Abstract In [8] we introduced the notion of a $k$-almost-quasifibration. In this article we update this definition and call it a $k$-$c$-quasifibration. This will help us to relate it to quasifibrations. We study some basic properties of $k$-$c$-quasifibrations. We also generalize a series of results on quasifibrations ([1]) to $k$-$c$-quasifibrations giving criteria for a map to be a $k$-$c$-quasifibration.

Keywords Quasifibration · Homotopy group

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1 Introduction

Recall that a surjective map $f : X \to Y$ is called a quasifibration ([1], [2], chap 4, p. 479) if for all $y \in Y$ and $x \in F_y$, the map $f : (X, F_y, x) \to (Y, y)$ is a weak equivalence. Hence, a quasifibration $f : X \to Y$ induces a long exact sequence of homotopy groups for all $y \in Y$. In [8] we introduced the notion of a $k$-almost-quasifibration for path connected spaces. For $k = \infty$, we called it an almost-quasifibration. It says that for some $y \in Y$ there exist an exact sequence of homotopy groups of the above type. Hence a quasifibration $f : X \to Y$ is an almost-quasifibration.

The main motivation behind the definition of a $k$-almost-quasifibration was that for computational purposes of homotopy groups, we need the long exact sequence of homotopy groups induced by $f$, for some $y \in Y$, instead of for all the points of $Y$. Also, since we constructed a class of examples in [8] supporting this definition. Furthermore, we had given many examples in [8] of 1-almost-quasifibrations and almost-quasifibrations which are not quasifibrations.

In this article we make a general definition of $k$-almost-quasifibration, and add an extra condition. We call it a $k$-$c$-quasifibration, and for $k = \infty$, we call it a $c$-quasifibration. This extra condition helps us to make the concept functorial and to give a necessary and sufficient condition for an almost-quasifibration to be a quasifibration. We also observe here that the examples of $k$-almost-quasifibrations we gave in [8] are all $k$-$c$-quasifibrations.

Here, we plan to study some basic properties of $k$-$c$-quasifibrations. We prove several criteria for a map to be a $k$-$c$-quasifibration. These results are analogous to the fundamental results proved in [1] in the context of quasifibrations. The methods used in the proofs of our results are not new, but we see that they are applicable in the case of $k$-$c$-quasifibrations also.
Throughout the paper for topological spaces $X$ and $Y$, $f : X \to Y$ will always denote a surjective continuous map.

**Definition 1.1** A subset of $Y$ is called exhaustive if it has a nonempty intersection with each path component of $Y$.

**Definition 1.2** Let $k \geq 0$ be an integer. Choose an exhaustive subset $\overline{Y}$ of $Y$. Then, $f$ is called a $k$-almost-quasifibration with respect to $\overline{Y}$, if for all $y \in \overline{Y}$ and $x \in F_y := f^{-1}(y)$ there are homomorphisms

$$\partial : \pi_{q+1}(Y, y) \to \pi_q(F_y, x),$$

for $q = 0, 1, 2, \ldots, k - 1$, so that the following sequence is exact.

$$1 \longrightarrow \pi_k(F_y, x) \overset{i_y}{\longrightarrow} \pi_k(X, x) \overset{f_*}{\longrightarrow} \pi_k(Y, y) \longrightarrow \pi_{k-1}(F_y, x) \longrightarrow \cdots$$

$$\cdots \longrightarrow \pi_1(Y, y) \overset{\partial}{\longrightarrow} \pi_0(F_y, x) \overset{i_*}{\longrightarrow} \pi_0(X, x) \overset{f_*}{\longrightarrow} \pi_0(Y, y) \longrightarrow 1.$$

$f$ is called an almost-quasifibration if the above sequence can be extended on the left up to infinity.

Denote the path components of $Y$ by $C_\alpha$, $\alpha \in I$. Let $X_\alpha = f^{-1}(C_\alpha)$. It is clear that $f$ is a $k$-almost-quasifibration with respect to an exhaustive subset $\overline{Y}$ if and only if for all $\alpha \in I$, $f|_{X_\alpha} : X_\alpha \to C_\alpha$ is a $k$-almost-quasifibration with respect to $\overline{Y} \cap C_\alpha$.

In Definition 1.2 we only assumed the existence of the connecting homomorphisms $\partial$. In Lemma 3.1 we observe that the commutativity of Diagram 1 is the key connection between an almost-quasifibration and a quasifibration. In the diagram, $\partial = \partial_\alpha$ is the connecting homomorphism coming from the long exact sequence of homotopy groups for the pair $(X, F_y)$. The commutativity of Diagram 1 makes $f_\alpha$ a bijection and hence $\partial$ becomes unique.

![Diagram 1](image)

Therefore, we now make the following definition.

**Definition 1.3** A $k$-almost-quasifibration $f : X \to Y$ with respect to an exhaustive subset $\overline{Y} \subseteq Y$ is called a $k$-$c$-quasifibration with respect to $\overline{Y}$, if Diagram 1 is commutative for $q = 0, 1, 2, \ldots, k - 1$, for all $y \in \overline{Y}$ and $x \in F_y$. When $k = \infty$ we call $f$ a $c$-quasifibration.

