Distribution of asset price movement and market potential

Dong Han Kim\(^1\) and Stefano Marmi\(^2,3\)

\(^1\) Department of Mathematics Education, Dongguk University-Seoul, 100-715 Korea
\(^2\) Scuola Normale Superiore, Piazza dei Cavalieri 7, 56126 Pisa, Italy
\(^3\) C.N.R.S. UMI 3483, Laboratorio Fibonacci, Piazza dei Cavalieri 7, 56126 Pisa, Italy
E-mail: kim2010@dongguk.edu and s.marmi@sns.it

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Abstract. In this article we discuss the distribution of asset price movements by introducing a market potential function. From the principle of free energy minimization we analyze two different kinds of market potentials. We obtain a U-shaped potential when market reversion (i.e. contrarian investors) is dominant. On the other hand, if there are more trend followers, flat and logarithmic potentials appear. By using the cyclically adjusted price-to-earning ratio, which is a common valuation tool, we empirically investigate the market data. By studying long term data we observe the historical change of the market potential of the US stock market. Recent US data show that the market potential looks more like a trend-following potential. Next, we compare the market potentials for 12 different countries. Though some countries have similar market potentials, there are specific examples like Japan which exhibits a very flat potential.

Keywords: models of financial markets
1. Introduction

In this paper we investigate the long-term behaviour of asset prices, inspired by an analogy with statistical mechanics. Stock market fluctuations exhibit several statistical peculiarities which are still awaiting for a satisfactory interpretation. A widely used approximation states that the movement of an asset price follows a random walk [1]. Real markets seem however to considerably deviate from this ideal behavior, exhibiting both significant positive serial correlation [2] for weekly and monthly holding-period returns and long-term negative serial correlation (mean-reversion) [3].

In addition the price of an asset moves much faster than its real value changes [4, 5]. According to the efficient markets hypothesis (EMH) the main engine of asset prices movement should be the arrival of some new piece of information which leads to a revision of the expectations of market participants. For most assets the intrinsic value will change under the influence of slowly varying macroeconomic conditions and/or under the effect of unanticipated new events (discoveries, technological innovations, acquisitions, etc). Noise trading should add high frequency mean-reverting noise between news, that should not contribute to the long term volatility of the price⁴.

⁴ It has even been suggested that the variation $\Delta P$ of an asset price may tentatively be decomposed into three terms: $\Delta P = \Delta M + \Delta I + \Delta N$ where $\Delta M$ is a slowly varying ‘trending’ term (e.g. fractional Brownian motion with Hurst exponent $> 1/2$) related to the macroeconomic environment, $\Delta I$ are jumps due to the arrival of new unexpected information and $\Delta N$ is is a rapidly varying ‘mean-reverting’ term (e.g. fractional Brownian motion with Hurst exponent $< 1/2$) due to ‘noise’ traders. See [6].
The price movement of an asset seems to be much more chaotic and rapid than any change of its intrinsic value. Although it is usually not possible to compute the present intrinsic value of an asset, it could be estimated by statistical means after a long period of time. The quoted price in the market moves very rapidly and it rarely equals to the intrinsic value (indeed there is a rather strong statistical evidence that the market overpays for superior growth expectations [7,8]). The relation between intrinsic value and price has been well summarized by Fisher Black in 1986 [9].

In this article we analyze this phenomenon by exploiting an analogy with some ideas of statistical mechanics. In this framework a system has a distribution which is not concentrated at the lowest energy position. Indeed the system should not only minimize its energy but also maximize its entropy. Thus it generates intrinsic ‘noise’. In a closed system with a constant temperature, the system will minimize its free energy. At high temperature, the contribution of the entropy to the free energy is larger than that of the internal energy. We try to understand the financial market via the market potential and the free energy minimizing distribution.

