Critical behaviour of the 1D q-state Potts model
with long-range interactions

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Abstract
The critical behaviour of the one-dimensional q-state Potts model with long-range interactions decaying with distance $r$ as $r^{-(1+\sigma)}$ has been studied in the wide range of parameters $0 < \sigma \leq 1$ and $\frac{1}{16} \leq q \leq 64$. A transfer matrix has been constructed for a truncated range of interactions for integer and continuous $q$, and finite range scaling has been applied. Results for the phase diagram and the correlation length critical exponent are presented.

Short title: Critical behaviour of the 1D LR q-state Potts model
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1 Introduction

Recently, finite range scaling (FRS) appeared as an efficient method to deal with various discrete one-dimensional models with long range (LR) interactions (Glumac and Uzelac, 1989, 1991). The aim of the present study is to apply it to the one-dimensional q-state Potts model with LR interactions, defined by the hamiltonian

\[ -\beta H = K \sum_{i<j} \frac{1}{(j-i)^{1+\sigma}} [\delta(s_i, s_j) - \frac{1}{q}], \]

(1)

where the interaction constant is chosen to be equal to \( k_B \) and the inverse temperature is denoted by \( K \). The symbol \( s_i \) denotes a Potts variable at site \( i \) which can assume \( q \) different states, while \( \delta(s_i, s_j) \) is the Kronecker symbol.

Such a model is expected to have, due to the LR of interactions, a nontrivial phase transition at finite temperature, like an Ising model (as its special case \( q=2 \)), although the problem becomes much more complex due to possibility of changing degrees of freedom through parameter \( q \).

The Potts model has been extensively studied in the case of short range (SR) interactions. In the two dimensional SR case for example, besides integer-\( q \) models which are equivalent to the classical \((q-1)/2\)-spin models, there are also non-integer \( q \) models which are connected with other critical problems such as percolation \((q=1\) limit), resistor network \((q=0\) limit), dilute spin glass \((q=1/2\) limit), branched polymers \((0\leq q \leq 1)\) (see review by Wu, 1982).

It would be interesting to extend those concepts to LR interactions. However, in contrast to the rich literature on the SR Potts model, the LR Potts model has been much less explored. The best explored member of the \( q \)-state Potts models is the \( q=2 \) model where both, analytical (Dyson 1969; Fisher, Ma and Nickel, 1972; Kosterlitz, 1976) and numerical (Nagle and Bonner, 1970; Glumac and Uzelac 1989,1991) results are available. Dyson has shown that, as a consequence of the LR forces, there is a critical temperature different from zero as long as \( 0 < \sigma \leq 1 \). Fisher et al. have performed \( \epsilon = 2\sigma - d \) and \( 1/n \) expansions in the context of the renormalization group (RG) for all dimensions \( d \) and \( \sigma \neq 2 \) on a more general system with an \( n \)-component
order parameter. They obtained the correlation function exponent \( \eta = 2 - \sigma \) to all orders in \( \epsilon \), \( \nu = 1/\sigma \) when \( \epsilon < 0 \) (classical regime), \( \nu = 1/\sigma + \frac{3}{2}\epsilon + \ldots \) when \( \epsilon > 0 \), and leading irrelevant exponent \( y_3 = -|\epsilon| \). A similar expansion, in \( 1-\sigma > 0 \), has been done by Kosterlitz (1976), who obtained: \( 1/\nu = -y_3 = \sqrt{2(1-\sigma)} \) when \( \sigma \to 1 \). Very recently, a new method, called cycle expansion, has been demonstrated by Mainieri (1992) on the example of the \( s=1/2 \) Ising model. Mainieri obtained very accurate estimates of the critical temperature and other thermodynamic quantities in the critical region.

The literature on \( q \neq 2 \) LR Potts models is relatively poor. By using the RG \( \epsilon \)-expansion, Priest and Lubensky (1976) have found a fixed point and critical exponent \( \nu \) to the first order in \( \epsilon = 3\sigma - d \). More recently, Theumann and Gusmão (1985), performed similar calculation to the second order of \( \epsilon = 3\sigma - d \). They obtained the following results for \( q < 3 \): \( \eta = 2 - \sigma \) to all orders of \( \epsilon \), and

\[
\nu^{-1} = \sigma - \frac{\alpha}{2\beta} \epsilon + \frac{\alpha \beta (\alpha - \beta) S(\sigma) + \frac{1}{8}\alpha (\beta^2 - \gamma) G(\sigma)}{8\beta^3} \epsilon^2 + O(\epsilon^3)
\]

where \( \alpha = q^2(q-2) \), \( \beta = q^2(q-3) \), \( \gamma = q^4(q^2-6q+10) \), \( S(\sigma) \equiv \Psi(\frac{3\sigma}{2}) - \Psi(\sigma) - \Psi(\sigma/2) + \Psi(1) \), and \( G(\sigma) \equiv \frac{\Gamma(\frac{3\sigma}{2})\Gamma(\frac{3}{2})}{\Gamma(\sigma)\Gamma(\frac{3}{2})} \). \( \Psi(z) \) is the logarithmic derivative of the \( \Gamma \) function. In the \( q \to 1 \) limit, their results are applicable to the description of percolation critical behaviour for a random Ising ferromagnet with a LR power-law interaction.

