Faddeev-Marchenko scattering for CMV matrices and the Strong Szegö Theorem

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July 25, 2008

Abstract

Simon proved the existence of the wave operators for the CMV matrices with Szegö class Verblunsky coefficients, and therefore the existence of the scattering function. Generally, there is no hope to restore a CMV matrix when we start from the scattering function, in particular, because it does not contain any information about the (possible) singular measure. Our main point of interest is the solution of the inverse scattering problem (the heart of the Faddeev–Marchenko theory), that is, to give necessary and sufficient conditions on a certain class of CMV matrices such that the restriction of this correspondence (from a matrix to the scattering function) is one to one. In this paper we show that the main questions on inverse scattering can be solved with the help of three important classical results: Adamyan-Arov-Krein (AAK) Theory, Helson-Szegö Theorem and Strong Szegö Limit Theorem. Each of these theorem states the equivalence of certain conditions. Actually, to each theorem we add one more equivalent condition related to the CMV inverse scattering problem.

*Partially supported by the Austrian Founds FWF, project number: P20413–N18
1 Introduction

To a given collection of numbers \( \{a_n\}_{n \geq 0} \) in the unit open disk \( \mathbb{D} \) and \( a_{-1} \) in the unit circle \( \mathbb{T} \) we associate the CMV matrix \( \mathfrak{A} = \mathfrak{A}_1 \mathfrak{A}_0 \), where

\[
\mathfrak{A}_0 = \begin{bmatrix} A_0 & \cdots \\ A_2 & \ddots \end{bmatrix}, \quad \mathfrak{A}_1 = \begin{bmatrix} -a_{-1} & \cdots \\ A_1 & \ddots \end{bmatrix}
\]

(1.1)

and the \( A_k \)'s are the \( 2 \times 2 \) unitary matrices

\[
A_k = \begin{bmatrix} a_k & \rho_k \\ \rho_k & -a_k \end{bmatrix}, \quad \rho_k = \sqrt{1 - |a_k|^2}.
\]

Note that \( \mathfrak{A} \) is a unitary operator in \( l^2(\mathbb{Z}_+) \). The initial vector \( e_0 \) of the standard basis is cyclic for \( \mathfrak{A} \), indeed by the definition

\[
\mathfrak{A}^{-1}\{\rho_{2n-1}e_{2n-1} - e_{2n}\bar{a}_{2n-1}\} = e_{2n}\bar{a}_{2n} + e_{2n+1}\rho_{2n} \\
\mathfrak{A}\{e_{2n}\rho_{2n} - e_{2n+1}a_{2n}\} = e_{2n+1}a_{2n+1} + e_{2n+2}\rho_{2n+1}.
\]

That is, acting in turn by \( \mathfrak{A}^{-1} \) and \( \mathfrak{A} \) on \( e_0 \) we can get in the linear combination any vector of the standard basis.

Let \( \sigma \) be the spectral measure of \( \mathfrak{A} \), i.e.,

\[
R(z) = \frac{1}{\mathfrak{A} - z} d\sigma = \langle \mathfrak{A} + z, e_0 \rangle. \tag{1.3}
\]

The matrix \( \mathfrak{A} \) is of the Szegö class, \( \mathfrak{A} \in \mathfrak{S}_z \), if \( \sum |a_k|^2 < \infty \). It holds if and only if the measure \( \sigma \) is of the form

\[
d\sigma = w(t)dm(t) + d\sigma_s, \quad \log w \in L^1,
\]

where \( dm(t) \) is the Lebesgue measure and \( \sigma_s \) is the singular component. Define the outer function

\[
D(z) = e^{\frac{1}{2} \int_{\mathbb{T}} \log w(t)dm(t)} \quad D_s(z) = D(1/\bar{z}).
\]

Simon [23, Sect. 10] proved the existence of the wave operators for the CMV matrices \( \mathfrak{A} \) with \( l^2 \) (Szegö class) Verblunsky coefficients, and therefore the existence of the scattering function, which is of the form

\[
s(t) = -a_{-1} \frac{D(t)}{D_s(t)}. \tag{1.4}
\]
Generally, there is no hope to restore $\mathfrak{A}$ when we start from $s(t)$, in particular, because it does not contain any information about the (possible) singular measure. Even if we assume that $\mathfrak{A} \in Sz_{a.c.}$, i.e., $\sigma_s = 0$, the correspondence $\mathfrak{A} \mapsto s$ is not one to one on its image. (An easy example: $s(t) = t^2$ corresponds simultaneously to $D(t) = (1 - t)^2$ and $D(t) = (1 + t)^2$). Our main point of interest is the solution of the inverse scattering problem (the heart of the Faddeev–Marchenko theory), that is, to give necessary and sufficient conditions on a certain class of CMV matrices and correspondingly on the associated scattering functions such that the restriction of the map $\mathfrak{A} \mapsto s$ is one to one. Naturally, we would like to have an explicit algorithm to get $\mathfrak{A}$. One of the key elements of the Faddeev–Marchenko construction is the so called Gelfand-Levitan-Marchenko (GLM) transformation operators, which transform an orthogonal standard basis into an intrinsic orthogonal basis related to a perturbed CMV matrix.

Usually in the Faddeev-Marchenko theory the scattering function appears in the following way.

**Theorem 1.1.** Let $\mathfrak{A} \in Sz$. Then there exists a unique generalized eigenvector $\Psi(t) = \{\Psi_n(t)\}_{n=0}^\infty$ and a so called scattering function $s(t)$, $|s(t)| = 1$, such that

$$[\Psi_0(t) \quad \Psi_1(t) \ldots] \mathfrak{A} = t [\Psi_0(t) \quad \Psi_1(t) \ldots], \quad t \in \mathbb{T}, \quad (1.5)$$

and the following asymptotics are satisfied

$$\Psi_{2n}(t) = t^n + o(1), \quad \Psi_{2n+1}(t) = \overline{s(t)}t^{-n-1} + o(1), \quad n \to \infty, \quad (1.6)$$

in the $L^2$-norm.

Also, $s(t)$ is called the nonlinear Fourier transform (NLFT) of the Verblunsky coefficients $\{a_k\}_{k=-1}^{\infty}$ [26].

In fact, Theorem 1.1 is a restatement of the classical Szegö theorem on the asymptotic behavior of orthogonal polynomials on the unit circle (OPUC).

Note, $\mathfrak{A}$ is unitary equivalent to the multiplication operator by the independent variable in $L^2_{d\sigma}$ with respect to the following orthonormal basis, see (1.2),

$$t^{-1}\{\rho_{2n-1}P_{2n-1}(t) - P_{2n}(t)\bar{a}_{2n-1}\} = P_{2n}(t)\bar{a}_{2n} + P_{2n+1}(t)\rho_{2n},
\quad t\{P_{2n}(t)\rho_{2n} - P_{2n+1}(t)a_{2n}\} = P_{2n+1}(t)a_{2n+1} + P_{2n+2}(t)\rho_{2n+1}, \quad (1.7)$$

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where \( P_n \) are Laurent polynomials

\[
P_0(t) = \pi_0(t), \quad P_1(t) = \pi_1(t) - 1 + \pi_0(t), \quad P_2(t) = \pi_2(t) + \pi_1(t) - 1 + \pi_0(t), \ldots
\]

Now we can see that the relations

\[
\lim_{n \to \infty} t^{-n}D_*(t)P_{2n}(t) = 1, \quad \lim_{n \to \infty} t^{n+1}D(t)P_{2n+1}(t) = -\bar{a}_{-1}
\]

are indeed consequences of Szegő’s Theorem. Moreover, \( \Psi_n(t) = D_*(t)P_n(t) \), which proves simultaneously the representation (1.4) of \( s \) in (1.6).

