A robust multi-dimensional sparse Fourier transform in the continuous setting

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Abstract

Sparse Fourier transform (Sparse FT) is the problem of learning an unknown signal, whose frequency spectrum is dominated by a small amount of \( k \) individual frequencies, through fast algorithms that use as few samples as possible in the time domain. The last two decades have seen an extensive study on such problems, either in the one-/multi-dimensional discrete setting [Hassanieh, Indyk, Katabi, and Price STOC’12; Kapralov STOC’16] or in the one-dimensional continuous setting [Price and Song FOCS’15]. Despite this rich literature, the most general multi-dimensional continuous case remains mysterious.

This paper initiates the study on the Sparse FT problem in the multi-dimensional continuous setting. Our main result is a randomized non-adaptive algorithm that uses sublinear samples and runs in sublinear time. In particular, the sample duration bound required by our algorithm gives a non-trivial improvement over [Price and Song FOCS’15], which studies the same problem in the one-dimensional continuous setting.

The dimensionality in the continuous setting, different from both the discrete cases and the one-dimensional continuous case, turns out to incur many new challenges. To overcome these issues, we develop a number of new techniques for constructing the filter functions, designing the permutation-then-hashing schemes, sampling the Fourier measurements, and locating the frequencies. We believe these techniques can find their applications in the future studies on the Sparse FT problem.

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1 Introduction

The Fourier transform (FT), since the introduction by Joseph Fourier in 1822 [Fou22], has become ubiquitous in areas such as signal processing, electrical engineering, applied mathematics and machine learning. This popularity basically stems from two rationales: (i) in various applications like signal processing, the Fourier spectrum is a convenient way for data compression, because it stores the signals in terms of the energies on merely a few components; and (ii) the seminal fast Fourier transform (FFT) algorithm by Cooley and Tukey [CT65] is extremely efficient in real-world applications, which computes in time $O(N \log N)$ the Fourier spectrum of a length-$N$ discrete signal in any dimension $d \geq 1$.

For a long while, the theoretical computer science (TCS) community has been obsessed by the Fourier transform as well, not only because of the interest in itself, but because of its applications to integer multiplication [Für09], Subset Sum and 3SUM [CLRS09, Bri17, KX17], fast Johnson-Lindenstrauss transform [LDFU13], linear programming [LSZ19, JSWZ20], distributional learning [DKS16a, DKS16b, DKS16c] and learning mixture of regressions [CLS20] etc.

Since Cooley and Tukey’s work, the TCS community has been struggling for an $O(N \log N)$-time FFT algorithm – without success yet. By contrast, as quoted from Indyk and Kapralov [IK14]:

“many of these applications rely on the fact that most of the Fourier coefficients of the signals are small or equal to zero, i.e., the signals are (approximately) sparse.”

In other words, when the Fourier spectrum is approximately $k$-sparse for some $k = o(N)$, we even hope for the more demanding algorithms that access sublinear samples and run in sublinear time. These practical and/or theoretical desires have motivated the recent research interest in the sparse Fourier transform problem (Sparse FT).

Over the last two decades, the Sparse FT problem has been investigated and extended in various directions. By now we can even say that it constitutes a “subarea” within sublinear algorithms. These former works can be classified into two lines: (i) those in the one-/multi-dimensional discrete settings [HIKP12a, HIKP12b, IKP14, IK14, Kap16, Kap17, NSW19, and follow-ups]; and (ii) those in the one-dimensional continuous setting [BCG+12, Moi15, PS15, CKPS16, and follow-ups]. Despite this rich literature, it is quite surprising that, in the (most general) multi-dimensional continuous setting, the research on the Sparse FT problem remains a huge blank! The target of this work is to fill some of these gaps. Concretely, we formulate the multi-dimensional continuous Sparse FT problem below (Section 1.1), and explore it in the bulk of this paper.

1.1 Formulation

The multi-dimensional continuous Sparse FT problem concerns the setting in which we observe a complex-valued signal function $x(t) = x^*(t) + g(t) \in \mathbb{C}$ over a certain duration $t \in [0, T]^d$. Here, the $k$-Fourier-sparse signal $x^*(t)$ for $t \in \mathbb{R}^d$ is what we wish to learn, whereas the extra term $g(t)$ is the noise. We assume $x^*(t)$ to have a $k$-sparse Fourier spectrum. Namely, it can be formulated as

$$x^*(t) := \sum_{i \in [k]} x^*_i(t) = \sum_{i \in [k]} v_i \cdot e^{2\pi i \cdot f_i^\top t},$$

for $k \geq 1$ magnitude-frequency pairs $\{(v_i, f_i)\}_{i \in [k]} \subseteq \mathbb{C} \times \mathbb{R}^d$ (a.k.a. the tones); where $i := \sqrt{-1}$. The Fourier transform is the sum of $k \geq 1$ shifted $d$-dimensional Dirac delta functions: for $f \in \mathbb{R}^d$,

$$\hat{x}^*(f) := \sum_{i \in [k]} \hat{x}^*_i(f) = \sum_{i \in [k]} v_i \cdot \text{Delta}_{f_i}(f).$$
To make the problem interesting, similar to the past works in the one-dimensional continuous case [PS15, CKPS16], we assume a bounded Fourier spectrum \( \text{supp}(\hat{x}) \subseteq [−F, F]^d \), for some \( F > 0 \). Especially, the frequencies \( \{f_i\}_{i \in [k]} = \text{supp}(\hat{x}^*) \subseteq [−F, F]^d \). There is no other requirement for the noise \( g(t) \) over the whole duration \( t \in [0, T]^d \).

Further, in order to distinguish the frequencies \( \{f_i\}_{i \in [k]} \), the past works either assume (in the discrete cases) that \( \{f_i\}_{i \in [k]} \) are “on-the-grid” [HIKP12a, Kap16], or assume (in the one-dimensional continuous case) that \( \{f_i\}_{i \in [k]} \) have some minimum distance \( \eta > 0 \) [PS15]. We need such a suitable assumption in the multi-dimensional continuous setting. Since later we will consider the so-called \( \ell_2/\ell_2 \)-guarantee (see Footnote 1), we naturally also consider the \( \ell_2 \)-distances among \( \{f_i\}_{i \in [k]} \).

**Assumption 1.1** (Separation among the frequencies). For some separation level \( \eta > 0 \), the \( k \geq 1 \) frequencies \( \{f_i\}_{i \in [k]} \) have the minimum \( \ell_2 \)-distance \( \min_{i \neq i', f_i \in [k]} \| f_i - f_{i'} \|_2 = \eta \).

The only way that an algorithm can access the signal \( x(t) = x^*(t) + g(t) \) is to sample \( x(\tau_j) \in \mathbb{C} \) at a certain amount of \( m \geq 1 \) time points \( \{\tau_j\}_{j \in [m]} \). And within the duration \( t \in [0, T]^d \), these time points \( \{\tau_j\}_{j \in [m]} \) can be chosen arbitrarily.

Under the above definitions and assumptions, our primary goal is to get a good approximation \( x'(t) := \sum_{i \in [k]} x'_i(t) \) to the \( k \)-Fourier-sparse signal \( x^*(t) \). If there were an oracle, we would achieve \( x'(t) \equiv x^*(t) \). Against the observed signal \( x(t) = x^*(t) + g(t) \), this gives an “ideal” error

\[
\frac{1}{T^d} \cdot \int_{t \in [0, T]^d} |x'(t) - x(t)|^2 \cdot dt = \|g\|_T^2.
\]

for \( \|g\|_T^2 := \frac{1}{\pi^d} \cdot \int_{t \in [0, T]^d} |g(t)|^2 \cdot dt \geq 0 \); we measure the approximation under the \( \ell_2/\ell_2 \) guarantee.  

Beyond the noise error, for some parameter \( \delta > 0 \), we shall incorporate one more term

\[
\delta \cdot \sum_{i \in [k]} \frac{1}{T^d} \int_{t \in [0, T]^d} |x_{i}^*(t)|^2 \cdot dt = \delta \cdot \sum_{i \in [k]} |v_i|^2.
\]

To conclude, let \( N^2 := \|g\|_T^2 + \delta \cdot \sum_{i \in [k]} |v_i|^2 > 0 \) be the noise level, then our algorithm should output a \( k \)-Fourier-sparse recovered signal \( x'(t) \) such that, for some approximation ratio \( C > 1 \),

\[
\text{signal estimation error} := \frac{1}{T^d} \cdot \int_{t \in [0, T]^d} |x'(t) - x(t)|^2 \cdot dt \leq C^2 \cdot N^2. \tag{3}
\]

Apart from a \( k \)-Fourier-sparse approximation \( x'(t) \approx x^*(t) \), we are interested in recovering the individual tones \( \{(v'_i, f'_i)\}_{i \in [k]} \approx \{(v_i, f_i)\}_{i \in [k]} \) also. To this end, our algorithm also needs to output \( k \geq 1 \) individual recovered signals \( x'_i(t) := v'_i \cdot e^{2\pi i f'_i t} \approx x_{i}^*(t) \) such that

\[
\text{tone estimation error} := \sum_{i \in [k]} \frac{1}{T^d} \cdot \int_{t \in [0, T]^d} |x'_i(t) - x_{i}^*(t)|^2 \cdot dt \leq C^2 \cdot N^2.
\]

In addition to those primary goals, we have three more secondary optimization goals, i.e. our algorithm should use a minimum amount of (sublinear) samples over a shortest possible duration \( t \in [0, T]^d \), and should have a fastest possible (sublinear) running time.

---

1. Most previous works choose one guarantee among the strongest \( \ell_\infty/\ell_2 \) [IK14, NSW19], the second strongest \( \ell_2/\ell_2 \) [GMS05, HIKP12a, PS15, Kap16, CKPS16, CKSZ17, Kap17], and the relatively weak \( \ell_2/\ell_1 \) [CT06, RV08, CGV13]. For an overview of these guarantees, the reader can refer to [NSW19, Table 1].
1.2 Our results

The next two theorems summarize our results, respectively for the tone estimation (Theorem 1.2) and for the signal estimation (Theorem 1.3). In short, our algorithm RECOVERYSTAGE (Algorithm 10) achieving a constant approximation to the noise level $N > 0$ in any constant dimension.

**Theorem 1.2** (Main result for tone estimation). Consider any signal $x(t) : [0, T]^d \rightarrow \mathbb{C}$ of the form

$$x(t) := x^*(t) + g(t) \quad \text{and} \quad x^*(t) := \sum_{i \in [k]} x_i(t) = \sum_{i \in [k]} v_i \cdot e^{2\pi i f_i^t t},$$

where $g(t) \in \mathbb{C}$ is the noise and the tones $\{(v_i, f_i)\}_{i \in [k]} \subseteq \mathbb{C} \times \mathbb{R}^d$ constitute a $k$-sparse Fourier spectrum. Assume (i) a bounded Fourier spectrum $\{f_i\}_{i \in [k]} \subseteq \text{supp}(\tilde{x}) \subseteq [-F, F]^d$ and (ii) a minimum $\ell_2$-norm frequency separation $\eta := \min_{i \neq i'} \|f_i - f_{i'}\|_2 > 0$.

For some parameter $\delta > 0$, define the $\ell_2$-norm noise level $N > 0$ by letting

$$N^2 := \frac{1}{T^d} \int_{t \in [0, T]^d} |g(t)|^2 \cdot dt + \delta \sum_{i \in [k]} |v_i|^2.$$

Whenever the (randomized non-adaptive) algorithm RECOVERYSTAGE (Algorithm 10) observes the signal $x(t) \in \mathbb{C}$ over a sufficiently long duration $t \in [0, T]^d$, where\(^2\)

$$T \gtrsim \eta^{-1} \cdot d^{1.5} \cdot \log(kd/\delta) \cdot \log(d),$$

it outputs $k \geq 1$ recovered tones $\{(v_i', f_i')\}_{i \in [k]} \subseteq \mathbb{C} \times \mathbb{R}^d$ that, with success probability $1 - 1/\text{poly}(k)$, approximate the true tones $\{(v_i, f_i)\}_{i \in [k]}$ up to an error proportional to the noise level $N > 0$. More concretely, the recovered tones $\{(v_i', f_i')\}_{i \in [k]}$ can be reindexed such that:

- All the “large” true tones are recovered well.\(^3\) Once a true tone $i \in [k]$ has a sufficiently large magnitude $|v_i| \gtrsim N$, both its magnitude $v_i \in \mathbb{C}$ and its $f_i \in [-F, F]$ are recovered up to\(^4\)

  $$\|f_i' - f_i\|_2 \lesssim_d N \cdot T^{-1} \cdot |v_i|^{-1} \quad \text{and} \quad |v_i' - v_i| \lesssim_d N.$$

- In addition, the total tone estimation error is bounded (which is a stronger guarantee)\(^5\):

  $$\sum_{i \in [k]} \frac{1}{T^d} \int_{t \in [0, T]^d} |x_i'(t) - x_i(t)|^2 \cdot dt \lesssim_d C^2 N^2, \quad \text{(4)}$$

  where $x_i'(t) := v_i' \cdot e^{2\pi i f_i't} \in \mathbb{C}$ for all $i \in [k]$ are the individual recovered signals.

Further, the algorithm RECOVERYSTAGE takes $\tilde{O}_d(k \cdot \log(F/\eta))$ samples and time.\(^6\)

---

\(^2\)We often denote $f \gtrsim g$ when $f \geq C_0 \cdot g$ for some universal constant $C_0 > 0$, and the notation $f \lesssim g$ has a similar meaning. Also, we denote $f \approx g$ when both equations $f \gtrsim g$ and $f \lesssim g$ hold.

\(^3\)If the magnitude of a true tone is too small $v_i \lesssim N$, then the corresponding signal $x_i(t) = v_i \cdot e^{2\pi i f_i^t t}$ can be erased by a particular noise $g(t) = -x_i(t)$, and no estimation guarantee is possible.

\(^4\)We say $f \lesssim_d g$ if $f \leq \text{poly}(d) \cdot g$.

\(^5\)We pick $C$ between $\Omega(1)$ and the signal-to-noise ratio $\rho \gg 1$ (see Definition H.1).

\(^6\)The notation $\tilde{O}_d(f)$ assumes a constant dimension $d \geq 1$ and hides the term $\text{poly}(\log f)$; similar for $\tilde{O}_d$ and $\tilde{O}_d$. 
Theorem 1.3 (Main result for signal estimation). In the same setting as Theorem 1.2, whenever the duration \( T \geq \eta^{-1} \cdot (d^{1.5} \cdot \log(d) + d^{1.5} \cdot k^{1-1/d}) \cdot \log(kd/\delta) \), the signal estimation error of the \( k \)-Fourier-sparse recovered signal \( x'(t) := \sum_{i \in [k]} x'_i(t) \in \mathbb{C} \) against the observed signal \( x(t) \in \mathbb{C} \), in terms of the \( \ell_2/\ell_2 \) guarantee,

\[
\frac{1}{T^d} \int_{t \in [0,T]^d} |x'(t) - x(t)|^2 \cdot dt \lesssim_d N^2.
\] (5)

Remark 1.4. In the one-dimensional case \( d = 1 \), our algorithm RECOVERYSTAGE works when the duration \( T \geq \eta^{-1} \cdot \log k \) is large enough, while the previous result \([PS15]\) requires \( T \gtrsim \eta^{-1} \cdot \log^2 k \) for the signal estimation. For more details about this improvement, the reader can refer to Section 2.5.

1.3 An overview of previous techniques

The Sparse FT problem falls into the “sparse recovery” paradigm. Among such problems, an example is to learn an approximately \( k \)-sparse length-\( N \) vector \( \hat{y} \in \mathbb{R}^N \), by just accessing the length-\( N' \) measurements \( y := \Phi \hat{y} \) given by an amount of \( N' \)-to-\( N \) sensing matrices \( \Phi \in \mathbb{R}^{N' \times N} \), for some \( N' \ll N \). Based on the measurements, an algorithm should output a \( k \)-sparse vector \( \hat{y}' \in \mathbb{R}^N \) to approximate \( \hat{y} \in \mathbb{R}^N \). E.g., under the \( \ell_2/\ell_2 \) guarantee, we aim at achieving

\[
\|\hat{y}' - \hat{y}\|_2 \lesssim \min_{k\text{-sparse } z} \|z - \hat{y}\|_2.
\]

If we allow ourselves to design the sensing matrices \( \Phi \in \mathbb{R}^{N' \times N} \), the above problem is known as compressed sensing, and the optimization goals are threefold: (i) to access the fewest measurements, i.e. sample complexity\(^7\); (ii) to fast extract the \( k \)-sparse approximation \( \hat{y}' \approx \hat{y} \), i.e. decoding time; and (iii) to use column-sparsest possible sensing matrices \( \Phi \), hence a faster encoding time.\(^8\)

We instead face the (discrete) sparse Fourier transform problem, if the above vector \( \hat{y} \in \mathbb{R}^N \) is replaced by a length-\( N \) Fourier spectrum \( \hat{x} \in \mathbb{C}^N \) (of any dimension \( d \geq 1 \)) and the measurements \( y \) are replaced by the signal samples \( x \in \mathbb{C} \). Again, the Fourier spectrum \( \hat{x} \in \mathbb{C}^N \) is unknown, and we can only leverage the signal samples \( x \in \mathbb{C} \). Now our optimization goals are to reduce the sample complexity and the decoding/running time.

Compressed sensing. To leverage the measurements, the earlier works on compressed sensing \([GLPS10, DBIPW10, IP11, IPW11, BIP+16, NS19]\) first get a bunch of pseudorandom hash functions \( h: [N] \mapsto [\mathcal{B}] \), where \( \mathcal{B} = \Theta_d(k) \) is the number of bins. Such a “hashing” is associated with a random sign function \( s: [n] \mapsto \{-1,+1\} \). In one hashing, we derive the linear combination of the form

\[
u_j := \sum_{i \in [m]: h(i) = j} y_i \cdot s(i),\]
(6)

for every bin \( j \in [\mathcal{B}] \), based on a certain amount of \( m = o(N) \) measurements \( \{y_i\}_{i \in [m]} \subseteq \mathbb{R}^N' \). This scheme is known as “hashing into \( \mathcal{B} \) bins”. Following such ideas, \( O(k \log(n/k)) \) samples suffice to get a desired \( k \)-sparse approximation \( \hat{y}' \approx \hat{y} \) \([GLPS10, NS19]\).

\(^7\)Only in the literature on compressed sensing, sample complexity is often called the number of measurements.

\(^8\)Optimizing encoding time only makes sense when we are allowed to design the sensing matrix, for more details of encoding time, we refer the readers to \([NS19]\).
Discrete Fourier transform. A most basic obstacle to adopting a compressed sensing algorithm to the discrete Sparse FT problem is that, how to implement the “hashing into \( \mathcal{B} \) bins” scheme by using the Fourier samples. Now we observe the signal \( x \) in the time domain, but wish to recover its Fourier spectrum \( \hat{x} \in \mathbb{C}^N \) in the frequency domain.

The approach in the past works [HIKP12a, IK14, Kap16, Kap17] is to mimic the transformation in Eq. (6). That is, we first permute a bunch of \( m = o(N) \) signal samples \( \{x_i\}_{i \in [m]} \) via a pseudo-random affine permutation \( \mathcal{P} \). Then, the permuted samples \( \{(\mathcal{P}x)_i\}_{i \in [m]} \) are respectively scaled by coefficients \( \{\mathcal{G}(l_i)\}_{i \in [m]} \), i.e. the values of a filter function \( \mathcal{G} : \mathbb{R}^d \to \mathbb{R} \) at \( m = o(N) \) many lattice points \( \{l_i\}_{i \in [m]} \subset \mathbb{R}^d \). Akin to Eq. (6), we use a transformation \( u_j = \sum_{i \in [m]: \mathcal{P}(i) = j} (\mathcal{P}x)_i \cdot \mathcal{G}(l_i) \).

The second difficulty is that the hashing is no longer perfect. For the compressed sensing, a coordinate \( i \in [N] \) contributes 100% to a target bin, and 0% to the other \( (\mathcal{B} - 1) \) bins. But for the discrete Fourier transform, besides the target bin (which still gets 100%), any other bin should get a \( \delta > 0 \) fraction of mass from a coordinate \( i \in [N] \). This modification (a.k.a. “leakage” [IK14]) is to make the “hashing into \( \mathcal{B} \) bins” efficient. Because of the imperfect hashing, the sample complexity must blow up by a \( \log(1/\delta) \) factor, which makes it more difficult to optimize.

The Sparse FT problem suffers from another hindrance. For the compressed sensing, from the measurements we can easily recover a heavy coordinate \( i \in [N] \). But for the Sparse FT problem, instead of a coordinate \( i \in [N] \) itself, we observe an angle \( \theta \approx 2\pi\sigma(i + b) \pmod{2\pi} \) for some choice of \( \sigma \) and \( b \). Clearly, observing a single \( \theta \) is insufficient to recover \( i \in [N] \). Therefore, the Sparse FT problem requires a number of observations for a single heavy coordinate \( i \in [N] \).

Continuous Fourier transform. The one-dimensional continuous model is proposed by [PS15]. Their approach is enabled by devising the “continuous” counterparts of the “discrete” permutations, hashes, filters and estimation algorithms. In contrast to the discrete cases, currently we also need to reason about the one-dimensional duration \( t \in [0, T] \) in which the samples (in the time domain) are taken from. This gives one more optimization goal: to reduce the duration \( T > 0 \).

In the discrete setting, ideally we can recover the heavy coordinates/frequencies with no error, since they are on-the-grid and just have finite possibilities. But we cannot achieve so in the continuous setting. Since the “continuous” frequencies are off-the-grid, (i) a pair of frequencies \( f_i \neq f_j \) can be too close to be distinguished [Moi15]; and (ii) even if a frequency \( f_i \) is well separated from the others, we can only recover it up to some precision.

The discrete Fourier transform preserves the \( \ell_2 \)-norm of the spectrum \( \hat{x} \in \mathbb{C}^N \) as a length-\( N \) vector, so the guarantees in either the frequency domain or the time domain are equivalent. But in the continuous setting, we even cannot define the best \( k \)-sparse Fourier spectrum. For this reason, the work [PS15] considers both tone estimation and signal estimation in the time domain.

2 Our techniques

This section outlines our recovery algorithm for the multi-dimensional continuous Sparse FT problem: (Section 2.1) the permutation \( \mathcal{P} \), the hashing \( \# \), and a new sampling scheme for the Fourier measurements; (Section 2.2) the multi-dimensional filter function \( (\mathcal{G}(t), \hat{\mathcal{G}}(f)) \), e.g. a non-trivial construction to reduce the approximation ratio; (Section 2.3) how to recover the tones \( \{(v_i, f_i)\}_{i \in [k]} \subset \mathbb{C} \times \mathbb{R}^d \), especially, a new “fine-grained” technique for frequency location; and (Sections 2.4 and 2.5) how to establish the duration bounds claimed in Theorems 1.2 and 1.3, especially, how to get the improvement over [PS15] by combining the tools from real analysis.

Different from the discrete cases, where the samples w.l.o.g. are taken “on the grid”, currently we may access a signal sample \( x(t) \) at any time point \( t \in [0, T]^d \) within the duration. For this reason,
we shall modify the transformation in Eq. (6) as follows:

\[ u_j = \sum_{i \in [m], \hat{h}(i) = j} \mathcal{P} x(\tau_i) \cdot \mathcal{G}(l_i), \]  

(7)

where \( \{ \tau_i \}_{i \in [m]} \subseteq [0, T]^d \) are the sampling time points for a single “hashing”. Recall that the lattice points \( \{ l_i \}_{i \in [m]} \subseteq \mathbb{R}^d \) determine the scaling coefficients \( \{ \mathcal{G}(l_i) \}_{i \in [m]} \). Also, the above transformation is taken separately for all bins \( j \in [B] \).

2.1 Permutation and hashing

A first key step in this paper and almost all previous works, is to map the frequency domain into \( B = B^d = \Theta_d(k) \) many bins, via a careful pseudorandom permutation \( \mathcal{P} \) together with a careful pseudorandom hashing scheme \( \mathcal{h} \). For ease of presentation, below we review the construction of \((\mathcal{P}, \mathcal{h})\) in [IK14, Kap16] for the multi-dimensional discrete case, and then elaborate on our construction for the multi-dimensional continuous case by comparing it with the “discrete” one.

The multi-dimensional discrete setting. Now the frequency domain \( \{ \xi_i \}_{i \in [n^d]} = [n]^d \) is “on-the-grid”. Partition this domain \( [n]^d = \text{HEAD} \sqcup \text{TAIL} \) into the head frequencies and the tail frequencies (i.e. \( |\text{HEAD}| = k \) and \( |\text{TAIL}| = n^d - k = N - k \) and denote the magnitudes by \( \hat{x}[\xi_i] \in \mathbb{C} \).

In the frequency domain, roughly speaking, the permutation by [IK14, Kap16] works as follows:

\[ \hat{\mathcal{P}} x[\Sigma \xi_i - b \pmod{n}] = \hat{x}[\xi_i] \cdot e^{-\frac{2n \pi}{\pi} \xi_i^\top a}, \]  

(8)

where the modulo operation is taken coordinate-wise, \( \Sigma \in [n]^{d \times d} \) is a random matrix, and \( b, a \in [n]^d \) are random vectors. When \( d = 1 \), this permutation degenerates to the ones in [HIKP12a, HIKP12b, IKP14, CKSZ17, Kap17].

The matrix \( \Sigma \in [n]^{d \times d} \) is sampled uniformly at random among all integer matrices with odd determinants. So the inverse \( \Sigma^{-1} \pmod{n} \) exists, making the permutation one-to-one. The vector \( b \sim \text{Unif}[n]^d \) is uniform random, i.e. the “anchor point” of the permuted frequency domain.

Also, \( \Sigma \) and \( b \) together determine the hashing \( \mathcal{h} \). Since \( \Sigma \) is invertible, the linear transformation \( \Sigma \xi_i - b \pmod{n} \) forms a bijection from the “grid” frequency domain \( \{ \xi_i \}_{i \in [n^d]} = [n]^d \) to itself. [IK14, Kap16] partition the codomain \( [n]^d \) into \( B = B^d \) isomorphic Cartesian sub-grid, each of which has \( (\frac{n}{B})^d = \frac{N}{B} \) grid points. The sub-grids are exactly the desired bins. For a uniform random “anchor point” \( b \sim \text{Unif}[n]^d \), a frequency \( \xi_i \in [n]^d \) is equally likely to fall into one of the bins.

Another crucial observation is that, any two different frequencies \( \xi_i \neq \xi_j \in [n]^d \) fall into the same bin with probability \( \leq 0.01 \cdot k^{-1} \) [IK14, Kap16]. Thus, 90% head frequencies \( \xi_i \in \text{HEAD} \) will not collide with other head frequencies, hence being isolated.

The above permutation samples a uniformly random vector \( a \sim \text{Unif}[n]^d \), and thus rotates a magnitude \( \hat{x}[\xi_i] \in \mathbb{C} \) by a certain angle \( -(2n/\pi) \cdot \xi_i^\top a \), i.e. the rotated magnitude \( \hat{x}[\xi_i] \cdot e^{-\frac{2n \pi}{\pi} \xi_i^\top a} \in \mathbb{C} \) has a random phase. This is crucial because, given any sufficiently large subset \( S \subseteq \text{TAIL} \) of the tail magnitudes, a uniform random \( a \sim \text{Unif}[n]^d \) makes the total rotated magnitude (over \( \xi \in S \)) much smaller than the sum of the individual magnitudes.

Let \( S = \{ \xi \in \text{TAIL} : \mathcal{h}(\xi) = j \} \) denote the tail frequencies hashed into a certain bin \( j \in [B]^d \). Given the above equation, the total tail magnitude \( z_j := \sum_{\xi \in S} \hat{x}[\xi] \cdot e^{-\frac{2n \pi}{\pi} \xi^\top a} \in \mathbb{C} \) is small enough such that (i) \( z_j \in \mathbb{C} \) will not be identified as a spurious head frequency, when no head frequency is hashed into the \( j \)-th bin; and (ii) \( z_j \in \mathbb{C} \) will not falsify an isolated head frequency \( \xi_i \in \text{HEAD} \) too much, when \( \xi_i \in \text{HEAD} \) is the unique head frequency in the \( j \)-th bin.
The multi-dimensional continuous setting. In this case, the frequency domain is no longer the hypergrid $[n]^d$, but is the hypercube $[-F,F]^d$ for some $F > 0$. The magnitudes are specified by a function $\hat{x}(f) \in \mathbb{C}$ for all $f \in [-F,F]^d$. Again, we partition this domain $[-F,F]^d = \text{HEAD} \sqcup \text{TAIL}$ into the head/tail frequencies, and assume that there is no dominant tail magnitude.

The head frequencies $\text{HEAD} = \{f_i\}_{i \in [k]}$, corresponding to the $k$-Fourier-sparse signal $x^*(t)$, have the $\ell_2$-norm separation $\min_{i \neq i'} \| f_i - f_{i'} \|_2 = \eta > 0$ (Assumption 1.1). The continuous Fourier transform of an individual head signal $x_i^*(t) := e^{2\pi i f_i t}$ is a shifted Dirac delta function (Eq. (2)).

Inspired by the “discrete” permutation by [IK14, Kap16] (Eq. (8)), for the current setting we will use a “continuous” permutation

$$\Sigma f - b \mapsto \hat{x}(f) \cdot \det(\Sigma)^{-1} \cdot e^{-2\pi i f^\top a},$$

where the function $\operatorname{frac}(x) : \mathbb{R}^d \mapsto [0,1)^d$ computes the coordinate-wise fractional part of the input. Below we will explain how to select the counterpart $d$-to-$d$ matrix $\Sigma$ and $d$-dimensional vectors $b, a$ and why we shall use the $\operatorname{frac}(\cdot)$ function.

As mentioned, in the discrete setting, (i) the “discrete” random matrix $\Sigma$ shall be invertible; and (ii) makes any two different head frequencies $\xi_i \neq \xi_{i'} \in \text{HEAD}$ hashed into the same bin with probability at most $0.01 \cdot k^{-1}$ (i.e. the collision probability). Our “continuous” random matrix $\Sigma$ possesses both properties as well, and we construct it in two steps.

Step I. We first sample an interim matrix $\Sigma' \sim \text{Unif}(\text{SO}(d))$ uniformly at random from the $d$-dimensional rotation group, leading to a rotation matrix $\Sigma'$ with determinant either $\det(\Sigma') = 1$ or $\det(\Sigma') = -1$. Clearly, such an interim matrix $\Sigma' \in \mathbb{R}^{d \times d}$ is invertible.

To explain Step II, we need to determine what the “continuous” bins stand for. Respecting the transformation $\operatorname{frac}(\Sigma f - b)$ in Eq. (9), we are interested in the codomain $[0,1)^d$. We simply partition this unit hypercube into $B = B^d = \Theta_d(k)$ isomorphic sub-hypercubes, each of which has the volume $1/B$. Such sub-hypercubes are exactly the bins in the continuous setting.

Step II. We sample a random scaling factor $\beta \sim \text{Unif}([\beta, 2\beta])$, where the parameter $\beta > 0$ is sufficiently large, and derive the ultimate random matrix by letting $\Sigma := \beta \Sigma'$. Clearly, $\Sigma \in \mathbb{R}^{d \times d}$ is invertible. We next explain why the resulting collision probability is small.

Given Eq. (9), whether two head frequencies $f_i \neq f_{i'} \in \text{HEAD}$ collides or not depends on the difference vector $\Sigma f_i - f_{i'} \in \mathbb{R}^d$. Since $\Sigma$ is a random rotation matrix scaled by $\beta \sim \text{Unif}([\beta, 2\beta])$, this difference vector is distributed almost uniformly within the $\ell_2$-norm “eggshell”

$$\{ z \in \mathbb{R}^d : \beta \cdot \| f_i - f_{i'} \|_2 \leq \| z \|_2 \leq 2\beta \cdot \| f_i - f_{i'} \|_2 \}.$$

The two frequencies $f_i \neq f_{i'}$ have an $\ell_2$-distance $\| f_i - f_{i'} \|_2 \geq \eta$ (see Assumption 1.1) and thus, the above “eggshell” is thick enough, giving a large enough support for the random difference vector $\Sigma(f_i - f_{i'})$. The rounded vector $\operatorname{frac}(\Sigma(f_i - f_{i'}))$ is distributed almost uniformly within the unit hypercube $[0,1)^d$, and the collision probability roughly equals the volume $1/B = \Theta_d(k^{-1})$ of a single bin. The range $\beta \sim \text{Unif}([\beta, 2\beta])$ is carefully chosen, hence a collision probability $\leq 0.01 \cdot k^{-1}$.

To conclude, the given matrix $\Sigma$ is likely to isolate at least $90\%$ head frequencies.

Recall that the $d$-dimensional vector $b$ serves as the “anchor point” of the hashing scheme $h$. In the continuous setting, we just sample a uniform random $b \sim \text{Unif}[0,1)^d$ from the unit hypercube (independently of $\Sigma$). Then due to Eq. (9), a certain frequency $f \in [-F,F]^d$ is equally likely to be hashed into one of the $B = B^d = \Theta_d(k)$ bins. (The sampling scheme for the vector $a \in \mathbb{R}^d$ is much more complicated, and is elaborated below.)

For more details about our permutation $P$ and hashing scheme $h$ (e.g. their performance guarantees), the reader can refer to Appendices E.5 to E.9. (It is noteworthy that our constructions give a nontrivial extension to the one-dimensional counterpart constructions by [PS15, CKPS16].)
Non-uniform sampling scheme for $a$. Our sampling scheme, which is illustrated in Figure 1, differs from all the previous schemes [HIKP12a, IK14, PS15, Kap16, Kap17, CKPS16, NSW19]. Especially, we sample a set of frequencies in a non-uniform way. This vector rotates any magnitude $\hat{x}(f) \in \mathbb{C}$ by a certain angle $-2\pi \cdot f^\top a \in \mathbb{R}$ (see Eq. (9)). Let $S = \{ f \in \text{Tail} : \hat{h}(f) = j \}$ be the tail frequencies hashed into a certain bin $j \in [B]^d$, then we hope a small total rotated magnitude $\| \int_{f \in S} \hat{x}(f) \cdot e^{-2\pi i f^\top a} \cdot df \| \ll \int_{f \in S} |\hat{x}(f)| \cdot df$.

In the continuous setting, the vector $a \in \mathbb{R}^d$ represents a sampling time point $t \in [0, T]^d$ over the duration. We have to sample this time point almost (but not exactly) uniformly from a constant proportion of the duration, such as $a \sim \text{Unif}[\frac{0}{d} \cdot T, (1 - \frac{0}{d}) \cdot T]^d$, given the next two reasons.

First, recall that the noise level $\mathcal{N}^2$ involves the term $\| g \|_T^2 = \frac{1}{2\pi} \int_{t \in [0,T]^d} |g(t)|^2 \cdot dt$, but we have no guarantee on the noise $g(t)$ at a specific time point $t \in [0, T]^d$. Hence, as long as the sampling range $a \in A$ is too small (namely $|A| \ll T^d$), the average noise $\frac{1}{|A|} \int_{t \in A} |g(t)|^2 \cdot dt \gg \| g \|_T^2$ can be intolerably large, and makes the samples $a \in A$ useless.

Second, unlike the discrete case, where the on-the-grid frequencies are perfectly separated, two “continuous” frequencies $f \neq f' \in [-F, F]^d$ can be arbitrarily close (as long as either $f \in \text{Tail}$ or $f' \in \text{Tail}$ or both are tail frequencies). If $\| f - f' \|_2 \ll 1/(\sqrt{d} \cdot T)$ and the magnitudes $\hat{x}(f) = \hat{x}(f')$, then the individual signals $\hat{x}(f) \cdot e^{-2\pi i f^\top t} \approx \hat{x}(f') \cdot e^{-2\pi i f'^\top t}$ are close over the whole duration $t \in [0, T]^d$ (i.e. $\| t \|_2 \leq \sqrt{d} \cdot T$). To distinguish $f \neq f' \in [-F, F]^d$, sampling the vector $a$ from the (nearly) whole duration achieves the best we can.

We often sample a pair of $a, a' \in [0, T]^d$ and are more interested in the time difference $\Delta_a := (a' - a)$ rather than $a, a'$ themselves. Over a period of $\Delta_a \in \mathbb{R}^d$, a signal with frequency $f \in \mathbb{R}^d$ rotates by an angle $2\pi \cdot f^\top \Delta_a \in \mathbb{R}$. Denote by $\| \theta \| := \min_{z \in \mathbb{Z}} |\theta + 2\pi z|$ the “circular distance”. Our actual observation is this circular distance $\| 2\pi \cdot f^\top \Delta_a \| \in [0, \pi]$.

To distinguish this frequency $f \in \mathbb{R}^d$ from the others, and to recover $f \in \mathbb{R}^d$ more accurately, we need a largest possible $\ell_2$-norm $\| \Delta_a \|_2$. Moreover, because we do not know the direction of the frequency $f \in \mathbb{R}^d$ (or the direction of the difference between $f$ and the interim estimation of it), the sampled $\Delta_a \in \mathbb{R}^d$ must have a uniform random direction.
The above two requirements for the time difference $\Delta = a' - a$ may violate our previous requirement that, both time points $a, a'$ shall be sampled almost uniformly from a constant proportion of the duration $t \in [0, T]^d$. In particular, the dimensionality $d \geq 1$ will incur many technical issues. To overcome these challenges, we sample $a, a'$ in a coupling fashion. We first determine a careful time difference $\Delta_n$ (i.e. a uniform random direction). However, the $\ell_2$-norm $\|\Delta_n\|_2$ cannot be too large; otherwise, we cannot guarantee a large enough sampling range for $A \ni a, a'$ such that $|A| \approx T^d$. Both the sampling range of the $\ell_2$-norm $\|\Delta_n\|_2$, and the sampling scheme for $a, a' \in [0, T]^d$ (given a specific $\Delta_n$) are carefully chosen.

By contrast, if we instead sample two uniform random $a, a' \sim \text{Unif}[0, T]^d$, then the time difference $\Delta_n$ has a non-uniform direction. This makes the analysis about the observed circular distance $\|2\pi \cdot f^\top \Delta_n\|_\infty \in [0, \pi]$ much more complicated. And more importantly, both the true observations $\|2\pi \cdot f^\top \Delta_n\|_\infty$ and the “fake” observations $\|2\pi \cdot f'^\top \Delta_n\|_\infty$ (due to another frequency $f' \neq f$) may concentrate in a small range like $[0, \frac{\pi}{10}]$. Then, it is impossible for us to distinguish $f \neq f'$. This issue does not exists in the one-dimensional continuous or in the multi-dimensional discrete cases.

- In the one-dimensional continuous case, $\Delta_n$ is just a random variable rather than a vector. We need not to concern the direction of $\Delta_n$, let alone whether this direction is uniform random.

- In the multi-dimensional discrete case, the frequencies are on-the-grid. Therefore, the observed circular distance just has finite possibilities, e.g. $\{0, \frac{1}{N} \cdot \pi, \ldots, \frac{N-1}{N} \cdot \pi, \pi\}$. It turns out that we can easily distinguish the true observations from the fake observations.

For more details about the sampling scheme, the reader can refer to Section F.6.

### 2.2 Filter function

Another key step in this paper and the past works is to design a careful filter function. We restrict our attention to the family of so-called “precise” filter functions [HIKP12a, PS15, CKPS16], and construct a desired filter in three steps: (Appendix C) an auxiliary univariate $(G(t_r), \hat{G}(f_r))$; then (Appendix D) an interim one-dimensional filter $(G(t_r), \hat{G}(f_r))$; and then (Appendix E) the ultimate multi-dimensional filter $(G(t), \hat{G}(f))$.

Akin to the “discrete” counterpart in [IK14, Kap16], our multi-dimensional filter is the product of the one-dimensional filters, namely $G(t) = \prod_{r \in [d]} G(t_r)$ and $\hat{G}(f) = \prod_{r \in [d]} \hat{G}(f_r)$. Moreover, our one-dimensional filter, although similar at the high level, differs from its counterpart by [CKPS16] in several remarkable components.

Indeed, a naive adoption of [CKPS16] is insufficient for our purpose. For example, recall Eq. (7) that $\{\{G(l_i)\}_{i \in [m]}$ are the scaling coefficients of the samples (at the time points $\{\tau_i\}_{i \in [m]} \subseteq [0, T]^d$) for a single “hashing”. We can regard $\sum_{i \in [m]} |G(l_i)|$ as the units of samples taken for one “hashing”. In fact, the one-dimensional filter by [CKPS16] leads to $\sum_{i \in [m]} |G(l_{i,r})| = \Theta(1)$, where $l_{i,r} \in \mathbb{R}$ is the $r$-th coordinate of a lattice point $l_i \in \mathbb{R}^d$. Since $G$ is the product of $G$’s over all dimensions $r \in [d]$, we essentially sample $\sum_{i \in [m]} |G(l_i)| = 2^{\Theta(d)}$ units of samples for one “hashing”, and will get a $2^{\Theta(d)}$-approximation algorithm.

To improve the approximation ratio, we modify the construction by [CKPS16] and, more importantly, refine the performance analysis of the filters. For other nontrivial modifications towards the ultimate filter $(G(t), \hat{G}(f))$, we add several remarks in Appendices C to E.

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9Roughly speaking, a “crude” filter function has values either $\hat{G}(f) = 1 \pm \Theta_d(1)$ or $\hat{G}(f) = 0 \pm \Theta_d(1)$ (e.g. [IK14]), whereas a “precise” filter function has values either $\hat{G}(f) = 1 \pm \Theta_d(1/k)$ or $\hat{G}(f) = 0 \pm \Theta_d(1/k)$ (e.g. [HIKP12a]). We briefly compare these two families in Section 3, especially their performance guarantees.

10Here, because of other technical reasons, we cannot choose a different set of lattice points $\{l'_{j}\}_{j \in [m]} \neq \{l_j\}_{j \in [m]}$ for the filter $(G(t), \hat{G}(f))$ in the sampling process, or naively scale the filter $(G(t), \hat{G}(f))$ by a suitable factor.
Below we will discuss how to overcome some of the challenges. Further, the new challenges, especially, resulting in the limit of the current methods. We will also explain why this hides a small error, which stems from the noise frequencies (i.e. $\Psi$-dependence). According to Eq. (9), a sampling time point $a \in [0,T]^d$ gives a measurement $y_i(a) \in \mathbb{C}$ such that $y_i(a) \approx v_i \cdot \det(\Sigma)^{-1} \cdot e^{-2\pi i f^\top_i a}$. Here, the "$\approx$" notation hides a small error, which stems from the noise frequencies (i.e. $g(t) \in \mathbb{C}$) hashed into the same bin $j := h(f_i) \in [B]^d$. To recover the frequency $f_i$, the idea is to leverage the difference $\Delta_a := (a' - a)$ between two time points $a, a' \in [0,T]^d$ and the relative phase

$$\psi_i(a, a') := \arg(y_i(a)/y_i(a')) \approx \arg(e^{2\pi i f^\top_i \Delta_a}) = 2\pi \cdot f^\top_i \Delta_a. \quad (10)$$

The above "$\approx$" notation hides an error phase of, say, $\pm (2\pi)/10^3$.

We recover the frequency $f_i' \approx f_i$ in two steps. First, the "coarse-grained" location (Algorithm 2) keeps track of a hypothesis region $\mathcal{H}_i \supseteq f_i$ for the frequency (e.g. at the beginning $\mathcal{H}_i = [-F, F]^d$) and shrinks $\mathcal{H}_i$ round by round. Second, the "fine-grained" locating (Algorithm 4) carefully derives

2.3 Sparse recovery

Using the permutations, hashing schemes and filters, the past works develop several kinds of algorithms to learn the Fourier spectrum. Our algorithm falls into the family of voting-based algorithms, which are used by [HIKP12, PS15] for the one-dimensional discrete/continuous cases. Our main task is to recover the frequencies $\{f_i\}_{i \in [k]} \subseteq [-F, F]^d$. As long as we achieve good approximations $\{f_i'\}_{i \in [k]} \approx \{f_i\}_{i \in [k]}$, the magnitudes $\{v_i'\}_{i \in [k]} \approx \{v_i\}_{i \in [k]} \subseteq \mathbb{C}$ can be easily recovered.

Unfortunately, a continuous multi-dimensional frequency domain $[-F, F]^d$ incurs a number of new challenges, especially, resulting in poly($d$)-approximations for the frequency/tone estimation. Below we will discuss how to overcome some of the challenges. Further, the poly($d$)-approximation of our algorithm originates from three places (namely the coarse-grained location, the fine-grained location, and the tone estimation). We will also explain why this poly($d$) dependence seems to be the limit of the current methods.

For ease of presentation, we will restrict our attention to a tone $(v_i, f_i) \in \mathbb{C} \times \mathbb{R}^d$ that is isolated by the permutation $\mathcal{P}$ and hashing $\hat{h}$. According to Eq. (9), a sampling time point $a \in [0,T]^d$ gives a measurement $y_i(a) \in \mathbb{C}$ such that $y_i(a) \approx v_i \cdot \det(\Sigma)^{-1} \cdot e^{-2\pi i f^\top_i a}$. Here, the "$\approx$" notation hides a small error, which stems from the noise frequencies (i.e. $g(t) \in \mathbb{C}$) hashed into the same bin $j := h(f_i) \in [B]^d$. To recover the frequency $f_i$, the idea is to leverage the difference $\Delta_a := (a' - a)$ between two time points $a, a' \in [0,T]^d$ and the relative phase

$$\psi_i(a, a') := \arg(y_i(a)/y_i(a')) \approx \arg(e^{2\pi i f^\top_i \Delta_a}) = 2\pi \cdot f^\top_i \Delta_a. \quad (10)$$

The above "$\approx$" notation hides an error phase of, say, $\pm (2\pi)/10^3$. We recover the frequency $f_i' \approx f_i$ in two steps. First, the "coarse-grained" location (Algorithm 2) keeps track of a hypothesis region $\mathcal{H}_i \supseteq f_i$ for the frequency (e.g. at the beginning $\mathcal{H}_i = [-F, F]^d$) and shrinks $\mathcal{H}_i$ round by round. Second, the "fine-grained" locating (Algorithm 4) carefully derives

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Figure 2: Comparison, in the frequency domain, of the ideal single-dimensional filter functions $\hat{G}'(f)$ and our single-dimensional filter functions $\hat{G}(f)$, where the integer $W \in \mathbb{N}_{\geq 1}$ is a carefully chosen parameter (see Definition D.1).
Figure 3: Demonstration for the coarse-grained location in two dimensions $d = 2$. The black points refer to the true frequencies. The blue/green/red circles show that we gradually shrink the hypothesis regions for the frequencies.

d linear equations of the form $2\pi \cdot f_i^T \Delta_a^r = \psi_i^r$ (for all $r \in [d]$) based on $d$ particular time differences $\Delta_a^r \in \mathbb{R}^d$, and solves these linear equations to find $f_i' \approx f_i$ within the hypothesis region $H_i$.

Coarse-grained location. Suppose that the frequency $f_i$ locates in some hypothesis region $H_i$. We carefully divide $H_i = \bigcup_{q \in Q} H_{i,q}$ into smaller sub-regions, and pick a candidate frequency $\xi_q$ for every sub-region. The particular sub-region $H_{i,q^*} \ni f_i$ that contains the frequency is called the true sub-region. Based on the measurements, we can prune some of the wrong sub-regions $H_{i,q} \not\ni f_i$, hence a smaller new hypothesis region. The coarse-grained location repeats this pruning process, which is illustrated in Figure 3.

Given a pair of sampling time points $a, a' \in [0, T]^d$, in view of Eq. (10), we will vote for all candidates frequencies $\xi_q$ such that

$$\|2\pi \cdot \xi_q^T \Delta_a - \psi_i(a, a')\|_\infty \leq (2\pi)/50,$$

(11)

where the RHS can be other suitable thresholds. By setting such a threshold, (i) the true candidate frequency $\xi_{q^*}$ (for which $H_{i,q^*} \ni f_i$) gets a vote with probability 90%, since $\xi_{q^*}$ is close enough to $f_i$. By contrast, (ii) if a wrong candidate frequency $\xi_q$ (for which $H_{i,q} \not\ni f_i$) is too far from $f_i$, then we hope $\xi_q$ to get a vote with probability < 50%. Given Eq. (10) and (11), the wrong candidate frequency $\xi_q$ loses a vote when $\|2\pi \cdot (\xi_q - f_i)^T \Delta_a\|_\infty \geq (2\pi)/40$. Namely, with probability > 50%, we hope the gap between $(\xi_q - f_i)^T \Delta_a \in \mathbb{R}$ and its closest integer to be at least

$$\min_{z \in \mathbb{Z}} |(\xi_q - f_i)^T \Delta_a - z| \geq 1/40,$$

(12)

To this end, we let the time difference $\Delta_a \in \mathbb{R}^d$ have a uniform random direction and a random $\ell_2$-norm $\|\Delta_a\|_2 \sim \text{Unif}[w, 2w]$, for some $w > 0$. In any dimension $d \geq 2$, we have

$$(\xi_q - f_i)^T \Delta_a = \|\xi_q - f_i\|_2 \cdot \|\Delta_a\|_2 \cdot \cos(\gamma),$$

(13)
where the random angle $\gamma := \langle \xi_q - f_i, \Delta_a \rangle$. Clearly, as long as a fixed $|\cos(\gamma)| \in [0, 1]$ (namely a fixed direction of $\Delta_a$), is not too small, a large enough sampling range $\|\Delta_a\|_2 \sim \text{Unif} [w, 2w]$ ensures Eq. (12) with probability > 50%. This is exactly what we desire.

