Compute-and-Forward for Block-Fading Channels via Algebraic Lattice Codes

Shanxiang Lyu, Antonio Campello and Cong Ling
Department of EEE, Imperial College London
London, SW7 2AZ, United Kingdom
Email: s.lyu14, a.campello, c.ling@imperial.ac.uk

Jean-Claude Belfiore
Mathematical and Algorithmic Sciences Lab
France Research Center
Huawei Technologies
belfiore@telecom-paristech.fr

Abstract—Previous approaches of compute and forward (C&F) are mainly based on static channel model, where the channel coefficients stay fixed during the whole transmission time span. In this work, we investigate the C&F strategy under block fading channels. Our technique is to design codes using construction A over rings, so as to allow better quantization for the channels. Advantages in terms of decoding error probabilities and computation rates are presented, and the construction is shown to strictly outperform, in this scenario, the compute-and-forward strategy over the integers \( \mathbb{Z} \).

Index Terms—compute and forward, block fading, construction A.

I. INTRODUCTION

Building upon the property that lattice codes are closed under integer combinations of codewords, the compute-and-forward (C&F) relaying protocol proposed by Nazer and Gaspar [1] has become a popular physical layer network coding framework. The protocol has been extended towards several direction. Since \( \mathbb{Z} \) may not be the most suitable space to quantize the actual channel, one line of work is to use more compact rings. If the message space and the lattice cosets are both \( \mathcal{O} \)-modules where \( \mathcal{O} \) refers to a ring, the linear labeling technique in [2] enables the decoding of a ring combination of lattice codewords. It has also been shown that using Eisenstein integers \( \mathbb{Z}[\omega] \) [3], [4] or rings from quadratic number fields [5] can have better computation rates for some complex channels than Gaussian integers \( \mathbb{Z}[i] \). The second line of work is to incorporate a MIMO model, e.g., MIMO compute and forward [6] and integer forcing (IF) linear receivers [7]. They both allow cooperation among receive antennas, and their equivalent channels are both multiple-access channels (MAC) so their rate analysis adheres to that of [1].

The third line of extension is to investigate non-static channels, and it has been argued that time-varying channels are more suitable to model realistic systems such as the orthogonal frequency-division multiplexing (OFDM) system [8], [9]. In [8], the computation rates have been analyzed and it shows the rationale of decoding an integer combination of lattice codewords stills works to some extent under block fading. Actual implementation of this idea is later investigated via root-LDA lattices [9], where full diversity is shown for two-way relay channels and multiple-hop line networks in simulations. As the channel coefficients in different fading blocks are not the same, it seems natural to employ different integer coefficients across different blocks so as to enjoy better quantizing performance, rather than approaches of [8], [9] that fix the integer coefficient for the whole duration of the codeword. However, the resulted combination may no longer be a lattice codeword, which draws us into a dilemma.

In [10], it was briefly suggested that number fields constructions as in [11], [5] could be advantageous for C&F in a block-fading scenario. Here we provide a precise analysis on its decoding error and rates, showing that this scheme can minimize the upper bound of the decoding error probability. Specifically, via these codes, integer coefficients among blocks belong to the embedding of rings into Euclidean space, and \( \mathbb{Z} \) is only a special case where its conjugates are the same. This type of lattices naturally suits block fading channels as it has been shown that algebraic lattice codes can be capacity achieving for compound block fading channels [11]. The contribution of this work is to demonstrate the error and rates advantages of this algebraic coding scheme for C&F in block fading channels, and to present a practical algorithm for finding equations that have high rates.

The rest of this paper is organized as follows. In Section II, we review some backgrounds about C&F and algebraic number theory. In Sections III and IV, we present our actual coding scheme, as well as analyzing its properties. Sections V presents the scheme to maximize the computation rate, and the last section provides some simulation results.

Notation: Matrices and column vectors are denoted by uppercase and lowercase boldface letters. \( x(i) \) and \( X(i, j) \) refer to scalars of \( x \) and \( X \) with indexes \( i \) and \( i, j \). The set of all \( n \times n \) matrices with determinant \( \pm 1 \) and integer coefficients will be denoted by \( \text{GL}_n(\mathbb{Z}) \). We denote \( \log^+(x) = \max(\log(x), 0) \).

