TOPOLOGICAL COMPLEXITY OF CONFIGURATION SPACES

MICHAEL FARBER AND MARK GRANT

Abstract. The topological complexity $TC(X)$ is a homotopy invariant which reflects the complexity of the problem of constructing a motion planning algorithm in the space $X$, viewed as configuration space of a mechanical system. In this paper we complete the computation of the topological complexity of the configuration space of $n$ distinct points in Euclidean $m$-space for all $m \geq 2$ and $n \geq 2$; the answer was previously known in the cases $m = 2$ and $m$ odd. We also give several useful general results concerning sharpness of upper bounds for the topological complexity.

1. Introduction

The motion planning problem is a central theme of robotics [14]. Given a mechanical system $S$, a motion planning algorithm for $S$ is a function associating with any pair of states $(A, B)$ of $S$ a continuous motion of the system starting at $A$ and ending at $B$. If $X$ denotes the configuration space of the system, one considers the path fibration

$$\pi : PX \to X \times X, \quad \pi(\gamma) = (\gamma(0), \gamma(1)), \quad (1)$$

where $PX = X'$ is the space of all continuous paths $\gamma : I = [0, 1] \to X$. In these terms, a motion planning algorithm for $S$ is a section (not necessarily continuous) of $\pi$.

The topological complexity of a topological space $X$, denoted $TC(X)$, is defined to be the genus, in the sense of Schwarz [15], of fibration (1). More explicitly, $TC(X)$ is the minimal integer $k$ such that $X \times X$ admits a cover by $k$ open subsets, on each of which there exists a continuous local section of fibration (1). One of the basic properties of $TC(X)$ is its homotopy invariance [6]. If $X$ is a Euclidean Neighbourhood Retract then the number $TC(X)$ can be equivalently characterized (see

2000 Mathematics Subject Classification. Primary 55M99, 55R80; Secondary 68T40.

Key words and phrases. Topological complexity, configuration spaces.

The research was supported by grants from the EPSRC and from the Royal Society.
as the minimal integer $k$ such that there exists a section $s : X \times X \to PX$ of (1) and a decomposition
\[ X \times X = G_1 \cup \cdots \cup G_k, \quad G_i \cap G_j = \emptyset, \quad i \neq j \]
where each $G_i$ is locally compact and such that the restriction $s|_{G_i} : G_i \to PX$ is continuous for $i = 1, \ldots, k$. A section $s$ as above can be viewed as a motion planning algorithm: given a pair of states $(A, B) \in X \times X$ the path $s(A, B)(t)$ represents a continuous motion of the system starting from $A$ and ending at $B$. The number $TC(X)$ is a measure of the complexity of motion planning algorithms for a system whose configuration space is $X$.

The concept $TC(X)$ was introduced and studied in [6], [7]. We refer the reader to surveys [9], [11] for detailed treatment of the invariant $TC(X)$. Computation of $TC(X)$ in various practically interesting examples has received much recent interest, see for instance papers [1], [2], [10], [12], [13].

In this paper we study the topological complexity $TC(F(R^m, n))$ of the space of configurations of $n$ distinct points in Euclidean $m$-space. Here $m, n \geq 2$, and
\[ F(R^m, n) = \left\{ (x_1, \ldots, x_n) \in (R^m)^n; x_i \neq x_j \text{ for } i \neq j \right\}, \]
topologised as a subspace of the Cartesian power $(R^m)^n$. This space appears in robotics when one controls multiple objects simultaneously trying to avoid collisions between them. Our main result in this paper is the following.

**Theorem 1.** One has
\[ \text{TC}(F(R^m, n)) = \begin{cases} 2n - 1 & \text{for all } m \text{ odd,} \\ 2n - 2 & \text{for all } m \text{ even.} \end{cases} \]

The cases $m = 2$ and $m \geq 3$ odd of Theorem 1 were proven by Farber and Yuzvinsky in [8], where it was conjectured that $TC(F(R^m, n)) = 2n - 2$ for all even $m$. Here we settle this conjecture in the affirmative. Note that the methods employed in [8] are not applicable in the case when $m > 2$ is even. We therefore suggest an alternative approach based on sharp upper bounds for the topological complexity.

