Two weight commutators for Beurling–Ahlfors operator

Xuan Thinh Duong, Ji Li, and Brett D. Wick

Abstract: We establish the equivalent characterisation of the weighted BMO space on the complex plane $\mathbb{C}$ via the two weight commutator of the Beurling–Ahlfors operator with a BMO function. Our method of proofs relies on the explicit kernel of the Beurling–Ahlfors operator and the properties of Muckenhoupt weight class.

Keywords: weighted BMO; commutator; two weights, Hardy space; factorization.

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1 Introduction and statement of main result

The theory of singular integrals in harmonic analysis has had its origin closely related to other fields of mathematics such as complex analysis and partial differential equations. A typical example is the Hilbert transform which has arisen from the complex conjugates of harmonic functions in the real and complex parts of analytic functions. The higher dimensional version of the Hilbert transform is the Riesz transform on the Euclidean space $\mathbb{R}^n$. The Hardy space $H^1$ and its dual space, the BMO space (BMO is abbreviation for bounded mean oscillation) have played an important role for the end-point estimates of singular integrals as they are known as the substitutes of the spaces $L^1$ and $L^\infty$.

For singular integrals, weighted estimates are important and the $A_p$ class of Muckenhoupt weights has provided the appropriate class of weights for the study of Calderón-Zygmund operators. One can use two weight estimates on commutators of BMO functions with certain singular integral operators to characterise BMO spaces. The recent paper [7] gives a notable result which characterised weighted BMO spaces through two weight estimates on commutators of BMO functions and the Riesz transform. More specifically, consider the $j$-th Riesz transform on $\mathbb{R}^n$ given by $R_j = \frac{2}{i\pi} \Delta^{-1/2}$, the weights $\lambda_1, \lambda_2$ in the Muckenhoupt class $A_p$, $1 < p < \infty$, and the weight $\nu = \frac{\lambda_1^{1/p}}{\lambda_2^{1/p}}$. Denote by $L^p_w(\mathbb{R}^n)$ the weighted $L^p$ space with the measure $w(x)dx$ and $[b, R_j](f)(x) = b(x)R_j(f)(x) - R_j(bf)(x)$ the commutator of the Riesz transform $R_j$ and the function $b \in \text{BMO}_\nu(\mathbb{R}^n)$, i.e., the Muckenhoupt–Wheeden weighted BMO space (introduced in [8]). The main result in [7], Theorem 1.2, says that there exist constants $0 < c < C < \infty$ such that

$$c\|b\|_{\text{BMO}_\nu(\mathbb{R}^n)} \leq \sum_{i=1}^{n} \| [b, R_i] : L^p_{\lambda_1}(\mathbb{R}^n) \to L^p_{\lambda_2}(\mathbb{R}^n) \| \leq C\|b\|_{\text{BMO}_\nu(\mathbb{R}^n)} \quad (1.1)$$

where the constants $c$ and $C$ depend only on $n, p, \lambda_1, \lambda_2$. This result extended previous results of Bloom [1], Coifman, Rochberg and Weiss [2] and Nehari [11].

While the upper bound of the two weight commutator can be obtained for a large class of singular integral operators, the lower bound is delicate and its proof for each specific operator can be quite different and depends on the nature of the operator. For example, the proof for the lower bound of the commutator with the Riesz transform used the spherical harmonic
expansions for the Riesz kernels, which relies on the property of the Fourier transform of the Riesz kernels.

In this paper, we consider the Beurling-Ahlfors operator $B$ (see for example [9, 10]) which plays a notable role in complex analysis and is given by convolution with the distributional kernel p.v. $\frac{1}{z^2}$, i.e., for $x \in \mathbb{C}$,

$$B(f)(x) = \text{p.v.} \frac{1}{\pi} \int_{\mathbb{C}} \frac{f(y)}{(x-y)^2} \ dy.$$  

Here, for simplicity, we just use $dy$ to denote Lebesgue measure on $\mathbb{C}$. For other works on the Beurling–Ahlfors operator, see for example [9] where they established a sharp weighted estimate of $B$, which is sufficient to prove that any weakly quasiregular map is quasiregular.