In this paper we show that the above posed questions on inverse scattering can be solved with the help of three important classical results: Adamyan-Arov-Krein (AAK) Theory, Helson-Szegő Theorem and Strong Szegő Limit Theorem. Each of these theorem states the equivalence of certain conditions. Actually, to each theorem we add one more equivalent condition related to the CMV inverse scattering problem. In this way we describe three subclasses of \( \text{Sz}_{a.c.} \):

- \( \text{Sz}^{\text{reg}} \subset \text{Sz}_{a.c.} \) on which \( \mathfrak{A} \mapsto s \) is one to one,
- \( \mathfrak{A} \in \text{HS} \subset \text{Sz}^{\text{reg}} \) if the corresponding GLM operator \( \mathcal{M} \) is bounded,
- \( \mathfrak{A} \in \text{GI} \subset \text{HS} \) if \( \det(\mathcal{M}^*\mathcal{M}) < \infty \).

The AAK theory describes the solutions of the so called Nehari problem, which are of the form

\[
-\frac{\psi \mathcal{E} + \bar{\phi}}{\psi 1 + \phi \mathcal{E}},
\]

where \( \phi \) depends on the data of the problem; \( \psi, \psi(0) > 0 \), is an outer function, such that \( |\psi|^2 + |\phi|^2 = 1 \); and \( \mathcal{E} \) is an arbitrary function of the Schur class (\( \mathcal{E} \in H^\infty, \|\mathcal{E}\| \leq 1 \)). The function \( \phi \) has the following properties

\[
\phi \in H^\infty, \quad \|\phi\| \leq 1, \quad \phi(0) = 0, \quad \log(1 - |\phi|^2) \in L^1.
\]

But not any function of the form (1.9) generates the set of all solutions of the Nehari problem. If \( \phi \) generates all solutions it is called regular. The AAK theory gives several necessary and sufficient conditions for regularity. Our contribution to them is as follows: Let \( R \) be given by (1.3) and let

\[
\phi(z) = a_{-1} \frac{R(0) - R(z)}{R(0) + R(z)}.
\]
Then $\mathfrak{A} \mapsto s$ is one to one if and only if $\phi$ is regular.

A unimodular function $s$ belongs to the Helson-Szeg"{o} class, $s \in HS$, if it possesses the representation
\[
s = e^{i(\tilde{u} + v)}, \quad u, v \in L^\infty, \quad \|v\| < \pi/2,
\] (1.10)
where $\tilde{u}$ is the harmonic function conjugated to $u$. The classical Helson-Szeg"{o} and Hunt-Muckenhoupt-Wheeden Theorems give several necessary and sufficient conditions for a function $s$ to be of Helson–Szeg"{o} class. Our contribution is the following: $\mathfrak{A} \mapsto s$ is one to one and the GLM operator is bounded if and only if $s \in HS$.

The B.Golinskii–Ibragimov Theorem is discussed in details in Sect. 6 of Simon’s book [22]. Our Theorem complements this theory from the point of view of scattering. Recall a function $f(t)$ belongs to the Sobolev space $B_{21/2}^2$ if its Fourier coefficients $\{c_k\}_{k=-\infty}^{\infty}$ satisfy the condition
\[
\sum_k |k||c_k|^2 < \infty.
\]

In a sense Theorem 5.3 says that the Nonlinear Fourier Transform (NLFT) of the Verblunsky coefficients (that is the scattering function $s(t)$) belongs to $B_{21/2}^2$ if and only if its Linear Fourier Transform (LFT) belongs to $B_{21/2}^2$. We believe that this Theorem is an essential improvement of the general Faddeev-Marchenko theory (see Remark 5.5).

Finally we demonstrate that the so-called Widom Formula has a natural proof in the frame of the scattering theory.

## 2 From spectral representation to scattering representation

First of all let us point out that (1.3) gives a one to one correspondence between the CMV matrix $\mathfrak{A}$ on one hand and the measure $\sigma$ and normalizing constant $a_{-1}$ on the other hand. In the orthogonalization procedure for the system
\[
1, t^{-1}, t, t^{-2}, t^2, t^{-3}, \ldots
\]
with respect to $d\sigma$ we choose
\[
\pi_0^{(0)} > 0, \quad \frac{\pi_1^{(1)}}{|\pi_1^{(1)}|} = -\bar{a}_{-1}.
\]
In this case
\[ \pi^{(2n)}_{2n} = \frac{1}{\rho_0 \rho_1 \cdots \rho_{2n-1}}, \quad \pi^{(2n+1)}_{2n+1} = -\bar{a}_{-1} \frac{1}{\rho_0 \rho_1 \cdots \rho_{2n}}. \] (2.1)

In the Szegö case we have the decomposition \( d\sigma = |D|^2 dm + \sigma_s \) and we define the scattering function \( s = -a_{-1} D/D_* \), thus we have the map
\[ \mathfrak{A} \leftrightarrow \{ d\sigma, a_{-1} \} \mapsto s. \] (2.2)

To clarify the uniqueness property of this map we define certain Hilbert spaces associated with the symbol \( s \). But first we give another description of the space \( L^2_{d\sigma} \). For \( f \in L^2_{d\sigma} \) we set
\[
F(z) = (F_1(z), F_2(z)) = (F_1(z; f), F_2(z; f)),
\] (2.3)
where
\[
F_1(z) = F_1(z; f) = \frac{1}{D(z)} \int_T \frac{t}{t - z} f(t) d\sigma(t), \quad |z| < 1,
\]
\[
F_2(z) = F_2(z; f) = \frac{a_{-1} D_*(z)}{D_*(z)} \int_T \frac{t}{t - z} f(t) d\sigma(t), \quad |z| > 1.
\] (2.4)

The first component is analytic inside the unit disk, the second in its exterior. Actually, they are functions of bounded characteristic with an outer denominator (of the Smirnov class).

Note that for \( z \in \mathbb{D} \)
\[
(DF_1)(z) - \bar{a}_{-1}(D_* F_2)(1/\bar{z}) = \int_T \frac{1 - |z|^2}{|t - z|^2} f(t) d\sigma(t). \] (2.5)

**Definition 2.1.** Let \( \mathcal{F} = (F_1(z), F_2(z)) \), where \( F_1(z) \) is analytic in \( \mathbb{D} \) and \( F_2(z), F_2(\infty) = 0 \), is analytic for \( |z| > 1 \). We say that \( \mathcal{F} \in M(\sigma, a_{-1}) \) if
\[
\| \mathcal{F} \|^2 = \sup_{r < 1} \int_T |(DF_1)(rt) - \bar{a}_{-1}(D_* F_2)(\bar{t}/r)|^2 \frac{dm(t)}{ReR(rt)} < \infty,
\] (2.6)
where \( R(z) \) is defined in (1.3).

**Theorem 2.2.** \( \mathcal{F} \in M(\sigma, a_{-1}) \) if and only if it is of the form (2.4). Moreover
\[
\| \mathcal{F} \|_{M(\sigma, a_{-1})} = \| f \|_{L^2_{d\sigma}}.
\] (2.7)
Proof. The proof is based on a concept of the so called Hellinger integral, see e.g. [25] or [14, 12]. We recall the corresponding construction here.

