COALGEBRA EXTENSIONS AND ALGEBRA COEXTENSIONS OF GALOIS TYPE

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Abstract

The notion of a coalgebra-Galois extension is defined as a natural generalisation of a Hopf-Galois extension. It is shown that any coalgebra-Galois extension induces a unique entwining map $\psi$ compatible with the right coaction. For the dual notion of an algebra-Galois coextension it is also proven that there always exists a unique entwining structure compatible with the right action.

1 Introduction

Hopf-Galois extensions can be viewed as non-commutative torsors or principal bundles with universal differential structure. From the latter point of view a quantum group gauge theory was introduced in [4] and developed in [9, 6]. It turns out that to develop gauge theory on quantum homogeneous spaces (e.g.,
the family of Podleś quantum spheres) or with braided groups (see [11] for a recent review), one needs to consider coalgebra bundles. Such a generalisation of gauge theory was proposed recently in [5]. It introduces the notions of an entwining structure with an entwining map $\psi$, and a $\psi$-principal coalgebra bundle. The latter can be viewed as a generalisation of a Hopf-Galois extension.

In the present paper we introduce the notion of a coalgebra-Galois extension for arbitrary coalgebras as a natural generalisation of a Hopf-Galois extension. It is obtained by giving up the condition that the coaction be an algebra map. Our main result is that such defined coalgebra-Galois extension always induces a unique compatible entwining map $\psi$. We dualise coalgebra-Galois extensions and thus introduce the notion of an algebra-Galois coextension. We then prove the dual version of the main result. Finally, we prove that if two quotient coalgebras $C/I_1$ and $C/I_2$ cogenerate (in the sense of Definition 5.1) the coalgebra $C$, then the coinvariants of $C$ are the same as the intersection of the coinvariants of $C/I_1$ with the coinvariants of $C/I_2$.

**Notation.** Here and below $k$ denotes a field. All algebras are over $k$, associative and unital with the unit denoted by 1. All algebra homomorphisms are assumed to be unital. We use the standard algebra and coalgebra notation, i.e., $\Delta$ is a coproduct, $m$ is a product, $\varepsilon$ is a counit, etc. The identity map from the space $V$ to itself is also denoted by $V$. The unadorned tensor product stands for the tensor product over $k$. To abbreviate notation, we identify the tensor products $k \otimes V$ and $V \otimes k$ with $V$. For a $k$-algebra $A$ we denote by $\mathcal{M}_A$, $_A \mathcal{M}$ and $\mathcal{M}_A$ the category of right $A$-modules, left $A$-modules and $A$-bimodules respectively. Similarly, for a $k$-coalgebra $C$ we denote by $\mathcal{M}^C$, $\mathcal{C} \mathcal{M}$ and $\mathcal{C} \mathcal{M}^C$ the category of right $C$-comodules, left $C$-comodules and $C$-bicomodules respectively. Also, by $\mathcal{M}^C_A$ ($\mathcal{C} \mathcal{M}_A$) we denote the category of left (right) $A$-modules with the action $\nu \mu$ ($\mu \nu$) and right (left) $C$-comodules with the coaction $\Delta_V$ ($\nu \Delta$) such that $\Delta_V \circ \nu \mu = (\nu \otimes C) \circ (A \otimes \Delta_V)$ ($\nu \Delta \circ \mu_V = (C \otimes \mu_V) \circ (\nu \Delta \otimes A)$), i.e., $\Delta_V$ ($\nu \Delta$) is right (left) $A$-linear. For coactions and coproducts we use Sweedler’s notation with suppressed summation sign: $\Delta_A(a) = a_{(0)} \otimes a_{(1)}$, $\Delta(c) = c_{(1)} \otimes c_{(2)}$. 

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2 \textbf{C-Galois extensions}

First recall the definition of a Hopf-Galois extension (see [12] for a review).

\textbf{Definition 2.1} Let $H$ be a Hopf algebra, $A$ be a right $H$-comodule algebra, and $B := A^{\text{co}H} := \{ a \in A \mid \Delta_A a = a \otimes 1 \}$. We say that $A$ is a (right) Hopf-Galois extension (or $H$-Galois extension) of $B$ iff the canonical left $A$-module right $H$-comodule map \textit{can} := $(m \otimes H) \circ (A \otimes_B \Delta_A) : A \otimes_B A \rightarrow A \otimes H$ is bijective.

Note that in the situation of Definition 2.1, both $A \otimes_B A$ and $A \otimes H$ are objects in $A \text{Mod}_H$ via the maps $m \otimes_B A$, $A \otimes_B \Delta_A$ and $m \otimes A$, $A \otimes \Delta$ respectively. The canonical map \textit{can} is a morphism in this category. Thus the extension $B \subseteq A$ is Hopf-Galois if $A \otimes_B A \cong A \otimes H$ as objects in $A \text{Mod}_H$ by the canonical map \textit{can}.

It has recently been observed in [2] that one can view quantum embeddable homogeneous spaces $B$ of a Hopf algebra $H$, such as the family (indexed by $c \in [0, \infty]$) of quantum two-spheres of Podleś [13], as extensions by a coalgebra $C$. It is known (see p.200 in [13]) that, except for the North Pole sphere ($c = 0$), no other spheres of this family are quantum quotient spaces of $SU_q(2)$. They escape the standard Hopf-Galois description. To include this important case in Galois extension theory, one needs to generalise the notion of a Hopf-Galois extension to the case of an algebra extended to an algebra by a coalgebra (cf. [7] for the dual picture). This is obtained by weakening the requirement that $\Delta_A$ be an algebra map, and leads to the notion of a coalgebra-Galois extension. (A special case of this kind was considered in [17, p.291].)

To formulate the definition of a coalgebra-Galois extension, first we need a general concept of coinvariants:

\textbf{Definition 2.2 ([19])} Let $A$ be an algebra and a right $C$-comodule. Then

$$A^{\text{co}C} := \{ b \in A \mid \Delta_A(ba) = b\Delta_A(a), \forall a \in A \}$$

is a subalgebra of $A$. We call it the subalgebra of (right) coinvariants.
Observe that when $\Delta_A$ is an algebra map, the above definition coincides with the usual definition of coinvariants as elements $b$ of $A$ such that $\Delta_A(b) = b \otimes 1$. Definition 2.2 does not require the existence of a group-like element in the coalgebra. This allows one to define coalgebra-Galois extension for arbitrary coalgebras.

**Definition 2.3** Let $C$ be a coalgebra, $A$ an algebra and a right $C$-comodule and let $B = A^{coC}$. We say that $A$ is a (right) coalgebra-Galois extension (or $C$-Galois extension) of $B$ iff the canonical left $A$-module right $C$-comodule map $\text{can} := (m \otimes C) \circ (A \otimes_B \Delta_A) : A \otimes_B A \to A \otimes C$ is bijective.