We now recall the Muckenhoupt–Wheeden type weighted $\text{BMO}$ space on $\mathbb{C}$. For $\nu \in A_2(\mathbb{C})$, $\text{BMO}_\nu(\mathbb{C})$ is defined (see [8]) as the set of all $f \in L^1_{\text{loc}}(\mathbb{C})$, such that

$$\|f\|_{\text{BMO}_\nu(\mathbb{C})} := \sup_Q \frac{1}{\nu(Q)} \int_Q |f - f_Q| dx < \infty,$$

where the supremum is taken over all cubes $Q \subset \mathbb{C}$ and

$$f_Q := \frac{1}{|Q|} \int_Q f(y) dy.$$

A natural question is as follows.

**Q:** Can we establish the characterisation of two weight commutator and the related weighted $\text{BMO}$ space for the Beurling–Ahlfors operator, i.e. obtain (1.1) with the Beurling–Ahlfors operator in place of the Riesz transform?

Our following main result gives a positive answer to this question.

**Theorem 1.1.** Suppose $1 < p < \infty$, $\lambda_1, \lambda_2 \in A_p(\mathbb{C})$ and $\nu = \lambda_1^{-\frac{1}{p}} \lambda_2^{-\frac{1}{p}}$. Suppose $b \in L^1_{\text{loc}}(\mathbb{C})$. Let $B$ be the Beurling–Ahlfors operator. Then we have

$$\|b\|_{\text{BMO}_\nu(\mathbb{C})} \approx \|[b, B] : L^p_{\lambda_1}(\mathbb{C}) \to L^p_{\lambda_2}(\mathbb{C})\|.$$  

We provide the proof of the above theorem in Section 2. Then in Section 3, we provide the application of Theorem 1.1 the weak factorisation of the weighted Hardy space via the bilinear form in terms of the Beurling–Ahlfors operator, which extends the classical result of Coifman, Rochberg and Weiss [2].

Throughout the paper, we denote by $C$ and $\tilde{C}$ positive constants which are independent of the main parameters, but they may vary from line to line. For every $p \in (1, \infty)$, we denote by $p'$ the conjugate of $p$, i.e., $\frac{1}{p} + \frac{1}{p'} = 1$. If $f \leq C g$, we then write $f \preceq g$ or $g \succeq f$; and if $f \preceq g \preceq f$, we write $f \approx g$.

2 Proof of the main theorem

We first recall the definition and some basic properties of the Muckenhoupt $A_p(\mathbb{C})$ weights.
**Definition 2.1.** Suppose \( w \in L^1_{\text{loc}}(\mathbb{C}), \) \( w \geq 0, \) and \( 1 < p < \infty. \) We say that \( w \) is a Muckenhoupt \( A_p(\mathbb{C}) \) weight if there exists a constant \( C \) such that

\[
\sup_Q \langle w \rangle_Q \left( |w|^{-\frac{1}{p'}} \right)^{p-1} \leq C < \infty, \tag{2.1}
\]

where the supremum is taken over all cubes \( Q \) in \( \mathbb{C}. \) We denote by \([w]_{A_p}\) the smallest constant \( C \) such that (2.1) holds.

The class \( A_1(\mathbb{C}) \) consists of the weights \( w \) satisfying for some \( C > 0 \) that

\[
\langle w \rangle_Q \leq C \text{ess inf}_{x \in Q} w(x)
\]

for any cubes \( Q \subset \mathbb{C}. \) We denote by \([w]_{A_1}\) the smallest constant \( C \) such that the above inequality holds.