II. PRELIMINARIES

A. Compute and forward

Consider a general real-valued AWGN network [1] with \( L \) source nodes and \( M \) relays. We assume that each source node \( l \) is operating at the same rate and define the message rate as \( R_{\text{mes}} = \frac{1}{n} \log(|W|) \), where \( W \) is the message space. A message \( w_l \in W \) is encoded, via a function \( \mathcal{E}(\cdot) \), into a point \( x_l \in \mathbb{R}^T \), satisfying the power constraint \( \sum_{t=1}^{T} x_l(i)^2 \leq TP \).
where \( P \) denotes the signal to noise ratio (SNR). The received signal at a relay is given by

\[
y = \sum_{i=1}^{L} h_i x_i + z,
\]

where the channel coefficients \( h_i \) remain constant over the whole time frame, and \( z \sim N(0, I_p) \).

In the C&F scheme [1], \( x_i \) is a lattice point representative of a coset in the quotient \( \Lambda / \Lambda' \), where \( \Lambda \) and \( \Lambda' \) are called the \textit{fine} and \textit{coarse} lattices. Instead of directly decoding the messages, a relay searches for an integer combination of \( w_i \), \( l = 1, \ldots, L \). To this purpose, the relay first estimates a linear combination of lattice codewords \( \tilde{x} = [Q(\alpha y)] \mod \Lambda' = \sum_{i=1}^{L} \alpha_i x_i \), where \( \alpha \in \mathbb{R} \) is a minimum mean square error (MMSE) constant, and \( Q(\cdot) \) is a nearest neighbor quantizer to \( \Lambda \). For certain coding schemes, there exists an isomorphic mapping \( g(\cdot) \) between the lattice cosets \( \Lambda / \Lambda' \) and the message space \( \mathcal{W} \). \( g(\Lambda / \Lambda') \cong \mathcal{W} \), which enables the relay to forward a message \( u = g(\tilde{x}) \) in the space \( \mathcal{W} \), explicitly given by

\[
u = \sum_{i=1}^{L} g(a_i) w_i,
\]

the decoding error event of a relay given \( h \in \mathbb{R}^L \) and \( a \in \mathbb{Z}^L \) as \([Q(\alpha y)] \mod \Lambda' \neq \sum_{i=1}^{L} \alpha_i x_i \) for optimized \( \alpha \). A computation rate is said to be achievable at a given relay if there exists a coding scheme such that the probability of decoding error tends to zero as \( T \to \infty \). The achievable computation rates by the computer-and-forward protocol are given in the following theorem.

\textbf{Theorem 1.} [1] The following computation rate is achievable:

\[
R_{\text{comp}}(h, a) = \frac{1}{2} \max_{\alpha \in \mathbb{R}} \log \left( \frac{P}{|\alpha|^2 + P \| \alpha h - a \|^2} \right).
\]

\textbf{B. Number fields and algebraic lattices}

A number field is a field extension \( \mathbb{K} = \mathbb{Q}(\zeta) \) that defines a minimum field containing both \( \mathbb{Q} \) and a primitive element \( \zeta \). The degree of the minimum polynomial of \( \zeta \) is called the degree of \( \mathbb{K} \). Any element in \( \mathbb{K} \) can be represented by using the power basis \( \{1, \zeta, \ldots, \zeta^{n-1}\} \), so that if \( c \in \mathbb{K} \), then \( c = c_1 + c_2 \zeta + \ldots + c_n \zeta^{n-1} \) with \( c_i \in \mathbb{Q} \). A number is called an algebraic integer if its minimal polynomial has integer coefficients. Let \( \mathbb{S} \) be the set of algebraic integers, then the integer ring is \( \mathcal{O}_\mathbb{K} = \mathbb{K} \cap \mathbb{S} \). For instance, \( \mathbb{K} = \mathbb{Q}(\sqrt{5}) \) is a quadratic field, its power basis is \( \{1, \sqrt{5}\} \), and the basis for \( \mathcal{O}_\mathbb{K} \) is \( \{1, \frac{1+\sqrt{5}}{2}\} \).

