Conditional expectation operators on $C(X)$

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Abstract
At the COSAEF conference in 2021, several participants asked the question whether a conditional expectation operator in the sense of Kuo, Labaushagne and Watson could be constructed in vector lattices other than $L_p$ spaces and in particular in $C(X)$. This work answers positively to this question and participates in an old discussion on integrals in $C(X)$ space.

1 Introduction
The study of probabilities in Riesz spaces is a very active research area. Several research schools are involved in this effort. We can mention in particular [1, 2, 3, 4, 5, 6, 7]. Generally, when it was necessary to illustrate this theory, the authors always give examples on $L_p$ spaces. The question of existence of a conditional expectation operator on spaces other than $L_p$ ones then became legitimate. In this work, we will give sufficient conditions for the existence of such operator in the space $C(X)$. In a future work, the authors will characterize this type of operators in finite dimensional spaces with the considerable contributions of Professors Kuo and Watson.

This work will be divided into three parts. A preliminary part which will list the useful notions for the sequel. A second part devoted to a dual approach and in which we will be interested in the dual of the space $C^\infty(X)$. In the third part, we will construct a non-trivial operator of the conditional expectation. Throughout this paper, all topological spaces are assumed to be Hausdorff.

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2 Preliminaries

We recall here some basic properties of Riesz spaces and for details and terminology see [8, 9].

Throughout this paper, $X$ denotes a Tychonoff space; that is, a completely regular Hausdorff space.

A Riesz space is an ordered vector space $(E, \leq)$ such that each pair of elements of $E$ has a supremum and an infimum element compatible with the order. We say that a Riesz space $E$ is Dedekind complete if every non-empty subset of $E$ which is bounded from above has a supremum. In particular, the space $C(X)$ is Dedekind complete if and only if $X$ is extremally disconnected (see for example [9]) i.e the closure of every open subset of $X$ is open-and-closed.

The study of the universal completion of $C(X)$ spaces has attracted the interest of several researchers. We find constructions from the early 40’s (see [10]) until very recently [11]. This notion plays an important role throughout the paper and so we recall the definition and some elementary properties for the reader’s convenience.

A Riesz space $E$ is said to be universally complete if $E$ is Dedekind complete and laterally complete i.e every subset of $E$ which consists of mutually disjoint elements, has a supremum in $E$. $E^u$ is a universal completion of $E$, if $E^u$ is universally complete and $E^u$ contains $E$ as an order dense Riesz subspace.

Every Archimedean Riesz space has, up to a Riesz isomorphism, a unique universal completion and if $e$ is a weak order unit for $E$ then $e$ is a weak order unit for $E^u$.

From now, $X$ will denote an extremally disconnected compact space.

We recall the definition of a conditional expectation operator on a Riesz space from [4].

**Definition 2.1.** Let $E$ be a Riesz space with weak order unit. A positive order continuous projection $T: E \to E$, with range, $\mathcal{R}(T)$, a Dedekind complete Riesz subspace of $E$, is called a conditional expectation operator on $E$ if $Te$ is a weak order unit of $E$ for each weak order unit $e$ of $E$.

3 A Dual approach

In [12], the authors proved that a Dedekind complete Riesz space $E$ with a weak order unit $e$ and who admits a strictly positive order continuous linear functional $\varphi$ can be in a certain way identified as $L_1$ space. The point is that it is well known that the
order continuous dual of $C(X)$, denoted by $C(X)\tilde{\sim}$, didn’t contain any strictly positive element if $X$ is a separable compact space that has no isolated points, (see for example [13]).

We decide thus to focus on a larger space and we choose the universal completion of $C(X)$. Indeed, the universal completion of a Riesz space plays a key role in the probability theory in Riesz spaces (See for example [14]).

It has been shown by Azouzi and Trabelsi in [2] that a conditional expectation $T$ on a Dedekind complete Riesz space $E$ can be extended to a largest Riesz subspace of the universal completion $E^u$ of $E$, called the natural domain of $T$ and denoted by $L^1(T)$, to which $T$ extends uniquely to a conditional expectation. That is

$$E \subset L^1(T) \subset E^u.$$  

The natural domain of $T$ enjoys several important properties. In fact, $L^1(T)$ is an order dense ideal of $E^u$. It is also a Dedekind complete Riesz space with $e$ as weak unit and it is $T$-universally complete (i.e., every increasing net $(x_\alpha)$ in $L^1(T)^+$ with $(Tx_\alpha)$ order bounded in $L^1(T)$ is order convergent).