Coming back to market dynamics, if we assume that the intrinsic value of an asset changes slowly in time and that the price of the asset should converge to the intrinsic value, the market should exhibit some mean reversion (see [10]). This implies the existence of a mean reverting ‘force’ which we assume to be a function of the deviation from the intrinsic value. Then the force is determined by a potential function. When the price is at the intrinsic value, it is at the lowest potential in the phase space of the asset price. In section 2, the detailed model for the mean reverting potential is discussed.

Whereas many long-term investors try to exploit this mean reversion there is another major investment strategy which is especially popular when the asset prices are rising: trend following. This means being long when the market is bullish and short when it is bearish. A typical trading rule to achieve this is to buy if the spot price is higher than a moving average and sell if it is lower [11].

Others (contrarian investors) will try to exploit market reversion, so they sell the asset when its price goes up and vice versa. We will investigate the movement of the asset price in these two different environments.

In the last section of this paper we investigate two historical datasets on market valuations: the first and most important has has been collected and made available by the well-known economist Robert J Shiller [12], the second has been used in reference [13]. Shiller’s dataset is very widely considered by the economics and finance research communities (and also by many market practitioners) as the reference dataset on valuation measures for the US stock market index and has been compiled over a very long period of

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time starting from 1872. To our best knowledge this is the deepest research grade historical
dataset of stock market valuations which is freely available to the scientific community.

There are works on the market potential from stochastic dynamics models applied
to high frequency financial time series [14, 15]. However, in this article we obtain the
market potential not by making an assumption on the microscopic structure of the
market dynamics but from the entropy maximization principle of statistical mechanics.
This allows us to distinguish different market behaviours on a time scale of many years
without making an a-priori assumption on the underlying stochastic process which governs
the market behavior (e.g. trending, mean-reverting, range-bounded, etc). Thus we do not
need to calibrate a given stochastic process: we simply infer the shape of the market
potential from the empirical distribution of price deviations. We only assume ergodicity
of the price deviation movement so that we can recover the potential from the empirical
distribution (see equation (20)).

This paper is composed as follows: In the next section, we introduce the mathematical
formulations used in this paper. In section 3 we discuss various kinds of market potentials
deduced by free energy minimization. Finally, the empirical investigations of historical
datasets are given in the last section.

2. Maximum entropy principle

If \( x \) is a dimensionless variable (as it will always be the case for us, see equation (10)) and
\( \rho(x) \) is a probability density function, the Shannon entropy of \( \rho(x) \) is defined by

\[
S(\rho) = -\int \rho(x) \log \rho(x) dx. \tag{1}
\]

We set the Boltzmann constant equal to 1. Entropy measures the uncertainty or the
randomness of a random variable. The maximum entropy principle was first introduced
by Jaynes [16, 17] in order to provide a new formulation of statistical mechanics based
on information theory (see [18] for the detailed discussion of the general dimensional
variables). A well-known maximum entropy principle states that when the variance of \( \rho(x) \)
is fixed, the distribution which has maximum entropy among all distributions supported
in the whole real line is the normal distribution [19].

Let \( \varphi(x) \) be a given potential function. Then the average energy for the density \( \rho(x) \) is

\[
E = \int \varphi(x) \rho(x) dx. \tag{2}
\]

The density function which maximizes entropy \( S(\rho) \) under the condition of fixed average
energy \( E \) is given by [19, section 12.1]

\[
\rho_E(x) = \frac{e^{-\beta \varphi(x)}}{\int e^{-\beta \varphi(x)} dx} \tag{3}
\]
for some constant $\beta$ satisfying
\[
\int \rho_E(x) \varphi(x) dx = \int \frac{\varphi(x) e^{-\beta \varphi(x)}}{e^{-\beta \varphi(x)}} dx = E.
\] (4)

Refer to [19,20] for the references of the maximum entropy distributions.

