New Solvable Potentials with Bound State Spectrum

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A new family of solvable potentials related to the Schroedinger-Riccati equation has been investigated. This one-dimensional potential family depends on parameters and is restricted to the real interval. It is shown that this potential class, which is a rather general class of solvable potentials related to the hypergeometric functions, can be generalized to even wider classes of solvable potentials. As a consequence the nonlinear Schroedinger-type equation has been obtained.

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1. Introduction

Though, we deal in the present paper with real equations the obtained results can be developed to the complex domain following the approaches \[1\]. Solvable problems of non-relativistic quantum mechanics have always attracted much attention \[2\,\,3\,\,4\]. The analytical methods to resolve the Schroedinger equation are very well-known \[5\]. A further remarkable development in solving the Schroedinger equation was the introduction of the concept of shape invariance \[6\]. Many of the potentials related by supersymmetry \[2\,\,3\,\,4\] were found to have similar shapes (i.e. to depend on the coordinate in similar way), only the parameters appearing in them were different. Although the number of potentials satisfying the shape invariance condition is limited, it turned out that the energy spectrum and the wavefunctions can be determined by elementary calculations in this case.

2. Schroedinger equation

In the present paper we consider the Schroedinger equation in one dimension, setting \(\hbar = 2m = 1\):

\[
1
\]
\[
\frac{d^2}{dx^2} \psi_n(x) + (E_n - V(x))\psi_n(x) = 0 \quad (1)
\]

then the function
\[
W_n(x) = -\frac{\psi_n'(x)}{\psi_n(x)}, \quad (2)
\]

where prime denotes differentiation with respect to \(x\), satisfies the corresponding Riccati equation
\[
W_n'(x) - W_n^2(x) = E_n - V(x). \quad (3)
\]

Assuming that the function \(W_0(x)\) has a zero inside interval \(I\) and
\[
W_0'(x) > 0 \quad \forall x \in I \subset \mathbb{R}, \quad (4)
\]

which is associated with normalization of the basic function \(\psi_0\), we get
\[
W_0'(x) = F(W_0), \quad (5)
\]

where \(F\) is an arbitrary function satisfying eq.(4). The last equation is obtained from reversibility of the function \(W_0(x)\) on interval \(I\). Taking eq.(5) into account and comparing it with eq.(3) we get the following result:
\[
W_0'(x) = W^2_0 + f(W_0), \quad (6)
\]

where
\[
E_0 - V(x) = f(W_0). \quad (7)
\]

Now we can express the potential \(V(x)\) in terms of \(W_0\) and we can use eq.(6) to generate potentials by choosing \(f(W_0)\).

A simplest and most obvious choice seems to be a second order polynomial
\[
W_0'(x) = AW_0^2 + BW_0 + C, \quad (8)
\]

where \(A, B, C\) are parameters. This differential equation is a first-order one and it can be solved in a straightforward way. The solution of the eq.(8) has the form
\[
W_0(x) = -\frac{B}{2A} + \frac{\sqrt{-B^2 + 4AC}}{2A} \tan \left( \frac{1}{2} \sqrt{-B^2 + 4AC}(x - x_0) \right), \quad (9)
\]

where
\[
x_0 - \frac{\pi}{\sqrt{-B^2 + 4AC}} \leq x \leq x_0 + \frac{\pi}{\sqrt{-B^2 + 4AC}}. \quad (10)
\]
Thus
\[ \psi_0(x) = e^{\frac{B(x-x_0)}{2A}} \left( \cos \left( \frac{1}{2} \sqrt{-B^2 + 4AC(x-x_0)} \right)^{\frac{1}{4}} \right) \] (11)
is the unnormalized form of the ground state.

3. Cascade of equations

In order to have the same potential function in the Riccati equation for
\( n = 1 \) we introduce expression for \( W_1 \):
\[ W_1 = W_0 - \frac{a_1}{b_1 W_0 - c_1}. \] (12)

Now, let us turn our attention to the explicit determination of the coefficients \( a_1, b_1, c_1 \) in terms of the \( A, B, C \). From the eq.(3) and eq.(8) we get
\[ a_1 = (A + 2)C - \frac{(A + 2)^2 B^2}{4(A + 1)^2}, \] (13)
\[ b_1 = A + 2, \] (14)
\[ c_1 = -\frac{(A + 2)B}{2(A + 1)}. \] (15)

Straightforward calculations lead us to the form of the first excited state wavefunction
\[ \psi_1(x) = e^{a_{11}x} (\cos \theta(x-x_0))^{\gamma_{11}} \left( \alpha_1 \cos \theta(x-x_0) + \beta_1 \sin \theta(x-x_0) \right)^{\gamma_1}, \] (16)
where all coefficients, denoted in the Greek letters, depend on the parameters \( A, B, C \). Although this relationship is rather complex but, by use of the equations (2), (9) and (12), easy to achieve.

