New applications of the Egorychev method of coefficients of integral representation and calculation of combinatorial sums

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Abstract

Here we present the new applications of the Egorychev method of coefficients of integral representations and computation of combinatorial sums developed by the author at the end of 1970’s and its recent applications to the algebra and the theory of holomorphic functions in $\mathbb{C}^n$ and others.

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Introduction

At the end of the 1970’s G.P. Egorychev developed the method of coefficients, which was successfully applied to many combinatorial problems [11, 13, 9, 14] and [42]. Here we present the (see section 2.1) short description of the Egorychev method and its recent applications to several problems of enumeration and summation in various fields of mathematics: algebra, the theory of integral representations in $\mathbb{C}^n$ and the theory of approximation.

Enumerative combinatorial problems in algebra have been considered for a long time starting from the well-known estimates for the number of Sylow subgroups, the number of fixed-order subgroups in a finite p-group (G. Frobenius, Ph. Hall, etc.). For example, the ranks of the $n$-factors $M_q(n)$ for the lower central series of the free group $\Phi$ with $q$ generators

$$M_q(n) = \frac{1}{n} \sum \eta(d)q^{n/d}$$

were computed by M.Jr Hall ([30], Theorem 11.2.2).
In section 2.2 we obtained two simple formulas for the number of quadrics and symmetric forms of modules over local rings [34, 8].

In section 3 we solved the interesting enumeration problem of the number of $D$-ideals of ring $R_n(K, J)$ in lattices by means of the inclusion-conclusion method [7, 5, 21]. Thus it was possible to us with the help of the Egorychev method to calculate the difficult 6-multiple (!) combinatorial sum in the closed form.

In section 4 we solved several summation problems. In sections 4.1, 4.2 and section 4.3 we found the simple new computation and generalization of the several multiple combinatorial sums, which originally arose in the theory of holomorphic functions in $\mathbb{C}^n$ [40, 11, 6].

In section 5 we found the new short computation of multiple sum to the theory of cubature formulas [10, 11].

During the work on the article we added a number of corrections and the additions improving earlier known results.

1 The method of coefficients and its algebraic applications

Hans Rademacher [19] noted, that the applications of the method of generating functions is usually connected with the use of operations over the Laurent series and the Dirichlet series. Earlier G.P. Egorychev had developed the method of integral representations and calculation of combinatorial sums of various type (the inference rules and the completeness Lemma; see, [9, 11, 12, 13, 14], and [42]) connected with the use of the theory of analytic functions, the theory of multiple residues in $\mathbb{C}^n$ and the formal power Laurent series over $\mathbb{C}$.

1.1 The Egorychev method of coefficients

1.1.1 The computational scheme of the method of coefficients

The general scheme of the Egorychev method of integral representation and computation of sums can be followed up by the following steps [9].

1. Assignment of a table of integral representations of combinatorial numbers.

For example, the binomial coefficients $\binom{n}{k}$, $n, k = 0, 1, \ldots$,

\[
\binom{n}{k} = \text{res}_w (1 + w)^n w^{-k-1} = \frac{1}{2\pi i} \int_{0<|w|=\rho} (1 + w)^n w^{-k-1} dw, \quad (1.1)
\]
\[
\binom{n + k - 1}{k} = \text{res}_w (1 - w)^{-n} w^{-k-1} = \frac{1}{2\pi i} \int_{0<|w|=\rho<1} (1 - w)^{-n} \frac{w^{k+1} - 1 + \exp w}{w^{k+1}} dw.
\] (1.2)

The Stirling numbers \(S_2(n, k)\) of second kind, \(n, k = 0, 1, \ldots\) ([9], p. 273): \(S_2(0, 0) := 1\) and

\[
S_2(n, k) = \text{res}_w (-1 + \exp w)^n w^{-k-1} = \frac{1}{2\pi i} \int_{0<|w|=\rho} \frac{(-1 + \exp w)^n}{w^{k+1}} dw.
\] (1.3)

The Kronecker symbol \(\delta(n, k)\), \(n, k = 0, 1, \ldots\),

\[
\delta(n, k) = \text{res}_w w^{-n+k-1}.
\] (1.4)

2. Representation of the summand \(a_k\) of the original sum \(\sum_k a_k\) by a sum of product of combinatorial numbers.

3. Replacement of the combinatorial numbers by their integrals.

4. Reduction of products of integrals to multiple integral.

5. Interchange of the order of summation and integration. This gives the integral representation of original sum with the kernel represented by a series. The use of this transformation requires to deform the domain of integration in such a way as to obtain the series under the integral which converges uniformly on this domain saving the value of the integral.

6. Summation of the series under the integral sign. As a rule, this series turns out to be a geometric progression [10]. This gives the integral representation of the original sum with the kernel in closed form.

7. The computation of the resulting integral by means of tables of integrals, iterated integration, the theory of one and multidimensional residues, or new methods.

1.1.2 Laurent power series and the inference rules: definition and properties of \(\text{res}\) operator

Using the \(\text{res}\) concept and its properties the idea of integral representations can be extended on sums that allow computation with the help of formal Laurent power series of one and several variables over \(\mathbb{C}\). The \(\text{res}\) concept is directly connected with the classic concept of residue in the theory of analytic functions and which may be used with series of various types. This connection enabled us to express properties of \(\text{res}\) operator analogous to properties of residue in the theory of analytic functions. This in turn allows us to unify the scheme of the method of integral representations of sums independently.
of what kind of series – convergent or formal – is used (separately, or jointly) in the process of computation of a particular sum.

In this section we shall restrict ourselves explaining only one-dimensional case, although in further computations the res concept will also be used for multiple series. Besides, the one-dimensional case is of interest both in itself and in the computation of multiple integrals (res) in terms of repeated integrals.

Let \( L \) be the set of formal Laurent power series over \( \mathbb{C} \) containing only finitely many terms with negative degrees. The order of the monomial \( c_k w^k \) is \( k \). The order of the series \( C(w) = \sum_k c_k w^k \) from \( L \) is minimal order of monomials with nonzero coefficient.

Let \( L_k \) denote the set of series of order \( k \), \( L = \bigcup_{k=-\infty}^{\infty} L_k \).

Two series \( A(w) = \sum_k a_k w^k \) and \( B(w) = \sum_k b_k w^k \) from \( L \) are equal if \( a_k = b_k \) for all \( k \). We can introduce in \( L \) operations of addition, multiplication, substitution, inversion and differentiation [25, 15, 31].

The ring \( L \) is a field [18].

Let \( f(w), \psi(w) \in L_0 \). Below we shall use the following notations:

\[ h(w) = w f(w) \in L_1, \quad l(w) = \frac{w}{\psi(w)} \in L_1, \quad z'(w) = \frac{d}{dw} z(w), \quad \overline{h} = h(z) \in L_1 \]

- the inverse series of the series \( z = h(w) \in L_1 \).

For \( C(w) \in L \) define the formal residue as

\[ \text{res}_w C(w) = c_{-1}. \quad (1.5) \]

Let \( A(w) = \sum_k a_k w^k \) be the generating function for the sequence \( \{a_k\} \). Then

\[ a_k = \text{res}_w A(w) w^{-k-1}, \quad k = 0, 1, \ldots \quad (1.6) \]

For example, one of the possible representations of the binomial coefficient is

\[ \binom{n}{k} = \text{res}_w (1 + w)^n w^{-k-1}, \quad k = 0, 1, \ldots, n. \quad (1.7) \]

There are several properties, (inference rules) for the res operator which immediately follow from its definition and properties of operations in formal Laurent power series over \( \mathbb{C} \).

We list only a few of them which will be used in this article. Let \( A(w) = \sum_k a_k w^k \) and \( B(w) = \sum_k b_k w^k \) be the generating functions from \( L \).

**Rule 1** (Removal).

\[ \text{res}_w A(w) w^{-k-1} = \text{res}_w B(w) w^{-k-1} \quad \text{for all } k \text{ iff } A(w) = B(w). \quad (1.8) \]
Rule 2 (Linearity). For any \( \alpha, \beta \) from \( \mathbb{C} \)

\[
\alpha \text{res}_w A(w) w^{-k-1} + \beta \text{res}_w B(w) w^{-k-1} = \\
= \text{res}_w ((\alpha A(w) + \beta B(w)) w^{-k-1}) . \quad (1.9)
\]

By induction from (1.9) follows, that operators \( \sum \) and \( \text{res} \) are commutative.

Rule 3 (Substitution).

a) For \( w \in L_k (k \geq 1) \) and \( A(w) \) any element of \( L \),

b) for \( A(w) \) polynomial and \( w \) any element of \( L \) including a constant

\[
\sum_k w^k \text{res}_z (A(z) z^{-k-1}) = [A(z)]_{z = w} = A(w). \quad (1.10)
\]

Rule 4 (Inversion).

For \( f(w) \) from \( L_0 \)

\[
\sum_k z^k \text{res}_w (A(w) f(w) w^{-k-1}) = [A(w) / f(w) h'(w)]_{w = h(z)}, \quad (1.11)
\]

where \( z = h(w) = w f(w) \in L_1 \).

Rule 5 (Change of variables).

If \( f(w) \in L_0 \), then

\[
\text{res}_w (A(w) f(w)^k w^{-k-1}) = \text{res}_z \left( [A(w) / f(w) h'(w)]_{w = h(z)} z^{-k-1} \right), \quad (1.12)
\]

where \( z = h(w) = w f(w) \in L_1 \).

Rule 6 (Differentiation).

\[
k \text{res}_w A(w) w^{-k-1} = \text{res}_w A'(w) w^{-k}. \quad (1.13)
\]

1.1.3 Connection with the theory of analytic functions

If a formal power series \( A(w) \in L \) converges in a punctured neighborhood of zero, then the definition of \( \text{res}_w A(w) \) coincides with the usual definition of \( \text{res}_{w=0} A(w) \), used in the theory of analytic functions.

The formula (1.6) is an analog of the well known integral Cauchy formula

\[
a_k = \frac{1}{2\pi i} \int_{|w|=\rho} A(w) w^{-k-1} dw
\]

for the coefficients of the Taylor series in a punctured neighborhood of zero. The substitution rule (1.10) of the \( \text{res} \) operator is a direct analog of the
famous Cauchy theorem. Similarly, it is possible to introduce the definition of formal residue at the point of infinity, the logarithmic residue and the theorem of residues (all necessary concepts and results in the theory of residues in one and several complex variables [25, 15, 1, 9]).

Moreover, it is easy to see that each rule of the \texttt{res} operator can be simply proven by reduction to the known formula in the theory of residues for corresponding rational function.

### 1.2 Simple formulas for the number of quadrics and symmetric forms of modules over local rings

Simple formulas for the number of quadrics and symmetric forms of modules over local rings are found analytically. This simplification of known formulas of Levchuk and Starikova is achieved using the method of coefficients and leads to the number-theoretical identities of new type. The problem of algebraic interpretation of new formulas is posed.

#### 1.2.1 The algebraic statement of a problem

Let $R$ be a local ring with $2 \in R^*$. In [34] quadrics of a projective space $RP_{n-1}$ associated to a free module of rank $n$ over $R$ were enumerated. Every invertible symmetric matrix over $R$ is congruent to a diagonal matrix [37]. Observe that the Mobius, Minkowski and Laguerre’s classical geometries have natural representations by projective lines over algebras [24]. The fundamental theorem of projective geometry over a field was extended to projective spaces over rings [38].

Consider the case of a local ring $R$ with main maximal ideal of nilpotent step $s$, where $2 \in R^*$ and $|R^* : R^{*2}| = 2$. The following enumerative formulas of classes of projectively congruence quadric and symmetric forms of projective space $RP_{n-1}$ are given [33, 34]. Denote by $N(n, s)$ the number of all such classes. Then

\[
N(n, s) = \sum_{m=1}^{n} \sum_{q=1}^{\min(m, s)} \binom{s}{q} \left\{ \left( -1 + \frac{m}{2} \right) q - 1 \right\} + \sum_{(n_1, \ldots, n_q) \in \Omega_q(m)} \left[ \frac{1}{2} \prod_{j=1}^{q} (n_j + 1) \right],
\]

if $(1 + R^{*2}) \subset R^{*2}$,

\[
N(n, s) = \sum_{m=1}^{n} \sum_{q=1}^{\min(m, s)} \binom{s}{q} 2^{q-1} \left\{ \left( -1 + \frac{m}{2} \right) q - 1 \right\} + \left( m - 1 \right)
\]

if $(1 + R^{*2}) \not\subset R^{*2}$.

\[\text{(1.14)}\]
if \( R^* \cap (1 + R^2) \not\subseteq R^2 \).

(1.15)

Here \( \binom{p}{q}' \) is equal to \( \binom{p}{q} \) for nonnegative integers \( p \) and \( q \), and 0 otherwise; \( \Omega_q(m) \) denotes the set of all ordered partitions of the number \( m \) in \( q \) parts: \( n_1 + \ldots + n_q = m \).

These formulas can be simplified using the integral representation technique.

1.2.2 The analytic solution of a problem

The main result of this paper is formulated in the following theorems.

**Theorem 1.1.** If \( (1 + R^2) \subset R^2 \) then

\[
N(n, s) = T(n, 2s) + T\left(\left\lfloor \frac{n}{2} \right\rfloor, s\right),
\]

where

\[
T(n, p) = -\frac{1}{2} + \frac{1}{2} \binom{n + p}{p}.
\]

**Theorem 1.2.** If \( R^* \cap (1 + R^2) \not\subseteq R^2 \) then

\[
N(n, s) = S(n, s) + S\left(\left\lfloor \frac{n}{2} \right\rfloor, s\right),
\]

where

\[
S(n, s) = -\frac{1}{2} + \sum_{q=0}^{s} 2^{q-1} \binom{s}{q} \binom{n}{q},
\]

and for fixed \( s \) and \( n \to \infty \)

\[
S(n, s) \propto -\frac{1}{2} + 2^{s-1} \frac{n^s}{s!},
\]

(1.18)

\[
N(n, s) \propto -1 + \frac{1}{2} (2^s + 1) \frac{n^s}{s!}
\]

(1.19)

The following section containing the proof of the theorem 1.1 in particular solves difficult triple summation problem with the inner sum taken by partitions and summand involving integer part function. The second section gives a proof of theorem 1.2 including asymptotic solution of the problem. Final remarks can be found in Conclusion.

**Proof of the theorem 1.1**
Proof of formula (1.16) for the sum (1.14) splits into the sequence of intermediate simpler statements of similar type, formulated in lemmas 1.1–1.4. In each of them we obtain with the help of the method of coefficients and combinatorial techniques integral representations for intermediate combinatorial expressions, that define sums (1.14). Write $N(n, s)$ in (1.14) as

$$N(n, s) = S_1(n, s) + S_2(n, s),$$

(1.20)

where

$$S_1(n, s) = \sum_{m=1}^{n} \sum_{q=1}^{\min(m, s)} \left( \binom{s}{q} \left( -1 + \frac{m}{q} \right) \right)^r,$$

(1.21)

$$S_2(n, s) = \sum_{m=1}^{n} \sum_{q=1}^{\min(m, s)} \left( \binom{s}{q} \right) S_3(n, s, m, q),$$

(1.22)

and

$$S_3(n, s, m, q) = \frac{1}{2} \sum_{(n_1, \ldots, n_q) \in \Omega_q(m)} \left[ \frac{1}{2} \prod_{j=1}^{q} (n_j + 1) \right],$$

(1.23)

where $\Omega_q(m)$ is a set of ordered sequences of integers $n_1, \ldots, n_q$, such that $n_1 + \ldots + n_q = m$, $n_i \geq 1$ for $i = 1, \ldots, q$.

**Lemma 1.1** .

$$S_1(n, s) = -1 + \left( s + [\frac{n}{2}] \right).$$

(1.24)

**Proof.** We have

$$S_1(n, s) = \sum_{k=1}^{\left[ \frac{n}{2} \right]} \sum_{q=0}^{\min(2k, s)} \left( \binom{s}{q} \right) \left( -1 + k \right) \left( \frac{1}{q} \right) = \sum_{k=0}^{\left[ \frac{n}{2} \right] - 1} \sum_{q=0}^{\min(2k+2, s)} \left( \binom{s}{q} \right) \left( k \right) \left( -1 + k \right) \left( \frac{1}{q} \right) =$$

(using (1.17) twice)

$$= \sum_{k=0}^{\left[ \frac{n}{2} \right] - 1} \sum_{q=0}^{\min(2k+2, s)} \text{res}_{x,y}\left\{ (1 + x)^s (1 + y)^k / x^{q+1} y^{k-q+2} \right\} =$$

$$= \sum_{k=0}^{\left[ \frac{n}{2} \right] - 1} \sum_{q=0}^{\min(2k+2, s)} \text{res}_{y}\left\{ (1 + y)^k / y^{k+2} \left[ \sum_{q=0}^{\infty} y^q \text{res}_x(1 + x)^s / x^{q+1} \right] \right\}.$$
In last expression using substitution rule within square brackets with the change \( x = y \) we have

\[
S_1(n, s) = \sum_{k=0}^{\left\lfloor \frac{n}{2} \right\rfloor - 1} \text{res}_y \left\{ \frac{(1 + y)^k}{y^{k+2}} \cdot [(1 + x)^s] \right\}_{x=y} = \\
= \sum_{k=0}^{\left\lfloor \frac{n}{2} \right\rfloor - 1} \text{res}_y \{(1 + y)^s(1 + y)^k/y^{k+2}\} = \\
= \text{res}_y \left\{ \sum_{k=0}^{\left\lfloor \frac{n}{2} \right\rfloor - 1} (1 + y)^s(1 + y)^k/y^{k+2}\right\} = \\
(\text{formula of the geometric progression in } k)
\]

Now we will obtain an integral representation for sum \( S_3(n, s, m, q) \) starting with its summand \( \left[ 1/2 \prod_{j=1}^{q} (n_j + 1) \right] \).

Note that \( \Omega_q(m) = \Omega'_q(m) \cup \Omega''_q(m) \), where

\[
\Omega'_q(m) = \{(n_1, \ldots, n_q) : n_1 + \ldots + n_q = m, n_i \geq 1, \text{ for } i = 1, \ldots, q, \text{ and } \exists j \text{ such that } n_j \text{ is odd}\},
\]

\[
\Omega''_q(m) = \{(n_1, \ldots, n_q) : n_1 + \ldots + n_q = m, n_i \geq 1 \text{ for } i = 1, \ldots, q, \text{ and } n_j \text{ is even for all } j \in \mathbb{N}\}.
\]

Then

\[
\left[ \frac{1}{2} \prod_{j=1}^{q} (n_j + 1) \right] = \begin{cases} 
\frac{1}{2} \prod_{j=1}^{q} (n_j + 1), & \text{if } n \in \Omega'_q(m), \\
\frac{1}{2}(-1 + \prod_{j=1}^{q} (n_j + 1)), & \text{if } n \in \Omega''_q(m). 
\end{cases} 
\] (1.25)

**Lemma 1.2**. The following equalities are valid

\[
\sum_{n \in \Omega_q(m)} \left[ 1/2 \prod_{j=1}^{q} (n_j + 1) \right] = \frac{1}{2} \sum_{n \in \Omega_q(m)} \prod_{j=1}^{q} (n_j + 1) - \frac{1}{2} |\Omega''_q(m)|, 
\] (1.26)

\[
\sum_{n \in \Omega_q(m)} \prod_{j=1}^{q} (n_j + 1) = \text{res}_x \left\{ f^q(x)/x^{m+q+1} \right\} = \text{res}_x \left\{ \frac{(-1 + (1 - x)^{-2})^q}{x^{m+1}} \right\}, 
\] (1.27)
\[ |\Omega_q'(m)| = \text{res}_x \{ (1 - x^2)^{-q} / x^{m-2q+1} \} = \begin{cases} \left( \frac{m-1}{q-1} \right), & \text{if } m \text{ is even}, \\ 0, & \text{if } m \text{ is odd}. \end{cases} \quad (1.28) \]

**Proof.** Using (1.25) we have

\[
\sum_{n \in \Omega_q(x)} \frac{1}{2} \prod_{j=1}^{q} (n_j + 1) = \sum_{n \in \Omega_q'(x)} \frac{1}{2} \prod_{j=1}^{q} (n_j + 1) + \sum_{n \in \Omega_q''(x)} \frac{1}{2} \prod_{j=1}^{q} (n_j + 1) =
\]

\[
= \sum_{n \in \Omega_q'(x)} \frac{1}{2} \prod_{j=1}^{q} (n_j + 1) + \sum_{n \in \Omega_q''(x)} \frac{1}{2} \left( -1 + \prod_{j=1}^{q} (n_j + 1) \right) =
\]

\[
= \frac{1}{2} \sum_{n \in \Omega_q'(x)} \prod_{j=1}^{q} (n_j + 1) - \frac{1}{2} |\Omega_q''(x)|,
\]

which gives (1.26).

