Based on the Liouville-von Neumann equation, we obtain a closed system of equations for the description of a qutrit or coupled qutrits in an arbitrary, time-dependent, external magnetic field. The dependence of the dynamics on the initial states and the magnetic field modulation is studied analytically and numerically. We compare the relative entanglement measure’s dynamics in bi-qudits with permutation particle symmetry. We find the magnetic field modulation which retains the entanglement in the system of two coupled qutrits. Analytical formulae for the entanglement measures in finite chains from 2 to 6 qutrits or 3 quartits are presented.

Keywords: Entanglement; qudit; multiqudit chain.

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1. Introduction

Multi-level quantum systems are studied extensively, since they have wide applications. Some of the existing analytical results for spin 1 are derived in terms of a coherent vector. The class of exact solutions for a three-level system is given in Ref. 3. The application of coupled multi-level systems in quantum devices is actively studied. The study of these systems is topical in view of possible applications for useful work in microscopic systems. Exact solutions for two uncoupled qutrits interacting with the vacuum are obtained in Ref. 5. For the case of qutrits interacting with a stochastic magnetic field, the exact solutions are obtained in Ref. 7. The exact solutions for coupled qudits in an alternating magnetic field, to our knowledge, have not yet been found. The entanglement in multi-particle coupled systems is an important resource for many problems in the quantum information science, but its quantitative value computing is difficult because of different types of entanglement. Multi-dimensional entangled states are interesting both for the study of the foundations of quantum mechanics and for the topicality of developing new protocols for quantum communication. For example, it was shown that for maximally entangled states of two quantum systems, the qudits break the local realism stronger than the qubits and the entangled qudits are less influenced by noise than the entangled qubits.
Using entangled qutrits or qudits instead of qubits is more protective from interception. From a practical point of view, it is clear that generating and saving the entanglement in a controlled manner is the primary problem for the realization of quantum computers. Maximally entangled states are best suited for the protocols of quantum teleportation and quantum cryptography.

The entanglement and the symmetry are two basic notions of quantum mechanics. We study the dynamics of multipartite systems, which are invariant at any subsystem permutation. The aim of this work is finding exact solutions for the dynamics of coupled qudits interacting with an alternating magnetic field as well as the comparative analysis of the entanglement measures in a finite chain of coupled qudits. The paper is organized as following. The Hamiltonian of the anisotropic qutrit in an arbitrary alternating magnetic field is described in Sec. II. Then the system of equations for the description of the qutrit dynamics is derived in the Bloch vector representation. We introduce a consistent magnetic field, which describes an entire class of field forms. In section III we find an analytical solution for the density matrix in the case of isotropic interaction. Analytical formulae, which describe the entanglement in finite spin chains of qutrits or quartits, are presented in Sec. IV. The results are demonstrated graphically in Sec. V at specific parameters. The conclusions are given in Sec. VI.

2. Qutrit

2.1. Qutrit Hamiltonian and Liouville-von Neumann equation

We take the qutrit Hamiltonian (for the spin particle with \( s=1 \)) in the space of one qutrit \( C^3 \) in the basis \( |1\rangle = (1, 0, 0), \ |0\rangle = (0, 1, 0), \ |−1\rangle = (0, 0, 1) \), in an external magnetic field \( \overrightarrow{h} = (h_1, h_2, h_3) \) with anisotropy, in the form

\[
\hat{H}(\overrightarrow{h}) = h_1 S_1 + h_2 S_2 + h_3 S_3 + Q(S_2^3 - \frac{s(s+1)}{3} E_{2s+1 \times 2s+1}) + d(S_1^2 - S_2^2), \quad (1)
\]

where \( h_1, h_2, h_3 \) are the Cartesian components of the external magnetic field in frequency units (we assume \( \hbar = 1 \), Bohr magneton \( \mu_B = 1 \)); \( S_1, S_2, S_3 \) are the spin-1 matrices; \( E_{2s+1 \times 2s+1} \) is the unity matrix; \( Q, d \) are the anisotropy constants. When the constants \( Q, d \) are zeros, then the two Hamiltonian eigenvalues are symmetrically placed with respect to the zero level.

