Composite-fermionization of bosons in rapidly rotating atomic traps

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The non-perturbative effect of interaction can sometimes make interacting bosons behave as though they were free fermions. The system of neutral bosons in a rapidly rotating atomic trap is equivalent to charged bosons coupled to a magnetic field, which has opened up the possibility of fractional quantum Hall effect for bosons interacting with a short range interaction. Motivated by the composite fermion theory of the fractional Hall effect of electrons, we test the idea that the interacting bosons map into non-interacting spinless fermions carrying one vortex each, by comparing wave functions incorporating this physics with exact wave functions available for systems containing up to 12 bosons. We study here the analogy between interacting bosons at filling factors \( \nu = n/(n+1) \) with non-interacting fermions at \( \nu^* = n \) for the ground state as well as the low-energy excited states and find that it provides a good account of the behavior for small \( n \), but interactions between fermions become increasingly important with \( n \). At \( \nu = 1 \), which is obtained in the limit \( n \to \infty \), the fermionization appears to overcompensate for the repulsive interaction between bosons, producing an attractive interactions between fermions, as evidenced by a pairing of fermions here.

I. INTRODUCTION

The experimental realization of Bose-Einstein condensation (BEC) of atomic gases has generated a rich variety of phenomena. In particular, it has allowed the possibility of testing the remarkable concept of “statistical transmutation,” namely the idea that interacting bosons may sometimes behave like spinless fermions. For contact interactions, it may seem rather sensible for bosons to emulate fermions, to the extent allowed by symmetry requirements, because the Pauli principle itself fully takes care of the repulsion. Of course, a conceptual understanding of how this precisely happens, what it means, and how bosons can behave like fermions while satisfying the constraints of bosonic exchange symmetry requires a detailed theory. The tendency for fermionization has been appreciated for quite some time for bosons in one dimension. Girardeau showed that for an infinitely strong delta function repulsion, the bosonic ground state wave function \( \Psi_B \) is related to the Slater determinant ground state wave function \( \Psi_F \) for spinless fermions in one dimension as:

\[
\Psi_B = |\Psi_F|
\]

The problem was solved exactly for an arbitrary strength of the interaction by Lieb and Liniger; the fermionic description is a useful starting point in the strong-coupling limit, when the interaction strength is large compared to the Fermi energy. Recent experiments are in excellent agreement with the Lieb-Liniger theory in the entire range of interaction strength, which can be varied in an optically confined one dimensional boson system by controlling the density and the confinement strength.

This work is concerned with the possibility of an emergence of fermion-like structures in a bosonic system in two dimensions, under conditions appropriate for a fractional quantum Hall effect (FQHE) of bosons. The familiar FQHE occurs when charged electrons are confined to two dimensions and exposed to a strong magnetic field. There is no realizable system of charged electrons where FQHE can be studied. However, a system of neutral atoms in a rotating trap is mathematically equivalent to a system of charged bosons in a magnetic field, which, with confinement to two dimensions, should create, for sufficiently rapid rotation, a FQHE state of bosons. BEC systems confined to two dimensions have been created and their properties have been studied under rotation, although the FQHE conditions have so far not been achieved. Rotation of a BEC produces vortices in the condensate. As the rotation frequency is increased, the BEC state is destroyed and, eventually, the FQHE state may be achieved (the latter has no off-diagonal long range order). These advances have motivated a number of studies of the FQHE of bosons interacting via a short-range interaction.

We will assume below that the Landau level (LL) spacing for bosons is sufficiently large that it is a good approximation to restrict the bosons to the lowest Landau level. The bosonic system is then always in the strong coupling limit, because the nature of the state is completely determined by the interaction. In fact, the solutions are independent of the strength of the interaction, which merely sets the energy scale. It is natural to appeal to the fractional quantum Hall effect of electrons for guidance. Laughlin’s wave function can be generalized for the ground state at the bosonic filling \( \nu = 1/2 \):

\[
\Psi_B^{\nu=1/2} = \prod_{j<k} (z_j - z_k)^2 \exp \left[ -\frac{1}{4} \sum_i |z_i|^2 \right]
\]

where \( z_j = x_j - iy_j \) denotes the position of the \( j \)th boson on the two-dimensional plane, and the magnetic length has been set to unity. More generally, the understanding of the electronic FQHE is based on the formation of quasiparticles known as composite fermions.
Reimann, to a parabolic quantum dot. Confined to a disk by a parabolic confinement; these composite fermion theory for interacting bosons in a magnetic field manifests through an incompressible state of bosons.

