Generation of the Complete Four-dimensional Bell Basis

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Here we demonstrate the experimental generation of a complete basis of four-dimensional Bell-like entangled states. Our experiment constitutes the first demonstration of a complete Bell basis beyond qubits. In the case of two-dimensional polarization entanglement, it is well known that one can rotate between all four Bell states with an half-wave plate (which performs a Pauli-X transformation) and quarter-wave plate (performing a Pauli-Z transformation). We generate all sixteen Bell states by generalizing this method to a higher-dimensional space.

Our states consist of two photons entangled in their orbital angular momentum (OAM) [20, 21]. The sixteen orthogonal states in this basis are created by applying high-dimensional generalizations of Pauli gates on an initial OAM-entangled state that is produced via spontaneous parametric down-conversion. A four-dimensional X-gate is applied to one photon of the entangled pair, which corresponds to a cyclic transformation between the four basis states [22,24]. The second photon sees a four-dimensional Z-gate, which imparts a mode-dependent phase shift on the photon. We quantify the quality of our generated states by measuring their overlap with ideal states from a four-dimensional Bell basis, and verify the presence of four-dimensional entanglement by measuring an appropriate entanglement witness.

Technique – The D-dimensional Bell basis of a bipartite system AB, as generalized in the original teleportation paper by Bennett et al [15], can be written in the form

$$|\psi\rangle_{AB}^{mn} = \frac{1}{\sqrt{D}} \sum_{k=0}^{D-1} e^{i \frac{\pi}{2} nk} |k\rangle_A |k \oplus m\rangle_B$$

where \( k \oplus m \equiv (k + m) \mod D \). For \( D = 2 \), this reduces to the four well-known maximally en-
The photons are produced in the state \( |\Psi\rangle \) where the photons are generated via a frequency-degenerate type-II spontaneous parametric down-conversion (SPDC) process in a periodically poled Potassium Titanyl Phosphate (ppKTP) crystal. Cyclically symmetric and six symmetric states, while the remaining eight states are neither symmetric nor antisymmetric. In the four-dimensional case, equation (1) involves sixteen orthogonal Bell states that can be categorized into four distinct groups, as shown in Figure 1a. The four states in each group are labeled by the variable \( n = 0, 1, 2, 3 \), which defines the phase relationships between the probability amplitudes. As defined in Eq. 1, the four-dimensional Bell basis contains two antisymmetric and six symmetric states, while the remaining eight states are neither symmetric nor antisymmetric.

In the first set of states \( \psi_n \) in Figure 1a, photons A and B share the same state, while the relative phase between the probability amplitudes varies according to \( n \). The three other sets of states are obtained by performing specific transformations on the first group. To obtain the second group \( \psi_1, \psi_2, \psi_3 \), photon B is transformed by a clockwise mode transformation \( X(-2 \rightarrow -1 \rightarrow 0 \rightarrow 1 \rightarrow -2) \). For the groups \( \psi_2, \psi_3 \), the state of photon B is transformed by an \( X^\dagger \) transformation \((-2 \rightarrow 1 \rightarrow 0 \rightarrow -1 \rightarrow -2)\), respectively. To transform among states within each group, photon A undergoes a mode-dependent phase transformation. In this manner, all sixteen states in the four-dimensional Bell basis are obtained (Fig. 1).

**Experiment** — In Fig. 2 photons entangled in orbital angular momentum (OAM) are generated via a frequency-degenerate type-II spontaneous parametric down-conversion (SPDC) process in a periodically poled Potassium Titanyl Phosphate (ppKTP) crystal. The photons are produced in the state

\[
|\Psi\rangle = \sum_{\ell=-\infty}^{+\infty} c_\ell |\ell\rangle_A |\ell\rangle_B,
\]

where \(|\ell\rangle\) represents a photon carrying an OAM of \( \ell \hbar \) and \( c_\ell \) is a complex probability amplitude. For the purposes of our experiment, we use a four-dimensional subset of this state consisting of OAM mode values \( \ell \) varying from \(-2\) to \(1\). One photon is vertically polarized and thus reflected by the polarizing beam splitter (PBS) into path A. The other photon is vertically polarized and therefore transmitted at the PBS into path B. The reflection at the PBS flips the sign of the OAM mode (\(|\ell\rangle \rightarrow |-\ell\rangle\)). This transforms the entangled photons into the first state of our basis \(|\Psi_\ell\rangle = (|\ell\rangle - 2, -2 + |\ell\rangle, -1, -1 + |\ell\rangle, 0 + |\ell\rangle + |1, 1\rangle) / 2\). In order to create the remaining three states in the first group, we apply a mode-dependent phase transformation with a Dove prism in arm A. In general, a Dove prism oriented at an angle \( \alpha \) introduces a phase \(|\ell\rangle \rightarrow \exp(i2\ell\alpha) |\ell\rangle\) that depends on the OAM value \( \ell \) of the incoming photon and the rotation angle \( \alpha \) of the prism. The effect of this element on the state can be written as

\[
|\Psi\rangle \xrightarrow{DP(\alpha)} |\Psi'\rangle = \frac{1}{2} \sum_{\ell \in \{-2,-1,0,1\}} e^{i2\ell\alpha} |\ell\rangle_A |\ell\rangle_B
\]

