QUANTUM $j$-INARIANT IN POSITIVE CHARACTERISTIC I: DEFINITION AND CONVERGENCE

L. DEMANGOS AND T.M. GENDRON

Abstract. We introduce the quantum $j$-invariant in positive characteristic as a multi-valued, modular-invariant function of a local function field. In this paper, we concentrate on basic definitions and questions of convergence.

1. Introduction

In [3], a notion of modular invariant for quantum tori was introduced, which may be viewed analytically as a certain extension of the usual $j$-invariant to the real numbers. For any $\theta \in \mathbb{R}$ and $\epsilon > 0$, one defines first the approximant $j_\epsilon(\theta)$, using Eisenstein-like series over the set of "$\epsilon$ diophantine approximations"$
\Lambda_\epsilon(\theta) = \{ n \in \mathbb{Z} \mid |n\theta - m| < \epsilon \text{ for some } m \in \mathbb{Z} \}.$
Then $j^{qt}(\theta)$ is defined to be the set of limits of the approximants as $\epsilon \rightarrow 0$.

PARI-GP experiments indicate that $j^{qt}(\theta)$ is multi-valued and yet remarkably, for all quadratics tested, the set of all approximants is finite (see the Appendix in [3]). However due to the chaotic nature of the sets $\Lambda_\epsilon(\theta)$, explicit expressions for $j^{qt}(\theta)$ have proven to be elusive: in [3], only a single value of $j^{qt}(\phi)$, where $\phi =$ the golden mean, was rigorously computed. Moreover the experimental observations made at the quadratics remain without proof.

This paper is the first in a series of two, in which we consider the analog of $j^{qt}$ for a function field over a finite field. In this case, the non archimedean nature of the absolute value simplifies the analysis considerably, allowing us to go well beyond what was obtained in [3]. In particular, for $f$ belonging to the function field analog of the reals, the set $\Lambda_\epsilon(f)$ has the structure of an $\mathbb{F}_q$-vector space and it is possible to describe a basis for it using the sequence of denominators of best approximations of $f$. With this in hand, explicit formulas for the approximants $j_\epsilon(f)$ and their absolute values can be given, see Theorem 3 and its proof. Using these formulas, we are able to prove the multi-valued property of $j^{qt}$, as well as give the characterization: $f$ is rational if and only if $j_\epsilon(f) = \infty$ eventually. In the sequel [2], we study the set of values of $j^{qt}(f)$ for $f$ quadratic.

Acknowledgements. We thank E.-U. Gekeler as well as the referee for a number of useful suggestions.

2. Quantum $j$ Invariant

Let $\mathbb{F}_q$ be the field with $q = p^r$ elements, $p$ a prime. Let $k = \mathbb{F}_q(T)$ where $T$ is an indeterminate and $\mathcal{A} = \mathbb{F}_q[T]$. Let $v$ be the place at $\infty$ i.e. that defined by

1991 Mathematics Subject Classification. Primary 11R58,11F03; Secondary 11K60.

Key words and phrases. quantum $j$-invariant, global function fields, Diophantine approximation.
\( v(f) = -\deg_{T}(f) \). The absolute value \( |\cdot| \) associated to \( v \) is \n
\[ |f| = q^{-v(f)} = q^{\deg_T(f)}. \]

Denote by \( k_{\infty} \) the local field obtained as the completion of \( k \) w.r.t. \( v \), and by \( C_{\infty} \) the completion of a fixed algebraic closure \( \overline{k_{\infty}} \) of \( k_{\infty} \) w.r.t. \( v \). For a review of basic function field arithmetic, see [6], [8], [9], [10].

Consider the compact \( A \)-module

\[ S^1 := k_{\infty}/A. \]

The character group of \( S^1 \) may be identified with the group of characters \( e : k_{\infty} \rightarrow \mathbb{C}^\times \) trivial on \( A \). Define the basic character \( e_0(g) = \chi(c_{-1}(g)) \), where \( \chi : (\mathbb{F}_q, +) \rightarrow \mathbb{C}^\times \) is a nontrivial character with values in the group of \( p \)-th roots of unity and \( c_{-1}(g) \) is the coefficient of \( T^{-1} \) in the Laurent series expansion of \( g \). Then \( e_0 \) induces a character of \( S^1 \) and moreover every character of \( S^1 \) is of the form \( e(g) = e_0(ag) \) for some \( a \in A \). Indeed, every character of \( k_{\infty} \) may be written in the form \( e(g) = e_0(rg) \) for some \( r \in k_{\infty} \) c.f. [10], sect. II.5. If \( e \) is trivial on \( A \) but \( r \notin A \), we could find an element \( a \in A \) with \( c_{-1}(ar) \notin \ker(\chi) \). But then \( e(a) \neq 1 \), contradiction.

We will need the following analog of the classical Kronecker’s Theorem (see [1]).

