Deformed photon-added nonlinear coherent states and their non-classical properties

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Abstract

In this paper, we will try to present a general formalism for the construction of deformed photon-added nonlinear coherent states (DPANCSs) $|\alpha, f, m\rangle$, which in a special case lead to the well-known photon-added coherent state (PACS) $|\alpha, m\rangle$. Some algebraic structures of the introduced DPANCSs are studied and particularly the resolution of the identity, as the most important property of generalized coherent states, is investigated. Meanwhile, it will be demonstrated that the introduced states can also be classified in the $f$-deformed coherent states, with a special nonlinearity function. Next, we will show that these states can be produced through a simple theoretical scheme. A discussion on the DPANCSs with negative values of $m$, i.e. $|\alpha, f, -m\rangle$, is then presented.

Our approach has the potentiality to be used for the construction of a variety of new classes of DPANCSs, corresponding to any nonlinear oscillator with known nonlinearity function, as well as arbitrary solvable quantum system with known discrete, non-degenerate spectrum. Finally, after applying the formalism to a particular physical system known as the Pöschl–Teller (P-T) potential and the nonlinear coherent states corresponding to a specific nonlinearity function $f(n) = \sqrt{n}$, some of the non-classical properties, such as the Mandel parameter, second-order correlation function, in addition to first- and second-order squeezing of the corresponding states, will be investigated numerically.

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1. Introduction

As was shown by Agarwal and Tara, ‘photon-added coherent states’ (PACSs) are obtained by iterated actions ($m$ times) of the bosonic creation operator $a^\dagger$ on the standard coherent states

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The explicit form of these states has been expressed as follows [1]:

\[ |α, m⟩ = \exp \left( -\frac{|α|^2}{2} \right) \sum_{n=0}^{∞} \frac{α^n \sqrt{(n+m)!}}{n!} |n+m⟩, \]

(1)

where \( m \) is a non-negative integer and \( L_m(x) \) is the \( m \)th order of the Laguerre polynomial. These states exhibit non-classical features like squeezing and sub-Poissonian statistics. Besides, the nonlinear coherent state is defined as the solution of the eigenvalue equation

\[ A |α, f⟩ = α |α, f⟩, \]

(2)

with the decomposition in the number states’ space as [2]

\[ |α, f⟩ = N(|α|^2)^{-\frac{1}{2}} \sum_{n=0}^{∞} \frac{α^n}{\sqrt{n!} [f(n)]!} |n⟩, \]

(3)

where \( n = a^†a \) is the number operator, \( A = α f(n) \) is the \( f \)-deformed annihilation operator, \( f(n) \) is an intensity-dependent function, \([f(n)]! = f(n) f(n-1) \cdots f(1)\) and by convention \([f(0)]! = 1\]. It is shown that the states in (1) may be considered as nonlinear coherent states with \( f(n,m) = 1 - m/(1+n) \) [3]. Consequently, the eigenvalue equation \( f(n,m) a |α, m⟩ = α |α, m⟩ \) is established. Using the Stieltjes power-moment problem, the over-completeness of PACSs is explicitly shown in [4]. Photon-added and photon-subtracted coherent states associated with inverse \( q \)-boson operators were introduced in [5]. Wave packet dynamics of PACSs has been investigated in [6]. Higher-order squeezing and higher-order sub-Poissonian statistics of PACSs have been studied in [7], and squeezing and higher-order squeezing of PACSs propagating in a Kerr-like medium have been discussed in [8]. Dynamical squeezing of PACSs was investigated in [9]. Generalized hypergeometric photon-added and photon-depleted coherent states were introduced in [10] and PACSs for exactly solvable Hamiltonian studied in [11]. Photon-added Barut–Girardello coherent states of the pseudo-harmonic oscillator have been constructed in [12] and recently generation of coherent states of photon-added type via pathway of eigenfunctions has been argued in [14].

On the other hand, the experimental scheme for the generation of PACSs may be found in the literature. Among them, we may refer to [15] and especially to the recent work of Zavatta et al in which \(|α, 1⟩\) has been produced experimentally by using a parametric amplifier consisting of a type I beta-barium borate down-conversion crystal [16].

The goal of the present contribution is to introduce a formalism for the construction of deformed photon-added nonlinear coherent states (DPANCSs) by iterated actions of an ‘\( f \)-deformed creation operator’ on a ‘nonlinear coherent state’. A deep insight into the works that have been performed previously [1–14] in comparison with our work shows that we indeed deformed both \( a^† \) (creation operator) and \(|α⟩\) (coherent state), respectively, to \( A^† \) and \(|α, f⟩\). Hence, our presentation is essentially different with respect to earlier ones, through which we will get new results.

This paper is organized as follows. After the introduction of the explicit form of DPANCSs in the next section, the algebraic structure of the states is investigated in section 3, from which we deduce an appropriate nonlinearity function associated with the introduced states. Section 4 deals with the resolution of the identity of the DPANCSs, and then a simple scheme for their generation is presented in section 5. Then, after applying the proposed approach to the Pöschl–Teller potential (P-T) and the nonlinear coherent states corresponding to the nonlinearity function \( f(n) = \sqrt{n} \), as some physical realizations of the formalism, the corresponding DPANCSs are introduced and the non-classicality features of the associated states are numerically investigated in section 6. Next, in section 7, DPANCSs with negative values of \( m \) are discussed. At last, we conclude the paper in section 8.
2. Introducing the general structure of DPANCSs

Recall that the actions of \( f \)-deformed annihilation and creation operators on the number states are expressed, respectively, by

\[
A|n\rangle = f(n)\sqrt{n}|n-1\rangle \quad \text{and} \quad A^\dagger|n\rangle = f(n+1)\sqrt{n+1}|n+1\rangle.
\]