The advantage of the FRS method is the capability to cover, unlike the \( \epsilon \)-expansion, a wide range of parameter space. In the present article the phase diagram and the correlation length critical exponent will be studied in the region \( 0 < \sigma \leq 1 \) and \( \frac{1}{16} \leq q \leq 64 \).

The main idea of our approach is the following: we truncate the originally infinite-range interaction to the \( L \) first neighbours and solve exactly (although numerically) the finite-range version of (1)

\[
-\beta H = \sum_{i=1}^{N} \sum_{j=1}^{L} K_j \delta(s_i, s_{i+j}) - \frac{1}{q} p.b.c.
\]
where $K_j$ denotes $\frac{K_j}{j^{\sigma}}$. Then, by using the FRS method (section 2.1) with the appropriate extrapolating procedure (section 3), the $L \to \infty$ behaviour will be deduced.

For this purpose a transfer matrix for a reduced model defined by eq. (3) will be constructed. Two different formalisms will be applied: one for integer $q$ values (section 2.2) and the other for non-integer $q$ values (section 2.3).

The results are presented in section 3, while section 4 contains a short discussion and some open questions.

2 Method

2.1 FRS

The FRS method has been constructed in analogy with the FSS (Fisher and Barber, 1972), where instead of finite-size, the finite-range of interactions is scaled (Glumac and Uzelac, 1989, 1991). The basic idea of FRS is that by studying the sequence of systems with their long-range interactions truncated above certain range, one can obtain, by using scaling properties, the information on the critical behaviour of the true infinite system.

Let $A_\infty(t)$ be the physical quantity of the true long-range system, which algebraically diverges, in the vicinity of the critical point $t=0$

$$A_\infty(t) \simeq A_0 t^{-\rho}$$

where $t=(T-T_c)/T_c$, $T_c$ is the critical temperature, $\rho$ is the related critical exponent, and $A_0$ is a constant. Then, analogous to the FSS hypothesis one can assume that for large finite range $M$ and small $t$, $A_M(t)$ can be written in the form

$$A_M(t) = A_\infty(t) \cdot f\left(\frac{M}{\xi_\infty}\right)$$

where $f$ is a homogeneous function with following properties

$$\lim_{x \to \infty} f(x) = 1, \quad \lim_{x \to 0} f(x) = \text{const} \cdot x^{\frac{4}{\nu}}$$

By applying the equation (5) to the correlation length $\xi_\infty(t) = \xi_0 t^{-\nu}$, the standard scaling procedure gives the condition for critical temperature through the fixed
point equation
\[ \frac{\xi_M(t^*)}{M} = \frac{\xi_M'(t^*)}{M'} \] (7)
and the expression for correlation length critical exponent \( \nu \)
\[ \nu^{-1} = \frac{\ln \frac{\xi_M'(t^*)}{\xi_M(t^*)}}{\ln \frac{M}{M'}} - 1 \] (8)
where \( M' = M - 1 \) in all calculations, and \( \xi' \) is temperature derivative of \( \xi \).

There are two important facts concerning the above method which are of interest.
First, since the critical behaviour is essentially dependent on the range of interaction, one can (unlike in the FSS (Brézin, 1982)) expect applicability of FRS in mean-field as well as in the non-trivial region (Glumac and Uzelac 1989, 1991). Second, the correlation length calculated from the transfer matrix presents the average distance between domain walls for an infinitely long strip so that relation (7) gives the transition temperature for both first and second-order phase transitions. This fact is common with FSS (Binder, 1987).

2.2 Transfer matrix: integer-q formalism

It is straightforward to construct the transfer matrix for model (3) with an integer number of Potts states.

The chain with a range of interaction \( L \), can be represented as a strip with columns of height \( L \) (Uzelac and Glumac, 1988). Each column then can be considered as an object with \( q^L \) possible states, interacting only with its first neighbour.

