Optimal Unitary Linear Processing for Amplify-and-Forward Cooperative OFDM systems

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Abstract—In this paper, we consider the amplified-and-forward relaying in an OFDM system with unitary linear processing at the relay. We proposed a general analytical framework to find the unitary linear processing matrix that maximizes the system achievable rate. We show that the optimal processing matrix is a permutation matrix, which implies that a subcarrier pairing strategy is optimal. We further derived the optimal subcarrier pairing schemes for scenarios with and without the direct source-destination path for diversity. Simulation results are presented to demonstrate the achievable gain of optimal subcarrier pairing compared with non-optimal linear processing and non-pairing.

I. INTRODUCTION

OFDM-based relaying combines the advantages of both OFDM and relaying techniques to improve network performance and efficiency. It is essential that the relaying techniques can be maximally explored in the OFDM system. In a narrow-band single carrier system, the relay retransmits the processed version of the received signal over the same carrier. For an OFDM system with multiple subcarriers, the relay has an additional frequency dimension available, which can be exploited to process the incoming signals adaptively based on subcarrier strength for relaying purpose, and thus can potentially improve the overall relay performance. Without such consideration, directly applying a relaying method optimized for single-carrier systems will not be optimal in an OFDM system. Constrained on the class of permutation matrices, this framework enables us to find the optimal permutation matrix, i.e., subcarrier pairing, for relaying in both scenarios, without and with the direct path. The former is consistent with previous works, and the latter has never been shown before to the best of our knowledge. Furthermore, we prove that the optimal unitary linear processing matrix is in fact a permutation matrix which gives an optimal subcarrier pairing strategy.

The rest of this paper is organized as follows. In Section II we present the system model and formulate the optimal linear processing matrix to maximize the achievable rate. Section III presents our analytical framework to determine the optimal permutation matrix and hence the optimal subcarrier pairing strategy. In Section IV we show that subcarrier pairing in fact is the optimal linear processing strategy. We present simulation results to demonstrate the performance gain achieved through optimal subcarrier pairing in Section V and finally conclude in Section VI.

II. PROBLEM STATEMENT

A. System Model

We consider a dual-hop relay network with a pair of source and destination nodes and a single AF relay node in an OFDM system with N subcarriers. We constrain ourselves to half-duplex transmission, where a relay node is either in transmission or reception but not simultaneously. The cooperative transmission takes place in two phases. In the first phase the source sends data through N subcarriers to the relay and
destination simultaneously (if the direct path is available). The relay then performs linear processing of the received signals over $N$ subcarriers, and forwards the amplified version of the processed signals to the destination.

We denote the channel gain over subcarrier $k$ from source to relay, from relay to destination, and from source to destination by $h_{1k}$, $h_{2k}$, and $h_{0k}$, respectively, and $s_k$ the source signal transmitted on subcarrier $k$ with power coefficient $d_{sk}$. The received signals at the relay and destination in the first stage are given by

$$\begin{align*}
y_r &= H_1 D_s s + n_r, \quad (1) \\
y_d^{(1)} &= H_0 D_s s + n_d^{(1)}, \quad (2)
\end{align*}$$

where $y_r = [y_{r1}, \ldots, y_{rN}]^T$ and $y_d^{(1)} = [y_d^{(1)}, \ldots, y_d^{(1)}]^T$ are the received signal vector at relay and destination, respectively, $H_1 = \text{diag}(h_{11}, \ldots, h_{1N})$ and $H_0 = \text{diag}(h_{01}, \ldots, h_{0N})$ are the corresponding channel matrices, and $D_s = \text{diag}(d_s)$ with $d_s = [d_{s1}, \ldots, d_{sN}]^T$ being the power coefficient vector. The signals are i.i.d. with average unit power $E[|s|^2] = 1$. Moreover, $n_r = [n_{r1}, \ldots, n_{rN}]^T$ and $n_d^{(1)} = [n_{d1}^{(1)}, \ldots, n_{dN}^{(1)}]$ are AWGN at the relay and the destination, with $n_r \sim \mathcal{CN}(0, \sigma_r^2 I)$ and $n_d^{(1)} \sim \mathcal{CN}(0, \sigma_d^2 I)$, respectively.