**Theorem 1.** For any \( f \in k_{\infty} - k \), the image of \( Af \) in \( S^1 \) is dense.

**Proof.** For any compact Hausdorff abelian group \( G \), the Weyl Criterion (c.f. [7], Corollary 1.2 of Chapter 4) is available. In particular, the \( A \)-sequence \( x_a := af \mod A, a \in A \), is uniformly distributed in \( S^1 \) if and only if

\[ \lim_{d \to \infty} \frac{1}{q^{d+1}} \sum_{\deg a \leq d} e(x_a) = 0 \]

for each non-trivial character \( e : S^1 \rightarrow \mathbb{C}^\times \). Since \( f \) is irrational, for each non-trivial character \( e \) there exists \( b \in A \) such that \( e(x_b) \neq 1 \). Indeed, if it were true that for some non-trivial character \( e \) and for all \( b \in A \),

\[ e(x_b) = e_0(ax_b) = e_0(x_{ab}) = e_0(bx_a) = 1, \]

this would imply \( af \in A \) i.e. \( f \in k \), contradiction. If \( \delta = \deg b \), then for \( d > \delta \) we have

\[ (1 - e(x_b)) \left( \sum_{\deg a \leq d} e(x_a) \right) = \sum_{\deg a \leq d} e(x_a) - \sum_{\deg a \leq d} e(x_{a+b}) \]

\[ = (1 - e(x_b)) \left( \sum_{\deg a \leq \delta} e(x_a) \right), \]

since the map \( a \mapsto a + b \) defines a bijection of the set of \( a \) satisfying \( \delta < \deg(a) \leq d \), i.e.,

\[ \sum_{\delta < \deg a \leq d} e(x_a) - \sum_{\delta < \deg a \leq d} e(x_{a+b}) = 0. \]

As \( 1 - e(x_b) \neq 0 \), it may be canceled in (2), (3) to give equality of the two summations with limits \( d, \delta \), i.e.,

\[ \lim_{d \to \infty} \frac{1}{q^{d+1}} \sum_{\deg a \leq d} e(x_a) = \lim_{d \to \infty} \frac{1}{q^{d+1}} \sum_{\deg a \leq \delta} e(x_a) = 0. \]
defines a bijection respecting vector space operations, making \( \Lambda \) vector space since multiplication by scalars leaves \( \Lambda \) invariant. The discriminant is the expression

\[
\Delta(\Lambda) := (T^q - T)E_{q-1}(\Lambda) + (T^q - T)^qE_{q-1}(\Lambda)^{q+1}
\]

and setting

\[
g(\Lambda) := (T^q - T)E_{q-1}(\Lambda),
\]

the (classical) \( j \) invariant is defined as

\[
j(\Lambda) := \frac{g(\Lambda)^{q+1}}{\Delta(\Lambda)} = \frac{1}{T^q - T} + J(\Lambda)
\]

where

\[
J(\Lambda) := \frac{T^q - T}{(T^q - T)^{q+1}} \cdot \frac{E_{q-1}(\Lambda)}{E_{q-1}(\Lambda)^{q+1}}.
\]

When we restrict to \( \Lambda = \Lambda(\omega) \) we obtain a well-defined modular function

\[
j : \text{PGL}_2(A) \backslash \Omega \rightarrow \mathbb{C}_\infty.
\]

See for example \([4, 5, \ldots]\).

Let \( f \in k_\infty \). The \( A \)-module \( \langle 1, f \rangle_A \subset k_\infty \) is a lattice only when \( f \in k \); when \( f \in k_\infty - k \), Theorem \([1]\) implies that \( \langle 1, f \rangle_A \) is dense in \( k_\infty \). In order to define \( j^q(f) \), we look for a suitable replacement for the lattice used to define \( j(\omega) \) for \( \omega \in \Omega \). We adapt the ideas found in \([8]\).

For any \( x \in k_\infty \) we denote the distance to the nearest element of \( A \) by

\[
\|x\| := \min_{a \in A} |x - a| \in [0, 1[.
\]

Given \( \varepsilon > 0 \), define

\[
\Lambda_\varepsilon(f) := \{ \lambda \in A \mid \|\lambda f\| < \varepsilon \}.
\]

Elements \( \lambda \in \Lambda_\varepsilon(f) \) are referred to as \( \varepsilon \)-approximations of \( f \) and the value \( \|\lambda f\| \) is called the error of \( \lambda \).

Since the distance between distinct elements of \( A \) is \( \geq 1 \), if we take \( \varepsilon < 1 \), then for all \( \lambda \in \Lambda_\varepsilon(f) \) there exists a unique \( \lambda^\perp \in A \) such that

\[
\|\lambda f\| = |\lambda f - \lambda^\perp| < \varepsilon.
\]

We call \( \lambda^\perp \) the \( f \)-dual of \( \lambda \), and we denote \( \Lambda_\varepsilon(f)^\perp = \{ \lambda^\perp \mid \lambda \in \Lambda_\varepsilon(f) \} \).

**Proposition 1.** For each \( 0 < \varepsilon < 1 \), \( \Lambda_\varepsilon(f), \Lambda_\varepsilon(f)^\perp \subset A \) are isomorphic \( \mathbb{F}_q \)-vector spaces. Moreover, they are \( A \)-ideals for \( \varepsilon \) sufficiently small if and only if \( f \in k \).

