A doubly exponential upper bound on noisy EPR states for binary games

Penghui Yao
State Key Laboratory for Novel Software Technology, Nanjing University
pyao@nju.edu.cn
April 23, 2019

Abstract

This paper initiates the study of a class of entangled-games, mono-state games, denoted by \((G, \psi)\), where \(G\) is a two-player one-round game and \(\psi\) is a bipartite state independent of the game \(G\). In the mono-state game \((G, \psi)\), the players are only allowed to share arbitrary copies of \(\psi\). This paper provides a doubly exponential upper bound on the copies of \(\psi\) for the players to approximate the value of the game to an arbitrarily small constant precision for any mono-state binary game \((G, \psi)\), if \(\psi\) is a noisy EPR state, which is a two-qubit state with completely mixed states as marginals and maximal correlation less than 1. In particular, it includes \((1 - \epsilon) |\Psi\rangle\langle\Psi| + \epsilon \frac{|0\rangle\langle0|}{2} \otimes \frac{|0\rangle\langle0|}{2}\), an EPR state with an arbitrary depolarizing noise \(\epsilon > 0\). This paper develops a series of new techniques about the Fourier analysis on matrix spaces and proves a quantum invariance principle and a hypercontractive inequality of random operators. The structure of the proofs is built the recent framework about the decidability of the non-interactive simulation of joint distributions [GKS16, DMN18, GKR18], which is completely different from all previous optimization-based approaches [CHTW04, KRT10, NPA08] or “Tsirelson’s problem”-based approaches [Fri12, Slo16, Slo19]. This novel approach provides a new angle to study the decidability of the complexity class \(\text{MIP}^*\), a longstanding open problem in quantum complexity theory.
Contents

1 Introduction
   1.1 Proof Overview .............................................................. 1
   1.2 Organization of the paper .............................................. 5

2 Preliminaries
   2.1 Gaussian spaces .............................................................. 6
   2.2 Quantum mechanics ....................................................... 8
   2.3 Matrix spaces ............................................................... 8
   2.4 Random operators ......................................................... 12
   2.5 Miscellaneous ............................................................. 14

3 Main results ................................................................. 14

4 Open questions ............................................................ 19

5 Markov super-operators, noise operators and maximal correlation 19

6 Smoothing operators ....................................................... 26

7 Joint regularity lemma .................................................... 28

8 Fréchet Derivative and Taylor expansion ................................ 30

9 Hypercontractive inequality of random operators ....................... 32

10 Quantum invariance principle ........................................... 35

11 Dimension reduction for random operators ................................ 40

12 Smoothing random operators ............................................ 43

13 Multilinearization of random operators ................................ 45

A Facts on Fréchet derivative .............................................. 52

B Facts on analysis ........................................................... 53

C Proofs in Section 8 .......................................................... 55

D Proofs in Section 10 ........................................................ 61
1 Introduction

The concept of interactive proof systems is nowadays fundamental to the theory of computing. It was first proposed by Babai [Bab85] and Goldwasser, Micali and Rackoff [GMR85] and later extended to the multi-prover setting in [BOGK88]. The study of the interactive proof systems from different lenses is at the heart of theory of computing, including the elegant characterizations $IP = PSPACE$ [Sha92, She92] for single-prover interactive proof systems and $MIP = NEXP$ [BFL91] for multiprover interactive proof systems. The latter one has further led to the celebrated PCP theorem [ALM+98, AS98].

The study on the power of interactive proof systems in the context of quantum computing also has a rich history. The model of single-prover quantum interactive proof systems was first studied by Watrous [Wat99], followed by a series of work [KW00, MW05, GW07, JJUW09], finally led to the seminar result $QIP = PSPACE$ [JJUW11]. Quantum multiprover interactive proof systems are more complicated. A key assumption on the classical multiprover interactive proof systems is that the provers are not allowed to communicate, which means that their only distributed resource is the shared randomness. In quantum multiprover interactive proof systems, this assumption is relaxed and allow the provers to share entanglement with the corresponding complexity class $MIP^*$ [CHTW04]. Surprisingly, understanding the power of $MIP^*$ turns out to be extremely difficult. A trivial lower bound on $MIP^*$ is $IP$, or equivalently $PSPACE$, which can be easily seen by ignoring all but one provers. By extending the techniques in [BFL91] to the quantum setting, Ito and Vidick proved the containment of $NEXP$ in $MIP^*$ [IV12]. The lower bound was later improved to $QMA_{EXP}$, a quantum computational complexity class analog to $NEXP$, and further to $NEEXP$, the class of nondeterministic double-exponential time by Ji [Ji16, Ji17]. In contrast, little is known about the upper bound on $MIP^*$. In his breakthrough results, Slofstra proved that it is undecidable to determine whether a multiprover interactive proof system has an entangled strategy that is accepted with probability $1$ [Slo16, Slo19]. His proof was later simplified by Dykema, Paulsen, and Prakash in [DPP19] and Fitzsimons, Ji, Vidick and Yuen in [FJVV19].

This paper concerns two-player one-round games, a core model precisely capturing the power of multiprover interactive proof systems. A two-player one-round game $G = (X, Y, A, B, \mu, V)$, where $X, Y, A, B$ are finite sets, $\mu$ is a distribution over $X \times Y$ and $V : X \times Y \times A \times B \rightarrow \{0, 1\}$ is a predicate and all of these are public, is run by three parties: a "referee" and two non-communicating players. The referee samples a pair of questions $(x, y)$ according to $\mu$, and sends $x$ and $y$ to the two players, separately. The two players have to provide an answer each to the referee from $A$ and $B$, respectively, say $(a, b)$. The referee accepts the answers he receives if and only if $V(x, y, a, b) = 1$. The only restriction to the players’ strategies is that the players are not allowed to exchange any information once the game has started. In the classical setting, the value of the game $\omega(G)$, the highest probability that the referee accept the game, is

$$\omega(G) = \max_{h_A : X \rightarrow A} \sum_{x,y} \mu(x,y) V(x,y,h_A(x),h_B(y)).$$

It is NP-hard to approximate $\omega(G)$ within a multiplicative constant thanks to the PCP theorem [ALM+98, AS98]. The entangled games, which are same as the classical games except that the players are allowed to share arbitrary entangled states before the they receive the questions, were first introduced by Cleve, Hoyer, Toner and Watrous [CHTW04] with the entangled value of a game, which is the highest probability that the referee accepts in a game when the players share entanglement, denoted by $\omega^*(G)$. It can be expressed as

$$\omega^*(G) = \lim_{n \rightarrow \infty} \max_{\psi_{AB} \in \mathcal{C}_{2^n}} \sum_{x,y} \mu(x,y) \sum_{a,b} V(x,y,a,b) \text{Tr}(P^x_a \otimes Q^y_b \psi_{AB}),$$

(1)
where $\mathcal{D}_n$ is the set of $n$-dimensional density operators, $\{P^x_a\}_a$ and $\{Q^y_b\}_b$ are POVM for any $x \in \mathcal{X}, y \in \mathcal{Y}$, respectively. Namely, $\sum_{a \in A} P^x_a = 1$, $\sum_{b \in B} Q^y_b = 1$, $P^x_a \geq 0$ and $Q^y_b \geq 0$.

In [CHTW04] Cleve et al. discovered that the model of entangled games gave a re-interpretation of Bell’s inequalities [Bel64] and an equivalent formulation CHSH games [CHSH69], a central role in quantum mechanics from all aspects. The CHSH game is a simple two-player one-round game and the violation of Bell’s inequalities by quantum mechanics implies that the classical value of CHSH game is strictly smaller than its entangled value. A large body of the subsequent work has been devoted to boost the gap between $\omega(G)$ and $\omega^*(G)$ and now we know of games of which $\omega^*(G) = 1$ while $\omega^*(G)$ can be arbitrary small [Raz98, Ara02].

However, the complexity of computing $\omega^*(G)$ is much more involved same as MIP*. It was shown in [KKM+08, IKM09] that approximating $\omega^*(G)$ to a inverse-polynomial precision is NP-hard. Later, Vidick proved that $\omega^*(G)$ of three players is NP-hard to approximate to a constant factor [Vid16]. Recently, Ji proved that it is QMA-hard to approximate $\omega^*(G)$ of multiplayer games to a inverse-polynomial precision [Ji16], which is further improved to be QMIP*-complete and thus NEXP-hard [Ji17, IV12]. Very recently, Natarajan and Vidick have proved that it is QMA-hard to approximate $\omega^*(G)$ to a constant precision under a randomized reduction [NV18].

Similar to the complexity class MIP*, the progress on the upper bound on the complexity of $\omega^*(G)$ is much less. For a few known classes of games, computing $\omega^*(G)$ is easier than computing $\omega(G)$. Cleve et al. in [CHTW04] gave a polynomial-time algorithm to exactly compute $\omega^*(G)$ of XOR games $G$ building on the work of Tsirelson [Cir80], Kempe, Regev and Toner later present a polynomial-time algorithm for unique games with a factor 6 approximation to $1 - \omega^*(G)$ [KRT10]. Interestingly, both of the two classes of games are believed to be NP-hard under certain complexity assumptions [Has01]. To the best of my knowledge, all the algorithms to decide $\omega^*(G)$ of certain class of games, including those mentioned above, are based on semidefinite programs. In particular, a hierarchy of semidefinite programs was proposed in [NPA08], whose optimal values converge to $\omega^*(G)$, while the rate of the convergence is unknown. On the other hand, Slofstra’s results [Slo16, Slo19] imply that determining whether $\omega^*(G) = 1$ is undecidable. Whether approximating $\omega^*(G)$ is decidable is still widely open.

The main difficulty in computing $\omega^*(G)$ is that there is no upper bound on the dimension of the preshared entangled states, because if we knew an upper bound, we could approximate the optimal value by using the $\epsilon$-net and brute force. On the other hand, It is known that a positive answer to a so-called 'Tsirelson’s problem' (see e.g. [Et12]) implies the existence of an algorithm approximating $\omega^*(G)$ of any entangled game, while Tsirelson’s problem is equivalent to Conne’s Embedding Conjecture [Con76], a longstanding open problem in functional analysis [JNP+11, Oza13].

This paper initiates the study of mono-state games, a new class of entangled games denoted by $(G, \psi)$, where $G$ is a two-player one-round game and $\psi$ is a bipartite state (possibly mixed) independent of the game $G$. In the mono-state game $(G, \psi)$, the players are only allowed to share arbitrary copies the state $\psi$. The value of the game, denoted by $\omega^*(G, \psi)$, can be expressed as

$$\omega^* \left( G, \psi \right) = \lim_{n \to \infty} \max_{\{ P^x_a \}_a} \sum_{x,y} \mu \left( x, y \right) \sum_{a,b} V(x, y, a, b) \text{Tr} \left( P^x_a \otimes Q^y_b \right) \psi \otimes n, $$

where $\{ P^x_a \}_a$ and $\{ Q^y_b \}_b$ are POVM for any $x \in \mathcal{X}$ and $y \in \mathcal{Y}$, respectively.

To the best of my knowledge, the decidability of mono-state games has not been studied yet. It is easy to see that the highest probability that the referee accepts is equal to the classical value if $\psi$ is a separable state. However, the situation is more involved when $\psi$ is entangled as the amount entanglement increases and tends to infinity when having more copies of $\psi$, which is potentially helpful for the referee to accept the game with higher probability. Indeed, Mančinska and Vidick in [MV15] constructed a mono-state game where the referee accepts with probability.
tending to 1 when the copies of the shared states tending to infinity, while sharing any bounded dimensional entanglement, the probability that the referee accepts is bounded away from 1.

This paper takes a step towards understanding the decidability of the mono-state games. The following is an informal statement of the main result.

**Theorem (Main result, informal).** Given a mono-state binary game \((G, \psi)\), where \(\psi\) is a noisy EPR state and a parameter \(\epsilon\), there exists an explicitly computable \(D\) such that it suffices for the players to share \(D\) copies of \(\psi\) to achieve the probability of winning at least \(\omega(G, \psi) - \epsilon\). Hence, the game \((G, \psi)\) is decidable.

The notion of noisy EPR states will be defined later, which include \((1 - \epsilon) |\Psi\rangle\langle \Psi| + \epsilon \frac{1}{2} \otimes \frac{1}{2}\), an EPR state with arbitrary small \(\epsilon > 0\) depolarizing noise. All the previous work studying the upper bound on the complexity of entangled games are either via convex optimization [CHTW04, KRT10, NPA08] or based on Tsirelson’s problem approaches [Fri12, Slo16, Slo19]. This paper adopts the Fourier analysis on matrix spaces and reduces the problem to a quantum non-interactive simulation of joint distributions. It shed a new light upon this problem and provides a new angle to study the decidability of \(\omega^*(G)\) and MIP*. Moreover, a series of results about the Fourier analysis on matrix spaces have been developed in this paper, which may be useful for other topics such as quantum property testing, quantum machine learning, etc.

Non-interactive simulations of joint distributions is a fundamental problem in information theory and communication complexity. Consider two non-communicating players Alice and Bob. Suppose they are provided a sequence of independent samples \((x_1, y_1), (x_2, y_2), \ldots\) from a joint distribution \(\mu\) on \(X \times Y\), where Alice observes \(x_1, x_2, \ldots\) and Bob observes \(y_1, y_2, \ldots\). Without communicating with each other, what joint distribution \(\nu\) can Alice and Bob jointly simulate? The research on this problem dates back to the classic works by Gács and Körner [GK73] and Wyner [Wyn75] and Witsenhausen [Wit75], followed by fruitful subsequent work (see, for example, [KA16] and the references therein). Recently, Ghazi, Kamath and Sudan in [GKS16] introduced a framework built on the theory of Fourier analysis on discrete functions and Hermite analysis on continuous functions [MOO10, Mos10, O'D13] to resolve the decidability of the non-interactive simulation of joint distributions, when the target distribution \(\nu\) is a binary distribution. The result was further generalized to an arbitrary distribution \(\nu\) in [DMN18, GKR18].

In quantum universe, it is natural to consider the non-interactive simulation of quantum states, which is also named the local state transformation. Suppose the two non-communicating players Alice and Bob are provided arbitrary copies of bipartite quantum states \(\psi_{AB}\). Without communicating with each other, what bipartite quantum state \(\phi_{AB}\) can Alice and Bob jointly create? Delgosha and Beigi first studied this problem and gave a criterion for the impossibility of local state transformation of \(\psi_{AB}\) to \(\phi_{AB}\) exactly [DB14]. Other than this result, not much about this problem is known. The proofs of the decidability of the non-interactive simulation of joint distributions in [GKS16, DMN18, GKR18] heavily use Fourier analysis, which has been intensively studied and has fruitful applications in theoretical computer science [O'D13].

The hypercontractive inequality, a key component in Fourier analysis and Hermite analysis, has also been extended to the quantum setting from various aspects and obtained several interesting applications [BRdW08, Mon12, TPK14, Kin14, DB14, CKMT15]. However, the understanding of the Fourier analysis on matrix spaces, in particular, the Fourier analysis on quantum operations is much less compared with the one on Boolean functions or real functions. This paper essentially resolves the decidability of local state transformation when \(\psi_{AB}\) is a noisy EPR state and \(\phi_{AB}\) is a classical two-bit distribution by proving a quantum invariance principle and a hypercontractive inequality of random operators, successfully generalizing the proofs in [GKS16, DMN18, GKR18] to the quantum setting. The tools developed in this paper are...
interesting on their own right and are believed to have further applications.

1.1 Proof Overview

To explain the ideas of the proof in high level, we start with a pair of measurements performed by Alice and Bob, denoted by \((\{P, 1 - P\}, \{Q, 1 - Q\})\), respectively, where \(P, Q \in \mathcal{H}_2^\otimes n\) and \(0 \leq P, Q \leq 1\) and \(n\) copies of noisy EPR states \(\psi_{AB}^\otimes n\). We would like to construct a universal bound \(n_0\) which is independent of the measurements, and a transformation \(f_n : \mathcal{H}_2^\otimes n \to \mathcal{H}_2^\otimes n_0\), such that

- \(0 \leq f_n(P) \leq 1\) and \(0 \leq f_n(Q) \leq 1\);
- \(\text{Tr} P \psi_A^\otimes n \approx \text{Tr} f_n(P) \psi_A^\otimes D\) and \(\text{Tr} Q_n \psi_B^\otimes n \approx \text{Tr} f_n(Q) \psi_B^\otimes D\);
- \(\text{Tr} (P \otimes Q) \psi_{AB}^\otimes n \approx \text{Tr} (f_n(P) \otimes f_n(Q)) \psi_{AB}^\otimes D\).

The first item implies \(\{f_n(P), 1 - f_n(P)\}\) and \(\{f_n(Q), 1 - f_n(Q)\}\) are valid measurements. The second item implies that the probability that Alice outputs 0 is almost unchanged under the transformation \(f_n\). Same for the probability that Bob outputs 0. The last item implies that the probability that both Alice and Bob output 0 is almost unchanged. As both Alice’s and Bob’s outputs are binary. This concludes that the distribution of the joint output is almost unchanged. The construction of the transformation \(f_n\) is based on the framework introduced in [GKS16], which, in turn, is built on the results in Fourier analysis and Hermite analysis developed in [MOO10, Mos10]. Analogously, we choose an orthonormal basis in \(M_d\) and apply the theory of Fourier analysis to the expansion of \(P\) and \(Q\) on this basis.

The construction consists of several steps.

1. Convert the operators \((P, Q)\) to have low degree via smoothing the operators.
2. Choose a bounded-sized subset of coordinates \(H\) such that all the coordinates not in \(H\) are low influential.
3. Replace all the low-influential coordinates by correlated Gaussian variables, resulting random operators via a quantum invariance principle and a hypercontractive inequality of random operators.
4. Apply the dimension reduction for random operators to reduce the number the Gaussian variables in the random operators to a constant.
5. Convert the random operators to low degree again via smoothing the random operators.
6. Apply the multilinearization lemma to convert the random operators to multilinear.
7. Again apply the quantum invariance principle to convert the random operators back to the measurement operators.

The main difficulty in this construction is proving the quantum invariance principle. The classical invariance principle in [MOO10] was proved via approximating a rounding map by a \(C^\infty\) function and then apply the Taylor expansion to reduce the difference between the original function and the new function to the third order of the Taylor expansion. Further apply a hypercontractive inequality to prove that the high order terms in the Taylor’s expansion is small for low-degree and low-influential functions. Generalizing such a mechanism to quantum operations is not an easy task due to the non-commutativity of the operators. To resolve the issues, we adopt Fréchet derivatives, a notion of derivatives in Banach space, for which a similar
form of Taylor expansion exists. After applying the quantum invariance principle, part of the
registers are replaced by Gaussian random variables, resulting random operators, a hybrid of
operators and Gaussian variables. But the high order terms in such an expansion are much
more involved. Borrowing techniques from matrix analysis, we manage to upper bound the
high order term with the $4$-norm of random operators. A following difficulty is to prove a
hypercontractive inequality of random operators. The proof is a delicate combination of the
hypercontractive inequality of unital channels due to King [Kin14] and the hypercontractive
inequality of Gaussian variables due to Wolff [Wol07].

1.2 Organization of the paper

Section 2 summarizes some useful definitions and basic facts on quantum mechanics, the analysis
on Gaussian spaces, matrices spaces and random operators. The main results and the proofs
are stated in Section 3. Further work and some open problems are listed in Section 4. Section 5
gives the definition of quantum maximal correlation, a key concept in this work, with several
crucial properties. The smoothing operators lemma, joint regularity lemma, hypercontractive
inequality of random operators, quantum invariance principle, dimension reduction for random
operators, smoothing random operators and multilinearization for random operators are proved
in Sections 9, 10, 11, 12 and 13, respectively. Section 8 introduces Féchet derivative. Some
basic facts on Féchet derivatives are summarized in Appendix A. Appendix B presents some
useful facts on the analytical properties of the functions $f: \mathbb{R}^n \to \mathcal{M}_d$.

Acknowledgments

This work is supported by the National Key R&D Program of China 2018YFB1003202 and
a China Youth 1000-Talent grant and Anhui Initiative in Quantum Information Technologies
Grant No. AHY150100. Part of the work was done when the author was a Hartree postdoctoral
fellow at QuICS, University of Maryland. The author thanks Thomas Vidick pointing out that
a union bound on the question sets was missing in the previous version. The author also thanks
Hong Zhang for helpful discussion and thanks Prithish Kamath and Ashley Montanaro for the
correspondence.

2 Preliminaries

For an integer $n \geq 1$, let $[n]$ and $[n]_{\geq 0}$ represent the sets $\{1, \ldots, n\}$ and $\{0, \ldots, n - 1\}$, re-
respectively. Given a finite set $\mathcal{X}$ and a natural number $k$, let $\mathcal{X}^k$ be the set $\mathcal{X} \times \cdots \times \mathcal{X}$, the
Cartesian product of $\mathcal{X}$, $k$ times. Given $a = a_1, \ldots, a_k$ and a set $S \subseteq [k]$, we write $a_S$ to
represent the projection of $a$ to the coordinates specified in $S$. For any $i \in [k]$, $a_{<i}$ represents
$a_1, \ldots, a_{i-1}$. $a_{\leq i}, a_{>i}, a_{\geq i}$ are defined similarly. Let $\mu$ be a probability distribution on $\mathcal{X}$, and
$\mu(x)$ represent the probability of $x \in \mathcal{X}$ according to $\mu$. Let $X$ be a random variable distributed
according to $\mu$. We use the same symbol to represent a random variable and its distribution
whenever it is clear from the context. The expectation of a function $f$ on $\mathcal{X}$ is defined as
$E[f(X)] \overset{\text{def}}{=} E_{x \sim X}[f(x)] = \sum_{x \in \mathcal{X}} \Pr[X = x] \cdot f(x) = \sum_x \mu(x) \cdot f(x)$, where $x \sim X$ represents
that $x$ is drawn according to $X$. For two distributions $p$ and $q$, the $\ell_1$-distance between $p$ and $q$
is defined to be $\|p - q\|_1 \overset{\text{def}}{=} \sum_x |p(x) - q(x)|$.

In this paper, the lower-cased letters in bold $x, y, \cdots$ are reserved for random variables. The
capital letters in bold, $P, Q, \ldots$ are reserved for random operators.
2.1 Gaussian spaces

For any integer \( n > 0 \), let \( \gamma_n \) represent the standard \( n \)-dimensional normal distribution. All the functions considered in this paper are in \( L^2(\mathbb{C}, \gamma_n) \) unless explicitly mentioned. We say \( f \in L^2(\mathbb{R}, \gamma_n) \) if \( f(x) \in \mathbb{R} \) for all \( x \). We equipped \( L^2(\mathbb{C}, \gamma_n) \) with an inner product \( \langle f, g \rangle_{\gamma_n} \equiv \mathbb{E}_{x \sim \gamma_n} \left[ \overline{f(x)} g(x) \right] \). Given \( p \geq 1 \) and \( f \in L^2(\mathbb{C}, \gamma_n) \), the \( p \)-norm of \( f \) is defined to be \( \|f\|_p \equiv (\int_{\mathbb{R}^n} |f(x)|^p \, \gamma_n(dx))^{1/p} \). The set of Hermite polynomials forms an orthonormal basis in \( L^2(\gamma_1) \) with respect to the inner product \( \langle \cdot, \cdot \rangle_{\gamma_1} \). The Hermite polynomial \( H_r : \mathbb{R} \to \mathbb{R} \) for \( r \in \mathbb{Z}_{\geq 0} \) is defined as

\[
H_0(x) = 1; \quad H_1(x) = x; \quad H_r(x) = \frac{(-1)^r}{\sqrt{r!}} \frac{d^r}{dx^r} e^{-x^2/2}.
\]

For any \( \sigma \in \{\sigma_1, \ldots, \sigma_n\} \in \mathbb{Z}_{\geq 0}^n \), define \( H_\sigma : \mathbb{R}^n \to \mathbb{R} \) as

\[
H_\sigma(x) \equiv \prod_{i=1}^n H_{\sigma_i}(x_i).
\]

And \( |\sigma| \equiv \sum_i \sigma_i \). It is easy to verify that the set \( \{H_\sigma : \sigma \in \mathbb{Z}_{\geq 0}^n\} \) forms an orthonormal basis in \( L^2(\mathbb{C}, \gamma_n) \). Every function \( f \in L^2(\mathbb{C}, \gamma_n) \) has a Hermite expansion as

\[
f(x) = \sum_{\sigma \in \mathbb{Z}_{\geq 0}^n} \hat{f}(\sigma) \cdot H_\sigma(x),
\]

where \( \hat{f}(\sigma) \)'s are the Hermite coefficients of \( f \), which can be obtained by \( \hat{f}(\sigma) = \langle H_\sigma, f \rangle_{\gamma_n} \). The degree of \( f \) is defined to be \( \text{deg}(f) \equiv \max \{ |\sigma| : \hat{f}(\sigma) \neq 0 \} \). Analogous to Fourier analysis, we have Parseval’s identity, that is, \( \|f\|_2^2 = \sum_{\sigma \in \mathbb{Z}_{\geq 0}^n} |\hat{f}(\sigma)|^2 \). We say \( f \in L^2(\mathbb{C}, \gamma_n) \) is multilinear if \( \hat{f}(\sigma) \) is non-zero only if \( \sigma \in \{0,1\}^n \). We need the following basic notions of variance and influence of a coordinate on a function.

**Definition 2.1.** Given a function \( f \in L^2(\mathbb{C}, \gamma_n) \), the variance of \( f \) is defined to be

\[
\text{Var}[f] \equiv \mathbb{E}_{x \sim N(0,1)^n} \left[ |f(x) - \mathbb{E}[f]|^2 \right].
\]

For any set \( S \subseteq [n] \), the conditional variance \( \text{Var}[f(x)|x_S] \) is defined via

\[
\text{Var}[f(x)|x_S] \equiv \mathbb{E}_{x \sim N(0,1)^n} \left[ |f(x) - \mathbb{E}[f(x)|x_S]|^2 \right].
\]

The influence of \( i \)-th coordinate on \( f \), denoted by \( \text{Inf}_i(f) \), is defined by

\[
\text{Inf}_i(f) \equiv \mathbb{E}_{x \sim N(0,1)^n} \left[ \text{Var}[f(x)|x_{-i}] \right].
\]

The total influence of \( f \) is defined by

\[
\text{Inf}(f) = \sum_i \text{Inf}_i(f).
\]

The following fact summarizes the basic properties of variance and influence. Readers may refer to [O’D13] for a thorough treatment.
Fact 2.2. \textbf{O’D13 MOO10} Given $f \in L^2(\mathbb{C}, \gamma_n)$, it holds that
1. $\hat{f}(\sigma) \in \mathbb{R}$ if $f: \mathbb{R}^n \to \mathbb{R}$;
2. $\text{Var}[f] = \sum_{\sigma \neq 0^n} |\hat{f}(\sigma)|^2$;
3. $\text{Inf}_i(f) = \sum_{\sigma_i \neq 0} |\hat{f}(\sigma)|^2$, and hence for all $i$, $\text{Inf}_i(f) \leq \text{Var}[f]$;
4. $\text{Inf}(f) = \sum_{\sigma} \{|i: \sigma_i > 0\}| \text{Inf}_i(f)$.
5. $\text{Inf}(f) \leq \deg(f) \text{Var}[f]$.

Definition 2.3. Given $0 \leq \rho \leq 1$ and $f \in L^2(\mathbb{C}, \gamma_n)$, we define the Ornstein-Uhlenbeck operator $U_\rho$ to be
$$U_\rho f(z) \overset{\text{def}}{=} \mathbb{E}_{x \sim N(0,1)^n} \left[ f\left( \rho z + \sqrt{1 - \rho^2} x \right) \right].$$

Fact 2.4. \textbf{O’D13} Page 338, Proposition 11.37] For any $0 \leq \rho \leq 1$ and $f \in L^2(\mathbb{C}, \gamma_n)$, it holds that
$$U_\rho f = \sum_{\sigma \in \mathbb{Z}_2^n} \hat{f}(\sigma) \rho^{\left| \sigma \right|} H_\sigma.$$

We will also be working on vector-valued functions in this paper. A vector-valued function $f = (f_1, \ldots, f_k): \mathbb{R}^n \to \mathbb{C}^k$ is in $L^2\left( \mathbb{C}^k, \gamma_n \right)$ if $f_i \in L^2(\mathbb{C}, \gamma_n)$ for all $i$. It is in $L^2\left( \mathbb{R}^k, \gamma_n \right)$ if $f_i \in L^2(\mathbb{R}, \gamma_n)$ for all $i$. For any $f, g \in L^2\left( \mathbb{C}^k, \gamma_n \right)$, the inner product $(f,g)_{\gamma_n} \overset{\text{def}}{=} \sum_{i=1}^k \langle f_i, g_i \rangle_{\gamma_n}$. The $p$-norm of $f$ is $\|f\|_p = \left( \sum_{i=1}^k \|f_i\|_p^p \right)^{1/p}$. For any $f \in L^2(\mathbb{C}^k, \gamma_n)$, the Hermite coefficients of $f$ are the vectors $\hat{f}(\sigma) \overset{\text{def}}{=} \langle \hat{f}_1(\sigma), \ldots, \hat{f}_k(\sigma) \rangle$. The degree of $f$ is $\deg(f) \overset{\text{def}}{=} \max_i \deg(f_i)$. We say $f$ is multilinear if each $f_i$ is multilinear. The variance of $f$ is $\text{Var}[f] \overset{\text{def}}{=} \sum_i \text{Var}[f_i]$. The influence of $i$-th coordinate on $f$ is $\text{Inf}_i(f) \overset{\text{def}}{=} \sum_i \text{Inf}_i(f_i)$. The total influence of $f$ is $\text{Inf}(f) = \sum_i \text{Inf}_i(f)$. The action of Ornstein-Uhlenbeck operator on $f$ is defined to be $U_\rho f \overset{\text{def}}{=} (U_\rho f_1, \ldots, U_\rho f_k)$.

For any vector $v \in \mathbb{C}^k$, the norm of $v$ is defined to be $\|v\|_2 \overset{\text{def}}{=} \sqrt{\sum_{i=1}^k |v_i|^2}$. It is easy to verify that Fact 2.2 and Fact 2.4 can be generalized to vector-valued functions by definitions.

Fact 2.5. Given $f \in L^2\left( \mathbb{C}^k, \gamma_n \right)$ and $0 \leq \rho \leq 1$, it holds that
1. $\hat{f}(\sigma) \in \mathbb{R}^k$ if $f: \mathbb{R}^n \to \mathbb{R}^k$;
2. $\text{Var}[f] = \sum_{\sigma \neq 0^n} \|\hat{f}(\sigma)\|_2^2$;
3. $\text{Inf}_i(f) = \sum_{\sigma_i \neq 0} \|\hat{f}(\sigma)\|_2^2$, and hence for all $i$, $\text{Inf}_i(f) \leq \text{Var}[f]$;
4. $\text{Inf}(f) = \sum_{\sigma} \{|i: \sigma_i > 0\}| \text{Inf}_i(f)$;
5. $\text{Inf}(f) \leq \deg(f) \text{Var}[f]$;
6. $U_\rho f = \sum_{\sigma \in \mathbb{Z}_2^n} \hat{f}(\sigma) \rho^{|\sigma|} H_\sigma$.

We are also working on the joint distribution $\rho$-correlated Gaussian distribution $(\mathbb{R} \times \mathbb{R}, \mathcal{G}_\rho)$. This is a 2-dimensional Gaussian distribution $(X,Y)$, where $X$ and $Y$ are marginal distributed according to $\gamma_1$ with $\mathbb{E}[XY] = \rho$. 