Odd, and

Where the sum is over all permutations \( \sigma \), \( \text{sgn}(\sigma) = +1 \) or \(-1\) depending on whether the permutation is even or odd, and \( N \) is an even integer. The Pfaffian has the same form as the projection of the real space Bardeen-Cooper-Schrieffer wave function into a fixed number of particles \( N \), and therefore represents a pair of fermions, as noted by Greiter, Wen and Wilczek\[26\] The fermion pairing manifests through an incompressible state of bosons.

The mapping into fermions for the bosonic FQHE problem is conceptually distinct from that applicable in one dimension (Eq. \( \text{(1)} \)). The modulus of the fermion wave function is a manifestly bad approximation for the former, because such a wave function has substantial mixing with higher Landau levels, and therefore a very high kinetic energy.

Much work has already been done toward testing the composite fermion theory for interacting bosons in a magnetic field. Many studies take bosons to be in a plane, confined to a disk by a parabolic confinement; these are analogous to the CF theory of electrons confined to a parabolic quantum dot\[26,27\] Viefers, Hansson and Reimann\[4\] Cooper and Wilkin\[18\], Wilkin, and Gunn\[15\], and Manninen et al\[17\] have found high overlaps between the exact solutions and Jain’s wave functions for up to \( N = 10 \) particles at the “magic” angular momenta of the yrast spectrum; further, they also found that the state at \( \nu = 1 \) is well described by Moore-Read’s wave function.

While a parabolic potential appears naturally for optically confined bosonic systems, the strength of confinement can be varied, and it may be useful to consider the situation without confinement. For a large number of bosons, it is natural, in the simplest approximation, to neglect the effect of boundaries and concentrate on the bulk properties. That is most conveniently accomplished in theory by studying bosons in the spherical geometry\[28\] in which the bosons move on the two-dimensional surface of the sphere, with a radial magnetic field produced by a magnetic monopole at the center. Exact diagonalization studies have been carried out in the spherical geometry. Regnault and Jolicoeur\[29\] have shown that the ground state at \( \nu = n/(n+1) \) is incompressible, consistent with the analogy to filled LL state at \( \nu^* = n \). Their results also show evidence of incompressibility at \( \nu = 1 \).

In addition, at filling factor \( \nu = 1 \), we will consider Moore and Read’s Pfaffian wave function\[24\], given by

\[
\Psi^B = \mathcal{P}_{LLL} \prod_{j<k}(z_j - z_k) \Phi^F, \tag{3}
\]

where \( \Phi^F \) is the wave function for non-interacting fermions (at the effective filling factor), and \( \mathcal{P}_{LLL} \) projects the wave function into the lowest Landau level.

Exactly matching between interacting bosons and non-interacting fermions should be noted. Eq. \( \text{(3)} \) produces wave functions for the ground and excited states at arbitrary filling in the range \( 1 \geq \nu \geq 1/2 \). This paper examines their accuracy by comparison with exact wave functions. If valid, a simplification of the problem is achieved through a mapping of a non-trivial interacting boson problem into a more amenable non-interacting fermion problem, and many essential properties of bosons in rapidly rotating traps should find an explanation in terms of almost free particles.

In addition, at filling factor \( \nu = 1 \), we will consider Moore and Read’s Pfaffian wave function\[24\], given by

\[
\Psi^B_{\text{PF}} = \text{Pf}\left\{ \frac{1}{z_j - z_k} \right\} \prod_{j<k}(z_j - z_k). \tag{4}
\]

\( \text{Pf}\{M_{jk}\} \) is the Pfaffian of an antisymmetric matrix \( M \) with elements \( M_{jk} \), defined as (up to an overall constant)

\[
\text{Pf}\{M_{jk}\} = \sum_{\sigma} \text{sgn}(\sigma) M_{\sigma(1)\sigma(2)} M_{\sigma(3)\sigma(4)} \cdots M_{\sigma(N-1)\sigma(N)}, \tag{5}
\]

where the sum is over all permutations \( \sigma \), \( \text{sgn}(\sigma) = +1 \) or \(-1\) depending on whether the permutation is even or odd, and \( N \) is an even integer. The Pfaffian has the same form as the projection of the real space Bardeen-Cooper-Schrieffer wave function into a fixed number of particles \( N \), and therefore represents a pair of fermions, as noted by Greiter, Wen and Wilczek\[26\] The fermion pairing manifests through an incompressible state of bosons.