Let

\[
\begin{bmatrix}
da & db \\
* & dc
\end{bmatrix}
\]

(2.8)

be 2 × 2 nonnegative matrix measure on \( T \). In fact, it means that \( db \) is absolutely continuous with respect to \( da \), \( db = fda \), moreover

\[ dc - |f|^2 da \geq 0. \]

Let us point out that \( \inf c(T) \) over all possible nonnegative matrix-functions of the form (2.8) with the fixed \( da \) and \( db \) corresponds precisely to the case

\[ dc = |f|^2 da. \]

(2.9)

Now, let

\[
\begin{bmatrix}
u(z) & v(z) \\
* & w(z)
\end{bmatrix} = \int_T \frac{1 - |z|^2}{|t - z|^2} \begin{bmatrix} da(t) & db(t) \\
* & dc(t) \end{bmatrix}.
\]

Since this matrix is nonnegative we have

\[ w_r(t) \geq \frac{|v_r(t)|^2}{u_r(t)}, \quad t \in T, \; r \in (0, 1), \]

where \( u_r(z) = u(rz) \), etc. Therefore

\[ c(T) = w(0) \geq \int_T \frac{|v_r(t)|^2}{u_r(t)} dm(t). \]

(2.10)

Thus we get

\[ c(T) \geq \sup_r I(r), \quad \text{where } I(r) = \int_T \frac{|v_r(t)|^2}{u_r(t)} dm(t), \]

(2.11)

for all \( dc \) which forms a nonnegative matrix function (2.8) with the given \( da \) and \( db \). Note that this already proves the easy part of the theorem. Indeed, set

\[ u(z) = \text{Re} R(z), \quad v(z) = (DF_1)(z) - \bar{a}_{-1}(D_2 F_2)(1/\bar{z}). \]

(2.12)

Due to (2.5) we have boundedness of sup in (2.6).
Now, for harmonic functions $u(z)$ and $v(z)$ let us prove that $I(r)$ increases with $r$. For $k \in (0, 1)$, the matrix measure

$$
\begin{bmatrix}
  u_k(t) & v_k(t) \\
  * & \frac{|v_k(t)|^2}{u_k(t)}
\end{bmatrix} dm(t)
$$

is nonnegative on $T$. Its harmonic extension in the disk is of the form

$$
\begin{bmatrix}
  u_k(z) & v_k(z) \\
  * & w(z, k)
\end{bmatrix}.
$$

(2.13)

Therefore, due to (2.10) we get

$$
I(k) = w(0, k) \geq \int \frac{|v_k(rt)|^2}{u_k(rt)} dm(t) = \int \frac{|v(rkt)|^2}{u(rkt)} dm(t) = I(rk).
$$

Thus, if $\sup I(r) < \infty$ then the limit $\lim_{r \to 1} I(r)$ exists. Consider the sequence of harmonic matrix functions (2.13). Due to the compactness principle there exists a limit (on a subsequence)

$$
\lim_{k \to 1} \begin{bmatrix}
  u_k(z) & v_k(z) \\
  * & w(z, k)
\end{bmatrix} = \begin{bmatrix}
  u(z) & v(z) \\
  * & w(z, 1)
\end{bmatrix} = \int_T \frac{1 - |z|^2}{|t - z|^2} \begin{bmatrix}
  da(t) & db(t) \\
  * & dc(t, 1)
\end{bmatrix}.
$$

Therefore, for this particular matrix measure we have $db(t) = f(t) da(t)$, and moreover,

$$
c(T, 1) = w(0, 1) = \lim I(k_n) = \sup I(r).
$$

Since generally we have inequality (2.11), this value of $c(T, 1)$ corresponds to the infimum, and, therefore $dc(t, 1)$ is of the form (2.9), that is,

$$
\int |f|^2 da = \sup I(r).
$$

Thus for $u$ and $v$ of the form (2.12) we get (2.7).

Lemma 2.3. $M(\sigma, a_{-1})$ is a space with the reproducing kernels $K_z = K_z(\sigma, a_{-1})$, $|z| < 1$, $|z| > 1$, in particular,

$$
\langle F, K_0 \rangle = F_1(0), \quad \langle F, K_\infty \rangle = (tF_2)(\infty),
$$

where

$$
\begin{align*}
K_0 &= \left( \frac{1}{DD(0)} \frac{R(0) + R}{2}, \frac{a_{-1}}{D_+D(0)} \frac{R(0) - \bar{R}}{2} \right), \\
K_\infty &= \left( \frac{a_{-1}}{DD(0)} \frac{R(0) - R}{2t}, \frac{1}{D_+D(0)} \frac{R(0) + \bar{R}}{2t} \right).
\end{align*}
$$

(2.14)
Proof. Use definitions (2.4) and (2.7).

Now we note the following identity for the boundary values

$$|D(t)|^2 f(t) = (DF_1)(t) - \bar{a}_{-1} (D_s F_2)(t), \quad t \in \mathbb{T}.$$ 

We introduce the scalar product

$$\|F\|_s^2 := \int |s(t)F_1(t) + F_2(t)|^2 \, dm(t) = \int |f|^2 |D|^2 \, dm \leq \int |f|^2 d\sigma.$$  (2.15)

Note that equality holds if and only if $f|_{\text{supp}(\sigma_s)} = 0$.

**Definition 2.4.** Let $M_s = M_s(D)$ be the Hilbert space of the functions (2.3), $f|_{\text{supp}(\sigma_s)} = 0$, with the scalar product (2.15).

Let us point out that the scalar product depends on $s$ but the collection of functions depends actually on $D$, that is why, it’s better to write $M_s(D)$.

**Definition 2.5.** We define $\tilde{M}_s = \text{clos}\{F = (F_1, F_2) : F_1 \in H^2, F_2 \in H^2\}$ with the norm,

$$\|F\|_s^2 = \int |s(t)F_1(t) + F_2(t)|^2 \, dm(t) = \langle \begin{bmatrix} I & \mathcal{H}_s \end{bmatrix} \begin{bmatrix} F_1 \\ F_2 \end{bmatrix}, \begin{bmatrix} F_1 \\ F_2 \end{bmatrix} \rangle,$$

where $\mathcal{H} = \mathcal{H}_s$ is the Hankel operator with the symbol $s$

$$\mathcal{H}_s : H^2 \rightarrow H^2_-, \quad \mathcal{H}_s F_1 = P_-(s F_1), \quad F_1 \in H^2.$$

The space $\tilde{M}_s$ depends only on $s$, more precisely on $\mathcal{H}_s$, that is, of the negative Fourier coefficients of $s$, so we write $\tilde{M}_s(\mathcal{H})$.

**Lemma 2.6.** Let $D \in H^2$, $D(0) > 0$, be an outer function and $s = -a_{-1} D/\bar{D}$, $a_{-1} \in \mathbb{T}$. Then $\tilde{M}_s(\mathcal{H}_s) \subset M_s(D)$.

**Proof.** For polynomials $P_1, P_2$ set $f = \frac{1}{D} P_1 + \frac{1}{\bar{D}} \bar{P}_2 \in L^2_{udm}$. Then $F_1(f) = P_1, F_2(f) = -a_{-1} \bar{P}_2$. This set is complete in $M_s$. 

\qed
3 AAK Theory and uniqueness in Inverse Scattering

The AAK Theory deals with the Nehari problem [3, 4, 5], see also [8].

Problem 3.1. Given a Hankel operator $\mathcal{H}$, $\|\mathcal{H}\| \leq 1$, describe the collection of symbols
\[\mathcal{N}(\mathcal{H}) = \{ f \in L^\infty : \mathcal{H} = \mathcal{H} f, \| f \|_{\infty} \leq 1 \}.\] (3.1)

The Nehari Theorem stays solvability of the problem, i.e., $\mathcal{N}(\mathcal{H}) \neq \emptyset$. The following lemma is related to the question of uniqueness of a solution.

