STRICHTZ ESTIMATES AND NONLINEAR WAVE EQUATION ON NONTRAPPING ASYMPTOTICALLY CONIC MANIFOLDS

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Abstract. We prove the global-in-time Strichartz estimates for wave equations on the nontrapping asymptotically conic manifolds. We obtain estimates for the full set of wave admissible indices, including the endpoint. The key points are the properties of the microlocalized spectral measure of Laplacian on this setting showed in [14] and a Littlewood-Paley squarefunction estimate. As applications, we prove the global existence and scattering for a family of nonlinear wave equations on this setting.

Key Words: Strichartz estimate, Asymptotically conic manifold, Spectral measure, Global existence, Scattering theory

AMS Classification: 35Q40, 35S30, 47J35.

1. Introduction and Statement of Main Results

Let \((M^\circ, g)\) be a Riemannian manifold of dimension \(n \geq 2\), and let \(I \subset \mathbb{R}\) be a time interval. Suppose \(u(t, z): I \times M^\circ \to \mathbb{C}\) to be the solutions of the wave equation

\[
\partial_t^2 u + Hu = 0, \quad u(0) = u_0(z), \quad \partial_t u(0) = u_1(z)
\]

where \(H = -\Delta_g\) denotes the minus Laplace-Beltrami operator on \((M^\circ, g)\). The general homogeneous Strichartz estimates read

\[
\|u(t, z)\|_{L^q_t L^r_z(I \times M^\circ)} \leq C \left( \|u_0\|_{H^s(M^\circ)} + \|u_1\|_{H^{s-1}(M^\circ)} \right),
\]

where \(H^s\) denotes the \(L^2\)-Sobolev space over \(M^\circ\), and \(2 \leq q, r \leq \infty\) satisfy

\[
s = n \left( \frac{1}{2} - \frac{1}{r} \right) - \frac{1}{q}, \quad \frac{2}{q} + \frac{n-1}{r} \leq \frac{n-1}{2}, \quad (q, r, n) \neq (2, \infty, 3).
\]

In the flat Euclidean space, where \(M^\circ = \mathbb{R}^n\) and \(g_{jk} = \delta_{jk}\), one can take \(I = \mathbb{R}\); see Strichartz [30], Ginibre and Velo [13], Keel and Tao [21], and references therein. In general manifolds, for instance the compact manifold with or without boundary, most of the Strichartz estimates are local in time. If \(M^\circ\) is a compact manifold without boundary, due to finite speed of propagation one usually works in coordinate charts and establishes local Strichartz estimates for variable coefficient wave operators on \(\mathbb{R}^n\). See for examples [23, 26, 32]. Strichartz estimates also are considered on compact manifold with boundary, see [3, 4] and references therein. When we consider the noncompact manifold with nontrapping condition, one can obtain global in time Strichartz estimates. For instance, when \(M^\circ\) is a exterior manifold in \(\mathbb{R}^n\) to a convex obstacle, for metrics \(g\) which agree with the Euclidean metric outside a compact set with nontrapping assumption, the global Strichartz estimates are obtained by Smith-Sogge [27] for odd dimension, and Burq [2] and Metcalfe [25] for even dimension. Blair-Ford-Marzuola [5] established global Strichartz estimates for the wave equation on flat cones \(C(S^1_\rho)\) by using the explicit representation of the fundamental solution.
In this paper, we consider the establishment of global in time Strichartz estimates on asymptotically conic manifolds which satisfy a nontrapping condition. Here, ‘asymptotically conic’ is meant in sense that $M^n$ can be compactified to a manifold with boundary $M$ such that $g$ becomes a scattering metric on $M$. For the more general case of nontrapping asymptotically conic manifolds, Hassell, Tao, and Wunsch first established an $L^q_{t,x}$ Strichartz estimates in [17] and then they [18] extended the estimates to full admissible Strichartz exponents except endpoint $q = 2$, that is, the local in time Strichartz inequalities for Schrödinger

$$\|e^{it\Delta}u_0\|_{L^q_t L^r_x([0,1]\times M^o)} \leq C\|u_0\|_{L^2(M^o)}.$$  

Recently, Hassell and the author [14] improved the Strichartz inequalities by replacing the interval $[0, 1]$ by $\mathbb{R}$. The purpose of this article is to extend the above investigations carried out for Schrödinger to wave equations.

Let us recall the asymptotically conic geometric setting or scattering manifold, which is the same as in [11, 12, 16, 18, 14]. Let $(M^o, g)$ be a complete noncompact Riemannian manifold of dimension $n \geq 2$ with one end, diffeomorphic to $(0, \infty) \times Y$ where $Y$ is a smooth compact connected manifold without boundary. Moreover, we assume $(M^o, g)$ is asymptotically conic or scattering manifold which means that $M^n$ is interior of its compactification $\bar{M}$ with boundary $\partial M = Y$ such that the metric $g$ becomes a asymptotically conic metric on $M$. In details, the metric $g$ in a collar neighborhood $[0, \epsilon) \times \partial M$ near $Y$ takes the form

$$(1.1) \quad g = \frac{dx^2}{x^4} + \frac{h(x)}{x^2} \frac{dx^2}{x^4} + \frac{\sum h_{jk}(x,y)dy^j dy^k}{x^2},$$

where $x \in C^\infty(M)$ is a boundary defining function for $\partial M$ and $h$ is a smooth family of metrics on $Y$. Here we use $y = (y_1, \ldots, y_{n-1})$ for local coordinates on $Y = \partial M$, and the local coordinates $(x, y)$ on $M$ near $\partial M$. Away from $\partial M$, we use $z = (z_1, \ldots, z_n)$ to denote the local coordinates. If $h_{jk}(x, y) = h_{jk}(y)$ independent of $x$, we say $M$ is perfectly conic near infinity. Moreover if every geodesic $z(s)$ in $M$ reaches $Y$ as $s \to \pm \infty$, we say $M$ is nontrapping. The function $r := 1/x$ near $x = 0$ can be thought of as a “radial” variable near infinity and $y$ can be regard as the $n-1$ “angular” variables; and the metric is asymptotic to the exact conic metric $((0, \infty), Y; dr^2 + r^2 h(0))$ as $r \to \infty$. The Euclidean space $M^o = \mathbb{R}^n$ is an example of an asymptotically conic manifold with $Y = \mathbb{S}^{n-1}$ and the standard metric. However a metric cone itself is not an asymptotically conic manifold because of its cone points. We remark that the Euclidean space is a perfectly metric nontrapping cone, where the cone point is a removable singularity.

Let $\dot{H}^s(M^o) = (-\Delta g)^{-\frac{s}{2}}L^2(M^o)$ be the homogeneous Sobolev space over $M^o$. Throughout this paper, pairs of conjugate indices are written as $r, r'$, where $\frac{1}{r} + \frac{1}{r'} = 1$ with $1 \leq r \leq \infty$. Our main result concerning Strichartz estimates is the following.

**Theorem 1.1 (Long-time Strichartz estimate).** Let $(M^o, g)$ be an asymptotically conic non-trapping manifold of dimension $n \geq 3$. Let $H = -\Delta g$ and suppose that $u$ is the
solution to the Cauchy problem
\begin{equation}
\begin{aligned}
\begin{cases}
\partial_t^2 u + Hu = F(t, z), & (t, z) \in I \times \mathbb{M}^o; \\
u(0) = u_0(z), \ & \partial_t u(0) = u_1(z),
\end{cases}
\end{aligned}
\end{equation}
for some initial data $u_0 \in \dot{H}^s, u_1 \in \dot{H}^{s-1}$, and the time interval $I \subseteq \mathbb{R}$, then
\begin{equation}
\|u(t, z)\|_{L^q(I; L^r(M^o))} + \|u(t, z)\|_{C(0; \dot{H}^s(M^o))} 
\lesssim \|u_0\|_{\dot{H}^s(M^o)} + \|u_1\|_{\dot{H}^{s-1}(M^o)} + \|F\|_{L^2(I; L^q(M^o))},
\end{equation}
where the pairs $(q, r), (\tilde{q}, \tilde{r}) \in [2, \infty]^2$ satisfy the wave-admissible condition
\begin{equation}
\frac{2}{q} + \frac{n-1}{r} < \frac{n-1}{2}, \quad (q, r, n) \neq (2, \infty, 3).
\end{equation}
and the gap condition
\begin{equation}
\frac{1}{q} + \frac{n}{r} = \frac{n}{2} - s = \frac{1}{q'} + \frac{n}{r'} - 2.
\end{equation}

**Remark 1.2.** We remark that the estimates are sharp from the sharpness in [21] for the Euclidean space. There is no loss of derivatives. We can take the interval $I = \mathbb{R}$ which means the estimates are global in time.

