Acoustic propagation in fluids:
an unexpected example of Lorentzian geometry

Matt Visser

Physics Department, Washington University, St. Louis, Missouri 63130-4899

(11 October 1993)

Abstract

It is a deceptively simple question to ask how acoustic disturbances propagate in a non–homogeneous flowing fluid. If the fluid is barotropic and inviscid, and the flow is irrotational (though it may have an arbitrary time dependence), then the equation of motion for the velocity potential describing a sound wave can be put in the (3 + 1)–dimensional form \( \Delta \psi = 0 \). The acoustic metric \( g_{\mu \nu}(t, \vec{x}) \) governing the propagation of sound depends on the density, flow velocity, and local speed of sound. Even though the underlying fluid dynamics is Newtonian, non–relativistic, and takes place in flat space + time, the fluctuations (sound waves) are governed by a Lorentzian spacetime geometry.

43.20.+g, 02.40.-k, 03.40.-t, 47.10.+g
I. INTRODUCTION

It is well known that for a static homogeneous inviscid fluid the propagation of sound waves is governed by the simple equation

$$\partial^2_t \psi = c^2 \nabla^2 \psi.$$  \hspace{1cm} (1)

($c \equiv$ speed of sound.) It is a deceptively simple question to ask what happens if the fluid is non–homogeneous, in motion, possibly even in non-steady motion. If the fluid is barotropic and inviscid, and the flow is irrotational (though it may have an arbitrary time dependence) then I shall show that the equation of motion for the velocity potential describing an acoustic disturbance can be put in the (3 + 1)–dimensional form

$$\Delta \psi \equiv \frac{1}{\sqrt{-g}} \partial_\mu \left( \sqrt{-g} \ g^{\mu\nu} \partial_\nu \psi \right) = 0.$$ \hspace{1cm} (2)

The propagation of sound is governed by the acoustic metric $g_{\mu\nu}(t, \vec{x})$. This acoustic metric describes a Lorentzian (pseudo–Riemannian) geometry and depends on the density, velocity of flow, and local speed of sound in the fluid. Specifically

$$g_{\mu\nu}(t, \vec{x}) \equiv \frac{\rho}{c} \begin{bmatrix} -(c^2 - \vec{v}^2) & -\vec{v} \\ \vdots & \ddots & \vdots \\ -\vec{v} & \vdots & \vec{I} \end{bmatrix}.$$ \hspace{1cm} (3)

In general, when the fluid is non–homogeneous and flowing, the acoustic Riemann tensor associated with this Lorentzian metric will be nonzero. It is quite remarkable that even though the underlying fluid dynamics is Newtonian, nonrelativistic, and takes place in flat space + time, the fluctuations (sound waves) are governed by a curved Lorentzian (pseudo–Riemannian) geometry. This connection between fluid dynamics and techniques more commonly encountered in the context of general relativity opens up many opportunities for cross–pollination between the two fields.

II. FLUID DYNAMICS
A. Fundamental equations

The fundamental equations of fluid dynamics are the equation of continuity

\[ \partial_t \rho + \nabla \cdot (\rho \vec{v}) = 0, \tag{4} \]

and Euler’s equation

\[ \rho \frac{d\vec{v}}{dt} \equiv \rho \left( \partial_t \vec{v} + (\vec{v} \cdot \nabla) \vec{v} \right) = \vec{F}. \tag{5} \]

Start by assuming the fluid to be inviscid, with the only forces present being those due to pressure and Newtonian gravity

\[ \vec{F} = -\nabla p - \rho \nabla \phi. \tag{6} \]

Standard manipulations yield

\[ \partial_t \vec{v} = \vec{v} \times (\nabla \times \vec{v}) - \frac{1}{\rho} \nabla p - \nabla (\frac{1}{2} v^2 + \phi). \tag{7} \]

Take the flow to be irrotational, introducing the velocity potential \( \psi \) such that \( \vec{v} = -\nabla \psi \). Take the fluid to be barotropic (\( \rho \) is a function of \( p \) only). Then define

\[ \zeta(p) = \int_0^p \frac{dp'}{\rho(p')}; \quad \text{so that} \quad \nabla \zeta = \frac{1}{\rho} \nabla p. \tag{8} \]

