POINCARÉ DUALITY ISOMORPHISMS IN TENSOR CATEGORIES

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Abstract. If for a vector space $V$ of dimension $g$ over a characteristic zero field we denote by $\wedge^i V$ its alternating powers, and by $V^\vee$ its linear dual, then there are natural Poincaré isomorphisms:

$$\wedge^i V^\vee \cong \wedge^{g-i} V.$$

We describe an analogous result for objects in rigid pseudo-abelian $\mathbb{Q}$-linear ACU tensor categories.

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1. Introduction

Let $V$ be a vector space of finite dimension $g$ over a characteristic zero field, let $I$ be the field of scalars, viewed as a vector space, and consider the alternating algebra $\wedge \cdot V$. Then the internal multiplication morphism defined by the formula

$$\iota_1 \left( x \wedge \cdots \wedge \omega_j \right) := \sum_{k=1}^j (-1)^j \langle x, \omega_k \rangle \omega_1 \wedge \cdots \wedge \hat{\omega}_k \wedge \cdots \wedge \omega_j$$

gives a map

$$\iota_1 : V \to Hom \left( \wedge^j V^\vee, \wedge^{j-1} V^\vee \right)$$

valued in the space of degree $-1$ anti-derivation. Since $\iota_1 \left( x \right)^2 = 0$, by the universal property of the alternating algebra the morphism $\iota_1$ extends to a morphism of algebras

$$\iota : \wedge V \to Hom \left( \wedge^j V^\vee, \wedge^{j-1} V^\vee \right)^{op},$$

where $(\cdot)^{op}$ means the opposite algebra, such that $\iota \left( x \right) : \wedge^j V^\vee \to \wedge^{j-1} V^\vee$ if $x \in \wedge^i V$ and $j \geq i$ (and it is zero otherwise). In order to match with the notations employed in the paper, it will be convenient to define, for every $j \geq i$:

$$\iota_{i,j} \left( x \right) := \frac{(j-i)!}{j!} \iota \left( x \right)_{\wedge^j V^\vee} : \wedge^j V^\vee \to \wedge^{j-1} V^\vee,$$ if $x \in \wedge^i V$.

This gives morphisms $\iota_{i,j} : \wedge^i V \to Hom \left( \wedge^j V^\vee, \wedge^{j-1} V^\vee \right)$ with the following property. If we identify $\wedge V^\vee \simeq (\wedge V)^\vee$ by means of

$$ev_{V,a}^i : \wedge^i V^\vee \otimes \wedge^i V \to I$$

obtained by the natural inclusions $\wedge^i (\cdot) \to \otimes^i (\cdot)$ followed by the perfect pairing

$$ev_V^i \left( \omega_1 \otimes \cdots \otimes \omega_i, x_1 \otimes \cdots x_i \right) := \prod_{k=1}^i \langle \omega_k, x_k \rangle,$$

then

$$ev_{V,a}^j \left( \omega_j, x_i \wedge x_{j-i} \right) = ev_{V,a}^{j-i} \left( \iota_{i,j} \left( x_i \right), \omega_j \right) \text{ for } x_i \in \wedge^i V, \ x_{j-i} \in \wedge^{j-i} V^\vee \text{ and } \omega_j \in \wedge^j V^\vee,$$

meaning that $\iota_{i,j} \left( x_i \right) : \wedge^j V^\vee \to \wedge^{j-1} V^\vee$ is dual to the multiplication map $x_i \wedge \cdot : \wedge^{j-i} V \to \wedge^j V$.  

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These internal multiplications morphisms allow for the definition of the Poincaré morphism
\[ D^{i,g} : \wedge^i V \to Hom (\wedge^g V^\vee, \wedge^{g-i} V^\vee) \simeq \wedge^{g-i} V^\vee \]
and using reflexivity after dualizing yields
\[ D_{i,g} : \wedge^i V \to Hom (\wedge^g V, \wedge^{g-i} V) \simeq \wedge^{g-i} V. \]
As it well known one has
\[ D_{g-i,g} \circ D^{i,g} = (-1)^{(g-i)} \left( \frac{g}{g-i} \right)^{-1} \quad \text{and} \quad D^{i,g} \circ D_{g-i,g} = (-1)^{(g-i)} \left( \frac{g}{g-i} \right)^{-1} \quad (3) \]

If the category of finite dimensional vector spaces is replaced by a more general neutral tannakian category, the fibre functor allows to extend this result to this category due to (3) and the existence of a faithful exact linear functor valued in the category of vector spaces, once the appropriate definition of the Poincaré morphism is given in such a way that it is preserved by tensor functors. The aim of this paper is to generalize this result to rigid pseudo-abelian and \( \mathbb{Q} \)-linear \( ACU \) tensor categories, with the aim of applications to Chow motives, and prove the analogue statement for the symmetric algebras \( \vee V \).

Suppose indeed that \( V \) is a supervector space of odd degree. Then the same formalism applies, replacing the alternating algebra with the symmetric algebra: the reason is that, by definition, the commutativity constraint \( \tau_{V,W} : V \otimes W \to W \otimes V \) in the category of supervector spaces is given by \( \tau_{V,W} (x \otimes y) = - (y \otimes x) \) if \( V \) and \( W \) have odd degree and, hence, the symmetrizer operates as an anti-symmetrizer on the underlying vector spaces.

The viewpoint taken in this paper is to use as the defining property of the internal multiplication morphisms. Suppose that \( C \) is a rigid pseudo-abelian and \( \mathbb{Q} \)-linear \( ACU \) tensor category with identity object \( \mathbb{I} \) and that we are given \( V \in C \) of rank \( r \in End (\mathbb{I}) \). If \( A \) denotes one of the alternating or symmetric algebras, the data of the multiplication morphisms \( \varphi_{i,j} : A_i \otimes A_j \to A_{i+j} \) is equivalent to that of the associated morphisms \( f_{i,j} : A_i \to Hom (A_{j-i}, A_{j+i}) \). When \( j \geq i \), we may consider the composite
\[ \iota_{i,j} : A_i \xrightarrow{f_{i,j}} \hom (A_{j-i}, A_j) \xrightarrow{d} \hom (A_{j-i}^{\vee}, A_{j+i}) \]
where \( d : \hom (X,Y) \to \hom (Y^{\vee}, X^{\vee}) \) is the internal duality morphism as defined in \( \S 2 \). Next we define
\[ D^{i,j} : A_i \xrightarrow{\iota_{i,j}} \hom (A_{j-i}^{\vee}, A_j) \xrightarrow{\alpha^{-1}} A_{j-i}^{\vee} \otimes A_j^{\vee}, \]
where \( \alpha : \hom (X,Y) \to Y \otimes X \) is the canonical morphism. Working dually and employing the reflexivity one also gets
\[ D_{i,j} : A_i^{\vee} \to A_{j-i} \otimes A_j^{\vee} \]

We say that \( V \) has alternating (resp. symmetric) rank \( g \in \mathbb{N}_{\geq 1} \) if \( L := \wedge^g V \) (resp. \( L := \vee^g V \)) is invertible and if \( (r+i-g) \) (resp. \( (r+g-1) \)) is invertible in \( End (\mathbb{I}) \) for every \( 0 \leq i \leq g \). Here, for an integer \( k \geq 1 \),
\[ \binom{T}{k} := \frac{1}{k!} T(T-1) \ldots (T-k+1) \in \mathbb{Q}[T] \quad \text{and} \quad \binom{T}{0} = 1. \]
Then we compute, for every \( i \leq g \), the compositions
\[ A_i \xrightarrow{D^{i,g}} A_{g-i}^{\vee} \otimes L \xrightarrow{D_{g-i}} A_i^{\vee} \otimes L^{-1} \otimes L \simeq A_i^{\vee}, \]
\[ A_{g-i}^{\vee} \xrightarrow{D_{g-i}} A_i \otimes L^{-1} \xrightarrow{D_{g-i}} A_{g-i} \otimes L \otimes L^{-1} \simeq A_{g-i}^{\vee} \]
and we prove in Theorem 5.3 (3) (resp. Theorem 6.2 (3)) that, when \( A = \wedge V \) (resp. \( A = \vee V \)), they are equal to
\[
(-1)^{(g-i)} \left( \frac{g}{g-i} \right)^{-1} \left( \frac{r-i}{g-i} \right) \quad \text{(resp.} \quad \left( \frac{g}{g-i} \right)^{-1} \left( \frac{r+g-1}{g-i} \right) \quad \text{)}, \\
(-1)^{(g-i)} \left( \frac{g}{i} \right)^{-1} \left( \frac{r+i-g}{i} \right) \quad \text{(resp.} \quad \left( \frac{g}{i} \right)^{-1} \left( \frac{r+g-1}{i} \right) \quad \text{).} \quad (4)
\]
In particular, the multiplication maps \( \varphi_{1,g-i} : A_i \otimes A_{g-i} \to A_g \) are perfect pairings for every \( 0 \leq i \leq g \) (see Corollaries 5.6 and 6.3). We remark that the same constants obtained in (3) and, more generally, for odd degree supervector spaces, matches those in (1) when \( r = g \) in the alternating case and, respectively, \( r = -g \) in the symmetric case. We say in this case that \( V \) has strong alternating or symmetric rank in these cases.

Some remarks are in order about the range of applicability of our results. First of all we note that, in general, the alternating or the symmetric rank may be not uniquely determined. Suppose, however, that we know that there is a field \( K \) such that \( r \in K \subset \text{End}(\mathcal{I}) \) admitting an embedding \( \iota : K \hookrightarrow \mathbb{R} \). Then it follows from the formulas \( \text{rank} (\wedge^k V) = \binom{g}{k} \) and \( \text{rank} (\vee^k V) = \binom{r+k-1}{k} \) (see [AKh] 7.2.4 Proposition [or [DK] (7.1.2)]) that we have \( r \in \{-1,g\} \) (resp. \( r \in \{-g,1\} \)) when \( V \) has alternating (resp. symmetric) rank \( g \). In particular, when \( r > 0 \) (resp. \( r < 0 \)) with respect to the ordering induced by \( \iota \), we deduce that \( r = g \) (resp. \( r = -g \)), so that \( g \) is a uniquely determined and \( V \) has strong alternating (resp. symmetric) rank \( g = r \) (resp. \( g = -r \)).

We recall that \( V \) is Kimura positive (resp. negative) when \( \wedge^{N+1} V = 0 \) (resp. \( \vee^{N+1} V = 0 \)) for \( N \geq 0 \) large enough. In this case, the formula \( \text{rank} (\wedge^k V) = \binom{g}{k} \) (resp. \( \text{rank} (\vee^k V) = \binom{r+k-1}{k} \)) implies that \( r \in \mathbb{Z}_{\geq 0} \) (resp. \( r \in \mathbb{Z}_{<0} \)) and the smallest integer \( N \) such that \( \wedge^{N+1} V = 0 \) (resp. \( \vee^{N+1} V = 0 \)) is \( r \) (resp. \( -r \)). Furthermore, it is known that in this case, when \( \text{End}(\mathcal{I}) \) does not have non-trivial idempotents, then \( \wedge^N V \) (resp. \( \vee^N V \)) is invertible (see [Ki1], 11.2 Lemma]: in other words \( V \) has strong alternating (resp. symmetric) rank \( g = r \) (resp. \( g = -r \)).

In particular, our results applies to the motives \( V = h^1(X) \) attached to abelian schemes \( X = A \) (see [DM] and [Kl]) or a smooth complete curve \( X = C \) over a field (see [Kl]), which are known to be Kimura negative, while products of an even number of such motives are Kimura positive (see [Kl] for applications of this notion to the product of two curves). In the subsequent paper [MS] we will apply these results in order to get a motive whose realizations affords two copies of odd weight modular forms on indefinite quaternion algebras. When the quaternion algebra is split, the construction due to Scholl refines and gives a motive whose realizations affords modular forms of both even or odd weight (see [Sc]). Working over an indefinite division quaternion algebra and employing ideas which goes back to [JL], a motive of even weight modular forms has been constructed in [IS] as the kernel of an appropriate Laplace operators. The results of this paper will be used in [MS] in order to show the existence of kernels of Dirac operators which are square-roots of these Laplace operators; the idea of constructing canonical models for the various incarnations of two copies of odd weight modular forms from square roots of the Laplace operators is due, once again, to Jordan and Livné. However, even for these realizations, it is not possible to canonically split them in a single copy: this is possible only including a splitting field for the quaternion algebra in the coefficients, but the resulting splitting depends on the choice of an identification of the base changed algebra with the split quaternion algebra.

Finally, we remark that the perfectness of the multiplication maps gives a Poincaré duality

\[ A_i \simeq \text{hom}(A_{g-i}, A_g) \simeq A_g \otimes A_{g-i}^\vee. \]  

Indeed, when \( V = h^1(A) \) for an abelian scheme \( A \) of dimension \( d \), we have that \( h^{2d}(A) \simeq \mathcal{I}(-d) \) is invertible and then it is known that

\[ \vee^i h^1(A) \simeq h^1(A) \simeq h^{2d-i}(A)^\vee(-d) \simeq h^{2d}(A) \otimes h^{2d-i}(A)^\vee \simeq \vee^{2i} h^1(A) \otimes \vee^{2d-i} h^1(A)^\vee, \]

where the canonical identifications \( h^k(A) \simeq \vee^k h^1(A) \) are proved in [Ki1] Remarks (3.1.2) (i)], while \( h^i(A) \simeq h^{2d-i}(A)^\vee(-d) \) is proved in [DM] (see also [Ki1] Remarks (3.1.2) (ii)]. This gives a refinement of the motivic Poincaré duality which states that, for a smooth projective scheme \( X \) of relative dimension \( d \), we have

\[ h(X) \simeq h(X)^\vee(-d). \]  

Applying (5) to the motive \( h^1(C) \) of a smooth complete curve over a field of genus \( e \), which is Kimura negative of Kimura rank \( 2e \) with \( \vee^{2e} h^1(A) \simeq \mathcal{I}(-e) \) (by [Kl] Theorem 4.2 and Remark 4.5), one gets

\[ \vee^i h^1(C) \simeq \vee^{2e} h^1(C) \otimes \vee^{2e-i} h^1(C)^\vee \simeq \vee^{2e-i} h^1(C)^\vee(-d) \]

which however, in this case, is not a refinement of (6). We also mention the fact that it is conjectured in [Kl] Conjecture 7.1 that Chow motives should be Kimura finite, i.e. they should be a direct sum of a Kimura positive and a Kimura negative motive.
The paper is organized as follows. In §2 we develop a general formalism of internal multiplication morphisms attached to a pairing \(\varphi : S \otimes X \to Y\) to be applied to the multiplication morphisms in some algebra object. In §3 we prove the prototype of our Poincaré isomorphism, which only depends on the data of \(\varphi_{S,X} : S \otimes X \to Y\), \(\varphi_{X,S} : X \otimes S \to Y\), \(\varphi_{X,Y} : S^\vee \otimes X^\vee \to Y^\vee\), \(\varphi_{X,Y,S} : X^\vee \otimes S^\vee \to Y^\vee\) subject to an appropriate commutativity constraint and one involving how the internal multiplications are related with respect to the Casimir elements: no associativity constraint is needed for these results. In §4 we apply the above results to the case of algebra objects and include results from §2 in order to get what is the effect of the associativity constraint on internal multiplication morphisms (see Proposition 4.1 and Corollary 4.2). We also make explicit the identifications of the associativity constraint on internal multiplication morphisms (see Proposition 4.1 and Corollary 4.2). These further results will be crucial for the applications given in [MS].

2. Linear algebra in tensor categories

In the first part of this paper we let \(\mathcal{C}\) be an \(ACU\) additive \(\otimes\)-biadditive category with unit object \((I, l, r)\) and internal homs. We will usually not write the associativity or unitary object constraints explicitly, while the commutativity constraint will be usually denoted by \(\tau\). We will usually not write the associativity or unitary object constraints explicitly, while the commutativity constraint will be usually denoted by \(\tau\). In Theorem 6.2 and Proposition 6.4. These further results will be crucial for the applications given in [MS].

To fix notations we recall that the existence of internal homs means that, if \(X, Y \in \mathcal{C}\) there is hom \((X, Y) \in \mathcal{C}\) such that

\[
\text{Hom}(S, \text{hom}(X, Y)) = \text{Hom}(S \otimes X, Y)
\]

holds as contravariant functors on \(\mathcal{C}\). Taking \(S = \text{hom}(X, Y)\) and \(1_{\text{hom}(X, Y)}\) yields

\[
ev_{X,Y} : \text{hom}(X, Y) \otimes X \to Y
\]

such that \(f : S \to \text{hom}(X, Y)\) uniquely corresponds to

\[
\varphi_f : S \otimes X \xrightarrow{f \otimes 1_X} \text{hom}(X, Y) \otimes X \xrightarrow{\text{ev}_{X,Y}} Y
\]

under the identification 7. The opposite evaluation is the composite

\[
ev^\tau_{X,Y} : X \otimes \text{hom}(X, Y) \xrightarrow{\tau \otimes \text{hom}(X, Y)} \text{hom}(X, Y) \otimes X \xrightarrow{\text{ev}_{X,Y}} Y
\]

and \((\text{hom}(X, Y), ev^\tau_{X,Y})\) represents Hom \((X \otimes S, Y)\). Then \((\text{hom}(X, Y), ev_{X,Y})\), uniquely determined up to a unique isomorphism, is called an internal hom pair for \((X, Y)\) and, when \(Y = I\), we write:

\[
(\text{hom}(X, Y), ev_{X,Y}) = (X^\vee, ev_X), \quad (\text{hom}(X, Y), ev^\tau_{X,Y}) = (X^\vee, ev^\tau_X)
\]

and we call \((X^\vee, ev_X)\) a dual pair for \(X\).

We remark that \((X, Y)\) is a bifunctor, contravariant in the first variable and covariant in the second variable as follows. If \(f : X_2 \to X_1\) and \(g : Y_1 \to Y_2\) we define

\[
\text{hom}(f, g) : \text{hom}(X_1, Y_1) \to \text{hom}(X_2, Y_2)
\]

as the unique morphism making the following diagram commutative:

\[
\begin{array}{ccc}
\text{hom}(X_1, Y_1) \otimes X_2 & \xrightarrow{\text{Log}(X_1, Y_1) \otimes f} & \text{hom}(X_1, Y_1) \otimes X_1 \\
\text{hom}(f, g) \otimes 1_X & \downarrow & \downarrow \text{deg}_{X_1, Y_1} \\
\text{hom}(X_2, Y_2) \otimes X_2 & \xrightarrow{\text{ev}_{X_2, Y_2}} & Y_2.
\end{array}
\]
Note that we have $\text{Hom}(1_S, \text{hom}(f, g)) = \text{Hom}(1_S \otimes f, g)$ via Yoneda’s embedding and (7), from which the functoriality of hom follows.

It follows from this functorial description that hom is biadditive. More explicitly, suppose that we have given biproduct decompositions $X = X^+ \oplus X^-$ and $Y = Y^+ \oplus Y^-$ which are given by injective morphisms $i_X^+: X^+ \to X$, $i_X^-: Y^+ \to Y$, surjective morphisms $p_X^+: X \to X^+$, $p_Y^+: Y \to Y^+$ and associated idempotents $e_X^+: X \to X$, $e_Y^+: Y \to Y$. The functorial description yields

$$\text{hom}(X, Y) = \text{hom}(X^+ +, Y^+) + \text{hom}(X^+, Y^-) + \text{hom}(X^-, Y^+) + \text{hom}(X^-, Y^-)$$

associated to the decomposition of $\text{Hom}(S \otimes X, Y)$. For $\varepsilon, \eta \in \{ \pm \}$, writing $i_{\text{hom}(X^\varepsilon, Y^\eta)}: \text{hom}(X^\varepsilon, Y^\eta) \to \text{hom}(X, Y)$, $p_{\text{hom}(X^\varepsilon, Y^\eta)}: \text{hom}(X, Y) \to \text{hom}(X^\varepsilon, Y^\eta)$ and $e_{\text{hom}(X^\varepsilon, Y^\eta)}: \text{hom}(X, Y) \to \text{hom}(X, Y)$ for the injective and surjective morphisms and the idempotents arising from the decomposition of $\text{Hom}(S \otimes X, Y)$ and Yoneda’s lemma, one checks

$$i_{\text{hom}(X^\varepsilon, Y^\eta)} = \text{hom}(p_{X^\varepsilon}^\eta, i_X^\varepsilon), \quad p_{\text{hom}(X^\varepsilon, Y^\eta)} = \text{hom}(i_X^\varepsilon, p_Y^{-\eta}) \quad \text{and} \quad e_{\text{hom}(X^\varepsilon, Y^\eta)} = \text{hom}(e_X^\varepsilon, e_Y^{-\eta})$$

as well as

$$\text{ev}_{X^\varepsilon, Y^\eta} = p_Y^{-\eta} \circ \text{ev}_{X, Y} \circ (i_{\text{hom}(X^\varepsilon, Y^\eta)} \otimes i_X^\varepsilon) = p_Y^{-\eta} \circ \text{ev}_{X, Y} \circ (\text{hom}(p_{X^\varepsilon}^\eta, i_X^\varepsilon) \otimes i_X^\varepsilon).$$

In particular, taking $f: X = X_2 \to X_1 = Y$ and $g = 1_I$ yields

$$f^{\vee} := Y^{\vee} \to X^{\vee}$$

and $X \leadsto X^{\vee}$ is a contravariant biadditive functor.

We proceed to define standard canonical morphisms. For a totally ordered finite set $I$, a family $(X_i, Y_i)_{i \in I}$ of objects $X_i, Y_i \in \mathcal{C}$ and a morphism $\varphi: \otimes_{i \in I} Y_i \to Y$, we may consider

$$\text{ev}_{X_i, Y_{i}} : (\otimes_{i \in I} \text{hom}(X_i, Y_i) \otimes (\otimes_{i \in I} X_i)) \tau_{i \in I}^{X_i Y_i} \otimes_{i \in I} (\text{hom}(X_i, Y_i) \otimes X_i) \otimes_{i \in I} Y_i \xrightarrow{\tau^{X_i Y_i}} Y,$$

where $\tau_{i \in I}^{X_i, Y_i}$ is obtained by appropriately switching the components. Then we may define

$$c_{X_i, Y_i} : \otimes_{i \in I} \text{hom}(X_i, Y_i) \to \text{hom}(X, Y)$$

as the unique morphism such that $\text{ev}_{X, Y} \circ (c_{X_i, Y_i} \otimes 1_X) = \text{ev}_{X_i, Y_i}^{X_i Y_i}$. When $I = \{1, \ldots, i\}$, $X_i = X$ for every $i$, $Y_i = \mathbb{I}$ for every $i$ and $\varphi: \otimes_{i \in I} \mathbb{I} \xrightarrow{\sim} \mathbb{I}$ is the canonical morphism we write $\tau_{X_i}^X := \tau_{X_i, Y_i}^X, \text{ev}_X := \text{ev}_{X_i, Y_i}^X$ and $c_{X_i, Y_i} := c_X$.

The morphisms

$$i_X: X \to X^{\vee \vee} \quad \text{and} \quad \alpha_{X, Y}: Y \otimes X^{\vee} \to \text{hom}(X, Y)$$

are defined, respectively, as the unique morphisms making the following diagrams commutative:

$$\begin{array}{ccc}
X \otimes X^{\vee} & \xrightarrow{i_X} & X^{\vee \vee} \\
\downarrow{\text{ev}_{X^{\vee \vee}}} & & \downarrow{\text{ev}_{X^{\vee \vee}}} \\
\text{hom}(X, Y) \otimes X & \xrightarrow{\text{ev}_{X, Y}} & Y \\
\end{array}$$

We may consider the morphism

$$\text{hom}(Y, Z) \otimes \text{hom}(X, Y) \otimes X \xrightarrow{1_{\text{hom}(Y, Z)} \otimes \text{ev}_{X, Y}} \text{hom}(Y, Z) \otimes Y \xrightarrow{\text{ev}_{Y, Z}} Z$$

and define the internal composition law

$$e = c_{X, Y, Z} = (\cdot) \circ (\cdot) : \text{hom}(Y, Z) \otimes \text{hom}(X, Y) \to \text{hom}(X, Z)$$

as the unique morphism such that

$$\text{ev}_{X, Z} \circ (c_{X, Y, Z} \otimes 1_X) = \text{ev}_{Y, Z} \circ (1_{\text{hom}(Y, Z)} \otimes \text{ev}_{X, Y}).$$

In symbols,

$$\tau_{X_i, Y_i}^X (((\otimes_{i \in I} f_i) \otimes (\otimes_{i \in I} x_i))) := \otimes_{i \in I} (f_i \otimes x_i).$$
The opposite internal composition law is defined as the composite
\[ c^{\tau}_{X,Y,Z} : \text{hom} (X, Y) \otimes \text{hom} (Y, Z) \xrightarrow{\tau_{\text{hom}(X,Y), \text{hom}(Y,Z)}} \text{hom} (Y, Z) \otimes \text{hom} (X, Y) \xrightarrow{c_{X,Y,Z}} \text{hom} (X, Z) \]

The following result is easily established.