Following the terminology of Solomon in [17], since we are also working in the deformed quantum optics field and the photons annihilate or create by the actions of the relevant \( f \)-deformed ladder operators, the notion of ‘deformed photon’ seems to be reasonable for distinguishing them from usual bosonic counterpart. In this section, we introduce a new family of coherent states, which we call DPANCSs, using the definition

\[
|\alpha, f, m\rangle = N_{\alpha}^{m,f}A^\dagger_m|\alpha, f\rangle, \quad m \in \mathbb{Z}^+,
\]

where \( |\alpha, f\rangle \) is in general any class of nonlinear coherent states introduced in (2) and \( N_{\alpha}^{m,f} \) is an appropriate normalization constant that may be determined. It is straightforward to obtain the explicit form of DPANCSs in terms of Fock states by a standard procedure. The final result reads

\[
|\alpha, f, m\rangle = N_{\alpha}^{m,f}\left(\sum_{n=0}^{\infty} |\alpha|^{2n}n^{f(2n)}!\right)^{-\frac{1}{2}}\sum_{n=0}^{\infty}\frac{\alpha^n[f(n+m)]!\sqrt{(n+m)!}f(n)^n}{n![f^2(n)]!}|n+m\rangle,
\]

with the normalization factor

\[
N_{\alpha}^{m,f} = \left(\sum_{n=0}^{\infty} |\alpha|^{2n}n^{f(2n)}!\right)^{-\frac{1}{2}}\left(\sum_{n=0}^{\infty}\frac{|\alpha|^{2n}(n+m)!f^2(n+m)!}{(n!)^2[f^2(n)]!}\right)^{-\frac{1}{2}}.
\]

In obtaining (5), we have utilized the relation

\[
A^\dagger_m = \frac{[f(n)]!}{[f(n-m)]!}a^\dagger_m.
\]

As a clear fact, note that the DPANCSs in (5) reduce to PACSs in (1), when \( f(n) = 1 \). It is worth mentioning that the number states \{\( |0\rangle, |1\rangle, \ldots, |m-1\rangle \} are absent from the DPANCSs in (5). This situation is exactly similar to PACSs of Agarwal and Tara [1].

3. The algebra structure of DPANCSs

Now, we show that DPANCSs can also be interpreted as \( f \)-deformed coherent states with a specific nonlinearity function. This may be done via demonstrating the fact that the DPANCS may be re-obtained from the eigenvalue equation

\[
f_{\alpha}(n, f, m)a|\alpha, f, m\rangle = \alpha|\alpha, f, m\rangle.
\]

Noting that the non-canonical commutation relation between the \( f \)-deformed ladder operators reads

\[
[A, A^\dagger] = (n+1)f^2(n+1) - nf^2(n);
\]

accordingly, it is convenient to show that

\[
[A, A^\dagger_m] = d^{m-1}\frac{[f(n+m-1)]!(n+m)f^2(n+m) - nf^2(n)}{[f(n)]!}.
\]

Next, due to the identity

\[
a^\dagger_m f(n) = f(n-m)a^\dagger_m,
\]

the right-hand side of equation (10) can be converted to

\[
[A, A^\dagger_m] = g(n, m)A^\dagger_{m-1},
\]

where
where relation (7) is used and we have set
\[ g(n, m) \equiv (n + 1) f^2(n + 1) - (n - m + 1) f^2(n - m + 1). \]
Multiplying both sides of equation (2) from the left by \( A^\dagger m \) yields
\[ A^\dagger m A |\alpha, f\rangle = \alpha A^\dagger m |\alpha, f\rangle. \] (13)

The commutation relation in (12) helps us to rewrite the latter equation as
\[ (A A^\dagger m - g(n, m) A^\dagger m - 1) |\alpha, f\rangle = \alpha A^\dagger m |\alpha, f\rangle. \] (14)

At last, making use of the identity \( A^\dagger - 1 = \frac{1}{(n+1)f(n+1)} \) \[ \text{cf.} \] [18] leads us to the following eigenvalue equation:
\[ (f(n + 1) - g(n, m)(n + 1)f(n + 1)) a |\alpha, f, m\rangle = \alpha |\alpha, f, m\rangle. \] (15)

Comparing equations (15) and (8) gives the form of the nonlinearity function associated with DPANCS as follows:
\[ f_d(n, f, m) = (n - m + 1)f^2(n - m + 1)(n + 1)f(n + 1), \] (16)

where the nonlinearity function \( f(\cdot) \) appearing on the right-hand side is determined by the nonlinearity of the original nonlinear coherent states, \( |\alpha, f, m\rangle \) in (5). So, we have finally succeeded in establishing the DPANCSs as \( f_d \)-deformed coherent states, too. Clearly, setting \( f(n) = 1 \) \[ \text{in} \] (16), one readily obtains \( f_d(n, f, m) = 1 - \frac{m}{(n+1)} \), which is the nonlinearity function of PACSs \[ \text{[3].} \]

4. Resolution of the identity of DPANCSs

We noted that the DPANCS in (5) is a superposition of all number states starting with \( |m\rangle \). Following the path of \[ \text{[4, 12],} \] the unity operator in such a subspace of the total Hilbert space spanned by the basis \( \{|n\rangle\}_{n=m}^{\infty} \) has been written as
\[ \hat{I}(m) = \sum_{n=m}^{\infty} |n\rangle \langle n| = \sum_{n=0}^{\infty} |n + m\rangle \langle n + m|. \] (17)

To be precise, the name unity operator for \( \hat{I}(m) \) seems to be unsuitable and it is more reasonable to be called the projection operator on the relevant subspace. This operator is bounded and positive valued with a densely defined inverse [13].

