On Alternating +Achiral Knots

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ABSTRACT

We summarize the results obtained in this paper and in \cite{5} as follows.

1) Let $K$ be an alternating $-\text{achiral}$ knot. Then the order of $-\text{achirality}$ is equal to 2 and the symmetry is visible on some minimal projection (such a projection is called an achiral minimal projection). This is Tait’s Conjecture, now proved in \cite{5}.

2) If $K$ is an alternating $+\text{achiral}$ knot, the order of $+\text{achirality}$ is equal to $2^\lambda$, for some integer $\lambda \geq 1$. If $K$ is moreover arborescent then the order is equal to 4.

3) For each integer $\lambda \geq 1$, there exists an alternating $+\text{achiral}$ knot which is not $-\text{achiral}$ and such that:
   (i) its order is equal to $2^\lambda$;
   (ii) there exists an achiral minimal projection.

4) There exist $+\text{achiral}$ alternating knots $K$ of order 4 without minimal achiral projection.

5) Let $K$ be a prime, alternating, arborescent knot. Suppose that $K$ is $+\text{achiral}$ (in short $K$ is a $+\text{AAA}$ knot). Then there exists a projection $\Pi_K \subset S^2$ of $K$ (non necessarily minimal nor alternating) and a diffeomorphism $\Phi : S^2 \rightarrow S^2$ of order 4 such that:
   1. $\Phi$ preserves the orientation of $S^2$;
   2. $\Phi(\Pi_K) = \hat{\Pi}_K$.
   3. $\Phi$ preserves the orientation of the projection.

1. Introduction

In this paper, knots and projections are assumed to be prime.

This paper is devoted essentially to the study of alternating, arborescent knots which are $+\text{achiral}$. We prove that the symmetry is visible on a projection (such a projection is called an achiral projection or, as in \cite{6}, a symmetry presentation) which is not always minimal. It is realized by a diffeomorphism of the projection of order 4. The square of this diffeomorphism is a periodic symmetry with an axis of order 2. A known particular case consists of rational knots. A precise statement is the following theorem:

Theorem 2.1. (Main Theorem)

Let $K \subset S^3$ be a prime, alternating, arborescent knot. Suppose that $K$ is $+\text{achiral}$ (in short $K$ is a $+\text{AAA}$ knot). Then there exists a projection $\Pi_K \subset S^2$ of $K$ (non necessarily minimal nor alternating) and a diffeomorphism $\Phi : S^2 \rightarrow S^2$ of order 4 such that:
1. $\Phi$ preserves the orientation of $S^2$;
2. $\Phi(\Pi_K) = \hat{\Pi}_K$.
3. $\Phi$ preserves the orientation of the projection.
Definition 1.1. Let \( \Pi \) be an oriented knot projection in \( S^2 \). We denote by \( \hat{\Pi} \) the mirror image of \( \Pi \) with respect to the sphere \( S^2 \) containing the projection. We denote by \( +\Pi \) (resp. \( -\Pi \)) the projection obtained by preserving (resp. reversing) the orientation of \( \Pi \).

The proof of the Main Theorem is given in Sections 4, 5 with the principal part in Section 6.

If we leave the area of arborescent knots, there exist alternating \( + \)-achiral knots with visible symmetry of any order equal to \( 2^k \). There exist such knots which are not \( - \)-achiral. We present a short proof of this presumably known fact in Section 8, Theorem 8.4. We also give in Theorem 8.5 an example of an alternating, non-arborescent knot which is \( + \)-achiral, of order 4, not \( - \)-achiral, with no minimal achiral projection.

We prove in Section 3 the following theorem which is related to a conjecture of Kauffman and Jablan [10].

**Theorem 3.1**

Let \( K \) be an alternating \( + \)-achiral knot with no minimal achiral projection. Then the order of \( + \)-achirality is equal to 4.

We recall that a flype is a modification of a projection as represented in Fig.1.

![Fig. 1. A flype](image)

The next theorem is a consequence of the Menasco-Thistlethwaite Flyping theorem, see [12]. It is essential for the study of symmetries of alternating knots and largely we rely on it.

**Theorem 1.2.** (Key Theorem) Let \( K \) be a prime alternating oriented knot. Let \( \Pi \) be an oriented minimal projection of \( K \). Then \( K \) is \( \pm \)-achiral if and only if one can transform \( \Pi \) into \( \pm \hat{\Pi} \) by a finite sequence of flypes and orientation preserving diffeomorphisms of \( S^2 \).
Our approach makes use of the technique we have already used in the proof of Tait’s Conjecture for achiral knots in [5]. The starting point is the Bonahon-Siebenmann canonical decomposition of a knot projection (assumed to be connected and prime) into twisted band diagrams and jewels. See [1] and [14].

The Appendix and Sections 4 and 5 are reminders of the results proved in [5]. The salient points of the Bonahon-Siebenmann decomposition are recalled in the Appendix.

In Section 4, we study the automorphism of the structure tree induced by the +achirality of the knot. In Section 5, we state a structure theorem stated as Theorem 5.2 for knots which are alternating, arborescent and +achiral.

**Theorem 5.2.**
Let $K$ be a +AAA knot and let $\Pi_K$ be a minimal alternating projection of $K$. Then the symmetry induces an automorphism of $\Pi_K \subset S^2$ which has an invariant Haseman circle $\gamma$. The two tangles $F$ and $F^*$ determined by $\gamma$ appear as in Fig.5 and $F \sim F^*$. Then there exists an adequate projection $\Pi_K$ where the symmetry is realized by a rotation of angle $\pi/2$ about an axis which intersects both tangles in their center, followed by a reflection through the Haseman circle $\gamma$.

Our method to obtain a projection with visible +achirality is illustrated in Section 7 by examples due to Dasbach-Hougardy and Stoimenow, for which no achiral projection was known.

Claude Weber wishes to thank Cameron Gordon and Steve Boyer for raising the following question during the meeting in honor of Michel Boileau’s 60th birthday in Toulouse (June 2013):
**Question:** Suppose that a minimal projection of an alternating knot has no flypes. Are the symmetries of the knot visible? And, if yes, how does one “see” them?

From the Key Theorem, we deduce that the + and − achirality are visible on a minimal projection if the projection has no flypes. The case of −achirality is yet treated in [5]. For +achirality, the method presented below will show how one can see the symmetry if there exists an invariant Haseman circle. Otherwise there exists an invariant jewel. From the classification of diffeomorphisms of finite order of the 2-sphere, this jewel must be invariant by a rotation-reflexion.

**Remark 1.3** Using the definitions given below, it is easy to see that a minimal alternating projection has no flypes if and only if the weight at each twisted band diagram is equal to zero. Note that this implies that each rational tangle is reduced to a spire.

2. Comments on the Main Theorem

We quote again the main theorem:
Theorem 2.1.

Let $K \subset S^3$ be a prime, alternating, arborescent knot. Suppose that $K$ is +achiral (in short $K$ is a $+\text{AAA}$ knot). Then there exists a projection $\Pi_K \subset S^2$ of $K$ (non necessarily minimal nor alternating) and a diffeomorphism $\Phi: S^2 \to S^2$ of order 4 such that:

1. $\Phi$ preserves the orientation of $S^2$;
2. $\Phi(\Pi_K) = \hat{\Pi}_K$.
3. $\Phi$ preserves the orientation of the projection.

Remark 2.2.

(1) Our constructive proof is done by induction on the number of essential Conway circles as defined in Definition 6.2. We propose an explicit procedure to obtain from a minimal alternating projection of $K$ a new projection $\Pi_K$ which satisfies the conditions stated in the Main Theorem. The projection $\Pi_K$ is by no means unique.

Hence one has a procedure for finding a symmetry presentation for the class of $+\text{AAA}$ knots, hence giving a partial positive answer to a question of E.Flapan ([6], p.23).

(2) The Main Theorem is somewhat a generalization of the well-known achiral projection of the figure eight knot. The $+\text{achirality}$ is realized by a rotation of angle $\pi/2$ with an axis perpendicular to the projection plane, intersecting this plane in the center of the picture, followed by a reflexion with respect to a 2-sphere intersecting the plane in a circle which meanders among the double points of the projection.

Fig. 2. An achiral projection of the figure eight knot

3. A discussion of diffeomorphisms of the 2-sphere, flypes, the Key Theorem and the Kauffman-Jablan Conjecture

Let us begin by quoting the Kauffman-Jablan Conjecture. See [10]. Since by [5], every prime alternating knot which is $-\text{achiral}$ has a minimal achiral projection, one can restate the Kauffman-Jablan conjecture under the following form:

Kauffman-Jablan Conjecture

Let $K$ be a prime alternating knot which is $+\text{achiral}$ but not $-\text{achiral}$. If $K$ has no
minimal achiral projection then $K$ is arborescent.

