Algebraic Solution of the Supersymmetric Hydrogen Atom

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Abstract
The $\mathcal{N} = 2$ supersymmetric extension of the Schrödinger-Hamiltonian with $1/r$-potential in $d$ dimension is constructed. The system admits a supersymmetrized Laplace-Runge-Lenz vector which extends the rotational $SO(d)$ symmetry to a hidden $SO(d+1)$ symmetry. It is used to determine the discrete eigenvalues with their degeneracies and the corresponding bound state wave functions.

1 Classical motion in Newton/Coulomb potential

For a closed system of two non-relativistic point masses interacting via a central force the angular momentum $L$ of the relative motion is conserved and the motion is always in the plane perpendicular to $L$. If the force is derived from a $1/r$-potential, there is an additional conserved quantity: the Laplace-Runge-Lenz¹ vector,

$$C = \frac{1}{m} p \times L - \frac{e^2}{r} r.$$

This vector is perpendicular to $L$ and points in the direction of the semi-major axis. For the hydrogen atom the corresponding Hermitian vector operator has the form

$$C = \frac{1}{2m} (p \times L - L \times p) - \frac{e^2}{r} r \quad (1)$$

with reduced mass $m$ of the proton-electron system. By exploiting the existence of this conserved vector operator, PAULI calculated the spectrum of the hydrogen

¹A more suitable name for this constant of motion would be HERMANN-BERNOULLI-LAPLACE vector, see [1].
atom by purely algebraic means [2, 3]. He noticed that the angular momentum $L$ together with the vector operator

$$K = \sqrt{-\frac{m}{2H}} C,$$  

which is well-defined and Hermitian on bound states with negative energies, generate a hidden $SO(4)$ symmetry algebra,

\[
\begin{align*}
[L_a, L_b] &= i\epsilon_{abc} L_c, \\
[L_a, K_b] &= i\epsilon_{abc} K_c, \\
[K_a, K_b] &= i\epsilon_{abc} L_c,
\end{align*}
\]

and that the Hamiltonian operator can be expressed in terms of $C_{(2)} = K^2 + L^2$, one of the two second-order Casimir operators of this algebra, as follows

$$H = \frac{me^4}{2} \frac{1}{C_{(2)} + \hbar^2}.$$  

One also notices that the second Casimir operator $\tilde{C}_{(2)} = L \cdot K$ vanishes and arrives at the bound state energies by purely group theoretical methods. The existence of the conserved vector $K$ also explains the accidental degeneracy of the hydrogen spectrum.

We generalize the Coulomb-problem to $d$ dimensions by keeping the $1/r$-potential. Distances are measured in units of the reduced Compton wavelength, such that the Schrödinger-operator takes the form

$$H = p^2 - \frac{\eta \cdot \epsilon}{r}, \quad p_a = \frac{1}{i} \partial_a, \quad a = 1, \ldots, d.$$  

$\eta$ is twice the fine structure constant. Energies are measured in units of $mc^2/2$.

The Hermitian generators $L_{ab} = x_{a} p_{b} - x_{b} p_{a}$ of the rotation group satisfy the familiar $so(d)$ commutation relations

$$[L_{ab}, L_{cd}] = i (\delta_{ac} L_{bd} + \delta_{bd} L_{ac} - \delta_{ad} L_{bc} - \delta_{bc} L_{ad}).$$  

It is not very difficult to guess the generalization of the Laplace-Runge-Lenz vector (1) in $d$ dimensions [4],

$$C_{a} = L_{ab} p_{b} + p_{b} L_{ab} - \frac{\eta x_{a}}{r}.$$  

These operators commute with $H$ in (5) and form a $SO(d)$-vector,

$$[L_{ab}, C_{c}] = i (\delta_{ac} C_{b} - \delta_{bc} C_{a}).$$  

The commutator of $C_{a}$ and $C_{b}$ is proportional to the angular momentum,

$$[C_{a}, C_{b}] = -4i L_{ab} H.$$
Now one proceeds as in three dimensions and defines on the negative energy subspace of $L_2(\mathbb{R}^d)$ the Hermitian operators

$$K_a = \frac{1}{2} \frac{C_a}{\sqrt{-H}}$$

with \([K_a, K_b] = iL_{ab}\). \hspace{1cm} (10)

