A Uniform Approach to Maximal Permissiveness in Modular Control of Discrete-Event Systems

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Abstract

In this paper, a uniform approach to maximal permissiveness in modular control of discrete-event systems is proposed. It is based on three important concepts of modular closed-loops: monotonicity, distributivity, and exchangeability. Monotonicity of various closed-loops satisfying a given property considered in this paper holds whenever the underlying property is preserved under language unions. Distributivity holds if the inverse projections of local plants satisfy the given property with respect to each other. Among new results, sufficient conditions are proposed for distributed computation of supremal relatively observable sublanguages.

1 Introduction

Discrete-event systems (DES) modeling real technological systems are typically represented as synchronous products of a large number of relatively small local components modeled as finite automata [17]. In order to guarantee safe operation of the resulting global system, a safety specification is given and it is required that the controlled system is included in this specification. As only controllable (and observable in presence of partial observations) specifications are achieved, the computation of sublanguages satisfying these conditions are of paramount importance. The synthesis of observable sublanguages is difficult, especially in the modular setting, where the number of states can grow exponentially with the number of local components.

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Unfortunately, observability is not preserved under language unions, unlike controllability. Therefore, the supremal observable sublanguage does not always exist, and there are only maximal observable sublanguages, which are not unique in general. A stronger notion, called normality, coincides with observability in the case when all controllable events are observable. Supremal normal sublanguages exist, but they are difficult to compute, especially in the modular framework. We have studied possibilities of local (modular) computations of supremal normal sublanguages in [7] for local specification languages and in [6] for global specification languages.

Relative observability was introduced and studied in [2] in the framework of partially observed DES as a condition stronger than observability and weaker than normality. It was shown to be closed under language unions, which makes it an interesting notion that can replace normality in practical applications.

In this paper, a unifying approach is presented for local computation of maximally permissive supervisors in modular DES. This approach is computationally attractive, but the optimality (supremality of the computed sublanguage) is only guaranteed under some additional conditions.

We emphasize that in case of global specification languages coordination control has been proposed in [8], which can reduce the supervisory control problem with global specification to the case of local specification based on the concept of conditional decomposability. Therefore we use a more general framework of coordination control rather than modular control in this paper.

In this paper a new approach to maximal permissiveness for modular and coordination control of DES is presented. It can be applied to computation of various supremal sublanguages having properties that are preserved under language unions, in particular to distributed computation of supremal normal and relatively observable sublanguages. Moreover, we show that mutual normality is equivalent to global mutual normality. Finally we present sufficient conditions for distributed computation of supremal relatively observable sublanguages based on the concept of global mutual observability between the local plants.

The paper is organized as follows. The next section recalls the basic results of supervisory control theory used further. Section 3 presents three algebraic concepts that enable distributed computation of supremal sublanguages. In Sections 4, 5, and 6, sufficient conditions for maximal permissiveness of local control synthesis are presented.

2 Preliminary Results

We first briefly recall some basic notations and concepts in supervisory control theory of DES. We use an alphabet $A$ to describe the set of event. The free monoid of words over $A$ is denoted by $A^*$. Each word is a string of events. Languages are subsets of $A^*$. The prefix closure of a language $L \subseteq A^*$ is $\overline{L} = \{w \in A^* : (\exists v \in A^*)wv \in L\}$. $L$ is said to be prefix-closed if $L = \overline{L}$. We only study prefix-closed languages in this paper.
A generator is a quadruple $G = (Q, A, \delta, q_0)$, consisting of a finite set of states $Q$, a finite set of events $A$, a partial transition function $\delta : Q \times A \rightarrow Q$, and an initial state $q_0 \in Q$. With a slight abuse of notation, the set of all possible transitions is also denoted by $\delta$: $\delta = \{(q, a, q') : \delta(q, a) = q'\}$. Transition function $\delta$ can be extended in the standard way to strings, that is, $\delta : Q \times A^* \rightarrow Q$. The language generated by $G$ is the set of all strings (trajectories) that can be generated by $G$ and is defined as $L(G) = \{s \in A^* : \delta(q_0, s) = s\}$, where $\delta(q_0, s)!$ means that $\delta(q_0, s)$ is defined.

A controlled generator over an alphabet $A$ is a triple $(G, A_c, \Gamma)$, where $G$ is a generator over $A$, $A_c \subseteq A$ is a set of controllable events, $A_u = A \setminus A_c$ is the set of uncontrollable events, and $\Gamma = \{\gamma \subseteq A : A_u \subseteq \gamma\}$ is the set of control patterns.

