RECONSTRUCTION OF RATIONAL RULED SURFACES FROM THEIR SILHOUETTES

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Abstract. We provide algorithms to reconstruct rational ruled surfaces in three-dimensional projective space from the “apparent contour” of a single projection to the projective plane. We deal with the case of tangent developables and of general projections to \( \mathbb{P}^3 \) of rational normal scrolls. In the first case, we use the fact that every such surface is the projection of the tangent developable of a rational normal curve, while in the second we start by reconstructing the rational normal scroll. In both instances we then reconstruct the correct projection to \( \mathbb{P}^3 \) of these surfaces by exploiting the information contained in the singularities of the apparent contour.

1. Introduction

Rational ruled surfaces are rational surfaces that contain a straight line through each point. They have been extensively investigated from the point of view of computer algebra, see for example [CZS01], [BEG09], or [SPD14]. When we project to \( \mathbb{P}^2 \) a rational ruled surface \( S \subset \mathbb{P}^3 \), we get a finite cover of \( \mathbb{P}^2 \) branched along a curve, which is the zero set of the discriminant of the equation of \( S \) along the direction of projection. In this paper, we study the problem of reconstructing the equation of the surface from the discriminant. We have already investigated this question in [GLSV18] for a wider class of surfaces, namely the ones admitting at worst ordinary singularities. However, for rational ruled surfaces we are able to create a faster algorithm for reconstruction, based on a completely different technique from the one in [GLSV18].

We follow the notation from [GLSV18] and we define the contour to be the locus of points on the surface \( S \) whose tangent space passes through the center of projection. It consists of the singular locus of the surface and of another curve, which we call the proper contour. The projection of the contour is the silhouette of \( S \), which splits into the singular image and the proper silhouette.

In the remainder of the introduction, we discuss the organization of the paper.

A special subcase of rational ruled surfaces is the one of developables, i.e., surfaces that are either a cone over a plane curve or the union of the tangents of a space curve. In Section 2 we consider the latter case. The main idea is that the projection of
the space curve appears as a component with multiplicity three in the discriminant. Our task is to lift this projected curve back to space; we do so by exploiting the information contained in particular singularities of the projection of the nodal curve of the surface.

In Section 3 we deal with general projections to \( \mathbb{P}^3 \) of rational normal scrolls. In this case, we first identify the rational normal scroll and construct a projection from it to \( \mathbb{P}^2 \) having the proper silhouette as branching locus. Secondly, we project the rational normal scroll to \( \mathbb{P}^3 \) so that we obtain a double curve as prescribed by the singular image.

We conclude the paper by recalling in Section 4, for the benefit of the reader, a known algorithm for the parametrization of curves which we implement \textit{ad hoc} in our algorithm, since currently available general-purpose algorithms for parametrizations of plane curves do not exploit the special structure of the curves we deal with. This determines a relevant speed-up of our algorithm, since the parametrization of a particular planar curve constitutes its computational bottleneck.

An implementation in Maple of the algorithms developed in this paper is available at

\[ \text{https://www.risc.jku.at/people/jschicho/pub/ChisiniRuled.mpl}. \]

2. Tangent developable surfaces

Among ruled surfaces, developable surfaces can be characterized as follows. A line of a ruled surface is called \textit{torsal} if all tangent planes at all smooth points of the line are equal; a \textit{developable surface} is then a ruled surface such that all lines of the surface are torsal.

A developable surface is either a \textit{tangent developable}, i.e., the union of all tangents of a space curve (Figure 1), or a cone over a planar curve — see [HC52, Chapter IV, Section 30] and [Ush99, Theorem 0], or [Arr, Proposition 2.12] for a proof in terms of algebraic geometry. In this section, we recognize general rational tangent developables from their silhouettes with respect to general projections. The case of cones is equivalent to the recognition of a planar algebraic curve from its branching points with respect to a projection to the projective line.

Let \( d \geq 3 \). Every tangent developable of a rational curve of degree \( d \) is a projection of the tangent developable \( T_d \subset \mathbb{P}^d \) of the rational normal curve of degree \( d \) in \( \mathbb{P}^d \).

The surface \( T_d \) has degree \( 2d - 2 \) and admits a parametrization \( \mathbb{C}^2 \longrightarrow T_d \) of the form

\[
(s, t) \mapsto \left( 1 : (t + s) : (t^2 + 2st) : (t^3 + 3t^2s) : \cdots : (t^d + dt^{d-1}s) \right).
\]

The surface \( T_d \) contains the rational normal curve as a cuspidal singularity, namely locally analytically the surface around such a curve looks like a cylinder over a plane cusp.
Here we describe the tangent developables in $\mathbb{P}^3$ we intend to recognize: we prescribe their singularities having in mind the situation of a general projection of $T_d$ to $\mathbb{P}^3$. We call a tangent developable in $\mathbb{P}^3$ good if it satisfies the following conditions:

- the cuspidal curve is smooth and irreducible;
- the nodal curve is irreducible and has only ordinary triple points, or singular points described in the next item;
- the nodal curve and the cuspidal curve to intersect in points of two types:
  - the local analytic equation of the surface at the point is equivalent to $(x^2 - y^3)z = 0$; in this local equation, the cuspidal curve is $x = y = 0$ and the nodal curve is $x^2 - y^3 = z = 0$; as the local equation shows, the nodal curve has a cusp at the intersection point (see Figure 2a);
  - the point is a transversal intersection of the two curves at a “cuspidal pinch point”; the local analytic equation at such a point is equivalent to $z^2y^3 - x^2 = 0$ (see Figure 2b);
- there are exactly $4(d - 3)$ cuspidal pinch points.

Now we examine the silhouette with respect to a general projection to $\mathbb{P}^2$ of a good tangent developable surface.

**Definition 2.1.** Let $S \subset \mathbb{P}^3$ be a good tangent developable surface. The image of the cuspidal curve of $S$ under a projection to $\mathbb{P}^2$ is called the *cuspidal image*; the image of the nodal curve of $S$ is called the *nodal image*.

**Proposition 2.2.** If $S \subset \mathbb{P}^3$ is a good tangent developable surface, then the discriminant of a general projection of $S$ to $\mathbb{P}^2$ has the following factors:

- a factor of multiplicity three, whose zero set is the cuspidal image;
- a factor of multiplicity two, whose zero set is the nodal image;
Figure 2. Two possible types of intersection between the cuspidal and the nodal curve of a tangent developable.

\[ \text{several linear factors of multiplicity one, which are images of tangent planes passing through the center of projection; their zero set is the proper silhouette of the projection.} \]

**Proof.** A general projection of a cusp is a triple zero of the discriminant, hence the cuspidal image is a triple component. Similarly, a general projections of a node is a double zero of the discriminant, hence the nodal image is a double component. Both these two results follow from a local analysis of the situation. Since all lines are torsal, namely all the tangent planes at points of one of these lines coincide, the proper contour is composed of lines, and their images determine the linear factors of the discriminant.

