HYPERTRANSCENDENCY OF PERTURBATIONS OF HYPERTRANSCENDENTAL FUNCTIONS

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Abstract. Inspired by the work of Bank on the hypertranscendence of $\Gamma^h$ where $\Gamma$ is the Euler gamma function and $h$ is an entire function, we investigate when a meromorphic function $f e^g$ cannot satisfy any algebraic differential equation over certain field of meromorphic functions, where $f$ and $g$ are meromorphic and entire on the complex plane, respectively. Our results (Theorem 1 and 2) give partial solutions to Bank’s Conjecture (1977) on the hypertranscendence of $\Gamma^h$. We also give some sufficient conditions for hypertranscendence of meromorphic function of the form $f + g$, $f \cdot g$ and $f \circ g$ in Theorem 3 and 4.

1. Introduction and main results

A meromorphic function $f$ on the complex plane is said to be hypertranscendental over a field $K$ of meromorphic functions, if $f$ does not satisfy any nontrivial algebraic differential equation whose coefficients are in the field $K$. We are interested in those $K$ which are related to the growth of $f$. Let $T(r, f)$ be the Nevanlinna characteristic function of $f$ (see Section 2 for the definitions and notations in Nevanlinna theory). We denote by $S(r, f)$ any quantity which is of growth $o(T(r, f))$ as $r \to \infty$ outside a set of finite measure $E \subset (0, \infty)$. By $M_0$ we mean the field of meromorphic functions $y$ with $T(r, y) = o(r)$ as $r \to \infty$ outside a set of finite measure and $S_f$ (resp. $S_f'$) the field of meromorphic functions $y$ satisfying the growth condition $T(r, y) = S(r, f)$ (resp. $T(r, y) = O(T(r, f))$ as $r \to \infty$ outside a set of finite measure).

In 1887, Hölder [9] established the hypertranscendence of the Euler gamma function $\Gamma$ over the field of rational functions, i.e., $\Gamma$ cannot satisfy any nontrivial algebraic differential equation whose coefficients are rational functions. Hilbert [8], in 1901, proved the hypertranscendence of Riemann zeta function using the functional equation of $\zeta$ and $\Gamma$. In 1976, Bank and Kaufman [4] extended the famous theorems of Hölder and Hilbert by showing that $\Gamma$ and $\zeta$ are hypertranscendental over the field $M_0$. One year later, Bank [2] asked to what extent the hypertranscendence of $\Gamma$ is due to the nature of its poles and zeros. In particular, he posed the following conjecture.

Bank’s Conjecture ([2]). For every entire function $h$, $\Gamma^h$ is hypertranscendental over $M_0$. 
Bank [2] gave an affirmative answer to the above conjecture when either $h$ or $h'$ has only finitely many zeros. In 1980, he [3] generalized this result to the following.

**Theorem A** ([3]). Let $h$ be an entire function with the property that for some nonnegative integer $j$ and some complex number $a$, the following condition holds:

$$
N(r, 1/(h^{(j)} - a)) = S(r, h^{(j)}),
$$

where as usual, $h^{(0)}$ denotes $h$. Then the function $\Gamma e^h$ is hypertranscendental over $M_0$.

Related to Theorem A, we obtained the following.

**Theorem 1.** Let $h$ be an entire function such that $T(r, \Gamma / \Gamma') = S(r, h^{(j)})$ and

$$
\delta(a, h^{(j)}) > 0,
$$

for some $a \in M_0$ and some nonnegative integer $j$. Then $\Gamma e^h$ is hypertranscendental over $M_0$.

Related to Bank’s Conjecture, we have the following partial result.

**Theorem 2.** For any entire function $h$, $P(z, \Gamma e^h, \ldots, (\Gamma e^h)^{(n)}) \not\equiv 0$ for any nontrivial distinguished polynomial $P(z, u_0, \ldots, u_n)$ over $M_0$.

**Remark 1.** The notion of distinguished polynomial was first introduced by B. Q. Li and Z. Ye in [12]. The definition is given as follow.