Lemma 3.2 (Adamyan-Arov-Krein). The point evaluation functional
\[\mathcal{F} \to F_1(0)\]
is bounded in $\tilde{M}_s(\mathcal{H})$ if and only if
\[\lim_{r \uparrow 1} \langle (I - r^2 \mathcal{H}^* \mathcal{H})^{-1} 1, 1 \rangle < \infty.\] (3.2)

Moreover,
\[\tilde{K}_0 = \tilde{K}^*_0 = \lim_{r \uparrow 1} \begin{bmatrix} I & r \mathcal{H}^* \\ r \mathcal{H} & I \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \tilde{K}_\infty = \tilde{K}^*_\infty = \lim_{r \uparrow 1} \begin{bmatrix} I & r \mathcal{H}^* \\ r \mathcal{H} & I \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ i \end{bmatrix}.\] (3.3)

Theorem 3.3 (Adamyan-Arov-Krein). A solution of the Nehari problem is not unique if and only if (3.2) holds. In this case the set $\mathcal{N}(\mathcal{H})$ is of the form
\[\mathcal{N}(\mathcal{H}) = \{ f = f_\mathcal{E} = -\frac{\psi_\mathcal{H}}{\psi_\mathcal{H}} \mathcal{E} + \phi_\mathcal{H} \psi_\mathcal{H} : \mathcal{E} \in H^\infty, \| \mathcal{E} \|_{\infty} \leq 1 \},\] (3.4)

where $\phi_\mathcal{H}$ is a Schur class function given by
\[\phi_\mathcal{H}(z) = \frac{(\tilde{K}_0)_2(1/z)}{(\tilde{K}_0)_1(z)} = z \frac{(\tilde{K}_\infty)_1(z)}{(\tilde{K}_0)_1(z)} = z \lim_{r \uparrow 1} \frac{-(I - r^2 \mathcal{H}^* \mathcal{H})^{-1} \mathcal{H}^* 1(z)}{((I - r^2 \mathcal{H}^* \mathcal{H})^{-1} 1)(z)}.\] (3.5)

and $\psi_\mathcal{H}$ is the outer function
\[\psi_\mathcal{H}(z) \psi_\mathcal{H}(0) = \lim_{r \uparrow 1} \frac{1}{((I - r^2 \mathcal{H}^* \mathcal{H})^{-1} 1)(z)}, \quad \psi_\mathcal{H}(0) > 0.\] (3.6)

Moreover, $|\psi_\mathcal{H}|^2 + |\phi_\mathcal{H}|^2 = 1$. 

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Similarly to (3.5), by (2.14) we define
\[ \phi := z \frac{(K_\infty)_1(z)}{(K_0)_1(z)} = \bar{a}_{-1} \frac{R(0) - R}{R(0) + R} \] (3.7)
and the outer function \( \psi, \psi(0) > 0 \), such that \( |\psi|^2 + |\phi|^2 = 1 \).

**Remark 3.4.** Let us note the important identity
\[ \frac{(K_\infty)_1(0)}{(K_0)_1(0)} = -\frac{\bar{a}_{-1}}{2} R'(0) = -\bar{a}_{-1} \langle \mathfrak{A}^{-1} e_0, e_0 \rangle = \langle \bar{a}_0 e_0 + \rho_0 e_1, e_0 \rangle = \bar{a}_0. \] (3.8)

As in (3.4) we consider the collection of functions
\[ f = -\frac{\psi}{\bar{\psi}} \frac{\mathcal{E} + \overline{\phi}}{1 + \phi \mathcal{E}}, \quad \mathcal{E} \in H^\infty, \|\mathcal{E}\| \leq 1. \] (3.9)
Let us note \( \psi/(1 + \mathcal{E}\phi) \in H^2 \). Indeed,
\[ \left| \frac{\psi}{1 + \mathcal{E}\phi} \right|^2 \leq \frac{1 - |\mathcal{E}\phi|^2}{|1 + \mathcal{E}\phi|^2} = \text{Re} \frac{1 - \mathcal{E}\phi}{1 + \mathcal{E}\phi} \in L^1, \]
and \( 1 + \mathcal{E}\phi \) is an outer function. Therefore, due to the identity
\[ f = -\frac{\psi}{\bar{\psi}} \frac{\mathcal{E}\psi^2}{1 + \phi \mathcal{E}}, \]
all of them correspond to the same Hankel operator.

**Definition 3.5.** \([1, 2]\) A function \( \phi \)
\[ \phi \in H^\infty, \|\phi\| \leq 1, \phi(0) = 0, \log(1 - |\phi|^2) \in L^1, \] (3.10)
is called Arov-regular if the set (3.9) describes the collection of all symbols with the same Hankel operator.

It is called Arov-singular if \( f_0 = -\frac{\psi}{\bar{\psi}} \phi \in H^\infty \). Equivalently, the entries of the unitary valued matrix function
\[ \begin{bmatrix} f_0 & \psi \\ \psi & \phi \end{bmatrix} \] (3.11)
belong to \( H^\infty \).
It means that (3.9) with a singular $\phi$ describes a proper subclass of the Schur class, $H_f = 0$ for all $f$. The Potapov-Ginzburg transform of the matrix (3.11)

$$A := \frac{1}{\psi} \begin{bmatrix} f_0 \phi - \psi^2 & f_0 \\ \phi & 1 \end{bmatrix} = \begin{bmatrix} -\frac{1}{\psi} \frac{\bar{\phi}}{\psi} \\ \frac{1}{\psi} \frac{\bar{\psi}}{\psi} \end{bmatrix}$$

(3.12)
is an (Arov-singular) $j$-inner matrix function,

$$A(z)^*jA(z) - j \geq 0, \quad z \in \mathbb{D}; \quad A(t)^*jA(t) - j = 0, \quad t \in \mathbb{T}, \quad j = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}.$$

**Theorem 3.6 (Arov).** Every function of the form (3.10) possesses (Arov-) singular-regular factorization:

$$\frac{1}{\psi} \begin{bmatrix} \phi & 1 \end{bmatrix} = \frac{1}{\psi_{\mathcal{H}}} \begin{bmatrix} \phi_{\mathcal{H}} & 1 \end{bmatrix} A.$$  

(3.13)

**Theorem 3.7.** [1], see also [13]. Let $\phi$ be a function of the form (3.10), equivalently, let the Herglotz class function

$$R(z) = \frac{1 - a_{-1}\phi(z)}{1 + a_{-1}\phi(z)} = \int \frac{t + z}{t - z} d\sigma, \quad a_{-1} \in \mathbb{T},$$  

(3.14)

be associated with the Szegö class measure $\sigma$, $\sigma(\mathbb{T}) = 1$. We set

$$\mathcal{H} = \mathcal{H}_s, \quad s = -a_{-1}\frac{D}{D_s}, \quad D(z) = \frac{\psi(z)}{1 + a_{-1}\phi(z)}.$$  

(3.15)

The function $\phi$ is Arov-regular if and only if one of the following equivalent conditions hold:

(i) $M(\sigma, a_{-1}) = M_s(D) = \tilde{M}_s(\mathcal{H})$.

(ii) The reproducing kernel $K_\phi$ belongs to $\tilde{M}_s(\mathcal{H})$.

(iii) $\phi = \phi_{\mathcal{H}}$, $\psi = \psi_{\mathcal{H}}$.