The lines in the proper silhouette are inflection tangent lines of the cuspidal image. The intersections of the cuspidal image and the nodal image are either cusps of the nodal image or transversal intersections coming from transverse intersections of nodal and cuspidal curve, or from pairs of distinct point, each on one of the two curves, collapsed by the projection. Apart from these intersections, and from triple points on the nodal image, we allow the cuspidal image and the nodal image to have only ordinary double points.

The recognition problem of a tangent developable reduces to the construction of the cuspidal curve from the factors of the discriminant. The cuspidal curve is a projection of a rational normal curve of the same degree $d$. This projection is given by four polynomials of degree $d$ of which we already know three, namely those that define the projection to the cuspidal image in $\mathbb{P}^2$. In order to find the fourth polynomial, we have to use the nodal image. Notice, in fact, that the inflection lines do not add any information: any choice of a fourth polynomial would lead to the same linear factors of the discriminant.
The basic idea of Algorithm \texttt{ReconstructTangentDevelopable} is to collect linear conditions for the fourth polynomial derived from cuspidal pinch points. Their projections to $\mathbb{P}^2$ are transversal intersections of the nodal image with the cuspidal image. Hence, we try every possible subset of cardinality $4(d - 3)$ of the set of all transversal intersections. Notice that if the cuspidal curve is defined over $\mathbb{Q}$, then the images of the cuspidal pinch points are all conjugated over $\mathbb{Q}$, so it is easy to extract them from the set of all intersections. Suppose that the three polynomials giving the parametrization of the cuspidal image are $H_0$, $H_1$ and $H_2$. Then the fourth polynomial $H_3$ can be found as follows: we make a symbolic ansatz for its coefficients and compute the determinant of the matrix

\begin{equation}
\begin{pmatrix}
\frac{\partial^3 H_0}{\partial t^3} & \frac{\partial^3 H_1}{\partial t^3} & \cdots & \frac{\partial^3 H_2}{\partial t^3} \\
\vdots & \vdots & \ddots & \vdots \\
\frac{\partial^3 H_3}{\partial t^3} & \frac{\partial^3 H_3}{\partial t^3} & \cdots & \frac{\partial^3 H_3}{\partial t^3}
\end{pmatrix}.
\end{equation}

This determinant vanishes at the parameters corresponding to images of the cuspidal pinch points (see Remark 2.3). Imposing this vanishing for all points of a subset of cardinality $4(d - 3)$ of transversal intersections of the nodal image with the cuspidal image provides linear conditions for the coefficients of $H_3$.

**Algorithm 1 \texttt{ReconstructTangentDevelopable}**

**Input:** A rational curve $C \subset \mathbb{P}^2$, the image of the cuspidal curve of a good tangent developable, and another curve $D$, the image of the nodal curve.

**Output:** The parametrization of the cuspidal curve of the tangent developable $S \subset \mathbb{P}^3$.

1. **Compute** a parametrization $(H_0 : H_1 : H_2)$ of $C$.
2. **Formulate** a symbolic ansatz for the coefficients of $H_3$ and compute the determinant of the matrix in Equation (1).
3. **Select** a set $T$ of $4(d - 3)$ transverse intersections of $C$ and $D$ (the candidates for the images of cuspidal pinch points).
4. **For** each point $x$ of $T$ **Do**
   5. **Evaluate** the determinant at $x$ and collect the linear equations in the coefficients of $H_3$.
5. **End For**
6. **Solve** the system of linear equations and obtain $H_3$.
7. **Return** the parametrization $(H_0 : H_1 : H_2 : H_3)$.

Lemma 2.4 proves that Algorithm \texttt{ReconstructTangentDevelopable} is correct.

**Remark 2.3.** Consider a nondegenerate rational curve $C$ of degree $d$ in $\mathbb{P}^n$, and a point $P \in C$, which we can suppose to be $(1 : 0 : \ldots : 0)$. A local parametrization of $C$ around $P$ is of the form $(f_0(t) : f_1(t) : \ldots : f_n(t))$ with $\text{ord}(f_i) = \alpha_i$. We may assume that $0 = \alpha_0 < \alpha_1 < \ldots < \alpha_n$ up to linear changes of coordinates. We say that $P$ is special if $\alpha_n > n$. The multiplicity of a special point is defined as
\[ \sum_{i=0}^{n} \alpha_i = \binom{n+1}{2}, \]
and one has that the sum of multiplicities of special points of \( C \) equals \((n+1)(d-n)\) (see [GS17, Definition 3.4 and Lemma 3.6]). A general rational curve \( C \) of degree \( d \) in \( \mathbb{P}^3 \) has exactly \( 4(d-3) \) special points of multiplicity \( 1 \) (see [GS17, Lemma 3.5]), and a local parametrization of \( C \) at these points is of the form \( (1 + \ldots : t + \ldots : t^2 + \ldots : t^4 + \ldots) \). These points determine the cuspidal pinch points of tangent developable of \( C \).

**Lemma 2.4.** The dimension of the solution space to the linear system in Algorithm `ReconstructTangentDevelopable` is 4.

*Proof.* The dimension is at least 4 since we suppose that we start from a projection of a tangent developable. In particular, there exists a rational curve \( C \) of degree \( d \) in \( \mathbb{P}^3 \) parametrized by \( (H_0 : H_1 : H_2 : H_3) \) where the \( H_i \) are linearly independent elements in the solution space. By assumption, the curve \( C \) admits exactly \( 4(d-3) \) special points of multiplicity 1: in fact, if \( C \) has a special point \( P \) of multiplicity 1, we see from its local parametrization that the tangent developable of \( C \) has a cuspidal pinch point at \( P \). Suppose, by contradiction, that the dimension of the solution space is at least 5. This implies that there exists a rational curve \( C' \) of degree \( d \) in \( \mathbb{P}^4 \) projecting to \( C \). The curve \( C' \) has at least \( 4(d-3) \) special points that remain special after projection to \( \mathbb{P}^3 \). Therefore, these points must be of multiplicity at least 2. This follows from the fact that special points of multiplicity 1 project to non-special points, since any multiplicity 1 point on the curve \( C' \) has local parametrization of type \( (1 : t + \ldots : t^3 + \ldots : t^4 + \ldots) \); these points project to points in \( \mathbb{P}^3 \) around which the parametrization is \( (1 : t + \ldots : t^2 + \ldots : t^3 + \ldots) \), and so they are non-special. The sum of multiplicities of special points in \( C' \) must give \( 5(d-4) \), but this is a contradiction, since \( 2 \cdot 4(d-3) > 5(d-4) \) for \( d \geq 3 \).

