Modulation instability of obliquely propagating ion acoustic waves in a collisionless magnetized plasma consisting of nonthermal and isothermal electrons

Sandip Dalui1 · Anup Bandyopadhyay1

Received: 26 June 2019 / Accepted: 15 October 2019 / Published online: 21 October 2019 © Springer Nature B.V. 2019

Abstract A nonlinear Schrödinger equation is derived to investigate the amplitude modulation of ion acoustic waves propagating obliquely to the direction of the uniform static magnetic field in a collisionless plasma consisting of warm adiabatic ions and two different species of electrons at different temperatures. We have investigated the relation between two nonlinear Schrödinger equations—one describes the amplitude modulation of ion acoustic waves propagating obliquely to the direction of the magnetic field and other describes the amplitude modulation of ion acoustic waves propagating along the magnetic field. The instability conditions of the modulated wave have been investigated with respect to different parameters. We have seen that the increase in the strength of the magnetic field tends to destabilize the modulated wave when \( \omega_c \) lies within the interval \( \omega_{c1} < \omega_c < \omega_{c2} \) whereas increase in the strength of the magnetic field tends to stabilize the modulated wave when \( \omega_c \) is restricted by the inequalities \( 0 < \omega_c < \omega_{c1} \) or \( \omega_{c2} < \omega_c < 1 \), where \( \omega_{c1} \) and \( \omega_{c2} \) are two critical values of the normalized ion cyclotron frequency \( \omega_c \). We have investigated the dependence of different parameters on \( \omega_{c1} \) and \( \omega_{c2} \). Again, the maximum growth rate of instability increases with increasing \( \cos \theta \) and it also increases with increasing \( \beta_e \) up to a critical value of the wave number, where \( \beta_e \) is the parameter associated with the nonthermal distribution of hotter electron species and \( \theta \) is the angle of propagation of the ion acoustic wave with the magnetic field.

Keywords Electron-ion plasma · Two temperature electrons · Cairns distribution · Nonlinear Schrödinger equation · Modulation instability

1 Introduction

Different Satellite observations (Dovner et al. 1994; Boström et al. 1988; Boström 1992; Ergun et al. 1998a,b; Delory et al. 1998; Pottelette et al. 1999; McFadden et al. 2003) indicate the presence of fast energetic population of electrons together with the isothermal population of electrons. The fast energetic population of electrons can be modeled by the nonthermal distribution of Cairns et al. (1995). Although there is no general procedure for the formation of the fast energetic electrons in the space plasma. So, the cooler electron species follows Maxwell-Boltzmann distribution and the hotter electron species can be taken as the nonthermal distribution of Cairns et al. (1995). Again, Yu and Luo (2008) reported that for phenomena on long time scales, one can consider electrons into two different species if the electrons are physically separated in space/time domain of interest. So, Maxwell-Boltzmann distributed electrons and Cairns (1995) distributed nonthermal electrons can be considered as two different electron species only when those electron species are physically separated in the phase space by external or self-consistent fields. On the basis of this assumption, several authors (Pakzad and Tribeche 2011; Abdelwahed 2012; Alam et al. 2013; Ghosh and Sekar Iyengar 2014; Singh and Lakhina 2015; Hou et al. 2016; Hossem and Mamun 2016) considered two populations of electrons at different temperatures. Several authors (Islam et al. 2008; Rufai et al. 2014; Singh and Lakhina 2015; Dalui et al. 2017a,b) considered different multi-species plasmas containing two different electron species at different temperatures, where the cooler electron species follows the Maxwell-Boltzmann distribution whereas the hotter electron species is modeled by the nonthermal distribution as prescribed by Cairns et al. (1995). In the present paper, we have investigated modulation instability (MI) of ion acoustic (IA) waves propagating...
along any arbitrary direction to the direction of the external uniform static magnetic field in a collisionless magnetized electron-ion plasma consisting of warm adiabatic ions and a superposition of isothermal cold electrons and nonthermal hot electrons.

Several authors (Sabry et al. 2012; El-Tantawy et al. 2017; Dalui et al. 2017b; Parkes 1987; Alam and Chowdhury 2000; Misra and Bhownik 2007; Bains et al. 2010) investigated MI of IA waves in different magnetized plasma systems. But, in the above mentioned papers, the direction of propagation of the IA wave is taken parallel to the direction of the magnetic field. Murtaza and Salahuddin (1982) studied the MI of obliquely propagating IA waves in magnetized plasma containing isothermal electrons and cold ions. Jehan et al. (2008) have investigated MI of low-frequency electrostatic ion waves in magnetized electron-positron-ion plasma. In both the papers (Murtaza and Salahuddin 1982; Jehan et al. 2008), authors have used Krylov-Bogoliubov-Mitropolsky method (Kakutani and Sugimoto 1974) to derive the nonlinear Schrödinger equation (NLSE). Recently, Ghosh and Banerjee (2016) derived a NLSE using the multiple scale perturbation method to describe the MI of obliquely propagating IA waves in a collisionless magnetized plasma.

Recently, Dalui et al. (2017b) have derived a three-dimensional nonlinear Schrödinger equation to describe the MI of IA waves propagating along the direction of the magnetic field in a collisionless magnetized warm plasma consisting of nonthermal and isothermal electrons. In the present paper, for the first time, we have investigated MI of IA waves propagating at an arbitrary angle to the direction of the external uniform static magnetic field in the same plasma system of Dalui et al. (2017b). So, in the present paper, we have generalized the earlier paper of Dalui et al. (2017b) in the following directions:

1. Starting from the appropriate basic equations, our aim is to make a systematic development to derive an appropriate nonlinear Schrödinger equation (NLSE) which can effectively describe the MI of IA waves propagating along any arbitrary direction with respect to the direction of the external uniform static magnetic field.

2. Our aim is to make a correspondence between the NLSE describing the amplitude modulation of IA waves propagating along the direction of the magnetic field and the NLSE describing the amplitude modulation of IA waves propagating along any arbitrary direction to the direction of the magnetic field. In fact, we have seen the following points:

   - The linear dispersion relation (LDR) of the IA wave obtained in the present paper is exactly same as that of Dalui et al. (2017b) if we put \( \theta = 0 \), where \( \theta \) is the angle between the direction of propagation of IA wave and the direction of the external uniform static magnetic field.
   - The expression of the group velocity of the IA wave obtained in the present paper is exactly same as that of Dalui et al. (2017b) if we put \( \theta = 0 \).
   - The NLSE obtained in the present paper for describing the amplitude modulation of IA waves propagating along any arbitrary direction to the direction of the magnetic field is exactly same as that of Dalui et al. (2017b) if we put \( \theta = 0 \) and if we ignore the perpendicular dispersive terms of the NLSE of Dalui et al. (2017b), i.e., if we ignore the weak dependence of the spatial coordinates perpendicular to the direction of the magnetic field.

3. Using the NLSE, our aim is to find the instability condition of the modulated IA waves propagating along any arbitrary direction.

4. Our aim is to present a through analysis of the instability condition with respect to \( n_2(= \cos \theta) \).

5. We want to analyze the characteristics of the growth rate of instability and the maximum growth rate of instability \( (\Gamma_{\text{max}}) \) with respect to \( n_2(= \cos \theta) \).

6. We want to analyze the existence region of \( n_2 \) with respect to \( k \) for instability of the modulated IA wave, where \( k \) is the wave number of IA wave.

It is important to note that this paper is not merely an extension of the earlier paper of Dalui et al. (2017b) because of the following reasons:

- In the paper of Dalui et al. (2017b), the LDR of the IA wave is free from effect of external uniform static magnetic field whereas in the present paper, the LDR of the IA wave not only depends on the intensity of the magnetic field but it also depends on the direction of propagation of the IA wave.
- In the paper of Dalui et al. (2017b), the group velocity of the IA wave is free from effect of external uniform static magnetic field whereas in the present paper, the group velocity of the IA wave not only depends on the intensity of the magnetic field but it also depends on the direction of propagation of the IA wave.
- In the paper of Dalui et al. (2017b), the nonlinear term of the NLSE equation describing the modulational instability of the IA wave is free from effect of external uniform static magnetic field whereas in the present paper, the nonlinear term of the NLSE equation depends on the intensity of the magnetic field and the direction of propagation of the IA wave.

So, the present problem can be considered as a new one.
2 Basic equations

In this paper, we have investigated the MI of obliquely propagating IA waves in a collisionless magnetized plasma containing isothermal and nonthermal electron species, whereas Dalui et al. (2017b) have investigated the MI of IA waves propagating along the direction of the magnetic field in the same plasma system. So, here the IA wave is propagating along a direction making an angle $\theta$ with the external uniform static magnetic field, whereas in Dalui et al. (2017b), the value of $\theta$ is equal to 0.

