Geometry of AdS black hole thermodynamics in extended phase space

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Abstract

We consider the geometry of anti-de-Sitter (AdS) black hole thermodynamics in four dimensions, where the equation of state in the extended phase space formalism allows explicit comparison with normal fluid systems. We show that for the two-dimensional parameter manifolds considered here, the scalar curvature is proportional to the thermodynamic volume. This allows us to critically examine the applicability of geometric methods in black hole thermodynamics in extended phase space. We show how several standard features that are expected to hold in normal fluid systems impose severe restrictions on the black hole parameters, leading to the fact that several results in the current literature on the geometry of thermodynamics in extended phase space may be physically invalid. These are true for both charged and rotating AdS black holes. As a by-product of our analysis, we examine a conjecture regarding the equality of the correlation lengths of co-existing phases near criticality, in charged AdS black hole backgrounds, and find reasonable validity.

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1 Introduction

In the absence of a fully understood quantum theory of gravity, coarse-grained approaches are often used to elucidate broad features of the physics of black holes. Understanding the thermodynamic properties of black holes is one such area of research, that is immensely popular in the current literature, with the hope being that the results obtained herein should be finally related to the micro structure of black holes derived in a more satisfactory quantum formalism. Black hole thermodynamics, which was formulated several decades ago, is based on the fact that the four laws of black hole mechanics are formally identical to the four laws of thermodynamics, with the black hole being an equilibrium thermal state with a non-zero Hawking temperature.

Ordinary thermodynamic systems on the other hand are by now known to be amenable to a Riemannian geometric treatment. The basic idea here is that the tunable parameters of such systems (for example the temperature, pressure etc.) form a non-trivial parameter manifold, whose geometry captures physical thermodynamic features, the most important aspect being phase transitions. In the geometric approach, the line element on the parameter manifold (which, for the purpose of the discussion here, will always be two-dimensional) is indicative of the distance between two thermodynamic states that are related by a fluctuation. Further, for systems that exhibit continuous second order phase transitions, the scalar curvature (i.e the Ricci scalar) on the parameter manifold is conjectured to be related to the correlation length of the system, and diverges near criticality. This approach (popularly known as thermodynamic geometry) has a long history (see e.g., the review [1]) and was formalized by Ruppeiner [2] (for a pedagogical exposition, see [3]) for thermodynamic systems, and provides a useful alternative method to study thermodynamics of fluids. It is indeed satisfying that, as is well known and as we will discuss in details, the geometric formalism inherits its own limits of applicability, due to some inherent physical features.

A natural question that arises in this context is the applicability of Riemannian geometric methods in black hole thermodynamics. This becomes particularly interesting in extended phase space thermodynamics, in theories that admit a cosmological constant. Here, following the proposal of [4], one identifies the variable cosmological constant as the pressure of the black hole system, with the conjugate volume being the thermodynamic volume, and the mass of the black hole is identified with its enthalpy. Once the formal notions of pressure and volume are established for a black hole, one is tempted to think that geometric methods applicable to fluids should be of relevance here, to glean insight into black hole thermodynamics in extended phase space. And since in ordinary fluid systems, there are stringent restrictions on the applicability of geometric methods as we have mentioned, it is natural to ask how far these restrict the applications of such geometric methods to black hole systems. This is the question we ask in this paper.

Here, one needs to be careful at the outset, given the fact that one of the basic premises of thermodynamic geometry, namely the extensivity of the entropy, is invalid for black holes. As is standard in the literature however, in constructing the geometry of black hole thermodynamics, one proceeds to construct the line element exactly in the same fashion as is done for usual thermodynamic systems. And, as is well known by now, thermodynamic geometry of black hole systems often exhibit the same behaviour as usual fluid systems, thus providing useful insights into the coarse grained nature of black hole microstates (see, e.g., the earlier works by one of the present authors [5] - [9] as well as the works of [10] - [12], and references therein).

In the era before the notion of extended phase space was introduced, thermodynamic geometry of various classes of black holes were worked out in scenarios that did not involve a pressure or volume. These were studied in various ensembles that typically include two fluctuating parameters, often taken as the mass and a charge, or the mass and the corresponding potential. For charged anti-de-Sitter (AdS) black holes, the pioneering works of [13], [14] established liquid-gas like phase transition properties, with an equation of state specifying the temperature as a function of the charge and the electric potential. Methods of thermodynamic geometry were used soon after these works, to understand such phase transitions from a geometric perspective, and useful scaling relations involving the black hole parameters were obtained in several works. In extended phase space however, as we will discuss in detail, one has on the other hand an explicit equation of state involving the black hole pressure and volume, and hence geometric notions typical to fluid systems are more directly applicable here.

Indeed, one of the main lessons from the study of thermodynamic geometry in fluids is the relationship between the scalar curvature $R$ and the correlation length $\xi$, via $R \sim \xi^d$ near criticality, with $d$ being the spatial dimension of the system, as can be shown via a renormalization group approach [15]. For two dimensional parameter manifolds that we will be interested in, the Riemann curvature tensor has a single independent component, and hence $R$, which fully characterizes the curvature of the manifold, plays a fundamental role. At the very outset, the relationship between $R$ and the correlation volume entails a dimensional constraint on $R$, namely that $R$ must scale as a volume dimension. This in fact is crucial in determining the applicability of the geometric method itself. Namely, in a coarse-grained approach (for example in a fluid system), the geometric analysis loses meaning if $R$ is of the order of, or less than, a typical molecular volume [16]. Also, by its very construction, the geometry assumes a Gaussian approximation to the expansion of the entropy around an equilibrium value, and it is important to understand if the validity of this approximation is ensured in suitable regions of the parameter space. All
these assume importance in black hole thermodynamics in extended phase space, with a well defined notion of thermodynamic volume as well as specific volume, i.e., volume per molecule [17], [18].