If \( w \in A_p(\mathbb{C}) \) with \( p > 1, \) then the “conjugate” weight

\[
w' = w^{1-p'} \in A_{p'}(\mathbb{C}) \tag{2.2}
\]

with \([w']_{A_{p'}} = [w]_{A_p}^{p'-1},\) where \( p' \) is the conjugate index of \( p, \) i.e., \( 1/p + 1/p' = 1. \) Moreover, suppose \( \lambda_1, \lambda_2 \in A_p(\mathbb{C}) \) with \( 1 < p < \infty. \) Set

\[
\nu = \lambda_1^{\frac{1}{p}} \lambda_2^{-\frac{1}{p}}. \tag{2.3}
\]

Then we have that \( \nu \in A_2(\mathbb{C}), \) see [7, Lemma 2.19]. Moreover, we have the following fundamental result (see [7, equation (2.21)]): for any ball \( B \subset \mathbb{C}, \)

\[
\frac{\lambda_1(B)}{|B|} \cdot \left( \frac{\lambda_2(B)}{|B|} \right)^{\frac{1}{p'}} \leq \frac{1}{\left( \frac{\lambda_1(B)}{|B|} \right)^{p'}) \left( \frac{\lambda_2(B)}{|B|} \right)^{\frac{1}{p}} \leq \frac{1}{\nu^{-1}(B)} \leq \frac{\nu(B)}{|B|}. \tag{2.4}
\]

Suppose \( 1 < p < \infty, \lambda_1, \lambda_2 \in A_p(\mathbb{C}) \) and \( \nu = \lambda_1^{\frac{1}{p}} \lambda_2^{-\frac{1}{p}}. \) Note that \( \nu \in A_2(\mathbb{C}). \) Since \( B \) is a Calderón–Zygmund operator, following the result in [7] we obtain that there exists a positive constant \( C \) such that for \( b \in BMO_p(\mathbb{C}), \)

\[
\| [b, B] : L^p_{\lambda_1}(\mathbb{C}) \to L^p_{\lambda_2}(\mathbb{C}) \| \leq C \| b \|_{BMO_p(\mathbb{C})}. \tag{2.5}
\]

When \( \lambda_1 = \lambda_2, \) then the upper bound with precise information about the constant \( C \) as a function of the \( A_p \) characteristic was obtained in [4].

We now prove the lower bound. Suppose that \( b \in L^1_{\text{loc}}(\mathbb{C}) \) and that \([b, B]\) is bounded from \( L^p_{\lambda_1}(\mathbb{C}) \) to \( L^p_{\lambda_2}(\mathbb{C}) \). It suffices to show that for every cube \( Q \subset \mathbb{C}, \) there exists a positive constant \( C \) such that

\[
\frac{1}{\nu(Q)} \int_Q |b(x) - b_Q| \, dx \leq C < \infty.
\]

To see this, without lost of generality, we now consider an arbitrary cube \( Q \subset \mathbb{C} \) centered at the origin. Then we have

\[
\int_Q |b(x) - b_Q| \, dx
\]
Again, for the term $I$ for any $Xuan Thinh Duong, Ji Li, and Brett D. Wick$ where in the first inequality we use Holder’s inequality with the index $b, B$:

\[
I := \frac{1}{|Q|} \int_{C} [b(x) - b_Q] \|\chi_Q(x)\|_{L^2_{\lambda_1}(C)} dx
\]

We now denote $\Gamma_Q(x) := \operatorname{sgn}(b(x) - b_Q)$. Then for the term $I_1$, we obtain that

\[
|I_1| = \frac{1}{|Q|} \left| \int_{C} [b, B](\chi_Q(x)) x^2 \Gamma_Q(x) \chi_Q(x) dx \right|
\leq \frac{1}{|Q|} \left\| [b, B](\chi_Q) \right\|_{L^p_{\lambda_1}(C)} \left\| x^2 \Gamma_Q(x) \chi_Q(x) \right\|_{L^p_{\lambda_2}(C)}
\leq C \left\| [b, B] : L^p_{\lambda_1}(C) \rightarrow L^p_{\lambda_2}(C) \right\| \left\| \chi_Q \right\|_{L^p_{\lambda_1}(C)} \left\| \chi_Q \right\|_{L^p_{\lambda_2}(C)}
\leq C \left\| [b, B] : L^p_{\lambda_1}(C) \rightarrow L^p_{\lambda_2}(C) \right\| \left\| \chi_Q \right\|_{L^p_{\lambda_1}(C)} \left\| \nu(Q) \right\|_{L^p_{\lambda_2}(C)}
\]