The operators \(\{L_{ab}, K_a\}\) form a closed symmetry algebra and can be combined to form generators \(L_{AB}\) of the orthogonal group\(^1\) \(SO(d+1)\),

$$L_{AB} = \begin{pmatrix} L_{ab} & K_a \\ -K_b & 0 \end{pmatrix}.$$ \hspace{1cm} (11)

They obey the commutation relations \((6)\) with indices running from 1 to \(d+1\).

One finds a relation similar to \((4)\) by solving

$$C_a C_a = -4K_a K_a H = \eta^2 + (2L_{ab}L_{ab} + (d-1)^2) H$$

for the Hamiltonian,

$$H = p^2 - \frac{\eta}{r} = -\frac{\eta^2}{(d-1)^2 + 4C_{(2)}}.$$ \hspace{1cm} (12)

\(C_{(2)}\) is the second-order Casimir operator of the dynamical symmetry group,

$$C_{(2)} = \frac{1}{2} L_{AB} L_{AB} = \frac{1}{2} L_{ab} L_{ab} + K_a K_a.$$ \hspace{1cm} (13)

It remains to find the admitted irreducible representations of \(SO(d+1)\). In three dimensions they are fixed by the condition \(\tilde{C}_{(2)} = 0\) on the Casimir operator not entering the relation \((4)\). In \(d = 2n-1\) and \(d = 2n\) dimensions there are \(n\) Casimir operators of the dynamical symmetry group and we expect \(n-1\) conditions. The analysis in \([5]\) lead to the following results:

- Only the completely symmetric representations of \(SO(d+1)\) are realized.
- As in three dimensions the energies, degeneracies and eigenfunctions are determined by group-theoretic methods.

2 Susy Quantum Mechanics

The Hilbert-Space of a supersymmetric system is the sum of its bosonic and fermionic subspaces, \(\mathcal{H} = \mathcal{H}_B \oplus \mathcal{H}_F\). Let \(A\) be a linear operator \(\mathcal{H}_F \rightarrow \mathcal{H}_B\). We shall use a block notation such that the vectors in \(\mathcal{H}_B\) have upper and those in \(\mathcal{H}_F\) lower components,

$$|\psi\rangle = \begin{pmatrix} |\psi_B\rangle \\ |\psi_F\rangle \end{pmatrix}.$$
Then the nilpotent supercharge and its adjoint take the forms
\[ Q = \begin{pmatrix} 0 & A \\ 0 & 0 \end{pmatrix}, \quad Q^\dagger = \begin{pmatrix} 0 & 0 \\ A^\dagger & 0 \end{pmatrix} \implies \{Q, Q\} = 0. \tag{14} \]

The block-diagonal super-HAMILTONian
\[ H \equiv \{Q, Q^\dagger\} = \begin{pmatrix} AA^\dagger & 0 \\ 0 & A^\dagger A \end{pmatrix} = \begin{pmatrix} H_B & 0 \\ 0 & H_F \end{pmatrix}, \tag{15} \]
commutes with the supercharge and the (fermion) number operator
\[ N = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}. \]
Bosonic states have \( N = 0 \) and fermionic states \( N = 1 \). The supercharge and its adjoint decrease and increase this conserved number by one.

In most applications in quantum mechanics \( A \) is a first order differential operator
\[ A = i\partial_x + iW(x) \tag{16} \]
and yields the isospectral partner-HAMILTONians
\[ H_B = p^2 + V_B, \quad H_F = p^2 + V_F, \quad \text{with} \quad V_{B/F} = W^\pm \pm W'. \tag{17} \]
Such one-dimensional systems were introduced by NICOLAI and WITTEN some time ago [6, 7]. See the texts [8, 9] for a discussion of such models and in particular their relation to isospectral deformations and integrable systems.