**Proof.** Given \( \lambda, \lambda' \in \Lambda_\varepsilon(f), \|\lambda + \lambda' f - (\lambda^\perp + \lambda'^\perp)\| = \|\lambda f - \lambda^\perp + (\lambda' f - \lambda'^\perp)\| \leq \max\{\|\lambda f - \lambda^\perp\|, \|\lambda' f - \lambda'^\perp\|\} < \varepsilon, \) which shows that \( \Lambda_\varepsilon(f) \) is a group; in fact a vector space since multiplication by scalars leaves \( \| \cdot \| \) invariant. The map \( \lambda \mapsto \lambda^\perp \) defines a bijection respecting vector space operations, making \( \Lambda_\varepsilon(f)^\perp \) a vector space isomorphic to \( \Lambda_\varepsilon(f) \). This takes care of the first statement. Concerning the second statement, suppose that \( f \in k \) and write \( f = a/b, \) where \( a, b \in A \)
are relatively prime. Then for \( \epsilon \leq |b|^{-1} \), the inequality \( |\lambda(a/b) - \lambda^\perp| < \epsilon \) implies that \( \lambda a - \lambda^\perp b = 0 \) or \( \lambda^\perp = \lambda a/b \). Since \( (a, b) = 1 \), this implies that \( b|\lambda \) so that \( \Lambda_\epsilon(f) \subset bA \). The opposite inclusion \( \Lambda_\epsilon(f) \supset bA \) is trivial. On the other hand, suppose \( f \in k_\infty - k \) and let \( \lambda \in \Lambda_\epsilon(f) \). By Theorem 1 for each \( c \in k_\infty \) such that \( |c| < 1 \) there exists \( \lambda' \in A \) such that \( |\lambda\lambda' f - c| < \epsilon \). If we assume \( \lambda\lambda' \in \Lambda_\epsilon(f) \) there exists \( (\lambda\lambda')^\perp \in A \) such that \( |\lambda\lambda' f - (\lambda\lambda')^\perp| < \epsilon \). However:

\[
|\lambda\lambda' f - (\lambda\lambda')^\perp| \leq \max\{|\lambda\lambda' f - c|, |(\lambda\lambda')^\perp - c|\}.
\]

As \( |c| < 1 \), we have

\[
|\lambda\lambda' f - c| \geq 1 > |\lambda\lambda' f - c|.
\]

Thus the inequality in (4) is an equality, i.e.

\[
\epsilon > |\lambda\lambda' f - (\lambda\lambda')^\perp| = |(\lambda\lambda')^\perp - c| \geq 1,
\]

contradiction. In particular, \( \Lambda_\epsilon(f) \) is not an ideal for all \( \epsilon \in (0, 1) \). Therefore, \( \Lambda_\epsilon(f) \) is an \( A \)-ideal for \( \epsilon \) sufficiently small if and only if \( f \in k \).

**Note 1.** In the number field setting, the analogous set \( \Lambda_\epsilon(\theta) \) for \( \theta \in \mathbb{R} - \mathbb{Q} \) is not a group, due to the archimedean nature of the absolute value of \( \mathbb{R} \), see [3].

**Note 2.** If \( f \in k_\infty - k \) then the minimal degree of the nonzero elements of \( \Lambda_\epsilon(f) \) tends to infinity as \( \epsilon \to 0 \) because there are only finitely many polynomials in \( T \) of fixed degree. In other words,

\[
\lim_{\epsilon \to 0} \min \{|\lambda| \mid \lambda \in \Lambda_\epsilon(f) - \{0\}\} = \infty.
\]

Thus

\[
\bigcap_{\epsilon > 0} \Lambda_\epsilon(f) = \{0\}
\]

and we could not use the intersection to define a lattice-like object with which one might hope to define the quantum \( j \)-invariant.

The idea now is to use \( \Lambda_\epsilon(f) \) as if it were a lattice to define an approximation to \( j^{q_1}(f) \). We define the \( \epsilon \)-zeta function of \( f \) via:

\[
\zeta_{f, \epsilon}(n) := \sum_{\lambda \in \Lambda_\epsilon(f) - \{0\}} \lambda^{-n}, \quad n \in \mathbb{N}.
\]

It is easy to see that \( \zeta_{f, \epsilon}(n) \) converges, since it is a subsum of the zeta function of \( A \) (see [3]) \( \zeta_A(n) = \sum_{\lambda \in A} \lambda^{-n} \). If we denote \( \zeta_{q_1}(n) = \sum_{\lambda \in \mathbb{F}_q - \{0\}} \lambda^{-n} \) then

\[
\zeta_{q_1}(n) \cdot \zeta_{f, \epsilon}(n) = \sum_{\lambda \in \Lambda_\epsilon(f) - \{0\}} \lambda^{-n}.
\]

Taking into account that \( \zeta_{q_1}(q - 1) = \zeta_{q_1}(q^2 - 1) = -1 \), we define

\[
\Delta_\epsilon(f) := (T^{q_1} - T) \sum_{\lambda \in \Lambda_\epsilon(f) - \{0\}} \lambda^{1-q^2} + (T^{q} - T)^q \left( \sum_{\lambda \in \Lambda_\epsilon(f) - \{0\}} \lambda^{1-q} \right)^{q+1}
\]

\[
= -(T^{q_1} - T)\zeta_{f, \epsilon}(q^2 - 1) + (T^{q} - T)^q \zeta_{f, \epsilon}(q - 1)^{q+1}
\]

and

\[
g_\epsilon(f) := (T^{q} - T) \sum_{\lambda \in \Lambda_\epsilon(f) - \{0\}} \lambda^{1-q} = -(T^{q} - T)\zeta_{f, \epsilon}(q - 1).
\]
Then the $\varepsilon$-modular invariant of $f$ is defined
\[
j_\varepsilon(f) := \frac{g_\varepsilon q^{\varepsilon+1}}{\Delta_\varepsilon(f)} = \frac{1}{T_{\varepsilon, \tau} - J_\varepsilon(f)}
\]
where
\[
J_\varepsilon(f) := \frac{T_{\varepsilon, \tau} - T}{(T_{\varepsilon, \tau} - T)^{\varepsilon+1}} \cdot \frac{\zeta_{f, \varepsilon}(q^2 - 1)}{\zeta_{f, \varepsilon}(q - 1)^{\varepsilon+1}}.
\]