The knot $K_2$ represented in Fig.25 is a counter-example to the conjecture. In Section 7 we announced that there exist knots which are counter-examples to the Conjecture for every order $2^\lambda$ and $\lambda \geq 2$. In fact the knots which are counter-examples to the Kauffman-Jablan conjecture exist only for the order 4. The following theorem indicates that the Kauffman-Jablan conjecture is almost true.

**Theorem 3.1.** Let $K$ be an alternating +achiral knot without minimal achiral projection. Then the order of +achirality is equal to 4.

To prove Theorem 3.1, let us go back to the Key Theorem.

### 3.1. Return to the Key Theorem

Let $K$ be an alternating oriented knot and let $\Pi$ be a minimal projection of $K$. The Key Theorem tells us that, if $K$ is ±achiral, we can proceed from $\Pi$ to $\pm\Pi$ by a sequence of flypes and diffeomorphisms. The approach we follow, via the automorphism of the structure tree, shows that only one diffeomorphism is sufficient, accompanied generally by several flypes. We choose to focus on the diffeomorphism which is the heart of the symmetry. Flypes are present to produce some local adjustments; however they are in general necessary.

### 3.2. The action of a diffeomorphism on a decomposition of the 2-sphere

Suppose that we have a finite decomposition $R = \{R_i\}$ of the 2-sphere $S^2$ in connected planar surfaces, with $R_i \cap R_j$ for $i \neq j$ either empty or a common boundary component. Suppose moreover that we have a diffeomorphism $g : S^2 \to S^2$ of finite order $n$ which respects the decomposition: for every index $i$, there exists an index $k(i)$ such that $g(R_i) = R_{k(i)}$.

Consider some $R_i$, and the images $g(R_i), g^2(R_i), \ldots, g^n(R_i) = R_i$. Two cases may happen.

1. $g(R_i), g^2(R_i), \ldots, g^n(R_i) = R_i$ are all distinct; we say that the orbit of $R_i$ is **generic**.
2. There exists an integer $m$ with $1 \leq m < n$ such that $g^m(R_i) = R_i$; we say that the orbit of $R_i$ is **short**.

Now, for a generic orbit the restriction $g^n|_{R_i} : R_i \to R_i$ is the identity. However for a short orbit the restriction $g^m|_{R_i} : R_i \to R_i$ is not the identity but a non-trivial automorphism of $R_i$. As we shall see the difficulties to find a projection where a symmetry is visible come from short orbits.
3.3. The action of a diffeomorphism of $S^2$ on the canonical decomposition

We apply the above description to a diffeomorphism $\psi$ responsible of a $\pm$achirality symmetry of an alternating knot. Let $\Sigma_i$ be a planar surface of the canonical decomposition and let $\Gamma_i = \Pi \cap \Sigma_i$. Hence $(\Sigma_i, \Gamma_i)$ is a diagram of the decomposition. Let $\Sigma_j = \psi(\Sigma_i)$. Then $\psi(\Gamma_i) \subset \Sigma_j$ is flype equivalent in $\Sigma_j$ to $\Gamma_j$. If $i \neq j$ we transform $\Gamma_j$ by flypes to $\psi(\Gamma_i)$.

We pursue these adjustments by flypes to $\psi^2(\Gamma_i)$, $\ldots$, $\psi^l(\Gamma_i)$ as long as these adjustments take place in different diagrams. Then two cases can happen.

1. The orbit of $\Sigma_i$ under $\psi$ is generic. Then we put an end to the adjustments when $l = n$. We do not need to make adjustments in the final step, since $\psi^n|\Sigma_i$ is the identity. Therefore, the union of diagrams encountered in the sequence of modifications contains a piece of the projection $\Pi$ which is invariant by $\psi$.

2. The orbit of $\Sigma_i$ is short. We put an end to the adjustments when $l = m$. But $\psi^m|\Sigma_i \rightarrow \Sigma_i$ is a non-trivial automorphism of $\Sigma_i$. We know by hypothesis that $\Gamma_i$ is flype equivalent to $\psi^m(\Gamma_i)$. But it is not certain that we can find a $\Gamma_i^\ell$ flype equivalent to $\Gamma_i$ such that $\Gamma_i^\ell = \psi^m(\Gamma_i^\ell)$. If it is possible, we are back to the preceding situation. If not, we have a knot with no minimal achiral projection.

3.4. An examination of the different diffeomorphisms $\psi$ involved in the case of $\pm$achirality

Case when a Haseman circle $\gamma$ is invariant.

Then $\psi$ exchanges the two tangles which are adjacent to $\gamma$, since $\psi$ is a rotation-reflexion with invariant circle $\gamma$.

Assertions.

1. If $\psi$ is responsible for $-achirality$ it is of order 2.
2. If $\psi$ is responsible for $+achirality$ it is of order 4.

These assertions are a direct consequence of the fact that the in and out strands alternate along $\gamma$. Hence:

1. In case of $-achirality$ there is no short orbit. This is the hidden reason why Tait’s Conjecture is true for $-achirality$.
2. In case of $+achirality$, the two tangles adjacent to $\gamma$ are exchanged by $\psi$ and hence they are the two components of a short orbit. This explains why Tait’s Conjecture for $+achirality$ may be incorrect in some instances. This strategy was beautifully exploited by Dasbach and Hougardy.
Case when a jewel is invariant.

First, it is necessary to make a comparison between the jewels (as defined below in Definition 9.6 of 9.2) and Conway’s basic polyhedra. Our notion of jewel is more restrictive than the notion of polyhedron since a jewel is a diagram with every Haseman circle trivial. For Conway (and others) a polyhedron is “lune free”, where a lune is a portion of a diagram with two edges connecting the same two vertices. In other words, in a polyhedron every vertex is connected to four different other vertices. Hence a polyhedron can be a tangle sum of several jewels. Thus it may contain non-trivial Haseman circles. Typically, the polyhedron 10*** is a tangle sum of two 6*.

Let us call $D$ the invariant jewel. We perform on $D$ the construction we already performed in [5] p.38, called the Filling Construction. For the convenience of the reader let us recall it. Let $\Pi$ a minimal projection of an alternating, + or −achiral knot with the jewel $D$ invariant by $\psi$. Let $\gamma_1, \ldots, \gamma_k$ be the boundary components of $D$. Each $\gamma_i$ bounds in $S^2$ a disc $\Delta_i$ which does not meet the interior of $D$. The projection $\Pi$ cuts $\gamma_i$ in four points. Inside $\Delta_i$ the projection $\Pi$ joins either opposite points or adjacent points. In the first case, we replace $\Pi \cap \Delta_i$ by a singleton and in second case by a 2-spire appropriately placed. We obtain in this manner a new projection $\Pi^*$. The Filling Construction is such that $\Pi^*$ is again a projection of a knot $K^*$. We choose the over/under crossings to get an alternating projection. An important fact is that $\Pi^*$ has no place for flypes.

Assume now that $K$ is +achiral. Then $K^*$ inherits the property of +achirality of $K$. The symmetry is realized by a diffeomorphism $\psi^*$ of $S^2$ which leaves $D$ invariant. The diffeomorphism reverses the orientation of $S^2$ while preserving the orientation of $\Pi$. Since it reverses the orientation of $S^2$ its set of fixed points is either a circle or empty. The first alternative is excluded since the knot is prime. Hence by Kerekjarto’s theorem, the diffeomorphism is conjugate to a rotation-reflexion of even order $n$.

If $n = 2$ there is no short orbit and hence we can proceed as explained above to obtain a minimal projection where the symmetry is visible.

Suppose now that $n \geq 4$. We consider $\psi^*$ as a diffeomorphism of $S^2$, with no consideration to the tangles. Then $\psi^*$ has one short orbit of cardinal 2; all the other orbits are generic.

**Question:** Where is the short orbit?

There are two possible answers.

1. The short orbit is in $D$. Hence the orbits of the discs $\Delta_i$ are generic, and again we can proceed as above by adjusting the various tangles by flypes in order to get a minimal projection invariant by $\psi$. Note that this is essentially the situation we shall encounter in the proof of Theorem 8.2.

2. The short orbit is contained in two tangles, which are exchanged by $\psi^*$. Let $F$ be one of them. Then $(\psi^*)^2$ sends $F$ to itself by a half turn. In other words
\((\psi^*)^2(F) = F^*\). But then \((\psi^*)^4|F = id|F\). Since we can see the order of +achirality everywhere, we have that \(\psi^4 = id : S^2 \to S^2\). Hence the order of +achirality is equal to 4. As in the example provided by Theorem 8.3 we can in this case and in this case only, construct knots where the +achirality is not visible on a minimal projection.

3.5. **Proof of Theorem 3.1**

**Proof.** The main ingredients of the proof are contained in 3.4. If there is an invariant Haseman circle (for instance if the knot is arborescent) we have a phenomenon which is a generalization of the Dasbach-Hougardy knot. If there is an invariant jewel, we have just proved the result. If there is no minimal achiral projection then \(\lambda = 2\). Such a knot can be arborescent as well as non-arborescent. In other words, Theorem 3.1 can be stated as follows:

**Theorem 3.2.** Let \(K\) be an alternating +achiral. Then if the order of its +achirality is different from 4, the symmetry is visible on some minimal projection.

