Trees in Wavelet analysis on Vilenkin groups
S. F. Lukomskii

N.G. Chernyshevskii Saratov State University
LukomskiiSF@info.sgu.ru
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Abstract
We consider a class of $(1, M)$-elementary step functions on the $p$-adic Vilenkin group. We prove that $(1, M)$-elementary step function generates a MRA on $p$-adic Vilenkin group iff it is generated by a rooted tree on the set of vertices \{0, 1, \ldots, p-1\} with 0 as a root. Bibliography: 14 titles.

keywords: zero-dimensional group, Vilenkin group, multiresolution analysis, wavelet bases, tree.

1 Introduction

In articles [1]-[4] first examples of orthogonal wavelets on the dyadic Cantor group ($p = 2$) are constructed and their properties are studied. Yu.Farkov [5]-[7] found necessary and sufficient conditions for a refinable function to generate an orthogonal MRA in the $L_2(\mathcal{G})$ -spaces on the $p$-adic Vilenkin group $\mathcal{G}$. These conditions use the Strang-Fix and the modified Cohen properties.

In [7] this construction is given in a concrete fashion for $p = 3$. In [8], some algorithms for constructing orthogonal and biorthogonal compactly supported wavelets on Vilenkin groups are proposed. In [5]-[8] two types of orthogonal wavelet examples are constructed: step functions and sums of Vilenkin series.

Khrennikov, Shelkovich, and Skopina [10],[11] introduced the concept of a $p$-adic MRA with orthogonal refinable function, and described a general pattern for their construction. This method was developed for an orthogonal refinable function $\varphi$ with condition $\text{supp}\hat{\varphi} \subset B_0(0)$, where $B_0(0) = \{x : |x|_p \leq 1\}$ is the unit ball in the field $\mathbb{Q}_p$. Similar results were obtained for arbitrary zero-dimensional group [13]. The condition $\text{supp}\hat{\varphi} \subset B_0(0)$ is very important. S. Albeverio, S. Evdokimov, M. Skopina [12] proved that if a refinable step function $\varphi$ generates an orthogonal $p$-adic MRA, then $\text{supp}\hat{\varphi}(\chi) \subset B_0(0)$.

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On the other hand on Vilenkin groups Yu.A.Farkov constructs examples of step refinable functions \( \varphi \) generating an orthogonal MRA with \( \text{supp} \hat{\varphi} \subset G_1^\perp \). In the author’s work [14] a necessary condition for a support of orthogonal refinable step function are found: if step refinable \((1, N)\)-elementary functions \( \varphi \) generated an orthogonal MRA on \( p \)-adic Vilenkin group, then \( \text{supp} \hat{\varphi} \subset G_{p-2}^\perp \).

In this work we study a structure of the set \( \text{supp} \hat{\varphi} \). We prove that \((1, N)\)-elementary function \( \varphi \) generates an orthogonal MRA on \( p \)-adic Vilenkin group iff the function \( \varphi \) is generated by means of some tree. For any tree we give an algorithm for constructing corresponding refinable function and orthogonal wavelets.

The paper is organized as follows. We consider \( p \)-adic Vilenkin group \( \mathfrak{G} \) as a zero-dimensional group \((G, \dot{+})\) with condition \( pg_n = 0 \). Therefore, in section 2, we recall some concepts and facts from the theory of zero-dimensional group. We will systematically use the notation and the results from [13],[14].

In section 3 and the following sections we consider MRA on \( p \)-adic Vilenkin group \( \mathfrak{G} \). In section 3 we study refinable step-functions which generate the orthogonal MRA. We define a class of \((N, M)\)-elementary set and prove that the shifts system \( \varphi(x - \hat{h})_{h \in H_0} \) is orthonormal if \( \text{supp} \hat{\varphi} \) is \((N, M)\)-elementary set.

In section 4 we introduce such concepts as ”a set generated by a tree” and ”a refinable step function generated by a tree” and prove, that any rooted tree generates a refinable step function that generate an orthogonal MRA on Vilenkin group.

In section 5 we give an algorithm for constructing orthogonal wavelets according to the tree.

2 Preliminaries

We will consider the Vilenkin group as a locally compact zero-dimensional Abelian group with additional condition \( p_n g_n = 0 \). Therefore we start with some basic notions and facts related to analysis on zero-dimensional groups. More information it is possible to find in [12],[14].

Let \((G, \dot{+})\) be a locally compact zero-dimensional Abelian group with the topology generated by a countable system of open subgroups

\[
\cdots \supset G_{-n} \supset \cdots \supset G_{-1} \supset G_0 \supset G_1 \supset \cdots \supset G_n \supset \cdots
\]
where
\[ \bigcup_{n=-\infty}^{+\infty} G_n = G, \quad \bigcap_{n=-\infty}^{+\infty} G_n = \{0\}, \]

\( p_n \) be an order of quotient group \( G_n/G_{n+1} \). We will always assume that all \( p_n \) are prime numbers. We will name such chain as basic chain. In this case, a base of the topology is formed by all possible cosets \( G_n + g, \ g \in G \).

We further define the numbers \((m_n)_{n=-\infty}^{+\infty}\) as follows:
\[ m_0 = 1, \quad m_{n+1} = m_n \cdot p_n. \]

Let \( \mu \) be a Haar measure on \( G \), we know that \( \mu G_n = \frac{1}{m_n} \). Further, let
\[ \int_G f(x) \, d\mu(x) \]
be the absolutely convergent integral of the measure \( \mu \).

Given an \( n \in \mathbb{Z} \), take an element \( g_n \in G_n \setminus G_{n+1} \) and fix it. Then any \( x \in G \) has a unique representation of the form
\[ x = \sum_{n=-\infty}^{+\infty} a_n g_n, \quad a_n = \overline{0, p_n - 1}. \] (2.1)

The sum (2.1) contain finite number of terms with negative subscripts, that is,
\[ x = \sum_{n=m}^{+\infty} a_n g_n, \quad a_n = \overline{0, p_n - 1}, \quad a_m \neq 0. \] (2.2)

We will name system \((g_n)_{n \in \mathbb{Z}}\) as a basic system.

Classical examples of zero-dimensional groups are Vilenkin groups and groups of \( p \)-adic numbers (see [12, Ch. 1, §2]). A direct sum of cyclic groups \( \mathbb{Z}(p_k) \) of order \( p_k, k \in \mathbb{Z} \), is called a Vilenkin group. This means that the elements of a Vilenkin group are infinite sequences \( x = (x_k)_{k=-\infty}^{+\infty} \) such that:

1) \( x_k = \overline{0, p_k - 1} \);

2) only a finite number of \( x_k \) with negative subscripts are different from zero;

3) the group operation \( + \) is the coordinate-wise addition modulo \( p_k \), that is,
\[ x + y = (x_k + y_k), \quad x_k + y_k = (x_k + y_k) \mod p_k. \]
A topology on such group is generated by the chain of subgroups
\[ G_n = \{ x \in G : x = (\ldots, 0, 0, \ldots, 0, x_n, x_{n+1}, \ldots) , \ x_\nu = 0, p_\nu - 1, \ \nu \geq n \} . \]
The elements \( g_n = (\ldots, 0, 0, 1, 0, 0, \ldots) \) form a basic system. From definition of the operation \( \dot{+} \) we have \( p_n g_n = 0 \). Therefore we will name a zero-dimensional group \((G, \dot{+})\) with the condition \( p_n g_n = 0 \) as Vilenkin group.