In the particular case where $T$ is the identity function, one can take

$$L^1(T) = E^u.$$

It has been proved in [11] that the universal completion of $C(X)$, where $X$ is an extremally disconnected compact Hausdorff space, is the space $C^\infty(X)$ of all extended real continuous functions $f$ on $X$ with the property that \( \{ x \in X \text{ such that } |f(x)| < \infty \} \) is dense in $X$.

Our problem could then be rephrased as asking for the existence of a strictly positive order continuous functional on $C^\infty(X)$.

**Proposition 3.1.** Let $X$ be an extremally disconnected compact space and $\varphi$ be an element of $(C^\infty(X)\tilde{\sim})_+$. Then the following assertions are equivalent:

a) $\varphi(1_A) = 0$ for every open-closed set $A \subset X$

b) $\varphi \equiv 0$

*Proof.* Only (a $\implies$ b) needs details. Suppose that there exists $0 < f \in C^\infty(X)$ such that $\varphi(f) > 0$. Put

$$A_f^\lambda = \{ x \in X \text{ such that } \lambda 1_X(x) - f(x) > 0 \}.$$  

We note that $A_f^\lambda$ is an open subset of $X$ for all $\lambda \in \mathbb{R}_+$ and $\overline{A_f^\lambda}$ is an open-closed set since $X$ is an extremally disconnected.
By the assumption above, \( \varphi\left(\frac{1}{A_\lambda}\right) = 0 \) for every \( \lambda \in \mathbb{R}_+ \) and by Freudenthal Spectral Theorem (Theorem 1.3.9 in [15]), we obtain \( \varphi(f) = 0 \).

The next two technical lemmas will be useful for the sequel. However, they are of interest independently of the rest of the article.

**Lemma 3.2.** Let \( X \) be an extremally disconnected compact space and \( A \) a subset of \( X \). If \( A \) verify
\[
A \cap C \neq \emptyset \implies A \subseteq C
\]
for every clopen set \( C \), then \( A \) is connected.

**Proof.** Suppose by the way of contradiction that \( A \) verify 3.1 and that \( A \) is disconnected. Then there exist two open sets \( O_1 \) and \( O_2 \) such that \( O_1 \cap A \neq \emptyset \), \( O_2 \cap A \neq \emptyset \), \( (O_1 \cap A) \cap (O_2 \cap A) = \emptyset \) and \( (O_1 \cup A) \cap (O_2 \cap A) = A \). Since \( X \) is extremally disconnected and compact, then by Theorem 16.17 in [16], the clopen sets form a base for the topology. That is \( O_1 = \bigcup_{\alpha \in \Gamma} C_\alpha \), where \( C_\alpha \) are clopen sets which gives the existence of \( \alpha_0 \) in \( \Gamma \) such that \( A \subseteq C_{\alpha_0} \). This yields to \( A \subseteq O_1 \) which contradicts the fact that \( (O_2 \cap A) \neq \emptyset \).

The following lemma is an easy consequence of lemma 3.2.

**Lemma 3.3.** Let \( X \) be an extremally disconnected compact space and \( A \) a subset of \( X \). If \( A \) is disconnected then there exists a clopen set \( C \) such that \( A \cap C \neq \emptyset \) and \( A \not\subseteq C \).

The next proposition will play a key role in the proof of the main theorem.

**Proposition 3.4.** Let \( X \) be an extremally disconnected compact space without isolated points and \( \varphi \) be a non null element of \( (C_\infty(X))_n^+ \). Then there exists a disjoint composition of clopen sets \( X = K_1 \sqcup K_2 \) such that \( \varphi(K_1) = \alpha_1 > 0 \) and \( \varphi(K_2) = \alpha_2 > 0 \).

**Proof.** Let us verify first that \( \varphi(1_X) = \alpha > 0 \). Suppose that this is not the case. Then \( \varphi(1_X) = 0 \) which implies \( \varphi(1_A) = 0 \) for every open-closed subset \( A \subseteq X \) and Proposition 3.1 gives \( \varphi = 0 \) which contradicts the fact that \( \varphi \) is strictly positive. Now, observe that, \( X \) being extremally disconnected, there exist two disjoint clopen sets \( K_1 \) and \( K_2 \) such that \( X = K_1 \sqcup K_2 \). In order to prove that \( \varphi(1_{K_1}) = \alpha_1 > 0 \) and \( \varphi(1_{K_2}) = \alpha_2 > 0 \), We will suppose by the way of contradiction that for every clopen set \( K \), we have \( \varphi(1_K) = \alpha \) or \( \varphi(1_K) = 0 \). Put
\[
\mathcal{K}_\alpha = \{ K \subseteq X; K \text{ is a clopen set and } \varphi(1_K) = \alpha \}.
\]