(iv) $\psi(0) = \psi_{\mathcal{H}}(0)$, that is the following limit exists

$$\lim_{r \uparrow 1} \langle (I - r^2\mathcal{H}_s^*\mathcal{H}_s)^{-1}, 1 \rangle = \frac{1}{D^2(0)} = \frac{1}{\psi^2(0)}.$$  

(3.16)
**Definition 3.8.** Let $\mathfrak{A} \in \text{Sz}$ correspond to the spectral data $\sigma, a_{-1}$. We say that $\mathfrak{A}$ is regular, $\mathfrak{A} \in \text{Sz}^{\text{reg}}$, if the associated function

$$
\phi(z) = \bar{a}_{-1} \frac{R(0) - R(z)}{R(0) + R(z)},
$$

$$
R(z) = \int \frac{t + z}{t - z} d\sigma(t),
$$

(3.17)

is Arov-regular.

**Theorem 3.9.** The correspondence $\mathfrak{A} \mapsto s$, $\mathfrak{A} \in \text{Sz}_{a.c.}$, is one to one precisely on the subclass $\text{Sz}^{\text{reg}}$.

**Proof.** Let $\phi$ be not regular. Then it is of the form (3.13), where $A$ is a non-constant $j$-inner matrix function with the entries $\phi_A, \psi_A$.

First we assume that the vector $A \begin{bmatrix} a_{-1} \\ 1 \end{bmatrix}$ is collinear to a constant, that is

$$
A(z) \begin{bmatrix} a_{-1} \\ 1 \end{bmatrix} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \omega(z), \quad |c_1|^2 + |c_2|^2 \neq 0.
$$

(3.18)

We claim that in this case the associated $R$ function

$$
R(z) = \frac{1 - a_{-1} \phi}{1 + a_{-1} \phi} = \int \frac{t + z}{t - z} |D(t)|^2 dm + \int \frac{t + z}{t - z} d\sigma_s(t)
$$

has the absolutely continuous component with the density proportional to the density of a canonical measure

$$
D = D_{\bar{\phi}_{\tilde{H}}} \psi_A(0), \quad \tilde{R}(z) = \frac{1 - a_{-1} \phi_{\tilde{H}}}{1 + a_{-1} \phi_{\tilde{H}}} = \int \frac{t + z}{t - z} |D_{\bar{\phi}_{\tilde{H}}}(t)|^2 dm,
$$

(3.19)

where $\phi_{\tilde{H}} = \bar{a}_{-1} \tilde{a}_{-1} \phi_{\tilde{H}}, \quad |\bar{a}_{-1}| = 1$. But also it has a non-trivial singular component $\sigma_s(T) = 1 - |\psi_A(0)|^2$, that is $\mathfrak{A} \notin \text{Sz}_{a.c.}$.

Let us show that, in fact, $\omega(z) = \text{const}$ in (3.18). We note that we have here a so called $j$-neutral vector, i.e.,

$$
[\bar{c}_1 \quad \bar{c}_2] j \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = -|c_1|^2 + |c_2|^2 = 0.
$$

Indeed, it follows from the fact that $A$ is $j$-unitary on the boundary

$$
(-|c_1|^2 + |c_2|^2) |\omega(t)|^2 = \begin{bmatrix} \bar{a}_{-1} \\ 1 \end{bmatrix} A(t)^* j A(t) \begin{bmatrix} a_{-1} \\ 1 \end{bmatrix}
$$

$$
= \begin{bmatrix} \bar{a}_{-1} \\ 1 \end{bmatrix} j \begin{bmatrix} a_{-1} \\ 1 \end{bmatrix} = -|a_{-1}|^2 + 1 = 0.
$$
Now we use the Schwartz Lemma for $j$-expanding matrix functions

\[ \left[ \frac{A(z)jA^*(z) - j}{1 - |z|^2} \star \frac{A(z) - A(0)}{z} \right] \geq 0. \quad (3.20) \]

Since

\[ \left[ \begin{array}{c} a_{-1} \\ 1 \end{array} \right]^* (A^*(0)jA(0) - j) \left[ \begin{array}{c} a_{-1} \\ 1 \end{array} \right] = (-|c_1|^2 + |c_2|^2)|\omega(0)|^2 - (-|a_{-1}|^2 + 1) = 0 \]

we get from (3.20)

\[ \left[ \begin{array}{c} a_{-1} \\ 1 \end{array} \right] \star \left[ \begin{array}{c} a_{-1} \\ 1 \end{array} \right]^* (A^*(0)jA(0) - j) \left[ \begin{array}{c} a_{-1} \\ 1 \end{array} \right] = \left[ \begin{array}{c} a_{-1} \\ 1 \end{array} \right] \star \left[ \begin{array}{c} a_{-1} \\ 1 \end{array} \right]^* \left( A(z)jA^*(z) - j \right) \left[ \begin{array}{c} a_{-1} \\ 1 \end{array} \right] \geq 0, \]

which implies $\omega(z) - \omega(0) = 0$.

Therefore we proved that

\[ A(z) \left[ \begin{array}{c} a_{-1} \\ 1 \end{array} \right] = A(0) \left[ \begin{array}{c} a_{-1} \\ 1 \end{array} \right] = \left[ \begin{array}{c} \tilde{a}_{-1} \\ 1 \end{array} \right] \frac{1}{\psi_A(0)} \]

(3.21)

with a certain $\tilde{a}_{-1} \in \mathbb{T}$. By (3.13) we get

\[ \frac{1}{D} = \frac{1}{\psi} \left[ \phi \ 1 \right] \left[ a_{-1} \ 1 \right] = \frac{1}{\psi_H} \left[ \phi_H \ 1 \right] A \left[ a_{-1} \ 1 \right] \]

\[ = \frac{1}{\psi_H} \left[ \phi_H \ 1 \right] \left[ \tilde{a}_{-1} \ 1 \right] \frac{1}{\psi_A(0)} = \frac{1}{\psi_A(0)D_H}. \]

(3.22)

Thus (3.19) is proved.

It remains to consider the case

\[ A(z) \left[ \begin{array}{c} a_{-1} \\ 1 \end{array} \right] = \left[ \mathcal{E}(z) \right] \omega(z), \]

(3.23)

where the inner function $\mathcal{E}$ is not a constant, that is, the symbol $s$ has a nontrivial inner function $\mathcal{E}$ in its canonical representation (3.4). We have to
show that there are at least two different CMV matrices from $\text{Sz}_{a.c.}$ with the given scattering function.

For $t_1, t_2 \in \mathbb{T}$, $t_1 \neq t_2$, let $e^{i\epsilon_k}$ be such that $e^{i\epsilon_k}(1 + \bar{t}_k\mathcal{E}(0)) > 0$. We define

$$D_k = C_k e^{i\epsilon_k}(1 + \bar{t}_k\mathcal{E}) \frac{\psi_{\mathcal{H}}}{1 + \mathcal{E}\phi_{\mathcal{H}}}, \quad (a_{-1})_k = e^{-2i\epsilon_k}t_k,$$

where the normalizing positive constants $C_k$ are chosen from the conditions $\int |D_k|^2 dm = 1$. Both data $\{|D_k|^2 dm, (a_{-1})_k\}$, $k = 1, 2$, produce the same function $s$ corresponding to the given $\mathcal{E}$. \hfill $\square$

Here is a sufficient condition known in the context of the AAK theory.

**Theorem 3.10.** Let $\mathfrak{A} \in \text{Sz}$ and let its spectral measure be absolutely continuous, $d\sigma = w dm$. If $1/w \in L^1$ then $\mathfrak{A} \in \text{Sz}_{\text{reg}}$.

