INELASTICITY OF SOLITON COLLISIONS FOR THE 5D ENERGY CRITICAL WAVE EQUATION

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Abstract. For the focusing energy critical wave equation in 5D, we construct a solution showing the inelastic nature of the collision of two solitons for any choice of sign, speed, scaling and translation parameters, except the special case of two solitons of same scaling and opposite signs. Beyond its own interest as one of the first rigorous studies of the collision of solitons for a non-integrable model, the case of the quartic gKdV equation being partially treated in [32, 33, 34], this result can be seen as part of a wider program aiming at establishing the soliton resolution conjecture for the critical wave equation. This conjecture has already been established in the 3D radial case in [10] and in the general case in 3, 4 and 5D along a sequence of times in [13].

Compared with the construction of an asymptotic two-soliton in [35], the study of the nature of the collision requires a more refined approximate solution of the two-soliton problem and a precise determination of its space asymptotics. To prove inelasticity, these asymptotics are combined with the method of channels of energy from [10, 23].

1. Introduction

1.1. Main result. We consider the focusing energy critical nonlinear wave equation in 5D

$$\partial_t^2 u - \Delta u - |u|^4 u = 0, \quad (t, x) \in \mathbb{R} \times \mathbb{R}^5. \quad (1.1)$$

Recall that the Cauchy problem for equation (1.1) is locally well-posed in the energy space $\dot{H}^1 \times L^2$, using suitable Strichartz estimates. See e.g. [24] and references therein. Note that equation (1.1) is invariant by the $\dot{H}^1$ scaling: if $u$ is solution of (1.1), then for any $\lambda > 0$, $u_\lambda$ defined by

$$u_\lambda(t, x) = \frac{1}{\lambda^{\frac{2}{5}}} u \left( \frac{t}{\lambda}, \frac{x}{\lambda} \right), \quad \|u_\lambda\|_{\dot{H}^1} = \|u\|_{\dot{H}^1},$$

is also solution of (1.1). The energy $E(u(t), \partial_t u(t))$ and the momentum $M(u(t), \partial_t u(t))$ of an $\dot{H}^1 \times L^2$ solution are conserved, where

$$E(u, v) = \frac{1}{2} \int v^2 + \frac{1}{2} \int |\nabla u|^2 - \frac{3}{10} \int |u|^{10}, \quad M(u, v) = \int v \nabla u.$$

Recall also that the function $W$ defined by

$$W(x) = \left(1 + \frac{|x|^2}{15}\right)^{-\frac{\frac{3}{6}}{2}}, \quad \Delta W + W = 0, \quad x \in \mathbb{R}^5, \quad (1.2)$$

is a stationary solution of (1.1), called here ground state, or soliton. By scaling, translation invariances and change of sign, we obtain a family of stationary solutions of (1.1) defined by $W_{\lambda, x_0, \pm}(x) = \pm \lambda^{-\frac{3}{10}} W(\lambda^{-1}(x - x_0))$, where $\lambda > 0$ and $x_0 \in \mathbb{R}^5$. 

Using the Lorentz transformation, we obtain traveling waves. For \( \ell \in \mathbb{R}^5 \), with \(|\ell| < 1\), let
\[
W_\ell(x) = W \left( \frac{1}{\sqrt{1 - |\ell|^2}} - 1 \right) \frac{\ell(x \cdot \ell)}{|\ell|^2} + x \right). \tag{1.3}
\]
Then, the functions \( w_{\ell, \pm}(t, x) = \pm W_\ell(x - \ell t) \), as well as rescaled and translated versions of \( w_{\ell, \pm} \), are solutions of (1.1). While the ground state \( W \) is the unique, up to scaling invariance and sign change, radial stationary solution of (1.1), there also exist non-radial solutions \( Q \in \dot{H}^1(\mathbb{R}^5) \) of the elliptic equation \( \Delta Q + |Q|^4 \frac{3}{2} Q = 0 \) on \( \mathbb{R}^5 \); see [7, 6] for explicit constructions. However, no classification result is known for such solutions.

The present paper addresses in the context of the wave equation (1.1) the classical question of the elastic or inelastic nature of the collision of traveling waves. Recall that such questions were first investigated by early numerical simulations [15, 50] on some nonlinear models, and then mathematically studied by integrability (see e.g. [50, 28, 18, 48, 40, 5]), using the inverse scattering transform. In such integrable cases, the collision of any number of solitons is elastic, meaning that neither the number of solitons, nor their speeds, are changed by the collision. For models perturbative to integrable models, few results are known (see e.g. [44, 42]) and it is generally observed that elasticity is lost.

For nonlinear equations that are not close to any known integrable model, the collision problem is widely open. To the authors’ knowledge, it was studied rigorously only for the quartic gKdV equation on the line
\[
\partial_t u + \partial_x \left( \partial_x^2 u + u^4 \right) = 0, \quad (t, x) \in \mathbb{R}^2,
\]
following Open Problem 4 in §11 of [40]. For two solitons with speeds \( 0 < c_2 < c_1 \), the authors of the present paper have addressed the collision problem for the quartic gKdV equation in the following two asymptotic situations:

(a) Solitons of very different speeds: \( 0 < c_2 \ll c_1 \). See [32]
(b) Solitons with almost equal speeds: \( 0 < 1 - c_2/c_1 \ll 1 \). See [33, 34].

Under condition (a) or (b), it is proved that in contrast with the integrable cases, the collision is always inelastic. In [32, 33], the explicit computation of an approximate two-soliton solution for all \((t, x) \in \mathbb{R}^2\) describes globally the collision and shows the presence of a non-trivial residual term after the collision. Moreover, as a consequence of the conservation of mass and energy, it is proved that the speeds and the sizes of the solitons are slightly altered by the interaction. In [34], the strategy is different and could in principle cover the whole range of parameters \( 0 < c_2 < c_1 \), though for technical reasons, the result is restricted to the case \( 1 - c_2/c_1 < 1/4 \). Indeed, an approximate solution of the two-soliton problem is computed only for large time, so that the solitons are decoupled regardless their respective speeds. Then, the defect due to the collision is propagated to any further time by special monotonicity properties of the gKdV equation. The present paper is partly inspired by this approach, replacing such monotonicity properties by the finite speed of propagation and the method of channels of energy introduced in [10].

Experimental and numerical results on collision are available for various physical contexts and nonlinear models, see e.g. [40, 47, 4, 17, 1, 29]. It seems that inelasticity is found in all non-integrable models studied, which supports the general belief that the existence of pure multi-solitons is tightly related to integrability. We refer the reader to the more extended discussions in [40] [4, 32].
In this paper, we prove the existence of a solution of (1.1) which shows the inelastic nature of the collision of any two solitons, except the special case of same scaling and opposite signs.

**Theorem 1.1.** For $k \in \{1, 2\}$, let $\lambda_k^\infty > 0$, $y_k^\infty \in \mathbb{R}^5$, $\epsilon_k \in \{\pm 1\}$, $\ell_k \in \mathbb{R}^5$ with $|\ell_k| < 1$, and

$$W_k^\infty(t, x) = \frac{\epsilon_k}{(\lambda_k^\infty)^{\frac{3}{2}}} W_{\ell_k} \left( \frac{x - \ell_k t - y_k^\infty}{\lambda_k^\infty} \right).$$

Assume that $\ell_1 \neq \ell_2$ and

$$\epsilon_1 = \epsilon_2 \quad \text{or} \quad \lambda_1^\infty \neq \lambda_2^\infty. \quad (1.4)$$

Then, there exists a solution $u$ of (1.1) in the energy space such that

(i) Two-soliton as $t \to +\infty$

$$\lim_{t \to +\infty} \|\nabla_{t,x} u(t) - \nabla_{t,x} (W_1^\infty(t) + W_2^\infty(t))\|_{L^2} = 0.$$

(ii) Dispersion as $t \to -\infty$. There exists $C > 0$ such that, for all $A > 0$ large enough,

$$\liminf_{t \to -\infty} \|\nabla u(t)\|_{L^2(|x| > |t| + A)} \geq CA^{-\frac{2}{5}}. \quad (1.5)$$

The solution constructed in Theorem 1.1 is a two-soliton asymptotically as $t \to +\infty$ and it does not necessarily exist for all $t \in \mathbb{R}$. However, by finite speed of propagation and small data Cauchy theory, it is straightforward to justify that it can be extended uniquely as a solution of (1.1) for all $t \in \mathbb{R}$ in the region $|x| > |t| + A$, provided that $A$ is large enough. Thus, the limit in (1.5) makes sense (see [6] for details). Since the estimate (1.5) gives an explicit lower bound on the loss of energy as dispersion as $t \to -\infty$, the solution $u$ is not a two-soliton asymptotically as $t \to -\infty$ and the collision is inelastic. Note that the two-soliton could have any global behavior, like dislocation of the solitons and dispersion, blow-up or a different multi-soliton plus radiation, but the property that we obtain is universal and independent of the behavior on compact sets. Note also that the only case left open by Theorem 1.1 corresponds, up to scaling and Lorentz invariance (and up to irrelevant translations), to the dipole case, i.e. $\lim_{t \to +\infty} \|u(t) - W_{\ell}(x - \ell t) + W_{-\ell}(x + \ell t)\|_{H^1} = 0$, for some $\ell \in \mathbb{R}^5$, $|\ell| < 1$. We expect that a similar dispersion phenomenon takes place but possibly at lower order due to cancellation of the tail asymptotics by symmetry.

In the case of $K$ solitons with $K \geq 3$, existence of an asymptotic multi-soliton at $+\infty$ still holds for collinear speeds from [35]. Applying the same strategy, inelasticity is proved under a simple explicit non-vanishing condition which generalizes (1.4). See details in [7].

The interest of this work is twofold. A main motivation is to continue the authors’ program on the collision of solitons for non-integrable equations. It is the first non-integrable model for which we are able to prove inelasticity without restriction on the relative sizes or speeds of the solitons except the dipole case $\epsilon_1 = -\epsilon_2$ and $\lambda_1^\infty = \lambda_2^\infty$. We also study the nature of soliton collisions because of its importance in the context of the soliton resolution conjecture for equation (1.1). A particular case of this conjecture says that any global and bounded solution of (1.1) in the energy space should decompose as $t \to +\infty$ as a finite sum of solitons plus a dispersive part. This conjecture was proved in [9, 10] for the 3D radial case. In [11, 13], the above version of the soliton resolution conjecture was proved in the non-radial case for a sequence of times $t_n \to +\infty$ in 3, 4 and 5D. We also refer to previous results of classification in [8, 43, 25, 26] and to constructions of special solutions in [27, 19, 20, 21]. We expect that, beyond its own interest, the full understanding of the collision problem will be a key to the proof of the soliton resolution conjecture for the whole sequence of time.
1.2. **Outline of the proof.** The strategy of the proof is to construct a refined approximate solution of the two-soliton problem that displays an explicit dispersive radial part at the leading order and then to propagate the dispersion for any negative time at the exterior of large cones by finite speed of propagation and the method of channels of energy.

First, we construct a **refined approximate solution** to the two-soliton problem for large $t > 0$ of the form $\vec{W} = \vec{W}_1 + \vec{W}_2 + \vec{v}_1 + \vec{v}_2$, where $\vec{W}_1$ and $\vec{W}_2$ are two solitons with time dependent scaling and translation parameters, and $\vec{v}_1$, $\vec{v}_2$ are correction terms improving the simpler approximate solution used in [35]. These correction terms of size $t^{-2}$ in the energy space are solutions of non-homogeneous wave equations whose source terms are the main order of the nonlinear interactions of size $t^{-3}$ between the two solitons. In this way, $\vec{W}$ is an approximate solution of the two soliton problem at order $t^{-4}$. Such refined approximate solutions were introduced in several other situations related to blow up or soliton interactions, see e.g. [41, 39, 45, 32, 33, 34, 19, 37, 16]. In the case of the gKdV equation [34], since solitons decay exponentially in space, the method of separation of variables applies and correction terms have simple expressions in terms of solutions of elliptic problems. In the present paper, this method would lead to correction terms not belonging to the energy space (see e.g. [19]). Since the strategy is based on a close examination of the asymptotics of the approximate solution, using cut-off to balance artificial growth cannot be successful. This is the reason why we define $\vec{v}_1$, $\vec{v}_2$ as solutions of linear evolution problems with source terms. Now, $\vec{v}_1$ and $\vec{v}_2$ are much less explicit but they belong to the energy space and their asymptotics contain the desired information. Because of the specific forms of the source terms, their equations cannot be reduced to radial ones by the Lorentz transformation.

The next step is to compute the **space asymptotics of the radial part of the approximate solution**. The main asymptotic part of $\vec{v}_1$, $\vec{v}_2$ is explicit but it turns out not to channel any energy (as a soliton). In view of the formula of the fundamental solution of the wave equation in 5D, it is not clear how to obtain manageable expressions for the next orders of $\vec{v}_1$ and $\vec{v}_2$. Our strategy is to compute only the radial part of their asymptotics using spherical means and reduction to a 1D problem. The computation reveals an explicit dispersive tail for the radial parts of $\vec{v}_1$ and $\vec{v}_2$ for large positive times. It is remarkable that understanding only the radial component of the approximate solution is sufficient to treat all cases of two solitons except the dipole.

Finally, we **propagate the dispersion by the method of channels of energy**, which is a refined characterization of dispersion for wave type equations introduced by Duyckaerts, Kenig and Merle in [10]. We check that under the non-vanishing condition [14], summing the dispersive tails of $\vec{v}_1$ and $\vec{v}_2$, the radial part of the approximate solution has itself a non-zero dispersive tail for large positive times. As in [35] and several other works related to the construction of multi-solitons (see references in [35]), the two-soliton is constructed by compactness using the approximate solution $\vec{W}$. We also prove that the non-zero dispersive tail of the approximate solution is greater than the error terms so that it is still visible in the two-soliton. The method of channels of energy (see [10, 11, 22, 23]) then allows us to propagate the dispersion for any negative time at the exterior of large cones. Moreover, from Theorem 2 of [12], the solution behaves asymptotically as $t \to -\infty$ as a non-zero solution of the linear wave equation in the region $|x| > |t| + A$ for $A$ large.

We expect that our method can solve the same problem for odd space dimensions larger than 5. The method should also extend to other wave type equations.
1.3. **Notation.** The canonical basis of $\mathbb{R}^5$ is denoted by $\{e_1, \ldots, e_5\}$. We denote for real-valued functions

$$(v, \tilde{v}) = \int v\tilde{v}, \quad \|v\|^2_{L^2} = \int |v|^2, \quad (v, \tilde{v})_{H^1} = \int \nabla v \cdot \nabla \tilde{v}, \quad \|v\|^2_{H^1} = \int |\nabla v|^2.$$ 

For

$$\tilde{v} = \left( \begin{array}{c} v \\ z \end{array} \right), \quad \hat{v} = \left( \begin{array}{c} \hat{v} \\ \hat{z} \end{array} \right),$$

set

$$\left( \hat{v}, \hat{\tilde{v}} \right) = (v, \tilde{v}) + (z, \hat{z}), \quad \left( \hat{v}, \hat{\tilde{v}} \right)_{H^1 \times L^2} = (v, \tilde{v})_{H^1} + (z, \hat{z}).$$

We denote by $d\omega$ the Lebesgue measure on the sphere, and by

$$\int_{|y-x|=r} v(y)d\omega(y) = \frac{3}{8\pi^2 r^{-4}} \int_{|y-x|=r} v(y)d\omega(y)$$

the average of a function $v$ over the sphere of $\mathbb{R}^5$ of center $x \in \mathbb{R}^5$ and radius $r > 0$.

Set $\langle x \rangle = (1 + |x|^2)^{1/2}$. Let

$$\Lambda = \frac{3}{2} + x \cdot \nabla, \quad \bar{\Lambda} = \frac{5}{2} + x \cdot \nabla, \quad \nabla \Lambda = \nabla \Lambda, \quad \bar{\nabla} = \left( \begin{array}{c} \bar{\Lambda} \\ \Lambda \end{array} \right). \quad (1.6)$$

When $x_1$ is seen as a specific coordinate, denote

$$\bar{x} = (x_2, \ldots, x_5), \quad \bar{\nabla} v = (\partial_{x_2} v, \ldots, \partial_{x_5} v), \quad \bar{\Delta} v = \sum_{j=2}^5 \partial^2_{x_j} v.$$ 

For $-1 < \ell < 1$,

$$(v, \tilde{v})_{H^1_\ell} = (1 - \ell^2) \int (\partial_{x_1} v)(\partial_{x_1} \tilde{v}) + \int \nabla v \cdot \nabla \tilde{v}, \quad \|v\|^2_{H^1_\ell} = (v, v)_{H^1_\ell},$$

$$x_\ell = \left( \frac{x_1 - \ell t}{\sqrt{1 - \ell^2}}, \bar{x} \right), \quad A_\ell = \partial_t + \ell \partial_{x_1}, \quad B_\ell = \partial^2_t - \ell^2 \partial^2_{x_1} = A_\ell^2 - 2\ell \partial_{x_1} A_\ell,$$

$$\Lambda_\ell = \frac{3}{2} + (x - \ell t e_1) \cdot \nabla, \quad \Delta_\ell = (1 - \ell^2) \partial^2_{x_1} + \bar{\Delta}.$$ 

For $\gamma > 0$ small to be fixed later, set

$$\varphi_\gamma(x) = (1 + |x|^2)^{-\gamma} \quad (1.7)$$

We recall standard Sobolev and Hölder inequalities

$$\|u\|_{L^{10/3}} \lesssim \|u\|_{H^1}, \quad \|u\|_{L^{10}} \lesssim \|u\|_{H^2}, \quad (1.8)$$

$$\int |u|^3 |v|^2 \lesssim \|u\|_{L^{10/3}} \|v\|_{L^{10/3}} \|w\|_{L^{10/3}} \|w\|^{4/3}_{L^{10/3}} \lesssim \|u\|_{H^1} \|v\|_{H^1} \|w\|^{4/3}_{H^1}, \quad (1.9)$$

$$\|uv\|_{L^{10/7}} \lesssim \|u\|_{L^{10/3}} \|v\|_{L^{5/2}}, \quad \|uvw\|_{L^{10/7}} \lesssim \|u\|_{L^{10/3}} \|v\|_{L^{10/3}} \|w\|_{L^{10}}. \quad (1.10)$$

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2. Preliminaries

We gather in this section preliminary results on the linearized operator around $W$, on the linear homogeneous and non-homogeneous wave equations in 5D, on the method of channels of energy and on the Cauchy problem for (1.1).

2.1. Linearized operator around the soliton. Let

$$L = -\Delta - \frac{7}{3}W^\frac{4}{5}, \quad (Lv, v) = \int \left( |\nabla v|^2 - \frac{7}{3}W^\frac{4}{5}v^2 \right),$$

$$H = \begin{pmatrix} L & 0 \\ 0 & \text{Id} \end{pmatrix}, \quad (H \vec{v}, \vec{v}) = (Lv, v) + \|v\|_{L^2}^2 \quad \text{for} \quad \vec{v} = \begin{pmatrix} v \\ z \end{pmatrix}.$$  

For $\vec{v}$ small in the energy space, we recall the expansion of the energy

$$E(W + v, z) = E(W, 0) + \frac{1}{2}(H \vec{v}, \vec{v}) + O(\|v\|_{H^1}^3).$$

Lemma 2.1 (Properties of $L$).

(i) Spectrum. The operator $L$ on $L^2$ with domain $H^2$ is a self-adjoint operator with essential spectrum $[0, +\infty)$, no positive eigenvalue and only one negative eigenvalue $-\lambda_0$, with a smooth radial positive eigenfunction $Y \in S(\mathbb{R}^5)$. Moreover,

$$L(\Lambda W) = L(\partial_x W) = 0, \quad \text{for any} \ j = 1, \ldots, 5.$$

(ii) Coercivity results. There exists $\mu > 0$ such that, for $\gamma > 0$ small enough, for all $v \in H^1$,

$$(Lv, v) \geq \mu \|v\|_{H^1}^2 - \frac{1}{\mu} \left\{ (v, \Lambda W)^2_{H^1} + |(v, \nabla W)_{H^1}|^2 + (v, W)^2_{H^1} \right\},$$

$$(Lv, v) \geq \mu \|v\|_{H^1}^2 - \frac{1}{\mu} \left\{ (v, \Lambda W)^2_{H^1} + |(v, \nabla W)_{H^1}|^2 + (v, W)^2_{H^1} \right\},$$

$$\int \left( |\nabla v|^2 \varphi^2 - \frac{7}{3}W^\frac{4}{5}v^2 \right) \geq \mu \int |\nabla v|^2 \varphi^2 - \frac{1}{\mu} \left\{ (v, \Lambda W)^2_{H^1} + |(v, \nabla W)_{H^1}|^2 + (v, W)^2_{H^1} \right\}.$$

(iii) Inversion of $L$. Let $F \in H^{-1}$ be such that $(F, \Lambda W) = |(F, \nabla W)| = 0$. Then, there exists a unique $V \in \dot{H}^1$ such that $(V, \Lambda W)_{\dot{H}^1} = |(V, \nabla W)_{\dot{H}^1}| = 0$ and $LV = F$. Moreover, if $F$ is of class $C^p$, $p \geq 1$ and satisfies, for some $0 < \delta < 1$, for all $\alpha \in \mathbb{N}^5$, $|\alpha| \leq p$, for all $x \in \mathbb{R}^5$,

$$|\partial^\alpha F(x)| \lesssim \langle x \rangle^{-5-\delta},$$

then $V$ is of class $C^{p+1}$ and satisfies, for all $\alpha' \in \mathbb{N}^5$, $2 \leq |\alpha'| \leq p + 1$, for all $x \in \mathbb{R}^5$,

$$|V(x)| \lesssim \langle x \rangle^{-3}, \quad |\nabla V(x)| \lesssim \langle x \rangle^{-4}, \quad |\partial^\alpha' V(x)| \lesssim \langle x \rangle^{-5}.$$

Proof. The spectral properties of $L$ in (i) are standard and easily checked. The coercivity properties (ii) are given respectively in [40], [8] and [35]. To prove (iii), we first define

$$\mathcal{Y}^\perp = \{ v \in \dot{H}^1, (v, \Lambda W)_{\dot{H}^1} = |(v, \nabla W)_{\dot{H}^1}| = (v, W)_{\dot{H}^1} = 0 \},$$

$$\mathcal{Y}_0^\perp = \{ v \in \dot{H}^1, (v, \Lambda W)_{\dot{H}^1} = |(v, \nabla W)_{\dot{H}^1}| = 0 \}.$$  

Denote $MV := V + \Delta^{-1}(\frac{7}{3}W^\frac{4}{5}V)$, so that $-\Delta M = L$. For $f \in \dot{H}^1$, by (1.9), we have $|(W^\frac{4}{5}V, f)| \lesssim \|V\|_{\dot{H}^1} \|f\|_{\dot{H}^1}$. It follows that $M$ is continuous in $\dot{H}^1$. We check that the image of $\mathcal{Y}^\perp$ by $M$ is included in $\mathcal{Y}^\perp$. Indeed, for any $V \in \dot{H}^1$, $(MV, \Lambda W)_{\dot{H}^1} = - (\Delta MV, \Lambda W) = \ldots$
(V, LAW) = 0, similarly (MV, ∇W)_{H^1} = 0, and (MV, W)_{H^1} = −(Δ MV, W) = (V, LW) = −\frac{4}{3} (V, W)_{H^1}, since LW = \frac{4}{3} Δ W and Δ W + W^{\frac{7}{3}} = 0. Moreover, M is coercive in Y^1 from (ii), since for μ > 0, for all V ∈ Y^1, (MV, V)_{H^1} = (LV, V) ≥ μ∥V∥^2_{H^1}. Thus, for any f ∈ Y^1, there exists a unique V ∈ Y^1 such that MV = f.

Let now f ∈ Y^0 and set f = f^+ + aW, where a is such that f^+ ∈ Y^1. Let V^1 ∈ Y^1 be such that MV^1 = f^+. Note that by Δ W + W^{\frac{7}{3}} = 0, one has MW = W + Δ^{-1}(\frac{7}{3} W^{\frac{7}{3}} V) = -\frac{4}{3} W. Let V = V^1 - \frac{4}{3} aW. Then V ∈ Y^0 and MV = f, in particular, LV = −Δ f. To conclude, note that setting F = −Δ f, the assumptions on F are equivalent to f ∈ Y^0.

Now, we prove decay properties of V assuming further that F is of class C^p, for p ≥ 1 and satisfies for some 0 < δ < 1, for all α ∈ N^5, |α| ≤ p, for all x ∈ R^5, |∂^α F(x)| ≲ ⟨x⟩^{-δ−δ}. Write −Δ V = F + \frac{7}{3} W^{\frac{4}{3}} V. First, recall that by the explicit expression of the fundamental solution 1/8π|x|^{-3} of the Laplace equation in R^5 (see e.g. §2.2 of [13]), the unique (in the class of functions going to 0 at ∞) solution U of −Δ U = F in R^5 is of class C^{p+1} and satisfies, for all α’ ∈ N^5, 2 ≤ |α’| ≤ p + 1, for all x ∈ R^5,

|U(x)| ≲ ⟨x⟩^{-3}, |∇U(x)| ≲ ⟨x⟩^{-4}, |∂^α’ U(x)| ≲ ⟨x⟩^{-5}.

Second, since F ∈ L^2 and W^{\frac{4}{3}} V ∈ L^2 (by the Hardy inequality), we have V ∈ \dot{H}^2 ⊂ L^10. Let

w(x) = c \int W^{\frac{4}{3}}(x − y)V(x − y)|y|^{-3} dy

be solution of −Δ w = \frac{7}{3} W^{\frac{4}{3}} V. Then, by Holder inequality,

|w(x)| ≲ ∥V∥_{L^{10}} \left( \int ⟨x − y⟩^{-\frac{10}{3}} |y|^{-\frac{10}{3}} dx \right)^{\frac{9}{10}}.

Since

\int_{|y| < \frac{1}{2}(x)} ⟨x − y⟩^{-\frac{10}{3}} |y|^{-\frac{10}{3}} dy ≤ ⟨x⟩^{-\frac{10}{3}} \int_{|y| < \frac{1}{2}(x)} |y|^{-\frac{10}{3}} dy ≤ ⟨x⟩^{-\frac{10}{3}},

and

\int_{|y| > \frac{1}{2}(x)} ⟨x − y⟩^{-\frac{10}{3}} |y|^{-\frac{10}{3}} dy ≤ ⟨x⟩^{-\frac{10}{3}} \int ⟨x − y⟩^{-\frac{10}{3}} |y|^{-\frac{10}{3}} dy ≤ ⟨x⟩^{-\frac{10}{3}},

this gives |V(x)| ≲ ⟨x⟩^{-\frac{12}{5}} and thus W^{\frac{4}{3}}(x)|V(x)| ≲ ⟨x⟩^{-\frac{32}{5}}. We bootstrap this estimate to we find the desired estimates on V and ∇V. For estimates on ∂^α V for |α’| ≥ 2, we write −Δ(∂^α V) = ∂^α(\frac{7}{3} W^{\frac{4}{3}} V + F) and proceed similarly by induction on |α’| ≤ p + 1. □

For −1 < ℓ < 1, let

W_\ell(x) = W \left( \frac{x_1}{\sqrt{1 − ℓ^2}}, x \right), \quad (1 − ℓ^2)\partial^2_{x_1} W_\ell + \Delta W_\ell + W_\ell^{\frac{7}{3}} = 0,

so that u(t, x) = W_\ell(x_1 − ℓt, x) is a solution of (1.1). Note that

E(W_\ell, −ℓ\partial_{x_1} W_\ell) = ℓ^2 \int |\partial_{x_1} W_\ell|^2 = (1 − ℓ^2)\frac{1}{2} E(W, 0).
Let
\[ L_\ell = -(1 - \ell^2)\partial_{x_1}^2 - \Delta - \frac{7}{3} W_\ell^\frac{4}{3}, \]
\[(L_\ell v, v) = \int \left( (1 - \ell^2) |\partial_{x_1} v|^2 + |\nabla v|^2 - \frac{7}{3} W_\ell^\frac{4}{3} v^2 \right), \]
\[ H_\ell = \left( -\Delta - \frac{7}{3} W_\ell^\frac{4}{3} - \ell \partial_{x_1} \right), \quad (H_\ell \bar{v}, \bar{v}) = (L_\ell v, v) + \|\partial_{x_1} v + z\|^2_{L^2}. \]

Let
\[ J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \]

The following functions appear when studying the properties of the operators \( H_\ell \) and \( H_\ell J \)
\[ \tilde{Z}_\ell^\Lambda = \left( -\ell \partial_{x_1} \Lambda W_\ell \right), \quad \tilde{Z}_\ell^{\nabla j} = \left( \partial_{x_1} W_\ell \right), \quad \tilde{Z}_\ell^W = \left( -\ell \partial_{x_1} W_\ell \right), \]
\[ Y_\ell(x) = Y \left( \frac{x_1}{\sqrt{1 - \ell^2}}, x \right), \quad \tilde{Z}_\ell^\pm = \left( \ell \partial_{x_1} Y_\ell \pm \frac{\sqrt{\lambda_0(1 - \ell^2)}}{\sqrt{1 - \ell^2}} Y_\ell \right) e^{\pm \frac{\ell}{\sqrt{1 - \ell^2}} x_1}. \]

We recall the following result from \([3]\) and \([35]\).

**Lemma 2.2.** Let \(-1 < \ell < 1\).

(i) Properties of \( L_\ell \).
\[ L_\ell(\Lambda W_\ell) = L_\ell(\partial_{x_1} W_\ell) = 0, \quad L_\ell Y_\ell = -\lambda_0 Y_\ell, \quad L_\ell W_\ell = -\frac{4}{3} W_\ell^\frac{2}{3}. \]

(ii) Properties of \( H_\ell \) and \( H_\ell J \).
\[ H_\ell \tilde{Z}_\ell^\Lambda = H_\ell \tilde{Z}_\ell^{\nabla j} = 0, \quad H_\ell \tilde{Z}_\ell^W = -\frac{4}{3} \left( \frac{W_\ell^\frac{2}{3}}{0} \right), \]
\[ \left( H_\ell \tilde{Z}_\ell^W, \tilde{Z}_\ell^W \right) = -\frac{4}{3} \int W_\ell^\frac{4}{3}, \quad -H_\ell J(\tilde{Z}_\ell^\pm) = \pm \sqrt{\lambda_0(1 - \ell^2)} \tilde{Z}_\ell^\pm, \]
\[ \left( \tilde{Z}_\ell^\Lambda, \tilde{Z}_\ell^W \right)_{H^1 \times L^2} = \left( \tilde{Z}_\ell^{\nabla j}, \tilde{Z}_\ell^W \right)_{H^1 \times L^2} = 0, \quad \left( \tilde{Z}_\ell^\Lambda, \tilde{Z}_\ell^\pm \right) = \left( \tilde{Z}_\ell^{\nabla j}, \tilde{Z}_\ell^\pm \right) = 0. \]

(iii) Coercivity. There exists \( \mu > 0 \) such that, for all \( \bar{v} \in H^1 \times L^2 \),
\[ (H_\ell \bar{v}, \bar{v}) \geq \mu \|\bar{v}\|_{H^1 \times L^2}^2 - \frac{1}{\mu} \left\{ (v, \Lambda W_\ell)_{H^1}^2 + (v, \nabla W_\ell)_{H^1}^2 \right\}, \]
\[ \int \left( |\nabla v|^2 \varphi_\gamma^2 - \frac{7}{3} W_\ell^\frac{4}{3} v^2 + z^2 \varphi_\gamma^2 + 2\ell(\partial_{x_1} v)z \varphi_\gamma^2 \right) \]
\[ \geq \mu \int \left( |\nabla v|^2 + z^2 \right) \varphi_\gamma^2 - \frac{1}{\mu} \left\{ (v, \Lambda W_\ell)_{H^1}^2 + (v, \nabla W_\ell)_{H^1}^2 \right\}. \]
We extend the above notation to any $\ell \in \mathbb{R}^5$ such that $|\ell| < 1$. The function $W_\ell$ defined in \((1.3)\) satisfies the equation $\Delta W_\ell - (\ell \cdot \nabla)^2 W_\ell - W_\ell^{7/3} = 0$. Let

$$L_\ell = -\Delta - \ell \cdot \nabla (\ell \cdot \nabla) - \frac{7}{3} W_\ell^\frac{4}{3}, \quad H_\ell = \begin{pmatrix} -\Delta - \frac{7}{3} W_\ell^\frac{4}{3} & -\ell \cdot \nabla \\ \ell \cdot \nabla & \text{Id} \end{pmatrix},$$

$$\hat{Z}_\ell = \begin{pmatrix} \Lambda W_\ell \\ -\ell \cdot \nabla (\ell W_\ell) \end{pmatrix}, \quad \hat{Z}_{\ell j} = \begin{pmatrix} \partial_{x_j} W_\ell \\ -\ell \cdot \nabla (\ell_{x_j} W_\ell) \end{pmatrix}, \quad \hat{Z}_\ell^W = \begin{pmatrix} W_\ell \\ -\ell \cdot \nabla W_\ell \end{pmatrix},$$

$$Y_\ell = Y \left( \left( \frac{1}{\sqrt{1-|\ell|^2}} - 1 \right) \frac{\ell (\ell \cdot x)}{|\ell|^2} + x \right), \quad \hat{Z}_\ell^{\pm} = \begin{pmatrix} \ell \cdot \nabla Y_\ell \pm \frac{\sqrt{2}}{\sqrt{1-|\ell|^2}} Y_\ell \pm \frac{\sqrt{2}}{\sqrt{1-|\ell|^2}} (\ell x) \\ Y_\ell \pm \frac{\sqrt{2}}{\sqrt{1-|\ell|^2}} (\ell x) \end{pmatrix}.$$

### 2.2. On the linear homogeneous and non-homogeneous wave equations.

For $g, h \in \mathcal{S}(\mathbb{R}^5)$, it is well-known (see, e.g. \cite{14} §2.4) that the solution $z$ of the homogeneous wave equation in $\mathbb{R} \times \mathbb{R}^5$

$$\begin{cases} \\
\partial_t^2 z - \Delta z = 0 & \text{on } \mathbb{R} \times \mathbb{R}^5, \\
z|_{t=0} = g, & \partial_t z|_{t=0} = h & \text{on } \mathbb{R}^5
\end{cases}$$

writes

$$z = \cos(t\sqrt{-\Delta}) g + \frac{\sin(t\sqrt{-\Delta})}{\sqrt{-\Delta}} h$$

$$= \frac{1}{3} \left[ \left( \frac{\partial}{\partial t} \right) \left( \frac{1}{t} \frac{\partial}{\partial t} \right) \left( t^3 \int_{|y-x|=t} g(y) d\omega(y) \right) + \left( \frac{1}{t} \frac{\partial}{\partial t} \right) \left( t^3 \int_{|y-x|=t} h(y) d\omega(y) \right) \right]. \quad (2.1)$$

