Abstract. Let $\Omega$ be a circular domain, that is, an open disk with finitely many closed disjoint disks removed. Denote by $H^\infty(\Omega)$ the Banach algebra of all bounded holomorphic functions on $\Omega$, with pointwise operations and the supremum norm. We show that the topological stable rank of $H^\infty(\Omega)$ is equal to 2. The proof is based on Suarez’s theorem that the topological stable rank of $H^\infty(D)$ is equal to 2, where $D$ is the unit disk. We also show that for domains symmetric to the real axis, the Bass and topological stable ranks of the real symmetric algebra $H^\infty_R(\Omega)$ are 2.

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1. Introduction

The aim of this short note is to prove that the topological stable rank of the Banach algebra $H^\infty(\Omega)$ of all bounded analytic functions on $\Omega$ is equal to 2, where $\Omega$ denotes a circular domain. By conformal equivalence, the same assertion will hold for any finitely connected, proper domain in $\mathbb{C}$ whose boundary does not contain any one-point components. We shall also show that for circular domains $\Omega$ that are symmetric to the real axis, the real algebra $H^\infty_R(\Omega)$ has the Bass and topological stable rank 2. Here $z^*$ denotes the complex conjugate of $z$. The precise definitions are given below.

The notion of the topological stable rank of a Banach algebra was introduced by M. Rieffel in [6], in analogy with the notion of the (Bass) stable rank of a ring defined by H. Bass [1]. We recall these definitions now.

Definition 1.1. Let $R$ be a commutative ring with identity element 1. An $n$-tuple $a := (a_1, \ldots, a_n) \in R^n$ is said to be invertible or unimodular, (for short $a \in U_n(R)$), if there exists a solution $(x_1, \ldots, x_n) \in R^n$ of the Bezout equation $\sum_{j=1}^n a_j x_j = 1$. We say that $a = (a_1, \ldots, a_n, a_{n+1}) \in U_{n+1}(R)$
is reducible if there exist \( h_1, \ldots, h_n \in \mathbb{R} \) such that \((a_1 + h_1 a_{n+1}, \ldots, a_n + h_n a_{n+1}) \in U_n(R)\).

The Bass stable rank of \( R \) (denoted by \( \text{bsr} \ R \)) is the least \( n \in \mathbb{N} \) such that every element \( a = (a_1, \ldots, a_n, a_{n+1}) \in U_{n+1}(R) \) is reducible, and it is infinite if no such integer \( n \) exists.

Let \( A \) be a commutative Banach algebra with unit element \( 1 \). The least integer \( n \) for which \( U_n(A) \) is dense in \( A^n \) is called the topological stable rank of \( A \) (denoted by \( \text{tsr} \ A \)) and we define \( \text{tsr} \ A = \infty \) if no such integer \( n \) exists.

It is well known that \( \text{bsr} \ A \leq \text{tsr} \ A \); see [6, Corollary 2.4].

In the case of the classical algebra \( H^\infty(\mathbb{D}) \) of the unit disk \( \mathbb{D} = \{ z \in \mathbb{C} : |z| < 1 \} \), D. Suarez [9] showed that the topological stable rank is 2. We will use this result in order to derive our result for \( H^\infty(\Omega) \) when \( \Omega \) is a circular domain.

**Theorem 1.1** (Suarez [9]). The topological stable rank of \( H^\infty(\mathbb{D}) \) is 2.

Let us recall that previously Tolokonnikov [10] showed that the Bass stable rank of \( H^\infty(\Omega) \) is 1. That was based on S. Treil’s [11] fundamental result that \( H^\infty(\mathbb{D}) \) has the Bass stable rank 1.

In [5] Mortini and Wick showed that the Bass and topological stable ranks of the real symmetric algebra

\[
H^\infty_\mathbb{R}(\mathbb{D}) = \{ f \in H^\infty(\mathbb{D}) : (f(z^*))^* = f(z) \ (z \in \mathbb{D}) \}
\]

are 2. Using this we will show that \( \mathbb{D} \) can be replaced by an arbitrary circular domain symmetric to the real axis.