3 SQM in Higher Dimensions

Supersymmetric quantum mechanical systems also exist in higher dimensions [7, 10]. The construction is motivated by the following rewriting of the supercharge
\[ Q = \psi \otimes A \quad \text{and} \quad Q^\dagger = \psi^\dagger \otimes A^\dagger \]
containing the fermionic operators
\[ \psi = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad \psi^\dagger = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \]
with anti-commutation relations
\[ \{\psi, \psi\} = \{\psi^\dagger, \psi^\dagger\} = 0 \quad \text{and} \quad \{\psi, \psi^\dagger\} = 1. \]
In [10] this construction has been generalized to \( d \) dimensions. Then one has \( d \) fermionic annihilation operators \( \psi_k \) and \( d \) creation operators \( \psi^\dagger_k \),
\[ \{\psi_k, \psi_\ell\} = \{\psi^\dagger_k, \psi^\dagger_\ell\} = 0 \quad \text{and} \quad \{\psi_k, \psi^\dagger_\ell\} = \delta_{k\ell}, \quad k, \ell = 1, \ldots, d. \tag{18} \]
For the supercharge one makes the ansatz

\[ Q = i \sum \psi_k (\partial_k + W_k(x)) . \]

It is nilpotent (i.e. \( Q^2 = 0 \)) if and only if \( \partial_k W_\ell - \partial_\ell W_k = 0 \) holds true. Locally this integrability condition is equivalent to the existence of a potential \( \chi(x) \) with \( W_k = \partial_k \chi \). Thus we are lead to the following nilpotent supercharge

\[ Q = e^{-\chi} Q_0 e^{\chi} \quad \text{with} \quad Q_0 = i \sum \psi_k \partial_k . \]

It acts on elements of the HILBERT-space

\[ \mathcal{H} = L^2(\mathbb{R}^d) \otimes \mathbb{C}^{2^d} , \]

which is graded by the 'fermion-number' operator \( N = \sum \psi_a \psi_a \),

\[ \mathcal{H} = H_0 \oplus H_1 \oplus \ldots \oplus H_d, \quad N|_{H_p} = p \mathbb{1} . \]

A state in \( \mathcal{H}_p \) has the expansion

\[ \Psi = \sum f_{a_1 \ldots a_p}(x) |a_1 \ldots a_p\rangle, \quad |a_1 \ldots a_p\rangle = \psi_{a_1}^\dagger \ldots \psi_{a_p}^\dagger |0\rangle \]

with antisymmetric \( f_{a_1 \ldots a_p} \). \( Q \) decreases \( N \) by one and its adjoint increases it by one. It follows that the super-HAMILTONian

\[ H = \{ Q, Q^\dagger \} = H_0 \otimes \mathbb{1}_{2^d} - 2 \sum \psi_k^\dagger \psi_\ell \partial_k \partial_\ell \chi = H_d \otimes \mathbb{1}_{2^d} + 2 \sum \psi_k \psi_\ell \partial_k \partial_\ell \chi \]

preserves the 'fermion-number'. The operators in the extreme sectors,

\[ H_0 \equiv H|_{H_0} = -\Delta + (\nabla \chi, \nabla \chi) + \Delta \chi \]
\[ H_d \equiv H|_{H_d} = -\Delta + (\nabla \chi, \nabla \chi) - \Delta \chi . \]

are ordinary SCHröDINGER-operators, whereas the restriction of \( H \) to any other sector is a matrix-SCHröDINGER-operator,

\[ H_p \equiv H|_{H_p} : 2^{(p)} \times 2^{(p)} \text{ matrix}. \]

