Learning Trembling Hand Perfect Mean Field Equilibrium for Dynamic Mean Field Games

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Abstract

Mean Field Games (MFG) are those in which each agent assumes that the states of all others are drawn in an i.i.d. manner from a common belief distribution, and optimizes accordingly. The equilibrium concept here is a Mean Field Equilibrium (MFE), and algorithms for learning MFE in dynamic MFGs are unknown in general due to the non-stationary evolution of the belief distribution. Our focus is on an important subclass that possess a monotonicity property called Strategic Complementarities (MFG-SC). We introduce a natural refinement to the equilibrium concept that we call Trembling-Hand-Perfect MFE (T-MFE), which allows agents to employ a measure of randomization while accounting for the impact of such randomization on their payoffs. We propose a simple algorithm for computing T-MFE under a known model. We introduce both a model-free and a model based approach to learning T-MFE under unknown transition probabilities, using the trembling-hand idea of enabling exploration. We analyze the sample complexity of both algorithms. We also develop a scheme on concurrently sampling the system with a large number of agents that negates the need for a simulator, even though the model is non-stationary. Finally, we empirically evaluate the performance of the proposed algorithms via examples motivated by real-world applications.

1 Introduction

Strategic complementarities refers to a well-established strategic game structure wherein the marginal returns to increasing one’s strategy rise with increases in the competitors’ strategies [16, 22, 1]. Many practical scenarios with this property of strategic alignment, including pricing in oligopolistic markets, adoption of network technology and standards, deposits and withdrawals in banking, weapons purchasing in arms races etc., have been identified, and our running example is that of dynamically adopting actions to prevent the spread of a computer or human virus, wherein stronger actions towards maintaining health (eg., installing patches, or wearing masks) by members enhances the returns (eg., system reliability or economic value) to a particular individual following suit.

The above examples are characterized by a large number of agents following the dynamics of repeated action, reward, and state transition that is a characteristic of stochastic games. However, analytical complexity implies that most work has focused on the static scenario with a small number of agents (see [16]). Recently, there have been attempts to utilize the information structure of a mean field game (MFG) [12, 21, 1, 10, 13, 14] to design algorithms to compute equilibria under a known model in the large population setting [11]. Here, each agent assumes that the states of all others are drawn in an i.i.d. manner from a common mean field belief distribution, and optimizes accordingly. However, the model is non-stationary due to the change in the mean field distribution at each time step, and provably convergent learning algorithms for identifying equilibrium strategies under an unknown model are currently unavailable.

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Main Contributions: We study the problem of learning under an unknown model in stochastic games with strategic complementarities under the mean field setting. Our main contributions are:

(i) We introduce the notion of trembling-hand perfection to the context of mean field games, under which a known randomness is introduced into all strategies. Unlike an ε-greedy policy in which randomization is added as an afterthought to the optimal action, under trembling-hand perfection, optimal value is computed consistently while accounting for the randomness in the strategy.

(ii) We describe TQ-value iteration, based on generalized value iteration, as a means of computing trembling-hand consistent TQ-values. We introduce a notion of equilibrium that we refer to as trembling-hand-perfect mean field equilibrium (T-MFE), and show existence of T-MFE and a globally convergent computational method in games with strategic complementarities.

(iii) We propose two learning algorithms—model-free and model-based—for learning T-MFE under an unknown model. The algorithms follow an iterative structure of identifying TQ-values to find a candidate strategy, and then taking only one-step McKean-Vlasov dynamics to update the mean field distribution. We show convergence and determine sample complexity bounds on each.

(iv) Finally, we develop a fully online learning scheme that utilizes the large population of agents to concurrently sample the real world. Our algorithm only needs a one-step McKean-Vlasov mean field update under each strategy, which automatically happens via agents applying the current strategy for one step. This obviates the need for a multi-step simulator required by typical RL approaches.

Related Work: There has recently been much interest in the intersection of machine learning and collective behavior in the form of stochastic games with a large population. Many of the issues faced in simultaneous learning and decision making are exemplified in [9], which presents an existence and model-free learning algorithm for MFE under contraction assumptions on the trajectories of state and action distributions over time. However, these conditions may not be verifiable for a given game in a straightforward manner. A more traditional game theoretic approach is to focus on specific classes of games that possess verifiable structural properties on information, payoffs and preferences. In line with this approach is learning in structured MFGs, such as in linear-quadratic, oscillator or potential game settings [11, 25, 5, 6]. Other approaches provide structure to the problem by considering localized effects, such as local interactions [24], local convergence [15], or a local version of Nash equilibrium [18]. While we are not aware of any work that considers learning in games with strategic complementarities, a survey on the algorithmic aspects of multi-agent RL, including a review of existing work in the mean field domain is available in [26].

Our focus on strategic complementarities aligns with the notion of structured games with verifiable properties. We develop a strategically consistent approach to learning via the trembling-hand concept, rather than arbitrarily adding a modicum of exploration to best responses as do many existing works. In turn, this allows us to explore both model-free and model-based methods to compute and learn optimal trembling-hand perfect MFE, including showing global convergence and determining their sample complexities. Furthermore, we exploit the large population setting to obtain samples without the aid of a simulator, which enables learning directly from the real system.

2 Mean Field Games and Trembling-Hand Perfection

Mean Field Games: An N-agent stochastic dynamic game is represented as \((S, A, P, (r^i)_{i=1}^N, \gamma)\), where \(S\) and \(A\) are the state and action spaces, respectively, both assumed to be finite. At time \(k\), agent \(i\) has state \(s^i_k \in S\), takes action \(a^i_k \in A\), and receives a reward \(r^i(s^i_k, a^i_k)\). Here, \(s^i_k = (s^i_k)_{i=1}^N\) is the system state and \(a_k = (a^i_k)_{i=1}^N\) is the joint action. The system state evolves according to
transition kernel $s_{k+1} \sim P(\cdot|s_k, a_k)$. Each agent aims at maximizing the infinite horizon cumulative discounted reward $E[\sum_{k=0}^{\infty} \gamma^k r^i(s_k, a_k)]$, with discount factor $\gamma \in (0, 1)$.

Identification of a best response is computationally hard under a Bayesian framework, and a more realistic approach, aligned with a typical agent’s computational capabilities is to reduce the information state of each agent to the so-called mean field state distribution [12] [21] [10] [13] [14].

We assume that $|r(s, a, z)| \leq 1$. The mean field $z_k$ evolves following the discrete time McKean-Vlasov equation

$$z_{k+1} = \Phi(z_k, \mu), \text{ where, } z_{k+1}(s') = \Phi(z_k, \mu)(s') = \sum_{s \in \mathcal{S}} \sum_{a \in \mathcal{A}} z_k(s) \mu(s, a) P(s'|s, a, z_k).$$

The value function $V_{\mu, z}$ corresponding to the strategy $\mu$ and the mean field $z$ is defined as $V_{\mu, z}(s) = E[\sum_{k=0}^{\infty} \gamma^k r(s_k, a_k, z)|s_0 = s]$, with, $a_k \sim \mu(s_k, \cdot)$, $s_{k+1} \sim P(\cdot|s_k, a_k, z)$.

**Mean Field Games with Strategic Complementarities (MFG-SC):** Structural assumptions on the nature of the MFG are needed in order to show existence of an equilibrium, and to identify convergent dynamics. Our focus is on a particular structure called strategic complementarities that aligns the increase of an agent’s strategy with increases the competitors’ strategies [17] [22] [1].

We introduce some concepts before defining MFG-SC. The partially ordered set $(\mathcal{X}, \succeq)$ is called a lattice if for all $x, y \in \mathcal{X}$, the elements $\sup\{x, y\}$ and $\inf\{x, y\}$ are in $\mathcal{X}$. $\mathcal{X}$ is a complete lattice if for any $S \subseteq \mathcal{X}$, both $\sup S$ and $\inf S$ are in $\mathcal{X}$. A function $f : \mathcal{X} \times \mathcal{X} \rightarrow \mathbb{R}$ is said to be supermodular if $f(\sup\{x, x'\}) + f(\inf\{x, x'\}) \geq f(x) + f(x')$ for any $x, x' \in \mathcal{X}$. A function $f : \mathcal{X} \times \mathcal{Y} \rightarrow \mathbb{R}$ is said to have increasing differences in $x$ and $y$ if for all $x' \succeq x, y' \geq y$, $f(x', y') - f(x', y) \geq f(x, y') - f(x, y)$. A correspondence $T : \mathcal{X} \rightarrow \mathcal{Y}$ is non-decreasing if $x' \succeq x, y \in T(x)$, and $y' \in T(x')$ implies that $\sup\{y, y'\} \in T(x')$ and $\inf\{y, y'\} \in T(x)$.

For probability distributions $p, p' \in \mathcal{P}(\mathcal{X})$, we say $p$ stochastically dominates $p'$, denoted as $p \succeq_{SD} p'$, if $\sum_{x \in \mathcal{X}} f(x)p(x) \geq \sum_{x \in \mathcal{X}} f(x)p'(x)$ for any bounded non-decreasing function $f$. The conditional distribution $p(\cdot|y)$ is stochastically non-decreasing in $y$ if for all $y' \succeq y$, we have $p(\cdot|y') \succeq_{SD} p(\cdot|y)$. Finally, the conditional distribution $p(\cdot|y, z)$ has stochastically increasing differences in $y$ and $z$ if $\sum_{x} f(x)p(x|y, z)$ has increasing differences in $y$ and $z$ for any bounded non-decreasing function $f$.

We now give the formal definition of mean field games with strategic complementarities [1].

**Definition 1 (MFG-SC).** Let $\Gamma$ be a stationary mean field game. We say that $\Gamma$ is a mean field game with strategic complementarities if it has the following properties:

(i) Reward function: $r(s, a, z)$ is non-decreasing in $s$, supermodular in $(s, a)$, and has increasing differences in $(s, a)$ and $z$. Also, $\max_a r(s, a, z)$ is non-decreasing in $s$ for all fixed $z$.