Since $\log \rho_E(x) = -\beta \varphi(x) - \log \left( \int e^{-\beta \varphi(x)} dx \right)$ from equation (3), we have by equations (1) and (4)
\[
S(\rho_E) = -\int \rho_E \log \rho_E dx = \int \frac{\beta \varphi e^{-\beta \varphi}}{e^{-\beta \varphi}} dx + \log \left( \int e^{-\beta \varphi} dx \right) = \beta E + \log \left( \int e^{-\beta \varphi(x)} dx \right).
\] (5)

Therefore, we have the inverse temperature
\[
\frac{dS(\rho_E)}{dE} = \beta + E \frac{d\beta}{dE} + \frac{1}{\int e^{-\beta \varphi} dx} \frac{d}{d\beta} \left( \int e^{-\beta \varphi} dx \right) \frac{d\beta}{dE} = \beta + \frac{d\beta}{dE} \left( E - \frac{1}{\int e^{-\beta \varphi} dx} \int \varphi e^{-\beta \varphi} dx \right) = \beta
\] (7)

by equation (4). Hence, we have $\beta = 1/T$.

The density $\rho_E(x) = \frac{e^{-\beta \varphi(x)}}{\int e^{-\beta \varphi(x)} dx}$ maximizes the quantity
\[
S - \beta E = -\int \rho(x) \log \rho(x) dx - \beta \int \varphi(x) \rho(x) dx.
\] (9)

In other words, the density $\rho_E(x)$ minimizes the (Helmholtz) free energy $F = E - TS$ when $\beta$ is constant.

### 3. Market potentials

Let $P$ and $I$ be the price of an asset and its intrinsic value, respectively. The price $P$ changes at high frequency whereas the intrinsic value changes in time at low frequency (apart from the rare occurrence of shocks). Define
\[
x = \log \left( \frac{P}{I} \right)
\] (10)

so that $x = 0$ corresponds to identity between price and intrinsic value (the market is perfectly efficient and the asset price is exactly equal to its intrinsic value). We assume that there is a potential $\varphi(x)$ which describes the forces which push $x$ toward the lowest energy point at $x = 0$.

We classify the potential $\varphi(x)$ into two cases; one is the case that the force becomes stronger as the price deviation $|x|$ becomes larger and the other case is that the force is weaker as $|x|$ goes bigger. We consider a system whose force increases proportionally to $|x|$ and another system whose force decreases proportionally to $1/|x|$ for large $|x|$.
We also could use similar idea of entropy maximizing principle to obtain the distribution of the market momentum.\(^6\)

### 3.1. Market reverting potential

Many market practitioners use a mean reverting trading strategy [22]. If the price goes up, then more people sell the asset and vice versa. The reverting force becomes stronger as price moves farther from the intrinsic value—the lowest energy position. Therefore, we assume that the mean reverting force, which is \(-\frac{d\varphi(x)}{dx}\), is proportional to the deviation \(-x\). Therefore, by integrating \(\frac{d\varphi(x)}{dx} = Cx\), we obtain

\[
\varphi(x) = Cx^2
\]

for some constant \(C > 0\). Then we have

\[
\rho(x) = \frac{e^{-\beta x^2}}{\int e^{-\beta x^2}dx},
\]

which is the Gaussian distribution.

The market temperature \(T = \beta^{-1} = \frac{dE}{dS}\) corresponds to the variance of \(\rho\). In this case, the energy is

\[
E = \int x^2 \rho(x)dx
\]

and \(E\) corresponds the variance of the density \(\rho(x)\) as well.

### 3.2. Market trending potential

Other investors use a trend-following trading strategy [23,24]. If the price becomes higher, then more people come to buy the asset and vice versa. The reverting force becomes weaker as price goes up or down from the intrinsic value—the lowest energy position. Therefore, we assume that the mean reverting force, \(-\frac{d\varphi(x)}{dx}\), is inversely proportional to the deviation \(-x\). Thus, by integrating \(\frac{d\varphi(x)}{dx} = \frac{C}{x}\), for large \(|x|\) we have

\[
\varphi(x) = C \log |x|
\]

for a constant \(C > 0\). Diffusion in logarithmic potentials has been recently investigated by various authors, see [25–27].

