On Greedy Clique Decompositions
and Set Representations of Graphs*

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Abstract

In 1994 S. McGuinness showed that any greedy clique decomposition of an $n$-vertex graph has at most $\lfloor n^2/4 \rfloor$ cliques (The greedy clique decomposition of a graph, J. Graph Theory 18 (1994) 427-430), where a clique decomposition means a clique partition of the edge set and a greedy clique decomposition of a graph is obtained by removing maximal cliques from a graph one by one until the graph is empty. This result solved a conjecture by P. Winkler. A multifamily set representation of a simple graph $G$ is a family of sets, not necessarily distinct, each member of which represents a vertex in $G$, and the intersection of two sets is non-empty if and only if two corresponding vertices in $G$ are adjacent. It is well known that for a graph $G$, there is a one-to-one correspondence between multifamily set representations and clique coverings of the edge set. Further for a graph one may have a one-to-one correspondence between particular multifamily set representations with intersection size at most one and clique partitions of the edge set. In this paper, we study for an $n$-vertex graph the variant of the set representations using a family of distinct sets, including the greedy way to get the corresponding clique partition of the edge set of the graph. Similarly, in this case, we obtain a result that any greedy clique decomposition of an $n$-vertex graph has at most $\lfloor n^2/4 \rfloor$ cliques.

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1 Background and Introduction

By an multigraph $M = (V(M), E(M), q)$ we mean a triple consisting of a set $V(M)$ of vertices, a set $E(M)$ of edges, and an integer-valued function $q$ defined on $V(M) \times V(M)$ in the following way. For each unordered pair $\{u, v\} \subset V(M)$, let $q(u, v)$ be the number of parallel edges joining $u$ with $v$. If $q(u, v) \neq 0$, then we say that $\{u, v\}$ is an edge of $M$ and $q(u, v)$ is called the multiplicity of the edge $\{u, v\}$. For the main results in this paper, we consider only finite, undirected, simple multigraphs, where simple means that $q(u, v) \leq 1$ for every $\{u, v\} \subset V$ and $q(u, u) = 0$ for every $u \in V(M)$. Therefore we simply call such multigraphs to be graphs for short throughout this article, unless otherwise stated.

For a vertex subset $S \subseteq V(M)$, $\langle S \rangle_V$ denotes the subgraph induced by $S$. For a vertex $v$ in $M$, $d_M(v)$ or $d(v)$ denote the degree of $v$ in $M$. Let $\mathcal{F} = \{S_1, ..., S_p\}$ be a family of distinct nonempty subsets of a set $X$. Then $S(\mathcal{F})$ denotes the union of sets in $\mathcal{F}$. The intersection multigraph of $\mathcal{F}$, denoted $\Omega(\mathcal{F})$, is defined by $V(\Omega(\mathcal{F})) = \mathcal{F}$, with $|S_i \cap S_j| = q(S_i, S_j)$ whenever $i \neq j$. Of course, so long as we are involved in this paper, $|S_i \cap S_j|$ always equal either 0 or 1 for all $i \neq j$, as appointed above.

We say that a multigraph $M$ is an intersection multigraph on a family (a multifamily, respectively) $\mathcal{F}$, if there exist a family (a multifamily, respectively) $\mathcal{F}$ such that $M \cong \Omega(\mathcal{F})$. We say that $\mathcal{F}$ is a representation (a multifamily representation, respectively) of the multigraph $M$. The intersection number, denoted $\omega(M)$ (multifamily intersection number, denoted $\omega_m(M)$, respectively), of a given multigraph $M$ is the minimum cardinality of a set $X$ such that $M$ is an intersection multigraph (multifamily intersection multigraph, respectively) on a family (a multifamily, respectively) $\mathcal{F}$ consisting of distinct (not necessarily distinct, respectively) subsets of $X$. In this case we also say that $\mathcal{F}$ is a minimum representation (multifamily representation, respectively) of $M$.

