NONUNIQUENESS OF WEAK SOLUTIONS FOR THE TRANSPORT EQUATION AT CRITICAL SPACE REGULARITY

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ABSTRACT. We consider the linear transport equations driven by an incompressible flow in dimensions $d \geq 2$. For divergence-free vector fields $u \in L^1 W^{1,q}$, the celebrated DiPerna-Lions theory of the renormalized solutions established the uniqueness of the weak solution in the class $L^\infty L^p$ when $\frac{1}{p} + \frac{1}{q} \leq 1$. For such vector fields, we show that in the regime $\frac{1}{p} + \frac{1}{q} > 1$, weak solutions are not unique in the class $L^1 L^p$. Crucial ingredients in the proof include the use of both temporal intermittency and oscillation in the convex integration scheme and a new family of stationary exact solutions to the transport equation driven by divergence-free vector fields with a small singular part.

1. INTRODUCTION

In this paper, we consider the linear transport equation on the torus $\mathbb{T}^d$

$$
\begin{cases}
\partial_t \rho + u \cdot \nabla \rho = 0 \\
\rho|_{t=0} = \rho_0,
\end{cases}
$$

(1.1)

where $\rho : [0, T] \times \mathbb{T}^d \to \mathbb{R}$ is a scalar density function, $u : [0, T] \times \mathbb{T}^d \to \mathbb{R}^d$ is a given vector field. We always assume $u$ is incompressible, i.e.,

$$\text{div } u = 0.$$ 

By the linearity of the equation, even for very rough vector fields it is not difficult to prove the existence of weak solutions that solves the equation in the sense of distributions

$$
\int_0^T \int_{\mathbb{T}^d} \rho (\partial_t \varphi + u \cdot \nabla \varphi) \, dx \, dt = 0 \quad \text{for all } \varphi \in C^\infty_c ([0, T] \times \mathbb{T}^d).
$$

(1.2)

Our main focus is the uniqueness/nonuniqueness issue of weak solutions to (1.1), more precisely, whether the DiPerna-Lions uniqueness result is sharp.

**Theorem 1.1** (DiPerna-Lions [DL89]). Let $p, q \in [1, \infty]$ and let $u \in L^1 (0, T; W^{1,q}(\mathbb{T}^d))$ be a divergence-free vector field. For any $\rho_0 \in L^p (\mathbb{T}^d)$, there exists a unique renormalized solution $\rho \in C([0, T]; L^p(\mathbb{T}^d))$ to (1.1). Moreover, if

$$
\frac{1}{p} + \frac{1}{q} \leq 1
$$

(1.3)

then this solution $\rho$ is unique among all weak solutions in class $L^\infty (0, T; L^p(\mathbb{T}^d))$.

Based on scaling analysis and a close examination of the proof in [DL89], one can speculate that if

$$
\frac{1}{p} + \frac{1}{q} > 1
$$

(1.4)

then the uniqueness may fail. More specifically,

**Conjecture 1.2.** Let $p, q \in [1, \infty]$. Let $u \in L^1 (0, T; W^{1,q}(\mathbb{T}^d))$ be a divergence-free vector field.

1. If $\frac{1}{p} + \frac{1}{q} \leq 1$, then there exists a unique weak solution $\rho \in L^\infty (0, T; L^p(\mathbb{T}^d))$ to (1.1).

2. If $\frac{1}{p} + \frac{1}{q} > 1$, then weak solutions in the class $L^\infty (0, T; L^p(\mathbb{T}^d))$ are not unique.

In this paper, we address the question (2) in Conjecture 1.2 and prove the following.

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Theorem 1.3. Let $d \geq 2$ and $p, q \in [1, \infty]$ satisfying $p > 1$ and (1.4). Then there exists a divergence-free vector field $u \in L^{1}(0,T;W^{1,q}(T^{d})) \cap L^{\infty}(0,T;L^{p'}(T^{d}))$, such that the uniqueness of (1.1) fails in the class $\rho \in L^{1}(0,T;L^{p}(T^{d})).$

This result is proved by convex integration, building upon the framework of [MS18]. New ingredients include:

(1) The use of both temporal intermittency and oscillations in the convex integration scheme, which is implemented by oscillating the stationary solutions intermittently in time.

(2) The construction of highly concentrated stationary solutions to the transport equation with incompressible vector fields in dimensions $d \geq 2$.

(3) A sharp estimate concerning vector potential with radial symmetry for the error of time derivative.

We will discuss these new devices more precisely later in the introduction.

1.1. Background and main results. It is known that for Lipschitz vector fields, smooth or classical solutions of (1.1) can be obtained by solving the ordinary differential equation for the flow map $X : [0,T] \times T^{d} \to T^{d}$

\[
\begin{aligned}
\partial_{t}X(t,x) &= u(t,X(t,x)) \\
X(0,x) &= x,
\end{aligned}
\]

and setting $\rho(t,X) = \rho_{0}(x)$. For instance, the wellposedness and uniqueness of (1.1) can be deduced from the Cauchy-Lipschitz theory for (1.5). Moreover, for such vector fields, the inverse flow map $X^{-1}(t)$ solves the transport equation

\[
\begin{aligned}
\partial_{t}X^{-1} + u \cdot \nabla X^{-1} &= 0 \\
X^{-1}(0) &= \text{Id}.
\end{aligned}
\]

For vector fields that are not necessarily Lipschitz, the link between the PDE (1.1) and the ODE (1.5) is less obvious. Even though one can prove the existence of weak solutions fairly easily by the linearity of the equation, the uniqueness issue of (1.1) becomes subtler for non-Lipschitz vector fields. The uniqueness class for the density is generally related to the Sobolev/BV regularity of the vector field. The first result in this direction dates back to the celebrated work of DiPerna-Lions [DL89] which used the method of renormalization. Since then a lot of effort has been devoted to determining how far the regularity assumption on the vector field can be relaxed. Profound ideas and complex theories, that are beyond the scope of this paper, have been developed, in particular, the notion of regular Lagrangian flows introduced by Ambrosio [Amb04]. We refer to the works [Amb04, LBL04, CLR03, BN18, CL02, CC16, CC18, CDL08] and the surveys [Amb17, DL08] for regularity/uniqueness results in this direction and for related results of the continuity equation.

Very roughly speaking, there are currently two distinct methods of proving nonuniqueness for (1.1). The first approach is Lagrangian, using the degeneration of the flow map to show nonuniqueness at the ODE level; while the second approach is Eulerian, using convex integration to prove nonuniqueness at the PDE level.

In regard to the Lagrangian approach, in their original work [DL89], DiPerna and Lions provided a counterexample $u \in W^{1,p}$ with unbounded divergence and a divergence-free counterexample $u \in W^{s,1}$ for all $s < 1$ but $u \notin W^{1,1}$. Much later, Depauw in [Dep03] constructed nonuniqueness in the class $\rho \in L^{\infty}_{t,x}$ for incompressible vector fields $L^{1}_{loc, BV}$ based on the example in [Aiz78]. This type of examples were revisited in [ACM19, CLR03, YZ17] in other contexts. More recently, in [DEIJ19], Drivas, Elgindi, Iyer and Jeong proved nonuniqueness in the class $\rho \in L^{\infty}_{t}L^{2}$ for $u \in L^{1}C^{1-}$ based on anomalous dissipation and mixing. We should emphasize that the Lagrangian approach is not suited for a construction of a divergence-free example with Sobolev regularity of one full derivative, say $u \in L^{1}W^{1,p}$.

On the Eulerian side, the first nonuniqueness result was obtained by Crippa, Gusev, Spirito, and Wiedemann in [CGSW15] using the framework of [DLS09]. However, the vector field $u$ was merely bounded and did not have an associated Lagrangian flow. The first breakthrough result for the Sobolev vector field was obtained by Modena and Székelyhidi [MS18]. Note that the Sobolev regularity $L^{1}W^{1,p}$ of the vector field implies the uniqueness of a regular Lagrangian flow, see for example [ACF15]. The contrast between the Lagrangian and Eulerian wellposedness has also been studied in various contexts, see for instance [DL08, RS09b, RS09a, CKV16].
Starting with the groundwork work of Modena and Székelyhidi [MS18], the Eulerian nonuniqueness issue of (1.1) has drawn a lot of research attention lately. Below are the functional classes where the nonuniqueness has been achieved:

1. [MS18] (Modena and Székelyhidi): $\rho \in C^L$ when $u \in C_t W^{1,q} \cap C_t L^{p'}$ for $\frac{1}{p} + \frac{1}{q} > 1 + \frac{1}{d-1}$, $p > 1$ and $d \geq 3$. Later in [MS19b]: extension to the endpoint $p = 1$ and $u$ also being continuous.

2. [MS19a] (Modena and Sattig): $\rho \in C_t L^p$ when $u \in C_t W^{1,q} \cap C_t L^{p'}$ for $\frac{1}{p} + \frac{1}{q} > 1 + \frac{1}{d}$ and $d \geq 3$.

3. [BCL20] (Brüé, Colombo, and De Lellis): positive $\rho \in C_t L^p$ when $u \in C_t W^{1,q}$ for $\frac{1}{p} + \frac{1}{q} > 1 + \frac{1}{d'}$, $p > 1$ and $d \geq 2$.

In light of the current state, it is then natural to ask whether one can close the gap between the DiPerna-Lions regime $\frac{1}{p} + \frac{1}{q} \leq 1$ and the Modena-Sattig-Székelyhidi regime $\frac{1}{p} + \frac{1}{q} > 1 + \frac{1}{d}$.

In this paper, we address this question and prove nonuniqueness in the full complement of the DiPerna-Lions regime

$$\frac{1}{p} + \frac{1}{q} > 1$$

for weak solutions in the class $\rho \in L^1_t L^p$, $p > 1$ for all dimensions $d \geq 2$.

**Theorem 1.4.** Let $d \geq 2$ and $p, q \in [1, \infty]$ satisfying $p > 1$ and (1.4). For any $\varepsilon > 0$ and any time-periodic\(^1\) $\tilde{\rho} \in C^\infty([0,T] \times \mathbb{T}^d)$ with constant mean

$$\int_{\mathbb{T}^d} \tilde{\rho}(x, t) \, dx = \int_{\mathbb{T}^d} \tilde{\rho}(x, 0) \, dx \quad \text{for all } t \in [0, T),$$

there exist a vector field $u : [0, T] \times \mathbb{T}^d \to \mathbb{R}^d$ and a density $\rho : [0, T] \times \mathbb{T}^d \to \mathbb{R}$ such that the following holds.

1. $u \in L^1(0, T; W^{1,q}(\mathbb{T}^d)) \cap L^\infty(0, T; L^{p'}(\mathbb{T}^d))$ and $\rho \in L^1(0, T; L^p(\mathbb{T}^d)).$

2. $(\rho, u)$ is a weak solution to (1.1) in the sense of (1.2).

3. The deviation of $L^p$ norm is small on average: $\|\rho - \tilde{\rho}\|_{L^1 L^p} \leq \varepsilon$.

4. $\rho(t)$ is continuous in the sense of distributions and for $t = 0, T$, $\rho(t) = \tilde{\rho}(t)$.

It is easy to deduce Theorem 1.3 from Theorem 1.4.

**Proof of Theorem 1.3.** Let $\rho_0 \in C^\infty_c(\mathbb{T}^d)$ with $\|\rho_0\|_p = 1$. We take $\tilde{\rho} = \chi(t)\rho_0(x)$ where $\chi \in C^\infty([0, T])$ is such that $\chi(t) = 1$ if $|t - \frac{T}{2}| \leq \frac{T}{2}$ and $\chi = 0$ if $|t - \frac{T}{2}| \geq \frac{T}{8}$. We apply Theorem 1.4 with $\varepsilon = \frac{T}{1000}$. The obtained solution $\rho$ cannot have a constant $L^p$ norm due to $\|\rho - \tilde{\rho}\|_{L^1 L^p} \leq \varepsilon$, and thus is different from the renormalized solution emerging from the same initial data. \qed

**Remark 1.5.** Several remarks are in order.

1. In fact, for any Sobolev space $W^{k,\tilde{p}}$ the vector field $u \in L^r W^{k,\tilde{p}}$ for some small $r > 0$ depending on $k, \tilde{p}$. The singularity of $u$ concentrates on a small “bad” set\(^2\) in $[0, T] \times \mathbb{T}^d$. The density $\rho$ also verifies $\rho \in L^r L^\infty$ for some $r > 0$.

2. $L^1 L^p$ is sharp in terms of the space regularity, however this is done at the expense of time regularity by using temporal intermittency. We discuss this below and in detail in Section 6. The question of whether the nonuniqueness holds in the class $\rho \in L^\infty L^p$ remains open.

3. It seems possible to also cover the border case $p = 1$ by utilizing the technique in [MS19b](see also [BDLIS15, BLJV18]).

4. A slight modification of the proof can also yield a positive density $\rho$ in the same $p, q$ regime, cf. [BCL20, Theorem 1.5].

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\(^1\) We identify $[0, T]$ with 1-dimensional torus.

\(^2\) In fact, the singular set of $u$ is dense, and as a result, there is no local regularity outside the singular set, cf. [CKN82, BCV18].
1.2. Continuity-defect equation and the convex integration scheme. Let us outline the main ideas and strategies of the proof. We follow the framework of [MS18] to treat both $\rho$ and $u$ as unknowns and construct a sequence of approximate solutions $(\rho_n, u_n, R_n)$ solving the continuity-defect equation

\[
\begin{align*}
&\partial_t \rho_n + u_n \cdot \nabla \rho_n = \text{div} R_n \\
&\text{div} u_n = 0.
\end{align*}
\]

The vectors $R_n$ are called the defect fields, which arise naturally when considering weak solutions of (1.1). This framework allows us to use the interplay between the density $\rho_n$ and the vector field $u_n$ as in a nonlinear equation.

The main goal is to design suitable perturbations $\theta_n := \rho_n - \rho_{n-1}$ and $w_n := u_n - u_{n-1}$ such that the defect fields $R_n \to 0$ in an appropriate sense. The most important step is to ensure the oscillation part

\[
d\text{div} R_{\text{osc}} := \text{div}(\theta_n w_n + R_{n-1})
\]

consists of only high frequencies so that the new defect field $R_n$ is much smaller than $R_{n-1}$. This technique is now considered standard among the experts, and we refer readers to [DLS09, DLS13, BDLIS15, BDLIS16, Ise18, BSV18, Nov18, Dai18] for more discussion on this technique in other models.

In previous works [MS18, MS19b, MS19a], perturbations $(\rho_n, w_n)$ are designed so that (1.7) has only high frequencies in space, and the error is canceled point-wise in time. In this work, the defect field $R_n \to 0$ in the norm $L_t^1 L_x^{1/2}$. In particular, the final solution is homogeneous in time.

In this paper, we use a convex integration scheme that features both spacial and temporal oscillations. This is done by adding in temporal oscillation when designing $(\rho_n, w_n)$ such that to the leading order (1.7) can be split into two parts, one with high spacial frequencies, and the other with high temporal frequencies. This idea is implicitly rooted in the work [BV19], but it was not formulated to encode temporal intermittency but rather to cancel part of the error caused by adding spatial intermittency.

Based on the above discussion, on the technical side, the defect fields $R_n$ shall be measured in $L_t^1 L_x^{1/2}$ instead of $L_t^1 L_x^{1/4}$. In other words, the defect fields $R_n$ are canceled weakly in space-time, rather than pointwise in time and weakly in space. This relaxation allows us to exploit temporal intermittency and design the perturbations $(\rho_n, w_n)$ with critical space regularity, which we discuss below.

1.3. Space-time intermittency in the convex integration. Even though the concept of intermittency and its theoretical studies has been around for many years [Man76, Fri95, CS14] in hydrodynamic turbulence, it was only implemented very recently with convex integration in the seminal work [BV19] of Buckmaster and Vicol. We could somehow summarize the difficulty as follows. At the heart of its argument, convex integration relies on adding highly oscillatory perturbations to obtain weakly converging solutions. A more intermittent perturbation carries a more diffuse Fourier side and introduces more interactions among oscillations. These harmful interactions are difficult to control and cause the iteration scheme to break down. We refer to [BV19, Luo19, CL19, MS18, MS19a] for recent developments of using intermittency in the convex integration and related discussions.

To fix ideas, let us denote by $D$ the intermittency dimension (in space), cf. [Fri95]. Roughly speaking, the solution is concentrated on a set of dimension $D$ in space. This is related to the development of “concentration” in the context of weak solutions, [DM87b, DM87a].

However, for the transport equation, using only spacial intermittency in a convex integration scheme is not enough to reach the full complement of DiPerna-Lions regime. If the solution $(\rho, u)$ is homogeneous in time, then by the duality $\rho \in L^{\infty} L^p$ and $u \in L^{\infty} L^{p'}$ imposed by the machinery of convex integration, we can see that

\[
u \in L^{\infty} L^{p'} \Rightarrow u \in L^{\infty} W^{1,q} \text{ for } \frac{1}{p} + \frac{1}{q} > 1 + \frac{1}{d-D}.
\]

In other words, the Sobolev regularity $u \in L^{\infty} W^{1,q}$ must come at the cost of integrability in space if the vector field is homogeneous in time. This simple heuristics works surprisingly well and explains the gap between the DiPerna-Lions regime $\frac{1}{p} + \frac{1}{q} \leq 1$ and the Modena-Sattig-Székelyhidi regime $\frac{1}{p} + \frac{1}{q} > 1 + \frac{1}{d}$ even when fully intermittent $D = 0$ building blocks were used in [MS19a, BCL20].

One of the most striking differences between previous schemes and the current one is that intermittency in space plays a very little role. In fact, one can use the “Mikado densities” and “Mikado fields” in [MS18] to implement the temporal intermittency and recover the same range of nonuniqueness result in $d \geq 3$. Even though our building blocks can be fully intermittent, the convex integration scheme goes through as long as they are not spatially homogeneous. The highly concentrated property of the building blocks was used to achieve negligible interference in $d \geq 2$. 

By contrast, in our case the Sobolev regularity \( u \in L^1 W^{1,q} \) comes completely from the temporal intermittency. We do not rely on the fundamental heuristics (1.8) in previous works. With temporal intermittency, we instead work with the duality \( \rho \in L^1 L^p \) and \( u \in L^\infty L^{p'} \). This is consistent with the decay of the defect field in \( L^1_{t,x} \) norm instead of \( L^\infty \) norm. The temporal intermittency of the vector field \( u \) allows us to improve the space regularity as the expense of time regularity, more precisely
\[
    u \in L^\infty L^{p'} \Rightarrow u \in L^1 W^{1,q} \text{ for } \frac{1}{p} + \frac{1}{q} > 1.
\] (1.9)

We will explain how to obtain (1.9) in the following.

If \( u \) is fully intermittent in time, then heuristically from \( L^\infty \) to \( L^1 \) one gains a full derivative in time. By a dimensional analysis for (1.9), as long as the associated temporal frequency is comparable to the spacial frequency, \( u \in L^1 W^{1,q} \) can be achieved since \( q < p' \) by \( \frac{1}{p} + \frac{1}{q} > 1 \). This also requires a sharper estimate when estimating the error with time derivative as the temporal frequencies are quite large and comparable to spacial frequencies. We also emphasis that the heuristics (1.9) encodes no information on spacial intermittency at all, which is fundamentally different from the heuristics (1.8). In fact, the convex integration scheme works for a wide range of choices of concentration and oscillation parameters. We refer to Section 7 for the specific choice of parameters and Lemma 9.1 for the sharp estimate.

1.4. Temporal intermittency by oscillating stationary solutions. We will now describe the implementation of temporal intermittency, where the basic idea is to oscillate stationary solutions of the transport equation intermittently in time. The stationary solutions are required to be intermittent in space as well.

Current spatially intermittent building blocks in the convex integration scheme are either not stationary [MS19a, BCL20], or stationary [MS18] but only work in \( d \geq 3 \). We also note that even though it seems to be theoretically possible to achieve temporal intermittency using non-stationary building blocks [MS19a, BCL20] in \( d \geq 2 \), such an approach would be less intuitive and significantly more complicated.

To overcome the lack of suitable stationary building blocks in 2D, we design a two-parameter family of exact stationary solutions \((\Phi, W)\) to (1.1) in \( \mathbb{R}^d \) for \( d \geq 2 \). To find such solutions, we start with the ansatz
\[
    W = \eta(z)\psi(r)e_z - \eta'(z)\phi(r)e_r, \quad \Phi = \eta(z)\psi(r),
\] (1.10)
in cylindrical coordinates in dimension \( d \geq 2 \). It turns out that the system
\[
    \begin{align*}
    W \cdot \nabla \Phi &= 0 \\
    \text{div} W &= 0
    \end{align*}
\] (1.11)
reduces to an integro-differential equation for the radial profiles \( \psi \) and \( \phi \), which, surprisingly, has closed-form Schwartz solutions \( \psi \in S(\mathbb{R}^d) \) in \( d \geq 2 \). The natural scaling of (1.11) also allows us to achieve desired concentrations in \( z \) and \( r \).

