Inverting the Achlioptas rule for explosive percolation

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In the usual Achlioptas processes the smallest clusters of a few randomly chosen (Achlioptas rule) have attracted much attention in the last years 1–16. These specific processes and models show a number of unusual features 17 distinguishing them sharply from ordinary aggregation processes and standard percolation, in which random clusters merge together with probability proportional to their sizes 13–20. Apart from the delayed percolation phase transition, which is continuous, as we found for a wide range of systems 4,9 and which was proven mathematically 12,13, these models demonstrate a uniquely small exponent \( \beta \) of the percolation cluster and unusual scaling functions. The smallness of \( \beta \) makes the transition so “sharp” that it is difficult to distinguish it from discontinuous in simulations, which resulted in the term “explosive percolation” 1. A few real-world applications of these processes were identified 21–23. The Achlioptas processes, generalizing percolation, constitute a wide class including the processes generated by the original “product rule” (two clusters with the smallest product of sizes are selected) 1, the sum rule (clusters with the smallest sum of sizes are selected), the rule selecting the smallest clusters 4, and many others, of which only a small number were explored. The problem is how far from the standard percolation scenario can these diverse rules and their variations lead? How easy can one deviate from the typical percolation behavior by exploiting the “power of choice” 24 in these processes? Notably, in these rules another kind of optimization can be considered, namely, selecting not the smallest but the largest clusters. In particular, the question is: what will happen if we invert the Achlioptas rule, that is, at each step merge the two largest clusters of a few randomly selected ones 25,26?

In the present article we answer to this question by considering a representative set of processes based on the inverse Achlioptas rule, for which we derive the Smoluchowski equation. By solving these equations numerically and analytically we find that this rule results in a percolation transition taking place at an earlier stage of the process, but with the same set of critical exponents as in ordinary percolation. We calculate the critical amplitudes for the relative size \( S \) of the percolation cluster and for the size distribution of finite clusters and obtain the scaling functions.

The paper is organized as follows. In Sec. II we introduce the model and describe the evolution equations. In Sec. III we obtain the critical singularity of the percolation cluster \( S \) by using the generating function technique. In Sec. IV we find the scaling functions and the critical exponent \( \tau \), \( P(s, t_c) \sim s^{1-\tau} \). Section V describes the order parameter and generalized susceptibility for these phase transitions. Finally in Sec. VI we obtain analytical estimates for the transition point and critical amplitude. Table I demonstrates a good agreement between the results of the numerical solution of evolution equations and these estimates.

I. THE MODEL

We consider the following model incorporating the inverse Achlioptas rule and convenient for treatment. We start from \( N \) isolated nodes. At each time step a new link connecting two nodes is added to the network as follows. At each step sample two times: (i) choose \( m \geq 1 \) nodes uniformly at random and compare the clusters to which these nodes belong; select the node within the largest of these clusters; (ii) similarly choose the second sample of \( m \) nodes and, again, as in (i), select the node belonging to the largest of the \( m \) clusters; (iii) add a link between the two selected nodes thus merging the two largest clusters. Note that the only difference from our previous works is that instead of selecting the two smallest clusters for merging 4,17,27, here we select the two largest. The probability distribution \( P(s, t) \), i.e., the probability that a uniformly randomly chosen node belongs to a cluster of size \( s \) at time \( t \), gives the complete description of the evolution of this system. Time \( t \) is the ratio of the number of steps (links) and the number of nodes. We emphasize
that, instead of the product rule \[4\], we select for merging the largest cluster from each of the two sets of \(m\) clusters, which makes our problem treatable analytically.

For infinite \(N\), this aggregation process is described by the following evolution equation:

\[
\frac{\partial P(s,t)}{\partial t} = s \sum_{u=1}^{s-1} Q(u,t) Q(s-u,t) - 2s Q(s,t),
\]

which is the Smoluchowski equation for this process \[28, 29\]. Here \(Q(s,t)\) is the probability that a cluster selected to merge is of size \(s\). The distribution \(Q(s,t)\) is expressed in terms of \(P(s,t)\) as

\[
Q(s) = \left[ \sum_{u=1}^{s} P(u) \right]^{m} - \left[ \sum_{u=1}^{s-1} P(u) \right]^{m-1},
\]

\[
\approx mP(s) \left[ \sum_{u<s} P(u) \right]^{m-1}.
\]

Notice the normalization conditions for these distributions, \(\sum_{s} P(s) = 1 - S\) and \(\sum_{s} Q(s) = (1 - S)^{m}\), where \(S\) is the relative size of the percolation cluster. For large \(s\) we have

\[
Q(s) \approx mP(s) (1 - S)^{m-1},
\]

both above and below \(t_c\). Equation \(1\) together with relation \(2\) describe the process exactly in the full range of \(t\), from 0 to infinity.