A natural projection $P : A^* \rightarrow B^*$, for $B \subseteq A$, is a homomorphism defined as $P(a) = \varepsilon$, for $a \in A \setminus B$, and $P(a) = a$, for $a \in B$. The inverse image of $P$, denoted by $P^{-1} : B^* \rightarrow \text{Pwr}(A^*)$ with $\text{Pwr}(A^*) = 2^{A^*}$ being the power set of $A^*$, is defined as $P^{-1}(v) = \{w \in A^* : P(w) = v\}$. These definitions can be extended to languages. A generator $G$ is said to be partially observed if only a proper subset of events $A_o \subset A$, called set of observable events, is observed. The partial observation is described by natural projection $O : A^* \rightarrow A^*_o$ defined as above with $B = A_o$.

A supervisor based on partial observations for a controlled generator $(G, A_c, \Gamma)$ is a map $S : O(L(G)) \rightarrow \Gamma$. The closed-loop system is denoted by $S/G$. The language generated by $S/G$, $L(S/G)$, is defined recursively as (1) $\varepsilon \in L(S/G)$ and, (2) for any $w \in L(S/G)$ and $a \in A$, $wa \in L(S/G)$ if and only if $wa \in L(G)$ and $a \in S(O(w))$.

Given a specification (language) $K \subseteq L(G)$, the aim of supervisory control under partial observations is to find a supervisor $S$ such that $L(S/G) = K$. The existence condition for such a supervisor is characterized by controllability and observability defined as follows. $K$ is controllable with respect to $L(G)$ and $A_u$ if $KA_u \cap L(G) \subseteq K$ [10]. $K$ is observable with respect to $L(G)$ and $A_u$ if $$(\forall w, w' \in K) O(w) = O(w') \Rightarrow (\forall a \in A)(wa \in K \land w'a \in L(G) \Rightarrow w'a \in K)$$ [13].

It is proved in [13] that there exists a supervisor that synthesizes $K$, that is, $L(S/G) = K$, if and only if $K$ is controllable with respect to $L(G)$ and $A_u$ and observable with respect to $L(G)$ and $A_o$.

For a generator $G$ and a projection $P$, $P(G)$ denotes the minimal generator such that $L(P(G)) = P(L(G))$. The reader is referred to [14] for a construction of $P(G)$. $P(G)$ is often called an observer of $G$.

In modular/coordination control, we consider local alphabets (event sets) $A_i, A_j, A_k \subseteq A$, we use $P^i_j$ to denote the projection from $(A_i \cup A_j)^*$ to $A_i^*$. If $A_i \cup A_j = A$, we simply write $P_i$.

The synchronous product of languages $L_i \subseteq A_i^*$, $i = 1, \ldots, n$, is defined as $\prod_{i=1}^n L_i = \cap_{i=1}^n P_i^{-1}(L_i) \subseteq A^*$, where $A = \cup_{i=1}^n A_i$ and $P_i : A^* \rightarrow A_i^*$ are projections to local alphabets. For the corresponding operation (also called synchronous product) in terms of generators $G_i$, it is known that $L\left(\prod_{i=1}^n G_i\right) = \prod_{i=1}^n L(G_i)$. 
We now consider control of modular DES with a global specification. The approach is based on the (relaxed) coordination control of [11], where conditional decomposability is used to bring the problem with global specification to the problems of local specification.

Consider generators $G'_1$ and $G'_2$ over the alphabets $A'_1$ and $A'_2$, respectively. Let $G = G'_1 \parallel G'_2$ and $A = A'_1 \cup A'_2$. Given a prefix-closed specification $K = K \subseteq L(G)$, $K$ is conditionally decomposable with respect to $A'_1$, $A'_2$, and $A'_k$ if $K = P_i(K) \parallel P_2(K)$, where $P_i : A^* \rightarrow (A'_i \cup A'_k)^*$, $i = 1, 2$.

The following algorithm [11] finds a coordinator $G'_k$ over $A'_k$ with $A'_1 \cap A'_2 \subseteq A'_k \subseteq A'_1 \cup A'_2 = A$ and $P_k$ projection to $A'_k$ such that (1) $G'_k = P_k(G'_1) \parallel P_k(G'_2)$, and (2) $K$ is conditionally decomposable with respect to $A'_1$, $A'_2$, and $A'_k$. Note that $G'_k = P_k(G'_1) \parallel P_k(G'_2)$ implies $G = G'_1 \parallel G'_2 = G'_1 \parallel G'_2 \parallel G'_k$.

Algorithm 1 (Construction of a Coordinator) Given $G'_1$ and $G'_2$ and $K \subseteq L(G)$, compute the event set $A'_k$ and the coordinator $G'_k$ as follows.