Therefore, we observe that a $c$-quasifibration $f : X \to Y$ with respect to $Y$ is a genuine quasifibration in the sense of [1] (Corollary 3.2). And the fibers of a $c$-quasifibration with respect to $\overline{Y}$, over the points of $\overline{Y}$ are all weak homotopy equivalent. We also show in Lemma 3.3 that the examples of $k$-almost-quasifibrations, (for $k = 1, \infty$) we gave in [8] are all $k$-$c$-quasifibrations (see Example 3.6).

In the next section we state our main results. In Section 3 we relate quasifibrations and $k$-$c$-quasifibrations and prove some basic results we need. Section 4 contains the proofs of the main results.


2 Statements of main results

Recall that for a surjective map \( f : X \to Y \), a subset \( Y_1 \subset Y \) is called distinguished ([1]) if \( f|_{f^{-1}(Y_1)} : f^{-1}(Y_1) \to Y_1 \) is a quasifibration. We need the following analogue of this definition, in the context of \( k \)-c-quasifibrations.

**Definition 2.1** A subset \( Y_1 \subset Y \) is called \((k)\)-\( c \)-distinguished with respect to an exhaustive subset \( \tilde{Y}_1 \subset Y_1 \), if \( f|_{f^{-1}(Y_1)} : f^{-1}(Y_1) \to Y_1 \) is a \((k)\)-c-quasifibration with respect to \( \tilde{Y}_1 \).

We begin with the following result giving a criterion for a map \( f \) to be a \( c \)-quasifibration from local data.

**Theorem 2.2** Let \( \{U_{\alpha}\}_{\alpha \in J} \) be an open covering of \( Y \), which is closed under taking finite intersections. Assume that, for any \( \alpha \in J \), there is an exhaustive subset \( \tilde{U}_{\alpha} \subset U_{\alpha} \), such that \( \tilde{U}_{\beta} \subset U_{\gamma} \) whenever \( \beta, \gamma \in J \), for \( \beta, \gamma \in J \). Furthermore, assume that \( U_{\alpha} \) is \( c \)-distinguished with respect to \( \tilde{U}_{\alpha} \), for any \( \alpha \in J \). Then \( f \) is a \( c \)-quasifibration with respect to some exhaustive subset of \( Y \).

Now, recall that a filtration of a space \( Y \) is an increasing sequence of subspaces \( Y_0 \subset Y_1 \subset \cdots \) such that \( Y = \bigcup_{i=0}^{\infty} Y_i \), and \( Y \) has the colimit topology.

For each \( i \in \mathbb{N} \), let \( \tilde{Y}_i \subset Y_i \) be an exhaustive subset of \( Y_i \), such that \( \tilde{Y}_i \subset \tilde{Y}_{i+1} \). Then clearly, the union \( \tilde{Y} := \bigcup_{i \in \mathbb{N}} \tilde{Y}_i \) is an exhaustive subset of \( Y \). We call \( \tilde{Y} \) an exhaustive subset of \( Y \) for the filtration \( \{Y_i\}_{i \in \mathbb{N}} \).

More generally, for any covering \( \{V_{\alpha}\}_{\alpha \in J} \) of \( Y \) by subsets with exhaustive subsets \( \tilde{V}_{\alpha} \subset V_{\alpha} \), for \( \alpha \in J \), \( \tilde{Y} := \bigcup_{\alpha \in J} \tilde{V}_{\alpha} \) is an exhaustive subset of \( Y \).

The following result shows that being a \( k \)-c-quasifibration is preserved under taking colimit.

**Theorem 2.3** Let \( Y, Y_1 \) and \( \tilde{Y} \) be as above. Assume that, for each \( i \), \( Y_i \) is \( T_1 \) and \( k \)-c-distinguished with respect to the exhaustive subset \( \tilde{Y}_1 \). Then, \( f \) is a \( k \)-c-quasifibration with respect to \( \tilde{Y} \).

In the next result we show that being a \( k \)-c-quasifibration is also preserved under deformation. For this, we need the following two definitions.

First, we make a variation of the definition of a \( k \)-equivalence.

**Definition 2.4** For pairs \( (X, X_1) \) and \( (Y, Y_1) \) of topological spaces, a map \( g : (X, X_1) \to (Y, Y_1) \) is called a \( k \)-\( c \)-equivalence with respect to a subset \( \tilde{X}_1 \subset X_1 \), if the following conditions are satisfied.

\[ g^{-1}\text{Image}(\pi_0(Y_1) \to \pi_0(Y)) = \text{Image}(\pi_0(X_1) \to \pi_0(X)) \]

- For all \( x \in \tilde{X}_1, g_* : \pi_q(X, X_1, x) \to \pi_q(Y, Y_1, g(x)) \) is a bijection for \( 1 \leq q \leq k - 1 \), and is a surjection for \( q = k \).

\( g \) is called a \( c \)-weak equivalence with respect to \( \tilde{X}_1 \), if the above conditions are satisfied for all \( k \).

We need this variation, since we will be considering only selected fibers of \( f : X \to Y \) in this work. Here, notice that a \( k \)-\( c \)-equivalence is defined with respect to a subset of the domain, and a \( k \)-c-quasifibration is defined with respect to a subset of the codomain.

**Remark 2.5** Recall that a \( k \)-equivalence demands that the second condition in Definition 2.4 must be satisfied for all \( x \in X_1 \). If \( \tilde{X}_1 \) is an exhaustive subset of \( X_1 \), then it has nonempty intersection with each path component of \( X_1 \), and hence a \( k \)-\( c \)-equivalence with respect to an exhaustive subset, is a \( k \)-equivalence. Similarly a \( c \)-weak equivalence with respect to an exhaustive subset is a weak equivalence in the standard sense.