Note that given a representation $\{S_v \mid v \in V(M)\}$ of $M$ and a vertex subset $S \subseteq V(M)$, then $\{S_v \mid v \in S\}$ form a representation of $\langle S \rangle_V$. Thus we know that $\omega(M)$ is not less than $\omega(\langle S \rangle_V)$ for any $S \subseteq V(M)$. Similarly for $\omega_m(M)$.

In 1966 P. Erdős et al. [1] proved that the edge set of any simple graph $G$ with $n$ vertices, no one of which is isolated vertex, can be partitioned using at most $\lceil n^2/4 \rceil$ cliques. In a couple decades S. McGuinness [3] showed that any greedy clique partition is such a partition.

A multifamily representation of a graph $G$ is a family of sets each member of which represent a vertex in $G$ and the intersection relation of two members of which represent the adjacency of the two corresponding vertices in $G$. P.
Erdős et al. [1] suggested a one-one correspondence between multifamily representations and clique coverings of a graph $G$. In fact, we may define a multifamily representation of a multigraph $M$ to be a family of sets for which each member represents a vertex in $M$, and the two vertices are adjacent with $q$ edges in $M$ if and only if the corresponding representation sets have an intersection of cardinality $q$. Then there is also a one-one correspondence between multifamily representations and clique partitions of $M$.

In next section we will narrate this correspondence in full detail. If a multifamily representation of a multigraph $M$ has pairwise distinct member sets, then it is called a representation of $M$. And then we turn the correspondence to prove that any $n$-vertex graph can be represented by at most $\lfloor n^2/4 \rfloor$ elements and we can accomplish such a representation from any greedy clique partition by a straightforward method based on this correspondence. In the end, certain future directions will be mentioned.

## 2 Partition Edge Set by Cliques

Given a multigraph $M = (V(M), E(M), q)$, $Q \subseteq V(M)$ is said to be a clique of $M$ if every pair of distinct vertices $u, v$ in $Q$ has $q(u, v) \neq 0$. A clique partition $Q$ of a multigraph is a set of cliques such that every pair of distinct vertices $u, v$ in $V(M)$ simultaneously appear in exactly $q(u, v)$ cliques in $Q$ and for each isolated vertex, that is, vertex with no edge incident to it, we need to use at least one trivial clique, that is, clique with only one vertex, in $Q$ to cover it. The minimum cardinality of a clique partition of $M$ is called the clique partition number of $M$, and is denoted by $cp(M)$. This number must exist as the edge set of $M$ forms a clique partition for $M$. We refer to a clique partition of $M$ with the cardinality $cp(M)$ as a minimum clique partition of $M$.

Note that a clique partition $Q$ of $M$ give rise to a clique partition of $M - v$ by deleting the vertex $v$ from each clique in $Q$. Thus $cp(M)$ is not less than the clique partition number of any induced subgraph of $M$.

P. Erdős et al. [1] proved the following theorem.

**Theorem 2.1.** The edge set of any simple graph $G$ with $n$ vertices no one of which is isolated vertex can be partitioned using at most $\lfloor n^2/4 \rfloor$ triangles and edges, and that the complete bipartite graph $K_{\lfloor n/2 \rfloor, \lceil n/2 \rceil}$ gives equality.

We somewhat modify their proof to prove the following theorem. We use $G^{(n)}$ to denote a graph $G$ with $n$ vertices.
Theorem 2.2. Any graph $G$ with $n \geq 4$ vertices (perhaps with isolated vertices) can be partitioned with at most $\lfloor n^2/4 \rfloor$ cliques $Q_1, \ldots, Q_N$ such that for any two vertices $u, v$ in $G$, we have

$$\{Q_i \mid u \in Q_i \in \{Q_1, \ldots, Q_N\}\} \neq \{Q_i \mid v \in Q_i \in \{Q_1, \ldots, Q_N\}\}. \quad (1)$$

Note that in such partition we need only use edges and triangles. Furthermore, the upper bound $\lfloor n^2/4 \rfloor$ is optimal.

