Computable Solutions of Fractional Partial Differential Equations Related to Reaction-Diffusion Systems

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Abstract

The object of this paper is to present a computable solution of a fractional partial differential equation associated with a Riemann-Liouville derivative of fractional order as the time-derivative and Riesz-Feller fractional derivative as the space derivative. The method followed in deriving the solution is that of joint Laplace and Fourier transforms. The solution is derived in a closed and computable form in terms of the H-function. It provides an elegant extension of the results given earlier by Debnath [4], Chen et al. [2], Haubold et al. [17], Mainardi et al. [27, 28], Saxena et al. [39], and Pagnini et al. [35]. The results obtained are presented in the form of four theorems. Some results associated with fractional Schrödinger equation and fractional diffusion-wave equation are also derived as special cases of the findings.

Key words: Mittag-Leffler function, quantum mechanics, Riesz-Feller space fractional derivative, H-function, Schrödinger equation, Caputo derivative, Feynman path.

Mathematics Subject Classification: 26A33, 44A10, 33C60, 35J10

1. Introduction

General models for reaction-diffusion systems are discussed by Henry and Wearne [18, 19], Henry et al. [20], Haubold et al. [16,17], Saxena et al. [38, 39, 40], Mainardi et al. [27, 28] and others. Stability in reaction-diffusion systems and nonlinear oscillation have been discussed by Gafiychuk et al. [12,13]. Recently, Engler [7] obtained the speed of spread for fractional reaction-diffusion. Distributed order sub-diffusion is discussed by Naber [32]. In a recent paper, Chen et al. [2] have derived the fundamental and numerical solution of a reaction diffusion equation associated with the Riesz fractional derivative as the space derivative. Reaction-diffusion models associated with Riemann-Liouville fractional derivative as the time fractional derivative and Riesz-Feller derivative as the space derivative are recently discussed by Haubold et al. [17]. Such equations in case of Caputo fractional derivative are also solved by Saxena et al. [39]. In connection with the evolution equations for the probabilistic generalization of the Voigt profile function, it is shown by Pagnini and Mainardi [35] that the solution of the following integro-differential equation

\[ \frac{\partial N}{\partial \tau} = D_0^{\alpha_1} N(x, t) + D_0^{\alpha_2} N(x, t), \quad N(x, 0) = \delta(x) \]  

in terms of its Fourier transform, where \( D_0^{\alpha_1} \) and \( D_0^{\alpha_2} \) are the Riesz fractional derivatives of orders \( \alpha_1 \) and \( \alpha_2 \) respectively, and \( \delta(x) \) is the Dirac-delta function, which is given in [35, p.1593]. Consider the following Fourier transform, with the Fourier parameter \( k \):

\[ N^*(k, \tau) = \exp\{-\tau(|k|^\alpha_1 + |k|^\alpha_2)\}. \]
This has motivated the authors to investigate the solutions of fractional partial differential equations (3.1) and (3.14). The technique used in deriving the results is that of joint Laplace and Fourier transforms. The results are obtained in a closed and computable form. Due to the general character of the derived results, many known results given earlier by Chen et al. [2], Haubold et al. [17], Saxena et al. [39], Pagnini et al. [35], and others readily follow as special cases of our derived results. Solutions of certain extended space-time fractional diffusion wave equation and generalized Schrödinger equations are also deduced from our findings.

The Schrödinger equation is a fundamental equation of quantum mechanics. Feynman and Hibbs [11] reconstructed the Schrödinger equation by making use of the path integral approach by considering a Gaussian probability distribution. This approach is further extended by Laskin [23-25] in formulating the fractional Schrödinger equation by generalizing the Feynman path integrals from Brownian-like to Levy-like quantum mechanical paths. In a similar manner, one obtains a time fractional equation if non-Markovian evolution is considered. In a recent paper, Naber [32] discussed certain properties of the time fractional Schrödinger equation by writing the Schrödinger equation in terms of fractional derivatives as dimensionless variables. The time fractional Schrödinger equations are discussed by Debnath [4,5], Bhatti and Debnath [1], Saxena et al. [38], Tofight [40], and others.

