Puncture Operator in $c=1$ Liouville Gravity

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ABSTRACT

We identify the puncture operator in $c=1$ Liouville gravity as the discrete state with spin $J = \frac{1}{2}$. The correlation functions involving this operator satisfy the recursion relation which is characteristic in topological gravity. We derive the recursion relation involving the puncture operator by the operator product expansion. Multiple point correlation functions are determined recursively from fewer point functions by this recursion relation.
Through recent progress in our understanding of 2 dimensional gravity by matrix models[1], we have gained precious insight into universal properties of quantum gravity. It becomes clear that the topological gravity point of view advocated by Witten[2] goes quite far to capture the essential features of the 2d gravity. In fact the most remarkable aspects of the 2d gravity, namely the KdV structure[3] and Virasoro like constraints[4][5] represent such topological aspects of the theory.

However we should also recognize that there are also geometrical aspects of the theory such as intrinsic and extrinsic fractal dimensions which is beyond the scope of the topological gravity. The Liouville theory approach to the 2d gravity and string theory has the advantage that it can capture both topological and geometrical aspects of the theory.

One of significant recent developments in Liouville gravity is the successful evaluation of correlation functions[6]. Certainly Liouville theory is more awkward to capture the topological aspects of the 2d gravity than the powerful matrix models. Nevertheless it is important to push the Liouville theory approach to see how far it can go to understand topological aspects of the theory from the continuum theory point of view. 2d gravity is our important foothold to study higher dimensional quantum gravity and general string theory. The significance of deeper understanding of Liouville gravity stems from it. We hope that such investigation is useful to make further progress in both more general quantum gravity theory and string theory.

The discrete states in $c = 1$ 2d Liouville gravity were interpreted as topological states in [7] and they were further studied in [8]. Recently Witten[9], Klebanov and Polyakov[10] have shown that the discrete states
in $c = 1$ 2d gravity form the operator algebra which is isomorphic to the area preserving diffeomorphism of the plane. The holomorphic part of the discrete states are

$$\Psi_{J,m}(z) = \psi_{J,m}(z) e^{i \sqrt{2} (1 + J) \phi(z)},$$

(1)

where $\psi_{J,m}(z)$ are well-known $SU(2)$ multiplets in $c = 1$ conformal field theory. After rescaling the operators, they are found to obey

$$\Psi_{J_1,m_1}(z) \Psi_{J_2,m_2}(0) = z^{-1} (J_2 m_1 - J_1 m_2) \Psi_{J_1 + J_2 - 1, m_1 + m_2}(0).$$

(2)

In order to make physical operators in the closed string theory, we need to combine the holomorphic and antiholomorphic sectors. Therefore the observables are

$$V_{J,m,m'} = \int d^2 z \Psi_{J,m}(z) \bar{\Psi}_{J,m'}(\bar{z}).$$

(3)

The theory poseses $SU(2)_L \times SU(2)_R$ symmetry.

Although Polyakov and Klebanov have studied most general ‘resonant’ three point functions of discrete states, it has not been made clear what principle determines general $n$ point amplitudes. The ‘resonant’ correlators conserve the Liouville charge. They are considered to be ‘bulk’ amplitudes if we regard the Liouville field as an extra dimension. Discrete states alone form closed operator algebra. It is consistent to consider only discrete states and put aside tachyons with general momentum. Furthermore we do not perturb the theory with the cosmological constant operator. This is the strategy what we shall adopt in this paper. The topological aspects of the theory may become more transparent in this truncation of the theory.
Since the discrete states are thought to be topological degrees of freedom, their correlation functions are expected to obey recursion relations which appear in general \( c < 1 \) matrix models and topological gravity. The essential feature of these topological degrees of freedom (gravitational descendents) can be understood if we consider the topological gravity of Witten. In term of the Liouville gravity, it is \( c = -2 \) matter coupled to 2d gravity.