Proof. For $n = 4$, it is easy to check the theorem holds for the 11 different graphs on 4 vertices. (Please see Figure 1)

Figure 1: The 11 Non-Isomorphic Graphs on 4 Vertices with Corresponding Family Representations
We proceed by mathematical induction from $n = 4$. First note that given any positive integer $n$,

$$\left\lfloor \frac{n^2}{4} \right\rfloor = \left\lfloor \frac{(n-1)^2}{4} \right\rfloor + \left\lfloor \frac{n}{2} \right\rfloor.$$

Hence we have to show that from $G^{(n-1)}$ to $G^{(n)}$, at most $\left\lfloor \frac{n}{2} \right\rfloor$ more cliques are needed. We have the following cases:

**Case 1**: In case $G^{(n)}$ has a vertex $v$ of degree $\leq \left\lfloor \frac{n}{2} \right\rfloor$, then first we delete $v$ and all edges incident with $v$. Then by induction hypothesis, we partition the resulting graph with at most $\left\lfloor \frac{(n-1)^2}{4} \right\rfloor$ cliques $K_2$ or $K_3$. Then from $G^{(n-1)}$ to $G^{(n)}$ we need only to use the edges joining the deleted vertex $v$ to other vertices of $G^{(n)}$, and then give rise to at most $\left\lfloor \frac{n}{2} \right\rfloor$ more cliques as $K_2$. Clearly the resulting clique partition of $G^{(n)}$ still satisfies (1).

**Case 2**: On the contrary, every vertex of $G^{(n)}$ is of degree $> \left\lfloor \frac{n}{2} \right\rfloor$. Let $x$ be the vertex with the minimum degree $t$, and set $t = \left\lfloor \frac{n}{2} \right\rfloor + r$, where $r > 0$. Let $x$ be adjacent to the vertices $y_1, \ldots, y_t$ and $G^{(t)}$ be the subgraph of $G^{(n)}$ induced by $\{y_1, \ldots, y_t\}$.

We claim that $G^{(t)}$ has $r$ edges and no two of which have a common vertex. Assume that $G^{(t)}$ has only $r-1$ such edges (note that it is similar to show the case $G^{(t)}$ has less than $r-1$ such edges), say

$$\{y_1, y_2\}, \{y_3, y_4\}, \ldots, \{y_{2r-3}, y_{2r-2}\}.$$

By $t = \left\lfloor \frac{n}{2} \right\rfloor + r = d(x) \leq n - 1$, we know that $r \leq \left\lfloor \frac{n}{2} \right\rfloor$ and thus $t \geq 2r$. Thus we may pick $y_{2r-1}$ from $\{y_1, \ldots, y_t\}$.

By hypothesis, $y_{2r-1}$ has degree $\geq \left\lfloor \frac{n}{2} \right\rfloor + r$. But it could be adjacent to at most $2r-2$ of the vertices $y_1, \ldots, y_{2r-2}$ and to at most $n-t$ of the vertices not in $G^{(t)}$, hence the degree of $y_{2r-1}$ is at most

$$(2r - 2) + (n-t) = (2r - 2) + (n - \left\lfloor \frac{n}{2} \right\rfloor + r)$$

$$= (n - \left\lfloor \frac{n}{2} \right\rfloor - 2) + r$$

$$< \left\lfloor \frac{n}{2} \right\rfloor + r.$$

However note that $\left\lfloor \frac{n}{2} \right\rfloor + r$ is the minimum degree, hence $y_{2r-1}$ is adjacent to some other vertex, say $y_{2r}$, in $G^{(t)}$ and

$$\{y_1, y_2\}, \{y_3, y_4\}, \ldots, \{y_{2r-3}, y_{2r-2}\}, \{y_{2r-1}, y_{2r}\}$$

are $r$ edges in $G^{(t)}$ and no two of which have a common vertex.