We sketch the proof as follows. Our strategy is to use the abstract Strichartz estimate proved in Keel-Tao [21] by closely following our previous argument [14] for Schrödinger. Thus, with $U(t)$ denoting the (abstract) propagator, we need to show uniform $L^2 \to L^2$ estimates for $U(t)$, and $L^1 \to L^\infty$ type dispersive estimate on the $U(t)U(s)^*$ with an bound of the form $O(|t-s|^{-(n-1)/2})$. In the flat Euclidean setting, the estimates are considerably simpler because of the explicit formula of the spectral measure. But in our general setting it turns out to be more complicated. It follows from [16] that the Schrödinger propagator $e^{it\Delta}$ fails to satisfy such an dispersive estimate at any pair of conjugate points $(z, z') \in \mathbb{M}^o \times \mathbb{M}^o$ (i.e. pairs $(z, z')$ where a geodesic emanating from $z$ has a conjugate point at $z'$), so we need localize the propagator such that the conjugating point are separated. Even though the half wave operator $U(t)(z, z')$ is finite speed of propagation, we can avoid the conjugated points in a sufficiently short time. If we do this, we would not obtain the long-time Strichartz estimates. We instead overcome difficulties caused by conjugate points by using microlocalizing the spectral measure [14] which is in the same spirt of the proof in [12] of a restriction estimate for the spectral measure, that is, an estimate of the form
\begin{equation}
\|dE_{\sqrt{\mathcal{L}}} (\lambda)\|_{L^p(M^o) \to L^{p'}(M^o)} \leq C\lambda^{\frac{n}{p} - \frac{1}{2} - 1}, \quad 1 \leq p \leq \frac{2(n+1)}{n+3}.
\end{equation}
However the microlocalized spectral measure $Q_i(\lambda) dE_{\sqrt{\mathcal{L}}}(\lambda) Q_i(\lambda)^*$ only has a size estimate in [12], where $Q_i(\lambda)$ is a member of a partition of the identity operator in $L^2(M^o)$. To obtain the dispersive estimate, the authors [14] refined the microlocalized spectral measure by capturing its oscillatory behavior. Then we efficiently exploit the oscillation of the ‘spectral multiplier’ $e^{it\Delta}$ and microlocalized spectral measure to prove the dispersive estimates for Schrödinger. However, the multiplier $e^{it\Delta}$ corresponding to the wave equation has much less oscillation than the Schrödinger multiplier $e^{it\Delta^2}$ at high frequency, we need modify the argument. Heuristically the wavefront of the solution to
the wave equation is microlocalized on the light cone in which the curvature vanishes at one direction, which is different from the Schrödinger’s paraboloid characteristic set. Because of this, we have to resort to a Littlewood-Paley squarefunction estimate on this setting. We remark that we avoid using the Littlewood-Paley squarefunction estimate in the Schrödinger case. We prove the Littlewood-Paley squarefunction estimate on this setting by using the spectral multiplier estimates in Alexopoulos [1] and Stein’s [28] classical argument involving Rademacher functions. The crucial ingredient is to obtain the Gaussian upper bounds on the heat kernel on this setting. We show the Gaussian upper bounds on the heat kernel by using the local-in-time heat kernel bounds in Cheng-Li-Yau [7], and Guillarmou-Hassell-Sikora’s [12] restriction estimate for low frequency which implies the long-time bounds. We use the squarefunction estimates to reduce Theorem 1.1 to a frequency-localized estimate, which allow us to define a microlocalized half wave propagator satisfying $L^2 \to L^2$ and dispersive estimates. We prove the homogeneous Strichartz estimates for the microlocalized half wave propagator by using a semiclassical version of Keel-Tao’s argument. The Strichartz estimate for $e^{it\sqrt{\Delta}}$ then follows by summing over each piece. The inhomogeneous Strichartz estimates follow from the homogeneous estimates and the Christ-Kiselev lemma. Compared with the establishment of Schrödinger inhomogeneous Strichartz estimate in [14], we do not require additional argument since if both $(q, r)$ and $(\tilde{q}, \tilde{r})$ satisfy (1.4) and (1.5) one must have $q > \tilde{q}$.

As an application of the Strichartz estimates, we note that these inequalities can be utilized to generalize a theorem of Lindblad-Sogge [24] on the asymptotically conic non-trapping manifolds. More precisely, we prove the well-posedness and scattering of the following semi-linear wave equation,

\begin{equation}
\begin{aligned}
\partial_t^2 u + Hu &= \gamma |u|^{p-1}u, \\
&:\quad (t, z) \in \mathbb{R} \times M^\circ, \gamma \in \{1, -1\}, \\
&:\quad u(t, z)|_{t=0} = u_0(z), \quad \partial_t u(t, z)|_{t=0} = u_1(z).
\end{aligned}
\end{equation}

(1.6)

In the case of flat Euclidean space, there are many results on the understanding of the global existence and scattering. We refer the readers to [24, 29] and references therein. Blair-Ford-Marzuola [19] also considered similar results for the wave equation on flat cones $C(S^1_p)$. Due to better understand the spectral measure corresponding to the fundamental solution, we can obtain the result for higher dimension. We here are mostly interested in the range of exponents $p \in [p_{\text{conf}}, 1 + \frac{4}{n-2}]$ and the initial data is in $\dot{H}^s(M^\circ) \times \dot{H}^{s-1}(M^\circ)$, where $p_{\text{conf}} = 1 + \frac{4}{n-2}$ and $s_c = \frac{n}{2} - \frac{2}{p-1}$.

Our main result concerning well-posedness and scattering is the following.

**Theorem 1.3.** Let $(M^\circ, g)$ be an non-trapping asymptotically conic manifold of dimension $n \geq 3$. Suppose $p \in [p_{\text{conf}}, 1 + \frac{4}{n-2}]$ and $(u_0, u_1) \in \dot{H}^s(M^\circ) \times \dot{H}^{s-1}(M^\circ)$, then there exists $T > 0$ and a unique solution $u$ to (1.6) satisfying

\begin{equation}
\begin{aligned}
&\quad u \in C([0, T]; \dot{H}^s(M^\circ)) \cap L^{q_0}([0, T]; L^{\rho_0}(M^\circ)), \\
&\quad \text{where } q_0 = (p-1)(n+1)/2.
\end{aligned}
\end{equation}

(1.7)

In addition, if there is a small constant $\epsilon(p)$ such that

\begin{equation}
\|u_0\|_{\dot{H}^s} + \|u_1\|_{\dot{H}^{s-1}} < \epsilon(p),
\end{equation}

(1.8)
then there is a unique global and scattering solution $u$ to (1.6) satisfying

$$u \in C_t(\mathbb{R}; H^s(M^\circ)) \cap L^{q_0}(\mathbb{R}; L^{q_0}(M^\circ)).$$

This paper is organized as follows. In Section 2 we review the results of the microlocalized spectral measure and prove the square function inequalities on this setting. Section 3 is devoted to the proofs of the microlocalized dispersive estimates and $L^2$-estimates. In the section 4, we prove the homogenous and inhomogeneous Strichartz estimates. Finally, we apply the Strichartz estimates to show Theorem 1.3.

Acknowledgments: The author would like to thank Jean-Marc Bouclet, Andrew Hassell and Changxing Miao for their helpful discussions and encouragement. This research was supported by the Fundamental Research Foundation of Beijing Institute of Technology(20111742015), PFMEC(20121101120044) and Discovery Grant DP120102019 from the Australian Research Council.