4. The automorphism of the structure tree for an alternating +achiral knot

For basic facts about the canonical decomposition of a projection and about the structure tree, see the Appendix.

Let \(\tilde{K}\) be the mirror image of \(K\). Let \(\tilde{\Pi}\) be the mirror image of a minimal projection \(\Pi\) of \(K\). It differs from \(\Pi\) by the sign at every crossing. Hence the structure tree \(A(\tilde{K})\) is obtained from \(A(K)\) by reversing the weight sign at each \(B\)-vertex. As abstract trees without signs at \(B\)-vertices, the two trees are canonically isomorphic ("equal").

Suppose now that \(K\) is an achiral knot (which may be + or −).

The Key Theorem says that there exists an isomorphism \(\psi : (S^2, \Pi) \to (S^2, \tilde{\Pi})\) which is a composition of flypes and orientation preserving diffeomorphisms. This
isomorphism induces an isomorphism \( \mathcal{A}(K) \to \mathcal{A}(\hat{K}) \). We interpret it as an automorphism \( \varphi: \mathcal{A}(K) \to \mathcal{A}(K) \) which, among other things, sends a \( B \)-vertex of weight \( a \) to a \( B \)-vertex of weight \( -a \). The Lefschetz Fixed Point Theorem implies that \( \varphi: \mathcal{A}(K) \to \mathcal{A}(K) \) has fixed points.

We prove in \( \text{[5]} \) the following result.

**Proposition 4.1.**

1) A twisted band diagram cannot be invariant by \( \psi: (S^2, \Pi) \to (S^2, \hat{\Pi}) \).
2) If a Haseman circle \( \gamma \) is invariant by \( \psi \) and if both diagrams adjacent to \( \gamma \) are jewels, then the two jewels are exchanged by \( \psi: (S^2, \Pi) \to (S^2, \hat{\Pi}) \).

**Corollary 4.2.** The automorphism \( \varphi: \mathcal{A}(K) \to \mathcal{A}(K) \) has exactly one fixed point.

The invariance properties of \( \psi: (S^2, \Pi) \to (S^2, \hat{\Pi}) \) can be stated as follows.

**Theorem 4.3.** Exactly one of the three situations occurs:

A) **Existence of an invariant jewel**: a jewel is invariant by \( \psi: (S^2, \Pi) \to (S^2, \hat{\Pi}) \).
B) **Existence of a polyhedral invariant circle**: a Haseman circle \( \gamma \) is invariant by \( \psi \) and both diagrams adjacent to \( \gamma \) are jewels. The two jewels are exchanged by \( \psi \).
C) **Existence of an arborescent invariant circle**: a Haseman circle \( \gamma \) is invariant by \( \psi \) and both diagrams adjacent to \( \gamma \) are twisted band diagrams. The two twisted band diagrams are exchanged by \( \psi \).

5. **Achirality when a Haseman circle is invariant**

We are now in Situation B) or C) above. Let \( \gamma \) be the Haseman circle invariant by \( \psi \). We split it into two circles joined by four strands as in Fig.3 below, where \( F_1 \) and \( F_2 \) are the two tangles exchanged by \( \psi \).

![Fig. 3. The splitting of a Haseman circle](image)
Let us orient \( K \) (and hence also \( \Pi_K \)) arbitrarily. We are interested in the orientation induced on the four strands which connect the two tangles \( F_1 \) and \( F_2 \). Let us proceed along the boundary \( \partial F_1 \) and observe whether the oriented strands leave or enter \( F_1 \). We observe that the orientations alternate when we follow the boundary of the tangle, see [5] Step 5 in Subsection 6.1. The proof given there used the hypothesis that \( K \) is a knot and not a link. Let us denote \( F_1 \) by \( F \). There are eight possibilities to place \( \hat{F} \) on \( F_2 \), since the set of gluing maps is in bijection with the mapping class group of a circle with four marked points, i.e. with the dihedral group \( D_4 \). A careful study of the situation for +achirality is summarized in next theorem, which can be considered as a structure theorem for +AAA knots. A detailed proof is in Section 6 of [5]. First we need the following definition.

**Definition 5.1.** Let \( F \) be a tangle. Let us denote by:
1) \( F^* = R^*(F) \) where \( R^* \) is the half-turn rotation centered in the middle of \( F \).
2) \( F^v = R^v(F) \) where \( R^v \) is the rotation of angle \( \pi \) in the projection plane about the vertical North-South axis of the tangle. Analogously the tangle \( F^h = R^h(F) \) is defined such that the rotation of angle \( \pi \) is about the West-East axis of \( F \).
3) \( F \sim G \) if \( F \) and \( G \) are two tangles which differ one from the other by flypes and diffeomorphisms fixing the boundary.
4) \( F \) is \( \ast \)-equivalent if \( F \sim F^* \). The \( \ast \)-equivalence plays an essential role in the proof of the Main Theorem given in Section 6.

Analogously \( F \) is \( h \)-equivalent (respectively \( v \)-equivalent) if \( F \sim F^h \) (respectively \( F \sim F^v \)).

![Fig. 4. Representations of \( F \), \( F^* \), \( F^h \) and \( F^v \)](image)

**Theorem 5.2.** (Structure Theorem for +AAA knots)
Let \( K \) be a +AAA knot and let \( \Pi_K \) be a minimal alternating projection of \( K \). Then the symmetry induces an automorphism of \( \Pi_K \subset S^2 \) which has an invariant Haseman circle \( \gamma \). The two tangles \( F \) and \( F^* \) determined by \( \gamma \) appear as in Fig. 5 and \( F \sim F^* \). Moreover there exists an adequate projection \( \Pi_K \) where the symmetry is realized by a rotation of angle \( \pi/2 \) about an axis which intersects both tangles in their center, followed by a reflection through the Haseman circle \( \gamma \).

Projections of Type I were defined and studied in [5].
Remark 5.3.

(1) Let $\Phi$ be the diffeomorphism of $S^2$ of order 4 described in Theorem 5.2. Then $\Phi^2$ sends each tangle into itself. Its restriction to each tangle is a half-turn centered in the middle of the tangle. Hence $\Phi^2(F) = F^*$. If we want $\Phi$ to be related to a symmetry of the projection, then the condition $F \sim F^*$ should be satisfied. In fact this condition is sufficient for $+\text{achirality}$ as can be seen by inspection (and $\Pi_K$ does not need to be alternating). If $\Pi_K$ is alternating, the condition is necessary, essentially from the Key Theorem. See the proof in Section 6, Type I projections. From the description of $\Phi$ and the fact that the orientations of the strands connecting the two tangle are alternating, we see that $\Phi$ preserves the orientation of the knot.

(2) We are still far from the theorem we are looking for, since $F \sim F^*$ means that flypes are involved. Roughly speaking, we must get rid of flypes. If we can replace $F \sim F^*$ by $F = F^*$ then we are finished. If $F \neq F^*$ this will be achieved as described in the next section, at the cost of increasing the number of crossings.

6. $+\text{Achirality}$ when the projection is arborescent: proof of the Main Theorem

We assume in this section that the knots are $+\text{AAA}$. 

6.1. Essential Conway circles

First let us rephrase the usual definition of a rational tangle (see for instance [2] or [6]) in terms of Conway circles as follows:
**Definition 6.1.** A *rational tangle* is a tangle where all the canonical Conway circles are concentric and such that the innermost circle bounds a disc containing exactly one spire.

Let $\tau$ be a tangle bounded by a Haseman circle $\gamma_0$ which is not necessarily a canonical Conway circle. Denote by $\Delta$ the disc bounded by $\gamma_0$. Let us consider maximal rational subtangles of $\tau$ (i.e. which are not included in bigger rational subtangles in the interior of $\Delta$) of the tangle $\tau$. Then one defines:

**Definition 6.2.**

1. An *essential* Conway circle of $\tau$ is a canonical Conway circle which is not properly contained in a maximal rational subtangle of $\tau$.
2. Consider $C^e_\tau$ the set of the essential Conway circles of $\tau$. The cardinal of $C^e_\tau$ is the *complexity* of $\tau$.
3. In the same lines, one defines the essential Conway circles and the complexity of a link projection.

The proof of the Main Theorem is done by induction on the complexity of the tangle $F$.

**Examples**

![Diagram](image-url)
(1) The rational tangle \( \tau \) described in Fig.7 has complexity equal to 1.
(2) The tangle \( \tau \) described in Fig.6 is of complexity 6 or 5 depending whether \( \gamma_0 \) is or is not a canonical Conway circle of the link diagram. The graph of Fig.6a describes schematically the tangle \( \tau \) as the boundary of a surface obtained by plumbing twisted bands by \( \gamma_0 \); which correspond respectively to the vertices. The associated weights are defined in Subsection 9.1. The edges correspond to the canonical Conway circles of \( \tau \). The edge (\( \delta \)) corresponds to a non-essential Conway circle.

