Unit Perturbations in Budgeted Spanning Tree Problems

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Abstract. The minimum spanning tree of a graph is a well-studied structure that is the basis of countless graph theoretic and optimization problem. We study the minimum spanning tree (MST) perturbation problem where the goal is to spend a fixed budget to increase the weight of edges in order to increase the weight of the MST as much as possible. Two popular models of perturbation are bulk and continuous. In the bulk model, the weight of any edge can be increased exactly once to some predetermined weight. In the continuous model, one can pay a fractional amount of cost to increase the weight of any edge by a proportional amount. Frederickson and Solis-Oba \cite{FS} have studied these two models and showed that bulk perturbation for MST is as hard as the $k$-cut problem while the continuous perturbation model is solvable in poly-time.

In this paper, we study an intermediate unit perturbation variation of this problem where the weight of each edge can be increased many times but at an integral unit amount every time. We provide an $(\text{opt}/2 - 1)$-approximation in polynomial time where \text{opt} is the optimal increase in the weight. We also study the associated dual targeted version of the problem where the goal is to increase the weight of the MST by a target amount while minimizing the cost of perturbation. We provide a 2-approximation for this variation. Furthermore we show that assuming the Small Set Expansion Hypothesis, both problems are hard to approximate.

We also point out an error in the proof provided by Frederickson and Solis-Oba in \cite{FS} with regard to their solution to the continuous perturbation model. Although their algorithm is correct, their analysis is flawed. We provide a correct proof here.

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1 Introduction

Classic problems in network optimization involve minimizing (or maximizing) the cost of interesting combinatorial objects such as cuts, paths or trees. The motivation for Ford and Fulkerson’s study of the minimum \(st\)-cut problem \([21,36,8]\) was to examine how much an enemy can interdict the network with a limited budget and hence reduce the capacity of the minimum cut to the lowest possible value under different costs for capacity reduction at the arcs. The minimum \(st\)-cut problem is posed in a directed graph with non-negative arc capacities and special source \(s\) and sink \(t\) with the goal of finding the minimum capacity of a cut separating all paths from \(s\) to \(t\). In early work, Fulkerson \([16]\) also introduced a budgeted version of this minimum \(st\)-cut problem. Here in addition to the capacities, each arc is also provided with a cost to increase its capacity by one unit and a global budget \(B > 0\) on the total cost that can be spent with the goal of maximizing the minimum cut after carrying out the increases within budget. Ahuja and Orlin \([1]\) later showed how this algorithm can be made to run in strongly polynomial time.

Fulkerson and Harding \([15]\) studied the analogue of this problem for the case of minimum-length \(st\)-paths. In the shortest \(st\)-path problem in a directed graph with non-negative arc lengths and special source \(s\) and destination \(t\), the goal is to find the minimum length path from \(s\) to \(t\). The budgeted version addressed in \([15]\) specifies an integral cost per unit increase of the length for each arc and a budget \(b\) and seeks to maximize the shortest \(st\)-path under this constraint. They reduce this problem to an instance of parametric minimum-cost flow which is also known to be solvable in polynomial time \([38]\).

The study of budgeted problems for the case of minimum weight spanning trees in undirected graphs was initiated by Frederickson and Solis-Oba \([14]\). The minimum spanning tree problem in a given undirected graph with non-negative edge weights is to find a spanning tree of minimum total edge weight. The budgeted version specifies costs per unit increase in the weight of the edges and a budget \(B > 0\) and requires spending the budget optimally to increase the edge-weights to maximize the weight of the resulting minimum-weight spanning tree. Frederickson and Solis-Oba gave a strongly polynomial algorithm for this problem. In follow-up work \([13]\), they extended this solution to that of maximizing the minimum weight base of matroids for which independence can be tested in polynomial time.

Juttner \([24]\) extended the study of strongly polynomial time algorithms for such budgeted optimization problems to a larger class extending the work of \([13]\) to the budgeted cases of minimum-cost circulations, matroid intersections and submodular optimization.

1.1 Budgeted versus Target Versions

In contrast to requiring that the total cost-spent is no more than a pre-specified budget \(B\) so as to maximize the minimum-weight solution, an alternate formulation is to specify a target final value \(T\) for the minimum weight solution and minimize the total cost spent in increasing the
weights of the edges to achieve this target. Note that a polynomial time algorithm for one version usually yields a polynomial time algorithm for the other, by carrying out a binary search over the specified parameter and optimizing the other; however, this strategy does not directly yield strongly polynomial algorithms. We will study both the original budgeted and the dual targeted versions of the problem in this paper.

1.2 Discrete Perturbation Models

We remark that all the above budgeted optimization formulations allow fractional changes, i.e., if we increase the weight of an edge $e$ by any $\delta > 0$, we pay $\delta \cdot c(e)$ where $c(e)$ is the cost per unit weight increase for the edge $e$. We may think of this as the continuous version of these problems. A natural direction to investigate is the discrete version of these problems where the changes to the elements (edges) are only allowed to be discrete units.

Frederickson and Solis-Oba [14] introduce an even simpler version of the discrete version of the robust minimum spanning tree problem: for a given value $k$, find the set of $k$ edges whose removal from the graph causes the largest increase in the weight of its minimum spanning trees. They point out that this problem already appears in the literature under the name of the most vital edges in the MST [19,20,23,28,39]. The most vital arcs problem has also been studied in the context of shortest $st$-paths [3,32].

1.3 Bulk versus Unit Perturbations

We distinguish between two types of discrete perturbation models: bulk and unit. In the bulk perturbation models, each edge is specified with a bulk reduction in its weight for which a modification cost must be paid in full for any modification to be effected at all. More generally, each edge has a series of integer-valued successive modifications in its weight that can be effected by a corresponding set of modification costs, each of which must be paid in full to effect these modifications. In contrast, the unit perturbation model specifies for each edge, a unit modification cost, and bounds (upper or lower, depending on whether the modification is an increase or decrease of weight respectively) and modifications can be made one unit at a time until the bound. In both models, in the budgeted version, we are given a budget $B$ on the total modification budget and the objective as before is to increase the cost of a minimum cost solution. Similarly, for the targeted version, we are given a target $T$ on the final value of the objective and the goal is to minimize the modification cost so as to reach the target value $T$ in the objective after these modifications.

As a simple example, consider the targeted unit-perturbation version of the $st$-minimum cut problem in a digraph with edge capacities. Assume that we have an estimate of fortification cost $r_a$ for every arc to increase its capacity by one unit; we assume that the fortification costs are valid for any starting capacity in a given range. Given a target $T$, the goal is to increase the capacities on a subset of arcs at minimum fortification cost.
so that the minimum $st$-cut in the fortified network is at least $T$. We may have individual limits on how much each arc can be fortified in total. We insist that the total capacity increase in each arc is integral to retains the combinatorial flavor in the discrete problem. This model is “smoother” than the bulk model where every edge has a fixed integral capacity increase (potentially greater than one) for which the full upgrading cost has to be paid: any smaller use of the budget does not upgrade the edge at all. In contrast to the more popular interdiction problems modeling an attacker’s concerted effort to weaken a network, fortification (such as infrastructure replenishment) is typically carried out over time. The unit-upgrading is more natural in this context of incremental upgrading of elements over time. Indeed, it is known that under the bulk model that requires us to pay a single cost to increase the capacity by an arbitrarily large amount, the minimum $st$-cut fortification problem is as hard as the notorious densest subgraph (DkS) problem (See [8] for a reduction). Henceforth we focus our attention in the rest of this paper on the unit perturbation models.

1.4 Problem Formulations

In this paper, we study both the targeted and budget version of the minimum weight spanning tree problem under unit perturbations. We use $(G, w)$ to denote the input graph $G$ with edge weights $w$. Given an upgrading scheme, let $c_x := c \cdot x = \sum_{e \in E(G)} c_e x_e$ is the total cost of this upgrading.

Problem 1. (Budgeted MMST) Given a graph $G$, functions $w, c : E(G) \to \mathbb{N}$ and a budget $B$, let $f_x$ be the weight of an MST in $(G, w + x)$. Then, find an upgrading scheme $x : E(G) \to \mathbb{N}$ that maximizes $f_x - f_0$ while ensuring the total cost, $c_x := c \cdot x$, is at most $B$. We call $f_G(B)$ this maximum value.

Note that we have modelled the increase in the weight of the MST as the target, which is more general/powerful than making the target the final weight of the MST.

We also consider the targeted version of this problem:

Problem 2. (Targeted MMST) Given a graph $G$, functions $w, c : E(G) \to \mathbb{N}$ and a target increase of $T - T_0$ where $T_0$ is the weight of the MST in $(G, w)$, find a perturbation scheme $x : E(G) \to \mathbb{N}$ that minimizes the cost $c_x := c \cdot x$ while ensuring that $f_x$ the weight of any MST in $(G, w + x)$, is at least $T$.

We define two related versions of the problems we study.

Problem 3. (Continuous Budgeted MMST) Given a graph $G$, functions $w, c : E(G) \to \mathbb{N}$ and a budget $B$, let let $f_x$ be the weight of an MST in $(G, w + x)$. Then, find a upgrading scheme $x : E(G) \to \mathbb{R}^+$ that maximizes $f_x - f_0$ while ensuring the total cost, $c_x := c \cdot x$, is at most $B$. We call $f_G(B)$ this maximum value.
Problem 4. (Continuous Targeted MMST) Given a graph $G$, functions $w, c : E(G) \to \mathbb{N}$ and a target increase value of $T - T_0$ where $T_0$ is the weight of the MST in $(G, w)$, find a upgrading scheme $x : E(G) \to \mathbb{R}^+$ that minimizes the cost $c_x := c \cdot x$ while ensuring that $f_x$ the weight of any MST in $(G, w + x)$, is at least $T$.

Frederickson and Solis-Oba study the above continuous version of the Budgeted MMST problem and provide an optimal polynomial time algorithm [12], where the key difference is that the upgrading amounts on the edges are allowed to be fractional rather than integral values. All of the above problems permit a natural generalization with individual upper bounds on the fortification or upgrading amounts on the edges.

