A Recursive Definition of the Holographic Standard Signature

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Abstract

We provide a recursive description of the signatures realizable on the standard basis by a holographic algorithm. The description allows us to prove tight bounds on the size of planar matchgates and efficiently test for standard signatures. Over finite fields, it allows us to count the number of $n$-bit standard signatures and calculate their expected sparsity.

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1 Introduction

Holographic algorithms have been a subject of much interest in the mathematical community since Leslie Valiant conceived of them in 2002 (see [12]). These algorithms can calculate certain exponential sums in polynomial time, skating dangerously close to \#P problems.

This paper will examine one small aspect of holographic algorithms; our narrow focus will allow us to avoid some of the details and much of the terminology surrounding the subject. However, to provide a little context for the reader unfamiliar with holographic computing, we will give an extremely rough sketch of the subject in the next paragraph. More precise and complete introductions can be found in [14] or [2].

We can think of holographic computing as follows. Fix a field \( \mathbb{F} \). Imagine that we build a circuit board out of special circuit components. Each component has a certain number of wires which we can attach to other components. We attach the wires so that none of them cross on the circuit board. (In other words, if we treat the circuit components as nodes and the wires as edges, we form a planar graph.) Each wire can take on only two values, either zero or one. If we specify the values of the wires attached to a component, it produces an output value lying in \( \mathbb{F} \). (If a component has \( n \) wires, this function from \( \{0,1\}^n \) to \( \mathbb{F} \) is the “signature” referred to in the title; we would call it an “\( n \)-bit standard signature”.) If we set all the wires on the entire circuit board, we define the entire circuit board as producing the product of the outputs of the individual circuit components. A holographic algorithm lets us compute the sum of these products over all (exponentially many) wire settings in polynomial time.

If the signatures could be chosen freely, it would follow that \( P = \#P \).

Sadly, if not surprisingly, we lack this freedom: only some functions are hospitable to holographic manipulations. These special functions are said to be realizable on the standard basis, or are simply called the standard signatures. It is possible to change our computational basis, which produces new sets of signatures. Much of the power of holographic algorithms arise from these changes of basis; however, this paper focuses only on the simpler case of the standard basis.

So, which functions are standard signatures? Three equivalent definitions are frequently used. Standard signatures were originally defined in terms of sums of weighted matchings on planar graphs by Valiant in [13]. However, Cai and Choudhary established an equivalence between standard signatures and the Pfaffians of certain matrices in [3] and [4], providing a second definition. One consequence of their result is a description of the standard signatures as an algebraic variety: a function is a standard signature if and only if a certain set of quadratic equations evaluate to zero. This provides a third definition of a standard signature.

Although the reader may think that three definitions is more than enough, we offer a fourth one. Our “new” definition is really a consequence of the Pfaffian definition, but it seems to highlight different properties than the other definitions. Our definition is recursive, i.e. we define \( n \)-bit standard signatures in terms of \((n - 1)\)-bit standard signatures. Here are some of the conclusions we draw:

- If we are operating over a finite field, we can count the exact number
of $n$-bit standard signatures. We can also calculate the asymptotics for large $n$. Over $\mathbb{F}_2$ and $\mathbb{F}_3$, the number of odd parity standard signatures coincides with the number of $n$-dimensional self-dual codes. (See Subsections 3.1, 3.2, and 3.7 respectively.)

- It is known that any $n$-bit standard signature can be represented by a planar matchgate with at most $O(n^2)$ nodes. We construct a matching lower bound showing that there exist standard signatures that require at least $\Omega(n^2)$ nodes to encode as a planar matchgate. (See Subsection 3.3.)

- Suppose we are given an $n$-bit function and we would like to determine if it is a standard signature. The naive approach takes $O(n^2)$ steps; using recursion and some structural properties, we can improve this bound to $O(n^{2^{\log_2 n}})$ steps. (See Subsection 3.4.)

- Suppose we are working over a finite field and we select an $n$-bit standard signature uniformly at random. We can calculate the expected sparsity of $f$, i.e., $\Pr[f(x) \neq 0]$. (See Subsection 3.5.)

The paper is structured in two halves. In the first half, Section 2, we present the four different definitions of a standard signature and a few lemmas. In the second half, Section 3, we illustrate various corollaries of the recursive definition. Subsections 3.6 and 3.7 are more speculative in nature. We also include two appendices: Appendix A lists the general form for a normalized 6-bit standard signature, and Appendix B illustrates one method of building recursion into planar matchgates.

2 Definitions

Let $V = \{0, 1\}$ be the field with 2 elements. We will be considering functions from $V^n \rightarrow \mathbb{F}$, where $\mathbb{F}$ is an arbitrary field. We refer to these as $n$-bit functions. (Other authors would call them $n$-arity functions.) Given $x \in V^n$, we often expand it in bits as $x = x_1 \cdots x_n$.

To keep our notation saner, if $\alpha$ is a bit string and we remove a bit from it, we will write $\alpha$. In a similar vein, given a function $f : V^n \rightarrow \mathbb{F}$, we can fix the last bit and define a new function $f_0 : V^{n-1} \rightarrow \mathbb{F}$ as $f_0(x_1 \cdots x_{n-1}) = f(x_1 \cdots x_{n-1}0)$ and $f_1(x_1 \cdots x_{n-1}) = f(x_1 \cdots x_{n-1}1)$.

Let $e_i \in V^n$ be the string all of whose bits equal zero except for the $i$-th bit. Also, for any two $n$-bit strings $x$ and $y$, let $x \oplus y$ represent the bitwise XOR of the two strings.

Given $x = x_1x_2 \cdots x_n \in V^n$, let $|x|$ be the Hamming weight of $x$, i.e., $|x| = x_1 + x_2 + \cdots + x_n$.

We define the partial Hamming weight as follows:

$$|x|^k_j = \sum_{i=j}^k x_i$$
Note that $|x|_1^1 = |x|$. If $k < j$, define $x_j^k = 0$.

If $f(x) = 0$ for all $x$, we call $f$ the constant zero function, and write $f \equiv 0$. We refer to other functions as non-zero functions, or write $f \not\equiv 0$.

We can interpret the input either as an $n$-bit string, or as (the binary representation of) an integer in the range $[0, 2^n - 1]$. Using the integer representation, we can specify a function $f : V^n \to F$ by listing its outputs (i.e. its “truth table”). That is, $f$ is fully determined by the ordered list

$$(f(0), f(1), f(2), ..., f(2^n - 1)) \in F^{2^n}$$

Viewed as elements of $F^{2^n}$, functions form a vector space over $F$: we can add together two functions, and we can multiply them by scalars in $F$.

We say that a function $f : V^n \to F$ has even parity if all odd weight codewords are sent to zero, that is

$$\text{if } |x| = 1 \mod 2 \text{ then } f(x) = 0$$

If $f$ has even parity and is not the constant zero function, then we say that $f$ is strictly even parity. We can define (strictly) odd parity functions in the same way. Note that the constant zero function is the unique $n$-bit function that has both even and odd parity.

### 2.1 Standard Signatures via Planar Matchgates

In this section, we will define a class of functions, the standard signatures, in terms of certain graphs and perfect matchings.

A planar matchgate over $F$ is a planar embedding of a planar graph $G$ with weighted edges $w_{i,j} \in F$, along with a set of special “input/output” nodes $v_1, ..., v_n$ on the outer face of the graph. We label the index of each $v_i$ consecutively; that is, if we start at node $v_i$, and proceed in an anti-clockwise direction around the outer face, the next input/output node we encounter is $v_{i+1}$.

We give an example below where $F = \mathbb{R}$. The small numbers are the edge weights, the large numbers are the labels of the input/output nodes. Two of the outer nodes are not input/output nodes (and thus are not labelled):

1In a more typical definition, as in [13], the input/output nodes are divided into distinct sets of “input” and “output” nodes. However, as long as we restrict our attention to the standard basis, that distinction is irrelevant, so we skip it for this paper.
A perfect matching is a collection of edges $E$ such that every node is adjacent to exactly one edge in $E$. The weight of a particular perfect matching is the product of the weights of the edges in $E$. Following Valiant, we will define $\text{PerfMatch}(G)$ to be the sum of the weight of every perfect matching in $G$ (or zero if there are none.) In other words,

$$\text{PerfMatch}(G) = \sum_{E} \prod_{(i,j) \in E} w_{i,j}$$

Next, specify a vector $x \in \{0, 1\}^n$. If the $i$-th bit of $x$ is a one, then suppose we remove node $v_i$ and all of its adjacent edges from $G$. This produces some subgraph, which we will call $G_x$. We can now define a function $f : \{0, 1\}^n \to \mathbb{F}$ by

$$f(x) = \text{PerfMatch}(G_x)$$

The set of functions that can be described in this fashion (for some $G$) form the $n$-bit standard signatures over $\mathbb{F}$.

Given a weighted planar graph $G'$, it is possible to calculate $\text{PerfMatch}(G')$ in time polynomial in the number of nodes using an object called a Pfaffian. This result was proved by Fisher, Kasteleyn and Temperley in 1961 (see [8] for a survey); this is sometimes called the FKT Theorem. We will examine Pfaffians in greater detail in Subsection 2.3.