For $f \in \mathcal{S}(\mathbb{R} \times \mathbb{R}^5)$, we define

$$v(t) = -\int_t^{+\infty} \frac{\sin((s-t)\sqrt{-\Delta})}{\sqrt{-\Delta}} f(s') ds' = \int_0^{+\infty} \frac{\sin(s\sqrt{-\Delta})}{\sqrt{-\Delta}} f(s+t) ds$$

the unique solution of the non-homogeneous wave equation

$$\partial_t^2 v - \Delta v = f \quad \text{on } \mathbb{R} \times \mathbb{R}^5$$

which converges to 0 in the energy norm as $t \to +\infty$. From \((2.1)\), one has

$$\frac{\sin(s\sqrt{-\Delta})}{\sqrt{-\Delta}} f(s+t) = \frac{1}{3} \left[ \left( \frac{1}{\sigma} \frac{\partial}{\partial \sigma} \right) \left( \sigma^3 \int_{|y-x|=\sigma} f(\zeta, y) d\omega(y) \right) \right]_{\sigma=s, \zeta=t+s} \frac{\sin(s\sqrt{-\Delta})}{\sqrt{-\Delta}} f(s+t)$$

$$= \frac{1}{3s} \left[ \frac{\partial}{\partial s} \left( s^3 \int_{|y-x|=s} f(t+s, y) d\omega(y) \right) - s^3 \int_{|y-x|=s} \partial_t f(t+s, y) d\omega(y) \right].$$

Thus, integrating by parts in the variable $s$ and then changing variable,

$$v(t, x) = \frac{1}{3} \int_0^{+\infty} \frac{1}{s} \left[ \frac{\partial}{\partial s} \left( s^3 \int_{|y-x|=s} f(t+s, y) d\omega(y) \right) - s^3 \int_{|y-x|=s} \partial_t f(t+s, y) d\omega(y) \right] ds$$

$$= \frac{1}{3} \int_0^{+\infty} \int_{|y|=s} \left[ s f(t+s, x+y) - s^2 \partial_t f(t+s, x+y) \right] d\omega(y) ds. \quad (2.2)$$

Now, we prove estimates on $v$ assuming bounds on $f, A_\ell f$ and $A_\ell^2 f$. 
Lemma 2.3 (Bounds for the non-homogeneous wave equation). Let \(-1 < \ell < 1, q \geq 2\) and \(p > 2\). Let \(f\) be a smooth function such that, for all \((t, x) \in (1, +\infty) \times \mathbb{R}^5\),

\[
|A_{\ell}^q f(t, x)| \lesssim t^{-(q+m)} \langle x_\ell \rangle^{-p} \quad \text{for } m = 0, 1, 2. \tag{2.3}
\]

Let \(v\) be given by (2.2). Then, for all \((t, x) \in (1, +\infty) \times \mathbb{R}^5\),

- if \(q = 2\) and \(2 < p < 5\),
  \[
  |v(t, x)| \lesssim (t + \langle x_\ell \rangle)^{-2} \langle x_\ell \rangle^{-(p-2)} \log \left(2 + \frac{\langle x_\ell \rangle}{t}\right),
  \]
  \[
  |\nabla v(t, x)| \lesssim (t + \langle x_\ell \rangle)^{-1} t^{-1} \langle x_\ell \rangle^{-(p-1)},
  \]

- if \(q > 2\) and \(2 < p < 5\),
  \[
  |v(t, x)| \lesssim (t + \langle x_\ell \rangle)^{-2} t^{-2} \langle x_\ell \rangle^{-(p-2)} \log(1 + \langle x_\ell \rangle),
  \]
  \[
  |\nabla v(t, x)| \lesssim (t + \langle x_\ell \rangle)^{-1} t^{-1} \langle x_\ell \rangle^{-(p-1)},
  \]

- if \(q \geq 2\) and \(p = 5\),
  \[
  |v(t, x)| \lesssim (t + \langle x_\ell \rangle)^{-2} t^{-2} \langle x_\ell \rangle^{-(p-2)} \log(1 + \langle x_\ell \rangle),
  \]
  \[
  |\nabla v(t, x)| \lesssim (t + \langle x_\ell \rangle)^{-1} t^{-1} \langle x_\ell \rangle^{-(p-1)},
  \]

- if \(q \geq 2\) and \(p > 5\),
  \[
  |v(t, x)| \lesssim (t + \langle x_\ell \rangle)^{-2} t^{-2} \langle x_\ell \rangle^{-(p-2)},
  \]
  \[
  |\nabla v(t, x)| \lesssim (t + \langle x_\ell \rangle)^{-1} t^{-1} \langle x_\ell \rangle^{-(p-1)}.\]

Remark 2.4. Note the particular space-time decay properties of \(v\): e.g., in the case \(q > 2\) and \(2 < p < 5\), we have \(|v| \lesssim t^{-q} \langle x_\ell \rangle^{-(p-2)}\) and \(|v| \lesssim t^{-(q-2)} \langle x_\ell \rangle^{-p}\) on \((1, +\infty) \times \mathbb{R}^5\).

Proof of Lemma 2.3. We set \(\ell = \ell e_1\). First, we claim that for \((t, x) \in (1, +\infty) \times \mathbb{R}^5\),

\[
|v(t, x)| \lesssim J(t, x - \ell t) \quad \text{where} \quad J(t, a) = \int |y|^{-3} (t + |y|)^{-q} |a + y - \ell |y|\rangle^{-p} dy \tag{2.4}
\]

\[
|\nabla v(t, x)| \lesssim K(t, x - \ell t) \quad \text{where} \quad K(t, a) = \int |y|^{-4} (t + |y|)^{-q} |a + y - \ell |y|\rangle^{-p} dy. \tag{2.5}
\]

Proof of (2.4). From (2.2), \(v\) writes

\[
v(t, x) = \frac{1}{8\pi^2} \int_0^{+\infty} \int_{|y| = s} \left[s f(t + s, x + y) - s^2 \partial_t f(t + s, x + y)\right] d\omega(y) s^{-4} ds
\]

\[
= \frac{1}{8\pi^2} \int \left[|y|^{-3} f(t + |y|, x + y) - |y|^{-2} \partial_t f(t + |y|, x + y)\right] dy.
\]

Set \(g(t, x, y) = f(t + |y|, x + y)\). Since

\[
\partial_{y_1} g(t, x, y) = \frac{y_1}{|y|} \partial_x f(t + |y|, x + y) + \partial_x f(t + |y|, x + y)
\]

and, using the definition of \(A_{\ell}, \partial_t f = A_{\ell} f - \ell \partial_{x_1} f\), we obtain

\[
\left(1 - \ell \frac{y_1}{|y|}\right) \partial_t f(t + |y|, x + y) = A_{\ell} f(t + |y|, x + y) - \ell \partial_{y_1} g(t, x, y).
\]
Thus, integrating by parts,

\[ 8\pi^2 v(t, x) = \int \left[ |y|^{-3} f(t + |y|, x + y) + \frac{\ell \partial_{y_1} g(t, x, y) - A_\ell f(t + |y|, x + y)}{|y|(|y| - \ell y_1)} \right] dy \]

\[ = \int \left[ k(y) f(t + |y|, x + y) - h(y) A_\ell f(t + |y|, x + y) \right] dy, \]

where \( h(y) = \frac{1}{|y|(|y| - \ell y_1)} \) and \( k(y) = |y|^{-3} - \ell \partial_{y_1} h(y) \). We note that \( \frac{1}{|y|(|y| - \ell y_1)} \lesssim |y|^{-2} \) and \( \left| \partial_{y_1} \left( \frac{1}{|y|(|y| - \ell y_1)} \right) \right| \lesssim |y|^{-3} \) on \( \mathbb{R}^5 \). Thus, using (2.3),

\[ |v(t, x)| \lesssim \int |y|^{-3} |f(t + |y|, x + y)| dy + \int |y|^{-2} |A_\ell f(t + |y|, x + y)| dy \]

\[ \lesssim \int \left[ |y|^{-3} (t + |y|)^{-q} + |y|^{-2} (t + |y|)^{-(q+1)} \right] \langle x + y - \ell (t + |y|) \rangle^{-p} dy \lesssim J(x - \ell t). \]

**Proof of (2.5).** For \( j = 1, \ldots, 5 \), we have

\[ 8\pi^2 \partial_{x_j} v(t, x) = \int \left[ k(y) \partial_{x_j} f(t + |y|, x + y) - h(y) (\partial_{x_j} A_\ell f)(t + |y|, x + y) \right] dy, \]

As before,

\[ \partial_{y_j} g(t, x, y) = \frac{y_j}{|y|} \partial_{x_j} f(t + |y|, x + y) + \partial_{x_j} f(t + |y|, x + y) \]

\[ = \frac{y_j}{|y|} A_\ell f(t + |y|, x + y) - \frac{y_j}{|y|} \partial_{x_j} f(t + |y|, x + y) + \partial_{x_j} f(t + |y|, x + y). \]

In particular, for \( j = 1 \), we obtain

\[ \partial_{x_1} f(t + |y|, x + y) = |y| (|y| - \ell y_1)^{-1} \left( \partial_{y_1} g(t, x, y) - \frac{y_1}{|y|} A_\ell f(t + |y|, x + y) \right) \]

and next, for all \( j = 1, \ldots, 5 \),

\[ \partial_{x_j} f(t + |y|, x + y) = \partial_{y_j} g(t, x, y) + \frac{\ell y_j}{|y| - \ell y_1} \partial_{y_1} g(t, x, y) - \frac{y_j}{|y| - \ell y_1} A_\ell f(t + |y|, x + y). \]

Integrating by parts, we obtain

\[ \int k(y) \partial_{x_j} f(t + |y|, x + y) dy = - \int \partial_{y_j} k(y) f(t + |y|, x + y) dy \]

\[ - \ell \int \partial_{y_1} \left( \frac{k(y)y_j}{|y| - \ell y_1} \right) f(t + |y|, x + y) dy - \int \frac{k(y)y_j}{|y| - \ell y_1} A_\ell f(t + |y|, x + y) dy. \]

Note that \( |\partial_{y_j} k(y)| \lesssim |y|^{-4} \), \( |\partial_{y_1} \left( \frac{k(y)y_j}{|y| - \ell y_1} \right) | \lesssim |y|^{-4} \) and \( \left| \frac{k(y)y_j}{|y| - \ell y_1} \right| \lesssim |y|^{-3} \). Thus, using (2.3),

\[ \int k(y) \partial_{x_j} f(t + |y|, x + y) dy \lesssim \int |y|^{-4} |f(t + |y|, x + y)| dy + \int |y|^{-3} \langle A_\ell f(t + |y|, x + y) \rangle^{-p} dy \]

\[ \lesssim \int \left[ |y|^{-4} (t + |y|)^{-q} \right. \int |y|^{-3} \langle A_\ell f(t + |y|, x + y) \rangle^{-p} dy \]

\[ \lesssim K(x - \ell t). \]
Proceeding similarly, we have
\[
\int h(y)(\partial_x A_f)(t + |y|, x + y)dy = - \int \partial_y h(y)A_f(t + |y|, x + y)dy \\
- \ell \int \partial_y h(y)A_f(t + |y|, x + y)dy - \int \frac{h(y)j}{|y| - y_1} A_f^2 f(t + |y|, x + y)dy.
\]
Note that $|\partial_x h(y)| \lesssim |y|^{-3}$, $|\partial_y h(y)| \lesssim |y|^{-3}$ and $|\frac{h(y)j}{|y| - y_1}| \lesssim |y|^{-2}$. Thus, using (2.3),
\[
\left| \int h(y)(\partial_x A_f)(t + |y|, x + y)dy \right| \\
\lesssim \int |y|^{-3}|A_f(t + |y|, x + y)|dy + \int |y|^{-2}|A_f^2 f(t + |y|, x + y)|dy \lesssim K(x - \ell t).
\]
Now, we estimate $J(t, a)$. We split $J$ as follows
\[
J = \int_{|y| < \frac{1}{4} \langle a \rangle} + \int_{\frac{1}{4} \langle a \rangle < |y| < \frac{1}{2} \langle a \rangle} + \int_{|y| > \frac{1}{2} \langle a \rangle} = J_1 + J_2 + J_3.
\]
First, we observe that if $|y| < \frac{1}{4} \langle a \rangle$ then $|y - \ell||y| < 2|y| < \frac{1}{2} \langle a \rangle$ and thus $\langle a + y - \ell|y| \rangle \gtrsim \langle a \rangle$.
It follows that
\[
J_1 \lesssim \langle a \rangle^{-p} \int_{|y| < \frac{1}{4} \langle a \rangle} |y|^{-3}(t + |y|)^{-q}dy \lesssim t^{-q-2} \langle a \rangle^{-p} \int_{|z| < \frac{1}{4} \langle a \rangle} |z|^{-3}(1 + |z|)^{-q}dz.
\]
For all $b > 0$, we have
\[
\int_{|z| < b} |z|^{-3}(1 + |z|)^{-q}dz \lesssim \int_0^b r(1 + r)^{-q}dr \lesssim \begin{cases} b^2(1 + b^2)^{-1} \log(2 + b) & \text{if } q = 2, \\
2(1 + b^2)^{-1} & \text{if } q > 2. \end{cases}
\]
Thus, we have, for any $p > 2$,
\[
J_1 \lesssim \begin{cases} (t + \langle a \rangle)^{-2} \langle a \rangle^{-q} \log \left( \frac{2}{1 - \ell} \right) & \text{if } q = 2, \\
(t + \langle a \rangle)^{-2} t^{-q-2} \langle a \rangle^{-q} & \text{if } q > 2. \end{cases}
\]
Second, we observe that
\[
J_2 \lesssim \langle t + \langle a \rangle \rangle^{-q} \langle a \rangle^{-3} \int_{\frac{1}{4} \langle a \rangle < |y| < \frac{1}{2} \langle a \rangle} \langle a + y - \ell|y| \rangle^{-p}dy.
\]
We change variable $z = \varphi(y) = a + y - \ell|y|$. Since $|D\varphi(y)| = 1 - \frac{\ell |y|}{|y|} \geq 1 - \ell$ and
\[
|y| < \frac{2}{1 - \ell} \langle a \rangle \quad \text{implies} \quad |z| \leq \langle a \rangle + \frac{2(1 + \ell)}{1 - \ell} \langle a \rangle \leq \frac{4}{1 - \ell} \langle a \rangle,
\]
we obtain, for any $q \geq 2$,
\[
J_2 \lesssim \langle t + \langle a \rangle \rangle^{-q} \langle a \rangle^{-3} \int_{|z| \leq \frac{4}{1 - \ell} \langle a \rangle} (z)^{-p}dz \lesssim \begin{cases} (t + \langle a \rangle)^{-q} \langle a \rangle^{-q} & \text{if } p < 5, \\
(t + \langle a \rangle)^{-q} \langle a \rangle^{-3} \log(1 + \langle a \rangle) & \text{if } p = 5, \\
(t + \langle a \rangle)^{-q} \langle a \rangle^{-3} & \text{if } p > 5. \end{cases}
\]
Third, for \( |y| > \frac{2}{1-\ell} \langle a \rangle \), we have \( |a + y - \ell| y| | \geq |y| - (\ell|y| + |a|) \geq (1 - \ell)|y| - |a| \geq \frac{1}{2} (1 - \ell)|y| \), and so, for any \( q \geq 2, p > 2 \),
\[
J_3 \lesssim \int_{|y| > \frac{2}{1-\ell} \langle a \rangle} |y|^{-3-p} (t + |y|)^{-q} dy \lesssim (t + \langle a \rangle)^{-q} \langle a \rangle^{-(p-2)}.
\]

The estimates on \( v(t, x) \) follow from gathering the above estimates on \( J_1, J_2 \) and \( J_3 \).

Finally, we estimate \( K(t, a) \). We split \( K \) as follows
\[
K = \int_{|y| < \frac{1}{4} \langle a \rangle} + \int_{\frac{1}{4} \langle a \rangle < |y| < \frac{2}{1-\ell} \langle a \rangle} + \int_{|y| > \frac{2}{1-\ell} \langle a \rangle} = K_1 + K_2 + K_3.
\]

First, as before,
\[
K_1 \lesssim \langle a \rangle^{-p} \int_{|y| < \frac{1}{4} \langle a \rangle} |y|^{-4} (t + |y|)^{-q} dy \lesssim t^{-q-1} \langle a \rangle^{-p} \int_{|z| < \frac{1}{4} \langle a \rangle} |z|^{-4} (1 + |z|)^{-q} dz.
\]

For all \( b > 0, q \geq 2 \), we have \( \int_{|z| < b} |z|^{-4} (1 + |z|)^{-q} dz \lesssim \int_0^b (1 + r)^{-q} dr \lesssim b (1 + b)^{-1} \). Thus, we have, for any \( q \geq 2, p > 2 \),
\[
K_1 \lesssim (t + \langle a \rangle)^{-q} \langle a \rangle^{-4} \int_{\frac{1}{4} \langle a \rangle < |y| < \frac{2}{1-\ell} \langle a \rangle} (a + y - \ell|y|)^{-p} dy,
\]
and thus, proceeding as before for \( J_2 \), we obtain, for any \( q \geq 2 \),
\[
K_2 \lesssim (t + \langle a \rangle)^{-q} \langle a \rangle^{-4} \int_{|z| \leq \frac{1}{4} \langle a \rangle} |z|^{-p} dz \lesssim \begin{cases} (t + \langle a \rangle)^{-q} \langle a \rangle^{-(p-1)} & \text{if } p < 5, \\ (t + \langle a \rangle)^{-q} \langle a \rangle^{-4} \log(1 + \langle a \rangle) & \text{if } p = 5, \\ (t + \langle a \rangle)^{-q} \langle a \rangle^{-4} & \text{if } p > 5. \end{cases}
\]

Third, for any \( q \geq 2, p > 2 \),
\[
K_3 \lesssim \int_{|y| > \frac{2}{1-\ell} \langle a \rangle} |y|^{-4-p} (t + |y|)^{-q} dy \lesssim (t + \langle a \rangle)^{-q} \langle a \rangle^{-(p-1)}.
\]

The estimates on \( \nabla v(t, x) \) follow from gathering the above estimates on \( K_1, K_2 \) and \( K_3 \).

2.3. Spherical means and reduction to 1D. We recall a standard property of spherical means of general solutions of the linear wave equation (see e.g. [14], §2.2 and §2.4), both in the homogeneous and non-homogeneous cases.

**Lemma 2.5.** Let \( u_1(t, x) \) be solution of the 5D linear wave equation. Then, the radial function
\[
U_1(t, x) = \int_{|y| = |x|} u_1(t, y) d\omega(y)
\]
also satisfies the 5D linear wave equation. For \( f \in S(\mathbb{R} \times \mathbb{R}^5) \), let \( v(t, x) \) be given by (2.2) and
\[
V(t, x) = \int_{|y| = |x|} v(t, x) d\omega(x), \quad F(t, x) = \int_{|y| = |x|} f(t, x) d\omega(x).
\]
Then,
\[
V(t) = \int_0^\infty \frac{\sin(s\sqrt{-\Delta})}{\sqrt{-\Delta}} F(s + t) ds. \tag{2.6}
\]
Remark 2.6. Note that for \( U(r) = \int_{|y|=r} u(t, x) d\omega(x) \), the following hold:

\[
\int u^2 = \frac{8\pi^2}{3} \int_0^{+\infty} U^2(r) r^4 dr, \quad \int |\nabla u|^2 \geq \frac{8\pi^2}{3} \int_0^{+\infty} (\partial_r U)^2 r^4 dr. \tag{2.7}
\]

Now, we recall a standard reduction of radial 5D to 1D in the non-homogeneous case.

Lemma 2.7 (Reduction to 1D). Let \( F \in \mathcal{S}(\mathbb{R} \times \mathbb{R}^5) \) be a radial function and \( V \) be the radial function defined by (2.6). Let

\[
h(t, r) = r^2 \partial_r F(t, r) + 3r F(t, r) \quad \text{and} \quad \phi(t, r) = r^2 \partial_r V(t, r) + 3r V(t, r).
\]

Then,

\[
\phi(t, r) = \frac{1}{2} \int_0^{+\infty} \int_{|r-\sigma|}^{r+\sigma} h(t, \sigma, a) d\sigma d\sigma.
\]

Remark 2.8. Note that \( \phi \) satisfies a non-homogeneous wave equation with zero Dirichlet conditions at \( r = 0 \). We refer to computations in §2.4 of [13].

2.4. Channels of energy. We recall a result on channels of energy for the linear radial wave equation in 5D from [23] (see also [9] and [22] for any odd space dimension).

Proposition 2.9 ([23], Proposition 4.1). There exists a constant \( C > 0 \) such that any radial energy solution \( U_1 \) of the 5D linear wave equation

\[
\begin{aligned}
&\partial_{tt}^2 U_1 - \Delta U_1 = 0, \quad (t, x) \in [0, \infty) \times \mathbb{R}^5, \\
&U_{1t=0} = U_0 \in \dot{H}^1, \quad \partial_t U_{1t=0} = U_1 \in L^2,
\end{aligned}
\]

satisfies, for any \( R > 0 \), either

\[
\liminf_{t \to +\infty} \int_{|x|>|t|+R} |\partial_t U_1(t, x)|^2 + |\nabla U_1(t, x)|^2 dx \geq C \| \Pi_R(U_0, U_1) \|_{(H^1 \times L^2)(|x|>R)}^2
\]

or

\[
\liminf_{t \to +\infty} \int_{|x|>|t|+R} |\partial_t U_1(t, x)|^2 + |\nabla U_1(t, x)|^2 dx \geq C \| \Pi_R(U_0, U_1) \|_{(H^1 \times L^2)(|x|>R)}^2
\]

where \( \Pi_R(U_0, U_1) \) denotes the orthogonal projection of \((U_0, U_1)^T\) onto the complement of the plane

\[
\text{span} \left\{ (|x|^{-3}, 0)^T, (0, |x|^{-3})^T \right\}
\]

in \((\dot{H}^1 \times L^2)(|x| > R)\).

Remark 2.10. Part of the proof of Proposition 4.1 in [23] relies on reduction to 1D and on the fact that for a radial function \( f \) on \( \mathbb{R}^5 \), and \( R > 0 \), the function \( \tilde{f}(x) = f(x) - \frac{R^3}{|x|^3} f(R) \) is the orthogonal projection perpendicular to \(|x|^{-3}\) in \( \dot{H}^1(|x| > R) \) and so

\[
\| \Pi_R(f, 0) \|_{(H^1 \times L^2)(|x|>R)}^2 = \| \tilde{f} \|_{(H^1 \times L^2)(|x|>R)}^2 = \int_{R}^{+\infty} (\tilde{f}'(r))^2 r^4 dr
\]

\[
= \int_{R}^{+\infty} (f'(r))^2 r^4 dr - 3R^2 f^2(R) = \int_{R}^{+\infty} \left( r^2 f'(r) + 3r f(r) \right)^2 dr.
\]
Remark 2.11. It follows from the proof of Proposition 4.1 in [23] that there exists $c > 0$ such that if
\[
\limsup_{t \to +\infty} \int_{|x| > |t| + R} |\partial_t U_\ell(t, x) + \nabla U_\ell(t, x)|^2 dx \leq c \|\pi_R(U_0, U_1)\|_{(H^1 \times L^2)(|x| > R)}^2,
\]
then, for some $C > 0$,
\[
\liminf_{t \to -\infty} \int_{|x| > |t| + R} |\partial_t U_\ell(t, x)|^2 dx \geq C \|\pi_R(U_0, U_1)\|_{(H^1 \times L^2)(|x| > R)}^2
\]
and
\[
\liminf_{t \to -\infty} \int_{|x| > |t| + R} |\nabla U_\ell(t, x)|^2 dx \geq C \|\pi_R(U_0, U_1)\|_{(H^1 \times L^2)(|x| > R)}^2.
\]

2.5. Lorentz transform. For $\beta \in \mathbb{R}^5$ with $|\beta| < 1$, the Lorentz transform of parameter $\beta$ of a function $u(t, x)$ is defined by
\[
u_\beta(t, x) = u \left( \frac{t - \beta x}{\sqrt{1 - |\beta|^2}}, \frac{x - \beta t}{\sqrt{1 - |\beta|^2}} \right), \quad x_\beta = \frac{\beta}{|\beta|} \left( x - \frac{\beta}{|\beta|} t \right).
\]
In particular, if $\beta = \beta e_1$, then the Lorentz transform of $u(t, x)$ is given simply by
\[
u_\beta(t, x) = u \left( \frac{t - \beta x}{\sqrt{1 - \beta^2}}, \frac{x - \beta t}{\sqrt{1 - \beta^2}}, x \right).
\]
Let $-1 < \ell < 1$ and $-1 < \beta < 1$ and set $\tilde{\ell} = \frac{\ell + \beta}{1 + \ell \beta}$. Then the soliton $w_\ell(t, x) = W_\ell(x - \ell e_1 t)$ is transformed into the soliton $w_{\tilde{\ell}}(t, x) = W_{\tilde{\ell}}(x - \tilde{\ell} e_1 t)$ by the Lorentz transform of parameter $\beta e_1$. Moreover, if $-1 < \ell_1 < \ell_2 < 1$, then $-1 < \tilde{\ell}_1 < \tilde{\ell}_2 < 1$. Indeed, the Lorentz transform of $w_\ell(t, x) = W_\ell(x - \ell e_1 t)$ of parameter $\beta e_1$ writes
\[
\tilde{w}(t, x) = w_{\tilde{\ell}} \left( \frac{t - \beta x}{\sqrt{1 - \beta^2}}, \frac{x - \beta t}{\sqrt{1 - \beta^2}}, x \right) = W \left( \frac{1}{\sqrt{(1 - \beta^2)(1 - \ell^2)}} \left( x_1 - \frac{\beta + \ell}{1 + \ell \beta} t \right), x \right) = w_{\tilde{\ell}}(t, x).
\]
For the second statement, we note that for fixed $\beta \in (-1, 1)$, $\frac{d}{dt} = \frac{1 - \beta^2}{1 + \ell \beta} > 0$, and $\tilde{\ell} \to 1$ as $\ell \to -1$.

2.6. On the nonlinear wave equation. We recall from Lemma 2.1 and Theorem 2.7 of [24] (see also references therein) the following fact concerning small solutions of equation (1.1).

Proposition 2.12 (Cauchy problem for small data in $\mathbb{R} \times \mathbb{R}^5$). There exists $\delta_0 > 0$ such that for any $(u_0, u_1)^T \in H^1 \times L^2$ with $\|u_0, u_1\|_{H^1 \times L^2} \leq \delta_0$, the unique global solution $\bar{u} = (u, \partial_t u)^T \in C(\mathbb{R}, H^1 \times L^2)$ of (1.1) with initial data $(u_0, u_1)$ satisfies $\sup_{t \in \mathbb{R}} \|\bar{u}(t)\|_{H^1 \times L^2} \lesssim \delta_0$. Moreover, if $\bar{u}_L = (u_\ell, \partial_t u_\ell)^T$ is the global solution of the linear wave equation $\partial_t^2 u_- \Delta u_L = 0$ with initial data $(u_0, u_1) \in H^1 \times L^2$, then
\[
\sup_{t \in \mathbb{R}} \|\bar{u}(t) - \bar{u}_L(t)\|_{H^1 \times L^2} \lesssim \|(u_0, u_1)\|_{H^1 \times L^2}.$
3. Non-homogeneous linearized problem related to soliton interaction

Following the sketch of proof given in [34] we study a non-homogeneous linearized wave equation related to the interaction of two solitons. We prove the existence of an approximate solution to this problem and then prove sharp asymptotic properties.

3.1. Approximate solution to a non-homogeneous linearized equation. Let $-1 \leq \ell < 1$ and $F, G$ be defined by

$$F = W^\frac{4}{3} + \kappa_\ell AW, \quad G = \ell(1 - \ell^2)^{-\frac{1}{2}} \kappa_\ell \partial_{x_1} AW, \quad \kappa_\ell = -(1 - \ell^2) \frac{\langle W^\frac{4}{3}, AW \rangle}{\|AW\|_{L^2}^2} > 0. \quad (3.1)$$

Set

$$w_\ell(t, x) = W(x_\ell), \quad F_\ell(t, x) = F(x_\ell), \quad G_\ell(t, x) = G(x_\ell), \quad x_\ell = \left( \frac{x_1 - \ell t}{\sqrt{1 - \ell^2}} \right).$$

**Lemma 3.1.** There exists a smooth function $v_\ell$ such that, for all $0 < \delta < 1$ and all $t \geq 1$,

$$\|v_\ell, \partial_{x_\ell}v_\ell\|_{H^1 \times L^2} \lesssim t^{-2}, \quad \|v_\ell(t)\|_{L^2} \lesssim t^{-\frac{1}{2} + \delta}, \quad \|\mathcal{E}_\ell(t)\|_{L^2} \lesssim t^{-4 + \delta}, \quad (3.2)$$

where

$$\mathcal{E}_\ell = \partial_{x_\ell}^2 v_\ell - \Delta v_\ell - \frac{7}{3} w_\ell^3 v_\ell - f_\ell - g_\ell, \quad f_\ell = \ell t^{-3} F_\ell, \quad g_\ell = \ell^2 G_\ell.$$

Moreover, for all $m \geq 0$, $|\alpha| = 1$, $|\alpha'| \geq 2$, $t \geq 1$, $x \in \mathbb{R}^5$,

$$|A^m_\ell v_\ell(t, x)| \lesssim (t + \langle x_\ell \rangle)^{-1} t^{-(4m+\delta)} \langle x_\ell \rangle^{2+\delta}, \quad |A^m_\ell \partial_x v_\ell(t, x)| \lesssim (t + \langle x_\ell \rangle)^{-1} t^{-(4m+\delta)} \langle x_\ell \rangle^{2+\delta}, \quad (3.3)$$

and

$$|A^m_\ell \partial_{x_\ell} \mathcal{E}_\ell(t, x)| \lesssim t^{-(4m+\delta)} \langle x_\ell \rangle^{-3}, \quad |A^m_\ell \partial_x \mathcal{E}_\ell(t, x)| \lesssim t^{-(4m+\delta)} \langle x_\ell \rangle^{-4}, \quad (3.4)$$

**Remark 3.2.** In contrast with the strategy used in [34] for the gKdV equation, we do not construct an approximate solution $v_\ell$ of the equation $\mathcal{E}_\ell = 0$ simply by separation of variables. Indeed, the decay properties in space of such approximate solution would not be sufficient for our needs. Rather, we solve alternatively the linear wave equation $\partial_t^2 v - \Delta v = K_1$, and the elliptic equation $Lv = K_2$, for various functions $K_1$ and $K_2$. For the linear wave equation we use the estimates of Lemma [2.3] see also Remark [2.1]. For the elliptic equation, we use Lemma [2.1]. Because of the existence of a non-trivial kernel for the operator $L$, specific relations on $F$ and $G$ are needed. To state them precisely, we introduce

$$D_0 = x_1 AW, \quad D_j = x_1 \partial_{x_1} W, \quad LD_0 = -2 \partial_{x_1} AW, \quad LD_j = -2 \partial_{x_1} \partial_{x_1} W.$$

We note that the following relations hold, for $j = 1, \ldots, 5$,

$$(G, AW) = (G, \partial_{x_1} W) = (G, D_j) = 0, \quad (F, \partial_{x_1} W) = 0 \quad (F, \partial_{x_1} W) = 2\ell(1 - \ell^2)^{-\frac{1}{2}} (G, D_1). \quad (3.5)$$

Indeed, first, for $j = 1, \ldots, 5$,

$$(\partial_{x_1} AW, AW) = (\partial_{x_1} W, \partial_{x_1} AW) = (\partial_{x_1} AW, \partial_{x_1} W) = (\partial_{x_1} AW, D_j) = 0.$$
Moreover, for $j = 1, \ldots, 5$, $(F, \partial_{x_j} W) = 0$. Now, we compute $(F, \Delta W)$ and $(G, D_0)$

$$
(F, \Delta W) = (W_3^4, \Delta W) + \kappa_\ell \|\Delta W\|^2_{L^2},
$$

$$
2(G, D_0) = 2\kappa_\ell \ell(1 - \ell^2)^{-\frac{1}{2}} (\partial_{x_j} \Lambda W, x_1 \Lambda W) = -\kappa_\ell \ell (1 - \ell^2)^{-\frac{1}{2}} \|\Delta W\|^2_{L^2}.
$$

Thus, the condition $(F, \Delta W) = 2\ell(1 - \ell^2)^{-\frac{1}{2}} (G, D_0)$ is equivalent to

$$
(W_3^4, \Delta W) = -\kappa_\ell \|\Delta W\|^2_{L^2} (1 + \ell^2 (1 - \ell^2)^{-1}) = -\frac{\kappa_\ell}{1 - \ell^2} \|\Delta W\|^2_{L^2}
$$

which is indeed the definition of $\kappa_\ell$. We define the operator $\mathcal{L}_\ell$ by

$$
\mathcal{L}_\ell G_\ell = -(1 - \ell^2) \partial_{x}^2 G_\ell - \Delta G_\ell - \frac{7}{3} \ell_\ell^4 G_\ell \quad \text{so that} \quad \mathcal{L}_\ell \ell = (L\ell G)(x_\ell).
$$

Proof of Lemma 3.7

Approximate solution at order $t^{-2}$. First, following §2.2 we set

$$
v_1(t) = \int_0^\infty \frac{\sin(s\sqrt{-\Delta})}{\sqrt{-\Delta}} g_\ell(s + t)\, ds.
$$

Note that for $m \geq 0$, $A^m_\ell v_1$ satisfies

$$
(A^m_\ell v_1)(t) = \int_0^\infty \frac{\sin(s\sqrt{-\Delta})}{\sqrt{-\Delta}} (A^m_\ell g_\ell)(s + t)\, ds.
$$

Since $|A^m_\ell g_\ell| \lesssim t^{-(2m)} \langle x_\ell \rangle^{-\delta}$, it follows from Lemma 2.3 that, for $m \geq 1$,

$$
|v_1(t, x)| \lesssim (t + \langle x_\ell \rangle)^{-2} \langle x_\ell \rangle^{-2+\delta}, \quad |A^m_\ell v_1(t, x)| \lesssim (t + \langle x_\ell \rangle)^{-2} t^{-m} \langle x_\ell \rangle^{-2}. \quad (3.6)
$$

