Existence and Nonlinear Stability of Stationary Solutions to the Viscous Two-Phase Flow Model in a Half Line

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Abstract. The outflow problem for the viscous two-phase flow model in a half line is investigated in the present paper. The existence and uniqueness of the stationary solution is shown for both supersonic state and sonic state at spatial far field, and the nonlinear time stability of the stationary solution is also established in the weighted Sobolev space with either the exponential time decay rate for supersonic flow or the algebraic time decay rate for sonic flow.

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1 Introduction

The two-phase flow model plays a practically important part in nuclear, engineering, oil-and-gas, the analysis of fluidization and the study of sedimentation phenomena which is used in medicine, chemical engineering or waste water treatment [5, 8, 15]. For strongly coupled motions of two phases, it is appropriate to simplify the two-phase flow model as the drift-flux model which is based on

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the mixture momentum equation with the constitutive equation [8]. Due to the simplicity, accurateness and practically broad applications, the drift-flux model is significantly important for two-phase flow models and was mainly derived by Zuber and Findlay [31], Ishii and Hibiki [8]. In this paper, we consider the drift-flux model which was formally obtained from a Vlasov-Fokker-Planck equation coupled with compressible Navier-Stokes equations by some asymptotic limits in [1, 15].

We are concerned with the initial boundary value problem for the drift-flux model as follows:

\[
\begin{aligned}
  n_t + (nu)_x &= 0, \\
  \rho_t + (\rho u)_x &= 0, \\
  \left[(\rho + n)u\right]_t + \left[(\rho + n)u^2 + p(n,\rho)\right]_x &= \mu u_{xx}, \quad x > 0, \quad t > 0,
\end{aligned}
\]

(1.1)

where \(u\) is the mixed velocity of the fluid and particle, \(\rho > 0\) and \(n > 0\) stand for the densities of two fluids. The constant \(\mu\) is the viscosity coefficient. The pressure satisfies

\[
p(n,\rho) = A_1 \rho^\gamma + A_2 n^\alpha
\]

(1.2)

with four constants \(A_1 > 0, A_2 > 0, \gamma \geq 1\) and \(\alpha \geq 1\). The initial data satisfy

\[
(n,\rho,u)(0,x) = (n_0,\rho_0,u_0)(x), \quad \inf_{x \in \mathbb{R}_+} n_0(x) > 0, \quad \inf_{x \in \mathbb{R}_+} \rho_0(x) > 0,
\]

(1.3)

and the outflow boundary condition is imposed

\[
u(t,0) = u_- < 0,
\]

(1.4)

where \(\rho_+, n_+, u_+\) and \(u_-\) are constants.

There have been important progress about incompressible inviscid limit, global existence, time decay rate of solutions made recently on the drift-flux model. For instance, the incompressible inviscid limit of the solution to Cauchy problem for (1.1)-(1.2) in 3D have been shown in [13]. The global existence of weak solution to Dirichlet’s problem in 3D for (1.1)-(1.2) have been proved in [22]. For (1.1)-(1.2) with magnetic field, Wen and Zhu in [23] obtained the well-posedness and time decay estimates of strong solution to Cauchy problem in 3D. For (1.1)-(1.2) with magnetic field, it should be emphasized that Yin and Zhu in [29] got the nonlinear stability with the exponential or the algebraic time decay rate to outflow problem for the supersonic case. The existence of global weak solutions to Cauchy problem for (1.1) was studied in [3, 25]. The existence of global weak solutions to free boundary value problem for (1.1) in 1D can be found in [2, 4, 26, 27]. The global well-posedness of strong solution to Cauchy problem in Besov space
or Sobolev space for (1.1) have been acquired in [5,24,30]. For (1.1) with the different pressure, the nonlinear stability and convergence rates of stationary solutions to outflow problem have been given in [28] for the supersonic flow in a half line.

However, there is no result about the nonlinear time stability of the stationary solution for the outflow problem (1.1)-(1.4). The main purpose of this paper is to prove the nonlinear time stability of the stationary solution in the suitable weighted Sobolev space with either the exponential time decay rate for supersonic flow or the algebraic time decay rate for sonic flow.

The stationary solution \((\bar{n},\bar{\rho},\bar{u})(x)\) corresponding to the problem (1.1)-(1.4) satisfies the following system

\[
\begin{align*}
(\bar{n}\bar{u})_x &= 0, \\
(\bar{\rho}\bar{u})_x &= 0, \\
\left[ (\bar{n}+\bar{\rho})\bar{u}^2 + \bar{p} \right]_x &= \mu\bar{u}_{xx}, \quad x > 0,
\end{align*}
\]

and the boundary condition and spatial far field condition

\[
\bar{u}(0) = u_-, \quad \lim_{x \to \infty} (\bar{n},\bar{\rho},\bar{u})(x) = (n_+,\rho_+,u_+), \quad \inf_{x \in \mathbb{R}_+} \bar{n}(x) > 0, \quad \inf_{x \in \mathbb{R}_+} \bar{\rho}(x) > 0,
\]

where \(\bar{p} = p(\bar{n},\bar{\rho})\). Integrating (1.5a) and (1.5b) over \((x, +\infty)\), one can obtain

\[
\bar{u} = \frac{n_+}{n} n_+, \quad \bar{u} = \frac{\rho_+}{\bar{\rho}} u_+,
\]

which implies the following relationship

\[
u_+ = \frac{\bar{n}(0)}{n_+} n_+ \bar{u} = \frac{\bar{\rho}(0)}{\rho_+} \rho_+ u_- < 0
\]

is necessary for the existence of the stationary solution to the problem (1.5)-(1.6).

We introduce the sound speed \(c_+\) and a generalized Mach number \(M_+\) about the drift-flux model as follows:

\[
c_+ = \sqrt{\frac{A_1\gamma\rho_+^{\gamma} + A_2\alpha n_+^{\alpha}}{\rho_+ + n_+}}, \quad M_+ = \left| \frac{u_+}{c_+} \right|.
\]

First, we have the following results about the existence and uniqueness of stationary solution.
**Theorem 1.1.** Assume that (1.8), $M_+ \geq 1$ and $w_c < \frac{u_+}{u_+}$ hold, where $w_c$ satisfies (2.7), i.e.,

$$ (n_+ + \rho_+) u_+^2 (w_c - 1) + A_1 \rho_+^{\gamma} (w_c - 1) + A_2 n_+^{\alpha} (w_c^{-\alpha} - 1) = 0 $$

and $w_c \neq 1$ if $M_+ \neq 1$. Then there exists a unique strong solution $(\tilde{n}, \tilde{\rho}, \tilde{u})(x)$ to the boundary value problem (1.1)-(1.4), which satisfies for $M_+ > 1$ that

$$ |\vartheta^k(x)| \leq C e^{-\sigma x}, \quad x \to +\infty, \quad k = 0, 1, 2, 3, 4, \quad (1.10) $$

and for $M_+ = 1$ that

$$ |\vartheta^k(x)| \leq C \frac{\delta^{k+1}}{(1 + \vartheta x)^{k+1}}, \quad x \to +\infty, \quad k = 0, 1, 2, 3, 4 \quad (1.11) $$

with $\delta := |u_- - u_+|$, where $C > 0$ and $\sigma > 0$ are constants.

Then, we have existence and uniqueness of the solution to the initial boundary value problem (1.1)-(1.4) and its convergence to the stationary solution.

**Theorem 1.2.** Assume that the same conditions in Theorem 1.1 hold. Then, the following results hold.

(i) Assume that $M_+ > 1$ and $\lambda \geq 2$. Then for arbitrary $v \in [2, \lambda]$, there exist positive constants $\kappa$ and $\varepsilon_0$ such that if

$$ \kappa^{-1} \left\| (1 + \kappa x)^{\frac{\gamma}{2}} (n_0 - \tilde{n}, \rho_0 - \tilde{\rho}, u_0 - \tilde{u}) \right\|_{H^1} + \frac{\delta}{\kappa} + \kappa \leq \varepsilon_0, \quad (1.12) $$

then the initial boundary value problem (1.1)-(1.4) has a unique global solution $(n, \rho, u)(t, x)$ which satisfies

$$ \left\{ \begin{array}{l}
(1 + \kappa x)^{\frac{\gamma}{2}} (n - \tilde{n}), (1 + \kappa x)^{\frac{\gamma}{2}} (\rho - \tilde{\rho}), (1 + \kappa x)^{\frac{\gamma}{2}} (u - \tilde{u}) \in C([0, +\infty); H^1), \\
(1 + \kappa x)^{\frac{\gamma}{2}} (n - \tilde{n} x), (1 + \kappa x)^{\frac{\gamma}{2}} (\rho - \tilde{\rho} x) \in L^2([0, +\infty); L^2), \\
(1 + \kappa x)^{\frac{\gamma}{2}} (u - \tilde{u} x) \in L^2([0, +\infty); H^1),
\end{array} \right. $$

and

$$ \| (n - \tilde{n}, \rho - \tilde{\rho}, u - \tilde{u})(t) \|_{H^1} \leq C \left\| (1 + \kappa x)^{\frac{\gamma}{2}} (n_0 - \tilde{n}, \rho_0 - \tilde{\rho}, u_0 - \tilde{u}) \right\|_{H^1} (1 + t)^{-\frac{\lambda - \gamma}{2}}, \quad (1.13) $$

where $C > 0$ is a constant independent of time.
(ii) Assume that \( M_+ = 1 \) and \( 2 \leq \lambda < 5 \). Then for arbitrary \( \nu \in [2, \lambda] \), there exists a positive constant \( \varepsilon_0 \) such that if

\[
\delta^{-1} \left\| (1 + Bx)^{\frac{\nu}{2}} (n_0 - \bar{n}_t, \rho_0 - \bar{\rho}, u_0 - \bar{u}) \right\|_{H^1} + \delta^{\frac{\nu}{2}} \leq \varepsilon_0 \tag{1.14}
\]

for \( B = \frac{A_1 \gamma (\gamma + 1) \rho_1^2 + A_2 \nu (\nu + 1) \rho_1^2}{2 \mu |u_+|^2} \), then the initial boundary value problem (1.1)-(1.4) has a unique global solution \((n, \rho, u) (t,x)\) which satisfies

\[
\left\{ \begin{array}{l}
(1 + Bx)^{\frac{\nu}{2}} (n - \bar{n}, \rho - \bar{\rho}, (1 + Bx)^{\frac{\nu}{2}} (u - \bar{u})) \in C([0, +\infty) ; H^1), \\
(1 + Bx)^{\frac{\nu}{2}} (n - \bar{n})_x, (1 + Bx)^{\frac{\nu}{2}} (\rho - \bar{\rho})_x) \in L^2 ([0, +\infty) ; L^2), \\
(1 + Bx)^{\frac{\nu}{2}} (u - \bar{u})_x \in L^2 ([0, +\infty) ; H^1),
\end{array} \right.
\]

and

\[
\left\| (n - \bar{n}, \rho - \bar{\rho}, u - \bar{u}) (t) \right\|_{H^1} \leq C \left\| (1 + Bx)^{\frac{\nu}{2}} (n_0 - \bar{n}_t, \rho_0 - \bar{\rho}, u_0 - \bar{u}) \right\|_{H^1} (1 + t)^{-\frac{\nu - \lambda}{4}}, \tag{1.15}
\]

where \( C > 0 \) is a constant independent of time.

**Remark 1.1.** For supersonic case \( M_+ > 1 \), the asymptotic stability of stationary solution \((\bar{n}, \bar{\rho}, \bar{u})\) with an exponential time decay rate can be also obtained. Indeed, assume that \( M_+ > 1 \) and \( \lambda > 0 \) hold, then for a certain positive constant \( \kappa \in (0, \lambda] \), there exists a positive constant \( \varepsilon_0 \) such that if

\[
\left( e^{\frac{\nu}{2}} (n_0 - \bar{n}, \rho_0 - \bar{\rho}, u_0 - \bar{u}) \right) \in H^1 (\mathbb{R}_+), \quad \kappa^{-1} \left\| (n_0 - \bar{n}_t, \rho_0 - \bar{\rho}, u_0 - \bar{u}) \right\|_{H^1} \leq \varepsilon_0, \tag{1.16}
\]

then the initial boundary value problem (1.1)-(1.4) has a unique global solution \((n, \rho, u)(t,x)\) which satisfies

\[
\left\{ \begin{array}{l}
e^{\frac{\nu}{2}} (n - \bar{n}), e^{\frac{\nu}{2}} (\rho - \bar{\rho}), e^{\frac{\nu}{2}} (u - \bar{u}) \in C([0, +\infty) ; H^1), \\
e^{\frac{\nu}{2}} (n - \bar{n})_x, e^{\frac{\nu}{2}} (\rho - \bar{\rho})_x \in L^2 ([0, +\infty) ; L^2), \\
e^{\frac{\nu}{2}} (u - \bar{u})_x \in L^2 ([0, +\infty) ; H^1),
\end{array} \right.
\]

and

\[
\left\| e^{\frac{\nu}{2}} (n - \bar{n}, \rho - \bar{\rho}, u - \bar{u}) (t) \right\|_{H^1} \leq C \left\| e^{\frac{\nu}{2}} (n_0 - \bar{n}_t, \rho_0 - \bar{\rho}, u_0 - \bar{u}) \right\|_1 e^{-\frac{\nu}{2} t}, \tag{1.17}
\]

where \( C > 0 \) and \( 0 < \kappa_1 \ll \kappa \) are constants independent of time.

The proof of (1.17) can be obtained by similar arguments as (1.13). We omit the details.
Notation 1.1. We denote by $\| \cdot \|_{L^p}$ the norm of the usual Lebesgue space $L^p = L^p(\mathbb{R}_+), 1 \leq p \leq \infty$. And if $p = 2$, we write $\| \cdot \|_{L^2(\mathbb{R}_+)} = \| \cdot \|$. $H^s(\mathbb{R}_+)$ stands for the standard $s$-th Sobolev space over $\mathbb{R}_+$ equipped with its norm $\| f \|_{H^s(\mathbb{R}_+)} = \left( \sum_{i=0}^{s} \| \partial^i f \|_2^2 \right)^{1/2}$.

$C([0,T]; H^1(\mathbb{R}_+))$ represents the space of continuous functions on the interval $[0,T]$ with values in $H^s(\mathbb{R}_+)$. $L^2([0,T];B)$ denotes the space of $L^2$ functions on the interval $[0,T]$ with values in the Banach space $B$. For $W_{\kappa,\nu} = (1+\kappa x)^\nu$ a scalar function with $\kappa,\nu > 0$, weighted $L^2(\mathbb{R}_+)$ and $H^1(\mathbb{R}_+)$ spaces are defined as follows:

$$L^2_{W_{\kappa,\nu}}(\mathbb{R}_+) := \left\{ f \in L^2(\mathbb{R}_+) | \| f \|_{\kappa,\nu} := \left( \int_{\mathbb{R}_+} W_{\kappa,\nu} f^2 dx \right)^{1/2} < +\infty \right\}, \quad (1.18)$$

$$H^1_{W_{\kappa,\nu}}(\mathbb{R}_+) := \left\{ f \in H^1(\mathbb{R}_+) | \| f \|_{\kappa,\nu,1} := \left( \int_{\mathbb{R}_+} W_{\kappa,\nu} (f^2 + f_x^2) dx \right)^{1/2} < +\infty \right\}. \quad (1.19)$$

Note that $\| f \|_{\kappa,\nu} = \| (1+\kappa x)^{\nu/2} f \|$.