There exist many useful bases. Allard and Hard (AH) formed the Hermitian basis \( C_\alpha \) from the linear combinations of the irreducible tensor operators. This basis is normalized so that \( S_1 = C_{1,x}, S_2 = C_{1,y}, S_3 = C_{1,z} \), irrespective of the spin quantum number \( s \). It is convenient to construct the spin Hamiltonian for any spin. Hereinafter we use the Hermitian basis. From the physical point of view, for important physical applications the basis is preferred. It is not necessary for the basis to be Hermitian since the results of the calculations are independent of the choice of base, but there is a significant advantage of the Hermitian basis. It is useful that the Liouville-von Neuman equation does not involve complex numbers.
and can be solved using real algebra. It makes numerical calculations faster and simplifies the interpretation of the equation system. The transition matrix determines the coupling between the generalized Gell-Mann and (AH) Hermitian matrix bases. This coupling for qutrit is presented in Ref. 9.

The qutrit dynamics in a magnetic field is described in the density matrix formalism using the Liouville-von Neumann equation

$$i\partial_t \rho = [\hat{H}, \rho], \quad \rho(t = 0) = \rho_0. \quad (2)$$

It is convenient to rewrite Eq. (2) presenting the density matrix $\rho$ in the decomposition with a full set $11$ of orthogonal Hermitian matrices

$$\rho = \frac{1}{\sqrt{6}} C_\alpha R_{\alpha}. \quad (3)$$

Since $\text{Tr} C_i = 0$ for $1 \leq i \leq 8$, then from the condition $\text{Tr} \rho = R_0$ it follows that $R_0 = 1$. And although the results are independent of the basis choice, in this basis the functions $R_i = \text{Tr} \rho C_i$ have the concrete physical meaning $11$. The values $R_1, R_2, R_3$ are the polarization vector Cartesian components; $R_4$ is the two-quantum coherence contribution in $R_2$; $R_5$ is the one-quantum anti-phase coherence contribution in $R_2$; $R_6$ is the contribution of the rotation between the phase and anti-phase one-quantum coherence; $R_7$ is the one-quantum anti-phase coherence contribution in $R_1$; $R_8$ is the two-quantum coherence contribution in $R_1$.

The Liouville-von Neumann equation in terms of the functions $R_i$ takes the form of a closed system of 8 real differential first-order equations. This system of equations can be written in a compact form as following $12, 13$

$$\partial_t R_i = e_{ijl} h_l R_j, \quad (4)$$

where $e_{ijl}$ are the structure constants, $h_i = (h_1, h_2, h_3, 0, 0, 0, Q \sqrt{3}, 0, d)$ are the Hamiltonian components Eq. (1) in the basis $C_\alpha$.

### 2.2. The consistent field

Let us consider the qutrit dynamics in an alternating field of the form

$$\vec{h}(t) = (\omega_1 \text{cn}(\omega t | k), \omega_1 \text{sn}(\omega t | k), \omega_0 \text{dn}(\omega t | k)), \quad (5)$$

where cn, sn, dn are the Jacobi elliptic functions $14$. Such field modulation under the changing of the elliptic modulus $k$ from 0 to 1 describes the whole class of field forms from trigonometric $15$ ($\text{cn}(\omega t | 0) = \cos \omega t, \text{sn}(\omega t | 0) = \sin \omega t, \text{dn}(\omega t | 0) = 1$ ) to the exponentially impulse ones ($\text{cn}(\omega t | 1) = 1, \text{sn}(\omega t | 1) = \text{th} \omega t, \text{dn}(\omega t | 1) = \text{ch} \omega t$) $16$. The elliptic functions $\text{cn}(\omega t | k)$ and $\text{sn}(\omega t | k)$ have the real period $\frac{4K}{\omega}$, while the function $\text{dn}(\omega t | k)$ has a period of half the duration. Here $K$ is the full elliptic integral of the first kind $14$. In other words, even though the field is periodic with a common real period $\frac{4K}{\omega}$, but as we can see, the frequency of the longitudinal field amplitude modulation is twice as high as that of the transverse field. We call such
field consistent.