In what follows, we will consider only right coalgebra-Galois extensions, and skip writing “right” for brevity. The conditions of Definition 2.3 suffice to make both $A \otimes_B A$ and $A \otimes C$ objects in $A\mathcal{M}^C$ via the maps $m \otimes_B A, A \otimes_B \Delta_A$ and $m \otimes A, A \otimes \Delta$, respectively. The canonical map is again a morphism in $A\mathcal{M}^C$. The extension $B \subseteq A$ is $C$-Galois if $\text{can}$ is an isomorphism in $A\mathcal{M}^C$.

By the reasoning as in the proof of Proposition 1.6 in [9], one can obtain an alternative (differential) definition of a coalgebra-Galois extension:

**Proposition 2.4** Let $C$ and $B \subseteq A$ be as above. Let $\Omega^1_A := \text{Ker} \ m$ denote the universal differential calculus on $A$, and $C^+ := \text{Ker} \ \varepsilon$ the augmentation ideal of $C$. The algebra $A$ is a $C$-Galois extension if and only if the following sequence of left $A$-modules is exact:

$$0 \to A(\Omega^1_B)A \to \Omega^1_A \xrightarrow{\text{can}} A \otimes C^+ \to 0, \quad (2.1)$$

where $\text{can} := (m \otimes C) \circ (A \otimes \Delta_A)$.

In the Hopf-Galois case, one can generalise the above sequence to a non-universal differential calculus in a straightforward manner:

$$0 \to A\Omega^1(B)A \to \Omega^1(A) \xrightarrow{\tilde{\chi}} A \otimes (H^+/R_H) \to 0, \quad (2.2)$$

where $R_H$ is the $ad_{R}$-invariant right ideal of the Hopf algebra $H$ defining a bi-covariant calculus on $H$ [21], and $\tilde{\chi}$ is defined by a formula fully analogous to the formula for $\text{can}$. Sequence (2.2) is a starting point of the quantum-group
gauge theory proposed in [4] and continued in [9]. The Hopf-Galois extension describes a quantum principal bundle with the universal differential calculus. Proposition 2.4 shows that the $C$-Galois extension can also be viewed as a generalisation of such a bundle, a principal coalgebra bundle. The theory of coalgebra bundles and connections on them (also for non-universal differential calculus) was developed in [5]. More specifically, the theory considered in [5] uses the notion of an entwining structure (closely connected with the theory of factorisation of algebras considered in [10]) and identifies coalgebra principal bundles with $C$-Galois extensions constructed within this entwining structure.

The aim of this section is to show that to each $C$-Galois extension of Definition 2.3 there corresponds a natural entwining structure. Therefore the notions of a $C$-Galois extension of Definition 2.3 and a $\psi$-principal coalgebra bundle of [5, Proposition 2.2] are equivalent to each other provided that there exists a group-like $e \in C$ such that $\Delta_A(1) = 1 \otimes e$. First we recall the definition of an entwining structure.

**Definition 2.5** Let $C$ be a coalgebra, $A$ an algebra and let $\psi$ be a $k$-linear map $\psi : C \otimes A \rightarrow A \otimes C$ such that

$$
\psi \circ (C \otimes m) = (m \otimes C) \circ (A \otimes \psi) \circ (\psi \otimes A), \\
\psi \circ (C \otimes \eta) = \eta \otimes C, \\
(A \otimes \Delta) \circ \psi = (\psi \otimes C) \circ (C \otimes \psi) \circ (\Delta \otimes A), \\
(A \otimes \varepsilon) \circ \psi = \varepsilon \otimes A,
$$

where $\eta$ is the unit map $\eta : \alpha \mapsto \alpha 1$. Then $C$ and $A$ are said to be entwined by $\psi$ and the triple $(A, C, \psi)$ is called an entwining structure.

Entwining structures can be also understood as follows. Given an algebra $A$ and a coalgebra $C$ we consider $A \otimes C$ as an object in $\text{A}\text{M}$ with the structure map $m \otimes C$. Similarly we consider $C \otimes A$ as an object in $\text{C}\text{M}$ via the map $\Delta \otimes A$. Then we have (see also [14, Theorem 5.2])

**Proposition 2.6** Let $\mu_{A \otimes C} : A \otimes C \otimes A \rightarrow A \otimes C$ and $\Delta_{C \otimes A} : C \otimes A \rightarrow C \otimes A \otimes C$ be the maps making $A \otimes C$ an object in $\text{A}\text{M}_A$ and $C \otimes A$ an object in $\text{C}\text{M}^C$ correspondingly, and such that

$$
(\varepsilon \otimes A \otimes C) \circ \Delta_{C \otimes A} = \mu_{A \otimes C} \circ (\eta \otimes C \otimes A).
$$
Then the pairs \((\mu_{A \otimes C}, \Delta_{C \otimes A})\) of such compatible maps are in one-to-one correspondence with the entwining structures \((A, C, \psi)\).

**Proof.** First assume that \(C \otimes A \in C\mathcal{M}\) and \(A \otimes C \in A\mathcal{M}_A\). Then

\[
(\Delta \otimes A \otimes C) \circ \Delta_{C \otimes A} = (C \otimes \Delta_{C \otimes A}) \circ (\Delta \otimes A),
\]

and

\[
\mu_{A \otimes C} \circ (m \otimes C \otimes A) = (m \otimes C) \circ (A \otimes \mu_{A \otimes C}).
\]

Define

\[
\psi = (\varepsilon \otimes A \otimes C) \circ \Delta_{C \otimes A} = \mu_{A \otimes C} \circ (\eta \otimes C \otimes A).
\]

Then

\[
(m \otimes C) \circ (A \otimes \psi) \circ (\psi \otimes A) = (m \otimes C) \circ (A \otimes \mu_{A \otimes C}) \circ (A \otimes \eta \otimes C \otimes A) \circ (\psi \otimes A) = \mu_{A \otimes C} \circ (m \otimes C \otimes A) \circ (A \otimes \eta \otimes C \otimes A) \circ (\psi \otimes A) = \mu_{A \otimes C} \circ (m \otimes C \otimes A) \circ (m \otimes C) \circ (\varepsilon \otimes C \otimes A) = \mu_{A \otimes C} \circ (m \otimes C \otimes A) \circ (m \otimes C) = \psi \circ (C \otimes m).
\]

Furthermore

\[
\psi \circ (C \otimes \eta) = \mu_{A \otimes C} \circ (\eta \otimes C \otimes A) \circ (C \otimes \eta) = \mu_{A \otimes C} \circ (\eta \otimes C \otimes \eta) = \eta \otimes C.
\]

Therefore \(\psi\) satisfies conditions (2.3). Dualising the above calculation (namely, interchanging \(\Delta\) with \(m\), \(C\) with \(A\), \(\varepsilon\) with \(\eta\) and \(\Delta_{C \otimes A}\) with \(\mu_{A \otimes C}\)) one easily finds that \(\psi\) satisfies conditions (2.4) too. Hence \((A, C, \psi)\) is an entwining structure.

