The Circulation of Money and Holding Time Distribution

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Abstract

We have studied the statistical mechanics of money circulation in a closed economic system. An explicit statistical formulation of the circulation velocity of money is presented for the first time by introducing the concept of holding time of money. The result indicates that the velocity is governed by behavior patterns of economic agents. Computer simulations have been carried out in order to demonstrate the shape of the holding time distribution. We find that, money circulation is a Poisson process in which the holding time probability distribution follows a type of Gamma distribution, and the velocity of money depends on the share for exchange and the number of agents.

Key words: money circulation, holding time, statistical distribution, Poisson process, velocity of circulation

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1 Introduction

Since there are so many analogies between some features of economy and objects of physics, some advanced physical methods could be applied into analyzing economic issues [1]. In particular, the application of statistical physics methods to financial market has achieved a great success [2]. Recently, many efforts are being made to expand the scope of such analogical applications to other economic problems [3,4]. Among these problems, the dynamics of money has attracted much attention because of its significance in economy [5,6,7]. By

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identifying money in a closed economy with energy and the average amount of money with temperature, Adrian Drăgulescu and Victor Yakovenko have demonstrated that the equilibrium probability distribution of money follows the exponential Boltzmann-Gibbs law due to the conservation of money [6]. The distribution of money in a real economy mainly depends on economic agents’ behaviors and the manner they interact between each other. According to Adrian Drăgulescu and Victor Yakovenko, if a model has time-reversal symmetry, then a stationary Boltzmann-Gibbs distribution can be obtained; otherwise, the system may have a non-Boltzmann-Gibbs distribution or no stationary distribution at all. Following this work, an example of stationary non-Boltzmann-Gibbs distribution is given by Anirban Chakraborti and Bikas K. Chakrabarti when the impact of the saving propensity of the agents on the statistical mechanics of money is taken into account [7].

Indeed, money does matter in the performance of economy. The statistical analysis of money is therefore essential and very intriguing. However, comparing with its distribution among economic agents, understanding of the motion of money in the economy is more significant. As medium of exchange and unit of accounting, money cannot be used up as consumption goods or worn off as capital goods, but can only be transferred between agents. So it can be assumed to be conserved after being injected into an economy. The picture that money moves from hand to hand between agents in sequence draws some physicists to use one-dimensional lattice or square lattice network models to deal with monetary issues [4,5]. Although this analogy might appear attractive intuitively, it is not a short-cut for addressing these issues. In fact, economists have been thinking about the money in an alternative way by placing themselves in an economic space instead of usual physical one. The economic space comprises only two parts: demand and supply. Correspondingly, the economic agents can be classified to two groups: buyers and sellers. The former who holds money is in the demand side and the latter who produces goods is in the supply side. In economic space, the motion of money then can be figured out as moving between these two regions. When exchange occurs in the market, money flows from demand region to supply region can be seen. After that, money in the supply region is taken back to the demand region immediately, because the seller who just participates in the exchange becomes a potential buyer. It’s so-called circulation of money that the process takes place continuously.

The most representative economic theory that copes with money in such a way is the quantity theory of money [8], which describes the relationship between money flow and product flow at macro level. As the total amount of money is conserved, the money flow originates from monetary circulation. Meanwhile money is circulating; goods are continuously produced, exchanged and consumed, in which the product flow arises. Thus it can be seen that both flows coexist all the time and the ratio of them is called price level. Given the price
level fixed, the money flow corresponds to the product flow. So the circulation of money can reflect the operation of production economy. Furthermore, in a market economy effective demand is always deficient and the real output is dominated by willing expending. In this case, the circulation of money is not only the reflection but also the driving force of production. This is the reason that the quantity theory of money has been of high reputation in economics since being set forth.

In the quantity theory of money, in order to express the relationship between money and production, a significant variable, the velocity of money circulation, has been introduced. Although its definition is explicit, the essence of this concept remains ambiguous. The velocity of money circulation is defined as the ratio of money flow to the money stock. It can be calculated with nominal gross domestic product (GDP) and total amount of money. There is however no consensus so far on what factors govern this variable. Our goal is to describe the velocity of money by utilizing the ideas and concepts from statistical physics and to show the statistical characteristics of money circulation by computer simulations.

