A Review on Deformation Quantization of Coadjoint Orbits of Semisimple Lie Groups

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Abstract

In this paper we make a review of the results obtained in previous works by the authors on deformation quantization of coadjoint orbits of semisimple Lie groups. We motivate the problem with a new point of view of the well known Moyal-Weyl deformation quantization. We consider only semisimple orbits. Algebraic and differential deformations are compared.

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1 Motivation

The difference between classical and quantum mechanics is the presence of a non commutative algebra substituting the commutative algebra of classical observables. This non commutative algebra has a representation in some Hilbert space. The approach of deformation quantization consists in obtaining the quantum algebra as a deformation of the classical one. The parameter of deformation, which measures the non commutativity of the algebra, is the Planck constant $\hbar$. This is in agreement with the Bohr correspondence principle which states that when $\hbar \to 0$ the behaviour of the quantum system becomes closer to the classical one.

To quantize a physical system defined on $\mathbb{R}^2$ one associates to the coordinate functions $p$ and $q$ on $\mathbb{R}^2$ the self-adjoint operators on $P$ and $Q$ on $\mathcal{H} = L^2(\mathbb{R})$:

$$P\psi(q) = -i\hbar \partial_q \psi(q), \quad Q\psi(q) = q\psi(q), \quad \psi \in L^2(\mathbb{R})$$

satisfying the commutation rule $[Q, P] = -i\hbar I$. For generic polynomials in $p$ and $q$ one should select an ordering rule. Using the symmetric (or Weyl) ordering rule the operator associated to the polynomial $q^n p^m$ is self-adjoint and given by [1]:

$$w(x_{i_1} \cdots x_{i_{n+m}}) = \frac{1}{(n+m)!} \sum_{\sigma \in S_{n+m}} X_{i_{\sigma(1)}} \cdots X_{i_{\sigma(n+m)}}$$

with $i_j = 1, 2$ and $x_1 = q, x_2 = p, X_1 = Q, X_2 = P$. This defines a non commutative algebra structure on the polynomial algebra on $\mathbb{R}^2$, $\text{Pol}(\mathbb{R}^2)$,

$$a \star b = w^{-1}(w(a) \cdot w(b)), \quad a, b \in \text{Pol}(\mathbb{R}^2)$$

whose associated Lie bracket was first written by Moyal [2].

This product can be expressed as an infinite power series in $\hbar$:

$$f \star g = fg + B_1(f, g)\hbar + B_2(f, g)\hbar^2 + \ldots$$

with the $B_i$ bioperators that coincide with bidifferential operators acting on polynomials. This means that one can extend $B_i$ to act on $C^\infty(\mathbb{R}^2)$. The degree of the bidifferential operators is $(i, i)$ (we will see the explicit form later), which means that the product of two polynomials gives another polynomial. When extending to $C^\infty(\mathbb{R}^2)$ the product of two functions may be
an infinite, non convergent power series in $\hbar$. The product is well defined only in the space of formal power series in $\hbar$ with coefficients in $\mathcal{C}^\infty(\mathbb{R}^2)$. This is usually denoted by $\mathcal{C}^\infty(\mathbb{R}^2)[[\hbar]]$. This was the approach to differential deformations taken in Ref.[3].

It is our intention to give another point of view on the product defined above by $w$ on $\text{Pol}(\mathbb{R}^2)$ and its generalization to the $\mathcal{C}^\infty$ functions.

Let $H$ be the three parameter group $H = \mathbb{R}^3$ with multiplication

$$(a_1, b_1, c_1) \cdot (a_2, b_2, c_2) = (a_1 + a_2, b_1 + b_2, c_1 + c_2 + a_1 b_2).$$

It is the Heisenberg group with Lie algebra $\mathfrak{h} = \mathbb{R}^3 = \text{span}\{Q, P, E' = -iE\}$ and commutation rules

$$[Q, P] = -iE \quad \text{(the rest trivial)}.$$

We consider the dual space $\mathfrak{h}^*$ with coordinates $(q, p, e' = -ie)$ in the basis dual to $\{Q, P, E' = -iE\}$. The algebra of real polynomials on $\mathfrak{h}^*$ is

$$\text{Pol}(\mathfrak{h}^*) = \text{span}_\mathbb{R}\{q^m p^n e'^r, \; m, n, r = 0, 1, 2, \ldots\}.$$