Let us make use of the substitution \( \rho = \alpha_1^{-1} r \alpha_1 \) with the diagonal matrix \( \alpha_1 = \text{diag}(f,1,f^{-1}) \), where \( f(\omega t|k) = \text{cn}(\omega t|k) + i \text{sn}(\omega t|k) \). Then we obtain the equation for the matrix \( r \) in the form

\[
i \partial_t r = [\alpha_1 \hat{H} \alpha_1^{-1} - i \alpha_1 \partial_k (\alpha_1^{-1}), r].
\]

The equation for the matrix \( r \) written as following

\[
i \partial_t r = [\omega_1 S_1 + \delta \text{dn}(\omega t|k) S_3], \quad r(t = 0) = \rho_0, \quad \delta = \omega_0 - \omega.
\]

At \( k = 0 \) equation (7) describes the dynamics of the qutrit in a circularly polarized field. The exact solutions of this equation are known, and under certain initial conditions the explicit formulae are given in Ref. [19]. At the exact resonance, \( \omega = \omega_0 \) it is straightforward to present \( \rho \) in the deformed field \( k \neq 0 \) for the given initial condition \( \rho = \rho_0 \):

\[
\rho(t) = \alpha_1^{-1} e^{-i \omega t S_1} \rho_0 e^{i \omega t S_1} \alpha_1.
\]

Explicit solutions for specific initial conditions are given in Ref. [9].

3. Bi-qutrit

In the space \( C^3 \otimes C^3 \) the bi-qutrit density matrix can be written in the Bloch representation

\[
\rho = \frac{1}{6} R_{\alpha \beta} C_\alpha \otimes C_\beta, \quad R_{00} = 1, \quad \rho(t = 0) = \rho_0.
\]

where \( \otimes \) denotes the direct product. The functions \( R_{nm}, R_{0m} \) characterize the individual qutrits and functions \( R_{nm} \) characterize their correlations.

Let us consider the Hamiltonian of the system of two qutrits with anisotropic and exchange interaction in a magnetic field in the following form

\[
H_2 = \hat{H}(\vec{h}) \otimes E_{2s+1} \otimes E_{2s+1} + E_{2s+1} \otimes \hat{H}(\vec{h}) + JS_1 \otimes S_1,
\]

where \( \vec{h} \) and \( \vec{h} \) are the magnetic field vectors in frequency units, which operate on the first and the second qutrits respectively, and \( J \) is the constant of isotropic exchange interaction. We study the dynamics of two qutrits in the consistent magnetic field \( \vec{h} = (\omega_1 \text{cn}(\omega t|k)), \omega_1 \text{sn}(\omega t|k), \omega_0 \text{dn}(\omega t|k)) \), \( \vec{h} = (\omega_1 \text{cn}(\omega t|k)), \omega_1 \text{sn}(\omega t|k), \omega_0 \text{dn}(\omega t|k)) \) at the anisotropy constants equal to 0.