In the second phase, the received signal $y_r$ is linearly processed at the relay with a unitary matrix $W$, and the relay retransmits the amplified version of the processed signals. The received signal vector at the destination is given by

$$\begin{align*}
y_{d}^{(2)} &= H_2 D_r W y_r + n_{d}^{(2)} \quad (3)
\end{align*}$$

where $y_{d}^{(2)} = [y_{d}^{(2)}, \ldots, y_{d}^{(2)}]^T$, $H_2 = \text{diag}(h_{21}, \ldots, h_{2N})^T$, and $n_{d}^{(2)} = [n_{d}^{(2)}, \ldots, n_{d}^{(2)}]^T \sim \mathcal{CN}(0, \sigma_d^2 I)$. The power coefficient vector for the processed signal at the relay is denoted as $d_r$, and we have $D_r = \text{diag}(d_r)$. Note that, $W$ is a unitary matrix with $WW^H = I$.

Let $P_r$ and $P_e$ be the maximum average total power at the source and the relay, respectively. Thus, $D_s$ must satisfy $E[|D_s|^2] \leq P_s$. Similarly, the linear processing matrix $W$ and the power coefficient matrix $D_r$ at the relay must satisfy

$$E[|D_r W(D_s H_1 s)|^2] + E[|D_r W n_1|^2] \leq P_r \quad (4)$$

To focus on the effect of the processing matrix $W$ on the relay performance, in this study, we assume a pre-determined power allocation over subcarriers at the source, $i.e.$, $D_s$ is given. At the relay, we assume that the relay equally amplifies the processed signal over the subcarriers, $i.e.$, $D_r$ is a scalar of identity matrix $D_r = d_r I$ with

$$d_r = \sqrt{\frac{P_r}{\sum_{k=1}^{N_f} d_{sk}^2 |h_{1k}|^2 + N_f \sigma_r^2}} \quad (5)$$

which is obtained based on (4). Note that since $W$ is a unitary matrix, it does not appear in (5).

In a relay system, the direct path between the source and destination may or may not be available. When the direct path is available, we assume the receiver uses the maximum ratio combining to improve the reception and maximize the received SNR.

### B. Linear Processing and Achievable Rate

We consider the achievable rate in such AF relay OFDM system. Regardless whether the direct path is available, we can rewrite the end-to-end system equation in the following general form

$$y' = H_{eq}(W)s + n_{eq} \quad (6)$$

where $H_{eq}(W)$ is the equivalent channel matrix and is a function of the processing matrix $W$, and $n_{eq}$ the equivalent noise term. Then the system achievable rate is given by

$$C(W) = \frac{1}{2} \log \det(I + R_n^{-1}H_{eq}(W)H_{eq}^H(W)) \quad (7)$$

where $R_n = E[n_{eq}^H n_{eq}]$ is the covariance matrix of the equivalent noise term. The factor $1/2$ reflects the half-duplex operation.

For the conventional OFDM relaying without linear processing, $i.e.$, $W = I$, the relay simply forwards the amplified signal to the destination over the same subcarrier. However, such forwarding is in general not optimal in terms of maximizing the achievable rate. A special class of $W$ is the permutation matrix $\Pi$, for which linear processing reduces to subcarrier pairing. Such a scheme would uniquely couple a subcarrier over the first hop with a possibly different subcarrier over the second hop for signal relaying. This technique was studied recently in a few specific relay models $[1], [2], [6]$ and was shown to improve the rate. However, the optimality of such an approach remained unknown.