So, in such a case which we deal with, the (generalized) resolution of the identity takes the form
\[ \frac{1}{\pi} \int_D d^2\alpha W(|\alpha|^2) |\alpha, f, m\rangle \langle \alpha, f, m| = \hat{I}(m), \] (18)
where \( W(|\alpha|^2) \) is a positive weight function and \( D \) expresses the domain of the coherent states centered at the origin of the complex plane, both of which may be appropriately determined. Generally, \( D \) may be an entire plane or a finite disk centered at the origin, depending on the particular \( f(n) \) chosen. However, since in the continuation of the paper we will deal with the first type, in what follows we have set infinity in the upper bounds of the integrals, i.e. the Stieltjes moment problem has been encountered. Here, \( \alpha = r e^{i\phi} \) and \( d^2\alpha = r dr d\phi \). By substituting equation (5) into (18), one obtains
\[ 2 \sum_{n=0}^{\infty} \int_0^{\infty} dr r^{2n+1} W(r^2) \left( N^m_a \right)^2 \left( \sum_{n=0}^{\infty} \frac{r^{2n}}{n! [f^2(n)]^n} \right)^{-1} \times \frac{(n + m)! [f^2(n + m)]!}{(n!)^2 [f^4(n)]!} |n + m\rangle \langle n + m| = \hat{I}(m), \] (19)
where we have utilized \( \int_0^{2\pi} d\phi e^{i(n-n')} = 2\pi \delta_{nn'} \). Considering the following expression for weight function:

\[
W(r^2) = (N_{m/f})^{-2} \exp \left( \sum_{n=0}^{\infty} \frac{r^{2n}}{n! [f^2(n)]!} \right) r^{2m} \tilde{W}(r^2),
\]

we may rewrite (19) as

\[
2 \sum_{n=\nu}^{\infty} \int_0^\infty dr \ r^{2n+2m+1} \tilde{W}(r^2) r^{2n} \tilde{W}(r^2) = \frac{(n!)[f^2(n)]!}{(n+m)[f^2(n+m)]!} |n+m\rangle \langle n+m| = \hat{I}^{(m)}.
\]

Obviously, to satisfy this equation, the following moment integral should hold:

\[
2 \int_0^\infty dr \ r^{2n+2m+1} \tilde{W}(r^2) = \frac{(n!)[f^2(n)]!}{(n+m)[f^2(n+m)]!}.
\]

Finally, after performing the change in variable \( r^2 = x \) and replacing \( n+m \) by \( k-1 \), we arrive at

\[
\int_0^\infty x^{k-1} \tilde{W}(x) \, dx = \frac{(k-m-1)![f^2(k-m-1)]]!(k-1)!}{(k-1)!(k+2)(k-1)!}.
\]

As is clear, prior to investigating this property, the explicit form of the nonlinearity function, i.e. the particular physical system, must be specified.

5. Generation of the DPANCSs

In order to produce the DPANCSs in (5) physically, we consider the slab of excited two-level atoms through a cavity. Let the initial state of the atom–field system is expressed by \(|\Psi(0)\rangle = |\alpha, f\rangle |e\rangle\), where \(|e\rangle\) is the excited state of the atom and \(|\alpha, f\rangle\) is the nonlinear coherent state field. The interaction Hamiltonian assumes to have the following configuration:

\[
\hat{H} = \hbar g (\sigma_+ A + A^\dagger \sigma_-),
\]

where \(A, A^\dagger\) are the \( f\)-deformed ladder operators and \(\sigma_+, \sigma_-\) are respectively the raising and lowering operators of atomic states. In other words, a deeper insight into our proposed Hamiltonian in (24) shows that we have changed the coupling constant \(g\) to an alternative coupling \(g f(n)\), i.e. our setup works with an intensity-dependent coupling. The initial state \(|\Psi(0)\rangle\) evolves in time according to

\[
|\Psi(t)\rangle = \exp[-i \eta (\sigma_+ A + A^\dagger \sigma_-)] |\Psi(0)\rangle,
\]

where we have set \( \eta \equiv gt \) and \(g\) is the coupling constant. For \( \eta \ll 1 \), one has

\[
|\Psi(t)\rangle \approx (1 - i \eta (\sigma_+ A + A^\dagger \sigma_-)) |\alpha, f\rangle |e\rangle.
\]

Thus, we will have the simple form of the state vector of the whole atom–field system as follows:

\[
|\Psi(t)\rangle = |\alpha, f\rangle |e\rangle - i \eta A^\dagger |\alpha, f\rangle |g\rangle.
\]

Therefore, if the atom is detected in the ground state \(|g\rangle\), then the state of the field is transferred to \( A^\dagger |\alpha, f\rangle\), which is indeed the DPANCS \(|\alpha, f, 1\rangle\). Hence, we could, in principle, produce the state \(|\alpha, f, 1\rangle\). Generalizing the above procedure, one can easily produce, in principle, DPANCSs with arbitrary values of \(m\), by using the Hamiltonian

\[
\hat{H}_m = \hbar g (\sigma_+ A^m + A^\dagger^m \sigma_-).
\]

Clearly, the state \(|\alpha, f, m\rangle\) can be produced using an appropriate \(m\)-photon medium.
6. Physical properties of the DPANCSs

In this section, we briefly explain some of the ordinarily helpful criteria in the relevant literature, which will be used for investigating the non-classicality exhibition of our introduced states. For this purpose, we refer to the sub-Poissonian statistics, antibunching phenomenon, quadrature squeezing and finally amplitude-squared squeezing. A common feature of all of the above criteria is that the corresponding Glauber–Sudarshan P-function of a non-classical state is not positive definite. But, we would like to imply that finding this function is usually a hard task to do. Altogether, each of the above effects, which will be considered in the paper, is in fact sufficient for a quantum state to belong to non-classical states.