We will need some notation to describe various sets of standard signatures. First, let $A_n$ be the set of $n$-bit standard signatures. (The set $A_n$ depends on $\mathbb{F}$ of course, but we will treat $\mathbb{F}$ as constant, so we will suppress the extra notation.) We can partition $A_n$ into three disjoint subsets, based on the parity of the function:

$$A_n = A_n^{\text{odd}} \cup A_n^{\text{even}} \cup A_n^0$$

where $A_n^{\text{odd}}$ consists of the strictly odd parity standard signatures, $A_n^{\text{even}}$ consists of the strictly even parity standard signatures, and $A_n^0$ is a one-element set consisting of the constant zero function.

We will find it useful to normalize the standard signatures. Let us define a normalized standard signature as a standard signature $f$ where $f(0\cdots0) = 1$. We let $B_n$ be the set of normalized standard signatures. Note that all the elements of $B_n$ are strictly even parity.

### 2.2 Basic Lemmas

Before continuing with our definitions, we mention a few lemmas that we will find useful later.

**Lemma 1** If $f \equiv 0$ then $f$ is a standard signature.

**Proof:** Given any $n$-bit planar matchgate, we can add two more nodes and an edge between them of weight 0; the resulting standard signature is identically zero. $\square$

For $n = 1$, we can write down $A_1^{\text{odd}}$ and $A_1^{\text{even}}$ explicitly. We will state it as a lemma for future reference.

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2It might seem more natural to define a function as normalized if $f(1\cdots1) = 1$. However, the parity would change as a function of $n$; our definition makes the parity of $B_n$ even for all $n$. 

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Lemma 2 We can characterize the 1-bit standard signatures over any field \( \mathbb{F} \):

\[
\begin{align*}
A_{1}^{\text{odd}} &= \{ f \in V \to \mathbb{F} \mid f(0) = 0 \text{ and } f(1) \neq 0 \} \\
A_{1}^{\text{even}} &= \{ f \in V \to \mathbb{F} \mid f(1) = 0 \text{ and } f(0) \neq 0 \} \\
A_{1}^{0} &= \{ f \in V \to \mathbb{F} \mid f(0) = 0 \text{ and } f(1) = 0 \}
\end{align*}
\]

Lemma 3 By flipping a fixed input bit, we can construct a bijection between strictly even and strictly odd standard signatures.

Proof: Suppose we have a planar matchgate and node \( v \) is labelled as the \( i \)-th input/output node. Suppose we add a new node \( v' \), an edge between \( v \) and \( v' \), and we relabel node \( v' \) as the \( i \)-th input/output node. If \( f(x) \) is the standard signature of the original planar matchgate, then \( f(x+e_i) \) is the standard signature of the new planar matchgate. Note that \( f(x) \) and \( f(x+e_i) \) have opposite parities. Since this operation (flipping the \( i \)-th bit) is invertible, we have established our bijection. \( \square \)

Next, we let us examine normalized functions more carefully. Normalization preserves the quality of being a standard signature:

Lemma 4 Suppose that \( f : V^n \to \mathbb{F} \) and there exists \( \hat{x} \) such that \( f(\hat{x}) = \beta \neq 0 \). Let \( g(x) = \beta^{-1} f(x + \hat{x}) \) (Note that \( g(0\cdots 0) = 1 \).) Then \( f \) is a standard signature if and only if \( g \) is a standard signature.

Proof: Suppose \( f \) is a standard signature and consider a planar matchgate for it. Consider the \( n \) input/output nodes. If \( \hat{x}_i = 1 \), add a new edge and a new node to input/output node \( i \). Move the \( i \)-th input/output node to the new node. This has the effect of switching the value of the \( i \)-th input bit. Finally, add two new nodes with an edge between them, and weight the edge by \( \beta^{-1} \). The standard signature of the resulting planar matchgate calculates \( g \). On the other hand, given a planar matchgate for \( g \), we can repeat the process (using \( \beta \) instead of \( \beta^{-1} \)) and build a planar matchgate for \( f \). Therefore, \( f \) is a standard signature if and only if \( g \) is. \( \square \)

In this paper, we are interested in decomposing standard signatures recursively. Recall that \( f_0 \) and \( f_1 \) are obtained by fixing the last bit of a function \( f \). We will repeatedly use the following fact:

Lemma 5 If \( f \) is an \((n+1)\)-bit standard signature, then \( f_0 \) and \( f_1 \) are standard signatures.

Proof: Consider a planar matchgate for \( f \). Let \( v \) be the \((n+1)\)-st input/output node. Consider a new planar matchgate that is identical, except that \( v \) is no longer labelled as an input/output node. This planar matchgate calculates \( f_0 \); if we add a new node \( v' \) and a new weight one edge between \( v \) and \( v' \), the resulting planar matchgate calculates \( f_1 \). \( \square \)
2.3 Standard Signatures via Pfaffians

The determinant of a matrix over a field $\mathbb{F}$ is a polynomial in the entries of the matrix. In the case of a strongly skew-symmetric $n \times n$ matrix $M$, this polynomial happens to be square, and the square root is called the Pfaffian. (We will define the Pfaffian more formally in a moment.) If we remove a set of rows and matching columns from $M$ and calculate the determinant, we produce an object called a principal minor; there are $2^n$ principal minors. We can think of this operation (converting a matrix into one of its principal minors) as a function from $V^n \to \mathbb{F}$, where the $i$-th bit of the input tells us whether or not to delete the $i$-th row and column.

Suppose, instead of taking the determinant of these submatrices, we take the Pfaffian. This will give us another function $f : V^n \to \mathbb{F}$, a sort of square root of the principal minors. In [3] and [4], Cai and Choudhary prove that $f$ is a normalized standard signature; even more amazingly, as we let $M$ vary over all strongly skew-symmetric matrices over $\mathbb{F}$, we produce all the normalized standard signatures.

We now state the previous facts and observations more formally. Let $m(i, j)$ be the entry of $M$ in the $i$-th row and $j$-th column. A matrix $M$ is strongly skew-symmetric if $m(i, j) = -m(j, i)$ for all $i, j$, and $m(i, i) = 0$ for all $i$. (Strong skew-symmetry only differs from skew-symmetry when the field has characteristic two.) Note that the set of strongly skew-symmetric matrices can be viewed as $\mathbb{F}^{n(n-1)/2}$, since we can determine $M$ by specifying $n(n-1)/2$ entries.

The Pfaffian of an $n \times n$ strongly skew-symmetric matrix $M$ is defined as zero if $n$ is odd, and one if $n = 0$. If $n = 2k$ is a positive even number, then we define the Pfaffian of $M$ as follows. Suppose we pair up all the numbers between 1 and $n$, producing $k$ pairs. We can encode such a pairing with a permutation that has the following two properties:

$$\pi(1) < \pi(2), \pi(3) < \pi(4), \ldots, \pi(n-1) < \pi(n)$$  \hspace{1cm} (1)

and

$$\pi(1) < \pi(3) < \pi(5) < \cdots < \pi(n-1)$$  \hspace{1cm} (2)

We then view $(\pi(2i-1), \pi(2i))$ as paired numbers for $i = 1, \ldots, k$.

Let $\epsilon_\pi$ be the sign of the permutation, i.e. $\epsilon_\pi = 1$ if we can produce $\pi$ from the identity permutation by composing an even number of transpositions, and $\epsilon_\pi = -1$ otherwise. Then

$$\text{Pf}(M) = \sum_\pi \epsilon_\pi \prod_{j=1}^k m(\pi(2j-1), \pi(2j))$$

where the sum runs over permutations $\pi$ satisfying the inequalities in Formulas 1 and 2.

There is an alternate definition of $\epsilon_\pi$ that can be useful. Suppose that we have two pairs of integers $i < j$ and $k < l$, and suppose that $i < k$. We say that the two pairs overlap if $i < k < j < l$. Suppose we consider all the pairs defined by $\pi$. If there are an odd number of overlapping pairs, then $\epsilon_\pi = -1$; otherwise, $\epsilon_\pi = 1$. 

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For \( x = (x_1 \cdots x_n) \in V^n \), let \( M_x \) be the submatrix of \( M \) obtained by removing row \( i \) and column \( i \) from \( M \) if \( x_i = 0 \). Then define \( f_M : V^n \to \mathbb{F} \) by
\[
f_M(x) = \text{Pf}(M_x)
\]
Cai and Choudhary showed that the set of such functions are precisely the normalized standard signatures. Let us state this result formally.

**Theorem 1 (Cai and Choudhary)** Let \( M \) be the set of strongly skew-symmetric \( n \times n \) matrices over a field \( \mathbb{F} \). Then
\[
\{ f_M \mid M \in \mathcal{M} \} = B_n
\]

**Proof:** See [3] and [4].

There is a common method of calculating a determinant by recursively combining minors. We mention a Pfaffian version of the same thing.

**Lemma 6** Let \( M \) be an \((n+1) \times (n+1)\) strongly skew-symmetric matrix. Let \( \hat{x} = 1 \cdots 1 \in V_n^{n+1} \). Then
\[
\text{Pf}(M) = \text{Pf}(M_{\hat{x}})
\]
\[
= \sum_{i=1}^n (-1)^{i-1} m(i, n+1) \text{Pf}(M_{\hat{x}+e_i+e_{n+1}})
\]

Suppose that there are \( s \) non-zero bits in \( x \), and let \( p_1, ..., p_s \) be the positions of those bits, in order. Then
\[
\text{Pf}(M_x) = \sum_{i=1}^s (-1)^{i-1} m(p_i, n+1) \text{Pf}(M_{x+e_{p_i}+e_{n+1}})
\]
\[
= \sum_{i=1}^n x_i (-1)^{|x_i|} m(p_i, n+1) \text{Pf}(M_{x+e_{p_i}+e_{n+1}})
\]

**Proof:** Equation 4 is standard (see, e.g. [6]); it can be proved by using the “overlapping pairs” definition of \( \epsilon_x \).

