Condensates and instanton - torus knot duality. Hidden Physics at UV scale.

A. Gorsky\textsuperscript{2,3} and A. Milekhin\textsuperscript{1,2,3}

\textsuperscript{1}Institute of Theoretical and Experimental Physics, B.Cheryomushkinskaya 25, Moscow 117218, Russia
\textsuperscript{2}Moscow Institute of Physics and Technology, Dolgoprudny 141700, Russia
\textsuperscript{3}Institute of Information Transmission Problems, B. Karetkti 15, Moscow, Russia

\texttt{gorsky@itep.ru, milekhin@itep.ru}

Abstract

We establish the duality between the torus knot superpolynomials or the Poincaré polynomials of the Khovanov homology and particular condensates in \( \Omega \)-deformed 5D supersymmetric QED compactified on a circle with 5d Chern-Simons(CS) term. This is the generalization of the Witten’s recipe of the evaluation of the knot polynomials via Wilson loops in 3d CS theory for case of the torus knots. It is explicitly shown that \( n \)-instanton contribution to the condensate of the massless flavor in the background of four-observable, which can be associated with some composite defect, exactly coincides with the superpolynomial of the \( T(n, nk+1) \) torus knot where \( k \) - is the level of CS term. In contrast to the previously known results, the particular torus knot corresponds not to the partition function of the gauge theory but to the particular instanton contribution and summation over the knots has to be performed in order to obtain the complete answer. The instantons are sitting almost at the top of each other and the physics of the ”fat point” where the UV degrees of freedom are slaved with point-like instantons turns out to be quite rich. Also also see knot polynomials in the quantum mechanics on the instanton moduli space. We consider the different limits of this correspondence focusing at their physical interpretation and compare the algebraic structures at the both sides of the correspondence. Using the AGT correspondence, we establish a connection between superpolynomials for unknots and q-deformed DOZZ factors which suggests the interpretation of the composite defect.
1 Introduction

The knot invariants were introduced into the QFT framework long time ago [1] however the subject gets new impact last decade. It turns out that the knot invariants should be considered in the QFT in much more broader context. They play the several interesting roles besides the original interpretation as the Wilson loop observables in CS theory. New approaches to their evaluation have been developed. It was recognized in [30] that
the open topological strings with Calabi-Yau target space provide the effective tool to derive the knot invariants and simultaneously knot invariants count the particular BPS states in the gauge theory. The target space for the topological string was identified with $\mathcal{O}(-1) \times \mathcal{O}(-1) \rightarrow \mathbb{C}P_1$ and the knot is selected by the Lagrangian brane wrapping the Lagrangian submanifold. The boundary of the open string worldsheet is fixed at the Lagrangian submanifold in the CY internal space. The recent discussion on the topological string approach to the knot invariants evaluation can be found in [29, 74]. The simplicity of the torus knots stimulated the derivation of the very explicit results and representations for them [103, 30].

The progress in the knot theory brings on the scene the Khovanov homologies which categorized the HOMFLY homologies. The Poincaré polynomial of the Khovanov homologies has been interpreted in the framework of the topological strings in [23] and it was shown that such Poincaré polynomial, called superpolynomial [22], provides the refine counting of the BPS states. The Khovanov homologies were also related with the space of the solutions to the topological fields theories in four and five dimensions [26, 90]. The way to evaluate the superpolynomials for some class of knots has been suggested in [67] via the refined Chern-Simons theory or equivalently the particular matrix model.

Another way the knot invariants are related with the gauge theories concerns the 3d/3d duality [13] which relates the 3d theory on the submanifold in the CY space and the 3d SUSY gauge theory. The knot K complement yields the particular 3d SUSY gauge theory with some matter content. The superpolynomial is related to the partition sum of 3d theory and the parameters $(a, q, t)$ were identified with the mass and two equivariant parameters with respect to two independent rotations in $\mathbb{R}^4$ [97]. The relation between the partition function on the vortex and the knot polynomials has been discussed in [108]. If we introduce the defects in the theory say 2d defect in 4d theory or 3d defect in 5d theory any physical phenomena should be recognized equivalently from the worldvolumes of all branes involved into configurations or more generally on any probe branes. This simple argument suggested long time ago [111] works well and provide some interesting crosschecks (see, for instance [104]). In particular all knot invariants should be recognized by all participants of the configuration.

There is one more important characteristics of the knot - A-polynomial and its generalization- super-A-polynomial [97] depending on the set of variables which becomes the operator upon the quantization of the $(x, y)$ symplectic pair. The A-polynomial defines the twisted superpotentials in 3d theory [97, 106]. The simplest interpretation of the $(x, y)$ variables concerns the realization of the 3d theory as the theory on the domain wall separating two 4d theories [114, 96]. They are identified with the Wilson and t-Hooft loops variables. The nice review on the subject can be found in [99].

There was some parallel progress in mathematics concerning the homologies of the torus knots and links. In what follows we shall use quite recent results concerning superpolynomials of the torus knots and their relation with the higher $(q,t)$ Catalan numbers [5, 43, 6, 105].
Another interesting line of development motivated our consideration concerns the UV completion of the different theories and the features of the decoupling of the heavy degrees of freedom. Some interesting phenomena can happen and we know from the textbooks that the heavy degrees of freedom decouple when the masses of the corresponding excitations get large enough with the only exception - anomaly which matters at any scale since it arises from the spectral flow. However one could wonder what happens at the nonperturbative level. The issue of the role of small size instantons in the RG was raised long time ago in QCD when the integrals over the instanton size tend to diverge. The issue of the point-like instantons is also very important in the consideration of the so-called contact terms which measure the difference between the products of the different observables in UV and IR regions (see in this context, for instance [94, 93]) Therefore some regularization is needed to handle with this region in the instanton moduli space.

There were several attempts to analyze the issue of the point-like instantons carefully imposing a kind of regularization. We could mention the freckled instantons in [91], abelian instantons in the non-commutative gauge theories [92] and the abelian instantons on the commutative $R^4$ blown up in a few points [44]. In all cases one could define the corresponding solutions to the equations of motion with the non-vanishing topological charges. The last example concerns the abelian instantons in the $\Omega$-deformed abelian theory where the point-like instantons can be defined as well [59]. The different deformations provide the possibility to work with the point-like instantons in a well-defined manner. In these cases we can pose the question concerning the role of these defects in the nonperturbative RG flows.

Moreover one could ask what is the fate of the extended nonperturbative configurations which involve the heavy degrees of freedom in a nontrivial way. This issue has been examined in [18, 108, 107] when the vortex solution involving the ”heavy fields” has been considered. The starting point is the superconformal theory then the matter in the bi-fundamental is added. One considers the nonabelian string in the emerging UV theory. At the next step the RG flow generated by the FI term has been analyzed and it was argued that the remnant of the UV theory is the surface operator that is the nonabelian string with the infinite tension. Hence the decoupling in the nonperturbative sector for the extended object is incomplete - we obtain at the end the defect with the infinite tension which provides the boundary conditions for the fields.

The third motivation for this study has more physical origin. When investigating the superfluidity it is very useful to rotate the system since the superfluid component of the current can be extracted in this way. In the first quantized approach the density of the superfluid component is related to the correlator of the winding numbers. The $\Omega$ deformation which is nothing but two independent rotations in $R^4$ introduced in the clever way was used by Nekrasov to regularize the instanton moduli space but on the other hand it allows to look at the response of the ground state of the system on the rotation like in superfluidity. Since the curvature of the graviphoton field is just the angular velocity we could consider the behavior of the partition sum at the small angular velocities $\epsilon_1, \epsilon_2$. It turns out that the dependence on the angular velocities
is very simple [4] and the derivative of the partition sum with respect to the angular velocity yields the average angular momentum of the system [54]. It can be seen immediately that there is non-vanishing density of the angular momentum and one could be interested in its origin. It is not a simple task to answer the question what is the elementary rotator like the roton in the superfluid case in the 5d gauge theory. However no doubts it should be identified as some nonperturbative configuration related with the instantons.

The starting point of our analysis is the observation made in [7] concerning the relationship between the 5d SUSY gauge theory in the Ω - background with CS term at level one and matter in fundamental and knot invariants. It was shown that the particular correlator in U(1) SQED coincides with the sum over the bottom rows of the superpolynomials of the $T_{n,n+1}$ torus knot. Contrary to the previous relations between knots and the gauge theories in this case the particular gauge theory involves the infinite sum over the torus knots.

In this paper we generalize the observation made in [7] and find the similar relation between the sums over the torus knot superpolynomials and the U(1) 5d gauge theories with CS term. In more general situation the second derivative of Nekrasov partition function for 5d SQED with CS term serves as the generating function for the superpolynomial of the torus knots. It is useful to interpret the 4-observable considered in [7] in a bit different manner. We start with the superconformal 5d theory and add matter in fundamental. Then the "Wilson loop" operator in [7] can be considered as the derivative with respect to the mass of the one-loop determinant of the matter in the fundamental at the infinite mass limit. To some extend the approach used in this paper is along the line of development elaborated in [26, 90] where the knot homologies were interpreted in terms of the particular solutions to the equations of motion in 5D SUSY gauge theories. However from our analysis it is clear that the proper generating function for the torus knot superpolynomials implies the particular matter content in the 5D theory.

There are several crucial lessons for the physical picture which can be learned from the exact answer derived via localization. First of all the appearance of the higher $(q,t)$ Catalan number tells that we dealing with the point-like instantons sitting at the top of each other instead of the randomly distributed in $R^4$. This effect is due to the regulator degrees of freedom which yields the nonlocal operator in the correlator. Secondly the non-locality of the operator induced by the UV regulator degree of freedom implies that we have to recognize the compact nonlocal object with the finite action. More arguments came from the AGT dual picture which suggests that in $R^4$ we have composite defect consisting from the closed domain wall, string and ensemble of instantons sitting inside the sphere. This is the candidate state for the "elementary rotator" and on the other hand it captures the information on the torus knots superpolynomials. Moreover the whole worldsheet of the candidate defect also has the part extended along the closed cycles in the CY internal space. The presence of the CS in the 5d gauge theory tells that the geometry of the CY is more complicated then $O(-1) \times O(-1) \to CP_1$ blown up in in few points.
The other lesson can be formulated as follows. The example in [18] demonstrates that the UV degrees of freedom can penetrate as the nonabelian strings with the infinite tension known as the surface operators. Similar logic can be applied for the domain walls which separates the region with the "UV" degrees of freedom and region with the IR degrees of freedom only. In the similar simplest case the domain wall tension becomes infinite when the mass of the regulator tends to be infinite and the situation is analogous to the string with the infinite tension. Such domain walls yields the boundary conditions only.

However one could consider the more interesting question; are there the compact defects involving the UV regulator degrees of freedom with the finite action. The candidates are the closed string and closed domain wall. In this case we have to find arguments preventing them from shrinking to a point and therefore escaping from the physical spectrum. If considering the closed string involving the regulators the first mechanism could be analogue of the mechanism considered by Shifman and Yung in the "instead-of-confinement" approach [115] when the quantum state of the monopole-antimonopole pair nested on the string prevent it from shrinking. This monopole-antimonopole pair transforms into the interesting state upon the Seiberg duality. Another way to stabilize the closed string is to add a kind of rotation due to the additional quantum number like for the Hopf string considered in [119]. Similar mechanisms can be applied to the closed domain wall as well. One can consider its stabilization via the defects of low dimension or by rotation induced by the additional quantum numbers. There is for instance the "monopole bag" configuration [116]. In our case we have a kind of such object which seems to be prevented from shrinking by the instantons inside. Moreover one could expect that the mass of the regulator which enters the domain wall tension is "dressed" by the point-like instantons and its transforms into the $\Lambda_{QCD}$-type scale via dimensional transmutation thus providing the some dynamical mechanism behind it.

One more lesson concerns the question about the mutual back reaction of IR and UV degrees of freedom. Our answer for the torus knot superpotential involves the derivatives of the partition function in 5D theory with respect to the masses of the light and the "regulator" flavors hence it allows the twofold interpretation. First, it can be treated as a kind of the point-like instanton renormalization of the vev $\frac{d}{dM} < \bar{Q}Q >$ of the light flavor in the $\Omega$-background and is treated as the back reaction of the UV degrees of freedom on the condensate of the massless flavor. The operator has non-vanishing anomalous dimension hence to some extend the superpolynomial yields the instanton contribution to the anomalous dimensions of the composite operator. Oppositely the same correlator can be read in the opposite order and can be thought of as a kind of the backreaction of the light flavor on the defect which involves the UV degrees of freedom.

Summarizing, we shall demonstrate that the gauge theory whose n- instanton contribution to the particular correlator coincides with the superpolynomial of $T_{n,nk+1}$ torus knot is the U(1) 5d gauge theory with one compact dimension, CS term at level k, 2 flavors in fundamental and one flavor in the anti-fundamental representation. One
mass of the fundamental tends to zero while the second tends to infinity. The mass of
the anti-fundamental is arbitrary. The remnant of the heavy flavor in the IR is the
composite defect we shall attempt to identify.

The paper is organized as follows. In Section 2 we briefly remind the main facts
concerning the 5d SQED with CS term and focus at the decoupling procedure in this
theory. In Section 3 we describe the relation between the instanton contribution to
the derivative of condensate of the light hyper with respect to the regulator scale and
the torus knot supepolynomials. In Section 4 we attempt to interpret the result of
calculation in terms of a kind of composite defect involving UV degrees of freedom.
Section 5 is devoted to the consideration of the different limits for parameters involved
in the picture. The interpretation of the correlator from the AGT dual Liouville theory
viewpoint has been considered in Section 6. The question concerning the identification
of the knot polynomials in the quantum mechanics on the instanton moduli space will
be analyzed in Section 7. Our findings and the lines for the further developments are
summarized in the Conclusion.

2 Supersymmetric QED with CS term

2.1 Fields, couplings, symmetries and Lagrangian with $\Omega$ deformation

Five-dimensional supersymmetric QED consists of vector field $A_A$, four-component
Dirac spinor $\lambda$ and Higgs field $\phi$, all lying in the adjoint representation of $U(1)$. The
Lagrangian reads as follows:

$$
\mathcal{L} = -\frac{1}{4g^2} F_{AB} F^{AB} + \frac{1}{g^2} (\partial_A \phi)^2 + \frac{1}{g^2} \bar{\lambda} \gamma^A \partial_A \lambda
$$

(1)

$\gamma^A, A = 1, \ldots, 5$ are five-dimensional gamma matrices. Since the adjoint action for the
$U(1)$ group is trivial, this is a free theory.

When looking for the superfluidity it is useful to rotate the system to feel the dis-
sipationless component of the liquid. The same trick was used by Nekrasov [4] when
introducing the $\Omega$-background which corresponds to switching on the graviphoton field
whose components of curvature are identified with two independent angular velocities
in $R^4$. The response of the partition function in the $\Omega$-deformed theory yields the grav-
imagnetization of the ground state or the average angular momentum of the system.
To some extend it measures the " superfluid" component of the vacuum state of the
4d gauge theory.