By \( X \) denote the collection of the characters of a group \((G, \dot{+})\); it is a group with respect to multiplication too. Also let \( G_n^\perp = \{ \chi \in X : \forall x \in G_n , \chi(x) = 1 \} \) be the annihilator of the group \( G_n \). Each annihilator \( G_n^\perp \) is a group with respect to multiplication, and the subgroups \( G_n^\perp \) form an increasing sequence
\[ \cdots \subset G_{-n}^\perp \subset \cdots \subset G_0^\perp \subset G_1^\perp \subset \cdots \subset G_n^\perp \subset \cdots \quad (2.3) \]
with
\[ +\infty \bigcup_{n=-\infty} G_n^\perp = X \quad \text{and} \quad -\infty \bigcap_{n=-\infty} G_n^\perp = \{1\} , \]
the quotient group \( G_{n+1}^\perp / G_n^\perp \) having order \( p_n \). The group of characters \( X \) is a zero-dimensional group with a basic chain (2.3). The group may be equipped with the topology using the chain of subgroups (2.3), the family of the cosets \( G_n^\perp \cdot \chi , \chi \in X \), being taken as a base of the topology. The collection of such cosets, along with the empty set, forms the semiring \( \mathcal{X} \).

Given a coset \( G_n^\perp \cdot \chi \), we define a measure \( \nu \) on it by \( \nu(G_n^\perp \cdot \chi) = \nu(G_n^\perp) = m_n \) (so that always \( \mu(G_n) \nu(G_n^\perp) = 1 \)). The measure \( \nu \) can be extended onto the \( \sigma \)-algebra of measurable sets in the standard way. One then forms the absolutely convergent integral \( \int_X F(\chi) \, d\nu(\chi) \) of this measure.

The value \( \chi(g) \) of the character \( \chi \) at an element \( g \in G \) will be denoted by \((\chi, g)\). The Fourier transform \( \hat{f} \) of an \( f \in L_2(G) \) is defined as follows
\[ \hat{f}(\chi) = \int_G f(x)(\chi, x) \, d\mu(x) = \lim_{n \to +\infty} \int_{G_{-n}} f(x)(\chi, x) \, d\mu(x) , \]
the limit being in the norm of \( L_2(X) \). For any \( f \in L_2(G) \), the inversion formula is valid
\[ f(x) = \int_X \hat{f}(\chi)(\chi, x) \, d\nu(\chi) = \lim_{n \to +\infty} \int_{G_n^\perp} \hat{f}(\chi)(\chi, x) \, d\nu(\chi) ; \]
here the limit also signifies the convergence in the norm of $L_2(G)$. If $f, g \in L_2(G)$ then the Plancherel formula is valid
\[
\int_G f(x)\overline{g(x)}\,d\mu(x) = \int_X \hat{f}(\chi)\overline{\hat{g}(\chi)}\,d\nu(\chi).
\]

Provided with this topology, the group of characters $X$ is a zero-dimensional locally compact group; there is, however, a dual situation: every element $x \in G$ is a character of the group $X$, and $G_n$ is the annihilator of the group $G_n^\perp$. The union of disjoint sets $E_j$ we will denote by $\bigcup E_j$.

For any $n \in \mathbb{Z}$ we choose a character $r_n \in G_{n+1}^\perp \setminus G_n^\perp$ and fixed it. $(r)_{n \in \mathbb{Z}}$ is called a Rademacher system. Let us denote
\[
H_0 = \{ h \in G : h = a_{-1}g_{-1} + a_{-2}g_{-2} \ldots + a_{-s}g_{-s}, s \in \mathbb{N}, a_j = 0, p - 1 \},
\]
\[
H_0^{(s)} = \{ h \in G : h = a_{-1}g_{-1} + a_{-2}g_{-2} \ldots + a_{-s}g_{-s}, a_j = 0, p - 1 \}, s \in \mathbb{N}.
\]
The set $H_0$ is an analog of the set $\mathbb{N} = \mathbb{N} \bigcup \{0\}$.

If in the zero-dimensional group $G \ p_n = p$ for any $n \in \mathbb{Z}$ then we can define the mapping $\mathcal{A}: G \to G$ by $\mathcal{A}x := \sum_{n=-\infty}^{+\infty} a_ng_{n-1}$, where $x = \sum_{n=-\infty}^{+\infty} a_ng_n \in G$. The mapping $\mathcal{A}$ is called a dilation operator if $\mathcal{A}(x+y) = \mathcal{A}x + \mathcal{A}y$ for all $x, y \in G$. By definition, put $(\chi \mathcal{A}, x) = (\chi, \mathcal{A}x)$.

**Lemma 2.1 ([14])** For any zero-dimensional group
1) $\int_{G_n^\perp} (\chi, x)\,d\nu(\chi) = 1_{G_0}(x)$, 2) $\int_{G_0} (\chi, x)\,d\mu(x) = 1_{G_0^\perp}(\chi)$.

**Lemma 2.2 ([14])** If $p_n = p$ for any $n \in \mathbb{Z}$ and the mapping $\mathcal{A}$ is additive then
1) $\int_{G_n^\perp} (\chi, x)\,d\nu(\chi) = p^n1_{G_n}(x)$,
2) $\int_{G_n} (\chi, x)\,d\mu(x) = \frac{1}{p^n}1_{G_n^\perp}(\chi)$.

**Lemma 2.3 ([14])** Let $\chi_{n,s} = r_{n}^{\alpha_{n}}r_{n+1}^{\alpha_{n+1}} \ldots r_{n+s}^{\alpha_{n+s}}$ be a character does not belong to $G_n^\perp$. Then
\[
\int_{G_n^\perp \chi_{n,s}} (\chi, x)\,d\nu(\chi) = p^n(\chi_{n,s}, x)1_{G_n}(x).
\]

**Lemma 2.4 ([14])** Let $h_{n,s} = a_{-1}g_{-1} + a_{-2}g_{-2} + \ldots + a_{-s}g_{-s} \notin G_n$. Then
\[
\int_{G_n^\perp h_{n,s}} (\chi, x)\,d\mu(x) = \frac{1}{p^n}(\chi, h_{n,s})1_{G_n^\perp}(\chi).
\]
Definition 2.1 ([14]) Let $M, N \in \mathbb{N}$. Denote by $\mathcal{D}_M(G_N)$ the set of step-functions $f \in L_2(G)$ such that 1) $\text{supp } f \subset G_N$, and 2) $f$ is constant on cosets $G_M + g$. Similarly is defined $\mathcal{D}_N(G_M^\perp)$.

Lemma 2.5 ([14]) Let $M, N \in \mathbb{N}$. $f \in \mathcal{D}_M(G_N)$ if and only if $\hat{f} \in \mathcal{D}_N(G_M^\perp)$.

3 MRA and refinable function on Vilenkin groups

In what follows we will consider groups $G$ for which $p_n = p$ and $pg_n = 0$ for any $n \in \mathbb{Z}$. We know that it is a Vilenkin group. We will denote a Vilenkin group as $\mathfrak{G}$.

In this group we can choose Rademacher functions in various ways. We define Rademacher functions by the equation

$$(r_n, \sum_{k \in \mathbb{Z}} a_k g_k) = \exp \left( \frac{2\pi i}{p} a_n \right).$$

In this case

$$(r_n, g_k) = \exp \left( \frac{2\pi i}{p} \delta_{nk} \right).$$

Our main objective is to find a simple algorithm to get a refinable step-function that generates an orthogonal MRA on Vilenkin group.

Definition 3.1 A family of closed subspaces $V_n, n \in \mathbb{Z}$, is said to be a multiresolution analysis of $L_2(\mathfrak{G})$ if the following axioms are satisfied:

A1) $V_n \subset V_{n+1}$;
A2) $\bigcup_{n \in \mathbb{Z}} V_n = L_2(\mathfrak{G})$ and $\bigcap_{n \in \mathbb{Z}} V_n = \{0\}$;
A3) $f(x) \in V_n \iff f(Ax) \in V_{n+1}$ ($A$ is a dilation operator);
A4) $f(x) \in V_0 \implies f(x-h) \in V_0$ for all $h \in H_0$; ($H_0$ is analog of $\mathbb{Z}$).
A5) there exists a function $\varphi \in L_2(\mathfrak{G})$ such that the system $(\varphi(x-h))_{h \in H_0}$ is an orthonormal basis for $V_0$.