There is a Poisson structure on $\mathcal{C}^\infty(\mathfrak{h}^*)$

$$\{f_1, f_2\} = e'\left(\frac{\partial f_1}{\partial q} \frac{\partial f_2}{\partial p} - \frac{\partial f_2}{\partial q} \frac{\partial f_1}{\partial p}\right) \quad f_i \in \mathcal{C}^\infty(\mathfrak{h}^*), \quad (2)$$

under which the polynomials $\text{Pol}(\mathfrak{h}^*) \subset \mathcal{C}^\infty(\mathfrak{h}^*)$ are closed. This is the Kirillov Poisson structure for $\mathfrak{h}$.

The Poisson structure is tangent to the planes $e' = \text{constant}$, where it restricts as a symplectic structure (maximal rank) provided $e' \neq 0$. On the plane $e' = 0$ the Poisson structure is identically 0. The planes $e' = \text{constant} \neq 0$ and the points of the form $(q, p, 0)$ are the leaves of the symplectic foliation of the non regular Poisson structure (2).

The adjoint representation of $g = (a, b, c) \in H$ in the ordered basis $\{Q, P, E'\}$ is

$$\text{Ad}_g = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -b & a & 1 \end{pmatrix}.$$ 

On $\mathfrak{h}^*$ we have the coadjoint action defined by given by

$$\text{Ad}^*_g \xi(X) = \xi(\text{Ad}_{g^{-1}}X) \quad \xi \in \mathfrak{h}^*, \; X \in \mathfrak{h}, \; g \in H.$$
In the dual basis,

$$\text{Ad}^*_{g} = \begin{pmatrix} 1 & 0 & b \\ 0 & 1 & -a \\ 0 & 0 & 1 \end{pmatrix}.$$ 

The orbits of the coadjoint action are the planes $e' = \text{constant} \neq 0$ and the single points $(q, p, 0)$. They coincide with the leaves of the symplectic foliation of $(\mathcal{G})$. (This is also the case for a general Lie algebra).

So we have that the coadjoint orbits are symplectic manifolds with an action of the group $\mathbf{H}$. Since the stability group (for $e' \neq 0$ is $\mathbb{R} \approx \{ (0, 0, c) \}$, they are in fact coset spaces $\mathbf{H}/\mathbb{R}$.

Let $\mathfrak{h}_\hbar$ be the Heisenberg algebra with bracket multiplied by $\hbar \in \mathbb{R}$ (Later on we will take it as formal parameter). We consider now the enveloping algebra of $\mathfrak{h}_\hbar$, $U(\mathfrak{h}_\hbar)$, which is the tensor algebra $T(\mathfrak{h}_\hbar)$ modulo the ideal generated by the relations of the Lie bracket, that is

$$Q \otimes P - P \otimes Q = \hbar E', \quad Q \otimes E' - E' \otimes Q = 0, \quad P \otimes E' - E' \otimes Q = 0.$$

An ordering rule in $U(\mathfrak{h}_\hbar)$ is a linear bijection $\text{Pol}(\mathfrak{h}_\hbar^*) \rightarrow U(\mathfrak{h}_\hbar)$. We can take for example the symmetric (or Weyl) ordering rule

$$W(x_1 \cdots x_p) = \frac{1}{p!} \sum_{s \in S_p} X_{i_{s(1)}} \otimes \cdots \otimes X_{i_{s(p)}}, \quad (3)$$

where $x_1 = q$, $x_2 = p$, $x_3 = e'$ and $X_1 = Q$, $X_2 = P$, $X_3 = E'$ and $i_j = 1, 2, 3$.

We can define an associative, non commutative product on $\text{Pol}(\mathfrak{h}_\hbar^*)$ as

$$f_1 \ast f_2 = W^{-1}(W(f_1)W(f_2)). \quad (4)$$

Explicitly,

$$f_1 \ast f_2(q, p, e') = \sum_{k=0}^{\infty} \frac{1}{k!} (\frac{\hbar}{2} e')^k P^k(f_1, f_2) = \exp(-\frac{i\hbar}{2} e' P)(f_1, f_2), \quad (5)$$

$$P^k(f_1, f_2) = P^{i_1 j_1} \cdots P^{i_k j_k} \frac{\partial f_1}{\partial x^{i_1}} \cdots \frac{\partial f_1}{\partial x^{i_k}} \frac{\partial f_2}{\partial x^{j_1}} \cdots \frac{\partial f_2}{\partial x^{j_k}},$$