Let us transform the matrix density \( \rho = \alpha_2^{-1} r_2 \alpha_2 \) with the matrix \( \alpha_2 = \alpha_1 \otimes \alpha_1 \). The equation for the matrix \( r_2 \) takes the form

\[
i \partial_t r_2 = [\hat{H}(\text{dn}(\omega t|k)), r_2] \text{ with the transformed Hamiltonian } \tilde{H}(\text{dn}(\omega t|k)).
\]

Since \( \text{dn}(\omega t|k) = 1 \), then the transformed Hamiltonian \( \tilde{H} \) does not depend on time, and the solution for the density matrix in the circularly polarized field has the form

\[
\rho(t) = \alpha_2^{-1} e^{-i \tilde{H} t} \rho_0 e^{i \tilde{H} t} \alpha_2 |k = 0.
\]
Entanglement dynamics in finite qudit chain

In the consistent field at resonance $\omega = \omega_0 = \omega_1 = \omega_0 = h$ at equal $\omega_1 = \omega_1$, the Hamiltonian eigenvalues equal to $-2J, -J, J - 2\omega_1, -J - \omega_1, J - \omega_1, J + \omega_1, J + 2\omega_1$. This allows to find the exact solution in the closed form for any initial condition since the matrix exponent $e^{iHt}$ in this case can be calculated analytically. For a larger number of the qudits with a pairwise isotropic interaction, the generalization is evident. In the case of interaction of qudits with a different dimensionality, the reduction of the original system to the system with constant coefficients can be done by choosing, for example, the transformation matrix for spin-$3/2$ and spin-$2$

\[ \text{diag} \left( f_{3/2}, f_{1/2}, f_{-1/2}, f_{-3/2} \right) \otimes \text{diag} \left( f_2, f, f_1, f_-1, f_-2 \right). \]  

(12)

However, the Hamiltonian eigenvalues cannot be found in a simple analytical form because of the lowering of the system’s symmetry.

4. Analytical formulae for entanglement measures

4.1. Entanglement in the bi-qutrit

For the initial maximally entangled state which is symmetrical at the particle permutation

\[ |\psi> = \frac{1}{\sqrt{3}} \sum_{i=-1}^{1} |i> \otimes |i>, \]  

(13)

in the consistent field at the resonance $\omega = \omega_0 = \omega_1 = h$ at equal $\omega_1 = \omega_1$, the exact solution for the correlation functions is given in Ref. 9. The correlation functions have the property $R_{\alpha\beta} = R_{\beta\alpha}$, i.e. the symmetry is conserved during the evolution, since the initial state and the Hamiltonian are symmetric with respect to the particle permutation.

Given the exact solution, one can find the negative eigenvalues of the partly transposed matrix $\varrho^T = (T \otimes E)\varrho$ (here $T$ denotes the transposition): $\epsilon_1 = \epsilon_2 = -\frac{1}{27} \sqrt{69 + 28 \cos 3Jt - 16 \cos 6Jt}$, $\epsilon_3 = -\frac{1}{27} (5 + 4 \cos 3Jt)$. The absolute value of the sum of these eigenvalues

\[ m_{VW} = \sum_{i=1}^{3} |\epsilon_i| \]  

(14)

defines the entanglement measure (negativity) between the qutrits.$^{20}$ The entanglement between the qudits can be described quantitatively with the measure$^{21}$

\[ m_{SM} = \sqrt{\frac{1}{D-1} (R_{ij} - R_{00}R_{0j})^2}, \]  

(15)

where $D$ is the basis dimension (for qutrit $D = 9$). This measure equals to 0 for the separable state and to 1 for the maximally entangled state, and it is applicable for both pure and mixed states.
That is why for the maximally entangled initial state of two qutrits, the entanglement in the consistent field is defined by the formulae with the found solution for the density matrix

$$m_{SM} = \frac{1}{81} \sqrt{4457 + 2776 \cos 3Jt - 632 \cos 6Jt - 56 \cos 9Jt + 16 \cos 12Jt}. \quad (16)$$

This measure is numerically equivalent to the measure $$m_{VW}$$ which is defined by the absolute value of the sum of the negative eigenvalues.

