On relations between transportation cost spaces and $L_1$

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Abstract

The present paper deals with some structural properties of transportation cost spaces, also known as Arens-Eells spaces, Lipschitz-free spaces and Wasserstein spaces. The main results of this work are: (1) A necessary and sufficient condition on an infinite metric space $M$, under which the transportation cost space on $M$ contains an isometric copy of $\ell_1$. The obtained condition is applied to answer the open questions asked by Cúth and Johanis (2017) concerning several specific metric spaces. (2) The description of the transportation cost space of an unweighted finite graph $G$ as the quotient $\ell_1(E(G))/Z(G)$, where $E(G)$ is the edge set and $Z(G)$ is the cycle space of $G$. This is a generalization of the previously known result to the case of any finite metric space.

Keywords. Arens-Eells space, Banach space, earth mover distance, Kantorovich-Rubinstein distance, Lipschitz-free space, transportation cost, Wasserstein distance

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1 Introduction

1.1 Definitions

Let $(M,d)$ be a metric space. Consider a real-valued finitely supported function $f$ on $M$ with a zero sum, that is, $\sum_{v \in \text{supp} f} f(v) = 0$. A natural and important interpretation of such function, which goes back to at least Kantorovich-Gavurin [18], is to consider it as a transportation problem: one needs to transport certain product from locations where $f(v) > 0$ to locations where $f(v) < 0$. More formally, we represent $f$ as $f = a_1(1_{x_1} - 1_{y_1}) + a_2(1_{x_2} - 1_{y_2}) + \cdots + a_n(1_{x_n} - 1_{y_n})$, where $a_i \geq 0$, $x_i, y_i \in M$, and $1_u(x)$ for $u \in M$ is the indicator function of $u$. The cost of the corresponding transportation plan (which consists in moving $a_i$ units
from $x_i$ to $y_i$) is defined as $\sum_{i=1}^{n} a_i d(x_i, y_i)$. We denote the real vector space of all transportation problems by $\text{TP}(M)$. We introduce the transportation cost norm $\|f\|_{TC}$ of a transportation problem $f$ as the minimal cost over all such transportation plans. It is easy to see that the minimum is attained - we consider finitely supported functions - and that $\| \cdot \|_{TC}$ is a norm. The completion of this normed space is called a transportation cost space and is denoted by $\text{TC}(M)$.

Transportation cost spaces are of interest in many areas and are studied under many different names (the most common ones are included in the keywords section). We prefer to use the term transportation cost space since it makes the subject of this work instantly clear to a wide circle of readers and it also reflects the historical approach leading to these notions. Interested readers can find a review of the main definitions, notions, facts, terminology and historical notes pertinent to the subject in [24].

In the theory of metric embeddings, transportation cost spaces are of interest due to the following observation by Arens and Eells [2]: The metric space $M$ admits a canonical isometric embedding into $\text{TC}(M)$, given by $v \mapsto 1_v - 1_O$, where $O$ is a base point in $M$.

For background information on the transportation cost spaces we refer to [25, Chapter 10] (where such spaces are called Lipschitz free spaces) and [26, Chapter 3] (where such spaces are called Arens-Eells spaces).

1.2 Motivation and statement of results

In this work, new results pertinent to the relations between the structure of transportation cost spaces and $L_1$ are obtained. Previously, such relations have been studied by many researches, see, for example, [1], [4]–[10], [13], [15], [16], [19], [20], [23], [24].

From the historical perspective, there are a few arguments in favor of studying the Banach-space-theoretical structure of transportation cost spaces. Below, some of them are provided:

(1) The linearization of the theory of cotype for metric spaces. This idea was put forth by Johnson (see [3] and the discussion in [22]).

(2) The program of using transportation cost spaces to solve some important problems of linear and nonlinear theory of Banach spaces suggested by Godefroy-Kalton [14], who used the name Lipschitz-free spaces. This program was substantially advanced by Kalton [17].

(3) The observation by Arens and Eells [2] (see above) shows that transportation cost spaces are natural target spaces for metric embeddings. See [25, Chapter 10].

The main results of this paper include the following outcomes:

1) A necessary and sufficient condition for containment of $\ell_1$ in $\text{TC}(M)$ isometrically (Theorem 2.1). This result relies on the previous studies in this direction,
namely, [7, 24, Theorem 3.1], and [19]. It is used to answer the questions in [7, Remark 10, p. 3416] which were left open in [7] and [24].

2) A generalization of the quotient over the cycle space description of TC(G) for an unweighted finite graph G (see [25, Proposition 10.10]) to the case of an arbitrary finite metric space. It has to be mentioned that somewhat similar descriptions for TC(R^n) were obtained in [8] and [15].

2 Isometric copies of ℓ_1 in TC(M) with infinite M

In what follows, the standard terminology of matching theory [21] is used. Consider M as an infinite weighted complete graph, where the weight of each edge is defined as the distance between its ends.

Theorem 2.1. The space TC(M) contains ℓ_1 isometrically if and only if there exists a sequence of pairs \{x_i, y_i\}^\infty_{i=1} in M, with all elements distinct, such that each set \{x_i y_i\}^n_{i=1} of edges is a minimum weight perfect matching in the subgraph spanned by \{x_i, y_i\}^n_{i=1}.