□

**Remark 2.5.** Notice that there is no nodal curve in the tangent developable of a twisted cubic, and thus no special points in its cuspidal curve. However, Lemma 2.4 is trivially true for \( d = 3 \) since the space of univariate polynomials of degree at most 3 is 4-dimensional.

3. General projections of rational normal scrolls

In this section, we provide a reconstruction algorithm for rational ruled surfaces that admit particularly simple singularities. We hence consider good rational ruled surfaces \( S \subset \mathbb{P}^3 \) and good projections \( S \rightarrow \mathbb{P}^2 \), namely we ask that:

- \( S \) has at most ordinary singularities: an irreducible self-intersection curve, self-intersection triple points, and pinch points;
- the proper silhouette has only nodes and cusps;
- the singular image has only nodes and ordinary triple points;
- the proper contour is irreducible and projects birationally to the proper silhouette;
- the singular curve projects birationally to the singular image.
Recall that every rational ruled surface is a projection of a rational normal scroll. Our assumptions imply that the restriction of this projection to its ramification locus is generically injective. Notice that all our assumptions are fulfilled when we consider projections from general centers (both when we project from the rational normal scroll to $\mathbb{P}^3$, and when we project from the surface $S$ to $\mathbb{P}^2$). In particular, irreducibility of the singular curve is Franchetta’s Theorem (see [MP97, Theorem 6]). The fact that good rational ruled surfaces have ordinary singularities implies that they have finitely many torsal lines, each passing through exactly one pinch point.

We divide the reconstruction process in two steps: first, we determine the rational normal scroll of which the surface is a projection, together with the projection from this rational normal scroll to $\mathbb{P}^2$, by computing a parametrization of the dual of the proper silhouette, which is a rational curve; second, we construct the projection map from the rational normal scroll to the surface in $\mathbb{P}^3$. To explain how the first part of the reconstruction works, we recall some facts about rational normal scrolls.

For our purposes, we use the description of rational normal scrolls provided by [CLS11, Example 2.3.16]: given two natural numbers $d_1, d_2 \in \mathbb{N}$, the rational scroll $\Sigma_{d_1, d_2}$ is the Zariski closure of the image of the map

$$p_{d_1, d_2}: (\mathbb{C}^*)^2 \longrightarrow \mathbb{P}^{d_1+d_2+1}, \quad (s, t) \mapsto (1 : t : \ldots : t^{d_2} : s : st : \ldots : st^{d_1}).$$

In this way, a linear projection $\rho: \Sigma_{d_1, d_2} \longrightarrow \mathbb{P}^2$ can be identified with two vectors of polynomials $Q_1, Q_2 \in \mathbb{C}[t]^3$ of degrees $d_1$ and $d_2$ so that the map $r: (\mathbb{C}^*)^2 \longrightarrow \mathbb{P}^2$

$$r(s, t) \mapsto (Q_{20}(t) + s Q_{10}(t) : Q_{21}(t) + s Q_{11}(t) : Q_{22}(t) + s Q_{12}(t))$$

fits into the commutative diagram

\[ \begin{array}{ccc} \Sigma_{d_1, d_2} & \xrightarrow{\rho} & \mathbb{P}^2 \\ \downarrow{p_{d_1, d_2}} & & \downarrow{r} \\ (\mathbb{C}^*)^2 & & \\
\end{array} \]
For our purposes, we collect information about the branching locus of a projection of a rational normal scroll to \( \mathbb{P}^2 \). The following result is well-known, but we report the proof, since we could not find a reference asserting exactly the fact we need.

**Lemma 3.1.** Let \( \rho: \Sigma_{d_1,d_2} \rightarrow \mathbb{P}^2 \) be a projection from a rational normal scroll whose restriction to the ramification locus \( R \subset \Sigma_{d_1,d_2} \) is generically injective. Then every line in \( \Sigma_{d_1,d_2} \) is projected by \( \rho \) to a tangent line of the branching locus.

**Proof.** Let \( L \subset \Sigma_{d_1,d_2} \) be a line. By hypothesis, we know that \( \rho|_{L} \) and \( \rho|_{R} \) are both generically injective because any projection whose center is not in \( \Sigma_{d_1,d_2} \) is generically injective on the lines of the surface. Let \( p \in L \cap R \) be a smooth point on \( R \), then \( \rho(L) \) intersects \( \rho(R) \) tangentially at \( \rho(p) \). In fact, by definition, the tangent plane of \( \Sigma_{d_1,d_2} \) at \( p \) intersects the center of projection. The tangent line of \( R \) at \( p \) and \( L \) are both contained in this tangent plane. The projection either collapses the tangent line of \( R \). In the first case, since \( \rho|_{R} \) is generically injective, and there are no birational maps between smooth curves collapsing tangent vectors, it follows that \( \rho(p) \) is singular in \( \rho(R) \). Therefore the statement is trivially true. In the second case, both \( L \) and the tangent line of \( R \) are projected to a single line in \( \mathbb{P}^2 \), which equals \( \rho(L) \). Hence \( \rho(L) \) is tangent to \( \rho(R) \) at \( \rho(p) \). \( \square \)

The next proposition provides the first part of the reconstruction algorithm. We are going to use the concept of \( \mu \)-bases, introduced by Cox, Sederberg and Chen in [CSC98]: these are particular sets of generators of the module of syzygies of a parametrization of a curve or of a surface; we refer to the original paper for a more precise definition and for their properties.