A function $\varphi$ occurring in axiom A5 is called a scaling function.
Next we will follow the conventional approach. Let \( \varphi(x) \in L_2(\mathcal{G}) \), and suppose that \( (\varphi(x-h))_{h \in H_0} \) is an orthonormal system in \( L_2(\mathcal{G}) \). With the function \( \varphi \) and the dilation operator \( \mathcal{A} \), we define the linear subspaces \( L_n = (\varphi(\mathcal{A}^n x-h))_{h \in H_0} \) and closed subspaces \( V_n = \overline{L_n} \). It is evident that the functions \( p^h \varphi(\mathcal{A}^n x-h)_{h \in H_0} \) form an orthonormal basis for \( V_n, n \in \mathbb{Z} \). If subspaces \( V_n \) form a MRA, then the function \( \varphi \) is said to generate an MRA in \( L_2(\mathcal{G}) \). If a function \( \varphi \) generates an MRA, then we obtain from the axiom A1

\[
\varphi(x) = \sum_{h \in H_0} \beta_h \varphi(\mathcal{A}^n x-h) \left( \sum |\beta_h|^2 < +\infty \right). \tag{3.1}
\]

Therefore we will look up a function \( \varphi \in L_2(\mathcal{G}) \), which generates an MRA in \( L_2(\mathcal{G}) \), as a solution of the refinement equation (3.1). A solution of refinement equation (3.1) is called a refinable function.

**Lemma 3.1** ([14]) Let \( \varphi \in \mathcal{D}_M(\mathcal{G}-N) \) be a solution of (3.1). Then

\[
\varphi(x) = \sum_{h \in H_0^{(N+1)}} \beta_h \varphi(\mathcal{A}^n x-h) \tag{3.2}
\]

The refinement equation (3.2) may be written in the form

\[
\hat{\varphi}(\chi) = m_0(\chi) \hat{\varphi}(\chi \mathcal{A}^{-1}), \tag{3.3}
\]

where

\[
m_0(\chi) = \frac{1}{p} \sum_{h \in H_0^{(N+1)}} \beta_h (\chi \mathcal{A}^{-1}; h) \tag{3.4}
\]

is a mask of the equation (3.3).

**Lemma 3.2** ([14]) Let \( \varphi \in \mathcal{D}_M(\mathcal{G}-N) \). Then the mask \( m_0(\chi) \) is constant on cosets \( \mathcal{G}_{\perp}^\perp \mathcal{G} \). If \( \hat{\varphi}(\mathcal{G}_{\perp}^\perp) \neq 0 \) then \( m_0(\mathcal{G}_{\perp}^\perp) = 1 \).

**Lemma 3.3** ([14]) The mask \( m_0(\chi) \) is a periodic function with any period \( r_1^{\alpha_1} r_2^{\alpha_2} \ldots r_s^{\alpha_s} (s \in \mathbb{N}, \alpha_j = 0, p-1, j = 1, s) \).

So, if \( m_0(\chi) \) is a mask of (3.3) then

T1) \( m_0(\chi) \) is constant on cosets \( \mathcal{G}_{\perp}^\perp \mathcal{G} \),

T2) \( m_0(\chi) \) is periodic with any period \( r_1^{\alpha_1} r_2^{\alpha_2} \ldots r_s^{\alpha_s} \), \( \alpha_j = 0, p-1 \),

T3) \( m_0(\mathcal{G}_{\perp}^\perp) = 1 \).

Therefore we will assume that \( m_0 \) satisfies these conditions.
Theorem 3.1 ([14]) \( m_0(\chi) \) is a mask of equation (3.3) on the class \( \mathcal{D}_N(\mathfrak{G}^\perp_M) \) if and only if
\[
m_0(\chi)m_0(\chi A^{-1}) \ldots m_0(\chi A^{-M-N}) = 0 \tag{3.5}
\]
on \( \mathfrak{G}^\perp_{M+1} \setminus \mathfrak{G}^\perp_M \). If, in addition, the system \( \varphi(x-h)_{h \in H_0} \) is orthonormal, then \( \varphi(x) \) generate an orthogonal MRA.

So, to find a refinable function that generates orthogonal MRA, we need take a function \( m_0(\chi) \) that satisfies conditions T1, T2, T3, (3.5), construct the function
\[
\hat{\varphi}(\chi) = \prod_{k=0}^{\infty} m_0(\chi A^{-k}) \in \mathcal{D}_N(\mathfrak{G}^\perp_M)
\]
and check that the system \( \varphi(x-h)_{h \in H_0} \) is orthonormal.

For any zero-dimensional group \( G \) the shifts system \( (\varphi(x-h))_{h \in H_0} \) is orthonormal if the condition \( |\hat{\varphi}(\chi)| = 1 \) \( G^\perp_0(\chi) \) is valid [14]. For Vilenkin group \( G \) we can give another condition.

Definition 3.2 Let \( N, M \in \mathbb{N} \). A set \( E \subset X \) is called \((N, M)\)-elementary if \( E \) is disjoint union of \( p^N \) cosets
\[
\mathfrak{G}^\perp_{-N} \xi_j = \mathfrak{G}^\perp_{-N} r_{-N}^{\alpha-N} r_{-N+1}^{\alpha-N+1} \ldots r_0^{\alpha-1} r_{-1}^{\alpha-1} \ldots r_{M-1}^{\alpha-M-1} \eta_j = \mathfrak{G}^\perp_{-N} \xi_j \eta_j,
\]
\( j = 0, 1, \ldots, p^N - 1, j = \alpha_{-N} + \alpha_{-N+1} p + \cdots + \alpha_{-1} p^{N-1} (\alpha_{-\nu} = \overline{0, p - 1}) \) such that
1) \( \bigcup_{j=0}^{p^N-1} \mathfrak{G}^\perp_{-N} \xi_j = \mathfrak{G}^\perp_0, \mathfrak{G}^\perp_{-N} \xi_0 = \mathfrak{G}^\perp_{-N} \),
2) for any \( l = 0, M + N - 1 \) the intersection \( (\mathfrak{G}^\perp_{-N+l+1} \setminus \mathfrak{G}^\perp_{-N+l}) \cap E \neq \emptyset \).

Lemma 3.4 The set \( H_0 \subset \mathfrak{G} \) is an orthonormal system on any \((N, M)\)-elementary set \( E \subset X \).

Proof. Using the definition of \((N, M)\)-elementary set we have
\[
\int_E (\chi, h)(\chi, g) \, d\nu(x) = \sum_{j=0}^{p^N-1} \int_{\xi_j} (\chi, h)(\chi, g) \, d\nu(x) = \sum_{j=0}^{p^N-1} \int_X 1_{\mathfrak{G}^\perp_{-N} \xi_j} (\chi, h)(\chi, g) \, d\nu(x) =
\]
\[
\begin{align*}
\sum_{j=0}^{p^N-1} \int_X 1_{\Theta_{j}^{\perp \xi_j}}(\chi \eta_j)(\chi \eta_j, h)(\chi \eta_j, g) \, d\nu(x) &= \\
= \sum_{j=0}^{p^N-1} \int_X 1_{\Theta_{j}^{\perp \xi_j}}(\chi)(\chi, h)(\chi, g)(\eta_j, h)(\eta_j, g) \, d\nu(x).
\end{align*}
\]

Since
\[
(\eta_j, h) = (r_0^{a_0} r_1^{a_1} \ldots r_{M-1}^{a_{M-1}}, a_1 g_1 + a_2 g_2 + \ldots + a_g g_g) = 1,
\]
\[
(\eta_j, g) = (r_0^{a_0} r_1^{a_1} \ldots r_{M-1}^{a_{M-1}}, b_1 g_1 + b_2 g_2 + \ldots + b_g g_g) = 1,
\]
then
\[
\int_E (\chi, h)(\chi, g) \, d\nu(x) = \sum_{j=0}^{p^N-1} \int_{\Theta_{j}^{\perp \xi_j}} (\chi, h)(\chi, g) \, d\nu(x) = \int_{\Theta_{j}^{\perp \xi_j}} (\chi, h)(\chi, g) \, d\nu(x) = \delta_{h, g}.
\]

**Theorem 3.2** Let \((\Theta, \dagger)\) be an \(p\)-adic Vilenkin group, \(E \subset \Theta_{M}^{\perp}\) an \((N, M)\)-elementary set. If \(|\hat{\varphi}(\chi)| = 1_E(\chi)\) on \(X\) then the system of shifts \((\varphi(x - h))_{h \in H_0}\) is an orthonormal system on \(\Theta\).

