Natural convection in a linearly heated vertical porous annulus

M. Sankar, S. Kiran and S. Sivasankaran

1Department of Mathematics, School of Engineering, Presidency University, Bangalore, India
2Department of Mathematics, Sapthagiri College of Engineering, Bangalore, India
3Department of Mathematics, King Abdulaziz University, Jeddah, Saudi Arabia

E-mail: sd.siva@yahoo.com; sdsiva@gmail.com

Abstract. The study reports influence of linear thermal conditions on buoyancy driven convection in an upright porous annular space, in which inner cylinder is linearly heated while the outer cylinder is maintained at uniform or non-uniform thermal condition. However, top boundary is insulated and bottom boundary is uniformly cooled. Using the finite difference based ADI scheme, the model equations are numerically solved to predict the flow and thermal patterns as well as the corresponding thermal transport rates from the heated boundaries. Nonuniform heating of inner and outer cylinders induces multicellular flow and hence increases the thermal transport. Among the heated boundaries, thermal transport from the bottom wall is more for case (II). In general, the heat dissipation from the boundaries are higher for case (II) as compared to case (I), and the presence of porosity reduces heat transport rate.

1. Introduction

Buoyancy-driven convection in an annular region formed from two vertical co-axial cylindrical tubes with non-uniform thermal boundary conditions is an important physical configuration portraying many practical applications. Unfortunately, the relevant studies on analogous geometry have constrained their investigation to the situation where the vertical bounding walls of the geometry are heated with uniform thermal condition. However, buoyant convection caused by the linear heating is more practical in the applications pertaining to storage of nuclear wastes and electronic equipment cooling.

A significant range of experimental and theoretical works are administered in the past decades in an endeavor to comprehend thermal transport in an enclosure. A complete and thorough review of natural convection in annular geometries is investigated, among them the problems which has many practical applications and has been widely studied. de Vahl Davis and Thomas [1] developed a heat transfer correlation on different flow regimes to measure the global heat transport rate and showed that the global Nusselt number is strongly dependent on the Rayleigh number and the geometry, but completely free from the Prandtl number. Kumar and Kalam [2] developed a new correlation and analyzed the inconsistencies in existing investigations for the thermal transport rates with the variance of aspect ratio and the Rayleigh number. Sankar et al. [3] investigated the importance of discontinuous heating on convection heat transfer in a cylindrical annular space. Further investigations are made by Lopez et al. [4] to find the location of discrete heaters and it was found that the thermal transport rate is greater at the lower source compared to the upper source.
In the upright porous annular space, convective thermal transport has been examined broadly, inferable from its significance in porous heat exchangers, insulation processes for building and numerous different applications. For the porous annulus, Prasad and Kulacki [5] numerically analyzed the curvature effect on thermal and velocity fields for which the vertical walls are maintained at constant temperature, with the horizontal walls being adiabatic. Hasmaoui et al. [6] considered Darcy-Boussinesq equations to solve analytically the flow and heat transfer for long aspect ratio ($A > 1$) annulus assuming parallel flow and analytical solutions are obtained and validated the results by numerically studying the same phenomenon. Shivakumara et al. [7] performed theoretical investigation of buoyant thermal transport in a porous annular space with Brinkman extended-Darcy (BED) model. Using the BED model, Sankar et al. [8] studied relative significance of partial thermal condition on buoyant thermal transport in the porous annulus. The work was further carried out for the same physical configuration by Sankar et al. [9] to understand the combined impacts of isolated source and internally generated heat on the convective flow and thermal transport rates.

Many scientific applications, namely nuclear waste storages, crystal growth manufacturing apparatus and cooling of electrical equipment are appropriately represented by the straight annulus geometry whose vertical walls are differentially heated, and insulated upper and lower boundaries. In a square enclosure, the impact of linear thermal condition on the buoyant convection is investigated by Sathiyamoorthy et al. [10] by considering various Prandtl numbers and determined the multicellular flow movements when the vertical boundaries are heated linearly. Numerical simulation of buoyant convection in a trapezoidal geometry having linearly varying thermal conditions is performed by Natarajan et al. [11]. Further, in a linearly heated square enclosure, Sathiyamoorthy and Chamkha [12] analyzed the magnetoconvective flows and associated thermal transport rates. In a porous square enclosure, Sivasankaran and Bhuvaneswari [13] performed buoyant convection from the sinusoidally changing temperature along vertical walls and concluded that the thermal transport rates are higher than a single sinusoidal temperature profile.