If \( K' \) and \( K'' \) in \( \mathcal{K}_\alpha \), then it would immediately follow that \( K' \cap K'' \) in \( \mathcal{K}_\alpha \). Indeed, if \( \varphi(1_{K' \cap K''}) = 0 \), then \( \varphi(1_{K' \cup K''}) = 2\alpha > \varphi(1_X) \) which is impossible. Now let
\[
K^\circ = \bigcap_{K \in \mathcal{K}_\alpha} K.
\]
If $K^\circ$ is disconnected then lemma 3.3 implies the existence of some clopen set $C_1$ such that $K^\circ \cap C_1 \neq \emptyset$ and $K^\circ \not\subseteq C_1$. It follows that there is a clopen set $C_2 = X \setminus C_1$ such that $X = C_1 \sqcup C_2$. Therefore, either $C_1$ or $C_2$ is in $\mathcal{R}_\alpha$. Suppose first that $C_1 \in \mathcal{R}_\alpha$. It follows that $K^\circ \subseteq C_1$ which is impossible. Moreover, if $C_2 \in \mathcal{R}_\alpha$, then $K^\circ \subset C_2$ which contradicts the fact that $C_1 \cap C_2 = \emptyset$. It follows that $K^\circ$ is connected. This later fact together with the total discontinuity of $X$ yield to the existence of $x^*$ in $X$ such that $K^\circ = \{x^*\}$. From the fact that $X$ has no isolated point, follows that

$$\inf_{K \in \mathcal{R}_\alpha} \{1_K\} \equiv 0.$$

This is impossible since $\varphi(1_K) = \alpha$ for all $K$ in $\mathcal{R}_\alpha$ and $\varphi$ is order continuous. \hfill $\Box$

Although the results of the following theorem seem to be known, we are not able to find a precise reference. However, we believe that the approach and the proof are new.

**Theorem 3.5.** Let $X$ be an extremally disconnected compact space without isolated points and $\text{card}(X) \geq \aleph_0$. Then $C^\infty(X)_{=\sim} = \{0\}$.

**Proof.** From Proposition 3.4 we claim that there exists a disjoint composition $X = K_1 \sqcup K_2$ such that $\varphi(1_{K_1}) = \alpha_1 > 0$ and $\varphi(1_{K_2}) = \alpha_2 > 0$. Since $K_1$ is an open set in an extremally disconnected space then $K_1$ is extremally disconnected. Moreover, $K_1$ is closed in a compact space gives that $K_1$ is compact. Applying 3.4 to $K_1$ gives the existence of a disjoint composition $K_1 = K_{1,1} \sqcup K_{1,2}$ such that $\varphi(1_{K_{1,1}}) = \alpha_{1,1} > 0$ and $\varphi(1_{K_{1,2}}) = \alpha_{1,2} > 0$. Proceeding further, we can find a sequence $(K_{i_1,\ldots, i_k})_{k \in \mathbb{N}}$ with $i_l \in \{1,2\}$ for every $l \in \{1,\ldots, k\}$ such that $\varphi(1_{K_{i_1,\ldots, i_k}}) = \alpha_{i_1,\ldots, i_k} > 0$. Then is defined an element

$$f = \sup_{k \in \mathbb{N}} \alpha_{i_1,\ldots, i_k} 1_{K_{i_1,\ldots, i_k}}$$

satisfying $\varphi(f) > M$ for every $M \in \mathbb{R}_+$. This trivial contradiction ends the proof. \hfill $\Box$

As an immediate corollary, we can state an analogous result of Theorem 5.2 in [13].

**Corollary 3.6.** Let $X$ be an extremally disconnected compact space with $\text{card}(X) \geq \aleph_0$. Then the following assertions are equivalent:

a) $X$ has no isolated point.

b) $C^\infty(X)_{=\sim} = \{0\}$.