(ii) Transition probability: $P(\cdot|s, a, z)$ is stochastically supermodular in $(s, a)$, has stochastically increasing differences in $(s, a)$ and $z$, and is stochastically non-decreasing in each of $s, a,$ and $z$. 

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MFG-SC Example—Infection Spread: We assume a large but fixed population of agents (computers or humans). At any time step, an agent may leave the system (network or town) with probability $ζ$, and is immediately replaced with a new agent. The state of an agent $s ∈ Z^+$ is its health level, and the agent can take action $a ∈ \{1, 2, 3, ..., |A|\}$ (installing security patches, wearing a mask etc.) to stay healthy. The susceptibility of each agent, $p(s)$ is a decreasing function in state (higher health implies lower susceptibility). At each time step, the agent interacts with the ensemble of agents who have a mean field state distribution $z$. We define infection intensity $i_s = c_p(\sum_{s ∈ S} sz(s))$, which can be interpreted as the probability of getting infected via interaction with the population.

The resultant next state $s’ = (s + a − w_1)_+1\{E_1\} + (s + a)1\{E_2\} + w_21\{E_3\}$, where $(.)_+$ denotes projection around zero (state is non-negative), $w_1, w_2 ∈ Z^+$ are realizations of non-negative random variables, and $E_1s$ are mutually exclusive events with probabilities $i_z(1 − ζ), (1 − i_z)(1 − ζ)$, and $ζ$, respectively. Events $E_1$ and $E_2$ correspond to the agent remaining in the system, and being infected (health may deteriorate) or not infected, respectively. $E_3$ is the event that the agent leaves and is replaced with an agent with random state (regeneration). An agent receives a reward that depends on its own immunity $1 − p(s)$ as well as that of the population (i.e., system value increases with immunity), but pays for its action. Hence, $r(s, a, z) = δ_1(1 − p(s)) + δ_2\sum_{s ∈ S} z(s)(1 − p(s)) − δ_3a$, where $δ_i$s are positive constants.

It is easy to verify that the model has non-decreasing differences in the transition matrix. Also, it encourages crowd seeking behavior, where the mean field positively affects rewards obtained. Showing that it has strategic complementarities is straightforward.

Trembling-Hand-Perfect Mean Field Equilibrium: The notion of trembling-hand-perfection is a means of refining the Nash equilibrium concept to account for the fact that equilibria that naturally occur are often those that are optimal when a known amount of randomness is introduced into the strategies employed to ensure that they are totally mixed, i.e., all actions will be played with some (however small) probability. Thus, the agent is restricted to only playing such mixed (randomized) strategies, but maintains strategic consistency in that it accounts for the probability of playing each action while calculating the expected payoff of such a totally mixed strategy.

Formalizing the above thoughts, we denote the set of trembling-hand strategies as $Π^ε$. A trembling-hand strategy $µ ∈ Π^ε$ is a mapping $µ : S → P(\mathcal{A})$, where $P(\mathcal{A})$ is the set of $ε$-randomized probability vectors over $\mathcal{A}$. Any probability vector in $P(\mathcal{A})$ has the value $(1 − ε)$ for one element and the value $ε/(|A| − 1)$ for all the other elements. Thus, any $µ ∈ Π^ε$ has the following form: $µ(s, a) = (1 − ε)$ for $a = a_ε$ for some $a_ε$ and $µ(s, a) = ε/(|A| − 1)$ for all other $a ∈ \mathcal{A}$. In the standard reinforcement learning parlance, $Π^ε$ is essentially the set of all $ε$-greedy policies.

The main difference between a trembling-hand strategy and an $ε$-greedy policy lies in the value function. Recall that under the $ε$-greedy idea, the agent computes the pure (deterministic) best response policy, and then arbitrarily adds randomization. However, under the strategic game setting, choosing a strategy that could result in an arbitrary loss of value is impermissible. Rather, the agent must compute the best trembling-hand strategy, i.e., it must account for the impact on value of the $ε$ randomness. Formally, we first define the optimal trembling-hand value function corresponding to the mean field $z$ as $V^*_z = \max_{µ ∈ Π^ε} V_{µ,z}$. The optimal trembling hand strategy $µ^*_z$ corresponding to the mean field $z$ is then defined as $µ^*_z ∈ Ψ(z)$, where $Ψ(z) = \arg \max_{µ ∈ Π^ε} V_{µ,z}$.

We define trembling-hand-prefect mean field equilibrium (T-MFE) in terms of a trembling-hand strategy $µ^*$ and a mean field distribution $z^*$ that must jointly satisfy, (i) optimality—the strategy $µ^∗$ must be superior to all other strategies, given the belief $z^*$, and (ii) consistency—given a candidate mean field distribution $z^*$, the strategy $µ^*$ must regenerate $z^*$ under the McKean-Vlasov dynamics.
Definition 2 (T-MFE). Let $\Gamma$ be a stationary mean field game. A trembling-hand-perfect mean field equilibrium $(\mu^*, z^*)$ of $\Gamma$ is a strategy $\mu^* \in \Pi^e$ and a mean field $z^*$ such that, (optimality condition) $\mu^* \in \Psi(z^*)$, and (consistency condition) $z^* = \Phi(z^*, \mu^*)$.

3 Existence and Computation of T-MFE

Existence of T-MFE: We first introduce a method to compute the optimal trembling-hand value function $V^*_z$ for a given mean field $z$. Note that classical value iteration for any given finite MDP will converge under determinstic policies (pure strategies)—something that is not possible under our restriction to trembling hand strategies $\Pi^e$, which only allows totally mixed strategies. We overcome this issue by using a generalized value iteration approach \cite{19}. Here, rather than the value function, we compute the $Q$-value function, which for a strategy $\mu$ and a given mean field $z$ is defined as $Q_{\mu,z}(s,a) = E[\sum_{t=0}^{\infty} \gamma^t r(s_t, a_t, z)|s_0 = s, a_0 = a]$, with, $a_t \sim \mu(s_t, \cdot)$, $s_{t+1} \sim P(\cdot|s_t, a_t, z)$. The optimal trembling-hand $Q$-value function (TQ-value function) for a given mean field $z$ is then defined as $Q^*_z = \max_{a \in \Pi^e} Q_{\mu,z}$. The optimal trembling-hand strategy $\mu^*_z$ for a given mean field $z$ can then be computed as $\mu^*_z = \pi^*_Q$.

For any given $Q$-value function, define the trembling-hand strategy $\pi^*_Q$ and the function $G(Q)$ as

$$\pi^*_Q(s,a) = \begin{cases} (1-\epsilon) & \text{for } a = \arg \max_b Q(s,b) \\ \epsilon/(|A| - 1) & \text{for } a \neq \arg \max_b Q(s,b) \end{cases}, \ G(Q)(s) = \sum_{a \in A} \pi^*_Q(s,a) \ Q(s,a). \quad (2)$$

Note that $\pi^*_Q$ is the usual $\epsilon$-greedy policy with respect to $Q$. Using the above notation, we define the TQ-value operator $F_z$ for a given mean field $z$ as

$$F_z(Q)(s,a) = r(s,a,z) + \gamma \sum_{s'} P(s'|s,a,z) \ G(Q)(s'). \quad (3)$$

The TQ-value operator $F_z$ has properties similar to the standard Bellman operator. In particular, we show below that $F_z$ is a contraction with $Q^*_z$ as its unique fixed point.

Proposition 1. (i) $F_z$ is a contraction mapping in sup norm for all $z \in \mathcal{P}(S)$. More precisely, $\|F_z(Q_1) - F_z(Q_2)\|_{\infty} \leq \gamma \|Q_1 - Q_2\|_{\infty}$ for any $Q_1, Q_2$, and for all $z \in \mathcal{P}(S)$.

(ii) The optimal trembling-hand $Q$-value function $Q^*_z$ for a given mean field $z$ is the unique fixed point of $F_z$, i.e., $F_z(Q^*_z) = Q^*_z$.

The contraction property of $F_z$ implies that the iteration $Q_{m+1,z} = F_z(Q_{m,z})$ will converge to the unique fixed point of $F_z$, i.e., $Q_{m,z} \to Q^*_z$. We call this procedure as TQ-value iteration.

From the above result, we can compute the optimal trembling-hand strategy for a given mean field $z$. However, it is not clear if there exists a T-MFE $(z^*, \mu^*)$ that simultaneously satisfies the optimality condition and consistency condition. We answer this question affirmatively below.

Theorem 1. Let $\Gamma$ be a stationary mean field game with strategic complementarities. Then, there exists a trembling-hand-perfect mean field equilibrium for $\Gamma$.

The proof of existence follows naturally from the monotonicity properties of MFG-SC. Thus, given this game structure, no additional conditions are needed to show existence.
Computing T-MFE: Given that a T-MFE exists, the next goal is to devise an algorithm to compute a T-MFE. A natural approach is to use a form of best-response dynamics as follows. Given the candidate mean field $z_k$, the trembling best-response strategy $\mu_k$ can be computed as $\mu_k = \Psi(z_k)$. While the typical approach would be to then compute the stationary distribution under $\mu_k$, we simply update the next mean field $z_{k+1}$ by using just one-step McKean-Vlasov dynamics as $z_{k+1} = \Phi(z_k, \mu_k)$, and the cycle continues. While the approach is intuitive and reminiscent of the best-response dynamics proposed in [1], it is not clear that it will converge to any equilibrium. We show that in mean field games with strategic complementarities, such a trembling best-response (T-BR) process converges to a T-MFE. Our computation algorithm, which we call the T-BR algorithm, is presented in Algorithm 1.

