Non-Abelian Duality Based on Non-Semi-Simple Isometry Groups

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Abstract

Non-Abelian duality transformations built on non-semi-simple isometry groups are analysed. We first give the conditions under which the original non-linear sigma model and its non-Abelian dual are equivalent. The existence of an invariant and non-degenerate bilinear form for the isometry Lie algebra is crucial for this equivalence. The non-Abelian dual of a conformally invariant sigma model, with non-semi-simple isometries, is then constructed and its beta functions are shown to vanish. This study resolves an apparent obstruction to the conformal invariance of sigma models obtained via non-Abelian duality based on non-semi-simple groups.

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1. Introduction

Duality transformations have been the centre of wide investigations recently. (see [1, 2] for a review and further references). They are crucial tools for understanding the structure of the moduli space resulting from compactifications of string theories. These transformations connect two apparently different conformal string backgrounds. It is in this sense that they are going to be treated in here. We will not be concerned with the global issues or the reversibility of these transformations. The implementation of duality relies heavily on the existence of isometries on target space-time backgrounds [3]. However, duality can also be defined without isometries. The latter case is known as Poisson-Lie T-duality [4].

The duality is Abelian if it is built on an Abelian isometry group [3, 4] and non-Abelian when its corresponding isometry group is non-Abelian [7]. In both cases, the starting point is a non-linear sigma model enjoying some global symmetries which form some isometry groups [3]. The dual sigma model is then obtained by gauging one (or more) isometry groups and at the same time constraining, by means of Lagrange multipliers, the field strength of each gauge field to vanish. Integrating out these Lagrange multipliers and fixing the gauge invariance yields the original model. On the other hand, integrating over the gauge fields and keeping the Lagrange multipliers results in the dual sigma model. One expects then that if the original backgrounds are conformal backgrounds (i.e. they satisfy the vanishing of the beta functions conditions) then the dual backgrounds are also conformal and form consistent backgrounds on which the string propagates [3, 3, 8].

This is indeed the case when dealing with Abelian duality. However, for non-Abelian duality some obstruction to the conformal invariance of the dual theory have been reported. The authors of [9] considered a cosmological solution to string theory (the beta functions vanish) based on a Bianchi V space-time. The application of non-Abelian duality transformations to this string solution does not lead to conformal string backgrounds. A similar conclusion was reached when considering a string solution based on Bianchi IV [10]. Both examples, however, share the same feature: their non-Abelian isometry groups are both non-semi-simple.

We examine, in this letter, the implementation of non-Abelian duality when the gauged isometry group is non-semi-simple. We show that the Lagrange multiplier, necessary for forcing the gauge field to be a pure gauge, fulfills his rôle only when the Lie algebra of the isometry group is endowed with an invariant and non-degenerate bilinear form. In other words, the original sigma model and its gauged version (with the Lagrange multiplier term) are equivalent only if the invariant bilinear form is invertible. The above mentioned examples,
where non-Abelian duality seems to fail, do not possess an invertible bilinear form.

We provide here a two-dimensional non-linear sigma model having a non-semi-simple isometry group whose Lie algebra possesses an invariant bilinear form. The model is a WZW model based on the centrally extended Euclidean group $E_2^c$. It is therefore conformally invariant to all order in perturbation theory. We construct then the non-Abelian dual theory and explicitly check that its one-loop beta functions vanish. This model is presented in section 3. The general procedure for obtaining dual sigma models is briefly outlined in section 2. The crucial function of the invariant bilinear form is also emphasized.