An ideal \( \mathcal{J} \) of \( \mathcal{O}_\mathbb{K} \) is a nonempty subset of \( \mathcal{O}_\mathbb{K} \) that has the following properties. 1) \( \mathcal{J}_1 \subset \mathcal{J}_2 \subset \mathcal{J} \) if \( \mathcal{J}_1 \subset \mathcal{J}_2 \subset \mathcal{J} \); 2) \( \mathcal{J}_1 \mathcal{J}_2 \subset \mathcal{J} \) if \( \mathcal{J}_1 \subset \mathcal{J} \), \( \mathcal{J}_2 \subset \mathcal{J} \). Ideal \( \mathcal{I} \) can be decomposed into a product of prime ideals, i.e., \( p\mathcal{O}_\mathbb{K} = \prod_{i=1}^{l} p_i^{e_i} \) for a prime \( p \), in which \( e_i \) is the ramification index. The inertial degree of \( p_i \) is \( r_i = [\mathcal{O}_\mathbb{K} / p_i : \mathbb{Z}/p\mathbb{Z}] \), and it satisfies \( n = \sum_{i=1}^{l} e_ir_i \). Each prime ideal \( p_i \) is said to be lying above \( p \). There exists an isomorphic mapping \( \mathcal{O}_\mathbb{K}/p \cong \mathbb{F}_{p^r} \) for the inertial degree \( r \).

We follow [12], [11], [5] to build lattices by construction A over rings, and \( p \) is lying above \( p \) and chosen to be totally ramified so that \( \mathcal{O}_\mathbb{K}/p \cong \mathbb{F}_{p^r} \). Let \( \mathbb{G} \) be a generator matrix of a \((T_0, t)\) linear code over \( \mathbb{F}_p \) and \( t < T_0 \). The algebraic lattice over \( \mathcal{O}_\mathbb{K} \) is given via the following procedure.

1) Construct a codebook \( \mathcal{C} = \{ x = Gc \mid c \in \mathbb{F}^n \} \) with multiplication over \( \mathbb{F}_p \).

2) Define a component-wise ring isomorphism \( \mathcal{M} : \mathbb{F}_p \to \mathcal{O}_\mathbb{K}/p \), so that \( \mathcal{C} \) is mapped to the coset leaders of \( \mathcal{O}_\mathbb{K}/p^{T_0} \) defined by \( \Lambda^* \triangleq \mathcal{M}(\mathcal{C}) \).

3) Expand \( \Lambda^* \) by tiling \( \Lambda = \Lambda^* + p^{T_0} \).

A lattice \( \mathcal{O}_\mathbb{K}(C) \) generated by such constructions is an \( \mathcal{O}_\mathbb{K} \)-module of rank \( T_0 \).

\textbf{III. ALGEBRAIC CODING FOR BLOCK FADING CHANNELS}

For a block fading scenario consisting of \( n \) blocks with coherence time \( T_0 \), the received message in a relay written in a matrix format is

\[
y = \sum_{i=1}^{L} H_i x_i + z,
\]

where the channel state information (CSI) \( H_i = \text{diag}(h_{i,1}, \ldots, h_{i,n}) \) is available at the relay, \( x_i = [x_{i,1}, \ldots, x_{i,n}]^T \in \mathbb{R}^{n \times T_0} \) denotes a transmitted codeword, and \( z = [z_1, \ldots, z_n]^T \) with \( z_i \sim N(0, I_{T_0}) \) is a Gaussian noise. A diagram for this block fading channel model is shown in Figure 1. In the figure \( X_2 \) consists of codes over time rather than over multiple antennas as in [6], while their MIMO C&F model, whose channel matrices are not restricted to be diagonal, represents a general extension.