Problem 5. (MMSTU) Given a graph $G$ and functions $w, c, u : E(G) \to \mathbb{N}$ and a budget $B$, find an upgrading scheme $x : E(G) \to \mathbb{N}$, with $\forall e \in E(G), x(e) \leq u(e)$, that maximizes $f_x - f_0$ while ensuring $c_x \leq B$.

We notice that the algorithm of Frederickson and Solis-Oba [12] is optimal even with upper bounds, since the proof is not changed with upper bounds.

1.5 Contributions

1. Frederickson and Solis-Oba [12] prove that for the budgeted version of Continuous MMST an optimal solution can be found in polynomial time. However, there is an error in the proof of the optimality of their algorithm. We point out the mistake and provide a correct proof. (Section 2)

2. We provide a 2-approximation for the targeted version of the unit-perturbation MMST. For the budgeted version, we provide a $\frac{\text{opt}}{2} - 1$-solution, an algorithm that gives an upgrading scheme where the increase in the weight of the MST is at least $\frac{\text{opt}}{2} - 1$, where opt is the optimal increase in the weight of the MST. (Section 3)

3. We show that both versions of MMST are NP-complete even with unit upgrading costs. We use reductions from the minimum-weight $k$-cut problem where the goal is to find a minimum weight set of edges to break the graph into at least $k$ connected components, and from the complementary Max Components problem of using a given budget to delete edges (of total weight within this budget) so as to maximize the number of resulting connected components. By using appropriate approximation-preserving reductions and the known hardness results for these two problems, we get the following implications that show that our approximation algorithms are (nearly) best possible: Assuming the Small Set Expansion Hypothesis (SSEH) [33], Targeted Discrete MMST is NP-hard to approximate to a $(2 - \varepsilon)$-factor for $\varepsilon > 0$. Similarly, assuming SSEH, Budgeted Discrete MMST is NP-hard to approximate to a $\left(\frac{1}{2} + \varepsilon\right)$-factor for $\varepsilon > 0$. (Section 4)

4. The complexity of MMST with unit costs for upgrading any edge and the same starting weights for all edges in the graph is unresolved. Hence we study this special case of the Targeted MMST on
an undirected graph with \( n \) nodes where all edges have the same initial starting weight and MST weight target \( T \), and give an optimal algorithm that runs in \( n^{O(T)} \) time. For this, we use a supermodularity property of a coverage function used in the analysis. For the analogous budgeted version with budget \( B \), this also implies an optimal algorithm running in time \( n^{O(B)} \). (Section 5)

1.6 Related Work

Somewhat tangentially related to our work is the attacker’s problem of interdicting the network to decrease the weight of the minimum weight structure in the network. Such interdiction problems have a rich and long history in Combinatorial Optimization including serving as the motivation for Ford and Fulkerson’s study of the minimum \( st \)-cut problem \([21, 36, 8]\). The goal there is typically to worsen the optimum value for a structure that an enemy is trying to build by attacking or interdicting edges or nodes of the graph. Examples of interdicted structures studied include matchings \([13]\), minimum spanning trees \([24, 46]\), shortest paths \([17, 22]\), \( st \)-flows \([34, 41, 44]\) and global minimum cuts \([45]\). \([7]\) contains a good overview of the current literature.

Related to the problem of increasing the weight of the MST that we study, prior work \([27, 29]\) has addressed the problem of decreasing the total cost or bottleneck cost of the MST by paying upgrading costs on the nodes that in turn reduce the weight of the incident edges.

2 Correctness of MST Upgrading

2.1 Introduction of the problem

Frederickson and Solis-Oba studied the continuous version of the Budgeted MMST problem \([12]\), and prove that an optimal solution can be found in polynomial time. However, there is an error in the proof of the optimality of their algorithm. We emphasize that their algorithm is correct. In this section, we point out the mistake and provide a correct proof.

Let us first introduce the vocabulary used in \([12]\). Given \( S \subseteq E \), let \( c(S) = \sum_{e \in S} c(e) \). We define \( \text{coverage}(S, G) \) as the minimum number of edges that any minimum spanning tree (under the current weights \( w \)) of \( G \) shares with \( S \). We say that a set \( S \) is lifted by \( \delta \) when the weight of every edge in \( S \) is increased by the same amount \( \delta \). Let \( \text{tolerance}(S, G) \) be the maximum amount that the weights of the edges in \( S \) can be lifted by until \( \text{coverage}(S, G) \) changes (It can be shown that the coverage will only decrease at this change). Define \( \text{inc\_cost}(S, G) := c(S)/\text{coverage}(S, G) \), the cost per unit increment of \( S \). Roughly, the algorithm in \([12]\) is greedy and chooses a set \( S \) of minimum \( \text{inc\_cost} \) value in the current weighted graph and lifts the weights of all the edges in it by its \( \text{tolerance} \) or until it runs out of budget (whichever occurs first).

Let \( G_{w_i} \) be the graph obtained from \( G \) by first deleting all edges of weight strictly larger than \( w_i \) and then contracting all edges of weight strictly
smaller than $w_i$. For an edge $e \in E(G)$, let $sm_{\leq q}(e, G)$ be the set of edges in $G$ whose weights are at most the weight of $e$.

We first point out the following observation. Let $T$ be an MST of $G$ and let $T_{w_i}$ be the set of edges of weight $w_i$ in $T$. Then, it is easy to check that $T_{w_i}$ forms a spanning forest in $\tilde{G}_{w_i}$. Thus, if $S$ is a set of edges in $G$ with the same weight $w_i$, then $\text{coverage}(S, G)$ is equal to the increase in the number of components of $\tilde{G}_{w_i}$ after deleting $S$.

### 2.2 Continuous Downgrading of MSTs

In [12], Frederickson and Solis-Oba proposed the following algorithm to maximize the weight of the MSTs in the resulting graph. Note that when there is a choice of sets $S$ that minimize $\text{inc\_cost}$, the algorithm is free to choose any.

**Algorithm 1** raise\_mst ($G, w, c, B$)

1: $balance \leftarrow B$
2: $wmst \leftarrow \text{weight of a minimum spanning tree of } G$
3: **while** $balance > 0$ **do**
4: Find a set $S$ that minimizes $\text{inc\_cost}(S, G)$;
5: $A \leftarrow \min\{\text{tolerance}(S, G), balance/c(S)\}$;
6: Lift the weights of the edges in $S$ by $A$;
7: $balance \leftarrow balance - A \times c(S)$;
8: **end while**
9: $increase \leftarrow (\text{weight of a minimum spanning tree of } G) - wmst$;
10: Output $increase$;

Unfortunately, the proof of optimality of the algorithm (Theorem 3.1 in [12]) is incorrect. The main idea of their proof is to first decompose any given optimal solution into a sequence of fractional lifts (where a lift corresponds to the action of increasing a set of edges by the same amount). Then, at any partial budget $b < B$, one can identify $S_b^*$, the set of arcs being lifted after spending $b$ budget according to the decomposition of the optimal solution. Then, one can compare $S_b^*$ to $S_b$, the set of arcs lifted by raise\_mst after spending $b$ budget. The error in their proof arises in their decomposition wherein their definition of $S_b^*$ is not a lift. We explain why this is important in the rest of the proof next. As a last step in their proof, they show that $S_b$ performs as well as $S_b^*$ for any $b$ by using the following lemma (Lemma 3.1 in [12]).

**Lemma 1** (Lemma 3.1 in [12]). Let $G, G'$ be two graphs on the same vertex and edge sets but has edge-weights $w, w'$ respectively. Let $S$ be a set of edges such that for every $e \in S$, $sm_{\leq q}(e, G') \subseteq sm_{\leq q}(e, G)$. Then, $\text{coverage}(S, G) \leq \text{coverage}(S, G')$.

Given the decomposition, let $G_b^*$ and $G_b$ be the state of the graph after spending budget $b$ according to the decomposition of the optimal
solution and \textit{raise\_mst} respectively. Let \( w_b \), \( w_b^* \) be the weight of the edges in \( G_e^b \) and \( G_b \) respectively. In the discussion following their proof of Lemma 1 they pointed out that if it is possible to guarantee that the decomposed \( S^*_e \) only includes edges \( e \) for which \( sm\_eq(e, G_b) \subseteq sm\_eq(e, G_b^*) \), then by Lemma 1 the following inequality holds:

\[
\text{inc\_cost}(S^*_e, G_b) \leq \text{inc\_cost}(S^*_e, G_b^*). \tag{1}
\]

Since \( S_b \) is chosen by \textit{raise\_mst} to be an optimal set to lift, it follows that \textit{raise\_mst} is as good as an optimal solution. More specifically, between budget \( b \) and \( b + \epsilon \) for some \( \epsilon > 0 \), spending \( \epsilon \) to uniformly lift the set \( S^*_e \) increases the MST in \( G_b^* \) by \( \text{inc\_cost}(S^*_e, G_b^*) \). From inequality (1), this value is at most \( \text{inc\_cost}(S^*_e, G_b^*) \), which is at most \( \text{inc\_cost}(S^*_e, G_b) \) due to the choice of \( S_b \) by \textit{raise\_mst}. Hence, spending \( \epsilon \) on \( S_b \) in \( G_b \) is as worthwhile as spending it on \( S^*_e \) in \( G_b^* \). However, this argument relies on the subtle fact that \( S^*_e \) is lifted by the same amount \( \epsilon/c(S^*_e) \) on every edge. We will see that in the proposed decomposition of the optimal solution in [12], the \( S^*_e \)’s are not lifting sets in this way.

**Counterexample showing \( S^*_e \) is not a lift.** Consider a path with three edges \( e_1, e_2 \) and \( e_3 \) whose initial weights are 0. Assume the unit cost of raising the weight of any edge is 1. For \( B = 4 \), let \( w^* = (1, 2, 1) \) and \( w = (2, 2, 0) \) be their final weight in an optimal solution and \textit{raise\_mst} respectively. Furthermore, assume \textit{raise\_mst} simply found the set \( \{e_1, e_2\} \) and raised its weight by 2 until all the budget is spent. Following the definition in [12], \( S_b = \{1, 2\} \) for all \( 0 \leq b \leq 4 \). Recall that \( w_b^* = \text{min}(w^*, w_b + \Delta_b) \) with \( \Delta_b \) such that the cost of increasing to these weights would be \( b \).

