Radial Two Weight Inequality for Maximal Bergman Projection Induced by a Regular Weight

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Abstract

It is shown in quantitative terms that the maximal Bergman projection

$$P^+_{\omega}(f)(z) = \int_{D} f(\zeta) |B^\omega_\zeta(\zeta)| \omega(\zeta) \, dA(\zeta),$$

is bounded from $L^p_\nu$ to $L^p_\eta$ if and only if

$$\sup_{0 < r < 1} \left( \int_0^r \left( \frac{\eta(s)}{\int_0^s \omega(t) \, dt} \right)^{\frac{1}{p'}} \, ds + 1 \right)^{\frac{1}{p'}} \left( \int_r^1 \left( \frac{\omega(s)}{v(s)^{\frac{1}{p}}} \right)^{p'} \, ds \right)^{\frac{1}{p'}} < \infty,$$

provided $\omega, \nu, \eta$ are radial regular weights. A radial weight $\sigma$ is regular if it satisfies $\sigma(r) = \int_0^r \sigma(t) \, dt / (1 - r)$ for all $0 \leq r < 1$. It is also shown that under an appropriate additional hypothesis involving $\omega$ and $\eta$, the Bergman projection $P_\omega$ and $P^+_{\omega}$ are simultaneously bounded.

Keywords Bergman projection · Bergman space · Regular weight · Two weight inequality

Mathematics Subject Classification (2010) 30H20 · 46E30 · 47

Dedicated to Fernando Pérez-González on the occasion of his retirement

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1 Introduction and Main Results

A function $\omega : \mathbb{D} \rightarrow [0, \infty)$, integrable over the unit disc $\mathbb{D}$, is called a weight. It is radial if $\omega(z) = \omega(|z|)$ for all $z \in \mathbb{D}$. For $0 < p < \infty$ and a weight $\omega$, the Lebesgue space $L^p_\omega$ consists of complex-valued measurable functions $f$ in $\mathbb{D}$ such that

$$\|f\|_{L^p_\omega} = \left( \int_\mathbb{D} |f(z)|^p \omega(z) \, dA(z) \right)^{\frac{1}{p}} < \infty,$$

where $dA(z) = \frac{dx \, dy}{\pi}$ denotes the element of the normalized Lebesgue area measure on $\mathbb{D}$. The weighted Bergman space $A^p_\omega$ is the space of analytic functions in $L^p_\omega$, and is equipped with the corresponding $L^p_\omega$-norm. If the norm convergence in the Hilbert space $A^2_\omega$ implies the uniform convergence on compact subsets of $\mathbb{D}$, the point evaluations are bounded linear functionals on $A^2_\omega$. Therefore there exist reproducing Bergman kernels $B_\omega^p \in A^2_\omega$ such that

$$f(z) = \langle f, B_\omega^p \rangle_{A^2_\omega} = \int_\mathbb{D} f(\xi) \overline{B_\omega^p(\xi)} \omega(\xi) \, dA(\xi), \quad z \in \mathbb{D}, \quad f \in A^2_\omega.$$

The Hilbert space $A^2_\omega$ is a closed subspace of $L^2_\omega$, and hence the orthogonal projection from $L^2_\omega$ to $A^2_\omega$ is given by

$$P_\omega(f)(z) = \int_\mathbb{D} f(\xi) \overline{B_\omega(\xi)} \omega(\xi) \, dA(\xi), \quad z \in \mathbb{D}.$$

The operator $P_\omega$ is the Bergman projection.

In this paper we will characterize the radial two-weight inequality

$$\|P_\omega^+(f)\|_{L^p_\nu} \leq C \|f\|_{L^p_\nu}, \quad f \in L^p_\nu, \tag{1.1}$$

for the maximal Bergman projection $P_\omega^+(f)(z) = \int_\mathbb{D} f(\xi) |B_\omega^p(\xi)| \omega(\xi) \, dA(\xi)$ under certain smoothness requirements on the three radial weights involved. The question of when (1.1) is satisfied is an open problem even in the very particular case $\omega = \nu = \eta$ if no preliminary hypotheses is imposed on the radial weight.

Two weight inequalities for classical operators have attracted a considerable amount of attention in Complex and Harmonic Analysis, and are closely connected to other interesting questions in the area [2, 5–7, 10, 11]. The most commonly known result on Bergman projection is due to Bekollé and Bonami [3, 4], and concerns the case when $\nu = \eta$ is an arbitrary weight and the inducing weight $\omega$ is standard, that is, of the form $\omega(z) = (1 - |z|^2)\alpha$ for some $\alpha > -1$; see [1, 11, 13] for recent extensions of this result. In this classical case, the Bergman reproducing kernel $B_\omega^p(\xi)$ is given by the neat formula $(1 - \overline{\xi} \zeta)^{-\frac{1}{2}+\alpha}$. However, for a general radial weight $\omega$ such explicit formulas for the kernels do not necessarily exist, and that is one of the main obstacles in tackling (1.1). Moreover, kernels induced by radial weights may have zeros, and that of course does not make things any easier. Nonetheless, (1.1) has been recently characterized in the particular case $\nu = \eta$ provided $\omega$ and $\nu$ are regular weights [10].

