Chiral Effective Lagrangian in the large-$N_c$ limit: the nonet case.

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Abstract

A $U_L(3) \otimes U_R(3)$ low-energy effective lagrangian for the nonet of pseudogoldstone bosons that appear in the large $N_c$ limit of QCD is presented including terms up to four derivatives and explicit symmetry breaking terms up to quadratic in the quark masses. The one-loop renormalization of the couplings is worked out using the heat-kernel technique and dimensional renormalization. The calculation is carried through for $U_L(n_l) \otimes U_R(n_l)$, thus allowing for a generic number $n_l$ of light quark flavours. The crucial advantages that the expansion in powers of $1/N_c$ bring about are discussed. Special emphasis is put in pointing out what features are at variance with the $SU_L \otimes SU_R$ results when the singlet $\eta'$ is included in the theory.

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1 The $U_L(3) \otimes U_R(3)$ symmetry.

The pattern of the lowest-lying states in the spectrum of strong interactions uncovers an approximate continuous symmetry of nature, the so-called chiral symmetry, which is spontaneously broken. The octet of pseudoscalar particles - $\pi, K,$ and $\eta$ - with masses much smaller than those of the next excited states - the octet of vector particles $\rho, \omega$ and $K^*$, the baryons-, are the accepted candidates for pseudo-goldstone bosons associated to the spontaneous breaking of the symmetry.

This approximate symmetry is well incorporated in QCD as three of the quarks happen to be light. In the (chiral) limit of vanishing for pseudo-goldstone bosons associated to the spontaneous breaking of the symmetry.

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involves the whole nonet of pseudo-goldstone bosons and is invariant under $U_L(3) \otimes U_R(3)$. The departures from this scenario, which stem from the explicit breaking of chiral symmetry by quark masses and from the $U_A(1)$ anomaly, are treated perturbatively, in powers of the quark masses and $1/N_c$. It is conceivable that a good picture of the lightest hadrons and their interactions at low energies could emerge from this approach.

Many authors [12] have discussed how to construct such a lagrangian, i.e., how to extend the symmetry to $U_L(3) \otimes U_R(3)$ and properly take into account the effects of the $U_A(1)$ anomaly at the same time, nicely organized in powers of $1/N_c$. In these articles the physical consequences to lowest orders in $1/N_c$ and the derivative expansion have been worked out as well. We closely follow their work.

The present study is devoted to extend the analysis to a full $O(p^4)$ chiral lagrangian, i.e., with terms kept up to four derivatives and quadratic in the quark masses. The conservative bookkeeping of quark masses as two chiral powers, $m_q = O(p^2)$, is adopted, as in [11] (see also [14]). The celebrated Gell-Mann Okubo relations amongst the light pseudoscalar masses squared follow from this chiral power counting in a rather natural way. The exact mechanism of chiral symmetry breaking being unknown, however, many different possibilities other than $m_q = O(p^2)$ have not been ruled out hitherto, either from the experiment or from QCD. None of them has been considered here. They can be adequately treated in the framework of Generalized Chiral Perturbation Theory as proposed in [15], if needed. In order to discriminate amongst the various possibilities, the nature of chiral symmetry breaking needs to be established on more solid grounds.

Our study is to all orders in the large-$N_c$ expansion - in a sense that will be later qualified. We calculate all one-loop divergences to the effective action using the heat-kernel technique and dimensional regularisation, and carry out to this approximation the renormalization of the couplings.

The article is organized as follows: in section 2 the method of external sources and the generating functional are briefly reviewed and the notation is set. In section 3 the chiral lagrangian including terms up to $O(p^4)$ and the one-loop effective action are put forward, before the $1/N_c$ expansion is performed. Next, in section 4, the calculation is organized in powers of $1/N_c$ and the first non-trivial terms are also given. The conclusions and the appendices follow.

2 The method of the external sources.

In this section we briefly review the symmetries of QCD that are relevant for the chiral lagrangian in the large-$N_c$ limit.

The QCD lagrangian can always be written with a diagonal quark mass term, as

$$\mathcal{L}_{QCD} = -\frac{1}{2} Tr(G_{\mu\nu}G^{\mu\nu}) + \sum_{f=1}^{n_l} \bar{q}_f (i\gamma_\mu D^\mu - m_f) q_f,$$

where $f$ labels the light quarks $q_f$ that appear in $n_l$ number of light flavours. In nature, $n_l = 3$ at most, for the flavours $u, d, s$. Although explicitly omitted, the quark fields also carry a colour index $q^c$ which labels the fundamental representation of the gauge group $SU(N_c)$ of colour. Nature has chosen $N_c = 3$. The covariant derivative $D_\mu$ acts on colour indices through the gluon matrix, $G_\mu$ which form an adjoint $(N_c^2 - 1)$-dimensional representation of $SU(N_c)$. On the quark fields it acts in the usual way, diagonal in flavour indices,

$$D_\mu q^c = \partial_\mu q^c - i \frac{g}{\sqrt{N_c}} (G_\mu)^c_d q^d.$$


The field strength matrix is \( G_{\mu\nu} = \partial_\mu G_\nu - \partial_\nu G_\mu - i \frac{q}{\sqrt{N_c}} [G_\mu, G_\nu] \), whereas \( G_\mu \equiv G^a_\mu (\lambda^a_T) \); the sum over \( a \) runs from 1 to \( N_c^2 - 1 \) and \((\lambda^a)\) stands for the \( N_c \times N_c \) Gell-Mann matrices of \( SU(N_c) \).

The heavy quarks \( c, b, t \) are also omitted from the lagrangian in (1) for they decouple from the strong low energy processes which only involve light pseudoscalars with \( u, d, s \) quantum numbers.

The \( N_c \) dependence that accompanies the coupling constant, \( \frac{q}{\sqrt{N_c}} \), is explicitly displayed. Apart from the usual combinatorial factors that appear in the Feynman diagrams, this extra dependence in \( N_c \) must be added in order to allow for a smooth, non-trivial \( N_c \rightarrow \infty \) limit of QCD. We shall revert to the issue of counting the leading powers of \( N_c \) of each effective coupling in section 4.

If the light quark masses are switched off, the lagrangian (1) becomes invariant under the symmetry group \( U_L(n_l) \otimes U_R(n_l) \), \( n_l = 3 \). Henceforth we keep the number of light flavours, \( n_l \), generic in the expressions. This is most easily seen by writing the lagrangian above in terms of the quark left and right components,

\[
q_L = \frac{1 - \gamma_5}{2} q, \quad q_R = \frac{1 + \gamma_5}{2} q.
\]

The terms in (1) that involve the quark fields read,

\[
\bar{q}_L (i\gamma_\mu D^\mu) q_L + \bar{q}_R (i\gamma_\mu D^\mu) q_R.
\]

The symmetry of rotating independently the components \((q_L)_f \) and \((q_R)_f \) in flavour space with unitary matrices is manifest. It is a global symmetry that is explicitly broken by the quark masses. However, if all the quark masses were the same there would still be an invariance under the diagonal vector subgroup \( U_{L+R}(n_l) \). In that case the subgroup coincides with the unbroken subgroup after spontaneous breaking of \( U_L(n_l) \otimes U_R(n_l) \).

As it is well known not all the symmetries of the classical action are maintained at the quantum level; the quantum theory thus generates anomalous contributions to the divergences of some currents - they are no longer conserved.

The low-energy effective action is a convenient bookkeeping device to encode the symmetries of the underlying theory - QCD - which automatically incorporates all the unitarity features of quantum field theory. In writing the effective lagrangian care must be taken that all the (chiral) Ward Identities (WI) among the Green’s functions are well implemented, including the anomalous ones. Actually, the method put forward in ref. [11] constructs a solution to these WI’s. It is based upon the transformation properties of the generating functional.

The probability amplitude of transition from the vacuum in the remote past to the vacuum in the far future, in the presence of terms in the lagrangian that couple the external sources linearly to\( Z \) the far future, is explicitly displayed. Apart from the usual combinatorial factors that appear in the Feynman diagrams, this extra dependence in \( N_c \) must be added in order to allow for a smooth, non-trivial \( N_c \rightarrow \infty \) limit of QCD. We shall revert to the issue of counting the leading powers of \( N_c \) of each effective coupling in section 4.

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\[
e^{iZ[f]} = \sum_{n,m} \frac{i^n}{n!} \int dx_1 dx_2 \ldots dx_n f_{i_1}^{\mu_1}(x_1) f_{i_2}^{\mu_2}(x_2) \ldots f_{i_n}^{\mu_n}(x_n) |0\rangle \langle 0| T J_{\mu_1}^{i_1}(x_1) J_{\mu_2}^{i_2}(x_2) \ldots J_{\mu_n}^{i_n}(x_n),
\]

\( J' \)’s and \( f' \)’s stand for generic currents and external sources, respectively.

We shall consider both bilinear quark operators (currents) and the topological charge operator coupled to external sources, added to the QCD lagrangian,

\[
\mathcal{L} = \mathcal{L}_{QCD} + \bar{q}_L \gamma_\mu \mu^\mu(x) q_L + \bar{q}_R \gamma_\mu \mu^\mu(x) q_R - \bar{q}_R (s(x) + ip(x)) q_L
\]

\[
- \bar{q}_L (s(x) - ip(x)) q_R - \frac{q^2}{16\pi^2 N_c} \theta(x) Tr \left( G_\mu G_\nu \right),
\]

(6)
The first two new terms correspond to sources for the $U_L \otimes U_R$ Noether currents of QCD, the non-singlets are the generators of Current Algebra; $s$ is a source for the quark mass term and $p$ for pseudoscalar bilinears with the quantum numbers of $\pi, K, \eta$ and $\eta'$. The sources $l_\mu, r_\mu, s$ and $p$ are hermitian $n_f \times n_f$ matrices; $\theta$ is a real function. The axial $a_\mu$ and the vector $v_\mu$ sources are defined so that $l_\mu = v_\mu - a_\mu$ and $r_\mu = v_\mu + a_\mu$. One can, formally, write the generating functional as a path integral,

$$
\exp\{iZ[l,r,s,p,\theta]\} = \int [d\bar{q}\, dq\, dG_\mu] \exp\{i \int dx \mathcal{L}\}.
$$

The connected Green’s functions are obtained by performing functional derivatives of $Z$ with respect to the sources.

In order to further constrain the form of the effective lagrangian it is a convenient trick to promote the global $U_L \otimes U_R$ transformations - that leave the QCD lagrangian invariant - to local ones by allowing the external sources to transform along with the dynamical fields. The combined set of local $U_L \otimes U_R$ transformations

$$
g_L(x) \in U_L, \quad g_R(x) \in U_R
$$

 naïvely becomes a local gauge symmetry for the lagrangian density in eq. (1). This is not quite so, due to the fact that the transformations in (8) also induce a non-trivial anomalous $U_A(1)$ transformation on the generating functional $Z[l,r,s,p,\theta]$. Its origin may be traced to the transformation properties of the fermionic integration measure $\int \, dq$ (or, if one wishes, of the fermion determinant) once it is properly regularized. This is reflected in the anomalous divergence in the ninth axial singlet current,

$$
\partial_\mu J_5^{\mu(0)} = \frac{g^2}{16\pi^2} \frac{1}{N_c} \text{Tr}_c \left( G_\mu \tilde{G}^{\mu
u} \right) ; \quad J_5^{\mu(0)} = \bar{q} \gamma_\mu \gamma_5 q.
$$

This anomaly-related effect turns off any potential advantage inherent to the existence of a gauge symmetry which eventually would severely constraint the form of the effective action. However, this drawback is obviated as the $U_A(1)$ anomaly contribution may be altogether eliminated by judiciously choosing the transformation law for the external field $\theta(x)$. Indeed, for infinitesimal $g_L = I + i(\beta - \alpha), \quad g_R = I + i(\beta + \alpha)$, the source $\theta(x)$ ought to change as

$$
\theta(x) \rightarrow \theta(x) - 2(\alpha(x)),
$$

(here we have switched to the standard notation and denote the trace operation over flavour indices by brackets $tr_F(... \equiv \langle ... \rangle$) in order for the $U_A(1)$ anomaly to cancel. The term generated by the anomaly in the fermion determinant is explicitly compensated by the shift in the $\theta$ source.

A subtlety is still to be analyzed. The set of local gauge transformations we have constructed in (8) also induces a non-abelian anomaly. This new drawback can not be circumvented and

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*Being unitary, $g_R$ and $g_L$ can be always parametrized as $g_R = \exp(i\beta)\exp(i\alpha)$, $g_L = \exp(i\beta)\exp(-i\alpha)$, with $\alpha$ and $\beta$ $n_f \times n_f$ hermitian matrices. A pure vector transformation has $\alpha = 0$, whereas an axial one has $\beta = 0$. 
needs explicit consideration. As discussed in [16], imposing upon regularization the requirement of conservation for the nine vector currents, the change in $Z$ under (8) reads

$$\delta Z \equiv - \int dx \langle \alpha(x) \Omega(x) \rangle,$$

where

$$\Omega(x) = \frac{N_c}{16\pi^2} \epsilon^{\alpha\beta\mu\nu} \left[ v_{\alpha\beta}v_{\mu\nu} + \frac{4}{3}(\nabla_\alpha a_\beta)(\nabla_\mu a_\nu) + \frac{2}{3}i\{v_{\alpha\beta}, a_\mu a_\nu\} + \frac{8}{3}ia_\mu v_{\alpha\beta}a_\nu + \frac{4}{3}a_\alpha a_\beta a_\mu a_\nu \right],$$

$$v_{\alpha\beta} = \partial_\alpha v_\beta - \partial_\beta v_\alpha - i[v_\alpha, v_\beta], \quad \nabla_\alpha a_\beta = \partial_\alpha a_\beta - \partial_\beta a_\alpha - i[v_\alpha, a_\beta].$$

The terms in $\delta Z$ appear due to triangle AVV Feynman diagrams which involve insertions of an axial and two vector quark bilinear operators. Higher polygon-shaped diagrams, quadrangles and pentagons are anomalous as well and give rise to the cubic and quartic terms in (10). Adler and Bardeen showed that the coefficients of the anomaly are not affected by higher-order radiative corrections, i.e., diagrams with more than one-loop do not contribute to the anomalous terms [18]. $\delta Z$ fulfills the Wess-Zumino consistency conditions [19].

The integrated form of this anomaly must be added by hand to the low-energy effective field theory that we shall construct. Such a theory will consist of interacting bosons and therefore will not be anomalous. Thus, the above effect will be contained in an additional term to the effective lagrangian.

It is worth pointing out that unlike for $SU_L \otimes SU_R$, here there is the possibility of combining one Wess-Zumino-Witten (WZW) vertex (which start out at $O(p^4)$) with the $O(p^6)$ pieces in the lagrangian and generate additional one-loop divergences at $O(p^4)$. They will be given elsewhere [20]. Notice that for $SU_L \otimes SU_R$ this cannot occur at $O(p^4)$ because the lowest chiral power counting are $O(p^2)$ terms, which generate divergences at $O(p^6)$.

### 3 The chiral lagrangian

The mere knowledge of the symmetries and the way they are realized provide an enormous insight for they are reflected in every aspect of the theory, e.g., in the spectrum, the Green’s functions, the interactions, to mention a few. In the case of the strong interactions the symmetries are the bulk of the information which is available at low energies, since QCD is non-perturbative there. Although a lot of effort has been put in numerical calculation projects and enormous progress in solving the technical difficulties has been achieved, so far the problem has remained too hard to tackle satisfactorily.

In this section we shall write an effective lagrangian for the soft interactions of the lightest particles in the spectrum, the nonet of pseudoscalar mesons, $\pi, K, \eta, \eta'$. The effective lagrangian is a combined statement about the degrees of freedom and the symmetries that are relevant for the processes under study. Effective refers to the choice of field variables, in this case fields for the $\pi, K, \eta, \eta'$ particles that are observed in the range of energies below the $\rho$-meson mass $m_\rho$. More generally, being the lightest particles in the spectrum the use of the effective lagrangian will be the determination of the long distance behaviour of any of the QCD Green’s functions, where they are expected to dominate.

Being a symmetry statement, the strategy consists of writing down for the effective lagrangian the most general expression that contains all the independent terms compatible with the symmetries, multiplied by unknown constants. The fate of many effective theories is to be of little practical use if the number of unknown constants blows up. They can always be fitted from experiment but
too often the number of experimental results available is of the same order (if not smaller) as that of constants, rendering the approach little predictive.

In the present case the spontaneously broken symmetry character and the consequent goldstone nature of the $\pi$, $K$, $\eta$, $\eta'$ impose severe constraints on the form of the interactions. The number of unknown constants reduces in a drastic manner if one restricts to the lower terms. Actually, they would reduce to a handful of them if there were no $U_A(1)$ anomaly effects to incorporate, as happens in the $SU_L \otimes SU_R$ lagrangian if one only keeps terms up to $O(p^4)$. Fortunately, the swarm of new terms that the $U_A(1)$ anomaly introduces carry high powers of $1/N_c$, and to lower orders in $1/N_c$ only a few survive.

Following [11], [13] we collect the nine pseudoscalar fields in a hermitian matrix $\Phi(x)$,

$$\Phi(x) = \eta^0(x) \lambda_0 + \pi(x)$$

where $\pi(x) = \vec{\pi}(x) \cdot \vec{\lambda}$, $\lambda_0 = \sqrt{2} I / n_l$ and $\vec{\lambda}$ are the Gell-Mann matrices of $SU(n_l)$. For $n_l = 3$ (see Appendix B),

$$\Phi(x) = \begin{pmatrix}
\frac{1}{\sqrt{2}} \pi^0 + \frac{1}{\sqrt{6}} \eta^8 + \sqrt{\frac{2}{3}} \eta^0 & \pi^+ & K^+ \\
-\frac{1}{\sqrt{2}} \pi^0 + \frac{1}{\sqrt{6}} \eta^8 + \sqrt{\frac{2}{3}} \eta^0 & \pi^0 & K^0 \\
K^- & K^0 & -\frac{2}{\sqrt{6}} \eta^8 + \sqrt{\frac{2}{3}} \eta^0
\end{pmatrix}. \tag{11}$$

The unitary $n_l \times n_l$ matrix $U(x)$ is the exponential of $\Phi(x)$,

$$U(x) \equiv e^{i\Phi(x)/f}, \tag{12}$$

$f$ is an order parameter of chiral symmetry breaking and gives the strength of the coupling between the goldstone bosons and the currents that are spontaneously broken and do not annihilate the vacuum. In this case $\det U$ is a phase,

$$\det U = \exp(i\sqrt{2n_l} \eta^0 / f).$$

Under $U_L \otimes U_R$, $U$ transforms linearly,

$$U \rightarrow g_R U g_L^\dagger. \tag{13}$$

The transformations induced by [13] on $\Phi$ are more involved. Under $U_{R+L}(1)$, $\Phi$ is automatically invariant, as it should, for mesons must carry baryon number equal to zero. Under $SU_{L+R}(3)$, $\Phi$ transforms linearly: it contains two irreducible representations, the octet $\pi$ and the singlet $\eta^0$.

