Vertical vibrations of rail track generated by random irregularities of rail head rolling surface

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Abstract. It is known that geometrical irregularities of the rail head rolling surface produce additional force when the train runs on track. This force can be quite significant and should not be neglected in the analysis, especially when one deals with high speed railways. In this paper, an analytical method of modelling of such irregularities is presented, along with procedure of their stochastic simulation. The detailed description of this approach is associated with its practical application to the analysis of the rail track dynamic response to moving train. The type of imperfections is chosen randomly, leading to stochastic function which describes their distribution along the track. The system response to moving train is described by two layer model which allows to consider all important elements of rail track such as rails, sleepers, rail pads or under sleeper pads. This previously developed model allowed to analyse the dynamic response of rail track in the case of four forces, mainly due to complex computational procedure needed to apply. In this paper, the improved procedure is used, which allows to calculate the track response to the whole passing train.

1. Introduction
The modelling of load generated by train is of importance in the rail track dynamic response analysis, especially when it comes to changes of foundation stiffness or quite significant degradation of track [1]. Various imperfections or damages of rails can be found in operational lines. Some of them need expensive reparations while some can be marginalized by careful maintenance. The influence of such damages, if not critical, can be sometimes omitted. However, in many cases these imperfections lead to the increase of noise and dynamically varying vibrations that can propagate to surroundings, including buildings near rail track.

Geometrical irregularities of the rail head rolling surface belong to such imperfections. They are dangerous for the environment protection mainly in the case of high speed rails. Modelling of the load generated by moving train is quite difficult. Except of the quasi-static load produced by the weight of train, other additional loads are generated. These regular ones, as e.g. the load coming from the rail deflection between sleepers, are relatively easy to model by harmonic functions with the use of experimentally determined additional static force. Others, like the rail head rolling surface damages, can be easily described only if their regular shape is assumed on the whole length of rails [2]. In reality, these imperfections vary along track structure and, unluckily, their nature changes in time during everyday use of rail track. Even more difficult task is to build the dynamic model of rail track.
which takes into account nonlinear properties of structure [2], which are confirmed experimentally, and, additionally, stochastic characteristics of railway [3].

In this paper, the problem of modelling of stochastically distributed rail imperfections is discussed. In the final stage of this preliminary work, the load including the effects of such irregularities will be represented by random function, leading to the stochastic differential equation of the fourth order. For this purpose, one must develop methodology of analytical modelling of such imperfections. Having this procedure, one might solve classical dynamic system of differential equations and then make extended statistical analysis of functions representing the track response to moving train.

The two layer model described in this paper allows to consider mechanical properties of construction elements of rail track and also to analyse interactions between rails and sleepers [1, 4]. This model assumes that the rail can be represented by the Euler-Bernoulli beam with parameters constant along the track, whereas sleepers are modelled as a rigid body. The fasteners are described by the viscoelastic continuous constrains, i.e. stiffness and damping, and the sleeper foundation is modelled as a viscoelastic subgrade. The improved computational procedure to represent the dynamic response of the track to moving train is used in this paper. This procedure allowed to analyse the track response to the whole passing train, however in simpler case than this associated with stochastic load. One should underline that this ability of the developed technique is very important for the analysis of systems with nonlinear and stochastic properties [5]. In the case of such models, it is not enough to consider separate forces generated by train axles [6].

Each force associated with axles of train has three components: the force constant in time, produced by weight of train, the part varying in time, produced by the rail deflection between sleepers, and the part of load generated by rail head rolling surface irregularities, which are modelled randomly by using the developed stochastic procedure. Vertical vibrations of rails and sleepers can be obtained by applying the Fourier transform and its approximation based on wavelet filters [2].

2. Modelling of the rail track

The rail track can be modelled in various ways. It is usually done by numerical approaches based on FEM methods. Analytical solution to mathematical models are relatively rare, although they are valued for their ability to perform parametrical analysis with relatively low cost.

Two main approaches to analytical modelling of rail track can be found in the literature. The first one is based on so called one layer model. It assumes that the rail track can be represented by either the Euler-Bernoulli beam or the Timoshenko beam supported by viscoelastic foundation which represents track bed. The second approach uses the two layer model which consists of two layers associated with rails and sleepers. The upper layer, associated with rails, is represented by a beam, usually the Euler-Bernoulli one, mainly because it is computationally easier. The lower layer, associated with sleepers, is modelled as a rigid body. The fasteners between rails and sleepers are modelled by the viscoelastic continuous constrains, i.e. stiffness and damping coefficients. The sleepers foundation is considered as a viscoelastic layer. The moving train load is represented by a set of forces in various configurations related to train axles [1, 2].

