Evaluations of Derivatives of Jacobi Theta Functions in the origin

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abstract
In this article using Ramanujan’s theory of Eisenstein series we evaluate completely the derivatives of the theta functions $\vartheta_{1}^{(2\nu+1)}(z)$ and $\vartheta_{4}^{(2\nu)}(z)$ in the origin in closed polynomials forms using only the first three Eisenstein series of weights 2, 4, and 6.

keywords: theta functions; Ramanujan; derivatives; evaluations;

1 Introduction
Following Berndt in [3], let $q = e^{2\pi i \tau}$, where $\tau$ is in the upper half-plane $H$, and write

$$\Phi_{\nu}(q) := \sum_{n=1}^{\infty} \frac{n^{\nu}q^{n}}{1-q^{n}}$$  \hfill (1)

and

$$E_{2}(\tau) = 1 - 24\Phi_{1}(q) - \frac{3}{\pi y}$$ \hfill (2)

$$E_{\nu}(\tau) = 1 - \frac{2\nu}{B_{\nu}}\Phi_{\nu-1}(q)$$ \hfill (3)

for $n > 2$ even and where $y = Im(\tau) > 0$, $B_{n}$ denotes the $n$th Bernoulli number (see [2]).

Ramanujan then defines

$$P = P(q) := 1 - 24 \sum_{n=1}^{\infty} \frac{nq^{n}}{1-q^{n}}$$ \hfill (4)

$$Q = Q(q) := 1 + 240 \sum_{n=1}^{\infty} \frac{n^{3}q^{n}}{1-q^{n}}$$ \hfill (5)
and

\[ R = R(q) := 1 - 504 \sum_{n=1}^{\infty} \frac{n^5 q^n}{1 - q^n} \]  \hspace{1cm} (6)

According to Ramanujan we can evaluate (see [3] pg. 319) every \( \Phi_{2\nu+1}(q) \) in polynomials of \( Q \) and \( R \).

Also we will need the Dedekind’s eta-function which is

\[ \eta(\tau) := q^{1/24} \prod_{n=1}^{\infty} (1 - q^n) \]  \hspace{1cm} (7)

In the literature (see [7],[1]) the first elliptic theta function is defined by

\[ \vartheta_1(z, t) := \sum_{n=-\infty}^{\infty} (-1)^{n-1/2} q_1^{(n+1/2)^2} e^{(2n+1)iz} \]  \hspace{1cm} (8)

\[ \vartheta_4(z, t) = \sum_{n=-\infty}^{\infty} (-1)^n q_1^{n^2} e^{2nz} \]  \hspace{1cm} (9)

where \( t = 2\tau \) and hence \( q = e^{2\pi i\tau} = q_1 = e^{\pi it} \). Also hold the following relations

\[ \vartheta_1(z, t) = 2 \sum_{n=0}^{\infty} (-1)^n q_1^{(n+1/2)^2} \sin[(2n+1)z] \]  \hspace{1cm} (10)

\[ \vartheta_4(z, q) = 1 + 2 \sum_{n=1}^{\infty} (-1)^n q_1^n \cos(2nz) \]  \hspace{1cm} (11)

Since our attention will be focused on \( z \) and we do not make extended use of \( q \) or \( q_1 \) the reader may check everytime he wants, the correspondence, numerically with the program he use.

Hence we work with the notations

\[ \vartheta_1(z) = \vartheta_1 \left( z, \frac{t}{2} \right) = \vartheta_1(z, \tau) \]

\[ \vartheta_4(z) = \vartheta_4 \left( z, \frac{t}{2} \right) = \vartheta_4(z, \tau) \]

The sine Fourier series of the first derivative of \( \vartheta_1(z) \) is:

If \( |q| < 1 \), then

\[ \frac{\vartheta_1'(z)}{\vartheta_1(z)} = \cot(z) + 4 \sum_{n=1}^{\infty} \frac{q^{2n}}{1 - q^{2n}} \sin(2nz). \]  \hspace{1cm} (12)

Here we take \( Im(z) = 0 \).