The transfer matrix \( T \) is given by:
\[ \langle i|T|j \rangle = \exp \left\{ \sum_{k=1}^{L} K_k \left[ \sum_{n=1}^{L-k} \delta(i_n, i_{n+k}) + \sum_{n=1}^{k} \delta(i_{L+n-k}, j_n) \right] \right\} \] (9)
where \( j_m = 0, 1, \ldots, q - 1 \) are the elements of the \( L \)-component vector of states \(|j\rangle\) of a column of height \( L \). The matrix \( T \) can be further decomposed into a product of \( L \) matrices \( T_n \), each one describing the addition of one more site to the column (Temperley and Lieb, 1971).
\[ T = T_1 \cdots T_L \] (10)
\[
\langle i | T_n | j \rangle = \prod_{\substack{l = 1 \\text{to} \ L \ \text{if}, \ \text{not} \ i}} \delta(i_l, j_l) \exp \left\{ \sum_{k=1}^{n-1} K_k \delta(j_{n-k}, j_n) + \sum_{k=n}^{L-1} K_k \delta(j_{L+n-k}, j_n) + K_L \delta(i_n, j_n) \right\}
\]

(11)

There is a simple relation between neighbouring one-site matrices:

\[
U^T T_{n+1} U = T_n
\]

(12)

where \( T_{L+1} = T_1 \) and \( U \) is a matrix of the translation operator in the vertical strip direction, with matrix elements given by

\[
\langle i | U | j \rangle = \delta(i_1, j_L) \delta(i_2, j_1) \delta(i_3, j_2) \ldots \delta(i_{L-1}, j_{L-2}) \delta(i_L, j_{L-1}).
\]

(13)

The matrix \( U \) satisfies the relations \( U^L = 1 \) and \( U^{L-1} = U^T = U^{-1} \). Consequently, the transfer matrix can be written as the \( L \)-th power of a single matrix as a peculiarity of the present long-range model:

\[
T = (U \cdot T_L)^L = \tilde{T}^L
\]

(14)

\[
\langle i | \tilde{T} | j \rangle = \delta(j_1, i_2) \delta(j_2, i_3) \ldots \delta(j_{L-1}, i_L) \exp \left\{ \sum_{m=1}^{L-1} K_{L+1-m} \delta(j_L, i_m) \right\}
\]

(15)

Notice that \( \tilde{T} \) has only \( q \) nonzero elements per row which greatly reduces computer memory.

Further reduction is obtained by using the symmetry properties of Potts interaction. In the present calculation, only the translation invariance in the space of Potts states has been used, which decomposes \( \tilde{T} \) into \( q \) submatrices (of order \( q^{L-1} \)). The \( q-1 \) of them have the identical eigenvalues as a consequence of Kronecker-\( \delta \) type of interaction. Thus we have to diagonalize only two submatrices or, more precisely, we have to find only the largest eigenvalue of each of the two submatrices, \( \mu_{1,L} \) and \( \mu_{2,L} \) respectively, to be able to construct the correlation length \( \xi_L \)

\[
\xi_L = \frac{L}{\ln \frac{\lambda_1}{\lambda_2}} = \frac{1}{\ln \frac{\mu_1}{\mu_2}}
\]

(16)

where \( \lambda_1 > \lambda_2 \) and \( \mu_1 > \mu_2 \) are the eigenvalues of \( T \) and \( \tilde{T} \) respectively.

For this purpose, the method of direct iteration with a single vector is suitable (Wilkinson, 1965). If one defines a set of vectors \( b_j \) in a following way

\[
A b_j = b_{j+1}
\]

(17)
for arbitrary initial value $b_0$, where $A$ is a real nonsymmetric matrix, then the largest eigenvalue $\mu$ of matrix $A$ is given by

$$\mu = \lim_{j \to \infty} \frac{b_j(n)}{b_{j-1}(n)}$$

for any vector’s component $n$. By applying this method for cases with $q=2, 3, 4$ and 5, we obtained the two largest eigenvalues of $\hat{T}$ for maximum ranges $L=20, 13, 10$ and $9$, respectively. A few ten to a few hundred iterations were required to obtain the leading eigenvalue with an accuracy of $10^{-15}$ for both submatrices.

### 2.3 Transfer matrix: continuous-q formalism

As mentioned in Introduction, there are some interesting cases ($q=0, 1/2, 1, ...$) to which the described integer-q formalism cannot be applied or becomes inefficient (limitation to small ranges for large $q$ models). Thus, it would be advantageous to formulate a $q$-independent transfer matrix, which would allow to treat $q$ as a continuous parameter. Such a type of transfer matrix has already been constructed for the 2D Potts model with SR interaction (Blöte, Nightingale, Derrida 1981; Blöte and Nightingale, 1982). We will shortly explain the application of that method to the considered LR model.