In this paper, we will examine these issues, with our focus being four dimensional charged black holes as well as rotating black holes in anti-de-Sitter space, called Reissner-Nordstrom-AdS (RN-AdS) and Kerr-AdS (KAdS) black holes, respectively. One of the first difficulties that one encounters here is the fact that for RN-AdS black holes, the (dimensionless) specific heat capacity at constant volume \( c_v = 0 \) identically, while it vanishes in the slow rotation approximation for Kerr-AdS black holes. This renders the line element, one of whose components is proportional to \( c_v \) (see Eq. (1) in the next section), meaningless at first sight. An important idea in this context was put forward in the work of [19]. These authors suggested that one should generically consider the limit \( c_v \to 0 \) in the final expression of the scalar curvature of the geometry, instead of beginning from it. In this approach, one can define a normalized scalar curvature \( R_{c_v} \) which diverges at criticality, is finite away from it, and can be related to the correlation volume near criticality. \( R_{c_v} \) is then the scalar curvature appropriate for extended phase space thermodynamics of RN-AdS or Kerr-AdS black holes. Since \( R_{c_v} \) scales as the correlation volume, and is the only scalar quantity in two dimensional parameter spaces that we consider, the inherited properties of this quantity naturally determines the limits of applicability of the Riemannian geometric method in theories with vanishing specific heat.

In this work, we show that \( R_{c_v} \) is a natural physical quantity in closed thermodynamic fluid systems, where the volume is treated as a fluctuating variable, and establish its relation with the corresponding quantity in open fluid systems. These are shown to be proportional, with the system volume being the proportionality constant, in the limit \( c_v \to 0 \). This simple observation then allows us to compare several features of the geometry of thermodynamics of standard fluids to that of AdS black holes. In particular, we examine the applicability of geometric methods in black hole systems, both for RN-AdS and Kerr-AdS black holes, and show why many of the results that have recently appeared in the literature might in fact be invalid. Further, as a by-product of our analysis, we examine a recent conjecture [16] that the correlation lengths are equal in coexisting phases, near criticality, and find reasonable validity, in the context of RN-AdS black holes. This paper is organized as follows. In section 2 we review some of the basic features of thermodynamic geometry. In section 3 we establish the relation between the scalar curvatures of open and closed thermodynamic fluid systems in the limit of vanishing specific heat. Section 4 addresses the relevant issues related to the applicability of geometric methods to RN-AdS black hole systems, and we briefly comment on the geometry of Kerr-AdS black holes in section 5. The paper ends with a summary and discussions in section 6.

2 Thermodynamic geometry

Classical thermodynamic geometry has its origin in fluctuation theory. As worked out in the book [20] by Landau and Lifshitz (section 114), the fluctuation probability \( w \) in a closed thermodynamic system, with the independent variables taken as the temperature \( T \) and the volume \( V \) can be written as

\[
w \sim \exp \left( -\frac{1}{2} d\ell_c^2 \right), \quad d\ell_c^2 = \frac{C_V}{k_B T^2} dT^2 - \frac{1}{k_B T} \left( \frac{\partial P}{\partial V} \right)_T dV^2,
\]

(1)

where \( k_B \) is Boltzmann’s constant, and \( P \) denotes the pressure. Here, \( C_V = T(\partial S/\partial T)_V \) is the specific heat at constant volume. This is also written in terms of the dimensionless specific heat \( c_v \) as \( C_V = N k_B c_v \), with \( N \) being the number of particles. An early work of Ruppeiner [21] in fact uses the \( d\ell_c^2 \) of Eq. (1) as the line element on the parameter manifold (with coordinates \( T \) and \( V \)) to show that the scalar curvature \( R_{c_v} \) arising out of this line element diverges near the second order critical point in fluid systems [21]. The exponent in Eq. (1) and hence \( d\ell_c^2 \) being naturally dimensionless, \( R_{c_v} \) also inherits the same property, and [21] considers the quantity \( R_{c_v} V \) and relates it to the correlation volume of the system near criticality.

Since this is somewhat ad-hoc, a more methodical approach elaborated upon in [2] and in subsequent literature, is to consider the line element obtained from an appropriate thermodynamic potential per unit volume, in an open thermodynamic system with a fixed volume \( V \). In this case, it is more convenient to consider the temperature and the density \( \rho = N/V = 1/v \) as the fluctuating variables. The line element for such an open thermodynamic system is given, for a single component fluid, by

\[
d\ell_o^2 = \frac{1}{k_B T} \left( \frac{\partial s}{\partial T} \right)_\rho dT^2 + \frac{1}{k_B T} \left( \frac{\partial \mu}{\partial \rho} \right)_T d\rho^2,
\]

(2)

\(^1\)The subscript (or superscript) \( c \) will be used to denote geometric quantities in a closed subsystem, as opposed to an open subsystem, where we will use the subscript or superscript \( o \).

\(^2\)That this will be the case for fluid systems that show a coexistence line culminating in a second order critical point can be seen by noting that the second term in Eq. (1) is the inverse of the isothermal compressibility and vanishes along the spinodal curve. Hence the determinant of the metric of Eq. (1) is undefined along this curve and in particular at the critical point.
where \( s = S/V = -(\partial f/\partial T)_\rho \) and \( \mu = (\partial f/\partial \rho)_T \) being the entropy per unit volume and the chemical potential respectively, with \( f = F/V \) the Helmholtz free energy per unit volume. The scalar curvature \( R_0 \) constructed out of \( dl_o^2 \) has the dimensions of volume and is therefore an appropriate candidate for the correlation volume.

### 3 The scalar curvature in closed and open thermodynamic systems

The scalar curvature constructed out of the metric of closed (Eq. [1]) and open (Eq. [2]) thermodynamic systems, when compared after transforming to the same coordinates, might generally not be the same. But for the cases of our interest in this paper, namely when \( c_v \) vanishes, the two are related, as we now show.