Then \( w_b^* = (1, 1, 0) \). When \( b = 2 + \epsilon \), \( w_b = (1 + \epsilon/2, 1 + \epsilon/2, 0) \). Due to the cap imposed by \( w \), \( w_{2+\epsilon}^* = \text{min}(1, 1 + \epsilon/2 + \Delta_{2+\epsilon}), \text{min}(2, 1 + \epsilon/2 + \Delta_{2+\epsilon}), \text{min}(0 + \Delta_{2+\epsilon}) \) = \( (1, 1 + \epsilon/2 + \Delta_{2+\epsilon}, \Delta_{2+\epsilon}) \). Thus, by setting \( \Delta_{2+\epsilon} = \epsilon/4 \), we achieve a weight of \( w_{2+\epsilon} = (1, 1 + \epsilon/2, \Delta_{2+\epsilon}) \), spending a total budget of \( 2 + \epsilon \). We see that the lifting set \( S_{2+\epsilon} \) should be \( \{e_2, e_3\} \). However, \( e_2 \) and \( e_3 \) are not lifted by the same amount. Thus, it is unclear how much improvement was achieved by the optimal solution in the range of \( (w, w + \epsilon) \), making it incomparable to \textit{raise\_mst}.

**Remark 1.** We notice that an analogous proof is given in [13] in section 3.1 in the context of matroids. This proof contains the same problem, and the following correction can also be applied.

**A Corrected Decomposition.** We now provide a valid decomposition of an optimal solution, leading to a correct proof of the same result.

For any real number \( 0 \leq b \leq B \), let \( S_b \) denote the edge set lifted by \textit{raise\_mst} after spending budget \( b \). Let \( G_b = (V, E, w_b) \), the state of the
graph produced by the algorithm at budget $b$. More precisely, $S_b$ is the set of edges lifted in the $j$-th iteration where $j$ is the largest integer such that the budget spent at the beginning of the iteration is less or equals to $b$. Note that $\text{raise\_mst}$ lifts an edge set until it reaches its tolerance. Since all initial weights are integral, the tolerance of a set can only change once its weight were increased to the next integral value. Then, let $b_0 = 0, b_1, \ldots, b_{k-1}, b_k = B$ be the sequence of budget values that the algorithm $\text{raise\_mst}$ spends such that between any consecutive values, the difference was spent by $\text{raise\_mst}$ to lift the weight of an edge set to the next integral amount (or until out of budget). This implies that after spending budget $b_i$, an additional $b_{i+1} - b_i$ budget is used to increase the weight of an edge set $S_{b_i}$ by one unit, or possibly less because we ran out of budget. Let $w_b(e)$ denote the weight of edge $e$ after spending budget $b$. Then for $e \in S_{b_i}$, $e' \notin S_{b_i}$, the weight of $e'$ does not change as budget increases from $b_i$ to $b_{i+1}$, i.e. $w_{b_i}(e') = w_{b_{i+1}}(e')$. Furthermore, since $w_{b_i}(e')$ is an integer, it is not strictly between the interval $(w_{b_i}(e), w_{b_{i+1}}(e))$. Given an optimal solution, let $w^*$ represent the final weights of the edges. On a high level, we will decompose this optimal solution into a sequence of lifts by following $\text{raise\_mst}$ as closely as possible. At some budget $b$, an edge might reach its final weight $w^*$ and we must find an alternative way to spend the excess budget. Thus, we raise the rest of the edges in $S_b$ faster, until they reach what they are supposed to be relative to how much $\text{raise\_mst}$ has spent its budget. Lastly, if any excess budget remains, it is spent on lifting all edges (that are not yet capped) by the same amount $\Delta$. Then, the weight of any edge after spending budget $b$ in our decomposition has two components, $w_b$, the amount from emulating $\text{raise\_mst}$, and a global increase of $\Delta$.

In order to explicitly define the weight of an edge in our decomposition after spending budget $b$, let $c_{\Delta,b}(e)$ be the cost to increase an edge $e$ with initial weight $w(e)$ to $\min(w^*(e), w_b(e) + \Delta)$. Lightly abusing the notation, let $c_{\Delta,b} = \sum_{e \in E(G)} c_{\Delta,b}(e)$. Note that when $\Delta = 0$, due to the cap imposed by $w^*$, we might no longer need to spend all of the budget $b$ and thus $c_{0,b} \leq b$. Also, when there are no caps imposed by $w^*$, we have $c_{0,b} = b$.

Ideally, we would like $c_{\Delta,b} = b$ because it is easier compare how the optimal and $\text{raise\_mst}$ spend the budget. Thus, for $0 \leq i < k$, let $\Delta_i$ be a value such that $c_{\Delta_i,b_i} = b_i$. Informally, since some edges get capped by $w^*$, simply raising the weight of edges to $w_b$ might cost less than $b_i$. Then, $\Delta_i$ represents how much extra global weight we have to increase every non-capped edge in order to match the spending of $b_i$.

First, we show the following claim:

Claim. $\{\Delta_i\}_{i=0}^{k-1}$ is a non-decreasing sequence.

Proof. We will show that $\Delta_{i+1} \geq \Delta_i$. Consider the difference between $b_{i} = c_{\Delta_{i},b_i}$ and $c_{\Delta_{i},b_{i+1}}$. To achieve the second cost, the weight of some of the edges is increased to $w_{b_{i+1}}$. Since some of edges might get capped, the change in weight is at most $w_{b_{i+1}} - w_{b_i}$. Since $\text{raise\_mst}$ spends $b_{i+1} - b_i$ to change the edges from weight $w_{b_i}$ to $w_{b_{i+1}}$, the difference in the two
above costs is at most \(b_{i+1} - b_i\). Then, \(c_{\Delta, b_{i+1}} \leq c_{\Delta, b_i} + (b_{i+1} - b_i) = b_{i+1} = c_{\Delta, b_{i+1}, b_{i+1}}\). Thus, it follows that \(\Delta_{i+1} \geq \Delta_i\). □

Note that after spending budget \(b_i\), we should expect our decomposition of the optimal solution to ensure each edge has weight \(\min\{w^*(e), w_b(e) + \Delta_i\}\). Then, between a budget spending of \(b_i\) and \(b_{i+1}\), the weight of an uncapped edge in the decomposition should grow from \(w_{b_i}(e) + \Delta_i\) to \(w_{b_{i+1}}(e) + \Delta_{i+1}\). To achieve this increase, we break each interval \([b_i, b_{i+1}]\) of the budget-spending process into two phases. In the first phase, we spend the increase in budget to augment the weight of edges from \(w_{b_i} + \Delta_i\) to \(w_{b_{i+1}} + \Delta_{i+1}\). Then, in the second phase, we increase their weight from \(w_{b_{i+1}}(e) + \Delta_{i+1}\) to \(w_{b_{i+1}}(e) + \Delta_{i+1}\). Let \(\beta_i = c_{\Delta, b_i, b_{i+1}}\), representing the transitioning point (in terms of the budget) between the two phases. We then define \(w^*_i\).

**Definition 1.** The edge weight \(w^*_i(e) :=\)

1. \(\min\{w^*(e), w_{f(b)}(e) + \Delta_i\}\), with \(f : \mathbb{R} \to \mathbb{R}\) such that \(c_{\Delta, f(b)} = b\), for \(b \in [b_i, \beta_i]\),
2. \(\min\{w^*(e), w_{b_{i+1}} + \Delta(b)\}\), with \(\Delta(b)\) such that \(c_{\Delta(b), b_{i+1}} = b\), for \(b \in [\beta_i, b_{i+1}]\),

**Claim.** The following holds:
1. \(f(b_i) = b_i\) and \(f(\beta_i) = b_{i+1}\),
2. \(f(b)\) is an increasing function in the interval \([b_i, \beta_i]\)
3. \(\Delta(\beta_i) = \Delta_i\) and \(\Delta(b_{i+1}) = \Delta_{i+1}\), and
4. \(\Delta(b)\) is an increasing function in the interval \([\beta_i, b_{i+1}]\).

**Proof.** The first statement follows from the definition of \(\Delta_i\) and \(\beta_i\). To prove the second statement, consider \(b, b'\) such that \(b \leq b < b' \leq \beta_i\). Note that \(c_{\Delta, f(b)}\) and \(c_{\Delta, f(b')}\) corresponds to the cost of raising edges to a weight of \(\min\{w^*(e), w_{f(b)} + \Delta_i\}\) and \(\min\{w^*(e), w_{f(b')} + \Delta_i\}\). Since by definition, \(c_{\Delta, f(b)} = b < b' = c_{\Delta, f(b')}\), it follows that there exists some edge \(e\) such that \(w_{f(b)}(e) < w_{f(b')}(e)\). Since \(w_{\Delta}^v(e)\) is non-decreasing with respect to \(b''\), it follows that \(f(b) < f(b')\), proving the second statement. The third statement follows from the definition of \(\beta_i\) and \(\Delta_{i+1}\). For the fourth statement, we use similar arguments as the proof for the second statement. Consider \(b, b'\) such that \(\beta_i \leq b < b' \leq b_{i+1}\). Since \(c_{\Delta(b), b_{i+1}}\) corresponds to the cost of raising edges to a weight of \(\min\{w^*(e), w_{b_{i+1}} + \Delta(b)\}\), does not increase if \(\Delta(b)\) does not increase. Then, to achieve a higher cost of \(b'\), it follows that \(\Delta(b') < \Delta(b)\).