According to the definition for $$N$$-qudit pure state, the entanglement measure equals to

$$\eta_N = \frac{1}{N} \sum_{i=1}^{N} S_i, \quad (17)$$

where $$S_i = -\text{Tr} \rho_i \log_b \rho_i$$ is the reduced von Neumann entropy, the index $$i$$ numerates the particles, i.e. the other particles are traced out. We use the logarithm to the base $$b$$ to ensure that the maximal measure is normalized to 1. The base $$b$$ equals to 3 in the qutrit case.

Since the qutrit reduced matrix eigenvalues equal to $$\lambda_1 = \lambda_2 = \frac{1}{27}(5 + 4 \cos 3Jt)$$, $$\lambda_3 = \frac{1}{27}(17 - 8 \cos 3Jt)$$, then the entanglement measure in the bi-qutrit takes the form

$$\eta_2 = -\sum_{i=1}^{3} \lambda_i \log_3 \lambda_i. \quad (18)$$

Normalized to unity the measure I-concurrence which is easy to calculate is defined by the formulae

$$m_I = \sqrt{\frac{d}{d-1} \left(1 - \text{Tr} \rho_1^2\right)} = \frac{1}{9} \sqrt{57 + 32 \cos 3Jt - 8 \cos 6Jt}, \quad (19)$$

where $$d = 3$$ for a qudit, $$\rho_1 = \sqrt{\frac{1}{3}} C_\alpha R_\alpha$$ is the reduced qutrit matrix.

The measures $$m_{VW}$$, $$m_{SM}$$, $$\eta_2$$, $$m_I$$ do not depend on the parameters of the consistent field, the sign of the exchange constant at zero anisotropy parameters. It should be noted that the Wootters entanglement measure (the concurrence) in the system of two qubits with an isotropic interaction in a circularly polarized field at resonance is also independent of the alternating field amplitude, but depends on the exchange constant $$J$$ and the initial conditions only.

The numerical solution of the Liouville-von Neumann equation shows that if an identical external field operates on every qudit, the free Hamiltonian and the interaction Hamiltonian are commutative operators, the measures considered are determined only by the symmetric two-body interaction with the interaction constant of $$J$$. If a different field operates on every qudit, there arises a broken permutation symmetry of the total Hamiltonian, which changes the entanglement dynamics. Thus it is possible to control entanglement by changing the parameters of an external field.
It is possible to show that the distance measure $\sqrt{\text{Tr}(\rho(t) - \rho_0)^2}$ depends on the parameters of the consistent field. At a zero external field the entanglement measure (15) takes the analytical form at equal non-zero anisotropy parameters $Q = d = d = Q_m(SM(Q)) = 1\left(9J^2 + 8QJ + 16Q^2\right)^{1/2} \sqrt{4\sum_{k=0}^{4} q_k \cos \left(k\sqrt{9J^2 + 8QJ + 16Q^2}t\right)}$, (20)

where $q_0 = 4457J^8 + 11616QJ^7 + 47392Q^2J^6 + 85888Q^3J^5 + 163072Q^4J^4 + 19456Q^5J^3 + 221184Q^6J^2 + 131072Q^7J + 65536Q^8$;

$q_1 = 8J^2(J + 2Q)^2(347J^4 + 518QJ^3 + 1440Q^2J^2 + 1504Q^3J + 1024Q^4)$;

$q_2 = -8J^2(J + 2Q)^2(79J^4 + 76QJ^3 + 320Q^2J^2 + 448Q^3J + 256Q^4)$;

$q_3 = -8J^3(7J - 4Q)(J + 2Q)^3(J + 4Q)$, $q_4 = 16J^4(J + 2Q)^4$.