**Proposition 3.2.** Let \( B \subset \mathbb{P}^2 \) be the branching locus of a projection \( \Sigma_{d_1,d_2} \rightarrow \mathbb{P}^2 \) whose restriction to the ramification locus is generically injective. Let \((Q_1, Q_2)\) be a \( \mu \)-basis of a parametrization of \( B \), the dual curve of \( B \). Then the projection \( \rho: \Sigma_{d_1,d_2} \rightarrow \mathbb{P}^2 \) induced by \((Q_1, Q_2)\) has \( B \) as branching locus. Moreover, every general projection \( \tilde{\rho}: \Sigma_{\delta_1,\delta_2} \rightarrow \mathbb{P}^2 \) having \( B \) as branching locus is projectively equivalent to \( \rho \) over \( B \), namely there exists a projective automorphism \( \alpha: \Sigma_{\delta_1,\delta_2} \rightarrow \Sigma_{d_1,d_2} \) such that \( \alpha \) fixes all linear spaces of dimension \( d_1 + d_2 - 1 \) through the center of projection, which has dimension \( d_1 + d_2 - 2 \).

**Proof.** We first show that \( B \) is the branching locus of the map \( \rho: \Sigma_{d_1,d_2} \rightarrow \mathbb{P}^2 \) obtained from \((Q_1, Q_2)\). By the properties of \( \mu \)-bases (see [SG09, Theorem 2]), the vector \( Q_1 \times Q_2 \) gives a parametrization of \( B \). This implies that the projection \( \rho \) sends the rulings of \( \Sigma_{d_1,d_2} \) to the family of tangent lines of \( B \). In fact, the Plücker coordinates of the line between \( Q_1(t) \) and \( Q_2(t) \) are given by \((Q_1 \times Q_2)(t)\). By Lemma 3.1, the curve \( B \) is the branching locus of \( \rho \). Suppose that \( \phi: \Sigma_{\delta_1,\delta_2} \rightarrow \mathbb{P}^2 \) is another general projection having \( B \) as branching locus. Let \( \tilde{Q}_1, \tilde{Q}_2 \in \mathbb{C}[t]^3 \) be the two vectors of polynomials defining \( \phi \) as explained in Equation (2). By Lemma 3.1, the images of the rulings of \( \Sigma_{\delta_1,\delta_2} \) are lines tangent to \( B \). Hence, by the same argument as before, \( \tilde{Q}_1 \times \tilde{Q}_2 \) is a parametrization of \( B \). Thus \((Q_1, Q_2)\)
is a $\mu$-basis of this parametrization. By the uniqueness of $\mu$-bases, it follows that $\delta_1 = d_1$ and $\delta_2 = d_2$, and that $\phi$ and $\rho$ differ by an automorphism over $B$. □

Notice that the branching locus $B$ as in Proposition 3.2 is a rational curve: in the algorithm we are going to start by parametrizing it and its dual, and then we compute a $\mu$-basis of this parametrization. Proposition 3.2 ensures that this allows us to determine the rational normal scroll.

This concludes the first part of the reconstruction process, and proves the correctness of Algorithm ReconstructRationalScroll: there, instead of computing the map $\rho$, we compute the map $r$ as in Equation (2), which provides the same information. In fact, the proper silhouette of a projection $S \to \mathbb{P}^2$ is the branching locus of the corresponding projection $\Sigma_{d_1,d_2} \to \mathbb{P}^2$.

Algorithm 2 ReconstructRationalScroll

**Input:** A curve $B \subset \mathbb{P}^2$, the proper silhouette of a projection $S$ of a rational normal scroll $\Sigma_{d_1,d_2}$ whose restriction to the ramification locus is generically injective.

**Output:** Two numbers $d_1, d_2 \in \mathbb{N}$ and a map $r : (\mathbb{C}^*)^2 \to \mathbb{P}^2$ as in Equation (2), whose branching locus is $B$.

1: Compute the dual curve $\tilde{D}$ of $D$.
2: Parametrize the curve $\tilde{D}$.
3: Compute a $\mu$-basis $(Q_1, Q_2)$ of the parametrization of $\tilde{B}$.
4: Set $(d_1, d_2) = (\deg(Q_1), \deg(Q_2))$.
5: Set $r$ to the map

$$ (s, t) \mapsto \left( Q_{20}(t) + s Q_{10}(t) : Q_{21}(t) + s Q_{11}(t) : Q_{22}(t) + s Q_{12}(t) \right). $$

6: Return $d_1, d_2$ and $r$.

As a consequence of Algorithm ReconstructRationalScroll, we obtain a characterization of proper silhouettes of projections $S \to \mathbb{P}^2$ satisfying our requirements.

**Proposition 3.3.** Proper silhouettes of projections $S \to \mathbb{P}^2$ satisfying our requirements are rational plane curves with maximal number of cusps for a given degree and only ordinary nodes as the other singularities.

**Proof.** Let $B$ be a proper silhouette of a projection $S \to \mathbb{P}^2$ satisfying our requirements. We know that the degree of the proper silhouette $B$ is $n = 2d - 2$, where $d = \deg(S)$. We can obtain this number by considering the degree of the silhouette, which is $d(d-1)$, and subtracting from it twice the degree of the singular image, which is $\frac{1}{2}(d-1)(d-2)$ (see [Pie05, Section 6]). If we denote by $\delta$ the number of nodes of $B$ and by $\kappa$ the number of cusps of $B$, then from the rationality of $B$ and the Plücker formulas we get

$$ \frac{(n-1)(n-2)}{2} - \delta - \kappa = 0 \quad \text{and} \quad n^* = n(n-1) - 2\delta - 3\kappa, $$

where $n^*$ is the degree of the dual of $B$. Since the dual of $B$ is in fact a plane section of the dual of $S$, and the dual of a ruled surface is a ruled surface of the same degree,
we conclude that \( n^* = d \). These two equations imply that \( \kappa = \frac{3}{2}(n - 2) \). On the other hand, we know that for a rational curve of degree \( n \) the number of cusps must be less than or equal to \( \frac{3}{2}(n - 2) \) (see [Lef13, Section 5] and [Hol37, Section 4]). Hence proper silhouettes of rational ruled surfaces are rational curves with maximal number of cusps for a given degree.

Conversely, if we are given a rational curve of degree \( n \), where \( n \) is of the form \( 2d - 2 \) for some \( d \), having \( \frac{3}{2}(n - 2) \) ordinary cusps and only ordinary nodes as the other singularities, then we can apply Algorithm \texttt{ReconstructRationalScroll} to it, thus showing that such a curve is the silhouette of the projection of a rational ruled surface whose restriction to the ramification locus is generically injective. □

Now that, recalling Equation (2), we reconstructed the map \( \rho: \Sigma_{d_1,d_2} \rightarrow \mathbb{P}^2 \) via Algorithm \texttt{ReconstructRationalScroll}, we proceed by recovering the projection \( \Sigma_{d_1,d_2} \rightarrow S \). To do so, we need to use the information provided by the singular image, namely the projection on \( \mathbb{P}^2 \) of the singular locus of \( S \). We formulate two different algorithms to construct this projection: one (Algorithm \texttt{CollapseMates}) imposes the projection to collapse pairs of points in \( \Sigma_{d_1,d_2} \) in order to create a prescribed double curve; the other (Algorithm \texttt{UsePinchPoints}), instead, forces the differential of the projection to be rank-deficient at the preimages of pinch points.