Let $I = (i_0, i_1, \ldots, i_k)$ be a multi-index with $|I| = i_0 + i_1 + \cdots + i_k$. A polynomial in the variables $u_0, u_1, \ldots, u_k$ with meromorphic function coefficients in a set $S$ can always be written as

$$
P(z, u_0, u_1, \ldots, u_k) = \sum_{I \in \Lambda} a_I(z) u_0^{i_0} u_1^{i_1} \cdots u_k^{i_k},
$$

where the coefficients $a_I$ are meromorphic functions in $S$ and $\Lambda$ is an index set. We call $P$ a distinguished polynomial in $u_0, u_1, \ldots, u_k$ with coefficients in $S$, or simply an $S$-distinguished polynomial, if the index set $\Lambda$ satisfies $|I_i| \neq |I_j|$ for any distinct indices $I_i, I_j$ in $\Lambda$. In other words, each homogeneous part of the distinguished polynomial $P$ contains one term only.

If $K$ is a field of meromorphic functions, we denote by $A(K)$ the set of all meromorphic functions which satisfy some algebraic differential equation over $K$. It is well known (see Chapter 14 of [10]) that $A(K)$ is a differential field, i.e., a field with an additional map $D : A(K) \rightarrow A(K)$ such that $D(a \cdot b) = (Da) \cdot b + a \cdot Db$ for any $a, b \in A(K)$.

To explain the difference between Theorem A and Theorem I, let us sketch the main idea of the proof of Theorem A (see Part B in [3] or Chapter 14 of [10]).
Under this assumption, the condition (2) is less restrictive than the one on
Theorem 1 considers a sort of complement assumption that
Γ can deduce a contradiction to the hypertranscendence of Γ over
r, g
As
nonnegative integer
q
of total degree
P
follows that
e
r
as
hypertranscendental over a differential field
S ⊂ S
A
C
morphic function on the complex plane. Then, if
f
hypertranscendental over
S
, we have
T(r, f) = O(\overline{N}(r, 1/f) + \overline{N}(r, f) + T(r, g))
outside of a possible exceptional set of finite measure.
In particular, if all
N(r, 1/f), \overline{N}(r, f)
and
T(r, g)
are
S(r, f),
then
f + g
must be
hypertranscendental over
S.

The proofs of Theorem A and B in [2, 3] depend on the following Lemma first appeared in [1].

Lemma A ([1]). Let
P(z, y, y',..., y(n))
be a polynomial in
y, y',..., y(n)
whose coefficients are meromorphic functions on
C. For each
r > 0,
let
Δ(r)
be the maximum of the Nevanlinna characteristics of the coefficients of
P. Let
f
be a nonzero meromorphic function on the complex plane satisfying the equation
P = 0,
but for some nonnegative integer
q,
P_q(f, f',..., f(n)) ≠ 0,
where
P_q
is the homogeneous part of
P
of total degree
q
in the indeterminates
y, y',..., y(n).
Then
T(r, f) = O(E(r)),
as
r → ∞,
outside of a possible exceptional set of finite measure, where
E(r) = \overline{N}(r, 1/f) + \overline{N}(r, f) + Δ(r) + log r.
In addition, for any
α > 1,
there exist positive constants
C
and
r_0
such that
T(r, f) ≤ cE(αr),
for all
r ≥ r_0.
In 1991, Y. Z. He and C. C. Yang proved that $\Gamma(g)$ is hypertranscendental over the field $\mathcal{M}^g$ of meromorphic functions $y$ with $T(r, y) = O(T(r, g))$ by using Steinmetz’s Reduction Theorem (Theorem C below). Their method can be applied to the general case (see Theorem 3). In 2007, Markus applied the method of differential algebra to obtain the hypertranscendence of $\zeta(\sin z)$ and $\Gamma(\sin z)$ over $\mathbb{C}$, and he proved the differential independence between $f_i$ and $f_j(\sin z)$ for $i, j = 1, 2$, where $f_1 = \Gamma$ and $f_2 = \zeta$.

Applying the same idea of He and Yang in [7], we obtain the following general result which covers the results of He and Yang.