7
2.2 Quantum mechanics

We briefly review the formalism of quantum mechanics over a finite dimensional system. For a more thorough treatment, readers may refer to \[NC00, Wat18\]. For a quantum system \(A\), we associate a finite dimensional Hilbert space, which, by abuse of notation, is also denoted by \(A\). We denote by \(\mathcal{M}(A)\) and \(\mathcal{H}(A)\) the set of all linear operators and the set of Hermitian operators in the space, respectively. The identity operator in \(A\) is denoted by \(\mathds{1}_A\). If the dimension of \(A\) is \(d\), then we write \(\mathcal{M}(A) = \mathcal{M}_d\), \(\mathcal{H}(A) = \mathcal{H}_d\) and \(\mathds{1}_A = \mathds{1}_d\). The subscripts may be dropped whenever it is clear from the context. A quantum state in the quantum system \(A\) is represent by a density operator \(\rho_A\), a positive semi-definite operator over the Hilbert space \(A\) with unit trace. We denote by \(\mathcal{D}(A)\) the set of all density operators in \(A\). A quantum state is pure if the density operator is a rank-one projector \(|\psi\rangle\langle\psi|\), which is also represented by \(|\psi\rangle\) for convenience.

Composite quantum systems are associated with the (Kronecker) tensor product space of the underlying spaces, i.e., for quantum systems \(A\) and \(B\), the composition of the two systems are represented by \(A \otimes B\) with the sets of linear operators, Hermitian operators and density operators denoted by \(\mathcal{M}(A \otimes B)\), \(\mathcal{H}(A \otimes B)\) and \(\mathcal{D}(A \otimes B)\), respectively. We sometimes use the shorthand \(AB\) for \(A \otimes B\). The sets of the linear operators and the Hermitian operators in the composition of \(n \times d\)-dimensional Hilbert spaces are denoted by \(\mathcal{M}^{\otimes n}_d\) and \(\mathcal{H}^{\otimes n}_d\), respectively.

With slight abuse of notations, we assume that \(\mathcal{M}^{\otimes n}_d = \mathbb{C}\) and \(\mathcal{H}^{\otimes n}_d = \mathbb{R}\) when \(n = 0\). A quantum channel from the input system \(A\) to the output system \(B\) is represented by a completely positive, trace-preserving linear map (CPTP map). The set of all quantum channels from a system \(A\) to a system \(B\) is denoted \(\mathcal{L}(A,B)\). A quantum operator on \(A\) is a channel acting \(A\). The set of quantum operators on \(A\) is denoted \(\mathcal{L}(A)\). An important operation on a composite system \(A \otimes B\) is the partial trace \(\text{Tr}_B \rho_{AB}\) which effectively derives the marginal state of the subsystem \(A\) from the quantum state \(\rho_{AB}\). The partial trace is given by \(\text{Tr}_B \rho_{AB} \overset{\text{def}}{=} \sum_i (\mathds{1}_A \otimes \langle i|) \rho_{AB} (\mathds{1}_A \otimes |i\rangle)\) where \(|i\rangle\) is an orthonormal basis for \(B\). The partial trace is a valid quantum channel in \(\mathcal{L}(A,B)\). Note that the action is independent of the choices of the basis chosen to represent it, so we unambiguously write \(\rho_A = \text{Tr}_B \rho_{AB}\). In this paper, we may also apply the partial trace to an arbitrary operator in \(\mathcal{M}(A \otimes B)\). A pure state evolution on system \(A\) with state \(|\psi\rangle\) is represented by a unitary operator \(U^A\), denoted by \(U^A|\psi\rangle\). An evolution of register \(B\) of a state \(|\psi\rangle_{AB}\) under the action of a unitary \(U^B\) is represented by \((U^A \otimes U^B)|\psi\rangle_{AB}\). The superscripts and the subscripts might be dropped whenever it is clear from the context. A quantum measurement is represented by a positive-operator valued measure (POVM), which is a set of positive semi-definite operators \(\{M_i\}_{i=1}^n\) satisfying \(\sum_{i=1}^n M_i = \mathds{1}\), where \(n\) is the number of possible measurement outcomes. Suppose the state of the quantum system is \(\rho\). Applying the measurement \(\{M_i\}_{i=1}^n\), the probability that outputs \(i\) is \(\text{Tr} M_i \rho\).

In this paper, we will frequently use the following fact.

**Fact 2.6.** Given registers \(A, B\), operators \(P \in \mathcal{H}(A), Q \in \mathcal{H}(B)\) and a bipartite state \(\psi_{AB}\), it holds that

1. \(\text{Tr}(P \otimes \mathds{1}_B) \psi_{AB} = \text{Tr}_A \psi_{A}\).
2. \(|\text{Tr}(P \otimes Q) \psi_{AB}| \leq (\text{Tr} P^2 \psi_{A})^{1/2} \cdot (\text{Tr} Q^2 \psi_{B})^{1/2}\).

2.3 Matrix spaces

Given an \(m \times n\) matrix \(M\), \(\text{abs}(M)\) represents the matrix obtained by substituting each entry of \(M\) by its absolute value. For \(1 \leq p \leq \infty\) the \(p\)-norm of \(M\) is defined to be \(\|M\|_p \overset{\text{def}}{=} \left(\sum_{i=1}^{\min(m,n)} s_i(M)^p\right)^{1/p}\), where \(s_1(M), s_2(M), \ldots\) are the singular values of \(M\) sorted in
a non-increasing order. \( \|M\| \eqdef \|M\|_\infty = s_1(M) \) when \( p = \infty \) It is easy to verify that \( \|M\|_p \leq \|M\|_q \) if \( p \geq q \). For \( M \in \mathcal{M}_d \), \( |M| \eqdef \sqrt{M^\dagger M} \). The normalized \( p \)-norm of \( M \) is defined as \( \|M\|_p \eqdef \left( \frac{1}{n} \sum_{i=1}^d s_i(M)^p \right)^{1/p} \) and \( \|M\|_2 \eqdef \|M\|_\infty = s_1(M) \). We have \( \|M\|_p \geq \|M\|_q \) if \( p \geq q \). For any \( M \in \mathcal{M}_d \), \( (\lambda_1(M), \ldots, \lambda_d(M)) \) represents the eigenvalues of \( M \) in a non-increasing order.

Given \( P,Q \in \mathcal{M}_d \), we define
\[
\langle P,Q \rangle \eqdef \frac{1}{d} \text{Tr} \, P^\dagger Q.
\]

**Fact 2.7.** \( \langle \cdot, \cdot \rangle \) is an inner product. Then \( \langle \cdot, \cdot \rangle, \mathcal{M}_d \) forms a Hilbert space of dimension \( d^2 \). For any \( M \in \mathcal{M}_d \), \( \|M\|_2^2 = \langle M,M \rangle \).

We say \( \{B_0, \ldots, B_{d^2-1}\} \) is a standard orthonormal basis in \( \mathcal{M}_d \), if it is an orthonormal basis with all elements being Hermitian and \( B_0 = \mathbb{I}_d \).

**Fact 2.8.** Given a standard orthonormal basis \( \{B_0, \ldots, B_{d^2-1}\} \), the set \( \{B'_0, \ldots, B'_{d^2-1}\} \) is also a standard orthonormal basis in \( \mathcal{M}_d \) if and only if \( B'_0 = \mathbb{I}_d \) and there exists a \( (d^2 - 1) \times (d^2 - 1) \) orthogonal matrix \( U \) such that \( B'_i = \sum_{j=1}^{d^2-1} U_{i,j} B_j \) for all \( 1 \leq i \leq d^2 - 1 \).

**Fact 2.9.** Let \( \{B_i\}_{i=0}^{d^2-1} \) be a standard orthonormal basis in \( \mathcal{M}_d \), then
\[
\{B_\sigma \eqdef \otimes_{i=1}^n B_{\sigma_i}\}_{\sigma \in [d^2]_{\geq 0}^n}
\]
is a standard orthonormal basis in \( \mathcal{M}_d^{\otimes n} \).

Given a standard orthonormal basis \( \mathcal{B} = \{B_i\}_{i=0}^{d^2-1} \) in \( \mathcal{M}_d \), every matrix \( M \in \mathcal{M}_d^{\otimes n} \) has a Fourier expansion with respect to the basis \( \mathcal{B} \) given by
\[
M = \sum_{\sigma \in [d^2]_{\geq 0}^n} \widehat{M} (\sigma) \, B_\sigma,
\]
where \( \widehat{M} (\sigma) \)'s are the Fourier coefficients of \( M \) with respect to the basis \( \mathcal{B} \), which can be obtained as \( \widehat{M} (\sigma) = \langle B_\sigma, M \rangle \). The basic properties of \( \widehat{M} (\sigma) \) are summarized in the following fact, which follow from the orthonormality of \( \{B_\sigma\}_{\sigma \in [d^2]_{\geq 0}^n} \).

**Fact 2.10.** Given a standard orthonormal basis \( \{B_i\}_{i=0}^{d^2-1} \) in \( \mathcal{M}_d \) and \( M, N \in \mathcal{M}_d \), it holds that
\begin{enumerate}
\item \( \widehat{M} (\sigma) \) is real if \( \sigma \) is Hermitian;
\item \( \langle M, N \rangle = \langle 1, M^\dagger N \rangle = \langle MN^\dagger, 1 \rangle = \sum_\sigma \widehat{M} (\sigma) \hat{N} (\sigma) \);
\item \( \|M\|_2^2 = \langle M, M \rangle = \langle M^\dagger M, 1 \rangle = \langle 1, M^\dagger M \rangle = \sum_\sigma \left| \widehat{M} (\sigma) \right|^2 \);
\item \( \langle 1, M \rangle = \widehat{M} (0) \).
\end{enumerate}

The variance of a matrix \( M \in \mathcal{M}_d \) is defined to be \( \text{Var}[M] \eqdef \langle M, M \rangle - \langle M, 1 \rangle \langle 1, M \rangle \). The following lemma is easily verified.

**Lemma 2.11.** Given a standard orthonormal basis \( \{B_i\}_{i=0}^{d^2-1} \) in \( \mathcal{M}_d \) and \( M \in \mathcal{M}_d \), it holds that \( \text{Var}[M] = \sum_{\sigma \neq 0} \left| \widehat{M} (\sigma) \right|^2 \).
Definition 2.12. Given an integer \( d > 0 \), we fix an orthonormal basis \( \{|i\rangle\}_{i=0}^{d-1} \) in \( \mathbb{C}^d \) and define the Heisenberg-Weyl operators \( \{X^jZ^k\}_{0 \leq j,k \leq d-1} \), where \( X^j |k\rangle = |(k+j) \mod d\rangle \), \( Z^k |k\rangle = e^{i2\pi\frac{jk}{d}} |k\rangle \).

The following lemma can be easily verified.

**Lemma 2.13.** The set \( \{e^{i2\pi\frac{jk}{d}} X^jZ^k\}_{0 \leq j,k \leq d-1} \) forms a standard orthonormal basis in \( M_d \), where \( X, Z \) are defined in Definition 2.12.

In this paper, we will be working on a particular basis in \( M_2 \), Pauli basis, defined as \( \mathcal{P} = \{\mathbb{1}_2, \sigma_x, \sigma_y, \sigma_z\} \), which is the set of Heisenberg-Weyl operators in \( M_2 \).

**Definition 2.14.** Let \( \mathcal{B} = \{B_i\}_{i=1}^{d^2-1} \) be a standard orthonormal basis in \( M_d \), \( P, Q \in M_d^{\otimes n} \) and a subset \( S \subseteq [n] \).

1. The degree of \( P \) is defined to be \( \deg P \overset{\text{def}}{=} \max \{|\sigma| : \hat{P}(\sigma) \neq 0\} \), where \( |\sigma| \) represents the number of nonzeros in \( \sigma \).
2. For any \( S \subseteq [n] \), \( P_S \overset{\text{def}}{=} \frac{1}{d^n} \text{Tr}_{S^c} P \);
3. For any \( S \subseteq [n] \), \( \langle P, Q \rangle_S \overset{\text{def}}{=} \frac{1}{d^n} \text{Tr}_{S^c} P^\dagger Q \);
4. For any \( S \subseteq [n] \), \( \text{Var}_S[P] = \langle P, P \rangle_S - (P_S, P_S) = (P^\dagger P)_S^c - (P_S^c) \dagger (P_S^c) \).
5. For any \( i \in [n] \), \( \text{Inf}_i(P) \overset{\text{def}}{=} \langle \mathbb{1}, \text{Var}_i[P] \rangle \).
6. \( \text{Inf}(P) = \sum_i \text{Inf}_i(P) \).

With the notion of degree, we define the low degree part and the high degree part of an operator.

**Definition 2.15.** Given integers \( d, t > 0 \), a standard orthonormal basis \( \mathcal{B} = \{B_i\}_{i=0}^{d^2-1} \) in \( M_d \) and \( P \in M_d^{\otimes n} \), we define \( P_{\leq t} \overset{\text{def}}{=} \sum_{|\sigma| \leq t} \hat{P}(\sigma) \mathcal{B}_\sigma \) and \( P_{\geq t} \overset{\text{def}}{=} \sum_{|\sigma| \geq t} \hat{P}(\sigma) \mathcal{B}_\sigma \), and \( P_{= t} \overset{\text{def}}{=} \sum_{|\sigma| = t} \hat{P}(\sigma) \mathcal{B}_\sigma \), where \( \hat{P}(\sigma) \)'s are the Fourier coefficients of \( P \) with respect to the basis \( \mathcal{B} \).

**Lemma 2.16.** The degree of \( P \) is independent of the choices of the basis. Moreover, \( P_{\leq t}, P_{\geq t} \) and \( P_{= t} \) are also independent of the choices of the basis.

**Proof.** Let \( \{\mathcal{B}_\sigma\}_{\sigma \in [d^2]} \) and \( \{\mathcal{B}_\sigma'\}_{\sigma \in [d^2]} \) be two standard orthonormal basis in \( M_d \). From Fact 2.8 there exists a \((d^2 - 1) \times (d^2 - 1)\) orthogonal matrix \( U \) satisfying that \( \mathcal{B}_\sigma = \sum_{\sigma' = 1}^{d^2-1} U_{\sigma, \sigma'} \mathcal{B}_{\sigma'} \) for any \( \sigma \in [d^2 - 1] \). Suppose \( P = P_{= t} \) with respect to the basis \( \{\mathcal{B}_\sigma\}_{\sigma \in [d^2]} \). By linearity, we may assume that \( P = \mathcal{B}_\sigma = \bigotimes_{i=1}^{n} \mathcal{B}_{\sigma_i} \) without loss of generality. It is easy to verify that each term in the expansion of \( P \) in terms of the basis \( \{\mathcal{B}_i\}_{i=0}^{d^2-1} \) is of degree \( |\sigma| \). □
Lemma 2.17. Given $P \in M_n^{\otimes d}$ a standard orthonormal basis $\mathbb{B} = \{ B_i \}_{i=0}^{d-1}$ in $M_n$ and a subset $S \subseteq [n]$, it holds that

1. $P_S = (1, P)_{S^c} = \sum_{\sigma, \sigma' = 0} \hat{P}(\sigma) B_{\sigma' S} S$, $\| P_S \| \leq \| P \|$ and $\| P_S \|_2 \leq \| P \|_2$;
2. $\langle 1_{S^c}, \text{Var}_S(P) \rangle = \sum_{\sigma, \sigma' \neq 0} | \hat{P}(\sigma) |^2$.
3. $\text{Inf}_i(P) = \sum_{\sigma, \sigma' \neq 0} | \hat{P}(\sigma) |^2$.
4. $\text{Inf}(P) = \sum_{\sigma} | \hat{P}(\sigma) |^2 \leq \deg P \cdot \| P \|_2^2$.

Proof. 1. $P_S = \frac{1}{d} \sum_{\sigma} \hat{P}(\sigma) \text{Tr}_{S^c} B_{\sigma} = \text{RHS}$.

For the second inequality, it suffices to show that $\| P_{i} \| \leq \| P \|$ for any $i \in [n]$. Without loss of generality, we may assume $i = 1$. For any unit vector $|v\rangle$

$$\langle v| P_{-1} |v\rangle = \frac{1}{d} \langle v| (\text{Tr}_{1} P) |v\rangle = \text{Tr}P \left( \frac{1}{d} \otimes |v\rangle \langle v| \right) \leq \| P \|,$$

where the last inequality is from the fact that $|\text{Tr}PQ| \leq \| P \| \| Q \|_1$.

For the last inequality,

$$\| P_S \|_2^2 = \sum_{\sigma, \sigma' = 0} | \hat{P}(\sigma) |^2 \leq \sum_{\sigma} | \hat{P}(\sigma) |^2 = \| P \|_2^2,$$

where the equalities are both from Fact 2.10 item 3.

2. From the item 1,

$$\left( P^† P \right)_{S^c} = \sum_{\sigma, \sigma'} \hat{P}(\sigma) \hat{P}(\sigma') (B_{\sigma} B_{\sigma'})_{S^c} = \sum_{\sigma, \sigma': \sigma S = \sigma' S} \hat{P}(\sigma) \hat{P}(\sigma') B_{\sigma S} B_{\sigma' S}.$$

Meanwhile,

$$(P_{S^c})^† (P_{S^c}) = \sum_{\sigma, \sigma': \sigma S = \sigma' S = 0} \hat{P}(\sigma) \hat{P}(\sigma') X_{\sigma S} X_{\sigma' S}.$$

Therefore,

$$\langle 1_{S^c}, \text{Var}_S(P) \rangle = \sum_{\sigma, \sigma' \neq 0} | \hat{P}(\sigma) |^2.$$

3. It follows from the item 2 and the definition of $\text{Inf}_i(\cdot)$.

4. It follows by a direct calculation.

\[ \square \]

Definition 2.18. (Efron-Stein decomposition) Given integers $n, d > 0$, an operator $P \in M_n^{\otimes d}$, a standard orthonormal basis $\{ B_i \}_{i=0}^{d-1}$ and $S \subseteq [n]$, set $P[S] \overset{\text{def}}{=} \sum_{\sigma \in [d]_{\geq 1}, \text{supp}(\sigma) = S} \hat{P}(\sigma) B_{\sigma}$, where $\text{supp}(\sigma) \overset{\text{def}}{=} \{ i \in [n] : \sigma_i > 0 \}$. The Efron-Stein decomposition of $P$ is $P = \sum_{S \subseteq [n]} P[S]$.

Again, the definition of $P[S]$ is independent of the choices of the basis $\{ B_i \}_{i=0}^{d-1}$, followed by the same argument for Lemma 2.16.

The following proposition follows from the orthogonality of $B_i$’s.

Proposition 2.19. Given integers $d, n > 0$, $S \neq T \subseteq [n]$ and $P, Q \in M_n^{\otimes d}$, it holds that $\langle P[S], Q[T] \rangle = 0$. 

11
Proposition 2.20. Given integers \( d, n > 0, P \in \mathcal{M}_d^\otimes n \) and \( S, T \subseteq [n], S \not\subseteq T \), it holds that
\[
\text{Tr}_{T^c} P[S] = 0.
\]

Proof.
\[
\text{Tr}_{T^c} P[S] = \text{Tr}_{T^c} \left( \sum_{\sigma, \text{supp}(\sigma) = S} \hat{P}(\sigma) B_\sigma \right) = 0,
\]
where the second equality is because \( S \cap T^c \neq \emptyset \).

Lemma 2.21. Given \( \psi_{AB} \) with \( \psi_A = \frac{1}{d_A} \) and \( \psi_B = \frac{1}{d_B} \), where \( d_A \) and \( d_B \) are the dimensions of \( A \) and \( B \), respectively, there exist standard orthonormal basis \( \{ X_\alpha \}_{\alpha \in [d_A^2]_{\geq 0}} \) and \( \{ Y_\beta \}_{\beta \in [d_B^2]_{\geq 0}} \) in \( \mathcal{M}(A) \) and \( \mathcal{M}(B) \), respectively, such that
\[
\text{Tr} (X_\alpha \otimes Y_\beta) \psi_{AB} = 0,
\]
whenever \( \alpha \neq \beta \).

Proof. Let \( \{ A_\alpha \}_{\alpha \in [d_A^2]_{\geq 0}} \) and \( \{ B_\beta \}_{\beta \in [d_B^2]_{\geq 0}} \) be arbitrary standard orthonormal basis in \( \mathcal{M}_{d_A} \) and \( \mathcal{M}_{d_B} \), respectively. Define \( (M_{\alpha,\beta})_{\alpha \in [d_A^2]_{\geq 0}, \beta \in [d_B^2]_{\geq 0}} \) where \( M_{\alpha,\beta} = \text{Tr} (A_\alpha \otimes B_\beta) \psi_{AB} \), is a \( d_A^2 \times d_B^2 \) real matrix. Then
\[
M = \begin{pmatrix}
1 & 0 & \cdots & 0 \\
0 & \ddots & & \\
0 & & \ddots & M' \\
0 & & & \ddots
\end{pmatrix}
\]
(9)

Let \( M' = U^\dagger DV^\dagger \) be a singular eigenvalue decomposition of \( M' \) where \( U, V \) are both orthogonal matrices and \( D \) is a diagonal matrix. For any \( \alpha \in [d_A^2]_{\geq 0} \) and \( \beta \in [d_B^2]_{\geq 0} \) set
\[
X_\alpha \overset{\text{def}}{=} \begin{cases}
\sum_{\alpha' = 1}^{d_A^2} U_{\alpha,\alpha'} A_{\alpha'} & \text{if } \alpha \neq 0 \\
0 & \text{otherwise},
\end{cases}
\]
and
\[
Y_\beta \overset{\text{def}}{=} \begin{cases}
\sum_{\beta' = 1}^{d_B^2} V_{\beta,\beta'} B_{\beta'} & \text{if } \beta \neq 0 \\
0 & \text{otherwise}.
\end{cases}
\]

From Fact 2.8, \( \{ X_\alpha \}_{\alpha = 0}^{d_A^2 - 1} \) and \( \{ Y_\beta \}_{\beta = 0}^{d_B^2 - 1} \) are standard orthonormal basis in \( \mathcal{M}(A) \) and \( \mathcal{M}(B) \), respectively. Then
\[
\text{Tr} (X_\alpha \otimes Y_\beta) \psi_{AB} = \begin{cases}
(UMV)_{\alpha,\beta} = \delta_{\alpha,\beta} M'_{\alpha,\alpha} & \text{if } \alpha, \beta > 0 \\
\delta_{(0,0),(\alpha,\beta)} & \text{otherwise}.
\end{cases}
\]

2.4 Random operators

From the previous subsections, we see that the matrix spaces \( \mathcal{M}_d^\otimes n \) and Gaussian space \( L^2(\mathbb{C}, \gamma_n) \) are both Hilbert spaces. In this subsection, we unify both spaces by random operators. In this paper, we only concern the case that the dimension \( d = 2 \). However, the results in this subsection can be extended to arbitrary dimension \( d \) directly.
Given integers $h, n > 0$, we say $P$ is a random operator if it is expressed as

$$P = \sum_{\sigma \in [4]^h_{\geq 0}} p_\sigma (g) B_\sigma, \quad (10)$$

where $\{B_i\}_{i=0}^3$ is a standard orthonormal basis in $M_2$, $p_\sigma \in L^2 (\mathbb{C}, \gamma_n)$ for all $\sigma \in [4]^h_{\geq 0}$ and $g \sim N (0, 1)^n$. $P \in L^2 \left( M_2^\otimes h, \gamma_n \right)$ if $p_\sigma \in L^2 (\mathbb{C}, \gamma_n)$ for all $\sigma \in [4]^h_{\geq 0}$. Moreover, $P \in L^2 \left( M_2^\otimes n, \gamma_n \right)$ if $p_\sigma \in L^2 (\mathbb{R}, \gamma_n)$. Define a vector-valued function $p \overset{\text{def}}{=} (p_\sigma)_{\sigma \in [4]^h_{\geq 0}} : \mathbb{R}^n \to \mathbb{C}^4^h$. We say $p$ is the associated vector-valued function of $P$ under the basis $\{B_i\}_{i=0}^3$.

The following is a generalization of $p$-norm in $L^2 \left( M_2^\otimes h, \gamma_n \right)$.

**Definition 2.23.** Given integers $n, h \geq 0$ and $P \in L^2 \left( M_2^\otimes h, \gamma_n \right)$, for $p \geq 1$, the normalized $p$-norm of $P$ is $N_p (P) \overset{\text{def}}{=} \mathbb{E} \left[ \|P\|_p^p \right]^{\frac{1}{p}}$. The degree of $P$, denoted by $\text{deg} (P)$, is $\max_{\sigma \in [4]^h_{\geq 0}} \text{deg} (p_\sigma)$. We say $P$ is multilinear if $p_\sigma (\cdot) = p_\sigma (\cdot \circ \cdot)$ is multilinear for all $\sigma \in [4]^h_{\geq 0}$.

**Lemma 2.24.** Given integers $n, h \geq 0$ let $P \in L^2 \left( M_2^\otimes n, \gamma_h \right)$ with the associated vector-valued function $p$. It holds that $N_2 (P) = \|p\|_2$.

**Proof.** Consider

$$N_2 (P)^2 = \mathbb{E} \left[ \|P\|_2^2 \right] = \mathbb{E}_{g \sim N (0, 1)^n} \left[ \sum_{\sigma \in [4]^h_{\geq 0}} |p_\sigma (g)|^2 \right] = \|p\|_2^2,$$

where the second equality is from Fact [2.10] item 3. 

**Lemma 2.25.** Given a multilinear random operator $P \in L^2 \left( M_2^\otimes h, \gamma_n \right)$ with degree $d$ and the associated vector-valued function $p$ under a standard orthonormal basis, it holds that

$$\text{Inf} (p) \leq \text{deg} (P) N_2 (P)^2.$$

**Proof.** Consider

$$\text{Inf} (p) = \sum_{i=1}^n \text{Inf}_i (p) \leq \text{deg} (p) \text{Var} [p] \leq \text{deg} (P) \|p\|_2^2 = \text{deg} (P) N_2 (P)^2,$$

where the first inequality is from Fact [2.5] item 5; the second equality is from Lemma [2.24].

We say a pair of random operators $(P, Q) \in L^2 \left( M_2^\otimes h, \gamma_n \right) \times L^2 \left( M_2^\otimes h, \gamma_n \right)$ are joint random operators if the random variables $(g, h)$ in $(P, Q)$ are drawn from the joint distribution $\mathcal{S}_\rho^\otimes n$ for $0 \leq \rho \leq 1$.

---

1. To clarify the potential ambiguity, we consider $\|P\|_p$ to be a random variable and use $N_p (\cdot)$ to represent the normalized $p$-norm of a random operator.
2.5 Miscellaneous

Throughout this paper, any function \( f : \mathbb{R} \to \mathbb{R} \) is also viewed as a map \( f : \mathcal{H}_d \to \mathcal{H}_d \) defined as \( f(P) = \sum_i f(\lambda_i) |v_i\rangle\langle v_i| \), where \( P = \sum_i \lambda_i |v_i\rangle\langle v_i| \) is a spectral decomposition of \( P \).

Given a convex body \( \Delta \subseteq \mathbb{R}^k \), we say a map \( \mathcal{R} : \mathbb{R}^k \to \mathbb{R}^k \) is the rounding map of \( \Delta \) if for any \( x \in \mathbb{R}^k \), \( \mathcal{R}(x) \) is the element in \( \Delta \) that is closest to \( x \) in \( \| \cdot \|_2 \) distance. The following well-known fact states that the Lipschitz coefficient of a rounding map is at most 1.

**Fact 2.26.** Let \( \Delta \) be a convex set in \( \mathbb{R}^k \) with the rounding map \( \mathcal{R} \). It holds that

\[
\| \mathcal{R}(x) - \mathcal{R}(y) \|_2 \leq \| x - y \|_2,
\]

for any \( x, y \in \mathbb{R}^k \).

Thus, if \( \Delta \) contains the element \((0, \ldots, 0)\), then \( \mathcal{R} \) is a contraction map. Namely, \( \| \mathcal{R}(x) \|_2 \leq \| x \|_2 \) for any \( x \in \mathbb{R}^k \).

3 Main results

**Theorem 3.1.** Given \( 0 \leq \rho < 1 \), \( \epsilon \in (0, 1) \), integers \( n, s > 0 \), a bipartite state \( \psi_{AB} \) with \( \psi_A = \psi_B = \frac{1}{\sqrt{2}} \) and the maximal correlation \( \rho = \rho(\psi_{AB}) \) defined in Definition 5.6, sequences of possibly repetitive Hermitian operators \( P_1, \ldots, P_n, Q_1, \ldots, Q_s \in \mathcal{H}_2 \) satisfying \( 0 \leq P_u, Q_u \leq 1 \) for any \( 1 \leq u \leq s \), there exists an explicitly computable \( D = D(\rho, \epsilon, s) \) and maps \( f, g : \mathcal{H}_2^\otimes n \to \mathcal{H}_2^\otimes D \) with \( \mathcal{P}_u = f(P_u) \) and \( \mathcal{Q}_u = g(Q_u) \) for \( 1 \leq u \leq s \), such that the following holds.

1. \( 0 \leq \mathcal{P}_u \leq 1 \) and \( 0 \leq \mathcal{Q}_u \leq 1 \).
2. \( \left| \frac{1}{2^n} \text{Tr} P_u - \frac{1}{2^n} \text{Tr} \mathcal{P}_u \right| \leq \epsilon \) and \( \left| \frac{1}{2^n} \text{Tr} Q_u - \frac{1}{2^n} \text{Tr} \mathcal{Q}_u \right| \leq \epsilon \).
3. \( \left| \text{Tr} (P_u \otimes Q_u) \psi_{AB}^\otimes n - \text{Tr} (\mathcal{P}_u \otimes \mathcal{Q}_u)\psi_{AB}^\otimes D \right| \leq \epsilon \).

In particular, one may choose \( D = \text{exp} \left( \text{poly} \left( s, \text{exp} \left( \text{poly} \left( \frac{1}{\epsilon}, \frac{1}{1-\rho} \right) \right) \right) \right) \).

**Proof.** Let \( \delta, \tau \) be parameters which are chosen later. The proof is composed of several steps.

- **Smoothing operators.** For \( u \in [s] \), we apply Lemma 6.1 to operators \( P_u \) and \( Q_u \) to get \( P_u^{(1)} \) and \( Q_u^{(1)} \) and \( d_1 = \frac{2\log^2(1/\delta)}{C(1-\rho)\delta^2} \) for some constant \( C \) satisfying that

1. \( 0 \leq P_u^{(1)} \leq 1 \) and \( 0 \leq Q_u^{(1)} \leq 1 \);
2. \( \text{Tr} P_u^{(1)} = \text{Tr} P_u \) and \( \text{Tr} Q_u^{(1)} = \text{Tr} Q_u \);
3. \( \| P_u^{(1)} \|_2 \leq \| P_u \|_2 \) and \( \| Q_u^{(1)} \|_2 \leq \| Q_u \|_2 \);
4. \( \left| \text{Tr} \left( P_u^{(1)} \otimes Q_u^{(1)} \right) \psi_{AB}^\otimes n - \text{Tr} \left( P_u \otimes Q_u \right) \psi_{AB}^\otimes n \right| \leq \delta \);
5. \( \| \left( P_u^{(1)} \right)^{>d_1} \|_2 \leq \delta \) and \( \| \left( Q_u^{(1)} \right)^{>d_1} \|_2 \leq \delta \).