By orienting the Dove prism at different angles (\( \alpha = 0, \pi/4, \pi/2, 3\pi/4 \)), we obtain all four states in one group. A four-fold clockwise cyclic transformation of OAM modes was recently developed through the use of the computer algorithm MELVIN [22] and implemented with coherent light as well as single photons [23, 24]. The principle idea of cyclic transformations is to split even and odd OAM modes into two different paths and manipulate them independently. Finally, the two paths are recombined coherently. In our experiment, we implement three such cyclic transformations \((X,X^2,X^3)\) at the single-photon level.

As shown in Fig. 3 we use a double-path Sagnac interferometer containing two Dove-prisms (DP) [25] to split even and odd OAM modes into two different paths (green frame). In the path for even OAM modes, different OAM manipulations are performed which are necessary for the three cyclic transformations. The two paths are probabilistically recombined with a beam-splitter (BS) which forms a Mach-Zehnder (MZ) interferometer. In principle the two
paths can be recombined with another parity sorter in a deterministic way. To perform the $X$ transformation ($-2 \rightarrow -1 \rightarrow 0 \rightarrow 1 \rightarrow -2$) a spiral phase plate (SPP) adds an OAM quantum of $+1$ before the OAM sorter. After the sorter, one of the paths of the MZ interferometer undergoes an additional reflection. For the $X^2$ transformation ($-2 \equiv 0, -1 \equiv 1$) an SPP is inserted within the MZ interferometer adding an OAM quanta of $+2$ for even OAM modes only. An additional reflection at the end completes the $X^2$ transformation. In the case of the $X^3$ transformation ($-2 \rightarrow 1 \rightarrow 0 \rightarrow -1 \rightarrow -2$), an additional reflection occurs within the MZ interferometer. Additionally, an SPP subtracting one OAM quantum the paths are recombined. Together, the three cyclic transformations on photon B and the three phase transformations on photon A allow us to obtain all four groups of states in the four-dimensional Bell basis.

The detection system consists of a spatial light modulator (SLM), a single-mode fiber (SMF) and a single-photon detector. The SLM is used to flatten the phase of an incoming photon, transforming it into an $\ell = 0$ mode that efficiently couples to the SMF [20]. In this manner, the OAM content of single photons can be measured for specific modes or mode superpositions.

Results – The sixteen experimentally generated Bell states are analyzed using two different quantitative measures: their overlap with the theoretically expected Bell states, and a witness of four-dimensional entanglement. The overlap allows us to estimate how close we are in our experiment to the ideal Bell basis, and the witness allows us to verify the presence of genuine four-dimensional entanglement in our generated states. Fig. 3 shows the overlap of states within each of the four groups $\psi_{1n} - \psi_{3n}$. The overlap between generated states and ideal Bell states. The overlap of 16 states is separated into four subgraphs as shown in the figure. The $x$ and $y$ axis in the figure represent the experimental and ideal Bell states, respectively. The diagonal elements exhibit exactly the fidelity $F_{exp}$ between the experimental states and their corresponding states.
The fidelity of 16 states is classified into four groups \( (|\psi_{0n}\rangle, |\psi_{1n}\rangle, |\psi_{2n}\rangle, |\psi_{3n}\rangle) \) in the figure. The error is calculated using Monte Carlo simulation, and the red line denotes the theoretical bound for four-dimensional witness.

The measured fidelity witnesses \( F_{\text{wit}} \) for all sixteen states are plotted in Fig. 4. Each of the sixteen Bell states individually exceeds the bound of 0.75 by at least three standard deviations, and is thus certified to be four-dimensionally entangled. The error in the fidelity is calculated by propagating the Poissonian error in the photon-counting rates via a Monte Carlo simulation.

High-dimensional quantum dense coding – By replacing the spiral phase plates in our experiment with computer generated holograms implemented on SLMs, our technique can be extended for rapidly switching on-demand between all sixteen states in the four-dimensional Bell basis. This would constitute the first step in a high-dimensional quantum dense coding protocol [14]; by implementing both phase and cyclic transformations on one photon, Bob can encode 4 bits of information using the two-photon four-dimensional Bell basis. The subsequent step where Alice must distinguish between all sixteen Bell states in order to decode this information provides a significant challenge. It has been shown that it is impossible to unambiguously discriminate a single high-dimensional Bell state from the others with just linear optics [30]. However, it is possible to sort 16 Bell states into 7 classes of states that can be distinguished with a linear optical setup, as was recently demonstrated with hyperentangled time-polarization states [31]. Our experimental technique solely involves the photonic spatial degree of freedom, and can thus be readily combined with well-developed techniques for polarization [32] and time-bin encoding [33], allowing for a significant increase in Hilbert space dimensionality.