For a radial weight $\omega$, we assume throughout the paper that $\hat{\omega}(z) = \int_{|z|}^1 \omega(s) \, ds > 0$ for all $z \in \mathbb{D}$, for otherwise the Bergman space $A^2_\omega$ would contain all analytic functions in $\mathbb{D}$. A radial weight $\omega$ belongs to the class $\tilde{\mathcal{D}}$ if there exists a constant $C = C(\omega) > 1$ such
that \( \hat{\omega}(r) \leq C \hat{\omega} \left( \frac{1+r}{2} \right) \) for all \( 0 \leq r < 1 \). Moreover, if there exist \( K = K(\omega) > 1 \) and \( C = C(\omega) > 1 \) such that
\[
\hat{\omega}(r) \geq C \hat{\omega} \left( 1 - \frac{1-r}{K} \right), \quad 0 \leq r < 1,
\]
then we write \( \omega \in \tilde{D} \). The intersection \( \tilde{D} \cap \tilde{D} \) is denoted by \( D \). A radial weight \( \omega \) is regular if \( \hat{\omega}(r) = \omega(r)(1-r) \) for all \( 0 \leq r < 1 \). The class of regular weights is denoted by \( R \), and \( R \subseteq \tilde{D} \). For basic properties of these classes of weights and more, see [8, 9, 12] and the references therein.

The main result of this study is the following theorem, which provides a quantitative description of the boundedness of \( P^+_{\omega} : L^p_\nu \rightarrow L^p_\eta \) in terms of a Muckenhoupt-type condition related to weighted Hardy operators.

**Theorem 1** Let \( 1 < p < \infty \), \( \omega, \nu \in R \) and \( \eta \in \tilde{D} \). Then \( P^+_{\omega} : L^p_\nu \rightarrow L^p_\eta \) is bounded if and only if
\[
M_p(\omega, \nu, \eta) = \sup_{0 < r < 1} \left( \int_0^r \frac{\eta(s)}{\omega(t)^p} \frac{ds}{s} + 1 \right)^{\frac{1}{p}} \left( \int_r^1 \left( \frac{\omega(s)^{1/p}}{\nu(s)^{1/p}} \right) ds \right)^{\frac{1}{p^{1/p}}} < \infty. \tag{1.3}
\]

Moreover, \( \| P^+_{\omega} \|_{L^p_\nu \rightarrow L^p_\eta} = M_p(\omega, \nu, \eta) \).

The key tools in the proof of Theorem 1 are the precise estimates for the \( L^p \)-means and \( A^p \)-norms of the Bergman kernel \( B^z_\omega \) obtained in [10, Theorem 1]. A special case of the said result is repeatedly used in the proof and it is stated for further reference as Theorem A below. For a function \( f \) analytic in \( D \) and \( 0 < r < 1 \), write
\[
M_p(r, f) = \left( \frac{1}{2\pi} \int_0^{2\pi} |f(re^{it})|^p dt \right)^{\frac{1}{p}}, \quad 0 < p < \infty.
\]

**Theorem A** Let \( 0 < p < \infty \) and \( \omega, \nu \in \tilde{D} \). Then the following assertions hold:

(i) \( M_p^\omega(r, B^w_\alpha) = \int_0^{[\alpha]} \frac{dt}{\omega(t)^p(1-t)^p} \), \( r, |a| \rightarrow 1^- \);

(ii) \( \| B^w_\alpha \|^p_{A^p_\nu} = \int_0^{[\alpha]} \frac{\tilde{\nu}(t)}{\omega(t)^p(1-t)^p} dt \), \( |a| \rightarrow 1^- \).

It is worth noticing that the upper estimate
\[
\| B^w_\alpha \|^p_{A^p_\nu} \leq \int_0^{[\alpha]} \frac{\tilde{\nu}(t)}{\omega(t)^p(1-t)^p} dt, \quad |a| \rightarrow 1^- \tag{1.4}
\]
holds for \( \omega \in \tilde{D} \) and any radial weight \( \nu \) [10, p. 106].

The argument used to establish the one weight inequality [10, Theorem 3] for regular weights does not carry over as such to the two weight case. The proof of the sufficiency in Theorem 1 is much more involved due to the presence of the second weight \( \eta \).

The operators \( P_\omega \) and \( P^+_{\omega} \) are simultaneously bounded under a natural additional hypothesis. This is the content of the other main result of this study.
Theorem 2 Let $1 < p < \infty$, $\omega, \nu \in \mathcal{R}$ and $\eta \in \mathcal{D}$ such that
\[
\sup_{0 < r < 1} \left( \int_0^r \left( \frac{\omega(s)}{\eta(s)^{1/p}} \right)^{p'} \left( \int_1^r \left( \frac{\omega(t)}{v(t)^{1/p}} \right)^{p'} \, dt \right)^{-p} \, ds \right)^{1/p} < \infty. \tag{1.5}
\]

Then the following statements are equivalent:

(i) $P_{\omega}^+ : L^p_\nu \to L^p_\nu$ is bounded;
(ii) $P_{\omega} : L^p_\nu \to L^p_\nu$ is bounded;
(iii) $N_p(\omega, \nu, \eta) = \sup_{0 < r < 1} \frac{\eta(r)}{\omega(r)} \left( \int_r^1 \left( \frac{\omega(s)}{v(s)^{1/p}} \right)^{p'} \, ds \right)^{1/p} < \infty$;
(iv) $M_p(\omega, \nu, \eta) < \infty$.

Theorem 2 is a generalization of [10, Theorem 3] because the hypothesis (1.5) is satisfied for $\nu = \eta$. Although, the conditions $N_p(\omega, \nu, \eta) < \infty$ and $M_p(\omega, \nu, \eta) < \infty$ are equivalent for many weights, for example the standard weights have this property, the condition $N_p(\omega, \nu, \eta) < \infty$ may be essentially weaker than $M_p(\omega, \nu, \eta) < \infty$ under the hypothesis of Theorem 1. Indeed, if we pick up an arbitrary $\omega \in \mathcal{R}$, and define $v(s) = \omega(s)\hat{\omega}(s)^{p'} \left( \log \frac{e}{1-s} \right)^{2p'}$ and $\eta(s) = \omega(s)\hat{\omega}(s)^{p'} \left( \log \frac{e}{1-s} \right)^{p'}$, then $\nu, \eta \in \mathcal{R}$ and one can show by using Lemmas B and C below that $N_p(\omega, \nu, \eta) < \infty$ but $M_p(\omega, \nu, \eta) = \infty$. Now of course (1.5) fails for these choices of weights.

