Quantum fidelity approach to detecting quantum phases: revisiting the bond alternating Ising chain

Hai Tao Wang,1 Sam Young Cho,1∗ and Murray T. Batchelor1,2

1Centre for Modern Physics and Department of Physics, Chongqing University, Chongqing 400044, The People’s Republic of China
2Mathematical Sciences Institute and Department of Theoretical Physics, Research School of Physics and Engineering, Australian National University, Canberra ACT 0200, Australia

We demonstrate the quantum fidelity approach for exploring and mapping out quantum phases. As a simple model exhibiting a number of distinct quantum phases, we consider the alternating-bond Ising chain using the infinite time evolving block decimation method in the infinite matrix product state representation. Examining the quantum fidelity with an arbitrary reference state in the whole range of the interaction parameters leads to the explicit detection of the doubly degenerate groundstates, indicating a $Z_2$ broken symmetry. The discontinuities of the fidelity indicate a first-order quantum phase transition between the four ordered phases. In order to characterize each phase, based on the spin configurations from the spin correlations, even and odd antiferromagnetic order parameters are introduced. The four defined local order parameters are shown to characterize each phase and to exhibit first-order quantum phase transitions between the ordered phases.

PACS numbers: 03.65.Vf, 64.70.Tg, 75.10.Pq, 75.30.Kz, 75.40.Mg

I. INTRODUCTION

Tensor network representations have enabled significant progress in the computational study of quantum phase transitions [1–10]. More specifically, a wave function represented by a tensor network is convenient for the simulation of quantum many-body systems. In one-dimensional spin systems, a wave function for infinite-size lattices can be described by the infinite matrix product state (iMPS) representation [4, 5]. The iMPS and tensor networks in general offer an understanding of critical phenomena in infinite and finite lattice systems from the perspective of quantum entanglement (see, e.g., Refs [8–10] for reviews). It has also been demonstrated that quantum fidelity is a useful tool to detect phase transition points and degenerate groundstates [11, 13] originating from a spontaneous symmetry breaking for a broken symmetry phase, without knowing what type of internal order is present in quantum many-body states. Quantum fidelity based approaches have been successfully implemented to investigate quantum phase transitions in a number of models. Examples in one-dimension include the Ising model in a transverse magnetic field [14], the XYX model in an external magnetic field [12], the bond alternating spin-1/2 Heisenberg chain [15], and the $q$-state Potts quantum chain [13].

In this study we further demonstrate the quantum fidelity approach for exploring and mapping out quantum phases. As a simple but illustrative model exhibiting a number of distinct quantum phases, we consider the Ising chain with alternating interaction strengths. This bond alternating model, also known as the dimerized Ising chain, has been used, for example, to investigate non-equilibrium spin dynamics at finite temperatures [16, 17], the so called Glauber dynamics [18]. More recently it has also been investigated in the context of additional Dzyaloshinskii-Moriya interactions [19, 21]. However, it has not been fully considered how Landau’s spontaneous symmetry breaking picture applies to this model. Here we address this issue using the quantum fidelity approach and demonstrate how to define the explicit order parameters quantifying the distinct quantum phases throughout the whole parameter range.

We calculate the groundstate wavefunction of the infinite spin-1/2 bond-alternating Ising chain by employing the infinite matrix product state (iMPS) representation [4, 5] with the infinite time evolving block decimation (iTEBD) method developed by Vidal [5]. In order to capture the symmetry-broken phases, we use period four matrix product states including characteristic eight tensors for the iMPS representation. The basic idea outlined in Sec. II is to use the quantum fidelity with an arbitrary reference state, allowing the doubly degenerate groundstates to be detected for the whole range of the two exchange interaction parameters. From the discontinuities of the quantum fidelity, we find that a first-order phase transition occurs between the ordered phases once one of the two interaction parameters changes sign. Further, from the spin correlation functions calculated from the degenerate ground states, we discuss in Sec. III the characteristic spin configuration for each phase and define the possible local order parameters in Sec. IV, including even and odd antiferromagnetic ordering. It is shown that the four defined local order parameters reveal four ordered phases and exhibit first-order transitions between the ordered phases in agreement with the results using the quantum fidelity. Concluding remarks are given in Sec. V.

