Electron radiative recombination with a hydrogen-like ion

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Abstract
We survey the results of a long-term study of the process of radiative recombination (RR). A rigorous theory of non-relativistic electron RR with a hydrogen-like ion is used to calculate the total cross section of the process, the effective radiation, the recombination rate coefficient, and the emission coefficient in a plasma with a Maxwellian electron distribution. The exact results are compared with the numerous known asymptotic and interpolation formulas. We propose interpolation formulas which ensure a uniform approximation of all mentioned quantities in a wide range of plasma temperatures.

Keywords: radiative recombination, rate and emission coefficients, cross section, effective radiation

(Some figures may appear in colour only in the online journal)

1. Introduction

The investigation of radiative recombination (RR) has a long history. The process was the object for several theoretical papers starting already in the 1920s [1–8]. Later, this process was investigated theoretically in numerous papers [9–16]. The main problem has been to find appropriate approximations to overcome essential difficulties in numerical calculations involved.

In 1923 Kramers [1] used semiclassical arguments to derive the cross section for RR of electron been captured to highly excited states of a hydrogen atom. At that time the level of quantum theory of radiation did not allow to solve the problem of RR consistently. Instead the idea of Bohr’s correspondence principle was used to obtain an approximate solution.

Using Schrödinger quantum mechanics, Oppenheimer suggested the solution of the problem in [2–4]. Note that the asymptotic approximations of his formulas are in error.

Since the initial and final wave functions are known analytically in the Coulomb field of a bare ion, the cross section of RR in this case may in principle be calculated exactly. For capture to the lowest levels, this was done in the early work of Stobbe [5]. As it was shown by Gaunt [7], the results of original work of Kramers are valid in the region of large quantum numbers.

In the pioneering work of Menzel and Pekeris [8], the process of radiation recombination was discussed in detail (their results were later corrected by Burgess [17]) though at that time a numerical analysis of the cross section of the process was a rather complicated problem, especially for the total cross section. Much later Bethe and Salpeter [9] derived an approximate analytical formula for the cross section of RR, which in fact coincides with that of Kramers. In their work, advantage was taken of the fact that the oscillator strength crosses the continuum limit smoothly. Thus, knowledge of the bound-bound transition amplitudes was used to derive the cross section of RR. The Bethe–Salpeter formula provides satisfactory accuracy even for the ground and low excited states (see below).

Various other asymptotic formulas are used in the applications, as well as approximate formulas obtained by interpolating the results that are valid for large and small energies of the incident electron (see [10]). An important step has been made by Katkov and Strakhovenko in their paper [13], where a simple expression was obtained for the cross section $\sigma_{\text{rr}}^{(n)}$ of a free electron radiative capture to a level with an arbitrary principle value $n$ of a hydrogen-like ion.
Finally, in two papers of Milstein [15, 16] the relatively simple expressions for the total cross section $\sigma_\text{rr}$ and the total effective radiation $\sigma_\text{rd}$ of the process were derived, where

$$\sigma_\text{rr} = \sum_n \sigma_\text{rr}^{(n)}, \quad \sigma_\text{rd} = \sum_n \hbar \omega_n \sigma_\text{rd}^{(n)},$$

(1)

$\omega_n$ is a photon frequency, $\hbar \omega_n = \varepsilon + J_2 / n^2$, $\varepsilon = m_e v^2 / 2$, $J_2 = Z^2 e^4 m_e / (2 \hbar^2)$, $v$ is the electron velocity, $Z$ is the ion charge number, $e$ and $m_e$ are the electron charge and mass, respectively. The results were obtained using a dipole approximation and the analytical properties of the electron Green’s function in a Coulomb field.

Although the RR process has been treated theoretically for many years, the first successful direct measurement of the process was not done until 1990s, when Anderson with coauthors succeeded in measuring the recombination rate coefficient $k_\text{rr}$ for electron RR on bare carbon ions [18]

$$k_\text{rr} = \langle v \sigma_{\text{rr}} \rangle = \int v \sigma_{\text{rr}} f_e(v) dv,$$

(2)

where $f_e(v)$ is the Maxwellian electron distribution function. They also measured $k_\text{rr}$ for hydrogen and a few hydrogen-like ions and compared data with a calculation based on the theory of Stobbe, Bethe and Salpeter [19, 20]. Direct observations of electron-ion recombination in colliding-beam experiment was reported in [21].

The study of RR of elementary particles in the Coulomb field of a nucleus or an ion is quite important in several areas of physics: particle accelerators, plasma physics, astrophysics, antimatter production, and laser-induced recombination. For instance, in the particle accelerator physics the electron cooling of proton beams should be mentioned first [22–27]. The cooling mechanism of protons by electrons consists of a redistribution of energy during collision of the two gases.

Laser-induced recombination has been investigated both theoretically and experimentally with special emphasis on an enhancement of the recombination cross sections to moderately-high excited states [28–34].