**Theorem 1.** (Ramanujan) (see [6])

Let

\[ S_3(m) = \sum_{\alpha \equiv 1 \pmod{4}} q^{\alpha^2/8} \alpha^m \]  \hspace{1cm} (13)
Then

\[ S_3(1) = \eta^3(\tau) \]  
\[ S_3(3) = \eta^3(\tau)P \]  
\[ S_3(5) = \eta^3(\tau) \frac{(5P^2 - 2Q)}{3} \]  
\[ S_3(7) = \eta^3(\tau) \frac{35P^3 - 42PQ + 16R}{9} \]

and in general

\[ S_3(2m + 1) = \eta^3(\tau) \sum_{i+2j+3k=m} b_{ijk}P^iQ^jR^k \]

In this article we will find a general way to evaluate \( b_{ijk} \) and consequently the derivatives \( \vartheta(2m+1)(z) \) in the origin using only \( P, Q \) and \( R \). The same thing we do and with \( \vartheta(2\nu)(z) \). For to complete our purpose we use the following Theorem of Ramanujan for the evaluation of Eisenstein series:

**Theorem 2.** (Ramanujan) (see [3] chapter [15])

Let \( n \) be integer greater of 1, then

\[ S_{1,2n} = \frac{(-1)^{n-1}B_{2n}}{4n} E_{2n}(\tau) \]  

Also if \( n \) is even and exceeding 4, then (see and relation (3))

\[ -\frac{(n+2)(n+3)}{2n(n-1)} S_{1,n+2} = -20 \binom{2}{2-2} S_{1,4} S_{1,n-2} + \]

\[ \sum_{k=1}^{[(n-2)/4]} \binom{n-2}{2k} [(n + 3 - 5k)(n - 8 - 5k) - 5(k - 2)(k + 3)] S_{1,2k+2} S_{1,n-2k} \]  

The prime on the summation sign indicates that if \((n-2)/4\) is an integer, then the last term of the sum is to be multiplied by \( \frac{1}{2} \).

## 2 The \( \vartheta_1^{(2\nu+1)}(0) \) derivatives

**Lemma 1.**

For \( \nu \) non negative integer

\[ \vartheta_1^{(2\nu+1)}(0, \tau/2) = 2(-1)^\nu S_3(2\nu + 1) \]  

3
Proof
Recall the Ramanujans Theorem 1 and differentiate (9) to write

\[ \vartheta_{2 ν}^{1}(2ν + 1)(0, τ/2) = 2(-1)^{ν} \sum_{m=0}^{∞} (-1)^{m} q^{m+1/2} (2m + 1)^{2ν+1} = \]

\[ = 2(-1)^{ν} \sum_{m=-∞}^{∞} q^{(4m+1)^{2}/8} (4m + 1)^{2ν+1} = 2(-1)^{ν} S_{3}(2ν + 1) = \]

\[ = 2(-1)^{ν} \eta^{3}(τ) \sum_{i+2j+3k=ν} b_{ijk} P^{i} Q^{j} R^{k} \]

and the proof is complete.

Lemma 2.
\[
\frac{d^{2ν}}{dz^{2ν}} (\sin(2nz) \sin((2m + 1)z))_{z=0} = \frac{(-1)^{ν}}{2} \left[ (2m + 1 - 2n)^{2ν} - (2m + 1 - 2n)^{2ν} \right]
\]

Proof.
It is

\[
\sin(z) = \sum_{n=0}^{∞} \frac{(-1)^{n} z^{2n+1}}{(2n + 1)!}
\]

(23)

Observe first that

\[
\sigma_{1}(c, n) = \frac{d^{ν}}{dz^{ν}} (\sin(cz)z^{n})_{z=0} = n! \cdot i^{-n-1} \frac{1 - (-1)^{n}}{2} \binom{ν}{n}
\]

(24)

The result follows from (23) and (24).