![Fig. 1. The block fading model at one relay.](image-url)
First we construct a pair a linear codes \( (C_f, C_c) \) to build the coding lattice \( \Lambda_f^2 \) and the shaping lattice \( \Lambda_c^2 \). Define 
\[
C_f = \{ G_f w \mid w \in F_p^l \} \quad \text{and} \quad C_c = \{ G_c w \mid w \in F_p^l \},
\]
where 
\[
G_f \in F_p^{l_T \times l_f} \quad \text{and} \quad G_c \in F_p^{l_T \times l_c}
\]
is contained in the first \( l_c \) columns of \( G_f \). Then our fine lattice and coarse lattice are given by 
\[
\Lambda_f^{O_c} = M(C_f) + p^{T_0} \quad \text{and} \quad \Lambda_c^{O_c} = M(C_c) + p^{T_0}.
\]
For the time being, we get \( \mathbf{x}_i \in \Lambda_f^{O_c} \cap \mathcal{N}(\Lambda_c^{O_c}) \in \mathcal{O}_K^c \). Since \( \mathbb{K} : \mathbb{Q} = n \), the transmitted vector \( x_i = \gamma \sigma_i(\mathbf{X}_i) \in R^{nT_0} \), where \( \gamma \) is a scaling constant such that the second moment of the shaping lattice \( \mathcal{N}(\Lambda_c^{O_c}) \) has a power smaller than \( P \). Now we have \( x_i \in \gamma \Lambda_f^2 \cap \gamma \mathcal{N}(\Lambda_c^2) \). By rearranging \( x_i \) into \( \mathbf{X}_i \), it represents the row composition of the conjugates of \( \mathbf{x}_i \), i.e.,
\[
\mathbf{X}_i = \gamma \begin{bmatrix}
\sigma_1(\mathbf{x}_i) \\
\sigma_2(\mathbf{x}_i) \\
\vdots \\
\sigma_n(\mathbf{x}_i)
\end{bmatrix}.
\]

Similar to \[5\] Theorem 5, there exists an isomorphism between \( \gamma \Lambda_f^2 / \gamma \Lambda_c^2 \) and the message space \( W \). The equivalent lattices of \( \gamma \Lambda_f^2 \) and \( \gamma \Lambda_c^2 \) have volumes \( p^{T_0 - 1} c_T^{\Delta_f / 2} \) and \( p^{T_0 - 1} c_T^{\Delta_c / 2} \) (\( \Delta_K \) is the discriminant of \( \mathbb{K} \)), so the message rate at every node is \( R_{\text{mes}} = \frac{c_T}{T_0 - l_c} \log p \).

### IV. ERROR PROBABILITY AND RATE ANALYSIS

The following lemma is the crux of our decoding algorithm, which means different rows of \( \mathbf{X}_i \) are not only closed in \( \gamma \Lambda_f^2 \) under \( Z \) but more generally under \( \mathcal{O}_K \).

**Lemma 2.** Let \( \sigma_i \in \mathcal{O}_K \), \( A_l = \text{diag}(\sigma_1(a_l), \ldots, \sigma_n(a_l)) \) for \( 1 \leq l \leq L \), the physical layer codewords are closed under summation of ring elements, i.e., \( \sum_{l=1}^L \left( A_l \mathbf{X}_l \right) \in \gamma \Lambda_f^2 \).

According to Lemma 2, the decoder for block fading channel \( 2 \) can be designed as the one that extracts an algebraic combination of lattice codewords:
\[
BY = \sum_{l=1}^L A_l X_l + B \sum_{l=1}^L H_l X_l - \sum_{l=1}^L A_l X_l + BZ,
\]
where \( B = \text{diag}(b_1, \ldots, b_n) \) is a constant diagonal matrix, to be optimized in the sequel. The following proposition uses a union bound argument to evaluate the decoding error w.r.t. model (3), whose proof can be found in the appendix.

**Proposition 3.** Let \( a = [a_1, \ldots, a_L] \) and keep the notation as above. The error probability of decoding the linear combination associated to \( A_l \), \( 1 \leq l \leq L \), is upper bounded as

\[
P_e(B, a) \leq \sum_{x \notin \Lambda_0^2} \frac{1}{2} \exp \left( -\frac{n \left( d_{n,T_0}(\gamma x) \right)^{1/n}}{8 \sum_{j=1}^n \nu_{\text{eff}, j}} \right),
\]

where
\[
\nu_{\text{eff}, j} = \left| b_j \right|^2 + P \left\| b_j \mathbf{h}_j - \sigma_j(a) \right\|^2,
\]
\[
\mathbf{h}_j \triangleq [H_1(j, j), \ldots, H_L(j, j)]^T \in \mathbb{R}^L,
\]
\[
\sigma_j(a) \triangleq [A_1(j, j), \ldots, A_L(j, j)]^T \in \mathcal{O}_K^L,
\]
and
\[
d_{n,T_0}(x) \triangleq \prod_{j=1}^n \left( \sum_{i=(j-1)T_0+1}^{jT_0} x(i)^2 \right)
\]
is the block-wise product distance of a lattice point.

