Bridged Hamiltonian Cycles in Sub-critical Random Geometric Graphs

Ghurumuruhan Ganesan  
HBNI, Chennai, China  
Indian Institute of Science Education and Research (IISER), Bhopal, India

Abstract
In this paper, we consider a random geometric graph (RGG) $G$ on $n$ nodes with adjacency distance $r_n$ just below the Hamiltonicity threshold and construct Hamiltonian cycles using additional edges called bridges. The bridges by definition do not belong to $G$ and we are interested in estimating the number of bridges and the maximum bridge length, needed for constructing a Hamiltonian cycle. In our main result, we show that with high probability, i.e. with probability converging to one as $n \to \infty$, we can obtain a Hamiltonian cycle with maximum bridge length a constant multiple of $r_n$ and containing an arbitrarily small fraction of edges as bridges. We use a combination of backbone construction and iterative cycle merging to obtain the desired Hamiltonian cycle.

AMS 2000 Subject Classification. Primary: 60J10, 60K35; Secondary: 60C05, 62E10, 90B15, 91D30.
Keywords and phrases. Random geometric graphs, Hamiltonian cycles with bridges.

1 Introduction
Hamiltonian cycles in Random Geometric Graphs (RGGs) are extremely important from both theoretical and application perspectives. Penrose (1997) obtained sharp bounds on the threshold adjacency distance for the RGG to become Hamiltonian with high probability and later Díaz et al (2007) generalized this to metrics other than Euclidean distance along with providing an algorithm for finding the Hamiltonian cycle. Nearly simultaneously, Müller et al. (2011), Balogh et al. (2011) investigated the problem of coincidence of 2—connectedness and Hamiltonicity of RGGs and study how the graph becomes Hamiltonian just as it also becomes 2—connected. More recently Bal et al. (2017) have explored the existence of rainbow Hamiltonian cycles in edge coloured RGGs.
In this paper, we study construction of Hamiltonian cycles in an RGG $G$ with adjacency distance $r_n$ just below the Hamiltonicity threshold, using a small number of extra edges called bridges that do not belong to $G$. We use dense component construction involving discretization of the unit square, to obtain a number of small cycles in $G$ and then “stitch” the cycles together using bridges to obtain the desired bridged cycle. We also obtain bounds on the adjacency distance that ensures that the bridge fraction in the resulting cycle is arbitrarily small.

The paper is organized as follows. In Section 2, we describe our main result Theorem 1 regarding the maximum bridge length and bridge fraction of bridged Hamiltonian cycles in random geometric graphs whose adjacency distance is just below the Hamiltonian threshold. Next, in Section 3, we collect the preliminary results used in the proof of Theorem 1 and finally, in Section 4, we prove Theorem 1.

2 Bridged Hamiltonian Cycles in RGGs

Consider $n$ nodes $X_1, X_2, \ldots, X_n$, independently distributed in the unit square $S = [-\frac{1}{2}, \frac{1}{2}]^2$ each according to a certain density $f$ satisfying

$$0 < \epsilon_1 \leq f(x) \leq \epsilon_2 < \infty. \quad (2.1)$$

We define the overall process on the probability space $(\Omega_X, \mathcal{F}_X, \mathbb{P})$ and let $K_n = K_n(X_1, \ldots, X_n)$ be the complete graph with vertex set $\{X_1, \ldots, X_n\}$. Let $G(r_n)$ be the graph formed by the set of all edges of $K_n$, each of whose length is strictly less than $r_n$. We define $G(r_n)$ to be the random geometric graph (RGG) formed by the nodes $\{X_i\}_{1 \leq i \leq n}$ with adjacency distance $r_n$.

A cycle in $K_n$ is a sequence of distinct nodes $D = (X_{i_1}, \ldots, X_{i_t})$ such that $X_{i_j}$ is adjacent to $X_{i_{j+1}}$ for $1 \leq j \leq t - 1$ and $X_{i_t}$ is adjacent to $X_{i_1}$. The length of $D$ is the number of edges in $D$. The cycle $D$ is said to be Hamiltonian if $t = n$; i.e. the cycle $D$ contains all the $n$ nodes $\{X_i\}_{1 \leq i \leq n}$. An edge $e \in K_n$ of length at least $r_n$ is said to be a bridge with respect to $G(r_n)$. Throughout, we consider only bridges with respect to $G(r_n)$ and so we suppress the phrase “with respect to $G(r_n)$”. A cycle $D \subset K_n$ is said to be a bridged cycle if $D$ contains at least one bridge.

**Definition 1.** Let $D \subset K_n$ be a Hamiltonian cycle. For $w > 0$ and $0 < \gamma < 1$ we say that $D$ is a $(w, \gamma)$-bridged Hamiltonian cycle if the following two properties hold:

(a) The maximum length of an edge in $D$ is less than $w$.

(b) If $n_{br}$ denotes the number of bridges in $D$, then the ratio $\frac{n_{br}}{n} \leq \gamma$. 