6. If \( P_u = P_v \), then \( P_u^{(1)} = P_v^{(1)} \). If \( Q_u = Q_v \), then \( Q_u^{(1)} = Q_v^{(1)} \).
• **Regularization.** For any $u \in [s]$, applying Lemma 7.4 to $P_u^{(1)}$ and $Q_u^{(1)}$ with $\delta \leftarrow \delta, \epsilon \leftarrow \tau, d \leftarrow d_1$, we obtain a set $H_u \subseteq [n]$ of size $h_u = |H| \leq \frac{2d_1}{\tau}$ such that

\[
(\forall i \notin H_u) \quad \text{Inf}_i \left( (P_u^{(1)})_{\leq d_1} \right) \leq \tau, \quad \text{and} \quad \text{Inf}_i \left( (Q_u^{(1)})_{\leq d_1} \right) \leq \tau.
\]

Set $H = \bigcup_u H_u$. Then $h = |H| \leq \frac{2d_1}{\tau^2}$. It holds that for any $u \in [s],$

1. \[
(\forall i \notin H) \quad \text{Inf}_i \left( (P_u^{(1)})_{\leq d_1} \right) \leq \tau, \quad \text{and} \quad \text{Inf}_i \left( (Q_u^{(1)})_{\leq d_1} \right) \leq \tau;
\]

2. \[
\text{Tr} \left( P_u^{(1)} \otimes Q_u^{(1)} \right) \psi_{AB}^{\otimes n} = \sum_{\sigma \in \{4\}^{|H|}_\sigma} c_{\sigma} \text{Tr} \left( P_{u,\sigma}^{(1)} \otimes Q_{u,\sigma}^{(1)} \right) \psi_{AB}^{\otimes (n-h)},
\]

where $(c_i)^3_{i=0}$ are the singular values of the matrix $\text{Corr}(\psi_{AB})$ defined in Definition 7.1 and $c_\sigma = c_{\sigma_1} \cdot c_{\sigma_2} \cdots c_{\sigma_h}$ and

\[
P_{\sigma}^{(1)} = \sum_{\tau \in \{4\}^{|H|}_\tau; \tau_H = \sigma} \hat{P}(\sigma) A_\sigma \quad \text{and} \quad Q_{\sigma}^{(1)} = \sum_{\tau \in \{4\}^{|H|}_\tau; \tau_H = \sigma} \hat{Q}(\sigma) B_\sigma,
\]

for standard orthonormal basis $\{A_i\}_{i=0}^3$ and $\{B_i\}_{i=0}^3$.

• **Invariance from $\mathcal{F}_2^\otimes n$ to $L^2\left( \mathcal{M}_2^{\otimes h}, \gamma_3(n-h) \right).** For any $u \in [s]$, applying Lemma 10.8 to $P_u^{(1)}$ and $Q_u^{(1)}$ and $H$, we obtain degree-$d_1$ multilinear joint random operators $\left( P_u^{(2)}, Q_u^{(2)} \right) \in L^2(\mathcal{M}_2^{\otimes h}, \gamma_3(n-h)) \times L^2(\mathcal{M}_2^{\otimes h}, \gamma_3(n-h))$ with joint random variables $(g_u, h_u)^{3(n-h)} \sim G_p^{3(n-h)}$ such that,

1. \[
2^{n-h} \mathbb{E} \left[ \text{Tr} \ P_u^{(2)} \right] = \text{Tr} \ P_u^{(1)} \quad \text{and} \quad 2^{n-h} \mathbb{E} \left[ \text{Tr} \ Q_u^{(2)} \right] = \text{Tr} \ Q_u^{(1)};
\]

2. \[
N_2 \left( P_u^{(2)} \right) \leq \| P^{(1)} \|_2 \quad \text{and} \quad N_2 \left( Q_u^{(2)} \right) \leq \| Q^{(1)} \|_2;
\]

3. \[
\mathbb{E} \left[ \text{Tr} \ \zeta \left( P_u^{(2)} \right) \right] \leq O \left( 2^h \left( (3^d \sqrt{d_1})^{2/3} + \sqrt{\delta} \right) \right)
\]

and

\[
\mathbb{E} \left[ \text{Tr} \ \zeta \left( Q_u^{(2)} \right) \right] \leq O \left( 2^h \left( (3^d \sqrt{d_1})^{2/3} + \sqrt{\delta} \right) \right),
\]

where $\zeta(\cdot)$ is defined in Eq. [19].

4. \[
\text{Tr} \left( P_u^{(1)} \otimes Q_u^{(1)} \right) \psi_{AB}^{\otimes n} = \mathbb{E} \left[ \text{Tr} \left( P_u^{(2)} \otimes Q_u^{(2)} \right) \psi_{AB}^{\otimes n} \right].
\]

5. If $P_u^{(1)} = P_u^{(1)}$, then $P_u^{(2)} = P_u^{(2)}$. If $Q_u^{(1)} = Q_u^{(1)}$, then $Q_u^{(2)} = Q_u^{(2)}$.

• **Dimension reduction.** For any $u \in [s]$, applying Lemma 11.1 to $\left( P_u^{(2)}, Q_u^{(2)} \right)$ with $\delta \leftarrow \delta/2s, d \leftarrow d_1, n \leftarrow 3(n-h), \alpha \leftarrow 1/4s$, and the union bound on the $u$’s, we obtain joint random operators $\left( P_u^{(3)}, Q_u^{(3)} \right) \in L^2(\mathcal{F}_2^{\otimes h}, \gamma_{mn}) \times L^2(\mathcal{F}_2^{\otimes h}, \gamma_{mn})$ with the random variables drawn from $G_p^{\otimes mn}$ such that for all $u \in [s]$ the following holds.

1. \[
\left| \mathbb{E} \left[ \text{Tr} \ P_u^{(3)} \right] - \mathbb{E} \left[ \text{Tr} \ P_u^{(2)} \right] \right| \leq \delta 2^h N_2 \left( P_u^{(2)} \right)
\]

and

\[
\left| \mathbb{E} \left[ \text{Tr} \ Q_u^{(3)} \right] - \mathbb{E} \left[ \text{Tr} \ Q_u^{(2)} \right] \right| \leq \delta 2^h N_2 \left( Q_u^{(2)} \right).
\]
2. \[ N_2 \left( \mathbf{P}_u^{(3)} \right) \leq (1 + \delta) N_2 \left( \mathbf{P}_u^{(2)} \right) \] and \[ N_2 \left( \mathbf{Q}_u^{(3)} \right) \leq (1 + \delta) N_2 \left( \mathbf{Q}_u^{(2)} \right). \]

3. \[ \mathbb{E} \left[ \text{Tr} \, \zeta \left( \mathbf{P}_u^{(3)} \right) \right] \leq 2 \sqrt{s} \mathbb{E} \left[ \text{Tr} \, \zeta \left( \mathbf{P}_u^{(2)} \right) \right] \] and \[ \mathbb{E} \left[ \text{Tr} \, \zeta \left( \mathbf{Q}_u^{(3)} \right) \right] \leq 2 \sqrt{s} \mathbb{E} \left[ \text{Tr} \, \zeta \left( \mathbf{Q}_u^{(2)} \right) \right]. \]

4. \[ \left| \mathbb{E} \left[ \text{Tr} \left( \mathbf{P}_u^{(3)} \otimes \mathbf{Q}_u^{(3)} \right) \psi_{AB}^{\otimes h} \right] - \mathbb{E} \left[ \text{Tr} \left( \mathbf{P}_u^{(2)} \otimes \mathbf{Q}_u^{(2)} \right) \psi_{AB}^{\otimes h} \right] \right| \leq \delta N_2 \left( \mathbf{P}_u^{(2)} \right) N_2 \left( \mathbf{Q}_u^{(2)} \right). \]

5. If \( \mathbf{P}_u^{(2)} = \mathbf{P}_v^{(2)} \), then \( \mathbf{P}_u^{(3)} = \mathbf{P}_v^{(3)} \). If \( \mathbf{Q}_u^{(2)} = \mathbf{Q}_v^{(2)} \), then \( \mathbf{Q}_u^{(3)} = \mathbf{Q}_v^{(3)} \).

Here \( n_0 = \frac{4^{h+1} \delta^{(d+1)}}{d^2} \).

- **Smoothing random operators.** For any \( u \in [s] \), Applying Lemma 12.1 to \( \left( \mathbf{P}_u^{(3)}, \mathbf{Q}_u^{(3)} \right) \) with \( h \leftarrow h, n \leftarrow n_0 \) we obtain joint random operators \( \left( \mathbf{P}_u^{(4)}, \mathbf{Q}_u^{(4)} \right) \in L^2 \left( \mathcal{H}_2^{\otimes h}, \gamma_{n_0} \right) \times L^2 \left( \mathcal{H}_2^{\otimes h}, \gamma_{n_0} \right) \) such that

1. \[ \text{deg} \left( \mathbf{P}_u^{(4)} \right) \leq d_2 \text{ and } \text{deg} \left( \mathbf{Q}_u^{(4)} \right) \leq d_2. \]

2. \[ \mathbb{E} \left[ \text{Tr} \left( \mathbf{P}_u^{(4)} \right) \right] = \mathbb{E} \left[ \text{Tr} \left( \mathbf{P}_u^{(3)} \right) \right] \text{ and } \mathbb{E} \left[ \text{Tr} \left( \mathbf{Q}_u^{(4)} \right) \right] = \mathbb{E} \left[ \text{Tr} \left( \mathbf{Q}_u^{(3)} \right) \right]. \]

3. \[ N_2 \left( \mathbf{P}_u^{(4)} \right) \leq N_2 \left( \mathbf{P}_u^{(3)} \right) \text{ and } N_2 \left( \mathbf{Q}_u^{(4)} \right) \leq N_2 \left( \mathbf{Q}_u^{(3)} \right). \]

4. \[ \mathbb{E} \left[ \text{Tr} \, \zeta \left( \mathbf{P}_u^{(4)} \right) \right] \leq 2 \left( \mathbb{E} \left[ \text{Tr} \, \zeta \left( \mathbf{P}_u^{(3)} \right) \right] + \delta^2 N_2 \left( \mathbf{P}_u^{(3)} \right)^2 \right) \]

and \[ \mathbb{E} \left[ \text{Tr} \, \zeta \left( \mathbf{Q}_u^{(4)} \right) \right] \leq 2 \left( \mathbb{E} \left[ \text{Tr} \, \zeta \left( \mathbf{Q}_u^{(3)} \right) \right] + \delta^2 N_2 \left( \mathbf{Q}_u^{(3)} \right)^2 \right). \]

5. \[ \left| \mathbb{E} \left[ \text{Tr} \left( \mathbf{P}_u^{(3)} \otimes \mathbf{Q}_u^{(3)} \right) \psi_{AB}^{\otimes h} \right] - \mathbb{E} \left[ \text{Tr} \left( \mathbf{P}_u^{(4)} \otimes \mathbf{Q}_u^{(4)} \right) \psi_{AB}^{\otimes h} \right] \right| \leq \delta N_2 \left( \mathbf{P}_u^{(3)} \right) N_2 \left( \mathbf{Q}_u^{(3)} \right). \]

6. If \( \mathbf{P}_u^{(3)} = \mathbf{P}_v^{(3)} \), then \( \mathbf{P}_u^{(4)} = \mathbf{P}_v^{(4)} \). If \( \mathbf{Q}_u^{(3)} = \mathbf{Q}_v^{(3)} \), then \( \mathbf{Q}_u^{(4)} = \mathbf{Q}_v^{(4)} \).

Here \( d_2 = O \left( \frac{\log^2 \frac{1}{\delta} \log \left( \frac{1}{1-\rho} \right) }{\delta (1-\rho)} \right) \).

- **Multilinearization.** For any \( u \in [s] \), suppose

\[
\left( \mathbf{P}_u^{(4)}, \mathbf{Q}_u^{(4)} \right) = \left( \sum_{\sigma \in [4]_0^h} p_{u,\sigma}^{(4)}(g) \mathcal{A}_\sigma, \sum_{\sigma \in [4]_0^h} q_{u,\sigma}^{(4)}(h) \mathcal{B}_\sigma \right)_{(g,h) \sim \mathcal{G}_{\rho^{n_0}}^{t h}}.
\]

Applying Lemma 13.1 with \( d \leftarrow d_2, h \leftarrow h, n \leftarrow n_0, \delta \leftarrow \tau \), we obtain multilinear joint random operators

\[
\left( \mathbf{P}_u^{(5)}, \mathbf{Q}_u^{(5)} \right) = \left( \sum_{\sigma \in [4]_0^h} p_{u,\sigma}^{(5)}(x) \mathcal{A}_\sigma, \sum_{\sigma \in [4]_0^h} q_{u,\sigma}^{(5)}(y) \mathcal{B}_\sigma \right)_{(x,y) \sim \mathcal{G}_{\rho^{n_0} t}}.
\]

with \( t = O \left( \frac{d_2}{\tau^2} \right) \) such that the following holds.
1. \( \deg(P_u^{(5)}) \leq d_2 \) and \( \deg(Q_u^{(5)}) \leq d_2 \).
2. For all \((i, j) \in [n_0] \times [t]\)

\[
\Inf_{in_0+j}(P_{u, \sigma}) \leq \tau \Inf_i(P_{u, \sigma})
\]

and

\[
\Inf_{in_0+j}(Q_{u, \sigma}) \leq \tau \Inf_i(Q_{u, \sigma}).
\]

3. 

\[
E[\Tr P_u^{(5)}] = E[\Tr P_u^{(4)}] \quad \text{and} \quad E[\Tr Q_u^{(5)}] = E[\Tr Q_u^{(4)}].
\]

4. 

\[
N_2(P_u^{(5)}) \leq N_2(P_u^{(4)}) \quad \text{and} \quad N_2(Q_u^{(5)}) \leq N_2(Q_u^{(4)}).
\]

5. 

\[
|E[\Tr \zeta(P_u^{(5)})] - E[\Tr \zeta(P_u^{(4)})]| \leq \tau 2^{h+2} N_2(P_u^{(4)})^2
\]

and

\[
|E[\Tr \zeta(Q_u^{(5)})] - E[\Tr \zeta(Q_u^{(4)})]| \leq \tau 2^{h+2} N_2(Q_u^{(4)})^2.
\]

6. 

\[
|E[\Tr (P_u^{(5)} \otimes Q_u^{(5)}) \psi_{AB}^{(h)}] - E[\Tr (P_u^{(4)} \otimes Q_u^{(4)}) \psi_{AB}^{(h)}]| \leq \tau N_2(P_u^{(4)}) N_2(Q_u^{(4)}).
\]

7. If \( P_u^{(4)} = P_u^{(5)} \), then \( P_u^{(5)} = P_u^{(5)} \). If \( Q_u^{(4)} = Q_u^{(5)} \), then \( Q_u^{(5)} = Q_u^{(5)} \).

- **Invariance from** \( L^2(\mathcal{H}_2^{\otimes h}, \gamma_{nt}) \) **to** \( \mathcal{H}_2^{\otimes h+nt} \). From the item 1 and item 2 above and Lemma \[2.17\] item 4 and Lemma \[2.24\] we have

\[
\sum_{\sigma} \Inf_i(P_{u, \sigma}) \leq \tau d_2 N_2(P_u^{(4)})^2.
\]

Similarly, we have

\[
\sum_{\sigma} \Inf_i(Q_{u, \sigma}) \leq \tau d_2 N_2(Q_u^{(4)})^2.
\]

For any \( u \in [s] \), we apply Lemma \[10.9\] to \((P_u^{(5)}, Q_u^{(5)})\) with \( n \leftarrow n_0 t, h \leftarrow h, d \leftarrow d_2, \)

\[
\tau \leftarrow \tau_0 \overset{\text{def}}{=} \max_u \left\{ \max \left\{ \tau d_2 N_2(P_u^{(4)})^2, \tau d_2 N_2(Q_u^{(4)})^2 \right\} : u \in [s] \right\}
\]

to get \((P_u^{(6)}, Q_u^{(6)}) \in \mathcal{H}_2^{\otimes h+nt} \times \mathcal{H}_2^{\otimes h+nt}\) satisfying that

1. 

\[
2^{nt} E[\Tr P_u^{(5)}] = \Tr P_u^{(6)} \quad \text{and} \quad 2^{nt} E[\Tr Q_u^{(5)}] = \Tr Q_u^{(6)};
\]

2. 

\[
N_2(P_u^{(6)}) = \|P_u^{(6)}\|_2 \quad \text{and} \quad N_2(Q_u^{(6)}) = \|Q_u^{(6)}\|_2;
\]

3. 

\[
|E[2^{nt} \Tr \zeta(P_u^{(6)})] - \Tr \zeta(P_u^{(6)})| \leq O \left(2^{nt+h} \left(3d_2d_2\sqrt{\tau_0}\right)^{2/3}\right),
\]

and

\[
|E[2^{nt} \Tr \zeta(Q_u^{(6)})] - \Tr \zeta(Q_u^{(6)})| \leq O \left(2^{nt+h} \left(3d_2d_2\sqrt{\tau_0}\right)^{2/3}\right).
\]

4. 

\[
\Tr \left(P_u^{(6)} \otimes Q_u^{(6)}\right) \psi_{AB}^{\otimes h+nt} = E\left[\Tr \left(P_u^{(5)} \otimes Q_u^{(5)}\right) \psi_{AB}^{(h)}\right].
\]
Theorem 3.2. Given parameters $0 < \epsilon, \rho < 1$ a mono-state binary game $(G, \psi_{AB})$ with question sets $X, Y,$ where $\psi_{AB}$ is a noisy EPR state, i.e., $\psi_A = \psi_B = \frac{1}{2}$ and the maximal correlation $\rho = \rho(\psi_{AB}) < 1$ as defined in Definition 5.4, there exists an explicitly computable bound $D = D([X], [Y], \epsilon, \rho)$ such that it suffices for the players to share $D$ copies of $\psi_{AB}$ to achieve the probability of winning the game at least $\omega(G, \psi_{AB}) - \epsilon$. In particular, one may choose $D = \exp \left( \operatorname{poly} \left( \left| X \right|, \left| Y \right|, \operatorname{exp} \left( \frac{1}{1 - \frac{1}{\epsilon^4}} \right) \right) \right)$.

Proof. Suppose the players share $n$ copies of $\psi_{AB}$ and employ the strategies

$$\left( \{ P_a^x \}_{x \in \mathbb{X}, a \in \{ 0, 1 \}}, \{ Q_a^y \}_{y \in \mathbb{Y}, b \in \{ 0, 1 \}} \right)$$

with the winning probability $\omega$. Without loss of generality, we assume that $\mathbb{X} = \{ 1, 2, \ldots, |X| \}$ and $\mathbb{Y} = \{ 1, 2, \ldots, |Y| \}$.

Apply Theorem 3.1 to the following two sequences of measurement operators

$$\left( P_0^1, \ldots, P_0^1, P_0^2, \ldots, P_0^2, \ldots, P_0^{|X|}, \ldots, P_0^{|X|} \right)$$

$|X|$ times

$|Y|$ times

$|X|$ times

$|Y|$ times
and
\[ (Q_0^1, \ldots, Q_0^{|y|}, Q_0^1, \ldots, Q_0^{|y|}, Q_0^1, \ldots, Q_0^{|y|}) , \]
with parameter \( \epsilon \leftarrow \epsilon / 8, s \leftarrow |X| \cdot |Y| \). Let \( f \) and \( g \) be the maps induced by Theorem 3.1. Set \( \widetilde{P}_0^x \overset{\text{def}}{=} f (P_0^x) \) and \( Q_0^y \overset{\text{def}}{=} g (Q_0^y) \) for \( x \in X \) and \( y \in Y \). We claim that the strategy
\[ \left\{ \{ \widetilde{P}_0^x, P_1 \overset{\text{def}}{=} 1_D - \widetilde{P}_0^x \} \right\}_{x \in X} \cdot \left\{ Q_0^y, Q_1 \overset{\text{def}}{=} 1_D - Q_0^y \right\}_{y \in Y} \]
wins the game with probability \( \tilde{\omega} \geq \omega - \epsilon \). Theorem 3.1 item 1 guarantees that the operators above are valid measurements. Let \( \nu_{xy} (a, b) \overset{\text{def}}{=} \text{Tr} \left( P_a^x \otimes Q_b^y \right) \psi_{AB} \otimes D \) and \( \tilde{\nu}_{xy} (a, b) \overset{\text{def}}{=} \text{Tr} \left( \tilde{P}_a^x \otimes \tilde{Q}_b^y \right) \psi_{AB} \otimes D \). From Theorem 3.1 for any \( x \in X \) and \( y \in Y \),
\[
|\nu_{xy} (0, 0) + \nu_{xy} (0, 1) - \tilde{\nu}_{xy} (0, 0) - \tilde{\nu}_{xy} (0, 1)| \leq \epsilon / 8; \\
|\nu_{xy} (0, 0) + \nu_{xy} (1, 0) - \tilde{\nu}_{xy} (0, 0) - \tilde{\nu}_{xy} (1, 0)| \leq \epsilon / 8; \\
|\nu_{xy} (0, 0) - \tilde{\nu}_{xy} (0, 0)| \leq \epsilon / 8.
\]
where the first and the second inequalities are implied by the item 2 in Theorem 3.1. The last inequality is from the item 3 in Theorem 3.1. The three inequalities above together imply that \(|\nu_{xy} (a, b) - \tilde{\nu}_{xy} (a, b)| \leq \epsilon / 4\) for any \( a, b \in \{0, 1\} \). Thus
\[
|\omega - \tilde{\omega}| = \left| \sum_{xy} \mu (x, y) (\nu_{xy} (a, b) - \tilde{\nu}_{xy} (a, b)) V(x, y, a, b) \right| \\
\leq \sum_{xy} \mu (x, y) \sum_{a,b} |\nu_{xy} (a, b) - \tilde{\nu}_{xy} (a, b)| \\
\leq \epsilon.
\]
\[ \square \]

4 Open questions

In this work, we prove the decidability of mono-state binary games \((G, \psi)\) for any noisy EPR state \(\psi\), by reducing the problem to the decidability of the quantum non-interactive simulation of joint distributions. Several interesting open questions are followed by this work.

An immediate open question is the decidability of general mono-state games. To remove the restrictions in the main result, it seems that several new ideas are required. For instance, if the shared state \(\psi\) has maximal correlation 1, we cannot use the the framework of the non-interactive simulation, because such a state possibly can generate any distribution without communication, such as EPR states. For the case that \(\psi\) is a high dimensional state, we need a hypercontractive inequality of qudit quantum channels. For non-binary games, we need to work on several-matrix-variable functions possibly with much more involved calculation.

There are many other "tensored" quantities in quantum information theory and quantum complexity theory not known to be computable, such as the regularisations of the various entanglement measures [PV07], quantum information complexity [Tou15], etc.

5 Markov super-operators, noise operators and maximal correlation

Given \(\psi \in \mathcal{D}_d\), \(\psi > 0\), \(P, Q \in \mathcal{M}_d\), we define
\[
\langle P, Q \rangle_\psi \overset{\text{def}}{=} \frac{1}{2} \text{Tr} \left( P^\dagger Q + QP^\dagger \right) \psi,
\]

19
for any $P, Q \in \mathcal{M}_d$.

**Fact 5.1.** $\langle \cdot, \cdot \rangle_\psi$ is an inner product and $\| \cdot \|_\psi \overset{\text{def}}{=} \sqrt{\langle P, P \rangle_\psi}$ is a norm whenever $\psi$ is a positive definite density operator.

The inner product defined in Eq. (8) can be viewed as the inner product above with $\psi = \frac{1}{d}$, where $d$ is the dimension. For any integer $n > 0$, $\psi^\otimes n$ induces an inner product in $\mathcal{M}_d^\otimes n$. To keep the notations short, the inner product is represented as

$$\langle P, Q \rangle_\psi = \frac{1}{2} \text{Tr} \left( P^\dagger Q + QP^\dagger \right) \psi^\otimes n,$$

for $P, Q \in \mathcal{M}_d^\otimes n$.

The following fact can be easily verified.

For any quantum state $\psi > 0$ in $\mathcal{M}_d$, we denote the space induced by the inner product defined in Eq. (8) by $(\mathcal{M}_d, \langle \cdot, \cdot \rangle_\psi)$. Note that $\| \mathbb{I}_d \|_\psi = 1$. Similar to Section 2.3, we say an orthonormal basis is standard if all the operators are Hermitian and it contains $\mathbb{I}$ as an element. The Efron-Stein decomposition in Definition 2.18 can be extended to this setting as well. Similar to Lemma 5.11, the terms in Efron-Stein decompositions are orthogonal to each other with respect to the inner product $\langle \cdot, \cdot \rangle_\psi$.

**Definition 5.2.** Given quantum systems $A$ and $B$ and a bipartite quantum state $\psi_{AB} \in \mathcal{D}(A \otimes B)$, we define the Markov super-operator $\mathcal{J} : \mathcal{M}(B) \to \mathcal{M}(A)$ as follows.

$$\text{Tr} \left( M^\dagger \otimes Q \right) \psi_{AB} = \langle M, \mathcal{J}(Q) \rangle_{\psi_A},$$

for any $M \in \mathcal{M}(A)$ and $Q \in \mathcal{M}(B)$.

**Lemma 5.3.** Given quantum systems $A$, $B$ a bipartite state $\psi_{AB}$ and $Q \in \mathcal{M}(B)$, it holds that

$$\mathcal{J}(Q) = 2L(\psi_A, 2\text{Tr}_B (\mathbb{I} \otimes Q) \psi),$$

where $L(\cdot, \cdot)$ is the solution to the Lyapunov equation given in Definition C.1. In particular, if $\text{dim} A = \text{dim} B = 2$ and $\psi_{AB}$ is a state satisfying that $\psi_A = \frac{1}{2} \mathbb{I}$. Then $\mathcal{J}(Q) = 2\text{Tr}_B (\mathbb{I} \otimes Q) \psi_{AB}$.

**Proof.** By Definition 5.2, $\mathcal{J}(Q)$ must satisfy that

$$\text{Tr} M^\dagger (\text{Tr}_B (\mathbb{I} \otimes Q) \psi) = \frac{\text{Tr} M^\dagger \mathcal{J}(Q) \psi_A + \psi_A \mathcal{J}(Q)}{2},$$

for any $M \in \mathcal{M}(A)$. Thus

$$\mathcal{J}(Q) \psi_A + \psi_A \mathcal{J}(Q) = 2\text{Tr}_B (\mathbb{I} \otimes Q) \psi_{AB}. \quad (11)$$

We conclude the first part of the lemma. The second part follows from Eq. (11) with $\psi_A = \frac{1}{2} \mathbb{I}$.\qed

**Definition 5.4.** For any quantum system $A$ with dimension $d$, a quantum state $\psi \in \mathcal{D}(A)$ with $\psi > 0$ and $\rho \in [0, 1]$, the noise operator $\mathcal{J}_\rho : \mathcal{M}(A) \to \mathcal{M}(A)$ on $(\mathcal{M}(A), \psi)$ is defined as follows. For any $M, P \in \mathcal{M}(A)$,

$$\langle M, \mathcal{J}_\rho (P) \rangle_\psi \overset{\text{def}}{=} \langle \rho M + (1 - \rho) \left( \text{Tr} M^\dagger \psi \right) \cdot \mathbb{I}_A, P \rangle_\psi.$$

For the space $\mathcal{M}(A^n)$, with slight abuse of notations, we define $\mathcal{J}_\rho \overset{\text{def}}{=} \otimes_{i=1}^n \mathcal{J}_\rho$.\quad 20
The noise operator $T_\rho$ is also named depolarizing channel \cite{NC00} in the quantum information community, which is an analog of the Bonami-Beckner operator in Fourier analysis \cite{Bec75, Bon70}.

**Lemma 5.5.** Given integers $d, n > 0$, $\rho \in [0, 1]$, space $(M_d, \psi)$ with a standard orthonormal basis $B = \{B_i\}_{i=0}^{d^2-1}$, the following holds.

1. $T_\rho (P) = \rho P + (1 - \rho) \text{Tr} P \psi \cdot B_0$.

2. For any $P \in M_d^\otimes n$ with the Fourier expansion $P = \sum_{\sigma \in [d]^{2n}} \hat{P}(\sigma) B_{\sigma}$, it holds that

$$T_\rho (P) = \sum_{\sigma \in [d]^{2n}} \rho |\sigma| \hat{P}(\sigma) B_{\sigma}.$$

3. For any $P \in M_d^\otimes n$ $\|T_\rho (P)\| \leq \|P\|$ and $\|T_\rho (P)\|_\psi \leq \|P\|_\psi$.

**Proof.** Note that item 1 is equivalent to

$$\langle M, T_\rho (P) \rangle = \langle \rho M + (1 - \rho) \text{Tr} M^\dagger \psi \cdot B_0, P \rangle_\psi = \rho \langle M, P \rangle_\psi + (1 - \rho) \text{Tr} M^\dagger \psi \langle B_0, P \rangle_\psi = \rho \langle M, P \rangle_\psi + (1 - \rho) \hat{P}(0) \text{Tr} M^\dagger \psi.$$

and

$$\langle M, \hat{P}(0) B_0 + \rho \sum_{\sigma \neq 0} \hat{P}(\sigma) B_{\sigma} \rangle_\psi = \langle M, (1 - \rho) \hat{P}(0) B_0 + \rho P \rangle_\psi = \rho \langle M, P \rangle_\psi + (1 - \rho) \hat{P}(0) \text{Tr} M^\dagger \psi.$$

Thus

$$\langle M, T_\rho (P) \rangle = \langle M, \hat{P}(0) B_0 + \rho \sum_{\sigma \neq 0} \hat{P}(\sigma) B_{\sigma} \rangle_\psi,$$

for any $M \in M_d$. We conclude the item 1.

Item 2 is implied by item 1.

For item 3, we define $T^{(i)}$ be the operator on $M_d^\otimes n$ which applies $T_\rho$ to the $i$-th system and leaves other systems untouched. Then $T_\rho = T^{(n)}_\rho \circ \cdots \circ T^{(1)}_\rho$. From item 1, we have

$$T^{(i)}_\rho (P) \overset{\text{def}}{=} \sum_{\sigma : \sigma_i = 0} \hat{P}(\sigma) B_{\sigma} + \rho \sum_{\sigma : \sigma_i \neq 0} \hat{P}(\sigma) B_{\sigma} = \rho P + (1 - \rho) \sum_{\sigma : \sigma_i = 0} \hat{P}(\sigma) B_{\sigma} = \rho P + (1 - \rho) P_{-i} \otimes B_{(i)}.$$
Note that the spectral norm of the first term is at most \( \rho \|P \| \). The spectral norm of the second term is at most \((1 - \rho) \|P \| \) by Lemma 2.17 item 1. Hence \( \| A_{\rho}^{(i)}(P) \| \leq \|P \| \). Thus the first inequality in item 3 follows. To prove the second inequality, consider
\[
\| A_{\rho}^{(P)} \|_2^2 = \sum_{\sigma \in \mathcal{D}^n_{\geq 0}} \rho(\sigma)^2 \| \tilde{P}(\sigma) \|^2 \leq \sum_{\sigma \in \mathcal{D}^n_{\geq 0}} \| \tilde{P}(\sigma) \|^2 = \|P \|_2^2.
\]

\[\square\]

The notion of quantum maximal correlation, first introduced by Beigi [Bei13], which generalizes the maximal correlation coefficients [Hir35, Geb41, Régn9] in classical information theory to the quantum setting, is crucial to our analysis.

**Definition 5.6 (Maximal correlation).** [Bei13] Given quantum systems \( A, B \) and a bipartite state \( \psi_{AB} \in \mathcal{D}(A \otimes B) \), the maximal correlation of \( \psi_{AB} \) is defined to be

\[
\rho(\psi_{AB}) = \sup \left\{ \left| \text{Tr} \left( \left( P^\dagger \otimes Q \right) \psi_{AB} \right) \right| : P \in \mathcal{M}(A) , Q \in \mathcal{M}(B) , \text{Tr} \psi_A = \text{Tr} Q \psi_B = 0 , \|P\|_{\psi_A} = \|Q\|_{\psi_B} = 1 \right\}.
\]

**Fact 5.7.** [Bei13] Given quantum systems \( A, B \) and a bipartite quantum state \( \psi_{AB} \), it holds that

1. \( 0 \leq \rho(\psi_{AB}) \leq 1 \).
2. \( \rho(\psi_{AB}) = 1 \) if and only if there exist local measurements \( \{ M_A, \mathbb{1} - M_A \} \) and \( \{ N_B, \mathbb{1} - N_B \} \) such that \( 0 < \text{Tr} (\psi_{AB} (M_A \otimes N_B)) < 1 \), and

\[
\text{Tr} (\psi_{AB} (M_A \otimes (\mathbb{1} - N_B))) = \text{Tr} (\psi_{AB} ((\mathbb{1} - M_A) \otimes N_B)) = 0.
\]

The following proposition characterizes all the two-qubit states with maximal correlation being 1.