Considerations presented above can be generalized if we take the explicit form of the function \( W_n \) in terms of \( W_0 \):
\[ W_n = W_0 - \frac{a_n}{b_n W_0 - c_n}. \] (17)

This function preserves the expression of equation
\[ W_n'(x) - W_n^2(x) = W_0'(x) - W_0^2(x) + E_n - E_0 \] (18)
for suitable values of coefficients which are involved in a system of non-linear equations (too complicated to be presented here). We should select
the appropriate values of $A, B, C$ parameters to simplify calculations. It will be done in the next chapter.

The equations (2), (3) and (8) enable us to obtain every wavefunctions $\psi_n$ and energies $E_n$. It is easy to prove [4] that the function $W_0$ fulfills the shape invariance condition, so this potential family, resulting from the eq.(8), is an example for shape-invariant solvable potentials. Using eq.(17) the wavefunctions can be written, without normalization, as

$$\psi_n(x) = e^{\alpha_0 x}(\cos \theta(x-x_0))^{\gamma_0} \prod_{i=1}^{n}(\alpha_i \cos \theta(x-x_0) + \beta_i \sin \theta(x-x_0))^{\gamma_i},$$  \hspace{1cm} (19)

where, as in the previous case, all coefficients written in Greek depend on the $A, B, C$ parameters.

4. The classic potentials

Equation (8) offers a convenient way to link this simple method with the well-known solutions of the Schroedinger equation. For instance, choosing $B=0, C=1$ we get the following results:

$$W_0' = AW_0^2 + 1$$  \hspace{1cm} (20)

and the first three wavefunctions

$$\psi_0(x) = (\cos (\sqrt{A}x))^{\frac{1}{A}},$$  \hspace{1cm} (21)

$$\psi_1(x) = \frac{(\cos (\sqrt{A}x))^{\frac{1}{A}} \sin (\sqrt{A}x)}{\sqrt{A}},$$  \hspace{1cm} (22)

$$\psi_2(x) = \frac{(\cos (\sqrt{A}x))^{\frac{1}{A}}((-1 + (1 + A) \cos (2\sqrt{A}x)))}{A},$$  \hspace{1cm} (23)

which are orthogonal on domain $I$ (eq.(10)), tend to very well-known solutions of the quantum oscillator $\psi_n(x) \to H_n(x)e^{-\frac{x^2}{2}}$ for $A \to 0$. It can be seen from the figures below that the wavefunctions have the same characteristic shapes but they differ in domains.
Basing on the procedure described above we are able to get solutions of the Schroedinger equation with the radial Coulomb potential (angular momentum is equal to zero). In this case

\[ W'_0 = AW_0^2 - BW_0 + \frac{B^2}{4} \]

whose basic solution is

\[ \psi_0(x) = \frac{(\sin(\frac{1}{2}\sqrt{A - 1}Bx))^{\frac{1}{4}}}{\sqrt{A - 1}}e^{-\frac{B}{2}x} \]

which tends to the radial part of the ground state eigenfunction of the Schroedinger equation for one-electron atom, \( \psi_0 \to \frac{1}{2}Bxe^{-\frac{1}{2}Bx} \) for \( A \to 1 \).

With help of the eq.(17) we are able to get the wavefunctions for the excited states.

Another example of the eq.(8) which lead us to the very well-known solution is

\[ W'_0 = -AW_0^2 - W_0 + C, \]

where parameters \( A > 0 \), and \( C > 0 \). Thus

\[ W_0(x) = -\frac{1}{2A} + \frac{\sqrt{1 + 4AC}}{2A} \coth \left( \frac{1}{2}\sqrt{1 + 4AC}(x - x_0) \right), \]

where the integration constant

\[ x_0 = \frac{1}{\sqrt{1 + 4AC}}(\ln A + i\pi) \]
is the complex number. In this case we have
\[ W_0(x) = -\frac{1}{2A} + \frac{\sqrt{1 + 4AC}}{2A} \tanh \left( \frac{1}{2} \left( \sqrt{1 + 4AC}x - \ln A \right) \right) \tag{29} \]
and
\[ \psi_0(x) = e^{x^2} \cosh \left( \frac{1}{2} \left( \sqrt{1 + 4AC}x - \ln A \right) \right)^{-\frac{1}{3}} \tag{30} \]
is the basic wavefunction without normalization constant. From eq.(29) we have
\[ W_0 = C - e^{-x^2} \text{ for } A \to 0 \text{ what is the standard expression for the Morse potential} \tag{31} \]
All potentials resulting from eq.(8) have a trigonometric form. It means that they are expressed in terms of the tangent function. If we wish to obtain potentials interesting from the physical point of view, like the Coulomb potential, or the Morse potential, we should follow the procedure outlined above or choose the proper, initial value of the parameters in eq.(8). It should be emphasized that every solution of the Schrödinger equation related to the orthogonal polynomials can be obtained by this method.