**Lemma 2.1.** Suppose that we have given
\[ f : S \to \text{hom} (X, Y) \text{ and } g : T \to \text{hom} (Y, Z) \]
which correspond, under (7), to morphisms
\[ \varphi_f : S \otimes X \to Y \text{ and } \varphi_g : T \otimes Y \to Z. \]

Then
\[ c_{X,Y,Z} \circ (g \otimes f) : T \otimes S \xrightarrow{g \otimes f} \text{hom} (Y, Z) \otimes \text{hom} (X, Y) \xrightarrow{c_{X,Y,Z}} \text{hom} (X, Z) \]
corresponds, under (7), to the morphism
\[ \varphi_g \circ (1_T \otimes \varphi_f) : T \otimes S \otimes X \to Z. \]

In addition to the "external" duality morphism \( \text{Hom} (X, Y) \to \text{Hom} (Y \vee, X \vee) \), the category \( C \) is endowed with an internal duality morphism
\[ d_{X,Y} : \text{hom} (X, Y) \to \text{hom} (Y \vee, X \vee), \]
which is by definition the unique morphism making the following diagram commutative:
\[
\begin{array}{ccc}
\text{hom} (X, Y) \otimes Y \xrightarrow{1_{\text{hom}(X,Y)} \otimes 1_{Y \vee}} Y \otimes \text{hom} (X, Y) & \xrightarrow{c_{X,Y}} & \text{hom} (Y \vee, X \vee) \otimes Y \vee \\
\downarrow{d_{X,Y} \otimes 1_{Y \vee}} & & \downarrow{c_{X,Y,1}^{Y \vee}} \\
\text{hom} (Y \vee, X \vee) \otimes Y \vee & \xrightarrow{1_{\text{hom}(Y \vee, X \vee)} \otimes 1_{Y \vee}} & Y \otimes X \vee
\end{array}
\]

It enjoys a number of expected properties, namely it makes commutative the following diagrams.

- It is the unique morphism making the following diagram commutative\(^2\):
\[
\begin{array}{ccc}
\text{hom} (X, Y) \otimes X \otimes Y \vee \xrightarrow{(1_X \otimes d_{X,Y} \otimes 1_{Y \vee}) \circ (\tau_{\text{hom}(X,Y), 1_X \otimes 1_{Y \vee}})} X \otimes \text{hom} (Y \vee, X \vee) \otimes Y \vee \\
\downarrow{ev_{X,Y} \otimes 1_{Y \vee}} & & \downarrow{1_X \otimes ev_{Y \vee, X \vee}^{Y \vee}} \\
Y \otimes Y \vee & \xrightarrow{ev_{Y \vee}} & X \otimes X \vee
\end{array}
\]

- The following diagram is commutative\(^3\):
\[
\begin{array}{ccc}
\text{hom} (Y, Z) \otimes \text{hom} (X, Y) \xrightarrow{c_{Y,Z,Y} \otimes d_{X,Y}} \text{hom} (Z \vee, Y \vee) \otimes \text{hom} (Y \vee, X \vee) \\
\downarrow{c_{X,Y,Z}^{Y \vee, Y \vee, X \vee}} & & \downarrow{c_{Z \vee, Y \vee, X \vee}} \\
\text{hom} (X, Z) & \xrightarrow{d_{X,Z}} & \text{hom} (Z \vee, X \vee).
\end{array}
\]

\(^2\)In symbols, setting \( f^\vee := d_{X,Y} (f) \) for \( f \in \text{hom} (X, Y) \),
\[ \langle f^\vee (x), y^\vee \rangle = \langle x, f (y^\vee) \rangle \text{ for } x \in X \text{ and } y^\vee \in Y \vee. \]

\(^3\)In symbols, for \( f \in \text{hom} (X, Y) \) and \( g \in \text{hom} (Y, Z) \),
\[ (g \circ f)^\vee = g^\vee \circ opp f^\vee = f^\vee \circ g^\vee. \]
• If we have given \( f : X_2 \to X_1 \) and \( g : Y_1 \to Y_2 \) the following diagram is commutative:

\[
\begin{array}{ccc}
\text{hom}(X_1,Y_1) & \xrightarrow{d_{X_1,Y_1}} & \text{hom}(Y_1^\vee,X_1^\vee) \\
\text{hom}(f,g) & \xrightarrow{\text{hom}(f,g)} & \text{hom}(g^\vee,f^\vee) \\
\text{hom}(X_2,Y_2) & \xrightarrow{d_{X_2,Y_2}} & \text{hom}(Y_2^\vee,X_2^\vee).
\end{array}
\]

(13)

• The following further diagrams are commutative:

\[
\begin{array}{ccc}
Y \otimes X^\vee & \xrightarrow{\alpha_{Y,X^\vee}} & \text{hom}(X,Y) \\
\xrightarrow{(1_Y \otimes \iota_Y)} & \xrightarrow{d_{X,Y}} & \xrightarrow{d_{Y^\vee,X^\vee} \circ d_{X,Y}} \\
X^\vee \otimes Y^\vee \otimes Y^\vee & \xrightarrow{\phi_{X,Y,Z}} & \text{hom}(Y^\vee,X^\vee) \\
\xrightarrow{\phi_{X,Y,Z}(1_Y \otimes \iota_Y \otimes \iota_Y)} & & \xrightarrow{\phi_{X,Y,Z}(1_Y \otimes \iota_Y \otimes \iota_Y)} \\
Y \otimes Y^\vee & \xrightarrow{\text{ev}_{Y^\vee}} & \text{hom}(X,Y^\vee) \\
\xrightarrow{\phi_{X,Y,Z}(1_Y \otimes \iota_Y \otimes \iota_Y)} & & \xrightarrow{\phi_{X,Y,Z}(1_X \otimes \iota_Y \otimes \iota_Y)} \\
Y^\vee & \xrightarrow{\iota_Y} & \\
\end{array}
\]

(14)

We recall that \( C \) is rigid whenever the morphisms \( \epsilon^Y_{X,Y} \) and \( \iota_X \) are isomorphisms and is said to be pseudo-abelian when idempotents have kernels (and then also cokernels).

We will employ the following notation: a label \( (\otimes) \) (resp. \( (\tau) \)) placed in the middle of a diagram will mean that the diagram is commutative by functoriality of \( \otimes \) (resp. the \( \tau \) constraint).

2.1. Abstract internal multiplication. Suppose that we have given a morphism

\[ f : S \to \text{hom}(X,Y) \]

Then we define the corresponding "internal multiplication" morphism as the composite:

\[ \iota_f : S \xrightarrow{f} \text{hom}(X,Y) \xrightarrow{d_{X,Y}} \text{hom}(Y^\vee,X^\vee) \]

One checks that (15) implies that the following diagram is commutative:

\[
\begin{array}{ccc}
S \otimes X \otimes Y^\vee & \xrightarrow{(1_X \otimes \varphi_f) \circ (\tau_s \otimes 1_Y)} & X \otimes X^\vee \\
\xrightarrow{\varphi_f \otimes 1_{X^\vee}} & \xrightarrow{\text{ev}_{X^\vee}} & \\
Y \otimes Y^\vee & \xrightarrow{\iota_Y} & \\
\end{array}
\]

(15)

Remark 2.2. The morphism \( \varphi_{\iota,f} \), and hence \( \iota_f \), is characterized by the property of making (15) commutative.

Suppose now that we have also given:

\[ g : T \to \text{hom}(Y,Z) \] corresponding to \( \varphi_g : T \otimes Y \to Z \),
\[ h : U \to \text{hom}(X,Z) \] corresponding to \( \varphi_h : U \otimes X \to Z \),
\[ k : T \to \text{hom}(S,U) \] corresponding to \( \varphi_k : T \otimes S \to U \).

As an application of Lemma 2.1, we have the equivalence:

\[
\begin{array}{ccc}
T \otimes S \otimes X & \xrightarrow{1_T \otimes \varphi_f} & T \otimes Y \\
\xrightarrow{\varphi_k \otimes 1_X} & \xrightarrow{\varphi_k \otimes 1_X} & \xrightarrow{\varphi_k \otimes 1_X} \\
U \otimes X & \xrightarrow{\varphi_h} & Z \\
\end{array}
\]

(16)

We also have the associated internal multiplication morphisms:

\[ \iota_g : T \xrightarrow{g} \text{hom}(Y,Z) \xrightarrow{d_{Y,Z}} \text{hom}(Z^\vee,Y^\vee) \] corresponding to \( \varphi_{\iota,g} : T \otimes Z^\vee \to Y^\vee \),
\[ \iota_h : U \xrightarrow{h} \text{hom}(X,Z) \xrightarrow{d_{X,Z}} \text{hom}(Z^\vee,X^\vee) \] corresponding to \( \varphi_{\iota,h} : U \otimes Z^\vee \to X^\vee \).

\[ \text{Remark 2.2. The morphism } \varphi_{\iota,f}, \text{ and hence } \iota_f, \text{ is characterized by the property of making (15) commutative.} \]

Suppose now that we have also given:

\[ g : T \to \text{hom}(Y,Z) \] corresponding to \( \varphi_g : T \otimes Y \to Z \),
\[ h : U \to \text{hom}(X,Z) \] corresponding to \( \varphi_h : U \otimes X \to Z \),
\[ k : T \to \text{hom}(S,U) \] corresponding to \( \varphi_k : T \otimes S \to U \).

As an application of Lemma 2.1, we have the equivalence:

\[
\begin{array}{ccc}
T \otimes S \otimes X & \xrightarrow{1_T \otimes \varphi_f} & T \otimes Y \\
\xrightarrow{\varphi_k \otimes 1_X} & \xrightarrow{\varphi_k \otimes 1_X} & \xrightarrow{\varphi_k \otimes 1_X} \\
U \otimes X & \xrightarrow{\varphi_h} & Z \\
\end{array}
\]

(16)

We also have the associated internal multiplication morphisms:

\[ \iota_g : T \xrightarrow{g} \text{hom}(Y,Z) \xrightarrow{d_{Y,Z}} \text{hom}(Z^\vee,Y^\vee) \] corresponding to \( \varphi_{\iota,g} : T \otimes Z^\vee \to Y^\vee \),
\[ \iota_h : U \xrightarrow{h} \text{hom}(X,Z) \xrightarrow{d_{X,Z}} \text{hom}(Z^\vee,X^\vee) \] corresponding to \( \varphi_{\iota,h} : U \otimes Z^\vee \to X^\vee \).

4In symbols, the second commutative diagram tells that \( f^\vee \circ f = f \) up to the identification \( X^\vee \circ X = X \) and \( Y^\vee \circ Y = Y \) whenever \( X \) and \( Y \) are reflexive.
Consider the morphism:

\[ \varphi_k^T : S \otimes T \xrightarrow{\varphi_k} T \otimes S \xrightarrow{\varphi_k} U. \]

The equivalence \([10]\), applied with \((f, g, h, \varphi_k)\) replaced by \((i_g, i_h, \iota_f, \varphi_k^T)\), easily translates into the equivalence:

\[
\begin{array}{c}
S \otimes T \otimes Z^\vee \xrightarrow{\varphi_k \otimes 1_{Z^\vee}} S \otimes Y^\vee \xrightarrow{1 \otimes \varphi_k} T \otimes S \xrightarrow{\varphi_k \otimes 1_f} \hom (Z^\vee, Y^\vee) \otimes \hom (Y^\vee, X^\vee) \\
marrow_{\varphi_k \otimes 1_{Z^\vee}} \quad \varphi_k \otimes 1_{Z^\vee} \quad \varphi_k \otimes 1_f \quad \varphi_k \otimes 1_f \\
U \otimes Z^\vee \xrightarrow{\varphi_k \otimes 1_{Z^\vee}} X^\vee \xrightarrow{1 \otimes \varphi_k} U \xrightarrow{\varphi_k \otimes 1_f} \hom (Z^\vee, X^\vee).
\end{array}
\]

Finally we remark that the second square of the following diagram is commutative by \([12]\):

\[
\begin{array}{c}
T \otimes S \xrightarrow{\varphi_k \otimes f} \hom (Y, Z) \otimes \hom (X, Y) \xrightarrow{\varphi \otimes \eta, \varphi \otimes \eta} \hom (Z^\vee, Y^\vee) \otimes \hom (Y^\vee, X^\vee) \\
\varphi_k \otimes f \quad \varphi_k \otimes f \\
U \xrightarrow{h} \hom (X, Z) \xrightarrow{c_X \otimes \eta, d_X \otimes \eta} \hom (Z^\vee, X^\vee).
\end{array}
\]

It follows that we have the implication

\[ \text{\([10]\) commutative } \Rightarrow \text{\([17]\) commutative} \]

We now turn to the consideration of how the formation of internal multiplication behaves with respect to biproduct decompositions. To this end we assume that, for all the objects \(W\) considered above, we have given a biproduct decomposition \(W = W^+ \oplus W^-\) obtained by means of injective morphisms \(\iota^+_W : W \to W^+\) and associated idempotents \(\iota^-_W : W \to W^\perp\).

For \((\iota_1, \iota_2, \iota_3, \iota_4) \in \{\pm\} \times \{\pm\} \times \{\pm\}\), define the following morphisms:

\[
\begin{align*}
\varphi_{f,1}^\perp : S^\perp &\to \hom (X, Y) \\
\iota_1^\perp &\to \hom (X, Y)^{\hom (X^\perp, Y)} \\
\varphi_{f,2}^\perp : S^\perp &\to \hom (Y^\perp, X) \\
\iota_2^\perp &\to \hom (Y^\perp, X)^{\hom (Y^\perp, X^\perp)}
\end{align*}
\]

as well as the morphisms \(\varphi_{f,1}^\perp, \varphi_{f,2}^\perp, \varphi_{f,3}^\perp, \varphi_{f,4}^\perp\) and the other defined similarly as for \(f\).

**Lemma 2.3.** Writing \(\varphi_{f,1}^\perp, \varphi_{f,2}^\perp : S^\perp \otimes X^\perp \to Y^\perp\) (resp. \(\varphi_{f,1}^\perp, \varphi_{f,2}^\perp : S^\perp \otimes Y^\perp \to X^\perp\)) for the morphism corresponding to \(f_{1,1}^\perp, \varphi_{f,2}^\perp : S^\perp \to \hom (X^\perp, Y^\perp)\) (resp. \(\iota_1^\perp, \iota_2^\perp : S^\perp \to \hom (Y^\perp, X^\perp)\)), we have \(\varphi_{f,1}^\perp, \varphi_{f,2}^\perp = \varphi_{f,1}^\perp, \varphi_{f,2}^\perp = \varphi_{f,1}^\perp, \varphi_{f,2}^\perp\) as well as \(\iota_{f,1}^\perp, \iota_{f,2}^\perp = \iota_{f,1}^\perp, \iota_{f,2}^\perp\) or, equivalently, \(\varphi_{f,1}^\perp, \varphi_{f,2}^\perp = \varphi_{f,1}^\perp, \varphi_{f,2}^\perp\).

**Proof.** The equality \(\varphi_{f,1}^\perp, \varphi_{f,2}^\perp = \varphi_{f,1}^\perp, \varphi_{f,2}^\perp\) is a consequence of \(p_{X^\perp, Y^\perp} = \hom (X^\perp, Y^\perp)\) given by \([9]\) and the characterizing property \((S)\) and \(\varphi_{f,1}^\perp, \varphi_{f,2}^\perp = \varphi_{f,1}^\perp, \varphi_{f,2}^\perp\) is proved in the same way. Next, consider the following diagram:

\[
\begin{array}{c}
S^\perp \xrightarrow{\iota_1^\perp, \iota_2^\perp} S \xrightarrow{f} \hom (X, Y) \xrightarrow{d_X, Y} \hom (Y^\perp, X^\perp) \\
\text{\((\iota_1, \iota_2, \iota_3, \iota_4)\)} \quad \text{\((\iota_1, \iota_2, \iota_3, \iota_4)\)} \\
\text{\(p_{X^\perp, Y^\perp}\)} \quad \text{\(p_{X^\perp, Y^\perp}\)}
\end{array}
\]

The square is commutative because

\[
\hom \left((\iota_1^\perp, \iota_2^\perp)_{X^\perp}, (\iota_3^\perp, \iota_4^\perp)_{Y^\perp}\right) = \hom \left((\iota_1^\perp, \iota_2^\perp)_{Y^\perp}, (\iota_3^\perp, \iota_4^\perp)_{X^\perp}\right) = \hom \left((\iota_1^\perp, \iota_2^\perp)_{Y^\perp}, (\iota_3^\perp, \iota_4^\perp)_{X^\perp}\right),
\]

and

\[
\begin{array}{c}
\text{\(p_{X^\perp, Y^\perp}\)} \quad \text{\(p_{X^\perp, Y^\perp}\)}
\end{array}
\]
again by (9) and because the duality is a contravariant and additive functor, so that we may apply (13). But we have
\[ p_{\text{hom}}(Y^{n_2}, X^{\tau_2}) \circ d_{X,Y} \circ f \circ \iota^f_S = p_{\text{hom}}(Y^{n_2}, X^{\tau_2}) \circ \iota_f \circ \iota^f_S = \iota^f_{e_1,n_2;\tau_2}, \]
\[ d_{X^{\tau_2}, Y^{n_2}} \circ p_{\text{hom}}(X^{\tau_2}, Y^{n_2}) \circ f \circ \iota^f_S = d_{X^{\tau_2}, Y^{n_2}} \circ \iota^f_{e_1,n_2;\tau_2} = \iota^f_{f_{e_1,n_2;\tau_2}}. \]

\[ \square \]

It follows from Lemma 2.3 (also applied to \( g, h, k \)) that we may apply the above considerations with \((f, g, h, \varphi_k)\) replaced by \((f_{e_1,n_2;\tau_2}, g_{n_1,n_2;\tau_2}, h_{n_1,n_2;\tau_2}, \varphi_k)\). Hence, we deduce that

\[
S^{e_1} \otimes X^{\tau_2} \otimes Y^{n_2} \xrightarrow{(1_{X^{\tau_2}} \otimes \varphi^{e_1,n_2;\tau_2}) \circ (t_{S^{e_1},X^{\tau_2}} \otimes 1_{Y^{n_2}})} X^{\tau_2} \otimes X^{\tau_2},
\]

is commutative and that we have the implications:

\[
T^{n_1} \otimes S^{e_1} \otimes X^{\tau_2} \xrightarrow{1_{T^{n_1}} \otimes \varphi^{e_1,n_2;\tau_2}} T^{n_1} \otimes Y^{n_2} \xrightarrow{\varphi^{n_1,n_2;\tau_2}} U^{n_1} \otimes X^{\tau_2} \xrightarrow{U^{n_1} \otimes X^{\tau_2}} Z^{\tau_2},
\]

\[
S^{e_1} \otimes T^{n_1} \otimes Z^{\tau_2} \xrightarrow{\varphi^{e_1,n_2,\tau_2} \otimes 1_{Z^{\tau_2}}} S^{e_1} \otimes Y^{n_2} \xrightarrow{1_{S^{e_1}} \otimes \varphi^{e_1,n_2;\tau_2}} T^{n_1} \otimes S^{e_1} \otimes \text{hom}(Y^{n_2}, Z^{\tau_2}) \otimes \text{hom}(X^{\tau_2}, Y^{n_2}) \quad (20)
\]

\[
T^{n_1} \otimes S^{e_1} \otimes X^{\tau_2} \xrightarrow{1_{T^{n_1}} \otimes \varphi^{e_1,n_2;\tau_2}} T^{n_1} \otimes Y^{n_2} \xrightarrow{\varphi^{n_1,n_2;\tau_2}} U^{n_1} \otimes X^{\tau_2} \xrightarrow{U^{n_1} \otimes X^{\tau_2}} Z^{\tau_2},
\]

\[
S^{e_1} \otimes T^{n_1} \otimes Z^{\tau_2} \xrightarrow{\varphi^{e_1,n_2,\tau_2} \otimes 1_{Z^{\tau_2}}} S^{e_1} \otimes Y^{n_2} \xrightarrow{1_{S^{e_1}} \otimes \varphi^{e_1,n_2;\tau_2}} T^{n_1} \otimes S^{e_1} \otimes \text{hom}(Z^{\tau_2,\tau_2}, Y^{n_2}) \otimes \text{hom}(Y^{n_2,\tau_2}, X^{\tau_2}) \quad (21)
\]

as well as

\[(20) \text{ commutative } \Rightarrow (21) \text{ commutative.} \quad (22)\]

The proof of the following Lemma is just a formal computation.

**Lemma 2.4.** Suppose that \( e^{e_2}_Z \circ \varphi_h \circ (e^{e_2}_i \otimes 1_X) = e^{e_2}_Z \circ \varphi_h \circ (1_T \circ e^{n_2}_Y) = e^{e_2}_Z \circ \varphi_g \). Then we have the implication \((10)\) commutative \(\Rightarrow (21)\) commutative.

The following result is now a combination of the commutativity of (19), Lemma 2.4 and (22).

**Proposition 2.5.** Suppose that \( e^{e_2}_Z \circ \varphi_h \circ (e^{e_2}_i \otimes 1_X) = e^{e_2}_Z \circ \varphi_h \circ (1_T \circ e^{n_2}_Y) = e^{e_2}_Z \circ \varphi_g \) and that \((10)\) is commutative. Then (19), (20) and (21) are commutative.