**Remark 2.** If we extend the definition of \(f(b)\) and \(\Delta(b)\) to \(f(b) = b\) for \(b \in [\beta_i, b_{i+1}]\) and \(\Delta(b) = \Delta_i\) for \(b \in [b_i, \beta_i]\), then \(f(b)\) and \(\Delta(b)\) become two non-decreasing continuous functions. Furthermore, we have that \(w^*_i(e) = \min\{w^*(e), w_{f(b)}(e) + \Delta(b)\}\) and \(w^*_i(e)\) is also a non-decreasing function. It is also clear that after spending a total budget of \(B\), all edges should reach its desired maximum weight of \(w^*\).
Here is an intuitive explanation of \(f(b)\) and \(\Delta(b)\). During the first phase, the decomposition would like to copy raise\_mst and lift all the edges in \(S_b\). However, at some point, an edge \(e \in S_b\) might have reached its full capacity \(w^*\). Then, in order to match the same amount of spending as raise\_mst, we would need to raise the rest of the edges in \(S_b\) at a faster rate. Thus, to actually spend budget \(b\), we had to lift some edges higher, to a point where if all edges in \(S_b\) were raised to that point would have costed us \(f(b)\). Similarly, \(\Delta(b)\) is an adjustment function, corresponding to a faster rate of increasing \(\Delta\) in the second phase, caused by some edges reaching its full capacity.

**Example of the Correct Decomposition**: Consider performing the above decomposition on our previous counter-example. Recall that \(G\) is a path of three edges whose initial weights are 0. With a total budget of 4, raise\_mst lifts \(e_1, e_2\) by 2 to a final weight of \((2, 2, 0)\) for edges \((e_1, e_2, e_3)\). An optimal solution has a final weight of \((1, 2, 1)\). By definition, \(b_0 = 0, b_1 = 2, b_2 = 4\) where the set \(S = \{e_1, e_2\}\) is first raised to weight 1 then raised to weight 2. Since \(c_{0,b_0} = 0 = b_0, \Delta_0 = 0\). Between \(b_0\) and \(b_1\), our decomposition would copy raise\_mst exactly, so \(f(b) = b, \Delta(b) = 0\) for \(b_0 \leq b \leq b_1\). Note that \(\beta_0 = b_1\) and \(\Delta_1 = 0\) since there is no second phase in this interval. See Figure 1.

Now, consider when \(b \in [b_1, b_2]\). Note that \(c_{1,b_2}\) is the cost of raising the weights to \((\min(1, 3), \min(2, 3), \min(1, 1)) = (1, 2, 1)\). Then, \(c_{1,b_2} = 4 = b_2\) and therefore \(\Delta_2 = 1\). Note that \(\beta_1 = c_{\Delta_1,b_2}\) is the cost of reaching a weight of \((1, 2, 0)\) and thus \(\beta_1 = 3\). In the first phase, since \(e_1\) is already capped, \(c_{0,b'}\) is the cost of reaching a weight of \((1, b'/2, 0)\). Therefore \(c_{\Delta_1,b'} = b'/2 + 1\). In raise\_mst, after spending budget \(b\) to ensure \(c_{\Delta_1,f(b)} = b\), \(f(b) = 2b - 2\). Then, \(w^*_b = \min(w^*, w_{f(b)}) = (1, b - 1, 0)\), corresponding to spending the current increase in budget to raise \(e_2\)'s weight from 1 to 2. In the second phase, for \(3 = \beta_1 \leq b \leq b_2 = 4\), note that \(c_{\Delta,b_2}\) is the cost of raising the weights to \((1, 2, \Delta')\). Then, \(c_{\Delta,b_2} = 3 + \Delta'\). To ensure \(c_{\Delta(b),b_2} = b\), we see that \(\Delta(b) = b - 3\). Then, \(w^*_b = (1, 2, b - 3)\) corresponding to lifting the weight of \(e_3\) from 0 to 1.

Note that now each step of the decomposition corresponds to a proper lift. More precisely, between \([b_1, \beta_1]\), only edge \(e_2\) is lifted and between \([\beta_1, b_2]\) only edge \(e_3\) is lifted.

**The new decomposition provides lifts**: We now show that if we view the optimal solution as a continuous process that increases the weights of edges according to \(w^*_b\), then the decomposition produces a sequence of lifts. For any \(0 \leq b < B\), let \(S^*_b\) be the set of edges whose weight changes at budget \(b\) according to \(w^*_b\). Formally, \(S^*_b = \{e : w^*_b(e) < w^*_{b+1}(e) \forall e > 0\}\).

However, when lifting from \(b_i\) to \(\beta_i\) (or from \(\beta_i\) to \(b_{i+1}\)), we may however be lifting different sets. Indeed, we could hit one of the upper bounds for one of the edges that we are lifting during this period, and so the set we are lifting would shrink to another, smaller set. This is why we now introduce new breakpoints \((p_i')\) in between the \(b_i s\) and \(\beta_i s\), in order
to always have proper lifts, which will guarantee that we can indeed compare the optimal solution to the solution given by \textit{raise.mst}.

Lemma 2. For every $0 \leq i < k$, there exists a finite sequence $\{p^*_j\}_{j=0}^l$ where $p^*_0 = b_i$, $p^*_j = b_{i+1}$ and there exists $0 \leq l \leq 1$ such that $p^*_k = \beta_i$. Furthermore, for any interval $[p^*_j, p^*_{j+1}]$, the set of edges whose weights changed according to $w^*_b$ when $b$ is within this interval corresponds to a proper lift. More precisely, for all $p^*_j \leq b < p^*_{j+1}$, we have that $S^*_b = S^*_{p^*_j}$ and $w^*_b(e) - w^*_b(e') = w^*_{p^*_j}(e') - w^*_j(e')$ for any $e, e' \in S^*_b$.

Proof. Fix $0 \leq i < k$. We find the sequence separately for each of the two phases. Suppose we are in the first phase where $b_i \leq b \leq \beta_i$. Let $b_i < p_1 < p_2 < \ldots < \beta_i$ be a sequence of budgets at which some edge reaches its cap. Formally, $p_j$ is in the sequence if there exists an edge $e$ such that $w^*_e(e) < w^*_e(e') = w^*(e)$ for all $e' > 0$. Note that since there are only finitely many edges and each edge can reach its maximum cap only once, this sequence is finite. We claim that within the interval $(p_j, p_{j+1})$, edges are being properly lifted with respect to $w^*_b$.

Let $p_j \leq b < b' \leq p_{j+1}$. Let $P^* = \{e : w^*_e(e) = w^*(e)\}$, representing the set of all edges that reached its maximum cap at budget $p_j$. First we show $S^*_b = S^*_{p_j} \setminus P^*$. Let $e \in S^*_b$. Since no edges becomes capped between $p_j$ and $p_{j+1}$, it follows that $w^*_e = w^*_f(e) + \Delta_i$ and $w^*_b = w^*_f(e') + \Delta_i$. By definition of $S^*_b$, it follows that $w^*_b < w^*_b$. From Claim 2.2 we know $b_i \leq f(b) < f(b') \leq b_{i+1}$. Then, $w^*_b(e) \leq w^*_f(e) < w^*_f(e') \leq w^*_b(e')$. Thus, the weight of edge $e$ changed with respect to $w^*_b$, proving $e \in S^*_b$. 

Fig. 1. The functions $f$ (in red) and $\Delta$ (in blue) in the case of the previously discussed counter-example.
Since \( w_i^*(e) < w_j^*(e) \), it also follows that \( e \) does not become capped at nor before \( b \), proving \( e \notin C^* \). Thus, \( S_b^n \subseteq S_b \setminus C^* \).

For the other direction, let \( e' \in S_b^n \setminus C^* \). Since \( e' \) does not get capped, \( w_i^*(e') = w_{f(b)}(e') + \Delta_i \) and \( w_j^*(e') = w_{f(b)}(e') + \Delta_i \). From Claim 2.2, since \( b_i \leq f(b) < f(b') \leq b_{i+1} \), it follows that \( w_{f(b)}(e') < w_{f(b')}(e') \) and thus \( w_i^*(e) < w_j^*(e) \). Since this inequality holds for any \( b' < b'' < b_{i+1} \), it follows that \( w_i^*(e) < w_{k+1}^*(e) \) for all \( e > 0 \), proving \( e \in S^n_b \). Thus, we conclude that \( S^n_b = S^p_{p_j} = S_b \setminus P^* \).

Let \( e, e' \in S^n_b \). Since \( e \) is not capped, \( w_i^*(e) - w_{p_j}(e) = w_{f(b)}(e) - w_{f(b)}(p_j) \).

Since \( e, e' \in S^n_b \subseteq S_b \), \( \text{raise.mst} \) lifts their weight by the same amount in the interval \([b_i, b_{i+1}] \). Since \( b \in [b_i, b_{i+1}] \), it follows that \( w_{f(b)}(e) - w_{f(b)}(e') = w_{f(b)}(e') - w_{f(b)}(e) \), proving our lemma holds for the first phase.

For the second phase of the interval when \( b_i \leq b < b_{i+1} \), let \( \beta_i < q_1 < q_2 < \ldots < b_{i+1} \) be a sequence of budgets such that some edge \( e \) becomes capped at budget \( q_i \). Once again, this sequence is finite. Let \( q_1 \leq q < q_{i+1} \). Since no edge becomes capped, \( w_i^*(e) - w_{p_j}(e) = w_{f(b)}(e) - w_{f(p_j)}(e) \) holds for any \( e \in S^n_{b_j} \). From Claim 2.2, since \( \Delta(b) \) is an increasing function, it follows that \( w_i^*(e) \) increases for all non-capped edge \( e \). Since no new edges become capped, \( S^n_{b_j} = S_{b_j}^* \). It also follows that for any edge \( e \in S^n_{b_j} \), \( w_i^*(e) - w_{p_j}(e) = \Delta(b) - \Delta(p_j) \), proving the lemma also holds for the second phase.

Let \( P^* = \{ e : w_i^*(e) = w^*(e) \} \), representing the set of edges that reached its cap at budget \( b \). The next corollary follows from the proof of the previous lemma.