Under an axial transformation

$$U \rightarrow e^{i\alpha} U e^{i\alpha},$$

$\Phi$ changes nonlinearly,

$$\Phi \rightarrow \Phi + 2\alpha + O(\alpha^2).$$

This can be understood on geometrical grounds since the fields in $\Phi$ may be regarded as coordinates that span the coset space $U_L \otimes U_R / U_{R+L}$, upon which the $U_L \otimes U_R$ acts: the fields themselves are, in a sense, parameters of a group element, and the nonlinearity reflects the group transformation law when written in terms of the continuous parameters. For $\langle \log U \rangle = i\sqrt{2n_l} \eta^0 / f$,

$$\langle \log U \rangle \rightarrow \langle \log U \rangle + \langle \log (g_R g_L^\dagger) \rangle = \langle \log U \rangle + 2i\langle \alpha \rangle ;$$
\( \eta^0 \) gets thus shifted only under an axial transformation and is invariant under any vector one. Under \( U_L \otimes U_R \) it never mixes with any of the \( \pi \) components.

With this choice of fields, the origin of the derivative couplings among goldstone-boson becomes transparent: a lagrangian invariant under global \( U_L \otimes U_R \) ought to reduce to zero if \( U \) were a constant matrix, for then, by virtue of the symmetry, it could always be transformed away with a global unitary rotation: the couplings need thus be derivative couplings \([25]\). The expansion will be in powers of momenta of the soft light mesons divided by a scale of chiral symmetry breaking, which is of order \( \sim 4\pi f \sim m_\rho \), the mass of the next excited state, the \( \rho \)-meson. In addition, in the present case the \( U_A(1) \) anomaly introduces novel couplings of the \( \eta_0 \) meson that are not derivative couplings.

The objects at hand to construct the chiral effective lagrangian are the matrix \( U \) and the external sources, \( r_\mu, l_\mu, (s + ip) \) and \( \theta \). It is useful to introduce covariant derivatives for the fields and the sources. The covariant derivative will act on flavour space and, as usual, its action will depend on the transformation law of the object it derives,

\[
D_\mu U = \partial_\mu U - ir_\mu U + iUl_\mu, \\
D_\mu(s + ip) = \partial_\mu(s + ip) - iv_\mu(s + ip) + i(s + ip)l_\mu, \\
D_\mu \langle \log U \rangle = \langle \log U \rangle - i(r_\mu - l_\mu), \\
iD_\mu \theta = i\partial_\mu \theta + i\langle r_\mu - l_\mu \rangle.
\] (14)

One is led to introduce field strengths for the vector and axial sources

\[
F^L_{\mu \nu} = \partial_\mu l_\nu - \partial_\nu l_\mu - i[l_\mu, l_\nu], \\
F^R_{\mu \nu} = \partial_\mu r_\nu - \partial_\nu r_\mu - i[r_\mu, r_\nu].
\] (15)

Under local \( U_L \otimes U_R \) they transform as

\[
D_\mu U \rightarrow g_R \langle D_\mu U \rangle g_L^\dagger, \\
D_\mu(s + ip) \rightarrow g_R \langle D_\mu(s + ip) \rangle g_L^\dagger, \\
F^L_{\mu \nu} \rightarrow g_L F^L_{\mu \nu} g_R^\dagger, \\
F^R_{\mu \nu} \rightarrow g_R F^R_{\mu \nu} g_R^\dagger.
\] (16)

The combination

\[
X(x) \equiv \langle \log U(x) \rangle + \hat{\theta}(x) = i \frac{\sqrt{2} n_i}{f} \eta^0 + \hat{\theta}(x), \quad \hat{\theta} \equiv i\theta,
\] (17)

is invariant, and so is any function of \( X \) \([8], [13]\). This is a novelty of \( U_L \otimes U_R \) and it is possible because \( \langle \log U \rangle \) does not vanish, as it does for \( SU_L \otimes SU_R \). Due to the \( U_A(1) \) anomaly, each invariant operator generates, in reality, an infinite set of invariant operators, since the symmetry allows to multiply it by any function of \( X \) and still remain invariant. Notice that this method of finding new operators by multiplying the old ones by functions of \( X \) never introduces new derivatives to the vertices. It is for the same reason that counting the number of derivatives and retaining operators that contain up to a certain number does not limit the number free constants, as it used to in \( SU_L \otimes SU_R \). At each order in the derivative expansion we find an infinite number of constants.

It is customary to introduce the source \( \chi(x) \)

\[
\chi \equiv 2B(s + ip),
\]
which transforms as the $U$ matrix. $B$ is a constant that is related to the quark condensate $\langle \bar{q}q \rangle$. Its relevance comes from the fact that the explicit symmetry breaking driven by the quark masses provides the former goldstone bosons also with a mass. If $m_q$ denotes a quark mass, to lowest chiral order there is a contribution to the meson masses which involves $B$ and is given by $m_{\pi}^2 = 2Bm_q$. We assume that it is the bulk of it: we make the hypothesis that further pieces which involve order parameters other than $B$ are negligible, comparatively. This leads to counting $\chi$ as $O(p^2)$. The $\eta'$ gets an additional piece to its mass through the $U_A(1)$ anomaly, which is $O(1/N_c)$, and which does not add to the masses in the octet $[24]$. The singlet-octet mass splitting is very big in practice, of $\sim 400$ MeV in the least extreme case.

All the constants that appear in the effective theory are free, to be fitted from experiment. Although we have not been able to compute them from QCD, there are exact inequalities that have to be verified, which are based on the vector structure of QCD. These relations are non-perturbative $[23]$. 

From the effective lagrangian, one can make contact with QCD through the generating functional $Z[l, r, s, p, \theta]$ introduced in section 2. One demands that the same functional - the same Green’s functions - should be obtained by starting from the effective theory as from QCD. It can be formally written in terms of the light pseudoscalar fields, collected in $U(x)$, as

$$e^{iZ[l, r, s, p, \theta]} = \int [dU] e^{i\int dx L_{\text{eff}} \bigg|_{\text{low modes}}}.$$ 

The chiral lagrangian only copes with the long distance behaviour of Green’s functions. Only the lowest modes have physical significance in $[18]$. For distances smaller than $1/m_\rho$, the approach is inappropriate and the integrals over loop momenta have a natural cutoff associated with them that is $m_\rho$. The higher modes, corresponding to more energetic pseudoscalars, and the rest of massive hadrons are integrated out and their effect manifests through the coupling constants of the effective theory $[23]$.

The most general lagrangian invariant under (local) $U_L \otimes U_R$ $[1]$ that includes terms with two derivatives or less, or one power of $\chi$, is the following $[11], [13]$.

$$L_{(0+2)} = - W_0(X) + W_1(X)\langle D_\mu U^\dagger D^\mu U \rangle + W_2(X)\langle U^\dagger \chi + \chi^\dagger U \rangle + iW_3(X)\langle U^\dagger \chi - \chi^\dagger U \rangle + W_4(X)\langle U^\dagger D_\mu U \rangle \langle U^\dagger D^\mu U \rangle + W_5(X)\langle U^\dagger (D_\mu U) \rangle D^{\mu\nu} \hat{\theta} + W_6(X)D_\mu \hat{\theta} D^{\mu\nu} \hat{\theta}. \quad (19)$$

Parity conservation implies that they are all even functions of $X$ except for $W_3$ that is odd. Furthermore, $W_4(0) = 0$, $W_1(0) = W_2(0) = \frac{\mu^2}{4}$ gives the correct normalization for the quadratic terms.

The first term $W_0$ has neither derivatives nor powers of $\chi$, and therefore counts as $O(p^0)$. The rest of the terms count as $O(p^2)$. The $1/N_c$ power counting of the diverse couplings is given in the next section. The term $W_0$ brings a mass terms for the $\eta^0$.

$$m_{\eta^0}^2 \bigg|_{U_A(1)} = - \frac{2\eta}{f^2}W_0''(0), \quad (20)$$

\[\text{Terms of the sort } \sum_{\alpha=0}^8 \langle \lambda^\alpha U^\dagger D_\mu U \rangle \langle \lambda^\alpha U^\dagger D^\mu U \rangle \text{ and alike, given the properties of the } \{\lambda^\alpha\} \text{ matrices, can be re-expressed in terms of the operators written in the text. In this particular case, with the relation } \sum_{\alpha=0}^8 \lambda^\alpha_\mu \lambda^\alpha_\nu = 2\delta_{\mu} \delta_{\nu}, \text{ it becomes } -2 \langle (D_\mu U^\dagger) (D^\mu U) \rangle.\]
whereas the first term in the expansion of $W_3$ gives a contribution to singlet-octet mixing from $U_A(1)$.

The $N_c \to \infty$ limit of QCD actually imposes more restrictions on the effective lagrangian than the symmetry $U_L \otimes U_R$ alone. Under $U_L \otimes U_R$, the fields $\eta^0$ and $\pi$ never mix their components. There is no reason why the particles created by them should bear any sort of relationship whatsoever, as though they belonged to the same irreducible representation, like the $\pi$ do. There is no reason, either, why in the definition of $U(x)$ in (11) $\eta^0$ and $\pi$ should appear in the exponent divided by the same constant $f$, i.e., normalised in the same way. One could have written instead

$$U(x) = e^{i \left( \sqrt{\frac{2}{f_0^2}} \frac{\eta}{f_0} + \pi \right)},$$

with $f$ and $f_0$ unrelated and, yet, (11) would be $U_L \otimes U_R$ invariant. The proper way to cast this issue requires to fix the normalization of each field by looking at the kinetic energy too, not at $U(x)$ only. In (19), the kinetic energy terms are $\frac{f^2}{4} \langle D_\mu U^\dagger D^\mu U \rangle + W_4(0) \langle U^\dagger D_\mu U \rangle \langle U^\dagger D^\mu U \rangle$, which read,

$$\frac{1}{2} (\partial_\mu \pi) \cdot (\partial^\mu \pi) + \frac{1}{2} \left( \frac{f^2}{f_0^2} - \frac{4n_4(0)}{f_0^2} \right) \partial_\mu \eta_0 \partial^\mu \eta_0.$$ 

The normalization condition $(f^2 - 4n_4(0))/f_0^2 = 1$ relates three constants; it may be viewed as an arbitrariness in their definition, for a change in $f_0$ can be always compensated for by an appropriate change in $W_4(0)$. Once this normalization is fixed, the strength that the singlet $\eta^0$ couples to the singlet axial current is $f_0$ whereas the octet couple to the octet axial current with strength $f$.

Unlike $U_L \otimes U_R$, which does not have a dimension nine irreducible representation, the $N_c \to \infty$ really enforces a nonet symmetry, with the $\pi$, $K$, $\eta$, $\eta'$ all having identical properties. Indeed, the planar Feynman diagrams that contribute to $\langle J_{5}^{(0)}(x) J_{5}^{(0)}(0) \rangle$ and to $\langle J_{5}^{(a)}(x) J_{5}^{(a)}(0) \rangle$ ($a = 1, \ldots, 8$), with $J_{5}^{(a)}_{\mu} = \bar{q}\gamma_{\mu}e^{\frac{1}{2}N_c}q$, are the same since the $\bar{q}q \to \text{gluon}$ annihilation diagrams that would only contribute to the singlet channel turn out to be $1/N_c$ suppressed (OZI violating processes). Barring for $1/N_c$ corrections the two decay constants coincide $f/f_0 = 1$. Moreover, if the same quark mass were switched on for all light quark species, a mass would be generated identical for the singlet $\eta_0$ as for the octet particles $\eta$.

The standard normalization of the kinetic energy and nonet symmetry require $W_4(0) = 0$. Without loss of generality, in that case the term $W_4(X)$ may be eliminated altogether by a change of field variables of the type $U \to U \exp[iF(X)]$, that maintains the transformation properties for $U$ (but changes the normalization conditions of $\eta^0$).

The methods of the background field and the steepest descent, when applied to the functional integral (13), provide the loop-wise expansion of the generating functional. The Green’s functions are read off from the generating functional, about its minimum when the external sources are switched off, $\chi = 2B \text{diag}(m_u, m_d, m_s)$ and $\theta = 0$. To lowest order we assume that the minimum is achieved at $U_0 = I$, which is compatible with the equation that minimizes the lowest order effective action (13).

In order to include one-loop corrections, one proceeds to introduce a background field matrix $\bar{U}(x)$, and expand (19) about this background configuration. For that we decompose $\bar{U}(x)$ as

$$U(x) = \bar{U}(x) \Sigma(x), \quad \Sigma(x) = e^{i\Delta(x)},$$

10
with $\Delta$ the matrix of quantum fields. By choosing $\bar{U}$ to transform under $U_L \otimes U_R$ as $U$, the transformation laws for $\Sigma$ and $\Delta$ become

$$\Delta \rightarrow g_L \Delta g_L^\dagger, \quad \Sigma \rightarrow g_L \Sigma g_L^\dagger,$$

and, therefore, their covariant derivatives are

$$D_\mu \Sigma = \partial_\mu \Sigma - i[l_\mu, \Sigma], \quad D_\mu \Delta = \partial_\mu \Delta - i[l_\mu, \Delta].$$

Next we expand the lagrangian in (19) about $\bar{U}$ and keep terms up to quadratic in $\Delta$. The one-loop effective action is obtained upon evaluation of the functional determinant of the differential operator that appears in the piece quadratic in $\Delta$. Finally the background field $\bar{U}(x)$, which is \textit{a priori} independent of the sources, is judiciously chosen so as to verify the equations $\delta Z[\bar{U}, l, r, s, p, \theta] = 0$ with the sources held fixed. The method ensures that with this $\bar{U}$, $Z$ is the effective action $\Gamma[\bar{U}]$ - the generator of the one-particle irreducible Green’s functions of fields gathered in $\bar{U}$, in the presence of the external sources. For the one-loop effective action it suffices to use the tree level equations of motion. The corrections to it would modify the two-loop effective action.

Expanding the effective lagrangian in (19) about $\bar{U}$, disregarding the terms linear in $\Delta(x)$ that vanish with the equations of motion, it yields:

$$L_{0+2}(\bar{U}\Sigma) = L_{0+2}(\bar{U}) + \mathcal{A} \langle \Delta \rangle^2 + 2W'_1(\bar{X})\langle \Delta \rangle \langle C^\mu \Delta \rangle + W_1(\bar{X})\langle D_\mu \Delta D^\mu \Delta \rangle + W_1(\bar{X})\langle C_\mu [\Delta, D_\mu \Delta] \rangle + W_2(\bar{X})\langle \Delta \rangle \langle \Delta N \rangle - \frac{1}{2} W_2(\bar{X})\langle \Delta^2 M \rangle + iW'_3(\bar{X})\langle \Delta \rangle \langle \Delta M \rangle - \frac{i}{2} W_3(\bar{X})\langle \Delta^2 N \rangle - W'_2(\bar{X})\langle \Delta \rangle \langle D_\mu \Delta \rangle D^\mu \hat{\theta} + O(\Delta)^3,$$

where

$$\mathcal{A} \equiv \frac{1}{2} W''_0(\bar{X}) + \frac{1}{2} W''_1(\bar{X})\langle C^\mu \rangle - \frac{1}{2} W''_2(\bar{X})\langle M \rangle - \frac{i}{2} W''_3(\bar{X})\langle N \rangle - \frac{1}{2} W''_3(\bar{X})\langle C_\mu \rangle D^\mu \hat{\theta} - \frac{1}{2} W''_4(\bar{X})\langle D^\mu \hat{\theta} \rangle D_\mu \hat{\theta}$$

and

$$C_\mu \equiv \bar{U}^\dagger D_\mu \hat{U}, \quad M \equiv \bar{U}^\dagger \chi + \chi^\dagger \bar{U}, \quad N \equiv \bar{U}^\dagger \chi - \chi^\dagger \bar{U};$$

all transform as $C_\mu, M \rightarrow g_L Mg_L^\dagger, N \rightarrow g_L Ng_L^\dagger$; also, $C^\dagger_\mu = -C_\mu, M^\dagger = M$ and $N^\dagger = -N$. The functions $W_i(X)$ have been Taylor expanded about the background value $X = \langle \log \bar{U} \rangle + \hat{\theta}$ as follows,

$$W_k((\log(\bar{U}\Sigma)) + \hat{\theta}) = W_k(\bar{X}) + iW'_k(\bar{X})\langle \Delta \rangle - \frac{1}{2} W''_k(\bar{X})\langle \Delta \rangle^2 + O(\Delta)^3.$$

\*\*\*The procedure is not manifestly \textit{left - right} symmetric. This is not a worrisome issue given that the quantization of scalar fields in four dimensions does not have any anomaly that would favor one over the other. Simplicity reasons have led to our choice. A more symmetric treatment would lead to the same final results.

11
Integrations by parts have been performed where necessary and the total divergences have been discarded. The equations of motion can be read off from the terms that are linear in the variation $\Delta$,

$$D_\mu C^\mu = \frac{1}{2} W'_1 + \frac{1}{2} W'_1 (C^\mu C_\mu) - \frac{W'_1}{W_1} (C_\mu + D_\mu \hat{\theta}) C^\mu$$

$$+ \frac{1}{2} W_2 N + \frac{i}{2} W_3 M - \frac{1}{2} W'_2 (M) - \frac{i}{2} W'_3 (N)$$

$$+ \left( \frac{1}{2} W_5 - \frac{1}{2} W_6 \right) (D_\mu \hat{\theta})(D^\mu \hat{\theta}) + \frac{1}{2} W_5 D_\mu D^\mu \hat{\theta}. \quad (23)$$

(Henceforth the arguments $X$ are omitted from the functions $W_k$’s and their derivatives that appear in the calculation; also, bars are suppressed from $X$ and $U$).