Proof. Sufficiency. Let \{x_i, y_i\}^\infty_{i=1} be such a sequence. Set f_i = (1, x_i - y_i)/d(x_i, y_i). Since each finite set \{f_i\}^n_{i=1}, n \in \mathbb{N} is isometrically equivalent to the unit vector basis of ℓ_1^n by the argument of [19], we are done.

Necessity. Recall that a metric space M is called uniformly discrete if there exists a constant \delta > 0 such that

\[ \forall u, v \in M \ (u \neq v) \Rightarrow (d(u, v) \geq \delta). \]

To prove the necessity, we shall consider the three cases:

(A) The space M has an accumulation point, which means that there is a sequence \{u_i\}^\infty_{i=1} of distinct elements in M and u \in M, such that \lim_{i \to \infty} d(u_i, u) = 0.

(B) The space M is not uniformly discrete, but does not have an accumulation point.

(C) The space M is uniformly discrete.

The proof of the necessity will be performed according to the following steps:

• First, we derive that in Cases (A) and (B), the space M contains a sequence of pairs \{x_i, y_i\}^\infty_{i=1} such that for each n \in \mathbb{N} the set \{x_i y_i\}^n_{i=1} of edges is a minimum weight perfect matching in the subgraph spanned by \{x_i, y_i\}^n_{i=1}.
Further, it will be shown that in Case (C) either $M$ contains a sequence of pairs 
\( \{x_i, y_i\}_{i=1}^{\infty} \) such that for each $n \in \mathbb{N}$ the set \( \{x_i, y_i\}_{i=1}^{n} \) of edges is a minimum weight perfect matching in the subgraph spanned by \( \{x_i, y_i\}_{i=1}^{n} \), or TC($M$) does not contain an isometric copy of $\ell_1$.

**Case (A).** If $M$ has an accumulation point $u$, and \( \{u_i\}_{i=1}^{\infty} \) is in (A), then either there are infinitely many disjoint pairs \( (i, j) \), \( i, j \in \mathbb{N}, j > i \), such that all of the triangle inequalities below are strict:

\[
d(u_i, u_j) < d(u_i, u_t) + d(u, u_j),
\]

or, after eliminating finitely many elements of the sequence, for all of the remaining ones, the equality is reached in the triangle inequalities:

\[
d(u_i, u_j) = d(u_i, u_t) + d(u, u_j).
\]

To finish the proof, it suffices to select two disjoint subsequences \( \{x_i\}_{i=1}^{\infty} \) and \( \{y_i\}_{i=1}^{\infty} \) of the sequence \( \{u_i\}_{i=1}^{\infty} \) in such a way that, for each $n \in \mathbb{N}$, the set of edges \( \{x_i y_i\}_{i=1}^{n} \) is the minimum weight perfect matching in the complete graph with vertices \( \{x_i, y_i\}_{i=1}^{n} \).

First, this will be done in the easier case where all triangles inequalities are equalities, see (2). In this case, one may let $x_i = u_{2i-1}$ and $y_i = u_{2i}$ and check that, by virtue of equalities (2), any perfect matching in the weighted graph with the vertices \( \{u_i\}_{i=1}^{2n} \) is a minimum weight matching since all of them have weight $\sum_{i=1}^{2n} d(u_i, u)$.

In the case where (1) is satisfied for infinitely many disjoint pairs \( \{(i_t, j_t)\}_{t=1}^{\infty} \), we combine the fact that the differences $d(u, u_{i_t}) + d(u, u_{j_t}) - d(u_{i_t}, u_{j_t})$ are strictly positive and $\lim_{t \to \infty} d(u, u_t) = 0$, and get that one can pass to a subsequence (preserving the notation \( \{(i_t, j_t)\}_{t=1}^{\infty} \) for the subsequence) in such a way that, for each $t$ and for any $k_1$ and $k_2$ which are $i_s$ or $j_s$ for some $s > t$, the next inequality holds:

\[
d(u_{i_t}, u_{k_1}) + d(u_{j_t}, u_{k_2}) > \sum_{m=t}^{\infty} d(u_{i_m}, u_{j_m}).
\]

Now, let $x_1 = u_{i_1}, y_1 = u_{j_1}, \ldots, x_n = u_{i_n}, y_n = u_{j_n}, \ldots$.

To complete the proof, it remains to check that \( \{x_i y_i\}_{i=1}^{n} \) is the minimum weight perfect matching in the complete graph spanned by \( \{x_i, y_i\}_{i=1}^{n} \). It will be proved inductively that $x_i y_i, i = 1, 2, \ldots$, should be in the minimum weight perfect matching. Assume that there is a minimum weight perfect matching which does not contain $x_i y_i$. Then we have to match $x_1 = u_{i_1}$ with some $u_{k_1}$ and $y_1 = u_{j_1}$ with some $u_{k_2}$, where $k_1$ and $k_2$ are $i_s$ or $j_s$ for some $s \in \{2, \ldots, n\}$. But then (3) implies that the sum $d(x_1, u_{k_1}) + d(y_1, u_{k_2})$ is strictly larger than the weight of the matching \( \{x_i y_i\}_{i=1}^{n} \). Thus, $x_1 y_1$ should be in any minimum weight perfect matching.
It is clear that the same argument can be repeated for \(x_2y_2\), and so on. This completes the proof in the case where (1) is satisfied for infinitely many disjoint pairs, and, thus, for the case where \(M\) has an accumulation point.