6.1. Non-classicality criteria

Now, we are ready to introduce some of the non-classicality signs which are widely used in the literature. They will help us to investigate the non-classicality features of the introduced states in (5), corresponding to any chosen physical system.

- **Photon-counting statistics of the states** is investigated by evaluating the Mandel parameter that has been defined as [24]

\[
Q = \frac{\langle n^2 \rangle - \langle n \rangle^2}{\langle n \rangle} - 1. \tag{29}
\]

The states for which \( Q = 0 \), \( Q < 0 \) and \( Q > 0 \), respectively, correspond to the Poissonian (standard coherent states), sub-Poissonian (non-classical states) and super-Poissonian (classical states) statistics.

- **Although there are quantum states in which super-/sub-Poissonian statistical behavior is appeared with bunching/antibunching effect, this is not absolutely true.** To investigate bunching or antibunching effects, the second-order correlation function is widely used, which is defined as follows [25]:

\[
g^2(0) = \frac{\langle a^\dagger a^2 \rangle}{\langle a^\dagger a \rangle^2}. \tag{30}
\]

Depending on the specific nonlinearity function \( f(n) \), which has been chosen for the construction of coherent states, \( g^2(0) > 1 \) (\( g^2(0) < 1 \)) indicates the bunching (antibunching) effect. The case \( g^2(0) = 1 \) corresponds particularly to the canonical coherent states.

- **In order to examine the quantum fluctuations of the quadratures of the field,** we consider the Hermitian operators \( x = (a + a^\dagger)/\sqrt{2} \) and \( p = (a - a^\dagger)/i\sqrt{2} \) with commutation relation \([x, p] = i\). With the help of the common definitions of the variances of position and momentum, the following parameters may be defined: \( s_x = \frac{\langle x^4 \rangle - \langle x^2 \rangle}{0.5} \) and \( s_p = \frac{\langle p^4 \rangle - \langle p^2 \rangle}{0.5} \), respectively, for quadrature squeezing in \( x \) and \( p \). These squeezing parameters can be re-written as follows:

\[
s_x = 2\langle a^\dagger a \rangle + \langle a^2 \rangle + \langle a^4 \rangle - \langle a \rangle^2 - \langle a^2 \rangle - 2\langle a \rangle \langle a^\dagger \rangle \tag{31}
\]

and similarly for \( p \) as

\[
s_p = 2\langle a^\dagger a \rangle - \langle a^2 \rangle - \langle a^4 \rangle + \langle a^2 \rangle + \langle a^4 \rangle - 2\langle a \rangle \langle a^\dagger \rangle. \tag{32}
\]

A state is squeezed in \( x \) or \( p \) if it satisfies the inequalities \(-1 \leq s_x < 0 \) or \(-1 \leq s_p < 0 \), respectively.
6.2. Physical properties of the DPANCSs associated with the Pöschl–Teller (P-T) potential

In this subsection, we apply the mathematical-physics structure of DPANCSs presented in section 2 to a well-known physical system, i.e. P-T potential, which has its specific importance in atomic and molecular physics (see [19] and references therein). This system possesses the following non-degenerate spectrum: $e_n = n(n + \nu), \quad \nu > 2$. The special case $\nu = 2$ characterizes the one-dimensional square potential well. The nonlinearity function corresponding to this system according to the formalism proposed in [20, 21] may be easily obtained as

$$f(n) = \sqrt{n + \nu}. \quad (36)$$

Inserting (36) in (5), one can easily create the explicit form of DPANCSs associated with the P-T potential. We continue our study by discussing some of the quantum statistical properties and non-classicality features of the DPANCSs associated with the mentioned system. This investigation seems to be necessary, due to the fact that even though the nonlinear coherent states mostly possess less or more of the non-classicality signs, but there exist nonlinear coherent states which do not show either of the usual non-classicality criteria [23]. Altogether, before paying attention to this subject, we would like to establish the resolution of the identity requirement for the introduced states.

6.2.1. Resolution of the identity for the DPANCSs of the P-T potential. Due to the central importance of the resolution of the identity for any class of coherent states, we examine this property according to (23) by using the nonlinearity function of the P-T potential, i.e.

$$\int_0^\infty x^{k-1} \tilde{W}(x) \, dx = \frac{(m + v)! [\Gamma(k - m)]^2 [\Gamma(k + m + v)]^2}{(v!)^2 \Gamma(k + v) \Gamma(k)}. \quad (37)$$

With the help of the definition of Meijer’s G-function, it follows that [22]

$$\int_0^\infty dx x^{k-1} G_{\nu,n}^{m,n} \left( \beta x \left\vert \begin{array}{c} a_1, \ldots, a_m, a_{m+1}, \ldots, a_p \\ b_1, \ldots, b_m, b_{m+1}, \ldots, b_q \end{array} \right. \right)
= \frac{1}{\beta^k} \frac{\prod_{j=m+1}^{q} \Gamma(b_j + k)}{\prod_{j=m+1}^{q} \Gamma(1 - b_j - k)} \frac{\prod_{j=m+1}^{q} \Gamma(1 - a_j - k)}{\prod_{j=m+1}^{q} \Gamma(a_j + k)}. \quad (38)$$

\[\]
Comparing equations (37) and (38), it is easy to obtain

$$\tilde{W}(x) = \frac{(m + \nu)!}{(\nu!)^2} G_{2,4}^{4,0} \left( x \left| \begin{array}{c} 0, \nu \\ -m, -m, \nu - m, \nu - m \end{array} \right. \right).$$

(39)