Nonetheless, the coarse-grained location recovers the frequencies by at most $\|\xi_q - f_i\|_2 \lesssim d/T$ (instead of $\|\xi_q - f_i\|_2 \lesssim 1/T$). To see so, note that when $\Delta_a \in \mathbb{R}^d$ has a uniform random direction in any dimension $d \geq 2$, the angle $\gamma \in [0, \pi]$ concentrates within $[\frac{\pi}{2} - \frac{\pi}{2\sqrt{d}}, \frac{\pi}{2} + \frac{\pi}{2\sqrt{d}}]$. That is, with high probability we have $|\cos(\gamma)| \lesssim 1/\sqrt{d}$. Given Eq. (12) and (13), in order to vote for a wrong candidate frequency $\xi_q$ with probability < 50%, we require $\|\xi_q - f_i\|_2 \|\Delta_a\|_2 \gtrsim \sqrt{d}$.

Furthermore, given any time difference $\Delta_a \in \mathbb{R}^d$, the largest possible range $A \ni a, a'$, from which we can sample the two time points, has the volume $|A| = T^d \cdot (1 - \|\Delta_a\|_1/T)$. As mentioned in Section 2.1, this sampling range must be a constant proportion of the whole duration $\Delta = [\Delta_a]_{r \in [d]} \in \mathbb{R}^{dxd}$ that has a bounded spectral norm, and $d$ observations $\psi = (\psi_r)_{r \in [d]}$. Then Eq. (14) implies that

$$\Pr[\|\psi - f_i^\top \Delta_a^r\|_2 \leq \sqrt{d}/\rho] \geq 1 - 0.1/d.$$  

When $d = 1$, we can just take $\psi^r/\Delta_a^r$ as an approximation of $f_i$. It suffices to get a good estimation $f_i' \approx f_i$ via a small number of samples. However, when $d \geq 2$, we cannot extract enough information from the inner product $f_i^\top \Delta_a^r \in \mathbb{R}$ of the two vectors. To handle this issue, as Figure 4b illustrates, we will use $d$ random vectors to form a random matrix $\Delta = [\Delta_a^r]_{r \in [d]} \in \mathbb{R}^{dxd}$ that has a bounded spectral norm, and $d$ observations $\psi = (\psi_r)_{r \in [d]}$. Then Eq. (14) implies that

$$\Pr[\|\psi - f_i\|_2 \leq \sqrt{d}/\rho] \geq 1 - 0.1 = 0.9.$$  

Now we obtain a good estimation $\Delta^{-1} \psi \approx f_i$ such that $\|\Delta^{-1} \psi - f_i\|_2 \leq \|\Delta^{-1}\| \cdot \|\psi - f_i\|_2 \leq \|\Delta^{-1}\| (d/\rho)$. We emphasize that this approach needs $d$ observations – to make the union bound applicable, we must blow up the estimation error by a factor $\sqrt{d}$.

Given our framework, $\|\Delta_a^r\|_2$ has to be at most $O(T/d)$. To make our estimation as accurate as possible, we sample $\Delta_a^r$ uniformly from a hypersphere such that $\|\Delta_a^r\|_2 \gtrsim T/d$. Another issue is how to analyze the random matrix $\Delta = [\Delta_a^r]_{r \in [d]} \in \mathbb{R}^{dxd}$. Fortunately, the vectors $\Delta_a^r \cdot \sqrt{d}/\|\Delta_a^r\|_2$ are sub-Gaussian isotropic, so we can upper bound the spectral norm $\|\Delta\|$ up to a small blow-up given by $d \geq 1$. Putting everything together gives $\|f_i' - f_i\|_2 = \|\Delta^{-1} \psi - f_i\|_2 \lesssim_d 1/(\rho \cdot T)$.

**Tone estimation.** Hitherto, we have explained why the frequencies $\{f_i\}_{i \in [n]}$ can only be recovered up to $\|f_i' - f_i\|_2 \lesssim 1/(\rho \cdot T)$ rather than $\lesssim 1/(\rho \cdot T)$. Beyond that, we suffer from another $\sqrt{d}$ loss in the tone estimation, which stems from the convert from $\ell_1$-norm to $\ell_2$-norm.

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11We say $a \lesssim_d b$ if $a \leq \text{poly}(d) \cdot b$. For a matrix $\Delta$, we use $\|\Delta\|$ to denote the spectral norm of $\Delta$. 

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Suppose that we have recovered an exact magnitude $v_i' = v_i \in \mathbb{C}$ and a rather precise frequency $\|f_i' - f_i\|_2 \ll 1/(\sqrt{d} \cdot T)$, but it is possible that $\ell_1$-norm frequency error $\|f_i' - f_i\|_1 \approx \sqrt{d} \cdot \|f_i' - f_i\|_2$.

Then over the duration $t \in [0,T]^d$, the relative phase between the recovered signal $x_i'(t) = v_i' e^{2 \pi i f_i' t}$ and the true signal $x_i(t) = v_i e^{2 \pi i f_i t}$ can be as large as

$$\max_{t \in [0,T]^d} |2\pi (f_i' - f_i)^\top t| = 2\pi \cdot \|f_i' - f_i\|_1 \cdot T \approx \sqrt{d} \cdot \|f_i' - f_i\|_2 \cdot T.$$ 

This factor-$\sqrt{d}$ blow-up in the relative phase gives one more $\sqrt{d}$ loss in the tone estimation.

### 2.4 Duration for tone estimation

To obtain the tone estimation guarantee, we require $T \gtrsim \eta^{-1} \cdot d^{4.5} \cdot \log(kd/\delta) \cdot \log d$ (Theorem 1.2). The poly$(d)$ term stems from several places.

(i) Filter function (Section E). In order to avoid the $2^{\Theta(d)}$ blow-up of the approximation ratio in the multi-dimensional setting, we suffer from a loss $d$. Concretely (as mentioned in Section 2.2), we can only take $\sum_{i \in [m]} |G(l_i)| = O(1)$ units of samples (instead of $2^{\Theta(d)}$ units) for one hashing. Thus in any single dimension $r \in [d]$, we need $\sum_{i \in [m]} |G(l_{ir})| = 1 + O(d^{-1})$. The extra term $O(d^{-1})$ incurs the factor-$d$ blow-up in the duration bound $T$. How to achieve $\sum_{i \in [m]} |G(l_i)| = O(1)$ and to avoid the blow-up in $T$ seems difficult.

(ii) We need a large sampling range $|\text{supp}(a)| \approx T^d$ (Algorithm 3). For a $d$-dimensional vector $a = (a_r)_{r \in [d]}$, we require $|\text{supp}(a_r)| \geq T - \Theta(T/d)$ in any single dimension. The second term $\Theta(T/d)$ (rather than $\Theta(T)$) will incur a factor-$d$ loss in the duration bound $T$.

(iii) The procedure HashToBins (Algorithm 1) switches $\ell_2$-norm to $\ell_\infty$-norm, and thus incurs another loss $\sqrt{d}$.

(iv) The way that we generate the random matrix $\Sigma \in \mathbb{R}^{d \times d}$ loses a factor $\sqrt{d}$, in order to ensure a small enough collision probability $\Pr[\hat{h}(f_i) = \hat{h}(f_{i'})] \leq 0.01 \cdot k^{-1}$ for any pair of frequencies $f_i \neq f_{i'} \in \text{supp}(\hat{x}^*)$. 

Figure 4: Demonstration of the fine-grained location respectively in one dimension $d = 1$ and two dimensions $d = 2$. In one dimension, we simply take the median of the candidates resulted from the observations $\varphi^r \in \mathbb{R}$. In two dimensions, every observation $\varphi^r \in \mathbb{R}$ gives a hypothesis line (i.e. a hypothesis one-dimensional hyperplane) that is close to the true frequency $f$, and thus a pair of observations/lines determines an approximation $f' \approx f$. 

(a) One dimension

(b) Two dimensions
(v) To select the $k$ exact tones $\{(v_i, f_i)\}_{i \in [k]}$ from $k' = \Theta_d(k)$ candidate tones (Algorithm 9), we blow up the duration bound $T$ by an extra factor $d^{1.5} \cdot \log d$. In precise, we first pay a factor $d \cdot \log d$ because there are $k' = 2^{\Theta(d \log d)} \cdot k$ candidate tones, and pay another factor $\sqrt{d}$ in the selection process, since we cannot afford the running time to query points in $l_2$-distance, even with the best data structure. We instead work in $l_\infty$-distance and choose the gap $\eta' = \eta/\sqrt{d}$, suffering from another loss $\sqrt{d}$.

To sum up, we need a duration $T \geq \eta'^{-1} \cdot d^3 \cdot \log(k'd/\delta) = \eta^{-1} \cdot d^{4.5} \cdot \log(kd/\delta) \cdot \log d := C_{\text{tone}} \cdot \eta^{-1}$.

### 2.5 Duration for signal estimation

To further derive the signal estimation guarantee (Theorem 1.3) from the tone estimation guarantee, we adopt the proof framework of [PS15] and provide result in any $d$-dimension, via a combination of advanced tools from real analysis (like Parseval’s theorem, convolution theorem etc). If the tone estimation works when $T \geq \eta^{-1} \cdot C_{\text{tone}}$, then for the signal estimation we need

$$T \geq \eta^{-1} \cdot (C_{\text{tone}} + d^{1.5} k^{-1/d} \log k).$$

When $d = 1$, the above bound is $\eta^{-1} \cdot \log(k)$, which improves the bound $\eta^{-1} \cdot \log^2(k)$ by [PS15] and thus answers an open question asked in the thesis [Son19].

Concretely, we reformulate the signal estimation error as LHS of Eq. (5) = $\sum_{i,j \in [k]} err_{i,j}$, where

$$err_{i,j} := \frac{1}{T^d} \cdot \int_{[0,T]^d} (x_i^i(t) - x_i^j(t)) \cdot (x_j^i(t) - x_j^j(t)) \cdot dt.$$  

The total tone-wise error $\sum_{i \in [k]} err_{i,i}$ is exactly the tone estimation error studied before (i.e. LHS of Eq. (4)). To acquire the signal estimation guarantee, the work [PS15] further proves that every cross-tone error for $i \neq j \in [k]$ diminishes to zero at the rate\footnote{In precise, the work [PS15] proves this convergence rate for the one-dimensional case $d = 1$. But following the arguments therein, we can easily derive this claimed convergence rate in the general case $d \geq 1$.}

$$|err_{i,j}| \lessapprox \sqrt{d} \cdot (\|f_i - f_j\|_2 \cdot T)^{-1} \cdot \sqrt{err_i} \cdot err_j \cdot \log(1 + \|f_i - f_j\|_2 \cdot T).$$

Later in Lemma I.6, we will show a faster convergence rate for these cross-tone errors:

$$|err_{i,j}| \lessapprox \sqrt{d} \cdot (\|f_i - f_j\|_2 \cdot T)^{-1} \cdot \sqrt{err_i} \cdot err_j.$$  

(15)

This improvement stems from a novel application of Parseval’s theorem and the convolution theorem. Recall that if the duration $t \in \mathbb{R}^d$ is infinite, Parseval’s theorem gives analytic formulas both for the tone-wise errors $err_{i,i}$ and for the cross-tone errors $err_{i,j}$. Inspired by this, we access the proof details of Parseval’s theorem. Extending those arguments (in a nontrivial way) as well as applying the convolution theorem, we acquire in Lemma I.4 the counterpart analytic formulas for $err_{i,i} \geq 0$ and $err_{i,j} \geq 0$ in the case of a finite duration $t \in [0,T]^d$. Together with other nontrivial observations, Eq. (15) gets established. By contrast, the past work [PS15] just gives the approximate formulas for $err_{i,i} \geq 0$ and $err_{i,j} \geq 0$, therefore suffering from a $\log(k)$ loss in their analysis of the duration bound $T > 0$.

Indeed, we have a concrete example in which the convergence rate matches Eq. (15) both in the duration $T$ and in the dimension $d \geq 1$. We believe that this tight convergence rate, as well as the analytic formulas in Lemma I.4, can find their applications in the future.
3 Conclusion, future directions, other related work, roadmap

In this paper, we designed a randomized non-adaptive algorithm for the multi-dimensional continuous sparse Fourier transform problem, which achieves a constant approximation under the $\ell_2/\ell_2$ guarantee and, in any constant dimension, takes sublinear samples and running time. Many attractive directions deserve exploring in the future, for which we give a short discussion below.

3.1 Future directions

**Approximation ratio.** First of all, whether we can achieve a better $O(1)$-approximation (namely making it independent of dimension $d \geq 1$) or even an $(1 + \varepsilon)$-approximation? To this end, we need to pin down what an $(1 + \varepsilon)$-approximation stands for. This notion is clear in the discrete settings, as the noises are “on the grid” (i.e. the whole Fourier spectrum without the top-$k$ frequencies), but is unclear in the continuous settings since we allow an arbitrary noise $g(t) \in \mathbb{C}$ over the duration $t \in [0,T]^d$. To make the problem well defined, a slightly modified setup may be necessary.

**Sample complexity.** Another potential direction is to reduce sample complexity. Indeed, up to iterated logarithmic factors, our algorithm uses $k \cdot \log^{d+O(1)} k \cdot \log(F/\eta) \cdot 2^{O(d \log d)}$ samples/running time (see Algorithm 10). Here the term $\log^d k$ is a consequence of our “precise” filter function. As quoted from Kapralov [Kap16, Kap17]:

“in the discrete settings ... the price to pay for the precision of the filter, however, is that each hashing becomes a $\log^d k$ factor more costly in terms of sample complexity and running time than in the idealized case ...”

To shave the term $\log^d k$ in the discrete case, the past works [IK14, Kap16] instead employ “crude” filters. But for some technical issues, it is highly nontrivial how to use “crude” filter functions even in the one-dimensional continuous setting [PS15, CKPS16], let alone the multi-dimensional case. Anyway, if we really wish to eliminate this bottleneck, the techniques developed in [IK14, Kap16, CKSZ17, Kap17] shall be useful.

**Deterministic algorithm.** It turns out that no deterministic algorithm using sublinear samples can achieve either the $\ell_\infty/\ell_2$- or the $\ell_2/\ell_2$-guarantee [DBIPW10]. By contrast, under the (weaker) $\ell_\infty/\ell_1$-guarantee, the past works [MZIC19, LN20] design an $\tilde{O}(k^2)$-sample deterministic algorithm for the discrete Fourier transform; both works reply on some tools from functional analysis. It would be also very interesting, even in the one-dimensional case $d = 1$, to see a deterministic algorithm for the continuous Fourier transform.

**Linear decoding time.** Several previous works improve the sample complexity or other performance guarantees by allowing a linear (rather than sublinear) decoding time Fourier transform. In the multi-dimensional discrete case, the past works [IK14, NSW19] implement the “point-query” idea (which originates from the sparse recovery/heavy hitter literature) via the Fourier measurements in a clever way, and thus optimize the sample complexity. Can we obtain such results in the continuous cases? The main difficulty is, different from the discrete cases where the “on-the-grid” frequencies can be checked coordinate by coordinate, the “continuous” frequencies have infinitely many possibilities.

**Set query.** A problem in the “sparse recovery” paradigm has two primary tasks: (i) to recover the heavy locations; and (ii) to pin down the masses/densities in those locations. Price [Pri11] pulls the second task out from the sparse recovery literature, and defines a clean problem called “set query”.

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Kapralov [Kap17] introduces and studies the Fourier set query problem in the discrete settings. It would be interesting to explore such problems in the continuous settings.

### 3.2 Other related works

Compressed sensing is initiated by [CT06, Don06]. Since then, there is a long line of works exploring and extending it in various directions [GLPS10, GLPS10, IP11, IPW11, IR13, PW13, Pri13, AZGR16, LNNT16, BIP16, KP19, NS19]. As mentioned, compressed sensing allows us to design the sensing matrices, which is the main difference between it and the Sparse FT problem.

Apart from the one-/multi-dimensional discrete/continuous Sparse FT problems that we have considered thus far, where the sampling is carried out in an arbitrary yet non-adaptive way, there are other meaningful adjustments to the model.

For example, there is (i) a line of works studying the model where the sampling is conducted in a (more restricted) uniform way [RV08, BD08, CGV13, Bou14, HR16, BLL19, and the references therein]; and (ii) another line of works studying the model that allows an algorithm to adaptively take the samples and recover the Fourier spectrum [PW13, CKSZ17, and the references therein].

Independently of this paper and concurrently, Chen and Moitra [CM20] investigate a mixture model learning problem which they reduce to the problem of continuous Sparse FT. The main difference between their model and our model, is the way how the noise hampers the frequency recovery. Recall that we consider a signal

\[ x(t) = x^*(t) + g(t) \in \mathbb{C} \quad \text{over a duration} \quad t \in [0, T]^d, \]

where \( x^*(t) \in \mathbb{C} \) is the actual signal that we aim to recover, and the noise \( g(t) \in \mathbb{C} \) has a small enough constant-proportional energy compared to \( x^*(t) \), that is, \(|g|_T \leq 10^{-3} \cdot |x^*|_T\).\(^{13}\) In particular, the noise magnitude \( |g(t_0)| \) at a certain time point \( t_0 \in [0, T]^d \) has no requirement, and can even be much larger than the signal magnitude \( |x^*(t_0)| \).

By contrast, [CM20] make a stronger assumption on the noise \( g(t) \in \mathbb{C} \). At any time point \( t_0 \in [0, T]^d \), the noise magnitude \( |g(t_0)| \) is always inverse-polynomially small, compared to the corresponding average signal energy \(|x^*|_T\).

For their model, Chen and Moitra focus on the low-dimensional case, and their primary emphasis is on refining the duration requirement in the low-dimensional case, i.e. on the exact constant in front of \( \eta^{-1} \) for constant \( d \). Chen and Moitra use tensor decomposition and get sample complexity \( \tilde{O}_d(k^2) \cdot \text{poly}(FT) \) and running time \( \tilde{O}_d(k^6) \cdot \text{poly}(FT) \). Also, it is worth mentioning that, [CM20] consider the “tone recovery” problem only, without studying the “signal recovery” problem, whereas our paper investigates the both problems.

### 3.3 Organization of appendices

Section A provides some basic notations and definitions. Section B provides a list of probability tools. Section C and D present our filter, permutation and hashing in any single dimension. Afterwards, Section E presents the counterpart filter, permutation and hashing in the multi-dimensional setting. In Section F and G, we show to how to give accurate estimations of the frequencies. In Section H, we present our sparse recovery algorithm. Finally in Section I, we show how to obtain the signal estimation by paying a slightly longer duration.

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\[^{13}\text{Recall that the average energy, e.g., of the noise } g(t) \text{ over duration } t \in [0, T]^d, \text{ is defined as } ||g||_T = \frac{1}{T^d} \int_0^T |g(t)|^2 dt.\]
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A Preliminaries

A.1 Notations

We denote by $[n]$ the set $\{0, 1, 2, \cdots, n-1\}$, by $\mathbb{R}$ the set of real numbers, by $\mathbb{Z}$ the set of integers, and by $\mathbb{C}$ the set of complex numbers. Also, $\mathbb{N}_{\geq a}$ refers to the set of integers no less than $a \geq 0$. Let $\text{supp}(f)$ denote the support of a function or vector $f$, and let $\|f\|_0 = |\text{supp}(f)|$ be the cardinality. For a random variable $X$, for convenience we may abuse the notation $\text{supp}(X)$ to denote the support of $X$’s probability density function (PDF).

We use $\max\{a, b\}$ or $\max(a, b)$ (resp. $\min\{a, b\}$ or $\min(a, b)$) to denote the maximum (resp. the minimum) between $a, b \in \mathbb{R}$. Given any $p \geq 1$, a vector $x = (x_i)_{i \in [n]} \in \mathbb{R}^n$ has the the $\ell_p$-norm $\|x\|_p := (\sum_{i \in [n]} |x_i|^p)^{1/p}$; in the case that $p = \infty$, we define $\|x\|_\infty := \max_{i \in [n]} |x_i|$.

We use the notations $i := \sqrt{-1}$ and $e^{i\theta} := \cos(\theta) + i \cdot \sin(\theta)$ for any phase $\arg(e^{i\theta}) = \theta \in \mathbb{R}$. For a complex number $z = a + i \cdot b \in \mathbb{C}$, let $a \in \mathbb{R}$ be the real part and let $b \in \mathbb{R}$ be the imaginary part. Also, $\bar{z} := a - i\cdot b \in \mathbb{C}$ denotes the conjugate, and $|z| := \sqrt{a^2 + b^2} \geq 0$ denotes the norm.

A.2 Fourier transform and convolution

For convenience, throughout this paper we use the shorthand CFT (the continuous Fourier transform), DFT (the discrete Fourier transform), DTFT (the discrete-time Fourier transform) and FFT (the fast Fourier transform).

- In the time domain, we often use the notations $t$ and $\tau$.
- In the frequency domain, we often use the notations $f$ and $\xi$.

Given a $d$-variate function $x(t)$ for $t = (t_s)_{s \in [d]} \in \mathbb{R}^d$, we have the CFT $\hat{x}(f)$ for $f = (f_t)_{t \in [d]} \in \mathbb{R}^d$ and the inverse CFT $x(t)$ for $t \in \mathbb{R}^d$:

$$\hat{x}(f) := \int_{\tau \in \mathbb{R}^d} x(\tau) \cdot e^{-2\pi i f^\top \tau} \cdot d\tau \quad \text{and} \quad x(t) := \int_{\xi \in \mathbb{R}^d} \hat{x}(\xi) \cdot e^{2\pi i t^\top \xi} \cdot d\xi.$$

**Definition A.1** ($k$-Fourier-sparse signal). Given any $k$-Fourier-sparse signal $x^s(t)$ with the tones $\{(v_i, f_i)\}_{i \in [k]} \subseteq \mathbb{C} \times \mathbb{R}^d$, the corresponding CFT $\hat{x}^s(f)$ is the combination of $k \geq 1$ many (scaled) $d$-dimensional Dirac delta functions, each of which has a point mass (i.e. the involved magnitude) $v_i \in \mathbb{C}$ at the corresponding frequency $f_i \in \text{supp}(x^s)$. Without ambiguity, we denote $\hat{x}^s[f_i] := v_i \in \mathbb{C}$ for convenience. Then the $k$-sparse Fourier spectrum $\hat{x}^s(f)$ for $f \in \mathbb{R}^d$ can be formulated as

$$\hat{x}^s(f) := \sum_{i \in [k]} v_i \cdot \text{Delta}_{= f_i}(f) = \sum_{i \in [k]} \hat{x}^s[f_i] \cdot \text{Delta}_{= f_i}(f).$$

**Definition A.2** (Convolution). The convolution $(f * g)(t)$ for $t \in \mathbb{R}^d$ of two $d$-variate continuous function $f(t)$ and $g(t)$ is given by

$$(f * g)(t) := \int_{\tau \in \mathbb{R}^d} f(\tau) \cdot g(t - \tau) \cdot d\tau,$$

And the discrete convolution $(f * g)[i]$ for $i \in \mathbb{Z}$ of two same-length vectors $f$ and $g$ is given by\(^{14}\)

$$(f * g)[i] = \sum_{j \in \mathbb{Z}} f[j] \cdot g[i - j].$$

\(^{14}\)We define $(f * g)[i] := 0$ in the case that $i \notin \text{supp}(f) = \text{supp}(g)$.
B Probability tools

In this section we present a number of classical probability tools to be used in this paper: the Chernoff bound (Lemma B.1), the Hoeffding bound (Lemma B.2) and the Bernstein bound (Lemma B.3) measure the tail bounds of random scalar variables. Further, Lemma B.4 is a concentration result about random matrices.

We state the classical Chernoff bound below, which is named after Herman Chernoff but is due to Herman Rubin. It gives exponentially decreasing bounds for the tail distributions of the sums of independent random variables.

**Lemma B.1** (Chernoff bound [Che52]). Let \( \{X_i\}_{i \in [n]} \) be \( n \geq 1 \) independent Bernoulli random variables, such that \( X_i = 1 \) with probability \( p_i \in [0, 1] \) and \( X_i = 0 \) with probability \( 1 - p_i \). Then the following hold for the random sum \( X := \sum_{i \in [n]} X_i \) and the expectation \( \mu := E[X] = \sum_{i \in [n]} p_i \).

**Part (a):** \( \Pr[X \geq (1 + \delta)\mu] \leq e^{\delta \mu \cdot (1 + \delta)^{-(1 + \delta)\mu}} \) for any \( \delta > 0 \).

**Part (b):** \( \Pr[X \leq (1 - \delta)\mu] \leq e^{-\delta \mu \cdot (1 - \delta)^{-(1 - \delta)\mu}} \) for any \( 0 < \delta < 1 \).

We state the Hoeffding bound below:

**Lemma B.2** (Hoeffding bound [Hoe63]). Let \( \{X_i\}_{i \in [n]} \) be \( n \geq 1 \) independent random variables bounded between \( \text{supp}(X_i) \subseteq [a_i, b_i] \), for some \( a_i \leq b_i \in \mathbb{R} \). Then the following holds for the random sum \( X := \sum_{i \in [n]} X_i \) and any \( t \geq 0 \).

\[
\Pr[|X - E[X]| \geq t] \leq 2 \cdot \exp \left( -\frac{2t^2}{\sum_{i \in [n]} (b_i - a_i)^2} \right).
\]

We state the Bernstein inequality below:

**Lemma B.3** (Bernstein inequality [Ber24]). Let \( \{X_i\}_{i \in [n]} \) be \( n \geq 1 \) independent zero-mean random variables \( E[X_i] = 0 \). Suppose that \( |X_i| \leq M \) almost surely, for every \( i \in [n] \) and some \( M \geq 0 \). Then the following holds for the random sum \( X := \sum_{i \in [n]} X_i \) and any \( t \geq 0 \).

\[
\Pr[X > t] \leq \exp \left( -\frac{t^2/2}{\sum_{i \in [n]} E[X_i^2] + Mt/3} \right).
\]

Matrix concentration inequalities have various applications. Below, we state a matrix Bernstein inequality by [Tro15], which can be regarded as a matrix version of Lemma B.3.

**Lemma B.4** (Matrix Bernstein [Tro15, Theorem 6.1.1]). Let \( \{X_i\}_{i \in [m]} \subseteq \mathbb{R}^{n_1 \times n_2} \) be a set of \( m \geq 1 \) i.i.d. matrices with the expectation \( E[X_i] = 0^{n_1 \times n_2} \). For some \( M \geq 0 \), assume \( \|X_i\| \leq M, \quad \forall i \in [m] \).

Let \( X = \sum_{i \in [m]} X_i \) be the random sum. Let \( \text{Var}[X] \) be the matrix variance statistic of the sum:

\[
\text{Var}[X] := \max \left\{ \left\| \sum_{i \in [m]} E[X_i X_i^\top] \right\|, \left\| \sum_{i \in [m]} E[X_i^\top X_i] \right\| \right\}.
\]

Then

\[
E[\|X\|] \leq \sqrt{2 \cdot \text{Var}[X] \cdot \log(n_1 + n_2)} + \frac{M}{3} \cdot \log(n_1 + n_2).
\]
Furthermore, the following holds for any \( t \geq 0 \).

\[
\Pr[\|X\| \geq t] \leq (n_1 + n_2) \cdot \exp\left(-\frac{t^2/2}{\Var[X] + Mt/3}\right).
\]

**Lemma B.5** (Sub-gaussian rows [Ver10, Theorem 5.39]). Let \( A \) be an \( N \times n \) matrix whose rows \( A_i \) for \( i \in [N] \) are independent sub-gaussian isotropic random vectors in \( \mathbb{R}^n \). Then for every \( t \geq 0 \), with probability at least \( 1 - 2\exp(-ct^2) \), we have

\[
\sqrt{N} - C\sqrt{n} - t \leq s_{\min}(A) \leq s_{\max}(A) \leq \sqrt{N} + C\sqrt{n} + t.
\]

where \( s_{\max}(A) \) (resp. \( s_{\min}(A) \)) represents the largest (resp. smallest) singular value of matrix \( A \), and absolute constants \( C = C_K \), \( c = c_K \) depend only on the sub-gaussian norm \( K = \max_{i \in [N]} \|A_i\|_{\psi_2} \) of the rows.
C Building-block function \((G(t), \hat{G}(f))\) in a single dimension

This appendix presents the construction of a basic function \((G(t), \hat{G}(f))\) as well as its properties, which serves as the building block of our single-dimensional filter function (see Appendix D) and multi-dimensional filter function (see Appendix E).

C.1 Construction of function \((G(t), \hat{G}(f))\)

To introduce the building-block function \((G(t), \hat{G}(f))\), we will employ the rectangular function \(\text{rect}_{s_1}(f)\) and the sinc function \(\text{sinc}_{s_1}(t)\). Both functions are widely used in the previous literature, and we shall be familiar with their properties given in Fact C.2 (e.g. see [CKPS16]).

**Definition C.1** (Two basic functions). Given any \(s_1 > 0\), the \(\text{rect}_{s_1}(f)\) function and the \(\text{sinc}_{s_1}(t)\) function are defined as follows:

- \(\text{rect}_{s_1}(f) = 1/s_1 \cdot \mathbb{I}\{|f| \leq s_1/2\}\) for any \(f \in \mathbb{R}\). When \(s_1 = 1\), we shorthand it as \(\text{rect}(f)\).

- \(\text{sinc}_{s_1}(t) = \frac{\sin(\pi s_1 t)}{\pi s_1 t}\) for any \(t \neq 0\) and \(\text{sinc}_{s_1}(0) = 1\). When \(s_1 = 1\), we shorthand it as \(\text{sinc}(t)\).

**Fact C.2** (Facts about basic functions [CKPS16, Appendix C]). Given any \(s_1 > 0\), the following hold for the functions \(\text{sinc}_{s_1}(t)\) and \(\text{rect}_{s_1}(f)\):

- **Part (a):** \(1 - \frac{s_1^2}{6} \cdot (s_1 t)^2 \leq |\text{sinc}_{s_1}(t)| \leq 1\) for any \(t \in \mathbb{R}\).

- **Part (b):** \(|\text{sinc}_{s_1}(t)| \leq 1 - \frac{s_1^2}{8} \cdot (s_1 t)^2\) for any \(|t| \leq \frac{2\sqrt{3}}{s_1}\).

- **Part (c):** \(|\text{sinc}_{s_1}(t)| \leq \min(1, \frac{1}{\pi|s_1 t|})\) for any \(t \in \mathbb{R}\).

- **Part (d):** \(\text{sinc}_{s_1}(t) = \text{rect}_{s_1}(t)\) for any \(t \in \mathbb{R}\), and \(\text{rect}_{s_1}(f) = \text{sinc}_{s_1}(f)\) for any \(f \in \mathbb{R}\).

Our building-block function \((G(t), \hat{G}(f))\) is constructed in the following Definition C.3. This construction is similar to [CKPS16, Definition C.11], and we carefully modify the involved parameters for our later use. We present several important properties of \((G(t), \hat{G}(f))\) in Section C.2, and then prove these properties in Section C.3.

**Definition C.3** (Building-block function in a single dimension). We set the parameters as follows:

- The number of bins in a single dimension \(B = \Theta(d \cdot k^{1/d})\) is a certain multiple of \(d \in \mathbb{N}_{\geq 1}\).

- The noise level parameter \(\delta \in (0, 1)\).

- \(\alpha = \Theta(1/d)\) is chosen such that \(\frac{1}{100(d+1)-\alpha} \in \mathbb{N}_{\geq 1}\) is an integer; clearly \(\alpha \leq \frac{1}{100(d+1)} \leq \frac{1}{200}\).

- \(s_1 = \frac{2\beta}{\alpha}\) and \(s_2 = \frac{1}{B + B/d}\).

- \(\ell = \Theta(\log(kd/\delta))\) is an even integer. We safely assume \(\ell \geq 1000\).

Then for any \(t, f \in \mathbb{R}\) the building-block function \((G(t), \hat{G}(f))\) is given by

\[
G(t) = s_0 \cdot \text{rect}_{s_1}^{\ell}(t) \cdot \text{sinc}_{s_2}(t) \\
= s_0 \cdot \text{rect}_{s_2}(t) \cdot \text{sinc}_{1/(B + B/d)}(t),
\]

\[
\hat{G}(f) = s_0 \cdot (\text{sinc}_{s_1}(f))^{\ell} \cdot \text{rect}_{s_2}(f) \\
= s_0 \cdot (\text{sinc}_{s_2}(f))^{\ell} \cdot \text{rect}_{1/(B + B/d)}(f),
\]

where the scalar \(s_0 > 0\) achieves the normalization \(\hat{G}(0) = 1\). Notice that both \(G(t)\) and \(\hat{G}(f)\) take real values, and are even functions.
Figure 5: Demonstration for the function \( \hat{G}(f) \) in Lemma C.4.

C.2 Properties of function \((G(t), \hat{G}(f))\)

Lemma C.4 (Building-block function in a single dimension). The function \((G(t), \hat{G}(f))[B, \delta, \alpha, \ell]\) given in Definition C.3 satisfies the following (as Figure 5 illustrates):

Property I: The scalar \( s_0 \approx s_1 s_2 \sqrt{\ell} \approx \sqrt{\ell}/\alpha \).

Property II: \( 1 - \frac{\delta}{\text{poly}(k, d)} \leq \hat{G}(f) \leq 1 \) when \(|f| \leq \frac{1-\alpha B}{2B} \).

Property III: \( \hat{G}(f) \in [0, 1] \) when \( \frac{1-\alpha}{2B} \leq |f| \leq \frac{1}{2B} \).

Property IV: \( 0 \leq \hat{G}(f) \leq (\pi B f)^{-\ell} \leq \frac{\delta}{\text{poly}(k, d)} \) when \(|f| \geq \frac{1}{2B} \).

Property V: \( \text{supp}(G) \subseteq [-\ell \cdot \frac{B}{\alpha}, \ell \cdot \frac{B}{\alpha}] \).

Property VI: \( \max_{t \in \mathbb{R}} |G(t)| = G(0) \in [\frac{1-\alpha-\delta/(4kd)}{B}, \frac{1+\delta/(4kd)}{B}] \).

Property VII: \( \sum_{i \in \mathbb{Z}} G(i + 1/2)^2 \leq (1 + \frac{\delta}{4kd})^2 \cdot (1 + \frac{1}{\alpha}) \cdot B^{-1} \lesssim B^{-1} \).

C.3 Proof of properties

Claim C.5 (Property I of Lemma C.4). The scalar \( s_0 \approx s_1 s_2 \sqrt{\ell} \approx \sqrt{\ell}/\alpha \).

Proof. Recall that the scalar \( s_0 > 0 \) achieves the normalization \( \hat{G}(0) = 1 \). By definition,

\[
\hat{G}(0) = s_0 \cdot \int_{-\infty}^{+\infty} (\text{sinc}_{s_1}(\xi))^\ell \cdot \text{rect}_{s_2}(0 - \xi) \cdot d\xi
\]

\[
= s_0 \cdot \int_{-s_2/2}^{s_2/2} (\text{sinc}_{s_1}(\xi))^\ell \cdot d\xi
\]

\[
= 2s_0 \cdot \int_{0}^{s_2/2} (\text{sinc}_{s_1}(\xi))^\ell \cdot d\xi
\]

\[
= 2s_0 \cdot \int_{0}^{s_2/2} |\text{sinc}_{s_1}(\xi)|^\ell \cdot d\xi,
\]

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where the second step follows because \( \text{rect}_{\frac{t}{2}}(\xi) = \frac{1}{\pi} \cdot \mathbb{1}(|\xi| \leq \frac{t}{2}) \) for any \( \xi \in \mathbb{R} \) (see Definition C.1); the third step follows because \( \text{sinc}_a(\xi) \) is an even function in \( \xi \in \mathbb{R} \); and the last step is because \( \ell \in \mathbb{N}_{\geq 1} \) is an even integer (see Definition C.3).

Given that \( s_1 = \frac{2B}{\alpha} \) and \( s_2 = \frac{1}{B+B/d} \) and \( 0 < \alpha \leq \frac{1}{100(d+1)} \) (see Definition C.3), one can easily check that \( \frac{2}{\pi s_1} \leq \frac{s_2}{2} \). Accordingly, we know from the additivity of integration that

\[
\hat{G}(0) = \frac{2s_0}{s_2} \cdot \int_0^{2/(\pi s_1)} |\text{sinc}_s(\xi)|^\ell \cdot d\xi + \frac{2s_0}{s_2} \cdot \int_{2/(\pi s_1)}^{s_2/2} |\text{sinc}_s(\xi)|^\ell \cdot d\xi
\]

where the second step follows because \( (\text{sinc}_s(\xi))^\ell \geq 0 \) for any \( \xi \in \mathbb{R} \) (note that \( \ell \) is an even integer; see Definition C.3), and thus \( \hat{G}(0) \cdot \frac{s_2}{2s_0} \) is bounded between \((A_1 - A_2)\) and \((A_1 + A_2)\).

We will verify respectively in Claims C.13 and C.14 (see Section C.5 for the proofs of both claims) that \( A_1 \approx \frac{1}{s_1} \cdot \ell^{-1/2} \) and \( A_2 = O\left(\frac{1}{s_1} \cdot 2^{-\ell}\right) \). Under our choice of \( \ell = \Theta(\log(kd/\delta)) \), it follows that \( A_1 \gg A_2 \) and thus, that

\[
1 = \hat{G}(0) \approx \frac{2s_0}{s_2} \cdot A_1 \approx \frac{s_0}{s_1 s_2} \cdot \ell^{-1/2},
\]

which implies \( s_0 \approx s_1 s_2 \sqrt{\ell} \approx \sqrt{\ell}/\alpha \) (since \( s_1 = \frac{2B}{\alpha} \) and \( s_2 = \frac{1}{B+B/d} \)).

This completes the proof of Claim C.5. \( \square \)

**Claim C.6** (Property II of Lemma C.4). \( 1 - \frac{\delta}{\text{poly}(kd)} \leq \hat{G}(f) \leq 1 \) when \( |f| \leq \frac{1-\alpha}{2B} \).

**Proof.** We first prove the upper-bound part that \( \hat{G}(f) \leq \hat{G}(0) = 1 \) for any \( f \in \mathbb{R} \). Since \( \hat{G}(f) \) is an even function (see Definition C.3), it suffices to deal with the case that \( f \geq 0 \). By definition,

\[
\hat{G}(f) - \hat{G}(0) = s_0 \cdot \int_{-\infty}^{+\infty} (\text{sinc}_s(\xi))^\ell \cdot (\text{rect}_{s_2}(f - \xi) - \text{rect}_{s_2}(0 - \xi)) \cdot d\xi
\]

where the second step follows because \( \text{rect}_{s_2}(\xi) = \frac{1}{s_2} \cdot \mathbb{1}(|\xi| \leq s_2) \) for any \( \xi \in \mathbb{R} \) (see Definition C.1); the third step is by substitution; the fourth step applies the additivity of integration; the fifth step follows because \( \text{sinc}(\xi) \) is an even function in \( \xi \in \mathbb{R} \); and the last step is by substitution.
Given Equation (16), it suffices to show that $A_3 \leq 0$ when $\xi \in [0, s_1 f]$. Recall that $\ell \in \mathbb{N}_{\geq 1}$ is an even integer, and $s_1 s_2 = \left(\frac{2B}{\alpha}\right) \cdot \left(\frac{1}{B + B/d}\right) \cdot \frac{1}{2} = \frac{100}{100(d+1)\alpha} \cdot (100 \cdot d) \in \mathbb{N}_{\geq 1}$ is an integer (see Definition C.3). Therefore, for any $\xi \in [0, s_1 f]$ we have

\[
A_3 = \left| \text{sinc} \left( \frac{s_1 s_2}{2} + \xi \right) \right|^\ell - \left| \text{sinc} \left( \frac{s_1 s_2}{2} - \xi \right) \right|^\ell
= \left| \frac{\sin(\pi \cdot \frac{s_1 s_2}{2} + \pi \cdot \xi)}{\pi \cdot \frac{s_1 s_2}{2} + \pi \cdot \xi} \right|^\ell - \left| \frac{\sin(\pi \cdot \frac{s_1 s_2}{2} - \pi \cdot \xi)}{\pi \cdot \frac{s_1 s_2}{2} - \pi \cdot \xi} \right|^\ell
= \left| \frac{\sin(\pi \cdot \xi)}{\pi \cdot \frac{s_1 s_2}{2} + \pi \cdot \xi} \right|^\ell - \left| \frac{\sin(-\pi \cdot \xi)}{\pi \cdot \frac{s_1 s_2}{2} - \pi \cdot \xi} \right|^\ell
= \left| \frac{\sin(\pi \cdot \xi)}{|s_1 s_2/2 + \xi|} \right|^\ell - \left| \frac{\sin(-\pi \cdot \xi)}{|s_1 s_2/2 - \pi \cdot \xi|} \right|^\ell
\]

where the first step follows because $\ell \in \mathbb{N}_{\geq 1}$ is an even integer; the third step follows because $|\sin(\pi \cdot \xi)|$ is a periodic function in $\xi \in \mathbb{R}$ and its basic period is 1 (notice that $\frac{s_1 s_2}{2}$ is an integer); and the fourth step follows because $|\sin(\pi \cdot \xi)|$ is a even function.

To see the lower-bound part, due to the normalization $\hat{G}(0) = 1$, we have

\[
1 - \hat{G}(f) = \hat{G}(0) - \hat{G}(f)
= \frac{s_0}{s_1 s_2} \cdot \int_0^{s_1 f} \left( \left( \text{sinc} \left( \frac{s_1 s_2}{2} + \xi \right) \right)^\ell - \left( \text{sinc} \left( \frac{s_1 s_2}{2} - \xi \right) \right)^\ell \right) \cdot d\xi
\]

where the second step follows from Equation (16); the third step follows because $\ell$ is an even integer (see Definition C.3), namely $(\text{sinc}(\xi))^\ell \geq 0$ for any $\xi \in \mathbb{R}$; the fourth step is by Part (c) of Fact C.2; and the sixth step is by $s_1 = \frac{2B}{\alpha}$ and $s_2 = \frac{1}{B + B/d}$ (see Definition C.3).

According to Claim C.5, for some universal constant $C_0 > 0$, we have $s_0 \leq C_0 \cdot \sqrt{\ell}/\alpha$. Also, as promised by the concerning claim, $|f| \leq \frac{1 - \alpha}{2B} \leq \frac{1}{2B}$. Plugging these into the above inequality:

\[
1 - \hat{G}(f) \leq C_0 \cdot \sqrt{\ell}/\alpha \cdot 2B \cdot \frac{1}{2B} \cdot \left( \frac{(d + 1) \cdot \alpha}{\pi \cdot d} \right)^\ell
= C_0 \cdot \sqrt{\ell} \cdot \frac{d + 1}{\pi \cdot d} \cdot \left( \frac{(d + 1) \cdot \alpha}{\pi \cdot d} \right)^{\ell - 1}
\leq C_0 \cdot \sqrt{\ell} \cdot \left( \frac{(d + 1) \cdot \alpha}{\pi \cdot d} \right)^{\ell - 1}
\]

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\[
\begin{align*}
&\leq C_0 \cdot \sqrt{\ell} \cdot \left(\frac{1}{100\pi \cdot d}\right)^{\ell - 1} \\
&\leq \frac{\delta}{\text{poly}(k,d)},
\end{align*}
\]
where the third step follows because \(\frac{d+1}{\pi d} \leq \frac{2}{\pi} \leq 1\); the fourth step follows because \(0 < \alpha \leq \frac{1}{100(d+1)}\) (see Definition C.3); and the last step holds for any large enough \(\ell = \Theta(\log(kd/\delta))\).

This completes the proof of Claim C.6. \(\square\)

**Claim C.7** (Property III of Lemma C.4). \(\hat{G}(f) \in [0, 1]\) when \(\frac{1-\alpha}{2B} \leq |f| \leq \frac{1}{2B}\).

**Proof.** The upper-bound part has been shown in the proof of Claim C.6, namely \(\hat{G}(f) \leq \hat{G}(0) = 1\) for any \(f \in \mathbb{R}\). The lower-bound part is trivial, since both functions \((\text{sinc}_{s_1}(f))^\ell\) and \(\text{rect}_{s_2}(f)\) are nonnegative (note that \(\ell \in \mathbb{N}_{\geq 1}\) is an even integer; see Definition C.3).

This completes the proof of Claim C.7. \(\square\)

**Claim C.8** (Property IV of Lemma C.4). \(0 \leq \hat{G}(f) \leq (\pi B f)^{-\ell} \leq \frac{\delta}{\text{poly}(k,d)}\) when \(|f| \geq \frac{1}{2B}\).

**Proof.** The lower-bound part has been shown in the proof of Claim C.7, namely \(\ell \in \mathbb{N}_{\geq 1}\) is an even integer (see Definition C.3) and thus both functions \((\text{sinc}_{s_1}(f))^\ell\) and \(\text{rect}_{s_2}(f)\) are nonnegative.

For the upper-bound part, since \(\hat{G}(f)\) is an even function, it suffices to handle the case \(f \geq \frac{1}{2B}\). By definition,

\[
\hat{G}(f) = s_0 \cdot \int_{-\infty}^{+\infty} (\text{sinc}_{s_1}(\xi))^\ell \cdot \text{rect}_{s_2}(f - \xi) \cdot d\xi
\]

\[
= \frac{s_0}{s_2} \cdot \int_{f-s_2/2}^{f+s_2/2} (\text{sinc}_{s_1}(\xi))^\ell \cdot d\xi
\]

\[
= \frac{s_0}{s_2} \cdot \int_{f-s_2/2}^{f+s_2/2} |\text{sinc}_{s_1}(\xi)|^\ell \cdot d\xi
\]

\[
\leq \frac{s_0}{s_2} \cdot \int_{f-s_2/2}^{f+s_2/2} \frac{1}{\pi^\ell \cdot |s_1\xi|^\ell} \cdot d\xi,
\]

where the second step follows because \(\text{rect}_{s_2}(\xi) = \frac{1}{s_2} \cdot \mathbb{I}\{|\xi| \leq \frac{s_2}{2}\}\) for any \(\xi \in \mathbb{R}\) (see Definition C.1); the third step follows because \(\ell \in \mathbb{N}_{\geq 1}\) is an even integer (see Definition C.3); and the last step is by Part (c) of Fact C.2.

Recall Definition C.3 that \(s_2 = \frac{1}{B+B/d}\). Given this and since we assume \(f \geq \frac{1}{2B}\), one can easily check that the above interval of integral is lower bounded by \(f - s_2/2 \geq f/(d+1)\). Hence,

\[
\hat{G}(f) \leq \frac{s_0}{s_2} \cdot s_2 \cdot \frac{1}{\pi^\ell \cdot |s_1\xi|^\ell} \bigg|_{\xi = f/(d+1)}
\]

\[
= s_0 \cdot \left(\frac{(d+1) \cdot \alpha}{2\pi B f}\right)^\ell,
\]

where the second step follows because \(s_1 = \frac{2B}{\alpha}\) and \(s_2 = \frac{1}{B+B/d}\). According to Claim C.5, for some universal constant \(C_0 > 0\), we have \(s_0 \leq C_0 \sqrt{\ell}/\alpha\). As a consequence,

\[
\hat{G}(f) \leq C_0 \cdot \frac{\sqrt{\ell}}{\alpha} \cdot \left(\frac{(d+1) \cdot \alpha}{2\pi B f}\right)^\ell
\]

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\[
\leq C_0 \cdot 100 \cdot (d + 1) \cdot \sqrt{\ell} \cdot (200\pi Bf)^{-\ell} \\
\leq (\pi Bf)^{-\ell},
\]
where the second step follows because, given that \( \ell \geq 1000 \), the concerning formula \( C_0 \cdot \frac{\sqrt{\ell}}{\alpha} \cdot \left(\frac{d+1}{2\pi Bf}\right)^{\ell} \) is an increasing function when \( 0 < \alpha \leq \frac{1}{200((d+1))} \) (see Definition C.3); and the last step, which is equivalent to \( \frac{C_0 \cdot 1000 \cdot (d+1) \cdot \sqrt{\ell}}{200} \leq 1 \), holds for any large enough \( \ell = \Theta(\log(kd/\delta)) \).