To perform convex integration on \( \mathbb{T}^d \), we use Poisson’s summation formula to transfer the exact solution \((\Phi, W)\) in \( \mathbb{R}^d \) to approximate solutions \((\Phi_k, W_k)\) on \( \mathbb{T}^d \). These stationary approximate solutions \((\Phi_k, W_k)\) are highly concentrated around different points in \( \mathbb{T}^d \) for \( d \geq 2 \), making them suitable for our convex integration scheme utilising temporal intermittency and oscillations in all dimensions \( d \geq 2 \).

With stationary building blocks \((\Phi_k, W_k)\) at hand, we implement the temporal intermittency as follows. On one hand, we use temporal oscillation to relax the convex integration procedure from pointwise in time to weakly in time. Given a solution \((\rho, u, R)\) of the continuity-defect equation, we design the perturbation \((\theta, w)\) so that to leading order it produces a high-high to low cascade in space-time that balances the old defect field \( R \) in the sense that
\[
    \text{div}(\theta w + R) = \text{High Temporal Fre. Terms} + \text{High Temporal Fre. Terms} + \text{Lower Order Terms}.
\]

On the other hand, the relaxation of convex integration to be done weakly in time allows us to add in temporal intermittency in the perturbations \((\theta, w)\). The key is to ensure that \((\theta, w)\) is almost fully intermittent in time, which determines the regularity of the final solution \( \rho \in L^1_t L^p \) and \( u \in L^\infty_t L^{p'} \cap L^1 W^{1,q} \).

To summarize, in the convex integration scheme the perturbations consist of these space-time intermittent oscillatory building blocks. The temporal intermittency is used to achieve the optimal range in (1.9), whereas the temporal oscillation allows us to cancel the defect fields on average in space-time, consistent with the decay of \( R_n \) in the norm \( L^1_{t,x} \). We refer to Section 6 for more details.
1.5. **Organization of the paper.** The rest of the paper is organized as follows.

- We introduce the notations and many technical tools used throughout the paper in Section 2.
- Section 3 is devoted to the proof of Theorem 1.4 by assuming the main proposition, Proposition 3.1.
- In Section 4 we derive the stationary solutions $(\Phi, W)$ in the whole space $\mathbb{R}^d$, $d \geq 2$.
- In Section 5 we obtain the periodic approximate solutions $(\Phi_k, W_k)$ by transferring the solution $(\Phi, W)$ on $\mathbb{R}^d$ to the torus $\mathbb{T}^d$. These pairs $(\Phi_k, W_k)$ will be the main building blocks of the convex integration scheme.
- Section 6 is a detailed explanation for the use of temporal oscillation and intermittency in the convex integration scheme. In particular, we will define the temporal oscillators $\tilde{g}_k, g_k$ that we use to oscillate the building blocks $(\Phi_k, W_k)$.
- Section 7, 8, 9 constitute the proof of Proposition 3.1:
  - In Section 7 we first define the perturbation density $\rho$ and vector field $w$ using the building blocks $(\Phi_k, W_k)$. And then the new defect field $R$ is derived from the perturbations $\theta$ and $w$, which is the core of our convex integration scheme.
  - The estimates for the perturbations $\rho$ and $w$ are done in Section 8. Then we conclude the proof of the perturbation part of Proposition 3.1.
  - The new defect field $R$ is estimated in Section 9. The rest of the proof of Proposition 3.1 will be completed in the end.

2. **Preliminaries**

The purpose of this section to collect the technical tools that will be used throughout the paper. We keep this section relatively concise so that we are not distracted from the main goal of proving the nonuniqueness result.

2.1. **Notations.** Throughout the manuscript, we use the following notations.

- $\mathbb{T}^d = \mathbb{R}^d / \mathbb{Z}^d$ is the $d$-dimensional torus. For any function $f : \mathbb{T}^d \rightarrow \mathbb{R}$ we denote by $f(\sigma \cdot)$ the $\sigma^{-1}\mathbb{T}^d$-periodic function $f(\sigma x)$.
- For any $p \in [1, \infty]$, its Hölder dual is denoted as $p'$. Throughout the paper, $p$ is fixed as in Theorem 1.4. We will use $r$ for general $L^r$ norm.
- For any $1 \leq r \leq \infty$, the Lebesgue space is denoted by $L^r$. For any $f \in L^1(\mathbb{T}^d)$, its spacial average is
  \[ \frac{\int_{\mathbb{T}^d} f \, dx}{\int_{\mathbb{T}^d} 1 \, dx} = \int_{\mathbb{T}^d} f \, dx. \]
- For any function $f : [0, T] \times \mathbb{T}^d \rightarrow \mathbb{R}$, denote by $\|f(t)\|_r$ the Lebesgue norm on $\mathbb{T}^d$ (in space only) at a fixed time $t$. If the norm is taken in space-time, we use $\|f\|_{L^r_{t,x}}$.
- The space $C^\infty_0(\mathbb{T}^d)$ is the set of periodic smooth functions with zero mean, and $C^\infty(\mathbb{R}^d)$ is the space of smooth functions with compact support in $\mathbb{R}^d$.
- We often use the same notations for scalar functions and vector functions. Sometimes we use $C^\infty_0(\mathbb{T}^d, \mathbb{R}^d)$ for the set of periodic smooth vector fields with zero mean.
- We use $\nabla$ to indicate full differentiation in space only, and space-time gradient is denoted by $\nabla_{t,x}$. Also, $\partial_t$ is the partial derivative in the time variable.
- For any Banach space $X$, the Banach space $L^r(0,T;X)$ is equipped with the norm
  \[ \left( \int_0^T \| \cdot \|^r_X \, dt \right)^{\frac{1}{r}}, \]
  and we often use the short notations $L^p_t X$ and $\| \cdot \|_{L^p_t X}$.
- We write $X \lesssim Y$ if there exists a constant $C > 0$ independent of $X$ and $Y$ such that $X \leq CY$. If the constant $C$ depends on quantities $a_1, a_2, \ldots, a_n$ we will write $X \lesssim_{a_1, \ldots, a_n} Y$ or $X \leq C_{a_1, \ldots, a_n} Y$. 


2.2. Antidivergence operators $\mathcal{R}$ and $B$ on $\mathbb{T}^d$. We will use the standard antidivergence operator $\Delta^{-1} \nabla$ on $\mathbb{T}^d$, which will be denoted by $\mathcal{R}$.

It is well known that for any $f \in C^\infty(\mathbb{T}^d)$ there exist a unique $u \in C_0^\infty(\mathbb{T}^d)$ such that

$$\Delta u = f - \int_{\mathbb{T}^d} f.$$ 

For any smooth scalar function $f \in C^\infty(\mathbb{T}^d)$, the standard anti-divergence operator $\mathcal{R} : C^\infty(\mathbb{T}^d) \to C_0^\infty(\mathbb{T}^d, \mathbb{R}^d)$ can be defined as

$$\mathcal{R} f := \Delta^{-1} \nabla f,$$

which satisfies

$$\text{div}(\mathcal{R} f) = f - \int_{\mathbb{T}^d} f \text{ for all } f \in C^\infty(\mathbb{T}^d).$$

and

$$\mathcal{R}(\text{div } u) = u - \int_{\mathbb{T}^d} u \text{ for all } u \in C^\infty(\mathbb{T}^d, \mathbb{R}^d).$$

The next result, which says that $\mathcal{R}$ is bounded on all Sobolev spaces $W^{k,p}(\mathbb{T}^d)$, is classical, see for instance [MS18, Lemma 2.2] for a proof.

Lemma 2.1. For every $m \in \mathbb{N}$ and $r \in [1, \infty]$, the antidivergence operator $\mathcal{R}$ is bounded on $W^{m,r}(\mathbb{T}^d)$ for any $m \in \mathbb{N}$:

$$\|\mathcal{R} f\|_{W^{m,r}} \lesssim \|f\|_{W^{m,r}}. \tag{2.1}$$

Throughout the paper, we use heavily the following fact about $\mathcal{R}$.

$$\mathcal{R} f(\sigma \cdot) = \sigma^{-1} \mathcal{R} f \text{ for any positive } \sigma \in \mathbb{N}.$$ 

We will also use its bilinear counterpart $B : C^\infty(\mathbb{T}^d) \times C^\infty(\mathbb{T}^d) \to C^\infty(\mathbb{T}^d, \mathbb{R}^d)$ defined by

$$B(a, f) := a \mathcal{R} f - \mathcal{R}(\nabla a \cdot \mathcal{R} f).$$

This bilinear version $B$ has the additional advantage of gaining derivative from $f$ when $f$ has a very small period. See also higher order variants of $B$ in [MS19a].

It is easy to see that $B$ is a left-inverse of the divergence,

$$\text{div}(B(a, f)) = af - \int_{\mathbb{T}^d} af \, dx$$

which can be proved easily using integration by parts a couple of times. The following estimate is a direct consequence of Lemma 2.1.

Lemma 2.2. Let $1 \leq r \leq \infty$. Then for any $a, f \in C^\infty(\mathbb{T}^d)$

$$\|B(a, f)\|_r \lesssim \|a\|_{C^1} \|\mathcal{R} f\|_r.$$ 

Proof. This follows from Hölder’s inequality and Lemma 2.1. \qed

Remark 2.3. The assumption on $f$ in Lemma 2.2 can be relaxed to $f \in L^r(\mathbb{T}^d)$.

2.3. Improved Hölder’s inequality on $\mathbb{T}^d$. We recall the following result due to Modena and Székelyhidi [MS18, Lemma 2.1], which extends the first type of such result [BV19, Lemma 3.7].

Lemma 2.4. Let $\sigma \in \mathbb{N}$ and $a, f : \mathbb{T}^d \to \mathbb{R}$ be smooth functions. Then for every $r \in [1, \infty]$,

$$\|a f(\sigma \cdot)\|_r - \|a\|_r \|f\|_r \simeq \sigma^{-\frac{1}{r}} \|a\|_{C^1} \|f\|_r. \tag{2.2}$$

This result allows us to achieve sharp $L^r$ estimates when estimating the perturbations in Section 8. Note that the error term on the right-hand side can be made arbitrarily small by increasing the oscillation $\sigma$. 
2.4. Mean values and oscillations. We use the following Riemann-Lebesgue type lemma.

**Lemma 2.5.** Let \( \sigma \in \mathbb{N} \) and \( a, f : \mathbb{T}^d \to \mathbb{R} \) be smooth functions such that \( f \in C_0^\infty (\mathbb{T}^d) \). Then for all even \( n \geq 0 \)
\[
\left| \int_{\mathbb{T}^d} a(x)f(\sigma x) \, dx \right| \lesssim_n \sigma^{-n} \|a\|_{C^n} \|f\|_2.
\] (2.3)

**Proof.** Since \( f \) has zero mean, by repeatedly integrating by parts we deduce that
\[
\int_{\mathbb{T}^d} a(x)f(\sigma x) \, dx = \sigma^{-n} \int_{\mathbb{T}^d} \Delta^{n/2} a \Delta^{-n/2} f(\sigma \cdot) \, dx.
\]
On one hand, we have
\[
\|\Delta^{n/2} a\|_{L^2(\mathbb{T}^d)} \lesssim \|a\|_{C^n(\mathbb{T}^d)}.
\]
On the other hand, since \( f \) is zero-mean, by the Plancherel theorem
\[
\|\Delta^{-n/2} f\|_{L^2(\mathbb{T}^d)} \lesssim \|f\|_{L^2(\mathbb{T}^d)}.
\]
Thus for any even \( n \) we have
\[
\left| \int_{\mathbb{T}^d} a(x)f(\sigma x) \, dx \right| \lesssim_n \sigma^{-n} \|a\|_{C^n} \|f\|_2.
\]
\( \square \)

2.5. Mollifier with small low frequencies. It is well known that for the standard mollification \( *\phi_\varepsilon \), the following bound holds
\[
\|f * \phi_\varepsilon - f\|_{L^r} \lesssim \varepsilon \|\nabla f\|_{L^r}.
\]

We would like to obtain an improved bound when the scale of \( f \) is relatively close to \( \varepsilon \). To this end, we can either use cutoffs on the Fourier side or use the standard mollifier arising from a bump function with vanishing moments. To avoid technicalities of the harmonic analysis, we choose the latter approach in the paper.

Let \( \varphi \in C_0^\infty (\mathbb{R}^d) \) be a bump function such that \( \int_{\mathbb{R}^d} \varphi = 1 \) and
\[
\int_{\mathbb{R}^d} x^\beta \varphi = 0 \quad \text{for all multi-index } \beta \text{ such that } 1 \leq |\beta| \leq N,
\]
for some integer \( N \in \mathbb{N} \). Such a bump function \( \varphi \) can be obtained by differentiating a standard bump function and then normalizing it to have integral 1.

Consider the mollifier \( \varphi_\varepsilon = \varepsilon^{-d} \varphi(x/\varepsilon) \) and denote the mollification by \( f_\varepsilon = f * \phi_\varepsilon \). We have the following.

**Lemma 2.6.** Let \( 1 \leq r \leq \infty \). For any \( f \in C_0^\infty (\mathbb{R}^d) \), there holds
\[
\|f_\varepsilon - f\|_{L^r} \lesssim \varepsilon^N \|\nabla^N f\|_{L^r}.
\] (2.4)

**Proof.** This simply follows by the Taylor expansion. Indeed, we have
\[
f_\varepsilon - f = \int_{\mathbb{R}^d} \left[ f(x-y) - f(x) \right] \varphi_\varepsilon(y) \, dy.
\] (2.5)
By Taylor’s theorem, we have
\[
f(x-y) - f(x) = \sum_{1 \leq |\beta| \leq N-1} \frac{D^\beta f(x)}{\beta!} (-y)^\beta + \sum_{|\beta| = N} R_\beta(x,y)(-y)^\beta
\] (2.6)
where the remainder
\[
R_\beta(x,y) = \frac{|\beta|}{\beta!} \int_0^1 (1-s)^{|\beta|-1} D^\beta f(x - sy) \, ds.
\]
By the vanishing moments of the mollifier \( \varphi_\varepsilon \), from (2.5) and (2.6) we get
\[
f_\varepsilon - f = \sum_{|\beta| = N} \int_{\mathbb{R}^d} R_\beta(x,y)(-y)^\beta \varphi_\varepsilon(y) \, dy.
\]
Since for any \( y \in \mathbb{R}^d \), \( |\beta| = N \) implies that
\[
\|R_\beta(\cdot,y)\|_{L^r(\mathbb{R}^d)} \lesssim \|\nabla^N f\|_r,
\]
by Minkowski’s inequality it follows that
\[ \|f_\varepsilon - f\|_{L^r(\mathbb{R}^d)} \lesssim \|\nabla^N f\|_r \sum_{|\beta|=N} \int_{\mathbb{R}^d} |y^\beta| |\varphi_\varepsilon| \, dy \lesssim \varepsilon^N \|\nabla^N f\|_{L^r}. \]

\[ \hfill \Box \hfill \]

2.6. Inverse Laplacian of Schwartz functions \( S(\mathbb{R}^d) \) with zero moments. At last, we need the following result concerning the inverse Laplacian, which is a crucial ingredient when estimating the error term with time derivative.

**Lemma 2.7.** Suppose \( f \in S(\mathbb{R}^d) \) and
\[ \int f(x) x^\alpha \, dx = 0 \quad \text{for } 0 \leq |\alpha| \leq 1, \tag{2.7} \]
then \( h := \Delta^{-1} f \in W^{m,r}(\mathbb{R}^d) \cap C^\infty(\mathbb{R}^d) \) for all \( 1 < r \leq \infty \) and \( m \geq 0 \).

The necessity of small low frequencies can be seen on the Fourier side: inverting the Laplacian needs a decay \( \sim |\xi|^2 \) near 0 frequency, and (2.7) guarantees exactly this. See appendix A for a proof.

As a corollary, we obtain the following key estimate that will be crucial when estimating the defect field of time derivative in Lemma 9.1.

**Corollary 2.8.** Suppose \( f \in S(\mathbb{R}^d) \) is radial, then
\[ h := R f \in W^{m,r}(\mathbb{R}^d) \cap C^\infty(\mathbb{R}^d), \]
for \( 1 < r \leq \infty \) and \( m \geq 0 \). Here \( R := \Delta^{-1} \nabla \) is defined via the Newtonian potential in \( \mathbb{R}^d \).

**Proof.** Since \( \nabla f \) has zero mean by default, we only need to verify that all first moments vanish.

Since \( f \) is radial, for any \( i = 1, \ldots, d \) we have by a change of variable \( x \mapsto -y \)
\[ \int_{\mathbb{R}^d} \nabla f(x)x_i \, dx = -\int_{\mathbb{R}^d} \nabla f(y)y_i \, dy, \]
which implies that \( \nabla f \) has zero first moments and hence satisfies the conditions of Lemma 2.7.

\[ \hfill \Box \hfill \]

3. The main proposition and proof of Theorem 1.4

3.1. Time-periodic continuity-defect equation. We follow the framework of [MS18] to obtain approximate solutions to the transport equation by solving

\[ \begin{cases} \partial_t \rho + \div(\rho u) = \div R \\ \div u = 0, \end{cases} \tag{3.1} \]

where \( R : [0, T] \times \mathbb{T}^d \to \mathbb{R}^d \) is called the defect field. In what follows, \( (\rho, u, R) \) will denote a solution to (3.1).

Throughout the paper, we assume \( T = 1 \) and identify the time interval \([0, 1]\) with 1-dimensional torus. As a result, we will only consider smooth solutions \( (\rho, u, R) \) to (3.1) that are time-periodic as well, namely
\[ \rho(t+k) = \rho(t), \quad u(t+k) = u(t), \quad R(t+k) = R(t) \quad \text{for any } k \in \mathbb{Z}. \]

For any \( r > 0 \), let
\[ I_r := [r, 1-r]. \]

We now state the main proposition of the paper and use it to prove Theorem 1.4.

**Proposition 3.1.** Let \( d \geq 2 \) and \( p, q \in [1, \infty] \) satisfying \( p > 1 \) and (1.4). There exist a universal constant \( M > 0 \) and a large integer \( N \in \mathbb{N} \) such that the following holds.

Suppose \( (\rho, u, R) \) is a smooth solution of (3.1) on \([0, 1]\). Then for any \( \delta, \nu > 0 \), there exists another smooth solution \( (\rho_1, u_1, R_1) \) of (3.1) on \([0, 1]\) such that the density perturbation \( \theta := \rho_1 - \rho \) and the vector field perturbation \( w = u_1 - u \) verify the estimates:

\[ \|\theta\|_{L^1 L^p} \leq \nu M \|R\|_{L^1 L^p}^{1/p}, \tag{3.2} \]
\[ \|w\|_{L^\infty L^{p'}} \leq \nu^{-1} M \|R\|_{L^1 L^p}^{1/p'}, \tag{3.3} \]
\[ \|w\|_{L^1 W^{1,q}} \leq \delta. \tag{3.4} \]
In addition, the density \( \theta \) has zero spacial mean and satisfies
\[
\left| \int_{\mathbb{T}^d} \theta(t, x) \varphi(x) \, dx \right| \leq \delta \|\varphi\|_{C^N}, \quad \text{for any} \ t \in [0, 1] \text{ and any} \ \varphi \in C^\infty(\mathbb{T}^d),
\] (3.5)
\[
sup \theta \subset I_r \times \mathbb{T}^d \quad \text{for some} \ r > 0.
\] (3.6)

Moreover, the new defect field \( R_1 \) verifies
\[
\|R_1\|_{L^1_t \chi} \leq \delta.
\] (3.7)

3.2. Proof of Theorem 1.4.

Proof. We assume \( T = 1 \) without loss of generality. We will construct a sequence \( (\rho_n, u_n, R_n), \ n = 1, 2 \ldots \) of solutions to (3.1) as follows. For \( n = 1 \), we set
\[
\rho_1(t) := \bar{\rho}, \\
u_1(t) = 0, \\
R_1(t) := \mathcal{R}(\partial_t \bar{\rho}).
\]

Then \( (\rho_1, u_1, R_1) \) solves (3.1) trivially by the constant mean assumption on \( \bar{\rho} \).

Next, we apply Proposition 3.1 inductively to obtain \( (\rho_n, u_n, R_n) \) for \( n = 2, 3 \ldots \) as follows. Let
\[
\nu = \frac{\varepsilon}{2M} \|R_1\|_{L^1_t \chi}^{-\frac{1}{p'}}, \quad \delta_n := 2^{-p(n-1)}\|R_1\|_{L^1_t \chi},
\]
where we note that \( 1 < p, p' < \infty \) by the assumptions on \( p, q \).

