Generalized Lorentz Transformations

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Generalized Lorentz transformations with modified velocity parameter are considered. Lorentz transformations depending on the mass of the observer are suggested. The modified formula for the addition of velocities remarkably preserves the constancy of the velocity of light for all observers. The Doppler red shift is affected and can provide a test of such generalisations.

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I. INTRODUCTION

Over the years, modification and generalization of Lorentz transformations has been considered by many authors. Specifically, addition of Lorentz invariance violating interactions to the Standard Model has been considered [1, 2]. Also, extended linear and non-linear Lorentz transformations have been considered [3]. These papers contain many references to other work in this area.

The usual Lorentz transformations involve the dimensionless velocity parameter $\beta = v/c$ and the dilatation factor $\gamma = \frac{1}{\sqrt{1-\beta^2}}$. The important point to note is that with a general velocity parameter $B$ (a function of $\beta = v/c$ and possibly other parameters) Lorentz invariance is guaranteed as long as the corresponding dilatation factor $G$ is the same function of the new velocity parameter $B$ as the old $\gamma$ was of $\beta = v/c$. In other words, $G = \frac{1}{\sqrt{1-B^2}}$.

An important constraint on any velocity parameter is that it be equal to zero for $v = 0$ and be equal to unity for $v = c$. The latter constraint guarantees Einstein’s second postulate, namely that the velocity of light $c$ is the same for all observers.

First we explore the possibility of more general Lorentz transformations with $\beta$ replaced

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by the mass dependent velocity parameter $\beta(m) = v/c(m)$. Section 2 contains the definition of $c(m)$ consistent with the present experimental constraints. Section 3 gives the modified law of addition of velocities. It is pointed out that this formula still gives that the velocity of light is the same for all observers. In Section 4, it is suggested that accurate measurements of the Doppler shift could provide a possible test of the mass dependent Lorentz transformations considered here. In section 5, Lorentz transformations depending on a general velocity parameter $B$ which is a function of only $\beta = v/c$ are considered.

II. CHOICE OF $c(m)$

The mass dependent velocity $c(m)$ is defined to be

$$c(m) = c[1 + F(\zeta)],$$

where $F$ is an analytic function of the dimensionless variable $\zeta$ defined as $\zeta = m/P_M$, where the Planck mass $P_M \approx 1.22 \times 10^{19}$ GeV/c$^2 \approx 2.18 \times 10^{-5}$ gr.

To conform with present knowledge the function $F$ must satisfy the following constraints:

A) For $m = 0$, that is $\zeta = 0$, $F(0) = 0$. This implies that $c(0) = c$. B) For sub-atomic particles (electron, proton etc.) $\zeta$ is very small. Since, our accelerators work, this implies that $F(\zeta)$ must be negligibly small for small $\zeta$. C) Our understanding of planetary motion, constrains $F(\zeta)$ to be negligibly small for large $\zeta$. With these constraints in mind we had earlier considered

$$F(\zeta) = \zeta^n e^{\zeta^n}$$

where $n$ is a positive real number. This function has a maximum value $e^{-1} = 0.367879$ for $\zeta = 1$ independent of the value of $n$. It is centered around $\zeta = 1$. As $n$ increases, the height remains the same, but it becomes narrower and narrower. For extremely large $n$ (tending to infinity) this sequence of functions tends to a vertical line of height $e^{-1}$ at $\zeta = 1$. In the limit it is like a “finite Dirac delta-function”. A plot of the function is given in reference [5].
III. LORENTZ TRANSFORMATIONS DEPENDING ON MASS

Consider parallel Cartesian coordinate systems, $S$ and $S'$ whose origin coincided at $t = t' = 0$. Let their relative velocity be $v$ along the x-axis as measured by $S'$. Then

$$ct' = \gamma(ct - \beta x), \quad x' = \gamma(x - \beta ct), \quad y' = y, \quad z = z',$$

where

$$\beta = \frac{v}{c}, \quad \gamma = \frac{1}{\sqrt{1 - \beta^2}}. \quad (4)$$

This is the usual Lorentz transformation. It leaves the space-time interval invariant, that is $S^2 = (ct)^2 - r^2 = (ct')^2 - r'^2 = S'^2$ where $r^2 = x^2 + y^2 + z^2$. The crucial point is that the invariance of $S^2$ is guaranteed for any $\beta$ as long as $\gamma$ is defined in terms of $\beta$ as above in Eq (4). Thus, if we replace the velocity parameter $\beta$ by the mass dependent velocity parameter

$$\beta(m) = \frac{v}{c(m)} \quad (5)$$

and $\gamma$ by

$$\gamma(m) = \frac{1}{\sqrt{1 - \beta(m)^2}} \quad (6)$$

in Eq.(4), then the space-time interval will be invariant under these mass dependent Lorentz transformations.