Following the above calculation, for any \( f \geq \frac{1}{2B} \) we have
\[
\hat{G}(f) \leq (\pi Bf)^{-\ell} \leq (\pi/2)^{-\ell} \leq \frac{\delta}{\text{poly}(k,d)},
\]
where the last step holds for any large enough \( \ell = \Theta(\log(kd/\delta)) \).

This completes the proof of Claim C.8. \( \square \)

**Claim C.9** (Property V of Lemma C.4). \( \text{supp}(G) \subseteq [-\ell \cdot \frac{B}{\alpha}, \ell \cdot \frac{B}{\alpha}] \).

*Proof.* Recall that \( s_1 = \frac{2B}{\alpha} \). By definition, the function \( \text{rect}_{s_1}(t) = \frac{1}{s_1} \cdot \mathbb{I}\{|t| \leq \frac{s_1}{2}\} \) is supported on the interval \( t \in [-\frac{s_1}{2}, \frac{s_1}{2}] \), and thus \( \text{rect}_{s_1}(t) \) is supported on \( t \in [-\ell \cdot \frac{s_1}{2}, \ell \cdot \frac{s_1}{2}] = [-\ell \cdot \frac{B}{\alpha}, \ell \cdot \frac{B}{\alpha}] \). Clearly, the later interval contains the support of the function \( G(t) = s_0 \cdot \text{rect}_{s_1}(t) \cdot \text{sinc}_{s_2}(t) \).

This completes the proof of Claim C.9. \( \square \)

**Claim C.10** (Property VI of Lemma C.4). \( \max_{t \in \mathbb{R}} |G(t)| = G(0) = \left[\frac{1-\delta/(4kd)}{B}, \frac{1+\delta/(4kd)}{B}\right] \).

*Proof.* Observe that \( \hat{G}(\xi) = s_0 \cdot (\text{sinc}_{s_1}(f))^{\ell} \ast \text{rect}_{s_2}(f) \) is an even function in \( \xi \in \mathbb{R} \), since both \( \text{sinc}_{s_1}(\xi) \) and \( \text{rect}_{s_2}(f) \) are even functions.

We first prove that \( \max_{t \in \mathbb{R}} |G(t)| = G(0) \). By the definition of the inverse CFT,
\[
\int_{-\infty}^{+\infty} \hat{G}(\xi) \cdot d\xi = G(0) \leq \max_{t \in \mathbb{R}} G(t) \leq \max_{t \in \mathbb{R}} |G(t)|.
\]

Also, for any \( t \in \mathbb{R} \) we can derive \( G(t) \) from \( \hat{G}(f) \) via the inverse CFT:
\[
|G(t)| = \left| \int_{-\infty}^{+\infty} \hat{G}(\xi) \cdot e^{2\pi i t \cdot \xi} \cdot d\xi \right| \\
= \left| \int_{-\infty}^{+\infty} \hat{G}(\xi) \cdot \left( \cos(2\pi t \cdot \xi) + i \cdot \sin(2\pi t \cdot \xi) \right) \cdot d\xi \right| \\
= \left| \int_{-\infty}^{+\infty} \hat{G}(\xi) \cdot \cos(2\pi t \cdot \xi) \cdot d\xi \right| \\
\leq \left| \int_{-\infty}^{+\infty} \hat{G}(\xi) \right| \cdot | \cos(2\pi t \cdot \xi) | \cdot d\xi \\
\leq \int_{-\infty}^{+\infty} \left| \hat{G}(\xi) \right| \cdot d\xi \\
= \int_{-\infty}^{+\infty} \hat{G}(\xi) \cdot d\xi
\]
where the third step follows because \( \hat{G}(\xi) \) is an even function in \( \xi \in \mathbb{R} \) (see Definition C.3), whereas \( \sin(2\pi t \cdot \xi) \) is an odd function; the fifth step is because \( | \cos(2\pi t \cdot \xi) | \leq 1 \) for any \( \xi \in \mathbb{R} \); and the last step follows as \( \hat{G}(\xi) \geq 0 \) for any \( \xi \in \mathbb{R} \) (see Claims C.6, C.7 and C.8).
We first prove by induction that

\[ \max_{t \in \mathbb{R}} |G(t)| = \int_{-\infty}^{+\infty} \hat{G}(\xi) \cdot d\xi = 2 \cdot \int_{0}^{+\infty} \hat{G}(\xi) \cdot d\xi = 2 \cdot (A_4 + A_5 + A_6), \]  

(17)

where the second step follows as \( \hat{G}(\xi) \) is an even function in \( \xi \in \mathbb{R} \); and for the third step we denote the terms \( A_4 \) and \( A_5 \) and \( A_6 \) as follows:

\[
A_4 = \int_{0}^{(1-\alpha)/(2B)} \hat{G}(\xi) \cdot d\xi,
A_5 = \int_{(1-\alpha)/(2B)}^{1/(2B)} \hat{G}(\xi) \cdot d\xi,
A_6 = \int_{1/(2B)}^{+\infty} \hat{G}(\xi) \cdot d\xi.
\]

Let us quantify the three terms \( A_4 \) and \( A_5 \) and \( A_6 \) respectively:

- \( A_4 \in [\frac{1-\alpha-\delta/(4kd)}{2B}, \frac{1-\alpha}{2B}] \). This is because \( 1 - \frac{\delta}{4kd} \leq \hat{G}(\xi) \leq 1 \) for any \( \xi \in [0, \frac{1-\alpha}{2B}] \) (Claim C.6); we shall notice that \( 0 < \alpha \leq \frac{1}{100(d+1)} < 1 \) and that \( B > 1 \) (see Definition C.3).

- \( A_5 \in [0, \frac{\alpha}{2B}] \). This is because \( \hat{G}(\xi) \in [0, 1] \) for any \( \xi \in [\frac{1-\alpha}{2B}, \frac{1}{2B}] \) (see Claim C.7).

- \( A_6 \in [0, \frac{\delta(4kd)}{2B}] \). Based on Claim C.8, we have \( 0 \leq \hat{G}(\xi) \leq (\pi B\xi)^{-\ell} \) for any \( \xi \geq \frac{1}{2B} \). Then the lower-bound part \( A_6 \geq 0 \) follows immediately. For the upper-bound part, we have

\[
A_6 \leq \int_{1/(2B)}^{+\infty} (\pi B\xi)^{-\ell} \cdot d\xi
= \frac{1}{\pi B} \cdot \int_{\pi/2}^{+\infty} \xi^{-\ell} \cdot d\xi
= \frac{1}{2B} \cdot \frac{1}{\ell-1} \cdot (\pi/2)^{-\ell}
\leq \frac{\delta/(4kd)}{2B},
\]

where the second step is by substitution; the third step is by elementary calculation; and the last step, given Definition C.3, holds for any large enough \( \ell = \Theta(\log(kd/\delta)) \).

Applying the above bounds to Equation (17) completes the proof of Claim C.10.

\[\square\]

Claim C.11 (Property VII of Lemma C.4). \( \sum_{i \in \mathbb{Z}} G(i + 1/2)^2 \leq (1 + \frac{\delta}{4kd})^2 \cdot (1 + \frac{1}{4}) \cdot B^{-1} \lesssim B^{-1} \).

Proof. We first prove by induction that \( \text{rect}_{s_1}(t) \) is an even function and is non-increasing for any \( t \geq 0 \). Obviously, \( \text{rect}_{s_1}(t) \) itself meets the both properties. Given any \( \ell' < \ell \), w.l.o.g. we assume \( \text{rect}_{s_1}(t) \) to satisfy the two properties as well. Then for any \( t \in \mathbb{R} \), it follows that

\[
\text{rect}_{s_1}^{s\ell' + 1}(-t) = \int_{-\infty}^{+\infty} \text{rect}_{s_1}(-t - \tau) \cdot \text{rect}_{s_1}^{s\ell'}(\tau) \cdot d\tau
= \int_{-\infty}^{+\infty} \text{rect}_{s_1}(t + \tau) \cdot \text{rect}_{s_1}^{s\ell'}(-\tau) \cdot d\tau
= \text{rect}_{s_1}^{s\ell' + 1}(t),
\]

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where the second step follows since 

\[
\text{rect}^{s_{1}^{t'}}(t') - \text{rect}^{s_{1}^{t}}(t) = \int_{-\infty}^{+\infty} \text{rect}_{s_{1}}(t') \cdot \text{rect}_{s_{1}}^{t'}(\tau) \cdot d\tau - \int_{-\infty}^{+\infty} \text{rect}_{s_{1}}(t) \cdot \text{rect}_{s_{1}}^{t'}(\tau) \cdot d\tau
\]

\[
= 1/s_1 \cdot \int_{t-s_{2}/2}^{t+s_{2}/2} \text{rect}_{s_{1}}^{t'}(\tau) \cdot d\tau - 1/s_1 \cdot \int_{t-s_{2}/2}^{t+s_{2}/2} \text{rect}_{s_{1}}^{t'}(\tau) \cdot d\tau
\]

\[
= 1/s_1 \cdot \int_{t-s_{2}/2}^{t+s_{2}/2} \text{rect}_{s_{1}}^{t'}(\tau) \cdot d\tau - 1/s_1 \cdot \int_{t-s_{2}/2}^{t+s_{2}/2} \text{rect}_{s_{1}}^{t'}(\tau) \cdot d\tau
\]

\[
= 1/s_1 \cdot \int_{t}^{t} \left( \text{rect}_{s_{1}}^{t'}(\tau + s_{2}/2) - \text{rect}_{s_{1}}^{t'}(\tau - s_{2}/2) \right) \cdot d\tau
\]

\[
< 0,
\]

where the second step follows since \( \text{rect}_{s_{1}}(\xi) = \frac{1}{s_1} \cdot I\{|\xi| \leq \frac{s_1}{2}\} \) for any \( \xi \in \mathbb{R} \) (see Definition C.1); the third step is by the additivity of integration; the fourth step is by substitution; and the last step uses our induction hypotheses that \( \text{rect}_{s_{1}}^{t'}(t) \) is an even function and is non-increasing when \( t \geq 0 \). Thus, \( \text{rect}_{s_{1}}^{t'+1}(t) \) also meets the properties, and our claim follows by induction.

Further, it is easy to see that \( \text{rect}_{s_{1}}^{t'}(t) \) is a non-negative function. Put everything together:

\[
\max_{\tau \in \mathbb{R}} \{ s_{0}^2 \cdot \text{rect}_{s_{1}}^{t'}(\tau)^2 \} = s_{0}^2 \cdot \text{rect}_{s_{1}}^{t'}(0)^2
\]

\[
= G(0)^2 / \text{sinc}_{s_{2}}(0)^2
\]

\[
= G(0)^2
\]

\[
\leq \left( 1 + \delta/(4kd) \right)^2 \cdot B^{-2},
\]

where the third step follows because \( \text{sinc}_{s_{2}}(0) = 1 \) (see Definition C.1); and the fourth step follows from Claim C.10.

For any \( t \in \mathbb{R} \), we infer from the above that

\[
G(t)^2 = s_{0}^2 \cdot \text{rect}_{s_{1}}^{t}(t)^2 \cdot \text{sinc}_{s_{2}}(t)^2
\]

\[
\leq \max_{\tau \in \mathbb{R}} \{ s_{0}^2 \cdot \text{rect}_{s_{1}}^{t}(\tau)^2 \} \cdot \text{sinc}_{s_{2}}(t)^2
\]

\[
\leq \left( 1 + \delta/(4kd) \right)^2 \cdot B^{-2} \cdot \text{sinc}_{s_{2}}(t)^2
\]

Further, given that \( s_{2} = \frac{1}{B+B/d} < 1 \) (see Definition C.3), we have

\[
\sum_{i \in \mathbb{Z}} G(i + 1/2)^2 \leq \left( 1 + \delta/(4kd) \right)^2 \cdot B^{-2} \cdot \sum_{i \in \mathbb{Z}} \text{sinc}_{1/(B+B/d)}(i + 1/2)^2
\]

\[
= \left( 1 + \delta/(4kd) \right)^2 \cdot B^{-2} \cdot (B + B/d)
\]

\[
= \left( 1 + \delta/(4kd) \right)^2 \cdot (1 + 1/d) \cdot B^{-1},
\]

where the second step, which is equivalent to \( \sum_{i=0}^{+\infty} \text{sinc}_{1/(B+B/d)}(i + 1/2)^2 = B + B/d \), can be directly inferred from [BJP73, Equation (1)].

This completes the proof of Claim C.11.

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C.4 Construction and properties of standard window function \((G'(t), \hat{G}'(f))\)

We associate the building-block function with the standard window function \((G'(t), \hat{G}'(f))[B, \delta, \alpha, \ell]\) (similar to the ones used in [HIKP12a, HIKP12b]), which is more convenient for our later use.

Lemma C.12 (Standard window function in a single dimension). Consider the building-block function \((G'(t), \hat{G}'(f))[B, \delta, \alpha, \ell]\) given in Definition C.3, there exists another function \((G'(t), \hat{G}'(f))[B, \delta, \alpha, \ell]\) such that:

Property I: \(\hat{G}'(f) = 1\) when \(|f| \leq \frac{1-\alpha}{2B}\).

Property II: \(\hat{G}'(f) \in [0, 1]\) when \(\frac{1-\alpha}{2B} \leq |f| \leq \frac{1}{2B}\).

Property III: \(\hat{G}'(f) = 0\) when \(|f| \geq \frac{1}{2B}\).

Property IV: \(\|\hat{G}' - \hat{G}\|_\infty = \max_{f \in \mathbb{R}} |\hat{G}'(f) - \hat{G}(f)| \leq \frac{\delta}{\text{poly}(k,d)}\).

Proof. We define \(\hat{G}'(f)\) as follows; note that, similar to \(\hat{G}(f)\), this is also an even function:

\[
\hat{G}'(f) = \begin{cases} 
1, & \forall |f| \leq \frac{1-\alpha}{2B}; \\
\hat{G}(f), & \forall |f| \in \left(\frac{1-\alpha}{2B}, \frac{1}{2B}\right]; \\
0, & \forall |f| > \frac{1}{2B}.
\end{cases}
\]

By construction, Properties I and III follows directly. Further, Property II follows from Property III of Lemma C.4, and Property IV follows from Properties II to IV of Lemma C.4.

This completes the proof of Lemma C.12.

C.5 Facts

The following facts are helpful in proving Claim C.5.

Claim C.13. \(\int_0^{2/(\pi s_1)} \left(\text{sinc}_{s_1}(\xi)\right)^\ell \cdot d\xi \approx \frac{1}{s_1} \cdot \ell^{-1/2}\).

Proof. Let \(i^* = \lceil 2/\pi \cdot \sqrt{\ell/8} \rceil - 1\). We safely that assume \(\ell = \Theta(\log(kd/\delta))\) is an integer larger than 1000 (see Definition C.3), which guarantees the following facts:

(a): The integrand \(\left(\text{sinc}_{s_1}(\xi)\right)^\ell \geq 0\) for any \(\xi \in \mathbb{R}\);

(b): \(i^* \geq 2/\pi \cdot \sqrt{\ell/8} - 1 \geq 2/\pi \cdot \sqrt{1000/8} - 1 \approx 6.118 \geq 6\);

(c): \(i^* \cdot \sqrt{8/\ell} \leq 2/\pi \cdot \sqrt{\ell/8} \cdot \sqrt{8/\ell} = 2/\pi\);

(d): \((i^* + 1) \cdot \sqrt{8/\ell} \geq 2/\pi \cdot \sqrt{\ell/8} \cdot \sqrt{8/\ell} = 2/\pi\); and

(e): \((i^* + 1) \cdot \sqrt{8/\ell} \leq (2/\pi \cdot \sqrt{\ell/8} + 1) \cdot \sqrt{8/\ell} \leq 2/\pi + \sqrt{8/1000} \approx 2.281 \leq \frac{2.3}{\pi}\).

These facts are useful in proving the current claim.

For the upper-bound part, we have

\[
\int_0^{2/(\pi s_1)} \left(\text{sinc}_{s_1}(\xi)\right)^\ell \cdot d\xi = \frac{1}{s_1} \cdot \int_0^{2/\pi} \left(\text{sinc}(\xi)\right)^\ell \cdot d\xi
\]

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where the first step is by substitution; the second step by because \( \ell \in \mathbb{N}_{\geq 1} \) is an even integer (see Definition C.3); the third step follows from the above Fact (d); and the last step follows from the additivity of integration.

Given the above Fact (e), the whole interval of integral \( \xi \in [0, (i^* + 1) \cdot \sqrt{8/\ell}] \) is a subset of \( \xi \in [0, \frac{2\pi}{\pi}] \), namely Parts (b) of Fact C.2 is applicable here. In particular, each \( i \)-th summand in Equation (18) equals

\[
\int_{i \cdot \sqrt{8/\ell}}^{(i+1) \cdot \sqrt{8/\ell}} |\text{sinc}(\xi)|^\ell \, d\xi \leq \int_{i \cdot \sqrt{8/\ell}}^{(i+1) \cdot \sqrt{8/\ell}} (1 - \pi^2/8 \cdot \xi^2)^\ell \, d\xi \\
\leq \int_{i \cdot \sqrt{8/\ell}}^{(i+1) \cdot \sqrt{8/\ell}} (1 - \pi^2 i^2/\ell)^\ell \, d\xi \\
= \sqrt{8/\ell} \cdot (1 - \pi^2 i^2/\ell)^\ell \\
\leq \sqrt{8/\ell} \cdot e^{-\pi^2 i^2},
\]  

(19)

where the first step is follows from Parts (b) of Fact C.2; the second step is because \( 1 - \pi^2/8 \cdot t^2 \leq (1 - \pi^2/8 \cdot t^2)_{t=i \cdot \sqrt{8/\ell}} = 1 - \pi^2 i^2/\ell \); and the last step is by \( 0 \leq 1 - \frac{t}{x} \leq e^{-x} \) for any \( x \in (0, 1) \).

Applying Equation (19) to Equation (18) over all \( i \in [0 : i^*] \) results in

\[
\int_0^{2/(\pi s_1)} (\text{sinc}_{s_1}(\xi))^\ell \, d\xi \leq \frac{1}{s_1} \cdot \sum_{i=0}^{i^*} \sqrt{8/\ell} \cdot e^{-\pi^2 i^2} \\
\leq \frac{1}{s_1} \cdot \sqrt{8/\ell} \cdot \sum_{i=0}^{+\infty} e^{-\pi^2 i^2} \\
\leq \frac{1}{s_1} \cdot \sqrt{8/\ell} \cdot \sum_{i=0}^{+\infty} \frac{1}{(1 + i)^2} \\
= \frac{1}{s_1} \cdot \sqrt{8/\ell} \cdot \frac{\pi^2}{6} \\
\leq 5/s_1 \cdot \ell^{-1/2},
\]

where the third step is because \( e^{-\pi^2 i^2} \leq e^{-2i} \leq e^{-2\ln(1+i)} = (1 + i)^{-2} \) for each \( i \in \mathbb{N}_{\geq 0} \); and the last step is because \( \sqrt{8} \cdot \pi^2/6 \approx 4.6526 < 5 \).

Further, we can infer the lower-bound part as follows:

\[
\int_0^{2/(\pi s_1)} (\text{sinc}_{s_1}(\xi))^\ell \, d\xi = \frac{1}{s_1} \cdot \int_0^{2/\pi} (\text{sinc}(\xi))^\ell \, d\xi \\
= \frac{1}{s_1} \cdot \int_0^{2/\pi} |\text{sinc}(\xi)|^\ell \, d\xi
\]

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\[ \geq \frac{1}{s_1} \int_0^{i^* \cdot \sqrt{8/\ell}} |\text{sinc}(\xi)|^\ell \cdot d\xi \]
\[ \geq \frac{1}{s_1} \int_0^{\sqrt{8/\ell}} |\text{sinc}(\xi)|^\ell \cdot d\xi, \quad (20) \]

where the first step is by substitution; the second step by because \( \ell \in \mathbb{N}_{\geq 1} \) is an even integer (see Definition C.3); the third step follows from the Fact (c) given in the beginning of this proof; and the last step is due to the above Fact (b) that \( i^* \geq 6 > 1 \).

Under the assumption \( \ell \geq 1000 \), we have
\[ 0 < \frac{\pi^2}{2} / \ell \leq \frac{\pi^2/6 \cdot \xi^2}{2} \leq \frac{\pi^2/6 \cdot 8}{1000} \approx 0.013 \leq 1 \]
for any \( \xi \in [0, \sqrt{8/\ell}] \). Then for any \( \xi \in [0, \sqrt{8/\ell}] \) we have
\[ |\text{sinc}(\xi)|^\ell \geq \left(1 - \frac{\pi^2/6 \cdot 8}{\ell}\right)^\ell \]
\[ \geq \left(1 - \frac{\pi^2/6 \cdot 8}{1000}\right)^{1000} \]
\[ = \left(1 - \frac{\pi^2/750}{1000}\right)^{1000}, \quad (21) \]

where the first step is by Part (a) of Fact C.2; and the third step is because \( 0 \leq \frac{\pi^2/6 \cdot 8}{\ell} \leq 1 \) and that \( y = (1 - z)^{1/z} \) is a decreasing function for any \( z \in (0, 1) \).

Plugging Equation (21) back into Equation (20) results in
\[ \int_0^{2/(\pi s_1)} (\text{sinc}_{s_1}(\xi))^\ell \cdot d\xi \geq \frac{1}{s_1} \cdot \sqrt{8/\ell} \cdot (1 - \pi^2/750)^{1000} \geq 1/2^{18} \cdot \frac{1}{s_1} \cdot \ell^{-1/2}, \]

where the last step follows from elementary calculation.

This completes the proof of Claim C.13.

Claim C.14. \( \int_{2/(\pi s_1)}^{+\infty} (\text{sinc}_{s_1}(\xi))^\ell \cdot d\xi = O\left(\frac{1}{s_1} \cdot 2^{-\ell}\right) \).

Proof. We check the claim as follows:
\[ \int_{2/(\pi s_1)}^{+\infty} (\text{sinc}_{s_1}(\xi))^\ell \cdot d\xi = \int_{2/(\pi s_1)}^{+\infty} |\text{sinc}_{s_1}(\xi)|^\ell \cdot d\xi \]
\[ \leq \int_{2/(\pi s_1)}^{+\infty} \frac{1}{\pi \ell} \cdot |\text{sinc}_{s_1}(\xi)|^\ell \cdot d\xi \]
\[ = \frac{1}{\pi s_1} \cdot \int_{2}^{+\infty} \xi^{-\ell} \cdot d\xi \]
\[ = \frac{1}{\pi s_1} \cdot \frac{2}{\ell - 1} \cdot 2^{-\ell} \]
\[ = O\left(\frac{1}{s_1} \cdot 2^{-\ell}\right), \]

where the first step is because \( \ell \in \mathbb{N}_{\geq 1} \) is an even integer (see Definition C.3); the second step is by Part (c) of Fact C.2; and the third step is by substitution.

This completes the proof of Claim C.14.

\[ \square \]
D Filter, permutation and hashing in a single dimension

In this section, we first construct our single-dimensional filter function $(G(t), \hat{G}(f))$ and investigate several properties of it, based on the building-block function $(G(t), \hat{G}(f))$ introduced in Section C. In particular:

- Sections D.1 to D.3. We first leverage the function $(G(t), \hat{G}(f))$ introduced in Definition C.3 to construct our ultimate single-dimensional filter $(G(t), \hat{G}(f))$, and then prove the properties of this filter (by applying Lemma C.4 and extra arguments).
- Section D.4. We associate the filter $(G(t), \hat{G}(f))$ with another standard window $(G(t'), \hat{G}(f'))$ (in a manner similar to Lemma C.12), which is more convenient for our later use.

D.1 Construction of filter $(G(t), \hat{G}(f))$

**Definition D.1 (The single-dimensional filter).** Recall the parameters defined in Definition C.3:

- The number of bins in a single dimension $B = \Theta(d \cdot k^{1/d})$ is a certain multiple of $d \in \mathbb{N}_{\geq 1}$.
- The noise level parameter $\delta \in (0, 1)$.
- $\alpha = \Theta(1/d)$ is chosen such that $\frac{1}{100(d+1)\alpha} \in \mathbb{N}_{\geq 1}$ is an integer; clearly $\alpha \leq \frac{1}{100(d+1)} \leq \frac{1}{200}$.
- $s_1 = \frac{2B}{\alpha}$ and $s_2 = \frac{1}{B+B/d}$.
- $\ell = \Theta(\log(kd/\delta))$ is an even integer. We safely assume $\ell \geq 1000$.

Further, the *width parameter* $W = \Theta(\frac{F}{B\eta})$ is chosen to be a sufficiently large integer. Based on the building-block function $(G(t), \hat{G}(f))$ given in Definition C.3, for $i \in \mathbb{Z}$, define the shifted function

$$
\hat{G}_i(f) := \hat{G}(f + i),
$$

$$
G_i(t) := \int_{-\infty}^{+\infty} \hat{G}(\xi) \cdot e^{2\pi i t \cdot \xi} \cdot d\xi.
$$

Then for any $t, f \in \mathbb{R}$ the single-dimensional filter $(G(t), \hat{G}(f))$ is given by

$$
\hat{G}(f) = e^{-\frac{\delta}{\text{poly}(k,d)}} \cdot \sum_{i \in [-W:W]} \hat{G}_i(f),
$$

$$
G(t) = \int_{-\infty}^{+\infty} \hat{G}(\xi) \cdot e^{2\pi i t \cdot \xi} \cdot d\xi.
$$

D.2 Properties of filter $(G(t), \hat{G}(f))$

Later we will employ another slightly different filter $(G(t), \hat{G}(f))$, just by shifting the one given in Definition D.1. For ease of presentation, the following Lemma D.2 is stated for the shifted filter, but we show in Section D.3 the counterpart claims for the unshifted filter.

**Lemma D.2 (The single-dimensional filter).** The filter $(G(t), \hat{G}(f))[B, \delta, \alpha, \ell, W]$ given in Definition D.1 satisfies the following (as Figure 6 illustrates):

**Property I:** $e^{-\frac{\frac{\delta}{\text{poly}(k,d)}}}{1} \leq \hat{G}(f) \leq 1$ when $|f - i| \leq \frac{1-\alpha}{2B}$ for some integer $|i| \leq W$. 38
Figure 6: Demonstration for Lemma D.2. $B_i = [i - 1/(2B), i + 1/(2B)]$, $A_i = [i - (1 - \alpha)/(2B), i + (1 - \alpha)/(2B)]$.

Property II: $0 \leq \hat{G}(f) \leq 1$ when $\frac{1 - \alpha}{2B} \leq |f - i| \leq \frac{1}{2B}$ for some integer $|i| \leq W$.

Property III: $0 \leq \hat{G}(f) \leq \frac{\delta}{\text{poly}(k,d)}$ when $|f - i| \geq \frac{1}{2B}$ for any integer $|i| \leq W$.

Property IV: $G(t) = (2W + 1) \cdot e^{-\frac{1}{\text{poly}(k,d)} \cdot G(t)} \cdot \frac{\sin\left(\frac{t}{2W + 1}\right)\left(t + \frac{1}{2}\right)}{\sin\left(t + \frac{1}{2}\right)}$ for any $t \in \mathbb{R}$.

Property V: $\text{supp}(G) \subseteq \text{supp}(G) \subseteq [-\ell \cdot \frac{B}{\alpha}, \ell \cdot \frac{B}{\alpha}]$.

Property VI: $\sum_{i \in \mathbb{Z}} G(i)^2 = e^{-\frac{\delta}{\text{poly}(k,d)}} \cdot \sum_{i \in \mathbb{Z}} G(i)^2 \leq (1 + \frac{\delta}{2}) \cdot B^{-1}$.

Remark D.3. The function values $G(i)$ are the scaling coefficients for our samples in the time domain. In fact, we just need the values $G(i)$ at a few fixed points. Thus, we can calculate and store those effective coefficients before the sampling process.

D.3 Proof of properties

Recall that $e^{\delta/(4kd)} \cdot \hat{G}(f) = \sum_{i \in [-W : W]} \hat{G}_i(f)$ (see Definition D.1) and, that every summand function $\hat{G}_i(f)$ has an axis of symmetry $f = -i$ (see Definition C.3). The next Claim D.4 suggests that, at any $f \in \mathbb{R}$, the value $e^{\delta/(4kd)} \cdot \hat{G}(f)$ is dominated by one particular summand $\hat{G}_i(f)$, and the other $2W$ summands are negligibly small. Given this observation, we can easily conclude Properties I to III of Lemma D.2 from Properties II to IV of Lemma C.4.

Claim D.4 (Auxiliary result for Lemma D.2). Given any $f' \in \mathbb{R}$, let $i^* = \text{argmin}_{i \in [-W : W]} |f' + i|$ be the particular summand function $\hat{G}_i(f)$ with the closest-to-$f'$ axis of symmetry, then

$$0 \leq e^{\frac{\delta}{\text{poly}(k,d)}} \cdot \hat{G}(f') = \sum_{i \in [-W : W] \setminus \{i^*\}} \hat{G}_i(f') \leq \frac{\delta}{\text{poly}(k,d)}.$$

Proof. The first part $\sum_{i \in [-W : W] \setminus \{i^*\}} \hat{G}_i(f') \geq 0$ follows because each summand function $\hat{G}_i(f')$ is non-negative (see Properties II to IV of Lemma C.4).

We now show the second part $\sum_{i \in [-W : W] \setminus \{i^*\}} \hat{G}_i(f) \leq \delta/(4kd)$. Clearly, the summand functions $\hat{G}_i(f)$ have axes of symmetry $f = -i \in [-W : W]$ (see Definition D.1). Since $f = -i^*$ is the axis closest to $f'$, the distances between $f'$ and either the left-hand-side axes (i.e. $f = -i \leq -i^* - 1$) or the right-hand-side axes (i.e. $f = -i \geq -i^* + 1$) are at least $(1 - \frac{1}{2^j}), (3 - \frac{1}{2}), \ldots$. Further, when the distance between $f'$ and an axis $f = -i$ is at least $j - \frac{1}{2} \geq \frac{j}{2} \geq \frac{1}{4B}$ (for some $j \in \mathbb{N}_{\geq 1}$;
recall Definition D.1 that $B = \Theta(d \cdot k^{1/d})$ is the number of bins in a single dimension, it follows from Property IV of Lemma C.4 that

$$
\hat{G}_i(f') \leq (\pi B \cdot (j - 1/2))^{-\ell} \\
\leq (\pi B \cdot j/2)^{-\ell}.
$$

(22)

Take all of the $2W$ remaining summands $i \in [-W:W] \setminus \{i^*_i\}$ into account:

$$
\sum_{i \in [-W:W] \setminus \{i^*_i\}} \hat{G}_i(f') \leq \sum_{i \in \mathbb{Z}\setminus \{i^*_i\}} \hat{G}_i(f') \\
\leq 2 \cdot \sum_{j \in \mathbb{N} \geq 1} (\pi B \cdot j/2)^{-\ell} \\
\leq \frac{\delta}{\text{poly}(k,d)},
$$

where the first step follows because each summand function $\hat{G}_i(f')$ is non-negative (see Properties II to IV of Lemma C.4); the second step follows from Inequality 22; and the last step holds whenever $B = \Theta(d \cdot k^{1/d})$ and $\ell = \Theta(\log(kd/\delta))$ are large enough.

This completes the proof of Claim D.4.

Claim D.5 (Property I of Lemma D.2). $e^{-\frac{\delta}{\text{poly}(k,d)}} \leq \hat{G}(f) \leq 1$ when $|f - i| \leq \frac{1-\delta}{2B}$ for some integer $|i| \leq W$.

Proof. We let $i^*_i = \arg\min_{i \in [-W:W]} |f + i|$ index the summand function with the closest-to-$f$ axis of symmetry. For the lower-bound part, we observe that

$$
\hat{G}(f) = e^{-\frac{\delta}{\text{poly}(k,d)}} \cdot \sum_{i \in [-W:W]} \hat{G}_i(f) \\
\geq e^{-\frac{\delta}{\text{poly}(k,d)}} \hat{G}_{i^*_i}(f) \\
\geq e^{-\frac{\delta}{\text{poly}(k,d)}} \cdot (1 - \frac{\delta}{\text{poly}(k,d)}) \\
\geq e^{-\frac{\delta}{\text{poly}(k,d)}},
$$

where the first step follows from Claim D.4; the third step applies Property II of Lemma C.4; and the last step is because $1 - z/4 \geq e^{-3z/4}$ for any $z \in [0,1]$ (recall Definition D.1 that the noise level parameter $0 < \delta < 1$).

In addition, for the upper-bound part we have

$$
\hat{G}(f) = e^{-\frac{\delta}{\text{poly}(k,d)}} \cdot \hat{G}_{i^*_i}(f) + e^{-\frac{\delta}{\text{poly}(k,d)}} \cdot \sum_{i \in [-W:W] \setminus \{i^*_i\}} \hat{G}_i(f) \\
\leq e^{-\frac{\delta}{\text{poly}(k,d)}} \cdot 1 + e^{-\frac{\delta}{\text{poly}(k,d)}} \cdot \frac{\delta}{\text{poly}(k,d)} \\
\leq 1,
$$

where the second step applies Property II of Lemma C.4 (to the first term) and Claim D.4 (to the second term); and the last step is because $1 + z \leq e^z$ for any $z \geq 0$.

This completes the proof of Claim D.5.
Claim D.6 (Property II of Lemma D.2). $0 \leq \hat{G}(f) \leq 1$ when $\frac{1-\rho}{2\delta} \leq |f-i| \leq \frac{1}{2\delta}$ for some integer $|i| \leq W$.

Proof. The first part $\hat{G}(f) \geq 0$ follows because each summand function $\hat{G}_i(f)$ is non-negative (see Properties II to IV of Lemma C.4). For the upper-bound part, by definition we have

$$\hat{G}(f) = e^{-\frac{\delta}{\text{poly}(k,d)}} \cdot \hat{G}_i(f) + e^{-\frac{\delta}{\text{poly}(k,d)}} \cdot \sum_{i \notin [-W:W] \setminus \{i^*\}} \hat{G}_i(f)$$

$$\leq e^{-\frac{\delta}{\text{poly}(k,d)}} \cdot 1 + e^{-\frac{\delta}{\text{poly}(k,d)}} \cdot \frac{\delta}{\text{poly}(k,d)}$$

where the second step applies Property III of Lemma C.4 (to the first term) and Claim D.4 (to the second term); and the last step is because $1 + z \leq e^z$ for any $z \geq 0$.

This completes the proof of Claim D.6.

Claim D.7 (Property III of Lemma D.2). $0 \leq \hat{G}(f) \leq \frac{\delta}{\text{poly}(k,d)}$ when $|f-i| \geq \frac{1}{2\delta}$ for any integer $|i| \leq W$.

Proof. The first part $\hat{G}(f) \geq 0$ has been justified in the proof of Claim D.6. For the upper-bound part, by definition we have

$$\hat{G}(f) = e^{-\frac{\delta}{\text{poly}(k,d)}} \cdot \hat{G}_i(f) + e^{-\frac{\delta}{\text{poly}(k,d)}} \cdot \sum_{i \notin [-W:W] \setminus \{i^*\}} \hat{G}_i(f)$$

$$\leq e^{-\frac{\delta}{\text{poly}(k,d)}} \cdot \frac{\delta}{\text{poly}(k,d)} + e^{-\frac{\delta}{\text{poly}(k,d)}} \cdot \frac{\delta}{\text{poly}(k,d)}$$

$$\leq \frac{\delta}{\text{poly}(k,d)}$$

where the second step applies Property IV of Lemma C.4 (to the first term) and Claim D.4 (to the second term); and the last step follows from elementary calculation.

This completes the proof of Claim D.7.

Claim D.8 (Property IV of Lemma D.2). $G(t) = (2W+1) \cdot e^{-\frac{\delta}{\text{poly}(k,d)}} \cdot G(t) \cdot \frac{\text{sinc}(2W+1)(t)}{\text{sinc}(t)}$ for any $t \in \mathbb{R}$.

Proof. For convenient, in this proof we ignore the $\frac{0}{0}$ issue; this can be easily remedied by applying L’Hospital’s rule. According to the definition of the inverse CFT,

$$G(t) = \int_{-\infty}^{+\infty} \hat{G}(\xi) \cdot e^{2\pi i t \cdot \xi} \cdot d\xi$$

$$= e^{-\frac{\delta}{\text{poly}(k,d)}} \cdot \sum_{i \in [-W:W]} \int_{-\infty}^{+\infty} \hat{G}(\xi + i) \cdot e^{2\pi i t \cdot \xi} \cdot d\xi$$

$$= e^{-\frac{\delta}{\text{poly}(k,d)}} \cdot \sum_{i \in [-W:W]} e^{-2\pi i t \cdot i} \cdot \int_{-\infty}^{+\infty} \hat{G}(\xi + i) \cdot e^{2\pi i t \cdot (\xi + i)} \cdot d\xi$$

$$= e^{-\frac{\delta}{\text{poly}(k,d)}} \cdot \sum_{i \in [-W:W]} e^{-2\pi i t \cdot i} \cdot \int_{-\infty}^{+\infty} \hat{G}(\xi) \cdot e^{2\pi i t \cdot \xi} \cdot d\xi$$

$$\leq \int_{-\infty}^{+\infty} \hat{G}(\xi) \cdot e^{2\pi i t \cdot \xi} \cdot d\xi$$
where the second step is by Definition D.1; the fourth step is by substitution; and the last step is by the definition of the CFT.

It remains to calculate the sum of the geometric sequence $\sum_{i \in [-W:W]} e^{-2\pi i (t+1/2) i}$. Concretely, for any $\tau \in \mathbb{R}$ we have

$$\sum_{i \in [-W:W]} e^{-2\pi i t i} = e^{2\pi i t W} \cdot \left( 1 - e^{-2\pi i (2W+1)} \right)$$

$$= e^{2\pi i t W} \cdot 2i \cdot \sin(\pi t (2W + 1))$$

$$= \frac{\sin(2W+1)(t)}{\sin(t)} \cdot (2W + 1),$$

where the fourth step is by Euler’s formula (note that $\cos(z)$ is an even function while $\sin(z)$ is an odd function); and the last step applies the fact that $e^{\pi i \delta} - e^{-\pi i \delta}$ is an odd function; and the last step follows from Definition C.1.

Combining everything together completes the proof of Claim D.8. "\]

Claim D.9 (Property V of Lemma D.2). $\text{supp}(G) \subseteq \text{supp}(G) \subseteq [-2 \cdot \frac{B}{\alpha}, 2 \cdot \frac{B}{\alpha}]$. Hence: $\forall \alpha \in \mathbb{R}$.

Proof. By Claim D.8, the filter $G(t) = (2W+1) e^{-\frac{\delta}{\text{poly}(k, d)}} \cdot G(t) \cdot \frac{\sin(2W+1)(t)}{\sin(t)}$ has the same support as the function $G(t)$. Thus we immediately infer this claim from Property V of Lemma C.4.

Claim D.10 (Property VI of Lemma D.2). It follows that

$$\sum_{i \in \mathbb{Z}} G(i + 1/2)^2 = e^{-\frac{\delta}{\text{poly}(k, d)}} \cdot \sum_{i \in \mathbb{Z}} G(i + 1/2)^2 \leq (1 + 2/d) \cdot B^{-1}.$$ 

Proof. The second part of the claim is a direct follow-up to Property VII of Lemma C.4. That is,

$$e^{-\frac{\delta}{\text{poly}(k, d)}} \cdot \sum_{i \in \mathbb{Z}} G(i + 1/2)^2 \leq e^{-\frac{\delta}{\text{poly}(k, d)}} \cdot \left( 1 + \frac{\delta}{4kd} \right)^2 \cdot \left( 1 + \frac{1}{d} \right) \cdot B^{-1} \leq (1 + 2/d) \cdot B^{-1},$$

where the first step follows from Property VII of Lemma C.4; and the last step applies the fact that $e^{-z} \cdot (1 + z) \leq 1$ for any $z \in \mathbb{R}_{\geq 0}$.

To see the first part, it suffices to show that $|G(i + 1/2)| = e^{-\frac{\delta}{\text{poly}(k, d)}} \cdot |G(i + 1/2)|, \forall i \in \mathbb{Z}$. Based on Claim D.8, this equation is equivalent to

$$\left| (2W + 1) \cdot \frac{\sin(2W+1)(i + 1/2)}{\sin(i + 1/2)} \right| = 1.$$
According to Definition C.1, we have

\[
\left| (2W + 1) \cdot \frac{\text{sinc}(2W+1)(i + 1/2)}{\text{sinc}(i + 1/2)} \right| = \left| \frac{\sin(\pi \cdot (2W + 1) \cdot (i + 1/2))}{\sin(\pi \cdot (i + 1/2))} \right|
\]

\[
= \left| \frac{\sin(\pi \cdot (2i \cdot W + W + i) + \pi/2)}{\sin(\pi \cdot i + \pi/2)} \right|
\]

\[
= \left| \frac{(-1)^{2i\cdot W+i+W}}{(-1)^{i}} \right|
\]

\[= 1,
\]

where the third step follows because both \((2i \cdot W + W + i)\) and \(i\) are integers; thus we can apply certain properties of the \(\sin(z)\) function.

This completes the proof of Claim D.10.

D.4 Construction and properties of standard window \((G'(t), \hat{G}'(f))\)

Now we associate the filter \((G(t), \hat{G}(f))\) introduced in Definition D.1 with another standard window \((G'(t), \hat{G}'(f))\) (in a manner similar to Lemma C.12), which is more convenient for our later use.

Lemma D.11 (The single-dimensional standard window). For the filter \((G(t), \hat{G}(f))\) given in Definition C.3, there exists another function \((G'(t), \hat{G}'(f))\) such that:

Property I: \(\hat{G}'(f) = 1\) when \(|f - i| \leq \frac{1-\alpha}{2B}\) for some integer \(|i| \leq W\).

Property II: \(\hat{G}'(f) \in [0, 1]\) when \(\frac{1-\alpha}{2B} \leq |f - i| \leq \frac{1}{2B}\) for some integer \(|i| \leq W\).

Property III: \(\hat{G}'(f) = 0\) when \(|f - i| \geq \frac{1}{2B}\) for any integer \(|i| \leq W\).

Property IV: \(\|\hat{G}' - \hat{G}\|_\infty = \max_{f \in \mathbb{R}} |\hat{G}'(f) - \hat{G}(f)| \leq \frac{\delta}{\text{poly}(k,d)}\).

Proof. Recall Definition D.1 for the parameters \(B, \delta, \alpha, \ell\) and \(W\). We define the single-dimensional standard window \(\hat{G}'(f)\) as follows:

- \(\hat{G}'(f) = 1\) when \(|f - i| \leq \frac{1-\alpha}{2B}\) for some integer \(|i| \leq W\);
- \(\hat{G}'(f) = \hat{G}(f)\) when \(\frac{1-\alpha}{2B} \leq |f - i| \leq \frac{1}{2B}\) for some integer \(|i| \leq W\); and
- \(\hat{G}'(f) = 0\) when \(|f - i| \geq \frac{1}{2B}\) for any integer \(|i| \leq W\).

By construction, Properties I and III follows directly. Further, Property II follows from Property II of Lemma D.2, and Property IV can be inferred from Properties I to III of Lemma D.2.

This completes the proof of Lemma D.11.
Filter, permutation and hashing in multiple dimensions

Different from the previous sections, in this section \( t = (t_s)_{s \in [d]} \in \mathbb{R}^d \) and \( f = (f_r)_{r \in [d]} \in \mathbb{R}^d \) will respectively denote the \( d \)-dimensional vectors in the time domain and in the frequency domain, and \( i \in \mathbb{N}_{\geq 0}^d \) and \( j \in \mathbb{N}_{\geq 0}^d \) will denote the vector indices.

E.1 Construction of filter \((G(t), \hat{G}(f))\)

Definition E.1 (The multi-dimensional filter). Recall the parameters defined in Definition D.1:

- The number of bins in a single dimension \( B = \Theta(d \cdot k^{1/d}) \) is a certain multiple of \( d \in \mathbb{N}_{\geq 1} \).
  Over all the \( d \in \mathbb{N}_{\geq 1} \) dimensions, we have \( B = B^d = 2^{\Theta(d \log d)} \cdot k \) many bins.

- The noise level parameter \( \delta \in (0, 1) \).

- \( \alpha = \Theta(1/d) \) is chosen such that \( \frac{1}{100(d+1)-\alpha} \in \mathbb{N}_{\geq 1} \) is an integer; clearly \( \alpha \leq \frac{1}{100(d+1)} \leq \frac{1}{200} \).

- \( s_1 = \frac{2B}{\alpha} \) and \( s_2 = \frac{B}{B + B/d} \).

- \( \ell = \Theta((\log(kd/\delta)) \) is an even integer. We safely assume \( \ell \geq 1000 \).

Further, the width parameter \( W = \Omega(d \cdot B \cdot \frac{\ell}{\log(kd/\delta)}) \) is chosen to be a sufficiently large integer. Then for any \( t = (t_s)_{s \in [d]} \in \mathbb{R}^d \) and any \( f = (f_r)_{r \in [d]} \in \mathbb{R}^d \), the filter function \((G(t), \hat{G}(f))\) is given by

\[
G(t) = \prod_{s \in [d]} G(t_s) \quad \text{and} \quad \hat{G}(f) = \prod_{r \in [d]} \hat{G}(f_r),
\]

where the single-dimensional filter \((G(t_s), \hat{G}(f_r))\) is constructed according to Definition D.1, under the same parameters \( B, \delta, \alpha, s_1, s_2, \ell \) and \( W \).

Definition E.2 (Hypercube grid). Define

\[
\Lambda_W(z) := \{ f \in \mathbb{R}^d : \| f - i \|_\infty \leq z \ \text{for some vector index} \ i \in [-W : W]^d \}.
\]

This denotes the union of all the hypercubes that (for the chosen \( i \)'s) have edge length \( 2z \geq 0 \) and are centered at \( i \in [-W : W]^d \). Notice that \( \Lambda_W(z) \supseteq \Lambda_W(z') \) for any \( z \geq z' \geq 0 \).

E.2 Properties of filter \((G(t), \hat{G}(f))\)

Lemma E.3 (The multi-dimensional filter). The filter \((G(t), \hat{G}(f))\) given in Definition E.1 satisfies the following:

**Property I:** \( e^{-\frac{d}{\text{poly}(k,d)}} \cdot \delta \leq \hat{G}(f) \leq 1 \) for any \( f \in \Lambda_W(\frac{1-\alpha}{2B}) \).

**Property II:** \( \hat{G}(f) \in [0, 1] \) for any \( f \in \Lambda_W(\frac{1}{2B}) \setminus \Lambda_W(\frac{1-\delta}{4B}) \).

**Property III:** \( 0 \leq \hat{G}(f) \leq \frac{\delta}{\text{poly}(k,d)} \) for any \( f \in \mathbb{R}^d \setminus \Lambda_W(\frac{1}{2B}) \).

**Property IV:** \( \text{supp}(G) \subseteq [-\ell \cdot \frac{B}{\alpha}, \ell \cdot \frac{B}{\alpha}]^d \).

**Property V:** \( \sum_{i \in \mathbb{Z}^d} G(i)^2 \leq e^2 \cdot B^{-d} = e^2 \cdot B^{-1} \).
Figure 7: Demonstration for the filter $\hat{G}(f)$ in two dimension $d = 2$. “yellow” refers to Property I and $f \in \Lambda_W(\frac{1-\alpha}{2B})$; “blue” refers to Property II and $f \in \Lambda_W(\frac{1}{2B}) \setminus \Lambda_W(\frac{1-\alpha}{2B})$, and the other "white" region means Property III and $f \in \mathbb{R}^d \setminus \Lambda_W(\frac{1}{2B})$. $A_i = [i-1/(2B), i+1/(2B)] \times \mathbb{R}$ and $B_i = \mathbb{R} \times [i-1/(2B), i+1/(2B)]$. 
E.3 Proof of properties

Below we only present the proofs of Properties III and V, and the other properties directly follow from the corresponding properties of the single-dimensional filter \((G(t_s), \hat{G}(f_r))\) that are given in Definition D.1 and Lemma D.2.

**Claim E.4** (Property III of Lemma E.3). \(0 \leq \hat{G}(f) \leq \frac{\delta}{\text{poly}(k,d)}\) for any \(f \in \mathbb{R}^d \setminus \Lambda_W(\frac{1}{2B})\).