$$P = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad i_l, j_l = 1, 2, \quad l = 1 \ldots k.$$
The product defined in (5) is of the general form

\[ f \star g = \sum_{i=0}^{\infty} C_i(f, g) h^i \]

with \( C_i \) bidifferential operators and satisfying the properties

1. \( \lim_{\hbar \to 0} f_1 \star f_2 = f_1 f_2 \),
2. \( \lim_{\hbar \to 0} \frac{1}{2\hbar} (f_1 \star f_2 - f_2 \star f_1) = \{f_1, f_2\} \). (6)

Since in (5) only derivatives \( \frac{\partial}{\partial q} \) and \( \frac{\partial}{\partial p} \) appear, the bidifferential operators are tangent to the orbits \( e = \text{constant} \neq 0 \). One can in fact restrict it to the orbit (say, \( e = 1 \)) without ambiguity and obtain a non commutative associative product on \( \text{Pol}(R^2) \). It is the Moyal-Weyl product on \( \text{Pol}(R^2) \), that we have described in (1). The differential property allows us to extend the product to \( C^\infty(R^2) \).

In general, given a real manifold \( M \), a product defined on \( C^\infty(M)[[h]] \) satisfying the properties (5) is a called a star product on \( M \) (or on \( C^\infty(M) \)). If the \( C_i \)'s are bidifferential we say that the star product is differential. If \( M \) is an affine algebraic variety, and the star product is defined on polynomials we say that it is an algebraic star product on \( M \) (or on \( \mathbb{C}[M] \)).

The Moyal-Weyl star product on \( R^2 \) is differential, when restricted to \( \text{Pol}(R^2) \) is algebraic and additionally it converges on polynomials for all real values of \( h \).

We want now to generalize this construction to the coadjoint orbits of a semisimple Lie algebra. The simplest non trivial example of a coadjoint orbit is the sphere \( S^2 \approx SU(2)/U(1) \). We can suitably generalize the algebraic construction using the enveloping algebra of \( su(2) = \text{Lie}(SU(2)) \). This was done in Refs.[4, 5], and we will make a review of it in Section 2 (another construction for certain types of algebraic manifolds can be seen in Ref.[3]). The other approach is to generalize the differential construction. This was done for arbitrary symplectic (in fact, for regular Poisson) manifolds in Refs.[4, 8, 9]. For arbitrary Poisson manifolds it was done first in Ref.[10], and with different approaches in Refs.[11, 12].

How can one write a differential star product on an arbitrary, symplectic manifold \( M \)? Let \( U \) be an open set in \( M \) where Darboux (canonical) coordinates exist, so \( U \approx \mathbb{R}^{2n} \). We can write a star product on \( U \) using the
Darboux coordinates and (5). Formula (1) is not invariant under symplectic transformations (unless they are linear symplectic transformations), but two star products given for different open sets $U, V$ (denoted by $⋆_U$ and $⋆_V$) are isomorphic in the intersection $U \cap V$. It is a general principle that one can “glue” the star products defined in every open set of a covering of $M$ into a globally defined star product on $M$, $⋆_M$ if the two following conditions are satisfied:

1. In the intersection $U \cap V$ there is an isomorphism of algebras

$$\phi_{UV} : C^\infty(U) \rightarrow C^\infty(V)$$

$$\phi_{UV}(f_1 \star_U f_2) = \phi_{UV}(f_1) \star_V \phi_{UV}(f_2), \quad f_1, f_2 \in C^\infty(U).$$

2. In the intersection of three open sets $U \cap W \cap V$ the cocycle condition

$$\phi_{UV} = \phi_{UW} \circ \phi_{WV}$$

is satisfied.

The first condition is satisfied in the symplectic case. The second condition can also be satisfied since it has been shown that a global star product exists [7, 8, 9]. In Appendix 1 we will write an explicit form of the gluing in terms of a partition of unity.

It is worth mentioning that in some non trivial cases global Darboux coordinates exist for the whole manifold. (This is happening for example in the case of coadjoint orbits of $SO(2) \times GL_+(2)$, Ref. [13]). In this case a star product is defined on $C^\infty(M)[[\hbar]]$ using the Moyal-Weyl deformation with Darboux coordinates.