In this theory, there are infinitely many operators \( \sigma_i \) with \( i = 0, 1, 2, \cdots \).

The connected Green’s functions on the sphere are

\[
< \sigma_{i_1} \cdots \sigma_{i_n} PPP > = n!, \tag{4}
\]

where \( P = \sigma_0 \) and \( \sum_{j=1}^{n} (i_j - 1) = 0 \) due to the resonant condition. We consider the generating function of the connected Green’s functions. Let us define a field

\[
\Phi(\phi) = t_0 e^{-\phi} + t_1 + t_2 e^\phi + \cdots, \tag{5}
\]

where \( \phi \) represents Liouville coordinate. The generating function is

\[
W = \int d\phi (1 + \Phi + \Phi^2 + \cdots)
\]

\[= \int d\phi \frac{1}{1 - \Phi}, \tag{6}
\]

where \( \phi \) integral imposes Liouville charge conservation. The connected Green’s functions are given by

\[
< \sigma_{i_1} \sigma_{i_2} \cdots \sigma_{i_n} PPP >
\]

\[= \frac{\partial}{\partial t_{i_1}} \frac{\partial}{\partial t_{i_2}} \cdots \frac{\partial}{\partial t_{i_n}} W|_{t_i=0}. \tag{7}
\]

The generating function neatly summarizes the symmetry of the theory associated with the target space. The target space in this case is the Liouville
dimension. In what follows we demonstrate that the Virasoro constraints of this theory can be interpreted as diffeomorphism invariance of the Liouville dimension.

It is convenient to introduce a new variable \( z = e^{\phi} \). Let us consider the following infinitesimal change of the integration variable

\[
z \rightarrow z + cz^n.
\]  

(8)

Then \( W \) changes as follows

\[
\int \frac{dz}{z} \left( 1 - \Phi - c((n - 1)z^{n-1} + (z^n \partial_z - (n - 1)z^{n-1})\Phi) \right).
\]  

(9)

Therefore the change of the integration variable is equivalent to

\[
\Phi \rightarrow \Phi + \delta \Phi = (n - 1)z^{n-1} + \sum_i (i - n) t_i z^{i+n-1}.
\]  

(10)

It is interpreted as the change of the coupling constants \( \{t_i\} \). Since the generating function is invariant under the change of the integration variable, the theory must be invariant under the change of the coupling constants. This requirement leads to the constraints on the correlation functions.

\[
L_n < \sigma_{i_1} \cdots \sigma_{i_n} PPP > 0,
\]  

(11)

where \( L_n = n \frac{\partial}{\partial \Pi_{n+1}} + \sum_i (i - n - 1)t_i \frac{\partial}{\partial \Pi_{i+n}} \) with \( n \geq -1 \). \( L_n \) operators form the Virasoro algebra

\[
[L_n, L_m] = (n - m)L_{n+m}.
\]  

(12)

In particular

\[
L_{-1} = -\frac{\partial}{\partial t_0} + \sum_i it_i \frac{\partial}{\partial t_{i-1}}.
\]  

(13)
$L_{-1}$ constraint leads to the following recursion relation

$$
< \sigma_0 \sigma_{i_1} \cdots \sigma_{i_n} PPP > = \sum_{k=1}^{n} < i_k \sigma_{i_{k-1}} \cdots \hat{\sigma}_{i_k} \cdots \sigma_{i_n} PPP > ,
$$

where $\hat{\sigma}$ implies the absence of such an operator. This is the famous recursion relation of Witten[2]. It is strong enough to determine all correlation functions on the sphere with just single input $< PPP > = 1$.

Next we would like to pose analogous questions concerning the discrete states in the $c = 1$ Liouville gravity. We hope to know some principle which relates different amplitudes which contain different numbers of discrete states. In particular we hope to identify the operator which acts like the puncture operator $P$ in topological gravity.