So, we have used the equations (1)–(14) of Dalui et al. (2017b) and consequently without repeating all those equations, we want to mention the final form of the equation of continuity of ions, equation of motions along $x$, $y$ and $z$ axis, and the Poisson equation of Dalui et al. (2017b) for easy readability of the present paper.

$$\frac{dn}{dt} + n \left( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} \right) = 0,$$

$$\frac{du}{dt} = -\frac{\partial H}{\partial x} + \omega_c v,$$

$$\frac{dv}{dt} = -\frac{\partial H}{\partial y} - \omega_c u,$$

$$\frac{dw}{dt} = -\frac{\partial H}{\partial z},$$

$$\nabla^2 \phi = h_0 + h_1 \phi + h_2 \phi^2 + h_3 \phi^3 - n,$$

where

$$H = \phi + \frac{\sigma \gamma n \gamma^{-1}}{\gamma - 1}.$$

$$\frac{d}{dt} = \frac{\partial}{\partial t} + u \frac{\partial}{\partial x} + v \frac{\partial}{\partial y} + w \frac{\partial}{\partial z},$$

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}.$$ 

Here, $h_0 = 1$, $h_1 = (1 - \beta_c n_{i0} \sigma_c)$, $h_2 = \frac{\beta_c}{2} [n_{i0} \sigma_c^2 + 2 \bar{n}_{i0} (1 + 3 \beta_c) \sigma_c^3]$, $h_3 = \frac{\beta_c}{2} [\bar{n}_{i0} \sigma_c^3 + \bar{n}_{i0} (1 + 3 \beta_c) \sigma_c^3]$ with $\bar{n}_{i0} = \frac{n_{i0}}{n_0}$, $\bar{n}_{e0} = \frac{n_{e0}}{n_0}$, $\sigma_c = \frac{T_{ce}}{T_{ce}^0}$, $\sigma_e = \frac{T_{ce}^0}{T_{ce}}$, and $T_{ce}$ is determined by

$$\frac{n_{i0} + n_{e0}}{T_{ce}} = \frac{n_{i0}}{T_{ce}} + \frac{n_{e0}}{T_{ce}},$$

where $n_0$, $n_{i0}$ and $n_{e0}$ are equilibrium number densities of ion species, nonthermal electron species and isothermal electron species respectively; $T_{ce}$ and $T_{se}$ are, respectively, average temperatures of nonthermal electron species and isothermal electron species, and $\beta_c$ is the parameter associated with the nonthermal distribution function of Cairns et al. (1995).

We have used the notations $t$ for time, $(x, y, z)$ for spatial variables, $u = (u, v, w)$ for ion fluid velocity vector, $\omega_c$ for ion cyclotron frequency, $\phi$ for electrostatic potential. Again, $n_{se}$, $n_{ce}$ and $n$ are number densities of isothermal electron species, nonthermal electron species and ion species respectively. Here $t$, $(x, y, z)$, $u = (u, v, w)$, $\omega_c$, $\phi$, $n_{se}$, $n_{ce}$ and $n$ are normalized variables and these quantities have been normalized by $\sigma_{pi}^{-1}$, $\lambda_D$, $\sqrt{K_B T_{ef}/4\pi n_0 e^2}$, $c_s = \sqrt{K_B T_{ef}/m}$, $\omega_{pi} = \sqrt{4\pi e^2 n_0/m}$, $K_B T_{ef}/e$, $n_0$, $n_0$ and $n_0$ respectively, where $\sigma = T_i/T_{ef}$, $\gamma = \frac{5}{3}$, $K_B$ = Boltzmann constant, $m$ = mass of an ion, $-e$ = charge of an electron, $T_i$ = average ion temperature. Again, the basic parameters of the present plasma system are as follows: $\gamma$, $\sigma$, $\sigma_{sc} = \frac{T_{sc}}{T_{se}}$, $n_{sc} = \frac{n_{sc}}{n_0}$, $\beta_c$ and $\omega_c$. With respect to $\sigma_{sc}$ and $n_{sc}$, one can use the following equations:

$$\frac{(\bar{n}_{i0}, \bar{n}_{e0})}{n_{sc}} = \frac{1}{1 + n_{sc}} (n_{sc}, 1),$$

$$\sigma_{sc} = \frac{1 + n_{sc}}{\sigma_{sc} + n_{sc}} (1, \sigma_{sc}),$$

where we have used the equilibrium (unperturbed) charge neutrality condition $(n_{i0} + n_{e0} = n_0)$ and equation (9) to find the expressions $\bar{n}_{i0}, \bar{n}_{e0}, \sigma_c$ and $\sigma_e$.

Dalui et al. (2017b) have discussed the MI of IA waves propagating along the direction of the uniform static magnetic field ($B = B_0 \hat{z}$) in a collisionless magnetized plasma. It is found that the LDR of the IA wave propagating along the direction of the magnetic field is free from the effect of magnetic field intensity. To get the effect of the magnetic field intensity in the LDR of the IA wave, it is essential to consider the oblique propagation of IA waves with the external uniform static magnetic field ($B = B_0 \hat{z}$). Considering IA waves propagating along a direction having direction cosines $(l_x, m_x, n_x)$ and assuming space-time dependence of the perturbed dependent variables to be of the form $\exp[i(k (l_x x + m_x y + n_x z) - \omega t)]$, one can obtain the LDR of the IA wave in the following form:

$$k_x^2 \omega^2 - k_y^2 \omega_x^2 + \frac{k_z^2}{\omega_x^2} = \frac{k^2 + h_1}{1 + \sigma \gamma (k^2 + h_1)},$$

where $\omega$ and $k$ are, respectively, the wave frequency and wave number of IA wave, $k_x^2 = k_1^2 + k_2^2$, $k_y^2 = k_2^2 (l_x^2 + m_x^2)$, $k_z^2 = k_z^2 n_x^2$ and $l_x^2 + m_x^2 + n_x^2 = 1$. 

\[\text{Springer}\]
If the direction of propagation of the IA wave makes an angle $\theta$ with the direction of the external uniform static magnetic field, then

$$n_2 = \cos \theta, \quad l_2^2 + m_2^2 = \sin^2 \theta.$$  \hfill (13)

If we put $\theta = 0$, i.e., $n_2 = 1$ ($\Rightarrow l_2^2 + m_2^2 = 0 \Rightarrow l_2 = m_2 = 0$) then the LDR (12) reduces to the LDR (15) of Dalui et al. (2017b) with slight modification in notations only.

Here, $\omega$ and $\omega_c$ both are normalized by $\omega_{pi}$ and consequently for IA wave we have the following inequality: $\omega \ll \omega_c$. Using this inequality ($\omega \ll \omega_c$), the LDR (12) can be written as

$$\omega = k \frac{k^2 + h_1}{\omega_c^2} \left[ 1 + \sigma \gamma (k^2 + h_1) \right]^{-\frac{1}{2}},$$  \hfill (14)

or

$$\frac{\omega}{k} = n_2 \left[ \frac{k^2 (1 - n_2^2)}{\omega_c^2} + \frac{k^2 + h_1}{1 + \sigma \gamma (k^2 + h_1)} \right]^{-\frac{1}{2}}.$$ \hfill (15)

Now, the presence of $\omega_c$ in the LDR $\omega = \omega(k)$ as given in equation (14) or (15) shows that the IA wave is not free from the effect of the magnetic field. So, considering suitable stretchings of the spatial variables and time, and appropriate perturbation expansions of the dependent variables, the equations (1)–(5) along with the equation (6) can be used to investigate the MI of IA waves propagating along any arbitrary direction.

### 3 Derivation of the NLSE

Here, we have assumed that the characteristic frequency is much less than the ion cyclotron frequency and the particle pressure is much less than the magnetic pressure. So, in IA time scale, $\omega \ll \omega_c$, we have used this inequality to derive the relation (14) and the second condition gives that the plasma beta is small. Now we are going to consider the carrier waves propagating along a direction having direction cosines $(l_2, m_2, n_2)$. Then it is necessary to consider an axis $X$ along a direction having direction cosines $(l_2, m_2, n_2)$ and another axis $Y$, where $Y$ axis is perpendicular to the direction of $X$ axis as well as the direction of the magnetic field. Consequently each dependent variable depends on spatial variables $X$ and $Y$, and time $t$. If $(l_3, m_3, n_3)$ are direction cosines of $Y$ axis, then it is simple to check that the following results hold good: $l_3 = -\frac{m_3}{\sqrt{1 - n_3^2}}$, $m_3 = \frac{l_3}{\sqrt{1 - n_3^2}}$, $n_3 = 0$.

So, one can consider the following transformations of the axes: $X = l_2 x + m_2 y + n_2 z$ and $Y = -\frac{1}{\sqrt{1 - n_2^2}} (l_2 y - m_2 x)$.

But from these transformations, we see that $Y$ is undefined when $n_2^2 = 1$, i.e., when the carrier wave is propagating along the direction of the magnetic field. So, to avoid this difficulty, one can use the following basic results of geometry: $l_3 = \sqrt{1 - n_2^2} \cos \delta$ and $m_3 = \sqrt{1 - n_2^2} \sin \delta$ and consequently we have the following transformations: $X = \sqrt{1 - n_2^2} (x \cos \delta + y \sin \delta) + n_2 z$ and $Y = y \cos \delta - x \sin \delta$, where $\delta$ is the angle between the $x$–$z$ plane and the plane through the direction of propagation of the wave and the direction of the magnetic field. It is also important to note that the $Y$ axis is perpendicular to the direction of the magnetic field as well as the direction of propagation of the carrier wave. The dependence of the field variables on $Y$ arises due to anisotropy of the medium in presence of magnetic field. So, in general, one can take the following stretchings of the spatial coordinates and time:

$$\xi = \epsilon (X - V_g t), \quad \eta = \epsilon Y, \quad \tau = \epsilon^2 t,$$ \hfill (16)

where $\epsilon$ is a small parameter, $V_g$ is a constant and we have used exactly the same method as discussed in section III of the earlier paper of Dalui et al. (2017b) to find the stretched spatial variables $\xi$, $\eta$ and stretched time $\tau$ as given in equation (16).