Given \( (\rho_n, u_n, R_n) \), we apply Proposition 3.1 with parameters \( \nu \) and \( \delta_n \) to obtain a new triple \( (\rho_{n+1}, u_{n+1}, R_{n+1}) \).

Then the perturbations \( \theta_n := \rho_{n+1} - \rho_n \) and \( w_n := u_{n+1} - u_n \) verify
\[
\|\theta_n\|_{L^1_t L^p} \leq M\nu \delta_n^{\frac{1}{p'}}, \\
\|w_n\|_{L^\infty L^{p'}} \leq M\nu^{-1} \delta_n^{\frac{1}{p'}},
\]
and
\[
\|w_n\|_{L^1 W^{1,q}} \leq \delta_n, \\
sup \theta_n \subset I_{r_n} \times \mathbb{T}^d \quad \text{for some} \ r_n > 0,
\]
for all \( n = 1, 2 \ldots \) So there exists \( (\rho, u) \in L^1 L^p \times L^\infty L^{p'} \) such that
\[
\rho_n \to \rho \quad \text{in} \ L^1 L^p, \\
u_n \to u \quad \text{in} \ L^\infty L^{p'} \cap L^1 W^{1,q}.
\] (3.8) (3.9)

It is standard to prove \( (\rho, u) \) is a weak solution to (1.1) since
\[
\rho_n u_n \to \rho u \quad \text{in} \ L^1_t \chi.
\]

Moreover,
\[
\|\rho - \bar{\rho}\|_{L^1 L^p} \leq \sum_{n \geq 1} \|\theta_n\|_{L^1 L^p} \leq \sum_{n \geq 1} \varepsilon 2^{-n} \leq \varepsilon.
\]

To show that \( \rho(t) \) is continuous in the sense of distributions, let \( \varphi \in C^\infty_c(\mathbb{T}^d) \). It follows that
\[
\langle \rho(t) - \rho(s), \varphi \rangle \leq \left| \langle \rho(t) - \rho_n(t), \varphi \rangle \right| + \left| \langle \rho_n(t) - \rho_n(s), \varphi \rangle \right| + \left| \langle \rho_n(s) - \rho(s), \varphi \rangle \right|.
\]

Since by (3.4)
\[
\left| \langle \rho(t) - \rho_n(t), \varphi \rangle \right| \leq \sum_{k \geq n+1} \delta_n \|\varphi\|_{C^N} \quad \text{for all} \ t \in [0, 1],
\]
the continuity of \( \rho \) in distribution follows from the smoothness of \( \rho_n \).

The claim that \( \rho(t) = \bar{\rho}(t) \) for \( t = 0, 1 \) follows from the fact that \( \rho_n(0) = \rho(0) \) and \( \rho_n(1) = \rho(1) \) for all \( n \) since
\[
sup \theta_n \subset I_{r_n} \times \mathbb{T}^d.
\]

\( \square \)
4. A Stationary solution of the transport equation

In this section, we introduce an incompressible stationary vector field $W$ and the associated scalar density $\Phi$ that are exact stationary solutions to the transport equation. Even though $(\Phi, W)$ does not have a compact support, it is highly concentrated in the whole space $\mathbb{R}^d$ with Schwartz decay and can be fully intermittent. The density $\Phi$ is, in fact, Schwartz, and the incompressible vector field $W$ is smooth away from a line segment.

4.1. General setup. Here and throughout this section, denote the unit vector fields

$$e_z = e_d, \quad e_r(x) = \frac{\sum_{1 \leq i \leq d-1} x_i e_i}{\sum_{1 \leq i \leq d-1} x_i e_i},$$

where $e_i, 1 \leq i \leq d$ are the standard Euclidean basis and $x_i$ the Euclidean coordinates.

We seek a divergence-free vector field $W$ and a scalar density $\Phi$ in the form of

$$W = \eta(z)\psi(r)e_z - \eta'(z)\phi(r)e_r, \quad \Phi = \eta(z)\psi(r),$$

with the goal to find radial profiles $\psi$ and $\phi$ so that

- $W$ is divergence-free: $\text{div}(W) = 0$;
- The transport term vanishes $(W \cdot \nabla)\Phi = 0$;

i.e., $(\Phi, W)$ is a stationary solution of the transport equation (1.1). This reduces to the system of nonlinear ODEs

$$\begin{cases}
  r^{2-d} \frac{d}{dr} (r^{d-2} \phi) - \psi = 0, \\
  \phi \frac{d\psi}{dr} - \psi^2 = 0.
\end{cases}$$

It turns out that this system has solutions with very fast decay in dimensions $d \geq 2$.

For the profile $\eta(z)$, there is no restriction at all, and one can simply use any bump function. We also note that the $e_z$-profile $\eta(z)\psi(r)$ in (4.1) is designed so that the scale of $\eta$ is much larger than $\psi$, as $e_r$ component $\eta'(z)\phi(r)$ serves as a small divergence-free corrector.

4.2. Solving the profiles $\psi$ and $\phi$. Suppose $\psi \in C^\infty([0, \infty))$ is a smooth decaying function and let

$$\phi(r) := -\frac{1}{r^{d-2}} \int_r^\infty \psi(s)s^{d-2} ds,$$

(4.3)

to ensure that $W$ in (4.1) is divergence-free. If $\psi(r)$ decays sufficiently fast at infinity, then $\phi \in C^\infty((0, \infty))$ with a singularity $r^{2-d}$ near the origin. In particular,

$$\phi \in L^r(r^{d-1} dr), \quad 1 \leq r < \frac{d-1}{d-2}.$$  \quad (4.4)

Remark 4.1. We remark that in 2D, both $\psi$ and $\phi$ will be Schwartz, and the constructed solutions are smooth on $\mathbb{R}^2$. However, we will not take advantage of this in the paper.

The exact form of $\psi$ is determined by the following integro-differential equation, reflecting the vanishing transport,

$$\phi \psi' - \psi^2 = 0,$$

(4.5)

which will be solved in the next lemma.

Lemma 4.2. Suppose $d \geq 2$. All solutions of (4.3) and (4.5) are given by

$$\psi(r) = Ce^{-r^{d-1}}, \quad \phi(r) = \frac{-C}{c(d-1)r^{d-2}} e^{-r^{d-1}}.$$

where $C \in \mathbb{R}$ and $c > 0$. 
Proof. By definitions, we start with
\[
\frac{r^{d-2} \psi^2}{\psi'} = - \int_r^\infty \psi(s)s^{d-2} \, ds. \tag{4.6}
\]
Differentiating gives
\[
\frac{[(d-2)r^{d-3} \psi^2 + 2r^{d-2} \psi \psi'] \psi'}{(\psi')^2} - r^{d-2} \psi'' = \psi r^{d-2}.
\]
For any nontrivial solution there exists \( r \) with \( \psi'(r) \neq 0 \), in which case we have
\[
(d-2) \psi^2 \psi' + r \psi(\psi')^2 - r \psi^2 \psi'' = 0. \tag{4.7}
\]
Let us introduce an auxiliary function
\[
h := \frac{\psi}{\psi'}.
\]
Then (4.7) is equivalent to
\[
(d-2)h + rh' = 0,
\]
or
\[
h' + \frac{d-2}{r} h = 0,
\]
which can be solved explicitly
\[
h = c_0 r^{2-d}.
\]
Thus we may now solve for \( \psi \) and get
\[
\psi(r) = Ce^{r^{d-1}}.
\]
Here \( C \in \mathbb{R} \), but \( c \) is an arbitrary negative constant since \( \psi(r) \) is required to decay at infinity by (4.3). The profile \( \phi \) is then obtained by (4.3). \( \Box \)

In the sequel, we fix the profiles \( \psi = e^{-r^{d-1}} \) and \( \phi = \frac{-1}{(d-1)r^{\frac{d-1}{2}}} e^{-r^{d-1}} \) provided by the above lemma and also a smooth function \( \eta \in C_c^\infty((0,1]) \) so that
\[
\int_{\mathbb{R}} \eta(z) \, dz = 0, \quad \int_{\mathbb{R}} \eta^2(z) \, dz \int_{\mathbb{R}^{d-1}} \psi^2(|y|) \, dy = 1. \tag{4.8}
\]

4.3. Rescaled profiles with two scales. We rescale the profiles \( \eta, \psi \) and \( \phi \) to \( \eta^\tau, \psi^\tau, \psi^\mu \) and \( \phi^\mu \) so that \( \psi^\mu \) and \( \phi^\mu \) still verify the integral-differential equation 4.7, and the final density \( \| \Phi \|_p \sim 1 \) and the final vector field \( \| W \|_{p'} \sim 1 \). Recall that throughout the paper, the value of \( p \) is always fixed.

Let us introduce the radial coordinate associated with \( e_r \), namely the distance to the \( x_d \)-axis,
\[
r = \left| \sum_{1 \leq i \leq d-1} x_i e_i \right| = \sqrt{x_1^2 + \cdots + x_{d-1}^2}.
\]

Definition 4.3. Let \( \eta^\tau, \psi^\mu, \psi^\mu, \phi^\mu \in S(\mathbb{R}^{d-1}) \) and \( \phi^\mu \in C_c^\infty(\mathbb{R}^{d-1} \setminus 0) \) be defined by
\[
\eta^\tau = \tau^{\frac{d-1}{2}} \eta(\tau x_d), \quad \psi^\mu = \mu^{\frac{d-1}{2}} \psi(\mu r),
\]
and
\[
\phi^\mu = \mu^{-1 + \frac{d-1}{r}} \phi(\mu r). \tag{4.10}
\]

Remark 4.4. The profiles \( \eta^\tau \) and \( \psi^\mu \) will be used for the density \( \Phi \) while the profiles \( \eta^\tau, \psi^\mu \) and \( \phi^\mu \) will be used for the vector field \( W \).

We can easily prove the following simple lemma regarding the \( L^r \) scalings of the profiles.
Lemma 4.5. The rescaled functions $\eta^\tau$, $\eta^\tau_\mu$, $\psi_\mu$, $\tilde{\psi}_\mu$ and $\phi^\mu$ verify the identities
\[
\frac{\partial (r^{d-2} \phi^\mu)}{\partial r} = r^{d-2} \psi^\mu, \quad (\psi^\mu \tilde{\psi}_\mu) - \phi^\mu (e_r \cdot \nabla) \tilde{\psi}_\mu = 0,
\] (4.11)
and
\[
\int_{\mathbb{R}} \eta^\tau (x_d) \eta^\tau (x_d) \ dx_d \int_{\mathbb{R}^{d-1}} \psi^\mu (x') \tilde{\psi}_\mu (x') \ dx' = 1
\] (4.12)
Moreover, for any $1 \leq r \leq \infty$, there hold
\[
\|\eta^\tau\|_{L^r(\mathbb{R})} \lesssim r^{1/p' - 1/r}, \quad \|\eta^\tau\|_{L^r(\mathbb{R})} \lesssim r^{1/p' - 1/r}
\] (4.13)
and if $1 \leq r < \frac{d-1}{d-2}$, then holds
\[
\|\phi^\mu\|_{L^r(\mathbb{R}^{d-1})} \lesssim r^{-1 + \frac{d-1}{d-2} - \frac{d-1}{d-2}}.
\] (4.14)
Proof. The first two identities (4.11) follow from the definitions (4.3),(4.8), (4.9), the ODE (4.5), and the fact that $(e_r \cdot \nabla)f = f'(r)$ for any scalar function $f$.

The third identity (4.12) follows from rescaling (4.8).
The first set of estimates (4.13) follows from rescaling and the the fact that $\eta, \psi \in L^r(\mathbb{R})$ for all $1 \leq r \leq \infty$ thanks to $\eta \in C^\infty(\mathbb{R}^+)$ and $\psi \in \mathcal{S}(\mathbb{R})$, while (4.14) follows from rescaling the bound (4.4).

4.4. Smooth density $\Phi$ and singular flow $W$ in $\mathbb{R}^d$. We are ready to define the exact solution $(\Phi, W)$ in $\mathbb{R}^d$.

Definition 4.6. For any $\mu \geq \tau \geq 1$, define the scalar density $\Phi : \mathbb{R}^d \to \mathbb{R}$ by
\[
\Phi(x) = \eta^\tau (x_d) \tilde{\psi}_\mu (r),
\]
and the vector field $W : \mathbb{R}^d \to \mathbb{R}^d$ by
\[
W = W_z + W_r,
\]
where the vector fields $W_z$ and $W_r$ are respectively defined by
\[
W_z = \eta^\tau (x_d) \tilde{\psi}_\mu (r) e_z, \quad W_r = -\frac{\partial \eta^\tau}{\partial x_d} \phi^\mu (r) e_r.
\]
We remark that from Definition 4.3 and Definition 4.6 it follows that
\[
\|\Phi\|_{L^r(\mathbb{R}^d)} \sim r^{\frac{1}{p'} - \frac{1}{\mu} \frac{d-1}{d-2} - \frac{d-1}{d-2}}.
\] (4.15)
\[
\|W_z\|_{L^r(\mathbb{R}^d)} \sim r^{\frac{1}{p'} - \frac{1}{\mu} \frac{d-1}{d-2} - \frac{d-1}{d-2}}.
\] (4.16)
and for $1 \leq r < \frac{d-1}{d-2}$
\[
\|W_r\|_{L^r(\mathbb{R}^d)} \sim r^{-1 + \frac{1}{p'} - \frac{1}{\mu} \frac{d-1}{d-2} - \frac{d-1}{d-2}}.
\] (4.17)
As discussed before, the role of each parameter is as follows.
- $\mu^{-1}$ is the scale of the radius to the $x_d$-axis.
- $\tau^{-1}$ is the scale of the length along the $x_d$-axis.

Note that $W$ is divergence-free by design. Indeed, using standard vector calculus we compute
\[
\text{div}(W) = \text{div}(\eta^\tau \psi^\mu e_z) - \frac{\partial \eta^\tau}{\partial z} \phi^\mu e_r
\]
\[
= \frac{\partial \eta^\tau}{\partial z} \psi^\mu - \frac{\partial \eta^\tau}{\partial z} \text{div}(\phi^\mu e_r)
\]
\[
= 0,
\]
thanks to (4.11) and the identity
\[
\text{div}(\phi^\mu e_r) = \frac{1}{r^{d-2}} \frac{\partial (r^{d-2} \phi^\mu)}{\partial r}
\]
As a direct consequence of Definition 4.3, we obtain the main result of this section, which characterizes the properties of the solution \((\Phi, W)\).

**Theorem 4.7** (Exact stationary solution \((\Phi, W)\)). The density \(\Phi : \mathbb{R}^d \to \mathbb{R}\) and the vector field \(W : \mathbb{R}^d \to \mathbb{R}^d\) verify the following.

1. \(\Phi \in \mathcal{S}(\mathbb{R}^d)\) and the vector field \(W \in L^r(\mathbb{R}^d)\) (provided \(1 \leq r < \frac{d+1}{2}\)) have zero mean
   \[
   \int_{\mathbb{R}^d} \Phi \, dx = \int_{\mathbb{R}^d} W \, dx = 0. \tag{4.18}
   \]

2. The pair \((\Phi, W)\) solves the stationary transport equation,
   \[
   \text{div}(\Phi W) = (W \cdot \nabla) \Phi = 0. \tag{4.19}
   \]

3. For any \(\mu \geq \tau \geq 1\), \((\Phi, W)\) verifies
   \[
   |\nabla \Phi| + |\nabla \Phi| \lesssim_{l, m, n} \mu^{-\tau} |x|^{-m} \quad \text{for all } |x| \geq 1/4. \tag{4.20}
   \]

4. Moreover, there exists a vector potential \(\Theta : \mathbb{R}^d \to \mathbb{R}^d\) such that
   \[
   \text{div} \, \Theta = \Phi,
   \]
   and for any \(1 < r \leq \infty\) and \(k \in \mathbb{N}\), the vector potential \(\Theta \in W^{m, r}(\mathbb{R}^d) \cap C^\infty(\mathbb{R}^d)\) with
   \[
   \|\Theta\|_{L^r(\mathbb{R}^d)} \lesssim r^{-\frac{d}{2} - 1} \mu^{-\frac{d+1}{2} - \frac{d}{2}} \quad \text{for all } 1 < r \leq \infty.
   \]

**Proof.** We shall proceed with the proof in several steps, proving the properties one by one.

1. **Proof of (4.18)**
   By definition, \(\Phi \in \mathcal{S}(\mathbb{R}^d)\). From Lemma 4.5 it follows that \(W \in L^r(\mathbb{R}^d)\) for \(1 \leq r < \frac{d+1}{2}\).
   The zero-mean property (4.18) follows from integrating along \(e_z\) and the fact that the profile function \(\eta \in C^\infty_c(\mathbb{R})\) used in (4.9) has zero mean, i.e. (4.8).

2. **Proof of (4.19)**
   Since \(W = W_z + W_r\) is divergence-free, by a direct computation we conclude
   \[
   \text{div}(\Phi W) = (W_z + W_r) \cdot \nabla \Phi
   \]
   (by definitions)
   \[
   = \left[\eta^\mu \frac{\partial \eta^\mu}{\partial z} \psi^\mu - \frac{\partial \eta^\mu}{\partial z} \phi^\mu \eta^\tau (e_r \cdot \nabla) \psi^\mu\right] e_z
   \]
   \[
   = \frac{1}{2} \frac{\partial (\eta^\mu)^2}{\partial z} \left(\psi^\mu \right)^2 - \phi^\mu \frac{\partial \psi^\mu}{\partial r} \right) e_z
   \]
   (by Lemma 4.5) = 0.

3. **Proof of (4.20)**
   Next, we show the decay for \(W_z\) and \(W_r\) since \(\Phi\) is just the profile of \(W_z\). For simplicity of presentation, we assume \(l = 0\) and other cases can be derived easily from the case \(l = 0\) since the proof merely relies on the superexponential of \(\psi\).
   By (4.13), we have
   \[
   |W_z| \lesssim_m \frac{\phi^\mu}{\mu} |r|^{-\mu |x|^{-m}} |\psi^\mu| \quad \text{for all } |x| \geq 1. \tag{4.21}
   \]
   Recall that \(\psi^\mu = \mu^\frac{d-1}{2} e^{-(\mu r)^{d-1}}\). Clearly by superexponential
   \[
   e^{-(\mu r)^{d-1}} \lesssim_m r^{-m} \quad \text{for all } r \geq 1.
   \]
   Then for \(\mu r \geq \mu \geq 1\) a direct computation shows
   \[
   e^{-(\mu r)^{d-1}} \lesssim_m |\mu r|^{-m}
   \]
   \[
   \lesssim_{m, m} \mu^{-n} |r|^{-m} \quad \text{(by varying } m)\tag{4.22}
   \]
   and hence
   \[
   |\psi^\mu| \lesssim_{n, m} \mu^{-n} |r|^{-m}. \tag{4.23}
   \]
Since $\tau \leq \mu$, this combined with (4.21) implies
\[ |W_z| \lesssim_{n,m} \mu^{-n}|x|^{-m} \quad \text{for all } |x| \geq 1/4. \]

For $W_r$, by the compact support of $\eta$ we also have
\[ |W_r| \lesssim_{\tau, \mu} \mu^{-1+\frac{1}{p'}}|x|^{-m}|\phi\mu|. \] (4.24)

Since the profile function $\phi$ also has superexponential decay at infinity, it then follows from the same argument that
\[ |\phi\mu| \lesssim_{n,m} \mu^{-\frac{d+1}{p'}}\mu^{-n}r^{-m} \quad \text{for all } |x| \geq 1/4, \] (4.25)

which finishes the proof of the decay estimates.

(4) The vector potential $\Theta$

Finally, we construct the vector potential $\Theta$. First, since $\psi\mu \in S(\mathbb{R}^{d-1})$ is radially symmetric, by Corollary 2.8, there exists a vector potential $\Theta \in W^{m,r}(\mathbb{R}^{d-1}) \cap C^\infty(\mathbb{R}^{d-1})$ for all $m \geq 0$ and $1 < r \leq \infty$ such that
\[ \text{div } \tilde{\Theta} = \psi\mu. \]

We can then define
\[ \Theta := \eta^\tau \tilde{\Theta}. \]

Since the vector $\tilde{\Theta}$ is orthogonal to $e_z$, we have
\[ \text{div } \Theta = \text{div}(\eta^\tau \tilde{\Theta}) = \eta^\tau \text{div } \tilde{\Theta} = \eta^\tau \psi\mu = \Phi. \]

The estimates for the potential $\Theta$ then follow from (4.15) and Lemma 2.1.

\[ \square \]

5. Construction of the periodic building blocks

In this section, we use the previously constructed solution $(\Phi, W)$ on $\mathbb{R}^d$ to obtain periodic approximate solutions $(\Phi_k, W_k)$ which will be used in the convex integration scheme on $T^d$. The advantage of this approach is that there are no infinitesimal frequencies on the torus and it is easier to achieve intermittent oscillations on $T^d$ than $\mathbb{R}^d$. We proceed in three steps.