**Definition 2.6** Let \( Y_1 \subset Y \) be a subset. Let \( X_1 = f^{-1}(Y_1) \). A deformation of \( f : X \to Y \) to \( f_1 := f|_{X_1} : X_1 \to Y_1 \) is a pair \((H, h)\) of maps defined by \( H : X \times I \to X \) and \( h : Y \times I \to Y \), such that the following conditions are satisfied.

- \( h|_{X \times 0} = id_X, h_1(y_1) := h(t, y_1) = y_1 \) for all \( y_1 \in Y_1, t \in I \) and \( h_1(y) \in Y_1 \) for all \( y \in Y \).
- \( H|_{X \times 0} = id_X, H_1(y_1) := H(t, x_1) = x_1 \) for all \( x_1 \in X_1, t \in I \) and \( H_1(x) \in X_1 \) for all \( x \in X \).
- \( f \circ H_1 = h_1 \circ f_1 \).

We are now in a position to state our next result.

**Theorem 2.7** Let \( Y_1 \subset Y \) be a subset and \( \tilde{Y} \subset Y, \tilde{Y}_1 \subset Y_1 \) be exhaustive subsets, such that \( \tilde{Y}_1 \subset \tilde{Y} \). Then \( f : X \to Y \) is a \( c \)-quasifibration with respect to \( \tilde{Y} \) if the following are satisfied.

- There is a deformation \((H, h)\) from \( f \) to \( f_1 \) such that \( h_1(\tilde{Y}) \subset \tilde{Y}_1 \).
\begin{itemize}
\item $H_1 : f^{-1}(y) \to f^{-1}(h_1(y))$ is a weak equivalence for all $y \in \tilde{Y}$.
\item $Y_1$ is $c$-distinguished with respect to $\tilde{Y}_1$.
\end{itemize}

Combining Theorems 2.2, 2.3 and 2.7, we can now deduce our final result. This is analogous to Theorem 2.6 in [4].

**Main Theorem** Let $\{Y_i\}_{i \in \mathbb{N}}$ be a filtration of $Y$ by closed and $T_1$ subspaces. Let, $\tilde{Y} = \bigcup_{i \in \mathbb{N}} \tilde{Y}_i$ be an exhaustive subset of $Y$ for the filtration $\{Y_i\}_{i \in \mathbb{N}}$. Assume that $\tilde{Y}_i$ intersects each open subset $U$ of $Y$ in an exhaustive subset $\tilde{U}$ (e.g., if $\tilde{Y}_i$ is dense in $Y_i$), and the following is satisfied.

- $Y_1$ is $c$-distinguished with respect to $\tilde{Y}_1$ and for each $i \geq 1$, each open subset $U$ of $Y_{i+1} - Y_i$ is $c$-distinguished with respect to the exhaustive subset $\tilde{U} \subset \tilde{Y}_{i+1}$. For each $i \in \mathbb{N}$, $Y_i$ has a neighborhood $U_{i+1}$ in $Y_{i+1}$ and a deformation $(H, h)$ from $f \mid f^{-1}(U_{i+1}) : f^{-1}(U_{i+1}) \to U_{i+1}$ to $f \mid f^{-1}(Y_i) : f^{-1}(Y_i) \to Y_i$, such that $h_1(U_{i+1}) \subset \tilde{Y}_{i+1}$ and $H_1 : f^{-1}(y) \to f^{-1}(h_1(y))$ is a weak equivalence for all $y \in \tilde{U}_{i+1}$.

Then, each $Y_i$ is $c$-distinguished with respect to $\tilde{Y}_i$ and $f$ is a $c$-quasifibration with respect to $\tilde{Y}$.

Here, we recall that the quasifibration versions of Theorems 2.2, 2.3 and 2.7 were the basic tools behind the proof of the Dold-Thom Theorem ([1]). In particular, using these theorems one shows that for a based connected CW-pair $(X, A)$, the map $SP(X) \to SP(X/A)$ is a quasifibration with fiber $SP(A)$. Here, $SP(\cdot)$ is the union of the $n$-fold symmetric products $SP_n(\cdot)$. Recall that, the functor $SP_n(\cdot)$ sends a space to the quotient of its $n$-fold product by the obvious action of the symmetric group $S_n$. Also see [2], p. 484.

### 3 Quasifibrations and k-c-quasifibrations

In this section we study properties of $k$-c-quasifibrations and prove some basic results needed for the proofs of the theorems.

#### 3.1 k-c-quasifibrations and its functoriality

In the following lemma we give a simple necessary and sufficient condition for an almost-quasifibration to be a quasifibration.

**Lemma 3.1** $f : X \to Y$ is a quasifibration if and only if $f$ is an almost-quasifibration with respect to $Y$, and Diagram 1 is commutative for all $y \in Y, x \in F_y$ and for $q = 0, 1, 2, \ldots$.

**Proof** We observe that if $f$ is an almost-quasifibration and Diagram 1 is commutative then, Diagram 2 is also commutative.