2. Mathematical Prerequisites

The Riemann-Liouville fractional derivative of order \( \alpha > 0 \) is defined as (Samko et al. [37, p.37]: also see, Kilbas et al. [21])

\[
\alpha D_f^\alpha f(x, t) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_0^t (t-\tau)^{n-\alpha-1} f(x, \tau) d\tau, \quad n = [\alpha] + 1, \quad n \in N, \quad t > 0,
\]

where \([\alpha]\) means the integer part of the number \( \alpha \). The Laplace transform of the Riemann-Liouville fractional derivative is given by Oldham and Spanier [34, eq.(3.1.3)]; (also see, Kilbas et al. [21]):

\[
L\{\alpha D_f^\alpha N(x, t); s\} = s^\alpha \tilde{N}(x, s) - \sum_{r=1}^n s^{r-1} \alpha D_f^{r-\alpha} N(x, t)|_{t=0}, \quad n - 1 < \alpha \leq n.
\]

This derivative is useful in deriving the solutions of integral equations of fractional order governing certain physical problems of anomalous reaction and anomalous diffusion. In this connection, one can refer to the monograph by Podlubny [36], Samko et al. [37], Oldham and Spanier [34], Miller and Ross [31], Kilbas et al. [21], Mainardi [26], Diethelm [6], and recent papers on the subject [27,28,30,33,34,40,41]. Following Feller [9,10], it is conventional to define the Riesz-Feller space-fractional derivative of order \( \alpha \) and skewness \( \theta \) in terms of its Fourier transform as

\[
F\{xD_\alpha^\theta f(x); k\} = -\Psi_\alpha^\theta(k)f^*(k),
\]

where

\[
\Psi_\alpha^\theta(k) = |k|^\alpha \exp[i(\text{sign}k)\theta \pi/2], \quad 0 < \alpha \leq 2, \quad |\theta| \leq \min\{\alpha, 2 - \alpha\}.
\]

Further when \( \theta = 0 \), we have a symmetric operator with respect to \( x \) that can be interpreted as

\[
xD_\alpha^\theta = -\left(-\frac{d^2}{dx^2}\right)^{\frac{\theta}{2}}.
\]

This can be formally deduced by writing \(-k^\alpha = -(k^2)^{\frac{\theta}{2}}\). For \( 0 < \alpha < 2 \) and \( |\theta| \leq \min\{\alpha, 2 - \alpha\} \), the Riesz-Feller derivative can be shown to possess the following integral representation in \( x \) domain:

\[
xD_\alpha^\theta f(x) = \frac{\Gamma(1+\alpha)}{\pi} \left\{ \sin[(\alpha+\theta)\pi/2] \int_0^\infty \frac{f(x+\zeta) - f(x)}{\zeta^{1+\alpha}} d\zeta \\
+ \sin[(\alpha-\theta)\pi/2] \int_0^\infty \frac{f(x-\zeta) - f(x)}{\zeta^{1+\alpha}} d\zeta \right\}
\]
For \( \theta = 0 \), the Riesz-Feller fractional derivative becomes the Riesz fractional derivative of order \( \alpha \) for \( 1 < \alpha \leq 2 \), defined by analytic continuation in the whole range \( 0 < \alpha \leq 2 \), \( \alpha \neq 1 \) (see, Gorenflo and Mainardi [14]) as

\[
x D_0^\alpha = -\lambda [I_+^{-\alpha} - I_-^{-\alpha}],
\]

where

\[
\lambda = \frac{1}{2 \cos(\alpha \pi / 2)}; \quad I_+^{-\alpha} = \frac{d^2}{dx^2} I_+^{2-\alpha}.
\]

The Weyl fractional integral operators are defined in the monograph by Samko et al. [37] as

\[
(I_+^\beta N)(x) = \frac{1}{\Gamma(\beta)} \int_{-\infty}^{\infty} (x - \zeta)^{\beta-1} N(\zeta) d\zeta, \quad \beta > 0
\]

and

\[
(I_-^\beta N)(x) = \frac{1}{\Gamma(\beta)} \int_{x}^{\infty} (\zeta - x)^{\beta-1} N(\zeta) d\zeta, \quad \beta > 0.
\]

**Note 1.** We note that \( x D_0^\alpha \) is a pseudo differential operator. In particular we have

\[
x D_0^2 = \frac{d^2}{dx^2}, \text{ but } x D_0^1 \neq \frac{d}{dx}.
\]

For \( \theta = 0 \), we have

\[
F \{ x D_0^\alpha f(x); k \} = -|k|^\alpha f^*(k).
\]