**Corollary 1.** For an interval \([b_i, b_{i+1}] \), given the finite sequence \( \{p_j\} \) from above, we can explicitly describe \( S_{p_j}^* \). In particular:

- if \( b_i \leq p_j < p_j + 1 \leq \beta_i \), then \( S_{p_j}^* = S_b \setminus P_{p_j}^* \);
- if \( \beta_i \leq p_j \leq p_j + 1 \leq b_{i+1} \), then \( S_{p_j}^* = E(G^*) \setminus P_{p_j}^* \).

Now, we prove the correctness of the algorithm.

**Theorem 1.** The algorithm \( \text{raise.mst} \) gives an optimal increase for continuous MMST.

**Proof.** The strategy of the proof is the same as the one in [12]. We compare the lifted sets \( S_b \) in \( \text{raise.mst} \) to the ones obtained from our decomposition, \( S^n_b \). Given \( 0 \leq i < k \), let \( \{p_k\} \) be the finite sequence defined in the previous lemma (we drop the superscript \( i \) for convenience). We now show that for any interval \([p_j, p_{j+1}] \), lifting \( S_{p_j} = S_b \) in \( \text{raise.mst} \) increases the MST of \( G_{p_j} \) by at least as much as the increase in MST of \( G_{p_{j+1}}^* \) caused by lifting \( S_{p_j}^* \) in our decomposition.

If the lift was performed in the first phase where \( b_i \leq p_j < p_j + 1 \leq \beta_i \), then by Corollary 1, \( S_{p_j}^* \subseteq S_{p_j} = S_b \). Let \( e \in S_{p_j}^* \), \( e' \in s_{\text{mmst}}(e, G_{p_j}) \). By Corollary 1, \( e \in S_b \) and thus \( w_b(e) = w_{b+1}(e) < w_{b+1}(e) \) for all \( b \leq b < b_{i+1} \). We claim that \( w_b(e') = w_{b+1}(e) \) for all \( b_i \leq b \leq b_{i+1} \). If \( e' \in S_b \), since \( \text{raise.mst} \) only lifts edges of the same weight, \( w_b(e') = w_b(e) \). If
\[ e' \notin S_i, \text{ then } w_b(e') = w_b(e'). \] By our choice of \( b, w_{b+1}(e) \leq w_b(e)+1. \) Then, \( e' \in sm_{eq}(e, G_{p_j}) \) implies \( w_{p_j}(e') \leq w_{p_j}(e) = w_b(e) \), proving our claim.

It follows by definition of \( w^*_p \) and \( S^*_p \) that \( w^*_{p_j}(e) = w_{f(p_j)}(e) + \Delta \) and \( w^*_{p_j}(e') \leq w_{f(p_j)}(e') + \Delta \). Note that since \( p_j \leq \beta, f(p_j) < b_{i+1} \) by Claim 2.2. Then, using the claim in the previous paragraph, \( w_{f(p_j)}(e') \leq w_{f(p_j)}(e) \), proving that \( w^*_{p_j}(e') \leq w^*_{p_j}(e) \). Therefore, \( e' \in sm_{eq}(e, G^*_{p_j}) \) and \( sm_{eq}(e, G^*_{p_j}) \subseteq sm_{eq}(e, G^*_{p_j}) \). Then, by Lemma 1, \( \text{coverage}(S^*_p, G^*_p) \leq \text{coverage}(S^*_p, G^*_p) \).

\[
\text{inc-cost}(S_{p_j}, G_{p_j}) \leq \text{inc-cost}(S^*_p, G^*_p)
\]

by the optimal choice of \( S_{p_j} = S_i \).

This implies after spending the same amount of budget \( p_j+1 - p_j \), \text{raise mst} increased the MST value by at least as much as what the optimal solution did in their respective graphs.

Now, suppose the lift occurred in the second phase where \( \beta_i \leq p_j \leq p_{j+1} \). Let \( e \in S_{p_j} \) and \( e' \in sm_{eq}(e, G_{p_j}) \). By Corollary 1, \( e \) is an edge that is not capped after spending budget \( p_j \). We claim that \( w_{b+1}(e') \leq w_{b+1}(e) \). If \( e' \in S_{p_j} \), then \( w_{b+1}(e') = w_{b+1}(e) \) since both edges are lifted together by \text{raise mst}. If \( e' \notin S_{p_j} \), then \( w_{b+1}(e') \leq w_{b+1}(e) \), proving our claim.

By definition of \( w^* \) and \( S^* \), it follows that \( w^*_{p_j}(e) = w_{b+1}(e) + \Delta(p_j) \) and \( w^*_{p_j}(e') \leq w_{b+1}(e') + \Delta(p_j) \). Thus it follows that \( w^*_{p_j}(e') \leq w^*_{p_j}(e) \) and \( w^*_{p_j}(e') \leq w^*_{p_j}(e) \). Then, by Lemma 1, we can conclude that \( \text{coverage}(S^*_p, G^*_p) \leq \text{coverage}(S^*_p, G^*_p) \).

As before,

\[
\text{inc-cost}(S_{p_j}, G_{p_j}) \leq \text{inc-cost}(S^*_p, G^*_p)
\]

by optimality of \( S_{p_j} \).

Then once again, by spending the same budget \( p_j+1 - p_j \), \text{raise mst} increases the MST value by at least as much as the optimal solution. Then, it follows the total increase of MST performed by \text{raise mst} is also optimal.

\[
\sqrt{\text{3 Approximating Unit Perturbation MMST}}
\]

In this section, we present approximation algorithms for both the targeted and budgeted versions of MMST.
3.1 A $2(1 - \frac{1}{n})$-Approximation Algorithm for Targeted MMST

**Theorem 2.** Targeted MMST permits a $2(1 - \frac{1}{n})$-approximation algorithm.

**Proof.** The algorithm in [12] computes in strongly polynomial time the function $f(B)$, the maximum weight of the minimum spanning trees of a graph attainable by spending a budget of value $B$ to increase the weights of its edges, potentially in fractional increments. Given a target $T \in \mathbb{N}$, let $B_T$ such that $f(B_T) = T$. Clearly, $B_T$ is a lower bound of the optimal value for (discrete) unit perturbation to reach MST target weight of $T$.

We will follow the algorithm in [12] with budget $B_T$. Note that the algorithm lifts edges to their tolerance as long as there is enough budget to do so. Since the tolerances are integral, all edges are lifted an integral amount except for possibly the very last lift. Let $S$ denote the last set of edges computed by this algorithm and $\text{balance}$ denote the remaining budget. The set $S$ induces a partition $P_1, \ldots, P_k$ of some graph $G_{wl}$. In the last iteration, the algorithm in [12] lifts all the edges in $S$ by $\lceil \text{balance} \rceil$. The weight of the minimum spanning trees increases by $\text{coverage}(S, G_{wl}) \text{balance} \in \mathbb{N}$. If $\text{balance} \in \mathbb{N}$, then we are done.

Otherwise, we first lift all the edges in $S$ by $\lfloor \text{balance} \rfloor$. Then, it remains to increase the weight of the minimum spanning trees by

$$\text{coverage}(S, G) \left( \text{balance} - \lfloor \text{balance} \rfloor \right) < \text{coverage}(S, G) = \text{coverage}(S, G_{wl}) = k-1.$$  

The remaining budget $R$ is

$$R = \text{balance} - c(S) \lfloor \text{balance} \rfloor = c(S) \left( \text{balance} - \lfloor \text{balance} \rfloor \right).$$  

(2)

Assume that the node sets $P_i$ corresponding to the shores of the partition are ranked by increasing costs $c(\delta(P_i))$. Choose the $q = \text{coverage}(S, G) \left( \text{balance} - \lfloor \text{balance} \rfloor \right)$ cheapest shores $P_1, \ldots, P_q$ and lift all the edges in $\delta(P_1), \ldots, \delta(P_k)$ by one. Since $\text{tolerance}(S, G) \geq \text{balance} > \lceil \text{balance} \rceil$, it follows that the weight of the minimum spanning trees increases by at least $q$. By the choice of the sets $P_1, \ldots, P_q$, we have

$$\frac{1}{q} \sum_{i=1}^{q} c(P_i) \leq \frac{1}{k} \sum_{i=1}^{k} c(P_i) = \frac{2c(S)}{k}.$$  

This yields
\[ \sum_{i=1}^{q} c(P_i) \leq \frac{2q}{k} c(S) \]
\[ = \frac{2}{k} \text{coverage}(S, G) \left( \frac{\text{balance}}{c(S)} - \left\lfloor \frac{\text{balance}}{c(S)} \right\rfloor \right) c(S) \]
\[ = \frac{2(k-1)}{k} \frac{\text{balance}}{c(S)} - \left\lceil \frac{\text{balance}}{c(S)} \right\rceil c(S) \]
\[ \leq \frac{2(n-1)}{n} \frac{\text{balance}}{c(S)} - \left\lfloor \frac{\text{balance}}{c(S)} \right\rfloor c(S) \]
\[ = \frac{2(n-1)}{n} R. \]

Therefore, the total cost used by the approximation algorithm to reach MST target weight of \( T \) is bounded by
\[ B_T - R + \sum_{i=1}^{q} c(P_i) \leq B_T - R + \frac{2(n-1)}{n} R \]
\[ = B_T + (1 - \frac{2}{n}) R \]
\[ \leq 2(1 - \frac{1}{n}) B_T. \]

The theorem follows immediately.

3.2 An \( \frac{opt}{2} - 1 \)-Solution for Budgeted MMST

Recall that in an \( \frac{opt}{2} - 1 \)-solution, the increase in the weight of the MST is at least \( \frac{opt}{2} - 1 \), where \( opt \) is the optimal increase in the weight of the MST for the given budget \( B \).

**Theorem 3.** Budgeted MMST has a polynomial-time algorithm that produces an \( \frac{opt}{2} - 1 \) solution, where \( opt \) is the optimal increase in weight of the MST.

**Proof.** Again, we follow the algorithm in [12]. Let \( S \) denote the last set of edges computed by this algorithm and \( \text{balance} \) denote the remaining budget. The set \( S \) induces a partition \( P_1, \ldots, P_k \) of some graph \( G_{w_1} \). In the last iteration, the algorithm in [12] lifts all the edges in \( S \) by \( \text{balance} \).