Laser-stimulated electron-proton recombination for the conditions encountered in ion storage rings was first examined in [28], and the first observations of laser-induced recombination obtained with merged beams of protons and electrons were reported in [29, 30]. The laser-mediated electron-proton recombination rate coefficient for a steady state situation is given in [35].

Lately, new attention has been paid to the RR process, since the process has been suggested as a method to produce antimatter [22, 36]. The RR of a positron and an antiproton will produce antihydrogen. In order to increase the rate of recombination, production it has been proposed to stimulate the recombination process by a laser [28].

In plasma physics, when an electron emits a photon in collision with ions, the electron energy decreases and free-free or free-bound transitions can be observed. In the latter case an electron can be captured into any energy level of the recombined atom. The study of recombination radiation is important, for instance, in determining the rate at which positive ions recapture electrons, although it is usually believed that at higher temperatures recombination radiation is negligible compared to bremsstrahlung [37–42].

Plasma physics operates, in addition to the recombination rate coefficient $k_\text{rr}$, with the emission coefficient $q_\text{rr}$

$$q_\text{rr} = \langle v \sigma_{\text{rr}} \rangle = \int v \sigma_{\text{rr}} f_e(v) dv.$$

(3)

The most well-known calculations of these quantities were made by Seaton [43]. Using the first three terms in the asymptotic expansion of the Gaunt factor, he calculated the recombination rate coefficient and the mean kinetic energy of the recombining electrons. Seaton has shown that if one put the Gaunt factor equal to unity, then the obtained relatively simple expressions will have errors not more than 20%. A considerable improvement was obtained by using the asymptotic expansion of the Gaunt factor, as derived by Menzel and Pekeris [8] and corrected by Burgess [17]. From the expansion of the Gaunt factor, Seaton obtained asymptotic expansions which enable the rate coefficients to be calculated with errors which do not exceed 2% for temperatures of order $10^4$ K or less but which may be greater for temperatures of order $10^5$ K. He presented a systematic tabulation of the various functions which occur in these asymptotic expansions and derived a parametrization for the rate coefficient. It is this parametrization that is cited by the well-known reference book of formulas for the physics of plasma [44].

Further studies of RR cross sections, effective radiation, rate and emission coefficients following Seaton’s work are given in [45–62], where the coefficients for hydrogen-like ions were calculated in [49, 56–62]. In particular, accurate piecewise-continuous parametrization for the total cross section is proposed by Erdas and coauthors in [60] (whereas their parametrization of the total effective radiation coefficient seems erroneous). Review paper [62] discusses also measurements of the recombination rate coefficient although for the temperatures below 1 eV. Apart from Seaton’s work, the recombination rate coefficient for hydrogen and some other ions in the form of four-parameter fits are presented in [59] for relatively narrow range of temperatures $0.004$ eV $< T < 2$ eV. A more complicated formula for the recombination rate coefficient $k_\text{rr}$ containing the summation of an infinite series, was used in [14, 51, 52, 54]. Tabulations of fits to the recombination rate coefficient $k_\text{rr}$ and/or emission coefficient $q_\text{rr}$ have been done by Tarter [47], Aldrovandi and Pequignot [48, 50, 53], Martin [58], Erdas and Quarati [61].

All known parametrizations of $\sigma_\text{rr}$, $\sigma_\text{rd}$, $k_\text{rr}$, and $q_\text{rr}$ were proposed at the time when Milstein’s formulae [15, 16] had not yet gained wide popularity. In the present paper we fill this gap by averaging Milstein’s formulae over the Maxwellian distribution of the electrons. The results of averaging are presented in the form of uniform parametrizations. In the entire temperature range of practical interest, from the smallest to the largest temperature values, these parametrizations ensure the accuracy of approximation at a level not exceeding several percents. Comparing the results of our calculation with the known parametrizations and tabulations, we refine their range of applicability.

The above-mentioned publications date back to the late 1990s. More recent studies of RR focus primarily on processes involving partially ionized atoms heavier than
hydrogen. We intentionally leave this vast area of research beyond the scope of this article.

Below we adhere to the following outline. In section 2 we present the main expressions for the cross section of electron RR with hydrogen-like ion using in our calculations. In section 3 we do the same for the effective radiation. In sections 4 and 5 we compute the rate and emission coefficients of the RR. Finally, in section 6 we summarize our results.

2. Cross section of RR

To begin with, we recall the results on the cross section of RR known from the literature.