In the case of the Heisenberg group the star product on $\mathfrak{h}^\ast$ is tangent to the orbits so that the restriction to the orbit is well defined. For a more general group this is not guaranteed [14], and we will see that in the semisimple case one cannot have a star product that is simultaneously algebraic and differential, that is a star product defined on the polynomials on the orbit via bidifferential operators. Nevertheless, we can ask if there is any relation between the star products constructed using the algebraic method and the differential one. This question was addressed in Refs.[15, 16]. We will review it in Section 3.
2 Algebraic construction of a star product

Let $G$ a real semisimple Lie group and $\mathfrak{g} = \text{Lie}(G)$. Let $\{X_1, \ldots, X_n\}$ be a basis of $\mathfrak{g}$ and $(\xi_1, \ldots, \xi_n)$ the coordinates on $\mathfrak{g}^*$ with respect to the dual basis. On $\mathfrak{g}^*$ we have the Kirillov Poisson bracket

$$\{f_1, f_2\}(\xi) = \langle [df_1, df_2], \xi \rangle = \sum_{ijk} c_{ij}^k \xi_k \frac{\partial f_1}{\partial \xi_i} \frac{\partial f_2}{\partial \xi_j}, \quad f_1, f_2 \in C^\infty(\mathfrak{g}^*)$$

(7)

where $c_{ij}^k$ are the structure constants of $\mathfrak{g}$. Since $\mathfrak{g}$ is semisimple, one can identify $\mathfrak{g} \approx \mathfrak{g}^*$ by means of the invariant Cartan-Killing form. From now on we will work in the complex field, so we will take the complexification of an algebra when needed. The deformed algebras that we will obtain have a conjugation and the space of fixed points of a conjugation is a real algebra. We will denote $\text{Pol}(\mathfrak{g}^*)_C = C[\mathfrak{g}^*]$. If we multiply the structure constants of the algebra $\mathfrak{g}$ by a parameter $h$, when we set $h$ equal to a real number we obtain an algebra isomorphic to $\mathfrak{g}$. Let us now treat $h$ as a formal parameter. Let $\mathfrak{g}_h$ denote the Lie algebra over $C[[h]]$ obtained from $\mathfrak{g}$ by multiplying the structure constants by the formal parameter $h$. Let $U_h(\mathfrak{g})$ (or for brevity $U_h$) denote its universal enveloping algebra.

The Weyl map $W : C[\mathfrak{g}^*][[h]] \rightarrow U_h$ is defined as in (3) and it is a linear bijection. It defines a star product on $\mathfrak{g}^*$ via formula (4). It is easy to see that this star product is differential. There are other linear bijections between $C[\mathfrak{g}^*]$ and $U(\mathfrak{g})$ that define other (equivalent) star products. We will make use of this freedom later. But $W$ has the following property (see Ref. [17] pg. 183): let $A$ be an automorphism (derivation) of $\mathfrak{g}$. It extends to an automorphism (derivation) of $U_h$ denoted by $\tilde{A}$. It also extends to an automorphism (derivation) $\bar{A}$ of $S(\mathfrak{g})_C[[h]] \simeq C[\mathfrak{g}^*][[h]]$ ($S(\mathfrak{g})_C$ denotes the complexification of the symmetric tensors over $\mathfrak{g}$). Then

$$W \circ \bar{A} = \tilde{A} \circ W.$$  

(8)

This intertwining property will be useful for the construction of the deformation on the orbits.

The leaves of the symplectic foliation of (7) coincide with the coadjoint orbits of $G$. We consider the complexified group $G_C$, and a semisimple element $X \in \mathfrak{g}$ (that is, $X$ has no nilpotent part in the Jordan decomposition). The coadjoint orbit of this element is an affine algebraic variety defined over
The real form of this orbit is a union of real orbits. If $G$ is compact then the real form of the complex orbit is the real orbit itself. We will construct a deformation quantization of the real form of the complex orbit. In the symplectic foliation of $g^*$ there are leaves of different dimensions. When the dimension is maximal we say that the orbit (or leaf) is regular. In this case, the polynomials that define the affine variety are invariant polynomials. In fact, they are of the form

$$p_i = c_i^0, \quad i = 1, \ldots, m = \text{rank}(g), \quad c_i^0 \in \mathbb{R},$$