In order to be able to use the expressions collected in the Appendix A for the evaluation of the one-loop divergent parts it proves useful to perform a change of integration variables so as to leave the operator that acts on the quadratic piece in (21) in the usual form, with the laplacian piece $\partial_\mu \partial^\mu$ multiplied by a constant, not by a function. For this purpose we change variables to

$$\Delta(x) = \frac{f}{2} \frac{1}{\sqrt{W_1}} \varphi(x), \quad (24)$$

and expand the hermitian matrix $\varphi(x)$ in the basis of matrices $\lambda^\alpha$ (see Appendix B):

$$\varphi = \varphi^\alpha \lambda^\alpha, \quad \varphi^\alpha = \frac{1}{2} \langle \varphi \lambda^\alpha \rangle.$$  

Retaining the piece quadratic in the quantum fluctuating fields $\varphi^\alpha$, one finds

$$L^{Quadratic}_{(0+2)} = -\frac{f^2}{2} \varphi^\alpha (d_\mu d^\mu + \sigma)^{\alpha\beta} \varphi^\beta, \quad (25)$$

where

$$[d_\mu \varphi]^\alpha = \partial_\mu \varphi^\alpha + \omega^{\alpha\beta}_\mu \varphi^\beta, \quad (26)$$

and

$$\omega^{\alpha\beta}_\mu = \frac{i}{2} \langle (l_\mu + \frac{i}{2} C_\mu)[\lambda^\alpha, \lambda^\beta] \rangle + \frac{1}{4 W_1} \left( \langle C_\mu \lambda^\alpha \rangle \langle \lambda^\beta \rangle - \langle C_\mu \lambda^\beta \rangle \langle \lambda^\alpha \rangle \right). \quad (27)$$

Notice that $\omega^{\alpha\beta}_\mu$ is antisymmetric in $\alpha, \beta$. It is this property what allows to integrate $d_\mu$ by parts as a whole, as though it were the single $\partial_\mu$. For that reason the operator $d_\mu d^\mu + \sigma$ is manifestly hermitian. The expression in (23) differs from that of (21), after the change of variables, by a total derivative. The evaluation of the Gaussian integral involves the expression (25), though: it is the determinant of the differential operator that is hermitian the one that has to be evaluated.\[11\]

\[11\] In practice this means that any term of the sort $\varphi^\alpha f^{\alpha\beta}_\mu(x) \partial^\mu \varphi^\beta$ (which, in general, is not hermitian as can be seen if one tries to bring the operator act on $\varphi^\alpha$, on the left) can always be written as $\varphi^\alpha a^{\alpha\beta}_\mu(x) \partial^\mu \varphi^\beta - \frac{i}{2} \varphi^\alpha \partial_\mu (s^{\alpha\beta}_\mu(x)) \varphi^\beta + \frac{1}{2} \partial_\mu \left( \varphi^\alpha s^{\alpha\beta}_\mu(x) \varphi^\beta \right)$. Notice that the last term is a total derivative which we shall discard. The remaining part $a^{\alpha\beta}_\mu(x) \partial^\mu$ is hermitian now. The $a^{\alpha\beta}_\mu(x)$ and $s^{\alpha\beta}_\mu(x)$ are the antisymmetric and symmetric parts of $f^{\alpha\beta}_\mu(x)$, respectively. This decomposition is unique.
The curvature associated to this connection is

\[ R_{\mu\nu}^{\alpha\beta} = \partial_\mu \omega_\nu^{\alpha\beta} - \partial_\nu \omega_\mu^{\alpha\beta} + \omega_\mu^\alpha \omega_\nu^\beta - \omega_\nu^\alpha \omega_\mu^\beta \]

\[ = \frac{i}{4} \langle Q_{\mu\nu}[\lambda^\alpha, \lambda^\beta] \rangle + \frac{1}{4} \left( \langle H_{\mu\nu} \lambda^\alpha \rangle \langle \lambda^\beta \rangle - \langle H_{\mu\nu} \lambda^\beta \rangle \langle \lambda^\alpha \rangle \right) \]

\[ - \frac{n_l}{8} \left( \frac{W_1'}{W_1} \right)^2 \left( \langle C_\mu \lambda^\alpha \rangle \langle C_\nu \lambda^\beta \rangle - \langle C_\mu \lambda^\beta \rangle \langle C_\nu \lambda^\alpha \rangle \right), \quad (28) \]

\( \mu, \nu \) are space-time indices, whereas \( \alpha, \beta, \gamma \) label the Gell-Mann matrices of \( U(n_l) \),

\[ Q^{\mu\nu} = F_L^{\mu\nu} + U^\dagger F_R^{\mu\nu} U - \frac{i}{2} [C^{\mu}, C^{\nu}], \]

and

\[ H^{\mu\nu} = \left( \frac{W_1''}{W_1} - \frac{3}{2} \left( \frac{W_1'}{W_1} \right)^2 \right) \langle (C^{\mu}) C^{\nu} - \langle C^{\nu} \rangle C^{\mu} \rangle + \frac{W_1''}{W_1} \left( \frac{W_1'}{W_1} \right)^2 \left( C^{\nu} D^{\mu, \hat{\theta}} - C^{\mu} D^{\nu, \hat{\theta}} \right) \]

\[ + \frac{W_1'}{W_1} \left( i F_L^{\mu\nu} - i U^\dagger F_R^{\mu\nu} U \right). \]

For \( \sigma \) we find

\[ \sigma^{\alpha\beta} = \frac{1}{8} \langle [C_\mu, \lambda^\alpha][C^{\mu}, \lambda^\beta] \rangle + \frac{1}{8} \langle R \{ \lambda^\alpha, \lambda^\beta \} \rangle + S\delta^{\alpha\beta} + \frac{n_l}{8} \left( \frac{W_1'}{W_1} \right)^2 \langle C_\mu \lambda^\alpha \rangle \langle C^{\mu} \lambda^\beta \rangle \]

\[ + \frac{1}{4} \left( \langle T\lambda^\alpha \rangle \langle \lambda^\beta \rangle + \langle T\lambda^\beta \rangle \langle \lambda^\alpha \rangle \right), \quad (29) \]

where

\[ S = -\frac{1}{2} \left( \frac{W_1''}{W_1} - \frac{1}{2} \left( \frac{W_1'}{W_1} \right)^2 \right) \langle (C_\mu) + D_\mu \hat{\theta} \rangle^2 - \frac{1}{2} \frac{W_1'}{W_1} \langle (D_\mu C^{\mu}) + D_\mu D^{\mu, \hat{\theta}} \rangle, \]

\[ T = -\frac{1}{2} \frac{W_1''}{W_1} + \left( \frac{W_1''}{W_1} - \frac{1}{2} \left( \frac{W_1'}{W_1} \right)^2 \right) \langle (C_\mu) C^{\mu} - \frac{1}{2} \langle C_\mu C^{\mu} \rangle \rangle + \frac{W_1'}{W_1} \langle D_\mu C^{\mu} \rangle \]

\[ + \frac{1}{2} \left( \frac{W_1''}{W_1} - \frac{3}{2} \frac{W_1'}{W_1} N \right) - \frac{1}{2} \frac{W_1''}{W_1} N - i \frac{W_1'}{W_1} M + \frac{W_1''}{W_1} C^{\mu} \langle D_\mu \hat{\theta} \rangle - \frac{1}{2} \frac{W_1'}{W_1} D_\mu D^{\mu, \hat{\theta}} \]

\[ + \frac{1}{2} \left( \frac{W_1''}{W_1} - \frac{3}{2} \frac{W_1'}{W_1} N \right) (D_\mu \hat{\theta})^2, \]

\[ R = \frac{W_1''}{W_1} M + i \frac{W_1'}{W_1} N. \quad (30) \]

The one-loop effective action is obtained by including the quadratic fluctuations about the configuration \( \hat{U} \) that, consistently, minimizes the effective action itself. One needs to evaluate the integral of a Gaussian functional, with the known formal result

\[ \int [d\varphi] e^{-i L^2 T \int d^4 x \varphi^\alpha (d_\mu d^\mu + \sigma) \bar{\varphi}^\beta} \sim \frac{1}{\sqrt{\det (d_\mu d^\mu + \sigma)}}, \]

which, upon exponentiation, contributes to the effective action as

\[ \Gamma_{\text{One-loop}}^{\text{eff}} \hat{U} = \int d^4 x \mathcal{L}_{(0+2)}(\hat{U}) + \frac{i}{2} \text{Tr} \log (d_\mu d^\mu + \sigma) + \int d^4 x \mathcal{L}_{(4)}(\hat{U}). \quad (31) \]
A word is needed on the proper definition of the previous expressions. The heat-kernel technique has been used to define the determinant (see Appendix A), and the divergences have been dealt with dimensional regularization. In order to absorb the infinities that result from the functional determinant, counter-terms of \( O(p^2) \) and \( O(p^0) \), as well as new terms of \( O(p^4) \) need be included.

The determinant from the change of functional integration variables in (24) gives a contribution which is proportional to a singularity \( \delta^{(4)}(0) \), and dimensional regularization sets it equal to zero. (A similar remark was in order when a change of field variables allowed to cross out \( W_4(X) \) from the lagrangian in (19).

We find,

\[
\frac{1}{2} \sigma^{\alpha\beta} \sigma^{\beta\alpha} = \left( \frac{1}{8} + \frac{n_t^2}{32} \left( \frac{W_1'}{W_1} \right)^4 \right) \langle C_\mu C_\nu \rangle \langle C^\mu C^\nu \rangle + \frac{1}{16} \langle C_\mu C^\mu \rangle \langle C_\nu C^\nu \rangle - \frac{1}{4} \langle C_\mu C^\mu C_\nu \rangle \langle C^\nu \rangle
\]

\[
+ \frac{n_t}{8} \left( \frac{W_1'}{W_1} \right)^2 \left( \langle C_\mu C_\nu C^\mu C^\nu \rangle - \langle C_\mu C^\mu C_\nu C^\nu \rangle \right) + \frac{1}{16} \langle R \rangle^2 + \frac{n_t}{16} \langle R^2 \rangle + \frac{n_t}{2} S^2 + \frac{1}{4} \langle T \rangle^2
\]

\[
+ \frac{n_t}{4} \langle T^2 \rangle + S \langle T \rangle + \frac{n_t}{8} \left( \left( \frac{W_1'}{W_1} \right)^2 - 1 \right) \langle C_\mu C^\mu R \rangle - \frac{1}{8} \langle C_\mu C^\mu \rangle \langle R \rangle + \frac{1}{4} \langle C_\mu \rangle \langle C^\mu R \rangle
\]

\[
+ \frac{1}{2} S \langle C_\mu \rangle \langle C^\mu \rangle + \frac{n_t}{2} \left( \left( \frac{W_1'}{W_1} \right)^2 - 1 \right) S \langle C_\mu C^\mu \rangle + \frac{n_t}{2} S \langle R \rangle + \frac{1}{2} \langle RT \rangle + \frac{n_t}{4} \left( \frac{W_1'}{W_1} \right)^2 \langle C_\mu \rangle \langle C^\mu T \rangle,
\]

and

\[
\frac{1}{12} R^{(\alpha\beta)} R^{\mu\nu \(\beta\alpha\)} = -\frac{n_t}{24} \langle Q_\mu Q_\nu \rangle + \frac{1}{24} \langle Q_\mu \rangle \langle Q^\mu \rangle + i \frac{n_t}{24} \left( \frac{W_1'}{W_1} \right)^2 \langle Q_\mu \langle C^\mu, C^\nu \rangle \rangle
\]

\[
+ \frac{1}{24} \langle H_{\mu\nu} H^{\mu\nu} \rangle - \frac{n_t}{24} \langle H_{\mu\nu} \rangle \langle H^{\mu\nu} \rangle + \frac{n_t^2}{96} \left( \frac{W_1'}{W_1} \right)^4 \left( \langle C_\mu C_\nu \rangle \langle C^\mu C^\nu \rangle - \langle C_\mu C^\mu \rangle \langle C_\nu C^\nu \rangle \right)
\]

\[
+ \frac{n_t}{12} \left( \frac{W_1'}{W_1} \right)^2 \langle C_\mu H^{\mu\nu} \rangle \langle C^\nu \rangle,
\]

which are the only structures that get divergent contributions at one-loop (see Appendix A).

Since quantum scalar fields in four dimensions do not generate any anomaly to the \( U_L \otimes U_R \) symmetry, the new terms needed to renormalize the one-loop result are necessarily in the list of all possible operators of \( O(p^4) \) invariant under \( U_L \otimes U_R \).

The list of independent operators is given below. The criteria used to select this particular set are the following: terms involving the derivatives \( D_\mu M, D_\mu N, D_\mu F^{\mu\nu}_L \) and alike; three derivatives of \( \tilde{\theta} \) \( (D_\mu D_\nu D_\rho \tilde{\theta}), (D_\mu D_\nu D_\rho \tilde{\theta}) \); \( D_\mu D_\nu C_\nu, (D_\mu D_\nu \tilde{\theta}) \) or \( D_\mu C_\nu \) can be removed as combinations of those in (22) plus terms with the piece \( D_\mu C^\mu \); finally, these can be eliminated with the equations of motion (23). The two derivatives of \( \tilde{\theta} \) can always be chosen to appear under the form \( D_\mu D_\nu \tilde{\theta} \).

The rest of operators that are not independent have been removed upon integration by parts and with the help of the identities

\[
D_\mu C^\nu - D^\nu C_\mu = -[C_\mu, C^\nu] + i F^{\mu\nu}_L - i U^\dagger F^{\mu\nu}_R U,
\]

\[
[D_\mu, D_\nu] C_\rho = -i [F^{\mu\nu}_L, C_\rho],
\]

\[
[D_\mu, D_\nu] \tilde{\theta} = i (F^{\mu\nu}_R - F^{\mu\nu}_L).
\]
All the following terms are real $O_i^\dagger = O_i$. The first ones correspond to the twelve $O(p^4)$ operators of $SU_L \otimes SU_R$ (recall that $C_\mu \equiv U^\dagger D_\mu U$, $M \equiv U^\dagger \chi + \chi^\dagger U$, $N \equiv U^\dagger \chi - \chi^\dagger U$ and $\theta = i\theta$),

\[
\begin{align*}
O_0 &= \langle D_\mu U \ D_\nu U^\dagger \ D^\mu U \ D^\nu U^\dagger \rangle = \langle C^\mu C^\nu C_\mu C_\nu \rangle, \\
O_1 &= \langle D_\mu U \ D^\mu U \rangle^2 = \langle C^\mu C_\mu \rangle \langle C^\nu C_\nu \rangle, \\
O_2 &= \langle D_\mu U \ D_\nu U \rangle \langle D^\mu U \ D^\nu U \rangle = \langle C^\mu C^\nu \rangle \langle C_\mu C_\nu \rangle, \\
O_3 &= \langle D_\mu U \ D^\mu U \ D_\nu U^\dagger \ D^\nu U \rangle = \langle C^\mu C_\mu C^\nu C_\nu \rangle, \\
O_4 &= \langle D_\mu U \ D^\mu U \rangle \langle U^\dagger \chi + \chi^\dagger U \rangle = -\langle C^\mu C_\mu \rangle \langle M \rangle, \\
O_5 &= \langle D_\mu U \ D^\mu U \ (U^\dagger \chi + \chi^\dagger U) \rangle = -\langle C^\mu C_\mu \rangle \langle M \rangle, \\
O_6 &= \langle U^\dagger \chi + \chi^\dagger U \rangle^2 = \langle M \rangle^2, \\
O_7 &= \langle U^\dagger \chi - \chi^\dagger U \rangle^2 = \langle N \rangle^2, \\
O_8 &= \langle \chi^\dagger U \chi^\dagger U + U^\dagger \chi U^\dagger \chi \rangle = \frac{1}{2} \langle M^2 + N^2 \rangle, \\
O_9 &= -i \langle F_R^{\mu \nu} D_\mu U \ D_\nu U^\dagger + F_L^{\mu \nu} D_\mu U^\dagger \ D_\nu U \rangle = i \langle C_\mu C_\nu \left( F_R^{\mu \nu} + U^\dagger F_R^{\mu \nu} \right) \rangle, \\
O_{10} &= \langle U^\dagger F_R^{\mu \nu} U F_L^{\mu \nu} \rangle, \\
O_{11} &= \langle F_R^{\mu \nu} F_R^{\mu \nu} + F_L^{\mu \nu} F_L^{\mu \nu} \rangle, \\
O_{12} &= \langle \chi^\dagger \chi \rangle = \frac{1}{4} \langle M^2 - N^2 \rangle,
\end{align*}
\]

The following eight operators are obtained from the previous twelve by splitting up single traces into products of traces: $\langle C_\mu \rangle$ does not vanish, $\langle C_\mu \rangle \neq 0$, for $U_L \otimes U_R$. They read,

\[
\begin{align*}
O_{13} &= \langle U^\dagger D_\mu U \rangle \langle U^\dagger D^\mu U \ D_\nu U^\dagger \ D^\nu U \rangle = -\langle C^\mu \rangle \langle C_\mu C^\nu C_\nu \rangle, \\
O_{14} &= \langle U^\dagger D_\mu U \rangle \langle U^\dagger D^\mu U \ D_\nu U \ D^\nu U \rangle = -\langle C^\mu \rangle \langle C_\mu \rangle \langle C^\nu C_\nu \rangle, \\
O_{15} &= \langle U^\dagger D_\mu U \rangle \langle U^\dagger D_\nu U \rangle \langle D^\mu U \ D^\nu U \rangle = -\langle C^\mu \rangle \langle C_\nu \rangle \langle C_\nu \rangle \langle C_\nu \rangle, \\
O_{16} &= \langle U^\dagger D_\mu U \rangle \langle U^\dagger D_\nu U \rangle \langle U^\dagger D^\mu U \ D_\nu U \rangle = \langle C^\mu \rangle \langle C_\mu \rangle \langle C_\nu \rangle \langle C_\nu \rangle, \\
O_{17} &= \langle U^\dagger D^\mu U \rangle \langle U^\dagger D_\nu U \rangle \langle U^\dagger \chi + \chi^\dagger U \rangle = \langle C^\mu \rangle \langle C_\mu \rangle \langle M \rangle, \\
O_{18} &= \langle U^\dagger D_\mu U \rangle \langle D^\mu U \ D_\nu U \chi^\dagger \rangle = -\langle C^\mu \rangle \langle C_\mu \rangle \langle M \rangle, \\
O_{19} &= \langle F_R^{\mu \nu} F_R^{\mu \nu} \rangle + \langle F_L^{\mu \nu} F_L^{\mu \nu} \rangle, \\
O_{20} &= \langle F_R^{\mu \nu} \rangle \langle F_L^{\mu \nu} \rangle.
\end{align*}
\]