II. BOND-ALTERNATING ISING CHAIN AND QUANTUM FIDELITY PER SITE

We consider the spin-1/2 bond-alternating Ising chain given by the Hamiltonian

$$H = \sum_{i=-\infty}^{\infty} \left( J S_{2i}^z S_{2i+1}^z + J S_{2i}^z S_{2i+2}^z \right),$$

(1)
where $S_i^z$ is the spin operator on the $i$th site. The exchange couplings are $J$ and $J'$, which we parametrise in terms of the variable $\theta$, with $J = \cos \theta$ and $J' = \sin \theta$. For $J = J' < 0$ ($\theta = 5\pi/4$) the system becomes the ferromagnetic (FM) Ising model and for $J = J' > 0$ ($\theta = \pi/4$) the antiferromagnetic (AFM) Ising model. In the both cases $J = J' > 0$ and $J = J' < 0$ the Hamiltonian is one-site translational invariant. Due to the bond alternation, then, the Hamiltonian is two-site translational invariant, except for the cases $J = J' > 0$ and $J = J' < 0$.

An iMPS ground state of the system can be obtained by using the iTEBD algorithm with a chosen initial state in the iMPS representation. We calculate $n$ groundstate values $|\Psi^{(n)}(\theta)\rangle$ corresponding to the $n$-th random initial state. In order to determine how many groundstates exists for a given parameter $\theta$, we consider the quantum fidelity $F(\langle \Psi^{(n)}(\theta) |, |\phi\rangle) = |\langle \Psi^{(n)}(\theta) |, |\phi\rangle|^2$ which is the overlap function between the $n$-th calculated ground state $|\Psi^{(n)}(\theta)\rangle$ and an arbitrary reference state $|\phi\rangle$. For our numerical study, the reference state $|\phi\rangle$ is chosen randomly. The quantum fidelity scales as $F(\langle \Psi^{(n)}(\theta) |, |\phi\rangle) \sim L^2$, where $L$ is the system size. Following, e.g., Ref.13 the fidelity per site can be defined as

$$\ln d(\langle \Psi^{(n)}(\theta) |, |\phi\rangle) = \lim_{L \to \infty} \frac{\ln F(\langle \Psi^{(n)}(\theta) |, |\phi\rangle)}{L}. \quad (2)$$

From the fidelity $F(\langle \Psi^{(n)}(\theta) |, |\phi\rangle)$, the fidelity per site satisfies the properties: (i) normalization $d(\langle \phi |, |\phi\rangle) = 1$ and (ii) range $0 \leq d(\langle \Psi^{(n)}(\theta) |, |\phi\rangle) \leq 1$.

A degenerate groundstate can be determined from the fidelity per site $d(\langle \Psi^{(n)}(\theta) |, |\phi\rangle)$ as a function of the random initial state trials $n$, as shown in Fig.1 for (a) $\theta = \pi/4$, (b) $\theta = 3\pi/4$, (c) $\theta = 5\pi/4$ and (d) $\theta = 7\pi/4$. For the iMPS representation, the truncation dimension $\chi$ used is $\chi = 32$. These plots show that there are two different values of the fidelity per site for the groundstates from the 30 random initial states. For a large enough number of random initial state trials, the probability $P(n)$ of each degenerate groundstate approaches 1/2, i.e., $\lim_{n \to \infty} P(n) = 1/2$. This implies that there are doubly degenerate groundstates for each given $\theta$ value.

In order to determine how many ordered phases there are in the model, we calculated the fidelity per site as a function of the interaction parameter $0 \leq \theta \leq 2\pi$ (see Fig.2). Here, we have chosen the same reference state $|\phi\rangle$ as in Fig.1 Fig.2 shows clearly that the fidelity is discontinuous at four points, i.e., $\theta = 0, \pi/2, \pi$, and $3\pi/2$. The discontinuous fidelities indicate that a first-order quantum phase transition occurs at each discontinuous point corresponding to each critical point. Consequently, the system has four ordered phases:

- I: $0 < \theta < \pi/2 (J > 0$ and $J' > 0)$,
- II: $\pi/2 < \theta < \pi (J < 0$ and $J' > 0)$,
- III: $\pi < \theta < 3\pi/2 (J < 0$ and $J' < 0)$,
- IV: $3\pi/2 < \theta < 2\pi (J > 0$ and $J' < 0)$.