Let us start from the discussion of pure gauge $\mathcal{N} = 2$ super Yang-Mills theory in
presence of $\Omega$-background in four Euclidean dimensions. Then we will lift the theory
to five-dimensions. The field content of the theory is the gauge field $A_m$, the complex
scalar $\varphi, \bar{\varphi}$ and Weyl fermions $A^I_m, A^I_{\dot{\alpha}}$ in the adjoint of the $U(1)$ group. Here $m = 1, \ldots, 4, I = 1, 2$ are $SU(2)_I$ R-symmetry index, $\alpha, \dot{\alpha}$ are the $SU(2)_L \times SU(2)_R$ spinor
indices. To introduce $\Omega$-background one can consider a nontrivial fibration of $\mathbb{R}^4$ over a torus $T^2$ [4],[21]. The six-dimensional metric is:

$$ds^2 = 2dzd\bar{z} + (dx^m + \Omega^m dz + \bar{\Omega}^m dz)^2,$$

where $(z, \bar{z})$ are the complex coordinates on the torus and the four-dimensional vector $\Omega^m$ is defined as:

$$\Omega^m = \Omega^{mn} x_n, \quad \Omega^{mn} = \frac{1}{2\sqrt{2}} \left( \begin{array}{cccc}
0 & i\epsilon_1 & 0 & 0 \\
-i\epsilon_1 & 0 & 0 & 0 \\
0 & 0 & 0 & -i\epsilon_2 \\
0 & 0 & i\epsilon_2 & 0
\end{array} \right).$$

In general if $\Omega^{mn}$ is not (anti-)self-dual the supersymmetry in the deformed theory is broken. However one can insert R-symmetry Wilson loops to restore some supersymmetry [21]:

$$A_F^I = -\frac{1}{2} \Omega_{mn} (\sigma^{mn})_d I d\bar{z} - \frac{1}{2} \bar{\Omega}_{mn} (\bar{\sigma}^{mn})_d I dz.$$

The most compact way to write down the supersymmetry transformations and the Lagrangian for the $\Omega$-deformed theory is to introduce 'long' scalars (do not confuse them with $\mathcal{N} = 1$ superfields):

$$\Phi = \varphi + i\Omega^m D_m, \quad \bar{\Phi} = \bar{\varphi} + i\bar{\Omega}^m D_m,$$

Then bosonic sector of the deformed Lagrangian reads as:

$$L_\Omega = -\frac{1}{4g^2} F_{mn} F^{mn} + \frac{1}{g^2} D_m \Phi D^m \bar{\Phi} + \frac{1}{2g^2} [\Phi, \bar{\Phi}]^2 =$$

$$-\frac{1}{4g^2} F_{mn} F^{mn} + \frac{1}{g^2} (\partial_m \phi + F_{mn} \Omega^n) (\partial_m \phi - F^{mn} \Omega^n) + \frac{1}{2g^2} (i\Omega^m \partial_m \bar{\phi} + i\bar{\Omega}^m \partial_m \phi)^2.$$

We can couple this theory to fundamental hypermultiplet, which consists of two scalars $q, \bar{q}$ and two Weyl fermions $\psi$ and $\bar{\psi}$ and characterized by two masses: $m$ and $\tilde{m}$, since $\mathcal{N} = 2$ hypermultiplet is built from two $\mathcal{N} = 1$ hypermultiplets with opposite charges. Now the bosonic part reads as:

$$L_m = -\frac{1}{4g^2} F_{mn} F^{mn} + \frac{1}{g^2} (\partial_m \phi + F_{mn} \Omega^n) (\partial_m \phi - F^{mn} \Omega^n) +$$

$$\frac{1}{2} |D_m q|^2 + \frac{1}{2} |D_m \bar{q}|^2 + \frac{2}{g^2} (i\partial_m (\Omega^m \phi + \Omega^m \bar{\phi}) + g^2 (\bar{q}q - q\bar{q}))^2 +$$

$$\frac{1}{2} |(\phi - m - i\Omega^m D_m)q|^2 + \frac{1}{2} |(\phi - \tilde{m} - i\Omega^m D_m)\bar{q}|^2 + 2g^2 |\bar{q}q|^2.$$

General $\Omega$-deformation preserves only one supersymmetry[21]. It is convenient to introduce topological twist[4] and take $SU_L(2)$ times diagonal subgroup of $SU_R(2) \times$
SU(2) to be Lorentz group. Then \( \Gamma^I \alpha \) becomes scalar \( \eta \) and self-dual tensor \( \chi_{IJ} \), \( \Lambda^I_\alpha \) becomes vector \( \psi_I \), and \( \bar{\psi}, \psi \) becomes \( \theta, \nu_m, \omega_{mn} \). Supercharges have similar fate. The scalar supercharge \( Q \) stays unbroken. If we denote by superscripts \( + \) and \( - \) self-dual and anti-self-dual parts of 2-tensors, supersymmetry transformation takes the following form [21]:

\[
\begin{align*}
\delta \phi &= \sqrt{2} \Omega^n \psi_n \\
\delta \bar{\phi} &= \sqrt{2} \eta \\
\delta A_m &= -i \psi_m - i \sqrt{2} \Omega_m \eta \\
\delta \psi_m &= i \sqrt{2} \Omega^n F^+_{mn} + i \Omega_m D + i \sqrt{2} D_m \phi \\
\delta \chi_{mn} &= F^+_{mn} + i \sqrt{2} (\Omega_m D_n \bar{\phi} - \Omega_n D_m \bar{\phi}) \\
\delta \eta &= -i D + i \sqrt{2} \Omega^m D_m \bar{\phi} \\
\delta \psi &= i \sqrt{2} D_m \bar{\phi} \\
\delta \bar{\phi} &= 2i \phi \bar{q} \\
\delta q &= \bar{\theta} \\
\delta \bar{q} &= \theta \\
\delta \nu_m &= -i \sqrt{2} D_m q \\
\delta \omega_{mn} &= (\Omega_m D_n \bar{q} - \Omega_n D_m \bar{q})
\end{align*}
\]

(8)

2.2 Observables and BPS states

In what follows we shall briefly remind the catalogue of BPS defects in 4d theory with \( \mathcal{N} = 2 \) SUSY in \( \Omega \)-background[83, 84, 51, 14]. The \( \mathcal{N} = 2 \) superalgebra in four dimensions admits three types of central charges which correspond one-, two- and three-dimensional defects, namely monopoles, strings and domain walls [75].

In most cases to find the tension of the defect we integrate the time component of the current because the defect is assumed to be static and hence stretched along time direction. But as was shown in [14], \( \Omega \)-background acts effectively as an external field which affects the motion of defects. Hence the static configuration is not realized. We assume the worldvolume of the defect to be curved and introduce unit vectors \( t^n \) tangential to the worldvolume and \( n^n \) normal to the worldvolume. The fields which solve the BPS equation are considered to be independent of the directions along the worldvolume,

\[
t^n D^n \varphi = 0.
\]

(9)

Then the tension of the defect is the component of the supervariation along the worldvolume integrated over the directions normal to the worldvolume. Namely, the tension of the string reads as:

\[
T_s = \frac{1}{\sqrt{2}} \int \left( -\frac{i}{2} \left[ \Phi, \bar{\Phi} \right] \left( F^{mn} + \bar{F}^{mn} \right) + i \varepsilon^{mnkl} D_k \Phi D_l \bar{\Phi} \right) dx^m dx^n,
\]

(10)
and the BPS equation which describes string is the following system:

\[
\begin{align*}
(F^{mn})n_1^mn_2^n &= i \left[ \Phi, \bar{\Phi} \right], \\
D_w \Phi &= 0, \\
D_z \Phi &= 0.
\end{align*}
\] (11)

\(z, w\) are complex coordinates for \(\mathbb{C}^2 = \mathbb{R}^4\).

The second and the third equations follow from the second term in the integrand of (10). Note that the vectors \(n_1, n_2\) in the first equation can be substituted by \(t_1, t_2\) since the combination \((F_{mn}^+)\) is self-dual.

The tension (or mass) of the monopole is given by:

\[
T_m = 2\sqrt{2} \int \left( \varepsilon_{ijkl} i \left[ \Phi, \bar{\Phi} \right] D^j \Phi + (F^{ij} + \bar{F}^{ij}) D^k \Phi \right) dx^i dx^j dx^k, \] (12)

and the BPS equation:

\[
D^m \Phi = -\frac{1}{\sqrt{2}} i[\Phi, \bar{\Phi}] t^m + \frac{1}{2\sqrt{2}} \left( F^{mn} + \bar{F}^{mn} \right) t^n. \] (13)

The term \([\Phi, \bar{\Phi}]\) is not seen for the static monopole hence the equation (13) implies the usual Bogomolny equation for the monopole.

The theory has several types of domain walls. The one we are interested in is build from fundamental matter scalar and Higgs field and requires FI term \(\xi\). This domain wall is the defect of codimension one, hence the scalar field which builds the wall depends only on one coordinate, \(\Phi = \Phi(y), y = \sqrt{\varepsilon_1 |z|^2 + \varepsilon_2 |w|^2}\). BPS equations read as:

\[
\begin{align*}
\sqrt{\varepsilon_1 \varepsilon_2} D_y \Phi &= \frac{\mu}{2\sqrt{2}} (|q|^2 - |\bar{q}|^2 - \xi), \\
\sqrt{\varepsilon_1 \varepsilon_2} D_y q &= \frac{1}{\sqrt{2}} (\Phi + \sqrt{2}m) q + \frac{1}{\sqrt{2}} (\Phi + \sqrt{2}\bar{m}) \bar{q}.
\end{align*}
\] (14)

In 5d theory the 4d defects are lifted one dimension higher - instanton becomes BPS particle, monopole - monopole string, domain wall remains to be domain wall. We shall be interested in the domain walls in 5d SQED which interpolate between the vacua where the scalar in the vector multiplet takes the values of the masses of light \(m\) and heavy \(\bar{m}\) flavors. This light-heavy domain wall exists in the undeformed theory is not deformed significantly by the \(\Omega\)-deformation. Its tension reads as

\[
T_{lh} = \xi (\bar{m} - m) \] (15)

where we assume that the other quantum numbers are the same. The vev of the adjoint scalar is

\[
< \phi >_1 = m, \quad < \phi >_2 = \bar{m} \] (16)

Note that there are nontrivial configurations involving several types of defects. In particular the configuration involving the domain wall string and monopole has been discussed in [51, 14]. We conjecture that such composite defect (see Figure 1) does not
break the scalar supersymmetry $Q$ and, therefore, is a realization of deformed chiral ring operators in the $\Omega$-deformed theory. Such deformation is needed, since the Higgs scalar $\phi$ is not invariant under the supersymmetry: $Q\phi = \Omega^m \psi_m$ (recall (8)). Although an explicit solution is not known, there is a non-trivial consistency check: trajectory of the monopole has to lie on the string worldsheet. And indeed, this is true[14]: string worldsheet in given by the Seifert surface:

$$z^{\epsilon_2} w^{-\epsilon_1} = \text{const}$$

(17)

whereas the trajectory of the monopole is a torus knot $(\epsilon_1, \epsilon_2)$ which is essentially the boundary of the Seifert surface.

Figure 1: Possible candidate for the composite defect with instantons sitting on it.

### 2.3 On Decoupling procedure

Decoupling of the heavy flavor in the 5d gauge theory is very delicate issue mainly due to the UV incompleteness of the theory. It was discussed in many studies that the naive field theory intuition fails and the purely stringy degrees of freedom like different D-branes emerge in the UV completion problem. It can be recognized in the different ways, for instance, from the viewpoint of the ADHM quantum mechanics which describes the UV physics from the viewpoint of the instanton particles. The ADHM quantum mechanics in this case involves the tiny issues at the threshold when the continuum spectrum opens. It was assumed that in this quantum mechanics the stringy degrees of freedom get manifested in the index calculations.

One more pattern of the nontrivial decoupling of the heavy degrees of freedom is provided by the 4d example of the decoupling of the heavy flavor [18]. The naive decoupling of the heavy flavor fails and one finds himself with the remnant surface operator supplemented by the operator acting in the flavor fugacity space. This operator was identified with the integrable Hamiltonian of the Calogero-Ruijsenaars type [18].
In our paper we shall meet the subtleties with the UV completion as well. We start with the theory with the heavy flavor and try to decouple it. During this process we get the particular 4-observable as the remnant which seems to be identified naturally with the domain wall in the Ω-deformed 5d SQED. This is to some extend analogous to the 4d case however the 4-observable emerges instead of 2-observable remnant. We shall also see how the information about the UV completion can be extracted from the ADHM quantum mechanics on the instanton moduli space. The knot invariants encodes the particular set of states near threshold.

On the quantitative level we shall get the following remnant of the heavy flavor in the following way. Although we start with three matter hypermultiplets, actually we need only two of them since one has infinite mass. Now we will show that the only effect from this heavy hypermultiplet is an insertion of the operator

\[ O = \int d^5x \exp(-\beta \phi), \]

where \( \phi \) is a vector multiplet scalar:

\[ \lim_{m_2 \to \infty} \frac{\exp(\beta m_2)}{\beta} \frac{\partial}{\partial m_2} Z^{U(1)}(m_1, m_2, m_3) = \langle O \rangle^{U(1)}_{m_1, m_3} \] (18)

Suppose for a while that we consider four-dimensional theory without omega-deformation. Then integrating out heavy hypermultiplet will produce usual Coleman-Weinberg potential

\[ (\phi + m_2)^2 \left( \log \left( \frac{\phi + m_2}{\Lambda_{UV}} \right) - 1 \right) = \int_0^\infty dt \frac{t}{t^3} \exp(-t(\phi + m_2)) \] (19)

In order to lift this expression to a five-dimensional theory[20], we have to sum over the Kaluza-Klein modes, that is, add \( \frac{2\pi in}{R} \) to \( \phi + m_2 \) and sum over \( n \). This will result in

\[ \text{Li}_3 \left( e^{-2\pi R(\phi + m_2)} \right) \] (20)

Which is for large \( m_2 \) is just \( \exp(-2\pi R(\phi + m_2)) \) So we have reproduced eq. (18) with the operator \( O = \int d^5x \exp(-\beta \phi(x)) \).

When switch to the Omega-deformed theory, almost all supersymmetries are broken and the chiral ring gets deformed. Appropriate deformation of operators \( \phi \) was build in [21] and we claim that the \( \exp(-\beta \phi) \) with deformed \( \phi \), is exactly the operator we need even in the Omega-deformed theory. In the next section we will demonstrate this statement by a direct computation.

We shall attribute this operator to the light-heavy domain wall very much in the spirit of [18] where the operator corresponding to the nonabelian string has been considered. In our case tension of the domain wall is proportional to the mass of the heavy flavor hence it becomes infinite in the decoupling limit. In the next Section we shall consider the instanton ensembles populating the domain wall.
### 3 Superpolynomial of torus knots and 5d SQED

In this Section we shall explore the localization formulas for the instanton Nekrasov-like partition sums in the 5d SUSY QED. Therefore we look for the proper physical theory which would involves the knot invariants in a rational clear-cut manner. In this paper we extend the proposal formulated in [7], which relates $q,t$-Catalan numbers represented the bottom row of the superpolynomials of $T_{n,n+1}$ torus knots.