2. The duality procedure

The original theory is given by the general ungauged bosonic two-dimensional non-linear sigma model

\[
S(\varphi) = \int_{\partial \Sigma} d^2x \sqrt{\gamma} \left( \gamma^{\mu \nu} G_{ij}(\varphi) \partial_\mu \varphi^i \partial_\nu \varphi^j + \Phi(\varphi) R^{(2)} \right) + \Gamma(\varphi)
\]

\[
\Gamma(\varphi) = \int_{\Sigma} d^3y \epsilon^{\mu \nu \rho} H_{ijk}(\varphi) \partial_\mu \varphi^i \partial_\nu \varphi^j \partial_\rho \varphi^k.
\]

(1)

In this equation $\gamma_{\mu \nu}$ is the metric on the two-dimensional world sheet $\partial \Sigma$, $\gamma$ is its determinant and $R^{(2)}$ is the scalar curvature. The metric $G_{ij}$ and the dilaton field $\Phi$ correspond to the set of massless modes of an associated string theory. The other massless field, the anti-symmetric tensor $B_{ij}$, is defined through its three-form $2H_{ijk} = \partial_i B_{jk} + \partial_j B_{ik} + \partial_k B_{ij}$. The latter is defined on the three-dimensional space $\Sigma$ whose boundary is $\partial \Sigma$.

The sigma model Lagrangian is manifestly invariant under global reparametrisation of the target space-time. However, the only global symmetries suitable for gauging are those for which the metric remains form invariant. Such symmetries form a Lie group $G$, namely the isometry group of the metric $G_{ij}$. A general infinitesimal isometry on the target space-time is given by the global transformation

\[
\delta \varphi^i = \alpha^a K_a^i(\varphi),
\]

(2)

where $\alpha^a$ is a constant. The generator of this transformation are $K_a = K_a^i \partial_i$ and they satisfy the Lie algebra of $G$

\[
[K_a, K_b] = f_{ab}^c K_c \quad \text{or} \quad K_a^i \partial_i K_b^j - K_b^i \partial_i K_a^j = f_{ab}^c K_c^j,
\]

(3)

where $f_{bc}^a$ are structure constants supposed to be field-independent.
Global invariance of the action implies some constraints on $G_{ij}$, $H_{ijk}$ and $\Phi$. The condition on the metric term are simply the Killing equations

$$G_{ij} \nabla_k K_a^j + G_{kj} \nabla_i K_a^j = 0 \ ,$$

where the covariant derivative $\nabla_i$ is with respect to the metric $G_{ij}$. The dilaton term is invariant when $K_a^i \partial_i \Phi = 0$.

The last term of the action is invariant under (4) provided that the torsion $H_{ijk}$ obeys

$$K_a^i \partial_i H_{jkl} + H_{ikl} \partial_j K_a^i + H_{jkl} \partial_l K_a^i = 0 \ .$$

(5)

Since the three-form $H_{ijk}$ is closed, the invariance condition requires, for every vector $K_a^i$, the existence of a globally defined one-form fulfilling

$$K_a^i H_{ijk} = \partial_j L_{ak} - \partial_k L_{aj} \ .$$

(6)

This last equation is at the heart of gauging the general sigma model.

The first step towards constructing the dual theory is to gauge the above global symmetry. We therefore introduce a gauge field $A^a_\mu$ taking values in the Lie algebra $\mathcal{G}$ and transforming as

$$\delta A^a_\mu = - \partial_\mu \alpha^a - f^{ac}_{bc} A^b_\mu \alpha^c_\mu \ .$$

(7)

The gauging is then possible only if we impose two further conditions [11, 12]

$$K_a^i \partial_i L_{bj} + L_{ba} \partial_j K_a^i = - f^{e}_{ab} L_{ej}$$

$$L_{ai} K_b^i + L_{bi} K_a^i = 0 \ .$$

(8)

The last condition ensures that the gauge fields appear at most in a quadratic form and live entirely on the two-dimensional manifold.