*Proof.* We let \(r^* \in [d]\) denote (one of) the coordinate that maximizes, over all \(r \in [d]\), the distance of \(f_r\) from the lattice \([-W : W]\). Because \(f \in \mathbb{R}^d \setminus \Lambda_W(\frac{1}{2B})\), that maximum distance is at least \(\frac{1}{2B}\). Then by construction (see Definition E.1),

\[
\hat{G}(f) = \prod_{r \in [d]} \hat{G}(f_r) \leq \hat{G}(f_{r^*}) \leq \frac{\delta}{\text{poly}(k,d)},
\]

where the second step follows because \(\hat{G}(f_r) \in [0,1]\) for each coordinate \(r \in [d] \setminus \{r^*\}\) (see Properties II to IV of Lemma D.2); and the last step follows from Property III of Lemma D.2.

This completes the proof of Claim E.4. \(\square\)

**Claim E.5** (Property V of Lemma E.3). \(\sum_{i \in \mathbb{Z}^d} G(i)^2 \leq e^2 \cdot B^{-d} = e^2 \cdot B^{-1}\).

*Proof.* Due to Definition E.1 that \(G(t) = \prod_{s \in [d]} G(t_s)\) for any \(t \in \mathbb{R}^d\), we have

\[
\sum_{i \in \mathbb{Z}^d} G(i)^2 = \sum_{i \in \mathbb{Z}^d} \left( \prod_{s \in [d]} G(i_s)^2 \right) \\
= \prod_{s \in [d]} \left( \sum_{i_s \in \mathbb{Z}} G(i_s)^2 \right) \\
\leq \prod_{s \in [d]} \left( \left(1 + \frac{2}{d}\right) \cdot B^{-1} \right) \\
= \left(1 + \frac{2}{d}\right)^d \cdot B^{-d} \\
\leq e^2 \cdot B^{-d},
\]

where the third step follows from Property VI of Lemma D.2 that \(\sum_{i \in \mathbb{Z}} G(i)^2 \leq (1 + \frac{3}{d}) \cdot B^{-1}\); and the last step follows because \((1 + \frac{1}{z})^z \leq e\) for any \(z > 0\).

This completes the proof of Claim E.5. \(\square\)

E.4 Construction and properties of standard window \((G'(t), \hat{G}'(f))\)

Now we associate our multi-dimensional filter \((G(t), \hat{G}(f))\) given in Definition E.1 with another standard window \((G'(t), \hat{G}'(f))\) in a similar manner as Lemma D.11 and the counterpart results in [HIKP12a, HIKP12b], which is more convenient for our later use.

**Lemma E.6** (The multi-dimensional standard window). Consider the filter function \((G(t), \hat{G}(f))\) given in Definition E.1, there is another function \((G'(t), \hat{G}'(f))\) such that:

- **Property I:** \(\hat{G}'(f) = 1\) for any \(f \in \Lambda_W(\frac{1-a}{2B})\).
- **Property II:** \(\hat{G}'(f) \in [0,1]\) for any \(f \in \Lambda_W(\frac{1}{2B}) \setminus \Lambda_W(\frac{1-a}{2B})\).
Property III: \( \hat{G}'(f) = 0 \) for any \( f \in \mathbb{R}^d \setminus \Lambda_W(\frac{1}{2\delta}) \).

Property IV: \( \|\hat{G}' - \hat{G}\|_\infty = \max_{f \in \mathbb{R}^d} |\hat{G}'(f) - \hat{G}(f)| \leq \frac{\delta}{\text{poly}(k,d)} \).

**Proof.** We define \( \hat{G}'(f) \) as follows; noticeably, similar to \( \hat{G}(f) \), this is also an even function in every coordinate \( r \in [d] \) given that the other \((d-1)\) coordinates are fixed:

\[
\hat{G}'(f) = \begin{cases} 
1 & \forall f \in \Lambda_W(\frac{1-\alpha}{2\delta}) \\
\hat{G}(f) & \forall f \in \Lambda_W(\frac{1}{2\delta}) \setminus \Lambda_W(\frac{1-\alpha}{2\delta}) \\
0 & \forall f \in \mathbb{R}^d \setminus \Lambda_W(\frac{1}{2\delta}) 
\end{cases}
\]

Then all the properties above can be inferred from Lemma E.3.

This completes the proof of Lemma E.6. \( \square \)

### E.5 Permutation and hashing

We adopt the following notations for convenience:

- Let \( \lfloor z \rfloor \in \mathbb{Z} \) denote the greatest integer that is less than or equal to a real number \( z \in \mathbb{R} \). In the case that \( z = (z_r)_{r=1}^d \in \mathbb{R}^d \) is a vector, we would abuse the notation \( \lfloor z \rfloor = ([z_r])_{r=1}^d \in \mathbb{Z}^d \).

- Let \( \text{frac}(z) = z - \lfloor z \rfloor \in [0, 1) \) denote the fractional part of a real number \( z \in \mathbb{R} \). In the case that \( z = (z_r)_{r=1}^d \in \mathbb{R}^d \) is a vector, we would abuse the notation \( \text{frac}(z) = ([\text{frac}(z_r)])_{r=1}^d \in [0, 1)^d \).

- Denote the set \( [n] = \{0, 1, \ldots, n-1\} \), for any positive integer \( n \in \mathbb{N} \geq 1 \).

- Let \( \overline{z} \in \mathbb{C} \) denote the conjugate of a complex number \( z \in \mathbb{C} \). Notice that \( |z|^2 = z\overline{z} \).

**Definition E.7** (Setup for permutation and hashing). We sample the random matrix \( \Sigma \in \mathbb{R}^{d \times d} \) and the random vectors \( a, b \in \mathbb{R}^d \), and define the parameter \( D \) as follows:

- The \( d \)-to-\( d \) random matrix \( \Sigma \) is constructed in two steps. First, we sample an interim matrix \( \Sigma' \sim \text{Unif}(SO(d)) \) uniformly at random from the rotation group, namely a rotation matrix of determinant either \( \det(\Sigma') = 1 \) or \( \det(\Sigma') = -1 \). Then, we define \( \Sigma := \beta \Sigma' \), where the scaling factor \( \beta \sim \text{Unif}([\sqrt{\frac{2\delta}{\beta_1}}; \sqrt{\frac{4\delta}{\beta_1}}]) \) is uniform random.

- The random vector \( a \in \mathbb{R}^d \) will be specified later in Section F.6. In this section, we only need the property of \( a \) given in Conditions E.8 and E.9, which also will be verified in Section F.6.

- The random vector \( b' = (b'_r)_{r \in [d]} \sim \text{Unif}[0, 1]^d \). Then, let \( b := \Sigma^{-1}b' \).

- The parameter \( D = \Theta(\ell/\alpha) = \Theta(d \cdot \log(kd/\delta)) \) is a sufficiently large integer. Also, let \( D := D^d \).

**Condition E.8** (Duration requirement). Given any \( i \in [BD]^d \) and any choice of the random matrix \( \Sigma \in \mathbb{R}^{d \times d} \) according to Definition E.7, any choice of \( a \) ensures that \( \Sigma^\top (i + a) \in [0, T]^d \) is within the duration.

**Condition E.9** (Sampling requirement). Given any \( i \in [BD]^d \) and any choice of the random matrix \( \Sigma \in \mathbb{R}^{d \times d} \) according to Definition E.7, the following hold for the random vector \( a \):

\[
\mathbb{E}_a \left[ g(\Sigma^\top (i + a))^2 \right] \lesssim \frac{1}{Td} \int_{t \in [0,T]^d} |g(t)|^2 \cdot dt,
\]
Definition E.10 (Hashing). Define the vector-valued function

\[ h_{\Sigma, b}(f) = \left\lfloor B \cdot \frac{1}{2B} \cdot 1 + \Sigma(f - b) \right\rfloor \in [B]^d. \]

This function “hashes” any frequency \( f \in [-F, F]^d \) into one of the \( B = B^d = 2^{\Theta(d \log d)} \cdot k \in \mathbb{N}_{\geq 1} \) bins. When \( B = \Theta(d \cdot k^{1/d}) \) is large enough, every bin \( j \in [B]^d \) is likely to have at most one heavy hitter (namely one tone frequency \( f \in \text{supp}(\hat{x}^*) \)) and if so, we can recover the tone from the hitting bins via an 1-sparse algorithm. See Figure 8 for a demonstration.

Figure 8: Demonstration for the hashing scheme (Definition E.10) in two dimensions \( d = 2 \), where \( p, q \in \mathbb{Z} \) are integers. The unit square \([p, p + 1) \times [q, q + 1)\) are divided into \( B = B^2 \) subsquares, and the subsquare in which the frequency \( f \) is hashed into, is exactly the index \( h_{\Sigma, b}(f) \in [B]^d \).

Definition E.11 (Offset). Define the vector-valued function

\[ o_{\Sigma, b}(f) = \frac{1}{2B} \cdot 1 + \Sigma(f - b) - \frac{1}{2B} \cdot 1 \in \left[ -\frac{1}{2B}, \frac{1}{2B} \right]^d, \]

which measures the coordinate-wise distance from the center of the \( h_{\Sigma, b}(f) \)-th bin to \( f \in [-F, F]^d \).

Definition E.12 (Collision). Consider a tone frequency \( f \in \text{supp}(\hat{x}^*) \), the event \( E_{\text{coll}}(f) \) occurs when \( h_{\Sigma, b}(f') = h_{\Sigma, b}(f) \) for some other tone frequency \( f' \in \text{supp}(\hat{x}^*) \setminus \{f\} \), namely both \( f \neq f' \in \text{supp}(\hat{x}^*) \) are hashed into the same bin. In this case, the algorithm cannot recover the two collided tone frequencies \( f \neq f' \). See Figure 9a for a demonstration.

Definition E.13 (Large offset). Consider a tone frequency \( f \in \text{supp}(\hat{x}^*) \), the event \( E_{\text{off}}(f) \) occurs when \( \|o_{\Sigma, b}(f)\|_\infty \geq \frac{1 - \alpha}{2B} \), i.e. the frequency \( f \in \text{supp}(\hat{x}^*) \) locates on the boundary of the \( h_{\Sigma, b}(f) \)-th bin. In this case, the algorithm also cannot recover the tone frequency \( f \). See Figure 9b for a demonstration.

Our multi-dimensional permutation scheme is a natural generalization of the single-dimensional one by [PS15, Definition A.5].

Definition E.14 (Multi-dimensional permutation). Let \( \mathcal{P}_{\Sigma, b, a} x(t) = x(\Sigma^\top(t + a)) \cdot e^{-2\pi i b^\top \Sigma^\top t} \) for any \( t \in \mathbb{R}^d \).

Lemma E.15 (Identities). The permutation given in Definition E.14 satisfies that:
Figure 9: Demonstration for the bad events “collision” (Definition E.12) and “large offset” (Definition E.13) in two dimensions $d = 2$. In Figure 9a, the two frequencies $f \neq f' \in \text{supp}(\hat{x})$ may be hashed into two different unit squares (i.e. possibly either $p \neq p'$ or $q \neq q'$ or both), but it is always the case that the two subsquares have the same index $h_{\Sigma,b}(f) = h_{\Sigma,b}(f') \in [B]^d$. In Figure 9b, the red region (that gives a large offset) covers $1 - (1 - \alpha)^2$ fractions of the whole plane.

**Property I:** $\widehat{P_{\Sigma,b,a}x}(\Sigma(f-b)) = \hat{x}(f) \cdot \det(\Sigma)^{-1} \cdot e^{2\pi i a^\top \Sigma f}$ for any $f \in \mathbb{R}^d$.

**Property II:** $\widehat{P_{\Sigma,b,a}x}(t) = \hat{x}(\Sigma^{-1}t + b) \cdot \det(\Sigma)^{-1} \cdot e^{2\pi i a^\top (t + \Sigma^\top b)}$ for any $t \in \mathbb{R}^d$.

**Proof.** For Property I, by the definition of the CFT, the LHS equals

$$
\widehat{P_{\Sigma,b,a}x}(\Sigma(f-b)) = \int_{t \in \mathbb{R}^d} P_{\Sigma,b,a}x(t) \cdot e^{-2\pi i (f^\top - b^\top) \Sigma^\top t} \cdot dt
$$

$$
= \int_{t \in \mathbb{R}^d} x((\Sigma^\top (t + a)) \cdot e^{-2\pi i b^\top \Sigma^\top t} \cdot e^{-2\pi i (f^\top - b^\top) \Sigma^\top t} \cdot dt
$$

$$
= e^{2\pi i f^\top \Sigma^\top a} \cdot \int_{t \in \mathbb{R}^d} x((\Sigma^\top (t + a)) \cdot e^{-2\pi i f^\top \Sigma^\top (t + a)) \cdot dt
$$

$$
= e^{2\pi i f^\top \Sigma^\top a} \cdot \det(\Sigma)^{-1} \cdot \int_{\tau \in \mathbb{R}^d} x(\tau) \cdot e^{-2\pi i f^\top \tau} \cdot d\tau
$$

$$
= e^{2\pi i f^\top \Sigma^\top a} \cdot \det(\Sigma)^{-1} \cdot \hat{x}(f)
$$

$$
= e^{2\pi i a^\top \Sigma f} \cdot \det(\Sigma)^{-1} \cdot \hat{x}(f),
$$

where the second step follows from Definition E.14; the fourth step is by substitution; and the last step is by the definition of the CFT.

We directly infer Property II from Property I by substitution. Lemma E.15 follows then. \qed

### E.6 HashToBins: algorithm

**Fact E.16** (Identities under DFT). The following holds for each $j \in [B]^d$:

$$
\widehat{u}_j = \widehat{y}_{Dj} = \hat{G} \ast \widehat{P_{\Sigma,b,a}x}(B^{-1} \cdot j).
$$

**Proof.** It is noteworthy that $y = (y_j)_{j \in [BD]^d}$ is a $(BD)^d$-dimensional vector, and $u = (u_j)_{j \in [B]^d}$ is a $B^d$-dimensional vector. For the first equality $\widehat{u}_j = \widehat{y}_{Dj}$, due to the definition of the DFT,

$$
\widehat{u}_j = \sum_{i \in [B]^d} u_i \cdot e^{-2\pi i j^\top i}
$$

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Algorithm 1 HashToBins in multiple dimensions

1: procedure HashToBins($\Sigma, b, a, D$)
2: Define $(\mathcal{G}(t), \hat{\mathcal{G}}(f))$ according to Definition D.1.
3: Define $\mathcal{P}_{\Sigma, b, a}$ according to Definition E.14.
4: Define $y = (y_j)_{j \in [BD]^d}$, where $y_j = \mathcal{G}(j) \cdot \mathcal{P}_{\Sigma, b, a} x(j)$.
5: Define $u = (u_j)_{j \in [B]^d}$, where $u_j = \sum_{i \in [D]^d} y_{Bi+j}$.
6: return the DFT $\hat{u} = (\hat{u}_j)_{j \in [B]^d}$.
7: end procedure

\[
\begin{align*}
    &= \sum_{i \in [BD]^d} \sum_{l \in [D]^d} y_{Bl+i} \cdot e^{-\frac{2\pi i}{B} j^\top l} \\
    &= \sum_{i \in [BD]^d} \sum_{l \in [D]^d} y_{Bl+i} \cdot e^{-\frac{2\pi i}{B} (Bl+i)^\top} \\
    &= \sum_{i \in [BD]^d} y_i \cdot e^{-\frac{2\pi i}{B} j^\top i} \\
    &= \sum_{i \in [BD]^d} y_i \cdot e^{-\frac{2\pi i}{B} (Dj)^\top i} \\
    &= \hat{y}_{Dj},
\end{align*}
\]

where the second step is by Line 5 of HashToBins; the third step follows since both $j \in [B]^d$ and $l \in [D]^d$ are $d$-dimensional integer vectors and therefore, $e^{-\frac{2\pi i}{B} (Bl)^\top} = e^{-2\pi i j^\top l} = 1$; the fourth step is by substitution; and the last step also applies the DFT.

For the second equality, again we know from the definition of the DFT that

\[
\begin{align*}
    \hat{y}_{Dj} &= \sum_{i \in [BD]^d} y_i \cdot e^{-\frac{2\pi i}{B} (Dj)^\top i} \\
    &= \sum_{i \in [BD]^d} \mathcal{G}(i) \cdot \mathcal{P}_{\Sigma, b, a} x(i) \cdot e^{-\frac{2\pi i}{B} j^\top i} \\
    &= \sum_{i \in \mathbb{Z}^d} \mathcal{G}(i) \cdot \mathcal{P}_{\Sigma, b, a} x(i) \cdot e^{-\frac{2\pi i}{B} j^\top i} \\
    &= \hat{G} \cdot \mathcal{P}_{\Sigma, b, a} (B^{-1} \cdot j) \\
    &= \hat{G} \cdot \mathcal{P}_{\Sigma, b, a} x(B^{-1} \cdot j),
\end{align*}
\]

where the second step is by Line 4 of HashToBins; the third step follows because $\mathcal{G}(t) = 0$ when $\|t\|_\infty \geq \ell \cdot B/\alpha$ (see Lemma E.3), given a large enough $D = \Theta(d \cdot \log(kd/\delta))$ (see Definition E.7); and the fourth step is by the definition of the DTFT.

This completes the proof of Fact E.16. \qed

Fact E.17 (Sample complexity and time complexity). The procedure HashToBins takes $O(BD) = 2^{O(d \cdot \log d)} \cdot k \cdot \log^2(k/\delta)$ samples and runs in $O(BD + B \log B) = 2^{O(d \cdot \log d)} \cdot k \cdot \log^2(kd/\delta)$ time.

Proof. Recall that $\mathcal{B} = B^d = 2^{\Theta(d \cdot \log d)} \cdot k$ (Definition E.1) and $\mathcal{D} = D^d = \log^d(kd/\delta)$. The sample complexity is easy to see, because we have exactly $\mathcal{B}$ bins, and each bin $j \in [\mathcal{B}]$ requires exactly $\mathcal{D}$ samples in Line 5 of HashToBins.
The $BD$-dimensional vector $y = (y_j)_{j \in [BD]^d}$ can be computed $O(BD)$ time (assuming $O(1)$-time query oracles to evaluating the filter $(G(t), \hat{G}(f))$ and to sampling the signal $x(t)$; see Remark D.3). Furthermore, the $B$-dimensional vector $u = (u_j)_{j \in [B]^{d}}$, where $u_j = \sum_{i \in [D]} y_{Bi+j}$, can be computed in $O(BD)$ time. Then we can derive its DFT $\hat{u} = (\hat{u}_j)_{j \in [B]^{d}}$ through any FFT algorithm in $O(B \log B)$ time. The claimed time complexity follows as well. \hfill $\square$

### E.7 HashToBins: probabilities of bad events

**Lemma E.18** (Probability of collision). Consider the random matrix $\Sigma \in \mathbb{R}^{d \times d}$ and the random vector $b \in \mathbb{R}^{d}$ given in Definition E.7, for any pair of tone frequencies $f \neq f' \in \text{supp}(\hat{x})$, the probability of collision

$$\Pr_{\Sigma, b} [h_{\Sigma, b}(f) = h_{\Sigma, b}(f')] \leq 0.01 \cdot k^{-1}.$$  

**Proof.** Recall Assumption 1.1 that $\eta = \min\{\|f - f'\|_2 : f \neq f' \in \text{supp}(\hat{x})\}$ is the minimum $\ell_2$-distance between any pair of tone frequencies. Due to Definition E.10, two distinct tone frequencies $f \neq f' \in \text{supp}(\hat{x})$ are hashed into bins

$$h_{\Sigma, b}(f) = \left[ B \cdot \frac{1}{2B} \cdot 1 + \Sigma(f - b) \right]$$

$$h_{\Sigma, b}(f') = \left[ B \cdot \frac{1}{2B} \cdot 1 + \Sigma(f - b) + \Sigma(f' - f) \right].$$
In order to hash $f$ and $f'$ into the same bin, a necessary condition (under any realization of the random vector $b$) is that $\|\Sigma(f' - f) - i\|_\infty \leq \frac{1}{B}$; in other words, $\Sigma(f' - f)$ locates in a hypercube that is centered at some integer vector $i = (ir)_{r \in [d]} \in \mathbb{Z}^d$ and has edge length $\frac{2}{B}$.

Given the above two equations and as Figure 10 suggests, a necessary condition for the collision $k_{\Sigma,b}(f) = k_{\Sigma,b}(f')$ is that $\|\Sigma(f' - f) - i\|_\infty < \frac{2}{B}$ for some integer vector $i \in \mathbb{Z}^d$. Below we upper bound this probability based on case analysis.

Case (i): when $\eta \leq \|f - f'\|_2 \leq \frac{B - 2}{4\sqrt{d}} \cdot \eta$. Recall Definition E.7 that $\Sigma$ is a rotation matrix scaled by a random factor $\beta \sim \text{Unif}[\frac{2\sqrt{d}}{B\eta}, \frac{4\sqrt{d}}{B\eta}]$. Given this, we have $\|\Sigma(f - f')\|_2 = \beta \cdot \|f - f'\|_2$. Further, since $\frac{2\sqrt{d}}{B\eta} \leq \beta \leq \frac{4\sqrt{d}}{B\eta}$ and $\eta \leq \|f - f'\|_2 \leq \frac{B - 2}{4\sqrt{d}} \cdot \eta$, we have

$$\|\Sigma(f - f')\|_2 \geq \frac{2\sqrt{d}}{B\eta} \cdot \eta = \frac{2\sqrt{d}}{B},$$
$$\|\Sigma(f - f')\|_2 \leq \frac{4\sqrt{d}}{B\eta} \cdot \frac{B - 2}{4\sqrt{d}} \cdot \eta = 1 - \frac{2}{B}.$$ 

Given these, we can easily see that $\|\Sigma(f' - f) - i\|_\infty \geq 2/B$ for any integer vector $i \in \mathbb{Z}^d$. Namely, the collision never occurs in this case.

Case (ii): when $\|f - f'\|_2 > \frac{B - 2}{4\sqrt{d}} \cdot \eta$.

As for the case (II), for simplicity, let $r \geq 1 - \frac{2\sqrt{d}}{B}$ denote $\|f - f'\|_2 \cdot (\frac{2d}{B\eta})$, then we know $\|\Sigma(f - f')\|_2$ is distributed uniformly on $[r, 2r]$. Let $V_d(r) = \frac{d^{d/2}}{\Gamma(d/2 + 1)} \cdot (r)^d$ represent the volume of $d$-dimensional Euclidean ball of radius $r$ and $S_d(r) = \frac{2\pi^{d/2}}{\Gamma(\frac{d}{2})} r^{d-1}$ represent the surface of area. The probability density function for $\Sigma(f - f') = x$ is $\text{PDF}(x) = \frac{1}{r \cdot S_d(||x||_2)}$ for $r \leq ||x||_2 \leq 2r$. Then the probability for $\Sigma(f - f')$ falls into any region with volume $V$ at most

$$\frac{V}{r \cdot S_d(r)} \leq \frac{V \cdot \Gamma(d/2)}{(\sqrt{\pi})^d}.$$ 

Then we have

$$\Pr_{\Sigma,b} \left[ k_{\Sigma,b}(f) = k_{\Sigma,b}(f') \right] \leq \left(2 \cdot \left[r + \frac{2\sqrt{d}}{B}\right]\right)^d \cdot \left(\frac{2}{B}\right)^d \cdot \frac{\Gamma(d/2)}{(\sqrt{\pi})^d} \leq 2^{O(d)} \cdot \frac{B^d}{\sqrt{\pi}} \cdot \frac{d}{2} \leq 2^{O(d \log d)} \frac{1}{B},$$

where the first step is because that $\Sigma(f' - f)$ must locate in a hypercube that is centered at some integer vector and has edge length $2/B$, and there are at most $\left(2 \cdot \left[r + \frac{2\sqrt{d}}{B}\right]\right)^d$ different hypercubes in the sphere. The second step is by the Stirling’s approximation.

This completes the proof of Lemma E.18.

**Lemma E.19 (Probability of large offset).** Consider the random matrix $\Sigma \in \mathbb{R}^{d \times d}$ and the random vector $b \in \mathbb{R}^d$ given in Definition E.7, for any tone frequency $f \in \text{supp}(\mathcal{X})$, the probability of large offset

$$\Pr_{\Sigma,b} \left[ \text{Eoff}(f) \right] = \Pr_{\Sigma,b} \left[ \|\Sigma_b(f)\|_\infty \geq \frac{1 - \alpha}{2B} \right] = 1 - (1 - \alpha)^d \leq 0.01.$$
Proof. Recall (see Definition E.10) that
\[ h_{\Sigma, b}(f) = \lfloor B \cdot \text{frac}(\frac{1}{2B} \cdot 1 + \Sigma(f - b)) \rfloor \]
and (see Definition E.11) that
\[ \alpha_{\Sigma, b}(f) = \text{frac}(\frac{1}{2B} \cdot 1 + \Sigma(f - b)) - \frac{1}{B} \cdot h_{\Sigma, b}(f) - \frac{1}{2B} \cdot 1 \in [-\frac{1}{2B}, \frac{1}{2B})^d. \]

It can be seen that a large offset \(|\alpha_{\Sigma, b}(f)|_\infty \geq \frac{\alpha}{2B}\) occurs if and only if \(\frac{1}{2B} \cdot 1 + \Sigma(f - b) \notin S^d\), where (in each coordinate) the union of intervals
\[ S = \bigcup_{j \in [B]} \left( \frac{j}{B} + \frac{\alpha/2}{B}, \frac{j}{B} + \frac{1-\alpha/2}{B} \right) \subseteq [0, 1). \]

Indeed, for any choice of \(\Sigma\) according to Definition E.7, because \(b \in \mathbb{R}^d\) is uniformly random, each \(i\)-th coordinate of the vector \((\frac{1}{2B} \cdot 1 + \Sigma(f - b))\) is distributed uniformly on an interval of length \(|\text{supp}(b)| = 1 \in \mathbb{N}_{\geq 1}\). Accordingly, each \(i\)-th coordinate of the fractional part \(\text{frac}(\frac{1}{2B} \cdot 1 + \Sigma(f - b))\) must be distributed independently and uniformly on \([0, 1)\).

To summarize, the conditional probability \(\Pr_b[E_{\text{off}}(f) \mid \Sigma]\) always equals the probability that, an \textit{coordinate-wise independent} uniform random vector \(b \sim \text{Unif}[0, 1)^d\) locates outside the region \(S^d \subseteq [0, 1)^d\). Thus, for any realized \(\Sigma\) given by Definition E.7, we have
\[ \Pr_b[E_{\text{off}}(f) \mid \Sigma] = 1 - |S|^d = 1 - \left( \left( \frac{1-\alpha/2}{B} - \frac{\alpha/2}{B} \right) \cdot B \right)^d = 1 - (1-\alpha)^d. \]

Since \(0 < \alpha \leq \frac{1}{100(d+1)} < 1\) (see Definition E.1), we also have \(1 - (1-\alpha)^d \leq d \cdot \alpha \leq \frac{d}{100(d+1)} \leq 0.01\). This completes the proof of Lemma E.19. \(\Box\)

Lemma E.20 (Hashing into same bin). For any tone frequency \(f \in \text{supp}(\hat{x}^*)\), if the event \(E_{\text{off}}(f)\) does not happen, then \(h_{\Sigma, b}(f) = h_{\Sigma, b}(f')\) for any other frequency \(f' \in \mathbb{R}^d\) that
\[ \|f - f'\|_2 < \frac{\alpha}{8\sqrt{d}} \cdot \eta = \Theta(d^{-1.5} \cdot \eta). \]

Proof. Since the event \(E_{\text{off}}(f)\) does not happen, the offset \(|\alpha_{\Sigma, b}(f)|_\infty < \frac{\alpha}{2B}\) (see Definitions E.13 and E.11). Given this, by construction (see Definition E.10) a sufficient condition for the function \(h_{\Sigma, b}\) to hash \(f\) and \(f'\) into the same bin is \(\|\Sigma(f - f')\|_\infty < \frac{\alpha}{2B}\). We verify this condition as follows:
\[ \|\Sigma(f - f')\|_\infty \leq \|\Sigma(f - f')\|_2 \leq \frac{4\sqrt{d}}{B\eta} \cdot \|f - f'\|_2 < \frac{4\sqrt{d}}{B\eta} \cdot \alpha \leq \frac{\alpha}{8\sqrt{d}} \cdot \eta = \frac{\alpha}{2B}, \]

where the second step follows from Definition E.7, i.e. \(\Sigma\) is a rotation matrix scaled by a random factor \(\beta \sim \text{Unif}[\frac{4\sqrt{d}}{B\eta}, \frac{4\sqrt{d}}{B\eta}];\) and the third step follows from our premise \(\|f - f'\|_2 < \frac{\alpha}{8\sqrt{d}} \cdot \eta\).

This completes the proof. \(\Box\)
E.8 HashToBins: error due to noise

Lemma E.21 (The error due to noise). Suppose that Condition E.8 is true for the random vector $a \in \mathbb{R}^d$ and that $x^*(t) = 0$ for any $t \in [0, T]^d$, then the following holds for any random matrix $\Sigma \in \mathbb{R}^{d \times d}$ given in Definition E.7:

$$E_{b,a}[\|\hat{u}\|^2_2] \leq \frac{1}{T^d} \cdot \int_{\tau \in [0, T]^d} |g(\tau)|^2 \cdot \text{d}\tau.$$  

Proof. We know from Parseval’s theorem that $\|\hat{u}\|^2_2 = B \cdot \|u\|^2_2 = B \cdot \sum_{j \in [B]^d} |u_j|^2$. To see the lemma, let us consider a specific coordinate $|u_j|^2$. By definition (see Line 5 of HashToBins),

$$E_{b,a}[|u_j|^2] = E_{b,a}\left[\sum_{i \in [D]^d} y_{Bi+j}\right]^2$$

$$= E_{b,a}\left[\sum_{i \in [D]^d} y_{Bi+j} \cdot \sum_{i \in [D]^d} \overline{y}_{Bi+j}\right]$$

$$= \sum_{i \in [D]^d} E_{b,a}[y_{Bi+j} \cdot \overline{y}_{Bi+j}] + \sum_{i \neq i' \in [D]^d} E_{b,a}[y_{Bi+j} \cdot \overline{y}_{Bi'+j}], \quad (23)$$

where the second step follows as $|z|^2 = z\overline{z}$ for any complex number $z \in \mathbb{C}$; and the last step follows from the linearity of expectation.

As a premise of the current lemma, the signal $x(t) = x^*(t) + g(t)$ for any $t \in [0, T]^d$. Due to Line 4 of HashToBins, for any pair $i \in [D]^d$ and any $j \in [B]^d$ we have

$$y_{Bi+j} = G(Bi + j) \cdot P_{\Sigma, b, a} g(Bi + j)$$

$$= G(Bi + j) \cdot g(\Sigma^\top (Bi + j + a)) \cdot e^{-2\pi i b^\top (Bi+j)}$$

$$= S_{i,j} \cdot e^{-2\pi i b^\top (Bi+j)},$$

where the second step is by Definition E.14; and in the last step we denote

$$S_{i,j} = G(Bi + j) \cdot g(\Sigma^\top (Bi + j + a))$$

for ease of notation. Notice that $S_{i,j} \in \mathbb{R}$ is a real number and is determined by the random matrix $\Sigma \in \mathbb{R}^{d \times d}$ and the random vector $a \in \mathbb{R}^d$ (see Definition E.7).

We can reformulate the first term in Equation (23) as follows: for each $i \in [D]^d$,

$$E_{b,a}[y_{Bi+j} \cdot \overline{y}_{Bi+j}] = E_a[S_{i,j} \cdot \overline{S}_{i,j}] = E_a[S_{i,j}^2]$$

Indeed, the second term in Equation (23) equals zero. Particularly, for any $i \neq i' \in [D]^d$ we have

$$E_{b,a}[y_{Bi+j} \cdot \overline{y}_{Bi'+j}] = E_{b,a}[S_{i,j} \cdot \overline{S}_{i',j} \cdot e^{-2\pi i B \cdot b^\top (i-i')}],$$

$$= E_a[S_{i,j} \cdot \overline{S}_{i',j}] \cdot E_{b,a}[e^{-2\pi i B \cdot b^\top (i-i')}], \quad (24)$$

where the second step follows since $b \in \mathbb{R}^d$ and $S_{i,j}$ are independent (see Definition E.7).

Let us investigate the second term in Equation (24). We have $b^\top (i - i') = \sum_{r \in [d]} b_r \cdot (i_r - i'_r)$, in which at least one summand is non-zero (since $i \neq i \in [D]^d$). Since both of $B$ and $i_r - i'_r$ are
integers and each coordinate \( b_r \sim \text{Unif}[0,1] \) is independently, the fractional part \( \text{frac}(B \cdot b^\top (i - i')) \) must follow the distribution \( \text{Unif}[0,1] \). Accordingly, the second term in Equation (24) is equal to

\[
E_b \left[ e^{-2\pi i B \cdot b^\top (i - i')} \right] = 0,
\]

Applying all of the above arguments to Equation (23) leads to

\[
E_{b,a}[|u_j|^2] = \sum_{i\in [D]^d} E_a[S^2_{i,j}].
\]

Taking all vector indices \( j \in [B]^d \) into account, we infer that

\[
E_{b,a}\left[ ||\hat{u}||^2_2 \right] = B \cdot \sum_{j\in [B]^d} E_{b,a}\left[ |u_j|^2 \right]
\]

\[
= B \cdot \sum_{j\in [B]^d} \sum_{i\in [D]^d} E_a[S^2_{i,j}]
\]

\[
= B \cdot \sum_{j\in [B]^d} \sum_{i\in [D]^d} E_a\left[ G(Bi + j)^2 \cdot g((\Sigma^\top (Bi + j + a))^2) \right]
\]

\[
= B \cdot \sum_{i\in [BD]^d} G(i)^2 \cdot E_a\left[ g((\Sigma^\top (i + a))^2) \right],
\]

where the third step is by the definition of \( S_{i,j} \); and the last step is by substitution.

Due to Condition E.8, given any \( i \in [BD]^d \) and any choice of the random matrix \( \Sigma \in \mathbb{R}^{d \times d} \) according to Definition E.7, the random vector \( a \in \mathbb{R}^d \) satisfies that

\[
E_a\left[ g((\Sigma^\top (i + a))^2) \right] \lesssim \frac{1}{T^d} \cdot \int_{t\in [0,T]^d} |g(t)|^2 \cdot dt,
\]

Plugging the above equation into Equation (25) gives

\[
E_{b,a}\left[ ||\hat{u}||^2_2 \right] \lesssim \frac{1}{T^d} \cdot \int_{t\in [0,T]^d} |g(t)|^2 \cdot \left( B \cdot \sum_{i\in [BD]^d} G(i)^2 \right)
\]

\[
\lesssim \frac{1}{T^d} \cdot \int_{t\in [0,T]^d} |g(t)|^2 \cdot dt,
\]

where the last step follows from Property V of Lemma E.3 that \( \sum_{i\in \mathbb{Z}^d} G(i)^2 \leq e^2 \cdot B^{-1} \).

This completes the proof of Lemma E.21. \( \square \)

### E.9 HashToBins: error due to bad events

**Lemma E.22** (The error due to bad events). Suppose \( g(t) = 0 \) for any \( t \in [0,T]^d \). Given the hash function \( h_{\Sigma,b} \) under any \( \Sigma \in \mathbb{R}^{d\times d} \) and any \( b \in \mathbb{R}^d \) (according to Definition E.7), denote by

\[
H = \{ f \in \text{supp}(\hat{x}^\Sigma) : \text{neither } E_{\text{coll}}(f) \text{ nor } E_{\text{off}}(f) \text{ happens} \}
\]

the set of “good” tone frequencies. Then the following hold:

\[
\forall f \in H : \quad E_a\left[ |\hat{u}_{h_{\Sigma,b}}(f) - \hat{x}^\Sigma[f] \cdot e^{2\pi i a^\top \Sigma f}|^2 \right] \leq \frac{\delta}{\text{poly}(k,d)} \cdot ||\hat{x}^\Sigma||^2_1,
\]

\[
\sum_{f \in H} E_a\left[ |\hat{u}_{h_{\Sigma,b}}(f) - \hat{x}^\Sigma[f] \cdot e^{2\pi i a^\top \Sigma f}|^2 \right] \leq \frac{\delta}{\text{poly}(k,d)} \cdot ||\hat{x}^\Sigma||^2_1,
\]
Proof. As promised by the lemma, the signal $x(t) = x^*(t) + g(t) = x^*(t)$ for any $t \in [0, T]^d$. For a specific frequency $f' \in H$, w.l.o.g. we assume that $f'$ is hashed into the $j$-th bin, namely $b_{\Sigma, b}(f') = j$ (see Definition E.10). According to Property II of Fact E.16,

$$\hat{u}_j = \hat{G} \ast \mathcal{P}_{\Sigma, b, a} x^*(B^{-1} \cdot j) = \mathcal{G}' \ast \mathcal{P}_{\Sigma, b, a} x^*(B^{-1} \cdot j) + (\hat{G} - \hat{G}') \ast \mathcal{P}_{\Sigma, b, a} x^*(B^{-1} \cdot j).$$

For the second summand in Equation (26), the corresponding function admits the $\ell_\infty$ norm of

$$\| (\hat{G} - \hat{G}') \ast \mathcal{P}_{\Sigma, b, a} x^* \|_\infty \leq \| (\hat{G} - \hat{G}') \|_\infty \cdot \| \mathcal{P}_{\Sigma, b, a} x^* \|_1 \leq \frac{\delta}{\text{poly}(k, d)} \cdot \| \mathcal{P}_{\Sigma, b, a} x^* \|_1$$

where the second step is due to Property IV of Lemma E.6.

We then have

$$\| \mathcal{P}_{\Sigma, b, a} x^* \|_1 = \int_{x \in \mathbb{R}^d} \left| \hat{x}^*(\Sigma^{-1} z + b) \cdot \det(\Sigma)^{-1} \cdot e^{2\pi i a^T (z + \Sigma b)} \right| \cdot \mathrm{d}z$$

$$= \int_{x \in \mathbb{R}^d} \left| \hat{x}^*(\Sigma^{-1} z + b) \right| \cdot \det(\Sigma)^{-1} \cdot \mathrm{d}z$$

$$= \int_{x \in \mathbb{R}^d} \| \hat{x}^*(\xi) \| \cdot \mathrm{d}\xi$$

$$= \| x^* \|_1,$$

where the first step applies Property II of Lemma E.15; the second step follows since $|e^{i\theta}| = 1$ for any $\theta \in \mathbb{R}$ and $\det(\Sigma) \neq 0$ (Definition E.7); and the third step is by substitution.

Putting the equations together, we get

$$\| (\hat{G} - \hat{G}') \ast \mathcal{P}_{\Sigma, b, a} x^* \|_\infty \leq \frac{\delta}{\text{poly}(k, d)} \cdot \| \hat{x}^* \|_1.$$  \hspace{1cm} (27)

Moreover, the first summand in Equation (26) equals

$$\mathcal{G}' \ast \mathcal{P}_{\Sigma, b, a} x^*(B^{-1} \cdot j) = \int_{\xi \in \mathbb{R}^d} \hat{G}'(B^{-1} \cdot j - \xi) \cdot \mathcal{P}_{\Sigma, b, a} x^*(\xi) \cdot \mathrm{d}\xi$$

$$= \int_{\xi \in \mathbb{R}^d} \hat{G}'(B^{-1} \cdot j - \xi) \cdot \hat{x}^*(\Sigma^{-1} \xi + b) \cdot \det(\Sigma)^{-1} \cdot e^{2\pi i a^T (\xi + \Sigma b)} \cdot \mathrm{d}\xi$$

$$= \int_{\xi \in \mathbb{R}^d} \hat{G}'(B^{-1} \cdot j - \Sigma (\xi - b)) \cdot \hat{x}^*(\xi) \cdot e^{2\pi i a^T \xi} \cdot \mathrm{d}\xi,$$  \hspace{1cm} (28)

where the first step applies the convolution operation; the second step follows from Property II of Lemma E.15; and the third step is by substitution.

Notably (see Properties I to III of Lemma E.6 and Definition E.2), the standard window $\hat{G}'(\xi)$ is supported within the hypercube grid

$$\Lambda_{\Sigma}(\frac{1}{2B}) = \{ \xi \in \mathbb{R}^d : \| \xi - i \|_\infty \leq \frac{1}{2B} \text{ for some vector index } i \in [-W : W]^d \}.$$

Recall that $\Sigma$ is a random rotation matrix scaled by a random factor $\beta \sim \text{Unif}[\frac{2\sqrt{d}}{B\eta}, \frac{4\sqrt{d}}{B\eta}]$ (see Definition E.7). Further, the Fourier spectrum $\text{supp}(\hat{x}^*) \subseteq [-F, F]^d$ is bounded. Under any choice
of the random matrix $\Sigma$ of the random vector $b \in \mathbb{R}^d$ (see Definition E.7), for any $j \in [B]^d$ and any $\xi \in [-F, F]^d$ we have

$$\|B^{-1} \cdot j - \Sigma(\xi - b)\|_\infty \lesssim \|\Sigma(F, F, \ldots, F)^\top\|_\infty \leq \|\Sigma(F, F, \ldots, F)^\top\|_2 \leq \frac{4\sqrt{d}}{B\eta} \cdot \|(F, F, \ldots, F)^\top\|_2 \lesssim d \cdot \frac{F}{B\eta}.$$  

Thus, a sufficiently large width parameter $W = \Theta(d \cdot \frac{F}{B\eta})$ (see Definition E.1) guarantees that

$$\left\{ \frac{1}{B} \cdot j - \Sigma(\xi - b) : \xi \in [-F, F]^d \right\} \subseteq [-W, W]^d,$$

for any choice of $\Sigma$ and $b$ (according to Definition E.7) and any $j \in [B]^d$.

Given the above arguments and as Figure 11 suggests, Equation (28) suffices to integrate the tone frequencies hashed into the $j$-th bin, namely

$$\Phi_j = \left\{ \xi \in \text{supp}(\hat{x}^*) : \|B^{-1} \cdot j - \Sigma(\xi - b) - i\|_\infty \leq \frac{1}{2B} \text{ for some } i \in \mathbb{Z}^d \right\} = \left\{ \xi \in \text{supp}(\hat{x}^*) : -\frac{1}{2B} \cdot 1 \preceq B^{-1} \cdot j - \Sigma(\xi - b) - i \preceq \frac{1}{2B} \cdot 1 \text{ for some } i \in \mathbb{Z}^d \right\}.$$
This equation, together with Equation (26) and Equation (27), implies that
\[ f \]
where the last step is by \( \Phi_j = \{ \xi \in \text{supp}(\hat{x}) : h_{\Sigma,b}(\xi) = j \} \).

In the above condition, \( \frac{1}{2B} \cdot 1 + \Sigma(\xi - b) + i \) must be bounded within \([0,1]^d\), as the concerning bin \( j \in [B]^d = \{0,1,\ldots,B-1\}^d \). In particular, the case \( \| \frac{1}{2B} + \Sigma(\xi - b) + i \|_\infty = 1 \) occurs with zero probability, since \( b \in \mathbb{R}^d \) follows a continuous uniform distribution (Definition E.7); we safely ignore this case. Given the hash function \( h_{\Sigma,b}(\xi) \in [B]^d \) in Definition E.10, we conclude that
\[
\Phi_j = \{ \xi \in \text{supp}(\hat{x}) : h_{\Sigma,b}(\xi) = j \}.
\]
The concerning tone frequency \( f' \in H \) ensures that neither \( E_{\text{coll}}(f) \) nor \( E_{\text{off}}(f) \) happens:
- \( E_{\text{coll}}(f') \) does not happen. No other tone frequencies \( f \in \text{supp}(\hat{x}) \setminus \{f'\} \) collide with \( f' \) after the hashing. That is, the \( j \)-th bin contains \( f' \) as the only tone frequency, namely \( \Phi_j = \{f'\} \).
- \( E_{\text{off}}(f') \) does not happen. The offset \( \| \alpha_{\Sigma,b}(f') \|_\infty < \frac{1-\alpha}{2B} \) is small enough, namely the frequency \( f' \) lies within the hypercube grid \( \Lambda_{W}(\frac{1}{2B}) \). We know from Property I of Lemma D.11 that
\[
\tilde{G}'(B^{-1} \cdot j - \Sigma(f' - b)) = 1
\]
Also, recall Observation A.1 that \( \hat{x}(\xi) \) is the combination of \( k \) many scaled \( d \)-dimensional Dirac delta functions (at the tone frequencies \( \xi \in \text{supp}(\hat{x}) \)). In precise, for any frequency \( \xi \in [-F,F]^d \),
\[
\hat{x}(\xi) = \sum_{f \in \text{supp}(\hat{x})} \hat{x}[f] \cdot \text{Delta}_f(\xi).
\]
Applying all of the above arguments to Equation (28) results in
\[
\tilde{G}' \ast P_{\Sigma,b,a}x^*(j/B) = \int_{\xi \in \mathbb{R}^d} \hat{x}[f'] \cdot \text{Delta}_f(\xi) \cdot e^{2\pi i a^\top \Sigma \xi} \cdot d\xi
\]
\[
= \hat{x}[f'] \cdot e^{2\pi i a^\top \Sigma f'}.
\]
This equation, together with Equation (26) and Equation (27), implies that
\[
\left| \hat{u}_j - \hat{x}[f'] \cdot e^{2\pi i a^\top \Sigma f'} \right| \leq \frac{\delta}{\text{poly}(k,d)} \cdot \|\hat{x}\|_1
\]
Taking square on the both sides:
\[
\left| \hat{u}_j - \hat{x}[f'] \cdot e^{2\pi i a^\top \Sigma f'} \right|^2 \leq \frac{\delta^2}{\text{poly}(k,d)} \cdot \|\hat{x}\|_1^2 \leq \frac{\delta}{\text{poly}(k,d)} \cdot \|\hat{x}\|_1^2,
\]
where the last step is by \( 0 < \delta < 1 \); note that \( j = h_{\Sigma,b}(f') \in [B]^d \).

Finally, we note that \( |H| \leq k \), since \( H \subseteq \text{supp}(\hat{x}) \) and there are just \( k \) many tone frequencies \( f' \in \text{supp}(\hat{x}) \). Apply the last inequality over all \( f' \in H \) and take the expectation over the random vector \( a \in \mathbb{R}^d \) (see Definition E.7), then Lemma E.22 follows.
E.10 Performance guarantees

Lemma E.23 (Performance guarantee for HashToBins). Recall Theorem 1.2 for the $\ell_2$-norm noise level

$$\mathcal{N}^2 := \frac{1}{T^d} \cdot \int_{t \in [0,T]^d} |g(t)|^2 \cdot dt + \delta \cdot \sum_{i \in [k]} |\hat{x}[f_i]|^2.$$

Sample the matrix $\Sigma \in \mathbb{R}^{d \times d}$ and the vectors $b \in \mathbb{R}^d$ according to Definition E.7, and suppose that Conditions E.8 and E.9 hold for the random vector $a \in \mathbb{R}^d$. Consider the “good” frequencies

$$H := \{ f \in \text{supp}(\hat{x}) : \text{neither } E_{\text{coll}}(\xi) \text{ nor } E_{\text{off}}(\xi) \text{ happens} \},$$

and the bins $I := [\mathcal{B}] \setminus \mathcal{H}_\Sigma(\text{supp}(\hat{x}))$ with no frequency $\{f_i\}_{i \in [k]} = \text{supp}(\hat{x})$ hashed into.

Then given any $\Sigma \in \mathbb{R}^{d \times d}$, the following holds for each good frequency $f \in H$:

$$\mathbb{E}_{b,a} \left[ |\hat{u}_{\mathcal{H}_\Sigma}(f) - \hat{x}[f] \cdot e^{2\pi i \cdot a^\top \Sigma f}|^2 \right] \lesssim B^{-1} \cdot \mathcal{N}^2_g + k^{-1} \cdot \frac{\delta}{\text{poly}(k,d)} \cdot \mathcal{N}^2_v.$$

And take all good frequencies $f \in H$ and all bins $j \in I$ into account:

$$\mathbb{E}_{b,a} \left[ \sum_{f \in H} |\hat{u}_{\mathcal{H}_\Sigma}(f) - \hat{x}[f] \cdot e^{2\pi i \cdot a^\top \Sigma f}|^2 + \sum_{j \in I} |\hat{u}_i|^2 \right] \lesssim \mathcal{N}^2.$$

Proof. This can be easily seen by combining Lemmas E.21 and E.22. \qed
F  Locate inner

| Statement                  | Section  | Algorithm | Comment                        |
|---------------------------|----------|-----------|--------------------------------|
| Definitions F.1 and F.2   | Section F.1 | Algorithm 2 | Definitions                   |
| Lemma F.4                 | Section F.2 | Algorithm 2 | Sample complexity and running time |
| Lemma F.5                 | Section F.3 | Algorithm 2 | Voting process                 |
| Lemma F.10                | Section F.4 | Algorithm 2 | Election process               |
| Lemma F.13                | Section F.5 | Algorithm 2 | Guarantees                     |
| Lemmas F.14 and F.15      | Section F.6 | Algorithm 3 | Sampling scheme               |
| Lemma F.16                | Section F.7 | Algorithm 4 | Stronger guarantees            |

Table 1: List of Lemmas/Algorithms in locate inner section.