In other words, the fraction of edges that are bridges in $D$ is at most $\gamma$. From the above definition, we see that if $D$ does not contain any bridges, then $D$ is a $(r_n, 0)$–bridged Hamiltonian cycle.

Given $w > 0$ and $0 \leq \gamma \leq 1$, let $E(w, \gamma)$ be the event that there exists a $(w, \gamma)$–bridged Hamiltonian cycle. We are interested in estimating the probability of occurrence of $E(w, \gamma)$ for various values of $w$ and $\gamma$. For example, using the fact that any edge of the complete graph $K_n$ is at most $\sqrt{2}$, we get that $E(\sqrt{2}, 1)$ occurs with probability one. On the other hand, if the nodes are uniformly distributed and the adjacency distance $r_n$ is larger than the Hamiltonicity threshold, i.e. if $\pi n r_n^2 = \log n + \delta \log \log n$ for some constant $\delta > 4$, then we know Penrose (1997) that $E(r_n, 0)$ occurs with high probability, i.e. with probability converging to one as $n \to \infty$.

For values of $r_n$ just below the Hamiltonicity threshold, we have the following result.

**Theorem 1.** Let $\epsilon_1$ be as in Eq. 2.1 and for a constant $\alpha > 0$, define

$$nr_n^2 = \frac{1}{4\epsilon_1} (\log n + \alpha \log \log n + \omega_n)$$

(2.2)

where $\omega_n \to \infty$ as $n \to \infty$. For every integer $L \geq 9$ and every $\alpha \geq 8L - 1$, there exists a constant $C > 0$ such that

$$\mathbb{P}(E(2r_n, \gamma)) \geq 1 - \frac{1}{n^{9\gamma / 8}} - \frac{Ce^{-3\omega_n / 8}}{(\log n)^{(\alpha - 8L + 1)}}$$

(2.3)

where $\gamma := \frac{16}{L-8}$.

Choosing $L$ large, the fraction of bridges $\gamma$ can be made arbitrarily close to 0. If the node distribution is uniform, then $\epsilon_1$ as defined in Eq. 2.1 equals one and so for any constant $0 < \epsilon < 1$ and all $n$ large, we have from Eq. 2.2 that

$$r_n < \sqrt{(1 - \epsilon) \log n / \pi n}.$$ 

This implies that the corresponding RGG with adjacency distance $r_n$ is in fact below the connectivity threshold and is therefore disconnected with high probability Penrose (2003). Theorem 1 says that with high probability, using at most $\gamma n$ bridges each of length at most $2r_n$, we can still “patch” together a Hamiltonian cycle.

In our proof of Theorem 1 below, we first construct a collection of small cycles formed by edges of $G(r_n)$ and then join these cycles together using bridges of length at most $2r_n$, to obtain the desired bridged Hamiltonian cycle.
3 Preliminaries

In this section, we collect a couple of preliminary results used in the proof of Theorem 1. Throughout we use the following discretization procedure. Divide the unit square $S$ into disjoint squares $\{S_j\}$ of side length $t_n$ as shown in Fig. 1, where

$$8nt_n^2 = \frac{1}{\epsilon_1} \left( \log n + \alpha \log \log n + \omega_n \right) - \gamma_n$$

(3.1)

and $\gamma_n \in (0,1)$ is such that $\frac{1}{t_n}$ is an integer for all $n$ large. Such a $\gamma_n$ always exists and for completeness we provide a small justification in the Appendix.

By choice $t_n$ is slightly less than $\frac{r_n}{\sqrt{2}}$ and so any two nodes within a square $S_i$ are adjacent in the graph $G(r_n)$. Moreover if squares $S_i$ and $S_j$ are adjacent (i.e., share a corner), then every node of $\{X_k\}$ in $S_i$ is joined to every node in $S_j$ by an edge of length less than $2r_n$. Say that $S_j$ is dense if it contains at least $L$ nodes of $\{X_k\}$ and sparse otherwise. Letting $E(j)$ be the event that $S_j$ is sparse, the following Lemma estimates the joint event that multiple squares are sparse. Throughout constants do not depend on $n$.

**Lemma 1.** Let $q \geq 1$ be any integer constant and let $S_{j_1}, \ldots, S_{j_q}$ be any set of $q$ squares in $\{S_i\}_{i \geq 1}$. We have that

$$\mathbb{P} \left( \bigcap_{i=1}^{q} E(j_i) \right) \leq \frac{C}{n^{q/8} \cdot (\log n)^{qa_1}} \exp \left( \frac{-q\omega_n}{8} \right)$$

(3.2)

for all $n$ large, where $a_1 := \frac{\alpha}{8} - L$ and $C = C(q) > 0$ is a positive constant.