**Proposition 5.8.** Given a bipartite state \( \psi_{AB} \in \mathcal{D}(M_2 \times M_2) \), \( \rho(\psi_{AB}) = 1 \) if and only if there exist local unitaries \( U_A \) and \( V_B \) such that

\[
\left( U_A^\dagger \otimes V_B^\dagger \right) \psi_{AB} (U_A \otimes V_B) = \frac{1}{2} \sum_{a,b=0} c_{a,b} |aa\rangle \langle bb|,
\]

where \( (c_{a,b})_{a,b \in \{0,1\}} \) satisfies that \([c_{00}, c_{01}, c_{10}, c_{11}]\) is a density operator.

Moreover, if \( \psi_A = \psi_B = \frac{\mathbb{1}_2}{2} \), then \( \rho(\psi_{AB}) = 1 \) if and only if there exist local unitaries \( U_A \) and \( V_B \) such that

\[
\left( U_A^\dagger \otimes V_B^\dagger \right) \psi_{AB} (U_A \otimes V_B) = p |\Phi\rangle \langle \Phi| + (1 - p) |\Psi\rangle \langle \Psi|,
\]

for \( 0 \leq p \leq 1 \), where \( |\Phi\rangle = \frac{1}{\sqrt{2}} (|00\rangle + |11\rangle) \) and \( |\Psi\rangle = \frac{1}{\sqrt{2}} (|00\rangle - |11\rangle) \).

**Proof.** Let \( \{ M_A, \mathbb{1} - M_A \} \) and \( \{ N_B, \mathbb{1} - N_B \} \) be the measurements induced by Fact 5.7. We may assume that \( M_A = a_0 |0\rangle \langle 0| + a_1 |1\rangle \langle 1| \) and \( N_B = b_0 |0\rangle \langle 0| + b_1 |1\rangle \langle 1| \) up to local unitaries. We claim that \( M_A \neq \mathbb{1}_2 \) and \( N_B \neq \mathbb{1}_2 \). Suppose \( M_A = \mathbb{1}_2 \). Then we have \( 0 < \text{Tr} \psi_B N_B < 1 \) and \( 1 - \text{Tr} \psi_B N_B = 0 \), contradicting the fact that \( \psi_B \) is a density operator. The argument for \( N_B \) is similar. Thus we may further assume that \( a_1 \leq a_0 \leq 1 \) and \( b_1 \leq b_0 \leq 1 \) and \( a_1, b_0 \leq 1, a_0, b_0 > 0 \). Then \( \text{Tr} (M_A \otimes (\mathbb{1} - N_B)) \psi_{AB} = 0 \) implies that \( 0 \) is a density operator. The argument for \( N_B \) is similar. Thus we may further assume that \( a_1 \leq a_0 \leq 1 \) and \( b_1 \leq b_0 \leq 1 \) and \( a_1, b_0 \leq 1, a_0, b_0 > 0 \). Then \( \text{Tr} ((\mathbb{1} - M_A) \otimes M_B) = 0 \) implies that \( 0 \) is a density operator. We conclude Eq. (12). The second part follows by elementary calculation. \[\square\]
Definition 5.9. A bipartite state $\psi_{AB} \in \mathcal{D}(M_2 \times M_2)$ is a noisy EPR state if $\psi_A = \psi_B = \frac{1_2}{2}$ and its maximal correlation $\rho = \rho (\psi_{AB}) < 1$.

Proposition 5.8 gives a tight characterization of noisy EPR states. Probably the most interesting case is the EPR state with an arbitrary depolarizing noise $\epsilon > 0$, $(1 - \epsilon) |\Phi\rangle \langle \Phi| + \epsilon \frac{1_2}{2} \otimes \frac{1_2}{2}$, where $|\Phi\rangle = \frac{1}{\sqrt{2}} (|00\rangle + |11\rangle)$ is an EPR state. Beigi proved that the maximal correlation of this state is $1 - \epsilon$ in [Bei13].

The following proposition provides a useful characterization of the quantum maximal correlation.

Proposition 5.10. Given quantum systems $A, B$ and a bipartite state $\psi_{AB}$, for any $Q \in \mathcal{M}(B)$,

$$\max \left\{ \| \text{Tr} \left( P \otimes Q \right) \psi_{AB} \| : P \in \mathcal{P}(A), \| P \|_{\psi_A} = 1 \right\}$$

is achieved by

$$P^* = \frac{T(Q)}{\| T(Q) \|_{\psi_A}},$$

with the maximum value $\| T(Q) \|_{\psi_A}$, where $T : \mathcal{M}(B) \to \mathcal{M}(A)$ is the Markov super-operator in Definition 5.2. Thus,

$$\rho (\psi_{AB}) = \max \left\{ \| T(Q) \|_{\psi_A} : Q \in \mathcal{M}(B), \langle 1, Q \rangle_{\psi_B} = 0, \| Q \|_{\psi_B} = 1 \right\}.$$ 

In particular, if $\psi_A = \frac{1_d A}{d_A}$ and $\psi_B = \frac{1_d B}{d_B}$, then

$$\rho (\psi_{AB}) = \max \left\{ \| T(Q) \|_2 : \text{Tr} Q = 0, \| Q \|_2 = 1 \right\},$$

where $d_A = \dim(A)$ and $d_B = \dim(B)$.

Moreover, the maximal correlation in Definition 5.6 can be achieved by a pair of Hermitian operators $(P, Q)$ if $\psi_A = \frac{1_d A}{d_A}$ and $\psi_B = \frac{1_d B}{d_B}$.

Proof. The proof follows closely to the one of Lemma 2.8 in [Mos10]. Let $P \in \mathcal{M}(A)$ achieves the maximum in Eq. (13). Then it satisfies that $\| P \|_{\psi_A} = 1$. Write $P = \alpha P^* + \beta P'$, where $\| \alpha \|^2 + \| \beta \|^2 = 1$, $\| P' \|_{\psi_A} = 1$ and $\langle P', P' \rangle_{\psi_A} = 0$. By the definition of the Markov super-operator

$$0 = \langle P', T(Q) \rangle_{\psi_A} = \text{Tr} (P' \otimes Q) \psi_{AB},$$

So we should set $|\alpha| = 1$. Moreover,

$$\text{Tr} \left( T(Q) \right) \otimes Q \psi_{AB} = \| T(Q) \|_{\psi_A}^2.$$ 

To prove that the maximal correlation in Definition 5.6 can be achieved by a pair of Hermitian operators $(P, Q)$, it suffices to prove that the maximum in Eq. (14) can be achieved by a Hermitian matrix $Q$ by Lemma 5.3 and Lemma C.2. Suppose $Q = Q_1 + i \cdot Q_2$ achieves the maximum in Eq. (14) with Hermitian matrices $Q_1 \neq 0$ and $Q_2 \neq 0$. Then $\text{Tr} Q_1 \psi_B = \text{Tr} Q_2 \psi_B = 0$ and $1 = \| Q_2 \|_2 = \| Q_1 \|_2^2 + \| Q_2 \|_2^2$. Then

$$\| T(Q) \|_2 = \left( \frac{\| T(Q_1) \|_2^2 + \| T(Q_2) \|_2^2}{\| Q_1 \|_2^2 + \| Q_2 \|_2^2} \right)^{1/2} \leq \max \left\{ \| T \left( \frac{Q_1}{\| Q_1 \|_2} \right) \|_2, \| T \left( \frac{Q_2}{\| Q_2 \|_2} \right) \|_2 \right\}.$$ 

Thus one of $\frac{Q_1}{\| Q_1 \|_2}$ and $\frac{Q_2}{\| Q_2 \|_2}$ also achieves the maximum in Eq. (14). □
Lemma 5.11. Given quantum systems $A, B$ with $\dim A = d_A$ and $\dim B = d_B$, a bipartite quantum state $\psi_{AB} \in \mathcal{D}(A \otimes B)$, let $\{A_\sigma\}_{\sigma \in [d^A_1]}$ and $\{B_\tau\}_{\tau \in [d^B_1]}$ be standard orthonormal basis in $\mathcal{M}(A)$ and $\mathcal{M}(B)$ with respect to the inner product $(\cdot, \cdot)_{\psi_A}$ and $(\cdot, \cdot)_{\psi_B}$, respectively. It holds that

\[ \text{Tr} (A_\sigma \otimes B_\tau) \psi_{AB}^\otimes = 0, \]

whenever $\text{supp} (\sigma) \neq \text{supp} (\tau)$. Thus

\[ \text{Tr} \left( A[S] \otimes Q[T] \right) \psi_{AB}^\otimes = 0, \]

whenever $S \neq T$.

Proof. It follows from the equalities $\text{Tr} (A_0 \otimes B_\tau) \psi_{AB} = \text{Tr} (A_\sigma \otimes B_0) \psi_{AB} = 0$ whenever $\sigma \neq 0$ and $\tau \neq 0$.

Proposition 5.12. Given an integer $n > 0$, quantum systems $A, B$ a bipartite quantum state $\psi_{AB}, Q \in \mathcal{M}(B^n)$ and $S \subseteq [n]$, it holds that

\[ \mathcal{T} (Q[S]) = \mathcal{T} (Q) [S], \]

where $\mathcal{T} : \mathcal{M}(B) \to \mathcal{M}(A)$ is the Markov super-operator in Definition 5.12 with respect to $\psi_{AB}$.

Proof. It suffices to show that

\[ \langle X, \mathcal{T} (Q[S]) \rangle_{\psi_A} = \langle X, \mathcal{T} (Q) [S] \rangle_{\psi_A}, \]

for any $X \in \mathcal{M}(A^n)$. By the definition,

\[ \text{LHS} = \text{Tr} \left( X^\dagger \otimes Q[S] \right) \psi_{AB}^\otimes. \]

By Definition 2.18 and Proposition 2.19 we have

\[ \text{RHS} = \langle X[S], \mathcal{T} (Q) \rangle_{\psi_A} = \text{Tr} \left( X[S] \otimes Q \right) \psi_{AB}^\otimes. \]

By Lemma 5.11 and Definition 2.18

\[ \text{Tr} \left( X[S] \otimes Q \right) \psi_{AB}^\otimes = \text{Tr} \left( X^\dagger \otimes Q[S] \right) \psi_{AB}^\otimes = \text{Tr} \left( X[S] \otimes Q[S] \right) \psi_{AB}^\otimes. \]

\[ \square \]

Proposition 5.13. Given an integer $n > 0$, quantum systems $A, B$, a bipartite quantum state $\psi_{AB} \in \mathcal{D}(A \otimes B)$, $Q \in \mathcal{M}(B^n)$ and $S \subseteq [n]$. It holds that

\[ \| \mathcal{T} (Q[S]) \|_{\psi_A} \leq \rho |S| \| Q[S] \|_{\psi_B}, \]

where $\rho = \rho (\psi_{AB})$.

Proof. We may assume that $Q = Q[S]$ without loss of generality. It suffices to show the case that $S = [n]$. Let $d_A = \dim (A), d_B = \dim (B), \{A_i\}_{i \in [d^A_1]}$ and $\{B_i\}_{i \in [d^B_1]}$ be standard orthonormal basis in $\mathcal{M}(A)$ and $\mathcal{M}(B)$, respectively. For $r \in [n]$, set $\mathcal{T}(r) \stackrel{\text{def}}{=} \mathbb{1}_{\mathcal{M}(A)} \otimes \mathcal{T} \otimes \mathbb{1}_{\mathcal{M}(B)}$, $\psi^{(r)} \stackrel{\text{def}}{=} \psi_{A}^\otimes \otimes \psi_{B}^{\otimes(n-r)}$, $Q^{(0)} \stackrel{\text{def}}{=} Q$ and $Q^{(r)} = \mathcal{T}(r) (Q^{(r-1)})$, where $\mathbb{1}_{\mathcal{M}(A)}$ and $\mathbb{1}_{\mathcal{M}(B)}$ are
the identity maps mapping $\mathcal{M}(A)$ to $\mathcal{M}(A)$ and $\mathcal{M}(B)$ to $\mathcal{M}(B)$, respectively. Note that $Q^{(n)} = \mathcal{T}(Q)$. Hence it suffices to show that

$$\text{Tr} \left( Q^{(r)} \right) \widetilde{Q}^{(r)} \psi^{(r)} \leq \rho^2 \text{Tr} \left( Q^{(r-1)} \right) \widetilde{Q}^{(r-1)} \psi^{(r-1)} ,$$

(15)

and

$$Q^{(r)} = Q^{(r)}[n],$$

(16)

where $Q^{(r)}[n]$ is defined by expanding $Q^{(r)}$ over $\{ A_{\sigma_{\leq r}} \otimes B_{\sigma_{> r}} \}_{\sigma \in [d_A]_{\leq r} \times [d_B]_{> r}}$, because $\mathcal{T} = \mathcal{T}^{(n)} \circ \cdots \circ \mathcal{T}^{(1)}$. Let $\psi_A = \sum_{i=1}^{d_A} \lambda_i A_i |u_i\rangle\langle u_i|$ and $\psi_B = \sum_{i=1}^{d_B} \lambda_i B_i |v_i\rangle\langle v_i|$ be spectral decompositions of $\psi_A$ and $\psi_B$, respectively. For any $s \in [d_A]_{r-1} \times [d_B]^{n-r}$, we define

$$\theta^{(r)}_s \overset{\text{def}}{=} \sqrt{\lambda_{s_1} \lambda_{s_2} \cdots \lambda_{s_{r-1}} \lambda_{s_{r+1}} \lambda_{s_{r+2}} \cdots \lambda_{s_n}},$$

and

$$|w_s\rangle \overset{\text{def}}{=} |u_{s_1}\rangle \otimes \cdots \otimes |u_{s_{r-1}}\rangle \otimes |v_{s_{r+1}}\rangle \otimes \cdots \otimes |v_{s_n}\rangle .$$

For any $s, t \in [d_A]_{r-1} \times [d_B]^{n-r}$, we define

$$P^{(r)}_{s,t} \overset{\text{def}}{=} \theta^{(r)}_t ((|w_s\rangle \otimes \mathbb{1}) Q^{(r)} (|w_t\rangle \otimes \mathbb{1}))$$

and

$$Q^{(r-1)}_{s,t} \overset{\text{def}}{=} \theta^{(r)}_t ((|w_s\rangle \otimes \mathbb{1}) Q^{(r-1)} (|w_t\rangle \otimes \mathbb{1})),
$$

where $|w_s\rangle$ and $|w_t\rangle$ lie in the registers $\{1, \ldots, r-1, r+1, \ldots, n\}$. Then

$$P^{(r)}_{s,t} = \mathcal{T} (Q^{(r-1)}_{s,t}).$$

(17)

And we have

$$\langle \mathbb{1}_B, Q^{(r-1)}_{s,t} \rangle_{\psi_B} = 0,$$

by the induction $Q^{(r-1)} = Q^{(r-1)}[n]$. By Proposition 5.10 $\|P^{(r)}_{s,t}\|_{\psi_A} \leq \rho \|Q^{(r-1)}_{s,t}\|_{\psi_B}$. Consider

$$\sum_{s,t} \|P^{(r)}_{s,t}\|_{\psi_A}^2 = \sum_{s,t} \left( \theta^{(r)}_t \right)^2 \text{Tr} \left( P^{(r)}_{s,t} \right)^\dagger P^{(r)}_{s,t} \psi_A = \text{Tr} \left( Q^{(r)} \right) \widetilde{Q}^{(r)} \psi^{(r)},$$

and

$$\sum_{s,t} \|Q^{(r-1)}_{s,t}\|_{\psi_B}^2 = \sum_{s,t} \left( \theta^{(r)}_t \right)^2 \text{Tr} \left( Q^{(r-1)} \right) \widetilde{Q}^{(r-1)} \psi^{(r-1)}.$$ 

Combining with Eq. (17) and Proposition 5.10, we conclude Eq. (15).

For Eq. (16), compute

$$\mathcal{T}^{(r)} (\sigma) = \mathcal{T}^{(r)} (\mathcal{T}^{(r-1)} (\sigma)) = \langle X_{\sigma_{\leq r}} \otimes Y_{\sigma_{> r}}, \mathcal{T}^{(r)} (Q) \rangle_{\psi^{(r)}}$$

$$= \sum_{\tau:|\tau|=n} \mathcal{T}^{(r)} (\tau) \langle X_{\sigma_{\leq r}} \otimes Y_{\sigma_{> r}}, \mathcal{T}^{(r)} (X_{\sigma_{\leq r}} \otimes Y_{\sigma_{> r}}) \rangle_{\psi^{(r)}}$$

$$= \sum_{\tau:|\tau|=n, \tau_{-r} = \sigma_{-r}} \mathcal{T}^{(r)} (\tau) \langle X_{\sigma_{\leq r}}, \mathcal{T} (Y_{\sigma_{r}}) \rangle_{\psi_A} .$$

Note that

$$\langle X_0, \mathcal{T} (Y_{\sigma_{r}}) \rangle_{\psi_A} = \text{Tr} (\mathbb{1} \otimes Y_{\sigma_{r}}) \psi_{AB} = 0,$$

as $|\tau| = n$. Therefore, $\mathcal{T}^{(r)} (Q) (\sigma) = 0$ if $|\sigma| < n$. We conclude Eq. (16).
A key property of the (classical) maximal correlation is tensorization, which states that the maximal correlation of multiple independent identical copies of a distribution is same as the one of one copy. The same property also holds for the quantum maximal correlation shown by Beigi in [Bei13]. Here we provide a different proof.

**Fact 5.14.** [Bei13] Given quantum systems $A, B$ with $\dim A = d_A$ and $\dim B = d_B$, a bipartite quantum state $\psi_{AB} \in D(A \otimes B)$, it holds that

$$
\rho\left(\psi_{AB}^{\otimes n}\right) = \rho(\psi_{AB}).
$$

**Proof.** Given $Q \in M(\mathcal{B}^n)$ with $\langle 1, Q \rangle_{\psi_B} = 0$ and $\|Q\|_{\psi_B} = 1$. Using the Efron-Stein decomposition $Q = \sum_{S \neq \emptyset} Q[S]$, we have

$$
\|\mathcal{T}(Q)\|_{\psi_B}^2 = \left\|\sum_{S \neq \emptyset} \mathcal{T}(Q[S])\right\|_{\psi_B}^2 \quad \text{(linearity of } \mathcal{T})
$$

$$
= \left\|\sum_{S \neq \emptyset} \mathcal{T}(Q) [S]\right\|_{\psi_B}^2 \quad \text{(Proposition 5.12)}
$$

$$
= \sum_{S \neq \emptyset} \|\mathcal{T}(Q) [S]\|_{\psi_B}^2 \quad \text{(orthogonality of the Efron-Stein decomposition)}
$$

$$
\leq \sum_{S \neq \emptyset} \rho^{2|S|} \|Q[S]\|_{\psi_B}^2 \quad \text{(Proposition 5.13)}
$$

$$
\leq \rho^2 \sum_{S \neq \emptyset} \|Q[S]\|_{\psi_B}^2 \quad \text{(orthogonality of the Efron-Stein decomposition)}
$$

$$
= \rho^2.
$$

From Proposition 5.10, $\rho\left(\psi_{AB}^{\otimes n}\right) \leq \rho(\psi_{AB})$. The other direction is trivial. \qed

## 6 Smoothing operators

The main lemma in this section is the following.

**Lemma 6.1.** Given parameters $0 \leq \rho < 1$, $0 < \delta < 1$, integer $n > 0$ a noisy EPR state $\psi_{AB}$ with the maximal correlation $\rho = \rho(\psi_{AB})$, there exist $d = d(\rho, \delta)$ and a map $f : \mathcal{H}^{\otimes n}_2 \to \mathcal{H}^{\otimes n}_2$, such that for any $P \in \mathcal{H}_2^{\otimes n}, Q \in \mathcal{H}_2^{\otimes n}$ satisfying $0 \leq P \leq 1$ and $0 \leq Q \leq 1$, the operators $P^{(1)} = f(P)$ and $Q^{(1)} = f(Q)$ satisfy that

1. \[ 0 \leq P^{(1)} \leq 1 \text{ and } 0 \leq Q^{(1)} \leq 1; \]

2. \[ \text{Tr } P^{(1)} = \text{Tr } P \text{ and } \text{Tr } Q^{(1)} = \text{Tr } Q. \]

3. \[ \|P^{(1)}\|_2 \leq \|P\|_2 \text{ and } \|Q^{(1)}\|_2 \leq \|Q\|_2; \]
In particular, we can take \( d = \frac{2\log^2 \frac{1}{\epsilon}}{C(1-\rho)^2} \) for some constant \( C \).

Before proving Lemma 6.1, we need the following lemma, which is a generalization of the smoothing of the strategies in the classical setting in [Mos10], to the quantum setting.

**Lemma 6.2.** Given a noisy EPR state \( \psi_{AB} \) with the maximal correlation \( \rho = \rho(\psi_{AB}) < 1 \), a parameter \( 0 < \epsilon < 1 \) and operators \( P \in \mathcal{H}_2^\otimes n, Q \in \mathcal{H}_2^\otimes n \), let \( \gamma \) be chosen sufficiently close to 0 so that

\[
\gamma \leq 1 - (1 - \epsilon) \log \rho/(\log \epsilon + \log \rho).
\]

Then

\[
\left| \text{Tr} \left( P \otimes Q \right) \psi_{AB}^{\otimes n} - \text{Tr} \left( \mathcal{T}_{1-\gamma} (P) \otimes \mathcal{T}_{1-\gamma} (Q) \right) \psi_{AB}^{\otimes n} \right| \leq 2\epsilon \sqrt{\text{Var}[P] \text{Var}[Q]}.
\]

In particular, there exists an absolute constant \( C \) such that it suffices to take

\[
\gamma = C(1 - \rho) \epsilon \log(1/\epsilon).
\]

**Proof.** Note that \( \|\mathcal{T}_{1-\gamma} (Q)\|_2 \leq \|Q\|_2 \) by Lemma 5.5. Thus, from Fact 2.6, it suffices to show

\[
\left| \text{Tr} \left( P \otimes Q \right) \psi_{AB}^{\otimes n} - \text{Tr} \left( P \otimes \mathcal{T}_{1-\gamma} (Q) \right) \psi_{AB}^{\otimes n} \right| = \left| \text{Tr} \left( P \otimes (1 - \mathcal{T}_{1-\gamma} (Q)) \right) \psi_{AB}^{\otimes n} \right| \leq \epsilon \sqrt{\text{Var}[P] \text{Var}[Q]}.
\]

By Lemma 5.5 item 2, we have

\[
(1 - \mathcal{T}_{1-\gamma}) (Q[S]) = \left( 1 - (1 - \gamma)^{|S|} \right) Q[S].
\]

Using Proposition 5.12 and Proposition 5.13

\[
\|\mathcal{T}' (Q[S])\|_2 = \|\mathcal{T} (Q) [S]\|_2 \leq \rho^{|S|} \|Q[S]\|_2,
\]

and \( \langle \mathcal{T}' (Q[S]), P[S'] \rangle = 0 \) if \( S \neq S' \). Let \( \mathcal{T}' \equiv \mathcal{T} \circ \left( \mathbf{1}_{M(B)} - \mathcal{T}_{1-\gamma} \right) \). From Definition 5.2

\[
\text{Tr} \left( P \otimes (1 - \mathcal{T}_{1-\gamma} (Q)) \right) \psi_{AB}^{\otimes n} = \langle P, \mathcal{T}' (Q) \rangle.
\]

Combining Lemma 5.5 and Eq. (18),

\[
\|\mathcal{T}' (Q[S])\|_2 \leq \min \left( \rho^{|S|}, 1 - (1 - \gamma)^{|S|} \right) \|Q[S]\|_2 \leq \epsilon \|Q[S]\|_2.
\]

From Lemma 5.5, Proposition 2.19 and Proposition 5.12 \( \langle \mathcal{T}' (Q[S]), P[S'] \rangle = 0 \) if \( S \neq S' \). Therefore,

\[
\| \langle P, \mathcal{T}' (Q) \rangle \| = \sum_{S \neq \emptyset} \| \langle P[S], \mathcal{T}' (Q[S]) \rangle \| \leq \sqrt{\text{Var}[P]} \sqrt{\frac{1}{2^n} \sum_{S \neq \emptyset} \|\mathcal{T}' (Q[S])\|_2^2}
\]

\[
\leq \epsilon \sqrt{\text{Var}[P]} \sqrt{\frac{1}{2^n} \sum_{S \neq \emptyset} \|Q[S]\|_2^2} \leq \epsilon \sqrt{\text{Var}[P] \text{Var}[Q]},
\]

where the last inequality is from the orthogonality of the Efron-Stein decomposition. \( \square \)
We are now ready to prove Lemma 6.1.

**Proof of Lemma 6.1.** Given $\rho$ and $\delta$, we choose $\epsilon = \delta/2$ and $\gamma$ in Lemma 6.2. We choose $d$ sufficiently large such that $(1 - \gamma)^{2d} \leq \delta$, that is, $d = \frac{\log \frac{1}{\epsilon}}{2\gamma}$. Given $P, Q$ as in Lemma 6.1, we set $P^{(1)} \overset{\text{def}}{=} T_{1-\gamma}(P)$ and $Q^{(1)} \overset{\text{def}}{=} T_{1-\gamma}(Q)$. By Lemma 5.5 item 1, $0 \leq P^{(1)} \leq 1$ and $0 \leq Q^{(1)} \leq 1$. $\text{Tr} P^{(1)} = \text{Tr} P$ and $\text{Tr} Q^{(1)} = \text{Tr} Q$ follows by Lemma 5.5 item 2. By Lemma 5.5 item 3, $\|P^{(1)}\|_2 \leq \|P\|_2$ and $\|Q^{(1)}\|_2 \leq \|Q\|_2$. From the inequalities that $\text{Var}[P] \leq \|P\|_2^2 \leq 1$, $\text{Var}[Q] \leq \|Q\|_2^2 \leq 1$ due to Lemma 2.17 and Lemma 6.2, we get
\[
\left|\text{Tr}(P \otimes Q) \psi_{AB}^{\otimes n} - \text{Tr}\left(P^{(1)} \otimes Q^{(1)}\right) \psi_{AB}^{\otimes n}\right| \leq 2\epsilon = \delta.
\]
Also, $\hat{P}^{(1)}(\sigma) = (1 - \gamma)|\sigma| \hat{P}(\sigma)$ and $\hat{Q}^{(1)}(\sigma) = (1 - \gamma)|\sigma| \hat{Q}(\sigma)$. Thus, we get that
\[
\sum_{|\sigma| > d} \hat{P}^{(1)}(\sigma)^2 \leq (1 - \gamma)^{2d} \sum_{|\sigma| > d} \hat{P}(\sigma)^2 \leq (1 - \gamma)^{2d} \|P\|_2^2 \leq \delta;
\]
\[
\sum_{|\sigma| > d} \hat{Q}^{(1)}(\sigma)^2 \leq (1 - \gamma)^{2d} \sum_{|\sigma| > d} \hat{Q}(\sigma)^2 \leq (1 - \gamma)^{2d} \|Q\|_2^2 \leq \delta,
\]
where the second inequalities in both equations are from Fact 2.10 item 3. \hfill \Box

### 7 Joint regularity lemma

In the proof the decidability of the classical non-interactive joint simulation [GKS16, DMN18, GKR18], a subset $H$ of $[n]$ with bounded size are chosen such that all the coordinates not in $H$ are low influential even if the values of the coordinates in $H$ are fixed. However, the same strategy cannot be applied in the quantum setting because there is no common basis to fix due to the non-commutativity. Instead of fixing values, we expand the operators in proper chosen standard orthonormal basis. Before getting into the details, we introduce the following notion.

**Definition 7.1.** Given quantum systems $A$ and $B$ with dimension $d_A$ and $d_B$, respectively, and a bipartite quantum state $\psi_{AB}$, Let $A = \{A_i\}_{j \in [d_A^2]}$ and $B = \{B_i\}_{i \in [d_B^2]}$ be standard orthonormal basis in the space $(\mathcal{M}(A), \psi_A)$ and $(\mathcal{M}(B), \psi_B)$ defined in Section 5, respectively. The correlation matrix of $(\psi_{AB}, A, B)$ is defined as
\[
\text{Corr} (\psi_{AB}, A, B)_{i,j} \overset{\text{def}}{=} \text{Tr}(A_i \otimes B_j) \psi_{AB}.
\]

for $i \in [d_A^2], j \in [d_B^2]$. The correlation matrix of $(\psi_{AB}^{\otimes n}, A, B, \sigma, \tau)$ is defined to be
\[
\text{Corr} (\psi_{AB}^{\otimes n}, A, B)_{\sigma, \tau} \overset{\text{def}}{=} \text{Tr}(A_\sigma \otimes B_\tau) \psi_{AB}^{\otimes n}.
\]

for $\sigma \in [d_A]^n$, $\tau \in [d_B]^n$.

The following lemma follows by the definition.

**Lemma 7.2.** Given an integer $n > 0$, quantum systems $A$ and $B$ with dimension $d_A$ and $d_B$, respectively, and a bipartite quantum state $\psi_{AB}$, let $A = \{A_i\}_{i \in [d_A^2]}$ and $B = \{B_i\}_{i \in [d_B^2]}$ be standard orthonormal basis in $(\mathcal{M}(A), \psi_A)$ and $(\mathcal{M}(B), \psi_B)$, respectively. It holds that
\[
\text{Corr} (\psi_{AB}^{\otimes n}, A \otimes B \otimes n) = \text{Corr} (\psi_{AB}, A, B)^{\otimes n}.
\]
Lemma 7.3. Given quantum systems \( A \) and \( B \) with dimension \( d_A \) and \( d_B \), respectively, and a bipartite quantum state \( \psi_{AB} \), for any standard orthonormal basis \( A = \{ A_i \}_{i \in [d_A]} \) and \( B = \{ B_i \}_{i \in [d_B]} \) in \( \mathcal{M}(A), \psi_A \) and \( \mathcal{M}(B), \psi_B \), respectively. It holds that \( s_1 \left( \text{Corr} (\psi_{AB}, A, B) \right) = 1 \) and \( s_2 \left( \text{Corr} (\psi_{AB}, A, B) \right) = \rho \), where \( \rho = \rho (\psi_{AB}); s_i (\cdot) \) is the \( i \)-th largest singular eigenvalue.

By Fact 2.8, there exist standard orthonormal basis \( A = \{ A_i \}_{i \in [d_A]} \) and \( B = \{ B_i \}_{i \in [d_B]} \) in \( \mathcal{M}(A), \psi_A \) and \( \mathcal{M}(B), \psi_B \), respectively, such that

\[
\text{Corr} (\psi_{AB}, A, B)_{i,j} = \begin{cases} 
    c_i & \text{if } i = j \\
    0 & \text{otherwise,}
\end{cases}
\]

where \( c_1 = 1, c_2 = \rho (\psi_{AB}) \) and \( c_1 \geq c_2 \geq c_3 \geq \ldots \).