The statistical description of the velocity of money circulation is based on holding time of money which is defined as time interval between two transactions. Although this concept is kept in mind when economists think of the velocity, even the term referring to this kind of time interval has been mentioned in several cases, it’s somewhat new to them since there has been no explicit specification of it in economics. While there has been a similar term in physics which is called waiting-time. In recent times, several efforts have been devoted to measure the waiting time distributions in financial markets, see e.g. [9,10]. In the process of money circulation, not only the amount of money each agent holds can be considered as random variables, but also the holding time between two transactions varies randomly. Thus there exists a holding time distribution of money in a closed system and the connection between holding time and the velocity is established. The holding time is not a physical time but an economic variable which is determined by economic agents’ behavior. Consequently, with the introduction of holding time the velocity can be expressed in a statistical way, and furthermore a bridge that links such a macroeconomic variable to individual agents’ choices might be set up.

We find that the velocity of money is the expectation value of the reciprocal of holding time. Thus the characteristics of the holding time distribution do matter. Due to absence of the data of money circulation, empirical study is not feasible. Therefore, we performed computer simulations to see the shape of the holding time probability distribution and demonstrated how the individual choices and other factors affect the velocity.

Our model is a simple extension of Adrian Drăgulescu and Victor Yakovenko's
work, in which random exchange approach remains. However, we placed emphasis on measuring holding time distribution instead of money distribution. The dependences of the velocity of money circulation on some governing factors were discussed theoretically and experimentally.

2 The Quantity Theory of Money

When physicists intend to study economic issues, it is helpful to see how economists deal with them. Having knowledge about relevant basic thoughts and their context that economists have already developed enables us to apply physics to those issues more effectively. The circulation of money has been talked about for hundreds of years; however, it is still a topic of great interest with some puzzles uncovered. Many publications about it can be found in economics literature. In this section we shall present a brief review of the basic knowledge and the main results of the theory.

The quantity theory of money is a well-known doctrine to economists which emphasizes that the money supply is the main determinant of nominal GDP. This theory is constituted of two branches: one is built on the Fisher exchange equation; the other is on the Cambridge cash balance equation. The Fisher equation places emphasis on the part as medium of transactions that money plays and states that [11]

\[ MV = PY, \]  

(1)

where \( M \) is the amount of money in circulation, \( V \) the velocity of circulation of money, \( P \) is the average price level and \( Y \) is the level of real income. The left hand side of the equation represents the amount spent on final goods and services in transactions while the right hand side represents the amount received for these goods and services. In other words, the left hand side is the total spending from a monetary perspective while the right hand side is that from a real view. By definition, these two sides must be equal. Thus this equation shows the relationship between money flow and product flow. The Cambridge equation proposes that money is desired as a store of value and regarded money demand as a function of nominal income, i.e. [12]

\[ M = kPY, \]  

(2)

where \( k \) is the famous “Cambridge constant”.
From Eq.(1) we can immediately obtain
\[
\frac{\Delta M}{M} + \frac{\Delta V}{V} = \frac{\Delta P}{P} + \frac{\Delta Y}{Y}.
\]

But it is only a matter of arithmetic, not of economics. Till now we have not made any assumption of the causality between left and right.

According to Fisher, the statement of the quantity theory requires three pillars: firstly, that \(V\) and \(Y\) are fixed with respect to the money supply. Secondly, that the supply of money is exogenous. Thirdly, the direction of causation runs from left \((MV)\) to right \((PY)\). The story of the quantity theory then runs like this: since \(V\) and \(Y\) are fixed and \(M\) is exogenous, then an increase in the supply of money will lead to an exactly proportionate increase in the price level. Thus, money supply expansions only cause price inflation. This is the so-called monetary neutrality. For reaching this, the last pillar is the most crucial one among the three mentioned above.