For future reference it will be convenient to introduce some more notation. When \( \alpha_{X,Y} \) is an isomorphism, we define
\[
D_f : S \xrightarrow{f} \text{hom}(X,Y) \xrightarrow{\alpha^{-1}_{X,Y}} Y \otimes X^\vee.
\]
Suppose now that we have given \( g : S \rightarrow \text{hom}(X^\vee,Y^\vee) \), so that we have \( \iota_g : S \rightarrow \text{hom}(Y^{\vee \vee},X^{\vee \vee}) \). When \( X \) is reflexive, we define
\[
\iota^*_g : \text{hom}(Y^{\vee \vee},X^{\vee \vee}) \xrightarrow{\text{hom}(y^*_X,1_X)} \text{hom}(Y,X).
\]
Suppose that \( X \) is reflexive and that both \( \alpha_{X^\vee,Y^\vee} \) and \( \alpha_{Y,X} \) are isomorphisms. Then it follows from the first commutative diagram of (14) that \( d_{Y,X} \) is an isomorphism and then, from the second commutative diagram of (14), we deduce that hom \( (i_Y,i^{-1}_X) \circ d_{X,Y} = d_{Y,X} \), so that
\[
\iota^*_g : = d_{Y,X} \circ i^*_g = d_{Y,X} \circ \text{hom}(i_Y,i^{-1}_X) \circ d_{X,Y} \circ g =
\]
\[
d_{Y,X} \circ d^{-1}_{Y,X} \circ g = g.
\]
It follows from [15] that the first of the subsequent equivalently commutative diagrams commutes:

\[
\begin{align*}
S \otimes Y \otimes X^\vee \xrightarrow{(1_Y \otimes \varphi_g) \circ (\tau_S \otimes X \otimes 1_Y)} Y \otimes X^\vee & \quad \Rightarrow \quad S \otimes X \otimes (1_Y \otimes \varphi_g) \circ (\tau_S \otimes X \otimes 1_Y) \\
X \otimes X^\vee \xrightarrow{\varphi_g \otimes 1_Y} Y \otimes X^\vee & \quad \Rightarrow \quad X^\vee \otimes X \xrightarrow{\varphi_g \otimes 1_Y} Y \otimes X^\vee
\end{align*}
\]

(23)

Here the equivalence is easily obtained by applying \(1_S \otimes \tau_{X^\vee,Y} \) (resp. \(1_S \otimes \tau_{X^\vee,Y} \)) to the first (resp. second) diagram to get the second (resp. first) diagram. Also, it follows from the functorial description \( \text{Hom} \ (1_S, \text{hom} (i_Y, i_X^{-1})) = \text{Hom} \ (1_S \otimes i_Y, i_X^{-1}) \) (up to [17]) of hom that the following diagram is commutative:

\[
\begin{align*}
S \otimes Y & \xrightarrow{\varphi_g \otimes 1_Y} X \\
1_S \otimes \iota_Y & \quad \Rightarrow \quad S \otimes Y^\vee \xrightarrow{\varphi_g \otimes 1_Y} X^\vee.
\end{align*}
\]

(24)

Finally, in addition to \( D_{i_g} : S \rightarrow X^\vee \otimes Y^\vee \) defined when \( \alpha_{Y^\vee,X^\vee} \) is an isomorphism, we may define, when \( X \) is reflexive and \( \alpha_{Y,X} \) is an isomorphism:

\[
D_{i_g} : S \xrightarrow{\iota_g} \text{hom} (Y, X) \xrightarrow{\alpha_{Y,X}^{-1}} X \otimes Y^\vee.
\]

The relationship between \( D_{i_g} \) and \( D_{i_g} \) can be made explicit as follows. Consider the following diagram:

\[
\begin{align*}
S \xrightarrow{g} \text{hom} (X^\vee, Y^\vee) \xrightarrow{d_{X^\vee,Y^\vee}} \text{hom} (Y^\vee, Y^\vee) \xrightarrow{\text{hom}(1_Y, 1_Y^{-1})} \text{hom} (Y, X) \\
Y^\vee \otimes X \xrightarrow{(1_Y \otimes \varphi_g) \circ (\tau_X \otimes X \otimes 1_Y)} X \otimes Y^\vee \xrightarrow{\alpha_{Y,X}} X \otimes Y^\vee.
\end{align*}
\]

(25)

The first square is commutative thanks to the first diagram in [14], while the second square is commutative by functoriality of \( \alpha \). The subsequent lemma, whose proof we leave to the reader, shows that, when \( Y \) is reflexive, \((i_Y)^\vee = i_Y^{-1}\) and we find, in this case,

\[
D_{i_g} = (i_X \otimes i_Y^{-1}) \circ D_{i_g}.
\]

Lemma 2.6. We have the equality \((i_X)^\vee \circ i_X = 1_{X^\vee}\). In particular, if \( X \) is reflexive, then \( X^\vee \) is reflexive and \( i_X^{-1} = (i_X)^\vee\).

The following lemma will be useful later.

Lemma 2.7. Suppose that we have given \( f : S \rightarrow \text{hom} (X, Y) \) and, respectively, \( g : S \rightarrow \text{hom} (X^\vee, Y^\vee) \), that \( \alpha_{Y^\vee,X^\vee} \) is an isomorphism, so that \( D_{i_f} \) is defined, and, respectively, that \( \alpha_{Y^\vee,X^\vee} \) is an isomorphism and that \( X \) is reflexive and \( \alpha_{Y,X} \) is an isomorphism, so that \( D_{i_g} \) is defined. Then the first and, respectively, the second of the following diagrams is commutative:

\[
\begin{align*}
S \otimes X & \xrightarrow{\varphi_f} Y \\
D_{i_f} \otimes 1_X & \quad \Rightarrow \quad X^\vee \otimes Y^\vee \otimes X \xrightarrow{\varphi_f} Y^\vee \\
S \otimes X^\vee & \xrightarrow{\varphi_g} Y^\vee \\
D_{i_g} \otimes 1_X & \quad \Rightarrow \quad X \otimes Y^\vee \otimes X \xrightarrow{\varphi_g} Y^\vee
\end{align*}
\]

where \( \text{ev}_{13,Y^\vee} := (1_Y \otimes \text{ev}_X) \circ (\tau_X \otimes Y^\vee \otimes 1_X) \) and \( \text{ev}_{13,Y^\vee} := (1_Y \otimes \text{ev}_X) \circ (\tau_X \otimes Y^\vee \otimes 1_X) \).
Proof. Consider the following diagram, where we set ṯ_{Y,X} := (1_{X^Y} \otimes \iota_Y) \circ \tau_{Y,X^Y}:

\[
\begin{array}{c}
\text{hom}(X,Y) \otimes X \\
\otimes Y \otimes X^Y \otimes X \\
d_X,Y \otimes 1_X
\end{array}
\xrightarrow{	ext{injectivity}}
\begin{array}{c}
\text{hom}(Y^Y,X^Y) \otimes X \\
\otimes Y^Y \otimes X^Y \\
i_Y \otimes 1_X
\end{array}
\xrightarrow{\text{surjectivity}}
\begin{array}{c}
\text{hom}(Y^Y \otimes X^Y) \otimes X \\
\otimes Y^Y \otimes X^Y \\
\tau_{Y^Y,X^Y} \otimes 1_X
\end{array}
\]

The region (A) is commutative by definition of \(\alpha_{X,Y}\) and (B) thanks to the first diagram in (14). Noticing that \(\tau_{X^Y,Y^Y} = (\tau_{Y^Y,X^Y})^{-1}\), we deduce

\[
i_Y \circ \varphi_f = \quad i_Y \circ (f \otimes 1_X) = \quad e_{13,Y^Y} \circ (i_Y \otimes 1_X) \circ (f \otimes 1_X)
\]

Next, we have \(i_Y \circ \varphi_g = e_{13,Y^Y} \circ (D_{i_y} \otimes 1_X)\) by the previous computation (because \(\alpha_{Y^Y,X^Y}\) is an isomorphism). Applying \((i_Y)^\ast\) we deduce, by Lemma 2.8:

\[
\varphi_g = \quad (i_Y)^\ast \circ e_{13,Y^Y} \circ (D_{i_y} \otimes 1_X)
\]

But it follows from (23) that we have \(D_{i_y} = (i_X \otimes (i_Y)^\ast) \circ D_{i_y}\), proving that the second diagram is commutative.

\[\square\]

2.2. Some commutative diagram involving the Casimir element. If we have given two objects \(X\) and \(Y\) and \(W = W_1 \otimes W_2 \otimes W_3 \otimes W_4\), where \((W_1, W_2, W_3, W_4)\) is a permutation of the string \((X^Y, X^Y, Y)\), we define morphisms \(e_{ij,kl} : W \to \mathbb{I}\), where \(i, j, k, l \in \{1, 2, 3, 4\}\) are such that \(i < j, k < l\) and \(i < k\) and \(\alpha, \beta \in \{\phi, \tau\}\), as follows. We let \(ij\) be one of the two pairs for which \(ev : W_1 \otimes W_j \to \mathbb{I}\) is defined and we write a corresponding superscript \(\alpha = \phi\) if \(W_j \in \{X,Y\}\) (so that \(ev = ev_X\) or \(ev_Y\)) or \(\alpha = \tau\) if \(W_j \in \{X^Y, Y^Y\}\) (so that \(ev = ev_X\) or \(ev_Y\)). The same rule is applied to the triple \((k,l,\beta)\). Then we define

\[
e_{ij,kl}^{\alpha,\beta} : W \xrightarrow{\tau} X^Y \otimes X \otimes Y^Y \otimes Y \xrightarrow{ev_X \otimes ev_Y} \mathbb{I} \otimes \mathbb{I} \xrightarrow{\eta} \mathbb{I}\]

where \(\tau\sigma\) is the morphism obtained from any permutation \(\sigma\) suitably reordering the factors. We have, for example,

\[
e_{12,34}^{\phi,\phi} : X^Y \otimes X \otimes X^Y \otimes X \xrightarrow{ev_X \otimes ev_X} \mathbb{I} \otimes \mathbb{I} \xrightarrow{\eta} \mathbb{I}\]

\[
e_{14,23}^{\tau,\phi} : X \otimes X^Y \otimes X \otimes X^Y \xrightarrow{\tau_{1,23} \otimes 1_{X^Y}} X^Y \otimes X \otimes X \otimes X^Y \xrightarrow{ev_X \otimes ev_X} \mathbb{I} \otimes \mathbb{I} \xrightarrow{\eta} \mathbb{I}\]

We say that an object \(X\) admits a Casimir element if \(X\) is a reflexive object such that \(\epsilon : X^Y \otimes X^Y \to (X^Y \otimes X)^Y\) is an isomorphism. Then we define the Casimir element:

\[
C_X : \mathbb{I} \xrightarrow{(\sigma \otimes \tau)} (X^Y \otimes X)^Y \xrightarrow{\sigma} X^Y \otimes X \xrightarrow{\tau \otimes 1_X} X \otimes X^Y.
\]

We collect in the following lemma well known properties of the Casimir element.

Lemma 2.8. Suppose that \(X\) has a Casimir element.

1. \(C_X\) is the unique morphism making one of the following diagrams commutative:

\[
ev_X : X^Y \otimes X \xrightarrow{1_{X^Y} \otimes C_X \otimes 1_{X^Y}} X^Y \otimes X \otimes X^Y \otimes X \xrightarrow{ev_X^{\phi,\phi}} \mathbb{I},
\]

\[
ev_X : X^Y \otimes X \xrightarrow{C_X \otimes ev_X} X \otimes X^Y \otimes X \otimes X^Y \xrightarrow{ev_X^{\tau,\phi}} \mathbb{I},
\]

\[
ev_X : X^Y \otimes X \xrightarrow{\tau \otimes C_X} X \otimes X^Y \otimes X \otimes X^Y \xrightarrow{ev_X^{\tau,\phi}} \mathbb{I}.
\]
(2) \( C_X \) is the unique morphism making one of the following diagrams commutative:

\[
\begin{align*}
1_X & : X \xrightarrow{\text{ev}_X} X \otimes X^\vee \otimes X^\vee, \\
1_{X^\vee} & : X^\vee \xrightarrow{\text{ev}_X} X^\vee \otimes X \otimes X^\vee.
\end{align*}
\]

(3) If \( X_1, X_2 \) and \( X_1 \otimes X_2 \) have a Casimir element, then

\[
C_{X_1 \otimes X_2} = (1_{X_1} \otimes 1_{X_2} \otimes \epsilon) \circ (1_{X_1} \otimes \tau_{X_1}, X_2 \otimes 1_{X_2}) \circ (C_{X_1} \otimes C_{X_2}).
\]

(4) If \( X = X^+ \oplus X^- \) is a biproduct decomposition inducing \( X^\vee = X^{\vee+} \oplus X^{\vee-} \), then \( X^\pm \) both have a Casimir element and \( C_{X^\pm} = (p_{X^\pm} \otimes p_{X^{\vee \pm}}) \circ C_X \) for the associated surjective morphisms \( p_{X^\pm} : X \to X^\pm \) and \( p_{X^{\vee \pm}} : X^\vee \to X^{\vee \pm} \).

(5) We have that \( X^\vee \) has a Casimir element and \( C_{X^\vee} = (1_{X^\vee} \otimes i_X) \circ \tau_{X^\vee} \circ C_X \).

When \( X \) has a Casimir element, an explicit inverse of the canonical map \( f \mapsto \varphi_f \) can be given. This is the content of the subsequent proposition.

**Proposition 2.9.** Suppose that we have given \( f : S \to \text{hom} \,(X,Y) \), which is associated to \( \varphi_f : S \otimes X \to Y \), and that \( X \) has a Casimir element. Then the following diagram is commutative

\[
\begin{array}{ccc}
S & \xrightarrow{f} & \text{hom} \,(X,Y) \\
\downarrow 1_S \otimes C_X & & \downarrow \alpha_{X,Y} \\
S \otimes X & \xrightarrow{\varphi_f \otimes 1_X} & Y \otimes X^\vee.
\end{array}
\]

**Proof.** Consider the following diagram

\[
\begin{array}{ccc}
S \otimes X & \xrightarrow{1_S \otimes C_X \otimes 1_X} & S \otimes X \otimes X^\vee \otimes X \xrightarrow{\varphi_f \otimes 1_X} Y \otimes X^\vee \otimes X \xrightarrow{\text{ev}_X \otimes 1_X} \text{hom} \,(X,Y) \otimes X \\
\downarrow 1_S \otimes \text{ev}_X & & \downarrow \text{ev}_X \otimes 1_X \\
S \otimes X & \xrightarrow{\varphi_f} & Y \xrightarrow{\text{ev}_X} \text{hom} \,(X,Y) \otimes X.
\end{array}
\]

The first triangle is commutative because \( 1_X : X \xrightarrow{\text{ev}_X} X \otimes X^\vee \otimes X \xrightarrow{1_X \otimes \text{ev}_X} X \) by Lemma 2.8, the square is commutative by functoriality of \( \otimes \) and the second triangle by definition of \( \alpha_{X,Y} \). But the map \( \varphi_a \) associated to \( a := \alpha_{X,Y} \circ (\varphi_f \otimes 1_X) \circ (1_S \otimes C_X) \) is obtained going from \( S \otimes X \to \text{hom} \,(X,Y) \otimes X \) in the upper row and then applying \( \text{ev}_{X,Y} \). The commutativity implies that this is the morphism \( \varphi_f \circ 1_{S \otimes X} = \varphi_f \). \( \square \)

We are mainly concerned with the following consequence of Proposition 2.9 when \( X \) has a Casimir element,

\[
D_f : S \xrightarrow{1_S \otimes C_X} S \otimes X \otimes X^\vee \xrightarrow{\varphi_f \otimes 1_X} Y \otimes X^\vee.
\]

### 2.3. Behavior of the internal multiplications with respect to tensor product constructions.

We suppose in this section that we have given \( f_i : S_i \to \text{hom} \,(X_i,Y_i) \) is associated to \( \varphi_i = \varphi_{f_i} : S_i \otimes X_i \to Y_i \) for \( i = 1, 2 \). Define the following morphisms

\[
\begin{align*}
& f_1 \otimes \epsilon_2 : S_1 \otimes S_2 \xrightarrow{f_1 \otimes \epsilon_2} \text{hom} \,(X_1,Y_1) \otimes \text{hom} \,(X_2,Y_2) \xrightarrow{\text{ev}_1 \otimes \text{ev}_2} \text{hom} \,(X_1 \otimes X_2,Y_1 \otimes Y_2), \\
& f_1 \otimes f_2 : S_1 \otimes S_1 \xrightarrow{\tau_{S_1} \otimes f_2} \text{hom} \,(X_1,Y_1) \otimes \text{hom} \,(X_2,Y_2) \xrightarrow{\text{ev}_1 \otimes \text{ev}_2} \text{hom} \,(X_1 \otimes X_2,Y_1 \otimes Y_2), \\
& \varphi_1 \otimes \epsilon_2 : S_1 \otimes S_2 \otimes X_1 \otimes X_2 \xrightarrow{1_S \otimes \epsilon_2 \otimes 1_X \otimes 1_X} \text{hom} \,(X_1,Y_1) \otimes \text{hom} \,(X_2,Y_2) \xrightarrow{\text{ev}_1 \otimes \text{ev}_2} \text{hom} \,(X_1 \otimes X_2,Y_1 \otimes Y_2), \\
& \varphi_1 \otimes f_2 : S_1 \otimes S_2 \otimes X_1 \otimes X_2 \xrightarrow{1_S \otimes \tau_{S_2} \otimes 1_X \otimes 1_X} \text{hom} \,(X_1,Y_1) \otimes \text{hom} \,(X_2,Y_2) \xrightarrow{\text{ev}_1 \otimes \text{ev}_2} \text{hom} \,(X_1 \otimes X_2,Y_1 \otimes Y_2), \\
& \varphi_1 \otimes f_2 : S_1 \otimes S_2 \otimes X_1 \otimes X_2 \xrightarrow{1_S \otimes \tau_{S_2} \otimes 1_X \otimes 1_X} \text{hom} \,(X_1,Y_1) \otimes \text{hom} \,(X_2,Y_2) \xrightarrow{\text{ev}_1 \otimes \text{ev}_2} \text{hom} \,(X_1 \otimes X_2,Y_1 \otimes Y_2).
\end{align*}
\]

It is easy to see that one has the following result.

**Lemma 2.10.** We have that \( \varphi_1 \otimes f_2 = \varphi_{f_1 \otimes f_2} \) (resp. \( \varphi_1 \otimes \epsilon_2 = \varphi_{f_1 \otimes \epsilon_2} \)) is the morphism associated to \( f_1 \otimes f_2 \) (resp. \( f_1 \otimes \epsilon_2 \)).
Next, we consider the associated internal multiplication morphisms \( \iota_i := \iota_{f_i} \), for \( i = 1, 2 \). The following lemma is easily deduced from the characterizing property (15) of \( \varphi_{i_1} \).

**Lemma 2.11.** The following diagram is commutative

\[
\begin{array}{c}
S_1 \otimes S_2 \otimes X_1 \otimes X_2 \otimes Y_1^\vee \otimes Y_2^\vee \xrightarrow{(1_{X_1 \otimes X_2} \otimes (\varphi_1 \otimes \varphi_2) \otimes \tau_{X_1 \otimes X_2} \otimes 1_{Y_1^\vee \otimes Y_2^\vee})} X_1 \otimes X_2 \otimes X_1^\vee \otimes X_2^\vee \\
Y_1 \otimes Y_2 \otimes Y_1^\vee \otimes Y_2^\vee \xrightarrow{\epsilon_{1,24}} S
\end{array}
\]

and the same with \( \otimes \) replaced by \( \otimes^\tau \) and \( S_1 \otimes S_2 \) by \( S_2 \otimes S_1 \).

**Remark 2.12.** It is easy to deduce from Lemma 2.11 that \( \iota_{f_1} \otimes_{\otimes} \iota_{f_2} : S_1 \otimes S_2 \to \text{hom} \left( (Y_1 \otimes Y_2)^\vee, (X_1 \otimes X_2)^\vee \right) \) is associated to \( \varphi_{i_1} \otimes_{\otimes} \varphi_{i_2} \) making the following diagram commutative:

\[
\begin{array}{c}
S_1 \otimes S_2 \otimes Y_1^\vee \otimes Y_2^\vee \xrightarrow{\varphi_{i_1} \otimes \varphi_{i_2}} X_1^\vee \otimes X_2^\vee \\
S_1 \otimes S_2 \otimes (Y_1 \otimes Y_2)^\vee \xrightarrow{\varphi_{i_1} \otimes \varphi_{i_2}} (X_1 \otimes X_2)^\vee.
\end{array}
\]

In particular, when the \( \epsilon \) morphisms are isomorphism, we deduce from Remark 2.12 that the commutativity of the diagram of Lemma 2.11 is characterizing for \( \varphi_{i_1} \otimes_{\otimes} \varphi_{i_2} \) (and similarly for \( \varphi_{i_1} \otimes^\tau \varphi_{i_2} \)).

From now on we specialize ourselves to the case where \( f_1 : S_1 \to \text{hom} (X, Y) \) and \( f_2 : S_2 \to \text{hom} (X^\vee, Y^\vee) \). We will assume, from now on, that \( \alpha_{X,Y} \) and \( \alpha_{X^\vee, Y^\vee} \) are isomorphisms, so that \( D_{f_1} \) and \( D_{f_2} \) are defined, and that \( X, X^\vee \) and \( Y \) have a Casimir element.

**Lemma 2.13.** The following diagram is commutative:

\[
\begin{array}{c}
S_1 \otimes S_2 \xrightarrow{C_Y \otimes 1_{S_2} \otimes C_X} X \otimes X^\vee \otimes S_1 \otimes S_2 \otimes X \otimes X^\vee \\
Y \otimes X^\vee \otimes Y^\vee \otimes X^\vee \xrightarrow{Y \otimes \epsilon_{24} \otimes Y^\vee} Y \otimes Y^\vee \xrightarrow{C_Y \otimes 1_{Y^\vee} \otimes Y^\vee} Y \otimes Y^\vee \otimes Y \otimes Y^\vee,
\end{array}
\]

where

\[
\epsilon_{24} : Y \otimes X^\vee \otimes Y^\vee \otimes X^\vee \xrightarrow{1_Y \otimes \tau_{X^\vee} \otimes \epsilon_{Y^\vee \otimes X^\vee}} Y \otimes Y^\vee \otimes X \otimes X^\vee \xrightarrow{1_Y \otimes \epsilon_{Y^\vee} \otimes \epsilon_{X^\vee}} Y \otimes Y^\vee.
\]

**Proof.** Define

\[
\begin{array}{c}
D_{12} : S_1 \otimes S_2 \xrightarrow{1_{S_1} \otimes_{\otimes} S_2 \otimes C_Y \otimes C_X} S_1 \otimes S_2 \otimes X \otimes X^\vee \otimes X^\vee \otimes X^\vee \otimes 1_{Y^\vee \otimes X^\vee} \otimes 1_{X^\vee} \\
S_1 \otimes S_2 \otimes X \otimes X^\vee \otimes X^\vee \otimes X^\vee \xrightarrow{\varphi_{f_1} \otimes \varphi_{f_2} \otimes 1_{Y^\vee \otimes X^\vee}} Y \otimes Y^\vee \otimes X \otimes X^\vee.
\end{array}
\]
and consider the following diagram:

\[
\begin{array}{c}
Y \otimes Y^\vee \otimes S_1 \otimes S_2 \otimes X \otimes X^\vee \xrightarrow{1_{Y \otimes Y^\vee} \otimes (\varphi_{f_1} \otimes \varphi_{f_2})} Y \otimes Y^\vee \otimes Y \otimes Y^\vee \\
\end{array}
\]

The region (A) is commutative because \((C_Y \otimes 1_{Y \otimes Y^\vee}) \circ (\varphi_{f_1} \otimes \varphi_{f_2}) = (1_{Y \otimes Y^\vee} \otimes (\varphi_{f_1} \otimes \varphi_{f_2})) \circ (C_Y \otimes 1_{S_1 \otimes S_2 \otimes X \otimes X^\vee})\) by functoriality of \(\otimes\) and then

\[
(C_Y \otimes 1_{Y \otimes Y^\vee}) \circ (\varphi_{f_1} \otimes \varphi_{f_2}) \circ (1_{S_1 \otimes S_2 \otimes C_X}) = (1_{Y \otimes Y^\vee} \otimes (\varphi_{f_1} \otimes \varphi_{f_2})) \circ (C_Y \otimes 1_{S_1 \otimes S_2 \otimes C_X}).
\]

The region (D) is commutative by definition of \(ev_{24}^{T \otimes Y \otimes Y^\vee}\). We claim that the regions (B) and (C) are commutative, from which we will deduce that the external portion of this diagram is commutative, giving us the claim.

**Region (B) is commutative.** Consider the following diagram

\[
\begin{array}{c}
S_1 \otimes S_2 \xrightarrow{1_{S_1 \otimes S_2 \otimes C_X}} S_1 \otimes S_2 \otimes X \otimes X^\vee \xrightarrow{\varphi_{f_1} \otimes \varphi_{f_2}} Y \otimes Y^\vee \otimes Y \otimes Y^\vee
\end{array}
\]

In light of the definition of \(D_{12}\) the commutativity of the region (B) follows once we show that the external portion of this diagram is commutative. We remark that, by functoriality of \(\tau\), by an explicit computation of the involved permutations and by definition of \(ev_{X^\vee}\) the following diagram is commutative:

\[
\begin{array}{c}
X^V \xrightarrow{\tau_{X^V, X^\vee}} X \otimes X^\vee \xrightarrow{1_{X^\vee} \otimes ev_{X^\vee}} X^\vee
\end{array}
\]

It follows that \((1_{X^\vee} \otimes ev_{X^\vee}) \circ (\tau_{X^V, X^\vee}) \circ (1_{X^\vee} \otimes C_{X^V}) = (1_{X^\vee} \otimes ev_{X^\vee}) \circ (C_{X^V} \otimes 1_{X^\vee}) = 1_{X^\vee}\), where the last equality follows from 2. of Lemma 2.3. The commutativity of the region (E) follows.