Conversely, let \((A, C, \psi)\) be an entwining structure. Define \(\Delta_{C \otimes A} = (C \otimes \psi) \circ (\Delta \otimes A)\) and \(\mu_{A \otimes C} = (m \otimes C) \circ (A \otimes \psi)\). Then

\[
(C \otimes A \otimes \Delta) \circ \Delta_{C \otimes A} = (C \otimes A \otimes \Delta) \circ (C \otimes \psi) \circ (\Delta \otimes A) = (C \otimes \psi \otimes C) \circ (C \otimes C \otimes \psi) \circ (C \otimes \Delta \otimes A) \circ (\Delta \otimes A) = (C \otimes \psi \otimes C) \circ (\Delta \otimes A \otimes C) \circ (C \otimes \psi) \circ (\Delta \otimes A) = (\Delta_{C \otimes A} \circ C) \circ C \otimes \psi \circ (\Delta \otimes A) = (\Delta_{C \otimes A} \circ C) \circ \Delta_{C \otimes A}.
\]
We used property (2.4) to derive the second equality, and then coassociativity of the coproduct to derive the third one. Similarly,

\[(C \otimes A \otimes \varepsilon) \circ \Delta_{C \otimes A} = (C \otimes A \otimes \varepsilon) \circ (C \otimes \psi) \circ (\Delta \otimes A)\]

\[= (C \otimes \varepsilon \otimes A) \circ (\Delta \otimes A) = C \otimes A.\]

Hence $\Delta_{C \otimes A}$ is a right coaction. By dualising the above argument one verifies that $\mu_{A \otimes C}$ is a right action. Finally, an elementary calculation shows that

\[(\varepsilon \otimes A \otimes C) \circ \Delta_{C \otimes A} = \mu_{A \otimes C} \circ (\eta \otimes C \otimes A) = \psi.\]

Thus the bijective correspondence is established. \[\Box\]

To each entwining structure $(A, C, \psi)$ one can associate the category $\mathcal{M}^C_A(\psi)$ of right $(A, C, \psi)$-modules, introduced and studied in [3]. The objects of $\mathcal{M}^C_A(\psi)$ are right $A$-modules and right $C$-comodules $V$ such that for all $v \in V$ and $a \in A$,

\[\Delta_V(v \cdot a) = v(0) \psi(v(1) \otimes a).\]

The morphisms in $\mathcal{M}^C_A(\psi)$ are right $A$-module right $C$-comodule maps. Some important categories well-studied in the Hopf algebra theory, such as the category of right $(A, H)$-Hopf modules for a right $H$-module algebra $A$ [8], or the category of right-right Yetter-Drinfeld modules [22] can be seen as examples of $\mathcal{M}^C_A(\psi)$.

The main result of the paper is contained in the following:

**Theorem 2.7** Let $A$ be a $C$-Galois extension of $B$. Then there exists a unique map $\psi : C \otimes A \to A \otimes C$ entwining $C$ with $A$ and such that $A \in \mathcal{M}^C_A(\psi)$ with the structure maps $m$ and $\Delta_A$. (The map $\psi$ is called the canonical entwining map associated to the $C$-Galois extension $B \subseteq A$.)

**Proof.** Assume that $B \subseteq A$ is a coalgebra-Galois extension. Then $\text{can}$ is bijective and there exists the translation map $\tau : C \to A \otimes_B A$, $\tau(c) := \text{can}^{-1}(1 \otimes c)$. We use the notation $\tau(c) = c^{(1)} \otimes c^{(2)}$ (summation understood). Using [1, Proposition 3.9] or [10, Remark 3.4] and their obvious generalisation to the present case, for all $c \in C$ and $a \in A$, one obtains that
We show that \( \psi \) have:

Furthermore, by property (i) of the translation map, where we used the property (ii) of the translation map to derive the third equality. Thus we have proven that the second equations in (2.3)–(2.4) hold. Next we have:

Define a map \( \psi : C \otimes A \to A \otimes C \) by

\[
\psi = \text{can} \circ (A \otimes_B m) \circ (\tau \otimes A), \quad \psi(c \otimes a) = c^{(1)}(c^{(2)}a)_{(0)} \otimes (c^{(2)}a)_{(1)}. \tag{2.5}
\]

We show that \( \psi \) entwines \( C \) and \( A \). By definition of the translation map, we have:

\[
\psi(c \otimes 1) = c^{(1)}c^{(2)}_{(0)} \otimes c^{(2)}_{(1)} = 1 \otimes c.
\]

Furthermore, by property (i) of the translation map,

\[
((\text{id} \otimes \varepsilon) \circ \psi)(c \otimes a) = c^{(1)}(c^{(2)}a)_{(0)} \otimes \varepsilon((c^{(2)}a)_{(1)}) = c^{(1)}c^{(2)}a = \varepsilon(c)a.
\]

Thus we have proven that the second equations in (2.3)–(2.4) hold. Next we have

\[
\left( (m \otimes C) \circ (A \otimes \psi) \circ (\psi \otimes A) \right)(c \otimes a \otimes a')
\]

\[
= \left( (m \otimes C) \circ (A \otimes \psi) \right)(c^{(1)}(c^{(2)}a)_{(0)} \otimes (c^{(2)}a)_{(1)} \otimes a')
\]

\[
= c^{(1)}(c^{(2)}a)_{(0)}(c^{(2)}a)_{(1)}((c^{(2)}a)_{(1)}(2)a'_{(0)} \otimes (c^{(2)}a)_{(1)}(2)a'_{(1)})
\]

\[
= c^{(1)}(c^{(2)}a)_{(0)} \otimes (c^{(2)}a)_{(1)} = (\psi \circ (C \otimes m))(c \otimes a \otimes a'),
\]

where we used the property (ii) of the translation map to derive the third equality. Hence the first of equations (2.3) holds. Similarly,

\[
\left( (\psi \otimes C) \circ (C \otimes \psi) \circ (\Delta \otimes A) \right)(c \otimes a)
\]

\[
= (\psi \otimes C)(c^{(1)} \otimes c^{(2)}_{(1)}(c^{(2)}_{(2)}a)_{(0)} \otimes (c^{(2)}a)_{(1)})
\]

\[
= c^{(1)}(c^{(2)}_{(1)}c^{(2)}_{(2)}(c^{(2)}_{(2)}a)_{(0)})_{(0)} \otimes (c^{(2)}_{(1)}c^{(2)}_{(2)}(c^{(2)}_{(2)}a)_{(0)})_{(1)}
\]

\[
= (c^{(2)}_{(1)}c^{(2)}_{(2)}(c^{(2)}_{(2)}a)_{(0)})_{(0)} \otimes (c^{(2)}_{(1)}c^{(2)}_{(2)}(c^{(2)}_{(2)}a)_{(0)})_{(1)}
\]

\[
= c^{(1)}((c^{(2)}a)_{(0)})_{(0)} \otimes ((c^{(2)}a)_{(0)})_{(1)} \otimes (c^{(2)}a)_{(1)}
\]

\[
= c^{(1)}(c^{(2)}a)_{(0)} \otimes (c^{(2)}a)_{(1)} \otimes (c^{(2)}a)_{(2)} = (A \otimes \Delta) \circ \psi)(c \otimes a).
\]

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We used property (iii) of the translation map to derive the third equality and then property (ii) to derive the fourth one. Hence \( C \) and \( A \) are entwined by \( \psi \) as required.