2.1. Recombination to the ground state, \( n = 1 \)

If the cross section \( \sigma_{\pi}^{(n)} \) of the photoionization of a quantum level with the principal quantum number \( n \) is known, then the RR cross section \( \sigma_{\pi}^{(n)} \) to this level can be found using the principle of detailed balance, according to which

\[
\sigma_{\pi}^{(n)} = \frac{2(\hbar \omega / e)^2}{(m_e \omega)^2} \sigma_{\pi}^{(1)},
\]

where \( e \) is the speed of light. A quantum theory of photoionization was developed by Stobbe in 1930 [5]. For reference, we remind some results of this theory [63], section 56, equations (56.13) and (56.14). The photoionization cross section of the ground level \( (n = 1) \) of the hydrogen-like ion at the ionization threshold at \( \hbar \omega = J_Z \) is equal to

\[
\sigma_{\pi}^{(1)} = \frac{2\pi^2 \alpha a_B^2}{3 \varepsilon Z^2},
\]

where \( a_B = h^2 / m_e e^2 \) is the Bohr radius, \( e = 2, 71 \ldots \) (do not confuse with elementary charge \( e \)). For \( \hbar \omega \gg J_Z \), the photoionization cross section of the ground level

\[
\sigma_{\pi}^{(1)} = \frac{2\pi^2 \alpha a_B^2 (J_Z / \omega)^{7/2}},
\]

rapidly decreases with increasing photon energy \( \hbar \omega \). Combining equations (5) and (6) with equation (4), we find the RR cross section to the ground level of the hydrogen-like ion

\[
\sigma_{\pi}^{(1)} = \frac{2\pi^2}{3 \varepsilon^4 \alpha^3 a_B^3 \frac{J_Z}{\varepsilon}},
\]

at \( \varepsilon \ll J_Z \), and

\[
\sigma_{\pi}^{(1)} = \frac{2\pi^2}{3 \varepsilon} \alpha^3 a_B^2 \left( \frac{J_Z}{\varepsilon} \right)^{5/2}.
\]

At \( \varepsilon \gg J_Z \). The exact solution of the problem of RR to the ground state of a hydrogen-like ion was found by Stobbe [5] (see also [63], section 56):

\[
\sigma_{\pi}^{(1)}(\eta) = \frac{2\pi^2}{3} \frac{\eta^6 e^{-4\eta} \arctan(1/\eta)}{(1 - e^{-2\eta})(\eta^2 + 1)^2} \alpha^3 a_B^2,
\]

where \( \eta = J_Z / \omega = \sqrt{J_Z / \varepsilon} \) is the parameter of Sommerfeld. This expression agrees with the approximate formulas (7) and (8) in the limiting cases \( \eta \gg 1 \) and \( \eta \ll 1 \), respectively.

2.2. Recombination to excited states, \( n > 1 \)

In the limit \( \varepsilon \ll J_Z \) \( (\eta \gg 1) \), there is a semiclassical formula of Kramers [1], p 861, equation (65)

\[
\sigma_{\pi}^{(n)}(\eta) = \frac{32\pi}{3\sqrt{3}} \frac{\eta^4 \alpha^3 a_B^2}{n(n^2 + n^2)}.
\]

It was a new obtained as a result of simplifying formulas of quantum theory in the monograph of Bethe and Salpeter [9], equation (75.7). A heuristic derivation of the Kramers formula is given by Kogan and Lisitsa in [41], p 210, equation (19). In the monographs of Raizer [64–66] a similar derivation is called ‘adventurous’.

Exact formulas in \( \eta \) were found by Katkov and Strakhovenko in [13]. After summing over the orbital quantum number \( l \), they derived the RR cross section to an arbitrary state with the principal quantum number \( n \)

\[
\sigma_{\pi}^{(n)}(\eta) = \frac{2\pi^2}{3} \frac{\eta^6 e^{-4\eta} \arctan(1/\eta)}{(1 - e^{-2\eta})(\eta^2 + 1)^2} S_n(\eta) \alpha^3 a_B^2,
\]

where the coefficient \( S_n(\eta) \) for several lower levels has the form

\[
S_1 = 1,
\]

\[
S_2 = 2 + \frac{3}{x_2} + \frac{1}{x_3^2},
\]

\[
S_3 = 3 + \frac{14}{x_3} + \frac{19}{x_3} + \frac{8}{x_3} + \frac{1}{x_3^2},
\]

\[
S_4 = 4 + \frac{38}{x_4} + \frac{346}{x_4} + \frac{409}{x_4} + \frac{622}{x_4} + \frac{43}{x_4} + \frac{1}{x_4^2},
\]

where \( x_n = (n^2 + \eta^2) / 4\eta^2 \). Figure 1 shows the dependence of the Gaunt factor

\[
g^{(n)}(\eta) = \sigma_{\pi}^{(n)}(\eta) / \sigma_{\pi}^{(1)}(\eta)
\]

on \( \eta \) for a few lowest levels. It clearly demonstrates that the Kramers formula gives a more or less correct result only for
\( \eta \gg 1 \). Although it was derived under the assumption \( n \gg 1 \), for \( n = 1 \) and \( \eta \gg 1 \) it gives the result which is less than that of Stobbe (7) only by 20%:

\[
g^{(1)}(\infty) = \frac{8\sqrt{3}\pi}{e^4} = 0.7973. \tag{14}
\]