Hence using the Five Lemma, we see that $f_* : \pi_{q+1}(X, F_y, x) \to \pi_{q+1}(Y, y)$ is a bijection. Here $j : (X, x) \to (X, F_y, x)$ is the inclusion map.

\[
\begin{array}{ccccccccc}
\pi_{q+1}(F_y, x) & \xrightarrow{i_*} & \pi_{q+1}(X, x) & \xrightarrow{j_*} & \pi_{q+1}(X, F_y, x) & \xrightarrow{\partial_*} & \pi_q(F_y, x) & \xrightarrow{i_*} & \pi_q(X, x) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
\pi_{q+1}(F_y, x) & \xrightarrow{i_*} & \pi_{q+1}(X, x) & \xrightarrow{f_*} & \pi_{q+1}(Y, y) & \xrightarrow{\partial} & \pi_q(F_y, x) & \xrightarrow{i_*} & \pi_q(X, x)
\end{array}
\]

Diagram 2

Conversely, assume that $f$ is a quasifibration, and hence $f_* : \pi_{q+1}(X, F_y, x) \to \pi_{q+1}(Y, y)$ is a bijection. Then, we can define $\partial$ as $\partial_\ast \circ f_*^{-1}$, which makes $f$ an almost-quasifibration and Diagram 1 is commutative. This proves the lemma.

**Corollary 3.2** $f : X \to Y$ is a $c$-quasifibration with respect to $Y$ if and only if $f$ is a quasifibration.

Therefore, if $f$ is a $c$-quasifibration with respect to an exhaustive subset $\tilde{Y}$, and $\tilde{Y}$ is $c$-distinguished with respect to $\tilde{Y}$, then $f \mid f^{-1}(\tilde{Y})$ is a quasifibration.
**Lemma 3.3** A 1-almost-quasifibration with respect to some exhaustive subset \( \widetilde{Y} \) is a 1-c-quasifibration with respect to \( \widetilde{Y} \), if \( i_* : \pi_0(F_y, x) \to \pi_0(X, x) \) is an injection, equivalently if \( f_* : \pi_1(X, x) \to \pi_1(Y, y) \) is a surjection, for all \( y \in \widetilde{Y} \) and \( x \in F_y \).

*Proof* The proof is clear from Diagram 3. Since we only have to show that the triangle is commutative. But that follows, as \( i_* \) is an injection if and only if \( \partial \) is trivial and also \( i_* \) is an injection if and only if \( \partial \) is trivial. \( \square \)

\[ \begin{array}{c}
\pi_1(X, F_y, x) \\
\downarrow f_* \\
\pi_1(F_y, x) \\
\downarrow i_* \\
1 \\
\end{array} \]

Diagram 3

Therefore, we see that if \( F_y \) is connected for all \( y \in \widetilde{Y} \), then there is no difference between a 1-almost-quasifibration and a 1-c-quasifibration with respect to \( \widetilde{Y} \).

More generally, we have the following criterion.

**Lemma 3.4** \( f : X \to Y \) is a \( k \)-c-quasifibration with respect to an exhaustive subset \( \widetilde{Y} \) if and only if \( j_* : \pi_{k+1}(X, x) \to \pi_{k+1}(F_y, x) \) is a surjection and the map \( f_* : \pi_{q+1}(X, F_y, x) \to \pi_{q+1}(Y, y) \) is a bijection, for all \( y \in \widetilde{Y}, x \in F_y \) and for \( q = 0, 1, 2, \ldots, k-1 \).

*Proof* Consider the general Diagram 4.

First note that \( j_* : \pi_{k+1}(X, x) \to \pi_{k+1}(F_y, x) \) is a surjection if and only if \( \partial : \pi_{k+1}(F_y, x) \to \pi_k(F_y, x) \) is the trivial homomorphism if and only if \( i_* : \pi_k(F_y, x) \to \pi_k(X, x) \) is an injection.

\[ \begin{array}{c}
\pi_{k+1}(X, F_y, x) \\
\downarrow j_* \\
\pi_{k+1}(F_y, x) \\
\downarrow i_* \\
\cdots \\
\end{array} \]

Diagram 4

If \( f \) is a \( k \)-c-quasifibration then \( i_* : \pi_k(F_y, x) \to \pi_k(X, x) \) is an injection by definition. Also since Diagram 1 is commutative for all \( q = 0, 1, 2, \ldots, k-1 \), by the Five Lemma applied to Diagram 2, we get \( f_* : \pi_{q+1}(X, F_y, x) \to \pi_{q+1}(Y, y) \) is a bijection.

Conversely, assume that \( i_* : \pi_k(F_y, x) \to \pi_k(X, x) \) is an injection and \( f_* : \pi_{q+1}(X, F_y, x) \to \pi_{q+1}(Y, y) \) is a bijection for \( q = 0, 1, 2, \ldots, k-1 \). Once again, as in Lemma 3.1 we can define the homomorphisms \( \partial \), so as to get the required exact sequence and Diagram 1 to be commutative. \( \square \)

**Corollary 3.5** Assume \( \pi_2(X, x) = \langle 1 \rangle \) for all \( x \in X \). Then \( f \) is a 1-c-quasifibration with respect to \( \widetilde{Y} \), if and only if \( \pi_2(X, F_y, x) = \langle 1 \rangle \) and \( f_* : \pi_1(X, F_y, x) \to \pi_1(Y, y) \) is a bijection, for all \( y \in \widetilde{Y} \) and \( x \in F_y \).