Here, and in the rest of the proof, $0 < \delta < 1$ is arbitrary. For $m \geq 0$, $|\alpha| \geq 1$, we have

$$
|\partial^\alpha A^m_\ell g_\ell| \lesssim t^{-(2m)} \langle x_\ell \rangle^{-5}, \quad \text{and thus by Lemma 2.3 for } m \geq 0, \quad |\alpha| = 1, \quad |\alpha'| \geq 2,
$$

$$
|\partial^\alpha A^m_\ell v_1| \lesssim (t + \langle x_\ell \rangle)^{-2} t^{-m} \langle x_\ell \rangle^{-3+\delta}, \quad |\partial^\alpha A^m_\ell v_1| \lesssim (t + \langle x_\ell \rangle)^{-1} t^{-(m+1)} \langle x_\ell \rangle^{-4+\delta}. \quad (3.7)
$$

Now, let

$$
R_1 = \frac{7}{3} w_\ell^4 - a_1 w_\ell^2 - b_1 \cdot \nabla(w_\ell^2),
$$

where, for $j = 1, \ldots, 5$,

$$
a_1 = \frac{(\frac{7}{3} v_1 w_\ell^4, \ell_\ell^4 w_\ell)}{(w_\ell^2, \ell_\ell^4 w_\ell)}, \quad b_{1,j} = \frac{(\frac{7}{3} v_1 w_\ell^4, \partial_{x_j} \ell_\ell^4 w_\ell)}{(\partial_{x_j}(w_\ell^2), \partial_{x_j} w_\ell)},
$$

so that $(R_1, \ell_\ell^4 w_\ell) = 0$, $(R_1, \nabla w_\ell) = 0$. Note that by (3.6), $|a_1(t)| + |b_1(t)| \lesssim t^{-2}$. Next, since

$$
\dot{a}_1 = \frac{(\frac{7}{3} A^m_\ell v_1, w_\ell^4 A^m_\ell w_\ell)}{(w_\ell^2, \ell_\ell^4 w_\ell)},
$$

by (3.6), we have $|\dot{a}_1| \lesssim t^{-3}$ and similarly, $|b_1| \lesssim t^{-3}$. More generally, $|\frac{d^m a_1}{dt^m}| + |\frac{d^m b_1}{dt^m}| \lesssim t^{-2-m}$. Thus, using (3.6) and (3.7), we obtain, for all $\alpha \in \mathbb{N}^5$, $|\partial^\alpha R_1| \lesssim t^{-2} \langle x_\ell \rangle^{-6+\delta}$. Moreover, by direct computations,

$$
A^m_\ell R_1 = \frac{7}{3} (A^m_\ell v_1) w_\ell^4 - \frac{d^m a_1}{dt^m} w_\ell^2 - \frac{d^m b_1}{dt^m} \cdot \nabla(w_\ell^2).
$$

Using (3.6) and (3.7) again, we obtain, for all $m \geq 0$, $\alpha \in \mathbb{N}^5$, $|\partial^\alpha A^m_\ell R_1| \lesssim t^{-2-m} \langle x_\ell \rangle^{-6+\delta}$. From (iii) of Lemma 2.1 there exists $v_2$ solution of $\mathcal{L}_\ell v_2 = R_1$, satisfying, for $|\alpha| = 1, |\alpha'| \geq 2$,

$$
(v_2, \ell_\ell^4 w_\ell) = 0, \quad (v_2, \nabla(w_\ell) = 0,$$
and
\[ |v_2| \lesssim t^{-2}(x_t)^{-3}, \quad |\partial^\alpha v_2| \lesssim t^{-2}(x_t)^{-4}, \quad |\partial^\alpha' v_2| \lesssim t^{-2}(x_t)^{-5}. \]

Next, we see that, for all \(m \geq 1\),
\[ 0 = \frac{d^m}{dt^m}(v_2, \Lambda w_\ell) = (A_\ell^m v_2, \Lambda w_\ell), \]
and similarly, \((A_\ell^m v_2, \nabla W(x_\ell)) = 0\). We also observe that for all \(m \geq 0\),
\[ \mathcal{L}_\ell(A_\ell^m v_2) = A_\ell^m (\mathcal{L}_\ell v_2) = A_\ell^m R_1. \]

Thus, by (iii) of Lemma 2.1 for all \(m \geq 0\), \(|\alpha| = 1\), \(|\alpha'| \geq 2\),
\[ |A_\ell^m v_2| \lesssim t^{-2-m}(x_\ell)^{-3}, \quad |\partial^\alpha A_\ell^m v_2| \lesssim t^{-2-m}(x_\ell)^{-4}, \quad |\partial^\alpha' A_\ell^m v_2| \lesssim t^{-2-m}(x_\ell)^{-5}. \tag{3.8} \]

We see that \(v_1 + v_2\) satisfies
\[ \left( \partial^2_t - \Delta - \frac{7}{3} w_\ell^2 \right) (v_1 + v_2) = g_\ell + B_\ell v_2 - a_1 w_\ell^2 - b_1 \cdot \nabla (w_\ell^2), \tag{3.9} \]
which also rewrites (since \(\partial^2_t - \Delta = B_\ell - \Delta_\ell\)),
\[ \mathcal{L}_\ell(v_1 + v_2) = g_\ell - B_\ell v_1 - a_1 w_\ell^2 - b_1 \cdot \nabla w_\ell^2. \tag{3.10} \]

**Estimates at order** \(t^{-3}\). First, from (3.10), using the orthogonality relations \((G, \Lambda W) = 0\) and \((G, \nabla W) = 0\), we claim that, for any \(m \geq 0\),
\[ \left| \frac{d^m a_1}{dt^m} \right| + \left| \frac{d^m b_1}{dt^m} \right| \lesssim t^{-3}, \tag{3.11} \]
improving by a factor \(t^{-1}\) the previous estimates on \(a_1\) and \(b_1\).

Proof of (3.11). For \(m \geq 0\), by direct computations from (3.10), we have
\[ \mathcal{L}_\ell (A_\ell^m (v_1 + v_2)) = A_\ell^m g_\ell - B_\ell (A_\ell^m v_1) - \frac{d^m a_1}{dt^m} w_\ell^2 - \frac{d^m b_1}{dt^m} \cdot \nabla (w_\ell^2). \tag{3.12} \]
We project this estimate on \(\Lambda w_\ell\) and \(\partial_\xi w_\ell\). First, note that \(A_\ell^m g_\ell = \frac{d^m}{dt^m}(t^{-2})G_\ell\), and since \((G, \Lambda W) = 0\) and \((G, \partial_\xi W) = 0\), one has
\[ (A_\ell^m g_\ell, \Lambda w_\ell) = 0, \quad (A_\ell^m g_\ell, \nabla w_\ell) = 0. \]
Next, since \(LAW = 0\) and \(L \nabla W = 0\), one has \(\mathcal{L}_\ell(\Lambda w_\ell) = 0\) and \(\mathcal{L}_\ell(\nabla w_\ell) = 0\). Thus
\[ (\mathcal{L}_\ell (A_\ell^m (v_1 + v_2)), \Lambda w_\ell) = 0, \quad (\mathcal{L}_\ell (A_\ell^m (v_1 + v_2)), \nabla w_\ell) = 0. \]
Now, we estimate \((B_\ell A_\ell^m v_1, \Lambda w_\ell)\). Since \(B_\ell = A_\ell^2 - 2\ell \partial_\xi A_\ell\),
\[ (B_\ell A_\ell^m v_1, \Lambda w_\ell) = (A_\ell^{m+2} v_1, \Lambda w_\ell) + 2\ell (A_\ell^{m+1} v_1, \partial_\xi \Lambda w_\ell). \]
By (3.6),
\[ |(A_\ell^{m+2} v_1, \Lambda w_\ell)| \lesssim t^{-m-4+\delta}, \quad |(A_\ell^{m+1} v_1, \partial_\xi \Lambda w_\ell)| \lesssim t^{-m-3}. \]
Thus, \(|(B_\ell A_\ell^m v_1, \Lambda w_\ell)| \lesssim t^{-m-3}\), and similarly, \(|(B_\ell A_\ell^m v_1, \nabla w_\ell)| \lesssim t^{-m-3}\). Projecting (3.12) on \(\Lambda w_\ell\) and \(\nabla w_\ell\) and gathering the above estimates, we find (3.11).
Again by $\partial_t^2 - \Delta = B_t - \Delta_t$ and $B_t = A_t^2 - 2\ell \partial_{x_1} A_t$, we rewrite (3.19) as follows
\begin{equation}
B_t(v_1 + v_2) + \mathcal{L}_t(v_1 + v_2) = g_t + \mathcal{E}_1 + R_2
\end{equation}
(3.13)
where
\begin{equation*}
\mathcal{E}_1 = A_t^2 v_2, \quad R_2 = -2\ell \partial_{x_1} A_t v_2 - a_1 w_t^2 - b_1 \cdot \nabla(w_t^2).
\end{equation*}
From (3.8), we have, for $m \geq 0, |\alpha| = 1, |\alpha'| \geq 2,$
\begin{equation}
|A_t^m \mathcal{E}_1| \lesssim t^{-4-m} \langle x_t \rangle^{-3}, \quad |\partial^\alpha A_t^m \mathcal{E}_1| \lesssim t^{-4-m} \langle x_t \rangle^{-4} \quad |\partial^\alpha' A_t^m \mathcal{E}_1| \lesssim t^{-4-m} \langle x_t \rangle^{-5}.
\end{equation}
In particular, since $\partial_t = A_t - \ell \partial_{x_1},$
\begin{align*}
|A_t^m \partial_t \mathcal{E}_1| & \lesssim t^{-5-m} \langle x_t \rangle^{-3} + t^{-4-m} \langle x_t \rangle^{-4}, \\
|A_t^m \partial_t \partial_t^\alpha \mathcal{E}_1| & \lesssim t^{-5-m} \langle x_t \rangle^{-4} + t^{-4-m} \langle x_t \rangle^{-5}, \quad |A_t^m \partial_t \partial_t^\alpha' \mathcal{E}_1| \lesssim t^{-4-m} \langle x_t \rangle^{-5}.
\end{align*}
Similarly, from (3.8) and (3.11), for $m \geq 0, |\alpha| \geq 1,$ we have
\begin{equation}
|A_t^m R_2| \lesssim t^{-3-m} \langle x_t \rangle^{-4}, \quad |\partial^\alpha A_t^m R_2| \lesssim t^{-3-m} \langle x_t \rangle^{-5}.
\end{equation}
Now, we claim, for $j = 1, \ldots, 5, m \geq 0,$
\begin{equation}
\frac{d^m}{dt^m} (R_{2, \ell} \Lambda_{\ell} w_{\ell}) = - \frac{d^{m+1}}{dt^{m+1}} (t^{-2} \langle G, D_0 \rangle) + O(t^{-m-4+\delta}),
\end{equation}
(3.16)
Proof of (3.16). From (3.13), we have
\begin{align}
A_t^2(v_1 + v_2) - 2\ell \partial_{x_1}(A_t(v_1 + v_2)) + \mathcal{L}_t(v_1 + v_2) &= g_t + \mathcal{E}_1 + R_2, \\
A_t^2(v_1 + v_2) - 2\ell \partial_{x_1}(A_t^2(v_1 + v_2)) + \mathcal{L}_t(A_t(v_1 + v_2)) &= A_t g_t + A_t \mathcal{E}_1 + A_t R_2.
\end{align}
(3.17) (3.18)
First, we project (3.17) on $\Lambda_{\ell} w_{\ell}$. We have $(g_t, \Lambda_{\ell} w_{\ell}) = 0$, and by (3.14), $|\langle \mathcal{E}_1, \Lambda_{\ell} w_{\ell} \rangle| \lesssim t^{-4}$.
By (3.6) and (3.8) (using also that $|\Lambda W(x)| \lesssim \langle x \rangle^{-3}$),
\begin{equation}
|\langle A_t^2 (v_1 + v_2), \Lambda_{\ell} w_{\ell} \rangle| \lesssim t^{-4+\delta}.
\end{equation}
Therefore, we have obtained
\begin{equation}
|\langle R_{2, \ell} \Lambda_{\ell} w_{\ell} \rangle + 2\ell (\partial_{x_1}(A_t(v_1 + v_2)), \Lambda_{\ell} w_{\ell})| \lesssim t^{-4+\delta}.
\end{equation}
(3.19)
In a similar way (using $|\partial_{x_j} W(x)| \lesssim \langle x \rangle^{-4}$), we find for $j = 1, \ldots, 5$,
\begin{equation}
|\langle R_{2, \ell} \partial_{x_j} \Lambda_{\ell} w_{\ell} \rangle + 2\ell (\partial_{x_1}(A_t(v_1 + v_2)), \partial_{x_j} w_{\ell})| \lesssim t^{-4}.
\end{equation}
Now, we compute $\langle \partial_{x_1}(A_t(v_1 + v_2)), \Lambda_{\ell} w_{\ell} \rangle$ and $\langle \partial_{x_1}(A_t(v_1 + v_2)), \partial_{x_j} w_{\ell} \rangle$ from (3.18). Recall that we set $D_0 = x_1 \Lambda W$ and $LD_0 = -2 \partial_{x_1} \Lambda W$. We cannot project equation (3.18) directly on $D_0(x_\ell)$ because we only know $|A_t^2 v_2| \lesssim t^{-5} \langle x_\ell \rangle^{-3}, |D_0| \lesssim \langle x \rangle^{-2}$ and $\langle x \rangle^{-5} \not\in L^1(\mathbb{R}^5)$. Thus, we consider $\tilde{\chi}(x) = \chi(|x|)$ a smooth cut-off function such that
\begin{equation*}
\tilde{\chi} \equiv 1 \text{ on } [-1, 1], \quad \tilde{\chi} \equiv 0 \text{ on } [-2, 2]^c, \quad 0 \leq \tilde{\chi} \leq 1 \text{ on } \mathbb{R},
\end{equation*}
and we set
\begin{equation*}
D_\ell(t, x) = D_0 (x_\ell), \quad \tilde{D}_\ell(t, x) = D_\ell(t, x) \tilde{\chi}\left(\frac{x_\ell}{\ell^{10}}\right).
\end{equation*}
We project (3.18) on $\tilde{D}_\ell$. By (3.6), $|\langle A_{\ell}^2 v_2, \tilde{D}_\ell \rangle| \lesssim t^{-4+\delta}$, and by (3.8), $|\langle A_{\ell}^2 v_2, \tilde{D}_\ell \rangle| \lesssim t^{-5+\delta}$. Also, by (3.6)-(3.8), $|\langle \partial_{x_1} A_{\ell}^2 (v_1 + v_2), \tilde{D}_\ell \rangle| \lesssim t^{-4+\delta}$. Next,
\begin{align*}
\langle \mathcal{L}_\ell (A_{\ell} (v_1 + v_2)), \tilde{D}_\ell \rangle &= \langle A_{\ell} (v_1 + v_2), \mathcal{L}_\ell \tilde{D}_\ell \rangle - \langle A_{\ell} (v_1 + v_2), \mathcal{L}_\ell \left[1 - \tilde{\chi}\left(\frac{x_\ell}{\ell^{10}}\right)\right] \tilde{D}_\ell \rangle.
\end{align*}
Note that \((A\ell(v_1 + v_2), \mathcal{L}_\ell \delta) = (A\ell(v_1 + v_2), (LD_0)(x_\ell)) = -2(1 - \ell^2) \frac{1}{t}(A\ell(v_1 + v_2), \partial_{x_1} \Lambda_\ell w_\ell)\).

Next,
\[
\mathcal{L}_\ell \left[ \left( 1 - \bar{\chi} \left( \frac{x_\ell}{t^{10}} \right) \right) D_\ell \right] = \left( 1 - \bar{\chi} \left( \frac{x_\ell}{t^{10}} \right) \right) \mathcal{L}_\ell D_\ell - t^{-20} \Delta_\ell \bar{\chi} \left( \frac{x_\ell}{t^{10}} \right) D_\ell
- 2t^{-10}(1 - \ell^2)\partial_{x_1} \bar{\chi} \left( \frac{x_\ell}{t^{10}} \right) (\partial_{x_1} D_\ell) - 2t^{-10} \nabla \bar{\chi} \left( \frac{x_\ell}{t^{10}} \right) \cdot \nabla D_\ell
\]
so that
\[
\left| \mathcal{L}_\ell \left[ \left( 1 - \bar{\chi} \left( \frac{x_\ell}{t^{10}} \right) \right) D_\ell \right] \right| \lesssim |x_\ell|^{-4} 1_{|x_\ell| > t^{10}} + t^{-10} |x_\ell|^{-2} 1_{t^{10} < |x_\ell| < 2t^{10}}.
\]
Thus,
\[
\left| (A\ell(v_1 + v_2), \mathcal{L}_\ell \left[ \left( 1 - \bar{\chi} \left( \frac{x_\ell}{t^{10}} \right) \right) D_\ell \right] \right| \lesssim t^{-10}.
\]

Next, since \((A\ell g_\ell)(t, x) = -2t^{-3} G_\ell\), we have
\[
(A\ell g_\ell, D_\ell) = -2t^{-3}(G_\ell, D_\ell) + O(t^{-4}) = -2(1 - \ell^2) \frac{1}{t}(G_\ell, D_\ell) + O(t^{-4}).
\]
Moreover, by (3.14)-(3.15),
\[
|\langle A\ell \mathcal{E}_1, \tilde{D}_\ell \rangle| \lesssim t^{-5} \int_{|x_\ell| < 2t^{10}} |\langle x_\ell \rangle|^{-2} d\ell \lesssim t^{-5+\delta},
\]
and
\[
|\langle A\ell R_2, \tilde{D}_\ell \rangle| \lesssim t^{-4} \int_{|x_\ell| < 2t^{10}} |\langle x_\ell \rangle|^{-4} d\ell \lesssim t^{-4}.
\]
Thus, the projection of (3.18) on \(\tilde{D}_\ell\) gives
\[
-2(\partial_{x_1}(A\ell(v_1 + v_2)), \Lambda_\ell w_\ell) = 2(A\ell(v_1 + v_2), \partial_{x_1}(\Lambda_\ell w_\ell)) = 2t^{-3}(G, D_0) + O(t^{-4+\delta}).
\]
Inserted in (3.19), it gives
\[
(R_2, \Lambda_\ell w_\ell) = 2t^{-3}(G, D_0) + O(t^{-4+\delta}).
\]

To obtain the estimate on \((R_2, \partial_{x_j} w_\ell)\), for \(j = 1, \ldots, 5\), we compute \((A\ell(v_1 + v_2), \partial_{x_j} w_\ell)\). We use \(D_j(x) = x_1 \partial_{x_j} W(x)\), so that \(LD_j = -2\partial_{x_j} \partial_{x_j} W\). We proceed as before, projecting (3.18) on \(D_j(x_\ell)\). The computations are similar and easier because of the better decay properties of \(D_j\) (the cut-off function \(\bar{\chi}\) is no longer needed). The proof of (3.16) for \(m \geq 1\) is similar and it is omitted.

**Approximate solution at order** \(t^{-3}\). Set
\[
v_3(t) = \int_0^\infty \frac{\sin(s\sqrt{-\Delta})}{\sqrt{-\Delta}} (f_\ell - R_2)(s + t) ds.
\]
We see that \(f_\ell\) satisfies (2.3) with \(q = 3\) and \(p = 3\). From (3.15), the function \(R_2\) satisfies (2.3) with \(q = 3\) and \(p = 4\). Thus, from Lemma 2.3 for \(m \geq 0\),
\[
|A\ell_m v_3| \lesssim (t + \langle x_\ell \rangle)^{-2} t^{-1-m} \langle x_\ell \rangle^{-1}. \quad (3.20)
\]
Moreover, from (3.15) and Lemma 2.3 for \(|\alpha| = 1, |\alpha'| \geq 2, m \geq 0\),
\[
|A\ell_m \partial^\alpha v_3| \lesssim (t + \langle x_\ell \rangle)^{-2} t^{-1-m} \langle x_\ell \rangle^{-2}, \quad |A\ell_m \partial^\alpha v_3| \lesssim (t + \langle x_\ell \rangle)^{-2} t^{-1-m} \langle x_\ell \rangle^{-3+\delta}. \quad (3.21)
\]
By construction, \(v_3\) verifies
\[
B_\ell v_3 + \mathcal{L} v_3 = f_\ell - R_2 - \frac{7}{3} w_\ell^3 v_3 \quad (3.22)
\]
and $v_1 + v_2 + v_3$ satisfies
\[
(\partial_t^2 - \Delta - \frac{7}{3}w_t^\frac{4}{3})(v_1 + v_2 + v_3) = g_t + f_t + E_1 - \frac{7}{3}w_t^\frac{4}{3}v_3.
\]

As before, set, for $j = 1, \ldots, 5$,
\[
a_3 = \frac{(\frac{7}{3}v_3w_t^\frac{4}{3}, \Lambda_\ell w_t)}{(w_t^2, \Lambda_\ell w_t)}, \quad b_{3,j} = \frac{(\frac{7}{3}v_3w_t^\frac{4}{3}, \partial_x \Lambda_\ell w_t)}{\partial_x (w_t^2), \partial_x \Lambda_\ell w_t}
\]

Let
\[
R_3 = \frac{7}{3}v_3w_t^\frac{4}{3} - a_3w_t^2 - b_3 \cdot \nabla (w_t^2),
\]
so that $R_3$ satisfies $(R_3, \Lambda_\ell w_t) = 0$ and $(R_3, \nabla w_t) = 0$. By the decay properties of $v_3$, $|A_\ell^m R_3| \lesssim t^{-3-m+\delta}(x_t)^{-3-\delta}$ and $|A_\ell^m \partial^\alpha R_3| \lesssim t^{-3-m}(x_t)^{-6}$ for all $\alpha \in \mathbb{N}^5$ with $|\alpha| \geq 1$. Thus, from (iii) of Lemma 2.1 there exists $v_4$ solution of $\mathcal{L}_\ell v_4 = R_3$, satisfying, for $|\alpha| = 1$, $|\alpha'| \geq 2$ and $m \geq 0$,
\[
(v_4, \Lambda_\ell w_t) = 0, \quad (v_4, \nabla w_t) = 0,
\]
\[
|A_\ell^m v_4| \lesssim t^{-3-m+\delta}(x_t)^{-3}, \quad |\partial^\alpha A_\ell^m v_4| \lesssim t^{-3-m+\delta}(x_t)^{-4}, \quad |\partial^\alpha' A_\ell^m v_4| \lesssim t^{-3-m+\delta}(x_t)^{-5}.
\]

In particular, $E_2 = B_t v_4 = A_\ell^2 v_4 - 2\ell \partial_x A_\ell v_3$ satisfies, for all $m \geq 0$,
\[
|A_\ell^m E_2| \lesssim t^{-4-m+\delta}(x_t)^{-3}, \quad |A_\ell^m \partial^\alpha E_2| \lesssim t^{-4-m+\delta}(x_t)^{-4}, \quad |A_\ell^m \partial^\alpha' E_2| \lesssim t^{-4-m+\delta}(x_t)^{-5}.
\]

Let $v_5 = v_1 + v_2 + v_3 + v_4$ satisfy
\[
\left(\partial_t^2 - \Delta - \frac{7}{3}w_t^\frac{4}{3}\right) v_5 = g_t + f_t + E_\ell,
\]
where $E_\ell = E_1 + E_2 + E_3$ and $E_3 = -a_3w_t^2 - b_3 \cdot \nabla (w_t^2)$.

**Estimates at order $t^{-4}$.** We claim that, for $m \geq 0$,
\[
\left|\frac{d^m a_3}{dt^m}\right| + \left|\frac{d^m b_3}{dt^m}\right| \lesssim t^{-4-m+\delta}. \tag{3.23}
\]

Proof of (3.23). By the estimate on $v_3$, we only have $|a_3(t)| + |b_3(t)| \lesssim t^{-3+\delta}$. We project (3.22) on $\Lambda_\ell w_t$. As before, $(\mathcal{L}_\ell v_3, \Lambda_\ell w_t) = 0$. Next, recall that $B_t v_3 = A_\ell^2 v_3 - 2\ell \partial_x A_\ell v_3$ and, using (3.21), we have
\[
|(A_\ell^2 v_3, \Lambda_\ell w_t)| \lesssim t^{-4+\delta}, \quad |(\partial_x A_\ell v_3, \Lambda_\ell w_t)| = |(A_\ell v_3, \partial_x \Lambda_\ell w_t)| \lesssim t^{-4+\delta}.
\]

Now, using (3.16) and then (3.3), we compute
\[
(f_t - R_2, \Lambda_\ell w_t) = t^{-3}(1 - t^2)^\frac{1}{4}(F, \Lambda W) - 2\ell t^{-3}(G, D_0) + O(t^{-4+\delta}) = O(t^{-4+\delta}).
\]

This is enough to obtain the estimate (3.23) on $a_3$. The estimate on $b_3$ is proved similarly using (3.16). For $m \geq 1$, apply $A_\ell^m$ to (3.22), so that
\[
B_t A_\ell^m A_\ell v_3 + \mathcal{L}_\ell (A_\ell^m v_3) = A_\ell^m f_t - A_\ell^m R_2 - \frac{7}{3}A_\ell^m (w_t^\frac{4}{3}v_3),
\]
then project on $\Lambda_\ell w_t$ (or $\nabla w_t$) and use (3.16) to find (3.23).

As a consequence of (3.23), we obtain, for $|\alpha| = 1$, $|\alpha'| \geq 2$ and $m \geq 0$,
\[
|A_\ell^m E_3| \lesssim t^{-4-m+\delta}(x_t)^{-6}, \quad |A_\ell^m \partial^\alpha E_3| + |A_\ell^m \partial^\alpha' E_3| \lesssim t^{-4-m+\delta}(x_t)^{-7}.
\]
Conclusion. Gathering the estimates on \( v_1, v_2, v_3 \) and \( v_4 \), we obtain
\[
|A^m_\ell v_\ell| \lesssim (t + \langle x_\ell \rangle)^{-2} t^{-m} \langle x_\ell \rangle^{-2+\delta} + t^{-(2+m)} \langle x_\ell \rangle^{3} + (t + \langle x_\ell \rangle)^{-2} t^{-(1+m)} \langle x_\ell \rangle^{-1},
\]
so that
\[
|A^{m+1}_\ell v_\ell| \lesssim t^{-(2+m)} \langle x_\ell \rangle^{-2+\delta}, \quad |A^m_\ell v_\ell| \lesssim t^{-(1+m)} \langle x_\ell \rangle^{-3+\delta}.
\]
Moreover, for \(|\alpha| = 1\),
\[
|A^{m}_\ell \partial^\alpha v_\ell| \lesssim t^{-(2+m)} \langle x_\ell \rangle^{-3+\delta}, \quad |A^{m}_\ell \partial^\alpha v_\ell| \lesssim t^{-(1+m)} \langle x_\ell \rangle^{-4+\delta},
\]
and for \(|\alpha'| \geq 2\),
\[
|A^{m}_\ell \partial^\alpha' v_\ell| \lesssim t^{-(2+m)} \langle x_\ell \rangle^{-4+\delta}, \quad |A^{m}_\ell \partial^\alpha' v_\ell| \lesssim t^{-(1+m)} \langle x_\ell \rangle^{-5+\delta}.
\]
Note that time estimates on \( v_\ell \) are easily obtained from (3.3) using \( \partial_t v = A_\ell v_\ell - \ell \partial_{x_\ell} v_\ell \). For example, we have
\[
|A^{m}_\ell \partial_t v_\ell| \lesssim |A^{m+1}_\ell v_\ell| + |A^{m}_\ell \partial_{x_\ell} v_\ell| \lesssim t^{-(2+m)} \langle x_\ell \rangle^{-3+\delta}.
\]

Gathering the estimates on \( \mathcal{E}_1, \mathcal{E}_2 \) and \( \mathcal{E}_3 \), we find, for \(|\alpha| = 1\), \(|\alpha'| \geq 2\) and \( m \geq 0\),
\[
|A^{m}_\ell \mathcal{E}_1| \lesssim t^{-4-m+\delta} \langle x_\ell \rangle^{-3}, \quad |A^{m}_\ell \partial^\alpha \mathcal{E}_1| \lesssim t^{-4-m+\delta} \langle x_\ell \rangle^{-4}, \quad |A^{m}_\ell \partial^\alpha' \mathcal{E}_1| \lesssim t^{-4-m+\delta} \langle x_\ell \rangle^{-5}.
\]
In particular, this proves \( \|\mathcal{E}_\ell\|_{L^2} \lesssim t^{-4+\delta} \), which is the estimate of \( \mathcal{E}_\ell \) in (3.2). \( \square \)

3.2. Asymptotics of solutions of non-homogeneous problems. To obtain explicitly the main order of the asymptotics of the radial part of the approximate solution \( v_\ell \) constructed in Lemma 3.1, we consider a simplified problem as \(|x| \to +\infty\). For \(-1 < \ell < 1\) and \( t > 0\), let
\[
f^m_\ell(t, x) = t^{-3} \langle x_\ell \rangle^{-3}, \quad g^m_\ell = \ell t^{-2} \partial_{x_\ell} \langle x_\ell \rangle^{-3}
\]
and
\[
v^m_\ell(t) = \int_0^\infty \frac{\sin(s \sqrt{-\Delta})}{\sqrt{-\Delta}} \left( f^m_\ell(t) + g^m_\ell \right) (t + u)du.
\]

Lemma 3.3 (Asymptotics for a non-homogeneous wave problem). For any \( 0 < \delta < 1 \), for all \( m \geq 0 \), \(|\alpha| = 1\), \(|\alpha'| \geq 2\), \( t > 1\), \( x \in \mathbb{R}^5\),
\[
|A^m_\ell v^m_\ell(t, x)| \lesssim (t + \langle x_\ell \rangle)^{-1} t^{-(1+m)} \langle x_\ell \rangle^{-2+\delta},
\]
\[
|A^m_\ell \partial^\alpha v^m_\ell(t, x)| \lesssim (t + \langle x_\ell \rangle)^{-1} t^{-(1+m)} \langle x_\ell \rangle^{-3+\delta},
\]
\[
|A^m_\ell \partial^\alpha' v^m_\ell(t, x)| \lesssim (t + \langle x_\ell \rangle)^{-1} t^{-(1+m)} \langle x_\ell \rangle^{-4+\delta}.
\]
Moreover, for \( r > 0\), \( t > 1\), let
\[
\phi_\ell(t, r) = r^2 \partial_r V^\ell_\ell(t, r) + 3rV^\ell_\ell(t, r), \quad V^\ell_\ell(t, r) = \int_{|x|=r} v^\ell_\ell(t, x) d\omega(x).
\]
Then, for all \( 1 < t < r^{1/2}\),
\[
\phi_\ell(t, r) = (1 - \ell^2)^{1/2} r^{-3} + O(r^{-1} t^{-3/2}).
\]

The first part of Lemma 3.3 is a consequence of Lemma 2.3 and the decay properties of the functions \( f^m_\ell \) and \( g^m_\ell \). The second part is proved in Appendix A. Note that we do not determine the asymptotic behavior of \( v^m_\ell \), but only the one of its spherical means.

Now, we check that the asymptotics of \( v_\ell \) defined in Lemma 3.1 and of \( v^m_\ell \) defined in Lemma 3.3 coincide at the main order, up to a multiplicative constant.
**Lemma 3.4** (Comparison of asymptotics). Let $-1 < \ell < 1$. For all $0 < \delta < 1$, $t > 1$, $x \in \mathbb{R}^5$,
\begin{align}
  \left| v_\ell(t, x) + \frac{3}{2}(15)^\frac{3}{2} \kappa_\ell v_\ell^2(t, x) \right| &\lesssim t^{-2}(x_\ell)^{-3+\delta}, \\
  \left| \nabla v_\ell(t, x) + \frac{3}{2}(15)^\frac{3}{2} \kappa_\ell \nabla v_\ell^2(t, x) \right| &\lesssim t^{-2}(x_\ell)^{-4+\delta}.
\end{align}
(3.29)

*Proof.* By the properties of $v_\ell$ in Lemma 3.1, we have
\begin{align*}
  v_\ell(t) + \frac{3}{2}(15)^\frac{3}{2} \kappa_\ell v_\ell^2(t) = \int_0^\infty \frac{\sin(s\sqrt{-\Delta})}{\sqrt{-\Delta}} \left( \mathcal{E}_\ell + \mathcal{E}_\ell^I + \mathcal{E}_\ell^{II} \right) (s + t) ds
\end{align*}
where $\mathcal{E}_\ell$ is defined in Lemma 3.1 and
\begin{align*}
  \mathcal{E}_\ell^I = f_\ell + g_\ell + \frac{3}{2}(15)^\frac{3}{2} \kappa_\ell \left( f_\ell^2 + g_\ell^2 \right) \quad \text{and} \quad \mathcal{E}_\ell^{II} = \frac{7}{3} w_\ell^4 v_\ell.
\end{align*}

First, we see from (3.4) that $\mathcal{E}_\ell$ satisfies (2.3) with $q = 4 - \delta$ and $p = 3$. Therefore, applying Lemma 2.3,
\begin{align*}
  \left| \int_0^\infty \frac{\sin(s\sqrt{-\Delta})}{\sqrt{-\Delta}} \mathcal{E}_\ell(s + t) ds \right| \lesssim t^{-2+\delta}(x_\ell)^{-1}(t + (x_\ell))^{-2} \lesssim t^{-2}(x_\ell)^{-3+\delta}.
\end{align*}

Second, we observe from the explicit expression of $W$
\begin{align*}
  AW(x) = -\frac{3}{2}(15)^\frac{3}{2} \langle x \rangle^{-3} + O(\langle x \rangle^{-5}), \quad \partial_{x_1}(AW)(x) = -\frac{3}{2}(15)^\frac{3}{2} \partial_{x_1}(\langle x \rangle^{-3}) + O(\langle x \rangle^{-6}).
\end{align*}

In particular, by the definitions of $f_\ell$ and $g_\ell$, we have
\begin{align*}
  |A_\ell^m \mathcal{E}_\ell^I| \lesssim t^{-3-m}(x_\ell)^{-4} + t^{-2-m}(x_\ell)^{-6}.
\end{align*}

Applying Lemma 2.3 with $q = 3$, $p = 4$ and $q = 2$, $p = 6$, we obtain
\begin{align*}
  \left| \int_0^\infty \frac{\sin(s\sqrt{-\Delta})}{\sqrt{-\Delta}} \mathcal{E}_\ell^I(s + t) ds \right| \lesssim (t + (x_\ell))^{-2} (t^{-1}(x_\ell)^{-2} + (x_\ell)^{-3}) \lesssim t^{-2}(x_\ell)^{-3}.
\end{align*}

Third, from (3.3) and the properties of $w_\ell$, $\mathcal{E}_\ell^{II}$ satisfies (2.3) with $q = 2$, $p = 6 - \delta$. Thus, by Lemma 2.3,
\begin{align*}
  \left| \int_0^\infty \frac{\sin(s\sqrt{-\Delta})}{\sqrt{-\Delta}} \mathcal{E}_\ell^{II}(s + t) ds \right| \lesssim t^{-2}(x_\ell)^{-3}.
\end{align*}

This proves the estimate on $v_\ell + \frac{3}{2}(15)^\frac{3}{2} \kappa_\ell v_\ell^2$. The estimate for the gradient is similar. \qed

4. **Refined approximate solution for the two-soliton problem**

For $k = 1, 2$, let
\begin{align*}
  \lambda_k^\infty > 0, \quad y_k^\infty \in \mathbb{R}^5, \quad \epsilon_k = \pm 1, \quad \ell_k = \ell_k e_1, \quad \text{where} \ -1 \leq \ell_1 < \ell_2 < 1.
\end{align*}

Indeed, by rotation invariance and the Lorentz transformation, we restrict ourselves without loss of generality to the case where $\ell_k = \ell_k e_1$, with $\ell_k \in (-1, 1)$ for $k = 1, 2$. See more details in §5 of [35].