The one layer dynamic model can be described by the following equation [1, 2]:

$$EI \frac{\partial^4 y_r}{\partial x^4} + N \frac{\partial^2 y_r}{\partial x^2} + m_r \frac{\partial^2 y_r}{\partial t^2} + c_r \frac{\partial y_r}{\partial t} + k_r y_r - k_N y_r^3 = P(x, t)$$

(1)

The parameters appearing in this equation are: $EI$ [Nm2] – bending stiffness of rail steel, $N$ [N] – axial force, $m_r$ [kg/m] – rail unit mass, $k_r$ [N/m3] – stiffness of rail foundation and $c_r$ [Ns/m2] – viscous damping of rail foundation. The parameter $k_N$ [N/m3] represents nonlinear part of rail track foundation stiffness.

The two layer dynamic model of rail track is represented by the system of two coupled differential equations [1, 4]:

$$EI \frac{\partial^4 y_r}{\partial x^4} + N \frac{\partial^2 y_r}{\partial x^2} + m_r \frac{\partial^2 y_r}{\partial t^2} + c_r \left( \frac{\partial y_r}{\partial t} - \frac{\partial y_s}{\partial t} \right) + k_r (y_r - y_s) - k_N y_s^3 = P(x, t)$$

(2)
\[
m_s \frac{\partial^2 y_s}{\partial t^2} + c_s \frac{\partial y_s}{\partial t} + k_s y_s + k_N y_s^3 = c_r \left( \frac{\partial y_r}{\partial t} - \frac{\partial y_s}{\partial t} \right) + k_r (y_r - y_s)
\]  

(3)

The additional parameters appearing in this equation are: \( m_s [\text{kg/m}] \) – unit mass of sleepers uniformly distributed along the track, \( k_s [\text{N/m}^2] \) – stiffness of sleepers foundation and \( c_s [\text{Ns/m}] \) – viscous damping of sleepers foundation.

3. Modelling of the train load

The load generated by train can be represented by a set of forces concentrated or distributed with density \( d(x, t) \):

\[
P(x, t) = P_c(x, t) + P_V(x, t) + P_R(x, t)
\]  

(4)

\[
P_c(x, t) = \sum_{i=0}^{l} \frac{P_0}{a} d(x, t) H(a^2 - (x - Vt - s_i)^2)
\]  

(5)

\[
P_D(x, t) = \sum_{i=0}^{l} \frac{dP}{a} e^{i\omega t} d(x, t) H(a^2 - (x - Vt - s_i)^2)
\]  

(6)

and the term \( P_R(x, t) \) denotes factor associated with random irregularities of rail head surface. \( H(.) \), \( 2a, s_i, L, \omega \) and \( V \) are the Heaviside function, the span of single load, the distance between consecutive forces produced by wheels, the number of axles, the frequency of load and the speed of train, respectively. The axles configuration can be introduced by inclusion of the phase shift associated with the wheel position on the surface of regular imperfection (e.g. cosine shape linked with rail deflection between sleepers). In the case of regular spacing between sleepers, the amplitude of this deflection is constant and its shape is regular, having a cosine form.

The density \( d(x, t) \) gives possibility of analysis of different types of loads. The load can be e.g. rectangular, Gaussian [1], cosine type or this one used in previous publications [2, 4]:

\[
d(x, t) = \cos^2 \left( \frac{x - Vt - s_i}{2a} \right)
\]  

(7)

which gives high regularity of the solution.

The random part \( P_R(x, t) \) is generated by irregularities appearing on the rolling surface of the rail head and modelling of this effect is described in the next section. Other factors influencing the load are omitted, e.g. sudden changes of foundation stiffness or characteristics of the train structure.

4. Random imperfections of the rail track

Random approach to modelling the dynamic response of the train track can be recognized in two different ways:

1. First approach (classical in Engineering):

   To perform an appropriate number of output realisations and then carry out „statistical” analysis of the obtained results. This means: to solve the system for deterministic input which is chosen randomly from a set of possible scenarios. The system itself still remains deterministic.

2. Second approach (based on stochastic analysis):

   To perform an appropriate number of input realisations, carry out „statistical” analysis and build the system of stochastic equations (in particular differential ones) which might be solved by using tools from stochastic analysis. The system becomes stochastic from the beginning.

Both approaches need computational solution to mathematically describe random changes of rolling surface along rail. For this purpose, a special algorithm is constructed and then implemented in Excel software in such a way that gives possibility to include the result in Mathematica code. Mathematica software was used previously to solve and to analyse the analytical systems described by equations (1-3). Therefore, it is crucial to keep the original path of investigations which will allow to follow analytical approach and, finally, to reach solution to the second approach mentioned above, i.e.
formulate random system described by the stochastic differential equations. This could be solved by using tools coming from mean-square analysis and hybrid approximation methods.