Now, using (ii), we have for all \( a, a' \in A \)
\[
a_{(0)} \psi (a_{(1)} \otimes a') = a_{(0)} a_{(1)}^{(1)} a_{(2)}^{(1)} a_{(0)}^{(2)} a_{(2)}^{(2)} (a_{(1)}^{(2)} a_{(2)}^{(2)})_{(0)} = (a a')_{(0)} \otimes (a a')_{(1)} = \Delta_A (a a'),
\]
i.e. \( A \) is an \((A, C, \psi)\)-module with structure maps \( m \) and \( \Delta_A \). It remains to prove the uniqueness of the entwining map \( \psi \) given by (2.5). Suppose that there is an entwining map \( \tilde{\psi} \) such that \( A \in \mathcal{M}_A^C(\tilde{\psi}) \) with structure maps \( m \) and \( \Delta_A \).

Then, for all \( a \in A, c \in C \),
\[
\psi(c \otimes a) = c^{(1)} (c^{(2)} a)_{(0)} \otimes (c^{(2)} a)_{(1)} = c^{(1)} c^{(2)}_{(0)} \tilde{\psi}(c^{(2)}_{(1)} \otimes a) = \tilde{\psi}(c \otimes a),
\]
where we used the definition of the translation map to obtain the last equality.

\( \blacksquare \)

### 3 \( A \)-Galois coextensions

The dual version of a Hopf-Galois extension can be viewed as a non-commutative generalisation of the theory of quotients of formal schemes under free actions of formal group schemes (cf. [15]). In this section we dualise \( C \)-Galois extensions and derive results analogous to the results discussed in the previous section.

First recall the definition of a cotensor product. Let \( B \) be a coalgebra and \( M, N \) a right and left \( B \)-comodule respectively. The cotensor product \( M \Box_B N \) is defined by the exact sequence
\[
0 \rightarrow M \Box_B N \hookrightarrow M \otimes N \xrightarrow{\ell} M \otimes B \otimes N,
\]
where \( \ell \) is the coaction equalising map \( \ell = \Delta_M \otimes N - M \otimes \Delta \). In particular, if \( C \) is a coalgebra, \( I \) its coideal and \( B = C/I \), then
\[
C \Box_B C = \left\{ \sum_i c^i \otimes \tilde{c}^i \in C \otimes C \middle| \sum_i c^i_{(1)} \otimes \pi(c^i_{(2)}) \otimes \tilde{c}^i = \sum_i c^i \otimes \pi(\tilde{c}^i_{(1)}) \otimes \tilde{c}^i_{(2)} \right\},
\]
where \( \pi : C \to B = C/I \) is the canonical surjection. The following definition [18, p.3346] dualises the concept of a Hopf-Galois extension:

**Definition 3.1** Let \( H \) be a Hopf algebra, \( C \) a right \( H \)-module coalgebra with the action \( \mu_C : C \otimes H \to C \). Then \( I := \{ \mu_C(c, h) - \varepsilon(h)c \mid c \in C, h \in H \} \) is a coideal in \( C \) and thus \( B := C/I \) is a coalgebra. We say that \( C \twoheadrightarrow B \) is a (right) Hopf-Galois coextension (or \( H \)-Galois coextension) iff the canonical left \( C \)-comodule right \( H \)-module map \( \text{cocan} := (C \otimes \mu_C) \circ (\Delta \otimes H) : C \otimes H \to C \square_B C \) is a bijection.

With the help of the property \( \Delta \circ \mu_C = (\mu_C \otimes \mu_C) \circ (C \otimes \text{flip} \otimes H) \circ (\Delta \otimes \Delta) \), it can be directly checked that the image of the map \( \text{cocan} \) is indeed contained in \( C \square_B C \). To see more clearly that Definition 3.1 is obtained by dualising Definition 2.1, one can notice that both \( C \otimes H \) and \( C \square_B C \) are objects in \( {}^C\mathcal{M}_H \), which is dual to \( {}_A\mathcal{M}_H \). The structure maps are \( \Delta \otimes C, C \otimes m \) and \( \Delta \square_B C, C \square_B \mu_C \) respectively. The canonical map \( \text{cocan} \) is a morphism in \( {}^C\mathcal{M}_H \). The coextension \( C \twoheadrightarrow B \) is Hopf-Galois if \( C \otimes H \cong C \square_B C \) as objects in \( {}^C\mathcal{M}_H \) by the canonical map \( \text{cocan} \).

The notion of a Hopf-Galois coextension can be generalised by replacing \( H \) by an algebra \( A \) and weakening the condition that the action \( \mu_C \) is a coalgebra map. This generalisation dualises the construction of a \( C \)-Galois extension of the previous section. First we prove

**Lemma 3.2** Let \( A \) be an algebra and \( C \) a coalgebra and right \( A \)-module with an action \( \mu_C : C \otimes A \to C \). Then the space

\[
\text{span}\{\mu_C(c, a)(1)\alpha(\mu_C(c, a)(2)) - c(1)\alpha(\mu_C(c(2), a)) \mid a \in A, c \in C, \alpha \in \text{Hom}(C, k)\}
\]

is a coideal of \( C \).

**Proof.** Let \( I \) denote the space defined in Lemma 3.2. We prove the lemma by showing that \( D := \{ \beta \in \text{Hom}(C, k) \mid \beta(I) = 0 \} \) is a subalgebra of the convolution algebra \( \text{Hom}(C, k) \) (see [20, Proposition 1.4.6 c])). Note first that \( \varepsilon \in D \). Furthermore,

\[
D = \{ \beta \in \text{Hom}(C, k) \mid (\beta \ast \alpha)(\mu_C(c, a)) = \beta(c(1))\alpha(\mu_C(c(2), a)), \forall a \in A, c \in C, \alpha \in \text{Hom}(C, k)\}.
\]
If $\beta_1, \beta_2 \in D$, then for any $a, c, \alpha$ we have
\[
((\beta_1 * \beta_2) * \alpha)(\mu_C(c, a)) = (\beta_1 * (\beta_2 * \alpha))(\mu_C(c, a))
\]
\[
= \beta_1(c(1))((\beta_2 * \alpha)(\mu_C(c(2), a)))
\]
\[
= \beta_1(c(1))\beta_2(c(2))\alpha(\mu_C(c(3), a))
\]
\[
= (\beta_1 * \beta_2)(c(1))\alpha(\mu_C(c(2), a)).
\]
Hence $\beta_1 * \beta_2 \in D$, and $D$ is a subalgebra of $\text{Hom}(C, k)$, as needed.