For other values of \( n \) in the limit \( \eta \to \infty \) the error is 12% for \( n = 2, 9\% \) for \( n = 3 \), and 7.5% for \( n = 4 \):

\[
g^{(2)}(\infty) = \frac{480\sqrt{3}\pi}{e^8} = 0.8762,
\]

\[
g^{(3)}(\infty) = \frac{2748\sqrt{3}\pi}{e^{12}} = 0.9075,
\]

\[
g^{(4)}(\infty) = \frac{13591\sqrt{3}\pi}{3\sqrt{3}e^{16}} = 0.9248. \tag{15}
\]

In other words, the Kramers formula can even be used for \( n = 1 \), if the error of the order of 20% is not burdensome.

Below we enlist some papers where Kramers’ formula (10) was used or cited with a mention of his name: [67], chapter 5, section 4, equation (5.27); [64], chapter 9, section 3, equation (9.5); [65], chapter 6, section 3, equation (6.5); [66], chapter 8.3.1, p 245; [41], p 210, equation (19); [16], equation (16); [62], equation (3.1). The formula (10) is cited by M Bell and J Bell in [24] with reference to Spitzer’s book [37], and neither Bell and Bell nor Spitzer mentioned the name of Kramers. Andersen attributed the formula (10) to Bethe and Salpeter [20], equation (3).

2.3. Total cross section of RR

The total cross section \( \sigma_{\text{rr}} \) of RR is defined as a sum of the recombination cross sections to individual levels. In an idealized model, the number of levels is infinite, so that the summation in the formula (1) goes from \( n = 1 \) to \( n = \infty \). Note that in a real plasma the number of levels is finite.

A contemporary quantum theory of RR for a hydrogen-like ion is described in Milstein’s papers [15, 16], where summation over \( n \) is performed and the following expression for the total cross section is obtained [16], equation (12):

\[
\sigma_{\text{rr}}(\eta) = -\frac{16}{3} \pi^2 \alpha^3 a_B^2 \eta^2 \times \left[ \int_0^\infty \frac{\text{sign}(\varepsilon - \varepsilon') \sinh(\pi \eta - \pi \varepsilon')}{\varepsilon - \varepsilon'} \frac{2 \sinh(\pi \eta) \sinh(\pi \varepsilon')}{2 \sinh(\pi \eta) \sinh(\pi \varepsilon')} \right] \times \left[ \frac{\xi}{\tilde{\xi}} \frac{d}{d\xi} F(\xi)^2 \right] d\varepsilon' + \coth(\pi \eta) - \frac{1}{\pi \eta}. \tag{16}
\]

Here \( \eta = \sqrt{J_c}/\varepsilon, \eta' = \sqrt{J_c}/\varepsilon', \xi = -4 \eta \eta'/(\eta - \eta')^2 \), and \( F(\xi) = \pFq{3}{2}{\xi}{i \eta, \eta'}{1} \). \( \xi \) is expressed in terms of the hypergeometric function \( \pFq{3}{2} \). Passing in this formula to integration over the variable \( \eta' \), we have

\[
\sigma_{\text{rr}}(\eta) = \frac{16}{3} \pi^2 \alpha^3 a_B^2 \eta^2 \times \left[ \int_0^\infty \frac{\eta^2 \sinh(\pi \eta - \pi \varepsilon')}{\eta^2 - \eta' \varepsilon'} \sinh(\pi \eta) \sinh(\pi \varepsilon') \right] \times \left[ \frac{\xi}{\tilde{\xi}} \frac{d}{d\xi} F(\xi)^2 \right] d\varepsilon' + \coth(\pi \eta) - \frac{1}{\pi \eta}. \tag{17}
\]

As stated in [16], in the limit \( \eta \ll 1 \) the result of integration coincides with the well-known formula

\[
\sigma_{\text{rr}}(\eta) = \frac{128 \pi \zeta(3)}{3} \eta^5 \alpha^3 a_B^2, \tag{18}
\]

where \( \zeta(x) = \sum_{k=1}^{\infty} 1/k^x \) is the Riemann zeta function. Using the fitting method at \( \eta < 0.05 \), we find a more accurate interpolation

\[
\sigma_{\text{rr}}(\eta) = \frac{128 \pi \zeta(3)}{3} \left[ 1 - 3.140 \eta 

+ 5.142 \eta^2 - 0.169 \eta^3 \right]. \tag{19}
\]

For the case \( \eta \gg 1 \) equation (17) is reduced to the asymptotic expression given in [16]

\[
\sigma_{\text{rr}}(\eta) = \frac{32 \pi}{3\sqrt{3}} \alpha^3 a_B^2 \eta^2 \ln(\eta). \tag{20}
\]