**Remark 2.2.** Our argument in Case (A) is close to the one in [7, Theorem 5]. For the convenience of the reader, we presented our argument in the form independent of [7].

Case (B). Since the space \(M\) is not uniformly discrete, there are sequences \(\{u_i\}_{i=1}^\infty\) and \(\{v_i\}_{i=1}^\infty\) in \(M\) such that \(u_i \neq v_i\) for all \(i \in \mathbb{N}\) and \(\lim_{i \to \infty} d(u_i, v_i) = 0\).

The following standard lemma will be applied.

**Lemma 2.3.** Each sequence \(\{w_i\}_{i=1}^\infty\) in a metric space either contains a Cauchy subsequence or a \(\delta\)-separated subsequence, where \(\delta\) is some positive number and \(\delta\)-separated means that any two elements are at distance at least \(\delta\).

We apply Lemma 2.3 to the sequence \(\{u_i\}_{i=1}^\infty\) and keep the notation \(\{u_i\}_{i=1}^\infty\) for the obtained subsequence, and the notation \(\{v_i\}_{i=1}^\infty\) for the corresponding subsequence of \(\{v_i\}_{i=1}^\infty\).

If \(\{u_i\}_{i=1}^\infty\) is a Cauchy sequence, we consider the completion \(\widetilde{M}\) of \(M\) and a point \(\tilde{u} \in \widetilde{M}\) such that \(\lim_{n \to \infty} d(u_i, \tilde{u}) = 0\), and construct, as in Case (A), sequences \(\{x_i\}_{i=1}^\infty\) and \(\{y_i\}_{i=1}^\infty\). Since these sequences are subsequences of \(\{u_i\}_{i=1}^\infty\), the constructed subspace (see the proof of **Sufficiency**) is not only in \(\text{TC}(M)\), but also in \(\text{TC}(\widetilde{M})\).

Now, assume that \(\{u_i\}_{i=1}^\infty\) is \(\delta\)-separated for some \(\delta > 0\). In this case, after omitting finite number of terms in the sequences \(\{u_i\}_{i=1}^\infty\) and \(\{v_i\}_{i=1}^\infty\), we may assume that \(d(u_i, v_i) < \delta/4\) for every \(i\). Denote the obtained subsequences of \(\{u_i\}_{i=1}^\infty\) and \(\{v_i\}_{i=1}^\infty\) by \(\{x_i\}_{i=1}^\infty\) and \(\{y_i\}_{i=1}^\infty\), respectively.

We have \(d(x_i, y_i) < \delta/4\) and also, due to the triangle inequality and \(\delta\)-separation of \(\{x_i\}_{i=1}^\infty\), the inequalities \(d(x_i, y_j) > 3\delta/4\) and \(d(y_i, y_j) > \delta/2\) for \(i \neq j\) hold. These inequalities immediately imply that, for every \(n \in \mathbb{N}\) the set \(\{x_i, y_i\}_{i=1}^n\) of edges is a minimum weight perfect matching in the subgraph spanned by \(\{x_i, y_i\}_{i=1}^n\). This completes our proof in Case (B).

Case (C). We show that if we assume both

(i) That \(\text{TC}(M)\) contains a sequence \(\{f_i\}_{i=1}^\infty\) which is isometrically equivalent to the unit vector basis of \(\ell_1\),

and

(ii) That \(M\) does not contain a sequence of pairs \(\{x_i, y_i\}_{i=1}^\infty\), such that each set \(\{x_i, y_i\}_{i=1}^n\) of edges is a minimum weight perfect matching in the subgraph spanned by \(\{x_i, y_i\}_{i=1}^n\),

we get a contradiction.
We start with a simple case where all \( \{f_i\} \) are in \( TP(M) \). Since our result is isometric, this will not complete the proof for the general case. However, in the easier case \( \{f_i\} \subset TP(M) \), the main ideas are more transparent.

For each element of the sequence \( f_i \), we pick an optimal transportation plan (it does not have to be unique):

\[
f_i = \sum_{j=1}^{m(i)} a_{j,i}(1_{x_{j,i}} - 1_{y_{j,i}}), \quad a_{j,i} > 0.
\]

We say that \( T_i := \{x_{j,i}, y_{j,i}\}_{j=1}^{m(i)} \) is the set of transportation pairs for \( f_i \).

**Lemma 2.4.** Each sequence \( \{f_i\}_{i=1}^{\infty} \subset TP(M) \) contains a subsequence \( \{f_{i_n}\}_{n=1}^{\infty} \) satisfying at least one of the two conditions:

1. In each \( T_{i_n} \), there exists a transportation pair such that the obtained set of transportation pairs is pairwise disjoint.

2. One can pick in each \( T_{i_n} \) a transportation pair such that, for the obtained set of transportation pairs, there is an element \( x \in M \) contained in each of them.