The H-function is defined by means of a Mellin-Barnes type integral in the following manner [29, p.2]:

\[
H_{p,q}^{m,n}(z) = H_{p,q}^{m,n} \left[ x \left( a_p, A_p \right) \right] = \frac{1}{2 \pi i} \int_{c} \Theta(\xi) z^{-\xi} d\xi,
\]

where \( i = (-1)^{\frac{1}{2}} \),

\[
\Theta(\xi) = \frac{\prod_{j=1}^{m} \Gamma(b_j + B_j \xi) \prod_{j=1}^{p} \Gamma(1 - a_j - A_j \xi)}{\prod_{j=m+1}^{p} \Gamma(1 - b_j - B_j \xi) \prod_{j=n+1}^{q} \Gamma(a_j + A_j \xi)},
\]

and an empty product is always interpreted as unity; \( m, n, p, q \in \mathbb{N} \) with \( 0 \leq n \leq p, 1 \leq m \leq q, A_j, B_j, a_j, b_j \in \mathbb{R} \) or \( C \), \( i = 1, ..., p; j = 1, ..., q \) such that

\[
A_i(b_j + k) \neq B_j(a_i - s - 1), \quad k, s \in \mathbb{N}_0 \quad i = 1, ..., n; \quad j = 1, ..., m,
\]

and these poles are separated, where we employ the usual notations: \( \mathbb{N}_0 = (0, 1, 2, ...) \); \( R = (-\infty, \infty) \), and \( C \) being the complex number field. A comprehensive account of the H-function is available from the monographs Mathai et al. [29] and Kilbas et al. [22]. We also need the following result in the analysis that follows; Haubold et al. [16] has shown that

\[
F^{-1}[E_{\beta,\gamma}(-at^\beta \Psi_\alpha^\theta(k); x)] = \frac{1}{\alpha \beta} H_{p,q}^{m,n} \left[ \frac{|x|}{(at^\beta)^\gamma} \prod_{i,j}^{(1, \frac{1}{\beta})}, (1, 1), (1, \rho) \right],
\]

where \( \Re(\alpha) > 0, \Re(\beta) > 0, \Re(\gamma) > 0 \).

3. Solution of Unified Fractional Partial Differential Equations
In this section, we will investigate the solution of fractional partial differential equations, which may be regarded as an extension of one-dimensional fractional reaction-diffusion equation, one-dimensional space-time fractional diffusion-wave equation and one-dimensional fractional Schrödinger equation. The results are presented in the form of the following four theorems.

**Theorem 1.** Consider the following one-dimensional non-homogeneous unified fractional differential equation:

\[
o D_t^\alpha N(x, t) = \lambda x D_0^\beta N(x, t) + \mu x D_0^\gamma U(x, t),
\]

where \( t > 0, \ x \in \mathbb{R}; \ \alpha, \ \theta, \ \beta, \ \gamma \) and \( \phi \) are real parameters with the constraints

\[
0 < \beta \leq 2, \ 0 < \gamma \leq 2, \ |\theta| \leq \min(\beta, 2 - \beta), \ |\phi| \leq \min(\gamma, 2 - \gamma), \ 0 < \alpha \leq 1,
\]

with the initial conditions

\[
o D_t^{\alpha-1} N(x, 0) = f(x), \ for \ x \in \mathbb{R}, \ \lim_{x \to \pm \infty} N(x, t) = 0, \ t > 0.
\]

Here \( o D_t^{\alpha-1} N(x, 0) \) means the Riemann-Liouville fractional partial derivative of \( N(x, t) \) with respect to \( t \) of order \( \alpha - 1 \) evaluated at \( t = 0; \ \lambda D_0^\beta \) and \( \mu D_0^\gamma \) are the Riesz-Feller space-fractional derivatives respectively of orders \( \beta \) and \( \gamma \) with asymmetries \( \theta \) and \( \phi \), \( o D_t^\alpha \) is the Riemann-Liouville time-fractional derivative of order \( \alpha; \ \lambda \) and \( \mu \) are arbitrary constants, and \( f(x) \) and \( U = U(x, t) \) are the given functions. Then for the solution of (3.1), subject to the above conditions, there holds the formula

\[
N(x, t) = \int_{-\infty}^{\infty} G(x - \xi, t)f(\xi)d\xi
\]

\[
- \mu \int_0^t (t - \tau)^{\alpha-1} \int_{-\infty}^{\infty} G_1(x - \tau, t - \tau) \psi(\xi, \tau)d\xi d\tau,
\]