Each phase has doubly degenerate groundstates denoted by $|\Psi^{(n)}_{\alpha,1}(\theta)\rangle$ and $|\Psi^{(n)}_{\alpha,2}(\theta)\rangle$, where $\alpha \in \{I, II, III, IV\}$ labels the phases. According to the spontaneous symmetry breaking theory, the doubly degenerate groundstates imply that a $Z_2$ symmetry is broken for each phase. Hence, for each phase, a different $Z_2$ symmetry is broken and the system state belongs to a different ordered phase.
The spin-spin correlation function is defined by

\[ O_{ss}^z(i, j) = \langle \sigma_i^z \sigma_j^z \rangle. \]  

(3)

where \( i \) and \( j \) denote locations along the chain. In Fig. 3, we plot the spin-spin correlation function \( O_{ss}^z(i, j) \) as a function of the separation distance \( |i - j| \) for the typical parameter values (a) \( \theta = \pi/4 \), (b) \( \theta = 3\pi/4 \), (c) \( \theta = 5\pi/4 \), and (d) \( \theta = 7\pi/4 \) in each of the phases.

## III. SPIN CORRELATIONS AND GROUNDSTATE WAVEFUNCTIONS

In order to gain further insight into the four ordered phases, we consider the spin-spin correlations for each phase. The spin-spin correlation function is defined by

\[ O_{ss}^z(i, j) = \langle \sigma_i^z \sigma_j^z \rangle. \]  

(3)

where \( i \) and \( j \) denote locations along the chain. In Fig. 3, we plot the spin-spin correlation function \( O_{ss}^z(i, j) \) as a function of the separation distance \( |i - j| \) for the typical parameter values (a) \( \theta = \pi/4 \), (b) \( \theta = 3\pi/4 \), (c) \( \theta = 5\pi/4 \), and (d) \( \theta = 7\pi/4 \) in each of the phases.

### A. Antiferromagnetic phase

The two groundstates for each phase give the same spin-spin correlation. Furthermore, the spin-spin correlation as a function of the lattice distance has only two values, i.e., +1 or -1. This result implies that the direction of the two spins \( i \) and \( j \) is either parallel for \( O_{ss}^z(|i - j|) = +1 \) or antiparallel for \( O_{ss}^z(|i - j|) = -1 \). For instance, for \( \theta = \pi/4 \), the spin configurations, using an obvious notation, are either \( \uparrow \uparrow \downarrow \downarrow \uparrow \uparrow \downarrow \downarrow \uparrow \uparrow \cdots \) or \( \downarrow \downarrow \uparrow \uparrow \downarrow \downarrow \uparrow \uparrow \downarrow \downarrow \cdots \). Consequently, for \( \theta = \pi/4 \), the groundstate wavefunctions can be written as

\[ |\Psi_{1,1}^z(\pi/4)\rangle = \prod_{i=-\infty}^{\infty} |\uparrow\rangle_{2i} |\downarrow\rangle_{2i+1}. \]  

(4a)

\[ |\Psi_{1,2}^z(\pi/4)\rangle = \prod_{i=-\infty}^{\infty} |\downarrow\rangle_{2i} |\uparrow\rangle_{2i+1}. \]  

(4b)

Thus, for \( 0 < \theta < \pi/2 \) (\( J > 0 \) and \( J' > 0 \)), the groundstate is in the antiferromagnetic (AFM) phase. The groundstate wavefunctions are two-site translational invariant. However, the Hamiltonian in Eq. \( 1 \) is two-site translational invariant for \( J \neq J' \) but is one-site translation invariant for \( J = J' \). Also, note that one of the two groundstates transforms to the other groundstate under the spin-flip transformation. Hence, for \( J \neq J' \), the two groundstates result from the spontaneous symmetry breaking of the spin-flip symmetry. We can think of the two groundstates in the AFM phase as losing more symmetry for \( J = J' \) than \( J \neq J' \) from the spontaneous symmetry breaking of the spin-flip symmetry.

### B. Odd antiferromagnetic phase

For \( \theta = 3\pi/4 \), the spin correlation in Fig. 3(b) implies that the spin configuration is either \( \cdots \uparrow \uparrow \downarrow \downarrow \downarrow \uparrow \uparrow \downarrow \downarrow \uparrow \uparrow \cdots \) or \( \cdots \downarrow \downarrow \uparrow \uparrow \downarrow \downarrow \uparrow \uparrow \downarrow \downarrow \cdots \). The groundstate wavefunctions are

\[ |\Psi_{1,1}^z(3\pi/4)\rangle = \prod_{i=-\infty}^{\infty} |\uparrow\rangle_{4i} |\downarrow\rangle_{4i+1} |\downarrow\rangle_{4i+2} |\uparrow\rangle_{4i+3}. \]  