Comparing equations (1) and (2), if we simply set \(k = 1/V\), there seems be no difference between these two. In fact, the Cambridge story, however, is fundamentally different from the Fisher story. The proposition that the Cambridge advances comes from an implied hypothesis that the direction of causation runs from right \((kPY)\) to left \((M)\). The reverse of causation leads to a possible unstable “Cambridge constant”, that is to say, change in money supply may have real effects.

Following this line of the Cambridge approach, monetarists present a restate-ment of the quantity theory of money [13]. But their fundamental contributions to the development of the theory are the empirical researches on this aspect [14]. Basing on this work, they believe that the velocity of money is stable.

Nowadays, as to the causality of money supply and the real output, most economists believe that the monetary neutrality holds in the long run (a few years). As Milton Friedman once made the statement, “Inflation is always and everywhere a monetary phenomenon.” In the short run, however, they are not in agreement about whether changes in the money supply lead to changes in the price level or in the real output.

What on earth is the causality of money supply and the real output? And why the monetary neutrality is correct in the long run but not in the short run? The finish of solving these questions lies on the thorough understanding to the nature of the circulation of money. In fact, with the development of the quantity theory of money, people understand the velocity step by step; in other words, the process of developing the quantity theory of money is actually that of comprehending the velocity. The earliest version of the quantity theory
of money takes the velocity as an exogenous variable and being determined by institutional arrangements and technologies of transaction. As a result, it can be presumed as a constant. Cambridge economists think that money is a kind of asset and the holders of them tend to optimize their portfolio. So the factors related to money demand are considered when investigating the velocity. Basing on that, Friedman dedicated precisely to unearthing the relationship between the velocity and those factors such as interest rate, expected inflation, permanent income, and the return on money, etc. However, all of analysis mentioned above is only qualitative, and no concrete function for expressing that relationship is offered. In the following section, we shall present a statistical expression of the velocity of money.

3 Statistical Analysis of The Circulation Velocity of Money

3.1 Holding time of money

The advantage of money as the medium of exchange, in that it overcomes the need to obtain coincidence of wants; it implies that an agent can sell his good at one time for “money” and then trading his “money” for the goods he finally wishes to purchase at another time. The divorce of sale and purchase results in a time interval within which money stays in the pocket of agents. Moreover, money is an asset which allows value to be stored easily, and there is also a time interval for agents to preserve it. In the real economy, money is actually changing hands all the time. In this process, if an agent receives money from others at one moment, he will hold it for a period, and eventually pays something to another agent. Now we introduce a new concept named as holding time which is defined the interval between the receipt of income and its disbursement.

Holding time is not an intrinsic character of money itself, but is a character of its holder’s behavior in utilizing the money for certain purpose. The reasons about why people hold money arise from the composite result of a number of different motives. Economists provide several variations of explanations about it. According to traditional approach, the motives are classified under three headings: the transactions-motive, the precautionary-motive and the speculative-motive. Keynes classified them under four headings by further dividing the transactions-motive into the income-motive and the business-motive [15]. Despite what motives are taken into account and how they are classified, the existence of holding time could not be denied. Their differences are only embodied in determinants of the holding time.

In the circulation story, money circulates round and round in economic space.
At the beginning of one cycle, money stays in the demand region till it participates in the transaction. When the exchange takes place, it moves to the supply side and immediately goes back to the initial position and gets to the end of this cycle or the beginning of the next. Since it is reasonable to assume that the transaction does not take any time, the holding time is equal to the period of one cycle that money goes.

3.2 Distribution of money over holding time

Let’s consider the economy in which the total quantity of money is $M$. At any given time, each unit of them must have ever participated in the exchange at certain previous moment and will repeat at a certain moment in the future. Thus it has its own specific holding time at present. As a result, the money may have different holding time at the same time, due to either different holders or various motives of the same holder. Therefore, the conserved money spreads over the holding times.