The quantum modular invariant or quantum $j$-invariant of $f$ is then
\[
j^{\text{qt}}(f) := \lim_{\varepsilon \to 0} j_\varepsilon(f) \subset k_\infty \cup \{\infty\}
\]
where by $\lim_{\varepsilon \to 0} j_\varepsilon(f)$ we mean the set of limit points of convergent sequences $\{j_\varepsilon(f); \varepsilon \to 0\}$. Thus, by definition, $j^{\text{qt}}(f)$ is in principle multi-valued, and so we denote it as such, writing

\[
j^{\text{qt}} : (k_\infty \cup \{\infty\}) \to (k_\infty \cup \{\infty\}), \quad j^{\text{qt}}(\infty) := \infty.
\]

In the number field case, $j^{\text{qt}}$ is only known experimentally to be multi-valued, having finitely many values for all real quadratics tested, see the Appendix in [3]. In this paper, we will prove that $j^{\text{qt}}$ is multi-valued and in the sequel [2], we will prove that $\#j^{\text{qt}}(f) < \infty$ for $f$ quadratic.

**Theorem 2.** If $f \in k$, then for $\varepsilon$ sufficiently small, $j_\varepsilon(f) = \infty$. In particular, $j^{\text{qt}}(f) = \infty$.

**Proof.** By Proposition 1, $\Lambda_\varepsilon(f)$ is an ideal for sufficiently small $\varepsilon$, hence it is a principal ideal of the form $(h)$, $h \in A$. Therefore, the quotient of $\varepsilon$ zeta functions occurring in (5) in the definition of $J_\varepsilon(f)$, being homogeneous of degree $1 - q^2$, can be written

\[
\frac{\zeta_{f, \varepsilon}(q^2 - 1)}{\zeta_{f, \varepsilon}(q - 1)^{\varepsilon+1}} = \frac{-\sum_{0 \neq \lambda \in \Lambda_\varepsilon(f)} \lambda^{-q^2+1}}{\left(\sum_{0 \neq \lambda \in \Lambda_\varepsilon(f)} \lambda^{1-q}\right)^{\varepsilon+1}} = \frac{-h^{q^2-1} \sum_{0 \neq \lambda \in A} \lambda^{-q^2+1}}{h^{q^2-1} \sum_{0 \neq \lambda \in A} \lambda^{1-q} \left(\lambda^{-q}\right)^{\varepsilon+1}} = \frac{\zeta_A(q^2 - 1)}{\zeta_A(q - 1)^{\varepsilon+1}}.
\]

This shows that $j_\varepsilon(f)$ is constant for small $\varepsilon$, hence $j^{\text{qt}}(f)$ is a single point. Moreover, since $A$ is the limit of the lattice $(1, \omega)_A$, $\omega \in \Omega$, for $\omega \to 0$, the above calculation shows that for $\varepsilon$ small, $j_\varepsilon(f) = j^{\text{qt}}(f)$ is the limit of classical $j$ at $\infty$. But it is well-known that classical $j$ has a pole at $\infty$, see [5].

**Proposition 2.** $j^{\text{qt}}$ is invariant w.r.t. the action of $\text{PGL}_2(A)$ and induces a multi-valued function

\[
j^{\text{qt}} : \text{PGL}_2(A) \setminus (k_\infty \cup \{\infty\}) \to (k_\infty \cup \{\infty\}).
\]

**Proof.** Essentially the same as the proof of Theorem 1 in [3]. By Theorem 2, it is enough to prove $\text{PGL}_2(A)$-invariance for $f \in k_\infty - k$. In brief, if $M = \begin{pmatrix} r & s \\ t & u \end{pmatrix} \in$ QUANTUM $j$-INVARIANT IN POSITIVE CHARACTERISTIC I 5
Every function eventually periodic. See [8], sect. 9.2.

Note that the continued fraction expansion is denoted \( M : \mathcal{A}_f(f) \rightarrow \mathcal{A}_f(M(f)), \) \( \varepsilon' = \frac{\varepsilon}{tf + u}. \)

Then one may check that \( \{ \varepsilon_i \} \) produces a limit point of \( j^0(f) \leftrightarrow \{ \varepsilon'_i \} \) produces a limit point of \( j^0(M(f)) \) and that these limit points coincide.