(5a)

\[ |\Psi_{1,2}^z(3\pi/4)\rangle = \prod_{i=-\infty}^{\infty} |\downarrow\rangle_{4i} |\uparrow\rangle_{4i+1} |\uparrow\rangle_{4i+2} |\downarrow\rangle_{4i+3}. \]  

(5b)

Thus, for \( \pi/2 < \theta < \pi \) (\( J < 0 \) and \( J' > 0 \)), the groundstate wavefunctions are four-site translational invariant because the exchange interaction on the even bonds is ferromagnetic (FM), i.e., \( J < 0 \), while the exchange interaction on the odd bonds is AFM, i.e., \( J' > 0 \). Note that one of the two groundstates transforms to the other groundstate under the spin-flip transformation and the two-site translational transformation. The Hamiltonian is invariant for the spin-flip transformation and the two-site translational transformation. Hence, the two degenerate ground states arise from the spontaneous symmetry breaking of the group \( G = Z_2 \times Z_2 \), i.e., the spin-flip \( Z_2 \) and the two-site translational \( Z_2 \) symmetries. The spin configuration shows two-spin alternating behavior. In order to distinguish this phase from the AFM phase in region I, we then call this phase the odd AFM phase.

### C. Ferromagnetic phase

The spin correlation for \( \theta = 5\pi/4 \) in Fig. 3(c) is FM. For \( \pi < \theta < 3\pi/2 \) (\( J < 0 \) and \( J' < 0 \)) the groundstate wavefunctions

\[ |\Psi_{1,1}^z(5\pi/4)\rangle = \prod_{i=-\infty}^{\infty} |\uparrow\rangle_{6i} |\downarrow\rangle_{6i+1} |\downarrow\rangle_{6i+2} |\uparrow\rangle_{6i+3} |\downarrow\rangle_{6i+4} |\uparrow\rangle_{6i+5} \]  

(5a)
are the FM wavefunctions

\[ |\Psi_{III,1}(5\pi/4)\rangle = \prod_{i=0}^{\infty} |\uparrow\rangle_i, \]  
\[ |\Psi_{III,2}(5\pi/4)\rangle = \prod_{i=0}^{\infty} |\downarrow\rangle_i. \]  

The groundstate wavefunctions are one-site translational invariant. However, the Hamiltonian in Eq. (1) is two-site translational invariant for \( J \neq J' \) but is one-site translation invariant for \( J = J' \). Also, note that one of the two groundstates transforms to the odd bonds interaction on the even bonds is AFM (\( \langle \sigma_i^z \rangle \neq \langle \sigma_{i+1}^z \rangle \)) while the other transforms to the even bonds interaction on the odd bonds is FM (\( \langle \sigma_i^z \rangle = \langle \sigma_{i+1}^z \rangle \)).

The two groundstates are four-site translationally invariant. Note that one of the two groundstates transforms to the exchange interaction on the even bonds is AFM (\( \langle \sigma_i^z \rangle \neq \langle \sigma_{i+1}^z \rangle \)) while the other transforms to the odd bonds interaction on the odd bonds is FM (\( \langle \sigma_i^z \rangle = \langle \sigma_{i+1}^z \rangle \)).

The groundstate wavefunctions are four-site translational invariant. Thus for \( 3\pi/2 < \theta < 2\pi \) (\( J > 0 \) and \( J' < 0 \)) the exchange interaction on the odd bonds is FM (\( J' < 0 \)) while the exchange interaction on the even bonds is AFM (\( J > 0 \)).

The groundstate wavefunctions are four-site translational invariant. Note that one of the two groundstates transforms to the other groundstate under the spin-flip transformation. Hence, for \( J = J' \), the two groundstates result from the spontaneous symmetry breaking of the spin-flip symmetry. For \( J = J' \), interestingly, the two groundstates in the FM phase have more symmetry than the Hamiltonian because the two groundstates have one-site translational symmetry but the Hamiltonian has two-site translational symmetry. Such a symmetry, which is not directly manifest in the Hamiltonian, is an example of an enhanced or emergent symmetry [22–28].