We remove these $r$ edges from $G^{(n)} - x$. Partition the resulting graph with at most $\left\lfloor \frac{(n-1)^2}{4} \right\rfloor$ cliques and (1) is satisfied. Then the $\left\lfloor \frac{(n-1)^2}{4} \right\rfloor$ cliques together with the triangles

$$\{x, y_1, y_2\}, \{x, y_3, y_4\}, \ldots, \{x, y_{2r-1}, y_{2r}\}$$

are
and the edges 
\[ \{x, y_k\}, \text{ where } 2r + 1 \leq k \leq t, \]
form a clique partition, which uses at most
\[
\left\lfloor \frac{(n - 1)^2}{4} \right\rfloor + r + (t - 2r)
= \left\lfloor \frac{(n - 1)^2}{4} \right\rfloor - r + \left\lfloor \frac{n}{2} \right\rfloor + r
= \left\lfloor \frac{n^2}{4} \right\rfloor
\]
clique.

Note that according to our convention in this paper, we need to use at least one trivial clique, even for each isolated vertex in the clique partition of the graph \( G^{(n)} - x \) with the \( r \) edges 
\[
\{y_1, y_2\}, \{y_3, y_4\}, \ldots, \{y_{2r-3}, y_{2r-2}\}, \{y_{2r-1}, y_{2r}\}
\]
removed. Thus the resulting clique partition of \( G^{(n)} \), obtained from that of \( G^{(n)} - x \) with the \( r \) edges removed, must agree with the requirement (1) of our theorem in the respect that for any two vertices \( u, v \) in \( G^{(n)} \),
\[
\{Q_i \mid u \in Q_i \in \{Q_1, \ldots, Q_N\}\} 
\neq \{Q_i \mid v \in Q_i \in \{Q_1, \ldots, Q_N\}\}.
\]

Last we show that the number \( \left\lfloor \frac{n^2}{4} \right\rfloor \) cannot be replaced by any smaller number by giving the following example. Let \( n = 2k \) or \( 2k + 1 \), we consider the complete bipartite graphs \( K_{k,k} \) and \( K_{k,k+1} \), which have \( 2k \) and \( 2k + 1 \) vertices, respectively. Clearly these two graphs have no triangle and their numbers of edges are
\[
k^2 = \left\lfloor \frac{(2k)^2}{4} \right\rfloor = \left\lfloor \frac{n^2}{4} \right\rfloor, \text{ if } n = 2k,
\]
and
\[
k(k + 1) = \left\lfloor \frac{(2k + 1)^2}{4} \right\rfloor = \left\lfloor \frac{n^2}{4} \right\rfloor, \text{ if } n = 2k + 1.
\]

Hence \( K_{k,k} \) and \( K_{k,k+1} \) always require \( \left\lfloor \frac{n^2}{4} \right\rfloor \) cliques for a clique partition.

Now we introduce the one-to-one correspondence between multifamily representations and clique partitions of a multigraph \( M \) as following.

Given a multigraph \( M^{(n)} = (V(M), E(M), q) \), we first construct a clique partition
\[
Q = \{Q_1, \ldots, Q_p\}
\]

Then with each clique $Q_k$ we associate an element $e_k$ and with each vertex $v_\alpha$ we associate a set $S_Q(v_\alpha)$ of elements $e_k$, where

$$e_k \in S_Q(v_\alpha) \iff v_\alpha \in Q_k,$$

i.e., $S_Q(v_\alpha)$ is the collection of elements for which the corresponding cliques contain $v_\alpha$. Thus we obtain

$$\mathcal{F}(Q) \equiv \{S_Q(v) : v \in V(M)\}.$$

Then clearly

$$S(\mathcal{F}(Q)) \equiv \bigcup_{v \in V(M)} S_Q(v)$$

contains $p$ elements. And

$$|S_Q(v_\alpha) \cap S_Q(v_\beta)| = q(v_\alpha, v_\beta),$$

since there is exactly $q(v_\alpha, v_\beta)$ cliques simultaneously containing the two vertices $v_\alpha, v_\beta$. Thus we have constructed a multifamily representation

$$\mathcal{F}(Q) = \{S_Q(v) : v \in V(M)\}$$

from the clique partition $Q$ of $M$, where

$$|S(\mathcal{F}(Q))| \equiv \left| \bigcup_{v \in V(M)} S_Q(v) \right| = p = |Q|.$$

Conversely, given a multifamily representation $\mathcal{F} = \{S_1, \ldots, S_n\}$ of $M$ with vertex set $V(M) = \{v_1, \ldots, v_n\}$, where $S_\alpha$ correspond to the set attaching to $v_\alpha$, then we can also construct a clique partition of $M$ by the following way.