We now give the precise definition of a circular domain, and also fix some convenient notation.

**Notation.** Let \( \Omega \) be a circular domain, of connectivity \( n \), that is, an open disk, \( D \), with \( n - 1 \) closed disjoint disks removed. Then \( \Omega \) is the intersection of \( n \) simply connected domains, \( \Omega = \Omega_0 \cap \Omega_1 \cap \cdots \cap \Omega_{n-1} \), where \( \Omega_i = \mathbb{C} \setminus D_i \), the \( D_i \) being open disks in the extended complex plane \( \overline{\mathbb{C}} = \mathbb{C} \cup \{ \infty \} \). We assume that \( \infty \in D_0 \). The boundary of a set \( \Omega \subset \mathbb{C} \) is denoted by \( \partial \Omega \).

Let \( H(\Omega) \) denote the set of all holomorphic functions on \( \Omega \), and let \( H^\infty(\Omega) \) be the Banach algebra of all bounded holomorphic functions on \( \Omega \), with pointwise operations and the supremum norm.

If \( \Omega \) is real symmetric (that is, \( z \in \Omega \) if and only if \( z^* \in \Omega \)), then we use the symbol \( H^\infty_\mathbb{R}(\Omega) \) to denote the set of functions \( f \) belonging to \( H^\infty(\Omega) \) that are real symmetric, that is, \( f(z) = (f(z^*))^* \ (z \in \Omega) \).

An example of a circular domain is the annulus \( \Lambda = \{ z \in \mathbb{C} : r_1 < |z| < r_2 \} \), where \( 0 < r_1 < r_2 \). In this case \( \Lambda = \Omega_0 \cap \Omega_1 \), where

\[
\Omega_0 := \{ z \in \mathbb{C} : |z| < r_2 \},
\]

\[
\Omega_1 := \{ z \in \mathbb{C} : |z| > r_1 \}.
\]

\(^1\)We tacitly assume that the closures of the removed disks are contained within \( D \).
Thus \( \Omega_0 = \mathbb{C} \setminus D_0 \) and \( \Omega_1 = \mathbb{C} \setminus D_1 \), where
\[
D_0 := \{ z \in \mathbb{C} : |z| > r_2 \}, \\
D_1 := \{ z \in \mathbb{C} : |z| < r_1 \}.
\]

Our main results are the following:

**Theorem 1.2.** Let \( \Omega \) be a circular domain. The topological stable rank of \( H^\infty(\Omega) \) is 2.

**Theorem 1.3.** Let \( \Omega \) be a circular domain symmetric to the real axis. Then the topological and Bass stable rank of \( H^\infty(\Omega) \) is 2.

### 2. Preliminaries

The following Cauchy decomposition is well known (for \( H^p(\Omega) \) functions, \( 1 \leq p \leq \infty \)) [4, Proposition 4.1, p. 86] or [3, Theorem 10.12, p.181].

**Lemma 2.1.** Let \( \Omega = \bigcap_{j=0}^{n-1} \Omega_j \) be a circular domain of connectivity \( n \), \( n \in \mathbb{N} \). Then any \( f \in H(\Omega) \) can be decomposed as \( f = f_0 + f_1 + \cdots + f_{n-1} \), where \( f_j \in H(\Omega_j) \). If additionally the real part of \( f \) is bounded above on \( \Omega \), then the same is true for the \( f_j \).

**Proof.** Apply Cauchy’s integral formula for a null homologic cycle, close to the boundary of \( \Omega \), and use the principle of analytic continuation. Now let us assume that the real part of \( f \) is bounded above on \( \Omega \). Fix \( k \in \{0, 1, \ldots, n-1\} \). Since \( f_j(\infty) = 0 \) for \( j = 1, 2, \ldots, n-1 \) and \( \sum_{j \neq k} f_j \) is holomorphic in a neighborhood of the set \( \overline{\mathbb{C}} \setminus \Omega_k \), we see that the real part of each \( f_j \) is bounded above on \( \Omega_j \), for \( j = 0, 1, \ldots, n-1 \). \( \square \)

We will use the following factorization result; the non-symmetric version appears in [10, Lemma 1]. Since in our viewpoint, the proof of the annulus-case by Tolokonnikov is not complete, we give a more general proof, that includes also the symmetric case.