Proof. We assume that both the dimensions of \( A \) and \( B \) are \( d \). The arguments are similar when the dimensions of the two systems are different. Let \( M \overset{\text{def}}{=} \text{Corr} (\psi_{AB}, A, B) \). It is easy to verify that \( M_{0,i} = M_{i,0} = 0 \) for all \( i \in [d^2] \). Note that \( M \) is a real Hermitian matrix. Thus we set \( M = U^\dagger DV \) to be a singular value decomposition of \( M \) where \( U \) and \( V \) are orthogonal matrices and \( U_{0,0} = V_{0,0} = 1, U_{0,i} = U_{i,0} = V_{0,i} = V_{i,0} = 0 \) for \( 1 \leq i \leq d^2 - 1 \). Let \( P_i \overset{\text{def}}{=} \sum_{j=1}^{d^2-1} U_{ij} A_j \) and \( Q_i \overset{\text{def}}{=} \sum_{j=1}^{d^2-1} V_{ij} B_j \) for \( 1 \leq i \leq d^2 - 1 \). Then \( \text{Tr} P_i = \text{Tr} Q_i = 0 \). \( \text{Var}[P_i] = \text{Var}[Q_i] = 1 \). Thus by the definition of the quantum maximal correlation,

\[
\rho \geq \text{Tr} \left( P_i \otimes Q_i \right) \psi_{AB} \\
= \sum_{j,k=1}^{d^2-1} U_{ij} V_{k'} \left( A_j \otimes B_k \right) \psi_{AB} \\
= \sum_{j,k=1}^{d^2-1} U_{ij} V_{k'} M_{jk} \left( U^\dagger V^\dagger \right)_{j'i'} \\
= \delta_{i',i} D_{ii'}.
\]

Hence \( \| M \| = 1 \) and \( s_2 \left( M \right) \leq \rho \).

From Proposition 5.10, we assume that \( P, Q \) are two Hermitian operators in \( \mathcal{H}(A), \mathcal{H}(B) \) which achieve \( \rho \). Then from Definition 5.6, both \( \{ I_{d_A}, P \} \) and \( \{ I_{d_B}, Q \} \) can be extended to orthonormal basis, say \( \{ P_i \}_{i=0}^{d^2-1} \) and \( \{ Q_i \}_{i=0}^{d^2-1} \) where \( P_1 = P \) and \( Q_1 = Q \). Let \( M \) be the corresponding correlated matrix. Then \( M(1,1) = 1, M(2,2) = \rho \). Thus, \( s_2 \left( M \right) = \rho \).

We reach the main lemma in this section.

Lemma 7.4. Given a noisy EPR state \( \psi_{AB} \) and operators \( P \in \mathcal{H}^{\otimes n}_2, Q \in \mathcal{H}^{\otimes n}_2 \), and parameters \( d, \delta, \epsilon > 0 \) satisfying that \( \| P \|_2, \| Q \|_2 \leq 1 \) and \( \| P^{>d} \|_2, \| Q^{>d} \|_2 \leq \delta \) and \( \| Q^{>d} \|_2 \leq \delta \), let \( (c_i)_{i=0}^{3} \) be the singular values of \( \text{Corr} (\psi_{AB}) \) in non-increasing order and \( A = \{ A_i \}_{i=0}^{3} \) and \( B = \{ B_i \}_{i=0}^{3} \) be the standard orthonormal basis induced by Lemma 7.3. Then there exists a subset \( H \subseteq [n] \) of size \( h \overset{\text{def}}{=} |H| \leq \frac{d^2 \epsilon}{\delta} \) such that for any \( i \notin H \), \( \text{Inf}_i \left( P \overset{\otimes d} \leq \epsilon, Q \overset{\otimes d} \leq \epsilon \right) \leq \epsilon \) and

\[
\text{Tr} \left( P \otimes Q \right) \psi_{AB}^{\otimes n} = \sum_{\sigma \in [4]^h} c_{\sigma} \text{Tr} \left( P_{\sigma} \otimes Q_{\sigma} \right) \psi_{AB}^{\otimes (n-h)},
\]

where \( c_{\alpha} \overset{\text{def}}{=} \prod_{i=1}^{d} c_{\alpha_i} \), for any \( \alpha \in [4]^h \); and

\[
P_{\sigma} = \sum_{\tau \in [4]^h, \text{P} = \sigma} \hat{P} (\tau) A_{\tau}
\]

29
and

\[
Q_\sigma = \sum_{\tau \in \{H = \sigma\}} \hat{Q}(\tau) B_\tau.
\]

Proof. Set \( H = \{ i : \text{Inf}_i \left( P^{\leq d} \right) \geq \epsilon \text{ or } \text{Inf}_i \left( Q^{\leq d} \right) \geq \epsilon \} \). From Lemma 2.17 item 4, \(|H| \leq \frac{2d}{\epsilon} \). Expanding \( P \) and \( Q \) in terms of the basis \( A \) and \( B \), respectively, we conclude the result. \( \square \)

8 Fréchet Derivative and Taylor expansion

In this section, we derive a Taylor expansion of matrix functions, for which we need to involve Fréchet derivatives. The Fréchet derivatives are derivatives defined on Banach spaces. In this paper, we only concern about the Fréchet derivatives on matrix spaces. Readers may refer to [Col97] for a more thorough treatment.

Definition 8.1. Given a map \( f : M_d \rightarrow M_d \) and Hermitian matrices \( P, Q \), the Fréchet derivative of \( f \) at \( P \) with direction \( Q \) is defined to be

\[
Df(P)(Q) \overset{\text{def}}{=} \frac{d}{dt}f(P + tQ)|_{t=0}.
\]

The \( k \)-th order Fréchet derivative of \( f \) at \( P \) with direction \( (Q_1, \ldots, Q_k) \) is defined to be

\[
D^k f(P)(Q_1, \ldots, Q_k) \overset{\text{def}}{=} \frac{d}{dt}D^{k-1}f(P + tQ_k)(Q_1, \ldots, Q_{k-1})|_{t=0}.
\]

The Fréchet derivatives share many common properties with the derivatives in Euclidean spaces, such as linearity, composition rules, etc. The most basic properties are summarized in Appendix A.

The function we are interested in this paper is the function \( \zeta : \mathbb{R} \rightarrow \mathbb{R} \) defined as follows, which was also studied in [Mos10, MOO10].

\[
\zeta(x) \overset{\text{def}}{=} \begin{cases} 
    x^2 & \text{if } x \leq 0 \\
    (x - 1)^2 & \text{if } x \geq 1 \\
    0 & \text{otherwise}
\end{cases}.
\]

(19)

The main reason to study \( \zeta(\cdot) \) is that \( \text{Tr } \zeta(\cdot) \) characterizes the minimum \( \|\cdot\|_2 \)-distance between a Hermitian matrix and the set of all positive semidefinite matrices with eigenvalues at most 1.

Lemma 8.2. Given \( M \in \mathcal{H}_d \), let \( \Delta = \{ X \in \mathcal{H}_d : 0 \leq X \leq 1 \} \) with the rounding map \( \mathcal{R} \). It holds that

\[
\text{Tr } \zeta(M) = \|M - \mathcal{R}(M)\|_2^2.
\]

Proof. Without loss of generality, we assume that \( M \) is diagonal. Let

\[
X_0 = \arg \min \left\{ \|M - X\|_2^2 : X \in \Delta \right\}.
\]

The lemma follows by easy calculation if \( X_0 \) is also diagonal. We now show that \( X_0 \) is indeed diagonal. Note that

\[
\|M - X\|_2^2 = \text{Tr } X_0^2 + \text{Tr } M^2 - \sum_i \lambda_i(M) X_0(i,i).
\]
It is known that $(X_0 (11), \ldots, X_0 (d, d))$ is majorized by $(\lambda_1 (X), \ldots, \lambda_n (X))$ \cite{Bha97}. Namely, $\sum_{j=1}^i \lambda_j (X) \geq \sum_{j=1}^i X_0 (j, j)$. Note that $X_0 \geq 0$. It is easy to verify that
\[
\sum_i \lambda_i (M) X_0 (i, i) \leq \sum_i \lambda_i (M) \lambda_i (X),
\]
the equality is achieved only if $X_0$ is also diagonal.

Note that the function $\zeta$ is in $\mathcal{C}^1$ but not in $\mathcal{C}^2$. We define a $\mathcal{C}^2$-approximation of $\zeta$ in the following, whose Fréchet derivatives are easier to calculate comparing with the $\mathcal{C}^\infty$-approximation considered in \cite{Mos10, MOO10}. For any $0 \leq \lambda < 1$, define $\zeta_{\lambda} : \mathbb{R} \to \mathbb{R}$ to be

\[
\zeta_{\lambda} (x) \overset{\text{def}}{=} \begin{cases} 
  x^2 + \frac{1}{3} \lambda^2 & \text{if } x \leq -\lambda \\
  \frac{(\lambda - x)^3}{6\lambda} & \text{if } -\lambda \leq x \leq \lambda \\
  0 & \text{if } \lambda \leq x \leq 1 - \lambda \\
  \frac{(x - 1 + \lambda)^3}{6\lambda} & \text{if } 1 - \lambda \leq x \leq 1 + \lambda \\
  (1 - x)^2 + \frac{1}{3} \lambda^2 & \text{if } x \geq 1 + \lambda.
\end{cases}
\] (20)

The following lemma can be verified by elementary calculus.

**Lemma 8.3.** For any $0 < \lambda < 1$, it holds that

1. $\|\zeta_{\lambda} - \zeta\|_\infty \leq 4\lambda^2$.
2. $\zeta_{\lambda} \in \mathcal{C}^2$. $\zeta_{\lambda}' (\cdot)$ is a piecewisely linear function. $\zeta_{\lambda}''' (\cdot)$ exists in $\mathbb{R}$ except for finite points. And $|\zeta_{\lambda}''' (x)| \leq \frac{1}{3}$ for any $x$ where $\zeta_{\lambda}''' (x)$ exists.

**Lemma 8.4.** For any Hermitian $P, Q$, $0 < \lambda < 1/2$, it holds that
\[
\text{Tr} \ zeta_{\lambda} (P + Q) = \text{Tr} \ zeta_{\lambda} (P) + \text{Tr} \ D\zeta_{\lambda} (P) (Q) + \frac{1}{2} \text{Tr} \ D^2 \zeta_{\lambda} (P) (Q) + O \left( \frac{\|Q\|_2 \|Q\|_4^2}{\lambda} \right). \quad (21)
\]

The proof is via calculating each order of the Fréchet derivatives, which is deferred to Appendix C.

The following lemma enables us to remove the part of an operator with low 2-norm without changing the value of $\text{Tr} \ zeta (\cdot)$ much. The proof is also deferred to Appendix C.

**Lemma 8.5.** For any Hermitian matrices $P$ and $Q$, it holds that $|\text{Tr} \ (\zeta (P + Q) - \zeta (P))| \leq 4 \left( \|P\|_2 \|Q\|_2 + \|Q\|_2^2 \right)$.

\footnote{The definition of $\zeta_{\lambda}$ is derived from the following construction.}

\[
\psi (x) \overset{\text{def}}{=} \begin{cases} 
  \frac{1}{2} & \text{if } -1 \leq x \leq 1 \\
  0 & \text{otherwise.}
\end{cases}
\]

\[
f (x) = x \cdot 1_{x \geq 0}.
\]

\[
\psi_{\lambda} (x) = \psi (x / \lambda) / \lambda,
\]

for $0 < \lambda \leq \frac{1}{2}$.

\[
f_{\lambda} \overset{\text{def}}{=} f \ast \psi_{\lambda}.
\]

\[
\zeta_{\lambda} (x) \overset{\text{def}}{=} \begin{cases} 
  2 \int_{-\infty}^{-} f_{\lambda} (t) \, dt & \text{if } x \leq 1/2 \\
  2 \int_{-1}^{-1} f_{\lambda} (t) \, dt & \text{if } x \geq 1/2.
\end{cases}
\]
9 Hypercontractive inequality of random operators

We first introduce a noise operator $\Gamma_\rho$ acting on $L^2 \left( \mathcal{M}_2^{\otimes h}, \gamma_n \right)$, which is a hybrid of the Ornstein-Uhlenbeck operator $U_\rho$ in Definition 2.3 and the noise operator $T_\rho$ in Definition 5.4. Recall that any $P \in L^2 \left( \mathcal{M}_2^{\otimes h}, \gamma_n \right)$ can be expressed as

$$P = \sum_{\sigma \in \{4\}^{h_{\geq 0}}} p_\sigma (g) B_\sigma,$$

where $\{B_i\}_{i=0}^3$ is a standard orthonormal basis in $\mathcal{M}_2$. $p_\sigma \in L^2 (\mathbb{C}, \gamma_n)$ and $g \sim N (0,1)^{\otimes h}$.

**Definition 9.1.** Given $0 \leq \rho \leq 1$ and integers $h, n \geq 0$, the noise operator $\Gamma_\rho : L^2 \left( \mathcal{M}_2^{\otimes h}, \gamma_n \right) \to L^2 \left( \mathcal{M}_2^{\otimes h}, \gamma_n \right)$ is defined to be

$$\Gamma_\rho (P) = \sum_{\sigma \in \{4\}^{h_{\geq 0}}} (U_\rho p_\sigma) (g) T_\rho (B_\sigma),$$

where $\{B_i\}_{i=0}^3$ is a standard orthonormal basis in $\mathcal{M}_2$.

The following lemma directly follows from Fact 2.5 and Lemma 5.5 item 2.

**Lemma 9.2.** Given $0 \leq \rho \leq 1$, integers $n, h \geq 0$ and a random operator $P \in L^2 \left( \mathcal{M}_2^{\otimes h}, \gamma_n \right)$ that has expansion in Eq. (22), it holds that

$$N_4 (\Gamma_\rho (P)) \leq N_2 (P),$$

where $\Gamma_\rho$ is the noise operator acting on $L^2 \left( \mathcal{M}_2^{\otimes h}, \gamma_n \right)$ defined in Definition 9.1 and $N_p$ is the normalized $p$-norm of a random operator in Definition 2.23.

The main result in this section is a hypercontractive inequality of random operators stated as follows.

**Lemma 9.3.** Given $0 \leq \rho \leq \frac{1}{\sqrt{3}}$, integers $n, h \geq 0$, for any multilinear random operator $P \in L^2 \left( \mathcal{M}_2^{\otimes h}, \gamma_n \right)$, it holds that

$$N_4 (\Gamma_\rho (P)) \leq N_2 (P),$$

where $\Gamma_\rho$ is the noise operator acting on $L^2 \left( \mathcal{M}_2^{\otimes h}, \gamma_n \right)$ defined in Definition 9.1 and $N_p$ is the normalized $p$-norm of a random operator in Definition 2.23.

The concept of random operators is a hybrid of the operators in $\mathcal{M}_2^{\otimes h}$ and the the functions in the Gaussian space $L^2 (\mathbb{C}, \gamma_n)$. Thus the proof of Lemma 9.3 is a combination of the hypercontractive inequality of unital quantum operators due to King [Kin14] and the hypercontractive inequality of Gaussian variables [Wol07, MOO10]. The proof is deferred to the end of this section. The following is an application of Lemma 9.3.

---

3From Lemma 2.8, the definition is independent of the choices of the basis as the Gaussian distribution $N (0,1)^n$ is invariant under orthogonal transformation.
Lemma 9.4. Given integers \( h, n \geq 0 \), for any multilinear random operator \( P \in L^2 \left( \mathcal{M}_2^h, \gamma_n \right) \) with the associated vector-valued polynomial \( p = (p_\sigma)_{\sigma \in [4]^h} \), it holds that

\[
N_4 (P) \leq 3^{d/2} N_2 (P),
\]

where \( d = \max_{\sigma \in [4]^h} (\deg (p_\sigma) + |\sigma|) \).

Proof. Suppose \( P = \sum_{\sigma \in [4]^h} p_\sigma (g) \mathcal{B}_\sigma \), where \( \{ \mathcal{B}_i \}_{i=0}^3 \) is a standard orthonormal basis in \( \mathcal{M}_2 \), \( p_\sigma \in L^2 (\mathbb{C}, \gamma_n) \) is multilinear and \( g \sim N (0, 1) \otimes n \). Set

\[
P = \sum_{(\sigma, \tau) \in [4]^h \times [4]^n : |\sigma| + |\tau| = i} p_\sigma (g) \mathcal{B}_\sigma.
\]

Applying Lemma 9.3

\[
N_4 (P) = N_4 \left( \frac{1}{\sqrt{3}} \left( \sum_{i=1}^d \left( \sqrt{3} \right)^i P^{=i} \right) \right) \leq N_2 \left( \sum_{i=1}^d \left( \sqrt{3} \right)^i P^{=i} \right) = \mathbb{E} \left[ \left\| \sum_{i=1}^d \left( \sqrt{3} \right)^i P^{=i} \right\|_2^2 \right]^{1/2}.
\]

Note that

\[
\mathbb{E} \left[ \text{Tr} \left( P^{=i} \right) \right] = 0,
\]

whenever \( i \neq j \). Therefore,

\[
N_4 (P) = \left( \sum_{i=1}^d \left( \sqrt{3} \right)^{2i} \mathbb{E} \left[ \left\| P^{=i} \right\|_2^2 \right] \right)^{1/2} \leq 3^{d/2} \left( \sum_{i=1}^d \mathbb{E} \left[ \left\| P^{=i} \right\|_2^2 \right] \right)^{1/2} = 3^{d/2} N_2 (P).
\]

Definition 9.5. For any map \( \Phi : \mathcal{M}_d \to \mathcal{M}_d \) and \( 1 \leq p \leq q \leq \infty \), the \( p \)-to-\( q \) norm of \( \Phi \) is defined to be

\[
\| \Phi \|_{p \to q} \overset{\text{def}}{=} \max_{M \in \mathcal{M}_d} \frac{\| \Phi (M) \|_q}{\| M \|_p}.
\]

The following fact is a quantum hypercontractive inequality due to King.

Fact 9.6. Let \( T_\rho : \mathcal{M}_2 \to \mathcal{M}_2 \) be a noise operator in Definition 5.4. For any integer \( n \geq 1 \), it holds that

\[
\left\| T_\rho ^{\otimes n} \right\|_{2 \to 4} = 1,
\]

for any \( 0 \leq \rho \leq \sqrt{\frac{1}{3}} \).

The following fact is a well known hypercontractive inequality in Gaussian space.
Fact 9.7. For any $0 \leq \rho \leq \frac{1}{\sqrt{3}}$, it holds that
\[ \sup_{f \in L^2(\mathbb{C}, \gamma_n)} \|U_\rho f\|_4 \leq \|f\|_2, \]
where $U_\rho$ is the Ornstein-Uhlenbeck operator acting on $L^2(\mathbb{C}, \gamma_n)$ Definition 2.3.

The following lemma is a generalization of Fact 9.7 for technical purposes.

Lemma 9.8. Given $p_1, \ldots, p_n \in L^2(\mathbb{C}, \gamma_n)$, it holds that
\[ \mathbb{E}_{x \sim N(0,1)^n} \left[ \left( \sum_{i=1}^n |(U_\rho p_i) (x)|^2 \right)^2 \right]^{\frac{1}{4}} \leq \mathbb{E}_{x \sim N(0,1)^n} \left[ \sum_{i=1}^n |p_i (x)|^2 \right]^{\frac{1}{2}}. \]

Proof. Let $q_i \overset{\text{def}}{=} U_\rho p_i$. Then
\[
\begin{align*}
\mathbb{E}_{x \sim N(0,1)^n} \left[ \left( \sum_{i=1}^n |(U_\rho p_i) (x)|^2 \right)^2 \right]^{\frac{1}{4}} &= \left( \sum_{i=1}^n \mathbb{E}_{x \sim N(0,1)^n} \left[ q_i (x) \right]^4 \right)^{\frac{1}{4}} + \sum_{i \neq j} \mathbb{E}_{x} \left[ |q_i (x)|^2 q_j (x) |^2 \right]^{\frac{1}{4}} \\
&\leq \left( \sum_{i=1}^n \|p_i\|_2^4 + \sum_{i \neq j} \|p_i\|_2^2 \|p_j\|_2^2 \right)^{\frac{1}{4}} \quad \text{(Cauchy-Schwarz inequality)} \\
&\leq \left( \sum_{i=1}^n \|p_i\|_2^2 \right)^{\frac{1}{2}} \quad \text{(Fact 9.7)} \\
&= \mathbb{E}_{x \sim N(0,1)^n} \left[ \sum_{i=1}^n |p_i (x)|^2 \right]^{\frac{1}{2}}.
\end{align*}
\]

It is now ready to prove Lemma 9.3.

The result in [Wol07, MOO10] is for $f \in L^2(\mathbb{R}, \gamma_n)$. But it can be extended to $L^2(\mathbb{C}, \gamma_n)$ easily. Let $f = f_1 + i \cdot f_2 \in L^2(\mathbb{C}, \gamma_n)$, where $f_i \in L^2(\mathbb{R}, \gamma_n)$. Then
\[
\begin{align*}
\|U_\rho f\|_4 &= \mathbb{E}_{x \sim N(0,1)^n} \left[ |(U_\rho f) (x)|^4 \right]^{\frac{1}{4}} \\
&= \mathbb{E}_{x \sim N(0,1)^n} \left[ |(U_\rho f_1) (x)|^4 + |(U_\rho f_2) (x)|^4 + 2 |(U_\rho f_1) (x)|^2 |(U_\rho f_2) (x)|^2 \right]^{\frac{1}{4}} \\
&\leq \left( \|U_\rho f_1\|_4^4 + \|U_\rho f_2\|_4^4 + 2 \|U_\rho f_1\|_2^2 \|U_\rho f_2\|_2^2 \right)^{\frac{1}{4}} \quad \text{(Cauchy-Schwarz inequality)} \\
&\leq \left( \|f_1\|_4^4 + \|f_2\|_4^4 + 2 \|f_1\|_2^2 \|f_2\|_2^2 \right)^{\frac{1}{4}} \quad \text{(Hypercontractive inequality in } L^2(\mathbb{R}, \gamma_n)) \\
&= \left( \|f_1\|_2^2 + \|f_2\|_2^2 \right)^{\frac{1}{2}} = \|f\|_2.
\end{align*}
\]
Proof of Lemma 9.3. Set $C = \Gamma_\rho (P)$. Let $P = \sum_{\sigma \in [4]_0} p_\sigma (g) B_\sigma$, where $\{B_i\}_{i=0}^3$ is a standard orthonormal basis. Set $Q = \sum_{\sigma \in [4]_0} (U_\rho p_\sigma) (g) B_\sigma$. Then by the definition of $\Gamma_\rho$,
\[ \Gamma_\rho (P) = T_\rho (Q). \]
Using Fact 9.6
\[ N_4 (\Gamma_\rho (P)) = E \left[ \left\| T_\rho (Q) \right\|_4^4 \right] \leq E \left[ \left\| Q \right\|_2^4 \right]^{1/4}. \]
Let $p_{ij} \in L^2 (C, \gamma_n)$ and $q_{ij} \in L^2 (C, \gamma_n)$ be the entries of $P$ and $Q$, respectively, for $1 \leq i, j \leq 2^h$. Then $q_{ij} = U_\rho p_{ij}$. Notice that
\[ N_4 (Q) = E \left[ \left\| Q \right\|_2 \right]^{1/4} = \frac{E}{x \sim N(0,1)^n} \left[ \sum_{ij} |q_{ij} (x)|^2 \right] \leq \frac{E}{x \sim N(0,1)^n} \left[ \sum_{ij} |p_{ij} (x)|^2 \right]^{1/2} = N_2 (P), \]
where the inequality is from Lemma 9.8. We conclude the result.

\[ \square \]

10 Quantum invariance principle

Throughout this section we define the following joint random variables.
\[ \left( g_{1,0}, g_{1,1}, g_{1,2}, g_{1,3}, \ldots, g_{n,0}, g_{n,1}, g_{n,2}, g_{n,3} \right) \sim \left( \{1\} \times N (0,1)^3 \right)^{\otimes h}. \]
For any $0 \leq i \leq n$, define the hybrid basis and the hybrid random operators
\begin{align*}
\mathcal{X}_\sigma^{(i)} &\overset{\text{def}}{=} (g_{1,\sigma_1}p_0) \otimes (g_{2,\sigma_2}p_0) \otimes \cdots \otimes (g_{i,\sigma_i}p_0) \otimes p_{\sigma > i} \quad (24) \\
M^{(i)} &\overset{\text{def}}{=} \sum_{\sigma \in [4]^n} \tilde{M} (\sigma) \mathcal{X}_\sigma^{(i)}. \quad (25)
\end{align*}

Lemma 10.1. $M^{(i)}$ is independent of the choices of the basis. Namely, for any standard orthonormal basis $\{B_i\}_{i=0}^3$ in $M_2$ and $M = \sum_{\sigma \in [4]^n} \lambda_\sigma B_\sigma$, set $N = \sum_{\sigma \in [4]^n} \lambda_\sigma \left( \prod_{j=1}^3 g_{j,\sigma_j} \right) B_0^{\otimes i} \otimes B_{\sigma > i}$. Then $N = M^{(i)}$.

Proof. From Fact 2.8, all orthonormal basis are equivalent up to orthogonal transformations. The lemma follows from the well known fact that the Gaussian distribution $N (0,1)^n$ is invariant under any orthogonal transformation. \[ \square \]

Lemma 10.2. For any integer $n > 0$ and $M \in M_2^{\otimes n}$, it holds that
\[ \left| E \left[ \text{Tr} \zeta_\lambda \left( M^{(i+1)} \right) - \text{Tr} \zeta_\lambda \left( M^{(i)} \right) \right] \right| \leq O \left( \frac{2^{nd} \lambda^{3/2}}{\lambda} \text{Inf}_1 (M)^{3/2} \right), \]
where $d = \deg M$.

Proof. Note that
\begin{align*}
M^{(i)} &= \sum_{\sigma, \sigma_i = 0} \tilde{M} (\sigma) \mathcal{X}_\sigma^{(i)} + \sum_{\sigma, \sigma_i \neq 0} \tilde{M} (\sigma) \mathcal{X}_\sigma^{(i)}, \\
M^{(i+1)} &= \sum_{\sigma, \sigma_i = 0} \tilde{M} (\sigma) \mathcal{X}_\sigma^{(i+1)} + \sum_{\sigma, \sigma_i = 0} \tilde{M} (\sigma) \mathcal{X}_\sigma^{(i+1)},
\end{align*}

with $d = \deg M$.\[ \square \]
and
\[ \sum_{\sigma, \sigma_i=0} \tilde{M} (\sigma) \chi_{(i)}^\sigma = \sum_{\sigma, \sigma_i=0} \tilde{M} (\sigma) \chi_{(i+1)}^\sigma. \]

Set
\[
A \overset{\text{def}}{=} \sum_{\sigma, \sigma_i=0} \tilde{M} (\sigma) \chi_{(i)}^\sigma,
\]
\[
B \overset{\text{def}}{=} \sum_{\sigma, \sigma_i=0} \tilde{M} (\sigma) \chi_{(i)}^\sigma,
\]
\[
C \overset{\text{def}}{=} \sum_{\sigma, \sigma_i=0} \tilde{M} (\sigma) \chi_{(i+1)}^\sigma.
\]

Then we have
\[ M^{(i)} = A + B; \quad M^{(i+1)} = A + C. \]

From Lemma 8.4
\[
\left\| E \left[ \text{Tr} \, \zeta \left( M^{(i+1)} \right) - \text{Tr} \, \zeta \left( M^{(i)} \right) \right] \right\|
\leq E \left[ \left( \text{Tr} \, D \zeta \left( A \right) \left( C \right) + \frac{1}{2} \text{Tr} \, D^2 \zeta \left( A \right) \left( C \right) \right) - \right]
\left( \text{Tr} \, D \zeta \left( A \right) \left( B \right) + \frac{1}{2} \text{Tr} \, D^2 \zeta \left( A \right) \left( B \right) \right]
+ O \left( E \left[ \| C \|_2 \| C \|_2^2 \left( \frac{\lambda}{\lambda} \right) \right] + O \left( E \left[ \| B \|_2 \| B \|_2^2 \left( \frac{\lambda}{\lambda} \right) \right] \right)
\leq E \left[ \left( \text{Tr} \, D \zeta \left( A \right) \left( C \right) + \frac{1}{2} \text{Tr} \, D^2 \zeta \left( A \right) \left( C \right) \right) - \right]
\left( \text{Tr} \, D \zeta \left( A \right) \left( B \right) + \frac{1}{2} \text{Tr} \, D^2 \zeta \left( A \right) \left( B \right) \right]
+ O \left( \frac{2n}{\lambda} \left( N_2 \left( C \right) N_4 \left( C \right)^2 + N_2 \left( B \right) N_4 \left( B \right)^2 \right) \right)
\leq O \left( \frac{2n}{\lambda} \left( N_2 \left( C \right) N_4 \left( C \right)^2 + N_2 \left( B \right) N_4 \left( B \right)^2 \right) \right). \quad \text{(Cauchy-Schwarz inequality)}
\]

Applying Lemma 9.4 we have
\[
\left\| E \left[ \text{Tr} \, \zeta \left( M^{(i+1)} \right) - \text{Tr} \, \zeta \left( M^{(i)} \right) \right] \right\| \leq O \left( \frac{2^{d_n/2} n}{\lambda} \left( N_2 \left( B \right)^3 + N_2 \left( C \right)^3 \right) \right). \]

Notice that
\[
N_2 \left( B \right) = N_2 \left( C \right) = \left( \sum_{\sigma, \sigma_i=0} \left| \tilde{M} (\sigma) \right|^2 \right)^{1/2} = \text{Inf}_1 \left( M \right)^{1/2}.
\]

Therefore,
\[
\left\| E \left[ \text{Tr} \, \zeta \left( M^{(i+1)} \right) - \text{Tr} \, \zeta \left( M^{(i)} \right) \right] \right\| \leq O \left( \frac{2^{d_n/2} n}{\lambda} \text{Inf}_1 \left( M \right)^{3/2} \right).
\]

\[ \Box \]

**Claim 10.3.**
\[
E \left[ \text{Tr} \, B f \left( A \right) \right] = E \left[ \text{Tr} \, C f \left( A \right) \right] \quad \text{(26)}
\]
\[
E \left[ \text{Tr} \, B f \left( A \right) B g \left( A \right) \right] = E \left[ \text{Tr} \, C f \left( A \right) C g \left( A \right) \right] \quad \text{(27)}
\]

for any \( f, g \in L^2 \left( \mathbb{R}, \gamma_1 \right) \).
Claim 10.4. It holds that
\[ \mathbb{E}[(\text{Tr } D\zeta_{\lambda}(A)(C))] = \mathbb{E}[(\text{Tr } D\zeta_{\lambda}(A)(B))]; \]
\[ \mathbb{E}[(\text{Tr } D^2\zeta_{\lambda}(A)(C))] = \mathbb{E}[(\text{Tr } D^2\zeta_{\lambda}(A)(B))]. \]

The proofs of the both claims above are deferred to Appendix D. Combining Lemma 10.2 and Lemma 8.3, we conclude the following lemma.

Lemma 10.5. Given \( M \in \mathcal{H}^{\otimes n} \), let \( \{B_i\}_{i=0}^3 \) be a standard orthonormal basis in \( \mathcal{M}_2 \). Then for any \( 0 < \lambda \leq \frac{1}{2} \) and \( H \subseteq [n] \), it holds that
\[ \left| \mathbb{E} \left[ \text{Tr} \left( \sum_{\sigma \in [4]_{\geq 0}^n} \tilde{M}(\sigma) \bigotimes_{i \in H} B_{i_{\sigma_i}} \otimes \bigotimes_{i \notin H} g_{i_{\sigma_i}} I_2 \right) \right] - \text{Tr} \zeta(M) \right| \leq 2^n \left( 8\lambda^2 + \frac{3d}{\lambda} \sum_{i \notin H} \text{Inf}_i(M)^{3/2} \right), \]
where \( d = \text{deg } M \).