4.2. Entanglement in the chain of qutrits

We consider the Hamiltonian of the chain of $N$ qutrits with the pairwise isotropic interaction in the consistent field $\vec{h}(t)$ at resonance in the following form

$$H_N = \sum_{\vec{S}} (\vec{h}(t) \vec{S} \otimes \vec{E} \otimes \cdots \otimes \vec{E} + J \vec{S} \otimes \vec{S} \otimes \vec{E} \otimes \cdots \otimes \vec{E}),$$

(21)

where the summation is over different possible positions of $\vec{S}$ in the direct products. Because the maximally entangled state of $N$ qutrits

$$|\phi> = \frac{1}{\sqrt{3}} \sum_{i=-1}^{1} |i> \otimes |N>$$

and the Hamiltonian (21) have a permutation symmetry, it follows that the density matrix of $N$ qutrits has symmetric correlation functions.

The entanglement measures for many-particle multi-level quantum systems have not been studied enough and are difficult to calculate in the analytical form, that is why we will present analytical formulae only for the entropy measure $\eta_N$, which is defined by the eigenvalues of the reduced one-particle matrices for each qutrit. As the result of the mentioned symmetry the reduced matrices are equal to each other. Therefore the entanglement measure for $N$ qutrits reads

$$\eta_N = -\sum_{i=1}^{3} r_i \log_3 r_i.$$  

(23)

The eigenvalues of the reduced matrices for 3, 4, 5, and 6 qutrits are presented in
the table below

| N \ r_i | r_1 = r_2 | r_3 |
|---|---|---|
| 3 | $29 - 4 \cos 5Jt$ | $17 + 8 \cos 5Jt$ |
| 4 | $905 - 98 \cos 3Jt - 72 \cos 7Jt$ | $395 + 156 \cos 3Jt + 144 \cos 7Jt$ |
| 5 | $16191 - 1944 \cos 5Jt - 800 \cos 9Jt$ | $8687 + 3888 \cos 5Jt + 1600 \cos 9Jt$ |
| 6 | $21977 - 1694 \cos 3Jt - 1936 \cos 7Jt - 560 \cos 11Jt$ | $9407 + 3388 \cos 3Jt + 3872 \cos 7Jt + 1120 \cos 11Jt$ |

(24)

The measures $\eta_3$, $\eta_4$, $\eta_5$, $\eta_6$ do not depend on the sign of the exchange constant like the measure $\eta_2$.

4.3. Entanglement in the bi-quartit

The applied approach for qutrits is translated to qudits. For a spin-$3/2$ particle or a four-level system, also denoted as a quartit, we take the Hamiltonian in the space $C^4$ in the basis $|3/2\rangle = (1, 0, 0, 0)$, $|1/2\rangle = (0, 1, 0, 0)$, $|-1/2\rangle = (0, 0, 1, 0)$, $|-3/2\rangle = (0, 0, 0, 1)$ and use the matrix representation of a complete set of the Hermitian orthogonal operators.

We will find the analytical formulae in the bi-quartit and in the 3 quartits (in bi-pentit, see below) with a pairwise isotropic interaction of the initial maximally entangled state in a consistent magnetic field at resonance without taking into account the anisotropy.

The negative eigenvalues of the partly transposed matrix $\rho^{pt}$ are equal to $\lambda_1 = \frac{1}{100}(-13 - 12 \cos 5Jt)$, $\lambda_2 = \lambda_3 = \lambda_4 = \lambda_5 = -\frac{1}{100} \sqrt{409 + 288 \cos 5Jt - 72 \cos 10Jt}$, $\lambda_6 = \frac{1}{100}(-37 + 12 \cos 5Jt)$. The entanglement measure in the bi-quartit equals

$$m_{bi-qrt}^{W} = \sum_{i=1}^{6} |\lambda_i|.$$  (25)

The entanglement between the quartits is described quantitatively with the measure[21]

$$m_{bi-qrt}^{SM} = \frac{\sqrt{1803365 + 191616 \cos 5Jt - 35808 \cos 10Jt - 6912 \cos 15Jt + 864 \cos 20Jt}}{625\sqrt{5}}.$$  (26)