Euler’s equation reduces to

\[ - \partial_t \psi + \zeta + \frac{1}{2} (\nabla \psi)^2 + \phi = 0. \tag{9} \]

B. Fluctuations

Linearize the equations of motion around some assumed background \( (\rho_0, p_0, \psi_0) \) by setting \( \rho = \rho_0 + \epsilon \rho_1, \ p = p_0 + \epsilon p_1, \) and \( \psi = \psi_0 + \epsilon \psi_1. \) The gravitational potential \( \phi \) is taken to be fixed and external. Sound is defined to be these linearized fluctuations in the dynamical quantities. The linearized continuity equation reads
\[ \partial_t \rho_1 + \nabla \cdot (\rho_1 \vec{v}_0 + \rho_0 \vec{v}_1) = 0, \] \hspace{1cm} (10)

while from the Euler equation, using \( \zeta = \zeta_0 + \epsilon(p_1/\rho_0) \),

\[ - \partial_t \psi_1 + \frac{p_1}{\rho_0} - \vec{v}_0 \cdot \nabla \psi_1 = 0. \] \hspace{1cm} (11)

Rearranging

\[ p_1 = \rho_0 (\partial_t \psi_1 + \vec{v}_0 \cdot \nabla \psi_1). \] \hspace{1cm} (12)

Substitute this linearized Euler equation into the linearized equation of continuity. Use \( \rho_1 = (\partial \rho/\partial p) p_1 \). One obtains, up to an overall sign,

\[ - \partial_t \left( \frac{\partial \rho}{\partial p} \rho_0 (\partial_t \psi_1 + \vec{v}_0 \cdot \nabla \psi_1) \right) + \nabla \cdot \left( \rho_0 \nabla \psi_1 - \frac{\partial \rho}{\partial p} \rho_0 \vec{v}_0 (\partial_t \psi_1 + \vec{v}_0 \cdot \nabla \psi_1) \right) = 0. \] \hspace{1cm} (13)

This wave equation describes the propagation of the scalar potential \( \psi_1 \), and thereby completely determines the quantities \( p_1 \) and \( \rho_1 \). The background fields \( p_0, \rho_0 \) and \( \vec{v}_0 \) are permitted to have arbitrary temporal and spatial dependencies. Now, written in this form, the physical import of the wave equation is somewhat less than pellucid. Define \( 1/c^2 = \partial \rho/\partial p \), and construct the symmetric \( 4 \times 4 \) matrix

\[ f^{\mu \nu} \equiv \frac{\rho_0}{c^2} \begin{bmatrix} -1 & \vdots & -v_0^j \\ \vdots & \ddots & \vdots \\ -v_0^i & \vdots & (c^2 \delta^{ij} - v_0^i v_0^j) \end{bmatrix}. \] \hspace{1cm} (14)

Then, using four dimensional coordinates \( x^\mu = (t, x^i) \) the wave equation is easily rewritten as

\[ \partial_{\mu} (f^{\mu \nu} \partial_\nu \psi_1) = 0. \] \hspace{1cm} (15)

This remarkably compact formulation is much more promising. The remaining steps are a straightforward application of the techniques of curved space Lorentzian geometry.
III. LORENTZIAN GEOMETRY

In any Lorentzian (pseudo–Riemannian) manifold the curved space scalar d’Alembertian
is given in terms of the metric \(g_{\mu\nu}(t, \vec{x})\) by 4–7

\[
\Delta \psi \equiv \frac{1}{\sqrt{-g}} \partial_\mu \left( \sqrt{-g} g^{\mu\nu} \partial_\nu \psi \right).
\] (16)

The inverse metric, \(g^{\mu\nu}(t, \vec{x})\) is pointwise the matrix inverse of \(g_{\mu\nu}(t, \vec{x})\), while \(g \equiv \det(g_{\mu\nu})\).
Thus we can rewrite our physically derived wave equation in terms of the d’Alembertian
provided we identify

\[
\sqrt{-g} g^{\mu\nu} = f^{\mu\nu}.
\] (17)