Proof of (1.27)–(1.28) uses combinatorial properties of generating functions of partitions of the set \( \Omega_q(x) \). Denote \( c_n = (n + 1) \geq 2 \). Since the generating function for sequence \( \{c_n\} \) is

\[
f(x) = \sum_{n=1}^{\infty} (n + 1)x^{n+1} = -x + \sum_{n=0}^{\infty} (n + 1)x^{n+1} =
\]

\[
= -x + x[(1 - x)]' = -x + x(1 - x)^{-2} = x \left( -1 + (1 - x)^{-2} \right),
\]

then we get

\[
\sum_{n \in \Omega_q(x)} \prod_{j=1}^{q} (n_j + 1) = \text{res}_x \{ f^q(x) / x^{m+q+1} \} = \text{res}_x \{ (-1 + (1 - x)^{-2})^q / x^{m+1} \}.
\]

It is easy to see that

\[ |\Omega_q''(x)| = \text{res}_x \{ R^q(x) / x^{m+q+1} \}, \]

where \( R^q(x) = x^3 + x^5 + \ldots = x^3(1 - x^2)^{-1}. \) Thus

\[ |\Omega_q'(x)| = \text{res}_x \{ x^3(1 - x^2)^{-1}q / x^{m+q+1} \} = \text{res}_x \{ ((1 - x^2)^{-q} / x^{m-2q+1} \},
\]

which is equivalent to (1.28). \( \blacksquare \)
Lemma 1.3. Let

\[ S_4(n, s) = \frac{1}{2} \sum_{m=1}^{n} \min(m, s) \frac{s}{q} \sum_{q=1}^{\min(m, s)} \prod_{j=1}^{q} (n_j + 1). \quad (1.29) \]

Then

\[ S_4(n, s) = -\frac{1}{2} + \frac{1}{2} \binom{2s + n}{n}. \quad (1.30) \]

Proof.

Since \( \left( \begin{array}{c} s \\ q \end{array} \right) = \text{res}_z \left\{ (1 + z)^s / z^{q+1} \right\} \)

from (1.29) and (1.27) we have

\[ S_4(n, s) = \frac{1}{2} \sum_{m=1}^{n} \min(m, s) \text{res}_z \left\{ (1 + (1 - x)^{-2})^s / x^{m+1} \right\} = \]

\[ = \frac{1}{2} \sum_{m=1}^{n} \text{res}_x \left\{ (1 + (1 - x)^2)^s / x^{m+1} \right\} = \]

\[ = \frac{1}{2} \text{res}_x \left\{ (1 - x)^{-2s} / x^{m+1} \right\} = \frac{1}{2} \text{res}_x \left\{ \sum_{m=1}^{n} (1 - x)^{-2s} / x^{m+1} \right\} = \]

(summation by the index \( m \))

\[ = \frac{1}{2} \text{res}_x \left\{ (1 - x)^{-2s} (1 - x^{-n} / (1 - x^{-1})) \right\} = \]

\[ = -\frac{1}{2} \text{res}_x \left\{ (1 - x)^{-2s-1} x^{-1} (1 - x^{-n}) \right\} = -\frac{1}{2} + \frac{1}{2} \left( \frac{2s + n}{n} \right). \]

Lemma 1.4. Let

\[ S_5(n, s) := -\frac{1}{2} \sum_{m=1}^{n} \min(m, s) \frac{s}{q} \Omega'_q(m). \quad (1.31) \]

Then

\[ S_5(n, s) = \frac{1}{2} - \frac{1}{2} \left( \frac{2s + [n/2]}{2s} \right). \quad (1.32) \]
Proof. Since
\[
\binom{s}{q} = \text{res}_z \{(1 + z)^s / z^{q+1}\}
\]
from (1.31) and (1.28) we have
\[
S_5(n, s) = -\frac{1}{2} \sum_{m=1}^{n} \sum_{q=0}^{\min(m, s)} \text{res}_z \{(1 + z)^s / z^{q+1}\} \text{res}_x \{(1 - x^2)^{-q} / x^{m-2q+1}\}
\]
(the substitution of the rule of the index q)
\[
= -\frac{1}{2} \sum_{m=1}^{n} \text{res}_x \{(1 + (1 - x^2)^{-1} x^2)^s / x^{m+1}\} =
\]
\[
= -\frac{1}{2} \sum_{m=1}^{n} \text{res}_x \{(1 - x^2)^{-s} / x^{m+1}\} =
\]
\[
= -\frac{1}{2} \text{res}_x \{\sum_{m=1}^{n} (1 - x^2)^{-s} / x^{m+1}\} = (a \text{ summation by the index } m) =
\]
\[
= -\frac{1}{2} \text{res}_x \{(1 - x^2)^{-s} x^{-2} (1 - x^{-n}) / (1 - x^{-1})\} =
\]
\[
= -\frac{1}{2} \text{res}_x \{(1 - x^2)^{-s-1} x^{-n} (1 + x) x^{-1}\} =
\]
\[
= \frac{1}{2} - \frac{1}{2} \text{res}_x (1 - x^2)^{-s-1} x^{-n-1} - \frac{1}{2} \text{res}_x (1 - x^2)^{-s-1} x^{-n}.
\]
Since
\[
-\frac{1}{2} \text{res}_x (1 - x^2)^{-s-1} x^{-n-1} = \begin{cases} -\frac{1}{2} \binom{s + \lfloor \frac{n}{2} \rfloor}{s}, & \text{if } n \text{ is even;} \\ 0, & \text{if } n \text{ is odd}, \end{cases}
\]
\[
-\frac{1}{2} \text{res}_x (1 - x^2)^{-2s-1} x^{-n} = \begin{cases} -\frac{1}{2} \binom{s + \lfloor \frac{n-1}{2} \rfloor}{s}, & \text{if } n \text{ is odd;} \\ 0, & \text{if } n \text{ is even}, \end{cases}
\]
we have
\[
S_5(n, s) = \frac{1}{2} - \frac{1}{2} \binom{s + \lfloor \frac{n}{2} \rfloor}{s}.
\]

Proof of the theorem 1.2

Analogously to the theorem 1.1 formula (1.17) for the sum (1.15) is immediately corollary of formulas (1.36), (1.37) and (1.34).
Formula (1.18) for the sum $S(n,s)$ follows from the classical tauberian theorem (see, for example, [29]): if the series $A(z) = a_0 + \sum_{k=1}^{\infty} a_k z^k$ converges for $|z| < 1$, the limit $\lim_{z \to 1} (1 - z)^{s+1} A(z) = B$ exists for some $s \geq 0$, and $k(a_k - a_{k-1}) > -c(k = 1, 2, \ldots)$ for some positive constant $c$, then $\lim_{k \to \infty} a_k k^{-s} = B/\Gamma(s + 1)$.

Formula (1.19) follows from (1.18) and (1.17).

Write $N(n,s)$ in (1.15) as
\[
N(n,s) = S_6(n,s) + S_7(n,s),
\]
(1.33)

where
\[
S_6(n,s) = \sum_{m=1}^{n} \sum_{q=1}^{\min(m,s)} \binom{s}{q} 2^{q-1} \binom{m-1}{q-1},
\]
and
\[
S_7(n,s) = \sum_{m=1}^{n} \sum_{q=1}^{\min(m,s)} \binom{s}{q} 2^{q-1} \binom{-1 + m/2}{q-1}.'
\]

Lemma 1.5. Let the sum
\[
S(n,s) = -\frac{1}{2} + \sum_{q=0}^{s} 2^{q-1} \binom{s}{q} \binom{n}{q}.
\]
(1.34)

Then the sum $S_6(n,s)$ has the following integral representations
\[
S_6(n,s) = -\frac{1}{2} + \frac{1}{2} \text{res}_y(1 + 2y)^s (1 + y)^n y^{-n-1} =
\]
\[
= -\frac{1}{2} + \frac{1}{2} \text{res}_z (1 + z)^s (1 - z)^{-s-1} z^{-n-1},
\]
(1.35)

and
\[
S_6(n,s) = S(n,s),
\]
(1.36)

\[
S_7(n,s) = S\left(\left\lfloor \frac{n}{2} \right\rfloor, s\right).
\]
(1.37)

Proof. We have
\[
S_6(n,s) = \sum_{m=1}^{n} \sum_{q=1}^{\min(m,s)} \binom{s}{q} 2^{q-1} \binom{m-1}{q-1} = \sum_{m=1}^{n} \sum_{q=0}^{\min(m,s)} \binom{s}{q} 2^{q-1} \binom{m-1}{m-q} =
\]

(using (1.17) twice)

14
\[ \sum_{m=1}^{\min(m, s)} \sum_{q=0}^{n} 2^{q-1} \text{res}_{x,y} \{(1 + x)^{s}(1 + y)^{m-1} / x^{q+1} y^{m-q+1} \} = \]

(the summation with respect to the index q, substitution rule and the change \( x = 2y \in L_1 \))

\[ = \frac{1}{2} \sum_{m=1}^{n} \text{res}_{y} \{(1 + 2y)^{s}(1 + y)^{m-1} / y^{m+1} \} = \]

\[ = \frac{1}{2} \text{res}_{y} \sum_{m=1}^{n} (1 + 2y)^{s}(1 + y)^{m-1} / y^{m+1} \]  

(the summation by the index m, formula of the geometric progression)

\[ = \frac{1}{2} \text{res}_{y} \{(1 + 2y)^{s}y^{-2}[1 - (1 + y)^{n}/y^{n}]/[1 - (1 + y)/y] \} = \]

\[ = -\frac{1}{2} \text{res}_{y}(1 + 2y)^{s}y^{-1} + 1/2 \text{res}_{y}(1 + 2y)^{s}(1 + y)^{n}y^{-n-1} = \]

\[ = -\frac{1}{2} + \frac{1}{2} \text{res}_{y}(1 + 2y)^{s}(1 + y)^{n}y^{-n-1}. \]

Further

\[ S_{6}(n, s) = -\frac{1}{2} + \frac{1}{2} \text{res}_{y}(1 + 2y)^{s}(1 + y)^{n}y^{-n-1} = \]

\[ = -\frac{1}{2} + \sum_{q=0}^{s} 2^{q-1} \binom{s}{q} \text{res}_{y}((1 + y)^{n}/y^{n-s+1}) = \]

\[ = -\frac{1}{2} + \sum_{q=0}^{s} 2^{q-1} \binom{s}{q} \binom{n}{q}, \]

and

\[ S_{6}(n, s) = -\frac{1}{2} + \frac{1}{2} \text{res}_{y}(1 + 2y)^{s}(1 + y)^{n}y^{-n-1}. \]

The second formula in (1.35) is obtained from the first one with the change of variables \( z = y/(1 + y) \) by Rule 3.

The proof of (1.34) is analogous to the proof of (1.36).

Corollary 1.1 . In notions of (1.13) the following combinatorial identity is valid

\[ \sum_{k=1}^{\min(2k, s)} \sum_{q=1}^{\binom{s}{q}} 2^{q-1} \binom{s}{q} (-1 + k) \binom{m - 1}{q - 1} + \sum_{m=1}^{n} \sum_{q=1}^{\min(m, s)} 2^{q-1} \binom{s}{q} \binom{m - 1}{q - 1} = \]
\[-1 + \sum_{q=0}^{s} 2^{q-1} \binom{s}{q} \left( \binom{n}{q} + \binom{\left\lfloor \frac{n}{2} \right\rfloor}{q} \right).\]

**Conclusion**

A simplification of formulas usually brings new information on the structure of objects of enumeration.

For example, simplification of known formulas for $R^3_q(n)$ from Sokolov (1969) allowed to understand better the structure of the enumerable regular words (commutators) on known Shirshov bases of a free Lie algebra. This allows to solve the Kargapolov problem of computing the ranks $R^k_q(n)$ of the factors for the lower central series of a free solvable group of step $k$ with $q$ generators for arbitrary $k$ (Egorychev, 1972). This, in turn, allowed to solve analogous problem for a free polynilpotent group (Gorchakov and Egorychev, 1972) and for free groups in varieties (Egorychev, 1977). Another answer to the same problem of Kargapolov was suggested in Petrogradsky in 1999. The reader can find more detailed statement of this question and corresponding references to the literature in ([9], pp. 129–132 and 207–222).

Simplicity found in formulae of Theorems 1 and 2 poses in [8] the following problem: Give an independent algebraic proof and interpretation of formulas (1.16) and (1.17) for the number of quadrics on the projective space $\mathbb{RP}_{n-1}$, $n > 2$, over local ring $R$ with the maximal ideal nilpotent of a class $s$.

The full answer to this problem was given O.V. Starikova and A.V. Sivistunova A.V. in the article (2011, [35]), in which the by algebraic proof (interpretation) of identities (1.16) and (1.17) is given in the theory of local rings of the specified type. This, in turn O.V. Starikova tried understood better the structure of studied objects and to solve more difficult enumeration problems for various classes of projective equivalent quadrics over local rings (2013, [36]).

2 The enumeration of $D$-invariant ideals of ring $R_n(K, J)$ on lattices

2.1 The combinatorial statement of a problem

In the description of ring ideals $R_n(K, J)$ uses the definition of $T$-border $A$, $A = A(T; L, L')$ ([23], Definition 2.1). The $T$-border depends on the $J$-
submodule $T$ in $K$ and the pair of sets of matrix elements:

\[ L = \{(i_1, j_1), (i_2, j_2), \ldots, (i_r, j_r)\}, \quad r \geq 1, \]

\[ 1 \leq j_1 < j_2 < \ldots < j_r \leq n, \quad 1 \leq i_1 < i_2 < \ldots < i_r \leq n; \] \hfill (2.1)

\[ L' = \{(1, j_r), (k_1, m_1), (k_2, m_2), \ldots, (k_q, m_q), (i_1, n)\}, \quad q \geq 0, \]

\[ j_r \leq m_1 < m_2 < \ldots < m_q \leq n, \quad 1 \leq k_1 < k_2 < \ldots < k_q \leq i_1. \] \hfill (2.2)

We define the pair $(L, L')$ as "the set of degree angles $n$".

Let $L(i, j), i, j \in \{1, n\}$, be the set of all sequences of type $L$ of arbitrary length $r, r \geq 1$, where $i_1 = i, j_r = j$, and $L'(i, j), i, j \in \{1, n\}$, is the set of all sequences of $L'$ type (including the empty set) of any length $q, q \geq 0$. It determined by the initial conditions $i_1 = i, j_r = j$.

In the article [23] it is shown

**Theorem 2.1** Let $n \in \mathbb{N}$. Then the number $N(R)$ of all ideals of the ring $R_n(K, J)$ is equal to

\[ \Omega(n) = |L \times L'| = \sum_{i=1}^{n} \sum_{j=1}^{n} |L(i, j)| \cdot |L'(i, j)|. \]

Then we prove

**Theorem 2.1** The number of all ideals of the ring $R_n(K, J)$ is equal to

\[ \Omega(n) = (2n-1) \left(\frac{2n-2}{n-1}\right). \] \hfill (2.3)

Then $\Omega(n)$ is the number of sets of angles $(L, L')$ of degree $n$, and $\Omega^+(n)$ is the number of all sets angles $(L, L')$ of degree $n$ with $i > j$ for all $(i, j) \in L$.

Note that an each sequence $\{(i_1, j_1), (i_2, j_2), \ldots, (i_r, j_r)\}$ of $L$ type (analogously of $L'$ type) of length $r, r \geq 1$, is one-to-one correspondence of the increasing path $(i_1, j_1), (i_2, j_2), \ldots, (i_r, j_r)$ with the $(r-1)$-diagonal steps on a rectangular lattice from the point $(i_1, j_1)$ to the point $(i_r, j_r)$, see Fig. 1.

According to [23, Theorem 2.2], for a strongly maximal ideal $J$ of $K$ any ideal of the ring $R_n(K, J)(n \geq 2)$ is generated by unique $T$-boundary $A$. Regard the $n \times n$ matrix as a square array of $n^2$ points $(i, j)$ (matrix positions). Associating "staircases" with $L$ and $L'$ we can compare with the ideal generated by the $T$-boundary $A$ the matrix.
We now distinguish $\mathcal{D}$-invariant ideals of the ring $R_n(K, J)$.

For any additive subgroup $F$ of $K$ we denote by $N_{ij}(F)$ (resp. by $Q_{ij}(F)$), the additive group of $R_n(K, J)$ generated by sets $Fe_{km}$ for all $(k, m) \geq (i, j)$ (resp. $(k, m) > (i, j)$) where $(i, j) < (k, m)$, if $i \leq k, m \leq j$ and $(i, j) \neq (k, m)$.

2.1.1 The enumeration of a number of ways in a rectangular lattice and the method of inclusion-conclusion

Theorem 2.2 For numbers $\Omega^+(n)$, $n = 3, 4, \ldots$ the following combinatorial formula is valid:

$$\Omega^+(n) = \sum_{i=2}^{n} \sum_{j=1}^{i-1} \left( \sum_{r \geq 1} \left( \sum_{k_1=0}^{r-1} \sum_{k_2=0}^{r-k_1-2} \sum_{s=r-k_2-1}^{2r-k_1-k_2-2} \binom{i-1}{k_1} \binom{j-i}{s} \binom{n-j}{k_2} \times \binom{s}{2r-s-k_1-k_2-2} \cdot \frac{(k_1-k_2+1)}{(s-r+k_1+2)} + \frac{2r-k_1-k_2-2}{s-r+k_2+2} \right) \right) \cdot \left( \binom{2s-2r+k_1+k_2+2}{i-1+n-j} \times \binom{2s-2r+k_1+k_2+2}{i-1+n-j} + \right)$$

Figure 1.
By the formulas (2.1), (2.2) and (2.6) we have
\[ r, r \geq 1 \]
\[ \text{from all } t \]
\[ \text{Let } \sum_{k_1=0}^{r-1} \sum_{k_2=0}^{2r-k_1-k_2-2} \binom{i-1}{k_1} \binom{j-i}{s} \binom{n-j}{k_2} \left( \frac{2r-s-k_1-k_2-2}{s} \right) \times \]
\[ \left( \frac{k_2-k_1+1}{s-r+k_2+2} \right) \cdot \left( \frac{2s-2r+k_1+k_2+2}{s-r+k_1+1} \right) \cdot \left( \frac{i-1+n-j}{i-1} \right) \] (2.4)

For the proof of the theorem we give a number of definitions, lemmas and theorems.