To this end, we start by considering the Helmholtz free energy per unit volume of a fluid system, given as

\[
f(T, \rho) = -\rho k_B T \log(e/\rho) + \rho h(T) - \rho k_B T \log(1 - b \rho) + d_1 \rho^4 + d_2 \rho^6,
\]

where the first two terms on the right hand side of Eq. [3] represent the ideal gas contribution to the free energy, and \( h(T) = -c_v k_B T \log(T/e) \). Here, \( a, b, d_1 \) and \( d_2 \) are constants. In \((T, v)\) coordinates, the Helmholtz free energy per unit volume takes the form

\[
f(T, v) = -\frac{k_B T}{v} \log(ev) - \frac{c_v k_B T}{v} \log(T/e) - \frac{a}{v^2} - \frac{k_B T}{v} \log \left( \frac{v-b}{v} \right) + \frac{d_1}{v^4} + \frac{d_2}{v^6},
\]

and results in the equation of state via \( P = -\partial F/\partial V \),

\[
P = \frac{k_B T}{v-b} - \frac{a}{v^2} + \frac{3d_1}{v^4} + \frac{5d_2}{v^6}.
\]

When \( a = b = d_1 = d_2 = 0 \), Eq. [3] describes the free energy per unit volume of the ideal gas, while \( d_1 = d_2 = 0 \) describes the Van der Waals (VdW) fluid. If \( b = d_2 = 0 \), the pressure in Eq. [5] has the same form as that of the RN-AdS black hole [22] and \( b = d_1 = 0 \) represents the slowly rotating Kerr-AdS black hole in the same sense [17, 23]. While for finite non-zero values of \( c_v \), these represent the RN-AdS fluid [24] or the Kerr-AdS fluid, it will be important for us to consider the limit \( c_v \to 0 \), since the specific heat vanishes for both the RN-AdS black hole and the slowly rotating Kerr-AdS black hole.

To compute the scalar curvature \( R_c \), we write down the relevant metric components from Eq. [1],

\[
g_{TT}(T, v) = \frac{C_v}{k_B T^2}, \quad g_{vv}(T, v) = -\frac{N}{k_B T} \left[ \frac{\partial P(T, v)}{\partial v} \right] = \frac{N}{k_B T} \left( \frac{k_B T}{(b-v)^2} + \frac{4(5d_2 + 6d_1 v^2 - av^4)}{v^3} \right).
\]

The scalar curvature in terms of \( T \) and \( \rho \) reads

\[
R_c(T, \rho) = \frac{2k_B \rho W(\rho) \left( k_B T + \rho \frac{W(\rho)}{\rho} \right)}{C_v \left( k_B T + 2\rho \frac{W(\rho)}{\rho} \right)^2},
\]

where \( W(\rho) = (1-b \rho)^2 \left( 6d_1 \rho^2 + 15d_2 \rho^4 - a \right) \).

Now for the open subsystem case, the metric components are calculated to be

\[
g_{TT}(T, \rho) = \frac{\rho c_v}{T^2}, \quad g_{\rho \rho}(T, \rho) = \frac{1}{k_B T} \left( \frac{k_B T}{\rho (1-b \rho)^2} + 2 \left( -a + 6d_1 \rho^2 + 15d_2 \rho^4 \right) \right),
\]

and we call the scalar curvature of this open system as \( R_o \). The expression for the normalized scalar curvature in this case is lengthy, but simplifies in the limit of \( c_v \to 0 \) to

\[
\lim_{c_v \to 0} \left( R_o(T, \rho)c_v \right) = \frac{2W(\rho) \left( k_B T + \rho \frac{W(\rho)}{\rho} \right)}{\left( k_B T + 2\rho \frac{W(\rho)}{\rho} \right)^2}.
\]

Comparing Eqs. [7] and [9], we obtain

\[
\frac{R_c C_v}{\rho} = \lim_{c_v \to 0} \left( R_o c_v k_B \right), \quad \text{i.e.,} \quad R_c c_v = \lim_{c_v \to 0} \left( \frac{R_o c_v}{V} \right).
\]
Now, following our earlier discussion, $R_c$ is dimensionless and $R_o$ carries the dimension of volume, so Eq. (10) yields a relation consistent with dimensional analysis. In fact, we can substantiate this result further. As we have mentioned in the introduction, near criticality, $R_o \sim \xi^3$, with $\xi$ being the correlation length (the same is not true for $R_c$, which is dimensionless). Then, we can write near criticality,

$$ \lim_{c_v \to 0} \frac{\xi^3 c_v}{V} = K ,$$

where $K = R_c c_v$ is a dimensionless quantity, independent of any scale of the system, and is in this sense universal. Therefore, a proper interpretation of $R_c c_v$ is that it is the (normalized) curvature of a corresponding open thermodynamic system, per unit system volume.

An important point to note here is that in $(T, V)$ coordinates, the metric obtained by simply dividing that of Eq. (1) by the system volume, i.e.,

$$ g_{TT} = \frac{C_V}{V k_B T^2} , \quad g_{VV} = \frac{1}{V k_B T} \left( \frac{\partial^2 F}{\partial V^2} \right) = - \frac{1}{V k_B T} \left( \frac{\partial P}{\partial V} \right) $$

(12)
gives the same scalar curvature $R_o$ for open subsystems (computed from Eq. (2)), as is readily checked. We note however that the form of the metric in Eq. (12) cannot be straightforwardly justified either from Eq. (1) or from Eq. (2). However, Eq. (12) will be important for two reasons.