Let us analyze the tangle \( F \) of Fig.3. Denote by \( \Delta \) the disc bounded by \( \gamma \). The disc contains half of the crossings of the projection \( \Pi_K \). Denote by \( \mathcal{C}_F^e \) the set of the essential Conway circles contained in \( \Delta \). These circles split \( \Delta \) into connected planar surfaces. Let \( \Sigma \) be such a connected planar surface. If the boundary is not connected, only one component of the boundary does not bound a disc in \( \Delta \); the other ones \( \{\gamma_1, \ldots, \gamma_n\} \) bound discs in \( \Delta \). We call them the inner discs of \( \Sigma \).

We denote by \( \mathcal{P}_F \) the set of planar connected components determined by \( \mathcal{C}_F^e \). Let us introduce an order relation in \( \mathcal{P}_F \).

**Definition 6.3.** Let \( \Sigma^1 \) and \( \Sigma^2 \) be elements of \( \mathcal{P}_F \). We write \( \Sigma^1 > \Sigma^2 \) if \( \Sigma^2 \) is contained in an inner disc of \( \Sigma^1 \).

**Remark 6.4.** A minimal element for this order relation is hence a disc \( \Delta \) which contains exactly one maximal rational tangle.

The maximal element for this order relation is represented in Fig.8.
6.2. Proof of the Main Theorem in some special cases

(1) Case when the tangle $F$ is a rational tangle. As one can see in its pillow-form (see for instance in [2, Theorem 8.2] or in an appropriated plumbing form (see below in Subsection 6.5), a rational tangle exhibits a lot of visible symmetries; rotating it by the angle $\pi$ about any principal axes (North-South, East-West) or orthogonal to the projection plane produces the same tangle. Hence a rational $+\!-\!$-achiral rational knot exhibits the visible symmetry as stated in the Structure Theorem.

Denote by $F_1, \ldots, F_n$ the inside tangles in the maximal element $F$ (Fig. 8).

(2) Case when the tangle $F$ is a Montesinos tangle.

Definition 6.5. A Montesinos tangle is obtained from a twisted band diagram by filling the inner discs with rational tangles.

Hence in this case, each $F_i$ is a rational tangle. In other terminology, one can consider the Montesinos tangle $F$ as a connected sum of rational tangles $F_1, \ldots, F_n$ and its complexity is equal to $n + 1$. A Bretzel tangle is the special case where the rational tangles are reduced to spires which are transversal to the twisted band.

Let $a = \sum a_i$. Without loss of generality, one can assume that $a = 0$ or $1$. By flypes on the twisted central band in $F$, one can move if necessarily the crossing point.

One has four cases with $n$ even or odd and $a = 0$ or $1$.

(a) $n = 2k$ and $a = 0$. The center of the half-turn is between the rational tangles $F_k$ and $F_{k+1}$. The condition $F \sim F^*$ implies that $F_i \sim F^*_{n+1-i}$ for $i = 1, \ldots, k$.

(b) $n = 2k$ and $a = 1$. Up to flypes one can place the crossing point at the center of the half-turn. With the half turn, one gets as in the first case $F_i \sim F^*_{n+1-i}$ for $i = 1, \ldots, k$.

(c) $n = 2k + 1$ and $a = 0$. The center of the half-turn is in the “middle” of the tangle $F_{k+1}$. Since it is rational, $F_{k+1}$ satisfies $F_{k+1} \sim F^*_{k+1}$ and the other ones $F_i \sim F^*_{n+1-i}$ for $i = 1, \ldots, k$. 

Fig. 8. The maximal element
(d) \( n = 2k + 1 \) and \( a = 1 \). Again up to flypes one can assume that the crossing point is on the North of the rational tangle \( F_{k+1} \). As illustrated in the example of Fig.9, it gives rise to another rational tangle \( +F_{k+1} \). The arguments and results are in the same lines as above.

**Remark 6.6.** For the example of Fig.9, the tangle \( +F_{k+1} \) is in the notation of subsection 6.5 a tangle \( P(1, -b) \) which is isotopic to the tangle \( P(2, 2, \ldots, 2) \) and hence exhibiting the desired symmetry.

![Fig. 9. An example of \( P(1, -b) \)](image)

#### 6.3. The \( \alpha \)-move and the cross-plumbing

**Notation.** Let \( Q \) be a tangle. We denote by \( +Q \) the tangle obtained by adding a crossing point of sign \( \pm 1 \) (actually this sign is immaterial) in the North of the tangle as described in Fig.10. Analogously, we denote by \( +Q \) the tangle obtained by adding a crossing point in the South.

![Fig. 10. \( +Q \) and \( +Q \)](image)

Consider the tangle \( +G \) described by Fig.11 where \( G \) is a connected sum of two subtangles \( G_1 \) and \( G_2 \) plumbed on the twisted central band (which is vertical in Fig.11). Now let us define an operation called the \( \alpha \)-move: first introduce two Reidemeister moves of type II one in the North of \( G_1 \) and the other in the South of \( G_2 \) of Fig.12 (giving rise to 4 supplementary crossings to the tangle); perform the isotopy of the strands as illustrated in the Fig.12. Finally one obtains a central non-twisted band appearing as vertical with two bands plumbed on it with a \( X \)-shape; these two bands support one \( +G_1 \) and the other \( +G_2 \). We call this kind of
plumbing a cross-plumbing of $G_1$ and $G_2$ (Fig. 13). The cross-plumbing picture brings to light the desired symmetry as stated in the Structure Theorem.

Then one has a straightforward but important lemma:

**Lemma 6.7.** $^+G$ is $\star$-equivalent if and only if $G$ is $h$-equivalent; analogously for $^+G$.

Let $^+G$ be described as on the left of Fig. 11. One has:

**Lemma 6.8.** $^+G$ is $\star$-equivalent if and only if $G_1$ and $G_2$ are $h$-equivalent.
Proof. The proof is given by Fig. 14

From these two above lemmas, one can deduce the following proposition which leads to $\star$-equivalent tangles of smaller complexity.

**Proposition 6.9.** $^+G$ is $\star$-equivalent if and only if $^+G_i$ and $^\star G_i$ are $\star$-equivalent.

6.4. **Proof of the Main Theorem in the general case**

Proof. The proof is done by induction on the complexity of the tangle $F$ which is $\star$-equivalent. Denote the complexity of a tangle $\tau$ by by $\mu(\tau)$. The case where
\( \mu(F) = 1 \) is already proved in 6.3.

Suppose that for every \( \ast \)-equivalent tangle \( \tau \) such that \( \mu(\tau) \leq m \), one has the symmetry under the rotation of angle \( \pi \) about an axis perpendicular to the projection plane cutting \( T \) in its middle.

Let us consider the maximal element \( F \) described by Fig. 8 with \( \mu(F) = m + 1 \). Without loss of generality as in the case of a Montesinos tangle, one can assume that \( a = 0 \) or \( 1 \).

1. Case with \( n \) even and \( a = 0 \) or \( 1 \); it can be treated in the same lines as in 6.2.
2. Case with \( n = 2k + 1 \) and \( a = 0 \). The tangle \( F_{k+1} \) is \( \ast \)-equivalent and the other subtangles are grouped two-by-two by the \( \ast \)-symmetry. As \( F_{k+1} \) is a subtangle of \( F \), its complexity \( \mu(F_{k+1}) \) is less than \( m \) and by the induction hypothesis, the \( \ast \)-symmetry is visible on \( F_{k+1} \). Again one obtains the visibility of the symmetry for \( F \) (Fig. 15).

![Fig. 15. Case with \( n = 2k + 1 \) and \( a = 0 \)](image)

![Fig. 16. Case \( n = 2k + 1 \) and \( a = 1 \)](image)
(3) Case \( n = 2k + 1 \) and \( a = 1 \). We have to handle the case \( +F_{k+1} \).

(a) Case where \( F_{k+1} \) is a rational tangle. As \( +F_{k+1} \) is also rational, the symmetry is automatic.

(b) Consider now the situation given by Fig.11 where the tangle \( +F_{k+1} \) is denoted by \( +G \) with two subtangles \( G_1 \) and \( G_2 \). By performing the \( \alpha \)-move, one gets a cross-plumbing of two subtangles which arise from \( G_1 \) and \( G_2 \). The tangles \( +G_1 \) and \( +G_2 \) are \( \star \)-equivalent by Proposition 6.8 with their complexities smaller than \( \mu(G) \). By the induction hypothesis, they satisfy the visibility of symmetry. Finally from the cross-plumbing which is put in evidence in the first step, one realizes the claimed symmetry.

(c) Case depicted by Fig.17 with \( G_1 \ldots G_m \) subtangles \( (m \geq 3) \).