**Region (C) is commutative.** Consider the following diagram

\[
\begin{array}{c}
S_1 \otimes S_2 \xrightarrow{1_{S_1 \otimes S_2 \otimes C_X} \otimes 1_{S_1 \otimes S_2 \otimes C_X}} S_1 \otimes S_2 \otimes X \otimes X^\vee \otimes X \otimes X^\vee \otimes X \otimes X^\vee \xrightarrow{1_{X^\vee} \otimes ev_{X^\vee}} X^\vee
\end{array}
\]
Going around this diagram clockwisely (resp. counter-clockwisely) from $S_1 \otimes S_2$ until $Y \otimes Y' \otimes X' \otimes X''$ we find $(1_Y \otimes \tau_{X',Y'} \otimes 1_{X''}) \circ (D_{f_1} \otimes D_{f_2})$ (resp. $D_{12}$) because, by (27), we have $D_{f_i} = (\varphi_{f_i} \otimes 1_{X'_{Y'}}) \circ (1_{S_i} \otimes C_{X_i})$ for $i = 1, 2$, where $X_1 = X$ and $X_2 = X'_{Y'}$ (resp. by definition of $D_{12}$). It follows that we have to show that the external portion of this diagram is commutative. The region $(F)$ is commutative by an explicit computation of the involved permutations, while $(G)$ by definition of $(\varphi_{f_1} \otimes \varphi_{f_2})$. □

In addition to the other assumptions we will assume in the following proposition that $\alpha_{Y',X'_{Y'}}$ and $\alpha_{Y''_{Y',X'_{Y'}}}$ are isomorphisms, so that $D_{\iota_{f_1}}$ and $D_{\iota_{f_2}}$ are defined, and that $Y', Y''_{Y'}$ have a Casimir element.

**Proposition 2.14.** The following diagrams are commutative.

(1)

\[
\begin{array}{ccc}
S_1 \otimes S_2 & \xrightarrow{C_{Y} \otimes 1_{S_1} \otimes S_2 \otimes C_X} & Y \otimes Y' \otimes S_1 \otimes S_2 \otimes X \otimes X' \\
D_{f_1} \otimes D_{f_2} & | & | \\
Y \otimes X' \otimes Y' \otimes X'' & \xrightarrow{ev_{1_{14},23}^r} & \xrightarrow{ev_{1_{14},23}^r} Y \otimes Y' \otimes Y \otimes Y''
\end{array}
\]

(2)

\[
\begin{array}{ccc}
S_1 \otimes S_2 & \xrightarrow{C_{X'} \otimes 1_{S_1} \otimes S_2 \otimes C_{Y'}} & X' \otimes X'' \otimes S_1 \otimes S_2 \otimes Y' \otimes Y'' \\
D_{\iota_{f_1}} \otimes D_{\iota_{f_2}} & | & | \\
X' \otimes Y'' \otimes X'' \otimes Y'' & \xrightarrow{ev_{1_{13},24}^r} & \xrightarrow{ev_{1_{13},24}^r} X' \otimes X'' \otimes X \otimes X''
\end{array}
\]

(3)

\[
\begin{array}{ccc}
S_1 \otimes S_2 & \xrightarrow{1_{S_1} \otimes S_2 \otimes C_X \otimes C_{Y'}} & S_1 \otimes S_2 \otimes X \otimes X' \otimes Y \otimes Y'' \\
D_{\iota_{f_1}} \otimes D_{\iota_{f_2}} & | & | \\
X' \otimes Y'' \otimes X'' \otimes Y'' & \xrightarrow{ev_{1_{13},24}^r} & \xrightarrow{ev_{1_{13},24}^r} Y \otimes Y' \otimes Y' \otimes Y''
\end{array}
\]

**Proof.** (1) According to Lemma 2.13 we have

\[
ev_{1_{14},23}^r \circ (1_Y \otimes \varphi_{f_1} \otimes \varphi_{f_2}) \circ (C_Y \otimes 1_{S_1} \otimes S_2 \otimes C_X) = ev_{1_{14},23}^r \circ (C_Y \otimes 1_Y \otimes \varphi_{f_1} \otimes \varphi_{f_2}) \circ (D_{f_1} \otimes D_{f_2}) .
\]

It follows from Lemma 2.8 (1) we have $ev_Y = ev_{1_{14},23}^r \circ (C_Y \otimes 1_Y \otimes \varphi_{f_1} \otimes \varphi_{f_2}) \circ (C_Y \otimes 1_{S_1} \otimes S_2 \otimes C_X) = ev_{1_{14},23}^r \circ (C_Y \otimes 1_Y \otimes \varphi_{f_1} \otimes \varphi_{f_2}) \circ (D_{f_1} \otimes D_{f_2})$.

(2) This is just our claim 1. applied to the couple $(\iota_{f_1}, \iota_{f_2})$ rather than $(f_1, f_2)$.
(3) Consider the following diagram:

We remark that our claim is the commutativity of the unlabeled region of (28) and that this commutativity follows once we show that the external part and the labeled regions of (28) are commutative. The external part of this diagram is commutative by Lemma 2.11.

**Commutativity of the labeled regions of (28).** The region (A) is commutative by our claim (2), the region (C) by the defining property of \(Ι_Χ\), the region (D) by the definitions of \(e_{V^1,1}\) and \(e_{1,1}^{V^1}\). The commutativity of the region (B) follows the equality \(C_{X^V} = (1_X \otimes i_X) \circ τ_{X,X^V} \circ C_X\) of Lemma 2.8 (5), from which we deduce that

\[
(C_{X^V} \otimes 1_{S_1 \otimes S_2} \otimes C_{Y^V}) = (1_X \otimes i_X \otimes 1_{S_1 \otimes S_2 \otimes Y^V} \otimes Y^V) \circ (τ_{X,X^V} \otimes 1_{S_1 \otimes S_2 \otimes Y^V} \otimes Y^V) \circ (C_X \otimes 1_{S_1 \otimes S_2} \otimes C_{Y^V}),
\]

together with the equality \((C_X \otimes 1_{S_1 \otimes S_2} \otimes C_{Y^V}) = (τ_{S_1 \otimes S_2,X^V} \otimes 1_{Y^V} \otimes Y^V) \circ (1_{S_1 \otimes S_2} \otimes C_X \otimes C_{Y^V})\) (by functoriality of \(τ\)) and the fact that \(τ_{X,X^V} = τ_1^{X^V} \circ τ_{X,X^V}\). □

3. A formal Poincaré duality isomorphism

We remark that, for every object \(W\), we have a natural map

\[
End(Ι) \to End(W)
\]

defined by the rule

\[
λ_W : W \overset{i_W}{\to} 1 \otimes W \overset{λ_1}{\otimes 1_W} 1 \otimes W \overset{i_W}{\to} W, \ λ \in End(Ι).
\]

It defines a left action of the commutative ring \(End(Ι)\) on \(W\) for which every \(f : W_1 \to W_2\) becomes \(End(Ι)\)-equivariant and such that, if we have given \(φ : U \otimes V \to W\), then \(φ \circ (λ_u \otimes 1_V) = λ_W \circ φ = φ \circ (1_U \otimes λ_V)\).

Suppose in this section that \(C\) is rigid. We assume that we have given morphisms \(f_{S,X} : S \to hom(X,Y)\), \(f_{S,Y} : Y \to hom(S,Y)\), \(f_{S,G} : X \to Y\) and \(f_{S,Y'} : Y \to X\). We write \(φ_{S,X} : X \otimes Y \to Y \otimes X\), \(φ_{S,Y} : Y \otimes X \to Y \otimes X\) and \(φ_{S,Y'} : Y \otimes X \to Y\). The associated morphisms. We set \(i_{S,X} := i_{f_{S,X}}\), \(i_{S,Y} := i_{f_{S,Y}}\), \(i_{S,Y'} := i_{f_{S,Y'}}\). We may consider:

\[
D_{S,X} := D_{i_{S,X}} : S \to X \otimes Y^V, \ D_{X,S} := D_{i_{X,S}} : X \to S \otimes Y^V,
\]
\[
D_{S,Y} : S \to X \otimes Y^V \otimes Y^V^V, \ D_{S,Y} = D_{i_{S,Y}} : S \to X \otimes Y^V \otimes Y^V^V,
\]
\[
D_{X,Y} : X \to S \otimes Y^V, \ D_{X,Y} = D_{i_{X,Y}} : X \to S \otimes Y^V.
\]

We note that all the results of the previous section available.

It is easy to deduce, from (24) and (28) (5), the following equivalence:

\[
(Cas)_{μ_{S,X}} : μ_{S,X} \cdot C_S = \left(φ_{i_{S,Y'}} \otimes e \cdot φ_{i_{S,X}}\right) \circ (C_X \otimes C_Y) \iff μ_{S,X} \cdot C_{S'} = \left(φ_{i_{S,X}} \otimes e \cdot φ_{i_{S,Y'}}\right) \circ (C_X \otimes C_Y^V)
\]

\[\text{6} \text{Since indeed } λ_U \otimes 1_V = λ_{U \otimes V} = 1_U \otimes λ_V, \text{ this second statement follows from the first.}\]
for some $\mu_{S,X} \in \text{End}(I)$. Exchanging the roles of $S$ and $X$ we also have, for some $\mu_{X,S} \in \text{End}(I)$,

$$(\text{Cas})_{\mu_{X,S}} : \mu_{X,S} \cdot C_X = \left(\varphi_{S \otimes X, v} \otimes \tau_{S,X,v} \right) \circ (C_S \otimes C_Y) \Leftrightarrow \mu_{X,S} \cdot C_X = \left(\varphi_{S \otimes X, v} \otimes \tau_{S,X,v} \right) \circ (C_S \otimes C_Y).$$

If $(V, W) = (S, X)$ or $(X, S)$ and $\lambda_{V,W}, \lambda_{V^\vee, W^\vee} \in \text{End}(I)$, we will consider the following diagrams:

\[
\begin{array}{ccc}
V \otimes W & \xrightarrow{\tau_{V,W}} & W \otimes V \\
\varphi_{V,W} & \downarrow & \varphi_{W,V} \\
Y & \rightarrow & Y,
\end{array}
\quad
\begin{array}{ccc}
V^\vee \otimes W^\vee & \xrightarrow{\tau_{V^\vee,W^\vee}} & W^\vee \otimes V^\vee \\
\varphi_{V^\vee,W^\vee} & \downarrow & \varphi_{W^\vee,V^\vee} \\
Y^\vee & \rightarrow & Y^\vee.
\end{array}
\]

It will be convenient to introduce the following shorthand. We set $[W] := V \otimes W^\vee$, $\varphi_{[V],[X]} := \varphi_{S,X} \otimes \varphi_{S^\vee,V^\vee}$, $\varphi_{[X],[S]} := \varphi_{S,X} \otimes \varphi_{S^\vee,V^\vee}$, $\varphi_{[V],[X]} := \varphi_{S,X} \otimes \varphi_{S^\vee,V^\vee}$, $\varphi_{[X],[S]} := \varphi_{S,X} \otimes \varphi_{S^\vee,V^\vee}$. If $(V, W) = (S, X)$ or $(X, S)$ and $\lambda_{[V],[W]} \in \text{End}(I)$, we will consider the following diagram:

\[
\begin{array}{ccc}
[V] \otimes [W] & \xrightarrow{\tau_{[V],[W]}} & [W] \otimes [V] \\
\varphi_{[V],[W]} & \downarrow & \varphi_{[W],[V]} \\
Y & \rightarrow & Y.
\end{array}
\]

**Remark 3.1.** We have

$$(\text{Com})_{\lambda_{V,W}} \text{ and } (\text{Com})_{\lambda_{V^\vee, W^\vee}} \text{ commutative } \Rightarrow (\text{Com})_{\lambda_{[V],[W]}} \text{ with } \lambda_{[V],[W]} = \lambda_{V,W} \cdot \lambda_{V^\vee, W^\vee}.$$

**Proposition 3.2.** If $(\text{Cas})_{\mu_{S,X}}$ is satisfied and $(\text{Com})_{\lambda_{[S],[X]}}$ is commutative, then the following diagrams are commutative:

\[
\begin{array}{ccc}
S \otimes S^\vee & \xrightarrow{D_{S,X} \otimes D_{S^\vee,X^\vee}} & S^\vee \otimes S \\
\mu_{S,X} \cdot \text{ev}_S & \downarrow & \mu_{S,X} \cdot \text{ev}_S \\
X^\vee \otimes Y^\vee \otimes X \otimes Y^\vee & \xrightarrow{\lambda_{[S],[X]} \cdot \text{ev}_{13,24}} & X \otimes Y^\vee \otimes X^\vee \otimes Y^\vee \\
\lambda_{[S],[X]} \cdot \text{ev}_{13,24} & \downarrow & \lambda_{[S],[X]} \cdot \text{ev}_{13,24} \\
\end{array}
\]

Similarly if $(\text{Cas})_{\mu_{X,S}}$ is satisfied and $(\text{Com})_{\lambda_{[X],[S]}}$ is commutative, we get the analogue commutative diagram where $(S, X)$ is replaced by $(X, S)$.

**Proof.** It is clear that the two diagrams are equivalently commutative, so that suffices to prove the commutativity of the first diagram. It follows from Proposition 2.14(3) that we have

$$\lambda_{[S],[X]} \cdot \text{ev}_{13,24} \circ \left( D_{S,X} \otimes D_{S^\vee,X^\vee} \right) = \lambda_{[S],[X]} \cdot \text{ev}_{13,24} \circ \left( \varphi_{S[X]} \otimes 1_{[V^\vee]} \right) \circ (1_{[S]} \otimes C_X \otimes C_{Y^\vee}).$$

In order to compute the right hand side, consider the following diagram:
Here \((A)\) is commutative by the adjoint property of Lemma \([2.11]\) \((B) = (\text{Com})_{\lambda_{[S],[X]} \otimes 1_{[Y^{\vee}]}}\) is commutative by assumption, the equality \(\mu_{S,X^\cdot 1[S]} \otimes C_{S^\vee} = (1[S] \otimes \varphi_{i[X],[S]}) \circ (1[S] \otimes C_X \otimes C_{Y^\vee})\) is assured by \((\text{Cas})_{\mu_{S,X^\cdot 1}}\), \((C)\) is commutative by definition of \(i_S\), \((D)\) is clearly commutative and \((E)\) by definition of \(ev^{2,\tau}_{13,24}\) and \(ev^{\tau,\phi}_{14,23}\). We deduce the first of the subsequent equalities, while that second follows from \(ev^{\tau,\phi}_{14,23} \circ (\tau_{S^{\vee},S^\vee} \otimes C_{S^\vee}) = ev_{S^\vee}\) granted by Lemma \([2.8]\) (1) and the third by definition of \(i_S\):

\[
\lambda_{[S],[X]} \cdot ev^{\tau,\tau}_{13,24} \circ (D_{i,S,X} \otimes D_{i,S^\vee,X^\vee}) = \mu_{S,X^\cdot 1} \cdot ev^{\tau,\phi}_{14,23} \circ (\tau_{S^{\vee},S^\vee} \otimes C_{S^\vee}) \circ (i_S \otimes 1_{S^\vee}) = \\
\mu_{S,X^\cdot 1} \cdot ev_{S^\vee} \circ (i_S \otimes 1_{S^\vee}) = \mu_{S,X^\cdot 1} \cdot ev_{S^\vee}.
\]

We end the proof of the proposition by rearranging the left hand side of this equality by looking at the following diagram:

Here \((F)\) is commutative by \([26]\), while \((G)\) is commutative by definition of \(i_W\) for \(W = X\) and \(Y\). The claimed commutativity follows.

Since the roles of \(S\) and \(X\) are symmetric, we get the same commutative diagram where \((S,X)\) is replaced by \((X,S)\) if \((\text{Cas})_{\mu_{X,S}}\) is satisfied and \((\text{Com})_{\lambda_{[X],[S]}}\) is commutative.

\begin{lemma}
If \((\text{Com})_{\lambda_{X^{\vee},S^\vee}}\) is commutative, the following diagrams are commutative:

\[
\begin{array}{ccc}
X^\vee \otimes S^\vee & \overset{D_{X^\vee,S^\vee} \otimes 1_{S^\vee}}{\longrightarrow} & S \otimes Y^\vee \otimes S^\vee \\
1_{X^\vee} \otimes D_{S^\vee,X^\vee} & \lambda_{X^\vee,S^\vee} \cdot ev^{1,1}_{X,X^\vee} & \\
X^\vee \otimes X \otimes Y^\vee & \overset{ev_X \otimes 1_{Y^\vee}}{\longrightarrow} & Y^\vee,
\end{array}
\]

\[
\begin{array}{ccc}
S^\vee \otimes X^\vee & \overset{1_S \otimes D_{X^\vee,S^\vee}}{\longrightarrow} & S^\vee \otimes S \otimes Y^\vee \\
D_{S^\vee,X} \otimes 1_{X^\vee} & \lambda_{X^\vee,S^\vee} \cdot ev_{X^\vee} \otimes 1_{Y^\vee} & \\
X \otimes Y^\vee \otimes X^\vee & \overset{ev_{1,1}_{Y,X^\vee}}{\longrightarrow} & Y^\vee.
\end{array}
\]

Similarly, if \((\text{Com})_{\lambda_{S^\vee,X^{\vee}}}\) is commutative, we get the analogue commutative diagram where \((S,X)\) is replaced by \((X,S)\).

If \((\text{Com})_{\lambda_{X,S}}\) is commutative, the following diagrams are commutative:

\[
\begin{array}{ccc}
X \otimes S & \overset{D_{X,S^\vee} \otimes 1_S}{\longrightarrow} & S^\vee \otimes Y^{\vee\vee} \otimes S \\
1_X \otimes D_{S^\vee,X^\vee} & \lambda_{X,S^\vee} \cdot ev^{1,1}_{X^\vee,Y^\vee} & \\
X \otimes X \otimes Y^{\vee\vee} & \overset{ev_X \otimes 1_{Y^{\vee\vee}}}{\longrightarrow} & Y^{\vee\vee},
\end{array}
\]

\[
\begin{array}{ccc}
S \otimes X & \overset{1_S \otimes D_{X^\vee,S^\vee}}{\longrightarrow} & S \otimes S^\vee \otimes Y^{\vee\vee} \\
D_{S,X} \otimes 1_X & \lambda_{X,S^\vee} \cdot ev_{X^\vee} \otimes 1_{Y^{\vee\vee}} & \\
X \otimes Y^{\vee\vee} \otimes X & \overset{ev_{1,1}_{Y,X^\vee}}{\longrightarrow} & Y^{\vee\vee}.
\end{array}
\]

Similarly, if \((\text{Com})_{\lambda_{S,X}}\) is commutative, we get the analogue commutative diagram where \((S,X)\) is replaced by \((X,S)\).
\end{lemma}
Proof. It is clear that the first two diagrams are equivalently commutative, so that suffices to prove the commutativity of the first diagram. Consider the following diagram:

\[
\begin{align*}
S^\vee \otimes X^\vee & \xrightarrow{\tau_{X^\vee \otimes X}^\vee} X^\vee \otimes S^\vee & \xrightarrow{\tau_{X^\vee \otimes X}} S^\vee \otimes X^\vee & \xrightarrow{\tau_{S^\vee \otimes X}} X^\vee \otimes S^\vee \\
1_{S^\vee \otimes X^\vee} \otimes \iota & \xrightarrow{(\tau)} 1_{X^\vee \otimes S^\vee} \otimes \iota & \xrightarrow{(\tau)} 1_{S^\vee \otimes X^\vee} \otimes \iota & \xrightarrow{(\tau)} 1_{X^\vee \otimes S^\vee} \otimes \iota \\
S^\vee \otimes X^\vee \otimes Y \otimes Y^\vee & \xrightarrow{\phi_{X^\vee \otimes Y} \otimes Y^\vee} X^\vee \otimes Y \otimes Y^\vee & \xrightarrow{\phi_{X^\vee \otimes Y} \otimes Y^\vee} S^\vee \otimes X^\vee \otimes Y \otimes Y^\vee & \xrightarrow{\phi_{S^\vee \otimes Y} \otimes Y^\vee} X^\vee \otimes S^\vee \otimes Y \otimes Y^\vee \\
1_{S^\vee \otimes \iota} \otimes \iota & \xrightarrow{(A)} \iota \otimes \iota & \xrightarrow{(B)} \iota \otimes \iota & \xrightarrow{1_{X^\vee \otimes \iota} \otimes \iota} \\
S^\vee \otimes S^\vee & \xrightarrow{\iota \otimes \iota} Y^\vee \otimes Y^\vee & \xrightarrow{\iota \otimes \iota} Y^\vee \otimes Y^\vee & \xrightarrow{\iota \otimes \iota} \\
\iota \otimes \iota & \xrightarrow{(A)} \iota \otimes \iota & \xrightarrow{(A)} \iota \otimes \iota & \\
Y^\vee & \xrightarrow{\iota \otimes \iota} Y^\vee & \xrightarrow{\iota \otimes \iota} Y^\vee & \\
\iota \otimes \iota & \xrightarrow{\iota \otimes \iota} \iota \otimes \iota & \\
Y^\vee & \xrightarrow{\iota \otimes \iota} Y^\vee & \\
\end{align*}
\]

The regions (A) are commutative by the adjoint property of \(\phi_{X^\vee \otimes Y} \otimes Y^\vee\) and \(\phi_{S^\vee \otimes Y} \otimes Y^\vee\), while (B) = (Com)\(\lambda_{X^\vee \otimes Y} \otimes 1_{Y^\vee}\) by our assumption. Recalling that \(D_{S^\vee \otimes X^\vee}^\ast = (\phi_{S^\vee \otimes X^\vee}^\ast \otimes 1_{Y^\vee}) \circ (1_{X^\vee \otimes Y^\vee} \circ C_Y)\) and \(D_{S^\vee \otimes X^\vee}^\ast = (\phi_{S^\vee \otimes X^\vee}^\ast \otimes 1_{Y^\vee}) \circ (1_{S^\vee \otimes Y^\vee} \circ C_Y)\) by (27), we deduce

\begin{align*}
(ev_X \otimes 1_{Y^\vee}) \circ (1_{X^\vee \otimes D_{S^\vee \otimes X^\vee}^\ast}) &= (ev_X \otimes 1_{Y^\vee}) \circ (1_{X^\vee \otimes \phi_{S^\vee \otimes X^\vee}^\ast} \otimes 1_{Y^\vee}) \circ (1_{X^\vee \otimes S^\vee} \otimes C_Y) \circ \tau_{S^\vee \otimes X^\vee} \circ \tau_{X^\vee \otimes S^\vee} \\
&= \lambda_{X^\vee \otimes S^\vee} \circ (ev_S \otimes 1_{Y^\vee}) \circ (1_{S^\vee \otimes \phi_{S^\vee \otimes X^\vee}^\ast} \otimes 1_{Y^\vee}) \circ (1_{S^\vee \otimes X^\vee} \circ C_Y) \circ \tau_{X^\vee \otimes S^\vee} \\
&= \lambda_{X^\vee \otimes S^\vee} \circ (ev_S \otimes 1_{Y^\vee}) \circ \tau_{S^\vee \otimes X^\vee} \circ (D_{S^\vee \otimes X^\vee}^\ast \otimes 1_{S^\vee}) \circ \tau_{X^\vee \otimes S^\vee} \\
&= \lambda_{X^\vee \otimes S^\vee} \circ ev_{Y^\vee,1} \circ (D_{S^\vee \otimes X^\vee}^\ast \otimes 1_{S^\vee}).
\end{align*}

The commutativity of the other diagrams is proved in a similar way. \(\square\)

**Remark 3.4.** If \(Y\) is a reflexive object the canonical morphism

\[\text{Hom}(X,Y) \to \text{Hom}(Y^\vee \otimes X, \iota)\]

mapping \(f : X \to Y\) to \(ev_{Y,1} \circ (1_{Y^\vee} \circ f) : Y^\vee \otimes X \xrightarrow{1_{Y^\vee} \otimes f} Y^\vee \otimes Y \xrightarrow{ev_{Y^\vee,1}} \iota\) is a bijection.

We can now prove the main result of this section.