Our goal in this study is to find the optimal linear processing matrix $W^*$ to maximize the achievable rate

$$W^* = \arg \max_{W, WW^H = I} \frac{1}{2} \log \det(I + R_n^{-1}H_{eq}(W)H_{eq}^H(W)). \quad (8)$$

In the following, we will first focus on the class of permutation matrices for relaying with and without direct path. We will then discuss the optimal $W^*$ for these scenarios.

### III. Linear Processing under Permutation: Subcarrier Pairing

To solve (8), we first focus on the class of permutation matrices $W = \Pi$, and propose a general framework to find the optimal subcarrier pairing. This framework relies on the following result

**Lemma 1:** Let $P$ and $Q$ be two diagonal matrices. For

$$\max_{\Pi} \log \det(I + (\Pi^I P Q^I)^H (\Pi^I P Q^I)),$$  \quad (9)

among all possible permutation matrices $\Pi$, the optimal $\Pi^*$ is the one that maps the sorted absolute values of the diagonal entries of $P$ to the sorted absolute value of the diagonal entries of $Q$.

To show the above, we see that the objective function in (9) can essentially be rewritten as $\prod_{i=1}^{N_f} (1 + |p_i|^2 |q_i|^2)$, where $p_i$
and \( q_i \) are respectively the diagonal elements of \( P \) and \( Q \). Let \( \{ |p_i|^2 \} \) and \( \{ |q_i|^2 \} \) be the corresponding ordered sequences. Then it is not difficult to show that \( \prod_{i=1}^{n} (1 + |p_i|^2 |q_i|^2) \leq \prod_{i=1}^{n} (1 + |p_i|^2 |q_i|^2) \).

### A. Optimal Pairing for Relay without Direct Path

In this case, the destination is out of the transmission zone of the source. From (3), the equivalent channel matrix, noise vector, and its covariance matrix in (6) in this case are given by

\[
\begin{align*}
H_{eq}(\Pi) &= H_{2D1} \Pi H_{1D1}, \\
n_{eq} &= H_{2D1} \Pi n_r + n_d^{(2)}, \\
R_n &= \sigma_d^2 H_{2D1}^2 H_2^H + \sigma_d^2 I.
\end{align*}
\]

Note that, except \( \Pi \), all matrices in (10) are diagonal. Using the property of the determinant, \( \det(I + AB) = \det(I + BA) \), from (7), we can write the end-to-end achievable rate as

\[
C(\Pi) = \frac{1}{2} \log \det(I + H_{eq}(\Pi) R_n^{-1} H_{eq}(\Pi)).
\]

Inserting the expression of \( R_n \), (11) can be rewritten as

\[
C(\Pi) = \frac{1}{2} \log \det(I + (R_n^{-1} H_{eq}(\Pi) R_n^{-1} H_{eq}(\Pi))^H)
\]

(12)

The \( i \)-th diagonal entries \( p_i \) and \( q_i \) of \( P \) and \( Q \) are

\[
p_i = \frac{h_{2p} d_s}{\sqrt{\sigma^2 + \sigma^2 |b_{2p} d_s|^2}} \quad \text{and} \quad q_i = h_{1d} d_s,
\]

respectively. Following Lemma 1, the optimal \( \Pi \) is to pair the ordered sequences of \( \{ p_i \} \) and \( \{ q_i \} \), or equivalently, to pairing the following two ordered sequences

\[
\{ \text{SNR}_{sr,i} \}, \quad \left\{ \frac{\text{SNR}_{rd,i}}{1 + \sigma_d^2 \text{SNR}_{rd,i}} \right\}
\]

where \( \text{SNR}_{sr,i} = \frac{|b_{2p} d_s|^2 d_s^2}{\sigma^2} \) and \( \text{SNR}_{rd,i} = \frac{|h_{1d} d_s|^2}{\sigma_d^2} \) are the received SNR from source to relay, and from relay to destination, over the \( i \)-th subcarrier, respectively.