This implies

\[
g = \det(f^{\mu\nu})
= \left(\frac{\rho_0}{c^2}\right)^4 \left[-1 \cdot (c^2 - v_0^2) - v_0^2(c^2)^2\right] = -\frac{\rho_0^4}{c^2}.
\] (18)

We can thus pick off the coefficients of the inverse metric

\[
g^{\mu\nu} \equiv \frac{1}{\rho_0 c} \begin{bmatrix}
-1 & -v_0^i \\
\vdots & \vdots \\
\cdots & \cdots \\
-v_0^j & (c^2 \delta^{ij} - v_0^i v_0^j)
\end{bmatrix}.
\] (19)

One could now determine the metric itself by inverting this \(4 \times 4\) matrix. On the other hand
it is even easier to recognize that we have in front of us an example of the Arnowitt–Deser–
Misner split of a \((3 + 1)\)–dimensional Lorentzian spacetime metric into space + time, more
commonly used in discussing the initial value data in general relativity (see, for example, 3
pp 505–508). The metric is

\[
g_{\mu\nu} \equiv \frac{\rho_0}{c} \begin{bmatrix}
-(c^2 - v_0^2) & -v_0^i \\
\vdots & \vdots \\
\cdots & \cdots \\
-v_0^j & \delta^{ij}
\end{bmatrix}.
\] (20)

Observe that the signature of this metric is in fact \((- , + , + , +)\), as it should be.
It should be emphasised that there are two distinct metrics relevant to the current discussion. The physical spacetime metric is just the usual flat metric of Minkowski space \( \eta_{\mu\nu} \equiv (\text{diag}[ -c_\infty^2, 1, 1, 1])_{\mu\nu} \). \( c_\infty \) = speed of light.) The fluid particles couple only to the physical metric \( \eta_{\mu\nu} \). In fact the fluid motion is completely non–relativistic — \( ||v_0|| \ll c_\infty \). Sound waves on the other hand, do not “see” the physical metric at all. Acoustic perturbations couple only to the acoustic metric \( g_{\mu\nu} \). The geometry determined by the acoustic metric does however inherit some key properties from the existence of the underlying flat physical metric.

For instance, the topology of the manifold does not depend on the particular metric considered. The acoustic geometry inherits the underlying topology of the physical metric — \( \mathbb{R}^4 \) with possibly a few regions excised (due to imposed boundary conditions).

Furthermore the acoustic geometry automatically inherits the property of “stable causality” \[7\]. Note that \( g^{\mu\nu}(\nabla_\mu t)(\nabla_\nu t) = -1/(\rho_0 c) < 0 \). This precludes some of the more entertaining pathologies that sometimes arise in general relativity.

Another concept that translates immediately is that of an “ergo–region”. Consider integral curves of the vector \( (\partial/\partial t)^\mu \equiv (1, 0, 0, 0)^\mu \). Then \( g_{\mu\nu}(\partial/\partial t)^\mu(\partial/\partial t)^\nu = g_{tt} = -[c^2 - v_0^2] \). This changes sign when \( ||\vec{v}_0|| > c \). Thus any region of supersonic flow is an ergo–region. The analogue of this behaviour in general relativity is the ergosphere surrounding any spinning black hole — it is a region where space “moves” with superluminal velocity relative to the fixed stars.

Observe that in a completely general Lorentzian geometry the metric has 6 degrees of freedom per point in spacetime. \( (4 \times 4 \text{ symmetric matrix} \Rightarrow 10 \text{ independent components}; \text{then subtract 4 coordinate conditions}) \). In contrast, the acoustic metric is more constrained. Being specified completely by the three scalars \( \psi_0(t, \vec{x}), \rho_0(t, \vec{x}), \) and \( c(t, \vec{x}) \), the acoustic metric has only 3 degrees of freedom per point in spacetime.

A point of notation: Where the general relativist uses the word “stationary” the fluid dynamicist uses the phrase “steady flow”. Where the general relativist uses the word “static” the fluid mechanic would translate this as “fluid at rest”.

6
The analogies I am invoking between acoustics in fluids and general relativity are very deep and very powerful — there is a lot of mathematical machinery available for use.