Let \( L^{(n)}(i,j) \) be the set of all sequences \( \{(i_1,j_1), (i_2,j_2), \ldots, (i_r,j_r)\} \), \( r \geq 1 \), of type \( (2.1) \) which \( i_1 = i, j_r = j \), and \( j_t \leq i_t \) from all \( t = 0,1,\ldots,r \).

Let \( L^{(n)}_r(i,j) \) be the set of all sequences \( \{(i_1,j_1), (i_2,j_2), \ldots, (i_r,j_r)\} \), \( r \geq 1 \), of type \( (2.1) \) by fixed \( r, r \geq 1 \).

Let \( L^{(n)}(i,j) \) be the set of all sequences \( \{(i_1,j_1), (i_2,j_2), \ldots, (i_r,j_r)\} \) of type \( (2.1) \) by fixed \( r, r \geq 1 \), of type \( (2.2) \) which \( i_1 = i, j_r = j \), and \( j_t \leq i_t \) from all \( t = 0,1,\ldots,r \). It is easy to note that
\[ L^{(n)}(i,j) = \bigcup_{r \geq 1} L^{(n-1)}_r(i-1,r) \] (2.6)

By the formulas (2.1), (2.2) and (2.6) we have
\[ \Omega^+(n) = \sum_{i,j=1}^{n} |L^{(n)}(i,j)| \cdot |L'(i,j)| = \]
\[ = \sum_{i=2}^{n} \sum_{j=1}^{n-1} |L^{(n-1)}(i-1,j)| \cdot \binom{i-1+n-j}{i-1} = \]
\[ = \sum_{i=2}^{n} \sum_{j=1}^{i-1} |L^{(n-1)}(i-1,j)| \cdot \binom{i-1+n-j}{i-1} + \]
\[ + \sum_{i=2}^{n} \sum_{j=i}^{n-1} |L^{(n-1)}(i-1,j)| \cdot \binom{i-1+n-j}{i-1} \] (2.7)
Let $A = \{j_1, j_2, \ldots, j_{r-1}, j_r\}$, $B = \{i, i_2, \ldots, i_r\}$, be two sequences of positive integers of type (1) which $1 \leq i_1 = i - 1$, $j_r = j \leq n - 1$ and $j_t \leq i_t$ from all $t = 0, 1, \ldots, r$.

Let us consider two cases: $n - 1 \geq i - 1 \geq j \geq 0$ or $0 \leq i - 1 < j \leq n - 1$.

1 case: $n - 1 \geq i - 1 \geq j \geq 0$.

**Lemma 2.1** If $i - 1 \geq j$, $i = 1, 2, \ldots, n$, then the following formula is valid

$$|L^{n-1}(i-1,j)| = \binom{n-i+j-1}{j-1}. \quad (2.8)$$

**Proof.**

As $i - 1 \geq j$ then $A \cap B = \emptyset$ and we have

$$|L^{n-1}(i-1,j)| = \binom{n-i}{r-1} \binom{j-1}{r-1}.$$

By the formula for (2.6) we get

$$|L^{n-1}(i-1,j)| = \sum_{r \geq 1} |L^{n-1}_r(i-1,j)| = \sum_{r \geq 1} \binom{n-i}{r-1} \binom{j-1}{r-1} =$$

$$= \binom{n-i+j-1}{j-1}.$$

2 case: $0 \leq i - 1 < j \leq n - 1$.

We use the known combinatorial result which in terms of rectangular paths in a rectangular lattice may be formulated as

**Lemma 2.2** (Feller, Ch. 3, paragraph 1, Theorem 1). The number of all increasing rectangular paths in a rectangular lattice (without diagonal steps) from the origin to the point $(X,Y)$, $X \geq Y$, is equal to

$$\Phi(X,Y) = \frac{X-Y+1}{X+1} \binom{X+Y}{Y}. \quad (2.9)$$

**Lemma 2.3** If $i - 1 < j$, $i = 1, 2, \ldots, n$, than the following formula is valid

$$|L^{n-1}_r(i-1,j)| = \sum_{s \geq 1} \left\{ \sum_{k_1=0}^{r-1} \sum_{k_2=0}^{r-k_1} \sum_{s=r-k_2-1}^{2r-k_1-k_2-2} \binom{i-1}{k_1} \binom{j-i}{s} \binom{n-j}{k_2} \times \right.$$

$$\times \left( \binom{s}{2r-s-k_1-k_2-2} \frac{k_1-k_2+1}{s-r+k_1+2} \binom{2s-2r+k_1+k_2+2}{s-r+k_2+1} \right)$$

$$+ \left. \cdots \right\}.$$
such that

where as usual the binomial coefficients \( \binom{a}{b} \) are equal zero, if the integers \( a, b \) such that \( a, b \geq a \geq 0 \) or \( b < 0 \).

**Proof.**

Indeed, let \( 0 \leq i - 1 < j \leq n - 1 \) and \( A = \{i, i_2, \ldots, i_r\} \) and \( B = \{i, i_2, \ldots, i_r\} \) be two sequences of positive integers of type (1) which \( 1 \leq i_1 = i - 1, j_r = j \leq n - 1 \) and \( j_t \leq i_t, \) from all \( t = 0, 1, \ldots, r. \)

Let \( I_1 = \{0, 1, \ldots, i - 2\}, I_2 = \{i - 1\}, I_3 = \{i, i + 1, \ldots, j - 1\}, I_4 = \{4\}, \)
\( I_5 = \{j + 1, j + 2, \ldots, n\}, |A \cap I_1| = k_1, 0 \leq k_1 \leq r - 1, |B \cap I_5| = k_2, \)
\( 0 \leq k_2 \leq r - 1. \)

Let \( |A \cap I_3| = r - k_1 - 1, |B \cap I_3| = r - k_2 - 1, |(A \cup B) \cap I_3| = s, \)
\( \max(r - k_1 - 1, r - k_2 - 1) \leq s \leq 2r - k_1 - k_2 - 2. \)

Then \( A \cap B \subseteq I_3, \) and \( |A \cap B| = |A \cap I_3| + |B \cap I_3| - |(A \cup B) \cap I_3| = 2r - s - k_1 - k_2 - 2. \) The number of different points of the sets \( A \cap I_3 \) and \( B \cap I_3 \) is equal to \( |A \cap I_3| + |B \cap I_3| - 2|A \cap B| = (r - k_1 - 1) + (r - k_2 - 1) - 2(2r - s - k_1 - k_2 - 2) = 2s - 2r + k_1 + k_2 + 2 \) including \( s - r + k_2 + 1 \) points from the set \( A \cap I_3 \) and \( s - r + k_1 + 1 \) points from the set \( B \cap I_3. \) It is clear that the number of possible choices of \( A \) and \( B \) from fixed integers \( k_1, k_2 \) and \( s, \) where \( k_1 = |A \cap I_1|, s = |(A \cup B) \cap I_3|, k_2 = |B \cap I_5|, 0 \leq k_1 \leq r - 1, \)
\( 0 \leq k_2 \leq r - 1, \max(r - k_1 - 1, r - k_2 - 1)s2r - k_1 - k_2 - 2, \) is equal to

\[
N_{r,n}(k_1, k_2, s) = \binom{i - 1}{k_1} \binom{j - i}{s} \binom{n - j}{k_2} \binom{s}{2r - s - k_1 - k_2 - 2} \times
\]
\[\times \binom{n - j}{k_2} \binom{s}{2r - s - k_1 - k_2 - 2} \Psi(s - r + k_1 + 1, s - r + k_2 + 1). \quad (2.11)\]

Here \( \Psi(s - r + k_2 + 1, s - r + k_1 + 1) \) is the number of possible partitions of the \( 2s - 2r + k_1 + k_2 + 2 \) distinct elements of the set \( (A \cup B) \cap I_3 \) among which \( X = s - r + k_2 + 1 \) elements belong to the set \( A \cap I_3 \) and \( Y = s - r + k_1 + 1 \) elements belong to the set \( B \cap I_3, \) while the elements of the sets \( A \) and \( B \)
satisfy to the condition (1).

It is clear that the number \( N_{r,n}(k_1, k_2, s) = 0 \) if the integer \( s \geq 0 \) satisfies the inequalities \( j - i < s < 2r - s - k_1 - k_2 - 2 \) as in this case the corresponding binomial coefficient \( \binom{j - i}{s} \) or \( \binom{s}{2r - s - k_1 - k_2 - 2} \) is equal to zero.
Determine the number $\Psi(s - r + k_1 + 1, s - r + k_2 + 1)$. To do this let us arrange selecting integers $l_1, l_2, \ldots, l_{2s - 2r + k_1 + k_2 + 2}$ of different elements of the sets $A \cap I_3$ and $B \cap I_3$ in increasing order.

Let $k_1 \geq k_2$.

Clearly, $l_1 \in A \cap I_2$. Correspondingly, we take a step from the point $(0, 0)$ to the point $(1, 0)$. If $l_2 \in A \cap I_3$, then the next step is the $(2, 0)$, while if $l_2 \in B \cap I_3$, then it is to the point $(1, 1)$, and so on.

The result is one possible path of the indicated form from the point $(0, 0)$ to the point $(X, Y)$. The total number of such paths is $\Phi(X, Y)$, if $X \geq Y$, and it is $\Phi(Y, X)$, if $X \leq Y$. Thus by the formula (2.9) we have

$$\Psi(s - r + k_1 + 1, s - r + k_2 + 1) = \Phi(s - r + k_1 + 1, s - r + k_2 + 1) =$$

$$= \frac{k_1 - k_2 + 1}{s - r + k_1 + 2} \left( \frac{2s - 2r + k_1 k_2 + 2}{s - r + k_2 + 1} \right), \text{ if } k_1 \geq k_2, \quad (2.12)$$

$$\Psi(s - r + k_1 + 1, s - r + k_2 + 1) = \Phi(s - r + k_1 + 1, s - r + k_1 + 1) =$$

$$= \frac{k_2 - k_1 + 1}{s - r + k_1 + 2} \left( \frac{2s - 2r + k_1 + k_2 + 2}{s - r + k_2 + 1} \right), \text{ if } k_2 \geq k_1. \quad (2.13)$$

By the formulas (2.1), (2.2) and (2.6) we get

$$N_{r,n}(k_1, k_2, s) = \binom{n}{k_1} \binom{n - j}{k_2} \left( \binom{s}{k_1} \binom{2r - s - k_1 - k_2 - 2}{s} \right) \times$$

$$\times \frac{k_1 - k_2 + 1}{s - r + k_1 + 2} \left( \frac{2s - 2r + k_1 + k_2 + 2}{s - r + k_2 + 1} \right), \text{ if } k_1 \geq k_2, \quad (2.14)$$

$$N_{r,n}(k_1, k_2, s) = \binom{n - 1}{k_1} \binom{n - j}{k_2} \left( \binom{s}{k_1} \binom{2r - s - k_1 - k_2 - 2}{s} \right) \times$$

$$\times \frac{k_2 - k_1 + 1}{s - r + k_1 + 2} \left( \frac{2s - 2r + k_1 + k_2 + 2}{s - r + k_2 + 1} \right), \text{ if } k_2 \geq k_1. \quad (2.15)$$

As

$$\mathbb{L}^{(n-1)}(i - 1, j) = \sum_{r \geq 1} \mathbb{L}^{(n-1)}(i - 1, r) =$$

$$= \sum_{r \geq 1} \sum_{k_1 = 1}^{r-1} \sum_{k_2 = 1}^{r-1} \sum_{s = r - k_2 + 1}^{2r - k_1 - k_2 - 2} N_{r,n}(k_1, k_2, s) =$$

$$= \sum_{r \geq 1} \left\{ \sum_{k_1 = 0}^{r-1} \sum_{k_2 = 0}^{r-1} \sum_{s = r - k_2 + 1}^{2r - k_1 - k_2 - 2} N_{r,n}(k_1, k_2, s) +$$
\[
N_r,n(k_1, k_2, s) = \left\{
\begin{array}{l}
\sum_{k_1=1}^{r-1} \sum_{k_2=0}^{r-2} N_r,n(k_1, k_2, s) \\
nen \sum_{s=r-k_2+1} N_r,n(k_1, k_2, s)
\end{array}
\right.
\]

then by the formulas (2.14) and (2.15) we get the formula (2.10)

\[
S_3(r, n) = \sum_{k=1}^{r-1} \binom{i-1}{k} \binom{n-j}{k} \cdot \frac{1}{j-i+1} \cdot \binom{j-i+1}{r-k} \binom{j-i+1}{r-k-1}.
\]

Further we will spend (2.10) summation on parameters \( s, k_1, k_2 \) and \( r \).

**Lemma 2.4** Let \( m, a, b \) be positive integers, such that \( m \geq a + b, a \geq b \). Then the following summation formula is valid

\[
S = \sum_{s=a}^{a+b} \binom{m}{s} \binom{s}{a+b-s} \cdot \frac{1}{s-b+1} \binom{2s-a-b}{s-a} =
\]

\[
= \frac{1}{m+1} \binom{m+1}{a+1} \binom{m+1}{b}; \tag{2.16}
\]

**Proof.** We have

\[
S = \sum_{s=a}^{a+b} \binom{m}{s} \binom{s}{a+b-s} \cdot \frac{1}{s-b+1} \binom{2s-a-b}{s-a} =
\]

\[
= \sum_{s=a}^{a+b} \frac{m!}{s!(m-s)!} \cdot \frac{s!}{(a+b-s)!(s-b+1)!} \cdot \frac{1}{s-b+1} \cdot \frac{(2s-a-b)!}{(s-a)!(s-b)!} =
\]

\[
= \sum_{s=a}^{a+b} \frac{m!}{(m-s)!(a+b-s)!(s-b+1)!(s-a)!} =
\]

\[
= \frac{1}{m+1} \binom{m+1}{a+1} \sum_{s=a}^{a+b} \binom{m-a}{s-a} \binom{a+1}{a+b-s} =
\]

\[
= \frac{1}{m+1} \binom{m+1}{a+1} \sum_{s=0}^{b} \binom{m-a}{s} \binom{a+1}{b-s} = \frac{1}{m+1} \binom{m+1}{a+1} \binom{m+1}{b}.
\]

Having spent a summation on \( s \) in right part of the (2.10) of the formula (2.16) we get.
Lemma 2.5. If \( i - 1 < j, i = 1, 2, \ldots, n \), then the following summation formula is valid
\[
\left| \mathcal{L}^{(n-1)}(i-1,j) \right| =
\sum_{r \geq 1} \left\{ \sum_{k_1=0}^{r-1} \sum_{k_2=0}^{k_1} \left( \frac{i - 1}{k_1} \right) \left( \frac{n - j}{k_2} \right) \frac{(k_1 - k_2 + 1)}{(j - i + 1)} \left( \frac{j - i + 1}{r - k_2} \right) \left( \frac{j - i + 1}{r - k_1 - 1} \right) \right. \\
+ \left. \sum_{k_1=0}^{r-1} \sum_{k_2=0}^{k_1} \left( \frac{i - 1}{k_1} \right) \left( \frac{n - j}{k_2} \right) \frac{(k_2 - k_1 + 1)}{(j - i + 1)} \left( \frac{j - i + 1}{r - k_1} \right) \left( \frac{j - i + 1}{r - k_2 - 1} \right) \right\}.
\]

2.1.2 The calculation of the difficult 6-multiple combinatorial sum \( \Omega^+(n) \) with the help of the Egorychev method of coefficients

Theorem 4 ([20]). For the number \( \Omega^+(n) \) the following equalities are valid:
\[
\Omega^+(n) = 2^{2n-1} + (n-1) \left( \frac{2n-2}{n-1} \right) - \frac{4}{n} \left( \frac{2n}{n-2} \right) - \left( \frac{2n}{n} \right) = \\
= 2^{2n-1} + \frac{2(2n-3)!}{(n-2)!(n+2)!} (n^4 - 2n^3 - 27n^2 + 20n - 4). \tag{2.17}
\]

Let \( \Omega^+(n) = T_1 + T_2 + T_3 \), where
\[
T_1 := \sum_{i=2}^{n} \sum_{j=1}^{i-1} \left( \frac{n - i + j - 1}{j - 1} \right) \left( \frac{i - 1 + n - j}{i - 1} \right), \tag{2.18}
\]
\[
T_2 := \sum_{i=2}^{n} \sum_{j=1}^{i-1} \left( \frac{i - 1 + n - j}{i - 1} \right) \left( \left( \frac{n - i + j}{j} \right) - \left( \frac{n - i + j}{j + 1} \right) \right) = \\
= \sum_{i=2}^{n} \sum_{j=1}^{i-1} \left( \frac{i - 1 + n - j}{i - 1} \right) \text{res}_u \left\{ \frac{(1+u)^{n+j-i} \cdot (1-u)}{u^{j+2}} \right\}, \tag{2.19}
\]
\[
T_3 := -\sum_{i=2}^{n} \sum_{j=1}^{i-3} \sum_{s=0}^{j-i-3} \left( \frac{i - 1 + n - j}{i - 1} \right) \left( \frac{i - 1 + n - j}{s + i} \right) \left( \frac{2j - 2i}{j - i - s - 3} \right) + \\
+ \sum_{i=2}^{n} \sum_{j=1}^{i-1} \sum_{s=0}^{j-i-1} \left( \frac{i - 1 + n - j}{i - 1} \right) \left( \frac{i - 1 + n - j}{s + i} \right) \left( \frac{2j - 2i}{j - i - s + 1} \right). \tag{2.20}
\]
Lemma 2.6. For any natural \( n > 2 \) the following combinatorial identity is valid

\[ T_1 = \sum_{i=2}^{n} \sum_{j=1}^{i-1} \binom{n-i+j-1}{j-1} \binom{n+i-j-1}{i-1} = (n-1) \binom{2n-2}{n-1}. \quad (2.21) \]

Proof. Using the known relations and integral representations for the corresponding binomial coefficients, we obtain

\[ T_1 = \sum_{i=2}^{n} \sum_{j=1}^{i-1} \binom{n-i+j-1}{j-1} \binom{n+i-j-1}{n-j} = \sum_{i=2}^{n} \sum_{j=0}^{i-2} \binom{n-i+j}{j} \binom{n+i-j}{n-j} \]

\[ = \sum_{i=2}^{n} \sum_{j=0}^{i-2} \left\{ \text{res}_y \frac{1-x}{x^{j+1}} \text{res}_x \frac{(1-y)^{-i-2}}{y^{n-j}} \right\}. \]

Using the formula of a geometric progression in the sum on the index \( j \), we get

\[ \sum_{i=2}^{n} \text{res}_{xy} \left\{ \left[ (1-x)^{-n+i-1} (1-y)^{-i-2} \frac{1-(y/x)^{i-1}}{1-y/x} \right] \right\} \]

\[ = \sum_{i=2}^{n} \text{res}_{xy} \left\{ \left[ (1-x)^{-n+i-1} (1-y)^{-i-2} \frac{1}{y^{n-i}} \right] \frac{1}{y-x} \right\} \]

\[ - \sum_{i=2}^{n} \text{res}_{xy} \left\{ \left[ (1-x)^{-n+i-1} (1-y)^{-i-2} \frac{1}{x^{i-1}y^{n-i+1}(x-y)} \right] \right\}. \]

\[ \text{(change of variables} \ X = x/(1-x), \ Y = y/(1-y) \text{ under the sign of the operator res in the first and second sums)} \]

\[ T_1 = \sum_{i=2}^{n} \text{res}_{xy} \left\{ \left[ (1+x)^{n-i} (1+y)^{n+i-1} \frac{1}{y^{n}(x-y)} \right] \right\} \]

\[ - \sum_{i=2}^{\infty} \text{res}_{xy} \left\{ \left[ (1+x)^{n-1} (1+y)^{n} \frac{1}{x^{i-1}y^{n-i+1}(x-y)} \right] \right\} = \]