- We first remove the singularity of $W$ by mollification.
- Then we use translations and rotations to obtain a total of $d$ pairs of solutions.
- Finally we apply the Poisson summation formula to obtain the periodic solutions.

The schematic procedure is the following.

- $(\Phi, W) \xrightarrow{\text{radial mollification}} (\Phi', W')$
- $(\Phi', W') \xrightarrow{\text{translation and rotation}} (\Phi_k, W_k)$
- $(\Phi_k, W_k) \xrightarrow{\text{periodization and normalization}} (\Phi_k, W_k)$

After these procedures, the main result in this section is summarized in Theorem 5.14.

In what follows, we will use the same small constant $\gamma > 0$ for various purposes. This small parameter $\gamma > 0$ will only depend on $d, p, q$ in the main proposition and will be fixed in Section 7.

5.1. Removing singularity by mollification. The flow $W$ and density $\Phi$ introduced above have many desired properties, but $W$ is not in $L^p$ due to the singularity of $W_r$. To remove the singularity, we need a mollification so that the resulting solution $(\Phi, W)$ does not deviate too much from $(\Phi, W)$. To this aim, we introduce a mollifier with small low frequencies.

Let us introduce the standard multiindex notation. Let $\beta = (\beta_1, \ldots, \beta_d) \in \mathbb{N}^d$ and we write
\[ x^\beta := x_1^{\beta_1}x_2^{\beta_2} \cdots x_d^{\beta_d}. \]

The order of $\beta$ is denoted by $|\beta| := \beta_1 + \beta_2 + \cdots + \beta_d$. 

4.7 \( R \).

\[ \Phi \leq R \] and \( \Phi \leq 5.5 \). Note that we will only mollify the first \( d - 1 \) coordinates, which means we only mollify the radial part of \( (\Phi, W) \).

**Definition 5.2.** Define mollified profiles \( \psi_{\gamma}^\mu, \phi_{\gamma}^\mu \in C^\infty(\mathbb{R}^{d-1}) \) by

\[
\begin{align*}
\psi_{\gamma}^\mu(r) &:= (\psi^\mu)_\gamma(r), \\
\phi_{\gamma}^\mu(r) &:= (\phi^\mu)_\gamma(r).
\end{align*}
\]

(5.1)

(5.2)

Let the mollified solution \( (\overline{\Phi}, \overline{W}) \) be

\[
\begin{align*}
\overline{\Phi} &= \Phi, \\
\overline{W} &= (W)_\gamma = \overline{W}_z + \overline{W}_r = \eta^\tau(x_d)\psi_{\gamma}^\mu(r)e_z - \frac{\partial \eta^\tau}{\partial z}(x_d)\phi_{\gamma}^\mu(r)e_r.
\end{align*}
\]

(5.3)

(5.4)

The next result shows that the transport error coming from the mollification is suitably small.

**Theorem 5.3.** There exists a vector \( \overline{R} \in C^\infty(\mathbb{R}^d \setminus \{x_d = 0\}) \) such that

\[ \text{div}(\overline{\Phi} \overline{W}) = \text{div} \overline{R}. \]

Moreover, for any \( 1 \leq r < \frac{d - 1}{d - 2} \) the vector \( \overline{R} \) verifies

\[
\| \overline{R} \|_{L^r(\mathbb{R}^d)} \lesssim_r \mu^{d - 1 - \frac{d - 1}{\tau}} \mu^{-1}.
\]

(5.5)

**Proof.** Let

\[
\overline{R} := (\overline{\Phi} \overline{W}_z - \Phi W_z) + (\overline{\Phi} \overline{W}_r - \Phi W_r).
\]

Thanks to Theorem 4.7, \( \text{div}(\Phi W) = 0 \), and hence we have

\[ \text{div}(\overline{\Phi} \overline{W}) = \text{div} \overline{R}. \]

The estimate (5.5) follows from Lemmas 5.4 and 5.5 below. \( \square \)

**Lemma 5.4.** There holds

\[
\| \overline{\Phi} \overline{W}_z - \Phi W_z \|_{L^\infty(\mathbb{R}^d)} \lesssim \mu^{-N\gamma \tau} \mu^{d - 1}.
\]

(5.6)

**Proof.** The estimate follows by applying Lemma 2.6. Indeed,

\[
\| \overline{\Phi} \overline{W}_z - \Phi W_z \|_{L^\infty(\mathbb{R}^d)} \lesssim \| \overline{\Phi} \|_{L^\infty(\mathbb{R}^d)} \| \overline{W}_z - W_z \|_{L^\infty(\mathbb{R}^d)}.
\]

(5.7)

Now using Lemma 2.6,

\[
\| \overline{W}_z - W_z \|_{L^\infty(\mathbb{R}^d)} \lesssim \| \eta \|_{L^\infty(\mathbb{R}^d)} \| \psi_{\gamma}^\mu - \psi^\mu \|_{L^\infty(\mathbb{R}^{d-1})} \lesssim \mu^{-N\gamma \tau} \mu^{d - 1},
\]

which implies that

\[
\| \overline{\Phi} \overline{W}_z - \Phi W_z \|_{L^\infty(\mathbb{R}^d)} \lesssim \mu^{-N\gamma \tau} \mu^{d - 1}.
\]

(5.8)

**Lemma 5.5.** For any \( 1 \leq r < \frac{d - 1}{d - 2} \),

\[
\| \overline{\Phi} \overline{W}_r - \Phi W_r \|_{L^r(\mathbb{R}^d)} \lesssim_r \mu^{-1} \mu^{-1} \mu^{d - 1 - \frac{d - 1}{\tau}}.
\]

(5.9)
Proof. By Hölder’s inequality we have
\[ \|\Phi W_r - \Phi W_r\|_{L^r(\mathbb{R}^d)} \lesssim \|\Phi\|_{L^\infty(\mathbb{R}^d)} \|W_r - W_r\|_{L^r(\mathbb{R}^d)}, \]
and by Young’s inequality and (4.14), for \(1 \leq r < \frac{4}{d-2}\) we get
\[ \|W_r - W_r\|_{L^r(\mathbb{R}^d)} \lesssim \|W_r\|_{L^r(\mathbb{R}^d)} \]
\[ \lesssim \left\| \frac{\partial \eta^T}{\partial x_d} \right\|_{L^r(\mathbb{R})} \times \|\phi^\mu\|_{L^r(\mathbb{R}^{d-1})} \]
\[ \lesssim r \tau^\frac{d-1}{r} \mu^{-\frac{d-1}{2d-4} - \frac{1}{r}} \mu^{-1}. \]
Thus
\[ \|\Phi W_r - \Phi W_r\|_{L^r(\mathbb{R}^d)} \lesssim \mu^{d-1-\frac{d-1}{2d-4} - \frac{1}{r}} \mu^{-1}. \]
\[ \square \]

5.2. Geometric setup and periodization. Next, we use the obtained smoothed solutions \((\Phi, \overline{W})\) to generate a family of \(d\) pairs \((\Phi_k, \overline{W}_k)\) by translation and rotations. The goal is to make sure \((\Phi_k, \overline{W}_k)\) centered at disjoint line segments that are parallel to the Euclidean basis \(e_i\). Due to the very simple geometry of the arrangement, we achieve this by a simple change of variables instead of using actual rotations.

We choose a collection of distinct points \(p_i \in [1/4, 3/4]^d\) for \(i = 1, \ldots, d\) and a number \(\varepsilon_0 > 0\) such that
\[ \bigcup_i B_{\varepsilon_0}(p_i) \subset [0, 1]^d, \]
and
\[ \text{dist}(p_i, p_j) \geq \varepsilon_0 \text{ if } i \neq j. \]
(5.10)
The points \(p_i\) will be the centers of our solutions \((\Phi_k, \overline{W}_k)\).

For \(k = 1, \ldots, d\), let us introduce the unit vector field \(e_{r_k} : \mathbb{R}^d \to \mathbb{R}^d\) by
\[ e_{r_k}(x) = \frac{\sum_{i \neq k} x_i e_i}{|\sum_{i \neq k} x_i e_i|}, \]
(5.11)
and its associated coordinates
\[ r_k = \left| \sum_{i \neq k} x_i e_i \right|. \]
(5.12)
We then choose translations \(\overline{x}_k : \mathbb{R}^d \to \mathbb{R}^d\) for \(k = 1, \ldots, d\),
\[ \overline{x}_k x := x + p_k. \]
We can then use the mollified radial profiles \(\psi^\mu_k\) and \(\phi^\mu_k\) to define \(d\)-pairs of translated and rotated solutions \((\Phi_k, \overline{W}_k)\) as follows.

Definition 5.6. For \(k = 1, \ldots, d\), we define \((\Phi_k, \overline{W}_k)\) via the translations and switching variables
\[ \overline{W}_k(\overline{x}_k x) = \eta^T(x_k) \psi^\mu_k(r_k) e_k - \frac{\partial \eta^T}{\partial x}(x_k) \phi^\mu_k(r_k) e_k, \]
\[ \overline{W}_k(\overline{x}_k x) = \eta^T(x_k) \tilde{\phi}^\mu(r_k). \]
We also write
\[ W_k = W_{2k} + W_{r_k}. \]

Theorem 4.7 also holds for \((\Phi_k, \overline{W}_k)\), and we collect them here for future reference.

Corollary 5.7. The smoothed density \(\Phi_k : \mathbb{R}^d \to \mathbb{R}\) and the smoothed vector field \(\overline{W}_k : \mathbb{R}^d \to \mathbb{R}^d\) verify the following.

1. The vector field \(W_k\) is divergence-free, and \(\Phi_k, \overline{W}_k \in \mathcal{S}(\mathbb{R}^d)\) both have zero mean
\[ \int_{\mathbb{R}^d} \Phi_k dx = \int_{\mathbb{R}^d} W_k dx = 0. \]
(5.13)

2. For any \(\mu \geq \tau \geq 1\), \((\Phi_k, \overline{W}_k)\) verifies
\[ |\nabla \Phi_k| + |\nabla \overline{W}_k| \lesssim t, n, \mu^{-n} |x - p_k|^{-m} \text{ for all } |x - p_k| \geq \varepsilon_0/4, \]
(5.14)
(3) There exist vector potentials $\overline{\Omega}_k \in C^\infty(\mathbb{R}^d) \cap W^{m,r}(\mathbb{R}^d)$ for all $1 < r \leq \infty$ and $m \in \mathbb{N}$ such that
\[
\text{div}(\overline{\Omega}_k) = \overline{F}_k,
\]
and
\[
\|\overline{\Omega}_k\|_{L^r(\mathbb{R}^d)} \lesssim_r \mu^{-1} \tau^\frac{d}{r} + \frac{d}{r} \frac{d-1}{r} \quad \text{for } 1 < r \leq \infty.
\] (5.16)

(4) Also there exist vector $\overline{R}_k \in L^r(\mathbb{R}^d)$ for all $1 \leq r < \frac{d-1}{2}$ such that
\[
\text{div}(\overline{F}_k, \overline{W}_k) = \text{div}(\overline{R}_k),
\]
\[
\|\overline{R}_k\|_{L^r} \lesssim_r \tau \mu^{-1} r^{\frac{d-1}{r}} \mu^{d-1} \frac{d-1}{r} \quad \text{for } 1 \leq r < \frac{d-1}{2}.
\] (5.17)

Proof. The first two properties follow from commuting differentiation with mollification and the fact that we have only mollified the radial profiles.

The last property was proved in Theorem 5.3.

For the third property, since the mollification we use is also radial, we can proceed as in Theorem 4.7 to obtain $\overline{\Omega}_k$. The estimate for $\overline{\Omega}_k$ then follows from that of $\Omega$ in Theorem 4.7.

\[\square\]

5.3. Periodization of the smoothed solutions $(\overline{F}_k, \overline{W}_k)$. We are now ready to transfer the smoothed solutions $(\overline{F}_k, \overline{W}_k)$ to periodic solution $(\Phi_k, W_k)$ via the Poisson’s summation formula. The highly concentrated property allows us to pass from $\mathbb{R}^d$ to $\mathbb{T}^d$ with ease.

Definition 5.8. Let $u \in S(\mathbb{R}^d)$. The periodization operator $P : S(\mathbb{R}^d) \to C^\infty(\mathbb{T}^d)$ is defined by
\[
P u := \sum_{m \in \mathbb{Z}^d} u(x + m).
\] (5.18)

We are ready to introduce the periodic solution $(\Phi_k, W_k)$. Note that we use a mild cutoff for the density $\Phi_k$ but not for the vector field $W_k$. This is to keep $W_k$ being divergence-free.

Definition 5.9 (Periodic solutions). Let $\chi_k$ be the cutoff defined by
\[
\chi_k = \chi(\varepsilon_0^{1/2} |x - p_k|),
\]
for some fixed cutoff $\chi \in C^\infty_c([0,1/2])$ such that $\chi(r) = 1$ for $r \leq \frac{1}{4}$.

Define periodic density $\Phi : \mathbb{T}^d \to \mathbb{R}$ and periodic vector fields $W_k : \mathbb{T}^d \to \mathbb{R}^d$ by
\[
\Phi_k := P(\chi_k \overline{F}_k),
\]
\[
W_k := W_{k_1} + W_{k_2},
\]
where $W_{k_1} : \mathbb{T}^d \to \mathbb{R}^d$ and $W_{k_2} : \mathbb{T}^d \to \mathbb{R}^d$ are given by
\[
W_{k_1} = c_\mu P \overline{W}_{k_1}, \quad W_{k_2} = c_\mu P \overline{W}_{k_2}.
\] (5.19)

Here $c_\mu \in [1/2,2]$ is a normalizing constant that will be fixed in Lemma 5.13 depending on $\mu$.

Finally, define the periodic potential $\Omega_k : \mathbb{T}^d \to \mathbb{R}$ by
\[
\Omega_k := P(\chi_k \overline{\Omega}_k),
\]

5.4. Estimates for the periodic solution $(\Phi_k, W_k)$. Here we derive several basic estimates for $(\Phi_k, W_k)$.

Thanks to Corollary 5.7, the “tail” part of the vector $W_k$ is negligible.

Lemma 5.10. Denote by
\[
W^t_k := c_\mu P (1 - \chi k) \overline{W}_k \quad \text{and} \quad \overline{W}_k := c_\mu P \chi_k \overline{W}_k.
\]

Then
\[
\|\nabla^m W^t_k\|_{L^\infty} \lesssim_{l,n,m} \mu^{-n}
\]

Proof. It follows from Corollary 5.7 that
\[
|\nabla^l ((1 - \chi_k) \overline{W}_k)| \lesssim_{l,n,m} \mu^{-n} |x - p_k|^{-m}.
\] (5.20)

Passing to $\mathbb{T}^d$ by taking $m = d + 1$, we have
\[
|\nabla^l W^t_k|_{L^\infty} \lesssim_{l,n} \mu^{-n}.
\] \[\square\]
Proposition 5.11. For any \( \tau, \mu \geq 1 \) such that \( \mu^{1-\gamma d} \geq \tau \), the following estimates hold for any \( 1 \leq r \leq \infty \):
\[
\mu^{-m} \| \nabla^m \Phi_k \|_{L^r(\mathbb{R}^d)} \lesssim_m \mu^{\frac{d+1}{r} - \frac{d+1}{\gamma d} - \frac{1}{\tau} - \frac{1}{r}} , \quad m \in \mathbb{N},
\]
\[
\mu^{-m} \| \nabla^m W_k \|_{L^r(\mathbb{R}^d)} \lesssim_m \mu^{\frac{d+1}{r} - \frac{d+1}{\gamma d} - \frac{1}{\tau} - \frac{1}{r}} 0 \leq m \leq 1.
\]

Remark 5.12. We limit the regularity of \( W_k \) due to a loss of optimal bound by the mollification scale \( \mu^{-1+\gamma} \) when estimating the radial part \( W_{r_k} \). By suitably choosing \( \gamma, N, \tau \) and \( \mu \), it is possible to extend the second estimate up to any fixed order \( m \). However, we only need \( m \leq 1 \) for applications in this paper.

Proof of Proposition 5.11. The bounds for \( \Phi_k \) can be deduced fairly easily as a corollary of the proof of \( W_k \). For this reason, we shall only prove the estimates for \( W_k \).

For the smoothed non-periodic vector field \( \overline{W}_k \), denote the principle part \( \overline{W}_k \) by \( \overline{W}_k \).

Then by linearity
\[
\nabla^m W_k = \nabla^m \overline{W}_{z_k} + \nabla^m W_{r_k} = \nabla^m P(\overline{W}_{z_k} + \overline{W}_{r_k}) + \nabla^m W_{r_k}.
\]

(5.21)

Since the tail part \( \nabla^m W_{r_k} \) has been estimated in Lemma 5.10, we focus on the first two terms in (5.21).

(1) **Bounding \( \nabla^m P(\overline{W}_{z_k}) \):**

Let us first estimate \( \nabla^m \overline{W}_{z_k} \). Due the extra cutoff \( \chi_k \), \( \overline{W}_k \) has compact support inside \([0,1]^d\) and thus by passing from \( \mathbb{T}^d \) to \( \mathbb{R}^d \), we have
\[
\| \nabla^m \overline{W}_{z_k} \|_{L^r(\mathbb{T}^d)} = \| \nabla^m (\chi_k \overline{W}_{z_k}) \|_{L^r(\mathbb{R}^d)},
\]

which can be computed by the product rule
\[
\| \nabla^m \overline{W}_{z_k} \|_{L^r(\mathbb{T}^d)} \leq \sum_i \| \nabla_i^\gamma \chi_k \nabla^m \overline{W}_{z_k} \|_{L^r(\mathbb{R}^d)}.
\]

Since \( \chi_k \) is a mild cutoff compared to \( W_{z_k} \), by a scaling analysis, it suffices to bound \( \| \nabla^m \overline{W}_{z_k} \|_{L^r(\mathbb{T}^d)} \), which can be estimated easily by Definition 4.3 and Lemma 4.5. Thus we obtain the desired bound
\[
\| \nabla^m \overline{W}_{z_k} \|_{L^r(\mathbb{T}^d)} \lesssim m \mu^{\frac{d+1}{r} - \frac{d+1}{\gamma d} - \frac{1}{\tau} - \frac{1}{r}} \quad \text{for all } 1 \leq r \leq \infty \text{ and } m \in \mathbb{N}.
\]

(2) **Bounding \( \nabla^m P(\overline{W}_{r_k}) \):**

Next, we bound \( \nabla^m \overline{W}_{r_k} \) in (5.21). Due to the compact support, by passing to \( \mathbb{R}^d \), we have
\[
\| \nabla^m \overline{W}_{r_k} \|_{L^r(\mathbb{T}^d)} = \| \nabla^m (\chi_k \overline{W}_{r_k}) \|_{L^r(\mathbb{R}^d)}.
\]

Since \( W_{r_k} \) is not smooth, the differentiation shall be estimated by the mollification. We use Young’s inequality to obtain for any \( 1 \leq r \leq \infty \) that
\[
\| \nabla^m W_{r_k} \|_{L^r(\mathbb{R}^d)} \lesssim \mu^{(1+\gamma)m} \| W_{r_k} \|_{L^r(\mathbb{R}^d)} \lesssim \mu^{(1+\gamma)m} \tau^{1-\frac{1}{r}} \mu^{(1+\gamma)(d-1-\frac{d-1}{r})} \| W_{r_k} \|_{L^1(\mathbb{R}^d)}.
\]

It follows from Definition 4.3 and Lemma 4.5 that
\[
\| \nabla^m W_{r_k} \|_{L^r(\mathbb{R}^d)} \lesssim \mu^{m \tau^{1-\frac{1}{r}} \mu^{\frac{d+1}{r} - \frac{d+1}{\gamma d} - \frac{1}{\tau} - \frac{1}{r}}} \quad \text{for all } 1 \leq r \leq \infty \text{ and } m \in \mathbb{N}.
\]

Note that by the assumption
\[
\mu^{\gamma (m+d-1-\frac{d-1}{r})} \leq \mu^\gamma d \leq \tau^{-1} \mu \quad \text{for } 0 \leq m \leq 1.
\]

So we have
\[
\mu^{-m} \| \nabla^m W_{r_k} \|_{L^r(\mathbb{T}^d)} \lesssim \tau^{-1} \mu^{\frac{1}{r} \mu^{\frac{1}{d} \frac{d+1}{r} - \frac{d+1}{\gamma d} - \frac{1}{\tau} - \frac{1}{r}}} \quad \text{for all } 1 \leq r \leq \infty \text{ and } 0 \leq m \leq 1.
\]

□
Lemma 5.13 (Normalized self-interactions). For any sufficiently large \( \tau, \mu \), we can choose constants \( c_\mu \in \left[ \frac{1}{2}, 2 \right] \), \( k = 1, \ldots, d \) in the definition of \( \Phi_k \) such that

\[
\int_{T^d} \Phi_k \Phi_k \, dx = e_k. \tag{5.25}
\]

Proof. Let us first show that for all sufficiently large \( \tau \)

\[
\int_{T^d} \Phi_k \Phi_k \, dx = 0.
\]

Indeed, by definitions we have for \( \tau \) sufficiently large that

\[
\int_{T^d} \Phi_k \Phi_k \, dx = c_\mu \int_{[0,1]^d} \chi_k^2 \chi_k \, dx
\]

(by Definition 5.2) \( = c_\mu \int_{R^d} \chi_k^2 \eta^\tau(x_k) \frac{\partial \eta^\tau}{\partial x_k}(x_k) \psi^\mu_l(x') \phi^\mu_l(x') \, e_{r_k} \, dx \)

(separating variables) \( = c_\mu \int_{R^d} \eta^\tau(x_k) \frac{\partial \eta^\tau}{\partial x_k}(x_k) \, dx_k \int_{R^{d-1}} \chi_k^2 \psi^\mu_l(x') \phi^\mu_l(x') \, e_{r_k} \, dx' \).