5. The new Hamiltonian

The results can be generalized to the form for which this method works \cite{7}. If we take eq.(8), in the form
\[ W'_0 = AW_0^2 + \frac{P_{l+1}(W_0)}{Q_l(W_0)} = R_{l+2,l}(W_0), \tag{31} \]
where \( P_{l+1}(W_0) \) is a polynomial in \( W_0 \) with degree no greater than \( l + 1 \), \( Q_l(W_0) \) is a polynomial in \( W_0 \) with degree equal to \( l \) and \( R_{l+2,l}(W_0) \) is a rational function such that both the numerator and the denominator are polynomials with degree \( l + 2 \) and \( l \) respectively. Substituting
\[ W_0(x) = R_{l+1,l}(\tan(\phi \cdot (x - x_0))) \tag{32} \]
in eq.(31) and adjusting the indices of the sums to get the same powers of \( \tan(\phi \cdot (x - x_0)) \), we get the explicit form of \( W_0 \). It should be emphasized that the condition of eq.(4) must be satisfied. The procedure outlined above can be applied to the function
\[ W_n(x) = R_{l+n+1,l+n}(\tan(\phi \cdot (x - x_0))) \tag{33} \]
and thus the excited state wavefunctions \( \psi_n(x) \) can be obtained.
Let us consider the simple equation, being the example of generalized eq. (31)

\[ W_0' = W_0^2 + 3\frac{W_0 - 1}{W_0 - 3} = \frac{(W_0 - 1)^3}{W_0 - 3}, \]

where all coefficients has been chosen to simplify calculations. Hence

\[ W_0(x) = \frac{2\sqrt{x + \frac{1}{4}} - 3}{2\sqrt{x + \frac{1}{4}} - 1}, \]

which gives the corresponding eigenvalue \( E_0 = -1 \) and the completely new potential is discovered:

\[ V(x) = -\frac{2}{\sqrt{x + \frac{1}{4}}} \text{ for } x \geq 0. \]

Thus the ground state function, without normalization constant, has the form

\[ \psi_0(x) = e^{-x + 2\sqrt{x + \frac{1}{4}}} \left( 2\sqrt{x + \frac{1}{4}} - 1 \right). \]

Substituting

\[ W_1 = \frac{P_2(W_0)}{Q_1(W_0)} \]

in eq. (33) and taking into account eq. (36), we obtain the unnormalized wavefunction

\[ \psi_1(x) = e^{-0.79x + 2.52\sqrt{x + \frac{1}{4}}} \left( 2\sqrt{x + \frac{1}{4}} - 1 \right) \left( 2\sqrt{x + \frac{1}{4}} - 3.74 \right), \]

where all decimal numbers are approximated and \( E_1 \approx -0.63 \). It is easy to show that the latest potential does not fulfill the shape invariance condition [4], so this new potential family is an example for non-shape-invariant solvable potentials.

Let us now discuss the question of the explicit form of the Schrödinger equation. Treating the eq. (31) not as a condition but rather as the transformed Schrödinger equation and substituting eq. (33) (for \( n = 0 \)) to eq. (31), we get

\[ -\psi_0''(x; \alpha)\psi_0(x; \alpha) - \alpha(\psi_0'(x; \alpha))^2 = \left[ E_0 - V \left( \frac{\psi_0(x; \alpha)}{\psi_0(x; \alpha)} \right) \right] \psi_0^2(x; \alpha), \]
where the parameter $\alpha$ is usually related to the parameter $A$ in eq. (31) and the potential $V$ has the form

$$V\left(\frac{-\psi_0'(x;\alpha)}{\psi_0(x;\alpha)}\right) = R_{l+2,l}\left(\frac{-\psi_0'(x;\alpha)}{\psi_0(x;\alpha)}\right).$$

(41)

As we see the eq. (40) is nonlinear differential equation. Taking into account the previous considerations regarding the quantum oscillator, the Coulomb potential and the Morse potential we get the following form of the eq. (40) in the $\alpha \to 0$ limit:

$$-\psi_0''(x;0) = \left[E_0 - V\left(\frac{-\psi_0'(x;0)}{\psi_0(x;0)}\right)\right]\psi_0(x;0)$$

(42)

what is the familiar form of the Schroedinger equation, and where nonlinearity is hidden in the form of the potential function.

6. Conclusions

The new method of obtaining solvable potentials has been reviewed in this paper. The main role in this method plays the Riccati equation which is a result of the transformed, one-dimensional, stationary Schroedinger equation. It allows us to emphasize the importance of function $W_0$ known in a literature as a ”superpotential” [2, 3]. By use of its features we can show that the potential is not an arbitrary function of $x$ but rather its form depends on the function $W_0$. As a consequence, we can find not only very well-known solutions of the Schroedinger equation but also a new class of the solvable potentials. These considerations may help us to identify new classes of the solvable potentials and may serve as an aid for further investigations concerning the relationship between solvability of the Schroedinger equation and the form of the potential.

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