25
\[= \sum_{i=2}^{n} \text{res}_y \left\{ \frac{(1 + y)^{n-i}(1 + y)^{n+i-1}}{y^n} \right\}
- \text{res}_{xy} \left\{ \left( (1 + x)^{n-1}(1 + y)^n \right) \cdot \frac{x}{(x-y)} \right\} |x| >> |y| \]
\[= \sum_{i=2}^{n} \text{res}_y \left( \frac{(1 + y)^{2n-1}}{y^n} \right) - \text{res}_{xy} \left\{ \left( (1 + x)^{n-1}(1 + y)^n \right) y^{-1} \right\} |x| >> |y|. \]

Proceeding similarly, we take the first sum res by \(y\), and the second by \(x\)

\[T_1 = \sum_{i=2}^{n} \left( \frac{2n-1}{n-1} \right) - \text{res}_y \left\{ \left[ \frac{d}{dx} \left( (1 + x)^{n-1}(1 + y)^n \right) \right] x = y \right\} \]
\[= (n-1) \left( \frac{2n-1}{n-1} \right) - (n-1) \text{res}_y \left\{ \left[ \frac{1 + x}{y^{n-1}} \right] x = y \right\} \]
\[= (n-1) \left( \frac{2n-1}{n-1} \right) - (n-1) \text{res}_y \left( \frac{1 + y}{y^{n-1}} \right)^{2n-2} \]
\[= (n-1) \left( \frac{2n-1}{n-1} \right) - (n-1) \left( \frac{2n-2}{n-2} \right) = (n-1) \left( \frac{2n-2}{n-1} \right). \]

Let
\[T_2 := \sum_{i=2}^{n} \sum_{j=i}^{n-1} \left( \frac{i-1+n-j}{n-j} \right) \left( \left( \frac{n-i+j}{j} \right) - \left( \frac{n-i+j}{j+1} \right) \right) = (2.22) \]
\[= \sum_{i=2}^{n} \left( \sum_{j=i}^{n} \ldots - \sum_{j=n} \ldots \right) = S_1 + S_2, \quad (2.23) \]

where
\[S_1 := \sum_{i=2}^{n} \left( \left( \frac{2n-i}{n+1} \right) - \left( \frac{2n-i}{n} \right) \right), \quad (2.24) \]
\[S_2 := \sum_{i=2}^{n} \sum_{j=i}^{n-1} \left( \frac{i-1+n-j}{n-j} \right) \left( \left( \frac{n-i+j}{j} \right) - \left( \frac{n-i+j}{j+1} \right) \right). \quad (2.25) \]

**Lemma 2.7.** For \(n = 3, 4, \ldots\) the following combinatorial identity is valid
\[S_1 = \sum_{i=2}^{n} \left( \left( \frac{2n-i}{n+1} \right) - \left( \frac{2n-i}{n} \right) \right) = \left( \frac{2n-1}{n+2} \right) - \left( \frac{2n-1}{n+1} \right). \quad (2.26)\]
Proof. We have

\[ S_1 = \sum_{i=2}^{n} \text{res}_x \left\{ (1 + x)^{2n-i}/x^{n+2} - (1 + x)^{2n-i}/x^{n+1} \right\} = \]

\[ = \text{res}_x \sum_{i=2}^{n} \left\{ (1 + x)^{2n-i}(1 - x)/x^{n+2} \right\} = \]

\[ = \text{res}_x \left\{ \frac{(1 + x)^{2n-2}(1 - x)}{x^{n+2}} \cdot \left( \frac{1 - (1 + x)^{-(n-1)}}{1 - (1 + x)^{-1}} \right) \right\} \]

\[ = -\text{res}_x \{(1 + x)^{n}(1 - x)/x^{n+3}\} + \text{res}_x\{(1 + x)^{2n-1}(1 - x)/x^{n+3}\} = \]

\[ = -0 + \text{res}_x(1 + x)^{2n-1}(1 - x)/x^{n+3} = \]

\[ = \binom{2n-1}{n+2} - \binom{2n-1}{n+1}. \] 

\[ (2.27) \]

The following statement establishes the proof of Lemma 2.7

Lemma 2.8. For \( n = 3, 4, \ldots \) the following identity is valid

\[ S_2 = \sum_{i=2}^{n} \sum_{j=i}^{n} \binom{i-1+n-j}{n-j} \left( \binom{n-i+j}{j} - \binom{n-i+j}{j+1} \right) = \]

\[ = -(n-1) - \binom{2n}{n} + 2 \binom{2n}{n+1}. \] 

\[ (2.27) \]

From the formulas (2.23), (2.26) and (2.27) follows

Lemma 2.9. For \( n = 3, 4, \ldots \) the following identity is valid

\[ T_2 := \sum_{i=2}^{n} \sum_{j=i}^{n-1} \binom{i-1+n-j}{n-j} \left( \binom{n-i+j}{j} - \binom{n-i+j}{j+1} \right) = \]

\[ = \binom{2n-1}{n+2} - \binom{2n-1}{n+1} - (n-1) - \binom{2n}{n} + 2 \binom{2n}{n+1}. \] 

\[ (2.28) \]
2.1.3 The calculation of the sum $T_3$

**Lemma 2.10.** For $n = 3, 4, \ldots$ the following identity is valid

$$F_1 = \sum_{i=2}^{n} \sum_{j=n}^i \left( \frac{i - 1 + n - j}{i - 1} \right) \text{res}_{yu} \frac{(1 + y)^{n-j+i-1}(1 + u)^{2j-2i}(1 - u^4)}{y^{i}u^{j-i+2}(y - u)} = 0.$$

**Proof.** We obtain with the method of coefficients successively

$$F_1 = \sum_{i=2}^{n} \sum_{j=n}^i \left( \frac{i - 1 + n - j}{i - 1} \right) \text{res}_{yu} \frac{(1 + y)^{n-j+i-1}(1 + u)^{2j-2i}(1 - u^4)}{y^{i}u^{j-i+2}(y - u)} =$$

$$= \sum_{i=2}^{n} \text{res}_{yu} \frac{(1 + y)^{i-1}(1 + u)^{2n-2i}(1 - u^4)}{y^{i}u^{n-i+2}(y - u)} =$$

$$= \sum_{i=2}^{n} \text{res}_{yu} \left\{ \frac{(1 + y)^{i-1}(1 + u)^{2(n-i)}(1 - u^4)}{y^{i+1}u^{n-i+2}(1 - u/y)} \right\}_{|y| \gg |u| = \rho} =$$

$$\text{res}_{u} \left\{ \frac{(1 + u)^{2(n-i)}(1 - u^4)}{u^{n-i+2}} \right\} \left\{ \sum_{i=2}^{n} \text{res}_{y} \frac{(1 + y)^{i-1}}{y^{i+1}} \left( 1 + \sum_{s=0}^{\infty} (u/y)^s \right) \right\} = 0$$

by the definition of $\text{res}_{u}$. \qed

**Lemma 2.11.** For $n = 3, 4, \ldots$ the following identity is valid

$$T_3 = -\sum_{i=2}^{n} \sum_{j=1}^{n-i-3} \sum_{s=0}^{2j-2i} \left( \frac{i - 1 + n - j}{i - 1} \right) \left( \frac{i - 1 + n - j}{s + i} \right) \left( \frac{2j - 2i}{j - i - s - 3} \right) +$$

$$+ \sum_{i=2}^{n} \sum_{j=i}^{n-i-1} \sum_{s=0}^{2j-2i} \left( \frac{i - 1 + n - j}{i - 1} \right) \left( \frac{i - 1 + n - j}{s + i} \right) \left( \frac{2j - 2i}{j - i - s + 1} \right)$$

$$= \sum_{i=2}^{n} \sum_{j=i}^{n} \left( \frac{i - 1 + n - j}{n - j} \right) \text{res}_{yu} \frac{(1 + y)^{n-j+i-1}(1 + u)^{2j-2i}(1 - u^4)}{y^{i}u^{j-i+2}(y - u)} \quad (2.29)$$

The proof will be divided into a series of statements.

**Lemma 2.12.** Let $|t| = \rho = 0, 01$, $|u| = 10$, $|y| = 10^4$. Then

$$T_3 = \text{res}_{yu} \left\{ \frac{t^{-n+1}(1 + y)^{n}(1 - u^4)(1 - t)^{-2}}{y - (-1 + t(1 + u)^2)} \right\} \left\{ yu^2(y - u)(y - \frac{t}{1-t}) \right\}_{|y| \gg |u| \gg |t| = \rho} \quad (2.30)$$
where the integrals are

\[
J_1 := \text{res}_{tu}\{ \frac{(1 - u^4)}{t^n(1 - t)u^2(u - t(1 + u)^2)} \}_{|u| \gg |t| = \rho}, \tag{2.32}
\]

\[
J_2 := \text{res}_{tu}\{ \frac{(1 - u^4)(1 + u)^{n-1}}{t^{n-1}(1 - t)u^2(u - t(1 + u)^2)} \}_{|u| \gg |t| = \rho}, \tag{2.33}
\]

\[
J_3 := \text{res}_{tu}\{ \frac{1}{t^{n+1}u} \cdot \frac{(1 - u)(1 - t)^{-n-1}}{(u - t(1 + u))^2 (u - \frac{1-t}{t})} \}_{|u| \gg |t| = \rho}, \tag{2.34}
\]

\[
J_4 := \text{res}_{tu}\{ \frac{(1 - u^4)(1 + u)^{2n-1}u - n(1 - t)^2}{(u - t(1 + u)^2) (u - \frac{1-t}{t})} \}_{|u| \gg |t| = \rho}. \tag{2.35}
\]

**Proof.** We have

\[
T_3 = \sum_{i=2}^{n} \sum_{j=1}^{n} (\text{res}_z(1 + z)^{i-1+n-j}z^{-n+j-1} \times
\]

\[
\times \text{res}_{zu}\{ \frac{(1 + y)^{n-j+i-1}(1 + u)^{2j-2i}(1 - u^4)}{y^ju^{j+i+2}(y - u)} \}_{|y| \gg |u| = \rho} =
\]

\[
= \sum_{i=2}^{n} \text{res}_{zu}\left\{ \sum_{j=1}^{\infty} (1 + z)^{i-1+n-j}z^{-n+j-1} \times
\]

\[
\times \left( \frac{(1 + y)^{n-j+i-1}(1 + u)^{2j-2i}(1 - u^4)}{y^ju^{j+i+2}(y - u)} \right)_{|y| \gg |u| = \rho} \right\}
\]

\[
= \sum_{i=2}^{n} \text{res}_{zu}\left\{ (1 + z)^{n-1}z^{-n+i-1} \left( 1 - \frac{z(1 + u)^2}{(1 + z)(1 + y)u} \right)^{-1} \cdot \frac{(1 + y)^{n-1}(1 - u^4)}{y^iu^2(y - u)} \right\}
\]

\[
= \text{res}_{zu}\left\{ \sum_{i=2}^{\infty} \frac{(1 + z)^{n-1}z^{-n+i-1}u}{(1 + z)(1 + y)u - z(1 + u)^2} \cdot \frac{(1 + y)^{n-1}(1 - u^4)}{y^iu(y - u)} \right\} =
\]

\[
= \text{res}_{zu}\left\{ \frac{(1 + z)^n z^{-n+1}}{(1 + z)(1 + y)u - z(1 + u)^2} \cdot \frac{(1 + y)^n(1 - u^4)}{y^2u(y - u)(1 - z/y)} \right\} =
\]

\[
= \text{res}_{zu}\left\{ \frac{(1 + z)^n z^{-n+1}}{((1 + z)(1 + y)u - z(1 + u)^2) yu(y - u)(y - z)} \right\}_{|y| \gg |u| \gg |z| = \rho}.
\]
If you make a change here $t = z(1 + z)^{-1} \in H_1$, $z = t(1 - t)^{-1}$, $dz/dt = (1 - t)^{-2}$, $y - z = y - t(1 - t)^{-1}$, then

$$T_3 = \text{res}_{zu} \left\{ \frac{(z/(1 + z))^{-n+1}(1 - u^4)(1 + y)^n}{(u(1 + y) - z(1 + u)^2/(1 + z))yu(y - u)(y - z)} \right\} =$$

$$= \text{res}_{tyu} \left\{ \frac{t^{-n+1}(1 - u^4)(1 + y)^n}{(1 - t)^2(u(1 + y) - t(1 + u)^2)yu(y - u)(y - t(1 - t)^{-1})} \right\} =$$

$$= \text{res}_{tyu} \left\{ \frac{t^{-n+1}(1 + y)^n(1 - u^4)(1 - t)^{-2}}{(y - (1 + t(1 + u)^2u^{-1})yu^2(y - u)(y - t(1 - t)^{-1})^2)} |y| > |y| > |t| = \rho \right\} =$$

$$= \text{res}_{tu} \left\{ \frac{(1 - u^4)t^{-n+1}}{(1 - t)^2u^2} \left[ \frac{(1 + y)^n(y - t(1 - t)^{-1})^{-1}}{y(y - (1 + t(1 + u)^2u^{-1}))} \right] |u| > |t| = \rho, y = u +$$

$$\left[ \frac{(1 + y)^n(y - u)^{-1}}{(y - t(1 - t)^{-1})(y - (1 + t(1 + u)^2u^{-1}))} \right] |u| > |t| = \rho, y = t(1 - t)^{-1} +$$

$$\left[ \frac{(1 + y)^n}{y(y - u)(y - t(1 - t)^{-1})} \right] |u| > |t| = \rho, y = -1 + t(1 + u)^2u^{-1} \right\}$$

We compute the last integral by the residue theorem, in which the choice is $|t| = \rho = 0, 1, |u| = 10, |y| = 10^3$ we have poles at the points $y = 0, y = u, y = t/(1 - t)$ and $y = -1 + t(1 + u)^2u^{-1}$. Then

$$T_3 = \text{res}_{tu} \left\{ \frac{(1 - u^4)t^{-n+1}}{(1 - t)^2u^2} \left[ \frac{(1 + y)^n(y - t(1 - t)^{-1})^{-1}}{y(y - (1 + t(1 + u)^2u^{-1}))} \right] |u| > |t| = \rho, y = u +$$

$$\left[ \frac{(1 + y)^n(y - u)^{-1}}{(y - t(1 - t)^{-1})(y - (1 + t(1 + u)^2u^{-1}))} \right] |u| > |t| = \rho, y = t(1 - t)^{-1} +$$

$$\left[ \frac{(1 + y)^n}{y(y - u)(y - t(1 - t)^{-1})} \right] |u| > |t| = \rho, y = -1 + t(1 + u)^2u^{-1} \right\} =$$

$$\text{res}_{tu} \left\{ \frac{(1 - u^4)}{t^{n-1}(1 - t)^2u^2} \left[ \frac{1 - t}{t(u - t(1 + u)^2)} \right] |u| > |t| = \rho +$$

$$\left[ \frac{(1 - t)(1 + u)^n}{u(1 - t) - t(u^2 + u - t(1 + u)^2)} \right] |u| > |t| = \rho +$$

$$\left[ \frac{u(1 - t)^{3-n}}{t(t - u(1 - t))(ut + (1 - t)u - (1 - t)t(1 + u)^2)} \right] |u| > |t| = \rho +$$

$$\left[ \frac{(1 - t)u^3(t(1 + u)^2u^{-1})^n(-u + t(1 + u)^2)^{-1}}{((u + t(1 + u)^2 - u^2)(-(1 - t)u + (1 - t)t(1 + u)^2) - ut)} \right] |u| > |t| = \rho \right\} \right\} =$$

$$= J_1 + J_2 + J_3 + J_4.$$
Lemma 2.13  For \(|u| \gg |t| = \rho\) the following identity is valid

\[ J_1 = \text{res}_{tu}\{\frac{(1-u^4)}{(1-t)tu^2(u - t(1 + u)^2)}\}_{|u| \gg |t| = \rho} = 0. \quad (2.36) \]

**Proof.** As \(|t| = \rho = 0.01, |u| = 10\) and \(|t(1 + u)^2/u| < 1\), the calculation of residues \(u\) and \(t\) is carried out using the method of coefficients similar analogously in Lemma 2.11. \(\blacksquare\)

Lemma 2.14  For \(|u| \gg |t| = \rho\) and \(n = 3, 4, \ldots\) the following identity is valid

\[ J_2 = \text{res}_{tu}\{\frac{t^{-n+1}(1-u^4)(1+u)^n}{(1-t)u^2(u(1-t) - t)(u^2 + u - t(1+u)^2)}\}_{|u| \gg |t| = \rho} = \]

\[ = (n - 1) - \frac{2}{n} \binom{2n}{n-2}. \quad (2.37) \]

The proof is analogical to Lemma 2.12.

\[ J_3 = \text{res}_{tu}\{\frac{1}{(1-t)^{n+1}tu^{n+1}} \cdot \frac{1-u^4}{u(1-t)(1+u)^2(u(1-t) - t)}\}_{|u| \gg |t| = \rho}, \]

\[ J_4 = \text{res}_{tu}\{\frac{(1-u^4)(1+u)^2n-1u^{-n+1}(1-t)^2}{(u-t(1+u)^2)(u-t(1-t))^2(u-t(1-t)/t)}\}_{|u| \gg |t| = \rho}. \quad (2.38) \]

Lemma 2.15  For \(n = 3, 4, \ldots\) and meeting the conditions \(|u| \gg |t| = \rho\) the following identity is valid

\[ J_3 = \text{res}_{tu}\{\frac{1}{(1-t)^{n+1}tu^{n+1}} \cdot \frac{(1-u^4)}{u(1-t)(1+u)^2(u(1-t) - t)}\}_{|u| \gg |t| = \rho} = \]

\[ = 2^{2n-1} - \binom{2n}{n+1} - \binom{2n+1}{n} + \binom{2n}{n}. \quad (2.39) \]

**Proof.** Let \(|t| = \rho_1 = 0.01, |u| = \rho_2 = 10\) and \(|t|/(1-t)| < 1 < \rho_2\) and \(|(1-t)|/|t| \approx 100 > 10 = \rho_2\), then compute the last integral as a residue for \(u\) at \(u = 0, u = t/(1-t)\), except for the point \(u = (1-t)/t\), we obtain

\[ J_3 = \text{res}_t\{\frac{1}{(1-t)^{n+1}tu^{n+1}} \cdot \frac{(1-u^4)}{(u-t(1-t))^2(u-t(1-t)/t)}\}_{u=0} \]
As a result of a simple calculation gives us the following expression for \(4t/(1-t)\):

\[
\begin{align*}
+\frac{d}{du}\left[\frac{(1-u^4)}{u(u-(1-t)/t)}\right]_{u=t/(1-t)} &= \\
= \text{res}_t\left\{\frac{1}{(1-t)^{n+1}t^{n+1}} \cdot \frac{1}{(-t/(1-t))^2(-(1-t)/t)^2}\right\} + \\
+\text{res}_t\left\{\frac{t^{-(n+1)1}}{(1-t)^{n+1}u(u-(1-t)/t)} - \frac{(1-u^4)(2u-(1-t)/t)}{u^2(u-(1-t)/t)^2}\right\}_{u=t/(1-t)}.
\end{align*}
\]

Thus

\[
[(1-u^4)]_{u=t/(1-t)} = [(1-u^2)(1+u^2)]_{u=t/(1-t)} = (1-2t)(2t^2-2t+1)/(1-t)^4,
\]

then

\[
J_3 = -\text{res}_t\left\{(1-t)^{-n}t^{n+2}\right\} - \text{res}_t\left\{\frac{4(t/(1-t))^2}{(1-t)^{n+1}t/(1-t)-(1-t)/t}\right\} = \\
= -\text{res}_t\left\{\frac{1}{(1-t)^{n+1}t^{n+1}} \cdot \frac{(1-2t)(2t^2-2t+1)(2t/(1-t)-(1-t)/t)}{(1-t)^4(t/(1-t))^2(t/(1-t)-(1-t)/t)^2}\right\} = \\
= -\left(\frac{2n}{n+1}\right) + \text{res}_t\left\{\frac{2t^4-2t^3+5t^2-4t+1}{(1-t)^{n+2}t^{n+2}(1-2t)}\right\}.
\]