The rest of this paper will be organized as follows. We prove the existence and uniqueness of the stationary solution in Subsection 2.1, reformulate the original problem (1.1)-(1.4) in Subsection 2.2 to gain the nonlinear stability for supersonic flow in Section 3 and sonic flow in Section 4.

## 2 Preliminaries

### 2.1 Existence of stationary solution

In this section, we prove Theorem 1.1 about the existence and uniqueness of the stationary solution to the boundary value problem (1.5)-(1.6). We use the similar arguments to [12, 28, 29] to obtain the existence and uniqueness of the stationary solution (1.5)-(1.6) of the two-phase flow model.

We denote

$$\bar{w}(x) := \frac{{\bar{u}(x)}}{u_+} = \frac{n_+}{\bar{n}(x)} = \frac{\rho_+}{\bar{\rho}(x)}.$$ (2.1)
By substituting (2.1) into (1.5c) and integrating the resultant equality over \((x, +\infty)\), one can obtain
\[
\mu u_+ \tilde{w}_x = F(\tilde{w}),
\]
\[
F(\tilde{w}) = (n_+ + \rho_+) u_+^2 (\tilde{w} - 1) + A_1 \rho_+^{\gamma} (\tilde{w}^{-\gamma} - 1) + A_2 n_+^a (\tilde{w}^{-a} - 1).
\]
(2.2)

The boundary condition of \(\tilde{w}(x)\) is obtained from (1.6),
\[
\tilde{w}(0) = \frac{u}{u_+} > 0, \quad \lim_{x \to +\infty} \tilde{w}(x) = 1.
\]
(2.4)

It is easy to have \(F(1) = 0\). More properties of function \(F(\tilde{w})\) should be investigated to gain the existence of solution to (1.5)-(1.6). Differentiate (2.3) about \(\tilde{w}\) to obtain
\[
F'(\tilde{w}) = (n_+ + \rho_+) u_+^2 - A_1 \gamma \rho_+^{\gamma} \tilde{w}^{-\gamma - 1} - A_2 n_+^a \tilde{w}^{-a - 1},
\]
\[
F''(\tilde{w}) = A_1 \gamma (\gamma + 1) \rho_+^{\gamma} \tilde{w}^{-\gamma - 2} + A_2 n_+^a (\alpha + 1) \tilde{w}^{-a - 2} > 0.
\]
(2.5)

(2.6)

Thanks to \(F''(\tilde{w}) > 0\), there exists a unique zero point \(w_*\) of \(F'\), i.e. \(F'(w_*) = 0\). If \(M_+ > 1\), i.e. \(F'(1) > F'(w_*) = 0\), \(F(\tilde{w}) = 0\) exists another root \(w_c\) due to \(\lim_{\tilde{w} \to 0} F(\tilde{w}) = +\infty\), \(F(1) = 0\) and the convexity of \(F\). That implies
\[
(n_+ + \rho_+) u_+^2 (w_* - 1) + A_1 \rho_+^{\gamma} (w_*^{-\gamma} - 1) + A_2 n_+^a (w_*^{-a} - 1) = 0.
\]
(2.7)

If \(M_+ = 1\), i.e. \(F'(1) = F'(w_*) = 0\), \(\tilde{w} = 1\) is the unique root of \(F(\tilde{w}) = 0\).

Using the method of phase plane analysis (see [12] for details), the result of (1.11) can be obtained similarly. The details will be omitted.

**Lemma 2.1.** Assume that (1.8) holds. Then the boundary value problem (1.5)-(1.6) has a unique strong solution \(\tilde{w}(x)\) if and only if \(M_+ \geq 1\) and \(w_* < \tilde{w}(0)\). Moreover, if \(\tilde{w}(0) \leq 1\), then \(\tilde{w}_x \geq 0\). If \(M_+ > 1\), then the solution \(\tilde{w}(x)\) satisfies the estimate (1.10). If \(M_+ = 1\), then the solution \(\tilde{w}(x)\) satisfies the estimate (1.11).

### 2.2 Reformulation of the evolutionary problem

Since the pressure is a nonlinear function of two densities, it is difficult to obtain the stability of the stationary solution by the ideas used in compressible Navier-Stokes equation for the outflow/inflow problems in [6,7,11,12,14,16-19,21]. Inspired by the idea dealing with two-phase flow model in [5,23,28,29], we first take a similar nonlinear variable transformation to divide the two mass variables from each other. Accurately, we take a new variable \(\varphi = a(x) \left( \frac{n}{p} - \frac{\tilde{n}}{p} \right)\) with
\[ a(x) = \frac{A_2 x^{\alpha - 1} \tilde{\rho}^x}{A_1 \gamma \tilde{\rho} + A_2 x^{\alpha} \tilde{\rho}^x}, \] which makes the original system a transport equation and a coupled hyperbolic-parabolic system.

Let
\[ k = \frac{n}{\rho}, \quad \tilde{k}(x) = \frac{n}{\tilde{\rho}}, \quad \tilde{n} = \tilde{k} \tilde{\rho}, \quad \tilde{\rho} = n + \rho, \quad (2.8) \]
where we have used (2.1). Then we can obtain
\[ n = k \rho, \quad \tilde{n} = \tilde{k} \tilde{\rho}. \quad (2.9) \]

We introduce the new variables
\[
\begin{aligned}
\varphi &= a(x) \left( \frac{n}{\rho} - \frac{\tilde{n}}{\tilde{\rho}} \right) = a(x) (k - \tilde{k}), \\
\eta &= \rho - \tilde{\rho} + b(x) \varphi, \\
\psi &= u - \tilde{u},
\end{aligned}
\]
where
\[ a(x) = \frac{A_2 \alpha k^{\alpha - 1} \tilde{\rho}^x}{A_1 \gamma \tilde{\rho} + A_2 \alpha \tilde{k} \tilde{\rho}^x}, \quad b(x) = \tilde{\rho}. \quad (2.11) \]

Hence, we have the equations for \((\varphi, \eta, \psi)\) as follows
\[
\begin{cases}
\varphi_t + u \varphi_x = -a \left( \frac{1}{a} \right)_x \varphi u, \\
\eta_t + u \eta_x + \tilde{\rho} \psi_x &= - (\eta \psi_x - b \varphi \psi_x) - \left[ \psi \tilde{\rho}_x + \eta \tilde{u}_x + ba \left( \frac{1}{a} \right)_x \varphi u - b_x \varphi u - b \varphi \tilde{u}_x \right], \\
\psi_t + u \psi_x + G(0, 0) \eta_x - \mu \frac{1}{(1 + k) \tilde{\rho}} \psi_{xx} &= - \left[ (G(\varphi, \eta) - G(0, 0)) \eta_x - \left( G(\varphi, \eta) - G(0, 0) \right) b \varphi_x \\ &+ (H(\varphi, \eta) - H(0, 0)) \frac{1}{a} \varphi_x \right] \\
&- \left[ \tilde{u}_x \varphi - \mu \tilde{u}_{xx} \left( \frac{1}{(1 + k) \tilde{\rho}} - \frac{1}{(1 + k) \tilde{\rho}} \right) + H(\varphi, \eta) \left( \frac{1}{a} \right)_x \varphi \\ &- G(\varphi, \eta) b_x \varphi + (G(\varphi, \eta) - G(0, 0)) \tilde{\rho}_x \right],
\end{cases}
\]
where

\[
\begin{align*}
G(\varphi, \eta) &= \frac{A_1 \gamma (\bar{\rho} + \eta - b \varphi)^{\gamma - 1} + A_2 \alpha \left( \tilde{k} + \frac{\varphi}{a} \right)^{\alpha} (\bar{\rho} + \eta - b \varphi)^{\alpha - 1}}{(1 + \tilde{k} + \frac{\varphi}{a}) (\bar{\rho} + \eta - b \varphi)}, \\
H(\varphi, \eta) &= \frac{A_2 \alpha \left( \tilde{k} + \frac{\varphi}{a} \right)^{\alpha - 1} (\bar{\rho} + \eta - b \varphi)^{\alpha}}{(1 + \tilde{k} + \frac{\varphi}{a}) (\bar{\rho} + \eta - b \varphi)}, \\
H(0,0) &= \frac{A_2 \alpha \tilde{k}^{\alpha - 1} \bar{\rho}^{\alpha}}{(1 + k) \bar{\rho}}.
\end{align*}
\] (2.13)

The initial and boundary conditions to the system (2.12) satisfy

\[
(\varphi, \eta, \psi)(0, x) := (\varphi_0, \eta_0, \psi_0)(x) = \left( a \left( \frac{n_0}{\rho_0} - \frac{\bar{n}}{\bar{\rho}} \right), \rho_0 - \bar{\rho} - b \varphi_0, u_0 - \bar{u} \right)(x), \quad x > 0, \\
\lim_{x \to +\infty} (\varphi_0, \eta_0, \psi_0)(x) = (0, 0, 0), \\
\psi(t, 0) = 0.
\] (2.14) (2.15) (2.16)

Letting \( W = (1 + k \bar{x})^2 \) or \( (1 + Bx)^2 \), we denote function space \( Y_W(0, T) \) for \( T > 0 \) by

\[
Y_W(0, T) = \left\{ (\varphi, \eta, \psi)(\varphi, \eta, \psi) \in C \left( [0, T]; H_W^1 \right), (\varphi_x, \eta_x) \in L^2 \left( [0, T]; L_W^2 \right), \psi_x \in L^2 \left( [0, T]; H_W^1 \right) \right\}.
\] (2.17)

3 Stability of supersonic steady-state

**Theorem 3.1.** Assume that \( M_+ > 1 \) and \( \lambda \geq 2 \) hold. Then for any \( \nu \in [2, \lambda] \), there exist positive constants \( \kappa \) and \( \epsilon_0 \) such that if

\[
\kappa^{-1} \left\| (1 + k \bar{x})^2 (\varphi_0, \eta_0, \psi_0) \right\|_1 + \frac{\delta}{\kappa} + \kappa \leq \epsilon_0,
\] (3.1)
then the initial boundary value problem (2.12)-(2.16) has a unique global solution \((\varphi, \eta, \psi)\) which satisfies \((\varphi, \eta, \psi) \in Y_{(1+\kappa)x)}(\mathbb{R}^+ )\) and

\[
\left\| (1+\kappa x)^\frac{2}{n}(\varphi, \eta, \psi) \right\|_1 \leq C \left\| (1+\kappa x)^\frac{2}{n}(\varphi_0, \eta_0, \psi_0) \right\|_1 (1+kt)^{-\frac{\lambda - \nu}{2}}, \quad (3.2)
\]

where \(C > 0\) is a constant independent of time.

**Remark 3.1.** Due to the lack of the estimate \(\int_0^t \| \varphi_x \|^2 dt\), it is necessary to use space weight functions compared with [12, 19]. Indeed, to get the weighted \(H^1(\mathbb{R}^+)\) norm, we let \(\nu \geq 2\) to estimate \(\mathcal{J}_2\), see (3.34) for details.

The essential proof of our main Theorem 3.1 is to obtain the uniform a priori estimates of solutions to the initial boundary value problem (2.12)-(2.16) in the weighted Sobolev space.

**Proposition 3.1.** Assume that the same conditions in Theorem 3.1 hold. Suppose that \((\varphi, \eta, \psi)\) is a solution to the problem (2.12)-(2.16) satisfying \((\varphi, \eta, \psi) \in Y_{(1+\kappa)x)}(0, T)\) for a certain constant \(T > 0\). Then for any \(\nu \in [2, \lambda]\), there exist positive constants \(\varepsilon\) and \(C\) independent of \(T\) such that if

\[
0 < \kappa^{-1} \delta + \kappa \leq \delta_0, \quad (3.3)
\]

then it holds for any \(t \in [0, T]\) and \(\theta \geq 0\) that

\[
(1+kt)^{\lambda - \nu + \theta} \left\| (\varphi, \eta, \psi, \varphi_x, \eta_x, \psi_x) \right\|^2_{v,x} \]
\[
+ \int_0^t (1+\kappa \tau)^{\lambda - \nu + \theta} \left\| (\psi_x, \eta_x, \psi_x) \right\|^2_{v,x} d\tau \]
\[
+ \kappa \int_0^t (1+\kappa \tau)^{\lambda - \nu + \theta} \left\| (\varphi, \eta, \psi, \varphi_x, \eta_x, \psi_x) \right\|^2_{v-1,x} d\tau \]
\[
\leq C (1+kt)^{\theta} \left\| (\varphi_0, \eta_0, \psi_0, \varphi_{0x}, \eta_{0x}, \psi_{0x}) \right\|^2_{\lambda,x}. \quad (3.4)
\]

The prior assumption with an algebraic space weight is defined as follows:

\[
\mathcal{N}_{\lambda,x}(T) := \sup_{0 \leq t \leq T} \left\| (\varphi, \eta, \psi, \varphi_x, \eta_x, \psi_x) \right\|_{\lambda,x} \leq \kappa \varepsilon, \quad (3.5)
\]

where \(\varepsilon\) is a small positive number and will be given later.