Figure 1: Tiling the unit square into disjoint $t_n \times t_n$ squares $\{S_i\}_{1 \leq i \leq N}$
Proof of Lemma 1: If \( \bigcap_{i=1}^{q} E(j_i) \) occurs, then the total number of nodes in \( \bigcup_{i=1}^{q} S_{j_i} \) is at most \( Lq \). The total area covered by \( \bigcup_{i=1}^{q} S_{j_i} \) is \( qt_n^2 \) and so the probability that node \( X_i \) belongs to one of these \( q \) squares is

\[
\int_{\bigcup_{i=1}^{q} S_{j_i}} f(x) dx \in [\epsilon_1 qt_n^2, \epsilon_2 qt_n^2],
\]

using the bounds for \( f(.) \) in Eq. 2.1. Thus

\[
\mathbb{P} \left( \bigcap_{i=1}^{q} E(j_i) \right) \leq \sum_{k=0}^{Lq} \binom{n}{k} (\epsilon_2 qt_n^2)^k (1 - \epsilon_1 qt_n^2)^{n-k}
\]

\[
\leq \sum_{k=0}^{Lq} (\epsilon_2 qnt_n^2)^k (1 - \epsilon_1 qt_n^2)^{n-k} \tag{3.3}
\]

\[
\leq \frac{1}{(1 - \epsilon_2 qt_n^2)Lq} \sum_{k=0}^{Lq} (\epsilon_2 qnt_n^2)^k (1 - \epsilon_1 qt_n^2)^n
\]

\[
\leq \frac{1}{(1 - \epsilon_2 qt_n^2)Lq} \sum_{k=0}^{Lq} (\epsilon_2 qnt_n^2)^k e^{-\epsilon_1 qnt_n^2}, \tag{3.4}
\]

where (3.3) follows from the fact that \( \binom{n}{k} \leq n^k \) and the estimate (3.4) is true since \( 1 - x < e^{-x} \) for all \( x > 0 \).

From Eq. 3.1, we have that \( t_n \to 0 \) as \( n \to \infty \) and so using the fact that \( q \) is a constant, we have that \( (1 - \epsilon_2 qt_n^2)Lq \leq 2 \) for all \( n \) large. Plugging this into Eq. 3.4 and using the fact that \( nt_n^2 \leq \frac{2}{e_1} \log n \) for all \( n \) large (see Eq. 3.1), we get that

\[
\mathbb{P} \left( \bigcap_{i=1}^{q} E(j_i) \right) \leq 2 \sum_{k=0}^{Lq} (\epsilon_2 qnt_n^2)^k e^{-\epsilon_1 qnt_n^2} \leq D_1 (\log n)^{Lq} e^{-\epsilon_1 qnt_n^2}, \tag{3.5}
\]

for some constant \( D_1 > 0 \). Setting \( \alpha_1 = \frac{\alpha}{8} - L \) and again using the expression for \( t_n \) in Eq. 3.1 we get that the final term in Eq. 3.5 is

\[
\frac{D_1}{n^{q/8} \cdot (\log n)^{q\alpha_1}} \cdot \exp \left( -\frac{q\omega_n}{8} + \frac{q\gamma_n}{8} \right) \leq \frac{D_2}{n^{q/8} \cdot (\log n)^{q\alpha_1}} \exp \left( -\frac{q\omega_n}{8} \right)
\]

for some constant \( D_2 > 0 \), since \( \gamma_n < 1 \). \( \square \)

3.1 Left-Right Crossings Two \( t_n \times t_n \) squares \( S_i \) and \( S_j \) are said to be adjacent if they share a corner and plus adjacent if they share a common side. A sequence of distinct squares \( Y = (Y_1, \ldots, Y_w) \subset \{ S_j \} \) is said to form
a $S$–path if $Y_i$ is adjacent to $Y_{i+1}$ for every $1 \leq i \leq w - 1$. If, in addition, $Y_1$ is also adjacent to $Y_w$, then $Y$ is said to form a $S$–connected cycle. If all the squares in a $S$–path $Y$ are dense, we say that $Y$ is a dense $S$–path. Analogous definitions as above hold for $S$–cycles and the plus connected case as well.

For future use, we are interested in obtaining a network of long dense $S$–paths that criss-cross each other. We therefore have a couple of additional definitions. For a constant $M > 0$, we divide the unit square $S$ into a set of horizontal rectangles $R_H$ each of size $1 \times Mt_n$ and also vertically into a set of rectangles $R_V$, each of size $Mt_n \times 1$. If $(Mt_n)^{-1}$ is an integer, we obtain a perfect tiling as in Fig. 2b. Otherwise we choose $M_n \in [M, M + \sqrt{t_n}]$ so that $(M_n t_n)^{-1}$ is an integer for all $n$ large and the tiling of $S$ into rectangles of size $1 \times M_n t_n$ is then perfect. This is possible since $t_n \to 0$ as $n \to \infty$ (see Eq. 3.1) and so

$$\frac{1}{Mt_n} - \frac{1}{(M + \sqrt{t_n}) t_n} = \frac{1}{M \sqrt{t_n}} \cdot \frac{1}{M(M + \sqrt{t_n})}$$

is bounded below by $\frac{1}{2M^2 \sqrt{t_n}} \to \infty$ as $n \to \infty$ For notational simplicity, we assume henceforth that $M_n = M$ is a constant such that the tiling is perfect.