**Proof.** Only a) $\Rightarrow$ b) needs some details. In order to prove this implication we will assume that the space $X$ admits isolated points. So, if $x^*$ is an isolated point in $X$ then any open dense subset of $X$ contains $x^*$. Since for every $f$ in $C^\infty(X)$, the set

$$\mathcal{F}_f = \{x \in X \text{ such that } |f(x)| < \infty\}$$

would contain an isolated point, we conclude that $\mathcal{F}_f = X$. Therefore, $f$ is constant. Since $\varphi$ is order continuous, it follows that $\varphi(f) = 0$. Hence, $f = 0$. This means that $f$ is constant, and therefore, $\varphi(f) = 0$. This trivial contradiction ends the proof. \hfill $\Box$
is an open dense subset of $X$, it follows that for any $f$ in $C^\infty(X)$, $f(x^*)$ is finite. Now, put $\delta_{x^*}$ as

$$\delta_{x^*} : \overset{\sim}{C^\infty(X)} \longrightarrow \mathbb{R} \quad f \mapsto f(x^*)$$

It is clear that $\delta_{x^*}$ is in $\overset{\sim}{C^\infty(X)}$, which makes an end to our proof. 

Recently, Mozo Carollo proved in [17], that if $X$ is an extremally disconnected $P$-space then $C(X)$ is universally complete. The next corollary follows immediately.

**Corollary 3.7.** Let $X$ be an extremally disconnected compact $P$-space with $\text{card}(X) \geq \aleph_0$. Then the following assertions are equivalent:

a) $X$ has no isolated point.

b) $\overset{\sim}{C(X)} = \{0\}$.

### 4 Conditional expectation on $C(X)$

In this section, We will provide an affirmative answer of the question of existence of non trivial strictly positive conditional expectation on spaces bigger than $L_p$ ones which is the most typical space $C(X)$.

We will recall throughout this section definitions from [18, 19].

**Definition 4.1.** Let $X$ be a Tychonoff space. We call a point $x \in X$ perfectly disconnected in $X$ if it is not simultaneously an accumulation point of two disjoint subsets. If every point of $X$ is perfectly disconnected, then $X$ called perfectly disconnected.

Notice that every perfectly disconnected space is extremally disconnected.

Following the notation in [18], we recall the definition of the Alexandroff duplicate of $X$.

**Definition 4.2.** Let $X$ be a topological space. The Alexandroff duplicate of $X$ is the space created by taking two (disjoint) copies of $X$, say $A(X) = X \cup X'$.

Recently, the authors in [18], proved in Theorem 2.4 that if $X$ is perfectly disconnected and the set of isolated points of $X$ is clopen then $A(X)$ is extremally disconnected.

We are able now, to prove the main result of this section.

**Theorem 4.3.** Let $X$ be perfectly disconnected and the set of isolated points of $X$ is clopen then there exists a non trivial conditional expectation operator on $C(A(X))$. 
Proof. Our prove will be constructive. Consider the map
\[ T : C(A(X)) \rightarrow C(A(X)) \]
\[ f \mapsto g \]
where \( g(x) = \frac{f(x) + f(x')}{2} \) for every \( x \) in \( A(X) \). We will prove that \( T \) is a conditional expectation on \( C(A(X)) \). In this order, we can notice first that \( T \) is strictly positive, order continuous and satisfies \( T1 = 1 \).

Furthermore, \( T \) is a projection since for all \( f \in C(A(X)) \) and for all \( x \) in \( A(X) \), we have
\[
T \circ T(f(x)) = T\left(\frac{f(x)+f(x')}{2}\right)
= \frac{1}{2}(T(f(x)) + T(f(x')))
= \frac{1}{2}\left(\frac{f(x)+f(x')}{2} + \frac{f(x')+f(x)}{2}\right)
= T(f(x)).
\]
It remains now to show that \( R(T) \) is a Dedekind complete Riesz subspace. There are two methods to prove it. The first one is a direct application of Theorem 3.1 in [20]. The second possibility is the observation that
\[
R(T) = \{ g \in C(A(X)) \text{ such that } g(x) = g(x') \text{ for all } x \in A(X) \}.
\]
Indeed, let \( g \) be in \( R(T) \). By definition, there exists \( f \in C(A(X)) \) such that \( T f = g \). Then \( g(x') = \frac{f(x') + f(x'')}{2} = \frac{f(x') + f(x)}{2} = g(x) \) for all \( x \in A(X) \). Conversely, it is clear that \( T g = g \) for any \( g \in C(A(X)) \) satisfying \( g(x) = g(x') \) for all \( x \) in \( A(X) \) which makes an end to the proof.

The important question that remains open is to characterize the spaces \( X \) for which there are conditional expectation operators on \( C(X) \).

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