3. Convergence

For \( f = f_0 \in k_\infty - k \) let \( a_0 \in A \) be its polynomial part. Then writing \( f = a_0 + 1/f_1, \) we let \( a_1 \in A \) be the polynomial part of \( f_1, \) and so forth: in this way we obtain the continued fraction expansion of \( f: \)

\[
f = a_0 + \frac{1}{a_1 + f_2} = a_0 + \frac{1}{a_1 + a_2 + f_3} = \cdots.
\]

The resulting series of iterated fractions

\[
a_0, \quad a_0 + \frac{1}{a_1}, \quad a_0 + \frac{1}{a_1 + a_2}, \quad \ldots
\]

converges to \( f \) and is infinite unless \( f \in k. \) Note that for all \( i \geq 1, \) \( |1/f_i| > 1 \) and therefore \( |a_i| > 1. \) In particular, \( a_i \notin \mathbb{F}_q \) for all \( i \geq 1. \) The continued fraction expansion is denoted

\[
f = [a_0, a_1, \ldots].
\]

Every \( f \in k_\infty - k \) may be so represented, and the representation is unique. Moreover, \( f \in k_\infty - k \) is quadratic if and only if its continued fraction expansion is eventually periodic. See [3], sect. 9.2.

The sequence of denominators of best approximations is defined recursively by

\[
q_0 = 1, q_1 = a_1, \ldots, q_i = a_i q_{i-1} + q_i - 2.
\]

Note that

\[
|q_n| = |a_1 \cdots a_n| = q^{\sum_{i=1}^{n} \deg(a_i)}.
\]

The corresponding sequence of duals \( \{ q^+_n \} \) is obtained by the recursion

\[
q^+_0 = a_0, q^+_1 = a_1 a_0 + 1, \ldots, q^+_i = a_i q^+_{i-1} + q^+_{i-2}.
\]

Indeed, by [3], page 307:

\[
\| q_n f \| = |q_n f - q^+_n| = \frac{1}{|q_n||a_{n+1}|} = \frac{1}{|q_{n+1}|} = q^{-\sum_{i=1}^{n+1} \deg(a_i)}.
\]

Since we will want to consider sums over monic polynomials, let us denote by

\[
\bar{q}_n = c_n q_n
\]

the unique scalar multiple of \( q_n \) which is monic. Clearly \( |\bar{q}_n| = |q_n|, \) \( \bar{q}^+_n = c_n q^+_n \) and \( \| \bar{q}_n f \| = \| q_n f \|. \) It is a consequence of the above remarks that the following set forms a basis of \( A \) as an \( \mathbb{F}_q \)-vector space:

\[
(6) \quad \left\{ T^{\deg(a_1) - 1}, \ldots, 1; T^{\deg(a_2) - 1}q_1, \ldots, q_1; T^{\deg(a_3) - 1}q_2, \ldots, q_2; \ldots \right\}.
\]

(The order in which the basis elements have been listed corresponds to decreasing errors, see [3] below.) Using the basis \( (6), \) a typical element of \( A \) can then be concisely written in the form

\[
a = c_m(T)q_m + c_{m+1}(T)q_{m+1} + \cdots + c_{m'}(T)q_{m'}, \quad 0 \leq m \leq m',
\]
where \( c_i(T) \in A \) is a polynomial of degree \( \leq \deg(a_{i+1}) - 1 \). Notice that \( a \) is monic if and only if \( c_m(T) \) is monic.

**Lemma 1.** Let \( \varepsilon = q^{d-\sum_{i=1}^{N+1} \deg(a_i)} \) for \( 0 < l \leq \deg(a_{N+1}) \). Then

\[
\Lambda_\varepsilon(f) = \text{span}_{\bar{q}_n} (T^{l-1} \bar{q}_N, \ldots, T \bar{q}_N, \bar{q}_N; T^{\deg(a_{N+1})-1} \bar{q}_{N+1}, \ldots).
\]

**Proof.** A general element \( T^r \bar{q}_n \) in the basis displayed in (6), where \( 0 \leq r \leq \deg(a_{n+1}) - 1 \), produces the error

\[
\|T^r \bar{q}_n f\| = q^{r - \sum_{i=1}^{n+1} \deg(a_i)}.
\]

Indeed we have \( (T^r \bar{q}_n)^2 = T^r \bar{q}_n \) and

\[
\|T^r \bar{q}_n f\| = \|T^r \bar{q}_n f - T^r \bar{q}_n^1\| = \|T^r\|\|\bar{q}_n f\| = q^{r - \sum_{i=1}^{n+1} \deg(a_i)}.
\]

From this the inclusion \( \supset \) follows. In view of the strictly decreasing pattern of errors of elements of the basis (6) and the non archimedean nature of \( |\cdot| \), no other basis elements may be involved in an element of \( \Lambda_\varepsilon(f) \), giving the equality claimed. □

**Theorem 3.** Let \( f \in k_{\infty} - k \). Then \( |j_\varepsilon(f)| = q^{2q-1} \) for all \( \varepsilon < 1 \).

**Proof.** Let \( \varepsilon = q^{d-\sum_{i=1}^{N+1} \deg(a_i)} \), \( 0 < l \leq \deg(a_{N+1}) \). Denote

\[
\hat{\Lambda}_\varepsilon(f) = \frac{\zeta_{l,\varepsilon}(q^2 - 1)}{\zeta_{l,\varepsilon}(q - 1)^{q+1}}
\]

the quotient of \( \varepsilon \)-zeta functions occurring in (9). We begin with the case \( l = \deg(a_{N+1}) \) i.e. \( \varepsilon = q^{d - \sum_{i=1}^{N} \deg(a_i)} \). Let us write

\[
\Lambda_\varepsilon^{\text{mon}}(f) = \{ \lambda \in \Lambda_\varepsilon(f) \text{ monic} \}.
\]