Let $R \in R_H \cup R_V$ be any rectangle. A distinct sequence of squares $Y = (Y_1, \ldots, Y_D) \subset\{S_j\}$ contained in $R$ is said to be a left right (top bottom) crossing of $R$ if $Y$ is a $S$–path, the square $Y_1$ intersects the left side (top side) of $R$ and the square $Y_D$ intersects the right side (bottom side) of $R$. The crossing $Y$ is said to be dense if every square in $L$ is dense. An analogous definition holds for the plus connected case and for an illustration of left right crossings, we refer to Fig. 2a.

For $R \in R_H$, let $F_n(R)$ be the event that the horizontally long rectangle $R \in R_H$ contains a dense left right crossing of $t_n \times t_n$ squares belonging to $\{S_j\}_{j \geq 1}$. Analogously, for $R \in R_V$, let $F_n(R)$ be the event that $R$ contains a dense top bottom crossing. Setting

$$F_n := \bigcap_{R \in R_H \cup R_V} F_n(R)$$

we have the following result.

**Lemma 2.** We have that

$$\mathbb{P}(F_n) \geq 1 - \frac{Dn}{n^{M/8}}$$

for some constant $D > 0$ and all $n$ large.
Figure 2: (a) Illustration of two left right crossings, where the top left right crossing is plus connected. (b) The backbone formed by the solid and dotted wavy lines due to the occurrence of the event $F_n$. Here $a = Mt_n$ is the width of the rectangles and each wavy line is a crossing of the form as shown in (a). The grey star represents a dense $S-$component distinct from the $S-$component containing the backbone.

**Proof of Lemma 2.** We first prove that

$$\min_{R \in \mathcal{R}_H \cup \mathcal{R}_V} \mathbb{P}(F_n(R)) \geq 1 - \frac{D \sqrt{n}}{n^{M/8}} \quad (3.8)$$

for some constant $D > 0$ and all $n$ large.

The proof is as in Lemma 2 of Ganesan (2013). To prove (3.8), let $R \in \mathcal{R}_H$ and suppose that $F_n(R)$ does not occur. Necessarily, there exists a sparse plus connected top bottom crossing of $R$ (see for example, Theorem 6 of Ganesan 2021) and we estimate the probability of such an event happening. Let $\pi = (Y_1, \ldots, Y_q)$ be any plus connected top bottom crossing of $R$ and let $A_\pi$ be the event that every square of $\pi$ is sparse. By estimate (3.2) of Lemma 1, we have that

$$\mathbb{P}(A_\pi) \leq \frac{D}{n^{q/8}} \quad (3.9)$$

for some constant $D > 0$.

The square $Y_1$ intersects the top edge of $R$ and since

$$t_n \geq \frac{r_n}{5} \geq C \cdot \sqrt{\frac{\log n}{n}} \quad (3.10)$$
for some constant $C > 0$, (see Eqs. 3.1 and 2.2), there are $\frac{1}{t_n} \leq C\sqrt{n}$ choices of $Y_1$. For each such choice of $Y_1$, the number of choices for $\pi$ is at most $8^q$ and since the height of the rectangle $R$ is $Mt_n$, it is also necessary that $q \geq M$. Therefore using the geometric summation formula we have that

$$\mathbb{P}(F_n^c(R)) \leq C\sqrt{n} \sum_{q \geq M} 8^q \cdot \frac{D}{n^{q/8}} = CD\sqrt{n} \sum_{q \geq M} \left(\frac{8}{n^{1/8}}\right)^q \leq \frac{D_1\sqrt{n}}{n^{M/8}}$$

(3.11)

for some constant $D_1 > 0$ and all $n$ large. This proves (3.8).

From Eqs. 3.8, 3.6, 3.11 and the union bound, we then get

$$\mathbb{P}(F_n) \geq 1 - \#(\mathcal{R}_H \cup \mathcal{R}_V) \frac{D_1\sqrt{n}}{n^{M/8}} \geq 1 - \frac{D_2n}{n^{M/8}}$$

(3.12)

for all $n$ large and some constant $D_2 > 0$, where the last estimate in Eq. 3.12 follows from

$$\#(\mathcal{R}_H \cup \mathcal{R}_V) = \frac{2}{Mt_n} \leq D_3\sqrt{n}$$

(3.13)

for some constant $D_3 > 0$, by Eq. 3.10.

4 Proof of Theorem 1

Proof Outline: Recalling the definition of the event $F_n$ in Lemma 2, we set the constant $M > 0$ to be sufficiently large so that

$$\mathbb{P}(F_n) \geq 1 - \frac{1}{n^{10}}.$$

(4.1)

If $F_n$ occurs, then by considering lowermost dense left right crossings of rectangles in $\mathcal{R}_H$ and leftmost dense top bottom crossings of rectangles in $\mathcal{R}_V$, we obtain a unique “backbone” of crossings which we denote as $\mathcal{B}$. This is illustrated in Fig. 2b, where the solid and dotted wavy lines together form the backbone $\mathcal{B}$.

Say that a set of squares $\mathcal{C} := \{Z_i\}_{1 \leq i \leq r} \subset \{S_j\}$ is a dense $S$–component if:

(a) For any $1 \leq i_1 \neq i_2 \leq r$, the squares $Z_{i_1}$ and $Z_{i_2}$ are connected by a dense $S$–path.