Recall that the double curve of \( S \) is irreducible by assumption. This implies that, in presence of pinch points, the curve in \( \Sigma_{d_1,d_2} \) that is mapped \( 2 : 1 \) to the singular curve of \( S \) is also irreducible, because the two sheets of the double cover meet precisely at the preimages of the pinch points. Let \( W \subset \mathbb{P}^2 \) be the singular image of a projection \( S \rightarrow \mathbb{P}^2 \) satisfying our requirements. Let \( W' \subset \rho^{-1}(W) \) be the curve that is mapped \( 2 : 1 \) to the singular curve \( Z \) of \( S \). The curve \( W' \) has bidegree

\[
(d_1 + d_2 - 2, (d_1 + d_2 - 2)(d_2 - 1)).
\]

In fact, by the so-called double point formula (see [Ful98, Section 9.3] for the general formula, and [Dol12, Chapter 10, Equation 10.52] for a specialization of the formula in our case) its divisor class is given by \((d_1 + d_2 - 2)(H - L)\), where \( H \) is the class of a hyperplane section of \( \Sigma_{d_1,d_2} \) and \( L \) is the class of a ruling; the bidegree of \( W' \) then follows from the fact that \( H \) has bidegree \((1, d_2)\) and \( L \) has bidegree \((0, 1)\).

For any general point \( p \in W' \) there is a unique \( q \in W' \) such that \( \{p, q\} \) is a fiber of the double cover of \( Z \). Note that the restriction \( \rho' \) of \( \rho \) to \( W' \) is also a \( 2 : 1 \) map whose fibers are equal to the fibers of \( W' \rightarrow Z \). This allows us to compute \( q \) in terms of \( p \) using \( \rho \). We say that \( q \) is the mate of \( p \).

In order to express the conditions imposed by our algorithm more explicitly, we can assume without loss of generality that the projection \( \mathbb{P}^3 \rightarrow \mathbb{P}^2 \) is the map forgetting the last coordinate. Notice that the map \( \Sigma_{d_1,d_2} \rightarrow \mathbb{P}^3 \) is defined by four linear forms, three of which we already know from the map \( \rho: \Sigma_{d_1,d_2} \rightarrow \mathbb{P}^2 \) obtained via Algorithm \texttt{ReconstructRationalScroll}. Let \( F_0, F_1, \) and \( F_2 \) be the known forms, and let \( F \) be the form to be determined. For every general point \( p \in W \) and its mate \( q \), we get a linear condition in the coefficients of \( F \) by requiring
that the map $\left(F_0 : F_1 : F_2 : F\right)$ collapses $p$ and $q$, namely by asking that
\[ F_i(q)F(p) - F_i(p)F(q) = 0 \quad \text{for every } i \in \{0, 1, 2\}. \]

The forms $F_0$, $F_1$, and $F_2$ also satisfy this linear condition, so we look for a solution of the linear system that is linearly independent from $F_0$, $F_1$, and $F_2$.

We claim that the dimension of the solution space for the linear system above is not bigger than 4. Assume, indirectly, that there is a fifth linearly independent form $G$ satisfying the linear equations defined above. Then, the image of the map $\Sigma_{d_1,d_2} \rightarrow \mathbb{P}^4$ defined by $F_0$, $F_1$, $F_2$, $F$, and $G$ is a nondegenerate rational ruled surface with a double curve of the same degree as the double curve of $S$, namely $\frac{1}{2}(d-1)(d-2)$. Moreover, the projection from $\mathbb{P}^4$ to $\mathbb{P}^3$ is birational once restricted to the two surfaces. This contradicts the following lemma.

**Lemma 3.4.** Let $f: \Sigma_{d_1,d_2} \rightarrow X \subset \mathbb{P}^3$ be a projection of a rational normal scroll satisfying our assumptions. If $f$ factors through a projection $\tilde{f}$ to $\mathbb{P}^4$, then $\tilde{X} := \tilde{f}(\Sigma_{d_1,d_2})$ has at most isolated singularities.

**Proof.** Assume by contradiction that $\tilde{X}$ has a singular curve. Consider a general plane section $C$ of $X$ and its preimage $\tilde{C} \subset \tilde{X}$. Both $\tilde{C}$ and $C$ have the same geometric genus because they are birational. By Bertini’s theorem, since $X$ is the projection of a smooth projective variety, the singularities of $C$ are exactly at the intersection with the singular locus $Z$ of $X$, and their number is $\deg(Z)$. Since $\deg(Z) = \frac{1}{2}(d-1)(d-2)$ (see [Pie05, Section 6]), and we know that $C$ is rational, it follows from the geometric genus formula (see [Har77, Chapter IV, Exercise 1.8(a)]) that the delta invariant of each of the singularities of $C$ is 1. The sum of delta invariants of singularities of the space curve $\tilde{C}$ must be strictly smaller than the sum of the delta invariants of the singularities of $C$. This follows from the geometric genus formula, since the arithmetic genus $\frac{1}{2}(d-1)(d-2)$ of $C$ is bigger than the arithmetic genus of $\tilde{C}$, which cannot be greater than $\frac{1}{2}d^2 - d + 1$ (see [Har77, Chapter IV, Theorem 6.4 and Figure 18]). It follows that the singular locus $\tilde{Z} \subset \tilde{X}$ must have strictly lower degree than $Z \subset X$, because every intersection of $\tilde{C}$ and $\tilde{Z}$ determines a singularity of $\tilde{C}$. This implies the image of $\tilde{Z}$ under the projection to $\mathbb{P}^3$ is a component of $Z$, but by assumption $Z$ is irreducible. This contradiction concludes the proof. \hfill \Box

From an algorithmic point of view, to obtain the desired projection $\Sigma_{d_1,d_2} \rightarrow \mathbb{P}^3$, we need to find a solution $F$ of the infinitely many linear equations above (one for each pair of mates) that is linearly independent from $F_0$, $F_1$, and $F_2$. We could just collect sufficiently many points on $W$ and solve the linear equations arising from them and their mates. However, finding points on $W$ is not trivial. What we do instead is to compute the mate of a point with coordinates in a transcendental field extension of the base field that is isomorphic to the function field of $W$. More concretely, the equations
\[ F_i(q)F_j(p) - F_i(p)F_j(q) = 0 \quad \text{for every } i, j \in \{0, 1, 2\} \]
allow one to write the coordinates \((u,v)\) of the mate \(q\) of a point \(p = (s,t)\) as rational functions of \(s\) and \(t\). This is a consequence of the fact that whenever we have a \(2 : 1\) map \(C \to D\) between two curves, then there exists a birational automorphism of \(C\) swapping the two points in any fiber. This leads to a single linear equation for \(F\) with coefficients in this function field. Using Gröbner bases, we can eliminate from this single equation the generators of the function field and obtain an equivalent system of linear equations with scalar coefficients.