Let

$$S(\mathcal{F}) \equiv \bigcup_{\alpha=1}^{n} S_\alpha = \{e_1, \ldots, e_p\}.$$

For each fixed $e_k$ in $S(\mathcal{F})$ we form a clique $Q_\mathcal{F}(e_k)$ using those vertices $v_\alpha$ such that the set $S_\alpha$ attaching to it contains $e_k$. Clearly each $Q_\mathcal{F}(e_k)$ is indeed a clique of $M$. Thus we obtain

$$Q(\mathcal{F}) = \{Q_\mathcal{F}(e_1), \ldots, Q_\mathcal{F}(e_p)\}.$$

And

$$q(v_\alpha, v_\beta) = |S_\alpha \cap S_\beta|$$

is the number of cliques in $Q(\mathcal{F})$ simultaneously containing $v_\alpha, v_\beta$. 7
since each element in $S_\alpha$ exactly represents a clique in $Q(F)$ containing $v_\alpha$. Thus we have constructed a clique partition $Q(F)$ of $M$ from the multifamily representation $F$ of $M$, where

$$|Q(F)| = p = |\bigcup_{\alpha=1}^{n} S_\alpha| \equiv |S(F)|.$$ 

Thus we have established a one-one correspondence between multifamily representations and edge clique partitions of the multigraph $M$.

From above we know that $\omega_m(M) = \psi_p(M)$. In particular we may consider the simple graphs as special classes of multigraphs:

**Theorem 2.3.** Let $G$ be a graph. Then we have $\omega_m(G) = \psi_p(G)$.

If we are given a graph $G^{(n)}$, then by Theorem 2.2 we may obtain a clique partition $Q$ with cardinality less or equal to $\lfloor n^2/4 \rfloor$, agreeing with the requirement (1). Then by the above method we may obtain a representation $F(Q) = \{S_Q(v) : v \in V(G)\}$ of $G$ consisting of distinct sets. Thus we have the following theorem.

**Theorem 2.4.** Let $G$ be a graph. Then $\omega(G^{(n)}) \leq \lfloor n^2/4 \rfloor$.

Again considering the two complete bipartite graphs $K_{k,k}$ and $K_{k,k+1}$, one can easily see that the bound $\lfloor n^2/4 \rfloor$ in Theorem 2.4 is sharp.

### 3 Greedy Clique Decomposition of Graphs

One may not be satisfied with the above theorem and would rather ask that how to obtain a representation of $G^{(n)}$ using at most $\lfloor n^2/4 \rfloor$ elements. S. McGuinness [3] proved the following theorem, which solved a conjecture by P. Winkler [9]:

**Theorem 3.1.** Every greedy clique decomposition of an $n$-vertex graph uses at most $\lfloor n^2/4 \rfloor$ cliques.

In the theorem, the so-called *clique decomposition* is a clique partition of the edge set, and *greedy clique decomposition* of a graph $G^{(n)}$ means an ordered set $Q = \{Q_1, ..., Q_m\}$ such that each $Q_i$ is a maximal clique in $G - \bigcup_{j<i} E(Q_j)$, where $G - \bigcup_{j<i} E(Q_j)$ is the subgraph of $G$ obtained by deleting all edges in the edge subset $\bigcup_{j<i} E(Q_j)$ while leaving all vertices in $G$ preserved.

For a representation $F$ of $G$, we referred as *monopolized elements* to those elements in $S(F)$ which appear in only one member of $F$. Here we prove the following main theorem, as a variant of S. McGuinness’s result:
Theorem 3.2. Every representation \( F \) of \( G^{(n)} \) with \( n \geq 4 \) obtained from \( F(Q) \), where \( Q \) is any greedy clique decomposition of \( G^{(n)} \) by successively attaching monopolized elements to the sets which repetitiously occur in \( F(Q) \), uses at most \( \lfloor n^2/4 \rfloor \) elements.