Although these aspects have been investigated in [11][7], our understanding is far from complete. Especially little is known concerning the puncture operator. In $(p, q)$ 2d gravity, the operator with gravitational scaling exponent (Liouville charge) $\beta = -\sqrt{\frac{2}{2p}}$ appears to play the role of the puncture operator. It is a so-called mod $p$ or missing state. Therefore one might expect that in the $c \to 1$ limit, it becomes a discrete state with isospin $J = 1/2$. Indeed we find that the operator with $J = 1/2$ acts like the puncture operator and it gives rise to the recursion relations among the amplitudes.

Since $(m, m')$ dependence of the correlation functions are governed by $SU(2)_L \times SU(2)_R$ symmetry, we confine our attention to tachyonic discrete states, $m = m' = \pm J$ and denote them by $V_{J, \pm J}$. Since these states constitute $SU(2)$ multiplets with general discrete states, it is natural to consider them
as part of discrete states. The recursion relation we have found is

\[
< \tilde{V}_{\frac{1}{2}+\epsilon, -\frac{1}{2}+\epsilon} \tilde{V}_{J_1, -J_1} \cdots \tilde{V}_{J_n, -J_n} \tilde{V}_{J'_1, J'_1} \cdots \tilde{V}_{J'_{n'}, J'_{n'}} >
= \sum_{i=1}^{n} (2J_i - 1) < \cdots \tilde{V}_{J_i, -J_i + \frac{1}{2}} \cdots \tilde{V}_{J'_{i'}, J'_{i'}} > ,
\]

where \( \tilde{V}_{J, \pm J} \) are rescaled operators

\[
V_{J, \pm J} = \frac{\Gamma(1 - 2J)}{\Gamma(2J)} \tilde{V}_{J, \pm J}. \tag{16}
\]

Needless to say, analogous recursion relation holds with the opposite sign of the \( J_z \)'s.

Since the correlation functions of the discrete states are singular, we need to rescale the operators to obtain finite correlation functions. In order to make the correlation functions well defined, we adopt infinitesimal shift of the momentum, namely \( J \rightarrow J - \epsilon \). In order to prove this recursion relation, we consider the following operator product expansion (OPE).

\[
e^{i\sqrt{2}(\frac{1}{2} - \epsilon)\varphi(z)} e^{i\sqrt{2}(\frac{1}{2} - \epsilon)\varphi(0)} \times e^{-i\sqrt{2}J\varphi(0)} e^{\sqrt{2}(J - 1)\varphi(0)}
= z^{-2J(\frac{1}{2} - \epsilon)} z^{2(J - 1)(\frac{1}{2} + \epsilon)}
\times e^{i\sqrt{2}(\frac{1}{2} - \epsilon)\varphi(z) - i\sqrt{2}J\varphi(0)} e^{\sqrt{2}(\frac{1}{2} - \epsilon)\varphi(z) + \sqrt{2}(J - 1)\varphi(0)}
= z^{-1 + 2\epsilon(2J - 1)} e^{i\sqrt{2}(-J + \frac{1}{2})\varphi(0)} e^{\sqrt{2}(J - \frac{1}{2} - 1)\varphi(0)} + \cdots . \tag{17}
\]

Therefore

\[
< V_{\frac{1}{2} - \epsilon, \frac{1}{2} - \epsilon} V_{J, -J} >
= \int dz^2 |z|^{-2 + 4\epsilon(2J - 1)} < V_{J - \frac{1}{2}, -J + \frac{1}{2}} >
= 2\pi \int_{0}^{\infty} dr r^{-1 + 4\epsilon(2J - 1)} e^{-\mu r} < V_{J - \frac{1}{2}, -J + \frac{1}{2}} >
\]
\[ = 2\pi (\mu)^{-4\epsilon(2J_1-1)} \Gamma(4\epsilon(2J_1 - 1)) < V_{J_1, J_1} > \]
\[ = \frac{\pi}{2\epsilon(2J - 1)} < V_{-\frac{1}{2}, -\frac{1}{2}} >, \tag{18} \]