We shall evaluate the K-theoric equivariant integral over the moduli space of the instantons. It is equal to equivariant Euler characteristic of the tautological line bundle $V$ over the Hilbert scheme:

$$C_n(q,t) = \chi^T(Hilb^n(\mathbb{C}^2), \Lambda^n V)$$  \hspace{1cm} (21)

where $q,t$ are equivariant parameters for the natural torus $T$ action on $\mathbb{C}^2$. $C_n(q,t)$ are called $q,t$-Catalan numbers.

First, recall some relevant mathematical results.

In [3] Haiman and Garsia introduced the following generalization of Catalan numbers:

$$C_n(q,t) = \sum_{\lambda : |\lambda| = n} t^2 \sum a(1-t)(1-q) \prod_{l,l',0,0} (1-q^{a'}t^{l'})(\sum q^{a'}t^{l'})$$  \hspace{1cm} (22)

where all sums and products are taken over partition $\lambda$. $l$ and $a$ denote leg and arm, whereas $l'$ and $a'$ denote coleg and coarm respectively. $\prod_{0,0}$ denote the omission of $(0,0)$ box. In case $q = t = 1$, $C_n(1,1) = \frac{1}{n+1} \binom{2n}{n}$

It is also useful to present the expressions for the so-called higher Catalan numbers introduced in [38]. They can be represented in terms of the Young diagrams as follows

$$C_n^k(q,t) = \sum_{\lambda : |\lambda| = n} t^{(k+1)} \sum a(1-t)(1-q) \prod_{l,l',0,0} (1-q^{a'}t^{l'})(\sum q^{a'}t^{l'})$$  \hspace{1cm} (23)

In what follows we shall identify the index $k$ with the level of 5d Chern-Simons term. The shift $k \rightarrow k+1$ corresponds to the decoupling of one flavor in the 5d supersymmetric SQED.

In [6] it was shown that these numbers calculate Poincaré polynomial for a plain curve singularity corresponding to $(n+1,n)$ torus knot. Furthermore, in [5] was conjectured the following expression for a superpolynomial for $(nk + 1, n)$ torus knot:

$$P(A,q,t)_{nk+1,n} = \sum_{\lambda : |\lambda| = n} t^{(k+1)} \sum a(1-t)(1-q) \prod_{l,l',0,0} (1 + Aq^{-a'}t^{-l'})(\sum q^{a'}t^{l'})$$  \hspace{1cm} (24)

$$\prod_{l,l',0,0} (1-q^{a'}t^{l'})(\sum q^{a'}t^{l'})$$  \hspace{1cm} (24)

$$\prod_{l,l',0,0} (1-q^{a'}t^{l'})(\sum q^{a'}t^{l'})$$  \hspace{1cm} (24)
In this paper we extend the proposal formulated in [7], which relates $q,t$-Catalan numbers and certain vacuum expectation value in five-dimensional $U(1)$ gauge theory in the $\Omega$-deformation. We claim that the above superpolynomial could be obtained via five-dimensional $U(1)$ gauge theory with 2 fundamental flavors with masses $m_f, M$, one anti-fundamental flavor with the mass $m_a$ and Chern-Simons term with the coupling $k$:

$$P(A, q, t)_{n, n+1} = t^{-n/2} q^{-n/2} \frac{1}{1 + A} \frac{\exp(\beta M)}{\beta^2} \frac{\partial}{\partial m_f} \frac{\partial}{\partial M} Z_{n}^{U(1)}(m_f, m_a, M), \ m_f = 0, \ M \rightarrow \infty$$

(25)

where $Z_{n}^{U(1)}$ is $n$-instanton contribution to the partition function.

**NB:** our choice of variables is different from one adopted in [5]. We will perform the identification of variables when we discuss various limits of these formulas.

\[Q_f, \mu_1\overrightarrow{\bigtriangledown} Q, \lambda \overrightarrow{\bigtriangledown} Q_a, \mu_2\]

Figure 2: $\mathcal{O}(-1) \times \mathcal{O}(-1) \rightarrow \mathbb{P}^1$ with two blow-ups corresponding to the 5D SQED with two flavors.

One of the dimensions is compactified on a circle with radius $\beta$. We denote the $\Omega$-background parameters by $\epsilon_1$ and $\epsilon_2$. Then

$$t = \exp(-\beta \epsilon_1) \quad \quad \quad (26)$$

$$q = \exp(-\beta \epsilon_2) \quad \quad \quad (27)$$

$$A = -\exp(\beta m_a) \quad \quad \quad (28)$$

We are going to prove this relation using the refined topological vertex technique[8]. According to [9], the full partition function in the case of one fundamental flavor and one anti-fundamental flavor is given by.a:

$$Z_{n}^{U(1)}(m_f, m_a) = \sum_{\lambda} (-Q)^{|\lambda|} t^{\lambda/2} q^{\lambda/2} \chi$$

\[a\text{note that in our notations } t \rightarrow 1/t\]
\[
\frac{t^\sum l \sum q \prod_{i=1,j=1}^\infty (1 - Q_f q^{i-1/2} t^{l_i -j +1/2})(1 - Q_a q^{-\lambda_i^j + 1/2} t^{1/2-i})}{\prod (t^l - q^{a+1})(t^{l+1} - q^a)}
\]  

(29)

where Kähler parameters: 
\[Q_f = \exp(-\beta m_f)/\sqrt{q^l}, \quad Q_a = \sqrt{q^l} \exp(\beta m_a), \quad Q_f \text{ defines the coupling constant via } Q = \exp(-\beta/g).\]

Corresponding three-dimensional Calabi-Yau geometry is represented on the Figure 2. Perturbative part is given by:

\[
Z^{U(1),\text{pert}}(m_f, m_a) = \prod_{i=1,j=1}^\infty (1 - Q_f q^{i-1/2} t^{-j +1/2})(1 - Q_a q^{-1/2} t^{1/2-i})
\]  

(30)

Then n-instanton contribution is given by:

\[
Z^{U(1)}_n(m_f, m_a) = \sum_{|\lambda|=n} (-Q)^{|\lambda|} t^{|\lambda|/2} q^{|\lambda|/2} \times
\]

\[
\frac{t^\sum l \sum q \prod (1 - \exp(-\beta m_f) t^l q^a')(1 - \exp(\beta m_a) t^{-l'} q^{-a'})}{\prod (t^l - q^{a+1})(t^{l+1} - q^a)}
\]  

(31)

Factors like

\[
\prod (1 - \exp(-\beta m_f) t^l q^a')
\]  

(32)

correspond to chiral matter contribution. \(U(1)\) gauge part contributes:

\[
\sum_{|\lambda|=n} (-Q)^{|\lambda|} t^{|\lambda|/2} q^{|\lambda|/2} \frac{t^\sum l \sum q \prod}{\prod (t^l - q^{a+1})(t^{l+1} - q^a)}
\]  

(33)

Therefore, for the superpolynomial we need:

- In order to obtain a factor \(\prod 0^0 (1 - q^{a'} t^l')\) in the superpolynomial we have add a zero mass chiral multiplet and differentiate with respect to its mass.
- To obtain \(\sum q^a' t^l'\) we take another chiral multiplet, differentiate with respect to its mass and after that we send the mass to infinity.
- Factor \(\prod 0^0 (1 + a q^{-a'} t^{-l'})\) comes from the anti-fundamental multiplet.

Finally, it is well-known that the Chern-Simons action with the coupling constant \(k\) contributes \(t^k \sum q^k \sum a\) - it can be easily seen in the above formulas if we remember that the Chern-Simons term emerges as a one-loop effect from \(k\) very heavy chiral multiplets. However, on the level of the corresponding Calabi-Yau manifold, Chern-Simons coupling affects the whole geometry. The coupling is actually given by the intersection number of two-cycles on the manifold[81, 82]. For example, the theory with coupling \(k = -1\) and without flavors is given by geometry \(\mathcal{O}(0) \times \mathcal{O}(-2) \rightarrow \mathbb{P}^1\) - see Figure 3. Recall, that the n-instanton contribution arises from the worldsheat instanton wrapping the base \(\mathbb{P}^1\) \(n\) times. This fact suggests that the shift \(1 \rightarrow kn + 1\) in the torus knot is actually an analogue of the Witten effect, since the instanton wraps around the other two-cycle \(kn\) additional times and acquires additional charge.
Figure 3: $O(0) \times O(-2) \rightarrow \mathbb{P}^1$ geometry corresponding to 5D SQED with the Chern-Simons coupling $k = -1$, but without any flavors.

Now let us return to the operator $\exp(-\beta \phi)$. Consider the term $(-1)(1-t)(1-q) \sum q^{\alpha_i t_i}$ in the original expression for a superpolynomial. If $\lambda_i$ is the length of $i$-th row, then we can rewrite it as

$$(1-t) \sum_{i=1} (q^{\lambda_i t_i - 1} - t^{i-1}) = \sum_{i=1} (q^{\lambda_i t_i - 1} - t^{i-1} - q^{\lambda_i t_i} + t^i)$$  \hspace{1cm} (34)$$

Where the sum is over rows in a particular Young diagram $\lambda$. But the last expression is exactly what we will obtain if we calculate the vev of $\exp(-\beta \phi)$ using Nekrasov formulas([21] eq. (4.19)) - in this approach one introduces the profile function:

$$f_{\lambda, \epsilon_1, \epsilon_2}(x) = |x| + \sum_{i=1} \left( |x + \epsilon_1 - \epsilon_2 \lambda_i - \epsilon_1 i| - |x - \epsilon_2 \lambda_i - \epsilon_1 i| - |x + \epsilon_1 - \epsilon_1 i| + |x - \epsilon_1 i| \right)$$ \hspace{1cm} (35)$$

Then contribution to the vev of $\phi^n$ is given by

$$\frac{1}{2} \int_{-\infty}^{+\infty} dx \ x^n f'_{\lambda, \epsilon_1, \epsilon_2}(x)$$ \hspace{1cm} (36)$$

We see that even in the presence of Omega-deformation decoupling of the heavy flavor leads to the insertion of $\exp(-\beta \phi)$.

Now we can evaluate the whole partition function summing over the instanton contributions

$$Z^{Nek}(m, \epsilon_1, \epsilon_2, \beta, Q) = \sum_n (-Q)^n Z_n(m, \epsilon_1, \epsilon_2, \beta)$$ \hspace{1cm} (37)$$

Where the whole partition function obeys some interesting equations as a function of its arguments. In the NS limit when $\epsilon_2 = 0 \rightarrow t = 1$ the summation of q-Catalan numbers can be performed explicitly and yields[14]:

$$P(q, Q) = \frac{\exp(\beta M)}{\beta^2} \frac{\partial}{\partial m_f} \frac{\partial}{\partial M} Z^{Nek}(m_f, M, q, Q/\sqrt{q}), \ m_f = 0, \ M \rightarrow \infty = \frac{A_q(Qq^2)}{A_q(Qq)} \hspace{1cm} (38)$$
where $A_q(s)$ is the $q$-Airy function:

$$A_q(s) = \sum_k s^k q^{k^2} \frac{k!}{(q; q)_k}$$

(39)

where $(z; q)_k = \prod_{l=0}^{k-1} (1 - zq^l)$ is Pochhammer symbol. This implies that the partition function obeys the following relation:

$$P(q, Q) = 1 - QP(q, Q)P(q, qQ)$$

(40)

Unfortunately, we do not know any field-theoretic explanation of this relation. We will return to this question when we will be discussing the stable limit $k \to \infty$.

4 The Attempt of interpretation

4.1 Point-like Abelian instantons. Ways to blowup

In this Section we shall consider the physical picture behind the duality found. It implies that we have in $Q = n$ sector $n$ point-like instantons sitting almost at the top of each other in the background provided by the nonlocal operator $\exp(-\beta \Phi)$. When the $\Omega$ deformation is switched off the operator becomes local therefore the physical picture we shall try to develop should respect this property. Another suggesting argument goes as follows. Consider the limit of $\epsilon_1, \epsilon_2 \to 0$ when the Nekrasov partition function is reduced to the form

$$Z_{Nek} \propto \exp\left(\frac{F}{\epsilon_1 \epsilon_2}\right)$$

(41)

Having in mind that $\epsilon_1, \epsilon_2$ are two angular velocities the simple argument shows that there is the average angular momentum $< J > \neq 0$ in the system [54] and one could say about the gravimagnetization of the ground state. Combining these arguments we could suspect that the microscopic state we are dealing with is built from the regulator degree of freedom, is nonlocal, has instanton charge $n$ and some angular momentum. This is the qualitative description of the part of this extended object in $R^4 \times S^1$.

Note that a somewhat similar situation occurs in the description of the nonperturbative effects in the ABJM model [69] where the membrane M2 instantons wrapping the $(m, n)$ cycle in the internal space yields the corresponding contribution to the partition function at strong coupling regime. The brane interpretation of the nonperturbative effects at weak coupling is not completely clarified in that case.

We have to combine two parts into the worldvolume of some brane. First of all let us comment what are natural configurations in D=5 which obey the required property. The first candidate is the dyonic instanton [70] or its supergravity counterpart - supertube. The dyonic instanton has the instanton charge $Q$, F1 charge $P$ and D2 dipole charge/ It has the geometry of the cylinder with the distributed charge densities and its angular momentum is proportional to the product of two charges $J \propto PQ$. 

16
In the other duality frame it is presented by the D3 brane with the KK momentum [102]. In this case the defect could be represented by the M5 brane supplemented by the instanton charges.

The second candidate is the D6-D0 state which corresponds to the rotating black hole [71]. This configuration can be BPS in some region of parameters [56]. From the field theory viewpoint it represents the domain wall configuration in D=5 gauge theory which carries the additional angular momentum. In the 4d dimensional Ω-deformed gauge theory such closed domain wall does exist [14] and has the geometry of the squashed sphere $S^3_b$ where $b^2 = \epsilon_1 \epsilon_2$. Therefore the candidate defect would have the worldsheet $S^3_b \times S^1 \times M$ in this case. Moreover more complicated defect involving the domain wall string and monopole can be considered in the Ω-deformed 4D theory [14] and presumably can be lifted to D=5. Note that as we have mentioned before there are different types of the domain walls. Some of them exist in the theory without Ω-deformation while some of them exist only we deformation is switched on.

In all cases we assume that the key contribution for the mechanism preventing the closed object from shrinking is the angular momentum. When the SUSY is broken in some way the additional source come from the difference between the energies providing the stabilizing pressure. It is natural to expect that the nonperturbative configuration is sensitive to the Ω background and moreover we assume that the defects like strings and domain wall have the infinite tension being proportional to the mass of the regulator. However the naive argument could fail if the very heavy object is dressed by the other instanton-like configurations which yields via dimensional transmutation the factor

$$\Lambda = M \exp(-c/g^2(M))$$  \hspace{1cm} (42)

and potentially could yield nonvanishing contribution.

The important question concerns the stability of the composite defect we are identifying. There are two separate questions- it can shrink to a point or expand. We have already mentioned that there are at least two mechanisms preventing the composite defect from shrinking; the additional defects of lower dimensions with their own quantum charges and/or the rotation. In our case it seems that both mechanisms are involved. The dyonic instanton and the rotating black hole both have the angular momentum and additional quantum numbers.