To simplify the analysis, we have ignored the dependence of the field variables on $Y$, i.e., we have assumed that each dependent variable depends on spatial variable $X$ and time $t$. So, to include the magnetic field intensity and the obliqueness of the IA wave in the expression of the coefficient of the nonlinear term of the NLSE, we have used the following stretchings of the spatial coordinate and time (Murtaza and Salahuddin 1982; Jehan et al. 2008; Ghosh and Banerjee 2016):

$$\xi = \epsilon (X - V_g t) = \epsilon (l_2 x + m_2 y + n_2 z - V_g t), \quad \tau = \epsilon^2 t.$$ \hfill (17)

Similar types of stretchings of spatial coordinate and time as given in (17) have been used by several authors (Yadav and Sharma 1990; Mishra et al. 1994; Malik 1996; Cairns et al. 1996; Mamun 1998; Mamun et al. 2002; Ahmadihojatabad et al. 2010; Kadijani et al. 2010; Alinejad and Mamun 2011; Javidan and Pakzad 2013; Pakzad and Javidan 2013; Ferdousi et al. 2015; Khalid and Rahman 2019) to study the oblique propagation of waves in magnetized plasma, where they have ignored the weak dependence of the spatial coordinates perpendicular to the direction of the uniform static magnetic field.

We can take the following perturbation of the dependent field quantities:

$$G = G^{(0)} + \sum_{l=1}^{\infty} e^l \sum_{a=-\infty}^{\infty} G_{a}^{(l)}(\xi, \tau) \exp[i a \psi],$$ \hfill (18)

where $\psi = kX - \omega t = k (l_2 x + m_2 y + n_2 z) - \omega t$, $G = n, u, v, w, \phi$ with $(n^{(0)}, u^{(0)}, v^{(0)}, w^{(0)}, \phi^{(0)}) = (1, 0, 0, 0, 0)$.  

© Springer
Following Dalui et al. (2017b), we have considered the relation $G^{(l)}_a = \tilde{G}^{(l)}_a$, where ‘bar’ denotes the complex conjugate.

Following the same analysis of Dalui et al. (2017b), it is reasonable to take $\tilde{G}^{(l)}_a = 0$ for any $l$ and one can use the consistency conditions: $(n_0^{(l)}, u_0^{(l)}, v_0^{(l)}, w_0^{(l)}, \phi_0^{(l)}) = (0, 0, 0, 0, 0)$ and $(n_a^{(l)}, u_a^{(l)}, v_a^{(l)}, w_a^{(l)}, \phi_a^{(l)}) = (0, 0, 0, 0, 0)$ for $l < |a|$.

Substituting the stretchings (17) and the perturbation expansions of $n, u, v, w$ and $\phi$ in equations (1)–(5) and collecting the terms of different powers of $\epsilon$, one can generate a sequence of equations of different orders and finally, changing the values of $a$, it is possible to obtain different equations for different harmonics.

### 3.1 First order: $O(\epsilon) = 1$

Because of the first consistency condition, the zeroth harmonic ($a = 0$) equations of (1)–(5) are trivially satisfied.

Solving the first harmonic ($a = 1$) equations of (1)–(4), we get

\[ n_1^{(1)} = \frac{\Lambda}{1 - \sigma \gamma \Lambda} \phi_1^{(1)}, \]  
\[ u_1^{(1)} = \frac{\omega k l_2 + i \omega v k m_2}{\omega^2 - \omega_c^2} \frac{1}{1 - \sigma \gamma \Lambda} \phi_1^{(1)}, \]  
\[ v_1^{(1)} = \frac{\omega k m_2 - i \omega v k l_2}{\omega^2 - \omega_c^2} \frac{1}{1 - \sigma \gamma \Lambda} \phi_1^{(1)}, \]  
\[ w_1^{(1)} = \frac{k n_2}{\omega} \frac{1}{1 - \sigma \gamma \Lambda} \phi_1^{(1)}, \]

where

\[ \Lambda = \frac{k^2(\omega^2 - n_c^2 \omega_c^2)}{\omega^2(\omega^2 - \omega_c^2)}. \]  

From equation (5) at the first harmonic ($a = 1$), we get

\[ n_1^{(1)} = (k^2 + h_1) \phi_1^{(1)}. \]  

Therefore, from equations (19) and (24), we get the following dispersion relation

\[ \frac{k^2}{\omega^2 - \omega_c^2} + \frac{h_1}{\omega^2} = \frac{k^2 + h_1}{1 + \sigma \gamma (k^2 + h_1)}, \]  

where we have used the expression of $\Lambda$ as given in (23) to get the equation (25). The equation (25) is exactly same as the LDR (12) and this equation will assume the LDR (14) or equivalently the equation (15) if we use the condition $\omega \ll \omega_c$ for the IA mode.

### 3.2 Second order: $O(\epsilon) = 2$

#### 3.2.1 First harmonic ($a = 1$)

Solving the first harmonic ($a = 1$) equations of (1)–(4), we get

\[ n_1^{(2)} = \frac{\Lambda}{1 - \sigma \gamma \Lambda} \phi_1^{(2)} + \left[ \frac{V_g k [\omega^4 - n_c^2 \omega_c^2 (2\omega^2 - \omega_c^2)]}{\omega^2(\omega^2 - \omega_c^2)(1 - \sigma \gamma \Lambda)^2} \right] \frac{\partial \phi_1^{(1)}}{\partial \xi}. \]  

\[ u_1^{(2)} = \frac{\omega k l_2}{\omega^2 - \omega_c^2} \frac{\phi_1^{(2)}}{\phi_1^{(1)} + \sigma \gamma n_1^{(2)}} \frac{\partial \phi_1^{(1)}}{\partial \xi}, \]  

\[ v_1^{(2)} = \frac{\omega k m_2 - i \omega v k l_2}{\omega^2 - \omega_c^2} \frac{\phi_1^{(2)}}{\phi_1^{(1)} + \sigma \gamma n_1^{(2)}} \frac{\partial \phi_1^{(1)}}{\partial \xi}, \]  

\[ w_1^{(2)} = \frac{k n_2}{\omega} \frac{\phi_1^{(2)}}{\phi_1^{(1)} + \sigma \gamma n_1^{(2)}} \frac{i(V_g k - \omega) n_2}{\omega^2(1 - \sigma \gamma \Lambda)} \frac{\partial \phi_1^{(1)}}{\partial \xi}. \]

From first harmonic ($a = 1$) equation obtained from (5), we get

\[ n_1^{(2)} = (k^2 + h_1) \phi_1^{(2)} - 2 i k \frac{\partial \phi_1^{(1)}}{\partial \xi}. \]  

From equations (26) and (30), we get

\[ \left[ \frac{\Lambda}{1 - \sigma \gamma \Lambda} - (k^2 + h_1) \right] \phi_1^{(2)} + \frac{\partial}{\partial \xi} \left[ \frac{V_g}{k [\omega^4 - n_c^2 \omega_c^2 (2\omega^2 - \omega_c^2)]} \frac{(\omega^2 - \omega_c^2)^2 (1 - \sigma \gamma \Lambda)^2}{\omega^2(\omega^2 - \omega_c^2)} \right] \frac{\partial \phi_1^{(1)}}{\partial \xi} = 0, \]  

where

\[ A = \frac{2k^2}{\omega^3} \left[ \frac{\omega^4 - n_c^2 \omega_c^2 (2\omega^2 - \omega_c^2)}{(\omega^2 - \omega_c^2)^2 (1 - \sigma \gamma \Lambda)^2} \right]. \]
Because of the LDR (25), the coefficient of $\phi_1^{(2)}$ in (31) is equal to zero. So, the equation (31) will be identically satisfied if we take

$$V_g = \frac{\omega}{k} \left[ \frac{(\omega^2 - \omega_c^2)(\omega^2 - n_2^2\omega_c^2)}{[\omega^2 - n_2^2\omega_c^2(2\omega^2 - \omega_c^2)]} \right] - \frac{\omega^2(\omega^2 - \omega_c^2)^2}{\omega_c^2(2\omega^2 - \omega_c^2)} \left( 1 - \sigma \gamma A \right)^2. \tag{33}$$

Now differentiating the LDR (25) with respect to $k$ and using the relation (23), we get

$$\frac{\partial \omega}{\partial k} = \frac{\omega}{k} \left[ \frac{(\omega^2 - \omega_c^2)(\omega^2 - n_2^2\omega_c^2)}{[\omega^2 - n_2^2\omega_c^2(2\omega^2 - \omega_c^2)]} \right] - \frac{\omega^2(\omega^2 - \omega_c^2)^2}{\omega_c^2(2\omega^2 - \omega_c^2)} \left( 1 - \sigma \gamma A \right)^2. \tag{34}$$

So, the equation (31) will be trivially satisfied if $V_g = \frac{\partial \omega}{\partial k}$, i.e., if $V_g$ coincides with the group velocity.

Again, it is simple to check that the expression of $V_g$ as given in equation (33) is exactly same as the expression of $V_g$ as given in equation (32) of Dalui et al. (2017b) if we put $\theta = 0$, i.e., $n_2 = 1$ ($\Leftrightarrow l_2^2 + m_2^2 = 0 \Leftrightarrow l_2 = m_2 = 0$).

### 3.2.2 Second harmonic ($a = 2$)

Solving the second harmonic ($a = 2$) equations obtained from (1)–(5), we get

$$\begin{align*}
(\phi_2^{(2)}, n_2^{(2)}, u_2^{(2)}, v_2^{(2)}, w_2^{(2)}) &= (C_{\phi}, C_n, C_u, C_v, C_w)[\phi_1^{(1)}]^2, \tag{35}
\end{align*}$$

where $C_{\phi}, C_n, C_u, C_v, C_w$ are given in Appendix A.