The gauged action is then found to be given by

$$S_{\text{gauge}} (\varphi, A) = \int_{\partial \Sigma} d^2 x \sqrt{\gamma} \left( \gamma^{\mu\nu} G_{ij} (\varphi) D_\mu \varphi^i D_\nu \varphi^j + \Phi (\varphi) R^{(2)} \right)$$

$$- 6 \int_{\partial \Sigma} d^2 x \epsilon^{\mu\nu} \left( L_{ai} A^a_\mu \partial_\nu \varphi^i + \frac{1}{4} \left( L_{ai} K_b^i - L_{bi} K_a^i \right) A^a_\mu A^b_\nu \right)$$

$$+ \int_{\Sigma} d^3 y \epsilon^{\mu\nu\rho} \left[ H_{ijk} (\varphi) \partial_\mu \varphi^i \partial_\nu \varphi^j \partial_\rho \varphi^k \right] \ .$$

(9)

The covariant derivative is such that $D_\mu \varphi^i = \partial_\mu \varphi^i + A^a_\mu K_a^i$.

The dual theory is then constructed by considering the first order action given by [3]

$$S_1 (\varphi, A, X) = S_{\text{gauge}} (\varphi, A) + \int_{\partial \Sigma} d^2 x \epsilon^{\mu\nu} \Omega_{ab} X^a F^b_{\mu\nu} \ .$$

(10)
Here $F_{\mu\nu}^a = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a + f_{bc}^a A_\mu^b A_\nu^c$ is the field strength and $X^a$ is the Lagrange multiplier. The Lagrange multiplier transforms as $\delta X^a = -f_{bc}^a X^b \alpha^c$ and takes values in the Lie algebra $G$. That is $X = X^a T_a$ with $[T_a, T_b] = f_{ac}^b T_c$. The added term is gauge invariant provided that

$$\Omega_{ac} f_{bd}^c + \Omega_{bc} f_{ad}^c = 0 \quad (11)$$

This means that $\Omega_{ab}$ is an invariant bilinear form of $G$.

The original theory is retrieved by integrating over the Lagrange multiplier $X^a$. This integration leads, in the path integral, to a delta function enforcing the condition

$$\Omega_{ab} F_{\mu\nu}^b = 0 \quad (12)$$

This last equation yields $F_{\mu\nu}^a = 0$ only if $\Omega_{ab}$ is invertible. In this case, and only in this case, that one can use gauge invariance to set $A_\mu^a$ to zero and hence to get the ungauged action. One does not realise this issue by writing the Lagrange multiplier term in the form $\epsilon_{\mu\nu} X_a F_{\mu\nu}^a$ [9, 10]. One must specify the invariant bilinear form used to lower the index of the Lagrange multiplier $X^a$.

The first example where non-Abelian duality failed was considered by the authors of [3]. There, a cosmological solution to string theory in the form of a four-dimensional Bianchi V was analysed. The string backgrounds contains a vanishing antisymmetric tensor field, a constant dilaton and a metric in the form

$$\begin{align*}
    ds^2 &= -dt^2 + a^2(t) \left[ dx^2 + e^{-2x} \left( dy^2 + dz^2 \right) \right].
\end{align*}$$

(13)

Conformal invariance then demands $a(t) = t$ and the metric becomes flat. This metric possesses a non-Abelian isometry group generated by

$$[K_1, K_2] = -K_2 \quad , \quad [K_1, K_3] = -K_3 \quad , \quad [K_2, K_3] = 0 \quad (14)$$

which is realised by the differential operators

$$K_1 = \frac{\partial}{\partial x} + y \frac{\partial}{\partial y} + z \frac{\partial}{\partial z} \quad , \quad K_2 = \frac{\partial}{\partial y} \quad , \quad K_3 = \frac{\partial}{\partial z} \quad (15)$$

The non-vanishing structure constants are $f_{12}^2 = f_{13}^3 = -1$. The corresponding invariant bilinear form is found to be

$$\Omega_{ab} = \begin{pmatrix}
    k & 0 & 0 \\
    0 & 0 & 0 \\
    0 & 0 & 0
\end{pmatrix},$$

(16)

where $k$ is an arbitrary constant. It is clear that this is degenerate. The integration over the Lagrange multiplier term in (10) leads to $F_{\mu\nu}^1 = 0$ only. This alone does not allow one to set
$A_\mu$ to zero and hence to arrive at the original theory. Therefore, the first order action in (10) is not equivalent to the original ungauged action. As a consequence, the action obtained by integrating out the gauge fields does not necessarily provide conformal string backgrounds.