(b) If $Z$ is any dense square adjacent to some square in $\mathcal{C}$, then $Z \in \mathcal{C}$ itself.

In other words, a dense $S$–component is a maximal connected set of dense squares. For a square $A \in \{S_j\}$, we define the dense $S$–component $\mathcal{T}(A)$
Bridged Hamiltonian Cycles in Sub-critical...

Following the above notation, we let $\mathcal{T}(B)$ be the dense $S$–component containing all the squares of $B$. Using $\mathcal{T}(B)$, we now proceed in three steps to obtain the bridged Hamiltonian cycle. In the first step, we show that with high probability the backbone component $\mathcal{T}(B)$ is the only dense component among the squares $\{S_j\}_{j \geq 1}$. In the second step, we show that with high probability, every sparse square is adjacent to some dense square of the backbone component. Finally, we construct the bridged Hamiltonian cycle iteratively by connecting cycles within dense and sparse squares, and then estimate its bridge fraction.

4.1. Isolated Dense Components

For a square $A \in \{S_j\}$, let

$$I(A) := F_n \bigcap \{\mathcal{T}(A) \neq \mathcal{T}(B)\}$$

be the event that the dense component containing the square $A$ is not the backbone component $\mathcal{T}(B)$. The event $F_n$ guarantees the existence of the backbone and therefore $B$ is well-defined and $\mathcal{T}(B) \neq \emptyset$. Defining

$$I_n := \bigcup_{A \in \{S_j\}} I(A),$$

we have that

$$\mathbb{P}(I_n) \leq \frac{C_f}{(\log n)^{a-8L+1}} e^{-\frac{3\omega_n}{8}}$$

for some constant $C_f > 0$ and for all $n$ large, where $\omega_n \to \infty$ is as in Eq. 2.2.

Proof of Eq. 4.4.: For the $t_n \times t_n$ square $A \in \{S_j\}$ let $U_i(A)$ be the set of all squares of $\{S_j\}$ adjacent to $A$ and for $i \geq 2$, let $U_i(A)$ be the set of all squares of $\{S_j\}$ adjacent to some square of $U_{i-1}(A)$ so that $U_i(A)$ has $(2i+1)^2$ squares of $\{S_j\}$.

Let $S(1+2t_n)$ be the square with same centre as the unit square $S$ and of side length $1+2t_n$. Divide the annulus $S(1+2t_n) \setminus S$ (of width $t_n$) into $t_n \times t_n$ squares $\{Q_j\}$. There necessarily exists a plus connected $S$–cycle $L_{\text{cyc}}(A) = (Z_1, \ldots, Z_q)$ containing $q$ distinct squares in $\{S_j\} \cup \{Q_j\}$ surrounding $A$ and satisfying the property that $Z_1$ is plus adjacent to $Z_t$ and $Z_i$ is plus adjacent to $Z_{i+1}$ for $1 \leq i \leq q-1$ (see for example Theorem 4 of Ganesan 2021). This is illustrated in Fig. 3, where the dark grey square is $A$ and the light and the dark grey squares together form $C(A)$. The sequence of dotted squares are sparse and form $L_{\text{cyc}}(A)$. 
Figure 3: The plus connected $S-$cycle $L_{\text{cyc}}(A)$ shown by the sequence of dotted squares for the square $A$ denoted by the dark grey square. The light grey squares together with the square $A$ form the dense $S-$component $C(A)$.

If the event $I(A)$ occurs, then $C(A)$ is distinct from the backbone component and consequently, the plus connected $S-$cycle $L_{\text{cyc}}(A)$ must be contained in $U_{2M+1}(A)$. Moreover, any square of $L_{\text{cyc}}(A)$ contained in the interior of the unit square $S$ shares a corner with some dense square in $C(A)$ and so is sparse. Thus

$$\mathbb{P}(I(A)) \leq \sum_{\pi} \mathbb{P}(L_{\text{cyc}}(A) = \pi), \quad (4.5)$$

where the summation is over all plus connected $S-$cycles contained in $U_{2M+1}(A)$. To evaluate $\mathbb{P}(L_{\text{cyc}}(A) = \pi)$ we consider three cases below depending on where the square $A$ is located.

**Case I** The square $A$ is within a distance $2t_n$ from one of the corners of the unit square $S$. There are at least three sparse squares $Y_{i_1}, Y_{i_2}$ and $Y_{i_3}$ of $\pi$ that lie in the interior of the unit square $S$, each of which is sparse. Letting $E(i_j)$ denote the event that $Y_{i_j}$ is sparse, we get from Eq. 3.2 of Lemma 1 that

$$\mathbb{P}(L_{\text{cyc}}(A) = \pi) \leq \mathbb{P}\left(\bigcap_{j=1}^{3} E(i_j)\right) \leq \frac{C}{n^{3/8} \cdot (\log n)^{3\alpha_1}} \exp\left(-\frac{3\omega_n}{8}\right)$$
for some constant $C > 0$, where $\alpha_1 = \frac{a}{8} - L$. Plugging this estimate into Eq. 4.5 and using the fact that the number of possibilities for $\pi$ depends only on $M$, we get that
\[
P(I(A)) \leq \frac{D}{n^{3/8} \cdot (\log n)^{3\alpha_1}} \exp\left( -\frac{3\omega_n}{8} \right)
\] (4.6)
for some constant $D > 0$.