5. The draw procedure

The assumptions made for irregularity of rail are as follows:

1. The length of the investigated area of beam with „surface imperfections” is finite, although the length of beam remains infinite.
2. The step of draw in \(x\) direction is fixed, although it can be changed in separated sections.
3. The amplitude of imperfection is limited (finite).
4. The amplitude increase between two consecutively selected points can be limited (and fixed) either unlimited (although remaining within the limits specified by the condition 3).
5. The amplitude of imperfection and the direction of its change at consecutive points is randomly chosen as many times as all conditions are fulfilled and, additionally, realistic physical character of imperfection is kept (all irregularities are below horizontal line – there are no elevations above the zero level).

Figures 1 and 2 show examples of imperfections chosen by using the above procedure without limitation of the single draw amplitude (Fig. 1) and with limitation (Fig. 2). The condition 5 is omitted in these examples.

**Figure 1.** Case 1: random imperfection without limits of randomly chosen amplitude increase.

Conditions for imperfections shown in Fig. 1 are as follows:

1. The length of the investigated area of beam with „surface imperfections“: 25 m;
2. The step of draw in \(x\) direction: 0.1 m;
3. The amplitude of imperfection: 0.5 mm;
4. The amplitude increase: without limits.

**Figure 2.** Case 2: random imperfection with fixed limit of randomly chosen amplitude increase.

Conditions for imperfections shown in Fig. 2 are as follows:

1. The length of the investigated area of beam with „surface imperfections“: 25 m;
2. The step of draw in x direction: 0.1 m;
3. The amplitude of imperfection: 0.5 mm;
4. The amplitude increase: 0.25 mm.

Another question is: how many times the amplitudes should be randomized? Is it enough to make only one draw or more realisations should be made in order to obtain real representation of physical situation?

Figures 3-4 shows the spectra amplitudes of imperfections related to cases 1 and 2. Figure 4 shows that indeed the increase in the number of draws leads to more realistic representation which is justified by several measurements done on operational rail lines. On the other hand, this result strongly depends on a number of parameters of the developed procedure, so high caution must be kept already at this stage in order to stay within limits of the model applicability.

**Figure 3.** Case 2, the length of rail: 5 m, the number of draws: 1.

**Figure 4.** Case 2, the length of rail: 5 m, the number of draws: 100.
6. Approximation of imperfections

The main idea presented in this paper deals with the problem of approximation of randomly chosen imperfections in a way that allows to use it in analytical computations leading to solution of the systems defined by equations (1) and (2-3). A very natural approach can be sought in application of the Fourier series. However, this kind of approximation leads to a very complex computations which are highly inefficient. Because of this, parametrical analysis of the system can be computationally impossible.

Figures 5-7 shows approximation of randomly modelled irregularities by using the Fourier series. One can see relatively good agreement of shape and amplitudes between approximated and chosen imperfection. Nevertheless, as expected, this kind of approximation gives formula which is ineffective when used in analytical computations, especially when one deals with the whole train loading. Therefore other ways of approximation must be sought.

Figure 5. Case 2; the length of rail: 25 m, random data - black line, Fourier approximation - red line, number of Fourier coefficients: 10.

Figure 6. Case 2; the length of rail: 25 m, random data - black line, Fourier approximation - red line, number of Fourier coefficients: 100.

One can see that the higher number of Fourier coefficients gives better approximation. In the same time, the approximate solution becomes extremely difficult to apply in analytical calculations related to the solution of the system described by equation (1) and even more in the case of two layer model: equations (2-3).

Figure 7 shows in details approximation applied to a short length of the beam. One can see another disadvantage of the proposed method: the approximate solution might not fit with zero level at the ends of area with random irregularities.
7. The track response
The found approximation procedure is still not good. Due to a high number of Fourier coefficients needed for exact enough approximation of randomly generated irregularities, the output function cannot be effectively used in analysis of the rail track response to passing train. Such a response can be relatively easy obtained in the case of linear and deterministic rail track models defined by equations (1-3) (with $k_N = 0$). Figure 8 presents example of rail deflection as a result of passing train EMU250 Pendolino, obtained for realistic parameters taken from operational lines of HSR in Poland. Inclusion of stochastic and nonlinear properties of rail track in this complex dynamic model still remains an open problem in the case of analytical solution.

Figure 8. Rail track response to moving train: rail deflection under load generated by EMU250 Pendolino train.
8. Conclusions

Preliminary results with regard to modelling of random irregularities of the rail head rolling surface are presented. Although the obtained results are satisfactory in some cases, they are still difficult to implement in previously developed analytical systems, mainly due to complex formulas and inappropriate boundary conditions. The response of the rail track to the whole passing train is shown in the case of linear and deterministic model. Combination of these two models still remains an open problem and it is left for future work.

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