Thus

\[
J_3 = -\left(\frac{2n}{n+1}\right) + \text{res}_t\left\{\frac{2t^4-2t^3+5t^2-4t+1}{(1-t)^{n+2}t^{n+2}(1-2t)}\right\}. \tag{2.40}
\]

If under the integral sign (2.40) we make the substitution \(t = (1-(1-4w)^{1/2})/2 \in H_1\), then

\[
w = t(1-t) \in H_1, \quad t = (1-(1-4w)^{1/2})/2 \in H_1, \quad 1-2t = (1-4w)^{1/2},
\]

\[
dt/dw = (1-4w)^{-1/2}, \quad (1-4w)^{-1/2} = \sum_{s=0}^{\infty} \binom{2s}{s} w^s
\]
as a result of a simple calculation gives us the following expression for \(J_3\).

\[
J_3 = -\left(\frac{2n}{n+1}\right) + \text{res}_w\left[\frac{2t^4-2t^3+5t^2-4t+1}{(1-t)^{n+2}t^{n+2}(1-2t)}\right]_{t=(1-(1-4w)^{1/2})/2} = \\
= -\left(\frac{2n}{n+1}\right) + \text{res}_w\left[\frac{1+10(1-4w)-2(1-4w)^{1/2}-2(1-4w)^{3/2}+(1-4w)^2}{8(1-4w)w^{n+2}}\right] = \\
= -\left(\frac{2n}{n+1}\right) + \frac{1}{8} \text{res}_w\left[\frac{1}{(1-4w)w^{n+2}}\right] + \frac{5}{4} \text{res}_w\left[\frac{1}{w^{n+2}}\right].
\]
\[-\frac{1}{4} \text{res}_w \frac{(1 - 4w)^{-1/2}}{w^{n+2}} - \frac{1}{4} \text{res}_w \frac{(1 - 4w)^{1/2}}{w^{n+2}} + \text{res}_w \frac{(1 - 4w)}{w^{n+2}} =
\]

\[-\left( \frac{2n}{n + 1} \right) + 4^{n+1} + 0 - \frac{1}{4} \text{res}_w \frac{(1 - 4w)^{-1/2}(1 + (1 - 4w))}{w^{n+2}} + 0 =
\]

\[-\left( \frac{2n}{n + 1} \right) + 2^{2n-1} - \frac{1}{4} \text{res}_w \frac{(1 - 4w)^{-1/2}(2 - 4w)}{w^{n+2}} =
\]

\[-\left( \frac{2n}{n + 1} \right) + 2^{2n-1} - \frac{1}{2} \text{res}_w \frac{(1 - 4w)^{-1/2}}{w^{n+2}} + \text{res}_w \frac{(1 - 4w)^{-1/2}}{w^{n+2}}.
\]

Using the well-known expansion \((1 - 4w)^{-1/2} = \sum_{s=0}^{\infty} \binom{2s}{s} w^s\) for \((2.41)\), we obtain

\[J_3 = -\left( \frac{2n}{n + 1} \right) + 2^{2n-1} - \frac{1}{2} \left( \frac{2n + 2}{n + 1} \right) + \left( \frac{2n}{n} \right) =
\]

\[= 2^{2n-1} - \left( \frac{2n}{n + 1} \right) - \left( \frac{2n + 1}{n} \right) + \left( \frac{2n}{n} \right).
\]

Lemma 2.16 The following identity is valid

\[J_4 = \text{res}_{tu}\left\{ \frac{(1 - u^4)(1 + u)^{2n-1}u^{-n+1}(u - t(1 + u^2))}{(u - t/(1 - t))(u - (1 - t)/t)(1 - t^2)} \right\}_{u \gg |t| = \rho} = 0. (2.42)
\]

Proof. Let

\[J_4 = \text{res}_u \left( \frac{(1 - u^4)(1 + u)^{2n-6}}{u^{n-1}} \times \right.
\]

\[\times \text{res}_t \left\{ \frac{t - u/(1 + u^2)}{(t - u/(1 + u^2))(t - 1/(1 + u))(t - u/(1 + u))^2} \right\},
\]

where \(|u| \gg |t| = \rho\).

If now, in accordance with the condition \(|u| \gg |t| = \rho\) for example, \(|t| = \rho_1 = 0.01, |u| = \rho_2 = 10\) and \(|u/(1 + u)^2| \approx 10 \gg \rho_1 = 0.01, |u/(1 + u)| \approx 1 \gg \rho_1 = 0.01, |1/(1 + u)| \approx 0.1 \gg \rho_1 = 0.01\), then the integrand in \(t\) and the integral \((2.34)\) has no singularities inside \(|t| = \rho_1 = 0.01\), and therefore the integral \((2.42)\) is zero.

Remark 2.1 Calculating the sum of \(J_1, J_2, J_3\) in closed form was held up by appropriate residue theorem, and the results were confirmed by the numerical test.

Two of them were calculated with the well-known theorem of residues, while the integrals \(J_1 = J_4 = 0\) are trivial by Theorem difficult to assess the full amount of the deduction, as the corresponding residue at infinity is zero.
From the formulas (2.36) – (2.39) and (2.42) we immediately obtain the following formula for \( T_3 = J_1 + J_2 + J_3 + J_4 \).

**Lemma 2.17** For \( n = 3, 4, \ldots \) the following identity is valid

\[
T_3 = 2^{2n-1} + (n-1) - \frac{2}{n} \binom{2n}{n-2} - \binom{2n}{n+1} + \binom{2n}{n}. \tag{2.43}
\]

From the formulas (2.21), (2.28) and (2.43) follows the validity of the Main theorem.

As a result, using the fundamental theorem, we get the general formula (2.3) for enumeration of \( \mathcal{D} \)-invariant ideals of ring \( R_n(K, J) \).

## 3 Calculation of multiple combinatorial sums in the theory of holomorphic functions in \( \mathbb{C}^n \)

In section 4 we solved the several summation problems. In sections 4.2, 4.3 and section 4.4 we found the simple new proof and generalization of the several multiple combinatorial sums, which originally arose in the theory of holomorphic functions in \( \mathbb{C}^n \) [40, 41, 6].

### 3.1 Introduction

**Definition 3.1** Domain \( G \subset \mathbb{C}^n \) is called a linearly convex ([1], §8), if for every point \( z_0 \) its boundary \( \partial G \) there is complex \((n-1)\)-dimensional analytic plane passing through \( z_0 \) and does not intersect \( G \).

Let in the space \( \mathbb{C}^n \) set linearly convex polyhedron, ie, bounded linearly convex domain \( G = \{ z : g^l(z, \bar{z}) < 0, \quad l = 1, \ldots, N \} \), where the functions \( g^l(z, \bar{z}) \) are twice continuously differentiable in a neighborhood of this area. The boundary \( \partial G \) of \( G \) consists of faces

\[
S^l = \{ z \in \overline{G} : g^l(z, \bar{z}) = 0 \}, \quad l = 1, \ldots, N.
\]

**Definition 3.2** If at any non-empty edge

\[
S^{j_1\ldots j_k} = S^{j_1} \cap \ldots \cap S^{j_k} = \{ \zeta \in \partial G : g^{j_1}(\zeta, \bar{\zeta}) = 0, \ldots, g^{j_k}(\zeta, \bar{\zeta}) = 0 \}
\]
following inequality holds \( \partial g^{j_1} \wedge \ldots \wedge \partial g^{j_k} \neq 0 \) or, that the same,

\[
\text{rank} \left( \begin{array}{ccc}
\frac{\partial g^{j_1}}{\partial \kappa_1} & \cdots & \frac{\partial g^{j_1}}{\partial \kappa_n} \\
\vdots & \ddots & \vdots \\
\frac{\partial g^{j_k}}{\partial \kappa_1} & \cdots & \frac{\partial g^{j_k}}{\partial \kappa_n}
\end{array} \right) = k.
\]

then \( G \) has a piecewise regular boundary.

In [10] we obtained a new integral representation for holomorphic functions on linear convex domains with piecewise regular boundary of a bounded linear convex domain.

We received a number of identities, when we considered an example of

\[
G = \{ z = (z_1, z_2, z_3) \in \mathbb{C}^3 : g^i(|z|) = -|z_1| + 1 < 0, \quad g^2(|z|) = -|z_2| + 1 < 0, \\
g^3(|z|) = -|z_3| + 1 < 0, \quad g^4(|z|) = a_1|z_1| + a_2|z_2| + a_3|z_3| - r < 0, \quad a_i > 0, \\
i = 1, 2, 3, \quad a_1 + a_2 + a_3 < r \}
\]

the most difficult of which in the notation \( \alpha_i = \frac{a_i}{r} \), \( i = 1, 2, 3 \) for integers \( s_1, s_2, s_3 \geq 0 \) has the form:

**Theorem 3.1** For \( 0 < \alpha_j < 1, j = 1, 2, 3; s_j \in \mathbb{Z}_+ \) following identity is valid

\[
(1 - \alpha_2 - \alpha_3)^{s_1+1} \sum_{k=0}^{s_2} \sum_{l=0}^{s_3} \frac{(s_1 + k + l)!}{s_1!k!!l!!} \alpha_2^k \alpha_3^l = \\
+(1 - \alpha_1 - \alpha_3)^{s_2+1} \sum_{k=0}^{s_1} \sum_{l=0}^{s_3} \frac{(s_2 + k + l)!}{s_2!k!!l!!} \alpha_1^k \alpha_3^l + \\
+(1 - \alpha_1 - \alpha_2)^{s_3+1} \sum_{k=0}^{s_1} \sum_{l=0}^{s_2} \frac{(s_3 + k + l)!}{s_3!k!!l!!} \alpha_1^k \alpha_2^l + \\
- \frac{(s_1 + s_2 + 1)!}{s_1!s_2!} \sum_{m=0}^{s_2} (-1)^m \binom{s_2}{m} \frac{(s_1 + m + l)!}{s_1 + m + 1} \left( \left( 1 - \frac{\alpha_2}{1 - \alpha_3} \right)^{s_1 + m + 1} - \left( \frac{\alpha_1}{1 - \alpha_3} \right)^{s_1 + m + 1} \right) \times \\
\times (1 - \alpha_3)^{s_1+s_2+2} \sum_{k=0}^{s_3} \binom{s_3}{k} \left( \frac{s_1 + s_2 + k + 1}{k} \right) \alpha_3^k + \\
- \frac{(s_2 + s_3 + 1)!}{s_2!s_3!} \sum_{m=0}^{s_3} (-1)^m \binom{s_3}{m} \frac{(s_2 + m + l)!}{s_2 + m + 1} \left( \left( 1 - \frac{\alpha_3}{1 - \alpha_1} \right)^{s_2 + m + 1} - \left( \frac{\alpha_2}{1 - \alpha_1} \right)^{s_2 + m + 1} \right) \times
\]

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\[(1 - \alpha_1)^{s_2 + s_3 + 2} \sum_{k=0}^{s_1} \frac{(s_2 + s_3 + k + 1)}{s_2} \alpha_1^k +\]
\[-\frac{(s_1 + s_3 + 1)!}{s_1!s_3!} \sum_{m=0}^{s_3} \left( \sum_{l=0}^m \frac{(s_3)}{s_1 - 1 + \alpha} \right) \left( 1 - \frac{\alpha_3}{1 - \alpha_2} \right)^{s_1 + m + 1} \times\]
\[-\frac{(s_1 + s_2 + s_3 + 2)!}{s_1!s_2!s_3!} \int_0^1 \int_0^x x^{s_1} y^{s_2} (1 - y)^{s_3} \ dx \wedge dy \equiv 1.\]

(3.1)

If \(\alpha_1 + \alpha_2 + \alpha_3 = 1\), then (3.1) is equivalent to the identity

\[\alpha_1^{s_1 + 1} \sum_{k=0}^{s_2} \sum_{l=0}^{s_3} \frac{(s_1 + k + l)!}{s_1!k!l!} \alpha_2^{k} \alpha_3^l +\]
\[+\alpha_2^{s_2 + 1} \sum_{k=0}^{s_1} \sum_{l=0}^{s_3} \frac{(s_2 + k + l)!}{s_2!k!l!} \alpha_1^{k} \alpha_3^l +\]
\[+\alpha_3^{s_3 + 1} \sum_{k=0}^{s_1} \sum_{l=0}^{s_2} \frac{(s_3 + k + l)!}{s_3!k!l!} \alpha_1^{k} \alpha_2^l = 1,\]

which allows us to formulate the following theorem:

**Theorem 3.2** If the complex parameters \(z_1, \ldots, z_n\) satisfy the relation

\[z_1 + \ldots + z_n = 1,\]

(3.2)

then for any values \(s_1, \ldots, s_n = 0, 1, 2, \ldots\) the following identity is valid

\[z_1^{s_1 + 1} \sum_{j_2=0}^{s_2} \ldots \sum_{j_n=0}^{s_n} \left( \frac{s_1 + \sum_{i \neq 1} j_i}{s_1, j_2, \ldots, j_n} \right) z_2^{j_2} \ldots z_n^{j_n} + \ldots +\]
\[+ z_n^{s_n + 1} \sum_{j_1=0}^{s_1} \ldots \sum_{j_{n-1}=0}^{s_{n-1}} \left( \frac{s_n + \sum_{i \neq n} j_i}{s_n, j_1, \ldots, j_{n-1}} \right) z_1^{j_1} \ldots z_{n-1}^{j_{n-1}} = 1.\]

(3.3)
In the article of D. Zeilberger [44] provides the following identity
\[
\sum_{i=1}^{k} \sum_{0 \leq \alpha_j \leq n-1 \atop j \neq i} \frac{(\alpha_1 + \ldots + \alpha_{i-1} + (n-1) + \alpha_{i+1} + \ldots + \alpha_k)!}{\alpha_1! \ldots \alpha_{i-1}! \alpha_{i+1}! \ldots \alpha_k!} p_1^{\alpha_1} \ldots p_{i-1}^{\alpha_{i-1}} p_i^{\alpha_i} p_{i+1}^{\alpha_{i+1}} \ldots p_k^{\alpha_k} = 1,
\]
for \( p_1 + \ldots + p_k = 1 \).

For \( n = 2 \) the identity (3.3) in a somewhat altered form can be proved in [43] (V. Shelkovich, 1982).

02 May 2015 professor S. LJ. Damjnovic from Belgrad paid attention of the authors to the articles [27] and [28]. From which it follows that identity (3.3) under \( n = 2 \) and real values of parameters \( z_1, z_2 \) from 1960 was known as identity of Chaundy and Bullard [26]. We note that in [28] a detailed story connected with this identity is given.

3.2 Lemmas

Lemma 3.1 The following integral representation is valid
\[
\sum_{k=0}^{s_3} \binom{s_1 + s_2 + k + 1}{k} \alpha^k = \text{res}_z \frac{(1 - z^{-s_3-1})}{(z-1)(1 - \alpha z)^{s_1+s_2+2}} = \text{res}_z \frac{(1 - \alpha z)^{-s_1-s_2-2}}{z^{s_3+1}(1 - z)}. \tag{3.4}
\]

Proof. We have directly
\[
\text{res}_z \frac{(1 - \alpha z)^{-s_1-s_2-2}}{z^{s_3+1}(1 - z)} = \text{res}_z \left( \sum_{k=0}^{\infty} \alpha^k \binom{s_1 + s_2 + k + 1}{k} z^{-s_3-1} \right) \left( \sum_{k=0}^{\infty} z^k \right) = \text{res}_z \sum_{n=0}^{\infty} z^n \left( \sum_{k=0}^{n} \alpha^k \binom{s_1 + s_2 + k + 1}{k} \right) z^{-s_3-1} = \sum_{k=0}^{s_3} \binom{s_1 + s_2 + k + 1}{k} \alpha^k.
\]

On the other side
\[
\sum_{k=0}^{s_3} \binom{s_1 + s_2 + k + 1}{k} \alpha^k = \sum_{k=0}^{s_3} \text{res}_z \{(1 - \alpha z)^{-s_1-s_2-2} z^{-k-1} \} = \text{res}_z \left( \sum_{k=0}^{\infty} (1 - \alpha z)^{-s_1-s_2-2} z^{-k-1} \right) = \text{res}_z \frac{(1 - z^{-s_3-1})}{(z-1)(1 - \alpha z)^{s_1+s_2+2}}.
\]
3.2.1 Calculation the sum $S$ represented in the form:

$$S(1, s_1, s_2, s_3; \alpha_2, \alpha_3) + S(s_2, s_1, s_3; \alpha_1, \alpha_3) + S(s_3, s_1, s_2; \alpha_1, \alpha_2) +$$

$$- (T(s_1, s_2, s_3; \alpha_1, \alpha_2, \alpha_3) + T(s_1, s_3, s_2; \alpha_2, \alpha_3, \alpha_1) + T(s_1, s_2, s_3; \alpha_1, \alpha_3, \alpha_2)) +$$

$$+ R(s_1, s_2, s_3) = 1, \quad (3.5)$$

where

$$S(s_1, s_2, s_3; \alpha_2, \alpha_3) = (1 - \alpha_2 - \alpha_3)^{s_1+1} \sum_{k=0}^{s_2} \sum_{l=0}^{s_3} \frac{(s_1 + k + l)!}{s_1!k!l!} \alpha_2^k \alpha_3^l, \quad (3.6)$$

$$T(s_1, s_2, s_3; \alpha_1, \alpha_2, \alpha_3) = (1 - \alpha_3)^{s_1+s_2+2} \frac{(s_1 + s_2 + 1)!}{s_1!s_2!} \sum_{m=0}^{s_3} \frac{(-1)^m \binom{s_3}{m}}{s_1 + m + 1} \times$$

$$\times \left(1 - \frac{\alpha_2}{1 - \alpha_3}\right)^{s_1+m+1} \left(-\frac{\alpha_1}{1 - \alpha_3}\right)^{s_1+m+1} \sum_{k=0}^{s_3} \binom{s_1 + s_2 + k + 1}{k} \alpha_3^k, \quad (3.7)$$

$$R(s_1, s_2, s_3) = \frac{(s_1 + s_2 + s_3 + 2)!}{s_1!s_2!s_3!} \int_{\alpha_1}^{1-\alpha_2-\alpha_3} \int_{\alpha_2}^{1-\alpha_3-x} x^{s_1} y^{s_2} (1 - x - y)^{s_3} dx \wedge dy. \quad (3.8)$$

3.2.1 Calculation the sum $S_1(s_1, s_2, s_3; \alpha_2, \alpha_3)$.

By direct verification it is easy to see that due to the symmetry of similar terms in the sum of its parameters relative to the latter identity can be represented in the form:

$$S_1(s_1, s_2, s_3) = (1 - \alpha_2 - \alpha_3) \text{res}_{z_1, z_2, z_3} \frac{(1 - (1 - \alpha_2 - \alpha_3)z_1 - \alpha_2z_2 - \alpha_3z_3)^{-1}}{z_1^{s_1+1}z_2^{s_2+1}z_3^{s_3+1}(1 - z_2)(1 - z_3)}.$$  

$$S_1(u_1, u_2, u_3) := \sum_{s_1, s_2, s_3 = 0}^{\infty} S_1(s_1, s_2, s_3) u_1^{s_1} u_2^{s_2} u_3^{s_3} =$$

$$= \frac{(1 - \alpha_2 - \alpha_3)}{(1 - u_2) \cdot (1 - u_3) \cdot (1 - (1 - \alpha_2 - \alpha_3)u_1 - \alpha_2u_2 - \alpha_3u_3). \quad (3.9)$$
**Lemma 3.3** The following relations are valid

\[
S_2(u_1, u_2, u_3) = \frac{(1 - \alpha_1 - \alpha_3)}{(1 - u_1) \cdot (1 - u_3) \cdot (1 - \alpha_1 u_1 - (1 - \alpha_1 - \alpha_3) u_2 - \alpha_3 u_3)}, \tag{3.12}
\]

\[
S_3(u_1, u_2, u_3) = \frac{(1 - \alpha_1 - \alpha_2)}{(1 - u_1) \cdot (1 - u_2) \cdot (1 - \alpha_1 u_1 - \alpha_2 u_2 - (1 - \alpha_1 - \alpha_2) u_3)}. \tag{3.13}
\]

The proof is analogous Lemma 3.2.
3.2.2 Calculation of sums \( T_i(s_1, s_2, s_3; \alpha) \), \( i = 1, 2, 3 \).