F.1 Definitions and algorithm

Definition F.1 (Setup for LOCATEINNER). We adopt the following notations:

- The guessed approximation ratio $C \in [120, \rho]$.
- Let $M = 4 \cdot [4\sqrt{d} \cdot C^{2/3}] \in \mathbb{N}_{\geq 1}$.
- Let $\varpi = C^{-2/3}$; this parameter will be used in the voting scheme (see Definition F.3).
- $\varphi_j = \arg(\hat{u}_j) - \arg(\hat{u}_j')$ denotes the phase difference between $\hat{u}_j$ and $\hat{u}_j'$;
- The number of iterations $R_{\text{vote}} = \Theta(d \cdot \log(C \cdot d) + \log \log(F/\eta))$ is sufficiently large.

Definition F.2 (Hyperball and sub-hyperballs). For any frequency $\text{List}[j] = f_{\text{grid}}[j] \in \mathbb{R}^d$ and any $L^{\text{dia}} \geq 0$, $\text{HB}(\text{List}[j], L^{\text{dia}})$ denotes the $\ell_2$-norm hyperball with center $\text{List}[j]$ and diameter $L^{\text{dia}}$.

$$\text{HB}(\text{List}[j], L^{\text{dia}}) := \left\{ \xi \in \mathbb{R}^d : \|\xi - \text{List}[j]\|_2 \leq L^{\text{dia}}/2 \right\}.$$  

Let $\bigcup_{q \in Q} \text{HB}(f_{\text{grid}}[j], \frac{1}{M} \cdot L^{\text{dia}})$ denote a cover of $\text{HB}(\text{List}[j], L^{\text{dia}})$, by using a minimum amount of sub-hyperballs that have the diameter $\frac{1}{M} \cdot L^{\text{dia}}$ each. At most $M := |Q| = (4M \cdot \sqrt{d})^d = 2^{\Theta(d \cdot \log(C \cdot d))}$ many sub-hyperballs can be used, namely the external covering number [SSBD14, Page 337].

Given that the hyperball $\text{HB}(\text{List}[j], L^{\text{dia}})$ contains the targeted tone frequency $f \in [-F, F]^d$, all the sub-hyperballs can be classified into three groups (as Figure 12 shows):

- The true sub-hyperball $\text{HB}(f_{q^*}^{\text{grid}}, \frac{1}{M} \cdot L^{\text{dia}})$ $\ni f$, for some index $q^* \in Q$. For convenience, assume that the true sub-hyperball is unique, namely the targeted tone frequency is not on the boundary of two or more sub-hyperballs.\(^\text{15}\)
- The wrong sub-hyperballs $q \in Q \setminus \{q^*\}$ have the $\ell_2$-distances
  $$\|f_{q^*}^{\text{grid}} - f_q^{\text{grid}}\|_2 \geq \frac{1}{M} \cdot L^{\text{dia}} \cdot [4\sqrt{d/\varpi} = \frac{1}{M} \cdot L^{\text{dia}} \cdot [4\sqrt{d} \cdot C^{2/3}]].$$
- The remaining sub-hyperballs are called the intermediate sub-hyperballs.

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Figure 12: Demonstration for Definition F.2 when $d = 2$. The “red” point means the true frequency $f$. The “lime” region means the true sub-hyperball $\mathbf{HB}(f_q^{\text{grid}[j]}, \frac{1}{M} \cdot L_{d\text{d}}) \ni f$. The “blue” regions mean the wrong sub-hyperballs, and the remaining “yellow” regions (with overlapping parts) represent the intermediate sub-hyperballs.

Figure 13: Demonstration for the voting scheme (Definition F.3)
Algorithm 2 \textsc{LocateInner}, Lemmas F.4, F.5, F.10, F.13

\begin{enumerate}
\item \textbf{procedure} \textsc{LocateInner}(\(\Sigma, b, D, \text{List}, L^\text{dia}, C, T\))
\item Define \(M \in \mathbb{N}_{\geq 1}\) according to Definition F.1. \hfill \triangleright Definition F.2
\item \textbf{for} \(j \in [B]^d\) that \(\text{List}[j] \neq \text{NIL}\) \textbf{do}
\item \hspace{1em} Cover \(\text{HB}(\text{List}[j], L^\text{dia})\) via sub-hyperballs \(\bigcup_{q \in Q} \text{HB}(f_q^{\text{grid}[j]}, \frac{1}{M} \cdot L^\text{dia})\).
\item \textbf{end for}
\item Initialize \(\mathcal{V}otej[q] = 0\) for all \(q \in Q\) and each \(j \in [B]^d\). \hfill \triangleright Voting process. Lemma F.5
\item Define \(\mathcal{R}_{\text{vote}} \in \mathbb{N}_{\geq 1}\) according to Definition F.1.
\item \textbf{for} \(r = 1, 2, \ldots, \mathcal{R}_{\text{vote}}\) \textbf{do}
\item \hspace{1em} \((a_r, \Delta_r^u) \leftarrow \text{SampleTimePoint}(M, L^\text{dia}, C, T)\). \hfill \triangleright Algorithm 3
\item \hspace{1em} \(\widehat{\mathcal{V}} \leftarrow \text{HashToBins}(x, \Sigma, b, a_r, D)\).
\item \hspace{1em} Let \(\varphi = (\varphi_j)_{j=1}^B\), where \(\varphi_j = \arg(\bar{u}_j) - \arg(\bar{u}_j)\).
\item \hspace{1em} \textbf{for} \(j \in [B]^d\) that \(\text{List}[j] \neq \text{NIL}\) \textbf{do}
\item \hspace{2em} \textbf{if} \((a_r, \Delta_r^u)\) is non-empty \textbf{then}
\item \hspace{3em} \textbf{let} \(\text{List}_{\text{new}}[j]\) be any frequency so that \(\text{HB}(\text{List}_{\text{new}}[j], \frac{1}{2} \cdot L^\text{dia}) \supseteq \text{Winner}[j]\).
\item \hspace{3em} \textbf{end if}
\item \hspace{2em} \textbf{end if}
\item \hspace{1em} \textbf{end for}
\item \hspace{1em} \textbf{return} the frequencies \(\text{List}_{\text{new}}\).
\item \textbf{end for}
\end{enumerate}

\textbf{Definition F.3} (Voting scheme). Let \(\|\|_{\circ} \in [0, \pi]\) denote the “phase distance” from \(e^{i\theta} = 1\) to any \(\theta \in \mathbb{R}\). As Figure 13 suggests, given any \(\varphi_j \in \mathbb{R}\) and any \(\Delta_a := \Delta^a_r\), let \(\mathcal{V}otej[q] \leftarrow \mathcal{V}otej[q] + 1\) (namely adding a vote to any sub-hyperball \(q \in Q\)) for which
\[
\|\varphi_j - 2\pi \cdot \Delta_a, f_q^\text{grid}[j]\|_{\circ} \leq \pi \cdot \varpi.
\]

\textbf{F.2 Sample complexity and running time}

The goal of this section is to prove Lemma F.4.

\textbf{Lemma F.4} (Sample complexity and running time of \textsc{LocateInner}). The procedure \textsc{LocateInner} (Algorithm 2) has the following performance guarantees:

\begin{itemize}
\item The sample complexity is
\[
\Theta(\mathcal{R}_{\text{vote}} \cdot \mathcal{B} \mathcal{D}) = 2^{\Theta(d \log d)} \cdot (\log C + \log \log(F/\eta)) \cdot k \cdot \mathcal{D}.
\]
\end{itemize}

\footnote{We make this assumption just to specify the true sub-hyperball; our proof does not rely on the assumption.}
The running time is
\[ \Theta(\mathcal{R}_{\text{vote}} \cdot (BD + B \log B + BM)) = 2^{\Theta(d \cdot \log d + \log C)} \cdot \log \log (F/\eta) \cdot k \cdot \mathcal{D} + \log k \]

Proof. Throughout the procedure \textsc{LocateInner}, the subroutine \textsc{HashToBins} (Algorithm 1) is invoked \(2 \cdot \mathcal{R}_{\text{vote}}\) times. Recall Definition F.1 that \(\mathcal{R}_{\text{vote}} = \Theta(d \cdot \log (C \cdot d) + \log \log (F/\eta))\).

Sample complexity. The procedure \textsc{LocateInner} takes samples only by invoking the subroutine \textsc{HashToBins}. Due to Fact E.17, \textsc{HashToBins} has the sample complexity \(O(BD)\). Recall Definition E.1 that \(B = 2^{\Theta(d \cdot \log d)} \cdot k\). Thus, \textsc{LocateInner} has the sample complexity
\[ \#_{\text{sample}}(\text{LocateInner}) = \Theta(\mathcal{R}_{\text{vote}} \cdot B \cdot D) = \Theta(d \cdot \log (C \cdot d) + \log \log (F/\eta)) \cdot 2^{\Theta(d \cdot \log d)} \cdot k \cdot D. \]

Running time. The running time of \textsc{LocateInner} is dominated by the \(2 \cdot \mathcal{R}_{\text{vote}}\) many loops for the voting process (namely Lines 8 to 16). Such a loop invokes the subroutine \textsc{HashToBins} twice, and then update \(V_{\text{vote}}[j][q]\) for all \(q \in Q\) and all \(j \in [B]^d\). (The subroutine \textsc{SampleTimePoint} runs in \(O(d)\) time; see Algorithm 3.)

Due to Fact E.17, \textsc{HashToBins} has the running time \(O(BD + B \log B)\) time. Thus, the time that \textsc{LocateInner} spends on hashing is
\[ \#_{\text{time}}(\text{hashing}) = \Theta(\mathcal{R}_{\text{vote}} \cdot (BD + B \log B)) = \Theta(d \cdot \log (C \cdot d) + \log \log (F/\eta)) \cdot 2^{\Theta(d \cdot \log d)} \cdot k \cdot (D + \log k). \]

Further, the time that \textsc{LocateInner} spends on voting is
\[ \#_{\text{time}}(\text{voting}) = \Theta(\mathcal{R}_{\text{vote}} \cdot B \cdot M) = \Theta(d \cdot \log (C \cdot d) + \log \log (F/\eta)) \cdot 2^{\Theta(d \cdot \log d)} \cdot k \cdot 2^{\Theta(d \cdot \log (C \cdot d))}. \]

In total, the procedure \textsc{LocateInner} has the running time
\[ \#_{\text{time}}(\text{LocateInner}) = \#_{\text{time}}(\text{hashing}) + \#_{\text{time}}(\text{voting}) = 2^{\Theta(d \cdot \log (C \cdot d))} \cdot k \cdot (D + \log k) \cdot \log \log (F/\eta). \]

This completes the proof.

F.3 Voting process

The goal of this section is to prove Lemma F.5.

Lemma F.5 (The voting process of \textsc{LocateInner}). Given any realized matrix \(\Sigma\) and any realized vector \(b\), assume three premises for a particular good tone frequency \(f \in H = \{\xi \in \text{supp}(\hat{x}^*) : \text{neither} E_{\text{coll}}(\xi) \text{ nor } E_{\text{off}}(\xi) \text{ happens}\):
- W.l.o.g. the tone frequency $f \in H$ is hashed into the bin $h_{\Sigma,b}(f) = j \in [B]^d$ (Definition E.10).
- The tone frequency $f \in H$ locates within the hyperball $HB(List[j], L^{da})$.
- Given the guessed approximation ratio $C \in [120, \rho]$, the following holds for both $a = a_r$ and $a = a_r + \Delta^r_a$, in every single iteration $r \in [R_{vote}]$ of the procedure LOCATEINNER:

$$E_a \left[ \left| \hat{u}_j - \hat{x}[f] \cdot e^{2\pi i a^\top \Sigma f} \right|^2 \right] \leq C^{-2} \cdot |\hat{x}[f]|^2.$$ 

Then the following hold in every single iteration of procedure LOCATEINNER (Algorithm 2):

**Property I:** The (unique) true sub-hyperball gets a vote with probability at least

$$1 - \frac{4}{(C \cdot \omega)^2} = 1 - \frac{4}{C^{2/3}} > \frac{1}{2}.$$ 

**Property II:** Any wrong sub-hyperball gets a vote with probability at most

$$8\omega + \frac{4}{(C \cdot \omega)^2} = \frac{12}{C^{2/3}} < \frac{1}{2}.$$ 

Claim F.6 (Property I of Lemma F.5). The (unique) true sub-hyperball gets a vote with probability at least

$$1 - \frac{4}{(C \cdot \omega)^2} = 1 - \frac{4}{C^{2/3}} > \frac{1}{2}.$$ 

Proof. For brevity, we rewrite $a_r$ and $\Delta^r_a$ respectively as $a$ and $\Delta_a$ in this proof. Note that all of the probabilities and the expectations given below are taken over the random vectors $a$ and $\Delta_a$.

Combining the second premise of the lemma and Chebyshev’s inequality together, we know that the following holds with probability at least $1 - \frac{2}{(C \cdot \omega)^2}$:

$$\left| \frac{\hat{u}_j}{\hat{x}[f]} \cdot e^{2\pi i a^\top \Sigma f} \right| \leq \omega / \sqrt{2} \cdot |\hat{x}[f]|,$$

which is equivalent to

$$\left| \frac{\hat{u}_j}{\hat{x}[f]} \cdot e^{-2\pi i a^\top \Sigma f} - 1 \right| \leq \omega / \sqrt{2}.$$ 

I.e., the complex number $\frac{\hat{u}_j}{\hat{x}[f]} \cdot e^{-2\pi i a^\top \Sigma f}$ lies in the circle $\{z \in \mathbb{C} : |z - 1| \leq \omega / \sqrt{2}\}$. Clearly, any complex number in this circle has the phase $\leq \sin^{-1}(\omega / \sqrt{2})$. In particular,

$$\left\| \arg(\hat{u}_j) - \arg(\hat{x}[f]) - 2\pi \cdot a^\top \Sigma f \right\|_{A_1} \leq \sin^{-1} \left( \frac{\omega}{\sqrt{2}} \right),$$

where $\|\theta\|_\circ \in [-\pi, \pi)$ denotes the “phase distance” from $e^{i\theta} = 1$ to any $\theta \in \mathbb{R}$.

Similarly, when $a$ is replaced with $(a + \Delta_a)$, with probability $1 - \frac{2}{(C \cdot \omega)^2}$ we also have

$$\left\| \arg(\hat{u}_j) - \arg(\hat{x}[f]) - 2\pi \cdot (a + \Delta_a)^\top \Sigma f \right\|_{A_2} \leq \sin^{-1} \left( \frac{\omega}{\sqrt{2}} \right),$$

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Put the above two inequalities together, (by the union bound) the following holds for the phase difference $\varphi_j = \arg(\hat{u}_j) - \arg(\tilde{u}'_j)$ with probability $1 - \frac{4}{(C \cdot \bar{\omega})^2}$:

$$
\left\| \varphi_j - 2\pi \cdot \Delta_a^\top \Sigma f \right\|_\infty = \left\| A_1 - A_2 \right\|_\infty \\
\leq \left\| A_1 \right\|_\infty + \left\| A_2 \right\|_\infty \\
\leq 2 \sin^{-1} \left( \frac{\bar{\omega}}{\sqrt{2}} \right) \\
\leq (\pi/2) \cdot \bar{\omega},
$$

(29)

where the second step applies the triangle inequality; and last step follows since for any $z \in (0, 1)$, we have $\sin^{-1}(z/\sqrt{2}) \leq (\pi/4) \cdot z$.

Let $q^* \in Q$ be the index of the true sub-hyperball $\mathbb{H}_B(f_{q^*}^{\text{grid}}[j], \frac{1}{M} \cdot L_{\text{dia}}) \ni f$. Compared with the center frequency $f_{q^*}^{\text{grid}}[j]$ of this sub-hyperball, the tone frequency $f$ differs by has the $\ell_2$-distance

$$
\left\| f - f_{q^*}^{\text{grid}}[j] \right\|_2 \leq \frac{1}{2M} \cdot L_{\text{dia}}.
$$

Equation (29) suggests that the phase $\varphi_j = \arg(\hat{u}_j) - \arg(\tilde{u}'_j) \in \mathbb{R}$ is likely to be a good approximation to $2\pi \cdot \Delta_a^\top \Sigma f$. Indeed, that inequality (if true) guarantees a vote for the true sub-hyperball $\mathbb{H}_B(f_{q^*}^{\text{grid}}[j], \frac{1}{M} \cdot L_{\text{dia}})$:

$$
\left| 2\pi \cdot \Delta_a^\top \Sigma (f - f_{q^*}^{\text{grid}}[j]) \right| \leq 2\pi \cdot \left\| \Sigma^\top \Delta_a \right\|_2 \cdot \left\| f - f_{q^*}^{\text{grid}}[j] \right\|_2 \\
\leq 2\pi \cdot \left\| \Sigma^\top \Delta_a \right\|_2 \cdot \left( \frac{1}{2M} \cdot L_{\text{dia}} \right) \\
\leq 2\pi \cdot \left( \frac{\bar{\omega} \cdot M}{2L_{\text{dia}}} \right) \cdot \left( \frac{1}{2M} \cdot L_{\text{dia}} \right) \\
= (\pi/2) \cdot \bar{\omega},
$$

(30)

where the second step uses the $\ell_2$-distance derived above; and the third step follows because the vector $\Delta_a$ is sampled such that $\left\| \Sigma^\top \Delta_a \right\|_2 \sim \text{Unif}[\frac{\bar{\omega} \cdot M}{4L_{\text{dia}}}, \frac{\bar{\omega} \cdot M}{2L_{\text{dia}}}]$ (see Algorithm 3).

Combining everything together, with probability at least $1 - \frac{4}{(C \cdot \bar{\omega})^2}$ we have

$$
\left\| \varphi_j - 2\pi \cdot \Delta_a^\top \Sigma f_{q^*}^{\text{grid}}[j] \right\|_\infty = \left\| \varphi_j - 2\pi \cdot \Delta_a^\top \Sigma f \right\|_\infty + \left| 2\pi \cdot \Delta_a^\top \Sigma (f - f_{q^*}^{\text{grid}}[j]) \right| \\
\leq (\pi/2) \cdot \bar{\omega} + (\pi/2) \cdot \bar{\omega} \\
= \pi \cdot \bar{\omega},
$$

where the first step follows from the triangle inequality; and the second step follows by applying inequalities (29) and (30).

Recall Definitions F.1 and F.3 that $\bar{\omega} = C^{-2/3}$ and $C \geq 120$. Via elementary calculation, it can be seen that

$$
1 - \frac{4}{(C \cdot \bar{\omega})^2} = 1 - \frac{4}{C^{2/3}} \geq 1 - \frac{4}{120^{2/3}} \approx 0.8356 > \frac{1}{2}.
$$

This completes the proof of Claim F.6. \qed

**Claim F.7** (Property II of Lemma F.5). Any wrong sub-hyperball gets a vote with probability at most

$$
8\bar{\omega} + \frac{4}{(C \cdot \bar{\omega})^2} = \frac{12}{C^{2/3}} < \frac{1}{2}.
$$
Proof. Once again, we rewrite \(a_r\) and \(\Delta_a\) respectively as \(a\) and \(\Delta_a\) for simplicity, and all of the probabilities and the expectations in this proof are taken over the random vectors \(a\) and \(\Delta_a\).

Let \(q^* \in Q\) be the index of the true sub-hyperball \(\mathbf{HB}(f_q^{\text{grid}[j]}, \frac{1}{M} \cdot L_{\text{dia}}) \ni f\). For a specific wrong sub-hyperball \(\mathbf{HB}(f_q^{\text{grid}[j]}, \frac{1}{M} \cdot L_{\text{dia}})\), where \(q \in Q \setminus \{q^*\}\), the next inequality turns out to hold with probability at least \(1 - 8\omega\):

\[
\left\| 2\pi \cdot \Delta_a^\top \Sigma(f_q^{\text{grid}[j]} - f_q^{\text{grid}[j]}) \right\|_\infty \geq 2\pi \cdot \omega. \tag{31}
\]

We assume this fact for a while, and will justify this fact in the last part of this proof.

As shown in the proof of Claim F.6, the following holds with probability at least \(1 - 4 \cdot (C\omega)^{-2}\):

\[
\left\| \varphi_j - 2\pi \cdot \Delta_a^\top \Sigma f_q^{\text{grid}[j]} \right\|_\infty \leq \pi \cdot \omega. \tag{32}
\]

Conditioned on both Inequalities (31) and (32), we must have

\[
\left\| \varphi_j - 2\pi \cdot \Delta_a^\top \Sigma f_q^{\text{grid}[j]} \right\|_\infty = \left\| 2\pi \cdot \Delta_a^\top \Sigma(f_q^{\text{grid}[j]} - f_q^{\text{grid}[j]}) + (\varphi_j - 2\pi \cdot \Delta_a^\top \Sigma f_q^{\text{grid}[j]}) \right\|_\infty \geq \pi \cdot \omega.
\]

Given this, we know from Definition F.3 that the \(q\)-th (wrong) sub-hyperball is guaranteed to lose a vote. And based on the union bound, we derive Claim F.7 as desired:

\[
\Pr[\text{\(q\)-th sub-hyperball gets a vote}] \leq \Pr[\text{Equation (31) does not hold}] + \Pr[\text{Equation (32) does not hold}]
\]

\[
= 8\omega + \frac{4}{(C\omega)^2}
\]

\[
= \frac{12}{C^{2/3}}
\]

\[
\leq \frac{120^{2/3}}{120^{2/3}}
\]

\[
\leq 0.4933
\]

\[
\leq \frac{1}{2}
\]

where the third step follows because \(\omega = C^{-2/3}\) (see Definition F.3); and the fourth step follows because \(C \geq 120\) (see Definition F.1).

To establish the claim, we are left to justify that Equation (31) holds with probability at least \(1 - 8\omega\). Indeed, an equivalent condition of Equation (31) is that

\[\text{“}\Delta_a^\top \Sigma(f_q^{\text{grid}[j]} - f_q^{\text{grid}[j]}) \text{ differs from its closest integer } \lfloor \Delta_a^\top \Sigma(f_q^{\text{grid}[j]} - f_q^{\text{grid}[j]}) + \frac{1}{2} \rfloor \text{ by at least } \omega,\text{”}\]

because \(\|z\|_\infty \in [-\pi, \pi)\) denotes the “phase distance” from \(0 = \arg(e^{i\theta})\) to any \(z \in \mathbb{R}\).

We then observe that the vector \(\Delta_a\) is sampled (see Algorithm 3) such that \(\Sigma^\top \Delta_a\) has a uniform random direction, and the \(\ell_2\)-norm follows the uniform distribution

\[
\left\| \Sigma^\top \Delta_a \right\|_2 \sim \text{Unif}\left[\frac{\omega \cdot M}{4L_{\text{dia}}}, \frac{\omega \cdot M}{2L_{\text{dia}}}\right].
\]

Given these, we infer (e.g. from [CFJ13]) that \(\Delta_a^\top \Sigma(f_q^{\text{grid}[j]} - f_q^{\text{grid}[j]})\) has the same distribution as the random variable \((\bar{w} \cdot \cos \bar{\theta})\), where
• \( \tilde{\theta} \in [0, \pi] \) is the angle between the vectors \( \Sigma^\top \Delta_a \) and \( (f_{q^*}^{\text{grid}[j]} - f_q^{\text{grid}[j]}) \). It is known that \( \tilde{\theta} \) has the following probability density function: for all \( \tilde{\theta} \in [0, \pi] \),

\[
\text{PDF}(\tilde{\theta}) = \frac{\sin^{d-2}(\tilde{\theta})}{\int_0^\pi \sin^{d-2}(z) \cdot dz}.
\]

• \( \tilde{w} \sim \text{Unif}[w, 2w] \) with the parameter

\[
w = \frac{\bar{\omega}}{4L_{\text{dia}}} \cdot \left\| f_{q^*}^{\text{grid}[j]} - f_q^{\text{grid}[j]} \right\|_2 \\
\geq \frac{\bar{\omega}}{4L_{\text{dia}}} \cdot \frac{1}{M} \cdot L_{\text{dia}} \cdot \frac{4\sqrt{d}}{\bar{\omega}} \\
\geq \sqrt{d},
\]

where the second step follows from the definition of a \textit{wrong} sub-hyperball (see Definition F.2).

We conclude from the above that

\[
\Pr_{\Delta_a} [\text{Equation (31) does not hold}] = \Pr_{\tilde{w}, \tilde{\theta}} \left[ \left| \tilde{w} \cdot \cos(\tilde{\theta}) - \left( \tilde{w} \cdot \cos(\tilde{\theta} + 1/2) \right) \right| \leq \bar{\omega} \right] \tag{33}
\]

It turns out that \( \bar{\omega} \leq 1/5 \) and that \( w \geq 1 \). Concretely, we know from Definitions F.1 and F.3 that

\[
\bar{\omega} = C^{-2/3} \leq 120^{-2/3} \approx 0.0411 < \frac{1}{5}.
\]

Further, we have shown that the parameter \( w \geq \sqrt{d} \). Given these, Claim F.8 (presented below) is applicable to the RHS of Equation (33). By doing so, we accomplish Claim F.7.

This completes the proof. \( \square \)

**Claim F.8** (Technical result for Claim F.7). \textit{Given any} \( u \in (d/2^d, 1/5] \) \textit{and any} \( w \geq \sqrt{d} \), it follows

\[
\Pr_{\tilde{w}, \tilde{\theta}} \left[ \left| \tilde{w} \cdot \cos(\tilde{\theta}) - \left( \tilde{w} \cdot \cos(\tilde{\theta} + 1/2) \right) \right| \leq u \right] \lesssim u,
\]

where \( \tilde{w} \sim \text{Unif}[w, 2w] \), and the random phase \( \tilde{\theta} \in [0, \pi] \) has the probability density function

\[
\text{PDF}(\tilde{\theta}) = \frac{\sin^{d-2}(\tilde{\theta})}{\int_0^\pi \sin^{d-2}(z) \cdot dz}, \quad \forall \tilde{\theta} \in [0, \pi].
\]

**Proof.** Fix \( \tilde{\theta} \) first. We know that

\[
\int_0^\pi \sin^{d-2}(z) \cdot dz = \pi \cdot \frac{(2(d-2) - 1)!!}{(2(d-2))!!} \approx 1/d,
\]

where the first step is by induction and the second step follows from the Wallis formula.

We need some asymptotic evaluations:

• \( |\sin(z)| \approx 1 - \frac{\cos^2 z}{2} \) when \( |\cos(z)| \ll 1 \).

• \( \sin x \approx x \) when \( |x| \ll 1 \).
In the next a few paragraphs, we discuss the three cases for $|\cos(\theta)|$.

- **Case 1.** $|\cos(\theta)| \leq c/\sqrt{d}$
- **Case 2.** $c/\sqrt{d} \leq |\cos(\theta)| \leq 1/2$
- **Case 3.** $|\cos(\theta)| > 1/2$

**Case 1.** If $|\cos(\theta)| \leq c/\sqrt{d}$. First consider the range that $|\cos(\theta)| \leq c/\sqrt{d}$ for some small constant $c > 1$. Then in this range we know that $|\sin z| \approx 1 - \frac{\cos^2(\theta)}{2} \geq 1 - \frac{c^2}{d}$, which implies that $|\sin(\theta)|^{d-2} = \Omega(1)$. In other word, we can treat $\theta$ as nearly uniform distributed in this range. By similar arguments in Claim F.9, we can prove in this range

$$\Pr_{\tilde{\theta}} \left[ |\tilde{w} \cdot \cos(\tilde{\theta}) - \tilde{w} \cdot \cos(\tilde{\theta}) + 1/2 | \leq u, |\cos(\tilde{\theta})| \leq c/\sqrt{d} \right] \lesssim u$$

**Case 2.** If $c/\sqrt{d} \leq |\cos(\theta)| \leq 1/2$. We fix an integer $i > c$. We have that $|\sin(z)| \geq 1/2$. We can use a straight line to simulate $\cos$ function. For any integer $i \geq c$, we have that

$$\cos^{-1}(\frac{i-u}{w}) - \cos^{-1}(\frac{i+u}{w}) \lesssim u \cdot (\cos^{-1}(\frac{i+1+u}{w}) - \cos^{-1}(\frac{i-1+u}{w})).$$

We know that PDF($\tilde{\theta}$) is increasing, then we have that

$$u \cdot \Pr_{\tilde{\theta}}[\tilde{\theta} \in [\cos^{-1}(\frac{i-u}{w}), \cos^{-1}(\frac{i-1+u}{w})]] \gtrsim \Pr_{\tilde{\theta}}[\tilde{\theta} \in [\cos^{-1}(\frac{i+u}{w}), \cos^{-1}(\frac{i-1+u}{w})]].$$

We have that

$$\Pr_{\tilde{\theta}} \left[ |\tilde{w} \cdot \cos(\tilde{\theta}) - i| \leq u, \frac{1}{2} \geq |\cos(\tilde{\theta})| \geq c/\sqrt{d} \right] \lesssim u \cdot \Pr_{\tilde{\theta}} \left[ |\tilde{w} \cdot \cos(\tilde{\theta}) - i| \leq 1, \frac{1}{2} \geq |\cos(\tilde{\theta})| \geq c/\sqrt{d} \right].$$

Combine this together, we know that

$$\Pr_{\tilde{\theta}} \left[ |\tilde{w} \cdot \cos(\tilde{\theta}) - \tilde{w} \cdot \cos(\tilde{\theta}) + 1/2 | \leq u, 1/2 \geq |\cos(\tilde{\theta})| \geq c/\sqrt{d} \right] \lesssim u.$$

**Case 3.** If $|\cos(\theta)| > 1/2$. Then we have

$$\Pr_{\tilde{\theta}} \left[ |\cos(\tilde{\theta})| \geq 1/2 \right] \leq \frac{\pi \cdot (1/2)^{d-2}}{\int_0^\pi \sin^{d-2}(z) \cdot dz} \lesssim \frac{d}{2^d} \leq u.$$

The first step is because $|\sin(\tilde{\theta})| < 1/2$.

**Combine three cases.** Then combine these three cases together, we have

$$\Pr_{\tilde{\theta}} \left[ |\tilde{w} \cdot \cos(\tilde{\theta}) - \tilde{w} \cdot \cos(\tilde{\theta}) + 1/2 | \leq u \right]$$

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Claim F.9 (Technical result for Claim F.7). Given any $u \in (0, 1/5]$ and any $w \geq 1$, the following holds for the random variables $\tilde{w} \sim \text{Unif}[w, 2w]$ and $\tilde{\theta} \sim \text{Unif}[-\pi, \pi)$:

$$\Pr_{\tilde{w}, \tilde{\theta}} \left[ |\tilde{w} \cdot \sin(\tilde{\theta}) - [\tilde{w} \cdot \sin(\tilde{\theta}) + 1/2]| \leq u \right] \leq 8u.$$  

Proof. To improve the readability, we provide Figure 14 for demonstration. We denote

$$\tilde{d}_\theta = |\tilde{w} \cdot \sin(\tilde{\theta}) - [\tilde{w} \cdot \sin(\tilde{\theta}) + 1/2]|$$

for ease of notation. Notice that $\tilde{d}_\theta$ represents the distance between $(\tilde{w} \cdot \sin(\tilde{\theta}))$ and its closest integer. By symmetry, the following random distance $\tilde{d}_\psi$ has the same distribution as $\tilde{d}_\theta$:

$$\tilde{d}_\psi = |\tilde{w} \cdot \sin(\tilde{\psi}) - [\tilde{w} \cdot \sin(\tilde{\psi}) + 1/2]|,$$

where the new random phase $\tilde{\psi}$ is distributed uniformly on $[0, \frac{\pi}{2}]$ rather than on $[-\pi, \pi)$.

Let us investigate the new random distance $\tilde{d}_\psi$ via case analysis.

This completes the proof. \hfill \blacksquare
Case (i): when \((1 - u)/(2w) \leq \sin(\tilde{\psi}) \leq 1\).

Notice that this case is non-empty, since \(u \in (0, 1/5]\) and \(w \geq 1\). Suppose the random phase \(\tilde{\psi}\) is fixed. Because the random variable \(\tilde{w} \sim \text{Unif}[w, 2w]\), the random closest integer \(\lfloor \tilde{w} \cdot \sin(\tilde{\psi}) + 1/2 \rfloor\) admits the following lower and upper bounds:

\[
\lfloor \tilde{w} \cdot \sin(\tilde{\psi}) + 1/2 \rfloor \geq \tilde{w} \cdot \sin(\tilde{\psi}) - 1/2 \geq w \cdot \sin(\tilde{\psi}) - 1/2 \\
\lfloor \tilde{w} \cdot \sin(\tilde{\psi}) + 1/2 \rfloor \leq \tilde{w} \cdot \sin(\tilde{\psi}) + 1/2 \leq 2w \cdot \sin(\tilde{\psi}) + 1/2
\]

Namely, the random closest integer \(\lfloor \tilde{w} \cdot \sin(\tilde{\psi}) + 1/2 \rfloor\) has at most \((w \cdot \sin(\tilde{\psi}) + 1)\) many possibilities. Consider the set \(\tilde{A}_\psi\) of all possible \((\tilde{w} \cdot \sin(\tilde{\psi}))\) such that the random distance \(\tilde{d}_\psi \leq u\):

\[
\tilde{A}_\psi = \{ \tilde{w} \cdot \sin(\tilde{\psi}) : \tilde{w} \in [w, 2w] \text{ and } \tilde{d}_\psi \leq u \}.
\]

Since the closed integer \(\lfloor \tilde{w} \cdot \sin(\tilde{\psi}) + 1/2 \rfloor\) has at most \((w \cdot \sin(\tilde{\psi}) + 1)\) many possibilities, the total length of this set \(\tilde{A}_\psi\) is at most

\[
|\tilde{A}_\psi| \leq 2u \cdot (w \cdot \sin(\tilde{\psi}) + 1).
\tag{34}
\]

Hence, under any choice of the random phase \(\tilde{\psi}\), the conditional probability (over the uniform random variable \(\tilde{w} \sim \text{Unif}[w, 2w]\)) below is at most

\[
\Pr_{\tilde{w}}[\tilde{d}_\psi \leq u \mid \text{case (i)}] = \frac{|\tilde{A}_\psi|}{(2w - w) \cdot \sin(\tilde{\psi})} \\
\leq \frac{2u \cdot (w \cdot \sin(\tilde{\psi}) + 1)}{(2w - w) \cdot \sin(\tilde{\psi})} \\
= 2u \cdot \left(1 + (w \cdot \sin(\tilde{\psi}))^{-1}\right) \\
\leq 2u \cdot \left(1 + \frac{2}{1 - u}\right) \\
\leq 7u,
\]

where the second step applies Equation (34); the fourth step follows because (in this case) we assume that \(\sin(\tilde{\psi}) \geq \frac{1 - u}{2w}\); and the last step is because \(u \in (0, 1/5]\).

Case (ii): when \(u/w < \sin(\tilde{\psi}) < (1 - u)/(2w)\).

Notice that this case is non-empty, since \(u < \frac{1 - u}{2}\) for any \(u \in (0, 1/5]\). Of course, any realized random variable \(\tilde{w} \sim \text{Unif}[w, 2w]\) satisfies \(\tilde{w} \geq w\) and \(\tilde{w} \leq 2w\). On the lower-bound part:

\[
\tilde{w} \cdot \sin(\tilde{\psi}) \geq w \cdot \sin(\tilde{\psi}) > w \cdot \frac{u}{w} = u.
\]

Further, on the upper-bound part:

\[
\tilde{w} \cdot \sin(\tilde{\psi}) \leq 2w \cdot \sin(\tilde{\psi}) < 2w \cdot \frac{1 - u}{2w} = 1 - u.
\]
Combining both inequalities together, regardless of the realized $\tilde{w} \sim \text{Unif}[w, 2w]$, the random variable $(\tilde{w} \cdot \sin(\tilde{\psi}))$ locates between $(u, 1 - u)$ and differs from its closest integer by at least $u$.

From the above arguments, we conclude that the next conditional probability equals zero.

$$\Pr_{\tilde{w}, \tilde{\psi}} \left[ d_{\psi} \leq u \mid \text{case (ii)} \right] = 0.$$  

**Case (iii):** when $0 \leq \sin(\tilde{\psi}) \leq u/w$.

Of course, the following conditional probability is at most one:

$$\Pr_{\tilde{w}, \tilde{\psi}} \left[ d_{\psi} \leq u \mid \text{case (iii)} \right] \leq 1.$$  

Observe that $u/w \leq u \leq 1/5$, because $u \in (0, 1/5]$ and $w \geq 1$. Then, since the random phase $\tilde{\psi}$ is distributed uniformly on $[0, \pi/2]$, this case happens with probability

$$\Pr_{\tilde{w}, \tilde{\psi}} [\text{case (iii)}] = \frac{2}{\pi} \cdot \sin^{-1}(u/w)$$

and

$$\leq \frac{2}{\pi} \cdot \sin^{-1}(u)$$

$$\leq \frac{2}{3} \cdot u,$$

where the last step follows since $z \leq \sin(\pi/3 \cdot z)$ for any $z \in [0, 1/2]$ and we have $u \in (0, 1/5] \subseteq [0, 1/2]$.

Putting all the three cases together, we conclude that

$$\Pr_{\tilde{w}, \tilde{\psi}} \left[ d_{\psi} \leq u \right] \leq 7u \cdot \Pr_{\tilde{w}, \tilde{\psi}} \left[ \text{case (i)} \right] + 0 \cdot \Pr_{\tilde{w}, \tilde{\psi}} \left[ \text{case (ii)} \right] + 1 \cdot \Pr_{\tilde{w}, \tilde{\psi}} \left[ \text{case (iii)} \right]$$

$$= 7u \cdot \Pr_{\tilde{w}, \tilde{\psi}} \left[ \text{case (i)} \right] + 1 \cdot \Pr_{\tilde{w}, \tilde{\psi}} \left[ \text{case (iii)} \right]$$

$$\leq 7u \cdot 1 + 1 \cdot \frac{2}{3} \cdot u$$

$$\leq 8u,$$

where the first step applies the bounds on the conditional probabilities derived before.

This completes the proof of Claim F.9.   

**F.4 Election process**

The goal of this section is to prove Lemma F.10.

**Lemma F.10** (The election process of LocateInner). For any matrix $\Sigma \in \mathbb{R}^{d \times d}$ and any vector $b \in \mathbb{R}^{d}$, assume three premises for a particular good tone frequency $f \in H = \{\xi \in \text{supp}(\hat{x}^*) : \text{neither } E_{\text{coll}}(\xi) \text{ nor } E_{\text{off}}(\xi) \text{ happens}\}:

- The tone frequency $f \in H$ is hashed into the bin $h_{\Sigma, b}(f) = j \in [B]^{d}$ (Definition E.10).
- The tone frequency $f \in H$ locates within the hyperball $\mathcal{H}_{\text{List}[j], L^{\text{dia}}}$. 

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• Given the guessed approximation ratio $C \in [120, \rho]$, the following holds for both $a = a_r$ and $a = a_r + \Delta a$, in every single iteration $r \in [\mathcal{R}_{\text{vote}}]$ of the procedure LOCATEINNER (Algorithm 2):

$$
E_a \left[ | \tilde{u}_j - \tilde{x}[f] | \cdot e^{2\pi a^{-1} \Sigma f} \right] \leq C^{-2} \cdot | \tilde{x}[f] |^2.
$$

Then with probability at least $1 - M \cdot 2^{-\Omega(\mathcal{R}_{\text{vote}})}$, the following hold for the algorithm LOCATEINNER:

**Property I:** The tone frequency $f \in H$ locates in one of the winning sub-hyperballs

$$
\text{Winner}[j] = \bigcup_{q \in Q: \forall \text{vote}_j |q| \geq \frac{1}{2} \mathcal{R}_{\text{vote}}} \mathbf{HB}(f_q^{\text{grid}}[j], M^{-1} \cdot L^{\text{dia}}).
$$

**Property II:** All the winning frequencies $\text{Winner}[j]$ can be included within the smaller hyperball

$$
\text{Winner}[j] \subseteq \mathbf{HB}(f_q^{\text{grid}}[j], L^{\text{dia}}_{\text{new}}),
$$

which is centered at $f_q^{\text{grid}}[j]$ and has the new diameter $L^{\text{dia}}_{\text{new}} := \frac{1}{2} \cdot L^{\text{dia}}$.

**Property III:** The output frequency $\text{List}_{\text{new}}[j] \in \mathbb{R}^d$ makes the tone frequency $f \in H$ locate in a new hyperball that is centered at $\text{List}_{\text{new}}[j]$ and has a new diameter $L^{\text{dia}}_{\text{new}} = \frac{1}{2} \cdot L^{\text{dia}}$.

$$
f \in \mathbf{HB}(\text{List}_{\text{new}}[j], L^{\text{dia}}_{\text{new}}).
$$

To make Lemma F.10 meaningful, later we will choose a large enough $\mathcal{R}_{\text{vote}} \in \mathbb{N}_{\geq 1}$ such that the failure probability $M \cdot 2^{-\Omega(\mathcal{R}_{\text{vote}})} \ll 1$. Furthermore, we observe that Property III of Lemma F.10 (see Figure 15 for demonstration) is a direct follow-up to Properties I and II. Below, we would show that Property I holds with probability $1 - 2^{-\Omega(\mathcal{R}_{\text{vote}})}$, and that Property II holds with probability $1 - (M - 1) \cdot 2^{-\Omega(\mathcal{R}_{\text{vote}})}$. Then, all the properties can be inferred via the union bound.

**Claim F.11** (Property I of Lemma F.10). With probability at least $1 - 2^{-\Omega(\mathcal{R}_{\text{vote}})}$, the tone frequency $f \in H$ locates in one of the winning sub-hyperballs

$$
\text{Winner}[j] = \bigcup_{q \in Q: \forall \text{vote}_j |q| \geq \frac{1}{2} \mathcal{R}_{\text{vote}}} \mathbf{HB}(f_q^{\text{grid}}[j], M^{-1} \cdot L^{\text{dia}}).
$$

**Proof.** Given the second premise of Lemma F.10 that, the tone frequency $f \in H$ locates within the hyperball $\mathbf{HB}(\text{List}[j], L^{\text{dia}})$, there is a unique true sub-hyperball $\mathbf{HB}(f_q^{*, \text{grid}}[j], \frac{1}{M} \cdot L^{\text{dia}})$ containing $f \in H$ (see Definition F.2), for some vector index $q^* \in Q$. Apparently, a necessary condition for $f \in H$ to locate in none of the winning sub-hyperballs is the event

$$
\overline{E}_{q^*} = \{ \text{the } q^*-\text{th sub-hyperball in total gets less than } \frac{1}{2} \cdot \mathcal{R}_{\text{vote}} \text{ votes} \}.
$$

Based on Property I of Lemma F.5, in every iteration $r \in [\mathcal{R}_{\text{vote}}]$, the $q^*$-th sub-hyperball independently loses a vote with probability at most $4 \cdot C^{-2/3} < 1/2$. Combining a simple coupling argument together with the Chernoff bound (see Part (a) of Lemma B.1), the event $\overline{E}_{q^*}$ happens with probability at most

$$
\Pr[\overline{E}_{q^*}] \leq \exp \left( - \frac{\mathcal{R}_{\text{vote}}}{2} \cdot \left( \ln \left( \frac{C^{2/3}}{8} \right) + \frac{8}{C^{2/3}} - 1 \right) \right)
$$

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Figure 15: Demonstration for Property III of Lemma F.10 in two dimensions $d = 2$. The given (black) circle $HB(List[j], L_{\text{dia}})$ narrows down into a smaller (red) circle $HB(List_{\text{new}}[j], L_{\text{new}}^{\text{dia}})$.

\[
\leq \exp \left( -\frac{R_{\text{vote}}}{2} \left( \ln \left( \frac{120^{2/3}}{8} \right) + \frac{8}{120^{2/3}} - 1 \right) \right) = \exp \left( -\Omega(R_{\text{vote}}) \right)
\]

where the second step follows because the formula $\ln z + \frac{1}{2}$ is increasing in $z \in \mathbb{R}_{>0}$ (and $C \geq 120$; see Definition F.1); and the last step follows as $\ln(\frac{120^{2/3}}{8}) + \frac{8}{120^{2/3}} - 1 \approx 0.1174 = \Omega(1)$.

This completes the proof of Claim F.11.

**Claim F.12 (Property II of Lemma F.10).** All the winning frequencies $Winner[j]$ can be included within the smaller hyperball

\[Winner[j] \subseteq HB(f_{q^*}^{\text{grid}[j]}, L_{\text{new}}^{\text{dia}}),\]

which is centered at $f_{q^*}^{\text{grid}[j]}$ and has the new diameter $L_{\text{new}}^{\text{dia}} := \frac{1}{2} \cdot L^{\text{dia}}$.

**Proof.** Recall Definition F.2 that we cover the hyperball $HB(List[j], L^{\text{dia}})$ by using $M = 2^{\Theta(d \log(C \cdot d))}$ many sub-hyperballs. Given the second premise of Lemma F.10, one particular sub-hyperball $q^* \in Q$ is the true sub-hyperball, and there are at most $(M - 1)$ many wrong sub-hyperballs.

We first demonstrate that, a specific wrong sub-hyperball $q \in Q$ “wins” with probability at most $2^{-\Omega(R_{\text{vote}})}$. By definition (see Line 19 of LOCATEINNER), this wrong sub-hyperball “wins” if and only if the following event happens:

\[E_q = \{ \text{the } q\text{-th sub-hyperball in total gets at least } \frac{1}{2} \cdot R_{\text{vote}} \text{ votes} \} .\]

According to Property II of Lemma F.5, in each iteration $r \in [R_{\text{vote}}]$, the $q$-th sub-hyperball independently gets a vote with probability at most $12 \cdot C^{-2/3} < 1/2$. Combining a simple coupling
argument together with the Chernoff bound (see Part (a) of Lemma B.1), the event $E_q$ happens with probability at most

$$\Pr [E_q] \leq \exp \left( -\frac{R_{\text{vote}}}{2} \cdot \left( \ln \left( \frac{C^{2/3}}{24} \right) + \frac{24}{C^{2/3}} - 1 \right) \right) \leq \exp \left( -\frac{R_{\text{vote}}}{2} \cdot \left( \ln \left( \frac{120^{2/3}}{24} \right) + \frac{24}{120^{2/3}} - 1 \right) \right) = \exp \left( -\Omega(R_{\text{vote}}) \right)$$

where the second step follows because the formula $\ln z + \frac{1}{2}$ is increasing in $z \in \mathbb{R}_{>0}$ (and $C \geq 120$; see Definition F.1); and the last step follows as $\ln(\frac{120^{2/3}}{24}) + \frac{24}{120^{2/3}} - 1 \approx 9.2161 \times 10^{-5} = \Omega(1)$.

Since there are at most $(M - 1)$ many wrong sub-hyperballs, we can apply the union bound for all of them. Hence, with probability at least $1 - (M - 1) \cdot 2^{-\Omega(R_{\text{vote}})}$, none of the wrong sub-hyperballs “win” in the election process.

According to Definition F.2, the diameter of a sub-hyperball is $\frac{1}{M} \cdot L^{\text{dia}}$, and any intermediate sub-hyperball $q \in Q$ (or the true sub-hyperball $q^* \in Q$ itself) satisfies that

$$\| f_{q^*}^{\text{grid}[j]} - f_q^{\text{grid}[j]} \|_2 \geq \frac{1}{M} \cdot L^{\text{dia}} \cdot \left[ 4\sqrt{d} \cdot C^{2/3} \right].$$

For these reasons, the $\ell_2$-distance between any $f \in \text{Winner}[j]$ and the center frequency $f_{q^*}^{\text{grid}[j]}$ of the true sub-hyperball is at most

$$\| f - f_{q^*}^{\text{grid}[j]} \|_2 \leq \left( \left[ 4\sqrt{d} \cdot C^{2/3} \right] - 1 + \frac{1}{2} \right) \cdot \frac{1}{M} \cdot L^{\text{dia}} \leq \frac{1}{4} \cdot L^{\text{dia}},$$

where the last step is because $M = 4 \cdot \left[ 4\sqrt{d} \cdot C^{2/3} \right]$ (see Definition F.1).