**Proof.** Let \(\tilde{H}_0 \subset H_0\) be an finite set. Using the Plancherel equation we have
\[
\int_{\Theta} \varphi(x - g) \overline{\varphi(x - g)} \, d\mu(x) = \int_X |\hat{\varphi}(\chi)|^2(\chi, g)(\chi, h) \, d\nu(\chi) = \int_X (\chi, h)(\chi, g) \, d\nu(\chi) = \sum_{j=0}^{p^N-1} \int_{\Theta_{j}^{\perp \xi_j}} (\chi, h)(\chi, g) \, d\nu(\chi).
\]

Transform the inner integral
\[
\int_{\Theta_{j}^{\perp \xi_j}} (\chi, h)(\chi, g) \, d\nu(\chi) = \int_X 1_{\Theta_{j}^{\perp \xi_j}}(\chi)(\chi, h)(\chi, g) \, d\nu(\chi) = \\
= \int_X 1_{\Theta_{j}^{\perp \xi_j}}(\chi \eta_j)(\chi \eta_j, h - g) \, d\nu(\chi) = \int_X 1_{\Theta_{j}^{\perp \xi_j}}(\chi \eta_j, h - g) \, d\nu(\chi) = \delta_{h, g}.
\]

□
\[ = \int_{\mathcal{G}^\perp \xi} (\chi \eta_j, h^{-g}) d\nu(\chi). \]

Repeating the arguments of lemma 3.4 we obtain
\[ \int_{\mathcal{G}} \varphi(x\cdot h) \overline{\varphi(x\cdot g)} d\mu(x) = \delta_{h,g}. \]

\textbf{Theorem 3.3 ([14])} Let \( \varphi(x) \in \mathcal{D}_M(\mathcal{G}_N) \). A shifts system \( (\varphi(x\cdot h))_{h \in H_0} \) will be orthonormal if and only if for any \( \alpha_{-N}, \alpha_{-N+1}, \ldots, \alpha_{-1} = (0, p - 1) \)
\[ \sum_{\alpha_0, \alpha_1, \ldots, \alpha_{M-1} = 0}^{p-1} |\hat{\varphi}(\mathcal{G}_N^{-N} r_{-N}^{\alpha_{-N}} \cdots r_0^{\alpha_0} \cdots r_{M-1}^{\alpha_{M-1}})|^2 = 1. \quad (3.6) \]

\textbf{Lemma 3.5 ([14])} Let \( \hat{\varphi} \in \mathcal{D}_N(\mathcal{G}_M) \) be a solution of the refinement equation
\[ \hat{\varphi}(\chi) = m_0(\chi) \hat{\varphi}(\chi \mathcal{A}^{-1}) \]
and \( (\varphi(x\cdot h))_{h \in H_0} \) be an orthonormal system.
Then for any \( \alpha_{-N}, \alpha_{-N+1}, \ldots, \alpha_{-1} = 0, p - 1 \)
\[ \sum_{\alpha_0 = 0}^{p-1} |m_0(\mathcal{G}_N^{-N} r_{-N}^{\alpha_{-N}} r_{-N+1}^{\alpha_{-N+1}} \cdots r_{-1}^{\alpha_{-1}} t_0^{\alpha_0})|^2 = 1. \quad (3.7) \]

\section{4 Trees and refinable functions}

In this section we reduce the problem of construction of step refinable function to construction of some tree.

We will consider some special class of refinable functions \( \varphi(\chi) \) for which \( |\hat{\varphi}(\chi)| \) is a characteristic function of a set. Define this class.

\textbf{Definition 4.1} A mask \( m_0(\chi) \) is called \( N \)-elementary \((N \in \mathbb{N}_0)\) if \( m_0(\chi) \) is constant on cosets \( \mathcal{G}_N^{\perp} \chi \), its modulus \( m_0(\chi) \) has two values only: 0 and 1, and \( m_0(\mathcal{G}_N^{\perp}) = 1 \). The refinable function \( \varphi(x) \) with Fourier transform
\[ \hat{\varphi}(\chi) = \prod_{n=0}^{\infty} m_0(\chi \mathcal{A}^{-n}) \]
is called \( N \)-elementary too. \( N \)-elementary function \( \varphi \) is called \((N,M)\)-elementary if \( \hat{\varphi}(\chi) \in \mathcal{D}_N(\mathcal{G}_M) \). In this case we will call the Fourier transform \( \hat{\varphi}(\chi) \) \((N,M)\)-elementary, also.
Definition 4.2 Let $\tilde{E} = \bigcup_{\alpha_1, \alpha_0} \mathcal{G}^{\perp} \alpha_1 r_1^{\alpha_1} \alpha_0 r_0^{\alpha_0} \subset \mathcal{G}^{\perp}$ be an $(1, 1)$-elementary set. We say that the set $\tilde{E}_X$ is a periodic extension of $\tilde{E}$ if

$$\tilde{E}_X = \bigcup_{s=1}^{\infty} \bigcup_{\alpha_1, \ldots, \alpha_s=0}^{p-1} \tilde{E}\alpha_1^{\alpha_1} \alpha_2^{\alpha_2} \ldots \alpha_s^{\alpha_s}.$$ 

We say that the set $\tilde{E}$ generates an $(1, M)$ elementary set $E$, if $\bigcap_{n=0}^{\infty} \tilde{E}_X A^n = E$.

Since $\tilde{E}_X \supset \mathcal{G}^{\perp}_N$ then $\bigcap_{n=0}^{M+1} \tilde{E}_X A^n = E$ and $\left( \bigcap_{n=0}^{M+1} \tilde{E}_X A^n \right) \cap (\mathcal{G}^{\perp}_{M+1} \setminus \mathcal{G}^{\perp}_M) = \emptyset$. The converse is also true. Since

$$\left( \bigcap_{n=0}^{M+1} \tilde{E}_X A^n \right) \cap (\mathcal{G}^{\perp}_{M+1} \setminus \mathcal{G}^{\perp}_M) = \emptyset.$$

Then we have

$$\left( \bigcap_{n=0}^{M+2} \tilde{E}_X A^n \right) \cap (\mathcal{G}^{\perp}_{M+2} \setminus \mathcal{G}^{\perp}_{M+1}) = \tilde{E}_X \cap \left( \bigcap_{n=0}^{M+1} \tilde{E}_X A^n \cap (\mathcal{G}^{\perp}_{M+1} \setminus \mathcal{G}^{\perp}_M) \right) A =$$

$$= \tilde{E}_X \cap \emptyset = \emptyset.$$

Let us write the set $\{0, 1, \ldots, p-1\}$ in the form

$$\{0, u_1, u_2, \ldots, u_q, \alpha_1, \alpha_2, \ldots, \alpha_{p-q-1}\} = V, \quad 0 = u_0,$$

where $1 \leq q \leq p - 1$. We will consider the set $V$ as a set of vertices. By $T(0, u_1, u_2, \ldots, u_q, \alpha_1, \alpha_2, \ldots, \alpha_{p-q-1}) = T(V)$ we will denote a rooted tree on the set of vertices $V$, where 0 is a root, $u_1, u_2, \ldots, u_q$ are first level vertices, $\alpha_1, \alpha_2, \ldots, \alpha_{p-q-1}$ are remaining vertices.