where $p_i$ are homogeneous polynomials that generate the whole algebra of invariant polynomials (Chevalley theorem)

$$\text{Inv}(g) = \mathbb{C}[p_1, \ldots, p_m].$$

c_i^0 are generic constants and $\{dp_i\}$ are independent on the regular orbit [18]. The ideal of the orbit is $\mathcal{I}_0 = (p_i - c_i^0, i = 1 \ldots m)$, and the ring of polynomials on the orbit is

$$\mathbb{C}[[\Theta]] = \mathbb{C}[g^*]/\mathcal{I}_0.$$

If the orbit is not regular, then one can show [5] that the ideal of the orbit is generated by a set of polynomials $\{r_\alpha, \alpha = 1, \ldots, l\}$ satisfying

$$r_\alpha(g^{-1}X) = \sum_{\beta=1}^l T(g)_{\alpha\beta} r_\beta(X), \quad (9)$$

where $T$ is a finite dimensional representation of $G$. This means that if not the polynomials, the set itself is invariant under the action of the group $G$.

It is not difficult to see that the star product defined on $g^*$ via the Weyl map $W$ is not tangent to the orbits, so the restriction is not well defined. But there are other isomorphisms $\psi : \mathbb{C}[g^*][[\hbar]] \to U_h$ and star products

$$f_1 *_{\psi} f_2 = \psi^{-1}(\psi(f_1)\psi(f_2)).$$

We are going to construct an isomorphism $\psi$ such that the star product $*_{\psi}$ is well defined on the orbits.

Let $R_\alpha = W(r_\alpha)$. Notice that (9) includes as a particular case the regular orbits, since one can take $\{r_\alpha\} = \{p_i - c_i^0\}$. We consider now the two sided
ideal in $U_h$ generated by $R_\alpha$, $\mathcal{I}_h = (R_\alpha)$. It is easy to show using property (8) that the left and right ideals are equal, and then equal to $\mathcal{I}_h$,

$$\mathcal{I}_h^{\text{left}} = \mathcal{I}_h^{\text{right}} = \mathcal{I}_h.$$ 

We look for an isomorphism $\psi$ such that $\psi(\mathcal{I}_0) = \psi(\mathcal{I}_h)$ so the diagram

$$\begin{array}{ccc}
\mathbb{C}[\mathfrak{g}^*][[h]] & \xrightarrow{\psi} & U_h \\
\downarrow \pi & & \downarrow \pi_h \\
\mathbb{C}[\mathfrak{g}^*][[h]]/\mathcal{I}_0 & \xrightarrow{\overline{\psi}} & U_h/\mathcal{I}_h \\
\end{array}$$ 

(10)

commutes and the induced map $\overline{\psi}$ is an isomorphism. Then the product defined by:

$$\tilde{f}_1 \star_{\psi} \tilde{f}_2 = \overline{\psi}^{-1}(\tilde{\psi}(f_1)\overline{\psi}(f_2))$$

is an algebraic star product on the orbit.

In [4] the case of regular orbits was solved and in [5] the regularity condition was removed. Here we show explicitly the case of SU(2), although the construction was done in complete generality.

### 2.1 Star products on the coadjoint orbits of SU(2)

We consider the Lie algebra $\mathfrak{g} = \mathfrak{su}(2) = \text{span}_\mathbb{R}\{X, Y, Z\}$ with commutation rules

$$[X, Y] = Z, \quad \text{and cyclic permutations}.$$

Let $\{x, y, z\}$ be coordinates on $\mathfrak{g}^*$ with the dual basis. The coadjoint orbits of SU(2) are spheres $S^2$ centered in the origin. As affine algebraic varieties they are defined by the polynomial constrain

$$p \equiv x^2 + y^2 + z^2 = r^2, \quad r \in \mathbb{R}.$$

The origin itself is the only non regular orbit.