It is easy to verify that

\[
\| (\varphi, \eta, \psi) \|_{L^\infty} \leq \sqrt{2} \mathcal{N}_{\lambda,x}(T) \leq \sqrt{2} \kappa \varepsilon, \quad (3.6)
\]
\[
\| (1+\kappa x) (\varphi, \eta, \psi) \|_{L^\infty} \leq \sqrt{2} (1+\kappa) \mathcal{N}_{\lambda,x}(T) \leq 2 \sqrt{2} \kappa \varepsilon. \quad (3.7)
\]

Our first goal is to gain the basic weighted energy estimates.
Lemma 3.1. Assume that the assumptions conditions in Proposition 3.1 and (3.5) hold with $\varepsilon \ll 1$. Then the solution $(\varphi, \eta, \psi)$ to the problem (2.12)–(2.16) satisfies the following estimate for $t \in [0, T)$:

\[
(1 + \kappa t)^{\frac{5}{2}} \| (\varphi, \eta, \psi) \|^2_{v,x} + \kappa \int_0^t (1 + \kappa \tau)^{\frac{5}{2}} \| (\varphi, \eta, \psi) \|^2_{v-1,x} d\tau + \int_0^t (1 + \kappa \tau)^{\frac{5}{2}} \| \psi_x \|^2_{v,x} d\tau \\
\leq C\| (\varphi_0, \eta_0, \psi_0) \|^2_{v,x} + C(\delta_0 + \varepsilon) \int_0^t (1 + \kappa \tau)^{\frac{5}{2}} \| (\varphi, \eta) \|^2_{v-1,x} d\tau \\
+ \kappa^2 \int_0^t (1 + \kappa \tau)^{\frac{5}{2}-1} \| (\varphi, \eta, \psi) \|^2_{v,x} d\tau.
\]  
(3.8)

Proof. We multiply (2.12a) by $\tilde{\rho}\varphi W_{v,x}$, (2.12b) by $G(0,0)\eta W_{v,x}$ and (2.12c) by $\tilde{\rho}\psi W_{v,x}$, where $W_{v,x}$ is a space weight function defined in Notation. We sum up resulting equations and then integrate over $\mathbb{R}_+$ to obtain

\[
\frac{d}{dt} \int W_{v,x} \left[ \frac{1}{2} \tilde{\rho} \varphi^2 + \frac{1}{2} G(0,0) \eta^2 + \frac{1}{2} \tilde{\rho} \psi^2 \right] dx \\
- \left[ W_{v,x} \left( \frac{u \varphi}{2} + \frac{u \eta}{2} + \frac{u \varphi}{2} \tilde{\rho} \psi^2 + G(0,0) \tilde{\rho} \eta \eta - \frac{1}{(1+k)\rho} \tilde{\rho} \psi_x \right) \right] (t,0) \\
+ \int v \kappa W_{v-1,x} \left[ \frac{(-u) \varphi}{2} + \frac{(-u) \eta}{2} - G(0,0) \eta^2 + \frac{(u) \varphi}{2} \tilde{\rho} \psi^2 - G(0,0) \tilde{\rho} \eta \psi \right] dx \\
+ \int v \kappa W_{v-1,x} \mu \tilde{\rho} \left( \frac{1}{(1+k)\rho} \right) \psi_x dx + \mu \int W_{v,x} \tilde{\rho} \left( \frac{1}{(1+k)\rho} \right) \psi_x^2 dx \\
= - \sum_{i=1}^7 I_i,
\]  
(3.9)

where

\[
I_1 = \int W_{v,x} \left[ \tilde{\rho} \tilde{u}_x \psi^2 + G(0,0) \tilde{u}_x \eta^2 - (G(0,0))_x \tilde{u} \eta^2 + a \left( \frac{1}{a} \right)_x \tilde{u} \varphi^2 \right] dx, \\
I_2 = \int W_{v,x} \left[ -G(0,0) b \varphi \eta \psi_x - \tilde{\rho} \psi_x \varphi^2 + G(0,0) \psi_x \eta^2 - \tilde{\rho} \psi_x \eta^2 \right] dx, \\
I_3 = \int W_{v,x} \mu \tilde{\rho} \left( \frac{1}{(1+k)\rho} \right)_x \psi_x dx, \\
I_4 = \int W_{v,x} \left[ -G(0,0) b \tilde{u}_x \eta \varphi - G(0,0) b \tilde{u}_x u \varphi \eta + G(0,0) b \tilde{a} \left( \frac{1}{a} \right)_x \tilde{u} \varphi \eta \\
+ \mu \tilde{\rho} \tilde{u}_x \left( \frac{1}{(1+k)\rho} \right) \psi_x - \mu \tilde{\rho} \tilde{u}_x \left( \frac{1}{(1+k)\rho} \right) \tilde{u} \varphi\psi \right] dx.
\]
\[-G(\varphi, \eta)\tilde{\rho}b_x \varphi \psi + H(\varphi, \eta)\tilde{\rho} \left( \frac{1}{a} \right)_x \varphi \psi - (G(0,0))_x \tilde{\rho} \eta \psi + (G(\varphi, \eta) - G(0,0))\tilde{\rho} \rho_x \psi \]

\[I_5 = \int W_{i,x} \left[ - (G(\varphi, \eta) - G(0,0))\tilde{\rho} b_x \varphi \psi + (H(\varphi, \eta) - H(0,0))\tilde{\rho} \left( \frac{1}{a} \right)_x \varphi \psi \right] dx, \]

\[I_6 = \int W_{i,x} (G(\varphi, \eta) - G(0,0))\tilde{\rho} \eta_x \psi dx, \]

\[I_7 = \int W_{i,x} \left[ a \left( \frac{1}{a} \right)_x \varphi \psi^2 - \tilde{\rho}_x \varphi \frac{\varphi^2}{2} - \tilde{\rho}_x \psi \frac{\psi^2}{2} - (G(0,0))_x \psi \frac{\eta^2}{2} \right] dx. \]

First, we estimate terms in the left side of (3.9). The second term in the left side is estimated as follows:

\[\left[ W_{i,x} \left( - \frac{u}{2} \tilde{\rho} \varphi^2 + \frac{u}{2} G(0,0) \eta^2 + \frac{u}{2} \tilde{\rho} \psi^2 + G(0,0) \tilde{\rho} \eta \psi - \mu \left( 1 + k_+ \right) \tilde{\rho} \tilde{\rho} \psi_x \right) \right] (t,0) = \left[ \frac{|u_-|}{2} \left( \tilde{\rho}(0) \varphi^2(t,0) + \frac{A_1 \gamma \tilde{\rho}^{\gamma-1}(0) + A_2 \sigma k_+^{a-1}(0)}{(1 + k_+) \tilde{\rho}(0)} \eta^2(t,0) \right) \right] \geq 0, \quad (3.10)\]

where we have used the boundary condition \( \psi(t,0) = 0 \). Under the condition of \( u_+ < 0 \) and \( M_+ > 1 \), we decompose \( u \) as \( u = \psi + (\tilde{u} - u_+) + u_+ \) and use (1.10) to gain

\[\left( - \frac{u}{2} \right) \tilde{\rho} \varphi^2 + \left( - \frac{u}{2} \right) G(0,0) \eta^2 + \left( - \frac{u}{2} \right) \tilde{\rho} \psi^2 + G(0,0) \tilde{\rho} \eta \psi \geq (\varphi, \eta, \psi) M_1 (\varphi, \eta, \psi)^T - C(\delta_0 + \epsilon) \kappa (\varphi^2 + \eta^2 + \psi^2), \quad (3.11)\]

where \((.)^T\) represents the transpose of a row vector and the symmetric matrix \( M_1 \) is denoted by

\[M_1 = \begin{pmatrix}
-\frac{\rho_+ u_+}{2} & 0 & 0 \\
0 & -\frac{(A_1 \gamma \rho_+^{\gamma-1} + A_2 \sigma k_+^{a-1}) u_+}{2(1 + k_+)} & \frac{A_1 \gamma \rho_+^{\gamma} + A_2 \sigma k_+^a}{2(1 + k_+)} \\
0 & \frac{A_1 \gamma \rho_+^{\gamma} + A_2 \sigma k_+^a}{2(1 + k_+)} & -\frac{\rho_+ u_+}{2}
\end{pmatrix}.\]

It is easy to verify that \( M_1 \) is a positive matrix by using the condition \( M_+ > 1 \) and \( u_+ < 0 \). Hence the estimate for the third term is obtained as follows:
\[ v \kappa \int W_{v-1,k} \left[ \frac{(u)}{2} \tilde{\rho} q^2 + \frac{(u)}{2} G(0,0) \eta^2 + G(0,0) \tilde{\eta} q + \frac{(u)}{2} \tilde{\rho} \eta^2 \right] dx \]
\[ \geq c \kappa \|(\varphi, \eta, \psi)\|_{v-1,k}^2, \] (3.12)

where we take \( \varepsilon \) and \( \delta_0 \) small enough. The forth term is estimated as follows:

\[ \int v \kappa W_{v-1,k} \mu \tilde{\rho} \frac{1}{(1+k)\rho} \psi_x^2 dx \]
\[ = -\frac{\mu}{2} v(v-1) \kappa^2 \int W_{v-2,k} \tilde{\rho} \frac{1}{(1+k)\rho} \psi^2 dx - \frac{\mu}{2} v \kappa \int W_{v-1,k} \tilde{\rho} \frac{1}{(1+k)\rho} \psi_x^2 dx \]
\[ = -\frac{\mu}{2(1+k_+)} v(v-1) \kappa^2 \| \varphi \|_{v-2,k}^2 + C \kappa (\delta + \| \psi \|_{L^\infty}) (\| (\varphi, \eta, \psi) \|_{v-1,k}^2 \]
\[ \geq -C \delta_0 \kappa \| \varphi \|_{v-1,k}^2 - C(\varepsilon + \delta_0) \delta_0 \kappa \| \eta \|_{v-1,k}^2, \] (3.13)

where we have used \( \psi(t,0) = 0 \), (1.10) and (3.6). The fifth term of left side is estimated as follows:

\[ \mu \int W_{v,\kappa} \tilde{\rho} \frac{1}{(1+k)\rho} \psi_x^2 dx \geq \left( \frac{\mu}{1+k_+} - \varepsilon \delta_0 \right) \| \psi_x \|_{v,\kappa}^2 \geq \frac{\mu}{2(1+k_+)} \| \psi_x \|_{v,\kappa}^2, \] (3.14)

where we have used (2.8), (3.6), Sobolev inequality and let \( \varepsilon \) and \( \delta_0 \) small enough in the last inequality. Next, we estimate each term in the right side of (3.9). Using Sobolev inequality, Cauchy-Schwarz inequality, (1.10) and (3.5), one has

\[ |I_1| + |I_2| \leq C \delta \int W_{v-1,k} e^{-\sigma x} (|\varphi|^2 + |\eta|^2 + |\psi|^2) dx \leq C \delta \kappa \|(\varphi, \eta, \psi)\|_{v-1,k}^2, \] (3.15)

\[ |I_5| \leq C (1+k_x) (\| \varphi \|_{L^\infty}) (\| (\varphi, \eta, \psi) \|_{v-1,k}^2 \]
\[ \leq C \kappa \| (\varphi, \eta, \psi) \|_{v-1,k}^2 + C e \kappa \| \eta \|_{v-1,k} + C e \kappa \| \psi \|_{v,\kappa}^2, \] (3.16)

\[ I_3 = -\int W_{v,\kappa} \tilde{\rho} \varphi_x L^2 (\varphi, \eta) (1+k)(\eta_x - b \varphi_x) + \rho \frac{1}{a} \varphi_x dx \]
\[ -\int W_{v,\kappa} \tilde{\rho} \varphi_x \left( \frac{1}{(1+k)\rho} \right)^2 (1+k)(\tilde{\rho}_x - b \varphi) + \rho \frac{1}{a} \varphi \right] dx \]
\[ =: \sum_{j=1}^{2} I_3^j, \] (3.17)

\[ |I_3^j| \leq C e \kappa \| \varphi_x \|_{v,\kappa}^2 + C e \kappa (\| \varphi_x \|_{v-1,k}^2 + \| \eta_x \|_{v,\kappa}^2), \] (3.18)
\[ |I_3^2| \leq C \delta_0 \kappa \| \psi \|_{V^{-1, \kappa}}^2 + C \delta_0 \kappa \| \psi_x \|_{V, \kappa}^2 \]  
\[ |I_4| \leq C \delta \int W_{v, \kappa} e^{-\kappa x} (|\varphi \eta| + |\psi \cdot \eta| + |\psi \cdot \varphi|) \, dx \]
\[ \leq C \delta_0 \kappa \| (\varphi, \eta, \psi) \|_{V^{-1, \kappa}}^2 + C \delta_0 \kappa \| \psi_x \|_{V, \kappa}^2, \]  
\[ |I_5| \leq \| (1 + \kappa \kappa) \psi \|_{L^\infty} \| (\varphi, \eta, \psi) \|_{V^{-1, \kappa}}^2 \leq C \epsilon \kappa \| (\varphi, \eta) \|_{V^{-1, \kappa}}^2 + C \epsilon \kappa \| \psi_x \|_{V^{-1, \kappa}}^2. \]  
Finally, with $\delta_0$, $\kappa$, and $\epsilon$ suitably small, the substitution of (3.10)-(3.21) into (3.9) leads to
\[
\frac{d}{dt} \int W_{v, \kappa} \left[ \frac{1}{2} \rho \varphi^2 + \frac{1}{2} G(0,0) \eta^2 + \frac{1}{2} \rho \psi^2 \right] \, dx + c \kappa \| (\varphi, \eta, \psi) \|_{V^{-1, \kappa}}^2 + c \| \psi_x \|_{V, \kappa}^2
\leq C(\delta_0 + \epsilon) \kappa \| (\varphi, \eta) \|_{V^{-1, \kappa}}^2. \]  
Multiplying (3.9) by $(1 + \kappa \kappa)^{\delta}$ and integrating in $\tau$ over $[0, t]$ yield that
\[
(1 + \kappa \kappa)^\delta \| (\varphi, \eta, \psi) \|_{V, \kappa}^2 + \kappa \int_0^t (1 + \kappa \kappa)^\delta \| (\varphi, \eta, \psi) \|_{V^{-1, \kappa}}^2 \, d\tau + \int_0^t (1 + \kappa \kappa)^\delta \| \psi_x \|_{V, \kappa}^2 \, d\tau
\leq C \| (\varphi_0, \eta_0, \psi \psi_0) \|_{V, \kappa}^2 + C \delta_0 \kappa \int_0^t (1 + \kappa \kappa)^\delta \| (\varphi, \eta, \eta_x) \|_{V^{-1, \kappa}}^2 \, d\tau
+ \kappa \int_0^t (1 + \kappa \kappa)^{\delta - 1} \| (\varphi, \eta, \psi) \|_{V, \kappa}^2 \, d\tau. \]  
Hence we obtain (3.8) and complete the proof of Lemma 3.1. \[ \square \]

Then, we need to obtain the weighted estimate of $(\varphi, \eta, \psi)$ in order to complete the proof of Proposition 3.1.