Let us translate the partition function of model (3), into the graph-theory language. First, we rewrite the basic expression $\exp\{K_j \cdot \delta(s_i, s_{i+j})\}$ as

$$e^{K_j \delta(s_i, s_{i+j})} = 1 + v_j \delta(s_i, s_{i+j}),$$

where $v_j = e^{K_j} - 1$, so that the partition function, with periodic boundary conditions, becomes

$$Z_{N,L} = \sum_{s_1=1}^{q} \sum_{s_2=1}^{q} \cdots \sum_{s_N=1}^{q} \prod_{j=1}^{L} \left[1 + v_j \delta(s_1, s_{1+j})\right] \left[1 + v_j \delta(s_2, s_{2+j})\right] \cdots \left[1 + v_j \delta(s_N, s_{N+j})\right]$$

If we perform the multiplications of the above terms, the one-to-one correspondence between every term of that expansion and the particular graph on the $m \times L$ lattice ($N=m \cdot L$) can be established in the standard way (Blöte and Nightingale, 1982).
Now, the partition function becomes

$$Z_{N,L} = q^N \cdot \sum_G u_1^{b_1(G)} \cdot u_2^{b_2(G)} \cdot \ldots \cdot u_L^{b_L(G)} \cdot q^{l(G)}.$$  \hfill (21)

The above summation goes over all possible graphs on the lattice with $L$ different types of bonds described by $u_j = v_j/q$, $b_j(G)$ is the number of bonds of type $j$, and $l(G)$ is the number of independent loops on the graph $G$.

A certain number of different bonds in every single graph, produce connections between sites. If one considers a strip of height $L$, then all possible interactions between all sites in that strip produce a particular state of connectivity $\alpha$ of the sites in the last column. The LR interactions produce a larger number of connectivities than in the 2D SR models.

The connectivity is represented by an integer, and it will be used as an index of the transfer matrix. The partition function $Z_N$ can be written as a vector with components $Z_N(\alpha)$, where each component $Z_N(\alpha)$ describes the contribution to the partition function of all graphs that produce one particular state of connectivity $\alpha$ in the last column. By adding a new column, we get $Z_{N+L}$ which again can be written as a vector $Z_{N+L}(\beta)$. The connection between those two partition functions is given by the transfer matrix in the following way

$$Z_{N+L}(\beta) = \sum_\alpha T(\beta, \alpha) Z_N(\alpha)$$ \hfill (22)

where the summation goes over all graphs, produced by the added column, that lead from connectivity $\alpha$ to $\beta$. A transfer matrix decomposition analogous to that of section 2.1, leads to

$$T = qT_1 \cdot \ldots \cdot qT_L = (qT_1 V^T)^L = \tilde{T}^L$$ \hfill (23)

where $T_j$ describes the addition of the $j$-th site to the column, and $V$ is the matrix of the translation operator in the vertical strip direction. Further decomposition of $T_1$ is possible

$$T_1 = T_{1,L}(u_L) \cdot \prod_{j=L-1}^1 T_{1,j}(u_j)$$ \hfill (24)
Each of the $T_{1,j}(u_j)$ matrices describes the presence of an $u_j$ type bond on the first site and has at most two nonzero elements per row. $T_{1,L}(u_L)$ is an upper triangular matrix, while the other matrices are lower triangular. It is easy to construct a matrix $M$ which relates the lower triangular matrices

$$M^T T_{1,j}(x) M = T_{1,j+1}(x)$$

The matrix $\tilde{T}$ then becomes

$$\tilde{T} = q T_{1,L}(u_L) \prod_{j=L-1}^{1} (T_{1,L-1} M)_{u_j} V^T$$

By using these decompositions, it is sufficient to keep in computer memory only four matrices: $T_{1,L}$, $T_{1,L-1}$, $M$ and $V$, with at most two nonzero elements per row, which strongly reduces amount of data to be stored.

Since we want to get information about LR correlations, we should also take care of the connectivity between sites in the last column and the sites in the first column. In order to do so, one can imagine the top site in the last column of $Z_N$ to be some kind of ”ghost” site representing the connection between the first site of last column of $Z_{N+L}$ and the sites of the first column. Thus we need a column of height $L$ to describe all connectivities in the model with the range of interaction equal to $L-1$.

In order to keep the ”ghost” site fixed, the matrix of the translation operator $V$ will be constructed by acting with operator $V$ on non-”ghost” sites only. Consequently, if we perform the following set of transformations

$$u_L \rightarrow u_{L-1}, u_{L-1} \rightarrow u_{L-2}, \ldots, u_2 \rightarrow u_1, u_1 \rightarrow 0,$$

we will obtain $\tilde{T}$ for the model with range $L-1$, with LR correlations included.

With this redefinition, $\tilde{T}$ has the following structure

$$\tilde{T} = \begin{pmatrix} \tilde{T_-} & \tilde{X} \\ 0 & \tilde{T_+} \end{pmatrix}$$

where $\tilde{T_+}$ and $\tilde{T_-}$ contain the largest $\mu_{1,L}$ and the second largest $\mu_{2,L}$ eigenvalue of the whole matrix $\tilde{T}$, respectively. The correlation length is given by the relation (16) again.
The order of $\tilde{T}$ for a model with range $L$ is given by $N_L$, and the order of $\tilde{T}_+$ is given by $N_{L-1}$, where

$$N_L = \sum_{i=1}^{L+1} \varphi_i^L$$

$$\varphi_i^L = i \cdot \varphi_i^{L-1} + \varphi_{i-1}^{L-1}$$

for $L \geq 2$, and $\varphi_2^2 = 3$, $\varphi_1^n = \varphi_{n+1}^n = 1$. Some values of $N_L$ are presented in table 1.