II. PERCOLATION CLUSTER SIZE

Let us define the generating functions

\[
\rho(z,t) = \sum_{s} z^{s} P(s,t)
\]

and

\[
\sigma(z,t) = \sum_{s} z^{s} Q(s,t).
\]

We multiply both sides of Eq. \(1\) by \(z^{s}\) and sum over \(s\), which gives

\[
\frac{\partial \rho(z,t)}{\partial t} = 2 \left[ \sigma(z,t) - 1 \right] \frac{\partial \sigma(z,t)}{\partial \ln z}.
\]

Using Eq. \(3\), we find the relation between the functions \(\sigma(z)\) and \(\rho(z)\) for \(z\) close to 1,

\[
\sigma(z) = \sum_{s} Q(s) + \sum_{s} Q(s)(z^{s} - 1)
\]

\[
\approx (1 - S)^{m} + m(1 - S)^{m-1} \sum_{s} P(s)(z^{s} - 1)
\]

\[
= m(1 - S)^{m-1} \rho(z) - (m - 1)(1 - S)^{m},
\]

and so

\[
\frac{\partial \rho}{\partial t} \approx 2 \left[ m(1 - S)^{m-1} \rho - (m - 1)(1 - S)^{m} - 1 \right]
\]

\[
\times m(1 - S)^{m-1} \frac{\partial \rho}{\partial \ln z}.
\]

By applying hodograph transformation \[29\] to this partial differential equation we get the ordinary differential equation

\[
\frac{d \ln z}{dt} \approx 2m(1 - S)^{m-1}
\]

\[
\times \left[ (m - 1)(1 - S)^{m} + 1 - m(1 - S)^{m-1} \rho \right],
\]

which leads to

\[
\ln z \approx 2m(m-1) \int_{t_c}^{t} dt (1 - S)^{2m-1} + 2m \int_{t_c}^{t} dt (1 - S)^{m-1}
\]

\[
-2m^{2}\rho \int_{t_c}^{t} dt (1 - S)^{2m-2} + g(\rho),
\]

where the function \(g(\rho)\) is the initial condition for Eq. \(10\), which can be found from the critical distribution \(P(s,t_c) \approx a_{0} s^{1-\tau}\). At \(t = t_c\), proceeding as in \[3, 17, 27\] for \(z\) close to 1, we find the singularity

\[
1 - \rho(z,t_c) \approx -a_{0} \Gamma(2 - \tau) (- \ln z)^{\frac{\tau}{\tau-2}}.
\]

Inverting the function \(\rho(z,t_c)\) in this equation we obtain

\[
g(\rho) = - \frac{1}{a_{0} \Gamma(2 - \tau)} \frac{1}{(- \ln z)^{\frac{\tau}{\tau-2}}},
\]

which we substitute into Eq. \(10\). Finally, we set \(z = 1\) in the resulting equation and obtain

\[
0 \approx 2m(m-1) \int_{t}^{t_c} dt (1 - S)^{2m-1} + 2m \int_{t_c}^{t} dt (1 - S)^{m-1}
\]

\[
-2m^{2}(1 - S) \int_{t_c}^{t} dt (1 - S)^{2m-2} - \left[ -S \right] \frac{1}{a_{0} \Gamma(2 - \tau)} \frac{1}{(- \ln z)^{\frac{\tau}{\tau-2}}}.\]

From this equation we find the critical singularity of \(S\),

\[
S \approx [-a_{0} \Gamma(2 - \tau)]^{1/(\tau-2)} [2m^{2}(t - t_{c})]^{(\tau-2)/(3-\tau)}.\]

III. SCALING PROPERTIES

In this section we find the scaling form of \(P(s,t)\) near \(t_c\) using the approach of our previous works \[18, 27\]. The form of the scaling function is determined by the critical exponent \(\tau\). The scaling function must decay faster than any power law, and must take only positive values. We show that these conditions are satisfied only for \(\tau = 5/2\), which enables us to find the scaling function for each \(m\).