We now give some non-trivial examples of \( k \)-c-quasifibrations.
Example 3.6 Let $S$ be a connected aspherical 2-manifold. Let $G$ be a discrete group acting on $S$, effectively and properly discontinuously with isolated fixed points. If $S/G$ has genus zero assume that it has a puncture. Consider the space $O_n(S, G)$ of $n$-tuples of points of $S$ with pairwise distinct orbits. Then, in [8], Theorem 1.3 and Proposition 1.6] we had shown that the projection $p_1 : O_n(S, G) \to O_1(S, G)$ to the first $l$ coordinates is a 1-almost-quasifibration, with respect to the subset $O_1(S, G)$ of points of $O_1(S, G)$, whose coordinates have trivial isotropy groups. And the projection is an almost-quasifibration with respect to $O_1(S, G)$, if either the action of $G$ on $S$ is free, or if $S/G$ has genus zero with at most two cone points of order 2 each. Using Lemma 3.3, now it follows that $p_1$ is a 1-c-quasifibration or a $c$-quasifibration with respect to $O_1(S, G)$, for the respective cases as above.

The next examples are from 3-manifold topology and of much importance, since they are the building blocks of 3-manifolds, other than the hyperbolic ones. The standard references for the background of these examples are [3] and [9].

Example 3.7 Let $M$ be a connected Seifert fibered space. Let $B$ be the base orbifold with infinite orbifold fundamental group. Let $q : M \to B$ be the quotient map. Then, it is well known that for any regular point $b \in B$, there is the following exact sequence (9, Lemma 3.2). Here, $S^1_b$ is the circle fiber over $b$ and $m \in S^1_b$.

$$
1 \longrightarrow \mathbb{Z} \cong \pi_1(S^1_b, m) \longrightarrow \pi_1(M, m) \longrightarrow \pi_1^{orb}(B, b) \longrightarrow 1.
$$

Note that $B$ is a good orbifold, since it has infinite orbifold fundamental group ([7], Proposition 3.15). That is, there is a 2-manifold $\hat{B}$ and a discrete group $G$ acting effectively and properly discontinuously on $\hat{B}$ with $B = \hat{B} / G$. Let $p : \hat{B} \to B$ be the orbifold covering map. Let $\hat{p} : \hat{M} \to \hat{B}$ be the pull back of $p$ by $q$ as shown in Diagram 5. Let $\hat{G}$ be the group of covering transformation of $\hat{p}$.

$$
\begin{array}{ccc}
\hat{M} & \xrightarrow{\hat{p}} & M \\
\downarrow {\hat{q}} & & \downarrow {q} \\
\hat{B} & \xrightarrow{p} & B \\
\end{array}
$$

Diagram 5

For the rest of the deduction we follow the reference [5]. Let $G(\hat{M}, \hat{G})$ and $G(\hat{B}, G)$ be the corresponding translation Lie groupoids. The above data induces a homomorphism $\hat{Q} : G(\hat{M}, \hat{G}) \to G(\hat{B}, G)$, which in turn defines a continuous map $BQ : B\hat{G}(\hat{M}, \hat{G}) \to B\hat{G}(\hat{B}, G)$ on their classifying spaces. From some generalities, we have the following two isomorphisms.

$$
\pi_1(B\hat{G}(\hat{M}, \hat{G}), \hat{m}) \cong \pi_1(M, m) \text{ and } \pi_1(B\hat{G}(\hat{B}, G), \hat{b}) \cong \pi_1^{orb}(B, b).
$$

Here, $b \in B$ is a regular point such that $p(\hat{b}) = b$, $\hat{p}(\hat{m}) = m$ and $q(m) = b$. Furthermore, $BQ$ induces the above exact sequence. Therefore, $BQ$ is a 1-c-quasifibration with respect to the set $\hat{B}$ of all points of $\hat{B}$ with trivial isotropy groups. Consequently, it is also a c-quasifibration with respect to $\hat{B}$, since the above two classifying spaces are aspherical, which is a consequence of the fact that both $\hat{M}$ and $\hat{B}$ are aspherical.

The following examples, although easy, show that there are $k$-c-quasifibrations for any $k$.

Example 3.8 One can also construct examples of $k$-c-quasifibrations from Example 3.6 by taking product with a space $Z$, that is by considering

$$
p_1 \times id_Z : O_n(S, G) \times Z \to O_1(S, G) \times Z.
$$

Then, $p_1 \times id_Z$ becomes a 1-c-quasifibration with respect to $O_1(S, G) \times Z$. More generally, for a $k$-c-quasifibration $f$ with respect $Y$, $f \times id_Z : X \times Z \to Y \times Z$ is a $k$-c-quasifibration with respect to $Y \times Z$. In addition, if $X$, $Y$ and $F_\vartheta$ are all aspherical for all $\vartheta \in \hat{Y}$ and if $f$ is a 1-c-quasifibration, then $f \times id_Z$ is a $k$-c-quasifibration for all $k$. Also, for any quasifibration $f$, if we change the sign of $\vartheta$ at some stage, say $q$ and for all $\vartheta \in Y$, then the long sequence of homotopy groups still remains exact, and hence $f$ has an almost-quasifibration structure which is not a c-quasifibration, since Diagram 1 at stage $q$ is not commutative.
We also asked in [8] if for any connected manifold $M$ with an effective and properly discontinuous action of a discrete group $G$, $p_1 : \mathcal{O}_a(M, G) \to \mathcal{O}_b(M, G)$ is an almost-quasifibration ($c$-quasifibration) with respect to $\mathcal{O}_b(M, G)$?