**Theorem 3.5.** Suppose that (Cas)\(\mu_{S,X}\) is satisfied and that (Com)\(\lambda_{S^\vee,[S]}\) is commutative. Then we have

\[\mu_{S,X} : S \xrightarrow{D_{S^\vee,X^\vee}} X^\vee \otimes Y^\vee \xrightarrow{D_{X^\vee,S^\vee} \otimes 1_{Y^\vee}} S \otimes Y^\vee \otimes Y^\vee \xrightarrow{1_{S} \otimes ev_{Y^\vee} \circ \lambda_{S^\vee,[S]} \circ \lambda_{X^\vee \otimes S^\vee} S,}
\]

if (Com)\(\lambda_{X^\vee \otimes S^\vee} \circ \lambda_{X^\vee \otimes S^\vee}\) is commutative,

\[\mu_{S,X} \lambda_{S^\vee,X^\vee} : S \xrightarrow{D_{S^\vee,X^\vee}} X^\vee \otimes Y^\vee \xrightarrow{D_{X^\vee,s^\vee} \otimes 1_{Y^\vee}} S \otimes Y^\vee \otimes Y^\vee \xrightarrow{1_{S} \otimes ev_{Y^\vee} \circ \lambda_{S^\vee,[S]} \circ \lambda_{X^\vee \otimes S^\vee} S,}
\]

if (Com)\(\lambda_{S^\vee,X^\vee} \circ \lambda_{X^\vee \otimes S^\vee}\) is commutative.

Suppose that (Cas)\(\mu_{X,S}\) is satisfied and that (Com)\(\lambda_{X,[S]}\) is commutative. Then we have

\[\mu_{X,S} : X^\vee \xrightarrow{D_{X^\vee,S^\vee}} S \otimes Y^\vee \xrightarrow{D_{S^\vee,X^\vee} \otimes 1_{Y^\vee}} X^\vee \otimes Y^\vee \otimes Y^\vee \xrightarrow{ev_{Y^\vee} \circ \lambda_{X,[S]} \circ \lambda_{X^\vee \otimes S^\vee} X^\vee,}
\]

if (Com)\(\lambda_{S^\vee,X^\vee} \circ \lambda_{X^\vee \otimes S^\vee}\) is commutative,

\[\mu_{X,S} \lambda_{X^\vee \otimes S^\vee} : X^\vee \xrightarrow{D_{X^\vee,S^\vee}} S \otimes Y^\vee \xrightarrow{D_{S^\vee,X^\vee} \otimes 1_{Y^\vee}} X^\vee \otimes Y^\vee \otimes Y^\vee \xrightarrow{ev_{Y^\vee} \circ \lambda_{X,[S]} \circ \lambda_{X^\vee \otimes S^\vee} X^\vee,}
\]

if (Com)\(\lambda_{X^\vee \otimes S^\vee} \circ \lambda_{X^\vee \otimes S^\vee}\) is commutative.
We have the similar statements exchanging the roles of $S$ and $X$ in the assumptions and the claims.

**Proof.** Suppose that $(\text{Cas})_{\mu_{S,X}}$ is satisfied, that $(\text{Com})_{\lambda_{[S],[X]}}$ is commutative and consider the following diagram:

Here the region $(A)$ is commutative by Proposition 3.2, $(B)$ by Lemma 3.3 when $(\text{Com})_{\lambda_X^{sv},sv}$ is commutative and $(C)$ by by definition of $ev^{T,2}_{13,24}$, $ev^{13,Y}_{Y}$ and $ev^{13}_Y$. We deduce, setting $a := \lambda_{[S],[X]}\lambda_X^{sv},sv \cdot (1_S \otimes ev^{13}_Y) \circ (D_X^{sv} \otimes 1_Y) \circ D_S$, the equality

$$ev_S \circ (1_S \otimes a) = ev_S \circ (1_S \otimes \mu_{S,X}).$$

Hence, by Remark 3.4 we get $a = \mu_{S,X}$. The commutativity of the other diagrams is proved in a similar way. \hfill \Box

As an immediate consequence of Theorem 3.5 we get, in light of Remark 3.4, the following result.

**Corollary 3.6.** Suppose that $(\text{Cas})_{\mu_{S,X}}$ and $(\text{Cas})_{\mu_{X,S}}$ are satisfied, that $(\text{Com})_{\lambda_{X},X}$, $(\text{Com})_{\lambda_{S},S}$ and $(\text{Com})_{\lambda_{X},sv}^{sv}$ are commutative, that $\mu_{S,X}, \mu_{X,S}, \lambda_{S,X}, \lambda_{S}^{sv}, \lambda_{X,S}$ and $\lambda_{X},sv$ are invertible and that $S$ is an invertible object. Then $D_{S,X}, D_{X,Y}, D_{S,Y}, f_{S,X}, f_{X,S}, f_{S,Y}$ and $f_{X,S}$ are isomorphisms.

Another important result for us will be the following corollary of Theorem 3.5. Define the following morphisms

$$\varphi_{X,Y}^{13} : X \otimes Y \otimes S \otimes Y \rightarrow X \otimes S \otimes Y \otimes Y,$$

$$\varphi_{X,Y}^{13} : X \otimes Y \otimes S \otimes Y \rightarrow X \otimes S \otimes Y \otimes Y,$$

as well as

$$\varphi_{X,Y}^{13} : X \otimes Y \otimes S \otimes Y \otimes Y \rightarrow X \otimes S \otimes Y \otimes Y,$$

$$\varphi_{X,Y}^{13} : X \otimes Y \otimes S \otimes Y \otimes Y \rightarrow X \otimes S \otimes Y \otimes Y.$$

**Corollary 3.7.** Suppose that $(\text{Cas})_{\mu_{S,X}}$ is satisfied and that $(\text{Com})_{\lambda_{[S],[X]}}, (\text{Com})_{\lambda_{X,S}}$ and $(\text{Com})_{\lambda_{X},sv}$ are commutative. Then, setting $\mu := \mu_{S,X}$ and $\lambda := \lambda_{[S],[X]}\lambda_X^{sv},sv \lambda_{X,S}$, the following diagrams are commutative:
Proof. Suppose that \((Cas) \nu_{s,x}\) is satisfied and that \((Com) \lambda_{[S],[X]}, \lambda_{X,s}\) and \((Com) \lambda_{X,s}\) are commutative. Consider the following diagram:

\[
\begin{array}{c}
S \otimes X \xrightarrow{D_{s,x} \otimes 1_Y} X^Y \otimes 1_Y \otimes X \xrightarrow{\mu_{s,x} \otimes 1_Y} Y^Y \\
S \otimes S^Y \otimes Y^Y \xrightarrow{(A)} S \otimes S^Y \otimes Y^Y \xrightarrow{\mu_{s,x} \otimes 1_Y} Y^Y \\
X^Y \otimes S \otimes S^Y \otimes Y^Y \xrightarrow{(C)} S \otimes Y^Y \otimes S^Y \otimes Y^Y \\
X^Y \otimes S^Y \otimes Y^Y \otimes Y^Y \xrightarrow{(D)} S \otimes Y^Y \otimes S^Y \otimes Y^Y \\
\end{array}
\]

The region \((A)\) is commutative by Lemma 3.3, the commutativity of \((B)\) is clear, \((C)\) is commutative by Theorem 3.5 and \((D)\) by definition of the evaluation maps (we have written \(ev_{14} := ev_{14,Y^Y \otimes Y^Y \otimes Y^Y}\) for shortness and similarly for \(ev_{13}\)). Recalling that we have, by definition, \(D_{s,x} = D_{s,x}\) and \(D_{X^Y,s} = D_{s,x}^{\otimes Y^Y}\), it follows from Lemma 2.7 we have \(\tau_{X^Y,s} \otimes 1_Y = (D_{s,x} \otimes 1_Y) \otimes \varphi_{X^Y,s^Y} \otimes 1_Y \otimes Y^Y = ev_{13,Y^Y \otimes Y^Y \otimes Y^Y} (D_{s,x} \otimes 1_Y \otimes Y^Y \otimes Y^Y)\). Hence the commutative diagram gives the claimed equality:

\[
\mu_{s,x} \cdot \varphi_{X^Y,s} = \lambda_{[S],[X]} \lambda_{X,s} \cdot (ev_{13,Y^Y \otimes Y^Y} \circ \varphi_{X^Y,s^Y} \otimes 1_Y \otimes Y^Y) \circ (1_X \otimes \tau_{X^Y,s^Y} \otimes 1_Y \otimes Y^Y) \circ (D_{s,x} \otimes D_{s,s^Y}) = \lambda_{[S],[X]} \lambda_{X,s} \cdot \varphi_{X^Y,s^Y} \circ (D_{s,x} \otimes D_{s,s^Y}).
\]

The commutativity of the other diagram is proved in the same way.

\[\square\]

Corollary 3.8. Suppose that \((Cas) \nu_{s,x}\) is satisfied, that \((Com) \lambda_{[S],[X]}, \lambda_{X,s}\) and \((Com) \lambda_{X,s}\) are commutative and that \(Y\) is invertible of rank \(r_Y\) (so that \(r_Y \in \{\pm 1\}\)). For every morphism \(g : A \to Y\) of \(Y\) and \(h : C \to Y\) of \(Y\), the following diagrams are commutative, where \(\mu\) and \(\lambda\) are as in Corollary 3.7:

\[
\begin{array}{c}
A \otimes S \otimes X \xrightarrow{D_{s} \otimes \varphi_{s,x}} B \otimes Y^Y \otimes Y \\
A \otimes C \otimes X \xrightarrow{D_{s} \otimes \varphi_{s,x}} C \otimes Y^Y \otimes Y \\
A \otimes Y \otimes Y^Y \otimes Y^Y \xrightarrow{\lambda_{13}} B \otimes Y^Y \otimes Y^Y \\
\end{array}
\]

Proof. Consider the following diagram:

\[
\begin{array}{c}
A \otimes S \otimes X \xrightarrow{1_{A} \otimes \varphi_{s,x}} A \otimes Y \xrightarrow{D_{s} \otimes 1_Y} B \otimes Y^Y \otimes Y \\
A \otimes X \otimes Y^Y \otimes S \otimes Y^Y \xrightarrow{(A)} A \otimes Y \otimes Y^Y \otimes Y \xrightarrow{(B)} A \otimes Y \otimes Y^Y \otimes Y \\
A \otimes Y \otimes Y^Y \otimes Y^Y \xrightarrow{\lambda_{13}} A \otimes Y \otimes Y^Y \otimes Y \\
\end{array}
\]

The region \((A)\) is commutative by Corollary 3.7, The region \((B)\) because \(r_Y = r_Y^Y := ev_{13,Y^Y \otimes Y^Y \otimes Y^Y}\), implying that \(C_{Y^Y} = ev_{13,Y^Y \otimes Y^Y \otimes Y^Y} \circ C_{Y^Y}\) and the functoriality of \(\otimes\). Finally \((C)\) is commutative by \(27\). The commutativity of the first diagram follows and the commutativity of the second diagram is proved in the same way. \[\square\]
4. Application to \(\Delta\)-graded algebras in \(C\)

If \(\Delta\) is a commutative integral semigroup, a \(\Delta\)-graded algebra in \(C\) is a family \(A = (A_i, \varphi_{i,j})_{i,j \in \Delta}\) where \(\varphi_{i,j} = \varphi^A_{i,j} : A_i \otimes A_j \rightarrow A_{i+j}\) are morphisms making the following diagrams commutative:

\[
\begin{array}{c}
A_i \otimes A_j \otimes A_k \xrightarrow{1_{A_i} \otimes \varphi_{j,k}} A_i \otimes A_{j+k} \\
\varphi_{i,j} \otimes 1_{A_k} \downarrow \quad \varphi_{i,j+k} \downarrow \\
A_{i+j} \otimes A_k \xrightarrow{\varphi_{i+j,k}} A_{i+j+k},
\end{array}
\]  

(29)

There is an obvious notion of morphisms of \(\Delta\)-graded algebras and of direct sum decomposition, given component-wise. If \(i, j \in \Delta\), we write \(j \geq i\) to mean that \(3(j-i) \in \Delta\) (unique because \(\Delta\) is integral) such that \((j-i) + i = j\).

If \(j \geq i\), we have \(\varphi_{i,j-i} = \varphi_{f_{i,j-i}} : A_i \otimes A_{j-i} \rightarrow A_j\), where \(f_{i,j-i} = f^A_{i,j-i} : A_i \rightarrow \text{hom}(A_{j-i}, A_j)\), so that we may consider the associated internal multiplication morphism

\[\epsilon_{i,j} := \epsilon_{f_{i,j-i}} : A_i \rightarrow \text{hom}(A^\vee_j, A^\vee_{j-i})\] corresponding to \(\varphi_{i,j} : A_i \otimes A_j^\vee \rightarrow A_{i+j}^\vee\).

Suppose that, for every \(i \in \Delta\), we have given a biproduct decomposition \(A_i = A_i^+ \oplus A_i^-\) and write \(i^+_W\) and \(e^+_W\) as usual for \(W = A_i\) or \(A_i^\vee\) and we also set \(i^+_W = i^+_W\), \(e^+_W = e^+_W\) when \(W = A_i\) and \(p^+_W = p^+_W\) when \(W = A_i^\vee\). We make a choice \(\epsilon : \Delta \rightarrow \{\pm\}\) of factors for every \(i \in \Delta\) that we may assume, without loss of generality, to be given by the constant function \(\epsilon_i = \pm\).

Next we consider

\[f_{i,j}^+: A_i^+ \rightarrow A_i \xrightarrow{f_{i,j}^+} \text{hom}(A_j, A_{i+j}),\]

\[\varphi_{i,j}^+ : A_i^+ \otimes A_j^+ \rightarrow A_i \otimes A_j \rightarrow A_{i+j},\]

and, for \(j \geq i\),

\[\epsilon_{i,j} : A_i^\vee \rightarrow A_i \xrightarrow{\epsilon_{i,j}} \text{hom}(A^\vee_j, A^\vee_{j-i}),\]

\[\varphi_{i,j}^\vee : A_i^\vee \otimes A_j^\vee \rightarrow A_i \otimes A_j \rightarrow A_{i+j}^\vee,\]

The following result is a restatement of Lemma 2.3 and Proposition 2.5.

**Proposition 4.1.** The following diagrams are commutative.

1. When \(j \geq i\)

\[
\begin{array}{c}
A_i \otimes A_{j-i} \otimes A_j^\vee \xrightarrow{(1_{A_{j-i}} \otimes \varphi_{i,j}) \circ (\tau_{A_i, A_{j-i}} \otimes 1_{A_j^\vee})} A_{j-i} \otimes A_j^\vee \\
\varphi_{i,j-i} \otimes 1_{A_j^\vee} \downarrow \quad \epsilon_{A_{j-i}} \downarrow \\
A_j \otimes A_j^\vee \\
\end{array}
\]

2. When \(k \geq i\) and \(k - i \geq j\),

\[
\begin{array}{c}
A_j \otimes A_i \otimes A_k^\vee \xrightarrow{1_{A_j} \otimes \varphi_{i,k}} A_j \otimes A_k^\vee \\
\varphi_{i,k} \otimes 1_{A_j^\vee} \downarrow \quad \varphi_{i,j-i} \downarrow \\
A_{i+j} \otimes A_k^\vee \xrightarrow{\varphi_{i+j,k}} A^\vee_{j-i} \rightarrow A_{i+j+k},
\end{array}
\]

where

\[\varphi_{i,j}^\vee : A_i \otimes A_j \rightarrow A_i \otimes A_j \rightarrow A_{i+j}^\vee,\]

\[\varphi_{i,j} : A_i \otimes A_j \rightarrow A_i \otimes A_j \rightarrow A_{i+j}^\vee,\]

\[\varphi_{i,j}^+: A_i \otimes A_j \rightarrow A_i \otimes A_j \rightarrow A_{i+j}^+,\]

\[\varphi_{i,j}^- : A_i \otimes A_j \rightarrow A_i \otimes A_j \rightarrow A_{i+j}^-,\]

\[\varphi_{i,j}^\vee : A_i \otimes A_j \rightarrow A_i \otimes A_j \rightarrow A_{i+j}^\vee,\]

Integrality means that we may left (and right) simplify.
Suppose that $e_{i,j}^+ \circ \varphi_{i,j} \circ (e_i^+ \otimes 1_{A_j}) = e_{i,j}^+ \circ \varphi_{i,j} \circ (1_{A_i} \otimes e_j^+) = e_{i,j}^+ \circ \varphi_{i,j}$. Then $A^+ := (A_{i,j}^+, \varphi_{i,j}^+)_{i,j} \in \Delta$ is a $\Delta$-graded algebra in $C$, we have $\varphi_{i,j}^+ = \varphi_{i,j}$, the internal multiplications (i.e. $f_{i,j}^+$ associated to this algebra structure are explicitly given by $\varphi_{i,j}^+ = \varphi_{i,j}$ and we have the analogue of the above commutative diagrams with the $+$ sign inserted. Furthermore, we have a biproduct decomposition $A = A^+ \oplus A^-$ as $\Delta$-graded algebras, where $A^- := (A_{i,j}^-, \varphi_{i,j}^-)_{i,j} \in \Delta$ satisfies the analogue results.

We will assume from now on that we have given $\Delta$-graded algebras $A = (A_{i,j}^+, \varphi_{i,j}^+)_{i,j} \in \Delta$ and $A^\gamma = (A_{i,j}^\gamma, \varphi_{i,j}^\gamma)_{i,j} \in \Delta$ and $C$ is rigid. Then we define a $\Delta \times \Delta$-graded family $A \otimes A^\gamma := (A_{i,j}^f, \varphi_{i,j}^f)_{i,j,(k,l)} \in \Delta$ by the rule $A_{i,j}^f := A_i \otimes A_j^f$ and

$$\varphi_{i,j}^{f,k} := \varphi_{i,k} \circ \varphi_{j,l} : A_i \otimes A_j^k \otimes A_l \rightarrow A_{i,j}^{k+l},$$

associated to $f_{i,j}^{l,m} := f_{i,k}^l \otimes f_{k,l}^m : A_{i,j}^l \rightarrow \hom (A_{i,j}^{l+m}, A_{i,j}^{k+l})$ by Lemma 2.10. It is easily checked that $A \otimes A^\gamma$ is indeed a $\Delta \times \Delta$-graded algebra.

Next we define, when $l \geq i$ and $k \geq j$,

$$\delta_{i,k}^A := \varphi_{i,j} \circ \varphi_{l,m} : A_{i,j}^l \otimes A_{j,m}^k \rightarrow A_{i,j}^{l+m},$$

$$\delta_{i,k}^A := \varphi_{i,j} \circ \varphi_{l,m} : A_{i,j}^l \otimes A_{j,m}^k \rightarrow A_{i,j}^{l+m},$$

and

$$\delta_{i,k}^{f,l} := \delta_{i,k}^{f,l} \circ \delta_{i,k}^{f,l} : A_{i,j}^l \otimes A_{j,m}^k \rightarrow A_{i,j}^{l+m},$$

associated to $f_{i,j}^{l,m} := f_{i,k}^l \otimes f_{k,l}^m : A_{i,j}^l \rightarrow \hom (A_{i,j}^{l+m}, A_{i,j}^{k+l})$ by Lemma 2.10. Applying Proposition 4.1 to $A \otimes A^\gamma$ we find, thanks to Corollary 2.12 and 24, the following result.

**Corollary 4.2.** The following diagrams are commutative.

1. When $l \geq i$ and $k \geq j$,

\[
\begin{array}{ccc}
A_{i,j}^l \otimes A_{j,m}^k & \xrightarrow{(1_{A_{i,j}^{l-i}} \otimes \delta_{i,k}^{f,l}) \circ \tau_{A_{i,j}^{l-i}}} & A_{i,j}^{l+m} \\
\phi_{j,l}^{i,k} \circ 1_{A_{i,j}^{l-i}} & \downarrow & \downarrow \psi_{j,l}^{i,k} \\
A_{i,j}^{l+m} & \xrightarrow{\psi_{j,l}^{i,k}} & A_{i,j}^{l+m} \\
\end{array}
\]

2. When $n \geq i$, $m \geq j$, $n-i \geq k$ and $m-j \leq l$,

\[
\begin{array}{ccc}
A_{i,j}^l \otimes A_{j,m}^k \otimes A_{n-m,l}^m & \xrightarrow{1_{A_{i,j}^{l+i}} \otimes \delta_{i,k}^{m,l}} & A_{i,j}^{l+k} \otimes A_{n-m,l}^m \\
\phi_{j,l}^{i,k} \circ 1_{A_{i,j}^{l+i}} & \downarrow & \downarrow \psi_{j,l}^{i,k} \\
A_{i,j}^{l+k} \otimes A_{n-m,l}^m & \xrightarrow{\psi_{j,l}^{i,k}} & A_{i,j}^{l+k} \otimes A_{n-m,l}^m \\
\end{array}
\]

where

$$\psi_{j,l}^{i,k} : A_{i,j}^{l+k} \otimes A_{n-m,l}^m \rightarrow A_{i,j}^{l+k} \otimes A_{n-m,l}^m$$

We define, when $g \geq i$,

$$D^{i,g} := D_{i,g}^{i-1} : A_i \rightarrow A_{i-1} \otimes A_i^g$$

and

$$D_{i,g} := D_{i,g}^g : A_i \rightarrow A_{i-1} \otimes A_i^g$$

We leave to the reader to restate the result of the previous section in this context.
4.1. **Symmetric and alternating algebras.** Suppose now that we have given a rigid and pseudo-abelian object \( V \in \mathcal{C} \) and that \( \mathcal{C} \) is \( \mathbb{Q} \)-linear. The associativity constraint of \( \mathcal{C} \) implies that we may define an \( N \)-graded tensor algebra \( \otimes^i V \) by the rule \( \varphi_{V,i} := 1_{i+1} \otimes V : (\otimes^i V) \otimes (\otimes^j V) \to \otimes^{i+j} V \).

The permutation group \( S_i \) acts on \( \otimes^i V \) and, for a character \( \chi \) of \( S_i \) and a subset \( S \subseteq S_i \), we define

\[
e_{\chi}^{\otimes^i V} := \frac{1}{\# S} \sum_{\sigma \in S} \chi^{-1}(\sigma) \sigma.
\]

There are exactly two characters of \( S_i \), namely \( \chi = \varepsilon \) (the sign character) and \( \chi = 1 \) (the trivial character), which are distinct when \( i \geq 2 \). We let \( \wedge^i V \) (resp. \( \vee^i V \)) the bidual of \( \otimes^i V \) which corresponds to the idempotent \( e_{\chi}^{\otimes^i V} := e_{\chi}^{\otimes^i V} \) (resp. \( e_{\varepsilon}^{\otimes^i V} := e_{\varepsilon}^{\otimes^i V} \)) which exists because we assume that \( V \) is pseudo-abelian. We write \( i_{V,a}^i : \wedge^i V \to \otimes^i V \) (resp. \( i_{V,a}^i : \vee^i V \to \otimes^i V \)) and \( p_{V,a}^i : \otimes^i V \to \wedge^i V \) (resp. \( p_{V,a}^i : \otimes^i V \to \vee^i V \)) for the associated injective and surjective morphisms. Next we claim that setting

\[
\varphi_{V,a}^{\otimes^i V} : \wedge^i V \otimes \wedge^j V \xrightarrow{i_{V,a}^{\otimes^i V} \otimes i_{V,a}^{\otimes^j V}} (\otimes^i V) \otimes (\otimes^j V) \xrightarrow{\varphi_{V,i}^{\otimes^i V}} \otimes^{i+j} V \Rightarrow \wedge^{i+j} V,
\]

\[
\varphi_{V,a}^{\vee^i V} : \vee^i V \otimes \vee^j V \xrightarrow{i_{V,a}^{\vee^i V} \otimes i_{V,a}^{\vee^j V}} (\otimes^i V) \otimes (\otimes^j V) \xrightarrow{\varphi_{V,i}^{\vee^i V}} \otimes^{i+j} V \Rightarrow \vee^{i+j} V
\]

give rise to \( N \)-graded algebras, called respectively the alternating and the symmetric algebras on \( V \). Since \( \varphi_{V,i}^{\otimes^i V} = 1_{i+1} \otimes V \) we may check, according to Proposition \ref{prop:symmetric-alternating-algebras}, that we have \( e_{\chi}^{\otimes^i V} \circ (e_{\varepsilon}^{\otimes^i V}) = e_{\varepsilon}^{\otimes^i V} \) and \( e_{\varepsilon}^{\otimes^i V} \circ (e_{\chi}^{\otimes^i V}) = e_{\chi}^{\otimes^i V} \) for \( \chi = \varepsilon \) or \( \chi = 1 \). But we have that the action of \( e_{\chi}^{\otimes^i V} \circ 1_{\otimes^i V} \in End \otimes^{i+j} V \) (resp. \( 1_{\otimes^i V} \circ e_{\chi}^{\otimes^i V} \in End (\otimes^{i+j} V) \)) is obtained by identifying \( S_i \simeq S_{i+1} \subset S_{i+j} \) (resp. \( S_j \simeq S_{i+1} \subset S_{i+j} \)) and it is given by \( e_{\chi}^{\otimes^i V} \in \mathbb{Q}[S_{i+j}] \) (resp. \( e_{\varepsilon}^{\otimes^i V} \in \mathbb{Q}[S_{i+j}] \)). Hence the claimed relation follows from the identity \( e_{\varepsilon}^{\otimes^i V} e_{\varepsilon}^{\otimes^j V} = e_{\varepsilon}^{\otimes^i V} + e_{\varepsilon}^{\otimes^j V} \) taking place in \( \mathbb{Q}[S_{i+j}] \).