**Proof.** This lemma can be proved by considering an alternative: either there is an element \( x \in M \) contained in infinitely many transportation pairs or there is no such element. \( \square \)

We apply Lemma 2.4 to the sequence \( \{f_i\}_{i=1}^{\infty} \) equivalent to the unit vector basis of \( \ell_1 \). Assume that condition (ii) is satisfied for one of its subsequences, which we still denote \( \{f_i\}_{i=1}^{\infty} \). Without loss of generality (and changing signs of \( \{f_i\} \) if needed), it may be assumed that the disjoint pairs are \( (x_{1,i}, y_{1,i}) \), denoted by \( (x_i, y_i) \), for short. By our assumption (ii), there is \( n \in \mathbb{N} \) such that \( \{x_i y_i\}_{i=1}^{n} \) is not a minimum weight perfect matching of the graph spanned by \( \{x_i, y_i\}_{i=1}^{n} \). Pick a minimum weight perfect matching for this graph. Interchanging the labels in some pairs \( (x_i, y_i) \) and changing the signs of the corresponding \( f_i \) if needed, one may assume that the minimum weight perfect matching is of the form \( \{x_i y_{\pi(i)}\}_{i=1}^{n} \) for some bijection \( \pi : \{1, \ldots, n\} \rightarrow \{1, \ldots, n\} \).

Let \( a_i > 0 \) be the quantity transported from \( x_i \) to \( y_i \) in the plans (4) (written in a different notation). Set \( a = \min_{1 \leq i \leq n} a_i > 0 \).

Now, consider the vector \( \sum_{i=1}^{n} f_i \) and construct the following transportation plan for it: the plan is close to being the sum of plans (4), but

\[
\sum_{i=1}^{n} a_i(1_{x_i} - 1_{y_i})
\]

in it is replaced by
\[
\sum_{i=1}^{n} (a_i - a)(1_{x_i} - 1_{y_i}) + \sum_{i=1}^{n} a(1_{x_i} - 1_{y_{\pi(i)}}).
\]  

(6)

Since \( \{x_i,y_i(i)\}_{i=1}^{n} \) is a minimum weight perfect matching while \( \{x_i,y_i\}_{i=1}^{n} \) is not, the obtained transportation plan has a strictly smaller cost than the sum of the plans (4). This leads to

\[
\left\| \sum_{i=1}^{n} f_i \right\|_{TC} < \sum_{i=1}^{n} \|f_i\|_{TC},
\]

which is a contradiction.

Next, suppose that condition (2) of Lemma 2.4 is satisfied for a subsequence of \( \{f_i\}_{i=1}^{\infty} \), which we still denote \( \{f_i\}_{i=1}^{\infty} \). Relabelling, it may be assumed that \( \{x_{1,i}, y_{1,i}\}_{i=1}^{\infty} \) are such that all \( x_{1,i} \) are the same, let us denote all of them by \( x \). We claim that if \( \{f_i\}_{i=1}^{\infty} \) are isometrically equivalent to the unit vector basis of \( \ell_1 \), then

\[
\forall i, j \in \mathbb{N} \quad d(x, y_{1,i}) + d(x, y_{1,j}) = d(y_{1,i}, y_{1,j}).
\]

(7)

Assume the contrary, that is,

\[
\exists i, j \in \mathbb{N} \quad d(x, y_{1,i}) + d(x, y_{1,j}) > d(y_{1,i}, y_{1,j}).
\]

(8)

Let \( a = \min \{a_{1,i}, a_{1,j}\} \). Consider the function \( f_i - f_j \), subtract the corresponding plans (4), and make the following modification in the resulting transportation plan. We replace the difference

\[
a_{1,i}(1_{x} - 1_{y_{1,i}}) - a_{1,j}(1_{x} - 1_{y_{1,j}})
\]

by

\[
(a_{1,i} - a)(1_{x} - 1_{y_{1,i}}) - (a_{1,j} - a)(1_{x} - 1_{y_{1,j}}) + a(1_{y_{1,j}} - 1_{y_{1,i}}).
\]

The strict inequality (8) implies that it is a strictly better plan. Thus, \( \|f_i - f_j\|_{TC} < \|f_i\|_{TC} + \|f_j\|_{TC} \), and this contradiction proves (7).

Now, we introduce a new sequence \( \{\tilde{x}_i, \tilde{y}_i\}_{i=1}^{\infty} \) by setting \( \tilde{x}_i = y_{1,2i-1} \) and \( \tilde{y}_i = y_{1,2i} \). It is easy to see that (7) implies that the sequence of pairs \( \{\tilde{x}_i, \tilde{y}_i\}_{i=1}^{\infty} \) satisfies the condition in (11), and we get a contradiction. This completes the proof in the case \( \{f_i\} \subset \text{TP}(M) \).

In the rest of this proof, our aim is to generalize the argument we just presented to the case where elements \( \{f_i\}_{i=1}^{\infty} \) are not in \( \text{TP}(M) \) but in its completion \( \text{TC}(M) \).

By the standard description of the completion (see [11], Section 3.11.4], each element \( f \in \text{TC}(M) \) can be presented as a series of the form

\[
f = \sum_{k=1}^{\infty} \left( \sum_{i=s_k+1}^{s_{k+1}} a_i (1_{x_i} - 1_{y_i}) \right)
\]

(9)
for some $0 = s_1 < s_2 < \cdots < s_k < \ldots$, $\{x_i\} \subset M$, $\{y_i\} \subset M$, and $\{a_i\} \subset \mathbb{R}^+$ with

$$\sum_{k=1}^{\infty} \left\| \sum_{i=s_k+1}^{s_{k+1}} a_i (1_{x_i} - 1_{y_i}) \right\|_{TC} < \infty.$$  \hfill (10)

Furthermore, the norm $\|f\|_{TC}$ is equal to the infimum of sums of the form (10) over all representations (9), in which we assume that the sums in brackets are optimal transportation plans for the corresponding elements of $\text{TP}(M)$.