**Theorem 3.** Let $f$ be hypertranscendental over the rational function field $\mathbb{C}(z)$ and $g$ be a nonconstant entire function. Then $f \circ g$ is hypertranscendental over the field $S^g$.

As a consequence, we can generalize a result of L. Markus (see Lemma 1 in [13]) by using a different method.

**Corollary 1.** Let $a$ be a nonzero complex number. Then both $\Gamma(\sin az)$ and $\zeta(\sin az)$ are hypertranscendental over the field of meromorphic functions $y$ with $T(r, y) = O(r)$ as $r \to \infty$ outside some set of finite measure.

It is natural to consider the hypertranscendency of $g \circ f$ over some fields for entire hypertranscendental $f$ and meromorphic $g$. This seems to be a more difficult problem as Steinmetz’s Reduction Theorem cannot be applied directly here (see Remark 2 in Section 3). However, we do obtain one related result in Theorem 4.

Inspired by the results of Bank, He-Yang and Markus, in this paper, we will first prove a result similar to Lemma A, that is $T(r, f)$ can be controlled by one counting function $N(r, 1/f)$ (see Lemma 2). Using Lemma 2 we then obtain the following results on the hypertranscendency of perturbations of hypertranscendental functions, including that of $\Gamma$ and $\Gamma e^h$.

**Theorem 4.** Let $g$ and $f$ be meromorphic functions and $S$ be the field of meromorphic functions $y$ with $T(r, y) = S(r, f'/f)$, i.e. $S = S_{f'/f}$. Let $O$ be the set of entire functions on $\mathbb{C}$. Suppose $f$ is hypertranscendental over $S$.

1. If $f \in O$, and $g - R$ has finitely many zeros, where $R$ is a non-constant rational function, then $g \circ f$ is hypertranscendental over $S$.
2. Assume that $f \in S_g$ and $\delta(a, g) > 0$ for some $a \in S \setminus \{0\}$, then $fg$ is hypertranscendental over $S$.
3. If there exists a non-negative integer $k$ such that $T(r, f) = S(r, g^{(k)})$ and $\delta(a, g^{(k)}) > 0$ for some $a \in S$, then $f + g$ is hypertranscendental over $S$.
4. Assume that $g \in O$, and if there exists a nonnegative integer $k$ such that $T(r, f'/f) = S(r, g^{(k)})$ and $\delta(a, g^{(k)}) > 0$
for some $a \in S$, then $fe^g$ is hypertranscendental over $S$.

(5) If $g \in O$ and $f \in S_{\exp(g)}$, then $P(z, fe^g, (fe^g)', \ldots, (fe^g)^{(n)}) \neq 0$ for any nontrivial distinguished polynomial $P(z, u_0, \ldots, u_n)$ over $S$.

In Section 5 we will use Theorem 4 to prove Theorem 1 and 2. Section 2 introduces the basics of Nevanlinna Theory. Theorem 3 and 4 will be proven in Section 3 and 4 respectively.

2. NEVANLINNA THEORY

We recall the basic notations and results of Nevanlinna theory [10] which are main tools for proving our results.

Let $f$ and $a$ be meromorphic functions in the complex plane $\mathbb{C}$ and $\mathbb{D}_r = \{|z| < r\}$. Denote the number of poles of $f$ in $\mathbb{D}_r$ by $n(r, f)$, and let $n(r, a) = n(r, a, f) = n(r, 1/(f - a))$. When the number of distinct poles of $f$ in $\mathbb{D}_r$ is denoted by $n(r, f)$, we then let $n(r, a) = n(r, 1/(f - a))$. Correspondingly we define the counting function and truncated counting function in Nevanlinna theory as follows:

$$N(r, a, f) := \int_0^r \frac{n(t, a) - n(0, a)}{t} dt + n(0, a) \log r;$$

$$\overline{N}(r, a, f) := \int_0^r \frac{\overline{n}(t, a) - \overline{n}(0, a)}{t} dt + \overline{n}(0, a) \log r.$$