where the Green functions \( G(x, t) \) and \( G_1(x, t) \) are given by

\[
G(x, t) = \frac{t^{\alpha-1}}{2\pi} \int_{-\infty}^{\infty} \exp(-ikx) E_{\alpha,\alpha}(-\lambda \Psi_k(\xi)t^{\alpha})dk,
\]

\[
= \frac{t^{\alpha-1}}{\beta |x|} H_{3.3}^{2.1} \left[ \frac{|x|}{(\lambda t^\alpha)^{\beta}} \right]_{(1,1,1,1,1,1,1,1,1)}(1,1,1,1,1,1,1,1,1)
\]

with \( \beta > 0, \ \rho = \frac{\beta - \alpha}{2\beta} \), and

\[
G_1(x, t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \exp(-ikx) \Psi_{\gamma}(k) E_{\alpha,\alpha}(-\lambda \Psi_k(\xi)t^{\alpha})dk
\]

and \( H_{3.3}^{2.1}(z) \) is the \( H \)-function defined by (2.12) and \( E_{\alpha,\beta}(z) \) is the Mittag-Leffler function, defined by [8, Section 18.1]

\[
E_{\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n + \beta)}, \ \alpha, \ \beta \in C, \ \Re(\alpha) > 0, \ \Re(\beta) > 0.
\]

In deriving the value of the integral in (3.5), the formula (2.15) has been used.

**Proof:** In order to derive the solution of (3.1), we introduce the joint Laplace-Fourier transform in the form

\[
\tilde{N}^*(k, s) = \int_0^\infty \int_{-\infty}^{\infty} e^{-st + ikx} N(x, t) dx \ dt,
\]

\[
(3.9)
\]
where $\Re(s) > 0$, $k > 0$. If we apply the Laplace transform with respect to the time variable $t$, Fourier transform with respect to space variable $x$ and use the initial conditions (3.2), (3.3) and the formula (2.2), then the given equation transforms into the form

$$s^\alpha \tilde{N}^*(k, s) - f^*(k) = -\lambda \Psi^\theta_\alpha(k) \tilde{N}^*(k, s) - \mu \Psi^\phi_\beta(k) \tilde{U}^*(k, s),$$

where according to the conventions followed, the symbol $\tilde{N}(k, s)$ will stand for the Laplace transform with respect to time variable $t$ and $*$ represents the Fourier transform with respect to space variable $x$. Solving for $\tilde{N}^*(k, s)$, it yields

$$\tilde{N}^*(k, s) = \frac{f^*(k)}{s^\alpha + \lambda \Psi^\theta_\alpha(k)} - \frac{\mu \Psi^\phi_\beta(k) \tilde{U}^*(k, s)}{s^\alpha + \lambda \Psi^\theta_\alpha(k)}.$$  \hspace{1cm} (**10**)

To invert (**10**), it is convenient to first invert the Laplace transform and then the Fourier transform. Thus to invert the Laplace transform we use the formula given in [39]

$$L^{-1}\left\{\frac{s^\beta-1}{a + s^\alpha}; t\right\} = t^{\alpha-\beta} E_{\alpha,\alpha-\beta+1}(-at^\alpha),$$ \hspace{1cm} (**11**)

where $\Re(s) > 0$, $\Re(\alpha) > 0$, $\Re(\alpha - \beta) > -1$ and the convolution theorem of the Laplace transform to obtain

$$\tilde{N}^*(k, t) = f^*(k)t^{\alpha-1} E_{\alpha,\alpha}(-\lambda t^\alpha) \Psi^\theta_\beta(k)$$

$$- t^\mu \Psi^\phi_\gamma(k) \int_0^t (t-\tau)^{\alpha-1} E_{\alpha,\alpha}(-\lambda \Psi^\phi_\beta(k)(t-\tau)^\alpha)U^*(k, \tau)d\tau.$$ \hspace{1cm} (**12**)

Now the application of the inverse Fourier transform gives the exact solution in the following form:

$$N^*(k, t) = f^*(k)t^{\alpha-1}E_{\alpha,\alpha}(-\lambda \Psi^\phi_\beta(k)t^\alpha)$$

$$- \mu \int_0^t (t-\tau)^{\alpha-1}F^{-1}[\Psi^\phi_\gamma(k)E_{\alpha,\alpha}(-\lambda \Psi^\phi_\beta(k)(t-\tau)^\alpha)U^*(k, \tau)]d\tau.$$ \hspace{1cm} (**13**)

Finally, the application of the convolution theorem of the Fourier transform yields the desired solution (3.4).