From the detailed literature review made in this work, it is found that the major efforts are paid towards exploring the influence of various non-uniform heating on buoyant thermal transport in rectangular or trapezoidal geometries. However, buoyant thermal transport in the porous annular space with different thermal profiles at vertical boundaries has not been examined in the current literature. Though this geometry has more practical relevance, it has not received enough attention and has not been thoroughly studied in the literature. Hence, keeping the important applications of this geometry in mind, it is proposed to investigate the impacts of irregular thermal profiles on buoyant convection in a porous annular space between two vertical, co-axial cylindrical tubes.

2. Mathematical Formulation

Figure 1 displays schematically the physical geometry of the chosen study. This consists of an upright annular space of elevation $H$, breadth $D$ and contains the fluid-saturated porous medium. The annular enclosure is constructed by upright, co-axial cylindrical tubes having internal radius $r_i$, and external radius $r_o$. In this analysis, two cases of thermal conditions are investigated by considering different temperature conditions at outer wall. For both cases (I and II), the inner cylinder is linearly heated in axial direction, bottom boundary is kept at higher temperature and upper surface is kept adiabatic. However, the outer cylinder too linearly heated for case (I), but the outer surface is uniformly cooled for case (II). In addition, the following assumptions are made to obtain the basic equations:

- The flow is taken as axisymmetric, laminar and constant thermo-physical conditions are assumed, apart from the Boussinesq approximation.
- In the porous medium, the Local Thermal Equilibrium (LTE) assumption is applied.
Figure 1. Physical configuration and coordinate system

Under these above assumptions the basic equations of motion are:

\[ \frac{\partial T}{\partial t} + U \frac{\partial T}{\partial R} + W \frac{\partial T}{\partial Z} = \nabla^2 T \]  
(1)

\[ \frac{\partial \zeta}{\partial t} + U \frac{\partial \zeta}{\partial R} + W \frac{\partial \zeta}{\partial Z} - \frac{U \zeta}{R} = Pr \left[ \nabla^2 \zeta - \frac{\zeta}{R^2} \right] - \Gamma \frac{Pr}{Da} \zeta - PrRa \frac{\partial T}{\partial R} \]  
(2)

\[ \zeta = \frac{1}{R} \left[ \frac{\partial^2 \psi}{\partial R^2} - \frac{1}{R} \frac{\partial \psi}{\partial R} + \frac{\partial^2 \psi}{\partial Z^2} \right] \]  
(3)

\[ U = -\frac{1}{R} \frac{\partial \psi}{\partial Z}, \quad W = \frac{1}{R} \frac{\partial \psi}{\partial R} \]  
(4)

In Eq. (2), the constant \( \Gamma \) can assume the values 0 or 1. For \( \Gamma = 0 \), the results for non-porous case and for \( \Gamma = 1 \), the results for porous case can be simulated. To obtain above equations, the following transformations are utilized:

\[ (R, Z) = \left( \frac{r, z}{D} \right), \quad (U, W) = \left( \frac{u, w}{\alpha/D} \right), \quad T = \frac{\theta - \theta_c}{\theta_h - \theta_c}, \quad t = \frac{t^*}{D^2/\alpha}, \quad P = \frac{P}{\rho_0 \alpha^2/D^2}, \quad \zeta = \frac{\zeta^*}{\alpha/D^2}, \quad \psi = \frac{\psi^*}{D \alpha}. \]