The proximity function is defined as

$$m(r, f) := \frac{1}{2\pi} \int_0^{2\pi} \log^+ |f(re^{i\theta})| \, d\theta$$

and

$$m(r, a, f) := m(r, 1/(f - a)),$$

where $\log^+ x = \max\{0, \log x\}$. The Nevanlinna characteristic function of $f$ is defined by

$$T(r, f) = m(r, f) + N(r, f).$$

The First Main Theorem of Nevanlinna theory for small functions [14] says that for any meromorphic function $a$ with $T(r, a) = S(r, f)$,

$$T(r, f) = T(r, a, f) + S(r, f)$$

where $T(r, a, f) := m(r, a, f) + N(r, a, f)$. Finally, we denote the Nevanlinna order of $f$ by

$$\rho(f) := \limsup_{r \to \infty} \frac{\log T(r, f)}{\log r},$$

and the deficiency of $a$ for $f$ by

$$\delta(a, f) := \liminf_{r \to \infty} \frac{m(r, a, f)}{T(r, f)} = 1 - \limsup_{r \to \infty} \frac{N(r, a, f)}{T(r, f)}.$$
If \( \delta(a, f) > 0 \), then we say \( a \) is a deficient function of \( f \).

The logarithmic derivative lemma states that

**Lemma 1 ([10])**. Let \( f \) be a transcendental meromorphic function and \( k \geq 1 \) be an integer. Then

\[
m \left( r, \frac{f^{(k)}}{f} \right) = S(r, f),
\]

and if \( f \) is of finite order of growth, then

\[
m \left( r, \frac{f^{(k)}}{f} \right) = O(\log r).
\]

3. Proof of Theorem 3

To prove Theorem 3, we first introduce the Steinmetz’s Reduction Theorem.

**Theorem C** (Steinmetz’s Reduction Theorem [6, 15]). Let \( F_j, 1 \leq j \leq N \) be meromorphic functions on \( \mathbb{C} \). Let \( h_j, 1 \leq j \leq N \) be meromorphic and \( g \) be entire on \( \mathbb{C} \) such that for each \( j \),

\[
T(r, h_j) = O(T(r, g))
\]
as \( r \to \infty \) outside some set of finite measures. Given a functional equation of the form

\[
F_1(g(z))h_1(z) + \cdots + F_N(g(z))h_N(z) = 0,
\]

then there exist polynomials \( p_j \), not all zeros, such that

\[
p_1(g(z))h_1(z) + \cdots + p_N(g(z))h_N(z) = 0.
\]

Furthermore, if \( h_j \neq 0 \) for some \( j \), then there exist polynomials \( Q_j \), not all zeros, such that

\[
F_1(z)Q_1(z) + \cdots + F_N(z)Q_N(z) = 0.
\]

**Proof of Theorem 3**. We will follow the idea of the proof of Theorem 4 in [17].

Suppose that \( f \circ g \) satisfies a nontrivial algebraic differential equation with coefficients in \( \mathcal{S}^g \), i.e., there exists a nontrivial differential polynomial \( P(z, w, w', \ldots, w^{(n)}) \) with coefficients in \( \mathcal{S}^g \) such that

\[
P(z, f \circ g, (f \circ g)', \ldots, (f \circ g)^{(n)}) = \sum_j (M_j(f) \circ g)(H_j(g)(z)) = 0
\]

where \( M_j(f) \) is a differential monomial of \( f \) with constant coefficients and \( H_j(g)(z) \) is a differential polynomial of \( g(z) \) whose coefficients are some linear combinations of the coefficients of the original differential polynomial \( P(z, w, w', \ldots, w^{(n)}) \).

Now, set \( F_j(z) = M_j(f)(z) \) and \( h_j(z) = H_j(g)(z) \), it follows from the second result of Theorem C that there exist polynomials \( Q_i \), not all zeros, such that

\[
F_1(z)Q_1(z) + \cdots + F_N(z)Q_N(z) = 0.
\]
which implies that $f$ satisfies a nontrivial algebraic differential equation with coefficients in $\mathbb{C}(z)$. This is a contradiction to our assumption that $f$ is hypertranscendental over $\mathbb{C}(z)$.

\[\square\]