It is readily seen that the methods used to prove Theorems 1 and 2 carry over to the case $p = 1$. In fact, the proof in this case turns out much more simple for obvious reasons. To be precise, one can show that in the case $p = 1$ the operators $P_{\omega}$ and $P_{\omega}^+$ are simultaneously bounded, and the uniform boundness of the quantity
\[
\frac{\omega(r)}{v(r)} \int_0^r \frac{\eta(t)}{\hat{\omega}(tr)} \, dt
\]
is the characterizing condition.

Throughout the paper $\frac{1}{p} + \frac{1}{p'} = 1$ for $1 < p < \infty$. Further, the letter $C = C(\cdot)$ will denote an absolute constant whose value depends on the parameters indicated in the parenthesis, and may change from one occurrence to another. We will use the notation $a \lesssim b$ if there exists a constant $C = C(\cdot) > 0$ such that $a \leq Cb$, and $a \gtrsim b$ is understood in an analogous manner. In particular, if $a \lesssim b$ and $a \gtrsim b$, then we will write $a \asymp b$.

2 Proof of Theorem 1

Throughout the proofs we will repeatedly use several basic properties of weights in the classes $\hat{\mathcal{D}}$ and $\hat{\mathcal{D}}$, gathered in the following two lemmas. For a proof of the first lemma, see [8, Lemma 2.1]; the second one can be proved by similar arguments. For each radial weight $\omega$ and $x \geq 1$, we write $\omega_x = \int_0^x s^x \omega(s) \, ds$.

Lemma B Let $\omega$ be a radial weight. Then the following statements are equivalent:

(i) $\omega \in \hat{\mathcal{D}}$;
(ii) There exist constants $C = C(\omega) > 0$ and $\beta = \beta(\omega) > 0$ such that

$$\hat{\omega}(r) \leq C \left( \frac{1-r}{1-t} \right)^{\beta} \hat{\omega}(t), \quad 0 \leq r \leq t < 1;$$

(iii) There exists a constant $C = C(\omega) > 0$ such that

$$\omega_x \leq C \hat{\omega} \left( 1 - \frac{1}{x} \right), \quad 1 \leq x < \infty.$$

**Lemma C** Let $\omega$ be a radial weight. Then $\omega \in \hat{D}$ if and only if there exist $C = C(\omega) > 0$ and $\gamma = \gamma(\omega) > 0$ such that

$$\hat{\omega}(t) \leq C \left( \frac{1-t}{1-r} \right)^{\gamma} \hat{\omega}(r), \quad 0 \leq r \leq t < 1.$$

Lemma B (ii) shows that if $\omega \in \hat{D}$, then there exists $\beta = \beta(\omega) > 0$ such that $\frac{\hat{\omega}(r)}{(1-r)^{\beta}}$ is essentially increasing on $[0, 1)$. Similarly, by Lemma C, $\frac{\hat{\omega}(r)}{(1-r)^{\gamma}}$ is essentially decreasing on $[0, 1)$ for $\gamma = \gamma(\omega) > 0$ sufficiently small if $\omega \in \hat{D}$.

### 2.1 Necessity

In this section we prove that $M_p(\omega, \nu, \eta) < \infty$ is a necessary condition for $P^+_\omega : L^p_\nu \to L^p_\eta$ to be bounded under the hypotheses of Theorem 1, and establish the desired lower estimate for the operator norm. This is done in the following result under slightly weaker hypotheses than those of the theorem, by using an appropriate family of test functions depending on the weights $\omega$ and $\nu$.

**Proposition 3** Let $1 < p < \infty$, $\omega \in \hat{D}$ and $\nu, \eta$ radial weights. If $P^+_\omega : L^p_\nu \to L^p_\eta$ is bounded, then

$$\sup_{0 < r < 1} \left( \int_0^r J_\omega(s)^p \eta(s) ds + 1 \right)^{1/p} \left( \int_r^1 \left( \frac{\omega(s)}{\nu(s)^{1/p}} \right)^{p'} ds \right)^{1/p'} \leq \|P^+_\omega\|_{L^p_\nu \to L^p_\eta} < \infty,$$

where $J_\omega(s) = \int_0^s \frac{dt}{\omega(t)^{1-1/p}}$ for all $0 \leq s < 1$.

**Proof** Assume that $P^+_\omega : L^p_\nu \to L^p_\eta$ is bounded, that is,

$$\int_D \int_D f(\xi) |B^\omega_\zeta(\xi)| \omega(\zeta) dA(\zeta) \leq \|P^+_\omega\|_{L^p_\nu \to L^p_\eta} \|f\|_{L^p_\nu}, \quad f \in L^p_\nu, \quad (2.1)$$

with $\|P^+_\omega\|_{L^p_\nu \to L^p_\eta} < \infty$. If $\nu$ vanishes on a set $E \subset D$ of positive measure, then by choosing $f = \chi_E$ the right side of (2.1) is zero. It follows that $\omega$ vanishes (almost everywhere) on $E$ or else $\eta \equiv 0$ (almost everywhere) on $D$. The latter option being unacceptable as $\hat{\eta}(r) > 0$ for all $0 \leq r < 1$, we deduce that $\omega dA$ is absolutely continuous with respect to $\nu dA$. Therefore $\omega/\nu$ is well defined almost everywhere. Hence, for each $n \in \mathbb{N}$ and $0 \leq t < 1$, the function $f_{n,t} = \min \left\{ n, \left( \frac{\omega}{\nu} \right)^{-1} \right\} \chi_{D \setminus D(0,t)}$ belongs to $L^p_\nu$. A direct calculation shows that
Theorem 1, we have
\[ J_\omega(r) + 1 = J_\omega(r^2) + 1 \] for all \( 0 \leq r < 1 \). Therefore