Lemma 3.
\[
\frac{d^{ν}}{dz^{ν}} (\cot(z) \sin(cz))_{z=0} = \]

\[ = \frac{i^{ν}(1 + (-1)^{ν})}{2} \sum_{n=1}^{ν-1} \frac{\sigma_{1}(c, n)}{n!} \left( \frac{d^{n}}{dz^{n}}(\cot(z) - 1/z) \right)_{z=0} + \frac{i^{ν}(1 + (-1)^{ν})}{2} \frac{e^{ν+1}}{ν + 1}
\]

(25)

Proof.
It is

\[
\cot(z) = \frac{1}{z} - \sum_{n=1}^{∞} \frac{(-1)^{n-1} 2^{2n} B_{2n}}{2n！} z^{2n-1}
\]

(26)

The result follows as in Lemma 2 using (24) and (26).
Theorem 3.

For \(\nu\) not negative integer holds

\[
S_3(2\nu + 1) = -\sum_{n=1}^{2\nu-1} \frac{|B_{n+1}| 2^{n+1} \binom{2\nu}{n} S_3(2\nu - n)}{n+1} + \frac{1}{2\nu + 1} S_3(2\nu + 1) -
2 \sum_{l=1}^{\nu} \binom{2\nu}{2l-1} 2^{2l} S_3(2(\nu - l) + 1) \Phi_{2l-1}(q)
\]

Moreover we have

\[\vartheta_1^{(2\nu+1)}(0) = 2(-1)^\nu S_3(2\nu + 1)\]

Proof.

From (10) we have

\[
\vartheta_1'(z) = \vartheta_1(z) \cot(z) + 4 \sum_{n=1}^{\infty} \frac{q^{2n}}{1 - q^{2n}} \sin(2nz) \vartheta_1(z)
\]

Expanding the \(\vartheta_1(z)\) in series (10) and taking the \(2\nu\) derivative with respect to \(z\) in zero we have

\[
\vartheta_1^{(2\nu+1)}(0) = 2 \sum_{m=0}^{\infty} (-1)^m q^{(m+1/2)^2} (\cot(z) \sin((2m + 1)z))_{z=0}^{(2\nu)} +
8 \sum_{n=1}^{\infty} \frac{q^{2n}}{1 - q^{2n}} \sum_{m=0}^{\infty} (-1)^m q^{(m+1/2)^2} (\sin(2zn) \sin((2m + 1)z))_{z=0}^{(2\nu)}
\]

Write

\[
A(\tau) = 2 \sum_{m=0}^{\infty} (-1)^m q^{(m+1/2)^2} (\cot(z) \sin((2m + 1)z))_{z=0}^{(2\nu)}
\]

and

\[
B(\tau) = 8 \sum_{n=1}^{\infty} \frac{q^{2n}}{1 - q^{2n}} \sum_{m=0}^{\infty} (-1)^m q^{(m+1/2)^2} (\sin(2zn) \sin((2m + 1)z))_{z=0}^{(2\nu)}
\]

Then from Lemma 3 we have

\[
A(\tau/2) = (-1)^\nu \sum_{n=1}^{2\nu-1} \frac{d^n}{dz^n} (\cot(z) - 1/z)_{z=0}^{-n-1} \left( \frac{1 - (-1)^n}{2} \right) \binom{2\nu}{n} S_3(2\nu - n) +
\]

\[
+ \frac{(-1)^\nu}{2\nu + 1} S_2(2\nu + 1)
\]

From Lemma 2 we have

\[
B(\tau/2) = (-1)^{\nu-1} \sum_{l=1}^{\nu} \binom{2\nu}{2l-1} 2^{2l} S_3(2(\nu - l) + 1) \sum_{n=1}^{\infty} \frac{q^n n^{2l-1}}{1 - q^n}
\]
Using the above identities we get the result.