Recall that a Blaschke product \( B \) with zeros \( (z_j) \) in the disk
\[
D(a, r) = \{ z \in \mathbb{C} : |z - a| < r \}
\]
has the form \( B(z) = b(\frac{z-a}{r}) \), where \( b \) is the usual Blaschke product of the unit disk with zeros \( w_j = \frac{z_j-a}{r} \). Similarly, the Blaschke product \( B_e \) with zeros \( (z_j) \) in the exterior of the disk \( D(a, r) \) has the form \( B_e(z) = b(\frac{r}{z-a}) \) where \( b \) is the usual Blaschke product of the unit disk with zeros \( w_j = \frac{r}{z_j-a} \).

We call these functions generalized Blaschke products.

**Proposition 2.2.** Let \( \Omega \) be a circular domain of connectivity \( n \), \( n \in \mathbb{N} \), and let \( \overline{D_j} \) denote the bounded components of \( \overline{\mathbb{C}} \setminus \Omega \), \( (j = 1, \ldots, n-1) \), that is, \( D_j \) is the open disk \( D(a_j, r_j) \). Define
\[
\Omega_j = \mathbb{C} \setminus \overline{D_j}, \quad j = 1, \ldots, n-1, \\
\Omega_0 = \Omega \cup \left( \bigcup_{j=1}^{n-1} D_j \right).
\]
Then every function $f$ in $H^\infty(\Omega)$, $f \neq 0$, can be decomposed as:

$$f = f_0 \cdot f_1 \cdot f_2 \cdots f_{n-1} \cdot r,$$

where

$$f_j \in H^\infty(\Omega_j) \cap \left( H^\infty\left( \bigcup_{k \neq j} D_k \right) \right)^{-1}, \quad j = 0, 1, 2, \ldots, n - 1,$$

and where $r$ is a rational function with poles and zeros contained in the set $\{a_1, \cdots, a_{n-1}\}$.

If $\Omega$ is a domain symmetric to the real axis, and $f \in H^\infty_R(\Omega)$, then each of the functions $f_j$ and $r$ above can be taken to be real symmetric themselves.

**Proof.** We may assume that $\Omega$ is the circular domain

$$\Omega = D(a_0, r_0) \setminus \bigcup_{j=1}^{n-1} D(a_j, r_j),$$

where $D_j = \overline{D(a_j, r_j)} \subseteq D(a_0, r_0)$ and where the closures of the $D_j$ ($j = 1, \ldots, n - 1$) are disjoint.

Let $D_0 := D(a_0, r_0)$. Set $\Omega_j = \overline{\Omega \setminus D_j}$, $(j = 0, 1, \ldots, n - 1)$. It is well known that the sequence $(z_k)$ of zeros of $f$ satisfies the generalized Blaschke condition; that is $\sum_k \text{dist}(z_k, \partial \Omega)$ converges (see [4, 8]). Split $(z_k)$ into $n$ sequences $(z_{k,j})_k$, $j = 0, 1, \ldots, n - 1$, so that the cluster points of $(z_{k,j})_k$ are exactly those of $(z_k)$ that belong to $\partial D_j$, $j = 0, 1, \ldots, n - 1$. Let $B_j$ be the generalized Blaschke product formed with the zeros $(z_{k,j})_k$ of $f$, $j = 0, 1, \ldots, n - 1$. It is clear that the zeros of $B_j$ cluster only at $\partial D_j$, $0 \leq j \leq n - 1$.

Then $f$ can be written as $f = B_0 \cdot B_1 \cdot B_{n-1} \cdot g$, where $g \in H^\infty(\Omega)$ and $g$ has no zeros in $\Omega$ (note that here we have used the fact that division by $B_j$ does not change the relative supremum of $f$ on the boundary of $\Omega_j$).