### 3.2.3 Zeroth harmonic ($a = 0$)

Solving the zeroth harmonic ($a = 0$) equations obtained from (1)–(5), we get

$$\begin{align*}
(\phi_0^{(2)}, n_0^{(2)}, u_0^{(2)}, v_0^{(2)}, w_0^{(2)}) &= (D_{\phi}, D_n, D_u, D_v, D_w)[\phi_1^{(1)}]^2, \tag{36}
\end{align*}$$

where $D_{\phi}, D_n, D_u, D_v, D_w$ are given in Appendix B.

### 3.3 Third order ($O(\epsilon) = 3$): first harmonic ($a = 1$)

Solving the first harmonic ($a = 1$) equations obtained from (1)–(4), we get

$$n_1^{(3)} = \frac{A}{1 - \sigma \gamma A} \phi_1^{(3)} - iA \frac{\partial \phi_1^{(1)}}{\partial \tau} + Q_1|\phi_1^{(1)}|^2 \phi_1^{(1)} + iA \left[ V_g - \frac{\omega((\omega^2 - \omega_c^2)(\omega^2 - n_2^2\omega_c^2))}{k[\omega^2 - n_2^2\omega_c^2(2\omega^2 - \omega_c^2)]} \right] \frac{\partial \phi_1^{(2)}}{\partial \xi}$$

$$- \left[ \frac{P_1 P_2 + P_3 + P_4 + P_5}{1 - \sigma \gamma A} \right] \frac{\partial^2 \phi_1^{(1)}}{\partial \xi^2}, \tag{37}$$

where $P_1, P_2, P_3, P_4, P_5, Q_1$ are given in Appendix C.

Again, from the third order first harmonic equation obtained from (5), we get

$$n_1^{(3)} = (k^2 + h_1)\phi_1^{(3)} - 2ik \frac{\partial \phi_1^{(2)}}{\partial \xi} - \frac{\partial^2 \phi_1^{(1)}}{\partial \xi^2}$$

$$+ \left[ 2h_2(C_{\phi} + D_{\phi}) + 3h_3 \right]|\phi_1^{(1)}|^2 \phi_1^{(1)}. \tag{38}$$

Now, eliminating $n_1^{(3)}$ from (37) and (38), we get

$$iA \frac{\partial \phi_1^{(1)}}{\partial \tau} + P \frac{\partial^2 \phi_1^{(1)}}{\partial \xi^2} + Q|\phi_1^{(1)}|^2 \phi_1^{(1)} = 0, \tag{39}$$

where

$$P = - \frac{1}{A} \left[ 1 - \frac{P_1 P_2 + P_3 + P_4 + P_5}{1 - \sigma \gamma A} \right], \tag{40}$$

$$Q = - \frac{1}{A} \left[ Q_1 - 3h_3 - 2h_2(C_{\phi} + D_{\phi}) \right]. \tag{41}$$

Here we have removed the terms $\phi_1^{(3)}$ and $\frac{\partial \phi_1^{(2)}}{\partial \xi}$ using the LDR (25) and the compatibility condition (33) respectively to simplify the equation (39).

It is simple to check that the expressions of $P$ and $Q$ as given in equations (40) and (41) are, respectively, same as the expressions of $P$ and $Q$ as given in equations (45) and (46) of Dalui et al. (2017b) if we put $\theta = 0$, i.e., $n_2 = 1$ ($\Leftrightarrow l_2^2 + m_2^2 = 0 \Leftrightarrow l_2 = m_2 = 0$). Therefore, the NLSE (39) of the present paper is exactly same as the NLSE (44) of Dalui et al. (2017b) if we remove the fourth term of equation (44) of Dalui et al. (2017b). In fact, the fourth term of equation (44) of Dalui et al. (2017b) is responsible for the weak dependence of the spatial coordinates perpendicular to the direction of the uniform static magnetic field as described by the first and second equations of (18) of Dalui et al. (2017b).

One can easily check that the expressions of $P$ and $Q$ as given in equations (60) and (61) of Dalui et al. (2017a) can also be obtained from the expressions of $P$ and $Q$ as given in equations (40) and (41) if we put $\theta = 0$ and $\gamma = 3$ and consequently the NLSE (59) of Dalui et al. (2017a) is exactly same as the NLSE (39) of the present paper for $\theta = 0$ and $\gamma = 3$. But Dalui et al. (2017a) have considered the MI.
of IA wave in a collisionless unmagnetized plasma whose constituents are exactly same as the present problem. So, in connection of MI, we can get all the results of unmagnetized plasma from the magnetized one.

4 Modulational instability

Following the same method as discussed in section IV of the paper of Dalui et al. (2017a,b), we have the following expression of \( \Omega^2 \):

\[
\Omega^2 = \left[ P K^2 \right]^2 \left( 1 - \frac{2Q|\phi_0|^2}{P K^2} \right). 
\] (42)

where \( \Omega \) and \( K \) are, respectively, the wave frequency and wave number of the modulated IA wave.

From this expression of \( \Omega^2 \), we can conclude the following points regarding the stability of the modulated IA waves:

- If \( PQ < 0 \), then \( \Omega^2 \) is strictly positive and consequently the IA wave is always modulationally stable.
- If \( PQ > 0 \), the IA wave is modulationally stable or unstable according as \( K \geq K_c \) or \( K < K_c \), where \( K_c = \sqrt{\frac{2Q|\phi_0|^2}{P}} \).
- For modulationally unstable IA wave, the analytical expression of the growth rate of instability \( [\Gamma (= Im(\Omega))] \) can be written in the following form:

\[
\Gamma^2 = \left[ P K^2 \right]^2 \left( \frac{2Q|\phi_0|^2}{PK^2} - 1 \right). 
\] (43)

- If the IA wave is modulationally unstable, \( \Gamma \) attains its maximum value \( \Gamma_{\text{max}} \) at \( K = \frac{K_c}{\sqrt{2}} = \sqrt{\frac{Q|\phi_0|^2}{P}} \), where \( \Gamma_{\text{max}} = |Q||\phi_0|^2 \).

For obliquely propagating IA waves in a magnetized plasma, we have \( \omega \ll \omega_c \), and consequently the LDR of IA wave is given by the equation (15). Using the equation (15), one can find that both the group velocity \( (V_g) \) and the phase velocity \( (\frac{\omega}{k}) \) decreases for increasing \( k \).

The MI of obliquely propagating IA waves depends on the coefficients of dispersive and nonlinear terms of the NLSE (39) and we have the following three conditions regarding the stability of the modulated IA waves: (i) modulated IA wave is stable if \( PQ < 0 \), (ii) modulated IA wave is stable if \( K \geq K_c \) whenever \( PQ > 0 \) and (iii) modulated IA wave is unstable if \( K < K_c \) whenever \( PQ > 0 \).

Here \( PQ \) depends on the parameters \( \sigma, \gamma, k, \beta_c, n_{sc}, \sigma_{sc}, \omega_c \) and \( n_2 \). Therefore, \( PQ \) can be taken as a function of \( k \) and \( n_2 \) for fixed values of the other parameters \( \sigma, \gamma, \beta_c, n_{sc}, \sigma_{sc} \) and \( \omega_c \). So, mathematically, we get a relation between \( k \) and \( n_2 \) if we set \( PQ = 0 \). This mathematical relationship between \( k \) and \( n_2 \) has been plotted in Fig. 1. From Fig. 1, we see that within the intervals \( 0 \leq k \leq 0.266 \) and \( 0.31 \leq k \leq 0.956 \), there is no relation between \( k \) and \( n_2 \) for \( PQ = 0 \). Here, we have used the terminology \( N \) to indicate that we are unable to determine the relationship between \( k \) and \( n_2 \) for \( PQ = 0 \) for the entire indicated rectangular regions enclosed by \( 0 \leq k \leq 0.266 \) and \( 0 \leq n_2 \leq 1 \), and \( 0.31 \leq k \leq 0.956 \) and \( 0 \leq n_2 \leq 1 \). In particular, in the rectangular region \( (0 \leq k \leq 0.266, 0 \leq n_2 \leq 1) \), the function \( PQ \) is strictly negative, i.e., \( PQ < 0 \). Therefore, within that region, there is no relation between \( k \) and \( n_2 \) for \( PQ = 0 \). To confirm our analysis, we draw Fig. 2 to explain the behavior of \( PQ \) within the rectangular region of \( k – n_2 \) plane defined by \( 0 \leq k \leq 0.266 \) and \( 0 \leq n_2 \leq 1 \). In Fig. 2, \( PQ \) has been plotted against \( k \) for different values of \( n_2 \) with \( \sigma = 0.001, \gamma = 5/3, \beta_c = 0.1, n_{sc} = 0.9, \sigma_{sc} = 0.25 \) and \( \omega_c = 0.3 \). From Fig. 2, we see that \( PQ < 0 \) for all \( 0 \leq k \leq 0.266 \) with \( n_2 = 0.25, n_2 = 0.5, n_2 = 0.7 \) and \( n_2 = 0.86 \). So, in the interval \( 0 \leq k \leq 0.266, PQ < 0 \) for any physically admissible value of \( n_2 \) and for this reason there is no relationship between \( k \) and \( n_2 \) when \( PQ = 0 \) within the interval \( 0 \leq k \leq 0.266 \). Finally, from Figs. 1 and 2, we see that there exists a region (an interval or union of more than one interval) of \( k \) such that \( PQ < 0 \) for any set of given values of the parameters and consequently, MI of the obliquely prop-
agating IA waves are stable for any value of \( k \) lying in that region of \( k \).