First we group the tangles as shown in Fig.18. The tangle \( \tilde{G}_1 \) is the connected sum of subtangles \( G_1 \ldots G_{m-1} \) enriched with a supplementary crossing in the North and the tangle \( \tilde{G}_2 \) is \( +G_m \). As proved earlier the tangles \( \tilde{G}_1 \) and \( \tilde{G}_2 \) are \( \star \)-equivalent. By the cross-plumbing construction and the induction hypothesis on the complexity, the proof is achieved.
6.5. Rational tangles and annuli

6.5.1. Rational tangles

According to Definition 6.3, let us consider a maximal chain $P(x_1) > P(x_2) > \cdots > P(x_u)$ which corresponds to a rational tangle where each $P(x_i)$ is a twisted band diagram with exactly one inner disc for $i = 1, \ldots, u - 1$; and $P(x_u)$ is a spire. The weight $x_i \neq 0$ for $i = 1, \ldots, u - 1$ and $|x_u| \geq 2$. We denote this chain by $P(x_1, \ldots, x_u)$. If we wish to have a symmetric tangle, the problem is created by odd weights. We could use $\alpha - \text{moves}$. But there is a better and more global way to proceed. This argument is present several times in the literature, more or less explicitly.

Since the projection is alternating, the signs of the weights alternate. Without real loss of generality, we assume that $x_1 > 0$. Let $a_i = (-1)^{i+1} x_i$. As usual we define the rational number $p/q$ by the continued fraction

$$\frac{p}{q} = a_1 + \frac{1}{a_2 + \frac{1}{a_3 + \cdots + \frac{1}{a_u}}}.$$  

We denote this continued fraction expansion by $C(a_1, \ldots, a_u)$. Let $C(b_1, \ldots, b_v)$ be the continued fraction expansion of $p/q$ with each $b_j$ even except for $b_v$ if $p$ and $q$ are both odd. Let $y_j = (-1)^{j+1} b_j$. It is possible to modify the plumbing $P(x_1, \ldots, x_u)$ to the plumbing $P(y_1, \ldots, y_v)$ by a sequence of operations which correspond to a $\pm$ blow-up in plumbing calculus (see Walter Neumann [13]) and also to Lagrange Formula (see Cromwell’s book p.204 [3]) in continued fraction expansions. The operation modifies the plumbing

$$P(z_1, \ldots, z_i, z_{i+1}, \ldots, z_w)$$

to the plumbing

$$P(z_1, \ldots, z_{i-1}, z_i \pm 1, \pm 1, z_{i+1} \pm 1, z_{i+2}, \ldots, z_w)$$

The corresponding tangles are isotopic in the 3-ball $B^3$ by an isotopy which is the identity on its boundary. We denote this isotopy relation by $\approx$. Note that the plumbing notation takes care of signs elegantly. The $\pm$ blow-up operation is defined as follows. It is the combination of a gimmick and a transfer move in the sense of Kauffman-Lambropoulou (see [11]) including a rotation of angle $\pi/2$ for the interior tangle. The gimmick introduces two crossing points of opposite sign by a Reidemeister move of type II at the extremity of a twist. The transfer move pushes a non-alternating arc. Its new position creates the $\pm 1$ between the $i$th and the $(i+1)$st entry in the plumbing. See Fig. 19.

At some stage of the sequence of operations, it is possible to encounter a weight equal to zero. In this case the following plumbings are isotopic.
The isotopies can easily be seen on the corresponding diagrams. Of course this can also be checked on continued fractions. These last equivalences are called 0-absorption by Walter Neumann.

To illustrate how this works, we transform the tangle $C(3, 3, 2, 2)$ with blow ups on the plumblings.

$$P(z_1, \ldots, z_{i-1}, 0, z_{i+1}, \ldots, z_w) \approx P(z_1, \ldots, z_{i-1} + z_{i+1}, \ldots, z_w)$$

$$P(z_1, \ldots, z_{w-2}, z_{w-1}, 0) \approx P(z_1, \ldots, z_{w-2})$$

The corresponding continued fraction expansion with even integers is $C(4, -2, 2, -4, 2)$. 

$P(3, -3, 2, -2) \approx P(4, +1, -2, 2, -2) \approx P(4, 2, +1, -1, 2, -2) \approx P(4, 2, 2, 1, 0, 2, -2) \approx P(4, 2, 2, 3, -2)$

$P(4, 2, 2, 4, +1, -1) \approx P(4, 2, 2, 4, 2, +1, 0) \approx P(4, 2, 2, 4, 2)$. 

The corresponding continued fraction expansion with even integers is $C(4, -2, 2, -4, 2)$. 

Fig. 19. A gimmick followed by a transfer move
A useful special case is $P(1, -b)$ with $b \geq 2$. We have $P(1, -b) \approx P(2, 2, \ldots, 2)$ with $b$ weights equal to 2.

Here is a quick one: $P(3, -1, 3, -2) \approx P(4, +1, 0, 3, -2) \approx P(4, 4, -2)$. The reader familiar with blow downs will find some shortcuts.

**Remark 6.10.**

1. We have assumed that the first weight $x_1$ is $> 0$, but there is no real difference in the arguments if we have $x_1 < 0$. It suffices to use $-\text{blow ups}$ instead.

2. The last weight $y_v$ is odd if $p$ and $q$ are both odd. This fact does not prevent $P(y_1, \ldots, y_v)$ from being invariant by a half turn since the innermost circle contains a spire.

### 6.5.2. Rational annuli

Let us consider again the connected planar surfaces in $\Delta$ bounded by essential Conway circles. Among them, consider a chain of maximal length made of concentric annuli. Contrary to the case of rational tangles we assume that the inner circle bounds a twisted band diagram $B$ with at least three boundary components. We call the union of the annuli in this chain a **rational annulus**.

Each successive annulus in the chain bears an integral weight $x_i \neq 0$. From these we obtain a rational number $p/q$ as above. If one of the integers $p$ or $q$ is even, the procedure used above for rational tangles produces a symmetric part of the projection, using only gimmicks and transfer moves.

If both $p$ and $q$ are odd, the same procedure produces a symmetric part of the corresponding projection, except for the last step. Since the inside circle does not bound a spire we cannot conclude as in the case of rational tangles. If necessary, we use instead a $\alpha$–move, which involves the twisted band diagram $B$ and one exterior crossing point.

**Conclusion**

Let us define the “rational part” of the projection to be the union of the rational tangles and of the rational annuli. Up to the exception we have just met, the rational part does not present serious difficulties. Only the twisted band diagrams with at least three boundary components need special care, treated with $\alpha$–moves. The induction argument involves them only.

### 7. Examples: the knots of Dasbach-Hougardy and Stoimenow

Knots (non-necessarily alternating) which are $+\text{achiral}$ but not $-\text{achiral}$ are rather rare among achiral knots. Among the 20 achiral knots (all alternating) with crossing
number \( c < 12 \), no one is only +achiral.

According to Hoste-Thistlethwaite-Weeks \([9]\) we have:

1) For \( c = 12 \) there are 54 alternating achiral knots. Exactly one of them is only +achiral. It was recognized by Haseman (it is her knot 59=60) and also earlier by Tait, with a vocabulary different from what is used today. See \([14]\). There are also 4 non-alternating achiral knots. No one is only +achiral.

2) For \( c = 14 \) there are 223 alternating achiral knots. Among them 5 are only +achiral. There are also 51 non-alternating achiral knots. Exactly one of them is only +achiral.

3) For \( c = 16 \) there are 1049 alternating achiral knots. Among them 40 are only +achiral. There are also 490 non-alternating achiral knots, with 25 only +achiral.

4) All in all there are 1’701’935 non-trivial knots with \( c \leq 16 \). There are 491’327 alternating knots and 1’201’608 non-alternating ones. There are 1’892 achiral knots (including a surprising one with 15 crossings). Among them 1’346 are alternating and 546 non-alternating. There are 82 knots which are only +achiral; 56 are alternating and 26 non-alternating.

5) Hence there are 1’290 alternating achiral knots with \( c \leq 16 \) which are −achiral. For all of them the −achirality is visible on a minimal projection, according to our result \([5]\).

A conclusion of this little statistics is that the knots we are interested in are rather rare individuals in the knot population.

**Question:** Is this conclusion biased by the small values of \( c \)?

From now let us focus in alternating knots which are +achiral but not −achiral. Of particular interest are those for which there exists no achiral minimal projection. The first ever discovered was given by Dasbach-Hougardy in \([4]\). Since it is arborescent, in accordance with \([5]\) it has a minimal projection of Type I as displayed in Fig.5. Its tangle \( F \) has the shape shown in Fig.20.

![Fig. 20. The shape of the Dasbach-Hougardy tangle](image)

**Definition 7.1.** An alternating knot is **suitable** if it has a minimal projection of Type I and \( F \) as shown by Fig. 20.

Obviously, a suitable knot is +achiral and not −achiral only if a clever choice is
made for the subtangles $F_1$ and $F_2$. The next proposition makes this explicit.