Thus, any frequency $f \in \text{Winner}[j]$ can be included in a smaller hyperball $\text{HB}(f_{q^*}^{\text{grid}[j]}, L^{\text{dia}}_{\text{new}})$ that is centered at $f_{q^*}^{\text{grid}[j]}$ and has the new diameter $L^{\text{dia}}_{\text{new}} := \frac{1}{2} \cdot L^{\text{dia}}$.

This accomplishes the proof of Claim F.12. \qed

### F.5 Performance guarantees

The goal of this section is to prove Corollary F.13.

**Corollary F.13 (The guarantee of $\text{LOCATEINNER}$).** Given $\Sigma$ and $b$ (according to Definition E.7), let $H \subseteq \supp(\hat{x}^\ast)$ be a subset of “good” tone frequencies:

$$H = \{ \xi \in \supp(\hat{x}^\ast) : \text{neither } E_{\text{off}}(\xi) \text{ nor } E_{\text{coll}}(\xi) \text{ happens} \}$$

Let $j := k_{\Sigma,b}(f) \in [B]^d$ where a good frequency $f \in H$ is hashed into. Suppose $f \in \text{HB}(\text{List}[j], L^{\text{dia}})$ at the beginning, then with failure probability at most $M \cdot 2^{-\Omega(R_{\text{vote}})}$, the procedure $\text{LOCATEINNER}$ outputs a new frequency $\text{List}_{\text{new}}[j] \in [-F,F]^d$ so that

$$f \in \text{HB}(\text{List}_{\text{new}}[j], L^{\text{dia}}_{\text{new}}),$$

where the new diameter $L^{\text{dia}}_{\text{new}} = L^{\text{dia}}/2$.

**Proof.** This follows immediately from Property III of Lemma F.10. \qed
Figure 16: Demonstration of the sampling for $\Delta a$, and then for $a$ and $(a + \Delta a)$, where in Figure 16a the parameter $r := \frac{\varpi}{4L^{2\varepsilon}}$.

F.6 Sampling time points

The procedure SAMPLETIMEPOINT is given in Algorithm 3, which is illustrated in Figure 16.

Algorithm 3 SAMPLETIMEPOINT, Lemmas F.14 and F.15

1: procedure SAMPLETIMEPOINT($M, L^{dia}, C, T$)
2: Define $\varpi \in (0, 1)$ according to Definition F.1.
3: Sample $\Delta a \in \mathbb{R}^d$ such that $\Sigma^T \Delta a \sim \text{Unif}\{z \in \mathbb{R}^d : \|z\|_2 = 1\}$. $\triangleright |\Sigma| \neq 0$; Definition E.7
4: Scale $\Delta a$ by a random factor $\beta \sim \text{Unif}\left[\frac{\varpi M}{4L^{2\varepsilon}}, \frac{\varpi M}{2L^{2\varepsilon}}\right]$.
5: Let $A := \{z \in \mathbb{R}^d : \{z, z + \Sigma^T \Delta a\} \subseteq [\frac{0.01}{d} \cdot T, (1 - \frac{0.01}{d}) \cdot T]^d\}$.
6: Sample $a \in \mathbb{R}^d$ such that $\Sigma^T a \sim \text{Unif}(A)$.
7: return $a$ and $\Delta a$.
8: end procedure

F.6.1 Duration requirement

The goal of this part is to prove Lemma F.14, and thus to obtain the duration bound required by Condition E.8.

Lemma F.14 (Duration of LOCATEINNER). To satisfy Condition E.8, the sampling duration requirement of the procedure LOCATEINNER (Algorithm 2) is

$$ T = \Omega\left(d^3 \cdot \eta^{-1} \cdot \log(kd/\delta)\right). $$

Proof. The procedure LOCATEINNER uses the samples in the time domain by invoking the subroutine HASHTOBINS (Algorithm 1) with a number of pairs $a' \in \{a, a + \Delta a\}$ output by SAMPLETIMEPOINT (Algorithm 3). In particular (see Line 4 of HASHTOBINS), we take the following sample for
all $i \in [BD]^d$ and both $a' \in \{a, a + \Delta a\}$:

$$P_{\Sigma, b, a'} x(i) = x(i + a') \cdot e^{-2\pi i b^\top i},$$

where the equation follows from Definition E.14.

To meet Condition E.8, we shall have

$$\Sigma^\top (i + a') \in [0, T]^d,$$

for all $i \in [BD]^d$ and both $a' \in \{a, a + \Delta a\}$, under any choice of the random matrix $\Sigma \in \mathbb{R}^{d \times d}$ (according to Definition E.7).

We know from Line 5 that both $a' \in \{a, a + \Delta a\}$ satisfy that

$$\Sigma^\top a' \in \left[\frac{0.01}{d}, T, (1 - \frac{0.01}{d}) \cdot T\right]^d.$$

Given this, a sufficient condition for Equation (35) is that

$$\|\Sigma^\top i\|_\infty \leq \frac{0.01}{d} \cdot T,$$

for all $i \in [BD]^d$, under any choice of the random matrix $\Sigma \in \mathbb{R}^{d \times d}$.

For the above equation, we deduce that

$$\|\Sigma^\top i\|_\infty \leq \|\Sigma^\top i\|_2 \leq \frac{4\sqrt{d}}{B\eta} \cdot \|i\|_2 \leq \frac{4\sqrt{d}}{B\eta} \cdot \sqrt{d} \cdot BD \lesssim \frac{d^2 \cdot \eta^{-1} \cdot \log(kd/\delta)}{d},$$

where the second step follows because $\Sigma \in \mathbb{R}^{d \times d}$ is a rotation matrix scaled by a random factor $\beta \sim \text{Unif}[\frac{2\sqrt{d}}{B\eta}, \frac{4\sqrt{d}}{B\eta}]$ (Definition E.7); the third step follows since $i \in [BD]^d = \{0, 1, \cdots, BD - 1\}^d$; and the last step follows because $D = \Theta(d \cdot \log(kd/\delta))$ (see Definition E.7).

Putting the above arguments together, we know that Condition E.8 holds for any sufficiently large $T = \Omega(d^2 \cdot \eta^{-1} \cdot \log(kd/\delta))$.

This completes the proof.

F.6.2 Performance guarantees

The goal of this part is to prove Lemma F.15, and thus to verify Condition E.9.

**Lemma F.15** (Performance guarantees). Suppose that Condition E.8 is true and that $L^{da} \geq \frac{20d}{T}$ (which will be ensured by Definition G.1), then Condition E.9 holds for both $a' \in \{a, a + \Delta a\}$ derived from the procedure $\text{SampleTimePoint}$ (Algorithm 3): 

$$\mathbb{E}_{a'} \left[g(\Sigma^\top (i + a'))^2\right] \lesssim \frac{1}{Td} \cdot \int_{t \in [0, T]^d} |g(t)|^2 \cdot dt,$$

for all $i \in [BD]^d$, under any choice of the random matrix $\Sigma \in \mathbb{R}^{d \times d}$ (according to Definition E.7).
Proof. Since both time points \( a' = a \) and \( a' = a + \Delta_a \) are constructed in a symmetric fashion (see Line 5 of \texttt{SampleTimePoint}), we only need to reason about the time point \( a \sim \text{Unif}(A) \) given in \texttt{SampleTimePoint}. Denote \( T' := (1 - 0.02/d) \cdot T \geq 0.98 \cdot T \). By construction (see Line 3),
\[
\|\Sigma^T \Delta_a\|_2 \leq \frac{\bar{w} \cdot M}{2L_{da}} \leq \frac{17 \sqrt{d}}{2L_{da}} \leq \frac{17}{40 \sqrt{d}} \cdot T \leq \frac{1}{2 \sqrt{d}} \cdot T',
\]
where the second step follows since \( C \geq 120 \) and \( \bar{w} = C^{-2/3} \) and \( M = 4 \cdot [4 \sqrt{d} \cdot C^{2/3}] \leq 17 \sqrt{d} \cdot C^{2/3} \) (see Definition F.1); the third step follows from the premise that \( L_{da} \geq \frac{20d}{T} \); and the last step holds because \( T' \geq 0.98T \).

We have
\[
\|\Sigma^T \Delta_a\|_\infty \leq \|\Sigma^T \Delta_a\|_1 \leq \sqrt{d} \cdot \|\Sigma^T \Delta_a\|_2 \leq \frac{1}{2} \cdot T'.
\]
Let \((\Sigma^T \Delta_a)_r\) denote the \( r \)-th coordinate of \( \Sigma^T \Delta_a \in \mathbb{R}^d \). For any choice of \( \Delta_a \) by \texttt{SampleTimePoint}, the volume of the sampling range \( \Sigma^T a \sim \text{Unif}(A) \) is
\[
\text{vol}(A) = \prod_{r \in [d]} (T' - |(\Sigma^T \Delta_a)_r|)
= T'^d \cdot \prod_{r \in [d]} (1 - |(\Sigma^T \Delta_a)_r|) \cdot T'^{-1}
\geq T'^d \cdot \prod_{r \in [d]} \exp\left(-2 \cdot |(\Sigma^T \Delta_a)_r| \cdot T'^{-1}\right)
= T'^d \cdot \exp\left(-2 \cdot \|\Sigma^T \Delta_a\|_1 \cdot T'^{-1}\right)
\geq T'^d \cdot e^{-1}
\geq T^d \cdot 0.98 \cdot e^{-1},
\]
where the first step is by Line 5 of \texttt{SampleTimePoint}; the third step follows since \( |(\Sigma^T \Delta_a)_r| \leq \|\Sigma^T \Delta_a\|_\infty \leq \frac{1}{2} \cdot T' \) and \( 1 - z \geq e^{-2z} \) when \( z \in [0, \frac{1}{2}] \); the fifth step follows since \( \|\Sigma^T \Delta_a\|_1 \leq \frac{1}{2} \cdot T' \); and the last step is because \( T'^d = (1 - 0.02/d)^d \cdot T^d \geq 0.98 \cdot T^d \).

We conclude from the above that, for any choice of \( \Delta_a \) by \texttt{SampleTimePoint}, the time point \( \Sigma^T a \sim \text{Unif}(A) \) is sampled uniformly from a constant proportion of the duration \( t \in [0, T]^d \). And because \( \Sigma^T (i + a) \) is guaranteed to be within the duration \( t \in [0, T]^d \), for any choice of \( \Sigma \in \mathbb{R}^{d \times d} \) and any \( i \in [BD]^d \), we have
\[
\mathbb{E}_a \left[ g(\Sigma^T (i + a))^2 \right] \leq \frac{1}{T^d} \cdot \int_{t \in [0,T]^d} |g(t)|^2 \cdot dt.
\]
This completes the proof. \qed

\subsection*{F.7 Stronger Guarantee}

\textbf{Lemma F.16 (Stronger guarantees).} Let
\[
\rho^2 = |\hat{x}[f]|^2 / \mathbb{E}_a[|\hat{u}_j - \hat{x}[f] / e^{2\pi i a^T \Sigma f}|^2]
\]
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Algorithm 4 A stronger version of LOCATEINNER when search range is small

1: procedure LOCATEINNER*(Σ, b, D, List, L\textsuperscript{dia}, C, T) \hfill \triangleright \text{Lemma F.16}
2: \hspace{1em} for \( r = 1, 2, \ldots, R_{\text{reg}} \) do
3: \hspace{2em} \( (a_r, \Delta^r_a) \leftarrow \text{SAMPLETIMEPOINT}(M, L_{\text{dia}}, C, T) \). \hfill \triangleright \text{Algorithm 3}
4: \hspace{2em} \( \hat{u} \leftarrow \text{HASHTOBINS}(x, \Sigma, b, a_r, D) \).
5: \hspace{2em} \( \hat{u}' \leftarrow \text{HASHTOBINS}(x, \Sigma, b, a_r + \Delta^r_a, D) \).
6: \hspace{2em} \( \varphi_{j,r} = \text{arg}(\hat{u}_j) - \text{arg}(\hat{u}_j) \).
7: \hspace{1em} end for
8: \hspace{1em} List\textsubscript{new} \leftarrow \emptyset
9: \hspace{1em} Form matrix \( \Delta^\top := [\Sigma^\top \Delta^1_a, \ldots, \Sigma^\top \Delta^r_{\text{reg}}] \in \mathbb{R}^{d \times R_{\text{reg}}} \).
10: \hspace{1em} for \( j \in [B] \) do
11: \hspace{2em} Form vector \( \varphi_j \in \mathbb{R}^{R_{\text{reg}}} \).
12: \hspace{2em} List\textsubscript{new} \leftarrow List\textsubscript{new} \cup \{ \frac{1}{2\pi} \cdot \Delta^1 \varphi_j \}.
13: \hspace{1em} end for
14: \hspace{1em} return List\textsubscript{new}.
15: end procedure

Let \( C_* = d^2 \). Let \( R_{\text{reg}} = C \cdot d \) for some constant \( C \). Let \( L_{\text{dia}} = 20d/T \). There is an algorithm (procedure LOCATEINNER* in Algorithm 4) that output a list of frequencies such that there is a mapping \( \pi : [k] \to [m] \),

\[ \| f'_{\pi(i)} - f_i \|_2 \lesssim \frac{C_*}{\rho T}, \ \forall i \in [k]. \]

Proof. We provide Figure 17 for demonstration. Let \( d' = R_{\text{reg}} = Cd \) for simplicity. By Markov inequality, we know that the following holds with probability at least \( 1 - \frac{1}{100} \):

\[ \left| \hat{u}_j - \hat{x}[f] \cdot e^{2\pi i a^\top \Sigma f} \right| \leq |\hat{x}[f]| \cdot 10 \sqrt{d'}/\rho, \]

which is equivalent to

\[ \left| \frac{\hat{u}_j}{\hat{x}[f]} \cdot e^{-2\pi i a^\top \Sigma f} - 1 \right| \leq 10 \sqrt{d'}/\rho. \]

Namely, the complex number \( \hat{u}_j/\hat{x}[f] \cdot e^{-2\pi i a^\top \Sigma f} \) lies in the circle \( \{ z \in \mathbb{C} : |z| = 1 \leq \omega/\sqrt{2} \} \). Clearly, any complex number in this circle has the phase less than \( \sin^{-1}(10 \sqrt{d'}/\rho) \). In particular,

\[ \left\| \frac{\text{arg}(\hat{u}_j) - \text{arg}(\hat{x}[f]) - 2\pi \cdot a^\top \Sigma f}{A_1} \right\| \leq \sin^{-1}(10 \sqrt{d'}/\rho), \]

where \( \|\theta\|_\circ \in [-\pi, \pi] \) denotes the “phase distance” \( \min_{z \in \mathbb{Z}} |\theta - 2\pi \cdot z| \).

Similarly, when \( a \) is replaced with \( (a + \Delta_a) \), with probability \( 1 - \frac{1}{100} \) we also have

\[ \left\| \frac{\text{arg}(\hat{u}_j') - \text{arg}(\hat{x}[f]) - 2\pi \cdot (a + \Delta_a)^\top \Sigma f}{A_2} \right\| \leq \sin^{-1}(10 \sqrt{d'}/\rho), \]

Put the above two inequalities together, (by the union bound) the following holds for the phase difference \( \varphi_{j,r} = \text{arg}(\hat{u}_j) - \text{arg}(\hat{u}_j') \) with probability \( 1 - \frac{2}{100} \):

\[ \| \varphi_{j,r} - 2\pi \cdot \Delta_a^\top \Sigma f \|_\circ = \| A_1 - A_2 \|_\circ \]
frequency \( f \in \text{supp}(\hat{x}^\ast) \)

approximation \( f' \approx f \)

\[ \|A_1\|_\infty + \|A_2\|_\infty \leq 2 \sin^{-1}(10\sqrt{d}/\rho) \leq 10\sqrt{d'}/\rho. \]

where the second step applies the triangle inequality; and last step follows since for any \( z \in (0, 1) \), we have \( \sin^{-1}(z/\sqrt{2}) \leq (\pi/4) \cdot z \).

Then with probability at least 0.8, we have that

\[ \|\varphi - 2\pi \cdot \Delta \cdot f\|_\infty \leq 10\sqrt{d'}/\rho, \]

where

\[ \varphi := \begin{bmatrix} \varphi_{j,1} \\ \varphi_{j,2} \\ \vdots \\ \varphi_{j,d'} \end{bmatrix} \in \mathbb{R}^{d'} \quad \text{and} \quad \Delta := \begin{bmatrix} \Delta_{a1}^T \Sigma \\ \Delta_{a2}^T \Sigma \\ \vdots \\ \Delta_{ad'}^T \Sigma \end{bmatrix} \in \mathbb{R}^{d' \times d}. \]

We deduce from the above that

\[ \|\varphi - 2\pi \cdot \Delta f\|_2 \leq \sqrt{d'} \cdot 10\sqrt{d'}/\rho = 10d'/\rho. \]

\( \{\Sigma^T \Delta_a^r\}_{r \in [d']} \) are uniformly distributed on a sphere. Consider any \( r \in [d'] \), by Theorem 3.4.6 in [Ver18], we know that \( \Sigma^T \Delta_a^r \) is sub-gaussian. Besides, the value of each coordinate of \( \Sigma^T \Delta_a^r \) follows a Beta-distribution, and \( d \cdot \frac{d}{\|\Sigma^T \Delta_a^r\|_2^2} \cdot \mathbb{E}[\Delta_a^r \Sigma \Sigma^T \Delta_a^r] = I \). Thus we know that \( \frac{\sqrt{d}}{\|\Sigma^T \Delta_a^r\|_2} \Sigma^T \Delta_a^r \) is a sub-gaussian isotropic random vector. By selecting \( \|\Sigma^T \Delta_a^r\|_2 \approx T/d \) and the constant \( C = d'/d \) large enough, then by Lemma B.5, we can show that with probability at least \( 1 - 1/\text{poly}(d) \), we
have \( s_{\min}(\Delta) \geq \sqrt{d \cdot \frac{T}{d^2}} = \frac{T}{d} \). Let \( \Delta^\dagger \) represent the Generalized inverse of \( \Delta \), let \( f_{LS} = \Delta^\dagger \varphi \) represent the least squares solution, then we have

\[
2\pi \| \Delta (f - f_{LS}) \|_2 \leq 2\| \varphi - \Delta^\dagger \varphi \|_2 \lesssim d/\rho.
\]

Then we have

\[
\| f - f_{LS} \|_2 \leq \frac{d}{\rho} \cdot \frac{d}{T} = C_* \cdot \frac{1}{T\rho}.
\]

where the first step is by \( \| \Delta x \|_2/\| x \|_2 \geq s_{\min}(\Delta) \); the second step follows from \( s_{\min}(\Delta) \geq T/d \) and the last step follows because we define \( C_* := d^2 \).

This completes the proof. \( \square \)
G Locate signal

| Statement         | Section | Algorithm | Comment                      |
|-------------------|---------|-----------|------------------------------|
| Definition G.1    | Section G.1 | Algorithm 5 | Definitions                   |
| Lemma G.2         | Section G.2 | Algorithm 5 | Sample complexity and running time |
| Lemma G.3         | Section G.3 | Algorithm 5 | Duration                      |
| Lemma G.4         | Section G.4 | Algorithm 5 | Guarantees, without Alg. 4    |
| Lemma G.5         | Section G.5 | Algorithm 5 | Stronger guarantees, with Alg. 4 |

Table 2: List of Lemmas/Algorithms in locate signal section.

G.1 Algorithm

Denote \( H := \{ \xi \in \text{supp}(\hat{x}^*) : \text{neither } E_{\text{coll}}(\xi) \text{ nor } E_{\text{off}}(\xi) \text{ happens} \} \). Recall the performance guarantees given in Corollary F.13:

Assume that a specific “good” tone frequency good frequency \( f \in H \) locates in a hyperball \( \text{HB}(\text{List}[j], L_{\text{dia}}) \) that is centered at some frequency \( \text{List}[j] \in \mathbb{R}^d \) and has the diameter \( L_{\text{dia}} > 0 \).

Then with probability at least \( 1 - M \cdot 2^{-\Omega(R_{\text{vote}})} \), then procedure LOCATEINNER (Algorithm 2) outputs \( \text{List}_{\text{new}}[j] \) for which

\[
f \in \text{HB}(\text{List}_{\text{new}}[j], L_{\text{dia}}_{\text{new}}).\]

where the new diameter \( L_{\text{dia}}_{\text{new}} := \frac{1}{2} \cdot L_{\text{dia}} \).

Given this, we would estimate the good frequencies by invoking the procedure LOCATEINNER repeatedly. This idea is implemented as the procedure LOCATESIGNAL (Algorithm 5).

Definition G.1 (Setup for LOCATESIGNAL). The procedure LOCATESIGNAL keeps track of a number of \( B = 2^{\Theta(d \cdot \log d)} \cdot k \) hyperballs. These hyperballs have

- The same initial diameter \( L_{\text{dia}} := 2\sqrt{d} \cdot F \).
- The final diameter \( L_{\text{dia}} \in (\frac{20d}{T}, \frac{40d}{T}) \) is chosen so that \( \log_2(\frac{\text{initial } L_{\text{dia}}}{\text{final } L_{\text{dia}}}) \) is an integer; clearly, this final \( L_{\text{dia}} \) is well defined and is unique.
- The number of iteration \( R_{\text{search}} := \log_2(\frac{\text{initial } L_{\text{dia}}}{\text{final } L_{\text{dia}}}) = O(\log(T \cdot F)) \).

That is, each hyperball is initialized to be \( \text{HB}(0, 2\sqrt{d} \cdot F) \supseteq [-F, F]^d \). Clearly, such a hyperball contains all the “good” tone frequencies \( f \in H \) at the beginning. Then, the subroutine LOCATEINNER is invoked \( R_{\text{search}} \) times, until the diameter shrinks to the final \( L_{\text{dia}} \in (\frac{20d}{T}, \frac{40d}{T}) \).

G.2 Sample complexity and running time

The goal of this section is to prove Lemma G.2.

Lemma G.2 (Sample complexity and running time of LOCATESIGNAL). The procedure LOCATESIGNAL (Algorithm 5) has the following performance guarantees:
Algorithm 5 LocateSignal, Lemmas G.2, G.3, G.4

1: procedure LocateSignal(Σ, b, D, C, T)
2:   List[j] ← 0 ∈ R^d for each j ∈ [B]^d. ▷ Initialize the center frequency
3:   \(L_{dia} = 2\sqrt{d} \cdot F.\) ▷ Initialize the diameter
4:   for r = 1, 2, · · · , R_{search} do
5:     List_{new} ← LocateInner(Σ, b, D, List, \(L_{dia}, C, T\)). ▷ Algorithm 2
6:     List ← List_{new}.
7:   end for
8:   \(L_{dia} ← \frac{1}{2} \cdot L_{dia}.\)
9: return List^*.

Proof. How many frequencies the output List^* contains is easy to see, since List^* is indexed by the bins \(j ∈ [B]^d\). Below we quantify the sample complexity and the running time.

Sample complexity. The procedure LocateSignal invokes the subroutine LocateInner \(R_{search}\) times. Due to Lemma F.4, the subroutine LocateInner has the sample complexity
\[
\#_{sample}(LocateInner) = 2^{\Theta(d \cdot \log d)} \cdot k \cdot D \cdot (\log C + \log \log(F/\eta)) \cdot \log(T \cdot F).
\]
Thus, LocateSignal has the sample complexity
\[
\#_{sample}(LocateSignal) = \#_{sample}(LocateInner) \cdot R_{search} = 2^{\Theta(d \cdot \log d)} \cdot k \cdot D \cdot (\log C + \log \log(F/\eta)) \cdot \log(T \cdot F).
\]

Running time. Due to Lemma F.4, the subroutine LocateInner has the running time
\[
\#_{time}(LocateInner) = 2^{\Theta(d \cdot \log(C \cdot d))} \cdot k \cdot (D + \log k) \cdot \log \log(F/\eta).
\]
Thus, LocateSignal has the running
\[
\#_{time}(LocateSignal) = \#_{time}(LocateInner) \cdot R_{search} = 2^{\Theta(d \cdot \log(C \cdot d))} \cdot k \cdot (D + \log k) \cdot \log \log(F/\eta) \cdot \log(T \cdot F).
\]
This completes the proof of Lemma G.2. □

G.3 Duration requirement

The goal of this section is to prove Lemma G.3.

Lemma G.3 (Duration of LocateSignal). The sampling duration requirement of the procedure LocateSignal (Algorithm 5) is
\[
T = \Omega(d^3 \cdot \eta^{-1} \cdot \log(kd/\delta)).
\]

Proof. This follows immediately from Lemma F.14. □

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Figure 18: Demonstration of Algorithm 5 for a single frequency \( f \in \text{supp}(\hat{x}^*) \) in two dimensions \( d = 2 \). The blue/green/red circles refer to the coarse-grained location, and the “orange” lines refer to the fine-grained location.

G.4 Performance guarantees

The goal of this section is to prove Lemma G.4.

**Lemma G.4 (Guarantees of LocateSignal).** Given \( \Sigma \in \mathbb{R}^{d \times d} \) and \( b \in \mathbb{R}^d \), the output list \( \text{List}^* \) of procedure LocateSignal (Algorithm 5) contains at most \( B = 2^{O(d \cdot \log d)} \cdot k \) many frequencies with minimum separation \( \Omega(\eta) \). Let \( H \subseteq \text{supp}(\hat{x}^*) \) be a subset of “good” tone frequencies:

\[
H = \{ \xi \in \text{supp}(\hat{x}^*) : \text{neither } E_{\text{off}}(\xi) \text{ nor } E_{\text{coll}}(\xi) \text{ happens} \}
\]

For any good frequency \( f \in H \), suppose that its signal-to-noise ratio \( \rho(i) \geq C \) (see Definition H.1), then with probability at least 99%, there exists an output frequency \( f' \in \text{List}^* \) such that

\[
\| f - f' \|_2 \sim \frac{d}{T}.
\]

**Proof.** The concerning frequency \( f \in H \) w.l.o.g. is hashed into the bin \( j := h_{\Sigma,b}(f) \in [B]^d \). Recall Definition G.1 that the procedure LocateSignal keeps track of a number of \( B = 2^{O(d \cdot \log d)} \cdot k \) hyperballs. The \( j \)-th hyperball is initialized to be \( \text{HB}(0, 2\sqrt{d} \cdot F) \supseteq [-F, F]^d \), and thus contains the frequency \( f \in H \). Then, the procedure LocateSignal invokes the subroutine LocateInner \( \mathcal{R}_{\text{search}} \) times, each of which shrinks the diameter of the \( j \)-th hyperball by half, until the diameter drops down to the final \( L_{\text{fin}} \in (\frac{20d}{T}, \frac{40d}{T}) \).

**Failure probability.** For the concerning frequency \( f \in H \), we know from Corollary F.13 that each invocation of LocateInner fails with probability at most \( M \cdot 2^{-\Omega(\mathcal{R}_{\text{vote}})} \). By the union bound, the failure probability of LocateSignal is at most

\[
\mathcal{R}_{\text{search}} \cdot M \cdot 2^{-\Omega(\mathcal{R}_{\text{vote}})} = \underbrace{O(\log(F \cdot T))}_{\mathcal{R}_{\text{search}}} \cdot \underbrace{2^{\Theta(d \cdot \log(\mathcal{C} \cdot d))}}_{M} \cdot 2^{-\Omega(\mathcal{R}_{\text{vote}})}
\]

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where the first step follows because the parameters $R_{\text{search}} = O(\log(T \cdot F))$ and $M = 2^{\Theta(d \cdot \log(C \cdot d))}$ (see Definitions G.1 and F.1); and the last step holds since we choose in Definition F.1 a sufficiently large $R_{\text{vote}} = \Theta(d \cdot \log(C \cdot d) + \log \log(F/\eta))$.

**Performance guarantee.** At the beginning, the initial $j$-th hyperball $\mathbf{H}(0, 2F) = [-F, F]^d$ contains the concerning frequency $f \in H$. If the procedure LOCATE SIGNAL succeeds in all of the first $r \in [R_{\text{search}}]$ iterations, then (Corollary F.13) we locate $f \in H$ within a hyperball that is centered at some frequency $\text{List}_{\text{new}}[j] \in [-F, F]^d$ and has the diameter $L_{\text{dia}}^{\text{new}} = 2\sqrt{d \cdot F \cdot 2^{-r}}$. Formally, we have

$$f \in \mathbf{H}(\text{List}_{\text{new}}[j], L_{\text{dia}}^{\text{new}}).$$

In particular, if all of the $R_{\text{search}}$ iterations succeed, the diameter drops down to the final $L_{\text{dia}} \in (\frac{20d}{T}, \frac{40d}{T}]$. As a consequence, we have

$$f \in \mathbf{H}(\text{List}^*[j], 40d/T).$$

That is, the $\ell_2$-distance between the concerning tone frequency $f \in H$ and the output frequency $\text{List}^*[j]$ is at most $\frac{1}{2} \cdot \frac{40d}{T} = \frac{20d}{T}$.

This completes the proof of Lemma G.4.

**G.5 Stronger guarantees**

The goal of this section is to improve Lemma G.4.

**Lemma G.5 (Stronger guarantees, compared to Lemma G.4).** Given $\Sigma \in \mathbb{R}^{d \times d}$ and $b \in \mathbb{R}^d$ (according to Definition E.7), the output list $\text{List}^*$ of procedure LOCATE SIGNAL (Algorithm 5) contains at most $B = 2^{O(d \cdot \log d)} \cdot k$ many frequencies with minimum separation $\Omega(\eta)$. Let $H \subseteq \text{supp}(\hat{x}^*)$ be a subset of “good” tone frequencies:

$$H = \{ \xi \in \text{supp}(\hat{x}^*) : \text{neither } E_{\text{off}}(\xi) \text{ nor } E_{\text{coll}}(\xi) \text{ happens} \}$$

For any good frequency $f \in H$, suppose that its signal-to-noise ratio $\rho(i) \geq \mathcal{C}$ (see Definition H.1), then with probability at least 99%, there exists an output frequency $f' \in \text{List}^*$ such that

$$\| f - f' \|_2 \lesssim C_* \cdot \frac{1}{\rho T}.$$  

**Proof.** The proof follows from Lemma F.16. 

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H Sparse recovery

| Statement | Section | Algorithm | Comment |
|-----------|---------|-----------|---------|
| Definition H.1 | Section H.1 | None | Definitions and facts |
| Lemma H.3 | Section H.2 | Algorithm 6 | Estimate signal |
| Lemma H.4 | Section H.3 | Algorithm 7 | One stage, sample complexity and running time |
| Lemma H.5 | Section H.4 | Algorithm 7 | One stage, guarantees |
| Lemma H.9 | Section H.5 | Algorithm 8 | Multi stage |
| Lemma H.11 | Section H.6 | Algorithm 9 | Merged stage, running time |
| Lemma H.14 | Section H.7 | Algorithm 9 | Merged stage, guarantees |
| Lemma H.16 | Section H.8 | Algorithm 10 | Running merged stage twice |
| Theorem H.18 | Section H.9 | Algorithm 10 | Recovery stage |

Table 3: List of Lemmas/Algorithms in sparse recovery section

H.1 Definitions and facts

Definition H.1 (Signal-to-noise ratio). For the $i$-th tone $(v_i, f_i)$, define the signal-to-noise ratio $ho_i := |v_i|/\mu_i \geq 0$, where the noise $\mu_i \geq 0$ is given by

$$\mu_i^2 = \mathbb{E}_{\Sigma, b, a} [\hat{u}_j \cdot e^{-2\pi i a^\top f_i} - v_i]^2,$$

where $j = h_{\Sigma, b}(f_i) \in [B]^d$ is the bin that the tone frequency $f_i \in [-F, F]^d$ is hashed into according to Definition E.10, and $\hat{u}_j = \hat{u} \cdot e^{-\pi i / B} \cdot \|j\|_1$.

Definition H.2 (Hypercube). For any frequency $f \in \mathbb{R}^d$ and any $L_{\text{edge}} \geq 0$, we denote by $\text{HC}(f, L_{\text{edge}})$ the $\ell_\infty$-norm hypercube with center $f \in \mathbb{R}^d$ and the edge length $L_{\text{edge}}$:

$$\text{HC}(f, L_{\text{edge}}) := \{\xi \in \mathbb{R}^d : \|\xi - f\|_\infty \leq L_{\text{edge}} / 2\}.$$

H.2 EstimateSignal

The goal of this section is to prove Lemma H.3.

Algorithm 6 EstimateSignal

1: procedure EstimateSignal($\Sigma, b, a, D, T, \text{List}$)
2: Sample $a \in \mathbb{R}^d$ according to Definition F.1.
3: Let $\hat{u} \leftarrow \text{HashToBins}(\Sigma, b, a, D).$ $\triangleright$ Algorithm 1
4: Let $v'(\xi) = \hat{u}_{h_{\Sigma, b}(\xi)} \cdot e^{-(\pi i / B) \cdot \|h_{\Sigma, b}(\xi)\|_1} \cdot e^{-2\pi i a^\top \xi}$ for $\xi \in \text{List}.$
5: return $\{v'(\xi)\}_{\xi \in \text{List}}$.
6: end procedure

Lemma H.3 (EstimateSignal). The procedure EstimateSignal (Algorithm 6) satisfies that:

- The sample complexity is upper bounded by the sample complexity of the procedure LocateSignal (Algorithm 5).
The running time is upper bounded by the running time of the procedure \textsc{LocateSignal} (Algorithm 5).

Denote by $H \subseteq [k]$ the indices of a subset of true tones $\{(v_i, f_i)\}_{i \in [k]}$ for which neither $E_{\text{coll}}(f_i)$ nor $E_{\text{off}}(f_i)$ happens. There is a subset $S \subseteq H$ and an injection $\pi : S \mapsto [k]$ such that

**Property I:** For the tones in set $S$, the (partial) tone estimation error
\[
\sum_{i \in S} E_{\Sigma,b} \left[ \frac{1}{T_d} \cdot \int_{\tau \in [0,T_d]} \left| v_i' \cdot e^{2\pi i f_i^T \tau} - v_{\pi(i)} \cdot e^{2\pi i f_{\pi(i)}^T \tau} \right|^2 \cdot d\tau \right] \lesssim (C^2 + dC_*)^2 \cdot N^2.
\]

**Property III:** For each tone $i \in S$, the (single) tone estimation error
\[
|v_i' - v_{\pi(i)}| \lesssim (C + \sqrt{dC_*}) \cdot N.
\]

**Proof.** The bounds on the sample complexity and the running time are direct follow-ups to the previous lemmas. Also, Properties I and II will be proved soon after in Lemma H.5; particularly, we will specify the subset $S \subseteq H$ therein.

### H.3 \textsc{OneStage}: algorithm, sample complexity and running time

The goal of this section is to prove Lemma H.4.

**Algorithm 7 \textsc{OneStage}, Lemma H.4, H.5**

1: procedure \textsc{OneStage}(x, $\Sigma, b, D, C, T$)
2: \hspace{1em} List $\leftarrow \text{LocateSignal}(\Sigma, b, D, M, C, T)$.
3: \hspace{1em} for $\xi \in \text{List}$ do
4: \hspace{2em} if either $E_{\text{coll}}(\xi)$ or $E_{\text{off}}(\xi)$ or both happen then
5: \hspace{3em} Remove $\xi$ from List.
6: \hspace{2em} end if
7: \hspace{1em} end for
8: \hspace{1em} $\{v'(\xi)\}_{\xi \in \text{List}} \leftarrow \text{EstimateSignal}(\Sigma, b, a, D, T, \text{List})$ \hspace{1em} $\triangleright$ Algorithm 6
9: \hspace{1em} Add an supplementary list List$_{\sup} = \{(0, \xi)\}_{i=1}^k$, for which $\min_{\xi' \in \text{List}_{\sup}} \|\xi - \xi'\| \geq \eta$, and $\min_{\xi, \xi' \in \text{List}_{\sup}} \|\xi - \xi'\| \geq \eta$.
10: \hspace{1em} return $\{(v'(\xi), \xi)\}_{\xi \in \text{List}} \cup \{(0, \xi)\}_{\xi \in \text{List}_{\sup}}$
11: end procedure

**Lemma H.4 (Sample complexity and running time of \textsc{OneStage}).** The procedure \textsc{OneStage} (Algorithm 7) has the following performance guarantees:

- The sample complexity is $2^{O(d \cdot \log d)} \cdot (\log C + \log \log(F/\eta)) \cdot k \cdot \log(F \cdot T) \cdot D$.
- The running time is $O(2^{O(d \cdot \log d \cdot \log C)} \cdot \log(F \cdot T) \cdot \log \log(F/\eta) \cdot k \cdot (D + \log k))$.
- The output $\{(v'(\xi), \xi)\}_{\xi \in \text{List}}$ contains at most $O(\mathcal{B}) = 2^{O(d \cdot \log d)} \cdot k$ many candidate tones.

**Proof.** It follows directly from previous Lemma.
H.4 OneStage: performance guarantees

The goal of this section is to prove Lemma H.5.

**Lemma H.5** (Guarantees of OneStage). The procedure OneStage (Algorithm 7) has the following performance guarantees. For each true tone \((v_i, f_i)\), it can “succeed” in LocateSignal (Algorithm 5) with probability at least 0.99. More specifically, let \(S \subseteq H\) denote the set of successful tones in LocateSignal (Algorithm 5). There exists an injection \(\pi : S \mapsto [k]\) such that

- **Property I:** Each true tone \((v_i, f_i)\) succeeds with probability \(\Pr[i \in S] \geq 0.9\) and if so, those tones whose signal-to-noise ratio \(\rho(i) \geq C\) (see Definition H.1) has the estimation error
  \[
  \|f'_i - f_{\pi(i)}\|_2 \lesssim C_\ast \frac{1}{\rho_{\pi(i)} \cdot T}. 
  \]

- **Property II:** For all the successfully recovered tones \(S\), the (partial) tone estimation error
  \[
  \sum_{i \in S} \mathbb{E}_{\Sigma,b} \left[ \frac{1}{T^d} \cdot \int_{\tau \in [0,T]^d} \left| v'_i \cdot e^{2\pi i f_{\pi(i)}^\top \tau} - v_{\pi(i)} \cdot e^{2\pi i f_{\pi(i)}^\top \tau}\right|^2 \cdot d\tau \right] \lesssim (C^2 + dC_\ast^2) \cdot N^2. 
  \]

- **Property III:** For each successfully recovered tone \(i \in S\), if its signal-to-noise ratio \(\rho(i) \geq C\), the (single) tone estimation error
  \[
  |v'_i - v_{\pi(i)}| \lesssim (C + \sqrt{d}C_\ast) \cdot N. 
  \]

The performance guarantees on the sample complexity, the duration, the success probability, and the running time are controlled by the counterpart performance guarantees of the subroutine LocateSignal. For ease of presentation, here we omit the formal proofs of these performance guarantees.

**Claim H.6** (Property I of Lemma H.5). For each successfully recovered tone \(i \in S\) with large enough signal-to-noise ratio, the frequency estimation error
\[
\|f'_i - f_{\pi(i)}\|_2 \lesssim C_\ast \frac{1}{\rho_{\pi(i)} \cdot T}. 
\]

**Proof.** This follows directly from Lemma G.5.

**Claim H.7** (Property II of Lemma H.5). For all the successfully recovered tones \(S\), the (partial) tone estimation error
\[
\sum_{i \in S} \mathbb{E}_{\Sigma,b} \left[ \frac{1}{T^d} \cdot \int_{\tau \in [0,T]^d} \left| v'_i \cdot e^{2\pi i f_{\pi(i)}^\top \tau} - v_{\pi(i)} \cdot e^{2\pi i f_{\pi(i)}^\top \tau}\right|^2 \cdot d\tau \right] \lesssim (C^2 + dC_\ast^2) \cdot N^2. 
\]

**Proof.** Consider a specific true tone \((v_{\pi(i)}, f_{\pi(i)})\) that \(i \in S \subseteq H\), for which neither \(E_{\text{coll}}(f_{\pi(i)})\) nor \(E_{\text{off}}(f_{\pi(i)})\) happens, and this tone “succeeds” in LocateSignal (Algorithm 5). Assume w.l.o.g. that the tone frequency is hashed into the bin \(j = h_{\Sigma,b}(f_{\pi(i)}) \in [B]^d\) (according to Definition E.10). For simplicity, we adopt the following notations in this proof:

- \(j' = h_{\Sigma,b}(f'_i) \in [B]^d\) is the bin where the estimation frequency \(f'_i\) hashed into;
Lemma E.20 is applicable. That is, the tone frequency holds. The procedure \( E_{\Sigma,a} \) is the estimation magnitude returned by the procedure \( \text{Estimatesignal} \) (Algorithm 6);

- \( v_i' = \hat{u}_{i'}' \cdot e^{-2\pi i a^T f_i'} \in \mathbb{C} \) is the estimation magnitude returned by the procedure \( \text{Estimatesignal} \) (Algorithm 6);

- \( v_i'' = \hat{u}_{i''} \cdot e^{-2\pi i a^T f_i} \in \mathbb{C} \); and

- \( \mu_{\pi(i)}^2 = \mathbb{E}_{\Sigma,a}[|\hat{u}_{i'}' \cdot e^{-2\pi i a^T f_{\pi(i)}} - v_{\pi(i)}|^2] \geq 0 \) according to Definition H.1.

We discuss two cases for the signal-to-noise ratio \( \rho_{\pi(i)} = \frac{|v_i|}{\mu_{\pi(i)}} \).

**Case (i):** when the signal-to-noise ratio \( \rho_{\pi(i)} = \frac{|v_i|}{\mu_{\pi(i)}} \geq C \), i.e. when the premise for Lemma F.5 holds.

According to Markov inequality, the equation below holds with probability at least \( 1 - C^{-2} \geq 0.9999 \) (given that \( C \geq 120 \); see Definition F.1). In what follows, We assume that this equation holds.

\[
|\hat{u}_{i'}' \cdot e^{-2\pi i a^T f_{\pi(i)}} - v_{\pi(i)}| \leq C^2 \cdot \mu_{\pi(i)}^2.
\]

(36)

The procedure \( \text{Locatesignal} \) and the follow-up procedures have the desired performance guarantees. In particular, we derive a "good" frequency estimation \( f_i' \in [-F,F]^d \) from the procedure \( \text{Locatesignal} \). Also, it follows from Lemma G.5 that

\[
||f_i' - f_{\pi(i)}||_2 \leq C \cdot \frac{1}{\rho_{\pi(i)} \cdot T}.
\]

(37)

Since we choose a large enough duration \( T = \Omega(\frac{d 5.5}{y} \log(dk/\delta) \log d) \) finally according to Theorem 1.2 and \( \rho_{\pi(i)} \geq C \geq 120 \), the \( \ell_2 \)-norm frequency error \( ||f_i' - f_{\pi(i)}||_2 \) is sufficiently small and thus Lemma E.20 is applicable. That is, the tone frequency \( f_{\pi(i)} \) and the estimation frequency \( f_i' \) are hashed into the same bin \( j = j' \in [B]^d \) (see Lemma E.20). Combining the above arguments together, we have

\[
\begin{align*}
&\frac{1}{Td} \cdot \int_{\tau \in [0,T]^d} \left| v_i' \cdot e^{2\pi i f_i' T \tau} - v_{\pi(i)} \cdot e^{2\pi i f_{\pi(i)} T \tau} \right|^2 \cdot d\tau \\
&= \frac{1}{Td} \cdot \int_{\tau \in [0,T]^d} \left| \hat{u}_{i'}' \cdot e^{-2\pi i a^T f_i'} \cdot e^{2\pi i f_i' T \tau} - v_{\pi(i)} \cdot e^{2\pi i f_{\pi(i)} T \tau} \right|^2 \cdot d\tau \\
&= \frac{1}{Td} \cdot \int_{\tau \in [0,T]^d} \left| \hat{u}_{i'}' \cdot e^{-2\pi i a^T f_i'} \cdot e^{2\pi i f_i' T \tau} - v_{\pi(i)} \cdot e^{2\pi i f_{\pi(i)} T \tau} \right|^2 \cdot d\tau \\
&\leq \frac{1}{Td} \cdot \int_{\tau \in [0,T]^d} \left| \hat{u}_{i'}' \cdot e^{-2\pi i a^T f_i'} \cdot e^{2\pi i f_i' T \tau} - v_{\pi(i)} \cdot e^{2\pi i f_{\pi(i)} T \tau} \right|^2 \cdot d\tau \\
&\leq \frac{1}{Td} \cdot \int_{\tau \in [0,T]^d} \left| v_{\pi(i)} \cdot e^{2\pi i f_i' T \tau} - v_{\pi(i)} \cdot e^{2\pi i f_{\pi(i)} T \tau} \right|^2 \cdot d\tau \\
&\leq A_1(\tau) + 2 \cdot |v_{\pi(i)}|^2 \cdot (1 - \text{sincT}(f_i' - f_{\pi(i)})) \\
&\leq A_1(\tau) + 2 \cdot |v_{\pi(i)}|^2 \cdot \left(\frac{\pi^2}{6} \cdot T^2 \cdot ||f_i' - f_{\pi(i)}||^2_2\right) \\
&\leq A_1(\tau) + C^2 \cdot |v_{\pi(i)}|^2 / \rho_{\pi(i)}^2
\end{align*}
\]

(38)

where the first step is by the definition of \( v_i' \) (see Algorithm 6); the second step follows because the tone frequency \( f_{\pi(i)} \) and the estimation frequency \( f_i' \) are hashed into the same bin \( j = j' \in [B]^d \);
the third step applies the triangle inequality; the forth step applies Property II of Lemma I.4 to the second summand; the fifth step applies Part (e) of Fact I.2; the last step applies Equation (37).

Then we consider $A_1(\tau)$. We have

$$A_1(\tau) = \frac{1}{T^d} \cdot \int_{\tau \in [0,T]^d} \left| \hat{u}_j \cdot e^{-2\pi i \cdot \mathbf{a}^T \tau} \cdot e^{2\pi i \cdot f_{\pi(i)}^T \tau} - v_{\pi(i)} \cdot e^{2\pi i \cdot f_{\pi(i)}^T \tau} \right|^2 \cdot d\tau$$

$$\leq \frac{1}{T^d} \cdot \int_{\tau \in [0,T]^d} \left| \hat{u}_j \cdot e^{-2\pi i \cdot \mathbf{a}^T \tau} \cdot e^{2\pi i \cdot f_{\pi(i)}^T \tau} - v_{\pi(i)} \cdot e^{2\pi i \cdot f_{\pi(i)}^T \tau} \cdot e^{-2\pi i \cdot \mathbf{a}^T (f'_{\pi(i)} - f_{\pi(i)})} \right|^2 \cdot d\tau$$

$$+ \frac{1}{T^d} \cdot \int_{\tau \in [0,T]^d} \left| v_{\pi(i)} \cdot e^{2\pi i \cdot f_{\pi(i)}^T \tau} \cdot e^{-2\pi i \cdot \mathbf{a}^T (f'_{\pi(i)} - f_{\pi(i)})} - v_{\pi(i)} \cdot e^{2\pi i \cdot f_{\pi(i)}^T \tau} \right|^2 \cdot d\tau$$

$$= \left| \hat{u}_j \cdot e^{-2\pi i \cdot \mathbf{a}^T \tau} - v_{\pi(i)} \right|^2 \cdot \frac{1}{T^d} \cdot \int_{\tau \in [0,T]^d} \left| e^{2\pi i \cdot f_{\pi(i)}^T \tau} \right|^2 \cdot d\tau$$

$$+ \frac{1}{T^d} \cdot \int_{\tau \in [0,T]^d} \left| v_{\pi(i)} \cdot e^{2\pi i \cdot f_{\pi(i)}^T \tau} - v_{\pi(i)} \right|^2 \cdot \frac{1}{T^d} \cdot \int_{\tau \in [0,T]^d} \left| e^{-2\pi i \cdot \mathbf{a}^T (f'_{\pi(i)} - f_{\pi(i)})} \right|^2 \cdot d\tau$$

$$\leq C^2 \cdot \mu_2^2 + \frac{1}{T^d} \cdot \int_{\tau \in [0,T]^d} \left| v_{\pi(i)} \right|^2 \cdot \left| e^{-2\pi i \cdot \mathbf{a}^T (f'_{\pi(i)} - f_{\pi(i)})} - 1 \right|^2$$

$$\leq C^2 \cdot \mu_2^2 + \left| v_{\pi(i)} \right|^2 \cdot \left| a^T (f'_{\pi(i)} - f_{\pi(i)}) \right|^2$$

$$\leq C^2 \cdot \mu_2^2 + \left| v_{\pi(i)} \right|^2 \cdot \left| a \right|_2 \cdot \left| f'_{\pi(i)} - f_{\pi(i)} \right|_2$$

$$\leq C^2 \cdot \mu_2^2 + \left| v_{\pi(i)} \right|^2 \cdot \left| a \right|_2 \cdot \left| f'_{\pi(i)} - f_{\pi(i)} \right|_2$$

$$\leq C^2 \cdot \mu_2^2 + \left| v_{\pi(i)} \right|^2 / \rho_{\pi(i)}^2$$

(39)

where the second step is by triangle inequality; the fourth step is by the integral equals to 1 and follows from Equation (36); the sixth step is because $|e^{2\pi i x} - 1| \leq |x|$; the last step is because $\|a\|_2 \leq \sqrt{dT}$ (see Algorithm 3) and $\|f'_{\pi(i)} - f_{\pi(i)}\|_2 \lesssim C_s / (T \rho_{\pi(i)})$, which implies $|a^T (f'_{\pi(i)} - f_{\pi(i)})| \lesssim \sqrt{dT} / \rho_{\pi(i)}$.