We are now considering Case (C) where the space $M$ is uniformly discrete. Let

$$\delta = \inf_{x,y \in M} d(x,y).$$ \hfill (11)

For $f \in \text{TP}(M)$, we introduce $\|f\|_1 = \sum_{v \in \text{supp} f} |f(v)|$ and extend this norm to functions on $M$ with countable supports and absolutely summable collections of values.

For $f \in \text{TP}(M)$ the amount of the product which is to be delivered is $\|f\|_1/2$ and the vector $f$ is in the kernel of the linear functional on finitely supported vectors defined as the sum of the values. By (11), we get that $\delta \|f\|_1/2 \leq \|f\|_{TC}$ for any $f \in \text{TP}(M)$. For this reason, the space $\text{TC}(M)$ is continuously embedded in $\ell_1(M)$, and is in the kernel of the functional defined as the sum of all coordinates (this functional is naturally defined for $f \in \ell_1(M)$).

**Lemma 2.5.** Let $g \in \text{TP}(M)$ and $h \in \text{TC}(M)$ be such that $\|h\|_{TC} < \|g\|_{TC}$, and the diameter of the support of $g$ be $\leq D$. Then $\|g + h\|_1 \geq \frac{2}{D} (\|g\|_{TC} - \|h\|_{TC})$.

**Proof.** If $h \notin \text{TP}(M)$, we can approximate $h$ arbitrarily well - both in $\|\cdot\|_{TC}$ and $\|\cdot\|_1$ - by vectors belonging to $\text{TP}(M)$. For this reason, we may assume that $h \in \text{TP}(M)$.

Let us write an optimal transportation plan for $h$:

$$h = a_1 (1_{u_1} - 1_{v_1}) + a_2 (1_{u_2} - 1_{v_2}) + \cdots + a_n (1_{u_n} - 1_{v_n}), \quad a_i > 0.$$ \hfill (12)

By combining the corresponding terms, we may and shall assume that none of the $v_i$ is equal to some of the $u_j$. We split the transportation plan in (12) into two sums: (a) The sum $h_1$ which contains the terms $a_i (1_{u_i} - 1_{v_i})$ with at most one of the elements $u_i, v_i$ being in the support of $g$; (b) The sum $h_2$ which contains those terms $a_i (1_{u_i} - 1_{v_i})$ for which both of the elements $u_i, v_i$ are in the support of $g$.

The important and easy-to-see observation is that $\|g + h_1 + h_2\|_1 \geq \|g + h_2\|_1$. This is because each term of $h_1$ can decrease the value of $g + h_2$ at some point, but it adds the same amount elsewhere.

It remains to estimate $\|g + h_2\|_1$. Notice that

$$\|g + h_2\|_{TC} \geq \|g\|_{TC} - \|h_2\|_{TC} \geq \|g\|_{TC} - \|h\|_{TC},$$

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and that - due to the choice of $h_2$ - the diameter of the support of $g + h_2$ also does not exceed $D$. It remains to observe that $\|g + h_2\|_{TC}$ does not exceed the amount of product which is to be moved - that is, $\|g + h_2\|_1/2$ - times the maximal distance which this product has to travel - that is, $D$. Therefore,

$$\|g + h_2\|_1 \geq \frac{2}{D} \|g + h_2\|_{TC},$$

whence the conclusion follows. □

It has to be pointed out that Lemma 2.5 together with equation (9) along with the fact that $\|f\|_{TC}$ is the infimum of sums (10) implies that $m_i := \|f_i\|_1 > 0$. Furthermore, it is easy to see that equation (9) and the fact that $\|f\|_{TC}$ is the infimum of sums (10) imply that, for each $i, m \in \mathbb{N}$, one can write $f_i = S_i^m + R_i^m$ where $S_i^m \in TP(M)$, $R_i^m \in TC(M)$, $1 - 2^{-m} \leq \|S_i^m\|_{TC} \leq 1 + 2^{-m}$, $\|R_i^m\|_{TC} \leq 2^{-m+1}$, and $\|R_i^m\|_1 \leq m_i/8$. Using the last inequality, one arrives at $\|S_i^m\|_1 \geq 7m_i/8$.

Writing $f_{i}^+$ and $f_{i}^-$ for the non-negative and non-positive parts of $f_i$, we have $\|f_{i}^+\|_1 = \|f_{i}^-\|_1 = m_i/2$, and therefore we can select in the support of $f_{i}^+$ a finite subset $V_{i}^+$ such that $\sum_{v \in V_{i}^+} f_{i}(v) \geq \frac{7m_i}{16}$ and in the support of $f_{i}^-$ a finite subset $V_{i}^-$ such that $\sum_{v \in V_{i}^-} f_{i}(v) \leq -\frac{7m_i}{16}$.

Consider optimal transportation plans for $\{S_i^m\}_{m=1}^\infty$. For each of them, create a matrix whose columns are labelled by elements of $V_{i}^+$ and whose rows are labelled by elements of $V_{i}^-$. In the intersection of the column corresponding to $x$ and the row corresponding to $y$, we record the amount of product which is moved in such an optimal plan from $x$ to $y$, while we put 0 if nothing is moved. We claim that the sum of all entries of the obtained matrix is at least $\frac{4m_i}{16} = \frac{m_i}{4}$.