Lemma 3.3. Let $C$ and $A$ be as above and let $I$ denote the coideal of Lemma 3.2. Put $B := C/I$, and denote by $\pi : C \to C/I$ the canonical surjection. Then the action $\mu_C$ is left $B$-colinear, i.e., $(\pi \otimes C) \circ \Delta \circ \mu_C = (B \otimes \mu_C) \circ ((\pi \otimes C) \circ \Delta \otimes A)$, and $((C \otimes \mu_C) \circ (\Delta \otimes A))(C \otimes A) \subseteq C \square_B C$.

Proof. Choose $a \in A, c \in C$. The above $B$-colinearity condition can be written as $\pi(\mu_C(c, a)_1) \otimes \mu_C(c, a)_2 = \pi(c_1) \otimes \mu_C(c_2, a)$ It is equivalent to the condition
\[
\pi(\mu_C(c, a)_1) \otimes \alpha(\mu_C(c, a)_2) = \pi(c_1) \otimes \alpha(\mu_C(c_2, a)), \forall \alpha \in \text{Hom}(C, k),
\]
which can be written as
\[
\pi(\mu_C(c, a)_1)\alpha(\mu_C(c, a)_2) - c_1\alpha(\mu_C(c_2, a))) = 0, \forall \alpha \in \text{Hom}(C, k).
\]
This proves our first assertion.

As for the second assertion, observe that it can be stated as
\[
c(c_1) \otimes \pi(c_2) \otimes \mu_C(c_3, a) = c_1 \otimes \pi(\mu_C(c_2, a)_1) \otimes \mu_C(c_2, a)_2.
\]
This is true by the left $B$-colinearity argument applied to the last two tensorands.

Now we can conclude that we have a well-defined map
\[
cocan := (C \otimes \mu_C) \circ (\Delta \otimes A) : C \otimes A \to C \square_B C,
\]
and can consider:
Definition 3.4 Let $A$ be an algebra, $C$ a coalgebra and right $A$-module, and $B = C/I$, where $I$ is the coideal of Lemma 3.2. We say that $C$ is a (right) algebra-Galois coextension (or $A$-Galois coextension) of $B$ iff the canonical left $C$-comodule right $A$-module map $\text{cocan} := (C \otimes \mu_C) \circ (\Delta \otimes A) : C \otimes A \rightarrow C \Box_B C$ is bijective.

Again, to see that Definition 3.4 dualises the notion of a $C$-Galois extension one can notice that both $C \otimes A$ and $C \Box_B C$ are objects in $\mathcal{C} \mathcal{M}_A$, which is dual to $A \mathcal{M}^C$. The structure maps are $\Delta \otimes C$, $C \otimes \mu$ and $\Delta \Box_B C$, $C \Box_B \mu_C$, respectively. The canonical map $\text{cocan}$ is a morphism in $\mathcal{C} \mathcal{M}_A$. The right coextension $C \rightarrow B$ is $A$-Galois if $C \otimes A \cong C \Box_B C$ as objects in $\mathcal{C} \mathcal{M}_A$ by the canonical map $\text{cocan}$. (In what follows, we consider only right coextensions and omit “right” for brevity.)

Very much as in the previous section, it turns out that every $A$-Galois extension is equipped with an entwining structure. More precisely, we have the following dual version of Theorem 2.7:

Theorem 3.5 Let $C$ be an $A$-Galois coextension of $B$. Then there exists a unique map $\psi : C \otimes A \rightarrow A \otimes C$ entwining $C$ with $A$ and such that $C \in \mathcal{C} \mathcal{M}_A(\psi)$ with the structure maps $\Delta$ and $\mu_C$. (The map $\psi$ is called the canonical entwining map associated to the $A$-Galois coextension $C \rightarrow B$.)

Proof. We dualise the proof of Theorem 2.7. Assume that $C \rightarrow B$ is an $A$-Galois coextension. Then $\text{cocan}$ is a bijection and there exists the cotranslation map $\check{\tau} : C \Box_B C \rightarrow A$, $\check{\tau} := (\varepsilon \otimes A) \circ \text{cocan}^{-1}$. By dualising properties of the translation map (or directly from the definition of $\check{\tau}$), one can establish the following properties of the cotranslation map:

(i) $\check{\tau} \circ \Delta = \eta \circ \varepsilon$,

(ii) $\mu_C \circ (C \otimes \check{\tau}) \circ (\Delta \otimes C) = \varepsilon \otimes C$ on $C \Box_B C$,

(iii) $\check{\tau} \circ (C \otimes \mu_C) = m \circ (\check{\tau} \otimes A)$ on $C \Box_B C \otimes A$.

It follows from (iii) that

$$m \circ (\check{\tau} \otimes \check{\tau}) = \check{\tau} \circ (C \otimes \mu_C \circ (C \otimes \check{\tau})) \text{ on } C \Box_B C \otimes C \Box_B C.$$
Consequently, we can conclude from (ii) that
\[ m \circ (\tilde{\tau} \otimes \tilde{\tau}) \circ (C \otimes \Delta \otimes C) = \tilde{\tau} \circ (C \otimes \varepsilon \otimes C) \] on \( C \square_B C \square_B C \).
\[ (3.6) \]
(Observe that \( (C \otimes \Delta \otimes C)(C \square_B C \square_B C) \subseteq C \square_B C \otimes C \square_B C \). Here and below one has to pay attention to the domains of certain mappings.)