1. Let $A'_k = A'_1 \cap A'_2$ be the set of all shared events of the generators $G'_1$ and $G'_2$.
2. Extend the alphabet $A'_k$ so that $K$ becomes conditionally decomposable with respect to $A'_1$, $A'_2$, and $A'_k$. (see [10] for a polynomial algorithm.)
3. Define the coordinator $G'_k$ as $G'_k = P_k(G'_1) \parallel P_k(G'_2)$.

It is well known that the computation of a projected generator (observer) can be exponential in the worst case. However, it is also known that if the projection satisfies the observer property [13], then the projected generator is of the same order as the original generator. Therefore, one might want to extend the event set $A'_k$ further so that the projection $P_k : A^* \rightarrow A'_k^*$ satisfies the observer property [13].

Denote $G_i = G'_i \parallel G'_k$, $i = 1, 2$. Local supervisors $S_i$ operate on $G_i$ over alphabets $A_i = A'_i \cup A'_k$. In this paper, we assume that controllability and observability of events are consistent over local supervisors, that is, $S_i$ can control events in $A_{i,c} = A_i \cap A_k$ and observe events in $A_{i,o} = A_i \cap A_o$. Let $A_{i,u} = A_i \setminus A_{i,c}$. Local observation mapping is the projection $O_i : A_i^* \rightarrow A_{i,o}^*$. The relation among $O$, $P_i$, and $O_i$ are shown in Figure [1]. Local languages $P_i(K) \subseteq (A'_i \cup A'_k)^*$, $i = 1, 2$ are then used as specifications for local supervisors in the coordination control, given simply by supremal controllable sublanguages or controllable and observable/normal sublanguages of $P_i(K)$ with respect to $L(G_i)$, $A_{i,u}$ and $A_{i,o}$. In the next section we will investigate algebraic conditions under which this local (coordinated) control synthesis is as permissive as the least restrictive monolithic synthesis.
3 Comparison of Monolithic and Coordination Control

If $K$ is not controllable and observable, then we would like to find some sublanguage that is controllable and observable and we would like to make such a sublanguage as large as possible. Let

$$(K, L)^\uparrow$$

denote a sublanguage of $K$ which is either controllable (for $A_o = A$) or observable (for $A_c = A$) or both (in general) with respect to $L$. If $L$ is understood, then we use $K^\uparrow$ to denote $(K, L)^\uparrow$. Note that $(K, L)^\uparrow$ corresponds to a closed-loop language, because controllability and observability characterize closed-loops. If $(K, L)^\uparrow$ is the supremal sublanguage (in several cases listed below it exists), we use $(K, L)^\uparrow$ to denote $(K, L)^\uparrow$.

We recall that controllability is preserved under language unions and hence the supremal controllable sublanguage exists. On the other hand, observability is not preserved under language unions. Therefore, supremal observable sublanguages do not exist in general. However, if all unobservable events are uncontrollable, observability is equivalent to normality, which is defined as follows. $K$ is normal with respect to $L$ and $A_o$ if $O^{-1}O(K) \cap L \subseteq K$. Normality is preserved under language unions and hence the supremal normal sublanguage exists.

More recently, relative observability has been introduced in [2, 3] as a property weaker than normality but preserved under language unions, which is defined as follows. Let $K \subseteq C \subseteq L$. $K$ is C-observable with respect to $L$ and $A_o$ if $(\forall w \in K)(\forall w' \in C) O(w) = O(w') \Rightarrow (\forall a \in A)(wa \in K \land w'a \in L \Rightarrow w'a \in K)$. The supremal relatively observable sublanguage exists.

Example 2 Based on the above discussions, examples of $(K, L)^\uparrow$ include:

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1Since $K \subseteq O^{-1}O(K) \cap L$ is automatic, normality is equivalent to $O^{-1}O(K) \cap L = K$. 

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1. the supremal controllable sublanguage of \( K \), denoted by \((K,L)^\uparrow_{\text{c}}\) or \( K_{\text{c}} \),
2. the supremal normal sublanguage of \( K \), denoted by \((K,L)^\uparrow_{\text{n}}\) or \( K_{\text{n}} \),
3. the supremal controllable and normal sublanguage of \( K \), denoted by \((K,L)^{\uparrow_{\text{cn}}}\) or \( K_{\text{cn}} \),
4. the supremal \( K \)-observable sublanguage of \( K \) with respect to \( L \), denoted by \((K,L)^{\uparrow_{\text{cr}}}\) or \( K_{\text{cr}} \),
5. the supremal \( L \)-observable sublanguage of \( K \) with respect to \( L \), denoted by \((K,L)^{\uparrow_{\text{c}R}}\) or \( K_{\text{c}R} \),
6. the supremal controllable and \( K \)-observable sublanguage of \( K \) with respect to \( L \), denoted by \((K,L)^{\uparrow_{\text{c}R}^+}\) or \( K_{\text{c}R}^\uparrow \).