For example for $p = 7, q = 2, u_1 = 3, u_2 = 5$ we have trees

```
\begin{figure}
\begin{center}
\begin{tikzpicture}
\node (0) at (0,0) {0};
\node (1) at (1,1) {1};
\node (2) at (2,1) {2};
\node (3) at (1,0) {3};
\node (4) at (2,0) {4};
\node (5) at (1,-1) {5};
\node (6) at (2,-1) {6};
\draw (0) -- (1);
\draw (0) -- (2);
\draw (1) -- (3);
\draw (2) -- (4);
\draw (3) -- (5);
\draw (4) -- (6);
\end{tikzpicture}
\end{center}
\end{figure}
```

or

```
\begin{figure}
\begin{center}
\begin{tikzpicture}
\node (0) at (0,0) {0};
\node (1) at (1,1) {1};
\node (2) at (2,1) {2};
\node (3) at (1,0) {3};
\node (4) at (2,0) {4};
\node (5) at (1,-1) {5};
\node (6) at (2,-1) {6};
\draw (0) -- (1);
\draw (0) -- (2);
\draw (1) -- (3);
\draw (2) -- (4);
\draw (3) -- (5);
\draw (4) -- (6);
\end{tikzpicture}
\end{center}
\end{figure}
```

and so on.
For any tree path \( P_j = (0, u_j, \alpha_{s-1}, \alpha_{s-2}, \ldots, \alpha_0, \alpha_{-1}) \) we construct the set of cosets

\[
G_{-1}^{\perp} r_{-1}^{u_j}, G_{-1}^{\perp} r_{-1}^{s-1} r_{0}^{u_j}, G_{-1}^{\perp} r_{-1}^{s-2} r_{s-1}^{u_j}, \ldots, G_{-1}^{\perp} r_{-1}^{s} r_{0}^{s}, G_{-1}^{\perp} r_{-1}^{0}. \tag{4.1}
\]

For example for the tree on Figure 2 and the path \((0, 3, 2, 6)\) we have 3 cosets

\[
G_{-1}^{\perp} r_{-1}^3, G_{-1}^{\perp} r_{-1}^2 r_{0}^3, G_{-1}^{\perp} r_{-1}^6 r_{0}^2,
\]

for the path \((0, 3, 1)\) we have two cosets

\[
G_{-1}^{\perp} r_{-1}^3, G_{-1}^{\perp} r_{-1}^1 r_{0}^3.
\]

We will represent the tree \( T(V) \) as the tree \( \overbrace{0 \rightarrow T_1 \cdots T_q}^{T} \) where \( T_j \) are tree branches of \( T(V) \) with \( u_j \) as a root. By \( E_j \) denote a union of all cosets (4.1) for fixed \( j \) and set

\[
\tilde{E} = \bigcup_{j=1}^{q} E_j \bigcup G_{-1}^{\perp}. \tag{4.2}
\]

It is clear that \( \tilde{E} \) is an \((1, 1)\) elementary set and \( \tilde{E} \subset G_{1}^{\perp} \).

**Definition 4.3** Let \( \tilde{E}_X \) be a periodic extension of \( \tilde{E} \). We say that the tree \( T(V) \) generates a set \( E \), if \( E = \bigcap_{n=0}^{\infty} \tilde{E}_X A^n \).

**Lemma 4.1** Let \( T(V) \) be a rooted tree with \( 0 \) as a root. Let \( E \subset X \) be a set generated by the tree \( T(V) \), \( H \) a height of \( T(V) \). Then \( E \) is an \((1, H - 2)\)-elementary set.

**Proof.** Let us denote

\[
m(\chi) = 1_{\tilde{E}_X}(\chi), \quad M(\chi) = \prod_{n=0}^{\infty} m(\chi A^{-n}).
\]

First we note that \( M(\chi) = 1_E(\chi) \).

Indeed

\[
1_E(\chi) = 1 \iff \chi \in E \iff \forall n, \chi A^{-n} \in \tilde{E}_X \iff \forall n, 1_{\tilde{E}_X}(\chi A^{-n}) = 1 \iff \forall n, m(\chi A^{-n}) = 1 \iff \prod_{n=0}^{\infty} m(\chi A^{-n}) = 1 \iff M(\chi) = 1.
\]
It means that \( M(\chi) = 1_E(\chi) \).

Now we will prove, that \( 1_E(\chi) = 0 \) for \( \chi \in \mathfrak{G}_{H-2}^+ \setminus \mathfrak{G}_{H-2}^+ \). Since \( \tilde{E}_X \supset \mathfrak{G}_{-1}^+ \) it follows that \( 1_{\tilde{E}_X}(\mathfrak{G}_{H-1}^+ A^{-H}) = 1_{\tilde{E}_X}(\mathfrak{G}_{-1}^+) = 1 \). Consequently
\[
\prod_{n=0}^{\infty} 1_{\tilde{E}_X}(\chi A^n) = \prod_{n=0}^{H-1} 1_{\tilde{E}_X}(\chi A^n)
\]
if \( \chi \in \mathfrak{G}_{H-1}^+ \setminus \mathfrak{G}_{H-2}^+ \). Let us denote \( m(\mathfrak{G}_{-1}^+ r_{-1}^i r_{-1}^j) = \lambda_{i + kp} \). By the definition of cosets (4.1) \( m(\mathfrak{G}_{-1}^+ r_{-1}^i r_{-1}^j) \neq 0 \) \( \Leftrightarrow \) the pair \((k, i)\) is an edge of the tree \( T(V) \).

We need prove that
\[
1_E(\mathfrak{G}_{-1}^+ r_{-1}^i r_{-1}^j r_{-1}^k) = 0
\]
for \( \alpha_{H-2} \neq 0 \). Since \( \tilde{E}_X \) is a periodic extension of \( \tilde{E} \) it follows that the function \( m(\chi) = 1_{\tilde{E}_X}(\chi) \) is periodic with any period \( r_{-1}^{\alpha_1} r_{-1}^{\alpha_2} \cdots r_{-1}^{\alpha_s} \), \( s \in \mathbb{N} \), i.e. \( m(\chi r_{-1}^{\alpha_1} r_{-1}^{\alpha_2} \cdots r_{-1}^{\alpha_s}) = m(\chi) \) when \( \chi \in \mathfrak{G}_{H-1}^+ \). Using this fact we can write \( M(\chi) \) for \( \chi \in \mathfrak{G}_{H-1}^+ \setminus \mathfrak{G}_{H-2}^+ \) in the form
\[
M(\mathfrak{G}_{-1}^+ \zeta) = M(\mathfrak{G}_{-1}^+ r_{-1}^{\alpha_{-1}} r_{-1}^{\alpha_0} \cdots r_{-1}^{\alpha_{H-2}}) = m(\mathfrak{G}_{-1}^+ r_{-1}^{\alpha_{-1}} r_{-1}^{\alpha_0}) \cdots m(\mathfrak{G}_{-1}^+ r_{-1}^{\alpha_{H-3}} r_{-1}^{\alpha_{H-2}}) m(\mathfrak{G}_{-1}^+ r_{-1}^{\alpha_{H-2}}) = \lambda_{\alpha_{-1} + p\alpha_0} \lambda_{\alpha_0 + p\alpha_1} \cdots \lambda_{\alpha_{H-3} + p\alpha_{H-2}} \lambda_{\alpha_{H-2}}, \alpha_{H-2} \neq 0.
\]
If \( \lambda_{\alpha_{H-2}} = 0 \) then \( M(\mathfrak{G}_{-1}^+ \zeta) \neq 0 \). Let \( \lambda_{\alpha_{H-2}} \neq 0 \). It means that \( \alpha_{H-2} = u_j \) for some \( j = 1, q \). If \( \lambda_{\alpha_{H-3} + p\alpha_{H-2}} = 0 \) then \( M(\mathfrak{G}_{-1}^+ \zeta) \neq 0 \). Therefore we assume that \( \lambda_{\alpha_{H-3} + p\alpha_{H-2}} \neq 0 \). It is true if \( (\alpha_{H-2}, \alpha_{H-3}) \) is an edge of \( T(V) \). Repeating these arguments, we obtain a path \((0, u_j = \alpha_{s-1}, \alpha_{s-2}, \ldots, \alpha_0, \alpha_{-1})\) defines the coset \( \mathfrak{G}_{-1}^+ r_{-1}^{\alpha_{-1}} r_{-1}^{\alpha_0} \cdots r_{-1}^{\alpha_{s-1}} \subset E \). But for any \( \alpha_{-1} = 0, p - 1 \) there exists unique path with endpoint \( \alpha_{-1} \) and starting point zero. It means that \( E \) is \((1, H - 2)\)-elementary set. \( \square \)