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We will consider below the spherical geometry and report on detailed and quantitative tests of the validity of the correspondence between interacting bosons in the FQHE regime and free fermions in the integral quantum Hall regime, which makes definite predictions for the quantum numbers of the low-energy states of the interacting boson system, their energies and their eigenfunctions. Various trial wave functions will be compared with the exact eigenstates and the predicted energies with the exact eigenenergies. It is well known that Laughlin’s wave function\[22\] which is also a special case of Eq. \( \text{(3)} \), is the exact solution for the ground state at \( \nu = 1/2 \) for bosons in the lowest Landau level interacting with a short range interaction. However, that by itself does not imply a correspondence between interacting bosons and free fermions; for that purpose it is necessary to verify the correspondence of Eq. \( \text{(3)} \) for the ground states and excitations in a broader range of filling factors. We will test it for the ground state and excitations at \( \nu = 1/2 \), \( \nu = 2/3 \) and \( \nu = 3/4 \).
II. THE HAMILTONIAN

We consider a system of \( N \) bosons with mass \( m \) in a harmonic trap that is rotating with frequency \( \omega \). In the rotating frame of reference, the system is described by the Hamiltonian:

\[
\mathcal{H} = \sum_{i=1}^{N} \left\{ \frac{1}{2m}(p_i - m\omega^2 x_i)^2 + \frac{m}{2} \left[ (\omega_s^2 - \omega^2)(x_i^2 + y_i^2) + \omega_z^2 z_i^2 \right] \right\} + \sum_{i<j}^{N} V(r_i - r_j),
\]

where \( \omega_s \) and \( \omega_z \) are the radial and axial trap frequencies respectively. Vectors \( r_i = (x_i, y_i, z_i) \) represent particle positions. In an ultra-cold dilute Bose gas, the scattering between particles is dominated by the s-wave scattering process. It is then an excellent approximation to describe the interaction by a delta function:

\[
V(r) = \frac{4\pi \hbar^2 a_s}{m} \delta^{(3)}(r),
\]

where \( a_s \) is the s-wave scattering length, assumed to be positive in this work. When \( \omega \) and \( \omega_s \) are identical, the Hamiltonian resembles that of particles with charge \( e \) in a magnetic field \( \mathbf{B} = 2m\omega/e\hat{z} \). An effective magnetic length is defined as \( \ell = \sqrt{\hbar/(2ma_s)} \). The effective cyclotron frequency is defined as \( \omega_c = eB/m = 2\omega \). If the axial trap is strong enough such that the wave function along the z direction is the ground state of the harmonic potential in the z axis, the system enters a two dimensional (2D) regime where the potential felt by particles is written as:

\[
V(r) = g \delta^{(2)}(r),
\]

with \( g = \hbar^2 a_s \sqrt{8\pi/(m\ell_z)} \), where \( \ell_z = \sqrt{\hbar/(m\omega_z)} \). The energy scale in the 2D regime is set by the effective coupling constant \( g \). We will assume below that the interaction strength is sufficiently small compared to the Landau level spacing that LL mixing is negligible. From our experience with electronic FQHE, we know that a modest amount of LL mixing does not significantly alter the results.

III. COMPOSITE FERMION THEORY

For bosons in the lowest Landau level, there are three situations. (i) For \( \nu < 1/2 \), there are many linearly independent wave functions that vanish upon coincidence of bosons, producing an enormous ground state degeneracy. (ii) For \( \nu = 1/2 \) there is a single wave function that has zero energy for the delta function interaction, giving a non-degenerate ground state here. It remains the ground state for arbitrarily high coupling \( g \), and may be considered to be the analog of the Girardeau wave function of the one dimensional problem. (iii) For the excitations at \( \nu = 1/2 \), or for any eigenfunctions at \( \nu > 1/2 \), there are no wave functions in the lowest Landau level that vanish when two particles coincide. While no exact results are available here, analogy to fermions gives plausible wave functions that we now describe.

A. \( \nu = 1/2 \)

For \( \nu = 1/2 \), Laughlin’s wave function for the ground state is given by (in the spherical geometry)

\[
\Psi^{\nu=1/2}_G = \Phi^{2}_1,
\]

where

\[
\Phi_1 = \prod_{j<k}(u_j v_k - v_j u_k),
\]

is the wave function of the lowest filled Landau level.

\[
(u_j, v_j) = (\cos(\theta_j/2), e^{i\phi_j/2}), \sin(\theta_j/2), e^{-i\phi_j/2})
\]

are the spinor coordinates describing the position of a particle on the surface of a sphere. It is the exact ground state for bosons at \( \nu = 1/2 \) interacting with a delta function interaction, which can be seen straightforwardly by noting that \( \Phi^{2}_1 \) is the only wave function at \( \nu = 1/2 \) that is confined to the lowest LL and has zero interaction energy for the delta function interaction. The wave functions \( \Phi^{2p}_G \) with \( p \geq 2 \) are not relevant for the short range interaction, as these are degenerate with a large number of other states.