Algorithm 1: T-BR Algorithm
1: Initialization: Initial mean field $z_0$
2: for $k = 0, 1, 2, \ldots$ do
3: For the mean field $z_k$, compute the optimal TQ-Value function $Q^*_z$ using the TQ-value iteration $Q_{m+1,z_k} = F(Q_{m,z_k})$.
4: Compute the strategy $\mu_k = \Psi(z_k)$ as the trembling-hand strategy w.r.t $Q^*_z$, i.e., $\mu_k = \pi^\epsilon_{Q^*_z}$.
5: Compute the next mean field $z_{k+1} = \Phi(z_k, \mu_k)$.
6: end for

We now give the formal convergence result of the T-BR algorithm.

Theorem 2. Let $\Gamma$ be a stationary mean field game with strategic complementarities. Let $\{z_k\}$ and $\{\mu_k\}$ be the sequences of mean fields and strategies generated according to Algorithm 1. Then $\mu_k \rightarrow \mu^*$ and $z_k \rightarrow z^*$ as $k \rightarrow \infty$ where $(\mu^*, z^*)$ constitutes a T-MFE of $\Gamma$.

Remark 1. Consider the sequence of mean fields $\{z_k\}$ generated by the T-BR algorithm. According to Theorem 2, there exists a finite $k_0 = k_0(\epsilon)$ such that $\|z_k - z^*\| \leq \epsilon$ for all $k \geq k_0$, where $z^*$ is the T-MFE. A precise characterization of $k_0$ is difficult because the convergence of the T-BR algorithm is based on monotonicity properties of MFG-SC, rather than on contraction arguments.

4 Learning T-MFE

We address the problem of learning T-MFE when the model is unknown. In this section, we assume the availability of a simulator, which, given the current state $s$, current action $a$ and current mean field $z$, can generate the next state $s' \sim P(\cdot|s, a, z)$. We discuss how to learn directly from real-world samples in the next section. We also assume that the reward function is known, as is common in the literature. We now introduce two reinforcement learning algorithms—a model-free algorithm and a model-based algorithm—for learning T-MFE.

4.1 Model-Free TMFQ-Learning Algorithm for learning T-MFE

We now describe the TMFQ-learning algorithm, which builds on the T-BR algorithm. Recall that the T-BR algorithm uses knowledge of the model, $P$ in two locations. The first is that at each step $k$, for a given mean field $z_k$, the optimal TQ-value function $Q^*_z$ is computed using TQ-value iteration. In the learning approach, we use the generalized Q-learning framework [19] as the basis for the model-free TQ-learning algorithm as follows:

$$Q_{t+1,z_k}(s_t, a_t) = (1 - \alpha_t)Q_{t,z_k}(s_t, a_t) + \alpha_t(r(s_t, a_t, z_k) + \gamma G(Q_{t,z_k})(s_{t+1}))$$

(4)

where $\alpha_t$ is the appropriate learning rate. Here, the state sequence $\{s_t\}$ is generated using the simulator by fixing the mean field $z_k$, i.e., $s_{t+1} \sim P(\cdot|s_t, a_t, z_k), \forall t$. Using the properties of the generalized Q-learning formulation [19], it can be shown that $Q_{t,z_k} \rightarrow Q^*_z$ as $t \rightarrow \infty$.  

6
There exists a $C$ such that

$|r(t, \cdot, z) - r(t, \cdot, \tilde{z})|_1 \leq C_1 \|z - \tilde{z}\|_1$, for all $z, \tilde{z} \in \mathcal{P}(S)$.

(ii) There exists a $C_2 > 0$ such that $\|P(\cdot, \cdot, z) - P(\cdot, \cdot, \tilde{z})\|_1 \leq C_2 \|z - \tilde{z}\|_1$, for all $z, \tilde{z} \in \mathcal{P}(S)$.

(iii) Let $P_{\mu, \tilde{z}}(s|s) = \sum_a \mu(s, a)P(s'|s, a)$, for all $z, \tilde{z} \in \mathcal{P}(S)$. Let $\mu_1, \mu_2$ be the trembling-hand policies corresponding to $Q_1, Q_2$, i.e., $\mu_1 = \pi_{Q_1}^{\epsilon}, \mu_2 = \pi_{Q_2}^{\epsilon}$. Then there exists a $C_3 > 0$ such that $\|P_{\mu_1, \tilde{z}} - P_{\mu_2, \tilde{z}}\|_1 \leq C_3 \|Q_1 - Q_2\|_\infty$ for all $z \in \mathcal{P}(S)$ and for any given $Q_1, Q_2$.

Assumption 1(i) and 1(ii) indicate that the reward function and transition kernel are Lipschitz with respect to the mean field, while Assumption 1(iii) indicates that the distance between the Markov chains induced by two policies on the same transition kernel are upper bounded by a constant times the distance between their respective Q functions. We then have the following result.

Theorem 4. Let Assumption 1 hold. Let $\{z_k\}$ and $\{\mu_k\}$ be the sequences of the mean fields and strategies generated by the TMFQ-learning algorithm. Assume that for each $z_k$, TQ-learning
(according to (4)) update is performed $T_0$ number of times, i.e., $Q_{z_k} = Q_{T_0, z_k}$. For any given $0 \leq \epsilon, \delta < 1$, let $k_0 = k_0(\epsilon)$ and $T_0$ be given as

$$T_0 = O\left(\left(\frac{B^2L^{1+3w}V_{\text{max}}^2}{\beta^2\epsilon^2} \ln \left(\frac{2Bk_0|S||A|V_{\text{max}}}{\delta\beta\epsilon}\right)\right)^{\frac{1}{\epsilon}} + \left(\frac{L}{\beta} \ln \left(\frac{BV_{\text{max}}}{\epsilon}\right)\right)^{\frac{1}{\beta}}\right),$$  

(5)

where $B = (1 + C_2 + C_3D)^{k_0+1}(C_3 + 1)$, $D = (C_1 + \gamma C_2)/(1 - \gamma)$, $V_{\text{max}} = 1/(1 - \gamma)$, $\beta = (1 - \gamma)/2$, $L$ is an upper bound on the covering time \footnote{Covering time of a state-action pair sequence is the number of steps needed to visit all state-action pairs starting from any arbitrary state-action pair \footnote{1}.} and $w \in (1/2, 1)$. Then,

$$\mathbb{P}(\|z_{k_0} - z^*\| \leq 2\epsilon) \geq (1 - \delta).$$  

(6)

We may also eliminate the dependence of the constant term $B$ on $k_0$ under a contraction assumption on the McKean-Vlasov dynamics $\Phi$ (for instance, following conditions similar to (4)).

**Assumption 2.** (i) Let $Q_1, Q_2$ be two arbitrary $Q$-value functions and let $\mu_1 = \pi_Q^1$, $\mu_2 = \pi_Q^2$. Let $z_1, z_2$ be two arbitrary mean fields. Then there exists positive constants $C_4$ and $C_5$ such that $\|\Phi(z_1, \mu_1) - \Phi(z_2, \mu_2)\|_1 \leq C_4\|z_1 - z_2\|_1 + C_5\|Q_1 - Q_2\|_\infty$. Also assume that $(C_4 + C_5D) < 1$, where $D = (C_1 + \gamma C_2)/(1 - \gamma)$.

**Corollary 1.** Let Assumption \footnote{1} and Assumption \footnote{2} hold. Then, the we obtain the bound on $T_0$ as in (5) with $B = (C_5 + 1)/(1 - (C_4 + C_5D))$, which does not depend on $k_0$.

### 4.2 Generative Model-Based Reinforcement Learning for T-MFE

We now develop a model-based variant of the T-BR algorithm. Non-stationarity in our system (model changes at each step) implies that there is no single model that can be learned. Our approach follows the generation of a new model each time that the mean field evolves under a T-BR like approach.

At each iteration, we first estimate the model $P(\cdot|\cdot, z_k)$ for the given $z_k$ using $n_0$ simulator samples for each $(s, a)$. We next define the approximate TQ-value operator $\hat{Q}_{z_k}$ as in (3) by replacing the actual model $P$ with the estimated model $\hat{P}$. It is straightforward to show that $\hat{Q}_{z_k}$ is also a contraction. This ensures that approximate TQ-value iteration will converge to an approximate TQ-value function $\hat{Q}_{z_k}$, which will, of course, have an error with respect to the true TQ-value function $Q^*_k$. We determine the trembling-hand best response strategy with respect to $\hat{Q}_{z_k}$. Finally, the next mean field $z_{k+1}$ is obtained using this strategy and the estimated model in the McKean-Vlasov update equation, denoted by $\hat{\Phi}(\cdot, \cdot)$. The GMBL algorithm is summarized in Algorithm \footnote{4} (supplementary material).

Note that we are not estimating the model for all the possible mean fields, but only for the number of times, i.e., $Q_{z_k} = Q_{T_0, z_k}$. For any given $0 \leq \epsilon, \delta < 1$, let $k_0 = k_0(\epsilon)$ and $T_0$ be such that

$$N_0 = O\left(|S||A| \max \left(\frac{2V_{\text{max}}^2}{\epsilon^2} \log \left(\frac{2|S||A|k_0}{\delta}\right), \frac{2B^2}{\epsilon^2} \log \left(\frac{2|S||A|k_0}{\delta}\right)\right)\right).$$  

(7)
where \( B = (1 + C_2 + C_3 D)^{k_0 + 1} \), \( D = (C_1 + \gamma C_2)/(1 - \gamma) \), \( V_{\text{max}} = 1/(1 - \gamma) \). Then,

\[
P(\|z_{k_0} - z^*\| \leq 2\bar{\epsilon}) \geq (1 - \bar{\delta}).
\]  

(8)

Note that a result similar to that of Corollary 1 can easily be shown here as well under the same assumption. We omit details due to page limitations.