D. Even antiferromagnetic phase

The spin correlation for \( \theta = 7\pi/4 \) in Fig. 3(d) is similar to the case \( \theta = 3\pi/4 \). Here the groundstate wavefunctions are

\[ |\Psi_{IV,1}(7\pi/4)\rangle = \prod_{i=0}^{\infty} |\uparrow\rangle_i |\downarrow\rangle_{i+1} |\downarrow\rangle_{i+2} |\uparrow\rangle_{i+3} \]  
\[ |\Psi_{IV,2}(7\pi/4)\rangle = \prod_{i=0}^{\infty} |\downarrow\rangle_i |\uparrow\rangle_{i+1} |\uparrow\rangle_{i+2} |\downarrow\rangle_{i+3}. \]

Thus for \( 3\pi/2 < \theta < 2\pi \) (\( J > 0 \) and \( J' < 0 \)) the exchange interaction on the odd bonds is FM (\( J' < 0 \)) while the exchange interaction on the even bonds is AFM (\( J > 0 \)).

The groundstate wavefunctions are four-site translational invariant. Note that one of the two groundstates transforms to the other groundstate under the spin-flip transformation and the two-site translational transformation. Also, the Hamiltonian is invariant for the spin-flip transformation and the two-site translational transformation. Hence, the two degenerate ground states arise from the spontaneous symmetry breaking of the group \( G = Z_z \times Z_\pi \times Z_\pi \), i.e., the spin-flip \( Z_\pi \) and the two-site translational \( Z_\pi \) symmetries. The spin configuration shows two-spin alternating behavior. In order to distinguish this phase from the AFM phase in the regions I and II, we now call this phase the even AFM phase. However, both of the odd and even AFM phases originate from the breaking of the same symmetry. Then one needs to consider how to distinguish the two phases. This point is clarified in the discussion of the appropriate order parameters in Sec. IV.

IV. ORDER PARAMETERS

To distinguish the four phases, one has to define characteristic local order parameters. Based on the spin configurations or the wavefunctions, we define the order parameters for the four phases. Normally, one uses the FM and the AFM order parameters defined by the magnetization \( M_i^F = \langle \sigma_i^z + \sigma_{i+1}^z \rangle / 2 \) and the staggered magnetization \( M_i^{AF} = \langle \sigma_i^z - \sigma_{i+1}^z \rangle / 2 \). However, for the bond-alternating model, these definitions for the order parameters do not distinguish between all four phases, because of the even AFM and odd AFM phases. Furthermore, the odd and even AFM phase originate from the spontaneous breaking of the same symmetries. To overcome this, one needs to consider the symmetries of the groundstates. One can notice that from the two degenerate groundstates for each phase, the defined FM order parameter should be one-site translational invariant, the AFM order parameter two-site translational invariant, and both the odd and even AFM order parameters four-site translational invariant. Then, proper order parameters should be four-site translational invariant. From the properties of the groundstates for each phase, in terms of the normal definitions of the magnetization and the staggered magnetization, we define four order parameters as

FIG. 4: (Color online) The four order parameters as a function of the interaction parameter \( \theta \). In (a)-(d), the blue triangles and the red circles denote the average value of the local order parameters, defined in the text, with each ground state, \( |\Psi_{a,1}\rangle \) and \( |\Psi_{a,2}\rangle \), respectively.
generate groundstates. However, we have detected only the
these points the groundstates are infinitely degenerate. As a
result, the spin-1
order by the sudden drop of the order parameters to zero at
dependence on the sign of the interaction strengths \( J \) and \( J' \).
We have investigated quantum fidelity in an infinite-size
bond-alternating Ising chain by employing the iMPS representa-
tion with the iTEBD method. By detecting the doubly de-
generate groundstates for each phase by means of the quantum
fidelity with an arbitrary reference state, it was shown that, for
each phase, a different \( Z_2 \) symmetry is broken and the system
state belongs to a different ordered phase. By also detecting
the discontinuities of the quantum fidelity, we demonstrated
that first-order quantum phase transitions occur between the
ordered phases as the interaction parameter \( \theta \) varies through
\( 0 < \theta < 2\pi \). Based on the spin configurations from the char-
acteristic properties of the spin correlations, the four defined
local order parameters, including the even and the odd AFM
order parameters, are shown to clearly characterize each phase
and the existence of the first-order quantum phase transitions
between the ordered phases. Consequently, by taking a sim-
ple and well known model as example, we have demonstrated
the usefulness of the quantum fidelity with an arbitrary refer-
ence state to investigate the nature of quantum phases without
knowing a priori what type of internal order is present in a
quantum many-body state.