**Lemma 3.2.** Assume that the same assumptions Proposition 3.1 and (3.5) hold with $\epsilon \ll 1$. Then the solution $(\varphi, \eta, \psi)$ to the problem (2.12)-(2.16) satisfies the following estimate for $t \in [0, T]$:
\[
(1 + \kappa \kappa)^\delta \| (\varphi, \eta, \psi) \|_{V, \kappa}^2 + \kappa \int_0^t (1 + \kappa \kappa)^\delta \| (\varphi, \eta, \psi) \|_{V^{-1, \kappa}}^2 \, d\tau + \int_0^t (1 + \kappa \kappa)^\delta \| \eta_x \|_{V, \kappa}^2 \, d\tau
\leq C \| (\varphi_0, \eta_0, \psi_0, \psi_0, \eta_0) \|_{V, \kappa}^2 + C \epsilon \kappa \int_0^t (1 + \kappa \kappa)^\delta \| \psi_x \|_{V, \kappa}^2 \, d\tau
+ C \epsilon \kappa \int_0^t (1 + \kappa \kappa)^{\delta - 1} \| (\varphi, \eta, \psi, \varphi_x, \eta_x) \|_{V, \kappa}^2 \, d\tau. \]  
(3.24)
Proof. Differentiating (2.12a) in $x$ and multiplying the resulting equation by $\varphi_x$, one can obtain that

$$\left( \frac{1}{2} \varphi_x^2 \right)_t + \left( \frac{u}{2} \varphi_x^2 \right)_x + \frac{1}{2} \varphi_x^2 \psi_x + \frac{1}{2} \tilde{u}_x \varphi_x^2$$

$$= - \left[ ua \left( \frac{1}{a} \right)_x \varphi_x^2 + \tilde{u}_x a \left( \frac{1}{a} \right)_x \varphi \varphi_x + u \left( a \left( \frac{1}{a} \right)_x \right) \varphi \varphi_x + a \left( \frac{1}{a} \right)_x \varphi \varphi_x \psi_x \right]. \quad (3.25)$$

Differentiating (2.12b) in $x$ and multiplying the resulting equation by $\eta_x$, we obtain that

$$\left( \frac{1}{2} \eta_x^2 \right)_t + \left( \frac{u}{2} \eta_x^2 \right)_x + \frac{3}{2} \tilde{u}_x \eta_x^2 + \tilde{\rho} \eta_x \psi_{xx}$$

$$= - \left[ (\eta - b \varphi) \eta_x \psi_{xx} + \frac{3}{2} \varphi_x \eta_x^2 - b \varphi_x \eta_x \psi_x \right]$$

$$- \left[ 2 \tilde{\rho}_x \eta_x \psi_x - 2 b_x \varphi \psi_x \eta_x - b \tilde{u}_x \varphi_x \eta_x - b_x u \varphi_x \eta_x \right.$$  

$$+ ba \left( \frac{1}{a} \right)_x \varphi \eta_x \psi_x + ba \left( \frac{1}{a} \right)_x u \varphi_x \eta_x$$

$$+ \left( ba \left( \frac{1}{a} \right)_x \right) u \varphi \eta_x + \left( ba \left( \frac{1}{a} \right)_x \right) - b_x \tilde{u}_x \varphi \eta_x$$

$$- b_x \tilde{u}_x \varphi \eta_x - b \tilde{u}_x \varphi \eta_x - b \tilde{u}_x \varphi \eta_x - b_x u \varphi \eta_x + \tilde{\rho} \tilde{\rho} \eta_x \psi_x \right]. \quad (3.26)$$

Multiplying (2.12c) by $(1 + \tilde{k}) \tilde{\rho}^2 \eta_x$, we obtain that

$$\left[ (1 + \tilde{k}) \tilde{\rho}^2 \eta_x \psi \right]_t - \left[ (1 + \tilde{k}) \tilde{\rho}^2 \eta_x \psi \right]_x + G(0,0) \eta_x^2 - \mu \tilde{\rho} \eta_x \psi_{xx}$$

$$= - \left[ (1 + \tilde{k}) \tilde{\rho}^2 (\eta_t + u \eta_x) \psi_x - \mu (1 + \tilde{k}) \tilde{\rho}^2 \eta_x \psi_{xx} \left( \frac{1}{(1 + \tilde{k}) \rho} - \frac{1}{(1 + \tilde{k}) \tilde{\rho}} \right) \right.$$

$$+ (G(\varphi, \eta) - G(0,0)) (1 + \tilde{k}) \tilde{\rho}^2 \eta_x^2 - (G(\varphi, \eta) - G(0,0)) (1 + \tilde{k}) \tilde{\rho}^2 b \varphi \eta_x$$

$$+ (H(\varphi, \eta) - H(0,0)) (1 + \tilde{k}) \tilde{\rho}^2 \tilde{\rho} \eta_x \psi_x$$

$$\left. - \left[ (1 + \tilde{k}) \tilde{\rho}^2 \tilde{\rho}_x \eta_x \psi - \mu (1 + \tilde{k}) \tilde{\rho}^2 \tilde{\rho}_x \eta_x \left( \frac{1}{(1 + \tilde{k}) \rho} - \frac{1}{(1 + \tilde{k}) \tilde{\rho}} \right) \right. \right] \eta_x$$

$$+ (G(\varphi, \eta) - G(0,0)) (1 + \tilde{k}) \tilde{\rho}^2 \tilde{\rho}_x \eta_x - G(\varphi, \eta) (1 + \tilde{k}) \tilde{\rho}^2 b \varphi \eta_x$$

$$\eta_x$$

$$+ (H(\varphi, \eta) - H(0,0)) (1 + \tilde{k}) \tilde{\rho}^2 \tilde{\rho}_x \eta_x \psi - G(\varphi, \eta) (1 + \tilde{k}) \tilde{\rho}^2 b \varphi \eta_x \psi_x$$

$$\right]. \quad (3.27)$$
\[ + H(\varphi, \eta)(1+\tilde{k})\rho^2 \left( \frac{1}{a} \right)_x \varphi \eta_x \right] - \left( (1+\tilde{k})\rho^2 \right)_x \eta \psi. \] (3.27)

Multiply (3.26) by \( \mu W_{v,x} \), (3.25) by \( W_{v,x} \) and (3.27) by \( W_{v,x} \). We sum up resulting equations and then integrate over \( \mathbb{R}_+ \), which can imply

\[
\frac{d}{dt} \int W_{v,x} \left[ \frac{1}{2} \varphi_x^2 + \frac{\mu}{2} \eta_x^2 + (1+\tilde{k})\rho^2 \eta_x \psi \right] dx \\
+ \left[ W_{v,x} \left( \frac{u}{2} \varphi_x^2 + \frac{\mu}{2} \eta_x^2 - (1+\tilde{k})\rho^2 \eta_x \psi \right) \right] (t,0) \\
+ \int \nu \kappa W_{v-1,x} \left[ \frac{(-u)}{2} \varphi_x^2 + \frac{\mu(-u)}{2} \eta_x^2 \right] dx \\
- \int \nu \kappa W_{v-1,x} (1+\tilde{k})\rho^2 \eta_x \psi dx + \int \mu G(0,0) \eta_x^2 dx \\
=: - \sum_{i=1}^{8} J_i, \] (3.28)

where

\[ J_1 = \int W_{v,x} \frac{1}{2} \bar{\mu}_x \left( \varphi_x^2 + 3 \mu \eta_x^2 \right) dx, \]
\[ J_2 = \int W_{v,x} \left[ \frac{1}{2} \left( \varphi_x^2 + 3 \mu \eta_x^2 \right) \psi_x - \mu b \varphi_x \eta_x \varphi_x \right] dx, \]
\[ J_3 = \int W_{v,x} \left[ \mu (\eta - b \varphi) \eta_x \varphi_{xx} - \mu (1+\tilde{k})\rho^2 \left( \frac{1}{(1+k)\rho} - \frac{1}{(1+k)\rho} \right) \eta_x \varphi_{xx} \right] dx, \]
\[ J_4 = \int W_{v,B} \left[ (G(\varphi, \eta) - G(0,0)) (1+\tilde{k})\rho^2 \eta_x^2 - (G(\varphi, \eta) - G(0,0)) (1+\tilde{k})\rho^2 b \varphi_x \eta_x \right. \\
\left. + (H(\varphi, \eta) - H(0,0)) (1+\tilde{k})\rho^2 \frac{1}{a} \varphi_x \eta_x \right] dx, \]
\[ J_5 = \int W_{v,x} (1+\tilde{k})\rho^2 (\eta_x + u \eta_x) \varphi_x dx, \]
\[ J_6 = \int W_{v,x} \left[ a \left( \frac{1}{a} \right)_x \varphi \varphi_x \left( \varphi_x + \mu b \eta_x \right) - 2 \mu b_x \varphi \eta_x \psi_x \right] dx, \]
\[ J_7 = \int W_{v,x} \left[ \mu \left( ba \left( \frac{1}{a} \right)_x - b \bar{\mu}_x - b_x \right) \varphi_x \eta_x + \left( u \left( \frac{1}{a} \right)_x \right) + \bar{\mu}_x a \left( \frac{1}{a} \right)_x \right] \varphi \varphi_x \\
+ \mu \left( b_x c \left( \frac{1}{a} \right)_x + b \left( \frac{1}{a} \right)_x \right) \bar{\mu}_x - 2 b_x \bar{\mu}_x - b_{xx} u \right] \varphi \eta_x \]
\[ +ua \left( \frac{1}{a} \right)_x \varphi_x^2 + \mu \ddot{u}_{xx} \eta_x + (G(\varphi, \eta) - G(0,0))(1 + \tilde{k}) \tilde{\rho}^2 \tilde{\rho}_x \eta_x \]
\[ + (1 + \tilde{k}) \tilde{\rho}^2 \left( H(\varphi, \eta) \left( \frac{1}{a} \right)_x - G(\varphi, \eta) b_x \right) \varphi \eta_x + (1 + \tilde{k}) \tilde{\rho}^2 \ddot{u}_x \eta_x \varphi \]
\[ - \mu \left( \frac{1}{(1 + k) \rho} - \frac{1}{(1 + \tilde{k}) \rho} \right) (1 + \tilde{k}) \tilde{\rho}^2 \ddot{u}_{xx} \eta_x + \mu (2 \tilde{\rho}_x + \tilde{\rho}_{xx}) \psi_x \eta_x \right] dx, \]
\[ J_8 = \int W_{v, \kappa} \left[ (1 + \tilde{k}) \tilde{\rho}^2 \right]_x \eta_1 \psi dx. \]

First, we estimate terms in the left side of (3.28). The second term in the left side is estimated as follows:
\[ - \left[ W_{v, \kappa} \left( \frac{u}{2} \varphi_x^2 + \frac{\mu}{2} \eta_x^2 - (1 + \tilde{k}) \tilde{\rho}^2 \eta_1 \psi \right) \right] (t, 0) \]
\[ = \frac{|u - u_+|}{2} \left( \varphi_x(t, 0)^2 + \mu \eta_x(t, 0)^2 \right) \geq 0, \quad (3.29) \]
where we have used the boundary condition \( \psi(t, 0) = 0 \). Using (1.10), (3.5), \( u_+ < 0 \), the third term is estimated as follows:
\[ \int v_\kappa W_{v, -1, \kappa} \left[ \frac{(u - u_+)}{2} \varphi_x^2 + \frac{\mu (u - u_+)}{2} \eta_x^2 \right] dx \]
\[ \geq \int v_\kappa W_{v, -1, \kappa} \left[ \frac{(u - u_+)}{2} \varphi_x^2 + \frac{\mu (u - u_+)}{2} \eta_x^2 \right] dx - C(\|\psi\|_{L^\infty} + \delta) \kappa \| (\varphi_x, \eta_x) \|_{v_\kappa, -1, \kappa} \]
\[ \geq c \kappa \| (\varphi_x, \eta_x) \|_{v_\kappa, -1, \kappa}^2, \quad (3.30) \]
where we let \( \epsilon \) and \( \delta_0 \) are small enough in the last inequality. The forth term is estimated as follows:
\[ - \int v_\kappa W_{v, -1, \kappa} (1 + \tilde{k}) \tilde{\rho}^2 \eta_1 \psi \eta_1 \psi dx \leq C \kappa \| (\varphi, \eta, \psi) \|_{v_\kappa, -1, \kappa}^2 + C \delta_0 \| (\eta_x, \varphi_x) \|_{v_\kappa, -1, \kappa}^2, \quad (3.31) \]
where we have used (2.12b), (1.10), (3.6) and Cauchy-Schwarz inequality. Using (1.10), (3.5), (3.6), we obtain the estimate for the fifth term as follows:
\[ \mu \int W_{v, \kappa} G(0,0) \eta_2^2 dx \]
\[ \geq \mu \frac{A_1 T^{-1} + A_2 \rho_+^{-1} k_+^k}{(1 + k_+) \rho_+} \int W_{v, \kappa} \eta_2^2 dx - C \delta_0 \kappa \| \eta_x \|_{v_\kappa, \kappa}^2 \geq c \| \eta_x \|_{v_\kappa, \kappa}^2, \quad (3.32) \]
where we take \( \delta_0 \) small enough in the last inequality. The next thing is to estimate terms in the right side of (3.28). Using (2.12), (1.10), (3.6), (3.7), Sobolev inequality
and Cauchy-Schwarz inequality, under the condition of (3.5) and $2 \leq \nu \leq \lambda$, one has

$$|\mathcal{J}_1| \leq C\delta \int W_{V,K} e^{-\nu x} (\varphi_x^2 + \eta_x^2) \, dx \leq C\delta_0 \kappa \| (\varphi_x, \eta_x) \|^2_{V^{-1,K}},$$  

(3.33)

$$|\mathcal{J}_2| \leq C\| (1 + \kappa \xi) \psi_x \|_L^\infty \left( \| \varphi_x \|^2_{V^{-1,K}} + \| \eta_x \|^2_{V^{-1,K}} \right)$$

$$\leq C\| (1 + \kappa \xi) \psi_x \| \left( \| \varphi_x \|^2_{V^{-1,K}} + \| \eta_x \|^2_{V^{-1,K}} \right)$$

$$+ C\| (\varphi_x, \eta_x) \|_{V^{-1,K}} \left( \| (\varphi_x, \eta_x) \|^2_{V^{-1,K}} + \| (1 + \kappa \xi) \psi_{xx} \|^2 \right)$$

$$\leq C\varepsilon \kappa \| \psi_{xx} \|^2_{V,K} + C\varepsilon \kappa \| (\varphi_x, \eta_x) \|^2_{V^{-1,K}},$$  

(3.34)

$$|\mathcal{J}_3| \leq C\| (\varphi, \eta) \|_L^\infty \int W_{V,K} (\eta_x^2 + \psi_{xx}^2) \, dx \leq C\varepsilon \kappa \| \eta_x \|^2_{V,K} + C\varepsilon \kappa \| \psi_{xx} \|^2_{V,K},$$  

(3.35)

$$|\mathcal{J}_4| \leq C\| (1 + \kappa \xi) (\varphi, \eta) \|_L^\infty \int W_{V^{-1,K}} (\varphi_x^2 + \eta_x^2) \, dx \leq C\varepsilon \kappa \| (\varphi_x, \eta_x) \|^2_{V^{-1,K}},$$  

(3.36)

$$|\mathcal{J}_5| \leq C \| \varphi_x \|^2_{V,K} + C\delta_0 \kappa \| (\varphi_x, \eta_x) \|^2_{V^{-1,K}},$$  

(3.37)

$$|\mathcal{J}_6| \leq C\delta_0 \varepsilon \kappa \| \varphi_x \|^2_{V^{-1,K}} + C\delta_0 \varepsilon \kappa \| (\varphi_x, \eta_x) \|^2_{V^{-1,K}},$$  