The same result can be obtained from the Kramers formula

\[
\sigma_{\text{Kramers}}(\eta) = \frac{32 \pi}{3\sqrt{3}} \alpha^3 a_B^2 \sum_{n=1}^{\infty} \frac{\eta^4}{n(\eta^2 + n^2)}, \tag{21}
\]

which is valid for \( \eta \gg 1 \). In this formula the main contribution to the sum is given by large \( n \sim \eta \). Replacing the summation over \( n \) by integration from 1 to \( \infty \), we arrive at the formula

\[
\sigma_{\text{rr}}(\eta) = \frac{16 \pi}{3\sqrt{3}} \eta^2 \ln(1 + \eta^2) \alpha^3 a_B^2, \tag{22}
\]

that almost coincides with (20). The latter formula is cited in [41], p 211, equation (23) and not only there. The sum in equation (21) can also be calculated without going over to integration. This is done by Bell and Bell in [24] (see equation (9) there); the result is expressed in terms of the digamma function \( \psi(\zeta) = \Gamma'(\zeta)/\Gamma(\zeta) \) and has the form

\[
\sigma_{\text{Kramers}}(\eta) = \frac{16 \pi}{3\sqrt{3}} \eta^2 \left[ \psi(1 + i\eta) + \psi(1 - i\eta) + 2\gamma \right] \alpha^3 a_B^2, \tag{23}
\]

where \( \gamma \approx 0.5772 \) is the Euler constant. In [24] cited above other works of 1939–1969, [17, 43, 68, 69] concerning calculations of the total recombination cross section, were discussed. Authors of [24] proposed the approximate formula

\[
\sigma_{\text{rr}}(\eta) = \frac{32 \pi}{3\sqrt{3}} \alpha^3 a_B^2 [\ln(\eta) + 0.1402 + 0.5250 \eta^{-2/3}] \tag{24}
\]

for the case \( \eta \gg 1 \). Our calculations has shown that it is valid with the accuracy better than 3.3% for \( \eta \geq 10 \) and better than 1% for \( \eta \geq 50 \).
Comparing the result of integration in the formula (17) with the Kramers formula (23), we found that in the interval $50 < \eta < 250$ the formula
\[
\sigma_\eta = 0.9248 \sigma_{\text{Kramers}}
\] (25) approximates exact solution (17) with accuracy better than 1%, and the exact value is only 3.2% less than the approximation (25) at $\eta = 10$.

Note that the asymptotics (19) and (25) are joined at the point $\eta = 0.2341$, where the value calculated by the exact formula (17) is 10% less. Figure 2 shows the exact ratio $\sigma_\eta/\eta^2\alpha^2a_0^2$, together with its asymptotics (19) and (25), as a function of $\eta$. This ratio determines the reaction rate of the RR (see section 4).

Figure 3 shows the ratio $\sigma_\eta/\sigma_\infty$ as a function of $\eta$. It is clearly seen that for $\eta \gg 1$ the main contribution to the total cross section is given by recombination to the numerous excited states of the recombined atom.

Concluding this section, we point out that an accurate piecewise-continuous parametrization of the total cross section for RR was proposed by Erdas et al. [60]

\[
\sigma_\eta(\eta) = \sigma_\infty(\eta) = \begin{cases} 
1.202, & \text{if } \eta < 10^{-1/2} \\
0.4804 + 1.200\ln\eta, & \text{if } \eta > 10 \\
1.396 + 0.3661\ln\eta + 0.1957\ln^2\eta, & \text{otherwise.}
\end{cases}
\] (26)

According to our calculations, the accuracy of this parametrization in the interval $\eta < 100$ is better than 1.5% but the error approaches 5% as $\eta \to \infty$. We found the uniform parametrization
\[
\sigma_\eta(\eta) = \sigma_\infty(\eta) = 
\begin{align*}
&1.202 + 0.5782\ln(\eta^2 + 1) + 0.2148\ln(\eta^2 + 1) \\
&\quad + 0.3425\ln(\eta^2 + 1)
\end{align*}
\] (27)
The error is less than one percent.

The RR of non-relativistic electrons is the most interesting energy region for plasma physics and astrophysics. In this region, there were a theoretical problem to take a sum over principal quantum numbers and angular momenta. The RR cross section of relativistic electron decreases very rapidly with the electron energy. The main contribution to the total cross section is given by the transition to the ground state. The detailed discussion of RR cross section of a relativistic electron is presented in [70]. It follows from the result of that paper that up to $\sim 100$ keV the non-relativistic dipole approximation describes the cross section with high accuracy.