Since the quartit reduced matrix eigenvalues equal to $\lambda_1 = \lambda_2 = \frac{1}{100}(13 + 12 \cos 5Jt)$, $\lambda_3 = \lambda_4 = \frac{1}{100}(37 - 12 \cos 5Jt)$, hence the measure $\eta_2$ reads

$$\eta_2^{bi-qrt} = -4 \sum_{i=1}^{4} \lambda_i \log_4 \lambda_i.$$  (27)

The I-concurrence is equal to

$$m_T^{bi-qrt} = \frac{1}{25} \sqrt{553 + 96 \cos 5Jt - 24 \cos 10Jt}.$$  (28)
4.4. Entanglement in the quartit chain

The eigenvalues of the reduced matrices for 3 quartits are equal to 
\[ r_1 = r_2 = 0.141 + 0.068 \cos \frac{5 \pi}{6} + 0.04 \cos 8 Jt, \]
\[ r_3 = r_4 = 0.359 - 0.068 \cos \frac{5 \pi}{6} - 0.04 \cos 8 Jt. \]

Therefore the entanglement measure for 3 quartits is following
\[ \eta_{3}^{qrt} = - \sum_{i=1}^{4} r_i \log_4 r_i. \]  

(29)

4.5. Entanglement in the bi-pentit

For the spin-2 particle or a 5-level system, also denoted as a pentit, we take the Hamiltonian in the space \( C^5 \) in the basis \[ |2> = (1, 0, 0, 0, 0), \]
\[ |1> = (0, 1, 0, 0, 0), \]
\[ |0> = (0, 0, 1, 0, 0), \]
\[ |-1> = (0, 0, 0, 1, 0), \]
\[ |-2> = (0, 0, 0, 0, 1). \]

The entanglement between the pentits is described using the measure \( 21 \)
\[ m_{SM}^{bi-pnt} = (0.802 + 0.106 \cos 3 Jt - 0.019 \cos 4 Jt + 0.242 \cos 7 Jt - 0.098 \cos 10 Jt - 0.088 \cos 14 Jt + 0.067 \cos 17 Jt - 0.014 \cos 20 Jt)^{1/2}. \]

The I-uncorrelation is determined by the formulae \[ m_{SM}^{bi-pnt} = (0.791 + 0.114 \cos 3 Jt - 0.018 \cos 4 Jt - 0.005 \cos 6 Jt + 0.230 \cos 7 Jt - 0.079 \cos 10 Jt - 0.079 \cos 14 Jt + 0.060 \cos 17 Jt - 0.015 \cos 20 Jt)^{1/2}. \]

We have replaced the exact bulky rational coefficients by its decimal approximations and the terms less than 0.001 have removed for inconvenience reduction.

The pentit reduced matrix eigenvalues are equal to \[ p_1 = p_2 = \frac{1}{6125} (1173 - 140 \cos 3 Jt + 640 \cos 7 Jt - 448 \cos 10 Jt), \]
\[ p_3 = p_4 = \frac{1}{6125} (513 + 280 \cos 3 Jt + 320 \cos 7 Jt + 112 \cos 10 Jt), \]
\[ p_5 = \frac{1}{6125} (2753 - 280 \cos 3 Jt - 1920 \cos 7 Jt + 672 \cos 10 Jt), \]

hence the measure \( \eta_{2} \) reads
\[ \eta_{2}^{bi-pnt} = - \sum_{i=1}^{5} p_i \log_5 p_i. \]  

(30)

All the measures do not depend on the sign of the exchange constant and the parameters of the consistent field at zero anisotropy parameters.

5. Numerical results

Although the analytical expressions for the measures in a bi-qutrit \( m_{VW}, \) \( m_{SM} \) are different, but the numerical values are practically identical. The maximal deviation in the rectangle \( (1 \geq J \geq 0.01) \times (100 \geq t \geq 0) \) equals 0.014.