The present calculations were performed for the maximum range $L=8$ with $q=1/16, 1/2, 1, 8, 16, 32$ and $64$. The largest eigenvalues of the matrices $\tilde{T}_\pm$, are extracted by using the method explained earlier (section 2.2) with a similar accuracy and number of iterations.

3 Results

3.1 Convergence and extrapolation procedures

The critical temperature and exponent $\nu$, given by the equations (7) and (8), still depend on the range $M$ and it is important to apply the right extrapolation procedure. A simple analysis done within FSS (Privman and Fisher, 1983) can be reproduced in the case of the FRS (Glumac and Uzelac, 1989) and shows that the convergence is dominantly governed by the leading irrelevant field and the corresponding critical exponent $y_3 < 0$. On the basis of such analysis, we expect power-law convergence of $T_c$ and $\nu$ in the large-$M$ limit

$$T_{c,M} = T_c + a \cdot M^{y_3 - \frac{1}{y}}$$

$$\nu_{M}^{-1} = \nu^{-1} + b(T_{c,M} - T_c) \cdot M^{\frac{1}{y}} + c \cdot M^{y_3} = \nu^{-1} + b' \cdot M^{y_3}$$

where $a$, $b$, $b'$ and $c$ are constants. Thus, we chose to fit the obtained results for $K_{c,M}$ and $\nu_M$ to the form

$$y_M = y_e + A \cdot M^{-x_y}$$

in the least-squares approximation (LSA), where $y_e$ denotes the extrapolated quantity. The calculations are performed by taking the five largest values of $M$. 

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It should be noticed, however, that Privman and Fisher (1983) analysis does not apply to the MF region and, consequently, the extrapolation form (32) becomes arbitrary there. On the other hand, in our previous analysis for the Ising model (Glumac and Uzelac, 1989) it turns out that the extrapolation form

\[ y_M = B + A\left(\frac{M - 1}{M}\right)^x \]  

(33)
gives a better agreement with the exactly known results for \( \nu \) in the MF region then the function of the form (32).

The error bars of the above extrapolations can be estimated only roughly, by looking for the remaining L dependence of the results. In tables (2) and (3) are presented results which last digit is modified under the change of maximal range from L-1 to L. The more careful examination of the errors, shows that the size of the error estimate changes with q and \( \sigma \), being less then ten percents at the most of q and \( \sigma \) region. Exception is the small q and \( \sigma \) region where the error is estimated to be a few times larger.

### 3.2 Critical temperature

The inverse critical temperature \( K_{c,M} \), defined by equation (7), is calculated with an accuracy higher than \( 10^{-10} \) for different values of Potts states and with \( \sigma \) as an parameter.

When increasing the exponent of LR interaction, there will be a value of \( \sigma \) beyond which the SR critical behaviour (with \( T_c = 0 \)) takes place. For the Ising model (q=2) it was analytically shown (Dyson, 1969) that this \( \sigma \) is equal to 1. As argued more generally (Fisher et al. 1972, Sak 1973) this exchange of regimes should occur when the correlation function exponent \( \eta \) of the LR system becomes smaller or equal to the SR one, \( \eta_{SR} \). Thus in present case, we expect that \( \sigma_c = 1 \) for all values of q. This expectation is confirmed by our numerical results. In our earlier works (Glumac and Uzelac, 1989, 1991), we have been able to detect the appearance of SR forces governed critical behaviour by the change of \( K_{c,M} \) from descending to ascending sequence and by the sudden increase of convergence exponent. In the
present calculations, both of these phenomena were observed for any q considered at $\sigma \approx 1$. The extrapolated values for the critical temperature are presented in figures (1a) and (1b).

We have used the LSA extrapolation method with the form (32) to calculate $K_{c,e}$ and the leading convergence exponent $x_K$ as a function of $q$ and $\sigma$ which are presented in tables (2a) and (2b) respectively. The only point where this could not be applied is $\sigma = 1$, where the simple convergence expression does not apply any more. In order to avoid non-monotonic behaviour, we have used only the data for the three largest values of $M$ fitting a linear relation to them ($x_K = 1$). The error should not be important, since for $\sigma = 1$ the variation of data with $M$ is rather weak.

### 3.3 Critical exponent $\nu$

The correlation length critical exponents $\nu_M$ has been calculated by using the equation (8). The accuracy of results is reduced (with respect to that of the critical temperature) to the order of $10^{-6}$ due to the numerical differentiation.