**Proof.** For the given scattering function $s$ we have the canonical representation

$$s = -a_{-1}D = -\mathcal{E}\frac{\psi_{\mathcal{H}} 1 + \bar{\phi}_{\mathcal{H}}\mathcal{E}}{\psi_{\mathcal{H}} 1 + \phi_{\mathcal{H}}\mathcal{E}}.$$

Therefore,

$$G := \frac{1}{D}\frac{\psi_{\mathcal{H}}}{1 + \phi_{\mathcal{H}}\mathcal{E}} = a_{-1}\frac{1}{D} \frac{\psi_{\mathcal{H}}}{1 + \phi_{\mathcal{H}}\mathcal{E}}. \quad (3.24)$$

The function $\frac{\psi_{\mathcal{H}}}{1 + \phi_{\mathcal{H}}\mathcal{E}}$ belongs to $H^2$ and, due to the assumption, $\frac{1}{D} \in H^2$. Thus $G$ belongs to $H^1$, that is, all its negative Fourier coefficients vanish. From the second representation $\bar{G} \in H^1$. That is, all positive Fourier coefficients of $G$ vanish. Therefore $G$ is constant. Since $\mathcal{E}$ is the inner part of $G$, we have $\mathcal{E} = \text{const}$. Using the normalization $D(0) > 0$, $\psi_{\mathcal{H}}(0) > 0$, we get $\mathcal{E} = a_{-1}$. \hfill $\square$

A similar sufficient condition $1/\psi \in H^2$ is given in Proposition 4.2. For a weaker condition on $|\psi|$, which ensures regularity of $\phi$, see [28].

## 4 Helson-Szego Theorem and Boundedness of the GLM Transform

We define the orthonormal system in $M(\sigma, a_{-1})$

$$\mathfrak{e}_n(z) := \mathcal{F}(z, P_n).$$
Recall, the first component \((e_n)_1(z)\) is holomorphic for \(|z| < 1\) and the second, \((e_n)_2(z)\), for \(|z| > 1\). Moreover, due to the orthogonality property of the Laurent polynomials \(P_n\) we have

\[
e_{2n} = \left( \frac{1}{D(0)n^{2n}}z^n + \ldots, O(\frac{1}{z^{n+1}}) \right),
\]

\[
e_{2n+1} = \left( O(z^{n+1}), \frac{1}{D(0)n^{2n+1}} \frac{1}{z^{n+1}} + \ldots \right).
\]

**Definition 4.1.** The (lower–triangular) matrix of the Gelfand-Levitan-Marchenko (GLM) transformation is defined by the following relation:

\[
\begin{bmatrix}
e_0 & e_1 & e_2 & \ldots
\end{bmatrix} = \begin{bmatrix} 1 & 0 & z & \ldots \\
0 & 1/z & 0 & \ldots
\end{bmatrix} M,
\]

that is,

\[
e_{2n} = M_{2n}^{2n} \begin{bmatrix} z^n \\ 0 \end{bmatrix} + M_{2n+1}^{2n} \begin{bmatrix} 0 \\ 1/z^n+1 \end{bmatrix} + M_{2n+2}^{2n} \begin{bmatrix} z^{n+1} \\ 0 \end{bmatrix} + \ldots
\]

\[
e_{2n+1} = M_{2n+1}^{2n+1} \begin{bmatrix} 0 \\ 1/z^n+1 \end{bmatrix} + M_{2n+2}^{2n+1} \begin{bmatrix} z^{n+1} \\ 0 \end{bmatrix} + M_{2n+2}^{2n+1} \begin{bmatrix} 0 \\ 1/z^{n+2} \end{bmatrix} + \ldots
\]

Let us point out that generally only matrix elements of \(M\) are well defined.

**Proposition 4.2.** The vector \(M e_0\) belongs to \(l^2(\mathbb{Z}_+)\) if and only if \(1/\psi \in H^2\). In particular, \(M e_0 \in l^2(\mathbb{Z}_+)\) implies \(A \in Sz^{reg}\).

**Proof.** By the definition, see (2.14),

\[
e_0 = \begin{bmatrix} 1/\psi(0) \\ \overline{\phi}/\psi(0) \end{bmatrix}.
\]

Therefore \(\sum_{j \geq 0} |M_{2j}^0|^2 < \infty\) is equivalent to \(\frac{1}{\psi(0)} \in H^2\) and \(\sum_{j \geq 0} |M_{2j+1}^0|^2 < \infty\) is equivalent to \(\frac{\overline{\phi}/\psi(0)}{\psi(0)} \in H^2\).

Let us chose \(f_0 = -\overline{\phi}/\psi\) as a symbol of the associated Hankel operator \(H\). Since \(1/\psi \in H^2\), we get

\[
(I-H^*H) \frac{1}{\psi(0)} = \frac{1}{\psi(0)} - P_+ \left( \frac{\overline{\phi}}{\psi} \right) \frac{\overline{\phi}}{\psi(0)} = \frac{1}{\psi(0)} - P_+ (1-|\psi|^2) \frac{1}{\psi(0)} = 1.
\]

Therefore, \(\psi = \psi_H\). By Theorem 3.7 \(A \in Sz^{reg}\). 

\[
\]
Theorem 4.3. Let $\mathfrak{A} \in \mathbf{Sz}^{reg}$, that is, $M(\sigma, a_{-1}) = \tilde{M}_s$. In this case the orthonormal basis (4.1) is of the form
\[
e_{2n} = \begin{bmatrix} t^n & 0 \\ 0 & \bar{t}^n \end{bmatrix} \frac{\tilde{\mathcal{K}}_{st}^{2n}}{\| \tilde{\mathcal{K}}_{st}^{2n} \|}, \quad e_{2n+1} = -a_{-1} \begin{bmatrix} t^{n+1} & 0 \\ 0 & \bar{t}^n \end{bmatrix} \frac{\tilde{\mathcal{K}}_{st}^{2n+1}}{\| \tilde{\mathcal{K}}_{st}^{2n+1} \|},
\]
where $\tilde{\mathcal{K}}_{st}^{2n}$ are the reproducing kernels (3.3) of $\tilde{\mathcal{M}}_{st}^{2n}$. In particular,
\[
\tilde{\mathcal{M}}_{2n}^{2n} = \sqrt{\langle \tilde{\mathcal{K}}_{st}^{2n} \rangle_1(0)} = \sqrt{\lim_{r \to 1} \langle (I - r^2 \mathcal{H}_{st}^{2n} \mathcal{H}_{st}^{2n})^{-1} 11, 1 \rangle} = \rho_0 \rho_1 \ldots \frac{1}{\rho_{2n-1}}
\]
\[
\tilde{\mathcal{M}}_{2n+1}^{2n+1} = -a_{-1} \sqrt{\langle \tilde{\mathcal{K}}_{st}^{2n+1} \rangle_2(\infty)} = -a_{-1} \sqrt{\lim_{r \to 1} \langle (I - r^2 \mathcal{H}_{st}^{2n+1} \mathcal{H}_{st}^{2n+1})^{-1} 1t, t \rangle} = -a_{-1} \frac{\rho_0 \rho_1 \ldots \rho_{2n}}{D(0)} = \frac{-a_{-1}}{\rho_{2n+2} \rho_{2n+1} \ldots}.
\]
(4.5)

Remark 4.4. Theorem 4.3 provides an algorithm for inverse scattering, indeed, (4.4) gives also an explicit formula for the Verblunsky coefficients, see Remark 3.4:
\[
\bar{a}_n = \frac{\langle \tilde{\mathcal{K}}_{st}^{2n} \rangle_1(0)}{\langle \tilde{\mathcal{K}}_{st}^{2n} \rangle_1(0)} = \frac{(\tilde{\mathcal{K}}_{st}^{2n})_1(0)}{(\tilde{\mathcal{K}}_{st}^{2n})_1(0)}.
\]
(4.6)