**Proposition 7.2.**

(1) A suitable knot is $+\text{achiral}$ if and only if $F_i \approx F_i^h$ for $i = 1, 2$.

(2) A suitable knot is not $-\text{achiral}$ if $F_1$ is very distinct from $F_2$.

We say that $F_1$ is **very distinct** from $F_2$ if $F_1$ is not flype equivalent to any of $F_2$, $F_2^h$, $F_2^v$ or $F_2^*$. 

**Proof.**

(1) It relies on Proposition 6.3 from [5] which states that an alternating knot with a projection of Type I is $+\text{achiral}$ if and only if $F \sim F^*$. Now a half-turn in $F$ sends to the South the crossing which is in the North and exchanges $F_1$ with $F_2$. A flype with moving crossing the crossing which is now in the South sends it back to the North and exchanges again $F_1$ with $F_2$. But after the two moves $F_i$ is transformed into $F_i^h$. This proves Proposition 7.2 (a)

(2) We prove in the same proposition 6.3 that a knot with a Type I projection is $-\text{achiral}$ if and only if $F \sim F^h$ or $F \sim F^v$. With an argument similar to the one for case (a) it is easily checked that if $F_1$ is very distinct from $F_2$ then $F \not\sim F^h$ and also $F \not\sim F^v$. 

**Remark 7.3.**

(1) Let us consider suitable knots, with $F_1$ and $F_2$ arborescent and satisfying the conditions of Proposition 7.2. From the proof of this proposition, one can see that the crossing point present on Fig.20 (which is in the North) prevents the minimal projection to be achiral. In other words, such a knot has no minimal projection which is $+\text{achiral}$. However our method provides a non-minimal achiral projection.

(2) We can admit non-arborescent tangles for the $F_i$. These knots will also have no minimal projection where the $+\text{achirality}$ is visible. But we do not have a general method to exhibit an achiral projection for every case. It is easy to exhibit tangles $F_i$ which satisfy the conditions of Proposition 7.2. Denote by $c_i$ the number of crossings of $F_i$. Hence the number $c$ of crossings of the minimal projection of the knot is $2(c_1+c_2+1)$. To avoid degeneracy we must have $c_i \geq 3$. Let us restrict $F_i$ to rational tangles. Rational tangles have several advantages: the condition $F \sim F^h$ is always satisfied and if the tangles $F_1$ and $F_2$ have different Conway words, they are very distinct. Let $C(a_1, \ldots, a_u)$ be the Conway word for a rational tangle. Recall that $a_1$ can be equal to 1 but that $a_u$ cannot. Note that the first twist $a_1$ must be placed vertically in $F_1$ since the central band in $F$ which connects $F_1$ to $F_2$ is horizontal. Hence up to flypes there is only one way to place a rational
tangle in $F_i$. There are two rational tangles with crossing number equal to 3: $C(1,2)$ and $C(3)$. Hence there is exactly one knot with $c = 14$. It is the original Dasbach-Hougardy knot with $F_1 = C(1,2)$ and $F_2 = C(3)$. Its HTW notation is $14 - 10435(a)$. This motivates the following definition.

Definition 7.4. A suitable knot with tangles $F_1$ and $F_2$ arborescent, satisfying the conditions of Proposition 7.2 is a DH-knot.

We now consider DH-knots with 16 crossings. There are 4 rational tangles with crossing number equal to 4: $C(1,1,2)$, $C(1,3)$, $C(2,2)$ and $C(4)$. But $C(1,3)$ and $C(4)$ must be discarded since, if we substitute one of them in one of the $F_i$ we get a link and not a knot. Hence we can construct DH-knots as follows.

1) $F_1 = C(1,2)$ and $F_2 = C(1,1,2)$; this is the knot $16 - 178893(a)$.
2) $F_1 = C(1,2)$ and $F_2 = C(2,2)$; this is the knot $16 - 125918(a)$.
3) $F_1 = C(3)$ and $F_2 = C(1,1,2)$; this is the knot $16 - 223267(a)$.
4) $F_1 = C(3)$ and $F_2 = C(2,2)$; this is the knot $16 - 223382(a)$.
5) There is a fifth knot obtained by choosing $F_1 = C(1,2)$, $F_2 = C(3)$ and by adding one crossing in the central band; this is the knot $16 - 220003(a)$.

Therefore there are 6 DH-knots with $c \leq 16$. These knots were listed by Alexander Stoimenow in [15] as knots for which no projection is known to be achiral. The method we present provides for each of these six knots a non-minimal achiral projection.

Fig.21 illustrates our procedure applying to the original Dasbach-Hougardy knot.

Fig. 21. The DH-tangle under symmetrized form

It is clear that there exist DH-knots for every even crossing number $\geq 14$. Moreover if one asks the tangles $F_i$ to be rational, an exhaustive list can be obtained. Alexander Stoimenow has also listed in [15] four achiral knots for which no achiral projection is known. However they are non alternating and hence our method cannot be used. Apparently, the existence of an achiral projection for these knots is still unknown.
By Knotscape, these knots are only +achiral with order equal to 4.

8. +Achirality when a jewel is invariant

Let \( K \subset S^3 \) be a knot in \( S^3 \). Let \( \pi_0 \text{Diff}(S^3, K) \) be the group of isotopy classes of diffeomorphisms \( f : S^3 \to S^3 \) such that \( f(K) = K \). By definition it is the group of symmetries of the knot \( K \).

**Theorem 8.1.** Suppose that \( K \) is hyperbolic. Then the subgroup of \( \pi_0 \text{Diff}(S^3, K) \) of symmetries which preserve the orientation of the knot is a cyclic group \( C_n \).

See Feng Luo \[7\].

We write \( n \) as the product \( n = 2^\mu m \) with \( m \) odd and \( \mu \geq 0 \).

**Definition 8.2.** Let \( K \) be an hyperbolic +achiral knot. We define the order of +achirality of \( K \) to be equal to \( 2^\mu \).

Recall that alternating knots are all hyperbolic, except for a few torus knots which are chiral.

**Proposition 8.3.** Suppose that we have a projection \( \Pi_K \) of the +achiral hyperbolic knot \( K \) which is invariant by a diffeomorphism \( \varphi : S^3 \to S^3 \) which reverses the orientation of \( S^3 \) and preserves the orientation of \( K \). Suppose that the order of \( \varphi \) in \( \text{Diff}(S^3, K) \) is \( 2^\lambda \) (i.e. \( \varphi^{2^\lambda} = \text{id} \) and \( \varphi^u \neq \text{id} \) for \( u = 1, \ldots, 2^\lambda - 1 \)). Then \( \lambda = \mu \).

**Proof.**

Let \( g \) be a generator of \( C_n \). Then \( g \) reverses the orientation of \( S^3 \), and the other elements of \( C_n \) that reverse the orientation of \( S^3 \) are of the form \( g^k \) with \( k \) odd. The order in \( C_n \) of such an element is \( 2^\mu m' \) with \( m'|m \).

Let \( [\varphi] \) be the image of \( \varphi \) in \( \pi_0 \text{Diff}(S^3, K) \). By Borel’s theorem (see Peter Conner’s paper \[2\] or the Borel seminar on Transformation Groups, Annals of Math. Studies 46) the order of \( [\varphi] \) in \( \pi_0 \text{Diff}(S^3, K) \) is also \( 2^\lambda \). Hence \( \lambda = \mu \).

Let us prove the following theorem.

**Theorem 8.4.** For every \( \lambda \geq 1 \) there exists an alternating (non-arborescent) +achiral knot \( K_\lambda \) such that:

1) the order of +achirality of \( K_\lambda \) is equal to \( 2^\lambda \);

2) \( K_\lambda \) is not –achiral.
Moreover there exists a minimal alternating projection of $K_\lambda$ on which the +achirality is visible.

If we drop Condition 2) it is very easy to realize any order of +achirality. The construction we propose to prove the theorem is also well known: consider a jewel with a large symmetry group and fill the holes with tangles chosen such that the desired symmetry is satisfied and that the resulting knot is alternating. Hence the projection of this alternating knot is achiral and of order $2^\lambda$. By Proposition 8.3, we know that we have obtained the correct order of +achirality. The only difficulty is to obtain such a knot which is not $-\text{achiral}$.

Let $u$ and $v$ be two positive integers. Let $K(u, v)$ be the torus knot/link of type $(u, v)$. Consider the usual projection of $K(u, v)$ as a closed braid with $u$ strings. We surround each crossing point with a little circle and remove its interior disc. We obtain a turban $J(u, v)$ with $(u - 1)v$ boundary circles. Indeed $J(u, v)$ is a jewel if and only if $u \geq 3$ and $v \geq 3$. Note that $J(3, 3)$ is Conway’s 6*. Turbans with $u = 3$ are particularly adapted to achirality, as we shall see. We shall mainly use the jewels $J(3, 2^{\lambda-1})$. For our purpose, let us represent $J(3, 4)$ as shown in Fig.22.