Since

\[
\int_{R^d} \eta^\tau(x_k) \frac{\partial \eta^\tau}{\partial x_k}(x_k) \, dx_k = 0,
\]

by the compact support of \( \eta^\tau \), we conclude

\[
\int_{T^d} \Phi_k \Phi_k \, dx = 0.
\]

Therefore, we have

\[
\int_{T^d} \Phi_k \Phi_k \, dx = c_\mu \int_{R^d} \chi_k^2 \chi_k \, dx + \int_{T^d} \Phi_k \Phi_k \, dx
\]

\[
 = c_\mu \int_{R^d} \chi_k \, dx + c_\mu \int_{R^d} (1 - \chi_k^2) \chi_k \, dx + \int_{T^d} \Phi_k \Phi_k \, dx.
\]

It follows from Corollary 5.7 that

\[
\left| \int_{R^d} (1 - \chi_k^2) \chi_k \, dx \right| \lesssim_n \mu^{-n}. \tag{5.27}
\]

and from Lemma 5.10 that

\[
\left| \int_{T^d} \Phi_k \Phi_k \, dx \right| \lesssim_n \mu^{-n}. \tag{5.28}
\]

Putting together (5.26), (5.27), and (5.28), for all sufficiently large \( \tau \) we have

\[
\left| \int_{T^d} \Phi_k \Phi_k \, dx - c_\mu \int_{R^d} \chi_k \, dx \right| \lesssim \mu^{-1}.
\]

Due to (4.12),

\[
\int_{R^d} \Phi \, dx = e_k.
\]

By Lemma 5.4,

\[
\left| \int_{R^d} \chi_k \, dx - \int_{R^d} \Phi \, dx \right| \lesssim \mu^{-1}, \tag{5.29}
\]

which implies

\[
\int_{T^d} \Phi_k \Phi_k \, dx = (c_\mu - f(\mu))e_k \quad \text{for some function } |f(\mu)| \leq C\mu^{-1}.
\]

For sufficiently large \( \mu \), the existence of \( c_\mu \) follows from the above identity.

The next theorem is the main result of this section, which quantifies the transport error due to mollification and periodization, the negligible interactions between distinct pairs \( \Phi_k \) and \( \Phi_k' \), and the fact that \( \Phi_k \) is almost a divergence of a vector potential.
Theorem 5.14 (Approximate periodic solution \((\Phi_k, W_k))\). The periodic solutions \(\Phi_k, W_k \in C^\infty_0(\mathbb{T}^d)\) verify the following.

1. The vector field \(W_k\) is divergence-free,
   \[
   \text{div } W_k = 0;
   \]

2. The density \(\Phi_k\) is almost a divergence of the potential \(\Omega_k\),
   \[
   \| \text{div } \Omega_k - \Phi_k \|_{L^r(\mathbb{T}^d)} \lesssim_r \mu^{-\frac{d}{2}} \rho^{-\frac{d}{p}} - \frac{d-1}{r} \quad \text{for } 1 < r \leq \infty; \tag{5.30}
   \]

3. If \(k \neq k'\), then
   \[
   \| \Phi_k W_{k'} \|_{L^\infty(\mathbb{T}^d)} \lesssim_n \mu^{-n}. \tag{5.31}
   \]

In addition, there exist vectors \(R_k : \mathbb{T}^d \rightarrow \mathbb{R}^d, k = 1, \ldots, d\) such that
   \[
   \text{div}(\Phi_k W_k) = \text{div } R_k
   \]
with estimates
   \[
   \| R_k \|_{L^r(\mathbb{T}^d)} \lesssim_r \tau \mu^{-\frac{d}{2}} \rho^{-\frac{d}{p}} - \frac{d-1}{r} \quad \text{for } 1 < \frac{d}{2}. \tag{5.33}
   \]

**Proof.** Smoothness, zero mean and zero divergence follow from Theorem 4.7, Definition 5.6 and 5.9. We will prove (5.30), (5.33) and (5.31) as follows.

1. **Proof of (5.30).**
   By definition,
   \[
   \text{div } \Omega_k = \text{P}(\nabla \chi_k \cdot \overline{\Omega}_k) + \text{P}(\chi_k \text{div } \overline{\Omega}_k).
   \]
   By Theorem 4.7, we have
   \[
   \text{div } \Omega_k - \Phi_k = \text{P}(\nabla \chi_k \cdot \overline{\Omega}_k).
   \]
   By passing to \(\mathbb{R}^d\), we have
   \[
   \| \text{div } \Omega_k - \Phi_k \|_{L^r(\mathbb{T}^d)} \lesssim \| \overline{\Omega}_k \|_{L^r(\mathbb{R}^d)} \lesssim_r \tau \mu^{-\frac{d}{2}} \rho^{-\frac{d}{p}} - \frac{d-1}{r},
   \]
   which concludes the proof of (5.30).

2. **Definition of \(R_k\):**
   Set
   \[
   R_k = \text{P}(\chi_k \overline{\Omega}_k) - \text{R} \text{P}(\nabla \chi_k \overline{\Omega}_k) + \text{R} \text{P}(\nabla \chi_k \overline{\Phi}_k W_k) + \overline{\Phi}_k W_k. \tag{5.34}
   \]
   Here \(\chi_k\) is the same cutoff as in Definition 5.9 and \(W_k^j\) is as in Lemma 5.10.
   Note that \(R_k : \mathbb{T}^d \rightarrow \mathbb{R}^d\) is well-defined since the terms inside the periodization have compact support.

3. **Proof of (5.33):**
   Taking divergence, ignoring all the intermediate constants, gives
   \[
   \text{div } R_k = \text{P}(\chi_k^2 \text{div } \overline{\Omega}_k) + \text{P}(\nabla \chi_k \overline{\Phi}_k W_k) + \text{div}(\Phi_k W_k^j)
   \]
   (by Corollary 5.7) = \[
   \text{P}(\chi_k^2 \text{div}(\overline{\Phi}_k W_k)) + \text{P}(\nabla \chi_k \overline{\Phi}_k W_k) + \text{div}(\Phi_k W_k^j)
   \]
   (by product rule) = \[
   \text{div } \text{P}(\chi_k \overline{\Phi}_k) \text{P}(\chi_k W_k) + \text{div}(\Phi_k W_k^j)
   \]
   (by compact supports) = \[
   \text{div } \left( \text{P}(\chi_k \overline{\Phi}_k) \text{P}(\chi_k W_k) \right) + \text{div}(\Phi_k W_k^j) = \text{div}(\Phi_k W_k).
   \]
   Next, we bound \(R_k\). Thanks to Lemma 2.1, by passing to \(\mathbb{R}^d\), we obtain
   \[
   \| R_k \|_{L^r(\mathbb{T}^d)} \lesssim \| \chi_k^2 \overline{\Omega}_k \|_{L^r(\mathbb{R}^d)} + \| \nabla \chi_k^2 \overline{\Omega}_k \|_{L^r(\mathbb{R}^d)} + \| \nabla \chi_k^2 \overline{\Phi}_k W_k \|_{L^\infty(\mathbb{R}^d)} + \| \Phi_k W_k^j \|_{L^r(\mathbb{T}^d)}.
   \]
   We can use the bounds from Corollary 5.7 that
   \[
   \| \nabla \chi_k^2 \overline{\Phi}_k W_k \|_{L^\infty(\mathbb{R}^d)} \lesssim_n \mu^{-n};
   \]
and Lemma 5.10 to obtain
\[ \|R_k\|_{L^r(T^d)} \lesssim \| \chi_k^2 R_k \|_{L^r(R^d)} + \mu^{-n} \lesssim r \mu^{-\frac{r}{2}} \frac{1}{\sqrt{r}} \frac{1}{2} \mu^{d-1-\frac{d}{2}} \] for \( 1 \leq r < \frac{d-1}{d-2} \).

(4) **Proof of (5.31):**

By definitions,
\[ \Phi_k W_k' = c_k \chi_k \chi_k' \Phi_k W_k' + \Phi_k W_k' \] for all \( x \in [0,1]^d \).

The conclusion follows immediately from
\[ \chi_k \chi_{k'} = 0 \text{ if } k \neq k' \]
due to (5.10) and Lemma 5.10.

\square

6. Temporal intermittency and oscillation

Here we introduce one of the key ingredients of this paper, the use of both temporal intermittency and oscillation. This allows us to kill the previous defect field in a space-time average fashion instead of point-wise in time. For convenience, we will treat the time interval \([0,1]\) as 1-dimensional torus \( T \). In what follows we always write \([0,1]\) as an interval in time to distinguish it from the periodicity in space.

6.1. Limitations of the previous schemes. We start with discussing how the Sobolev regularity was obtained in previous convex integration schemes [MS18, MS19b, MS19a]. Assume that we have \((\rho, u, R)\) the solution to the defect equation
\[ \partial_t \rho + u \cdot \nabla \rho = \text{div} R, \]
the goal is to design suitable perturbations \((\theta, w)\) such that \((\rho + \theta, u + w)\) is a new solution to the defect equation with a smaller defect field \( R_1 \).

Typical in the convex integration scheme, the principle part of the perturbation \((\theta, w)\) takes the form
\[ \theta = \sum_k a_k \Phi_k \quad w = \sum_k b_k W_k. \]

The coefficients \( a_k, b_k \), depending on the previous defect field \( R \), are chosen such that the leading order high-high to low interaction balance the defect field \( R \)
\[ \sum_k a_k b_k \int_{T^d} \Phi_k W_k dx + R \sim 0. \]

Heuristically, without temporal intermittency, the duality given by the perturbation is
\[ \theta \in L^\infty L^p \quad w \in L^\infty L^{p'}. \]

To require Sobolev regularity of the vector field \( w \), one has to trade in some integrability in space to obtain
\[ w \in L^\infty W^{1,q} \text{ for } q \text{ such that } \frac{1}{p} + \frac{1}{q} > 1 + \frac{1}{d-D} \] (6.3)
where \( D \) is the intermittency dimension of \((\Phi_k, W_k)\). This has been done in [MS18, MS19b] for \( D = 1 \) and in [MS19a, BCL20] for \( D = 0 \).

We emphasize that with this approach of using only spacial intermittency, one can only obtain the nonuniqueness in the range
\[ \frac{1}{p} + \frac{1}{q} > 1 + \frac{1}{d} \]
6.2. Convex integration with space-time intermittency and oscillation. Our approach is to add in temporal intermittency and oscillation to the perturbation \((\theta, w)\),

\[
\theta = \tilde{g}_\kappa \sum_k a_k \Phi_k, \quad w = g_\kappa \sum_k b_k W_k,
\]

(6.4)

where \(\tilde{g}_\kappa, g_\kappa : [0, 1] \to \mathbb{R}\) are intermittent functions in time with oscillations.

By imposing the duality

\[
\int_{[0,1]} \tilde{g}_\kappa g_\kappa dt = 1,
\]

we anticipate that the defect field is canceled weakly in space-time

\[
\sum_k a_k b_k \int_{[0,1] \times \mathbb{T}^d} \tilde{g}_\kappa g_\kappa \Phi_k W_k \, dx dt + R \sim 0.
\]

(6.5)

This would allow us to obtain additional regularity in space at the expense of regularity in time, which means that \(\tilde{g}_\kappa\) and \(g_\kappa\) have different scaling for each \(L^p\) norm. Indeed, with intermittency in time, we impose the duality between \(\theta\) and \(w\) to be

\[
\theta \in L^1 L^p \quad w \in L^\infty L^{p'},
\]

(6.6)

which is also consistent with the ansatz (6.5).

The hope is that with enough temporal intermittency, we get

\[
w \in L^\infty L^{p'} \implies w \in L^1 W^{1,q} \quad \text{for} \quad \frac{1}{p} + \frac{1}{q} > 1.
\]

(6.7)

Note that temporal intermittency is the key difference between (6.3) and (6.7).

After performing convex integration in space, modulo an error term of high spacial frequencies, the remaining error in (6.5) reduces to

\[
R(\tilde{g}_\kappa g_\kappa - 1),
\]

(6.8)

which is a term of high temporal frequency and can thus be canceled by adding a temporal corrector \(\theta_o\) such that to the leading order

\[
\partial_t \theta_o = -(\tilde{g}_\kappa g_\kappa - 1) \, \text{div} \, R.
\]

(6.9)

To see this temporal corrector \(\theta_o\) is indeed small compared with \(\theta\), we note that the error term (6.8) has low frequencies in space, and thus if \(\theta_o\) oscillates much faster in time than the old defect field \(R\), we have

\[
\|\theta_o\|_{L^1_t L^p} \ll 1.
\]

(6.10)

6.3. Other considerations. Introducing temporal intermittency and oscillations comes the cost of worse bounds in time for the perturbations \(\theta\) and \(w\). Of particular importance is whether the iteration scheme will go through, i.e. the defect field \(R\) can be made small in \(L^1_t L^p\). The most relevant part in the scheme is the term \(R_{\text{term}}\), solving the equation

\[
\text{div} \, R_{\text{term}} = \partial_t \theta.
\]

(6.10)

It is clear that this term will impose certain constraints on the size of temporal frequencies. In the end, it is the potential theory and Lemma 2.7 that saves the day: writing \(\theta\) as a divergence of a potential allows us to gain one full derivative in space. We can also infer from (6.10) that the temporal frequency should be comparable to the spacial frequency.

Notice that (6.7) does not require any intermittency in space but only intermittency in time. It turns out that as long as \(\theta\) and \(w\) are not homogeneous, i.e. a little intermittent in space, (6.7) can be achieved. The spacial intermittency is used to reconcile (6.7) and (6.10) which is impossible when \(\theta\) and \(w\) are completely homogeneous.

This is quite surprising and very different than the idea used in [MS18, MS19b, MS19a, BCL20], where a more intermittent solution implies a larger regime of nonuniqueness.
6.4. Intermittent functions in time $\bar{g}_κ$ and $g_κ$. We shall define the intermittent oscillatory functions $\bar{g}_κ$ and $g_κ$ in this subsection. We take a profile function $g \in C_0^∞([0, 1])$ such that

$$\int_{[0, 1]} g^2 \, dt = 1.$$ 

Let $κ ≥ 1$ to be the temporal concentration parameter that will be fixed in the next section. We introduce the temporal intermittency by adding concentration using $κ$ as follows.

Define $g_κ : \mathbb{R} \to \mathbb{R}$ by

$$g_κ(t) = g(κt). \quad (6.11)$$

By a slight abuse of notation, we still denote by $g_κ$ the 1-periodic extension of $g_κ$ by means of the Poisson summation. Note that $κ ≥ 1$ implies $\text{supp} \, g_κ \subset [0, 1]$.

Next, we define

$$\bar{g}_κ = κg_κ, \quad (6.12)$$

so that $g_κ, \bar{g}_κ : [0, 1] \to \mathbb{R}$ are both 1-periodic. We will use $\bar{g}_κ$ to oscillate the density building blocks $Φ_k$ and $g_κ$ for the vectors $W_k$. Note that the important intermittency estimates

$$\|\bar{g}_κ\|_{L^r([0,1])} \lesssim κ^{1-\frac{1}{r}}, \quad (6.13)$$

and the normalization identity

$$\int_{[0, 1]} \bar{g}_κg_κ \, dt = 1. \quad (6.14)$$

Because of (6.13), for any Sobolev space $W^{k, \tilde{r}}$ we may choose $r > 0$ such that

$$\|w\|_{L^r_t W^{k, \tilde{r}}} \ll 1$$

which confirms Remark 1.5 that the vector field $u$ concentrates on a small “bad” set in $[0, 1] \times \mathbb{T}^d$.

6.5. Temporal correction function $h_κ$. Finally, concerning the temporal corrector $θ_o$ in (6.9), we define a periodic function $h_κ : [0, 1] \to \mathbb{R}$ by

$$h_κ(t) := \int_0^t (\bar{g}_κg_κ - 1) \, dτ, \quad (6.15)$$

so that

$$\partial_t h_κ = \bar{g}_κg_κ - 1. \quad (6.16)$$

Note that by (6.14), $h_κ$ is well-defined and an approximation of a saw-tooth function, and we have the estimate

$$\|h_κ\|_{L^∞[0, 1]} \leq 1, \quad (6.17)$$

which holds uniformly in $κ$.

In other words, $h_κ$ is not intermittent at all for any $κ > 0$, and it will be used to design the temporal corrector $θ_o$ in the next section.

7. Proof of Proposition 3.1: defining perturbations and the defect field

The main aim of this section is to define the perturbation density $θ$ and velocity $w$, as well as solve for the new defect field $R_1$. This section is the core of the proof of Proposition 3.1.

Let us summarize the main steps in this section as follows.

1. We first fix all the parameters in the building blocks $(Φ_k, W_k)$ and $\bar{g}_κ, g_κ$ as explicit powers of $λ$, whose value we shall fix in the end.

2. Next, we define a partition of the old defect field $R$ to ensure the smoothness of the perturbation.

3. Then we define the perturbation $(θ, w)$ which, to the leading order, consists of linear combinations of the building blocks $(Φ_k, W_k)$ with suitable coefficients that oscillate intermittently in time using functions $\bar{g}_κ, g_κ$ defined in Section 6.

4. Having defined the perturbation, we finally design the new defect field $R_1$ so that the new density $ρ + θ$ and the new vector field $u + w$ solve the continuity-defect equation with the new defect field $R_1$. 

7.1. **Defining the parameters.** Given $p, q$ as in Proposition 3.1, there exists $\gamma > 0$ such that
\[
\min \left\{ 1 - \frac{1}{p}, \frac{1}{q} - \frac{1}{p'} \right\} > 2\gamma
\]
(7.1)

Let $\lambda_0$ be the lower bound of $\tau, \mu$ given by Lemma 5.13. We fix the following frequency parameters $\lambda, \mu, \kappa, \sigma > 0$ as follows:

- The major frequency parameter
  \[ \lambda \geq \lambda_0 \]
  will be fixed at the end depending on the previous solution $(\rho, u, R)$ and the given parameters $\delta, \nu$ in Proposition 3.1.

- Concentration parameters $\mu, \tau, \kappa$:
  \[ \mu = \kappa = \lambda, \tau = \lambda_0. \]

- Oscillation parameter $\sigma \in \mathbb{N}$:
  \[ \sigma = \sigma = \lfloor \lambda^{\gamma} \rfloor. \]

Note that $\tau$ is now a fixed constant that the implicit constants in what follows can depend upon. Also, note that space and time periodicity require $\sigma$ and $\sigma'$ are integers.

By fixing $\tau = \lambda_0$, we effectively make $(\Phi_k, W_k)$ only $D = d - 1$ intermittent. In fact, both $\tau$ and $\mu$ can be any positive powers of $\lambda$. In contrast, the temporal concentration $\kappa$ has to be almost a full spacial derivative.

Below is a direct consequence of the choice of parameters.

**Lemma 7.1.** There exists $r > 1$ such that for any $\lambda \geq \lambda_0$, there holds
\[
\sigma \mu^{\frac{d+1}{r} - \frac{d+1}{q}} \leq \lambda^{-\gamma},
\mu^{\frac{d+1}{r} - \frac{d+1}{q}} \leq \lambda^{-\gamma}.
\]

7.2. **Defect field cutoff.** To ensure smoothness of the perturbation $(\theta, w)$, we shall avoid the region where $R$ is small. To this end, we introduce cutoffs based on each component of $R$. Denote by $R_k$ the components of old defect field $R$
\[
R(t, x) = \sum_{1 \leq k \leq d} R_k(t, x)e_k.
\]
(7.2)

We specify the constant $r > 0$ in Proposition 3.1 as follows. Fix $r > 0$ sufficiently small so that
\[
||R||_{L^\infty([0,1] \times \mathbb{T}^d)} \leq \frac{1}{4d}.
\]
(7.3)

Next, we define smooth cutoff functions $\chi_k \in C_c^\infty([0,1] \times \mathbb{T}^d)$ such that
\[
0 \leq \chi_k \leq 1, \quad \chi_k(t, x) = \begin{cases} 0 & \text{if } |R_k| \leq \frac{\delta}{8d} \text{ or } t \notin I_r/2 \\ 1 & \text{if } |R_k| \geq \frac{\delta}{8d} \text{ and } t \in I_r. \end{cases}
\]
(7.4)

where we recall the notation $I_r = [r, 1 - r] \subset [0,1]$. Note that by design each $\chi_k$ is also time-periodic.