The discussion so far proves the correctness of Algorithm CollapseMates.

**Algorithm 3 CollapseMates**

**Input:** A map \(r : (C^*)^2 \to \mathbb{P}^2\) as in Equation (2), whose branching locus is \(B\), and the singular image \(W\) of a projection \(S \to \mathbb{P}^2\) with proper silhouette \(B\) satisfying our requirements.

**Output:** A parametrization of the surface \(S\).

1: **Compute** the preimage of the singular image \(W\) under \(r\). Let \(h\) be the polynomial defining such preimage.

2: **Select** a factor \(H\) of \(h\) of bidegree \((d_1 + d_2 - 2, (d_1 + d_2 - 2)(d_2 - 1))\).

3: **Construct** the system of equations for the mate \(q = (u,v)\) of a point \(p = (s,t)\). Let \(F_0, F_1\) and \(F_2\) be the components of the map \(r\). The equations for mates are

\[
\begin{align*}
F_0(u,v)F_2(s,t) - F_2(u,v)F_0(s,t) &= 0 \\
F_1(u,v)F_2(s,t) - F_2(u,v)F_1(s,t) &= 0 \\
H(u,v) &= 0
\end{align*}
\]

4: **Write** \(u\) and \(v\) as rational functions \(U(s,t)\) and \(V(s,t)\) using the previous equations by computing a Gröbner basis with an elimination term order.

5: **Set up** a system of equations for the coefficients of a polynomial \(F_3\) of the form \(F_{23}(t) + sF_{13}(t)\) with \(F_{i3}\) of degree \(d_i\) with indeterminate coefficients as follows:

\[
F_0(U(s,t), V(s,t)) F_3(s,t) - F_3(U(s,t), V(s,t)) F_0(s,t) = 0
\]

6: **Solve** the linear system for the coefficients of \(F_3\).

7: **Return** the parametrization \((F_0 : F_1 : F_2 : F_3)\).

The second algorithm we propose is based on the observation that the fourth unknown polynomial \(F\) satisfies particularly simple equations coming from pinch points: the Jacobian of the projection \(\Sigma_{d_1,d_2} \to S\) is rank-deficient at the preimages of pinch points, more precisely the tangent line of \(W'\) at those points is collapsed to a point by the map defined by \((F_0 : F_1 : F_2 : F)\).

For this algorithm to work, we need to suppose that the images of pinch points under the projection \(S \to \mathbb{P}^2\) are transversal intersections of the proper silhouette and the singular image. This is true for projections from general centers (see [GLSV18, Proposition 2.1] and the discussion before for a more thorough analysis).
The algorithm works as follows: for each intersection $P$ of the proper silhouette and the singular image coming from a pinch point, one considers the corresponding point $P' \in \Sigma_{d_1,d_2}$ that is sent to $P$ by $\rho: \Sigma_{d_1,d_2} \rightarrow \mathbb{P}^2$. Notice that $P'$ is unique by assumption, since $P$ comes from a pinch point. The point $P'$ lies on the curve $W'$ which is mapped to the singular image by $\rho$. We compute the tangent line of $W'$ at $P'$, and we impose that the differential of the map $(F_0 : F_1 : F_2 : F_3)$ sends it to zero. In this way, we obtain linear conditions for the coefficients of $F$. We show that the solution space for these linear equations is exactly four-dimensional, thus proving the correctness of the algorithm. The dimension of this solution space is at least 4 because we know by hypothesis that there is a projection $\Sigma_{d_1,d_2} \rightarrow \mathbb{P}^3$ satisfying the requirements. Proposition 3.2 implies that the dimension cannot be bigger than 4.

Algorithm 4 UsePinchPoints

Input: A map $r: (\mathbb{C}^*)^2 \rightarrow \mathbb{P}^2$ as in Equation (2), whose branching locus is $B$, the singular image $W$ of a projection $S \rightarrow \mathbb{P}^2$ satisfying our requirements, and the images in $\mathbb{P}^2$ of the pinch points of $S$.

Output: A parametrization of the surface $S$.

1: Compute the preimages in $(\mathbb{C}^*)^2$ of the images of the pinch points.
2: Compute the preimage $W'$ in $(\mathbb{C}^*)^2$ of the singular image $W$.
3: Define a polynomial $F_3$ of the form $F_{32}(t) + s F_{31}(t)$ with $F_{3i}$ of degree $d_i$ with indeterminate coefficients.
4: For each preimage $P'$ of the images of the pinch points Do
5: Compute a tangent vector of $W'$ at $P'$.
6: Add linear equations for the coefficients of $F_3$ obtained by imposing that the map $(F_0 : F_1 : F_2 : F_3)$ sends the tangent vector to zero.
7: End For
8: Solve the linear system for $F_3$.
9: Return the parametrization $(F_0 : F_1 : F_2 : F_3)$.

We propose an alternative proof of this fact, based on the torsal lines on the ruled surface, and so reveals some of the underlying geometry of these surfaces. The proof that the dimension cannot be bigger goes in two steps: first, we show that lines in $S$ passing through critical values of the projection are torsal, and secondly we prove that a surface in $\mathbb{P}^4$ with too many torsal lines must be degenerate.