(3.38)

$$|\mathcal{J}_7| \leq C\delta_0 \kappa \| (\varphi, \eta, \psi) \|^2_{V^{-1,K}} + C\delta_0 \kappa \| (\varphi_x, \eta_x) \|^2_{V^{-1,K}} + C\delta_0 \kappa \| \psi_x \|^2_{V,K},$$  

(3.39)

$$|\mathcal{J}_8| \leq C\delta_0 \kappa \| (\eta_x, \psi_x) \|^2_{V,K} + C\delta_0 \kappa \| (\varphi_x, \eta_x, \psi) \|^2_{V^{-1,K}}.$$  

(3.40)

Finally, the substitution of (3.29)-(3.40) into (3.9) leads to

$$\frac{d}{dt} \int W_{V,K} \left[ \frac{1}{2} \varphi_x^2 + \frac{\mu}{2} \varphi_x^2 + (1 + \bar{k}) \rho^2 \eta_x^2 \psi \right] \, dx + c\kappa \| (\varphi_x, \eta_x) \|^2_{V^{-1,K}} + c \| \eta_x \|^2_{V,K}$$

$$\leq C \| \varphi_x \|^2_{V,K} + C\kappa \| (\varphi_x, \eta_x, \psi) \|^2_{V^{-1,K}} + C\varepsilon \kappa \| \psi_{xx} \|^2_{V,K},$$  

(3.41)

if $\delta_0$ and $\varepsilon$ are small enough. Multiplying (3.41) by $(1 + \kappa \tau)^\xi$ and integrating in $\tau$ over $[0,t]$, one can obtain

$$(1 + \kappa \tau)^\xi \| (\varphi_x, \eta_x) \|^2_{V,K} + \kappa \int_0^t (1 + \kappa \tau)^\xi \| (\varphi_x, \eta_x) \|^2_{V^{-1,K}} \, d\tau + \int_0^t (1 + \kappa \tau)^\xi \| \eta_x \|^2_{V,K} \, d\tau$$

$$\leq C \| (\varphi_0, \eta_0, \psi_0, \varphi_0x, \eta_0x) \|^2_{V,K} + C\varepsilon \kappa \int_0^t (1 + \kappa \tau)^\xi \| \psi_{xx} \|^2_{V,K} \, d\tau$$

$$+ C\xi \kappa \int_0^t (1 + \kappa \tau)^{\xi-1} \| (\varphi, \eta, \psi, \varphi_x, \eta_x) \|^2_{V,K} \, d\tau,$$  

(3.42)

where we have used Cauchy-Schwarz inequality.
Lemma 3.3. Assume that the same assumptions in Proposition 3.1 and (3.5) hold with \( \varepsilon \ll 1 \). Then the solution \((\varphi, \eta, \psi)\) to the problem (2.12)-(2.16) satisfies the following estimate for \( t \in [0, T)\):

\[
(1 + \kappa t)^{3} \| \psi_x \|_{v, \kappa}^2 + \int_0^t (1 + \kappa \tau)^{3} \| \psi_{xx} \|_{v, \kappa}^2 d\tau \\
\leq C \| (\varphi_0, \eta_0, \varphi_0, \varphi_0, \eta_0) \|_{\tilde{H}_0}^2 \\
+ C \xi_k \int_0^t (1 + \kappa \tau)^{3 - 1} \| (\varphi, \eta, \psi, \varphi_x, \eta_x, \psi_x) \|_{v, \kappa}^2 d\tau.
\]

(3.43)

Proof. We multiply (2.12c) by \(-\psi_{xx} W_{v, \kappa}\) and integrate the resulting equality in \(x\) over \(\mathbb{R}_+\) with the boundary condition \(\psi(t, 0) = 0\) to gain

\[
\frac{d}{dt} \int W_{v, \kappa} \frac{\psi_x^2}{2} dx + \nu \kappa \int W_{v - 1, \kappa} \psi_x dx + \mu \int W_{v, \kappa} \frac{1}{(1 + k)^p} \psi_{xx}^2 dx =: -\sum_{i=1}^3 \mathcal{K}_i, \quad (3.44)
\]

where

\[
\mathcal{K}_1 = \int W_{v, \kappa} \left[ -u \psi_x \psi_{xx} - G(0, \kappa) \eta_x \psi_{xx} \right] dx,
\]

\[
\mathcal{K}_2 = \int W_{v, \kappa} \left[ -\bar{u} \psi \psi_{xx} + \mu \bar{u} \psi \left( \frac{1}{(1 + k)^p} - \frac{1}{(1 + k)^p} \right) \psi_{xx} - H(\varphi, \eta) \left( \frac{1}{a} \right) \eta_x \psi_{xx} \right. \\
\left. + G(\varphi, \eta) b_x \varphi \psi_{xx} - (G(\varphi, \eta) - G(0, \kappa) \bar{\rho}) \eta_x \psi_{xx} \right] dx,
\]

\[
\mathcal{K}_3 = \int W_{v, \kappa} \left[ (H(\varphi, \eta) - H(0, \kappa)) \left( \frac{1}{c} \varphi_x \psi_{xx} + (G(\varphi, \eta) - G(0, \kappa)) b_x \varphi \psi_{xx} \right. \right. \\
\left. \left. + (G(\varphi, \eta) - G(0, \kappa)) \eta_x \psi_{xx} \right] dx.
\]

First, we estimate terms in the left side of (3.44). The second term is estimated as follows:

\[
\nu \kappa \int W_{v - 1, \kappa} \psi_x dx \leq C \xi \| (\varphi, \eta, \psi, \varphi_x, \eta_x, \psi_x, \psi_{xx}) \|_{\tilde{H}_0 - 1, \kappa}^2.
\]

(3.45)

The third term is estimated as follows:

\[
\mu \int W_{v, \kappa} \frac{1}{(1 + k)^p} \psi_{xx}^2 dx \geq \frac{\mu}{(1 + k_+)^p} \| \psi_{xx} \|_{v, \kappa}^2 - C(\delta_0 + \varepsilon) \| \psi_{xx} \|_{v, \kappa}^2 \\
\geq \frac{\mu}{2(1 + k_+)^p} \| \psi_{xx} \|_{v, \kappa}^2.
\]

(3.46)
where we let $\delta_0$ and $\epsilon$ small enough in the last inequality. Next, we estimate each term in the right side of (4.58). Using (1.10), (2.13), (3.7), Sobolev inequality and Cauchy-Schwarz inequality, under the condition of $2 \leq \nu \leq \lambda$, one has

$$|K_1| \leq \frac{\mu}{8(1+k_+)} \|\psi\|_{v,x}^2 + C \|\eta_x, \psi_x\|_{v,x}^2,$$  \hspace{1cm} (3.47)

$$|K_2| \leq C\delta \int W_{v,x} e^{-\sigma x} \left( \varphi^2 + \eta^2 + \psi^2 + \psi_{xx}^2 \right) dx$$
$$\leq C\delta_0 \kappa \|\varphi, \eta, \psi\|_{v-1,x}^2 + C\delta_0 \kappa \|\psi_{xx}\|_{v,x}^2,$$  \hspace{1cm} (3.48)

$$|K_3| \leq C\epsilon \|\psi_{xx}\|_{v,x}^2 + C\epsilon \kappa \|\varphi_x\|_{v-1,x}^2.$$  \hspace{1cm} (3.49)

Finally, by $\delta_0$ and $\epsilon$ small enough and the substitution of (3.45)-(3.49) into (3.44), we obtain

$$\frac{d}{dt} \int W_{v,x} \psi_x^2 dx + \frac{\mu}{4(1+k_+)} \int W_{v,x} \psi_{xx}^2 dx$$
$$\leq C \|\eta_x, \psi_x\|_{v,x}^2 + C\delta_0 \kappa \|\varphi, \eta, \psi\|_{v-1,x}^2 + C\delta_0 \kappa \|\psi_{xx}\|_{v,x}^2.$$  \hspace{1cm} (3.50)

Multiplying (3.50) by $(1 + \kappa \tau)^\xi$ and integrating in $\tau$ over $[0,t]$, one gains

$$(1 + \kappa t)^\xi \|\psi_x\|_{v,x}^2 + \int_0^t (1 + \kappa \tau)^\xi \|\psi_{xx}\|_{v,x}^2 d\tau$$
$$\leq C \|\varphi_0, \eta_0, \psi_0, \varphi_0, \eta_0, \psi_0\|_{\lambda, x}^2$$
$$+ C\gamma \kappa \int_0^t (1 + \kappa \tau)^\xi - 1 \|\varphi, \eta, \psi, \varphi_x, \eta_x, \psi_x\|_{v,x}^2 d\tau,$$  \hspace{1cm} (3.51)

where we use (3.8), (3.42) and the smallness of $\epsilon$. Thus, we complete the proof of Lemma 3.3. \hspace{1cm} $\Box$

**Proof of Proposition 3.1.** With the three lemmas above, we are ready to prove Proposition 3.1 (ii). We sum up the estimates (3.8), (3.42) and (3.50) and take $\epsilon$ and $\delta_0$ suitably small to obtain

$$(1 + \kappa t)^\xi \|\varphi, \eta, \psi\|_{v,x,1}^2 + \kappa \int_0^t (1 + \kappa \tau)^\xi \|\varphi, \eta, \psi\|_{v-1,x,1}^2 d\tau$$
$$+ \int_0^t (1 + \kappa \tau)^\xi \|\eta_x, \psi_x, \psi_{xx}\|_{v,x}^2 d\tau$$
$$\leq C \|\varphi_0, \eta_0, \psi_0, \varphi_0, \eta_0, \psi_0\|_{\lambda, x}^2$$
$$+ C\kappa \gamma \int_0^t (1 + \kappa \tau)^\xi - 1 \|\varphi, \eta, \psi, \varphi_x, \eta_x, \psi_x\|_{v,x}^2 d\tau,$$  \hspace{1cm} (3.52)
where $C > 0$ is a constant independent of $T, \nu, \kappa, \varepsilon$, and $\delta$. Hence, applying induction arguments similar to those used in [10, 20] to (3.52) gives the desired estimate (3.2).

4 Stability of sonic steady-state

Theorem 4.1. Suppose that $M_+ = 1$ and $2 \leq \lambda < 5$ hold. Then, for arbitrary $\nu \in [2, \lambda]$, there exists a positive constant $\varepsilon_0$ such that if

$$\delta^{-1} \left\| (1 + Bx) \frac{\nu}{2} (\varphi_0, \eta_0, \psi_0) \right\|_1 + \delta^\frac{1}{2} \leq \varepsilon_0$$

(4.1a)

with

$$B = \frac{A_1 \gamma (\gamma + 1) \rho_+^{\gamma} + A_2 \alpha (\alpha + 1) k_2 \rho_+^\alpha}{2 \mu |u_+|^2} \delta,$$

(4.1b)

then the initial boundary value problem (2.12)-(2.16) has a unique global solution $(\varphi, \eta, \psi)$ which satisfies $(\varphi, \eta, \psi) \in Y_{(1 + Bx)} (\mathbb{R}_+)$ and

$$\left\| (1 + Bx) \frac{\nu}{2} (\varphi, \eta, \psi) \right\|_1 \leq C \left\| (1 + Bx) \frac{\nu}{2} (\varphi_0, \eta_0, \psi_0) \right\|_1 (1 + \delta t)^{-\frac{\lambda - \nu}{4}},$$

(4.2)

where $C > 0$ is a constant independent of time.

We define

$$N_{\lambda, B}(T) = \sup_{0 \leq t \leq T} \left\| (\varphi, \eta, \psi, \varphi_x, \eta_x, \psi_x) \right\|_{B, \lambda}.$$  

(4.3)

Proposition 4.1. Assume that the assumptions conditions in Theorem 4.1 hold. Suppose that $(\varphi, \eta, \psi)$ is a solution to the problem (2.12)-(2.16) which satisfies $(\varphi, \eta, \psi) \in Y_{(1 + Bx)} (0, T)$ for a certain constant $T > 0$ and given by (4.1b). Then for arbitrary $\nu \in [2, \lambda]$, there exist positive constants $C$ and $\varepsilon_1$ independent of $T$ such that if $\delta^{-1} N_{\lambda, B}(T) + \delta^{\frac{2}{n}} \leq \varepsilon_1$ is satisfied, it holds for any $t \in [0, T]$ and $\beta \geq 0$ that

$$\begin{align*}
(1 + \delta t)^{\frac{\lambda - \nu}{4} + \beta} &\left\| (\varphi, \eta, \psi, \varphi_x, \eta_x, \psi_x) \right\|^2_{V, B} \\
+ \delta^2 &\int_0^t (1 + \delta \tau)^{\frac{\lambda - \nu}{4} + \beta} \left\| (\varphi, \eta, \psi, \varphi_x, \eta_x, \psi_x) \right\|^2_{V - 2, B} \\
+ \delta &\int_0^t (1 + \delta \tau)^{\frac{\lambda - \nu}{4} + \beta} \left\| (\varphi, \varphi_x) \right\|^2_{V - 1, B} + \int_0^t (1 + \delta \tau)^{\frac{\lambda - \nu}{4} + \beta} \left\| (\eta_x, \psi_x, \psi_{xx}) \right\|^2_{V, B} \\
\leq C (1 + \delta t)^{\beta} &\left\| (\varphi_0, \eta_0, \psi_0, \varphi_{0x}, \eta_{0x}, \psi_{0x}) \right\|^2_{B, \lambda}.
\end{align*}$$

(4.4)
It is easy to gain the following estimates

\[ \| (\varphi, \eta, \psi) \|_{L^\infty} \leq \sqrt{2} N_{\lambda, B}(T) \leq \sqrt{2} \delta \varepsilon_1 \]  

(4.5)

\[ \| (1+ Bx)(\varphi, \eta, \psi) \|_{L^\infty} \leq \sqrt{2} (1 + B) N_{\lambda, B}(T) \leq 2 \sqrt{2} \delta \varepsilon_1 \]  

(4.6)

In order to prove Proposition 4.1, we need the following lower bound estimates.

**Lemma 4.1.** The stationary solution \( \tilde{u}(x) \) satisfies

\[ \tilde{u}_x(x) \geq D \left( \frac{\delta^2}{(1+Bx)^2} - C \frac{\delta^3}{(1+Bx)^3} \right), \]  

(4.7)

\[ D = \frac{A_1 \gamma (\gamma + 1) \rho_+^{\gamma - 1} + A_2 \alpha (\alpha + 1) k_+ \rho_+^{\alpha - 1}}{2 \mu |u_+|^2}, \quad B = D \delta \]  

(4.8)

for \( x \in (0, \infty) \).