Note that, because \( f(x) = \frac{x}{\sqrt{\sigma^2 + \sigma^2 x^2}} \) is monotonically increasing for \( a > 0 \), ordering \( \{ |p_i| \} \) is equivalent to ordering \( |b_{2p} d_s|^2 \). Under the noise-free relay assumption, \( \sigma_d^2 = 0 \), the author of [2] proved that the sorted subcarrier pairing, which couples the relay subcarriers \( \{ d_{rj} h_{1d} \} \) with the transmit subcarriers \( \{ d_s h_{2p} \} \), is the optimal pairing scheme. Our result shows that this pairing scheme is optimal for noisy relaying as well.

### B. Optimal Pairing Relay with Full Diversity

We now consider the case when the direct path is available, \( i.e. \), relaying with full diversity. Through the maximum ratio combining technique, the observations from the direct path and the relay node can be coherently added. Define \( \Upsilon_2 \) as

\[
\Upsilon_2 = \frac{R_n^{-1} H_{2D1} \Upsilon_1 \frac{1}{\sigma_d^2} H_{1D1}}{1 + \frac{1}{\sigma_d^2} H_{1D1}}, \quad \text{and} \quad \Upsilon_0 = \frac{1}{\sigma_d^2} H_{0D1}, \quad \text{where} \quad R_n
\]

is given in (10). The achievable rate in this case is given by

\[
C(\Pi) = \frac{1}{2} \log \det \left( I + (\Upsilon_2 \Pi \Phi_1) \Upsilon_2 \Pi \Phi_2 \Upsilon_1 + \Upsilon_0 \Upsilon_0^H \right).
\]

To find the optimum \( \Pi \) to maximize \( C(\Pi) \), we again apply the result in Lemma 1 and all we need is to find the equivalent \( P \) and \( Q \) to express (13) as the form in (9). To do so, we re-arrange (13) as

\[
C(\Pi) = \frac{1}{2} \log \det \left( I + (\Upsilon_2 \Pi \Phi_1) (I + \Upsilon_0 \Upsilon_0^H)^{-1} (\Upsilon_2 \Pi \Phi_1)^H \right) + \frac{1}{2} \log \det(I + \Upsilon_0 \Upsilon_0^H) + \frac{1}{2} \log \det(I + (\Upsilon_2 \Pi \Phi_1) (I + \Upsilon_0 \Upsilon_0^H)^{-1} (\Upsilon_2 \Pi \Phi_1)^H).
\]

where the second term of (14) follows from the property \( \det(I + AB) = \det(I + BA) \). Since the first term of (14) is independent of \( \Pi \), we are only interested in the second term as a function of \( \Pi \), which can be written as

\[
C_2(\Pi) = \frac{1}{2} \log \det \left( I + \Upsilon_2 \Pi \Phi_1 (I + \Upsilon_0 \Upsilon_0^H)^{-1/2} (\Upsilon_2 \Pi \Phi_1)^H \right).
\]

Again using \( \det(I + AB) = \det(I + BA) \), we can set \( P = \Upsilon_2 \) and \( Q = \Upsilon_1 (I + \Upsilon_0 \Upsilon_0^H)^{-1/2} \), and (15) can then be transformed into the form of (9). Based on this, we obtain the optimal subcarrier pairing scheme for relaying with full diversity. Examining the diagonal entries of \( P \) and \( Q \), we conclude that the optimal pairing is essentially to order the following quantities over the input and output subcarriers of the relay, respectively,

\[
\left\{ \frac{\text{SNR}_{sr,i}}{1 + \sigma_d^2 \text{SNR}_{rd,i}} \right\}, \quad \left\{ \frac{\text{SNR}_{rd,i}}{1 + \sigma_d^2 \text{SNR}_{rd,i}} \right\}
\]