Such cutoffs $\chi_k$ can easily be constructed by first cutting according to the size of $|R_k|$ and then multiplying by an additional cutoff in time.

Note that the bounds of $\chi_k$ depends on $R$ and $\delta$. Let us cut off $R_k$ by introducing
\[
\tilde{R}_k = \chi_k R_k.
\]
(7.5)

In what follows we often use the crude bounds
\[
|\nabla_{t, x}^{n, \delta} \tilde{R}_k| \lesssim_{R, n, \delta} 1.
\]
(7.6)

These cutoffs $\chi_k$ shall not be confused with the ones in Section 5. From now on, $\chi_k$ refers to this definition only.
7.3. Density and velocity perturbation \((\theta, w)\). The idea of defining the perturbation \((\theta, w)\) is to use \(d\)-pairs of almost disjoint \((\Phi_k, W_k)\) to cancel each component \(R_k\) on average in time by using the intermittent oscillating factors in Section 6.

We first define the principle part of the perturbations. Let

\[
\theta_p(t, x) := \nu^{-1}g_\kappa(\sigma t) \sum_{1 \leq k \leq d} \frac{\|\tilde{R}_k(t)\|_1^{1 - \frac{d}{p}}}{\|\tilde{R}_k\|_1^{1 - \frac{d}{p}}} \text{sign}(-R_k) \chi_k |R_k|^\frac{d}{p} \Phi_k(\sigma x), \tag{7.7}
\]

\[
w_p(t, x) := \nu g_\kappa(\sigma t) \sum_{1 \leq k \leq d} \frac{\|\tilde{R}_k(t)\|_1^{1 - \frac{d}{p}}}{\|\tilde{R}_k\|_1^{1 - \frac{d}{p}}} \chi_k |R_k|^\frac{d}{p} W_k(\sigma x). \tag{7.8}
\]

The smoothness of \(\theta_p\) and \(w_p\) will be proved in Lemma 8.1. We take a moment to analyze the role of each part involved in the definition.

- The factors \(\|\tilde{R}_k(t)\|_1^{1 - \frac{d}{p}}\) and \(\|\tilde{R}_k(t)\|_1^{1 - \frac{d}{p}}\) are for the normalization when using the high-high to low interactions in space to kill the old defect field \(R\).
- The cutoffs \(\chi_k\) is to ensure smoothness by avoiding the regime where \(R_k\) is small. Note that if \(\|\tilde{R}_k\|_{L^\infty_{t,x}} = 0\), then \(\|R\|_{L^\infty_{t,x}} \leq \delta\) and there is nothing to prove.
- The building blocks \(\Phi_k(\sigma x)\) and \(W_k(\sigma x)\) are used to perform the convex integration in space, similar to the previous works.
- Finally \(\tilde{g}_\kappa(\sigma t)\) and \(g_\kappa(\sigma t)\) are the factors that encode the temporal intermittency and oscillation. We will then perform a “convex integration in time” to kill the error of high temporal frequency.

For brevity, let us introduce shorthand notations

\[
\theta_p(t, x) = \nu^{-1}g_\kappa(\sigma t) \sum_{1 \leq k \leq d} A_k(t, x) \Phi_k(\sigma x), \tag{7.9}
\]

\[
w_p(t, x) = \nu g_\kappa(\sigma t) \sum_{1 \leq k \leq d} B_k(t, x) W_k(\sigma x), \tag{7.10}
\]

where

\[
A_k(t, x) = \frac{\|\tilde{R}_k(t)\|_1^{1 - \frac{d}{p}}}{\|\tilde{R}_k\|_1^{1 - \frac{d}{p}}} \chi_k \text{sign}(-R_k) |R_k|^\frac{d}{p}, \tag{7.11}
\]

\[
B_k(t, x) = \frac{\|\tilde{R}_k(t)\|_1^{1 - \frac{d}{p}}}{\|\tilde{R}_k\|_1^{1 - \frac{d}{p}}} \chi_k |R_k|^\frac{d}{p}. \tag{7.12}
\]

Note that the important identity that motivates our choice of \(A_k\) and \(B_k\),

\[
A_k B_k = -\chi_k^2 R_k \quad \text{for all } k = 1, \ldots, d. \tag{7.13}
\]

In view of the zero-mean requirement for \(\theta\) and the divergence-free condition for \(w\), we introduce correctors

\[
\theta_c(t, x) := \int_{\mathbb{R}^d} \theta_p(t, x) \, dx \tag{7.14}
\]

\[
w_c(t, x) := -\nu g_\kappa(\sigma t) \sum_{1 \leq k \leq d} B(\nabla B_k, W_k(\sigma \cdot)). \tag{7.15}
\]

where \(B\) is the bilinear antidivergence operator in Lemma 2.2.

Since \(\nabla B_k \cdot W_k = \text{div}(B_k W_k)\) has zero mean, by a direct computation

\[
\text{div} w_c = -\nu g_\kappa(\sigma t) \sum_{1 \leq k \leq d} \text{div} B(\nabla B_k, W_k(\sigma \cdot)) = - \text{div} w_p.
\]

Thanks to Theorem 5.14 and Lemma 2.2, these two correctors are small compared to the principle part \(\theta_p\) and \(w_p\).
Finally we take advantage of the temporal oscillation and define a temporal oscillator

\[ \theta_o(t, x) := \sigma^{-1} h(\sigma t) \sum_{1 \leq k \leq d} \left( \int_{\mathbb{T}^d} \Phi_k W_k \, dx \right) \cdot \nabla (\chi_k^2 R_k) \]  

(7.16)

which thanks to Lemma 5.14 is equivalent to

\[ \theta_o = \sigma^{-1} h(\sigma t) \text{div} \sum_{1 \leq k \leq d} \chi_k^2 R_k e_k. \]  

(7.17)

The role of this temporal oscillator \( \theta_o \) is to balance the high temporal frequency error by its time derivative in the convex integration scheme, which will be done in Lemma 7.5.

### 7.4. The new defect field \( R_1 \)

Our next goal is to define a suitable defect field \( R_1 \) such that the new density \( \rho_1 \) and vector field \( u_1 \),

\[ \rho_1 := \rho + \theta \quad u_1 := u + w \]

solve the continuity-defect equation

\[ \partial_t \rho_1 + u_1 \cdot \nabla \rho_1 = \text{div} \, R_1. \]  

(7.18)

To do so, we will solve the divergence equations

\[
\begin{align*}
\text{div} \, R_{\text{osc}} &= \text{div}(\theta_p w_p + R) + \partial_t \theta_o \\
\text{div} \, R_{\text{tem}} &= \partial_t (\theta_p + \theta_c) \\
\text{div} \, R_{\text{lin}} &= \text{div}(\theta u + \rho w) \\
\text{div} \, R_{\text{cor}} &= \text{div} \, (\theta w_c) + \text{div} \, ((\theta_o + \theta_c)w)
\end{align*}
\]

such that \( R_1 = R_{\text{osc}} + R_{\text{tem}} + R_{\text{lin}} + R_{\text{cor}} \).

The choice for \( R_{\text{lin}} \) and \( R_{\text{cor}} \) is relatively straightforward.

**Definition 7.2.** The new defect field \( R_1 \) is defined by

\[ R_1 = R_{\text{osc}} + R_{\text{lin}} + R_{\text{cor}} + R_{\text{tem}} \]

where \( R_{\text{lin}} \) and \( R_{\text{cor}} \) are defined by

\[
\begin{align*}
R_{\text{lin}} &= \theta u + \rho w \\
R_{\text{cor}} &= \theta w_c + (\theta_o + \theta_c)w,
\end{align*}
\]

while \( R_{\text{tem}} \) and \( R_{\text{osc}} \) are defined respectively in Lemma 7.3 and Lemma 7.5.

Next, we specify the choice for \( R_{\text{tem}} \), which utilizes the bilinear antidivergence operator \( \mathcal{B} \).

**Lemma 7.3.** Let

\[ R_{\text{tem}} := \nu^{-1} \partial_t \left( \tilde{\rho}_c(\sigma t) \sum_{1 \leq k \leq d} \mathcal{B}(A_k, \Phi_k(\sigma \cdot)) \right). \]

Then

\[ \partial_t (\theta_p + \theta_c) = \text{div} \, R_{\text{tem}}. \]

**Proof.** Note that

\[ \theta_p + \theta_c = \nu^{-1} \tilde{\rho}_c(\sigma t) \sum_{1 \leq k \leq d} (A_k \Phi_k(\sigma \cdot) - \int_{\mathbb{T}^d} A_k \Phi_k(\sigma \cdot)). \]

Then the conclusion follows immediately from the definition of \( \mathcal{B} \). \( \square \)
7.5. **Convex integration in space-time: designing** $R_{osc}$. This subsection is the core of our convex integration scheme. The main goal is to design a suitable oscillation part $R_{osc}$ of the defect field so that

$$\text{div } R_{osc} = \text{div}(\theta_p w_p + R) + \partial_t \theta_o.$$  

To this end, we first isolate terms in the nonlinearity $\text{div}(\theta_p w_p + R)$ according to their roles, and then use the temporal corrector $\partial_t \theta_o$ to balance the part with high temporal frequencies in $\text{div}(\theta_p w_p + R)$.

**Lemma 7.4** (Space-time oscillations). The following identity holds

$$\text{div}(\theta_p w_p + R) = \text{div} \left( R_{osc,x} + R_{hi,t} + R_{far} + R_{appr} + R_{rem} \right),$$  

where $R_{osc,x}$ is the oscillation error in space

$$R_{osc,x} = \tilde{g}_x(\sigma t) g_x(\sigma t) \sum_{1 \leq k \leq d} B \left( \nabla(A_k B_k), (\Phi_k \mathbf{W}_k(\sigma x) - \int_{T^d} \Phi_k \mathbf{W}_k \, dx) \right),$$  

$R_{hi,t}$ is the error of high frequency in time

$$R_{hi,t} = \left( \tilde{g}_x(\sigma t) g_x(\sigma t) - \int_{[0,1]} \tilde{g}_x g_x \right) \sum_{1 \leq k \leq d} A_k B_k \int_{T^d} \Phi_k \mathbf{W}_k \, dx,$$

$R_{appr}$ is the approximation error

$$R_{appr} = \tilde{g}_x(\sigma t) g_x(\sigma t) \sum_{1 \leq k \leq d} A_k B_k \Phi_k \mathbf{W}_k \cdot (\sigma x),$$

$R_{far}$ is the far field error

$$R_{far} = \tilde{g}_x(\sigma t) g_x(\sigma t) \sum_{k \neq k'} A_k B_k \Phi_k \mathbf{W}_{k'} \cdot (\sigma x),$$

and $R_{rem}$ is the remainder error

$$R_{rem} = \sum_{1 \leq k \leq d} (1 - \chi_k^2) R_k e_k.$$

**Proof.** By the definition of $\theta_p$ we have (5.31),

$$\theta_p w_p = \tilde{g}_x(\sigma t) g_x(\sigma t) \sum_{1 \leq k \leq d} A_k B_k \Phi_k \mathbf{W}_k(\sigma \cdot) + R_{far}.$$  

(7.20)

Taking divergence, we have

$$\text{div}(\theta_p w_p + R) = \tilde{g}_x(\sigma t) g_x(\sigma t) \sum_{1 \leq k \leq d} \text{div} \left( A_k B_k \Phi_k \mathbf{W}_k(\sigma \cdot) \right) + \text{div } R + \text{div } R_{far}.$$  

(7.21)

Notice that by Lemma 5.13

$$\text{div} \left( A_k B_k \Phi_k \mathbf{W}_k(\sigma \cdot) \right) = \text{div} \left( A_k B_k \left( \Phi_k \mathbf{W}_k(\sigma \cdot) - \int_{T^d} \Phi_k \mathbf{W}_k \right) \right) + \text{div}(A_k B_k e_k)$$

$$= A_k B_k \text{div} \left( \Phi_k \mathbf{W}_k(\sigma \cdot) \right) + \nabla(A_k B_k) \cdot (\Phi_k \mathbf{W}_k(\sigma \cdot) - \int_{T^d} \Phi_k \mathbf{W}_k) + \text{div}(A_k B_k e_k),$$

where the first two terms combined together have zero mean.

For this reason and by the definition of $B$, we may write

$$\text{div} \left( A_k B_k \Phi_k \mathbf{W}_k(\sigma \cdot) \right) = \text{div} B \left( A_k B_k, \text{div} \left( \Phi_k \mathbf{W}_k(\sigma \cdot) \right) \right)$$

$$+ \text{div} B \left( \nabla(A_k B_k), (\Phi_k \mathbf{W}_k(\sigma \cdot) - \int_{T^d} \Phi_k \mathbf{W}_k) \right) + \text{div}(A_k B_k e_k).$$  

(7.22)

It follows from (7.21) and (7.22) that

$$\text{div}(\theta_p w_p + R) = \tilde{g}_x(\sigma t) g_x(\sigma t) \sum_{1 \leq k \leq d} \text{div}(A_k B_k e_k) + \text{div } R + \text{div } R_{osc,x} + \text{div } R_{appr} + \text{div } R_{far}.$$  

(7.23)
To see $\text{div}(\theta p w_p + R) = \text{div}(R_{\text{osc,x}} + R_{\text{hl,t}} + R_{\text{far}} + R_{\text{appr}} + R_{\text{rem}})$, by an examination of (7.23) we need to show that

$$R_{\text{rem}} = R + \int_{[0,1]} \tilde{g}_n g_n \sum_{1 \leq k \leq d} A_k B_k \Phi_k \mathbf{W}_k \, dx.$$  

Using (6.14), Theorem 5.14 and (7.13) we obtain that

$$\int_{[0,1]} \tilde{g}_n g_n \sum_{1 \leq k \leq d} A_k B_k \Phi_k \mathbf{W}_k \, dx = \sum_{1 \leq k \leq d} A_k B_k \mathbf{e}_k,$$

which implies that

$$R + \sum_{1 \leq k \leq d} A_k B_k \mathbf{e}_k = R - \sum_{1 \leq k \leq d} \chi_k^2 R_k \mathbf{e}_k = \sum_{1 \leq k \leq d} (1 - \chi_k^2) R_k \mathbf{e}_k = R_{\text{rem}}.$$  

Due to the designed temporal corrector $\theta_o$, the error of high frequency in time $R_{\text{hl,t}}$ is canceled to the leading order by $\partial_t \theta_o$. We conclude the design of the oscillation error $R_{\text{osc}}$ in the following lemma.

**Lemma 7.5.** Let

$$R_{\text{osc}} := R_{\text{osc,x}} + R_{\text{osc,t}} + R_{\text{far}} + R_{\text{appr}} + R_{\text{rem}},$$

where $R_{\text{osc,x}}, R_{\text{far}}, R_{\text{appr}}, R_{\text{rem}}$ are as in Lemma 7.4, and $R_{\text{osc,t}}$ is the oscillation error in time

$$R_{\text{osc,t}} = \sigma^{-1} h(\sigma t) \mathcal{R} \sum_{1 \leq k \leq d} \mathbf{e}_k \cdot \partial_t \nabla (\chi_k^2 R_k).$$

Then the oscillation error $R_{\text{osc}}$ verifies the identity

$$\text{div} R_{\text{osc}} = \text{div}(\theta p w_p + R) + \partial_t \theta_o.$$  

**Proof.** By the previous lemma, we only need to verify that

$$\partial_t \theta_o + \text{div} R_{\text{hl,t}} = \text{div} R_{\text{osc,t}}.$$

By the definition of $\theta_o$ (7.17), we have

$$\partial_t \theta_o = -\partial_t h(\sigma t) \sum_{1 \leq k \leq d} \mathbf{e}_k \cdot \nabla (\chi_k^2 R_k) - \sigma^{-1} h(\sigma t) \sum_{1 \leq k \leq d} \mathbf{e}_k \cdot \partial_t \nabla (\chi_k^2 R_k).$$

It follows from (6.16) that

$$\partial_t h(\sigma t) \sum_{1 \leq k \leq d} \mathbf{e}_k \cdot \nabla (\chi_k^2 R_k) = (\tilde{g}_n(\sigma t) g_n(\sigma t) - \int_{[0,1]} \tilde{g}_n g_n) \sum_{1 \leq k \leq d} \mathbf{e}_k \cdot \nabla (-A_k B_k) = -\text{div} R_{\text{hl,t}},$$

which implies that

$$\partial_t \theta_o + \text{div} R_{\text{hl,t}} = \text{div} R_{\text{osc,t}}.$$  

7.6. **Verification of** $(u_1, \rho_1, R_1)$ **as a solution of the continuity-defect equation.** We conclude this section by showing that the new solution $(u_1, \rho_1, R_1)$ is indeed a solution to the continuity-defect equation.

**Lemma 7.6.** The density $\rho_1 = \rho + \theta$, vector field $u_1 = u + w$, and defect field $R_1 = R_{\text{lin}} + R_{\text{rem}} + R_{\text{cor}} + R_{\text{osc}}$ solve the equation

$$\partial_t \rho_1 + u_1 \cdot \nabla \rho_1 = \text{div} R_1.$$  

**Proof.** We compute that

$$\partial_t \rho_1 + u_1 \cdot \nabla \rho_1 = (\partial_t \rho + u \cdot \nabla \rho) + (\partial_t \theta + \text{div}(\theta u)) + \text{div}(\theta w) + \text{div}(\rho w)) = \text{div} R + \partial_t \theta + \text{div}(\theta u) + \text{div}(\theta w) + \text{div}(\rho w).$$

The claim follows from Definition 7.2, Lemma 7.3, and Lemma 7.5.

To complete the proof of Proposition 3.1, it remains to verify the estimates for the perturbation $(\theta, w)$ and the new defect field $R_1$. We do this in Section 8 for the perturbation and respectively in Section 9 for the new defect field.
8. Proof of the Proposition 3.1: Estimates on the Perturbation

In this section, we will derive estimates for the perturbation \((\theta, w)\). The main tools have been listed in Section 2. The main idea is to take the frequency parameter \(\lambda\) sufficiently large depending on the previous solution \((\rho, u, R)\) so that the error terms are negligible. It is also worth noting that all implicit constants will not depend on \((\rho, u, R)\) unless otherwise indicated.

We start with the smoothness and time periodicity of the coefficients \(A_k, B_k\), which are necessary conditions for Lemma 2.4 and 2.5.

**Lemma 8.1** (Smoothness of \(A_k, B_k\)). The coefficients \(A_k, B_k \in C^\infty([0, 1] \times \mathbb{T}^d)\) are time-periodic on \([0, 1]\), and the map
\[
t \mapsto \|\tilde{R}_k(t)\|_{L^1(\mathbb{T}^d)}
\]
is smooth on \([0, 1]\). In particular, all the perturbations \(\theta_p, \theta_c, \theta_o\) and \(w_p, w_c\) are smooth and time-periodic.

Moreover, the following estimates hold uniformly in time
\[
\|A_k(t)\|_{L^p(\mathbb{T}^d)} \leq \|\tilde{R}_k\|_{L^\infty_x}^{-1/p} \|\tilde{R}_k(t)\|_{L^1(\mathbb{T}^d)},
\]
\[
\|B_k(t)\|_{L^{p'}(\mathbb{T}^d)} \leq \|\tilde{R}_k\|_{L^\infty_x}^{p'}/p'.
\]

**Proof.** Denote by \(R_k^+ = \max\{R_k, 0\}\) and \(R_k^- = \min\{R_k, 0\}\). Due to the cutoff \(\chi_k\), the functions
\[
\chi_k R_k^+ - \chi_k R_k^-
\]
are smooth on \([0, 1] \times \mathbb{T}^d\). Thus the map
\[
t \mapsto \|\tilde{R}_k(t)\|_{L^1(\mathbb{T}^d)} = \int \chi_k R_k^+ - \chi_k R_k^- dx
\]
is smooth on \([0, 1]\).

Next, let us show that the coefficients \(A_k\) and \(B_k\) are smooth on \([0, 1] \times \mathbb{T}^d\). Indeed, due to the smoothness of \(\|\tilde{R}_k(t)\|_1\), the coefficients \(A_k, B_k\) are automatically smooth at all points where \(\|\tilde{R}_k(t)\|_1 > 0\). On the other hand, for any point \((t, x)\), where \(\|\tilde{R}_k(t)\|_1 = 0\), there is a neighborhood of that point where \(\chi_k \equiv 0\). Hence, \(A_k \equiv B_k \equiv 0\) in that neighborhood. Therefore, \(A_k, B_k \in C^\infty([0, 1] \times \mathbb{T}^d)\). Their time-periodicity follows simply from their definitions.