Following a similar procedure, it is not difficult to establish the following theorems:

**Theorem 2.** Consider the same equation in (3.1) with the same condition on the parameters except that $0 < \alpha \leq 2$, instead of $0 < \alpha \leq 1$. In addition to the initial conditions in (3.3), assume that

$$0 D_t^{\alpha-2}N(x, 0) = g(x).$$ \hspace{1cm} (**14**)

Then the solution of (3.1) under the conditions (3.2), (3.3) and (3.14) is given by

$$N(x, t) = \int_{-\infty}^\infty G(x - \xi, t)f(\xi)d\xi + \int_{-\infty}^\infty G_2(x - \xi, t)g(\xi)d\xi$$

$$- \mu \int_0^t (t-\tau)^{\alpha-1}\int_{-\infty}^\infty G_1(x - \tau, t-\tau)U(\xi, \tau)d\xi d\tau,$$ \hspace{1cm} (**15**)

where the Green functions $G(x, t)$ and $G_1(x, t)$ are defined in (3.6) and (3.7) respectively and $G_2(x, t)$ is given by

$$G_2(x, t) = \frac{t^{\alpha-2}}{2\pi} \int_{-\infty}^\infty \exp(-ikx)E_{\alpha,\alpha-1}(-\lambda \Psi^\phi_\beta(k)t^\alpha)dk$$

$$= \frac{t^{\alpha-2}}{\beta|x|} H_{4,3}^{2,1} \left[ \frac{|x|}{(\lambda t^\alpha)^{\frac{1}{3}}} \left( \frac{(1,\frac{1}{2})(\alpha-1,\frac{1}{2})(1,2)}{(1,1)(1,\frac{1}{2})(1,1)} \right), \beta > 0 \right]$$ \hspace{1cm} (**16**).
with $\rho = \frac{\beta - \theta}{\alpha}$ and $H^{2,1}_{3,3}(z)$ is the $H$-function defined by (2.17) and $E_{\alpha,\beta}(z)$ is the Mittag-Leffler function, defined in (3.8).

Now, we set $U(x, t) = N(x, t)$, where $N(x, t)$ is the unknown function to arrive at the next result.

**Theorem 3.** Consider the same equation in (3.1) with $U(x, t)$ replaced by $N(x, t)$. Assume that the conditions (3.2) and (3.3) hold. Then solution of (3.1) with $U(x, t)$ replaced by $N(x, t)$ is given by

$$N(x, t) = \int_{-\infty}^{\infty} G_3(x - \xi, t)f(\xi)d\xi,$$

where the Green function $G_3(x, t)$ is given by

$$G_3(x, t) = \frac{t^{\alpha-1}}{2\pi} \int_{-\infty}^{\infty} \exp(-ikx)E_{\alpha,\alpha}[-(\lambda \Psi_{\beta}(k) + \mu \Psi_{\gamma}(k))t^\alpha]dk. \tag{3.18}$$

Similarly, we find that the following theorem holds true:

**Theorem 4.** Consider the same equation in Theorem 3 with $0 < \alpha \leq 2$ and under the conditions (3.2), (3.3), and (3.14). Then the solution of (3.1) with $U(x, t)$ replaced by $N(x, t)$ with $0 < \alpha \leq 2$ is given by

$$N(x, t) = \int_{-\infty}^{\infty} G_3(x - \xi, t)f(\xi)d\xi,$$

where the Green function $G_4(x, t)$ is defined in (3.18) and the other Green function $G_4(x, t)$ is given by

$$G_4(x, t) = \frac{t^{\alpha-2}}{2\pi} \int_{-\infty}^{\infty} \exp(-ikx)E_{\alpha,\alpha-1}[-(\lambda \Psi_{\beta}(k) + \mu \Psi_{\gamma}(k))t^\alpha]dk. \tag{3.20}$$

**Note 2.** It is interesting to observe that for $g(x) = 0$, Theorems 2 and 4 yield similar types of results as Theorems 1 and 3 respectively.