Now we introduce the probability distribution function of money $P(\tau)$, which is defined the amount of money whose holding time is between $\tau$ and $\tau + d\tau$ is equal to $MP(\tau)\,d\tau$. For any unit of money, $P(\tau)$ is actually the probability of taking part in exchange after an interval of $\tau$. Thus we can give the normalization condition as follows:

$$\int_{0}^{\infty} P(\tau) \, d\tau = 1. \quad (4)$$

And we can also have the expectation of the holding time of money, $\bar{\tau}$.

$$\bar{\tau} = \int_{0}^{\infty} P(\tau) \, \tau \, d\tau. \quad (5)$$

We call it the average holding time of money.

Each unit of money injected into the economy changes hands for countless times. The time interval between any two times is various because the holders may be different. That is to say, the holding time of the same unit may be altered at different cycle due to the change of holder. It follows that the distribution of money over the holding time evolves with time. If the behavior pattern of the economic agents keeps unchanged, we can get an equilibrium state where the distribution of money is stationary. In other words, any single money’s holding time may strongly fluctuate, but the overall probability distribution does not change.

In the stationary state, the fraction of money $MP(\tau)\,d\tau$ participates in the
exchange after a period of $\tau$. The money flow generated by this fraction then is

$$F(\tau) = MP(\tau) \frac{1}{\tau} d\tau. \quad (6)$$

The total money flow $F$ in the economy is equal to the sum of the money flows due to the contribution of all parts whose holding time is equal to $\tau$ ($\tau \in [0, \infty)$), then we have

$$F = M \int_0^\infty P(\tau) \frac{1}{\tau} d\tau. \quad (7)$$

The velocity indicates the speed at which money circulates. From its definition $V = F/M$, we get

$$V = \int_0^\infty P(\tau) \frac{1}{\tau} d\tau. \quad (8)$$

This is the statistical expression of the circulation velocity of money with respect to holding time. The result shows the velocity $V$ is the mathematical expectation of $1/\tau$. From Eqs.(5) and (8), it is obvious that the velocity and the average holding time are inversely related. It also indicates that all the impacts of factors related on the velocity take effect through influencing the economic agents’ choices of holding time. Thus, an effective measure for increasing (decreasing) the velocity of money is to motivate the economic agents to shorten (lengthen) the average holding time.

The result that $P(m)$ follows Boltzmann-Gibbs’ distribution arise from non-negativity of the money amount possessed by each unit of agent and conservation of the total money. But till now we can not say anything about what the distribution $P(\tau)$ of money is likely to be. In order to see the characteristics of $P(\tau)$, we performed several computer simulations which will be described in next section.

### 4 Model and Computer Simulations

The model we used is very similar to that of Adrian Drăgulescu and Victor Yakovenko. The economic system is closed where the total money $M$ and the number of economic agents $N$ are fixed. Each of agents has a certain amount of money initially. The money exchange is performed by agents in sequence. In every round an arbitrary pair of agents is chosen to exchange, and the amount
of money that changes hands is given according to the trading rule. The non-

negativity of any agent’s money and the conservation of the total money in
each round are ensured.

We carried out the measurement by employing the trading rule which has
been used by Adrian Drăgulescu and Victor Yakovenko in their simulations.
Initially, the total money \( M \) is divided amongst \( N \) agents equally so that each
agent possesses the same amount of money \( M/N \). We choose a pair of agents
randomly at a time; one of them is randomly picked to be the “receiver”,
then the other one becomes the “payer”, and the amount \( \Delta m \) is transferred
from the payer to the receiver. The trading rule lets the exchange amount in
each round be of the following form: \( \Delta m = \frac{1}{2} \nu (m_1 + m_2) \), where \( \nu (0 < \nu < 1) \)
is a random fraction, \( m_1 \) and \( m_2 \) are the amount of money possessed by the
payer and the receiver, respectively. If the payer can’t offer the amount, the
transfer does not take place, and we turn to another pair of agents.

Following this procedure, the final stationary distribution of money among
agents \( P(m) \) is Boltzmann-Gibbs’ distribution, which is shown in Fig. 1.
Whether the system is at stationary state or not is judged by observing the
evolution of the entropy \( S = - \int_0^\infty dm P(m) \ln P(m) \). This is illustrated in the
inset of Fig. 1. All the records are obtained after \( S \) reaches the maximal value
denoted as the vertical line in the inset of Fig. 1, and average over 400 such
stationary distributions was taken to obtain a smooth distribution.