So far, all are parity even operators. The next seven are similar but have odd parity,

\[
\begin{align*}
O_{21} &= -i \langle D_\mu U \ D^\mu U \rangle \langle U^\dagger \chi - \chi^\dagger U \rangle = i \langle NC^\mu C_\mu \rangle, \\
O_{22} &= -i \langle D_\mu U \ D^\mu U \rangle \langle U^\dagger \chi - \chi^\dagger U \rangle = i \langle N \rangle \langle C^\mu C_\mu \rangle, \\
O_{23} &= i \langle U^\dagger D_\mu U \rangle \langle D^\mu U \chi - D^\mu U \chi^\dagger \rangle = i \langle NC^\mu \rangle \langle C_\mu \rangle, \\
O_{24} &= i \langle U^\dagger D^\mu U \rangle \langle U^\dagger D_\mu U \rangle \langle U^\dagger \chi - \chi^\dagger U \rangle = i \langle N \rangle \langle C^\mu \rangle \langle C_\mu \rangle, \\
O_{25} &= i \langle U^\dagger \chi U^\dagger \chi - \chi^\dagger U \chi^\dagger \rangle = i \langle NM \rangle, \\
O_{26} &= i \left( \langle U^\dagger \chi \rangle^2 - \langle \chi^\dagger U \rangle^2 \right) = i \langle N \rangle \langle M \rangle, \\
O_{27} &= \langle U^\dagger D_\mu U \rangle \langle F_L^{\mu \nu} U^\dagger \ D_\nu U - F_R^{\mu \nu} D_\nu U \ U^\dagger \rangle = \langle C_\mu \rangle \langle C_\nu \rangle \left( F_L^{\mu \nu} - U^\dagger F_R^{\mu \nu} U \right) \rangle.
\end{align*}
\]
Three operators involve the $\epsilon_{\mu\nu\rho\sigma}$ tensor. The first two of them are odd under parity and $O_{30}$ is even.

\[
\begin{align*}
O_{28} &= \epsilon_{\mu\nu\rho\sigma} \langle F_L^{\mu\nu} U_R^{\rho\sigma} \rangle, \\
O_{29} &= i \epsilon_{\mu\nu\rho\sigma} \left( \left( F_L^{\mu\nu} + U_R^{\mu\nu} \right) C^\rho C^\sigma \right), \\
O_{30} &= \epsilon_{\mu\nu\rho\sigma} \left( \left( F_L^{\mu\nu} - U_R^{\mu\nu} \right) C^\rho \right) \langle C^\sigma \rangle,
\end{align*}
\]

(32)

From $O_{31}$ to $O_{57}$ they involve derivatives of the source $\hat{\theta}$, and are given in Appendix C.

The operators that appear at $O(p^2)$ in (21) are

\[
\begin{align*}
E_1 &= -\langle C^\mu C_\mu \rangle \\
E_2 &= \langle M \rangle \\
E_3 &= i \langle N \rangle \\
E_4 &= \langle C^\mu \rangle \langle C_\mu \rangle \\
E_5 &= \langle C^\mu D_\mu \hat{\theta} \rangle \\
E_6 &= D_\mu \hat{\theta} D_\mu \hat{\theta}
\end{align*}
\]

(33)

The effective lagrangian including up to one-loop corrections is, thus,

\[
L^{One-loop} = -W_0^r(X, \mu) + \sum_{i=1}^{6} W_i^r(X, \mu) E_i + \sum_{i=0}^{57} \beta_i^r(X, \mu) O_i + \text{finite non-local}.
\]

(34)

The following structure of counter-terms renders it finite (see Appendix A):

\[
\begin{align*}
\delta W_i(X) &= h W_i^{(1)}(X, \mu) + h \Omega_i(X, \mu) \lambda_\infty + O(h^2), \quad i = 0, ..., 6, \\
\beta_i(X) &= \beta_i^r(X, \mu) + h B_i(X, \mu) \lambda_\infty + O(h^2), \quad i = 0, ..., 57.
\end{align*}
\]

(35)

with

\[
\lambda_\infty = \mu \frac{D-4}{(4\pi)^2} \left( \frac{1}{D-4} - \frac{1}{2} \left( \log 4\pi - \gamma + 1 \right) \right),
\]

(36)

so that

\[
W_i^r(X, \mu) = W_i(X) + h W_i^{(1)}(X, \mu), \quad i = 0, ..., 6.
\]

(37)

At this point we have reinserted the powers of $\hbar$ to help the counting of loops. Recall that the one-loop effective action carries one power of $\hbar$.

The roster of functions $\Omega_i$’s and $B_i$’s is given in Appendix D; they are the main result of our paper. The counter-terms in (35) are written in terms of the functions $\Omega_i$’s and $B_i$’s for the sake of concision. This notation, however, may seem a bit contrived. It should be read in the usual way of perturbation theory, namely with the functions $\Omega_i$’s and $B_i$’s understood as their series in powers of $X$. The renormalization of the functions means the renormalization of the coupling constants which are the coefficients in these expansions.

Parity, charge conjugation and time reversal ought to be conserved. Only the operators that are invariant under charge conjugation themselves can appear in the lagrangian. This is because $X$ is invariant under $C$ as well (see Appendix E). The list of $C$-violating operators is given in Appendix C.

The result is valid for any value of $n_l$. However, depending on the specific $n_l$ considered, there are $n_l$-dependent factorization relations among the traces of products of $n_l \times n_l$ matrices that are of relevance to us since some operators in the list become redundant, i.e., some can be written in
terms of a smaller subset. These relations follow from the Cayley-Hamilton theorem and have been extensively used in [1]. For \( n_l = 3 \) they boil down to [21]

\[
A^3 - \langle A \rangle A^2 + \frac{1}{2} \left( \langle A \rangle^2 - \langle A^2 \rangle \right) A - \det(A) = 0,
\]

for any \( 3 \times 3 \) matrix, be it hermitian or not, and to

\[
\sum_{6 \text{ perm}} \langle A_1 A_2 A_3 A_4 \rangle - \sum_{8 \text{ perm}} \langle A_1 A_2 A_3 \rangle \langle A_4 \rangle - \sum_{3 \text{ perm}} \langle A_1 A_2 \rangle \langle A_3 A_4 \rangle + \sum_{6 \text{ perm}} \langle A_1 A_2 \rangle \langle A_3 \rangle \langle A_4 \rangle - \langle A_1 \rangle \langle A_2 \rangle \langle A_3 \rangle \langle A_4 \rangle = 0.
\]  

(38)

The first relation ensures that the determinant of a matrix can always be expressed in terms of traces and justifies why the determinants of products of \( U \) matrices and their derivatives need not be considered independently. From the second relation, with

\[
A_1 = A_2 = C_\mu, \quad A_3 = A_4 = C_\nu,
\]

and summing over the indices \( \mu \) and \( \nu \), one gets

\[
2O_0 - O_1 - 2O_2 + 4O_3 + 8O_{13} - 2O_{14} - 4O_{15} - O_{16} = 0.
\]  

(40)

Thus, for \( n_l = 3 \) we can spare \( O_0 \) and the \( O(p^4) \) lagrangian will be written as

\[
\mathcal{L}_4 = \sum_{i=1}^{57} L_i(X)O_i,
\]

(41)

where

\[
\begin{align*}
L_1 &= \beta_1 + \frac{1}{2} \beta_0 & L_{13} &= \beta_{13} - 4 \beta_0 \\
L_2 &= \beta_2 + \beta_0 & L_{14} &= \beta_{14} + \beta_0 \\
L_3 &= \beta_3 - 2 \beta_0 & L_{15} &= \beta_{15} + 2 \beta_0 \\
L_{16} &= \beta_{16} + \frac{1}{2} \beta_0
\end{align*}
\]

(42)

and \( L_i = \beta_i \) for the rest.

From \( O_1 \) to \( O_9 \) the same notation as in \( SU_L \times SU_R \) [1] has been kept, also for the coefficient functions. The constants \( H_1, H_2 \) in [1] have turned into the functions \( L_{11}(X), L_{12}(X) \). The new operators that appear in this \( U_L \otimes U_R \) lagrangian are labeled from \( O_{13} \) onwards and the coefficient functions \( L_i(X) \)'s follow suit.

If one disregards all the coefficients associated to \( U_A(1) \) one finds for the \( B_i \)'s in [15]

\[
\begin{align*}
B_0 &= \frac{m_1}{48} & B_1 &= \frac{1}{16} & B_2 &= \frac{1}{8} & B_3 &= \frac{m_1}{24} & B_4 &= \frac{1}{8} & B_5 &= \frac{m_1}{8} & B_6 &= \frac{1}{16} \\
B_7 &= 0 & B_8 &= \frac{m_1}{16} & B_9 &= \frac{m_1}{12} & B_{10} &= -\frac{m_1}{24} & B_{11} &= -\frac{m_1}{12} & B_{12} &= \frac{m_1}{8} & B_{13} &= \frac{1}{4} \\
B_{14} &= 0 & B_{15} &= 0 & B_{16} &= 0 & B_{17} &= 0 & B_{18} &= -\frac{1}{4} & B_{19} &= \frac{1}{24} & B_{20} &= \frac{1}{172}
\end{align*}
\]

(43)

\( O_{30} \) involves an \( \epsilon_{\mu\nu\rho\sigma} \) and does not need to be renormalized.

In the case of \( n_l = 3 \), taking into account that the same relations from [12] should hold, and renaming the constants as \( \Gamma_i \)'s, we obtain

\[
\begin{align*}
\Gamma_1 &= \frac{3}{32} & \Gamma_2 &= \frac{3}{16} & \Gamma_3 &= 0 & \Gamma_4 &= \frac{1}{8} & \Gamma_5 &= \frac{3}{8} & \Gamma_6 &= \frac{1}{16} & \Gamma_7 &= 0 \\
\Gamma_8 &= \frac{3}{16} & \Gamma_9 &= \frac{1}{4} & \Gamma_{10} &= -\frac{1}{4} & \Gamma_{11} &= -\frac{1}{8} & \Gamma_{12} &= \frac{3}{8} & \Gamma_{13} &= 0 & \Gamma_{14} &= \frac{1}{16} \\
\Gamma_{15} &= \frac{1}{8} & \Gamma_{16} &= \frac{1}{32} & \Gamma_{17} &= 0 & \Gamma_{18} &= -\frac{1}{4} & \Gamma_{19} &= \frac{1}{24} & \Gamma_{20} &= \frac{1}{172}
\end{align*}
\]

(44)
They coincide to those of $SU_L(3) \otimes SU_R(3)$ except for the terms that involve $\langle M^2 \rangle$, $\langle M^2 \rangle$, $\langle N^2 \rangle$, $\langle N^2 \rangle$: $\Gamma_6$, $\Gamma_8$, $\Gamma_{12}$. The reason is that among the building blocks that have been used to write the chiral lagrangian, $C_\mu$ and $F_{\mu \nu}^{L,R}$ have vanishing traces in the case of $SU_L(3) \otimes SU_R(3)$, whereas neither $M$ nor $N$ do. Although it is less immediate to retrieve the $SU_L(3) \otimes SU_R(3)$ coefficients

$$
\Gamma_6^{[SU]} = \frac{11}{144}, \quad \Gamma_8^{[SU]} = \frac{5}{16}, \quad \Gamma_{12}^{[SU]} = \frac{5}{8},
$$

from our result, there are simple relations that have to be verified. For instance, if one adds all the divergent pieces that go with the operators $O_6$, $O_8$, $O_{12}$, sets $\chi = m^2 I$ for simplicity, and expands the operators, it is easy to check that the $SU_L(3) \otimes SU_R(3)$ and the $U_L(3) \otimes U_R(3)$ coefficients of $\langle \chi^4 \chi \rangle$ and $\bar{\pi}^2$ verify, respectively,

$$
\frac{9}{8} \left( 12 \Gamma_6^{[SU]} + 2 \Gamma_8^{[SU]} + \Gamma_{12}^{[SU]} \right) = (12 \Gamma_6 + 2 \Gamma_8 + \Gamma_{12}) ,
$$

and similarly

$$
\frac{9}{8} \left( 3 \Gamma_6^{[SU]} + \Gamma_8^{[SU]} \right) = (3 \Gamma_6 + \Gamma_8) .
$$

One recognizes the ratio $\frac{9}{8}$ as the fraction of degrees of freedom in the two theories, for these coefficients multiply one-loop divergent pieces which in the two cases stem from tadpole diagrams, which give a constant divergent contribution for all the virtual mesons that travel around the loop. Therefore, the total result is proportional to the number of degrees of freedom that in each case can circulate. (Of course the same argument goes through for any number of flavours $n_l$ and a similar result holds in general. The $SU_L(n_l) \otimes SU_R(n_l)$ coefficients are $\frac{24}{n_l} \Gamma_6^{[SU]} = \frac{2 + n_l^2}{16 n_l^2}$, $\Gamma_8^{[SU]} = \frac{n_l^2 - 4}{16 n_l}$.

The second relation holds in the form $\frac{n_l \Gamma_6 + \Gamma_{12}}{n_l^2} = \frac{n_l \Gamma_6^{[SU]} + \Gamma_{12}^{[SU]}}{n_l^2 - 1}$. The first relation, that now involves the combination $4 n_l \Gamma_6 + 2 \Gamma_8 + \Gamma_{12}$, reduces to the previous one if one realizes that $\Gamma_{12} = 2 \Gamma_8$ in either case.)

One important difference between the $SU_L \otimes SU_R$ case and ours is that in the first theory the meson masses do not get any infinite contribution and in this case they do. This statement needs some qualification for the language it uses is the customary of renormalizable field theories, where the divergences that are generated require a fixed number of counter-terms, of same type as the terms in the lagrangian only. The chiral expansion in increasing number of derivatives is not a theory of this kind, rather it is non-renormalizable, because at each higher loop new terms are required to absorb the new infinities. In $SU_L \otimes SU_R$ one only needs counter-terms of a chiral order higher than the terms involved in the loops. In $U_L \otimes U_R$ we find a combination of both previous cases. It is non-renormalizable and there is a $O(p^0)$ term, included to reproduce the $U_A(1)$ anomaly, which at one-loop induces a mixture of chiral orders in the divergent parts, as can be seen from the heat-kernel expressions: the divergences are proportional to $\sigma^2$ and $\sigma = \sigma_0 + \sigma_2$ decomposes in (2) in two pieces, $O(p^0)$ and $O(p^2)$, respectively. (There are divergences proportional to $R^2$ too in (2), but the curvature $R$ associated to the connexion (2) does not get any $O(p^0)$ contribution.)

There is a lot of freedom in deciding the prescriptions of what removes which divergences, all of them equally acceptable from the point of view of rendering the final result finite. They are not completely arbitrary, though, since the nesting of divergences when higher loops are considered imposes some constraints among the results at different orders, of the Gell-Mann and Low type in QED. Furthermore, some of them appear more natural than others.
Let us analyse the question of the renormalization of the pseudo-goldstone boson masses, induced by a quark-mass term. Let us first disregard the $U_A(1)$ anomaly by freezing the functions of $X$ to their constant values at $X = 0$ and, for simplicity, consider the symmetric case where the three quark species are degenerate in mass, and switch the external sources off: $\chi$ is a constant that multiplies the unit matrix; at tree-level $\chi$ gives the nonet mass. Now, let us write all the terms quadratic in the fields with at most two derivatives, having added the one-loop divergences to the tree-level result (prior to renormalization). There are contributions from the operators $E_1$, $E_2$ in (B3) for the tree-level parts, whereas the divergent parts come with the operators $O_4$, $O_5$, $O_6$, $O_8$, $O_{17}$ and $O_{18}$, and can be read off from (B5) and (B3). It yields,

$$
\left(1 - 2n_I \frac{\chi}{f^2} h\lambda_\infty\right) \left(\frac{1}{2} \partial_\mu \bar{\pi} \cdot \partial^\mu \pi - \frac{1}{2} \chi \bar{\pi} \cdot \pi\right) + \frac{1}{2} \partial_\mu \eta^0 \partial^\mu \eta^0 - \frac{1}{2} \left(1 - 2n_I \frac{\chi}{f^2} h\lambda_\infty\right) \chi(\eta^0)^2,
$$

where $\lambda_\infty$ is the ultraviolet divergent amount, that in dimensional regularization is essentially $\frac{1}{1-D}$, (see [36]). The difference between the singlet and the octet is apparent. The piece $\partial_\mu \eta^0 \partial^\mu \eta^0$ does not get any divergent contribution while the octet counterpart $\partial_\mu \bar{\pi} \cdot \partial^\mu \pi$ does, and exactly by the same amount as the mass term $\bar{\pi} \cdot \pi$ does as well. In the octet sector, one can pull out the common factor from the entire kinetic term and render a finite result by a field redefinition $\pi \to (1 - h n_I \frac{\chi}{f^2} \lambda_\infty)^\pi$. There is no infinity left over that could require mass renormalization.

In chiral perturbation theory it is often simpler to talk about a renormalization of an $O(p^4)$ operator rather than a wave-function renormalization, but in our case this is what it corresponds to, and this precision is required to qualify the mass renormalization issue. The same result also holds in $SU_L \otimes SU_R$.

For the $\eta^0$, though, the divergent contributions to $\partial_\mu \eta^0 \partial^\mu \eta^0$ - which are none -, and to $(\eta^0)^2$ are different, and therefore the mass gets necessarily renormalized by an infinite amount. This is a remarkable difference between $SU_L \otimes SU_R$ and $U_L \otimes U_R$.