Conclusion – Here we have shown the application of recently developed high-dimensional quantum gates to photonic quantum entanglement. By doing so, we were able to create very general high-dimensional quantum states, for which no method of creation was known. The quantum states we created are a high-dimensional generalisation of the Bell-basis, arguably the most commonly used set of entangled quantum states in two dimensions. Access to the complete high-dimensional Bell basis allows for the exploration of strong non-classical correlations and their application in quantum information protocols such as quantum dense-coding. Furthermore, our technique can be used for the generation of complete sets of high-dimensional Greenberger-Horne-Zeilinger states [6].

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APPENDIX

Deviations from the ideal states

Deviations from the ideal states can be explained by three main effects: Non-equal distribution of modes form the original state, cross-talk between modes, and loss of coherence in the interferometers: The spiral bandwidth of the OAM distribution is not flat, thus the state created in the down-conversion process is not maximally entangled. In our experiment we measure an initial state of $\ket{\psi} = \alpha\ket{0,0} + \beta\ket{1,1} + \beta - 1, -1) + \gamma\ket{2,2}$ with $\alpha/\beta = 0.69$ and $\alpha/\gamma = 0.45$. Thus the maximum possible fidelity with a maximally entangled Bell state is limited by 93 percent. This inherent unbalancing of the created modes can be overcome with a procrustean filtering technique \[2,34\]. Another issue that lowers the fidelity is the cross-talk counts between different modes, and we find that the cross-talk limits the fidelity to 91 percent. These impurities mainly stem from misalignments of the OAM sorter and the Mach-Zehnder interferometer, which can be reduced by active stabilisation. The cross-talk limits the fidelity to 91 percent. These impurities mainly stem from misalignments of the OAM sorter and the Mach-Zehnder interferometer, which can be reduced by active stabilisation. The coherence of the off-diagonal elements in this experiment has been measured to be 0.97(6) percent on average. Taking these three limiting factors into account the expected fidelity witness values are given by $F_{\text{wit}} = 0.81(5)$. Hence, the observed average fidelity witness of $F_{\text{wit}} = 0.808 \pm 0.016$ is mainly due to unbalancing and cross-talk in the diagonal elements.

Overlap between states

Here we show the data from which Figure 3 has been created. It shows the overlap between different states from the same class, with the same OAM values but different phases.

| $|\psi_{0,0}$ | $|\psi_{0,1}$ | $|\psi_{0,2}$ | $|\psi_{0,3}$ |
|---|---|---|---|
| $|\psi_{0,0}$ | 0.810 | 0.063 | 0.011 | 0.048 |
| $|\psi_{0,1}$ | 0.024 | 0.823 | 0.041 | 0.002 |
| $|\psi_{0,2}$ | 0.015 | 0.060 | 0.818 | 0.049 |
| $|\psi_{0,3}$ | 0.027 | 0.006 | 0.032 | 0.835 |

| $|\psi_{1,0}$ | $|\psi_{1,1}$ | $|\psi_{1,2}$ | $|\psi_{1,3}$ |
|---|---|---|---|
| $|\psi_{1,0}$ | 0.762 | 0.004 | 0.058 | 0.046 |
| $|\psi_{1,1}$ | 0.053 | 0.748 | 0.002 | 0.074 |
| $|\psi_{1,2}$ | 0.030 | 0.053 | 0.780 | 0.009 |
| $|\psi_{1,3}$ | 0.005 | 0.025 | 0.044 | 0.811 |

From there, the average expected fidelity can be calculated to be $F_{\text{exp}} = 0.78 \pm 0.03$.

Entanglement Witness

First, we calculate the overlap $F_{\text{wit}}$ between our state and a $d$-dimensional maximally entangled target state. Then, we compute a $d$-dimensional entanglement bound $B(d) = \sum_{i=0}^{d-1} \lambda_i^2$, which is the sum of the squares of all but the smallest Schmidt coefficient of the target state. If the overlap $F_{\text{wit}}$ exceeds the bound for a $d$-dimensional entangled state, then the measurement data can only be explained with a $(d+1)$-dimensionally entangled state.

Combining the beams probabilistically

In our experiments, we combine the two photon paths for photon B probabilistically. The beam splitter in Figure 2 is implemented via a half-wave plate at $45^\circ$ in the horizontal arm (after which the polarisation is diagonal), and polarising beam splitter. In order to erase the which-path information, we could use a polariser at $45^\circ$. However, we use half-wave plate at $45^\circ$ which rotates horizontal to diagonal, and vertical to anti-diagonal; and afterwards use the SLM as an effective polariser as the SLM only works with horizontally polarised light.