Equation 4 follows by simply applying Equation 4 to the submatrix defined by the rows and columns specified by \( x \).

Equation 5 follows from Equation 4 since the terms in the sum corresponding to irrelevant rows are zeroed out by the \( x_i \) terms, and the \((-1)^{|x_i|} \) term alternates signs at every non-zero bit in \( x \).

For a fixed \( n \), we can expand the Pfaffian as a multivariate polynomial and write down a parameterized expression for the general form of a normalized standard signature. The number of terms in the longest polynomial is of size \( O(\sqrt{n!}) \), but for small \( n \) this size is manageable. To see the case of \( n = 6 \) bits, please refer to Appendix A.

### 2.4 Standard Signatures via Algebraic Varieties

The Pfaffian definition of a standard signature above is quite powerful, and illuminates other interesting structural features of the standard signatures. It allows us to describe the set of \( n \)-bit standard signatures as an algebraic variety in \( \mathbb{F}(2^n) \). In other words, \( f \) is a standard signature if and only if
the set of outputs \(f(0\cdots00), f(0\cdots01), ..., f(1\cdots1)\) satisfy a collection of polynomial (in fact quadratic) equalities.

We proceed with this alternate definition. A function \(f : V^n \to \mathbb{F}\) is a standard signature if and only if it satisfies the following two classes of constraints:

- First, there is a Parity Constraint: \(f\) must be an even parity or odd parity function.
- Second, there are the Matchgate Identities, also known as the useful Grassmann-Plücker equations. Let \(p\) be an \(n\)-bit string. (The "\(p\)" stands for "position vector".) Let \(L = |p|\). Let \(p_1, ..., p_L\) be the positions of the \(L\) non-zero bits of \(p\), in order. Then for all \(\alpha, p \in V^n\), the following equation holds:

\[
\sum_{i=1}^{L} (-1)^i f(\alpha + e_{p_i}) f(\alpha + e_{p_i} + p) = 0
\]

The equivalence of these constraints with the Pfaffian definition of a standard signature was proved by Cai and Choudhary in [3] and [4]. We can now prove a few more lemmas. First, remember that polynomial images of affine spaces are not necessarily algebraic varieties (see e.g. the exercises in Chapter 3, Section 3 of [5]). In the case of normalized standard signatures, however, we are lucky:

**Lemma 7** The set \(B_n\) of normalized standard signatures is an algebraic variety isomorphic to \(\mathbb{F}_{(2^n)}(n-1)/2\). If \(\mathbb{F}\) is an infinite field, then \(B_n\) has dimension \(n(n-1)/2\).

**Proof:** Since \(A_n\) is an algebraic variety, we can intersect it with \(f(0\cdots0) = 1\) and conclude that \(B_n\) is an algebraic variety.

Now, we turn to the isomorphism. First, since the Pfaffian is a polynomial in the entries of the matrix \(M\), there exists a map \(K : \mathbb{F}^{n(n-1)/2} \to \mathbb{F}^{(2^n)}\) that is surjective on \(B_n\). Next, fix \(a < b \leq n\). Suppose that \(\hat{x} = (\hat{x}_1 \cdots \hat{x}_n)\), where \(\hat{x}_i = 1\) iff \(i = a\) or \(i = b\). Then note that \(\text{Pf}(M_{\hat{x}}) = m(a,b)\). Therefore, if we project the coordinates corresponding to weight two codewords, we get a map \(K' : B^n \to \mathbb{F}^{n(n-1)/2}\) that recovers \(M\). Note that \(K' \circ K\) is the identity in \(\mathbb{F}^{n(n-1)/2}\), and \(K \circ K'\) is the identity on \(B^n\). Therefore, \(B_n\) is isomorphic (as an algebraic variety) to \(\mathbb{F}^{n(n-1)/2}\), and hence they share the same dimension. If \(\mathbb{F}\) is infinite, \(\mathbb{F}^{n(n-1)/2}\) is \(n(n-1)/2\) dimensional. \(\square\)

Suppose we take a matchgate \(G\) and let the edge weights vary. Each choice of edge weights will define a standard signature. Let \(J_G\) be the collection of such standard signatures, viewed as a subset of \(\mathbb{F}^{(2^n)}\). Then the following lemma holds:

**Lemma 8** Assume that our field \(\mathbb{F}\) is infinite. Suppose that \(G\) is an \(n\)-bit planar matchgate. Suppose that the underlying planar graph of \(G\) has \(X\) nodes and \(E\) edges. Then the set \(J_G\) is contained in an algebraic variety of dimension at most \(E\).

**Proof:** Given a weighted \(X\) node planar graph, we can calculate the sum of all its weighted perfect matchings using the FKT Theorem
This theorem expresses the sum as the Pfaffian of a particular \( X \times X \) matrix \( M \), namely a polynomial in the edge weights.

If we consider all the \( 2^X \) principal submatrices of the planar graph, each one corresponds to removing or including a particular node in the graph (not just the input/output nodes). The underlying planar graph forces some of the entries of the matrix to be zero. If we ignore that restriction, we have exactly described the set of normalized standard signatures on \( X \) bits. From Lemma 7, this object is an algebraic variety in \( \mathbb{F}^{(2^X)} \). We will now restrict this variety to recover \( J_G \).

For each edge \( e_{i,j} \) that does not appear in the underlying graph, we set matrix entries \( m(i,j) = m(j,i) = 0 \). This results in an intersection of algebraic varieties, so adding these constraints for all the missing edges gives us another algebraic variety \( P \). Since \( P \) is parameterized by \( E \) variables over \( \mathbb{F} \), it follows that \( \dim(P) \leq E \).

We are interested in projecting \( P \) down to the \( 2^n \) variables (where we are only allowed to remove rows and columns corresponding to the input/output nodes from \( M \)). We can now use polynomial implicitization (see Chapter 3, Section 3, Theorem 1 of [5]) to find the smallest variety \( P' \) in \( \mathbb{F}^{(2^n)} \) containing the projection. (Note that this theorem assumes that \( \mathbb{F} \) is infinite.) We construct \( P' \) by eliminating variables (i.e. intersecting ideals), so \( \dim(P') \leq \dim(P) \leq E \). This establishes our theorem.

### 2.5 Standard Signatures via Recursion

We will present a recursive definition of a standard signature which makes no explicit reference to Pfaffians or planar matchgates. We begin by defining a new set of functions. Suppose we are given a non-zero function \( f : V^n \rightarrow \mathbb{F} \) and we choose a base point \( \hat{x} \) such that \( f(\hat{x}) \neq 0 \). (We will see in Corollary 1 that the choice of base point is irrelevant for standard signatures; for now, let us choose \( \hat{x} \) to be the lexicographically smallest \( x \) such that \( f(x) \neq 0 \).) Let us define the shift basis functions \( s^f_i \) (where \( i = 1, \ldots, n \)) as

\[
 s^f_i(x) = \begin{cases} 
 0 & \text{if } x_i = \hat{x}_i \\
 (-1)^{|x+e_i|} f(x + e_i) & \text{if } x_i \neq \hat{x}_i 
\end{cases}
\]

Next let us define the shift set as the set of functions formed by linear combinations of the shift basis functions, i.e.

\[
 \mathcal{S}_f = \left\{ \sum_{i=1}^n \lambda_i s^f_i \bigg| \lambda_i \in \mathbb{F} \right\}
\]

Note that the elements of the shift set all have the opposite parity as \( f \).

We point out two properties of the shift set.

**Lemma 9** The shift basis functions for \( f \), viewed as vectors over \( \mathbb{F}^{(2^n)} \), are linearly independent (i.e. they actually form a basis for \( \mathcal{S}_f \)). Therefore, \( \mathcal{S}_f \) can be viewed as an \( n \) dimensional subspace of \( \mathbb{F}^{(2^n)} \).

**Proof:** Notice that

\[
 s_i(\hat{x} + e_j) = \begin{cases} 
 0 & \text{if } i \neq j \\
 \pm f(\hat{x}) & \text{if } i = j 
\end{cases}
\]
so $s_i^f(x + e_j)$ is non-zero if and only if $i = j$. It follows that the $s_i^f$ are linearly independent, and hence that the shift set $S_f$ has dimension $n$. □

We can now introduce our new definition:

**Theorem 2** The set of normalized standard signatures can be defined recursively:

$$B_{n+1} = \left\{ f : V^{n+1} \to \mathbb{F} \mid f_0 \in B_n \text{ and } f_1 \in S_{f_0} \right\}$$  \hspace{1cm} (6)

The set of all strictly odd or strictly even standard signatures can be similarly defined:

$$A^{odd}_{n+1} = \left\{ f : V^{n+1} \to \mathbb{F} \mid f_0 \in A^{odd}_n \text{ and } f_1 \in S_{f_0} \right\} \cup \left\{ f : V^{n+1} \to \mathbb{F} \mid f_0 \equiv 0 \text{ and } f_1 \in A^{even}_n \right\}$$ \hspace{1cm} (7)

$$A^{even}_{n+1} = \left\{ f : V^{n+1} \to \mathbb{F} \mid f_0 \in A^{even}_n \text{ and } f_1 \in S_{f_0} \right\} \cup \left\{ f : V^{n+1} \to \mathbb{F} \mid f_0 \equiv 0 \text{ and } f_1 \in A^{odd}_n \right\}$$ \hspace{1cm} (8)

**Proof:** We start by proving Equation (6). First, Lemma □ shows that if $f$ is a standard signature, then $f_0$ is a standard signature.