If we ignore the first summand on the last expression and apply the monotone convergence theorem, we deduce \( \int_0^1 \left( \int_r^1 f_{n,t}(s) \omega(s) \, ds \right)^p \eta(r) \, dr \) < \( \infty \). This explains why in the statement

the supremum over \( (0, r_0) \) only, with a prefixed \( r_0 \in (0, 1) \), is bounded by a constant \( C = C(r_0) > 0 \) times \( \| P_{\omega}^+ \|_{L^p_{\omega} \to L^p} \). Further, an application of Lemma B (ii) to \( \omega \in \hat{D} \) gives

\[ J_\omega(r) + 1 = J_\omega(r^2) + 1 \] for all \( 0 \leq r < 1 \). Therefore

because \( f_{n,t} \leq f_{n,t} \omega \) on \( D \). This together with the monotone convergence theorem shows that

\[ \| P_{\omega}^+ \|_{L^p_{\omega} \to L^p} \geq \left( \int_0^1 \eta(r) (J_\omega(r) + 1)^p r \, dr \right)^{\frac{1}{p}} \left( \int_0^1 \left( \frac{\omega(s)}{v(s)^\frac{1}{p}} \right)^{p'} \, ds \right)^{\frac{1}{p'}} , \quad 0 < t < 1 , \]

and the proposition is proved.

If \( \omega \in \hat{D} \), then by Lemma B (ii) there exists \( \beta(\omega) > 0 \) such that \( J_\omega(r) \geq \tilde{\omega}(r)^{-1}(1 - (1 - r)^\beta) \) for all \( 0 \leq r < 1 \). Therefore, under the hypotheses of Theorem 1, we have

\[ \| P_{\omega}^+ \|_{L^p_{\omega} \to L^p} \geq M_p(\omega, v, \eta) , \]

and thus the necessity part is proved.

2.2 Sufficiency

The proof of the sufficiency of \( M_p(\omega, v, \eta) < \infty \) for \( P_{\omega}^+ : L^p_{\omega} \to L^p \) to be bounded is more involved than that of the necessity. We begin with the following technical lemma.
Lemma 4 Let $1 < p < \infty$ and $\omega \in \mathcal{D}$ and $\nu \in \tilde{\mathcal{D}}$ such that

$$\int_0^1 \left( \frac{\hat{\omega}(s)}{\hat{\nu}(s)^{1/p}} \right)^{p'} \frac{ds}{1-s} < \infty.$$ 

Then

$$\int_0^r \left( \int_t^1 \left( \frac{\hat{\omega}(s)}{\hat{\nu}(s)^{1/p}} \right)^{p'} \frac{ds}{1-s} \right)^{\frac{1}{p'}} \frac{dt}{\hat{\omega}(t)(1-t)} \leq \left( \int_r^1 \left( \frac{\hat{\omega}(s)}{\hat{\nu}(s)^{1/p}} \right)^{p'} \frac{ds}{1-s} \right)^{\frac{1}{p'}} \frac{1}{\hat{\omega}(r)(1-t)}, \quad 0 \leq r < 1.$$ 

Proof Let $\alpha = \alpha(\nu, \omega, p) \in (0, 1)$ to be appropriately fixed later. Then Hölder’s inequality and Lemma C yield

$$\int_0^r \left( \int_t^1 \left( \frac{\hat{\omega}(s)}{\hat{\nu}(s)^{1/p}} \right)^{p'} \frac{ds}{1-s} \right)^{\frac{1}{p'}} \frac{dt}{\hat{\omega}(t)(1-t)} \leq \left( \int_0^r \frac{dt}{\hat{\omega}(t)^p(1-\alpha)(1-t)} \right)^{\frac{1}{p'}} \left( \int_0^r \frac{dt}{\hat{\omega}(t)^p(1-\alpha)(1-t)} \right)^{1 - \frac{1}{p'}} \frac{1}{\hat{\omega}(r)(1-\alpha)}, \quad 0 \leq r < 1, \quad (2.3)$$

where, by Fubini’s theorem and Lemma C,

$$\int_0^r \left( \int_t^1 \left( \frac{\hat{\omega}(s)}{\hat{\nu}(s)^{1/p}} \right)^{p'} \frac{ds}{1-s} \right)^{\frac{1}{p'}} \frac{dt}{\hat{\omega}(t)^p(1-\alpha)(1-t)} = \int_0^r \left( \frac{\hat{\omega}(s)}{\hat{\nu}(s)^{1/p}} \right)^{p'} \left( \int_0^s \frac{dt}{\hat{\omega}(t)^p(1-\alpha)(1-t)} \right) \frac{ds}{1-s}$$

$$+ \left( \int_r^1 \left( \frac{\hat{\omega}(s)}{\hat{\nu}(s)^{1/p}} \right)^{p'} \frac{ds}{1-s} \right) \left( \int_0^r \frac{dt}{\hat{\omega}(t)^p(1-\alpha)(1-t)} \right)$$

$$\leq \int_0^r \left( \frac{\hat{\omega}(s)^{1-\alpha}}{\hat{\nu}(s)^{1/p}} \right)^{p'} \frac{ds}{1-s} + \left( \int_r^1 \left( \frac{\hat{\omega}(s)}{\hat{\nu}(s)^{1/p}} \right)^{p'} \frac{ds}{1-s} \right) \frac{1}{\hat{\omega}(r)^p(1-\alpha)}, \quad (2.4)$$