Recall from Equation (2) that a projection $\alpha: \Sigma_{d_1,d_2} \rightarrow \mathbb{P}^4$ can be encoded in two vectors of polynomials $Q_1, Q_2 \in \mathbb{C}[t]^5$ via the map

$$(\mathbb{C}^*)^2 \rightarrow \mathbb{P}^4, \quad (s,t) \mapsto Q_2(t) + s Q_1(t)$$

in such a way that $Q_1(t)$ and $Q_2(t)$ are linearly independent for every $t$. The map $\alpha$ is singular at the point $(s,t)$ if and only if the matrix

$$
\begin{pmatrix}
Q_1(t) & Q_2(t) \\
\frac{\partial Q_2}{\partial t}(t) + s \frac{\partial Q_1}{\partial t}(t)
\end{pmatrix}
$$

has rank at most 2.
Lemma 3.5. Let $P' \in \Sigma_{d_1,d_2}$ and let $\alpha: \Sigma_{d_1,d_2} \rightarrow \mathbb{P}^4$ be a projection entering in a commutative diagram of projections

$$\xymatrix{ \Sigma_{d_1,d_2} \ar[r]^-{\alpha} \ar[d]_-{\beta} & \mathbb{P}^4 \ar[d] \cr \mathbb{P}^3 \ar[ur] & }$$

where the image of $\beta$ satisfies our general requirements. Suppose that $\alpha$ is singular at $P'$. Then $\alpha(P')$ lies on a torsal line of $\alpha(\Sigma_{d_1,d_2})$.

Proof. Recalling notation introduced above and the fact that a line is torsal when the tangent planes of the ruled surface at each of its points coincide, we have to prove that the matrix

$$\begin{pmatrix} Q_1(t_0) & Q_2(t_0) & \partial Q_1/\partial t(t_0) & \partial Q_2/\partial t(t_0) \end{pmatrix}$$

has rank 3, where $(s_0, t_0)$ are the coordinates of $P'$. Since $\alpha$ is singular at $P'$, this matrix cannot have rank 4. On the other hand, if this matrix had rank 2, then the line in $\alpha(\Sigma_{d_1,d_2})$ passing through $\alpha(P')$ would be singular. However, there are no singular lines in $\alpha(\Sigma_{d_1,d_2})$, since otherwise also the image of $\beta$ would have singular lines, and this is not allowed by our general assumption. \square

Lemma 3.6. Let $T$ be a rational ruled surface in $\mathbb{P}^4$ with at least $2(\deg(T) - 2)$ torsal lines. Suppose that there exists a projection $T \rightarrow S$ where $S \subset \mathbb{P}^3$ is a rational ruled surface with at most ordinary singularities. Then $T$ is degenerate, namely it lies on a hyperplane.

Proof. The surface $T$ is the image of a projection $\alpha: \Sigma_{d_1,d_2} \rightarrow T$ from a rational normal scroll. Let $Q_1, Q_2 \in \mathbb{C}[t]^5$, as above, be the two vectors of polynomials of degree $d_1$ and $d_2$, respectively, encoding the projection $\alpha$. Torsal lines in $T$ correspond to values $t_0 \in \mathbb{C}$ such that the matrix

$$M := \begin{pmatrix} Q_1(t) & Q_2(t) & \partial Q_1/\partial t(t) & \partial Q_2/\partial t(t) \end{pmatrix}$$

has rank 3 at $t_0$. These values are precisely the common zeros of the determinants of the submatrices $M_0, \ldots, M_4$ obtained by removing a row from the previous matrix. The degree of these determinants (as polynomials in $t$) is at most $2(\deg(T) - 2)$.

In fact, elementary column operations transform the previous matrix into

$$\begin{pmatrix} \partial Q_1/\partial t(t) & \partial Q_2/\partial t(t) & d_1Q_1 - t\partial Q_1/\partial t(t) & d_2Q_2 - t\partial Q_2/\partial t(t) \end{pmatrix}$$

and $\deg(T) = d_1 + d_2$. Hence all the five determinants $M_i := \det(M_i)$ are of the form $\lambda_i M$ for $\lambda_i \in \mathbb{C}$ and $M \in \mathbb{C}[t]$. The kernel of $M^5$ contains the element

$$(M_0, -M_1, \ldots, M_4) = M(\lambda_0, -\lambda_1, \ldots, \lambda_4),$$

thus $\lambda_0Q_{10}(t) - \lambda_1Q_{11}(t) + \cdots + \lambda_4Q_{14}(t) = 0$ and similarly for $Q_2(t)$. Hence all the points of the form $Q_2(t) + sQ_1(t)$ are contained in a hyperplane, namely $T$ is degenerate. \square
We can now prove that the solution space for the polynomial $F$ defining the map $\Sigma_{d_1,d_2} \to \mathbb{P}^3$ is exactly four-dimensional. In fact, if it were bigger, we would get a projection $\alpha: \Sigma_{d_1,d_2} \to \mathbb{P}^4$ which is singular at all points $P'$ that are preimages of the transversal intersections of the proper silhouette and the singular image coming from pinch points of $S$. By [Pie05, Section 6], the surface $S$ has $2(\deg(S) - 2)$ pinch points. By Lemma 3.5, they determine $2(\deg(S) - 2)$ torsal lines in $T := \alpha(\Sigma_{d_1,d_2})$. Lemma 3.6 shows the contradiction.

The discussion so far proves the correctness of Algorithm UsePinchPoints.

4. A faster parametrization for the silhouette

The bottleneck of our algorithm is the computation of the parametrization of the dual of the silhouette. Our situation is quite special: by assumption the silhouette admits only nodes and cusps. General-purpose algorithms for parametrizing curves (as, for example, the one implemented in Maple), do not have the possibility to take into account this special structure of the curve. This is why we implement an ad hoc procedure for the parametrization of the silhouette that uses the fact that we only have nodes and cusps. Although the methods used are all known, we believe it could be beneficial for the reader to have an overview of this algorithm.

We use the well-known technique of adjoints to compute the parametrization, see [SWPD08, Section 4.7]. Given a planar curve $C$ of degree $d$ with only nodes and cusps, the linear system of adjoints is given by those homogeneous forms of degree $d - 2$ that pass through the singularities of $C$. In order to get the adjoint forms, we have to take the homogeneous component of degree $d - 2$ of the radical of the Jacobian ideal of $C$. One way to obtain this radical ideal is the following: consider the discriminant of the curve $C$ along a random projection; this is a bivari- ate homogeneous polynomial whose factors $H_{\text{nodes}}$ of order 2 correspond to nodes of $C$ and whose factors $H_{\text{cusps}}$ of order 3 correspond to cusps of $C$. If we add the form $H_{\text{nodes}} \cdot H_{\text{cusps}}$ to the Jacobian ideal of $C$, then we get its radical. The image of $C$ under the rational map induced by the linear system of adjoints is a rational normal curve $R_{d-2}$ in $\mathbb{P}^{d-2}$ of degree $d - 2$. Suppose now that a smooth point $P \in C$ is known. Then we get a point $P_{d-2}$ in $R_{d-2}$, and the projection from $P_{d-2}$ maps $R_{d-2}$ to a rational normal curve $R_{d-3}$ in $\mathbb{P}^{d-3}$ of degree $d - 3$. Since the projection is a map between smooth curves, it can be extended also to $P_{d-3}$, which gets mapped to the image of the tangent line $T_{P_{d-2}}R_{d-2}$ under the projection. In this way we obtain a point $P_{d-3} \in R_{d-3}$, so we can repeat the procedure until we land on $\mathbb{P}^1$: 