3. Effective radiation

Let us now turn to the discussion of effective radiation $\varkappa_\text{eff}$, see equation (1), which also is an important characteristic of the process of RR; it determines the emission coefficient (see section 5). Using the analytical properties of the Green’s function of electron in a Coulomb field, Milstein in his paper [16] has obtained the following result:

\[
\varkappa_\text{eff} = -\frac{16}{3}\pi^2\alpha^2a_0^2J_Z\eta^2
\begin{align*}
&\times \left\{ \int_0^\infty \frac{\sinh(\pi\eta - \pi\eta')}{2\sinh(\pi\eta)}\frac{\eta^2}{\xi^2}\left[ \frac{\xi}{d\xi}F(\xi) \right]^2 \right. \\
&\quad - \frac{4\pi}{\pi} + \frac{4\pi^2\eta^2}{\eta^2 + \eta'}^2 \coth(\pi\eta) \left[ \frac{2\eta'}{\eta^3} \right] \\
&\quad + 8 \left[ \frac{2 - \ln(4) + \psi(1) - \psi(1 + i\eta))\coth(\pi\eta)}{\pi} \right. \\
&\quad + \left. \frac{1}{\pi} \frac{\ln\psi(1 + i\eta)}{\pi} \right] \right\}
\end{align*}
\] (28)

where $\psi(x) = \Gamma'(x)/\Gamma(x)$, $\psi'(x) = d\psi(x)/dx$, and other notation are given after equation (16). In the limiting case $\eta \ll 1$ this yields

\[
\varkappa_\text{eff}(\eta) = \frac{128\pi}{3}\zeta(3)\alpha^2a_0^2J_Z\eta^3.
\] (29)
The asymptotics (29) corresponds to the fact that $\hbar\omega \approx \varepsilon$ for $\eta \ll 1$, and the formula (18) can be used.
For \( \eta \gg 1 \), numerical integration gives the following asymptotic behavior:

\[
\kappa_{\eta}(\eta) = \frac{128\pi}{3} \zeta(3) 0.1174 \alpha^3 Z a J Z \eta^2. \tag{30}
\]

Using the Kramers formula, another numerical coefficient in front of \( \eta^2 \) was obtained. This is due to the fact that for \( \eta \gg 1 \) the main contribution to the effective radiation is given by transitions to levels with a principal quantum number \( n \sim 1 \), and for these transitions the Kramers formula is not applicable. The result of numerical integration in equation (28) is shown in figure 4 together with the asymptotics (29) and (30).

A piecewise-continuous parametrization of the total effective radiation for recombination was proposed by Erdas et al. [60]. However, their formula seems to contain a typo, since it describes a function with discontinuities. We have obtained a simpler parametrization

\[
\kappa_{\eta} = J Z \sigma_{\eta}(\eta)(1.2321 + 1.2025 \eta^{-2}), \tag{31}
\]

which has an accuracy better than 1%.

4. Recombination rate coefficient

To obtain the recombination rate coefficient \( k_{\eta} \), equation (2), in plasma with a Maxwellian electron distribution function \( f_e(n) \), it is necessary to calculate the integral

\[
k_{\eta} = \int_0^\infty \frac{2 \zeta}{m} \sqrt{2} \exp\left(-\zeta/T\right) \sqrt{\pi} T^{3/2} \text{d}\zeta.
\]

Passing here to integration over the dimensionless variable \( \eta = \sqrt{J Z/Z} \), we arrive at the expression

\[
k_{\eta} = \frac{4}{\sqrt{\pi}} \alpha Z \left(\frac{J Z}{T}\right)^{3/2} \int_0^\infty \frac{\sigma_{\eta}(\eta)}{\eta^3} \exp\left(-\frac{J Z}{T \eta^2}\right) \text{d}\eta. \tag{32}
\]

The result of integration is a function of the dimensionless parameter \( T/J Z \). Substitution of the small-\( \eta \) asymptotics (18) for the RR cross section in equation (32) leads to divergence of the integral at the upper limit where this asymptotics is not applicable. Consequently, for any value of the parameter \( T/J Z \), it is necessary to take into account correctly the contribution of electrons with low energy \( \varepsilon \ll J Z \) (i.e., \( \eta \gg 1 \)). If we use the asymptotics (22) in equation (32), we obtain the integral which can be taken analytically. At \( T/J Z \ll 1 \) the result of this integration reads

\[
k_{\eta} = \frac{32\sqrt{\pi}}{3\sqrt{3}} \alpha^4 Z a J Z \left(\frac{J Z}{T}\right)^{1/2} \left[ \ln\left(\frac{J Z}{T}\right) + \gamma \right], \tag{33}
\]

where \( \gamma = 0.5772 \) is the Euler constant. Previously, this formula was obtained in [24].