where \( \text{SNR}_{sr,i} = \frac{|b_{2p} d_s|^2 d_s^2}{\sigma^2} \), \( \text{SNR}_{rd,i} = \frac{|h_{1d} d_s|^2}{\sigma_d^2} \), and \( \text{SNR}_{rd,i} = \frac{|b_{2p} d_s|^2 d_s^2}{\sigma^2} \) are the received SNR from source to relay, from relay to destination, and from source to destination, over the \( i \)-th subcarrier, respectively. Again, sorting \( \left\{ \frac{\text{SNR}_{sr,i}}{1 + \sigma_d^2 \text{SNR}_{rd,i}} \right\} \) is equivalent to sorting \( \{ \text{SNR}_{rd,i} \} \). Thus, the optimal pairing for relaying with full diversity has a very clear strategy based on the SNR on each path: it is to match incoming and outgoing subcarriers of the relay, according to the SNR strength on the relay-destination subcarrier, and the relative ratio of SNR strengths on the source-relay and source-destination paths. Various sorting algorithms can be employed at the relay node with the computational complexity of \( O(N \log_2 N) \) [7].

### IV. The Optimal Linear Processing

In previous section, we have considered the special class of permutation for linear processing, and have obtained the optimal permutation \( \Pi^* \) in different relay scenarios using the
proposed framework. Back to the general problem in (8), the following result shows that the optimal permutation $\Pi^*$ is in fact the optimal linear processing matrix $W^*$.

**Theorem 1:** Let $P$ and $Q$ be two diagonal matrices. The solution to the following maximization

$$\max_{W:W^H W = I} \det(I + (PWQ)^H (PWQ))$$  \hspace{1cm} (17)

is $W^* = \Pi^*$, where $\Pi^*$ is the solution in Lemma I.

**Proof:** See Appendix A.

Combining Theorem I and the results in Section III, the following result immediately follows:

**Corollary 1:** The optimal $W^*$ for (8) in the relaying problem with or without direct path considered in Section III is $W^* = \Pi^*$.

In other words, the optimal linear processing is essentially subcarrier pairing with the optimal pairing strategy demonstrated in Section III.

V. SIMULATION RESULTS

In this section, we compare the performance of the optimal subcarrier pairing (SP) scheme with other non optimal schemes through Monte-Carlo simulations. We consider the scenario of relaying with direct path. We assume a 5MHz OFDM system with $N = 128$. The achievable rate is averaged over randomly generated multi-tap frequency selective channels (with fixed channel order $L = 11$). A source-destination pair is placed at a distance $d_{sd}$ apart, and the distances between source and relay, and relay and destination are set at $d_{sr}$ and $d_{rd}$, respectively. The pathloss exponent of 2 is assumed. We assume $P_s = P_r$, and $P_s$ is equally allocated across subcarriers, i.e., $d_{si} = \sqrt{\frac{P_s}{N}}$. We denote $\text{SNR} = \frac{P_s d_{sd}^{-2}}{N \sigma^2}$ as the average per subcarrier received SNR over the direct path.

We first consider $d_{sd} = 20m$, $d_{sr} = 6m$, and $d_{rd} = 16m$. Fig. 1 depicts the performance of the average rate per subcarrier vs. SNR, averaged over random channel realizations under the following four schemes: optimal SP scheme $\Pi^*$; no SP used; a random $W$ used; using $\Pi^*$ that is obtained assuming no direct path (i.e., (12)). The reason we consider the fourth scheme is that, in some cases, it may be easier for the relay to compute the optimal SP only based on the SNRs obtained on the two relay paths, although the receiver may use direct path signals for processing. We see that the optimal SP scheme outperforms all other schemes. When compared with the non SP scheme, it provides about 1dB gain. We expect to see a substantial gap between the optimal SP and other schemes in the highly asymmetric and frequency-selective channels.

Next, we study when subcarrier pairing is the most beneficial. In Fig 2, we show the effect of the relative SNR strengths, between relay and direct paths, on the achievable rate under different linear processing schemes. With fixed $d_{sd} = 20m$, we vary the relay position between source and destination. We plot the average rate vs. the relative distance $d_{sr}/d_{rd}$ for $\text{SNR} = 14\text{dB}$. This figure shows that the performance of all schemes coincide, when relay is very close to destination, i.e., $d_{sr}/d_{rd}$ is very high. In this case, SP does not provide much performance gain. On the other hand, when the relay moves closer to source, the gain of using the optimal SP over other schemes become more substantial.