We now go to the complexification. Let $\mathcal{I}_0 = (p - r^2) \subset \mathbb{C}[x, y, z]$. Consider the following basis of $\mathbb{C}[x, y, z]$,

$$B = B_0 \cup B_1$$

$$B_0 = \{x^m y^n z^q(p - r^2), \quad m, n, q = 0, 1, 2, \ldots \}$$

$$B_1 = \{x^m y^n z^\nu, \quad \nu = 0, 1, \quad m, n = 0, 1, 2, \ldots \}.$$
$B_0$ is a basis of $\mathcal{I}_0$ and the set of equivalence classes in $\mathbb{C}[g^*]/\mathcal{I}_0$ of the elements of $B_1$ is a basis of $\mathbb{C}[\Theta]$. Consider the map

$$
\begin{align*}
\psi(x^m y^n z^q (p - r^2)) &= X^m Y^n Z^q (P - r^2) \\
\psi(x^m y^n z^\nu) &= X^m Y^n Z^\nu.
\end{align*}
$$

where $P = X^2 + Y^2 + Z^2$ is the Casimir operator of $\mathfrak{su}(2)$. Then $\psi(\mathcal{I}_0) = \mathcal{I}_h$ with $\mathcal{I}_h = (P - r^2) \subset U_h$ and the induced map $\tilde{\psi}$ is an isomorphism, so an algebraic star product on $S^2$ is defined. In [15] it was proven that it is not differential. Notice also that the star product is tangential only to the orbit with radius $r$, and not to the orbit with radius $r + \delta r$.

In the general case it is tricky to show that the images under $\psi$ of $B_1$ are linearly independent in $U_h/\mathcal{I}_h$. In Ref.[4] this was done for regular orbits and in Ref.[5] we were able to remove the regularity hypothesis.

There are other isomorphisms that one can use. In Ref.[19] it was explicitly constructed a star product on the regular orbits by finding another such isomorphism. The construction is based in the decomposition [20]

$$
\text{Pol}(g^*) = \text{Inv}(g^*) \otimes \text{Harm}(g^*).
$$

$\text{Inv}(g^*)$ are the polynomials on $g^*$ that are invariant under the action of the group and $\text{Harm}(g^*)$ are the harmonic polynomials, which can be identified with the polynomials on the orbit, $\text{Pol}(\Theta)$. Then, a basis of $\text{Pol}(g^*)$ is

$$
\{ p^r f, \quad f \in \text{Harm}(g^*), \quad r = 0, 1, 2, \ldots \}.
$$

The isomorphism is given in terms of this basis as

$$
\psi(p^r f) = P^r W(f). \tag{11}
$$

The corresponding star product is not differential [19], but it has the property that it is tangent to all the orbits in a neighborhood of the orbit $p = r^2$.

We want to make the observation that in the case of regular orbits, being $p$ an invariant polynomial, we can slightly generalize the construction and choose $\psi(p - r^2) = P - c(h)$ where $c(0) = r^2$. This allows to chose $c(h) = r(r + h)$. For each irreducible representation of $\mathfrak{su}(2)$ and its enveloping algebra we can find special values of $r$ in such a way that the representation descends to the quotient $U_h/\mathcal{I}_h$. The image of this algebra under the irreducible representation is a finite dimensional algebra that was studied in
Ref. [21]. It has been recently called a “fuzzy” sphere in the physics literature. It is also the algebra of geometric quantization [4]. In this way we can construct a family of algebraic deformations. It turns out that they are not all isomorphic [15].

3 Comparison between the algebraic and the differential methods for regular orbits

We have seen in the case of SU(2) how to construct a family of deformations of the regular orbits. The construction can be generalized to all the regular orbits of a compact semisimple Lie group. (For the non compact case and semisimple orbits, we recall that the algebraic variety is indeed a union of orbits). The deformations are of the form $U_h/I_h$ where $I_h = (P_i - c_i(h))$ with $P_i = W(p_i)$, the Casimir operators of $g$. Different choices for $c_i(h)$ (but always with $c_i(0) = c_i^0$) may give non isomorphic algebras. The question is if any of these algebras is somehow equivalent to a differential one. The first thing we note is that we may perhaps get an injective homomorphism of the star product algebra of polynomials into the $C^∞$ functions with a differential star product algebra, so “equivalence” in this context will mean to have such embedding.

If $g$ is a semisimple Lie algebra, it is not possible to have a differential star product that is tangential to all the orbits of $g^*$ [14]. Instead, we can consider a regularly foliated neighborhood of the orbit, where a differential, tangential star product always exists [22].

We will consider three different star products:

**Star product $*$ on $g^*$**. It is the one induced by the Weyl map. It is algebraic, differential, defined on all $g^*$ and not tangential.