Seaton in his famous article [43], equation (36) gave an approximation formula, which in our notations reads

\[
k_{\eta} = \frac{32\sqrt{\pi}}{3\sqrt{3}} \alpha^4 Z a J Z \left(\frac{J Z}{T}\right)^{1/2} \times \left[ \ln\left(\frac{J Z}{T}\right) + 0.8576 + 0.9380 \left(\frac{J Z}{T}\right)^{-1/2}\right]. \tag{34}
\]

Seaton’s calculations were confirmed in [24]. Note that the coefficient in front of the leading term \( \ln(J Z/T) \) in equations (33) and (34) coincide, but the next-to-leading terms differ noticeably. Seaton also derived his result from the Kramers formula, but took into account the deviation of the Gaunt factor from unity. Recall that the Gaunt factor is a ratio of the result following from the exact quantum theory and the classical Kramers formula, equation (13). Milstein’s formula (17) contains this factor de facto, because it was obtained by methods of quantum physics, while in derivation of equation (33) we actually used the classical Kramers formula. Thus, Seaton’s expression (34) is more accurate than equation (33). This statement was checked and confirmed in [47]. A similar parametrization of the recombination rate coefficient to the excited levels \( k \geq 2 \) was proposed in the article [45], equation (3.11) by Hummer and Seaton and then confirmed by Brown and Mathews in their paper [46], p 491. The latter authors stated that Seaton’s result are valid within an error less than 1%, but our calculations give slightly different values for the coefficients in Seaton’s formula (34). We have found that in the interval \( 10^{-3} < T/J Z < 10 \) the parametrization

\[
k_{\eta} = \frac{32\sqrt{\pi}}{3\sqrt{3}} \alpha^4 Z a J Z \left(\frac{J Z}{T}\right)^{1/2} \times \left[ \ln\left(\frac{J Z}{T}\right) + 0.9004 + 0.8932 \left(\frac{J Z}{T}\right)^{-1/2}\right]. \tag{35}
\]

gives a uniform approximation to the exact result with the error less than 1.3%, and the maximum error is reached on the edge of the interval at \( T/J Z = 10 \). Note that Seaton’s formula (34) is given in the well-known NRL formulay on plasma physics [44], p 55, equation (13), where its region of applicability is cited as \( T/J Z \ll 400 \) eV. However, we found that the error exceeds 50% at the upper edge of the interval and is as large as 9% at \( T/J Z = 10 \).

Using in equation (32) the exact formula (17) and fitting the result of numerical integration, we obtained the following
asymptotics for \( T/J_2 \gg 1 \)

\[
k_{\text{rr}} = \frac{139.0 (J_2/T)^{1/2} \alpha c Z a_0^2}{1 + 5.787 (J_2/T)^{0.5885}}.
\]  

(36)

The accuracy of this formula is better than 2.4\% in the interval \( 1.5 < T/J_2 < 10^4 \) (note that non-relativistic approximation discussed in our paper is applicable at \( T \ll m_e c^2 \)). The temperature dependence \( k_{\text{rr}} \propto T^{-3/2} \) for high temperatures and \( k_{\text{rr}} \propto T^{-1/2} \) for low temperatures was predicted in monograph [49], but the numerical coefficients in [49] differ significantly from those found by us.

In multiple editions of his widely cited monograph [64–66] Raizer quotes the formula \( k_{\text{rr}} = 2.7 \times 10^{-1} (J_2/T)^{1/2} [\text{cm}^3 \text{eV}^{1/2} \text{s}^{-1}] \) referring to the English edition of his other book coauthored with Zel’ dovich [67]. We found that his formula in applicable at very narrow interval \( 0.002 < T/J_2 < 6 \) where the error varies from \(-30\%\) in the mid of the interval to \(+30\%\) at its ends.

The result of numerical integration over a wide range of values of the ratio \( T/J_2 \) is shown in figure 5, where the asymptotics (35) and (36) are also plotted. Combining these asymptotics we derived the following uniform parametrization

\[
k_{\text{rr}} = \frac{8.414 [\ln (1 + J_2/T) + 3.499] \alpha c Z a_0^2}{(T/J_2)^{1/2} + 0.6517 (T/J_2) + 0.2138 (T/J_2)^{3/2}}.
\]  

(37)

In the range \( 10^{-4} < T/J_2 < 10^4 \), its accuracy is \( 3\% \). For the first bound state, a uniform parametrization of the recombination rate coefficient reads

\[
k_{\text{rr}}^{(1)} = \frac{17.41 \alpha c Z a_0^2}{(T/J_2)^{1/2} + 0.3593 (T/J_2)^{3/2} + 0.1471 (T/J_2)^{3/2}}.
\]  

(38)

Its accuracy for \( T/J_2 < 10^4 \) is better than \( 3\% \). The ratio \( k_{\text{rr}}/k_{\text{rr}}^{(1)} \) is shown in figure 6. Note that the limit of \( k_{\text{rr}}^{(1)} = 17.5 \alpha c Z a_0^2 \) at \( T \to 0 \) was calculated in [71], section 24, problem 1.

Finally, we mention the work [72], equation (4) where a four-parametric fit is proposed for the recombination rate coefficients for various ions, which, according to the authors, has an accuracy better than \( 3\% \) in the interval \( 3 \times 10^{-5} < T/J_2 < 10^7 \). The same fit is reproduced by the authors of the review article [73], equation (6). A bit different numerical values for the fit parameters was reported later in [74]. Our calculations has shown that at least in the case of hydrogen ions, this parametrization overestimates the recombination rate coefficient by many times.