5 Online Learning of T-MFE

The availability of a large population of agents that regenerate to occupy all states, and explore all actions via trembling hand strategies suggests that we can do away with the need for a system simulator via an algorithm that simply aggregates these concurrently generated samples, and computes a new strategy that is then pushed to all agents. While this sounds attractive, a typical RL approach would require that given the current strategy \( \mu_k \) and mean field \( z_k \), we would need to compute the stationary distribution of the model \( P_{z_k, \mu_k} \), and set it as the next mean field. This would preclude learning without a simulator, since running one step in the real world would immediately cause a mean field update to \( z_{k+1} \), and induce a new model \( P_{z_{k+1}, \mu_k} \), making the system non-stationary.

The fact that our RL algorithms only need a one-step McKean-Vlasov update under each strategy implies that we can indeed simply apply \( \mu_k \) to the system, and set the resultant state distribution as \( z_{k+1} \). However, we face the issue that the samples obtained pertain to \( P_{z_k, \mu_k} \), whereas the system model is now \( P_{z_{k+1}} \), and so an online sample-based trembling-hand strategy \( \mu_{k+1} \) will lag the current system model by one step. Fortunately, convergence of the TMFQ-learning approach is robust to this lag, and we present its online version in Algorithm 5 (supplementary material). The PAC result is similar TMFQ-learning, with accuracy increasing in the number of agents, rather than with the number of samples as in Theorem 4, and is hence omitted.

6 Experiments

We consider the Infection Spread example described in Section 2. State space is \( S = \{0, \ldots, 24\} \) with 0 being the lowest health level. Action space is \( A = \{0, \ldots, 4\} \) with 4 being the strongest preventive action. Details of the parameters, as well as more experiments are presented in the supplementary materials. In all the figures shown below, the quantities on the y-axis are averaged over 20 runs, with the thick line showing the averaged value and the band around showing the standard deviation. Figure 1 shows the final mean field distribution obtained by each algorithm, simulated with 1000 agents. Note that all of them converge to the same mean field. Figure 2 is a heat map of the converged TQ-value function. Observe that the values are non-decreasing in state, showing alignment of incentives across agents. Figure 3 shows the final mean field distribution obtained by the online TMFQ-learning for different values of \( c_f \). As expected, more agents are in the lower health state for larger \( c_f \). Figure 4 shows the evolution of average health level of the population, \( \sum_s s z_k(s) \), with iteration number \( k \), for online TMFQ-learning. This plot indicates that the convergence of the algorithm is fairly fast. Figure 5 also shows the rate of convergence of of the mean field with different number of agents. Here, \( z^* \) is the final mean field obtained by the GMBL algorithm with \( n_0 = 500 \). Figure 6 shows the evolution of the corresponding mean population state. Figure 5 and 6 show poor convergence with 100 agents, which is too small for the mean field assumption to hold.


7 Conclusions

This paper introduced the notion of trembling-hand perfection to MFG, as a means of providing strategically consistent exploration. We showed existence of T-MFE in MFG with strategic complementarities, and developed an algorithm for computation. Based on this algorithm, we developed model-free, model-based and fully online learning algorithms, and provided PAC bounds on their performance. Experiments illustrated the accuracy and good convergence properties of our algorithms.
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A Algorithms Description

A.1 Next-MF Function for One-Step McKean-Vlasov Update

Algorithm 3: Next-MF

1: **Input:** Mean field \( z \) and strategy \( \mu \)
2: Initialize \( \hat{z} \), sample number \( j \leftarrow 0 \)
3: repeat
4: Sample the state \( s \sim z(\cdot) \), action \( a \in \mu(s, \cdot) \), next state \( s' \sim P(\cdot|s, a, z) \)
5: \( \hat{z}(s') \leftarrow \hat{z}(s') + 1 \)
6: \( \bar{z}_j = \text{normalize}(\hat{z}) \)
7: \( j \leftarrow j + 1 \)
8: until \( \|\bar{z}_j - \bar{z}_{j-1}\|_1 \leq \epsilon_2 \)
9: \( \bar{z} = \bar{z}_j \)
10: return \( \bar{z} \)

Algorithm 3 estimates the next mean field according to equation (1). We maintain a frequency estimate of \( s' \) in line 5 and normalize the frequency estimate to obtain a density function in line 6. Note that \( s' \) is sampled according to line 4.

A.2 GMBL Algorithm for T-MFE

Algorithm 4: GMBL Algorithm for T-MFE

1: Initialization: Initial mean field \( z_0 \)
2: for \( k = 0, 1, 2, \ldots \) do
3: For the mean field \( z_k \) estimate the model to get \( \hat{P}(\cdot|\cdot, z_k) \) by taking \( n_0 \) next-state samples for each state-action pair \((s, a)\)
4: Compute \( \hat{Q}^{*}_{z_k} \) using the approximate TQ-value iteration \( \hat{Q}_{m+1, z_k} = \hat{F}_{z_k}(\hat{Q}_{m, z_k}) \)
5: Compute the strategy \( \mu_k = \pi_{\hat{Q}^{*}_{z_k}} \). Then compute the next mean field \( z_{k+1} = \Phi(z_k, \mu_k) \)
6: end for

A.3 Online TMFQ-learning Algorithm for T-MFE

In Algorithm 3 at time \( k \) the current mean field is \( z_k \) and agents execute strategy \( \mu_{k-1} \). This causes a transition of mean field to \( z_{k+1} \). For learning the TQ-value function, we aggregate the samples from all the agents. The execution of trembling hand strategies ensures that we have samples of each \((s, a)\) pair, assuming that the mean field has non-zero mass in each state and number of agents is sufficiently large. For faster learning, we bootstrap the initial TQ-value for each \( k \) to the final TQ-value corresponding to the previous mean field, i.e., \( Q_{0, z_k} = Q_{k-1} \). Note that \( t \) in the above algorithm description does not refer to time, but rather the samples collected from the agents. In the limit of large number of agents, the algorithm will converges.

B Proofs of Results in Section 3

We use the following result from [2]
Lemma 1 ([2]). For any $Q_1, Q_2$, and for any $s \in S$,

$$|G(Q_1)(s) - G(Q_2)(s)| \leq \max_a |Q_1(s, a) - Q_2(s, a)|.$$  

B.1 Proof of Proposition [1]

Proof of Proposition [1]. For any given $(s, a) \in S \times A$ and mean field $z$,

$$|F_z(Q_1)(s, a) - F_z(Q_2)(s, a)| \leq \gamma \sum_{s'} P(s'|s, a, z)(G(Q_1)(s') - G(Q_2)(s'))$$

$$\leq \gamma \sum_{s'} P(s'|s, a, z) \max_b |Q_1(s', b) - Q_2(s', b)| \leq \gamma \|Q_1 - Q_2\|_{\infty}$$

where (a) follows from Lemma [1]. Since $(s, a) \in S \times A$ was arbitrary, we have $\|F_z(Q_1) - F_z(Q_2)\|_{\infty} \leq \gamma \|Q_1 - Q_2\|_{\infty}$. Existence of a unique fixed point for $F_z$ follows directly from the Banach’s fixed point theorem since $F_z$ is a contraction. The claim that this unique fixed point is equal to $Q_z^*$ follows from the Bellman optimality principle. \hfill \Box

B.2 Proof of Theorem [1] and Theorem [2]

We follow a proof approach that uses the strategic complementarity conditions to establish some monotone properties, and exploit that to show the existence of T-MFE and the convergence of T-BR algorithm. We first state some useful results from [1]. Note that, however, since we are considering trembling-hand polices, proofs in [1] are not directly applicable to our setting.

Lemma 2 (Lemma 4 in [1]). Suppose that $V_z(s)$ is a non-decreasing bounded function in $s$ and has increasing differences in $s$ and $z$. Then, $\sum_{s' \in S} P(s'|s, a, z) V_z(s')$ is non-decreasing in $(s, a)$ and $z$ and has increasing differences in $(s, a)$ and $z$. Moreover, the function $V_z^*(s)$ defined as, $V_z^*(s) = \max_a (r(s, a, z) + \gamma \sum_{s'} P(s'|s, a, z) V_z(s))$, is non-decreasing in $s$ and has increasing differences in $s$ and $z$. 

Algorithm 5: Online TMFQ-learning Algorithm for T-MFE

1: Initialize mean field $\mu_0$ and strategy $\mu_0$
2: for $k = 1, 2, \ldots$ do
3:   for agents $i = 1, 2, \ldots, I$ do
4:     Observe the current state $s_k^i$, take action $a_k^i \sim \mu_{k-1}(s_k^i)$,
5:     get reward $r_k^i$, observe the next state $s_k^{i+1}$
6:   end for
7:   $t = 0$, $Q_{0,z_k}(s, a) = Q_{k-1}(s, a), \forall (s, a)$
8:   for $(s, a, r, s') \in$ memory buffer do
9:     $Q_{t+1,z_k}(s, a) = (1 - \alpha_t)Q_{t,z_k}(s, a) + \alpha_t(r + \gamma G(Q_{t,z_k}(s', \cdot)))$
10:    $t \leftarrow t + 1$
11: end for
12: $Q_{zk} = Q_{t,z_k}$
13: end for
14: $\mu_k = \pi_{Q_{zk}}$
Lemma 3 (Lemma 6 in [1]). Suppose that $V_z(s)$ is non-decreasing in $s$ and has increasing differences in $s$ and $z$. Define a correspondence

$$
\Omega(s, z) = \arg \max_{\alpha \in \mathcal{A}} (r(s, a, z) + \gamma \sum_{s' \in S} P(s'|s, a, z)V_z(s')).
$$

Then, $\Omega$ is a non-decreasing correspondence in $(s, z)$.