In Figs. 3 and 4, \( n_2 \) has been plotted against \( k \) when \( PQ = 0 \) for different values of \( \beta_c \). In fact, in Fig. 3, \( n_2 \) has been plotted against \( k \) when \( PQ = 0 \) for (a) \( \beta_c = 0 \), (b) \( \beta_c = 0.1 \), (c) \( \beta_c = 0.15 \) and (d) \( \beta_c = 0.2 \) whereas in Fig. 4, \( n_2 \) has been plotted against \( k \) when \( PQ = 0 \) for (a) \( \beta_c = 0.35 \), (b) \( \beta_c = 0.4 \), (c) \( \beta_c = 0.5 \) and (d) \( \beta_c = 0.57 \). From Fig. 3, we see that the interval of \( k \) for the existence of the curve described by the relation \( PQ = 0 \) in \( k - n_2 \) plane increases with increasing \( \beta_c \) and it is simple to check that there exists a critical value \( \beta_c^{(e)} = 0.2296 \) (approximately) of \( \beta_c \) such that the interval of \( k \) for the existence of the curve described by \( PQ = 0 \) in \( k - n_2 \) plane increases with increasing \( \beta_c \) for \( 0 \leq \beta_c < \beta_c^{(e)} \). Also, from Fig. 3, we can conclude that the stable region of modulated IA waves described by the inequality \( PQ < 0 \) increases with increasing \( \beta_c \) for \( 0 \leq \beta_c < \beta_c^{(e)} \). From Fig. 4, we see that the interval of \( k \) for the existence of the curve defined by \( PQ = 0 \) decreases with increasing \( \beta_c \) and it is simple to check that the interval of \( k \) for the existence of the curve obtained from the relation \( PQ = 0 \) decreases with increasing \( \beta_c \) for \( \beta_c^{(c)} < \beta_c < \beta_c^{(e)} \). Also, from Fig. 4, we see that both the regions \( PQ < 0 \) and \( PQ > 0 \) decreases with increasing \( \beta_c \) for \( \beta_c^{(c)} < \beta_c < \beta_c^{(e)} \). So, we can conclude that the stable region of modulated IA wave described by the inequality \( PQ < 0 \) decreases with increasing \( \beta_c \) for \( \beta_c^{(c)} < \beta_c < \beta_c^{(e)} \). But for both the figures (Figs. 3-4), for any point \((k, n_2)\) lying within the region described by the inequality \( PQ > 0 \), the modulated IA wave is stable or unstable according as \( K \geq K_c \) or \( K < K_c \).

In Fig. 5, \( n_2 \) has been plotted against \( k \) when \( PQ = 0 \) for different values of \( n_{sc} \): (a) \( n_{sc} = 0.01 \), (b) \( n_{sc} = 0.3 \), (c) \( n_{sc} = 0.6 \) and (d) \( n_{sc} = 0.9 \). From Fig. 5, we see that the curve in \( k - n_2 \) plane obtained from the relation \( PQ = 0 \) is not a continuous curve, there exist at least two discontinuities. In Fig. 5(a), we have seen that the curve \( PQ = 0 \) exist within the intervals \( 0.168 < k < 0.192 \) and \( 1.138 < k < 1.818 \). In Fig. 5(b), we have seen that the curve \( PQ = 0 \) exist within the intervals \( 0.172 < k < 0.198 \) and \( 0.718 < k < 1.59 \). In Fig. 5(c), we have seen that the curve \( PQ = 0 \) exist within the intervals \( 0.174 < k < 0.2 \) and \( 0.84 < k < 1.694 \). And finally in Fig. 5(d), we have seen that the curve \( PQ = 0 \) exist within the intervals \( 0.174 < k < 0.2 \) and...
and $0.956 < k < 1.768$. Therefore, Fig. 5 shows that the modulated IA wave is stable for any point $(k, n_2)$ lying within the region(s) described by the inequality $PQ < 0$. Again, for any point $(k, n_2)$ lying within the region defined by $PQ > 0$, modulated IA wave is stable or unstable according as $K \geq K_c$ or $K < K_c$.

In Fig. 6, $n_2$ has been plotted against $k$ when $PQ = 0$ for different values of $\omega_c$: (a) $\omega_c = 0.1$, (b) $\omega_c = 0.2$, (c) $\omega_c = 0.3$ and (d) $\omega_c = 0.4$. From this figure, we can conclude that the interval of $k$ for the existence of the curve defined by the relation $PQ = 0$ in $k - n_2$ plane decreases with increasing $\omega_c$. We also observe that the stable region of modulated IA waves defined by the inequality $PQ < 0$ increases with increasing $\omega_c$ whereas the region in $k - n_2$ plane described by $PQ > 0$ decreases with increasing $\omega_c$ and for any point $(k, n_2)$ lying within the region defined by $PQ > 0$, modulated IA wave is stable or unstable according as $K \geq K_c$ or $K < K_c$.

In Figs. 7(a), 7(b) and 7(c), we have plotted $P$, $Q$ and $\Gamma_{\text{max}}/|\phi_0|^2$ against $k$ respectively for different $\beta_e$ with $\gamma = 5/3$, $\sigma = 0.001$, $n_{sc} = 0.25$, $\sigma_{sc} = 0.25$, $\omega_c = 0.2$ and $n_2 = 0.25$. In Fig. 7(a) there is a zoomed region of the Fig. 7(a) within the rectangle: $0.6 \leq k \leq 2$, $0 \leq P \leq 0.0025$. From Fig. 7, we see that $\Gamma_{\text{max}}$ increases with increasing $\beta_e$. There are two different types of curves: Type-I and Type-II. Type-I looks like green curve whereas red and blue curves are of Type-II. In fact, there exists a certain value $\beta_e^{(c)} = 0.1286$ (approximately) of $\beta_e$ such that if $\beta_e < \beta_e^{(c)}$ then the curve is of Type-I and if $\beta_e > \beta_e^{(c)}$ then the curve is of Type-II. In this figure, we see that the green curve exists within the interval $0.816 < k < 2$ only and consequently, for $k$ lying within the interval $0 < k < 0.816$, $PQ < 0$, and consequently, modulated IA wave is stable. Again, for $k$ lying within the intervals $0 < k < 0.08$ and $0.594 < k < 0.752$, the red curve does not exist because within these intervals of $k$ ($0 < k < 0.08$ & $0.594 < k < 0.752$), $PQ < 0$, and consequently, modulated IA wave is stable. Also, for $k$ lying within the intervals $0 < k < 0.004$ and $0.546 < k < 0.69$, the blue curve does not exist because within these intervals of $k$ ($0 < k < 0.004$ & $0.546 < k < 0.69$), $PQ < 0$, i.e., the modulated IA wave is stable. From this figure, we see that there exists a critical value $k_c$ of $k$ such that for any fixed $k$ lying within the interval $0 < k \leq k_c$, $\Gamma_{\text{max}}$ (if exists) increases with increasing values of $\beta_e$. Again, for any fixed $\beta_e$, $\Gamma_{\text{max}}$ (if exists) increases with increasing $k$.

In Figs. 8(a), 8(b) and 8(c), we have plotted $P$, $Q$ and $\Gamma_{\text{max}}/|\phi_0|^2$ against $k$ respectively for different $n_{sc}$ with $\sigma = 0.001$, $\beta_e = 0.3$, $\sigma_{sc} = 0.25$, $\omega_c = 0.2$ and $n_2 = 0.25$. In Fig. 8(a) there is a zoomed region of the Fig. 8(a) within the rectangle: $0.67 \leq k \leq 2$, $0 \leq P \leq 0.002$. From this figure, we see that the green curve does not exist within the intervals $0 < k < 0.15$ and $0.468 < k < 0.688$ and consequently, for $k$ lying within the intervals $0 < k < 0.15$ and $0.468 < k < 0.688$, we have $PQ < 0$, and consequently, the modulated IA wave is stable. Again, for $k$ lying within the
intervals $0 < k < 0.08$ and $0.594 < k < 0.752$, the red curve does not exist because within these intervals of $k$, $PQ < 0$, and consequently the modulated IA wave is stable. Also, for the blue curve, for $k$ lying within the intervals $0 < k < 0.494$ and $0.658 < k < 0.788$, the modulated IA wave is stable. From this figure, we see that for any fixed $n_{sc}$, $\Gamma_{\text{max}}$ (if exists) increases with increasing $k$.

For $\beta_c = 0$, i.e., when both the electron species are isothermally distributed then for this case, in Figs. 9(a), 9(b) and 9(c), we have plotted $P$, $Q$ and $\Gamma_{\text{max}}/|\phi_0|^2$ against $k$ respectively for different $\omega_c$. From this figure, we see that the region of existence of $\Gamma_{\text{max}}$ decreases with increasing $\omega_c$. There exists a critical value $\omega_c^{(c)} = 0.4595$ (approximately) of $\omega_c$ such that $\Gamma_{\text{max}}$ does not exist for $\omega_c > \omega_c^{(c)}$. From this figure, we see that for any fixed $\omega_c$, $\Gamma_{\text{max}}$ (if exists) increases with increasing $k$ whereas for any fixed $k$, $\Gamma_{\text{max}}$ decreases with increasing $\omega_c$.

In Figs. 10(a), 10(b) and 10(c), we have plotted $P$, $Q$ and $\Gamma_{\text{max}}/|\phi_0|^2$ against $k$ respectively for different $n_2$ with $\sigma = 0.001$, $\beta_0 = 0.1$, $\sigma_{sc} = 0.25$, $\omega_c = 0.2$ and $n_{sc} = 0.25$. From this Fig. 10, we see that the green curve exists within the intervals $0.427 < k < 0.507$ and $0.662 < k < 2$ as $PQ > 0$ for all values of $k$ lying within the interval $0.427 < k < 0.507$ or $0.662 < k < 2$. For all values of $k$ lying within the interval $0.507 < k < 0.662$, $PQ < 0$, and consequently, the modulated IA wave is stable for $0.507 < k < 0.662$. From Fig. 10, for any fixed value of $k$, $\Gamma_{\text{max}}$ (if exists) increases with increasing $n_2$ whereas for any fixed value of $n_2$, $\Gamma_{\text{max}}$ increases with increasing $k$.