The plan of the paper is as follows. In the next section we state Theorems 2 and 3 about sharp upper bounds; their proofs appear in section §3. The concluding section §4 contains the proof of Theorem 1.

2. Sharp upper bounds for the topological complexity

Let $X$ be a CW-complex of finite dimension $\text{dim}(X) = n \geq 1$. We denote by $\Delta_X \subset X \times X$ the diagonal $\Delta_X = \{(x, x); x \in X\}$. Let $A$ be
a local system of coefficients on $X \times X$. A cohomology class
\[ u \in H^\ast(X \times X; A) \]
is called a zero-divisor if its restriction to the diagonal is trivial, i.e.
\[ u|\Delta_X = 0 \in H^\ast(X; A|X). \]
The importance of zero-divisors stems from the following fact (see [11], Corollary 4.40):

If the cup-product of $k$ zero-divisors $u_i \in H^\ast(X \times X; A_i)$, where $i = 1, \ldots, k$, is nonzero then $\text{TC}(X) > k$.

Theorem 2 below supplements the general dimensional upper bound of [6] by giving necessary and sufficient conditions for its sharpness.

**Theorem 2.** For any $n$-dimensional cell complex $X$ one has

(a) $\text{TC}(X) \leq 2n + 1$;
(b) $\text{TC}(X) = 2n + 1$ if and only if there exists a local coefficient system $A$ on $X \times X$ and a zero-divisor $\xi \in H^1(X \times X; A)$ such that the $2n$-fold cup product
\[ \xi^{2n} = \xi \cup \cdots \cup \xi \neq 0 \in H^{2n}(X \times X; A^{2n}) \]
is nonzero. Here $A^{2n}$ denotes the tensor product of $2n$ copies $A \otimes \cdots \otimes A$ of $A$ (over $\mathbb{Z}$).

Next we state a similar sharp upper bound result for $(s - 1)$-connected spaces $X$ where $s > 1$. We use the following notation. If $B$ is an abelian group and $v \in H^r(X; B)$ is a cohomology class then the class
\[ \bar{v} = v \times 1 - 1 \times v \in H^r(X \times X; B) \]
is a zero-divisor, where $1 \in H^0(X; \mathbb{Z})$ is the unit and $\times$ denotes the cohomological cross-product.

We say that a finitely generated abelian group is square-free if it has no subgroups isomorphic to $\mathbb{Z}_{p^2}$, where $p$ is a prime.

**Theorem 3.** Let $X$ be a $(s - 1)$-connected $n$-dimensional finite cell complex where $s \geq 2$. Assume additionally that $2n = rs$ where $r$ is an integer\(^\dagger\). Then

(a) $\text{TC}(X) \leq r + 1$;
(b) $\text{TC}(X) = r + 1$ if and only if there exists a finitely generated abelian group $B$ and a cohomology class $v \in H^s(X; B)$ such that the $n$-fold cup-product of the corresponding zero-divisors (3) is nonzero
\[ \bar{v}^r = \bar{v} \cup \cdots \cup \bar{v} \neq 0 \in H^{2n}(X \times X; B^r). \]

Here $B^r$ denotes the $r$-fold tensor power $B \otimes \cdots \otimes B$;

\(^\dagger\)This last assumption is automatically satisfied (with $r = n$) for $s = 2$, i.e. when $X$ is simply connected.
(c) If $H_*(X;\mathbb{Z})$ is square-free, then $\text{TC}(X) = r + 1$ if and only if there exists a field $k$ and cohomology classes $v_1, \ldots, v_r \in H^s(X; k)$ such that
$$\bar{v}_1 \cup \cdots \cup \bar{v}_r \neq 0 \in H^{2n}(X \times X; k);$$

(d) If $H_*(X;\mathbb{Z})$ is free abelian, then $\text{TC}(X) = r + 1$ if and only if there exist classes $v_1, \ldots, v_r \in H^s(X; \mathbb{Z})$ such that
$$\bar{v}_1 \cup \cdots \cup \bar{v}_r \neq 0 \in H^{2n}(X \times X; \mathbb{Z}).$$