4. Selected Special Cases

If we set $\theta = \phi = 0$ then by virtue of the identity (2.11), Riesz-Feller derivative reduces to Riesz derivative and consequently Theorem 1 yields the following results:

**Corollary 1.** Consider the one-dimensional non-homogeneous unified fractional differential equation (3.1) for $\theta = \phi = 0$:

$$\alpha D_t^\alpha N(x, t) = \lambda x D_x^\beta N(x, t) + \mu x D_0^\gamma U(x, t), \tag{4.1}$$

where the conditions (3.2) and (3.3) hold under $\phi = 0$, $t > 0$, $x \in R$, $\alpha$, $\beta$, $\gamma$ are real. Then for the solution of (4.1) there holds the formula

$$N(x, t) = \int_{-\infty}^{\infty} G_5(x - \xi, t)f(\xi)d\xi$$

$$- \mu \int_{0}^{t} (t - \tau)^{\alpha-1}F^{-1}\left[ \int_{-\infty}^{\infty} G_6(x - \tau, t - \tau)U(\xi, \tau)d\xi \right]d\tau,$$

where the Green functions $G_5(x, t)$ and $G_6(x, t)$ are given by

$$G_5(x, t) = \frac{t^{\alpha-1}}{2\pi} \int_{-\infty}^{\infty} \exp(-ikx)E_{\alpha,\alpha}[-(\lambda |k|^{\alpha}t^\alpha)]dk$$

$$= \frac{1}{\beta|x|} H^{2,1}_{3,3}\left[ \frac{|x|}{(\lambda t^\alpha)^{\beta}} \right]^{(1, \frac{1}{2}), (\alpha, \frac{1}{2}), (1, \frac{1}{2})}, \beta > 0 \tag{4.4}$$
\[ G_6(x, t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \exp(-ikx)|k|^\gamma E_{\alpha,1}(-\lambda|k|^\beta t^\alpha)dk. \] (4.5)

For \( \mu = \phi = 0 \), Theorem 1 reduces to the following result given by Haubold et al. [10].

**Corollary 2.** Consider the following one-dimensional fractional reaction-diffusion model of (3.1) with \( \mu = \phi = 0 \)

\[ 0D_t^\alpha N(x, t) = \lambda x D_x^\theta N(x, t), \] (4.6)

where \( t > 0, x \in \mathbb{R}, \alpha, \theta \) real, and conditions (3.2) and (3.3) hold under \( \mu = \phi = 0 \). Then for the solution of (4.6) there holds the formula

\[ N(x, t) = \int_{-\infty}^{\infty} G(x - \xi, t)f(\xi)d\xi, \] (4.7)

where the Green function \( G(x, t) \) is defined in (3.6).

If we further set \( f(x) = \delta(x) \), where \( \delta(x) \) is the Dirac-delta function, we obtain the fundamental solution of the space-time fractional diffusion equation given by Haubold et al. [17].

By setting \( \theta = \phi = 0 \) and using (2.11), the following Corollaries 3-5 can be easily deduced from Theorems 2-4.

**Corollary 3.** Consider the following one-dimensional non-homogeneous unified fractional differential equation:

\[ 0D_t^\alpha N(x, t) = \lambda x D_x^\theta N(x, t) + \mu x D_0^\gamma U(x, t), \] (4.8)

where (3.2), (3.3), (3.14) hold under \( \theta = \phi = 0 \). Then for the solution of (4.8), there holds the formula

\[ N(x, t) = \int_{-\infty}^{\infty} G_5(x - \xi, t)f(\xi)d\xi + \int_{-\infty}^{\infty} G_7(x - \xi, t)g(\xi)d\xi, \]
\[ -\mu \int_0^t (t - \tau)^{\alpha-1}F^{-1} \left[ \int_{-\infty}^{\infty} G_6(x - \tau, t - \tau)U(\xi, \tau)d\xi \right]d\tau, \] (4.9)

where the Green functions \( G_5(x, t) \) and \( G_6(x, t) \) are defined in (4.3) and (4.5) respectively and \( G_7(x, t) \) is given by

\[ G_7(x, t) = \frac{t^{\alpha-2}}{2\pi} \int_{-\infty}^{\infty} \exp(-ikx)E_{\alpha,1}(-\lambda|k|^\beta t^\alpha)dk \]
\[ = \frac{t^{\alpha-2}}{\beta|x|} H_{3,3}^{\alpha,1} \left[ \frac{|x|}{(\lambda t^\alpha)^\beta}, (1,1), (1,1), (1,1), (1,1) \right], \beta > 0, \] (4.10)

where \( \rho = \frac{\beta - \theta}{2\beta} \).