The dimensionless boundary conditions are

\[ t = 0: \quad U = W = T = 0, \quad \psi = \zeta = 0; \quad \text{at} \ 0 \leq Z \leq A \text{ and } \frac{1}{\lambda - 1} \leq R \leq \frac{\lambda}{\lambda - 1}. \]

\[ t > 0: \quad \psi = \frac{\partial \psi}{\partial R} = 0, \quad T = 1 - Z; \quad \text{at} \ 0 \leq Z \leq A \text{ and } R = \frac{1}{\lambda - 1}. \]

\[ \psi = \frac{\partial \psi}{\partial R} = 0, \quad \begin{cases} T = 1 - Z & \text{for case (I)} \\ T = 0 & \text{for case (II)} \end{cases} \quad \text{at} \ 0 \leq Z \leq A \text{ and } R = \frac{\lambda}{\lambda - 1}. \]

\[ \psi = \frac{\partial \psi}{\partial Z} = 0, \quad T = 1 \text{ at } Z = 0. \]
\[ \psi = \frac{\partial \psi}{\partial Z} = 0, \frac{\partial T}{\partial Z} = 0 \text{ at } Z = A. \]

In the above conditions, \( A = \frac{H}{D} \) is the aspect ratio and \( \lambda = \frac{\omega}{\alpha} \) is the radius ratio. The flow depends on two non-dimensional parameters, namely the Rayleigh (\( Ra \)) and Prandtl (\( Pr \)) numbers. They are \( Ra = \frac{g\beta\Delta \theta D^3}{\nu \alpha} \) and \( Pr = \frac{\nu}{\alpha} \). The local thermal transport rate is calculated through the local Nusselt number described by \( Nu = -\frac{\partial T}{\partial n} \), where \( n \) is the perpendicular direction to boundary. However, the overall thermal transport rate is measured from the average Nusselt numbers at hot wall (\( Nu_H \)), cold wall (\( Nu_C \)) and bottom wall (\( Nu_B \)) and are given by

\[ Nu_H = \frac{1}{A} \int_0^A Nu_H dZ; \quad Nu_C = \frac{1}{A} \int_0^A Nu_C dZ \quad \text{and} \quad Nu_B = \frac{1}{A} \int_0^A Nu_B dZ \quad (5) \]

3. Solution Methodology

The dimensionless partial differential equations representing the vorticity transport and energy principles are solved using the ADI method, based on finite difference techniques. On contrast, the block relaxation technique known as the SLOR method is applied to the stream function equation after several trial runs for optimum value of relaxation parameter. The algebraic equations obtained in tri-diagonal structure are solved from the Thomas algorithm. The values boundary vorticity are calculated from Taylor series expansion of stream function values. Using second-order central difference expressions, the stream function-velocity relations are evaluated. Using Simpson’s rule, the average transport rates are determined. Along the radial and axial directions, uniform grids are chosen and all the simulations have carefully undergone the grid independency tests and the grid sizes of 126 \( \times \) 126 has been chosen. An in-house code is written for the problem investigated in this work, validated with standard benchmark results in the literature and is shown in Fig. 2.

![Fig 2](image_url)

*Figure 2. Comparison of present streamlines and isotherms with Sathiyamoorthy et al. [10] for the case of linear heating on both the walls for \( Pr = 0.7 \) and \( Ra = 10^5 \).*

4. Results and Discussion

Figures 3 and 4 illustrates the flow and thermal contours for the case of linear heating by fixing the values of \( Pr = 0.7, A = 1 \) and \( \lambda = 2 \). \( Da = 10^{-2} \) as representative cases. The impact
of Darcy number on the fluid flow pattern and temperature distribution is presented in Fig 3 for case (I). For a non-porous case and higher Darcy number \((Da = \infty \text{ and } Da = 10^{-1})\), Fig. 3(a, b) shows a three cellular structure with two small eddies at the diagonal corners of top right and bottom left of the annulus and a single elongated elliptical cellular structure is found along the lower left to right top corner. Two small eddies are expected due to the linear heating of walls. The isotherm plots of the two cases show similar heat transfer patterns. As the value of Darcy number is lowered to \(Da = 10^{-5}\), the three cellular structure disappears and two symmetric counter clock wise rotating cells are formed (see Fig. 3c). Also, it can be observed that a symmetric form of parallel lines are formed indicating a conduction type of heat transfer as set in the annulus. In the case of linearly heated inner wall and cooled outer wall (case(II)), the effect of linear heating has no significant impact on the streamlines and isotherms. Figure 4 (a and b) reveals a main single cellular structure is observed with its core at the center of the cavity and a small cell at the left top corner of the annuli for the higher values of \(Da(\infty \text{ and } 10^{-1})\). For the higher value of \(Da\), the flow strength is reduced considerably and the main flow is towards the lower right diagonal. In general, for case(I), the streamlines indicate a tri-cellular flow structure with hot fluid eddies are near the top left and bottom right corners, whereas the cold fluid is along the diagonal. A similar result simulated for case (II) reveals that the major portion of the annulus is occupied by main flow with a small eddy near top left corner. The isotherms show significant variations with the value of \(Da\).