**Remark 2.** Here, we will explain the reason why the Steinmetz’s Reduction Theorem does not work for the hypertranscendency of $g \circ f$. We use the same idea of proof of Theorem 3. Suppose that $g \circ f$ satisfies a nontrivial algebraic differential equation over a suitable field such that we can apply the Steinmetz’s Reduction Theorem, thus we have

\[
p_1(f(z))h_1(z) + \cdots + p_N(f(z))h_N(z) = 0
\]
or

\[
F_1(z)Q_1(z) + \cdots + F_N(z)Q_N(z) = 0
\]

where $h_j(z)$ is a differential polynomial of $f(z)$ whose coefficients are some linear combinations of the coefficients of the algebraic differential equation $g \circ f$ satisfied, and $F_j(z)$ is a differential monomial of $g$ with constant coefficients. From these two equalities, we cannot deduce any contradictions even though we have known the hypertranscendency of $f$.

### 4. Proof of Theorem 4

In this section, we are now going to prove our main result (Theorem 4). To prove Theorem 4, we need the following lemmata.

**Lemma 2.** Let $f$ be a nonzero meromorphic function on the complex plane and $P(z,y,y',\ldots,y^{(n)})$ be a polynomial in $y,y',\ldots,y^{(n)}$ whose coefficients are in the field $S_f$. Suppose $f$ satisfies the equation $P = 0$. Rewrite $P = 0$ as $P_q = \sum_{j=k}^m P_j$, for some nonnegative integers $q$ and $k(\geq q)$ such that $P_q \neq 0$ for each $j \geq k$, where $P_j$ is the homogeneous part of $P$ of total degree $j$ in the indeterminates $y,y',\ldots,y^{(n)}$. Then for any integer $N$ with $q \leq N \leq k$,

\[
m(r,P_q/f^N) = S(r,f).
\]

In addition if $q = 0$, then

\[
T(r,f) = N(r,0,f) + S(r,f).
\]

**Remark 3.** Lemma 2 is essentially B.Q. Li’s Lemma 4.1 in [11].

**Proof of Lemma 2.** Let $P(z,u_0,\ldots,u_n)$ be a polynomial in $u_0,\ldots,u_n$ with coefficients in $S_f$. Assume that

\[
I = \{i := (i_0,i_1,\ldots,i_n) \mid i_j \text{ is a nonnegative integer and } 0 \leq j \leq n\}
\]
is an index set with finite cardinal numbers. Define

\[
|i| = \sum_{j=0}^n i_j \quad \text{and} \quad I_p = \{i \in I : |i| = p\}.
\]
For each \( l \geq q \), let
\[
P_l = \sum_{i \in I_l} a_i(z)u_0^i \ldots u_n^i
\]
where \( a_i \in S_f \).

Take any point \( z \in \mathbb{C} \), we consider several cases.

Case (i) \( |f(z)| \geq 1 \). Since \( P_q = \sum_{i \in I_q} a_i(z)u_0^i \ldots u_n^i \),
\[
\left| \frac{P_q(f, f', \ldots, f^{(n)})}{f^n} \right| (z) \leq \left| \frac{P_q(f, f', \ldots, f^{(n)})}{f^q} \right| (z)
\]
\[
\leq \sum_{i \in I_q} \left| a_i(z) \frac{f^{i_0}(f')^{i_1} \ldots (f^{(n)})^{i_m}}{f^q} \right| := G_1(z).
\]

Case (ii) \( |f(z)| \leq 1 \). Then by \( P_q = \sum_{j=k}^m P_j, q \leq N \leq k \), we have
\[
\left| \frac{P_q(f, f', \ldots, f^{(n)})}{f^n} \right| (z) = \left| \sum_{j=k}^m P_j(f, f', \ldots, f^{(n)})f^j f^{-N} \right|
\]
\[
\leq \sum_{j=k}^m \left| P_j(f, f', \ldots, f^{(n)})f^j f^{-N} \right|
\]
\[
\leq \sum_{j=k}^m \left| a_i(z) \frac{f^{i_0}(f')^{i_1} \ldots (f^{(n)})^{i_m}}{f^j} \right| := G_2(z).
\]