Further define the minimum block-wise product distance of a lattice as \( d_{n,\text{min}}(\Lambda) \triangleq \min_{x \in \Lambda \setminus \{0\}} d_{n,T_0}(x) \). It follows from \( 4 \) that the decoding error is dictated by \( d_{n,\text{min}}(\gamma \Lambda_f^2) \) and the power of the effective noise. The first advantage of coding over algebraic lattices is to bring a lower bound to \( d_{n,\text{min}}(\gamma \Lambda_f^2) \). To be concise, we have \( \text{Nr}(x(i)) \in Z \) for \( x(i) \in \mathcal{O}_K \), so that for a \( \gamma x \in \gamma \Lambda_f^2 \neq 0 \), it yields
\[
d_{n,T_0}(\gamma x) = \gamma^{2n} \prod_{j=1}^n \left( \sum_{i=(j-1)T_0+1}^{jT_0} \sigma_j(x(i))^2 \right),
\]
\[
\geq \gamma^{2n} \prod_{j=1}^n \left( T_0 \left( \sum_{i=(j-1)T_0+1}^{jT_0} \sigma_j(x(i))^2 \right)^{1/T_0} \right),
\]
\[
= \gamma^{2n} T_0^n \left( \prod_{j=1}^n \text{Nr}(x(i))^2 \right)^{1/T_0} \geq \gamma^{2n} T_0^n.
\]

The second advantage of our scheme is that it often yields smaller effective noise power due to finer quantization than \( Z^L \). This can be reflected by the computation rate analysis and verified by simulations. Depending on how to bound the effective noise term in \( 8 \), we can use the same arguments as \( 8 \) to establish the computation rates for our arithmetic mean (AM) decoder and an geometric mean (GM) decoders. We will concentrate on analyzing the AM decoder whose proof is omitted due to its similarity with \( 8 \) (the only difference is how to represent the integer coefficients \( A_l \)).

**Theorem 4.** With properly chosen lattice codebooks, given channel \( H_l \) and the desired quantization coefficient \( A_l \) for \( 1 \leq l \leq L \) in a relay, the computation rate of our AM decoder is given by

\[
R_{\text{AM}}(H_l, A_l) = \max_B \frac{1}{2} \log \left( \frac{1}{\left\| B \right\|^2 + P \sum_{l=1}^n \left\| B H_l - A_l \right\|^2} \right).
\]

The denominator inside (5) can be described by a function \( \nu_{\text{eff}}(B, a) \). By assuming \( a \) to be fixed, the minimum mean
square error (MMSE) principle for optimizing $\nu_{\text{eff}}^2(B, a)$ is to pick the diagonal elements of $B$ in the following way:

$$b_j = \frac{P\sigma_j(a)\top h_j}{P\|h_j\|^2 + 1}.$$  \hfill (6)

Plugging (6) back yields

$$\nu_{\text{eff}}^2(B, a) = \sum_{j=1}^n \left(\frac{P\|h_j\|^2 + 1}{P\|h_j\|^2 + 1} - \sigma_j(a)\right)^2 + \left(\frac{P\|h_j\|^2 + 1}{P\|h_j\|^2 + 1}\right)^2.$$  

Further define a Gram matrix

$$M_j = I - \frac{P}{P\|h_j\|^2 + 1}h_jh_j\top,$$

then the computation rate of our AM decoder becomes

$$R_{\text{AM}}(M_j, a) = \frac{1}{2}\log^+ \left(\sum_{j=1}^n \sigma_j(a)^2 M_j\sigma_j(a)\right).$$  \hfill (7)

Its achievable rate is therefore maximized by optimizing $a \in O_K^L$. Since $\mathbb{Z}^L \subseteq O_K^L$, the achievable rate in (7) is no smaller than that of $\mathbb{Z}$-lattices.