2. The microlocalized spectral measure and Littlewood-Paley squarefunction estimate

In this section, we briefly recall the key elements of the microlocalized spectral measure, which was constructed by Hassell and the author [14] to capture both its size and the oscillatory behavior. We also prove the Littlewood-Paley squarefunction estimates on this setting that we require in subsequent section.

2.1. The microlocalized spectral measure. In the free Euclidean space, the half wave propagator has a explicit formula by using the Fourier transform, but in the asymptotically conical manifold it turns out to be quite complicated. From the results of [11, 15], we have known that the Schwartz kernel of the spectral measure can be described as a Legendrian distribution on the compactification of the space $M \times M$ uniformly with respect to the spectral parameter $\lambda$. As pointed out in introduction, we really need to choose an operator partition of unity to microlocalize the spectral measure such that the spectral measure can be expressed in a formula capturing not only the size also the oscillatory behavior. This was constructed and proved in [14]. For convenience, we recall and slightly modify the statement to adapt our following application.

**Proposition 2.1.** Let $(M^\circ, g)$ and $H$ be in Theorem 1.1. For fixed $\lambda_0 > 0$, then there exists an operator partition of unity on $L^2(M)$

$$\text{Id} = \sum_{i=0}^{N_l} Q^\text{low}_i(\lambda) \quad \text{for} \quad 0 < \lambda \leq 2\lambda_0;$$

$$\text{Id} = \left( \sum_{i=1}^{N'} + \sum_{i=N'+1}^{N_h} \right) Q^\text{high}_i(\lambda) \quad \text{for} \quad \lambda \geq \lambda_0/2,$$

where the $Q^\text{low}_i$ and $Q^\text{high}_i$ are uniformly bounded as operators on $L^2$ and $N_l$ and $N_h$ are bounded independent of $\lambda$, such that
Remark 2.3. Here $d \cdot$ have function inequality which was given by for $1 < p < \infty$ the general noncompact manifolds with ends, Bouclet [6] proved a weak version square estimate for low frequency and the spectral multiplier estimates in Alexopoulos [1]. In heat kernel bounds in Cheng-Li-Yau [7], Guillarmou-Hassell-Sikora’s [12] restriction asymptotically conic manifolds, though the proof is considerably simpler due to the totally hyperbolic manifolds and improved (2.7) for asymptotically conic manifolds Bouclet also pointed out that the usual square function inequalities may fail on asymptotically hyperbolic manifolds and improved (2.7) for asymptotically conic manifolds by showing

$$\| \varphi (2^{-2j}H) f \|_{L^p} + \| \left( \sum_{j \geq 0} |\varphi (2^{-2j}H) f|^2 \right)^{\frac{1}{2}} \|_{L^p} \sim \| f \|_{L^p}.$$  

2.2. The Littlewood-Paley squarefunction estimate. In this subsection, we prove the Littlewood-Paley squarefunction estimate for the asymptotically conic manifold, which allows us to reduce Theorem 1.1 to a frequency-localized estimate (see Proposition 1.2).

Let $\varphi \in C_0^\infty (\mathbb{R} \setminus \{0\})$ take values in $[0, 1]$ and be supported in $[1/2, 2]$ such that

$$1 = \sum_{j \in \mathbb{Z}} \varphi (2^{-j} \cdot), \quad \lambda > 0.$$  

Define $\varphi_0 (\lambda) = \sum_{j \leq 0} \varphi (2^{-j} \cdot)$. Then the result about the Littlewood-Paley squarefunction estimate reads as follows:

**Proposition 2.2.** Let $(M^o, g)$ be a asymptotically conic manifold, trapping or not, and $H = -\Delta_g$ is the Laplace-Beltrami operator on $(M^o, g)$. Then for $1 < p < \infty$, there exist constants $C_p$ and $C_p$ depending on $p$ such that

$$c_p \| f \|_{L^p (M^o)} \leq \left( \sum_{j \in \mathbb{Z}} |\varphi (2^{-j} \sqrt{H}) f|^2 \right)^{\frac{1}{2}} \| \| f \|_{L^p (M^o)} \leq C_p \| f \|_{L^p (M^o)}.$$  

**Remark 2.3.** To our knowledge, such squarefunction estimates are new in the case of asymptotically conic manifolds, though the proof is considerably simpler due to the heat kernel bounds in Cheng-Li-Yau [7], Guillarmou-Hassell-Sikora’s [12] restriction estimate for low frequency and the spectral multiplier estimates in Alexopoulos [1]. In the general noncompact manifolds with ends, Bouclet [6] proved a weak version square function inequality which was given by for $1 < p < \infty$

$$\| f \|_{L^p} \leq \left( \sum_{j \geq 0} |\varphi (2^{-2j}H) f|^2 \right)^{\frac{1}{2}} \| f \|_{L^2} + \| f \|_{L^2}.$$  

Bouclet also pointed out that the usual square function inequalities may fail on asymptotically hyperbolic manifolds and improved (2.7) for asymptotically conic manifolds by showing

$$\| \varphi_0 (2^{-2j}H) f \|_{L^p} + \left( \sum_{j \geq 0} |\varphi (2^{-2j}H) f|^2 \right)^{\frac{1}{2}} \| f \|_{L^p} \sim \| f \|_{L^p}.$$  

• when $Q(\lambda)$ is equal to either $Q^{\text{low}}_0 (\lambda)$ or $Q^{\text{low}}_1 (\lambda)$; or $Q(\lambda)$ is equal to $Q^{\text{high}}_1 (\lambda)$, we have

$$\left| \frac{d}{d\lambda}^\alpha (Q(\lambda) dE_{\sqrt{\Pi}} (\lambda) Q^* (\lambda)) \right| \leq C_\alpha \lambda^{n-1-\alpha} \quad \forall \alpha \in \mathbb{N}.$$  

• when $Q(\lambda)$ is equal to $Q^{\text{low}}_i (\lambda)$ or $Q^{\text{high}}_i (\lambda)$ for $i \geq 2$, we have

$$Q(\lambda) dE_{\sqrt{\Pi}} (\lambda) Q^* (\lambda) (z, z') = \lambda^{n-1} e^{i \lambda d(z, z')} a(\lambda, z, z').$$

Here $d(\cdot, \cdot)$ is the Riemannian distance on $M^o$, and a satisfies

$$|\partial_i^k a(\lambda, z, z')| \leq C_\alpha \lambda^{-\alpha} (1 + \lambda d(z, z'))^{-\frac{n-1}{2}}.$$  

Having this result, we can exploit the oscillations both in the multiplier $e^{i(t-s)\lambda}$ and in $e^{\pm i \lambda d(z, z')}$ to obtain the required dispersive estimate for the $TT^*$ version of the microlocalized propagator.
One can see that the squarefunction estimate in (2.6) involves the low frequency in contrast to (2.8).

Proof. This proof follows from the Stein’s [28] classical argument (in $\mathbb{R}^n$) involving Rademacher functions and an appropriate Mikhlin-Hörmander multiplier theorem. Now we provide details as follows. We notice that the asymptotically conic manifolds are a relatively well-behaved class of manifolds. In particular, all section curvatures of $(M^o, g)$ approach zero as $x$ goes to zero, and thus that $(M^o, g)$ has bounded sectional curvature and has low bounds for the injectivity radius. Now we need a theorem in Cheng-Li-Yau [7] and recall it for convenience. For complete Riemannian manifolds $M^o$ of bounded sectional curvature and injectivity radius bounded below, Cheng-Li-Yau’s theorem gives the following local-in-time Gaussian upper bound for the heat kernel

**Lemma 2.4.** There exists nonzero constants $c$ and $C$ such that the heat kernel on $M^o$, denoted $H(t, z, z')$, satisfies the Gaussian upper bound of the form for $t \in [0, T]$

\[
H(t, z, z') \leq Ct^{-\frac{n}{2}} \exp \left( -\frac{d(z, z')^2}{ct} \right),
\]

where $d(z, z')$ is the distance between $z$ and $z'$ on $M^o$.