Let $C_0 \gg 1$ and $T_0 \gg 1$ to be fixed and $I \subset [T_0, +\infty)$ be an interval of $\mathbb{R}$. For $k = 1, 2$, we consider $C^1$ functions $\lambda_k > 0$, $y_k \in \mathbb{R}^5$ defined on $I$. We assume that these functions satisfy, for all $t \in I$,
\begin{align}
  |\lambda_k(t) - \lambda_k^\infty| + |y_k(t) - y_k^\infty| \leq C_0 t^{-1}, \quad |\dot{\lambda}_k| + |\dot{y}_k| \leq C_0 t^{-2}.
\end{align}
(4.1)
For \( \tilde{G} = (G, H) \), define

\[
(\theta_k G)(t, x) = \frac{\epsilon_k}{\lambda_k^2(t)} G \left( \frac{x - \ell_k t - y_k(t)}{\lambda_k(t)} \right), \quad \tilde{\theta}_k G = \left( \begin{array}{c} \theta_k G \\ \theta_k H \end{array} \right), \quad \tilde{\theta}_k G = \left( \begin{array}{c} \theta_k G \\ \theta_k H \end{array} \right).
\]

In particular, set

\[
W_k = \theta_k W_{\ell_k}, \quad X_k = -\ell_k \cdot \nabla W_k, \quad \bar{W}_k = \left( \begin{array}{c} W_k \\ X_k \end{array} \right).
\]

Define also

\[
(\theta_k^G)(t, x) = \frac{\epsilon_k}{(\lambda_k^G)^{\frac{3}{2}}} G \left( \frac{x - \ell_k t - y_k^G(t)}{\lambda_k^G(t)} \right), \quad \tilde{\theta}_k^G = \left( \begin{array}{c} \theta_k^G \\ \theta_k^H \end{array} \right),
\]

\[
W_k^G = \theta_k^G W_{\ell_k}, \quad X_k^G = -\ell_k \cdot \nabla W_k^G, \quad \bar{W}_k^G = \left( \begin{array}{c} W_k^G \\ X_k^G \end{array} \right).
\]

4.1. **Main interaction terms.** Expanding the nonlinearity \(|u|^4 u\) at \(u = W_1 + W_2\), we identify the two main order interaction terms of the form \(t^{-3} \sum_k c_k |W_k|^\frac{4}{3} \). The remaining error term is of size \(t^{-4}\).

**Lemma 4.1.** For \(k, k' = 1, 2, k \neq k'\), let

\[
\sigma_{k,k'} = \left( \frac{1}{\sqrt{1 - |\ell_{k'}|^2}} - 1 \right) \frac{\ell_{k'} \cdot (\ell_k - \ell_{k'})}{|\ell_{k'}|^2} + \ell_k - \ell_{k'},
\]

and \(c_k = \frac{7}{3}(15)^{\frac{3}{2}} \epsilon_{k'} (\lambda_{k'}^G)^{\frac{3}{2}} |\sigma_{k,k'}|^{-3} \). Then,

\[
\left| \sum_k W_k \right|^\frac{4}{3} - \sum_k |W_k|^\frac{4}{3} W_k = t^{-3} \sum_k c_k |W_k|^\frac{4}{3} + R_\Sigma,
\]

where, for all \(t \in I\),

\[
\|R_\Sigma\|_{H^1} \lesssim t^{-4}. \tag{4.3}
\]

**Remark 4.2.** Lemma 4.1 holds for general \(\ell_k, \ell_{k'}\). For the special case \(\ell_k = \ell_k e_1\) and \(\ell_{k'} = \ell_{k'} e_1\), we obtain

\[
\sigma_{1,2} = \frac{\ell_1 - \ell_2}{(1 - \ell_2^2)^{\frac{3}{2}}} e_1, \quad \sigma_{2,1} = \frac{\ell_2 - \ell_1}{(1 - \ell_1^2)^{\frac{3}{2}}} e_1, \tag{4.4}
\]

and

\[
c_1 = \frac{7}{3}(15)^{\frac{3}{2}} \epsilon_2 (\lambda_2^G)^{\frac{3}{2}} \frac{(1 - \ell_2^2)^{\frac{3}{2}}}{|\ell_1 - \ell_2|^3}, \quad c_2 = \frac{7}{3}(15)^{\frac{3}{2}} \epsilon_1 (\lambda_1^G)^{\frac{3}{2}} \frac{(1 - \ell_1^2)^{\frac{3}{2}}}{|\ell_1 - \ell_2|^3}. \tag{4.5}
\]

**Proof of Lemma 4.1.** Let \(\sigma = \frac{1}{|\ell_1 - \ell_2|} |x - \ell_k t| \leq \sigma t\), \(B_k(t) = \{x, |x - \ell_k t| \leq \sigma t\}\), \(B(t) = \bigcup_k B_k(t)\).

We prove the \(L^2\) estimate of \(R_\Sigma\), the proof of the \(H^1\) estimate is similar. First, we claim

\[
\|W_k^G\|_{L^2(B_k^G)} \lesssim t^{-4}, \quad \|W_k^G\|_{L^2(B_k^G)} \lesssim t^{-1}.
\]
Indeed, for all $x \not\in B_k$,

$$|W_k|^\frac{4}{3} \lesssim \langle x - \ell_k t \rangle^{-7} \lesssim t^{-4} \langle x - \ell_k t \rangle^{-3}, \quad |W_k|^\frac{4}{3} \lesssim \langle x - \ell_k t \rangle^{-4} \lesssim t^{-1} \langle x - \ell_k t \rangle^{-3},$$

and $x \mapsto \langle x \rangle^{-3} \in L^2(\mathbb{R}^5)$.

Second, we claim, for $k, k' = 1, 2, k' \neq k$,

$$\left\|W_1 + W_2|^{\frac{4}{3}} (W_1 + W_2) - \frac{7}{3} W_k|^{\frac{4}{3}} W_k'\right\|_{L^2(B_k)} \lesssim t^{-\frac{2}{3}}. \quad (4.6)$$

Indeed, for $x \in B_k$ and $k' \neq k$, $|W_k|^\frac{4}{3} |W_{k'}|^2 \lesssim t^{-6} \langle x - \ell_k t \rangle^{-1}$, $|W_{k'}|^\frac{4}{3} \lesssim t^{-7}$, and so

$$\left\|W_k|^\frac{4}{3} |W_{k'}|^2\right\|_{L^2(B_k)} + \left\|W_k|^\frac{4}{3}\right\|_{L^2(B_k)} \lesssim t^{-\frac{2}{3}}.$$

Thus, (4.6) follows from the Taylor expansion

$$|W_k + W_{k'}|^\frac{4}{3} (W_k + W_{k'}) = \left|W_k|^\frac{4}{3} W_k\right| 1 + \frac{W_{k'}}{W_k} \left(1 + \frac{W_{k'}}{W_k}\right)
\quad = \left|W_k|^\frac{4}{3} W_k + \frac{7}{3} |W_k|^\frac{4}{3} W_{k'} + O\left(|W_k|^\frac{4}{3} |W_{k'}|^2\right) + O\left(|W_{k'}|^\frac{4}{3}\right).$$

Third, we claim, $k' \neq k$,

$$\left\|W_k|^\frac{4}{3} (W_{k'}(t, x) - W_{k'}(t, \ell_k t))\right\|_{L^2(B_k)} \lesssim t^{-4}. \quad (4.7)$$

Indeed, for $x \in B_k$,

$$|W_{k'}(t, x) - W_{k'}(t, \ell_k t)| \lesssim \sup_{B_k} |\nabla W_{k'}(t)| \cdot |x - \ell_k t| \lesssim t^{-4} |x - \ell_k t|,$$

and so,

$$|W_k|^\frac{4}{3} |W_{k'}(t, x) - W_{k'}(t, \ell_k t)| \lesssim t^{-4} \langle x - \ell_k t \rangle^{-3},$$

which implies (4.7).

Last, note from the explicit expression (1.2) of $W$ the following asymptotics for $|x| \gg 1$,

$$|W(x) - 15^{\frac{2}{3}} |x|^{-3} | \lesssim |x|^{-5}.$$  Thus, using the assumptions of the parameters (4.1) and the definition of $W_{k'}$ from (1.3) and (4.2), we have

$$W_{k'}(t, \ell_k t) = \frac{\epsilon_{k'}}{\lambda_{k'}^\frac{4}{3}(t)} W_{\ell_{k'}} \left(\frac{(\ell_k - \ell_{k'}) t - y_{k'}(t)}{\lambda_{k'}^\frac{4}{3}(t)}\right) = \frac{\epsilon_{k'}}{\lambda_{k'}^\frac{4}{3}(t)} W_{\ell_{k'}} \left(\frac{(\ell_k - \ell_{k'}) t}{\lambda_{k'}^\frac{4}{3}(t)}\right) + O(t^{-4})
\quad = \frac{\epsilon_{k'}}{\lambda_{k'}^\frac{4}{3}(t)} W \left(\frac{\sigma_{k', k} t}{\lambda_{k'}^\frac{4}{3}(t)}\right) + O(t^{-4}) = 15^{\frac{2}{3}} \epsilon_{k'} (\lambda_{k'}^\frac{4}{3})^\frac{2}{3} |\sigma_{k', k}|^{-3} t^{-3} + O(t^{-4}).$$

Gathering these estimates, we find (4.3). \qed
4.2. The approximate solution $\vec{W}$. To remove the main interaction terms $c_1 t^{-3} |W_1|^{\frac{4}{3}}$ and $c_2 t^{-3} |W_2|^{\frac{4}{3}}$ computed in Lemma 4.1, we define suitably rescaled versions of the function $v_{t_k}$ given by Lemma 3.1. Let

$$v_k(t, x) = \frac{1}{\lambda_k^3} v_{t_k} \left( t \lambda_k, \frac{x - y_k}{\lambda_k} \right), \quad (4.8)$$

$$z_k(t, x) = \frac{1}{\lambda_k^3} (\partial_t v_{t_k}) \left( t \lambda_k, \frac{x - y_k}{\lambda_k} \right) + \frac{\kappa_{t_k} \varepsilon_k}{2 \lambda_k^2 t^2} \Lambda_k W_k(t, x) \quad (4.9)$$

and

$$\vec{v}_k = \begin{pmatrix} v_k \\ z_k \end{pmatrix}, \quad \kappa_{t_k} = -(1 - t_k^2) \frac{(W^{\frac{4}{3}}, \Lambda W)}{\|\Lambda W\|_{L^2}^2}, \quad a_k = -\frac{c_k \kappa_{t_k} \varepsilon_k}{2}.$$

Set

$$\vec{W} = \begin{pmatrix} W \\ X \end{pmatrix} = \sum_{k=1,2} \left( \vec{W}_k + c_k \vec{v}_k \right). \quad (4.10)$$

Lemma 4.3. Assume (4.1). Then, the function $\vec{W}$ satisfies on $I \times \mathbb{R}^5$

$$\begin{cases}
\partial_t W = X - \text{Mod}_W - R_W \\
\partial_t X = \Delta W + \frac{1}{2} W^{\frac{4}{3}} W - \text{Mod}_X - R_X
\end{cases} \quad (4.11)$$

where $\Lambda_k = \frac{3}{2} + (x - \ell_k t - y_k) \cdot \nabla$.

$$\text{Mod}_W = \sum_k \left( \frac{\lambda_k}{\lambda_k} - \frac{a_k}{\lambda_k^2 t^2} \right) \Lambda_k W_k + \sum_k \dot{y}_k \cdot \nabla W_k$$

$$\text{Mod}_X = -\sum_k \left( \frac{\lambda_k}{\lambda_k} - \frac{a_k}{\lambda_k^2 t^2} \right) (\ell_k \cdot \nabla) \Lambda_k W_k - \sum_k (\dot{y}_k \cdot \nabla)(\ell_k \cdot \nabla) W_k,$$

$$\vec{R} = \begin{pmatrix} R_W \\ R_X \end{pmatrix}, \quad \|\vec{R}\|_{\dot{H}^1 \times L^2} + \|\nabla \vec{R}\|_{\dot{H}^1 \times L^2} \lesssim t^{-4+\delta}. \quad (4.12)$$

Moreover, for all $0 < \delta < 1$,

$$|W| + \langle x - \ell_k t \rangle |\nabla W| \lesssim \sum_k \left( \langle x - \ell_k t \rangle^{-3} + t^{-1} \langle x - \ell_k t \rangle^{-3+\delta} \right),$$

$$|X| \lesssim \sum_k \left( \langle x - \ell_k t \rangle^{-4} + t^{-2} \langle x - \ell_k t \rangle^{-3+\delta} \right). \quad (4.13)$$

Proof. Proof of (4.13). The estimates (4.13) on $W$ and $X$ are consequences of the decay of the function $W$ and of the estimates (3.3) of $v_{t_k}$. See also (3.24) for estimates on time derivatives.
Equation of $\tilde{v}_k$. We claim that $\tilde{v}_k$ satisfies the following system
\[
\begin{align*}
\partial_t v_k &= z_k - \frac{\kappa \ell_k \epsilon_k}{2 t^2 \lambda_k^3} \Lambda_k W_k + O_{H^1 \cap H^2}(t^{-4}) \\
\partial_t z_k &= \Delta v_k + \frac{7}{3} |W_k|^4 v_k + \frac{1}{t^3} |W_k|^4 + \frac{\kappa \ell_k \epsilon_k}{2 t^2 \lambda_k^3} (\ell_k \cdot \nabla) \Lambda_k W_k + O_{H^1}(t^{-4+\delta})
\end{align*}
\]

(4.14)

First, note that
\[
\partial_t v_k(t, x) = \frac{1}{\lambda_k^4} (\partial_t u_k) \left( \frac{t}{\lambda_k}, \frac{x - y_k}{\lambda_k} \right) - 3 \frac{\lambda_k}{\lambda_k^4} \lambda_k \partial_t v_k \left( \frac{t}{\lambda_k}, \frac{x - y_k}{\lambda_k} \right) - \frac{\lambda_k}{\lambda_k^4} \lambda_k \partial_t u_k \left( \frac{t}{\lambda_k}, \frac{x - y_k}{\lambda_k} \right)
\]
\[
- \frac{\lambda_k}{\lambda_k^4} \lambda_k \cdot \nabla \partial_t u_k \left( \frac{t}{\lambda_k}, \frac{x - y_k}{\lambda_k} \right) - \frac{\lambda_k}{\lambda_k^4} \lambda_k \cdot \nabla \partial_t v_k \left( \frac{t}{\lambda_k}, \frac{x - y_k}{\lambda_k} \right).
\]

Thus, by the definition of $z_k$ in (4.9) and using the notation $A_\ell = \partial_\ell + \ell \partial_{x_\ell}$, we obtain
\[
\partial_t v_k(t, x) = z_k - \frac{\kappa \ell_k \epsilon_k}{2 t^2 \lambda_k^3} \Lambda_k W_k - 3 \frac{\lambda_k}{\lambda_k^4} \lambda_k \partial_t v_k \left( \frac{t}{\lambda_k}, \frac{x - y_k}{\lambda_k} \right) - \frac{\lambda_k}{\lambda_k^4} \lambda_k A_\ell \partial_t u_k \left( \frac{t}{\lambda_k}, \frac{x - y_k}{\lambda_k} \right)
\]
\[
- \frac{\lambda_k}{\lambda_k^4} \lambda_k \cdot \nabla \partial_t u_k \left( \frac{t}{\lambda_k}, \frac{x - y_k}{\lambda_k} \right) - \frac{\lambda_k}{\lambda_k^4} \lambda_k \cdot \nabla \partial_t v_k \left( \frac{t}{\lambda_k}, \frac{x - y_k}{\lambda_k} \right).
\]

By (4.11) and (5.3), we obtain the first line of (4.14).

For the second line, we compute using the definition of $z_k$,
\[
\partial_t z_k = \frac{1}{\lambda_k^5} (\partial^2_t v_k) \left( \frac{t}{\lambda_k}, \frac{x - y_k}{\lambda_k} \right) - \frac{\lambda_k}{\lambda_k^5} \lambda_k \cdot \nabla \partial_t u_k \left( \frac{t}{\lambda_k}, \frac{x - y_k}{\lambda_k} \right) - \frac{\lambda_k}{\lambda_k^5} \lambda_k \cdot \nabla \partial_t v_k \left( \frac{t}{\lambda_k}, \frac{x - y_k}{\lambda_k} \right)
\]
\[
- \frac{\lambda_k}{\lambda_k^5} \lambda_k \cdot \nabla \partial_t u_k \left( \frac{t}{\lambda_k}, \frac{x - y_k}{\lambda_k} \right) - \frac{\lambda_k}{\lambda_k^5} \lambda_k \cdot \nabla \partial_t v_k \left( \frac{t}{\lambda_k}, \frac{x - y_k}{\lambda_k} \right).
\]

Thus, as before, by (4.11) and (5.3) (see also (3.24))
\[
\partial_t z_k = \frac{1}{\lambda_k^5} (\partial^2_t v_k) \left( \frac{t}{\lambda_k}, \frac{x - y_k}{\lambda_k} \right) - \frac{\kappa \ell_k \epsilon_k}{t^3 \lambda_k^2} \Lambda_k W_k - \frac{\kappa \ell_k \epsilon_k}{2 t^2 \lambda_k^3} \Lambda_k W_k - \frac{\kappa \ell_k \epsilon_k \ell_k}{2 t^2 \lambda_k^3} \Lambda_k W_k + O_{H^1}(t^{-4+\delta}).
\]

Therefore, inserting now (3.2) for $v_k$,
\[
\partial_t z_k = \frac{1}{\lambda_k^5} (\partial^2_t v_k) \left( \frac{t}{\lambda_k}, \frac{x - y_k}{\lambda_k} \right) + \frac{7}{3} |W_k|^4 v_k \left( \frac{t}{\lambda_k}, \frac{x - y_k}{\lambda_k} \right) + \frac{1}{t^3} |W_k|^4 + \frac{\kappa \ell_k \epsilon_k}{\lambda_k^2} \Lambda_k W_k
\]
\[
+ \frac{\kappa \ell_k \epsilon_k \ell_k}{t^3 \lambda_k^2} \partial_{x_\ell} \Lambda W_k - \frac{\kappa \ell_k \epsilon_k \ell_k}{2 t^2 \lambda_k^3} \partial_{x_\ell} \Lambda W_k + O_{H^1}(t^{-4+\delta}),
\]

which gives the second line of (4.14).

Equation of $\tilde{W}$. By direct computations, we check
\[
\partial_t W_k = -\ell_k \cdot \nabla W_k - \frac{\lambda_k}{\lambda_k} \Lambda_k W_k - \frac{\lambda_k}{\lambda_k} \cdot \nabla W_k.
\]
Thus, using also (4.14),
\[ \partial_t W = X - \text{Mod}_W + O_{H^1 \cap H^2}(t^{-4}). \]  
(4.15)

Moreover, using (4.16) and (4.14)
\[ \partial_t X = -\partial_t \left( \sum_k \ell_k \cdot \nabla W_k \right) + \sum_k c_k \partial_t z_k \]
\[ = \sum_k (\ell_k \cdot \nabla)^2 W_k + \sum_k \left( \frac{\lambda_k}{\lambda_k} + \frac{c_k \ell_k \cdot \lambda_k}{2 \lambda_k t^2} \right) (\ell_k \cdot \nabla) \Lambda_k W_k + \sum (\tilde{y}_k \cdot \nabla) (\ell_k \cdot \nabla) W_k \]
\[ + \sum_k c_k \left( \Delta v_k + \frac{7}{3} |W_k|^\frac{4}{3} v_k + t^{-3} |W_k|^\frac{1}{3} \right) + O_{H^1}(t^{-4+\delta}) \]

Note that \((\ell_k \cdot \nabla)^2 W_k = \Delta W_k + |W_k|^\frac{4}{3} W_k\), and thus \(\partial_t X = \Delta W + |W|^\frac{4}{3} W - \text{Mod}_X - R_X\)
with
\[ R_X = |W|^\frac{4}{3} W - \sum_k |W_k|^\frac{4}{3} W_k - t^{-3} \sum_k c_k |W_k|^\frac{4}{3} - \frac{7}{3} \sum_k c_k |W_k|^\frac{4}{3} v_k + O_{H^1}(t^{-4+\delta}). \]

Note that \(R_X = R_v + R_\Sigma + O_{H^1}(t^{-4+\delta})\) where
\[ R_v = |W|^\frac{4}{3} W - \left| \sum_k W_k \right|^\frac{4}{3} \left( \sum W_k \right) - \frac{7}{3} \sum_k c_k |W_k|^\frac{4}{3} v_k, \]
and from Lemma 4.14 \(\|R_\Sigma\|_{H^1} \lesssim t^{-4}\). Now, we prove \(\|R_v\|_{H^1} \lesssim t^{-4+\delta}\). We decompose
\[ R_v = \left| \sum W_k + c_k v_k \right|^\frac{4}{3} \left( \sum W_k + c_k v_k \right) - \left| \sum W_k \right|^\frac{4}{3} \left( \sum W_k \right) - \frac{7}{3} \left| \sum W_k \right|^\frac{4}{3} \left( \sum c_k v_k \right) \]
\[ + \frac{7}{3} \sum_k \left( c_k v_k \left( |W_1 + W_2|^{\frac{4}{3}} - |W_k|^{\frac{4}{3}} \right) \right). \]

First, we observe from (3.3)
\[ \left| \sum W_k + c_k v_k \right|^\frac{4}{3} \left( \sum W_k + c_k v_k \right) - \left| \sum W_k \right|^\frac{4}{3} \left( \sum W_k \right) - \frac{7}{3} \left| \sum W_k \right|^\frac{4}{3} \left( \sum c_k v_k \right) \]
\[ \lesssim \sum |v_k|^2 \lesssim t^{-4} \sum (x - \ell_k t)^{-4+\delta} \]
and so this term is \(O_{H^1}(t^{-4})\). Second, we estimate \(v_k \left( |W_1 + W_2|^{\frac{4}{3}} - |W_k|^{\frac{4}{3}} \right)\) for \(k = 1, 2\).

For \(|x - \ell_1 t| \leq \frac{|\ell_1 - \ell_2|}{10} t\), using (3.3), we have
\[ |v_1 \left( |W_1 + W_2|^{\frac{4}{3}} - |W_1|^{\frac{4}{3}} \right) | \lesssim |v_1| |W_2| \left( |W_1|^{\frac{4}{3}} + |W_2|^{\frac{4}{3}} \right) \lesssim t^{-4} \left| |W_1|^{\frac{1}{3}} - |W_2|^{\frac{1}{3}} \right| \left( |W_1|^{\frac{4}{3}} + |W_2|^{\frac{4}{3}} \right). \]

For \(|x - \ell_1 t| > \frac{|\ell_1 - \ell_2|}{10} t\), also using (3.3), we have
\[ |v_1 \left( |W_1 + W_2|^{\frac{4}{3}} - |W_1|^{\frac{4}{3}} \right) | \lesssim t^{-4+\delta} \left( |W_1|^{\frac{4}{3}} + |W_2|^{\frac{4}{3}} \right). \]

The same holds for the term \(v_2 \left( |W_1 + W_2|^{\frac{4}{3}} - |W_2|^{\frac{4}{3}} \right)\) and we obtain \(\|R_v\|_{H^1} \lesssim t^{-4+\delta}. \) \(\qed\)
5. Refined construction of a two-soliton solution

To construct the two-soliton solution at $+\infty$, we follow the strategy of [35] using the refined approximate solution $\vec{W}$ defined in the previous section. As in [35] and several other previous papers on multiple solitons, see e.g. [38, 30, 31, 2, 3], we argue by compactness and obtain the solution $u(t)$ as the limit of a sequence of approximate multi-solitons $u_n(t)$.

**Proposition 5.1.** There exist $T_0 > 0$ and a solution $u(t)$ of (1.1) on $[T_0, +\infty)$ satisfying, for all $t \in [T_0, +\infty)$,

$$\|\nabla u(t) - \nabla \vec{W}(t)\|_{L^2} + \|\partial_t u(t) - \mathbf{X}(t)\|_{L^2} \lesssim t^{-3 + \frac{1}{10}}$$

(5.1)

where $\lambda_k(t), y_k(t)$ are such that, for all $t \in [T_0, +\infty)$,

$$|\lambda_k(t) - \lambda_k^\infty| + |y_k(t) - y_k^\infty| \lesssim t^{-1}. \quad (5.2)$$

This section is devoted to the proof of Proposition 5.1. Since the ansatz $\vec{W}$ takes into account the consequence of the main order of the interactions of the two waves, the two-soliton solution $u(t)$ is computed in (5.1) up to order $t^{-3 + \frac{1}{10}}$ (the loss of the exponent $\frac{1}{10}$ has no special meaning here) to be compared with [35], where the corresponding error is of size $t^{-2}$ (see (4.9) in [35]). A computation at order $t^{-3 + \frac{1}{10}}$ will allow us to justify the non-zero dispersive part and thus to finish the proof of Theorem 1.1 in the next section.

Let $S_n \to +\infty$. Let $\zeta_{k,n}^\pm \in \mathbb{R}$ small to be determined later. These free parameters correspond to two exponentially stable/unstable directions for each soliton - see statements of Proposition 5.2, Claim 5.5 and Lemma 5.8. For any large $n$, we consider the solution $u_n$ of

$$\left\{ \begin{array}{l}
\partial_t^2 u_n - \Delta u_n - |u_n|^4 u_n = 0 \\
(u_n(S_n), \partial_t u_n(S_n))^T = \sum_{k=1,2} \left( \vec{W}_k^\infty(S_n) + c_k \vec{v}_k(S_n) + \sum_{\pm} \zeta_{k,n}^\pm (\vec{v}_k \vec{Z}_{\pm})(S_n) \right) \quad (5.3)
\end{array} \right.$$

Note that since $(u_n(S_n), \partial_t u_n(S_n)) \in H^1 \times L^2$, the solution $\vec{u}_n$ is well-defined in $H^1 \times L^2$ at least on a small interval of time around $S_n$.

Now, we state uniform estimates on $u_n$ backwards in time up to some uniform $T_0 \gg 1$.

**Proposition 5.2.** There exist $n_0 > 0$ and $T_0 > 0$ such that, for any $n \geq n_0$, there exist $(\zeta_{k,n}^\pm)_{k=1,2} \in \mathbb{R}^2 \times \mathbb{R}^2$, with

$$\sum_k |\zeta_{k,n}^\pm|^2 \lesssim S_n^{-7},$$

and such that the solution $\vec{u}_n = (u_n, \partial_t u_n)^T$ of (5.3) is well-defined in $H^1 \times L^2$ on the time interval $[T_0, S_n]$ and satisfies, for all $t \in [T_0, +\infty)$,

$$\left\| \vec{u}_n(t) - \vec{W}_n(t) \right\|_{H^1 \times L^2} \lesssim t^{-3 + \frac{1}{10}} \quad (5.4)$$

where $\vec{W}_n(t, x) = \vec{W}(t, x; \{\lambda_{k,n}(t)\}, \{y_{k,n}(t)\})$ is defined in (4.2) and

$$|\lambda_{k,n}(t) - \lambda_k^\infty| + |y_{k,n}(t) - y_k^\infty| \lesssim t^{-1}, \quad |\dot{\lambda}_{k,n}(t)| + |\dot{y}_{k,n}(t)| \lesssim t^{-2}. \quad (5.5)$$

Moreover, $\vec{u}_n \in C([T_0, S_n], H^1 \times H^1)$ and satisfies, for all $t \in [T_0, S_n]$, $\|\vec{u}_n(t)\|_{H^1 \times H^1} \lesssim 1$. 


5.1. Decomposition around $\tilde{W}$.

**Lemma 5.3** (Properties of the decomposition). There exist $T_0 \gg 1$ and $0 < \delta_0 \ll 1$ such that if $u(t)$ is a solution of (1.1) which satisfies on $I$,

$$
\| \tilde{u} - \sum_{k=1,2} \left( \tilde{W}_k^\infty + c_k \tilde{v}_k \right) \|_{H^1 \times L^2} < \delta_0,
$$

then there exist $C^1$ functions $\lambda_k > 0$, $y_k$ on $I$ such that, $\tilde{\varepsilon}(t)$ being defined by

$$
\tilde{\varepsilon} = \begin{pmatrix} \varepsilon \\ \eta \end{pmatrix}, \quad \tilde{u} = \begin{pmatrix} u \\ \partial_t u \end{pmatrix} = \tilde{W} + \tilde{\varepsilon},
$$

the following hold on $I$, for $k = 1,2$.

(i) First properties of the decomposition. For $j = 1, \ldots, 5$,

$$(\varepsilon, \Lambda_k W_k)_{H^1_k} = (\varepsilon, \partial_x W_k)_{H^1_k} = 0,$n

$$
|\lambda_k - \lambda_k^\infty| + |y_k - y_k^\infty| + \|\tilde{\varepsilon}\|_{H^1 \times L^2} \lesssim \left\| \tilde{u} - \sum_{k=1,2} \left( \tilde{W}_k^\infty + c_k \tilde{v}_k \right) \right\|_{H^1 \times L^2}.
$$

(ii) Equation of $\tilde{\varepsilon}$.

$$
\begin{cases}
\partial_t \varepsilon = \eta + \text{Mod}_W + R_W \\
\partial_t \eta = \Delta \varepsilon + |W + \varepsilon|^{\frac{3}{2}}(W + \varepsilon) - |W|^{\frac{3}{2}}W + \text{Mod}_X + R_X.
\end{cases}
$$

(iii) Parameter estimates. For any $0 < \delta < 1$,

$$
\frac{\dot{\lambda}_k}{\lambda_k} - \frac{a_k}{\lambda_k^2} + |\dot{y}_k| \lesssim \|\tilde{\varepsilon}\|_{H^1 \times L^2} + t^{-4+\delta}.
$$

(iv) Unstable directions. Let $z_k^\pm = (\tilde{\varepsilon}, \tilde{\theta}_k, \tilde{Z}_k^\pm)$. Then, for any $0 < \delta < 1$,

$$
\left| \frac{d}{dt} z_k^\pm + \frac{\sqrt{\lambda_0}}{\lambda_k} (1 - |k|^2)^{\frac{1}{2}} z_k^\pm \right| \lesssim \|\tilde{\varepsilon}\|_{H^1 \times L^2}^2 + t^{-1} \|\tilde{\varepsilon}\|_{H^1 \times L^2} + t^{-4+\delta}.
$$

*Proof. Decomposition.* The existence of parameters $\lambda_k$ and $y_k$ such that (5.8) and (5.9) hold is proved similarly as (i) of Lemma 3.1 in [35].

**Equation of $\tilde{\varepsilon}$**. The equation of $\tilde{\varepsilon}(t)$ is easily derived from the equation (1.1) of $u$ and (3.11). Indeed, first, since $\varepsilon = u - W$, we have

$$
\partial_t \varepsilon = \partial_t u - \partial_t W = \eta + X - \partial_t W = \eta + \text{Mod}_W + R_W.
$$

Second, since $\eta = \partial_t u - X$, we have

$$
\partial_t \eta = \partial_t^2 u - \partial_t X = \Delta u + |u|^{\frac{3}{2}}u - \Delta W - |W|^{\frac{3}{2}}W + \text{Mod}_X + R_X \\
= \Delta \varepsilon + |W + \varepsilon|^{\frac{3}{2}}(W + \varepsilon) - |W|^{\frac{3}{2}}W + \text{Mod}_X + R_X.
$$
We also denote
\[
R_{\text{NL}} = |W + \epsilon|^{\frac{4}{3}} (W + \epsilon) - |W|^{\frac{4}{3}} W - \frac{7}{3} \sum_k |W_k|^{\frac{4}{3}} \epsilon = R_1 + R_2;
\]
\[
R_1 = \frac{7}{3} \left( |W|^{\frac{4}{3}} - \sum_k |W_k|^{\frac{4}{3}} \right) \epsilon, \quad R_2 = |W + \epsilon|^{\frac{4}{3}} (W + \epsilon) - |W|^{\frac{4}{3}} W - \frac{7}{3} |W|^{\frac{4}{3}} \epsilon,
\]
and
\[
\tilde{L}' = \left( \Delta + \frac{7}{3} \sum_k |W_k|^{\frac{4}{3}} \right), \quad \tilde{\mathcal{M}} = \begin{pmatrix} \text{Mod}_W \\ \text{Mod}_X \end{pmatrix}, \quad \tilde{R} = \begin{pmatrix} R_W \\ R_X \end{pmatrix}, \quad \tilde{R}_{\text{NL}} = \begin{pmatrix} 0 \\ \tilde{R}_{\text{NL}} \end{pmatrix}, \quad \tilde{R}_1 = \begin{pmatrix} 0 \\ \tilde{R}_1 \end{pmatrix}, \quad \tilde{R}_2 = \begin{pmatrix} 0 \\ \tilde{R}_2 \end{pmatrix}.
\]
With this notation, the system (5.11) rewrites
\[
\partial_t \tilde{\epsilon} = \tilde{L}' \tilde{\epsilon} + \tilde{\mathcal{M}} \tilde{\epsilon} + \tilde{R} + \tilde{R}_{\text{NL}} = \tilde{L}' \tilde{\epsilon} + \tilde{\mathcal{M}} \tilde{\epsilon} + \tilde{R}_1 + \tilde{R}_2 + \tilde{R}_1 + \tilde{R}_2. \tag{5.13}
\]
We claim the following estimates on \(R_1\) and \(R_2\)
\[
\|R_1\|_{L^{\frac{14}{9}}} \lesssim t^{-1} \|\epsilon\|_{L^{\frac{14}{9}}}, \quad \|R_2\| \lesssim |W|^{\frac{4}{3}} |\epsilon|^{\frac{2}{3}} + |\epsilon|^{\frac{7}{3}}, \quad \|R_2\|_{L^{\frac{14}{9}}} \lesssim \|\epsilon\|_{L^{\frac{14}{9}}}^{14} \lesssim \|\epsilon\|_{H^1}^{14}. \tag{5.14}
\]
The estimate on \(R_2\) follows from (5.10). To prove the estimate on \(R_1\), we first recall the inequality, for \(p > 1\), for any reals \(r_k\),
\[
\left| \sum_k r_k^p - \sum_k |r_k|^{p} \right| \lesssim \sum_k |r_k|^{p-1} |r_k|^{p} \quad (5.15)
\]
Therefore,
\[
|W|^{\frac{4}{3}} - \sum_k |W_k|^{\frac{4}{3}} \lesssim \left| W \right|^{\frac{4}{3}} - \left| \sum_k W_k \right|^{\frac{4}{3}} + \left| \sum_k W_k \right|^{\frac{4}{3}} - \sum_k |W_k|^{\frac{4}{3}} \lesssim \left( \sum_k |W_k| \right) \left( \sum_k |W_k| + |v_k| \right)^{\frac{4}{3}} + \sum_k |W_k| |W_k|^{\frac{1}{3}}
\]
and thus
\[
|R_1| \lesssim |\epsilon| \left( \sum_k |v_k| \right) \left( \sum_k |W_k| + |v_k| \right)^{\frac{4}{3}} + |\epsilon| \sum_k |W_k||W_k|^{\frac{1}{3}}
\]
By (1.10), we obtain
\[
\|R_1\|_{L^{\frac{14}{9}}} \lesssim \|\epsilon\|_{L^{\frac{14}{9}}} \left( \sum_k |v_k| \right) \left( \sum_k \|W_k\|_{L^{\frac{14}{9}}} + \|v_k\|_{L^{\frac{14}{9}}} \right) + \|\epsilon\|_{L^{\frac{14}{9}}} \sum_k |W_k||W_k|^{\frac{1}{3}}_{L^{\frac{14}{9}}}
\]
By (3.3), we have \(\|v_k\|_{L^{\frac{14}{9}}} \lesssim t^{-2}\). Moreover \(\|W_k|W_k|^{\frac{1}{3}}\|_{L^{\frac{14}{9}}} \lesssim t^{-1}\) is a consequence of the following technical result.