**Proof.** Since we have \( F(1) = F'(1) = 0, F'''(\bar{w}) < 0 \) and \( 1 < \bar{w} \leq \frac{u_-}{u_+} \), the function \( F(\bar{w}) \) defined in (2.3) satisfies

\[ \frac{1}{2} F'' \left( \frac{u_-}{u_+} \right) (\bar{w} - 1)^2 \leq F(\bar{w}) \leq \frac{1}{2} F''(1)(\bar{w} - 1)^2, \]  

(4.9)

\[ \frac{1}{2} F''(1)(\bar{w} - 1)^2 - C(\bar{w} - 1)^3 \leq F(\bar{w}) \leq \frac{1}{2} F''(1)(\bar{w} - 1)^2. \]  

(4.10)

Substituting the equality \( F(\bar{w}) = \mu u_+(\bar{w} - 1) \) and solving (4.9) with respect to \( \bar{w} - 1 \) yield

\[ \frac{\delta}{|u_+|} \frac{1}{1+Bx} \leq (\bar{w} - 1) \leq \frac{\delta}{|u_+|} \frac{2 \mu |u_+|^2}{2 \mu |u_+|^2 + F''(\frac{u_-}{u_+}) \delta x}. \]  

(4.11)

Then, the substitution (4.11) into (4.10) with the aid of (2.2) leads to the desired estimate (4.7).

**Lemma 4.2.** There exists a constant \( C > 0 \) such that

\[ A_1 \gamma (\gamma - 1) \rho_+^{\gamma - 1} + A_2 \alpha (\alpha - 1) k_+ \rho_+^{\alpha - 1} \]  

\[ \leq - \left[ \frac{A_1 \rho_+^{\gamma} + A_2 \alpha k_+}{(1+k_+)} - |u_+|^2 \right]. \]  

(4.12)
Proof. Owing to $M_+ = 1$ and (2.1), we have
\[
 \frac{A_1 \gamma \tilde{\rho}^{\gamma} + A_2 \alpha \tilde{\rho}^{\alpha} \tilde{k}^\alpha}{(1+k) \tilde{\rho}} - |u_+|^2 \\
= - \frac{A_1 \gamma (\gamma-1) \rho_+^{\gamma-1} + A_2 \alpha (\alpha-1) \rho_+^{\alpha-1} k_+^{\alpha-1}}{1+k_+} (\tilde{w} - 1) + O((\tilde{w} - 1)^2). \tag{4.13}
\]
By substituting (4.11) into (4.13), we can get the desired estimate (4.12).

Lemma 4.3. There exists a constant $C > 0$ such that
\[
 \frac{A_1 \gamma (3-\gamma) \rho_+^{\gamma-2} + A_2 \alpha (3-\alpha) k_+^{\alpha-2}}{(1+k_+)(1+Bx)} \delta - C \frac{\delta^2}{(1+Bx)^2} \\
\leq - \frac{A_1 \gamma \tilde{\rho}^{\gamma-1} + A_2 \alpha \tilde{\rho}^{\alpha-1} \tilde{k}^{\alpha}}{(1+k) \tilde{\rho}} \tilde{u} + \frac{u_+^3}{\rho_+}. \tag{4.14}
\]
Proof. Due to $M_+ = 1$ and (2.1), we can get the following result
\[
 \frac{A_1 \gamma \tilde{\rho}^{\gamma-1} + A_2 \alpha \tilde{\rho}^{\alpha-1} \tilde{k}^\alpha}{(1+k) \tilde{\rho}} \tilde{u} - \frac{u_+^3}{\rho_+} \\
= - \frac{A_1 \gamma (3-\gamma) \rho_+^{\gamma-2} + A_2 \alpha (3-\alpha) k_+^{\alpha} \rho_+^{\alpha-2}}{(1+k_+)} |u_+| (\tilde{w} - 1) + O((\tilde{w} - 1)^2). \tag{4.15}
\]
With the help of the substitution of (4.11) into (4.15), we can obtain the estimate (4.14).

This paper will use the following Hardy type inequality to get the upper 5 for the index $\lambda$ as in [18]. For a rigorous proof of this lemma the reader is referred to [9].

Lemma 4.4. Let $\zeta \in C^1[0,\infty)$ and assume $\zeta > 0$, $\zeta_x > 0$ and $\zeta(x) \to \infty$ for $x \to \infty$. Then we have
\[
 \int_{\mathbb{R}^+} \psi^2 \zeta_x dx \leq 4 \int_{\mathbb{R}^+} \psi_x^2 \zeta_x^2 dx \tag{4.16}
\]
for $\psi_x \in L^2_{W_{v,B}}$ with $W_{v,B} = \zeta^2 / \zeta_x$. Here the coefficient 4 is the best possible constant and there is no function $\psi_x \in L^2_{W_{v,B}}$ with $W_{v,B} = \zeta^2 / \zeta_x$, $\psi \neq 0$, which attains the equality in (4.16).

By using Lemmas 4.1-4.4, we obtain the weighted $L^2$ estimate of $(\varphi, \eta, \psi)$. 

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Lemma 4.5. Assume that the same assumptions in Proposition 4.1 hold. Then the solution \((\varphi, \eta, \psi)\) to the problem (2.12)-(2.16) satisfies the following estimate for \(t \in [0, T)\)

\[
(1 + \delta t)^{\xi} \| (\varphi, \eta, \psi) \|^2_{V,B} + \delta^2 \int_0^t (1 + \delta \tau)^{\xi} \| (\varphi, \eta, \psi) \|^2_{V-2,B} d\tau \\
+ \delta \int_0^t (1 + \delta \tau)^{\xi} \| \varphi \|^2_{V-1,B} d\tau + \int_0^t (1 + \delta \tau)^{\xi} \| \psi_x \|^2_{V,B} d\tau \\
\leq C \| (\varphi_0, \eta_0, \psi_0) \|_{\lambda, B} + C \delta^{\xi} \int_0^t (1 + \delta \tau)^{\xi - 1} \| (\varphi, \eta, \psi) \|^2_{V,B} d\tau \\
+ C \int_0^t (1 + \delta \tau)^{\xi} (\delta \varepsilon_1 \| \varphi_x \|_{V-2,B} + \varepsilon_1 \| \eta_x \|_{V,B}) d\tau.
\]

(4.17)

Proof. We multiply (2.12a) by \(\tilde{\varphi} \eta W_{v,B}\), (2.12b) by \(G(0,0) \eta W_{v,B}\) and (2.12c) by \(\tilde{\varphi} \psi W_{v,B}\), where \(W_{v,B}\) is defined in Notation 1.1. The summation of resulting equations, and then integrating over \(\mathbb{R}_+\) imply

\[
\frac{d}{dt} \int W_{v,B} \left[ \frac{1}{2} \tilde{\varphi} \varphi^2 + \frac{1}{2} G(0,0) \eta^2 + \frac{1}{2} \tilde{\varphi} \psi^2 \right] dx \\
- \left[ W_{v,B} \left( \frac{u \tilde{\varphi} \varphi^2 + u G(0,0) \eta^2 + u \tilde{\varphi} \psi^2 + G(0,0) \tilde{\varphi} \eta \psi - \mu \tilde{\varphi} \frac{1}{(1 + \kappa) \rho} \psi \varphi}{2} \right) \right] (t, 0) \\
+ \int v BW_{v-1,B} \left[ \frac{(-u) \tilde{\varphi} \varphi^2 + (-u) G(0,0) \eta^2 + (-u) \tilde{\varphi} \psi^2 - G(0,0) \tilde{\varphi} \eta \psi}{2} \right] dx \\
+ \mu \int v BW_{v-1,B} \left( \frac{1}{(1 + \kappa)^2} \right) \psi \varphi dx + \mu \int W_{v,B} \left( \frac{1}{(1 + \kappa)^2} \right) \varphi^2 dx \\
+ \int W_{v,B} \left[ \tilde{\rho} \tilde{\varphi} x \psi^2 + G(0,0) \tilde{\varphi} x \eta^2 \right] dx \\
= -\sum_{i=1}^8 I_i,
\]

(4.18)

where

\[
I_1 = \mu \int v BW_{v-1,B} \left( \frac{1}{(1 + \kappa) \rho} \right) \psi \varphi dx \\
+ \mu \int W_{v,B} \left( \frac{1}{(1 + \kappa) \rho} \right) \psi^2 dx, \\
I_2 = \int W_{v,B} \left[ a \left( \frac{1}{a} \right) \tilde{\varphi} \varphi^2 - G(0,0) \tilde{\varphi} \eta \psi - \tilde{\varphi} \psi \frac{\varphi^2}{2} + G(0,0) \psi \frac{\eta^2}{2} - \tilde{\varphi} \psi \frac{\varphi^2}{2} \right] dx,
\]
First, we estimate terms in the left side of (4.18). The second term is estimated as

\[ I_3 = \int W_{v,B} \mu \tilde{\rho} \left( \frac{1}{(1+k)\rho} \right)_x \psi \psi_x dx, \]

\[ I_4 = \int W_{v,B} \left[ -G(0,0) \tilde{u}_x \phi \eta - G(0,0) b_x u \phi \eta + G(0,0) b \left( \frac{1}{a} \right)_x u \phi \eta \right. \]

\[ + \mu \tilde{\rho}_x \frac{1}{(1+k)\rho} \psi \psi_x - \mu \tilde{\rho} \tilde{u}_{xx} \psi \left( \frac{1}{(1+k)\rho} - \frac{1}{(1+k)\tilde{\rho}} \right) \]

\[ + \left( H(\phi, \eta) \left( \frac{1}{a} \right)_x - G(\phi, \eta) b_x \right) \tilde{\rho} \phi \psi \]

\[ dx, \]

\[ I_5 = \int W_{v,B} \left[ -(G(0,0))_x \tilde{\rho} \eta \psi + (G(\phi, \eta) - G(0,0)) \tilde{\rho} \tilde{\rho}_x \psi \right] dx, \]

\[ I_6 = \int W_{v,B} \left[ (H(\phi, \eta) - H(0,0)) \tilde{\rho} \tilde{\rho}_x \psi - (G(\phi, \eta) - G(0,0)) \tilde{\rho} b \phi \psi_x \right] dx, \]

\[ I_7 = \int W_{v,B} \left( G(\phi, \eta) - G(0,0) \right) \tilde{\rho} \eta \phi \psi dx, \]

\[ I_8 = \int W_{v,B} \left[ a \left( \frac{1}{a} \right)_x \psi \phi^2 + \tilde{\rho}_x \psi \phi \frac{\psi^2}{2} - \tilde{\rho}_x \psi \phi \frac{\psi^2}{2} - (G(0,0))_x \psi \phi \frac{\psi^2}{2} \right] dx. \]

First, we estimate terms in the left side of (4.18). The second term is estimated as follows:

\[- \left[ \int W_{v,B} \left( \frac{u}{2} \rho \phi^2 + \frac{u}{2} G(0,0) \eta^2 + \frac{u}{2} \tilde{\rho} \phi^2 + G(0,0) \tilde{\rho} \eta \phi - \mu \psi \psi_x \tilde{\rho} \frac{1}{(1+k)\rho} \right) \right] (t,0) \]

\[ = \frac{|u|}{2} \left[ \tilde{\rho}(0) \phi^2(t,0) + \frac{A_2 \gamma \mu^{\gamma-1}(0) + A_2 \kappa^{\alpha+1}(0)}{(1+k)\tilde{\rho}(0)} \eta^2(t,0) \right] \geq 0, \quad (4.19) \]

where we have used the boundary condition \( \psi(t,0)=0 \). Using \( M_+ = 1 \) and \( u_+ < 0 \), we can obtain

\[ \frac{(-u)}{2} \rho \phi^2 + \frac{(-u)}{2} G(0,0) \eta^2 + \frac{(-u)}{2} \tilde{\rho} \phi^2 - G(0,0) \tilde{\rho} \eta \phi \]

\[ \geq \rho_+ \frac{|u_+|}{2} \phi^2 + \frac{1}{2} (\eta, \psi) M_2 (\eta, \psi)^T - \frac{1}{2} \left( G(0,0) \tilde{u} - \frac{u_+^3}{\rho_+} \right) \eta^2 \]

\[- \left( G(0,0) \tilde{\rho} - u_+^2 \right) \eta \phi - C \eta \left( \phi^2 + \eta^2 + \psi^2 \right), \quad (4.20) \]

where \( (\cdot)^T \) represents the transpose of a row vector and \( M_2 \) is a symmetric matrix.
defined as follows

\[ M_2 = \begin{pmatrix} |u_+|^3 & -u_+^2 \\ \rho_+ & \rho_+ |u_+| \end{pmatrix}. \] (4.21)

It is easy to check that \( M_2 \) has two eigenvalues: 0, \( \frac{|u_+|^3}{\rho_+} + \rho_+ |u_+| \). There exists a change of coordinate

\[ \begin{pmatrix} \eta \\ \psi \end{pmatrix} = P \begin{pmatrix} \bar{\rho} \\ \bar{u} \end{pmatrix}, \] (4.22)

for which

\[ \begin{pmatrix} \eta \\ \psi \end{pmatrix}^T M_2 \begin{pmatrix} \eta \\ \psi \end{pmatrix} = \begin{pmatrix} \frac{|u_+|^3}{\rho_+} & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \bar{\rho} \\ \bar{u} \end{pmatrix}^2, \] (4.23)

where \( P \) is a matrix defined as follows

\[ P = \begin{pmatrix} 1 & -\frac{\rho_+}{u_+} \\ \rho_+ & u_+ \end{pmatrix}. \] (4.24)

By using (1.11), (4.6), (4.12), (4.14) and (4.20), we can gain

\[
\int v B W_{v-1,B} \left[ \frac{(-u)}{2} \bar{\rho} \phi^2 + \frac{(-u)}{2} G(0,0) \eta^2 + \frac{(-u)}{2} \bar{\rho} \psi^2 - G(0,0) \bar{\rho} \eta \psi \right] \, dx \\
\geq \frac{\rho_+ |u_+|}{2} v B \| \phi \|_{v-1,B}^2 + \left( \frac{|u_+|^3}{2 \rho_+} + \rho_+ |u_+| \right) v B \| \bar{\rho} \|_{v-1,B}^2 + \frac{\mu v B}{(1 + k_+)} \| \bar{u} \|_{v-2,B}^2 \\
- C \delta^2 \varepsilon_1 (\phi, \bar{\rho}, \bar{u}) \| \phi \|_{v-2,B}^2 - C \delta^2 \| \bar{\rho} \|_{v-1,B}^2 - C \delta^2 \| \bar{u} \|_{v-2,B}^2. \]

(4.25)

Using (4.22) and \( \psi(t,0) = 0 \), one can obtain the estimate for the forth term as follows:

\[
\frac{\mu}{(1 + k_+)} \int v B W_{v-1,B} \psi \phi \, dx \\
\geq \frac{\mu v (v-1) B^2}{2(1 + k_+)} \| \bar{u} \|_{v-2,B}^2 - C \delta^2 \| \bar{\rho} \|_{v-1,B}^2 - C \delta^2 \| \bar{u} \|_{v-2,B}^2. \]