We claim that the global-in-time Gaussian upper bound for the heat kernel also holds, that is

\[
H(t, z, z') \leq \frac{1}{|B(z, \sqrt{t})|} \exp \left( -\frac{d(z, z')^2}{ct} \right)
\]

holds for all $t > 0$, where $|B(z, \sqrt{t})|$ be the volume of the ball of radius $\sqrt{t}$ at $z$. By (2.9), we only consider the case $t \geq 1$. To prove this, we write

\[
H(t, z, z') = e^{-tH}(z, z') = \int_0^\infty e^{-t\lambda^2} dE_{\sqrt{\Pi}}(\lambda).
\]

Choose $\chi \in C^\infty_c(\mathbb{R})$, such that $\chi(\lambda) = 1$ for $\lambda \leq 1$, we decompose

\[
H(t, z, z') = \int_0^\infty e^{-t\lambda^2} \chi(\lambda) dE_{\sqrt{\Pi}}(\lambda) + \int_0^\infty e^{-t\lambda^2} (1 - \chi)(\lambda) dE_{\sqrt{\Pi}}(\lambda)
\]

=: $I + II$.

By using [12, Theorem 1.3], we see for $\lambda \leq 1$

\[
|dE_{\sqrt{\Pi}}(\lambda)(z, z')| \leq C\lambda^{n-1}.
\]

Hence $I \leq Ct^{-\frac{n}{2}}$. To treat $II$, we need the following lemma

**Lemma 2.5.** The following statements are equivalent:

1. Local-in-time heat kernel bound $\|e^{-tH}\|_{L^1 \rightarrow L^2} \leq Ct^{-\frac{n}{2}}$ for $t \leq 1$;
2. Spectral projection bound $\|E_{\sqrt{\Pi}}([0, \mu])\|_{L^1 \rightarrow L^2} \leq C\mu^{n/2}$ for $\mu \geq 1$. 


Proof. We first prove (1) implies (2). Let $t = \mu^{-2}$, (2) is proved by noticing $1_{[0,\mu]}(s) \leq e \exp(-\frac{e^2}{\mu^2})$ and writing

$$E_{\sqrt{\Pi}}([0,\mu]) = E_{\sqrt{\Pi}}([0,\mu])e^{H/\mu^2}e^{-H/\mu^2},$$

$$\implies \|E_{\sqrt{\Pi}}([0,\mu])\|_{L^1 \to L^2} \leq \|E_{\sqrt{\Pi}}([0,\mu])e^{H/\mu^2}\|_{L^2 \to L^2}\|e^{-H/\mu^2}\|_{L^1 \to L^2}.$$  

Next we show (2) implies (1). Choose $\phi_k \in C_c^\infty([k,k+1])$ taking value in $[0,1]$ such that $\sum_{k \geq 1} \phi_k(\mu) = 1$ for all $\mu \geq 1$, we write

$$e^{-t\Pi} = \int_0^\infty e^{-t\mu^2}dE_{\sqrt{\Pi}}(\mu) = \sum_{k \geq 0} \int_0^\infty e^{-t\mu^2} \phi_k(\mu) dE_{\sqrt{\Pi}}(\mu).$$

Using the scaling argument, we have by (2) and $t \leq 1$

$$\|e^{-t\Pi}\|_{L^1 \to L^2} \leq \sum_{k \geq 0} \int_0^\infty d\mu \left(e^{-t\mu^2} \phi_k(\mu)\right) \left\|E_{\sqrt{\Pi}}(\mu)\right\|_{L^1 \to L^2} d\mu$$

$$\leq \sum_{k \geq 0} \int_k^{k+1} e^{-t\mu^2} \mu^2 (t\mu + \mu^{-1}) d\mu \leq C t^{-\frac{n}{2}}.$$

This implies (1).

Now we turn to estimate $II$. From local-in-time heat kernel estimate \([2.10]\), one has by duality $\|e^{-t\Pi}\|_{L^1 \to L^2} \leq C t^{-\frac{n}{2}}$ for $t \leq 1$. Hence by Lemma \([2.5]||E_{\sqrt{\Pi}}([0,\lambda])||_{L^1 \to L^2} \leq C\lambda^{n/2}$ for $\lambda \geq 1$, which implies $\|E_{\sqrt{\Pi}}([0,\lambda])\|_{L^1 \to L^\infty} \leq C\lambda^n$. Therefore we have for $t \geq 1$

$$\|II\|_{L^1 \to L^\infty} \leq \sum_{k \geq 0} \int_0^\infty d\lambda \left(e^{-t\lambda^2} \phi_k(\lambda) (1 - \chi)(\lambda)\right) \left\|E_{\sqrt{\Pi}}(\lambda)\right\|_{L^1 \to L^\infty} d\lambda$$

$$\leq C e^{-t/2} \leq C t^{-\frac{n}{2}}.$$

Hence we have proved for all $t > 0$

$$H(t, z, z') \lesssim t^{-\frac{n}{2}}.$$

We use a theorem of Grigor’yan \([16]\) Theorem 1.1] that establishes Gaussian upper bounds for arbitrary Riemannian manifolds. His conclusion implies that if $H(t, z, z')$ satisfies on-diagonal bounds

$$H(t, z, z) \lesssim t^{-\frac{n}{2}}, \quad H(t, z', z') \lesssim t^{-\frac{n}{2}},$$

then we have

$$H(t, z, z') \lesssim t^{-\frac{n}{2}} \exp\left(-\frac{d(z, z')^2}{ct}\right).$$

Since $|B(z, \sqrt{t})| \sim t^{\frac{n}{2}}$, this gives

$$H(t, z, z') \lesssim \frac{1}{|B(z, \sqrt{t})|} \exp\left(-\frac{d(z, z')^2}{ct}\right).$$

Now we need a result of Alexopoulos \([11]\) Theorem 6.1], which outlines how his results on Markov chains can be extended to treat differential operators on manifolds
where the associated heat kernel satisfies Gaussian upper bounds. We remark here that the asymptotically conic manifold satisfies the doubling condition in contrast to the hyperbolic case. Given (2.11), Alexopoulos’ theorem implies that any spectral multiplier \( m(\sqrt{H}) \) satisfying the usual Hörmander condition maps \( L^p(M) \rightarrow L^p(M) \) for any \( p \in (1, \infty) \). Furthermore, this boundedness holds true for function \( m \in C^N(\mathbb{R}) \) which satisfies the weaker Mihlin-type condition for \( N \geq \frac{d}{2} + 1 \)

\[
(2.12) \quad \sup_{0 \leq k \leq N} \sup_{\lambda \in \mathbb{R}} \left| \left( \lambda \partial_\lambda \right)^k m(\lambda) \right| \leq C < \infty.
\]

We now want to apply this result to a family of multipliers \( m^\pm(s, \sqrt{H}), 0 \leq s \leq 1 \) defined using the Rademacher functions. Let us introduce the Rademacher functions defined as follows:

(i) the function \( r_0(s) \) is defined by \( r_0(s) = 1 \) on \([0, 1/2] \) and \( r_0(s) = -1 \) on \((1/2, 1) \), and then extended to the real line by periodicity, i.e. \( r_0(s + 1) = r_0(s) \);

(ii) for \( k \in \mathbb{N} \setminus \{0\} \), \( r_k(s) = r_0(2^k s) \).