3. Proofs of Theorems 2 and 3

Proof of Theorem 2. The first statement follows from [6], Theorem 4. If there exists a local system coefficient system $A$ and a zero-divisor $\xi \in H^1(X \times X; A)$ such that $\xi^{2n} \neq 0$ then $\text{TC}(X) \geq 2n+1$, by Corollary 4.40 of [11]. The remaining part of Theorem 2 was proven in [3], Theorem 7. More precisely, let $G = \pi_1(X, x_0)$ denote the fundamental group of $X$ and let $I \subset \mathbb{Z}[G]$ denote the augmentation ideal. $I$ can be viewed as a left $\mathbb{Z}[G \times G]$-module via the action
$$(g, h) \cdot \sum n_i g_i = \sum g g_i h^{-1};$$
where $g, h, g_i \in G$ and $\sum n_i g_i \in I$; this defines a local system with stem $I$ on $X \times X$, see [16], chapter 6. A crossed homomorphism $f : G \times G \to I$ given by the formula
$$f(g, h) = gh^{-1} - 1, \quad g, h \in G$$
determines a cohomology class $v \in H^1(X \times X; I)$. This class is a zero-divisor and has the property that $v^{2n} \neq 0$ assuming that $\text{TC}(X) = 2n + 1$ according to Theorem 7 from [3].

Proof of Theorem 3. Statement (a) follows directly from Theorem 5.2 of [7] which states that
$$(4) \quad \text{TC}(X) < \frac{2n + 1}{s} + 1.$$ for any $(s - 1)$-connected CW-complex $X$ of dimension $n$.

(b) One part of statement (b) follows from Corollary 4.40 of [11]; indeed if $\bar{v}^r \neq 0$ then $\text{TC}(X) \geq r + 1$ since each $\bar{v}$ is a zero-divisor.

The proof of the remaining part of statement (b) is derived from obstruction theory and results of A. S. Schwarz [15] centered around the notion of genus of a fibration. We assume that $X$ is $(s - 1)$-connected, $s \geq 2$, and $n$-dimensional and $2n = rs$ where $r$ is an integer. The case $n = 1$ is trivial, therefore we will assume that $n \geq 2$. We want to show that $\text{TC}(X) = r + 1$ implies that $\bar{v}^r \neq 0 \in H^{2n}(X \times X; B^r)$ for some class $v \in H^s(X; B)$.
Recall that $\text{TC}(X)$ is defined as the genus of the path fibration and according to Theorem 3 from [15] one has $\text{TC}(X) \leq r$ if and only if the $r$-fold fiberwise join
\[
\pi_r : P_r X \to X \times X
\]
of the original fibration $\pi : PX \to X \times X$ admits a continuous section. Hence our assumption $\text{TC}(X) = r + 1$ implies that $\pi_r$ has no continuous sections. The fibre $F_r$ of (1) is the $r$-fold join
\[
F_r = \Omega X \ast \Omega X \ast \cdots \ast \Omega X
\]
where $\Omega X$ denotes the space of loops in $X$ starting and ending at the base point $x_0 \in X$. Note that $\Omega X$ is $(s - 2)$-connected and therefore the fibre $F_r$ is $(2n - 2)$-connected since $r(s - 2) + 2(r - 1) = 2n - 2$.

The primary obstruction to the existence of a section of (1) is an element $\theta_r \in H^{2n}(X \times X; \pi_{2n-1}(F_r))$. It is in fact the only obstruction since the higher obstructions land in zero groups. Thus we obtain that $\theta_r \neq 0$. By the Hurewicz theorem
\[
\pi_{2n-1}(F_r) = H_{2n-1}(F_r) = B \otimes B \otimes \cdots \otimes B = B^r
\]
where $B$ denotes the abelian group $H_{s-1}(\Omega X) = H_s(X)$. Here we have used the Künneth theorem for joins, see for instance [15], chapter 1, §5. By Theorem 1 from [15] the obstruction $\theta_r$ equals the $r$-fold cup-product
\[
\theta_r = \theta \cup \cdots \cup \theta = \theta^r
\]
where $\theta \in H^s(X \times X; B)$ is the primary obstruction to the existence of a section of $\pi : PX \to X \times X$. Writing $\theta = v \times 1 + 1 \times w$ and observing that $\theta|_{\Delta X} = 0$ (since there is a continuous section of (1) over the diagonal $\Delta X \subset X \times X$) shows that $v + w = 0$ and therefore $\theta = v \times 1 - 1 \times v = \bar{v}$. Hence we have found a cohomology class $v \in H^s(X; B)$ with $\bar{v}^r \neq 0$.