So, assume that $f_0$ is a standard signature. Recall, from our Pfaffian definition, that for any normalized standard signature $f : V^{n+1} \to \mathbb{F}$, there exists some strongly skew-symmetric matrix $M$ such that

$$f(x_1 \cdots x_{n+1}) = \text{Pf}(M_{x_1 \cdots x_{n+1}})$$

Now, suppose that bit $x_{n+1} = 1$, and let $\underline{x} = x_1 \cdots x_n$. Recall Equation □

$$\text{Pf}(M_{\underline{x}}) = \sum_{i=1}^{n} x_i (-1)^{|x_i|-1} m(p_i, n+1) \text{Pf}(M_{\underline{x} + e_i + e_{n+1}})$$

Expressing this in terms of our function $f$, this equation becomes

$$f(x) = \sum_{i=1}^{n} x_i (-1)^{|x_i|-1} m(p_i, n+1) f(x + e_i + e_{n+1})$$

$$= \sum_{i=1}^{n} m(p_i, n+1) s_i^{f_0}$$

where $s_i^{f_0}$ is a shift basis function (with base point $0 \cdots 0$).

Finally, since the $(n+1)$st bit of $x = 1$, we can write

$$f(x) = f_1(x) = \sum_{i=1}^{n} m(i, n+1) s_i^{f_0}$$

In other words, the set of valid $f_1$ is exactly $S_{f_0}$. In other words,

$$B_{n+1} = \left\{ f : V^{n+1} \to \mathbb{F} \mid f_0 \in B_n \text{ and } f_1 \in S_{f_0} \right\}$$

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which establishes Equation 6.

Next, let us turn to proving Equation 7. Consider \( f \in A^{\text{odd}}_{n+1} \) where \( f \neq 0 \). Let \( \hat{x} = \hat{x}_1 \cdots \hat{x}_{n+1} \in V^{n+1} \) such that \( f(\hat{x}) \neq 0 \) and \( \hat{x}_{n+1} = 0 \). Let \( g(x) = (1/f(\hat{x}))f(x + \hat{x}) \), i.e. \( f \) normalized around \( \hat{x} \). From Equation 6 we know that \( g \) is a standard signature if and only if \( g_0 \in C_B g \). If we translate the elements of \( C_B g \) by adding \( \hat{x} \) to the inputs, notice that the resulting set is exactly \( C_B f \). Using Lemma 4, we can conclude that \( f \) is a standard signature if and only if \( f_1 \in C_B f_0 \). This establishes the first half of Equation 7.

On the other hand, if \( f \in A_{n+1} \) but \( f \equiv 0 \), then \( f_1 \in A_n \). We know that \( f_1 \subseteq A_n \) from Lemma 5. Conversely, given any \( f_1 \in A_n \), we can construct a planar matchgate for \( f \) by adding a new disconnected node and labelling it as input/output node \( n+1 \). This establishes the second half of Equation 7.

Equation 8 follows symmetrically to Equation 7.

Theorem 2 gives us a recursive procedure to determine if an \( n \)-bit function is a standard signature. If the function is the constant zero function, it is a standard signature. Otherwise, we can normalize it to a function \( f \). We can now check if \( f_1 \in C_B f_0 \) and if \( f_0 \in B_n \). The first condition can be checked by linear algebra, and the second condition can be checked recursively.

Finally, we justify our earlier comment about the irrelevance of our choice of base point for normalization.

Corollary 1 If \( g \) is a non-zero standard signature, then the set \( C_B g \) is independent of the choice of base point.

Proof: Suppose we have a standard signature \( f \) where there exist two base points around which we can normalize \( f_0 \) (i.e. there exist \( b \neq c \) such that \( b_{n+1} = c_{n+1} = 0 \), where \( f(b) \neq 0 \) and \( f(c) \neq 0 \).) These different definitions of “normalization” produce two possibly different sets \( C_B f \) and \( C_B f' \). If we applied the proof of Theorem 2 to each case, we would conclude that \( f \) is standard signature iff \( f_0 \in C_B f_0 \) iff \( f_0 \in C_B f_0' \). Therefore, \( C_B f_0 = C_B f_0' \). Now, for any \( g \in A_n \) there exists an \((n+1)\)-bit standard signature \( f \) such that \( f_0 = g \). Therefore, \( C_B g \) is independent of the choice of base point.

3 Consequences of Recursion

3.1 Counting Standard Signatures

Over a finite field \( \mathbb{F} \), there are only finitely many \( n \)-bit standard signatures for any fixed \( n \). In other words, \( |A_n| \) is finite. The recursive structure described in Theorem 2 allows us to find a formula to count \( |A_n| \).

Corollary 2 If we are operating in a finite field \( \mathbb{F} \), where \( |\mathbb{F}| = s \), then we can calculate the cardinality of the set of normalized standard signatures, odd parity standard signatures, and general standard signatures:

\[
|B_n| = \prod_{i=1}^{n-1} s^i = s^{(n-1)/2} \tag{9}
\]
\[ |A_n^{\text{odd}}| = (s - 1) \prod_{i=1}^{n-1} (s^i + 1) \]  \hspace{1cm} (10)

\[ |A_n| = 1 + 2 \times \left[ (s - 1) \prod_{i=1}^{n-1} (s^i + 1) \right] \]  \hspace{1cm} (11)

where we interpret the empty product \( \prod_{i=1}^{0} \) as evaluating to one.

**Proof:** We first consider Equation (9). For any \( f_0 \in B_n \), \( f_1 \) can be chosen freely from \( \mathcal{S}_{f_0} \). Lemma 9 shows that \( \mathcal{S}_{f_0} \) is \( n \)-dimensional, so

\[ |\mathcal{S}_{f_0}| = s^n \]

regardless of which particular (non-zero) \( f_0 \) we pick. If \( n > 1 \), then Theorem 2 implies that \( |B_n+1| = |B_n| \times s^n \).

Since \( |B_1| = 1 \), Equation (9) follows by induction.

Next, consider Equation (10). If \( n > 1 \), then note that Equation 7 of Theorem 2 is a disjoint union of two sets. Therefore,

\[ |A_{n+1}^{\text{odd}}| = |\left\{ f : V^{n+1} \rightarrow F \mid f_0 \in A_n^{\text{odd}} \right\}| \]

\[ + |\left\{ f : V^{n+1} \rightarrow F \mid f_0 \equiv 0 \right\}| \]

\[ = |A_n^{\text{odd}}| \times s^n + |A_n^{\text{even}}| \]

\[ = (s^{n} + 1) |A_n^{\text{odd}}| \]

By Lemma 2, \( |A_n^{\text{odd}}| = s - 1 \). Therefore, by induction, we have proved Equation (10).

Finally, we turn to Equation (11). By Lemma 3, \( |A_n^{\text{odd}}| = |A_n^{\text{even}}| \). If we account for the zero function, we can conclude that

\[ |A_n| = 1 + 2 \times |A_n^{\text{odd}}| \]

\[ = 1 + 2 \times \left[ (s - 1) \prod_{i=1}^{n-1} (s^i + 1) \right] \]

### 3.2 Asymptotics of \( |A_n| \)

If we want to evaluate \( |A_n| \) or \( |A_n^{\text{odd}}| \) for small \( s \) and \( n \), we can just plug in to Equations (10) or (11). However, we might also be interested in the behavior for fixed \( s \) as \( n \) grows.
In order to study this regime, we will introduce the (partial) function 
\( \gamma : \mathbb{C} \to \mathbb{C} \), where 
\[
\gamma(x) = \prod_{i=1}^{\infty} \left(1 + \left(\frac{1}{x}\right)^i\right)
\]

It is not a priori clear that \( \gamma \) converges. However, if we expand in \( 1/x \), 
then \( \gamma \) is the generating function for the number of ways of partitioning 
a set into unequal parts. We can then use the following lemma:

**Lemma 10** If \( x \in \mathbb{C} \) lies outside the unit circle, then \( \gamma(x) \) converges.

**Proof:** A proof can be found in [1], Section 14.4. \( \square \)

We are interested in integer values of \( x \) where \( x \geq 2 \), so \( \gamma(x) \) will converge. We can now express the asymptotics of \( |A_n| \) more precisely.