Combining the above results, we see that in any case
\[
\left| \frac{P_q(f, f', \ldots, f^{(n)})}{f^n} \right| (z) \leq G_1(z) + G_2(z)
\]
for any \( z \in \mathbb{C} \). By the well-known Logarithmic Derivative Lemma and \( a_i \in S_f \), we deduce that
\[
m(r, P_q/f^N) \leq m(r, G_1 + G_2) = S(r, f).
\]

Now if \( q = 0 \), then by taking \( N = 1 \), we have
\[
m(r, 1/f) \leq m(r, P_0/f) + m(r, 1/P_0) + O(1) = S(r, f)
\]
as \( T(r, P_0) = S(r, f) \). Hence the result follows from the First Main Theorem of Nevanlinna theory.

As a consequence, one can also obtain the following lemma first proved by A. Mohon'ko in 1982.

**Lemma 3** ([14]). Let \( f \) be a transcendental meromorphic solution of an algebraic differential equation \( P(y) = P(z, y, y', \ldots, y^{(k)}) = 0 \) with coefficients in \( S_f \). If a
meromorphic function $\phi$ with $T(r, \phi) = S(r, f)$ does not solve $P(z, y, y', \ldots, y^{(k)}) = 0$ i.e. $P(z, \phi, \phi', \ldots, \phi^{(k)}) \neq 0$, then

$$m \left( r, \frac{1}{f - \phi} \right) = S(r, f)$$

Proof. Let $g = f - \phi$, then $T(r, g) = T(r, f) + S(r, f)$. Since $P(f) \equiv 0$, we have

$$P(f) = P(g + \phi) = Q(g) + P(\phi) \equiv 0$$

where $Q$ is a differential polynomial over $S_f$ with lowest degree at least one, as $T(r, \phi) = S(r, f)$. The result follows immediately from Lemma 2 as $P(\phi) \neq 0$. □

Lemma 4 ([5]). Let $f$ be a transcendental entire function and let $g$ be a transcendental meromorphic function in the complex plane, then $T(r, f) = o(T(r, g \circ f))$ as $r \to \infty$.

Proof of Theorem 4 [11]. Without loss of generality, we can assume $R(z) = z$, since if $f$ is hypertranscendental over $S$, it is easy to show that $R \circ f$ is also hypertranscendental over $S$.

Suppose $g(z) - z = 0$ has $d$ roots, then $g(z) - z = Q(z)A(z)$ where $Q$ is a polynomial with degree $d$, and $A$ is a transcendental meromorphic function which is nowhere zero. Hence if $f$ is an entire function, we have

$$N(r, 0, g \circ f - f) = N(r, 0, Q(f)A(f)) = N(r, 0, Q(f)) \leq dT(r, f) + S(r, f).$$

By Lemma 4, we have $T(r, f) = o(T(r, g \circ f))$. Suppose $g \circ f$ is not hypertranscendental over $S$, that is, $g \circ f$ is a solution of an algebraic differential equation $P(z, y, y', \ldots, y^{(k)}) = 0$ with coefficients in $S$ (hence in $S_{gof}$ as well). By Lemma 3 and the assumption that $f$ is hypertranscendental over $S$, we have

$$m \left( r, \frac{1}{g \circ f - f} \right) = S(r, g \circ f).$$

By the First Main Theorem of Nevanlinna Theory for small functions [14],

$$T(r, g \circ f) = T(r, g \circ f - f) + S(r, g \circ f)$$

$$= m(r, 0, g \circ f - f) + N(r, 0, g \circ f - f) + S(r, g \circ f)$$

$$\leq S(r, g \circ f) + dT(r, f) = S(r, g \circ f)$$

which is a contradiction. This completes the proof of the first part.

2. If $a \neq 0$, since $f$ is hypertranscendental over $S$, it is easy to show that $af$ is also hypertranscendental over $S$, as $a \in S$.

Since $T(r, f) = S(r, g), T(r, a) = S(r, f'/f) = S(r, f)$, one can obtain that $T(r, af) = T(r, f) + S(r, f) = S(r, fg)$.