Before proving the theorem, we need the following lemma:

Lemma 3.3. Let \( Q \) be an edge clique partition of a graph \( G \), then we have that if \( F(Q) = \{ S_Q(v) : v \in V(G) \} \) has two identical sets, say \( S_Q(u) \) and \( S_Q(v) \), then the clique \( Q_{uv} \) in \( Q \) simultaneously containing \( u, v \) is a maximal clique in \( G \). Note that \( Q_{uv} \) has \( u \) and \( v \) as its monopolized elements, that is, \( u, v \) are in no clique of \( Q \) except \( Q_{uv} \).

Proof. If there is a clique \( Q' \) properly containing \( Q_{uv} \) in \( G \), say vertex \( w \) being in \( Q' \) but not in \( Q_{uv} \), then no clique in \( Q \) can simultaneously contain the three vertices \( u, v, w \). Thus the clique in \( Q \) simultaneously containing \( u, w \) doesn’t contain \( v \) and the clique in \( Q \) simultaneously containing \( v, w \) doesn’t contain \( u \), and therefore we must have \( S_Q(u) \neq S_Q(v) \).

If \( u \), say, belongs to one clique \( Q'' \) in \( Q \) other than \( Q_{uv} \), then there is a vertex, say \( u' \), adjacent to \( u \) and not in \( Q_{uv} \). In case that \( u' \) is not adjacent to \( v \) we must have \( S_Q(u) \neq S_Q(v) \). In case that \( u' \) is adjacent to \( v \), then no clique in \( Q \) can simultaneously contain \( u, v, u' \). Thus the clique in \( Q \) simultaneously containing \( u, u' \) doesn’t contain \( v \) and the clique in \( Q \) simultaneously containing \( u', v \) doesn’t contain \( u \), and therefore we must have \( S_Q(u) \neq S_Q(v) \). \( \square \)

Then we are in a position to proceed the proof of Theorem 3.2.

Proof. We use induction on \( n \).

When \( n = 4 \), it is easy to draw all the eleven different graphs on four vertices, and to check that every representation of each of them uses at most \( \lfloor n^2/4 \rfloor \) elements.

For the case \( n = 5 \), note that \( \lfloor 5^2/4 \rfloor - \lfloor 4^2/4 \rfloor = 6 - 4 = 2 \) and therefore we have two new elements in proceeding from \( n = 4 \) to \( n = 5 \). We have the following four cases:

Case 1: If \( G^{(5)} \) has one vertex with degree 2 or less, then we reduce \( G^{(5)} \) to \( G^{(4)} \) by deleting this vertex and all edges incident to it. Note that this vertex form a maximal clique in \( G^{(5)} \) along with some edge in \( G^{(4)} \) only if \( G^{(5)} \) is one of 13 non-isomorphic graphs in Figure 2 where hollow circle denote this vertex and dashed lines denote the edges incident to it. It is easy to check that every representation of each one of them uses at most \( \lfloor 5^2/4 \rfloor = 6 \) elements.
Figure 2: One Case for Representations of Graphs on 5 Vertices
Case 2: As for the case that there is no maximal clique in $G^{(5)}$ simultaneously containing this vertex and some edge in $G^{(4)}$, then in any greedy clique partition of $G^{(5)}$ we must use all the edges incident to this vertex as members of this greedy clique partition. Thus in this case, we may at first take a representation of $G^{(4)}$, and then go back to $G^{(5)}$ using the available two new elements to represent at most two edges incident to this vertex. Then we may confirm that in this case all representations of $G^{(5)}$ use at most $\left\lfloor \frac{5^2}{4} \right\rfloor = 6$ elements.

Case 3: As for the case that there is no edge in $G^{(5)}$ incident to this vertex, we may at first take a representation of $G^{(4)}$ and then go back to $G^{(5)}$ using one new monopolized element.

Case 4: Due to above, now we need to consider only those graphs on 5 vertices for which every vertex has degree greater than or equal to 3. There are only three such graphs and they are easy to be checked. (Please see Figure 3) Thus the case $n = 5$ is done, and we have proved the theorem for $n = 4$ and $n = 5$.

