been observed numerically that for these modes, the effect of boundary wave scattering from a minute low-order harmonic boundary deformation is most dramatic in the mesoscopic regime, i.e., the crossover regime between the macroscopic regime ($\lambda \sim R$) and the macroscopic regime (semiclassical limit $\lambda \ll R$), where $\lambda = 2\pi/\text{Re}[k]$ is the wavelength; the effect becomes very weak in the semiclassical limit. Below we point out that the key to understanding this phenomenon lies in the spectral function $S_p^{-1}(K_{m,1} R)$ for TM waves [or $T_p^{-1}(K_{m,1} R)$ for TE waves; see Eq. (14)], which in turn depends on the frequency spacing between $K_{m,1}$ and the nearest resonance of angular momenta $p_m$, as mentioned in Sec. II. We denote this distance $\Delta_{m,p}$, i.e.,

$$\Delta_{m,p} = |K_{m,1} - K_{p,\eta}| \forall \eta, \tag{A1}$$

and note that it is determined mostly by the real part of the high-Q resonant frequencies we are interested in, whose $|\text{Im}[K R]| \ll \text{Re}[K R]$. Thus $\text{Re}[\Delta_{m,p}] \approx \Delta_{m,p}$ is the quantity we will focus on here.

To find the frequency dependence of $\Delta_{m,p}$, we first note that all $K$’s of the same $\eta$ form a band in the $[m, \text{Re}[K R]]$ plane [see Fig. 10(a)]. These $\eta$ bands do not cross, and the slope of the $\eta = 1$ band is well approximated by $\text{Re}[K R]/m \approx 1/n$ [45]. It is straightforward to find that $\Delta_{m,p}$ is capped at about $|m - p|/n$ for a given $m$ and $p$, which is the distance between $K_{m,1}$ and $K_{p,1}$. For $p > m$, this is in fact the value of $\Delta_{m,p}$, since $K_{p,\eta < 1}$ are farther away from $K_{m,1}$ as can be seen in Fig. 10(a). For such a relatively large $\Delta_{m,p}$, the spectral function $S_p^{-1}(K_{m,1} R)$ is typically subunitary [see Fig. 10(b)], and the associated scattering processes, such as the first-order scattering $m \to p$, are very weak. For $p < m$, however, $K_{p,\eta < 1}$ can be much closer to $K_{m,1}$ when compared with $K_{p,1}$, which then leads to a very large spectral function and a very strong scattering strength. To find out when this situation occurs, we employ the approximation for $\text{Re}[K R]$ given in Ref. [25], which applies to both TM and TE modes of a small $\eta$:

$$\text{Re}[K R] = \frac{m}{n} + \frac{\beta_\eta}{n} \left(\frac{m}{2}\right)^{1/3} - \frac{1}{\tau \sqrt{n^2 - 1}} + O\left(\frac{1}{m}\right)^{1/3}. \tag{A2}$$

Here $\tau = 1$ for TM modes and $n^2$ for TE modes, and $\beta_\eta$ is the $\eta$th zero of the Airy function, the first three of which are 2.34, 4.09, and 5.52. $\Delta_{m,p}$ can then be approximated by

$$\Delta_{m,p} \approx \min\left[\frac{m - p}{n} + \frac{1}{n} \left[\beta_1 \left(\frac{m}{2}\right)^{1/3} - \beta_\eta \left(\frac{p}{2}\right)^{1/3}\right]\right]$$

$$= \min\left[\frac{m - p}{n} + \frac{1}{n} \left[\beta_1 \left(\frac{m}{2}\right)^{1/3} + \beta_\eta \left(\frac{p}{2}\right)^{1/3}\right]ight]$$

$$\approx \frac{1}{n} \min\left|m - p - (\beta_\eta - \beta_1) \left(\frac{m}{2}\right)^{1/3}\right| \tag{A3}$$

for $m - p \ll m^{2/3}$. Therefore we see that whenever

$$L \equiv \frac{m - p}{\left(\frac{n}{2}\right)^{1/3}} + \beta_1 \tag{A4}$$

approaches one $\beta_{\eta > 1}$, $\Delta_{m,p}$ approaches zero, and the spectral function enhances the scattering strength. From this expression, we can then estimate the upper bound of $m$ for this to occur at a given $v \equiv m - p > 0$, i.e.,

$$m_{\text{max}} \approx 2 \left[\frac{v}{\beta_2 - \beta_1}\right]^3 \approx 0.37 v^3. \tag{A5}$$

which is independent of the polarization and the refractive index. For example, $m_{\text{max}}$ from Eq. (A5) is 10 and 81 for $v = 3.6$, respectively, which agrees qualitatively with the numerical value of 12 and 76 shown in Fig. 10(b). For $m > m_{\text{max}}$, the spectral function $S_m^{-1}(K_{m,1} R)$ tails off and

FIG. 10. (Color online) (a) TM spectrum of a circular cavity with refractive index $n = 3.13$ near $\text{Re}[K R] = 27$. Dashed boxes indicate the first three bands with radial quantum number $\eta = 1, 2, 3$. $K_{mn,76,\eta = 1}$ and $K_{mn,70,\eta = 2}$ are marked by the horizontal arrows. They are the closest quasidegenerate pair in the frequency range. (b) Spectral function $|S_m^{-1}(K_{m,1} R)|$ for the first-order scattering $m \to m \pm v$ as a function of $m$ for $v = 3$ and 6. The peak of $S_m^{-1}(K_{m,1} R)$ at $m = 76$ is due to the quasidegeneracy shown in (a). Its other peak at $m = 12$ is due to another pair of quasidegenerate modes $K_{12,1}$ and $K_{6,3}$. The single peak of $S_m^{-1}(K_{m,1} R)$ at $m = 9$ is due to the quasidegeneracy between $K_{9,1}$ and $K_{6,2}$. 043801-8
eventually becomes comparable to the small $S_{m+1}^{-1}(K_{m,1} R)$ we have discussed since now $K_{m-n,1}$ is the closest resonance of angular momentum $m - n$ to $K_{m,1}$ and $\Delta_{m,m-n} \approx \nu / n$, just as $K_{m+1,1}$ is the closest resonance of angular momentum $m + \nu$ to $K_{m,1}$ and $\Delta_{m,m+\nu} \approx \nu / n$.

Equation (A5) explains why the sensitivity of the high-$Q$ modes of $\eta = 1$ on a low-order harmonic boundary deformation maximizes in the mesoscopic regime and becomes weak in the semiclassical limit. We note that quasidegeneracy also occurs among modes of much larger $\eta$'s, such as the TM resonances $K_{30,10} = 30.187 - 9.6157 \times 10^{-6} i$ and $K_{47,11} = 30.186 - 9.6122 \times 10^{-6} i$ in a circular cavity of $n = 3.13$. We do not study them here because their relatively low quality factors make them difficult to observe experimentally. Their coupling, nevertheless, gives an alternative explanation to the contrasting intracavity and far-field intensity patterns found in Ref. [46], similar to what we have shown in Figs. 4 and 7.

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