**Star products $*$ and $*_P$.** It is the star product on $g^*$ defined by

$$\psi((p_1 - c_1^0)^{q_1} \cdots (p_m - c_m^0)^{q_m} f) = (P_1 - c_1(h))^{q_1} \cdots (P_m - c_m(h))^{q_m} W(f),$$

where $c_i(h)$ is still to be determined and $f$ is an harmonic polynomial (see (11)). It is algebraic, not differential, defined on all $g^*$ and tangential. We will denote $*_P$ the restriction to the orbit.
**Star products** $\star_T$ and $\star_{T\Theta}$. We consider $\mathcal{N}_\Theta$, a regularly foliated neighborhood of the orbit, and an atlas given by a good covering with Darboux coordinates. On each open set, we construct the star product given by Kontsevich’s local formula \cite{Kontsevich} in the Darboux coordinates. From Kontsevich’s theorem, it follows that it is equivalent to the restriction of $\star_S$ to that open set. The star products obtained with the Darboux coordinates can be glued into a global star product on $\mathcal{N}_\Theta$ (see Appendix 1).

It is differential, tangential and not algebraic. Also, it is equivalent to $\star_S$ on $\mathcal{N}_\Theta$. Its restriction to $\Theta$ is denoted by $\star_{T\Theta}$.

We have the following maps:

\[
(C[g][[h]], \star_P) \xrightarrow{\eta \cong} (C[g][[h]], \star_S) \subset (C^\infty(\mathcal{N}_\Theta)[[h]], \star_S)
\]

\[
(C^\infty(\mathcal{N}_\Theta)[[h]], \star_S) \xrightarrow{\rho \text{ injective}} (C^\infty(\mathcal{N}_\Theta)[[h]], \star_T).
\]

So we have that $\rho \circ \eta$ is an injective homomorphism. Since both, $\star_P$ and $\star_T$ are tangential, we may ask if $\rho \circ \eta$ descends as an injective homomorphism to the algebras on the orbits. In \cite{Kontsevich} it was shown that $c_i(h)$ can be chosen in such a way that $\rho \circ \eta(I_0) = I_0$ so we have that the polynomial algebra $(C[\Theta][[h]], \star_P)\Theta)$ inject homomorphically into $(C^\infty(\Theta)[[h]], \star_{T\Theta})$.

**Appendix 1. Gluing of star products**

Let $M$ be a Poisson manifold and fix an open cover $\mathcal{U} = \{U_r\}_{r \in J}$ where $J$ is some set of indices. Assume that in each $U_r$ there is a differential star product

\[
\star_r : C^\infty(U_r)[[h]] \otimes C^\infty(U_r)[[h]] \to C^\infty(U_r)[[h]]
\]

and isomorphisms

\[
T_{sr} : C^\infty(U_{rs}) \to C^\infty(U_{sr}), \quad U_{r_1 \ldots r_k} = U_{r_1} \cap \cdots \cap U_{r_k}
\]

\[
T_{sr}(f) \star_s T_{sr}(g) = T_{sr}(f \star_r g)
\]

such that the following conditions are satisfied

\[
1. \quad T_{rs} = T_{sr}^{-1} \quad \text{on } U_{sr},
\]

\[
2. \quad T_{ts} \circ T_{sr} = T_{tr} \quad \text{on } U_{rst}.
\]
Then there exists a global star product on $M$ isomorphic to local star product on each $U_r$.

Let $\phi_i : U_i \to \mathbb{R}$ be a partition of unity of $M$ subordinate to the covering $\mathcal{U}$ and $f_r \in C^\infty(U_r)$ such that $f_r = T_{rs}f_s$. Let $f \in C^\infty(M)$ be the function

$$f = \sum_{r \in J} \phi_r f_r.$$

On $U_r$, $f$ becomes

$$f = (\phi_r \text{Id} + \sum_s \phi_s T_{sr}) f_r = A_r f_r.$$

We define a star product on $U_r$ as

$$f \star g = A_r(A_r^{-1}(f) \star_r A_r^{-1}(g)).$$

(13)

it is equivalent to $\star_r$. Using conditions (12) one has

$$A_r T_{rt} = A_t.$$

Then, the star products (13) on each $U_r$ coincide in the intersections, so they define a unique star product on $M$. The restriction of this star product to $U_r$ is equivalent to $\star_r$. Also, using different partitions of unity one obtains equivalent star products.

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