**Examples of** \((K,L)^{\uparrow_i}\) **include:**

1. a maximal observable sublanguage of \( K \), denoted by \((K,L)^{\uparrow_o}\) or \( K_{\uparrow_o} \),
2. a maximal controllable and observable sublanguage of \( K \), denoted by \((K,L)^{\uparrow_{\text{co}}}\) or \( K_{\uparrow_{\text{co}}} \),
3. a unique controllable and observable sublanguage of \( K \) introduced in [12], denoted by \((K,L)^{\uparrow_{\text{hl}}}\) or \( K_{\uparrow_{\text{hl}}} \).

Let us compare the computational complexity and performance of coordination control with those of monolithic control. It is well known that the computational complexity of most algorithms for partially observed DES is of the order \( O(2^{|K|}) \), where \(|K|\) denotes the number of states of the automaton generating the language \( K \) with the minimal number of states. To reduce the computational complexity, we use modular computation of supervisors as follows. Assume that the specification language is conditionally decomposable, that is, \( K = P_1(K) \| P_2(K) \). Denote

\[
K_1 = P_1(K), \quad K_2 = P_2(K), \\
L_1 = L(G_1), \quad L_2 = L(G_2).
\]

Note \( K = K_1 \| K_2 \) and \( L = L(G) = L_1 \| L_2 \). We compute \( K_i^{\uparrow_o} = (K_i,L_i)^{\uparrow_o} \), \( i = 1,2 \) and use coordination supervisors \( S_i \) such that \( L(S_i/G_i) = K_i^{\uparrow_o} \). We investigate the computational complexity and performance of \( S_1 \) and \( S_2 \) vs the monolithic supervisor \( S \) as follows.

**Computational complexity of** \( S_1 \) **and** \( S_2 \) **vs** \( S \): Computational complexity of \( S_1 \) and \( S_2 \) is \( O(2^{|K_1|} + 2^{|K_2|}) \) and computational complexity of \( S \) is \( O(2^{|K|}) \). If \(|K| = |K_1| \times |K_2| \) (by proper construction of \( G_k \)), then \( O(2^{|K_1|} + 2^{|K_2|}) \) is much smaller than \( O(2^{|K|}) \).

**Performance of** \( S_1 \) **and** \( S_2 \) **vs** \( S \): Since

\[
L(S_1/G_1) = K_1^{\uparrow_o} \quad \text{and} \quad L(S_2/G_2) = K_2^{\uparrow_o},
\]
the closed-loop system under coordination control is described by

\[ L(S_1/G_1) \parallel L(S_2/G_2) = K_1^\parallel K_2^\parallel. \]

For monolithic supervisor \( S \),

\[ L(S/G) = K^\parallel = (K_1 \parallel K_2)^\parallel. \]

We first show the following obvious result for the sake of completeness of the presentation. It states that both monolithic supervisor and coordination supervisor ensure the safety, that is, the language generated by the closed-loop system is within the specification language \( K \).

**Proposition 3 (Safety)** Given \( G_1, G_2, \) and \( K \subseteq L \). Let \( K = K_1 \parallel K_2 \) be conditionally decomposable, then

\[ L(S/G) \subseteq K \]
\[ L(S_1/G_1) \parallel L(S_2/G_2) \subseteq K. \]

**Proof 4** By the definition of the operation \((.)^\parallel\), \( K^\parallel \subseteq K, K_i^\parallel \subseteq K_i, i = 1, 2. \) Hence

\[ L(S/G) = K^\parallel \subseteq K \]
\[ L(S_1/G_1) \parallel L(S_2/G_2) = K_1^\parallel K_2^\parallel \subseteq K_1 \parallel K_2 = K. \]

To compare \( K_1^\parallel K_2^\parallel \) with \( K^\parallel = (K_1 \parallel K_2)^\parallel \), we note that under some very general conditions such as languages are prefix-closed, which we assume in this paper,

\[ (K_1, L_1)^\parallel (K_2, L_2)^\parallel \subseteq (K, L)^\parallel \]

is always true \[ -7 \]. Hence the key to the comparison is to find conditions under which

\[ (K, L)^\parallel \subseteq (K_1, L_1)^\parallel (K_2, L_2)^\parallel \]

is true. To this end, let us first define some properties of the operation \((.)^\parallel\) as follows.