If \( \frac{\text{balance}}{c(S)} \in \mathbb{N} \), then we are done. Otherwise, we lift all the edges in \( S \) by \( \left\lfloor \frac{\text{balance}}{c(S)} \right\rfloor \).

After this lifting operation, the remaining budget \( R \) is defined in [2] and the weight of the minimum spanning trees increases by
\[ A + \text{coverage}(S, G) \left\lfloor \frac{\text{balance}}{c(S)} \right\rfloor, \]
where \( A \) denotes the total increase in the MST weight before lifting \( S \). Therefore, the remaining increase \( I \) of the weight of the minimum spanning trees in comparison with the algorithm in [12] is
\[ I = \text{coverage}(S, G) \left( \frac{\text{balance}}{c(S)} - \left\lfloor \frac{\text{balance}}{c(S)} \right\rfloor \right). \]

Since the algorithm in [12] optimally uses the budget fractionally, the optimal increase in the MST using continuous expenditure of the budget \( B \) is

\[ A + \text{coverage}(S, G) \left( \frac{\text{balance}}{c(S)} - \left\lfloor \frac{\text{balance}}{c(S)} \right\rfloor \right), \]

which is an upper bound on \( \text{opt} \), the optimal increase in the weight of the MST for the given budget \( B \).

Assume that the shores \( P_i \) are ranked by increasing costs \( c(\delta(P_i)) \). Suppose first that there exists an index \( q \) such that \( c(\delta(P_1 \cup \cdots \cup P_q)) \leq R < c(\delta(P_1 \cup \cdots \cup P_{q+1})) \). Lift all the edges in \( \delta(P_1), \ldots, \delta(P_q) \) by one.

Since \( \text{tolerance}(S, G) \geq \text{balance}(S) > \left\lfloor \frac{\text{balance}}{c(S)} \right\rfloor \), it follows that the weight of the minimum spanning trees increases by at least \( q \) in this step.

By the ordering of the node sets \( P_i \), we have

\[ \frac{1}{q + 1} \sum_{i=1}^{q+1} c(\delta(P_i)) \leq \frac{1}{k} \sum_{i=1}^{q+1} c(\delta(P_i)) = \frac{2c(S)}{k}. \]

Therefore,

\[ q + 1 \geq \frac{k}{q + 1} \sum_{i=1}^{q+1} c(\delta(P_i)) \geq \frac{kc(\delta(P_1 \cup \cdots \cup P_{q+1}))}{2c(S)} \]

\[ > \frac{kR}{2c(S)} \]

\[ = \frac{k}{2} (\text{balance} - \left\lfloor \frac{\text{balance}}{c(S)} \right\rfloor) \]

\[ \geq \frac{\text{coverage}(S, G)}{2} \left( \frac{\text{balance}}{c(S)} - \left\lfloor \frac{\text{balance}}{c(S)} \right\rfloor \right) \]

\[ = I. \quad (3) \]

This implies \( q > \frac{I}{2} - 1 \), and hence the approximation algorithm increases the weight of the minimum spanning trees increases by at least

\[ A + \text{coverage}(S, G) \left( \frac{\text{balance}}{c(S)} - \left\lfloor \frac{\text{balance}}{c(S)} \right\rfloor \right) \]

\[ + q > \frac{1}{2} \left( A + \text{coverage}(S, G) \left( \frac{\text{balance}}{c(S)} - \left\lfloor \frac{\text{balance}}{c(S)} \right\rfloor \right) + I \right) - 1 \]

\[ \geq \frac{\text{opt}}{2} - 1. \]

In the case where \( c(\delta(P_1)) > R \), since \( c(\delta(P_1)) \leq \frac{1}{k} \sum_{i=1}^{k} c(\delta(P_i)) \), we have

\[ \frac{kR}{2c(S)} < \frac{kc(\delta(P_1))}{2c(S)} \leq 1. \]
The inequalities in (3) imply that $\frac{I}{c(S)} \leq \frac{kR^2}{c(S)}$ and thus $\frac{I}{c(S)} \leq 1$. Therefore, by doing nothing we are still within an additive one of the optimal value of $I$.

$$A + \text{coverage}(S, G) \left\lfloor \frac{\text{balance}(S)}{c(S)} \right\rfloor \geq A + \text{coverage}(S, G) \left\lfloor \frac{\text{balance}(S)}{c(S)} \right\rfloor + \frac{I}{2} - 1$$

$$\geq \frac{1}{2} \left( A + \text{coverage}(S, G) \left\lfloor \frac{\text{balance}(S)}{c(S)} \right\rfloor \right) + 1 - 1$$

$$\geq \frac{\text{opt}}{2} - 1.$$

4 Hardness of MMST

If we impose upper bounds on the extent to which edge weights can be increased, we can encode a $k$-cut problem using MMSTU (the version of MMST with such upper bounds).

Lemma 3. The Budgeted MMSTU problem is NP-complete.

Proof. The problem is clearly in NP. We now reduce min $k$-cut to this problem. Let $(G, k)$ be an instance of $k$-cut. For every edge $e \in E(G)$, consider assigning a cost $c(e) \equiv 1$, a weight $w(e) \equiv 0$ and an upperbound $u(e) \equiv 1$. For $B \in [0, n]$, consider solving the instance of MMSTU with budget $B$ and let $E_B$ be the set of edges giving by the solution that raises the MST to weight $T_B$. Since $G$ starts with a MST of weight 0 and every edge in $S_B$ has a final weight of 1, it follows that $\text{coverage}(S_B, G) = T_B$. Then, it follows that the smallest $B$ such that $T_B \geq k - 1$ provides a minimum $k$-cut for $G$.

What is perhaps a bit surprising is that even without upper bounds and all upgrade costs being unit, MMST is NP-hard.

Theorem 4. Budgeted MMST is NP-complete even with unit downgrading costs.

Proof. Given an instance of Min $k$-Cut, $(G, k)$, consider the following auxiliary graph $G'$. For every edge $e = uv \in E(G)$, we add a clique of size $n^2$ and add edges from the vertices of the clique to the vertices $u$ and $v$. Original edges start with weight 0 and all newly added edges have weight 1. The upgrading cost $c$ for all edges is 1. Note that an initial MST needs to connect the newly added $n^2|E(G)|$ vertices and thus has weight at least $n^2|E|$. An MST of weight $n^2|E(G)|$ exists by simply taking a spanning tree of the original zero-weight edges and greedily appending the newly added vertices to this spanning tree.

Suppose $F \subseteq E(G)$ is an optimal min $k$-cut and $|F| = b$. Consider spending $b$ budget to upgrade the edges in $F$ to weight 1. Since $G \setminus F$ has $k$ connected components in $G$, all the edges with final weight of 0,
$E(G) \setminus F$, also forms $k$ connected components in $G'$. Then it follows that has final weight of an MST in $G'$ is at least $2n|E| + k − 1$. One can also easily construct an MST with such weight by first taking a maximal forest using the edges in $E(G) \setminus F$, then add $k − 1$ edges in $F$ to form a tree that spans all the original vertices and lastly greedily append the newly added vertices in the clique. Therefore under budget $b$, one can increase the MST value by $k − 1$. Then, our goal is to find the least amount of budget $b$ that improves the MST value by $k − 1$ and show such solution can be translated to a $k$-cut.

Consider greedily trying all values of $b$ where $0 \leq b \leq |E(G)| < n$ and solve the instances of MMST on $G'$ with budget $b$. Let $b'$ be the smallest $b$ such that the solution to MMST increased the MST value by $k − 1$. Such $b'$ exists since raising the weight of every original edge by 1 increases the MST by at least $n ≥ k − 1$. —Let $F'$ be the set of edges whose weight was changed by such an optimal solution.

First, we claim that no newly added edge is in $F'$. Suppose for the sake of contradiction that there exists such edge $e' \in F'$ where $e'$ is also in $E(G') \setminus E(G)$. Let $T$ be an MST after upgrading the edges in $F'$ to their final weight. Since $e'$ is a newly-added edge, let $e \in E(G)$ be its associated original edge. Note that the endpoints of $e$ along with the newly-added clique forms a large clique $K'$ of size $n^2 + 2$. Since $|F'| ≤ b' < n^2$, $K' \setminus F'$ is connected. Then, one can construct an MST $T'$ from $T$ that avoids any edges in $E(K) \cap F'$. Consider the path $P'$ in $T'$ that connects the endpoints of $e'$. Note that $P'$ lies in $K'$ and the final weight of every edge in $P'$ is 1. Then, if we did not upgrade $e'$, the weight of every edge in $P'$ is still at most the weight of $e'$. Then, $T'$ is also an MST if we do not upgrade $e'$, contradicting the minimum choice of $b'$.

Next, we claim that no original edge is upgraded more than once. The argument is similar to the one before. Suppose for the sake of contradiction that an edge $e \in E(G)$ is upgraded more than once. Let $K'$ be the clique of size $n^2 + 2$ containing $e$. By similar argument, one can show that there exists an MST $T'$ in $G'$ after upgrading such that $e \notin T'$. Then, one can similarly argue that $T'$ remains an MST if $e$ is only upgraded to 1, contradicting the minimum choice of $b'$.

Then, it follows that $|F'| = b'$. Let $T'$ be an MST of $G'$ before upgrading the edges in $F'$. Note that $|E(T') \cap F'| ≥ k − 1$ since the weight of the final MST is increased by at least $k − 1$. We claim that $T = T'(V(G') \setminus V(G))$ is a spanning tree of $G$. Suppose for the sake of contradiction that $T$ is not connected and contains a cut $\delta(V_1)$. Since $G$ is connected, there exists $e \in \delta(V_1) \cap E(G)$. Let $e' \in \delta(V_1) \cap E(T')$. Note that $e'$ has weight 1 while $e$ has weight 0. Then substituting $e'$ with $e$ in $T'$ creates a better MST, a contradiction.