**Theorem 3** Suppose we are operating on a finite field \( \mathbb{F} \) of size \( |\mathbb{F}| = s \). Then

\[
\lim_{n \to \infty} \frac{|A_{n, odd}|}{s^{n(n-1)/2+1}} = \gamma(s)
\]

\[
\lim_{n \to \infty} \frac{|A_n|}{s^{n(n-1)/2+1}} = 2\gamma(s)
\]

Therefore, the growth rate is

\[
|A_{n, odd}| = \Theta \left( s^{n(n-1)/2+1} \right)
\]

\[
|A_n| = \Theta \left( s^{n(n-1)/2+1} \right)
\]

**Proof:** From Theorem 2, we can write

\[
|A_{n, odd}| = (s-1) \prod_{i=1}^{n-1} \left( s^i + 1 \right)
\]

\[
= (s-1) s^{n(n-1)/2} \prod_{i=1}^{n-1} \left( \frac{s^i + 1}{s^i} \right)
\]

\[
= (s-1) s^{n(n-1)/2} \prod_{i=1}^{n-1} \left( 1 + \frac{1}{s^i} \right)
\]

Therefore,

\[
\lim_{n \to \infty} \frac{|A_{n, odd}|}{s^{n(n-1)/2+1}} = \lim_{n \to \infty} \frac{|A_{n, odd}|}{(s-1)s^{n(n-1)}}
\]

\[
= \lim_{n \to \infty} \prod_{i=1}^{\infty} \left(1 + \left(\frac{1}{s}\right)^i\right)
\]

\[
= \gamma(s)
\]

It follows that \( |A_{n, odd}| = \Theta \left( s^{n(n-1)/2+1} \right) \).

Since \( |A_n| = 1 + 2|A_{n, odd}| \), the results on \( |A_n| \) follow. \( \square \)
In practice, the product form for $\gamma$ converges somewhat slowly. However, there is a trick for evaluating $\gamma$ more efficiently. Recall Euler’s Pentagonal Formula (see [1]): for any $|\sigma| < 1$,

$$\prod_{i=1}^{\infty} (1 - \sigma^i) = \sum_{i=-\infty}^{\infty} (-1)^i \sigma^{\omega(i)}$$

where $\omega(i) = (3i^2 - i)/2$. The sum formulation converges much more rapidly. If we let $\sigma = 1/s$, we can write

$$\gamma(1/s) = \prod_{i=1}^{\infty} (1 + \sigma^i)$$

$$= \prod_{i=1}^{\infty} \frac{1 - \sigma^{2i}}{1 - \sigma^i}$$

(12)

$$= \frac{\prod_{i=1}^{\infty} (1 - \sigma^{2i})}{\prod_{i=1}^{\infty} (1 - \sigma^i)}$$

(13)

$$= \sum_{i=-\infty}^{\infty} (-1)^i \sigma^{2\omega(i)}$$

$$\sum_{i=-\infty}^{\infty} (-1)^i \sigma^{\omega(i)}$$

Since the products are infinite, the step from Equation 12 to Equation 13 requires justification, but it is straightforward.

It now becomes computationally simple to calculate $\gamma(s)$ to high precision; here is a table for a few values:

| $s = |F|$ | $\gamma(s)$ |
|---|---|
| 2 | 2.384231 |
| 3 | 1.564934 |
| 4 | 1.355910 |
| 5 | 1.260501 |
| 7 | 1.170149 |
| 8 | 1.145129 |
| 9 | 1.126565 |

So, for instance, for large $n$, there are about

$$2\gamma(2)2^{\omega(n-1)/2+1} = 4.768 \times 2^{\omega(n-1)/2+1}$$

$n$-bit standard signatures over $F_2$. These calculations will also enable us to calculate the table of probabilities in Subsection 3.5.

### 3.3 Bounds on Planar Matchgate Sizes

If we are given an $n$-bit standard signature, by definition there exists some planar matchgate that computes it. However, it is not a priori clear how large the planar matchgate must be to simulate the standard signature. An upper bound of size $O(n^4)$ on the number of nodes and edges has been constructed by Li and Xia (see Theorem 3.3 in [9]), and in Appendix B we mention a recursive construction that would require $O(n^3)$ nodes and edges. However, these bounds are both beaten by Cai and Choudhary’s
original constructions in [3] and [4], which establish an $O(n^2)$ upper bound on the number of nodes and edges required.

In this subsection, we present a matching lower bound showing that the $O(n^2)$ upper bound is tight.

**Theorem 4** There exist standard signatures that can only be represented on graphs with at least $\Omega(n^2)$ nodes. More specifically, there exist standard signatures that require $X$ nodes, where

$$X + O(\log(X)) > n^2/16.015 - O(n \log(n))$$

**Proof:** First, suppose that $\mathbb{F}$ is an infinite field. Suppose we choose:

- an unweighted planar graph with at most $X$ nodes, where $X \geq n$, along with
- some planar embedding for the graph, and
- a choice of $n$ input/output nodes on the outer face.

We will call such an object a *stripped matchgate*, since we have stripped off the edge weights. If we take a stripped matchgate and add edge weights, we get a planar matchgate.

We will consider two planar embeddings to be isomorphic if they produce the same set of nodes on the outer face, in the same order. Note that there are only finitely many non-isomorphic planar embeddings for any graph. Since the other properties of a stripped matchgate are also finitary, it follows that there are only finitely many stripped matchgates with non-isomorphic planar embeddings. Let $\mathcal{G}$ be a set of planar matchgates representing each of the possible stripped matchgates with non-isomorphic planar embeddings; our comments above show that $|\mathcal{G}|$ is finite.

If $G$ is a representative planar matchgate then recall from Subsection 2.4 that $J_G$ is the set of all standard signatures sharing the same stripped matchgate.

Suppose our graph has $E$ edges. Since our graph is planar, $E \leq 3X$. Lemma 8 shows that $J_G$ is contained in an algebraic variety $P_G$ with $\dim(P_G) \leq E \leq 3X$ for any $G$. Therefore, the set of standard signatures definable on graphs with at most $X$ nodes is contained in a finite union of varieties: $\cup_{G \in \mathcal{G}} P_G$. This finite union is itself a variety; since each component has dimension at most $E$, the union has dimension at most $E \leq 3X$.

However, recall from Lemma 7 that $B_n$ is also an algebraic variety, and $\dim(B_n) = n(n-1)/2$. Therefore, if $3X < n(n-1)/2$, then

$$\dim(B_n) = n(n-1)/2 > 3X \geq \dim(\cup_{G \in \mathcal{G}} P_G)$$

Therefore,

$$B_n \not\subseteq \cup_{G \in \mathcal{G}} P_G$$

In fact, if we apply the switch planar matchgates in Appendix B to Cai and Choudhary’s construction, we can produce a planar matchgate for an $n$-bit standard signature on any field that uses at most $20n(n-1) + n + 2$ nodes. For fields of characteristic two, $7n(n-1) + n + 2$ nodes suffice.
and hence

\[ B_n \not\subseteq \bigcup_{G \in \mathcal{G}} J_G \]

Therefore, there exist standard signatures in \( B_n \) (and thus \( A_n \)) that require at least \( n(n - 1)/6 = n^2 - O(n) = \Omega(n^2) \) signatures to represent them.

Next, suppose that \( F \) is a finite field. Roughly speaking, we will repeat the argument above, but the finiteness of the number of planar matchgates is no longer sufficient—we need to count the number of planar matchgates explicitly, which is a more delicate operation.

Suppose we consider a planar matchgate with underlying (weighted) graph \( G \) on \( X \) nodes. We are going to represent the planar matchgate as a planar graph on \( X + 1 \) nodes with certain special labels. We proceed as follows: we take \( G \) and add a new node \( v \). We label this node as “extra”. We add an edge from \( v \) to each of the input/output nodes, and give the new edges weight one. We label each of the \( n \) input/output nodes by a distinct number from 1 to \( n \), namely the number of the node.

Let \( T_X \) be the set of labelled planar graphs with a node labelled “extra”, which has \( n \) neighbors, each labelled with a distinct number between 1 and \( n \). (So the elements of \( T_X \) are graphs with \( X + 1 \) nodes.) Note that \( T_X \) is larger than the set of planar matchgates, because we are not enforcing the input/output nodes to be on the outer face of the graph. However, every different \( X \) node matchgate maps to a distinct one of these labelled planar graphs, so by counting \( |T_X| \), we will get an upper bound on the number of standard signatures that can be represented with \( X \) node graphs. Note also that we are counting planar graphs, not planar embeddings (a different embedding of the same matchgate will produce the same standard signature, assuming that the input/output nodes are still on the outer face, and we orient the embedding to make the node labels run anti-clockwise.)

The reader may wonder how we can add the “extra” node \( v \) and its edges and be confident that our graph remains planar. The input/output nodes all lie on the outer face of some planar embedding; therefore, it is possible to place a node in the outer face and attach it to all the input/output nodes without crossing any edges.

Suppose we are given a planar matchgate with \( X - 2Y \) nodes. Then we can add disconnected 2-node subgraphs with edges of weight 1 at will without changing the standard signature. If we add \( Y \) of those subgraphs, we build a planar matchgate with \( X \) nodes. Therefore, all standard signatures representable on planar matchgates with \( X - 2Y \) nodes are representable on planar matchgates with exactly \( X \) nodes.

Therefore, all standard signatures on planar matchgates with at most \( X \) nodes can be represented by unique elements of \( T_X \) or \( T_X - 1 \).