Therefore, via using the above results in (20) and after setting $|\alpha|^2 = x$, the weight function finally takes the form

$$W(x) = G_{2,4}^{1,2} \left( -x \left| \begin{array}{c} -m, -\nu - m \\ 0, 0, -\nu, -\nu \end{array} \right. \right) G_{2,4}^{4,0} \left( x \left| \begin{array}{c} m, \nu + m \\ 0, 0, \nu, \nu \end{array} \right. \right),$$

(40)

where we have utilized the relation [12]

$$x^s G_{\alpha_1,\beta_1}^{m,n} \left( x \left| \begin{array}{c} \alpha_p \\ \beta_q \end{array} \right. \right) = G_{\alpha_1,\beta_1}^{m,n} \left( x \left| \begin{array}{c} \alpha_p + s \\ \beta_q + s \end{array} \right. \right),$$

(41)

and

$$N_{\alpha_1}^{m,n} = [\nu! (\nu^2) I_{\nu}(2\sqrt{x})]^{1/2} \left[ \frac{(\nu!)^2}{(m + \nu)!} G_{2,4}^{1,2} \left( -x \left| \begin{array}{c} -m, -\nu - m \\ 0, 0, -\nu, -\nu \end{array} \right. \right) \right]^{\nu/2},$$

(42)

which is indeed the closed form of the normalization factor for the DPANCSs corresponding to the P-T potential, and $I_\nu(x)$ is the modified Bessel function of the first kind.

6.2.2. Numerical results of the DPANCSs associated with the P-T potential. Using the criteria mentioned in subsection 6.1, we will argue and investigate the non-classicality of DPANCSs associated with the P-T potential. This can be considered as a physical realization of our proposed formalism. Firstly, figure 1 displays the weight function versus $x$ for different values of $m$ and a fixed value of $\nu = 3$. These results signify the positivity of $W(x)$. It is seen that $W(x)$ has a singularity at $x = 0$, but it tends to zero for $x \to \infty$. In figure 2, $W(x)$ has been plotted versus $x$ for fixed $m = 1$ and various values of $\nu$. The general behavior of the weight function in this case is the same as in figure 1. We continue with investigating the non-classicality of...
Figure 2. The plot of $W(x)$ as a function of $x$, with a fixed parameter $m = 1$ and different values of $\nu$ for the DAPNCS associated with the P-T potential. The continuous line is for $\nu = 7$, the dotted line is for $\nu = 5$ and the dashed line is for $\nu = 3$.

Figure 3. The variation of $Q$ as a function of $\alpha \in \mathbb{R}$, with a fixed parameter $\nu = 3$ and different values of $m$ for the DPANCS associated with the P-T potential. The continuous line is for DPANCSs and $m = 2$, the dashed line is for DPANCSs and $m = 5$, the dot-dashed line is for PACSs and $m = 2$, and the dotted line is for PACSs and $m = 5$.

the introduced states. For this purpose, by using the required mean values (see appendix), we have plotted the Mandel parameter, second-order correlation function, quadrature squeezing and amplitude-squared squeezing for the DPANCSs of the P-T potential versus real $\alpha$, for various values of $m$ and fixed $\nu = 3$. As shown in figure 3, the Mandel parameter is always negative, and so the sub-Poissonian behavior is visible. It is clearly seen that this parameter for the DPANCSs of the P-T potential is more negative than that for the PACSs ($f(n) = 1$). So, our deformation increases the depth of the non-classicality of these states. Besides, increasing $m$ results in an increase in the non-classicality of DPANCSs analogously to the results of the PACSs. Meanwhile, for large values of $\alpha$ in both PACSs and DPANCSs, the $Q$ parameters coincide with each other for different chosen values of $m$. It indeed tends to a finite negative
value for large $\alpha$. According to figure 4, it is visible that $g^2(0) < 1$ for small values of $\alpha$ and so the antibunching effect occurs. This observation illustrates that the sub-Poissonian statistics and antibunching effect occur simultaneously in this finite range, as one may compare figures 3 and 4. But our further calculations for larger $\alpha$ (and certainly the same fixed parameters) show that while the Mandel parameter tends to a finite negative value ($\approx -0.5$), the correlation function tends to $\approx 1$, which corresponds to the correlation function of the canonical coherent state. So, while the states in hand have sub-Poissonian statistics, they do not show the antibunching effect for large $\alpha$. It is noticeable that, in this case, comparing the two distinct non-classicality criteria, the Mandel parameter is more sensitive than the second-order correlation function. Squeezing parameters have been plotted in figure 5. We conclude from our numerical results which are presented in figure 5(a) that the DPANCSs are squeezed in the $x$-quadrature, in a wide region of $\alpha$, with no squeezing in the $p$-quadrature (see figure 5(b)). It is evident that for large values of $\alpha$, $s_x$ and $s_p$ respectively tend to $-0.5$ and 1. From figure 5(c) it is visible that in some regions of space, $S_X$ gets negative values, i.e. amplitude-squared squeezing in $X$ appears. But, as observed from figure 5(d), $S_P$ is always positive, i.e. no amplitude-squared squeezing in $P$ may be seen. Our further calculations confirm that with an increase in the values of $\alpha$, $S_X$ and $S_P$ respectively tend to $\approx -0.5$ and 1. Obviously, all of the limiting quantities are correct for the mentioned fixed parameters.