First, it is useful in situations where the exact analytical form of the Helmholtz free energy $F$ might be difficult to obtain, and one has to resort to approximations to write down the analytic form of the equation of state by which $P = P(V,T)$, within the limits of such approximations. Such a situation occurs in the study of Kerr-AdS black holes, which we will undertake in section 5. Secondly, as is well known, the concept of a specific volume is somewhat challenged in black hole thermodynamics in extended phase space. Since the number of degrees of freedom $N$ of a black hole (which, in Planck units, is proportional to its horizon area) and its thermodynamic volume $V$ are both functions of the radius of the horizon, these are not independent. While for the RN-AdS black hole, the specific volume appearing in its equation of state is $\rho = N/V$, the same is not true for Kerr-AdS black holes, with the specific volume diverging for ultra-spinning cases. Eq. (12) avoids this ambiguity and provides a generic way to compute the scalar curvature for black holes, with the appropriate dimension of volume. Henceforth, Eq. (12) will be our line element for the black hole cases, and by a slight abuse of notation, we will continue to refer to the scalar curvature obtained from the metric of Eq. (12) as $R_o$, for the black hole cases, so as not to clutter the notation.

We pause here to mention a few important details. First of all, we note that close to the second order critical point, fluid systems are solely characterized by the critical exponents (mean field exponents in classical systems that we consider), and therefore exhibit certain universal properties. For example, for the VdW fluids with finite $c_v$, the quantity $R_o c_v / V \sim t^{-2}$ is universal, near criticality, with $t = T/T_c - 1$, $T_c$ being the critical temperature. Eq. (10) is on the other hand a relation that is valid for systems with vanishing $c_v$, even away from criticality. Also, we note that for two dimensional parameter spaces, $R_o c_v$ is the only normalized scalar that has dimensions of volume, and scales at the correlation length near criticality. Hence, it is natural to demand that this normalized scalar curvature, which carries information about the scale of the system, carries the limits of applicability too, like theories with non-zero heat capacity that we mentioned in the introduction. For example, whenever $R_o c_v$ becomes comparable to, or smaller than the specific volume, the formalism itself should be understood to have broken down. There are some further compatibility features, as we will discuss in the next section.

For the moment, as an example, we consider the simplest non-trivial classical fluid system that is amenable to a geometric analysis, namely the VdW fluid. The VdW equation of state and its reduced form read

$$ P = \frac{k_B T}{(v - b)} - \frac{a}{v^2} , \quad P_r = \frac{8 T_r}{(3 v_r - 1)} - \frac{3}{v_r^2} , $$

(13)

respectively, where $v$ is the volume per molecule, and the reduced pressure, volume and temperature are given as $P_r = P/P_c$, $v_r = v/v_c$ and $T_r = T/T_c$, with $P_c = a/(27 b^2)$, $v_c = 3 b$ and $T_c = 8 a/(27 b k_B)$ denoting the critical values. The scalar curvature computed from Eq. (1) is called $R_c$, and given by

$$ R_c = \frac{k_B (3 v_r - 1)^2 \left( (1 - 3 v_r)^2 - 8 T_r v_r^3 \right)}{2 C_V \left( (1 - 3 v_r)^2 - 4 T_r v_r^3 \right)^2} . $$

(14)
On the other hand, the scalar curvature computed from Eq. (2) (or equivalently from Eq. (12)) is \( R_o = A.B \), where we have defined \[ 16 \]

\[
A = \frac{3b v_r (3v_r - 1)}{18c_v \left( (1 - 3v_r)^2 - 4T_r v_r^3 \right)^2},
\]

\[
B = 9 (3v_r - 1) \left( 8T_r v_r^3 - (1 - 3v_r)^2 \right) + 4c_v T_r v_r \left( 1 - 9v_r + 27v_r^2 - 27v_r^3 + 8T_r v_r^3 \right). \tag{15}
\]

Clearly then we have the relation \( \lim_{c_v \to 0} R_o c_v / V = R_c c_v \), confirming Eq. (10). While this holds away from criticality, note that the coefficient of \( c_v \) in the second term of \( B \) vanishes close to criticality, i.e., in the limit \( T_r, v_r \to 1 \). This is as we have mentioned before – close to the second order critical point, \( R_o c_v / V \) is universal, even for non-zero heat capacity.

We have thus established that for an open thermodynamic fluid system, the scalar curvature on the parameter space is related to that of the same system considered as closed, by a factor of the system volume, in the limit that the specific heat goes to zero. As mentioned before, [21] used Eq. (1) as the line element, and inserted the system volume appropriately, in order to obtain a scalar curvature of appropriate volume dimension. What we have shown here is that this will yield the scalar curvature of the corresponding open thermodynamic system, in the limit of vanishing \( c_v \). In this context, we mention that similar results hold for the RN-AdS fluid [24] which is a two-parameter family of fluid solutions having the same equation of state as the RN-AdS black hole. Of course for real fluids (VdW or otherwise), this is more of a mathematical curiosity, as \( c_v \) for such fluids is always non-zero.

## 4 RN-AdS black holes

Recall that in [4], it was proposed that a varying cosmological constant can be identified with the pressure of a black hole, with the volume of the event horizon being its conjugate. In this formalism, the mass of the black hole is identified with its enthalpy (rather than its internal energy), and this can be shown to satisfy a consistent Smarr relation. Following [4], the work of [22] derived a relation between the pressure, volume, and temperature of an RN-AdS black hole with electric charge \( Q \). This shows similar phase behaviour as that of the VdW fluid and is often used fruitfully to understand the coarse grained structure of these black holes.