The issue of instability which would result in expanding is more complicated. It is necessary to identify the presence or absence of the negative modes at the composite defect which is not a simple task. Is there are the odd number of negative modes at the configuration it would mean that this defect corresponds to the bounce describing the Schwinger-type process of the creation of the extended object in the graviphoton field.

### 4.2 From UV to IR on the defect

Recently the interesting approach for the evaluation of the superconformal indexes with the surface defects of has been suggested in [18, 108, 107]. The idea is based on
the realization of the bootstrapping program via particular pattern of RG flow. The aim is to evaluate the index in some quiver-like IR theory with the defect. Instead one enlarges the theory adding the hypermultiplet in the bifundamental representation Q with respect to the say $SU(N) \times SU(N)$ and consider the UV theory first. Since the initial IR theory is conformal the addition hyper brings the Landau pole into the problem. The FI term is added to the Lagrangian which forces the hypermultiplet to condense and fixes the scale in the model.

At the next step one would like to decouple additional flavor in bifundamental at the scales much lower then one fixed by its condensate. The decoupling could proceed in two different ways. If the background in UV theory is trivial the decoupling goes smoothly and we return down to the initial IR theory. However one can select more elaborated way and start with the nontrivial configuration at UV scale. In [18] one selects the nonabelian vortex configuration (see [75, 70, 101] for the review). The corresponding condensate of Q becomes inhomogeneous and at IR the theory becomes the same IR theory without additional hyper but with additional surface operator which now is the nonabelian string with the infinite tension.

This general picture of RG flows with the nonperturbative defects turns out to be very useful and provides new tool for the evaluation of indexes. It turns out that the index of the UV theory allows the integral representation which has the interesting pole structure. The residues of the particular poles in the index can be identified with the indexes of IR theory supplemented by the surface operators with some flux r. There are poles corresponding to the surface operators with the different fluxes. Moreover the index in the IR theory with the defect with flux r can be identified with the action of the particular difference operator $G_r$ with respect to the flavor fugacities acting on the IR index without the defects

$$I_r \propto G_r I_0$$

(43)

This operator was identified with the Ruijsenaars-Schneider (RS) operator known to be integrable. The trigonometric RS model is nothing but the CS theory perturbed by two Wilson loops in different directions [78](see Appendix D).

Upon deriving of the superconformal index in IR theory with defect one could be interested in the additional algebraic structure behind. It was found in [108] that the operators $G_r$ form a nontrivial algebra. The surface operators are realized by D2 branes with $R^2 \times S^1$ worldsheets and it was demonstrated that the Wilson loop along this $S^1$ emerges in the CS theory on $S^1 \times C$ where C is the curve defining the superconformal theory. The following correspondence takes place

$$G_r \leftrightarrow <W_r>$$

(44)

where the Wilson loop in the representation r is evaluated. Algebra of the operators $G_r$ gets mapped into the Verlinde algebra in the CS theory.

Let us combine together all arguments above and add some additional viewpoints which will be suggested later by the AGT dual picture. We are not able to describe precisely the composite state which is a kind of blow up of the point-like instantons
now but our candidate state looks as follows. In $R^4$ there is a closed domain wall with the geometry of the squashed sphere $S^3_b$, there is the short nonabelian string inside the sphere and the point-like instantons at the center of the sphere. The wall seems to be the compound of two walls one which exists without the $\Omega$ deformation and another one which separates two vacua with the different Bohr-Sommerfeld quantum numbers. Since it involves the wall between the light and regulator degrees of freedom we could to some extend tell that the regulator degrees of freedom are slaved inside the domain wall. It is not clear if the nonabelian string inside the wall is open or closed. If it is open then there are the boojum-like states at the intersection of the string and domain walls. This would be a kind of the "rigid boundary condition" in the terminology of [96]. If it is closed it should involve the monopole-antimonopole pair which provide it from shrinking. In this case the picture looks like the meson state in the "bag model". Remind that the instantons inside the bag are the crucial ingredient of the picture in both cases. At the CY side the defect is extended as well and wraps the compact cycles. It this respect it is the analogue of the membrane instanton in the ABJM model.

4.3 Analogy with QCD and CP(N) model

Let us comment on the related questions which can be raised in QCD and its two-dimensional "counterpart" which shares many features of QCD [64]- $CP(N)$ sigma-model. Now we know well the origin of this correspondence - it is just the matching condition between the theory in the bulk and the worldsheet theory on the defect. The analogous problem in QCD would concern the Casher-Banks relation for the chiral condensate relating it with the spectral density of the Dirac operator $\rho(0)$.

$$<\bar{\Psi}\Psi> = -\pi \rho(0)$$

(45)

The fermionic zero mode at the individual instanton is senseless in the QCD vacuum since we have strongly interacting instanton ensemble however the collective effect from the instanton ensemble yields the nonvanishing density at the origin.

Now the question parallel our study would concern the response of the quark condensate on the "mass of the regulator $M$" since we are looking at the derivative of the condensate with respect to the mass of the heavy flavor $M$. Speaking differently we are trying to evaluate the effect of the point-like instantons at the Dirac operator spectral density. The possible arguments could look as follows. We can consider the well-known path integral representation for the quark condensate in terms of the Wilson loops (see, for instance [86])

$$<\bar{\Psi}\Psi> = \sum_{\text{paths}} [DC] <W(C)>$$

(46)

where the measure over the paths is fixed by the QCD path integral and the vev of the Wilson loop involves the averaging over all configurations of the gauge field. Therefore using this representation we could say that we are searching for the back reaction of the UV degrees of freedom at the vev of the Wilson loop.
Since from our analysis we know that the key players are the point-like instantons we could wonder how they could affect the Wilson loop. The natural conjecture sounds as follows. It is known that the Wilson loop renormalization involves the specific UV contribution from the cusps [87]. Therefore one could imagine that the point-like instantons placed at the Wilson loop induces the cusps or self-intersections of the Wilson loops and therefore yield the additional UV renormalization of the quark condensate. If this interpretation is correct it would imply that the cusp anomalous dimension which on the other hand carries the information about the anomalous dimensions of the QCD operators with the large Lorentz spin should be related with the torus knot invariants.

The first quantized picture of the condensate is useful in another respect as well. We know from the old Witten’s picture that the knot invariants are evaluated from the vev of the Wilson loops in the 3d CS theory. How could we reproduce this picture from our consideration? The possible line of reasoning goes as follows. First, we use the first quantized picture for the condensate and represent it as the infinite sum over the vev of Wilson loops of the arbitrary shape. Then via a kind of “localization” which we do not know yet the sum over the paths gets localized at the infinite set of the torus knots, the 5d CS term works as the 3d CS term and the derivative of the condensate plays the role of the generating function for the torus knot superpolynomials where the instanton counting parameter plays the role of the generating parameter. To get the superpolynomial of the fixed torus $T_{n,nk+1}$ knot it is necessary to perform a kind of “Fourier transform” with the derivative of the condensate with respect to the coupling constant.

Another way to approach the question is to use the low-energy theorems [73]. The correlator we are looking at the SUSY theory is now the correlator of the bilinears of the massless and the regulator fields. Due to the low-energy theorems we get

$$\int d^4x \langle \bar{\Psi}(0)\bar{\Psi}_R\psi_R(x) \rangle \propto \langle \bar{\Psi}\Psi \rangle$$  \hspace{1cm} (47)

which however knows about the “perturbative” dilatational Ward identity and one could be interested how the point-like instantons affect this relation. We shall see later that the similar result in the SUSY case can be reformulated in the Liouville AGT side in terms of the similar low-energy theorem “dressed” by the small instantons.

How similar problem could be posed in the non-SUSY CP(N) model which can appear as the theory on the defect [72]? The analogue of the closed domain wall considered above is the kink-antikink bound state which is true excitation at the large N [63]. The analogous picture looks as follows. We have one vacuum in the non-SUSY CP(N) model however there is the excited vacuum between the kink-antikink pair. Following our analysis we could conjecture that this excited vacuum is the analogue of the ”regulator vacuum” in the SUSY case separated by the kinks. Naively it involves the large scale and can just decouple but the kink and antikink can be dressed by the point-like instantons similar to the dressing of the domain walls by the instantons. As a result of dressing the finite $\Lambda$ scale emerges and the kink-antikink state remains in the spectrum.
Finally note that the chiral condensate gets generated in the QED in the external magnetic field \[118\]. Naively it can be traced from the summation over the lowest Landau level. The analogous question sounds as follows: Is there the interplay between the fermion condensate and the high Landau levels which are a kind of regulators in this problem. Apparently more involved analysis demonstrated that the higher Landau levels matter for the condensation and there is interplay between the IR and UV physics once again.

5 Different limits

5.1 Down to HOMFLY, Jones and Alexander

In order to compare the superpolynomial with other knot invariants, let us rewrite formulas from the previous section in a bit different notation. It is convenient to change to the following variables:

\[
\frac{1}{\tilde{q}^2\tilde{t}^2} = q = \exp(-\beta \epsilon_1) \tag{48}
\]

\[
\tilde{q}^2 = t = \exp(-\beta \epsilon_2) \tag{49}
\]

\[
\tilde{a}^2\tilde{t} = A = -\exp(\beta m_a) \tag{50}
\]

And inverse:

\[
\tilde{t} = -\frac{1}{\sqrt{\tilde{q}t}} = -\exp(\beta(\epsilon_1 + \epsilon_2)/2) \tag{51}
\]

\[
\tilde{q} = \sqrt{\tilde{t}} = \exp(-\beta \epsilon_2/2) \tag{52}
\]

\[
\tilde{a} = \sqrt{-\sqrt{\tilde{q}}A} = \exp(\beta \frac{2m_a - \epsilon_1 - \epsilon_2}{4}) \tag{53}
\]

- $\tilde{t} = -1$: The superpolynomial reduces to HOMFLY. On the field theory side we have $\epsilon_1 + \epsilon_2 = 0$

- $\tilde{a} = \tilde{q}^N$ corresponds to the quantization condition in NS limit when there is no vev of the scalar. The mass of the antifundamental gets quantized $m_a = \frac{(2N - 1)\epsilon_2 - \epsilon_1}{2}$ and reduction to the bottom raw or Catalans corresponds to the semiclassical limit in NS quantization. What is more, if we take $\tilde{t} = -1$ we obtain Jones polynomial for the fundamental representation of $sl(N)$

- $\tilde{a} = 1$: we obtain Poincare polynomial for Hegaard-Floer homologies and mass of the anti-fundamental multiplet reads as $m_a = -\frac{\epsilon_1 + \epsilon_2}{2}$. Further specification $\tilde{t} = -1$ yields Alexander polynomial. Hence the sum over Alexanders corresponds to the condensate in the case of massless antifundamental. It has some interesting realization at the CY side summarized in the Appendix A.
5.2 Up to stable limit

Consider the limit \( k \to \infty \) which yields the \( T_{n,\infty} \) torus knot. At the gauge theory side it corresponds to the dominance of CS in the action.

The additional unexpected structure emerges in the stable limit [42]. It turns out that the superpolynomial allows two different "bosonic" and "fermionic" representations. Moreover it has the structure of the character of the very special representation in \( S\hat{L}(2) \) at level 1 introduced by Feigin and Stoyanovsky.

Where \( S\hat{L}(2) \) at level 1 could appear from? The possible conjecture sounds as follows. We have to recognize the knot invariants from the viewpoint of the theory on the flavor branes as well. We have two hypermultiplets, one in fundamental and one in antifundamental. The theory on their worldvolumes should enjoy \( SL(2) \) gauge group instead of \( SU(2) \) for two fundamentals. It is the analogue of the Chiral Lagrangian realized as the worldvolume theory on the flavor branes. Similar to the QCD we have the CS term here as well and the coefficient in front of it equals to the number of colors. In our abelian case we immediately arrive at the level 1 as expected. The "Chiral Lagrangian" in our case could have Skyrmion solutions like in QCD and we conjecture that the Feigin-Stoyanovsky representation is just representation in terms of Skyrmions.

Another question which can be asked in the stable limit is the A-polynomial. The point is that superpolynomial of uncolored \( T_{n,\infty} \) torus knot gives superpolynomial of unknot colored by the n-th symmetric representation \( M_n \):

\[
P_{n,\infty}(a,q,t) = \frac{P_n(a,q,t)}{M_1(a,q,t)}
\]

These are given by the so-called MacDonald dimensions[47]. Therefore we could investigate the dependence of the superpolynomial on the representation which is governed by the super-A-polynomial[97] of the knot. For the unknot in the symmetric representation, in our normalization it reads as

\[
\hat{A}(a,q,t,\hat{x},\hat{y}) = \frac{ta\hat{x} - a^{-1}\hat{x}^{-1}}{\hat{x}q - \hat{x}^{-1}q^{-1}} + \sqrt{-t}\hat{y}
\]

where operators \( \hat{x}, \hat{y} \) act as

\[
\hat{y}M_n = M_{n+1}, \quad \hat{x}M_n = (-tq)^n M_n
\]

and quantum A-polynomial annihilates superpolynomial:

\[
\hat{A}M_n = 0
\]

Note that this equation actually connects two different instanton contributions \( M_n \) and \( M_{n+1} \). Recall that in the NS limit \( \epsilon_2 = 0 \) but for a generic \( k \), we have found similar relation (40) which relates different instanton contributions too. These relations are similar in the spirit to non-perturbative Dyson-Schwinger equations introduced recently by N. Nekrasov[59]. We hope to discuss this issue elsewhere[85].
5.3 Nabla - Shift - Cut-and-join operator and the decoupling of heavy flavor

Let us make few comments concerning the role of operator providing the transformation $k \rightarrow k + 1$ in our picture. It has different reincarnations and different names in the several problems. It is known as Nabla operator in the theory of the symmetric functions, as the shift operator in the rational DAHA and Calogero model and cut-in-join operator in the context of some counting problems in geometry. Since we have identified this parameter as the level of 5D CS term we could enjoy this knowledge and see the different interpretations of this shift.

From the physical side it can be immediately recognize as the effect of the decoupling of the heavy flavor since it is know for a while [46] that one-loop effect provides this shift of the level. It provides the shift of the Calogero coupling constant in the quantum mechanics on the instanton moduli space and more formally it corresponds to the multiplication of the integrand over the instanton moduli by the determinant bundle [45]. It can be also seen at the CY side when it corresponds to the change of the geometry.

In the consider action of the $T_{n, nk+1}$ torus knots this operator was used [47] to generate a kind of the discrete Hamiltonian evolution in $k$ with the simple boundary condition for $k=0$, corresponding to unknot. Having in mind that the dynamics of the Calogero coupling can be interpreted as the realization of the RG evolution with the limit cycles [53] it would be interesting to look for the cyclic solutions to this discrete Hamiltonian dynamics and possible Efimov-like states in this framework.