Figure 3: Graphs on 5 Vertices Whose Vertices Have Degree $\geq 3$ with Corresponding Representations

Now let $F$ be a representation of $G^{(n)}$ with $n \geq 6$ derived from $F(Q)$, where $Q = \{Q_1, ..., Q_m\}$ is a greedy clique partition of $G^{(n)}$. Note that deleting $Q_j$ from the set $Q$ leaves a greedy clique partition of $G - E(Q_j)$.

In case that each $Q_j$ has at least three edges, we have $m \leq \left( \binom{n}{3} / 3 \right) < n^2 / 6$. Assume for the time being that every $Q_i$ has exactly three edges, that is, is exactly a triangle. Now if every triangle in $Q$ has at most one of its three vertices of degree 2, then by Lemma 3.3 we do not need to use any monopolized element for this greedy clique partition. If there is a triangle
in \( Q \) with at least two of its three vertices of degree 2, then recall that \( G^{(n)} \) have at least six vertices, two vertices of degree 2 in this triangle make \( m \) to be less than or equal to \( \binom{n}{3}/3 - 2 < (n^2/6) - 2 \). Thus although we might need two more monopolized elements for this triangle, yet in the same time we also have two less cliques (as \( K_3 \)) in \( Q \). Besides, if there is a clique of cardinality \( 3 + r \) where \( r > 0 \) in \( Q \), then despite that maybe we need \( r \) more monopolized elements for this clique, yet in the same time we also have two less cliques (as \( K_3 \)) in \( Q \). By Lemma 3.3, we need to use \( r + 2 \) more monopolized elements for this clique only when either this clique is an isolated clique or \( G^{(n)} \) is itself a clique. For the latter case, we use \( n \) elements to represent \( G^{(n)} \) and note that \( n^2/6 \geq n \) for \( n \geq 6 \). As for the former case, we lose all the edges joining this isolated clique to all the vertices not on this isolated clique, therefore we lose at least 5 edges from the calculated \( \binom{n}{2} \) edges and hence further lose at least two cliques from the calculated \( n^2/6 \) cliques (as \( K_3 \)). By Lemma 3.3 we need to use \( r + 1 \) more monopolized elements for this clique only when this clique has exactly \( r + 2 \) vertices of degree \( (3 + r) - 1 \). In this case, this clique has a vertex \( u \) adjacent to one vertex, say \( u' \), not in this clique, and all vertices in this cliques other than \( u \) are not adjacent to \( u' \). Therefore in \( G^{(n)} \) we have \( r + 2 \geq 3 \) less edges than complete graph \( K_n \), and thus we have still one less triangle in \( Q \). Now we have brought to the conclusion that in case that each \( Q_j \) has at least three edges, we never use more than \( n^2/6 \) elements to form a representation of \( G^{(n)} \). Now we have justified assuming some \( Q_j \) is an edge \( xy \). In case that \( d(x) = d(y) = 1 \), we may first take a representation of \( G^{(n)} - x - y \) by the method of Theorem 2.2 using at most \( \lceil (n-2)^2/4 \rceil \) elements, and then use two new elements for the isolated edge \( xy \) to form a representation for \( G^{(n)} \) with at most \( \lceil n^2/4 \rceil \) elements. Thus in this case every representation of \( G^{(n)} \) derived from \( F(Q) \), where \( Q \) is any greedy clique partition of \( G^{(n)} \), uses at most \( \lceil n^2/4 \rceil \) elements.