As we have already mentioned, a verbatim application of thermodynamic geometry in black hole thermodynamics is rendered difficult by its very nature. Recall that in the static case considered here, the number of degrees of freedom \( N \) and the system volume \( V \) are not independent, and both depend on the radius of the event horizon \[ 17 \]. One can nonetheless construct \( R \) with appropriate dimensions of volume, starting from the Helmholtz free energy \( F \) per unit thermodynamic volume \( V \), as is the case with ordinary thermodynamic systems. With \( r_+ \) denoting the radius of the event horizon, and the black hole volume \( V = (4/3)\pi r_+^3 \), this reads from Eq. (3.29) of [22],

\[
\frac{l_p^2}{\hbar} \frac{F}{V} = \left( \frac{\pi}{6} \right)^{1/3} \frac{Q^2}{V^{4/3}} + \left( \frac{3}{32\pi} \right)^{1/3} \frac{1}{V^{2/3}} - \left( \frac{9\pi}{16} \right)^{1/3} \frac{T}{V^{1/3}}, \tag{16}
\]

where \( l_p \) is the Planck length, \( h \) is Planck’s constant and \( c \) the speed of light. Here, \( Q \) and \( T \) have the dimensions of length and inverse length respectively, and the physical charge and temperature are given in terms of \( Q \) and \( T \) as \( Q^2 G / (4\pi \epsilon_0 c^4) \) and \( h c T / k_B \), respectively, with \( G \) being the four-dimensional gravitational constant and \( \epsilon_0 \) the permittivity of vacuum. Here, \( v = 2l_p^2 r_+ \) is the specific volume with its critical value \( v_c = 2l_p^2 \sqrt{6Q} \). In terms of the reduced variable \( v_r = v / v_c = V r_+^{1/3} \), with \( V_r = V / V_c \) (the critical values of the temperature, pressure and volume being \( T_c = \sqrt{6} / (18\pi Q) \), \( P_c = 1 / (96\pi Q^2) \), \( V_c = 8 \sqrt{6\pi Q^3} \)), the equation of state is

\[
P_r = \frac{8T_r}{3v_r} - \frac{2}{v_r^2} + \frac{1}{3v_r^4}. \tag{17}
\]

If we use the Helmholtz free energy \( F \) instead of the free energy per unit volume of Eq. (16), we obtain [19] using Eq. (1),

\[
R_c c_v = \left( \frac{3v_r^2 - 1}{2} \right) \left( \frac{3v_r^2 - 4T_r v_r^3 - 1}{2(1 - 3v_r^2 + 2T_r v_r^3)^2} \right). \tag{18}
\]
On the other hand, using Eq. (12)³ the scalar curvature per unit specific volume is given in the limit \( c_v \to 0 \) by

\[
\frac{R_o c_v}{v} = \left( \frac{Q^2}{l_p^2} \right) \frac{2 \pi v_r^2 (3v_r^2 - 1)(3v_r^2 - 4 T_r v_r^2 - 1)}{(1 - 3v_r^2 + 2 T_r v_r^2)^2}.
\]

(19)

Clearly then, as mentioned in Eq. (10), \( R_o c_v = \lim_{c_v \to 0} R_o c_v/V \). This is in lines with our previous argument: if one uses the line element of Eq. (1), then the scalar curvature computed from this agrees with the one obtained using Eq. (12) by an overall factor of the system volume, in the limit \( c_v \to 0 \), which is indeed the case here. Note that the scalar curvature \( R_o c_v \) (with appropriate dimensions of volume) is not universal, but the curvature per system volume is.

With a dimensionally consistent definition (Eq. (19)), we are in a position to examine the validity of the geometric approach to black hole thermodynamics in this case. There are a few important issues that need to be examined. First of all, we emphasize that in a coarse-grained approach, any quantity that has magnitude lower than the specific volume loses significance, as it becomes lower than the lowest scale allowed by the system. Indeed, as discussed in details in [16], for the VdW fluid system, this restricts the usage of the geometric methods discussed there (in a somewhat different context of locating first order phase transitions and the Widom line [26] in the supercritical region using geometric methods) to some suitable limits near criticality. Namely, in the sub-critical region, one is restricted to \( T_r \gtrsim 0.8 \) and in the super-critical region, to \( P_r \lesssim 10 \). It is important to understand such restrictions in the current scenario, which would imply that the right hand side of Eq. (19) should be greater than unity, namely that we should demand \( R_o c_v/v > 1 \).

A further significant issue is that from the equation of state of Eq. (17), one readily finds that in order to avoid negative temperatures, there exists a minimum value of the reduced volume for a given pressure, namely

\[
v_r^{\min} = \frac{1}{\sqrt{3 + \sqrt{9 + 3P_r}}} , \quad v^{\min} = \frac{v_c}{\sqrt{3 + \sqrt{9 + 3P_r}}}.
\]

(20)

This would imply the possibly weaker constraint \( R_o c_v/v^{\min} > 1 \).

Finally, we note that the geometric formalism is based on a Gaussian approximation. This implies that the entropy of the system is expanded about an equilibrium (extremal) value, and one retains only the lowest order correction to the same. As explained in [2] (see also the pedagogical exposition in [3]), this means that one should reasonably expect that the scalar curvature is much less than the system volume, which in this case translates to \( R_o c_v \ll V \). Near a second order phase transition, this condition is violated, and signals the expected breakdown of the Gaussian approximation near criticality. This constraint is often overlooked in standard analyses, namely because the system volume can always be chosen to be large. However the subtlety for RN-AdS black holes is that this is also determined by the radius of the outer event horizon, which also fixes the specific volume. Specifically then, from our previous discussion, we expect that the dimensionless quantity on the right hand side of Eq. (18) is much less than unity. To summarize, the conditions that we expect to hold in our coarse grained analysis are

\[
\frac{R_o c_v}{v} > 1 , \quad \frac{R_o c_v}{v^{\min}} > 1 , \quad \frac{R_o c_v}{V} \ll 1 ,
\]

(21)

where the second condition is a weaker one that follows naturally from the first, but will be useful for us in what follows.