Using the cotranslation map and noticing that \((C \otimes \Delta)(C \square_B C) \subseteq C \square_B C \otimes C \), one defines a map \( \psi : C \otimes A \rightarrow A \otimes C \) by
\[
\psi = (\tilde{\tau} \otimes C) \circ (C \otimes \Delta) \circ \text{cocan},
\]
\[
\psi(c \otimes a) = \tilde{\tau}(c(1), \mu_C(c(2), a)(1)) \otimes \mu_C(c(2), a)(2).
\]
We now show that \( \psi \) entwines \( C \) and \( A \). For any \( c \in C \) we find
\[
\psi(c \otimes 1) = \tilde{\tau}(c(1), \mu_C(c(2), 1)(1)) \otimes \mu_C(c(2), 1)(2) = \tilde{\tau}(c(1), c(2)) \otimes c(3) = 1 \otimes c,
\]
where we used property (i) to derive the last equality. Furthermore,
\[
(A \otimes \varepsilon) \circ \psi = ((\varepsilon \otimes A) \circ \text{cocan}^{-1} \otimes \varepsilon) \circ (C \otimes \Delta) \circ \text{cocan}
\]
\[
= (\varepsilon \otimes A) \circ \text{cocan}^{-1} \circ \text{cocan} = \varepsilon \otimes A.
\]
Thus we have proven that the second conditions of (2.3) and (2.4) are fulfilled by \( \psi \). To prove the first of equations (2.3), we compute
\[
(m \otimes C) \circ (A \otimes \psi) \circ (\psi \otimes A)
\]
\[
= (m \circ (A \otimes \tilde{\tau}) \otimes C) \circ (A \otimes C \otimes \Delta) \circ (\tilde{\tau} \otimes \text{cocan}) \circ (C \otimes \Delta \otimes A) \circ (\text{cocan} \otimes A)
\]
\[
= (m \circ (\tilde{\tau} \otimes C) \circ (C^{\otimes 3} \otimes \Delta) \circ (C^{\otimes 2} \otimes \text{cocan}) \circ (C \otimes \Delta \otimes A) \circ (\text{cocan} \otimes A)
\]
\[
= (m \circ (\tilde{\tau} \otimes C) \circ (C^{\otimes 3} \otimes \Delta) \circ (C \otimes (C \otimes \text{cocan}) \otimes (\Delta \otimes A)) \circ (\text{cocan} \otimes A)
\]
\[
= (m \circ (\tilde{\tau} \otimes C) \circ (C^{\otimes 3} \otimes \Delta) \circ (C \otimes (\Delta \otimes C) \otimes \text{cocan}) \circ (\text{cocan} \otimes A)
\]
\[
= (m \circ (\tilde{\tau} \otimes C) \circ (C \otimes \Delta \otimes C) \otimes C) \circ (C^{\otimes 2} \otimes \Delta) \circ (C \otimes \text{cocan}) \circ (\text{cocan} \otimes A).
\]
Taking advantage of (3.3), we obtain
\[
(m \otimes C) \circ (A \otimes \psi) \circ (\psi \otimes A)
\]
\[
= (\tilde{\tau} \circ (C \otimes \varepsilon \otimes C) \otimes C) \circ (C^{\otimes 2} \otimes \Delta) \circ (C \otimes \text{cocan}) \circ (\text{cocan} \otimes A).
\]
\[
= (\tilde{\tau} \otimes C) \circ (C \otimes (\varepsilon \otimes C^{\otimes 2}) \circ (C \otimes \Delta) \circ \text{cocan}) \circ (\text{cocan} \otimes A).
\]
\[ (3.7) \]
On the other hand, for any $c \in C$, $a \in A$, we have
\[
(\varepsilon \otimes C) \circ (C \otimes \Delta) \circ \text{coran}(c \otimes a) = \mu_C(c, a)(1) \otimes \mu_C(c, a)(2) \\
= (\Delta \otimes \mu_C)(c \otimes a).
\]  
(3.8)

Combining this with (3.7) yields
\[
(m \otimes C) \circ (A \otimes \psi) \circ (\psi \otimes A) = (\check{\tau} \otimes C) \circ (C \otimes \Delta) \circ (C \otimes \mu_C) \circ (\text{coran} \otimes A) \\
= (\check{\tau} \otimes C) \circ (C \otimes \Delta) \circ \text{coran} \circ (C \otimes m) \\
= \psi \circ (C \otimes m),
\]
as desired. Here we used the property that $\mu_C$ is an action to derive the penultimate equality. To prove the first of equations (2.4), first we observe that
\[
(\Delta \otimes \Delta) \circ \text{coran} = (C \otimes \Delta) \circ (C \otimes \Delta) \circ \text{coran}. 
\]  
(3.9)

Hence we obtain
\[
(\psi \otimes C) \circ (C \otimes \psi) \circ (\Delta \otimes A) \\
= (\psi \otimes C) \circ (C \otimes \check{\tau} \otimes C) \circ (C \otimes \Delta) \circ (C \otimes \text{coran}) \circ (\Delta \otimes A) \\
= (\psi \otimes C) \circ (C \otimes \check{\tau} \otimes C) \circ (C \otimes \Delta) \circ (\text{coran} \circ C \otimes \Delta) \circ (C \otimes \Delta) \circ (C \otimes \text{coran}) \\
= (\check{\tau} \otimes C \otimes \Delta) \circ (C \otimes \Delta) \circ (C \otimes \Delta) \circ (C \otimes \text{coran}) \circ (C \otimes \Delta) \circ (C \otimes \text{coran}).
\]

To finish the calculation, we note that $(C \otimes \check{\tau}) \circ (\Delta \otimes C) = \text{coran}^{-1}$. Indeed, thanks to (3.9), we have
\[
(C \otimes \check{\tau}) \circ (\Delta \otimes C) \circ \text{coran} = (C \otimes \check{\tau}) \circ (C \otimes \text{coran}) \circ (\Delta \otimes A) \\
= (C \otimes \varepsilon \otimes A) \circ (\Delta \otimes A) \\
= C \otimes A.
\]

Hence
\[
(\psi \otimes C) \circ (C \otimes \psi) \circ (\Delta \otimes A) = (\check{\tau} \otimes C \otimes \Delta) \circ (C \otimes \Delta) \circ (C \otimes \text{coran}) \\
= (\check{\tau} \otimes C \otimes \Delta) \circ (C \otimes \Delta) \circ (C \otimes \text{coran}) \\
= (A \otimes \Delta) \circ (\check{\tau} \otimes C) \circ (C \otimes \Delta) \circ (C \otimes \text{coran}) \\
= (A \otimes \Delta) \circ \psi,
\]
as needed. Thus we have proved that \((A,C,\psi)\) is an entwining structure.

The next step is to show that \(C \in \mathcal{M}_A^C(\psi)\), i.e., \(\Delta \circ \mu_C = (\mu_C \otimes C) \circ (C \otimes \psi) \circ (\Delta \otimes A)\). With the help of (3.9), property (ii) of the cotranslation map \(\tilde{\tau}\) and then (3.8), we compute:

\[
(\mu_C \otimes C) \circ (C \otimes \psi) \circ (\Delta \otimes A) \\
= (\mu_C \otimes C) \circ (C \otimes \tilde{\tau} \otimes C) \circ (C^{\otimes 2} \otimes \Delta \otimes C \otimes \text{cogcan}) \circ (\Delta \otimes A)
\]

As for the uniqueness of \(\psi\), suppose that there exists another entwining map \(\tilde{\psi}\) such that \(C \in \mathcal{M}_A^C(\tilde{\psi})\). Then

\[
\begin{align*}
\psi &= (\tilde{\tau} \otimes C) \circ (C \otimes \Delta) \circ (C \otimes \mu_C) \circ (\Delta \otimes A) \\
&= (\tilde{\tau} \otimes C) \circ (C \otimes (\mu_C \otimes C)) \circ (C \otimes \psi) \circ (\Delta \otimes A) \circ (\Delta \otimes A) \\
&= (\tilde{\tau} \circ (C \otimes \mu_C) \otimes C) \circ (C^{\otimes 2} \otimes \tilde{\psi}) \circ (C \otimes \Delta \otimes A) \circ (\Delta \otimes A) \\
&= (\tilde{\tau} \circ (C \otimes \mu_C) \otimes C) \circ (C \otimes \Delta \otimes A) \circ (\Delta \otimes A) \\
&= (\epsilon \otimes A \otimes C) \circ (C \otimes \tilde{\psi}) \circ (\Delta \otimes A) = \tilde{\psi}.
\end{align*}
\]

\[
\square
\]

4  Galois entwining structures

Theorem 2.7 allows one to view a \(\psi\)-principal bundle as a coalgebra-Galois extension. More precisely, recall from (3) the following

**Definition 4.1** Let \((A,C,\psi)\) be an entwining structure, and let \(e \in C\) be a group-like element. Then \(B := \{b \in A \mid \psi(e \otimes b) = b \otimes e\}\) is an algebra, and we say that \(A(B,C,\psi,e)\) is a coalgebra \(\psi\)-principal bundle iff the map 
\[\text{can}_\psi : A \otimes_B A \to A \otimes C, \ a \otimes a' \mapsto a\psi(e \otimes a')\] is bijective.
Proposition 4.2 For a given entwining structure \((A,C,\psi)\) and a group-like \(e \in C\), the following statements are equivalent:

(1) \(A(B,C,\psi,e)\) is a coalgebra \(\psi\)-principal bundle.