Suppose $fg$ is not hypertranscendental over $S$, that is, $fg$ is a solution of an algebraic differential equation $P(z, y, y', \ldots, y^{(k)}) = 0$ with coefficients in $S$ (hence
in $S_{fg}$ also). By Lemma 3 and the hypertranscendence of $af$ over $S$, we have

$$m\left(r, \frac{1}{fg - af}\right) = S(r, fg) = S(r, g).$$

On the other hand, by the First Main Theorem of Nevanlinna Theory for small functions, as $T(r, af) = S(r, fg)$,

$$T(r, fg) = T(r, fg - af) + S(r, fg)$$

$$= m(r, 0, fg - af) + N(r, 0, fg - af) + S(r, fg)$$

$$\leq N(r, 0, g - a) + N(r, 0, f) + S(r, g)$$

$$= N(r, 0, g - a) + S(r, g).$$

Since $T(r, fg) = T(r, g) + S(r, g)$, it follows that $T(r, g) = N(r, a, g) + S(r, g)$ which is a contradiction to the assumption that $\delta(a, g) > 0$.

If $f + g \in A(S)$, so does $f^{(k)} + g^{(k)}$, that is, there exists a nontrivial algebraic differential equation $P(z, y, y', \ldots, y^{(n)}) = 0$ over $S$ such that

$$P(z, f^{(k)} + g^{(k)}, f^{(k+1)} + g^{(k+1)}, \ldots, f^{(k+n)} + g^{(k+n)}) \equiv 0.$$

Set

$$Q(z, g^{(k)}, g^{(k+1)}, \ldots, g^{(n+k)}) := P(z, f^{(k)} + g^{(k)}, f^{(k+1)} + g^{(k+1)}, \ldots, f^{(k+n)} + g^{(k+n)}),$$

then

$$Q(z, g^{(k)}, g^{(k+1)}, \ldots, g^{(n+k)}) \equiv 0.$$ It is easy to check that all the Nevanlinna characteristic functions of the coefficients of $Q(z, g^{(k)}, g^{(k+1)}, \ldots, g^{(n+k)})$ are $S(r, g^{(k)})$, as $T(r, f) = S(r, g^{(k)})$ and $T(r, f^{(k)}) \leq (k + 1)T(r, f) + S(r, f)$.

On the other hand, since $f$ is hypertranscendental over $S$, so is $f^{(k)} + a$ for any $a \in S$, hence

$$Q(z, a, a', \ldots, a^{(n)}) = P(z, f^{(k)} + a, f^{(k+1)} + a', \ldots, f^{(k+n)} + a^{(n)}) \neq 0.$$ By Lemma 3, we have

$$m\left(r, \frac{1}{g^{(k)} - a}\right) = S(r, g^{(k)})$$

which is a contradiction to the assumption that $\delta(a, g^{(k)}) > 0$ for some $a \in S$.

If $fe^{g} \in A(S)$, then clearly, $\left(\frac{f'}{f}\right)^{(k)} + g^{(k+1)} \in A(S)$, and hence so does

$$\left(\frac{f'}{f}\right)^{(k)} + g^{(k+1)}$$

for any nonnegative integer $k$, that is, there exists an algebraic differential equation $P(z, y, y', \ldots, y^{(n)}) = 0$ over $S$ such that

$$P\left(z, \left(\frac{f'}{f}\right)^{(k)} + g^{(k+1)}, \left(\frac{f'}{f}\right)^{(k+1)} + g^{(k+2)}, \ldots, \left(\frac{f'}{f}\right)^{(k+n)} + g^{(k+n+1)}\right) \equiv 0.$$
Set
\[ Q(z, g^{(k)}, g^{(k+1)}, \ldots, g^{(n+k+1)}) := P\left(z, \left(\frac{f'}{f}\right)^{(k)} + g^{(k+1)}, \ldots, \left(\frac{f'}{f}\right)^{(k+n)} + g^{(k+n+1)}\right), \]
then
\[ Q(z, g^{(k)}, g^{(k+1)}, \ldots, g^{(n+k+1)}) \equiv 0 \]
and all the Nevanlinna characteristic functions of the coefficients of \( Q(z, g^{(k)}, g^{(k+1)}, \ldots, g^{(n+k+1)}) \) are \( S(r, g^{(k)}) \) from
\[ T(r, f'/f) = S(r, g^{(k)}) \]
and
\[ T(r, (f'/f)^{(j)}) \leq (j + 1)T(r, f'/f) + S(r, f'/f) \]
for any nonnegative integer \( j \).