Given any square integrable sequence of scalars \( \{a_k\}_{k \geq 0}, \) consider the function \( m(s) = \sum_{k \geq 0} a_k r_k(s) \). By a lemma in the [28, Appendix D], for any \( p \in (1, \infty) \) there exists constants \( c_p \) and \( C_p \) such that

\[
(2.13) \quad c_p \|m(s)\|_{L^p([0,1])} \leq \|m(s)\|_{L^2([0,1])} = \left( \sum_{k \geq 0} |a_k|^2 \right)^{1/2} \leq C_p \|m(s)\|_{L^p([0,1])}.
\]

Now define

\[
m^\pm(s, \lambda) = \sum_{j \geq 0} r_j(s) \varphi_{\pm j}(\lambda)
\]

where \( \varphi_{\pm j}(\lambda) = \varphi(2^{\pm j} \lambda) \). Then we define the operator \( m^\pm(s, \sqrt{H}) \) through the spectral measure \( dE_{\sqrt{H}}(\lambda) \):

\[
(2.14) \quad m^\pm(s, \sqrt{H}) = \int_0^\infty m^\pm(s, \lambda) dE_{\sqrt{H}}(\lambda).
\]

We note that this is well-defined by the spectral theory. It can be verified that \( m^\pm(s, \lambda) \) satisfies the condition (2.11), and we can take the constant \( C \) independent of \( s \). Therefore we have that for \( 1 < p < \infty \) and \( f \) in \( L^p \) by (2.13)

\[
\left\| \left( \sum_{j \geq 0} |\varphi_{\pm j}(\sqrt{H}) f|^2 \right)^{1/2} \right\|_{L^p} \lesssim \left\| \sum_{j \geq 0} \varphi_{\pm j}(\sqrt{H}) f r_k(s) \right\|_{L^p(M; L^p([0,1]))}^p \\
\lesssim \int_{M^p} \int_0^1 \left| m^\pm(s, \sqrt{H}) f(z) \right|^p d|s|dz(z) \lesssim \|f\|_{L^p}^p.
\]

Therefore we prove

\[
(2.15) \quad \left\| \left( \sum_{j \in \mathbb{Z}} |\varphi_j(\sqrt{H}) f|^2 \right)^{1/2} \right\|_{L^p} \lesssim \|f\|_{L^p}.
\]

To see the other inequality, we first define \( \tilde{\varphi}_j(\lambda) = \sum_{i=j-1}^{j+1} \varphi_i(\lambda) \), then the above also is true when \( \varphi_j(\lambda) \) is replaced by \( \tilde{\varphi}_j(\lambda) \). Let \( f_1 \in L^p \) and \( f_2 \in L^p \), we see by Hölder’s
inequality and (2.15)
\[
\left| \int_{M_0} f_1(z) f_2(z) dg(z) \right| = \left| \int_{M_0} \sum_{j \in \mathbb{Z}} \left( \tilde{\psi}_j(\sqrt{H}) f_1(z) \tilde{\psi}_j(\sqrt{H}) f_2(z) \right) dg(z) \right|
\lesssim \left\| \left( \sum_{j \in \mathbb{Z}} |\tilde{\psi}_j(\sqrt{H}) f_1|^2 \right)^{\frac{1}{2}} \right\|_{L^p} \left\| \left( \sum_{j \in \mathbb{Z}} |\tilde{\psi}_j(\sqrt{H}) f_2|^2 \right)^{\frac{1}{2}} \right\|_{L^{p'}}
\lesssim \| f_1 \|_{L^p} \left( \sum_{j \in \mathbb{Z}} |\psi_j(\sqrt{H}) f_2|^2 \right)^{\frac{1}{2}} \|_{L^{p'}}.
\]

By duality, we hence prove (2.6).

\[\square\]

3. L^2-estimates and dispersive estimates

In this section, we prove the L^2-estimates and dispersive estimates needed for the abstract Keel-Tao argument. We begin by defining microlocalized propagators and then show the definition is well-defined. We do this by showing that each microlocalized propagator is a bounded operator on L^2. This serves both to show the well-definedness of each microlocalized propagator, and to establish the L^2 \to L^2 estimate needed for the abstract Keel-Tao argument. We point out here that the microlocalized propagators are different from the ones defined in [14], which allow us to easily show the Strichartz estimate by spectral theory on Hilbert space but we need a square function inequality in the establishment of the Strichartz estimate. Since the microlocalized propagators avoid the conjugate points, we can prove the TT^* version dispersive estimates.

3.1. Microlocalized propagator and L^2-estimates. We start by dividing the half wave propagator into a low-energy piece and a high-energy piece. Choose \( \chi \in C_c^\infty(\mathbb{R}) \), such that \( \chi(t) = 1 \) for \( t \leq 1 \). We define

\[
U_{\text{low}}(t) = \int_0^\infty e^{it\lambda} \chi(\lambda) dE_{\sqrt{\Pi}}(\lambda), \quad U_{\text{high}}(t) = \int_0^\infty e^{it\lambda} (1 - \chi)(\lambda) dE_{\sqrt{\Pi}}(\lambda).
\]

Using the partition of unity \( 1 = \sum_{j \in \mathbb{Z}} \varphi(2^{-j} \lambda) \) we define

\[
U_{\text{low}}^j(t) = \int_0^\infty e^{it\lambda} \varphi(2^{-j} \lambda) \chi(\lambda) dE_{\sqrt{\Pi}}(\lambda),
\]

\[
U_{\text{high}}^j(t) = \int_0^\infty e^{it\lambda} \varphi(2^{-j} \lambda) (1 - \chi)(\lambda) dE_{\sqrt{\Pi}}(\lambda).
\]

Further using the low-energy and high-energy operator partition of identity operator in Proposition 2.1, we define

\[
U_{i,j}(t) = \int_0^\infty e^{it\lambda} \varphi(2^{-j} \lambda) \chi(\lambda) Q_i^{\text{low}}(\lambda) dE_{\sqrt{\Pi}}(\lambda), \quad 0 \leq i \leq N_l;
\]

\[
U_{i,j}(t) = \int_0^\infty e^{it\lambda} \varphi(2^{-j} \lambda) (1 - \chi)(\lambda) Q_i^{\text{high}}(\lambda) dE_{\sqrt{\Pi}}(\lambda), \quad N_l + 1 \leq i \leq N := N_l + N_h.
\]

Now we show this definition is well-defined. To do so, it suffices to show the above integrals are well-defined over any compact interval in \((0, \infty)\). Suppose that \( A(\lambda) \) is
a family of bounded operators on $L^2(M^0)$, compactly supported in $[a, b]$ and $C^1$ in $\lambda \in (0, \infty)$. Integrating by parts, the integral of

$$\int_a^b A(\lambda)dE_{\sqrt{\Pi}}(\lambda)$$

is given by

$$E_{\sqrt{\Pi}}(b)A(b) - E_{\sqrt{\Pi}}(a)A(a) - \int_a^b \frac{d}{d\lambda} A(\lambda)E_{\sqrt{\Pi}}(\lambda) d\lambda.$$  \hfill (3.4)

Now we need the following lemma which is the consequence of [14] Lemma 2.3, Lemma 3.1.

**Lemma 3.1.** Each $Q_i^{\text{low}}(\lambda)$ and each operator $\lambda \partial_{\lambda} Q_i^{\text{low}}(\lambda)$ is bounded on $L^2(M^0)$ uniformly in $\lambda$. The same statements are true for the high energy operators $Q_i^{\text{high}}(\lambda)$.

**Proof.** We use the notation in [11, 14, 15]. The uniform boundedness of the scattering pseudodifferential operator $Q_i^{\text{low}}(\lambda) \in \Psi^{-\infty}_k(M, M_{k,b}^2)$ is straightforward to prove using the fact that the order is $-\infty$. This implies that the kernel is smooth and uniformly bounded on iterated blowup space $M_{k,sc}^2$ as a multiple of the half density bundle $\Omega^\frac{1}{2}_{k,b}$. This bundle has a nonzero section given, in the region where $x \leq C\lambda$, by $\lambda^n|dgdg'|^{1/2}|d\lambda/\lambda|^{1/2}$, where the $|d\lambda/\lambda|^{1/2}$ is a purely formal factor, included to make a half-density on the whole space $M_{k,b}^2$, including in the $\lambda$-direction. On the other hand, the kernels are chosen to have support in a neighbourhood of the diagonal, which is equivalent to the region where $d(z, z') \leq C\lambda^{-1}$. It follows that the kernel is bounded by a multiple of the characteristic function of the set $\{(z, z') \mid d(z, z') \leq C\lambda^{-1}\}$ times the Riemannian half-density. Moreover, the same is true for $\lambda\partial_{\lambda} Q_i^{\text{low}}(\lambda)$, due to the smoothness of the kernel on $M_{k,sc}^2$. Since the volume of each ball of radius $r$ on $M^0$ is between $cr^n$ and $C r^n$, Schur’s test shows that such kernels are bounded on $L^2(M^0)$ uniformly in $\lambda$.