6.3. Numerical results of the DPANCSs for a nonlinear coherent state with $f(n) = \sqrt{n}$

As a second example, we work with the original nonlinear coherent states corresponding to the nonlinearity function $f(n) = \sqrt{n}$. The physical interest in the nonlinear coherent states constructed by this function comes out from the fact that it, indeed, rises in a natural way in Hamiltonians illustrating the interaction with intensity-dependent coupling between a two-level atom and an electromagnetic radiation field [27, 28]. Considering this nonlinearity function, the weight function may be straightforwardly obtained as follows:

$$W(x) = (m!)^2 \, _2F_3(1 + m, 1 + m; 1, 1, 1; x) G^{4,0}_{2,4} \left( x \begin{array}{c} m, m \\ 0, 0, 0, 0 \end{array} \right).$$

Figure 4. The variation of $g^2(0)$ as a function of $\alpha \in \mathbb{R}$, with a fixed parameters $\nu = 3$ and different values of $m$ for the DPANCS associated with the P-T potential. The continuous line is for $m = 1$, the dashed line is for $m = 2$ and the dotted line is for $m = 3$. 
Figure 5. Squeezing parameters as a function of $\alpha \in \mathbb{R}$, with a fixed parameter $\nu = 3$ and different values of $m$ for the DPANCS associated with the P-T potential. (a) The variation of $s_x$: the continuous line is for $m = 1$, the dotted line is for $m = 2$ and the dashed line is for $m = 3$; (b) the same as (a) except that it is plotted for $s_p$; (c) the variation of $S_x$: the continuous line is for $m = 1$, the dotted line is for $m = 3$ and the dashed line is for $m = 5$; (d) the same as (c) except that it is plotted for $S_p$.

where $\,_pF_q(a; b; x)$ is the generalized hypergeometric function. Figure 6 displays the weight function versus $x$ for different values of $m$. The positivity of $W(x)$ is revealed which confirms that the obtained DPANCSs are in fact of coherent state type, in its exact meaning.

Our aim is to produce the DPANCSs associated with this particular system and investigate their physical properties. Inserting the function $f(n) = \sqrt{n}$ in (5), one can easily create the explicit form of the associated DPANCSs. To proceed further, one needs to use the relations which are presented in the appendix, setting $f(n) = \sqrt{n}$. For this purpose, the normalization factor of the related DPANCSs is required, which may be determined as

$$N^m,f = \sqrt{\frac{I_0(2\sqrt{x})}{m!}} [\,_2F_3(1 + m, 1 + m; 1, 1, 1; x)]^{-1/2},$$

where $I_0(x)$ is the modified Bessel function of the first kind and $\,_pF_q(a; b; x)$ is the generalized hypergeometric function. Henceforth, we are ready to continue with investigating the non-classicality of the associated states. We have plotted the Mandel parameter, second-order correlation function, quadrature squeezing and amplitude-squared squeezing for the corresponding DPANCSs versus real $\alpha$, for various values of $m$. The Mandel parameter, shown in figure 7, is always negative and so the sub-Poissonian behavior is visible. It is clearly seen that this parameter for the DPANCSs of the chosen nonlinearity function is more negative than for PACSs ($f(n) = 1$). Therefore, our new deformation also increases the depth.
of the non-classicality. Besides, increasing $m$ results in an increase of the non-classicality of DPANCSs analogously to the numerical results of PACSs and DPANCSs for the P-T potential. With increasing $\alpha$ in both PACSs and DPANCSs, the corresponding $Q$ parameters coincide with each other for different chosen values of $m$. Interestingly, it is worth noticing that, while in the case of PACSs for large $\alpha$, $Q$ tends to zero (non-classicality disappears), this is not so for DPANCSs, again showing the strong non-classicality behavior of the introduced states. According to figure 8, it is visible that $g^2(0) < 1$ for small enough values of $\alpha$ and so the antibunching effect will appear. This observation illustrates that the sub-Poissonian statistics and antibunching effect occur simultaneously in this finite range, as one may compare figures 7 and 8. But our further calculations for larger $\alpha$ show that while the Mandel parameter tends to a finite negative value ($\approx -0.6$), the correlation function tends to $\approx 1$,
which corresponds to the correlation function of a canonical coherent state. So the presented states have sub-Poissonian statistics with no antibunching effect for large $\alpha$. Therefore, we may conclude that comparing the above two non-classicality criteria, the Mandel parameter is more sensitive than the second-order correlation function. Squeezing parameters have been plotted in figure 9. It is obvious from our numerical results presented in figure 9(a) that the corresponding DPANCSs are squeezed in the $x$-quadrature, in a wide region of $\alpha \geq 1.75$, while no squeezing is seen in the $p$-quadrature (see figure 9(b)). It is also evident that for large values of $\alpha$, $s_x$ and $s_p$ respectively tend to $-0.5$ and $1$, for those chosen values of $m$. From figure 9(c) it is visible that in some regions of space, especially large values of $\alpha$, $S_X$ gets negative values, i.e. amplitude-squared squeezing in $X$ appears. But, as observed from figure 9(d), $S_P$ is always positive, i.e. no amplitude-squared squeezing in $P$ may be seen. Our further calculations show that, at least for the chosen parameters, with an increase in the values of $\alpha$, $S_X$ and $S_P$ respectively tend to $\approx -0.5$ and $\approx 1$.