**Proposition 2.** Let $V^*_z$ and $\mu^*_z$ be the optimal trembling-hand value function and optimal trembling-hand strategy corresponding to mean field $z$. Then, $V^*_z(s)$ is non-decreasing in $s$ and has increasing differences in $s$ and $z$. Moreover, $\mu^*_z$ is stochastically non-decreasing in $s$ and $z$.

**Proof.** Let $\{Q_{m,z}, m \geq 0\}$ be the TQ-value iterates corresponding to a mean field $z$. If $\arg \max_b Q_{m,z}(s, b)$ is not unique, then $\pi^\epsilon_{Q_{m,z}}$ can be defined in more than one way. We define it as

$$
\pi^\epsilon_{Q_{m,z}}(s, a) = \begin{cases} 
1 - \epsilon & \text{for } a = \sup \{\arg \max_b Q_{m,z}(s, b)\} \\
\epsilon/(|\mathcal{A}| - 1) & \text{for } a \neq \sup \{\arg \max_b Q_{m,z}(s, b)\}
\end{cases}
$$

(9)

Note that the sup of a set is well defined with respect to a lattice according to the definition of MFG-SC. In the following we denote $\pi^\epsilon_{Q_{m,z}}$ simply as $\mu_{m,z}$.

By definition, $V^*_z = G(Q^*_z)$ and define $V^*_m = G(Q_{m,z})$. To show that $V^*_z(s)$ is non-decreasing in $s$ and has increasing differences in $s$ and $z$, it suffices to show $V^*_m(s)$ has the same properties for all $m$. This is because, since $G$ is continuous, monotonicity and increasing differences are preserved under limits.

Letting $Q_{0,z}(s, a) = 0$ for all $(s, a) \in \mathcal{S} \times \mathcal{A}$, we have $Q_{1,z}(s, a) = r(s, a, z)$. Then, $Q_{1,z}(s, a)$ is non-decreasing in $s$ and has increasing differences in $s$ and $z$ by Definition [1]. Define the correspondence $\Omega_m$ as

$$
\Omega_m(s, z) = \arg \max_{\alpha \in \mathcal{A}} (r(s, a, z) + \gamma \sum_{s' \in S} P(s'|s, a, z)V_{m-1,z}(s')).
$$

By Lemma 3, $\Omega_1(s, z) = \arg \max Q_{1,z}(s, a)$ is non-decreasing in $s$ and $z$. Using this, we can conclude that $\mu_{1,z} = \pi^\epsilon_{Q_{1,z}}$ is stochastically non-decreasing in $s$ and $z$ for $\epsilon$ such that $(1 - \epsilon) > \epsilon/(|\mathcal{A}| - 1)$. To see this, first note that for $\epsilon = 0$, the deterministic strategy $\pi^0_{Q_{1,z}}$ is non-decreasing in $s$ and $z$ by Lemma 3. Now, for $\epsilon$ with $(1 - \epsilon) > \epsilon/(|\mathcal{A}| - 1)$, $\pi^\epsilon_{Q_{1,z}}$ is stochastically non-decreasing in $s$ and $z$ because the probability of the maximizing action is greater than all other actions. Thus, $\mu_{m,z}$ is stochastically non-decreasing in $s$ and $z$ for all $m \in \mathbb{N}$ if each $Q_{m,z}$ is non-decreasing in $s$ and has increasing differences in $s$ and $z$.

Now, recall that $V_{1,z}(s) = \sum_a \mu_{1,z}(s, a)Q_{1,z}(s, a) = (1 - \epsilon)Q_{1,z}(s, \bar{a}) + \epsilon \sum_a Q_{1,z}(s, a)$ where $\bar{a} = \sup \{\arg \max Q_{1,z}(s, a)\}$. Since $Q_{1,z}(s, a)$ is non-decreasing in $s$ and has increasing differences in $s$ and $z$ for all $a$, we can conclude that $V_{1,z}(s)$ is non-decreasing in $s$ and has increasing differences in $s$ and $z$.

As the induction hypothesis, suppose that both $Q_{m,z}(s, a)$ and $V_{m,z}(s)$ are non-decreasing in $s$ and $z$ and have increasing differences in $s$ and $z$ and $\mu_{m,z}$ is stochastically non-decreasing in $s$ and $z$.

Induction step is as follows: By Definition [1] and Lemma 2, both $r(s, a, z)$ and $\gamma \sum_{s'} P(s'|s, a, z)V_{m,z}(s')$ are non-decreasing in $s$ and have increasing differences in $(s, a)$ and $z$. Therefore, $Q_{m+1,z}(s, a) = r(s, a, z) + \gamma \sum_{s'} P(s'|s, a, z)V_{m,z}(s')$ also satisfies the same properties. By Lemma 3, notice that $\Omega_{m+1}(s, z) := \arg \max_{\alpha \in \mathcal{A}} \{Q_{m+1,z}(s, a)\}$ is non-decreasing in $(s, z)$ and therefore $\mu_{m+1,z} = \pi^\epsilon_{Q_{m+1,z}}$ defined as in (9) is stochastically non-decreasing in $s$ and $z$ as argued before. Finally, $V_{m+1,z}(s) = \sum_a \mu_{m+1,z}(s, a)Q_{m+1,z}(s, a) = (1 - \epsilon)Q_{m,z}(s, \bar{a}) + \epsilon \sum_a Q_{m,z}(s, a)$ where $\bar{a} = \sup \{\arg \max Q_{m+1,z}(s, a)\}$.
is non-decreasing in $s$ and has increasing differences in $s$ and $z$ because $Q_{m+1,z}(s,a)$ is non-decreasing in $s$ and has increasing differences in $s$ and $z$ for all $a$.

Since TQ-value iteration converges, $Q^*_z(s,a)$ and $V^*_z(s)$ are non-decreasing in $s$ and have increasing differences in $s$ and $z$. By repeating the same argument again, $\Omega(s,z) = \arg\max_{a \in \mathcal{A}} \{Q^*_z(s,a)\}$ is non-decreasing in $(s,z)$. So, $\mu_z = \pi^{\star}_{Q^*_z}$ is stochastically non-decreasing in $s$ and $z$.

Tarski’s fixed-point theorem [20] ensures that monotone functions on a lattice have a fixed point. We use that result to prove the existence of T-MFE. We first state Tarski’s fixed-point theorem for completeness.

**Theorem 6** (Tarsi’s Fixed-point Theorem [20]). Suppose that $\mathcal{L}$ is a nonempty complete lattice, and $T : \mathcal{L} \rightarrow \mathcal{L}$ is a non-decreasing function. Then the set of fixed points of $T$ is a nonempty complete lattice.

We now give the proof of Theorem 1.

**Proof of Theorem 1**. For any strategy $\mu$ such that $\mu$ is stochastically non-decreasing in $s$, and for any given mean fields $z$ and $z'$, define the function $K_{\mu,z}(z')(s') = \sum_{s \in S} \sum_{a \in \mathcal{A}} z'(s) \mu(s,a) P(s'|s,a,z)$.

From [1] Lemma 7, for $z_2 \succeq_{SD} z_1$ and $z'_2 \succeq_{SD} z'_1$ and $\mu_2(s,\cdot) \succeq_{SD} \mu_1(s,\cdot)$ with $\mu_2$ and $\mu_1$ are stochastically non-decreasing in $s$, we have $K_{\mu_2,z_2}(z'_2) \succeq_{SD} K_{\mu_1,z_1}(z'_1)$.

Observe that McKean-Vlasov update function $\Phi(z,\mu)$ is a special case of $K_{\mu,z}(z')$ by setting $z' = z$. By the above argument, $\Phi(z,\mu) = K_{\mu,z}(z)$ is stochastically non-decreasing in $(z,\mu)$ provided that $\mu$ is stochastically non-decreasing in $s$.

Define $\Upsilon : \mathcal{P}(S) \rightarrow \mathcal{P}(S)$ as $\Upsilon(z) = \Phi(z,\mu^*_z)$ where $\mu^*_z$ is the optimal trembling-hand strategy corresponding to the mean field $z$. Recall that, by Proposition 2, $\mu^*_z$ is stochastically non-decreasing in $s$ and $z$. Then, for $z_2 \succeq_{SD} z_1$, $\Upsilon(z_2) = \Phi(z_2,\mu^*_z) \succeq_{SD} \Phi(z_1,\mu^*_z) = \Upsilon(z_1)$. From this, we can conclude that $\Upsilon$ is a stochastically non-decreasing function in $z$ and hence $\Upsilon(\cdot)$ has a fixed point by Tarsi’s theorem (Theorem 6). In other words, there exists a $z^*$ such that $z^* = \Upsilon(z^*) = \Phi(z^*,\mu^*_z)$. This implies that there exists a mean field $z^*$ and strategy $\mu^*$ such that they satisfy the optimality condition (i.e., $\mu^* = \mu^*_z$) and the consistency condition (i.e. $z^* = \Phi(z^*,\mu^*_z)$). Thus, there exists a trembling-hand-perfect mean field equilibrium for mean field games with strategic complementarities.

**Proof of Theorem 2**. We exploit two key monotonicity properties established before. First, from proof of Theorem 1 $\Upsilon(z)$ is stochastically non-decreasing in $z$. Second, from Proposition 2 an optimal trembling-hand strategy $\mu^*_z$ corresponding to a mean field $z$ is stochastically non-decreasing in $s$ and $z$.

Let $z_0$ be the smallest distribution by initialization in the $\preceq_{SD}$ ordering and let $\{Q_{z_k}, z_k, k \geq 0\}$ be the TQ-value functions and mean fields generated corresponding to Algorithm 1. In the following, we denote an optimal trembling-hand strategy $\mu^*_z$ corresponding to a mean field $z_k$ simply as $\mu_k$. By Proposition 2 $\mu_0 = \pi^{\star}_{Q^*_0}$ is stochastically non-decreasing in $s$ and $z$. Hence we take the following as our induction base: $z_0 \succeq_{SD} \Upsilon(z_0) = z_1$ and $\mu_0(s,\cdot) \preceq_{SD} \mu_1(s,\cdot)$ for all $s$ where $\mu_1$ is stochastically non-decreasing in $s$ and $z$.