By [2] p. 111-112], there exist $k_j \in \mathbb{Z}$ and $h$ holomorphic in $\Omega$, such that

$$g(z) = \prod_{j=1}^{n-1} (z - a_j)^{k_j} e^{h(z)}.$$

Note that the real part of $h$ is bounded above on $\Omega$.

By Lemma 2.1 there exist $h_j \in H(\Omega_j)$ such that $h = h_0 + h_1 + \cdots + h_{n-1}$ and the real part of each $h_j$ is bounded above on $\Omega_j$, for $j = 0, 1, \ldots, n - 1$.

Hence the functions $e^{h_j} \in H^\infty(\Omega_j)$.

Now $f = r \prod_{j=0}^{n-1} B_j e^{h_j}$, where $r(z) = \prod_{j=1}^{n-1} (z - a_j)^{k_j}$ gives the desired factorization.

In case of a symmetric domain $\Omega$ and $f \in H^\infty_R(\Omega)$, we can choose $a_j$ to be real if the disk $D(a_j, r_j)$ meets the real line, and the other $a_j$ in pairs
Thus we can ensure that \( r \) is real symmetric, because the exponents \( k_j \) are the same for \( a_j \) and \( a_j^* \) due to the fact that

\[
k_j = \frac{1}{2\pi i} \oint_{\Gamma} \frac{g'(z)}{g(z)} \, dz,
\]

where \( \Gamma \) denotes a suitable small circle around \( a_j \).

The Blaschke products above are easily seen to be choosable in a real symmetric fashion. Hence, since \( f \) is real symmetric, we conclude that \( g \) is real symmetric as well. Therefore, \( e^h \) is real symmetric; that is

\[
e^{h(z)} = (e^{h(z^*)})^* = e^{(h(z^*))^*}.
\]

Since \( \Omega \) is a domain, \( h(z) - (h(z^*))^* \) equals a constant \( 2k\pi i \) for some \( k \in \mathbb{Z} \). Therefore

\[
h(z) = \frac{h(z) + (h(z^*))^*}{2} + \frac{h(z) - (h(z^*))^*}{2} = \frac{h(z) + (h(z^*))^* + k\pi i}{2},
\]

Now in Cauchy’s decomposition, we simply consider the symmetric functions \( H_j(z) := \frac{h_j(z) + (h_j(z^*))^*}{2} \), and derive

\[
h(z) = \sum_{j=0}^{n-1} H_j(z) + k\pi i.
\]

Thus we have one of the following cases

\[
e^{h(z)} = e^{\sum_{j=0}^{n-1} H_j(z)} \quad (z \in \Omega)
\]

or

\[
e^{h(z)} = -e^{-\sum_{j=0}^{n-1} H_j(z)} \quad (z \in \Omega).
\]

In the latter case we take \(-r\) instead of \( r \). Thus all the factors in

\[
f = r \prod_{j=0}^{n-1} B_j e^{H_j}
\]

are symmetric.

We recall that the corona theorem holds for \( H^\infty(\Omega) \) when \( \Omega \) is a circular domain; see for example [4, Theorem 6.1, p.195].

**Proposition 2.3.** Let \( \Omega \) be a circular domain. Then \((f_1, \ldots, f_n)\) is invertible in \( H^\infty(\Omega) \) if and only if there exists a \( \delta > 0 \) such that

\[
\sum_{j=1}^{n} |f_j(z)| \geq \delta \quad (z \in \Omega).
\]

This corona-theorem is of course true for \( H^\infty_{\mathbb{R}}(\Omega) \). Indeed, if \( f_j \in H^\infty_{\mathbb{R}}(\Omega) \) and \((g_1, \ldots, g_n)\) is a solution of \( \sum_{j=1}^{n} g_j f_j = 1 \) in \( H^\infty(\Omega) \), then \((\bar{g}_1, \ldots, \bar{g}_n)\) is a solution of the Bezout equation \( \sum_{j=1}^{n} \bar{g}_j f_j = 1 \) in \( H^\infty_{\mathbb{R}}(\Omega) \), where \( \bar{g}_j(z) := \frac{g(z) + (g_j(z^*))^*}{2} \quad (z \in \Omega) \).
We will need two technical results, which are proved below. In the following, the notation $M(R)$ is used to denote the maximal ideal space of the unital commutative Banach algebra $R$. Also the complex homomorphism from $H^\infty(\Omega)$ to $\mathbb{C}$ of point evaluation at a point $z \in \Omega$ will be denoted by $\varphi_z$, that is, $\varphi_z(f) = f(z)$, $f \in H^\infty(\Omega)$.