7. A discussion on DPANCSs with negative $m$

The form of $f_d(n, f, m)$ in (16) suggests that a nonlinearity function can be constructed also for negative integer values of $m$. The corresponding coherent states associated with this nonlinearity function, denoted by $|\alpha, f, -m\rangle$, will be called ‘DPANCSs with negative $m$’. In order to construct these states, consider the following eigenvalue equation:

\[
\frac{(n + m + 1)f^2(n + m + 1)}{(n + 1)f(n + 1)}a|\alpha, f, -m\rangle = \alpha|\alpha, f, -m\rangle,
\]

which is obtained simply by replacing $m$ with $-m$ in (15) together with (16). Following the usual procedure, i.e. by expanding $|\alpha, f, -m\rangle$ in terms of the number states and finding the expansion coefficients, one straightforwardly arrives at

\[
|\alpha, f, -m\rangle = N_{-m, f}^{-\alpha m, f} \sum_{n=0}^{\infty} \frac{\alpha^n m! \sqrt{n}! [f(n)]! [f^2(m)]!}{(n + m)! [f^2(n + m)]!} |n\rangle.
\]
Figure 9. Squeezing parameters as a function of $\alpha \in \mathbb{R}$ for different values of $m$ for the DPANCS associated with $f(n) = \sqrt{n}$. (a) The variation of $s_x$: the continuous line is for $m = 1$, the dotted line is for $m = 2$ and the dashed line is for $m = 3$; (b) the same as (a) except that it is plotted for $s_y$, (c) the variation of $S_x$: the continuous line is for $m = 1$, the dotted line is for $m = 3$ and the dashed line is for $m = 5$; (d) the same as (c) except that it is plotted for $S_y$.

The constant $N_{\alpha}^{-m,f}$ is determined by the normalization condition as

$$N_{\alpha}^{-m,f} = \left( \sum_{n=0}^{\infty} \frac{|\alpha|^{2m}n!(m!)^2[f^2(n)]!/[f^4(n+m)]!}{[(n+m)!][f^4(n+m)]!} \right)^{-\frac{1}{2}}. \quad (47)$$

Unlike the DPANCSs in (5), the states $|\alpha, f, -m \rangle$ contain a superposition of all Fock states starting with the vacuum state $|0 \rangle$. In the limit $\alpha \to 0$, the state $|\alpha, f, -m \rangle$ reduces to the vacuum state, but in the same limit, irrespective of the value of $m$, the DPANCS reduces to the number state $|m \rangle$. Also, in the limit $m \to 0$, both of the states $|\alpha, f, \pm m \rangle$ recover trivially the original nonlinear coherent state $|\alpha, f \rangle$. It is worth adding the point that the states $|\alpha, -m \rangle$ which previously argued in [3] may be reobtained by setting $f(n) = 1$ in $|\alpha, f, -m \rangle$ in (46).

The procedure which we followed in section 6 for investigating the resolution of the identity and obtaining the appropriate weight function of DPANCSs can be used for the DPANCSs with negative $m$ in (46). Note that for these states the well-defined unity operator is as usual

$$I^{(-m)} = \hat{1} = \sum_{n=0}^{\infty} |n \rangle \langle n |. \quad (48)$$

So, in such a case which we deal with, the resolution of the identity requirement takes the form

$$\frac{1}{\pi} \int d^2 \alpha \ W_{\alpha}^{(-m)}(|\alpha|^2) |\alpha, f, -m \rangle \langle \alpha, f, -m | = \hat{1}, \quad (49)$$
Figure 10. The plot of $W^{(-m)}(x)$ as a function of $x$, with fixed $m$ parameters and $\nu = 3$ for the DPANCS with negative $m$ associated with the P-T potential. The continuous line is for $m = 1$, the dashed line is for $m = 2$ and the dotted line is for $m = 3$.

where $W^{(-m)}(|\alpha|^2)$ denotes the non-negative weight function which should be determined. In this way, one straightforwardly obtains

$$\int_0^\infty dx x^n \tilde{W}^{(-m)}(x) = \frac{(n+m)!^2[f^4(n+m)]!}{n![f^2(n)]!},$$

where

$$\tilde{W}^{(-m)}(x) = (N_n^{-m,\frac{\nu}{2}})^2 x^{-m}(m!)^2[f^4(m)]!W^{(-m)}(x).$$

Investigating the case for a particular physical system, i.e. the P-T potential, we finally arrive at

$$W^{(-m)}(x) = \frac{(\nu!)^2}{[(m+\nu)!]^4(N_n^{-m,\frac{\nu}{2}})^2(m!)^2}G_{2,4}^{4,0}\left(x \left| \begin{array}{c} 0, \nu \\ m, m, \nu + m, \nu + m \end{array} \right. \right).$$

where $N_n^{-m,\frac{\nu}{2}}$ has been introduced in (47). The DPANCSs with negative $m$ may be called ‘coherent states’ (according to the Klauder definition), if the weight function $W^{(-m)}(x)$ will be positive in all space. To check this requirement, in figure 10 we have plotted $W^{(-m)}(x)$ versus $x$ for a fixed value of $\nu = 3$ and different values of $m$. As is shown, unfortunately $W^{(-m)}(x)$ in some region of space gets negative values. So, our results in figure 10 indicate that the DPANCSs with negative $m$, associated with the P-T potential cannot be known as coherent state.

The latter results motivated us to check the above procedure for the PACSs with negative $m$, as introduced in [3] (the states which may be reproduced by setting $f(n) = 1$ in (46)). Unfortunately, the same conclusion has been obtained, i.e. $W_{\text{PACS}}^{(-m)}(x)$ will get negative values in some regions of space (see figure 11). We have also examined our conclusion for the DPANCSs with negative $m$ associated with other nonlinearity functions. For this purpose we worked with $f(n) = 1/\sqrt{n}$ (harmonious states [29]), $f(n) = 1/\sqrt{n+2\kappa - 1}$ (Barut–Girardello coherent states of $SU(1, 1)$ group [30]) and $f(n) = \sqrt{n}$. In all the latter cases we obtained the same result, i.e. the positivity of the weight function, and so the overcompleteness relation does not justify. We should mention that we investigated the uniqueness of the solution of the moment integrals by examining the Carleman criterion [31].
Figure 11. The plot of $W_{\text{PACS}}^{(-m)}(x)$ as a function of $x$, with fixed $m$ parameters for the PACS with negative $m$. The continuous line is for $m = 1$, the dotted line is for $m = 2$ and the dashed line is for $m = 3$.