![Fig. 22. The Jewel J(3,4)](image)

We fill the holes of $J(3, 2^{\lambda-1})$ with singletons in such a way to obtain an alternating knot denoted by $A(3, 2^{\lambda-1})$. Since it is alternating, it is not a torus knot.

**Claim.** The symmetry group of $J(3, 2^{\lambda-1})$ (hence of $A(3, 2^{\lambda-1})$) is isomorphic to the dihedral group $D_{2\lambda}$ of order $2^{\lambda+1}$.

We briefly describe the symmetries in the case of $J(3, 4)$. The important feature of Fig.22 is the circle $\sigma$ which meanders among the Haseman circles and is not part of the jewel. We consider it as the intersection of a 2-sphere $\Sigma^2$ with the projection sphere.

Consider one of the four lines, say $\delta$, which goes across two opposite subtangles. The rotation of angle $\pi$ with $\delta$ as axis is a diffeomorphism $\varphi_\delta$ describing the invertibility of $A(3, 2^{\lambda-1})$. Note that $\delta$ cuts the circle $\sigma$ in two points.

Consider one of the four lines, say $\Delta$ which goes through the center of the picture and misses the subtangles. The rotation of angle $\pi$ with axis $\Delta$ followed by a reflex-
ion with respect to the 2-sphere $\Sigma^2$ is a diffeomorphism $\varphi_\Delta$ of order 2, responsible for the $-\text{achirality}$ of the knot. Note that $\Delta$ has two intersection points with the circle $\sigma$ which are also intersection points of $\sigma$ with the projection. They are the two fixed points of the diffeomorphism, which hence reverses the orientation of the knot.

We thus have the eight reflexions of the dihedral group $D_8$.

Consider then the rotation of angle $2\pi/8$ with an axis perpendicular to the projection plane through the center of the picture, followed by a reflexion with respect to $\Sigma^2$. This diffeomorphism $\varphi_C$ realizes the $+\text{achirality}$ of the knot. It is of order 8 and is a generator of the subgroup $C_8 \subset D_8$.

The same arguments work with the order 8 replaced by the order $2^\lambda$ for any $\lambda \geq 3$.

**Convention.** It is necessary to decide where the North is in the boundary of every subtangle. Consider the line which passes through the center of the figure and the center of a given subtangle. This line intersects the boundary circle of the subtangle at two points. The one which is farther from the center of the figure is the North.

Now we consider the action on subtangles of the various diffeomorphisms which realize the symmetries of the turban. A rotation of angle $\pi$ with axis $\delta$ or $\Delta$ transforms a subtangle $F$ to $F^v$. A rotation around an axis perpendicular to the projection plane transforms $F$ to $F$. The reflexion through $\Sigma^2$ transforms $F$ to $\hat{F}^h$.

Let $G$ be an alternating tangle with the following properties:
1) the arcs inside $G$ connect opposite points from the boundary, as in a singleton;
2) $G \not\simeq G^v$.

It is easy to construct such tangles as Bretzel or Montesinos tangles; inside $G$ we must have an odd number of subtangles which satisfy Condition 1). Here is an example in Fig.23.

![Fig. 23. A tangle satisfying the conditions 1) and 2)](image)

With such a tangle $G$, we construct an alternating $+\text{achiral}$ knot of period $2^\lambda$ which is not $-\text{achiral}$ as follows. Consider the turban $J(3, 2^\lambda-1)$. In the holes outside the disc bounded by the circle $\sigma$ we place $G$; in those which are inside this disc we place $\hat{G}^h$. The fillings are made by taking account the position of the North in our convention. From the analysis of the symmetries of the jewel we deduce that the corresponding knot is $+\text{achiral}$, with the desired order. Furthermore, the knot is not $-\text{achiral}$. This is a consequence of Tait’s Conjecture (see [5]) and from the complete
list of the diffeomorphisms of the jewel which are able to produce \( -\)achirality. The fact that we have a knot and not a link follows from Condition 1) (by using the singletons, one can deduce easily this fact). From a simple analysis of the black and white regions near the boundary of \( G \) and \( \hat{G}_h \), one can conclude that the projection is alternating.

We have now proved the theorem for any \( \lambda \geq 3 \). An example for \( \lambda = 1 \) was discovered by Mary Haseman in [8]. See her figure 59. For Haseman, this knot was remarkable since it is the only alternating knot with no more than 12 crossings which is \(+\)achiral, but not \(-\)achiral. In fact, this knot was already discovered by Tait in one of his rare visit to knots with 12 crossings.

Fig. 24 pictures an example for \( \lambda = 2 \).

![Fig. 24. An example for \( \lambda = 2 \)](image)

Now let us prove a version of Theorem 8.4 where the \(+\)achirality is not visible on a minimal projection.

**Theorem 8.5.** For \( \lambda = 2 \) there exists an alternating \(+\)achiral knot \( L_\lambda \) which is not \(-\)achiral and such that:

(1) the order of \(+\)achrality of \( L_\lambda \) is equal to \( 2^\lambda = 4 \);
(2) the knot \( L_\lambda \) is not arborescent.

Moreover there exists no achiral minimal projection of \( L_\lambda \).

**Proof.**
First, consider the knot $K_2$ represented in Fig. 25. We claim that this knot (in fact these knots) satisfies all the conditions stated in Theorem 8.5.

Define the knot $K_1$ as the one obtained from $K_2$ by deleting the tangle $F$ and $\hat{F}$. $K_1$ is a polyhedral knot constructed with four "large" 6-tangles. By a 6-tangle we mean a tangle surrounded by a circle which intersects the knot projection in 6 points. From the situation of the 6-tangles, we see that $K_1$ is +achiral of order 4. The symmetry is essentially realized by the diffeomorphism (say $g$) of the 4-crossing knot $A(3, 2)$. Let us denote one of these 6-tangles by $G$. Since $G$ is not flype-equivalent to $G^*$, one can deduce that $K_1$ is not $-$achiral.

As $K_2$ is obtained from $K_1$ by reinserting $F$ and $\hat{F}$, the diffeomorphism $g$ acts almost on $K_2$. In fact, we have $g(F) = \hat{F}$ and $g^2(F) = F^*$. Hence $g$ induces a symmetry of $K_2$ if and only if $F \sim F^*$.

Now, suppose that we have a tangle $F$ such that:
1) $F \sim F^*$;
2) there is no minimal projection $F'$ flype-equivalent to $F$ such that $F' = (F')^*$.

The simplest tangle which satisfies the conditions is $P(1, 2)$.

Then the knot $K_2$ is polyhedral, +achiral of order 4 with no achiral minimal projection.
9. Appendix: The canonical decomposition of a projection

In the first three subsections we do not assume that link projections are alternating.

9.1. Diagrams

**Definition 9.1.** A planar surface $\Sigma$ is a compact connected surface embedded in the 2-sphere $S^2$. We denote by $k+1$ the number of connected components of the boundary $b\Sigma$ of $\Sigma$.

We consider compact graphs $\Gamma$ embedded in $\Sigma$ and satisfying the following four conditions:
1) vertices of $\Gamma$ have valency 1 or 4.
2) let $b\Gamma$ be the set of vertices of $\Gamma$ of valency 1. Then $\Gamma$ is properly embedded in $\Sigma$, i.e. $b\Sigma \cap \Gamma = b\Gamma$.
3) the number of vertices of $\Gamma$ contained in each connected component of $b\Sigma$ is equal to 4.
4) a vertex of $\Gamma$ of valency 4 is called a crossing point. We require that at each crossing point an over and an under thread be chosen and pictured as usual. We denote by $c$ the number of crossing points.

**Definition 9.2.** The pair $D = (\Sigma, \Gamma)$ is called a diagram.

**Definition 9.3.** A singleton is a diagram diffeomorphic to Fig. 26.

![Fig. 26. A singleton](image)

**Definition 9.4.** A band diagram is a diagram diffeomorphic to Fig. 27.

The sign of a crossing point sitting on a band is defined according to Fig. 28.

**First hypothesis.** Crossing points sitting side by side along the same band have the same sign. In other words we assume that Reidemeister move of type II cannot be applied to reduce the number of crossing points along a band.

Let us picture again a twisted band diagram with more details in Fig. 29.
In Fig. 29 the boundary components of $\Sigma$ are denoted by $\gamma_1, \ldots, \gamma_{k+1}$ where $k+1 \geq 1$. The $a_i$ are integers. $|a_i|$ denotes the number of crossing points sitting side by side between $\gamma_{i-1}$ and $\gamma_i$. The sign of $a_i$ is the sign of the crossing points. The integer $a_i$ will be called an intermediate weight. The corresponding portion of the diagram is called a twist.

Second hypothesis. If $k+1 = 1$ we assume that $|a_1| \geq 2$. If $k+1 = 2$ we assume that $a_1$ and $a_2$ are not both 0.