In both cases, of course, the divergences disappear if the quark-mass is turned off $\chi \to 0$, which reflects the fact that the spontaneously broken symmetry is built-in, loop by loop, in the quantum theory and prevents the goldstone particles from acquiring a mass. When the quark-mass is turned on, it is not true that the symmetry structure prevents the $\eta^0$ mass from being infinitely renormalized, as happens with the octet mass. The difference can be rooted to the terms in the lagrangian that are responsible for the wave-function renormalization. To one-loop, this only includes the terms that are quartic in the fields with two derivatives, which are obtained by expanding the operator $\langle C^\mu C_\mu \rangle$; by contracting the two fields that carry no derivatives a tadpole diagram is generated and its divergence multiplies $\partial_\mu \bar{\pi} \cdot \partial^\mu \pi$. The interesting point is that in the chiral lagrangian such a term involving the $\eta^0$ field, like $(\eta^0)^2 \partial_\mu \eta^0 \partial^\mu \eta^0$ or $\bar{\pi} \cdot \pi \partial_\mu \eta^0 \partial^\mu \eta^0$, does not exist at all, as can be immediately seen by making the invariant decomposition of the matrix $U$ in $U = e^{i \sqrt{\frac{1}{n_I}} \frac{\chi}{f^2} \eta^0} U_s$, where $U_s$ contains only the octet fields and has det $U_s = 1$. $C_\mu$ then reads

$$
C_\mu \equiv U^\dagger \partial_\mu U = i \sqrt{2} \frac{1}{n_I f} \partial_\mu \eta^0 + U_s^\dagger \partial_\mu U_s,
$$

and $\langle C^\mu C_\mu \rangle$

$$
\langle C^\mu C_\mu \rangle = \frac{2}{f^2} \partial_\mu \eta^0 \partial^\mu \eta^0 + \langle U_s^\dagger \partial_\mu U_s U_s^\dagger \partial^\mu U_s \rangle,
$$

and no crossed singlet-octet term survive for $\langle U_s^\dagger \partial_\mu U_s \rangle = 0$. $\langle C^\mu C_\mu \rangle$ provides the $\eta^0$ with a kinetic term and nothing else, and no term can generate a one-loop wave-function divergence for
it. Whereas for the octet, one learns from

\[ U_s^\dagger \partial_\mu U_s = i \frac{1}{f} \partial_\mu \pi + \frac{1}{2f^2} [\pi, \partial_\mu \pi] - i \frac{1}{6} \frac{1}{f^3} [\pi, [\pi, \partial_\mu \pi]] + \ldots \]

that

\[ -\frac{f^2}{4} (U_s^\dagger \partial_\mu U_s U_s^\dagger \partial_\mu U_s) = \frac{1}{2} \partial_\mu \pi \cdot \partial^\mu \pi + \frac{1}{48f^2} ([\pi, \partial_\mu \pi] [\pi, \partial^\mu \pi]) + \ldots . \quad (48) \]

It is the second term in (48) which is responsible for the one-loop wave-function renormalization. The structure of commutators makes one realize again that such terms vanish for the singlet.

This same piece generates also a divergence to the mass term. It gets additional \( O(p^2) \) contributions from operator \( E_2 \) in (33) which come from tadpole diagrams too that arise from four meson interaction terms \( \sim \frac{1}{Nc} \langle \Phi^4 \rangle \). It is a well-know result that a lagrangian for scalar fields with no derivative couplings other than the kinetic energy with a quartic interaction has an effective action that needs a wave-function renormalization that starts at two loops \( \sim \frac{1}{Nc} \). At one-loop it requires mass renormalization and this is what we find for the \( \eta^0 \) field. (Vertices with more than four fields from \( E_2 \) at one-loop do not participate in the renormalization of the kinetic terms).

We see that the singlet vs. octet difference in mass renormalization is imbued by the structure of the symmetry group.

If one includes the \( U_A(1) \)-anomaly effects, both the octet and the singlet get their masses renormalized by an infinite amount.

In this section we have presented the complete one-loop calculation of the divergent part of the effective action. It is a calculation to all orders in \( 1/N_c \), in the sense that the functions of \( X \) have been kept generic through the end. There are many unknown parameters in this approach (potentially, all the coefficients of the functions of \( X \)) and without any further restrictions we would not know how to eliminate any of them. We invoke the \( 1/N_c \) expansion of QCD and the restrictions it imposes on the chiral lagrangian to classify the coefficients according to the maximum \( 1/N_c \) power allowed for each, so as to estimate their size, and with this criterion try to select the fewer terms that allegedly bring the main contributions. This will be done in the next section.

4 The \( 1/N_c \) expansion.

The systematic expansion of QCD in powers of \( 1/N_c \) provides a way of effectively reducing the number of constants that intervene in a certain process, once it is decided where to truncate the series in \( 1/N_c \). If \( N_c \) is large enough, a few terms will suffice to give a good account of the exact result. How much large is \textit{large enough} is a question hard to assess, for a good reason, that despite the simplification the large \( N_c \) limit entails technically, it remains too difficult to sum the subclass of diagrams that survive in the limit and it worsens, if anything, for the sub-leading contributions. It is argued that, conceivably, big numerical factors might be accompanying the powers of \( N_c \) in the denominator; in that case a few terms would give accurate predictions even for \( N_c = 3 \). That would explain the remarkable resemblance of many qualitative features and patterns of the leading terms to those observed in hadron physics, with \( N_c = 3 \) \( \frac{3}{2} \). At any rate, lacking of any analytical result, it is the accuracy of the predictions in explaining the data what could give an ultimate justification for the expansion. It is this perspective what launched this project, to set out the basis for the systematic study of the predictions that come out from such scenario for soft \( \pi, K, \eta, \eta' \) so as to discern in what processes and to what extent an agreement with experiment holds.
On general grounds, Witten \cite{witten} showed that in the $N_c \to \infty$ limit, if QCD confines it has a mesonic spectrum that consists of an infinite number of noninteracting, stable states, with masses that have smooth and finite limits. Furthermore, the strong interaction scattering amplitudes are given, to lowest order in $1/N_c$, by sums of tree diagrams with mesons exchanged, which can be derived from an effective lagrangian with local vertices and local meson fields. The decay constants $f$’s are of order $\sqrt{N_c}$ and a coupling constant for a vertex with $k$ mesons attached to it is of order $N_c^{-(k-2)/2}$, i.e., it decreases with the multiplicity of mesons in the vertex, each new meson bringing in one additional power of $1/\sqrt{N_c}$. The $N_c$ counting rules imply that while large-$N_c$ QCD is a strong interacting theory in terms of quarks and gluons, it is equivalent to a weakly interacting theory of mesons. The higher order corrections in the $1/N_c$ expansion, in addition to new couplings in the effective theory, also include the loop diagrams of mesons, which as in any quantum field theory reestablish the unitarity constraints on the amplitudes (cuts, discontinuities across, etc...). Each loop of mesons, in the effective theory, contributes an extra power of $1/N_c$.

In addition to the mesons there are infinitely many glueball states, which at $N_c \to \infty$ are stable and noninteracting. The amplitude for a glueball to mix with a meson is of order $O(1/\sqrt{N_c})$, and the vertices in which they are involved are even more suppressed than the meson vertices: one power of $1/N_c$ for each glueball.

However, only $\pi$, $K$, $\eta$, $\eta'$ remain massless in the chiral limit and $N_c \to \infty$, their masses being precluded by Goldstone’s theorem. The rest of excited mesons and glueballs remain massive, with a typical hadronic mass of about 1 GeV. They decouple from the soft processes that involve the goldstone bosons by virtue of their masses and they are integrated out from the effective theory. The baryons have masses much higher for large $N_c$, for they are known to grow like $N_c^3$.

Within the large-$N_c$, the $U_A(1)$ anomaly effects can be accommodated in a rather natural way in the framework of the chiral lagrangian. Actually, the identification of each ingredient that has been taken into account in writing down the effective theory is clearcut: the constraints imposed on the interactions among goldstone bosons are contained in the operators that involve the $U$ matrices (but not log $U$); the quark masses enter through the terms in $\chi$; and the $U_A(1)$ enters through the functions of $X$. One can talk of switching off the $U_A(1)$ anomaly by freezing the functions to constants, in a similar way as one can take the chiral limit by sending the quark masses to zero. In this language, one can say that the $\eta^0$ has two ways of manifesting itself: either as a goldstone boson or as the argument of the functions of $X$, breaking chiral symmetry as dictated by the $U_A(1)$ anomaly. When it manifests as a goldstone boson, its couplings follow the rules of the meson couplings. This is its the mesonic part, associated to the content in quark degrees of freedom. Its other presence in the chiral lagrangian, imposed by $U_A(1)$, involves couplings that are more suppressed, $1/N_c^2$ per $\eta^0$ in a vertex. This is associated to the special gluonic content of the $\eta^0$, put forward by the anomaly. Although there would be no such a thing as the $\eta^0$ in a world without quarks - nor chiral symmetry -, in the chiral limit the $\eta^0$ mass is

$$m_{\eta^0}^2\bigg|_{U_A(1)} = \frac{4\pi f^2}{\tilde{f}^2} \left( \frac{d^2 E}{d\tilde{\theta}^2} \right)_{\tilde{\theta}=0}^{\text{no quarks}} + O\left(\frac{1}{N_c^2}\right)$$

where $E$ is the vacuum energy in a world without quarks and with a coupling to the topological charge $Tr G^{\mu\nu} \tilde{G}_{\mu\nu}$ in the lagrangian, as in \cite{witten}. The fixed proportion of $\eta^0$ and $\tilde{\theta}$ that appear in the combination $X = i\epsilon^{\mu\nu\rho\sigma} \eta^0 + \tilde{\theta}$ actually relates glueball and $\eta^0$ anomalous couplings to operators involving goldstone bosons. The vertex suppression of $1/N_c^2$ per $\eta^0$ is a combination of $1/N_c$-
glueball and \(1/\sqrt{N_c}\)-meson suppression, the former originated in the anomaly equation, the second carried by the \(f\) factor that divides the \(\eta^0\) field.

In order to obtain the \(1/N_c\) power counting of the sub-leading pieces too, one might proceed by comparison of Green’s functions, as evaluated from the chiral lagrangian and from the diagrams in QCD. In the effective theory, given an operator multiplied by a function \(G(X)\), the \(1/N_c\) power counting can be established on the basis of two distinguishing features: the number of traces over flavour indices (\(#(tf)\)) and the number of powers of the source \(\hat{\theta}(x)\), (\(#(\hat{\theta})\)). As a rule of thumb, the leading dependence on \(1/N_c\) of the various couplings follows from the simple prescription \([13]\):

\[
G(X) = N^2c^{-\#(tf) - #(\hat{\theta})} g(X/N_c),
\]

where \(g\) is a function whose expansion in powers of \(X/N_c\) has coefficients of order 1. The origin of each factor can be easily traced: in relation to the vacuum energy which is \(O(N_c)^2\), there is one power suppression of \(1/N_c\) for each flavour trace and one for each source - recall that in QCD the sources for \(1/N_c\) suppression are the loops of quarks and the non-planarity of the diagrams. Each trace taken over flavour indices amounts to a sum over quark flavours, which in turn can arise only in a quark loop in QCD and adds a factor of \(1/N_c\). As for the source \(\hat{\theta}(x)\), it couples to the topological charge in \([3]\) with strength \(1/N_c\); each derivative of the generating functional with respect to \(\hat{\theta}(x)\) will bring one power of \(1/N_c\) to the Green’s function. In the effective theory this is achieved by pulling out an explicit power of \(1/N_c\) for each \(\hat{\theta}(x)\). Finally, it was already mentioned that the ubiquitous factor of \(f\) count as \(\sqrt{N_c}\). In particular, the \(\eta^0\) that appears in \(X\) is always suppressed by a factor \(1/f\) \([17]\). There are no additional powers of \(1/N_c\) for the leading contributions: once all the previous factors of \(1/N_c\) have been pulled out, also from \(X\) as \(X/N_c\), the expansion of \(g\) in powers of \(X/N_c\) has coefficients that are order 1.

Recall at this point that the \(1/N_c\) counting should be done in a chiral lagrangian with a generic number \(n_l\) of light flavours. This is because of the \(n_l\)-dependent factorization relations already mentioned in section 3, that give linear relations among the \(a\ priori\) independent operators for particular values of \(n_l\). The mismatch of powers of \(1/N_c\) and the departure from the general rule \([19]\) are avoided by allowing for a generic \(n_l\). The correct counting is thus obtained for the functions \(\beta_i(X)\) in \([14],[15]\). The implications for the \(L_i(X)\) can be read off from \([12]\).

Furthermore, as pointed out in \([24]\), it is the \(U_L \otimes U_R\) lagrangian that provides the suitable basis to establish the correct \(1/N_c\) power counting of the constants that involve the nonet of mesons. The \(\eta'\) in the large-\(N_c\) limit gives a contribution to the large-distance behaviour of the Green’s functions that should not be overlooked; in the limit its properties are the same as for the rest of goldstone bosons in the nonet. In next to leading order, a topological mass term appears for the \(\eta'\), but it is \(O(1/N_c)\) and is treated perturbatively. In counting powers of \(1/N_c\), the \(\eta'\) cannot be integrated out from the nonet, for it is when \(N_c \to \infty\), when the \(1/N_c\) expansion is more sensible, that the \(\eta'\) becomes massless and does not decouple.

Expanding the \(W_i\)’s functions in \([13]\) in power series in \(X\),

\[
W_k(X) = W_{k0} + W_{k2}X^2 + W_{k4}X^4 + ... = \frac{f^2}{4} \left(v_{k0} + v_{k2}X^2 + v_{k4}X^4 + ...\right),
\]

for \(k = 0, 1, 2, 4, 5, 6\), and for \(W_3\)

\[
W_3(X) = W_{31}X + W_{33}X^3 + ... = -i\frac{f^2}{4} \left(v_{31}X + v_{33}X^3 + ...\right),
\]
For the parity-odd terms, the first contribution is given by the linear term of the required lagrangian starting with more than four fields are eliminated. This kind of terms would be only of the $\eta$.

As mentioned, the $\eta^0$ gets a contribution to its mass that is $O(1/N_c)$, in the notation of $m^2_{\eta^0}|_{U_A(1)} = -n_1 v_0^2$. Counting $m^2_{\eta^0}$ as two chiral powers $O(p^2)$, in the multiple expansion we shall count $1/N_c$ also as $O(p^2)$ \[\text{(53)}\]. However, to be fully consistent with it, if terms $O(p^4) \times \frac{1}{N_c}$ are kept, then the chiral $O(p^6)$ order should be also included. This would require a two-loop chiral perturbation theory calculation which is far beyond the scope of this article.

Following these criteria and using \[\text{(50)}\], \[\text{(51)}\] and \[\text{(52)}\], we expand the $B_i$ and $\Omega_i$ functions from Appendix D, first in powers of $X$ and then in powers of $\frac{1}{N_c}$, keeping corrections up to $\frac{1}{N_c^2}$ for the $O(p^4)$ terms, up to $\frac{1}{N_c^3}$ for the $O(p^2)$ terms and up to $\frac{1}{N_c^4}$ for the $O(p^0)$ one. Notice that none of the $\eta^0$ fields that appeared through the $X$ has survived, which means that all the terms in the lagrangian starting with more that four fields are eliminated. This kind of terms would be only required for calculating processes that are very difficult to measure experimentally.

To this order, only two $\Omega_i$’s survive:

$$\Omega_0 = \frac{m^2}{2} v_0^2 + O\left(\frac{1}{N_c}\right), \quad \Omega_2 = -\frac{1}{2} v_0^2 + n_1 v_0^2 v_31 + O\left(\frac{1}{N_c^3}\right).$$

The $B_i$’s for $i=0$ to 20 are the same as in \[\text{(13)}\] except for two new contributions:

$$B_8 = \frac{n_1}{16} - \frac{1}{2} v_31 + O\left(\frac{1}{N_c}\right), \quad B_{12} = \frac{n_1}{8} - v_31 + O\left(\frac{1}{N_c^2}\right).$$

\[\text{(55)}\]
The rest of coefficients either vanish exactly or do not contribute to this order in the expansion. There are also some contributions that are proportional to $\hat{\theta}$.

The use of the equations of motion does not ruin the $1/N_c$ power counting in (49), although it introduces additional products of traces and powers of derivatives of $\hat{\theta}$ in (23). The very structure of (23) complies with (49) since the functions of $X$ that multiply the various operators carry their own $1/N_c$ suppression required by (49): for the identity, $\frac{W'^2}{W_1} = \frac{1}{N_c} g(X/N_c)$; $\langle C^\mu C_\mu \rangle$, $\langle C^\mu \rangle C_\mu$ and $D^\mu \hat{\theta} C_\mu$ are multiplied by $\frac{W'^2}{W_1}$ and $\frac{W'^3}{W_1}$, respectively, which are $\frac{1}{N_c} g(X/N_c)$; $\frac{W_2}{W_1}$ multiplies $D^\mu D_\mu \theta$ whereas $\frac{W_2 - W_1}{W_1} = \frac{1}{N_c^2} g(X/N_c)$. Here $g$ denotes, in each case, some function that has $N_c$-independent Taylor expansion coefficients.

Finally, let us comment on the regeneration through quantum corrections of a term $\langle U^\dagger D_\mu U \rangle \langle U^\dagger D^\mu U \rangle$, which had been removed from the tree level effective lagrangian in (19). It is a confirmation that the effects of meson loops bring to the effective action contributions that are $1/N_c$ suppressed in relation to the leading tree level. It is only at $N_c = \infty$ that $\pi$, $K$, $\eta$, $\eta'$ form a nonet. When sub-leading corrections in $1/N_c$ are taken into account, the enlarged symmetry with respect to $U_L \otimes U_R$ no longer holds. The first instance is the $O(1/N_c)$ mass piece exclusive for the $\eta_0$. The reappearance of that term is another example: the $\eta^0$ (singlet) and the $\pi$ (octet) fields are normalized differently by sub-leading contributions.

5 Conclusion and outlook.

In this article the effective theory developed in [11] for the strong interactions at low energies among the lightest pseudoscalars is extended to include the $\eta'$ particle. The approach that has been adopted exploits the fact that, as the number of colours $N_c$ grows bigger, the mass difference (topological mass) between the $\eta'$ and the octet vanishes as $1/N_c$, and the $U_A(1)$ puzzle can be treated as a series in inverse powers on $N_c$. We exploit the possibility that a good description could emerge by taking the nonet of soft pseudoscalars as the goldstone bosons of the spontaneous breaking of $U_L \otimes U_R \rightarrow U_V$; thus benefitting from the perks that such theories feature, in terms of constraints on the form of the couplings and relations among the coupling constants. The departures from the results in the real world are dealt with as corrections in powers of the quark masses and $1/N_c$.