Now as the induction hypothesis, suppose that $z_0 \succeq_{SD} \Upsilon(z_0) = z_1 \succeq_{SD} \Upsilon(z_1) = z_2 \succeq_{SD} \cdots \succeq_{SD} \Upsilon(z_{k-1}) = z_k$ and that $\mu_0(s,\cdot) \preceq_{SD} \mu_1(s,\cdot) \preceq_{SD} \cdots \preceq_{SD} \mu_k(s,\cdot)$ for all $s$ where $\mu_i$ are stochastically non-decreasing in $s$ and $z$. 

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Then as an induction step, we have $\Upsilon(z_{k-1}) = z_k \preceq_{SD} z_{k+1} = \Upsilon(z_k)$ because $\Upsilon(z)$ is stochastically non-decreasing in $z$. Now, since $z_k \preceq_{SD} z_{k+1}$ and both $\mu_k$ and $\mu_{k+1}$ are stochastically non-decreasing in $s$ and $z$, it follows that $\mu_k(s, \cdot) \preceq_{SD} \mu_{k+1}(s, \cdot)$ for all $s$.

Observe that $\Pi^\epsilon$ is compact for a fixed $\epsilon > 0$ and since $(\mu_k)_{k \in \mathbb{N}} \subset \Pi^\epsilon$ is a stochastically non-decreasing (monotone) sequence, there must be a pointwise limit $\mu^*$ such that $\mu_k \to \mu^*$ as $k \to \infty$. Moreover, $\mathcal{P}(S)$ is also compact since we assume that $|S|$ is finite. Since $(z_k)_{k \in \mathbb{N}} \subset \mathcal{P}(S)$ is a stochastically non-decreasing (monotone) sequence, there must be a limit $z^*$ such that $z_k \to z^*$ as $k \to \infty$.

It is straightforward to show that the optimal trembling-hand strategy $\mu^*_z$ and $\Upsilon(z)$ are continuous in $z$. So, since $\mu_k \to \mu^*_z$ and $z_k \to z^*$, we can conclude that $\mu^* = \mu^*_z$ and $z^* = \Upsilon(z^*) = \Phi(z^*, \mu^*)$.

This concludes the proof that T-BR converges to a T-MFE.

C Proof of the Results in Section 4

C.1 Proof of Theorem 3

Proof. We only sketch the proof since it is almost the same to the proof of Theorem 2. At each time $k$ with a fixed $z_k$, the value of generalized Q-learning converges, i.e., $Q_{t,z_k} \to Q^*_z$ as $t \to \infty$. Then, under assumptions of the model, $Q^*_{z_k}$ satisfies the same complementarity properties for each $k \in \mathbb{N}$. Thus, we can conclude that, for all $k \in \mathbb{N}$, $\mu_k(s, \cdot) \preceq_{SD} \mu_{k+1}(s, \cdot)$ for all $s \in S$ where each $\mu_k$ is stochastically non-decreasing in $s$ and $z$ and that $z_k \preceq_{SD} \Upsilon(z_k) = z_{k+1}$. The rest of the proof that the limits exist follows the same as in the proof of Theorem 2.

C.2 Proof of Theorem 4

We first prove some useful lemmas.

Lemma 4. Let $z_1$ and $z_2$ be two arbitrary mean fields and $Q^*_{z_1}$ and $Q^*_{z_2}$ be the optimal TQ-value functions corresponding to them. Then, under Assumption 4,

$$||Q^*_{z_1} - Q^*_{z_2}||_\infty \leq D||z_1 - z_2||_1$$

where $D = (C_1 + \gamma C_2)/(1 - \gamma)^2$. 
Then, under Assumption 1, 
where \( D \) follows from Assumption 1.(i), \((b) \) from Lemma 1, \((c) \) from the fact that the maximum Q-value for a finite MDP is \( 1/(1-\gamma) \), and \((d) \) from Assumption 1.(ii). Since \((11) \) is true for any \((s, a)\), by taking the maximum on the left hand side and rearranging, we get \( \|Q_{z_1} - Q_{z_2}\|_\infty \leq D\|z_1 - z_2\|_1 \), where \( D = \frac{C_1}{1-\gamma} + \frac{\gamma C_2}{(1-\gamma)^2} \).

Lemma 5. Let \( Q_1, Q_2 \) be two arbitrary TQ-value functions and let \( \mu_1 \) and \( \mu_2 \) be the trembling-hand strategies corresponding to them, i.e., \( \mu_1 = \pi_{Q_1}^\epsilon, \mu_2 = \pi_{Q_2}^\epsilon \). Let \( z_1, z_2 \) be two arbitrary mean fields. Then, under Assumption 1:
\[
\|\Phi(z_1,\mu_1) - \Phi(z_2,\mu_2)\|_1 \leq (1 + C_2)\|z_1 - z_2\|_1 + C_3\|Q_1 - Q_2\|_\infty. \tag{12}
\]

Proof. We have,
\[
\|\Phi(z_1,\mu_1) - \Phi(z_2,\mu_2)\|_1 = \sum_{s'} |\Phi(z_1,\mu_1)(s') - \Phi(z_2,\mu_2)(s')|
\leq \sum_{s'} \left| \sum_{s\in S} \sum_{a\in A} z_1(s)\mu_1(s,a)P(s'|s,a,z_1) - \sum_{s\in S} z_2(s)\mu_2(s,a)P(s'|s,a,z_2) \right|
\leq \sum_{s'} \left| \sum_{s\in S} z_1(s)P_{z_1,\mu_1}(s'|s) - z_2(s)P_{z_2,\mu_2}(s'|s) \right| + \sum_{s'} \left| \sum_{s\in S} z_2(s)(P_{z_1,\mu_1}(s'|s) - P_{z_2,\mu_2}(s'|s)) \right|
\leq \|z_1 - z_2\|_1 + \|P_{z_1,\mu_1} - P_{z_2,\mu_2}\|_1
\leq \|z_1 - z_2\|_1 + \|P_{z_1,\mu_1} - P_{z_2,\mu_1}\|_1 + \|P_{z_2,\mu_1} - P_{z_2,\mu_2}\|_1
\leq \|z_1 - z_2\|_1 + C_2\|z_1 - z_2\|_1 + C_3\|Q_1 - Q_2\|_\infty
\]
where \((a) \) follows from Assumption 1.

We use the following Q-learning sample complexity results from [7].
Theorem 7 (Theorem 4 in [7]). Let $Q_t$ be the $t$-th update in Q-learning algorithm using a polynomial time learning rate given as $\alpha_t(s,a) = 1/(n_t(s,a) + 1)^w$, where $n_t(s,a)$ is the number of times the state-action pair $(s,a)$ is visited until time $t$ and $w \in (1/2, 1)$. Let $L$ be the upper bound on the covering time. Then, $\mathbb{P}(\|Q_{T_0} - Q^*\|_{\infty} \leq \epsilon_3) \geq (1 - \delta_3)$, for any $0 < \epsilon_3, \delta_3 < 1$, given that

$$T_0 = \Omega\left(\left(\frac{L^{1+3w} \max_{z \in \mathcal{S}} |A| \log \left(\frac{|\mathcal{S}| |A| \max_{\phi \in \Phi} \epsilon z_{\max}^{\infty}}{\beta^2 \epsilon_3^2} \right)}{\epsilon_3}\right)^{1/4} + \left(\frac{L}{\beta} \ln \left(\frac{\max_{z \in \mathcal{S}} \epsilon z_{\max}^{\infty}}{\epsilon_3}\right)\right)^{1/4}\right).$$

(13)

where $V_{\max} = 1/(1 - \gamma), \beta = (1 - \gamma)/2$.

Here the covering time of a state-action pair sequence is the number of steps needed to visit all state-action pairs starting from any arbitrary state-action pair.

We note that the Q-learning update used in TQ-learning algorithm satisfies all the conditions necessary for the above theorem. So, we will use the above result. We refer the reader to [7] for the details.

We now give the proof Theorem 4.

Proof of Theorem 4. Let $\{z_k\}$ and $\{\mu_k\}$ be the sequences of mean fields and strategies generated by the TMFQ-learning algorithm. Let $\{Q_{t,k}, t \geq 0\}$ be TQ-learning iterates corresponding to the mean field $z_k$ and let $Q_k = Q_{T_0,k}$ where $T_0$ is as shown in (13). We assume that the number of samples used in the Next-MF function is such that, for any given mean field $z$ and strategy $\mu$, Next-MF function returns a mean field $z'$ such that $\mathbb{P}(\|z' - \Phi(z,\mu)\|_1 \leq \epsilon_3) \geq (1 - \delta_3)$.

Define the event $E_k = \{\|Q_k - Q_{z_k}^*\|_{\infty} \leq \epsilon_3 \text{ and } \|z_{k+1} - \Phi(z_k,\mu_k)\|_1 \leq \epsilon_3\}$. Then, according to Theorem 7 and the assumption on the Next-MF function, $\mathbb{P}(E_k) \geq 1 - 2\delta_3$. Define the event $E = \cap_{k=1}^{k_0} E_k$. So, $\mathbb{P}(E) \geq (1 - 2k_0 \delta_3)$. We will now analyze the TMFQ-learning algorithm conditioned on the event $E$.