**Definition 5 (Monotonicity)** The operation \((.)^\parallel\) is monotonically increasing if, for all \( K \subseteq L \) and \( K' \subseteq L \),

\[ K \subseteq K' \Rightarrow (K, L)^\parallel \subseteq (K', L)^\parallel. \]

**Definition 6 (Exchangeability)** Given \( K_i \subseteq L_i \subseteq A_i^* \), \( (P_i^{-1}(K_i))^\parallel \) is exchangeable with respect to \( P_i \) if

\[ (P_i^{-1}(K_i))^\parallel = P_i^{-1}(K_i^\parallel). \]

That is,

\[ (P_i^{-1}(K_i), P_i^{-1}(L_i))^\parallel = P_i^{-1}((K_i, L_i)^\parallel). \]
**Definition 7** (Distributivity) Given three languages $L$, $M$, and $K \subseteq L$, $(K, L)\hat{=}$ is distributable with respect to $M$ if

$$(K \cap M, L \cap M)\hat{=} = (K, L)\hat{=} \cap M.$$ 

We now compare the performance of $S_1$ and $S_2$ vs $S$.

**Theorem 8** (Comparison) Given $K_i \subseteq L_i \subseteq A_i^*$, $i = 1, 2$, $K = K_1||K_2$, and $L = L_1||L_2$. Assume that (1) $(\cdot)\hat{=}$ is monotonically increasing; (2) $(P_i^{-1}(K_i))\hat{=}$ is exchangeable with respect to $P_i$; and (3) $(P_1^{-1}(K_1), P_1^{-1}(L_1))\hat{=}$ is distributable with respect to $P_2^{-1}(L_2)$ and $(P_2^{-1}(K_2), P_2^{-1}(L_2))\hat{=}$ is distributable with respect to $P_1^{-1}(L_1)$. Then

$$(K, L)\hat{=} \subseteq (K_1, L_1)\hat{=}|(K_2, L_2)\hat{=}.$$ 

**Proof 9**

$$(K, L)\hat{=} = (K_1||K_2, L)\hat{=}$$

$$(K_1||K_2, L)\hat{=} \cap (K_1||K_2, L)\hat{=}$$

$$(K_1||L_2, L)\hat{=} \cap (L_1||K_2, L)\hat{=}$$

(by monotonicity)

$$= (P_1^{-1}(K_1) \cap P_2^{-1}(L_2), P_1^{-1}(L_1) \cap P_2^{-1}(L_2))\hat{=}$$

$$\cap (P_1^{-1}(L_1) \cap P_2^{-1}(K_2), P_1^{-1}(L_1) \cap P_2^{-1}(L_2))\hat{=}$$

(by distributivity)

$$= (P_1^{-1}(K_1), P_1^{-1}(L_1))\hat{=} \cap P_2^{-1}(L_2)$$

$$\cap (P_2^{-1}(K_2), P_2^{-1}(L_2))\hat{=} \cap P_1^{-1}(L_1)$$

(by exchangeability)

$$= (P_1^{-1}(K_1), P_1^{-1}(L_1))\hat{=} \cap (P_2^{-1}(K_2), P_2^{-1}(L_2))\hat{=}$$

(by exchangeability)

$$= (K_1, L_1)\hat{=} \cap (K_2, L_2)\hat{=}.$$ 

Theorem 8 provides three conditions on the closed-loops $(K, L)\hat{=}$ that jointly guarantee that local closed-loops do not yield smaller result than the global closed-loop. If $(K, L)\hat{=}$ is the supremal sublanguages of $K$ that satisfies the conditions in Proposition 10 below, then monotonicity is satisfied automatically.

**Proposition 10** Assume that the operation $(\cdot)\hat{=}$ satisfies the following two conditions.

1. $(K')\hat{=} = K' \
2. (\forall K' \subseteq K'')((K'')\hat{=} = K'' \Rightarrow K'' \subseteq (K')\hat{=})$

Then $(\cdot)\hat{=}$ is monotonically increasing, that is, for all $M \subseteq L$ and $M' \subseteq L$, $M \subseteq M' \Rightarrow M\hat{=} \subseteq (M')\hat{=}$.
Proof 11 For $M \subseteq M', M^\uparrow \subseteq M \subseteq M'$. Take $K'' = M^\uparrow$ and $K' = M'$ in Condition (2). Since $(M^\uparrow)^\uparrow = M^\uparrow$ (by Condition (1)), Condition (2) gives 

$$M^\uparrow \subseteq (M')^\uparrow.$$ 

Since any supremal sublanguage of $K$ discussed in Example 2 satisfies two conditions in Propositions 10, we have the following corollary.

Corollary 12 Any supremal sublanguage of $K$ discussed in Example 2 is monotonically increasing, that is, for all $K \subseteq L$ and $K' \subseteq L$, $K \subseteq K' \Rightarrow K^\uparrow \subseteq (K')^\uparrow$.