To illustrate these arguments let us consider the limit $m_a \rightarrow \infty$, that is $A \rightarrow +\infty$. From physical viewpoint, we integrate out antifundamental multiplet, so the Chern-Simons coupling should reduce by one. Indeed, the following limit is well-defined:

$$\lim_{A \rightarrow +\infty} \frac{1}{A^{n-1}} P(A, q, t)_{nm+1, n} = P(A = 0, q, t)_{n(m-1)+1, n}$$

(58)

Actually, this relation is known in the knot theory and is quite general:

$$\lim_{A \rightarrow +\infty} \frac{1}{A^{n-1}} P(A, q, t)_{k, n} = \text{const} P(A = 0, q, t)_{k-n, n}$$

(59)

The cut-and-join operator was identified in [47] as the $W_3^0$ generator from $W_3$ algebra. This fits well with our consideration since the CS term written in superfield looks as follows

$$\delta L_{CS} = \int d^5 x d^4 \Phi^3$$

(60)

It is possible to develop the matrix model of the Dijkgraaf-Vafa type for the 5D gauge theory [82] which can be considered as the generation function for the superpolynomials of the $T_{n, nK+1}$ torus knots. The matrix model evaluation of the particular observable in the general case of all nonvanishing masses looks as

$$Z_{matr} = \int [dM] O(m_1) O(m_2) O(m_3) exp(t_2 Tr M^2 + t_3 Tr M^3)$$

(61)
with some measure probably suggested in [67] and the coefficient \( t_3 \) corresponds to the CS term. It is clear that in the matrix model framework it is coupled to the corresponding \( W_3^0 \) generator. The knot invariants presumably can be evaluated upon taking derivatives with respect to \( m_1, m_2 \) and the corresponding limits. The operators \( O(m) \) are conjectured to be

\[
O(m) = \text{det}(M - m)
\]

which can be evidently related with the resolvents. The type of the knot presumably is selected by the corresponding term of expansion in \( t_2 \) with fixed value of \( t_3 \).

Therefore the shift operator can be thought of as one of the consequences from the generalized Konishi relation in the 5D gauge theory yielding the W-constraints in the matrix model. If one introduces more times more general Virasoro and W-constraints can be formulated for the torus knots superpolynomials (see [55] for the related discussion). Let us emphasize that the matrix model with the cubic potential is different from the matrix model developed for the evaluation of the torus knot invariants from the type B topological strings in [29, 74].

### 6 AGT conjecture prospective

In this section we will continue our study of the five-dimensional SQED with two fundamental flavors in the \( \Omega \) deformation. We will establish a connection between 5D SQED and the three-point function in the q-deformed Liouville theory on a sphere. We argue that the three-point function is equal to the combination of four instanton partition functions.

In the pioneer work [68] it was shown that the perturbative part of the Nekrasov partition function for the four-dimensional \( SU(2) \) theory with four fundamental flavors actually coincides with the three-point function \( \langle e^{2\alpha_1 \phi} e^{2\alpha_2 \phi} e^{2\alpha_3 \phi} \rangle \) in the Liouville theory on a sphere, also known as a DOZZ factor [65, 66]:

\[
C(\alpha_1, \alpha_2, \alpha_3) = (\pi \mu \gamma(b^2) b^{2-2b^2})^{(Q-\alpha_1-\alpha_2-\alpha_3)/b} \times
\]

\[
\Upsilon'(0) \Upsilon(2\alpha_1) \Upsilon(2\alpha_2) \Upsilon(2\alpha_3)
\]

\[
\Upsilon(\alpha_1 + \alpha_2 + \alpha_3 - Q_c) \Upsilon(\alpha_1 + \alpha_2 - \alpha_3) \Upsilon(\alpha_1 + \alpha_3 - \alpha_2) \Upsilon(\alpha_2 + \alpha_3 - \alpha_1)
\]

where we have adopted the standard notation for the Liouville theory: the central charge is given by \( c = 1 + 6Q_c^2, Q_c = b + 1/b \). And \( \Upsilon \) is a combination of two Barnes' functions:

\[
\Upsilon(x) = \frac{1}{\Gamma_2(x|b, b^{-1}) \Gamma_2(Q - x|b, b^{-1})}
\]

One can think about the Barnes' function \( \Gamma_2 \) as a regularized product:

\[
\Gamma_2(x|\epsilon_1, \epsilon_2) = \prod_{n,m=0}^{+\infty} (x + m\epsilon_1 + n\epsilon_2)^{-1}
\]
- for the case of the combination \( \frac{\Gamma_2(x)\Gamma_2(y)}{\Gamma_2(x+z)\Gamma_2(y-z)} \) this is a precise prescription.

In [60] and later in [61, 62, 10] it was argued that the lift to the five-dimensional theory corresponds to the q-deformation on the CFT side. Now we are going to extend the proposal of [10], which connects the full (including both perturbative and non-perturbative) Nekrasov partition function for the 5D Abelian theory with two flavors with the q-deformed DOZZ factor in the case \( c = 1 \). We argue that the same relation holds for the general central charge. However, in our approach we will need the combination of several instanton partition functions in order to obtain a single DOZZ factor.

First of all, let us recall the q-deformation of Barnes’ double gamma function, which is closely related to the MacMahon function. Again, we will not need a precise definition, since we are interested in ratios of four such functions:

\[
\Gamma_2^\beta(x|\epsilon_1, \epsilon_2) = \prod_{i,j=0} (1 - \exp(-\beta(x + i\epsilon_1 + j\epsilon_2)))^{-1} \tag{66}
\]

And correspondingly:

\[
\Upsilon^\beta(x) = \frac{1}{\Gamma_2^\beta(x|b, b^{-1})\Gamma_2^\beta(Q-x|b, b^{-1})} \tag{67}
\]

Following [10], we define q-deformed DOZZ factor by substituting Barnes’ functions by their q-analogues. However, we will omit the factor \((\pi \mu \gamma(b^2))^{b^2-2b^2}(Q-\alpha_1-\alpha_2-\alpha_3)/b\) since it can be absorbed into the definition of vertex operators:

\[
C^\beta(\alpha_1, \alpha_2, \alpha_3) = \frac{\Upsilon^\beta(0)\Upsilon^\beta(2\alpha_1)\Upsilon^\beta(2\alpha_2)\Upsilon^\beta(2\alpha_3)}{\Upsilon^\beta(\alpha_1 + \alpha_2 + \alpha_3 - Qc)\Upsilon^\beta(\alpha_1 + \alpha_2 - \alpha_3)\Upsilon^\beta(\alpha_1 + \alpha_3 - \alpha_2)\Upsilon^\beta(\alpha_2 + \alpha_3 - \alpha_1)} \tag{68}
\]

Returning to the 5D partition function, it will be useful to consider a bit different representation for the partition function from the section 3[9]:

\[
Z_{\text{inst}} = \prod_{i,j=0} \frac{(1 - QQ_a\sqrt{qt}q^{i+1/2}t^{j+1/2})(1 - QQ_f\sqrt{qt}q^{i+1/2}t^{j+1/2})}{(1 - QQ_f^{i+1/2}t^{j+1/2})(1 - QQ_a^{i+1/2}t^{j+1/2})} \tag{69}
\]

where we have used an analytic continuation:

\[
\prod_{i,j=1} (1 - Qq^{-1/2}t^{i-j}) = \prod_{i,j=1} (1 - Qq^{-1/2}t^{1/2-j})^{-1} \tag{70}
\]

Kähler parameters for the coupling constant, fundamental and antifundamental masses read as:

\[
Q = \exp(-\beta/g), \quad Q_f = \exp(-\beta m_f)/\sqrt{qt} = \mu_f/\sqrt{qt}, \quad Q_a = \sqrt{qt} \exp(\beta m_a) = \sqrt{qt} \mu_a \tag{71}
\]
Finally, the partition function can be rewritten as:

\[
Z_{\text{inst}}(1/g, m_f, m_a) = \frac{\Gamma^\beta(1/g - m_a + \epsilon_1/2 + \epsilon_2/2|\epsilon_1, \epsilon_2)\Gamma^\beta(1/g + m_f + \epsilon_1/2 + \epsilon_2/2|\epsilon_1, \epsilon_2)}{\Gamma^\beta(1/g + \epsilon_1/2 + \epsilon_2/2|\epsilon_1, \epsilon_2)\Gamma^\beta(1/g + m_f - m_a + \epsilon_1/2 + \epsilon_2/2|\epsilon_1, \epsilon_2)}
\]

(72)

We see that the DOZZ factor and the 5D partition function are strikingly similar. First of all, we can establish usual relation in AGT correspondence:

\[
b = \epsilon_1, b^{-1} = \epsilon_2, Q_c = \epsilon_1 + \epsilon_2
\]

(73)

Then, it is straightforward to obtain the following expression:

\[
\Upsilon^\beta(0) = \frac{1}{\beta} \prod_{i,j=0} (1 - q^i t^j)_{0,0} (1 - \exp(-\beta Q_c) q^i t^j) (74)
\]

where the subscript 0, 0 denotes the omission of the \(i = j = 0\) term.

After trivial manipulations with various factors we arrive at the following identification:

\[
C^\beta(\alpha_1, \alpha_2, \alpha_3) = \frac{1}{Q \sqrt{qt}} \frac{\partial Z_1^3}{\partial m_f} Z_2^2 Z_4^4
\]

(75)

where

\[
Z_1^1 = Z_{\text{inst}}(g^{-1} = \frac{Q_c}{2} + \alpha_1 - \alpha_2 - \alpha_3, m_f = -g^{-1} - \frac{Q_c}{2}, m_a = g^{-1} + \frac{Q_c}{2} - 2\alpha_1)
\]

(76)

\[
Z_2^2 = Z_{\text{inst}}(\frac{Q_c}{2} - 2\alpha_1, \alpha_2 + \alpha_3 - \alpha_1 - g^{-1} - \frac{Q_c}{2}, \alpha_1 + \alpha_2 + \alpha_3 - 3\frac{Q_c}{2})
\]

(77)

\[
Z_3^3 = Z_{\text{inst}}(\alpha_1 + \alpha_2 - \alpha_3 - \frac{Q_c}{2}, 2\alpha_2 - g^{-1} - \frac{Q_c}{2}, g^{-1} + 2\alpha_3 - \frac{Q_c}{2})
\]

(78)

\[
Z_4^4 = Z_{\text{inst}}(2\alpha_3 - \frac{Q_c}{2}, \alpha_3 + \alpha_1 - \alpha_2 - g^{-1} - \frac{Q_c}{2}, g^{-1} + \alpha_2 + \alpha_1 - \alpha_3 - \frac{Q_c}{2})
\]

(79)

Equation (75) suggests that the DOZZ function is equal to the composite defect wave function, since the derivative with respect to \(m_f\) corresponds to the insertion of this defect, whereas the wave function is literary equal to the partition function in the presence of the defect. Terms in the denominator are conjugate wave functions.

Now let us return to the torus knot superpolynomial. It is clear that the derivative with respect to the fundamental mass corresponds to correlators of the form \(\langle \phi e^{2\alpha_1} e^{2\alpha_2} e^{2\alpha_3} \rangle\) on the Liouville side. However, the role of the operator \(\exp(-\beta \phi)\) is not quite clear. We will show now that in the absence of the CS term \((k = 0)\), it is not necessary to consider the VEV of \(\exp(-\beta \phi)\), since this VEV and the partition function is actually proportional. This observation establishes a bridge between Liouville correlators and torus knots.
Let us consider the following peculiar observable[59]:

\[
\langle Y(qt z) + Q \frac{z^k(z - \exp(\beta m_f))(1 - z \exp(-\beta m_f))}{Y(z)} \rangle
\]

(80)

where \( Y(z) \) is a generating function for chiral ring observables \( \exp(-\beta \phi), \exp(-2\beta \phi), \ldots \).

Also, in [110, 120] it was conjectured that the operator \( Y(z) \) corresponds to the insertion of a domain wall. For a particular instanton configuration, defined by a Young diagram \( \lambda \), \( Y(z) \) equals to

\[
Y(z) = \prod_{\partial_+ \lambda} (z - q^a t^\alpha) \prod_{\partial_- \lambda} (z - qtq^a t^\alpha) = z \prod_{\partial_+ \lambda} (1 - q^a t^\alpha / z) \prod_{\partial_- \lambda} (1 - qtq^a t^\alpha / z)
\]

(81)

where \( \partial_+ \lambda \) defines cells which can be added to the Young diagram and \( \partial_- \lambda \) are those which we can remove.

According to N. Nekrasov[59], (80) is a regular function as a function of ”spectral parameter” \( z \). This is an analogue of Baxter TQ-equation for general Omega-deformation. In our case it is a polynomial of degree \( k + 1 \). If we use the identities

\[
1 - (1 - q)(1 - t) \sum_{\square} q^a t^\alpha = \sum_{\partial_+ \lambda} q^a t^\alpha - qt \sum_{\partial_- \lambda} q^a t^\alpha
\]

\[
Y(0) = -1
\]

(82)

Then we see that

\[
\langle Y(z) \rangle = z Z_{\text{inst}} - Z_{\text{inst}} + \langle \exp(-\beta \phi) \rangle + O(1/z), \ z \to \infty
\]

(83)

Let us concentrate on the case \( k = 0 \) which corresponds to the unknot. We can find the constant term in (80) by taking \( z = 0 \):

\[
-Z - QZ \exp(\beta m_f)
\]

(84)

On the other hand we obtain this term by taking \( z \to \infty \). Finally, we arrive at:

\[
\langle \exp(-\beta \phi) \rangle = QZ \frac{e^{-\beta m_f} - e^{\beta m_a} - e^{\beta m_a} e^{-\beta m_f} - 1}{1 + Q e^{-\beta m_f}}
\]

(85)

Recalling that the instanton partition function \( Z = \langle 1 \rangle \) is equal to 1 if \( m_f \) or \( m_a \) equal to zero, we obtain:

\[
-\frac{\partial}{\partial (\beta m_f)} \langle \exp(-\beta \phi) \rangle \bigg|_{m_f = 0} = Q \frac{\partial Z}{\partial (\beta m_f)} \frac{2e^{\beta m_a}}{1 + Q} - Q \frac{e^{\beta m_a} (1 - Q) - (1 + Q)}{(1 + Q)^2}
\]

(86)

The problem with \( k = 1 \) is that one has to consider higher-order terms in the expansion of \( Y(z) \).
7 Knot invariants from quantum mechanics on the instanton moduli space and $n \leftrightarrow m$ duality

There are complicated consistency conditions for the branes of different dimensions to live together happily and they are formulated differently in their worldvolume theories. All physical phenomena have to equivalently described from the viewpoints of the worldvolume theories on the defects of the different dimensions involved. Therefore we have to recognize the knot invariants in the corresponding quantum mechanics on the instanton moduli space. In this Section we consider the NS limit postponing the case of the general $\Omega$- background for the further study. As we have mentioned before one could expect that the states near threshold should matter for the UV completion problem and we shall see that it is indeed the case.

Let us remind, following [19] how the Poincare polynomial of the HOMFLY homologies of the torus knots are obtained in the Calogero model. To this aim it is useful to represent the quantum Calogero Hamiltonian in terms of the Dunkl operators

$$H_{\text{cal}} = \sum_n \partial_n^2 + \sum_{i \neq n} \frac{c(c-1)}{(z_i - z_n)^2}$$  \hspace{1cm} (87)

$$H_{\text{cal}} = \sum_n D_n^2 \quad D_n = \partial_n + c \sum_{i \neq n} \frac{1 - \sigma_{i,n}}{z_i - z_n}$$  \hspace{1cm} (88)

The Dunkl operators enters as generators in the rational DAHA algebra [77]. It is important that for rational Calogero coupling $c = n/m$ there is the finite-dimensional representation of DAHA [76]. It is this finite-dimensional representation does the job.