Remark 9.5. Using flypes and then Reidemeister II move, we can reduce the number of crossing points of a twisted band diagram in such a way that either $a_i \geq 0$ for all $i = 1, \ldots, k+1$ or $a_i \leq 0$ for all $i = 1, \ldots, k+1$. This reduction process is not quite canonical, but any two diagrams reduced in this manner are equivalent by flypes. This is enough for our purposes.

Third hypothesis. We assume that in any twisted band diagram, all the non-zero $a_i$ have the same sign.
Notation. The sum of the $a_i$ is called the weight of the twisted band diagram and is denoted by $a$. If $k + 1 \geq 3$ we may have $a = 0$.

9.2. Haseman circles

Definition 9.6. A Haseman circle for a diagram $D = (\Sigma, \Gamma)$ is a circle $\gamma \subset \Sigma$ meeting $\Gamma$ transversally in four points, far from crossing points. A Haseman circle is said to be compressible if:

i) $\gamma$ bounds a disc $\Delta$ in $\Sigma$.

ii) There exists a properly embedded arc $\alpha \subset \Delta$ such that $\alpha \cap \Gamma = \emptyset$ and such that $\alpha$ is not boundary parallel. The arc $\alpha$ is called a compressing arc for $\gamma$.

Fourth hypothesis. Haseman circles are incompressible.

Two Haseman circles are said to be parallel if they bound an annulus $A \subset \Sigma$ such that the pair $(A, A \cap \Gamma)$ is diffeomorphic to Fig.30.

![Fig. 30. Parallel Haseman circles](image)

Analogously, we define a Haseman circle $\gamma$ to be boundary parallel if there exists an annulus $A \subset \Sigma$ such that:

1) the boundary $bA$ of $A$ is the disjoint union of $\gamma$ and a boundary component of $\Sigma$;

2) $(A, A \cap \Gamma)$ is diffeomorphic to Fig.30.

Definition 9.7. A jewel is a diagram which satisfies the following four conditions:

a) it is not a singleton.

b) it is not a twisted band diagram with $k + 1 = 2$ and $a = \pm 1$.

c) it is not a twisted band diagram with $k + 1 = 3$ and $a = 0$.

d) every Haseman circle in $\Sigma$ is either boundary parallel or bounds a singleton.

Comment. The diagrams listed in a), b) and c) satisfy condition d) but we do not wish them to be jewels. As a consequence, a jewel is neither a singleton nor a twisted band diagram.

9.3. Families of Haseman circles for a projection

Definition 9.8. A link projection $\Pi$ (also called a projection for short) is a diagram in $\Sigma = S^2$. 

Fifth hypothesis. The projections we consider are connected and prime.

**Definition 9.9.** Let \( \Pi \) be a link projection. A **family of Haseman circles** for \( \Pi \) is a set of Haseman circles satisfying the following conditions:
1. any two circles are disjoint.
2. no two circles are parallel.

Note that a family is always finite, since a projection has a finite number of crossing points.

Let \( \mathcal{H} = \{ \gamma_1, \ldots, \gamma_n \} \) be a family of Haseman circles for \( \Pi \). Let \( R \) be the closure of a connected component of \( S^2 \setminus \bigcup_{i=1}^{n} \gamma_i \). We call the pair \((R, R \cap \Gamma)\) a **diagram** of \( \Pi \) determined by the family \( \mathcal{H} \).

**Definition 9.10.** A family \( \mathcal{C} \) of Haseman circles is an **admissible family** if each diagram determined by it is either a twisted band diagram or a jewel. An admissible family is **minimal** if the deletion of any circle transforms it into a family which is not admissible.

The next theorem is the main structure theorem about link projections proved in [14]. It is essentially due to Bonahon and Siebenmann.

**Theorem 9.11.** (*Existence and uniqueness theorem of minimal admissible families*) Let \( \Pi \) be a link projection in \( S^2 \). Then:

i) there exist minimal admissible families for \( \Pi \).

ii) any two minimal admissible families are isotopic, by an isotopy which respects \( \Pi \).

**Definition 9.12.** “The” minimal admissible family will be called the **canonical Conway family** for \( \Pi \) and denoted by \( \mathcal{C}_{\text{can}} \). The decomposition of \( \Pi \) into twisted band diagrams and jewels determined by \( \mathcal{C}_{\text{can}} \) will be called the **canonical decomposition** of \( \Pi \).

It may happen that \( \mathcal{C}_{\text{can}} \) is empty. The next proposition tells us when this occurs.

**Proposition 9.13.** Let \( \Pi \) be a link projection. Then \( \mathcal{C}_{\text{can}} = \emptyset \) if and only if \( \Pi \) is either a jewel with empty boundary (i.e. \( v = 0 \)) or the minimal projection of the torus knot/link of type \((2, m)\).

**Comment.** A jewel with empty boundary is nothing else than a polyhedron in John Conway’s sense which is indecomposable with respect to tangle sum. The minimal projection of the torus knot/link of type \((2, m)\) can be considered as a
Definition 9.14. The arborescent part of a graph $\Gamma \subset S^2$ is the union of the twisted band diagrams determined by the canonical Conway family. The polyhedral part of $\Gamma \subset S^2$ is the union of the jewels determined by the canonical Conway family. A knot is arborescent if it has a projection such that all diagrams determined by the canonical Conway family are twisted band diagrams.

Remark 9.15. The adjective “arborescent” (or equivalently “algebraic”) has several meanings in the literature. We have adopted the more restrictive one, based on 2-dimensional diagrams. As a consequence of their 3-dimensional viewpoint, Bonahon-Siebenmann have a more permissive definition. For example, Conway has shown that some diagrams based of his polyhedron (= jewel) $6^*$, which are polyhedral in our sense, can be transformed into algebraic diagrams in his sense by adding some more crossing points. Typically the knot $10_{99}$ in Rolfsen’s notations is such a knot. Note that this knot is $+\text{achiral}$, and also $-\text{achiral}$. Both symmetries can easily be seen on a $6^*$ projection.

9.4. The structure tree $\mathcal{A}(K)$

Now we assume that knots and links are alternating.

Construction of $\mathcal{A}(K)$. Let $K$ be an alternating link and let $\Pi$ be a minimal projection of $K$. Let $C_{\text{can}}$ be the canonical Conway family for $\Pi$. We construct the tree $\mathcal{A}(K)$ as follows. Its vertices are in bijection with the diagrams determined by $C_{\text{can}}$. Its edges are in bijection with the Haseman circles of $C_{\text{can}}$. The extremities of an edge (representing a Haseman circle $\gamma$) are the vertices which represent the two diagrams containing the circle $\gamma$ in their boundary. Since the diagrams are planar surfaces of a decomposition of the 2-sphere $S^2$ and since $S^2$ has genus zero, the graph we have constructed is a tree. This tree is “abstract”, i.e. it is not embedded in the plane.

We label the vertices of $\mathcal{A}(K)$ as follows. If a vertex represents a twisted band diagram we label it with the letter $B$ and by the weight $a$. If the vertex represents a jewel we label it with the letter $J$.

Remark 9.16.

(1) The tree $\mathcal{A}(K)$ is independent of the minimal projection chosen to represent $K$. This is an immediate consequence of the Flyping Theorem. Indeed, as we have seen, the flypes modify the decomposition of the weight $a$ of a twisted band diagram as the sum of intermediate weights, but the sum remains constant. A flype also modifies the way in which diagrams are embedded in $S^2$. Since the tree is abstract a flype has no effect on it, see [14] Section 6. This is why we
call it the **structure tree** of $K$ (and not of $\Pi$).

(2) $\mathcal{A}(K)$ contains some information about the decomposition of $S^2$ in diagrams determined by $C_{can}$ but we cannot reconstruct the decomposition from it. However one can do better if no jewels are present. In this case the link (and its minimal projections) are called **arborescent** by Bonahon-Siebenmann. They produce a planar tree which actually encodes a given arborescent projection. See [1] for details.

(3) If $K$ is oriented, we do not encode the orientation in $\mathcal{A}(K)$.

**10. When do we have to add new crossings?**

We conclude this paper by examining how much the method we propose is “expensive”. More precisely: how many new crossings do we have to add in order to obtain an achiral projection for a given +AAA knot? A look at Section 6 reveals that $\alpha-$moves are costly and may give rise to complicated pictures. This case happens only if we meet along the way a twisted band diagram with odd half-twist number $a$ and odd number of inner discs $n$.

For instance, if all half-twist numbers are even no $\alpha-$move is needed. In fact, we can find a minimal projection with visible +achirality, by adjusting the twists in a balanced way.

Furthermore, if odd half-twist numbers are present in a rational tangle, we can cope by adding a few crossings. See 6.5. In a rational annulus one can also proceed without $\alpha-$move if $p$ and $q$ are not both odd (where $p/q$ denotes the rational number which classifies the tangle).

The “first case” where an $\alpha-$move is needed is represented in Fig.20 in Section 6 as the DH-tangle. Hence it is not surprising that Dasbach-Hougardy and Stoimenow knots are the first knots on the list where the problem of visible +achirality is present.
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