V. EFFICIENT SEARCH ALGORITHM

The optimization target in (7) is to find $a \in O_K^L$ to reach the minimum of $f(a) = \sum_{j=1}^n \sigma_j(a)^2 M_j\sigma_j(a)$. Our approach is to take advantage of the generator matrix of $O_K$, so that $f(a)$ represents the square distance of a lattice vector, and (7) is turned into a shortest vector problem (SVP). Let $\{\phi_1, ..., \phi_n\}$ be a $\mathbb{Z}$-basis of $O_K$, then its generator matrix is given by

$$\Phi = \begin{bmatrix} \sigma_1(\phi_1) & \cdots & \sigma_1(\phi_n) \\ \sigma_2(\phi_1) & \cdots & \sigma_2(\phi_n) \\ \vdots & & \vdots \\ \sigma_n(\phi_1) & \cdots & \sigma_n(\phi_n) \end{bmatrix}. $$

With Cholesky decomposition $M_j = M_j\top M_j$, we have $f(a) = \sum_{j=1}^n \|M_j\sigma_j(a)\|^2$. The lattice associated with $f(a)$ is indeed a $\mathbb{Z}$-submodule of $\mathbb{R}^n_L$, with a generator matrix $\Phi = M_m\top\Phi_m$,

$$M_m = \begin{bmatrix} M_1 & \cdots & 0 \\ 0 & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & M_n \end{bmatrix},$$

and $\Phi_m = U(I_L \otimes \Phi)$ where $U \in \text{GL}_{mL}(\mathbb{Z})$ is a row-shuffling operation. For instance, when $n = 2$, $L = 2$, we can visualize $\Phi_m$ as

$$\Phi_m = \begin{bmatrix} \sigma_1(\phi_1) & \sigma_1(\phi_2) & 0 & 0 \\ 0 & 0 & \sigma_1(\phi_1) & \sigma_1(\phi_2) \\ \sigma_2(\phi_1) & \sigma_2(\phi_2) & 0 & 0 \\ 0 & 0 & \sigma_2(\phi_1) & \sigma_2(\phi_2) \end{bmatrix}. $$

Finally, it yields $f(a) = f(\tilde{a}) = \|\Phi\tilde{a}\|^2$, with $\tilde{a} \in \mathbb{Z}^n_L$. The classic sphere decoding algorithm [13] can help to obtain this solution with reasonable complexity.

Remark 5. For a block fading channel with $n = 2$ blocks, codeword length $nT_0 \to \infty$, and the number of source nodes $L$ is large, we observe that $Q(\sqrt{5})$ is the best quadratic number field to use on the average. Intuitively, this can be explained as follows. Let the successive minima of $\Phi$ be $\lambda_1, ..., \lambda_L$. If $L$ is large, then the Gaussian heuristic [14] claims that asymptotically $\lambda_1 \approx |\det(\Phi)|^{1/n}F(1+\nu/2)^{1/n}$, which means $\lambda_1$ can be reflected by $|\det(\Phi)| = |\det(M_{mix})||\det(\Phi)|^L$. For a quadratic field $Q(\sqrt{D})$ with $D$ being positive and square free, $D = 5$ attains the smallest discriminant ($\Delta_5 = |\det(\Phi)|^2$). Note that the normalizing factor $\gamma$ has no effect on this analysis since $nT_0 \to \infty$ helps to claim the existence of a lattice that is good for shaping.

VI. NUMERICAL RESULTS

In this section, we will numerically verify the validness of the AM computation rate (7) and the optimality of $Q(\sqrt{5})$. In the example, we let $n = 2$, $L = 2$, and compare the average achievable rates at one relay. Domains for $a$ in (7) are chosen to be $\mathbb{Z}^2$, $\mathbb{Z}[\sqrt{3}]^2$, $\mathbb{Z}[\sqrt{5}]^2$, $\mathbb{Z}[\sqrt{7}]^2$ and $\mathbb{Z}[\sqrt{13}]^2$, respectively. We generate the channels from 1000 instance, where both $h_1, h_2$ have $\mathcal{N}(0, 1)$ entries, and calculate the average of the obtained rates.