The main purpose of our simulation is to show the holding time distribution
of money \( P(\tau) \) instead of \( P(m) \). In order to obtain the holding times of all
money, we numbered all of the money and tracked each of them. The moment
at which each unit of money participated exchange for the latest time was
memorized. After majority (99%) of money had ever taken part in trade, we
began to measure holding times by recording the corresponding moment at
which each unit of money participated exchange for the first time hereafter.
The holding time of each unit is the difference between the two moments. Due
to the random exchange, there are always a few units whose holding time is
too long to be recorded. Thus we just sampled the majority of total money,
and the remainder was omitted. After that we got the distribution of money
over holding time.

5 Results and Discussion

We performed several simulations of \( P(\tau) \) by altering values of \( N \) and \( M \).
The results show that the holding time distributions of money we observed
are stationary and have remarkably similar form independent of the values
given. The typical one is illustrated in Fig. 2 where the size was taken to be
\( N = 2500 \) and \( M = 250000 \). However, the profile of this distribution cannot provide any clue to make sure which kind of distribution it belongs to.

To examine the origination of this distribution, we turned to observe another kind of temporal distribution of money. Taking the time when we began to record as zero point, hereafter, we set the moment at which the unit of money is transferred for the first time to be latency time of this unit. Please notice that the latency time we defined here is different from the holding time of money. Fig. 3 shows the latency time distribution of money for this case. It can be seen from the inset of Fig. 3 that the distribution follows exponential law: \( P(t) = \frac{1}{T} e^{-\frac{t}{T}} \). The result we obtained here indicates that the transferring process of money is a Poisson process with the rate of changes \( 1/T \). Thus the average latency time is equal to \( T \) and its value obtained in Fig. 3 is about 5550.

Since the whole process is an independent stationary stochastic one, when we begin to observe one unit of money, if we look forward from the zero point of time, the probability of that it takes part in exchange during the period between \( t \) and \( t + dt \) for the first time is \( P(t) dt \); if we look backward, the probability of that the unit of money has ever been traded during the period between \( -t \) and \( -t + dt \) and has been held till \( t = 0 \) is also \( P(t) dt \). Suppose the unit of money takes part in exchange at \( t_1 (t_1 > 0) \) and its holding time is \( \tau (\tau \geq t_1) \), then it must be transferred at moment \( t_1 - \tau \). As a result, the probability of this case is \( P(t_1) P(\tau - t_1) dt_1 \), and the probability of that the holding time is equal to \( \tau \) can be obtained

\[
P(\tau) = \int_0^\tau P(t_1) P(\tau - t_1) dt_1 = \frac{1}{T^2} \tau e^{-\frac{\tau}{T}}.
\]  

(9)

This result shows the probability distribution of money over holding time follows a type of Gamma distribution.

Substituting Eq. (9) into (8), we get

\[
V = \frac{1}{T}.
\]

It’s not surprising to this result for the velocity of money is just the rate parameter of the circulation process. We further more substitute Eq. (9) into (5), then

\[
\bar{\tau} = 2T.
\]

Thus the relationship between the velocity and the average holding time can

\[
V = \frac{1}{\bar{\tau}}.
\]
be simply expressed as follows:

\[ \tau = \frac{2}{V}. \]

We proceeded a fitting of the simulation result in Fig. 2 with the theoretical expression of Eq. (9) and found a good agreement. The fitting value of the average latency time is about 5548, which is almost the same as the result derived from Fig. 3. The deviation may be due to nonuniform distribution of random number generated in the process of simulation.