Lemma 10.6. Given \( M \in \mathcal{H}^{\otimes n} \), \( 0 \leq M \leq 1 \) \( H \subseteq [n] \), an integer \( d > 0 \) and standard orthonormal basis \( \mathcal{B} = \{B_i\}_{i=0}^3 \), suppose \( \text{Inf}_i(M) \leq \tau \) for all \( i \notin H \) and \( \|M^{>d}\|_2 \leq \delta 2^n \), where \( M^{>d} \) is defined in Definition 2.15. Set
\[ M \overset{\text{def}}{=} \sum_{\sigma \in [4]_{>0}^n; |\sigma| \leq d} \tilde{M}(\sigma) \bigotimes_{i \in H} B_{i_{\sigma_i}} \otimes \bigotimes_{i \notin H} g_{i_{\sigma_i}} I_2. \]
Then it holds that
\[ \mathbb{E} \left[ \text{Tr} \zeta(M) \right] \leq O \left( 2^n \left( \left( 3d\sqrt{\tau d} \right)^{2/3} + \sqrt{\delta} \right) \right). \]

Proof. Without loss of generality, we may assume that \( H = \{1, 2, \ldots, s\} \). Then
\[ \mathbb{E} \left[ \text{Tr} \zeta(M) \right] \leq \mathbb{E} \left[ \text{Tr} \left( \zeta(M) - \zeta(M^{\leq d}) \right) \right] + \text{Tr} \zeta(M^{\leq d}) \tag{28} \]
Applying Lemma 10.5,

\[ \left| \mathbb{E} \left[ \text{Tr} \left( \zeta(M) - \zeta(M^{\leq d}) \right) \right] \right| \leq O \left( 2^n \left( \lambda^2 + \frac{3d}{\lambda} \sum_{i \notin H} \text{Inf}_i(M^{\leq d})^{3/2} \right) \right) \]
\[ \leq O \left( 2^n \left( \lambda^2 + \frac{3d\sqrt{\tau d}}{\lambda} \text{Inf}(M^{\leq d}) \right) \right) \]
\[ \leq O \left( 2^n \left( \lambda^2 + \frac{3d\sqrt{\tau d}}{\lambda} \right) \right), \tag{29} \]
where the last inequality is from Lemma 2.17 item 4.

Note that \( \zeta(M) = 0 \) since \( 0 \leq M \leq 1 \). Again applying Lemma 8.5,
\[ \text{Tr} \zeta(M^{\leq d}) = \left| \text{Tr} \zeta(M^{\leq d}) - \text{Tr} \zeta(M) \right| \leq 4 \left( \|M\|_2 \|M^{>d}\|_2 + \|M^{>d}\|_2^2 \right) \leq \sqrt{\delta} 2^{n+3}. \tag{30} \]
Combing Eqs. (28)–(30), we have

$$
E \left[ \text{Tr} \left( M^{\leq d} \right) \right] \leq O \left( 2^n \left( \frac{3d\sqrt{d}}{\lambda} + 8\sqrt{\delta} \right) \right).
$$

Choosing \( \lambda = \left( \frac{3d\sqrt{d}}{\tau} \right)^{1/3} \), we conclude the result.

The following lemma converts random operators to operators.

**Lemma 10.7.** Given integers \( d, h, n > 0 \) and a degree-\( d \) multilinear random operators \( M \in L^2 (\mathcal{H}^{\otimes h}_2, \gamma_n) \) with the associated vector-valued function \( p : \mathbb{R}^n \to \mathbb{R}^d \) under a standard orthonormal basis \( \{\mathcal{B}_i\}_{i=0}^3 \). Then there exists \( M^{(1)} \in \mathcal{F}^{\otimes (n+h)}_2 \) satisfying that

$$
\left| E[2^h \cdot \text{Tr} (M)] - \text{Tr} \left( M^{(1)} \right) \right| \leq O \left( 2^{n+h} \left( \frac{3d}{\tau^{3/2}} \right)^{2/3} \right).
$$

In particular, if for all \( i, \text{Inf}_i (p) \leq \tau \) for some \( \tau \in (0, 1) \), then from Lemma 2.25,

$$
\left| E[2^h \cdot \text{Tr} (M)] - \text{Tr} \left( M^{(1)} \right) \right| \leq O \left( 2^{n+h} \left( \frac{3d\sqrt{d}}{\tau} \right)^{2/3} \right).
$$

**Proof.** Let \( \{\mathcal{B}_i\}_{i=0}^3 \) be an arbitrary standard orthonormal basis in \( \mathcal{B}_2 \). Let \( \{\mathcal{g}_i\}_{i=1}^n \) be the random variables in \( M \). Substitute each random variable by \( \mathcal{B}_1 \) and the products of random variables by tensor products. The following proof is same as the one of Lemma 10.5.

We finally reach the main lemma in this section.

**Lemma 10.8.** Given \( 0 < \tau, \delta, \rho < 1 \), integers \( n > h > 0, d > 0 \), \( P, Q \in \mathcal{F}^{\otimes n}_2, 0 \leq P, Q \leq 1 \), \( H \subseteq [n] \) of size \( |H| = h \), a noisy EPR state \( \psi_{AB} \) with the maximal correlation \( \rho = \rho (\psi_{AB}) \), suppose \( \text{Inf}_i (P) \leq \tau \) and \( \text{Inf}_i (Q) \leq \tau \) for all \( i \notin H \) and \( \| P^{>d} \|_2^2 \leq 2^n, \| Q^{>d} \|_2^2 \leq 2^n \). Then there exist maps \( f, g : \mathcal{F}^{\otimes n}_2 \times \mathbb{R}^{n-h} \to \mathcal{F}^{\otimes h}_2 \) satisfying that

$$
\left( P^{(1)}, Q^{(1)} \right) \overset{def}{=} (f(P, g(Q)), g(Q, h))_{(\mathcal{g}_i)_{i=1}^{\otimes (n-h)}} \in L^2 \left( \mathcal{F}^{\otimes h}_2, \gamma_3 (n-h) \right) \times L^2 \left( \mathcal{F}^{\otimes h}_2, \gamma_3 (n-h) \right)
$$

are degree-\( d \) multilinear joint random operators with the joint random variables drawn from \( \mathcal{G}^{\otimes (n-h)}_p \). And

1. \( N_2 \left( P^{(1)} \right) \leq \| P \|_2 \) and \( N_2 \left( Q^{(1)} \right) \leq \| Q \|_2 \);
2. \( \text{Tr} (P \otimes Q) \psi_{AB}^{\otimes h} = E \left[ \text{Tr} \left( (P \otimes Q) \psi_{AB}^{\otimes h} \right) \right] \).
3. \( 2^{n-h} E[|P^{(1)}|] = E[|P^{(1)}|] = E[|P|] \).
4. \( \text{Tr} \left( P^{(1)} \right) \leq O \left( 2^h \left( \frac{3d\sqrt{d}}{\tau} \right)^{2/3} + \sqrt{\delta} \right) \) and \( \text{Tr} \left( Q^{(1)} \right) \leq O \left( 2^h \left( \frac{3d\sqrt{d}}{\tau} \right)^{2/3} + \sqrt{\delta} \right) \).

**Proof.** Let \( \{A_i\}_{i=0}^3 \) and \( \{\mathcal{B}_i\}_{i=0}^3 \) be standard orthonormal basis in \( \mathcal{B}_2 \) satisfying that \( \text{Tr} (A_i \otimes \mathcal{B}_j) \psi_{AB} = c_i c_j \) for \( 0 \leq i, j \leq 3 \), where \( 1 = c_0 > c_1 = \rho \geq c_2 \geq c_3 \). Set joint random variables

$$
\left( (\mathcal{g}_{i,0}^{(0)}, \mathcal{h}_{i,0}^{(0)}), (\mathcal{g}_{i,1}^{(0)}, \mathcal{h}_{i,1}^{(0)}), (\mathcal{g}_{i,2}^{(0)}, \mathcal{h}_{i,2}^{(0)}), (\mathcal{g}_{i,3}^{(0)}, \mathcal{h}_{i,3}^{(0)}) \right)_{i=1}^h \sim ((1, 1)) \times \mathcal{G}_c \times \mathcal{G}_c \times \mathcal{G}_c \otimes \mathcal{G}^{\otimes h}_p.
$$

Define

$$
P^{(0)} \overset{def}{=} \sum_{\sigma \in [4]_0^h} \hat{P} (\sigma) \prod_{i \notin H} g_{i,\sigma}^{(0)} A_{i,\sigma}^{H},
$$

38
and
\[ Q^{(0)} \overset{\text{def}}{=} \sum_{\sigma \in [4]_{\geq 0}}^{\sigma} \hat{Q}(\sigma) \prod_{i \notin H} h_{i,\sigma}, B_{\sigma}. \]

Then
\[ N_2 \left( P^{(0)} \right)^2 = \sum_{\sigma} |\hat{P}(\sigma)|^2 = \| P \|_2^2, N_2 \left( Q^{(0)} \right)^2 = \sum_{\sigma} |\hat{Q}(\sigma)|^2 = \| Q \|_2^2, \]

and
\[ \text{Tr} \ (P \otimes Q) \psi_{AB}^{\otimes h} = \mathbb{E} \left[ \text{Tr} \left( \left( P^{(0)} \otimes Q^{(0)} \right) \psi_{AB}^{\otimes h} \right) \right] = \sum_{\sigma \in [4]_{\geq 0}} c_{\sigma}, \]

and
\[ 2^n \mathbb{E} \left[ \text{Tr} \ P^{(0)} \right] = \text{Tr} \ P, 2^n \mathbb{E} \left[ \text{Tr} \ Q^{(0)} \right] = \text{Tr} \ Q. \]

From Lemma 10.6,
\[ \mathbb{E} \left[ \text{Tr} \ \zeta \left( P^{(0)} \right) \right] \leq O \left( 2^h \left( 3^d \sqrt{Q} + \sqrt{d} \right) \right) \quad \text{and} \quad \mathbb{E} \left[ \text{Tr} \ \zeta \left( Q^{(0)} \right) \right] \leq O \left( 2^h \left( 3^d \sqrt{Q} + \sqrt{d} \right) \right). \]

However, the correlation of \( \bigl( g^{(0)}_{i,j}, h^{(0)}_{i,j} \bigr) \) is not exactly the one we need. Given \( \bigl( g_t, h_t \bigr)_{i=1}^{3(n-h)} \sim \mathcal{G}_{\rho}^{\otimes (n-h)} \), we perform the following substitution in \( P^{(0)} \) and \( Q^{(0)} \)
\[ g^{(0)}_{i,b} \left\{ \begin{array}{ll} 1, & \text{if } b = 0 \\ g_{3(i-1)+b}, & \text{otherwise} \end{array} \right. \quad \text{and} \quad h^{(0)}_{i,b} \left\{ \begin{array}{ll} 1, & \text{if } b = 0 \\ h_{3(i-1)+b}, & \text{otherwise} \end{array} \right. \]

to get \( P \) and \( Q \), respectively. The items 1, 2, 3 still hold obviously. To argue item 4, note that the absolute values of all Fourier coefficients do not increase. By Lemma 10.5 item 4 follows. \( \square \)

Analogously, the following lemma converts joint random operators back to operators.

Lemma 10.9. Given \( 0 \leq \rho < 1, \delta, \tau \in (0,1) \), integers \( n, h > 0 \), a noisy EPR state \( \psi_{AB} \) with the maximal correlation \( \rho = \rho(\psi_{AB}) \), standard orthonormal basis \( \{ A_i \}_{i=0}^3 \) and \( \{ B_i \}_{i=0}^3 \) in \( \mathcal{M}_2 \), there exist maps \( f, g : L^2(\mathcal{H}_2^\otimes, \gamma_2) \rightarrow L^2(\mathcal{H}_2^\otimes, \gamma_2) \) such that for any degree-\( d \) multilinear joint random operators
\[ (P, Q) = \left( \sum_{\sigma \in [4]_{\geq 0}}^{\sigma} p_{\sigma} (g) A_{\sigma}, \sum_{\sigma \in [4]_{\geq 0}}^{\sigma} q_{\sigma} (h) B_{\sigma} \right) \in L^2(\mathcal{H}_2^\otimes \gamma_2) \times L^2(\mathcal{H}_2^\otimes \gamma_2), \]

satisfying that
\[ (\forall i \in [n]) : \sum_{\sigma \in [4]_{\geq 0}}^{\sigma} \inf_i (p_{\sigma}) \leq \tau \quad \text{and} \quad \sum_{\sigma \in [4]_{\geq 0}}^{\sigma} \inf_i (q_{\sigma}) \leq \tau. \]

Let \( (P, Q) = (f(P), g(Q)). \) The following holds.

1. \( \text{Tr} \ P = 2^n \mathbb{E}[\text{Tr} \ P] \) and \( \text{Tr} \ Q = 2^n \mathbb{E}[\text{Tr} \ Q]. \)

2. \( \text{Tr} \ (P \otimes Q) \psi_{AB}^{\otimes(n+h)} = \mathbb{E} \left[ \text{Tr} \ (P \otimes Q) \psi_{AB}^{\otimes h} \right]. \)

3. \( N_2(P) = \| P \|_2 \) and \( N_2(Q) = \| Q \|_2. \)
4. \[ |E[2^n \cdot \text{Tr} \, \zeta(P)] - \text{Tr} \, \zeta(P)| \leq O \left( 2^{n+h} \left( 3^d d^2 \right)^{2/3} \right). \]

and
\[ |E[2^n \cdot \text{Tr} \, \zeta(Q)] - \text{Tr} \, \zeta(Q)| \leq O \left( 2^{n+h} \left( 3^d d^2 \right)^{2/3} \right). \]

Proof. From Proposition [5.8], let \( A \) and \( B \) be the Hermitian matrices achieved the maximal correlation of \( \psi_{AB} \) in Definition [5.6]. Substitute each pair \( (g_i, h_i) \) by \((A, B)\) and the products of random variables by tensor products. The following proof is same as the one of Lemma [10.8]. \( \square \)

11 Dimension reduction for random operators

The following is the main lemma in this section.

**Lemma 11.1.** Given parameters \( \rho \in [0, 1], \delta, \alpha > 0 \), integers \( d, n, h > 0 \), a noisy EPR state \( \psi_{AB} \) with the maximal correlation \( \rho = \rho(\psi_{AB}) \), and degree-\( d \) multilinear joint random operators

\[
(P, Q) = \left( \sum_{\sigma \in [4]^b} p_{\sigma}(g) A_{\sigma}, \sum_{\sigma \in [4]^b} q_{\sigma}(h) B_{\sigma} \right) \quad \in \mathcal{L}^2(L^2(\mathcal{H}_2^n, \gamma_n)) \times \mathcal{L}^2(L^2(\mathcal{H}_2^n, \gamma_n)),
\]

where \( \{A_i\}_{i=0}^3 = \{B_i\}_{i=0}^3 \) are both standard orthonormal basis in \( \mathcal{M}_2 \), then there exists an explicitly computable \( n_0 = n_0(d, h, \delta) \), and joint random operators

\[
(P_{M}^{(1)}, Q_{M}^{(1)}) = \left( \sum_{\sigma \in [4]^b} p_{\sigma,M}(x) A_{\sigma}, \sum_{\sigma \in [4]^b} q_{\sigma,M}(y) B_{\sigma} \right) \quad \in \mathcal{L}^2(L^2(\mathcal{H}_2^n, \gamma_n)),
\]

for any \( M \in \mathbb{R}^{n \times D} \) such that the following holds.

If we sample \( M \sim N(0, 1)^{n \times D} \), then with probability at least \( 1 - \delta - 2\alpha \), it holds that

1. \( N_2\left(P_{M}^{(1)} \right) \leq (1 + \delta) N_2(P) \) and \( N_2\left(Q_{M}^{(1)} \right) \leq (1 + \delta) N_2(Q) \);
2. \( |E_P[\text{Tr} \, P_{M}^{(1)}] - E_P[\text{Tr} \, P]| \leq \delta 2^h N_2(P) \) and \( |E_Q[\text{Tr} \, Q_{M}^{(1)}] - E_Q[\text{Tr} \, Q]| \leq \delta 2^h N_2(Q) \);
3. \( |E_P[\text{Tr} \, \zeta \left(P_{M}^{(1)} \right)] \leq \frac{1}{\sqrt{m}} E_P[\text{Tr} \, \zeta \left(P \right)] \) and \( |E_Q[\text{Tr} \, \zeta \left(Q_{M}^{(1)} \right)] \leq \frac{1}{\sqrt{m}} E_Q[\text{Tr} \, \zeta \left(Q \right)] \);
4. \( |E_P.Q[\text{Tr} \, \left(P_{M}^{(1)} \otimes Q_{M}^{(1)} \right)] - E_P.Q[\text{Tr} \, \left(P \otimes Q \right) \psi_{AB}^{\otimes h}]| \leq \delta N_2(P) N_2(Q) \).

In particular, one may take \( n_0 = \frac{4^{4d+5}d^{O(d)}}{\delta} \).

**Fact 11.2.** [GKR15] Given parameters \( n, d \in \mathbb{Z}_{>0}, \rho \in [0, 1] \) and \( \delta > 0 \), there exists an explicitly computable \( D = D(d, \delta) \) such that the following holds.

For any \( n \) and any degree-\( d \) multilinear polynomials \( f, g : \mathbb{R}^n \to \mathbb{R} \), and \( M \in \mathbb{R}^{n \times D} \), define functions \( f_M, g_M : \mathbb{R}^D \to \mathbb{R} \) as
\[
f_M(x) \equiv f \left( \frac{Mx}{\|x\|_2} \right) \quad \text{and} \quad g_M(x) \equiv g \left( \frac{Mx}{\|x\|_2} \right). \tag{32}
\]

Then
\[
\Pr_{M \sim N(0, 1)^{l(n \times D)}} \left( |\langle f_M, g_M \rangle_{\ell^2} - \langle f, g \rangle_{\ell^2} | < \delta \|f\|_2 \|g\|_2 \right) \geq 1 - \delta.
\]

In particular, one may take \( D \geq \frac{1}{d^{O(d)}} \).
Choosing \( g = 1 \), we get

\[
\Pr_{M \sim N((0,1) \otimes (n \times D))} \left[ \left| \hat{f}_M(0) - \hat{f}(0) \right| < \delta \|f\|_2 \right] \geq 1 - \delta. \tag{33}
\]

If \( f \) and \( g \) are identical and \( \rho = 1 \), we have

\[
\Pr_{M \sim N((0,1) \otimes (n \times D))} \left[ \|f_M\|^2_2 - \|f\|^2_2 \leq \delta \|f\|^2_2 \right] \geq 1 - \delta; \tag{34}
\]

\[
\Pr_{M \sim N((0,1) \otimes (n \times D))} \left[ \|g_M\|^2_2 - \|g\|^2_2 \leq \delta \|g\|^2_2 \right] \geq 1 - \delta. \tag{35}
\]

**Fact 11.3.** \([GKR18]\) Given integers \( n, k, D > 0 \), let \( f \in L^2(\mathbb{R}^k, \gamma_n) \), and \( \Delta \) be a convex body in \( \mathbb{R}^k \) with rounding map \( \mathcal{R} \) defined in Section 2.5. Let \( f_M : \mathbb{R}^D \to \mathbb{R}^k \) be defined analogously to Eq. (32). It holds that,

\[
\Pr_{M \sim N((0,1) \otimes (n \times D))} \left[ \|f_M \circ f - f\|_2 \leq \frac{1}{\delta} \|f_M \circ f - f\|_2 \right] \geq 1 - \delta^2,
\]

for \( 0 < \delta < 1 \).

We are now ready to prove the main lemma.

**Proof of Lemma 11.1.** From Lemma 7.3 and Lemma 10.1, we may assume that the basis \( \{A_i\}_{i=0}^3 \) and \( \{B_i\}_{i=0}^3 \) satisfy \( \text{Tr} (A_i \otimes B_j) \psi_{AB} = c_i \delta_{i,j} \) without loss of generality. Then from Lemma 7.2,

\[
\mathbb{E}_{(g,h) \sim G^\otimes n - \mathcal{H}} \left[ \text{Tr} \left( P(g) \otimes Q(h) \right) (\psi_{AB})^\otimes n \right] = \sum_{\sigma \in [4]^\otimes n} c_\sigma \langle p_\sigma, q_\sigma \rangle_{\mathcal{G}^\otimes n},
\]

where \( c_\sigma \equiv c_{\sigma_1} \cdots c_{\sigma_n} \).

Applying Fact 11.2 with \( \delta \leftarrow \frac{\delta^2}{4^{k+2}}, n_0 \leftarrow \frac{4^{3k+4D^2(d)} \delta^2}{\delta^2} \) and the union bound, it holds

\[
\Pr_{M \sim N((0,1)^n \times n_0)} \left[ \left( \forall \sigma \in [4]^b \right) \left( \|p_{\sigma,M}, q_{\sigma,M}\|_{\mathcal{G}^\otimes n_0} - \langle p_{\sigma}, q_\sigma \rangle_{\mathcal{G}^\otimes n} \right) \leq \delta \|p_\sigma\|_2 \|q_\sigma\|_2 \right] \geq 1 - \delta/16, \tag{36}
\]

and

\[
\Pr_{M \sim N((0,1)^n \times n_0)} \left[ \left( \forall \sigma \in [4]^b \right) \left( \|p_{\sigma,M}(0) - \hat{p}_\sigma(0)\| \leq \delta \|p_\sigma\|_2 \right] \geq 1 - \delta/16, \tag{37}
\]

and

\[
\Pr_{M \sim N((0,1)^n \times n_0)} \left[ \left( \forall \sigma \in [4]^b \right) \left( \|q_{\sigma,M}(0) - \hat{q}_\sigma(0)\| \leq \delta \|q_\sigma\|_2 \right] \geq 1 - \delta/16, \tag{38}
\]

and

\[
\Pr_{M \sim N((0,1)^n \times n_0)} \left[ \left( \forall \sigma \in [4]^b \right) \left( \|p_{\sigma,M}\|^2_2 - \|p_\sigma\|^2_2 \leq \delta \|p_\sigma\|^2_2 \right] \geq 1 - \delta, \tag{39}
\]

and

\[
\Pr_{M \sim N((0,1)^n \times n_0)} \left[ \left( \forall \sigma \in [4]^b \right) \left( \|q_{\sigma,M}\|^2_2 - \|q_\sigma\|^2_2 \leq \delta \|q_\sigma\|^2_2 \right] \geq 1 - \delta, \tag{40}
\]

41
Define
\[
(\mathbf{P}^{(1)}_M, \mathbf{P}^{(1)}_M) \overset{\text{def}}{=} \left( \sum_{\sigma \in [4]^h \geq 0} p_{\sigma,M} (g) A_{\sigma}, \sum_{\sigma \in [4]^h \geq 0} q_{\sigma,M} (h) B_{\sigma} \right)_{(g,h) \sim S^n_{\rho} \otimes n^0}.
\]

For any \( M \) satisfying Eq. (36), we have
\[
\Pr_{M \sim N(0,1)^n \times n^0} \left[ \left\| \mathbf{P}^{(1)}_M - \mathbf{Q}^{(1)}_M \right\| \leq \delta N_2 (\mathbf{P}) N_2 (\mathbf{Q}) \right] \geq 1 - \frac{\delta}{16}. \tag{41}
\]

For any \( M \) satisfying Eq. (37),
\[
\Pr_{M \sim N(0,1)^n \times n^0} \left[ \left\| \mathbf{P}^{(1)}_M \right\| \leq \delta 2^h \| p_0 \|_2 \right] \geq 1 - \frac{\delta}{16}. \tag{42}
\]

Symmetrically, we have
\[
\Pr_{M \sim N(0,1)^n \times n^0} \left[ \left\| \mathbf{Q}^{(1)}_M \right\| \leq \delta 2^h \| q_0 \|_2 \right] \geq 1 - \frac{\delta}{16}. \tag{43}
\]

For any \( M \) satisfying Eq. (39),
\[
N_2 \left( \mathbf{P}^{(1)}_M \right)^2 = \sum_{\sigma \in [4]^h \geq 0} \| p_{\sigma,M} \|_2^2 \leq (1 + \delta) \sum_{\sigma \in [4]^h \geq 0} \| p_{\sigma} \|_2^2 = (1 + \delta) N_2 (\mathbf{P})^2,
\]

where the both equalities are from Lemma 2.24. Hence
\[
\Pr_{M \sim N(0,1)^n \times n^0} \left[ N_2 \left( \mathbf{P}^{(1)}_M \right) \leq (1 + \delta) N_2 (\mathbf{P}) \right] \geq 1 - \frac{\delta}{16}. \tag{44}
\]

Symmetrically, we have
\[
\Pr_{M \sim N(0,1)^n \times n^0} \left[ N_2 \left( \mathbf{Q}^{(1)}_M \right) \leq (1 + \delta) N_2 (\mathbf{Q}) \right] \geq 1 - \frac{\delta}{16}. \tag{45}
\]
In particular, one may take such that, for any joint random operators $\mathbf{P} \in L^2\left(3\mathcal{C}_{2}^{\otimes h}, \gamma_2\right)$ with the associated vector-valued function $p$, \(\|\mathcal{R} \circ p - p\|_2^2 = \frac{1}{2\pi} \text{Tr} \, \zeta\left(\mathbf{P}\right)\) due to Lemma 8.2. Hence Fact 11.3 implies that

\[
\Pr_{\mathbf{M} \sim N(0,1)^{n \times n_0}}\left[\text{Tr} \, \zeta\left(\mathbf{P}\right) \leq \frac{1}{\sqrt{\mathcal{G}}} \text{Tr} \, \zeta\left(\mathbf{P}^{(1)}\right)\right] \geq 1 - \alpha. \tag{46}
\]

Applying the same argument to $\mathbf{Q}$ and $\mathbf{Q}^{(1)}$, we have

\[
\Pr_{\mathbf{M} \sim N(0,1)^{n \times n_0}}\left[\text{Tr} \, \zeta\left(\mathbf{Q}\right) \leq \frac{1}{\sqrt{\mathcal{G}}} \text{Tr} \, \zeta\left(\mathbf{Q}^{(1)}\right)\right] \geq 1 - \alpha. \tag{47}
\]

Again applying the union bound to Eqs. (41) (42) (43) (44) (46) (47), with probability at least $1 - \delta - 2\alpha$ over $\mathbf{M} \sim N(0,1)^{n \times D}$ all the events in Eqs. (41) (42) (43) (44) (45) (46) (47) occur. Setting $p_{\sigma,\mathbf{M}} = p_{\sigma,\mathbf{M}}$ and $q_{\sigma,\mathbf{M}} = q_{\sigma,\mathbf{M}}$, we conclude the lemma.

\[\square\]

12 Smoothing random operators

The main result in this section is the following, which is a generalization of Lemma 6.1 to random operators.

**Lemma 12.1.** Given integers $n, h > 0$, a noisy EPR state $\psi_{AB}$ with the maximal correlation $\rho_{AB}^{\text{def}} = \rho(\psi_{AB}) < 1$, there exist an explicit $d = d(\rho, \delta)$ and a map $f : L^2\left(3\mathcal{C}_{2}^{\otimes h}, \gamma_n\right) \to L^2\left(3\mathcal{C}_{2}^{\otimes h}, \gamma_n\right)$ such that, for any joint random operators $(\mathbf{P}, \mathbf{Q}) \in L^2\left(3\mathcal{C}_{2}^{\otimes h}, \gamma_n\right) \times L^2\left(3\mathcal{C}_{2}^{\otimes h}, \gamma_n\right)$ with random variables drawn from $\mathcal{C}_{\rho}^{\otimes n}$, $(\mathbf{P}^{(1)}, \mathbf{Q}^{(1)}) \overset{\text{def}}{=} (f(\mathbf{P}), f(\mathbf{Q})) \in L^2\left(3\mathcal{C}_{2}^{\otimes h}, \gamma_n\right) \times L^2\left(3\mathcal{C}_{2}^{\otimes h}, \gamma_n\right)$ satisfy the following.

1. $\deg(\mathbf{P}^{(1)}) \leq d$ and $\deg(\mathbf{Q}^{(1)}) \leq d$.
2. $E\left[\text{Tr} \, \mathbf{P}^{(1)}\right] = E[\text{Tr} \, \mathbf{P}]$ and $E\left[\text{Tr} \, \mathbf{Q}^{(1)}\right] = E[\text{Tr} \, \mathbf{Q}]$.
3. $N_2\left(\mathbf{P}^{(1)}\right) \leq N_2(\mathbf{P})$ and $N_2\left(\mathbf{Q}^{(1)}\right) \leq N_2(\mathbf{Q})$.
4. $E\left[\text{Tr} \, \zeta\left(\mathbf{P}^{(1)}\right)\right] \leq 2 \left(E[\text{Tr} \, \zeta(\mathbf{P})] + \delta N_2(\mathbf{P})^2\right)$
   and $E\left[\text{Tr} \, \zeta\left(\mathbf{Q}^{(1)}\right)\right] \leq 2 \left(E[\text{Tr} \, \zeta(\mathbf{P})] + \delta N_2(\mathbf{Q})^2\right)$
5. $E\left[\text{Tr} \, (\mathbf{P} \otimes \mathbf{Q}) \psi_{AB}^{\otimes h}\right] - E\left[\text{Tr} \, (\mathbf{P}^{(1)} \otimes \mathbf{Q}^{(1)}) \psi_{AB}^{\otimes h}\right] \leq \delta N_2(\mathbf{P}) N_2(\mathbf{Q})$.

In particular, one may take $d = O\left(\frac{\log^2 \frac{1}{\delta} (1 - \rho)}{\delta (1 - \rho)}\right)$.

**Fact 12.2.** [GKR18] Let $\rho \in [0,1], \delta > 0, k, n \in \mathbb{Z}_{>0}$ be any given constant parameters, $f, g \in L^2\left(\mathbb{R}^k, \gamma_n\right)$; $\Delta_1, \Delta_2 \subseteq \mathbb{R}^k$ be convex bodies. Set $\mathcal{R}_1$ and $\mathcal{R}_2$ be the rounding maps with respect to $\Delta_1$ and $\Delta_2$, respectively. There exist an explicit $d = d(\rho, \delta)$ and functions $f^{(1)}, g^{(1)} \in L^2\left(\mathbb{R}^k, \gamma_n\right)$, where $f^{(1)}$ only depends on $f$ and $g^{(1)}$ only depends on $g$, such that the following hold.
1. \(f^{(1)}\) and \(g^{(1)}\) are degree at most \(d\).
2. \(\mathbb{E}[f^{(1)}] = \mathbb{E}[f]\) and \(\mathbb{E}[g^{(1)}] = \mathbb{E}[g]\).
3. For any \(i \in [k]\), it holds that \(\|f_i^{(1)}\|_2 \leq \|f_i\|_2\) and \(\|g_i^{(1)}\|_2 \leq \|g_i\|_2\).
4. \(\|R \circ f^{(1)} - f^{(1)}\|_2 \leq \|R \circ f - f\|_2 + \delta \|f\|_2\) and \(\|R \circ g^{(1)} - g^{(1)}\|_2 \leq \|R \circ g - g\|_2 + \delta \|g\|_2\).
5. For every \(i \in [k]\),
\[
\left|\langle f_i(x), g_i(y)\rangle_{\mathbb{P}^{\otimes n}} - \langle f_i^{(1)}(x), g_i^{(1)}(y)\rangle_{\mathbb{P}^{\otimes n}}\right| \leq \delta \|f_i\|_2 \|g_i\|_2.
\]

In particular, one may take \(d = O\left(\frac{\log^2 \frac{1}{\delta(1-\rho)}}{\delta}\right)\).