The high energy operators $Q_i^{\text{high}}(\lambda)$ are semiclassical pseudodifferential operators of semiclassical order $0$ and differential order $-\infty$. Therefore, they take the form

$$\lambda^n \int e^{i\lambda(z-z') \cdot \zeta} a(z, \zeta, \lambda^{-1}) d\zeta$$

in the interior, or

$$\lambda^n \int e^{i\lambda(y-y') \cdot \eta + (\sigma-1)\nu/z} a(x, y, \eta, \nu, \lambda^{-1}) d\eta d\nu$$

near the boundary. Here $a$ is smooth and compactly supported in its arguments. Integration by parts in $\zeta$, or in $\eta$, $\nu$, shows that the kernel is rapidly decreasing in $\lambda|z-z'|$, respectively $\lambda\sqrt{|y-y'|^2/x^2 + (\sigma-1)^2/x^2}$. Equivalently, the kernel is rapidly decreasing in $\lambda d(z, z')$. We see that the kernel is pointwise bounded by $C\lambda^n(1 + \lambda d(z, z'))^{-N}$ for any $N$. The same is true for $\lambda\partial_{\lambda} Q_i^{\text{high}}(\lambda)$. Again Schur’s test shows that such kernels are bounded on $L^2(M^0)$ uniformly in $\lambda$. \hfill $\square$

In view of this lemma, we can take $A(\lambda) = e^{i\mu_k \lambda(\lambda)}\varphi(2^{-j})Q_i^{\text{low}}(\lambda)$ (for $0 \leq i \leq N_1$), or $e^{i\mu_k \varphi(2^{-j})(1 - \chi)(\lambda)}Q_i^{\text{high}}(\lambda)$ (for $N_1 + 1 \leq i \leq N$), this means that the integrals
are well-defined over any compact interval in $(0, \infty)$, hence the operators $U_{i,j}(t)$ are well-defined. Now we see these operators are bounded on $L^2$. We only consider the low frequency part since the similarly argument also gives the boundedness on $L^2$ for high energy part. We have for $0 \leq i \leq N_i$, by [14, Lemma 5.3],

$$U_{i,j}(t)U_{i,j}(t)^* = \int \chi(\lambda)^2 \varphi(\frac{\lambda}{2^j}) \varphi(\frac{\lambda}{2^j}) Q_{i}^{low}(\lambda) dE_{\sqrt{\Pi}}(\lambda)Q_{i}^{low}(\lambda)^*$$

(3.5)

$$+ \int \chi(\lambda)^2 \varphi(\frac{\lambda}{2^j}) \varphi(\frac{\lambda}{2^j}) Q_{i}^{low}(\lambda) dE_{\sqrt{\Pi}}(\lambda)Q_{i}^{low}(\lambda)^*$$

We observe that this is independent of $t$ and we also note that the integrand is a bounded operator on $L^2$, with an operator bound of the form $C/\lambda$ where $C$ is uniform, as we see from Lemma 5.3 and the support property of $\varphi$. The integral is therefore uniformly bounded, as we are integrating over a dyadic interval in $\lambda$. Hence we have shown that

**Proposition 3.2** (**$L^2$-estimates**). Let $U_{i,j}(t)$ be defined in [3.3]. Then there exists a constant $C$ independent of $t, z, z'$ such that $||U_{i,j}(t)||_{L^2 \to L^2} \leq C$ for all $i \geq 0, j \in \mathbb{Z}$.

3.2. **Dispersive estimates**. Next we aim to establish the dispersive estimates for the microlocalized $U_{i,j}(t)U_{i,j}^*(s)$. We need the following proposition.

**Proposition 3.3** (**Microlocalized dispersive estimates**). Let $Q(\lambda)$ be the operator $Q_{i}^{low}$ or $Q_{i}^{high}$ constructed as in Proposition [2.1] and suppose $\phi \in C^{\infty}_c([1/2, 2])$ and takes value in $[0, 1]$. Then the kernel estimate

$$\left| \int_0^{\infty} e^{it\lambda} \phi(2^{-j}\lambda) \left( Q(\lambda) dE_{\sqrt{\Pi}}(\lambda)Q^*(\lambda) \right) (z, z') d\lambda \right| \leq C 2^{j(n+1)/2} (2^{-j} + |t|)^{-(n-1)/2}$$

holds for a constant $C$ independent of $j \in \mathbb{Z}$ and points $z, z' \in M^o$.

**Proof.** The key to the proof is to apply Proposition [2.1]. For $Q = Q_{i}^{low}$ for $i = 0, 1$, or $Q = Q_{i}^{high}$, we have by Proposition [2.1]

$$\left| \int_0^{\infty} e^{it\lambda} \phi(2^{-j}\lambda) \left( Q(\lambda) dE_{\sqrt{\Pi}}(\lambda)Q^*(\lambda) \right) (z, z') d\lambda \right| \leq C 2^{jn}.$$

We use the $N$-times integration by parts to obtain by [2.2]

$$\left| \int_0^{\infty} e^{it\lambda} \phi(2^{-j}\lambda) \left( Q(\lambda) dE_{\sqrt{\Pi}}(\lambda)Q^*(\lambda) \right) (z, z') d\lambda \right|$$

$$\leq \left| \int_0^{\infty} \left( \frac{1}{it} \frac{\partial}{\partial \lambda} \right)^N (e^{it\lambda}) \phi(2^{-j}\lambda) \left( Q(\lambda) dE_{\sqrt{\Pi}}(\lambda)Q^*(\lambda) \right) (z, z') d\lambda \right|$$

$$\leq C_N |t|^{-N} \int_{2^{-j-1}}^{2^{j+1}} \lambda^{n-1-N} d\lambda \leq C_N |t|^{-N} 2^j (n-N).$$

Therefore we obtain

$$\left| \int_0^{\infty} e^{it\lambda} \phi(2^{-j}\lambda) \left( Q(\lambda) dE_{\sqrt{\Pi}}(\lambda)Q^*(\lambda) \right) (z, z') d\lambda \right| \leq C_N 2^{jn} (1 + 2^j |t|)^{-N}.$$
By choosing \( N = (n - 1)/2 \), we prove (3.6). When \( Q \) is equal to \( Q^\text{low}_i \) or \( Q^\text{high}_i \) for \( i \geq 2 \), we see by Proposition 2.1

\[
\left| \int_0^\infty e^{it\lambda} \phi(2^{-j}\lambda)(Q(\lambda)dE_{\sqrt{H}}(\lambda)Q^*(\lambda))(z, z')d\lambda \right|
= \left| \int_0^\infty \left( \frac{1}{i(t - d(z, z'))} \frac{\partial}{\partial \lambda} \right)^N (e^{i(t-d(z,z'))\lambda}) \phi(2^{-j}\lambda)\lambda^{n-1}a(\lambda, z, z')d\lambda \right|
\leq C_N|t - d(z, z')|^{-N} \int_{2^{j-1}}^{2^j} \lambda^{n-1-N} (1 + \lambda d(z, z'))^{-\frac{n-1}{2}}d\lambda
\leq C_N2^{j(n-N)}|t - d(z, z')|^{-N}(1 + 2^j d(z, z'))^{-(n-1)/2}.
\]

It follows that

\[
(3.8) \quad \left| \int_0^\infty e^{it\lambda} \phi(2^{-j}\lambda)(Q(\lambda)dE_{\sqrt{H}}(\lambda)Q^*(\lambda))(z, z')d\lambda \right|
\leq C_N2^{jn}(1 + 2^j|t - d(z, z')|)^{-N}(1 + 2^j d(z, z'))^{-(n-1)/2}.
\]

If \( |t| \sim d(z, z') \), it is clear to see (3.6). Otherwise, we have \( |t - d(z, z')| \geq c|t| \) for some small constant \( c \), then choose \( N = (n - 1)/2 \) to prove (3.6).