Combine Equation (38) and Equation (39) together, we prove that

$$\frac{1}{T^d} \cdot \int_{\tau \in [0,T]^d} \left| v'_{\pi(i)} \cdot e^{2\pi i \cdot f_{\pi(i)}^T \tau} - v_{\pi(i)} \cdot e^{2\pi i \cdot f_{\pi(i)}^T \tau} \right|^2 \cdot d\tau \lesssim C^2 \cdot \mu_2^2 + \left| v_{\pi(i)} \right|^2 / \rho_{\pi(i)}^2$$

$$\lesssim (C^2 + dC_s^2) \cdot \mu_2^2$$

The last step follows because the signal-to-noise ratio $\rho_{\pi(i)} = |v_{\pi(i)}| / \mu_{\pi(i)}$ (see Definition H.1). This accomplishes Case (i).

**Case (ii):** when the signal-to-noise ratio $\rho_{\pi(i)} = |v_{\pi(i)}| / \mu_{\pi(i)} \leq C$, i.e. when the premise for Lemma F.5 does not hold. Under this case, we can use $(0, f'_{\pi})$ to recover the true tone $(v_{\pi}, f_{\pi})$, as

$$\frac{1}{T^d} \cdot \int_{\tau \in [0,T]^d} \left| v'_{\pi} \cdot e^{2\pi i \cdot f_{\pi}^T \tau} - v_{\pi} \cdot e^{2\pi i \cdot f_{\pi}^T \tau} \right|^2 \cdot d\tau = |v_{\pi(i)}|^2 \leq C^2 \cdot \mu_2^2.$$

Recall that we add a supplementary list $\mathbb{List}_{sup}$ in the Line 9 of OneStage, and there are enough candidates with zero magnitude and minimum separation in the list. We can let $S$ include some tones $(0, \xi)$ from $\mathbb{List}_{sup}$ when needed.

This completes the proof. $\square$

**Claim H.8** (Property III of Lemma H.5). For each successfully recovered tone $i \in S$, the (single) tone estimation error

$$|v'_{i} - v_{\pi(i)}| \lesssim (C + \sqrt{d}C_s) \cdot N.$$
Proof. This can be directly inferred from Claim H.7.

H.5 MultiStage

The goal of this section is to prove Lemma H.9.

Lemma H.9 (MultiStage). The procedure MultiStage (Algorithm 8) satisfies the following:

- The sample complexity is $R_{\text{merge}}$ times the sample complexity of OneStage (Algorithm 7).
- The running time is $R_{\text{merge}}$ times the running time of OneStage (Algorithm 7).

Proof. All these properties can be easily inferred from Lemma H.4.

Algorithm 8 MultiStage

1: procedure MultiStage($x, D, C, T, R_{\text{merge}}$) \hspace{1cm} $\triangleright$ Lemma H.9
2: \hspace{1.4em} Let $L_{\text{ist}}^* \leftarrow \emptyset$.
3: \hspace{1.4em} for $r = 1, 2, \ldots, R_{\text{merge}}$ do
4: \hspace{2.8em} Sample $\Sigma \in \mathbb{R}^{d \times d}$ and $b \in \mathbb{R}^d$ according to Definition E.7.
5: \hspace{2.8em} $L_{\text{ist}}^{\text{new}} \leftarrow \text{OneStage}(x, \Sigma, b, D, C, T)$ \hspace{1.4em} $\triangleright$ Algorithm 7
6: \hspace{2.8em} $L_{\text{ist}}^* \leftarrow L_{\text{ist}}^* \cup L_{\text{ist}}^{\text{new}}$.
7: \hspace{1.4em} end for
8: \hspace{1.4em} return the tones $L_{\text{ist}}^*$.
9: end procedure

Lemma H.10 (Guarantees of MultiStage). The procedure MultiStage (Algorithm 8) repeat the procedure OneStage (Algorithm 7) for $R_{\text{merge}} = \Theta(d \cdot \log d \cdot \log k)$ times, and returns a set $L_{\text{ist}} = \{(v^*_i, f^*_i)\}_{i \in [m]}$ of $m = |L_{\text{ist}}| = 2^{O(d \cdot \log d) \cdot k \cdot \log k} \in \mathbb{N}_{\geq 1}$ many candidate tones. With probability at least $1 - 1/\text{poly}(k)$, there are at least $R_{\text{merge}}$ different disjoint subsets $\{S_r\}$ of $L_{\text{ist}}$, where for each $r$ we have that $S_r \subset L_{\text{ist}}$ and $|S_r| \leq k$, and for $r \neq r'$ we have that $S_r \cap S_{r'} = \emptyset$. For each $S_r$, there is an injective projection $\pi_r : S_r \to [k]$ and has the following properties:

Property I: For each true tone $(v_i, f_i)$, $\Pr[i \in S_r] \geq 0.9$ and if the signal-to-noise ratio is large enough, the frequency estimation error

$$
\|f^*_i - f_{\pi_r(i)}\|_2 \lesssim C_* \frac{1}{\rho_{\pi_r(i)} \cdot T}.
$$

Property II: For all the successfully recovered tones $S_r$, the (partial) tone estimation error

$$
\sum_{i \in S_r} \mathbb{E}_{\Sigma, b} \left[ \frac{1}{T^2} \cdot \int_{\tau \in [0, T]} |v^*_i \cdot e^{2\pi i f^*_r \tau} - v_{\pi_r(i)} \cdot e^{2\pi i f^*_{\pi_r(i)} \tau}|^2 \cdot d\tau \right] \lesssim (C^2 + dC_*^2) \cdot N^2.
$$

Property III: For each successfully recovered tone $i \in S_r$, if its signal-to-noise ratio is large enough, then we can bound the (single) tone estimation error

$$
|v^*_i - v_{\pi_r(i)}| \lesssim (C + \sqrt{dC_*}) \cdot N.
$$

Proof. This lemma can be proved directly by Lemma H.5.
H.6 MergedStage: algorithm and running time

The goal of this section is to prove Lemma H.11. The concerning Algorithm 9 (MergedStage) is demonstrated in Figure 19.

Lemma H.11 (MergedStage, Input size and Running time). The procedure MergedStage (Algorithm 9) has the following properties:

- The input $\text{List} = \{(v_j, f_j)\}_{j \in [m]}$ is a multi-set of $m = |\text{List}| = 2^{O(d \log d)} \cdot k \cdot R_{\text{merge}}$ many candidate tones, where $R_{\text{merge}} = \Theta(d \cdot \log d \cdot \log k)$ is sufficiently large.
- The running time is $2^{O(d \log d)} \cdot k \cdot R_{\text{merge}} \cdot \log^d (k \cdot R_{\text{merge}}) = 2^{O(d \log d)} \cdot k \cdot \log^O(d) k$.

Proof. The first property about the input $\text{List}$ is guaranteed by Lemma H.9.

The second property is proved by using a well-known data-structure. We use a textbook $d$-dimensional range tree data-structure (see section 5 in [KSBO00]).

Theorem H.12 (Theorem 5.11 in [KSBO00]). Let $P$ be a set of $n$ points in $d$-dimensional space, with $d \geq 2$. A layered range tree for $P$ uses $O(n \log^{d-1} n)$ storage and it can be constructed in $O(n \log^{d-1} n)$ time. With this range tree one can report the points in $P$ that lie in a rectangular query range in $O(\log^{d-1} n + q)$ time, where $q$ is the number of reported points.

The above theorem works for $\ell_{\infty}$-norm. By choosing

$$n = m = |\text{List}|$$

we complete the proof of running time. 



Algorithm 9 MergedStage, Lemmas H.11 and H.14

1: procedure MergedStage($\text{List}, R_{\text{merge}}$)
2: Denote $\text{List} = \{(v_j, f_j)\}_{j \in [m]}$ for $m = |\text{List}|$.
3: Build a $d$-dimensional segment tree $\text{Tree}$ on the frequencies $\{f_j\}_{j \in [m]} \subseteq [-F, F]^d$.
4: All these frequencies $\{f_j\}_{j \in [m]}$ are unmarked.
5: $\text{List}^* \leftarrow \emptyset$.
6: while $\text{Tree}$ has at least one unmarked frequency $\xi_i$ from $\text{Tree}$. do
7: Choose an arbitrary unmarked frequency $\xi_i$ from $\text{Tree}$.
8: if $\text{Tree}.\text{COUNT}(\text{HC}(\xi_i, \eta/d^3)) \geq 8/10 \cdot R_{\text{merge}}$ then $\triangleright$ Theorem H.12
9: $f^* \leftarrow \xi_i$.
10: $v^* \leftarrow \text{median}\{v_j : j \in [m] \text{ and } f_j \in \text{HC}(\xi_i, \eta/d^3)\}$.
11: $\text{List}^* \leftarrow \text{List}^* \cup (v^*, f^*)$.
12: Delete $\{f_j : j \in [m] \text{ and } f_j \in \text{HC}(\xi_i, \eta/(10\sqrt{d}))\}$ from $\text{Tree}$.
13: Delete $\{(v_j, f_j) : j \in [m] \text{ and } f_j \in \text{HC}(\xi_i, \eta/(10\sqrt{d}))\}$ from $\text{List}$.
14: else
15: Mark the chosen frequency $\xi_i$ in $\text{Tree}$.
16: end if
17: end while
18: return the tones $\text{List}^*$.
19: end procedure
Figure 19: Demonstration for Algorithm 9 in one dimension \((d = 1)\) and two dimensions \((d = 2)\).
H.7 MergedStage: performance guarantees

The goal of this section is to prove Lemma H.14.

Claim H.13 (Approximate formula for tone-wise error). The following holds for any pair of tones 
\((v, f) \in \mathbb{C} \times \mathbb{R}^d\) and \((v^*, f^*) \in \mathbb{C} \times \mathbb{R}^d\):

\[
err := \frac{1}{T^2} \int_{\tau \in [0,T]^d} |v \cdot e^{2\pi i f^T \tau} - v^* \cdot e^{2\pi i f^* T \tau}|^2 \cdot d\tau
\approx |v - v^*|^2 + |v^*|^2 \cdot (1 - \text{sinc}_T(f - f^*))
\approx |v - v^*|^2 + |v^*|^2 \cdot \min\{1, T^2 \cdot \|f - f^*\|_2^2\}.
\]

Proof. We first prove the second part of the claim, which is equivalent to

\[1 - \text{sinc}_T(f - f^*) \approx \min\{1, T^2 \cdot \|f - f^*\|_2^2\}. \tag{40}\]

Indeed, when \(T^2 \cdot \|f - f^*\|_2^2 \geq \left(\frac{2.05}{\pi}\right)^2 \approx 1\), we know from Part (d) of Fact I.2 that

\[1 - \text{sinc}_T(f - f^*) = 1 \pm \frac{1}{2} \approx 1.\]

And when \(T^2 \cdot \|f - f^*\|_2^2 < \left(\frac{2.05}{\pi}\right)^2 \approx 1\), we know from Part (c) of Fact I.2 that

\[1 - \text{sinc}_T(f - f^*) \geq 1 - \exp\left(-\left(\frac{\pi^2}{6}\right) \cdot T^2 \cdot \|f - f^*\|_2^2\right) \gtrsim T^2 \cdot \|f - f^*\|_2^2,\]

and that

\[1 - \text{sinc}_T(f - f^*) \leq 1 - \exp\left(-\left(\frac{\pi^2}{5}\right) \cdot T^2 \cdot \|f - f^*\|_2^2\right) \lesssim T^2 \cdot \|f - f^*\|_2^2.\]

Combining the above arguments together implies Equation (40).

In what follows, we prove the first part of the claim that

\[err \approx |v - v^*|^2 + |v^*|^2 \cdot (1 - \text{sinc}_T(f - f^*)). \tag{41}\]

We know Property II of Lemma I.4 that

\[err = |v|^2 + |v^*|^2 - (v \cdot \overline{v^*} + \overline{v} \cdot v^*) \cdot \text{sinc}_T(f - f^*).\]

For brevity, we denote \(w_1 \cdot e^{i\theta} = v/v^*\) for some norm \(w_1 \geq 0\) and some phase \(\theta \in [0,2\pi]\), and denote \(w_2 = \text{sinc}_T(f - f^*) \in [-\frac{1}{4}, 1]\) (see Part (e) of Fact I.2). We notice that the formula

\[|v - v^*|^2 + |v^*|^2 \cdot (1 - \text{sinc}_T(f - f^*))\]

is non-negative. As a consequence, to verify Equation (41), it suffices to show that the following function \(L(w_1, w_2, \theta) \approx 1\), for any \(w_1 \geq 0\), any \(w_2 \in [-\frac{1}{4}, 1]\) and any \(\theta \in [0,2\pi]\):

\[
L(w_1, w_2, \theta) := \frac{err}{\text{RHS of (41)}}
= \frac{|v^* \cdot w_1 \cdot e^{i\theta}|^2 + |v^*|^2 - 2 \cdot |v^* \cdot w_1 \cdot e^{i\theta}| \cdot |v^*| \cdot \cos(\theta) \cdot w_2}{|v^* \cdot w_1 \cdot e^{i\theta} - v^*|^2 + |v^*|^2 \cdot (1 - w_2)}.
\]
For any fixed $w_1$, $w_2$ and $\theta$; the third step divides both the numerator and the denominator by $|v'_1|^2$; and the last step can be seen via elementary calculation.

Let us investigate the partial derivative $\frac{\partial L}{\partial \theta}$ in $\theta \in [0, 2\pi)$:

$$
\frac{\partial L}{\partial \theta} = \frac{2 \cdot w_1 \cdot w_2 \cdot \sin(\theta)}{w_1^2 - w_2 + 2 - 2 \cdot w_1 \cos(\theta)} - \frac{w_1^2 + 1 - 2 \cdot w_1 \cdot w_2 \cdot \cos(\theta)}{(w_1^2 - w_2 + 2 - 2 \cdot w_1 \cos(\theta))^2} \cdot 2 \cdot w_1 \cdot \sin(\theta)
$$

$$
= -\sin(\theta) \cdot 2 \cdot w_1 \cdot \frac{w_1^2 \cdot (1 - w_2) + (1 - w_2)^2}{(w_1^2 - w_2 + 2 - 2 \cdot w_1 \cos(\theta))^2}.
$$

where the second step can be seen via elementary calculation.

Because $w_1 \geq 0$ and $w_2 \in [-\frac{1}{4}, 1]$, we must have $A_4 \geq 0$. As a result, for any fixed $w_1$ and $w_2$, the function $L(w_1, w_2, \theta)$ is non-increasing when $\theta \in [0, \pi]$, and is non-decreasing when $\theta \in [\pi, 2\pi)$.

The functions $L_{\min}(w_1, w_2) := \min_{\theta \in [0, 2\pi)} L(w_1, w_2, \theta)$ and $L_{\max}(w_1, w_2) := \max_{\theta \in [0, 2\pi]} L(w_1, w_2, \theta)$ for any $w_1 \geq 0$ and any $w_2 \in [-\frac{1}{4}, 1]$ are given by

$$
L_{\min}(w_1, w_2) = L(w_1, w_2, \pi) = \frac{A_5(w_1, w_2)}{A_6(w_1, w_2)},
$$

$$
A_5(w_1, w_2) := w_1^2 + 1 - 2 \cdot w_1 \cdot w_2,
$$

$$
A_6(w_1, w_2) := w_1^2 + 2 \cdot w_1 - w_2 + 2,
$$

and

$$
L_{\max}(w_1, w_2) = L(w_1, w_2, 0) = \frac{A_7(w_1, w_2)}{A_8(w_1, w_2)},
$$

$$
A_7(w_1, w_2) := w_1^2 + 1 - 2 \cdot w_1 \cdot w_2,
$$

$$
A_8(w_1, w_2) := w_1^2 - 2 \cdot w_1 - w_2 + 2.
$$

We now justify the lower-bound part of Equation (41) by exploring the function $L_{\min}(w_1, w_2)$. For any fixed $w_1 \geq 1$, the numerator $A_5(w_1, w_2)$ is a non-decreasing function in $w_2 \in [-\frac{1}{4}, 1]$, while the denominator $A_6(w_1, w_2)$ is a non-increasing non-negative function in $w_2 \in [-\frac{1}{4}, 1]$. Given these, we can infer the lower-bound part of Equation (41) as follows:

$$
L(w_1, w_2, \theta) \geq \min_{w_1 \in [0, 1]} \min_{w_2 \in [-\frac{1}{4}, 1]} L_{\min}(w_1, w_2)
$$

$$
= \min_{w_1 \in [0, 1]} L_{\min}(w_1, -1/4)
$$

$$
= \min_{w_1 \in [0, 1]} \frac{w_1^2 + 1 - (1/2) \cdot w_1}{w_1^2 + 2 \cdot w_1 - (1/4) + 2}
$$

$$
\approx 0.3107,
$$

where the last step can be seen via numeric calculation.

We next show the upper-bound part of Equation (41) by exploring the function $L_{\max}(w_1, w_2)$. For any fixed $w_1 \geq 1$, both of the numerator $A_7(w_1, w_2)$ and the denominator $A_8(w_1, w_2)$ are linear
functions in \(w_2 \in [-\frac{1}{4}, 1]\). Accordingly, \(L_{\text{max}}(w_1, w_2)\) itself is a monotone function in \(w_2 \in [-\frac{1}{4}, 1]\). We can infer the upper-bound part of Equation (41) as follows:

\[
\max_{w_1 \in [0, 1]} \max_{w_2 \in [-\frac{1}{4}, 1]} L_{\text{max}}(w_1, w_2) = \max_{w_1 \in [0, 1]} \max \left\{ L_{\text{max}}(w_1, -1/4), L_{\text{max}}(w_1, 1) \right\}
\]

\[
= \max_{w_1 \in [0, 1]} \max \left\{ \frac{w_1^2 + 1 + (1/2) \cdot w_1}{w_1^2 - 2 \cdot w_1 + (9/4)}, 1 \right\}
\approx 2.7247,
\]

where the last step can be seen via numeric calculation.

This completes the proof. \(\square\)

**Lemma H.14 (Guarantees for MergedStage).** The procedures MergedStage (Algorithm 9) returns a set \(\text{List} = \{ (v_i', f_i') \}_{i \in [m]}\) of \(m = |\text{List}| = 2^{O(d \log d)} \cdot k \in \mathbb{N}_{\geq 1}\) many candidate tones. With probability at least \(1 - 1/\text{poly}(k)\), the outputs \(\text{List} = \{ (v_i', f_i') \}_{i \in [m]}\) satisfies the following:

**Property I:** The set size \(m = 2^{O(d \log d)} \cdot k\), and the frequency separation

\[
\min_{i,j \in [m]} \| f_i' - f_j' \|_2 \gtrsim \eta/\sqrt{d}.
\]

**Property II:** For the true tones \(\{ (v_i, f_i) \}_{i \in [k]}\), there is an injection \(\pi : [k] \to [m]\) such that

\[
\sum_{i \in [k]} \frac{1}{T^d} \cdot \int_{\tau \in [0,T]^d} \left| v'_{\pi(i)} \cdot e^{2\pi i f_i' \tau} - v_i \cdot e^{2\pi i f_i \tau} \right|^2 \cdot d\tau \leq (C^2 + dC_s^2) \cdot N^2.
\]

**Proof.** The size of output is straightforward from Algorithm 9. If we add one candidate tone into the output, we will delete at least \(8/10 \cdot \mathcal{R}_{\text{merge}}\) tones.

**Property I:** The set size can be induced from proof of the Property II. As for the frequency separation, it comes from that if we choose to take the median of \(\text{HC}(\xi_i, \eta/d^3)\) for \(\xi_i\), we will clear a larger region \(\text{HC}(\xi_i, \eta/(10\sqrt{d}))\).

It is safe to clear the larger region, as we have an assumption that \(\min_{i \neq j} \| f_i - f_j \|_2 \geq \eta\), which implies that \(\min_{i \neq j} \| f_i - f_j \|_\infty \geq \eta/\sqrt{d}\) for true tones \(\{ (v_i, f_i) \}\). Suppose \(\xi_i\) is a successful recovery of true tone \(f_i\). Then if we find a cluster of successful recovered tones \(\text{HC}(\xi_i, \eta/d^3)\), for all other successful recovered tones \(\xi_j\) where \(j \neq i\), we have that

\[
\| \xi_i - \xi_j \|_\infty = \| \xi_i - f_i + f_i - f_j + f_j - \xi_j \|_\infty \\
\geq \| f_i - f_j \|_\infty - \| \xi_i - f_i \|_\infty - \| f_j - \xi_j \|_\infty \\
\geq \| f_i - f_j \|_2/\sqrt{d} - \| \xi_i - f_i \|_2 - \| f_j - \xi_j \|_2 \\
\geq \eta/\sqrt{d} - 2C_s/(\rho T) \\
\geq \eta/\sqrt{d} - 2C_s/T \\
\geq \eta/\sqrt{d}
\]

where the second step follows from triangle inequality, the third step follows from \(\| \cdot \|_2/\sqrt{d} \leq \| \cdot \|_\infty \leq \| \cdot \|_2\), the last step follows from \(T \geq C_s \sqrt{d}/\eta\).

This means that \(\xi_j \notin \text{HC}(\xi_i, \eta/(10\sqrt{d}))\) and proves the safety of the operation.

**Property II:**
For each true tone \((v_i, f_i)\), by Lemma H.10, with probability at least \(1 - 1/\text{poly}(k)\), there are at least \(0.8\mathcal{R}_{\text{merge}}\) successful recovery \(\{(v'_i, f'_i)\}\) of it, where \(\|f'_i - f_i\|_2 \lesssim C_i/(\rho T)\). By the choice of duration \(T = \Omega(d^3 \cdot \eta^{-1} \cdot \log(kd/\delta))\) by Lemma F.14, we know that \(\|f'_i - f_i\|_\infty \leq \|f'_i - f_i\|_2 \ll \eta/d^3\).

And let \(\mu_i^2(f_i)\) denote the expected error of successful recovery \((v'_i, f'_i)\):

\[
\mu_i^2(f_i) = \mathbb{E}_{\Sigma, b, v'_i, f'_i} \left[ \frac{1}{T^d} \int_{t \in [0, T]^d} |v'_i \cdot e^{2\pi i f_i^\top t} - v_i \cdot e^{2\pi i f_i^\top t}|^2 \cdot dt \right]
\]

Then by Markov Inequality, we know that

\[
\Pr \left[ \int_{t \in [0, T]^d} |v'_i \cdot e^{2\pi i f_i^\top t} - v_i \cdot e^{2\pi i f_i^\top t}|^2 \cdot dt \geq 10\mu_i^2(f_i) \right] \leq 1/10. \tag{42}
\]

By Lemma H.5, we can bound the summation of expected errors of successful recovery:

\[
\sum_{i \in [k]} \mu_i^2(f_i) \lesssim (C^2 + dC^2) \cdot \mathcal{N}^2. \tag{43}
\]

As a summary, for each true tone \((v_i, f_i)\), we have shown that there are at least \(0.8\mathcal{R}_{\text{merge}}\) successful recovery \(\{(v'_i, f'_i)\}\) of it, i.e. \(\text{Tree.count(HC}(\xi_i, \eta/d^3)) \geq 8/10 \cdot \mathcal{R}_{\text{merge}}\). Then we will take the any frequency \(f_i^*\) in \(\text{HC}(f_i, \eta/d^3)\) in Line 9 and coordinate-wise median of magnitude \(v_i^*\) of successful recovery in \(\text{HC}(f_i, \eta/d^3)\) in Line 10.

Among the successful recovery \(\{(v'_i, \xi_i)| f_i' \in \text{HC}(\xi_i, \eta/d^3)\}\) of \((v_i, f_i)\), with probability \(1 - 1/\text{poly}(k)\), at least half of them will have error less than \(10\mu_i^2(f_i)\). Note that \(f^* = \xi_i\). To be more specific, with probability at least \(1 - 1/\text{poly}(k)\),

\[
\frac{1}{T^d} \int_{t \in [0, T]^d} |v_i^* \cdot e^{2\pi i f_i^*^\top t} - v_i \cdot e^{2\pi i f_i^\top t}|^2 \cdot dt \lesssim \mu_i^2(f_i),
\]

Then we have

\[
\sum_{i \in [k]} \frac{1}{T^d} \int_{t \in [0, T]^d} |v_i' \cdot e^{2\pi i f_i'^\top t} - v_i \cdot e^{2\pi i f_i^\top t}|^2 \cdot dt \lesssim \sum_{i \in [k]} \mu_i^2(f_i) \leq (C^2 + dC^2) \cdot \mathcal{N}^2.
\]

This completes the proof. \(\square\)

### H.8 Running MergedStage twice

The goal of this section is to prove Lemma H.16.

**Definition H.15** (Setup for RecoveryStage). Given two sets

\[
\mathcal{L}_{\text{List}}^1 = \{(v'_i, f'_i)\}_{i \in [k']}
\]

and

\[
\mathcal{L}_{\text{List}}^2 = \{(v''_i, f''_i)\}_{i \in [k'']}
\]

of sizes \(k', k'' = 2^{O(d \cdot \log d)} \cdot k \in \mathbb{N}_{\geq 1}\), output each pair \((v''_i, f''_i)\) in the second set (for \(i \in [k'']\)) that has a small frequency distance \(\|f''_i - f'_i\|_2 \leq \epsilon/T\), against some frequency \(f'_j\) in the first set (for \(j \in [k']\)). Denote the resulting set by \(\{(v''_i, f''_i)\}_{i \in S} \subseteq \{(v''_i, f''_i)\}_{i \in [k'']}\) of size \(|S| = k^* \leq k'' = 2^{O(d \cdot \log d)} \cdot k\).

\[\footnote{Note that we only need to take coordinate wise median for \(v\), for frequency \(f\), using \(\xi_i\) is good enough. Since \(\xi_i\) is close to the true \(f\).} \]
Algorithm 10 RECOVERYSTAGE

1: procedure RECOVERYSTAGE\(x, D, C, T\) \> Theorem H.18
2: \> \>
3: \> \>
4: \> \>
5: \> \>
6: \> \>
7: \> \>
8: \> \>
9: \> \>
10: \> \>
11: end procedure

Lemma H.16 (Running MERGEDSTAGE twice). Given two sets \(\{(v'_i, f'_i)\}_{i \in [k']}\) and \(\{(v''_i, f''_i)\}_{i \in [k'']}\) of sizes \(k', k'' = \mathcal{O}(d \log d) \cdot k \in \mathbb{N}_{\geq 1}\), assume w.l.o.g. that Definition H.15 selects \(k' \leq k''\) pairs \(\{(v''_i, f''_i)\}_{i \in [k']}\) of the second set, then these \(k' = \mathcal{O}(d \log d) \cdot k\) pairs can be reindexed such that

\[
\sum_{i \in \mathbb{N}_{\geq 1}} \frac{1}{Td} \cdot \int_{\tau \in [0,T]d} \left| v_i \cdot e^{2\pi i f'_i \tau} - v'_i \cdot e^{2\pi i f''_i \tau} \right|^2 \cdot d\tau + \sum_{i \in \mathbb{N}_{\geq 1}} |v''_i|^2 \leq C^2 \cdot \mathcal{N}^2.
\]

Proof. By Claim H.13, the following holds for any pair of tones \((v, f) \in \mathbb{C} \times \mathbb{R}^d\) and \((v^*, f^*) \in \mathbb{C} \times \mathbb{R}^d\):

\[
\text{err}((v, f), (v^*, f^*)) = \frac{1}{Td} \cdot \int_{\tau \in [0,T]d} \left| v \cdot e^{2\pi i f \tau} - v^* \cdot e^{2\pi i f^* \tau} \right|^2 \cdot d\tau \approx |v - v^*|^2 + (|v^*|^2 + |v|^2) \cdot \min(1, T^2 \cdot \|f - f^*\|_2^2)
\]

Then by Lemma H.14, with probability at least \(1 - 1/\text{poly}(k)\), there is a permutation of the output of the first run \(\{(v'_i, f'_i)\}_{i \in [k']}\) and an injective projection \(\pi : [k'] \to [k']\), subject to

\[
\sum_{i \in \mathbb{N}_{\geq 1}} \left( (|v'_i|^2 + |v_{\pi(i)}|^2) \cdot \min(1, T^2 \cdot \|f'_i - f_{\pi(i)}\|_2^2) + |v'_i - v_{\pi(i)}|^2 \right) \leq (C^2 + dC^2)\mathcal{N}^2
\]

If \(\|f'_i - f_{\pi(i)}\|_2 > 1/T\), then \(\text{err}((0, f'_i), (v_{\pi(i)}, f_{\pi(i)})) \leq \text{err}((v'_i, f'_i), (v_{\pi(i)}, f_{\pi(i)}))\). Let \(S = \{i \in \mathbb{N}_{\geq 1} : \|f'_i - f_{\pi(i)}\|_2 \leq c/T\}\) for any \(c = O(1)\). We can rewrite the result:

\[
\sum_{i \in S} \left( (|v'_i|^2 + |v_{\pi(i)}|^2) \cdot \min(1, T^2 \cdot \|f'_i - f_{\pi(i)}\|_2^2) + |v'_i - v_{\pi(i)}|^2 \right) + \sum_{i \in \mathbb{N}_{\geq 1}/S} \left( |v_{\pi(i)}|^2 + |v'_i|^2 \right) \leq (C^2 + dC^2)\mathcal{N}^2.
\]

If we can know the set \(S\) and the right permutation of the output of the first run, we can output a set of tones that meet this lemma easily. But the problem is that we do not have the information. This is why we run the MERGEDSTAGE twice. Recall that the signal \(x^*\) we want to recover is defined by \(\{v_i, f_i\}_{i=1}^k\), then it is equivalent to define \(x^*\) by \(\{v_i, f_i\}_{i=1}^k \cup \{0, f'_i\}_{i=1}^{k'}\), where \(f'_i\) is the output of the first run of MERGEDSTAGE. Then the number of frequencies is \(\mathcal{O}(d \log d)k\) and the separation gap is \(\Omega(\eta/\sqrt{d})\), then Lemma H.14 applies again.

Define

\[
S' = \{i \in [k''] : \exists j \in [k'], \|f'_i - f''_j\|_2 \leq 1/T\},
\]

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and we can reindex \( \{v_i'', f_i''\} \) such that:

\[
(C^2 + dC^2)N^2 \gtrsim \sum_{S' \subseteq [k]} (|v''_i|^2 + |v_i|^2) \min \{1, T^2 \cdot \|f''_i - f_i\|_2^2\} + (|v''_i|^2 + |v_{\pi(i)}|^2)
+ \sum_{i \in S' \setminus [k]} (|v''_i|^2 + |v_i|^2) \cdot \min \{1, T^2 \cdot \|f''_i - f_i\|_2^2\} + |v''_i| - |v_i| 0^2
+ \sum_{i \in [k] \setminus S'} (|v''_i|^2 + |v_i|^2) + \sum_{i \in [k'] \setminus (S' \cup [k])} |v''_i|^2
\geq \sum_{S' \subseteq [k]} (|v''_i|^2 + |v_i|^2) \min \{1, T^2 \cdot \|f''_i - f_i\|_2^2\} + (|v''_i|^2 + |v_{\pi(i)}|^2)
+ \sum_{i \in S' \setminus [k]} |v''_i|^2 + \sum_{i \in [k] \setminus S'} |v_i|^2
\]

This is exactly the summation of error of \( \{(v_i'', f_i'')\}_{i \in S'} \), and \( |S'| = k^* \leq k'' = 2^{O(d \log d)} k \) which complete the proof.

\[ \square \]

### H.9 RecoveryStage

The goal of this section is to prove Theorem H.18. Before the proof of the main result, we need the following lemma:

**Lemma H.17.** The following holds for any three tones \((v_{\pi(i)}, f_{\pi(i)}) \in \mathbb{C} \times \mathbb{R}^d \) and \((v_i^*, f_i^*) \in \mathbb{C} \times \mathbb{R}^d \) and \((v_i', f_i') \in \mathbb{C} \times \mathbb{R}^d \) that \( |v_i'| \gtrsim |v_i^*| \approx |v_{\pi(i)}|\):

\[
\begin{align*}
1/d \cdot \int_{\tau \in [0,T]^d} &\left| v_i' \cdot e^{2\pi i f_i' \tau} - v_{\pi(i)} \cdot e^{2\pi i f_{\pi(i)}' \tau}\right|^2 \cdot d\tau \\
&\lesssim \frac{1}{T^d} \cdot \int_{\tau \in [0,T]^d} \left| v_i^* \cdot e^{2\pi i f_{\pi(i)}' \tau} - v_{\pi(i)} \cdot e^{2\pi i f_{\pi(i)}' \tau}\right|^2 \cdot d\tau + |v_i'|^2.
\end{align*}
\]

**Proof.** We will show in Property II of Lemma I.4 (see Section I.2) that

\[
\begin{align*}
A_1 &= |v_i'|^2 + |v_{\pi(i)}|^2 - (v_i' \cdot \overline{v_{\pi(i)}} + v_{\pi(i)} \cdot \overline{v_i'}) \cdot \text{sinc}_T(f_i' - f_{\pi(i)}), \\
A_2 &= |v_i^*|^2 + |v_{\pi(i)}|^2 - (v_i^* \cdot \overline{v_{\pi(i)}} + v_{\pi(i)} \cdot \overline{v_i^*}) \cdot \text{sinc}_T(f_i^* - f_{\pi(i)}).
\end{align*}
\]

Because \( |v_i'| \gtrsim |v_i^*| \approx |v_{\pi(i)}| \), we can easily verify the lemma by elementary calculation (notice that \( |\text{sinc}_T(f_i^* - f_{\pi(i)})| \leq 1 \) and \( |\text{sinc}_T(f_i^* - f_{\pi(i)})| \leq 1 \) for any \( f_{\pi(i)}, f_i^*, f_i' \in \mathbb{R}^d \)).

\[ \square \]

**Theorem H.18 (RecoveryStage, formal of Theorem 1.2).** Let

\[
T \geq \frac{d^{4.5} \log(kd/\delta) \log d}{\eta}.
\]

The procedure RecoveryStage (Algorithm 10) takes

\[
2^{O(d \log d)} \cdot k \cdot \log^{d+1}(k/\delta) \cdot \log(F/\eta) \cdot \log \log(F/\eta)
\]

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samples over \([0, T]\), runs in
\[
2^{O(d \log d)} \cdot k \cdot \log^O(d)(k/\delta) \cdot \log(F/\eta) \cdot \log\log(F/\eta).
\]
time and outputs a set \(\{(v'_i, f'_i)\}_{i \in [k]} \subset \mathbb{C} \times \mathbb{R}^d\) of size \(k \in \mathbb{N}_{\geq 1}\) such that the following hold with probability \(1 - 1/\text{poly}(k)\)

**Property I** Magnitude estimation
\[
|v_i - v'_i| \leq (C + \sqrt{dC_*}) \cdot N, \forall i \in [k].
\]

**Property II** Frequency estimation
\[
\|f_i - f'_i\|_2 \leq C_* \frac{1}{\rho \cdot T}, \forall i \in [k].
\]

**Property III** Tone estimation (Total)
\[
\sum_{i \in [k]} \frac{1}{T^d} \cdot \int_{\tau \in [0, T]^d} \left| v_i \cdot e^{2\pi i f_i^\top \tau} - v'_i \cdot e^{2\pi i f'_i^\top \tau} \right|^2 \cdot d\tau \lesssim (C^2 + dC_*^2) \cdot N^2.
\]

**Property IV** The frequency separation of output frequencies
\[
\min_{i \neq j \in [k]} \|f'_i - f'_j\|_2 \geq \eta/2.
\]

**Claim H.19** (Sample complexity, running time and duration of Theorem H.18).

*Proof. Sample complexity.*
\[
\mathcal{R}_{\text{merge}} : 2^{\Theta(d \log d)} \cdot (\log C + \log \log(F/\eta)) \cdot k \cdot \log(F \cdot T) \cdot D
= 2^{\Theta(d \log d)} \cdot k \cdot \log^{d+1}(k/\delta) \cdot \log\log(F/\eta) \cdot \log(F/\eta)
\]

*Running time.*
\[
\mathcal{R}_{\text{merge}}(2^{O(d \log d)} \cdot k \cdot \log^d(k \cdot \mathcal{R}_{\text{merge}}) + 2^{\Theta(d \log d + \log^2 C)}) \cdot \log(F \cdot T) \cdot \log\log(F/\eta) \cdot k \cdot (D + \log k)
= 2^{O(d \log d)} \cdot k \cdot \log^d(k) + 2^{\Theta(d \log d)} \cdot \log(F/\eta) \cdot k \cdot \log\log(F/\eta) \cdot \log^{d+1}(k/\delta)
= 2^{O(d \log d)} \cdot k \cdot \log^d(k/\delta) \cdot \log(F/\eta) \cdot \log\log(F/\eta).
\]

*Duration.* As we run MERGEDSTAGE twice, the first run for \(k\)-sparsity signal and the second run for \(k' = 2^{O(d \log d)}k\)-sparsity signal, \(\eta' = \frac{\eta}{\sqrt{d}}\), by Lemma G.3, the duration is
\[
T = \Omega\left(\frac{d^3 \log(dk'/\delta)}{\eta'}\right) = \Omega\left(\frac{d^{4.5} \log(dk/\delta) \log d}{\eta}\right).
\]

**Claim H.20** (Property I of Lemma H.18).
\[
|v_i - v'_i| \leq (C + \sqrt{dC_*}) \cdot N, \forall i \in [k]
\]

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Proof. This proof is a direct application of previous Lemma. □

Claim H.21 (Property II of Lemma H.18).

\[ \| f_i - f'_i \|_2 \leq C \cdot \frac{1}{P} \cdot \frac{1}{T}, \forall i \in [k] \]

Proof. The proof is a direct application of previous Lemma. □

Claim H.22 (Property III of Lemma H.18).

\[ \sum_{i \in [k]} \frac{1}{T_d} \cdot \int_{\tau \in [0,T]^d} \left| v_i \cdot e^{2\pi i f_i^T \tau} - v'_i \cdot e^{2\pi i f'_i^T \tau} \right|^2 \cdot d\tau \lesssim (C^2 + dC^2) \cdot N^2. \]

Proof. We denote by \( \{(v^*_i, f^*_i)\}_{i \in [k']} \) the set of tones derived according to Definition H.15, where \( k'' = 2^{O(d \log d)} \cdot k \) (and we safely assume \( k'' \geq k \) in view of Lemma H.14). We assume w.l.o.g. that each \( (v^*_i, f^*_i) \in \mathbb{C} \times \mathbb{R}^d \) of the top-k largest-magnitude tones (for each \( i \in [k] \)) is mapped to a true tone \( (v_{\pi(i)}, f_{\pi(i)}) \in \mathbb{C} \times \mathbb{R}^d \) according to Lemma H.16.

Let \( \{(v'_i, f'_i)\}_{i \in [k]} \) be a subset of the recovered tones \( \{(v^*_i, f^*_i)\}_{i \in [k']} \) that have the top-k largest magnitudes; these \( k \in \mathbb{N}_{\geq 1} \) tones together form the output of the procedure RECOVERYSTAGE (Algorithm 10). Upon reindexing, we safely assume that \( |v'_i| \geq |v^*_i| \) for each \( i \in [k] \). Also, we know from Lemma H.16 that \( \min_{i \neq j \in [k]} \|f'_i - f'_j\| \gtrsim \eta \).

For each \( i \in [k] \), let us consider these three tones \( (v_{\pi(i)}, f_{\pi(i)}) \) and \( (v^*_i, f^*_i) \) and \( (v'_i, f'_i) \). In the case \( i \in S \) for which \( (v^*_i, f^*_i) \neq (v'_i, f'_i) \), it follows from Lemma H.17 that

\[ \frac{1}{T_d} \cdot \int_{\tau \in [0,T]^d} \left| v'_i \cdot e^{2\pi i f'_i^T \tau} - v_{\pi(i)} \cdot e^{2\pi i f_{\pi(i)}^T \tau} \right|^2 \cdot d\tau \lesssim \frac{1}{T_d} \cdot \int_{\tau \in [0,T]^d} \left| v^*_i \cdot e^{2\pi i f^*_i^T \tau} - v_{\pi(i)} \cdot e^{2\pi i f_{\pi(i)}^T \tau} \right|^2 \cdot d\tau + |v'_i|^2. \]

And in the other case \( i \in S \subseteq [k] \) for which \( (v^*_i, f^*_i) = (v'_i, f'_i) \), of course we have

\[ \frac{1}{T_d} \cdot \int_{\tau \in [0,T]^d} \left| v'_i \cdot e^{2\pi i f'_i^T \tau} - v_{\pi(i)} \cdot e^{2\pi i f_{\pi(i)}^T \tau} \right|^2 \cdot d\tau = \frac{1}{T_d} \cdot \int_{\tau \in [0,T]^d} \left| v^*_i \cdot e^{2\pi i f^*_i^T \tau} - v_{\pi(i)} \cdot e^{2\pi i f_{\pi(i)}^T \tau} \right|^2 \cdot d\tau. \]

Taking all the indices \( i \in [k] \) into account, we know from the above two equations that

\[ \sum_{i \in [k]} \frac{1}{T_d} \cdot \int_{\tau \in [0,T]^d} \left| v'_i \cdot e^{2\pi i f'_i^T \tau} - v_{\pi(i)} \cdot e^{2\pi i f_{\pi(i)}^T \tau} \right|^2 \cdot d\tau \lesssim \sum_{i \in [k]} \frac{1}{T_d} \cdot \int_{\tau \in [0,T]^d} \left| v^*_i \cdot e^{2\pi i f^*_i^T \tau} - v_{\pi(i)} \cdot e^{2\pi i f_{\pi(i)}^T \tau} \right|^2 \cdot d\tau + \sum_{i \in S} |v'_i|^2. \]

Because \( \{(v'_i, f'_i)\}_{i \in [k]} \) are chosen to be the top-k largest-magnitude recovered tones among \( \{(v^*_i, f^*_i)\}_{i \in [k']} \), the set \( S \) involved in the summation \( \sum_{i \in S} |v'_i|^2 \) only includes those small-magnitude recovered tones. We thus conclude that

\[ \sum_{i \in [k]} \frac{1}{T_d} \cdot \int_{\tau \in [0,T]^d} \left| v'_i \cdot e^{2\pi i f'_i^T \tau} - v_{\pi(i)} \cdot e^{2\pi i f_{\pi(i)}^T \tau} \right|^2 \cdot d\tau \lesssim (C^2 + dC_s^2) \cdot N^2. \]

This completes the proof of Property III of Theorem H.18. □
Claim H.23 (Property IV of Lemma H.18).

\[
\min_{i \neq j \in [k]} \| f'_i - f'_j \|_2 \geq \eta/2.
\]

Proof. Since \( \min_{i \neq j \in [k]} \| f_i - f_j \|_2 \geq \eta \) and for all \( i \in [k] \), \( \| f_i - f'_i \|_2 \leq \min(C, \frac{1}{\delta T}, d/T) \leq \eta/10 \) (given Lemma G.4, Claim H.21 and the duration

\[
T = \Omega\left(\frac{d^{1.5} \log(kd/\delta) \log d}{\eta}\right),
\]

) we can infer the current claim. \( \square \)
I Converting tone estimation into signal estimation

This section is structured in the following way:

- Section I.1 provides some basic definitions and mathematical facts.
- Section I.2 split the signal estimation error into the tone-wise errors (which we call the diagonal terms) and the cross-tone errors (which we call the off-diagonal terms).
- Section I.3 provides an upper bound for the cross-tone errors (i.e. the off-diagonal terms) via some advanced analytic tools.
- Section I.4 combines everything together, converting the tone estimation error into the signal estimation error as desired.

For ease of presentation, throughout this section we would shift the sampling time domain from \( t \in [0,T]^d \) to \( t \in [-T/2, T/2]^d \).

### I.1 Preliminaries and mathematical facts

In this part, we introduce some useful notations and mathematical facts. Recall Definition C.1 for the functions \( \text{rect}_{s_1}(\xi) \) and \( \sin_{s_1}(\tau) \) in the single-dimensional setting (namely \( \xi, \tau \in \mathbb{R} \)). Below, we define in Definition I.1 two counterpart functions (renoted by \( \text{rect}_{s_1}(\xi) \) and \( \sin_{s_1}(\tau) \) for convenience) when \( \xi, \tau \in \mathbb{R}^d \) are \( d \)-dimensional vectors, and then show in Fact I.2 several properties of these functions (which can be easily inferred from Fact C.2 or the previous literature like [CKPS16]).

**Definition I.1** (Two basic functions). Given any \( s_1 > 0 \), for all \( \xi, \tau \in \mathbb{R}^d \), the \( \text{rect}_{s_1}(\xi) \) function and the \( \sin_{s_1}(\tau) \) function are defined as follows:

- \( \text{rect}_{s_1}(\xi) = \prod_{r \in [d]} \text{rect}_{s_1}(\xi_r) \) for any \( \xi \in \mathbb{R}^d \). When \( s_1 = 1 \), we shorthand it as \( \text{rect}(\xi) \).
- \( \sin_{s_1}(\tau) = \prod_{r \in [d]} \sin_{s_1}(\tau_r) \) for any \( \tau \in \mathbb{R}^d \). When \( s_1 = 1 \), we shorthand it as \( \sin(\tau) \).

**Fact I.2** (Facts about basic functions). Given any \( s_1 > 0 \), the following hold for the functions \( \sin_{s_1}(\tau) \) and \( \text{rect}_{s_1}(\xi) \) in the \( d \)-dimensional setting, as Figure 20 suggests:

- **Part (a):** \(|\sin_{s_1}(\tau)| \leq \prod_{r \in [d]} \min\{1, \frac{1}{\pi \cdot s_1 \cdot \tau_r}\} \) for any \( \tau \in \mathbb{R}^d \).
- **Part (b):** \( \sin_{s_1}(\tau) = \overline{\text{rect}_{s_1}}(\tau) \) for any \( \tau \in \mathbb{R}^d \), and \( \text{rect}_{s_1}(\xi) = \overline{\sin_{s_1}}(\xi) \) for any \( \xi \in \mathbb{R}^d \).
- **Part (c):** \( \exp(-\frac{\pi^2 s_1^2}{6} \cdot \|\tau\|^2) \leq \sin_{s_1}(\tau) \leq \exp(-\frac{\pi^2 s_1^2}{6} \cdot \|\tau\|^2) \) for any \( \tau \in \mathbb{R}^d \) that \( \|\tau\| \leq \frac{0.05}{\pi s_1} \).
- **Part (d):** \( |\sin_{s_1}(\tau)| \leq \exp(-\frac{0.05^2}{6}) < \frac{1}{2} \) for any \( \tau \in \mathbb{R}^d \) that \( \|\tau\| \geq \frac{0.05}{\pi s_1} \).
- **Part (e):** \( -\frac{1}{4} \leq \sin_{s_1}(\tau) \leq 1 \) and \( \sin_{s_1}(\tau) \geq 1 - \frac{\pi^2}{6} \cdot \sin_{s_1}(\tau) \) for any \( \tau \in \mathbb{R}^d \).

*Also, in the single-dimensional setting:

- **Part (f):** \( \left| \frac{d}{d\tau} \sin_{s_1}(\tau) \right| = \frac{\cos(\pi s_1 \cdot \tau)}{\tau} - \frac{\sin_{s_1}(\tau)}{\tau} \leq \frac{7}{5} \cdot \min\{s_1, \frac{1}{|\tau|}\} \) for any \( \tau \in \mathbb{R} \).
I.2 Tone-wise errors and cross-tone errors

The goal of this section is to prove Lemma I.4. We first start with the following definitions.

**Definition I.3** (Tone-wise errors in time domain). Given any pair of tones $(v_i, f_i) \in \mathbb{C} \times \mathbb{R}^d$ and $(v'_i, f'_i) \in \mathbb{C} \times \mathbb{R}^d$, where $i \in [k]$, the error is given by the complex-valued function $a_i(\tau) \in \mathbb{C}$:

- Define $a_i(\tau) = v_i \cdot e^{2\pi i f_i \tau} - v'_i \cdot e^{2\pi i f'_i \tau}$ for all $\tau \in [-T/2, T/2]^d$ for notational brevity.
- The CFT is given by $\hat{a}_i(\xi) = v_i \cdot \delta_{f_i}(\xi) - v'_i \cdot \delta_{f'_i}(\xi)$ for all $\xi \in \mathbb{R}^d$.