5. The emission coefficient

A power of radiation in the recombination process (i.e. the energy emitted per unit volume per unit time) is given by the formula

\[
P_{\text{rr}} = n_{\text{H}} n_{\text{e}} q_{\text{rr}},
\]  

(39)

where

\[
q_{\text{rr}} = \int_0^\infty \frac{2 \sqrt{\varepsilon}}{m c} \exp(-\varepsilon/T) \frac{\varepsilon}{\sqrt{\pi} T^{3/2}} \mathrm{d}\varepsilon
\]

is the emission coefficient (2). It can also be written in the form

\[
q_{\text{rr}} = 4 \alpha c Z \left( \frac{J_2}{T} \right)^{3/2} \int_0^\infty \frac{\varepsilon_{\text{e}}(\eta)}{\eta^5} \exp(-J_2/\eta) \eta^2 \mathrm{d}\eta.
\]  

(40)

Substituting in the last expression the asymptotics (29) for \( \eta \ll 1 \), we arrive at the asymptotic expression

\[
q_{\text{rr}} = \frac{256}{3} \pi \zeta(3) \alpha c Z J_2 a_0^2 \left( \frac{J_2}{T} \right)
\]  

for the emission coefficient at \( T \gg J_2 \). We did not find an analogue of this formula in the literature. This may be due to the fact that a power of recombination radiation at high temperatures is significantly smaller than the power of bremsstrahlung.

In the opposite case \( T \ll J_2 \), the dependence of \( q_{\text{rr}} \) on temperature is different, namely:

\[
q_{\text{rr}} = 21.49 \alpha c Z J_2 a_0^2 \sqrt{\frac{J_2}{T}}.
\]  

(42)
Our result is approximately 18% less than that of equation (25) in the review article of Kogan and Lisitsa [41]. That result was derived in the Kramers approximation; the authors also refer to numerical calculations in early paper by Kogan [10], equation (9). Our formula (42) should also be compared with equation (33) on p 58 of the reference book [44] for which conditions of applicability are not specified. Although the temperature and Z dependence are the same, our coefficient is 1.8 times larger.

Seaton in the above-mentioned paper [43], equation (52) derived an asymptotic formula for what he called the total kinetic energy loss due to RR. His equation resembles equation (34) for the recombination rate coefficient at low electron energies ε. Unfortunately, a direct comparison of this Seaton’s result with our calculation is not possible as the total emission coefficient of RR is larger by definition (emitted photon brings out the energy ε of free electron minus the energy −JZ/n^2 of the electron in a recombined state).

The result of evaluation of equation (40) is shown in figure 7. It is seen that the asymptotics (41) works well enough only for an extremely large ratio T/JZ ≥ 10^6. We suggest a uniform parametrizations for the total emission coefficient

\[ q_{rr} = \frac{[21.09 + 9.474(T/JZ)^{1/2} + 0.7748(T/JZ)]c^2ZJZ}{(T/JZ)^{3/2} + 0.2271 (T/JZ) + 0.06348 (T/JZ)^{3/2} + 0.002404 (T/JZ)^2}, \]

and for transition to the first level

\[ q_{rr}^{(1)} = \frac{[17.19 + 6.396(T/JZ)^{1/2} + 0.7421(T/JZ)]c^2ZJZ}{(T/JZ)^{3/2} + 0.1694 (T/JZ) + 0.06201 (T/JZ)^{3/2} + 0.002768 (T/JZ)^2}. \]

They have the accuracy better than 1.4% and 1.6%, respectively, in the entire range of temperatures. In [61], Erdas and Quarati presented results of tabulation of the emission coefficient in a wider interval 10^{-6} < T/JZ < 10^8, but we concluded that these authors overestimated q_{rr} by 5%–70% in the range 3 \times 10^{-4} < T/JZ < 1.

Note also that evaluating the rate and emission coefficients for T/JZ ≥ 10^7 have no practical meaning since non-relativistic theory of recombination is not applicable at so large temperatures.

The ratio q_{rr}/q_{rr}^{(1)} is shown in figure 8. It is seen that transition of an electron to the first level gives the main contribution to q_{rr} for any temperatures. Transition to excited levels gives the contribution to q_{rr} from 20% to 23.5%. In contrast, the contribution of the excited levels to the recom- bination rate coefficient k_{rr} at T ≤ JZ is essentially greater than the contribution of the ground level, see figure 6.

6. Summary

Our work brings a line under the long-term research of the process of RR of the electrons with hydrogen-like ions. Using the exact convenient expressions for the cross section and the effective radiation, we have formulated a status of numerous approximate results concerning the process under discussion. For the total RR cross section, the total recombination effective radiation, the recombination rate coefficient and the emission coefficient, we have also suggested new uniform interpolation formulas having high accuracy in a wide range of electron energies and plasma temperatures.
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