**Remark 3.4.** If \( N = \frac{n-1}{2} \) is not an integer, one may need geometric mean argument to modify the proof.

As a consequence of Proposition 3.3, we immediately have

**Proposition 3.5.** Let \( U_{i,j}(t) \) be defined in (3.3). Then there exists a constant \( C \) independent of \( t, z, z' \) for all \( i \geq 0, j \in \mathbb{Z} \) such that

\[
(3.9) \quad \|U_{i,j}(t)U^*_{i,j}(s)\|_{L^1 \to L^\infty} \leq C2^{j(n+1)/2}(2^{-j} + |t-s|)^{-(n-1)/2}.
\]

**4. Strichartz estimates**

In this section, we show the Strichartz estimates in Theorem 1.1. To obtain the Strichartz estimates, we need a variant of Keel-Tao’s abstract Strichartz estimate for wave equation.

**4.1. Semiclassical Strichartz estimates.** We need a variety of the abstract Keel-Tao’s Strichartz estimates theorem. This is an analogue of the semiclassical Strichartz estimates for Schrödinger in [22, 33].

**Proposition 4.1.** Let \((X, \mathcal{M}, \mu)\) be a \( \sigma \)-finite measured space and \( U : \mathbb{R} \to B(L^2(X, \mathcal{M}, \mu)) \) be a weakly measurable map satisfying, for some constant \( C, \alpha \geq 0, \sigma, h > 0 \),

\[
\|U(t)\|_{L^2 \to L^2} \leq C, \quad t \in \mathbb{R},
\]

\[
\|U(t)U(s)^*f\|_{L^\infty} \leq Ch^{-\alpha}(h + |t-s|)^{-\sigma}\|f\|_{L^1}.
\]

Then for every pair \( q, r \in [1, \infty] \) such that \((q, r, \sigma) \neq (2, \infty, 1) \) and

\[
\frac{1}{q} + \frac{\sigma}{r} \leq \frac{\sigma}{2}, \quad q \geq 2,
\]

\[
\|U(t)U(s)^*f\|_{L^q} \leq Ch^{-\alpha}(h + |t-s|)^{-\sigma}\|f\|_{L^r}.
\]

\[
\|U(t)U(s)^*f\|_{L^q} \leq Ch^{-\alpha}(h + |t-s|)^{-\sigma}\|f\|_{L^r}.
\]
there exists a constant $\tilde{C}$ only depending on $C$, $\sigma$, $q$ and $r$ such that

\begin{equation}
(4.2) \quad \left( \int_{\mathbb{R}} \|U(t)u_0\|_{L^q_r}^q \, dt \right)^{\frac{1}{q}} \leq \tilde{C} \Lambda(h) \|u_0\|_{L^2}
\end{equation}

where $\Lambda(h) = h^{-(\alpha + \sigma)(\frac{1}{q} - \frac{1}{r}) + \frac{1}{2}}$.

**Proof.** If $(q, r, \sigma) \neq (2, \infty, 1)$ is on the line $\frac{1}{q} + \frac{\sigma}{r} = \frac{\sigma}{2}$, we replace $\|t - s + h\|^{-\sigma}$ by $\|t - s\|^{-\sigma}$ and then we closely follow Keel-Tao’s argument [21, Section 3-7] to show (4.2).

So we only consider $\frac{1}{q} + \frac{\sigma}{r} < \frac{\sigma}{2}$. By the $TT^*$ argument, it suffices to show

\begin{equation}
\left| \int \int (U(s)^* f(s), U(t)^* g(t)) \, ds \, dt \right| \leq \Lambda(h)^2 \|f\|_{L^{q'}_t L^{r'}_x} \|g\|_{L^{q'}_t L^{r'}_x}.
\end{equation}

By the interpolation of the bilinear form of (1.1), we have

\begin{equation}
\langle U(s)^* f(s), U(t)^* g(t) \rangle \leq Ch^{-\alpha(1 - \frac{\sigma}{2})}(h + |t - s|)^{-\sigma(1 - \frac{\sigma}{2})} \|f\|_{L^{r'}_t} \|g\|_{L^{r'}_x}.
\end{equation}

Therefore we see by Hölder’s and Young’s inequalities for $\frac{1}{q} + \frac{\sigma}{r} < \frac{\sigma}{2}$

\begin{equation}
\left| \int \int (U(s)^* f(s), U(t)^* g(t)) \, ds \, dt \right| \leq h^{-\alpha(1 - \frac{\sigma}{2})} \int \left( h + |t - s| \right)^{-\sigma(1 - \frac{\sigma}{2})} \|f(t)\|_{L^{r'}_t} \|g(s)\|_{L^{r'}_x} \, dt \, ds \leq h^{-\alpha(1 - \frac{\sigma}{2})} \int \|f\|_{L^{r'}_t} \|g\|_{L^{r'}_x} \, dt \, ds.
\end{equation}

This proves (4.2). \hfill \Box

### 4.2. Homogeneous Strichartz estimates

To prove the homogeneous Strichartz estimates, we first reduce the estimates to frequency localized estimates. Using the Littlewood-Paley frequency cutoff $\varphi_k(\sqrt{H})$, we define

\begin{equation}
(4.3) \quad u_k(t, \cdot) = \varphi_k(\sqrt{H}) u(t, \cdot).
\end{equation}

Notice the frequency cutoffs commute with the operator $H = -\Delta_g$, the frequency localized solutions $\{u_k\}_{k \in \mathbb{Z}}$ satisfy the family of Cauchy problems

\begin{equation}
(4.4) \quad \partial_t^2 u_k + Hu_k = 0, \quad u_k(0) = f_k(z), \quad \partial_t u_k(0) = g_k(z),
\end{equation}

where $f_k = \varphi_k(\sqrt{H}) u_0$ and $g_k = \varphi_k(\sqrt{H}) u_1$. By the squarefunction estimates (2.6) and Minkowski’s inequality, we obtain for $q, r \geq 2$

\begin{equation}
(4.5) \quad \|u\|_{L^q_t L^r_x(M^\circ)} \lesssim \left( \sum_{k \in \mathbb{Z}} \|u_k\|_{L^q_t L^r_x(M^\circ)}^2 \right)^{\frac{1}{2}}.
\end{equation}

Let $U(t) = e^{it\sqrt{H}}$ be the half wave operator, then we write

\begin{equation}
(4.6) \quad u_k(t, z) = \frac{U(t) + U(-t)}{2} f_k + \frac{U(t) - U(-t)}{2i\sqrt{H}} g_k.
\end{equation}

To prove the homogeneous estimates in Theorem 1.1 that is $F = 0$, it suffices to show by (4.5) and (4.6)
Proposition 4.2. Let \( f = \varphi_k(\sqrt{t}) \) for \( k \in \mathbb{Z} \), we have

\[
\|U(t)f\|_{L_t^q L_x^r(\mathbb{R} \times \mathbb{R}^n)} \lesssim 2^{ks} \|f\|_{L_t^2(M^n)},
\]

where the admissible pair \((q, r) \in [2, \infty]^2\) and \( s \) satisfy (4.1) and (1.5).

Now we prove this Proposition. By using Proposition 3.2 and Proposition 3.3, we have the estimates (3.1) for \( U_{i,j}(t) \), where \( \alpha = (n+1)/2, \sigma = (n-1)/2 \) and \( h = 2^{-j} \). Then it follows from Proposition 4.1 that

\[
\|U_{i,j}(t)f\|_{L_t^q L_x^r(\mathbb{R} \times \mathbb{R}^n)} \lesssim 2^{j\|n(\frac{1}{2} - \frac{1}{r}) - \frac{1}{q}\|} \|f\|_{L_t^2(M^n)}.
\]

Notice that

\[
U(t) = \sum_{i=0}^{N} \sum_{j \in \mathbb{Z}} U_{i,j}(t),
\]

we can write

\[
U(t)f = \sum_{i} \sum_{j \in \mathbb{Z}} \int_0^\infty e^{it\lambda} \varphi(2^{-j} \lambda) Q_i(\lambda)dE_{\sqrt{t}}(\lambda) \tilde{\varphi}(2^{-j} \sqrt{t}) f
\]

where \( \tilde{\varphi} \in C_0^\infty(\mathbb{R}) \) takes values in \([0, 1]\) such that \( \tilde{\varphi} \varphi = \varphi \). In view of the condition \( f = \varphi(2^{-k} \sqrt{t}) f \), then \( \tilde{\varphi}(2^{-j} \sqrt{t}) f \) vanishes if \( |j - k| \gg 1 \). Hence we obtain

\[
\|U(t)f\|_{L_t^q L_x^r(\mathbb{R} \times \mathbb{R}^n)} \lesssim 2^{j\|n(\frac{1}{2} - \frac{1}{r}) - \frac{1}{q}\|} \|f\|_{L_t^2(M^n)},
\]

which implies (4.7).