- Define the error $\|a_i\|_T = \left( \frac{1}{T^d} \int_{\tau \in [-T/2, T/2]^d} |a_i(\tau)|^2 \cdot d\tau \right)^{1/2}$, where $\tau \sim \text{Unif}[-T/2, T/2]^d$ is uniformly random, or equivalently,

\[
\|a_i\|_T = \left( \frac{1}{T^d} \int_{\tau \in [-T/2, T/2]^d} a_i(\tau) \cdot \overline{a_i(\tau)} \cdot d\tau \right)^{1/2}
\]

**Lemma I.4** (Tone-wise and cross-tone errors in time domain). Respecting a pair of error functions $a_i(\tau) \in \mathbb{C}$ and $a_j(\tau) \in \mathbb{C}$ given in Definition I.3, where $i, j \in [k]$, the following hold:

**Property I:** When $\tau \sim \text{Unif}[-T/2, T/2]^d$ is uniformly random,

\[
\mathbb{E}_\tau [a_i(\tau) \cdot \overline{a_j(\tau)}] = v_i \cdot \overline{v'_j} \cdot \text{sinc}_T(f_i - f_j) - v_i \cdot v'_j \cdot \text{sinc}_T(f_i - f'_j)
-
\overline{v'_i} \cdot \overline{v'_j} \cdot \text{sinc}_T(f'_i - f_j) + \overline{v'_i} \cdot v'_j \cdot \text{sinc}_T(f'_i - f'_j).
\]
Property II: In the special case that \( i = j \),

\[
\|a_i\|_T^2 = \mathbb{E}\left[ a_i(\tau) \cdot \overline{a_i(\tau)} \right] = |v_i|^2 + |v_i'|^2 - (v_i \cdot \overline{v_i} + \overline{v_i} \cdot v_i') \cdot \text{sinc}T(f'_i - f_i).
\]

Proof. Assume Property I to be true, then we can infer Property II by elementary calculation.

Before proving Property I, let us consider the following function \( y_i(\tau) \) for all \( \tau \in \mathbb{R}^d \):

\[
y_i(\tau) = a_i(\tau) \cdot \mathbb{I}\{ \tau \in [-T/2, T/2]^d \} = T^d \cdot a_i(\tau) \cdot \text{rect}_T(\tau),
\]

as well as its CFT \( \tilde{y}_i(\xi) \) for all \( \xi \in \mathbb{R}^d \):

\[
\tilde{y}_i(\xi) = T^d \cdot a_i \ast \text{rect}_T(\xi) = T^d \cdot \tilde{a}_i \ast \text{rect}_T(\xi) = T^d \cdot v_i \cdot \text{sinc}_T(f_i - \xi) - T^d \cdot v_i' \cdot \text{sinc}_T(f'_i - \xi),
\]

(44)

where the second step applies the convolution theorem; the third step is due to Part (d) of Fact C.2; and the last step follows from Definition I.3 that \( \tilde{a}_i(\xi) = v_i \cdot \Delta_{-f_i}(\xi) - v_i' \cdot \Delta_{-f'_i}(\xi) \) for \( \xi \in \mathbb{R}^d \).

Similar to Equation (44), we also have

\[
\overline{y}_j(\xi) = T^d \cdot v_j \cdot \text{sinc}_T(f_j - \xi) - T^d \cdot v_j' \cdot \text{sinc}_T(f'_j - \xi).
\]

(45)

Based on the above arguments, we deduce that when \( \tau \sim \text{Unif}[-T/2, T/2]^d \) is uniformly random,

\[
\mathbb{E}\left[ a_i(\tau) \cdot \overline{a_j(\tau)} \right] = \frac{1}{T^d} \cdot \int_{\tau \in [-T/2, T/2]^d} a_i(\tau) \cdot \overline{a_j(\tau)} \cdot d\tau
\]

\[
= \frac{1}{T^d} \cdot \int_{\tau \in \mathbb{R}^d} \mathbb{R}(y_i(\tau) \cdot y_j(\tau)) \cdot d\tau
\]

\[
= \frac{1}{T^d} \cdot \int_{\xi \in \mathbb{R}^d} \tilde{y}_i(\xi) \cdot \overline{\tilde{y}_j(\xi)} \cdot d\xi
\]

\[
= T^d \cdot v_i \cdot v_j \cdot \int_{\xi \in \mathbb{R}^d} \text{sinc}_T(f_i - \xi) \cdot \text{sinc}_T(f_j - \xi) \cdot d\xi
\]

\[
- T^d \cdot v_i \cdot \overline{v_j} \cdot \int_{\xi \in \mathbb{R}^d} \text{sinc}_T(f_i - \xi) \cdot \text{sinc}_T(f'_j - \xi) \cdot d\xi
\]

\[
- T^d \cdot v_i' \cdot v_j \cdot \int_{\xi \in \mathbb{R}^d} \text{sinc}_T(f'_i - \xi) \cdot \text{sinc}_T(f_j - \xi) \cdot d\xi
\]

\[
+ T^d \cdot v_i' \cdot \overline{v_j} \cdot \int_{\xi \in \mathbb{R}^d} \text{sinc}_T(f'_i - \xi) \cdot \text{sinc}_T(f'_j - \xi) \cdot d\xi,
\]

where the second step follows because \( y_i(\tau) = T^d \cdot a_i(\tau) \cdot \text{rect}_T(\tau) = 0 \) for any \( \tau \notin [-T/2, T/2]^d \); the third step applies Parseval’s theorem; and the last step employs Equations (44) and (45).
We next give in Equation (46) an explicit formula for $A_1$, and similar formulas respectively for $A_2$ and $A_3$ and $A_4$ can be obtained in the same way. Concretely, we have

$$A_1 = \int_{\xi \in \mathbb{R}^d} \text{sinc}_T(f_i - \xi) \cdot \text{sinc}_T(f_j - \xi) \cdot d\xi$$

$$= \frac{1}{T^d} \int_{\xi \in \mathbb{R}^d} \text{sinc}(Tf_i - \xi) \cdot \text{sinc}(Tf_j - \xi) \cdot d\xi$$

$$= \frac{1}{T^d} \int_{\xi \in \mathbb{R}^d} \text{sinc}(\xi + Tf_i - Tf_j) \cdot \text{sinc}(\xi) \cdot d\xi$$

$$= \frac{1}{T^d} \cdot \text{sinc}(Tf_i - Tf_j)$$

$$= \frac{1}{T^d} \cdot \text{sinc}_T(f_i - f_j), \quad (46)$$

where the second step follows by substitution; the third step also follows by substitution; the fourth step follows from Part (b) of Fact I.5; and the last step follows by substitution.

Applying Equation (46) and the counterpart formulas for $A_2$ and $A_3$ and $A_4$, we conclude that when $\tau \sim \text{Unif}[-T/2, T/2]^d$ is uniformly random,

$$E_{\tau} \left[ a_i(\tau) \cdot a_j(\tau) \right] = v_i \cdot \overline{v_j} \cdot \text{sinc}_T(f_i - f_j) - v_i \cdot \overline{v_j'} \cdot \text{sinc}_T(f_i - f_j')$$

$$- v_i' \cdot \overline{v_j} \cdot \text{sinc}_T(f_i' - f_j) + v_i' \cdot \overline{v_j'} \cdot \text{sinc}_T(f_i' - f_j').$$

This completes the proof. □

I.3 Upper bounding cross-tone errors

Fact I.5. The following hold for the single-/multi-dimensional sinc function:

Part (a): Single dimension. $\text{sinc}(\Delta_r) = \int_{\xi_r \in \mathbb{R}} \text{sinc}(\xi_r + \Delta_r) \cdot \text{sinc}(\xi_r) \cdot d\xi$ for any $\Delta_r \in \mathbb{R}$.

Part (b): Multi dimension. $\text{sinc}(\Delta) = \int_{\xi \in \mathbb{R}^d} \text{sinc}(\xi + \Delta) \cdot \text{sinc}(\xi) \cdot d\xi$ for any $\Delta \in \mathbb{R}^d$.

Proof. Part (b) can be easily inferred from Part (a), since $\text{sinc}(\Delta) = \prod_{r \in [d]} \text{sinc}(\Delta_r)$ is a product and we deal with all the coordinates $r \in [d]$ separately.

We deduce Part (a) as follows:

$$\int_{\xi_r \in \mathbb{R}} \text{sinc}(\xi_r + \Delta_r) \cdot \text{sinc}(\xi_r) \cdot d\xi_r = \int_{\xi_r \in \mathbb{R}} \text{sinc}(\Delta_r - \xi_r) \cdot \text{sinc}(\xi_r) \cdot d\xi_r$$

$$= \text{sinc} * \text{sinc}(\Delta_r)$$

$$= \int_{\tau \in \mathbb{R}} \text{sinc} * \text{sinc}(\tau) \cdot e^{-2\pi i \Delta_r \cdot \tau} \cdot d\tau$$

$$= \int_{\tau \in \mathbb{R}} \text{rect}^2(\tau) \cdot e^{-2\pi i \Delta_r \cdot \tau} \cdot d\tau$$

$$= \int_{-1/2}^{1/2} e^{-2\pi i \Delta_r \cdot \tau} \cdot d\tau$$

$$= \text{sinc}(\Delta_r),$$

where the first step follows by substitution and the fact that $\text{sinc}(\xi_r)$ is an even function; the second step follows from Definition A.2; the third step follows the definition of the CFT; the fourth step

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applies the convolution Theorem as $\text{sinc} * \text{sinc}(\tau) = \text{rect}(\tau) \cdot \text{rect}(\tau) = \text{rect}^2(\tau)$; the fifth follows because $\text{rect}(\tau) = I\{\tau \leq 1/2\}$; and the last step can be seen via elementary calculation.

This completes the proof.

\textbf{Lemma I.6} (Upper bounds on the cross-tone errors). For any pair of indices $i < j \in [k]$. Assume the following for both $(v, f, v', f') = (v_i, f_i, v'_i, f'_i)$ and $(v, f, v', f') = (v_j, f_j, v'_j, f'_j)$:

\begin{itemize}
  \item $\|f - f'\|_2 \leq \Delta f_{i,j}$, where the distance $\Delta f_{i,j} \geq 0$ is given by

  \[ \Delta f_{i,j} := \min \left\{ \|f' - f''\|_2 : f' \in \{f_i, f'_i\} \text{ and } f'' \in \{f_j, f'_j\} \right\}. \]

\end{itemize}

Then for the functions $a_i(\tau) : \mathbb{R}^d \to \mathbb{C}$ and $a_j(\tau) : \mathbb{R}^d \to \mathbb{C}$ given in Definition I.3, the cross-tone error satisfies the following when $T = \Omega\left(\frac{d}{\Delta f_{i,j}}\right)$ is large enough:

\[ |E_\tau[a_i(\tau) \cdot \overline{a_j(\tau)} + \overline{a_i(\tau)} \cdot a_j(\tau)]| \lesssim \frac{\sqrt{d}}{\Delta f_{i,j} \cdot T} \cdot \|a_i\|_T \cdot \|a_j\|_T, \]

where $\tau \sim \text{Unif}[-T/2, T/2]^d$ is uniformly random.

\textbf{Proof.} In this proof, we use $f_{i,s} \in \mathbb{R}$ to denote the $s$-th coordinate of the $i$-th frequency $f_i \in \mathbb{R}^d$.

For simplicity of notation, we define

\[ \text{err}_{i,j} := E_\tau[a_i(\tau) \cdot \overline{a_j(\tau)} + \overline{a_i(\tau)} \cdot a_j(\tau)]. \]

According to Property II of Lemma I.4, we can rewrite $\text{err}_{i,j}$ as follows:

\[ \text{err}_{i,j} = E_\tau\left[a_i(\tau) \cdot \overline{a_j(\tau)} + \overline{a_i(\tau)} \cdot a_j(\tau)\right] \]

\[ = (v_i \cdot \overline{v_j} + \overline{v_i} \cdot v_j) \cdot \text{sinc}(f_i - f_j) - (v_i \cdot \overline{v_j} + \overline{v_i} \cdot v_j) \cdot \text{sinc}(f'_i - f'_j) \]

\[ - (v'_i \cdot \overline{v_j} + \overline{v'_i} \cdot v_j) \cdot \text{sinc}(f'_i - f_j) + (v'_i \cdot \overline{v_j} + \overline{v'_i} \cdot v_j) \cdot \text{sinc}(f'_i - f'_j). \]

Below, we would prove the lemma based on case analysis.

In total there are four cases:

\begin{itemize}
  \item $T \cdot \|f'_i - f_i\|_2 \geq \frac{2.05}{\pi}$ and $T \cdot \|f'_j - f_j\|_2 \geq \frac{2.05}{\pi}$ (see Claim I.7).
  \item $T \cdot \|f'_i - f_i\|_2 < \frac{2.05}{\pi}$ and $T \cdot \|f'_j - f_j\|_2 < \frac{2.05}{\pi}$ (see Claim I.8).
  \item $T \cdot \|f'_i - f_i\|_2 \geq \frac{2.05}{\pi}$ and $T \cdot \|f'_j - f_j\|_2 < \frac{2.05}{\pi}$ (see Claim I.9).
  \item $T \cdot \|f'_i - f_i\|_2 < \frac{2.05}{\pi}$ and $T \cdot \|f'_j - f_j\|_2 \geq \frac{2.05}{\pi}$ (see Claim I.9).
\end{itemize}

Combining all the four cases completes the proof.

\textbf{I.3.1 Both pairs are far}

\textbf{Claim I.7} (Case (i) for Lemma I.6). If $T \cdot \|f'_i - f_i\|_2 \geq \frac{2.05}{\pi}$ and $T \cdot \|f'_j - f_j\|_2 \geq \frac{2.05}{\pi}$, then we have

\[ |\text{err}_{i,j}| \lesssim \frac{1}{\Delta f_{i,j} \cdot T} \cdot \|a_i\|_T \cdot \|a_j\|_T. \]
Proof. Let us first bound the tone-wise errors \(\|a_i\|^2_T\) and \(\|a_j\|^2_T\) from below. Respecting the \(i\)-th pair of tones \((v_i, f_i)\) and \((v'_i, f'_i)\), we know from Part (d) of Fact I.2 that
\[
|\text{sinc}_T(f'_i - f_i)| < \frac{1}{2}.
\]
Then, the tone-wise error \(\|a_i\|^2_T\) between \((v_i, f_i)\) and \((v'_i, f'_i)\) admits the lower bound
\[
\|a_i\|^2_T = |v_i|^2 + |v'_i|^2 - (v_i \cdot \overline{v}_i + \overline{v}_i \cdot v_i') \cdot \text{sinc}_T(f'_i - f_i)
\]
\[
= |v_i|^2 + |v'_i|^2 - 2 |v_i| \cdot |v'_i| \cdot \cos(\arg(v'_i/v_i)) \cdot \text{sinc}_T(f'_i - f_i)
\]
\[
\geq |v_i|^2 + |v'_i|^2 - 2 |v_i| \cdot |v'_i| \cdot \left|\cos(\arg(v'_i/v_i))\right| \cdot |\text{sinc}_T(f'_i - f_i)|
\]
\[
\geq |v_i|^2 + |v'_i|^2 - 2 |v_i| \cdot |v'_i| \cdot 1 \cdot \frac{1}{2}
\]
\[
\geq \frac{1}{4} \cdot (|v_i| + |v'_i|)^2,
\]
(48)
where the first step is by Property II of Lemma I.4; the fourth step follows since \(|\cos(\arg(v'_i/v_i))| \leq 1\) and \(|\text{sinc}_T(f'_i - f_i)| < \frac{1}{2}\); and the last step applies the AM-GM inequality.

Applying the same arguments to the \(j\)-th tone-wise error \(\|a_j\|^2_T\), we also have
\[
\|a_j\|^2_T \geq \frac{1}{4} \cdot (|v_j| + |v'_j|)^2.
\]
(49)

We next establish an upper bound on the cross-tone error \(|E_T[a_i(\tau) \cdot a_j(\tau) + a_i(\tau) \cdot a_j(\tau)]|\), where \(\tau \sim \text{Unif}[-\frac{T}{2}, \frac{T}{2}]^2\) is uniformly random. For simplicity, we denote
\[
\text{sinc}^\text{max}_{T,i,j} = \max \left\{ \left| \text{sinc}_T(f''_i - f''_j) \right| : f''_i \in \{f_i, f'_i\} \text{ and } f''_j \in \{f_j, f'_j\} \right\} \geq 0.
\]

Following Equation (47), we deduce that
\[
\left|\text{err}_{i,j}\right| = \left| (v_i \cdot \overline{v}_i + \overline{v}_i \cdot v_j) \cdot \text{sinc}_T(f_i - f_j) - (v_i \cdot \overline{v}_i + \overline{v}_i \cdot v_j') \cdot \text{sinc}_T(f_i - f'_j)
\right|
\]
\[
- (v'_i \cdot \overline{v}_i + \overline{v}_i \cdot v_j) \cdot \text{sinc}_T(f'_i - f_j) + (v'_i \cdot \overline{v}_i + \overline{v}_i \cdot v_j') \cdot \text{sinc}_T(f'_i - f'_j)\right|
\]
\[
\leq 2 |v_i| \cdot |v_j| \cdot \left|\text{sinc}_T(f_i - f_j)\right| + 2 |v_i| \cdot |v_j'| \cdot \left|\text{sinc}_T(f_i - f'_j)\right|
\]
\[
+ 2 |v'_i| \cdot |v_j| \cdot \left|\text{sinc}_T(f'_i - f_j)\right| + 2 |v'_i| \cdot |v_j'| \cdot \left|\text{sinc}_T(f'_i - f'_j)\right|
\]
\[
\leq 2 |v_i| \cdot |v_j| \cdot \text{sinc}^\text{max}_{T,i,j} + 2 |v_i| \cdot |v_j'| \cdot \text{sinc}^\text{max}_{T,i,j}
\]
\[
+ 2 |v'_i| \cdot |v_j| \cdot \text{sinc}^\text{max}_{T,i,j} + 2 |v'_i| \cdot |v_j'| \cdot \text{sinc}^\text{max}_{T,i,j}
\]
\[
= 2 \cdot (|v_i| + |v_i'|) \cdot (|v_j| + |v_j'|) \cdot \text{sinc}^\text{max}_{T,i,j}
\]
\[
\leq 2 \cdot (2 \cdot \|a_i\|_T) \cdot (2 \cdot \|a_j\|_T) \cdot \text{sinc}^\text{max}_{T,i,j}
\]
\[
= 8 \cdot \|a_i\|_T \cdot \|a_j\|_T \cdot \text{sinc}^\text{max}_{T,i,j},
\]
(51)
where the second step uses the triangle inequality; the third step follows from the definition of \(\text{sinc}^\text{max}_{T,i,j}\) (see Equation (50)); the fifth step follows from Equations (48) and (49); and the last step follows from the AM-GM inequality.

To accomplish Case (i), given Equation (51), we are left to justify that \(\text{sinc}^\text{max}_{T,i,j} \geq 0\) diminishes to zero when \(T > 0\) goes to the infinity (at the claimed rate). We safely assume \(T = \Omega\left(\frac{d}{\lambda T_{\text{sr},i,j}}\right)\) to be large enough, and consider a specific pair of frequencies \(f''_i \in \{f_i, f'_i\}\) and \(f''_j \in \{f_j, f'_j\}\). For
simplicity, we denote \( \delta_r = \max(0, \pi \cdot T \cdot |f''_{i,r} - f''_{j,r}| - 1) \geq 0 \) for each coordinate \( r \in [d] \). Given these, one can easily see that

\[
\sum_{r \in [d]} \delta_r \geq \sum_{r \in [d]} (\pi \cdot T \cdot |f''_{i,r} - f''_{j,r}| - 1) \\
= \pi \cdot T \cdot \|f''_{i} - f''_{j}\|_1 - d \\
= \pi \cdot T \cdot \|f''_{i} - f''_{j}\|_2 - d \\
\geq \pi \cdot T \cdot \Delta f_{i,j} - d \geq 0.
\] (52)

In addition, we have

\[
|\text{sinc}_T(f''_{i} - f''_{j})| \leq \prod_{r \in [d]} \min \left\{ 1, \frac{1}{\pi \cdot T \cdot |f''_{i,r} - f''_{j,r}|} \right\} \\
= \prod_{r \in [d]} \frac{1}{1 + \delta_r} \\
\leq \frac{1}{1 + \sum_{r \in [d]} \delta_r} \\
\leq \frac{1}{\pi \cdot T \cdot \Delta f_{i,j} - (d - 1)} \\
\leq \frac{1}{T \cdot \Delta f_{i,j}},
\] (53)

where the first step applies Part (a) of Fact I.2; the second step is due to the definition of \( \delta_r \)'s; the third step follows because \( \delta_r \geq 0 \) for each \( r \in [d] \); the fourth step follows from Equation (52); and the last step holds whenever \( T = \Omega\left(\frac{d}{\Delta f_{i,j}}\right) \) is large enough.

We observe that Equation (53) holds for any pair of frequencies \( f''_{i} \in \{f_{i}, f'_{i}\} \) and \( f''_{j} \in \{f_{j}, f'_{j}\} \). In other words,

\[
\text{sinc}_{\text{max}}^{\text{T,i,j}} \lesssim \frac{1}{T \cdot \Delta f_{i,j}}.
\]

Combining the above equation and Equation (51) together completes the proof. □

### I.3.2 Both pairs are close

**Claim I.8** (Case (ii) for Lemma I.6). If \( T \cdot \|f_{i} - f_{i}\|_2 < \frac{2.05}{\pi} \) and \( T \cdot \|f'_{j} - f_{j}\|_2 < \frac{2.05}{\pi} \), then we have

\[
\text{err}_{i,j} \lesssim \sqrt{\frac{\Delta}{\|f_{i,j}\|}} \cdot \|a_{i}\|_T \cdot \|a_{j}\|_T.
\]

**Proof.** Let us first bound the tone-wise errors \( \|a_{i}\|_T^2 \) and \( \|a_{j}\|_T^2 \) from below. Respecting the \( i \)-th pair of tones \((v_{i}, f_{i})\) and \((v'_{i}, f'_{i})\), we know from Part (d) of Fact I.2 that

\[
\exp\left( -\frac{\pi^2}{5} \cdot T^2 \cdot \|f'_{i} - f_{i}\|_2^2 \right) \leq \text{sinc}_T(f'_{i} - f_{i}) \leq \exp\left( -\frac{\pi^2}{6} \cdot T^2 \cdot \|f'_{i} - f_{i}\|_2^2 \right).
\]

Then, the tone-wise error \( \|a_{i}\|_T^2 \) between \((v_{i}, f_{i})\) and \((v'_{i}, f'_{i})\) admits the lower bound

\[
\|a_{i}\|_T^2 = |v_{i}|^2 + |v'_{i}|^2 - \left( v_{i} \cdot \overline{v'_{i}} + \overline{v_{i}} \cdot v'_{i} \right) \cdot \text{sinc}_T(f'_{i} - f_{i})
\]

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where the first step applies Property II of Lemma I.4.

Given Equation (54), we would prove that $||a_i||_F^2$ is lower bounded by

$$\|a_i\|_F^2 \geq \frac{3}{13} \left( |v_i - v_i'|^2 + (|v_i|^2 + |v_i'|^2) \cdot (1 - \sin_c(f_i' - f_i)) \right).$$ (55)

To see so, we denote $w_1 \cdot e^{i\theta} = v_i / v_i'$ for some norm $w_1 \geq 0$ and some phase $\theta \in [0, 2\pi)$, and denote $w_2 = \sin_c(f_i' - f_i) \in [-\frac{1}{4}, 1]$ (see Part (e) of Fact I.2). We notice that the RHS of Equation (55) is non-negative. Thus, it suffices to show that the following function $L(w_1, w_2, \theta) \geq \frac{3}{17}$, for any $w_1 \geq 0$, any $w_2 \in [-\frac{1}{4}, 1]$ and any $\theta \in [0, 2\pi)$:

$$L(w_1, w_2, \theta) := \frac{\text{RHS of (54)}}{\text{RHS of (55)}}$$

where the second step is by the definition of $w_1, w_2$ and $\theta$; the third step divides both the numerator and the denominator by $|v_i'|^2$; and the last step can be seen via elementary calculation.

Let us investigate the partial derivative $\frac{\partial L}{\partial \theta}$ in $\theta \in [0, 2\pi)$:

$$\frac{\partial L}{\partial \theta} = \frac{2 \cdot w_1 \cdot w_2 \cdot \sin(\theta)}{(w_1^2 + 1) \cdot (2 - w_2) - 2 \cdot w_1 \cdot \cos(\theta)} - \frac{(w_1^2 + 1 - 2 \cdot w_1 \cdot w_2 \cdot \cos(\theta)) \cdot (2 \cdot w_1 \cdot \sin(\theta))}{((w_1^2 + 1) \cdot (2 - w_2) - \cos(\theta))^2}$$

$$= -\sin(\theta) \cdot \frac{2 \cdot w_1 \cdot (w_1^2 + 1) \cdot (w_2 + 1)^2}{(w_1^2 + 1) \cdot (2 - w_2) - 2 \cdot w_1 \cdot \cos(\theta))^2} \cdot A_5$$

where the second step can be seen via elementary calculation.

Because $w_1 \geq 0$ and $w_2 \in [-\frac{1}{4}, 1]$, we must have $A_5 \geq 0$. Hence, for any fixed $w_1$ and $w_2$, the function $L(w_1, w_2, \theta)$ is non-increasing when $\theta \in [0, 2\pi)$. Then we conclude that the function $L_1(w_1, w_2) := \min_{\theta \in [0,2\pi]} L(w_1, w_2, \theta)$ for any $w_1 \geq 0$ and any $w_2 \in [-\frac{1}{4}, 1]$ is given by

$$L_1(w_1, w_2) = L(w_1, w_2, \pi) = \frac{A_6(w_1, w_2)}{A_7(w_1, w_2)}.$$ (56)

$$A_6(w_1, w_2) := w_1^2 + 1 + 2 \cdot w_1 \cdot w_2,$$

$$A_7(w_1, w_2) := (w_1^2 + 1) \cdot (2 - w_2) + 2 \cdot w_1.$$ (57)

Clearly, for any fixed $w_1 \geq 1$, the numerator $A_6(w_1, w_2)$ is a non-decreasing function in $w_2 \in [-\frac{1}{4}, 1]$, while the denominator $A_7(w_1, w_2)$ is a non-increasing non-negative function in $w_2 \in [-\frac{1}{4}, 1]$. Given these, we deduce that

$$\min_{w_1 \in [0,1]} \min_{w_2 \in [-\frac{1}{4}, 1]} L_1(w_1, w_2) = \min_{w_1 \in [0,1]} L_1(w_1, -1/4)$$

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which implies Equation (55) immediately.

Following Equation (55), we further have

\[
\|a_i\|^2 \geq |v_i - v'_i|^2 + (|v_i|^2 + |v'_i|^2) \cdot (1 - \text{sinc}_T(f'_i - f_i))
\]

\[
\geq |v_i - v'_i|^2 + (|v_i|^2 + |v'_i|^2) \cdot \left(1 - \exp\left(-\frac{\pi^2}{6} \cdot T^2 \cdot \|f'_i - f_i\|^2\right)\right)
\]

\[
\geq |v_i - v'_i|^2 + (|v_i|^2 + |v'_i|^2) \cdot T^2 \cdot \|f'_i - f_i\|^2
\]

\[
\geq \frac{1}{2} \left(|v_i - v'_i| + \sqrt{|v_i|^2 + |v'_i|^2} \cdot T \cdot \|f'_i - f_i\|^2\right)^2
\]

\[
\geq \frac{1}{4} \left(|v_i - v'_i| + (|v_i| + |v'_i|) \cdot T \cdot \|f'_i - f_i\|^2\right)^2,
\]

(56)

where the first step applies Equation (55); the second step applies Part (d) of Fact 1.2; the third step follows from the premise that \(T^2 \cdot \|f'_i - f_i\|^2 < \left(\frac{2.05}{6}\right)^2\), together with the fact that, for any \(0 \leq z < \left(\frac{1}{\pi}\right)^2 \approx 0.4258\), we have \(\exp(-\frac{\pi^2}{6} \cdot z) \leq 1 - z\); and both of the fourth step and the fifth step apply the AM-GM inequality.

Applying the same arguments to the \(j\)-th tone-wise error \(\|a_j\|^2\), we also have

\[
\|a_j\|^2 \geq \frac{1}{9} \left((v_j - v'_j) + (|v_j| + |v'_j|) \cdot T \cdot \|f'_j - f_j\|^2\right)^2.
\]

(57)

Following Equation (47), we deduce that

\[
|\text{err}_{i,j}| = \left| (v_i \cdot \overline{v}_j + \overline{v}_i \cdot v_j) \cdot \text{sinc}_T(f_i - f_j) - (v_i \cdot \overline{v}'_j + \overline{v}_i \cdot v'_j) \cdot \text{sinc}_T(f'_i - f'_j) - (v'_i \cdot \overline{v}_j + \overline{v}'_i \cdot v_j) \cdot \text{sinc}_T(f_i - f_j) + (v'_i \cdot \overline{v}'_j + \overline{v}'_i \cdot v'_j) \cdot \text{sinc}_T(f'_i - f'_j) \right|
\]

\[
= \left| \left((v_i - v'_i) \cdot \overline{v}_j + (v'_i - v_i) \cdot v_j \right) \cdot \text{sinc}_T(f_i - f_j) \right|
\]

\[
+ \left| (v_i \cdot \overline{v}_j + \overline{v}_i \cdot v_j) \cdot \left(\text{sinc}_T(f_i - f_j) - \text{sinc}_T(f'_i - f'_j)\right) \right|
\]

\[
+ \left| (v'_i \cdot \overline{v}_j + \overline{v}'_i \cdot v_j) \cdot \left(\text{sinc}_T(f_i - f_j) - \text{sinc}_T(f'_i - f'_j)\right) \right|
\]

\[
- \left| (v'_i \cdot \overline{v}_j + \overline{v}'_i \cdot v_j) \cdot \left(\text{sinc}_T(f'_i - f_j) - \text{sinc}_T(f'_i - f'_j)\right) \right|
\]

\[
\leq \left| \left((v_i - v'_i) \cdot \overline{v}_j + (v'_i - v_i) \cdot v_j \right) \cdot \text{sinc}_T(f_i - f_j) \right|
\]

\[
+ \left| (v_i \cdot \overline{v}_j + \overline{v}_i \cdot v_j) \cdot \left(\text{sinc}_T(f_i - f_j) - \text{sinc}_T(f'_i - f'_j)\right) \right|
\]

\[
+ \left| (v'_i \cdot \overline{v}_j + \overline{v}'_i \cdot v_j) \cdot \left(\text{sinc}_T(f_i - f_j) - \text{sinc}_T(f'_i - f'_j)\right) \right|
\]

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+ \left| (v'_i \cdot v'_j + v'_i \cdot v'_l) \cdot \left( \text{sinc}_T(f_i - f_j) - \text{sinc}_T(f'_i - f'_j) \right) \right|
\leq 2 \cdot |v_i - v'_i| \cdot |v_j - v'_j| \cdot \left| \text{sinc}_T(f_i - f_j) \right|
A_8
+ 2 \cdot |v_i| \cdot |v'_j| \cdot \left| \text{sinc}_T(f_i - f_j) - \text{sinc}_T(f'_i - f'_j) \right|
A_9
+ 2 \cdot |v'_i| \cdot |v_j| \cdot \left| \text{sinc}_T(f_i - f_j) - \text{sinc}_T(f'_i - f'_j) \right|
A_{10}
+ 2 \cdot |v_i| \cdot |v'_j| \cdot \left| \text{sinc}_T(f_i - f_j) - \text{sinc}_T(f'_i - f'_j) \right|
A_{11}
(58)

where the second step follows by elementary calculation; and the third step follows from the triangle inequality.

In what follows, we safely assume \( T = \Omega(\Delta_{i,j}) \) to be large enough, and upper bound the terms \( |A_8| \) and \( |A_9| \) and \( |A_{10}| \) and \( |A_{11}| \) one by one.

Bound on \( |A_8| \). Recall the quantity \( \text{sinc}^{\max}_{T,i,j} \) defined in Equation (50). We have

\[
|A_8| \leq \frac{1}{T \cdot \Delta_{i,j}} \text{sinc}^{\max}_{T,i,j}
\]

where the last step is by Equation (53), and holds whenever \( T = \Omega(\Delta_{i,j}) \) is large enough.

Bound on \( |A_9| \). Under the premises \( \|f'_j - f_j\|_2 \leq \Delta_{i,j} \) and \( \|f_i - f_j\|_2 \geq \Delta_{i,j} \) and \( \|f_i - f'_j\|_2 \geq \Delta_{i,j} \) (see the statement of Lemma 1.6), via a standard geometric argument, we know that the next equation holds for any \( \lambda \in [0, 1] \):

\[
\|f_i - f''_j(\lambda)\|_2 \geq \frac{\sqrt{3}}{2} \cdot \Delta_{i,j},
\]

where \( f''_j(\lambda) = \lambda \cdot f_j + (1 - \lambda) \cdot f'_j \).

Due to the mean value theorem, there exists a particular \( \lambda \in [0, 1] \) such that

\[
A_9 = \text{sinc}_T(f_i - f_j) - \text{sinc}_T(f_i - f'_j) = \left( \nabla(f_i - f''_j) \right)^\top (f'_j - f_j),
\]

where the gradient \( \nabla(f_i - f''_j) \in \mathbb{R}^d \) is given by \( \nabla_l(f_i - f''_j) = \left( \frac{\partial}{\partial \xi_l} \text{sinc}_T(\xi) \right)_{\xi = f_i - f''_j} \) for each coordinate \( l \in [d] \).

Consider a specific coordinate \( l \in [d] \). The corresponding partial derivative is

\[
\left| \nabla_l(f_i - f''_j) \right| = \left| \left( \frac{d}{d \xi_l} \text{sinc}_T(\xi_l) \right)_{\xi_l = f_i,f_j,f''_j} \cdot \prod_{r \in [d] \setminus \{l\}} |\text{sinc}_T(f_i,r - f''_{j,r})| \right|
\leq \frac{7}{5} \cdot T \cdot \min \left\{ 1, \frac{1}{T \cdot |f_i,l - f''_{j,l}|} \right\} \cdot \prod_{r \in [d] \setminus \{l\}} |\text{sinc}_T(f_i,r - f''_{j,r})| \]
\leq \frac{7}{5} \cdot T \cdot \min \left\{ 1, \frac{1}{T \cdot |f_i,l - f''_{j,l}|} \right\} \cdot \prod_{r \in [d] \setminus \{l\}} \min \left\{ 1, \frac{1}{\pi \cdot T \cdot |f_i,r - f''_{j,r}|} \right\}
\leq T \cdot \min \left\{ 1, \frac{1}{T \cdot |f_i,l - f''_{j,l}|} \right\} \cdot \prod_{r \in [d] \setminus \{l\}} \min \left\{ 1, \frac{1}{\pi \cdot T \cdot |f_i,r - f''_{j,r}|} \right\}
\leq T \cdot \min \left\{ 1, \frac{1}{T \cdot |f_i,l - f''_{j,l}|} \right\} \cdot \prod_{r \in [d] \setminus \{l\}} \min \left\{ 1, \frac{1}{\pi \cdot T \cdot |f_i,r - f''_{j,r}|} \right\}
\]

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\[
\lesssim T \cdot \frac{1}{T \cdot \|f_i - f_j''\|_2} 
\]
\[
\lesssim \frac{1}{\Delta f_{i,j}},
\]
where the second step uses Part (f) of Fact I.2; the third step uses Part (a) of Fact I.2; the fifth step holds whenever \( T = \Omega(\frac{d}{\Delta f_{i,j}}) \) is large enough, and can be seen by reusing the arguments for Equation (53); and the last step follows from Equation (60).

We emphasize that Equation (62) holds for any coordinate \( l \in [d] \), and therefore holds for the \( \ell_\infty \)-norm \( \|\nabla (f_i - f'_j)\|_\infty \) as well. Putting everything together,
\[
|A_9| = \langle \nabla (f_i - f_j) \rangle^\top (f'_j - f_j) 
\]
\[
\lesssim \frac{1}{\Delta f_{i,j}} \cdot \|f'_j - f_j\|_1
\]
\[
\lesssim \frac{\sqrt{d}}{\Delta f_{i,j}} \cdot \|f'_j - f_j\|_2,
\]
where the first step is by Equation (61); the third step is by Equation (62); and the last step follows because \( \sqrt{d} \cdot \|f'_j - f_j\|_2 \geq \|f'_j - f_j\|_1 \).

Reapplying the above arguments for \(|A_9|\), we also have
\[
|A_{10}| \lesssim \frac{\sqrt{d}}{\Delta f_{i,j}} \cdot \|f'_i - f_i\|_2,
\]
\[
|A_{11}| \lesssim \frac{\sqrt{d}}{\Delta f_{i,j}} \cdot (\|f'_i - f_i\|_2 + \|f'_j - f_j\|_2).
\]

Plugging Equations (59) and (63) and (64) and (65) into Equation (58) results in
\[
|\text{err}_{i,j}| \lesssim |v_i - v'_i| \cdot |v_j - v'_j| \cdot |A_8| 
+ |v_i| \cdot |v'_i| \cdot |A_9| 
+ |v'_i| \cdot |v_j| \cdot |A_{10}| 
+ |v'_i| \cdot |v'_j| \cdot |A_{11}|
\]
\[
\lesssim |v_i - v'_i| \cdot |v_j - v'_j| \cdot \frac{1}{T \cdot \Delta f_{i,j}}
\]
\[
+ (|v_i| + |v'_i|) \cdot (|v_j| + |v'_j|) \cdot \frac{\sqrt{d}}{\Delta f_{i,j}} \cdot (\|f'_i - f_i\|_2 + \|f'_j - f_j\|_2)
\]
\[
\lesssim \|a_i\|_T \cdot \|a_j\|_T \cdot \frac{\sqrt{d}}{T \cdot \Delta f_{i,j}}.
\]
where the second step uses Equations (59) and (63) and (64) and (65); and the last step uses Equations (56) and (57).

This completes the proof.

\( \square \)

I.3.3 One pair is far and one pair is close

Claim I.9 (Case (iii) for Lemma I.6). If \( T \cdot \|f'_i - f_i\|_2 \geq \frac{2.05}{\pi} \) and \( T \cdot \|f'_j - f_j\|_2 < \frac{2.05}{\pi} \), then we have
\[
|\text{err}_{i,j}| \lesssim \frac{\sqrt{d}}{\Delta f_{i,j} T} \cdot \|a_i\|_T \cdot \|a_j\|_T.
\]
Proof. We have shown in Equation (48) that
\[ \|a_i\|_T \gtrsim |v_i| + |v'_i|, \]  
and have shown in Equation (57) that
\[ \|a_j\|_T \gtrsim |v_j - v'_j| + (|v_j| + |v'_j|) \cdot T \cdot \|f'_j - f_j\|_2. \]  
Following Equation (58), we deduce that
\[ \|a_i\|_T \gtrsim |v_i - v'_i| \cdot |v_j - v'_j| \cdot \left| \text{sinc}_T(f_i - f_j) \right| \]
\[ + |v_i| \cdot |v'_j| \cdot \left| \text{sinc}_T(f_i - f_j) - \text{sinc}_T(f_i - f'_j) \right| \]
\[ + |v'_i| \cdot |v_j| \cdot \left| \text{sinc}_T(f_i - f_j) - \text{sinc}_T(f'_i - f_j) \right| \]
\[ + |v'_i| \cdot |v'_j| \cdot \left( \left| \text{sinc}_T(f_i - f_j) \right| + \left| \text{sinc}_T(f'_i - f'_j) \right| \right) \]
\[ \lesssim |v_i - v'_i| \cdot |v_j - v'_j| \cdot \left| \text{sinc}_T(f_i - f_j) \right| \]
\[ + |v_i| \cdot |v'_j| \cdot \left| \text{sinc}_T(f_i - f_j) - \text{sinc}_T(f_i - f'_j) \right| \]
\[ + |v'_i| \cdot |v_j| \cdot \left( \left| \text{sinc}_T(f_i - f_j) \right| + \left| \text{sinc}_T(f'_i - f'_j) \right| \right) \]
\[ \lesssim |v_i - v'_i| \cdot |v_j - v'_j| \cdot \frac{1}{T \cdot \Delta f_{i,j}} \]
\[ + |v_i| \cdot |v'_j| \cdot \frac{\sqrt{d}}{\Delta f_{i,j}} \cdot \|f'_j - f_j\|_2 \]
\[ + |v'_i| \cdot |v_j| \cdot \frac{1}{T \cdot \Delta f_{i,j}} \]
\[ + |v'_i| \cdot |v'_j| \cdot \frac{1}{T \cdot \Delta f_{i,j}}, \]  
where the first step applies the triangle inequality; the second step applies Equation (53) to $A_{12}$, applies Equation (63) to $A_{13}$, applies Equation (53) to $A_{14}$, and applies Equation (53) to $A_{15}$.

Combining Equations (66) and (67) and (68) together, it can be easily seen that
\[ |\text{err}_{i,j}| \lesssim \frac{\sqrt{d}}{T \cdot \Delta f_{i,j}} \cdot \|a_i\|_T \cdot \|a_j\|_T. \]

This completes the proof. 

\[ \square \]

I.4 Combining tone-wise errors and cross-tone errors

Let $\Re(z) \in \mathbb{R}$ denote the real part of a complex number $z \in \mathbb{C}$. 

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Lemma I.10. Let \( \{(v_i, f_i)\}_{i \in [k]} \) and \( \{(v'_i, f'_i)\}_{i \in [k]} \) be two sets of \( k \in \mathbb{N}_{\geq 1} \) tones, for which

\[
\min_{i \neq j} \|f_i - f_j\|_1 \geq \eta \quad \text{and} \quad \min_{i \neq j} \|f'_i - f'_j\|_1 \geq \eta \quad \text{and} \quad \min_{i \in [k]} \|f_i - f'_i\|_1 \leq \eta/100
\]

Then these two sets can be reindexed such that

\[
\frac{1}{T^d} \int_{\tau \in [-T/2, T/2]^d} \left| \sum_{i \in [k]} a_i(\tau) \right|^2 \cdot d\tau \leq (1 + \alpha) \cdot \sum_{i \in [k]} \frac{1}{T^d} \int_{\tau \in [-T/2, T/2]^d} |a_i(\tau)|^2 d\tau.
\]

where

\[
\alpha := O(\eta^{-1} \cdot T^{-1}) \cdot \sqrt{d} \cdot \min \left\{ k, \min_{j=1}^{k-1} \sqrt{d} \cdot j^{-1/d} \right\},
\]

which further implies

\[
\alpha = \begin{cases} O(\eta^{-1} \cdot T^{-1}) \cdot \log k, & \text{if } d = 1; \\ O(\eta^{-1} \cdot T^{-1}) \cdot \sqrt{d} \cdot \min \{ k, \sqrt{d} \cdot k^{1-1/d} \}, & \text{if } d \geq 2. \end{cases}
\]

Proof. It follows that

\[
\text{LHS of (69)} = \frac{1}{T^d} \int_{\tau \in [-T/2, T/2]^d} \left( \sum_{i \in [k]} a_i(\tau) \cdot \sum_{i \in [k]} \overline{a_i(\tau)} \right) \cdot d\tau
\]

where

\[
\text{diagonal terms} = \sum_{i \in [k]} \frac{1}{T^d} \int_{\tau \in [-T/2, T/2]^d} |a_i(\tau)|^2 \cdot d\tau = \sum_{i \in [k]} \|a_i\|_T^2,
\]

\[
\text{off-diagonal terms} = \sum_{i < j} \frac{1}{T^d} \int_{\tau \in [-T/2, T/2]^d} \left[ a_i(\tau) \cdot \overline{a_j(\tau)} + a_j(\tau) \cdot \overline{a_i(\tau)} \right] \cdot d\tau
\]

First we can simplify the off-diagonal terms in the following sense:

\[
|\text{off-diagonal terms}| \leq \sum_{i < j} \left| \int_{\tau \sim \text{Unif}[-T/2, T/2]^d} \left[ a_i(\tau) \cdot \overline{a_j(\tau)} + a_j(\tau) \cdot \overline{a_i(\tau)} \right] \cdot d\tau \right|
\]

\[
\leq \sum_{i < j} \left( \frac{\sqrt{d}}{T \cdot \Delta f_{i,j}} \cdot \|a_i\|_T \cdot \|a_j\|_T \right)
\]

where the second step uses the triangle inequality, and the third step applies Lemma I.6.
We consider two cases. Case 1. $d = 1$. Case $d \geq 2$. The reason we consider $d = 1$ separately because, for $d = 1$ we can get a much better bound than general $d$.

Case 1. $d = 1$.

We have

$$|\text{off-diagonal terms}| \lesssim \frac{1}{T_\eta} \sum_{i<j} \frac{1}{|i-j|} \cdot (\|a_i\|_T^2 + \|a_j\|_T^2)$$

$$\leq \frac{1}{T_\eta} \sum_{i=1}^k \|a_i\|_T^2 \sum_{j=1}^k \frac{1}{j}$$

$$\leq \frac{1}{T_\eta} \cdot \log k \cdot \sum_{i=1}^k \|a_i\|_T^2.$$

Case 2. $d \geq 2$. We give two bounds which are not comparative.

Case 2a.

We have

$$|\text{off-diagonal terms}| \lesssim \sqrt{d} \cdot \sum_{i<j} \left( \|a_i\|_T^2 + \|a_j\|_T^2 \right)$$

$$\leq \sqrt{d} \cdot k \cdot \sum_{i=1}^k \|a_i\|_T^2.$$

Case 2b.

We have

$$|\text{off-diagonal terms}| \lesssim \sqrt{d \cdot \sum_{i<j} \sqrt{d} \cdot \frac{1}{|i-j|^{1/d}} \cdot (\|a_i\|_T^2 + \|a_j\|_T^2)}$$

$$\leq \sqrt{d \cdot \sum_{i=1}^k \|a_i\|_T^2 \sum_{j=1}^k \frac{1}{j^{1/d}}}$$

$$\leq \frac{d}{T_\eta} \cdot k^{1-1/d} \cdot \sum_{i=1}^k \|a_i\|_T^2.$$

where the first step follows from Lemma I.11.

This completes the proof.

\[ \square \]

I.5 Geometric property

Lemma I.11 (Geometric property). Given a set $\{f_j\}_{j \in [k]} \subseteq \mathbb{R}^d$ of $k \in \mathbb{N}_{\geq 1}$ many $d$-dimensional frequencies with the minimum $\ell_2$-norm separation $\eta := \min_{i \neq j \in [k]} \|f_i - f_j\|_2 > 0$. Consider any particular frequency $f$ in the set, then these frequencies can be reindexed such that $f_1 = f$ and

$$\|f_1 - f_j\|_2 \geq (|1-j|^{1/d} / \sqrt{d}) \cdot \eta, \quad \forall j \in [2 : k]$$

which further implies

$$\sum_{j \geq 2} \frac{1}{\|f_1 - f_j\|_2} \leq k^{1-1/d} \sqrt{d} / \eta.$$
Proof. Choose frequency $f_1$ arbitrarily from the set and arrange other elements according to the $\ell_2$ distance to $f_1$, i.e. $\|f_1 - f_2\|_2 \leq \|f_1 - f_3\|_2 \leq \cdots \leq \|f_1 - f_i\|_2 \leq \|f_1 - f_{i+1}\|_2 \leq \cdots \leq \|f_1 - f_k\|_2$. Then we prove $\|f_1 - f_j\|_2 \gtrsim \frac{|1 - j|^{1/d}}{\sqrt{d}} \cdot \eta$, $\forall j \in [2, \cdots, k]$.

The proof relies on the notion of covering number and packing number. Let $L$ denote the length of maximum vectors in the set $\in \mathbb{R}^d$. Let $m$ denote the number of points. Due to the geometry properties of covering number and packing number [SSBD14, Page 337],

$$(2\sqrt{dL/\eta})^d \geq m,$$

which implies that

$$L \gtrsim m^{1/d} \eta / \sqrt{d}$$

Replacing $m = |1 - j|$ and $L = \|f_1 - f_j\|_2$, we obtain the bound that

$$\|f_1 - f_j\|_2 \gtrsim |1 - j|^{1/d} \eta / \sqrt{d}.$$ 

Then it is easy to get

$$\sum_{j \geq 2} \frac{1}{\|f_1 - f_j\|_2} \leq k^{1-1/d} \sqrt{d/\eta}.$$ 

\[ \square \]