(2) There exists a unique coaction \(\Delta_A : A \to A \otimes C\) such that \(B \subseteq A\) is a coalgebra-Galois extension of \(B\) by \(C\), \(\psi\) is the canonical entwining map, and \(\Delta_A(1) = 1 \otimes e\).

Proof. (1) \(\Rightarrow\) (2). By [3, Proposition 2.2], \(A\) is a right \(C\)-comodule with the coaction \(\Delta_A : a \mapsto \psi(e \otimes a)\). Using the second of conditions (2.3) one verifies that \(\Delta_A(1) = 1 \otimes e\). The first of conditions (2.3) implies that \(A \in \mathcal{M}_A^C(\psi)\) with the structure maps \(\Delta_A\) and \(m\). Therefore, by [3, Lemma 3.3], \(B\) as defined in Definition 4.1 is identical with the set \(\{b \in A \mid \forall a \in A, \Delta_A(ba) = b\Delta_A(a)\}\) considered in Definition 2.2. Hence \(\text{can} = \text{can}_\psi\) and \(B \subseteq A\) is a \(C\)-Galois extension. By Theorem 2.7, \(\psi\) is the associated canonical entwining map. Assume now that there exists another coaction \(\Delta'_A\) satisfying condition (2). Then, by Theorem 2.7, \(A \in \mathcal{M}_A^C(\psi)\) with the structure maps \(\Delta'_A\) and \(m\). Consequently, as \(\Delta'_A(1) = 1 \otimes e\), we have

\[\Delta'_A(a) = \Delta'_A(1 \cdot a) = ((m \otimes C) \circ (A \otimes \psi))((\Delta'_A(1) \otimes a) = \psi(e \otimes a) = \Delta(a), \quad \forall a \in A.\]

(2) \(\Rightarrow\) (1). Since \(\psi\) is the canonical entwining map, \(A \in \mathcal{M}_A^C(\psi)\) by Theorem 2.7, so that [3, Lemma 3.3] implies that \(B\) as defined in Definition 2.2 is identical with the set \(\{b \in A \mid \psi(e \otimes b) = b \otimes e\}\) as required in Definition 4.1. Furthermore, since the normalisation condition \(\Delta_A(1) = 1 \otimes e\) implies that \(\tau(e) = 1 \otimes_B 1\), we have \(\psi(e \otimes b) = e(1)(e(2)b)_{(0)} \otimes (e(2)b)_{(1)} = \Delta_A(b)\), for all \(a \in A\). Therefore \(\text{can} = \text{can}_\psi\) is bijective as required. \(\square\)

Remark 4.3 A slightly different definition of a \(C\)-Galois extension was proposed in [4]. Let \(A\) be an algebra, \(C\) a coalgebra and \(e\) a group-like element of \(C\). One assumes that \(A \otimes C \in \mathcal{M}_A\), \(A \in \mathcal{M}_C\), and the action and coaction are such that \(\Delta_A \circ m = \mu_{A \otimes C}(\Delta_A \otimes A)\) and \(\mu_{A \otimes C}(a \otimes e, a') = aa'_{(0)} \otimes a'_{(1)}\) for any \(a, a' \in A\). Then \(B := \{b \in A \mid \Delta_A(b) = b \otimes e\}\) is an algebra and the canonical map \(\text{can} : A \otimes_B A \to A \otimes C\) is well-defined. \(B \subseteq A\) is a \(C\)-Galois extension if the canonical map \(\text{can}\) is a bijection. One easily finds, however, that given \(A\) and
satisfying the above conditions, \( B = \{ b \in A \mid \forall a \in A, \ \Delta_A(ba) = b\Delta_A(a) \} \), so that \( A \in B\mathcal{M}^C \) via the maps \( \mu_A = m \) and \( \Delta_A \). Hence, by Theorem 2.7, this definition of a \( C \)-Galois extension is equivalent to the one introduced in [3] and in Definition 2.3 provided that \( \Delta_A(1) = 1 \otimes e \).  

**Remark 4.4** In the Hopf-Galois case, the formula for \( \psi \) becomes quite simple: \( \psi(h \otimes a) = a(0) \otimes ha(1) \). If the Hopf algebra \( H \) has a bijective antipode, \( \psi \) is an isomorphism, and its inverse is given by \( \psi^{-1}(a \otimes h) := hS^{-1}(a(1)) \otimes a(0) \). (In fact, \( \psi \) is an isomorphism if and only if \( H \) has a bijective antipode [3, Theorem 6.5].) Furthermore, the coaction \( \Delta_A : A \to H^{\text{op}} \otimes A \),

\[
\Delta_A(a) := a_{(-1)} \otimes a_{(0)} := S^{-1}(a(1)) \otimes a(0)
\]

makes \( A \) a left \( H^{\text{op}} \)-comodule algebra, where \( H^{\text{op}} \) stands for the Hopf algebra with the opposite multiplication. One can define the following left version of the canonical map: \( \text{can}_L : A \otimes_B A \to H^{\text{op}} \otimes A \), \( \text{can}_L(a \otimes_B a') := a_{(-1)} \otimes a_{(0)}a' \). It is straightforward to verify that \( \psi \circ \text{can}_L = \text{can} \). Since \( \psi \) and \( \text{can} \) are isomorphisms, we can immediately conclude that so is \( \text{can}_L \).

In parallel to the theory of coalgebra \( \psi \)-principal bundles, the notion of a dual \( \psi \)-principal bundle was introduced in [3]. This is recalled in the following

**Definition 4.5** Let \( (A, C, \psi) \) be an entwining structure, \( \kappa : A \to k \) an algebra homomorphism (character), and

\[
I_\kappa := \text{span}\{((\kappa \otimes C) \circ \psi)(c \otimes a) - c\kappa(a) \mid a \in A, c \in C\}.
\]

Then \( B := C/I_\kappa \) is a coalgebra, and we say that \( C(B, A, \psi, \kappa) \) is a dual \( \psi \)-principal bundle iff the map \( \text{cocoan}_\psi = (C \otimes \kappa \otimes C) \circ (C \otimes \psi) \circ (\Delta \otimes A) : C \otimes A \to C \square_B C \) is bijective.

Using Theorem 3.5 we can relate dual \( \psi \)-principal bundles with \( A \)-Galois coextensions.