It was shown in [43] that the particular twisted character of this finite-dimensional representation coincides with the Poincare polynomial of the HOMFLY homology of the $T_{n,m}$ torus knot

$$P_{n,m}(a, q) = a^{(n-1)(m-1)} \sum_{i=0}^{n-1} a^{2i} tr(q^i; Hom_{S_n}(\Lambda^i h, L_{m/n}^m))$$  \hspace{1cm} (89)

where the following objects are involved. The $h$ is the (n-1) dimensional reflection representation of $S_n$, $C[h], C[h^*], C[S_n]$ generates the whole Cherednik algebra (we present its definition in Appendix). The $L_{m/n}$ is the finite-dimensional representation of the Cherednik algebra. The $n \leftrightarrow m$ symmetry is not evident however it was proved in [19] via comparison with the arc spaces on the Seifert surfaces of the torus knots. The arc space is nothing but the space of open topological string instantons in the physical language. The element $\rho$ belongs to the algebra and acts semisimply. Its eigenvalues provides the q-grading in the character representation. It can be thought of the Cartan element of $SL(2, \mathbb{R})$ subalgebra of the Cherednik algebra which is known for the Calogero model for a while and plays the role of the spectrum generating algebra. This Cartan element corresponds to one of the U(1) rotations of the $C^2$ where the instantons live.
Therefore from the Calogero viewpoint the HOMFLY invariant can be considered as a kind of generalization of the Witten index. The HOMFLY torus knot invariants are captured by the subspace of the rational complexified Calogero model Hilbert space. As we argued before the torus knot invariants are derived upon the integration of the determinants over the instanton moduli space in 5d gauge theory and the integral is localized at the centered instantons at one point. This fits with the relevance of the case when all Calogero particles are concentrated around the origin and we consider a kind of the ”falling at the center” problem. Note that the rational Calogero system is the conformal quantum mechanical model and we effectively impose the restriction on the spectrum.

Where the Calogero model with the particular coupling comes from in our instanton problem? The answer comes from the quantum mechanics on n-instantons moduli space. The small abelian instantons yield the Calogero model indeed if we think about the theory on the commutative space when some number of the points are blow-uped [44]. If the abelian instantons are restricted on the complex line one gets the Calogero model for the elongated instantons indeed as shown in [44].

We have to explain why the coupling constant in the Calogero model equals to $\frac{n}{m}$. The key point is that the CS term induces the magnetic field on the ADHM moduli space [16, 17] equals to the level of CS term, in our case $B_{\text{eff}} = k$. Immediately we can recognize that the coupling constant in the Calogero model corresponding to the $T_{n, nk+1}$ is the CS level $k = \frac{nk+1}{n}$ at least at large $n$ as required from the DAHA representation. Therefore we could claim that it is CS term which generates the correct interaction of Calogero particles. The next remark concerns the fact that the reduction of the superpolynomial to HOMFLY polynomial implies the NS limit $t = 1$ in the gauge theory. Indeed the two-dimensional plane is selected by the external graviphoton field.

The proper framework to explain $n \leftrightarrow m$ duality in the Calogero coupling is suggested by the QHE which can be approximately described by the n-body Calogero or RS models depending on geometry [35, 36, 37]. In the Calogero approach to QHE the Calogero coupling is equal to the filling factor which is related to the coefficient in front of the 3d CS term in the effective theory of the integer QHE. This is parallel to our case where the Calogero interaction is induced via the reduction of 5d CS term to 3d and the Calogero coupling is the level of CS term again. Fermions in the IQHE get substituted by the abelian instantons in our case. With this identification of the Calogero model we could expect that duality in the torus knot problem gets mapped into the similar duality in the integer QHE. The $\nu \rightarrow \nu^{-1}$ duality in IQHE corresponds to the substitution of quasiparticles by holes and vise versa.

It is in order to make some digression which can be interesting by its own. In our study we have started with the theory with $N_f = 3$ which has Landau pole. The theory with $N_f = 2$ has vanishing $\beta$-function while theory with $N_f = 1$ is asymptotically free. They are very different therefore we could look for the origin of this difference in the
our framework. We know that one flavor is massless therefore we can not decouple it however the limit \( a \to 0 \) provides the decoupling of one flavor. The transition from \( N_f = 3 \to N_f = 2 \) from the point-like instantons looks quite smooth Starting with all massive flavors and sending one mass to infinity we see the clear picture of the knotting and formation of some small compact UV defect where the small instantons are nested. That is transition from the theory with Landau pole to the conformal theory involves the formation of some small size UV defect.

We can look at this point from the slightly different angle, namely from the realization of the HOMFLY polynomial in terms of the finite-dimensional representation of rational DAHA. When \( a \neq 0 \) we have conformal regime perturbed by the particular nonlocal operator. In this case the parameter \( a \) which we have identified with the mass of the fundamental measures the representation of \( S^n \) on the n-point-like instantons- Calogero particles. Since mass is related to the Casimir of the translation \( (p^2 = m^2) \) we could claim that the symmetric group and the Lorentz group are related in the nontrivial way. Indeed the SL(2,R) rotations are built in the rational DAHA nontrivially.

However the momentum is not the generator of the DAHA therefore the mixing of the N-particle state under the translations occurs and the different representations of the symmetric group emerge. When \( a \to 0 \) a bit unusual counting of the representation of the symmetric group by mass serving as fugacity disappears that is in some sense the group of space-time translations and symmetric group decouple from each other. This seem to be some important property of the asymptotically free theories which has to be elaborated in more details.

8 Conclusion

In this paper we have formulated the explicit instanton- torus knot duality between the n-instanton contribution to the particular condensate in the 5D SQED and superpolynomial of the particular torus knot. The second derivative of the Nekrasov partition function plays the role of the generating function for the torus knot superpolynomials. The condensate is evaluated in the background of the 4-observable - which can be considered as the result of the incomplete decoupling of the regulator degree of freedom. Hence to some extend we could say that the knot invariants govern the delicate UV properties of the gauge theory when the point-like instantons interact with the UV degrees of freedom.

What are the lessons we could learn from this correspondence? Some of them have been already mentioned in the Introduction. One more important physical lesson to be learned is as follows. In the conventional QCD there is the fermionic zero mode at the individual instanton, however the fermionic chiral condensate is not due to it. The chiral condensate is determined via Casher-Banks relation by the density of the quasi-zero modes in the instanton-anti-instanton ensemble. The details of the interaction of the instantons can not be recognized in this case and only the collective effect can be
seen in Casher-Banks relation. In our case due to supersymmetry we can say more on the microscopic structure behind the condensate. The CS term induces the attractive interaction of Calogero-type between point-like instantons and the falling at the center phenomena occurs. It turns out that the accurate treatment of this phenomena is performed in terms of the knot invariants. In our case the torus knots are selected by the choice of the matter content, however in general more complicated knots can be expected. The summation over the torus knot superpolynomials amounts to the nonvanishing condensate and corresponds to the summation over instantons which to some extend yields the microscopic picture for the analogue of ”Casher-Banks” relation.

One should not forget that we have considered \( R^4 \times S^1 \) geometry that is from the 4d viewpoint we have worked in the Euclidean space. Therefore one could try to interpret the composite defect we have found in \( R^4 \) as a kind of bounce configuration describing a kind of Schwinger-like process in the (3+1) Minkowski space-time. Indeed, if we have a closed domain wall, we could firmly claim that we have found the Euclidean bounce describing the creation of the closed objects with the geometry \( S^2 \) in the Minkowski space. It is well-known that the Euclidean bounce corresponds to the evaluation of the imaginary part of the effective actions in the external field. In our framework our configuration could correspond to the bounce for the Schwinger-like process for the extended objects in the external graviphoton field in the spirit of [89, 88]. However since we have composite defect it is necessary to prove the existence of the odd number of negative modes on this configuration which would support its interpretation as the bounce solution. This point certainly deserves further clarification.

It seems that we just have touched the tip of the iceberg and there are immediate questions to be formulated.

- How the matter content of the 5D has to be extended to fit with the general \( T_{n,m} \) torus knots and links? We expect that the superpotential in the 5d theory is in one-to-one correspondence with the type of the knot. The torus knots correspond to the simplest case when only the CS term is involved.

- How the instanton-knot correspondence will be modified in the nonabelian case and for the general quiver theory?

- We have not discuss in this study the differentials in the Khovanov homologies postponing this issue for the separate work. They should be related to the effect of surface operators. Indeed, it was shown in [43] that the differentials are attributed to the complex lines in \( R^4 \). The related question concerns the derivation of the Hall algebra in the 5d theory. The results in [98] certainly should be of some use.

- What is the meaning of the fourth grading which seems to be under the carpet for the torus knots [95] in the 5D gauge theory?

- What amount of this duality survives in the 4d case?
• We have not touched in this paper the theory on the worldvolume of the flavor brane at all postponing this issue for the future work. This would involve the analogue of the Chiral Lagrangian and we will have to recognize the torus knot invariants from this perspective as well. It is known that the Skyrmion is represented as the instanton trapped by the domain wall in $D=5$ gauge theory [101] realizing dynamically the Atiyah-Manton picture. It seems that our composite defect could have a relation with a kind of Skyrmion or dyonic Skyrmion in the "Chiral Lagrangian".

• It seems that our considerations have common features with the Higher dimensional [100] QHE effect in the refined case and the conventional QHE in the unrefined case. Is it possible to clarify the role of the torus knot invariants in that context?

• Is it possible to have the simple interpretation of $n \leftrightarrow (nk + 1)$ duality in terms of the topological strings and in the theory on the surface operator?

• Recently the relation between the RG cycles and the decoupling of the heavy degrees of freedom has been found in the $\Omega$-deformed SQCD. The similar RG cycles were found in the Calogero model [53, 52] in this context. How these RG cycles can be formulated in terms of knot invariants? Are there the Efimov-like states?

• There is an interesting duality between the pair of integrable systems. One classical integrable system belongs to the Toda-Calogero-RS family while the second quantum integrable model is a kind of the spin chain. The mapping between two sides of the correspondence is quite nontrivial [117, 112, 113]. In our case we see that the knot invariants are related with the spectrum of the quantum Calogero system. Is it possible to recognize the knot invariants at the spin chain side when the additional deformation is included? Some step in this direction was made in [14].

• Recently an additional clarification of the 2d-4d duality has been obtained. Using the representation of the nonabelian string in terms of the resolvent [33] in $N=1$ SYM theory the issue of the gluino condensate has been reconsidered in [34]. It was shown, using the interplay between 2d and 4d generalized Konishi anomalies, that the gluino condensate in $N=1$ theory penetrates the worldsheet theory and deforms the chiral ring and corresponding Bethe ansatz equations in the worldsheet theory in the nontrivial manner. Is it possible to use the 5d-3d correspondence to recognize the knot invariants on the defect "inside the condensate"?

• How the critical behavior discussed in [7] will be generalized in our situation [85] and what is its proper physical interpretation?
• The elementary "rotator" we have discussed in the paper has some similarity
with the microstate at the horizon of the black hole. Is there some holographic
relation between them?

• Is it possible to make the arguments concerning the explanations of the dimen-
sional transmutation phenomena via the composite defect precise?

• How these composite defects interact?

We hope to discuss these issues elsewhere.

Acknowledgment

We would like to thank K. Bulycheva, I. Danilenko, M. Gorsky, S.Gukov, S. Nechaev,
A. Vainshtein and especially to E. Gorsky and N. Nekrasov for the useful discussions.
A.G. thanks FTPI at University of Minnesota and IPhT at Saclay where the part of the
work has been carried out for hospitality and support. We are gratefully acknowledge
support from the Simons Center for Geometry and Physics, Stony Brook University
during the program "Gauge Theory, Integrability, and Novel Symmetries of Quantum
Field Theory" (A.G. and A. M) and during the Simons Workshop on Physics and
Mathematics 2014 (A.G) where part of the research was performed. The work was
supported in part by the grants RFBR-12-02-00284 and PICS-12-02-91052. The work
of A.M. was also supported by the Dynasty fellowship program.

9 Appendix A. Torus knots

In this Appendix we will sketch some properties of torus knots. For a comprehensive
review, see [79]. The general definition of a knot is a continuous embedding of $S^1$ into
$S^3$ up to a homotopy. The trivial example in unknot: this is just a circle lying inside
$S^3$. According to the Thurston theorem[80], every knot is either:

• Satellite. Such knots can be obtained by taking a non-trivial knot lying inside a
solid 2-torus(in this situation non-trivial means that the knot neither lying in a
3-ball inside the solid torus nor just wrapping one of the torus cycles) and then
embedding the solid torus into the $S^3$ as another non-trivial knot.

• Hyperbolic. In this case the compliment of the knot $S^3 \setminus \gamma$ is a hyperbolic space.

• Torus. This family is very well-studied. Such knots are characterized by two
numbers: $n$ and $m$. They can be obtained by wrapping the $S^1$ $n$ times over one
cycle on a two-torus and $m$ times over the other cycle without self-intersections.
Obvious property of the torus knot is \( K_{n,m} = K_{m,n} \). Actually, if \( n \) and \( m \) are not co-prime, it will be a link rather than a knot: link is a collection of knots which do not intersect. Also, \((n, 1)\) and \((1, m)\) represent unknot. Therefore the most simple example is \((3, 2)\) knot, so-called trefoil knot (see Figure 4).