**Lemma 3.4** Integral representation for \( T_1(s_1, s_2, s_3; \alpha_1, \alpha_2, \alpha_3) \) is equal

\[
T_1 = \frac{(s_2 + s_3 + 1)!}{s_2!s_3!} \cdot \left\{ (1 - \alpha_1 - \alpha_3)^{s_2+1} \int_0^1 t^{s_2} (1 - 1 \alpha - t(1 - 1 \alpha - 1 \alpha_3))^{s_3} dt + \right.
\]
\[
- \alpha_2^{s_2+1} \int_0^1 t^{s_2} (1 - 1 \alpha_2 t)^{s_3} dt \cdot \text{res}_{z_1}(1 - 1 \alpha_1 z_1) \frac{s_2 - s_3 - 2 z_1 - s_1 - 1}{1 - z_1} \right\}.
\]

**Proof.**

\[
T_1 = \sum_{m=0}^{s_3} \frac{(-1)^m}{s_2!s_3!} \left\{ \int_0^1 \sum_{m=0}^{s_3} \frac{s_3}{m} (1 - \alpha_1 - \alpha_3)^{s_2+1+m} (1 - 1 \alpha - 1 \alpha_3)^{s_3-m} dt + \right.
\]
\[
- \alpha_2^{s_2+1} \int_0^1 t^{s_2} (1 - 1 \alpha - 1 \alpha_2 t)^{s_3} dt \cdot \text{res}_{z_1}(1 - 1 \alpha_1 z_1) \frac{s_2 - s_3 - 2 z_1 - s_1 - 1}{1 - z_1} \right\}.
\]

\[
= \frac{(s_2 + s_3 + 1)!}{s_2!s_3!} \left\{ \int_0^1 \sum_{m=0}^{s_3} \frac{s_3}{m} (1 - \alpha_1 - \alpha_3)^{s_2+1+m} (1 - 1 \alpha - 1 \alpha_3)^{s_3-m} dt + \right.
\]
\[
- \alpha_2^{s_2+1} \int_0^1 t^{s_2} (1 - 1 \alpha - 1 \alpha_2 t)^{s_3} dt \cdot \text{res}_{z_1}(1 - 1 \alpha_1 z_1) \frac{s_2 - s_3 - 2 z_1 - s_1 - 1}{1 - z_1} \right\}.
\]

Analogously we find integral representations for \( T_2(s_1, s_2, s_3; \alpha_1, \alpha_2, \alpha_3) \) and \( T_3(s_1, s_2, s_3; \alpha_1, \alpha_2, \alpha_3) \):

\[
T_2 = \frac{(s_1 + s_3 + 1)!}{s_1!s_3!} \left\{ (1 - \alpha_2 - \alpha_3)^{s_1+1} \int_0^1 t^{s_1} (1 - 1 \alpha_2 - t(1 - 1 \alpha - 1 \alpha_3))^{s_3} dt + \right.
\]
\[
- \alpha_1^{s_1+1} \int_0^1 t^{s_1} (1 - 1 \alpha - t1 \alpha_1)^{s_3} dt \cdot (1 - 1 \alpha_2) \cdot \text{res}_{z_2}(1 - 1 \alpha_2 z_2) \frac{s_1 - s_3 - 2 z_2 - s_2 - 1}{1 - z_2} \right\}.
\]

\[
T_3 = \frac{(s_1 + s_2 + 1)!}{s_1!s_2!} (1 - \alpha_3) \left\{ (1 - \alpha_3 - \alpha_2)^{s_1+1} \int_0^1 t^{s_1} (1 - 1 \alpha_3 - t(1 - 1 \alpha - 1 \alpha_2))^{s_2} dt + \right.
\]
\[
- \alpha_1^{s_1+1} \int_0^1 t^{s_1} (1 - 1 \alpha - 1 \alpha_1 t)^{s_2} dt \times \text{res}_{z_3}(1 - 1 \alpha_3 z_3) \frac{s_1 - s_2 - 2 z_3 - s_3 - 1}{1 - z_3} \right\}.
\]
Lemma 3.5 The following identity is valid

\[
T_1(u_1, u_2, u_3) = \sum_{s_1, s_2, s_3 = 0}^{\infty} T_1(s_1, s_2, s_3) u_1^{s_1} u_2^{s_2} u_3^{s_3} = \frac{(1 - \alpha_1)}{(1 - u_1)} \times \\
= \times \frac{(1 - \alpha_1 - \alpha_2 - \alpha_3)}{1 - \alpha_1 u_1 - (1 - \alpha_1 - \alpha_3) u_2 - \alpha_3 u_3} \cdot \frac{1}{1 - \alpha_1 u_1 - \alpha_2 u_2 - u_3(1 - \alpha_1 - \alpha_2)} 
\]  

(3.17)

Proof. The generating function for \(T_1(s_1 s_2, s_3)\)

\[
T_1(u_1, u_2, u_3) = \sum_{s_1, s_2, s_3 = 0}^{\infty} T_1(s_1, s_2, s_3) u_1^{s_1} u_2^{s_2} u_3^{s_3} = \\
= (1 - \alpha_1) \sum_{s_2, s_3 = 0}^{\infty} (s_2 + s_3 + 1)! \text{res}_{z_1} \left[ (1 - \alpha_1 z_1)^{-s_2-s_3-2} \frac{z_1^{-s_1-1}}{1 - z_1} \right] \times \\
\times \int_0^1 \{(1 - \alpha_1 - \alpha_3)^{s_2+1} u_1^{s_2}(1 - \alpha_1 - t(1 - \alpha_1 - \alpha_3))^{s_3} - \alpha_2^{s_2+1} t^{s_2}(1 - \alpha_1 - \alpha_2 t)^{s_3}\} dt \cdot u_1^{s_2} u_2^{s_3} = \\
= (1 - \alpha_1) \sum_{s_2, s_3 = 0}^{\infty} \frac{(s_2 + s_3 + 1)!}{s_2! s_3!} \cdot \frac{(1 - \alpha_1 u_1)^{-s_2-s_3-2}}{1 - u_1} \times \\
\times \int_0^1 \{(1 - \alpha_1 - \alpha_3)^{s_2+1} t^{s_2}(1 - \alpha_1 - t(1 - \alpha_1 - \alpha_3))^{s_3} - \alpha_2^{s_2+1} t^{s_2}(1 - \alpha_1 - \alpha_2 t)^{s_3}\} dt \cdot u_1^{s_2} u_2^{s_3} = \\
= \frac{(1 - \alpha_1)}{(1 - u_1)(1 - \alpha_1 u_1)^2} \sum_{s_2, s_3 = 0}^{\infty} \text{res}_{z_2,z_3} \left( \frac{1}{z_2^{s_2+1} z_3^{s_3+1}} \right) \cdot \frac{u_2^{s_2} u_3^{s_3}}{1 - \alpha_1 u_1} \times \\
\times \int_0^1 \{(1 - \alpha_1 - \alpha_3)^{s_2+1} t^{s_2}(1 - \alpha_1 - t(1 - \alpha_1 - \alpha_3))^{s_3} - \alpha_2^{s_2+1} t^{s_2}(1 - \alpha_1 - \alpha_2 t)^{s_3}\} dt = \\
= \frac{(1 - \alpha_1)(1 - \alpha_1 - \alpha_3)}{(1 - u_1)(1 - \alpha_1 u_1)} \int_0^1 \left( 1 - \frac{u_2 t(1 - \alpha_1 - \alpha_3)}{(1 - \alpha_1 u_1)} - \frac{u_3(1 - \alpha_1 - t(1 - \alpha_1 - \alpha_3))}{(1 - \alpha_1 u_1)} \right)^{-2} dt + \\
- \frac{(1 - \alpha_1) \alpha_2}{(1 - u_1)(1 - \alpha_1 u_1)^2} \int_0^1 \left( 1 - \frac{u_2 \alpha_2 t}{(1 - \alpha_1 u_1)} - \frac{u_3(1 - \alpha_1 - \alpha_2 t)}{(1 - \alpha_1 u_1)} \right)^{-2} dt = \\
= \frac{(1 - \alpha_1)(1 - \alpha_1 - \alpha_3)}{(1 - u_1)(1 - \alpha_1 u_1)} \times \frac{1}{1 - \alpha_1 u_1 - (1 - \alpha_1 - \alpha_3) u_2 - \alpha_3 u_3} \times \\
\times \frac{1}{1 - \alpha_1 u_1 - \alpha_2 u_2 - u_3(1 - \alpha_1 - \alpha_2)} \tag{3.17} 
\]
Lemma 3.6 The following relations are valid

\[ T_2 = \frac{(1 - \alpha_2)(1 - u_2)^{-1}(1 - \alpha_1 - \alpha_2 - \alpha_3)}{1 - (1 - \alpha_2 - \alpha_3)u_1 - \alpha_2u_2 - \alpha_3u_3} \frac{1}{1 - \alpha_1u_1 - \alpha_2u_2 - u_3(1 - \alpha_1 - \alpha_2)}. \]  

(3.18)

\[ T_3 = \frac{(1 - \alpha_3)(1 - u_3)^{-1}(1 - \alpha_1 - \alpha_2 - \alpha_3)}{1 - (1 - \alpha_2 - \alpha_3)u_1 - \alpha_2u_2 - \alpha_3u_3} \frac{1}{1 - \alpha_1u_1 - (1 - \alpha_1 - \alpha_3)u_2 - \alpha_3u_3}. \]  

(3.19)

The proof is analogous Lemma 3.5.

3.2.3 Calculation the sum \( R(s_1, s_2, s_3) \)

Lemma 3.7 The following identities are valid

\[
R(s_1, s_2, s_3) = 2 \int_{\alpha_1}^{1-\alpha_2-\alpha_3} \int_{\alpha_2}^{1-\alpha_3-x} \text{res}_{t_1,t_2,t_3} \frac{x^{s_1}y^{s_2}(1 - x - y)^{s_3}dxdy}{(1 - t_1 - t_2 - t_3)^{-3}t_1^{s_1+1}t_2^{s_2+1}t_3^{s_3+1}}.
\]

(3.20)

\[
R(u_1, u_2, u_3) = \frac{1}{(1 - \alpha_1u_1 - u_2(1 - \alpha_1 - \alpha_3) - \alpha_3u_3)} \times \\
(\alpha_1 + \alpha_2 + \alpha_3 - 1)^2 \\
(1 - u_1(1 - \alpha_2 - \alpha_3) - \alpha_2u_2 - \alpha_3u_3)(1 - \alpha_1u_1 - \alpha_2u_2 - u_3(1 - \alpha_1 - \alpha_2)).
\]

(3.21)

**Proof.** We find an integral representation for \( R(s_1, s_2, s_3) \)

\[
R(s_1, s_2, s_3) := \frac{(s_1 + s_2 + s_3 + 2)!}{s_1!s_2!s_3!} \int_{\alpha_1}^{1-\alpha_2-\alpha_3} \int_{\alpha_2}^{1-\alpha_3-x} x^{s_1}y^{s_2}(1 - x - y)^{s_3}dxdy = \\
= 2 \int_{\alpha_1}^{1-\alpha_2-\alpha_3} \int_{\alpha_2}^{1-\alpha_3-x} \text{res}_{t_1,t_2,t_3} \frac{x^{s_1}y^{s_2}(1 - x - y)^{s_3}dxdy}{(1 - t_1 - t_2 - t_3)^{-3}t_1^{s_1+1}t_2^{s_2+1}t_3^{s_3+1}}.
\]

Generating function for \( R(s_1, s_2, s_3) \) is

\[
R(u_1, u_2, u_3) := \sum_{s_1,s_2,s_3=0}^{\infty} R(s_1, s_2, s_3) u_1^{s_1} u_2^{s_2} u_3^{s_3} = \\
= 2 \int_{\alpha_1}^{1-\alpha_2-\alpha_3} \int_{\alpha_2}^{1-\alpha_3-x} \text{res}_{t_1,t_2,t_3} \sum_{s_1,s_2,s_3=0}^{\infty} \frac{x^{s_1}y^{s_2}(1 - x - y)^{s_3}u_1^{s_1} u_2^{s_2} u_3^{s_3}dxdy}{(1 - t_1 - t_2 - t_3)^{3}t_1^{s_1+1}t_2^{s_2+1}t_3^{s_3+1}} = \\
= 2 \int_{\alpha_1}^{1-\alpha_2-\alpha_3} \int_{\alpha_2}^{1-\alpha_3-x} \frac{dx dy}{(1 - xu_1 - yu_2 - (1 - x - y)u_3)^3} =
\]

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\[ \begin{align*}
\text{Lemma 3.8} & \quad \text{The following formula is valid} \\
S_1 + S_2 + S_3 - (T_1 + T_2 + T_3) + R &= \frac{1}{(1-u_1)(1-u_2)(1-u_3)}. \quad (3.22)
\end{align*} \]

\textbf{Proof.} Substitute in (3.5) formulas (3.11)–(3.13), (3.17)–(3.19), (3.21) and get a true equality. We note that methods were used to overcome the technical difficulties encountered in the proof. Denote

\[ A := 1 - (1 - \alpha_2 - \alpha_3)u_1 - \alpha_2u_2 - \alpha_3u_3 = 1 - u_1 + \alpha_2(u_1 - u_2) + \alpha_3(u_1 - u_3) = 1 - u_1 + A_1; \]

\[ B := 1 - \alpha_1u_1 - (1 - \alpha_1 - \alpha_3)u_2 - \alpha_3u_3 = 1 - u_2 - \alpha_1(u_1 - u_2) + \alpha_3(u_2 - u_3) = 1 - u_2 + B_1; \]

\[ C := 1 - \alpha_1u_1 - \alpha_2u_2 - (1 - \alpha_1 - \alpha_2)u_3 = 1 - u_3 - \alpha_1(u_1 - u_3) - \alpha_2(u_2 - u_3) = 1 - u_3 + C_1. \]

Then

\[ S_1 + S_2 + S_3 - (T_1 + T_2 + T_3) + R = \frac{M}{(1-u_1)(1-u_2)(1-u_3)ABC}; \]

where after some simple calculations we get

\[ M = ABC - A_1B_1C_1 + \]

\[ +((-\alpha_1 - \alpha_2 - \alpha_3)(\alpha_1(1-u_2)(1-u_3))A_1 + \alpha_2(1-u_1)(1-u_3)B_1 + \alpha_3(1-u_1)(1-u_2)C_1) + \]

\[ +(-\alpha_2 - \alpha_3)(1-u_1)B_1C_1 + (-\alpha_1 - \alpha_3)(1-u_2)A_1C_1 + (-\alpha_1 - \alpha_2)(1-u_3)A_1B_1. \]

It is easy to show that

\[ \alpha_1A_1 + \alpha_2B_1 + \alpha_3C_1 = 0. \]

Somewhat more complicated was to see that

\[ (\alpha_1 + \alpha_2 + \alpha_3)((\alpha_1u_2 + u_3)A_1 + \alpha_2(u_1 + u_3)B_1 + \alpha_3(u_1 + u_2)C_1) \]

\[ -(\alpha_2 + \alpha_3)B_1C_1 - (\alpha_1 + \alpha_3)A_1C_1 - (\alpha_1 + \alpha_2)A_1B_1 = \]

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\[
\alpha_1 \alpha_2 \alpha_3 (u_1 - u_2 + u_2 - u_3 + u_3 - u_1)^2 = 0.
\]

After that, representing
\[
A_1 = u_1 (\alpha_2 + \alpha_3) - \alpha_2 u_2 - \alpha_3 u_3 = pu_1 - \alpha_2 u_2 - \alpha_3 u_3;
\]
\[
B_1 = -\alpha_1 u_1 + u_2 (\alpha_1 + \alpha_3) - \alpha_3 u_3 = -\alpha_1 u_1 + qu_2 - \alpha_3 u_3;
\]
\[
C_1 = -\alpha_1 u_1 - \alpha_2 u_2 + u_3 (\alpha_1 + \alpha_2) = -\alpha_1 u_1 - \alpha_2 u_2 + ru_3
\]

after some calculations we find that
\[
(-\alpha_1 - \alpha_2 - \alpha_3)(\alpha_1 u_2 u_3 A_1 + \alpha_2 u_1 u_3 B_1 + \alpha_3 u_1 u_2 C_1) +
\]
\[
(\alpha_2 + \alpha_3)u_1 B_1 C_1 + (\alpha_1 + \alpha_3)u_2 A_1 C_1 + (\alpha_1 + \alpha_2)u_3 A_1 B_1 = A_1 B_1 C_1,
\]

which implies that
\[
M = ABC.
\]

**Remark 3.1** This lemma allows us to assert a further conclusions about our results that we did not only check hard combinatorial identities, but also demonstrated a new multiple combinatorial identity is from the theory of functions in \( \mathbb{C}^n \).