Following the form (31) for the scaling correction in non-MF region, one can expect that the correction terms to $\nu^{-1}_M$ would be smaller while calculating the $\nu_M$ on the temperature $K_{c,e}$ instead of $K_{c,M}$. This is indeed the case for the $q=2$ ($s=1/2$ Ising) model, where detailed comparison between $\nu_M(K_{c,M})$ and $\nu_M(K_{c,e})$ has been given (Glumac and Uzelac, 1989). In the present model we observe better convergence of the data calculated from $K_{c,e}$ in the LR $\sigma$-region, for all $q$.

According to the discussion at the end of section 3.1, the extrapolations in the low $\sigma$ region where MF behaviour is expected were performed at the temperature $K_{c,M}$ using the extrapolation function of the form (33). Since the $\sigma_{MF}$ border is still an open question for $q > 2$ (see later in text), we have used eq. (33) for $\sigma < .3, q \leq 1$, and for $\sigma < .5, q = 2$.

The extrapolated results are presented in figure (2). In tables (3a) and (3b) are results for $\nu^{-1}_e$ and the corresponding convergence exponents respectively. Similarly to preceding section, for $\sigma = 1$ the LSA procedure was performed by imposing
\( x_\nu = 1 \) in eq. (32).

The special case \( q=2 \) which corresponds to the Ising model was extensively studied in our earlier work (Glumac and Uzelac, 1989) with a maximum range equal 10. The present work permits to reach the range of 20. Similar values of \( \nu_e \) were obtained in both works, which suggests that an increase of range from 10 to 20 does not change the accuracy significantly.

On table 4, our results are compared to the \( \epsilon \)-expansion results of Theumann and Gusmão (1985). A difference \( \Delta \) of only few percent is obtained. Figure (2a) suggests that \( \sigma = \frac{1}{3} \) as the MF border and a q-independent value of \( \nu_{MF}^{-1} = \sigma \) for all \( q \leq 1 \).

We have not found any analytical or numerical estimate for the values of \( \sigma_{MF} \) or \( \nu \) for \( q \geq 3 \). In that situation we decided to calculate \( \nu \) at the temperature \( K_{c,e} \) since the scaling law \( \frac{\xi}{M} = const. \) is more accurate at that temperature than at \( K_{c,M} \). Also, we decided to make the extrapolation using function (32) which possesses a more transparent \( L \)-dependence.

For large \( q \) (\( q \geq 16 \)), only the continuous formalism could be applied, which does not permit to go beyond the range of \( L=8 \). Contrary to the small \( q \) case, for large \( q \) this range was not sufficient to find a good extrapolation. Namely, when applying the LSA there, with the assumed form (31), one obtains \( x_\nu \to 0 \) so that the correction terms become of the same order as the leading term and \( \nu^{-1} \) diverges. This behaviour of \( \nu \) suggests a change of regime of convergence and brings up the question of the order of transition.

For all values of \( q \) the exponent \( \nu \) increases as \( \sigma \to 1 \), which in the case \( q=2 \) appeared (Glumac and Uzelac, 1989) as an indication of the essential singularity in this limit (Kosterlitz 1976).

4 Conclusion and discussion

By using a FRS method combined with transfer matrix calculations we have been able to study the long-range Potts model on the one-dimensional lattice in a wide
range of values for the number of states \( q \) and the interaction exponent \( \sigma \). We have shown that for this problem the transfer matrix can be decomposed into matrices of much simpler form. A different transfer matrix procedure can be done also in the case of continuous \( q \), so that our analysis could be extended to some non-integer \( q \) cases of interest.

The study is concentrated on the phase diagram and the correlation length critical exponent \( \nu \).

The critical temperature shows monotonic decrease with \( q \) and \( \sigma \) and has a finite value at \( \sigma = 1 \). The change of behaviour of \( K_{c,M} \) at \( \sigma \approx 1 \) indicates the exchange from LR to SR regime, established for any \( q \). The critical exponent was calculated at the extrapolated critical temperature, and good agreement with values obtained by other authors was observed. Generally, a good accuracy is harder to obtain in the small-\( \sigma \) region due to the very long-range of interaction and in the \( \sigma \rightarrow 1 \) region where the range of interaction ceases to be a good scaling variable.

There are two points that require some further discussion.

First is a question of mean-field border. The present method, unlike the FSS, has the advantage that it can be applied within the mean-field region, but on the other hand, passes smoothly between the two regions (Glumac and Uzelac 1989, 1991), which does not permit to point out the mean-field border \( \sigma_{MF} \). Another way to detect this border could be numerical, by observing the change of behaviour of \( \nu \) as a function of \( \sigma \). But, our previous results on the Ising model where \( \sigma_{MF} \) is known, show that close to it the numerical results for \( \nu \) are not sharp enough to locate this point with precision. In the present case, we can only confirm that our results are consistent with \( \sigma_{MF} \) as known from the literature for a few particular cases: \( q=2 \) (\( \sigma_{MF} = \frac{1}{2} \) Fisher et al 1972), \( q=1 \) (\( \sigma_{MF} = \frac{1}{3} \) Priest and Lubensky 1976), \( q = \frac{1}{2} \) (\( \sigma_{MF} = \frac{1}{3} \) Aharony 1978, Aharony and Pfeuty 1979), but the precise location of \( \sigma_{MF} \) for arbitrary \( q \) is left unestimated.

