SUPPLEMENTARY MATERIAL

Insights into traumatic brain injury from MRI of harmonic brain motion

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Mathematical details of the data analysis methods to estimate propagation direction and strain are included here for completeness.

S1. Amplitude-weighted propagation direction

To identify prominent directions of propagation for harmonic waves with frequency $\omega$, the following steps are implemented.

1. Each scalar component $u(X,t)$ is Fourier transformed in time to extract the Fourier coefficient field, $U(X)$, where $u(X,t) = \text{Re} \left[ U(X) \exp(i\omega t) \right]$ (the physical displacement is the real part of the complex expression).

2. The $U(X)$ field is further decomposed into harmonic functions of space, each with a different 3D wavenumber vector, $k$, as:

$$ U(X) = \sum_{m=1}^{M} \sum_{n=1}^{N} \sum_{p=1}^{P} A_{mnp} \exp(-i k_{mnp} \cdot X) \quad (S1.1) $$

or, alternatively

$$ U(X) = \sum_{m=1}^{M} \sum_{n=1}^{N} \sum_{p=1}^{P} A_{mnp} \exp(-i (k_{m}x + k_{n}y + k_{p}z)) \cdot (S1.2) $$

The coefficients $A_{mnp}$ (the “k-space” components of the image) are obtained by spatial Fourier transform of the $U(X)$ field.

3. Then, a directional spatial filter is used to isolate plane wave components inside a conical sector centered around the vector, $n_q$. First the angle of every wavenumber vector, $k_{mnp}$, relative to this direction is defined as $\theta_{mnpq} = \cos^{-1}(n_q \cdot n_{mnp})$ where $n_{mnp} = \frac{k_{mnp}}{|k_{mnp}|}$. The directional spatial filter is defined by,

$$ f(\theta_{mnpq}) = \begin{cases} \cos^2 4\theta_{mnpq}, & |\theta_{mnpq}| \leq \pi/8 \\ 0, & |\theta_{mnpq}| > \pi/8 \end{cases} \quad (S1.3) $$

4. The directionally-filtered scalar field

$$ U_q(X) = \sum_{m=1}^{M} \sum_{n=1}^{N} \sum_{p=1}^{P} f(\theta_{mnpq}) A_{mnp} \exp(-i k_{mnp} \cdot X) \quad (S1.4) $$

is the part of the original field explained by propagating waves with wavenumber vectors within the conical sector centered on $n_q$. It is obtained by inverse Fourier transformation of the filtered k-space field.
5. Finally, the amplitude-weighted propagation direction vector field, \( N^{(u)}(X) \), is defined by from the vector sum of multiple unit vectors, \( n_q \), for \( q = 1,2,3,\ldots,Q \), evenly distributed on the unit sphere, each weighted by the amplitude of the field filtered about that direction

\[
N^{(u)}(X) = \sum_{q=1}^{Q} n_q |U_q(X)|
\]  

(S1.5)

S2. Strain tensor, octahedral shear strain and axonal strain

Displacement fields \( \mathbf{u}(X,t) = [u_{LR}(x,y,z,t), u_{AP}(x,y,z,t), u_{SI}(x,y,z,t)] \), were differentiated with respect to spatial coordinates \( (x,y,z) \) by analytically calculating the derivatives of polynomial functions fitted to the displacement data (26). A matrix (3x3) representation of the strain tensor in Voigt notation (27) was constructed at each voxel from the appropriate derivatives at that voxel. Using the shorthand notation: \( u = u_{LR}, v = u_{AP}, \) and \( w = u_{SI} \), we write the strain tensor as:

\[
\varepsilon(x,y,z) = \begin{bmatrix}
\varepsilon_{xx} & \varepsilon_{xy} & \varepsilon_{xz} \\
\varepsilon_{xy} & \varepsilon_{yy} & \varepsilon_{yz} \\
\varepsilon_{xz} & \varepsilon_{yz} & \varepsilon_{zz}
\end{bmatrix} = \begin{bmatrix}
\frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\
\frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\
\frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z}
\end{bmatrix}
\]  

(sym.) (sym.) (sym.) (S2.1)

Octahedral shear strain (OSS) is then defined as:

\[
\varepsilon_{oss} = \frac{2}{3} \sqrt{\left(\varepsilon_{xx} - \varepsilon_{yy}\right)^2 + \left(\varepsilon_{yy} - \varepsilon_{zz}\right)^2 + \left(\varepsilon_{zz} - \varepsilon_{xx}\right)^2 + 6\left(\varepsilon_{xy}^2 + \varepsilon_{yz}^2 + \varepsilon_{xz}^2\right)}
\]  

(S2.3)

Strain, \( \varepsilon_a \), in the axonal fiber direction was estimated from the strain tensor and the unit vector in the fiber direction. The unit vector in the fiber direction, \( \mathbf{a} = [a_x, a_y, a_z] \), was estimated as the direction of maximal diffusivity obtained from diffusion tensor imaging (DTI). The fiber stretch is then defined by the expression:

\[
\varepsilon_a = \mathbf{a}^T \cdot \varepsilon \cdot \mathbf{a} = a_x \varepsilon_{xx}^2 + a_y \varepsilon_{yy}^2 + a_z \varepsilon_{zz}^2 + 2a_x \varepsilon_{xy} \varepsilon_{xz} + 2a_y \varepsilon_{xy} \varepsilon_{yz} + 2a_z \varepsilon_{xz} \varepsilon_{yz}
\]  

(S2.4)