VI. CONCLUSION

In this paper, AF relaying with unitary linear processing at the relay in an OFDM system is considered. We proposed a general framework to analyze how to select the unitary processing matrix to maximize the system achievable rate. We first considered the class of permutation matrices, and based on the proposed framework, we derived the corresponding optimal permutation matrix, or subcarrier pairing scheme, for relaying scenarios with and without direct path for diversity. We further show the optimality of the so obtained permutation matrix among all unitary linear processing matrices for system
achievable rate maximization. Simulation results also demonstrate the gain can be achieved through optimal subcarrier pairing as compared to the non-optimal linear processing and non-pairing cases.

APPENDIX A
PROOF OF THEOREM

We prove this by induction. We provide a brief description of the steps leading to the conclusion.

For $N = 2$, we need to show that the permutation matrix obtained from sorted subcarrier pairing is optimal. In order to do so, we first parameterize a $2 \times 2$ unitary matrix and then find its optimal entries. From the general parametrization of unitary matrices [8], the $2 \times 2$ unitary matrix $W$ can be represented by 4 independent parameters

$$W = \begin{pmatrix} e^{j\theta_1} & 0 \\ 0 & e^{j\theta_2} \end{pmatrix} \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & e^{j\phi_1} \end{pmatrix}.$$  (18)

Substituting (18) in (17) and taking derivative of the determinant with respect to four existing variables $\{\phi_i\}_{i=1}^3$ and $\theta$ of $W$, we realize that the optimal $\theta^* = 0$, when $p_1 > p_2$ and $q_1 > q_2$, or $p_1 > p_2$ and $q_1 > q_2$. The other variables are canceled out in the determinant. This observation shows that in the case of $N = 2$, $W^*$ is a permutation matrix.

Assume this is true for $N = n - 1$. For $N = n$, let $A_n = (P_n, W_n Q_n)_{\Pi}$ $(P_n, W_n Q_n)_{\Pi}$, where subscript $n$ denotes the matrix dimension. Since $(I_n + A_n)$ is a positive definite matrix, it has the following property [9]

$$\det(I_n + A_n) \leq (1 + a_{nn}) \det(I_{n-1} + A_{n-1})$$  (19)

where $a_{nn} = (A_n)_{nn}$. Re-arranging the expression of the determinant, we have

$$\det(I_n + A_n) = \det(I_n + W^H Q^H Q W P P^H)$$  (20)

Let $W = (w_1 \ldots w_n)$, $P = \text{diag}(p)$ with $p = [p_1, \ldots, p_n]^T$, and $Q = \text{diag}(q_1, \ldots, q_n)$. We have

$$A_n = \begin{pmatrix} (|p|^2 \cdot w_1)^H |q_1|^2 w_1 & \cdots & (|p|^2 \cdot w_1)^H |q_n|^2 w_n \\ \vdots & \ddots & \vdots \\ (|p|^2 \cdot w_n)^H |q_1|^2 w_1 & \cdots & (|p|^2 \cdot w_n)^H |q_n|^2 w_n \end{pmatrix} = \begin{pmatrix} \mathbf{A}_{n-1}^H & \mathbf{A}_{n-1} \\ \mathbf{A}_{n-1}^H & \mathbf{A}_{n-1} \end{pmatrix}$$  (21)

From (19), we have

$$\det(I_n + A_n) \leq (1 + \sum_i |p_i|^2 |w_i|^2 |q_i|^2) \det(I_{n-1} + A_{n-1})$$  (22)

Note that

$$\det(I_n + A_n) = \det(I_n + \Pi^H A_n \Pi)$$  (23)

for any permutation matrix $\Pi$. Let $|q_{(n)}|^2 = \max\{|q_i|^2\}$; then w.l.o.g., we let $|q_1|^2 = |q_{(n)}|^2$. Thus

$$\det(I_n + A_n) \leq (1 + \sum_i |p_i|^2 |w_i|^2 |q_{(n)}|^2) \det(I_{n-1} + A_{n-1})$$

$$\leq (1 + |p_{(n)}|^2 |q_{(n)}|^2) \det(I_{n-1} + A_{n-1})$$  (24)

where $|p_{(n)}|^2 = \max\{|p_i|^2\}$.

