Yet another generalization of 
Postnikov’s hook length formula for binary trees

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Abstract. — We discover another one-parameter generalization of
Postnikov’s hook length formula for binary trees. The particularity of
our formula is that the hook length $h_v$ appears as an exponent. As an
application, we derive another simple hook length formula for binary trees
when the underlying parameter takes the value $1/2$.

1. Introduction

Consider the set $B(n)$ of all binary trees with $n$ vertices. For each
vertex $v$ of $T \in B(n)$ the hook length of $v$, denoted by $h_v$, or just $h$
short, is the number of descendants of $v$ (including $v$). The following hook
length formula for binary trees

\begin{equation}
\sum_{T \in B(n)} \prod_{v \in T} (1 + \frac{1}{h_v}) = \frac{2^n}{n!} (n + 1)^{n-1}
\end{equation}

was discovered by Postnikov [Po04]. Further combinatorial proofs and
extensions have been proposed by several authors [CY08, GS06, MY07,
Se08]. In particular, Lascoux conjectured the following one-parameter
generalization

\begin{equation}
\sum_{T \in B(n)} \prod_{v \in T} (x + \frac{1}{h_v}) = \frac{1}{(n + 1)!} \prod_{k=0}^{n-1} ((n + 1 + k)x + n + 1 - k),
\end{equation}

which was subsequently proved by Du-Liu [DL08]. The latter generalization
appears to be very natural, because the left-hand side of (2) can be
obtained from the left-hand side of (1) by replacing 1 by $x$.

It is also natural to look for an extension of (1) by introducing a new
variable $z$ in the right-hand side, namely by replacing $2^n(n + 1)^{n-1}/n!$
by $2^n z(n + z)^{n-1}/n!$! It so happens that the corresponding left-hand side is
also a sum on binary trees, but this time the hook length $h_v$ appears as
an exponent. The purpose of this Note is to prove the following Theorem.

Theorem 1. For each positive integer $n$ we have

\begin{equation}
\sum_{T \in B(n)} \prod_{v \in T} \frac{(z + h)^{h-1}}{h(2z + h - 1)^{h-2}} = \frac{2^n z}{n!} (n + z)^{n-1}.
\end{equation}

With $z = 1$ in (3) we recover Postnikov’s identity (1). The following
corollary is derived from our identity (3) by taking $z = 1/2$. 

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Corollary 2. For each positive integer \( n \) we have

\[
\sum_{T \in B(n)} \prod_{v \in T} \left(1 + \frac{1}{2h}\right)^{h-1} = \frac{(2n+1)^{n-1}}{n!}.
\]

2. Proof of the Theorem

Let us take an example before proving the Theorem. There are five binary trees with \( n = 3 \) vertices:

\[
\begin{array}{c}
T_1 & T_2 & T_3 & T_4 & T_5 \\
\begin{array}{ccc}
1 & 2 & 3 \\
\end{array} & \begin{array}{ccc}
2 & 3 \\
1 \\
\end{array} & \begin{array}{ccc}
3 & 2 \\
1 \\
\end{array} & \begin{array}{ccc}
3 & 2 \\
1 \\
\end{array} & \begin{array}{ccc}
3 & 1 \\
1 \\
\end{array}
\end{array}
\]

The hook lengths of \( T_1, T_2, T_3, T_4 \) are all the same \( 1, 2, 3 \); but the hook lengths of \( T_5 \) are \( 1, 1, 3 \). The left-hand side of (3) is then equal to

\[
4 \times \frac{1}{(2z)^{-1}} \frac{(z+2)^1}{2} \frac{(z+3)^2}{3(2z+1)} + \frac{1}{(2z)^{-1}} \frac{1}{(2z)^{-1}} \frac{(z+3)^2}{3(2z+1)} = \frac{2^3 z(z+3)^2}{3!}.
\]

Let \( y(x) \) be a formal power series in \( x \) such that

\[
y(x) = e^{xy(x)}.
\]

By the Lagrange inversion formula \( y(x)^z \) has the following explicit expansion:

\[
y(x)^z = \sum_{n \geq 0} z(n+z)^{n-1} \frac{x^n}{n!}.
\]

Since \( y^{2z} = (y^z)^2 \) we have

\[
\sum_{n \geq 0} 2z(n+2z)^{n-1} \frac{x^n}{n!} = \left( \sum_{n \geq 0} z(n+z)^{n-1} \frac{x^n}{n!} \right)^2.
\]

Comparing the coefficients of \( x^n \) on both sides of (7) yields the following Lemma.
Lemma 3. We have

\[
\frac{2z(n+2z)^{n-1}}{n!} = \sum_{k=0}^{n} \frac{z(k+z)^{k-1}}{k!} \times \frac{z(n-k+z)^{n-k-1}}{(n-k)!}.
\]

Proof of the Theorem. Let

\[
P(n) = \sum_{T \in \mathcal{B}(n)} \prod_{v \in T} \frac{(z+h)^{h-1}}{h(2z+h-1)^{h-2}}.
\]

With each binary tree \(T \in \mathcal{B}(n)\) \((n \geq 1)\) we can associate a triplet \((T', T'', u)\), where \(T' \in \mathcal{B}(k)\) \((0 \leq k \leq n-1)\), \(T'' \in \mathcal{B}(n-1-k)\) and \(u\) is a vertex of hook length \(h_u = n\). Hence

\[
P(n) = \sum_{k=0}^{n-1} P(k)P(n-1-k) \times \frac{(z+n)^{n-1}}{n(2z+n-1)^{n-2}}.
\]

It is routine to verify that \(P(n) = 2^n z(z+n)^{n-1}/n!\) for \(n = 1, 2, 3\). Suppose that \(P(k) = 2^k z(z+k)^{k-1}/k!\) for \(k \leq n-1\). From identity (9) and Lemma 3 we have

\[
P(n) = \sum_{k=0}^{n-1} \frac{2^k z(z+k)^{k-1}}{k!} \times \frac{2^{n-k-1} z(z+n-k-1)^{n-k-2}}{(n-k-1)!} \times \frac{(z+n)^{n-1}}{n(2z+n-1)^{n-2}} = \frac{2^n z}{n!} (z+n)^{n-1}.
\]

By induction, formula (3) is true for any positive integer \(n\).

3. Concluding and Remarks

The right-hand sides of (3) and (4) have been studied by other authors [GS06, DL08, MY07], but our formula has the following two major differences: (i) the hook length \(h_u\) appears as an exponent; (ii) the underlying set remains the set of binary trees, whereas in the above mentioned papers the summation has been changed to the set of \(m\)-ary trees or plane forests. It is interesting to compare Corollary 2 with the following results obtained by Du and Liu [DL08]. Note that the right-hand sides of formulas (4), (10) and (11) are all identical!
Proposition 4. For each positive integer $n$ we have

\[
\sum_{T \in \mathcal{T}(n)} \prod_{v \in I(T)} \left( \frac{2}{3} + \frac{1}{3h} \right) = \frac{(2n + 1)^{n-1}}{n!},
\]

where $\mathcal{T}(n)$ is the set of all 3-ary trees with $n$ internal vertices and $I(T)$ is the set of all internal vertices of $T$.

Proposition 5. For each positive integer $n$ we have

\[
\sum_{T \in \mathcal{F}(n)} \prod_{v \in (T)} \left( 2 - \frac{1}{h} \right) = \frac{(2n + 1)^{n-1}}{n!},
\]

where $\mathcal{F}(n)$ is the set of all plane forests with $n$ vertices.

References

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