**Case II** The square $A$ does not belong to case (I) but is within a distance of $3t_n$ from the boundary of $S$. In this case at least 5 squares in the $S$–cycle $L_{cyc}(A)$ lie in the interior of the unit square $S$. Arguing as in Case (I) above and using Eq. 3.2 with $q = 5$ gives
\[
P(I(A)) \leq \frac{D}{n^{5/8} \cdot (\log n)^{5\alpha_1}} \exp\left( -\frac{5\omega_n}{8} \right).
\] (4.7)

**Case III** The square $A$ is at a distance of $3t_n$ away from the boundary of $S$. In this case at least 8 squares in the $S$–cycle $L_{cyc}(A)$ lie in the interior of the unit square $S$. Arguing as in Case (I) above and using Eq. 3.2 with $q = 8$ gives
\[
P(I(A)) \leq \frac{D}{n \cdot (\log n)^{8\alpha_1}} e^{-\omega_n}.
\] (4.8)

Next if $N_j$ is the number of squares satisfying Case $(j)$ for $j \in \{I, II, III\}$, then we have that
\[
N_I \leq 16, N_{II} \leq C \cdot \sqrt{n} \quad \text{and} \quad N_{III} \leq \frac{Cn}{\log n}
\] (4.9)
for some constant $C > 0$. Indeed, the first estimate on $N_I$ is true since there are four corners of $S$. To obtain $N_{II}$, use the fact that the number of squares intersecting the boundary of $S$ and contained in the interior of $S$ is at most $\frac{4}{t_n}$. Therefore the number of squares at a distance of at most $3t_n$ from the boundary of $S$ is at most $\frac{12}{t_n} \leq C \sqrt{n}$, by Eq. 3.1. The final estimate on $N_{III}$ is true since the total number of squares in $\{S_j\}$ contained in the interior of $S$ is $\frac{1}{t_n} \leq \frac{Cn}{\log n}$, again by Eq. 3.1. This proves (4.9).

Plugging the estimates (4.6), (4.7), (4.8) and (4.9) into (4.5) we get
\[
P(I_n) \leq \frac{16D}{n^{3/8} \cdot (\log n)^{3\alpha_1}} e^{-\frac{3\omega_n}{8}} + \frac{CD\sqrt{n}}{n^{5/8} \cdot (\log n)^{5\alpha_1}} e^{-\frac{5\omega_n}{8}}
\]
\[+ \frac{Cn}{\log n} \cdot \frac{D}{n \cdot (\log n)^{8\alpha_1}} e^{-\omega_n}.
\]
For all \( n \) large both \( n^{3/8} \) and \( n^{5/8} \) are much larger than any fixed power of \( \log n \) and so we get that

\[
P(I_n) \leq \frac{D_1}{(\log n)^{8\alpha_1 + 1}} e^{-\frac{3\omega_n}{8}}
\]

for all \( n \) large and some constant \( D_1 > 0 \). Using \( \alpha_1 = \frac{q}{8} - L \) we get (4.4).

4.2. Isolated Sparse Squares

Let \( A \in \{S_j\} \) be any square and let \( J(A) \) be the event that all the squares adjacent to \( A \) and contained in the unit square \( S \) are sparse. Defining

\[
J_n = \bigcup_{A \in \{S_j\}} J(A)
\]

we have that

\[
P(J_n) \leq \frac{C_J}{(\log n)^{\alpha - 8L + 1}} e^{-\frac{3\omega_n}{8}}
\]

for some constant \( C_J > 0 \) and for all \( n \) large. In particular if the event \( J^c_n \) occurs, then every sparse square is adjacent to some dense square.

**Proof of Eq. 4.11.** : We consider cases (I), (II) and (III) as in the previous subsection. In case (I) there are at least three squares adjacent to \( A \) and contained in the unit square. Using Eq. 3.2 with \( q = 3 \) gives

\[
P(J(A)) \leq \frac{C}{n^{3/8} \cdot (\log n)^{3\alpha_1}} e^{-\frac{3\omega_n}{8}} \text{ for Case I}
\]

where \( C > 0 \) is a constant. Similarly for case (II), there are at least 5 squares adjacent to \( A \) and again using Eq. 3.2 with \( q = 5 \) gives

\[
P(J(A)) \leq \frac{C}{n^{5/8} \cdot (\log n)^{5\alpha_1}} e^{-\frac{5\omega_n}{8}} \text{ for Case II.}
\]

Finally, for case (III), there are 8 squares adjacent to \( A \) and so using Eq. 3.2 with \( q = 8 \) gives

\[
P(J(A)) \leq \frac{C}{n \cdot (\log n)^{8\alpha_1}} e^{-\omega_n} \text{ for Case III.}
\]