Denote

\[
\Lambda_\varepsilon^{\text{bas}}(f) = \{ T^r \bar{q}_j | 0 \leq r \leq \deg(a_{i+1}) - 1, i \geq N \} \subset \Lambda_\varepsilon^{\text{mon}}(f)
\]

and write as well

\[
\Lambda_\varepsilon^{\text{non-bas}}(f) = \Lambda_\varepsilon^{\text{mon}}(f) - (\Lambda_\varepsilon^{\text{bas}}(f) \cup \{0\}).
\]

Note that \( \Lambda_\varepsilon^{\text{non-bas}}(f) \) consists of \( \lambda \in \Lambda_\varepsilon(f) \) written in the concise form (7)

\[
\lambda = c_m(T)\bar{q}_m + \cdots + c_{m'}(T)\bar{q}_{m'}
\]

where \( m \leq m' \), \( c_m(T), c_{m'}(T) \neq 0 \). \( c_i(T) \in A \) is a polynomial of degree \( \leq \deg(a_{i+1}) - 1 \), \( c_m(T) \) is monic, and if \( m = m' \), then \( c_m(T) \neq T^r \) for \( 0 \leq r \leq \deg(a_{m+1}) - 1 \). If we introduce the notation

\[
\zeta_{T,i}(n) := 1 + T^{-n} + \cdots + T^{-n(\deg(a_{i+1}) - 1)}
\]

then clearly

\[
\sum_{\lambda \in \Lambda_\varepsilon^{\text{bas}}(f)} \lambda^{-n} = \sum_{i=N}^{\infty} \zeta_{T,i}(n)\bar{q}_i^{-n}.
\]

It follows that \( \hat{\Lambda}_\varepsilon(f) \) can be written

\[
\sum_{i=N}^{\infty} \zeta_{T,i}(q^2 - 1)\bar{q}_i^{1-q^2} + \sum_{i=m}^{m'} c_i(T)\bar{q}_i^{1-q^2} - q^{2q-1}.
\]
To ease notation, in what follows we omit the summation limit $\Lambda_{\epsilon}^{\text{non-bas}}(f)$. Dividing out the numerator and denominator by $\bar{q}N^{-q}$ and writing $\alpha_i := \bar{q}_{N+i}/\bar{q}_N$, we obtain

$$\tilde{J}_\epsilon(f) = \frac{\sum_{i=0}^{\infty} \zeta_{T,N+i}(q^2 - 1)\alpha_i^{1-q} + \sum (\sum_{i=m}^{m'} c_{N+i}(T)\alpha_i) \bar{q}^{1-q} \left( \sum_{i=0}^{\infty} \zeta_{T,N+i}(q - 1)\alpha_i^{1-q} + \sum (\sum_{i=m}^{m'} c_{N+i}(T)\alpha_i) \right)^{q+1}}{\sum_{i=0}^{\infty} \zeta_{T,N+i}(q - 1)\alpha_i^{1-q} + \sum (\sum_{i=m}^{m'} c_{N+i}(T)\alpha_i) \bar{q}^{1-q} \left( \sum_{i=0}^{\infty} \zeta_{T,N+i}(q - 1)\alpha_i^{1-q} + \sum (\sum_{i=m}^{m'} c_{N+i}(T)\alpha_i) \right)^{q+1}}.$$  

In the above, we now have $0 \leq m \leq m'$. Note that $\alpha_0 = 1$,

$$|\alpha_1| = |\bar{q}_{N+1}/\bar{q}_N| = |q_{N+1}/q_N| = |(a_{N+1}q_N + q_{N-1})/q_N| = |a_{N+1}|$$

and that in general, $|\alpha_i| = |a_{N+i} \cdots a_{N+1}|$ for $i > 0$. It is clear then that both the numerator and denominator of the above expression have absolute value 1: since $\alpha_0 = 1$ is a summand in both the numerator and the denominator, and all other summands have absolute value < 1. In particular, $|\tilde{J}_\epsilon(f)| = 1$. In order to calculate $|\tilde{J}_\epsilon(f)|$, it suffices to calculate the absolute value of

$$\frac{1}{T^q - T} - \frac{T^q - T}{(T^q - T)^{q+1}} \tilde{J}_\epsilon(f).$$

Writing (10) in the form of a single fraction, we obtain

$$\frac{U - V}{(T^q - T)(T^q - T)^{q+1}} \tilde{J}_\epsilon(f)_{\text{denom}}$$

where $\tilde{J}_\epsilon(f)_{\text{denom}}$ is the denominator of $\tilde{J}_\epsilon(f)$,

$$U = \left\{ (T^q - T) \left( \sum_{i=0}^{\infty} \zeta_{T,N+i}(q - 1)\alpha_i^{1-q} + \sum (\sum_{i=m}^{m'} c_{N+i}(T)\alpha_i) \bar{q}^{1-q} \right)^{q+1} \right\}$$

and

$$V = \left( T^q - T \right) \left( \sum_{i=0}^{\infty} \zeta_{T,N+i}(q^2 - 1)\alpha_i^{1-q} + \sum (\sum_{i=m}^{m'} c_{N+i}(T)\alpha_i) \bar{q}^{1-q} \right)^{q+1}.$$  