**Corollary 4.** Consider the following one-dimensional non-homogeneous unified fractional differential equation where \( \alpha, \beta \) and \( \gamma \) are real parameters with the constraints

\[ 0D_t^\alpha N(x, t) = \lambda x D_x^\theta N(x, t) + \mu x D_0^\gamma N(x, t), \] (4.11)

where (3.2), (3.3) hold for \( \theta = \phi = 0 \). Then
\[ N(x,t) = \int_{-\infty}^{\infty} G_8(x-\xi,t)f(\xi)d\xi, \quad (4.12) \]

where the Green function \( G_8(x,t) \) is given by
\[
G_8(x,t) = \frac{t^{\alpha-1}}{2\pi} \int_{-\infty}^{\infty} \exp(-ikx)E_{\alpha,\alpha}[-(\lambda|k|^\beta + \mu |k|^\gamma)t^\alpha]dk. \quad (4.13)
\]

**Corollary 5.** Consider the following one-dimensional non-homogeneous unified fractional differential equation
\[
_{0}D_t^{\alpha}N(x,t) = \lambda x D_{0}^{\beta}N(x,t) + \mu x D_{0}^{\gamma}N(x,t), \quad (4.14)
\]
where (3.2), (3.3), (3.14) hold. Then
\[
N(x,t) = \int_{-\infty}^{\infty} G_8(x-\xi,t)f(\xi)d\xi + \int_{-\infty}^{\infty} G_9(x-\xi,t)g(\xi)d\xi, \quad (4.15)
\]
where the Green functions \( G_8(x,t) \) is defined in (4.13) and the other Green function \( G_9(x,t) \) is given by
\[
G_9(x,t) = \frac{t^{\alpha-2}}{2\pi} \int_{-\infty}^{\infty} \exp(-ikx)E_{\alpha,\alpha-1}[-(\lambda|k|^\beta + \mu |k|^\gamma)t^{\alpha}]dk. \quad (4.16)
\]

**Note 3.** It is observed that for \( g(x) = 0 \), Theorems 2 and 3 yield similar types of results as Theorems 1 and 3, respectively.

5. Further Special Cases

In this section we discuss some consequences of the main results relating to the fractional Schrödinger equation. In the following corollaries 6 and 7, \( h \) is the Planck constant.

If we set \( \gamma - \phi = 0 \) and \( \lambda = \frac{ih}{2m} \), then Theorems 1 and 2 give the solutions of non-homogeneous fractional Schrödinger equations as shown below.

**Corollary 6.** Consider the following one-dimensional non-homogeneous unified fractional Schrödinger equation of mass \( m \):
\[
_{0}D_t^{\alpha}N(x,t) = \left(\frac{ih}{2m}\right)x D_0^{\beta}N(x,t) + \mu U(x,t), \quad (5.1)
\]
where \( t > 0, \ x \in \mathbb{R}, \ \alpha, \ \beta \) real, with the constraints
\[
0 < \beta \leq 2, \ 0 < \alpha \leq 1, \quad (5.2)
\]
with the initial conditions (3.3), and \( N(x,t) \) as the wave function.

Then for the solution of (5.1), subject to the above constraints, there holds the formula
\[
N(x,t) = \int_{-\infty}^{\infty} G(x-\xi,t)f(\xi)d\xi \\
+ \int_{-\infty}^{\infty} (t-\xi)^{\alpha-1} \left[ \int_{-\infty}^{\infty} G_{10}(x-\xi,t-\tau)U(\xi,\tau)d\xi \right]d\tau, \quad (5.3)
\]
where the Green function \( G(x,t) \) is defined in (3.6) and \( G_{10}(x,t) \) is given by
\[
G_{10}(x,t) = \int_{-\infty}^{\infty} \exp(-ikx)E_{\alpha,\alpha}[-(\lambda \Psi_0^\beta(k))t^\alpha]dk \\
= \frac{1}{|x|^\beta} H_{3.3}^{2.1} \left[ \frac{|x|}{(at)^{\frac{\alpha}{2}}} \right]_{(1,\frac{\alpha}{2})}^{(1,1,\frac{\alpha}{2})}, \quad (5.4)
\]
where $\beta > 0$, $a = \frac{i\hbar}{2m}$, $\rho = \frac{\beta - \theta}{2\beta}$.