4 Distributed Computation of Supremal Controllable Sublanguages

Recall $K$ is controllable with respect to $L$ and $A_u$ if $KA_u \cap L \subseteq K$. For local languages, $K_i$ is controllable with respect to $L_i$ and $A_{i,u}$ if $K_iA_{i,u} \cap L_i \subseteq K_i$. The supremal control sublanguages are denote by $(K,L)^c$ and $(K_i,L_i)^c$ respectively. By Corollary 12 $(.)^c$ is monotonically increasing. We show that $(P_i^{-1}(K_i))^c$ is exchangeable in the following proposition.

Proposition 13 (Exchangeability of $(.)^c$)

Given $K_i \subseteq L_i \subseteq A_i^*$, $(P_i^{-1}(K_i))^c$ is exchangeable with respect to $P_i$, that is

$$(P_i^{-1}(K_i))^c = P_i^{-1}(K_i^c).$$

Or,

$$(P_i^{-1}(K_i), P_i^{-1}(L_i))^c = P_i^{-1}((K_i, L_i)^c).$$

Proof 14 Let the automata generating $K_i$ and $L_i$ be $H_i = (Q_H, A_i, \delta_H, q_0)$ and $G_i = (Q_i, A_i, \delta, q_0)$ respectively, with $H_i$ being a subautomaton of $G_i$, that is, $Q_H \subseteq Q$ and $\delta_H = \delta|_{Q_H}$. It is well-known that the subautomaton

$$H_i^c = (Q_H^c, A_i, \delta_H^c, q_0)$$

generating $K_i^c$ can be obtained from $H_i$ by removing the states from $Q_H$ that are co-accessible to $Q \setminus Q_H$ via uncontrollable events, that is,

$$Q_H^c = Q_H \setminus \{q \in Q_H : (\exists s \in A^*_i,u)\delta(q,s) \in Q \setminus Q_H\}$$

and $\delta_H^c = \delta|_{Q_H^c}$.

The automata for $P_i^{-1}(K_i)$ and $P_i^{-1}(L_i)$ can be obtained from $H_i$ and $G_i$ by adding self-loops of $A \setminus A_i$ to all states. Denote the resulting automata by $H_i = (Q_H, A, \delta_H, q_0)$ and $G_i = (Q_i, A, \delta, q_0)$ respectively. Since the only difference
between \( H_i \) and \( \tilde{H}_i \) (\( G_i \) and \( \tilde{G}_i \)) is the self-loops of \( A \setminus A_i \); the states that are co-accessible to \( Q \setminus Q_H \) via uncontrollable events in \( H_i \) and \( \tilde{H}_i \) are same. Hence,

\[
\tilde{H}^c_i = (Q^c_H, A, \tilde{\delta}^c_H, q_0),
\]

where \( \tilde{\delta}^c_H = \tilde{\delta}_H|_{Q^c_H} \). Therefore,

\[
L(\tilde{H}^c_i) = P^{-1}_i(L(H^c_i))
\]

\[
\Rightarrow (P^{-1}_i(K_i))^c = P^{-1}_i(K^c_i).
\]

The following proposition gives a sufficient condition for distributivity of \( (.)^c \).

**Proposition 15 (Distributivity of \( (.)^c \))**

Given three languages \( L, M, \) and \( K \subseteq L \), if \( M \) is controllable with respect to \( L \), then \( (K, L)^c \) is distributable with respect to \( M \), that is,

\[
(K \cap M)^c = K^c \cap M.
\]

Or,

\[
(K \cap M, L \cap M)^c = (K, L)^c \cap M.
\]

**Proof 16** \((\subseteq)\) Clearly,

\[
(K \cap M)^c \subseteq K \cap M \subseteq M.
\]

To prove \( (K \cap M)^c \subseteq K^c \), we need to show the following. (1) \( (K \cap M)^c \subseteq K \), which is clearly true. (2) \( (K \cap M)^c \) is controllable with respect to \( L \). To prove this, note that \( (K \cap M)^c \) is controllable with respect to \( L \cap M \). On the other hand, \( L \) is controllable with respect to itself, \( L \). By the assumption, \( M \) is controllable with respect to \( L \). Since all languages are closed, \( L \cap M \) is controllable with respect to \( L \). Hence, by the chain property of controllability\(^2\), \( (K \cap M)^c \) is controllable with respect to \( L \).