Although the denominator in (11) is not the least common denominator, this way of writing the fraction gives us a presentation of $U$ as a $(q+1)$th power, which will be convenient in the analysis that follows. The denominator in (11) has absolute value $q^q \cdot q^q(q^q+1)$, so what remains is to calculate $|U - V|$. The high degree term of each of $U$ and $V$ is $T^q(q^q+1)$, which therefore drops out of $U - V$. So to calculate $|U - V|$ we must identify and compare terms of next highest degree in each of $U$ and $V$. We begin with $U$. Telescopic cancellation gives

$$(T^q - T)\zeta_{T,N}(q - 1) = T^q - T^{(1-q)\deg(a_{N+1})+q},$$

so the product inside the $q+1$ power develops as

$$T^q - T^{(1-q)\deg(a_{N+1})+q}$$

$$+(T^q - T)\zeta_{T,N+1}(q - 1)\alpha_1^{1-q} + \cdots$$

$$+(T^q - T)\sum_{c(T) \neq 0, c(T) \text{monic}} c(T)^{1-q} + (T^q - T)\sum_{c(T) \neq 0} (c(T) + \alpha_1)^{1-q} + \cdots.$$
where the \( c(T) \) terms (which are the coefficients of \( \alpha_0 = 1 \)) satisfy \( \text{deg}(c(T)) \leq \text{deg}(a_{N+1}) - 1 \) and where the additional conditions “\( c(T) \neq 0, T^i, \text{monic} \)” in the first sum in (14) come from the conditions placed on the non basic expressions (10). In lines (13), (14) we have listed terms in order of decreasing absolute value. Note that lines (12) and (13) come from “basic sums” and line (14) comes from the non-basic sum. Let us assume first that \( \text{deg}(a_{N+1}) > 1 \). The sum occurring in the first term in (14) can be written more explicitly as

\[
\sum_{c(T) \neq 0, T^i \text{monic}} c(T)^{1-q} = \sum_{c \neq 0} (c + T)^{1-q} + \sum_{(c_0, c_1) \neq (0, 0)} (c_0 + c_1 T + T^2)^{1-q} + \cdots .
\]

Note then that

\[
\left| \sum_{c \neq 0} (c + T)^{1-q} \right| = \sum_{c \neq 0} \left| c \right| \left( d(T) + T^q \right)^{q-1} = \frac{T^{(q-1)(q-2)} + T^{\text{lower order}}}{T^{(q-1)^2} + T^{\text{lower order}}} = q^{1-q}.
\]

It is straightforward to see that the remaining summands in the right hand side of (15) have absolute value \( q^{1-q} \). Thus

\[
(T^q - T) \sum_{c(T) \neq 0, T^i \text{monic}} c(T)^{1-q} = q.
\]

Aside from \( T^q \), the remaining high order displayed terms in equations (12)–(14),

\[
-T^{(1-q) \text{deg}(a_{N+1})+q}, T^q \alpha_1^{1-q}, T^q \sum_{c(T) \neq 0} (c(T) + \alpha_1)^{1-q},
\]

all have absolute value \( q^{(1-q) \text{deg}(a_{N+1})+q} < q \). Therefore for \( \text{deg}(a_{N+1}) > 1 \), the term of next highest absolute value (in the product within the \( q + 1 \) power) is

\[
\Omega(q - 1) := (T^q - T) \sum_{c(T) \neq 0, T^i \text{monic}} c(T)^{1-q}.
\]

Upon performing the \( q + 1 \) power, we see that after the term \( T^{q(q+1)} \), the term \( T^{q^2} \cdot \Omega(q - 1) \) has next highest absolute value, and this is \( q^{q^2+1} \). We have a similar analysis of \( V \): since

\[
(T^q - T) \Omega_{T,N}(q^2 - 1) = T^{q^2} - T^{(1-q^2) \text{deg}(a_{N+1})+q^2},
\]

we may write \( V \) as \( (T^q - T) \times \) the expression

\[
T^{q^2} - T^{(1-q^2) \text{deg}(a_{N+1})+q^2}
\]

\[
+ (T^q - T) \Omega_{T,N+1}(q^2 - 1) \alpha_1^{1-q^2} + \cdots
\]

\[
+ (T^q - T) \sum_{c(T) \neq 0, T^i \text{monic}} c(T)^{1-q^2} + (T^q - T) \sum_{c(T) \neq 0} (c(T) + \alpha_1)^{1-q^2} + \cdots .
\]

The term \( (T^q - T)(T^{q^2} - T) \sum_{c(T) \neq 0, T^i \text{monic}} c(T)^{1-q^2} \) has absolute value \( q^{q+1} \) and the remaining possible candidates for next highest absolute value are

\[
-T^{q(q+1)} \cdot T^{(1-q^2) \text{deg}(a_{N+1})}, T^{q(q+1)} \cdot \alpha_1^{1-q^2}, T^{q(q+1)} \cdot \sum_{c(T) \neq 0} (c(T) + \alpha_1)^{1-q^2}.
\]
each having absolute value

\[ q^{l(q+1)} \cdot q^{(1-q^2) \deg(a_{N+1})} < q^{q+1} < q^{q^2+1}. \]