Fig. 2. Comparison of achievable rates with different rings.
APPENDIX A

PROOF OF PROPOSITION 3

Proof: We first follow [11] to find the effective noise. With chosen \( B \) and \( A_j \), it first computes \( S = B Y + \sum_{i=1}^{L} A_i D_i \), where \( D_i \) is the dither from a source node which is uniformly distributed on the Voronoi region \( \gamma V_{A_i} \). To get an estimate of the lattice equation \( V = \sum_{i=1}^{L} A_i X_i \), mod \( \gamma A_i \), \( S \) is first quantized w.r.t. the fine lattice \( \gamma A_i \) denoted by \( Q(\cdot) \) and then modulo the coarse lattice \( \gamma A_c \). Since

\[
[Q(S)] \mod \gamma A_c = [Q(S) \mod \gamma A_i] \mod \gamma A_c,
\]

if \([S] \mod \gamma A_i\) has an effective noise falls within the Voronoi of the fine lattice, then the noise effect can be canceled. Now we show that \([S] \mod \gamma V_{A_i} \) is equivalent to \( V \) plus a block wise noise. Denote \( \Theta_i = BH_i - A_i \) and \( X_i = [X_i + D_i] \mod \gamma A_c \), then

\[
[S] \mod \gamma A_i = [V + \sum_{i=1}^{L} (\Theta_i X_i) + B Z] \mod \gamma A_i. \tag{8}
\]

As \( X_i \) and \( Z \) are still uniform and Gaussian across different time blocks, the probability density function (PDF) of the \( j \)th row of \( Z_{\text{eff}} \) can be shown to be upper bounded by a Gaussian \( N(0, \sigma_{Z_{\text{eff}}}) \) in which

\[
\nu_{Z_{\text{eff}}, j}^2 = |b_j|^2 + P \|b_j h_j - \sigma_j(a)\|^2, \tag{9}
\]

\( h_j \triangleq [H_1(j, j), \ldots, H_L(j, j)]^T \in \mathbb{R}^L \), and \( \sigma_j(a) = a_j \triangleq [A_1(j, j), \ldots, A_L(j, j)]^T \in O_{\mathbb{Z}}^L \). It turns out to be a non-AWGN lattice decoding problem, whose decoding error probability is

\[
\Pr(\{B, A_j\}) = \sum_{V' \notin V + \gamma A_c} \Pr(V \rightarrow V'),
\]

\[
= \sum_{V' \notin V + \gamma A_c} \Pr\left(\|V + Z_{\text{eff}} - V'\|^2 \leq \|Z_{\text{eff}}\|^2\right),
\]

\[
= \sum_{V' \notin V + \gamma A_c} \Pr\left(\sum_{j=1}^{n}\|v_j - v'_j\|^2 + 2(v_j - v'_j)^T z_{\text{eff}, j} \leq 0\right), \tag{10}
\]

in which \( v_j^T, v_j'^T \) and \( z_{\text{eff}, j} \) are the \( j \)th rows of \( V, V' \) and \( Z_{\text{eff}} \), respectively. Further define \( \Upsilon \triangleq \sum_{j=1}^{n} 2(v_j - v'_j)^T z_{\text{eff}, j} \). Similar to the analysis of [8], the PDF of \( \Upsilon \) is upper bounded by a zero mean Gaussian with variance \( \sum_{j=1}^{n} 4v_{eff, j}^2 \|v_j - v'_j\|^2 \). It then follows from the property of a Q function \( Q_g(x) = \frac{1}{\sqrt{2\pi}} \int_{x}^{\infty} \exp\left(-\frac{z^2}{2}\right) \) du that

\[
\Pr(V \rightarrow V') \leq Q_g\left(\frac{\sum_{j=1}^{n} \|v_j - v'_j\|^2}{2\sum_{j=1}^{n} 4v_{eff, j}^2 \|v_j - v'_j\|^2}\right),
\]

\[
\leq \left\{\begin{array}{ll}
\frac{1}{2} & \frac{1}{2} \exp\left(-\frac{(\sum_{j=1}^{n} \|v_j - v'_j\|^2)^2}{8 \sum_{j=1}^{n} 4v_{eff, j}^2 \|v_j - v'_j\|^2}\right), \\
\frac{1}{2} & \frac{1}{2} \exp\left(-\frac{(\prod_{j=1}^{n} \|v_j - v'_j\|^2)^{1/n}}{8 \sum_{j=1}^{n} 4v_{eff, j}^2 \|v_j - v'_j\|^2}\right),
\end{array}\right. \tag{11}
\]

where (a) has used the bound \( Q_g(x) \leq 1/2 \exp(-x^2/2) \), (b) comes after using \( v_{eff, j}^2 \leq \sum_{j=1}^{n} 4v_{eff, j}^2 \|v_j - v'_j\|^2 \) and the AM-GM inequality. The relaxation in (b) serves the purpose of bounding the error probability via the block-wise product distance of our algebraic lattice. Combining [8], [10] and [11] proves the proposition. ■