for all $0 \leq r < 1$. The latter term is of the desired form. To deal with the first term, observe first that by Lemma C (ii) there exists a constant $\beta = \beta(\omega) > 0$ such that $\frac{\hat{\omega}(r)}{(1-r)^p}$ is essentially increasing on $[0, 1)$. Further, for each sufficiently small $\gamma = \gamma(\nu) > 0$ the
function \( \frac{\nu(r)}{(1-r)^\gamma} \) is essentially decreasing on \([0, 1]\) by Lemma C. Pick up such a \( \gamma \) from the interval \((0, \beta\gamma)\), and fix \( \alpha \in (1 - \frac{\gamma}{p\beta}, 1) \). Then

\[
\int_0^r \left( \frac{\omega(s)}{\nu(s)} \right)^{p' \alpha} \frac{ds}{1-s} \leq \left( \frac{\omega(r)}{\nu(r)} \right)^{p' \alpha} \int_0^r \left( \frac{(1-s)^{\beta(1-\alpha)}}{\nu(s)} \right)^{p'} \frac{ds}{1-s}
\]

\[
\leq \left( \frac{\omega(r)}{\nu(r)} \right)^{p' \alpha} \left( \frac{1}{\nu(r)^{p' \alpha(1-\alpha)}} \right) \int_0^r \frac{ds}{1-s} \leq \frac{1}{\omega(r)^{p' \alpha}}.
\]

which together with (2.4) gives

\[
\int_0^r \left( \int_t^1 \left( \frac{\omega(s)}{\nu(s)} \right)^{p' \alpha} \frac{ds}{1-s} \right) dt \leq \frac{1}{\omega(r)^{p' \alpha}}.
\]

Finally, by combining the above inequality with (2.3) we obtain the claim. □

We are now ready to prove the sufficiency part of Theorem 1. To do this, assume \( M_p(\omega, \nu, \eta) < \infty \), and observe that then the function \( h(z) = \nu(z)^{\frac{1}{p'}} \)

\[
\left( \int_{|z|} \left( \frac{\omega(s)}{\nu(s)} \right)^{p'} v(s) ds \right)^{\frac{1}{p'}}
\]

is well defined for all \( z \in \mathbb{D} \). Hence an integration shows that

\[
\int_t^1 \left( \frac{\omega(s)}{h(s)} \right)^{p'} \frac{ds}{v(s)} = p' \left( \int_t^1 \left( \frac{\omega(s)}{v(s)} \right)^{p'} v(s) ds \right)^{\frac{1}{p'}} = \frac{1}{\omega(t)^{p' \alpha}}, \quad 0 \leq t < 1.
\] (2.5)

Hölder’s inequality yields

\[
\| P_{\omega} (f) \|_{L^p_0} \leq \int_{\mathbb{D}} \left( \int_{\mathbb{D}} |f(\xi)|^p h(\xi)^p |B^\omega(\xi)| dA(\xi) \right)^{\frac{1}{p}} \cdot \left( \int_{\mathbb{D}} |B^\omega(\xi)| \left( \frac{\omega(\xi)}{h(\xi)} \right)^{p'} dA(\xi) \right)^{\frac{1}{p'}} \eta(z) dA(z),
\] (2.6)

where, by (2.2), Fubini’s theorem and (2.5),

\[
\int_{\mathbb{D}} |B^\omega(\xi)| \left( \frac{\omega(\xi)}{h(\xi)} \right)^{p'} dA(\xi) \leq \int_0^1 \left( \int_{|z|} \left( \frac{\omega(s)}{h(s)} \right)^{p'} \left( \int_0^{|z|} \frac{dt}{\omega(t)(1-t)} \right) ds + M_p(\omega, v, \eta) \right) \left( \int_t^1 \left( \frac{\omega(s)}{v(s)} \right)^{p'} v(s) ds \right)^{\frac{1}{p'}} \frac{dt}{\omega(t)(1-t)} + M_p(\omega, v, \eta).
\]
This together with (2.6), Fubini’s theorem and another application of Theorem A (i) gives
\[ \|P_{\omega}^+(f)\|_{L^p} \lesssim \int_D \left( \int_D |f(\xi)|^{p^*} \, dA(\xi) \right) \left( \int_0^{\infty} \left( \int_0^1 \left( \frac{\omega(s)}{v(s)} \right)^{\rho'} v(s) \, ds \right)^{\frac{1}{\rho'}} \frac{dt}{\omega(t)(1-t)} \right)^{\frac{p}{p'}} \eta(z) \, dA(z) + M_{p^*}^{-1}(\omega, \nu, \eta) I_1(f), \]
where
\[ I_1(f) = \int_D \left( \int_D |f(\xi)|^{p^*} \, dA(\xi) \right) \eta(z) \, dA(z) \]
and
\[ I_2(f) = \int_D |f(\xi)|^{p^*} \left( \int_0^1 \left( \frac{\omega(s)}{v(s)} \right)^{\rho'} v(s) \, ds \right)^{\frac{1}{p'}} \frac{dt}{\omega(t)(1-t)} \eta(\tau r) \, dA(\xi). \]
By (1.4),
\[ \|B_{\omega}^\nu\|_{A^1_p} \lesssim 1 + \int_0^{\infty} \frac{\eta(t)}{\omega(t)(1-t)} \, dt, \quad \xi \in \mathbb{D}. \tag{2.8} \]
This together with Fubini’s theorem and Hölder’s inequality, implies
\[ I_1(f) = \int_D \left( \int_D |f(\xi)|^{p^*} \, dA(\xi) \right) \eta(z) \, dA(z) \]
\[ \lesssim \int_D |f(\xi)|^{p^*} \nu(\xi) \left( \int_0^1 \left( \frac{\omega(s)}{v(s)} \right)^{\rho'} v(s) \, ds \right)^{\frac{1}{p'}} \left( \int_0^{\infty} \frac{\eta(t)}{\omega(t)(1-t)} \, dt \right) \, dA(\xi) + M_{p^*}(\omega, \nu, \eta) \|f\|_{L^{\infty}}^{p^*} \]
\[ \lesssim \int_D |f(\xi)|^{p^*} \nu(\xi) \left( \int_0^1 \left( \frac{\omega(s)}{v(s)} \right)^{\rho'} v(s) \, ds \right)^{\frac{1}{p'}} \left( \int_0^{\infty} \frac{\eta(t)}{\omega(t)^{p^*}(1-t)} \, dt \right)^{\frac{1}{p^*}} \]
\[ + M_{p^*}(\omega, \nu, \eta) \|f\|_{L^{\infty}}^{p^*} \tag{2.9} \]
because \( \frac{\eta(\xi)}{(1-|\xi|)} \) is a weight by the hypothesis \( \eta \in \mathcal{D} \) and Lemma C. Fubini’s theorem and Lemma C for \( \omega \in \mathcal{D} \) give
\[ \int_0^{\infty} \frac{\eta(t)}{\omega(t)^{p^*}(1-t)} \, dt \lesssim \frac{\eta(\xi)}{\omega(\xi)^{p^*}} + \int_0^{\infty} \frac{\eta(t)}{\omega(t)^{p^*}} \, dt, \quad \xi \in \mathbb{D}. \]
But since \( \eta \in \mathcal{D} \) by the hypothesis, there exists a constant \( K = K(\eta) > 1 \) such that \( \eta \) satisfies (1.2). Hence, by using Lemma B (ii) for \( \omega \in \mathcal{D} \), we deduce
\[ \frac{\eta(s)}{\omega(s)^p} \, ds \geq \frac{\eta(s)}{\omega(1-K(1-r))^p} \, ds \geq \frac{\eta(s)}{\omega(1-K(1-r))^p} \, ds \geq \frac{\eta(r)}{\omega(r)^p}, \quad r \geq 1 - K^{-1}, \]
and hence
\[
\sup_{0 < r < 1} \frac{\hat{\eta}(r)}{\hat{\omega}(r)} \left( \int_r^1 \left( \frac{\omega(s)}{v(s)} \right)^{p'} \frac{1}{p'} \, ds \right)^{1/p'} \leq M_p(\omega, v, \eta) < \infty. \tag{2.10}
\]