Proof of Theorem 4.3. Let us note that $(e_n)_1(0) = 0$ for $n \geq 1$. That is all vectors spanned by this system are orthogonal to $\tilde{\mathcal{K}}_0$. Therefore $\tilde{\mathcal{K}}_0$ is collinear to $e_0$ and coincides with the initial basis vector after the appropriate normalization. A dense set in the orthogonal complement in $\tilde{M}_s$ is of the form $\mathcal{F} = (tF_1, F_2)$, $F_1 \in H^2$, $F_2 \in H^2$. Now, by definition $(tF_1, F_2) \in \tilde{M}_s$ means that $(F_1, F_2) \in \tilde{M}_{ts}$. Thus we have the decomposition
\[
\tilde{M}_s = \{ \tilde{\mathcal{K}}_0 \} \oplus \begin{bmatrix} t & 0 \\ 0 & 1 \end{bmatrix} \tilde{\mathcal{M}}_{ts}.
\]
Continue in this way we get (4.4).

To prove (4.5) we use (4.1) and (2.1).
Theorem 4.5. Let $A \in \text{Sz}^{reg}$. Then the following upper-lower triangular factorization holds true
\[
\lim_{r \uparrow 1} U^* \begin{bmatrix} I & r\mathcal{H}^* \\ r\mathcal{H} & I \end{bmatrix}^{-1} U = \mathcal{M}\mathcal{M}^*,
\] (4.7)

where $U : l^2(\mathbb{Z}_+) \to H^2 \oplus H^2_-$ is the reordering of the standard basis
\[
Ue_{2n} = t^n \oplus 0, \quad Ue_{2n+1} = 0 \oplus t^{-n-1}.
\]

In particular, $\|\mathcal{M}\| < \infty$ if and only if $\|\mathcal{H}\| < 1$.

Proof. Let $\mathcal{M}_r$ be the GLM transform that corresponds to the Hankel operator $r\mathcal{H}$. In this case directly from (4.2) we have
\[
\mathcal{M}_r^* U^* \begin{bmatrix} I & r\mathcal{H}^* \\ r\mathcal{H} & I \end{bmatrix} U \mathcal{M}_r = I.
\]

We rewrite this into the form
\[
U^* \begin{bmatrix} I & r\mathcal{H}^* \\ r\mathcal{H} & I \end{bmatrix}^{-1} U = \mathcal{M}_r \mathcal{M}_r^*.
\]

We can pass here to the limit since only a finite number of elements of $\mathcal{M}_r$ are involved in the entry of the product (recall it is a lower triangular matrix) and $(\mathcal{M}_r)^k \to \mathcal{M}_r^k$ for fixed $k$ and $l$.

Definition 4.6. We say that $A$ belongs to the Helson-Szegő class, $A \in \text{HS}$ if the matrix of the GLM transform generates a bounded operator in $l^2(\mathbb{Z}_+)$, $\|\mathcal{M}\| < \infty$.

Proposition 4.7. The GLM transformation is a bounded operator in $l^2(\mathbb{Z}_+)$ if and only if $\sigma$ is absolutely continuous and the Riesz projections $P_\pm$ are bounded operators in $L^2_{w^{-1}} dm$.

Proof. It was proved in Proposition 4.2 that $\sigma$ is absolutely continuous and $M_s(D) = \tilde{M}_s$. Let
\[
\mathcal{F} = (F_1, F_2) = \sum \tilde{f}_k e_k, \quad \tilde{f} \in l^2(\mathbb{Z}_+).
\]

Then
\[
\langle \mathcal{M}\tilde{f}, \mathcal{M}\tilde{f} \rangle \leq C(\tilde{f}, \tilde{f})
\]
means
\[ \|F_1\|^2 + \|F_2\|^2 \leq C\|\mathcal{F}\|_s^2, \quad (4.8) \]
where in the LHS we have the standard \(L^2\) norm.

On the other hand, according to definition (2.4)
\[ F_1 = \frac{1}{D} P_+ g, \quad F_2 = -\frac{a-1}{D_s} P_- g, \quad g = w f. \]

We put these in (4.8). Due to \(\|\mathcal{F}\|_s^2 = \|f\|_{L_{wdm}^2}^2\), we get
\[ \langle w^{-1} P_+ g, P_+ g \rangle + \langle w^{-1} P_- g, P_- g \rangle \leq C\langle w f, f \rangle = C\langle w^{-1} g, g \rangle. \]

Recall that a weight \(w\) satisfies \(A_2\) (or Hunt-Muckenhoupt-Wheeden) condition if for all arcs \(I \subset \mathbb{T}\) the following supremum is finite
\[ \sup_I \langle w \rangle_I \langle w^{-1} \rangle_I < \infty, \quad (4.9) \]
where \(\langle w \rangle_I = \frac{1}{|I|} \int_I wdm\).

We combine the classical Helson-Szegő and Hunt-Muchenhoupt-Wheeden Theorems, see e.g. [19], with Proposition 4.7.

**Theorem 4.8.** The following are equivalent:

(i) \(\mathfrak{A} \in \text{HS}\),

(ii) \(w \in A_2\),

(iii) \(\|\mathcal{H}_s\| < 1, \ w^{-1} \in L^1\),

(iv) \(s \in \text{HS}\).

5 Golinskii-Ibragimov Theorem and Faddeev-Marchenko type scattering theorem

The following theorem was proved in [11], see also [20] and [16], where the general case \(B^{1/p}_p\) was considered.
Theorem 5.1. Let \( s(t) \) be an unimodular function of the class \( B_2^{1/2} \). Then there exists a unique representation

\[
s(t) = t^N e^{iv(t)}
\]

(5.1)

where \( v(t) \in B_2^{1/2} \).

The integer \( N \) in the representation (5.1) is called the index of \( s(t) \). It can be computed as the winding number of the harmonic extension \( s(rt) \) in the unit disk of the given function for \( r \) sufficiently close to 1.

Definition 5.2. We say that \( A \) belongs to the Golinskii-Ibragimov class, \( A \in GI \) if

\[
\sum k|a_k|^2 < \infty
\]

(5.2)

Theorem 5.3. A unimodular function \( s(t) \) is the scattering function for a unique \( A \in GI \) if and only if \( s(t) \in B_2^{1/2} \) and its index \( N = 0 \). Moreover \( GI \subset HS \).

Note, statements like NLFT belongs to a certain class if and only if LFT belongs to the same class is typical for the Faddeev–Marchenko theory, see e.g. [17, Theorem 3.3.3], or the newest results of this type [7].

Theorem 5.3 is the scattering counterpart of the (spectral) Golinskii-Ibragimov version of the Strong Szegő Limit Theorem (see [9] and [22]).

Theorem 5.4. A measure \( d\sigma = wdm + d\sigma_s \) is the spectral measure of \( A \in GI \) if and only if \( \log w(t) \in B_2^{1/2} \) and \( \sigma_s = 0 \).