ACKNOWLEDGMENT

The authors acknowledge Y.-C. Huang (Jerry) for fruitful discussions.

REFERENCES

[1] B. Nazer and M. Gastpar, “Compute-and-forward: Harnessing interference through structured codes,” IEEE Trans. Inf. Theory, vol. 57, no. 10, pp. 6463–6486, 2011.
[2] C. Feng, D. Silva, and F. R. Kschischang, “An Algebraic Approach to Physical-Layer Network Coding,” IEEE Trans. Inf. Theory, vol. 59, no. 11, pp. 7576–7596, nov 2013.
[3] Q. T. Sun, J. Yuan, T. Huang, and K. W. Shum, “Lattice network codes based on Eisenstein integers,” IEEE Trans. Commun., vol. 61, no. 7, pp. 2713–2725, 2013.
[4] N. E. Tunali, Y. C. Huang, J. J. Boutros, and K. R. Narayanan, “Lattices over Eisenstein Integers for Compute-and-Foward,” IEEE Inf. Theory, vol. 61, no. 10, pp. 5306–5321, 2015.
[5] Y.-C. Huang, K. R. Narayanan, and P.-C. Wang, (2015). “Adaptive Compute-and-Forward with Lattice Codes Over Algebraic Integers.” [Online]. Available: http://arxiv.org/abs/1501.07740
[6] J. Zhan, B. Nazer, M. Gastpar, and U. Erez, “MIMO compute-and-forward,” in 2009 IEEE Int. Symp. Inf. Theory. IEEE, jun 2009, pp. 2848–2852.
[7] J. Zhan, B. Nazer, U. Erez, and M. Gastpar, “Integer-forcing Linear Receivers,” IEEE Trans. Inf. Theory, vol. 60, no. 12, pp. 7661–7685, dec 2014.
[8] E. I. Bakouy and B. Nazer, “The impact of channel variation on integer-forcing receivers,” in 2015 IEEE Int. Symp. Inf. Theory. IEEE, jun 2015, pp. 576–580.
[9] C. Wang, Y. C. Huang, K. R. Narayanan, and J. J. Boutros, “Physical-layer network-coding over block fading channels with root-LDA lattice codes,” 2016 IEEE Int. Conf. Commun. ICC 2016, pp. 1–6, 2016.
[10] Y.-C. Huang, (2016). “Construction \& A Lattices : A Review and Recent Results.” [Online]. Available: https://www.york.ac.uk/media/mathematics/documents/Jerry_[_Huang[_York2016.pdf
[11] A. Campello, C. Ling, and J.-C. Belfiore, “Algebraic lattice codes achieve the capacity of the compound block-fading channel,” in 2016 IEEE Int. Symp. Inf. Theory. IEEE, jul 2016, pp. 910–914.
[12] W. Kositwattanarerk, S. S. Ong, and F. Oggier, “Construction of a lattices over number fields and block fading (wiretap) coding,” IEEE Trans. Inf. Theory, vol. 61, no. 5, pp. 2273–2282, 2015.
[13] B. Hassibi and H. Vikalo, “On the sphere-decoding algorithm I: Expected complexity,” IEEE Trans. Signal Process., vol. 53, no. 8, pp. 2806–2818, aug 2005.
[14] N. Gama and P. Q. Nguyen, “Predicting Lattice Reduction,” Eurocrypt, vol. 4965, pp. 31–51, 2008.
[15] U. Niesen and P. Whiting, “The degrees of freedom of compute-and-forward,” IEEE Trans. Inf. Theory, vol. 58, no. 8, pp. 5214–5232, 2012.