It follows that \( I_1(f) \leq M_p(\omega, v, \eta) \left\| f \right\|_{L_p}^p \).

To deal with the remaining terms, we split the integral over \((0, 1)\) in (2.7) into two parts at \(|\xi|\). On one hand, since \(\omega, \eta, \nu \), and hence \(8\), yield
\[
M_{\hat{\omega}}(\omega, \nu, \eta, \hat{\eta}(\tau)) \leq M^p_p(\omega, v, \eta, \hat{\eta}(\xi)^{p/2}, \xi \in \mathbb{D}. \tag{2.11}
\]
Therefore, by using (2.10) again we deduce
\[
\hat{h}(\xi) \int_{|\xi|}^{r} \left( \int_0^1 \left( \frac{\omega(s)}{v(s)} \right)^{p'} \frac{1}{p'} \, ds \right)^{1/p'} \frac{dt}{\hat{\omega}(t)(1 - t)} \left( \int_0^{r|\xi|} \frac{dx}{\hat{\omega}(x)(1 - x)} \right) \eta(r) \, dr
\]
\[
\leq M^p_p(\omega, v, \eta) \int_{|\xi|}^{r} \left( \int_0^1 \left( \frac{\omega(s)}{v(s)} \right)^{p'} \frac{1}{p'} \, ds \right)^{1/p'} \frac{dt}{\hat{\omega}(t)(1 - t)} \left( \int_0^{r|\xi|} \frac{dx}{\hat{\omega}(x)(1 - x)} \right) \eta(r) \, dr
\]
\[
\leq M^p_p(\omega, v, \eta) \int_{|\xi|}^{r} \left( \int_0^1 \left( \frac{\omega(s)}{v(s)} \right)^{p'} \frac{1}{p'} \, ds \right)^{1/p'} \frac{\hat{\eta}(\xi)^{p/2}}{\hat{\omega}(\xi)} \leq M^p_p(\omega, v, \eta)v(\xi), \xi \in \mathbb{D}.
\]

On the other hand, since \(\omega, v \in \mathcal{R} \), Lemma 4, Lemma C and the hypothesis \(M_p(\omega, v, \eta) < \infty\) yield
\[
\int_0^{r|\xi|} \left( \int_0^1 \left( \frac{\omega(s)}{v(s)} \right)^{p'} \frac{1}{p'} \, ds \right)^{1/p'} \frac{dt}{\hat{\omega}(t)(1 - t)} \left( \int_0^{r|\xi|} \frac{dx}{\hat{\omega}(x)(1 - x)} \right) \eta(r) \, dr
\]
\[
\leq \int_0^{r|\xi|} \left( \int_0^1 \left( \frac{\omega(s)}{v(s)} \right)^{p'} \frac{1}{p'} \, ds \right)^{1/p'} \frac{1}{\hat{\omega}(r)} \left( \int_0^{r} \frac{dx}{\hat{\omega}(x)(1 - x)} \right) \eta(r) \, dr
\]
\[
\leq M^p_p(\omega, v, \eta) \int_0^{r|\xi|} \frac{\eta(r)}{\hat{\omega}(r)p} \left( \int_0^{r|\xi|} \frac{\omega(s)}{v(s)} \, ds \right)^{1/p} \frac{dr}{\hat{\omega}(\xi)^{p/2}}, \xi \in \mathbb{D}.
\]
This together with the hypothesis $M_p(\omega, \upsilon, \eta) < \infty$ gives