Claim 5.4 (Claim 2 in [35]). Let \(0 < r_2 \leq r_1\) be such that \(r_1 + r_2 > \frac{5}{3}\). For \(t\) large, if \(r_1 > \frac{5}{3}\) then \(\int |W_1|^{r_1}|W_2|^{r_2} \lesssim t^{-3r_2}\), whereas if \(r_1 \leq \frac{5}{3}\) then \(\int |W_1|^{r_1}|W_2|^{r_2} \lesssim t^{5-3(r_1+r_2)}\).
In conclusion, the orthogonality condition \( (\varepsilon, \Lambda_1 W_1)_{H^1_{t_1}} = (\partial_t \varepsilon, \Lambda_1 W_1)_{H^1_{t_1}} + (\varepsilon, \partial_t (\Lambda_1 W_1))_{H^1_{t_1}} = 0 \)

Thus, using the first line of (5.10), and the expression of \( \text{Mod}_W \) in Lemma [4.3],

\[
0 = (\eta, \Lambda_1 W_1)_{H^1_{t_1}} - (\varepsilon, \mathcal{L}_1 \cdot \nabla (\Lambda_1 W_1))_{H^1_{t_1}} - \frac{\lambda_1}{\lambda_1} (\varepsilon, \Lambda_1^2 W_1)_{H^1_{t_1}} - (\varepsilon, \dot{y}_1 \cdot \nabla \Lambda_1 W_1)_{H^1_{t_1}} + \left( \frac{\lambda_1}{\lambda_1} - \frac{a_1}{\lambda_1^2 t^2} \right) (\Lambda_1 W_1, \Lambda_1 W_1)_{H^1_{t_1}} + (\dot{y}_1 \cdot \nabla W_1, \Lambda_1 W_1)_{H^1_{t_1}},
\]

Next, \( (\Lambda_1 W_1, \Lambda_1 W_1)_{H^1_{t_1}} = (1 - |\mathcal{L}_1|^2)^2 \| \Lambda W \|_{H^1}^2 \), and by parity, \( (\nabla W_1, \Lambda_1 W_1)_{H^1_{t_1}} = 0 \). Using Claim [5.4], we have

\[
\left| (\Lambda_2 W_2, \Lambda_1 W_1)_{H^1_{t_1}} \right| + \left| (\nabla W_2, \Lambda_1 W_1)_{H^1_{t_1}} \right| \lesssim t^{-3}
\]

In conclusion, the orthogonality condition \( (\varepsilon, \Lambda_1 W_1)_{H^1_{t_1}} = 0 \), gives

\[
\frac{\lambda_1}{\lambda_1} - \frac{a_1}{\lambda_1^2 t^2} \leq \| \varepsilon \|_{H^1 \times L^2} + |\dot{y}_1| \| \varepsilon \|_{H^1 \times L^2} + t^{-3} \sum_{k=1,2} \left( \left| \frac{\lambda_k}{\lambda_k} - \frac{a_k}{\lambda_k^2 t^2} \right| + |\dot{y}_k| \right) + t^{-4+\delta}.
\]

Using the other orthogonality conditions, we obtain

\[
\sum_{k=1,2} \left( \left| \frac{\lambda_k}{\lambda_k} - \frac{a_k}{\lambda_k^2 t^2} \right| + |\dot{y}_k| \right) \leq \| \varepsilon \|_{H^1 \times L^2} + (\| \varepsilon \|_{H^1 \times L^2} + t^{-3}) \sum_{k=1,2} \left( \left| \frac{\lambda_k}{\lambda_k} - \frac{a_k}{\lambda_k^2 t^2} \right| + |\dot{y}_k| \right) + t^{-4+\delta}.
\]

Therefore, for \( \delta_0 \) small enough and \( T_0 \) large enough, we find (5.11).
Equations of the unstable directions. Recall that $\tilde{Z}_{k,\ell,t}^\pm \in \mathcal{S}$ by their definition in §2.1. By (5.13), we have
\[
\frac{d}{dt} \tilde{z}_{1,t}^\pm = \frac{d}{dt} (\varepsilon, \tilde{\theta}_1 \tilde{Z}_{k,\ell,t}^\pm) = (\partial_t \varepsilon, \tilde{\theta}_1 \tilde{Z}_{k,\ell,t}^\pm) + (\varepsilon, \partial_t (\tilde{\theta}_1 \tilde{Z}_{k,\ell,t}^\pm))
\]
\[
= \left( \mathcal{L} \varepsilon, \tilde{\theta}_1 \tilde{Z}_{k,\ell,t}^\pm \right) - \frac{\ell_1}{\lambda_1} \cdot (\varepsilon, \tilde{\theta}_1 \nabla \tilde{Z}_{k,\ell,t}^\pm) - \frac{\lambda_1}{\lambda_1} \left( \varepsilon, \tilde{\theta}_1 \tilde{\Lambda} \tilde{Z}_{k,\ell,t}^\pm \right) - \frac{\dot{\gamma}_1}{\lambda_1} \cdot (\varepsilon, \tilde{\theta}_1 \nabla \tilde{Z}_{k,\ell,t}^\pm)
\]
\[
+ \left( \text{Mod}, \tilde{\theta}_1 \tilde{Z}_{k,\ell,t}^\pm \right) + \left( \tilde{\mathbf{R}}, \tilde{\theta}_1 \tilde{Z}_{k,\ell,t}^\pm \right) + \left( \tilde{\mathbf{R}}_{\text{NL}}, \tilde{\theta}_1 \tilde{Z}_{k,\ell,t}^\pm \right)
\]
First, by direct computations, using (ii) of Lemma 2.2,
\[
\left( \mathcal{L} \varepsilon, \tilde{\theta}_1 \tilde{Z}_{k,\ell,t}^\pm \right) - \frac{\ell_1}{\lambda_1} \cdot (\varepsilon, \tilde{\theta}_1 \nabla \tilde{Z}_{k,\ell,t}^\pm) = \frac{1}{\lambda_1} \left( \varepsilon, \tilde{\theta}_1 (-H_\ell, J \tilde{Z}_{k,\ell,t}^\pm) \right) + \left( \varepsilon, f'(W_2)(\theta_1 Z_{k,\ell,t,2}^\pm) \right)
\]
\[
= \pm \frac{\sqrt{\lambda_0}}{\lambda_1} (1 - |\ell_1|^2) \tilde{z}_{1,t}^\pm + \left( \varepsilon, f'(W_2)(\theta_1 Z_{k,\ell,t,2}^\pm) \right).
\]
By the decay properties of $\tilde{Z}_{k,\ell,t}^\pm$ and Claim 5.4
\[
\left| \left( \varepsilon, f'(W_2)(\theta_1 Z_{k,\ell,t,2}^\pm) \right) \right| \lesssim t^{-4} \|\varepsilon\|_{H^1}.
\]
Next, by (5.11),
\[
\left| \frac{\dot{\lambda}_1}{\lambda_1} \right| \left| \left( \varepsilon, \tilde{\theta}_1 \tilde{\Lambda} \tilde{Z}_{k,\ell,t}^\pm \right) \right| + \left| \frac{\dot{\gamma}_1}{\lambda_1} \cdot (\varepsilon, \tilde{\theta}_1 \nabla \tilde{Z}_{k,\ell,t}^\pm) \right| \lesssim t^{-1} \|\varepsilon\|_{H^{1/2} \times L^2} \lesssim \|\varepsilon\|_{H^1 \times L^2}^2 + t^{-4+\delta} \|\varepsilon\|_{H^{1/2} \times L^2}.
\]
Concerning the term with Mod, (ii) of Lemma 2.2 yields $(\tilde{\theta}_1 \tilde{Z}_{k,\ell,t}^\pm, \tilde{\theta}_1 \tilde{Z}_{k,\ell,t}^\pm) = (\tilde{Z}_{k,\ell,t}^\pm, \tilde{Z}_{k,\ell,t}^\pm) = 0$. Moreover, by Claim 5.3 we have
\[
\left| \left( \tilde{\theta}_2 \tilde{Z}_{k,\ell,t}^\pm, \tilde{\theta}_1 \tilde{Z}_{k,\ell,t}^\pm \right) \right| + \left| \left( \tilde{\theta}_1 \tilde{Z}_{k,\ell,t}^\pm, \tilde{\theta}_1 \tilde{Z}_{k,\ell,t}^\pm \right) \right| \lesssim t^{-3},
\]
and thus, by (5.11),
\[
\left| \frac{\dot{\lambda}_2}{\lambda_2} - \frac{a_2}{\lambda_2^2} \right| \left| \left( \tilde{\theta}_2 \tilde{Z}_{k,\ell,t}^\pm, \tilde{\theta}_1 \tilde{Z}_{k,\ell,t}^\pm \right) \right| + \left| \frac{\dot{\gamma}_2}{\lambda_2} \cdot (\tilde{\theta}_2 \tilde{Z}_{k,\ell,t}^\pm, \tilde{\theta}_1 \tilde{Z}_{k,\ell,t}^\pm) \right| \lesssim t^{-3} \|\varepsilon\|_{H^1 \times L^2}.
\]
Finally, we claim
\[
\left| \left( \tilde{\mathbf{R}}, \tilde{\theta}_1 \tilde{Z}_{k,\ell,t}^\pm \right) \right| + \left| \left( \tilde{\mathbf{R}}_{\text{NL}}, \tilde{\theta}_1 \tilde{Z}_{k,\ell,t}^\pm \right) \right| \lesssim t^{-4+\delta} + t^{-1} \|\varepsilon\|_{H^1} + \|\varepsilon\|_{H^1}^2.
\]
Indeed, from (4.12), we have $|\left( \tilde{\mathbf{R}}, \tilde{\theta}_1 \tilde{Z}_{k,\ell,t}^\pm \right)| \lesssim t^{-4+\delta}$. Second, by (5.14) and the decay of $Y$, $|\left( \tilde{\mathbf{R}}_1, \tilde{\theta}_1 \tilde{Z}_{k,\ell,t}^\pm \right)| \lesssim \|	ilde{\mathbf{R}}_1\|_{L^{\infty}} \lesssim t^{-1} \|\varepsilon\|_{H^1}$ and $|\left( \tilde{\mathbf{R}}_2, \tilde{\theta}_1 \tilde{Z}_{k,\ell,t}^\pm \right)| \lesssim \|	ilde{\mathbf{R}}_2\|_{L^{\infty}} \lesssim \|\varepsilon\|_{H^1}^2$.

The computation for $\tilde{z}_{2,t}^\pm$ is the same and we obtain for $k = 1, 2$,
\[
\left| \frac{d}{dt} \tilde{z}_{2,t}^\pm (t) + \frac{\sqrt{\lambda_0}}{\lambda_k(t)} (1 - |\ell_k|^2) \tilde{z}_{2,t}^\pm (t) \right| \lesssim t^{-4+\delta} + \|\varepsilon(t)\|_{H^1 \times L^2}^2 + t^{-1} \|\varepsilon(t)\|_{H^1 \times L^2}.
\]
The proof of Lemma 5.3 is complete. \qed
5.2. Bootstrap setting. We denote by $B_{R^2}(r)$ (respectively, $S_{R^2}(r)$) the open ball (respectively, the sphere) of $R^2$ of center 0 and of radius $r > 0$, for the norm $|\xi_k| = (\sum_{k=1,2} \xi_k^2)^{1/2}$.

For $t = S_n$ and for $t < S_n$ as long as $u_n(t)$ is well-defined in $H^1 \times L^2$ and satisfies (5.6), we will consider the decomposition of $\bar{u}_n(t)$ from Lemma 5.3. For simplicity of notation, we will denote the parameters $\lambda_{k,n}, y_{k,n}$ and $\tilde{\varepsilon}$ of this decomposition by $\lambda_k, y_k$ and $\tilde{\varepsilon}$.

We start with a technical result similar to Lemma 3 in [2]. This claim will allow us to adjust the initial values of $(z^\pm_n(S_n))_k$ from the choice of $\zeta_{k,n}^\pm$ in (5.3).

Claim 5.5 (Choosing the initial unstable modes). There exist $n_0 > 0$ and $C > 0$ such that, for all $n \geq n_0$, for any $(\xi_k)_{k \in\{1,2\}} \in B_{R^2}(S_n^{-7/2})$, there exists a unique $(\zeta_{k,n}^\pm, k \in\{1,2\}) \in B_{R^2}(C S_n^{-7/2})$ such that the decomposition of $u_n(S_n)$ satisfies

$$z^-(S_n) = \xi_k, \quad z^+(S_n) = 0,$$

$$|\lambda_k(S_n) - \lambda_k^\infty| + |y_k(S_n) - y_k^\infty| + \|\tilde{\varepsilon}(S_n)\|_{H^1 \times L^2} \lesssim S_n^{-7/2}. \tag{5.17}$$

Sketch of the proof of Claim 5.5. The proof of existence of $(\zeta_{k,n}^\pm)_k$ in Claim 5.5 is similar to Lemma 3 in [2] and we omit it. Estimates in (5.17) are consequences of (5.9).

From now on, for any $(\xi_k)_k \in B_{R^2}(S_n^{-7/2})$, we fix $(\zeta_{k,n}^\pm)_k$ as given by Claim 5.5 and the corresponding solution $u_n$ of (5.3). We fix $\delta = 1/20$.

The proof of Proposition 5.2 is based on the following bootstrap estimates, for $k = 1, 2$,

$$\begin{align*}
|\lambda_k(t) - \lambda_k^\infty| &\leq C_0 t^{-1}, \quad |y_k(t) - y_k^\infty| \leq C_0 t^{-1}, \\
|z_k^+(t)|^2 &\leq t^{-7}, \quad \|\tilde{\varepsilon}(t)\|_{H^1 \times L^2} \leq t^{-3+\frac{1}{10}}. \tag{5.18}
\end{align*}$$

Set

$$T^* = T^*_n((\xi_k)_k) = \inf\{t \in [T_0, S_n] : u_n \text{ satisfies (5.6) and (5.18) holds on } [t, S_n]\}. \tag{5.19}$$

Note that by Claim 5.5, estimate (5.18) is satisfied at $t = S_n$. Moreover, if (5.18) is satisfied on $[\tau, S_n]$ for some $\tau \leq S_n$ then by the well-posedness theory and continuity, $u_n(t)$ is well-defined and satisfies the decomposition of Lemma 5.3 on $[\tau', S_n]$, for some $\tau' < \tau$. In particular, the definition of $T^*$ makes sense. In what follows, we will prove that there exists $T_0$ large enough and at least one choice of $(\xi_k)_k \in B_{R^2}(S_n^{-7/2})$ so that $T^* = T_0$, which is enough to finish the proof of Proposition 5.2. For this, we derive general estimates for any $(\xi_k)_k \in B_{R^2}(S_n^{-7/2})$ (see Lemma 5.7) and use a topological argument (see Lemma 5.8) to control the unstable directions, in order to strictly improve (5.18) on $[T^*, S_n]$.

As a consequence of the bootstrap estimates (5.18) and (5.11), we have, for $k = 1, 2$,

$$|\frac{\lambda_k}{\lambda_k^\infty} - \frac{a_k}{\lambda_k^\infty t^2}| + |y_k| \lesssim \|\tilde{\varepsilon}(t)\|_{H^1 \times L^2} + t^{-4+\delta} \lesssim t^{-3+\frac{1}{10}}. \tag{5.20}$$

In particular, from the expression of $\text{Mod}_W$ and $\text{Mod}_X$ in Lemma 4.3 for all $\alpha \in \mathbb{N}^5$,}

$$|\partial_x^\alpha \text{Mod}_W(t)| \lesssim t^{-3+\frac{1}{10}} \sum |W_k|^{1+\frac{|\alpha|}{3}}, \quad |\partial_x^\alpha \text{Mod}_X(t)| \lesssim t^{-3+\frac{1}{10}} \sum |W_k|^{1+\frac{|\alpha|}{3}}. \tag{5.20}$$
5.3. **Energy functional.** One of the main points of the proof of Proposition 5.2 is to derive suitable estimates in the energy norm that will strictly improve the bound on $\|\tilde{\varepsilon}(t)\|_{H^{1} \times L^{2}}$ from (5.18); the other estimates then follow easily. In this section, for brevity of notation, we denote $f(u) = |u|^{4d}u$ and $F(u) = \frac{3}{10}|u|^{10}$. For $0 < \sigma < \frac{t_{1} - t_{0}}{10}$ small enough to be fixed, set
\[ \ell_{1}^{+} = \ell_{1} + \sigma(t_{2} - \ell_{1}), \quad \ell_{2}^{-} = \ell_{2} - \sigma(t_{2} - \ell_{1}), \quad \ell_{2} < \ell < 1, \]
and for $t > 0$,
\[ \Omega(t) = (\ell_{1}^{+} t, \ell_{2}^{-} t) \times \mathbb{R}^{4}, \quad \Omega^{c}(t) = \mathbb{R}^{5} \setminus \Omega(t). \]

We consider the continuous function $\chi(t, x) = \chi(t, x_{1})$ defined as follows, for all $t > 0$,
\[
\begin{cases}
\chi(t, x) = \ell_{1} \text{ for } x_{1} \in (-\infty, \ell_{1}^{+} t], \\
\chi(t, x) = \ell_{2} \text{ for } x_{1} \in [\ell_{2}^{-} t, +\infty), \\
\chi(t, x) = \frac{x_{1}}{(1 - 2\sigma)t} - \frac{\sigma}{1 - 2\sigma}(\ell_{2} + \ell_{1}) \text{ for } x_{1} \in [\ell_{1}^{+} t, \ell_{2}^{-} t].
\end{cases}
\tag{5.21}
\]

In particular,
\[
\begin{align*}
\partial_{t}\chi(t, x) &= 0, \quad \nabla \chi(t, x) = 0 \quad \text{on } \Omega^{c}(t), \\
\partial_{x_{1}} \chi(t, x) &= \frac{1}{(1 - 2\sigma)t} \quad \text{for } x \in \Omega(t), \\
\partial_{x_{1}} \chi(t, x) &= -\frac{x_{1}}{t} \frac{1}{(1 - 2\sigma)t} \quad \text{for } x \in \Omega(t).
\end{align*}
\tag{5.22}
\]

We define (see [36, 30, 31, 3, 35, 37] for similar functionals)
\[
\mathcal{H} = \int \{ |\nabla \varepsilon|^{2} + |\eta|^{2} - 2(F(W + \varepsilon) - F(W) - f(W)\varepsilon) \} + 2 \int \chi(\partial_{x_{1}} \varepsilon) \eta,
\]

**Lemma 5.6.** There exists $\mu > 0$ such that, for all $t > 1$, the following hold.

(i) **Bound.**
\[
|\mathcal{H}(t)| \leq \frac{\|\tilde{\varepsilon}(t)\|_{H^{1} \times L^{2}}^{2}}{\mu}. \tag{5.23}
\]

(ii) **Coercivity.**
\[
\mathcal{H}(t) \geq \mu \|\tilde{\varepsilon}(t)\|_{H^{1} \times L^{2}}^{2} - \frac{t^{-7}}{\mu}. \tag{5.24}
\]

(iii) **Time variation.** For all $0 < \delta < 1$,
\[
- \frac{d}{dt} (t^{2}\mathcal{H}) \lesssim t^{-5 + \frac{1}{10} + \delta}. \tag{5.25}
\]

**Proof of Lemma 5.6.** Proof of (5.23). Since
\[
|F(W + \varepsilon) - F(W) - f(W)\varepsilon | \lesssim |\varepsilon|^{10} + |\varepsilon| |W|^{4},
\]
estimate (5.23) on $\mathcal{H}$ follows (1.18), (1.9) and $\|\tilde{\varepsilon}\|_{H^{1} \times L^{2}} + \|W\|_{H^{1}} \lesssim 1$.

**Proof of (5.24).** Set
\[
\mathcal{N}_{\Omega}(t) = \int_{\Omega} \left( |\nabla \varepsilon(t)|^{2} + \eta^{2}(t) + 2(\chi(t)\partial_{x_{1}} \varepsilon(t)) \eta(t) \right), \quad \mathcal{N}_{\Omega^{c}}(t) = \int_{\Omega^{c}} \left( |\nabla \varepsilon(t)|^{2} + \eta^{2}(t) \right).
\]

...
Note that, since $|\chi| < 7$,  
\begin{equation}
N_\Omega = 7 \int_{\Omega} \left| \frac{x}{\ell} \partial_{x_1} \varepsilon + \eta \right|^2 + \int_{\Omega} |\nabla \varepsilon|^2 + \int_{\Omega} \left( 1 - \frac{\varepsilon^2}{\ell} \right) (\partial_{x_1} \varepsilon)^2 + (1 - 7) \int \eta^2  
\end{equation}

\begin{equation}
\geq 7 \int_{\Omega} \left| \frac{x}{\ell} \partial_{x_1} \varepsilon + \eta \right|^2 + (1 - 7) \int (|\nabla \varepsilon|^2 + \eta^2) .
\end{equation}

We claim the following estimate, for some small $\gamma > 0$,  
\begin{equation}
\mathcal{H}(t) \geq N_\Omega(t) + \mu \mathcal{N}_\Omega C(t) - \frac{t^{-7}}{\mu} - \frac{t^{-4\gamma}}{\mu} \| \varepsilon \|^2_{H^1 \times L^2} - \frac{1}{\mu} \| \varepsilon \|^3_{H^1 \times L^2} .
\end{equation}

Note that (5.27) implies (5.24), since \(\bar{\varepsilon}(t) \geq N_\Omega \). To prove (5.27), we decompose $\mathcal{H} = f_1 + f_2 + f_3$, where  
\begin{equation}
f_1 = \int |\nabla \varepsilon|^2 - \sum_k \int f'(W_k) \varepsilon^2 + \int \eta^2 + 2 \int (\chi \partial_{x_1} \varepsilon) \eta ,
\end{equation}

\begin{equation}
f_2 = -2 \int \left( F(W + \varepsilon) - F(W) - f(W) \varepsilon - \frac{1}{2} f'(W) \varepsilon^2 \right) ,
\end{equation}

\begin{equation}
f_3 = \int \left( \sum_k f'(W_k) - f'(W) \right) \varepsilon^2 .
\end{equation}

We claim the following estimates  
\begin{equation}
f_1 \geq N_\Omega + \mu \mathcal{N}_\Omega C - \frac{t^{-7}}{\mu} - \frac{t^{-4\gamma}}{\mu} \| \varepsilon \|^2_{H^1 \times L^2} ,
\end{equation}

\begin{equation}|f_2| + |f_3| \lesssim \| \varepsilon \|^3_{H^1 \times L^2} + t^{-1} \| \varepsilon \|^2_{H^1 \times L^2} ,
\end{equation}

which imply (5.27) for $T_0$ large enough.

**Proof of (5.28).** For $\varphi_\gamma$ defined in (1.7), set  
\[
\varphi_k(t, x) = \varphi_\gamma \left( \frac{x - \ell_k e_1 t - y_k(t)}{\lambda_k(t)} \right) .
\]

We decompose $f_1$ as follows  
\[
f_1 = N_\Omega + \sum_k \left( \int |\nabla \varepsilon|^2 \varphi_k^2 - \int f'(W_k) \varepsilon^2 + \int \eta^2 \varphi_k^2 + 2 \int (\chi \partial_{x_1} \varepsilon) \eta \varphi_k^2 \right) + \int_{\Omega^{C}} \left| \nabla \varepsilon \right|^2 + \eta^2 + 2 \chi (\partial_{x_1} \varepsilon) \eta \left( 1 - \sum_k \varphi_k^2 \right) - \int_{\Omega} \left( |\nabla \varepsilon|^2 + \eta^2 + 2 \chi (\partial_{x_1} \varepsilon) \eta \right) \sum_k \varphi_k^2 + 2 \sum_k \int (\chi - \ell_k) (\partial_{x_1} \varepsilon) \eta \varphi_k^2 = N_\Omega + f_{1,1} + f_{1,2} + f_{1,3} + f_{1,4} .
\]

By Lemma 2.2 (iii), the orthogonality conditions on $\varepsilon$ and a change of variable, we have  
\[
f_{1,1} \geq \mu \int \left( |\nabla \varepsilon|^2 + \eta^2 \right) \left( \sum_k \varphi_k^2 \right) - \frac{1}{\mu} \sum_k \left( \left( z_k^- \right)^2 + \left( z_k^+ \right)^2 \right) .
\]
Thus, using \((5.18)\),
\[
f_{1,1} \geq \mu \int (|\nabla \varepsilon|^2 + \eta^2) \left( \sum_k \varphi_k^2 \right) - \frac{1}{\mu} t^{-7} \geq \mu \int_{\Omega^C} (|\nabla \varepsilon|^2 + \eta^2) \left( \sum_k \varphi_k^2 \right) - \frac{1}{\mu} t^{-7}.
\]

Next, note that if \(x\) is such that \(\varphi_k(t, x) > \frac{1}{2}\), then \(\varphi_k^2(x) \leq t^{-4\gamma}\) for \(k' \neq k\). Thus, the estimate \(1 - \sum_k \varphi_k^2 \leq -t^{-4\gamma}\) holds on \(\mathbb{R}\). By direct computations (with the notation \(v_+ = \max(0, v)\)),
\[
f_{1,2} = 7 \int_{\Omega^C} \left| \frac{1}{7} \partial_x \varepsilon + \eta \right|^2 \left( 1 - \sum_k \varphi_k^2 \right) + \int_{\Omega^C} |\nabla \varepsilon|^2 \left( 1 - \sum_k \varphi_k^2 \right)
+ \int_{\Omega^C} \left( 1 - \frac{\eta^2}{7} \right) |\partial_x \varepsilon|^2 \left( 1 - \sum_k \varphi_k^2 \right) + (1 - 7) \int_{\Omega^C} \eta^2 \left( 1 - \sum_k \varphi_k^2 \right)
\geq (1 - 7) \int_{\Omega^C} (|\nabla \varepsilon|^2 + \eta^2) \left( 1 - \sum_k \varphi_k^2 \right) - \frac{\|\varepsilon\|_{H^1 \times L^2}^2}{t^{4\gamma}}.
\]

Also, we see easily that \(|f_{1,3}| \lesssim t^{-4\gamma} \|\varepsilon\|_{H^1 \times L^2}^2\). Last, by the definition of \(\chi\) in \((5.21)\), the decay property of \(\varphi\) and \((5.18)\) (for a bound on \(y_k\)), we have
\[
\|\chi - \ell_k \varphi_k\|_{L^\infty} \leq t^{-4\gamma}.
\]

Thus, \(|f_{1,4}| \lesssim t^{-4\gamma} \|\varepsilon\|_{H^1 \times L^2}^2\).

In conclusion, for some \(\mu > 0\), and \(T_0\) large enough, it holds
\[
f_{1,1} + f_{1,2} + f_{1,3} + f_{1,4} \geq \mu N_{\Omega^C} - \frac{1}{\mu} t^{-7} - t^{-4\gamma} \|\varepsilon\|_{H^1 \times L^2}^2.
\]

**Proof of \((5.29)\).** Using \((1.8)\), \((1.9)\), \((4.13)\) and \((5.18)\), we have
\[
|f_2| \lesssim \int |\varepsilon|^{\frac{10}{3}} + |\varepsilon|^3 \|W\|^{\frac{1}{3}} \lesssim \|\varepsilon\|_{H^1 \times L^2}^2.
\]

Last, we observe that by \((5.14)\), \(|f_3| \lesssim \|R_1\|_{L^\infty} \|\varepsilon\|_{L^2}^2 \lesssim t^{-1} \|\varepsilon\|_{H^1 \times L^2}^2\).