B. \( \nu \geq 1/2 \)

For electrons in the lowest LL, the CF theory hypothesis that strongly interacting electrons map into weakly interacting fermions of a new kind, called composite fermions. The composite fermions experience an effective magnetic field given by \( B^\ast = B - m\rho_0 \phi_0 \), where \( B \) is the external magnetic field, \( \phi_0 = \hbar c/e \), and \( m \) is an even integer. Equivalently, the filling factor of composite fermions, \( \nu^\ast \), is related to the electron filling factor, \( \nu \), by \( \nu = \nu^\ast/(m\nu^\ast + 1) \). This is interpreted in terms of electrons having captured an even number \( (m) \) of flux quanta of the external magnetic field to become composite fermions, which no longer see the magnetic flux that they have assimilated into themselves. This physics suggests the wave functions \( \Psi^\nu_G = P_{LLL} \Psi^\nu \Phi_{\nu^\ast} \), where \( \Phi_{\nu^\ast} \) is the Slater determinant wave function for non-interacting electrons at \( \nu^\ast \), \( \Phi_1 \) is the wave function of one filled Landau level, and \( P_{LLL} \) projects the wave function into the lowest Landau level, as appropriate for very large magnetic fields. These wave function explicitly relate the eigenfunctions of interacting electrons at
\( \nu \) to those of non-interacting fermions at \( \nu^* \), and have been tested both in the spherical geometry\(^{26,27,34}\) and the disk geometry\(^{26,27,34}\).

The considerations in the preceding paragraph are readily generalized to bosons by taking \( m \) to be an odd integer. We specialize to \( m = 1 \) (other odd integer values not being relevant to the problem of our interest) and filling factors

\[
\nu = \frac{n}{n + 1}.
\]

which correspond to \( \nu^* = n \) of fermions. The wave function at \( \nu \) is now given by

\[
\Psi^B_\nu = \mathcal{P}_{LLL} \Phi_1 \Phi_n.
\]

which is the spherical analog of Eq. (11). The Jastrow factor \( \Phi_1 \) now attaches a single vortex to each fermion in \( \Phi_n \).

These two equations define the mapping between interacting bosons and non-interacting fermions in microscopic detail. The first equation has implications about the structure of the low-energy eigenstates of the interacting boson system, whereas the last gives trial wave functions for the eigenstates, and also the eigenenergies. There are two ways to physically think about the above equations. (i) Bosons have captured an odd number of vortices each to convert into a composite fermion\(^{26,33}\) (ii) The bosons are represented as bound states of fermions and an odd number of vortices. The tests below, of course, are independent of the interpretation.

We note that the ground state and excitations of interacting bosons at \( \nu = n/(n + 1) \) are images of the ground state and excitations of fermions at \( \nu^* = n \) according to Eq. (13). The wave function for the ground state at \( \nu = n/(n + 1) \) is given by Eq. (15) with \( \Phi_n \) taken as the wave function of the ground state at \( \nu^* = n \), i.e., the \( n \) filled Landau level state. The wave function for the excited state is similarly related to the lowest energy particle-hole excitation, i.e. an exciton at \( \nu^* = n \). The eigenstates of the spherical geometry are labeled by the total orbital angular momentum, \( L \). The ground state has \( L = 0 \), which implies uniform density. It has no adjustable parameters, given that the wave function of \( n \) filled Landau levels is unique. The wave function for the exciton for any given \( L \) is also determined completely by group theory, and therefore is free of any adjustable parameters.

A subtle feature of the composite fermion theory ought to be noted. The ground state wave function at \( \nu = 1/2 \) \( (\nu^* = 1) \) is given by \( \Psi^B_{1/2} = \Phi_1^2 \) (no lowest-Landau level projection is required here, because the wave function is already in the lowest Landau level). It manifestly eliminates spatial coincidence of particles, and thus has zero interaction energy for the contact interaction potential. As mentioned earlier, no such wave functions can be written, even in principle, for the excited states at \( \nu = 1/2 \) or for any states at \( \nu > 1/2 \). The CF theory circumvents this problem by first neglecting the lowest LL constraint to write wave functions \( (\Phi_1 \Phi_n) \) in which bosons do not occupy the same spatial position, and then projecting Them into the lowest Landau level, hoping that this would capture the actual correlations within the lowest LL. The wave functions \( \Psi^B \) are in general much more complicated than Laughlin’s wave function at \( \nu = 1/2 \). Their validity is far from obvious, and their confirmation would provide a non-trivial evidence for the composite-fermionization of the bosonic system.