Measures \( \eta_{2} \) and \( m_{t} \) qualitatively coincide with the measures \( m_{VW}, \) \( m_{SM}. \)

We have found that the anisotropy of the qutrits disentangles them, namely the entanglement is decreased down to 0.001 (see graphs 1 and 2 in Fig[1].

In the constant longitudinal field \( \vec{h} = - \vec{h} = (0, 0, \omega_0) \) (the bi-qutrit Hamiltonian eigenvalues are equal to \( J, J, x_1, x_2, x_3, -p, -p, p, p, \) where \( x_1, x_2, x_3 \) are the roots of the equation \( x^3 + 2x^2 J - p^2 x - 2J^3 = 0, \) \( p = \sqrt{J^2 + \omega_0^2} \) the Hamiltonian contains the asymmetric part, thus it follows that the density matrix for the initial symmetric state will not be symmetric because of the breaking of the symmetry of the particle
permutations. The analytical solution is cumbersome. In the constant longitudinal impulse field \( \vec{h} = - \vec{h} = (0, 0, 2(\theta((t - 17)(t - 60)) + \theta((40 - t)(57 - t)(t - 60)))) \) the entanglement dynamics is blocked at \( \omega_0 \gg J \). This points to the possibility to control the entanglement.

In Fig. 2 we present the comparative dynamics of the entropy measure in the finite qutrit chain. The disentanglement dynamics of the measures \( \eta_3, \eta_4, \eta_5, \eta_6 \) is similar to the one in the case of two qutrits, but with smaller oscillation amplitude, i.e. larger number of the qutrits disentangles less than two qutrits (0.889 \( \leq \eta_3 \leq 1 \)).

The measures in bi-quartit, as shown in Fig. 3 qualitative coincide, almost completely \( m_{b1-qrt}^{bi-qrt} \) and \( m_{VW}^{bi-qrt} \). The disentanglement 3 quartits is insignificant less than in bi-quartit.

The disentanglement measures in bi-pentit, as shown in Fig. 4 qualitative coincide, almost completely \( m_{I}^{bi-pnt} \) and \( m_{SM}^{bi-pnt} \).

6. Conclusion

The comparative analysis of the bi-qutrit entanglement measures on the base of the analytical solution for the density matrix demonstrates that, in spite of the different approaches to the derivation of the formulae for the entanglement, all the formulae yield quite close results (Fig. 1), and the measures \( m_{VW} \) and \( m_{SM} \) are practically equal. This is in accordance with the general results for the entanglement in the systems with a permutational symmetry.

The analytical formulae for the measures \( \eta_3, \eta_4, \eta_5, \eta_6 \) are similar to the measure for two qutrits \( \eta_2 \), but with a numerically smaller oscillation amplitude, i.e. the larger number of the qutrits disentangles fewer than two qutrits.

Nevertheless, the comparison of measures in two coupled qutrits, quartits, and...
Entanglement dynamics in finite qudit chain

Fig. 3. Disentanglement of the maximally entangled state in the chain of 2, 3 qudrits with $J = 0.1$. The measures $m_{M}^{bi-qudit}$, $m_{M}^{hi-qudit}$, $\eta_{2}^{bi-qudit}$, $m_{W}^{bi-qudit}$, are presented by the curves 1, 2, 3, 4 respectively; $\eta_{3}^{qudit}$ is the curve 5.

pentits

\begin{align}
\text{bi - qudit} & \quad m_{VW}^{bi-qudit} \equiv m_{SM}^{bi-qudit} \\
\text{bi - quartit} & \quad \eta_{2}^{bi-qudit} \equiv m_{W}^{bi-qudit} \\
\text{bi - pentit} & \quad m_{SM}^{bi-pentit} \equiv m_{I}^{bi-pentit}
\end{align}

(31)

on the base of analytical solutions shows the absence of a full coincidence of the measures even in a particular case of disentangling a maximally entangled state. In other words, it is impossible to prefer any measure, there remains therefore the question concerning the quantitative determination of entanglement even in case of two multi-level particles.

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