Due to the nilpotency of \( Q \) and \( [Q, H] = 0 \) one has a HODE-type decomposition of the HILBERT-space [5],

\[ \mathcal{H} = Q \mathcal{H} \oplus Q^\dagger \mathcal{H} \oplus \text{Ker} \mathcal{H} . \]

Actually, the graded HILBERT-space is a \( Q \)-complex of the following structure,

\[ \mathcal{H}_0 \xrightarrow{Q^\dagger} \mathcal{H}_1 \xrightarrow{Q} \mathcal{H}_2 \xrightarrow{Q^\dagger} \ldots \xrightarrow{Q^\dagger} \mathcal{H}_d \]
Similarly as in the one-dimensional case one has a pairing of all \( H \)-eigenstates with non-zero energy. Every excited state is degenerate and the eigenfunctions are mapped into each other by \( Q \) and its adjoint. The situation is depicted in the following figure.

4 The supersymmetric H-Atom

We supersymmetrized the H-atom along these lines and showed that it admits supersymmetric generalizations of the angular momentum and LAPLACE-RUNGE-LENZ vector [5]. As for the ordinary COULOMB problem the hidden \( SO(d+1) \)-symmetry allows for a purely algebraic solution. Here we discuss the construction for the 3-dimensional system and sketch the generalization to arbitrary dimensions.

To construct the supersymmetrized H-atom in 3 dimensions we choose \( \chi = -\lambda r \) in (19) and obtain the super-HAMILTONian [5]

\[
H = (-\Delta + \lambda^2) \mathbb{I}_8 - \frac{2\lambda}{r} B, \quad B = \mathbb{I} - N + S^\dagger S, \quad S = \hat{\mathbf{x}} \cdot \psi \quad (25)
\]

on the HILBERT-space

\[
\mathcal{H} = L_2(\mathbb{R}^3) \times \mathbb{C}^8 = \mathcal{H}_0 \oplus \mathcal{H}_1 \oplus \mathcal{H}_2 \oplus \mathcal{H}_3. \quad (26)
\]

We defined the triplet \( \psi \) containing the 3 annihilation operators \( \psi_1, \psi_2, \psi_3 \). States in \( \mathcal{H}_0 \) are annihilated by \( S \) and states in \( \mathcal{H}_3 \) by \( S^\dagger \). With \( \{S^\dagger, S\} = 1 \) we
find the following HAMILTON-operators in these extreme subspaces,

\[ H_0 = -\triangle + \lambda^2 - \frac{2\lambda}{r}, \]
\[ H_3 = -\triangle + \lambda^2 + \frac{2\lambda}{r}. \]

\( H_0 \) describes the proton-electron and \( H_3 \) the proton-positron system.

The conserved angular momentum contains a spin-type term,

\[ J = L + S = x \wedge p - i\psi \wedge \psi. \]  \hspace{1cm} (27)

The operators \( x \) and \( \psi \) are both vectors such that \( S \) and \( B \) in (25) commute with this total angular momentum. To find the susy extension of the RUNGE-LENZ vector is less simple. It reads [5]

\[ C = p \wedge J - J \wedge p - 2\lambda \hat{x}B \]  \hspace{1cm} (28)

with \( J \) from (27) and \( B \) from (25). The properly normalized vector

\[ K = \frac{1}{2\sqrt{\lambda^2 - H}} C \]  \hspace{1cm} (29)

together with \( J \) form an SO(4) symmetry algebra on the subspace of bound states for which \( H < \lambda^2 \).