We end this section by mentioning another example for the latter result which may be found in the literature, where the following state has been introduced by Klauder et al.\cite{32}:

$$|\alpha\rangle = \left[ \frac{\text{$_2F_2$(1, 1; 2, 2; $|\alpha|^2)$}}{2} \right]^{-1/2} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{(n+1)^2 n!}} |n\rangle,$$

(53)

with $_pF_q$ the generalized hypergeometric function. As illustrated there, the weight function for this set of states has been derived as $W(x) = x e^{-x(x-1)}$, which is not trivially a non-negative function in all space. So, even though this class of states is normalizable and continuous in the label, it is not known as a coherent state in its exact meaning.

8. Summary and conclusion

In summary, we presented a general formalism for the construction of DPANCSs with the explicit form in (5), which recovers, in special cases, PACSs (setting $f(n) = 1$ in (5)) and canonical coherent states (setting $f(n) = 1$ in (5), together with $m = 0$). The algebraic structure and the resolution of the identity requirement of the introduced states are also illustrated. As in the case of PACSs, we established that the DPANCSs can be specified with a nonlinearity function denoted by $f_d(n, f, m)$. Therefore, the introduced DPANCSs are of the $f$-deformed type, too. We briefly argued that their physical generation is possible. Then, after applying the formalism to the P-T potential, the non-classical properties of the DPANCSs associated with the P-T potential are checked through evaluating the Mandel parameter, second-order correlation function, as well as first- and second-order squeezing, numerically. Along the physical realization of the formalism, we also briefly applied the same procedures which have been used for the P-T potential to a well-known class of nonlinear coherent states with $f(n) = \sqrt{n}$. Generally, we observed much intensity (in depth and domain) of non-classicality signs for the DPANCSs associated with such a system in comparison with that for the PACSs of \cite{1}. Then, a discussion on the ‘DPANCSs with negative $m$’ is presented. According to our results, it is deduced that these latter states do not satisfy the resolution of the identity, appropriately. Although the function satisfies the related moment integral uniquely determined, altogether the positivity of the obtained function was not confirmed. Also, we
further investigated the existence and positivity of the weight function for the PACSs with negative values of \( m \) which have been introduced in [3]. Unfortunately, we found the same conclusion. So, recalling the minimal requirements of any quantum state to be exactly named ‘coherent state’ [32], we may conclude that the ‘PACSs and DPANCSs with negative values of \( m \)’ are not strictly known as coherent states.

Finally, it is worth mentioning that even though we have used only the P-T potential and a particular class of nonlinear coherent state as some physical realizations of our proposed structure, its essential potentiality to be used for any class of nonlinear coherent states with a known nonlinearity function, in addition to any solvable quantum system with arbitrary discrete spectrum, should be clear. So, in this way, a vast new family of DPANCSs can, in principle, be constructed. Apart from the generalized structure of our proposal, it is noticeable that it is a rather different formalism with new outputs in resultant coherent states and their non-classicality aspects, in comparison with earlier works [1–12].

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Appendix

The mean values of the relevant operators over the state \(|a, f, m\rangle\), required for our numerical calculations, may be easily obtained as follows:

\[
\langle a \rangle = a \left( \bar{N}_a \right) \sum_{n=0}^{\infty} \frac{|a|^{2n} (n+m)! f^2(n+m)!}{n!(f^2(n))!} \frac{(n+m+1)f(n+m+1)}{(n+1)f^2(n+1)},
\]

(A.1)

\[
\langle a^2 \rangle = a^2 \left( \bar{N}_a \right)^2 \sum_{n=0}^{\infty} \frac{|a|^{2n} (n+m)! f^2(n+m)!}{n!(f^2(n))!} \frac{(n+m+1)f(n+m+1)}{(n+1)f^2(n+1)} \frac{(n+1)(n+m+2)f(n+m+1)f(n+m+2)}{(n+1)(n+2)f^2(n+1)f^2(n+2)},
\]

(A.2)

\[
\langle a^4 \rangle = a^4 \left( \bar{N}_a \right)^2 \sum_{n=0}^{\infty} \frac{|a|^{2n} (n+m+4)! f^2(n+m+4)!}{n!(n+4)!f^2(n)!f^2(n+4)!} f(n+m)!
\]

(A.3)

\[
\langle a^4 \rangle = a^4 \left( \bar{N}_a \right)^2 \sum_{n=0}^{\infty} \frac{|a|^{2n} (n+m)! f^2(n+m)!}{n!(f^2(n))!} f(n+m),
\]

(A.4)

\[
\langle a^6 \rangle = a^6 \left( \bar{N}_a \right)^2 \sum_{n=0}^{\infty} \frac{|a|^{2n} (n+m)! f^2(n+m)!}{n!(f^2(n))!} (n+m)(n+m-1)
\]

(A.5)

\[
\langle a^8 \rangle = a^8 \left( \bar{N}_a \right)^2 \sum_{n=0}^{\infty} \frac{|a|^{2n} (n+m)! f^2(n+m)!}{n!(f^2(n))!} (n+m)^2.
\]

(A.6)

where we have set \( \bar{N}_a = N_a^{m,f} \left( \sum_{n=0}^{\infty} \frac{|a|^{2n}}{n!f^2(n)!} \right)^{-\frac{1}{2}} \) and \( N_a^{m,f} \) determined in (6). Note that \( \langle a \rangle, \langle a^2 \rangle \) and \( \langle a^4 \rangle \) can be obtained by taking the complex conjugate of \( \langle a \rangle, \langle a^2 \rangle \) and \( \langle a^4 \rangle \), respectively.
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