### 3.3 Proof of Theorems 3.1 and 3.2

**Theorem 3.3** The following formula is valid
\[
R + S_1(s_1, s_2, s_3; \alpha_2, \alpha_3) + S_1(s_2, s_1, s_3; \alpha_1, \alpha_3) + S_1(s_3, s_1, s_2; \alpha_1, \alpha_2) -
\]
\[-(T_1(s_1, s_2, s_3; \alpha_1, \alpha_2, \alpha_3) + T_1(s_1, s_3, s_2; \alpha_2, \alpha_3, \alpha_1) + T_1(s_1, s_3, s_2; \alpha_1, \alpha_3, \alpha_2)) = 1
\]

**Proof Theorem 3.1.** Using the generating function (3.22) we have
\[
\left( \sum_{i=1}^{3} S_i - \sum_{j=1}^{3} T_j + R \right) (\alpha) = \text{res}_{u_1 u_2 u_3} \frac{u_1^{-s_1} u_2^{-s_2} u_3^{-s_3}}{(1 - u_1)(1 - u_2)(1 - u_3)} =
\]
\[
= \text{res}_{u_1 u_2 u_3} \left( 1 + \sum_{j_1=1}^{\infty} u_1^{j_1} \right) \left( 1 + \sum_{j_2=1}^{\infty} u_2^{j_2} \right) \left( 1 + \sum_{j_3=1}^{\infty} u_3^{j_3} \right) u_1^{-s_1} u_2^{-s_2} u_3^{-s_3} = 1.
\]

\[\blacksquare\]
Theorem 3.4 If the complex parameters $z_1,\ldots,z_n$ satisfy the relation

$$z_1 + \ldots + z_n = 1,$$

then for any values $s_1,\ldots,s_n = 0,1,2,\ldots$ the following identity is valid

$$z_1^{s_1+1} \sum_{j_2=0}^{s_2} \ldots \sum_{j_n=0}^{s_n} \left( \frac{s_1 + \sum_{i \neq 1} j_i}{s_1,j_2,\ldots,j_n} \right) z_2^{j_2} \ldots z_n^{j_n} + \ldots +$$

$$+ z_n^{s_n+1} \sum_{j_1=0}^{s_1} \ldots \sum_{j_n-1=0}^{s_n-1} \left( \frac{s_n + \sum_{i \neq n} j_i}{s_n,j_1,\ldots,j_n-1} \right) z_1^{j_1} \ldots z_{n-1}^{j_{n-1}} = 1.$$

Proof Theorem 3.2 Denoting the sum on the left side (3.3) by $S_s(z) := S_{s_1\ldots s_n}(z_1,\ldots,z_n)$, we find in closed form the generating function for the sequence $\{S_s(z)\}_{s_i \geq 0}$

$$T(t,z) := \sum_{s_i \geq 0} S_s(z) t_1^{s_1} \ldots t_n^{s_n}. \quad (3.23)$$

To do this, using the coefficients we find the beginning of integral representation for each summand of the left side (3.3). For example,

$$R_1 := z_1^{s_1+1} \sum_{j_2=0}^{s_2} \ldots \sum_{j_n=0}^{s_n} \left( \frac{s_1 + \sum_{i \neq 1} j_i}{s_1,j_2,\ldots,j_n} \right) z_2^{j_2} \ldots z_n^{j_n} =$$

$$= z_1^{s_1+1} \sum_{j_2=0}^{s_2} \ldots \sum_{j_n=0}^{s_n} \left( \frac{s_1 + \sum_{i \neq 1} (s_i - j_i)}{s_1,s_2-j_2,\ldots,s_n-j_n} \right) z_2^{s_2-j_2} \ldots z_n^{s_n-j_n} =$$

$$= z_1 \sum_{j_2=0}^\infty \ldots \sum_{j_n=0}^\infty \text{res}_{x_1=\ldots=x_n} \frac{(1 - \sum_i z_i x_i)^{-1}}{x_1^{s_1+1} x_2^{s_2-j_2+1} \ldots x_n^{s_n-j_n+1}} =$$

(summation over the indices $j_2,\ldots,j_n$ by the formula of the sum of a geometric progression)

$$= z_1 \text{res}_{x_1=\ldots=x_n} \frac{1}{1 - \sum_i z_i x_i} \prod_{i \neq 1} (1 - x_i)^{-1} \times \prod_i x_i^{-s_i-1}. \quad (3.24)$$

Then

$$P_1(t) := \sum_{s_i \geq 0} R_1 t_1^{s_1} \ldots t_n^{s_n} =$$

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\[
\sum_{s_i \geq 0} \frac{1}{1 - \sum_i z_i x_i} \prod_{i \neq 1} (1 - x_i)^{-1} \times \prod_i x_i^{-s_i-1} t_i^{s_i} =
\]

(summation over the indices \(s_1, \ldots, s_n\) : rule of substitution, replace \(x_1 = t_1, \ldots, x_n = t_n\))

\[
= \frac{z_1}{1 - \sum_i z_i t_i} \prod_{i \neq 1} (1 - t_i)^{-1}. \tag{3.25}
\]

Thus

\[
T(t) = \frac{\{z_1(1 - t_1) + \ldots + z_n(1 - t_n)\}}{1 - \sum_i z_i t_i} \prod_i (1 - t_i)^{-1}
\]

\[
= \frac{\sum_i z_i - \sum_i z_i t_i}{1 - \sum_i t_i z_i} \prod_i (1 - t_i)^{-1} \tag{considering (3.2)}
\]

\[
= \frac{1 - \sum_i z_i t_i}{1 - \sum_i t_i z_i} \prod_i (1 - t_i)^{-1} = \prod_i (1 - t_i)^{-1}. \tag{3.26}
\]

Thus, in accordance with (3.3) and (3.26) we have for all \(s_i \geq 0, i = 1, \ldots, n,\)

\[
S_s(z) := \text{res}_{t_1 \ldots t_n} T(t, z) t_1^{-s_1-1} \ldots t_n^{-s_n-1} = \text{res}_{t_1 \ldots t_n} \prod_i (1 - t_i)^{-1} t_i^{-s_i-1} =
\]

\[
= \text{res}_{t_1 \ldots t_n} \left( 1 + \sum_{j_1=1}^\infty t_1^{j_1} \right) \ldots \left( 1 + \sum_{j_n=1}^\infty t_n^{j_n} \right) \prod t_i^{j_i} t_i^{-s_i-1} = 1. \]

**Remark 3.2** The Theorem 3.1 is a special case of the Theorem 3.2 for \(n = 3.\)

### 3.4 The new short calculation of Zeilberger and Krivokolesko combinatorial sums and its some applications

In [40] a theory of integral representations of holomorphic functions in a linearly convex domains \(D \subset \mathbb{C}^n\) with a piecewise regular boundary, which allowed V.P. Krivokolesko in [41] to find a series of combinatorial identities for a certain family of integral parameters of \(D.\) For example,

\[
\alpha_1^{s_1+1} \sum_{k=0}^{s_2} \sum_{l=0}^{s_3} \frac{(s_1 + k + l)!}{s_1! k!! l!!} \alpha_2^{k} \alpha_3^{l} + \alpha_2^{s_2+1} \sum_{k=0}^{s_1} \sum_{l=0}^{s_3} \frac{(s_2 + k + l)!}{s_2! k!! l!!} \alpha_1^{k} \alpha_3^{l} +
\]

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\[ + \alpha_3 s_3 + 1 \sum_{k=0}^{s_2} \sum_{l=0}^{s_1} \frac{(s_3 + k + l)!}{s_3! k! l!} \alpha_1^k \alpha_2^l = 1, \tag{3.27} \]

where numeric parameters \( \alpha_1, \alpha_2, \alpha_3 \) satisfy equality \( \alpha_1 + \alpha_2 + \alpha_3 = 1 \). V.P. Krivokolesko raised the question of finding a simple proof of the following identity (3.28), which generalizes the identity (3.27):

\[
\begin{align*}
&z_1^{s_1 + 1} \sum_{j_2=0}^{s_2} \ldots \sum_{j_n=0}^{s_n} \left( s_1 + \sum_{i \neq 1} j_i \right) \frac{z_2^{j_2} \ldots z_n^{j_n}}{s_1, j_2, \ldots, j_n} + \\
&+ z_n^{s_n + 1} \sum_{j_1=0}^{s_1} \ldots \sum_{j_{n-1}=0}^{s_{n-1}} \left( s_n + \sum_{i \neq n} j_i \right) \frac{z_1^{j_1} \ldots z_{n-1}^{j_{n-1}}}{s_n, j_1, \ldots, j_{n-1}} = 1, \tag{3.28} \end{align*}
\]

for all values of the parameters \( s_1, \ldots, s_n = 0, 1, 2, \ldots \), where the numerical parameters \( z_1, \ldots, z_n \) satisfy the equation

\[ z_1 + \ldots + z_n = 1. \tag{3.29} \]

Particular cases (3.28) is the identity of V.L. Shelkovich in quantum field theory [43] and probability of identity in the theory of D. Zeilberger dilations (the wavelet theory) [44].

Here in Theorem 1 using the coefficients we found elegant formula integral representation (generating function) for the sum of (3.30), which is a natural generalization of the sum in the left-hand side of (3.28). It is possible to find a short analytic proof of the desired identity (3.28) (Lemma 3.9). Moreover found some interesting recurrence relations for the calculation of multiple combinatorial sums of different types (Lemma 3.10 and 3.11).

**Theorem 3.5** Let

\[
S_s(z; \alpha) := z_1^{s_1 + 1} \sum_{j_2=0}^{s_2} \ldots \sum_{j_n=0}^{s_n} \left( \alpha + s_1 + \sum_{i \neq 1} j_i \right) \frac{z_2^{j_2} \ldots z_n^{j_n}}{s_1, j_2, \ldots, j_n} + \\
+ z_n^{s_n + 1} \sum_{j_1=0}^{s_1} \ldots \sum_{j_{n-1}=0}^{s_{n-1}} \left( \alpha + s_n + \sum_{i \neq n} j_i \right) \frac{z_1^{j_1} \ldots z_{n-1}^{j_{n-1}}}{s_n, j_1, \ldots, j_{n-1}}, \tag{3.30} \]

where the real parameter \( \alpha \geq 0 \) and the complex parameters \( z_1, \ldots, z_n \) satisfy the condition. (3.29) Then the generating function

\[
T_\alpha(t) := \sum_{s_i \geq 0} S_s(z; \alpha) t_1^{s_1} \ldots t_n^{s_n} \tag{3.31} \]

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for the sequence \( \{S_s(z;\alpha)\}_{s \geq 0} \) has the form

\[
T_\alpha(t) = (1 - \sum_{i} z_i t_i)^{-\alpha} \prod_{i}(1 - t_i)^{-1}.
\]  

(3.32)

**Proof.** Following the scheme of computation method of integral first find an integral representation for each summand \( R_k(z), k = 1, \ldots, n \) in (3.30). For example,

\[
R_1(z) := z^s_1 \sum_{j_2=0}^{s_2} \cdots \sum_{j_n=0}^{s_n} \left( \alpha + s_1 + \sum_{i \neq 1} j_i \right) z^{j_2} \cdots z^{j_n} =
\]

\[
= z^s_1 \sum_{j_2=0}^{s_2} \cdots \sum_{j_n=0}^{s_n} \left( \alpha + s_1 + \sum_{i \neq 1} (s_i - j_i) \right) z^{j_2} \cdots z^{j_n} =
\]

\[
= z_1 \sum_{j_2=0}^{s_2} \cdots \sum_{j_n=0}^{s_n} R_{i,j_2,\ldots,j_n} \left( \frac{1 - \sum_i z_i x_i)^{-\alpha-1}}{x_1^{-s_1} x_2^{s_2-j_2+1} \cdots x_n^{s_n-j_n+1}} \right) =
\]

(the rule linearly \( \text{res}_x \) : entry under the sign of the sum sign \( \text{res}_x \))

\[
= z_1 \text{res}_x \left\{ \sum_{j_2=0}^{\infty} \cdots \sum_{j_n=0}^{\infty} \frac{1 - \sum_i z_i x_i)^{-\alpha-1}}{x_1^{-s_1} x_2^{s_2-j_2+1} \cdots x_n^{s_n-j_n+1}} \right\} =
\]

(summation over indices \( j_2, \ldots, j_n \) formula of the sum of a geometric progression)

\[
= z_1 \text{res}_x (1 - \sum_i z_i x_i)^{-\alpha-1} \prod_{i \neq 1} (1 - x_i)^{-1} \prod_{i} x_i^{-s_i-1}.
\]

Thence

\[
P_1(t, z) := \sum_{s_1 \geq 0} R_1(z) t^{s_1} \cdots t^{s_n} =
\]

\[
= z_1 \sum_{s_1 \geq 0} \text{res}_x (1 - \sum_i z_i x_i)^{-\alpha-1} \prod_{i \neq 1} (1 - x_i)^{-1} \prod_{i} x_i^{-s_i-1} t^{s_1} =
\]

(summation over indices \( s_1, \ldots, s_n \): rule of substitution for the operator \( \text{res}_x \) change \( x_i = t_i, i = 1, \ldots, n \))

\[
= z_1 (1 - \sum_i z_i t_i)^{-\alpha-1} \prod_{i \neq 1} (1 - t_i)^{-1} = \frac{z_1 (1 - t_1)}{1 - \sum_i z_i t_i} \prod_{i} (1 - t_i)^{-1}.
\]

Thus,

\[
T(t, z) := \sum_{i} P_i(t, z) = \frac{z_1(1 - t_1) + \ldots + z_n(1 - t_n)}{(1 - \sum_i z_i t_i)^{\alpha+1}} \prod_{i} (1 - t_i)^{-1} =
\]

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\[
\sum_i z_i - \sum_i z_i t_i \prod_i (1 - t_i)^{-1},
\]
and subject to (3.29) we obtain the required formula (3.32):

\[
T_\alpha(t) := [T(t, z)]_{z_1 + \ldots + z_n = 1} = \frac{1 - \sum_i z_i t_i}{(1 - \sum_i z_i t_i)^{\alpha+1}} \prod_i (1 - t_i)^{-1} = \\
= (1 - \sum_i z_i t_i)^{-\alpha} \prod_i (1 - t_i)^{-1}.
\]

Lemma 3.9 The identity (3.28) is valid.

Proof. If we denote the left-hand side of the identity (3.28) after \(S_s(z)\), then

\[
S_s(z) := \text{res}_t T_0(t) t_1^{-s_1-1} \ldots t_n^{-s_n-1} = \text{res}_t \prod_i (1 - t_i)^{-1} t_i^{-s_i-1} = \\
= \text{res}_t (1 + \sum_{j_1=1}^\infty t_1^{j_1}) \ldots (1 + \sum_{j_n=1}^\infty t_n^{j_n}) \prod_i t_i^{-s_i-1} = 1.
\]

Lemma 3.10 The following identity is valid for \(S_s(z; t)\):

\[
S_s(z; \alpha + 1) - z_1 S_{s_1-1, s_2, \ldots, s_n}(z; \alpha + 1) - \ldots - z_n S_{s_1, s_2, \ldots, s_n-1}(z; \alpha + 1) = \\
= S_s(z; \alpha), \quad \alpha = 0, 1, 2, \ldots \tag{3.34}
\]

In particular, (3.34) for \(\alpha = 0\) by 3.9 and relation \(S_s(z; 0) := S_s(z) = 1\) we have the following recursive formula:

\[
S_s(z; 1) = 1 + z_1 S_{s_1-1, s_2, \ldots, s_n}(z; 1) + \ldots + z_n S_{s_1, s_2, \ldots, s_n-1}(z; 1). \tag{3.35}
\]

Proof. According to (3.31) and (3.32) for any \(s_i = 0, 1, 2, \ldots\) we have

\[
S_s(z; \alpha) := \text{res}_t T_\alpha(t) \left( \prod_i t_i^{-s_i-1} \right) = \\
= \text{res}_{t_1, \ldots, t_n} (1 - \sum_i z_i t_i)^{-\alpha} \prod_i (1 - t_i)^{-1} t_i^{-s_i-1} = \\
= \text{res}_{t_1, \ldots, t_n} \left( (1 - \sum_i z_i t_i)^{-\alpha-1} \prod_i (1 - t_i)^{-1} \right) (1 - \sum_i z_i t_i) (\prod_i t_i^{-s_i-1}) =
\]
\[
\begin{align*}
= \text{res}_{t_1, \ldots, t_n} T_{\alpha+1}(t) (1 - \sum_i z_i t_i) (\prod_i t_i^{-s_i-1}) = \\
(by \ linearity \ of \ the \ operator \ \text{res}_t) \\
= \text{res}_{t_1, \ldots, t_n} T_{\alpha+1}(t) \prod_i t_i^{-s_i-1} - z_1 \text{res}_{t_1, \ldots, t_n} T_{\alpha+1}(t) t_1^{-(s_1-1)-1} (\prod_{i \neq 1} t_i^{-s_i-1}) - \ldots - \
-z_n \text{res}_{t_1, \ldots, t_n} T_{\alpha+1}(t) t_n^{-(s_n-1)-1} (\prod_{i \neq n} t_i^{-s_i-1}) = \\
(\text{according \ to \ the \ definition} \ (3.36)) \\
= S_s(z; \alpha + 1) - z_1 S_{s_1-1, s_2, \ldots, s_n}(z; \alpha + 1) - \ldots - z_1 S_{s_1, s_2, \ldots, s_n-1}(z; \alpha + 1).
\end{align*}
\]

**Lemma 3.11** Let \( S_s(z; \alpha, \beta) \) be the numbers determined by the formula (3.30), where the parameter \( \alpha \geq 0 \), and the complex parameters \( z_1, \ldots, z_n, \beta \) satisfy the condition
\[
z_1 + \ldots + z_n = \beta. \tag{3.37}
\]
Obviously,
\[
S_s(z; \alpha, 1) := S_s(z; \alpha), \quad S_s(z; 0, 1) := S_s(z), \tag{3.38}
\]
\[
S_s(z_1, \ldots, z_n; \alpha, \beta) = S_s(z_1/\beta, \ldots, z_n/\beta; \alpha), \quad \text{if} \ \beta \neq 0. \tag{3.39}
\]
Then the generating function
\[
T_{\alpha, \beta}(t) := \sum_{s_i \geq 0} S_s(z; \alpha, \beta) t_1^{s_1} \ldots t_n^{s_n} \tag{3.40}
\]
for the sequence \( \{S_s(z; \alpha, \beta)\}_{s_i \geq 0} \) has the form
\[
T_{\alpha, \beta}(t) = (\beta - \sum_i z_i t_i) (1 - \sum_i z_i t_i)^{-\alpha - 1} \prod_i (1 - t_i)^{-1}, \tag{3.41}
\]
and the following formula for the number of \( S_s(z; \alpha, \beta) \):
\[
S_s(z; \alpha, \beta) = (\beta - 1) S_s(z; \alpha + 1, 1) + S_s(z; \alpha, 1), \quad \alpha = 1, 2, \ldots. \tag{3.42}
\]
In other words, if \( \beta \notin \{0, 1\} \), then in view of (3.38) and (3.39) the following recursive formula for the number of \( S_s(z; \alpha) \) is valid:
\[
S_s(\beta z; \alpha + 1) = \frac{1}{\beta - 1} (S_s(z; \alpha) - S_s(\beta z; \alpha)), \quad \alpha = 1, 2, \ldots. \tag{3.43}
\]
Proof. The formula (3.41) immediately from (3.33) using the equation (3.37). Thus, we have
\[ T_{\alpha,\beta}(t) = (\beta - \sum_i z_i t_i)(1 - \sum_i z_i t_i)^{-\alpha - 1} \prod_i (1 - t_i)^{-1} = \]
\[ = (\beta - 1)(1 - \sum_i z_i t_i)^{-\alpha - 1} \prod_i (1 - t_i)^{-1} + (1 - \sum_i z_i t_i)^{-\alpha} \prod_i (1 - t_i)^{-1} := \]
(by formula (3.32))
\[ = (\beta - 1)T_{\alpha+1}(t) + T_{\alpha}(t). \] (3.44)
Equating the coefficients of the monomials \( t_{s_1} \ldots t_{s_n} \) in the right and left sides of (3.44), we obtain (3.42). ■

4 Integral representation and computation of a multiple sum in the theory of cubature formulas

Let \( \alpha = (\alpha_0, \alpha_1, \ldots, \alpha_d) \), \( \beta = (\beta_0, \beta_1, \ldots, \beta_d) \) be vectors from \( E^{d+1} \) with integer non-negative coordinates, and the vector \( \gamma = (\gamma_0, \gamma_1, \ldots, \gamma_d) \in E^{d+1} \). Denote
\[ |\alpha| := \alpha_0 + \alpha_1 + \ldots + \alpha_d = 2s + 1, \quad \alpha! := \alpha_0! \alpha_1! \ldots \alpha_d!, \quad \binom{\alpha}{\beta} := \binom{\alpha_0}{\beta_0} \ldots \binom{\alpha_d}{\beta_d}, \]
where
\[ \binom{a}{b} := \frac{\Gamma(a + 1)}{\Gamma(b + 1) \Gamma(a - b + 1)}, \text{ and } \binom{a}{b} := 0, \text{ if } b \geq a + 1. \]

Moreover, we write \( \alpha - 1/2 := (\alpha_0 - 1/2, \alpha_1 - 1/2, \ldots, \alpha_d - 1/2) \).