A second point is the possible appearance of a first order transition for some \( q > q_c \). Within the transfer matrix formalism this problem was considered for the 2D SR case where \( q_c \) exists and is exactly known (\( q_c = 4 \), Baxter 1973). Iglói
and Sólyom (1983) have shown that the first order transition is connected with the crossing of the largest and the third largest eigenvalue of the transfer matrix for finite length chain and large q. We have applied a similar calculation in our model considering two groups of parameters (q=100, \( \sigma = .8, L=4 \) and q=300, \( \sigma = .4, L=4 \)). Both cases give a negative result; the first and third eigenvalues do not intersect as a function of temperature. By analogy with Iglói and Sólyom, that result can be interpreted as the absence of a first-order transitions for any q in the 1D LR Potts model. Since this is opposite to the indications of the behaviour of \( \nu \) for large q (at the end of section 3.3), the question of the order of transition for large q is still left open.
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Figure captions:

Figure 1a:
The extrapolated critical temperature as a function of $\sigma$.

Figure 1b:
The extrapolated critical temperature as a function of $\sigma$. Figures (a) and (b) have common q=1 line which allows the comparison of scales on both figures.
solid filled circle;

Figure 2a:
The exponent $\nu e^{-1}$ as a function of $\sigma$. solid filled circle.

Figure 2b:
The exponent $\nu e^{-1}$ as a function of $\sigma$. Figures (a) and (b) have common q=2 line which allows the comparison of scales on both figures.
Table captions:

Table 1:
$N_L$ is the order of transfer matrix for continuous-q model with maximum range equal $L$.

Table 2a:
The extrapolated values of inverse critical temperature as a function of $q$ and $\sigma$. For the error bars see in text.

Table 2b:
The critical temperature convergence exponent $x_K$ as a function of $q$ and $\sigma$.

Table 3a:
The extrapolated values of $\nu^{-1}$ as a function of $q$ and $\sigma$. For the error bars see in text.

Table 3b:
The convergence exponent $x_\nu$ as a function of $q$ and $\sigma$.

Table 4:
The comparison between extrapolated values $\nu^{-1}$ and Theumann and Gusmão values $\nu^{-1}_{TG}$ for $\sigma = .4$. 
| $L$ | 2   | 3   | 4   | 5   | 6   | 7   | 8   | 9   |
|-----|-----|-----|-----|-----|-----|-----|-----|-----|
| $N_L$ | 5   | 15  | 52  | 203 | 877 | 4140| 21147| 115975|

| $\sigma \backslash q$ | $\frac{1}{16}$ | $\frac{1}{4}$ | 1   | 2   | 3   | 4   | 5   | 8   | 16  | 32  | 64  |
|-----------------------|-----------------|---------------|-----|-----|-----|-----|-----|-----|-----|-----|-----|
| .1                    | .004            | .031          | .061| .0927| .136| .203| .28 | .48 | .78  | 1.00| 1.14|
| .2                    | .007            | .052          | .102| .1831| .2701| .362| .45 | .64 | .91  | 1.12| 1.31|
| .3                    | .010            | .077          | .149| .2717| .3862| .489| .576| .76 | 1.02 | 1.24| 1.45|
| .4                    | .014            | .108          | .204| .3625| .4939| .601| .690| .875| 1.14 | 1.37| 1.59|
| .5                    | .020            | .147          | .270| .4590| .6013| .713| .803| .99 | 1.26 | 1.53| 1.75|
| .6                    | .028            | .198          | .351| .5644| .7143| .829| .920| 1.12| 1.40 | 1.69| 1.94|
| .7                    | .041            | .269          | .452| .6833| .8374| .954| 1.046| 1.24| 1.54 | 1.84| 2.15|
| .8                    | .064            | .375          | .584| .8231| .9774| 1.093| 1.185| 1.384| 1.69 | 2.00| 2.336|
| .9                    | .121            | .552          | .763| .9973| 1.144| 1.255| 1.343| 1.540| 1.84 | 2.173| 2.519|
| 1.0                   | .430            | .815          | .990| 1.230| 1.348| 1.440| 1.518| 1.697| 1.997 | 2.325| 2.677|