Finally, we show the pointwise \(L^p\) and \(L^{p'}\) estimates for \(A_k\) and \(B_k\). For \(A_k\) we have

\[
\|A_k(t)\|_{L^p(\mathbb{T}^d)} \leq \frac{\|\tilde{R}_k(t)\|_1^{-1/p}}{\|\tilde{R}_k\|_{L^\infty_x}^{1+1/p}} \|\chi_k R_k\|_p^{1/p} \|\tilde{R}_k\|_{L^1(\mathbb{T}^d)}^{1+1/p},
\]

where we have used the fact that \(p \in (1, \infty)\). The estimate for \(B_k\) can be deduced in the same way:

\[
\|B_k(t)\|_{L^{p'}(\mathbb{T}^d)} \leq \frac{\|\tilde{R}_k(t)\|_1^{-1/p}}{\|\tilde{R}_k\|_{L^\infty_x}^{1+1/p}} \|\chi_k R_k\|_p^{1/p} \|\tilde{R}_k\|_{L^1(\mathbb{T}^d)}^{1+1/p}.
\]
8.1. **Estimates for the density \( \theta \).** Here and in what follows, \( C_R \) represents a positive constant that depends on the old defect field \( R \) that may change from line to line.

**Lemma 8.2** (Estimate on \( \theta_p \)). There holds
\[
\| \theta_p \|_{L^1_t L^p_x} \lesssim \nu \| R \|_{L^1_t L^\frac{2}{1+\frac{1}{p}}} + C_R \sigma^{-1}.
\]
In particular, for \( \lambda \) sufficiently large,
\[
\| \theta_p \|_{L^1_t L^p_x} \lesssim \nu \| R \|_{L^1_t L^\frac{2}{1+\frac{1}{p}}}.
\]

**Proof.** We first take \( L^p \) norm in space, using the shorthand notation
\[
| \Delta k(t) \Phi_k (\sigma \cdot) \|_{L^p(T^d)} \lesssim \left| \sum_{1 \leq k \leq d} \| A_k(t) \Phi_k (\sigma \cdot) \|_{L^p(T^d)} \right|.
\]

Since \( A_k(t, x) \) is smooth on \( \mathbb{T}^d \), by Lemma 2.4, we have
\[
\left| \sum_{1 \leq k \leq d} \| A_k(t) \Phi_k (\sigma \cdot) \|_{L^p(T^d)} \right| \lesssim \| A_k(t) \|_{L^p(T^d)} \| \Phi_k (\sigma \cdot) \|_{L^p} + C_R \sigma^{-1} \| \Phi_k \|_{L^p}.
\]

By Proposition 5.11 and Lemma 8.1, combining (8.3) and (8.4) we obtain
\[
\| \theta_p \|_{L^1_t L^p_x} \lesssim \nu^{-1} \| \bar{g}_{\kappa} (\sigma t) \| \left[ \sum_{1 \leq k \leq d} \| \bar{R}_k \|_{L^1_t L^\frac{1}{1+\frac{1}{p}}} + C_R \sigma^{-1} \right].
\]

We take \( L^1 \) in time to obtain
\[
\| \theta_p \|_{L^1_t L^p_x} \lesssim \nu^{-1} \sum_{1 \leq k \leq d} \| \bar{R}_k \|_{L^1_t L^\frac{1}{1+\frac{1}{p}}} \int_{[0,1]} | \bar{g}_{\kappa} (\sigma t) | \| \bar{R}_k (t) \|_1 dt + C_R \sigma^{-1}.
\]

With the smoothness of \( t \rightarrow \| \bar{R}_k (t) \|_1 \) proven in Lemma 8.1, applying Lemma 2.4 once again (in time) gives
\[
\int_{[0,1]} | \bar{g}_{\kappa} (\sigma t) | \| \bar{R}_k (t) \|_1 dt \lesssim \| \bar{R}_k \|_{L^1_t L^\frac{1}{1+\frac{1}{p}}} \| \bar{g}_{\kappa} \|_{L^1([0,1])} + C_R \sigma^{-1}.
\]

It follows from (8.5) and (8.6) that
\[
\| \theta_p \|_{L^1_t L^p_x} \lesssim \nu^{-1} \sum_{1 \leq k \leq d} \| \bar{R}_k \|_{L^1_t L^\frac{1}{1+\frac{1}{p}}} + C_R \| \bar{g}_{\kappa} \|_{L^1([0,1])} + C_R \sigma^{-1},
\]
where we have aso used (6.13).

Once we take \( \lambda \) sufficiently large so that the error term
\[
C_R \sigma^{-1} \leq \nu^{-1} \| R \|_{L^1_t L^\frac{2}{1+\frac{1}{p}}},
\]
the desire bound follows
\[
\| \theta_p \|_{L^1_t L^p_x} \lesssim \nu^{-1} \| R \|_{L^1_t L^\frac{2}{1+\frac{1}{p}}},
\]
with an implicit constant independent of \( \lambda, R \) and \( \nu \).

**Lemma 8.3** (Estimate on \( \theta_c \)). There holds
\[
\| \theta_c \|_{L^1_t L^p_x} \lesssim C_R \nu^{-1} \sigma^{-1}.
\]
In particular, for \( \lambda \) sufficiently large
\[
\| \theta_c \|_{L^1_t L^p_x} \lesssim \nu^{-1} \| R \|_{L^1_t L^\frac{2}{1+\frac{1}{p}}}.
\]

**Proof.** Since
\[
\theta_c = \nu^{-1} \bar{g}_{\kappa} (\sigma \cdot) \sum_{1 \leq k \leq d} \int_{T^d} A_k(t, x) \Phi_k (\sigma x) dx,
\]
this follows directly from Lemma 2.5. \( \square \)
Lemma 8.4 (Estimate on $\theta_o$). There holds
$$\|\theta_o\|_{L^\infty_t L^\infty_x} \leq C_R \sigma^{-1}.$$  
In particular, for $\lambda$ sufficiently large
$$\|\theta_o\|_{L^1_t L^p_x} \leq \nu^{-1} \|R\|_{L^1_t L^p_x}^{1/\gamma}.$$  

Proof. By (7.17), Hölder’s inequality and (6.17) we have
$$\|\theta_o\|_{L^\infty_t L^\infty_x} \leq \sigma^{-1} \|h(\sigma)\|_{L^\infty([0,1])} \sum_{1 \leq k \leq d} \|e_k \cdot \nabla R_k\|_{L^\infty_t L^\infty_x} \leq C_R \sigma^{-1}.$$ 

8.2. Estimates for the vector field $w$. The vector field $w$ can also be estimated using the tools in Section 2.

Lemma 8.5 (Estimate on $w_p$). There holds
$$\|w_p\|_{L^\infty_t L^{p'}_x} \lesssim \nu \|R\|_{L^1_t L^p_x}^{1/\gamma} + C_R \sigma^{-1}$$
$$\|w_p\|_{W^{1,q}} \leq \nu C_R \lambda^{-\gamma}.$$  

In particular, for $\lambda$ sufficiently large,
$$\|w_p\|_{L^\infty_t L^{p'}_x} \lesssim \nu \|R\|_{L^1_t L^p_x}^{1/\gamma}$$
$$\|w_p\|_{L^1_t W^{1,q}_x} \leq \delta/2.$$  

Proof. We first prove the $L^\infty L^{p'}$ estimate, and then the Sobolev estimate $L^1 W^{1,q}$.

1) $L^\infty L^{p'}$ estimates:
Taking the $L^{p'}$ norm in space yields
$$\|w_p(t)\|_{p'} \lesssim \nu |g_n(\sigma t)| \sum_{1 \leq k \leq d} \|B_k(t) W_k(\sigma \cdot)\|_{p'}.$$  

(8.7)

Since $x \mapsto B_k(t, x)$ is smooth on $\mathbb{T}^d$ for all fixed $t \in [0, T]$, by Lemma 2.4, we have
$$\|B_k(t) W_k(\sigma \cdot)\|_{p'} \lesssim \|B_k(t)\|_{p'} \|W_k\|_{p'} + \sigma^{-1} C_R \|W_k\|_{p'}.$$  

(8.8)

From (8.7), (8.8), Lemma 8.1, and the fact that $\|W_k\|_{p'} \sim 1$, it follows that
$$\|w_p(t)\|_{p'} \lesssim \nu |g_n(\sigma t)| \sum_{1 \leq k \leq d} \|R_k\|_{L^1_t L^p_x}^{1/\gamma} + \sigma^{-1} C_R.$$  

(8.9)

We simply take $L^\infty$ in time to obtain
$$\|w_p\|_{L^\infty_t L^{p'}(\mathbb{T}^d)} \lesssim \nu \|R\|_{L^1_t L^p_x}^{1/\gamma} + C_R \sigma^{-1},$$
where we have used (6.13).

Once we take $\lambda$ sufficiently large so that the error term
$$C_R \sigma^{-1} \lesssim \nu \|R\|_{L^1_t L^p_x}^{1/\gamma},$$
the desire bound follows
$$\|w_p\|_{L^\infty_t L^{p'}(\mathbb{T}^d)} \lesssim \nu \|R\|_{L^1_t L^p_x}^{1/\gamma}.$$  

2) Sobolev estimate $L^1 W^{1,q}$:
Taking Sobolev norm $W^{1,q}$ in space we have
$$\|w_p(t)\|_{W^{1,q}(\mathbb{T}^d)} \lesssim \nu |g_n(\sigma t)| \sum_{1 \leq k \leq d} \|B_k(t) W_k(\sigma \cdot)\|_{W^{1,q}(\mathbb{T}^d)}. $$  

(8.10)
Direct computation using Hölder’s inequality gives
\[ \left\| B_k(t) W_k(\sigma) \right\|_{W^{1,\gamma}(\mathbb{T}^d)} \leq C_R \left( \left\| W_k(\sigma) \right\|_{L^4(\mathbb{T}^d)} + \sigma \left\| \nabla W_k(\sigma) \right\|_{L^2(\mathbb{T}^d)} \right) . \]

From this, by Proposition 5.11 and the fact that \( \tau = \lambda_0 \) is fixed, we get
\[ \left\| B_k(t) W_k(\sigma) \right\|_{W^{1,\gamma}(\mathbb{T}^d)} \lesssim C_R \sigma^{\mu \frac{1}{p'} - \frac{1}{q'}}. \] (8.11)

Thus from (8.10) and (8.11) we get
\[ \left\| w_p(t) \right\|_{W^{1,\gamma}(\mathbb{T}^d)} \leq C_R \sigma^{\mu \frac{1}{p'} - \frac{1}{q'}} \left| g_\kappa(\sigma t) \right| . \] (8.12)

Integrating (8.12) in time and using (6.13) we have
\[ \left\| w_p \right\|_{L^1 W^{1,q}} \leq C_R \kappa^{-1} \sigma^{\mu \frac{1}{p'} - \frac{1}{q'}} = \sigma^{\mu \frac{1}{p'} - \frac{1}{q'}}. \]

Thanks to Lemma 7.1, it follows from the above that
\[ \left\| w_p \right\|_{L^1 W^{1,q}} \leq C_R \lambda^{-\gamma}. \]

\[ \square \]

**Lemma 8.6 (Estimate on \( w_c \)).** There holds
\[ \left\| w_c \right\|_{L^\infty L^{p'}} \leq C_R \nu \sigma^{-1}, \]
\[ \left\| w_c \right\|_{L^1 W^{1,q}} \leq C_R \nu \kappa^{-1}. \]

In particular, for \( \lambda \) sufficiently large
\[ \left\| w_c \right\|_{L^\infty L^{p'}} \leq C_R \nu \left\| R \right\|_{L^\infty_{k,x}}, \]
\[ \left\| w_c \right\|_{L^1 W^{1,q}} \leq \delta/2. \]

**Proof.** We first prove the \( L^\infty L^{p'} \) estimate, and then the Sobolev estimate \( L^1 W^{1,q} \).

1. **\( L^\infty L^{p'} \) estimates:** Taking \( L^{p'} \) norm in space we have
\[ \left\| w_c(t) \right\|_{p'} \leq \nu \left| g_\kappa(\sigma t) \right| \sum_{1 \leq k \leq d} \left\| B(\nabla B_k, W_k(\sigma)) \right\|_{p'}. \] (8.13)

   By Lemma 2.2 we get
\[ \left\| B(\nabla B_k, W_k(\sigma)) \right\|_{p'} \leq C_R \left\| R W_k(\sigma) \right\|_{p'}. \] (8.14)

   Since the assumption on \( p, q \) implies that for \( 1 < p' < \infty \), we have
\[ \left\| R W_k(\sigma) \right\|_{p'} \lesssim \sigma^{-1}. \]

   Then it follows from (8.13) and (8.14) that
\[ \left\| w_c(t) \right\|_{p'} \leq C_R \nu \sigma^{-1} \left| g_\kappa(\sigma t) \right|, \]
which implies the desire bound thanks to (6.13).

2. **\( L^1 W^{1,q} \) estimates:** We take \( W^{1,q} \) norm in space to obtain
\[ \left\| w_c(t) \right\|_{W^{1,q}} \leq \nu \left| g_\kappa(\sigma t) \right| \sum_{1 \leq k \leq d} \left\| B(\nabla B_k, W_k(\sigma)) \right\|_{W^{1,q}}. \]

   By Poincare’s inequality, we have
\[ \left\| w_c(t) \right\|_{W^{1,q}} \lesssim \nu \left| g_\kappa(\sigma t) \right| \sum_{1 \leq k \leq d} \left\| \nabla B(\nabla B_k, W_k(\sigma)) \right\|_{q}. \] (8.15)

   In fact, a slight modification of the proof of Lemma 2.2 gives
\[ \left\| \nabla B(\nabla a, f) \right\|_{r} \lesssim \left\| a \right\|_{C^2} \left( \left\| R f \right\|_{r} + \left\| \nabla R f \right\|_{r} \right) \quad \text{for all } 1 \leq r < \infty \text{ and } a, f \in C^\infty(\mathbb{T}^d). \]
Due to the assumptions on \( p, q, 1 \leq q < p' < \infty \), which in particular implies that
\[
\left\| \nabla B_k(\nabla B_k, W_k(\sigma \cdot)) \right\|_{p'} \leq C_R,
\] (8.16)
where we used the fact that \( \nabla R \) is a Calderón-Zygmund operator on \( \mathbb{T}^d \).

Combining (8.15) and (8.16) we have
\[
\|w_o(t)\|_{W^{1,q}} \leq C_{R} \nu |g_\alpha(\sigma t)|,
\]
which implies the desire bound after integrating in time thanks to (6.13).

8.3. Proof of the perturbation part of Proposition 3.1. We finish proving (3.2)–(3.7) of Proposition 3.1 in the lemma below.

**Lemma 8.7.** There exists a universal constant \( M \) and a large \( N \in \mathbb{N} \) such that for all \( \lambda(\nu, \delta, R) \) sufficiently large, the following hold.

1. The density perturbation \( \theta \) verifies
\[
\nu^{-1} \|\theta\|_{L^1 L^p} \leq M \|R\|_{L^1_{1,\nu}}^{1/p} \quad \text{and} \quad \text{supp} \, \theta \subset I_r \times \mathbb{T}^d.
\]

2. The vector field perturbation \( w \) verifies
\[
\nu \|w\|_{L^\infty L^{p'}} \leq M \|R\|_{L^1_{1,\nu}}^{1/p'} \quad \text{and} \quad \|w\|_{L^1 W^{1,q}} \leq \delta.
\]

3. The density perturbation \( \theta \) has zero mean, and for all \( t \in [0, T] \) and \( \varphi \in C^\infty(\mathbb{T}^d) \)
\[
\int_{\mathbb{T}^d} \theta(t, x)\varphi(x) \, dx \leq \delta \|\varphi\|_{C^N}.
\]

**Proof.** By Lemmas 8.2.8.3.8.4 and Lemmas 8.5 and 8.6, for \( \lambda \) sufficiently large, we conclude that
\[
\|\theta\|_{L^1 L^p} \lesssim \nu \|R\|_{L^1_{1,\nu}}^{1/p}
\]
\[
\|w\|_{L^\infty L^{p'}} \lesssim \nu^{-1} \|R\|_{L^1_{1,\nu}}^{1/p'}
\]
with implicit constants independent of \( \lambda \) and \( (\rho, u, R) \). We thus choose the constant \( M \) to be maximum of the two implicit constants.

To see that \( \text{supp} \, \theta \subset I_r \times \mathbb{T}^d \), we simply note that by (7.4), the coefficients \( A_k \) and \( B_k \) in the definitions of \( \theta_p, \theta_c \) and \( \theta_o \) all verify this property.

By Lemmas 8.5 and 8.6 again, for \( \lambda \) sufficiently large, we have
\[
\|w\|_{L^1 W^{1,q}} \leq \delta.
\]

Finally, let us show the last property. Noticing that \( \theta_p + \theta_c \) has zero mean by default and \( \theta_o \) is a divergence, we conclude that the density perturbation \( \theta \) is mean-free. To show the last estimate, fix a test function \( \varphi \in C^\infty(\mathbb{T}^d) \). By definitions, we have
\[
\left| \int_{\mathbb{T}^d} \theta_p \varphi \, dx \right| \leq \left| \int_{\mathbb{T}^d} \theta_p \varphi \, dx \right| + \left| \int_{\mathbb{T}^d} \theta_c \varphi \, dx \right| + \left| \int_{\mathbb{T}^d} \theta_o \varphi \, dx \right|.
\]
We show the bounds for \( \theta_p \) and \( \theta_c \) since the argument can be adapted to bound \( \theta_o \) as well.

On one hand, applying Lemma 2.5 we have
\[
\left| \int_{\mathbb{T}^d} \theta_p \varphi \, dx \right| \lesssim \sigma^{-N} \|\bar{g}_\kappa\|_{\infty} \sum_{1 \leq k \leq d} \|A_k \varphi\|_{C^\infty} \|\Phi_k\|_2.
\]
Recall that \( \gamma N > d + 1 \), and then
\[
\left| \int_{\mathbb{T}^d} \theta_p \varphi \, dx \right| \leq C_R \lambda^{-d-1} \|\Phi_k\|_2 \|\varphi\|_{C^\infty}
\]
\[
\leq C_R \lambda^{-1} \|\varphi\|_{C^\infty}.
\] (8.17)
On the other hand, by Lemma 8.4, we have
\[
\left| \int_{T^d} \theta_o \varphi \, dx \right| \leq \| \theta_o \|_{L^2_{loc}} \| \varphi \|_\infty \leq C R \sigma^{-1} \| \varphi \|_\infty.
\] (8.18)

Putting (8.17) and (8.17) together and increasing the value of \( \lambda \) if necessary, we obtain
\[
\left| \int_{T^d} \theta \varphi \, dx \right| \leq \delta \| \varphi \|_{C^\infty}. \]

\( \square \)

9. PROOF OF THE PROPOSITION 3.1: ESTIMATES ON THE NEW DEFECT FIELD

We now turn to the final step of proving Proposition 3.1. Recall that we need to estimates the terms that solve the divergence equations
\[
\begin{align*}
\text{div } R_{\text{osc}} &= \partial_t \theta_o + \text{div}(\theta_p w_p) + \text{div } R \\
\text{div } R_{\text{tem}} &= \partial_t \theta_p + \partial_t \theta_c \\
\text{div } R_{\text{lin}} &= \text{div}(\theta u + \rho w) \\
\text{div } R_{\text{cor}} &= \text{div}(\theta u_c + (\theta_o + \theta_c) w).
\end{align*}
\]
The linear error \( R_{\text{lin}} \) and correction error \( R_{\text{cor}} \) can be estimated easily by standard methods. The temporal error \( R_{\text{tem}} \) is subtler and we need to exploit the derivative gain given by the potential \( \Theta_k \) in Theorem 4.7 and Definition 5.6. Such a difficulty is not present in [MS18, MS19a].

For the oscillation error \( R_{\text{osc}} \) we will use the decomposition done at the end of Section 7, which reads
\[
R_{\text{osc}} = R_{\text{osc},x} + R_{\text{osc},t} + R_{\text{far}} + R_{\text{appr}} + R_{\text{tem}}.
\]

We summarize how each part of \( R_{\text{osc}} \) will be estimated as follows.

1. As typical in the literature, \( R_{\text{osc},x} \) can be shown to be small due to a gain of \( \sigma^{-1} \) given by the antidivergence.
2. The term \( R_{\text{osc},t} \) is small by itself since it is the outcome of a temporal cancellation.
3. The far field error \( R_{\text{far}} \) is the interference between different building blocks \( (\Phi_k, W_k) \) and is very small by Theorem 5.14.
4. The approximation error \( R_{\text{appr}} \) is caused by the building blocks not being exact solutions of the transport equation and can be estimated by Theorem 5.14.
5. Finally, \( R_{\text{tem}} \) is the leftover old defect field that is small due to our choice of cutoffs \( \chi_k \) in (7.4).