Heo S. and Xu Y. ([46], the identity (2.9), p. 631–635) with the help of the method of generating functions and the difference-differential operators of various type, found not simple proof the following multiple combinatorial identity ([46], identity (2.9)):
\[ 2^{2s} \alpha! \binom{\alpha + \gamma}{\alpha} = \sum_{j=0}^{s} (-1)^j \binom{d + \sum_{i=0}^{d} (\alpha_i + \gamma_i)}{j} \times \]
\[ \times \sum_{\beta_0 + \beta_1 + \ldots + \beta_d = s - j} \prod_{i=0}^{d} \binom{\beta_i + \gamma_i}{\beta_i} (2\beta_i + \gamma_i + 1)^{\alpha_i}. \] (4.1)

The purpose of the given section is finding a new simple proof of identity (4.1) by means of the method of coefficients and multiple applying of known theorem on the total sum of residues.
4.1 The proof of the identity (4.1)

It is possible to copy the identity (4.1) in the form of

\[
\sum_{j=0}^{s} (-1)^j \left( d + \sum_{i=0}^{d} (\alpha_i + \gamma_i) \right) \times
\]

\[
\sum_{\beta_0 + \beta_1 + \ldots + \beta_d = s - j} \prod_{i=0}^{d} \left( \beta_i + \gamma_i \right) \frac{(2\beta_i + \gamma_i + 1)^{\alpha_i}}{\alpha_i!} = 2^s \prod_{i=0}^{d} \left( \alpha_i + \gamma_i \right). \tag{4.2}
\]

Let’s enter use following designations for the right part of identity (4.2):

\[
T(s; \alpha, \beta) := \sum_{j=0}^{s} (-1)^j \left( \sum_{i=0}^{d} (\alpha_i + \gamma_i) + d \right) \cdot S_j, \tag{4.3}
\]

where

\[
S_j := \sum_{\beta_0 + \beta_1 + \ldots + \beta_d = s - j} \prod_{i=0}^{d} \left( \frac{\beta_i + \gamma_i}{\beta_i} \right) \frac{(2\beta_i + \gamma_i + 1)^{\alpha_i}}{\alpha_i!}. \tag{4.4}
\]

Then by means of the method of coefficients we receive

\[
S_j = \sum_{\beta_0 + \beta_1 + \ldots + \beta_d = s - j} \prod_{i=0}^{d} \left( \frac{\beta_i + \gamma_i}{\beta_i} \right) \frac{(2\beta_i + \gamma_i + 1)^{\alpha_i}}{\alpha_i!} = \\
= \sum_{\beta_0=0}^{\infty} \ldots \sum_{\beta_d=0}^{\infty} \text{res}_{z_0, \ldots, z_d, t} (t^{-s+j-1} \prod_{i=0}^{d} (1 - tz_i)^{-\gamma_i-1} z_i^{-\beta_i-1}) \times \\
\times \text{res}_{w_0, \ldots, w_d} \left( \prod_{i=0}^{d} w_i^{-\alpha_i-1} \exp(w_i(2\beta_i + \gamma_i + 1)) \right) = \\
= \text{res}_{w_0, \ldots, w_d} \left\{ t^{-s+j-1} \prod_{i=0}^{d} w_i^{-\alpha_i-1} \exp(w_i(\gamma_i + 1)) \times \\
\times \prod_{i=0}^{d} \left( \sum_{\beta_i=0}^{\infty} \exp(\beta_i(2w_i)) \text{res}_{z_i} (1 - tz_i)^{-\gamma_i-1} z_i^{-\beta_i-1} \right) \right\} = \\
\text{(the summation by each } \beta_i, \text{ and res}_{z_i} i = 0, \ldots, d : \text{)}
\]

the substitution of the rule of changes \( z_i = \exp(2w_i), i = 0, \ldots, d \)
According to (4.3)–(4.5) we received

\[ \text{res}_{\mathbf{w}_0, \ldots, \mathbf{w}_d, t} \{ t^{-s+j-1} \prod_{i=0}^{d} w_i^{-\alpha_i} \exp(w_i(\gamma_i + 1)) \cdot \prod_{i=0}^{d} (1 - t \exp(2w_i))^{-\gamma_i-1} \} = \]

\[ \text{res}_{\mathbf{w}_0, \ldots, \mathbf{w}_d, t} \{ t^{-s+j-1} \prod_{i=0}^{d} w_i^{-\alpha_i} (\exp(-w_i) - t \exp(w_i))^{-\gamma_i-1} \}, \]

i.e.

\[ S_j = \text{res}_{\mathbf{w}_0, \ldots, \mathbf{w}_d, t} \{ t^{-s+j-1} \prod_{i=0}^{d} w_i^{-\alpha_i} (\exp(-w_i) - t \exp(w_i))^{-\gamma_i-1} \}. \quad (4.5) \]

According to (4.3)–(4.5) we received

\[ T(s; \alpha, \beta) = \sum_{j=0}^{s} \text{res}_{\mathbf{w}_0, \ldots, \mathbf{w}_d, t} \{ t^{-s+j-1} \prod_{i=0}^{d} w_i^{-\alpha_i} (\exp(-w_i) - t \exp(w_i))^{-\gamma_i-1} \} \times \]

\[ \times \text{res}_x \{ x^{-j-1}(1 - x)^{d+\sum_{i=0}^{d}(\alpha_i+\gamma_i)} \} = \]

\[ \text{res}_{\mathbf{w}_0, \ldots, \mathbf{w}_d, t} \{ t^{-s-1} \prod_{i=0}^{d} w_i^{-\alpha_i} (\exp(-w_i) - t \exp(w_i))^{-\gamma_i-1} \times \]

\[ \times \left( \sum_{j=0}^{\infty} t^j \text{res}_x \{ x^{-j-1}(1 - x)^{d+\sum_{i=0}^{d}(\alpha_i+\gamma_i)} \} \right) \]

(\text{the summation by } j, \text{ and } \text{res}_x : \text{the substitution rule, the change } x = t)

\[ = \text{res}_{\mathbf{w}_0, \ldots, \mathbf{w}_d, t} \{ t^{-s-1} \prod_{i=0}^{d} w_i^{-\alpha_i} (\exp(-w_i) - t \exp(w_i))^{-\gamma_i-1}(1-t)^{d+\sum_{i=0}^{d}(\alpha_i+\gamma_i)} \}. \]

Thus we proved

Lemma 4.1 Let parameters \( s, \alpha_0, \alpha_1, \ldots, \alpha_d, \beta_0, \beta_1, \ldots, \beta_d \) be the integer non-negative numbers, for which \( \alpha_0 + \cdots + \alpha_d = 2s+1 \), and the vector \((\mu_0, \mu_1, \ldots, \mu_d) \in \mathbb{R}^{d+1}\). Then the following integral formula is valid:

\[ \sum_{j=0}^{s} (-1)^j \left( d + \sum_{i=0}^{d} (\alpha_i + \gamma_i) \right) \sum_{\beta_0 + \beta_1 + \cdots + \beta_d = s-j} \prod_{i=0}^{d} \frac{(\beta_i + \gamma_i + 1)^{\alpha_i}}{(\alpha_i)!} = \]

\[ = \text{res}_{\mathbf{w}_0, \ldots, \mathbf{w}_d, t} \left\{ t^{-s-1} \prod_{i=0}^{d} w_i^{-\alpha_i} (\exp(-w_i) - t \exp(w_i))^{-\gamma_i-1} \right. \times \]

\[ \left. \times (1-t)^{d+\sum_{i=0}^{d}(\alpha_i+\gamma_i)} \right\}. \quad (4.6) \]
Remark 4.1. It is easy to see, that a calculation of multiple integral in the right part (4.6) on variables $t, w_0, \ldots, w_d$ sequentially gives the multiple sum of the left part (4.6).

We will spend a new proof of identity (4.2) similarly by calculation of multiple residue of a zero point in the right part of the formula (4.6) it is on each variable $w_0, \ldots, w_d$ and $t$ sequentially (see lemmas 4.1–4.3 and the theorem 4.1).

Let’s enter necessary designations. Denote

$$f = f(w, t) := e^{-w} - te^w, \quad g = g(w, t) := e^{-w} + te^w.$$  \hfill (4.7)

where $\alpha$ is the fixed integer and $\gamma \in \mathbb{R}$. Obviously

$$f' := \frac{df}{dw} = -g, \quad g' := \frac{dg}{dw} = -f,$$

$$g^2 - f^2 = 4t, \quad (f - \gamma)^' = \gamma f - \gamma - 1g.$$  \hfill (4.8)

$$\left(g^\alpha\right)^' = -\alpha g^{\alpha-1}f, \quad f(0) = 1 - t, \quad g(0) = 1 + t. \hfill (4.9)$$

Lemma 4.2 If $s$ is the integer non-negative number and $\gamma \in \mathbb{R}$, in designations (4.7) and (4.8) the following expansion is a derivative

$$\left(f^{-\gamma-1}\right)^{(\alpha)} = (\gamma + 1) \cdot \ldots \cdot (\gamma + \alpha) f^{-\gamma-\alpha-1}g^\alpha + \sum_{k=1}^{[\alpha/2]} c_k(\gamma) f^{-\gamma+2k-\alpha-1}g^{\alpha-2k}, \hfill (4.10)$$

with integer coefficients $c_1, c_2, \ldots, c_{[\alpha/2]}$ is valid.

According (4.9) the formula (4.10) generates the following formula

$$\text{res}_{w_i} w_i^{-\alpha-1}(\exp(-w_i) - t \exp(w_i))^{-\gamma-1}:=[(\exp(-w_i) - t \exp(w_i))^{-\gamma-1}]_{w=0}^{w=\alpha} =$$

$$= \left(\frac{\alpha + \gamma}{\alpha}\right)(1-t)^{-\gamma-\alpha-1}(1+t)^\alpha(1 + \sum_{k=1}^{[\alpha/2]} h_k(\alpha, \gamma)(1-t)^{2k}(1+t)^{-2k}), \hfill (4.11)$$

where the rational coefficients $h_k(\alpha, \gamma) := c_k(\gamma)/\alpha!$, $k = 1, \ldots, [\alpha/2]$.

Proof. The formula (4.10) most easier to prove an induction on parameter $\alpha$. Really with the help (4.8) we have for initial values $\alpha = 1, 2, 3$:

$$(f^{-\gamma-1})' = -(\gamma + 1)f^{-\gamma-2}f' = (\gamma + 1)f^{-\gamma-2}g.$$  

$$(f^{-\gamma-1})'' = ((f^{-\gamma-1})')' = ((\gamma + 1)f^{-\gamma-2}g)' = (\gamma + 1)(f^{-\gamma-2})'g + (\gamma + 1)f^{-\gamma-2}f =$$
\[ f^{-\gamma -1} = (\gamma + 1)(\gamma + 2)f^{-\gamma -3}g^2 - (\gamma + 1)f^{-\gamma -1}. \]

\[ (f^{-\gamma -1})'' = ((f^{-\gamma -1})')' = (\gamma + 1)(\gamma + 2)f^{-\gamma -3}g^2 - (\gamma + 1)f^{-\gamma -1} \]

Further on an induction if the formula \((4.10)\) is valid for current value \(\alpha\), with the help \((4.8)\) we have

\[ (f^{-\gamma -1}) = (\gamma + 1)(\gamma + 2)(\gamma + 3)f^{-\gamma -4}g^3 + \]

\[ + (\gamma + 1)(\gamma + 2)f^{-\gamma -3}2gf - (\gamma + 1)^2f^{-\gamma -2}g = \]

\[ = (\gamma + 1)(\gamma + 2)(\gamma + 3)f^{-\gamma -4}g^3 - (\gamma + 1)(3\gamma + 2)f^{-\gamma -2}g. \]

as was shown. 

**Lemma 4.3** If \(s\) is the integer non-negative number then the following formulas are valid:

\[ J = \text{res}_t(1 - t)^{-1}(1 + t)^{2s+1}t^{-s-1} = 2^{2s}, \quad (4.12) \]

\[ J_k = \text{res}_t(1 - t)^{k-1}(1 + t)^{2s-k+1}t^{-s-1} = 0, \quad \forall k = 1, \ldots, 2s. \quad (4.13) \]
Proof. We have

$$J = \text{res}_{t}(1 - t)^{-1}(1 + t)^{2s+1}t^{-s-1} = \text{res}_{t=0}(1 - t)^{-1}(1 + t)^{2s+1}t^{-s-1} =$$

(\text{the theorem of the full sum of residuís})

$$= -\text{res}_{t=1}(1 - t)^{-1}(1 + t)^{2s+1}t^{-s-1} - \text{res}_{t=\infty}(1 - t)^{-1}(1 + t)^{2s+1}t^{-s-1} =$$

(directly by definition of a deduction in a corresponding point)

$$= [(1 + t)^{2s+1}t^{-s-1}]_{t=1} - \text{res}_{t=0}(1 - 1/t)^{-1}(1 + 1/t)^{2s+1}(1/t)^{-s}(-1/t)^2 =$$

$$= 2^{2s+1} - \text{res}_{t=0}(1 - 1)^{2s+1}t^{-s} = 2^{2s+1} - J \iff J = 2^{2s+1} - J \Rightarrow J = 2^s.$$

Let \(k\) be the any fixed number from set \(\{1, \ldots, 2s\}\). Acting just as in the previous case, we have

$$J_k = \text{res}_{t}(1 - t)^{k-1}(1 + t)^{2s-k+1}t^{-s-1} =$$

$$\text{res}_{t=0}(1 - t)^{k-1}(1 + t)^{2s-k+1}t^{-s-1} - \text{res}_{t=\infty}(1 - t)^{k-1}(1 + t)^{2s+1}t^{-s-1} =$$

( as powers of binomials \((1 - t)^{k-1}\) and \((1 + t)^{2s-k+1}t^{-s-1}\) non negative at any \(k\) from set \(\{1, \ldots, 2s\}\))

$$= 0 - \text{res}_{t=0}(1 - 1/t)^{k-1}(1 + 1/t)^{5}(1/t)^{-3}(-1/t)^2 =$$

$$= -\text{res}_{t=0}(1 - t)^{-s-1}(1 + t)^{2s-k+1}t^{-s-1} = -J_k \iff J_k = -J_k \iff J_k = 0.$$

\(\blacksquare\)

**Theorem 4.1** The identity \((4.2)\) is valid.

**Proof.** By \((4.6)\) we have following integrated representation for sum \(T(s; \alpha, \beta)\) in the left part of identity \((4.2)\):

$$T(s; \alpha, \beta) = \text{res}_{w_0, \ldots, w_d,t}\left\{ \left(\frac{(1 - t)^{d+\sum_{i=0}^{d}(\alpha_i+\gamma_i)}}{t^{s+1}} \prod_{i=0}^{d} \left(\frac{\exp(-w_i) - t \exp(w_i))^{-\mu_i}}{w_i^{\alpha_i+1}} \right) \right\} =$$

$$= \text{res}_{t}\{t^{-s-1}(1 - t)^{d+\sum_{i=0}^{d}(\alpha_i+\gamma_i)}(\prod_{i=0}^{d} \text{res}_{w_i} w_i^{-\alpha_i-1} (\exp(-w_i) - t \exp(w_i))^{-\mu_i-1})\}.$$

Calculating in last expression each of deductions on variables \(w_0, \ldots, w_d\) by the formula \((4.11)\) we have

$$T(s; \alpha, \beta) = \text{res}_{t}\{t^{-s-1}(1 - t)^{d+\sum_{i=0}^{d}(\alpha_i+\gamma_i)} \times$$

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\[ \times \prod_{i=0}^{d} \left( \frac{\alpha_i + \gamma_i}{\alpha_i} \right) (1 - t)^{-\gamma_i - \alpha_i - 1} (1 + t)^{\alpha_i} \left( \sum_{k=1}^{[\alpha_i/2]} h_k(\alpha_i, \gamma_i) (1 - t)^{2k} (1 + t)^{-2k} \right) \]

(4.14)

(trivial reductions in a sing of product

\[ \prod_{i=0}^{d} \ldots \text{in according of the assumption } \sum_{i=0}^{d} \alpha_i = 2s + 1 \]

\[ = \left( \frac{\alpha + \gamma}{\alpha} \right) \text{res}_t \left\{ \frac{(1 + t)^{2s+1}}{t^{s+1}(1 - t)} \prod_{i=0}^{d} (1 + \sum_{k=1}^{[\alpha_i/2]} h_k(\alpha_i, \gamma_i) (1 - t)^{2k} (1 + t)^{-2k}) \right\} . \]

As \([\alpha_0/2] + [\alpha_1/2] + \ldots + [\alpha_d/2] \leq \left[ \sum_{i=0}^{d} \alpha_i/2 \right] = s\), it is easy, that after disclosing brackets and reduction of similar members of product

\[ \prod_{i=0}^{d} (1 + \sum_{k=1}^{[\alpha_i/2]} h_k(\alpha_i, \gamma_i) (1 - t)^{2k} (1 + t)^{-2k}) , \]

in a sign of \(\text{res}_t\) in (4.14) is representable in the form of a multinominal of a kind

\[ 1 + \sum_{k=2}^{2s} \lambda_k (1 - t)^k (1 + t)^{-k} , \]

where coefficients \(\lambda_1, \ldots, \lambda_{2s-1}\) are some fixed rational numbers. Thus

\[ T(s; \alpha, \beta) = \left( \frac{\alpha + \gamma}{\alpha} \right) \text{res}_t \left\{ \frac{(1 + t)^{2s+1}}{t^{s+1}(1 - t)} \left( 1 + \sum_{k=1}^{2s} \lambda_k (1 - t)^k (1 + t)^{-k} \right) \right\} = \]

\[ = \left( \frac{\alpha + \gamma}{\alpha} \right) \left( \text{res}_t \left( \frac{(1 + t)^{2s+1}}{(1 - t)t^{s+1}} + \sum_{k=1}^{2s} \lambda_k \text{res}_t \left( \frac{(1 - t)^{k-1} (1 + t)^{2s-k+1}}{t^{s+1}} \right) \right) \right) = \]

(calculation of residius in last expression under formulas (4.12) and (4.13))

\[ = \left( \frac{\alpha + \gamma}{\alpha} \right) \left\{ 2^{2s} + \sum_{k=1}^{2s} \lambda_k \times 0 \right\} = \left( \frac{\alpha + \gamma}{\alpha} \right) 2^{2s} . \]
Remark 4.2 It is to our interest to answer this question

\[
J = \text{res}_{w_0, \ldots, w_d, t} \left\{ t^{s-1} \prod_{i=0}^{d} \frac{w_i^{-\alpha_i-1} (\exp(-w_i) - t \exp(w_i))^{-\mu_i-1}}{(1-t)^{d+\sum_{i=0}^{d} (\alpha_i + \gamma_i)}} / \alpha! \right\},
\]

in the left hand of initial identity (4.1). For example, the integral (4.15) can be resulted to in the following kind

\[
J = \text{res}_t \left\{ (t^{s-1} \prod_{i=0}^{d} \frac{\text{res}_{w_i} w_i^{-\alpha_i-1} (\exp(-\lambda_i w_i) - t \exp(\lambda_i w_i))^{-\gamma_i-1}}{\alpha!} \right\} / \alpha!.
\]

The calculation of integral (4.16) is connected with studying hyperbolic t-sine

\[
\sinh_t(x) := \frac{\exp(-x) - t \exp(x)}{2},
\]

and the functions \( \sinh_t^{-\gamma}(x), \gamma \in \mathbb{N}, \) and

\[
J_{a,\gamma}(t) := \text{res}_z (z^{-\alpha-1} (\exp(-z) - t \exp(z))^{-\gamma-1}) / \alpha! = \text{res}_z (z^{-\alpha-1} (\exp(-z) - t \exp(z))^{-\gamma-1}) / \alpha!.
\]

In my opinion, the study of these functions is interesting, including their combinatorial interpretation and various relations with them.

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