A similar model was also studied in [10]. It is based on the Bianchi IV type cosmological metric

$$ds^2 = -dt^2 + a^2(t) dx^2 + b^2(t) e^{-x} [dy^2 + dz^2] .$$

(17)

With a zero torsion and a constant dilaton field, conformal invariance at the one loop level imposes $a(t) = t/2$ and $b(t) = t$. This metric has also a non-Abelian isometry group having the Lie algebra

$$[K_1, K_2] = 0 , \ [K_2, K_3] = 0 , \ [K_1, K_3] = K_1$$

(18)

with the differential representation

$$K_1 = \frac{\partial}{\partial y} , \ K_2 = \frac{\partial}{\partial z} , \ K_3 = \frac{\partial}{\partial x} + y \frac{\partial}{\partial y} .$$

(19)

The only non-zero structure constants are $f^{13}_{13} = 1$. Their unique invariant bilinear form is given by

$$\Omega_{ab} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & k & m \\ 0 & m & n \end{pmatrix}$$

(20)

with $k, m, n$ being arbitrary constants. Again this is not invertible and the integration over the Lagrange multiplier gives $F^2_{\mu\nu} = F^3_{\mu\nu} = 0$ when $(kn - m^2) \neq 0$. This is not sufficient for obtaining the original theory.

Therefore, the equivalence of the first order action and the original sigma model can be established only when the isometry group owns an invertible bilinear form. We present below a non-semi-simple isometry group having a non-degenerate bilinear form. The non-Abelian theory is constructed and the resulting sigma model is shown to be conformally invariant at the one loop level.

### 3. Duality with non-semi-simple isometries

The model we would like to consider is a Wess-Zumino-Witten (WZW) model defined on the group manifold $\mathcal{M}_G$. It is based on the four-dimensional non-semi-simple Lie algebra $\mathcal{G}$

$$[J, P_i] = \epsilon_{ij} P_j , \ [P_i, P_j] = \epsilon_{ij} T , \ [T, J] = 0 , \ [T, P_i] = 0 .$$

(21)
The algebra \( \mathcal{G} \), generated by \( T_a = \{P_1, P_2, J, T\} \), has an invariant bilinear form \( \Omega_{ab} \) satisfying (11). Furthermore, it is invertible (there is an inverse matrix \( \Omega^{ab} \) obeying \( \Omega^{ab} \Omega_{bc} = \delta^a_c \)). We have
\[
\Omega_{ab} = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & b & 1 \\
0 & 0 & 1 & 0
\end{pmatrix}, \quad \Omega^{ab} = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 1 & -b
\end{pmatrix},
\]
where \( b \) is a constant.

In general, given some structure constants \( f^a_{ab} \) and their corresponding non-degenerate and invariant bilinear form \( \Omega_{ab} \), one constructs the WZW action
\[
S(g) = \frac{k}{8\pi} \int_{\partial \Sigma} d^2x \sqrt{-\gamma} \gamma^{\mu\nu} \Omega_{ab} B^a_\mu B^b_\nu + \frac{k}{24\pi} \int_{\Sigma} d^3x \sqrt{-\gamma} \epsilon^{\mu\nu\rho} f^a_{ad} f^b_{bc} B^a_\mu B^b_\nu B^c_\rho
\]
where \( g \) is defined on the group manifold \( \mathcal{M}_G \). The quantities \( B^a_\mu \) are defined through \( B^a_\mu T_a = g^{-1} \partial_\mu g \).