**Proof of Lemma 12.1.** From Lemma 7.3, there exist standard orthonormal basis \(\{A_i\}_{i=0}^3\) and \(\{B_i\}_{i=0}^3\) satisfying that \(\text{Tr}(A_i \otimes B_j) \psi_{AB} = c_i \delta_{i,j}\), where \(1 = c_0 > c_1 = \rho \geq c_2 \geq c_3 \geq 0\). Let \(p, q : \mathbb{R}^n \to \mathbb{R}^4\) be the associated vector-valued functions of \(P\) and \(Q\) under the basis \(\{A_i\}_{i=0}^3\) and \(\{B_i\}_{i=0}^3\), respectively. Then
\[
P = \sum_{\sigma \in [4]^h_{\geq 0}} p_{\sigma}(g) A_\sigma, \quad \text{and} \quad Q = \sum_{\sigma \in [4]^h_{\geq 0}} q_{\sigma}(h) B_\sigma,
\]
where \((g, h) \sim \mathcal{S}_{\mathbb{P}^{\otimes n}}\). From Lemma 7.2
\[
\mathbb{E}\left[\text{Tr} (P \otimes Q) (\psi_{AB})^{\otimes h}\right] = \sum_{\sigma \in [4]^h_{\geq 0}} c_{\sigma} \langle p_{\sigma}, q_{\sigma} \rangle_{\mathcal{S}_{\mathbb{P}^{\otimes n}}}.
\]
Applying Fact 12.2 to \((p, q)\), we obtain \((p^{(1)}, q^{(1)})\). Define
\[
P^{(1)} \overset{\text{def}}{=} \sum_{\sigma \in [4]^h_{\geq 0}} p_{\sigma}^{(1)}(g) A_\sigma, \quad \text{and} \quad Q^{(1)} \overset{\text{def}}{=} \sum_{\sigma \in [4]^h_{\geq 0}} q_{\sigma}^{(1)}(h) B_\sigma.
\]
Item 1 follows directly.

Note that \(\mathbb{E}\left[\text{Tr} P^{(1)}\right] = 2^h \mathbb{E}\left[p_0^{(1)}\right]\) and \(\mathbb{E}[\text{Tr} P] = 2^h \mathbb{E}[p_0]\). Thus the first equality in item 2 follows from Fact 12.2 item 2. The second equality follows similarly.

To prove item 3, we have that
\[
N_2\left(P^{(1)}\right) = \|p^{(1)}\|_2 \leq \|p\|_2 = N_2\left(P\right),
\]
where the both equalities are from Lemma 2.24, the inequality is from Fact 12.2 item 3. The second part of item 2 in Lemma 12.1 follows by the same argument. To prove item 3, we define
\[
\Delta_1 \overset{\text{def}}{=} \left\{x \in \mathbb{R}^4 : 0 \leq \sum_{\sigma \in [4]^h_{\geq 0}} x_{\sigma} A_\sigma \leq 1\right\},
\]

44
and
\[ \Delta_2 \overset{\text{def}}{=} \left\{ x \in \mathbb{R}^4 : 0 \leq \sum_{\sigma \in \{0\}^4} x_\sigma \mathbb{B}_\sigma \leq 1 \right\}, \]

It is easy to verify that both \( \Delta_1 \) and \( \Delta_2 \) are convex bodies. From Lemma 8.2,
\[
\begin{align*}
\| R \circ p - p \|_2^2 &= \frac{1}{2^n} \mathbb{E}[\text{Tr} \, \zeta (P)] , \\
\| R \circ q - q \|_2^2 &= \frac{1}{2^n} \mathbb{E}[\text{Tr} \, \zeta (Q)] , \\
\| R \circ p^{(1)} - p^{(1)} \|_2^2 &= \frac{1}{2^n} \mathbb{E}[\text{Tr} \, \zeta (P^{(1)})] , \\
\| R \circ q^{(1)} - q^{(1)} \|_2^2 &= \frac{1}{2^n} \mathbb{E}[\text{Tr} \, \zeta (Q^{(1)})] .
\end{align*}
\]

The Fact 12.2 item 4 implies that
\[ \left( \frac{1}{2^n} \mathbb{E}[\text{Tr} \, \zeta (P^{(1)})] \right)^{1/2} \leq \left( \frac{1}{2^n} \mathbb{E}[\text{Tr} \, \zeta (P)] \right)^{1/2} + \delta \|p\|_2 . \]

Note that \( \|p\|_2^2 = N_2 (P)^2 \) by Lemma 2.24. Taking square on both sides of the inequality above, we conclude the first inequality in Lemma 12.1 item 3. The second inequality follows exactly same.

To prove item 4, consider
\[
\begin{align*}
&\left| \mathbb{E} \left[ \text{Tr} (P \otimes Q) \psi_{AB}^{\otimes n} \right] - \mathbb{E} \left[ \text{Tr} (P^{(1)} \otimes Q^{(1)}) \psi_{AB}^{\otimes n} \right] \right| \\
&= \left| \sum_{\sigma \in \{0\}^4} c_\sigma \left( \langle p_\sigma, q_\sigma \rangle_{G^{\otimes n}} - \langle p^{(1)}_\sigma, q^{(1)}_\sigma \rangle_{G^{\otimes n}} \right) \right| \\
&\leq \delta \sum_{\sigma \in \{0\}^4} \|p_\sigma\|_2 \|q_\sigma\|_2 \quad \text{(Fact 12.2 item 5) and } c_\sigma \leq 1 \text{ due to Lemma 7.3} \\
&\leq \delta \left( \sum_{\sigma \in \{0\}^4} \|p_\sigma\|_2^2 \right)^{1/2} \left( \sum_{\sigma \in \{0\}^4} \|q_\sigma\|_2^2 \right)^{1/2} \\
&= \delta N_2 (P) N_2 (Q) \quad \text{(Lemma 2.24).}
\end{align*}
\]

\[ \square \]

13 Multilinearization of random operators

**Lemma 13.1.** Given \( 0 \leq \rho < 1 \), integers \( d,h,n > 0 \), a noisy EPR state \( \psi_{AB} \) with the maximal correlation \( \rho = \rho (\psi_{AB}) \), there exists a map \( f : L^2 \left( \mathcal{H}_2^{\otimes h}, \gamma_n \right) \to L^2 \left( \mathcal{H}_2^{\otimes h}, \gamma_{\text{int}} \right) \) such that, for any degree-\( d \) joint random operators
\[
(P, Q) = \left( \sum_{\sigma \in \{0\}^4} p_\sigma (g) A_\sigma, \sum_{\sigma \in \{0\}^4} q_\sigma (h) B_\sigma \right) \in L^2 \left( \mathcal{H}_2^{\otimes h}, \gamma_n \right) \times L^2 \left( \mathcal{H}_2^{\otimes h}, \gamma_n \right) ,
\]
where $\{A_i\}_{i=0}^3$ and $\{B_i\}_{i=0}^3$ are standard orthonormal basis in $M_2$,  
\[ \left( P^{(1)}, Q^{(1)} \right) \overset{\text{def}}{=} (f(P), f(Q)) \]
\[ = \left( \sum_{\sigma \in [4]^h} p^{(1)}_{\sigma}(x) A_{\sigma}, \sum_{\sigma \in [4]^h} q^{(1)}_{\sigma}(y) B_{\sigma} \right) \in L^2 \left( \mathcal{H}_2^{\otimes \rho}, \gamma_{nt} \right) \times L^2 \left( \mathcal{H}_2^{\otimes \rho}, \gamma_{nt} \right) \]
are multilinear joint random operators. It further holds that  
1. Both $\deg \left( P^{(1)} \right)$ and $\deg \left( Q^{(1)} \right)$ are at most $d$.  
2. For all $(i, j) \in [n] \times [t]$ and $\sigma \in [4]^h \geq 0$, $\Inf_{in+j} \left( p^{(1)}_{\sigma} \right) \leq \delta \cdot \Inf_{i} \left( p_{\sigma} \right)$ and $\Inf_{in+j} \left( q^{(1)}_{\sigma} \right) \leq \delta \cdot \Inf_{i} \left( q_{\sigma} \right)$.
3. $N_2 \left( P^{(1)} \right) \leq N_2 \left( P \right)$ and $N_2 \left( Q^{(1)} \right) \leq N_2 \left( Q \right);$  
4. $E \left[ \Tr P^{(1)} \right] = E[\Tr P]$ and $E \left[ \Tr Q^{(1)} \right] = E[\Tr Q]$.  
5. $\left| E \left[ \Tr \zeta \left( P^{(1)} \right) \right] - E \left[ \Tr \zeta \left( P \right) \right] \right| \leq \delta 2^{h+2} N_2 \left( P \right)^2$ and $\left| E \left[ \Tr \zeta \left( Q^{(1)} \right) \right] - E \left[ \Tr \zeta \left( Q \right) \right] \right| \leq \delta 2^{h+2} N_2 \left( Q \right)^2$.  
6. $\left| E \left[ \Tr \left( P^{(1)} \otimes Q^{(1)} \right) \psi_{AB}^{\otimes h} \right] - E \left[ \Tr \left( P \otimes Q \right) \psi_{AB}^{\otimes h} \right] \right| \leq \delta N_2 \left( P \right) N_2 \left( Q \right)$.

In particular, we may take $t = O \left( \frac{d^2}{\delta^2} \right)$.  

**Definition 13.2.** Suppose $f \in L^2 \left( \mathbb{R}^n, \gamma_n \right)$ is given by the Hermite expansion $f = \sum_{\sigma \in \mathbb{Z}_{\geq 0}^2} \hat{f} (\sigma) H_{\sigma}$. The multilinear truncation of $f$ is defined to be the function $f^{ml} \in L^2 \left( \mathbb{R}^n, \gamma_n \right)$ given by
\[ f^{ml} \overset{\text{def}}{=} \sum_{\sigma \in \{0,1\}^n} \hat{f} (\sigma) H_{\sigma} (x). \]

**Fact 13.3.** [GKR15] Given parameters $\rho \in [0, 1], \delta > 0$ and $d \in \mathbb{Z}_{\geq 0}$, there exists $t = t \left( d, \delta \right)$ such that the following holds:

Let $f, g \in L^2 \left( \mathbb{R}^n, \gamma_n \right)$ be degree-$d$ polynomials. There exist polynomials $\tilde{f}, \tilde{g} \in L^2 \left( \mathbb{R}^{nt}, \gamma_{nt} \right)$ over variables $\tilde{x} \overset{\text{def}}{=} \left\{ x^{(i)} : (i, j) \in [n] \times [t] \right\}$ and $\tilde{y} \overset{\text{def}}{=} \left\{ y^{(i)} : (i, j) \in [n] \times [t] \right\}$, respectively as $\tilde{f} \left( \tilde{x} \right) \overset{\text{def}}{=} f \left( x^{(1)}, \ldots, x^{(n)} \right)$ and $\tilde{g} \left( \tilde{y} \right) \overset{\text{def}}{=} g \left( y^{(1)}, \ldots, y^{(n)} \right)$.

Let $\tilde{f}^{ml}$ and $\tilde{g}^{ml}$ be the multilinear truncations of $\tilde{f}$ and $\tilde{g}$, respectively. Then the following holds:
1. $\tilde{f}^{ml}$ and $\tilde{g}^{ml}$ are multilinear with degree $d$.  
2. $\Var \left[ \tilde{f}^{ml} \right] \leq \Var[f]$ and $\Var \left[ \tilde{g}^{ml} \right] \leq \Var[g]$.  
3. $\left\| \tilde{f}^{ml} \right\|_2 \leq \left\| \tilde{f} \right\|_2 = \left\| f \right\|_2$ and $\left\| \tilde{g}^{ml} \right\|_2 \leq \left\| \tilde{g} \right\|_2 = \left\| g \right\|_2$.  

46
4. Given two independent distributions $g \sim N(0,1)^n$ and $x \sim N(0,1)^n$. The distributions of $f(g)$ and $f(x)$ are identical. The distributions of $g(g)$ and $g(x)$ are identical.

5. $\|\hat{f} - \hat{f}^m\|_2 \leq \frac{\delta}{2} \|f\|_2$ and $\|\hat{g} - \hat{g}^m\|_2 \leq \frac{\delta}{2} \|g\|_2$

6. For all $(i,j) \in [n] \times [t]$, it holds that $\inf_{x_j^{(i)}} \left( \hat{f}^m \right) \leq \delta \cdot \inf_i (f)$ and $\inf_{y_j^{(i)}} \left( \hat{g}^m \right) \leq \delta \cdot \inf_i (g)$.

7. $\hat{f}(0) = \hat{f}^m(0)$ and $\hat{g}(0) = \hat{g}^m(0)$

8. $\left( \hat{f}^m, \hat{g}^m \right)_{\rho_\alpha^m} \leq \left( f, g \right)_{\rho_\alpha}$

In particular, we may take $t = O \left( \frac{\eta^2}{\delta^2} \right)$.

**Proof of Lemma 13.1** Applying Fact 13.3 to $\{ p_\sigma \}_{\sigma \in \mathbb{R}^{[4]}_{\geq 0}}$ and $\{ q_\sigma \}_{\sigma \in \mathbb{R}^{[4]}_{\geq 0}}$, we get $\{ p_\sigma \}_{\sigma \in \mathbb{R}^{[4]}_{\geq 0}}$ and $\{ q_\sigma \}_{\sigma \in \mathbb{R}^{[4]}_{\geq 0}}$. Define joint random operators

$$\left( P, Q \right) \overset{\text{def}}{=} \left( \sum_{\sigma \in \mathbb{R}^{[4]}_{\geq 0}} p_\sigma(x) A_\sigma, \sum_{\sigma \in \mathbb{R}^{[4]}_{\geq 0}} q_\sigma(y) B_\sigma \right) \in L^2 \left( \mathcal{G}_{\rho_\alpha^m} \times \mathcal{G}_{\rho_\alpha} \right)$$

Let $p_\sigma^{(1)}(\cdot) \overset{\text{def}}{=} p_{\rho_\alpha^m}(\cdot)$ and $q_\sigma^{(1)}(\cdot) \overset{\text{def}}{=} q_{\rho_\alpha}(\cdot)$. Define

$$\left( P^{(1)}, Q^{(1)} \right) \overset{\text{def}}{=} \left( \sum_{\sigma \in \mathbb{R}^{[4]}_{\geq 0}} p_\sigma^{(1)}(x) A_\sigma, \sum_{\sigma \in \mathbb{R}^{[4]}_{\geq 0}} q_\sigma^{(1)}(y) B_\sigma \right)$$

Items 1 and item 2 are implied by Fact 13.3 item 1 and item 5, respectively. Item 3 is from Lemma 2.24 and the item 3 in Fact 13.3. Item 4 follows from the fact that $E[P] = 2^{h_0}(0)$ and $E[Q] = 2^{h_0}(0)$ and the item 7 in Fact 13.3.

We will prove the first inequality in item 5. The second one can be proved similarly. Define a convex body

$$\Delta \overset{\text{def}}{=} \left\{ x \in \mathbb{R}^{[4]}_{\geq 0} : 0 \leq \sum_{\sigma \in \mathbb{R}^{[4]}_{\geq 0}} x_\sigma A_\sigma \leq 1 \right\}$$

with the rounding map $\mathcal{R}$. Set $p \overset{\text{def}}{=} (p_\sigma)_{\sigma \in \mathbb{R}^{[4]}_{\geq 0}}$, $\bar{p} \overset{\text{def}}{=} (\bar{p}_\sigma)_{\sigma \in \mathbb{R}^{[4]}_{\geq 0}}$ and $p^{(1)} \overset{\text{def}}{=} (p_\sigma^{(1)})_{\sigma \in \mathbb{R}^{[4]}_{\geq 0}}$ to be vector-valued functions. Then by Lemma 8.2

$$\|p - \mathcal{R} \circ p\|_2^2 = \frac{1}{2^h} E[\text{Tr} \, \zeta \left( P \right)] \quad \text{and} \quad \|p^{(1)} - \mathcal{R} \circ p^{(1)}\|_2^2 = \frac{1}{2^h} E[\text{Tr} \, \zeta \left( P^{(1)} \right)]$$
Hence

\[
\frac{1}{2^n} |\mathbb{E}\left[ \text{Tr} \left( \xi \left( P^{(1)} \right) \right) - \mathbb{E}[\text{Tr} \left( \xi (P) \right)] \right] = \left| \left\| p^{(1)} - \mathcal{R} \circ p^{(1)} \right\|^2_2 - \left\| p - \mathcal{R} \circ p \right\|^2_2 \right|
\]

\[
= \left| \left\| p^{nl} - \mathcal{R} \circ \bar{p}^{nl} \right\|^2_2 - \left\| \bar{p} - \mathcal{R} \circ \bar{p} \right\|^2_2 \right| \quad (\text{Fact 13.3 item 4})
\]

\[
= \left( \left\| p^{nl} - \mathcal{R} \circ p^{nl} \right\|_2 - \left\| \bar{p} - \mathcal{R} \circ \bar{p} \right\|_2 \right) \left( \left\| p^{nl} - \mathcal{R} \circ p^{nl} \right\|_2 + \left\| \bar{p} - \mathcal{R} \circ \bar{p} \right\|_2 \right)
\]

\[
\leq 4 \| p \|_2 \left\| \bar{p} - p^{nl} \right\|_2 \quad (\text{Triangle inequality})
\]

\[
\leq 8 \| p \|_2 \| \bar{p} - p^{nl} \|_2 \quad (\text{Fact 2.26} \text{ and } \mathcal{R} \text{ is a contraction map})
\]

\[
\leq 4\delta \| p \|_2^2 \quad (\text{Fact 13.3 item 5})
\]

\[
= 4\delta N_2 (P)^2 \quad (\text{Lemma 2.24}).
\]

To prove item 7, consider

\[
\frac{1}{2^n} |\mathbb{E}\left[ \text{Tr} \left( \psi_{AB} \right) \psi_{\mathcal{H}} \right] - \mathbb{E}\left[ \text{Tr} \left( \mathcal{P}^{(1)} \otimes \mathcal{Q}^{(1)} \right) \psi_{\mathcal{H}} \right] | = \sum_{\sigma \in [4]^h \geq 0} c_\sigma \left( \langle p_\sigma, q_\sigma \rangle_{\mathcal{G}^A} - \langle p_\sigma^{(1)}, q_\sigma^{(1)} \rangle_{\mathcal{G}^A} \right)
\]

\[
\leq \delta \sum_{\sigma \in [4]^h \geq 0} \| p_\sigma \|_2 \| q_\sigma \|_2 \quad (\text{Fact 13.3 item 8})
\]

\[
\leq \delta \left( \sum_{\sigma \in [4]^h \geq 0} \| p_\sigma \|_2 \right)^{1/2} \left( \sum_{\sigma \in [4]^h \geq 0} \| q_\sigma \|_2 \right)^{1/2}
\]

\[
= \delta N_2 (P) N_2 (Q) \quad (\text{Lemma 2.24}).
\]

\[\square\]

References

[ALM⁺98] Sanjeev Arora, Carsten Lund, Rajeev Motwani, Madhu Sudan, and Mario Szegedy. Proof verification and the hardness of approximation problems. J. ACM, 45(3):501–559, May 1998.

[Ara02] P.K. Aravind. A simple demonstration of Bell’s theorem involving two observers and no probabilities or inequalities. arXiv preprint arXiv:0206070, 2002.

[AS98] Sanjeev Arora and Shmuel Safra. Probabilistic checking of proofs: A new characterization of np. J. ACM, 45(1):70–122, January 1998.

[Bab85] László Babai. Trading group theory for randomness. In Proceedings of the Seventeenth Annual ACM Symposium on Theory of Computing, STOC ’85, pages 421–429, New York, NY, USA, 1985. ACM.

48
[Bec75] William Beckner. Inequalities in Fourier analysis. *Annals of Mathematics*, 102(1):159–182, 1975.

[Bei13] Salman Beigi. A new quantum data processing inequality. *Journal of Mathematical Physics*, 54(8):082202, 2013.

[Bel64] J. S. Bell. On the Einstein Podolsky Rosen paradox. *Physics Physique Fizika*, 1:195–200, Nov 1964.

[BFL91] László Babai, Lance Fortnow, and Carsten Lund. Non-deterministic exponential time has two-prover interactive protocols. *Computational Complexity*, 1(1):3–40, Mar 1991.

[Bha97] Rajendra Bhatia. *Matrix Analysis*. Springer, New York, New York, NY, 1997.

[BOGKW88] Michael Ben-Or, Shafi Goldwasser, Joe Kilian, and Avi Wigderson. Multi-prover interactive proofs: How to remove intractability assumptions. In *Proceedings of the Twentieth Annual ACM Symposium on Theory of Computing*, STOC ’88, pages 113–131, New York, NY, USA, 1988. ACM.

[Bon70] Aline Bonami. Étude des coefficients de Fourier des fonctions de $L^p(g)$. *Annales de l’Institut Fourier*, 20(2):335–402, 1970.

[BRdW08] A. Ben-Aroya, O. Regev, and R. d. Wolf. A hypercontractive inequality for matrix-valued functions with applications to quantum computing and lds. In *2008 49th Annual IEEE Symposium on Foundations of Computer Science*, pages 477–486, Oct 2008.

[CHSH69] John F. Clauser, Michael A. Horne, Abner Shimony, and Richard A. Holt. Proposed experiment to test local hidden-variable theories. *Phys. Rev. Lett.*, 23:880–884, Oct 1969.

[CHTW04] Richard Cleve, Peter Hoyer, Benjamin Toner, and John Watrous. Consequences and limits of nonlocal strategies. In *Proceedings of the 19th IEEE Annual Conference on Computational Complexity*, CCC ’04, pages 236–249, Washington, DC, USA, 2004. IEEE Computer Society.

[Cir80] B. S. Cirel’son. Quantum generalizations of Bell’s inequality. *Letters in Mathematical Physics*, 4(2):93–100, Mar 1980.

[CKMT15] Toby Cubitt, Michael Kastoryano, Ashley Montanaro, and Kristan Temme. Quantum reverse hypercontractivity. *Journal of Mathematical Physics*, 56(10):102204, 2015.

[Col97] Rodney Coleman. *Calculus on Normed Vector Spaces*. Springer-Verlag, New York, New York, NY, 1997.

[Con76] A. Connes. Classification of injective factors cases $\Pi_1$, $\Pi_\infty$, III$_\lambda$, $\lambda \neq 1$. *Annals of Mathematics*, 104(1):73–115, 1976.

[DB14] Payam Delgosha and Salman Beigi. Impossibility of local state transformation via hypercontractivity. *Communications in Mathematical Physics*, 332(1):449–476, Nov 2014.

[DMN18] Anindya De, Elchanan Mossel, and Joe Neeman. Non interactive simulation of correlated distributions is decidable. In *Proceedings of the Twenty-Ninth Annual ACM-SIAM Symposium on Discrete Algorithms*, SODA ’18, pages 2728–2746, Philadelphia, PA, USA, 2018. Society for Industrial and Applied Mathematics.

[DPP19] Ken Dykema, Vern I. Paulsen, and Jitendra Prakash. Non-closure of the set of quantum correlations via graphs. *Communications in Mathematical Physics*, 365(3):1125–1142, Feb 2019.
Joseph Fitzsimons, Zhengfeng Ji, Thomas Vidick, and Henry Yuen. Quantum proof systems for iterated exponential time, and beyond. In Proceedings of the 51st Annual ACM SIGACT Symposium on Theory of Computing, STOC 2019, New York, NY, USA, 2019. ACM.

Tobias Fritz. Tsirelson’s problem and Kirchberg’s conjecture. Reviews in Mathematical Physics, 24(05):1250012, 2012.

Hans Gebelein. Das statistische problem der korrelation als variations- und eigenwertproblem und sein zusammenhang mit der ausgleichsrechnung. ZAMM - Journal of Applied Mathematics and Mechanics, 21(6):364–379, 1941.

Peter Gacs and J Körner. Common information is far less than mutual information. Problems of Control and Information Theory, 2, Jan 1973.

Badih Ghazi, Prithish Kamath, and Prasad Raghavendra. Dimension reduction for polynomials over gaussian space and applications. In Proceedings of the 33rd Computational Complexity Conference, CCC ’18, pages 28:1–28:37, Germany, 2018. Schloss Dagstuhl–Leibniz-Zentrum fuer Informatik.

Badih Ghazi, Prithish Kamath, and Madhu Sudan. Decidability of non-interactive simulation of joint distributions. In 2016 IEEE 57th Annual Symposium on Foundations of Computer Science (FOCS), pages 545–554, Los Alamitos, CA, USA, Oct 2016. IEEE Computer Society.

Shafi Goldwasser, Silvio Micali, and Charles Rackoff. The knowledge complexity of interactive proof-systems. In Proceedings of the Seventeenth Annual ACM Symposium on Theory of Computing, STOC ’85, pages 291–304, New York, NY, USA, 1985. ACM.

Gus Gutoski and John Watrous. Toward a general theory of quantum games. In Proceedings of the Thirty-ninth Annual ACM Symposium on Theory of Computing, STOC ’07, pages 565–574, New York, NY, USA, 2007. ACM.

Johan Hastad. Some optimal inapproximability results. J. ACM, 48(4):798–859, July 2001.

H. O. Hirschfeld. A connection between correlation and contingency. Mathematical Proceedings of the Cambridge Philosophical Society, 31(4):520–524, 1935.

Tsuyoshi Ito, Hirotada Kobayashi, and Keiji Matsumoto. Oracularization and two-prover one-round interactive proofs against nonlocal strategies. In Proceedings of the 2009 24th Annual IEEE Conference on Computational Complexity, CCC ’09, pages 217–228, Washington, DC, USA, 2009. IEEE Computer Society.

Tsuyoshi Ito and Thomas Vidick. A multi-prover interactive proof for nexp sound against entangled provers. In Proceedings of the 2012 IEEE 53rd Annual Symposium on Foundations of Computer Science, FOCS ’12, pages 243–252, Washington, DC, USA, 2012. IEEE Computer Society.

Zhengfeng Ji. Classical verification of quantum proofs. In Proceedings of the Forty-eighth Annual ACM Symposium on Theory of Computing, STOC ’16, pages 885–898, New York, NY, USA, 2016. ACM.

Zhengfeng Ji. Compression of quantum multi-prover interactive proofs. In Proceedings of the 49th Annual ACM SIGACT Symposium on Theory of Computing, STOC 2017, pages 289–302, New York, NY, USA, 2017. ACM.

Rahul Jain, Zhengfeng Ji, Sarvagya Upadhyay, and John Watrous. QIP = PSPACE. J. ACM, 58(6):30:1–30:27, December 2011.
[JNP+11] M. Junge, M. Navascues, C. Palazuelos, D. Perez-Garcia, V. B. Scholz, and R. F. Werner. Connes’ embedding problem and Tsirelson’s problem. *Journal of Mathematical Physics*, 52(1):012102, 2011.

[JUW09] Rahul Jain, Sarvagya Upadhyay, and John Watrous. Two-message quantum interactive proofs are in PSPACE. In *Proceedings of the 2009 50th Annual IEEE Symposium on Foundations of Computer Science*, FOCS ’09, pages 534–543, Washington, DC, USA, 2009. IEEE Computer Society.

[KA16] S. Kamath and V. Anantharam. On non-interactive simulation of joint distributions. *IEEE Transactions on Information Theory*, 62(6):3419–3435, June 2016.

[Kin14] Christopher King. Hypercontractivity for semigroups of unital qubit channels. *Communications in Mathematical Physics*, 328(1):285–301, May 2014.

[KKM+08] Julia Kempe, Hirotada Kobayashi, Keiji Matsumoto, Ben Toner, and Thomas Vidick. Entangled games are hard to approximate. In *Proceedings of the 2008 49th Annual IEEE Symposium on Foundations of Computer Science*, FOCS ’08, pages 447–456, Washington, DC, USA, 2008. IEEE Computer Society.

[KRT10] J. Kempe, O. Regev, and B. Toner. Unique games with entangled provers are easy. *SIAM Journal on Computing*, 39(7):3207–3229, 2010.

[KW00] Alexei Kitaev and John Watrous. Parallelization, amplification, and exponential time simulation of quantum interactive proof systems. In *Proceedings of the Thirty-second Annual ACM Symposium on Theory of Computing*, STOC ’00, pages 608–617, New York, NY, USA, 2000. ACM.

[Lya92] Aleksandr M. Lyapunov. The general problem of the stability of motion. *International Journal of Control*, 55(3):531–534, 1992.

[Mon12] Ashley Montanaro. Some applications of hypercontractive inequalities in quantum information theory. *Journal of Mathematical Physics*, 53(12):122206, 2012.

[MOO10] Elchanan Mossel, Ryan O’Donnell, and Krzysztof Oleszkiewicz. Noise stability of functions with low influences: Invariance and optimality. *Annals of Mathematics*, 171:295–341, Mar 2010.

[Mos10] Elchanan Mossel. Gaussian bounds for noise correlation of functions. *Geometric and Functional Analysis*, 19(6):1713–1756, Mar 2010.

[MV15] Laura Mancinska and Thomas Vidick. Unbounded entanglement in nonlocal games. *Quantum Info. Comput.*, 15(15-16):1317–1332, November 2015.

[MW05] Chris Marriott and John Watrous. Quantum Arthur—Merlin games. *Comput. Complex.*, 14(2):122–152, June 2005.

[NC00] Michael A Nielsen and Isaac L Chuang. *Quantum computation and quantum information*. Cambridge University Press, Cambridge, UK, 2000.

[NPA08] Miguel Navascués, Stefano Pironio, and Antonio Acín. A convergent hierarchy of semidefinite programs characterizing the set of quantum correlations. *New Journal of Physics*, 10(7):073013, jul 2008.

[NV18] Anand Natarajan and Thomas Vidick. Low-degree testing for quantum states, and a quantum entangled games PCP for QMA. In *Proceedings of the 2018 59th Annual IEEE Symposium on Foundations of Computer Science*, FOCS ’18, pages 731–742, Washington, DC, USA, 2018. IEEE Computer Society.

[O’D13] Ryan O’Donnell. *Analysis of Boolean Functions*. Cambridge University Press, Cambridge, UK, 2013.
A Facts on Fréchet derivative

In this section, we summarize some basic facts on Fréchet derivatives.

Fact A.1. Given $f_1, f_2, g : \mathcal{M}_d \to \mathcal{M}_d$ and $P, Q_1, \ldots, Q_k \in \mathcal{M}_d$, it holds that
1. \( D(f_1 + f_2)(P)(Q) = Df_1(P)(Q) + Df_2(P)(Q) \).
2. \( D(f_1 \cdot f_2)(P)(Q) = Df_1(P)(Q) \cdot f_2(Q) + f_1(P) \cdot Df_2(P)(Q) \).
3. \( D(g \circ f)(P)(Q) = (Dg(f(P)) \circ Df(P))(Q) \).
4. \( D^k f(P)(Q_1, \ldots, Q_k) = D^k f(P) \left( Q_{\sigma(1)}, \ldots, Q_{\sigma(k)} \right) \) for any integer \( k > 0 \) and permutation \( \sigma \in S_k \).

The following fact follows from elementary matrix calculations. Readers who are interested may refer to [Bha97, Chapter X.4].

**Fact A.2.** [Bha97, Page 311, Example X.4.2]
- Let \( f(x) = x^2 \). Then
  \[ Df(P)(Q) = \{P, Q\}. \]
- Let \( f(x) = x^{-1} \). Then for any invertible \( P \),
  \[ Df(P)(Q) = -P^{-1}QP^{-1}. \]

**B Facts on analysis**

In this section, we list some basic results of matrix-valued functions. Most of the proofs are the direct generalization of the analysis of one-variable real functions. Here we include proofs for completeness.