\[
\begin{align*}
R_{d-2} \subset \mathbb{P}^{d-2} \xrightarrow{\pi_{d-2}} R_{d-2} \subset \mathbb{P}^{d-3} \xrightarrow{\pi_{d-3}} \ldots \xrightarrow{\pi_1} \mathbb{P}^1
\end{align*}
\]
In this way, we get a map $\varphi : C \rightarrow \mathbb{P}^1$, whose inverse is the desired parametrization. Notice that the map $\varphi$ can be computed by selecting those adjoint forms that vanish with multiplicity $d - 3$ at $P$.

This discussion leads to the following algorithm:

**Algorithm 5 ParametrizeSilhouette**

**Input:** A rational curve $C \subset \mathbb{P}^2$ with only nodes and cusps and a smooth point $P \in C$.

**Output:** A parametrization $\psi : \mathbb{P}^1 \rightarrow C$ of $C$.

1. **Compute** the radical $J$ of the Jacobian ideal of $C$: for example, consider the discriminant $H$ of a general projection of $C$ on a line, factor $H$ and add to the ideal of derivatives of $C$ the factors of $H$ of order 2 and 3.
2. **Let** $L$ be the saturation of the ideal generated by the equation of $C$ and by the $(d - 3)^{rd}$ power of the ideal of the point $P$.
3. **Let** $K := J \cap L$.
4. **Compute** a basis $B$ of the homogeneous component of degree $d - 2$ of $K$.
5. **Compute** the inverse $\psi$ of the map $C \rightarrow \mathbb{P}^1$ induced by $B$.
6. **Return** $\psi$.

We implemented the algorithms in Maple and tested it on a computer with an Intel I7-5600 processor (1400 MHz). We report the timings in Table 1.

**Table 1.** The table shows the degree $d$ of the surface $S$, the degree of the proper silhouette $B$, its number of nodes $n$ and of cusps $c$ in the ruled case, and the degree of the nodal curve $N$, of the cuspidal curve $C$, and the number $i$ of inflection lines in the tangent developable case, and the computing time in CPU seconds. Notice that the general algorithm developed in [GLSV18] takes 4s and 130s in the cases of ruled surfaces of degree 4 and 5, respectively.

| $d$ | $B$ | $n$ | $c$ | $N$ | $C$ | $i$ | time | type    |
|-----|-----|-----|-----|-----|-----|-----|------|---------|
| 4   | -   | -   | 6   | 4   | 6   | 2s  |       | developable |
| 5   | -   | -   | 16  | 5   | 9   | 28s |       | developable |
| 6   | -   | -   | 30  | 6   | 12  | 145s|       | developable |
| 4   | 6   | 4   | 6   | -   | -   | <1s |       | ruled    |
| 5   | 8   | 12  | 9   | -   | -   | 90s |       | ruled    |

**References**

[Arr] E. Arrondo, Subvarieties of Grassmannians, Available at http://www.mat.ucm.es/~arrondo/trento.pdf.

[BEG09] L. Busé, M. Elkadi, and A. Galligo, A computational study of ruled surfaces, J. Symbolic Comput. 44 (2009), no. 3, 232–241.
D. A. Cox, J. B. Little, and H. K. Schenck, *Toric varieties*, Graduate Studies in Mathematics, vol. 124, American Mathematical Society, Providence, RI, 2011.

D. A. Cox, T. W. Sederberg, and F. Chen, *The moving line ideal basis of planar rational curves*, Comput. Aided Geom. Design 15 (1998), no. 8, 803–827.

F. Chen, J. Zheng, and T. W. Sederberg, *The mu-basis of a rational ruled surface*, Comput. Aided Geom. Design 18 (2001), no. 1, 61–72.

I. V. Dolgachev, *Classical algebraic geometry. A modern view*, Cambridge University Press, Cambridge, 2012.

W. Fulton, *Intersection theory*, second ed., Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics, vol. 2, Springer-Verlag, Berlin, 1998.

M. Gallet, N. Lubbes, J. Schicho, and J. Vršek, *Reconstruction of surfaces with ordinary singularities from their silhouettes*, Available at https://arxiv.org/abs/1810.05559.

M. Gallet and J. Schicho, *Counting projections of rational curves*, Available at https://arxiv.org/abs/1707.05264.

R. Hartshorne, *Algebraic geometry*, Graduate Texts in Mathematics, vol. 52, Springer-Verlag, New York-Heidelberg, 1977.

D. Hilbert and S. Cohn-Vossen, *Geometry and the imagination*, Chelsea Publishing Company, New York, 1952, Translated by P. Neményi.

T. R. Hollcroft, *The existence of algebraic plane curves*, Bull. Amer. Math. Soc. (1937), no. 8, 503–521.

S. Lefschetz, *On the existence of loci with given singularities*, Trans. Amer. Math. Soc. 14 (1913), no. 1, 23–41.

E. Mezzetti and D. Portelli, *A tour through some classical theorems on algebraic surfaces*, An. Științ. Univ. Ovidius Constanța Ser. Mat. 5 (1997), no. 2, 51–78.

R. Piene, *Singularities of some projective rational surfaces*, Computational methods for algebraic spline surfaces, Springer, Berlin, 2005, pp. 171–182.

N. Song and R. Goldman, *µ-bases for polynomial systems in one variable*, Comput. Aided Geom. Design 26 (2009), no. 2, 217–230.

L.-Y. Shen and S. Pérez-Díaz, *Characterization of rational ruled surfaces*, J. Symbolic Comput. 63 (2014), 21–45.

J. R. Sendra, F. Winkler, and S. Pérez-Díaz, *Rational algebraic curves. A computer algebra approach*, Algorithms and Computation in Mathematics, vol. 22, Springer, Berlin, 2008.

V. Ushakov, *Developable surfaces in Euclidean space*, J. Austral. Math. Soc. Ser. A 66 (1999), no. 3, 388–402.

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