In the real world the observer and the observed have masses and the mass dependent Lorentz transformations defined by $\beta(m)$ may be relevant. The mass which enters here is the mass of the observer in frame $S'$. The mass dependent velocity parameter will give a modified relative velocity formula. For simplicity, consider three systems $S_1$, $S_2$ and $S_3$ with aligned x-axes. Let $S_2$ be moving along the x-direction with velocity $v$ respect to $S_1$ and let $S_3$ be moving with velocity $v'$ along the x-direction with respect to $S_2$. Let the corresponding mass dependent velocity parameters be

$$\beta(m) = \frac{v}{c(m)} \quad \text{and} \quad \beta(m') = \frac{v'}{c(m')} \quad (7)$$

Then the relative speed of $S_3$ with respect to $S_1$ will be

$$\beta''(m, m') = \frac{\beta(m) + \beta(m')}{1 + \beta(m)\beta(m')} \quad (8)$$

This mass-dependent relativistic formula for addition of velocities reduces to the usual formula with the replacements $\beta(m) = \beta$ and $\beta(m') = \beta'$. Note that, since for $m = 0$(photons),
\[ \beta(0) = \beta = 1 \text{, the relative velocity } \beta'' = 1 !. \] Also, if \( m = m' = 0 \) then \( \beta'' = 1! \) In other words, the mass-dependent relative velocity formula above respects Einstein’s postulate that the velocity of light is same for all observers!

**IV. DOPPLER SHIFT**

An experimental test of the mass-dependent Lorentz transformations considered here can come from extremely accurate measurements of the Doppler shift, especially for objects with masses of the order of the Planck mass. For the frames \( S \) and \( S' \) moving with relative velocity \( v \) in the \( x \) direction (considered above), the energy (in \( S' \)), namely

\[ E' = \gamma(m)(E - \beta(m)cp_x). \tag{9} \]

In terms of the frequencies \( \nu \) and \( \nu' \), this gives

\[ \nu' = \gamma(m)(1 - \frac{v}{c})\nu. \tag{10} \]

This is the usual formula with \( \gamma \) replaced by \( \gamma(m) \). Since, \( F(\varsigma) \) is chosen to be extremely small except for masses of the order of the Planck mass \((\varsigma \approx 1)\) and since \( m \) is the mass of the observer, different observers with different masses should see different red shift from the same source. One needs new experiments to test this, particularly for observers with masses in the Planck mass range.

**V. GENERALIZED LORENTZ TRANSFORMATIONS DEPENDING ONLY ON THE VELOCITY PARAMETER \( \beta \)**

In this case, the general velocity parameter \( B \) is a function of \( \beta \) only. Clearly, there are many possible choices for \( B(\beta) \) which satisfy \( B(0) = 0 \) and \( B(1) = 1 \).

A possible choice is that \( B(\beta) \) is a curve passing through the points \((0, 0)\) and \((1, 1)\) in the \( \beta-B \) plane. An obvious choice is \( B = Sin(\frac{\pi\beta}{2}) \). For small velocities this is approximately equal to \( \beta \). However, for \( \beta = 0.5 \) its value is approximately equal to 0.7.

Another simple possibility is that the curve is a circle in the \( \beta-B \) plane passing through the points \((0, 0)\) and \((1, 1)\), so that the part of the straight line \( B = \beta \) between the points \((0, 0)\) and \((1, 1)\) is a chord. The equation of such a circle is

\[ [\beta - A]^2 + [B - (1 - A)]^2 = A^2 + (1 - A)^2. \tag{11} \]
Larger the radius of the circle, that is larger the value of $A$, more closely the circle will approximate the straight line $B = \beta$ between the points $(0, 0)$ and $(1, 1)$. Quantitatively, for large $A > 0$, the centre of the circle will be at $(A, -A)$ in the fourth quadrant and the usual midway point $(\beta, B) = (0.5, 0.5)$ is replaced by the point $(\beta, B) = (0.5 - (4A)^{-1}, 0.5 + (4A)^{-1})$. Thus, such a circle would lie above but very close to the usual line $\beta = B$ between the points $(0, 0)$ and $(1, 1)$. An alternative possibility is obtained by changing $A$ to $-A$. Such a circle will have its centre in the second quadrant at $(-A, A)$ and lie below but very close to the usual line $\beta = B$ between the points $(0, 0)$ and $(1, 1)$. Since, the usual $\gamma$ is replaced by $G = \frac{1}{\sqrt{1 - B^2}}$, again the Doppler shift will be affected and would provide tests of such generalizations.

VI. CONCLUDING REMARKS

We have considered generalized Lorentz transformations such that the velocity of light is the same for all observers. It is pointed out that detailed measurements of the Doppler effect can provide a test of the generalizations presented here. Experiments are needed over a large range of velocities to check that the velocity parameter is indeed $\beta = v/c$.

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