\((\subseteq)\) To prove \( K^c \cap M \subseteq (K \cap M)^c \), we need to show the following. (1) \( K^c \cap M \subseteq K \cap M \), which is clearly true. (2) \( K^c \cap M \) is controllable with respect to \( L \cap M \). Indeed, this is true because

\[
(K^c \cap M, A_u \cap L \cap M)
\]

\[
\subseteq (K^c)A_u \cap M \quad \subseteq (K^c)A_u \cap L \cap M
\]

\[
\subseteq K^c \quad \cap \quad (K^c \cap M)
\]

(because \( K^c \) is controllable with respect to \( L \))

\(^2\)If \( M_1 \) is controllable with respect to \( M_2 \) and \( M_2 \) is controllable with respect to \( M_3 \), then \( M_1 \) is controllable with respect to \( M_3 \).
From Proposition 15, we conclude that in order to use Theorem 8 for the supremal controllable sublanguage \((.)^\mathcal{C}\), we need that (1) \(P_i^{-1}(L_1)\) is controllable with respect to \(P_2^{-1}(L_2)\) and (2) \(P_2^{-1}(L_2)\) is controllable with respect to \(P_1^{-1}(L_1)\). These two conditions are equivalent to global mutual controllability (GMC) introduced in [6]. Let us recall the definition of GMC.

**Definition 17** The modular plant languages \(L_1\) and \(L_2\) are globally mutually controllable if \(\forall i, j \in \{1, 2\}, i \neq j,\)
\[
P_i^{-1}(L_i) A_{i,u} \cap P_j^{-1}(L_j) \subseteq P_i^{-1}(L_i).
\]

Note that in the definition of GMC, local uncontrollable events \(A_{i,u}\) are used. While in the definition of \(P_i^{-1}(L_i)\) being controllable with respect to \(P_j^{-1}(L_j)\), global uncontrollable events \(A_u\) are used, that is,
\[
P_i^{-1}(L_i) A_u \cap P_j^{-1}(L_j) \subseteq P_i^{-1}(L_i).
\]
However, it is proved in [6] that they are equivalent as re-stated in the following Proposition.

**Proposition 18** The modular plant languages \(L_1\) and \(L_2\) are globally mutually controllable if and only if (1) \(P_1^{-1}(L_1)\) is controllable with respect to \(P_2^{-1}(L_2)\) and \(A_u\) and (2) \(P_2^{-1}(L_2)\) is controllable with respect to \(P_1^{-1}(L_1)\) and \(A_u\).

Therefore, we have the following result for the supremal controllable sublanguage \((.)^\mathcal{C}\).

**Theorem 19** (Comparison for \((.)^\mathcal{C}\))

Given \(K_i \subseteq L_i \subseteq A_i^\bullet, i = 1, 2,\) and \(K = K_1 || K_2, L = L_1 || L_2.\) If \(L_1\) and \(L_2\) are globally mutually controllable, then
\[
K^\mathcal{C} \subseteq K_1^\mathcal{C} || K_2^\mathcal{C}.
\]
That is,
\[
(K, L)^\mathcal{C} \subseteq (K_1, L_1)^\mathcal{C} || (K_2, L_2)^\mathcal{C}.
\]

**Proof 20** By Corollary 12, \((.)^\mathcal{C}\) is monotonically increasing. By Proposition 13, \((P_i^{-1}(K_i))^\mathcal{C}\) is exchangeable with respect to \(P_i.\) By Propositions 15 and 18, \((P_i^{-1}(K_1), P_i^{-1}(L_1))^\mathcal{C}\) is distributable with respect to \(P_2^{-1}(L_2)\) and \((P_2^{-1}(K_2), P_2^{-1}(L_2))^\mathcal{C}\) is distributable with respect to \(P_1^{-1}(L_1).\) Therefore, by Theorem 8
\[
K^\mathcal{C} \subseteq K_1^\mathcal{C} || K_2^\mathcal{C}.
\]
Note that the above proof is based on arguments that do not depend on the particular property (in this case, controllability). Therefore, we can extend this
result for distributed computation of languages arising in supervisory control with partial observations in the next two sections.

In the literature, there exists a well known concept of mutual controllability (MC) \cite{14} that also ensures \((K_1, L_1)^c \parallel (K_2, L_2)^c = (K, L)^c\). In the rest of this section we compare GMC with MC. Let us recall the definition of MC.

**Definition 21** The modular plant languages \(L_1\) and \(L_2\) are mutually controllable if for all \(i, j \in \{1, 2\}, i \neq j\),

\[
L_i(A_{i,u} \cap A_j) \cap P_i(P_j^{-1}(L_j)) \subseteq L_i.
\]

To compare GMC with MC, we show that MC is equivalent to the following weakly globally mutual controllability (WGMC).

**Definition 22** The modular plant languages \(L_1\) and \(L_2\) are weakly globally mutually controllable if all \(i, j \in \{1, 2\}, i \neq j\),

\[
P^{-1}_i(L_i)(A_{i,u} \cap A_j) \cap P_j^{-1}(L_j) \subseteq P^{-1}_i(L_i).
\]

The following proposition states the relation between WGMC and MC.