Let $z_0 \in \Omega$. The set

$$M_{z_0}(H^\infty(\Omega)) = \{ \varphi \in M(H^\infty(\Omega)) : \varphi(z) = z_0 \}$$

is called the fiber of $M(H^\infty(\Omega))$ over $z_0$. It is well known (see [4]), that we have $\varphi(f) = 0$ for some $\varphi \in M_{z_0}(H^\infty(\Omega))$ if and only if $\liminf_{z \to z_0} |f(z)| = 0$. The zero set of $f \in H^\infty(\Omega)$ is the set $\{ \varphi \in M(H^\infty(\Omega)) : \varphi(f) = 0 \}$.

We need a Lemma that lets us decompose two functions that live on different circular domains. To this end, let $D_1, D_2$ be open disks in $\mathbb{C}$ such that $\overline{D_1} \cap \overline{D_2} = \emptyset$. Next, define $\Omega_j := \mathbb{C} \setminus \overline{D_j}$ for $j = 1, 2$. Suppose that $f_j \in H^\infty(\Omega_j)$ for $j = 1, 2$ are non-zero functions. Next, set

\[
Z_1 = \left\{ \xi \in \partial D_1 = \partial \Omega_1 : f_2(\xi) = 0 \text{ and } \liminf_{z \to \xi} |f_1(z)| = 0 \right\},
\]

\[
Z_2 = \left\{ \xi \in \partial D_2 = \partial \Omega_2 : f_1(\xi) = 0 \text{ and } \liminf_{z \to \xi} |f_2(z)| = 0 \right\}, \text{ and}
\]

\[
Z_3 = \left\{ a \in \Omega_1 \cap \Omega_2 : f_1(a) = f_2(a) = 0 \right\}.
\]

**Lemma 2.4.** Let $D_1, D_2$ be open disks in $\mathbb{C}$ such that $\overline{D_1} \cap \overline{D_2} = \emptyset$. Define $\Omega_j := \mathbb{C} \setminus \overline{D_j}$. Let $f_j \in H^\infty(\Omega_j)$ be nonzero functions. Then the zero sets of $f_1$ and $f_2$ meet in at most a finite number of fibers of $H^\infty(\Omega_1 \cap \Omega_2)$. In other words, there exist at most finitely many $z_j \in \Omega_1 \cap \Omega_2$ for which

$$\liminf_{z \to z_j} |f_1(z)| = \liminf_{z \to z_j} |f_2(z)| = 0.$$

Moreover, $f_1$ and $f_2$ can be written as

\[
f_1 = \prod_{z_j \in Z_2 \cup Z_3} (z - z_j)^{m_j} F_1, \text{ and}
\]

\[
f_2 = \prod_{z_j' \in Z_1 \cup Z_3} (z - z_j')^{m_j'} F_2
\]

where $F_j$ is in $H^\infty(\Omega_1 \cap \Omega_2)$ and has the property that for any element $\varphi \in M(H^\infty(\Omega_1 \cap \Omega_2))$ either $\varphi(F_1) \neq 0$ or $\varphi(F_2) \neq 0$.

Additionally, when $\lambda \in \Omega_1 \cap \Omega_2$, each $\varphi \in M_\lambda(H^\infty(\Omega_1 \cap \Omega_2))$ is such that $\varphi = \varphi_\lambda \in M(H^\infty(\Omega_1))$ whenever $\lambda \in \Omega_2$, or $\varphi = \varphi_\lambda \in M(H^\infty(\Omega_2))$ whenever $\lambda \in \Omega_1$.