For larger values of $\omega_c$, we have observed the interesting change in the maximum growth rate of instability. In fact, in general, we have seen that there are two critical values $\omega_{c1}$ and $\omega_{c2}$ of $\omega_c$ such that the maximum growth rate of instability decreases with increasing $\omega_c$ for $0 < \omega_c < \omega_{c1}$ whereas the maximum growth rate of instability increases with increasing $\omega_c$ for $\omega_{c1} < \omega_c < \omega_{c2}$ but for $\omega_{c2} < \omega_c < 1$, again, the maximum growth rate of instability decreases with increasing $\omega_c$. Therefore, the increase in the strength of the magnetic field tends to destabilize the wave only when $\omega_c$ lies within the interval $\omega_{c1} < \omega_c < \omega_{c2}$ whereas increase in the strength of the magnetic field tends to stabilize the wave when $\omega_c$ is restricted by the inequalities $0 < \omega_c < \omega_{c1}$ or $\omega_{c2} < \omega_c < 1$. This fact is clear from Fig. 11. From Fig. 11(b), it is clear that the values of $\omega_{c1}$ and $\omega_{c2}$ are, respectively, 0.575 and 0.77 for $k = 1.5$ and the other fixed values of the parameters as given in the figure. Similarly, for $k = 1.25$, one can find $\omega_{c1} = 0.555$ and $\omega_{c2} = 0.83$ whereas for $k = 1$, the critical values of $\omega_c$ are...
is exactly same as those properties which have been pointed earlier by Ghosh and Banerjee (2016).

The dependence of different parameters of the present plasma system on \( \omega_{c1} \) and \( \omega_{c2} \) can be described as follows:

(A) Dependence of the nonthermal parameter \( \beta_e \): Here we have observed that \( \omega_{c1} \) increases with increasing \( \beta_e \) whereas \( \omega_{c2} \) decreases with increasing \( \beta_e \) and consequently \( \omega_{c2} - \omega_{c1} \) decreases for increasing \( \beta_e \). Therefore, the values of \( \omega_{c1} \) and \( \omega_{c2} \) are also dependent on the nonthermal parameter \( \beta_e \).

For \( k = 1.5, n_{sc} = 0.9, \sigma_{sc} = 0.25 \) and \( n_2 = 0.5 \), the values of \( \omega_{c1} \) and \( \omega_{c2} \) are, respectively, 0.555 and 0.845 for \( \beta_e = 0 \) whereas the values of \( \omega_{c1} \) and \( \omega_{c2} \) are, respectively, 0.575 and 0.77 for \( \beta_e = 0.3 \), and also the values of \( \omega_{c1} \) and \( \omega_{c2} \) are, respectively, 0.621 and 0.677 for \( \beta_e = 0.57 \). It is important to note that the physically admissible range of \( \beta_e \) is \( 0 \leq \beta_e \leq \frac{4}{3} \) although mathematically \( \beta_e \) is restricted by the inequality \( 0 \leq \beta_e \leq \frac{2}{3} \) (Verheest and Pillay 2008).

(B) Dependence of \( n_{sc} \): There exists a critical value \( n_{sc}^c \) of \( n_{sc} \) such that for \( n_{sc} > n_{sc}^c \), there does not exist any critical value of \( \omega_{c} \), in other words, the maximum growth rate of instability decreases with increasing \( \omega_{c} \) for all physically admissible continuous range of \( \omega_{c} \). But for \( n_{sc} \geq n_{sc}^c \), the maximum growth rate of instability increases with increasing \( \omega_{c} \) for \( \omega_{c1} < \omega_{c} < \omega_{c2} \) whereas the maximum growth rate of instability decreases with increasing \( \omega_{c} \) for \( 0 < \omega_{c} < \omega_{c1} \) as well as for \( \omega_{c2} < \omega_{c} < 1 \). For \( k = 1.5, \beta_e = 0.3, \sigma_{sc} = 0.25 \) and \( n_2 = 0.5 \), the value of \( n_{sc}^c \) is equal to 0.512 (approx.). The values of \( \omega_{c1} \) are, respectively, 0.644, 0.592 and 0.575 for \( n_{sc} = 0.513, n_{sc} = 0.7 \) and \( n_{sc} = 0.9 \) whereas the values of \( \omega_{c2} \) are, respectively, 0.652, 0.732 and 0.77 for \( n_{sc} = 0.513, n_{sc} = 0.7 \) and \( n_{sc} = 0.9 \). Therefore, \( \omega_{c1} \) decreases with increasing \( n_{sc} \) whereas \( \omega_{c2} \) increases with increasing \( n_{sc} \), and consequently \( \omega_{c2} - \omega_{c1} \) increases for increasing \( n_{sc} \).

\( \omega_{c1} = 0.502 \) and \( \omega_{c2} = 0.874 \). Therefore, we find that the values \( \omega_{c2} - \omega_{c1} \) are, respectively, 0.372, 0.275 and 0.205 for \( k = 1, k = 1.25 \) and \( k = 1.5 \), which shows that \( \omega_{c2} - \omega_{c1} \) decreases for increasing \( k \). Therefore, we see that the qualitative behavior regarding the instability condition of the modulated IA waves on the strength of the magnetic field

is exactly same as those properties which have been pointed earlier by Ghosh and Banerjee (2016).

The dependence of different parameters of the present plasma system on \( \omega_{c1} \) and \( \omega_{c2} \) can be described as follows:

(A) Dependence of the nonthermal parameter \( \beta_e \): Here we have observed that \( \omega_{c1} \) increases with increasing \( \beta_e \) whereas \( \omega_{c2} \) decreases with increasing \( \beta_e \) and consequently \( \omega_{c2} - \omega_{c1} \) decreases for increasing \( \beta_e \). Therefore, the values of \( \omega_{c1} \) and \( \omega_{c2} \) are also dependent on the nonthermal parameter \( \beta_e \).

For \( k = 1.5, n_{sc} = 0.9, \sigma_{sc} = 0.25 \) and \( n_2 = 0.5 \), the values of \( \omega_{c1} \) and \( \omega_{c2} \) are, respectively, 0.555 and 0.845 for \( \beta_e = 0 \) whereas the values of \( \omega_{c1} \) and \( \omega_{c2} \) are, respectively, 0.575 and 0.77 for \( \beta_e = 0.3 \), and also the values of \( \omega_{c1} \) and \( \omega_{c2} \) are, respectively, 0.621 and 0.677 for \( \beta_e = 0.57 \). It is important to note that the physically admissible range of \( \beta_e \) is \( 0 \leq \beta_e \leq \frac{4}{3} \) although mathematically \( \beta_e \) is restricted by the inequality \( 0 \leq \beta_e \leq \frac{2}{3} \) (Verheest and Pillay 2008).

(B) Dependence of \( n_{sc} \): There exists a critical value \( n_{sc}^c \) of \( n_{sc} \) such that for \( n_{sc} > n_{sc}^c \), there does not exist any critical value of \( \omega_{c} \), in other words, the maximum growth rate of instability decreases with increasing \( \omega_{c} \) for all physically admissible continuous range of \( \omega_{c} \). But for \( n_{sc} \geq n_{sc}^c \), the maximum growth rate of instability increases with increasing \( \omega_{c} \) for \( \omega_{c1} < \omega_{c} < \omega_{c2} \) whereas the maximum growth rate of instability decreases with increasing \( \omega_{c} \) for \( 0 < \omega_{c} < \omega_{c1} \) as well as for \( \omega_{c2} < \omega_{c} < 1 \). For \( k = 1.5, \beta_e = 0.3, \sigma_{sc} = 0.25 \) and \( n_2 = 0.5 \), the value of \( n_{sc}^c \) is equal to 0.512 (approx.). The values of \( \omega_{c1} \) are, respectively, 0.644, 0.592 and 0.575 for \( n_{sc} = 0.513, n_{sc} = 0.7 \) and \( n_{sc} = 0.9 \) whereas the values of \( \omega_{c2} \) are, respectively, 0.652, 0.732 and 0.77 for \( n_{sc} = 0.513, n_{sc} = 0.7 \) and \( n_{sc} = 0.9 \). Therefore, \( \omega_{c1} \) decreases with increasing \( n_{sc} \) whereas \( \omega_{c2} \) increases with increasing \( n_{sc} \), and consequently \( \omega_{c2} - \omega_{c1} \) increases for increasing \( n_{sc} \).