Let us obtain the Taylor expansion of \(P(s,t)\),

\[
P(s,t) = A_{0}(s) + A_{1}(s)(t-t_{c}) + A_{2}(s)(t-t_{c})^{2} + \ldots
\]
in terms of $\tau$, by sequentially differentiating the evolution equation (11) and the relation (13) at $t = t_c$ with respect to $t$. Recall that the critical distribution $Q(s,t_c) \cong mP(s,t_c)$ for large $s$, and so the derivatives are

$$\frac{\partial^n Q(s,t)}{\partial t^n} \bigg|_{t=t_c} \cong m \frac{\partial^n P(s,t)}{\partial t^n} \bigg|_{t=t_c}. \quad (15)$$

Differentiating both sides of Eq. (11) $n - 1$ times and replacing the right-hand side with Eq. (15) we find the asymptotics of the coefficient $A_n(s) \equiv \partial_t^{(n)} P(s,t)|_{t_c}/n!$. Due to Eq. (15) the equations for $A_n$ for each $m \geq 1$ differ only by the factor $m^{2n}$ on the right-hand side. So the asymptotics $A_n(s)$ is

$$A_n(s) \cong a_n m^{2n} s^{1-\tau+n(3-\tau)}, \quad (16)$$

with the prefactor $a_n$ that we have calculated in [27] for ordinary percolation ($m = 1$),

$$a_n = \frac{2^n [a_0 \Gamma(2-\tau)]^{n+1}}{(n+1)! \Gamma[(n+1)(2-\tau)]}. \quad (17)$$

The scaling form of the distribution $P(s,t)$ is

$$P(s,t) \cong s^{1-\tau} \sum_n a_n m^{2n} s^{3-\tau}(t-t_c)]^n \cong s^{1-\tau} \sum_n f[s^{3-\tau}(t-t_c)], \quad (18)$$

where $f(x)$ is the series

$$f(x) \cong a_0 \Gamma(2-\tau) \sum_{n=0}^{\infty} \frac{[2a_0 m^2 \Gamma(2-\tau)]^n}{(n+1)! \Gamma[(n+1)(2-\tau)]}. \quad (19)$$

The parameter $m$ appears only as a factor of $x$. The function $f(x)$ depends essentially on $\tau$. For $x \gg 1$ this function approaches 0 exponentially, staying positive, for any $2 < \tau < 3$. In the phase $t < t_c$, i.e., $x < 0$, only one value of the exponent $\tau$ results in a scaling function $f(x)$ with the proper decay to 0 as $x$ approaches $-\infty$. For $\tau < 5/2$, the function $f(x)$, Eq. (19), oscillates around 0 in the region $x < 0$, see Fig. 1(a). Since the scaling function cannot take negative values, we exclude the range $\tau < 5/2$ from the possible values of $\tau$. For $\tau > 5/2$, the function $f(x)$ stays positive but approaches 0 as a power-law as $x \to -\infty$, see Fig. 1(b). The scaling function must decay more rapidly than any power law for $x \to \pm \infty$, and so we also exclude the range $\tau > 5/2$.

At $\tau = 5/2$ the function $f(x)$ takes the form

$$f(x) \cong a_0 \sum_{n=0}^{\infty} \frac{[-4\pi (a_0 m^2 x^2)^2]^n}{n!} \cong a_0 \exp \left[-4\pi (a_0 m^2 x^2)^2\right], \quad (20)$$

that is, decays exponentially with $x^2$ on the both sides of the transition. Thus $\tau = 5/2$ for all $m \geq 1$, and so Eq. (20) gives the scaling function of this transition. Figure 2(a) shows that near $t_c$ at large $s$ the numerical solution $P(s,t)$ of Eqs. (11) and (2) agrees completely with the scaling functions (20). Inserting $\tau = 5/2$ into Eq. (13) we find

$$S \cong 8\pi m^2 a_0^2 (t-t_c) \quad (21)$$

near $t_c$. Figure 2(b) presents the evolution of the relative size of the percolation cluster for each $m$, which we found numerically from Eqs. (11) and (2). $S(t) = 1 - \sum_s P(s,t)$. The curves $S(t,m)$ in Fig. 2(b) intersect with each other, and so $S(t)$ grows slower for larger $m$ above $t_c$. Recall that in our model the percolation cluster can be selected more than once at the same step. This corresponds to adding a new link between two nodes in the percolation cluster, which does not changes cluster sizes and happens.
with probability $[1 - (1 - S)^m]^2$. This probability grows rapidly with $m$ effectively delaying the aggregation process above $t_c$ compared with $m = 1$. If we forbid the same cluster from being selected more than once at each step, the delay effect disappears and a larger $m$ results in a faster growth of $S$, which approaches $S \approx t$ when $m \to \infty$. In Table I we show precise results for $\tau$, $t_c$, and $a_0$, which are computed from $P(s \leq 10^5, t)$ using the method of Ref. [9]. The numerical results for $\tau$ agree with the exact result $\tau = 5/2$. Above the upper critical dimension, which is the case for our models, all critical exponents can be expressed in terms of a single one. So we arrive at the same critical exponents as in ordinary percolation.