**Theorem 4.1** Let \( M, p \in \mathbb{N}, p \geq 3 \). Let \( E \subset \mathfrak{G}_{M}^+ \) be an \((1, M)\)-elementary set, \( \hat{\varphi} \in \mathfrak{D}_{-1}(\mathfrak{G}_{M}^+), |\hat{\varphi}(\chi)| = 1_E(\chi), \hat{\varphi}(\chi) \) the solution of the equation
\[
\hat{\varphi}(\chi) = m_0(\chi)\hat{\varphi}(\chi A^{-1}), \quad (4.3)
\]
where \( m_0(\chi) \) is a \( 1\)-elementary mask. Then there exists a rooted tree \( T(V) \) with \( \text{height}(T) = M + 2 \) that generates the set \( E \).
**Prof.** Since the set \( E \) is \((1, M)\)-elementary set and \(|\hat{\varphi}(\chi)| = 1_E(\chi)\), it follows from theorem 3.2 that the system \((\varphi(x \cdot h))_{h \in H_0}\) is an orthonormal system in \(L_2(\mathcal{G})\). Using the theorem 3.3 we obtain that for \(\alpha_1 = 0, p - 1\)

\[
\sum_{\alpha_0, \alpha_1, \ldots, \alpha_{M-1}} |\hat{\varphi}(G_{-1}^{\alpha_1}r_{\alpha_0}^{-1}r_0^{\alpha_0} \ldots r_{M-1}^{\alpha_{M-1}})|^2 = 1.
\]

Since \(\hat{\varphi}\) is a solution of refinement equation \((4.3)\) it follows from lemma 3.5 that for \(\alpha_1 = 0, p - 1\)

\[
\sum_{\alpha_0=0}^{p-1} |m_0(\mathcal{G}_{-1}^{\alpha_1}r_0^{-1}r_0^{\alpha_0})|^2 = 1. \tag{4.4}
\]

Let as denote \(\lambda_{\alpha-1+p_{\alpha_0}} := m_0(\mathcal{G}_{-1}^{\alpha_1}r_0^{-1}r_0^{\alpha_0})\). Then we write \((4.4)\) in the form

\[
\sum_{\alpha_0=0}^{p-1} |\lambda_{\alpha_1+p_{\alpha_0}}|^2 = 1. \tag{4.5}
\]

Since the mask \(m_0(\chi)\) is \(1\)-elementary it follows that \(|\lambda_{i+p_j}|\) take two value only: 0 or 1.

Now we will construct the tree \(T\). Let \(\mathfrak{U}\) be a family of cosets \(\mathcal{G}_{-1}^{\alpha_1} \subset \mathcal{G}_{-1}^{\alpha_1}\) such that \(\hat{\varphi}(\mathcal{G}_{-1}^{\alpha_1}) \neq 0\) and \(\mathcal{G}_{-1}^{\alpha_1} \notin \mathfrak{U}\). We can write a coset \(\mathcal{G}_{-1}^{\alpha_1} \in \mathfrak{U}\) in the form

\[
\mathcal{G}_{-1}^{\alpha_1} = \mathcal{G}_n^{\alpha_1} r_0^{\alpha_0} \ldots r_{M-1}^{\alpha_{M-1}}.
\]

If \(\mathcal{G}_{-1}^{\alpha_1} \in \mathcal{G}_n^{\alpha_1} \setminus \mathcal{G}_{n-1}^{\alpha_1} (n \leq M)\) then \(\mathcal{G}_{-1}^{\alpha_1} = \mathcal{G}_n^{\alpha_1} r_0^{\alpha_0} \ldots r_{n-1}^{\alpha_{n-1}}, \alpha_{n-1} = 1, p - 1\).

Let \(u \in \mathfrak{U}, p - 1\). By \(T_u\) we denote the set of vectors \((u, \alpha_{n-1}, \ldots, \alpha_0, \alpha_{-1})\) for which \(\mathcal{G}_n^{\alpha_1} r_0^{\alpha_0} \ldots r_{n-1}^{\alpha_{n-1}} r_u^{\alpha_{n-1}} \in \mathfrak{U}\). We will name the vector \((u, \alpha_{n-1}, \ldots, \alpha_0, \alpha_{-1})\) as a path too. So \(T_u\) is the set of paths with starting point \(u\), for which \(\hat{\varphi}(\mathcal{G}_{-1}^{\alpha_1} r_0^{\alpha_0} \ldots r_{n-1}^{\alpha_{n-1}} r_u^{\alpha_{n-1}}) \neq 0\). We will show that \(T_u\) is a rooted tree with \(u\) as a root.

1) All vertices \(\alpha_j, u\) of the path \((u, \alpha_{n-1}, \ldots, \alpha_0, \alpha_{-1})\) are pairwise distinct. Indeed

\[
\hat{\varphi}(\mathcal{G}_{-1}^{\alpha_1} r_0^{\alpha_0} \ldots r_{n-1}^{\alpha_{n-1}} r_u^{\alpha_{n-1}}) = \lambda_{\alpha_{-1}+p_{\alpha_0}} \lambda_{\alpha_0+p_u} \ldots \lambda_{\alpha_{n-1}+p_{\alpha_0}} \lambda_u \neq 0, u \neq 0.
\]

If \(\alpha_{n-1} = u\) then \(|\lambda_{u+p_{\alpha_0}}| = |\lambda_{u+p_0}| = 1\) that contradicts the equation \((4.5)\). If \(\alpha_{n-1} = 0\) then \(|\lambda_{0+p_{\alpha_0}}| = |\lambda_{0+p_0}| = 1\) that contradicts the equation \((4.5)\) too. Consequently \(\alpha_{n-1} \notin \{0, u\}\). By analogy we obtain that \(\alpha_i \notin \{0, u, \alpha_{n-1}, \ldots, \alpha_{i+2}, \alpha_{i+1}\}\).
2) If two patches \((u, \alpha_{n-1}, \ldots, \alpha_0, \alpha_{-1})\) and \((u, \beta_{l-1}, \ldots, \beta_0, \beta_{-1})\) have the common subpath \((u, \alpha_{k-1}, \ldots, \alpha_{k-j+1}, \alpha_{k-j}) = (u, \beta_{l-1}, \ldots, \beta_{l-j+1}, \beta_{l-j})\) and \(\alpha_{k-j-1} \neq \beta_{l-j-1}\) then \(\{\alpha_{-1}, \alpha_0, \ldots, \alpha_{k-j-1}\} \cap \{\beta_{-1}, \beta_0, \ldots, \beta_{l-j-1}\} = \emptyset\).

Indeed, assume

\[
\{\alpha_{-1}, \alpha_0, \ldots, \alpha_{k-j-1}\} \cap \{\beta_{-1}, \beta_0, \ldots, \beta_{l-j-1}\} \neq \emptyset.
\]

Then there exists \(v \in \{\alpha_{-1}, \alpha_0, \ldots, \alpha_{k-j-1}\} \cap \{\beta_{-1}, \beta_0, \ldots, \beta_{l-j-1}\}\).