### C. \( \nu = 1 \)

We will also be interested in the nature of the state in the limit of \( n \to \infty \), i.e. at \( \nu = 1 \). Let us recall what happens for electrons in this limit, which corresponds to \( \nu_e = 1/2 \) for electrons. If the residual interactions between composite fermions are negligible, a Fermi sea of composite fermions is obtained here (the state with an infinite number of filled Landau levels is another representation of a Fermi sea), as proposed by Halperin, Lee, and Read\(^{36}\). That provides a good description of the compressible state at \( \nu_e = 1/2 \), also explaining why there is no FQHE here\(^{36}\). However, in the second Landau level, electrons form an incompressible state when the Landau level is half full (which corresponds to a total filling of \( \nu_e = 5/2 \), which appears to be best described by Moore-Read’s wave function. This implies that the mapping into noninteracting composite fermions is no longer valid, and one must consider the residual interactions between them, which presumably cause a pairing instability of the CF Fermi sea\(^{26}\).

If bosons behaved like non-interacting fermions in the limit of \( n \to \infty \), the system at \( \nu = 1 \) would be analogous to the Halperin-Lee-Read Fermi sea. On the other hand, if the bosons map into interacting fermions, Moore-Read’s wave function becomes a plausible candidate. In the spherical geometry, it is given by

\[
\Psi^B_{Pf} = \text{Pf} \left\{ \frac{1}{u_j v_k - u_j v_k} \right\} \prod_{j<k} (u_j v_k - u_j v_k).
\]

### IV. CALCULATION

We will study the wave function in Eq. (15) for the ground states and excitations at \( \nu = 1/2 \), \( \nu = 2/3 \) and \( \nu = 3/4 \). For technical convenience, we will define the lowest LL projection as follows:

\[
\Psi^B_{\nu/(n+1)} = \Phi_1^{-1} \mathcal{P}_{LLL} \Phi_1^2 \Phi_n
\]
are very close to the exact eigenstates. The more convenient of these two methods relies on having at least two factors of $\Phi_1$; it does not work for $\Phi_1\Phi_2$, but requires $\Phi_1^2\Phi_n$. That is the reason for defining the projection as in Eq. (15). We refer the reader to the literature for the explicit construction of the lowest Landau level projected wave functions $P_{LLL}\Phi_1^2\Phi_n$, which can be used here without change. The presence of $\Phi_1$ in the denominator is not a cause for concern, because $P_{LLL}\Phi_1^2\Phi_n$, being anti-symmetric, also contains the factor $\Phi_1$ in it. We have not tested the relative importance of this method of projection as opposed to a direct projection, but, based on our experience with fermions, we expect them to produce more or less the same lowest LL wave function.

To compare $\Psi^B$ with the exact wave functions $\Psi^{ex}$, we will calculate their overlap:

$$O^2 = \frac{|\langle \Psi^{ex}_{n/(n+1)} | \Psi^B_{n/(n+1)} \rangle|^2}{|\langle \Psi^{ex}_{n/(n+1)} | \Psi^B_{n/(n+1)} \rangle|^2}.$$  

For the Metropolis Monte Carlo evaluation, it is convenient to rewrite it as

$$O^2 = \frac{|\langle \Psi^{ex}_{n/(n+1)} | \Psi^B_{n/(n+1)} \rangle|^2}{|\langle \Psi^{ex}_{n/(n+1)} | \Psi^{ex}_{n/(n+1)} \rangle|^2}.$$  

Then, using the wave function $\Psi^B_{n/(n+1)}$ as the sampling weight, both the numerator and the denominator can be calculated simultaneously: $\Psi^B_{n/(n+1)}$ represents either the ground state wave function or the CF exciton wave function at $\nu = n/(n+1)$. The corresponding exact wave functions are obtained by the Lánčzos algorithm.