We will now examine these constraints one by one. In this section, we will study Eq. (21) for RN-AdS black holes. In the next section, we will comment upon these in the context of Kerr-AdS black holes. To make contact with existing literature, we recall that in [27] it was shown that for the RN-AdS black hole, one can derive an analytical formula for the coexistence curve. In terms of the reduced temperature, the reduced volume on the liquid \( (v_r^l) \) and gas \( (v_r^g) \) sides at the first order phase transition where the Helmholtz free energy becomes equal for the two phases can be written as [27]

\[
v_r^{l,g} = \frac{\sqrt{1 + y^2 + \sqrt{9y^2}}}{1 - 2y}, \quad y = \cos \left( \frac{1}{3} \arccos(1 - T_r^2) + \pi \right).
\]

(22)

It is then readily checked that in the limit \( T_r \to 0 \), \( v_r^{\min} = v_r^{l}|_{T_r \to 0} = 1/\sqrt{6} \). At this point, we recall that for the VdW fluid, the reduced minimum volume is numerically equal to 1/3, independent of the pressure, which is the hard sphere cutoff in that case. In our case, to make physical sense of the scalar curvature, one is thus forced to restrict to situations where \( |R_o| c_v > v_r^{\min} \). To make the above more concrete, we first write Eq. (19) in terms of

³Note that if we used Eq. (16) in Eq. (2) with \( \rho = 1/v \), the scalar curvature evaluates to \( \sqrt{6} Q c_v^2 \), which does not capture any critical behavior, and the analogy to fluid systems is lost. We therefore use Eq. (12) for computing the scalar curvature.
log($R_o c_v / v$) as a function of $Q$ for different $T_r$. The solid blue and the dashed red lines correspond to log($|R_o c_v / v^g|$) and log($|R_o c_v / v^θ|$) respectively, for $T_r = 0.99$. The dot-dashed black and the dotted brown lines correspond to the same quantities for $T_r = 0.7$.

$P_r$ and $v_r$, yielding

$$R_o c_v = \frac{32\sqrt{2} \pi Q^3 v_r^3 (1 - 3v_r^2) (1 + 3P_r v_r^4)}{3\sqrt{3} (1 - 2v_r^2 + P_r v_r^4)^2}.$$  \hspace{1cm} (23)

Corresponding to the minimum value of $v$, i.e., $v_{\text{min}} = v_r v_{\text{min}}^o$, we obtain

$$\frac{R_o c_v}{v_{\text{min}}} = \frac{2\pi Q^2 (\sqrt{9 + 3P_r} - 3)}{3l_p^2 P_r}.$$  \hspace{1cm} (24)

The condition $R_o c_v > v_{\text{min}}$ then implies that

$$P_r < \frac{4\pi Q^2}{3l_p^2} \left( \frac{\pi Q^2}{l_p^2} - 3 \right), \hspace{1cm} \text{and} \hspace{1cm} \frac{Q}{l_p} > \frac{\sqrt{3}}{\sqrt{\pi}}.$$  \hspace{1cm} (25)

Note that the constraint of Eq. (25) is minimal, i.e., based on the minimum volume at a given pressure. It is useful to check the validity of the geometric approach in more general cases, for example at the saturation curve and in the super-critical region. For this, one has to check that the quantity $|R_o c_v / v| > 1$ in both these regions, with the constraint on the charge given in Eq. (25).

In Fig. [1], we plot the quantity log($|R_o c_v / v|$) as a function of the charge $Q$ (after setting $l_p = 1$), for different values of the reduced temperature, at saturation, using Eq. (22). In the figure, the solid blue and the dashed red lines correspond to log($|R_o c_v / v^g|$) and log($|R_o c_v / v^θ|$) respectively, for $T_r = 0.99$. The dot-dashed black and the dotted brown lines correspond to log($|R_o c_v / v^g|$) and log($|R_o c_v / v^θ|$) respectively, for $T_r = 0.7$. We note that for the last case, the geometric analysis becomes invalid on the liquid side of the saturation curve, even for $Q \sim 1$, which satisfies the constraint of Eq. (25). Broadly, our analysis implies that at saturation, we are restricted to $T_r \gtrsim 0.75$ for the geometric analysis to be valid.

In Fig. [2], we plot the quantity $R_o c_v / v$ in the super-critical region $T_r = 2$ as a function of $v_r$, for different values of the charge, again setting $l_p = 1$. The solid blue, dashed red and dot-dashed black curves correspond to $Q = 1, 1.2$ and 1.5 respectively, all of which satisfy the bound given in Eq. (25). We see however that in the region $v_r < 0.25$, all the curves indicate $R_o c_v / v < 1$, i.e the geometric analysis is strictly invalid in these regions. In Fig. [3], we plot $R_o c_v / V$ as a function of $v_r$ for different values of $T_r$. The solid blue, dashed red and dot-dashed black here correspond to $T_r = 2, 4$ and 10, respectively. We note that for all these values of $T_r$, $R_o c_v / V \sim 0.5$, for small $v_r$, a fact that readily follows from the limit of small $v_r$ in Eq. (18). This seems to indicate from our arguments above that the geometric formalism possibly does not have a proper physical interpretation in these regions. In fact, it can be checked that if we strictly enforce $R_o c_v \ll V$, the allowed region of parameter space is indeed very small.