**Corollary 7.** Consider the following one-dimensional non-homogeneous unified fractional Schrödinger equation of a particle of mass $m$:

$$D_t^\alpha N(x,t) = \left(\frac{i\hbar}{2m}\right)_x D_\theta^\beta N(x,t) + \mu U(x,t), \quad (5.5)$$

where $t > 0, x \in \mathbb{R}, \alpha, \theta, \beta$ real, with the constraints

$$0 < \beta \leq 2, \ |\theta| \leq \min(\beta, 2 - \beta), \ 1 < \alpha \leq 2,$$

and with the initial conditions (3.3). Then for the solution of (5.5), subject to the above constraints, there holds the formula

$$N(x,t) = \int_{-\infty}^{\infty} G(x - \xi, t)f(\xi)d\xi + \int_{-\infty}^{\infty} tG_2(x - \xi, t)g(\xi)d\xi$$

$$+ \mu \int_{0}^{t} (t - \tau)^{\alpha - 1} \left[ \int_{-\infty}^{\infty} G_10(x - \xi, t - \tau)U(\xi, \tau)d\xi \right]d\tau,$$

(5.7)

where the Green functions $G(x,t), G_2(x,t)$ and $G_{10}(x,t)$ are respectively given by (3.6), (3.16), and (5.4), respectively.

On the other hand, if we set $\gamma = \phi = 0$, Theorems 1 and 2 provide the solutions of non-homogeneous fractional generalized diffusion-wave equations as given below:

**Corollary 8.** Consider the following one-dimensional non-homogeneous unified fractional diffusion-wave equation:

$$\partial_t^\alpha N(x,t) = c^2 D_\theta^\beta N(x,t) + \mu U(x,t), \quad (5.8)$$

where $t > 0, x \in \mathbb{R}, \alpha, \beta$ real, with the constraints

$$0 < \beta \leq 2, \ 0 < \alpha \leq 1,$$

and with the initial conditions (3.3) and (5.2), where $\mu$ and $c$ are arbitrary constants. Then for the solution of (5.8), subject to the above conditions, there holds the formula

$$N(x,t) = \int_{-\infty}^{\infty} G(x - \xi, t)f(\xi)d\xi$$

$$+ \int_{-\infty}^{t} (t - \tau)^{\alpha - 1} \left[ \int_{-\infty}^{\infty} G_{10}(x - \xi, t - \tau)U(\xi, \tau)d\xi \right]d\tau,$$

(5.10)

where the Green function $G(x,t)$ and $G_{10}(x,t)$ are respectively given in (3.6) and (5.4).

**Corollary 9.** Consider the following one-dimensional non-homogeneous unified fractional diffusion-wave equation:

$$\partial_t^\alpha N(x,t) = c^2 D_\theta^\beta N(x,t) + \mu U(x,t), \quad (5.11)$$

where $t > 0, x \in \mathbb{R}, \alpha, \theta, \beta$ real, with the constraints $0 < \beta \leq 2, \ |\theta| \leq \min(\beta, 2 - \beta), \ 1 < \alpha \leq 2$, and with the initial conditions (3.3) and (3.14). Then for the solution of (5.11), subject to the above conditions, there holds the formula
\[ N(x,t) = \int_{-\infty}^{\infty} G(x-\xi,t)f(\xi)d\xi + \int_{-\infty}^{\infty} t G_2(x-\xi,t)g(\xi)d\xi \]
\[ + \mu \int_{0}^{t} (t-\tau)^{\alpha-1} \left[ \int_{-\infty}^{\infty} G_{10}(x-\tau,t-\tau)U(\xi,\tau)d\xi \right]d\tau, \quad (5.12) \]

where the Green functions \( G(x,t), G_2(x,t) \) and \( G_{10}(x,t) \) are, respectively, given by (3.6), (3.16), and (5.4).

Now, setting \( \mu = 1 \) and \( \theta = \beta = 0 \), we find that the Corollaries 8 and 9 give rise to the expressions for the solution of non-homogeneous one-dimensional fractional generalized wave equations as given by Debnath [4, p.141].

Finally, for \( \alpha = 1 \) and \( \theta = \phi = 0 \), Theorem 3 provides a solution of generalized diffusion equation with two space-fractional derivatives, which were recently studied by Pagnini and Mainardi [35].

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