$$h^p(\zeta) \left( \int_0^{\infty} \left( \int_0^{x(t)} \frac{\omega(s)}{v(s)} v(t) \, dt \right)^{\frac{1}{p'}} \frac{dt}{\omega(t)(1-t)} \right) \left( \int_0^{\infty} \frac{dx}{\omega(x)(1-x)} \right) \eta(r) \, dr$$

$$\leq M_p^p(\omega, \upsilon, \eta, \nu, \eta) \left( \int_0^{\infty} \left( \int_0^{z(t)} \frac{\eta(s)}{\omega(s)^p} \, ds \right)^{\frac{1}{p'}} \left( \int_0^{\infty} \frac{\eta(s)}{\omega(s)^p} \, ds \right)^{\frac{1}{p'}} \right) \approx M_p^p(\omega, \upsilon, \eta, \nu) \nu(\zeta), \quad \zeta \in \mathbb{D}.$$ 

Consequently, by combining the previous estimates, we deduce that the third to last term in (2.7) is bounded by a constant times $M_p^p(\omega, \upsilon, \eta) \| f \|_{L_p^p}$. In order to bound $I_2(f)$, let us observe that the third to last term in (2.7) differs from $I_2(f)$ only by the extra factor $\left( \int_0^{\infty} \frac{dx}{\omega(x)(1-x)} \right)$, which has been bounded by a constant times $\max\{\frac{1}{\omega(r)}, \frac{1}{\omega(|x|)}\}$ in the calculations above. Since $\max\{\frac{1}{\omega(r)}, \frac{1}{\omega(|x|)}\}$ is uniformly bounded away from zero on $\mathbb{D}$, the same reasoning shows that $I_2(f) \approx M_p^p(\omega, \upsilon, \eta) \| f \|_{L_p^p}$. These inequalities, together with (2.7) and (2.9) give $\| P_{\omega}^+ (f) \|_{L_p^p} \lesssim M_p(\omega, \upsilon, \eta) \| f \|_{L_p^p}$ as claimed.

## 3 Proof of Theorem 2

To prove Theorem 2, we first show that $N_p(\omega, \upsilon, \eta) \lesssim \| P_\omega \|_{L_p^p}$ under the hypotheses of the theorem.

**Proposition 5** Let $1 < p < \infty$, $\omega \in \widehat{\mathcal{D}}$, $\eta \in \mathcal{D}$ and $\upsilon$ a radial weight. If $P_\omega : L_v^p \rightarrow L_\eta^p$ is bounded, then

$$\sup_{0 < r < 1} \frac{\hat{\eta}(r)}{\omega(r)} \left( \int_0^1 \left( \frac{\omega(t)}{v(t)} \right)^{\frac{1}{p'}} \upsilon(t) \, dt \right)^{\frac{1}{p'}} \lesssim \| P_\omega \|_{L_p^p \rightarrow L_p^p}.$$ 

**Proof** The adjoint of $P_\omega$ is defined by

$$\langle P_\omega (f), g \rangle_{L_\eta^p} = \langle f, P_\omega^* (g) \rangle_{L_v^p}, \quad f \in L_v^p, \quad g \in L_\eta^p.$$ 

Now [10, Theorem 1(i)] and Lemma C, applied to $\eta \in \widehat{\mathcal{D}}$, yield

$$\int_{\mathbb{D}} \left( \int_{\mathbb{D}} |B_\zeta^\omega(z)| \omega(\zeta) \, dA(\zeta) \right) \eta(\zeta) \, dA(\zeta) \lesssim \int_0^1 \eta(r) \log \frac{e}{1-r} \, dr = \int_0^1 \frac{\hat{\eta}(r)}{1-r} \, dr < \infty. \quad (3.1)$$

If $f$ and $g$ are bounded functions, then (3.1) shows that we may apply Fubini’s theorem to deduce

$$\langle P_\omega (f), g \rangle_{L_\eta^p} = \int_{\mathbb{D}} P_\omega (f)(z) \bar{g}(z) \eta(\zeta) \, dA(\zeta)$$

$$= \int_{\mathbb{D}} \left( \int_{\mathbb{D}} f(\zeta) B_\zeta^\omega(z) \omega(\zeta) \, dA(\zeta) \right) \bar{g}(z) \eta(\zeta) \, dA(\zeta)$$

$$= \int_{\mathbb{D}} f(\zeta) \left( \int_{\mathbb{D}} \bar{g}(z) B_\zeta^\omega(z) \eta(\zeta) \, dA(\zeta) \right) \omega(\zeta) \, dA(\zeta)$$

$$= \int_{\mathbb{D}} f(\zeta) \left( \frac{\omega(\zeta)}{v(\zeta)} \int_{\mathbb{D}} g(z) B_\zeta^\omega(z) \eta(\zeta) \, dA(\zeta) \right) \nu(\zeta) \, dA(\zeta) = \langle f, P_\omega^* (g) \rangle_{L_v^p}.
Since the simple functions are dense in $L^p_0$ for each $1 \leq p < \infty$ and radial $\sigma$, this shows that

$$P^*_\omega(g)(\xi) = \frac{\omega(\xi)}{v(\xi)} \int_\mathbb{D} g(z) B^\omega_\xi(\xi) \eta(z) \, dA(z), \quad \xi \in \mathbb{D}, \quad g \in L^p_0.$$

The adjoint operator $P^*_\omega : L^p_0' \rightarrow L^p_0$ is bounded by the hypothesis, and $\|P^*_\omega\|_{L^p_0' \rightarrow L^p_0} = \|P_\omega\|_{L^p_0 \rightarrow L^p_0}$. Thus

$$\int_\mathbb{D} \left( \frac{\omega(\xi)}{v(\xi)} \right)^{p'} \left| \int_\mathbb{D} g(z) B^\omega_\xi(\xi) \eta(z) \, dA(z) \right|^{p'} v(\xi) \, dA(\xi) \leq \|P_\omega\|_{L^p_0' \rightarrow L^p_0} \|g\|_{L^p_0'}, \quad g \in L^p_0'. \quad (3.2)$$