V. CONCLUSION

We have investigated quantum fidelity in an infinite-size
bond-alternating Ising chain by employing the iMPS representa-
tion with the iTEBD method. By detecting the doubly de-
generate groundstates for each phase by means of the quantum
fidelity with an arbitrary reference state, it was shown that, for
each phase, a different \( Z_2 \) symmetry is broken and the system
state belongs to a different ordered phase. By also detecting
the discontinuities of the quantum fidelity, we demonstrated
that first-order quantum phase transitions occur between the
ordered phases as the interaction parameter \( \theta \) varies through
\( 0 < \theta < 2\pi \). Based on the spin configurations from the char-
acteristic properties of the spin correlations, the four defined
local order parameters, including the even and the odd AFM
order parameters, are shown to clearly characterize each phase
and the existence of the first-order quantum phase transitions
between the ordered phases. Consequently, by taking a sim-
ple and well known model as example, we have demonstrated
the usefulness of the quantum fidelity with an arbitrary refer-
ence state to investigate the nature of quantum phases without
knowing a priori what type of internal order is present in a
quantum many-body state.

Acknowledgments

It is a pleasure to acknowledge Professor Huan-Qiang Zhou
for encouragement and support. This work was supported by
the National Natural Science Foundation of China (Grant No.
11374379). M.T.B. is supported by the 1000 Talents Program
of China. His work is also partially supported by the Aus-
tralian Research Council.

[1] M. Fannes, B. Nachtergaele and R. F. Werner, Commun. Math.
Phys. 144, 443 (1992).
[2] S. R. White, Phys. Rev. Lett. 69, 2863 (1992); Phys. Rev. B 48,
10345 (1993).
[3] S. Östlund and S. Rommer, Phys. Rev. Lett. 75, 3537 (1995).
[4] G. Vidal, Phys. Rev. Lett. 91, 147902 (2003).
[5] G. Vidal, Phys. Rev. Lett. 98, 070201 (2007).
[6] V. Murg, F. Verstraete and J. I. Cirac, Phys. Rev. A 75, 033605
[7] H.-Q. Zhou, R. Orůs and G. Vidal, Phys. Rev. Lett. 100, 080601 (2008).
[8] U. Schollwöck, Rev. Mod. Phys. 77, 259 (2005).
[9] D. Perez-Garcia, F. Verstraete, M. M. Wolf and J. I. Cirac, Quantum Inf. Comput. 7, 401 (2007).
[10] R. Orús, Annals of Physics 349, 117 (2014).
[11] H.-Q. Zhou and J. P. Barjaktarevic, J. Phys. A 41, 412001 (2008).
[12] J.-H. Zhao, H.-L. Wang, B. Li and H.-Q. Zhou, Phys. Rev. E 82, 061127 (2010).
[13] Y. H. Su, B.-Q. Hu, S.-H. Li and S. Y. Cho, Phys. Rev. E 88, 032110 (2013).
[14] H.-Q. Zhou, J.-H. Zhao and B. Li, J. Phys. A 41, 492002 (2008).
[15] H.-T. Wang, B. Li and S. Y. Cho, Phys. Rev. B 87, 054402 (2013).
[16] S. Cornell, M. Droz and N. Menyhárd, J. Phys. A 24, L201 (1991).
[17] S. J. Cornell, K. Kaski and R. B. Stinchcombe, J. Phys. A 24, L865 (1991).
[18] R. J. Glauber, J. Math. Phys. 4, 294 (1963).
[19] B. Li, S. Y. Cho, H.-L. Wang and B.-Q. Hu, J. Phys. A 44, 392002 (2011).
[20] N. Amiri and A. Langari, Phys. Status Solidi B 250, 537 (2013).
[21] G.-H. Liu, W. Li, W.-L. You, G. Su and G.-S. Tian, Eur. Phys. J. B 86, 227 (2013).
[22] C. D. Batista and G. Ortiz, Adv. Phys. 53, 1 (2004).
[23] C. D. Batista and Z. Nussinov, Phys. Rev. B 72, 045137 (2005).
[24] J. Schmalian and C. D. Batista, Phys. Rev. B 77, 094406 (2008).
[25] C. D. Batista, Phys. Rev. B 80, 180406(R) (2009).
[26] D. E. Liu, S. Chandrasekharan and H. U. Baranger, Phys. Rev. Lett. 105, 256801 (2010).
[27] P. Silvi, G. De Chiara, T. Calarco, G. Morigi and S. Montangero, Ann. Phys. (Berlin) 525, 827 (2013).
[28] P. Chen, Z.-L. Xue, I. P. McCulloch, M.-C. Chung, C.-C. Huang and S.-K. Yip, Phys. Rev. Lett. 114, 145301 (2015).