As for the case that one of \( x, y \) has degree more than one, in any representation of \( G^{(n)} \) we can not use any monopolized element on \( x \) or \( y \). Now let \( R \) consist of the members of \( Q - \{Q_j\} \) that are incident to \( x \), and \( S \) consist of those incident to \( y \). Then the set

\[
Q' = Q - (R \cup S \cup \{Q_j\})
\]

is a greedy clique partition of

\[
G' = (G^{(n)} - x - y) - \bigcup_{Q_i \in R \text{ or } S} E(Q_i),
\]

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except possibly leaving some isolated vertices in $G'$ uncovered by any members of $Q'$. Recall that for the present case, in $F$ we never use any monopolized element on $x, y$. Now if we can prove that every monopolized element in $S(F)$ is always necessary for deriving a representation of $G'$ from $F(Q')$, then by induction hypothesis we prove that

$$|Q(F) - (R \cup S \cup \{Q_j\})| \leq \lfloor (n - 2)^2 / 4 \rfloor. \quad (2)$$

If in $F$ we used one monopolized element on some vertex $v$ not belonging to any member of $R \cup S$, then in $F(Q)$, the set $S_Q(v)$ must be identical with some $S_Q(u)$ where $u$ is also a vertex not belonging to any member of $R \cup S$. Since both $u$ and $v$ do not belong to any member of $R \cup S$, then $S_Q'(u) = S_Q'(v)$ in $F(Q')$. Thus this monopolized element is necessary for deriving a representation of $G'$ from $F(Q')$.

If in $F$ we used one monopolized element on some vertex $v$ belonging to one member, say $Q_v$, of $R \cup S$. Then in $F(Q)$, the set $S_Q(v)$ must be identical with some $S_Q(u)$ where $u$ is also a vertex belonging to $Q_v$. Now by Lemma 3.3 $v$ must have all its neighbors in $Q_v$. Thus $v$ is an isolated vertex in $G'$. Thus this monopolized element is necessary for deriving a representation of $G'$ from $F(Q')$. Thus we have proved the statement (2).

Now it suffices to prove that

$$|R \cup S| \leq n - 2,$$

since

$$n - 2 \leq \lfloor n^2 / 4 \rfloor - \lfloor (n - 2)^2 / 4 \rfloor - 1.$$

We prove this by choosing distinct vertices in $V(G) \setminus \{x, y\}$ from the vertex sets of the members of $R \cup S$. Note that since each edge is covered exactly once in a clique partition, each $v \notin \{x, y\}$ appears once in $R$ if $v$ is adjacent to $x$ and once in $S$ if $v$ is adjacent to $y$. Consider $Q_1 \in R$. If $Q_1$ contains a vertex $v$ not adjacent to $y$, then we choose such a $v$ for $Q_1$. If all vertices in $Q_1$ are adjacent to $y$, then we choose a vertex $v \in Q_1$ such that $vy$ belongs to the first member of $Q$, say $Q_2$, which contains both $y$ and some vertex of $Q_1$. Note that $Q_2$ is the only member of $S$ containing $v$.

Now we have two cases, that is, either that $Q_1$ precedes $xy$ in $Q$ or that $xy$ precedes $Q_1$ in $Q$. For the first case, since $Q_1$ and $xy$ are maximal while chosen, $Q_2$ must precede $Q_1$ in $Q$ for otherwise from the aforementioned hypothesis that all vertices in $Q_1$ are adjacent to $y$ and $Q_1$ precedes $xy$ in $Q$, $Q_1$ should have contained $y$ and hence $xy$. For the second case, since $xy$ is maximal while chosen, one of $Q_1, Q_2$ precedes $xy$ or otherwise $xy$ should have contained $v$. Thus in this case $Q_2$ precedes $Q_1$ in $Q$. Note that in both cases, we have that $Q_2$ precedes both of $Q_1, xy$ in $Q$. 

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For the members of $S$, similarly as above choose vertices by reversing the roles of $x$ and $y$.

In above we have shown that if $v$ belongs to some $Q_1 \in R$ and to some $Q_2 \in S$, and $v$ is chosen for one of them, then the one for which it is chosen occurs after the other one in the ordered set $Q$. Hence no vertex is chosen twice. Thus we conclude that

$$|R \cup S| \leq n - 2.$$  

\[ \square \]

4 Conclusion Remarks

The edge clique partitions, as a special case of edge clique covers, are served as great classifying and clustering tools in many practical applications, therefore it is interesting to explore the concept in more details.

One may work on the cases besides multifamily and family, say antichain, uniform family etc. The greedy way to obtain these variants also naturally gives the optimal upper bounds for the corresponding intersection numbers.

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