**Proof of \((5.25)\).** We decompose
\[
\frac{d}{dt} \mathcal{H} = \int \partial_1 \left\{ |\nabla \varepsilon|^2 + |\eta|^2 - 2(F(W + \varepsilon) - F(W) - f(W)\varepsilon) \right\}
+ 2 \int \chi \partial_1 ((\partial_{x_1} \varepsilon)\eta) + 2 \int (\partial_1 \chi)(\partial_{x_1} \varepsilon)\eta = g_1 + g_2 + g_3.
\]

We claim the following estimates
\[
g_1 = 2 \int \varepsilon (-\Delta \text{Mod}_W - f'(W) \text{Mod}_W) + 2 \int \eta \text{Mod}_X
+ 2 \int \left( \sum_k \ell_k \partial_{x_1} W_k \right) (f(W + \varepsilon) - f(W) - f'(W)\varepsilon) + O \left( t^{-7 + \frac{1}{10} + \delta} \right),
\]

\[
g_2 = 2 \int \text{Mod}_X \left( \sum_k \ell_k \partial_{x_1} W_k \right) (\varepsilon f(W + \varepsilon) - \varepsilon f(W) - \varepsilon f'(W)\varepsilon) + O \left( t^{-7 + \frac{1}{10} + \delta} \right),
\]

\[
g_3 = 2 \int \text{Mod}_X \left( \sum_k \ell_k \partial_{x_1} W_k \right) (\eta f(W + \varepsilon) - \eta f(W) - \eta f'(W)\varepsilon) + O \left( t^{-7 + \frac{1}{10} + \delta} \right).
\]
We integrate by parts terms in $g$. Using (4.11) and (5.10),

$$
g_2 = -\frac{1}{(1-2\sigma)t} \int_\Omega (\eta^2 + (\partial_{x_1}\varepsilon) - |\nabla \varepsilon|^2)$$

$$-2 \int \chi \left( \sum_k \partial_{x_1} W_k \right) \left( f(W + \varepsilon) - f(W) - f'(W)\varepsilon \right)$$

$$+ 2 \int (\chi \partial_{x_1} \text{Mod} W) \eta - 2 \int \varepsilon \partial_{x_1} \text{Mod} X + O(t^{-7}),$$

$$g_3 = -\frac{2}{(1-2\sigma)t} \int_\Omega \frac{x_1}{t} (\partial_{x_1}\varepsilon) \eta.$$

**Estimate on $g_1$.** From direct differentiation and integration by parts, we have

$$g_1 = 2 \int (\partial_{\varepsilon})(-\Delta \varepsilon - (f(W + \varepsilon) - f(W))) + 2 \int (\partial_{\eta}) \eta$$

$$- 2 \int (\partial_W) (f(W + \varepsilon) - f(W) - f'(W)\varepsilon).$$

Using (4.11) and (5.10),

$$g_1 = 2 \int (-\Delta \varepsilon - f'(W)\varepsilon) \text{Mod} W + 2 \int \eta \text{Mod} X$$

$$+ 2 \int (-\Delta \varepsilon - f'(W)\varepsilon) R_W + \int \eta R_X$$

$$- 2 \int X (f(W + \varepsilon) - f(W) - f'(W)\varepsilon) = g_{1.1} + g_{1.2} + g_{1.3}$$

We integrate by parts terms in $g_{1.1}$. Next, by (1.8), (1.9), (4.12), (4.13), and (5.18), we obtain

$$|g_{1.2}| \lesssim \|\varepsilon\|_{H^1} \|R_W\|_{H^1} + \|\varepsilon\|_{L^\infty} \|W\|_{L^4} \|R_W\|_{L^\infty} + \|\eta\|_{L^2} \|R_X\|_{L^2} \lesssim t^{-7+\frac{1}{10}+\delta}.$$
Note that by integration by parts and \(5.22\)
\[
2 \int (\chi \partial_x \eta) + 2 \int (\chi \partial_x \varepsilon) \Delta \varepsilon = - \int \partial_x \chi \left( \eta^2 + (\partial_x \varepsilon)^2 - |\nabla \varepsilon|^2 \right) = - \frac{1}{(1 - 2\sigma)t} \int_\Omega (\eta^2 + (\partial_x \varepsilon)^2 - |\nabla \varepsilon|^2).
\]

Next, we observe
\[
\int (\chi \partial_x \varepsilon) (f(W + \varepsilon) - f(W)\varepsilon)
= \chi \partial_x \left( F(W + \varepsilon) - F(W) \right) - \int \chi (\partial_x(W) (f(W + \varepsilon) - f(W) - f'(W)\varepsilon) \right).
\]
Integrating by parts and using \(5.22\),
\[
- \int \chi \partial_x (F(W + \varepsilon) - F(W) - f(W)\varepsilon) = \frac{1}{(1 - 2\sigma)t} \int_\Omega (F(W + \varepsilon) - F(W) - f(W)\varepsilon).
\]
Thus, by \(5.18\) and
\[
\|W\|_{L^\infty(\Omega)} \lesssim \sum_k \left( \|W_k\|_{L^\infty(\Omega)} + \|v_k\|_{H^1} \right) \lesssim t^{-\frac{3}{2}},
\]
we obtain
\[
\left| \int \chi \partial_x \left( F(W + \varepsilon) - F(W) - f(W)\varepsilon \right) \right| \lesssim t^{-\frac{3}{2}} \int_\Omega \left( |\varepsilon|^\frac{4}{3} + W_\frac{4}{3} |\varepsilon|^2 \right) \lesssim t^{-10}.
\]
Second, again by \(3.3\) and \(5.18\)
\[
\left| \int \chi \left( \partial_x W - \partial_x \sum W_k \right) (f(W + \varepsilon) - f(W) - f'(W)\varepsilon) \right|
= \left| \int \chi \partial_x \left( \sum c_k v_k \right) (f(W + \varepsilon) - f(W) - f'(W)\varepsilon) \right| \lesssim \sum_k \int |\partial_x v_k| \left( |\varepsilon|^2 W_\frac{4}{3} + |\varepsilon|^{\frac{3}{2}} \right)
\lesssim \left( \sum_k \|\partial_x v_k\|_{L^\infty} \right) \|\varepsilon\|_{L^{10}}^\frac{1}{3} \lesssim t^{-8 + \frac{3}{2}} \lesssim t^{-7}.
\]
Last, integrating by parts,
\[
2 \int (\chi \partial_x \varepsilon) \text{Mod}_X = -2 \int (\chi \varepsilon) \partial_x \text{Mod}_X + O(t^{-7}),
\]
since by \(5.18\), \(5.22\) and \(5.20\)
\[
\left| \int (\partial_x \chi) \varepsilon \text{Mod}_X \right| \lesssim t^{-4 + \frac{1}{10}} \int_\Omega |\varepsilon| \left( \sum_k |W_k|^{\frac{3}{2}} \right) \lesssim t^{-4 + \frac{1}{10}} \|\varepsilon\|_{L^\infty} \left( \sum_k \|W_k\|_{L^\infty(\Omega)} \right) \lesssim t^{-7}.
\]
Last, we finish the estimate of \(g_2\) by observing that \(4.12\) and \(5.18\) yield
\[
\left| \int (\chi \partial_x \eta) \right| + \left| \int (\chi \partial_x \varepsilon) R_W \right| \lesssim \|\eta\|_{L^2} \|\partial_x \eta\|_{L^2} + \|\varepsilon\|_{L^2} \|R_W\|_{L^2} \lesssim t^{-7 + \frac{1}{10} + \delta}.
\]
Estimate on \(g_3\). This estimate is a direct consequence of \(5.22\).
Gathering the above estimates, we rewrite
\[
\frac{d}{dt} \mathcal{H} = h_1 + h_2 + h_3 + h_4 + O\left(t^{-7+\frac{1}{10}+\delta}\right),
\]
where
\[
h_1 = -\frac{1}{(1-2\sigma)t} \int_{\Omega} \left( \eta^2 + (\partial_x \varepsilon)^2 + 2\frac{x}{t}(\partial_x \varepsilon)\eta - |\nabla \varepsilon|^2 \right),
\]
\[
h_2 = 2 \int \left( \sum_k (\ell_k - \chi) \partial_{x_1} W_k \right) \left( f(W + \varepsilon) - f(W) - f'(W)\varepsilon \right),
\]
\[
h_3 = 2 \int \eta (\text{Mod}_X + \chi \partial_{x_1} \text{Mod}_W),
\]
\[
h_4 = 2 \int \varepsilon \left(-\Delta \text{Mod}_W - \chi \partial_{x_1} \text{Mod}_X - f'(W) \text{Mod}_W \right).
\]

First, by (5.26) and the definition of \(\chi\) in (5.21),
\[
-((1-2\sigma)t)h_1 = 7 \int_{\Omega} \left( \frac{\chi}{t} \partial_{x_1} \varepsilon + \eta \right)^2 + \left( 1 - \frac{x}{t} \right) \left( \partial_x \varepsilon \right)^2 + (1 - 7) \int \eta^2 
+ 2 \int \left( \frac{x}{t} - \chi \right) \left( \partial_x \varepsilon \right) \eta \leq N_\Omega + C\sigma \int \left( |\partial_x \varepsilon|^2 + \eta^2 \right) \leq (1 + C\sigma) N_\Omega.
\]

Second, we observe that by the definition of \(\chi\) in (5.22) and the decay of \(\partial_{x_1} W\) and \(W\),
\[
\| (\ell_k - \chi) \partial_{x_1} W_k \|_{L^{\frac{3}{2}}} \lesssim t^{-\frac{3}{2}}.
\]
Thus, by (1.5), \(|h_2| \lesssim t^{-\frac{5}{2}}\|\varepsilon\|^2_{L^\infty} \lesssim t^{-\frac{15}{2}+\frac{1}{6}} \lesssim t^{-8}.

Denote
\[
M_k = \left( \frac{\lambda_k}{\lambda_k} - \frac{a_k}{\lambda_k t^2} \right) \Lambda W_k + \dot{y}_k \cdot \nabla W_k
\]
so that \(\text{Mod}_W = \sum_k M_k\) and \(\text{Mod}_X = -\sum_k \ell_k \partial_{x_1} M_k\). Using (4.13) and the definition of \(\chi\) (see (5.22)), we have \(\|(\ell_k - \chi) \partial_{x_1} M_k\|_{L^2} \lesssim t^{-\frac{3}{2}+\frac{1}{6}}\). It follows from (5.11)
\[
\|\text{Mod}_X + \chi \partial_{x_1} \text{Mod}_W\|_{L^2} \lesssim t^{-\frac{3}{2}+\frac{1}{6}},
\]
and thus
\[
|h_3| = \left| \int \eta (\text{Mod}_X + \chi \partial_{x_1} \text{Mod}_W) \right| \lesssim t^{-\frac{3}{2}+\frac{1}{6}} \|\eta\|_{L^2} \lesssim t^{-\frac{15}{2}+\frac{1}{6}} \lesssim t^{-7}.
\]

Last, we see that by (i) of Lemma 2.2, \(-\Delta M_k + \ell_k \partial_{x_1}^2 M_k - f'(W_k)M_k = 0\). Thus,
\[
|\Delta M_k + \ell_k \chi \partial_{x_1}^2 M_k - f'(W)M_k| \lesssim |(\chi - \ell_k) \partial_{x_1}^2 M_k| + |f'(W) - f'(W_k)| \|M_k\|.
\]
As before, by \((4.13)\), \(\|(\chi - \ell_k)\partial_x^2 M_k\|_{L^\infty} \lesssim t^{-\frac{3}{2} + \frac{\beta}{16}}\). Moreover, by \((3.3)\) and Claim \(5.4\),

\[
\left|W_k^\frac{1}{7} - W_k^\frac{1}{7}\right| |M_k| \lesssim \left(\sum |W_k| + \sum |v_k|\right)^\frac{1}{7} \left(\sum |W_k'| + \sum |v_k|\right) |W_k|
\]

\[
\lesssim \sum_{k' \neq k''} |W_k'||W_k''| + t^{-\frac{3}{2}} \sum_{k' \neq k''} |W_k'|^2 + t^{-\frac{3}{2}} \sum_{k' \neq k''} |W_k'|^{1 + \frac{2}{7}} |W_k''| + t^{-\frac{3}{2}} \sum_{k'} |W_k'|^{\frac{17}{3}}
\]

and so

\[
\|(f'(W) - f'(W_k)) M_k\|_{L^\infty} \lesssim t^{-2}
\]

Therefore, using \((5.11)\),

\[
\| - \Delta \text{Mod}_W - \chi \partial_x \text{Mod}_x f'(W) \text{Mod}_W\|_{L^\infty} \lesssim t^{-\frac{3}{2} + \frac{\beta}{16}}.
\]

It follows that by \((5.18)\),

\[
\|h_4\| \lesssim \|\varepsilon\|_{L^\infty} \|- \Delta \text{Mod}_W - \chi \partial_x \text{Mod}_x f'(W) \text{Mod}_W\|_{L^\infty} \lesssim t^{-7}.
\]

In conclusion, using \((5.27)\), for \(\sigma\) small, and \(T_0\) large,

\[
- \frac{d}{dt} \mathcal{H} \leq \frac{(1 + C\sigma)}{t} \mathcal{N}_0 + O(t^{-7 + \frac{1}{10} + \delta}) \leq 2 \mathcal{H} + O(t^{-7 + \frac{1}{10} + \delta})
\]

and the proof of Lemma \(5.6\) is complete. \(\square\)

5.4. **Parameters and energy estimates.** The following result, mainly based on Lemma \(5.6\), improves all the estimates in \((5.18)\), except the ones on \((z_k^\pm)\).

**Lemma 5.7** (Closing estimates except \((z_k^\pm)\)). For \(C_0 > 0\) large enough, for all \(t \in [T^*, S_n]\),

\[
\begin{cases}
|\lambda_k(t) - \lambda_k^\infty| \leq \frac{C_0}{2} t^{-1}, & |y_k(t) - y_k^\infty| \leq \frac{C_0}{2} t^{-1}, \\
|z_k^\pm(t)|^2 \leq \frac{1}{2} t^{-7}, & \|\varepsilon(t)\|_{H^1 \times L^2} \leq \frac{1}{2} t^{-3 + \frac{\beta}{16}}.
\end{cases}
\] (5.30)

**Proof. Parameters estimates.** From \((5.11)\) and \((5.18)\), we have \(\|\frac{\lambda_k^\infty}{\lambda_k^\infty}\| + |y_k^\infty| \leq Ct^{-2}\) where the constant \(C\) depends on the parameters of the two solitons, but not on \(C_0\). By integration on \([t, S_n]\) for \(T^* \leq t \leq S_n\), and \((5.17)\), we obtain

\[
|\lambda_k(t) - \lambda_k^\infty| \leq |\lambda_k(t) - \lambda_k(S_n)| + |\lambda_k(S_n) - \lambda_k^\infty| \leq C't^{-1},
\]

and similarly, \(|y_k(t) - y_k^\infty| \lesssim C't^{-1}\), where \(C'\) is also independent of \(C_0\). We choose \(C_0 = 2C'\).

Now, we prove the bound on \(z_k^\pm(t)\). Let \(\beta_k = \lambda_k^\pm(1 - |\ell_k|^2)^{1/2} > 0\). Then, from \((5.12)\) and \((5.18)\),

\[
\frac{d}{dt} \left(e^{-\beta_k t} z_k^\pm\right) \lesssim e^{-\beta_k t} t^{-4 + \frac{\beta}{16}}.
\]

Integrating on \([t, S_n]\) and using \((5.16)\), we obtain \(-z_k^\pm(t) \lesssim t^{-4 + \frac{\beta}{16}}\). Doing the same for \(-e^{-\beta_k t} z_k^\pm\), we obtain the conclusion for \(T_0\) large enough.

**Bound on the energy norm.** To prove the estimate on \(\|\varepsilon(t)\|_{H^1 \times L^2}\), we use Lemma \(5.6\). Recall from \((5.17)\) and then \((5.23)\) that \(\mathcal{H}(S_n) \lesssim S_n^{-7}\). Integrating \((5.23)\) on \([t, S_n]\), we obtain, for all \(t \in [T^*, S_n]\), \(\mathcal{H}(t) \lesssim t^{-6 + \frac{\beta}{16}}\). Using \((5.24)\), we conclude that \(\|\varepsilon\|_{H^1 \times L^2} \lesssim t^{-3 + \frac{\beta}{16} + \frac{\beta}{16}}\).

The estimate follows from the choice \(\delta = \frac{1}{16}\) and for \(T_0\) large enough. \(\square\)
As in [2, 3, 15], the parameters $z_k^-$ require a specific argument.

**Lemma 5.8 (Control of unstable directions).** There exist $(\xi_{k,n})_k \in B_{\mathbb{R}^2}(S_n^{-7/2})$ such that, for $C^* > 0$ large enough, $T^*((\xi_{k,n})_k) = T_0$. In particular, let $(\xi_{n}^+)$ be given by Claim 5.3 from such $(\xi_{k,n})_k$, then the solution $u_n$ of (5.3) satisfies (5.4).

**Proof.** We follow the strategy of Lemma 6 in [2]. The proof is by contradiction, we assume that for any $(\xi_k)_{k \in \{1,2\}} \in B_{\mathbb{R}^2}(S_n^{-7/2})$, $T^*((\xi_k)_k)$ defined by (5.19) satisfies $T^* \in (T_0, S_n)$. In this case, by Lemma 5.7 and continuity, it holds necessarily

$$\sum |z_k^-(T^*)|^2 = \frac{1}{(T^*)^2}.$$

Let $\bar{\beta} = \min_k \beta_k$. From (5.12) and (5.18), for all $t \in [T^*, S_n]$, one has

$$\frac{d}{dt} \left( t^2 (z_k^-)^2 \right) = 2t^2 z_k^- \frac{d}{dt} z_k^- + 7t^6 (z_k^-)^2 \leq -2t^2 \beta_k (z_k^-)^2 + \frac{C}{t} \leq -2\beta t^7 (z_k^-)^2 + \frac{C}{t}.$$

Thus, for $T_0$ large enough, and any $t_0 \in [T^*, S_n]$,

$$\sum |z_k^-(t_0)|^2 = \frac{1}{t_0} \implies \frac{d}{dt} \left( t^2 \sum |z_k^-(t)|^2 \right) \bigg|_{t=t_0} \leq -2\beta + \frac{C}{T_0} \leq -\beta.$$

As a standard consequence of this transversality property, the maps

$$(\xi_k)_k \in B_{\mathbb{R}^2}(S_n^{-7/2}) \mapsto T^*((\xi_k)_k)$$

and

$$(\xi_k)_k \in B_{\mathbb{R}^2}(S_n^{-7/2}) \mapsto M((\xi_k)_k) = \left( \frac{T^*}{S_n} \right)^{7/2} (z_k^-((T^*))_k) \in S_{\mathbb{R}^2}(S_n^{-7/2})$$

are continuous. Moreover, $M$ restricted to $S_{\mathbb{R}^2}(S_n^{-7/2})$ is the identity and this is contradictory with Brouwer’s fixed point theorem.

Estimates (5.4) follow directly from the estimates (5.18) on $\varepsilon(t)$, $\lambda_k(t)$, $y_k(t)$.

5.5. Proof of the $\dot{H}^2 \times \dot{H}^1$ bound. We introduce a functional of energy type for $\partial_x \varepsilon$, for any $j = 1, \ldots, 5$,

$$F_j = \left( |\partial_{x_j} \eta|^2 + |\nabla \partial_{x_j} \varepsilon|^2 - \frac{7}{3} (\partial_{x_j} \varepsilon)^2 \right) (W + \varepsilon)^{\frac{4}{3}}.$$

Note that by (1.9), $\|W\|_{L^\infty} \leq 1$ and (5.18),

$$\int |W|^{\frac{4}{3}} + |\varepsilon|^{\frac{4}{3}} (\partial_{x_j} \varepsilon)^2 \leq \|\varepsilon\|_{H^2}^2 + \|\varepsilon\|_{H^1}^{\frac{4}{3}} \|\partial_{x_j} \varepsilon\|_{H^1}^{\frac{2}{3}} \leq t^{-6+\frac{1}{5}} + t^{-4+\frac{7}{15}} \|\nabla \partial_{x_j} \varepsilon\|_{L^2}^2,$$

and so, for $T_0$ large enough, $F_j \geq \frac{1}{2} \int (|\partial_{x_j} \eta|^2 + |\nabla \partial_{x_j} \varepsilon|^2) - C t^{-6+\frac{1}{5}}$. By (5.20) and (4.12), we rewrite (4.11) and (5.10) as follows

$$\begin{cases}
\partial_t W = X - R_\varepsilon \\
\partial_t X = \Delta W - |W|^{\frac{4}{3}} W - R_\eta \\
\partial_t \varepsilon = \eta + R_\varepsilon \\
\partial_t \eta = \Delta \varepsilon + |W + \varepsilon|^{\frac{4}{3}} (W + \varepsilon) - |W|^{\frac{4}{3}} W + R_\eta
\end{cases}.$$
where $\|R_\eta\|_{H^1 \cap H^2} + \|R_\eta\|_{H^1} \lesssim t^{-3 + \frac{1}{10}}$. We compute

$$
\frac{d}{dt} F_j = \frac{14}{3} \int (\partial_{x_j} \eta)(\partial_{x_j} W) \left( |W + \varepsilon|^\frac{4}{3} - |W|^\frac{4}{3} \right) - \frac{28}{9} \int (X + \eta)(W + \varepsilon)|W + \varepsilon|^{-\frac{2}{3}}(\partial_{x_j} \varepsilon)^2
$$

$$
+ 2 \int (\partial_{x_j} R_\eta)(\partial_{x_j} \eta) + 2 \int (\nabla \partial_{x_j} R_\varepsilon)(\nabla \partial_{x_j} \varepsilon) - \frac{14}{3} \int (\partial_{x_j} R_\varepsilon)(\partial_{x_j} \varepsilon)|W + \varepsilon|^{\frac{4}{3}}.
$$

Thus,

$$
\left| \frac{d}{dt} F_j \right| \lesssim \int |\partial_{x_j} \eta||\partial_{x_j} W| \left( |W|^\frac{4}{3}|\varepsilon| + |\varepsilon|^{\frac{4}{3}} \right) + \int |X + \eta||W + \varepsilon|^\frac{1}{7}(\partial_{x_j} \varepsilon)^2
$$

$$
+ \int |\partial_{x_j} R_\eta||\partial_{x_j} \eta| + \int |\nabla \partial_{x_j} R_\varepsilon||\nabla \partial_{x_j} \varepsilon| + \int |\partial_{x_j} R_\varepsilon||\partial_{x_j} \varepsilon| \left( |W|^\frac{4}{3} + |\varepsilon|^{\frac{4}{3}} \right)
$$

Using Holder inequality (in particular, (1.9)), Sobolev inequality and (5.18), we check the following estimates

$$
\int |\partial_{x_j} \eta||\partial_{x_j} W| \left( |W|^\frac{4}{3}|\varepsilon| + |\varepsilon|^{\frac{4}{3}} \right) \lesssim \|\varepsilon\|_{H^1\cap H^2} \lesssim t^{-3 + \frac{1}{10}} \|\eta\|_{H^1},
$$

$$
\int |X||W|^\frac{1}{7}(\partial_{x_j} \varepsilon)^2 \lesssim \|\varepsilon\|^2_{H^1} \lesssim t^{-6 + \frac{1}{3}},
$$

$$
\int |\eta||W|^\frac{1}{7}(\partial_{x_j} \varepsilon)^2 \lesssim \int |\eta|(\partial_{x_j} \varepsilon)^2 \lesssim \|\eta\|_{L^\infty}^\frac{1}{7} \|\eta\|_{H^2}^\frac{2}{7} \|\varepsilon\|_{H^2}^\frac{2}{7} \lesssim t^{-\frac{6}{7} + \frac{4}{35}} \|\eta\|_{H^1}^\frac{2}{7} \|\varepsilon\|^2_{H^2}
$$

$$
\int |X||\varepsilon|^\frac{1}{7}(\partial_{x_j} \varepsilon)^2 \lesssim \int |\varepsilon|^\frac{1}{7}(\partial_{x_j} \varepsilon)^2 \lesssim \|\varepsilon\|_{L^\infty}^\frac{1}{7} \|\varepsilon\|_{H^2}^\frac{2}{7} \lesssim t^{-\frac{6}{7} + \frac{4}{35}} \|\eta\|_{H^1} \|\varepsilon\|^2_{H^2},
$$

$$
\int |\eta||\varepsilon|^\frac{1}{7}(\partial_{x_j} \varepsilon)^2 \lesssim \|\eta\|_{H^1} \|\eta\|_{H^2} \|\varepsilon\|^2_{H^2} \lesssim t^{-\frac{6}{7} + \frac{4}{35}} \|\eta\|_{H^1} \|\varepsilon\|^2_{H^2},
$$

$$
\int |\partial_{x_j} R_\eta||\partial_{x_j} \eta| + \int |\nabla \partial_{x_j} R_\varepsilon||\nabla \partial_{x_j} \varepsilon| \lesssim t^{-3 + \frac{1}{10}} \left( \|\eta\|_{H^1} + \|\varepsilon\|_{H^2} \right),
$$

$$
\int |\partial_{x_j} R_\varepsilon||\partial_{x_j} \varepsilon| \left( |W|^\frac{4}{3} + |\varepsilon|^{\frac{4}{3}} \right) \lesssim t^{-3 + \frac{1}{10}} \|\varepsilon\|_{H^1} + t^{-3 + \frac{1}{10}} \|\varepsilon\|_{H^2} \lesssim t^{-6 + \frac{1}{3}} + t^{-7 + \frac{2}{35}} \|\varepsilon\|_{H^2}.
$$

We deduce from these estimates that, for $F = \sum_{j=1}^5 F_j$,

$$
\left| \frac{d}{dt} F \right| \lesssim t^{-4} + \|\eta\|^3_{H^1} + \|\varepsilon\|^3_{H^2} \lesssim t^{-4} + |F|^\frac{3}{2}.
$$

By (5.17), we know that $|F_j(S_n)| \lesssim S_n^{-7}$ and thus, by integration, we obtain the uniform bound $|F| \lesssim t^{-3}$ on $[T_0, S_n]$. It follows that $\|\tilde{u}_n\|_{H^2 \times H^1} \lesssim \|\tilde{W}\|_{H^2 \times H^1} + \|\tilde{\varepsilon}\|_{H^2 \times H^1} \lesssim 1$.

5.6. End of the proof of Proposition 5.1. We claim the following property

for all $\nu > 0$, there exists $K > 0$ such that, for all $n \geq n_0$, $\|\tilde{u}_n(T_0)\|_{(H^1 \times L^2)(|x| > K)} < \nu$.

(5.31)

Proof of (5.31). Let $0 < \nu < 1$. First, fix $T_1 > T_0$ independent of $n$ such that from (5.4), $\|\tilde{u}_n(T_1) - \tilde{W}(T_1)\|_{H^1 \times L^2} \lesssim T_1^{-3 + \delta} < \nu$. Second, by (4.13), let $K_1 > 1$ independent of $n$ be such that $\|\tilde{W}(T_1)\|_{(\dot{H}^1 \times L^2)(|x| > K_1)} < \nu$. In particular, it holds $\|\tilde{u}_n(T_1)\|_{(\dot{H}^1 \times L^2)(|x| > K_1)} < \nu$. 
Now, for $0 < \gamma < 1$ and $K \gg 1$ we consider the function $g_K$ defined on $\mathbb{R}^5$ by
\[ g_K(x) = \left(1 + \frac{|x|^2}{K^2 + |x|^2}\right)^{-\gamma}, \]
so that $|\nabla g_K(x)| \leq 2\gamma \frac{g_K(x)}{|x|}$.

Note that for any function $v \in \dot{H}^1$,
\[ \int |\nabla (vg_K)|^2 \leq 2 \int |v|^2 g_K^2 + 2 \int |v|^2 |\nabla g_K|^2 \leq 2 \int |\nabla v|^2 g_K^2 + 8\gamma^2 \int \frac{|vg_K|^2}{|x|^2}. \]

By the Hardy inequality, $\frac{|v g_K|^2}{|x|^2} \lesssim \int |\nabla (vg_K)|^2$ and so we can fix $\gamma > 0$ small, independently of $K$ and $v$, such that
\[ \int |\nabla (vg_K)|^2 \leq 4 \int |\nabla v|^2 g_K^2. \tag{5.32} \]

From now on, $\gamma > 0$ is fixed to such value. Let $K > \max(K_1^2, \nu^{-2/\gamma})$. In particular,
\[ \int |\nabla_{t,x} u_n(T_1)|^2 g_K \lesssim \int |\nabla_{t,x} u_n(T_1)|^2 + g_K(\sqrt{K}) \int |\nabla_{t,x} u_n(T_1)|^2 \lesssim \nu^2. \]

By usual computations using (1.1), one has
\[ \frac{d}{dt} \int \left( (\partial_t u_n)^2 + |\nabla u_n|^2 - \frac{3}{5} |u_n|^{10/3} \right) g_K^{10/3} = -2 \int \left( \nabla (g_K^{10/3}) \cdot \nabla u_n \right) \partial_t u_n. \]

From the expression of $g_K$, one has
\[ \nabla (g_K^{10/3}) = \frac{10}{3} g_K \nabla g_K = \frac{20}{3} \gamma x (K^2 - 1)(1 + |x|^2)^{-1} (K^2 + |x|^2)^{-\gamma - 1} g_K. \]

and so $|\nabla (g_K^{10/3})| \lesssim K^{-2\gamma}$, which implies, by the uniform estimates in (5.4)-(5.5)
\[ \left| \frac{d}{dt} \int \left( (\partial_t u_n)^2 + |\nabla u_n|^2 - \frac{3}{5} |u_n|^{10/3} \right) g_K^{10/3} \right| \lesssim K^{-2\gamma}. \]

Therefore, integrating on $[T_0, T_1]$ and using the properties of the function $g$,
\[ \int \left( (\partial_t u_n)^2 + |\nabla u_n|^2 - \frac{3}{5} |u_n|^{10/3} \right) (T_0) g_K^{10/3} \lesssim \int \left( (\partial_t u_n)^2 + |\nabla u_n|^2 \right) (T_1) g_K^{10/3} + \frac{T_1 - T_0}{K^{2\gamma}} \lesssim \nu^2, \]

by choosing in addition $K$ such that $K^{2\gamma} > \frac{T_1 - T_0}{\nu^2}$.

To finish the proof of (5.31), we recall that $\tilde{u}_n = \tilde{W} + \tilde{\varepsilon}$, where $\tilde{W}$ satisfies (4.13) and $\tilde{\varepsilon}$ satisfies (5.18). In particular, for $K$ large depending on $\nu$, but independent of $n$,
\[ \int \left( (\eta(T_0))^2 + |\nabla \varepsilon(T_0)|^2 \right) \frac{10}{3} g_K^{10/3} \lesssim \int \left( (\partial_t u_n(T_0))^2 + |\nabla u_n(T_0)|^2 \right) \frac{10}{3} g_K^{10/3} + \nu^2, \]

and
\[ \int |u_n(T_0)|^{10/3} \frac{10}{3} g_K^{10/3} \lesssim \int |\tilde{W}(T_0)|^{10/3} \frac{10}{3} g_K^{10/3} + \int |\varepsilon(T_0)|^{10/3} \frac{10}{3} g_K^{10/3} \lesssim \nu + \int |\varepsilon(T_0)|^{10/3} \frac{10}{3} g_K^{10/3}. \]

Therefore,
\[ \int \left( (\eta(T_0))^2 + |\nabla \varepsilon(T_0)|^2 \right) \frac{10}{3} g_K^{10/3} \lesssim \int |\varepsilon(T_0)|^{10/3} \frac{10}{3} g_K^{10/3} + \nu^2. \]
Now, by (5.32),
\[
\int |\varepsilon(T_0)|^{\frac{10}{3}} g_{K}^{\frac{10}{3}} \leq \left( \int |\nabla \varepsilon(T_0) g_{K}|^2 \right)^{\frac{5}{6}} \leq \left( \int |\nabla \varepsilon(T_0)|^2 g_{K}^{\frac{10}{3}} \right)^{\frac{5}{6}} \leq \left( \int |\nabla \varepsilon(T_0)|^2 g_{K}^{\frac{10}{3}} \right)^{\frac{5}{6}} \leq T_0^{-4+\frac{10}{3}} \int |\nabla \varepsilon(T_0)|^2 g_{K}^{\frac{10}{3}}.
\]

Taking $T_0$ larger than a universal constant, we obtain
\[
\int (|\partial_t u_n(T_0)|^2 + |\nabla u_n(T_0)|^2) g_{K}^{\frac{10}{3}} \leq \int (|\eta(T_0)|^2 + |\nabla \varepsilon(T_0)|^2) g_{K}^{\frac{10}{3}} + \nu^2 \leq \nu^2,
\]
and (5.31) follows from the properties of $g_K$. \qed

From the estimates of Proposition 5.2 on $(\tilde{u}_n(T_0))$ and (5.31), it follows that up to the extraction of a subsequence (still denoted by $(\tilde{u}_n)$), the sequence $(\tilde{u}_n(T_0))$ converges to some $(u_0, u_1)^T$ in $\dot{H}^1 \times L^2$ as $n \to +\infty$. Consider the solution $u(t)$ of (1.1) associated to the initial data $(u_0, u_1)^T$ at $t = T_0$. Then, by the continuous dependence of the solution of (1.1) with respect to its initial data in the energy space $\dot{H}^1 \times L^2$ (see e.g. [24] and references therein) and the uniform bounds (5.4), the solution $u$ is well-defined in the energy space on $[T_0, \infty)$.

Recall that we denote by $\lambda_{k,n}$ and $y_{k,n}$ the parameters of the decomposition of $u_n$ on $[T_0, S_0]$. By the uniform estimates in (5.5), using Ascoli’s theorem and a diagonal argument, it follows that there exist continuous functions $\lambda_k$ and $y_k$ such that up to the extraction of a subsequence, $\lambda_{k,n} \to \lambda_k$, $y_{k,n} \to y_k$ uniformly on compact sets of $[T_0, +\infty)$, and on $[T_0, +\infty)$,
\[
|\lambda_k(t) - \lambda_k^\infty| \lesssim t^{-1}, \quad |y_k(t) - y_k^\infty| \lesssim t^{-1}.
\]

Passing to the limit in (5.4) for any $t \in [T_0, +\infty)$, we finish the proof of Proposition 5.1

6. NON-ZERO DISPERSION

In this section, we finish the proof of Theorem 1.1 by proving (1.5). Let $R \gg 1$ to be fixed large enough, $t_R = R^{\frac{15}{2}}$, and $\Sigma_R = \{ (t,x) \in \mathbb{R} \times \mathbb{R}^5 \text{ such that } |x| > R + |t-t_R| \}$. Let $u(t)$ be the solution constructed in Proposition 5.1.

6.1. Approximate cut-off problem. Let $\chi_1 : \mathbb{R}^5 \to \mathbb{R}$ be a smooth radially symmetric function such that $\chi_1 \equiv 1$ for $|y| > 1$ and $\chi_1 \equiv 0$ for $|y| < \frac{1}{R}$. Let $\chi_R(x) = \chi_1(x/R)$. We define $\tilde{u}_R = (u_R, \partial_t u_R)^T$ the solution of (1.1) with the following data at the time $t_R$
\[
u(t_R) = u(t_R) \chi_R, \quad \partial_t u_R(t_R) = \partial_t u(t_R) \chi_R.
\]

Claim 6.1. For large $R$, $\|\tilde{u}_R(t_R)\|_{\dot{H}^{1}\times L^2} \lesssim R^{-\frac{3}{2}}$.

Proof. First, by direct computations, using Hardy inequality, at $t = t_R$,
\[
\int |\nabla u_R|^2 = \int |\nabla u|^2 \chi_R^2 - \int u^2 \chi_R \Delta \chi_R \lesssim \int_{|x| > \frac{R}{2}} |\nabla u|^2 + \int_{\frac{R}{2} < |x| < R} \frac{u^2}{|x|^2} \lesssim \int_{|x| > \frac{R}{2}} \left( |\nabla W|^2 + \frac{W^2}{|x|^2} \right) + \|\nabla (u - W)\|_{L^2}^2.
\]
Note that by (4.13), we have, for $|x| > \frac{R}{2}$ and $t = t_R \ll R$,
\[ |\nabla W(t_R)|^2 + \frac{W^2(t_R)}{|x|^2} \lesssim |x|^{-8} + t_R^{-2}|x|^{-8+\frac{3}{16}}, \]
and so,
\[ \int_{|x| > \frac{R}{2}} \left( |\nabla W(t_R)|^2 + \frac{W^2(t_R)}{|x|^2} \right) \lesssim R^{-3} + t_R^{-2}R^{-3+\frac{3}{16}} \lesssim R^{-3}. \]
Using also (5.1), $\int |\nabla u_R(t_R)|^2 \lesssim R^{-3} + t_R^{-6+\frac{1}{4}} \lesssim R^{-3}$.

The estimate for $\|\partial_t u(t)\|_{L^2}$ is similar and easier. Indeed, at $t = T_R$,
\[ \int |\partial_t u_R|^2 = \int |\partial_t u|^2 \lesssim \int_{|x| > \frac{R}{2}} |\partial_t u|^2 \lesssim \int_{|x| > \frac{R}{2}} |X|^2 + \|\partial_t u - X\|^2_{L^2}. \]
By (4.13), for $|x| > \frac{R}{2}$ and $t = t_R \ll R$, it holds $|X(t_R)|^2 \lesssim |x|^{-8} + t_R^{-4}|x|^{-6+\frac{1}{4}}$, and so $\int_{|x| > \frac{R}{2}} |X(t_R)|^2 \lesssim R^{-3}$. Using also (5.1), $\int |\partial_t u_R(t_R)|^2 \lesssim R^{-3} + t_R^{-6+\frac{1}{4}} \lesssim R^{-3}$. □

Using this claim, by the small data Cauchy theory, for $R$ large enough, the solution $\tilde{u}_R$ is global and bounded in $\dot{H}^1 \times L^2$. Moreover, since $\tilde{u}_R(t) = u(t_R)|_{\Sigma}$, for $|x| > R$, by the property of finite speed of propagation of the wave equation, we can define globally $\tilde{u}(t, x)$ on $\Sigma_R$ by setting $u(t, x) = u_R(t, x)$. This extension makes sense even if $u(t)$ is not global in $\dot{H}^1 \times L^2$ in negative times. We will prove in this section the following statement, for $R$ large,
\[ \lim_{t \to -\infty} \|\nabla u(t)\|_{L^2(|x| > R + |t-t_R|)} \gtrsim R^{-\frac{7}{4}}, \tag{6.1} \]
which implies, for $A = R + t_R$ large enough,
\[ \lim_{t \to -\infty} \|\nabla u(t)\|_{L^2(|x| > |t| + A)} \gtrsim A^{-\frac{5}{2}}. \]

6.2. Reduction to a linear problem. We define $\tilde{u}_L = (u_L, \partial_t u_L)^T$ the (global) solution of the 5D linear wave equation with initial data at $t = t_R$,
\[ \begin{cases} \partial_t^2 u_L - \Delta u_L = 0 & \text{on } \mathbb{R} \times \mathbb{R}^5, \\ u_L(t_R) = u_R(t_R) = u(t_R)\chi_R, \quad \partial_t u_L(t_R) = \partial_t u_R(t_R) = \partial_t u(t_R)\chi_R & \text{on } \mathbb{R}^5. \end{cases} \tag{6.2} \]
Using Claim 6.1 and Proposition 2.12 it follows that for $R$ large enough,
\[ \sup_{t \in \mathbb{R}} \|\tilde{u}_L - \tilde{u}_R\|_{\dot{H}^1 \times L^2} \lesssim R^{-\frac{7}{4}}, \quad R^{-\frac{7}{4}} = R^{-\frac{7}{4}}. \tag{6.3} \]

Therefore it suffices to prove (6.1) on $\tilde{u}_L$ instead of $\tilde{u}_R$.