where we introduced another infinitesimal regularization parameter \( \mu \). Let us rescale the operator as eq. (14). If we note that
\[ \frac{\Gamma(1 - 2(\frac{1}{2} - \epsilon))}{\Gamma(2(\frac{1}{2} - \epsilon))} = \frac{1}{2\epsilon} \]
\[ \frac{\Gamma(1 - 2J)}{\Gamma(2J)} = \frac{1}{(2J - 1)^2} \frac{\Gamma(1 - 2(J - \frac{1}{2}))}{\Gamma(2(J - \frac{1}{2}))}, \tag{19} \]

we obtain the recursion relation eq. (15).

Let us check our recursion relation by known results. The \( n + 1 \) point tachyon amplitudes with one negative chirality state insertion is
\[ < V_{\frac{n-1}{2}, \frac{n-1}{2}} V_{J_1, J_1} \cdots V_{J_n, J_n} > = (n - 2)! \tag{20} \]

The resonant condition fixes the negative chirality states to be \( J = \frac{n-1}{2}, m = -\frac{n-1}{2} \). It leads to another condition \( \sum_{i=1}^{n} J_i = \frac{n-1}{2} \). In order to satisfy the latter condition, we need at least one insertion of the cosmological constant operator \( V_{0,0} \) in the correlator. Let \( J_1 = \frac{1}{2} \), then the recursion relation (15) implies
\[ < \tilde{V}_{\frac{n-2}{2}, \frac{n-2}{2}} \tilde{V}_{\frac{1}{2}, \frac{1}{2}} \tilde{V}_{J_2, J_2} \cdots \tilde{V}_{J_n, J_n} > \]
\[ = (n - 2) < \tilde{V}_{\frac{n-2}{2}, \frac{n-2}{2}} \tilde{V}_{J_2, J_2} \cdots \tilde{V}_{J_n, J_n} >, \tag{21} \]

which agrees with eq. (20).

How about the correlation functions which involve more than single negative chirality states. Such amplitudes with generic momenta are not singular
unlike the amplitudes with single negative chirality state. If we insist on fac-
torizing a pole due to the Liouville charge conservation, the residue vanishes
for the former unlike the latter[7]. However if we adopt the rescaling pro-
cedure eq.(16), the correlation functions are nonvanishing in both cases. Such
amplitudes have been investigated in [12][13]. They allow the physical inter-
pretation in terms of the effective Lagrangian of massless scalar field(tachyon)
by introducing 1PI vertices. Therefore such correlation functions are rather
complicated. 1PI vertices could arise if we integrate out discrete states and
consider tachyon field only. The strategy we adopt in this paper is rather
different. We consider the whole discrete states and look for recursion re-
lations among the correlators. We have checked that the general resonant
correlators of tachyonic discrete states satisfy the recursion relation eq.(15)
up to 6 point functions.

Since we have rescaled the discrete states, all correlation functions become
finite. Before rescaling, the degree of the divergence of a n point function
with \(n_0\) insertions of \(V_{0,0}\) operator is \(n - 2n_0\). The only resonant three point
function of the discrete states is

\[
\langle \tilde{V}_{1/2, -1/2} \tilde{V}_{1/2, 1/2} \tilde{V}_{0,0} \rangle = 1. \tag{22}
\]