We conclude that \(|U - V| = q^{q^2+1}\) and therefore \(|j_k(f)| = q^{2q-1}\). Now we consider the case \( l = \deg(a_{N+1}) = 1 \). In line (14), the \( c(T) \) have degree 0 i.e. they are constants, which means that the non-basic summand \( \Omega(q - 1) \) no longer appears. We are thus left with the terms (16), each having absolute value \( q^2 \) as constants. The sum of these terms continues to have absolute value \( q^1 \) indeed, their sum is the rational expression

\[ T^q - \prod_{c}(c + \alpha_i)^{q-1} + T^{q-1} \sum_{c} \prod_{d \neq c}(d + \alpha_i)^{q-1} \frac{Tq^{-1} \prod_{c}(c + \alpha_i)^{q-1}}{Tq^{-1} \prod_{c}(c + \alpha_i)^{q-1}}. \]

In the above, we have incorporated the term \( \alpha_i \) into the family of terms \( c + \alpha_1 \) by allowing \( c = 0 \). The term of \( - \prod_{c}(c + \alpha_i)^{q-1} \) having highest absolute value is \( \alpha_i^1(q-1)q \). The term of the next expression in the numerator having highest absolute value is in principle \( Tq^{-1} \cdot \alpha_i^1(q-1)^2 \). But this term has coefficient \( \#F_q = q \equiv 0 \).

Therefore, the terms of next highest absolute value in \( U \) are obtained as mixed products of the form \( (T^n) \cdot q^{1-q^2} = q^{q+1} < q^{q^2+1} \). Once again we conclude that \(|U - V| = q^{q^2+1}\) and \(|j_k(f)| = q^{2q-1}\). What remains is the case \( \varepsilon = q^{l-\sum_{i=1}^{N+1} \deg(a_i)} \) for \( 0 < l < \deg(a_{N+1}) \). Here we note that the change in \( J_k(f) \) is to replace \( \zeta_{T,N}(n) \) by

\[ \zeta_{T,N,l}(n) := 1 + \cdots + T^{-(l-1)n} \]

and to replace \( \sum_{c(T) \neq 0, T^l} c(T)^n \sum_{c(T) \neq 0}(c(T) + \alpha_1)^n \) by

\[ \sum_{c(T) \neq 0, T^l} c(T)^n \sum_{c(T) \neq 0, \deg(c(T)) \leq l-1}(c(T) + \alpha_1)^n. \]

Denote \( \Omega(q - 1)_l = (T^n - T) \sum_{c(T) \neq 0, T^l} c(T)^1-q \). Then \( (T^n - T) \zeta_{T,N,l}(n) = Tq - T^{1-q}l^q+q \). When \( l > 1 \), we have

\[ |T^{1-q}l^q+q| = q^{1-q}l^q+q < q = |\Omega(q - 1)_l|. \]

Therefore, in the counterpart of \( U - V \), the term determining the absolute value is once again \( \Omega(q - 1)_l \), and so we arrive at an identical computation of \(|j_k(f)|\). If \( l = 1 \), there is no \( \Omega(q - 1)_l \) term and using an argument identical to that given above in the case \( l = \deg(a_{N+1}) = 1 \), we again conclude that \(|j_k(f)| = q^{2q-1}\). \qed

**Corollary 1.** \( f \in k \Leftrightarrow j_k(f) = \infty \text{ for } \varepsilon \text{ sufficiently small} \Leftrightarrow \infty \in j(f) \).

**References**

[1] Cassels, J.W.S., An Introduction to Diophantine Approximation. Cambridge Tracts in Mathematics and Mathematical Physics, 45. Cambridge University Press, New York, 1957.

[2] Demangos, L & Gendron, T.M., Quantum \( j \)-Invariant in Positive Characteristic II: Formulas and Values at the Quadratics.

[3] Castaño Bernard, C. & Gendron, T.M., Modular invariant of quantum tori. Proc. Lond. Math. Soc. 109 (2014), Issue 4, 1014–1049.
[4] E. U. Gekeler, Zur Arithmetik von Drinfeld Moduln, Math. Ann. 262 (1983), 167–182.
[5] E. U. Gekeler, On the coefficients of Drinfeld modular forms, Invent. Math. 93 (1988), Issue 3, 667–700.
[6] D. Goss, Basic structures of Drinfeld modular forms, Invent. Math. 93 (1988), Issue 3, 667–700.
[7] Kuipers, L & Niederreiter,H., Uniform Distribution of Sequences. Dover, N.Y., 2012.
[8] Thakur, D.S., Function Field Arithmetic, World Scientific, Singapore, 2004.
[9] Villa Salvador, Gabriel Daniel, Topics in the Theory of Algebraic Function Fields, Birkhäuser, Boston, 2006.
[10] Weil, André, Basic Number Theory, Springer-Verlag, Berlin, 1974.

Instituto de Matemáticas – Unidad Cuernavaca, Universidad Nacional Autónoma de México, Av. Universidad S/N, C.P. 62210 Cuernavaca, Morelos, México

Current address: Department of Mathematical Sciences (Mathematics Division), University of Stellenbosch, Private Bag X1, Matieland 7602, South Africa

E-mail address: l.demangos@gmail.com

Instituto de Matemáticas – Unidad Cuernavaca, Universidad Nacional Autónoma de México, Av. Universidad S/N, C.P. 62210 Cuernavaca, Morelos, México

E-mail address: tim@matcuer.unam.mx