IV. SUSCEPTIBILITY AND ORDER PARAMETER

According to Ref. [17], the order parameter and generalized susceptibility for this class of problems are related to the probability $c_2$ that two nodes selected by the model rules fall within the same cluster,

$$c_2 = \sum_s sQ(s)^2 [1 - (1 - S)^m]^2 \equiv \chi/N + O^2. \quad (22)$$

The first term on the right-hand side is the probability that both selected nodes belong to the same finite cluster, which is equal to the susceptibility $\chi$ divided by $N$. The second term is the probability that both nodes belong in the percolation cluster, which is equal to the square of the order parameter $O$. In the models under consideration, the susceptibility near the critical threshold $t_c$ is

$$\chi = \sum_s sQ(s)^2 \frac{P(s)}{N} \approx m^2 \sum_s sP(s) = m^2 \langle s \rangle_P, \quad (23)$$

where we used Eq. (23), and $\langle s \rangle_P$ is the first moment of the distribution $P(s)$. For $t > t_c$, summing both sides of Eq. (1) over $s$ we get

$$\frac{\partial S}{\partial t} = 2[1 - (1 - S)^m] \sum_s sQ(s) \approx 2m^2 S \langle s \rangle_P. \quad (24)$$

Similarly, for $t < t_c$, multiplying both sides of Eq. (1) by $s$ and summing over $s$ we obtain

$$\frac{\partial \langle s \rangle_P}{\partial t} = 2 \left( \sum_s sQ(s) \right)^2 \approx 2m^2 \langle s \rangle_P^2. \quad (25)$$

Using Eqs. (24), (25), and (26) we find that the first moment of the distribution $P(s)$ is symmetric below and above $t_c$, namely $\langle s \rangle_P \approx (2m^2)^{-1}[t - t_c]^{-1}$. Then, the asymptotics of the susceptibility is independent of $m$,

$$\chi \approx \frac{1}{2}[t - t_c]^{-1}. \quad (26)$$

The critical singularity of the order parameter $O = 1 - (1 - S)^m \approx mS$ is

$$O \approx 8\pi m^3 a_0^2 (t - t_c), \quad (27)$$

where we have used Eq. (24). Notice that, in contrast to $\chi$, the critical amplitude of the order parameter $O$ depends on $a_0$ and $m$.

V. ESTIMATES

Let us estimate $P(s, t)$ by substituting the approximated relation (23) into the evolution equation (4).

$$\frac{\partial P(s, t)}{\partial t} \approx m^2 s \sum_{u=1}^{s-1} P(u, t)P(s-u, t) - 2msP(s, t). \quad (28)$$

which is valid for large $s$ and $t$ close to $t_c$. Let us rewrite the last equation in terms of the rescaled distribution $P(\tilde{s}, \tilde{t}) = mP(s, t)$ and time $\tilde{t} \equiv mt$,

$$\frac{\partial \tilde{P}(\tilde{s}, \tilde{t})}{\partial \tilde{t}} = \sum_{u=1}^{s-1} \tilde{P}(u, \tilde{t})\tilde{P}(s-u, \tilde{t}) - 2\tilde{s}\tilde{P}(\tilde{s}, \tilde{t}). \quad (29)$$

We assume that Eq. (29), being asymptotically exact near the critical point, describes approximately $P(s, t)$ in the full range of cluster sizes and time. This equation coincides with the exact Eq. (1) for ordinary percolation ($m = 1$), which, for the initial condition $P(s, 0) = \delta_{s,1}$, has the solution $P(s, t_c) = 1/(\sqrt{2}\pi)s^{-3/2}$ at the critical point $t_c = 1/2$. This readily leads to the following estimates for $t_c$ and $a_0$ of our problem:

$$t_c^e = \frac{1}{2m}, \quad (30)$$

and

$$a_0^e = \frac{1}{\sqrt{2\pi m}}. \quad (31)$$

We also estimate the critical amplitude $B$ of the percolation cluster relative size, $S \approx B(t - t_c)$. Inserting $a_0^e$ into Eq. (21) gives $B^e = 4$, independently of $m$. Table I shows the numerical results for $t_c$, $a_0$, and $B$ for different $m$, and compares them with the estimates of this section. Notice that our simple estimate produces surprisingly accurate results for $m > 1$. The estimate $t_c^e$ is especially good, with an error of only 5 to 10%, while the estimate $a_0^e$ has a relative error about 2 times larger.

VI. CONCLUSIONS

In the present paper we have demonstrated that two types of the local optimization rule result in contrasting effects. The original Achlioptas rule based on selection of the smallest clusters for merging together drastically changes the critical features of continuous phase
For instance, at each step apply one of the two rules at random. Our results suggest that this is similar to the combination of the Achlioptas rule and the interconnection of random nodes, which leads to the usual explosive percolation effects \cite{49, 33}. We based our conclusions on a set of models convenient for analytical treatment. We expect however that these conclusions are qualitatively valid for a much wider class of processes with inverse Achlioptas rules.

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