**IV. GEOMETRIC ACOUSTICS**

Taking the short wavelength/high frequency limit to obtain geometrical acoustics is now easy. Sound rays (phonons) follow the *null geodesics* of the acoustic metric. Compare this to general relativity where in the geometrical optics approximation light rays (photons) follow *null geodesics* of the physical spacetime metric. Since null geodesics are insensitive to any overall conformal factor in the metric \[6,7\] one might as well simplify life by considering the metric

\[
h_{\mu\nu} \equiv \begin{pmatrix}
-(c^2 - v_0^2) & -v_0^i \\
\cdot & \cdot & \cdot \\
-v_0^i & \delta^{ij}
\end{pmatrix}.
\] (21)

Thus, in the geometric acoustics limit, sound propagation is insensitive to the density of the fluid, and depends only on the local speed of sound and the velocity of the fluid. It is only for specifically wave related properties that the density of the medium becomes important.

One can rephrase this in a language more familiar to the acoustics community. Take \(\psi_1 \sim ae^{i\varphi}\). Then, neglecting variations in the amplitude \(a\), the wave equation reduces to the *Eikonal equation*

\[
h^{\mu\nu} \partial_\mu \varphi \partial_\nu \varphi = 0.
\] (22)

This Eikonal equation is blatantly insensitive to any overall multiplicative prefactor.

As a sanity check on the formalism, let the null geodesic be parameterized by \(X^\mu(t) \equiv (t, \vec{x}(t))\). Then the null condition implies

\[
h_{\mu\nu} \frac{dX^\mu}{dt} \frac{dX^\nu}{dt} = 0
\]

\[\iff -(c^2 - v_0^2) - 2v_0^i \frac{dx^i}{dt} + \frac{dx^i}{dt} \frac{dx^i}{dt} = 0
\]

\[\iff \left\| \frac{d\vec{x}}{dt} - \vec{v}_0 \right\| = c.
\] (23)
Here the norm is taken in the flat physical metric. This has the obvious interpretation that the ray travels at the speed of sound relative to the moving medium.

If the geometry is stationary, one can do slightly better. Let $X^\mu(s) \equiv (t(s); \vec{x}(s))$ be some null path from $\vec{x}_1$ to $\vec{x}_2$ parameterized in terms of physical arc length (i.e. $||d\vec{x}/ds|| \equiv 1$). Then the condition for the path to be null (though not yet necessarily a null geodesic) is

$$-(c^2 - v_0^2) \left( \frac{dt}{ds} \right)^2 - 2v_0^i \left( \frac{dx^i}{ds} \right) \left( \frac{dt}{ds} \right) + 1 = 0. \quad (24)$$

Solving the quadratic

$$\frac{dt}{ds} = -v_0^i \left( \frac{dx^i}{ds} \right) + \sqrt{c^2 - v_0^2 + \left( v_0^i \frac{dx^i}{ds} \right)^2} \over c^2 - v_0^2. \quad (25)$$

The total time taken to traverse the path is thus

$$T[\gamma] = \int_{\vec{x}_1}^{\vec{x}_2} (dt/ds) ds = \int_{\gamma} \left\{ \sqrt{(c^2 - v_0^2)ds^2 + (v_0^i dx^i)^2} - v_0^i dx^i \right\} / (c^2 - v_0^2). \quad (26)$$

Extremizing the total time taken is Fermat’s principle for sound rays. One has thus checked the formalism for stationary geometries (steady flow) by reproducing the discussion on p 262 of Landau and Lifshitz [2].

V. DISCUSSION

A. Limitations

The derivation of the wave equation made two key assumptions — the flow is irrotational flow and the fluid is barotropic.

The d’Alembertian equation of motion for acoustic disturbances, though derived only under the assumption of irrotational flow, continues to make perfectly good sense in its own right if the background velocity field $\vec{v}_0$ is given some vorticity. This leads one to hope that it might be possible to find a suitable generalization of the present derivation that might
work for flows with nonzero vorticity. In this regard, note that if the vorticity is everywhere confined to thin vortex filaments, the present derivation already works everywhere outside the vortex filaments themselves.

The restriction to a barotropic fluid ($\rho$ a function of $p$ only) is in fact related to issues of vorticity. Examples of barotropic fluids are:

- Isothermal fluids subject to isothermal perturbations.
- Fluids in convective equilibrium subject to adiabatic perturbations.