The resulting sigma model is found by choosing an explicit parametrisation of the group manifold. In the case at hand this is taken to be
\[
g = \exp (a_1 P_1 + a_2 P_2) \exp (uJ + vT). \tag{24}
\]
The gauge-like quantities \( B^a_\mu \) are then given by
\[
B^1_\mu = \cos (u) \partial_\mu a_1 - \sin (u) \partial_\mu a_2 \\
B^2_\mu = \cos (u) \partial_\mu a_2 + \sin (u) \partial_\mu a_1 \\
B^3_\mu = \partial_\mu u \\
B^4_\mu = \partial_\mu v + \frac{1}{2} \epsilon_{ij} a_j \partial_\mu a_i.
\]
The sigma model reads then
\[
S(g) = \frac{k}{8\pi} \int_{\partial \Sigma} d^2x \sqrt{-\gamma} \gamma^{\mu\nu} (\partial_\mu a_i \partial_\nu a_i + \epsilon_{ij} a_j \partial_\mu a_i \partial_\nu u + b \partial_\mu u \partial_\nu u + 2 \partial_\mu u \partial_\nu v) \\
+ \frac{k}{8\pi} \int_{\Sigma} d^2x \epsilon^{\mu\nu} (2u \partial_\mu a_1 \partial_\nu a_2). \tag{26}
\]
One can read off the metric and the antisymmetric tensor field. The dilaton field vanishes in this case.

The space-time backgrounds corresponding to (26) satisfy, as expected, the one-loop conformal invariance conditions

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2 String backgrounds based on non-semi-simple groups have also been considered in [14].
\[
R_{MN} - \frac{1}{4} H_{MPR} H_{N}^{PR} + 2 \nabla_N \nabla_M \Phi = 0
\]
\[
\nabla^L \left( e^{-2\Phi} H_{LMN} \right) = 0
\]
\[
R - \frac{1}{12} H_{MNP} H^{MNP} + 4 \nabla^N \nabla_N \Phi - 4 \partial^N \Phi \partial_N \Phi - \frac{2}{3 \alpha'} (c - d) = 0 .
\] (27)

Here \( M, N, \ldots = 1, \ldots, 4 \) and correspond to the space-time coordinates \( \varphi^N = \{a_1, a_2, u, v\} \).

The central charge \( c \) equals to 4 and is the same as the dimension of the target space-time \( d \).

Other solutions to the above equations can be generated through duality transformations. The WZW action is invariant under the chiral symmetry

\[
g \longrightarrow L g L^{-1} .
\] (28)

Under an infinitesimal chiral transformation \( L = \alpha^a T_a \) we have

\[
\delta B^a_\mu = \partial_\mu \lambda^a + f^a_{bc} B^b_\mu \lambda^c , \quad \lambda^a = - \alpha^a + W^a_b \alpha^b .
\] (29)

Here the quantity \( W^a_b \) is defined via \( W^a_b T_b = g^{-1} T_a g \) and it has the following useful properties

\[
\Omega_{ab} W^a_c W^b_d = \Omega_{cd}
\]
\[
\partial_\mu W^a_b = f^a_{ce} W^e_\mu B^c_b
\]
\[
\delta W^a_b = \alpha^c \left( f^a_{be} W^b_d - f^a_{ce} W^b_e \right)
\] (30)

This isometry group is an anomaly-free subgroup and its corresponding gauged action is written as

\[
S_1 (g, A, X) = S (g) + \frac{k}{4 \pi} \int_{\partial \Sigma} \Omega_{ab} \left[ P^{\mu \nu} B^a_\mu W^b_\nu A^c_\nu + P^{\mu \nu} B^a_\mu A^b_\nu - P^{\mu \nu} W^a_\mu A^b_\nu A^c_\nu \right]
\]
\[
+ \frac{k}{4 \pi} \int_{\partial \Sigma} \epsilon^{\mu \nu} \Omega_{ab} X^a F^b_{\mu \nu} .
\] (31)