(c) In one direction the statement of (c) follows from the upper bound (a) and [6], Thm. 7, i.e. the existence of classes $v_1, \ldots, v_r \in H^s(X; \Bbbk)$ with $\bar{v}_1 \cup \cdots \cup \bar{v}_r \neq 0$ combined with (a) gives $\text{TC}(X) = r + 1$. Suppose now that $H_*(X)$ is square free. Write $B = H_*(X)$ as a direct sum
\[
B = \oplus_{i \in I} B_i
\]
where each $B_i$ is either $\Bbbk$ or a cyclic group of prime order $\Bbbk_p$ and $I$ is an index set. The $r$-fold tensor power $B^r = B \otimes \cdots \otimes B$ is a direct sum
\[
B^r = \bigoplus_{(i_1, \ldots, i_r) \in I^r} B_{i_1} \otimes B_{i_2} \otimes \cdots \otimes B_{i_r}
\]
One knows that the join a $p$-connected complex and a $q$-connected complex is $(p + q + 2)$-connected.
and each tensor product $B_{i_1} \otimes B_{i_2} \otimes \cdots \otimes B_{i_r}$ is either $\mathbb{Z}$, $\mathbb{Z}_p$ or trivial. As we know from the proof of (b) there is a class $v \in H^s(X; B)$ such that $\bar{v}^r \neq 0 \in H^{2n}(X \times X; B^r)$. For any index $i \in I$ denote by $v_i$ the image of $v$ under the coefficient projection $B \to B_i$. Since $\bar{v}^r \neq 0$ there exists a sequence $(i_1, \ldots, i_r) \in I^r$ such that the product

$$\bar{v}_{i_1} \cup \cdots \cup \bar{v}_{i_r} \in H^{2n}(X \times X; B_{i_1} \otimes B_{i_2} \otimes \cdots \otimes B_{i_r}).$$

is nonzero. If the product $B_{i_1} \otimes B_{i_2} \otimes \cdots \otimes B_{i_r}$ is $\mathbb{Z}_p$ then each $B_{i_j}$ is either $\mathbb{Z}$ or $\mathbb{Z}_p$ and taking $k = \mathbb{Z}_p$ and reducing all these classes $v_{i_k}$ mod $p$ we obtain that (c) is satisfied. In the case when the product $B_{i_1} \otimes B_{i_2} \otimes \cdots \otimes B_{i_r}$ is infinite cyclic each of the groups $B_{i_k}$ is $\mathbb{Z}$ and the class

$$(7) \quad \bar{v}_{i_1} \cup \cdots \cup \bar{v}_{i_r} \neq 0 \in H^{2n}(X \times X; \mathbb{Z})$$

is integral and nonzero.

Since the group $H^{2n}(X \times X; \mathbb{Z})$ is square-free the cup-product (7) is indivisible by some prime $p$. Indeed, the group $H^{2n}(X \times X; \mathbb{Z})$ is direct sum of cyclic groups of prime order and infinite cyclic groups and the product (7) has a nontrivial component in at least one of these groups. A nonzero element of $\mathbb{Z}$ is divisible by finitely many primes and a nonzero element of $\mathbb{Z}_p$ is divisible by all primes except $p$.

Therefore, as follows from the exact sequence

$$\cdots \to H^{2n}(X \times X; \mathbb{Z}) \xrightarrow{p} H^{2n}(X \times X; \mathbb{Z}) \to H^{2n}(X \times X; \mathbb{Z}_p) \to \cdots,$$

the mod $p$ reduction of the product (7) is nonzero. Now, taking $k = \mathbb{Z}_p$ and reducing the classes $v_{i_k}$ mod $p$ gives a sequence of classes $\bar{w}_{j_k} \in H^s(X; k)$ such that $\prod \bar{w}_{j_k} \neq 0$ where $k = 1, \ldots, r$.