Finally, it is of interest to consider how the correlation length indicates first order phase transitions near criticality. Following [26], it was suggested in [16] that close to the second order critical point, first order phase transitions can be predicted from the equality of the correlation length in the liquid and the gas phases. Since the scalar curvature is proportional to the correlation volume, the equality of this in the liquid and gas phases was conjectured to predict first order phase transitions, close to criticality. In [16], this conjecture was checked for
5 Kerr-AdS black holes

We will now briefly consider the extended phase space geometry of the four dimensional Kerr-AdS black holes, whose thermodynamics was considered in [23], [17]. Exact computations in this case are known to be complicated, and we will simply comment upon an approximate equation of state obtained in the slow rotation limit of the black hole. This is given in (Eq. (3.35) of [17])

\[
P = \frac{T}{v} - \frac{1}{2\pi v^2} + \frac{48J^2}{\pi v^6} - \frac{384J^4 (8\pi Tv + 7)}{\pi v^{10} (\pi Tv + 1)^2} + \frac{36864J^6 (13\pi Tv + 11)}{\pi v^{14} (\pi TV + 1)^3} + \mathcal{O}(J^8),
\]

where \( J \) is the angular momentum of the black hole and \( P, T \) and \( v \) denote the pressure, temperature and specific volume respectively, as before. Note that similar to the RN-AdS case, we have \( V = (1/6)\pi v^3 \).

We will mainly focus on the equation of state up to \( \mathcal{O}(J^2) \), i.e., retaining the first three terms on the right hand side of Eq. (26), and note that the specific heat at constant volume, \( c_v = 0 \) to this order [17], [23]. In a similar fashion as illustrated in [24], the geometry of the system at this order is conveniently studied by assuming a Kerr-AdS fluid that has an equation of state of the form

\[
P = \frac{T}{v} - \frac{a}{v^2} + \frac{d}{v^6},
\]

where \( a \) and \( d \) are two constants of appropriate dimensions, that characterize the departure of the fluid from an ideal gas form. For the slowly rotating KAdS black hole, we have \( a = 1/(2\pi) \) and \( d = 48J^2/\pi \). Following our discussion in section 3, one can check that the free energy per unit volume of the KAdS fluid can be written as

\[
f(T, \rho) = f_{id} - a\rho^2 + \frac{d}{5}\rho^6,
\]

where \( \rho = 1/v \), and \( f_{id} \) is the ideal gas part of the free energy, given by the first two terms on the right hand side of Eq. (2). The critical values of the thermodynamic quantities are

\[
T_c = \frac{8}{5k_B} \left( \frac{a^5}{15d} \right)^{1/4}, \quad v_c = \left( \frac{15d}{a} \right)^{1/4}, \quad P_c = \frac{2a^{3/2}}{3\sqrt{15d}}.
\]

Curiously, we find that equality of \( R_c c_v \) provides a somewhat better approximation to the Maxwell construction away from criticality.
Denoting $T = T_r T_c$, $v = v_r v_c$, $P = P_r P_c$, the reduced form of the equation of state is

$$P_r = \frac{12 T_r}{5 v_r} - \frac{3}{2v_r^2} + \frac{1}{10 v_r^6}.$$  

(30)

$R_o$ is computed using Eq. (12), and we find that

$$\lim_{c_v \to 0} \frac{R_o c_v}{V} = \frac{(1 - 5v_c^4) (1 - 5v_c^4 + 8T_r v_c^2)}{2 (1 - 5v_c^4 + 4T_r v_c^2)^2} = R_c c_v,$$

with $R_c$ computed from Eq. (1). Note that the minimum volume for a given pressure here has a more complicated expression than that in the RN-AdS case, but in the limit $P_r \to 0$, this equals $1/(15)^{1/4}$ and goes to zero as $P_r \to \infty$. This in particular implies that much like the RN-AdS case, there is a lower bound on the geometric analysis will break down.

The analysis above is based on a slow rotation approximation where one only keeps terms up to $O(J^2)$ in the expression for the pressure in Eq. (26). Indeed, the general case, without such an approximation, is difficult to handle analytically, the broad reason being that the exact dependence of the outer horizon radius as a function of the angular momentum is difficult to obtain. Hence, the free energy has to be computed order by order via a perturbation expansion of the outer horizon radius in terms of the rotation parameter (for an elaboration of this method, see [17]). Fortunately however, in [28], an exact expression for the critical point was given via the temperature - entropy criticality and it was found that the critical point obtained in the $O(J^2)$ expansion of the equation of state in Eq. (26) was a reasonably good approximation, with the absolute values of the relative deviations in the critical thermodynamic quantities being maximally $\sim 2.2\%$. This can be gleaned from the approximate values of the critical thermodynamic variables obtained from Eq. (26) at $O(J^2)$, and their exact values which read (with a superscript $e$ denoting the exact values),

$$(T_c, v_c, P_c) = \left(\frac{23/4}{55/4 \sqrt{3\pi} J}, 10^{1/4} \sqrt{3\sqrt{J}}, \frac{1}{36 \sqrt{10\pi} J}\right), \quad (T^{e}_c, v^{e}_c, P^{e}_c) = \left(\frac{0.04175}{\sqrt{J}}, 6.04736 \sqrt{J}, \frac{0.00286}{J}\right).$$

(32)

Note that at $O(J^2)$, one can begin with the correct constants, i.e., adjust $a$ and $d$ in Eq. (27) so that two of the critical values above are exact. Namely, we use the expressions for the critical values of $T_c$ and $v_c$ from Eq. (29), and equate them to $T^{e}_c$ and $v^{e}_c$ given in the second relation of Eq. (32). Then, instead of $a = 1/(2\pi) = 0.159$ and $d = 48 J^2/\pi = 15.279 J^2$, we obtain the slightly different values $a^{e} = 0.1578$ and $d^{e} = 14.0694 J^2$. With our chosen values of $a = a^{e}$ and $d = d^{e}$, the reduced form of the equation of state in Eq. (30) is of course unchanged. Then from Eq. (29), we obtain the critical pressure $P_c = 0.002877/J$, with the absolute value of the relative error $(P^{e}_c - P_c)/P^{e}_c \sim 0.66\%$, indeed a small deviation from the exact value. It is thus tempting to think that near criticality, the effect of the exact equation of state of the KAdS black hole is to effectively “renormalize” the constants $a$ and $d$ appearing in Eq. (27), but we do not have a convincing proof for this.