We prove a similar result for truncations of solitons. For any fixed $\ell \in \mathbb{R}^5, |\ell| < 1, \lambda > 0, \ y \in \mathbb{R}^5$ and $\epsilon = \pm 1$, set $\beta = (\ell, \lambda, y, \epsilon)$. Denote
\[ w_{\beta}(t, x) = \frac{\epsilon}{\lambda^\frac{1}{3}} W_\ell \left( \frac{x - \ell t - y}{\lambda} \right), \quad \tilde{w}_\beta = \left( \begin{array}{c} w_{\beta} \\ \partial_t w_{\beta} \end{array} \right). \]
Define also $\tilde{w}_{\beta, R} = (w_{\beta, R}, \partial_t w_{\beta, R})^T$ the solution of (1.1) with truncated data at $t_R$
\[ w_{\beta, R}(t_R) = w_{\beta}(t_R)\chi_R, \quad \partial_t w_{\beta, R}(t_R) = \partial_t w_{\beta}(t_R)\chi_R, \]
and $\tilde{w}_{\beta, L} = (w_{\beta, L}, \partial_t w_{\beta, L})^T$ the solution of the 5D linear wave equation with data at $t_R$
\[ w_{\beta, L}(t_R) = w_{\beta, R}(t) = w_{\beta}(t)\chi_R, \quad \partial_t w_{\beta, L}(t_R) = \partial_t w_{\beta, R}(t) = \partial_t w_{\beta}(t)\chi_R. \]
We claim the following on \( \tilde{w}_{\beta,L} \).

**Claim 6.2.** For any \( R \) large enough, for all \( t \in \mathbb{R} \),

\[
\| \nabla w_{\beta,L}(t) \|_{L^2(|x|>R+|t-t_R|)} + \| \partial_t w_{\beta,L}(t) \|_{L^2(|x|>R+|t-t_R|)} \lesssim (R+|t|)^{-\frac{3}{2}} + R^{-\frac{3}{2}}.
\]

**Proof.** First, as in the proof of Claim 6.1 we see that \( \| \tilde{w}_{\beta,R}(t_R) \|_{\dot{H}^1 \times L^2} \lesssim R^{-\frac{3}{2}} \). In particular, for \( R \) large enough, the solution \( \tilde{w}_{\beta,R} \) is global in \( \dot{H}^1 \times L^2 \) and, by Proposition 2.12

\[
\sup_{t \in \mathbb{R}} \| \tilde{w}_{\beta,L} - \tilde{w}_{\beta,R} \|_{\dot{H}^1 \times L^2} \lesssim R^{-\frac{3}{2}} = R^{-\frac{3}{2}}.
\] (6.4)

Second, by direct computations, we see that for all \( t \)

\[
\| \nabla \beta(t) \|_{L^2(|x|>R+|t-t_R|)} + \| \partial_t \beta(t) \|_{L^2(|x|>R+|t-t_R|)} \lesssim \int_{|x|>R+|t-t_R|} |x-\ell t|^{-8} \, dx
\]

\[
\lesssim \int_{|y|>R+|t-t_R|-|\ell||t|} |y|^{-8} \, dy \lesssim \int_{r>R-|t_R|+(1-|\ell||t|)} r^{-4} \, dr \lesssim (R+|t|)^{-3},
\]

where we have used \( t_R = \frac{H}{4 R} \ll R \). For \( x \in \Sigma_R \), \( w_{\beta} \) and \( w_{\beta,R} \) coincide by finite speed of propagation. \( \Box \)

**6.3. Reduction to a radial linear problem.** To use the method of channels of energy, we work on a radial solution. Since the solitons \( W_1 \) and \( W_2 \) at time \( t = t_R \) are not centered at \( x = 0 \), we remove their contribution from the linear solution \( U_L \) before reducing to a radial problem using Claim 6.2.

For \( k = 1, 2 \), set

\[
\beta_k = (\ell_k, \lambda_k(t_R), y_k(t_R), \epsilon_k) \quad \text{so that} \quad \tilde{W}_k(t_R,x) = \tilde{w}_{\beta_k}(t_R).
\]

In view of Lemma 2.5, we introduce the radial solution \( U_L \) of the 5D linear wave equation, defined by, for all \( t, x \in \mathbb{R}^5, r = |x| \),

\[
U_L(t,x) = \int_{|y|=|x|} \left( U_L - \sum_k w_{\beta_k,L} \right) (t,y) d\omega(y), \quad \tilde{U}_L = \left( U_L \partial_t U_L \right).
\] (6.5)

Our goal is to apply Proposition 2.9 to \( \tilde{U}_L \). By (1.4),

\[
\Psi = \frac{(1-\ell_1^2)^{\frac{3}{2}}(1-\ell_2^2)^{\frac{3}{2}}}{|\ell_1 - \ell_2|^3} |\lambda_1^\infty \lambda_2^\infty | e_1 (\lambda_1^\infty)^{\frac{1}{2}} + e_2 (\lambda_2^\infty)^{\frac{1}{2}} \neq 0.
\]

**Lemma 6.3.** For \( R \) large enough, it holds

\[
\| \pi_R \tilde{U}_L(t_R) \|_{\dot{H}^1 \times L^2(|x|>R)} \gtrsim \Psi^2 R^{-5}.
\]

**Proof.** Define the radial function \( V_L \) as follows

\[
V_L(x) = \int_{|y|=|x|} \sum_k c_k v_k(t_R,y) d\omega(y), \quad \tilde{V}_L = \left( V_L \partial_t V_L \right).
\]

We claim the following result on \( V_L \).

**Claim 6.4.** For \( R \) large enough, it holds

\[
\| \pi_H(V_L,0) \|_{\dot{H}^1 \times L^2(|x|>R)} \gtrsim \Psi^2 R^{-5}.
\]
Proof of Claim 6.4: By the definition of $v_k$ in (4.8), we have
\[ v_k(t_R, x) = \lambda_k^{-3}(t_R) v_{\ell_k} \left( \frac{t_R}{\lambda_k(t_R)}, \frac{x - y_k(t_R)}{\lambda_k(t_R)} \right). \]

Similarly, set
\[ v_k^\sharp(t_R, x) = (\lambda_k^\infty)^{-3} v_{\ell_k} \left( \frac{t_R}{\lambda_k}, \frac{x}{\lambda_k} \right), \quad \hat{V}_L^\sharp(t, x) = -\frac{3}{2} (15)^{\frac{3}{2}} \int_{|y|=r} \kappa_{\ell_k} c_k v_k^\sharp(t_R, y) d\omega(y) \]
and
\[ \tilde{v}_k^\sharp(t_R, x) = (\lambda_k(t_R))^{-3} v_{\ell_k} \left( \frac{t_R}{\lambda_k(t_R)}, \frac{x - y_k(t_R)}{\lambda_k(t_R)} \right), \]
where $v_{\ell_k}$ is defined in (3.25). By (3.29), we have
\[ \int_{|x|>R} \left| \nabla \left( v_k + \frac{3}{2} (15)^{\frac{3}{2}} \kappa_{\ell_k} \tilde{v}_k^\sharp \right) \right|^2 dx \lesssim t_R^{-4} R^{-3+2\delta} = R^{-\frac{20}{3}+2\delta} \lesssim R^{-6}. \]
Moreover, from $|\lambda_k(t_R) - \lambda_k^\infty| \lesssim t_R^{-1}$, $|y_k| \lesssim 1$ and (3.29), we check that
\[ \int_{|x|>R} \left| \nabla \left( v_k^\sharp - \tilde{v}_k^\sharp \right) \right|^2 dx \lesssim R^{-6}. \]
It follows that, for $R$ large,
\[ \int_{|x|>R} \left| \nabla \left( v_k + \frac{3}{2} (15)^{\frac{3}{2}} \kappa_{\ell_k} \hat{V}_L^\sharp \right) \right|^2 dx \lesssim R^{-6}, \]
and thus
\[ \int_{|x|>R} \left| \nabla \left( \hat{V}_L - V_L^\sharp \right) \right|^2 dx \lesssim R^{-6}. \]

Let $\phi_{\ell_k}$ be defined as in (3.27) for $\ell = \ell_k$, i.e.
\[ \phi_k(t, r) = (\lambda_k^\infty)^{-2} \phi_{\ell_k} \left( \frac{t}{\lambda_k^\infty}, \frac{r}{\lambda_k^\infty} \right) \]
and
\[ \phi(t, r) = \frac{3}{2} (15)^{\frac{3}{2}} \sum_k \kappa_{\ell_k} c_k \phi_k(t, r) = r^{-1} \partial_r \left( r^3 \hat{V}_L^\sharp(t, r) \right). \]

Then, from (3.28) and the definition of $\kappa_{\ell}$ in (3.1), for $r > R$,
\[ \phi(t_R, r) = -\frac{3}{2} (15)^{\frac{3}{2}} \frac{\langle W^3, \Lambda W \rangle}{\langle W, W \rangle} r^{-3} + O(r^{-1} t_R^{-\frac{3}{2}}), \quad \Psi = \sum_k (1 - \ell_k^2)^{\frac{3}{2}} c_k \lambda_k^\infty. \]
Using the values of $c_1$, $c_2$ from (4.5), we see that $\Psi \neq 0$ under the assumption (1.4). In particular, for $R$ large enough,
\[ \int_{r>R} \phi^2(t_R, r) dr \geq C \Psi^2 R^{-5} - C' R^{-1} t_R^{-\frac{3}{2}} \gtrsim \Psi^2 R^{-5}. \]
From Remark 2.10 we have
\[ \| \pi_R(V_L^\sharp, 0) \|^2_{(H^1 \times L^2)(|x|>R)} = \int_{r>R} \phi^2(t_R, r) dr \]
which finishes the proof of the claim. \qed
For \(|x| > R\), we have \(\bar{u}_L(t_R, x) = \bar{u}(t_R, x)\) and \(\bar{w}_{\beta_k, L}(t_R) = \bar{W}_k(t_R, x)\) and thus, by the definition of \(\bar{W} = \sum_k (\bar{W}_k + c_k \bar{v}_k)\), one has using Proposition 5.1

\[
\left\| \left( \bar{u}_L - \sum \bar{w}_{\beta_k, L} \right) (t_R) - \sum c_k \bar{v}_k (t_R) \right\|_{(H^1 \times L^2)(|x| > R)} = \left\| \left( \bar{u} - \sum \bar{W}_k \right) (t_R) - \left( \bar{W} - \sum W_k \right) (t_R) \right\|_{(H^1 \times L^2)(|x| > R)} \\
= \left\| \bar{u}(t_R) - \bar{W}(t_R) \right\|_{(H^1 \times L^2)(|x| > R)} \lesssim t_R^{-\frac{6+\frac{1}{5}}{3}} = R^{-\frac{31}{60}}.
\]

Thus, \(\left\| \bar{U}_L(t_R) - \bar{W}_L \right\|_{(H^1 \times L^2)(|x| > R)} \lesssim R^{-\frac{31}{60}}\), which, combined with Claim 6.3 finishes the proof of the lemma.

\[\square\]

### 6.4 Channels of energy

We finish the proof of Theorem 1.1. Using Lemma 6.3 and applying Proposition 2.9 to the function \(\bar{U}_L\), we find that for \(R\) large enough, either

\[
\liminf_{t \to -\infty} \left\| \bar{U}_L(t) \right\|_{(H^1 \times L^2)(|x| > R + |t|)} \gtrsim R^{-\frac{5}{2}},
\]

or

\[
\liminf_{t \to +\infty} \left\| \bar{U}_L(t) \right\|_{(H^1 \times L^2)(|x| > R + |t|)} \gtrsim R^{-\frac{5}{2}}.
\]

Now, we transfer this information back to \(u(t)\), using \(u_L(t)\). By the definition of \(\bar{U}_L\) in (6.3) and Claim 6.2 we have

\[
\left\| \bar{u}_L(t) \right\|_{(H^1 \times L^2)(|x| > R + |t|)} \geq \left\| \bar{U}_L(t) \right\|_{(H^1 \times L^2)(|x| > R + |t|)} - \sum \left\| \bar{w}_{\beta_k, L}(t) \right\|_{(H^1 \times L^2)(|x| > R + |t|)}
\]

\[
\geq \left\| \bar{U}_L(t) \right\|_{(H^1 \times L^2)(|x| > R + |t|)} - C_1 R^{-\frac{5}{2}} - C_2 (R + |t|)^{-\frac{3}{2}}.
\]

Thus, either

\[
\liminf_{t \to +\infty} \left\| \bar{u}_L(t) \right\|_{(H^1 \times L^2)(|x| > R + |t|)} \gtrsim R^{-\frac{5}{2}},
\]

or

\[
\liminf_{t \to -\infty} \left\| \bar{u}_L(t) \right\|_{(H^1 \times L^2)(|x| > R + |t|)} \gtrsim R^{-\frac{5}{2}}.
\]

By (6.3), it follows that, for large \(R\), either

\[
\liminf_{t \to +\infty} \left\| \bar{u}(t) \right\|_{(H^1 \times L^2)(|x| > R + |t|)} \gtrsim R^{-\frac{5}{2}} \quad \text{or} \quad \liminf_{t \to -\infty} \left\| \bar{u}(t) \right\|_{(H^1 \times L^2)(|x| > R + |t|)} \gtrsim R^{-\frac{5}{2}}.
\]

Moreover, by (5.1) and (4.13), we have, for any large \(R\),

\[
\lim_{t \to +\infty} \left\| \bar{u}(t) \right\|_{(H^1 \times L^2)(|x| > R + |t|)} = 0.
\]

Therefore, from Remark 2.11, we have both, for large \(R\),

\[
\liminf_{t \to -\infty} \left\| \nabla u(t) \right\|_{L^2(|x| > R + |t|)} \gtrsim R^{-\frac{5}{2}} \quad \text{and} \quad \liminf_{t \to -\infty} \left\| \partial_t u(t) \right\|_{L^2(|x| > R + |t|)} \gtrsim R^{-\frac{5}{2}}.
\]
7. Extensions to the case $K \geq 3$

7.1. Collinear speeds. For $K \geq 3$ collinear speeds $\ell_k = \ell_k e_1$ where $-1 < \ell_1 < \cdots < \ell_K < 1$, the existence of a multi-soliton at $+\infty$ is proved in [35]. The method used in the present paper to prove Theorem 1.1 can be extended to this case, using a refined approximate solution $\tilde{W}$ of the form $\tilde{W} = \sum_{k=1}^K (\tilde{W}_k + c_k \tilde{v}_k)$. Similarly as in Lemma 6.3, for $j, k \in \{1, \ldots, K\}$ with $j \neq k$, define

$$
\Psi_{j,k} = \frac{(1 - \ell_j^2)^{\frac{3}{2}}(1 - \ell_k^2)^{\frac{3}{2}}}{|\ell_j - \ell_k|} \lambda_j^\infty \lambda_k^\infty \left( \epsilon_j (\lambda_j^\infty)^{\frac{3}{2}} + \epsilon_k (\lambda_k^\infty)^{\frac{3}{2}} \right).
$$

Then, the collision is inelastic under the non-vanishing condition $\sum_{j \neq k} \Psi_{j,k} \neq 0$. Note that this condition is Lorentz invariant since using the notation of 2.5 for any $-1 < \beta < 1$,

$$
\frac{(1 - \ell_j^2)^{\frac{3}{2}}(1 - \ell_k^2)^{\frac{3}{2}}}{|\ell_j - \ell_k|} = \frac{(1 - \ell_j^2)^{\frac{3}{2}}(1 - \ell_k^2)^{\frac{3}{2}}}{|\ell_j - \ell_k|^3}.
$$

7.2. Non-collinear speeds. The arguments in [35] do not apply to $K \geq 3$ for non-collinear speeds. However, under the smallness condition $|\ell_k| < \frac{2}{3}$, the existence of a multi-soliton with speeds $\{\ell_k\}_{1 \leq k \leq K}$ can be proved using a refined approximate solution similar to the function $W$ defined in §4 and a variant of the energy estimates of §5.3. Actually, any further improvement in the approximate solution $\tilde{W}$ would lead to a existence result with a weaker condition on the speeds. Inelasticity of the collisions then holds under the following general non-vanishing condition

$$
\sum_{k=1}^K (1 - |\ell_k|^2)^{\frac{3}{2}} c_k \lambda_k^\infty \neq 0,
$$

where the coefficients $c_k$ are explicitly defined in Lemma 4.1.

Appendix A. End of the proof of Lemma 3.3

By Lemmas 2.5 and 2.7 we have

$$
\phi_\ell(t, r) = \frac{1}{2} \int_0^{+\infty} \int_{|r - \sigma|} x \cdot \nabla \left( f_\ell^3 + g_\ell^3 \right) + 3 \left( f_\ell^3 + g_\ell^3 \right) (t + \sigma) d\omega(x) \ d\sigma.
$$

Computation of $x \cdot \nabla \left( f_\ell^3 + g_\ell^3 \right) + 3 \left( f_\ell^3 + g_\ell^3 \right)$. We compute

$$
x \cdot \nabla f_\ell^3 = -3t^{-3} \left( \frac{x_1 (x_1 - \ell t)}{1 - \ell^2} + |\bar{x}|^2 \right) \langle x_\ell \rangle^{-5}
$$

$$
= -3t^{-3} \left( \frac{(x_1 - \ell t)^2}{1 - \ell^2} + |\bar{x}|^2 \right) \langle x_\ell \rangle^{-5} - \frac{3\ell}{1 - \ell^2} t^{-2} (x_1 - \ell t) \langle x_\ell \rangle^{-5}
$$

$$
= -3t^{-3} (\langle x_\ell \rangle)^{-3} + 3t^{-3} (\langle x_\ell \rangle)^{-5} - \frac{3\ell}{1 - \ell^2} t^{-2} (x_1 - \ell t) \langle x_\ell \rangle^{-5},
$$

and so

$$
x \cdot \nabla f_\ell^3 + 3 f_\ell^3 = 3t^{-3} \langle x_\ell \rangle^{-5} - \frac{3\ell}{1 - \ell^2} t^{-2} (x_1 - \ell t) \langle x_\ell \rangle^{-5}.
$$
Next, \( g_\ell^* = -\frac{3\ell}{1 - \ell^2} t^{-2} (x_1 - \ell t) \langle x_\ell \rangle^{-5} \),

\[
x \cdot \nabla g_\ell^* = -\frac{3\ell}{1 - \ell^2} t^{-2} \left( x_1 \langle x_\ell \rangle^{-5} - \frac{5}{1 - \ell^2} x_1 (x_1 - \ell t)^2 \langle x_\ell \rangle^{-7} - 5 (x_1 - \ell t) |x|^2 \langle x_\ell \rangle^{-7} \right)
\]

\[
= -\frac{3\ell}{1 - \ell^2} t^{-2} \left( (x_1 - \ell t) \langle x_\ell \rangle^{-5} - 5 (x_1 - \ell t) \left( \frac{(x_1 - \ell t)^2}{1 - \ell^2} + |x|^2 \right) \langle x_\ell \rangle^{-7} \right)
\]

\[
+ \ell t \langle x_\ell \rangle^{-5} - \frac{5\ell t}{1 - \ell^2} (x_1 - \ell t)^2 \langle x_\ell \rangle^{-7} \right)
\]

\[
= \frac{12\ell}{1 - \ell^2} t^{-2} (x_1 - \ell t) \langle x_\ell \rangle^{-5} - \frac{15\ell t}{1 - \ell^2} t^{-2} (x_1 - \ell t) \langle x_\ell \rangle^{-7} - \frac{3\ell^2}{1 - \ell^2} t^{-1} (x_1 - \ell t)^{-5}
\]

\[
+ \frac{15\ell^2}{(1 - \ell^2)^2} t^{-1} (x_1 - \ell t)^{-5},
\]

and

\[
x \cdot \nabla g_\ell^* + 3g_\ell^* = \frac{3\ell}{1 - \ell^2} t^{-2} (x_1 - \ell t) \langle x_\ell \rangle^{-5} - \frac{15\ell t}{1 - \ell^2} t^{-2} (x_1 - \ell t) \langle x_\ell \rangle^{-7}
\]

\[
- \frac{3\ell^2}{1 - \ell^2} t^{-1} \langle x_\ell \rangle^{-5} + \frac{15\ell^2}{(1 - \ell^2)^2} t^{-1} (x_1 - \ell t)^{-5}.
\]

Summing up, we find

\[
x \cdot \nabla \left( f_\ell^* + g_\ell^* \right) + 3 \left( f_\ell^* + g_\ell^* \right) = 3t^{-3} \langle x_\ell \rangle^{-5} - \frac{15\ell t}{1 - \ell^2} t^{-2} (x_1 - \ell t) \langle x_\ell \rangle^{-7}
\]

\[
- \frac{3\ell^2}{1 - \ell^2} t^{-1} \langle x_\ell \rangle^{-5} + \frac{15\ell^2}{(1 - \ell^2)^2} t^{-1} (x_1 - \ell t)^{-5}.
\]

**Case \( \ell = 0 \).** In this case, we claim that, for \( 1 \ll t \leq r^{\frac{1}{2}} \), \( \phi_0(t, r) = r^{-3} + O(r^{-1} t^{-\frac{3}{2}}) \). Note that \( x \cdot \nabla f_0^* + 3f_0^* = 3t^{-3} \langle x \rangle^{-5} \) and \( g_0^* = 0 \). Thus,

\[
\phi(t, r) = \frac{3}{2} \int_0^{+\infty} (t + \sigma)^{-3} \int_{|\sigma|}^{+\sigma} a \left( \frac{\langle x \rangle^{-5}}{|x| = a} \right) d\omega(x) d\sigma
\]

\[
= \frac{3}{2} \int_0^{+\infty} (t + \sigma)^{-3} \left( \int_{|\sigma|}^{+\sigma} \frac{a}{(1 + a^2)^{\frac{7}{2}}} d\sigma \right) = \frac{1}{2} \int_0^{+\infty} (t + \sigma)^{-3} \left[ (1 + a^2)^{-\frac{3}{2}} \right]_{|\sigma|}^{+\sigma} d\sigma
\]

\[
= \frac{1}{2} \int_0^{+\infty} (t + \sigma)^{-3} \left( 1 + (r - \sigma)^2 \right)^{-\frac{3}{2}} d\sigma - \frac{1}{2} \int_0^{+\infty} (t + \sigma)^{-3} \left( 1 + (r + \sigma)^2 \right)^{-\frac{3}{2}} d\sigma.
\]

First, we estimate

\[
\int_0^{+\infty} (t + \sigma)^{-3} \left( 1 + (r + \sigma)^2 \right)^{-\frac{3}{2}} d\sigma \lesssim r^{-3} \int_0^{+\infty} (t + \sigma)^{-3} d\sigma \lesssim r^{-3} t^{-2}.
\]

Second, we compare

\[
\left| \int_0^{+\infty} (t + \sigma)^{-3} \left( 1 + (r - \sigma)^2 \right)^{-\frac{3}{2}} d\sigma - (t + r)^{-3} \int_0^{+\infty} (1 + (r - \sigma)^2)^{-\frac{3}{2}} d\sigma \right|
\]

\[
\lesssim r^{-1} t^{-3} \int_0^{+\infty} |r - \sigma| (1 + (r - \sigma)^2)^{-\frac{3}{2}} d\sigma \lesssim r^{-1} t^{-3} \int_0^{+\infty} |\sigma| (1 + (\sigma')^2)^{-\frac{3}{2}} d\sigma' \lesssim r^{-1} t^{-3}.
\]
Third, we compute
\[ \int_0^{+\infty} (1 + (r - \sigma)^2)^{-\frac{3}{2}} d\sigma = \int_{-\infty}^{+\infty} (1 + \sigma^2)^{-\frac{3}{2}} d\sigma = 2 + O(r^{-2}). \]

In conclusion, we have obtained, for \( r, t \) large, with \( t < r \frac{11}{12} \),
\[ \phi(t, x) = r^{-3} + O(r^{-1} t^{-3}) + O(r^{-4} t) = r^{-3} + O(r^{-1} t^{-\frac{9}{2}}). \]

From now on, we focus on the case \( 0 < \ell < 1 \).

**Rewriting** \( x \cdot \nabla \left( f_\ell^x + g_\ell^x \right) + 3 \left( f_\ell^x + g_\ell^x \right) \). First, we compute \( \Delta(\langle x_\ell \rangle^{-3}) \). We have
\[ \partial_{x_1}(\langle x_\ell \rangle^{-3}) = -\frac{3}{1 - \ell^2} (x_1 - \ell t) \langle x_\ell \rangle^{-5}, \]
\[ \partial_{x_1}^2(\langle x_\ell \rangle^{-3}) = -\frac{3}{1 - \ell^2} \langle x_\ell \rangle^{-5} + \frac{15}{(1 - \ell^2)^2} (x_1 - \ell t)^2 \langle x_\ell \rangle^{-7}, \]
and
\[ \tilde{\Delta}(\langle x_\ell \rangle^{-3}) = -12 \langle x_\ell \rangle^{-5} + 15|\bar{x}|^2 \langle x_\ell \rangle^{-7}. \]

Thus,
\[ \Delta(\langle x_\ell \rangle^{-3}) = -15 \langle x_\ell \rangle^{-5} + 3 \left( 1 - \frac{1}{1 - \ell^2} \right) \langle x_\ell \rangle^{-5} + 15 \left( \frac{(x_1 - \ell t)^2}{(1 - \ell^2)^2} + |\bar{x}|^2 \right) \langle x_\ell \rangle^{-7}, \]

which we rewrite as follows
\[ \Delta(\langle x_\ell \rangle^{-3}) = -15 \langle x_\ell \rangle^{-5} - \frac{3 \ell^2}{1 - \ell^2} \langle x_\ell \rangle^{-5} + 15 \left( \langle x_\ell \rangle^2 - 1 - \frac{(x_1 - \ell t)^2}{(1 - \ell^2)^2} \right) \langle x_\ell \rangle^{-7} \]
\[ = -3 \frac{\ell^2}{1 - \ell^2} \langle x_\ell \rangle^{-5} + 15 \frac{\ell^2}{(1 - \ell^2)^2} (x_1 - \ell t)^2 \langle x_\ell \rangle^{-7} - 15 \langle x_\ell \rangle^{-7}. \]

We rewrite
\[ x \cdot \nabla \left( f_\ell^x + g_\ell^x \right) + 3 \left( f_\ell^x + g_\ell^x \right) = f^I + f^{\text{II}} + f^{\text{III}}, \]
where
\[ f^I(t, x) = t^{-1} \Delta(\langle x_\ell \rangle^{-3}), \quad f^{\text{II}}(t, x) = 15 t^{-1} \langle x_\ell \rangle^{-7}, \]
\[ f^{\text{III}}(t, x) = 3 t^{-3} \langle x_\ell \rangle^{-5} - 15 \frac{\ell}{1 - \ell^2} t^{-2} (x_1 - \ell t) \langle x_\ell \rangle^{-7}, \]
and set
\[ \phi^{\text{I,II,III}}(t, r) = \frac{1}{2} \int_0^{+\infty} \int_{|r-\sigma|}^{r+\sigma} a \left( \int_{|x|=a} f^{\text{I,II,III}}(t + \sigma, x) d\omega(x) \right) d\sigma. \]

**Computation of \( \phi^I \).** It is a standard fact that for a smooth function \( h \),
\[ \int_{|x|=r} \Delta h(x) d\omega(x) = \left( \frac{d^2}{dr^2} + \frac{4}{r} \frac{d}{dr} \right) \left( \int_{|x|=r} h(x) d\omega(x) \right) \]
\[ = \frac{1}{r} \frac{d}{dr} \left( r^{-2} \frac{d}{dr} \left( r^3 \int_{|x|=r} h(x) d\omega(x) \right) \right). \]
We set $N(x) = \langle x \rangle^{-3}$, $N_\ell(t, x) = N(x_\ell)$, $M(x) = \langle x \rangle^{-5}$, $M_\ell(t, x) = M(x_\ell)$, $K(x) = \langle x \rangle^{-7}$ and $K_\ell(t, x) = K(x_\ell)$. We have

\[
\phi^1(t, r) = \frac{1}{2} \int_0^{+\infty} (t + \sigma)^{-1} \int_{|r - \sigma|}^{r + \sigma} \frac{d}{da} \left( a^{-2} \frac{d}{da} \left( a^3 \int_{|x| = a} N_\ell(t + \sigma, x) d\omega(x) \right) \right) d\sigma \\
= \frac{1}{2} \int_0^{+\infty} (t + \sigma)^{-1} \left[ a^{-2} \frac{d}{da} \left( a^3 \int_{|x| = a} N_\ell(t + \sigma, x) d\omega(x) \right) \right]_{a = r + \sigma}^{r - \sigma} d\sigma = \phi^{1,1} + \phi^{1,2} + \phi^{1,3}
\]

where

\[
\phi^{1,1}(t, r) = \frac{1}{2} \int_0^{+\infty} (t + \sigma)^{-1} \left[ a^{-2} \frac{d}{da} \left( a^3 \int_{|x| = a} N_\ell(t + \sigma, x) d\omega(x) \right) \right]_{a = r + \sigma} d\sigma \\
\phi^{1,2}(t, r) = -\frac{1}{2} \int_0^r (t + \sigma)^{-1} \left[ a^{-2} \frac{d}{da} \left( a^3 \int_{|x| = a} N_\ell(t + \sigma, x) d\omega(x) \right) \right]_{a = r - \sigma} d\sigma \\
\phi^{1,3}(t, r) = -\frac{1}{2} \int_r^{+\infty} (t + \sigma)^{-1} \left[ a^{-2} \frac{d}{da} \left( a^3 \int_{|x| = a} N_\ell(t + \sigma, x) d\omega(x) \right) \right]_{a = \sigma - r} d\sigma
\]

To compute $\phi^{1,1}$, $\phi^{1,2}$ and $\phi^{1,3}$, we will use the following identity

\[
a^{-2} \frac{d}{da} \left( a^3 \int_{|x| = a} N_\ell(t + \sigma, x) d\omega(x) \right) = \int_{|x| = a} \left( 3N_\ell(t + \sigma, x) + x \cdot \nabla N_\ell(t + \sigma, x) \right) d\omega(x) \\
= 3 \int_{|x| = a} M_\ell(t + \sigma, x) d\omega(x) + \ell(t + \sigma) \int_{|x| = a} \partial_{x_1} N_\ell(t + \sigma, x) d\omega(x),
\]

since by direct computations

\[
3N_\ell + x \cdot \nabla N_\ell = 3\langle x_\ell \rangle^{-3} - 3|x_\ell|^2\langle x_\ell \rangle^{-5} + \ell \ell t \partial_{x_1} N_\ell = 3M_\ell + \ell \ell t \partial_{x_1} M_\ell.
\]

To compute $\phi^{1,1}$, we observe as above that

\[
(r + \sigma)^{-2} \frac{d}{d\sigma} \left( (r + \sigma)^3 \int_{|x| = r + \sigma} N_\ell(t + \sigma, x) d\omega(x) \right) \\
= \int_{|x| = r + \sigma} \left( 3N_\ell(t + \sigma, x) + x \cdot \nabla N_\ell(t + \sigma, x) \right) d\omega(x) - \ell(r + \sigma) \int_{|x| = r + \sigma} \partial_{x_1} N_\ell(t + \sigma, x) d\omega(x) \\
= 3 \int_{|x| = r + \sigma} M_\ell(t + \sigma, x) d\omega(x) + \ell(t - r) \int_{|x| = r + \sigma} \partial_{x_1} N_\ell(t + \sigma, x) d\omega(x),
\]

and thus eliminating the terms containing $\partial_{x_1} N_\ell$, we find

\[
\left[ a^{-2} \frac{d}{da} \left( a^3 \int_{|x| = a} N_\ell(t + \sigma, x) d\omega(x) \right) \right]_{a = r + \sigma} = 3 \left( \frac{r + \sigma}{r - \ell} \right) \int_{|x| = r + \sigma} M_\ell(t + \sigma, x) d\omega(x) \\
- (r + \sigma)^{-2} \frac{d}{d\sigma} \left( \int_{|x| = r + \sigma} M_\ell(t + \sigma, x) d\omega(x) \right).
\]
Therefore, we have obtained

$$\phi^{1,1} = -\frac{1}{2} (r-t)^{-1} \int_0^{+\infty} (r + \sigma)^{-2} \frac{d}{d\sigma} \left( (r + \sigma)^3 \int_{|x|=r+\sigma} N_\ell(t + \sigma, x) d\omega(x) \right) d\sigma$$

$$+ \frac{3}{2} (r-t)^{-1} \int_0^{+\infty} \left( \frac{r + \sigma}{t + \sigma} \right) \int_{|x|=r+\sigma} M_\ell(t + \sigma, x) d\omega(x) d\sigma.$$ 

Integrating by parts, we find

$$\phi^{1,1} = -(r-t)^{-1} \int_0^{+\infty} \int_{|x|=r+\sigma} N_\ell(t + \sigma, x) d\omega(x) d\sigma + \frac{r}{2} (r-t)^{-1} \int_{|x|=r} N_\ell(t, x) d\omega(x)$$

$$+ \frac{3}{2} (r-t)^{-1} \int_{|x|>r} \left( t + |x| - r \right) N_\ell(t + |x| - r, x) dx + \frac{r^{-3}}{2} (r-t)^{-1} \int_{|x|=r} N_\ell(t, x) d\omega(x)$$

which rewrites

$$\frac{8\pi^2}{3} \phi^{1,1} = -(r-t)^{-1} \int_{|x|>r} |x|^{-4} N_\ell(t + |x| - r, x) dx + \frac{r^{-3}}{2} (r-t)^{-1} \int_{|x|=r} N_\ell(t, x) d\omega(x)$$

$$+ \frac{3}{2} (r-t)^{-1} \int_{|x|>r} (t + |x| - r)^{-1} |x|^{-3} M_\ell(t + |x| - r, x) dx. \quad (A.1)$$