We now need to determine the size of \( T_X \). Planarity is a very restrictive condition on a graph; there at most \( 2^{0.007X + O(\log X)} \) planar graphs with \( X \) (unlabelled) nodes (see [7]). There are at most \( 3X \) edges on a planar graph, so we have at most \((s - 1)^{3X}\) labellings. There are \( X \) possible choices for the “extra” node. The neighbors of the extra node are all labelled by distinct numbers between 1 and \( n \), so there are \( n! \) possible
numberings. Therefore,

\[ |T_X| \leq (X + 1)(n!)(s - 1)^3(X+1)2^{5.007(X+1)+O(\log(X+1))} \]

Therefore,

\[ |T_X| + |T_{X-1}| \leq 2(X + 1)(n!)(s - 1)^3(X+1)2^{5.007(X+1)+O(\log(X+1))} \]

Bringing all the terms into the exponent and absorbing extraneous ones into the \( O(\log(X)) \) term, (and remembering that \( n! = 2^{n \log_2 \left( \frac{n}{e} \right)} + O(\log(n)) \)), we can rewrite this as

\[ |T_X| + |T_{X-1}| \leq 2^{5.007X+3X \log_2(s-1)+n \log_2(n/e)+O(\log(X))} \]

However, we know that there are

\[ 1 + 2(s - 1) \prod_{i=1}^{n-1} (s^i + 1) > s^{n(n-1)/2} = 2^{n(n-1)/2} \]

\( n \)-bit standard signatures. Therefore, in order to express all these standard signatures, we need \( X \) to be at least large enough that

\[ 2^{5.007X+3X \log_2(s-1)+n \log_2(n/e)+O(\log(X))} > 2^{\log_2(s)n(n-1)/2} \]

Comparing exponents, we therefore need

\[ 5.007X + 3X \log_2(s-1) + n \log_2(n/e) + O(\log(X)) > \log_2(s)n(n-1)/2 \]

Replacing \( s - 1 \) by \( s \) on the left hand side and solving for \( X \), we get

\[ X + O(\log(X)) > \frac{n(n-1) - \frac{2n \log_2(n/e)}{\log_2(s)}}{6 + (10.014/\log_2(s))} + O(\log(n)) \]

So, there must exist some standard signature that requires at least

\[ \frac{n(n-1) - \frac{2n \log_2(X)}{\log_2(s)}}{6 + (10.014/\log_2(s))} + O(\log(n)) \]

nodes. This lower bound is \( \Omega(n^2) \), so we have established the rough bound for the theorem. To obtain the specific bound, note that the denominator is maximized when \( s = 2 \), at which point the denominator becomes \( 16.014 \ldots \). Conservatively rounding it up to 16.015 gives the result. □

### 3.4 Efficiently Detecting Standard Signatures

Suppose we are given a function \( f : V^n \rightarrow \mathbb{F} \), and we would like to determine if \( f \) is a standard signature. What is the complexity of deciding that question?

First, let us find a lower bound. The function \( f \) has \( 2^n \) inputs. Suppose that \( f(0 \cdots 0) \neq 0 \). At the very least, we need to check that all the \( 2^{n-1} \) odd-parity strings map to zero. Therefore, deciding if \( f \) is a standard signature takes at least

\[ 2^{n-1} = \Omega(2^n) \]
steps to evaluate.

But how should we actually verify that \( f \) is a standard signature?

One reasonable approach would be to use the algebraic variety defining \( A_n \). Recall that \( f \) is a standard signature iff it satisfies the Parity Constraint and the Matchgate Identities for every \( p, \alpha \in V^n \). We can verify the Parity Constraint by running through the output values once and checking for non-zero values, which takes \( 2^n \) steps. For the Matchgate Identities, each equation has \( n \) terms, and there are \( 2^n \) choices for both \( p \) and \( \alpha \). Assuming the Parity Constraint holds, we only need to check the Matchgate Identities for even parity \( p \) and \( \alpha \) of opposite parity to \( f \). This approach would take

\[
2^n + n2^{2(n-1)} = O(n2^n)
\]

steps to evaluate.

The recursive structure of the standard signatures allows us to use a much more efficient approach. The general outline of our technique is to assume that \( f \) is a standard signature. This assumption lets us recover a unique fingerprint for \( f \) by examining only a small subset of the output values. We then use this fingerprint to reconstruct an actual standard signature \( f' \); this reconstruction takes \( n2^n \) steps. Finally, \( f \) is a standard signature iff \( f \equiv f' \), which we can check in another \( 2^n \) steps. This approach takes only \( O(n2^n) \) steps to evaluate. We now analyze this process more carefully.

**Theorem 5** Suppose we are given a function \( f : V^n \to \mathbb{F} \) (that is, we are given a list of \( f(x) \) for all \( x \in V^n \), sorted by \( x \)). Then we can determine if \( f \) is a standard signature in time \( O(n2^n) \).

**Proof:** We begin by determining if \( f \) is identically zero. This takes \( O(2^n) \) steps; if \( f \equiv 0 \) then it is a standard signature, and we are done. Otherwise, we will discover a string \( \hat{x} \in V^n \) such that \( f(\hat{x}) \neq 0 \). Let us normalize our function at \( \hat{x} \) by constructing the new function \( g(x) = (1/f(\hat{x}))f(x + \hat{x}) \). From Lemma 4, \( f \) is a standard signature if and only if \( g \) is, so we will henceforth focus on \( g \). Constructing \( g \) takes another \( O(2^n) \) steps.

Suppose that we have a standard signature \( h \). Recall from the Pfaffian definition of the standard signature that there is some matrix \( M \) such that

\[
h(x) = Pf(M_x)
\]

Let us use \( m(i,j) \) to represent the entry of \( M \) in the \( i \)-th row and \( j \)-th column. Suppose that \( x \) has Hamming weight two, i.e. \( x = e_i + e_j \), where \( i < j \). Then \( M_x \) is a \( 2 \times 2 \) matrix of the form

\[
\begin{pmatrix}
0 & m(i,j) \\
-m(i,j) & 0
\end{pmatrix}
\]

If our function \( f \) happens to be sparse, with only \( k \) non-zero values, then we only need to check at most \( n \binom{k}{2} \) Matchgate Identities. Therefore, we can determine if \( f \) is a standard signature in only \( n^2 \binom{k}{2} \) steps.
In particular,
\[ h(x) = \Pr(M_x) = m(i, j) = -m(j, i) \]
In other words, the \( m(n - 1)/2 \) weight 2 codewords completely specify \( M \).

So, given \( g \), let \( M \) be the matrix determined by the value of \( g \) on all the weight-two codewords. Now that we have \( M \), let us construct a standard signature \( h \) from it. We do this recursively. Define \( h^1(0) = 1, h^1(1) = 0 \).

Define
\[
h^j(x_1 \cdots x_j) = \begin{cases} h^{j-1}(x_1 \cdots x_{j-1}) & \text{if } x_j = 0 \\ \sum_{i=1}^{j-1} -m(i, j)s^{j-1}_i & \text{if } x_j = 1 \end{cases}
\]
where \( s^{j-1}_i \) is a shift-basis function of \( h^{j-1} \). Recovering \( h^n \) takes
\[
\sum_{i=2}^{n} (i - 1)2^{i-1} = (n - 2)2^n + 2 = O(n2^n)
\]
steps. From our recursive definition of the standard signatures (cf. the proof of Theorem 2), it follows that \( h^n \) is a standard signature, and by construction \( h^n(x) = g(x) \) for all weight-two codewords \( x \).

Since each standard signature defines a unique \( M \), \( g \) is a standard signature if and only if \( g \equiv h^n \). We can compare their outputs in \( 2^n \) steps; they are identical if and only if \( g \) (and hence \( f \)) is a standard signature.

### 3.5 Expected Sparsity

How large is the support of a typical standard signature? That is, if we choose \( f \in A_n \) “randomly”, what fraction of the entries are non-zero? To put it another way, if we view \( f \) as a vector in \( \mathbb{F}^{(2^n)} \), how sparse is the vector?

For infinite fields, it is not clear which measure we should use to select our function \( f \). But if \( \mathbb{F} \) is a finite field, it seems natural to choose \( f \) uniformly at random from, say, \( A_n^{even} \), and the problem is well-defined. It turns out that we can prove a slightly stronger result– we can calculate the expected sparsity for each individual input bit.

**Theorem 6** Assume we are operating over a finite field \( \mathbb{F} \) of size \( s = |\mathbb{F}| \).

Suppose we choose \( f \in A_n^{even} \) uniformly at random, and select any fixed even parity \( n \)-bit string \( \hat{x} \). Then
\[
\Pr(f(\hat{x}) \neq 0) = \left[ \prod_{i=1}^{n-1} \left(1 + s^{-1}\right) \right]^{-1} \tag{14}
\]

The analogous result also holds for strictly odd parity standard signatures.