Another measure of the quantitative accuracy of the CF description is the comparison between the predicted energy with the exact one. The CF prediction for the energy of the ground or excited states is given by

$$E = \frac{\langle \Psi^B_{n/(n+1)} | V | \Psi^B_{n/(n+1)} \rangle}{\langle \Psi^{ex}_{n/(n+1)} | \Psi^{ex}_{n/(n+1)} \rangle}.$$  

Even though the wave functions are rather complicated, the integral can be evaluated by the Metropolis Monte Carlo method. We find it convenient to write the numerator as

$$\sum_{i<j} \langle \Psi^B_{n/(n+1)} | \delta^{(2)}(\Omega_i, \Omega_j) | \Psi^B_{n/(n+1)} \rangle \langle \Psi^{ex}_{n/(n+1)} | \delta^{(2)}(\Omega_i, \Omega_j) | \Psi^{ex}_{n/(n+1)} \rangle$$

$$= \frac{N(N-1)}{2} \int d\Omega_1 \cdots d\Omega_N \langle \Psi^{ex}_{n/(n+1)} | \delta^{(2)}(\Omega_1, \Omega_2) | \Psi^{ex}_{n/(n+1)} \rangle \langle \Psi^{ex}_{n/(n+1)} | \delta^{(2)}(\Omega_1, \Omega_2) | \Psi^{ex}_{n/(n+1)} \rangle$$

$$= \frac{N(N-1)}{4\pi R^2} \int d\Omega_1 \cdots d\Omega_N \langle \Psi^{ex}_{n/(n+1)} | \delta^{(2)}(\Omega_1, \Omega_2) | \Psi^{ex}_{n/(n+1)} \rangle \langle \Psi^{ex}_{n/(n+1)} | \delta^{(2)}(\Omega_1, \Omega_2) | \Psi^{ex}_{n/(n+1)} \rangle.$$  

where, in the spherical geometry, the unit vector $\Omega_i = (\sin \theta_i \cos \varphi_i, \sin \theta_i \sin \varphi_i, \cos \theta_i)$ describes the position of particles on the surface of the sphere, and we have used $\int d\Omega_1/4\pi R^2 = 1$ in the last step ($R$ is the radius of the sphere), which expresses the integral in a form where $|\Psi^B_{n/(n+1)}(\Omega_1)|^2$ can be used as the sampling function.

For the Monte Carlo evaluation of the overlap, occupation number basis states are transformed into real space basis wave functions, which are permanent. The permanent is an analog of a determinant of a square matrix $M$ with elements $M_{jk}$ in which all signs are taken as positive in the expansion of minors. In general, it can be written as $\text{per} (M_{jk}) = \sum_{\sigma} \prod_{j=1}^N M_{j,\sigma(j)}$, where $\sigma$ are permutations of $N$ indices. We evaluate the permanents using the Ryser algorithm. Typically we perform $10^3 \sim 10^5$ iterations in one Monte Carlo run. For larger systems, the majority of the computational time is spent on evaluating permanents. For example, we need to evaluate 61108 permanents at each Monte Carlo step for a system of $N = 12$ bosons at $\nu = 3/4$ which takes approximately 480 CPU hours for $10^3$ iterations on a single node of a PC cluster, consisting of dual 2.4GHz PentiumIV processors, to accumulate the desired accuracy. We use as many as 10 nodes to increase the efficiency.

In the energy calculation, the wave function in Eq. (15) consists of a linear combination of several determinants for an excited state at a given angular momentum $L$. (For the ground state, only one determinant needs to be evaluated.) The calculation for energy is far less time consuming than that for the overlap. We perform about $1.2 \times 10^7$ iterations in a single Monte Carlo run. The quoted statistical uncertainty in the calculation reflects one standard deviation from 10 independent runs. To give an idea of the computation time, approximately 40 CPU hours are needed for the ground state energy of $\nu = 3/4$ at $N = 12$. In Eq. (19), the positions of the first two particles are identical. To avoid numerical division by zero, we set $\delta \Omega_1 = \delta \Omega_2 + \delta \Omega$. The results are independent of $\delta \Omega$ provided it is sufficiently small: we typically use $|\delta \Omega| = 10^{-6}$.

The Moore-Read wave function is known to be the exact ground state for a three body interaction.

$$H_{\text{pfaff}} = \sum_{i<j<k} \delta^{(2)}(r_i - r_j) \delta^{(2)}(r_i - r_k).$$  

It can therefore be obtained by exact diagonalization using Lánčzos algorithm in the spherical geometry. That provides a direct method to evaluate the scalar product involved in the overlap calculation.