**Proposition 4.6** For a given entwining structure \( (A, C, \psi) \) and an algebra map \( \kappa : A \to k \), the following statements are equivalent:
(1) $C(B, A, \psi, \kappa)$ is a dual $\psi$-principal bundle.

(2) There exists a unique action $\mu_C : C \otimes A \to C$ such that $C \to B$ is an $A$-Galois coextension of $B$ by $A$, $\psi$ is the canonical entwining map, and $\varepsilon \circ \mu_C = \varepsilon \otimes \kappa$.

Proof. (1) $\Rightarrow$ (2). By [5, Proposition 2.6], $C$ is a right $A$-module with the action $\mu_C = (\varepsilon \otimes C) \circ \psi$. Taking advantage of the second of conditions (2.4) one verifies that $\varepsilon \circ \mu_C = \varepsilon \otimes \kappa$. The first of conditions (2.4) implies that $C \in \mathcal{M}_A^C(\psi)$ with structure maps $\Delta$ and $\mu_C$. Using this fact, explicit form of $\mu_C$, and (2.4) one can show that $I_\kappa = I$, the latter being defined in Lemma 3.2. Hence $\text{cocan} = \text{cocan}_\psi$ and $C \to B$ is an $A$-Galois coextension. By Theorem 3.5, $\psi$ is the associated canonical entwining map. Assume now that there exists another action $\mu'_C$ of $A$ on $C$ satisfying conditions (2). Then, by Theorem 3.5, $C \in \mathcal{M}_A^C(\psi)$ with the structure maps $\Delta$ and $\mu'_C$. Consequently, since $\varepsilon \circ \mu_C = \varepsilon \otimes \kappa$, we have

$$
\mu'_C = (\varepsilon \otimes C) \circ \Delta \circ \mu'_C
= (\varepsilon \otimes C) \circ (\mu'_C \otimes C) \circ (C \otimes \psi) \circ (\Delta \otimes A)
= (\varepsilon \otimes \kappa \otimes C) \circ (C \otimes \psi) \circ (\Delta \otimes A)
= (\kappa \otimes C) \circ \psi = \mu_C.
$$

(2) $\Rightarrow$ (1). Since $\psi$ is the canonical entwining map, $C \in \mathcal{M}_A^C(\psi)$ by Theorem 3.5. The normalisation condition $\varepsilon \circ \mu_C = \varepsilon \otimes \kappa$ implies that $(\varepsilon \otimes \varepsilon) \circ \text{cocan} = \varepsilon \otimes \kappa$ and, consequently, $\kappa \circ \tau = \varepsilon \otimes \varepsilon$. This, in turn, leads to the equality $\mu_C = (\kappa \otimes C) \circ \psi$. Using this equality, the fact that $C \in \mathcal{M}_A^C(\psi)$, and (2.4) one shows that $I_\kappa = I$. Hence $\text{cocan}_\psi = \text{cocan}$ and $\text{cocan}_\psi$ is bijective, as required. \(\Box\)

5 Appendix

We say that two subgroups $G_1$ and $G_2$ of a group $G$ generate $G$ if any element of $G$ can be written as a finite length word whose letters are elements of $G_1$ or $G_2$. The (dual) coalgebra version of this concept is given in the following definition:
Definition 5.1 Let $C$ be a coalgebra and $I_1$, $I_2$ its coideals. Let $\varphi_{(i)}$ denote the composite map

$$C \xrightarrow{\Delta_n} C^\otimes n+1 \xrightarrow{\pi_{i1} \otimes \cdots \otimes \pi_{in+1}} C/I_{i1} \otimes \cdots \otimes C/I_{in+1},$$

where $\Delta_n(c) := c^{(1)} \otimes \cdots \otimes c^{(n+1)}$, $(i) := (i_1, \ldots, i_n) \in \{1, 2\}^n$ is a finite multi-index, and each $\pi_{ik}$ is a canonical surjection. We say that the quotient coalgebras $C/I_1$ and $C/I_2$ cogenerate $C$ iff $\bigcap_{(i) \in \mathcal{M}_f} \ker \varphi_{(i)} = 0$, where $\mathcal{M}_f$ is the space of all finite multi-indices. We write then $(C/I_1) \cdot (C/I_2) = C$.

Observe that the above construction is closely related to the wedge construction of $[21]$. In the group situation it is clear that what is invariant under both generating subgroups is invariant under the whole group, and vice-versa. Below is the (dual) coalgebra version of this classical phenomenon.

Proposition 5.2 Let $C$ be a coalgebra, $I_1$ and $I_2$ coideals of $C$, and $A$ a right $C$-comodule. Then, defining coinvariants as in Definition 2.2, we have:

$$(C/I_1) \cdot (C/I_2) = C \implies A^{coC} = A^{co(C/I_1)} \cap A^{co(C/I_2)}.$$ 

Proof. Clearly, we always have $A^{coC} \subseteq A^{co(C/I_1)} \cap A^{co(C/I_2)}$. Assume now that there exists $b \in A^{co(C/I_1)} \cap A^{co(C/I_2)}$ such that $b \notin A^{coC}$. Then there also exists $a \in A$ such that $0 = \Delta_A (ba) - b \Delta_A (a) =: \sum_{j \in \mathcal{J}} f_j \otimes h_j$, where $\{f_\alpha\}_{\alpha \in A}$ is a basis of $A$ and $\{h_j\}_{j \in \mathcal{J}}$ is a non-empty set which does not contain zero. Furthermore, for any $(i) \in \mathcal{M}_f$, we have:

$$(A \otimes \varphi_{(i)}) (\sum_{j \in \mathcal{J}} f_j \otimes h_j)$$

$$= ((A \otimes \pi_{i1} \otimes \cdots \otimes \pi_{in}) \circ (A \otimes \Delta_{n-1}))(ba)_0 \otimes (ba)_{(1)} - ba_0 \otimes a_{(1)})$$

$$= (ba)_0 \otimes \pi_{i1}((ba)_{(1)}) \otimes \cdots \otimes \pi_{in}(ba)_{(n)} - ba_0 \otimes \pi_{i1}(a_{(1)}) \otimes \cdots \otimes \pi_{in}(a_{(n)})$$

$$= (((A \otimes \pi_{i1}) \circ \Delta_A \otimes C^{\otimes (n-1)} \circ \cdots \circ (A \otimes \pi_{in}) \circ \Delta_A)(ba)$$

$$- ba_0 \otimes \pi_{i1}(a_{(1)}) \otimes \cdots \otimes \pi_{in}(a_{(n)})$$

$$= 0$$

Consequently, by the linear independence of $f_j$, $j \in \mathcal{J}$, we have $\varphi_{(i)}(h_j) = 0$ for $j \in \mathcal{J}$. Hence, as this is true for any $(i) \in \mathcal{M}_f$, we obtain $\bigcap_{(i) \in \mathcal{M}_f} \ker \varphi_{(i)} \neq 0$, as needed. \qed
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