The last term is the usual Lagrange multiplier term. Since \( \Omega_{ab} \) is invertible, the integration over \( X^a \) leads to \( F^a_{\mu \nu} = 0 \) which, owing to gauge invariance, yields \( A^a_\mu = 0 \). Substituting this back in the gauged action gives the original WZW theory. The gauge field \( A^a_\mu \) transforms as in equation (28). We have, for convenience, defined the projection matrix

\[
P^{\pm \mu \nu} = \sqrt{-g} \gamma^{\mu \nu} \pm \epsilon^{\mu \nu} .
\] (32)
The individual infinitesimal coordinate transformations are written as

\[\begin{align*}
\delta a_1 &= \alpha_1 (1 - \cos(u)) + \alpha_2 \sin(u) - \alpha_3 a_2 \\
\delta a_2 &= -\alpha_1 \sin(u) + \alpha_2 (1 - \cos(u)) + \alpha_3 a_1 \\
\delta u &= 0 \\
\delta v &= \alpha_1 \left(-\frac{1}{2}a_2 + \frac{1}{2}a_1 \sin(u) + \frac{3}{2}a_2 \cos(u)\right) \\
&\quad + \alpha_2 \left(-\frac{3}{2}a_1 + \frac{1}{2}a_1 \cos(u) - \frac{3}{2}a_2 \sin(u)\right) + \alpha_3 \left(a_2^2 - a_1^2\right) \\
\delta X_1 &= \alpha_2 X_3 - \alpha_3 X_2 \\
\delta X_2 &= -\alpha_1 X_3 + \alpha_3 X_1 \\
\delta X_3 &= 0 \\
\delta X_4 &= \alpha_1 X_2 - \alpha_2 X_1 .
\end{align*}\]

(33)

The fourth gauge parameter, \(\alpha_4\), appears only in the transformations of the gauge fields \(A^c_\mu\).

We are now at a stage where we can eliminate the gauge fields. There is, however, a new feature regarding the integration over the gauge fields. This is due to the presence of the central element \(T\) in the Lie algebra \(G\). In order to see this, the gauged action is written as

\[S(g, A, X) = S(g) + \frac{k}{4\pi} \int_{\partial \Sigma} d^2x \left\{ \frac{1}{2} \left( \sqrt{\gamma} \gamma^{\mu \nu} M_{ij} + \epsilon^{\mu \nu} N_{ij} \right) A^i_\mu A^j_\nu \\
+ \left[ P^{\mu \nu} \Omega_{cd} W^c_i B^b_\nu - P^{\mu \nu} \Omega_{db} B^b_i + 2\epsilon^{\mu \nu} \Omega_{ib} \partial_b X^b \right] A^i_\mu \\
+ 2\epsilon^{\mu \nu} \left( \partial_\nu X_3 - B^3_\nu \right) A^i_\mu \right\} , \]

(34)

where the indices \(i, j = 1, 2, 3\) and we have defined

\[\begin{align*}
M_{ij} &= 2\Omega_{ij} - \Omega_{ci} W^c_j - \Omega_{cj} W^c_i \\
N_{ij} &= 2\Omega_{cd} X^c f_{ij}^c - \Omega_{ci} W^c_j + \Omega_{cj} W^c_i .
\end{align*}\]

(35)

Notice that the integration over \(A^4_\mu\) leads, in the path integral, to the constraint

\[\epsilon^{\mu \nu} \left( \partial_\nu X_3 - B^3_\nu \right) = 0 \]

(36)

which can be solved by setting \(X_3 = u\). This is consistent with the fact that \(\delta u = \delta X_3 = 0\).