We compute $\phi^{1,2}$ similarly. First, for $0 < \sigma < r$,

$$(r - \sigma)^{-2} \frac{d}{d\sigma} \left( (r - \sigma)^3 \int_{|x|=r-\sigma} N_\ell(t + \sigma, x) d\omega(x) \right)$$

$$= - \int_{|x|=r-\sigma} (3 N_\ell(t + \sigma, x) + x \cdot \nabla N_\ell(t + \sigma, x)) d\omega(x) - \ell(r - \sigma) \int_{|x|=r-\sigma} \partial_x N_\ell(t + \sigma, x) d\omega(x)$$

$$= -3 \int_{|x|=r-\sigma} M_\ell(t + \sigma, x) d\omega(x) - \ell(r + t) \int_{|x|=r-\sigma} \partial_x N_\ell(t + \sigma, x) d\omega(x),$$

and thus

$$\left[ a^{-2} \frac{d}{da} \left( a^3 \int_{|x|=a} N_\ell(t + \sigma, x) d\omega(x) \right) \right]_{a=r-\sigma} = 3 \left( \frac{r - \sigma}{r + t} \right) \int_{|x|=r-\sigma} M_\ell(t + \sigma, x) d\omega(x)$$

$$- (r - \sigma)^{-2} \left( \frac{t + \sigma}{r + t} \right) \frac{d}{d\sigma} \left( (r - \sigma)^3 \int_{|x|=r-\sigma} N_\ell(t + \sigma, x) d\omega(x) \right).$$

Therefore,

$$\phi^{1,2} = \frac{1}{2} (r + t)^{-1} \int_0^{r} (r - \sigma)^{-2} \frac{d}{d\sigma} \left( (r - \sigma)^3 \int_{|x|=r-\sigma} N_\ell(t + \sigma, x) d\omega(x) \right) d\sigma$$

$$- \frac{3}{2} (r + t)^{-1} \int_0^{r} \left( \frac{r - \sigma}{t + \sigma} \right) \int_{|x|=r-\sigma} M_\ell(t + \sigma, x) d\omega(x) d\sigma.$$
Integrating by parts, we find

\[ \phi^{1,2} = -(r + t)^{-1} \int_0^r \int_{|x| = r - \sigma} N_\ell(t + \sigma, x) d\omega(x) d\sigma - \frac{r}{2} (r + t)^{-1} \int_{|x| = r} N_\ell(t, x) d\omega(x) \]

\[ - \frac{3}{2} (r + t)^{-1} \int_0^r \left( \frac{r - \sigma}{r + \sigma} \right) \int_{|x| = r - \sigma} M_\ell(t + \sigma, x) d\omega(x) d\sigma, \]

and thus

\[ \frac{8\pi^2}{3} \phi^{1,2} = -(r + t)^{-1} \int_{|x| < r} |x|^{-4} N_\ell(t + r - |x|, x) dx - \frac{r^{-3}}{2} (r + t)^{-1} \int_{|x| = r} N_\ell(t, x) d\omega(x) \]

\[ - \frac{3}{2} (r + t)^{-1} \int_{|x| < r} (t + r - |x|)^{-1} |x|^{-3} M_\ell(t + r - |x|, x) dx. \quad (A.2) \]

Finally, we compute \( \phi^{1,3} \). For \( \sigma > r \),

\[ (\sigma - r)^{-2} \frac{d}{d\sigma} \left( (\sigma - r)^3 \int_{|x| = \sigma - r} N_\ell(t + \sigma, x) d\omega(x) \right) \]

\[ = \int_{|x| = \sigma - r} (3N_\ell(t + \sigma, x) + x \cdot \nabla N_\ell(t + \sigma, x)) d\omega(x) - \ell(\sigma - r) \int_{|x| = \sigma - r} \partial_{x_1} N_\ell(t + \sigma, x) d\omega(x) \]

\[ = 3 \int_{|x| = \sigma - r} M_\ell(t + \sigma, x) d\omega(x) + \ell(r + t) \int_{|x| = \sigma - r} \partial_{x_1} N_\ell(t + \sigma, x) d\omega(x), \]

and thus

\[ \left[ a^{-2} \frac{d}{da} \left( a^3 \int_{|x| = a} N_\ell(t + \sigma, x) d\omega(x) \right) \right]_{a = \sigma - r} = -3 \left( \frac{\sigma - r}{r + t} \right) \int_{|x| = \sigma - r} M_\ell(t + \sigma, x) d\omega(x) \]

\[ + (\sigma - r)^{-2} \left( \frac{t + \sigma}{r + t} \right) \frac{d}{d\sigma} \left( (\sigma - r)^3 \int_{|x| = \sigma - r} N_\ell(t + \sigma, x) d\omega(x) \right). \]

Therefore, we write

\[ \phi^{1,3} = -\frac{1}{2} (r + t)^{-1} \int_r^{+\infty} (\sigma - r)^{-2} \frac{d}{d\sigma} \left( (\sigma - r)^3 \int_{|x| = \sigma - r} N_\ell(t + \sigma, x) d\omega(x) \right) d\sigma \]

\[ + \frac{3}{2} (r + t)^{-1} \int_r^{+\infty} \left( \frac{\sigma - r}{r + \sigma} \right) \int_{|x| = \sigma - r} M_\ell(t + \sigma, x) d\omega(x) d\sigma \]

and by integration by parts,

\[ \phi^{1,3} = -(r + t)^{-1} \int_r^{+\infty} \int_{|x| = \sigma - r} N_\ell(t + \sigma, x) d\omega(x) d\sigma \]

\[ + \frac{3}{2} (r + t)^{-1} \int_r^{+\infty} \left( \frac{\sigma - r}{r + \sigma} \right) \int_{|x| = \sigma - r} M_\ell(t + \sigma, x) d\omega(x) d\sigma. \]
We obtain the following expression concerning $\phi^{1,3}$

$$\frac{8\pi^2}{3} \phi^{1,3} = -(r + t)^{-1} \int |x|^{-4} N_\ell(t + r + |x|, x) dx$$

$$+ \frac{3}{2} (r + t)^{-1} \int (t + r + |x|)^{-1} |x|^{-3} M_\ell(t + r + |x|, x) dx. \quad (A.3)$$

**Asymptotics of $\phi^1$.** We extract the asymptotics of $\phi^1$ for $r \gg 1$, $1 \ll t \leq r^{1/2}$ from the exact expressions $(A.1)$, $(A.2)$ and $(A.3)$. First, in view of $(A.1)$, we set

$$\Gamma_1(\ell) = \frac{3}{8\pi^2} \int_{|y|>1} |y|^{-4} \left( \frac{(y_1 - \ell |y| + \ell y_2)^2}{1 - \ell^2} + |\tilde{y}|^2 \right)^{-\frac{3}{2}} dy,$$

$$\Theta_1(\ell) = \frac{1}{2} \int_{|y|=1} \left( \frac{|y_1|^2}{1 - \ell^2} + |\tilde{y}|^2 \right)^{-\frac{1}{2}} d\omega(y).$$

Observe that $\Gamma_1 < +\infty$. Indeed, if $|y| > 1$ and $y_1 < 0$, then $y_1 - \ell |y| + \ell < y_1$ and so 

$$\frac{(y_1 - \ell |y| + \ell y_2)^2}{1 - \ell^2} + |\tilde{y}|^2 \geq |y|^2.$$ 

If $|y| > 1$ and $y_1 > 0$, then $y_1 - \ell |y| + \ell > y_1 - \ell (y_1 + |\tilde{y}|) = (1 - \ell) y_1 - |\tilde{y}|$ and so 

$$\frac{(y_1 - \ell |y| + \ell y_2)^2}{1 - \ell^2} + |\tilde{y}|^2 \geq \frac{(1-\ell)^2}{4} |y_1|^2 - \frac{1}{2} |\tilde{y}|^2 + |\tilde{y}|^2 \gtrsim |y|^2.$$ 

Thus, $\Gamma_1(\ell) \lesssim \int_{|y|>1} |y|^{-7} dy < +\infty$. Using the inequality $|A^{-\frac{3}{2}} - B^{-\frac{3}{2}}| \lesssim (A^{-\frac{5}{2}} + B^{-\frac{5}{2}}) |A - B|$ and the lower bounds

$$\frac{(y_1 - \ell |y| + \ell y_2)^2}{1 - \ell^2} + |\tilde{y}|^2 + r^{-2} \gtrsim |y|^2,$$ 

$$\frac{(y_1 - \ell |y| + \ell y_2)^2}{1 - \ell^2} + |\tilde{y}|^2 \gtrsim |y|^2$$

we estimate, for $r \gg 1$, $t \leq r^{1/2}$ large,

$$r^2 \int_{|x|>r} |x|^{-4} N_\ell(t + |x| - r, x) dx - \frac{8\pi^2}{3} \Gamma_1(\ell) \lesssim \int_{|y|>1} |y|^{-9} \left( \frac{t |y|}{r} + \frac{t^2}{r^2} \right) dy \lesssim \frac{t}{r} \int_{|y|>1} |y|^{-8} dy \lesssim \frac{t}{r}.$$

It follows that

$$- \frac{3}{8\pi^2} (r - t)^{-1} \int_{|x|>r} |x|^{-4} N_\ell(t + |x| - r, x) dx = -\Gamma_1(\ell) r^{-3} + O(tr^{-4})$$

$$= -\Gamma_1(\ell) r^{-3} + O(r^{-1} t^{-\frac{9}{4}}).$$

Similarly,

$$\frac{1}{2} t^{-1} \int_{|x|=r} N_\ell(t, x) d\omega(x) - \frac{8\pi^2}{3} \Theta_1 \lesssim \int_{|y|=1} \left( \frac{(y_1 - \ell y_2)^2}{1 - \ell^2} + |\tilde{y}|^2 + r^{-2} \right)^{-\frac{3}{2}} - \left( \frac{y_1^2}{1 - \ell^2} + |\tilde{y}|^2 \right)^{-\frac{3}{2}} \left( \frac{t |y|}{r} + \frac{t^2}{r^2} \right) d\omega(y) \lesssim \frac{t}{r}.$$
and thus the second term in (A.1) is estimated as

\[
\frac{3}{8\pi^2} \frac{r^{-3}}{2} (r-t)^{-1} \int_{|x|=r} N_e(t,x) d\omega(x) = r^{-3} \Theta_1(\ell) + O(r^{-1} t^{-\frac{3}{2}}).
\]

Now, we bound the last term in (A.1) as follows

\[
(r-t)^{-1} \int_{|x|>r} (t+|x|-r)^{-1} |x|^{-3} M_e(t+|x|-r,x) dx \lesssim r^{-4} t^{-1} \int_{|y|>1} |y|^{-3} \left( |y_1 - \ell| |y| + \ell - \ell \frac{t}{r} \right)^2 + |\hat{y}|^2 + r^{-2} \right)^{-\frac{3}{2}} dy \lesssim r^{-4} t^{-1} \int_{|y|>1} |y|^{-8} dy \lesssim r^{-4} t^{-1}.
\]

Thus, \( \phi^{1,1} = -\Gamma_1(\ell)r^{-3} - \Theta_1(\ell)r^{-3} + O(r^{-1} t^{-\frac{3}{2}}) \).

Second, in view of (A.2), we set

\[
\Gamma_2(\ell) = \frac{3}{8\pi^2} \int_{|y|<1} |y|^{-4} \left( \frac{(y_1 - \ell |y| + \ell)^2}{1 - \ell^2} + |\hat{y}|^2 \right)^{-\frac{3}{2}} dy.
\]

Observe that \( \Gamma_2(\ell) < +\infty \) for \( 0 < \ell < 1 \). Indeed, for \( |y| < 1 \), we have if \( y_1 > 0, |y_1 - \ell| |y| + \ell \geq |(1 - \ell)y_1 + \ell| - |y_1 - |y|| \geq (1 - \ell)y_1 + \ell - |\hat{y}| \) and so \( |y_1 - \ell| |y| + \ell + |\hat{y}|^2 \geq |y|^2 + \ell^2 \). For \( |y| < 1 \), if \( y_1 < 0, |y_1 - \ell| |y| + \ell \geq |(1 + \ell)y_1 + \ell| - |y_1| \geq |y_1| + \ell \) and so \( |y_1 - \ell| |y| + \ell + |\hat{y}|^2 \geq |y_1| + \ell \). Thus, for \( 0 < \ell < 1 \),

\[
\Gamma_2(\ell) \lesssim \int_{|y|<1} |y|^{-4} \left( \frac{(y_1 + \ell (1 - \ell)}{1 + \ell} \right)^2 + |\hat{y}|^2 \right)^{-\frac{3}{2}} dy < +\infty.
\]

Moreover, using the inequality \( |A^{-\frac{3}{2}} - B^{-\frac{3}{2}}| \lesssim (A^{-\frac{3}{2} + \frac{1}{4}} + B^{-\frac{3}{2} + \frac{1}{4}})|A - B|^{\frac{3}{4}} \), we obtain

\[
\left| r^2 \int_{|x|<r} |x|^{-4} N_e(t+r-|x|, x) dx - \frac{8\pi^2}{3} \Gamma_2(\ell) \right| \lesssim \int_{|y|<1} |y|^{-4} \left( \frac{(y_1 - \ell |y| + \ell - \ell \frac{t}{r})^2}{1 - \ell^2} + |\hat{y}|^2 + r^{-2} \right)^{-\frac{3}{2}} \left( \frac{(y_1 - \ell |y| + \ell)^2}{1 - \ell^2} + |\hat{y}|^2 \right)^{-\frac{3}{2}} dy \lesssim \left( \frac{t}{r} \right)^{\frac{3}{4}} \int_{|y|<1} |y|^{-4} \left( \frac{(y_1 + \ell (1 - \ell)}{1 + \ell} \right)^2 + |\hat{y}|^2 \right)^{-\frac{3}{2} + \frac{1}{4}} dy \lesssim \left( \frac{t}{r} \right)^{\frac{3}{4}}.
\]

The second term in (A.2) writes as before

\[
- \frac{3}{8\pi^2} \frac{r^{-3}}{2} (r+t)^{-1} \int_{|x|=r} N_e(t,x) d\omega(x) = -r^{-3} \Theta_1(\ell) + O(r^{-1} t^{-\frac{3}{2}}).
\]
Now, we bound the last term in (A.3) as follows
\[
\begin{aligned}
(r + t)^{-1} \int_{|x| < r} (t + r - |x|)^{-1} |x|^{-3} M_\ell(t + r - |x|) dx \\
\lesssim r^{-4 + \frac{1}{2}} t^{-1} \int_{|y| < 1} |y|^{-3} \left( \left( y_1 - \ell |y| - \ell - \frac{4}{r} \right)^2 + |\bar{y}|^2 + r^{-2} \right)^{-\frac{5}{2} + \frac{1}{4}} dy \\
\lesssim r^{-\frac{7}{2}} t^{-1} \int_{|y| < 1} |y|^{-3} \left( \left( y_1 + \frac{\ell (1 + \frac{r}{2})}{1 + \ell} \right)^2 + |\bar{y}|^2 \right)^{-\frac{5}{2} + \frac{1}{4}} dy \lesssim r^{-\frac{7}{2}} t^{-1}.
\end{aligned}
\]

In conclusion of these estimates, we obtain \( \phi^{1,2} = -\Gamma_2(\ell) r^{-3} - \Theta_1(\ell) r^{-3} + O(r^{-1} t^{-\frac{5}{2}}). \)

Third, in view on (A.3), we set
\[
\Gamma_3(\ell) = \frac{3}{8\pi^2} \int |y|^{-4} \left( \left( \frac{y_1 - \ell |y| - \ell}{1 - \ell^2} \right)^2 + |\bar{y}|^2 \right)^{-\frac{3}{2}} dy.
\]

Observe that \( \Gamma_3(\ell) < +\infty \) for \( 0 < \ell < 1 \). Indeed, we have if \( y_1 < 0, |y_1 - \ell |y| - \ell|^2 + |\bar{y}|^2 \gtrsim |y|^2 + \ell^2 \), and if \( y_1 > 0, |y_1 - \ell |y| - \ell| \geq |1 - \ell| y_1 - \ell - |y| - |y| \geq |(1 - \ell) y_1 - \ell - |y| \) and so \( |y_1 - \ell |y| - \ell|^2 + |\bar{y}|^2 \gtrsim |y_1 - \frac{\ell}{1 - \ell}|^2 + |\bar{y}|^2 \). Thus, \( 0 < \ell < 1 \),
\[
\Gamma_3(\ell) \lesssim \int |y|^{-4} \left( \left( \frac{y_1 - \ell}{1 - \ell} \right)^2 + |\bar{y}|^2 \right)^{-\frac{3}{4}} dy < +\infty.
\]

As before, we estimate the first term in (A.3),
\[
\begin{aligned}
\left| r^2 \int |x|^{-4} N_\ell(t + r + |x|, x) dx - \frac{8\pi^2}{3} \Gamma_3(\ell) \right| \\
\lesssim \int |y|^{-4} \left( \left( \frac{y_1 - \ell |y| - \ell}{1 - \ell^2} \right)^2 + |\bar{y}|^2 + r^{-2} \right)^{-\frac{3}{2}} \left( \left( \frac{y_1 - \ell |y| - \ell}{1 - \ell^2} \right)^2 + |\bar{y}|^2 \right)^{-\frac{3}{4}} dy \\
\lesssim \left( \frac{t}{r} \right)^{\frac{3}{4}} \int |y|^{-4} (1 + |y|^{\frac{3}{4}}) \left( \left( \frac{y_1 - \ell}{1 - \ell} \right)^2 + |\bar{y}|^2 \right)^{-\frac{5}{2} + \frac{1}{4}} dy \\
+ \left( \frac{t}{r} \right)^{\frac{3}{4}} \int |y|^{-4} (1 + |y|^{\frac{3}{4}}) \left( \left( \frac{y_1 - \ell (1 + \frac{r}{2})}{1 - \ell} \right)^2 + |\bar{y}|^2 \right)^{-\frac{5}{2} + \frac{1}{4}} dy \lesssim \left( \frac{t}{r} \right)^{\frac{3}{4}}.
\end{aligned}
\]

Now, we bound the last term in (A.3) as follows
\[
\begin{aligned}
(r + t)^{-1} \int (t + r + |x|)^{-1} |x|^{-3} M_\ell(t + r + |x|, x) dx \\
\lesssim r^{-4 + \frac{1}{2}} t^{-1} \int |y|^{-3} \left( \left( y_1 - \ell |y| - \ell - \frac{4}{r} \right)^2 + |\bar{y}|^2 + r^{-2} \right)^{-\frac{5}{2} + \frac{1}{4}} dy \\
\lesssim r^{-\frac{7}{2}} t^{-1} \int |y|^{-3} \left( \left( y_1 - \frac{\ell (1 + \frac{r}{2})}{1 - \ell} \right)^2 + |\bar{y}|^2 \right)^{-\frac{5}{2} + \frac{1}{4}} dy \lesssim r^{-\frac{7}{2}} t^{-1}.
\end{aligned}
\]
Thus, \( \phi^{1,3} = -\Gamma_3(\ell)r^{-3} + O(r^{-1}t^{-\frac{9}{2}}) \).

 Gathering these estimates, we obtain \( \phi^1 = -\Gamma(\ell)r^{-3} + O(r^{-1}t^{-\frac{9}{2}}) \), where

\[
\Gamma(\ell) = \frac{3}{8\pi^2\ell^2} \int \left[ \left( \frac{(x_1 - \ell|x| + 1)^2}{1 - \ell^2} + |x|^2 \right)^{\frac{3}{2}} + \left( \frac{(x_1 + \ell|x| + 1)^2}{1 - \ell^2} + |x|^2 \right)^{\frac{3}{2}} \right] \frac{dx}{|x|^4}
\]

**Computation and asymptotics of \( \phi^{11} \).** Now, we compute the asymptotic of \( \phi^{11} \) for \( r \) large and \( t < \frac{r^4}{l^2} \) large,

\[
\phi^{11} = \frac{45}{16\pi^2} \int_0^{t+\sigma} (t+\sigma)^{-1} \int_{|r-\sigma|}^{r+\sigma} a^{-3} \int_{|x|=a} K_\ell(t+\sigma) d\omega(x) d\sigma \]

\[
= \frac{45}{16\pi^2} \int_0^{t+\sigma} (t+\sigma)^{-1} \int_{|r-\sigma|}^{r+\sigma} |x|^{-3} K_\ell(t+\sigma) dx d\sigma
\]

Note that for \( |x| > r + \sigma \), we have \( |x - \ell e_1(t+\sigma)| \geq |x - \ell(t+\sigma)| \geq (1 - \ell)|x| \), and so

\[
\int_{|x| > (r+\sigma)} |x|^{-3} K_\ell(t+\sigma) dx \lesssim \int_{|x| > (r+\sigma)} |x|^{-10} dx \lesssim (r+\sigma)^{-5}.
\]

Thus,

\[
|\phi^{11} - \frac{45}{16\pi^2} \int_0^{t+\sigma} (t+\sigma)^{-1} \int_{|r-\sigma|}^{r+\sigma} |x|^{-3} K_\ell(t+\sigma) dx d\sigma| \lesssim r^{-4} t^{-1}.
\]

We remark that \( |r - \sigma| < \ell(t+\sigma) \) is equivalent to \( \frac{r-\ell t}{1+\ell} < \sigma < \frac{r+\ell t}{1+\ell} \). Thus it is natural to decompose the integral according to the three regions \( 0 < \sigma < \frac{r-\ell t}{1+\ell} \), \( \frac{r+\ell t}{1+\ell} < \sigma < \frac{r+\ell t}{1+\ell} \).

First, for \( 0 < \sigma < \frac{r-\ell t}{1+\ell} < \frac{r}{1+\ell} < r \) and \( |x| > r - \sigma \geq r \), we observe that

\[
(|x_1 - \ell(t+\sigma)|^2 + |\bar{x}|^2)^{\frac{3}{2}} \geq |x_1 - \ell(t+\sigma)| \geq r - \sigma - \ell(t+\sigma) \geq \frac{r - \ell t}{1+\ell} - \sigma.
\]

Thus, using the change of variable \( \sigma = \frac{r-\ell t}{1+\ell} \sigma' \),

\[
\int_0^{r-\ell t} (t+\sigma)^{-1} \int_{|r-\sigma|}^{r+\sigma} |x|^{-3} K_\ell(t+\sigma) dx d\sigma
\]

\[
\lesssim t^{-1} r^{-3} \int_0^{\frac{r-\ell t}{1+\ell}} \left( \frac{r-\ell t}{1+\ell} - \sigma \right)^{-\frac{3}{2}} \int \left( (x_1 - \ell(t+\sigma))^2 + |\bar{x}|^2 + 1 \right)^{-\frac{3}{2}} dx d\sigma
\]

\[
\lesssim t^{-1} r^{-3} \int_0^{1} (1 - \sigma')^{-\frac{3}{2}} d\sigma' \lesssim r^{-1} t^{-\frac{2}{3}}.
\]

Second, for \( \sigma > \frac{r+\ell t}{1+\ell} \geq \frac{r}{1+\ell} \geq r \), \( |x| > r - \sigma \), we observe that

\[
(|x_1 - \ell(t+\sigma)|^2 + |\bar{x}|^2)^{\frac{3}{2}} \geq |x| - \ell(t+\sigma) \geq \sigma - \ell(t+\sigma) \geq \sigma - \frac{r+\ell t}{1-\ell}.
\]
Thus,
\[
\int_{\frac{r-\ell t}{1+\ell}}^{+\infty} (t+\sigma)^{-1} \int_{|r-\sigma|<|x|} |x|^{-3} K_\ell(t+\sigma) dx d\sigma \\
\lesssim r^{-1} \int_{\frac{r-\ell t}{1+\ell}}^{+\infty} \sigma^{-3} \left( \sigma - \frac{r + \ell t}{1 - \ell} \right)^{-\frac{3}{2}} \int \left( (x_1 - \ell(t+\sigma))^2 + |\vec{x}|^2 + 1 \right)^{-3 - \frac{3}{8}} dx d\sigma \\
\lesssim r^{-4 + \frac{4}{7}} \int_{1}^{+\infty} (\sigma')^{-3} (1 - \sigma')^{-\frac{4}{7}} d\sigma' \lesssim r^{-\frac{15}{7}} \lesssim r^{-1} t^{-\frac{3}{4}}.
\]

Third, we consider the region \( r-\ell t < \sigma < \frac{r+\ell t}{1+\ell} \). We observe that for \( |x| > 10(t+\sigma) \), we have \( |x - \ell e_1(t+\sigma)| \geq |x - \ell(t+\sigma)| \geq \frac{1}{2}|x| \), and so
\[
\int_{\frac{r-\ell t}{1+\ell}}^{\frac{r+\ell t}{1+\ell}} (t+\sigma)^{-1} \int_{|x|>10(t+\sigma)} \left( |x|^{-3} + (t + \sigma)^{-3} \right) K_\ell(t+\sigma) dx d\sigma \\
\lesssim \int_{\frac{r-\ell t}{1+\ell}}^{\frac{r+\ell t}{1+\ell}} (t+\sigma)^{-4} \int_{|x|>10(t+\sigma)} |x|^{-7} dx d\sigma \lesssim \int_{\frac{r-\ell t}{1+\ell}}^{\frac{r+\ell t}{1+\ell}} (t + \sigma)^{-6} d\sigma \lesssim r^{-5}.
\]

Next, using the inequality \( |A^3 - B^3| \lesssim (A^4 + B^4)|A - B| \), we observe that
\[
|x|^{-3} - (\ell(t+\sigma))^{-3} \lesssim (|x|^{-4} + (t+\sigma)^{-4}) \left( |x_1 - \ell(t+\sigma)| + |\vec{x}| \right)
\]
and thus using the change of variable \( x = (t+\sigma)y \),
\[
\int_{\frac{r-\ell t}{1+\ell}}^{\frac{r+\ell t}{1+\ell}} (t+\sigma)^{-1} \int_{|r-\sigma|<|x|<10\ell(t+\sigma)} \left| |x|^{-3} - (\ell(t+\sigma))^{-3} \right| K_\ell(t+\sigma) dx d\sigma \\
\lesssim \int_{\frac{r-\ell t}{1+\ell}}^{\frac{r+\ell t}{1+\ell}} (t+\sigma)^{-1} \int_{|r-\sigma|<|x|<10\ell(t+\sigma)} \left( |x|^{-4} + (t+\sigma)^{-4} \right) \left( |x_1 - \ell(t+\sigma)| + |\vec{x}| \right)^{-\frac{10}{7}} dx d\sigma \\
\lesssim \int_{\frac{r-\ell t}{1+\ell}}^{\frac{r+\ell t}{1+\ell}} (t+\sigma)^{-\frac{10}{7}} \int_{|y|<10} \left( |y|^{-4} + 1 \right) \left( |y_1 - \ell_1| + |\vec{y}| \right)^{-\frac{10}{7}} dy d\sigma \lesssim r^{-\frac{15}{7}}.
\]

Fourth, we observe that for \( \frac{r-\ell t}{1+\ell} < \sigma < r \), we have \( \ell(t+\sigma) > r - \sigma \) and so for \( |x| < r - \sigma \), \( (|x_1 - \ell(t+\sigma)|^2 + |\vec{x}|^2)^{\frac{1}{2}} \geq \ell(t+\sigma) - |x| \geq \sigma - \frac{\ell t}{1+\ell} \). Thus, the following holds
\[
\int_{\frac{r-\ell t}{1+\ell}}^{r} (t+\sigma)^{-4} \int_{|x|<|r-\sigma|} K_\ell(t+\sigma) dx d\sigma \\
\lesssim r^{-4} \int_{\frac{r-\ell t}{1+\ell}}^{r} \left( \sigma - \frac{r - \ell t}{1 + \ell} \right)^{-\frac{3}{2}} \int \left( (x_1 - \ell(t+\sigma))^2 + |\vec{x}|^2 + 1 \right)^{-3 - \frac{3}{8}} dx d\sigma' \lesssim r^{-\frac{15}{7}},
\]
and similarly,
\[
\int_{\frac{r-\ell t}{1+\ell}}^{\frac{r+\ell t}{1+\ell}} (t+\sigma)^{-4} \int_{|x|<|r-\sigma|} K_\ell(t+\sigma) dx d\sigma' \lesssim r^{-\frac{15}{7}}.
\]

Therefore,
\[
\phi_{11} = \frac{45}{16\pi^2\ell^3} \int_{\frac{r-\ell t}{1+\ell}}^{\frac{r+\ell t}{1+\ell}} (t+\sigma)^{-4} \int K_\ell(t+\sigma) dx d\sigma + O(r^{-1} t^{-\frac{9}{4}}).
\]
By change of variable, we see that

\[
\Theta(\ell) = \frac{15(1 + 3\ell^{-2})}{16\pi^2} \int \left( \frac{|y|^2}{1 - \ell^2} + |\bar{y}|^2 + 1 \right)^{-\frac{7}{2}} dy.
\]

Moreover, for \(1 \ll t < r^\frac{11}{12}\),

\[
\ell^{-3} \int_{r^\frac{11}{12}}^{r^\frac{4}{3}} (t + \sigma)^{-4} = \ell^{-3} \int_{r^\frac{11}{12}}^{r^\frac{4}{3}} \sigma^{-4} d\sigma + O(r^{-3} - \frac{1}{12})
\]

\[
= \frac{\ell^{-3}r^{-3}}{3} \left( (1 + \ell)^3 - (1 - \ell)^3 \right) + O(r^{-3} - \frac{1}{12}) = \frac{2}{3}(1 + 3\ell^{-2})r^{-3} + O(r^{-3} - \frac{1}{12}).
\]

It follows that \(\phi^H = \Theta(\ell)r^{-3} + O(r^{-1}t^{-\frac{3}{4}})\).

**Estimate of \(\phi^III\).** We observe that for \(t\) large, \(|f^{III}(t, x)| \lesssim t^{-2}(\ell)^{-5}\). For \(\sigma > 0\) fixed,

\[
\left| \int_{|r-\sigma|}^{r+\sigma} a \left( \int_{|x|=a} f^{III}(t + \sigma, x)d\omega(x) \right) da \right| \lesssim (t + \sigma)^{-2} \int_{|r-\sigma|}^{+\infty} a^{-3} \int_{|x|=a} M_\ell(t + \sigma, x)d\omega(x) da
\]

\[
\lesssim (t + \sigma)^{-2} \int_{|x|>|r-\sigma|} |x|^{-3} |x_1 - \ell(t + \sigma)|^2 + |\bar{x}|^2 \left( \frac{1}{r^3} + \frac{1}{x^3} \right) dx
\]

\[
\lesssim (t + \sigma)^{-2} (r + \sigma + |r - \sigma|)^{-3/2} \lesssim (t + \sigma)^{-2} r^{-3/2}.
\]

Thus, \(|\phi^III(t, r)| \lesssim r^{-3/2} \int_{0}^{+\infty} (t + \sigma)^{-2} d\sigma \lesssim r^{-1}t^{-\frac{3}{4}}\).

**Conclusion.** For \(1 \ll t \leq r^\frac{11}{12}\), we have obtained \(\phi_\ell(t, r) = (\Theta(\ell) - \Gamma(\ell))r^{-3} + O(r^{-1}t^{-\frac{3}{4}})\).

First,

\[
\Theta(\ell) = \frac{15}{8\pi^2} (3\ell^{-2} + 1)(1 - \ell^2)^\frac{3}{2} \int (|y|^2 + |\bar{y}|^2 + 1)^{-\frac{7}{2}} dy
\]

\[
= 5(3\ell^{-2} + 1)(1 - \ell^2)^\frac{3}{2} \int_{0}^{+\infty} r^4(1 + r^2)^{-\frac{7}{2}} dr = (3\ell^{-2} + 1)(1 - \ell^2)^\frac{3}{2}.
\]

Second, we compute \(\Gamma(\ell)\). Note that

\[
\frac{1}{2\pi^2} \int \left( \frac{(x_1 - \ell|x| + 1)^2}{1 - \ell^2} + |\bar{x}|^2 \right)^{-\frac{3}{2}} dx
\]

\[
= \int_{0}^{+\infty} \int_{-r}^{r} \left( \frac{(a - \ell r + 1)^2}{1 - \ell^2} + r^2 - a^2 \right)^{-\frac{3}{2}} \left( 1 - \frac{a^2}{r^2} \right) da dr
\]

\[
= (1 - \ell^2)^{\frac{3}{2}} \int_{-\infty}^{1} \int_{-1}^{1} \left( (rb + \ell r + 1)^2 + r^2(1 - b^2)(1 - \ell^2) \right)^{-\frac{3}{2}} (1 - b^2) db dr,
\]
and similarly
\[
\frac{1}{2\pi^2} \int \left( \frac{(x_1 + \ell |x| + 1)^2}{1 - \ell^2} + |\hat{x}|^2 \right)^{-\frac{3}{2}} \frac{dx}{|x|^4} = (1 - \ell^2)^{\frac{3}{4}} \int_0^{+\infty} \int_{-1}^{1} \left( (rb + \ell r + 1)^2 + r^2(1 - b^2)(1 - \ell^2) \right)^{-\frac{3}{2}} (1 - b^2) \, db \, dr.
\]
Thus, by direct computation
\[
\Gamma(\ell) = \frac{3}{4} \ell^{-2} (1 - \ell^2)^{\frac{3}{2}} \int_{-\infty}^{+\infty} \int_{-1}^{1} \left( (rb + \ell r + 1)^2 + r^2(1 - b^2)(1 - \ell^2) \right)^{-\frac{3}{2}} (1 - b^2) \, db \, dr
\]
\[
= \frac{3}{4} \ell^{-2} (1 - \ell^2)^{\frac{3}{2}} \int_{-\infty}^{+\infty} \int_{-1}^{1} \left( (r(1 + b\ell) + b + \ell)^2 + (1 - b^2)(1 - \ell^2) \right) \frac{1}{(1 + b\ell)^2} \, db \, dr
\]
\[
= \frac{3}{4} \ell^{-2} (1 - \ell^2)^{\frac{3}{2}} \left( \int_{-1}^{1} (1 + b\ell) \, db \right) \left( \int_{-\infty}^{+\infty} (u^2 + 1)^{-\frac{3}{2}} \, du \right) = \ell^{-2} (1 - \ell^2)^{\frac{3}{2}}
\]
Therefore, \( \phi(t, r) = (1 - \ell^2)^{\frac{3}{4}} r^{-3} + O(r^{-1} \ell^{-\frac{3}{2}}) \) and Lemma 3.3 is proved.

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