To solve for the spectrum we would like to find a relation similar to (3). However, one soon realizes that there is no algebraic relation between the conserved operators \( 1, N, J^2, K^2 \) and \( H \). However, we can prove the equation

\[ \lambda^2 C_{(2)} = K^2 H + (J^2 + (1 - N)^2) QQ^\dagger + (J^2 + (2 - N)^2) Q^\dagger Q, \]  \hspace{1cm} (30)

where \( C_{(2)} \) is the second-order CASIMIR (4). This relation is sufficient to obtain the energies since each of the three subspaces in the HODGE-decomposition (24) is left invariant by \( H \) and thus we may diagonalize it on each subspace separately. Since \( H_{\mathcal{Q}\mathcal{H}} = QQ^\dagger \) and \( H_{Q^\dagger\mathcal{H}} = Q^\dagger Q \) we can solve (30) for \( H \) in both subspaces,

\[ H_{\mathcal{Q}\mathcal{H}} = \lambda^2 \frac{C_{(2)}}{(1 - N)^2 + C_{(2)}}, \]
\[ H_{Q^\dagger\mathcal{H}} = \lambda^2 \frac{C_{(2)}}{(2 - N)^2 + C_{(2)}}, \]  \hspace{1cm} (31)

States with zero energy are annihilated by both \( \mathcal{Q} \) and \( Q^\dagger \), and according to (30) the second-order CASIMIR must vanish on these modes, such that

\[ C_{(2)}|_{\text{Ker } H} = 0. \]

We conclude that every supersymmetric ground state of \( H \) is an SO(4) singlet.
In the figure below we have plotted the spectrum of the supersymmetric $H$-atom in 3 dimensions. The bound states reside in the sectors with fermion numbers 0 and 1. In the sectors with fermion numbers 2 and 3 there are only scattering states. All bound states transform according to the symmetric representations of $SO(4)$. This is particular to 3 dimensions. The energies with degeneracies and the wave functions for all bound states can be found in [5].

5 Higher dimensions

The super-HAMILTONian with $\chi = -\lambda r$ describes a supersymmetrized COULOMB-problem in $d$ dimensions. As in 3 dimensions it can be solved with the help of a supersymmetrized angular momentum and RUNGE-LENS vector generating a dynamical symmetry $SO(d+1)$. The supersymmetric extension of the angular momenta reads

$$J_{ab} = L_{ab} + S_{ab} \quad \text{with} \quad S_{ab} = \frac{1}{i} \left( \psi_a^\dagger \psi_b - \psi_b^\dagger \psi_a \right).$$

(32)

The supercharge, HAMILTONian and $S = \hat{x} \cdot \psi$ are scalars with respect to the rotations generated by the $J_{ab}$. The supersymmetric extension of LAPLACE-RUNGE-LENS vector

$$C_a = J_{ab} p_b + p_b J_{ab} - 2\lambda \hat{x}_a B$$

(33)

and the super-HAMILTONian

$$H = -\triangle + \lambda^2 - \frac{2\lambda}{r} B$$

(34)

both contain the scalar operator

$$B = \frac{1}{2} (d - 1) \mathbb{1} - N + S^\dagger S.$$  

(35)
Again the **FOCK-BARGMANN** symmetry group $SO(d + 1)$ is generated by

$$L_{AB} = \begin{pmatrix} L_{ab} & K_a \\ -K_b & 0 \end{pmatrix}, \quad K_a = \frac{C_a}{\sqrt{4(\lambda^2 - H)}}$$

and the second-order **CASIMIR**

$$C_{(2)} = \frac{1}{2} J_{AB} J_{AB}, \quad (36)$$

together with $\lambda, d, N$ enter the formulas for

$$H|_{QH} \quad \text{and} \quad H|_{Q^1H}.$$ 

The analysis parallels the one in 3 dimensions. To find the allowed representations one uses the branching-rules from the dynamical symmetry $SO(d + 1)$ to the rotational symmetry $SO(d)$ generated by the $J_{ab}$. Only those representation for which the **YOUNG-diagram** has exactly one row and exactly one column give rise to normalizable states. The construction of the bound state wave function uses the realization of the **CARTAN-** and step operators $H_\alpha, E_\alpha$ as differential operators. This way one finds the highest weight state in each representation [5].