\( \omega_{c1} = 0.502 \) and \( \omega_{c2} = 0.874 \). Therefore, we find that the values \( \omega_{c2} - \omega_{c1} \) are, respectively, 0.372, 0.275 and 0.205 for \( k = 1, k = 1.25 \) and \( k = 1.5 \), which shows that \( \omega_{c2} - \omega_{c1} \) decreases for increasing \( k \). Therefore, we see that the qualitative behavior regarding the instability condition of the modulated IA waves on the strength of the magnetic field

is exactly same as those properties which have been pointed earlier by Ghosh and Banerjee (2016).
We have seen that there exist two critical values $n_{c1}^2$ and $n_{c2}^2$ of $n_2$ such that for $n_2 < n_{c1}^2$, we have found two critical values $\omega_{c1}$ and $\omega_{c2}$ of $\omega_c$ such that the maximum growth rate of instability increases with increasing $\omega_c$ for $\omega_{c1} < \omega_c < \omega_{c2}$ whereas the maximum growth rate of instability decreases with increasing $\omega_c$ for $0 < \omega_c < \omega_{c1}$ as well as for $\omega_{c2} < \omega_c < 1$. For $n_{c1}^2 < n_2 < n_{c2}^2$, there does not exist any critical value of $\omega_c$. In other words, the maximum growth rate of instability decreases with increasing $\omega_c$ for all physically admissible continuous range of $\omega_c$ whereas for $n_2 > n_{c2}^2$, there may exist more than one maximum (minimum) values of the curve $\Gamma_{\max}/|\phi_0|^2$ when it is plotted against $\omega_c$. For example, for $k = 1.5$, $\beta_e = 0.3$, $n_{sc} = 0.9$, $\sigma_{sc} = 0.25$, the critical values $n_{c1}^2$ and $n_{c2}^2$ are, respectively, 0.64 and 0.74 (approx.). The dependence of $n_2$ on the optimum values of the curve $\Gamma_{\max}/|\phi_0|^2$ against $\omega_c$ can be easily described by Fig. 12. This figure shows that $\Gamma_{\max}$ increases with increasing $\omega_c$ for $\omega_{c1} < \omega_c < \omega_{c2}$ and also for $\omega_{c3} < \omega_c < \omega_{c4}$, where $\omega_{c1} = 0.202$, $\omega_{c2} = 0.284$, $\omega_{c3} = 0.958$ and $\omega_{c4} = 0.983$. There are actually two intervals of $\omega_c$ where the maximum growth rate of instability increases with increasing $\omega_c$. Again, it is important to note that there is a singularity of $\Gamma_{\max}$ at $\omega_{c2} = 0.29$ (approx.). If we increase the value of $n_2 = 0.75$, then there are more than two intervals of $\omega_c$ where the maximum growth rate of instability increases with increasing $\omega_c$.

5 Conclusions

In this paper, we have studied the MI of obliquely propagating IA waves in a magnetized collisionless plasma consisting of adiabatic warm ions and a superposition of isothermal cold electrons and nonthermal hot electrons, immersed in a uniform static magnetic field.

Analytically we have observed the following results:

- The LDR of the IA wave of Dalui et al. (2017b) can be obtained from the LDR of the present paper if we put $\theta = 0$ whereas the LDR of the IA wave of Dalui et al. (2017a) can be obtained from the LDR of the present paper if we put $\theta = 0$ along with $\gamma = 3$.
- The group velocity of the IA wave propagating at an arbitrary angle to the direction of the magnetic field is exactly same as that of the IA wave propagating parallel to the direction of the magnetic field if $\theta = 0$.
- The NLSE of IA waves of Dalui et al. (2017a) can also be obtained from the NLSE of the present paper if we put $\theta = 0$ and $\gamma = 3$. So, all the results of Dalui et al. (2017a) can be obtained from the present paper. So, in connection of MI, one can get all the results of unmagnetized plasma from the magnetized one.
- The NLSE of IA waves of Dalui et al. (2017b) can also be obtained from the NLSE of the present paper if we put $\theta = 0$ and if we remove the fourth term of equation (44) of Dalui et al. (2017b). In fact, the fourth term of equation (44) of Dalui et al. (2017b) is responsible for the weak dependence of the spatial coordinates perpendicular to the direction of propagation of the wave as described by the first and second equations of (18) of Dalui et al. (2017b). Therefore, analytically it is possible to establish a relation between the MI of IA waves propagating at an arbitrary angle to the direction of the magnetic field and the MI of IA waves propagating parallel to the direction of the magnetic field.
- Considering appropriate values or limiting values of the parameters of the present plasma system, it can be shown that our results correspond to the results of Murtaza and Salahuddin (1982) and Jehan et al. (2008). The results of Ghosh and Banerjee (2016) can also be compared with our result by taking $\beta_e = 0$, $q \rightarrow 1$ and $n_{sc} \rightarrow \infty$ along with positron concentration is equal to 0, where $q$ is the nonextensive parameter associated with the $q$-nonextensive distribution of electron of Ghosh and Banerjee (2016).

Numerically we have seen the following points:

- The phase velocity ($\omega/k$) is a decreasing function of $k$ for the fixed values of other parameters but for a fixed value of $k$, (i) it increases with increasing $\omega_c$, (ii) it increases with increasing $\beta_e$ and (iii) it also increases with increasing $n_2$. Similar observations can also be obtained for the group velocity ($V_g$).
- The increase in the strength of the magnetic field tends to destabilize the wave when $\omega_c$ lies within the interval $\omega_{c1} < \omega_c < \omega_{c2}$ whereas the increase in the strength of the magnetic field tends to stabilize the wave when $\omega_c$
is restricted by the inequalities $0 < \omega_c < \omega_{c1}$ or $\omega_{c2} < \omega_c < 1$, where $\omega_{c1}$ and $\omega_{c2}$ are two critical values of $\omega_c$. But there may exist more than two intervals of $\omega_c$ where the maximum growth rate of instability increases with increasing $\omega_c$. We have extensively investigated the effects of different parameters of the present plasma system on the critical values of $\omega_c$. Therefore, the qualitative behavior regarding the instability condition of the modulated IA waves on the strength of the magnetic field is exactly same as those properties which have been pointed earlier by Ghosh and Banerjee (2016).

- There are two types of regions of the parameter space where the modulated IA waves are stable, viz., (i) the region defined by the inequality $PQ < 0$ and (ii) the region defined by the inequality $PQ > 0$ along with the condition $K \geq K_c$. We have extensively studied both the regions described by the inequalities $PQ < 0$ and $PQ > 0$. We have observed the following facts: (a) the stable region of modulated IA waves defined by the inequality $PQ < 0$ increases with increasing $\omega_c$ if $0 < \omega_c < \omega_{c1}$ or $\omega_{c2} < \omega_c < 1$, whereas this stable region of modulated IA waves described by the inequality $PQ < 0$ decreases with increasing $\omega_c$ for $0 < \omega_c < \omega_{c1}$ or $\omega_{c2} < \omega_c < 1$ and (b) the stable region of modulated IA waves described by the inequality $PQ < 0$ increases with increasing $\omega_c$ for $0 < \omega_c < \omega_{c1}$ or $\omega_{c2} < \omega_c < 1$, whereas this stable region of modulated IA waves described by the inequality $PQ < 0$ decreases with increasing $\omega_c$ for $0 < \omega_c < \omega_{c1}$ or $\omega_{c2} < \omega_c < 1$.

- The maximum growth rate of instability decreases with increasing $\cos \theta$.

In this paper we have not considered the effect of Landau damping on the MI of obliquely propagating IA waves. In unmagnetized collisionless plasma, effect of wave-particle interaction on MI of electrostatic wave was investigated by Ichikawa and his collaborators (Ichikawa et al. 1972; Ichikawa and Taniuti 1973; Ichikawa et al. 1973; Ichikawa 1974; Weiland et al. 1978). Chatterjee and Misra (2015) have investigated nonlinear Landau damping and modulation of electrostatic waves in a nonextensive collisionless unmagnetized electron-positron-pair plasma by considering the Vlasov-Poisson model. To consider the kinetic effects of both electron and ion species in magnetized plasma, it is necessary to consider full kinetic description of ions and electrons, but this problem is beyond the scope of the present paper.

Finally, one can consider the same problem by considering the stretchings of the spatial coordinates and time as given in equation (16) to investigate the weak dependence of the spatial coordinates perpendicular to the direction of the external uniform static magnetic field.

**Acknowledgements**  The authors are grateful to the reviewer for his constructive comments, without which this paper could not have been written in its present form. The authors are grateful to Prof. Basudeb Ghosh, Department of Physics, Jadavpur University for his helpful suggestions.

**Publisher’s Note**  Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.

### Appendix A

\[ C_{\phi} = \frac{2h_2(1 - \sigma \gamma \chi)(1 - \sigma \gamma \Lambda)^2 - C_1 A^2}{2C_2}, \]
\[ C_\sigma = (4k^2 + h_1)C_{\phi} + h_2, \quad C_u = \omega b_1 C_\sigma + i \omega m_2 C_s, \]
\[ C_v = \omega m_2 C_s - i \omega b_1 C_s, \quad \chi = \frac{k^2(4\omega^2 - n^2\omega_c^2)}{\omega^2(4\omega^2 - \omega_c^2)}, \]
\[ C_w = \frac{k h_2}{12}\left[ C_{\phi} + \sigma \gamma C_n + A + \sigma \gamma (\gamma - 2) A^2 \right], \]
\[ C_1 = 2 + \frac{12\gamma^2(\gamma - 2)\chi}{(4\omega^2 - \omega_c^2)(\omega^2 - n^2\omega_c^2)}, \]
\[ C_2 = (1 - \sigma \gamma \Lambda)^2 \left[ \chi - (1 - \sigma \gamma \chi)(4k^2 + h_1) \right], \]
\[ C_\sigma = \frac{k}{4\omega^2 - \omega_c^2} \left[ \frac{A(2\omega^2 + \omega_c^2)}{(\omega^2 - \omega_c^2)(1 - \sigma \gamma \Lambda)^2} + 2\sigma \gamma (\gamma - 2) A^2 \right], \]
\[ C_s = \frac{k}{4\omega^2 - \omega_c^2} \left[ \frac{3\omega^2 A}{(\omega^2 - \omega_c^2)(1 - \sigma \gamma \Lambda)^2} + \sigma \gamma (\gamma - 2) A^2 \right], \]
\[ D_{\phi} = \frac{D_1}{D_2}, \quad D_2 = h_1(V_g^2 - \sigma \gamma n_2^2) - n_2^2, \]
\begin{align*}
D_1 &= \frac{[\sigma \gamma (\gamma - 2) A + 3] \Delta n^2}{(1 - \sigma \gamma A)^2} - 2k^2 n^2 - 2h_2 (V_g^2 - \sigma n^2), \\
D_n &= h_1 D_\phi + 2h_2, \quad D_a = \frac{-2\Lambda_0 k l_2}{(\omega^2 - \omega_c^2)(1 - \sigma \gamma A)^2}, \\
D_v &= \frac{-2\Lambda_0 km_2}{(\omega^2 - \omega_c^2)(1 - \sigma \gamma A)^2}, \\
D_w &= \frac{n^2}{V_g} [D_\phi + \gamma D_a - 2k^2 \\
&+ \{\gamma (\gamma - 2) A + 3 - \frac{2V_k}{\omega} \} A] \frac{1}{(1 - \sigma \gamma A)^2}].
\end{align*}