This implies any MST $T'$ in $G'$ restricted to $G$ is a spanning tree in $G$. Then, $\text{coverage}(F', G) ≥ k − 1$ and thus $F'$ is a $k$-cut in $G$. By our choice of $b'$, it follows that $F'$ is an optimal $k$-cut.

**Corollary 2.** If there exists an $\alpha$-approximation of Budgeted Discrete MMST then there exists an $\alpha$-approximation of Max Components. If there exists an $\alpha$-approximation of Targeted Discrete MMST then there exists an $\alpha$-approximation of $k$-cut.
Proof. It follows from the proof of Theorem 4 that given the auxiliary graph \( G' \), any feasible solution that spends budget \( 0 \leq b \leq n \) to upgrade a set \( F' \) and increases the MST by \( T_b \) can be transformed into a feasible solution that spends budget at most \( b \), only upgrades original edges by at most once, does not upgrade any newly added edges and increases the MST by at least \( T_b \). Then, it follows that an \( \alpha \)-approximation to \( \text{MMST} \) also produces an \( \alpha \)-approximation to the Max Component problem. Similarly, an \( \alpha \)-approximation to the Targeted Discrete \( \text{MMST} \) also provides an \( \alpha \) approximation to the Minimum \( k \)-Cut Problem.

Hardness of Approximation. Let \( G \) be a \( d \)-regular undirected unweighted graph. The edge expansion \( \phi(S) \) of \( S \subset V(G) \) is defined as :

\[
\phi(S) = \frac{|E(S, V(G) \setminus S)|}{d \min\{|S|, |V \setminus S|\}}
\]

where \( E(S, V(G) \setminus S) \) is the set of edges across the partition \((S, V(G) \setminus S)\).

Problem 6 (Small Set Expansion (SSE)).
Given a regular graph \( G \), let \( \delta, \eta \in (0, 1) \). The \( \text{SSE}(\delta, \eta) \) problem is to distinguish between:

1. (Completeness) There exists \( S \subset V \) of size \( \delta |V| \) such that \( \phi(S) \leq \eta \).
2. (Soundness) For every \( S \subset V \) of size \( \delta |V| \), \( \phi(S) \geq 1 - \eta \).

The author in [33] showed that assuming SSE is hard given any \( \eta \) for some \( \delta \), which is called the SSE hypothesis, or SSEH, \( k \)-cut is also hard to approximate to a \( 2 - \varepsilon \) factor. We then provide a similar result for the Max Component Problem.

Theorem 5. Assuming SSEH, it is also hard to approximate Maximum Components to within \((\frac{1}{2} + \varepsilon)\) factor of the optimum for every constant \( \varepsilon > 0 \).

Proof. We reduce SSE to Max Components. Given an instance of \( \text{SSE}(\delta, \eta) \) on a \( d \)-regular graph \( G \) with sufficiently large \( n = |V(G)| \), let \( \varepsilon > 0 \) be a value dependent on \( n, \delta \) and \( \eta \). We specify later their relationship. We will show that if \( B = (\frac{1}{2} + \eta)d \delta n \), then solving Maximum Components with approximation ratio \((\frac{1}{2} + \varepsilon)\) on \( G \) with budget \( B \) is sufficient to solve this instance of \( \text{SSE} \). First, we make the following observations about Completeness and Soundness.

(Completeness) If there exists \( S \subset V \) of size at most \( \delta n \) such that \( \phi(S) \leq \eta \), then consider partition the graph into \( |S| + 1 \) groups where the first group is \( V \setminus S \) and each of the remaining groups contains a single vertex from \( S \). The edges between the partitions are those in \( E(S, V \setminus S) \) and the edges within the set \( S \). There are \( d|S|\phi(S) \leq \eta d|S| \) edges of the former type and only at most \( d|S|/2 \) of the latter. Hence, the number of edges between this partition is at most \((1/2 + \eta)d|S| \leq (1/2 + \eta)d\delta n \). This implies there exists a \( k' \)-cut with at most \( B \) edges where \( k' = \delta n + 1 \).

(Soundness) Suppose that for every \( S \subset V \) of size \( \delta n \), \( \phi(S) \geq 1 - \eta \). Let \( k > \delta n(\frac{1}{2} + \frac{1}{2}) \) and \( T_1, ..., T_k \subset V \) be any \( k \)-partition of the graph.
Assume without loss of generality that $|T_1| \leq \ldots \leq |T_k|$. Let $A = T_1 \cup \ldots \cup T_i$ where $i$ is the maximum index such that $|T_1 \cup \ldots \cup T_i| \leq \delta n$. We can then add up to $x\delta n$ nodes to $A$, where $(1-x)\delta n$ is the size of $A$, to obtain a set $A'$ of size exactly $\delta n$. Note that, since $|A'| = \delta n$, $\phi(A') \geq 1 - \eta$, and so $E(A', V \setminus A') \geq (1 - \eta)d[A'] = (1 - \eta)d\delta n$. This implies that $E(A, V \setminus A) \geq (1 - \eta)d\delta n - x\delta nd$. We would like to prove that we always have $(1 - \eta)d\delta n - x\delta nd > B = \left(\frac{1}{2} + \eta\right)d\delta n$. That is, that $x \leq (1 - \eta) - \left(\frac{1}{2} + \eta\right) = \frac{1}{2} - 2\eta$. Thus we wish to prove that $|A| \geq \left(\frac{1}{2} + 2\eta\right)d\delta n$.

Suppose for the sake of contradiction that $|A| < \left(\frac{1}{2} + 2\eta\right)d\delta n$. Then, $n = |T_1 \cup \ldots \cup T_k| > |T_{i+1} \cup \ldots \cup T_k| \geq (k - i) \cdot |T_{i+1}| > \left(\frac{1}{2} + \frac{\epsilon}{2}\right)d\delta n - \left(\frac{1}{2} + 2\eta\right)d\delta n = \left(\frac{\epsilon}{2} - 2\eta\right)d\delta n^2 \left(\frac{1}{2} - 2\eta\right)$

However, if $\eta < \min(\frac{\epsilon}{4}, 1/4)$, then this is positive and tends towards infinity as $n$ goes to infinity. Thus for sufficiently large $n$, this inequality is false, a contradiction. This proves that indeed, $|A| \geq \left(\frac{1}{2} + 2\eta\right)d\delta n$.

Thus this means that $E(A, V \setminus A) \geq B$, which means that any $k$-cut has at least $B$ edges when $k > \delta n\left(\frac{3}{2} + \frac{\epsilon}{2}\right)$.

In other words, if we are in the completeness case, then there exists a $B$ cost cut with $\delta n + 1$ components, whereas in the soundness case, all $B$ cost cuts give at most $\delta n\left(\frac{3}{2} + \frac{\epsilon}{2}\right)$ components. The gap between the two is more than $\frac{1}{2} + \epsilon$ and so an approximation of Maximum Components to within $\left(\frac{3}{2} + \epsilon\right)$ can distinguish between the two cases. This concludes the proof.

\[ \square \]

**Corollary 3.** Assuming SSEH, Budgeted Discrete MMST is NP-hard to approximate to a $\frac{1}{2} + \epsilon$-factor for $\epsilon > 0$.

Assuming SSEH, Targeted Discrete MMST is NP-hard to approximate to a $2 - \epsilon$-factor for $\epsilon > 0$.

### 5 MST Fortification with all edges starting with the Same Weight

In the previous section, we have shown that the unit MST upgrading problem is hard even if all the costs are unitary. However, it is unknown if the problem remains hard if all initial weights start off the same. Thus, consider the special targeted version of unit MST upgrading where all the initial weights of the graph start with the same weight. Without loss of generality, assume the weights all start with 0. First we provide a polytime algorithm that solves this problem with a fixed target MST value of $T$. For this section, we drop the term $G$ and use $\text{coverage}(F)$ to denote the coverage of set $F$ in $G$ with initial weights of zero.
Theorem 6. Let $G$ be a graph whose edges have weight 0 and upgrading costs of $c_e$. Given a fixed target MST value of $T$, there exists an algorithm that runs in $n^{O(T)}$ time and upgrades edges to produce a final MST value of $T$ while minimizing the total cost of upgrading.

Note that since all the costs are integral, if it takes budget $B$ to raise the MST by $T$, then $B \geq T$. Thus, given a bounded constant budget $B$, the special budgeted version of unit MST upgrading can also be solved exactly in polytime.

To prove the theorem, we require a supermodular property of coverage.

Lemma 4. (Supermodularity of coverage)
For any $F, F' \subseteq E(G)$, $\text{coverage}(F) + \text{coverage}(F') \leq \text{coverage}(F \cup F') + \text{coverage}(F \cap F')$.

Proof. First we decompose our sets by edge weight $F = \bigcup F_{w_i}$ and $F = \bigcup F'_{w_i}$, where $F_{w_i}$ and $F'_{w_i}$ are the set of edges with weight $w_i$ that are also in $F$ and $F'$ respectively. Then

$$\text{coverage}(F) + \text{coverage}(F') = \text{coverage}(\cup F_{w_i}) + \text{coverage}(\cup F'_{w_i})$$

and

$$\text{coverage}(F \cup F') + \text{coverage}(F \cap F') = \text{coverage}(\cup (F_{w_i} \cup F'_{w_i})) + \text{coverage}(\cup (F_{w_i} \cap F'_{w_i}))$$

Note that the second equality follows from the fact that $F_{w_i} \cap F'_{w_j} = \emptyset$ for $i \neq j$. It now suffices to show that for $F$ and $F'$ of a single weight class, the inequality holds.