(4.26)
Adding (4.26) and (4.27) to the third term in the left side of (4.18), we can gain

\[
\begin{align*}
\lambda \text{ will consider the case } \nu \text{ where we have used } \\
\nu \geq 1, 5 \geq \nu - 1 \text{ and } \\
C W W_1 \text{ for } \nu \geq 1 + k_+ \text{, and } \\
\int W_{v,B} \left[ \rho \tilde{u}_x \psi^2 + G(0,0) \tilde{u}_x \frac{\eta_1^2}{2} - (G(0,0))_x \tilde{u} \frac{\eta_1^2}{2} \right] dx \\
\geq \int W_{v,B} \left[ \rho + \psi^2 + \frac{A_1 \gamma (\gamma - 1) \rho_1 ^2 - 2 + A_2 \rho (a - 1) \rho_1 ^{a - 1} k_+ ^2 \eta_1 ^2}{(1 + k_+)} \right] \tilde{u}_x dx \\
- C \delta^2 \varepsilon_1 \| (\eta, \psi) \|_{v-3,B}^2 \\
\geq \frac{\mu}{(1 + k_+)} B^2 \| \tilde{u} \|_{v-2,B} - C \delta \left( \delta + \varepsilon_1 \right) \| \rho \|_{v-1,B}^2 - C \delta^2 \left( \delta + \varepsilon_1 \right) \| \tilde{u} \|_{v-2,B}^2. 
\end{align*}
\]

(4.27)

Using (1.15), (4.22) and (4.24), the fifth term in the left side of (4.18) is estimated as follows:

\[
\begin{align*}
\frac{\mu \psi \psi_x}{(1 + k_+)} dx \\
\int W_{v,B} \left[ \frac{(-u)}{2} \tilde{u} \psi^2 + \frac{(-u)}{2} G(0,0) \eta_1 ^2 + \frac{(-u)}{2} \tilde{u} \psi^2 - G(0,0) \tilde{u} \eta_1 \right] dx \\
+ \int W_{v,B} \left[ \tilde{u}_x \psi^2 + G(0,0) \tilde{u}_x \frac{\eta_1^2}{2} - (G(0,0))_x \tilde{u} \frac{\eta_1^2}{2} \right] dx \\
- vB \cdot \int W_{v,B} \frac{\mu \psi \psi_x}{(1 + k_+)} dx \\
\geq \frac{\rho + |u|}{2} \| \psi \|_{v-1,B}^2 + \frac{1}{2} \left( \frac{|u|}{\rho_+} + \rho + |u| \right) \delta \| \tilde{u} \|_{v-1,B}^2 \\
+ \frac{\mu}{1 + k_+} B^2 \left[ 1 - \frac{v(v - 3)}{2} \right] \| \tilde{u} \|_{v-2,B}^2 - C \delta^2 \| \tilde{u} \|_{v-1,B}^2 \\
- C \delta^2 \| \tilde{u} \|_{v-2,B}^2 - C \delta^2 \varepsilon_1 \| (\varphi, \tilde{u}) \|_{v-1,B}^2 \\
\geq c \delta \| \varphi \|_{v-1,B}^2 + c \delta^2 \| (\varphi, \psi) \|_{v-2,B}^2. 
\end{align*}
\]

(4.28)

Adding (4.26) and (4.27) to the third term in the left side of (4.18), we can gain

where we have used \( \nu \in [2, 3] \) and \( \delta, \varepsilon_1 \) suitably small in the last inequality. We will consider the case \( \lambda \in [3, 5] \) by using Hardy type inequality in Lemma 4.4 with \( \zeta = (\nu - 1)B(1 + Bx)^{\nu-1} \). Therefore, the sixth term is estimated as follows:

\[
\int W_{v,B} \frac{1}{1 + k_+} \psi^2 dx \geq \frac{\mu B^2 (v-1)^2}{4(1 + k_+)} \| \psi \|_{v-2,B}^2. 
\]

(4.29)

Let \( f_1(v) = 1 - \frac{v(v-3)}{2} \) and \( f_2(v) = \frac{(v-1)^2}{4} \). It is easy to check \( f_1(v) + f_2(v) > 0 \) for \( v \in (-1, 5) \). For \( v \in (3, \lambda] \), adding (4.28) to (4.29) and using \( c |(\eta, \psi)| \leq |(\tilde{\rho}, \tilde{u})| \leq C |(\eta, \psi)| \),
we can get
\[
vB \int W_{v-1,B} \left[ \frac{(-u)}{2} \bar{\rho} \varphi^2 + \frac{(-u)}{2} G(0,0) \eta^2 + \frac{(-u)}{2} \bar{\rho} \psi^2 - G(0,0) \bar{\rho} \eta \psi \right] dx \\
- vB \int W_{v-1,B} \bar{\rho} \mu \psi \psi_x \left( \frac{1}{1 + k} \right) dx + \int W_{v,B} \bar{\rho} \tilde{u} \psi^2 dx + \mu \int W_{v,B} \frac{1}{1 + k} \psi_x^2 dx \\
\geq c \delta \| \varphi \|^2_{v-2,B} + c \delta^2 \| (\eta, \psi) \|^2_{v-2,B} + c \| \psi_x \|^2_{v,B}.
\]

(4.30)

Next, we estimate each term in the right side of (4.18). Using (1.11), (2.13), (4.7), Sobolev inequality, Cauchy-Schwarz inequality, under the conditions \( \gamma \geq 1, \alpha \geq 1 \) and \( 2 \leq v \leq \lambda \), one has
\[
|I_1| \leq C \delta \| (\varphi, \eta) \|_{L^\infty} \left( \| \varphi \|^2_{v-2,B} + \| \psi_x \|^2_{v,B} \right) + C \| (\varphi, \eta) \|_{L^\infty} \| \psi_x \|_{v,B} \\
\leq C \delta \varepsilon_1 \| \psi_x \|_{v,B} + C \delta^2 \| \varphi \|^2_{v-2,B},
\]
\[
|I_2| \leq C \varepsilon_1 \| \psi_x \|^2_{v,B} + C \delta^2 \| \varphi \|^2_{v-2,B} + C \delta^2 \| (\eta, \psi) \|^2_{v-2,B},
\]
\[
|I_3| \leq C \delta \| \psi_x \|^2_{v,B} + C \delta \varepsilon_1 \| (\varphi_x, \eta_x) \|^2_{v-2,B} + C \delta^3 \| (\varphi, \psi) \|^2_{v-2,B},
\]
\[
|I_4| \leq C \delta \sqrt{\delta} \| \varphi_x \|^2_{v,B} + C \delta \sqrt{\delta} \| \varphi \|^2_{v-2,B} + C \delta^2 \sqrt{\delta} \| (\eta, \psi) \|^2_{v-2,B},
\]
\[
|I_5| \leq \left| \int W_{v,B} \left\{ [G(\varphi, \eta) - G(0,0) - G_\eta(0,0) \eta - G_\varphi(0,0) \varphi] \bar{\rho} \varphi_x \psi + G_\varphi(0,0) \bar{\rho} \varphi_x \psi \right\} dx \right| \\
\leq C \delta \sqrt{\delta} \| \varphi \|^2_{v-2,B} + C \delta^2 \sqrt{\delta} \| \varphi \|^2_{v-2,B} + C \delta^3 \varepsilon_1 \| \eta \|^2_{v-2,B},
\]
\[
I_6 = \int W_{v,B} \left[ G(\varphi, \eta) - G(0,0) - G_\varphi(0,0) \varphi - G_\eta(0,0) \eta \right] \bar{\rho} b \varphi_x \psi dx \\
+ \int W_{v,B} \left[ H(\varphi, \eta) - H(0,0) - H_\varphi(0,0) \varphi - H_\eta(0,0) \eta \right] \bar{\rho} \frac{1}{a} \varphi_x \psi dx \\
+ \int W_{v,B} G_\varphi(0,0) \bar{\rho} b \varphi_x \psi + \int W_{v,B} H_\varphi(0,0) \bar{\rho} \frac{1}{a} \varphi_x \psi \\
+ \int W_{v,B} G_\eta(0,0) \bar{\rho} b \eta \varphi_x \psi + \int W_{v,B} H_\eta(0,0) \bar{\rho} \frac{1}{a} \eta \varphi_x \psi =: \sum_{i=1}^{6} I_{6,i},
\]
\[
|I_6^1| + |I_6^2| \leq C \| (1 + Bx)(\varphi, \eta) \|_{L^\infty} \| (\varphi_x, \psi) \|^2_{v-2,B} \\
\leq C \delta^2 \varepsilon_1 \| \varphi \|^2_{v-2,B} + C \delta^2 \varepsilon_1 \| \varphi_x \|^2_{v-2,B},
\]
\[
|I_6^3| + |I_6^4| \leq C \| (1 + Bx) \psi \|_{L^\infty} \| (\varphi, \varphi_x) \|^2_{v-1,B} \leq C \delta \varepsilon_1 \| (\varphi, \varphi_x) \|^2_{v-1,B},
\]
\[
|I_6^5| + |I_6^6| \leq \left| \int vBW_{v-1,B} \left[ G_\eta(0,0) b \bar{\rho} + H_\eta(0,0) \frac{1}{a} \bar{\rho} \right] \eta \varphi \psi dx \right|
\]
Finally, combining (4.19)-(4.38) and (4.18), we can obtain

\[
\frac{d}{dt} \int W_{v,B} \left( \rho \frac{\partial}{\partial t} + G(0,0) \frac{\eta^2}{2} + \rho \frac{\psi^2}{2} \right) dx \\
+ c\delta \mu (\eta, \eta) \| \psi_x \|_{L^2}^2 + C \delta \| \psi \|_{L^2}^2 + c \| \psi_x \|_{L^2}^2 \\
\leq C \delta \epsilon_1 \| \psi \|_{L^2}^2 + C \epsilon_1 \| \eta_x \|_{L^2}^2
\]

with \( \delta \) and \( \epsilon_1 \) small enough. Multiplying (4.39) by \((1 + \delta \tau)^{\xi}\) and integrating in \( \tau \) over \([0,t]\), we obtain

\[
(1 + \delta \tau)^{\xi} \| (\eta, \eta) \|_{L^2}^2 + \delta^2 \int_0^t (1 + \delta \tau)^{\xi} \| (\eta, \eta) \|_{L^2}^2 d\tau \\
+ \int_0^t (1 + \delta \tau)^{\xi} \left( \delta \| \psi \|_{L^2}^2 + \| \psi_x \|_{L^2}^2 \right) d\tau \\
\leq C \| (\eta_0, \eta_0, \eta_0) \|_{L^2}^2 + C \delta \xi \int_0^t (1 + \delta \tau)^{\xi-1} \| (\eta, \eta) \|_{L^2}^2 d\tau \\
+ C \delta \epsilon_1 \int_0^t (1 + \delta \tau)^{\xi} \| \psi_x \|_{L^2}^2 d\tau + C \epsilon_1 \int_0^t (1 + \delta \tau)^{\xi} \| \eta_x \|_{L^2}^2 d\tau.
\]

Hence (4.17) is gained and the proof of Lemma 3.1 is completed. \(\square\)

In order to get the proof of Proposition 4.1, we need to obtain the weighted estimate of \((\eta_x, \eta_x, \psi_x)\).

**Lemma 4.6.** Assume that the same assumptions in Proposition 4.1 hold. Then the solution \((\eta, \eta, \psi)\) to (2.12)-(2.16) satisfies the following energy estimate for \(t \in [0,T]\):

\[
+ \int W_{v,B} \left[ \frac{G(0,0)}{\frac{\eta^2}{2} + \rho \frac{\psi^2}{2}} \right] \eta \psi d\tau
\]
Proof. We multiply (3.26) by $\mu W_{v,B}$, (3.25) by $W_{v,B}$ and (3.27) by $W_{v,B}$. The summation of resulting equations, and then integrating over $\mathbb{R}_+$ imply

$$\frac{d}{dt} \int W_{v,B} \left[ \frac{1}{2} \dot{q}_x^2 + \frac{\mu}{2} \ddot{q}_x^2 + (1 + \bar{k}) \bar{\rho}^2 \ddot{q}_x \mathcal{P} \right] dx$$

$$- \left[ W_{v,B} \left( \frac{\mu}{2} \dot{q}_x^2 + \frac{\mu}{2} \ddot{q}_x^2 - (1 + \bar{k}) \bar{\rho}^2 \ddot{q}_x \mathcal{P} \right) \right] (0,0)$$

$$- \int v B W_{v-1,B} \left[ \frac{\mu}{2} \dot{q}_x^2 + \frac{\mu}{2} \ddot{q}_x^2 \right] dx$$

$$+ \int v B W_{v-1,B} (1 + \bar{k}) \bar{\rho}^2 \eta \dot{q}_x dx + \int G(0,0) \eta^2 dx$$

$$=: - \sum_{i=1}^{a} I_i \quad (4.42)$$

where

$$I_1 = \int W_{v,B} \frac{1}{2} \ddot{q}_x \left( \dot{q}_x^2 + 3 \mu \ddot{q}_x^2 \right) dx,$$

$$I_2 = \int W_{v,B} \left[ \frac{1}{2} \left( \dot{q}_x^2 + 3 \mu \ddot{q}_x^2 \right) \psi_x - \mu b \dot{q}_x \dot{q}_x \psi_x \right] dx,$$

$$I_3 = \int W_{v,B} \left[ \mu (\eta - b \dot{q}_x) \eta \psi_{xx} - \mu (1 + \bar{k}) \bar{\rho}^2 \left( \frac{1}{(1 + \bar{k}) \bar{\rho}^2} - \frac{1}{(1 + \bar{k}) \bar{\rho}^2} \right) \eta \psi_{xx} \right] dx,$$

$$I_4 = \int W_{v,B} \left[ (G(\psi, \eta) - G(0,0)) (1 + \bar{k}) \bar{\rho}^2 \left( \eta^2 - b \dot{q}_x \eta \right) \right.$$

$$\left. + (H(\psi, \eta) - H(0,0)) (1 + \bar{k}) \bar{\rho}^2 \frac{1}{a} \dot{q}_x \eta \right] dx,$$

$$I_5 = \int W_{v,B} (1 + \bar{k}) \bar{\rho}^2 (\eta + u \eta \dot{q}_x) \psi_x dx,$$

$$I_6 = \int W_{v,B} \left[ -2 \mu b_x \varphi \psi_x \psi_x + a \left( \frac{1}{a} \right) \varphi \psi_x (\varphi_x + \mu b \dot{q}_x) \right] dx,$$

$$I_7 = \int W_{v,x} \left[ \mu \left( ba \left( \frac{1}{a} \right) u - b \ddot{u}_x - b \dot{u} \right) \varphi_x \right] \psi_x \eta \left( 1 \right)$$
follows:

First, we estimate terms in the left side of (4.42). The second term is estimated as follows:

where we let \( \delta \) small enough in the last inequality. The forth term is estimated as follows:

\[
\int v B W_{v-1,B} (1 + \tilde{k}) \tilde{\rho}^2 \tilde{\eta} \psi dx
\]

\[
= v(v-1)B^2 \int W_{v-2,B} (1 + \tilde{k}) \tilde{\rho}^2 \left[ (\rho u - \tilde{\rho} \tilde{u}) + ba(u - \tilde{k}) \right] \psi dx
\]
\begin{equation}
+ vB \int W_{v-1, B} \left[ (1+k)^2 \right] \left[ (\rho u - \bar{\rho} \bar{u}) + ba u(k - \bar{k}) \right] \psi \, dx \\
+ vB \int W_{v-1, B} (1+k)^2 \{ (\rho u - \bar{\rho} \bar{u}) + ba u(k - \bar{k}) \} \psi_x \\
+ \left[ (ba)_x u + ba \bar{u}_x + ba \psi_x \right] (k - \bar{k}) \psi \, dx \\
\leq C \delta^2 \| (\varphi, \eta, \psi) \|^2_{v-2, B} + C \| \psi_x \|^2_{v, B}.
\end{equation}

(4.45)

where we have used (1.11), (2.12b) and Cauchy-Schwarz inequality. The fifth term is estimated as follows:

\begin{equation}
\int W_{v, B} G(0, 0) \eta_x^2 \geq \left( \frac{A_1 \gamma \rho^{-1} + A_2 \alpha k^2 \rho^{-1}}{1+k} \right) \| \eta_x \|^2_{v, B} \\
\geq \frac{A_1 \gamma \rho^{-1} + A_2 \alpha k^2 \rho^{-1}}{2(1+k)} \| \eta_x \|^2_{v, B},
\end{equation}

(4.46)

where we have used (1.11) and the condition \( \delta \) small enough. Next, we estimate each term in the right side of (4.42). Using (1.11), (2.12), Sobolev inequality, Young inequality, Cauchy-Schwarz inequality and \( 2 \leq v \leq \lambda \), one has

\begin{align}
|J_1| & \leq C \delta^2 \| (\varphi_x, \eta_x) \|^2_{v-2, B}, \\
|J_2| & \leq C \| (1+Bx) \psi_x \|_{L^\infty} \| (\varphi_x, \eta_x) \|^2_{v-1, B} \\
& \leq C \| (1+Bx) \psi_x \| + \| (1+Bx) \psi_x \| \| (\varphi_x, \eta_x) \|^2_{v-1, B} \\
& \leq C \delta \epsilon_1 \| \psi_{xx} \|^2_{v, B} + C \delta \epsilon_1 \| \psi_x \|^2_{v, B} + C \delta \epsilon_1 \| (\varphi_x, \eta_x) \|^2_{v-1, B}, \\
|J_3| + |J_4| & \leq C \| (1+Bx) (\varphi, \eta) \|_{L^\infty} \left( \| (\varphi_x, \eta_x) \|^2_{v-1, B} + \| \psi_{xx} \|^2_{v-1, B} \right) \\
& \leq C \delta \epsilon_1 \| \psi_{xx} \|^2_{v-1, B} + C \delta \epsilon_1 \| (\varphi_x, \eta_x) \|^2_{v-1, B}, \\
|J_5| & \leq C \| \psi_x \|^2_{v, B} + C \delta^2 \| (\eta, \varphi) \|^2_{v-2, B}, \\
|J_6| & \leq C \delta^2 \| (\varphi_x, \eta_x, \psi_x) \|^2_{v-2, B} \leq C \delta^3 \epsilon_1 \| (\varphi_x, \eta_x, \psi_x) \|^2_{v-2, B}, \\
|J_7| & \leq C \delta^2 \| (\varphi_x, \eta_x, \psi_x) \|^2_{v-1, B} + C \delta^2 \| \varphi \|^2_{v-2, B} + C \delta^3 \| \eta \|^2_{v-2, B}, \\
|J_8| & \leq C \delta^2 \| (\varphi, \eta, \psi) \|^2_{v-2, B} + C \delta^2 \| \psi \|^2_{v-2, B} + C \delta^2 \| (\varphi, \eta) \|^2_{v-2, B}, \\
|J_9| & \leq C \delta^2 \| (\eta_x, \psi_x) \|^2_{v, B} + C \delta^2 \| \psi \|^2_{v-2, B} + C \delta^4 \| (\varphi, \eta) \|^2_{v-2, B}.
\end{align}

(4.47)-(4.54)

Finally, the substitution (4.43)-(4.54) into (4.42) leads to

\begin{equation}
\frac{d}{dt} \int W_{v, B} \left[ \frac{\varphi_x^2}{2} + \frac{\mu \eta_x^2}{2} + (1+k) \rho^2 \eta_x \psi \right] \, dx + \delta \| (\varphi_x, \eta_x) \|^2_{v-1, B}
\end{equation}

(4.54)
Proof. We multiply (2.12c) by $R$ where we have used Cauchy-Schwarz inequality and (4.40). Assume that the same assumptions in Proposition 3.1 hold. Then the solution $(\phi, \eta, \psi)$ to the problem (2.12)-(2.16) satisfies the following energy estimate for $t \in [0, T)$

$$
(1 + \delta \tau)^\xi \| \phi_x \|^2_{v,B} + \int_0^t (1 + \delta \tau)^\xi \| \phi_{xx} \|^2_{v,B} d\tau
\leq C \| (\phi_0, \eta_0, \phi_{0x}, \eta_{0x}) \|^2_{\lambda, B} + C\delta^2 \int_0^t (1 + \delta \tau)^\xi \| \phi_{xx} \|^2_{v,B} d\tau
+ C\delta^2 \int_0^t (1 + \delta \tau)^\xi^{-1} \| (\phi, \eta, \psi, \phi_x, \eta_x) \|^2_{v,B} d\tau,
$$

(4.56)

where we have used Cauchy-Schwarz inequality and (4.40). □

Lemma 4.7. Assume that the same assumptions in Proposition 3.1 hold. Then the solution $(\phi, \eta, \psi)$ to the problem (2.12)-(2.16) satisfies the following energy estimate for $t \in [0, T)$

$$
(1 + \delta \tau)^\xi \| \phi_x \|^2_{v,B} + \int_0^t (1 + \delta \tau)^\xi \| \phi_{xx} \|^2_{v,B} d\tau
\leq C \| (\phi_0, \eta_0, \phi_{0x}, \eta_{0x}) \|^2_{\lambda, B}
+ C\delta^2 \int_0^t (1 + \delta \tau)^\xi^{-1} \| (\phi, \eta, \psi, \phi_x, \eta_x, \psi_x) \|^2_{v,B} d\tau.
$$

(4.57)

Proof. We multiply (2.12c) by $-\psi_{xx} W_{v,B}$ and integrate the resulting equality in $x$ over $R_+$, with the boundary condition $\psi(t, 0) = 0$ to obtain

$$
\frac{d}{dt} \int W_{v,B} \frac{\psi_x^2}{2} dx + \nu B \int W_{v-1,B} \psi_x \psi_x dx + \mu \int W_{v,B} \frac{\phi_{xx}^2}{(1 + \kappa \rho)} dx =: -\sum_{i=1}^3 K_i,
$$

(4.58)

where

$$
K_1 = \int W_{v,B} \left[ -u \phi_x \phi_{xx} - G(0, 0) \eta_x \phi_{xx} \right] dx,
K_2 = \int W_{v,B} \left[ -\bar{u}_x \psi_x \phi_{xx} + \mu \bar{u}_x \left( \frac{1}{(1 + \kappa \rho)} - \frac{1}{(1 + \kappa \rho)} \right) \psi_{xx} - H(\phi, \eta) \left( \frac{1}{a} \right) \phi \psi_{xx}
+ G(\phi, \eta) b_x \phi \psi_{xx} - (G(\phi, \eta) - G(0, 0)) \bar{\rho}_x \phi_{xx} \right] dx,
$$
$K_3 = \int W_{\nu,B} \left[ (H(\phi,\eta) - H(0,0)) \frac{1}{a} \phi_x \psi_{xx} + (G(\phi,\eta) - G(0,0)) b \phi_x \psi_{xx} 
right. \\
\left. + \left[ G(\phi,\eta) - G(0,0) \right] \eta_x \psi_{xx} \right] dx.$

First, we estimate terms in the left side of (4.58). The second term is estimated as follows:

$$|\nu B \int W_{\nu-1,B} \psi_x \psi_t| \leq C \delta \|(\eta_x, \psi_x, \psi_{xx})\|^2_{v-1,B} + C \delta^2 \varepsilon_1 \|\phi_x\|^2_{v-1,B} + C \delta^3 \|(\phi, \eta, \psi)\|^2_{v-2,B},$$

(4.59)

where we have used (2.12c), (1.11), (4.6), Sobolev inequality and Cauchy-Schwarz inequality. The third term is estimated as follows:

$$\mu \int W_{\nu,B} \frac{1}{(1+k)\rho} \psi_{xx}^2 \geq \left( \frac{\mu}{(1+k)\rho} - C \delta \varepsilon_1 - C \delta \right) \|\psi_{xx}\|^2_{v,B}$$

$$\geq \frac{\mu}{2(1+k)\rho} \|\psi_{xx}\|^2_{v,B},$$

(4.60)

if $\delta$ and $\varepsilon_1$ are small enough. Next, we estimate each term in the right side of (4.58). Using (1.11), (4.6), Sobolev inequality and Cauchy-Schwarz inequality, under the condition of $2 \leq \nu \leq \lambda$, we can obtain

$$|K_1| \leq C \|(\eta_x, \psi_x)\|^2_{v,B} + \frac{1}{8(1+k)\rho} \|\psi_{xx}\|^2_{v,B},$$

(4.61)

$$|K_2| \leq C \delta^2 \|(\phi, \eta, \psi)\|^2_{v-2,B} + C \delta^2 \|\psi_{xx}\|^2_{v-2,B},$$

(4.62)

$$|K_3| \leq C \delta \varepsilon_1 \|(\phi_x, \eta_x)\|^2_{v-1,B} + C \delta \varepsilon_1 \|\psi_{xx}\|^2_{v,B},$$

(4.63)

Finally, we substitute (4.59)-(4.63) into (4.58), which yields

$$\frac{d}{dt} \int W_{\nu,B} \frac{1}{2} \psi_x^2 dx + \mu \frac{1}{4(1+k)\rho} \|\psi_{xx}\|^2_{v,B} dx$$

$$\leq C \|(\eta_x, \psi_x)\|^2_{v,B} + C \delta^2 \|\phi_x\|^2_{v-2,B} + C \delta \varepsilon_1 \|(\phi, \eta, \psi)\|^2_{v-2,B},$$

(4.64)

if $\delta$ is small enough. Multiplying (4.64) by $(1+\delta \tau)^\xi$ and integrating the resultant inequality in $\tau$ over $\tau \in [0,t]$, we can obtain

$$(1+\delta t)^\xi \|\psi_x\|^2_{v,B} + \int_0^t (1+\delta \tau)^\xi \|\psi_{xx}\|^2_{v,B} d\tau$$
\[
\begin{align*}
\leq C \left\| \langle \varphi_0, \eta_0, \varphi_0, \varphi_{0x}, \eta_{0x}, \varphi_{0x} \rangle \right\|_{\lambda, B}^2 + C \delta^2 \int_0^t (1 + \delta \tau)^{\varepsilon - 1} \left\| \langle \varphi, \eta, \varphi_x, \eta_x, \varphi_x \rangle \right\|_{\nu, B}^2 d\tau,
\end{align*}
\]

where we have used (4.17), (4.41) and the smallness of \( \delta \) and \( \varepsilon_1 \). \( \square \)

We sum up the estimates (4.17), (4.41) and (4.57) and take \( \delta \) and \( \varepsilon_1 \) suitably small, which yields

\[
\begin{align*}
(1 + \delta t)^{\varepsilon_1} \left\| \langle \varphi, \eta, \varphi_x \rangle \right\|_{\nu, B, 1}^2 + \delta^2 \int_0^t (1 + \delta \tau)^{\varepsilon_1} \left\| \langle \varphi, \eta, \varphi_x \rangle \right\|_{\nu - 2, B}^2 &+ \delta \int_0^t (1 + \delta \tau)^{\varepsilon_1} \left\| \langle \varphi, \varphi_x \rangle \right\|_{\nu - 1, B}^2 d\tau + \int_0^t (1 + \delta \tau)^{\varepsilon_1} \left\| \langle \eta_x, \varphi_x, \varphi_{xx} \rangle \right\|_{\nu, B}^2 d\tau \\
&\leq C \left\| \langle \varphi_0, \eta_0, \varphi_{0x}, \eta_{0x}, \varphi_{0x} \rangle \right\|_{\lambda, B}^2 + C \delta^2 \int_0^t (1 + \delta \tau)^{\varepsilon_1 - 1} \left\| \langle \varphi, \eta, \varphi_x, \eta_x, \varphi_x \rangle \right\|_{\nu, B}^2 d\tau,
\end{align*}
\]

where \( C > 0 \) is a constant independent of \( T, \nu \). Applying induction arguments to (4.66), we can acquire

\[
\begin{align*}
(1 + \delta t)^{\varepsilon_1 + \beta} \left\| \langle \varphi, \eta, \varphi_x, \eta_x, \varphi_x \rangle \right\|_{\nu, B}^2 + \delta^2 \int_0^t (1 + \delta \tau)^{\varepsilon_1 + \beta} \left\| \langle \varphi, \eta, \varphi_x, \eta_x, \varphi_x \rangle \right\|_{\nu - 2, B}^2 &+ \delta \int_0^t (1 + \delta \tau)^{\varepsilon_1 + \beta} \left\| \langle \varphi, \varphi_x \rangle \right\|_{\nu - 1, B}^2 d\tau + \int_0^t (1 + \delta \tau)^{\varepsilon_1 + \beta} \left\| \langle \eta_x, \varphi_x, \varphi_{xx} \rangle \right\|_{\nu, B}^2 d\tau \\
&\leq C (1 + \delta t)^{\beta} \left\| \langle \varphi_0, \eta_0, \varphi_{0x}, \eta_{0x}, \varphi_{0x} \rangle \right\|_{\lambda, B}^2.
\end{align*}
\]

Thus,

\[
\left\| \langle \varphi, \eta, \varphi_x, \eta_x, \varphi_x \rangle \right\|_{\nu, B} \leq C (1 + \delta t)^{-\frac{\varepsilon_1 + \beta}{\nu - 1}} \left\| \langle \varphi_0, \eta_0, \varphi_{0x}, \eta_{0x}, \varphi_{0x} \rangle \right\|_{\lambda, B}.
\]

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