The integration over the rest of the gauge fields is Gaussian and can be carried out. However, one needs first to choose a gauge fixing condition. A suitable gauge is provided by the choice

\[a_1 = a_2 = X_1 = 0 . \]

(37)
When taking into account the constraint (36), the number of fields is equal to four \( \varphi^N = \{ u, v, X_2, X_4 \} \). The matrix in front of the quadratic term in the gauge fields is invertible. Therefore the integration is straightforward and we get the following non-linear sigma model

\[
S = \frac{k}{8\pi} \int_{\partial \Sigma} d^2x \sqrt{-\gamma} \gamma^{\mu\nu} \left[ b \partial_\mu u \partial_\nu u + 2 \partial_\mu u \partial_\nu v - \frac{2}{\cos(u) - 1} \partial_\mu X_2 \partial_\nu X_2 \\
+ \frac{4 \sin(u) - u}{X_2 (\cos(u) - 1)} (\partial_\mu X_2 \partial_\nu X_4 - \partial_\mu X_2 \partial_\nu v) \\
+ \frac{2 u \sin(u) + 2 \cos(u) - u^2 - 2}{X_2^2 (\cos(u) - 1)} (\partial_\mu v \partial_\nu v + \partial_\mu X_4 \partial_\nu X_4 - 2 \partial_\mu X_2 \partial_\nu v) \right].
\] (38)

The torsion vanishes in the dual theory. The integration over the gauge fields leads also to an extra local determinant. The regularisation of the latter yields a contribution to the dilaton field given by [5, 15]

\[
\Phi = -\frac{1}{2} \ln \left[ \det (M_{ij} + N_{ij}) \right] + \text{constant} \\
= -\frac{1}{2} \ln \left[ X_2^2 (1 - \cos(u)) \right] + \text{constant}.
\] (39)

We have explicitly checked that the resulting space-time backgrounds do indeed satisfy the conformal invariance condition given in (27). The gauge fixing conditions lead to a non-trivial Faddeev-Popov factor in the path integral measure. As can be verified, this factor combines with the left-right invariant Haar measure, \( da_1 da_2 du dv \), of the original WZW model to give the expected measure of the dual theory \( e^{-2\Phi} \sqrt{G} dX_2 dX_4 du dv \) [16].

It is worth mentioning that the metric of the original theory describes a plane wave. This plane is monochromatic and has, in a special coordinate system, two singularities when \( \cos(u) = \pm 1 \). The metric of the dual theory has, in addition to these singularities, a further singularity at \( X_2 = 0 \). The resulting geometry is not that of a plane wave. A similar model was obtained by considering, in a different context and using a different group manifold parametrisation, the non-Abelian dual of the above WZW model [17].

4. Conclusions

We have resolved in this paper a standing problem concerning the implementation of non-Abelian duality based on non-semi-simple isometry groups. It is shown that the construction of non-Abelian dual sigma models is possible only when the isometry group possesses a non-degenerate invariant bilinear form. We confirm our analyses by constructing a non-Abelian dual of a sigma model having non-semi-simple isometries. The backgrounds of the resulting theory fulfill the one loop conformal invariance conditions.
Our analyses can be generalised to other WZW models based on more complicated non-semi-groups. Notice also that in the example we have studied, the metric of the original theory has the translation symmetry \( v \rightarrow v + \epsilon \). This translation is generated by a null vector. In general, bosonic and supersymmetric string solutions with covariantly constant null Killing vectors have played a rôle in the construction of dyonic Bogomol’nyi-Prasad-Sommerfield (BPS) states \[18\]. It would be of interest to examine the effects of non-Abelian duality on these BPS states.

Abelian duality has proved to be crucial in the understanding of string theory and membranes. However, its non-Abelian counterpart has not been fully explored. One of the areas where non-Abelian duality might be of interest is in cosmological and inflationary models based on string effective theories \[19\]. This is due to the fact that most of the relevant cosmological models have a tendency to possess non-Abelian rather than Abelian isometries. This is certainly the case in the two examples considered here and which do not have non-degenerate bilinear forms (Bianchi IV and V). In order to explore the cosmological implications of non-Abelian duality, one is forced to centrally extend the Lie algebras of the two isometries. This would provide us with an invertible bilinear form. Physically, this is achieved by extending the dimension of space-time. This issue is currently under investigation.

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