Let $\{\bar{z}_k\}$ and $\{\bar{\mu}_k\}$ be the sequences of mean fields and strategies generated by the T-BR algorithm. We assume that T-BR algorithm and TMFQ-learning algorithm have the same initialization, i.e., $z_0 = z_0$. Now, conditioned on the event $E$,

$$\|\bar{z}_{k+1} - z_{k+1}\|_1 \leq \|\Phi(\bar{z}_k, \bar{\mu}_k) - \Phi(z_k, \mu_k)\|_1 + \|\Phi(z_k, \mu_k) - z_{k+1}\|_1$$

$$\leq (1 + C_2)\|\bar{z}_k - z_k\|_1 + C_3\|Q_{\bar{z}_k}^* - Q_{z_k}^*\|_{\infty} + \epsilon_3$$

$$\leq (1 + C_2)\|\bar{z}_k - z_k\|_1 + C_3\|Q_{\bar{z}_k}^* - Q_{z_k}^*\|_{\infty} + C_3\|Q_{\bar{z}_k}^* - Q_{z_k}^*\|_{\infty} + \epsilon_3$$

(a)

$$\leq (1 + C_2)\|\bar{z}_k - z_k\|_1 + C_3D\|\bar{z}_k - z_k\|_1 + (C_3 + 1)\epsilon_3$$

(b)

$$\leq (1 + C_2)\|\bar{z}_k - z_k\|_1 + (C_3 + 1)\epsilon_3.$$

Here (a) follows from Lemma 4 and the assumption on the Next-MF function and (b) follows from Lemma 4.

Iteratively applying the above inequality, we get $\|\bar{z}_{k_0} - z_{k_0}\|_1 \leq B\epsilon_3$, where $B = (1 + C_2 + C_3D)^{k_0+1}(C_3 + 1)$. Now, $\|z^* - z_{k_0}\|_1 \leq \|z^* - \bar{z}_{k_0}\|_1 + \|\bar{z}_{k_0} - z_{k_0}\|_1 \leq \tilde{\epsilon} + B\epsilon_3$ because $\|z^* - \bar{z}_{k_0}\|_1 \leq \tilde{\epsilon}$ by the definition of $k_0$.

So, $\mathbb{P}(\|z_{k_0} - z_{k_0}\|_1 \leq \tilde{\epsilon} + B\epsilon_3) \geq \mathbb{P}(E) = 1 - 2k_0\delta_3$.

Setting $\epsilon_3 = \tilde{\epsilon}/B$ and $\delta_3 = \delta/2k_0$, and using the corresponding $T_0$ from (13), we get the desired result.\[\square\]
Proof of Corollary 1. Conditioned on the event $E$ as defined in the proof Theorem 4 we get,

$$
\|\tilde{z}_{k+1} - z_{k+1}\|_1 \leq \|\Phi(\tilde{z}_k, \hat{\mu}_k) - \Phi(z_k, \mu_k)\|_1 + \|\Phi(z_k, \mu_k) - z_{k+1}\|_1
$$

(a) \leq C_4\|\tilde{z}_k - z_k\|_1 + C_5\|Q^*_{\tilde{z}_k} - Q^*_z\|_\infty + \epsilon_3

\leq C_4\|\tilde{z}_k - z_k\|_1 + C_5\|Q^*_{\tilde{z}_k} - Q^*_z\|_\infty + C_5\|Q^*_{\tilde{z}_k} - Q^*_z\|_\infty + \epsilon_3

(b) \leq C_4\|\tilde{z}_k - z_k\|_1 + C_5D\|\tilde{z}_k - z_k\|_1 + (C_5 + 1)\epsilon_3

\leq (C_4 + C_5D)\|\tilde{z}_k - z_k\|_1 + (C_5 + 1)\epsilon_3

Here (a) follows from Assumption 2 and (b) follows from Lemma 4.

Iteratively applying the above inequality, we get $\|\tilde{z}_{k_0} - z_{k_0}\|_1 \leq B\epsilon_3$, where $B = (C_5 + 1)/(1 - ((C_4 + C_5D)))$. Rest of the proof is similar to that of Theorem 4. \qed

C.3 Proof of Theorem 5

For a given mean field $z$, let $\tilde{P}(|\cdot|, \cdot, z)$ be the estimate of the model obtained by taking $n_0$ next-state samples for each $(s, a)$. Let $\tilde{F}_z$ be the approximate TQ-value operator obtained by replacing $P$ by $\tilde{P}$ in (3). Similar to the proof of Proposition 1 it is straightforward to show that $\tilde{F}_z$ is a contraction. Let $Q^*_z$ be its unique fixed point. Since $\tilde{P}$ is different from $P$, $\tilde{Q}^*_z$ and $Q^*_z$ will also be different. However, for sufficiently large $n_0$, we can give the following bound.

Lemma 6. For any $0 < \epsilon_4, \delta_4 < 1$,

$$
\mathbb{P}(\|\tilde{Q}^*_z - Q^*_z\|_\infty \leq \epsilon_4) \geq (1 - \delta_4), \quad \text{for} \quad n_0 \geq \frac{2V_{\text{max}}^4}{\epsilon_4^2} \log \left(\frac{2|S||A|}{\delta_4}\right) (14)
$$

where $V_{\text{max}} = 1/(1 - \gamma)$.

Proof.

$$
|\tilde{Q}^*_z(s, a) - Q^*_z(s, a)| = |\tilde{F}_z(\tilde{Q}^*_z)(s, a) - F_z(Q^*_z)(s, a)|
\leq \gamma|\sum_{s'} \tilde{P}(s'|s, a, z)G(Q^*_z)(s') - \sum_{s'} P(s'|s, a, z)G(Q^*_z)(s')|
\leq \gamma|\sum_{s'} \tilde{P}(s'|s, a, z)(G(\tilde{Q}^*_z)(s') - G(Q^*_z)(s'))) + \gamma|\sum_{s'} (\tilde{P}(s'|s, a, z) - P(s'|s, a, z))G(Q^*_z)(s')|
\leq \gamma\|\tilde{Q}^*_z - Q^*_z\|_\infty + \gamma|\sum_{s'} (\tilde{P}(s'|s, a, z)) - P(s'|s, a, z))G(Q^*_z)(s')|, (15)
$$

where (a) follows from Lemma 1.

For bounding $|\sum_{s'} (\tilde{P}(s'|s, a, z)) - P(s'|s, a, z))G(Q^*_z)(s')|$, note that $\sum_{s'} \tilde{P}(s'|s, a, z))G(Q^*_z)(s')$ is an unbiased estimated of $\sum_{s'} P(s'|s, a, z))G(Q^*_z)(s')$. Also note that $\max_{s'} |G(Q^*_z)(s')| \leq 1/(1 - \gamma) = V_{\text{max}}$. So, by applying Hoeffding’s inequality, for a given $(s, a)$, we get

$$
\mathbb{P}(\sum_{s'} (\tilde{P}(s'|s, a, z)) - P(s'|s, a, z))G(Q^*_z)(s')) \leq 2 \exp \left(\frac{-n_0\epsilon^2}{2V_{\text{max}}^2}\right).
$$
Using the union bound argument, for all \((s,a)\), we get

\[
\mathbb{P}(\sum_{s'}(\tilde{P}(s'|s,a,z) - P(s'|s,a,z))G(Q_z^*)(s')| \leq \epsilon) \geq 1 - 2|S||A| \exp\left(\frac{-n_0\epsilon^2}{2V_{\max}^2}\right).
\]

So, with \(n_0 \geq \frac{2V_{\max}^2}{\epsilon^2} \log \left(\frac{2|S||A|}{\delta}\right)\),

\[
\mathbb{P}(\sum_{s'}(\tilde{P}(s'|s,a,z) - P(s'|s,a,z))G(Q_z^*)(s')| \leq \epsilon) \leq 1 - \delta, \quad \forall(s,a) \in S \times A. \tag{16}
\]

Now, for the above \(n_0\), from (15) and (16), \(|\tilde{Q}_z^*(s,a) - Q_z^*(s,a)| \leq \gamma\|\tilde{Q}^* - Q_z^*\|_\infty + \epsilon\), with a probability greater than \((1 - \delta)\) for all \((s,a)\). This implies that \(\|\tilde{Q}^* - Q_z^*\|_\infty \leq \epsilon/(1 - \gamma) = \epsilon V_{\max}\). Now, using \(\epsilon = \epsilon_4/V_{\max}\) and \(\delta = \delta_4\) in the expression for \(n_0\) above, we get the desired result. \(\square\)

We now bound the error in the McKean-Vlasov update due to replacing \(P\) by \(\tilde{P}\).