(d) The proof of statement (d) of Theorem 3 is similar to that of (c), with the simplification that all the groups $B_i$ are in this case infinite cyclic.

4. Proof of Theorem 1

The cases $m = 2$ and $m \geq 3$ odd of Theorem 1 were dealt with by Farber and Yuzvinsky in [8]. Their arguments also show that if $m \geq 4$ is even, then $\text{TC}(F(\mathbb{R}^m, n))$ equals either $2n - 1$ or $2n - 2$. Hence to prove Theorem 1 it suffices to show that $\text{TC}(F(\mathbb{R}^m, n)) \neq 2n - 1$ when $m \geq 4$ is even.

Fix $n \geq 2$. For any $m \geq 2$ the space $F(\mathbb{R}^m, n)$ is $(m - 2)$-connected, since it is the complement of an arrangement of codimension $m$ subspaces of $\mathbb{R}^{mn}$. Its integral cohomology ring is shown in [5] to be graded-commutative algebra over $\mathbb{Z}$ on generators $e_{ij} \in H^{m-1}(F(\mathbb{R}^m, n))$, $1 \leq i < j \leq n$,
subject to the relations
\[ e_{ij}^2 = 0, \quad e_{ij}e_{ik} = (e_{ij} - e_{ik})e_{jk} \]
for any triple \(1 \leq i < j < k \leq n\). In particular, \(H^*(F(\mathbb{R}^m, n))\) is nonzero only in dimensions \(i(m-1)\) where \(i = 0, 1, \ldots, (n-1)\). Applying the result of Eilenberg and Ganea [4] we obtain that for \(m \geq 3\) the space \(F(\mathbb{R}^m, n)\) is homotopy equivalent to a finite complex of dimension \(\leq (m-1)(n-1)\). Now we may apply statement (d) of Theorem 3 which gives, firstly, that \(\text{TC}(F(\mathbb{R}^m, n)) \leq 2n - 1\) and, secondly, \(\text{TC}(F(\mathbb{R}^m, n)) = 2n - 1\) if and only if there exist cohomology classes
\[ \bar{v}_1, \ldots, \bar{v}_{2(n-1)} \in H^{m-1}(F(\mathbb{R}^m, n)) \]
such that the product of the corresponding zero-divisors
\[ \bar{v}_1 \cup \bar{v}_2 \cup \cdots \cup \bar{v}_{2(n-1)} \]
is nonzero; recall that the notation \(\bar{v}\) is introduced in (3). We show below that such classes \(\bar{v}_1, \ldots, \bar{v}_{2(n-1)}\) do not exist if \(m \geq 4\) is even.

We recall the result of [3] stating that \(\text{TC}(F(\mathbb{C}, n)) = 2n - 2\). It is shown in the proof of Theorem 6 in [9], that \(F(\mathbb{C}, n)\) is homotopy equivalent to the product \(X \times S^1\) where \(X\) is a finite polyhedron of dimension \(\leq n - 2\). This argument uses the algebraic structure of \(\mathbb{C} = \mathbb{R}^2\) and does not generalize to \(F(\mathbb{R}^m, n)\) with \(m > 2\). Using the product inequality (Theorem 11 in [6]) one obtains
\[ \text{TC}(F(\mathbb{C}, n)) \leq \text{TC}(X) + \text{TC}(S^1) - 1 \leq (2n - 2 + 1) + 2 - 1 = 2n - 2. \]
Hence there exist no \(2(n-1)\) cohomology classes \(v_1, \ldots, v_{2(n-1)} \in H^1(F(\mathbb{C}, n))\) such that the product of the zero-divisors \(\bar{v}_1 \cup \cdots \cup \bar{v}_{2(n-1)}\) is nonzero, as this would contradict Theorem 7 from [6].