where in the first inequality we use Holder’s inequality with the index $b, B$ from $L^p_{\lambda_1}(C)$ to $L^p_{\lambda_2}(C)$ and the fact that $|\Gamma_Q(x)| \leq 1$ for any $x \in C$, and in the last inequality we use the fundamental fact in $[24]$.

As for the term $I_2$, similarly, we have

\[
|I_2| = \frac{2}{|Q|} \left| \int_{C} [b, B](y \chi_Q(y)) x \Gamma_Q(x) \chi_Q(x) dx \right|
\leq \frac{1}{|Q|} \left\| [b, B](y \chi_Q(y)) \right\|_{L^p_{\lambda_1}(C)} \left\| x \Gamma_Q(x) \chi_Q(x) \right\|_{L^p_{\lambda_2}(C)}
\leq C \left\| [b, B] : L^p_{\lambda_1}(C) \rightarrow L^p_{\lambda_2}(C) \right\| \frac{1}{|Q|} \left\| y \chi_Q(y) \right\|_{L^p_{\lambda_1}(C)} \left\| x \chi_Q(x) \right\|_{L^p_{\lambda_2}(C)}
\leq C \left\| [b, B] : L^p_{\lambda_1}(C) \rightarrow L^p_{\lambda_2}(C) \right\| \left\| \chi_Q \right\|_{L^p_{\lambda_1}(C)} \left\| \nu(Q) \right\|_{L^p_{\lambda_2}(C)}
\]

Again, for the term $I_3$, using similar argument, we get that

\[
|I_3| = \frac{1}{|Q|} \left| \int_{C} [b, B](y^2 \chi_Q(y)) x \Gamma_Q(x) \chi_Q(x) dx \right|
\]
We recall the weighted Hardy space, then prove that it is the predual of BMO. Applications: Weak factorization of the weighted Hardy space

\[ \sum_{\lambda} \lambda \text{ satisfies } \| \sum_{\lambda} \lambda \] with \( \lambda \). Definition 3.1. A function \( a \in L^2(\mathbb{C}) \) is called an \( \nu \)-weighted \( (1,2) \)-atom if it satisfies

1. \( \text{supp } a \subset B \), where \( B \) is a ball in \( \mathbb{C} \);
2. \( \int_{\mathbb{C}} a(x) dx = 0 \);
3. \( \|a\|_{L^2(\mathbb{C})} \leq \nu(B)^{-\frac{1}{2}} \).

We say that \( f \) belongs to the weighted Hardy space \( H^1_\nu(\mathbb{C}) \) if \( f \) can be written as

\[ f = \sum_j \alpha_j a_j \] (3.1)

with \( \sum_j |\alpha_j| < \infty \). The \( H^1_\nu(\mathbb{C}) \) norm of \( f \) is defined as

\[ \|f\|_{H^1_\nu(\mathbb{C})} := \inf \left\{ \sum_j |\alpha_j| : f \text{ has the representation as in (3.1)} \right\} \]

We also recall the John–Nirenberg inequality for the BMO(\( \nu(\mathbb{C}) \). According to [5], we know that for \( \nu \in A_2(\mathbb{C}) \) and for \( 1 \leq r \leq 2 \),

\[ \|b\|_{\text{BMO}, \nu(\mathbb{C})} \leq \|b\|_{\text{BMO}, \nu, \nu(\mathbb{C})} \leq C_{n,p,r}[\nu] A_2 \|b\|_{\text{BMO}, \nu(\mathbb{C})}, \] (3.2)

where

\[ \|b\|_{\text{BMO}, \nu, \nu(\mathbb{C})} := \left( \sup_Q \frac{1}{\nu(Q)} \int_Q |b(x) - b_Q|^r \nu^{1-r}(x) dx \right)^{\frac{1}{r}}. \] (3.3)
Theorem 3.2. Suppose $\nu \in A_2(\mathbb{C})$. The dual of $H^1_{\nu}(\mathbb{C})$ is $\text{BMO}_\nu(\mathbb{C})$.