We note that it follows from the definitions that \( \varphi_{V,a}^{\otimes^i V} \) and \( \varphi_{V,a}^{\vee^i V} \) are uniquely characterized by the property of making the following diagrams commutative:

\[
\begin{array}{c}
(\otimes^i V) \otimes (\otimes^j V) \xrightarrow{\varphi_{V,a}^{\otimes^i V} \otimes \varphi_{V,a}^{\otimes^j V}} \otimes^{i+j} V \\
\downarrow p_{V,a}^i \otimes p_{V,a}^j \downarrow \\
\wedge^i V \otimes \wedge^j V \xrightarrow{i_{V,a}^{\otimes^i V} \otimes i_{V,a}^{\otimes^j V}} (\otimes^i V) \otimes (\otimes^j V) \xrightarrow{\varphi_{V,i}^{\otimes^i V}} \wedge^{i+j} V
\end{array}
\]

\[
\begin{array}{c}
(\otimes^i V) \otimes (\otimes^j V) \xrightarrow{\varphi_{V,a}^{\otimes^i V} \otimes \varphi_{V,a}^{\otimes^j V}} \otimes^{i+j} V \\
\downarrow p_{V,a}^i \otimes p_{V,a}^j \downarrow \\
\wedge^i V \otimes \wedge^j V \xrightarrow{i_{V,a}^{\otimes^i V} \otimes i_{V,a}^{\otimes^j V}} (\otimes^i V) \otimes (\otimes^j V) \xrightarrow{\varphi_{V,i}^{\otimes^i V}} \wedge^{i+j} V
\end{array}
\]

Next we remark that we may use rigidity of \( V \) to canonically identify \( \varphi_{V,a}^{\otimes^i V} : (\otimes^i V)^\vee \xrightarrow{\varphi_{V,a}^{\otimes^i V}} \varphi_{V,\vee}^{\otimes^i V} \) in other words \( (\otimes^i V)^\vee, \varphi_{V,a}^{\otimes^i V} \) is a dual pair for \( \otimes^i V \). Next, we define

\[
e_{\chi}^{\otimes^i V} : \wedge^i V \otimes \wedge^j V \xrightarrow{i_{V,a}^{\otimes^i V}} (\otimes^i V) \otimes (\otimes^j V) \xrightarrow{\varphi_{V,i}^{\otimes^i V}} \wedge^{i+j} V,
\]

\[
e_{\chi}^{\otimes^i V} : \vee^i V \otimes \wedge^j V \xrightarrow{i_{V,a}^{\otimes^i V}} (\otimes^i V) \otimes (\otimes^j V) \xrightarrow{\varphi_{V,i}^{\otimes^i V}} \vee^{i+j} V
\]

and we claim that \( (\wedge^i V, e_{\varepsilon}^{\otimes^i V}) \) is a dual pair for \( \wedge^i V \) and \( (\vee^i V, e_{\varepsilon}^{\otimes^i V}) \) is a dual pair for \( \vee^i V \). Indeed we have \( e_{\varepsilon}^{\otimes^i V} \circ (\sigma \otimes \sigma) = e_{\varepsilon}^{\otimes^i V} \) for every \( \sigma \in S_i \), because the canonical morphism \( \otimes^i \mathbb{I} \to \mathbb{I} \) appearing in the definition of \( e_{\varepsilon}^{\otimes^i V} \) is \( S_i \)-invariant. Equivalently, \( e_{\varepsilon}^{\otimes^i V} \circ (\sigma^{-1} \otimes 1_{\wedge^i V}) = e_{\varepsilon}^{\otimes^i V} \circ (1_{\vee^i V} \otimes \sigma) \) proving that \( e_{\varepsilon}^{\otimes^i V} \) is \( S_i \)-invariant. Then our claim follows from \ref{prop:symmetric-alternating-algebras} (with \( Y = Y^\pm = \mathbb{I} \)).

5. A "Poincaré" Duality Isomorphism for the Alternating Algebras

In this section we suppose that \( \mathcal{C} \) is rigid, \( \mathbb{Q} \)-linear and pseudo-abelian. We consider an object \( V \in \mathcal{C} \) and we apply the results on \( \Delta \)-graded algebras with \( A = (\wedge^i V, \varphi_{V,a}^{\otimes^i V}) \) and \( A^\vee = (\vee^i V, \varphi_{V,a}^{\vee^i V}) \). We will use the shorter notations \( i_{V,a}^i = i_{V,a}^{\otimes^i V}, p_{V,a}^i = p_{V,a}^{\otimes^i V}, e_{V,a}^i = e_{V,a}^{\otimes^i V, \varphi_{V,a}^{\otimes^i V}} \). In order to make explicit the internal multiplications morphisms we define, for every \( j \geq i \),

\[
\varphi_{V,a}^{i,j} : (\otimes^i V) \otimes (\otimes^j V) \to \otimes^{i+j} V.
\]
Lemma 5.1. Setting $\varphi_i, \varphi_j$, satisfies the characterizing property \([\text{15}]\) of Proposition \([\text{22}]\) with $\varphi_f = \varphi_{i,j}^{V,t} = 1_{\otimes V}$. Since the alternating algebra is obtained from the tensor algebra as in Proposition \([\text{44}]\), it follows that the internal multiplication $\iota_{i,j} = \iota_{i,j}^{V,a}$ of the alternating algebra is explicitly given by

$$
\varphi_{i,j} : \wedge^i V \otimes \wedge^j V \xrightarrow{\iota_{i,j}} (\otimes^i V) \otimes (\otimes^j V) \xrightarrow{ev_{1 \otimes \cdots \otimes 1}} \wedge^{j-i} V.
$$

In order to make this morphism completely explicit, we note that $S_i \times S_j$ acts on $(\otimes^i V) \otimes (\otimes^j V)$ and we may identify $S_{j-i} \simeq S_{(i+1, \ldots, j)} \subset S_j$ acting on $(\otimes^i V) \otimes (\otimes^j V)$. With this identification, $\varphi_{i,j} \circ \sigma = \sigma \circ \varphi_{i,j}$ for every $\sigma \in S_{j-i}$.

Lemma 5.1. Setting $\tilde{\varphi}_{i,j} := (\text{ev}^{t \otimes 1}_{V} \otimes 1_{\otimes V}) \circ (\text{ev}^{t \otimes 1}_{V} \otimes 1_{\otimes V}) = \left(\frac{(j-i)!}{j!} \sum_{p \in \mathcal{P}^{i \leq j}} \varepsilon^{-1}(\delta^{i \leq j}_p) \delta^{i \leq j}_p \right)(\text{ev}^{t \otimes 1}_{V} \otimes 1_{\otimes V}) \circ (\text{ev}^{t \otimes 1}_{V} \otimes 1_{\otimes V})$ we have that $\tilde{\varphi}_{i,j}$ is the unique morphism making the following diagram commutative:

$$
\begin{array}{ccc}
(\otimes^i V) \otimes (\otimes^j V) & \xrightarrow{\tilde{\varphi}_{i,j}} & \wedge^{j-i} V \\
\downarrow \text{ev}^{t \otimes 1}_{V} \otimes 1_{\otimes V} & & \downarrow \text{ev}^{t \otimes 1}_{V} \\
\wedge^i V \otimes \wedge^j V & \xrightarrow{\varphi_{i,j}} & \wedge^{j-i} V.
\end{array}
$$

In particular, when $i = 1$, setting $\text{ev}^{V}_{V} := (\text{ev}^{t \otimes 1}_{V} \otimes 1_{\otimes V}) \circ e_1$, we may take

$$
\tilde{\varphi}_{1,j} = \frac{1}{j} \sum_{p=1}^{j} (-1)^{p-1} \text{ev}_{0,p}.
$$

Proof. Suppose that we have given a subgroup $H \subset G$ of a finite group $G$, that $G$ acts on $X$, $H$ acts on $Y$ and that we have given $f : X \to Y$ which is $H$-equivariant. If $\chi$ is a character of $G$ and $R_{H \backslash G}$ is a set of coset representatives for $H \backslash G$, we may consider the elements $e^X_G$, $e^Y_H$ and $e^X_H$ defined as in \([\text{30}]\) with $S = G$, $H$ or $R_{H \backslash G}$ and $S_i$ replaced by a more general group $G$. We have that $e^X_G$ and $e^Y_H$ are idempotents and we let $p_G^X : X \to X^X$ and $p_Y^X : Y \to Y^X$ be the associated surjective morphisms and $i^X_X$ the associated injective morphism. Then it is a general fact that, setting $f_{R_{H \backslash G}} := f \circ e^X_{R_{H \backslash G}}$ and $f^X := p_Y^X \circ f \circ i^X_X$, the morphism $f^X$ is characterized as the unique morphism such that $f^X \circ p_G^X = p_Y^X \circ f_{R_{H \backslash G}}$.

We may apply this general remark with $X = (\otimes^i V) \otimes (\otimes^j V)$, $Y = \otimes^{j-i} V$, $f = \text{ev}^{t \otimes 1}_{V} \otimes 1_{\otimes V}$, $H = S_{(i+1, \ldots, j)}$ and $G = S_i \times S_j$; since $f$ is $H$-equivariant by \([\text{22}]\) and $R$ is a set of coset representatives

$^8$We have, symbolically,

$$
ev_{0,p}(f \otimes x_1 \otimes \cdots \otimes x_p \otimes \cdots \otimes x_j) = (f, x_p) \chi_1 \otimes \cdots \hat{x}_p \otimes \cdots \otimes x_j
$$
for \( S_{i+1,\ldots,j} \setminus S_i \times S_j \), we deduce that \( f^\epsilon = \varphi_{i,j} \) is the unique morphism such that \( f^\xi \circ p_1^X = p_1^Y \circ f_{R_\mu \cap G} \), where thanks to Corollary 4.3 and the equality \( \# R = \frac{(j-i)!}{j!} \),

\[
 f_{R_{\mu \cap G}} = (ev_1^{\nu,\tau} \otimes 1_{\otimes^{j-i}V}) \circ (e_1^{i} \otimes e^{i-j} \otimes^{j-i}V). 
\]

□

Beside the properties encoded in Proposition 4.4, the internal multiplication morphisms \( \iota_{i,j} \) has another key property. In symbols it says that the normalized family \( \iota_j = j \cdot \iota_{1,j} \) is an antiderivation, i.e.

\[
 \iota_{j+1}(x)(\omega \wedge \omega_j) = \iota_{j+1}(x)(\omega_j) + (-1)^j \iota_{j+1}(x)(\omega_i) \text{ for } x \in V, \omega_j \in \wedge^j V^\vee \text{ and } \omega \in \wedge^l V^\vee.
\]

This is the content of the following proposition whose proof, based on Lemma 5.1 we leave to the reader.

**Proposition 5.2.** The following diagram is commutative, when \( j, l \geq 1 \):

\[
\begin{array}{ccc}
V \otimes \wedge^j V^\vee \otimes \wedge^l V^\vee & \xrightarrow{(j+l)\varphi_{i,j+l}} & \wedge^{j+l} V^\vee \\
1_V \otimes \varphi_{j,l} \downarrow & & \downarrow \varphi_{j+1,l-1} \\
V \otimes \wedge^{j+l} V^\vee & \xrightarrow{(j+l)\varphi_{i,j+l}} & \wedge^{j+l} V^\vee.
\end{array}
\]

Working with the dual algebras one easily sees that, setting

\[
\varphi_{i,j}, \ast : = ev_1^{\nu} \otimes 1_{\otimes^{j-i}V} : \left( \otimes^j V^\vee \right) \otimes \left( \otimes^i V \right) \rightarrow \otimes^{j-i} V,
\]

the morphism \( \varphi_{i,j}, \ast \), satisfies the property 23 with \( \varphi_g = \varphi_{i,j}, t = 1 \otimes^i V \), which is of course characterizing.

It follows that \( \iota_{i,j}^{1} = \iota_{i,j}, \ast = \iota_{i,j}^{A} \ast \) is obtained in the analogous way as \( \iota_{i,j} \) was obtained and the analogous of Proposition 5.2 is true.

Exactly as we did with more general \( \Delta \)-graded algebras, we can now define, when \( l \geq i \) and \( k \geq j \), \( \delta_{i,l}^A := \varphi_{i,l} : \wedge^i V \otimes \wedge^l V^\vee \rightarrow \wedge^{l-i} V^\vee \), \( \delta_{j,k}^A := \varphi_{j,l} : \wedge^j V^\vee \otimes \wedge^k V \rightarrow \wedge^{k-j} V^\vee \) and

\[
\delta_{i,j}^{i,k} := \delta_{j,k} \otimes \varphi_{i,l} : \wedge^i V \otimes \wedge^j V^\vee \otimes \wedge^k V \rightarrow \wedge^{j-i} V^\vee \otimes \wedge^{k-j} V^\vee, \text{ associated to } \delta_{i,j}^{i,k} := \iota_{i,j} : \wedge^i V \rightarrow \hom \left( \wedge^i V, \wedge^{j-i} V^\vee \right),
\]

where \( \wedge^j V := \wedge^j V \otimes \wedge^j V^\vee \). Beside the properties encoded in Corollary 4.2, the following property is enjoyed by the families \( \delta_{i,l}^{A, p} \) and \( \delta_{l,q}^{A, p} \); the proof is just an application of Proposition 5.2 and its dual statement for the second diagram.

**Corollary 5.3.** If \( j, l \geq 1 \) then the following diagram is commutative:

\[
\begin{array}{ccc}
\Lambda^0_1 V \otimes \Lambda^j_1 V \otimes \Lambda^k_1 V & \xrightarrow{(j+k)\delta_{i,j}^{i,k}} & \Lambda^{j+k}_1 V \\
1_{\Lambda^0_1 V} \otimes \varphi_{j+k}^{i,k} \downarrow & & \downarrow \varphi_{j+k+1}^{i,k} \\
\Lambda^0_1 V \otimes \Lambda^{j+k} V & \xrightarrow{(j+k)\delta_{i,j}^{i,k}} & \Lambda^{j+k} V.
\end{array}
\]

The following diagram is commutative, when \( i, k \geq 1 \):

\[
\begin{array}{ccc}
\Lambda^0_1 V \otimes \Lambda^i_1 V \otimes \Lambda^k_1 V & \xrightarrow{(i+k)\delta_{i,j}^{i,k}} & \Lambda^{i+k} V \\
1_{\Lambda^0_1 V} \otimes \varphi_{j+k}^{i,k} \downarrow & & \downarrow \varphi_{j+k+1}^{i,k} \\
\Lambda^0_1 V \otimes \Lambda^{i+k} V & \xrightarrow{(i+k)\delta_{i,j}^{i,k}} & \Lambda^{i+k} V.
\end{array}
\]

The proof of the following lemma, which is postponed to the subsequent subsection, is based on Corollaries 4.2 and 5.3.
Lemma 5.4. Let $r := \text{rank} (V)$ be the rank of $V$, defined as the composition
\[ r : I \xrightarrow{C_V} V \otimes V^\vee \xrightarrow{ev_V^r} I. \]

For every $g \geq i$ we have the equality
\[ \binom{g}{i}^{-1} \binom{r + i - g}{i} C_{\Lambda^i V} = \delta_{i,g} \circ (C_{\Lambda^i V} \otimes C_{\Lambda^g V}), \]
where, for every $k \in \mathbb{N}_{\geq 1},$
\[ \binom{T}{k} := \frac{1}{k!} T (T - 1) \ldots (T - k + 1) \in \mathbb{Q}[T] \text{ and } \binom{T}{0} = 1. \]

We define, when $g \geq i,$
\[ D^{i,g} := D_{i,g} : \Lambda^i V \rightarrow \Lambda^{g-i} V^\vee \otimes \Lambda^g V^\vee \text{ and } D_{i,g} : D^{i,g} : \Lambda^i V \rightarrow \Lambda^{g-i} V \otimes \Lambda^g V^\vee. \]

Thanks to Lemma 5.4, the commutative diagrams of Proposition 3.2, Lemma 3.3, Theorem 3.5, and, respectively, Corollary 3.7, translate into the following result.

Theorem 5.5. The following diagrams are commutative, for every $g \geq i \geq 0.$

\begin{enumerate}
  \item \[
  \begin{array}{c}
  \Lambda^i V \otimes \Lambda^i V \\
  \xrightarrow{D^{i,g} \otimes D_{i,g}} \\
  \Lambda^{g-i} V^\vee \otimes \Lambda^g V^\vee \xrightarrow{ev_{13,24}} I.
  \end{array}
  \]
  \item \[
  \begin{array}{c}
  \Lambda^i V \otimes \Lambda^{g-i} V \\
  \xrightarrow{D^{g-i} \otimes D_{g-i}} \\
  \Lambda^{g-i} V^\vee \otimes \Lambda^g V^\vee \xrightarrow{ev_{13,24}} I.
  \end{array}
  \]
  \item \[
  \begin{array}{c}
  \Lambda^i V \\
  \xrightarrow{D^{i,g}} \\
  \Lambda^{g-i} V \otimes \Lambda^g V^\vee \xrightarrow{D_{g-i} \otimes 1_{\Lambda^g V^\vee}} \\
  \Lambda^{g-i} V \otimes \Lambda^g V^\vee \xrightarrow{ev_{13,24}} \Lambda^i V
  \end{array}
  \]
  and
  \[
  \begin{array}{c}
  \Lambda^{g-i} V \\
  \xrightarrow{D_{g-i}} \\
  \Lambda^i V \otimes \Lambda^{g-i} V \\
  \xrightarrow{D^{i,g} \otimes 1_{\Lambda^g V}} \Lambda^{g-i} V \otimes \Lambda^g V^\vee \\
  \xrightarrow{ev_{13,24}} \Lambda^{g-i} V
  \end{array}
  \]
  \item \[
  \begin{array}{c}
  \Lambda^i V \otimes \Lambda^{g-i} V \\
  \xrightarrow{\varphi_{i,g^{-1}}} \\
  \Lambda^{g-i} V \\
  \xrightarrow{D^{i,g} \otimes \varphi_{i,g}} \Lambda^{g-i} V \otimes \Lambda^g V^\vee \\
  \xrightarrow{ev_{13,24}} \Lambda^{g-i} V \\
  \xrightarrow{\varphi_{i,g^{-1}}} \Lambda^g V
  \end{array}
  \]
  and
  \[
  \begin{array}{c}
  \Lambda^i V \otimes \Lambda^{g-i} V \\
  \xrightarrow{D_{i,g} \otimes \varphi_{i,g^{-1}}} \\
  \Lambda^{g-i} V \otimes \Lambda^g V^\vee \\
  \xrightarrow{D^{i,g} \otimes \varphi_{i,g^{-1}}} \Lambda^{g-i} V \otimes \Lambda^g V^\vee \\
  \xrightarrow{ev_{13,24}} \Lambda^{g-i} V \otimes \Lambda^g V^\vee
  \end{array}
  \]
\end{enumerate}
Proof. The commutative diagrams (1), (2), (3) and (4) are just Proposition 3.2, Lemma 3.3, Theorem 3.5 and, respectively, Corollary 3.7 with \((S, X, Y) = (\wedge^i V, \wedge^g V, \varphi_{g-i})\), \(\varphi_{S, X} = \varphi_{g-i}, \varphi_{X, S} = \varphi_{g-i}, \varphi_{S, X} = \varphi_{g-i}, \varphi_{X, S} = \varphi_{g-i}\). The morphisms \((\varphi_{S, X})_V, \varphi_{X, S})_V, \varphi_{S, X} = \varphi_{g-i}, \varphi_{X, S} = \varphi_{g-i}\) are isomorphisms and the multiplication maps \(\varphi_{g-i}^V, \varphi_{g-i}^V, \varphi_{g-i}^V, \varphi_{g-i}^V\) are perfect pairings (meaning that the associate hom valued morphisms are isomorphisms). For further, we have \((\varphi_{g-i}^V)^{-1} = (\varphi_{g-i}^V)^{-1}(g-i)^i\) is a field or \((g-i)^i\) is a path diagram in the commutative diagrams of Theorem 3.5.

We say that \(V\) has alternating rank \(g \in \mathbb{N}\) if \(\wedge^g V\) is an invertible object and \((\overline{r_i})\) and \((\overline{r_i}+1)\) are invertible for every \(0 \leq i \leq g\). For example, when \(End(1)\) is a field or \(r \in \mathbb{Q}\), the second condition means that \(r\) is not a root of the polynomials \((\overline{r_i}^i) \in \mathbb{Q}[T]\) and \((\overline{r_i}+1) \in \mathbb{Q}[T]\) for every \(0 \leq i \leq g\), i.e. that \(r \neq i, i+1, \ldots, g-1\) and \(r \neq g-i, g-i+1, \ldots, g-1\) for every \(1 \leq i \leq g\).

We say that \(V\) has strong alternating rank \(g \in \mathbb{N}\) if \(\bigwedge^g V\) is an invertible object and \(r = (g-i)\) (hence \(V\) has alternating rank \(g\)). With these notations Corollary 3.6 specializes to the following result.

Corollary 5.6. If \(V\) has weak geometric rank \(g \in \mathbb{N}\) then, for every \(0 \leq i \leq g\), the morphisms \(D^i g\), \(D^g-i g\), \(D^g-i g\) and \(D^g-i g\) are isomorphisms and the multiplication maps \(\varphi_{g-i}^V, \varphi_{g-i}^V, \varphi_{g-i}^V, \varphi_{g-i}^V\) are perfect pairings (meaning that the associate hom valued morphisms are isomorphisms). Furthermore, when \(V\) has geometric rank \(g\), we have \((\overline{r_i}^i) = (\overline{r_i}+1) = 1\) in the commutative diagrams of Theorem 5.5.

We end this section with the following result.