**Proof.** It is clear that if $\varphi \in M(H^\infty(\Omega_1 \cap \Omega_2))$, then $\varphi \in M(H^\infty(\Omega_1))$ and $\varphi \in M(H^\infty(\Omega_2))$. 

Now the set \( Z_3 = \{ z \in \Omega_1 \cap \Omega_2 \mid f_1(z) = f_2(z) = 0 \} \) is finite, for otherwise, there is an accumulation point of zeros in \( \partial \Omega_1 \) or in \( \partial \Omega_2 \). But \( \partial \Omega_1 \) is contained in \( \Omega_2 \), and \( \partial \Omega_2 \) is contained in \( \Omega_1 \). So either \( f_1 \) or \( f_2 \) is identically 0, a contradiction.

Consider the set \( Z_2 \) and let \( \lambda \in Z_2 \). There are only finitely many zeros of \( f_1 \) on the circle \( \partial D_2 \subset \Omega_1 \), since \( f_1 \) is not identically zero. Similarly, we can argue in the case when \( \lambda \in Z_1 \). Thus, \( Z_1 \) is finite as well. This completes the proof.

It is clear that an analogous version holds true for the symmetric case.

**Lemma 2.5.** Let \( D_1, D_2 \) be open disks in \( \mathbb{C} \) such that \( \overline{D_1} \cap \overline{D_2} = \emptyset \). Define \( \Omega_1 := \mathbb{C} \setminus \overline{D_1} \) and \( \Omega_2 := \mathbb{C} \setminus \overline{D_2} \). Let \( f_1, g_1 \in H^\infty(\Omega_1) \) and \( f_2, g_2 \in H^\infty(\Omega_2) \) be nonconstant functions such that there exists \( \delta > 0 \) such that the following hold:

\[
\begin{align*}
\text{(P1)} & \quad \text{For all } z \in \Omega_1, \ |f_1(z)| + |g_1(z)| \geq \delta. \\
\text{(P2)} & \quad \text{For all } z \in \Omega_2, \ |f_2(z)| + |g_2(z)| \geq \delta.
\end{align*}
\]

Then, for every \( \varepsilon > 0 \), there exist \( F_1, G_1 \in H^\infty(\Omega_1) \), \( F_2, G_2 \in H^\infty(\Omega_2) \) such that

\[
\begin{align*}
\text{(C1)} & \quad (F_1, G_2) \text{ is invertible in } H^\infty(\Omega_1 \cap \Omega_2), \\
\text{(C2)} & \quad (G_1, F_2) \text{ is invertible in } H^\infty(\Omega_1 \cap \Omega_2), \\
\text{(C3)} & \quad (F_1, G_1) \text{ is invertible in } H^\infty(\Omega_1), \\
\text{(C4)} & \quad (F_2, G_2) \text{ is invertible in } H^\infty(\Omega_2), \text{ and} \\
\text{(C5)} & \quad \|f_1 - F_1\| + |g_1 - G_1| + |f_2 - F_2| + |g_2 - G_2| < \varepsilon.
\end{align*}
\]

In particular, \((F_1F_2, G_1G_2)\) is invertible in \( H^\infty(\Omega_1 \cap \Omega_2) \).

**Proof.** Consider the pair \((f_1, g_2) \in H^\infty(\Omega_1) \times H^\infty(\Omega_2)\). By Lemma 2.4 we may perturb the finitely many zeros of \( f_1 \) belonging to \( S_2 \cup S_3 \) and those of \( g_2 \) that lie in \( S_1 \) so that the new functions \( F_1 \) and \( G_2 \) form an invertible pair in \( H^\infty(\Omega_1 \cap \Omega_2) \). Now we do the same with the pair \((g_1, f_2) \in H^\infty(\Omega_1) \times H^\infty(\Omega_2)\). This gives an invertible pair \((G_1, F_2) \in H^\infty(\Omega_1 \cap \Omega_2)\). By choosing these perturbations sufficiently small, we see that the pairs \((F_1, G_1)\) and \((F_2, G_2)\) stay invertible in the associated space \( H^\infty(\Omega_1) \), respectively \( H^\infty(\Omega_2) \). This yields that \((F_1F_2, G_1G_2)\) is invertible in \( H^\infty(\Omega_1 \cap \Omega_2) \).