**Lemma B.1.** Suppose \( f: [a, b] \to \mathcal{H}_d \) is a continuous mapping and is differentiable in \((a, b)\). Then there exists \( x \in (a, b) \) such that

\[ ||f(a) - f(b)||_2 \leq (b - a) ||f'(x)||_2. \]

*Proof.* Set \( Z \overset{\text{def}}{=} f(b) - f(a) \) and define

\[ \psi(t) \overset{\text{def}}{=} \text{Tr} \ Z f(t), \]

for \( t \in [a, b] \). Then \( \psi \) is a real-valued continuous function on \([a, b]\) which is differentiable in \((a, b)\). The mean value theorem shows that

\[ \psi(b) - \psi(a) = (b - a) \psi'(x) = (b - a) \text{Tr} \ Z f'(x), \]

for some \( x \in (a, b) \). On the other hand,

\[ \psi(b) - \psi(a) = \text{Tr} \left( Z \cdot (f(b) - f(a)) \right) = ||Z||_2^2. \]

Thus

\[ ||Z||_2^2 = (b - a) \text{Tr} \ Z f'(x) \leq (b - a) ||Z||_2 ||f'(x)||_2. \]

\[ \square \]

**Lemma B.2.** Let \( \{f_n\}_{n \in \mathbb{N}} \) be a sequence of functions mapping \( \mathbb{R} \) to \( \mathcal{H}_d \), differentiable in \([a, b]\) and \( \lim_{n \to \infty} f_n(x_0) = f(x_0) \) for some point \( x_0 \in [a, b] \). Suppose \( \{f'_n(x)\}_{n \in \mathbb{N}} \) converges uniformly on \([a, b]\). Namely for any \( \epsilon > 0 \), there exists \( n_0 = n(\epsilon) \) such that for any \( m \geq n > n_0 \) and \( x \in [a, b] \), \( ||f'_n(x) - f'_m(x)||_2 \leq \epsilon \). Then \( \{f_n\}_{n \in \mathbb{N}} \) converges uniformly on \([a, b]\) to a function \( f \) and

\[ \lim_{n \to \infty} f'_n(x) = f'(x) \quad (a < x < b). \]
Remark B.3. Note that all p-norms of matrices are topologically equivalent. Thus the norm $\|\cdot\|_2$ used in Lemma B.2 can be replaced by $\|\cdot\|_p$ for any $1 \leq p \leq +\infty$.

Proof. Let $\epsilon > 0$ be given. Choose $N$ such that for any $m, n \geq N$,
\[ \| f_n (x_0) - f_m (x_0) \|_2 < \epsilon/2, \] (48)
and for any $t \in [a, b]$
\[ \| f'_n (t) - f'_m (t) \|_2 < \frac{\epsilon}{2 (b - a)}. \]

Applying Lemma B.1 to $f_n - f_m$, we have
\[ \| f_n (x) - f_m (x) \|_2 + \| f_m (t) - f_n (t) \|_2 \leq \frac{|x - t| \epsilon}{2 (b - a)} \leq \epsilon/2, \] (49)
for any $x, t$ on $[a, b]$ if $m, n \geq N$. Then from Eqs. (48) (49)
\[ \| f_n (x) - f_m (x) \|_2 \leq \| f_n (x) - f_m (x) - f_n (x_0) + f_m (x_0) \|_2 + \| f_n (x_0) - f_m (x_0) \|_2 \leq \epsilon, \]
for any $m, n \geq N$ and $x \in [a, b]$. So $\{ f_n \}_{n \in \mathbb{N}}$ converges uniformly on $[a, b]$. Let $f (x) \stackrel{\text{def}}{=} \lim_{n \to \infty} f_n (x)$ for $x \in [a, b]$.

Fix $x \in [a, b]$ and define
\[ \psi_n (t) \stackrel{\text{def}}{=} \frac{f_n (t) - f_n (x)}{t - x}, \quad \psi (t) \stackrel{\text{def}}{=} \frac{f (t) - f (x)}{t - x}, \]
for $t \in [a, b], t \neq x$. Then Eq. (49) implies that
\[ \| \psi_n (t) - \psi_m (t) \|_2 \leq \frac{\epsilon}{2 (b - a)}. \]
Thus $\{ \psi_n \}_{n \in \mathbb{N}}$ uniformly converges for $t \neq x$. Note that $\lim_{x \to t} \psi_n (x) = f'_n (t)$. Thus
\[ \lim_{n \to \infty} \psi_n (t) = \psi (t), \]
uniformly for $a \leq t \leq b, t \neq x$. Thus From Theorem 7.17 in [Rud76], we conclude
\[ f' (x) = \lim_{t \to x} \psi (t) = \lim_{n \to \infty} f'_n (x). \]

Lemma B.4. Let $f$ be a real function on $[a, b]$, $f^{(n-1)}$ is continuous on $[a, b]$ and $f^{(n)} (t)$ exists for all $t \in (a, b)$ except finite points $\{ t_1, \ldots, t_m \} \subseteq (a, b)$. Moreover, $| f^{(n)} (t) | \leq M$ for all $t \in (a, b)$ and $t \notin \{ t_1, \ldots, t_m \}$. Then for any distinct points $\alpha, \beta$ in $[a, b]$, we have
\[ | f (\beta) - P (\beta) | \leq \frac{M}{n!} | \beta - \alpha |^n, \]
where
\[ P (t) \stackrel{\text{def}}{=} \sum_{k=0}^{n-1} \frac{f^{(k)} (\alpha)}{k!} (t - \alpha)^k. \]
Proof. Let \( L \) be the number satisfying that
\[
 f ( \beta ) = P ( \beta ) + \frac{L}{n!} (\beta - \alpha)^n.
\]
It suffices to show that \(|L| \leq M\). Set
\[
g (t) \overset{\text{def}}{=} f (t) - P (t) - \frac{L}{n!} (t - \alpha)^n.
\]
We assume that \( t_1 < t_2 < \ldots < t_m \), without loss of generality. Then
\[
g (\alpha) = g' (\alpha) = \ldots = g^{(n - 1)} (\alpha) = 0.
\]
Note that \( g (\beta) = 0 \). By the mean value theorem, \( g' (\beta_1) = 0 \) for some \( \beta_1 \in (\alpha, \beta) \). Repeat this for \( n - 1 \) steps, we get \( \beta_{n - 1} \in (\alpha, \beta) \) such that \( g^{(n - 1)} (\beta_{n - 1}) = 0 \). Note that
\[
g^{(n - 1)} (t) = f^{(n - 1)} (t) - f^{(n - 1)} (\alpha) - L (t - \alpha).
\]
Set \( t_0 = \alpha \). Let \( i_0 \) be the largest integer such that \( t_{i_0} < \beta_{n - 1} \). Then
\[
g^{(n - 1)} (\beta_{n - 1}) = f^{(n - 1)} (\beta_{n - 1}) - f^{(n - 1)} (t_{i_0}) + \sum_{i = 0}^{i_0 - 1} f^{(n - 1)} (t_{i + 1} - f^{(n - 1)} (t_i)) - L (t - \alpha).
\]
Applying the mean value theorem, we have
\[
g^{(n - 1)} (\beta_{n - 1}) = f^{(n)} (\xi_{i_0}) (\beta_{n - 1} - t_{i_0}) + \sum_{i = 0}^{i_0 - 1} f^{(n)} (\xi) (t_{i + 1} - t_i) - L (\beta - \alpha),
\]
where \( \xi_{i_0} \in [t_{i_0}, \beta] \) and \( \xi_i \in [t_i, t_{i + 1}] \). As \( g^{(n - 1)} (\beta_{n - 1}) = 0 \) and \( \left| f^{(n)} (t) \right| \leq M \) for any \( t \) where \( f^{(n)} (t) \) is defined, we have
\[
|L| (\beta - \alpha) \leq |M (\beta - \alpha)|.
\]
Thus \(|L| \leq M\).

\[\square\]

C Proofs in Section 8

Before proving Lemma 8.4 and Lemma 8.5, we first introduce Lyapunov equation, a well studied equation in control theory \[\text{Lya92}\].

Definition C.1. Let \( P, Q \) be two Hermitian matrices in \( \mathcal{H}_d \). We define Lyapunov equation.
\[
P X + X P = Q.
\]
(50)

The solution to Eq. (50) is denoted by \( L (P, Q) \).

Lemma C.2. Given Hermitian matrices \( P, Q \in \mathcal{H}_d \), the Lyapunov equation \( \text{(50)} \) has an unique solution if and only if \( P \) and \( -P \) has no common eigenvalues. Namely, \( I_d \otimes P + P \otimes I_d \) is invertible.

Moreover, let \( P = U D U^\dagger \) be a spectral decomposition of \( P \), where \( D = \text{Diag} (d_1, \ldots, d_n) \) satisfies that \( d_i + d_j \neq 0 \) for any \( 0 \leq i, j \leq n \). Then Eq. (50) has a unique solution \( X_0 \) and it satisfies that
\[
(U^\dagger X_0 U)_{i,j} = \frac{(U^\dagger Q U)_{i,j}}{d_i + d_j}.
\]

55
Proof. Let \( X' \stackrel{\text{def}}{=} U^\dagger X U \) and \( Q' \stackrel{\text{def}}{=} U^\dagger Q U \). Then we have

\[
DX' + X'D = Q',
\]

which is equivalent to

\[
(d_i + d_j) X'_{ij} = Q'_{ij},
\]

for \( 1 \leq i, j \leq n \). Hence there is a unique solution if and only if \( d_i + d_j \neq 0 \) for all \( i, j \). \( \square \)

**Fact C.3.** [Bha97, Page 205, Theorem VII.2.3] Let \( P \) be a positive definite matrix. Then

\[
L(P, Q) = \int_0^\infty e^{-tP}Qe^{-tP}dt.
\]

**Fact C.4.** [Sen07] Let \( g \) be a \( k \)-times differentiable real-valued function defined on a set \( I \subseteq \mathbb{R} \) which is a union of a constant number of open intervals. Let \( X \in \mathcal{H}_d \) have eigenvalues in \( I \). Then the \( k \)-th order Fréchet derivative \( \text{Tr} D^k f(X)(Y, \ldots, Y) \) exists for any \( Y \in \mathcal{H}_d \).

**Definition C.5.** For any Hermitian matrices \( P, Q \) that \( P \) is invertible, we define

\[
\ell_Q(P) \stackrel{\text{def}}{=} L(|P|, PQ + QP).
\]

It is easy to verify that \( \ell_Q(P) = Q \) if \( P > 0 \).

**Definition C.6.** For any Hermitian matrices \( P \) and \( Q \) and \( P \) is invertible,

\[
\kappa_Q(P) \stackrel{\text{def}}{=} \{ P, \ell_Q(P) \} = \{ P, L(|P|, PQ + QP) \}.
\]

**Lemma C.7.** Let \( P, Q \) be Hermitian matrices, where \( P \) is invertible. The following holds.

1. Let \( f(x) \stackrel{\text{def}}{=} \sqrt{x} \) for \( x \geq 0 \). Then \( Df(P)(Q) = L\left(\sqrt{P}, Q\right) \) if \( P \) is positive definite.
2. Let \( f(x) \stackrel{\text{def}}{=} |x| \). Then \( Df(P)(Q) = \ell_Q(P) \).
3. Let \( f(x) = x|x| \). Then \( Df(P)(Q) = \frac{1}{2} \left(\{ |P|, Q \} + \kappa_Q(P) \right) \).
4. Let \( p(x) = \begin{cases} x^2 & \text{if } x \geq 0 \\ 0 & \text{otherwise} \end{cases} \). Then
   \[
   Dp(P)(Q) = \frac{1}{2} \{ P, Q \} + \frac{1}{4} \{ |P|, Q \} + \frac{1}{4} \kappa_Q(P).
   \]

**Proof.**

1. Let \( g(x) \stackrel{\text{def}}{=} x^2 \) and \( Df(P)(Q) = X \). Applying the composition rule in Fact A.1, we have

\[
Q = Dg \circ f(P)(Q) = (Dg(f(P)) \circ Df(P))(Q) = Dg\left(\sqrt{P}\right)(X) = \left\{ \sqrt{P}, X \right\}.
\]

Hence \( X = L\left(\sqrt{P}, Q\right) \).

2. Let \( g(x) = x^2 \) and \( h(x) = \sqrt{x} \). Then \( f = h \circ g \).

\[
Df(P)(Q) = Dh(g(P)) \circ Dg(P)(Q) = Dh\left(P^2\right)(PQ + QP) = \ell_Q(P).
\]

The theorem in [Sen07] is stated for real symmetric matrices and \( I \) is an interval. But the proofs can be directly generalized to Hermitian matrices and a union of constant number of open intervals.
3. Let $g(x) = x$ and $h(x) = |x|$. From Fact A.1 item 2,

$$Df(P)(Q) = Dg(P)(Q)h(P) + g(P)Dh(P)(Q) = Q|P| + P\ell_Q(P).$$

Using $f = h \cdot g$, we have

$$Df(P)(Q) = Dg(P)(Q)h(P) + g(P)Dh(P)(Q) = |P|Q + \ell_Q(P)P.$$ 

Then

$$Df(P)(Q) = \frac{1}{2}((Q|P| + P\ell_Q(P)) + (|P|Q + \ell_Q(P)P)) = \frac{1}{2}(|P|,Q) + \kappa_Q(P)).$$

4. It follows from that $f(x) = \frac{1}{2}x^2 + \frac{1}{2}x|x|$. \[\square\]

**Lemma C.8.** Let $P, Q$ be Hermitian matrices where $P$ is invertible. It holds that

$$D\ell_Q(P)(Q) = L\left(|P|, 2Q^2 - 2\ell_Q(P)^2\right).$$

Moreover, if $P = \text{Diag}(a_1, \ldots, a_d)$ diagonal, then

$$(\ell_Q(P))_{i,j} = \frac{Q_{ij}(a_i + a_j)}{|a_i| + |a_j|}. $$

$$(D\ell_Q(P)(Q))_{i,j} = 2\sum_k Q_{ik}Q_{kj}\left(1 - \frac{(a_i + a_k)(a_i + a_j)}{|a_i| + |a_k|(|a_k| + |a_j|)}\right).$$

**Proof.** From the definition of $\ell_Q(\cdot)$ in Definition C.5, we have

$$|P|\ell_Q(P) + \ell_Q(P)|P| = PQ + QP.$$ 

Taking Fréchet derivative on both sides with respect to $Q$, we have

$$|P| D\ell_Q(P)(Q) + D\ell_Q(P)(Q)|P| = 2Q^2 - 2\ell_Q(P)^2.$$ 

We conclude Eq. (51). \[\square\]

**Lemma C.9.** Let $P$ and $Q$ be Hermitian matrices where $P$ is invertible. It holds that

$$\|\ell_Q(P)\|_2 \leq \|Q\|_2,$$

and

$$\|\ell_Q(P)\|_4 \leq 2\|Q\|_4.$$ 

57
Proof. To prove the first inequality, we assume that \( P = \text{Diag}(a_1, \ldots, a_d) \) is a diagonal matrix. Then by Eq. (52)
\[
\|Q\|_2^2 = \sum_{ij} |Q_{ij}(a_i + a_j)|^2 \leq \sum_{ij} |Q_{ij}|^2 = \|Q\|_2^2.
\]

To prove the second inequality, from Fact C.3, we have
\[
\kappa \text{ of the function } Q \text{'s kernel is a zero space. Let } P = \text{Diag}(a_1, \ldots, a_d) \text{ be a diagonal matrix. Then by Eq. (52)}
\]
\[
\|Q\|_2^2 = \sum_{ij} |Q_{ij}(a_i + a_j)|^2 \leq \sum_{ij} |Q_{ij}|^2 = \|Q\|_2^2.
\]

Proof. Without loss of generality, we assume that \( P = \text{Diag}(a_1, \ldots, a_d) \) is a diagonal matrix. Then \( \kappa \) from the definition of \( \kappa \). Thus have
\[
\text{Tr} \kappa(Q) = 2\text{Tr} |P| Q.
\]

For the second equality, consider
\[
\text{Tr} P \kappa(Q) = 2\text{Tr} P^2 \ell_Q(P) = 2 \sum_{i} |a_i| Q_{ii} = 2\text{Tr} |P|Q.
\]

Before proving Lemma 8.4, we need to compute the the first three orders of Fréchet derivatives of the function
\[
q(x) = \begin{cases} x^3 & \text{if } x \geq 0 \\ 0 & \text{otherwise.} \end{cases}
\]

Lemma C.11. Given an integer \( d > 0 \) and \( P, Q \in \mathcal{H}_d \) satisfying that intersection of \( P \)’s kernel and \( Q \)’s kernel is a zero space. Let \( f(t) = \text{Tr} q(P + tQ) \). Then \( f' \), \( f'' \) exist on \( \mathbb{R} \) and \( f''' \) exists except for finite points.

Moreover, it holds that
\[
f'(t) = \text{Tr} \left( QP + P^2Q + P|P|Q \right);
\]
\[
f''(t) = \text{Tr} \left( 4PQ^2 + \frac{3}{2}|P|Q^2 + \frac{3}{4}Q\kappa(Q) \right);
\]
\[
f'''(t) = \text{Tr} \left( 4PQ^2 + \frac{3}{2}|P|Q^2 + \frac{3}{4}Q\kappa(Q) \right);
\]

58
If $P$ is invertible, then
\[
 f'''(t) = \text{Tr} \left( 4 Q^2 + 3 Q^2 \ell_Q (P) + \frac{3}{4} Q \{ P, D\ell_Q (P) (Q) \} \right). \tag{58}
\]

**Proof.** Note that $q', q''$ exist in $\mathbb{R}$ and $q'''$ exists on $\mathbb{R} \setminus \{0\}$. $P + tQ$ has eigenvalue 0 only for finite choices of $t$. Thus the lemma follows from Fact C.4. Note that $q(x) = xp(x)$, where $p(\cdot)$ is defined in Lemma C.7 item 4.

\[
 \text{Tr} \ Dq (P) (Q) = \text{Tr} \ P Q p (P) + \text{Tr} \ P^2 Q + \text{Tr} \ P |P| Q + \frac{1}{4} \text{Tr} \ P \kappa_Q (P) \defeq g_1, Q (P) + g_2, Q (P) + g_3, Q (P), \tag{59}
\]

where the second equality is from Lemma C.10.

Further taking derivates of $g_1, Q$, $g_2, Q$, and $g_3, Q$ we have
\[
 Dg_1, Q (P) (Q) = \text{Tr} \left( \frac{1}{2} Q \{ P, Q \} + \frac{1}{4} \{ |P| , Q \} + \frac{1}{4} Q \kappa_Q (P) \right)
 = \text{Tr} \ P Q^2 + \frac{1}{2} \text{Tr} \ |P| Q^2 + \frac{1}{4} \text{Tr} \ P \kappa_Q (P). \tag{60}
\]

\[
 Dg_2, Q (P) (Q) = 2 \text{Tr} \ P Q^2; \tag{61}
\]

\[
 Dg_3, Q (P) (Q) = \text{Tr} \left( |P| Q^2 + Q \ell_Q (P) \right). \tag{62}
\]

By symmetry,
\[
 Dg_3, Q (P) (Q) = \text{Tr} \left( |P| Q^2 + Q \ell_Q (P) \right). \tag{63}
\]

Thus
\[
 Dg_3, Q (P) (Q) = \text{Tr} \left( |P| Q^2 + \frac{1}{2} Q \kappa_Q (P) \right). \tag{64}
\]

Combining Eqs. (60)(61)(62) we conclude
\[
 f''(t) = \text{Tr} \left( 4 P Q^2 + \frac{3}{2} |P| Q^2 + \frac{3}{4} Q \kappa_Q (P) \right)
 \defeq g_4, Q (P) + g_5, Q (P) + g_6, Q (P). \tag{65}
\]

Further taking derivative on Eq. (60),
\[
 Dg_4, Q (P) (Q) = 4 \text{Tr} \ Q^3. \tag{66}
\]

Applying Lemma C.7 item 2,
\[
 Dg_5, Q (P) (Q) = \frac{3}{2} \text{Tr} \ \ell_Q (P) Q^2. \tag{67}
\]

From Definition C.6
\[
 D\kappa_Q (P) (Q) = \{ Q, \ell_Q (P) \} + \{ P, D\ell_Q (P) (Q) \}
\]

Thus
\[
 Dg_6, Q (P) (Q) = \text{Tr} \left( \frac{3}{2} Q^2 \ell_Q (P) + \frac{3}{4} Q \{ P, D\ell_Q (P) (Q) \} \right). \tag{68}
\]

Combining Eqs. (64)(65)(66), we conclude Eq. (58). \qed
**Lemma C.12.** Given an integer \( d > 0 \) and \( P, Q \in \mathcal{H}_d \) and \( P \) is invertible, let \( f(t) = \text{Tr} q(P + tQ) \). It holds that 
\[
|f'''(0)| = 11 \|Q\|_2^2 \|Q\|_4^2.
\]

**Proof.** We upper bound each term in Eq. (58). For the first term, consider

\[
|4 \text{Tr} Q^3| \leq 4 \|Q\|_2 \|Q^2\|_2 = 4 \|Q\|_2 \|Q\|_4^2. \tag{67}
\]

For the second term,

\[
|3 \text{Tr} Q^2 \ell_Q(P)| \leq 3 \|Q\|_2^2 \|\ell_Q(P)\|_2 \leq 3 \|Q\|_4^2 \|Q\|_2, \tag{68}
\]

where the second inequality is from Lemma [C.9]. For the final term, assuming that \( P = \text{Diag}(a_1, \ldots, a_d) \) is a diagonal matrix and applying Lemma [C.8], we have

\[
\left| \frac{3}{4} \text{Tr} Q \left\{ P, D\ell_Q(P)(Q) \right\} \right| = \frac{3}{4} \sum_{ijk} Q_{ij} Q_{jk} Q_{ki} \left( \frac{a_i + a_j}{|a_i| + |a_j|} - \frac{(a_i + a_j)(a_j + a_k)(a_k + a_i)}{(|a_i| + |a_j|)(|a_j| + |a_k|)(|a_k| + |a_i|)} \right)
\leq \frac{3}{4} \sum_{ijk} Q_{ij} Q_{jk} Q_{ki} \left( \frac{a_i + a_j}{|a_i| + |a_j|} \right) + \frac{3}{4} \sum_{ijk} Q_{ij} Q_{jk} Q_{ki} \left( \frac{(a_i + a_j)(a_j + a_k)(a_k + a_i)}{(|a_i| + |a_j|)(|a_j| + |a_k|)(|a_k| + |a_i|)} \right)
= \frac{3}{4} \text{Tr} \left( \ell_Q(P) Q^2 \right) + \frac{3}{4} \text{Tr} \ell_Q(P)^3 \tag{Eq. (52)}
\leq \frac{3}{4} \|\ell_Q(P)\|_2 \|Q\|_4^2 + \frac{3}{4} \|\ell_Q(P)\|_4 \|\ell_Q(P)\|_2^2
\leq 4 \|Q\|_2 \|Q\|_4^2 \tag{Lemma [C.9]}. \tag{69}
\]

Combining Eqs. (67), (68), (69), the result follows.

It is now ready to prove Lemma 8.4.

**Proof of Lemma 8.4.** We assume \( P \) is invertible. The general case follows by the continuity. Then \( P + tQ \) is invertible except for the finite \( t \)'s.

From the definition of \( \zeta_{\lambda} \), we have

\[
\zeta_{\lambda}(x) = x^2 + \frac{x^2}{3} - \frac{q(x + \lambda)}{6\lambda} + \frac{q(x - \lambda)}{6\lambda} + \frac{q(x - 1 + \lambda)}{6\lambda} - \frac{q(x - 1 - \lambda)}{6\lambda}. \tag{70}
\]

Note that \( q(\cdot) \) is the 1st order and 2nd order differentiable. \( q''(\cdot) \) exists except for finite points. Thus from Lemma [B.4] and Fact [C.4], it suffices to upper bound \( \text{Tr} D^3 \zeta_{\lambda}(P)(Q) \), which is directly implied by Lemma [C.12].

**Proof of Lemma 8.5.** Note that \( \zeta(x) = p(x - 1) + p(-x) \). Then from Item 4 of Lemma [C.7]

\[
\text{Tr} D\zeta(P)(Q) = \text{Tr} (2P - I) Q + \frac{1}{2} \text{Tr} \left( |P - I| - |P| \right) Q + \frac{1}{4} (\kappa_Q(P - I) - \kappa_Q(-P))
= \text{Tr} (2P - I) Q + \text{Tr} \left( |P - I| - |P| \right) Q,
\]

where the second equality is from Lemma [C.10].
Assuming that $P = \text{Diag}(a_1, \ldots, a_d)$ is a diagonal matrix, we have
\[
|\text{Tr} \ D\zeta (P) (Q)| = \left| \sum_i (2a_i - 1 + |a_i| - |a_i|) Q_{ii} \right| \leq 4 \sum_i |a_i Q_{ii}|. 
\]
Thus for any $P, Q \in \mathcal{H}_d$, there exists a unitary $U$ such that
\[
|\text{Tr} \ D\zeta (P) (Q)| \leq 4 \text{Tr} \left| UPU^\dagger \right| \left| UQU^\dagger \right|.
\]
Then by the mean value theorem,
\[
|\text{Tr} (\zeta (P + Q) - \zeta (P))| = (\text{Tr} \ D\zeta (P + \theta Q) (Q)) \leq 4 \text{Tr} \left| U (P + \theta Q) U^\dagger \right| \left| UQU^\dagger \right|,
\]
for some $\theta \in [0, 1]$ and unitary $U$. Moreover,
\[
\text{Tr} \left| U (P + \theta Q) U^\dagger \right| \left| UQU^\dagger \right| \\
\leq \left\| U (P + \theta Q) U^\dagger \right\|_2 \left\| UQU^\dagger \right\|_2 \\
= \|P + \theta Q\|_2 \|Q\|_2 \\
\leq \|P\|_2 \|Q\|_2 + \|Q\|_2^2.
\]

**D Proofs in Section 10**

Proof of Claim 10.3. A crucial observation is that
\[
A = \mathbf{1}_{2^i} \otimes \mathbf{1}_2 \otimes \tilde{A} \tag{71}
\]
\[
B = \mathbf{1}_{2^i} \otimes \tilde{B} \tag{72}
\]
\[
C = \mathbf{1}_{2^i} \otimes \tilde{C}. \tag{73}
\]
The both equalities can be proved by expanding the both sides. For the first equality,
\[
\mathbb{E}[\text{Tr} \ Bf (A)] = 2^i \mathbb{E} \left[ \text{Tr} \ B \left( \mathbf{1}_2 \otimes f (\tilde{A}) \right) \right] \\
= 2^i \sum_{\sigma \in [4]^n} \tilde{M} (\sigma) \mathbb{E} \left[ \text{Tr} \left( \prod_{j=1}^i g_{i,\sigma_j} \right) \mathcal{P}_{\sigma_{>i}} \left( \mathbf{1}_2 \otimes f (\tilde{A}) \right) \right] \\
= 2^{i+1} \sum_{\sigma \in [4]^n, \sigma_{i+1} = 0} \tilde{M} (\sigma) \mathbb{E} \left[ \text{Tr} \left( \prod_{j=1}^i g_{j,\sigma_j} \right) \mathcal{P}_{\sigma_{>i+1}} f (\tilde{A}) \right].
\]
And
\[
\mathbb{E}[\text{Tr} \ C f (A)] \\
= 2^i \mathbb{E} \left[ \text{Tr} \ \tilde{C} \left( \mathbf{1}_2 \otimes f (\tilde{A}) \right) \right] \\
= 2^i \sum_{\sigma \in [4]^n} \tilde{M} (\sigma) \mathbb{E} \left[ \text{Tr} \left( \mathbf{1}_2 \otimes \left( \prod_{j=1}^{i+1} g_{j,\sigma_j} \right) \mathcal{P}_{\sigma_{>i+1}} \right) \left( \mathbf{1}_2 \otimes f (\tilde{A}) \right) \right] \\
= 2^{i+1} \sum_{\sigma \in [4]^n, \sigma_{i+1} = 0} \tilde{M} (\sigma) \mathbb{E} \left[ \text{Tr} \left( \prod_{j=1}^i g_{j,\sigma_j} \right) \mathcal{P}_{\sigma_{>i+1}} f (\tilde{A}) \right].
\]
For the second equality,
\[
\mathbb{E}[\text{Tr } Bf(A)Bg(A)] = 2^t \mathbb{E}[\text{Tr } B (I_2 \otimes f(\tilde{A}))B (I_2 \otimes g(\tilde{A}))]
\]
\[
= 2^i \sum_{\sigma, \tau \in [4]^n} \tilde{M}(\sigma) \tilde{M}(\tau) \mathbb{E}\left[\text{Tr} \left( \prod_{j=1}^{i} g_{j,\sigma,j,\tau} \right) P_{\sigma > i} \left( I_2 \otimes f(\tilde{A}) \right) P_{\tau > i} \left( I_2 \otimes g(\tilde{A}) \right) \right]
\]
\[
= 2^{i+1} \sum_{\sigma, \tau \in [4]^n: \sigma_{i+1} = \tau_{i+1}} \tilde{M}(\sigma) \tilde{M}(\tau) \mathbb{E}\left[\text{Tr} \left( \prod_{j=1}^{i} g_{j,\sigma,j,\tau} \right) P_{\sigma > i} \left( I_2 \otimes f(\tilde{A}) \right) P_{\tau > i} \left( I_2 \otimes g(\tilde{A}) \right) \right].
\]
And
\[
\mathbb{E}[\text{Tr } Cf(A)Cg(A)] = 2^t \mathbb{E}[\text{Tr } C (I_2 \otimes f(\tilde{A}))C (I_2 \otimes g(\tilde{A}))]
\]
\[
= 2^i \sum_{\sigma, \tau \in [4]^n} \tilde{M}(\sigma) \tilde{M}(\tau) \mathbb{E}\left[\text{Tr} \left( \prod_{j=1}^{i} g_{j,\sigma,j,\tau} \right) \left( I_2 \otimes P_{\sigma > i} \right) \left( I_2 \otimes f(\tilde{A}) \right) \right]
\]
\[
= 2^{i+1} \sum_{\sigma, \tau \in [4]^n: \sigma_{i+1} = \tau_{i+1}} \tilde{M}(\sigma) \tilde{M}(\tau) \mathbb{E}\left[\text{Tr} \left( \prod_{j=1}^{i} g_{j,\sigma,j,\tau} \right) P_{\sigma > i} \left( I_2 \otimes f(\tilde{A}) \right) P_{\tau > i} \left( I_2 \otimes g(\tilde{A}) \right) \right].
\]

\(\square\)

Proof of Claim 10.4. To prove the first equality, from Eqs. (59) (70), it suffices to show that
\[
\mathbb{E}[\text{Tr } t(A, B)] = \mathbb{E}[\text{Tr } t(A, C)],
\]
for
\[
t(A, B) \in \left\{ p(A)B, A^2B, A|A|B \right\},
\]
which directly follows by Eq. (26) in Claim 10.3.

To prove the second equality, from Eqs. (70) (63), we first prove that
\[
\mathbb{E}[\text{Tr } t(A, B)] = \mathbb{E}[\text{Tr } t(A, C)]
\]
for
\[
t(A, B) \in \left\{ AB^2, |A|B^2, B\kappa_B(A) \right\}.
\]
when \(A\) is invertible. Then the second equality in Claim 10.4 follows by the continuity of \(D^2\zeta_\lambda(\cdot)\) due to Lemma C.1 and Fact C.3.

The first two cases directly follow from Eq. (27) in Claim 10.3. To prove the final case, we use Fact C.3

\[
\text{Tr } B\kappa_B(A)
\]
\[
= \text{Tr } (AB + BA) \int_0^\infty e^{-t|A|} (AB + BA) e^{-t|A|} dt
\]
\[
= 2 \int_0^\infty \text{Tr } \left( Ae^{-t|A|} B e^{-t|A|} + A^2 e^{-t|A|} B e^{-t|A|} B \right) dt
\]
when \(A\) is invertible. Then the result follows from Eq. (27) in Claim 10.3. \(\square\)