Proposition 5.7. The following diagrams are commutative, when \(\bigwedge^g V\) is invertible of rank \(r_{\wedge^g V}\) (hence \(r_{\wedge^g V} \in \{1, -1\}\)):

\[
\lhd V \otimes \bigwedge^g V \otimes V \xrightarrow{T_{\lhd V \otimes \bigwedge^g V \otimes V}} V \otimes \lhd V \otimes \bigwedge^g V
\]

\[
(1_{\lhd V} \otimes \varphi_{g-i}, 1_{\bigwedge^g V} \otimes \varphi_{1_{V}}) \circ (T_{\lhd V \otimes \bigwedge^g V \otimes V})
\]

\[
\bigwedge^g V \otimes \bigwedge^g V \otimes \bigwedge^g V \otimes \bigwedge^g V \otimes \bigwedge^g V
\]

\[
D^i_{\bigwedge^g V} \otimes D^g-i_{\bigwedge^g V} \otimes D^g-i_{\bigwedge^g V} \otimes D^g-i_{\bigwedge^g V}
\]

\[
\bigwedge^g V \otimes \bigwedge^g V \otimes \bigwedge^g V \otimes \bigwedge^g V \otimes \bigwedge^g V
\]

where

\[
M = r_{\wedge^g V} \otimes \bigwedge^g V \otimes \bigwedge^g V \otimes \bigwedge^g V \otimes \bigwedge^g V \otimes \bigwedge^g V
\]

and

\[
\lhd V \otimes \bigwedge^g V \otimes \bigwedge^g V \otimes \bigwedge^g V \otimes \bigwedge^g V \otimes \bigwedge^g V
\]

\[
(1_{\lhd V} \otimes \varphi_{1_{V} \otimes (\bigwedge^g V \otimes \bigwedge^g V \otimes \bigwedge^g V \otimes \bigwedge^g V \otimes \bigwedge^g V)}
\]

\[
\bigwedge^g V \otimes \bigwedge^g V \otimes \bigwedge^g V \otimes \bigwedge^g V \otimes \bigwedge^g V \otimes \bigwedge^g V
\]

\[
D_{1_{\bigwedge^g V} \otimes \bigwedge^g V \otimes \bigwedge^g V \otimes \bigwedge^g V \otimes \bigwedge^g V \otimes \bigwedge^g V}
\]

\[
\bigwedge^g V \otimes \bigwedge^g V \otimes \bigwedge^g V \otimes \bigwedge^g V \otimes \bigwedge^g V \otimes \bigwedge^g V
\]

\[
(-1)^{g-i} \varphi_{1_{\bigwedge^g V} \otimes \bigwedge^g V \otimes \bigwedge^g V \otimes \bigwedge^g V \otimes \bigwedge^g V \otimes \bigwedge^g V}
\]

Proof. We first apply Corollary 3.8 with

\[
g = i, g : V \rightarrow \text{hom}(\bigwedge^g V, \bigwedge^g V)\text{ and } (S, X, Y) = (\lhd V, \bigwedge^g V, \bigwedge^g V).\]

Since \(r_{\wedge^g V} = r_{\wedge^g V}^{-1}, \text{ the result is that, setting } \mu := (g-i)^{-1}(g-i),\)

\[
r_{\wedge^g V} \mu \circ (1_{\lhd V \otimes \bigwedge^g V \otimes \bigwedge^g V} \otimes \varphi_{i, g}) \circ (D_{\bigwedge^g V} \otimes \varphi_{g-i, g}) \circ T_{\lhd V \otimes \bigwedge^g V \otimes \bigwedge^g V, V} \]

\[
= (\varphi_{i, g}) \circ (1_{\lhd V \otimes \bigwedge^g V \otimes \bigwedge^g V} \otimes (1 \otimes D_{\bigwedge^g V} \otimes \varphi_{g-i, g}) \circ (1 \otimes D_{\bigwedge^g V} \otimes \varphi_{g-i, g}) \circ T_{\lhd V \otimes \bigwedge^g V \otimes \bigwedge^g V, V, V}. \quad (34)
\]
Here we recall that, by definition,

\[ \varphi_{g-i,g}^{13} := (\varphi_{g-i,i} \otimes 1_{\Lambda^g V^V \otimes \Lambda^g V^V}) \circ (1_{\Lambda^g V^V} \otimes \tau_{\Lambda^g V^V, \Lambda^i V^V} \otimes 1_{\Lambda^g V^V}) . \]

Hence we find

\[
g \cdot \left( \varphi_{1,i,g} \otimes 1_{\Lambda^g V^V \otimes \Lambda^g V^V} \right) \circ (1_V \otimes \varphi_{g-i,g}^{13}) = g \cdot \left( \varphi_{1,i,g} \otimes 1_{\Lambda^g V^V \otimes \Lambda^g V^V} \right) \\
\circ (1_V \otimes \varphi_{g-i,i} \otimes 1_{\Lambda^g V^V \otimes \Lambda^g V^V}) \circ (1_V \otimes \tau_{\Lambda^g V^V, \Lambda^i V^V} \otimes 1_{\Lambda^g V^V}) \\
= g \cdot \left( \varphi_{1,i,g} \circ (1_V \otimes \varphi_{g-i,i}) \otimes 1_{\Lambda^g V^V \otimes \Lambda^g V^V} \right) \circ (1_V \otimes \tau_{\Lambda^g V^V, \Lambda^i V^V} \otimes 1_{\Lambda^g V^V}) = a + b(35)
\]

where

\[
a := (g - i) \cdot (\varphi_{g-i-1,i} \otimes 1_{\Lambda^g V^V \otimes \Lambda^g V^V}) \circ \left( \varphi_{1,i,g-1} \otimes \tau_{\Lambda^g V^V, \Lambda^i V^V} \otimes 1_{\Lambda^g V^V} \right), \\
b := (-1)^{g-i} \cdot (\varphi_{g-i-1} \otimes 1_{\Lambda^g V^V \otimes \Lambda^g V^V}) \circ \left( (1_{\Lambda^g V^V} \otimes \varphi_{1,i}) \otimes 1_{\Lambda^g V^V \otimes \Lambda^g V^V} \right) \\
\circ (\tau_{\Lambda^g V^V, \Lambda^i V^V} \otimes 1_{\Lambda^g V^V})
\]

and where we have used the equality

\[
g \cdot \varphi_{1,i,g} \circ (1_V \otimes \varphi_{g-i,i}) = (g - i) \cdot \varphi_{g-i-1,i} \circ \left( \varphi_{1,i,g-1} \otimes 1_{\Lambda^i V^V} \right) \\
+ (-1)^{g-i} \cdot \varphi_{g-i-1} \circ \left( 1_{\Lambda^g V^V} \otimes \varphi_{1,i} \right) \circ (\tau_{\Lambda^g V^V, \Lambda^i V^V} \otimes 1_{\Lambda^i V^V})
\]

of Proposition 5.2 at the end. Inserting (35) in (34) yields

\[
r_{\Lambda^g V^V} g \cdot (D^{i,g} \otimes \varphi_{i,g}) \circ \tau_{\Lambda^i V^V, \Lambda^g V^V} = a \circ (1_V \otimes D^{i,g} \otimes D^{g-i,g}) \circ \tau_{\Lambda^i V^V, \Lambda^g V^V} \\
+ b \circ (1_V \otimes D^{i,g} \otimes D^{g-i,g}) \circ \tau_{\Lambda^i V^V, \Lambda^g V^V}. \tag{36}
\]

We now compute \( a \circ (1_V \otimes D^{i,g} \otimes D^{g-i,g}) \circ \tau_{\Lambda^i V^V, \Lambda^g V^V} \), using the following formulas:

\[
D^{i,g} \otimes D^{g-i,g} \\
= \left( \varphi_{i,g} \otimes 1_{\Lambda^i V^V} \otimes \varphi_{g-i,g} \otimes 1_{\Lambda^g V^V} \right) \circ (1_{\Lambda^i V^V} \otimes C_{\Lambda^g V^V} \otimes 1_{\Lambda^g V^V} \otimes C_{\Lambda^i V^V}) \quad \text{(by 27)} \tag{37}
\]

\[
(\varphi_{g-i-1,i} \otimes 1_{\Lambda^g V^V \otimes \Lambda^g V^V}) \circ (1_{\Lambda^g V^V} \otimes \tau_{\Lambda^i V^V, \Lambda^i V^V} \otimes 1_{\Lambda^g V^V}) \\
= \varphi_{g-i-1,i}^{13} \quad \text{(by definition)} \tag{38}
\]

\[
\varphi_{1,i,g} \circ (1_V \otimes \varphi_{i,g}) = \varphi_{1,i,g+1} \circ (\varphi_{1,i} \otimes 1_{\Lambda^i V^V}) \\
= (-1)^i \cdot \varphi_{1,i+1,g} \circ (\varphi_{1,i} \otimes 1_{\Lambda^i V^V}) \quad \text{(by Prop. 11.1 (2))} \tag{39}
\]

\[
(\varphi_{i+1,g} \otimes 1_{\Lambda^i V^V} \otimes \varphi_{g-i,g} \otimes 1_{\Lambda^g V^V}) \circ (1_{\Lambda^i V^V} \otimes C_{\Lambda^g V^V} \otimes 1_{\Lambda^g V^V} \otimes C_{\Lambda^i V^V}) \\
= D^{i+1,g} \otimes D^{g-i,g} \quad \text{(by 24)}. \tag{40}
\]
We have

\[
a \circ (1_V \otimes D^{i,g} \otimes D^{g-i,g}) \circ \tau_{\Lambda^i V \otimes \Lambda^{g-i} V, V} = (g - i) \cdot (\varphi_{g-i-1,i} \otimes 1_{\Lambda^g V \otimes \Lambda^g V})
\]

\[
\circ (1_V \otimes D^{i,g} \otimes D^{g-i,g}) \circ \tau_{\Lambda^i V \otimes \Lambda^{g-i} V, V} \quad \text{(by 37)}
\]

\[
= (g - i) \cdot (\varphi_{g-i-1,i} \otimes 1_{\Lambda^g V \otimes \Lambda^g V}) \circ (1_{\Lambda^g V \otimes \Lambda^g V} \otimes \tau_{\Lambda^g V \otimes \Lambda^g V, V})
\]

\[
\circ \left( \varphi_{g-i-1,i} \otimes 1_{\Lambda^g V \otimes \Lambda^g V} \otimes 1_{\Lambda^g V \otimes \Lambda^g V} \right) \circ \left( 1_V \otimes \varphi_{i,g} \otimes 1_{\Lambda^g V \otimes \varphi_{g-i-1,g} \otimes 1_{\Lambda^g V} \otimes \varphi_{g-i-1,g} \otimes 1_{\Lambda^g V} \right)
\]

\[
= (g - i) \cdot \varphi_{g-i-1,i} \circ \left( \varphi_{i,g} \otimes 1_{\Lambda^g V \otimes \varphi_{g-i-1,g} \otimes 1_{\Lambda^g V} \otimes \varphi_{g-i-1,g} \otimes 1_{\Lambda^g V} \right)
\]

\[
\circ (1_V \otimes C_{\Lambda^g V} \otimes 1_{\Lambda^g V} \otimes C_{\Lambda^g V}) \circ \tau_{\Lambda^i V \otimes \Lambda^{g-i} V, V} \quad \text{(by 38)}
\]

\[
= (g - i) \cdot \varphi_{g-i-1,i} \circ \left( \varphi_{i,g} \otimes 1_{\Lambda^g V \otimes \varphi_{g-i-1,g} \otimes 1_{\Lambda^g V} \otimes \varphi_{g-i-1,g} \otimes 1_{\Lambda^g V} \right)
\]

\[
\circ (1_V \otimes C_{\Lambda^g V} \otimes 1_{\Lambda^g V} \otimes C_{\Lambda^g V}) \circ \tau_{\Lambda^i V \otimes \Lambda^{g-i} V, V} \quad \text{(by 39)}
\]

\[
= (-1)^i (g - i) \cdot \varphi_{g-i-1,i} \circ \left( \varphi_{i,g} \otimes 1_{\Lambda^g V \otimes \varphi_{g-i-1,g} \otimes 1_{\Lambda^g V} \otimes \varphi_{g-i-1,g} \otimes 1_{\Lambda^g V} \right)
\]

\[
\circ (1_{\Lambda^i V} \otimes C_{\Lambda^g V} \otimes 1_{\Lambda^g V} \otimes C_{\Lambda^g V}) \circ \tau_{\Lambda^i V \otimes \Lambda^{g-i} V, V} \quad \text{(by 40)}
\]

\[
= (-1)^i (g - i) \cdot \varphi_{g-i-1,i} \circ \left( \varphi_{i,g} \otimes 1_{\Lambda^g V \otimes \varphi_{g-i-1,g} \otimes 1_{\Lambda^g V} \otimes \varphi_{g-i-1,g} \otimes 1_{\Lambda^g V} \right)
\]

\[
\circ (1_{\Lambda^i V} \otimes C_{\Lambda^g V} \otimes 1_{\Lambda^g V} \otimes C_{\Lambda^g V}) \circ \tau_{\Lambda^i V \otimes \Lambda^{g-i} V, V} \quad \text{(by 41)}
\]

We compute \( b \circ (1_V \otimes D^{i,g} \otimes D^{g-i,g}) \circ \tau_{\Lambda^i V \otimes \Lambda^{g-i} V, V} \), using the following formulas:

\[
\varphi_{g-i-1,i} \otimes 1_{\Lambda^g V \otimes \Lambda^g V} \circ \varphi_{i,g} \circ 1_{\Lambda^g V \otimes \Lambda^g V} \circ \varphi_{g-i-1,g} \otimes 1_{\Lambda^g V \otimes \Lambda^g V} \circ 1_{\Lambda^g V \otimes \Lambda^g V} \quad \text{(by definition)}
\]

\[
\varphi_{i,g} \circ 1_{\Lambda^g V \otimes \Lambda^g V} \circ \varphi_{g-i-1,g} \circ 1_{\Lambda^g V \otimes \Lambda^g V} \quad \text{(by definition)}
\]

\[
\varphi_{g-i-1,g} \circ 1_{\Lambda^g V \otimes \Lambda^g V} \circ \varphi_{i,g} \circ 1_{\Lambda^g V \otimes \Lambda^g V} \quad \text{(by definition)}
\]

\[
= (-1)^{g-i} \cdot \varphi_{g-i-1,g} \circ 1_{\Lambda^g V \otimes \Lambda^g V} \quad \text{(by Prop. 1.11 (2))}
\]

\[
\varphi_{g-i-1,g} \circ 1_{\Lambda^g V \otimes \Lambda^g V} \circ \varphi_{i,g} \circ 1_{\Lambda^g V \otimes \Lambda^g V} \circ \varphi_{g-i-1,g} \circ 1_{\Lambda^g V \otimes \Lambda^g V} \circ 1_{\Lambda^g V \otimes \Lambda^g V} \quad \text{(by 27)}
\]

\[
\circ (1_{\Lambda^i V} \otimes C_{\Lambda^g V} \otimes 1_{\Lambda^g V} \otimes C_{\Lambda^g V}) \circ \tau_{\Lambda^i V \otimes \Lambda^{g-i} V, V} \quad \text{(by 41)}
\]

\[
= (-1)^{g-i} \cdot \varphi_{g-i-1,g} \circ 1_{\Lambda^g V \otimes \Lambda^g V} \circ \varphi_{i,g} \circ 1_{\Lambda^g V \otimes \Lambda^g V} \circ \varphi_{g-i-1,g} \circ 1_{\Lambda^g V \otimes \Lambda^g V} \circ 1_{\Lambda^g V \otimes \Lambda^g V}
\]

\[
\circ \left( \varphi_{i,g} \otimes 1_{\Lambda^g V \otimes \Lambda^g V} \otimes \varphi_{g-i-1,g} \otimes 1_{\Lambda^g V \otimes \Lambda^g V} \otimes \varphi_{g-i-1,g} \otimes 1_{\Lambda^g V \otimes \Lambda^g V} \right)
\]

\[
\circ (1_{\Lambda^i V} \otimes C_{\Lambda^g V} \otimes 1_{\Lambda^g V} \otimes C_{\Lambda^g V}) \circ \tau_{\Lambda^i V \otimes \Lambda^{g-i} V, V} \quad \text{(by 41)}
\]

\[
= (-1)^{g-i} \cdot \varphi_{g-i-1,g} \circ \left( \varphi_{i,g} \otimes 1_{\Lambda^g V \otimes \Lambda^g V} \otimes \varphi_{g-i-1,g} \otimes 1_{\Lambda^g V \otimes \Lambda^g V} \otimes \varphi_{g-i-1,g} \otimes 1_{\Lambda^g V \otimes \Lambda^g V} \right)
\]

\[
\circ (1_{\Lambda^i V} \otimes C_{\Lambda^g V} \otimes 1_{\Lambda^g V} \otimes C_{\Lambda^g V}) \circ \tau_{\Lambda^i V \otimes \Lambda^{g-i} V, V} \quad \text{(by 41)}
\]

\[
= (-1)^{g-i} \cdot \varphi_{g-i-1,g} \circ \varphi_{i,g} \circ 1_{\Lambda^g V \otimes \Lambda^g V} \circ \varphi_{g-i-1,g} \circ 1_{\Lambda^g V \otimes \Lambda^g V} \circ \varphi_{g-i-1,g} \circ 1_{\Lambda^g V \otimes \Lambda^g V}
\]

\[
\circ (1_{\Lambda^i V} \otimes C_{\Lambda^g V} \otimes 1_{\Lambda^g V} \otimes C_{\Lambda^g V}) \circ \tau_{\Lambda^i V \otimes \Lambda^{g-i} V, V} \quad \text{(by 41)}
\]
We have

\[ b \circ (1_V \otimes D^{i,g} \otimes D^{g-1,i,g}) \circ \tau_{\lambda^i V \otimes \lambda^g V \otimes V} = (-1)^{g-i} \cdot \varphi_{g-i,i-1} \otimes 1_{\lambda^g V \otimes \lambda^g V \otimes V} \]

\[ \circ \left( 1_{\lambda^g V \otimes \lambda^g V \otimes V} \otimes \varphi_{i,i,g} \otimes 1_{\lambda^g V \otimes \lambda^g V \otimes V} \right) \circ (\tau_{V \otimes \lambda^g V \otimes V} \circ \tau_{\lambda^g V \otimes \lambda^g V \otimes V} \otimes 1_{\lambda^g V \otimes \lambda^g V \otimes V}) \]

\[ \circ (1_V \otimes D^{j,g} \otimes D^{g-1,j,g}) \circ \tau_{\lambda^j V \otimes \lambda^g V \otimes V} \quad \text{(by (37))} \]

\[ = (-1)^{g-i} \cdot \varphi_{g-i,i-1} \otimes 1_{\lambda^g V \otimes \lambda^g V \otimes V} \otimes \varphi_{i,i,g} \otimes 1_{\lambda^g V \otimes \lambda^g V \otimes V} \]

\[ \circ \left( 1_{\lambda^g V \otimes \lambda^g V \otimes V} \otimes \varphi_{i,i,g} \otimes 1_{\lambda^g V \otimes \lambda^g V \otimes V} \otimes \varphi_{g-i,i} \otimes 1_{\lambda^g V \otimes \lambda^g V \otimes V} \right) \circ (1_V \otimes \varphi_{i,i,g} \otimes 1_{\lambda^g V \otimes \lambda^g V \otimes V}) \]

\[ \circ (1_V \otimes \varphi_{i,i,g} \otimes 1_{\lambda^g V \otimes \lambda^g V \otimes V} \otimes \varphi_{g-i,i} \otimes 1_{\lambda^g V \otimes \lambda^g V \otimes V}) \circ \tau_{\lambda^j V \otimes \lambda^g V \otimes V} \quad \text{(by (42))} \]

\[ = (-1)^{g-i} \cdot \varphi_{g-i,i-1} \circ \left( 1_{\lambda^g V \otimes \lambda^g V \otimes V} \otimes \varphi_{i,i,g} \otimes 1_{\lambda^g V \otimes \lambda^g V \otimes V} \otimes \varphi_{g-i,i} \otimes 1_{\lambda^g V \otimes \lambda^g V \otimes V} \right) \circ (1_V \otimes \varphi_{i,i,g} \otimes 1_{\lambda^g V \otimes \lambda^g V \otimes V}) \]

\[ \circ (1_V \otimes \varphi_{i,i,g} \otimes 1_{\lambda^g V \otimes \lambda^g V \otimes V} \otimes \varphi_{g-i,i} \otimes 1_{\lambda^g V \otimes \lambda^g V \otimes V}) \circ \tau_{\lambda^j V \otimes \lambda^g V \otimes V} \quad \text{(by (23))} \]

\[ \circ \left( 1_{\lambda^g V \otimes \lambda^g V \otimes V} \otimes \varphi_{i,i,g} \otimes 1_{\lambda^g V \otimes \lambda^g V \otimes V} \otimes \varphi_{g-i,i} \otimes 1_{\lambda^g V \otimes \lambda^g V \otimes V} \right) \circ (1_V \otimes \varphi_{i,i,g} \otimes 1_{\lambda^g V \otimes \lambda^g V \otimes V}) \]

\[ \circ (1_V \otimes \varphi_{i,i,g} \otimes 1_{\lambda^g V \otimes \lambda^g V \otimes V} \otimes \varphi_{g-i,i} \otimes 1_{\lambda^g V \otimes \lambda^g V \otimes V}) \circ (1_V \otimes \varphi_{i,i,g} \otimes 1_{\lambda^g V \otimes \lambda^g V \otimes V}) \circ (1_{\lambda^g V \otimes \lambda^g V \otimes V}) \quad \text{(by (43))} \]

\[ = (-1)^{g-i} \cdot \varphi_{g-i,i-1} \circ \left( 1_{\lambda^g V \otimes \lambda^g V \otimes V} \otimes \varphi_{i,i,g} \otimes 1_{\lambda^g V \otimes \lambda^g V \otimes V} \otimes \varphi_{g-i,i} \otimes 1_{\lambda^g V \otimes \lambda^g V \otimes V} \right) \circ (1_V \otimes \varphi_{i,i,g} \otimes 1_{\lambda^g V \otimes \lambda^g V \otimes V}) \]

\[ \circ (1_V \otimes \varphi_{i,i,g} \otimes 1_{\lambda^g V \otimes \lambda^g V \otimes V} \otimes \varphi_{g-i,i} \otimes 1_{\lambda^g V \otimes \lambda^g V \otimes V}) \circ (1_{\lambda^g V \otimes \lambda^g V \otimes V}) \quad \text{(by (44))} \]

\[ = (-1)^{g-i} \cdot \varphi_{g-i,i-1} \circ \left( D^{i,g} \otimes D^{g-i+1,j,g} \right) \circ (1_{\lambda^g V \otimes \lambda^g V \otimes V}) \quad \text{(46)} \]

Inserting (11) and (40) in (39) gives

\[ r_{\lambda^g V \otimes \lambda^g V \otimes V} \cdot \left( D^{i,g} \otimes \varphi_{g-i,i-1} \right) = (-1)^i (g - i) \cdot \varphi_{g-i,i-1} \circ \left( D^{i+1,g} \otimes D^{g-i,g} \right) \circ (1_{\lambda^g V \otimes \lambda^g V \otimes V}) \]

\[ + (-1)^{g-i} \cdot \varphi_{g-i,i-1} \circ \left( D^{i,g} \otimes D^{g-i+1,j,g} \right) \circ (1_{\lambda^g V \otimes \lambda^g V \otimes V}) \quad \text{(47)} \]

Another computation involving the functoriality of the \( \otimes \)-operation, that of the \( \tau \)-constraint and the anti-commutativity constraint in the alternating algebra reveals that:

\[ (-1)^i (g - i) \cdot \varphi_{g-i,i-1} \circ \left( D^{i+1,g} \otimes D^{g-i,g} \right) \circ (1_{\lambda^g V \otimes \lambda^g V \otimes V}) \]

\[ = (1)^{g-i} (g - i) \cdot \varphi_{g-i,i-1} \circ \left( D^{i,g} \otimes D^{g-i+1,j,g} \right) \circ (1_{\lambda^g V \otimes \lambda^g V \otimes V}) \quad \text{(48)} \]

The commutativity of the first diagram now follows from (47) and (48).

The second commutative diagram is obtained in a similar way, starting with Corollary 3.8 applied with \( h = t^*_V : \lambda^1 V \to \lambda^g V \lambda^g V \) and \( (S, X, Y) = (\lambda^i V, \lambda^g - i V, \lambda^g V) \) and employing the appropriate dual statements. \( \square \)

5.1. **Proof of Lemma 5.4** The proof of Lemma 5.4 will be divided into several steps. We will use the shorthand \( C_p := C_{\lambda^p V} : \mathbb{I} \to \lambda^p V \) in the sequel.

**Step 1**
We claim the commutativity of the following diagrams for every \( m \geq 1 \):

\[
\begin{array}{c}
\Lambda^m_1 V \otimes \Lambda^m_1 V \\
\downarrow \delta_{0,m+1} \otimes 1_{\Lambda^m_1 V} \\
V \\
\end{array}
\quad \quad
\begin{array}{c}
\Lambda^m_1 V \otimes \Lambda^m_1 V \\
\downarrow \delta_{0,m+1} \otimes 1_{\Lambda^m_1 V} \\
V \\
\end{array}
\]

(49)

The proof of the commutativity of the second diagram is identical to that of the first one, so we will concentrate on the first. The case \( m = 1 \) is trivial and the general case is done by induction, assuming it true for \( m \).