Figure 5: Coexistence curve for the Kerr-AdS fluid on the $T_r - P_c$ plane. The solid red line is the fit given in [28] and the dotted blue line is the approximate curve obtained from Eq. (27).

Now, instead of the above, if we use the equation of state in Eq. (26) up to $O(J^6)$, and make a reasonable assumption that the critical values of the thermodynamic quantities are now given by their exact values appearing in the second relation of Eq. (32), then we obtain the reduced equation of state at this order,

$$P_r = \frac{2.41 T_r}{v_r} - \frac{1.52 v_r^3}{v_r^2} + \frac{3.43 	imes 10^{-1}}{v_r^6} - \frac{6.49 	imes 10^{-4} (7 + 6.34 T_r v_r)}{v_r^{10} (1 + 0.79 T_r v_r)^2} + \frac{4.68 	imes 10^{-5} (11 + 10.31 T_r v_r)}{v_r^{14} (1 + 0.79 T_r v_r)^3}.$$  

(33)
Clearly then, close to criticality where $v_r$ and $T_r$ are $O(1)$, terms higher than $O(J^2)$ in the equation of state of Eq. (26) are strongly suppressed, and hence the quadratic order equation of state of Eq. (27) is sufficiently robust. This can be further substantiated as follows. Using Eq. (30), one can numerically plot the coexistence curve in the $T_r - P_r$ plane. In Fig. (5), we show this with the dotted blue line corresponding to the coexistence curve obtained via the Maxwell construction. In this figure, the analytical coexistence curve given in Eq. (43) of [23] is shown by the solid red line. One can clearly see that close to criticality, i.e., for $T_r \geq 0.85$, the two curves almost merge, implying that in this region, the small $J$ equation of state provides an excellent approximation to the exact one.

In fact, our analysis for small $J$ indicates that the geometric method is valid only in the region close to criticality, for the Kerr-AdS fluid case. We arrive at this conclusion by computing $R_o c_v / v$ using Eq. (31). At saturation, we find that in order for the first condition of Eq. (21) to be true on the liquid side of the saturation curve, we require, for $J = 0.1$, $T_r \gtrsim 0.91$. For smaller values of $J$, we are restricted even closer to the critical point. Since the small $J$ approximation is excellent near criticality as we have just discussed, we can conclude that our results are robust and will not change significantly if one relaxes the assumption of small $J$. Although this analysis is justified for the Kerr-AdS fluid, we note that for the Kerr-AdS black hole, it is somewhat challenged, as the notion of the specific volume being the volume per degree of freedom does not hold [17]. However, purely from the equation of state for the slowly rotating Kerr-AdS black hole of Eq. (26), if we identify $v$ as an analog of the specific volume, then the interpretation of $R_o c_v / v$ will be similar to that of the Kerr-AdS fluid.

6 Summary and Discussions

In the study of classical phase transitions driven by thermal fluctuations, the formalism of thermodynamic geometry has been popular of late. This Riemannian geometric formalism begins by constructing a metric on the parameter manifold of the theory, and ultimately relates the Ricci scalar curvature of this manifold to the correlation length, near criticality. For two-dimensional parameter manifolds where the Riemann tensor has a single independent component, the Ricci scalar fully characterizes the curvature of the manifold and in that sense uniquely specifies the geometry. The conventional approach then is to treat the system as an open system, whose volume is held fixed, and this is in turn in equilibrium with a sufficiently larger system, with which it can exchange energy. This point of view provides several natural restrictions on the applicability of the geometric method itself. These are often to do with constraints imposed by the system volume [2], [16], [3]. These constraints are listed here in Eq. (21).

In black hole thermodynamics, the Riemannian geometric method is far more restricted than examples involving conventional fluids, the prime reason being that one of key the assumptions in the latter, namely that entropy is extensive, does not hold for black holes. Nevertheless, it has been common to use the geometric method purely as a mathematical tool prior to the advent of extended phase space thermodynamics [4], when the idea of the thermodynamic black hole volume was somewhat obscure. However, with the recent understanding that a variable cosmological constant can indeed be identified the black hole pressure and has a well defined conjugate volume, it is important to ask how far one can proceed with geometric methods, given that the restrictions due to the system volume given in Eq. (21) are now far more transparent.

In this spirit, we have studied the geometric formalism for the four dimensional RN-AdS and Kerr-AdS black holes in extended phase space thermodynamics – systems that have received considerable attention of late. We have shown that in a dimensionally consistent analysis, the scalar curvature is proportional to the thermodynamic volume of the black hole. In this sense, it is not universal, as the thermodynamic volume of the RN-AdS (Kerr-AdS) black hole necessarily depends on its charge (angular momentum). However, the scalar curvature per unit volume is an universal quantity, much like fluid systems, as was pointed out in early studies of the geometry of fluid systems. Here, since the specific heat at constant volume vanishes, one has to more appropriately consider a normalized scalar curvature, following [19].

We have considered three physical constraints on the normalized scalar curvature $R_o c_v$ in the geometry of extended phase space thermodynamics, as summarized in Eq. (21): a) It must be larger than the minimum volume allowed at a certain pressure, b) It must be larger than the specific volume, which sets the lowest scale of the system and c) Away from criticality, it must be small compared to the overall volume of the system, for the Gaussian approximation on which the geometric formulation rests, to be valid. We find that all three might be seriously challenged in extended phase space black hole thermodynamics. This points to the fact that the application of geometric methods in this formalism as elaborated upon by several authors of late, has to be undertaken with great care and this is the main takeaway message from the results presented in this paper.

Acknowledgments: The work of T. S. is supported in part by Science and Engineering Research Board (India) via Project No. EMR/2016/008037.
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