To consider how individual agent’s choice affects the velocity of money, we furthermore added a multiplier \( k \) and denominated it share for exchange, the exchange amount then becomes \( \Delta m = \frac{k}{2} \nu (m_1 + m_2) \). As a further check we have determined the dependencies of the velocity of money on the share for exchange \( k \), the number of agents \( N \) and the average amount of money in the system \( \bar{m} \), which are shown in Figs. 4(a)-(c) respectively. For each case, we performed the simulation and got a holding time distribution of money \( P(\tau) \), after that the corresponding velocity was simply deduced by using Eq.(8).

In Fig. 4(a) we set \( N = 2500 \) and \( \bar{m} = 100 \). It can be seen that the velocity is proportional to the share \( k \). For larger share for exchange the velocity of money in the trade process increases. So we can conclude that the circulation velocity is determined by the agents’ behaviors in the exchange since the share for exchange \( k \) reflects economic agent’s choice. Fig. 4(b) shows the velocity of circulation \( V \) plotted versus \( 1/N \), for \( k = 1 \) and \( \bar{m} = 100 \). As shown, \( V \) vs \( 1/N \) is linear for the whole range of \( N \). Fig. 4(c) shows the variation of the velocity of circulation with average amount of money \( \bar{m} \) for \( k = 1 \) and \( N = 2500 \), from which we can see that \( V \) remains constant for different values of \( \bar{m} \).

These results can be understood by a simple deduction. According to the trading rule: \( \Delta m = \frac{k}{2} \nu (m_1 + m_2) \) and non-negativity of \( m \), the average money flow, \( F \), can be derived as follows:

\[ F \propto k \bar{m}. \]

From its definition it can also be written as

\[ F = MV = N \bar{m} V. \]

Comparing these two equations above, we immediately get

\[ V \propto \frac{k}{N}. \]
Thus the velocity of money is proportional to the share for exchange, re-
versely proportional to number of agents, and independent of average amount
of money. This theoretical result is in good agreement with that of simulations.

6  Summary

In this paper, we have considered money circulation in a closed economy by
applying the statistical approach. We present the quantity theory of money
briefly to argue that the circulation velocity of money is an essential variable
for our understanding of the dynamics of money. By introducing the concept
of holding time of money, we provide a statistical expression of the circulation
velocity. The result indicates that the velocity and the average holding time are
reversely related and the main determinant of them is the agents’ behaviors.

We have performed several computer simulations based on random exchange
model. We find the money involved in exchange process possesses not only
a stationary probability distribution among agents but also a stationary one
over holding time. The holding time probability distribution is found to be
a type of Gamma distribution because the circulation of money is a kind
of Poisson process. The dependence of the circulation velocity of money on
agents’ choices are demonstrated by changing the share for exchange. We also
investigated the dependence of the distribution on the number of agents and
the average amount of money per agent. The theoretical results we derived
according to the model show good agreements with the simulations. We believe
that this study promises a fresh insight into the velocity of money circulation
and opens a way to a firm microfoundation of it.

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Figure Captions

Figure 1 The stationary probability distribution of money among agents \( P(m) \) versus money \( m \). Solid curve: fit to the Boltzmann-Gibbs law \( P(m) \propto \exp(-m/\bar{m}) \). The vertical line indicates the initial distribution of money. The inset shows time evolution of entropy \( S \) and the vertical line denotes the moment after which the measurements are performed.

Figure 2 The stationary probability distribution of money versus holding time \( P(\tau) \). The solid curve is fit to the equation \( P(\tau) = \frac{1}{T} \tau \exp(-\tau/T) \) with \( T = 5548 \).

Figure 3 The stationary distributions versus latency time \( t \). The fitting in the inset indicates that the distribution follows the exponential law: \( P(t) = \frac{1}{T} \exp(-t/T) \).

Figure 4 Dependencies of the velocity of money circulation \( V \) on (a) the share for exchange \( k \); (b) the reciprocal of number of agents \( 1/N \); (c) the average amount of money in the system \( \bar{m} \).
Wang and Ding, Fig. 1
Wang and Ding, Fig. 2
Wang and Ding, Fig. 3
Wang and Ding, Fig. 4(a)
The reciprocal of number of agents, $1/N \times 10^4$

Wang and Ding, Fig. 4(b)
Wang and Ding, Fig. 4(c)