Proof. This duality result follows from a standard argument, see for example [3]. By completeness, we provide the proof as follows. We first show that
\[
\text{BMO}_\nu(\mathbb{C}) \subset (H^1_{\nu}(\mathbb{C}))^*.
\]
In fact, for any $g \in \text{BMO}_\nu(\mathbb{C})$, define
\[
L_g(a) := \int_{\mathbb{C}} a(x) g(x) dx,
\]
where $a$ is an $\nu$-weighted $(1,2)$-atom.

Assume that $a$ is supported in a cube $Q$. Then from Hölder’s inequality and $\nu \in A_2(\mathbb{C})$, we see that
\[
\left| \int_{\mathbb{C}} a(x) g(x) dx \right| = \left| \int_Q a(x) [g(x) - g_Q] dx \right|
\leq \left( \int_Q |g(x) - g_Q|^2 \nu^{-1}(x) dx \right)^{\frac{1}{2}} \left( \int_Q |a(x)|^2 \nu(x) dx \right)^{\frac{1}{2}}
\leq \frac{1}{\nu(B)} \int_Q |g(x) - g_Q|^2 \nu^{-1}(x) dx \right)^{\frac{1}{2}}
\leq C \|g\|_{\text{BMO}_\nu(\mathbb{C})}.
\]
Thus $L_g$ extends to a bounded linear functional on $H^1_{\nu}(\mathbb{C})$.

Conversely, assume that $L \in (H^1_{\nu}(\mathbb{C}))^*$. For any cube $Q$, let
\[
L^2_{0,\nu}(Q) = \{ f \in L^2_{\nu}(Q) : \text{supp}(f) \subset Q, \int_Q f(x) dx = 0 \}.
\]
Then we see that for any $f \in L^2_{0,\nu}(Q)$, the function $\frac{1}{\nu(Q)^{\frac{1}{2}} \|f\|_{L^2_{\nu}(Q)}} f$ is an $H^1_{\nu}(\mathbb{C})$-atom. This implies that
\[
|L(a)| \leq \|L\| \|a\|_{H^1_{\nu}(\mathbb{C})} \leq \|L\|.
\]
Moreover, we see that
\[
|L(f)| \leq \|L\| \nu(Q)^{\frac{1}{2}} \|f\|_{L^2_{\nu}(Q)}
\]
From the Riesz representation theorem, there exists $[\varphi] \in [L^2_{0,\nu}(Q)]^* = L^2_{0,\nu-1}(Q)/\mathbb{C}$, and $\varphi \in [\varphi]$, such that for any $f \in L^2_{0,\nu}(Q)$,
\[
L(f) = \int_Q f(x) \varphi(x) dx
\]
and
\[
\|\varphi\| = \inf_{c} \|c + \varphi\|_{L^2_{\nu-1}(Q)} \leq \|L\| \nu(Q)^{\frac{1}{2}}.
\]
Let $Q$ fixed and $Q_j = 2^j Q$, $j \in \mathbb{N}$. Then we have that for all $f \in L^2_{0,\nu}(Q)$ and $j \in \mathbb{N}$,
\[
\int_Q f(x) \varphi_Q(x) dx = \int_Q f(x) \varphi_{Q_j}(x) dx.
\]
It follows that for almost every \( x \in Q \), \( \varphi_{Q_j}(x) - \varphi_{Q_j}(x) = C_j \) for some constant \( C_j \). From this we further deduce that for all \( j, l \in \mathbb{N}, j \leq l \) and almost every \( x \in Q_j \),

\[
\varphi_{Q_j}(x) - C_j = \varphi_{Q_j}(x) = \varphi_{Q_l}(x) - C_l.
\]