| $\sigma \backslash q$ | $\frac{1}{16}$ | $\frac{1}{4}$ | 1   | 2   | 3   | 4   | 5   | 8   | 16  | 32  | 64  |
|-----------------------|-----------------|---------------|-----|-----|-----|-----|-----|-----|-----|-----|-----|
| .1                    | 1.5             | 1.4           | 1.3 | .86 | .81 | .8  | .9  | 1.0 | 1.1 | 1.0 | .8  |
| .2                    | 1.5             | 1.4           | 1.3 | 1.02| .99 | 1.1 | 1.1 | 1.2 | 1.2 | 1.0 | .9  |
| .3                    | 1.6             | 1.5           | 1.4 | 1.13| 1.13| 1.2 | 1.3 | 1.3 | 1.2 | 1.0 | .9  |
| .4                    | 1.6             | 1.5           | 1.4 | 1.19| 1.21| 1.3 | 1.3 | 1.3 | 1.2 | 1.0 | .9  |
| .5                    | 1.6             | 1.4           | 1.3 | 1.22| 1.25| 1.3 | 1.4 | 1.3 | 1.2 | 1.1 | .9  |
| .6                    | 1.6             | 1.4           | 1.3 | 1.22| 1.26| 1.3 | 1.4 | 1.4 | 1.3 | 1.2 | 1.0 |
| .7                    | 1.5             | 1.3           | 1.3 | 1.20| 1.26| 1.3 | 1.4 | 1.4 | 1.3 | 1.2 | 1.2 |
| .8                    | 1.4             | 1.2           | 1.3 | 1.22| 1.31| 1.4 | 1.5 | 1.5 | 1.5 | 1.4 | 1.4 |
| .9                    | 1.1             | 1.5           | 1.7 | 1.88| 1.70| 1.7 | 1.7 | 1.6 | 1.6 | 1.6 | 1.6 |
| 1.0                   | 1.1             | 1.1           | 1   | 1   | 1   | 1   | 1   | 1   | 1   | 1   | 1   |
### Table 1

| \( \sigma \) \( q \) | \( \frac{1}{16} \) | \( \frac{1}{2} \) | 1 | 2 | 3 | 4 | 5 | 8 | 16 | 32 | 64 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| .1 | .11 | .12 | .12 | .101 | .091 | .16 | .21 | .31 | - | - | - |
| .2 | .18 | .20 | .21 | .202 | .215 | .24 | .30 | .45 | .75 | - | - |
| .3 | .32 | .31 | .31 | .301 | .323 | .41 | .53 | 1.1 | 4.1 | - | - |
| .4 | .34 | .34 | .35 | .373 | .481 | .59 | .89 | 3.0 | - | - | - |
| .5 | .33 | .34 | .37 | .430 | .577 | .77 | 1.2 | 5.0 | - | - | - |
| .6 | .30 | .32 | .37 | .501 | .664 | .83 | 2.3 | - | - | - | - |
| .7 | .25 | .29 | .35 | .518 | .636 | .78 | 1.0 | 1.4 | 4.6 | - | - |
| .8 | .18 | .24 | .32 | .483 | .574 | .67 | .80 | .99 | 1.3 | 1.4 | - |
| .9 | .10 | .21 | .30 | .405 | .491 | .56 | .62 | .76 | .97 | 1.1 | - |
| 1.0 | .05 | .18 | .25 | .309 | .393 | .46 | .52 | .63 | .79 | .92 | 1.0 |

### Table 2

| \( \sigma \) \( q \) | \( \frac{1}{16} \) | \( \frac{1}{2} \) | 1 | 2 | 3 | 4 | 5 | 8 | 16 | 32 | 64 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| .1 | - | - | - | - | .94 | .89 | .84 | .83 | .61 | - | - |
| .2 | - | - | - | - | .75 | .84 | .76 | .56 | .24 | - | - |
| .3 | .72 | .74 | .75 | - | .78 | .65 | .50 | .24 | .04 | - | - |
| .4 | .71 | .72 | .72 | - | .52 | .48 | .30 | .08 | - | - | - |
| .5 | .72 | .72 | .71 | .70 | .48 | .37 | .23 | .05 | - | - | - |
| .6 | .75 | .74 | .72 | .49 | .40 | .36 | .24 | .12 | - | - | - |
| .7 | .78 | .77 | .77 | .43 | .46 | .41 | .30 | .24 | .06 | - | - |
| .8 | .81 | .92 | .99 | .43 | .59 | .57 | .47 | .42 | .31 | .32 | - |
| .9 | .87 | 1.00 | 1.00 | 1.41 | 2.16 | 1.32 | 1.08 | .84 | .62 | .62 | - |
| 1.0 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |

### Table 3

| \( q \) | \( \nu_e^{-1} \) | \( \nu_{TG}^{-1} \) | \( \Delta \) (\%) |
|---|---|---|---|
| \( \frac{1}{16} \) | .340 | .3250 | 4.5 |
| \( \frac{1}{2} \) | .338 | .3300 | 2.5 |
| 1 | .349 | .3387 | 3 |
| 2 | .373 | .4 | 7 |
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