To prove this, let us introduce the following functions on $V_{i}^+$ and $V_{i}^-$, respectively:

$$P_{i}^m(v) = \begin{cases} S_{i}^m(v) & \text{if } S_{i}^m(v) > 0 \text{ and } v \in V_{i}^+ \\ 0 & \text{for all other } v \in V_{i}^+ \end{cases}$$

$$N_{i}^m(v) = \begin{cases} S_{i}^m(v) & \text{if } S_{i}^m(v) < 0 \text{ and } v \in V_{i}^- \\ 0 & \text{for all other } v \in V_{i}^- \end{cases}$$

It has to be shown that in any optimal transportation plan for $S_i^m$, a nontrivial part of product which is available at points of $V_{i}^+$, that is $P_{i}^m$, should be transported to satisfy the need at points of $V_{i}^-$, that is $N_{i}^m$. Evidently, each unit of product available in the transportation problem $S_i^m$ at points where $P_{i}^m > 0$ should be moved to the points where $S_{i}^m(v) < 0$. These points can be the ones where $N_{i}^m(v) < 0$, but can be also some other points where $S_{i}^m(v) < 0$. To estimate the amount which

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should be moved to the points where \( N_i^m(v) < 0 \) we need some inequalities. We start with

\[
\|P_i^m\|_1 \geq \|f_i^+\|_1 - \frac{m_i}{16} - \|(R_i^m)^+\|_1 \geq \frac{6m_i}{16}.
\]

To see that some of these \( \|P_i^m\|_1 \) units have to go to the points where \( N_i^m < 0 \), we need to prove that outside \( V_i^- \) the total amount of negative values of \( S_i^m \) is relatively small. In fact, since \( S_i^m = f_i - R_i^m \) such values can occur either at the points \( v \) where \( f_i(v) < 0 \), but \( v \notin V_i^- \), or because of subtraction of \( (R_i^m)^+ \). Therefore, the total amount of this need is \( \leq \frac{2m_i}{16} \). Thus at least \( \frac{4m_i}{16} = \frac{m_i}{4} \) of the product available at the points where \( P_i^m(v) > 0 \) should be transported to the points where \( N_i^m(v) < 0 \), as claimed.

Since the matrix described in the paragraph where we introduced \( V_i^+ \) and \( V_i^- \), is finite and a sum of its entries is at least \( \frac{m_i}{4} \), there exists a subsequence in \( m \) such that, for some choice of \( x_i \in V_i^+ \) and \( y_i \in V_i^- \), the amount of transported product from \( x_i \) to \( y_i \) will be at least \( \varepsilon_i > 0 \), which is a positive number depending only on \( i \). Since subsequences \( \{S_i^m\}_{m=1}^\infty \) and \( \{R_i^m\}_{m=1}^\infty \) also satisfy the defining inequalities for \( \{S_i^m\}_{m=1}^\infty \) and \( \{R_i^m\}_{m=1}^\infty \), we may assume without loss of generality, that the subsequences are \( \{S_i^m\}_{m-1}^\infty \) and \( \{R_i^m\}_{m=1}^\infty \) themselves.

From here on, we follow the same line of argument as in the first part of the proof (for Case (C)). Using the same reasoning as in Lemma 2.4 one obtains, after taking subsequences in \( i \) the following alternatives: (1) the pairs \( \{x_i, y_i\} \) are disjoint; (2) they all have a common element (changing signs of \( f_i \) we can assume that all of \( x_i \) are the same).

If alternative (1) holds, using the assumption (ii), we conclude that there exists a finite subcollection \( \{x_i, y_i\}_{i=1}^n \) such that in the subgraph spanned by it, there is a perfect matching with a smaller weight. After that one can apply the same “improvement of a transportation plan” as we used when replacing (5) by (6). This improvement shows that there is \( \tau_n > 0 \) such that

\[
\|S_1^m + \cdots + S_n^m\|_{TC} \leq \|S_1^m\|_{TC} + \cdots + \|S_n^m\|_{TC} - \tau_n.
\]

A crucial issue is that this holds for every \( m \in \mathbb{N} \). More precisely, formula (i) shows that \( \tau_n \) can be chosen to be the product of \( \min_{1 \leq i \leq n} \varepsilon_i \) and the difference between the weights of the matching \( \{x_i y_i\}_{i=1}^n \) and the minimum weight perfect matching in the subgraph spanned by \( \{x_i, y_i\}_{i=1}^n \).

Therefore, one obtains

\[
\|f_1 + \cdots + f_n\|_{TC} \leq \|S_1^m + \cdots + S_n^m\|_{TC} + \|R_1^m + \cdots + R_n^m\|_{TC} \\
\leq n \cdot (1 + 2^{-m}) - \tau_n + n2^{-m+1}.
\]

Since this inequality holds for every \( m \) and \( \tau_n \) does not depend on \( m \), we can pick \( m \) in such a way that the number in the rightmost side of the last inequality
is < n. This gives a contradiction with the assumption that \( \{f_i\} \) is isometrically equivalent to the unit vector basis of \( \ell_1 \).