As before, we let \( N_j \) be the number of squares in \( \{S_k\} \) satisfying Case \((j)\) for \( j \in \{I, II, III\} \). Using the estimates for \( N_I, N_{II} \) and \( N_{III} \) in Eq. 4.9 and arguing as before, we get (4.11).
4.3. Constructing the Hamiltonian Cycle

Define the event

$$H_n = F_n \cap (I_n \cup J_n)^c$$

where $F_n$ is the “backbone” event defined in Eq. 3.6 and the events $I_n$ and $J_n$ are as in Eqs. 4.3 and 4.10, respectively. From Eqs. 3.12, 4.4 and 4.11, we have that

$$\mathbb{P}(H_n) \geq 1 - \frac{1}{n^9} - \frac{(C_I + C_J)e^{-32n}}{(\log n)^{\alpha-8L+1}}$$

(4.13)

for all $n$ large. If the event $H_n$ occurs, then there is a backbone $B$ containing dense squares. Recall that $T(B)$ is the dense $S$–component containing all the squares of $B$. Since the event $I_n^c$ occurs, there is no dense star connected $S$–component other than $T(B)$. Moreover the event $J_n^c$ also occurs and so every sparse square is adjacent (i.e. shares a corner) with some dense square in $T(B)$.

We obtain the desired Hamiltonian cycle as follows. Let $T(B) = \{W_i\}_{1 \leq i \leq t}$ be the set of dense squares in the backbone $B$ and for $1 \leq i \leq t$, let $\eta_i$ be the small cycle of edges containing all the nodes of $\{X_j\}$ present in the square $W_i$. We now obtain a “long” cycle $\tau(B)$ containing all nodes present in the squares of $T(B)$ as follows. We set $\tau_1 = \eta_1$ and get a series of cycles with increasing lengths, using the small cycles $\{\eta_i\}_{1 \leq i \leq t}$. The final cycle $\tau_t$ would then be the desired long cycle in $G$. First we merge the cycles $\eta_1$ and $\tau_2$ as in Fig. 4 by removing one edge each from $\tau_1$ and $\eta_2$ shown by dotted edges and adding the cross edges shown by straight line segments. The resulting cycle is denoted as $\tau_2$.

To continue the iteration, we now argue that for any $2 \leq i \leq t - 1$, the intermediate cycle $\tau_i$ still contains an edge of the small cycle $\eta_{(i)}$ contained in the square $W_{l(i)}$, adjacent to $W_{i+1}$. This would then allow us to perform the merging between $\eta_{i+1}$ and $\tau_i$ as described above, and get the cycle $\tau_{i+1}$. The dense square $W_{l(i)}$ adjacent to $W_{i+1}$ contains at least $L \geq 8$ nodes of $\{X_j\}$ and so the corresponding small cycle $\eta_{l(i)}$ containing all the nodes of $W_{l(i)}$ has at least $L \geq 8$ edges of $G$. There are exactly 8 squares of $\{S_j\}$ adjacent to $W_{l(i)}$ and so apart from $W_{i+1}$ there are at most 7 squares in the intermediate $S$–component $B_i$ that are adjacent to $W_{l(i)}$. This means that at most 7 edges from the small cycle $\eta_{l(i)}$ have been removed so far in the iteration process above.

Continuing this way iteratively for $t$ iterations, we get the final cycle $\tau_t$ that contains all the nodes of the backbone component. We now iteratively expand the cycle $\chi_0 := \tau_t$ by considering sparse squares attached to dense squares in the component $T(B)$. More precisely, let $\{Z_1, \ldots, Z_b\} \subset \{S_j\}$ be
the set of all sparse squares. For $1 \leq j \leq b$, let $\xi_j, 1 \leq j \leq b$ be a path in $G$ containing all the nodes of $\{X_k\}$ in the square $Z_j$. As before, we call $\{\xi_j\}$ as small paths. Starting from $\chi_0$, we iteratively construct a sequence of intermediate cycles $\{\chi_i\}_{1 \leq i \leq b}$ using the paths $\{\xi_j\}_{1 \leq j \leq b}$.

Suppose $Z_1$ is adjacent to the square $W_{k(1)} \in \mathcal{T}(\mathcal{B})$. As argued above, there exists at least $L - 7$ edges of the small cycle $\eta_{k(1)}$ still present in $\chi_0$. Removing one such edge and adding cross edges as in Fig. 4, we join the path $\xi_1$ and $\chi_0$ to get the new cycle $\chi_1$. Repeating the above procedure until all sparse squares are exhausted, we get the final desired Hamilton cycle $H$.