Using Theorems 2 and 3 the relations (15),(16) and (17) follow from (14). We give also as examples evaluations for higher order of \( \nu \):

### 3 Evaluations

From Theorem 3 with \( \nu = 4 \) we have to evaluate \( \Phi_7(q) = \Phi_7 \). For \( \nu = 5 \) we have the value of \( S_3(11) \) but first we have to evaluate \( \Phi_9(q) = \Phi_9 \). This will be done by using Ramanujan’s Theorem 2.

Here are the first examples of Ramanujan’s Theorem 2

\[
\begin{align*}
\Phi_7 &= \frac{1}{480}(Q^2 - 1) \\
\Phi_9 &= \frac{1}{264}(1 - QR) \\
\Phi_{11} &= \frac{-691 + 441Q^3 + 250R^2}{65520} \\
\Phi_{13} &= \frac{1 - Q^2R}{24} \\
\Phi_{15} &= \frac{-3617 + 1617Q^4 + 2000QR^2}{16320} \\
\Phi_{17} &= \frac{43867 - 38367Q^3R - 5500R^3}{28728}
\end{align*}
\]

...etc

Hence using Theorem 3 we get (where we have set \( h = \eta(\tau) \))

\[
\begin{align*}
&\frac{1}{2}g_1^{(9)}(0) = S_3(9) = \frac{1}{9}h^3(35P^4 - 84P^2Q - 12Q^2 + 64PR) \\
&\frac{1}{2}g_1^{(11)}(0) = S_3(11) = \frac{1}{9}h^3(385P^5 - 1540P^3Q - 30030P^2Q - 25740P^2Q^2 + 552Q^3 + 45760P^3R^2 + 4992PQR - 512R^3) \\
&\frac{1}{2}g_1^{(13)}(0) = S_3(13) = \frac{1}{27}h^3(5005P^6 - 30030P^4Q - 25740P^4Q^2 + 552Q^3 + 45760P^3R + 87360P^2QR - 3648QR^2) \\
&\frac{1}{2}g_1^{(15)}(0) = S_3(15) = \frac{1}{27}h^3(25025P^7 - 210210P^5Q - 300300P^5Q^2 + 19320PQ^3 + 400400P^4R + 87360P^2QR - 3648Q^2R - 17920PR^2)
\end{align*}
\]

...etc
4 The $\vartheta_4^{(2\nu)}(0)$ derivatives

Using the identities (see [7]):

$$\frac{\vartheta_4'(z)}{\vartheta_4(z)} = 4 \sum_{n=1}^{\infty} \frac{q^n \sin(2nz)}{1-q^{2n}}$$ (29)

and the obvious

$$\frac{q^n}{1-q^{2n}} + \frac{q^{2n}}{1-q^{2n}} = \frac{q^n}{1-q^n}$$

we have after derivating $2\nu - 1$-times the relation (29):

$$2 \cdot 4^\nu \sum_{n=1}^{\infty} \frac{n^{2\nu-1}q^{2n}}{1-q^{2n}} + \left( \frac{d^{2\nu} \log(\vartheta_4(z))}{dz^{2\nu}} \right)_{z=0} = 2 \cdot 4^\nu \sum_{n=1}^{\infty} \frac{q^n n^{2\nu-1}}{1-q^n}$$ (30)

But in general holds the following:

**Theorem 4.** (Faa di Bruno)

Let $f$, $g$ be sufficiently differentiable functions, then

$$\frac{d^n f(g(x))}{dx^n} = \sum_{m_1!(1!)^{m_1}m_2!(2!)^{m_2} \ldots m_n!(n!)^{m_n}} f^{(m_1+m_2+\ldots+m_n)}(g(x)) \cdot \prod_{j=1}^{n} (g^{(j)}(x))^{m_j}$$ (31)

where the sum is over all $n$-tuples of nonnegative integers $(m_1, m_2, \ldots, m_n)$ satisfying the constraint

$$1 \cdot m_1 + 2 \cdot m_2 + 3 \cdot m_3 + \ldots + n \cdot m_n = n$$ (32)