4.3. Inhomogeneous Strichartz estimates. In this subsection, we prove the inhomogeneous Strichartz estimates including the endpoint \( q = 2 \) for \( n \geq 4 \). Let \( U(t) = e^{iH_\tau} : L^2 \to L^2_\tau \). We have already proved that

\[
\|U(t)u_0\|_{L_t^q L_x^r} \lesssim \|u_0\|_{H^s}
\]

holds for all \((q, r, s)\) satisfying (1.4) and (1.5). For \( s \in \mathbb{R} \) and \((q, r)\) satisfying (1.4) and (1.5), we define the operator \( T_s \) by

\[
T_s : L^2 \to L^2_t L^r_x, \quad f \mapsto H^{-\frac{s}{2}} e^{it\sqrt{H}} f.
\]

Then we have by duality

\[
T_{1-s}^*: L^q_t L^r_x \to L^2, \quad F(\tau, z) \mapsto \int_\mathbb{R} H^{\frac{s-1}{2}} e^{-i\tau \sqrt{H}} F(\tau) d\tau,
\]

where \( 1 - s = n(\frac{1}{2} - \frac{1}{q}) - \frac{1}{2} \). Therefore we obtain

\[
\left\| \int_\mathbb{R} U(t)U^*(\tau) H^{-\frac{s}{2}} F(\tau) d\tau \right\|_{L_t^q L_x^r} = \left\| T_s T_{1-s}^* F \right\|_{L_t^q L_x^r} \lesssim \|F\|_{L_t^q L_x^r}.
\]

Since \( s = n(\frac{1}{2} - \frac{1}{q}) - \frac{1}{2} \) and \( 1 - s = n(\frac{1}{2} - \frac{1}{q}) - \frac{1}{2} \), thus \((q, r)\) satisfy (1.5). By the Christ-Kiselev lemma \[9\], we thus obtain for \( q > \tilde{q}' \),

\[
\left\| \int_{\tau < t} \frac{\sin(t - \tau) \sqrt{H}}{\sqrt{H}} F(\tau) d\tau \right\|_{L_t^q L_x^r} \lesssim \|F\|_{L_t^q L_x^r}.
\]

Notice that for all \((q, r)\) satisfying (1.4) and (1.5), we must have \( q > \tilde{q}' \). Therefore we have proved all inhomogeneous Strichartz estimates including the endpoint \( q = 2 \).
5. Wellposedness and Scattering

In this section, we prove Theorem 1.3. We prove the result by a contraction mapping argument. The key point is the application of Strichartz estimates. Let \( q_0 = (n+1)(p-1)/2, q_1 = 2(n+1)/(n-1) \) and \( \alpha = s_c - \frac{1}{2} \). For any small constant \( \epsilon > 0 \) such that \( 2 \epsilon < \epsilon(p) \) given by (1.3), there exists \( T > 0 \) such that

\[
X := \left\{ u : u \in C_t(\dot{H}^s) \cap L^{q_0}([0, T]; L^{q_0}(M)) \cap L^{q_1}([0, T]; \dot{H}^s_{q_1}(M^0)), \right. \\
\left. \|u\|_{L^{q_0}([0, T]; L^{q_0}(M))} + \|u\|_{L^{q_1}([0, T]; \dot{H}^s_{q_1}(M^0))} \leq C\epsilon \right\}.
\]

(5.1)

Consider the solution map \( \Phi \) defined by

\[
\Phi(u) = \cos(t\sqrt{H})u_0(z) + \frac{\sin(t\sqrt{H})}{\sqrt{H}}u_1(z) + \int_0^t \frac{\sin((t-s)\sqrt{H})}{\sqrt{H}}F(u(s, z))ds
\]

where \( F(u) = \gamma|u|^{p-1}u \). We claim the map \( \Phi : X \to X \) is contracting. Indeed, by Theorem 1.1, we obtain

\[
\|\Phi(u)\|_{C_t(\dot{H}^s)\cap L^{q_0}([0, T]; L^{q_0}(M))\cap L^{q_1}([0, T]; \dot{H}^s_{q_1}(M^0))} \leq C(\|u_0\|_{\dot{H}^s} + \|u_1\|_{\dot{H}^s_{q_1}}).
\]

(5.2)

Hence we must have

\[
\|\Phi(u)\|_{C_t(\dot{H}^s)\cap L^{q_0}([0, T]; L^{q_0}(M))\cap L^{q_1}([0, T]; \dot{H}^s_{q_1}(M^0))} \leq \frac{1}{2} C\epsilon
\]

for \( T = \infty \) if the initial data has small norm \( \epsilon(p) \), or, if not, this inequality will be satisfied for some \( T > 0 \) by the dominated convergence theorem. We first note that the Sobolev embedding \( L^{q_0}_{t,z} \dot{H}^s_{q_0} \hookrightarrow L^{q_0}_{t,z} \) where \( r_0 = 2n(n+1)(p-1)/[(n^2-1)(p-1)-4] \).

Under the condition \( p \in \left[p_{\text{conf}}, 1 + \frac{4}{n-2}\right] \), it is easy to check that the pairs \((q_0, r_0), (q_1, q_1)\) satisfy (1.4) and (1.5) with \( s = 1/2 \). Applying Theorem 1.1 with \( q' = \tilde{r}' = \frac{2(n+1)}{n+3} \), one has

\[
\|u_{\text{inh}}\|_{C_t(\dot{H}^s)\cap L^{q_0}([0, T]; L^{q_0}(M))\cap L^{q_1}([0, T]; \dot{H}^s_{q_1}(M^0))} \leq C\|F(u)\|_{L^{q_0}_{t,z} \dot{H}^{s_{q_0}}}
\]

(5.3)

By the assumption on \( p \), we have \( 0 \leq s \leq 1 \). By using the fraction Liebniz rule for Sobolev spaces on the asymptotically conic manifold [3, Theorem 27], we have

\[
\|F(u)\|_{L^{q_0}_{t,z} \dot{H}^{s_{q_0}}} \leq C\|u\|^{p-1}_{L^{q_0}_{t,z}} \|u\|_{L^{q_1}(M)} \dot{H}^s_{q_1} \leq C^2(C\epsilon)^{p-1} \epsilon \leq \frac{C\epsilon}{2}.
\]

(5.4)

A similar argument as above leads to

\[
\|\Phi(u_1) - \Phi(u_2)\|_{L^{q_0}([0, T]; \dot{H}^s_{q_0}(M^0))\cap L^{q_0}([0, T]; L^{q_0}(M^0))}
\]

(5.5)

\[
\leq C\|F(u_1) - F(u_2)\|_{L^{q_0}_{t,z} \dot{H}^{s_{q_0}}} \\
\leq C^2(C\epsilon)^{p-1} \|u_1 - u_2\|_{L^{q_1}([0, T]; \dot{H}^s_{q_1}(M^0))\cap L^{q_0}([0, T]; L^{q_0}(M^0))} \\
\leq \frac{1}{2} \|u_1 - u_2\|_{L^{q_0}([0, T]; \dot{H}^s_{q_0}(M^0))\cap L^{q_0}([0, T]; L^{q_0}(M^0))}.
\]

(5.6)

Therefore the solution map \( \Phi \) is a contraction map on \( X \) under the metric \( d(u_1, u_2) = \|u_1 - u_2\|_{L^{q_0}([0, T]; \dot{H}^s_{q_0}(M^0))\cap L^{q_0}([0, T]; L^{q_0}(M^0))} \). The standard contraction argument completes the proof of Theorem 1.3.
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