The density function \(\rho\) is

\[
\rho(x) = \frac{e^{-\beta \log |x|}}{\int e^{-\beta \varphi(x)}dx} = \frac{|x|^{-\beta}}{\int e^{-\beta \varphi(x)}dx}.
\]

\(^6\) Usually, momentum (but we will prefer the terminology quantity of motion or impetus so as to avoid confusion with the momentum effect in finance [21]) is given by \(mv\) and the kinetic energy is given by \(K = \frac{1}{2}mv^2\).

The momentum is proportional to the velocity of the price movement if many investors sell when price goes up and vice versa. This situation is similar to mean reverting case treated above and leads to a normal distribution for the returns \(v\) with the variance \((i.e.\) the volatility of the returns\) which can be interpreted as the market temperature.

However, one can also consider the opposite scenario: here investors are prevalently trend-followers, so that they prefer to invest in momentum stocks, i.e. in stocks with a recent past of high returns. Then the momentum is inversely proportional to the velocity and the kinetic energy will be

\[
K = C \log |v|
\]

for large \(|v|\).

By the similar calculation, we have same distributions for the potential energy.
Figure 1. The historical US market data: inflation adjusted S&P500 stock price index from January 1871 to February 2011 (left) and cyclically adjusted price-earnings ratio from January 1881 to February 2011 (CAPE) (right). On the CAPE graph (right) the data is categorized into 4 time period, (i) 1881–1913, (ii) 1913–1945, (iii) 1946–1977, (iv) 1978–2010. The 4 horizontal lines in CAPE graph represent the average of CAPE value in each period of time.

Note that the density $\rho$ decreases in a power law as many researches on stylized facts of market movement suggest [28].

The inverse of the market temperature $\frac{dS}{dE} = \beta = \frac{1}{T}$ gives the exponent of the decay rate of $\rho(x)$ as a function of $x$. In this case the energy is obtained as

$$E = \frac{1}{\int e^{-\beta x} dx} \int |x|^{-\beta} \log |x| dx. \quad (16)$$

3.3. Constant force

If we assume that the mean reverting force is constant, Then we have the density function

$$\rho(x) = \frac{e^{-\beta |x|}}{\int e^{-\beta |x|} dx}. \quad (17)$$

Such a potential for short time scales was considered by some authors [29].

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Figure 2. Graphs of the market potential from historical US Market data. Each graph represents the estimated potential \((x_i, \varphi_i)\) from the relative frequencies of figure 1 in one of 4 time periods (i) 1881–1913, (ii) 1913–1945, (iii) 1946–1977, (iv) 1978–2010. The abscissa of the graphs indicate the deviation from the average of CAPE (the horizontal line in the CAPE graph of figure 1) and the ordinate is the negative logarithm of the relative frequency as in equation (20).

4. Empirical data

In this section, we visualize the market potential deduced from various historical market datasets. In order to do this we will use a common valuation tool to extract the potentials from the market data. In section 2, we showed how the market potential function \(\varphi(x)\) can be obtained from the distribution function \(\rho(x)\) by equation (3). The distribution function gives the relative frequency of the deviations of the price from the lowest energy position or the intrinsic value.

It is a highly challenging problem to determine the intrinsic value of financial assets. One of the major insights provided by the maximum entropy approach is the recommendation to focus on the behaviour of mean values of a rather small set of relevant quantities, instead of trying to follow the full details of the time evolution of a complex system. In a very similar spirit in financial economics the use of simple valuation tools has been proposed in order to have an approximate valuation measure for a market index. The cyclically adjusted price-to-earnings ratio (CAPE) is a valuation measure developed by Shiller [5]. It is the price-to-earning ratio computed by using the ten years average of inflation adjusted earnings, i.e.

\[
\text{CAPE} = \frac{\text{price}}{10 \text{ years average of inflation adjusted earning}}.
\]  