Proof of Theorem 5.3. Let \( A \in GI \). By Theorem 5.4

\[
D = e^{\frac{u+iv}{2}}, \ u := \log w.
\]

Therefore

\[
s = -a_{-1}e^{iu} \in B_2^{1/2}.
\]

Conversely, let \( s = e^{iu} \in B_2^{1/2} \). We define

\[
a_{-1} = -e^{i\int vdm}, \ w = Ce^{-\bar{v}}, \ C > 0 : \int wdm = 1.
\]

Therefore \( A \) with the spectral data \((wdm, a_{-1})\) belongs to \( GI \).
To show that this is the only CMV matrix of the Szegő class $Sz_{a,c}$, which corresponds to the given scattering function we note that $P_-s \in B_{1/2}^2$ means precisely that $H_s^*H_s$ belongs to the trace class. Therefore we have the alternative: 1) $||H_s|| < 1$, or, 2) there exists $g \in H^2$, $g \neq 0$, such that

$$(I - H_s^*H_s)g = 0.$$ 

In the second case we have

$$||P_-sg|| = ||sg||^2 - ||P_-sg||^2 = 0.$$ 

Thus $h := sg \in H^2_-$, and we get

$$-a_{-1}Dg = \bar{D}h.$$ 

The LHS is in $H^1$ and the RHS has all nonnegative Fourier coefficients equal to zero. Therefore $Dg = 0$, but this contradicts to $g \neq 0$.

Thus $||H_s|| < 1$ and $w^{-1} = 1/Ce^v \in L^1$. By Theorem 4.8 $\mathcal{A} \in \text{HS}$ which guarantees uniqueness of the inversion problem.

**Remark 5.5.** To our best knowledge scattering for CMV matrices was not studied in the frame of the standard Faddeev-Marchenko approach, that is we cannot compare Theorem 5.3 with a ”traditional” one. But we can discuss certain conditions related to coefficients of Jacobi matrices. The first one is the classical scattering theory condition, see [6, 10] and also [27],

$$\sum n(|p_n - 1| + |q_n|) < \infty. \quad (5.3)$$

The second condition was obtained by E. Ryckman [21] who proved a counterpart of the Strong Szegő Theorem for Jacobi matrices. It corresponds to $\sum n|a_n|^2 < \infty$ in the CMV case and is of the form

$$\sum_{n=1}^{\infty} n\{|\sum_{k=n}^{\infty} (p_k - 1)|^2 + |\sum_{k=n}^{\infty} q_k|^2\} < \infty. \quad (5.4)$$

Evidently, (5.3) implies (5.4) (we discuss only the behavior of the $q_k$’s):

$$\sum_{n=1}^{\infty} n\sum_{k=n}^{\infty} |q_k|^2 \leq \sum_{n=1}^{\infty} \sum_{k=n}^{\infty} (\sum_{l=n}^{\infty} n|q_l|)|q_k| = \sum_{k=1}^{\infty} \sum_{n=1}^{k} (\sum_{l=n}^{\infty} n|q_l|)|q_k|$$

$$\leq \sum_{k=1}^{\infty} \sum_{n=1}^{k} (\sum_{l=1}^{\infty} l|q_l|)|q_k| = (\sum_{l=1}^{\infty} l|q_l|)(\sum_{k=1}^{\infty} k|q_k|).$$
Note that for a typical example $q_n = \frac{1}{n^\beta}$ both conditions (5.3), (5.4) require $\beta > 2$. However, for the oscillating sequence $q_n = (\frac{-1}{n})^\beta$ we have $\beta > 2$ for (5.3) and $\beta > 1$ for (5.4).

**Remark 5.6.** Note that for a typical $A_2$ weight $w(t) = |t - 1|^{2\gamma_1}|t + 1|^{2\gamma_2}$, $\gamma_k > -1/2$, the so called Jacobi (OPUC) Verblunsky coefficients are of the form

$$a_n = -\frac{\gamma_1 - (-1)^n\gamma_2}{n + 1 + \gamma_1 + \gamma_2},$$

which violates (5.2). That is, a characterization of the class HS in terms of the Verblunsky coefficients looks like an extremely interesting challenging problem.

## 6 Widom’s Formula

The so-called Widom Formula has a natural proof in the frame of the scattering theory. Roughly speaking it says, see (4.7) and (4.5),

$$\det \left[ I - H^*H \right]^{-1} = \prod_{n=0}^{\infty} |\mathcal{M}_n|^2 = \frac{1}{\rho_0^2\rho_1^4\rho_2^6\ldots}.$$  

(6.1)

Here the determinant has sense if $H$ is of the trace class. But, in fact, a stronger statement holds true.

**Theorem 6.1.** Let $H$ be of the Hilbert-Schmidt class. Then

$$\det(I - H^*H) = \rho_0^2\rho_1^4\rho_2^6\ldots$$  

(6.2)

To get (6.2) we prove a counterpart of the factorization formula (4.7) for the matrix $\lim_{r \to 1}(I - r^2H^*H)^{-1}$. Consider a subspace of $\tilde{M}_s$ consisting of the vectors of the form

$$\text{clos}\{F = (F, -H^*F) : F \in H^2\} \subset \tilde{M}_s.$$

In fact, $F$ belongs to this subspace if and only if $F$ belongs to

$$\tilde{M}_s^+ = \text{clos}\{F \in H^2 : \|F\|^2_s = \langle(I - H^*H)F, F\rangle\}.$$  

By the way, we have the orthogonal decomposition

$$\tilde{M}_s = \tilde{M}_s^+ \oplus H_2.$$
in the following sense

\[ F = (F, -HF) \oplus (0, G), \quad F \in \tilde{M}_s^+, \ G \in H^2. \]

Also,

\[ \tilde{K}_0 = ((\tilde{K}_0^s)_1, -\mathcal{H}(\tilde{K}_0^s)_1), \]

and therefore \((\tilde{K}_0^s)_1 \in \tilde{M}_s^+\) is the reproducing kernel in this subspace.

Similar to Theorem 4.3 we have

**Theorem 6.2.** Let \( \mathfrak{A} \in Sz^{reg} \). The system of vectors

\[ f_n = \frac{t^n(\tilde{K}_{st}^n)_1}{\|\tilde{K}_{st}^n\|} \]

forms an orthonormal basis in \( \tilde{M}_s^+ \).

Similar to (4.2) we define the (lower–triangular) matrix

\[ [f_0 \ f_1 \ f_2 \ \ldots] = [1 \ z \ z^2 \ \ldots] \mathcal{L}. \]

In this case, similar to (4.7), we have

\[ \lim_{r \uparrow 1} (I - r^2 H*H)^{-1} = \mathcal{L}\mathcal{L}^*. \]

Finally, we note that

\[ \langle (I - r^2 H_{st2n+1} H^*_{st2n+1})^{-1} \tilde{t}, \tilde{t} \rangle = \langle (I - r^2 H^*_{st2n+1} H_{st2n+1})^{-1} 1, 1 \rangle \]

and therefore, by (4.5), (6.3) and (6.4), we have

\[ \mathcal{L}^n_n = \frac{1}{\rho_n \rho_{n+1} \ldots} \]

for both even and odd \( n \)’s.

**Proof of Theorem 6.1.** Since \( \|H\| < 1 \) and \( \text{tr}(H^*H) < \infty \) we have

\[ (I - H^*H)^{-1} = I + \Delta, \]

where \( \Delta \geq 0, \text{tr}\Delta < \infty \). Let \( \Delta^{(n)} \) be the initial \( n \times n \) block of the matrix \( \Delta \). Then

\[ \det(I + \Delta) = \lim_{n \to \infty} \det(I + \Delta^{(n)}). \]

Due to the triangular factorization (6.5)

\[ \det(I + \Delta^{(n)}) = \prod_{k=0}^{n-1} |\mathcal{L}_k^k|^2, \]

where \( \mathcal{L}_k^k \) is given by (6.6). \( \square \)
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