Finally, consider alternative (2): the pairs \( \{x_i, y_i\} \) have a common point. As above, one may assume that this common point coincides with all of \( x_i \) and denote it by \( x \). Similarly to the argument above, the goal is to prove the equalities in some of the triangle inequalities. Now the desired equalities are:

\[
d(x, y_i) + d(x, y_j) = d(y_i, y_j) \quad \text{for } i \neq j. \tag{14}
\]

The proof goes according to the same steps as above. If one of the triangle inequalities is strict, that is \( d(x, y_i) + d(x, y_j) > d(x_i, x_j) \), we can find \( \tau_{i,j} > 0 \) such that \( \|S_i^m + S_j^m\|_{TC} \leq \|S_i^m\|_{TC} + \|S_j^m\|_{TC} - \tau_{i,j} \) for every \( m \). From here, we derive \( \|f_i + f_j\|_{TC} < \|f_i\|_{TC} + \|f_j\|_{TC} \), contrary to the assumption that \( \{f_i\} \) is isometrically equivalent to the unit vector basis of \( \ell_1 \).

After establishing (14), we complete the proof as in the previous case. \( \square \)

**Example 2.6.** As an application of Theorem 2.1, we use it to answer the questions on isometric presence of \( \ell_1 \) in \( TC(M) \) for metric spaces \( M \) listed in [7, Remark 10, p. 3416], for which the answer has not been known. In all of the examples \( M = \{v_n\}_{n=1}^\infty \). The metrics on \( M \) are defined for \( n > k \) as follows:

(a) \( \rho(v_k, v_n) = k + n - \frac{1}{k} \)
(b) \( \rho(v_k, v_n) = 2 - \frac{1}{k} + \frac{1}{n} \)
(c) \( \rho(v_k, v_n) = 2 - \frac{1}{k} - \frac{1}{2n} \)
(d) \( \rho(v_k, v_n) = 1 + \frac{1}{n} \)
(e) \( \rho(v_k, v_n) = 1 + \frac{1}{2k} + \frac{1}{n} \)

Using Theorem 2.1 we can prove that in all of the examples the answer is negative - the corresponding transportation cost spaces do not contain isometric copies of \( \ell_1 \).

In each of the cases, we prove that, for any selected sequence \( \{x_i, y_i\}_{i=1}^\infty \) of pairs of distinct elements in the metric space, one can find \( m \) such that the set \( \{x_i y_i\}_{i=1}^m \) of edges is not a minimum weight perfect matching in the complete graph spanned by \( \{x_i, y_i\}_{i=1}^m \) (with weight of each edge equal to the distance between its ends).

The main observation here is that no matter how the sequence \( \{x_i, y_i\}_{i=1}^\infty \) is selected, it is possible to pick two pairs \( (x_j, y_j) \) and \( (x_m, y_m) \), \( j < m \), such that the indices of vertices \( x_m \) and \( y_m \) in the sequence \( \{v_n\}_{n=1}^\infty \) are larger than the indices of \( x_j \) and \( y_j \). Without loss of generality we may assume that indices of \( x_j, y_j, x_m, y_m \) are \( q_1 < q_2 < q_3 < q_4 \), respectively.

This will immediately imply the desired conclusion of the previous paragraph as soon as it will be derived that \( d(x_j, x_m) + d(y_j, y_m) < d(x_j, y_j) + d(x_m, y_m) \). Hence,
it remains to verify this inequality in cases (a)-(e). Indeed, direct calculations lead to the following inequalities, which are obviously true:

(a) \( q_1 + q_3 - \frac{1}{q_1} + q_2 + q_4 - \frac{1}{q_1} < q_1 + q_2 - \frac{1}{q_1} + q_4 - \frac{1}{q_2} \) or \( \frac{1}{q_1} < \frac{1}{q_2} \).

(b) \( 2 - \frac{1}{q_1} + \frac{1}{q_3} + 2 - \frac{1}{q_2} + \frac{1}{q_4} < 2 - \frac{1}{q_1} + \frac{1}{q_2} + 2 - \frac{1}{q_3} + \frac{1}{q_4} \) or \( \frac{1}{q_1} < \frac{1}{q_2} \).

(c) \( 2 - \frac{1}{q_1} - \frac{1}{q_2} + 2 - \frac{1}{q_3} - \frac{1}{q_4} < 2 - \frac{1}{q_1} - \frac{1}{q_2} + 2 - \frac{1}{q_3} - \frac{1}{q_4} \) or \( \frac{1}{q_1} < \frac{1}{q_2} \).

(d) \( 1 + \frac{1}{q_3} + 1 + \frac{1}{q_4} < 1 + \frac{1}{q_2} + 1 + \frac{1}{q_4} \) or \( \frac{1}{q_1} < \frac{1}{q_2} \).

(e) \( 1 + \frac{1}{2q_1} + \frac{1}{q_3} + 1 + \frac{1}{2q_2} + \frac{1}{q_4} < 1 + \frac{1}{2q_1} + \frac{1}{q_2} + 1 + \frac{1}{2q_3} + \frac{1}{q_4} \) or \( \frac{1}{2q_1} < \frac{1}{2q_2} \).

3 Canonical description of \( \text{TC}(\mathcal{M}) \) as a quotient of \( \ell_1 \) for finite metric space \( \mathcal{M} \)

The goal of this section is a generalization for an arbitrary finite metric space of the known description [25, Proposition 10.10] of transportation cost spaces for finite unweighted graphs as quotients of finite-dimensional \( \ell_1 \).