See for example [1], §311, pp 547–548, and §313 pp 554–556. Failure of the barotropic condition implies that the perturbations cannot be vorticity free and requires more sophisticated analysis.

B. Precursors

It is perhaps surprising that anything new can be said about so venerable a subject as fluid dynamics. Certainly there are precursors to the discussion of this letter in the fluid dynamics literature. For instance, take the background to be static, so that $\vec{v}_0 = 0$, while $\partial_t \rho_0 = 0 = \partial_t p_0$, though $p_0$ and hence $c$ are permitted to retain arbitrary spatial dependencies. Then the wave equation derived in this letter reduces to

$$\partial_t^2 \psi = c^2 \frac{1}{\rho_0} \nabla \cdot (\rho_0 \nabla \psi).$$  \hspace{1cm} (27)

This equation is in fact well known. It is equivalent, for instance to eq. (13) of §313 of Lamb’s classic *Hydrodynamics* [1]. See also eq. (1.4.5) of the recent book by DeSanto [8]. The superficially similar wave equations discussed by Landau and Lifshitz [2] (see §74, eq. (74.1)), and by Skudrzyk [9] (see p 282), utilize different physical assumptions concerning the behaviour of the fluid. The novelty I am claiming for the current letter is firstly, inclusion of nonzero background velocities and arbitrary time dependencies, and secondly and more importantly, the interpretation of these results in terms of Lorentzian geometry.
VI. CONCLUSIONS

I have shown that acoustic waves in an inviscid fluid can (under the assumptions of irrotational barotropic flow) be described by the scalar d’Alembertian of a suitable Lorentzian geometry. For inhomogeneous flows this Lorentzian geometry will exhibit nonzero curvature.

Prior to this observation, Lorentzian geometries have been of interest to physics only within the confines of Einstein’s theory of gravitation (general relativity). A large quantity of technical mathematical machinery currently utilized only within the context of general relativity may thus become of interest to the fluid dynamics community.

On the other hand, the results of this letter give the general relativists a very down to earth physical model for certain classes of Lorentzian geometry.

Particularly intriguing is the fact that while the underlying physics of fluid dynamics is completely nonrelativistic, Newtonian, and sharply separates the notions of space and time, one nevertheless sees that the fluctuations couple to a full–fledged Lorentzian spacetime.

Note Added: After this paper was submitted for publication I was informed that similar results can be found in the interesting but little–known work of Unruh [10]. In that Letter, Unruh investigated the acoustic equivalent of Hawking radiation arising from the fluid dynamical analogue of a black hole. I wish to thank Ted Jacobson and John Friedman for bringing this reference to my attention. Further work on acoustic black holes may be found in [11].

ACKNOWLEDGMENTS

This research was supported by the U.S. Department of Energy.
REFERENCES

* Electronic mail: visser@kiwi.wustl.edu

[1] Sir Horace Lamb, *Hydrodynamics*, Sixth edition, (Dover, New York, 1932) {originally published 1879}.

[2] L. D. Landau and E. M. Lifshitz, *Fluid Mechanics*, (Pergamon, London, 1959).

[3] L. M. Milne-Thomson, *Theoretical Hydrodynamics*, Fifth edition, (MacMillan, London, 1968).

[4] V. Fock, *The theory of space, time and gravitation*, Second edition, (Pergamon, New York, 1964).

[5] C. Møller, *The theory of relativity*, Second edition, (Clarendon, Oxford, 1972).

[6] C. W. Misner, K. S. Thorne, and J. A. Wheeler, *Gravitation* (Freeman, San Francisco, 1973).

[7] S. W. Hawking and G. F. R. Ellis, *The large scale structure of space–time*, (Cambridge University Press, Cambridge, 1973).

[8] J. A. DeSanto, *Scalar wave theory*, (Springer–Verlag, Berlin, 1992).

[9] E. Skudrzyk, *The foundations of Acoustics*, (Springer-Verlag, Wien, 1971) {see especially pp 270–282}.

[10] W. G. Unruh, Phys. Rev. Lett, 46, 1351 (1981).

[11] T. Jacobson, Phys. Rev. D 44, 1731 (1991).