V. QUANTITATIVE COMPARISONS

The results of our study are summarized in Table I and Figs. 1-3. The Table I gives the overlaps of exact eigenstates $\Psi^{ex}$ with the wave functions $\Psi^B$ for the ground state and first excited state. We make the following remarks.
The quantum numbers of the low-energy excitations at \( \nu = 1/2 \) and \( 2/3 \) can also be understood by analogy to \( \nu^* = 1 \) and \( 2 \), although the correspondence for excitations is poor for \( \nu = 3/4 \). Thus, an inspection of the low-energy spectrum of interacting bosons at \( \nu \) clearly shows similarity with fermions at \( \nu^* \).

(ii) At \( \nu = 1/2 \), the wave function \( \Phi^*_2 \) is known to be exact. Our calculations of the overlap and energy for this filling constitute a non-trivial test of the correctness of our computer codes.

(iii) The low-energy excited states at \( \nu = 1/2 \) are extremely well described, quantitatively, as excitons of composite fermions.\(^{41,42}\) The composite-fermion theory predicts that there is a single state at orbital angular momenta from \( L = 2 \) to \( L = N \), which is clearly the case in exact diagonalization studies. (At \( \nu^* = 1 \) there is also an exciton at \( L = 1 \), but its wave function is annihilated upon projection into the lowest Landau level\(^{41} \).) Further, the energy of the CF exciton is in excellent agreement with the exact energy. Previous studies\(^{30,39} \) studied the excited states at \( \nu = 1/2 \) by exact diagonalization, but did not provide a microscopic understanding.

(iv) For \( \nu = 2/3 \), the CF theory provides an excellent approximation for the ground state, with very high overlaps and very good energies for 10 and 12 particles. The CF theory again correctly predicts the quantum numbers of the low energy excitations, and also the qualitative shape of the exciton dispersion curve, but the energies are now off by up to \( \sim 50\% \). At \( \nu = 3/4 \), the situation becomes worse. In accordance with the prediction of the CF theory, the ground state has \( L = 0 \), but no well defined branch of excitations may be identified with the CF exciton branch; furthermore, the energies predicted by the CF theory are quite inaccurate, for both the ground and excited states. These studies show that the CF description worsens with increasing \( n \) along the sequence \( \nu = n/(n + 1) \).

(v) At \( \nu = 1 \), a good account of the ground state is obtained through analogy to a paired fermion state, as can be seen from the overlaps given in the last column of the Table I. This result is consistent with the earlier studies in the toroidal and disk geometries.\(^{13,15,16} \)

One may ask to what extent the difference between bosons at \( \nu = n/(n + 1) \) and electrons at \( \nu_e = n/(2n + 1) \) has to do with the fact that the bosons are interacting via a short-range, contact interaction, as opposed to the long-range Coulomb interaction for the electrons. To investigate this issue, we obtain exact wave functions for a system of charged bosons interacting via the Coulomb interaction. Table II presents their overlaps with various wave functions. The CF theory is in better agreement with the Coulomb states at \( \nu = n/(n + 1) \), but the overall behavior is qualitatively similar. For example, the overlaps for \( \nu = 3/4 \) are not high, and much smaller than those at \( \nu_e = 3/7 \) for the electron FQHE. The paired wave function is also a better approximation for the Coulomb ground state at \( \nu = 1 \) than it is for the hard-core interaction; in contrast, it is not valid for

(i) For \( \nu = 1/2 \), \( 2/3 \), and \( 3/4 \), the structure of the low-energy states is in clear correspondence with that of fermions at \( \nu^* = 1 \), \( 2 \), and \( 3 \). In all these cases, the ground state is a uniform state (\( L = 0 \)), well separated from the other states by a gap, as shown earlier.\(^{20} \)
TABLE I: The overlap of the exact wave functions for the ground state and the first excited state at \( \nu = 1/2, 2/3, 3/4 \), and 1 with the trial wave functions of Laughlin (\( \nu = 1/2 \) ground state), Moore and Read (\( \nu = 1 \) ground state), and Jain (other states), for several particle numbers \( N \). \( O^2_{\text{gr}} \) is the overlap for the ground state, and \( O^2_{\text{ex}} \) for the first excited state, which occurs at the orbital angular momentum \( L \). The definition of the overlap is given in Eq. (16). The statistical uncertainty in the last two digits is shown in parentheses when it is larger than \( 10^{-5} \). For \( \nu = 1 \), only the ground state overlap is shown, which has been evaluated exactly.