Next, we observe that the exact sequence in the definition of $k$-$c$-quasifibration is functorial, in the sense described in the following lemma. The lemma is easy to verify.

**Lemma 3.9** Let $f : X \to Y$ and $\hat{f} : \hat{X} \to \hat{Y}$ be two $k$-$c$-quasifibrations with respect to $\hat{Y}$ and $\tilde{Y}$, respectively. Assume that there are continuous maps $g : X \to \hat{X}$ and $h : Y \to \hat{Y}$ making Diagram 6 commutative and that $h(\tilde{Y}) \subset \tilde{Y}$.

\[
\begin{array}{c}
X \\ \downarrow g \\
\hat{X} \\
\end{array} \rightarrow 
\begin{array}{c}
Y \\ \downarrow h \\
\hat{Y} \\
\end{array}
\]

**Diagram 6**

Then, for all $y \in \tilde{Y}$ and $x \in F_y$, Diagram 7 is commutative. Here, $\hat{x} = g(x)$, $\hat{y} = h(y)$ and $\hat{F}_y = \hat{f}^{-1}(\hat{y})$.

\[
\begin{array}{c}
1 \longrightarrow \pi_k(F_y, x) \longrightarrow \pi_k(X, x) \longrightarrow \pi_k(Y, y) \longrightarrow \pi_{k-1}(F_y, x) \longrightarrow \cdots \longrightarrow \pi_0(Y, y) \longrightarrow 1 \\
\downarrow g_* \quad \quad \downarrow g_* \quad \quad \downarrow h_* \quad \quad \downarrow g_* \quad \quad \downarrow h_* \\
1 \longrightarrow \pi_k(\tilde{F}_y, \hat{x}) \longrightarrow \pi_k(\hat{X}, \hat{x}) \longrightarrow \pi_k(\tilde{Y}, \hat{y}) \longrightarrow \pi_{k-1}(\tilde{F}_y, \hat{x}) \longrightarrow \cdots \longrightarrow \pi_0(\tilde{Y}, \hat{y}) \longrightarrow 1
\end{array}
\]

**Diagram 7**

3.2 Some basic results

We will need the following local to global type lemma for the proofs of the theorems.

**Lemma 3.10** Let $\mathcal{O} = \{Y_a\}_{a \in I}$ be a covering of $Y$ by subsets. Consider the partial ordering on $I$ induced by set inclusions of the members of $\mathcal{O}$, and let this order (denoted by $\preceq$) be a total ordering. Assume that any compact subset of $\tilde{Y}$ is contained in some $Y_a \in \mathcal{O}$. Let $Y_a$ be an exhaustive subset of $Y_\alpha \in \mathcal{O}$, with the property that if $a \preceq b$ in $I$, then $\tilde{Y}_a \subset \tilde{Y}_b$. If each $Y_a \in \mathcal{O}$ is $k$-$c$-distinguished with respect to $\tilde{Y}_a$, then $f$ is a $k$-$c$-quasifibration with respect to $\bigcup_{a \in I} \tilde{Y}_a$.

**Proof** Let $y \in \bigcup_{a \in I} \tilde{Y}_a$ and choose $\beta \in I$ so that $y \in \tilde{Y}_\beta$. Let $X_{\alpha} = f^{-1}(Y_{\alpha})$ for all $\alpha \in I$. For $\beta \preceq y \preceq \delta$ in $I$, consider Diagram 8.

\[
\begin{array}{c}
\pi_{q+1}(X_\delta, F_y, x) \\
\downarrow f_* \quad \quad \downarrow a_* \\
\pi_{q+1}(X_\delta, y) \longrightarrow \pi_q(F_y, x) \longrightarrow \pi_{q+1}(X_\delta, F_y, x) \\
\downarrow f_* \quad \quad \downarrow a_* \\
\pi_{q+1}(Y_\gamma, y) \longrightarrow \pi_q(F_y, x)
\end{array}
\]

**Diagram 8**
The two vertical triangles are commutative for $q = 0, 1, 2, \ldots, k - 1$, since each $f|_{X_a}$ is a $k$-c-quasifibration with respect to $\tilde{Y}_a$, and $Y_{\beta} \subset \tilde{Y}_\gamma \subset \tilde{Y}_\delta$. And the whole diagram is commutative by functoriality of boundary maps. Now, note that, since any compact subset of $Y$ is contained in some member of $O$, it is contained in $Y_a$ for some $\alpha \in J$ with $\beta \not\approx \alpha$, as $(J, \not\approx)$ is totally ordered. Therefore, a colimit argument on Diagram 8 gives the commutative Diagram 1, and hence we obtain a definition of $\partial$ for $f : X \to Y$.

\[
\begin{array}{ccccccc}
1 & \longrightarrow & \pi_k(F_y, x) & \overset{i} \longrightarrow & \pi_k(X_y, x) & \overset{f_r} \longrightarrow & \pi_k(Y_y, y) & \overset{\partial} \longrightarrow & \pi_{k-1}(F_y, x) & \longrightarrow & \cdots \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \cdots \\
1 & \longrightarrow & \pi_k(F_y, x) & \overset{i} \longrightarrow & \pi_k(X_y, x) & \overset{f_r} \longrightarrow & \pi_k(Y_y, y) & \overset{\partial} \longrightarrow & \pi_{k-1}(F_y, x) & \longrightarrow & \cdots
\end{array}
\]

Diagram 9

Next, consider Diagram 9. This diagram is commutative by functoriality of $k$-$c$-quasifibration (see Lemma 3.9). Hence, again by taking colimit we see that, since $f|_{X_a}$ is a $k$-$c$-quasifibration with respect to $\tilde{Y}_a$, for each $\alpha$, $f$ is a $k$-$c$-quasifibration with respect to $\bigcup_{a \in J} \tilde{Y}_a$. \hfill $\square$

The following proposition is the main ingredient for the proof of Theorem 2.2. This result is analogous to [1], Satz 2.2.