Assume that \(v \neq \alpha_{k-j-1}\). Then \(v = \alpha_{\nu}, -1 \leq \nu \leq k - j - 2\) and \(v = \beta_{\mu}, -1 \leq \mu \leq l - j - 1\). It follows that

\[
(u = \alpha_k, \ldots, \alpha_{k-j}, \alpha_{k-j-1}, \ldots, \alpha_{\nu+1}, \alpha_{\nu} = \beta_{\mu}, \beta_{\mu-1}, \ldots, \beta_0, \beta_{-1}) \in T_u
\]

\[
(u = \beta_l, \ldots, \beta_{l-j} = \alpha_{k-j}, \beta_{l-j-1}, \ldots, \beta_{\mu+1}, \beta_{\mu}, \beta_{\mu-1}, \ldots, \beta_0, \beta_{-1}) \in T_u.
\]

So we have two different patches with the same sheet \(\beta_{-1}\). But this contradicts theorem 3.3. This means that \(T_u\) has no cycles, consequently \(T_u\) is a graph with \(u\) as a root.

3) By analogy we can proof that different trees \(T_u\) an \(T_v\) has no common vertices. It follows that the graph \(T = (0, T_{u_1}, \ldots, T_{u_q})\) is a tree with 0 as a foot.

4) It is evident that this tree generates refinable function \(\hat{\phi}\) with a mask \(m_0\). Show that \(\text{height}(T) = M + 2\). Indeed, since \(\hat{\phi} \in \mathcal{D}_{-1}(\mathcal{G}_M)\) it follows that there exists a coset \(\mathcal{G}_M^{-1}r_{-1}^{α_{-1}}r_0^{α_0} \ldots r_{M-1}^{α_{M-1}}\), \(α_{M-1} \neq 0\) for which \(|\hat{\phi}(\mathcal{G}_M^{-1}r_{-1}^{α_{-1}}r_0^{α_0} \ldots r_{M-1}^{α_{M-1}})| = 1\). This coset generates a path \((0, α_{M-1} = u, α_{M-2}, \ldots, α_0, α_{-1})\) of \(T\). This path contain \(M + 2\) vertex. It means that \(\text{height}(T) \geq M + 2\). On the other hand there isn’t coset \(\mathcal{G}_M^{-1} \subset \mathcal{G}_{M+1} \setminus \mathcal{G}_M\), consequently there isn’t path with \(L > M + 2\). So \(\text{height}(T) = M + 2\). Since \(\text{supp} \hat{\phi}(\chi)\) is \((1, M)\)-elementary set, it follows that the set of all vertices of the tree \(T\) is the set \(\{0, 1, \ldots, p - 1\}\). The theorem is proved.□

**Definition 4.4** Let \(T(V)\) be a rooted tree with 0 as a root, \(H\) a height of \(T(V)\), \(V = \{0, 1, \ldots, p - 1\}\). Using cosets (1.1) we define the mask \(m_0(\chi)\) in the subgroup \(\mathcal{G}_1^\perp\) as follows: \(m_0(\mathcal{G}_1^\perp) = 1, m_0(\mathcal{G}_1^\perp r_i^{-1}r_0^j) = \lambda_{i+pj}, |\lambda_{i+pj}| = 1\) when \(\mathcal{G}_1^\perp r_i^{-1}r_0^j \subset \tilde{E}\), (1.2), \(|\lambda_{i+pj}| = 1\) when \(\mathcal{G}_1^\perp r_i^{-1}r_0^j \subset \tilde{G}_1^\perp \setminus \tilde{E}\). Let us extend the mask \(m_0(\chi)\) on the \(X \setminus \mathcal{G}_1^\perp\) periodically, i.e \(m_0(\chi r_1^{α_1}r_2^{α_2} \ldots r_s^{α_s}) = m_0(\chi)\). Then we say that the tree \(T(V)\) generates the
mask \( m_0(\chi) \). Set \( \hat{\varphi}(\chi) = \prod_{n=0}^{\infty} m_0(\chi A^{-n}) \). It follows from lemma 4.1 that

1) \( \text{supp} \hat{\varphi}(\chi) \subset \mathcal{S}_{H-2}^\perp \),
2) \( \hat{\varphi}(\chi) \) is \((1, H - 2)\) elementary function,
3) \( (\varphi(x-h))_{h \in H_0} \) is an orthonormal system.

In this case we say that the tree \( T(V) \) generates the refinable function \( \varphi(x) \).

**Theorem 4.2** Let \( p \geq 3 \) be a prime number,

\[
V = \{0, u_1, u_2, \ldots, u_q, a_1, a_2, \ldots, a_{p-q-1}\}
\]
a set of vertices, \( T(V) \) a rooted tree, \( 0 \) the root, \( u_1, u_2, \ldots, u_q \) a first level vertices. Let \( H \) be are height of \( T(V) \). By \( \varphi(x) \) denote the function generated by the \( T(V) \). Then \( \varphi(x) \) generate an orthogonal MRA on \( p \)-adic Vilenkin group.

**Proof.** Since \( T(V) \) generates the the function \( \varphi \) then 1) \( \hat{\varphi} \in \mathcal{D}_{-1}(\mathcal{G}_1^\perp) \),
2) \( \hat{\varphi}(\chi) \) is \((1, H - 2)\) elementary function, 3) \( \hat{\varphi}(\chi) \) is a solution of refinable equation (3.3), 4) \( (\varphi(x-h))_{h \in H_0} \) is an orthonormal system. From the theorem 3.1 it follows that \( \varphi(x) \) generates an orthogonal MRA. \( \square \)

**Remark.** It is possible to give an algorithm for constructing the refinable function \( \varphi(x) \). Let \( T(V) \) be a tree on the set \( \{0, 1, \ldots, p - 1\} \).

Construct a finite sequence \( (\lambda_i+jp)_{i,j=0}^{p-1} \) as follows: \( \lambda_0 = 1, |\lambda_i+jp| = 1 \) if the pair \( (j, i) \) is an edge of \( T(f) \). For any vertex \( \alpha_{-1} \) we take the path \( (0 = \alpha_{s+1}, u_j = \alpha_s, \alpha_{s-1}, \ldots, \alpha_0, \alpha_{-1}) \) and suppose

\[
\hat{\varphi}(\mathcal{S}_{-1}^\perp r_1^{\alpha_{-1}} r_0^{\alpha_0} \ldots r_{s-1}^{\alpha_s} r_s^{\alpha_s} r_{s+1}^{\alpha_{s+1}}) = \lambda_{\alpha_{-1}+\alpha_0p} \cdot \lambda_{\alpha_0+\alpha_1p} \cdot \ldots \cdot \lambda_{\alpha_{s-1}+\alpha_sp} \cdot \lambda_{\alpha_s}.
\]

Otherwise we suppose \( \hat{\varphi}(\mathcal{G}_1^\perp \zeta) = 0 \). Then \( \varphi \) generates an orthogonal MRA on \( p \)-adic Vilenkin group \( \mathcal{G} \).

5 **Construction of wavelet bases**

In [6] and [7] Yu.A.Farkov reduces the problem of \( p \)-wavelet decomposition into a problem of matrix extension. We will use more simple method [13].

As usual, \( W_n \) stands for the orthogonal complement of \( V_n \) in \( V_{n+1} \); that is \( V_{n+1} = V_n \oplus W_n \) and \( V_n \perp W_n \) \( (n \in \mathbb{Z}, \text{and } \oplus \text{ denotes the direct sum}) \).

It is readily seen that
1) \( f \in W_n \iff f(Ax) \in W_{n+1} \),
2) \( W_n \perp W_k \) for \( k \neq n \),
3) \( \oplus W_n = L_2(\mathcal{G}), n \in \mathbb{Z} \).
From theorems 4.1, 4.2 we derive an algorithm for constructing wavelet bases.

**Step 1.** Choose an arbitrary tree $T(V) = T(0, u_1, \ldots, u_q, \alpha_1, \ldots, \alpha_{p-q-1})$ on the set $V = \{0, 1, \ldots, p-1\}$. Let $H$ be a height of the tree $T(V)$.

**Step 2.** Choose a finite sequence $(\lambda_{i+jp})_{i,j=0}^{p-1}$ such that $\lambda_0 = 1$, $|\lambda_{i+jp}| = 1$ if the pair $(j, i)$ is the edge of the tree $T(V)$, $|\lambda_{i+jp}| = 0$ otherwise.