The most general effective action is given that includes terms up to $O(p^4)$, with quantum corrections included to one-loop. The $U_A(1)$ anomaly is conveniently incorporated.

This is the first step of a project towards a systematic study to spell out the predictions that could emerge from such scenario. It will shed light on what processes can be accomodated within the approach.

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Appendix A: The Heat-Kernel Technique.

This appendix is a reminder of some results that have been used in the text, which are obtained with the help of the heat-kernel technique. It provides a convenient way to evaluate the one-loop effective action.

Let $\hat{H}$ be an operator acting on a Hilbert space. The heat-kernel two-point function associated to $\hat{H}$ is defined as a function of the parameter $\tau$,

$$K(x, y; \tau) = \langle x | e^{-i\tau \hat{H}} | y \rangle \theta(\tau).$$

It verifies the equation

$$i \frac{\partial}{\partial \tau} K(x, y; \tau) = \int d^D z \langle x | \hat{H} | z \rangle K(z, y; \tau) + i\delta(\tau)\delta^D(x - y),$$

with the boundary condition,

$$\lim_{\tau \to 0} K(x, y; \tau) = \delta^D(x - y),$$

which follows from its definition. We do not specify the dimensionality of space-time and allow for a generic $D$. Often, $\hat{H}$ is a local operator, i.e., $\langle x | \hat{O} | y \rangle = O_x \delta^D(x - y)$, where $O_x$ is a differential operator with a finite number of terms. We shall limit ourselves to this case. Then, $K(x, y; \tau)$ is a Green’s function that verifies a partial differential equation of the Schrödinger type,

$$i \frac{\partial}{\partial \tau} K(x, y; \tau) = O_x K(x, y; \tau) + i\delta(\tau)\delta^D(x - y).$$

When the operator $\hat{H}$ is elliptic the definition of $K$ differs somewhat from the one given above. In that case there is no need for a factor of $i$ in the exponent, since all the eigenvalues of $\hat{H}$ are non-negative. There, the equation is a heat-transport-like equation, from which the technique shares its name. In the present case we are dealing with operators of the sort $\partial_\mu \partial^\mu$, which in Minkowski space is hyperbolic. It is convenient to pick out the mass term from $\hat{H}$, if the theory is massive, or, else, to introduce an infrared regulator by adding a constant term $M^2$ to $\hat{H}$, $\hat{H} + M^2$. For small values of $\tau$, $K(x, y; \tau)$ admits an asymptotic expansion of the form,

$$K(x, y; \tau) \sim i \frac{1}{(4\pi i\tau)^D} \sum_{n=0}^{\infty} h_n(x, y)(i\tau)^n.$$  \hspace{1cm} (56)

The functions $h_n(x, y)$ are known as the Seeley-DeWitt coefficients [28].

Given that

$$K(x, y; \tau \to 0) \sim i \frac{1}{(4\pi i\tau)^D} e^{\frac{(x-y)^2}{4\tau}} \to \delta^D(x - y),$$

the boundary condition translates into

$$h_0(x, x) = 1.$$  \hspace{1cm} (56)

The computation of the one-loop effective action using the background field method entails the evaluation of

$$\log \det(\hat{H}) = Tr \log \hat{H}.$$
$Tr$ stands for the trace over space-time as well as all the internal indices. This can be written in terms of the kernel $K(x, y; \tau)$ if one uses the following integral representation for the logarithm

$$\log \hat{H} - \log \hat{H}_0 = - \int_0^\infty \frac{d\tau}{\tau} \left( e^{-i\tau \hat{H}} - e^{-i\tau \hat{H}_0} \right).$$

Apart from an unessential $\hat{H}$-independent, divergent constant $C$

$$Tr \log \hat{H} = - \int d^D x \int_0^\infty \frac{d\tau}{\tau} \, tr \langle x | e^{-i\tau \hat{H}} | x \rangle + C.$$

This expansion will be used to depict the divergent short-distance contributions so as to be able to subtract them away. Given $\tau$, only those points separated an interval $(x - y)^2$ that is less or of the order of $\tau$ give non-negligible (non-oscillatory) values for $K(x, y; \tau)$. The short distance contribution to the effective action is thus contained in the integral about $\tau = 0$; it will not be affected by $M$, which only modifies the contribution from large values of $\tau$ (large wavelength modes).

The particular form of the operator $\hat{H}$ which corresponds to our computation is

$$\hat{H} = d_\mu d^\mu + \sigma,$$

where $d_\mu = \partial_\mu + \omega_\mu(x)$. A naïve application of eq. (57) could lead to infrared divergences associated to the large $\tau$ integration region. For that reason we have changed it to $\hat{H} + M^2 = d_\mu d^\mu + \sigma + M^2$.

Each term in the Seeley-DeWitt expansion yields a finite contribution, which give

$$i \, tr \langle x | \log \left( \hat{H} + M^2 \right) | x \rangle = \frac{1}{(4\pi)^{D/2}} \sum_{n=0}^\infty (M^2)^{\frac{D}{2} - n} \Gamma \left( n - \frac{D}{2} \right) \langle h_n(x, x) \rangle,$$

except for an unessential additive constant. The Seeley-DeWitt coefficients $h_n$ are known in this case,

$$h_0(x, x) = I,$$
$$h_1(x, x) = -\sigma,$$
$$h_2(x, x) = \frac{1}{2} \sigma^2 + \frac{1}{12} R_{\mu\nu} R^{\mu\nu} + \frac{1}{6} d_\mu d^\mu \sigma.$$

where $R_{\mu\nu} = \partial_\mu \omega_\nu - \partial_\nu \omega_\mu + [\omega_\mu, \omega_\nu]$. Higher order $h_n(x, x)$ may be found in [28].

The singularities that arise in the calculation have been regulated by analytic continuation over the $D$-complex plane. In a four-dimensional theory the ultraviolet divergences appear as poles about $D = 4$, and get contributions from $h_0, h_1$ and $h_2$, as $\Gamma \left( - \frac{D}{2} \right), \Gamma \left( 1 - \frac{D}{2} \right)$ and $\Gamma \left( 2 - \frac{D}{2} \right)$, respectively. Retaining in (58) these three terms only, it reads

$$i \, tr \langle x | \log \left( \hat{H} + M^2 \right) | x \rangle_{\text{div}} = \frac{M^{D-4}}{(4\pi)^{\frac{D}{2}}} \left( \frac{2}{D - 4} + \log 4\pi - \gamma + 1 \right) \left( \frac{1}{2} (\sigma + M^2)^2 + \frac{1}{12} R_{\mu\nu} R^{\mu\nu} \right) + \frac{1}{12} R_{\mu\nu} R^{\mu\nu} \right) + O(D - 4),$$

$^\dagger$In general $\sigma(x)$ and $\omega(x)$ can be matrices. In that case the coefficients $h_n$ are matrices as well and a trace over these internal indices is also understood in [28].
Notice that in (60) it is the combination $\sigma + M^2$ the one that multiplies the pole $\frac{1}{D-4}$, i.e., it is the whole operator that adds to $d_\mu d^\mu$ in $\hat{H} + M^2$. If $\sigma$ has its own mass terms - the light pseudoscalar masses in our case - there is no need to introduce any new infrared regulator $M$. We see that only $h_2$ is involved in the residue of the pole $\frac{1}{D-4}$.

Therefore, a finite expression is obtained by subtracting

$$\frac{\mu^{D-4}}{(4\pi)^2} \left( -\frac{2}{D-4} + \log 4\pi - \gamma + 1 \right) \langle h_2(x,x) \rangle$$

to the effective action. This copes with the ultraviolet divergences and it is the procedure used in the text. $\mu$ is a parameter with mass units. The total derivative $\langle d_\mu d^\mu \sigma \rangle$ has been discarded.

8 Appendix B: U(3).

This appendix is devoted to present some of the properties of the $U(3)$ group which have been used in the text. Whenever the generalization is straightforward we give the result for $U(n)$.

The explicit form of the $U(3)$ hermitian generators, $\lambda_\mu = \lambda^\dagger_\mu$, $\mu = 0, 1, 2, ..., 8$, is

$$\lambda_0 = \frac{\sqrt{2}}{3} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \lambda_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \lambda_2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad (61)$$

$$\lambda_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \lambda_4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad \lambda_5 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix},$$

$$\lambda_6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad \lambda_7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \quad \lambda_8 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}.$$  

For $U(n)$ there are $n^2$ such matrices, and $\mu = 0, 1, ..., n^2 - 1$. Also $\lambda_0 = \sqrt{\frac{2}{n}} I$, with $I$ the $n \times n$ identity. These matrices obey, in general, the following basic trace properties:

$$Tr(\lambda_\mu) \equiv \langle \lambda_\mu \rangle = \sqrt{2n} \delta_{\mu 0}, \quad \langle \lambda_\mu \lambda_\nu \rangle = 2\delta_{\mu\nu}.$$  

(62)

The product of two matrices verifies:

$$[\lambda_\mu, \lambda_\nu] = 2i f_{\mu\nu\rho} \lambda_\rho, \quad \{\lambda_\mu, \lambda_\nu\} = 2d_{\mu\nu\rho} \lambda_\rho,$$  

(63)

$$\lambda_\mu \lambda_\nu = (d_{\mu\nu\rho} + if_{\mu\nu\rho}) \lambda_\rho.$$  

(64)

For $U(3)$, the non-zero entries of the antisymmetric $f_{\mu\nu\rho}$ and symmetric $d_{\mu\nu\rho}$ constants are

$$f_{123} = 1, \quad f_{458} = f_{678} = \frac{\sqrt{2}}{2}, \quad f_{147} = -f_{156} = f_{246} = f_{257} = f_{345} = -f_{367} = \frac{1}{2},$$  

(65)
\[ d_{0\mu\nu} = \sqrt{\frac{2}{3}} \delta_{\mu\nu} , \]
\[ d_{118} = d_{228} = d_{338} = -d_{888} = \frac{1}{\sqrt{3}} , \]
\[ d_{146} = d_{157} = -d_{247} = d_{256} = d_{344} = d_{355} = -d_{366} = \frac{1}{2} , \]
\[ d_{448} = d_{558} = d_{668} = d_{778} = -\frac{1}{2\sqrt{3}} . \]

These tensors are not independent as they satisfy the following relations (repeated indices are summed up):
\[ d_{\mu\nu\nu} = n\sqrt{2n} \delta_{\mu0} , \]
\[ d_{\mu\nu\lambda} d_{\rho\nu\lambda} = n \left( \delta_{\mu\rho} + \delta_{\mu0} \delta_{\rho0} \right) , \]
\[ f_{\mu\nu\lambda} f_{\rho\nu\lambda} = n \left( \delta_{\mu\rho} - \delta_{\mu0} \delta_{\rho0} \right) , \]
\[ f_{\mu\nu\tau} f_{\lambda\rho\tau} + f_{\mu\lambda\tau} f_{\nu\rho\tau} + f_{\mu\rho\tau} f_{\nu\lambda\tau} = 0 , \]
\[ f_{\mu\nu\tau} d_{\lambda\rho\tau} + f_{\mu\lambda\tau} d_{\nu\rho\tau} + f_{\mu\rho\tau} d_{\nu\lambda\tau} = 0 , \]
\[ f_{\mu\nu\sigma} f_{\rho\tau\sigma} = d_{\mu\rho\sigma} d_{\tau\nu\sigma} - d_{\mu\tau\sigma} d_{\nu\rho\sigma} . \]

The identity
\[ (\lambda_{\alpha})_{ab} (\lambda_{\alpha})_{cd} = 2 \delta_{ad} \delta_{bc} , \]
has been extensively used. It yields the properties
\[ \lambda^a \lambda^a = 2nI , \]
\[ \langle \lambda_{\alpha} A \lambda_{\alpha} B \rangle = 2 \langle A \rangle \langle B \rangle , \]
\[ \langle \lambda_{\alpha} A \rangle \langle \lambda_{\alpha} B \rangle = 2 \langle AB \rangle . \]

9 Appendix C: Operators with derivatives of \( \hat{\theta} \) and some C-violating operators.

In this appendix to give list of the operators that involve derivatives of \( \hat{\theta} \), that we have not included in the text (see (32)).

\[ O_{31} = D_\mu \hat{\theta} \langle C^\mu C^\nu C_\nu \rangle , \]
\[ O_{32} = D_\mu \hat{\theta} \langle C^\mu \rangle \langle C^\nu C_\nu \rangle , \]
\[ O_{33} = D_\mu \hat{\theta} \langle C^\mu C^\nu \rangle \langle C_\nu \rangle , \]
\[ O_{34} = D_\mu \hat{\theta} \langle C^\mu \rangle \langle C^\nu \rangle \langle C_\nu \rangle , \]
\[ O_{35} = D_\mu \hat{\theta} \langle C^\mu \rangle \langle C^\nu C_\nu \rangle , \]
\[ O_{36} = D_\mu \hat{\theta} D_\nu \hat{\theta} \langle C^\mu C^\nu \rangle , \]
\[ O_{37} = D_\mu \hat{\theta} D^\mu \hat{\theta} \langle C^\nu \rangle \langle C_\nu \rangle , \]
\[ O_{38} = D_\mu \hat{\theta} D_\nu \hat{\theta} \langle C^\mu \rangle \langle C^\nu \rangle , \]
\[ O_{39} = D_\mu \hat{\theta} D^{\mu} \hat{\theta} D_\nu \hat{\theta} \langle C^\nu \rangle, \]
\[ O_{40} = D_\mu \hat{\theta} D^{\mu} \hat{\theta} D_\nu \hat{\theta} D^\nu \hat{\theta}, \]
\[ O_{41} = iD^\mu D_\mu \hat{\theta} \langle C^\mu C_\nu \rangle, \]
\[ O_{42} = iD^\mu D_\mu \hat{\theta} \langle C^\mu \rangle \langle C_\nu \rangle, \]
\[ O_{43} = i\langle C^\mu \rangle D_\mu \hat{\theta} D^\nu D_\nu \hat{\theta}, \]
\[ O_{44} = iD_\mu \hat{\theta} D^\mu \hat{\theta} D^\nu D_\nu \hat{\theta}, \]
\[ O_{45} = D^\mu D_\mu \hat{\theta} D^\nu D_\nu \hat{\theta}, \]
\[ O_{46} = D_\mu \hat{\theta} (C^\mu M), \]
\[ O_{47} = D_\mu \hat{\theta} \langle C^\mu \rangle \langle M \rangle, \]
\[ O_{48} = iD_\mu \hat{\theta} \langle C^\mu N \rangle, \]
\[ O_{49} = iD_\mu \hat{\theta} \langle C^\mu \rangle \langle N \rangle, \]
\[ O_{50} = D_\mu \hat{\theta} D^\mu \hat{\theta} \langle M \rangle, \]
\[ O_{51} = iD_\mu \hat{\theta} D^\mu \hat{\theta} \langle N \rangle, \]
\[ O_{52} = iD^\mu D_\mu \hat{\theta} \langle M \rangle, \]
\[ O_{53} = D^\mu D_\mu \hat{\theta} \langle N \rangle, \]
\[ O_{54} = D_\mu \hat{\theta} \langle C_\nu \left( F^{\mu \nu}_L - U^\dagger F^{\mu \nu}_R U \right) \rangle, \]
\[ O_{55} = D_\mu \hat{\theta} \langle C_\nu \rangle \langle F^{\mu \nu}_L - F^{\mu \nu}_R \rangle, \]
\[ O_{56} = \epsilon_{\mu \nu \rho \sigma} \langle \left( F^{\mu \nu}_L - U^\dagger F^{\mu \nu}_R U \right) C^\rho \rangle D^\sigma \hat{\theta}, \]
\[ O_{57} = \epsilon_{\mu \nu \rho \sigma} \langle F^{\mu \nu}_L - F^{\mu \nu}_R \rangle \langle C^\rho \rangle D^\sigma \hat{\theta}. \]
which should be taken into account and added to $\Omega_5$ and $\Omega_6$.