In consequence, we assume without loss of generality that $F$ and $F'$ both contain only edges of weight $w_i$. The coverage of any edge-set is the same as the number of additional components created from its deletion in $G_{w_i}$. Let $\{G_i\}_{i=1}^k$ be the connected components of $G_{w_i} \setminus (F \cap F')$. Let $F_i, F'_i$ be the set of edges in $G_i$ that belongs in $F, F'$ respectively. Let $a_i, a'_i, b_i$ represent the number of additional components created from $G_i$ by deleting $F_i, F'_i, F_i \cup F'_i$ respectively. Note that $a_i, a'_i, b_i$ also represent the coverage of $F_i, F'_i, F_i \cup F'_i$ respectively in $G_i$. Since $F_i \cap F'_i = \emptyset$ by construction, it follows that any spanning tree of $G_i$ must contain $a_i, a'_i$ edges from $F_i, F'_i$ respectively. Thus $b_i \geq a_i + a'_i$. In $G_{w_i}$, the coverage of $F$ can be viewed as first deleting $F \cap F'$ and then deleting the $F_i$’s in sequence and counting how many additional components it creates. In other words $\text{coverage}(F) = k - 1 + \sum a_i$. We can obtain similar equations for $\text{coverage}(F')$, $\text{coverage}(F \cup F')$.

Then,

$$\text{coverage}(F) + \text{coverage}(F') = (k - 1 + \sum a_i) + (k - 1 + \sum a'_i)$$

$$\leq (k - 1) + (k - 1 + \sum b_i) = \text{coverage}(F \cap F') + \text{coverage}(F \cup F')$$

\qed
To prove Theorem 6, we also need to solve the following variant of the knapsack problem:

**Problem 7 (Unbounded Knapsack Problem).** For \( i \in [n] \), let \( w_i \) and \( p_i \) be respectively the weight and the profit of item \( i \). Given fixed capacity \( W \), find \( x_i \in \mathbb{Z}_{\geq 0} \) that maximizes \( \sum_{i=1}^{n} p_i x_i \) subject to \( \sum_{i=1}^{n} w_i x_i \leq W \).

**Lemma 5.** Given an instance of the Unbounded Knapsack Problem, if each profit \( p_i \) is of order \( O(n) \), then a solution can be found in time \( O(n^2 p^3) \) where \( p = \max_i p_i \).

A solution for this above lemma is proposed in [42]; we formally reproduce the proof below for completeness.

**Proof.** Given an instance the Unbounded Knapsack Problem, let \( x^* \) be an optimal solution and \( P^* \) be the optimal profit. Assume without loss of generality that \( p_1/w_1 = \max_{i=1}^{n} \{p_i/w_i\} \). In other words, item 1 gives the most bang-for-buck. Let \( P = \sum_{i=2}^{n} p_i p_i \). We break into two cases depending on how \( P^* \) compares to \( P \).

**Case 1** Suppose \( P^* \leq P \). Then, consider the following dynamic programming. For \( i \in [n], 1 \leq p \leq P \), let \( f(i, p) \) be the least possible weight of a solution such that it contains at least one copy of item \( i \) and its profit is exactly \( p \). For \( i \in [n] \), let \( f(i, p_i) = w_i \) and let \( f(i, \infty) = \infty \) for all \( p < p_i \). This corresponds to the fact that if a solution contains item \( i \), its profit must be at least \( p_i \). For \( i \in [n], p_i < p \leq P \), let \( f(i, p) = \min_{j \in [n]} \{ f(j, p - p_1) \} + w_i \). Intuitively, this recursion says that if a solution has profit \( p \) and contains item \( i \), then removing it results in a solution with profit \( p - p_i \) and weight \( f(i, p) - w_i \). Thus, searching for the best way to achieve profit \( p - p_i \) also leads to a solution for \( f(i, p) \).

Since \( i, p \) are polynomially bounded, all values of \( f \) can be computed in polytime. For \( 1 \leq p \leq P \), let \( w(p) = \min_{i \in [n]} \{ f(i, p) \} \). Then, simply find the largest value of \( p \) such that \( w(p) \leq W \).

**Case 2** Suppose \( P^* > P \). Without loss of generality, assume \( x^* \) is an optimal solution that maximizes the value \( x^*_1 \). First we claim that \( x^*_1 \neq 0 \).

Since \( \sum_{i=1}^{n} x^*_i p_i = P^* > P = \sum_{i=1}^{n} p_i p_i \), there exists \( i \in [n] \) such that \( x^*_i \geq p_i \). If \( i = 1 \), then we are done.

Otherwise, since \( p_1/w_1 \geq p_i/w_i \), it follows that by swapping out \( p_1 \) copies of item \( i \) and replace it with \( p_i \) copies of item 1, the profit does not change and the weight does not increase. Since \( x^* \) is an optimal solution that maximizes \( x^*_1 \), it follows that \( x^*_1 > 0 \).

Then, it follows that removing a single copy of item 1 from \( x^* \) is an optimal solution for profit \( P^* - p_1 \). Therefore one can recursively remove \( p_1 \) from \( P^* \) until the profit falls below \( P \) and use the solutions from Case 1 to build \( x^* \).

In particular, consider \( W' \) such that \( W - w_1 < W' \leq W \). Let \( w' \) be the largest integer such that \( W' \leq w(P) \) and \( W' = W - k' w_1 \) for some integer \( k' \).

Note that \( x^* \) has total weight \( W' \) if and only if there exists \( p' \) such that \( w(p') = w' \). Then, let \( \mathcal{W} \) be the set of weights \( W' \) where such \( p' \) exists for the associated \( w' \). Since \( x^* \) contains a copy of item 1, its total
weight must be one of the values in \( \mathcal{W} \). For each \( W' \in \mathcal{W} \), its associated solution must be \( x' \) and an additional \( k' \) copies of item 1. Therefore, its associated profit is \( p' + k'p_1 \). Then, taking the solution with the largest profit \( p_1 + k_1p_1 \) is an optimal solution.

**Proof of Theorem 6.** Let \( w^* \) be an optimal solution representing the final weights of every edge. Let \( b^* \) be the upgrading cost to reach the final weights \( w^* \). Let \( E_i^* \) represent the set of edges whose final weight is at least \( i \). Then, we can decompose \( w^* \) into a sequence of lifts where we first lift \( E_i^* \) by one unit, then \( E_2^* \) and so on. Note that at the time of lifting \( E_i^* \), the set \( E_i^* \) has the highest weight among all edges of the graph. Then, it follows from Lemma 1 that every lift increases the MST by exactly the same amount as the coverage of \( E_i^* \) at the beginning where all weights are the same. Thus, it motivates us to only look at the coverage of sets at the initial stage where all weights are 0.

Define \( F_i \) to be a set of edges in \( G \) with the least upgrading cost such that \( G \setminus F_i \) creates \( i \) additional components for \( i \leq T \). Note that finding these sets is equivalent to solving \( T \) iterations of the Min \( i \)-Cut problem, which can be solved in \( n^{O(T)} \) time. Let \( c(F_i) \) represent the cost to upgrade the set \( F_i \) by one unit. It follows that an optimal solution is a combination of these cuts that maximizes the sum of the coverage while ensuring the total cost is within budget.

Then, consider the following instance of the Unbounded Knapsack Problem. Consider \( T \) items \( F_1, \ldots, F_T \) where \( F_i \) has weight \( c(F_i) \) and profit \( i \). It follows from Lemma 5 that a solution can be obtained in polynomial time.

Let \( x^* \) be an optimal solution to the knapsack problem. Now, we slightly modify the knapsack solution so that the subsets form a chain. Suppose there exists \( i, j \) where neither \( F_i, F_j \) are subsets of the other one and their \( x^* \) values are non-zero. Then, by Lemma 5, we can use \( F_i \cap F_j, F_i \cup F_j \) instead since it does not change the total upgrading cost. Then, we can perform these uncrossing operations to obtain a family of edge-sets \( F_k^* \) and integers \( y_k^* \) such that \( \sum_{i \leq k} y_k^* \text{coverage}(F_k^*) \geq \sum_{i \leq k} x^*_i \text{coverage}(F_i) \). It is worth mentioning that since there are at most \( T \) edge-sets with non-zero \( x^* \), \( y^* \) values, it takes at most \( T^2 \) many applications of Lemma 5 to obtain the sets \( F_k^* \). Since \( F_k^* \) forms a chain, we can lift them in sequence from the largest to the smallest by \( y_k^* \) amount at a time. It follows from Lemma 4 that every lift increases the MST by exactly \( \text{coverage}(F_k^*) \). Thus, it follows that this process reaches target \( T \) and minimizes the downgrading cost.

**Corollary 4.** For every constant \( \varepsilon > 0 \), there exists a \((1/2 - \varepsilon)\)-approximation algorithm for MMST with uniform starting weights.

**Proof.** Let us fix \( \varepsilon > 0 \). We now describe an algorithm that runs in polynomial time and that gives a \((1/2 - \varepsilon)\)-approximation algorithm for MMST with uniform starting weights. Let \( (G, 0, c, B) \) be an instance of MMST with 0 weight for all edges. Let \( opt \) be the optimal increase for the MST in an optimal solution. We first run the greedy algorithm in Theorem 3 in polynomial time, giving us a \( t \) increase in the MST weight, with \( opt \geq t \geq \frac{2\varepsilon}{\varepsilon} - 1 \). If \( t \geq \frac{\varepsilon}{2} \), then \( opt \geq \frac{1}{2} \) and so \( t \geq opt(\frac{1}{2} - \frac{1}{opt}) \geq \frac{\varepsilon}{4} \).
opt(\(\frac{1}{2} - \varepsilon\)). If \(t < \frac{1}{2}\), then \(\frac{1}{2} > \frac{\text{opt}}{2} - 1\), and \(\text{opt} < 2 + \frac{\varepsilon}{2}\) and so since \(\varepsilon\) is fixed, by Theorem 5 an optimal solution can be obtained in this case in time \(n^{O(2+2/\varepsilon)}\) which is polynomial in \(n\) for any fixed constant \(\varepsilon\).

Thus, in all cases, the algorithm runs in polynomial time and finds a \((1/2 - \varepsilon)\)-approximation.

\[\square\]

6 Extension and Open Problem

We expect our modifications to work in a straightforward manner for the extension of the main problems from MSTs (graphical matroid bases) to general matroid bases following the framework in [13]. The main open problem from our work is to extend it to the case of directed graphs.
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