**Step 3.** Construct the mask $m_0(\chi)$ and Fourier transform $\hat{\varphi}(\chi)$ using definition 4.4. It is clear that $E = \text{supp}(\hat{\varphi}(\chi))$ is $(1, H - 2)$-elementary set.

**Step 4.** Find coefficients $\beta_n$ for which

$$m_0(\chi) = \frac{1}{p} \sum_{h \in H_0^{(2)}} \beta_h(\chi A^{-1}h). \quad (5.1)$$

To find coefficients $\beta_h$, we write this equation in the form

$$m_0(\chi_k) = \frac{1}{p} \sum_{j=0}^{p^2-1} \beta_j(\chi_k, A^{-1}h_j) \quad (5.2)$$

where

$h_j = a_1 g_1 + a_2 g_2, \quad j = a_1 + a_2 p, \quad a_1, a_2 = \overline{0, p-1}$.

$\chi_k \in \mathcal{G}^{\perp}_{-1} r_{-1}^{a_1} r_0^{a_0}, \quad k = a_1 + a_0 p, \quad a_1, a_0 = \overline{0, p-1}$.

Since the matrix $\frac{1}{p}(\chi_k, A^{-1}h_j)$ of this system is unitary it follows that the system (5.2) has a unique solution.

**Step 5.** We set $m_l(\chi) = m_0(\chi r_0^{-l})$, $l = \overline{1, p-1}$, $X_0 = \{\chi : |m_0(\chi)| = 1\}$. Clearly, $m_l(\chi)$ may be written as

$$m_l(\chi) = \frac{1}{p} \sum_{h \in H_0^{(2)}} \beta_h(\chi r_0^{-l}, A^{-1}h) = \frac{1}{p} \sum_{h \in H_0^{(2)}} \beta_h^{(l)}(\chi, A^{-1}h)$$

where $\beta_h^{(l)} = \beta_h(r_0^l, A^{-1}h)$. By the construction of $\hat{\varphi}(\chi)$ we have $|m_l(X_0 r_0^l)| = 1$, $|m_l(X_0 r_0^\nu)| = 0$ for $\nu \neq l$, $m_l(\chi)m_k(\chi) = 0$ when $k \neq l$.

**Step 6.** Define the functions

$$\psi_l(x) = \sum_{h \in H_0^{(2)}} \beta_h^{(l)}(\varphi(A \cdot x^l \cdot h))$$

**Theorem 5.1** The functions $\psi_l(x^l \cdot h)$, where $l = \overline{1, p-1}$, $h \in H_0$, form an orthonormal basis for $W_0$. 
**Proof.** a) We claim that \((\varphi(\cdot g^{(1)}), \psi_l(\cdot g^{(2)})) = 0\) for any \(g^{(1)}, g^{(2)} \in H_0\). Since
\[
\hat{\varphi}_h(\chi) = \overline{(\chi, h)\hat{\varphi}(\chi)}, \quad \hat{\varphi}_{A^l g}(\chi) = \frac{1}{p(\chi, A^{-1} g)}\hat{\varphi}(\chi A^{-1}),
\]
it follows that
\[
(\varphi(\cdot g^{(1)}), \psi_l(\cdot g^{(2)})) = \int_X \hat{\varphi}(\chi)\overline{\hat{\varphi}(\chi A^{-1})(\chi, g^{(1)}) (\chi, g^{(2)}) m_l(\chi)} \, d\nu(\chi) = 0
\]
because \(\text{supp} \hat{\varphi}(\chi) = E\) and \(m_l(E) = 0, l = 1, p - 1\).

b) By analogy
\[
(\psi_k(\cdot g^{(1)}), \psi_l(\cdot g^{(2)})) = \int_X |\hat{\varphi}(\chi A^{-1})|^2 (\chi, g^{(2)} - g^{(1)}) m_k(\chi) m_l(\chi) \, d\nu(\chi) = 0
\]
when \(k \neq l\).

c) We verify that \((\psi_l(\cdot g^{(1)}), \psi_l(\cdot g^{(2)})) = 0\), provided that \(g^{(1)}, g^{(2)} \in H_0\) and \(g^{(1)} \neq g^{(2)}\). Write this scalar product in the form
\[
(\psi_l(\cdot g^{(1)}), \psi_l(\cdot g^{(2)})) = \int_X |\hat{\varphi}(\chi A^{-1})|^2 (\chi, g^{(2)} - g^{(1)}) m_l(\chi)^2 \, d\nu(\chi) = \int_{E_A \cap X_0 r^l_0} (\chi, g^{(2)} - g^{(1)}) \, d\nu(\chi).
\]
Show that \(E_A \cap X_0 r^l_0\) is an \((1, H - 1)\)-elementary set. By the definition
\[
E = \bigsqcup_{(0, \alpha_s, \alpha_{s-1}, \ldots, \alpha_0, \alpha_{-1}) \in T(V)} \mathcal{G}^\perp_{-1} r_{-1}^{\alpha_{-1}} r_0^{\alpha_0} \ldots r_s^{\alpha_s} r_{s+1}^{0} \quad (s \leq H - 3) \quad (5.3)
\]
where the union is taken over all paths \((0, \alpha_s, \alpha_{s-1}, \ldots, \alpha_0, \alpha_{-1})\) of the tree \(T(V)\). It means that for any \(\alpha_{-1} = 0\), \(p - 1\) the union \((5.3)\) contains unique coset \(\mathcal{G}^\perp_{-1} r_{-1}^{\alpha_{-1}} r_0^{\alpha_0} \ldots r_s^{\alpha_s} r_{s+1}^{0}\).

Consequently
\[
E_A = \bigsqcup_{(0, \alpha_s, \alpha_{s-1}, \ldots, \alpha_0, \alpha_{-1}) \in T(V)} \mathcal{G}^\perp_{0} r_0^{\alpha_0} r_1^{\alpha_1} \ldots r_{s+1}^{0} r_{s+2} = \sum_{\alpha_{-2} = 0}^{p-1} \bigsqcup_{(0, \alpha_s, \alpha_{s-1}, \ldots, \alpha_0, \alpha_{-1}) \in T(V)} \mathcal{G}^\perp_{-1} r_{-1}^{\alpha_{-2}} r_0^{\alpha_{-1}} \ldots r_s^{\alpha_s} r_{s+1}^{0} r_{s+2}.
\]
On the other hand
\[ X_{0\gamma 0} = \bigcup_{j \in \mathbb{N}} \bigcup_{(\gamma^{j-1}, \gamma^{j}) \in T(V)} \bigcup_{b_1, b_2, \ldots, b_j = 0} \mathcal{G}_{-1}^{-1} r_0^{-1} r_1^{b_1} \ldots r_j^{b_j} \cdot \]

Therefore
\[ EA \bigcap X_{0\gamma 0} = \bigcup_{\gamma = 0}^{p-1} \bigcup_{\alpha_0, \alpha_{s-1}, \ldots, \alpha_{s-1} = \gamma^{s+1}, \gamma^{-1}) \in T(V)} \mathcal{G}_{-1}^{-1} r_0^{-1} r_1^{\alpha_0} \ldots r_s^{1} r_{s+2}. \]

It means that \( EA \bigcap X_{0\gamma 0} \) is \((1, H - 1)\)-elementary set. By lemma 3.4 it follows that
\[ \int_{EA \bigcap X_{0\gamma 0}} (\chi, g^{(2)} - g^{(1)}) d\nu(\chi) = 0. \]

d) We claim that any function \( f \in W_0 \) can be expanded uniquely in a series in \((\psi_l(x - g))_{l = 1, p-1, \alpha_0 \in H_0}\). The proof of this fact may be found in [13], theorem 5.1.

**Step 7.** Since the subspaces \((V_j)_{j \in \mathbb{Z}}\) form an MRA in \( L_2(\mathcal{G}) \), it follows that the functions
\[ (\psi_l(A^n x - h)) \quad l = 1, p-1, n \in \mathbb{Z}, h \in H_0 \]
form a complete orthogonal system in \( L_2(\mathcal{G}) \).

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