For the sake of brevity, we use the following short notation:

$$\omega_i \equiv \frac{W_i}{W_1}, \quad \omega'_i \equiv \frac{W'_i}{W_1}, \quad \omega''_i \equiv \frac{W''_i}{W_1},$$

and, similarly, to include as many primes as necessary. Let us emphasize that $\omega'$ is not the derivative of $\omega_i$, and so forth. Also, recall that $W_1(0) = W_2(0) = \frac{L^2}{4}$ so that $\omega_1(0) = \omega_2(0) = 1$.

\[
\begin{align*}
\Omega_0 &= \frac{n^2_0}{8} \omega''_0 - \frac{n^2_0}{4} \omega''_0 \omega'_1 - \frac{n^2_0}{2} \omega''_0 \omega_1 - \frac{n^2_0}{8} \omega''_0 \omega'_1^2 - \frac{n^2_0}{32} \omega''_0 \omega'_1^2, \\
\Omega_1 &= -\frac{n^2_1}{8} \omega'_0 - \frac{n^2_1}{4} \omega'_0 \omega'_1 - \frac{n^2_1}{4} \omega''_0 \omega'_1^2 + \frac{n^2_1}{8} \omega'_0 \omega'_1 \omega'_1 - \frac{n^2_1}{16} \omega'_0 \omega'_1 \omega_1^3, \\
\Omega_2 &= \left( -\frac{n^3_2}{8} \omega''_0 \omega'_1 - \frac{n^3_2}{4} \omega''_0 \omega'_1 \omega'_2 + \frac{i n_1}{4} \omega''_0 \omega'_1 \omega_3 - \frac{i n_1}{8} \omega''_0 \omega'_1 \omega_3 - \frac{i n_1}{8} \omega''_0 \omega'_1 \omega_3 - \frac{n^3_2}{16} \omega''_0 \omega'_1 \omega'_2 \right), \\
\Omega_3 &= \left( -\frac{i n_1}{2} \omega''_0 \omega'_2 - \frac{1}{4} \omega''_0 \omega_3 - \frac{n^3_2}{8} \omega''_0 \omega'_1 \omega_2 + \frac{i n_1}{8} \omega''_0 \omega'_1 \omega_3 + \frac{i n_1}{8} \omega''_0 \omega'_1 \omega_3 - \frac{n^3_2}{16} \omega''_0 \omega'_1 \omega'_3 \right), \\
\Omega_4 &= -\frac{n^3_4}{8} \omega''_0 \omega'_1 - \frac{m_1}{4} \omega''_0 \omega'_1 \omega'_1 - \frac{m_1}{4} \omega''_0 \omega'_1 \omega_1 + \frac{n^3_4}{8} \omega''_0 \omega'_1 \omega'_1 + \left( \frac{n_1}{8} - \frac{n^3_4}{16} \right) \omega''_0 \omega'_1 \omega_1^3, \\
\Omega_5 &= \left( -\frac{3}{4} \left( n^3_5 - n_1 \right) \omega'_0 \omega'_1^3 + \frac{1}{4} \left( n^3_5 - n_1 \right) \omega''_0 \omega'_1 \omega'_1 \right), \\
\Omega_6 &= \left( \frac{n^3_6}{8} \omega''_0 \omega'_1 \omega'_1 - \frac{n^3_6}{8} \omega''_0 \omega'_1 \omega'_5 + \frac{n^3_6}{8} \omega''_0 \omega'_1 \omega'_5 - \frac{n^3_6}{8} \omega''_0 \omega'_1 \omega'_5 + \frac{n^3_6}{8} \omega''_0 \omega'_1 \omega'_5 - \frac{n^3_6}{8} \omega''_0 \omega'_1 \omega'_5 \right), \\
\Omega_7 &= -\left( -\frac{n^3_7}{8} \omega''_0 \omega'_1 \omega'_1 - \frac{n^3_7}{8} \omega''_0 \omega'_1 \omega'_5 - i \left( \frac{n_1}{8} - \frac{n^3_7}{4} \right) \omega''_0 \omega'_1 \omega'_1 + \frac{n^3_7}{8} \omega''_0 \omega'_1 \omega'_5 \right),
\end{align*}
\]

(72)

The $O(p^4)$ is renormalized with the following $B_i$'s,

$$B_0 = \frac{n_1}{48} + \frac{n_1}{6} \omega'_1^2,$$
\[ B_1 = \frac{1}{16} + \frac{n_1^2}{8} \omega_1' \omega_1'' + \frac{n_1^2}{4} \omega_1' \omega_1'' + \left( \frac{n_1^2}{48} + \frac{n_1^4}{32} \right) \omega_1''', \]

\[ B_2 = \frac{1}{8} + \frac{n_1^2}{24} \omega_1^4, \]

\[ B_3 = \frac{n_1}{24} - \frac{n_1}{6} \omega_1', \]

\[ B_4 = \frac{\omega_2}{8} - \frac{i}{n_1} \omega_1' \omega_3 + \frac{n_1^2}{4} \omega_1' \omega_2 + \frac{1}{4} \omega_1'' \omega_2 + \frac{n_1^2}{8} \omega_1' \omega_2' - i \frac{n_1^3}{2} \omega_1'' \omega_3' + \left( \frac{n_1^2}{8} - \frac{3}{8} \right) \omega_1' \omega_2 - \frac{n_1^2}{8} \omega_1' \omega_2' - \frac{n_1^2}{4} \omega_1' \omega_2'' + i \frac{n_1}{8} \omega_1' \omega_3', \]

\[ B_5 = \frac{n_1}{8} \omega_2 - \frac{n_1}{8} \omega_1' \omega_2, \]

\[ B_6 = \frac{1}{16} \omega_2^2 + \frac{1}{4} \omega_2 \omega_2'' - \frac{n_1^2}{8} \omega_2'' - \frac{1}{4} \omega_2' \omega_3' - i \frac{n_1}{2} \omega_2' \omega_3' + \left( \frac{n_1^2}{8} - \frac{1}{4} \right) \omega_1' \omega_2 \omega_2', \]

\[ B_7 = \frac{1}{8} \omega_1' \omega_2' \omega_3' + \left( \frac{1}{4} - \frac{n_1^2}{8} \right) \omega_1' \omega_3' \omega_3' + i \frac{n_1}{8} \omega_1' \omega_2 \omega_3' + \frac{n_1^2}{8} \omega_1' \omega_3', \]

\[ B_8 = \frac{n_1^4}{16} \omega_2^2 - \frac{n_1^4}{16} \omega_3^2 - \frac{n_1^4}{4} \omega_3^2 - i \frac{n_1}{2} \omega_2' \omega_3' - i \frac{n_1}{8} \omega_1' \omega_2 \omega_3 + \left( \frac{n_1^2}{32} - \frac{1}{16} \right) \omega_1' \omega_2' \omega_2' - i \frac{n_1}{16} \omega_1' \omega_2 \omega_3 - \frac{n_1^3}{32} \omega_1' \omega_2' \omega_3', \]

\[ B_9 = \frac{n_1}{12} + \frac{n_1}{12} \omega_1', \]

\[ B_{10} = -\frac{n_1}{12} - \frac{n_1}{12} \omega_1', \]

\[ B_{11} = -\frac{n_1}{24} + \frac{n_1}{24} \omega_1', \]

\[ B_{12} = \frac{n_1}{4} \omega_2 + \frac{n_1}{2} \omega_3 - \frac{n_1}{2} \omega_3' + i \omega_2' \omega_3 - i \omega_2 \omega_3' + \frac{n_1}{4} \omega_1' \omega_2 \omega_2' + \frac{n_1}{2} \omega_1' \omega_3', \]

\[ B_{13} = \frac{1}{4}, \]

\[ B_{14} = -\frac{n_1}{4} \omega'' + \frac{n_1}{2} \omega_1' \omega_2 + \frac{n_1}{3} \omega'' + \left( \frac{5 n_1}{12} - \frac{n_1^3}{8} \right) \omega_1' \omega_2'' + \left( -\frac{n_1}{8} + \frac{3 n_1^3}{16} \right) \omega_1' \omega_4. \]
\[ B_{15} = -\frac{n_1}{3} \omega''^2 + \frac{2}{3} \frac{n_1}{4} \omega'' \omega' - \frac{n_1}{4} \omega' \omega''^4, \]
\[ B_{16} = -\frac{1}{4} \omega'' + \frac{3}{8} \omega' \omega + \left( \frac{n_2^2}{8} - \frac{1}{4} \right) \omega'' \omega' + \left( \frac{3}{4} - \frac{3}{8} n_2^2 \right) \omega''^2 \omega + \left( \frac{9 n_2^4}{32} - \frac{9}{16} \right) \omega' \omega''^4, \]
\[ B_{17} = \frac{n_1}{8} \omega'' \omega' + \frac{n_1}{4} \omega'' \omega' + \frac{n_1}{4} \omega'' \omega' - \frac{i}{8} \omega_1 \omega_2 + \frac{n_1}{8} \omega'' \omega_2 - \frac{n_1}{8} \omega'' \omega' + \frac{i}{8} \omega'' \omega_3 - \frac{n_1}{8} \omega'' \omega_3 + \frac{3}{4} \omega'' \omega_2 - \frac{n_1}{4} \omega'' \omega_2 - \frac{n_1}{4} \omega'' \omega'_2 \]
\[ + 3i \left( \frac{1}{8} - \frac{n_2^4}{16} \right) \omega' \omega_3^3, \]
\[ B_{18} = -\frac{1}{4} \omega'' - \frac{i}{2} \omega'' \omega' + \frac{3}{4} \frac{n_1}{4} \omega'' \omega_2 - \frac{i}{4} \omega'' \omega_2 - \frac{i}{4} \omega'' \omega' + \frac{n_1}{2} \omega'' \omega_2 + \frac{n_1}{2} \omega'' \omega'_2 \]
\[ + \frac{i}{4} \omega'' \omega_3 - \frac{i}{4} \omega'' \omega'_3, \]
\[ B_{19} = \frac{1}{24} + \frac{1}{24} \omega'' \omega', \]
\[ B_{20} = \frac{1}{12} + \frac{1}{12} \omega'' \omega', \]
\[ B_{21} = -\frac{n_1}{8} \omega_3 + \frac{n_1}{8} \omega' \omega_3, \]
\[ B_{22} = \frac{1}{8} \omega_3 - \frac{i}{8} \omega_1 \omega_2 - i \frac{n_1}{4} \omega'' \omega_2 - \frac{1}{4} \omega_1 \omega_3 - \frac{n_2^2}{8} \omega'_1 \omega_3 - \frac{n_1}{4} \omega'' \omega'_3 \]
\[ + \frac{n_1}{4} \omega'' \omega'_2 + \frac{i}{8} \left( 3 - n_2^2 \right) \omega''^2 \omega_3 + \frac{n_2^2}{8} \omega'' \omega'_3 \omega'_3 \]
\[ + \frac{n_2^2}{16} \omega_1 \omega'' \omega_3 - \frac{1}{16} \omega'' \omega'_3 \omega'_3, \]
\[ B_{23} = \frac{1}{4} \omega_3 + \frac{i}{2} \omega'' \omega_3 + \frac{n_1}{4} \omega'' \omega_2 - \frac{i}{4} \omega'' \omega'_2 - \frac{3}{4} \omega'' \omega_3 + \frac{i}{4} \omega'' \omega'_3, \]
\[ B_{24} = \frac{i}{8} \omega'' \omega_2 - \frac{n_1}{4} \omega'' \omega_3 + \frac{n_1}{8} \omega'' \omega'_3 + \frac{i}{8} \left( 1 - \frac{n_2^2}{2} \right) \omega'' \omega'_3 \omega_2 \]
\[ + \frac{3}{8} \omega''^2 \omega_3 - \frac{n_1}{8} \omega'' \omega'_3 - \frac{n_1}{4} \omega'' \omega'_3 + \frac{3i}{8} \left( n_2^2 - 1 \right) \omega'' \omega_3 + \left( \frac{n_3^2}{8} + \frac{n_3^2}{16} \right) \omega'' \omega'_3 \omega'_3, \]
\[ B_{25} = \frac{i}{2} \omega'' \omega_2 - \frac{i}{2} \omega'' \omega_3 + \frac{n_1}{2} \omega'' \omega'_3 + \frac{i}{4} \omega'' \omega'_3 + i \omega'' \omega'_3 \]
\[ - \frac{n_1}{4} \omega'' \omega'_3 + \frac{n_1}{4} \omega'' \omega'_3 - \frac{1}{4} \omega'' \omega'_3 = \frac{n_1}{4} \omega'' \omega'_3 + \frac{i}{4} \omega'' \omega'_3, \]
\[ B_{26} = \frac{i}{2} \omega'' \omega'_2 - \frac{n_1}{2} \omega'' \omega'_3 + \frac{1}{8} \omega'' \omega_3 + \frac{i}{2} \omega'' \omega'_2 - \frac{n_1}{4} \omega'' \omega'_2 \omega_3 + \frac{1}{4} \omega'' \omega'_3 \omega_3 \]
\[ + \frac{1}{4} \omega'' \omega'_2 + \frac{i}{8} \omega'' \omega'_3 - \frac{1}{4} \omega'' \omega'_2 + \frac{i}{16} \omega'' \omega'_2 \omega_2 + \frac{i}{8} \omega'' \omega'_3, \]

33
\[
\begin{align*}
&+ \frac{1}{4} \left( \frac{n_1^2}{2} - 1 \right) \omega_1' \omega_2' \omega_3' - i \frac{n_1}{8} \omega_1' \omega_2^2 + \frac{1}{4} \left( \frac{n_1^2}{2} - 1 \right) \omega_1' \omega_2 \omega_3' - \frac{n_1^2}{8} \omega_1' \omega_2' \omega_3' \\
&+ i \frac{n_1}{4} \omega_1' \omega_3^2 - \frac{n_1^2}{8} \omega_1' \omega_2 \omega_3' + i \frac{n_1}{8} \omega_1' \omega_3 \omega_3'' + \frac{1}{8} \left( \frac{n_1^2}{2} - 1 \right) \omega_1' \omega_2 \omega_3 \\
&+ \frac{n_1^4}{16} \omega_1' \omega_2' \omega_3' - i \frac{n_1^3}{16} \omega_1' \omega_3 \omega_3', \\
B_{27} &= -i \frac{n_1}{6} \omega_1' \omega_2' + i \frac{n_1}{6} \omega_1' \omega_3', \\
B_{28} &= 0, \\
B_{29} &= 0, \\
B_{30} &= 0, \\
B_{31} &= 0, \\
B_{32} &= \frac{n_1}{2} \omega_1'' - \frac{n_1}{2} \omega_1' \omega_2 - \frac{n_1}{6} \omega_1' \omega_2^2 + \left( \frac{n_1}{6} + \frac{n_1^3}{4} \right) \omega_1' \omega_1'' + \left( \frac{n_1}{3} - \frac{n_1^3}{4} \right) \omega_1' \omega_1^4, \\
B_{33} &= \frac{2 n_1}{3} \omega_1'' - \frac{4 n_1}{3} \omega_1' \omega_1'' + \frac{2 n_1}{3} \omega_1' \omega_1^4, \\
B_{34} &= -\frac{1}{2} \omega_1'' + \frac{1}{2} \omega_1' + \left( \frac{n_1^2}{2} - 1 \right) \omega_1'' + \frac{5}{2} \left( 1 - \frac{n_1^2}{2} \right) \omega_1' \omega_1'' + \frac{3}{2} \left( 1 - \frac{n_1^2}{2} \right) \omega_1' \omega_1^4, \\
B_{35} &= \frac{n_1}{4} \omega_1' - \frac{n_1}{8} \omega_1^2 + \frac{n_1}{6} \omega_1' \omega_2 - \frac{n_1^2}{8} \omega_1' \omega_5' + \frac{n_1^2}{4} \omega_1'' \omega_5' - \frac{n_1^2}{8} \omega_1' \omega_6' \\
&+ \frac{n_1^2}{4} \omega_1' \omega_6' + \left( \frac{n_1}{24} + \frac{n_1^3}{8} \right) \omega_1^2 \omega_6' - \frac{n_1}{8} \omega_1' \omega_1'' \omega_5' - \frac{n_1}{4} \omega_1' \omega_2'' \omega_5' \\
&+ \frac{n_1^2}{8} \omega_1' \omega_1'' \omega_6' + \left( \frac{n_1}{6} - \frac{n_1^3}{16} \right) \omega_1' + \frac{n_1^3}{16} \omega_1' \omega_6', \\
B_{36} &= \frac{n_1}{3} \omega_1'' - \frac{2 n_1}{3} \omega_1' \omega_1'' + \frac{n_1}{3} \omega_1' \omega_1^4, \\
B_{37} &= -\frac{1}{4} \omega_1'' + \frac{1}{8} \omega_1' + \left( \frac{n_1^2}{4} - \frac{5}{12} \right) \omega_1'' - \frac{n_1}{8} \omega_1' \omega_5' - \frac{n_1}{4} \omega_1' \omega_5'' + \frac{n_1}{8} \omega_1' \omega_6 \\
&+ \frac{n_1}{4} \omega_1'' + \left( \frac{5}{6} - \frac{n_1^2}{2} \right) \omega_1' \omega_1'' + \frac{n_1^3}{8} \omega_1' \omega_1'' \omega_5' - \frac{n_1^3}{8} \omega_1' \omega_1' \omega_6' \\
&+ \frac{1}{4} \omega_1^2 \omega_5'' - \frac{n_1}{4} \omega_1^2 \omega_6'' + \left( \frac{3 n_1^2}{16} - \frac{7}{24} \right) \omega_1' \omega_6'' + \left( \frac{n_1}{8} - \frac{3 n_1^3}{16} \right) \omega_1' \omega_1' \omega_6' \\
&+ \left( \frac{n_1}{8} + \frac{3 n_1^3}{16} \right) \omega_1' \omega_1' \omega_6', \\
B_{38} &= \left( \frac{n_1^2}{2} - \frac{5}{6} \right) \omega_1'' + \left( \frac{5}{3} - n_1^2 \right) \omega_1^2 \omega_1'' + \left( \frac{n_1^2}{2} - \frac{5}{6} \right) \omega_1' \omega_1^4, \\
B_{39} &= \frac{1}{2} \left( n_1^2 - 1 \right) \omega_1' \omega_2 - \frac{3}{4} \left( n_1^2 - 1 \right) \omega_1' \omega_1'' + \frac{1}{4} \left( n_1^3 - n_1 \right) \omega_1' \omega_1'' \omega_5',
\end{align*}
\]
\[
\begin{align*}
- \frac{1}{4} (n_i^3 - n_i) \omega_1' \omega_1'' \omega_6' + \frac{1}{4} (n_i^2 - 1) \omega_1' \omega_6' + \frac{1}{4} (n_i^3 - n_i) \omega_1' \omega_6' \\
+ \frac{1}{4} (n_i^3 - n_i) \omega_1' \omega_6', \\
B_{40} &= \frac{n_i^2}{8} \omega_1''^2 - \frac{n_i^2}{8} \omega_5' \omega_6' + \frac{m_i}{4} \omega_1' \omega_5' - \frac{m_i}{4} \omega_1' \omega_6' + \frac{n_i^2}{8} \omega_5' + \frac{n_i^2}{8} \omega_6'^2 \\
&- \frac{n_i^2}{8} \omega_1' \omega_5' + \frac{1}{4} \left( \frac{n_i^3}{2} - n_i \right) \omega_1'' \omega_5' - \frac{1}{4} \left( \frac{n_i^3}{2} - n_i \right) \omega_1'' \omega_6' \\
&- \frac{n_i}{8} \omega_1' \omega_5' + \frac{m_i}{8} \omega_1' \omega_6' - \frac{n_i^2}{8} \omega_1' \omega_5' \omega_6' - \frac{n_i^2}{8} \omega_1' \omega_6' + \frac{n_i}{8} \omega_1' \omega_5' \omega_6' \\
+ \frac{n_i^2}{8} \omega_1' \omega_5' \omega_6' + \frac{n_i^2}{32} \omega_1' \omega_5' + \frac{1}{8} \left( \frac{n_i^3}{2} - n_i \right) \omega_1' \omega_5' + \frac{1}{8} \left( \frac{n_i^3}{2} - n_i \right) \omega_1' \omega_6' \\
+ \frac{n_i}{32} \omega_1' \omega_5' \omega_6' - \frac{n_i^3}{16} \omega_1' \omega_5' \omega_6', \\
B_{41} &= -i \frac{n_i}{4} \omega_1' - i \frac{n_i}{4} \omega_1' \omega_5' - i \frac{n_i}{8} \omega_1' \omega_5' - i \frac{n_i}{4} \omega_1' \omega_6' + \frac{1}{4} \left( \frac{n_i^3}{2} - n_i \right) \omega_1' \omega_5' \\
+ i \frac{n_i}{8} \omega_1' \omega_1'' \omega_5' + i \frac{n_i}{8} \omega_1' \omega_5' + i \frac{1}{2} \left( \frac{n_i^3}{2} - n_i \right) \omega_1' \omega_5', \\
B_{42} &= \frac{i}{4} \omega_1' + \frac{i}{2} \left( 1 - \frac{n_i^3}{2} \right) \omega_1' \omega_1'' + \frac{n_i}{4} \omega_1' \omega_5' + i \frac{n_i}{4} \omega_1' \omega_5' + i \left( \frac{3}{4} + \frac{3 n_i^3}{16} \right) \omega_1' \omega_5' \\
- i \frac{n_i}{8} \omega_1' \omega_1'' \omega_5' - i \frac{n_i}{4} \omega_1' \omega_5' - i \frac{1}{2} \left( \frac{n_i^3}{2} - n_i \right) \omega_1' \omega_5', \\
B_{43} &= \frac{i}{4} \left( 1 - n_i \right) \omega_1' \omega_1'' - \frac{i}{4} \left( 1 - n_i \right) \omega_1' \omega_1'' - \frac{i}{4} \left( n_i^3 - n_i \right) \omega_1' \omega_1'' \\
+ i \frac{1}{4} \left( n_i^3 - n_i \right) \omega_1' \omega_1'' \omega_5', \\
B_{44} &= -i \frac{n_i^2}{4} \omega_1' \omega_1'' - i \frac{m_i}{4} \omega_1' \omega_1'' \omega_5' - i \frac{m_i}{4} \omega_1' \omega_1'' \omega_6' - i \frac{n_i}{4} \omega_1' \omega_1'' \omega_5' \omega_6' + i \frac{m_i}{4} \omega_1' \omega_1'' \omega_6' \\
+ i \frac{n_i^2}{8} \omega_1' \omega_1'' + i \frac{n_i^2}{8} \omega_1' \omega_1'' + i \left( \frac{n_i^3}{2} - n_i \right) \omega_1' \omega_1'' \omega_5' - i \frac{1}{8} \left( n_i^3 - 3 n_i \right) \omega_1' \omega_1'' \omega_6' \\
+ i \frac{n_i^2}{8} \omega_1' \omega_1'' + i \frac{n_i^2}{8} \omega_1' \omega_1'' + i \frac{n_i^2}{8} \omega_1' \omega_1'' + i \left( \frac{n_i^3}{2} - n_i \right) \omega_1' \omega_1'' \omega_5' - i \frac{n_i}{4} \omega_1' \omega_1'' \omega_6' \\
+ i \frac{n_i^2}{8} \omega_1' \omega_1'' + i \frac{1}{8} \left( n_i^3 - 3 n_i \right) \omega_1' \omega_1'' \omega_5' - i \frac{n_i^2}{4} \omega_1' \omega_1'' \omega_6', \\
B_{45} &= \frac{n_i^2}{8} \omega_1''^2 + \frac{m_i}{4} \omega_1' \omega_1'' + \frac{n_i^2}{8} \omega_1''^2 + \frac{1}{4} \left( \frac{n_i^3}{2} - n_i \right) \omega_1' \omega_1'' + \frac{n_i}{8} \omega_1' \omega_1'' \omega_5' \omega_6' \\
+ \frac{n_i^2}{32} \omega_1' \omega_1'' \omega_6', \\
B_{46} &= \frac{1}{2} \omega_1'' \omega_2 - i \frac{m_i}{2} \omega_1'' \omega_3 - \frac{1}{2} \omega_1'' \omega_2 + i \frac{m_i}{4} \omega_1' \omega_1'' \omega_3 + i \frac{m_i}{4} \omega_1' \omega_1'' \omega_3 \\
B_{47} &= -i \frac{m_i}{2} \omega_1' \omega_2 + i \omega_1'' \omega_3 + \frac{m_i}{2} \omega_1' \omega_2 + \frac{1}{4} \left( n_i - n_i^3 \right) \omega_1' \omega_1'' \omega_2
\end{align*}
\]
\[ B_{48} = i \frac{m}{2} \omega'' \omega' + i \frac{m}{2} \omega' \omega_3 - i \frac{m}{4} \omega' \omega_2 - i \frac{m}{2} \omega'' \omega_2 - \frac{1}{2} \omega'' \omega_3 + i \frac{m}{4} \omega' \omega_2, \]
\[ B_{49} = -\frac{i}{2} \omega'' \omega_2 - \frac{m}{4} \omega'' \omega_3 + i \frac{2}{m} (1 - \frac{m}{2}) \omega' \omega_2 + \frac{i}{2} \omega'' \omega_2 + \frac{m}{4} \omega'' \omega_3, \]
\[ B_{50} = -\frac{m}{4} \omega' \omega_2 - \frac{m}{4} \omega' \omega'' + i \frac{2}{m} \omega' \omega_3 - i \frac{m}{8} \omega'' \omega_3 + i \frac{m}{8} \omega' \omega_2 + \frac{m}{4} \omega'' \omega_2 + \frac{m}{4} \omega'' \omega_2 + \frac{m}{4} \omega'' \omega_2, \]
\[ B_{51} = -\frac{m}{4} \omega' \omega_3 - \frac{m}{4} \omega' \omega'' - \frac{1}{4} \omega' \omega_2 - \frac{m}{4} \omega'' \omega_3 - \frac{m}{4} \omega'' \omega_3 - \frac{1}{4} \omega'' \omega_3, \]
\[ B_{52} = i \frac{m}{4} \omega' \omega_2 + i \frac{m}{4} \omega' \omega'' + \frac{1}{4} \frac{m}{2} \omega' \omega_3 + i \frac{m}{4} \omega' \omega_5 + i \frac{m}{4} \omega' \omega'' + \frac{m}{4} \omega'' \omega_5, \]