On the other hand, since \( f \) is hypertranscendental over \( S \), so is \( (f'/f)^{(k)} \) for any nonnegative integer \( k \), and hence so is \( (f'/f)^{(k)} + a' \) for any \( a \in S \). Therefore,
\[ Q(z, a', \ldots, a^{(n+1)}) = P\left(z, \left(\frac{f'}{f}\right)^{(k)} + a', \left(\frac{f'}{f}\right)^{(k+1)} + a'', \ldots, \left(\frac{f'}{f}\right)^{(k+n)} + a^{(n+1)}\right) \neq 0. \]
By Lemma 3 we have
\[ m\left(\frac{1}{r, g^{(k)} - a}\right) = S(r, g^{(k)}) \]
which is a contradiction to the inequality (3).

Let
\[ P(z, u_0, u_1, \ldots, u_n) = \sum_{i=0}^{m} P_i(z, u_0, u_1, \ldots, u_n) \]
be a distinguished polynomial over \( S \), where \( P_i(z, u_0, u_1, \ldots, u_n) \) contains only one term \( a_i(z)u_0^{i_0}u_1^{i_1}\cdots u_n^{i_n} \) with coefficient \( a_i \in S \) and \( i = i_0 + i_1 + \cdots + i_n \).

We first notice that the assumption \( f \in S_{\exp(g)} \) and Lemma 4 imply that
\[ T\left(r, \frac{f e^g}{f e^g}\right) = T\left(r, \frac{f'}{f} + g'\right) \leq 2T(r, f) + S(r, f) + 2T(r, g) + S(r, g) = S(r, f e^g) \]
Assume to the contrary that \( P(z, f e^g, (f e^g)', \ldots, (f e^g)^{(n)}) \equiv 0 \). Let \( q \) be a nonnegative integer such that \( a_q \neq 0 \) and \( a_j \equiv 0, j = 0, 1, \ldots, q - 1 \). Applying Lemma 2 to \( N = q + 1 \), one can conclude that
\[ m(r, P_q/(f e^g)^{q+1}) = S(r, f e^g) \]
and
\[ m(r, 1/(f e^g)) = S(r, f e^g) \text{ as } T(r, P_q/(f e^g)^{q}) = S(r, f e^g). \] However, \( m(r, 0, f e^g) + N(r, 0, f e^g) = S(r, f e^g) + N(r, 0, f) \leq T(r, f) = S(r, f e^g) \), which is impossible, thus \( a_q \equiv 0 \). Repeating the above argument, one can obtain that \( a_i \equiv 0 \) for all \( i = 0, 1, \ldots, m \). Hence the result follows.
This completes the proof of Theorem 4. □

5. Proof of Theorem 1 and 2

In this section, we will prove Theorem 1 and 2 by using Theorem 4.

Proof of Theorem 1. This follows immediately from part (4) of Theorem 4 and the fact that $T(r, \Gamma'/\Gamma) = r + o(r)$ in [4]. □

Proof of Theorem 2. We consider the following two cases.

Case 1. If $\rho(e^h) < \infty$, then $\Gamma e^h$ is hypertranscendental over $\mathcal{M}_0$ (see p.271 of [3]). Actually, in this case, $h$ is a polynomial, hence it is not hard to see that $e^h \in A(\mathcal{M}_0)$ as $h' = (e^h)'/e^h$. If $\Gamma e^h \in A(\mathcal{M}_0)$, one can conclude that $\Gamma \in A(\mathcal{M}_0)$ which is a contradiction to the hypertranscendence of $\Gamma$ over $\mathcal{M}_0$.

Case 2. If $\rho(e^h) = \infty$, then $\Gamma \in \mathcal{S}_{\exp(h)}$, hence the result follows immediately from Theorem 4(5). □

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