Now we observe that for any even \(m \geq 2\) there is an algebra isomorphism
\[ (8) \quad \phi : H^*(F(\mathbb{C}; n)) \to H^{*(m-1)}(F(\mathbb{R}^m, n)) \]
mapping classes of degree \(i\) to classes of degree \((m-1)i\) where \(i = 0, 1, \ldots, n-1\), see [5]. Thus we conclude that there exist no cohomology classes \(w_1, \ldots, w_{2(n-1)} \in H^{m-1}(F(\mathbb{R}^m, n))\) such that the product of the corresponding zero-divisors \(\bar{w}_1 \cup \cdots \cup \bar{w}_{2(n-1)}\) is nonzero. Theorem 3 (statement (d)) gives now that \(\text{TC}(F(\mathbb{R}^m, n)) \leq 2(n-1)\).

On the other hand, it is proven in [8] that one may find \(2n - 3\) cohomology classes \(v_1, \ldots, v_{2n-3} \in H^1(F(\mathbb{C}, n))\) such that the cup-product \(\bar{v}_1 \cup \cdots \cup \bar{v}_{2n-3}\) is nonzero. Hence, repeating the above argument we see that for \(m\) even there exists classes \(w_1, \ldots, w_{2n-3} \in H^{m-1}(F(\mathbb{R}^m, n))\)
(where \(w_i = \phi(v_i)\)) with nonzero product \(\bar{w}_1 \cup \cdots \cup \bar{w}_{2n-3}\); this gives the opposite inequality \(\text{TC}(F(\mathbb{R}^m, n)) \geq 2n - 2\).

Hence, \(\text{TC}(F(\mathbb{R}^m, n)) = 2n - 2\) as stated. \(\square\)

**REFERENCES**

[1] D. C. Cohen and G. Pruidze, *Motion planning in tori*, to appear in Bull. London Math. Soc. [arXiv:math/0703069]

[2] D. C. Cohen and G. Pruidze, *Topological complexity of basis-conjugating automorphism groups*, preprint. [arXiv:0804.1825]

[3] A. Costa and M. Farber, *Motion planning in spaces with "small" fundamental groups*, Preprint 2008.

[4] S. Eilenberg, T. Ganea, *On the Lusternik - Schnirelmann category of abstract groups*. (65)(1957), 517 - 518.

[5] E. R. Fadell, S. Y. Husseini, *Geometry and Topology of Configuration Spaces*. Springer Monographs in Mathematics, Springer - Verlag, 2001.

[6] M. Farber, *Topological complexity of motion planning*, Discrete Comput. Geom. 29 (2003), 211–221.

[7] M. Farber, *Instabilities of robot motion*, Topology Appl. 140 (2004), 245–266.

[8] M. Farber, S. Yuzvinsky, *Topological robotics: subspace arrangements and collision free motion planning*, in “Geometry, topology, and mathematical physics”, Amer. Math. Soc. Transl. (2004), 145–156.

[9] M. Farber, *Topology of robot motion planning*, in: Morse Theoretic Methods in Nonlinear Analysis and in Symplectic Topology (P. Biran et al (eds.)) (2006), 185–230.

[10] M. Farber and M. Grant, *Robot motion planning, weights of cohomology classes, and cohomology operations*, Proc. Amer. Math. Soc. 136 (2008), 3339-3349.

[11] M. Farber, *Invitation to topological robotics*, EMS, to appear.

[12] J. González, *Topological robotics in lens spaces*, Math. Proc. Cambridge Philos. Soc. 139 (2005), no. 3, 469–485.

[13] M. Grant, *Topological complexity of motion planning and Massey products*, to appear in Proceedings of the M. M. Postnikov Memorial Conference 2007, Banach Centre Publications [arXiv:0709.2287].

[14] J.-C. Latombe, *Robot Motion Planning*, Kluwer, Dordrecht, 1991.

[15] A. S. Schwarz, *The genus of a fiber space*, Amer. Math. Soc. Transl.(2) 55 (1966), 49–140.

[16] G.W. Whitehead, *Elements of homotopy theory*, Springer - Verlag, 1978.

DEPARTMENT OF MATHEMATICAL SCIENCES, DURHAM UNIVERSITY, SOUTH ROAD, DURHAM, DH1 3LE, UK

E-mail address: michael.farber@durham.ac.uk

E-mail address: mark.grant@durham.ac.uk