\[ + \frac{i}{4} \left( \frac{n_i}{2} - n_i \right) \omega' \omega_2 + \frac{1}{4} \left( \frac{n_i}{2} - 1 \right) \omega' \omega_3 + \frac{i}{4} \frac{n_i}{2} \omega' \omega_3 + \frac{1}{4} \frac{n_i}{2} \omega' \omega_3, \]
They verify \( \xi_C \) only. In the Dirac representation sources that appear in the article under the discrete symmetries, \( C, P \) and \( T \). In order to specify them one needs to first define how they act on the space-time coordinates \( x^\mu = (t, \vec{x}) \).

The Parity (\( P \)) operation transforms \( \vec{x} \rightarrow -\vec{x} \) while leaving the time component unchanged. Using the Minkowski space notation, \( x^\mu \overset{P}{\rightarrow} x'^\mu = p^\mu x'^\nu \), where \( p^\mu = \text{diagonal} \ (1, -1, -1, -1) \) is a matrix.

Time-Reversal (\( T \)) reverses the flow of the time-component \( t \rightarrow -t \) while leaving the space components unchanged. \( x^\mu \overset{T}{\rightarrow} x'^\mu = t^\mu x'^\nu \), where \( t^\mu = \text{diagonal} \ (-1, 1, 1, 1) \).

Charge-Conjugation (\( C \)) does not act on space-time-indices, it interchanges the rôle of particles and anti-particles.

In Quantum Mechanics they are implemented with operators acting on a Hilbert space that are unitary for \( C, P \); and anti-unitary for \( T \).

Acting on the (Dirac) quark fields \( q_a(x) \), where \( a \) labels any colour or flavour index, they read

\[
\begin{align*}
q_a(x) &\overset{C}{\rightarrow} q_a^{(C)}(x) = \xi_C C q_a T(x), & C \gamma_\mu^T C^{-1} &= -\gamma_\mu; \\
&\overset{P}{\rightarrow} q_a^{(P)}(x) = \xi_P P q_a(x_p), & P \gamma_\mu^P P^{-1} &= \gamma_\mu; \\
&\overset{T}{\rightarrow} q_a^{(T)}(x) = \xi_T T q_a(x_t), & T \gamma_\mu^T T^{-1} &= \gamma_\mu.
\end{align*}
\]

(74)

The \( \xi_C, \xi_P, \xi_T \) are arbitrary phase factors, \( |\xi_C|^2 = |\xi_P|^2 = |\xi_T|^2 = 1 \). The matrices \( C, P, T \) act on Dirac indices only. In the Dirac representation \( C = i\gamma^0 \gamma^2, P = \gamma^0 \). Once \( C, P \) are fixed, \( T = -i\gamma_5 C \). They verify \( C^{-1} = C^\dagger = C^T = -C \), and \( T^{-1} = T^\dagger = -T^T = T \). Acting on \( \gamma_5 \) they yield

\[
C \gamma_5^T C^{-1} = \gamma_5, \quad \gamma_5^0 \gamma_5^0 = -\gamma_5, \quad T \gamma_5^T T = \gamma_5.
\]

11 Appendix E: The Discrete Symmetries: \( C, P \) and \( T \).

For the sake of completeness we give in this appendix the transformation laws for the fields and the sources that appear in the article under the discrete symmetries, \( C, P \) and \( T \). In order to specify them one needs to first define how they act on the space-time coordinates \( x^\mu = (t, \vec{x}) \).

- \( B_{53} = -i \frac{n_l}{4} \omega_1' \omega_3 - i \frac{n_l}{4} \omega_1' \omega_3 + \frac{1}{2} \omega_1' \omega_2 - i \frac{1}{4} \omega_3 \omega_5' - i \frac{n_l}{4} \omega_3' \omega_5' + \frac{1}{2} \omega_2' \omega_5' \)
- \( B_{54} = i \frac{n_l}{6} \omega_1' \omega_3' - i \frac{n_l}{6} \omega_1' \omega_1'' \)
- \( B_{55} = -i \frac{1}{6} \omega_1' \omega_3' + i \frac{1}{6} \omega_1' \omega_1'' \)
- \( B_{56} = 0 \)
- \( B_{57} = 0 \)

(73)
For $\bar{q}_a(x)$,

$$
\begin{align*}
\bar{q}_a(x) & \overset{C}{\rightarrow} -\xi_C^* \bar{q}_a^T(x) C^{-1}, \\
& \overset{P}{\rightarrow} \xi_P \bar{q}_a(x_p) \gamma_0, \\
& \overset{T}{\rightarrow} \xi_T \bar{q}_a(x_t) T, \\
\end{align*}
$$

(75)

The phase factors $\xi_C$, $\xi_P$, $\xi_T$ shall be omitted henceforth. The quark bilinears $\bar{q}_a(x) \Gamma q_b(x)$ transform as

$$
\begin{align*}
\overset{C}{\rightarrow} & \quad \bar{q}_b(x)[\Gamma]C \ q_a(x), \\
& \quad [\Gamma]C = (C^{-1} \Gamma C)^T, \\
\overset{P}{\rightarrow} & \quad \bar{q}_a(x_p)[\Gamma]P \ q_b(x_p), \\
& \quad [\Gamma]P = \gamma^0 \Gamma \gamma^0, \\
\overset{T}{\rightarrow} & \quad \bar{q}_a(x_t)[\Gamma]T \ q_b(x_t), \\
& \quad [\Gamma]T = \overline{T^*} \Gamma T.
\end{align*}
$$

(76)

The star $(\ast)$ in the last line of (76) denotes complex conjugation. There is a minus sign in the bilinear transformed under $C$ which comes from the anti-commutation of two quark fields.

$$[\Gamma]_C = I, \quad [\Gamma]_P = I, \quad [\Gamma]_T = I,$$

$$[i\gamma_5]_C = i\gamma_5, \quad [i\gamma_5]_P = -i\gamma_5, \quad [i\gamma_5]_T = -i\gamma_5,$$

$$[\gamma^\mu]_C = -\gamma^\mu, \quad [\gamma^\mu]_P = \gamma^\mu, \quad [\gamma^\mu]_T = -t^\mu [\gamma^\mu],$$

$$[\gamma^\mu \gamma_5]_C = [\gamma^\mu \gamma_5], \quad [\gamma^\mu \gamma_5]_P = -t^\mu [\gamma^\mu \gamma_5], \quad [\gamma^\mu \gamma_5]_T = -t^\mu [\gamma^\mu \gamma_5].$$

For the gluon field (hermitian) matrix $G^\mu(x)$ in colour-space,

$$
\overset{C}{\rightarrow} -G^\mu_T(x), \quad \overset{P}{\rightarrow} p^\mu_G G^\mu_G(x_p), \quad \overset{T}{\rightarrow} -t^\mu_G G^\mu_G(x_t).
$$

It is easy to verify that the QCD action is invariant under $C$, $P$ and $T$.

For the topological charge density $Q(x) \sim x_{\mu \nu \rho \sigma} Tr_c G^{\mu \nu}(x) G^{\rho \sigma}(x)$, which is also real,

$$
\overset{C}{\rightarrow} Q(x), \quad \overset{P}{\rightarrow} \det(p^\mu_G) Q(x_p) = -Q(x_p), \quad \overset{T}{\rightarrow} \det(t^\mu_G) Q(x_t) = -Q(x_t).
$$

So far for operators involving the dynamical fields.

The (hermitian) operator $i\bar{q}_a(x) \gamma_5 q_b(x)$ has the same quantum as the light pseudoscalar matrix $\Phi_{ab}(x)$, and, since the vacuum is invariant under $C$, $P$, and $T$ the transformation laws of the latter are taken from those of the former. Let us write them down for the simpler case of $U_L(2) \otimes U_R(2)$, where

$$
\Phi = \begin{pmatrix} \frac{\pi_0 - \eta}{\sqrt{2}} & \pi^+ \\ \pi^- & \frac{-\pi_0 + \eta}{\sqrt{2}} \end{pmatrix}.
$$

(77)

Under the discrete symmetries

$$
\overset{C}{\rightarrow} \begin{pmatrix} \frac{\pi_0 - \eta}{\sqrt{2}} \\ \pi^+ \end{pmatrix} = \Phi^T,
$$

(78)

and, similarly,

$$
\overset{P}{\rightarrow} -\Phi(x_p), \quad \overset{T}{\rightarrow} -\Phi(x_t),
$$

(79)
which generalize immediately for $U_L(n_l) \otimes U_R(n_l)$. It translates into

$$U(x) \xrightarrow{C} U^{(C)}(x) = U^T(x),$$

$$U(x) \xrightarrow{P} U^{(P)}(x) = U^\dagger(x_p),$$

$$U(x) \xrightarrow{T} U^{(T)}(x) = U(x_l),$$

for the $U(x)$ matrix, as defined in \[12\].

As for the external sources, we shall choose their transformation so as to leave the action invariant.

The real source $\theta(x)$, that couples to the topological charge $Q(x)$, transforms as:

$$\xrightarrow{C} \theta(x), \quad \xrightarrow{P} -\theta(x_p), \quad \xrightarrow{T} -\theta(x_t).$$

In the text, the combination $\theta = i\theta$ appeared in a natural way. It transforms accordingly, with an extra minus sign for the $T$ transformation because it anti-commutes with the imaginary number $i$ due to its anti-unitary character. The combination $X$, defined in \[17\], transforms as $\theta$ does.

The (hermitian) source matrices $s, p, v, a, \mu$, in flavour space transform as

$$s(x) \xrightarrow{C} s^T(x), \quad \xrightarrow{P} s(x_p), \quad \xrightarrow{T} s(x_l),$$

$$p(x) \xrightarrow{C} p^T(x), \quad \xrightarrow{P} -p(x_p), \quad \xrightarrow{T} -p(x_t),$$

$$v^\mu(x) \xrightarrow{C} -v^\mu T(x), \quad \xrightarrow{P} p^\mu T v^\mu'(x_p), \quad \xrightarrow{T} -t^\mu T v^\mu'(x_t),$$

$$a^\mu(x) \xrightarrow{C} a^\mu T(x), \quad \xrightarrow{P} -p^\mu a^\mu'(x_p), \quad \xrightarrow{T} -t^\mu a^\mu'(x_t).$$

The combination $\chi = 2B(s + ip)$ transforms as the $U$ fields.

The left and right combinations of the vector and axial sources, $l_\mu = v_\mu - a_\mu$ and $r_\mu = v_\mu + a_\mu$, transform as

$$l^\mu(x) \xrightarrow{C} -l^\mu T(x), \quad \xrightarrow{P} p^\mu T r^\mu'(x_p), \quad \xrightarrow{T} -t^\mu T r^\mu'(x_t),$$

$$r^\mu(x) \xrightarrow{C} -l^\mu T(x), \quad \xrightarrow{P} p^\mu T r^\mu'(x_p), \quad \xrightarrow{T} -t^\mu T r^\mu'(x_t).$$

Both the $C$ and the $P$ transformations interchange left and right.

For the field strengths $F_L^{\mu\nu}, F_R^{\mu\nu}$ associated to $l_\mu, r_\mu$,

$$F_L^{\mu\nu}(x) \xrightarrow{C} -F_R^{\mu\nu T}(x), \quad \xrightarrow{P} p^\mu T r^\nu F_R^{\mu\nu'}(x_p), \quad \xrightarrow{T} -t^\mu T r^\nu F_R^{\mu\nu'}(x_t),$$

$$F_R^{\mu\nu}(x) \xrightarrow{C} -F_L^{\mu\nu T}(x), \quad \xrightarrow{P} p^\mu T r^\nu F_R^{\mu\nu'}(x_p), \quad \xrightarrow{T} -t^\mu T r^\nu F_R^{\mu\nu'}(x_t),$$

Finally, for the combination $C^\mu = U^\dagger D^\mu U$, that is anti-hermitian $C^{\mu\dagger} = -C^\mu$,

$$C^\mu(x) \xrightarrow{C} [U(x)C^\mu(x)U^\dagger(x)]^T, \quad \xrightarrow{P} -p^\mu T U(x_p)C^\mu'(x_p)U^\dagger(x_p), \quad \xrightarrow{T} t^\mu T C^\mu'(x_t).$$
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