Hence from

$$\frac{d^{2\nu} \log(z)}{dz^{2\nu}} = -\frac{\Gamma(2\nu)}{z^{2\nu}}, \nu = 1, 2, \ldots$$ (33)

we get

$$\left( \frac{d^{2\nu} \log(\vartheta_4(z))}{dz^{2\nu}} \right)_{z=0} = -\sum m_1!(1!)^{m_1}m_2!(2!)^{m_2} \ldots m_n!(n!)^{m_n} \cdot \prod_{j=1}^{n} (\vartheta_4^{(j)}(0))^{m_j}$$

$$\times \frac{1}{[\vartheta_4(0)]^{2(m_1+m_2+\ldots+m_{2\nu})}} \cdot \prod_{j=1}^{2\nu} \left( \vartheta_4^{(j)}(0) \right)^{m_j}$$ (34)

Let $x$ be the elliptic modulus and $K$ the complete elliptic integral of the first kind as defined in [4] pages 101-102, then if we set

$$w := \sqrt{\frac{2}{\pi}(1-x)K}$$ (35)
Theorem 5.

The derivatives $\vartheta^{(2\nu)}(0)$, $\nu = 1, 2, 3, \ldots$ can be evaluated from

$$\vartheta^{(2\nu)}(0) = \frac{1}{2} \cdot 4^\nu \sum_{m_1![(m_1)!m_2![(2\nu)!]^m_2}} \frac{(2\nu)!}{\Gamma[2(m_1 + m_2 + \ldots + m_{2\nu})]} \times$$

$$\times \prod_{j=1}^{2\nu} \left(\vartheta^{(j)}(0)\right)^{m_j} = \frac{B_{2\nu}}{4^\nu} (E_{2\nu}(2\tau) - E_{2\nu}(\tau))$$

(36)

$$\vartheta^{(2\nu)}(0) = w \sum_{0 \leq i,j,k,l,m,n \leq \nu} c_{ijklmn} P^i Q^j R^k P_1^l Q_1^m R_1^n$$

(37)

where $P_1 = P(q)$, $P_2 = P(q^2)$, $Q_1 = Q(q^2)$, $Q_2 = Q(q^2)$, $R_1 = R(q)$, $R_2 = R(q^2)$ and $c_{ijkl}$ are rationals.

Proof.

i) Relation (36) follows from (34),(30) and (1),(2),(3).

ii) For relation (37) observe that (36) is a recurrence relation both in $\vartheta^{(2\nu)}(0)$, $E_{2\nu}(\tau)$ and $E_{2\nu}(2\tau)$. Hence the $\vartheta^{(2\nu)}(0)$ can be evaluated using Theorem 2 from $P(q)$, $Q(q)$, $R(q)$ and $P(q^2)$, $Q(q^2)$, $R(q^2)$ (except for $\nu = 1$ which we use (4) instead of (2)).

Examples

$$\vartheta^{(2)}(0) = \frac{1}{3} (P_1 - P)w$$

$$\vartheta^{(3)}(0) = \frac{1}{15} \left(5(P - P_1)^2 - 2Q + 2Q_1\right)w$$

$$\vartheta^{(6)}(0) = \frac{1}{63} (-35P^3 + 105P^2 P_1 - 105PP_1^2 + 35P_1^3 + 42PQ - 42P_1Q - 84P_1Q_1 + 42P_1Q_1 + 16R + 16R_1)w$$

$$\vartheta^{(8)}(0) = \frac{1}{135} (175P^4 - 700P^3 P_1 + 1050P^2 P_1^2 - 700PP_1^3 + 175P_1^4 - 420P^2 Q + 840P_1P_1Q - 420P_1^2 Q - 60Q^2 + 420P^2 Q_1 - 840P_1P_1Q_1 + 42P_1^2 Q_1 - 168QQ_1 + 228Q_1^2 + 320P R - 320P_1 R - 320P_1R_1 + 320P_1 R_1)w$$
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