We will first need a simple lemma, whose proof is just an application of Lemma 2.8 (3), Lemma 2.8 (4) and (5).

**Lemma 5.8.** The following diagram is commutative, for every \( p \geq 0 \),

\[
\begin{array}{c}
\Lambda^p_1 V \otimes \Lambda^p_1 V \\
\downarrow \delta_{0,m+1} \otimes 1_{\Lambda^p_1 V} \\
V \\
\end{array}
\]

We now consider the following diagram, where

\[
f := m \cdot \delta_{0,m+1} \otimes 1_{\Lambda^m_1 V} \quad \text{and} \quad g := (-1)^{m} \cdot \left( \tau_{\Lambda^m_1 V, \Lambda^m_1 V} \otimes 1_{\Lambda^m_1 V} \right)
\]

We deduce that we have:

\[
(m + 1) \cdot \delta_{0,m+1} \otimes 1_{\Lambda^m_1 V} \otimes 1_{\Lambda^m_1 V} \circ (a, b) = a + b, \quad (50)
\]

where

\[
a = m \cdot \delta_{0,m+1} \otimes 1_{\Lambda^m_1 V} \circ (1_V \otimes C_m) \circ C_1, \quad b = (-1)^{m} \cdot \delta_{0,m+1} \otimes 1_{\Lambda^m_1 V} \circ (1_V \otimes C_m) \circ C_1.
\]

We will now derive an alternative expression for \( a \) by looking at the following diagram:

\[
\begin{array}{c}
\Lambda^m_1 V \otimes \Lambda^m_1 V \\
\downarrow \delta_{0,m+1} \otimes 1_{\Lambda^m_1 V} \\
V \\
\end{array}
\quad \quad
\begin{array}{c}
\Lambda^m_1 V \otimes \Lambda^m_1 V \\
\downarrow \delta_{0,m+1} \otimes 1_{\Lambda^m_1 V} \\
V \\
\end{array}
\]

Here the region \( A \) is again commutative by Lemma 5.8, the region \( D \) is commutative by our induction assumption \( 49 \) and the functoriality of \( \otimes \). We deduce the associativity constraint. We deduce

\[
a = m \cdot \delta_{0,m+1} \otimes 1_{\Lambda^m_1 V} \circ (1_V \otimes C_1 \otimes C_m).
\]

(51)
We now compute \( b \), noticing that \( \tau_{\Lambda^1V,\Lambda^1V} \otimes 1_{\Lambda^1V} = \tau_{\Lambda^1V \otimes \Lambda^1V, \Lambda^1V} \circ \left( 1_{\Lambda^1V} \otimes \tau_{\Lambda^1V, \Lambda^1V} \right) \) in the first of the subsequent equalities, employing the functoriality of \( \tau \) in the second one and finally appealing to the commutativity constraint \( \varphi_{m,0} \circ \tau_{\Lambda^1V, \Lambda^1V} = (-1)^m \varphi_{0,m} \) in third equality:

\[
\begin{align*}
\quad b &= (-1)^m \cdot \varphi_{m,1} \circ \left( 1_{\Lambda^1V} \otimes \tau_{\Lambda^1V, \Lambda^1V} \right) \circ \left( 1_{\Lambda^1V} \otimes \tau_{\Lambda^1V, \Lambda^1V} \right) \circ \left( 1_{\Lambda^1V} \otimes C_m \otimes C_1 \right) \\
&= (-1)^m \cdot \varphi_{m,1} \circ \tau_{\Lambda^1V, \Lambda^1V} \circ \left( \delta_{m,1} \otimes 1_{\Lambda^1V} \right) \circ \left( 1_{\Lambda^1V} \otimes C_1 \otimes C_m \right) \\
&= \varphi_{0,m} \circ \left( \delta_{0,1} \otimes 1_{\Lambda^1V} \right) \circ \left( 1_{\Lambda^1V} \otimes C_1 \otimes C_m \right). 
\end{align*}
\]

(52)

Inserting (51) and (52) in (50) we deduce

\[
(m + 1) \cdot \delta_{0,m+1} \circ (1_{\Lambda^1V} \otimes C_{m+1}) = (m + 1) \cdot \varphi_{0,m} \circ \left( \delta_{0,1} \otimes 1_{\Lambda^1V} \right) \circ (1_{\Lambda^1V} \otimes C_1 \otimes C_m),
\]

from which the claim follows.

**Step 2**

Noticing that \( \delta_{1,1}^{1,1} = (1_{\Lambda^1V} \otimes \text{ev}_V) \circ (\tau_{\Lambda^1V} \otimes 1_{\Lambda^1V}) \) and \( \delta_{1,1}^{0,1} = \text{ev}_V \otimes 1_{\Lambda^1V} \), we deduce from Lemma 2.8 (2) that we have

\[
\begin{align*}
\delta_{0,1}^{1,1} \circ (1_{\Lambda^1V} \otimes C_1) &= (1_{\Lambda^1V} \otimes \text{ev}_V) \circ (1_{\Lambda^1V} \otimes \tau_{\Lambda^1V} \otimes 1_{\Lambda^1V}) \circ (1_{\Lambda^1V} \otimes C_1) \\
&= (1_{\Lambda^1V} \otimes \text{ev}_V) \circ \tau_{\Lambda^1V \otimes \Lambda^1V} \circ (1_{\Lambda^1V} \otimes C_1) = (1_{\Lambda^1V} \otimes \text{ev}_V) \circ (C_1 \otimes 1_{\Lambda^1V}) = 1_{\Lambda^1V},
\end{align*}
\]

\[
\delta_{1,1}^{1,1} \circ (1_{\Lambda^1V} \otimes C_1) = (\text{ev}_V \otimes 1_{\Lambda^1V}) \circ (1_{\Lambda^1V} \otimes C_1) = 1_{\Lambda^1V}.
\]

Hence it follows from (49) that the following diagrams are commutative:

\[
\begin{align*}
\Lambda^1_0 V \otimes \Lambda^1_{m-1} V & \xrightarrow{\delta_{0,m-1}^{1,1}} \Lambda^1_m V \otimes \Lambda^1_{m-1} V \\
\Lambda^1_0 V \otimes \Lambda^1_{m} V & \xrightarrow{\delta_{0,m}^{1,1}} \Lambda^1_m V \otimes \Lambda^1_{m-1} V.
\end{align*}
\]

(53)

**Step 3**

Next we claim that the following diagram is commutative for every \( m \geq 2 \):

\[
\begin{align*}
\Lambda^1_0 V \otimes \Lambda^1_{m} V & \xrightarrow{\delta_{0,m}^{1,1}} \Lambda^1_m V \otimes \Lambda^1_{m-1} V \\
\Lambda^1_0 V \otimes \Lambda^1_{m} V & \xrightarrow{\delta_{0,m}^{1,1}} \Lambda^1_m V \otimes \Lambda^1_{m-1} V.
\end{align*}
\]

(54)

Consider the following diagram, where

\[
\begin{align*}
f := \delta_{1,0}^{1,1} \otimes 1_{\Lambda^1_{m-1} V} \quad \text{and} \quad g := (1 - m) \cdot \left( 1_{\Lambda^1_0 V} \otimes \delta_{1,m-1}^{0,1} \right) \circ \left( \tau_{\Lambda^1_0 V, \Lambda^1_0 V} \otimes 1_{\Lambda^1_{m-1} V} \right):
\end{align*}
\]

\[
\begin{align*}
\Lambda^1_0 V & \xrightarrow{\tau_{\Lambda^1_0 V, \Lambda^1_0 V}} \Lambda^1_0 V \otimes \Lambda^1_0 V \otimes \Lambda^1_{m-1} V \\
\Lambda^1_0 V \otimes \Lambda^1_{m} V & \xrightarrow{\tau_{\Lambda^1_0 V, \Lambda^1_{m}} \otimes \Lambda^1_0 V} \Lambda^1_0 V \otimes \Lambda^1_{m} V \otimes \Lambda^1_{m-1} V \\
\Lambda^1_0 V & \xrightarrow{\delta_{0,1}^{1,1} \otimes \Lambda^1_0 V} \Lambda^1_{m} V \otimes \Lambda^1_{m-1} V.
\end{align*}
\]

The region (A) is commutative by Corollary 4.2 (2) with \( i = l = 1, \quad j = k = 0 \), and \( m = n \), the region (B) = \( 1_{V^*} \otimes \left[ \Lambda^1_0 V \right] \) is commutative by the commutativity of the first diagram in (50) and the functoriality of
with \( i = 1, j = 0, k = l = m - 1 \). Noticing that \( \varphi_{0,1}^{1,0} = 1_{V \otimes V} \), \( \varphi_{0,m-1}^{0,m-1} = 1_{\wedge_{m-1}^{m-1} V} \) and \( \delta_{1,0}^{0,1} = ev_V \), we deduce the equality

\[
m \cdot \delta_{1,m}^{1,m} \circ (1_{V \otimes V} \otimes C_m) = \left( 1_{\wedge_{m-1}^{m-1} V} \oplus 1_{\wedge_{m-1}^{m-1} V} \right) \circ (a, b) = a + b,
\]

where

\[
a = (ev_V \otimes 1_{\wedge_{m-1}^{m-1} V}) \circ (1_{V \otimes V} \otimes C_{m-1}) \circ \tau_{V \otimes V} = (ev_V \otimes 1_{\wedge_{m-1}^{m-1} V}) \circ \left( 1_{\wedge_{m-1}^{m-1} V} \otimes C_{m-1} \right),
\]

\[
b = (1 - m) \cdot \varphi_{0,m-1}^{1,m-2} \circ \left( 1_{\wedge_{m-1}^{m-1} V} \otimes \delta_{1,m-1}^{0,m-1} \right) \circ \left( \tau_{V \otimes V} \otimes \delta_{1,m-1}^{0,m-1} \otimes \varphi_{0,m-1}^{0,m-1} \otimes (1_{V \otimes V} \otimes C_{m-1}) \otimes \tau_{V \otimes V} \right)
\]

\[
= (1 - m) \cdot \varphi_{0,m-1}^{1,m-2} \circ \left( 1_{\wedge_{m-1}^{m-1} V} \otimes \delta_{1,m-1}^{0,m-1} \right) \circ (1_{V \otimes V} \otimes \tau_{V \otimes V} \otimes C_{m-1}).
\]

Next we remark that, by the commutativity of \( 1_V \otimes \wedge^m \) (second diagram of \( 53 \)) with \( m \) replaced by \( m - 1 \), \( (1_V \otimes \delta_{1,m-1}^{0,m-1} \otimes (1_V \otimes C_{m-1}) = (1_V \otimes \varphi_{0,m-1}^{0,m-2} \otimes (1_V \otimes C_{m-2}) \) and that, by definition of the multiplication in the mixed algebra, \( \varphi_{0,m-1}^{1,m-2} \circ \left( 1_V \otimes \varphi_{1,m-2}^{0,m-2} \right) = \varphi_{1,m-2}^{1,m-2} \), so that

\[
b = (1 - m) \cdot \varphi_{1,m-2}^{1,m-2} \circ (1_{V \otimes V} \otimes C_{m-2}).
\]

Inserting \( 55 \) in \( 55 \) we find the claimed commutativity.

**Step 4**

We now claim that

\[
C_1 \otimes C_m \xrightarrow{(m-1)} \wedge^m \otimes \wedge^m \otimes \cdots \otimes \wedge^m \otimes C_{m-1}
\]

is commutative for \( m \geq 1 \), where \( r := \text{rank}(V) \). According to \( 55 \) we have, for \( m \geq 2 \),

\[
m \cdot \delta_{1,m}^{1,m} \circ (C_1 \otimes C_m) = m \cdot \delta_{1,m}^{1,m} \circ \left( 1_{\wedge^m V} \otimes C_m \right) \circ C_1 =
\]

\[
= \left( 1_{\wedge_{m-1}^{m-1} V} \oplus 1_{\wedge_{m-1}^{m-1} V} \right) \circ (a \circ C_1, b \circ C_1)
\]

where \( a = ev_V \otimes C_{m-1} \) and \( b = (1 - m) \cdot \varphi_{1,m-2}^{1,m-2} \circ (1_{\wedge^m V} \otimes C_{m-2}) \). We have

\[
a \circ C_1 = (ev_V \otimes C_{m-1}) \circ C_1 = C_{m-1} \circ ev_V \otimes C_1 = r \cdot C_{m-1},
\]

because \( r = ev_V \circ C_V \). On the other hand, by Lemma 5.8

\[
b \circ C_1 = (1 - m) \cdot \varphi_{1,m-2}^{1,m-2} \circ (C_1 \otimes C_{m-2}) = (1 - m) \cdot \varphi_{1,m-2}^{1,m-2} \circ (C_{m-2} \otimes C_1)
\]

\[
= (1 - m) \cdot C_{m-1}.
\]

The claimed commutativity of \( 57 \) follows for \( m \geq 2 \). When \( m = 1 \) we have, by definition, \( \delta_{1,1}^{1,1} = (ev_V \otimes ev_V) \circ (\tau_{V \otimes V} \otimes 1_{1_V} \otimes V) \), so that

\[
\delta_{1,1}^{1,1} \circ (C_1 \otimes C_1) = (ev_V \otimes ev_V) \circ (\tau_{V \otimes V} \otimes 1_{1_V} \otimes V) \circ (C_V \otimes C_V)
\]

\[
= ev_V \otimes (ev_V \otimes 1_{1_V} \otimes V) \circ (\tau_{V \otimes V} \otimes 1_{1_V} \otimes V) \circ (C_V \otimes 1_{1_V} \otimes V) \circ C_V
\]

\[
= ev_V \otimes (ev_V \otimes 1_{1_V} \otimes V) \circ (\tau_{V \otimes V} \otimes 1_{1_V} \otimes V) \circ (C_V \otimes 1_{1_V}) \circ C_V
\]

\[
= ev_V \circ C_V = r,
\]

because \( (ev_V \otimes 1_{1_V}) \circ \tau_{V \otimes V} \circ (C_V \otimes 1_{1_V}) = (1_V \otimes ev_V) \circ (C_V \otimes 1_{1_V}) = 1_V \) by Lemma 2.8 (2).
We can now prove that, for $0 \leq k \leq m$, we have

$$C_k \otimes C_m \rightarrow \bigwedge^k V \otimes \bigwedge^m V = \bigwedge^{k+m} V.$$  

(58)

When $k = 0$ the claim is reduced to a triviality: we have $C_k \otimes C_m = C_m$, $(\binom{r+m}{k} C_{m-k} = C_m$ and $(\binom{m}{k}) \cdot \delta_{k,m} = 1_{\bigwedge^m V}$. In particular we may assume $1 \leq k \leq m$. For $k = 1$ this is precisely (57), so that we may assume that the commutativity is known for $1 \leq k \leq m$ and that we would like to prove it for $2 \leq k+1 \leq m$.

Consider the following diagram

The region (A) is commutative by Lemma (5.8)

$$(\varphi_{k,1}^{k_1} \otimes 1_{\bigwedge^m V}) \circ (C_k \otimes C_1 \otimes C_m) = (\varphi_{k,1}^{k_1} \circ (C_k \otimes C_1)) \otimes C_m = C_k+1 \otimes C_m.$$  

The region (B) is commutative by induction. We deduce

$$(\binom{m}{k}) \cdot \delta_{k+1,m}^{m} \circ (C_{k+1} \otimes C_m) = (\binom{r+k-m}{k}) \cdot \delta_{1,m-k}^{1} \circ (C_1 \otimes C_{m-k}).$$  

(59)

We now note that we have $k+1 \leq m$ if and only if $m-k \geq 1$, so that (57), with $m$ replaced by $m-k$ gives the equality

$$(m-k) \cdot \delta_{k+1,m}^{m} \circ (C_{k+1} \otimes C_m) = (r - m + k + 1) \cdot C_{m-k}.$$  

(60)

Noticing that $\binom{m}{k+1} = \frac{m-k}{k+1} \binom{m}{k}$ we deduce, inserting (60) in (59), that we have

$$(\binom{m}{k+1}) \cdot \delta_{k+1,m}^{m} \circ (C_{k+1} \otimes C_m) = \frac{1}{k+1} \left( \binom{r+k-m}{k} \cdot (r - m + k + 1) \cdot C_{m-k} \right).$$  

The claim follows because $\binom{r+k-m}{k} = \frac{1}{k+1} \cdot \binom{r+k-m}{k}$

6. A Poincaré duality isomorphism for the symmetric algebras

In this section we suppose that $C$ is rigid, $\mathbb{Q}$-linear and pseudo-abelian. We consider an object $V \in C$ and we apply the results on $\Delta$-graded algebras with $A = \left(\bigvee V, \varphi_{i,j}^{V} \right)$ and $A^{\vee} = \left(\bigvee V^{\vee}, \varphi_{i,j}^{V^{\vee}} \right)$. We will use the shorter notation $i_{V}^{p} := i_{V}^{p_{V}, e_{V}}$, $p_{V}^{p_{V}, e_{V}} := e_{V}^{p_{V}, e_{V}}$, $\varphi_{i,j} := \varphi_{i,j}^{V}$ and $\varphi_{i,j}^{V^{\vee}}$. The same argument employed in the alternating case shows that the internal multiplication morphisms are given, for every $j \geq i$, by the composite

$$\varphi_{i,j} : \bigvee^i V \otimes \bigvee^j V^{\vee} \xrightarrow{\varphi_{i,j}} \bigvee^{i+j} V \otimes \bigvee^{i-j} V^{\vee} \otimes \bigvee^{i-j} V^{\vee} \xrightarrow{\varphi_{j-i}^{V^{\vee}}} \bigvee^j V^{\vee}.$$  

These morphisms can then be lifted to the tensor algebras as in Lemma (5.11) the only difference being that the character $\varepsilon$ has to be replaced by the trivial character. The effect of this change is that the resulting normalized family $t_{j} := j \cdot t_{i,j}$ is now a derivation, rather than being an anti-derivation, i.e. it satisfies the symbolic theoretic formula

$$t_{j+1} (x) (\omega_j \cdot \omega_1) = t_{j+1} (x) (\omega_j) \wedge \omega_1 + \omega_j \wedge t_{j+1} (x) (\omega_1) \quad \text{for } x \in V, \omega_j \in \bigvee^j V^{\vee} \text{ and } \omega_1 \in \bigvee^1 V^{\vee},$$

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which has a formal diagram theoretic formulation analogue to Proposition 5.2. Then the analogue of Corollary 5.3 that we leave to the reader to precisely formulate, is just a formal consequence and the proof of Lemma 5.4 suitable modified employing the analogue of this corollary, lead to the following result.

**Lemma 6.1.** Let \( r := \text{rank}(V) \) be the rank of \( V \), defined as the composite
\[
r : I \xrightarrow{\nabla V} V \otimes V^\vee \xrightarrow{\text{ev}_V} I.
\]
For every \( g \geq i \) we have the equality
\[
(g \choose i)^{-1} (r + g - 1) \cdot C_{Vg-i, V} = \delta_{i,g}^i \circ (C_{V \otimes C} \otimes C_{V^\vee V}),
\]
where, for every \( k \in \mathbb{N}_{\geq 1} \),
\[
\begin{pmatrix} T \\ k \end{pmatrix} := \frac{1}{k!} (T(T-1) \cdots (T-k+1) \in \mathbb{Q}[T] \text{ and } \begin{pmatrix} T \\ 0 \end{pmatrix} = 1.
\]

As in the alternating case we may define, for every \( g \geq i \), the Poincaré morphisms
\[
D^{i,g} := D_{i,g} : \nabla^i V \rightarrow V^{g-i} \otimes V^{g}\quad \text{and} \quad D^{i,g} := D_{i,g} : \nabla^i V \rightarrow V^{g-i} \otimes V^{g}.
\]
The following result is obtained from Lemma 6.1 in the same way as Theorem 5.5 has been obtained from Lemma 5.4 with .

**Theorem 6.2.** The following diagrams are commutative, for every \( g \geq i \geq 0 \).

1. 
\[
\begin{array}{c}
\nabla V \otimes \nabla V \\
\xrightarrow{D^{i,g} \otimes D_{i,g}} \\
\nabla^{g-i} V \otimes \nabla^{g-i} V \\
\text{ev}_{V^{g-i}, V} \\
\text{ev}_{V^{g-i}, V}.
\end{array}
\]

2. 
\[
\begin{array}{c}
\nabla V \otimes \nabla^{g-i} V \\
\xrightarrow{D^{i,g} \otimes D_{i,g}} \\
\nabla V \otimes \nabla^{g-i} V \\
\text{ev}_{V^{g-i}, V} \\
\text{ev}_{V^{g-i}, V}.
\end{array}
\]

3. 
\[
\begin{array}{c}
\nabla V \\
\xrightarrow{D^{i,g}} \\
\nabla^{g-i} V \\
\text{ev}_{V^{g-i}, V} \\
\text{ev}_{V^{g-i}, V}.
\end{array}
\]

and

4. 
\[
\begin{array}{c}
\nabla V \otimes \nabla^{g-i} V \\
\xrightarrow{D^{i,g} \otimes D_{i,g}} \\
\nabla V \otimes \nabla^{g-i} V \\
\text{ev}_{V^{g-i}, V} \\
\text{ev}_{V^{g-i}, V}.
\end{array}
\]

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We say that $V$ has **symmetric rank** $g \in \mathbb{N}_{\geq 1}$ if $\varphi^g V$ is an invertible object and $(r^g g - 1)$ and $(r^+ g - 1)$ are invertible for every $0 \leq i \leq g$. For example, when $\text{End}(l)$ is a field or $r \in \mathbb{Q}$, the second condition means that $r$ is not a root of the polynomials $(r^g g - 1) \in \mathbb{Q}[T]$ and $(r^+ g - 1) \in \mathbb{Q}[T]$ for every $0 \leq i \leq g$, i.e. that $r \neq 1 - g, 2 - g, \ldots, -i$ and $r \neq 1 - g, 2 - g, \ldots, -i - g$ for every $1 \leq i \leq g$.

We say that $V$ has **strong symmetric rank** $g \in \mathbb{N}_{\geq 1}$ if $\varphi^g V$ is an invertible object and $r = -g$ (hence $V$ has symmetric rank $g$). With these notations Corollary 5.4 specializes to the following result.

**Corollary 6.3.** If $V$ has symmetric rank $g \in \mathbb{N}$ then, for every $0 \leq i \leq g$, the morphisms $D^{i,g}$, $D_{g-i,g}$, $D^{g-i,g}$ and $D_{i,g}$ are isomorphisms and the multiplication maps $\varphi^g_{i,g}$, $\varphi^{g-i}_{g-i}$, $\varphi^{g}_{i,g}$ and $\varphi^{g}_{g-i}$ are perfect pairings (meaning that the associate valued morphisms are isomorphisms). Furthermore, when $V$ has strong symmetric rank $g$, we have $(r^+ g - 1) = (-1)^{g-i}$ and $(r^g g - 1) = (-1)^g$ in the commutative diagrams of Theorem 6.2.

We end this section with the analogue of Proposition 5.7 in this setting. This a technical result that will be crucial for the computation of $[V, S]$. The proof is just a copy of that of Proposition 5.7.

**Proposition 6.4.** The following diagrams are commutative when $\varphi^g V$ is invertible of rank $r_{\varphi^g V}$ (hence $r_{\varphi^g V} \in \{\pm 1\}$):

$$
\begin{array}{ccc}
\varphi^g_{i,g} \otimes \varphi^{g-i}_{g-i} \otimes V & \xrightarrow{T_{\varphi^g_{i,g} \otimes \varphi^{g-i}_{g-i} \otimes V}} & V \otimes \varphi^g_{i,g} \otimes \varphi^{g-i}_{g-i} \\
\end{array}
$$

where

$$
M = \varphi^{g-i}_{g-i} \otimes \varphi^{g-i}_{g-i} \otimes \varphi^{g-i}_{g-i} \otimes V
$$

and

$$
\begin{array}{ccc}
\varphi^g_{i,g} \otimes \varphi^{g-i}_{g-i} \otimes V & \xrightarrow{T_{\varphi^g_{i,g} \otimes \varphi^{g-i}_{g-i} \otimes V}} & V \otimes \varphi^g_{i,g} \otimes \varphi^{g-i}_{g-i} \\
\end{array}
$$

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