Proposition 3.11 Let $(Y_1; Y_1, Y_2)$ be an excisive triad. Let $\tilde{Y}_{12} \subset Y_1 \cap Y_2$, $\tilde{Y}_1 \subset Y_1$ and $\tilde{Y}_2 \subset Y_2$ be exhaustive subsets such that $\tilde{Y}_{12} \subset \tilde{Y}_1 \cap \tilde{Y}_2$. If $Y_1, Y_2$ and $Y_1 \cap Y_2$ are all c-distinguished with respect to the corresponding exhaustive subsets above, then $f$ is a c-quasifibration with respect to $\tilde{Y}_1 \cup \tilde{Y}_2$.

To prove the proposition we need the following two lemmas.

Lemma 3.12 Let $Y_1 \subset Y$ be c-distinguished with respect to an exhaustive subset $\tilde{Y}_1 \subset Y_1$. Then $f : (X, F_y, x) \to (Y, y)$ is a weak equivalence for all $y \in \tilde{Y}_1$ if and only if $f : (X, f^{-1}(Y_1), x) \to (Y, Y_1, y)$ is a c-weak equivalence with respect to $f^{-1}(\tilde{Y}_1)$.

Proof For $y \in \tilde{Y}_1$, consider the following map of triples

$$f : (X, f^{-1}(Y_1), f^{-1}(y)) \to (Y, Y_1, y).$$

Then, the proof follows by applying the Five Lemma on the commutative diagram of the long exact sequences of homotopy groups for triples, induced by the above map of triples. \hfill $\square$

The following lemma is analogous to Proposition 4K.1 of [2], for our variation of the definition of a $k$-equivalence.

Lemma 3.13 Let $f : (X; X_1, X_2) \to (Y; Y_1, Y_2)$ be a map of excisive triads. Let $\tilde{X}_{12} \subset X_1 \cap X_2$, $\tilde{X}_1 \subset X_1$ and $\tilde{X}_2 \subset X_2$ be exhaustive subsets. Assume that, $f : (X_i, X_1 \cap X_2) \to (Y_i, Y_1 \cap Y_2)$ is a $k$-$c$-equivalence with respect to $\tilde{X}_{12}$, for $i = 1, 2$. Then $f : (X, X_1) \to (Y, Y_1)$ is a $k$-$c$-equivalence with respect to $\tilde{X}_i$, for $i = 1, 2$.

Proof First note that, since $\tilde{X}_{12}$ is exhaustive in $X_1 \cap X_2$, $f : (X_i, X_1 \cap X_2) \to (Y_i, Y_1 \cap Y_2)$ is a $k$-$c$-equivalence, for $i = 1, 2$, by Remark 2.5. Therefore, by Proposition 4K.1 of [2], $f : (X, X_1) \to (Y, Y_1)$ is a $k$-$c$-equivalence, and hence, in particular, $k$-$c$-equivalence with respect to $\tilde{X}_i$, for $i = 1, 2$. \hfill $\square$

Now, we can prove Proposition 3.11.

Proof of Proposition 3.11 First note that, since $Y_1 \cap Y_2$ is c-distinguished with respect to $\tilde{Y}_{12}$, for $i = 1, 2$, $f : (f^{-1}(Y_1), F_y) \to (Y_i, y)$ is a weak equivalence for all $y \in \tilde{Y}_{12}$. See Corollary 3.2. Hence by Lemma 3.12, for $i = 1, 2$,

$$f : (f^{-1}(Y_1), f^{-1}(Y_1) \cap f^{-1}(Y_2)) \to (Y_i, Y_1 \cap Y_2)$$

is a c-weak equivalence with respect to $f^{-1}(\tilde{Y}_{12})$. Now using Lemma 3.13, we get that $f : (X, f^{-1}(Y_1)) \to (Y, Y_1)$ is a c-weak equivalence with respect to $f^{-1} (\tilde{Y}_1)$ for $i = 1, 2$. Applying the converse of Lemma 3.12 we get that $f : (X, F_y) \to (Y, y)$ is a weak equivalence for all $y \in \tilde{Y}_1, i = 1, 2$, and hence for all $y \in \tilde{Y}_1 \cup \tilde{Y}_2$. This completes the proof of the proposition. \hfill $\square$
4 Proofs of the Theorems

We begin with the proof of Theorem 2.2 which says that a $c$-quasifibration can be deduced from local data.