Figure 4: trefoil knot (this figure courtesy of Wikipedia)

The algebraic knot in \( S^3 \) which is the main object in this section can be described by the intersection of \( S^3 \) with some algebraic curve. If we realize the sphere as
\[
|z_1|^2 + |z_2|^2 = 1
\]
then the simplest \((p, q)\) torus knots which can be obtained from the unknot by the \( SL(2, \mathbb{Z}) \) action corresponds to the curve
\[
z_1^p = z_2^q
\]
which is called Seifert surface of the knot. The sphere is invariant under
\[
z_1 \rightarrow e^{i\theta} z_1 \quad z_2 \rightarrow e^{i\theta} z_2
\]
while the Seifert surface under
\[
z_1 \rightarrow e^{iq\theta} z_1 \quad z_2 \rightarrow e^{ip\theta} z_2
\]

In the CS theory the knot invariants one can evaluated from the corresponding Wilson loop vacuum expectation value. The useful tool is the knot operator introduced in [103]
\[
W_R^{n,m} |p\rangle = \sum_{\mu \in M_R} \exp[-i\mu^2 \frac{nm}{k+N} - 2\pi i \frac{m}{k+N} p\mu] |p + n\mu\rangle
\]
where \( M_R \) is the set of weights corresponding to the irreducible representation \( R \) and \(|p\rangle\) is the element of the basis of the Hilbert space of the SU(N) CS theory on the torus labeled by the weights \( p \). When evaluating the vev of Wilson loop one performs the Heegaard cut of \( S^3 \) into two solid tori. Then the torus knot is introduced on the surface of one of the solid tori by the action of the knot operator on the corresponding vacuum state. In the standard framing the vev of Wilson loop is given by
\[
< W_R^{n,m} > = E^{2\pi} \frac{< p | SW_R^{n,m} |p \rangle}{< p | S | p \rangle}
\]
where \( S \) - is the operator of S-transformation from \( SL(2, \mathbb{Z}) \).
Appendix B. Higher (q,t) Catalan numbers

In this Appendix we briefly review higher (q,t) deformed Catalan numbers $C_n^k(q,t)$ which enter the expression for the $T_{n,nk+1}$ superpolynomial at $a = 0$. There are several definitions of the higher Catalan numbers related to the geometry of the Hilbert schemes of points, symmetric functions, representation theory and combinatorics of paths. To orient the reader we provide a few of them

- Let us introduce the elementary symmetric functions $e_n$, Macdonald polynomial $H_\mu$ for the partition $\mu$, the Hall product $<,>$ on the symmetric functions and $\Lambda$ - the ring of the symmetric functions. There is the so-called Nabla operator $\nabla$ which acts on the Macdonald basis as

$$\nabla H_\mu = T_\mu H_\mu, \quad T_\mu = q^{n(\mu')} t^{n(\mu)}, \quad n(\mu) = \sum_{x \in d(\mu)} l(x)$$

where $\mu'$ denotes the transpose of $\mu$. The $C_n^k(q,t)$ in terms of the symmetric functions are defined as follows

$$C_n^k(q,t) = \langle \nabla^k(e_n), e_n \rangle$$

- One can define the higher Catalans in terms of the so-called diagonal harmonics. To this aim consider the polynomial ring $\mathbb{C}(x_1, y_1, \ldots, x_n, y_n)$. The symmetric group $S_n$ acts diagonally $\omega x_i = x_{\omega(i)}$, $\omega y_i = y_{\omega(i)}$, $\omega \in S_n$. Introduce the ideal generated by all alternating polynomials and let be $\mathfrak{m}$ the maximal ideal generated by $x_1, y_1, \ldots, x_n, y_n$. Let $M^k = I^k/\mathfrak{m}I^k$. It is possible to introduce the double grading is the space of polynomials according the degrees in $x$ and $y$ variables. The grading tells that bi-degree $(d_1, d_2)$ corresponds to the situation when all monomials of the polynomial have equal bi-degree $(d_1, d_2)$. The higher Catalans are now defined as

$$C_n^k(q,t) = \sum_l \sum_s q^l t^s \dim M^k_{l,s}$$

where $M^k_{l,s}$ is the bihomogeneous component of $M^k$ of bi-degree $(l,s)$.

- The last definition which is the closest to our context is based on the Hilbert scheme of $n$ points on $\mathbb{C}^2$ Hilb$^n(\mathbb{C}^2)$. Let us define $O(k) = O(1)^{\otimes k}$, where $O(1) = \det T$ and $T$ is the tautological rank $n$ bundle. The double grading in this case is introduced on the set of global section $H^0(Z_n, O(k))$ where $Z_n$ tells that all points are sitting at the top of each other. The $C_n^k(q,t)$ are now defined as follows

$$C_n^k(q,t) = \sum_l \sum_s q^l t^s \dim H^0(Z_n, O(k))_{l,s}$$

Let us emphasize that the key property of the higher Catalans which is important in our study is that they provide the description of the properties of the set of points sitting at the top of each other.
11 Appendix C. The Dunkl operators and rational DAHA

In this Appendix we briefly describe the rational double affine Hecke algebras (DAHA) $H_c$ and their finite-dimensional representations relevant for the invariants of the torus knots. The rational DAHA of type $A_{n-1}$ with parameter $c$ is generated by the $V = C^{n-1}, V^*$ and the permutation group $S_n$ with the following relations

\begin{align*}
\sigma x \sigma^{-1} &= \sigma(x), \quad \sigma y \sigma^{-1} = \sigma(y) \quad (100) \\
x_1x_2 &= x_2x_1, \quad y_1y_2 = y_2y_1 \quad (101) \\
yx - xy &= \langle y, x \rangle - c \sum_{s \in \mathcal{S}} \langle \alpha_s, x \rangle < \alpha^v_s, y > \quad (102)
\end{align*}

where $\mathcal{S}$ is the set of all transpositions, and $\alpha_s, \alpha^v_s$ are the corresponding roots and coroots.

It is convenient to introduce the Dunkl operators

\begin{equation}
D_i = \frac{\partial}{\partial x_i} - c \sum_{i \neq j} \frac{s_{ij} - 1}{x_i - x_j} 
\end{equation}

and introduce the space of polynomial functions on $V$, where elements of $V$ acts by the multiplications and $V^*$ by the Dunkl operators. This representation is denoted by $M_c$. It is known [76] that for rational $c = m/n$ DAHA has unique finite dimensional representation $L_{m/n}$ which was identified as the factor $L_c = M_c/I_c$ where $I_c$ is the ideal generated by the following set of the homogeneous polynomials $f_i$ of degree $m$

\begin{equation}
f_i = \text{Coef}_{m}[(1 - zx_i)^{-1}\prod_{i=1}^{n}(1 - zx_i)^{m/n}] 
\end{equation}

They are annihilated by the Dunkl operators

\begin{equation}
D_k(f_i) = 0 
\end{equation}

and therefore are invariants under the DAHA action. The dimension of the finite-dimensional representation is

\begin{equation}
dim L_{m/n} = m^{n-1} 
\end{equation}

There is $SL(2, R)$ subalgebra of DAHA which involves the Hamiltonian of the rational complexified Calogero model

\begin{equation}
H_{Cal} = \sum_i D_i^2 
\end{equation}

the rest of the operators from this subalgebra are

\begin{equation}
K = \frac{1}{2} \sum_i (x_iD_i + D_ix_i) \quad J_1 = \sum_i x_i^2 
\end{equation}
Let us also note that there is the so-called shift operator which acts by changing \( c \rightarrow c + 1 \) and is the counterpart of the Nabla operator in the theory of the symmetric functions and the cut- and-join operator. It corresponds to the shift of the level of 5d CS term.

12 Appendix D. The Ruijsenaars-Shneyder many-body integrable model from the perturbed CS theory

The surface operator which survives at the road from UV to IR [18] induces the action of the quantum trigonometric RS Hamiltonian which is simply described in terms of the perturbed 3d CS theory [78]. The phase space in this model is identified with the space of flat connections on the torus with the marked point and particular monodromy around while the Hamiltonian can be described as the Wilson loop around one cycle.

Equivalently is can be obtained via the Hamiltonian reduction. To perform the Hamiltonian reduction replace the space of two dimensional gauge fields by the cotangent space to the loop group:

\[
T^* \hat{G} = \{(g(x), k_x + P(x))\}
\]

The relation to the two dimensional construction is the following. Choose a non-contractible circle \( S^1 \) on the two-torus which does not pass through the marked point \( p \). Let \( x, y \) be the coordinates on the torus and \( y = 0 \) is the equation of the \( S^1 \). The periodicity of \( x \) is \( \beta \) and that of \( y \) is \( R \). Then

\[
P(x) = A_x(x, 0), g(x) = P \exp \int_0^R A_y(x, y) dy.
\]

The moment map equation looks as follows:

\[
k g^{-1} g + g^{-1} P g - P = J \delta(x),
\]

with \( k = \frac{1}{R\beta} \). The solution of this equation in the gauge \( P = \text{diag}(q_1, \ldots, q_N) \) leads to the Lax operator \( A = g(0) \) with \( R, \beta \) exchanged. On the other hand, if we diagonalize \( g(x) \):

\[
g(x) = \text{diag} \left( z_1 = e^{\sqrt{-1} q_1 x}, \ldots, z_N = e^{\sqrt{-1} q_N x} \right)
\]

then a similar calculation leads to the Lax operator

\[
B = P \exp \oint \frac{1}{k} P(x) dx = \text{diag}(e^{\sqrt{-1} \theta_i}) \exp \sqrt{-1 R\beta \nu x}
\]

with

\[
r_{ij} = \frac{1}{1 - e^{\sqrt{-1} q_{ij}}}, i \neq j; \quad r_{ii} = -\sum_{j \neq i} r_{ij}
\]

thereby establishing the duality \( A \leftrightarrow B \) explicitly.
References

[1] E. Witten, “Quantum Field Theory and the Jones Polynomial,” Commun. Math. Phys. 121, 351 (1989).

[2] T. Dimofte, S. Gukov and L. Hollands, “Vortex Counting and Lagrangian 3-manifolds,” Lett. Math. Phys. 98, 225 (2011) [arXiv:1006.0977 [hep-th]].

[3] A. M. Garsia, M. Haiman, ”A remarkable q, t-Catalan sequence and q-Lagrange inversion,” Journal of Algebraic Combinatorics, 5(3), 191-244.

[4] N. A. Nekrasov, “Seiberg-Witten prepotential from instanton counting,” Adv. Theor. Math. Phys. 7, 831 (2004) [hep-th/0206161].

[5] A. Oblomkov, J. Rasmussen, V. Shende, ”The Hilbert scheme of a plane curve singularity and the HOMFLY polynomial of its link,” Duke Mathematical Journal, 03/2010 [arXiv:1201.2115].

[6] E. Gorsky, ”q,t-Catalan numbers and knot homology,” Zeta functions in algebra and geometry, 213-232, Contemp. Math., 566, Amer. Math. Soc., Providence, RI, 2012 [arXiv:1003.0916]

[7] K. Bulycheva, A. Gorsky and S. Nechaev, “Critical behavior in topological ensembles,” arXiv:1409.3350 [hep-th].

[8] A. Iqbal, C. Kozcaz and C. Vafa, “The Refined topological vertex,” JHEP 0910, 069 (2009) [hep-th/0701156].

[9] A. Iqbal, C. Kozcaz and K. Shabbir, “Refined Topological Vertex, Cylindric Partitions and the U(1) Adjoint Theory,” Nucl. Phys. B 838, 422 (2010) [arXiv:0803.2260 [hep-th]].

[10] L. Bao, E. Pomoni, M. Taki and F. Yagi, “M5-Branes, Toric Diagrams and Gauge Theory Duality,” JHEP 1204, 105 (2012) [arXiv:1112.5228 [hep-th]].

[11] H. J. Chung, T. Dimofte, S. Gukov and P. Sulkowski, “3d-3d Correspondence Revisited,” arXiv:1405.3663 [hep-th].

[12] H. Fuji, S. Gukov, M. Stosic and P. Sulkowski, “3d analogs of Argyres-Douglas theories and knot homologies,” JHEP 1301, 175 (2013) [arXiv:1209.1416 [hep-th]].

[13] T. Dimofte, D. Gaiotto and S. Gukov, “Gauge Theories Labelled by Three-Manifolds,” Commun. Math. Phys. 325, 367 (2014) [arXiv:1108.4389 [hep-th]].

[14] K. Bulycheva and A. Gorsky, “BPS states in the Omega-background and torus knots,” JHEP 1404, 164 (2014) [arXiv:1310.7361 [hep-th]].
[15] S. Gukov, A. Iqbal, C. Kozcaz and C. Vafa, “Link Homologies and the Refined Topological Vertex,” Commun. Math. Phys. 298, 757 (2010) [arXiv:0705.1368 [hep-th]].

[16] B. Collie and D. Tong, “Instantons, Fermions and Chern-Simons Terms,” JHEP 0807, 015 (2008) [arXiv:0804.1772 [hep-th]].

[17] S. Kim, K. M. Lee and S. Lee, “Dyonic Instantons in 5-dim Yang-Mills Chern-Simons Theories,” JHEP 0808, 064 (2008)

[18] D. Gaiotto, L. Rastelli and S. S. Razamat, “Bootstrapping the superconformal index with surface defects,” JHEP 1301, 022 (2013) [arXiv:1207.3577 [hep-th]].

[19] E. Gorsky, ” Arc spaces and DAHA representations”, Selecta Mathematica 19 (2013) 125, [arxiv:1110.1674]

[20] A. E. Lawrence and N. Nekrasov, “Instanton sums and five-dimensional gauge theories,” Nucl. Phys. B 513, 239 (1998) [hep-th/9706025].

[21] N. Nekrasov and A. Okounkov, “Seiberg-Witten theory and random partitions,” hep-th/0306238.

[22] N. Dunfield, S. Gukov and J. Rasmussen, The Superpolynomial for Knot Homologies, arxiv:math/0505662

[23] S. Gukov, A. S. Schwarz and C. Vafa, “Khovanov-Rozansky homology and topological strings,” Lett. Math. Phys. 74, 53 (2005) [arXiv:hep-th/0412243].

[24] T. Dimofte and S. Gukov, “Refined, Motivic, and Quantum,” Lett. Math. Phys. 91, 1 (2010) [arXiv:0904.1420 [hep-th]].

[25] S. Gukov, “Surface Operators and Knot Homologies,” arXiv:0706.2369 [hep-th].

[26] E. Witten, “Fivebranes and Knots,” arXiv:1101.3216 [hep-th].

[27] E. Witten, “A New Look At The Path Integral Of Quantum Mechanics,” arXiv:1009.6032 [hep-th].

[28] N. A. Nekrasov and S. L. Shatashvili, “Quantization of Integrable Systems and Four Dimensional Gauge Theories,” arXiv:0908.4052 [hep-th].

[29] A. Brini, B. Eynard and M. Marino, “Torus knots and mirror symmetry,” arXiv:1105.2012 [hep-th].

[30] H. Ooguri and C. Vafa, “Knot invariants and topological strings,” Nucl. Phys. B 577, 419 (2000) [arXiv:hep-th/9912123].
[31] E. Witten, “Chern-Simons Gauge Theory As A String Theory,” Prog. Math. 133, 637 (1995) [arXiv:hep-th/9207094].

[32] A. Oblomkov and V. Shende, The Hilbert scheme of a plane curve singularity and the HOMFLY polynomial of its link, arxiv:math/1003.1568

[33] D. Gaiotto, S. Gukov and N. Seiberg, “Surface Defects and Resolvents,” JHEP 1309, 070 (2013) [arXiv:1307.2578].

[34] M. Shifman and A. Yung, “Quantum Deformation of the Effective Theory on Non-Abelian string and 2D-4D correspondence,” Phys. Rev. D 89, 065035 (2014) [arXiv:1401.1455 [hep-th]].

[35] L. Susskind, “The Quantum Hall fluid and noncommutative Chern-Simons theory,” hep-th/0101029.

[36] A. P. Polychronakos, “Quantum Hall states on the cylinder as unitary matrix Chern-Simons theory,” JHEP 0106, 070 (2001) [hep-th/0106011].

[37] A. Gorsky, I. I. Kogan and C. Korthels-Altes, “Dualities in quantum Hall system and noncommutative Chern-Simons theory,” JHEP 0201, 002 (2002) [hep-th/0111013].

[38] M. Haiman, (q,t) Catalan numbers and the Hilbert scheme, Discrete Mathematics, 193, (1998), 201

[39] E. Carlsson, N. Nekrasov and A. Okounkov, “Five dimensional gauge theories and vertex operators,” arXiv:1308.2465 [math.RT].

[40] H. C. Kim, S. Kim, E. Koh, K. Lee and S. Lee, “On instantons as Kaluza-Klein modes of M5-branes,” JHEP 1112, 031 (2011) [arXiv:1110.2175 [hep-th]].

[41] H. C. Kim, S. S. Kim and K. Lee, “5-dim Superconformal Index with Enhanced En Global Symmetry,” JHEP 1210, 142 (2012) [arXiv:1206.6781 [hep-th]].