| \( \nu \) | \( N \) | \( O^2_{\text{gr}} \) | \( \nu \) | \( N \) | \( O^2_{\text{gr}} \) | \( \nu \) | \( N \) | \( O^2_{\text{gr}} \) | \( \nu \) | \( N \) | \( O^2_{\text{gr}} \) |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 1/2 | 4 | 0.9999 | 2/3 | 4 | 1.0000 | 3/4 | 9 | 0.8163(76) | 1 | 4 | 1.0000 |
| 5 | 1.0000 | 4 | 0.9965 | 6 | 0.9850 | 4 | 0.7544(05) | 12 | 0.735(84) | 6 | 0.480(62) | 6 | 0.97279 |
| 6 | 1.0000 | 5 | 0.9959 | 8 | 0.9820(10) | 5 | 0.8701(14) | 8 | 0.96878 |
| 7 | 1.0000 | 5 | 0.9954 | 10 | 0.9724(89) | 6 | 0.855(12) | 10 | 0.95922 |
| 8 | 1.0000 | 6 | 0.9945 | 12 | 0.88435 |
| 9 | 1.0000 | 6 | 0.9954 (2) | 14 | 0.88580 |

TABLE II: The overlap of the Coulomb ground state wave function at \( \nu = 1/2, 2/3, 3/4 \), and 1 with the trial wave functions of Laughlin (\( \nu = 1/2 \) ground state), Moore and Read (\( \nu = 1 \) ground state), and Jain (\( \nu = 2/3 \) and 3/4 ground states), for several particle numbers \( N \). (Table I dealt with the exact wave functions for a short range interaction.) The statistical uncertainty in the last two digits is shown when it is larger than \( 10^{-5} \). The overlaps for \( \nu = 1 \) (last column) are exact.

| \( \nu \) | \( N \) | \( O^2_{\text{gr}} \) | \( \nu \) | \( N \) | \( O^2_{\text{gr}} \) | \( \nu \) | \( N \) | \( O^2_{\text{gr}} \) | \( \nu \) | \( N \) | \( O^2_{\text{gr}} \) |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 1/2 | 4 | 0.9999 | 2/3 | 4 | 1.0000 | 3/4 | 9 | 0.8163(76) | 1 | 4 | 1.0000 |
| 5 | 1.0000 | 4 | 0.9965 | 6 | 0.9850 | 4 | 0.7544(05) | 12 | 0.735(84) | 6 | 0.480(62) | 6 | 0.97279 |
| 6 | 1.0000 | 5 | 0.9959 | 8 | 0.9820(10) | 5 | 0.8701(14) | 8 | 0.96878 |
| 7 | 1.0000 | 5 | 0.9954 | 10 | 0.9724(89) | 6 | 0.855(12) | 10 | 0.95922 |
| 8 | 1.0000 | 6 | 0.9945 | 12 | 0.91645 |
| 9 | 1.0000 | 6 | 0.9954 (2) | 14 | 0.92133 |

The above results allow us to make the following conclusions: (i) The mapping into free fermions is qualitatively valid for a range of parameters. It correctly captures the incompressibility of the ground state at \( \nu = 1/2, 2/3 \) and 3/4. (ii) The mapping is also quantitatively very accurate for the ground state and excitations at \( \nu = 1/2 \) and for the ground state at \( \nu = 2/3 \), but becomes progressively worse with increasing \( n \). This implies that the residual interactions between fermions become increasingly more important with \( n \), and must be considered for a more complete understanding. (iii) A qualitative indication of the breakdown of the free-fermion model is the appearance of a paired state at \( \nu = 1/2 \); no interacting fermions would have produced a Fermi sea here. The residual interactions between fermions thus cause a qualitative change in the nature of the state beyond certain \( n \), presumably through a pairing instability; we cannot, however, ascertain from our study for what \( n \) a phase transition occurs.

Overall, these results establish that the mapping into strictly free fermions is valid only for a limited range of parameters, but the mapping into weakly interacting fermions provides a useful starting point for a wider range of parameters. One might ask why the interacting fermion language is to be preferred over the original interacting boson model. The reason is that the interaction between the fermions is much weaker, with a large part of the repulsive interaction taken care of by the Pauli avoidance.

The bosonic FQHE should be contrasted with the FQHE of electrons along the sequence \( \nu_e = n/(2n+1) \), for which the mapping into free composite fermions remains valid for the entire parameter range. The wave functions of Eq. (13) are known to provide an excellent description of the state for 1/3, 2/5, and 3/7, where exact results are available, and presumably also for higher \( n \), as evidenced by the experimental observation of many fractions along the sequence as well as of the Fermi sea of composite fermions at \( \nu = 1/2 \).
emergent particles may be attractive. That appears to be the case at $v = 1$. Here, bosons dress themselves with vortices to turn into fermions, but that presumably overcompensates for the repulsive interaction, thereby producing an attractive interaction between the fermions.

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