It is clear that an analogous version holds true for the symmetric case.

### 3. Proof of \( \text{tsr}(H^\infty(\Omega)) = 2 \)

**Proof of Theorem 3.2.** Let \( f, g \in H^\infty(\Omega) \). By Proposition 2.2 we can write

\[
\begin{align*}
f &= f_0 \cdot f_1 \cdots \cdot f_{n-1} \cdot r, \\
g &= g_0 \cdot g_1 \cdots \cdot g_{n-1} \cdot s.
\end{align*}
\]

where \( f_j \) and \( g_j \in H^\infty(\Omega_j) \). We note that since the rational functions \( r, s \) have zeros and poles only in the set \( \{a_1, \ldots, a_{n-1}\} \), it follows that \( r, s \) are invertible in \( H^\infty(\Omega) \). Since each \( \Omega_i \) is simply connected, it follows from
the fact that the topological stable rank of $H^\infty(\mathbb{D})$ is 2 and the Riemann mapping theorem, that also the topological stable rank of $H^\infty(\Omega_i)$ is equal to 2. Hence the pairs $(f_0, g_0), \ldots, (f_{n-1}, g_{n-1})$ can be replaced by unimodular pairs $(\tilde{f}_0, \tilde{g}_0), \ldots, (\tilde{f}_{n-1}, \tilde{g}_{n-1})$ such that for every $i = 0, 1, \ldots, n - 1$
\[ \|f_i - \tilde{f}_i\|_\infty + \|g_i - \tilde{g}_i\|_\infty < \epsilon. \]
By a repeated application of Lemma 2.5 to the pairs $(\tilde{f}_k, \tilde{g}_j)$ with $j \neq k$, we get the existence of $F_0, \ldots, F_{n-1}, G_0, \ldots, G_{n-1}$, such that
\[ \|F_k - f_k\|_\infty + \|G_k - g_k\|_\infty < \epsilon, \]
and the pair $(F_k, G_j)$ is unimodular in $H^\infty(\Omega_k \cap \Omega_j)$ for all $0 \leq k, j \leq n - 1$.

By the elementary theory of Banach algebras, it follows that there exists a $\delta > 0$ such that
\[ |F_k(z)| + |G_j(z)| \geq \delta \quad (z \in \Omega_k \cap \Omega_j). \]
Thus there exists a $\delta' > 0$ such that with
\[ \tilde{f} := F_0 \cdot F_1 \cdots \cdot F_{n-1} \cdot r, \]
\[ \tilde{g} := G_0 \cdot G_1 \cdots \cdot G_{n-1} \cdot s, \]
we have for all $z \in \Omega = \Omega_0 \cap \cdots \cap \Omega_{n-1}$,
\[ |\tilde{f}(z)| + |\tilde{g}(z)| \geq \delta'. \]
By the corona theorem for $H^\infty(\Omega)$, we obtain that $(\tilde{f}, \tilde{g})$ is a unimodular pair in $H^\infty(\Omega)$. Also, it can be seen that given $\epsilon' > 0$, we can choose $\epsilon > 0$ small enough at the outset so that
\[ \|f - \tilde{f}\|_\infty + \|f - \tilde{g}\|_\infty \leq \epsilon'. \]
This completes the proof. \hfill $\square$

The same proof shows that the topological stable rank of $H^\infty(\Omega)$ is 2 as well. Since the unimodular pair $(z, 1 - z^2)$ is not reducible (here we assume that $]-1, 1[ \subseteq \Omega$, $-1, 1 \notin \Omega_i$) we have that the Bass stable rank of $H^\infty(\Omega)$ is not one. Since the Bass stable rank is always less than the topological stable rank, we obtain that it must be 2.

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