[42] E. Gorsky, A. Oblomkov and J. Rasmussen ”On stable Khovanov Homology of the torus knots”, arXiv: 1206.2226

[43] E. Gorsky, A. Oblomkov, J. Rasmussen and V. Shende ”Torus knots and DAHA representations” arXiv: 1207.4523

[44] H. W. Braden and N. A. Nekrasov, “Space-time foam from noncommutative instantons,” Commun. Math. Phys. 249, 431 (2004) [hep-th/9912019].

[45] Y. Tachikawa, “Five-dimensional Chern-Simons terms and Nekrasov’s instanton counting,” JHEP 0402, 050 (2004) [hep-th/0401184].
[46] E. Witten, “Phase transitions in M theory and F theory,” Nucl. Phys. B 471, 195 (1996) [hep-th/9603150].

[47] P. Dunin-Barkowski, A. Mironov, A. Morozov, A. Sleptsov and A. Smirnov, “Superpolynomials for toric knots from evolution induced by cut-and-join operators,” JHEP 03, 021 (2013) [JHEP 1303, 021 (2013)] [arXiv:1106.4305 [hep-th]].

[48] N. Seiberg, “Five-dimensional SUSY field theories, nontrivial fixed points and string dynamics,” Phys. Lett. B 388, 753 (1996) [hep-th/9608111].

[49] D. R. Morrison and N. Seiberg, “Extremal transitions and five-dimensional supersymmetric field theories,” Nucl. Phys. B 483, 229 (1997) [hep-th/9609070].

[50] N. Drukker, M. Marino and P. Putrov, “Nonperturbative aspects of ABJM theory,” JHEP 1111, 141 (2011) [arXiv:1103.4844 [hep-th]].

[51] K. Bulycheva, H. Y. Chen, A. Gorsky and P. Koroteev, “BPS States in Omega Background and Integrability,” JHEP 1210, 116 (2012) [arXiv:1207.0460 [hep-th]].

[52] K. M. Bulycheva and A. S. Gorsky, “Limit cycles in renormalization group dynamics,” Phys. Usp. 57, 171 (2014) [arXiv:1402.2431 [hep-th]].

[53] A. Gorsky, “SQCD, Superconducting Gaps and Cyclic RG Flows,” arXiv:1202.4306 [hep-th].

[54] A. Gorsky, “Angular Momentum and Gravimagnetization of the $\mathcal{N} = 2$ SYM vacuum,” Theor. Math. Phys. 171, 616 (2012) [arXiv:1102.1841 [hep-th]].

[55] O. Dubinkin, “On the Virasoro constraints for torus knots,” J. Phys. A 47, no. 48, 485203 (2014) [arXiv:1307.7909 [hep-th]].

[56] E. Witten, “BPS Bound states of D0 - D6 and D0 - D8 systems in a B field,” JHEP 0204, 012 (2002) [hep-th/0012054].

[57] C. j. Kim, K. M. Lee and S. H. Yi, “Tales of D0 on D6-branes: Matrix mechanics of identical particles,” Phys. Lett. B 543, 107 (2002) [hep-th/0204109].

[58] G. T. Horowitz and M. M. Roberts, “Counting the Microstates of a Kerr Black Hole,” Phys. Rev. Lett. 99, 221601 (2007) [arXiv:0708.1346 [hep-th]].

[59] N. Nekrasov, “Non-Perturbative Schwinger-Dyson Equations: From BPS/CFT Correspondence to the Novel Symmetries of Quantum Field Theory,” “Pomeranchuk 100”, World Scientific 2014.

[60] H. Awata and Y. Yamada, “Five-dimensional AGT Conjecture and the Deformed Virasoro Algebra,” JHEP 1001, 125 (2010) [arXiv:0910.4431 [hep-th]].
[61] C. Kozcaz, S. Pasquetti and N. Wyllard, “A & B model approaches to surface operators and Toda theories,” JHEP 1008, 042 (2010) [arXiv:1004.2025 [hep-th]].

[62] R. Schiappa and N. Wyllard, “An A(r) threesome: Matrix models, 2d CFTs and 4d N=2 gauge theories,” J. Math. Phys. 51, 082304 (2010) [arXiv:0911.5337 [hep-th]].

[63] E. Witten, “Instantons, the Quark Model, and the 1/n Expansion,” Nucl. Phys. B 149, 285 (1979).

[64] V. A. Novikov, M. A. Shifman, A. I. Vainshtein and V. I. Zakharov, “Two-Dimensional Sigma Models: Modeling Nonperturbative Effects of Quantum Chromodynamics,” Phys. Rept. 116, 103 (1984) [Sov. J. Part. Nucl. 17, 204 (1986)] [Fiz. Elem. Chast. Atom. Yadra 17, 472 (1986)].

[65] A. B. Zamolodchikov and A. B. Zamolodchikov, “Structure constants and conformal bootstrap in Liouville field theory,” Nucl. Phys. B 477, 577 (1996) [hep-th/9506136].

[66] H. Dorn and H. J. Otto, “Two and three point functions in Liouville theory,” Nucl. Phys. B 429, 375 (1994) [hep-th/9403141].

[67] M. Aganagic and S. Shakirov, “Knot Homology from Refined Chern-Simons Theory,” arXiv:1105.5117 [hep-th].

[68] L. F. Alday, D. Gaiotto and Y. Tachikawa, “Liouville Correlation Functions from Four-dimensional Gauge Theories,” Lett. Math. Phys. 91, 167 (2010) [arXiv:0906.3219 [hep-th]].

[69] F. Calvo and M. Marino, “Membrane instantons from a semiclassical TBA,” JHEP 1305, 006 (2013) [arXiv:1212.5118 [hep-th]].

[70] N. D. Lambert and D. Tong, “Dyonic instantons in five-dimensional gauge theories,” Phys. Lett. B 462, 89 (1999) [hep-th/9907014].

[71] R. P. Kerr, “Gravitational field of a spinning mass as an example of algebraically special metrics,” Phys. Rev. Lett. 11, 237 (1963).

[72] A. Gorsky, M. Shifman and A. Yung, “Non-Abelian meissner effect in Yang-Mills theories at weak coupling,” Phys. Rev. D 71, 045010 (2005) [hep-th/0412082].

[73] M. A. Shifman, “Anomalies and Low-Energy Theorems of Quantum Chromodynamics,” Phys. Rept. 209, 341 (1991) [Sov. Phys. Usp. 32, 289 (1989)] [Usp. Fiz. Nauk 157, 561 (1989)].

[74] H. Jockers, A. Klemm and M. Soroush, “Torus Knots and the Topological Vertex,” Lett. Math. Phys. 104, 953 (2014) [arXiv:1212.0321 [hep-th]].
[75] M. Shifman and A. Yung, “Supersymmetric Solitons and How They Help Us Understand Non-Abelian Gauge Theories,” Rev. Mod. Phys. 79, 1139 (2007) [hep-th/0703267].

[76] Y. Berest, P. Etingof, V. Ginzburg, ”Finite-dimensional representations of rational Cherednik algebras,” Int. Math. Res. Not. 2003, no. 19, 1053–1088 [arXiv:math/0208138].

[77] I. Cherednik, ”A unification of Knizhnik-Zamolodchikov and Dunkl operators via affine Hecke algebras,” Invent. Math. 106 (1991), 411–431

[78] A. Gorsky and N. Nekrasov, “Relativistic Calogero-Moser model as gauged WZW theory,” Nucl. Phys. B 436, 582 (1995) [hep-th/9401017].

[79] L. H. Kauffman, (2013). Knots and physics (Vol. 53). World scientific.

[80] W. Thurston, “Three-Dimensional Manifolds, Kleinian Groups and Hyperbolic Geometry,” Bull. Amer. Math. Soc. (N.S.) 6 (1982) 357–381.

[81] K. A. Intriligator, D. R. Morrison and N. Seiberg, “Five-dimensional supersymmetric gauge theories and degenerations of Calabi-Yau spaces,” Nucl. Phys. B 497, 56 (1997) [hep-th/9702198].

[82] A. Klemm and P. Sulkowski, “Seiberg-Witten theory and matrix models,” Nucl. Phys. B 819, 400 (2009) [arXiv:0810.4944 [hep-th]].

[83] K. Ito, S. Kamoshita and S. Sasaki, “BPS Monopole Equation in Omega-background,” JHEP 1104, 023 (2011) [arXiv:1103.2589 [hep-th]].

[84] K. Ito, S. Kamoshita and S. Sasaki, “Deformed BPS Monopole in Omega-background,” Phys. Lett. B 710, 240 (2012) [arXiv:1110.1455 [hep-th]].

[85] K. Bulycheva, A Gorsky, S. Nechaev and A. Milekhin, in preparation

[86] A. A. Migdal, “Loop Equations and 1/N Expansion,” Phys. Rept. 102, 199 (1983).

[87] A. M. Polyakov, “Gauge Fields as Rings of Glue,” Nucl. Phys. B 164, 171 (1980).

[88] M. Dedushenko and E. Witten, “Some Details On The Gopakumar-Vafa and Ooguri-Vafa Formulas,” arXiv:1411.7108 [hep-th].

[89] R. Gopakumar and C. Vafa, “M theory and topological strings. 2.,” hep-th/9812127.

[90] D. Gaiotto and E. Witten, “Knot Invariants from Four-Dimensional Gauge Theory,” Adv. Theor. Math. Phys. 16, no. 3, 935 (2012) [arXiv:1106.4789 [hep-th]].
[91] A. Losev, N. Nekrasov and S. L. Shatashvili, “Freckled instantons in two-dimensions and four-dimensions,” Class. Quant. Grav. 17, 1181 (2000) [hep-th/9911099].

[92] N. Nekrasov and A. S. Schwarz, “Instantons on noncommutative R**4 and (2,0) superconformal six-dimensional theory,” Commun. Math. Phys. 198, 689 (1998) [hep-th/9802068].

[93] G. W. Moore, N. Nekrasov and S. Shatashvili, “Integrating over Higgs branches,” Commun. Math. Phys. 209, 97 (2000) [hep-th/9712241].

[94] A. Losev, N. Nekrasov and S. L. Shatashvili, “Issues in topological gauge theory,” Nucl. Phys. B 534, 549 (1998) [hep-th/9711108].

[95] E. Gorsky, S. Gukov and M. Stosic, “Quadruply-graded colored homology of knots,” arXiv:1304.3481 [math.QA].

[96] A. Gadde, S. Gukov and P. Putrov, “Walls, Lines, and Spectral Dualities in 3d Gauge Theories,” JHEP 1405, 047 (2014) [arXiv:1302.0015 [hep-th]].

[97] H. Fuji, S. Gukov and P. Sulkowski, “Super-A-polynomial for knots and BPS states,” Nucl. Phys. B 867, 506 (2013) [arXiv:1205.1515 [hep-th]].

[98] S. Gukov and M. Stosic, “Homological Algebra of Knots and BPS States,” arXiv:1112.0030 [hep-th].

[99] S. Gukov and I. Saberi, “Lectures on Knot Homology and Quantum Curves,” arXiv:1211.6075 [hep-th].

[100] J. Hu and S. Zhang ,Collective excitations at the boundary of a 4D Quantum Hall droplet, cond-mat 0112432

[101] M. Eto, M. Nitta, K. Ohashi and D. Tong, “Skyrmions from instantons inside domain walls,” Phys. Rev. Lett. 95, 252003 (2005) [hep-th/0508130].

[102] H. Y. Chen, M. Eto and K. Hashimoto, “The Shape of Instantons: Cross-Section of Supertubes and Dyonic Instantons,” JHEP 0701, 017 (2007) [hep-th/0609142].

[103] J. M. F. Labastida and M. Marino, “Polynomial invariants for torus knots and topological strings,” Commun. Math. Phys. 217, 423 (2001) [hep-th/0004196].

[104] M. Aganagic, N. Haouzi and S. Shakirov, “A_n-Triality,” arXiv:1403.3657 [hep-th].

[105] E. Gorsky and A. Negut, “Refined knot invariants and Hilbert schemes,” arXiv:1304.3328 [math.RT].

[106] M. Aganagic and C. Vafa, “Large N Duality, Mirror Symmetry, and a Q-deformed A-polynomial for Knots,” arXiv:1204.4709 [hep-th].
[107] S. S. Razamat and B. Willett, “Down the rabbit hole with theories of class $S$,” JHEP 1410, 99 (2014) [arXiv:1403.6107 [hep-th]].

[108] M. Bullimore, M. Fluder, L. Hollands and P. Richmond, “The superconformal index and an elliptic algebra of surface defects,” JHEP 1410, 62 (2014) [arXiv:1401.3379 [hep-th]].

[109] N. A. Nekrasov and S. L. Shatashvili, “Quantization of Integrable Systems and Four Dimensional Gauge Theories,” arXiv:0908.4052 [hep-th].

[110] N. A. Nekrasov and S. L. Shatashvili, “Supersymmetric vacua and Bethe ansatz,” Nucl. Phys. Proc. Suppl. 192-193, 91 (2009) [arXiv:0901.4744 [hep-th]].

[111] N. Dorey, “The BPS spectra of two-dimensional supersymmetric gauge theories with twisted mass terms,” JHEP 9811, 005 (1998) [hep-th/9806056].

[112] D. Gaiotto and P. Koroteev, “On Three Dimensional Quiver Gauge Theories and Integrability,” JHEP 1305, 126 (2013) [arXiv:1304.0779 [hep-th]].

[113] A. Gorsky, A. Zabrodin and A. Zotov, “Spectrum of Quantum Transfer Matrices via Classical Many-Body Systems,” JHEP 1401, 070 (2014) [arXiv:1310.6958 [hep-th]].

[114] Y. Terashima and M. Yamazaki, “SL(2,R) Chern-Simons, Liouville, and Gauge Theory on Duality Walls,” JHEP 1108, 135 (2011) [arXiv:1103.5748 [hep-th]].

[115] M. Shifman and A. Yung, “Lessons from supersymmetry: ”Instead-of-Confinement” Mechanism,” Int. J. Mod. Phys. A 29, no. 27, 1430064 (2014) [arXiv:1410.2900 [hep-th]].

[116] S. Bolognesi, “Instanton Bags, High Density Holographic QCD and Chiral Symmetry Restoration,” Phys. Rev. D 90, no. 10, 105015 (2014) [arXiv:1406.0205 [hep-th]].

[117] A. Givental and B. s. Kim, “Quantum cohomology of flag manifolds and Toda lattices,” Commun. Math. Phys. 168, 609 (1995) [hep-th/9312096].

[118] V. P. Gusynin, V. A. Miransky and I. A. Shovkovy, “Dimensional reduction and catalysis of dynamical symmetry breaking by a magnetic field,” Nucl. Phys. B 462, 249 (1996) [hep-ph/9509320].

[119] A. Gorsky, M. Shifman and A. Yung, “Revisiting the Faddeev-Skyrme Model and Hopf Solitons,” Phys. Rev. D 88, 045026 (2013) [arXiv:1306.2364 [hep-th]].

[120] N. Nekrasov, ”Supersymmetric gauge theories and quantization of integrable systems”, Conference “Strings 2009”.