Let \( \mathcal{M} \) be a finite metric space with \( n \) elements. It can be viewed as a weighted complete graph \( K_n \), where the weight of the edge joining \( u \) and \( v \) is the distance \( d(u, v) \). The weighted \( \ell_1 \)-space on the edge set \( E(K_n) \) will be introduced as follows. Given \( f : E(K_n) \to \mathbb{R} \), denote by \( f_{uv} \) the value of this function on the edge \( uv \). The norm of \( f \) is defined as:

\[
\|f\|_{1,d} := \sum_{uv \in E(K_n)} |f_{uv}|d(u, v).
\]

The normed space obtained in this way will be denoted by \( \ell_{1,d} = \ell_{1,d}(E(K_n)) \). It can be readily seen that it is an \( \frac{n(n-1)}{2} \)-dimensional space isometric to \( \ell_1^{n(n-1)/2} \).

Further, let us fix an orientation on the edges of \( K_n \). Notice that only intermediate objects and results rather than final outcomes will depend on it. For this reason, it is customary to say that we select a reference orientation. Consider a cycle \( C \) in \( K_n \) and pick one of the two possible orientations of \( C \) satisfying the following condition: each vertex of \( C \) is a head of exactly one edge and a tail of exactly one edge. Having done so, we introduce the signed indicator function \( \chi_C \in \ell_{1,d} \) of the cycle \( C \) by

\[
\chi_C(e) = \begin{cases} 
1 & \text{if } e \in C \text{ and its orientations in } C \text{ and } G \text{ are the same} \\
-1 & \text{if } e \in C \text{ but its orientations in } C \text{ and } G \text{ are different} \\
0 & \text{if } e \notin C,
\end{cases}
\]
where $e$ is used to denote edges in $K_n$.

The span of this set of functions in $\ell_{1,d}$ is denoted by $Z$ and called the cycle space (or the flow space in some sources). The following assertion holds.

**Theorem 3.1.** $\text{TC}(M) = \ell_{1,d}/Z$.

**Proof.** Since the spaces $\ell_{1,d}/Z$ and $\text{TC}(M)$ are finite-dimensional, it suffices to show that the dual space $(\ell_{1,d}/Z)^*$ can be in a natural way identified with the space $\text{Lip}_0(M)$, which is known to coincide with $(\text{TC}(M))^*$, see [25, Theorem 10.2].

To begin with, let us introduce the spaces $\ell_\infty,d$ and $\ell_2,d$ as spaces of real-valued functions on $E(K_n)$ with the norms

$$\|f\|_\infty,d = \max_{uv \in E(K_n)} \frac{|f_{uv}|}{d(u,v)}$$

and

$$\|f\|_2,d = \left( \sum_{uv \in E(K_n)} |f_{uv}|^2 \right)^{\frac{1}{2}}, \quad (16)$$

respectively.

It is clear that $\ell_{2,d}$ is an inner product space, in which the notion of orthogonality is naturally defined. We denote the inner product inducing the norm $(16)$ by $\langle \cdot, \cdot \rangle$.

The subspace of $\ell_{2,d}$ orthogonal to the cycle space $Z$ is denoted by $B$ and is called the cut space or cut subspace. Observe that by virtue of $(16)$, $B$ does not really depend on the distance $d$, but only on the size of $M$. One has a direct, orthogonal in $\ell_2,d$, decomposition

$$\ell_{2,d} = Z \oplus B. \quad (17)$$

Next, we apply the standard duality result, which, generally speaking, states that that the dual of the quotient space $X/Y$ is isometric to the subspace $Y^\perp := \{f \in X^*: \forall y \in Y, f(y) = 0\}$. Observing that our choice of norms on $\ell_{1,d}$ and $\ell_{\infty,d}$ is such that $\ell_{\infty,d} = (\ell_{1,d})^*$ with the pairing given by $g(f) = \langle g, f \rangle$, one concludes that the dual space of the quotient space $\ell_{1,d}/Z$ is naturally isometric to the space $B_\infty$, where $B_\infty$ stands for the space $B$ endowed with its $\ell_{\infty,d}$-norm.

To complete our argument it is convenient to use another description of $B$. Denote by $\ell_2(M)$ the space $\mathbb{R}^M$ with its Euclidean norm. Let $D$ is defined as a matrix whose rows are labelled using elements of $M$, whose columns are labelled using (oriented) edges of $K_n$ and the $ve$-entry is given by

$$d_{ve} = \begin{cases} 1, & \text{if } v \text{ is the head of } e, \\ -1, & \text{if } v \text{ is the tail of } e, \\ 0, & \text{if } v \text{ is not incident to } e. \end{cases}$$
The description of $B$ which we are going to use is that that $B$ is the image of $\ell_2(M)$ under the action of $D^T$ with $D^T$ being the transpose of the matrix $D$. See [25] p. 315.

Therefore, each $b \in B$ can be represented as $b = D^T f$, implying that $b(uv) = h(u) - h(v)$ for some $h : M \to \mathbb{R}$ and all oriented edges $uv$, where $u$ is the head and $v$ is the tail. It is clear that addition of a constant to the function $h$ does not change $D^T h$, so one may assume $h(O) = 0$, that is, $h \in \text{Lip}_0(M)$. Clearly, that the Lipschitz constant of $f$ is equal to

$$\text{Lip}(h) = \max_{uv \in E(K_n)} \frac{|h(u) - h(v)|}{d(u,v)} = \|b\|_{\infty,d}.$$  

Thus, we have established a natural isometry between $B_\infty$ and Lip$_0(M)$. \hfill \Box

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