Let \(p_k\) be the monthly CAPE value. We estimate the distribution \(\rho(x)\) by the relative frequency of the deviation of \(p_k\) from the average \(\bar{p} = \frac{1}{M} \sum_{k=1}^{M} p_k\), where \(M\) is the number of monthly CAPE data \(p_k\). We divide the range of \(p_k - \bar{p}\) by equally distanced points \(a_0 < a_1 < \ldots < a_n\) and set \(x_i, \rho_i\) to be respectively the midpoint of the \(i\)-th interval \([a_{i-1}, a_i]\) and of the relative frequency of \(p_k - \bar{p}\) belonging of the \(i\)-th interval, i.e.

\[
x_i = \frac{a_{i-1} + a_i}{2}, \quad \rho_i = \frac{\#\{k : a_{i-1} \leq p_k - \bar{p} < a_i\}}{M},
\]
Then, as it is discussed in section 2, we estimate the market potential \( \varphi \) at \( x_i \) by
\[
\varphi_i = -\log \rho_i. \tag{20}
\]

4.1. Market potential from historical US data

We use the data set constructed by Shiller [12] to investigate the chronological changes of the market potential. This US data set consists of monthly stock price, dividends, and earnings data and the consumer price index (to correct for inflation), all starting January 1871\(^7\) (see figure 1). For our purposes we will not use the dividends data.

In figure 2, we present how the potential in the US stock market evolves. The market data is categorized into 4 time period, (i) 1881–1913, (ii) 1913–1945, (iii) 1946–1977, (iv) 1977 onwards.

\(^{7}\) According to Shiller: ‘Monthly dividend and earnings data are computed from the S and P four-quarter totals for the quarter since 1926, with linear interpolation to monthly figures. Dividend and earnings data before 1926 are from [30], interpolated from annual data. Stock price data are monthly averages of daily closing prices through January 2000, the last month available as this book goes to press. The CPI-U (Consumer Price Index-All Urban Consumers) published by the US Bureau of Labor Statistics begins in 1913; for years before 1913 I spliced to the CPI Warren and Pearson’s price index, by multiplying it by the ratio of the indexes in January 1913. December 1999 and January 2000 values for the CPI-U are extrapolated. See [31]. Data are from their table 1, pp 11–14.’

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Figure 4. Graphs of the market potential from various countries—Canada, Denmark, Japan, Sweden, and Switzerland by recent data. Each graph represents the estimated potential \((x_i, \phi_i)\) from the relative frequency. The abscissa indicate the deviation from the historical average of CAPE and the ordinate is the negative logarithm of the relative frequency as before as in equation (20).

(iv) 1978–2010. During the periods of (i) and (iii), the market potential in the US seems to be mean reverting and U-shaped as we discussed in section 3.1, thus, we may conclude that the market reversion strategy was dominant during the time span. On the other hands, in periods of (ii) and (iv), the market potential became rather flat. It suggests that there were more market trending investors as it is considered in section 3.2.

4.2. Market potential for various countries

For Australia, Belgium, Canada, Denmark, Germany, Japan, the Netherlands, Norway, Sweden, Switzerland, and United Kingdom we use the CAPE dataset\(^8\) built in [13]. At variance with US dataset, these series span a narrower time window running from December 1969 to December 2010. The estimated market potential \((x_i, \phi_i)\) for these 12 countries are shown in figures 3 and 4. Among them Norway, UK, Belgium and Netherlands can be categorized as countries with market reversion potentials.

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\(^8\) Country indexes are identified with MSCI Indexes. MSCI Indexes as well as MSCI Dividend Yields, and MSCI Price over Earning Ratios were provided by FactSet \url{http://www.factset.com/} The nominal series given in local currencies have been deflated using Consumer Price Indexes available at the Federal Reserve Economic Data repository \url{http://research.stlouisfed.org/fred2/}.

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Other countries like USA, Switzerland, Germany and Denmark have similar market potential. The most dramatic example is observed in Japan, which has a very flat potential.

5. Conclusion

We discussed market potentials of financial markets corresponding to the regimes of a mean reverting market and of a trend followers market. Each of the markets exhibits a U-shaped or logarithmic flat potential depending on the regimes. We empirically investigated the shape of market potentials of US financial market according to the time period and various country’s financial market by calculating the relative frequency of CAPE—deviation.

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