**Proof:** Let \( C_n^{even} \subseteq A_n^{even} \) such that for any \( g \in C_n^{even} \), \( g(0 \cdots 0) \neq 0 \). Note that if we take an element of \( f \) and divide its outputs by \( f(0 \cdots 0) \), we obtain a normalized standard signature in \( B_n \). Each element of \( B_n \) is the image of exactly \( s - 1 \) elements of \( C_n \). Therefore,
\[
|C_n| = (s - 1)|B_n| = (s - 1) \prod_{i=2}^{n} s^{i-1}
\]
Now, if we choose $f \in A_{n}^{\text{even}}$ uniformly at random, notice that

$$\Pr[f(0 \cdots 0) \neq 0] = \Pr[f \in C_{n}^{\text{even}}]$$

Since we are selecting functions uniformly, it follows that

$$\Pr[f(0 \cdots 0) \neq 0] = \frac{|C_{n}^{\text{even}}|}{|A_{n}^{\text{even}}|} = \frac{(s - 1) \prod_{i=1}^{n-1} s^i}{(s - 1) \prod_{i=1}^{n-1} (1 + s^i)} = \prod_{i=1}^{n-1} \frac{s^i}{1 + s^i} = \prod_{i=1}^{n-1} \frac{1}{1 + s^{-i}} = \left[ \prod_{i=1}^{n-1} (1 + s^{-i}) \right]^{-1}$$

Suppose we take the classes of functions above and add $\hat{x}$ to their inputs (i.e. we translate them by $\hat{x}$). The sizes of the sets, and thus the probabilities, do not change. Therefore, we can conclude that for any fixed $\hat{x}$,

$$\Pr[f(\hat{x}) \neq 0] = \left[ \prod_{i=1}^{n-1} (1 + s^{-i}) \right]^{-1}$$

as desired. \(\Box\)

As a simple consequence, we can calculate the expected sparsity:

**Corollary 3** If we choose non-zero $f \in A_{n}$ uniformly at random, then

$$\text{Expected Sparsity} := E \left[ \frac{|\{x | f(x) \neq 0\}|}{2^n} \right] = \left[ \prod_{i=1}^{n-1} (1 + s^{-i}) \right]^{-1}$$

**Proof:**

$$\text{Expected Sparsity} := E \left[ \frac{|\{x | f(x) \neq 0\}|}{2^n} \right] = \frac{1}{2^n} E \left[ \sum_{x \in V^n} \Pr(f(x) \neq 0) \right] = \frac{2^n}{2^n} \prod_{i=1}^{n-1} (1 + s^{-i})^{-1} = \left[ \prod_{i=1}^{n-1} (1 + s^{-i}) \right]^{-1}$$
as desired. □

For a fixed field of size $s$, the probability converges as $n \to \infty$:

\[
\lim_{n \to \infty} \left( \text{Expected Sparsity of } A_n^{\text{even}} \right) = \lim_{n \to \infty} \left( \Pr(f(0 \cdots 0) \neq 0) \right) = \lim_{n \to \infty} \left[ \prod_{i=1}^{n-1} (1 + s^{-i}) \right]^{-1} = 1/\gamma(s)
\]

(See Subsection 3.2 for details and computational issues.) We include a table of these limiting probabilities for a few small fields. For comparison, we also list the expected sparsity of an arbitrary function $g : V^n \to \mathbb{F}$ selected uniformly at random, which equals $1 - (1/s)$.

| $s = |\mathbb{F}|$ | $1/\gamma(s)$ | $1 - (1/s)$ |
|-------------------|---------------|-------------|
| 2                 | 0.419422      | 0.5         |
| 3                 | 0.639005      | 0.666666    |
| 4                 | 0.737512      | 0.75        |
| 5                 | 0.793335      | 0.8         |
| 7                 | 0.854592      | 0.857142    |
| 8                 | 0.873264      | 0.875       |
| 9                 | 0.887654      | 0.888888    |

### 3.6 Expressiveness of Holographic Algorithms

If our base field $\mathbb{F}$ is finite, then there are only a finite number of $n$-bit standard signatures. In addition to being of intrinsic interest, the number of standard signatures gives us some intuition about the expressiveness of a holographic algorithm: the more signatures, the more expressive the algorithms could possibly be. We have found it instructive to compare the relative sizes of a few classes of functions.

Let $|\mathbb{F}| = s$.

- The number of functions from $V^n$ to $\mathbb{F}$:

\[s^{2^n}\]

- The number of functions from $V^n$ to $\mathbb{F}$ with even or odd parity:

\[2s^{2^{n-1}} - 1\]

- The number of standard signatures:

\[|A_n| = 1 + 2 \times \left( (s - 1) \prod_{i=1}^{n-1} (s^i + 1) \right) = \Theta \left( s^{n(n-1)/2+1} \right)\]

- The number of symmetric realizable functions (assuming the characteristic of the field is odd, and the characteristic doesn’t divide $n$) on any basis of size 1 (not just the standard basis):

\[s(s - 1)^3(s + 3) + 1 = \Theta(s^5)\]

See Theorem 4.2 in [2] for more details.
So, based only on cardinality, we could argue that general functions are exponentially more expressive than standard signatures, which, in turn, are exponentially more expressive than symmetric realizable functions.

### 3.7 Cardinality of Self-dual Codes

It would be extremely interesting to find an isomorphism between the $n$-bit standard signatures and other, better studied mathematical objects. Having an exact count of the number of standard signatures over various finite fields can facilitate this hunt; if an isomorphic object exists, it will necessarily have the same cardinality. Do any such objects exist?

We can find an example in the world of self-dual codes. Recall that over $\mathbb{F}_2$, $|A_n^{\text{odd}}| = \prod_{i=1}^{n-1} (2^i + 1)$. Surprisingly, this equals the number of dimension $n$ self-dual codes over $\mathbb{F}_2$ (i.e. self-dual codes in $\mathbb{F}_2^{2n}$). Moreover, over $\mathbb{F}_3$, it turns out that $|A_n^{\text{odd}}| = 2 \prod_{i=1}^{n-1} (3^i + 1)$ equals the number of dimension $n$ self-dual codes over $\mathbb{F}_3$. See Chapter 3 of [11] for these results; consider “type $q^k$” self-dual codes.

When discussing a self-dual code, we implicitly assume some particular inner product; for the two results above, we used the Euclidean inner product. Frustratingly, if we continue to use the same inner product, the cardinalities diverge for all other finite fields. The agreement over $\mathbb{F}_2$ and $\mathbb{F}_3$ seems like a fairly spectacular coincidence, though.

How can we circumvent this divergence? We might look for a better inner product, but no obvious candidates suggest themselves. (See [10] for a thorough examination of many alternate possibilities.) If we stick to the Euclidean inner product, though, we can match the cardinalities with a little normalization gimmick. For any $f \in A_n^{\text{odd}}$, since $f \not\equiv 0$, there exists a lexicographically smallest $\hat{x} \in V^n$ such that $f(\hat{x}) \neq 0$. Call a standard signature semi-normalized if $f(\hat{x}) = 1$. Let $H_n^{\text{odd}}$ be the set of semi-normalized standard signatures. Let $\#SD(\mathbb{F}, n)$ be the number of $n$ dimensional self-dual codes over $\mathbb{F}$ with the Euclidean inner product. If $|\mathbb{F}|$ is even, it turns out that

$$|H_n^{\text{odd}}| = \#SD(\mathbb{F}, n)$$

while if $|\mathbb{F}|$ is odd, then

$$|H_n^{\text{odd}}| + |H_n^{\text{even}}| = \#SD(\mathbb{F}, n)$$

Although numerically surprising, the above observations do not suggest how we might actually construct an isomorphism between the standard signatures and the self-dual codes. Until we can build a non-trivial isomorphism, these cardinality results remain only curiosities.

### A The Six-bit Normalized Standard Signature

Recall that a normalized standard signature is a standard signature $f$ where $f(0 \cdots 0) = 1$. Since $f$ is an even function, we only need to specify
the output for even-weight inputs; all the odd-weight inputs evaluate to zero.

As we discussed in Subsections 2.3 and 3.5, all the outputs can be expressed as polynomials in the \( f(\hat{x}) \), for \( \hat{x} \) with Hamming weight two. More generally, if \( x \) has Hamming weight \( 2k \), then \( f(x) \) can be expressed in terms of \( f(x') \) where \( x' \) has weight \( 2k - 2 \). Each monomial term in a polynomial has coefficient \( \epsilon_{x} = \pm 1 \). It is straightforward to show by induction on the weight of the input string that if \( |x| = 2k \), then there are

\[
(2k - 1)!! = \frac{(2k)!}{k!2^k} = O\left(\sqrt{(2k)!}\right)
\]

monomial terms in \( f(x) \).

Here is the set of polynomials for the six-bit normalized standard signature. Note that if we fix the first bit as zero, we produce the general form for all the five-bit normalized standard signatures, and so forth.