Define

\[
\varphi(x) = \varphi_j(x) - C_j
\]
on \( B_j \) for \( j \in \mathbb{N} \). Thus, \( \varphi \) is well defined. Moreover, since \( C = \bigcup_j Q_j \), by Hölder’s inequality and \( \nu \in A_2(\mathbb{C}) \), we see that for any \( c \) and any cube \( Q \),

\[
\left[ \int_Q |\varphi(x) - \varphi_Q|^2 \nu^{-1}(x) \, dx \right]^{\frac{1}{2}} = \sup_{\|f\|_{L^2(\nu(Q))} \leq 1} \|f, \varphi - \varphi_Q\| = \sup_{\|f\|_{L^2(\nu(Q))} \leq 1} \left| \int_Q f(x)[\varphi(x) - \varphi_Q] \, dx \right|
\]

\[
= \sup_{\|f\|_{L^2(\nu(Q))} \leq 1} \left| \int_Q [f(x) - f_Q]\varphi(x) + c] \, dx \right|
\]

\[
\leq \sup_{\|f\|_{L^2(\nu(Q))} \leq 1} \left[ \|f\|_{L^2(\nu(Q))} + |f_Q|\nu(Q)^{\frac{1}{2}} \|\varphi(x) + c]\chi_Q\|_{L^2_{\nu^{-1}}(Q)} \right]
\]

\[
\leq \|\varphi(x) + c]\chi_Q\|_{L^2_{\nu^{-1}}(Q)}.
\]

Taking the infimum over \( c \), we have that \( \varphi \in \text{BMO}_\nu(\mathbb{C}) \) and \( \|\varphi\|_{\text{BMO}_\nu(\mathbb{C})} \leq C\|L\|. \]

The main result of this section is as follows.

**Theorem 3.3.** Suppose \( 1 < p < \infty \), \( \lambda_1, \lambda_2 \in A_p(\mathbb{C}) \) and \( \nu = \frac{\lambda_1}{\lambda_2^\theta} \lambda_2^{-\frac{1}{p'}} \). For every \( f \in H^1_\nu(\mathbb{C}) \), there exist sequences \( \{\alpha_j^k\} \in \ell^1 \) and functions \( h_j^k \in L^p_{\lambda_1}(\mathbb{C}) \), \( g_j^k \in L^{p'}_{\lambda_2}(\mathbb{C}) \) with \( p' = \frac{p}{p-1} \) and \( \lambda_2 = \frac{\lambda_1}{\lambda_2^\theta} \) such that

\[
f(x) = \sum_{k=1}^{\infty} \sum_{j=1}^{\infty} \alpha_j^k \Pi(g_j^k, h_j^k)(x)
\]

in the sense of \( H^1_\nu(\mathbb{C}) \), where \( \Pi(g_j^k, h_j^k)(x) \) is the bilinear form defined as

\[
\Pi(g_j^k, h_j^k)(x) := h_j^k(x)B(g_j^k)(x) - g_j^k(x)B^*(h_j^k)(x).
\]

Moreover, we have that

\[
\inf \left\{ \sum_{k=1}^{\infty} \sum_{j=1}^{\infty} |\alpha_j^k|\|g_j^k\|_{L^p(\mathbb{C})}\|h_j^k\|_{L^p_{\lambda_1}(\mathbb{C})} \right\} \approx \|f\|_{H^1_\nu(\mathbb{C})},
\]

where the infimum is taken over all possible representations of \( f \) from (3.4).
It is well known that this theorem follows from the duality between $H^1_\nu(C)$ and $\text{BMO}_\nu(C)$ and the equivalence between $\text{BMO}_\nu(C)$ and the boundedness of the commutator, provided in Theorem 1.1. We omit the details of this proof.

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