Since $W_{n-1}^* = \Pi_{n-1}$, by the Hardmark inequality, we have

$$\max_{W} \det(I_{n-1} + A_{n-1}(W)) = \prod_{i=1}^{n-1} (1 + |p_{(i)}|^2 |q_{(i)}|^2)$$  (25)

where both $p_{(i)}^2$ and $q_{(i)}^2$ are sorted in ascending order. Thus

$$\det(I_n + A_n) \leq (1 + |p_{(n)}|^2 |q_{(n)}|^2) \prod_{i=1}^{n-1} (1 + |p_{(i)}|^2 |q_{(i)}|^2)$$

with equality if and only if $w_n = [0, \ldots, 0, 1, 0, \ldots, 0]^T$ has entry 1 at the $(n)$th position. This implies that

$$W_n = [W_{n-1}^* w_n],$$  (26)

where $W_{n-1}^*$ is $W_{n-1}$ with an additional row of zeros inserted at the $(n)$th row. Hence, $W_n$ is a permutation matrix. Therefore $|I_n + A_n|$, i.e., the objective in (17) is maximized by $W_n^* = \Pi_{n}$.

REFERENCES

[1] A. Hottinen and T. Heikkinen, “Subchannel assignment in OFDM relay nodes,” in Annual Conf. on Information Sciences and Systems, Princeton, NJ, Mar. 2006, pp. 1314–1317.

[2] M. Herdin, “A chunk based OFDM amplify-and-forward relaying scheme for 4g mobile radio systems,” in Proc. IEEE Int. Conf. Communications (ICC), vol. 10, June 2006, pp. 4507–4512.

[3] Y. Li, W. Wang, J. Kong, W. Hong, X. Zhang, and M. Peng, “Power allocation and subcarrier pairing in OFDM-based relay networks,” in Proc. IEEE Int. Conf. Communications (ICC), Beijing, May 2008, pp. 2602–2606.

[4] I. Hammerstrom and A. Wittneben, “Power allocation schemes for amplify-and-forward MIMO-OFDM relay links,” IEEE Trans. Wireless Commun., vol. 6, no. 8, pp. 2798–2802, Aug. 2007.

[5] Y. Wang, P. Wang, F. Liu, X. Wang, and X. Yin, “Multilayer subchannel matching algorithms for two-hop OFDM relay networks,” in MICNET ‘09: Proc. of the 1st ACM workshop on Mob. internet through cellular networks. New York, NY, USA: ACM, 2009, pp. 37-42.

[6] Z. Shen, X. Wang, and H. Zhang, “Power allocation and subcarrier pairing for OFDM-based AF cooperative diversity systems,” in Vehicular Technology Conference, 2009. VTC Spring 2009. IEEE 69th, Barcelona, Apr. 2009, pp. 1–5.

[7] T. H. Cormen, C. E. Leiserson, R. L. Rivest, and C. Stein, Introduction to Algorithms, 2nd ed. The MIT Press, September 2001. [Online]. Available: http://www.amazon.com/Introduction-Algorithms-Second-Thomas-Cormen/dp/0262531968

[8] P. Dita, “Factorization of unitary matrix,” Jour. of Physics :Mathematical and General, vol. 36, pp. 2781 – 2789, Mar. 2003.

[9] F. Holland, “Another proof of Hadamard’s determinantal inequality,” Bulletin of the Irish Mathematical Society, no. 59, pp. 61–64, 2007.