**Proof of Theorem 2.2** The proof is a consequence of Lemma 3.10, Proposition 3.11 and the Zorn’s lemma.

Let $\mathcal{O} = \{U_a\}_{a \in J}$ be the open covering of $Y$ and $(\tilde{U}_a)_{a \in J}$ be the exhaustive subsets of $\mathcal{O}$. Recall that $\bigcup_{a \in J} \tilde{U}_a$ is denoted by $\bar{Y}$. Next, we define a set $\mathcal{U}$ as follows.

$$\mathcal{U} = \{V_K \mid V_K \subset Y, \ V_K = \bigcup_{a \in K \subset J} U_a, \text{ for some } K \subset J, \ V_K \text{ is } c\text{-distinguished with respect to } \tilde{V}_K := \bigcup_{a \in K} \tilde{U}_a, \text{ and for } K_1, K_2 \subset J, \ V_{K_1} \cap V_{K_2} \text{ is } c\text{-distinguished with respect to } \bigcup_{a \in K_1 \cap K_2} \tilde{U}_a\}.$$ 

By hypothesis, $\mathcal{U}$ is nonempty, since it contains $\mathcal{O}$. Now, we partially order $\mathcal{U}$ using set inclusions of its members. Then, given any chain $\mathcal{C}$ in $\mathcal{U}$, Lemma 3.10 shows that $\bigcup_{V \in \mathcal{C}} V$ is $c$-distinguished with respect to $\bigcup_{V \in \mathcal{C}} \bar{V}$. This is because any compact subset of $\bigcup_{V \in \mathcal{C}} V$ is contained in one of the members of $\mathcal{C}$.

Therefore, by the Zorn’s lemma $\mathcal{U}$ has a maximal element, say $V_{K_0}$. We claim that $V_{K_0} = Y$. If not, then choose $U \in \mathcal{O}$ such that $U$ is not contained in $V_{K_0}$. Now since, $\mathcal{O} \subset \mathcal{U}$, by definition of $\mathcal{U}$, $V_{K_0} \cap U$ is $c$-distinguished with respect to $(\bigcup_{a \in K_0} \tilde{U}_a) \cap \bar{U}$. Next, using Proposition 3.11, we get that $V_{K_0} \cap U$ is a $c$-distinguished with respect to $(\bigcup_{a \in K_0} \tilde{U}_a) \cup \bar{U}$. Hence $V_{K_0} \cup U$ is an element of $\mathcal{U}$, which is a contradiction to the maximality of $V_{K_0}$.

Therefore, $V_{K_0} = Y$ and hence $f$ is a $c$-quasifibration with respect to the exhaustive subset $\bigcup_{a \in K_0} \tilde{U}_a$. This completes the proof of the theorem. \hfill $\Box$

Next, we prove Theorem 2.3 which says that under a colimit $k$-$c$-quasifibration is preserved.

**Proof of Theorem 2.3** Theorem 2.3 is an immediate application of Lemma 3.10. We just have to note that under the colimit topology on $Y$, induced by the filtration $\{Y_1\}_{k \in \mathbb{N}}$ of $T_1$ subspaces, any compact subset of $\bar{Y}$ is contained in one of the members of the filtration. \hfill $\Box$

Recall that Theorem 2.7 says that a certain kind of deformation preserves $c$-quasifibration.

**Proof of Theorem 2.7** Let us first recall the notations and the hypothesis.

$f : X \to Y$ is a surjective map, $Y_1 \subset Y$ and $X_1 = f^{-1}(Y_1)$. $\bar{Y} \subset Y$ and $\bar{Y}_1 \subset Y_1$ are exhaustive subsets, such that $\bar{Y}_1 \subset \bar{Y}$. $(H, h)$ is a deformation from $f$ to $f_1 = f|_{Y_1} : X_1 \to Y_1$ such that $h_1(\bar{Y}) \subset \bar{Y}_1$. Furthermore, for all $y \in \bar{Y}$, $H_{1|f^{-1}(y)} : f^{-1}(y) \to f^{-1}(h_1(y))$ is a weak equivalence.

Now, consider Diagrams 10 and 11 for all $y \in \bar{Y}$.

Note that, since $H_{1|f^{-1}(y)} : f^{-1}(y) \to f^{-1}(h_1(y))$ is a weak equivalence, an application of the Five Lemma to the long exact sequence of homotopy groups of pairs proves that the composition $i \circ H_1$ is a weak equivalence. Next, note that since $H$ is a deformation, the inclusion map $i$ is a weak equivalence. Therefore, we conclude that the vertical map $H_1$ in Diagram 10 is a weak equivalence.

$$\begin{array}{ccc}
(X, f^{-1}(y)) & \xrightarrow{h_1} & (X_1, f^{-1}(h_1(y))) \\
\downarrow & & \downarrow \quad i \\
(X_1, f^{-1}(h_1(y))) & \xrightarrow{i} & (X, f^{-1}(h_1(y)))
\end{array}$$

Diagram 10

Hence, in Diagram 11, the top horizontal map is a weak equivalence. Also, the right hand side vertical map is a weak equivalence, since $Y_1$ is $c$-distinguished with respect to $\bar{Y}_1$. Finally, the below horizontal map is a weak equivalence since $h$ is a deformation.
\[(X, f^{-1}(y)) \xrightarrow{H_1} (X_1, f^{-1}(h_1(y))) \]
\[
\downarrow f \hspace{1cm} \downarrow f_1 \\
(Y, y) \xrightarrow{h_1} (Y_1, h_1(y))
\]

Diagram 11

Therefore, we conclude that the left hand side vertical map in Diagram 11 is a weak equivalence for all \(y \in \tilde{Y}\), that is \(f\) is a \(c\)-quasifibration with respect to \(\tilde{Y}\).
This completes the proof of the theorem. \(\square\)

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