Appendix C

\begin{align*}
P_1 &= \frac{V_g k \omega}{\omega^2 - \omega_c^2} + \frac{V_g k - \omega}{\omega} - \sigma \gamma A \\
&+ \frac{\gamma V_g k^3}{\omega^3} (\omega^2 - \omega_c^2^2) (2\omega^2 - \omega_c^2), \\
P_2 &= \frac{2 V_g k \omega^2}{\omega^3} \left\{ \frac{2\omega^2 - \omega_c^2^2}{\omega^2 - \omega_c^2^2} (1 - \sigma \gamma A)^2 \right\} \\
&- \frac{2(\omega^2 - \omega_c^2^2)}{\omega^2 (\omega^2 - \omega_c^2^2)(1 - \sigma \gamma A)^2}, \\
P_3 &= \frac{V_g k \omega^2}{\omega^2 - \omega_c^2^2} \left\{ \frac{2 \omega^2 - \omega_c^2^2}{\omega^2 - \omega_c^2^2} (3\omega^2 - 2\omega^2 \omega_c^2 + \omega_c^2) \right\} \\
&- \frac{V_g k \omega^2}{\omega^2 - \omega_c^2^2} \left\{ \frac{2 \omega^2 - \omega_c^2^2}{\omega^2 - \omega_c^2^2} \right\}, \\
P_4 &= \frac{(V_g k - \omega) (\omega^2 - \omega_c^2^2)}{\omega^2 - \omega_c^2^2} (1 - \sigma \gamma A), \\
P_5 &= -\frac{V_g k \omega^2}{\omega^2 (\omega^2 - \omega_c^2^2)(1 - \sigma \gamma A)}, \\
Q_1 &= \frac{A}{(1 - \sigma \gamma A)^2} \left\{ \frac{\gamma (\gamma - 2) (\gamma - 3) A^3}{2(1 - \sigma \gamma A)^2} \right\} \\
&+ \left\{ 1 + \gamma (\gamma - 2) A \right\} (C_n + D_n) + \frac{2k n^2}{\omega} C_w.
\end{align*}

References

Abdelwahed, H.G.: Astrophys. Space Sci. 341, 491 (2012)
Ahmadhijojatabad, N., Abbasi, H., Pajouh, H.H.: Phys. Plasmas 17, 112305 (2010)
Alam, M.K., Chowdhury, A.R.: Aust. J. Phys. 53, 289 (2000)
Alam, M.S., Masud, M.M., Mamun, A.A.: Chin. Phys. B 22, 115202 (2013)
Alinejad, H., Mamun, A.A.: Phys. Plasmas 18, 112103 (2011)
Bains, A.S., Misra, A.P., Saini, N.S., Gill, T.S.: Phys. Plasmas 17, 021203 (2010)
Boström, R.: IEEE Trans. Plasma Sci. 20, 756 (1992)
Boström, R., Gustafsson, G., Holback, B., Holmgren, G., Koskinen, H., Kintner, P.: Phys. Rev. Lett. 61, 82 (1988)
Cairns, R.A., Mamun, A.A., Bingham, R., Boström, R., Dendy, R.O., Nairn, C.M.C., Shukla, P.K.: Geophys. Res. Lett. 22, 2709 (1995)
Cairns, R.A., Mamun, A.A., Bingham, R., Shukla, P.K.: Phys. Scr. T 63, 80 (1996)
Chatterjee, D., Misra, A.P.: Phys. Rev. E 92, 063110 (2015)
Dalui, S., Bandyopadhyay, A., Das, K.P.: Phys. Plasmas 24, 042325 (2017a)
Dalui, S., Bandyopadhyay, A., Das, K.P.: Phys. Plasmas 24, 102310 (2017b)
Delory, G.T., Ergun, R.E., Carlson, C.W., Muschietti, L., Chaston, C.C., Peria, W., McFadden, J.P., Strangeway, R.: Geophys. Res. Lett. 25, 2069 (1998)
Dovner, P.O., Eriksson, A.I., Boström, R., Holback, B.: Geophys. Res. Lett. 21, 1827 (1994)
El-Tantawy, S.A., Wazzwaz, A.M., Rahman, A.: Phys. Plasmas 24, 022126 (2017)
Ergun, R.E., Carlson, C.W., McFadden, J.P., Mozer, F.S., Delory, G.T., Peria, W., Chaston, C.C., Tenerim, M., Elphic, R., Strangeway, R., et al.: Geophys. Res. Lett. 25, 2052 (1998a)
Ergun, R.E., Carlson, C.W., McFadden, J.P., Mozer, F.S., Delory, G.T., Peria, W., Chaston, C.C., Tenerim, M., Elphic, R., Strangeway, R., et al.: Geophys. Res. Lett. 25, 2061 (1998b)
Ferdousi, M., Sultana, S., Mamun, A.A.: Phys. Plasmas 22, 032117 (2015)
Ghosh, B., Banerjee, S.: Turk. J. Phys. 40, 1 (2016)
Ghosh, S.S., Sekar Iyengar, A.N.: Phys. Plasmas 21, 082104 (2014)
Hossen, M.A., Mamun, A.A.: IEEE Trans. Plasma Sci. 44, 643 (2016)
Hou, Y.W., Chen, M.X., Yu, M.Y., Wu, B.: Plasma Phys. Rep. 42, 900 (2016)
Ichikawa, Y.H.: Prog. Theor. Phys. Suppl. 55, 212 (1974)
Ichikawa, Y.H., Taniuti, T.: J. Phys. Soc. Jpn. 34, 513 (1973)
Ichikawa, Y.H., Imamura, T., Taniuti, T.: J. Phys. Soc. Jpn. 33, 189 (1972)
Ichikawa, Y.H., Suzuki, T., Taniuti, T.: J. Phys. Soc. Jpn. 34, 1089 (1973)
Islam, S.A., Bandyopadhyay, A., Das, K.P.: J. Plasma Phys. 74, 765 (2008)
Javidan, K., Pakzad, H.R.: Indian J. Phys. 87, 83 (2013)
Jehan, N., Salahuddin, M., Saleem, H., Mirza, A.M.: Phys. Plasmas 15, 092301 (2008)
Kadijani, M.N., Abbasi, H., Pajouh, H.H.: Plasma Phys. Control. Fusion 53, 025004 (2010)
Kakutani, T., Sugimoto, N.: Phys. Fluids 17, 1617 (1974)
Khalid, M., Rahman, A.: Astrophys. Space Sci. 364, 28 (2019)
Malik, H.K.: Phys. Rev. E 54, 5844 (1996)
Mamun, A.A.: Astrophys. Space Sci. 260, 507 (1998)
Mamun, A.A., Shukla, P.K., Stenflo, L.: Phys. Plasmas 9, 1474 (2002)
McFadden, J.P., Carlson, C.W., Ergun, R.E., Mozer, F.S., Muschietti, L., Roth, I., Moebius, E.: J. Geophys. Res. 108, 8018 (2003)
Mishra, M.K., Chhabra, R.S., Sharma, S.R.: J. Plasma Phys. 52, 409 (1994)
Misra, A.P., Bhowmik, C.: Phys. Plasmas 14, 012309 (2007)
Murtaza, G., Salahuddin, M.: Plasma Phys. 24, 451 (1982)
Pakzad, H.R., Javidan, K.: Nonlinear Process. Geophys. 20, 249 (2013)
Pakzad, H.R., Tribeche, M.: Astrophys. Space Sci. 334, 45 (2011)
Parkes, E.: J. Phys. A, Math. Gen. 20, 3653 (1987)
Pottelette, R., Ergun, R.E., Treumann, R.A., Berthomier, M., Carlson, C.W., McFadden, J.P., Roth, I.: Geophys. Res. Lett. 26, 2629 (1999)
Rufai, O.R., Bharuthram, R., Singh, S.V., Lakhina, G.S.: Phys. Plasmas 21, 082304 (2014)
Sabry, R., Moslem, W., Shukla, P.: Plasma Phys. Control. Fusion 54, 035010 (2012)
Singh, S.V., Lakhina, G.S.: Commun. Nonlinear Sci. Numer. Simul. 23, 274 (2015)
Verheest, F., Pillay, S.R.: Phys. Plasmas 15, 013703 (2008)
Weiland, J., Ichikawa, Y.H., Wilhelmsson, H.: Phys. Scr. 17, 517 (1978)
Yadav, L.L., Sharma, S.R.: Phys. Lett. A 150, 397 (1990)
Yu, M.Y., Luo, H.: Phys. Plasmas 15, 024504 (2008)