### 6 Conclusions

We have succeeded in supersymmetrizing the celebrated construction of **PAULI, FOCK and BARGMANN**. For the **COULOMB**-problem with extended $N = 2$ supersymmetry we have found the conserved angular momentum and conserved **RUNGE-LENS** vector. Together they generate the **FOCK-BARGMANN** symmetry group $SO(d + 1)$. A general relation of the type

$$QQ^1 = f_1 (\lambda, d, N, C_{(2)}) \quad \text{and} \quad Q^1 Q = f_2 (\lambda, d, N, C_{(2)}) \quad (37)$$

has been derived which allows for an algebraic treatment of the supersymmetrized hydrogen atom in $d$ dimensions. The energies depend on the fine structure constant, the dimension of space, the fermion number and the second order **CASIMIR**-operator. The bound states transform according to particular irreducible $SO(d + 1)$-representations. The allowed representations, the explicit form of the bound states and their energies have been determined.

We have not discussed the scattering problem. It is well-known how to extend supersymmetric methods from bound to scattering states in supersymmetric quantum mechanical systems [12]. Thus one may expect that the construction generalizes to the scattering problem, for which the non-compact dynamical symmetry group will be $SO(d,1)$.

**ITZYKSON** and **BANDER** [13] distinguished between the infinitesimal and the global method to solve the **COULOMB** problem. The former is based on the **LAPLACE-RUNGE-LENS** vector and is the method used here. In the second method one performs a stereographic projection of the $d$-dimensional momentum space to the unit sphere in $d+1$ dimensions which in turn implies a $SO(d+1)$
symmetry group. It would be interesting to perform a similar global construction for the supersymmetrized systems.

Every multiplet of the dynamical symmetry group appears several times [5] and there is a new 'accidental' degeneracy: in higher dimensions some eigenvalues of the Hamiltonian appear in many different particle-number sectors. It may very well be, that the algebraic structures discussed in the present work have a more natural setting in the language of superalgebras or the $SO(d,2)$-setting in [4]. We have not investigated this questions.

There exist earlier results on the supersymmetry of both the non-relativistic and relativistic hydrogen atom. In [14] the Runge-Lenz vector or its relativistic generalization, the Johnson-Lippmann operator, enter the expressions for the supercharges belonging to the ordinary Schrödinger- or Dirac-operators with $1/r$ potential. This should be contrasted with the present work, where the Coulomb-problem is only a particular channel of a manifestly supersymmetric matrix-Schrödinger operator. Our Hamiltonians incorporate both the proton-electron and the proton-positron systems as particular subsectors.

The supercharge [19] and super-Hamiltonian [22] describe a wide class of supersymmetric systems, ranging from the supersymmetric oscillator in $d$ dimensions to lattice Wess-Zumino-models with $\mathcal{N} = 1$ or $\mathcal{N} = 2$ supersymmetries in 2 dimensions [11]. In passing we mention, that the supercharge in $d$ dimension is actually a dimensionally reduced Dirac operator in $2d$ dimensions. During the reduction process the Abelian gauge potential $A_\mu$ in $2d$ dimensions transforms into the potential $\chi$ in [19], see [11].

More generally, one may ask for which gauge- and gravitational background field the Dirac-operator admits an extended supersymmetry. This question has been answered in full generality in [15]. For example, on a 4-dimensional hyper-Kähler space with self-dual gauge field the Dirac-operator admits an $\mathcal{N} = 4$ supersymmetry. The extended supersymmetry may be used to construct possible zero-modes of the Dirac-operator. Earlier results on the supersymmetries of Dirac-type operators can be found in [16], for example. Comtet and Horvathy investigated the solutions of the Dirac-equation in the hyper-Kähler Taub-Nut gravitational instanton [17]. The spin 0 case can be solved with the help of a Kepler-type dynamical symmetry [18] and the fermion case by relating it to the spin 0 problem with the help of supersymmetry.

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