Summarizing, we begin with $\sum_{i=1}^t \# \eta_i$ edges belonging to the small cycles and after the iterative procedure described above, we obtain a Hamiltonian cycle containing $n$ edges. Therefore if $A$ and $R$ denote the total number of edges added and removed in the above process, respectively, then

$$n = \sum_{i=1}^t \# \eta_i - R + A \geq \sum_{i=1}^t \# \eta_i - R.$$  \hspace{1cm} (4.14)

By definition, the edges in the small cycles belong to the graph $G(r_n)$ since $t_n < \frac{r_n}{\sqrt{2}}$ (see Eq. 3.1). Therefore it suffices to find an upper bound for $A$, the total number of cross edges added. In each iteration, we remove exactly one edge belonging to the small cycle $\eta_i$ in some dense square $W_i$
and add two cross edges connecting nodes in \( W_i \) with nodes in a square adjacent to \( W_i \). There are at most eight \( t_n \times t_n \) squares adjacent to \( W_i \) and therefore \( R \leq 8t \) and \( A \leq 16t \). Consequently we also get from Eq. 4.14 that

\[
 n \geq \sum_{i=1}^{t} \#\eta_i - 8t \geq (L - 8)t, \tag{4.15}
\]

since each dense square contains at least \( L \) nodes and therefore the small cycle \( \eta_i \) contains at least \( L \) edges. Thus from Eq. 4.15 we see that \( t \leq \frac{n}{L-8} \) and so the total number of cross edges added is \( A \leq \frac{16n}{L-8} \). By construction, the number bridges in the Hamiltonian cycle \( \mathcal{H} \) is no more than the number of cross edges added and so \( \mathcal{H} \) has a bridge fraction of at most \( \frac{16}{L-8} \).

Acknowledgements. I thank Professors Rahul Roy and Federico Camia for crucial comments and for my fellowships.

Appendix

Writing \( 8nt_n^2 = \theta_n - \gamma_n \) there \( \theta_n = \frac{1}{\epsilon_1} \left( \log n + \alpha \log \log n + \omega_n \right) \), it suffices to show that

\[
 \sqrt{\frac{8n}{\theta_n - 1}} - \sqrt{\frac{8n}{\theta_n}} \geq 1.
\]

Indeed we have that

\[
 \sqrt{\frac{8n}{\theta_n - 1}} - \sqrt{\frac{8n}{\theta_n}} = \sqrt{8n} \left( \frac{\sqrt{\theta_n - \sqrt{\theta_n - 1}}}{\sqrt{\theta_n}} \right) \geq C_1 \sqrt{\frac{n}{\theta_n^2}} \left( \sqrt{\theta_n} - \sqrt{\theta_n - 1} \right) \tag{A.1}
\]

\[
 = C_1 \sqrt{\frac{n}{\theta_n^2}} \left( \frac{1}{\sqrt{\theta_n} + \sqrt{\theta_n - 1}} \right) \geq C_2 \sqrt{\frac{n}{\theta_n^2}} \cdot \frac{1}{\sqrt{\theta_n}} \tag{A.2}
\]

\[
 = C_2 \cdot \sqrt{\frac{n}{\theta_n^3}} \tag{A.3}
\]

for some constants \( C_1, C_2 > 0 \), where Eqs. A.1 and A.2 follow from the fact that \( \theta_n \geq \frac{\log n}{\epsilon_1} \) and so \( \theta_n - 1 \geq \frac{\theta_n}{2} \) for all \( n \) large. Using \( \theta_n \leq D_2 \log n \) for some constant \( D_2 > 0 \) we then get that the final expression in Eq. A.3 is at least 1 for all \( n \) large.
References

BAL, D., BENNETT, P., P-GIMÉNEZ, X. and PRLAT, P. (2017). Rainbow perfect matchings and hamilton cycles in the random geometric graph. Random Structures and Algorithms 51, 587–606.

BALOGH, J., BOLLOBÁS, B., KRIVELEVICH, M., MÜLLER, T. and WALTERS, M. (2011). Hamiltonian cycles in random geometric graphs. Annals of Applied Probability 21, 3, 1053–1072.

DÍAZ, J., MITSCHE, D. and PÉREZ, X. (2007). Sharp threshold for hamiltonicity of random geometric graphs. SIAM Journal of Discrete Mathematics 21, 57–65.

GANESAN, G. (2013). Size of the giant component in a random geometric graph. Annales de l’Institut Henri Poincaré 49, 1130–1140.

GANESAN, G. (2021). Duality and Outermost Boundaries in Generalized Percolation Lattices, Algorithms, Computing and Mathematics (ACM) Conference, Chennai, India (2021), pp. 1–20, Available at: http://ceur-ws.org/Vol-3010/.

MÜLLER, T., P-GIMÉNEZ, X. and WORMALD, N. (2011). Disjoint hamilton cycles in the random geometric graph. Journal of Graph Theory 68, 299–322.

PENROSE, M. (1997). The longest edge of the random minimal spanning tree. Annals of Applied Probability 7, 7340–7361.

PENROSE, M. (2003). Random Geometric Graphs. Oxford University Press.

Publisher’s Note. Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.

Ghurumuruhan Ganesan
Institute of Mathematical Sciences,
HBNI, Chennai, China
Present address: Indian Institute of Science Education and Research (IISER), Bhopal, India
E-mail: gganesan82@gmail.com

Paper received: 2 June 2021; accepted 23 November 2021.