**Lemma 7.** Let \(\tilde{\Phi}\) be the approximate McKean-Vlasov update function as defined in (1) but by replacing \(P\) by \(\tilde{P}\). Then,

\[
\mathbb{P}(\|\tilde{\Phi}(z,\mu) - \Phi(z,\mu)\| \leq \epsilon_4) \geq (1 - \delta_4), \quad \text{for } n_0 \geq \frac{2}{\epsilon_4^2} \log \left(\frac{2|S||S||A|}{\delta_4}\right) \tag{17}
\]

**Proof.** From [23], for a given \((s,a)\),

\[
\mathbb{P}(\|\tilde{P}(\cdot|s,a,z) - P(\cdot|s,a,z)\| \geq \epsilon_4) \leq 2|S|\exp\left(\frac{-n_4\epsilon^2}{2}\right).
\]

By the union bound argument, for all \((s,a)\), we get

\[
\mathbb{P}(\|\tilde{P}(\cdot|s,a,z) - P(\cdot|s,a,z)\| \leq \epsilon_4) \geq 1 - |S||A|2|S|\exp\left(\frac{-n_4\epsilon^2}{2}\right). \tag{18}
\]

So, with \(n_0 \geq \frac{2}{\epsilon_4^2} \log \left(\frac{2|S||S||A|}{\delta_4}\right)\),

\[
\mathbb{P}(\|\tilde{P}(\cdot|s,a,z) - P(\cdot|s,a,z)\| \leq \epsilon_4) \geq 1 - \delta_4, \quad \forall(s,a) \in S \times A.
\]

Now, with the above \(n_0\), with a probability greater than \((1 - \delta_4)\), we get

\[
\|\tilde{\Phi}(z,\mu) - \Phi(z,\mu)\|_1 = \sum_{s'}|\tilde{\Phi}(z,\mu)(s') - \Phi(z,\mu)(s')|
\]

\[
\leq \sum_s \sum_a \sum_{s'} z(s)\mu(s,a)|\tilde{P}(s'|s,a,z) - P(s'|s,a,z)|
\]

\[
= \sum_s \sum_a z(s)\mu(s,a)\|\tilde{P}(\cdot|s,a,z) - P(\cdot|s,a,z)\|_1 \leq \epsilon_4
\]

where the last inequality follows from (18) and the fact that \(\sum_s \sum_a z(s)\mu(s,a) = 1\). \(\square\)

We now give the proof Theorem 5.
Proof of Theorem 5. Let \( \{z_k\} \) and \( \{\mu_k\} \) be the sequences of mean fields and strategies generated by the GMBL algorithm. Let \( n_0 \) be the maximum of the two values given by Lemma 6 and Lemma 7, i.e.,

\[
 n_0 = \max \left( \frac{2V_{\max}^4}{\epsilon_4^2} \log \left( \frac{2|S||A|}{\delta_4} \right), \frac{2|S||A|}{\epsilon_4^2} \log \left( \frac{2|S||A|}{\delta_4} \right) \right)
\]

Define the event \( E_k = \{\|\hat{Q}^*_{z_k} - Q^*_{z_k}\|_\infty \leq \epsilon_4 \text{ and } \|\hat{\Phi}(z_k, \mu_k) - \Phi(z_k, \mu_k)\|_1 \leq \epsilon_4\} \). Then, from Lemma 6 and Lemma 7, we get \( \mathbb{P}(E_k) \geq (1 - 2\delta_4) \). Define the event \( E = \cap_{k=1}^{n_0} E_k \). So, \( \mathbb{P}(E) \geq (1 - 2k_0\delta_4) \). We will now analyze the GMBL algorithm conditioned on the event \( E \).

Let \( \{\bar{z}_k\} \) and \( \{\bar{\mu}_k\} \) be the sequences of mean fields and strategies generated by the T-BR algorithm. We assume that T-BR algorithm and GMBL algorithm have the same initialization, i.e., \( \bar{z}_0 = z_0 \). Now, conditioned on the event \( E \)

\[
\|\bar{z}_{k+1} - z_{k+1}\|_1 \leq \|\hat{\Phi}(\bar{z}_k, \bar{\mu}_k) - \Phi(z_k, \mu_k)\|_1
\]
\[
\leq \|\hat{\Phi}(\bar{z}_k, \bar{\mu}_k) - \Phi(z_k, \mu_k)\|_1 + \|\Phi(z_k, \mu_k) - \hat{\Phi}(z_k, \mu_k)\|_1
\]
\[
\leq (1 + C_2)\|\bar{z}_k - z_k\|_1 + C_3\|Q^*_{z_k} - \hat{Q}^*_{z_k}\| + \epsilon_4
\]
\[
 = (1 + C_2)\|\bar{z}_k - z_k\|_1 + C_3\|Q^*_z - Q^*_{z_k}\|_\infty + C_3\|Q^*_{z_k} - \hat{Q}^*_{z_k}\| + \epsilon_4
\]
\[
\leq (1 + C_2)\|\bar{z}_k - z_k\|_1 + C_3\|D\|\bar{z}_k - z_k\|_1 + (C_3 + 1)\epsilon_4
\]
Here (a) follows from Lemma 5 and (b) follows from Lemma 4.

Iteratively applying the above inequality, we get \( \|\bar{z}_{k_0} - z_{k_0}\|_1 \leq B\epsilon_4 \), where \( B = (1 + C_2 + C_3D_1)k_0 + 1(C_3 + 1) \). Now, conditioned on the event \( E \), we have

\[
\|\bar{z}^* - z_{k_0}\|_1 \leq \|\bar{z}^* - \bar{z}_{k_0}\|_1 + \|\bar{z}_{k_0} - z_{k_0}\|_1 \leq \bar{\epsilon} + B\epsilon_4
\]
because \( \|\bar{z}^* - \bar{z}_{k_0}\|_1 \leq \bar{\epsilon} \) by the definition of \( k_0 \).

Setting \( \epsilon_4 = \bar{\epsilon}/B \) and \( \delta_4 = \bar{\delta}/2k_0 \) in the expression for \( n_0 \), we get the desired result. \( \square \)

D Experiments

D.1 Parameters for Infection Spread Model

We use the following parameters for simulations

\[
|S| = 25 \quad |A| = 5 \quad k = 0.05 \quad \delta_1 = 1 \quad \delta_2 = 0.2 \quad \delta_3 = 0.01 \quad \zeta = 0.1 \quad \epsilon = 0.3 \quad \gamma = 0.75
\]
\[
w_1 \sim \begin{cases} \text{uniform}\{1, 2, 3\} & \text{w.p} 0.9 \\ 0 & \text{w.p} 0.1 \end{cases} \quad w_2 \sim \begin{cases} \text{uniform}\{0, \ldots, s\} & \text{w.p} 0.9 \\ 0 & \text{w.p} 0.1 \end{cases}
\]

For GMBL, we set \( n_0 = 500 \) and we run the outer loop for 500 iterations. For TMFQL we run Q Learning for 1000 time steps, and perform 5000 iterations of the outer loop. For O-TMFQ Learning, we run the outer loop for 5000 iterations with different number of agents. For better sample efficiency, in both TMFQL and O-TMFQ Learning we initialize the Q function at each iteration with Q of the previous iteration. We used a logarithmically decaying learning rate, we decay the learning rate from \( 10^{-3} \) to \( 10^{-2} \). 

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D.2 Amazon Mechanical Turk (MTurk) and other Gig Economy Marketplaces

We consider Amazon Mechanical Turk (MTurk) as an example of a Gig economy marketplace, which includes firms like Uber and Airbnb. MTurk is a crowd sourcing market, wherein human workers are recruited to perform so-called Human Intelligence Tasks (HITs). These HITs may take the form of labeling data sets or other tasks that are simple from a human’s perspective, but might be difficult for machine learning to directly undertake. The workers are called Turkers, and each has a quality score that depends on previous HITs undertaken. The firm that originates these HITs may specify the price that it is willing to pay, as well as the minimum quality of the Turkers that it desires.

There is a natural alignment of effort employed by Turkers in MTurk, since higher efforts translate into HITs done right, which in turn results in a higher quality, which finally results in firms willing to pay more per HIT. Thus, if mean field quality is high, there is an incentive to perform HITs well and enhance ones’ own quality. This notion of incentive alignment applies to essentially all Gig economy marketplaces—the reputation of the agent directly enhances its reward, while the reputation of the marketplace as a whole (i.e., its mean field) draws customers willing to pay into the system, and so enhances the reward of the agent.

The formal system description is analogous to Infection Spread considered earlier, but we focus here on the application scenario. Thus, we have that each agent (Turker) has his/her quality state, and the strategic action is the choice of how much work to put into a HIT assigned to that Turker. Higher effort implies higher cost, but also implies a higher improvement in the quality. The overall reward is a combination that depends on the Turker’s quality as well as the mean field quality.

Let $s$ denote the quality of a Turker’s profile, and let action $a$ denote the effort the turker puts in to maintain the quality of the profile; this may include number of jobs successfully completed, time taken to complete a job etc. Let $c(a) = \delta_3 a$ denote the cost incurred in performing action $a$. The quality perceived by an entity offering jobs depends on both the quality of the individual Turker and the population as a whole (via the mean field). Thus, the reward to a Turker is a function of the perceived quality and cost incurred in taking action. We define state transition and reward as follows,

$$s' = (s + a - w_1) + \mathbb{1}\{E_1\} + w_2 \mathbb{1}\{E_2\}$$

$$r = \delta_1 s + \delta_2 \sum_{s \in S} s z(s) - \delta_3 a,$$

where $E_1, E_2$ are mutually exclusive events that occur with probabilities $1 - \zeta, \zeta$, respectively, and $w_1, w_2$ are realizations of non-negative integer random variables.

We use the following parameters for simulations,

$$|S| = 100 \quad |A| = 5 \quad \delta_1 = 0.5 \quad \delta_2 = 0.2 \quad \zeta = 0.1 \quad \epsilon = 0.3 \quad \gamma = 0.75$$

$$w_1 \sim \text{uniform}\{0, 1, 2, 3\} \quad w_2 \sim \begin{cases} \text{uniform}\{0, \cdots, |S|\} & \text{w.p 0.9} \\ 0 & \text{w.p 0.1} \end{cases}$$

For O-TMFQ Learning, we run the outer iteration for 7000 steps. As before, for better sample efficiency we initialize the Q function at each iteration with Q of the previous iteration. We used a logarithmically decaying learning rate, we decay the learning rate from $10^{-3}$ to $10^{-2}$. The behavior of our RL algorithms is much the same as the earlier case, and is shown in Figures 7–10. Figure 7 shows the pdf of the final mean field distribution obtained by performing O-TMFQL with 2000 agents with different values of $\delta_3$ while figure 8 shows the average state of the population. Observe that as the cost of action $\delta_3$ increases, agents take lower actions and are hence distributed towards lower states. Figure 9 is a heat map of the final TQ value function. Figure 10 shows the mean state
evolution with different number of agents for $\delta_3 = 0.1$. Observe that the convergence is poor with lesser number of agents.