Figure 3. (Case I) Effect of Darcy number on streamlines and isotherms for \(Ra = 10^5\). (Left) \(Da = \infty\), (Middle) \(Da = 10^{-1}\) and (Right) \(Da = 10^{-5}\)

This part of discussion deals with the effect of Darcy number and the Rayleigh number on the global Nusselt numbers on hot, cold and bottom boundaries. Figure 4 depicts the effect of \(Da\) on the average \(Nu\) for increasing Rayleigh numbers for the hot, cold and bottom wall. It is observed that, for \(Da = 10^{-5}\) and \(Da = 10^{-4}\), the thermal transport rate remains constant for all Rayleigh numbers. Hence, for low Darcy number, there is negligible heat transfer and for \(Da = 10^{-3}\) it can be seen that average \(Nu\) remains constant till \(Ra = 10^5\) and further increase of Rayleigh number, increases thermal transport in the porous medium and reaches a maximum value. For higher values of \(Da\), an increasing trend of average \(Nu\) is observed with increasing the Rayleigh number. A similar observation is found on the cold wall, as shown in Fig. 3c, where an increasing trend of average \(Nu\) is observed for increasing \(Da\) and \(Ra\) except
Figure 4. (Case II) Effect of Darcy number on streamlines and isotherms for $Ra = 10^5$. (Left) $Da = \infty$, (Middle) $Da = 10^{-1}$ and (Right) $Da = 10^{-5}$ for the lower values of $Da = 10^{-5}$ and $Da = 10^{-4}$. For bottom wall, the thermal transport increases with $Ra$ and higher thermal transport rate is achieved for lower values of $Da$. In Fig. 5, for case (II), the effect of $Ra$ and $Da$ on the global $Nu$ is illustrated for hot (a), cold (b) and bottom (c) boundaries. It is detected that the variation of global Nusselt number due to linear heating of inner hot wall follows a smooth increase with an increase in Rayleigh number. In contrast, the average $Nu$ for outer wall does not show any significant increase as expected, which is maintained at cold temperature shown in Fig. 5(b). The overall Nusselt number along the bottom surface, as shown in Fig. 5(c), smoothly increases with an increase in $Ra$ except for $Da = 10^{-5}$ and $Da = 10^{-4}$ and remains constant for increasing Rayleigh number.

5. Conclusions
In this paper, the impacts of non-uniform thermal conditions imposed at the inner, outer and lower boundaries on the flow and thermal distributions, the associated thermal transport is investigated in detail. Two cases of nonuniform thermal conditions are formed by considering different temperature conditions at outer wall. For both case (I and II), the inner cylinder is linearly heated in axial direction, bottom boundary is kept at higher temperature and upper surface is kept adiabatic. However, in case (I), the outer cylinder is also linearly heated and for case (II) the outer wall is uniformly cooled. Through the numerical simulations, it has been observed that non-uniform thermal profiles initiate the multi-cellular flow structure. Also, the thermal transport is found to be better for non-uniform heating rather than uniform heating. Further, the presence of porosity causes the reduction in flow movement and hence suppressing the heat transport rates.

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Figure 5. Effect of Rayleigh and Darcy numbers on the average Nusselt number for Case (I) (a) Hot wall (b) Cold wall and (c) Bottom wall

Figure 6. Effect of Rayleigh and Darcy numbers on the average Nusselt number for Case (II) (a) Hot wall (b) Cold wall and (c) Bottom wall

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