By considering the standard orthonormal basis $\{z^j / \sqrt{2\omega_{2j+1}}\}, j \in \mathbb{N} \cup \{0\}$, of the Hilbert space $A^2_\omega$, one deduces

$$B^\omega_\xi(\xi) = \sum_{n=0}^{\infty} \frac{(\xi \bar{z})^n}{2\omega_{2n+1}}, \quad z, \xi \in \mathbb{D}.$$

By testing (3.2) with monomials $g_n(\xi) = \xi^n$ we obtain

$$\left( \frac{\eta_{2n+1}}{\omega_{2n+1}} \right)^{p'} \int_\mathbb{D} |\xi|^{np'} \left( \frac{\omega(\xi)}{v(\xi)} \right)^{p'} v(\xi) \, dA(\xi) = \|P^*_\omega(g_n)\|_{L^p_0'} \leq \|P_\omega\|_{L^p_0' \rightarrow L^p_0} \|g_n\|_{L^p_0'} = 2\|P_\omega\|_{L^p_0' \rightarrow L^p_0 \eta_{np'+1}}, \quad n \in \mathbb{N} \cup \{0\}, \quad (3.3)$$

from which Lemma B(ii)(iii), applied to $\omega$, $\eta \in \mathcal{D}$, yields

$$\|P_\omega\|_{L^p_0' \rightarrow L^p_0} \gtrsim \sup_{n \in \mathbb{N}} \left( \frac{\eta_{2n+1}}{\omega_{2n+1} \eta_{np'+1}} \right) \int_0^1 t^{np'} \left( \frac{\omega(t)}{v(t)} \right)^{p'} v(t) \, dt$$

$$\gtrsim \sup_{n \in \mathbb{N}} \left( \frac{\hat{\eta}(1 - \frac{1}{n})^{p'-1}}{\hat{\omega}(1 - \frac{1}{n})^{p'}} \int_0^1 t^{np'} \left( \frac{\omega(t)}{v(t)} \right)^{p'} v(t) \, dt \right)$$

Let $\frac{1}{2} \leq r < 1$ and fix $n \in \mathbb{N}$ such that $1 - \frac{1}{n} \leq r < 1 - \frac{1}{n+1}$. By applying Lemma B (ii) again we finally deduce

$$\|P_\omega\|_{L^p_0' \rightarrow L^p_0} \gtrsim \frac{\hat{\eta}(r)^{\frac{1}{p}}}{\hat{\omega}(r)} \left( \int_r^1 \left( \frac{\omega(t)}{v(t)} \right)^{p'} v(t) \, dt \right)^{\frac{1}{p'}}, \quad \frac{1}{2} \leq r < 1.$$

The assertion follows from this inequality because the last integral converges for $r = 0$ by (3.3) with $n = 0$. \qed
Proof of Theorem 2. By Theorem 1 and Proposition 5, it suffices to show that (iii) implies (iv) under the hypothesis (1.5). To see this, first observe that an integration by parts and Hölder’s inequality give

\[
\int_0^r \frac{\eta(s)}{\omega(s)^p} \, ds \leq \frac{\hat{\eta}(0)}{\hat{\omega}(0)^p} + p \int_0^r \frac{\hat{\eta}(s)\omega(s)}{\hat{\omega}(s)^{p+1}} \, ds
\]

\[
= \frac{\hat{\eta}(0)}{\hat{\omega}(0)^p} + p \int_0^r \frac{\hat{\eta}(s)\omega(s)}{\eta(s)^{1/p}} \frac{1}{\hat{\omega}(s)} \, ds
\]

\[
\leq \frac{\hat{\eta}(0)}{\hat{\omega}(0)^p} + p \left( \int_0^r \left( \frac{\hat{\eta}(s)\omega(s)}{\eta(s)^{1/p}} \right) s\omega(s)^{p'} \eta(s)^{-p} \, ds \right)^{\frac{1}{p'}}
\]

Moreover, since \( N_p(\omega, v, \eta) < \infty \), we have

\[
\left( \int_0^r \left( \frac{\omega(s)}{\eta(s)^{1/p}} \right) s\omega(s)^{p'} \eta(s)^{-p} \, ds \right)^{\frac{1}{p'}} \leq N_p^p(\omega, v, \eta) \left( \int_0^r \left( \frac{\omega(s)}{\eta(s)^{1/p}} \right) s\omega(s)^{p'} \eta(s)^{-p} \, ds \right)^{\frac{1}{p'}}.
\]

By combining (3.4) and (3.5), we deduce

\[
\int_0^r \eta(s) \omega(s)^p \, ds
\]

\[
\leq \frac{\hat{\eta}(0)}{\hat{\omega}(0)^p} + p N_p^p(\omega, v, \eta) \left( \int_0^r \left( \frac{\omega(s)}{\eta(s)^{1/p}} \right) s\omega(s)^{p'} \eta(s)^{-p} \, ds \right)^{\frac{1}{p'}}.
\]

Therefore

\[
\left( \int_0^r \frac{\eta(s)}{\omega(s)^p} \, ds \right)^{\frac{1}{p}} \leq N_p^p(\omega, v, \eta) \left( \int_0^r \left( \frac{\omega(s)}{\eta(s)^{1/p}} \right) s\omega(s)^{p'} \eta(s)^{-p} \, ds \right)^{\frac{1}{p'}}
\]

for all \( \frac{1}{2} \leq r < 1 \). This together with the hypothesis (1.5) completes the proof. \( \square \)

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