| Input       | Polynomial                              |
|-------------|-----------------------------------------|
| 000000      | 1                                       |
| 000011      | \( \lambda_{2,1} \)                     |
| 000101      | \( \lambda_{3,1} \)                     |
| 000110      | \( \lambda_{3,2} \)                     |
| 001001      | \( \lambda_{4,1} \)                     |
| 001010      | \( \lambda_{4,2} \)                     |
| 001100      | \( \lambda_{4,3} \)                     |
| 001111      | \( \lambda_{4,3} \lambda_{3,2} - \lambda_{4,2} \lambda_{3,1} + \lambda_{4,3} \lambda_{2,1} \) |
| 010001      | \( \lambda_{5,1} \)                     |
| 010010      | \( \lambda_{5,2} \)                     |
| 010100      | \( \lambda_{5,3} \)                     |
| 010111      | \( \lambda_{5,3} \lambda_{3,2} - \lambda_{5,2} \lambda_{3,1} + \lambda_{5,3} \lambda_{2,1} \) |
| 011000      | \( \lambda_{5,4} \)                     |
| 011011      | \( \lambda_{5,4} \lambda_{4,2} - \lambda_{5,2} \lambda_{4,1} + \lambda_{5,4} \lambda_{2,1} \) |
| 011101      | \( \lambda_{5,4} \lambda_{4,3} - \lambda_{5,3} \lambda_{4,1} + \lambda_{5,4} \lambda_{3,1} \) |
| 011110      | \( \lambda_{5,4} \lambda_{4,3} - \lambda_{5,3} \lambda_{4,2} + \lambda_{5,4} \lambda_{3,2} \) |
| 100001      | \( \lambda_{6,1} \)                     |
| 100010      | \( \lambda_{6,2} \)                     |
| 100100      | \( \lambda_{6,3} \)                     |
| 100111      | \( \lambda_{6,3} \lambda_{4,2} - \lambda_{6,2} \lambda_{3,1} + \lambda_{6,3} \lambda_{2,1} \) |
| 101000      | \( \lambda_{6,4} \)                     |
| 101011      | \( \lambda_{6,4} \lambda_{4,2} - \lambda_{6,2} \lambda_{4,1} + \lambda_{6,4} \lambda_{2,1} \) |
| 101101      | \( \lambda_{6,4} \lambda_{4,3} - \lambda_{6,3} \lambda_{4,1} + \lambda_{6,4} \lambda_{3,1} \) |
| 101110      | \( \lambda_{6,4} \lambda_{4,3} - \lambda_{6,3} \lambda_{4,2} + \lambda_{6,4} \lambda_{3,2} \) |
| 110000      | \( \lambda_{6,5} \)                     |
| 110011      | \( \lambda_{6,5} \lambda_{5,2} - \lambda_{6,2} \lambda_{5,1} + \lambda_{6,5} \lambda_{2,1} \) |
\begin{align*}
\text{f}(110101) &= \lambda_{6,1}\lambda_{5,3} - \lambda_{6,3}\lambda_{5,1} + \lambda_{6,5}\lambda_{3,1} \\
\text{f}(110110) &= \lambda_{6,2}\lambda_{5,3} - \lambda_{6,3}\lambda_{5,2} + \lambda_{6,5}\lambda_{3,2} \\
\text{f}(111001) &= \lambda_{6,2}\lambda_{5,4} - \lambda_{6,4}\lambda_{5,2} + \lambda_{6,5}\lambda_{4,2} \\
\text{f}(111010) &= \lambda_{6,3}\lambda_{5,4} - \lambda_{6,4}\lambda_{5,3} + \lambda_{6,5}\lambda_{4,3} \\
\text{f}(111100) &= \lambda_{6,1}\lambda_{5,2}\lambda_{4,3} - \lambda_{6,1}\lambda_{5,3}\lambda_{4,2} + \lambda_{6,1}\lambda_{5,4}\lambda_{3,2} \\
&\quad - \lambda_{6,2}\lambda_{5,1}\lambda_{4,3} + \lambda_{6,2}\lambda_{5,3}\lambda_{4,1} - \lambda_{6,2}\lambda_{5,4}\lambda_{3,1} \\
&\quad + \lambda_{6,3}\lambda_{5,1}\lambda_{4,2} - \lambda_{6,3}\lambda_{5,2}\lambda_{4,1} + \lambda_{6,3}\lambda_{5,4}\lambda_{2,1} \\
&\quad - \lambda_{6,4}\lambda_{5,1}\lambda_{3,2} + \lambda_{6,4}\lambda_{5,2}\lambda_{3,1} - \lambda_{6,4}\lambda_{5,3}\lambda_{2,1} \\
&\quad + \lambda_{6,5}\lambda_{5,1}\lambda_{3,3} - \lambda_{6,5}\lambda_{5,2}\lambda_{3,1} + \lambda_{6,5}\lambda_{5,3}\lambda_{2,1} \\
&- \lambda_{6,6}\lambda_{5,1}\lambda_{2,3} + \lambda_{6,6}\lambda_{5,2}\lambda_{2,1} - \lambda_{6,6}\lambda_{5,3}\lambda_{1,1} + \lambda_{6,6}\lambda_{5,4}\lambda_{1,2} \\
&+ \lambda_{6,7}\lambda_{5,1}\lambda_{1,3} - \lambda_{6,7}\lambda_{5,2}\lambda_{1,2} + \lambda_{6,7}\lambda_{5,3}\lambda_{1,1} - \lambda_{6,7}\lambda_{5,4}\lambda_{2,1} \\
\text{f}(111111) &= \lambda_{6,1}\lambda_{5,2}\lambda_{4,3} - \lambda_{6,1}\lambda_{5,3}\lambda_{4,2} + \lambda_{6,1}\lambda_{5,4}\lambda_{3,2} \\
&\quad - \lambda_{6,2}\lambda_{5,1}\lambda_{4,3} + \lambda_{6,2}\lambda_{5,3}\lambda_{4,1} - \lambda_{6,2}\lambda_{5,4}\lambda_{3,1} \\
&\quad + \lambda_{6,3}\lambda_{5,1}\lambda_{4,2} - \lambda_{6,3}\lambda_{5,2}\lambda_{4,1} + \lambda_{6,3}\lambda_{5,4}\lambda_{2,1} \\
&\quad - \lambda_{6,4}\lambda_{5,1}\lambda_{3,2} + \lambda_{6,4}\lambda_{5,2}\lambda_{3,1} - \lambda_{6,4}\lambda_{5,3}\lambda_{2,1} \\
&\quad + \lambda_{6,5}\lambda_{5,1}\lambda_{3,3} - \lambda_{6,5}\lambda_{5,2}\lambda_{3,1} + \lambda_{6,5}\lambda_{5,3}\lambda_{2,1} \\
&\quad - \lambda_{6,6}\lambda_{5,1}\lambda_{2,3} + \lambda_{6,6}\lambda_{5,2}\lambda_{2,1} - \lambda_{6,6}\lambda_{5,3}\lambda_{1,1} + \lambda_{6,6}\lambda_{5,4}\lambda_{1,2} \\
&\quad + \lambda_{6,7}\lambda_{5,1}\lambda_{1,3} - \lambda_{6,7}\lambda_{5,2}\lambda_{1,2} + \lambda_{6,7}\lambda_{5,3}\lambda_{1,1} - \lambda_{6,7}\lambda_{5,4}\lambda_{2,1} \\
&\quad - \lambda_{6,8}\lambda_{5,1}\lambda_{0,3} + \lambda_{6,8}\lambda_{5,2}\lambda_{0,2} - \lambda_{6,8}\lambda_{5,3}\lambda_{0,1} + \lambda_{6,8}\lambda_{5,4}\lambda_{0,2} \\
&\quad + \lambda_{6,9}\lambda_{5,1}\lambda_{0,3} - \lambda_{6,9}\lambda_{5,2}\lambda_{0,2} + \lambda_{6,9}\lambda_{5,3}\lambda_{0,1} - \lambda_{6,9}\lambda_{5,4}\lambda_{0,2}
\end{align*}

B Matchgate Recursion

The body of this paper has focussed on an algebraic recursion that allowed us to construct \((n+1)\)-bit standard signatures out of \(n\)-bit standard signatures. One may wonder if there is a planar matchgate counterpart—that is, is there some sort of recursive planar matchgate structure that reflects this. There is, and we offer one such possibility below. We begin by reviewing a particularly useful 4-bit standard signature.

**Lemma 11** Define the switch function \(f_{\text{switch}} : V^4 \rightarrow \mathbb{F}\) as:

\[
\begin{align*}
\text{f}_{\text{switch}}(0000) &= f_{\text{switch}}(0101) = f_{\text{switch}}(1010) = 1 \\
\text{f}_{\text{switch}}(1111) &= -1
\end{align*}
\]

Then the switch function is a standard signature.

**Proof:** It is possible to prove this result only using algebra, but it is simpler to construct the switch matchgate directly. These planar matchgates are modified versions of Figure 8 from Valiant [14].

First, suppose that \(\mathbb{F}\) is not characteristic two. Then \(\frac{1}{2} \in \mathbb{F}\), and we can consider the following planar matchgate (where unmarked edges have weight 1):

On the other hand, if \(\mathbb{F}\) has characteristic two, we can use the following planar matchgate (where all edges have weight 1):

\[
\begin{align*}
\text{On the other hand, if } \mathbb{F} \text{ has characteristic two, we can use the following planar matchgate (where all edges have weight 1):}
\end{align*}
\]
If we count up the weighted perfect matchings for these two graphs over their respective fields, they produce the switch function, as desired.

To simplify our diagrams, we will (following Valiant) adopt the following emblem for the switch matchgate, where the underlying planar matchgate is chosen from the two above depending on the base field’s characteristic:

Since the outputs are symmetric under rotation in the plane, we can ignore the labels without causing any ambiguity. If we consider input/output nodes 1 and 3, observe that either they must both be saturated, or neither of them is saturated. In other words, the planar matchgate acts as though there were a “virtual edge” between the nodes. The same principle applies to nodes 2 and 4.

Notice that over fields of characteristic two, the switch matchgate is equivalent to letting two edges cross each other (since $f_{\text{switch}}(1111) = -1 \equiv 1 \mod 2$). In other words, for those particular fields, the planarity requirement in a planar matchgate is redundant; we can simply take any non-planar crossings and replace them with planar switch matchgates. We can state this corollary formally:

**Corollary 4** If we operate over a field $\mathbb{F}$ of characteristic 2, we can remove the planarity restriction from the definition of a planar matchgate without changing the resulting set of standard signatures.

In any event, the purpose of this section is to provide a recursive planar matchgate construction that mirrored the algebraic recursion from Subsection 2.5. Here is one example, where we choose arbitrary $\lambda_i \in \mathbb{F}$, and all unmarked edges have weight one.
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