The degree of the divergence of this correlator is 1. The simple pole is due
to the Liouville charge conservation. For \(n\) point functions, the maximum
possible degree of divergence is \(n - 2\). This is because \(n - 3\) extra pole
could arise due to on shell intermediate states. It explains why need at least
single insertion of \(V_{0,0}\) operator for the rescaled correlation functions to be
nonvanishing.
The recursion relation eq.(15) imposes very strong constraints on the correlation functions. Let us consider the correlation functions with single insertion of $V_{0,0}$. The Liouville charge conservation requirement is $\sum (J_i - 1) = -2$. Let us take aside one $V_0$ operator and two $V_{1/2}$ operators in the correlator. For each operator with $J > 1$, we need $2J - 2$ numbers of $V_{1/2}$ operators to balance the Liouville charge. Therefore the relevant correlator is

$$
< \tilde{V}_{J_1,-J_1} \cdots \tilde{V}_{J_n,-J_n} \tilde{V}_{J'_1,J'_1} \cdots \tilde{V}_{J'_n,J'_n} \tilde{V}_{1,1/2} \cdots \tilde{V}_{1,1/2} \tilde{V}_0 > .
$$

(23)

where $\tilde{V}_{1,1/2} \cdots$ imply $\sum_{i=1}^{n+n'} (2J_i - 2)$ insertions of $\tilde{V}_{1/2}$. Let us assume that $N$ point functions are known. Then by the mathematical induction $N + 1$ functions are determined by the recursion relation eq.(15), since there always exist $V_{1/2}$ operators in the correlation functions we are considering. By using the recursion relations the correlation function eq.(23) is determined to be

$$
(\sum_{i=1}^{n} (2J_i - 1))!(\sum_{j=1}^{n'} (2J'_j - 1))!.
$$

(24)

For general discrete states we can write down the analogous recursion relation

$$
< \tilde{V}_{1/2,\pm 1/2, \pm 1/2} \tilde{V}_{J_1,m_1,m'_1} \cdots \tilde{V}_{J_n,m_n,m'_n} >
= \sum_{i=1}^{n} (2J_1 - 1)
C(1/2, \pm 1/2, J_i, m_i | J_i - 1/2, m_i \pm 1/2)C(1/2, \pm 1/2, J_i, m'_i | J_i - 1/2, m'_i \pm 1/2)
\times < \tilde{V}_{J_1,m_1,m'_1} \cdots \tilde{V}_{J_{i-1},m_{i-1},m'_{i-1}} \pm 1/2, m'_i \pm 1/2 \cdots \tilde{V}_{J_n,m_n,m'_n} > ,
$$

(25)

where $C(1/2, \pm 1/2, J_i, m_i | J_i - 1/2, m_i \pm 1/2)$ is the Crebsh-Gordan coefficients. Let us consider $V_{J,J-1}$ states which form Virasoro like closed operator algebra.
for example. The correlation functions of these states obey the following
recursion relation.

\[
\langle \tilde{V}_{\frac{1}{2}, \frac{1}{2}} \tilde{V}_{J_1, J_1 - 1} \cdots \tilde{V}_{J_n, J_n - 1} \tilde{V}_{\frac{1}{2}, \frac{1}{2}} \rangle = \sum_{i=1}^{n} \frac{(2J_i - 1)^2}{2J_i} \langle \cdots \tilde{V}_{J_i, \frac{1}{2}, J_i - \frac{1}{2}} \cdots \tilde{V}_{\frac{1}{2}, \frac{1}{2}} \rangle .
\]  

(26)

By this recursion relation, the correlation functions of these operators are
found to be

\[
\langle \tilde{V}_{J_1, J_1 - 1} \cdots \tilde{V}_{J_n, J_n - 1} \tilde{V}_{\frac{1}{2}, \frac{1}{2}} \tilde{V}_{0, 0} \rangle = (\prod_{i=1}^{n} \frac{1}{2J_i}) (n - 1)! \]  

(27)

which is of the very familiar type.

In this paper we investigated the algebraic relations among correlation
functions of the discrete states in \( c = 1 \) Liouville gravity. They are found
to obey the recursion relation which is characteristic in topological gravity.
Certainly it is still just a beginning. In order to make the full contact with
the recursion relations which are found in the matrix model, we need to
relate amplitudes of different genus. We also need to derive other recursion
relations which form \( W_\infty \) algebra. We would like to report progress in these
aspects in the near future.

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