9.1. Temporal error.

Lemma 9.1 (\( R_{\text{tem}} \) estimate). For \( \lambda \) sufficiently large,
\[
\| R_{\text{tem}} \|_{L^1_{t,x}} \leq \frac{\delta}{16}.
\]

Proof. We may rewrite it as
\[
R_{\text{tem}} = \nu^{-1} \sum_{1 \leq k \leq d} \partial_t (\tilde{g}_n(\sigma t)) \mathcal{B}(A_k, \Phi_k(\sigma)) + \tilde{g}_n(\sigma t) \mathcal{B}(\partial_t A_k, \Phi_k(\sigma))
\]
\[= R_{\text{tem},1} + R_{\text{tem},2}.
\]

We will treat the second term \( R_{\text{tem},2} \) as an error.

1. \( R_{\text{tem},1} \) estimate:
   Taking \( L^1 \) in space, we have
   \[
   \| R_{\text{tem},1}(t) \|_1 \leq \nu^{-1} \sigma |\partial_t \tilde{g}_n(\sigma t)| \sum_k \mathcal{B}(A_k, \Phi_k(\sigma)) \|_1.
   \] (9.1)

   By linearity of \( \mathcal{B} \), we introduce the split
   \[
   \| \mathcal{B}(A_k, \Phi_k(\sigma)) \|_1 \leq \| \mathcal{B}(A_k, \text{div } \Omega_k(\sigma)) \|_1 + \| \mathcal{B}(A_k, (\Phi_k - \text{div } \Omega_k)(\sigma)) \|_1,
   \] (9.2)
which will allow us to take advantage of the potential \( \Omega_k \).

For the first term in (9.2), we apply Lemma 2.2 to obtain
\[
\| B(A_k, \text{div} \Omega_k(\sigma \cdot)) \|_1 \leq C_R \| \mathcal{R}(\text{div} \Omega_k(\sigma \cdot)) \|_1
\]
(by periodic rescaling)
\[
\leq C_R \sigma^{-1} \| \mathcal{R} \text{div} \Omega_k(\sigma \cdot) \|_1
\]
(by definition of \( \mathcal{R} \))
\[
\leq C_R \sigma^{-1} \| \Omega_k \|_1
\]
(by Theorem 5.14)
\[
\leq C_{r,R} \sigma^{-1} \mu^{-1} \frac{d}{r} \mu^{-1} \frac{d}{r},
\]
for any \( 1 < r \leq \infty \). Then we fix \( r > 1 \) as in Lemma 7.1 so that
\[
\| B(A_k, \text{div} \Omega_k(\sigma \cdot)) \|_1 \leq C_R \sigma^{-1} \mu^{-1} \lambda^{-\gamma}.
\]

For the second term in (9.2), recall from Theorem 5.14 that
\[
\| \Phi_k - \text{div}(\Omega_k) \|_{L^r(T^d)} \lesssim \mu^{-1} \frac{d}{r} \mu^{-1} \frac{d}{r} \mu^{-1} \lambda^{-\gamma}.
\]

Thanks to this bound, we simply have
\[
\| B(A_k, \Phi_k(\sigma \cdot)) \|_1 \lesssim \mu^{-1} \frac{d}{r} \mu^{-1} \frac{d}{r} \mu^{-1} \lambda^{-\gamma},
\]
where we have bounded the \( L^1 \) norm by \( L^r \) norm. Thus
\[
\| B(A_k, \Phi_k(\sigma \cdot)) \|_1 \lesssim \mu^{-1} \frac{d}{r} \mu^{-1} \frac{d}{r} \mu^{-1} \lambda^{-\gamma},
\]
which together with the bound
\[
\int_{[0,1]} |\partial_t \tilde{g}_k(\sigma t)| \, dt \lesssim \kappa,
\]
implies that
\[
\| R_{\text{tem},1} \|_{L^1_{t,x}} \leq C_R \sigma \mu^{-1} \lambda^{-\gamma}
\]
\[
\leq C_R \lambda^{-\gamma},
\]
where we have also used Lemma 7.1.

For \( \lambda \) sufficiently large, we have
\[
\| R_{\text{tem},1} \|_{L^1_{t,x}} \leq \frac{\delta}{32}.
\]

(2) \( R_{\text{tem},2} \) estimate:
We treat the second term \( R_{\text{tem},2} \) as an error and use Lemma 2.2 to obtain that
\[
\| R_{\text{tem},2} \|_1 \leq C_R |\tilde{g}_k(\sigma t)| \sum_k \| \Phi_k(\sigma \cdot) \|_1.
\]

Using Proposition 5.11 and (6.13), integrating in time gives
\[
\| R_{\text{tem},2} \|_{L^1_{t,x}} \leq C_R \tau^{\frac{d}{2} - 1} \mu^{-1} \frac{d}{r} \mu^{-1} \frac{d}{r} \mu^{-1} \lambda^{-\gamma}.
\]

Thanks to Lemma 7.1, for \( \lambda \) sufficiently large, we have
\[
\| R_{\text{tem},2} \|_{L^1_{t,x}} \leq \frac{\delta}{32}.
\]

\[ \square \]

9.2. Linear error.

**Lemma 9.2** (\( R_{\text{lin}} \) estimate). For \( \lambda \) sufficiently large,
\[
\| R_{\text{lin}} \|_{L^1_{t,x}} \leq \frac{\delta}{16}.
\]
Proof. We start with Hölder’s inequality
\[ \| R_{\text{lin}} \|_{L^1_{t,x}} \leq \| \theta \|_{L^1_{t,x}} \| u \|_{L^\infty_{t,x}} + \| \rho \|_{L^\infty_{t,x}} \| w \|_{L^1_{t,x}}. \]

On one hand, by Hölder’s inequality we get
\[ \| \theta_p + \theta_c \|_{L^1_{t,x}} \leq C_R \nu^{-1} \sum_{1 \leq k \leq d} \| \tilde{g}_k(\sigma t) A_k(t,x) \Phi_k(\sigma x) \|_{L^1_{t,x}} \]
\[ \leq C_R \nu^{-1} \sum_{1 \leq k \leq d} \| \tilde{g}_k \|_{L^1([0,1])} \| \Phi_k \|_1 \quad (9.4) \]
\[ \leq C_R \nu^{-1} \mu^{d-1 - d - 1}. \]

By definition of \( \theta_o \) (7.16) we have
\[ \| \theta_o \|_{L^1_{t,x}} \lesssim C_R \sigma^{-1} \| h \|_{L^1([0,1])} \lesssim R \sigma^{-1}. \quad (9.5) \]

On the other hand, since \( 1 \leq q < \infty \), by Lemma 8.5 and Lemma 8.6
\[ \| w \|_{L^1_{t,x}} \leq \| w_p \|_{L^1_{t,x} W^{1,q}} + \| w_c \|_{L^1_{t,x} W^{1,q}} \]
\[ \leq C_R \nu(\sigma^{-1} + \kappa^{-1}). \quad (9.6) \]

Combining (9.4) and (9.6) we have
\[ \| R_{\text{lin}} \|_{L^1_{t,x}} \leq C_{\rho, u, R, \nu}(\mu^{\frac{d-1}{p}} - d - 1 + \kappa^{-1} + \sigma^{-1}). \]

Thanks to Lemma 7.1, for sufficiently large \( \lambda \) we have
\[ \| R_{\text{lin}} \|_{L^1_{t,x}} \leq \frac{\delta}{16}. \quad (9.7) \]

9.3. Correction error.

Lemma 9.3 (\( R_{\text{cor}} \) estimate). For \( \lambda \) sufficiently large,
\[ \| R_{\text{cor}} \|_{L^1_{t,x}} \leq \frac{\delta}{16}. \]

Proof. By Hölder’s inequality we have
\[ \| R_{\text{cor}} \|_{L^1_{t,x}} \leq \| \| \theta \|_{L^1_{t,x} W^{1,p}} \| \| w_c \|_{L^\infty_{t,x} L^\nu'} + (\| \theta_o \|_{L^1_{t,x} L^p} + \| \theta_c \|_{L^1_{t,x} L^p}) \| w \|_{L^\infty_{t,x} L^\nu'}. \]

All terms have been estimated before, and by Lemma 8.2, 8.3, 8.4, 8.5, 8.6 we have
\[ \| R_{\text{cor}} \|_{L^1_{t,x}} \lesssim R \nu^{-1} \| w_c \|_{L^\infty_{t,x} L^\nu'} + \nu(\| \theta_o \|_{L^1_{t,x} L^p} + \| \theta_c \|_{L^1_{t,x} L^p}) \]
\[ \leq C_R \sigma^{-1} (\nu^{-1} + \nu), \]
which concludes the proof.

9.4. Oscillation errors. We will estimate \( R_{\text{osc}} \) according to the decomposition in Lemma 7.5.

For reference, we recall that
\[ R_{\text{osc}} = R_{\text{osc,x}} + R_{\text{osc,t}} + R_{\text{far}} + R_{\text{app}} + R_{\text{rem}}, \]
where \( R_{\text{osc,x}} \) is the error of high frequency in space
\[ R_{\text{osc,x}} = \tilde{g}_c(\sigma t) g_c(\sigma t) \sum_{1 \leq k \leq d} B(A_k B_k, (\Phi_k \mathbf{W}_k(\sigma x) - \int_T \Phi_k \mathbf{W}_k \, dx)), \]
\( R_{\text{osc,x}} \) is the error of high frequency in time
\[ R_{\text{osc,t}} = \sigma^{-1} h(\sigma t) \mathcal{R} \sum_{1 \leq k \leq d} \mathbf{e}_k \cdot \partial_t \nabla (\chi^2_k R_k), \]
\( R_{\text{osc,t}} \) is the error of high frequency in time
$R_{\text{far}}$ is the far field error

$$R_{\text{far}} = \tilde{g}_\kappa(\sigma t) g_\kappa(\sigma t) \sum_{k \neq k'} A_k B_{k'}, \Phi_k W_k(\sigma x),$$

$R_{\text{appr}}$ is the approximation error

$$R_{\text{appr}} = \tilde{g}_\kappa(\sigma t) g_\kappa(\sigma t) \sum_{1 \leq k \leq d} B(A_k B_k, \text{div} R_k(\sigma x)),$$

and $R_{\text{rem}}$ is the remainder error

$$R_{\text{rem}} = \sum_{1 \leq k \leq d} (1 - \chi^2_k) R_k \mathbf{e}_k.$$

We start with $R_{\text{osc},x}$.

**Lemma 9.4** ($R_{\text{osc},x}$ estimate). For $\lambda$ sufficiently large, $\|R_{\text{osc},x}\|_{L^1_{t,x}} \leq \frac{\delta}{16}$.

**Proof.** Denote $\Theta_k \in C^\infty_0(\mathbb{T}^d)$ by

$$\Theta_k \equiv \Phi_k W_k - \int_{\mathbb{T}^d} \Phi_k W_k \, dx$$

so that $R_{\text{osc},x}$ reads

$$R_{\text{osc},x} = \tilde{g}_\kappa(\sigma t) g_\kappa(\sigma t) \sum_{1 \leq k \leq d} B\left(\nabla(A_k B_k), \Theta_k(\sigma \cdot)\right).$$

We take $L^1$ norm in space to obtain

$$\|R_{\text{osc},x}(t)\|_{1} \leq \|\tilde{g}_\kappa(\sigma t) g_\kappa(\sigma t)\| \sum_{1 \leq k \leq d} \|B(\nabla(A_k B_k), \Theta_k(\sigma \cdot))\|_1.$$ 

Applying Lemma 2.2 gives

$$\left\|B(\nabla(A_k B_k), \Theta_k(\sigma \cdot))\right\|_1 \lesssim C_R \sigma^{-1} \|R\Theta_k\|_1.$$

It follows that

$$\|R_{\text{osc},x}(t)\|_{1} \leq C_R \|\tilde{g}_\kappa(\sigma t) g_\kappa(\sigma t)\| \sigma^{-1}.$$

So for $L^1_{t,x}$ norm, we have

$$\|R_{\text{osc},x}\|_{L^1_{t,x}} \leq C_R \sigma^{-1}.$$

**Lemma 9.5** ($R_{\text{osc}}$ estimate). For $\lambda$ sufficiently large, $\|R_{\text{osc}}\|_{L^1_{t,x}} \leq \frac{\delta}{16}$.

**Proof.** By Lemma 2.1, we have

$$\|R_{\text{osc}}\|_{L^1_{t,x}} \lesssim \sigma^{-1} \|h(\sigma) \sum_{1 \leq k \leq d} e_k \cdot \partial_k \nabla(\chi_k^2 R_k)\|_{L^1_{t,x}}.$$ 

It follows from Hölder’s inequality that

$$\|R_{\text{osc}}\|_{L^1_{t,x}} \leq C_R \sigma^{-1} \|h(\sigma)\|_{L^1([0,1])} \leq C_R \sigma^{-1}.$$

**Lemma 9.6** ($R_{\text{rem}}$ estimate). There holds $\|R_{\text{rem}}\|_{L^1_{t,x}} \leq \frac{\delta}{2}$. 

Proof. We need to estimate
\[ \| R_{\text{rem}} \|_{L^1_t L^x} \leq \sum_{1 \leq k \leq d} \| (1 - \chi_k^2) R_k \epsilon_k \|_{L^1_t L^x}. \]

Note that
\[(t, x) \in \text{supp}(1 - \chi_k^2) \Rightarrow |R_k| \leq \frac{\delta}{4d} \quad \text{or} \quad t \in I^c_r, \]
and thus by (7.3) we have
\[ \| R_{\text{rem}} \|_{L^1_t L^x} \leq \sum_{1 \leq k \leq d} \delta \frac{4d}{d} \times \| (1 - \chi_k^2) R_k \|_{L^1_t L^x} (1 - \chi_k^2) |R_k| \text{d}x \text{d}t \]
\[ \leq d \times (|0, 1 \times \mathbb{T}^d| \times \delta \frac{4d}{d} + 2r \times \| R \|_{L^\infty_t L^x}) = \frac{\delta}{2}. \]

\[ \square \]

Lemma 9.7 (R_{far} estimate). For \( \lambda \) sufficiently large,
\[ \| R_{\text{far}} \|_{L^1_t L^x} \leq \frac{\delta}{16}. \]

Proof. Taking \( L^1 \) norm in space gives
\[ \| R_{\text{far}} \|_1 \leq |\tilde{g}_\kappa(\sigma t) g_\kappa(\sigma t)| \sum_{k \neq k'} \| A_k B_{k'}, \Phi_k W_{k'}(\sigma x) \|_1. \]

Using Theorem 5.14 we get
\[ \| R_{\text{far}} \|_1 \leq C_R |\tilde{g}_\kappa(\sigma t) g_\kappa(\sigma t)| \mu^{-d}. \]

Integrating in time and using (6.13) we have
\[ \| R_{\text{far}} \|_{L^1_t L^x} \leq C_R \mu^{-d}, \]
which concludes the proof once taking \( \lambda \) sufficiently large. \( \square \)

Lemma 9.8 (R_{appr} estimate). For \( \lambda \) sufficiently large,
\[ \| R_{\text{appr}} \|_{L^1_t L^x} \leq \frac{\delta}{16}. \]

Proof. We need to estimate
\[ \| R_{\text{appr}} \|_{L^1_t L^x} \leq |\tilde{g}_\kappa(\sigma t) g_\kappa(\sigma t)| \sum_{1 \leq k \leq d} \| \mathcal{B} (A_k B_k, \text{div} (R_k(\sigma x))) \|_{L^1_t L^x}. \]  

By Lemma 2.2 we have
\[ \| \mathcal{B} (A_k B_k, \text{div} (R_k(\sigma x))) \|_{L^1(\mathbb{T}^d)} \leq C_R \| \text{div} (R_k(\sigma x)) \|_{L^1(\mathbb{T}^d)} \leq C_R \| R_k \|_{L^1(\mathbb{T}^d)}. \]

Thanks to Theorem 5.14 and (6.13), it follows from (9.8) and (9.9) that
\[ \| R_{\text{appr}} \|_{L^1_t L^x} \leq C_R \mu^{-1}, \]
which completes the proof. \( \square \)
9.5. Conclusion of the proof of Proposition 3.1. We can finish the proof of Proposition 3.1 by showing (3.7).

We take \( \lambda \) sufficiently large so that all lemmas in this section and Lemma 8.7 hold. Then the new defect field \( R \) verifies

\[
\| R \|_{L^1([0,1 \times T^d])} \leq \| R_{\text{term}} \|_{L^1_{t,x}} + \| R_{\text{lin}} \|_{L^1_{t,x}} + \| R_{\text{cor}} \|_{L^1_{t,x}} + \| R_{\text{osc},x} \|_{L^1_{t,x}} + \| R_{\text{osc},t} \|_{L^1_{t,x}} + \| R_{\text{far}} \|_{L^1_{t,x}} + \| R_{\text{appr}} \|_{L^1_{t,x}} + \| R_{\text{rem}} \|_{L^1_{t,x}} \\
\leq 7 \times \frac{\delta}{16} + \frac{\delta}{2} \\
\leq \delta.
\]

**APPENDIX A. INVERSE LAPLACIAN FOR SCHWARTZ FUNCTIONS**

In this section, we prove that the inverse Laplacian of a Schwartz function has certain decay at infinity provided it has zero mean and zero first moments.

**Lemma A.1.** Suppose \( f \in \mathcal{S}([R^d]) \) and

\[
\int f(x)x^\alpha \, dx = 0 \quad \text{for} \quad 0 \leq |\alpha| \leq 1,
\]

then \( h := \Delta^{-1} f \in W^{m,p}([R^d]) \cap C^\infty([R^d]) \) for \( 1 < p \leq \infty \) and \( m \geq 0 \).

**Proof.** We provide a proof using homogeneous Littlewood-Paley decomposition on \( [R^d] \). For any \( q \in \mathbb{Z} \), let \( \Delta_q \) be the \( q \)-th Littlewood-Paley projection onto frequencies \( \lambda_q = 2^q \) so that

\[
f = \sum_{q=\infty}^{\infty} \Delta_q f.
\]

To show that \( h = \Delta^{-1} f \in W^{m,p}, \) for \( k \geq 0 \) and \( 1 < p \leq \infty \), by the Littlewood-Paley theorem, it suffices to show there exists a constant \( C_p > 0 \) for \( 1 < p \leq \infty \) such that

\[
\| \Delta_q f \|_{L^p([R^d])} \lesssim \lambda_q^{2+C_p} \quad \text{for all} \quad q \leq 0,
\]

namely, low frequencies are suitably small.

By the representation formula, we have

\[
\Delta_q f(x) = \int_{[R^d]} \varphi_q(x - y)f(y) \, dy. \quad (A.1)
\]

We use the Taylor theorem to expand \( \varphi_q(x - y) \) to the first order

\[
\varphi_q(x - y) = \varphi_q(x) - \nabla \varphi_q(x) \cdot y + \sum_{|\beta|=2} R_\beta(x,y)y^\beta, \quad (A.2)
\]

where the remainder is given by

\[
R_\beta(x,y) = \int_{0}^{1} (1-t)D^\beta \varphi(x - ty) \, dt. \quad (A.3)
\]

By assumptions of zero mean and zero first moments of \( f \), combined (A.1) and (A.2) we obtain

\[
\Delta_q f(x) = \sum_{|\beta|=2} \int_{[R^d]} R_\beta(x,y)y^\beta f(y) \, dy. \quad (A.4)
\]

By Minkowski’s inequality,

\[
\| \Delta_q f \|_{L^p([R^d])} \lesssim \sum_{|\beta|=2} \left( \int_{[R^d]} \left| R_\beta(x,y) \right|^p \, dx \right)^{\frac{1}{p}} |f(y)| \|y^2| \, dy.
\]
We can use Minkowski’s inequality to estimate \( R_\beta(x,y) \) as well to obtain

\[
\left( \int |R_\beta(x,y)|^p \, dx \right)^{\frac{1}{p}} \lesssim \int_0^1 \left( \int |\nabla^2 \varphi_q(x-ty)|^p \, dx \right)^{\frac{1}{p}} \, dt \\
\lesssim \|\nabla^2 \varphi_q\|_p \\
\lesssim \lambda_q^{2 + d - \frac{d}{p}},
\]

for \( |\beta| = 2 \). Therefore,

\[
\|\Delta_q f\|_{L^p(\mathbb{R}^d)} \lesssim \sum_{|\beta|=2} \int \lambda_q^{\beta + d - \frac{d}{p}} |f(y)||y|^2 \, dy \\
\lesssim \lambda_q^{2 + d - \frac{d}{p}}.
\]

\[\Box\]

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