**Proposition 23** Weak global mutual controllability is equivalent to mutual controllability.

**Proof 24** First we show that MC implies WGMC. Let MC be true, that is, for all \(i, j \in \{1, 2\}, i \neq j\),

\[
L_j(A_{j,u} \cap A_i) \cap P_j(P_i^{-1}(L_i)) \subseteq L_j.
\]

By applying the inverse projection \(P_j^{-1}\), we get, by monotonicity of \(P_j^{-1}\), that

\[
P_j^{-1}[L_j(A_{j,u} \cap A_i) \cap P_j(P_i^{-1}(L_i))] \subseteq P_j^{-1}(L_j).
\]

Because inverse projection preserves catenation and intersections, we obtain

\[
P^{-1}_j(L_j)P^{-1}_j(A_{j,u} \cap A_i) \cap P_j(P_i^{-1}(L_i)) \subseteq P^{-1}_j(L_j).
\]

Since \(A_{j,u} \cap A_i \subseteq P_j^{-1}(A_{j,u} \cap A_i)\) and \((P_j)^{-1}(L_i) \subseteq P_j^{-1}(P_j)^{-1}(L_i)\) we have

\[
P^{-1}_j(L_j)(A_{j,u} \cap A_i) \cap P_j^{-1}(L_i) \subseteq P_j^{-1}(L_j),
\]

Hence, WGMC is true.

The inverse implication can be proved as follows. Let WGMC be true, that is, for all \(i, j \in \{1, 2\}, i \neq j\),

\[
P_j^{-1}(L_j)(A_{j,u} \cap A_i) \cap P_i^{-1}(L_i) \subseteq P_j^{-1}(L_j).
\]

Note that
that at least one local supervisor does not have complete observations. We assume in this section we show how structural conditions proposed in Section 3 can be used for distributed computation of supremal normal sublanguages. We assume that at least one local supervisor does not have complete observations.

\[ P_j^{-1}(L_j)(A_{ju} \cap A_i) \cap P_i^{-1}(L_i) \subseteq P_j^{-1}(L_j)P_j^{-1}(A_{ju} \cap A_i) \cap P_i^{-1}(L_i) = L_j(A_{ju} \cap A_i) \parallel L_i. \]

WGMC implies that

\[ P_j P_j^{-1}(L_j)(A_{ju} \cap A_i) \cap P_i^{-1}(L_i) \subseteq P_j P_j^{-1}(L_j) = L_j. \]

We first show that \( \forall i, j \in \{1, 2\}, i \neq j \), \( P_j L_j(A_{ju} \cap A_i) \parallel L_i \subseteq L_j \) already implies MC.

Indeed, by conditional independence property we get that the last statement implies \( P_j L_j(A_{ju} \cap A_i) \parallel L_i = P_j L_j(A_{ju} \cap A_i) \cap P_i^{-1}P_j(L_i) = L_j(A_{ju} \cap A_i) \cap P_i^{-1}(L_i) \subseteq L_j \), which is MC.

It remains to show that the obviously strict inclusion of (1) becomes equality when \( P_j \) is applied to both sides. In fact,

\[ P_j P_j^{-1}(L_j)(A_{ju} \cap A_i) \cap P_i^{-1}(L_i) = P_j P_j^{-1}(L_j)P_j^{-1}((A_{ju} \cap A_i) \cap P_i^{-1}(L_i)). \]

The nontrivial inclusion is proven below. Let \( s \in P_j P_j^{-1}(L_j)P_j^{-1}((A_{ju} \cap A_i) \cap P_i^{-1}(L_i)) \). Then \( s = P_j(t) \) for some \( t \in P_j^{-1}(L_j)P_j^{-1}((A_{ju} \cap A_i) \cap P_i^{-1}(L_i)) \). Since \( P_j \circ P_j^{-1} \) is identity, this simply means that \( s = s'u \) for \( s' \in L_j \) and \( u \in A_{ju} \cap A_i \).

But then for any \( t' \in P_j^{-1}(s') \) we have that \( su = P_j(t'u) = P_j(t')P_j(u) = P_j(t'u)u \), hence \( s \in P_j P_j^{-1}(L_j)P_j^{-1}((A_{ju} \cap A_i) \cap P_i^{-1}(L_i)) \) as well. Therefore, we get from WGMC by applying \( P_j \) to both sides

\[ P_j P_j^{-1}(L_j)(A_{ju} \cap A_i) \cap P_i^{-1}(L_i) \subseteq P_j P_j^{-1}(L_j) = L_j, \]

also that \( P_j P_j^{-1}(L_j)(A_{ju} \cap A_i) \cap P_i^{-1}(L_i) \subseteq P_j P_j^{-1}(L_j), \forall i, j \in \{1, 2\}, i \neq j \), from which we have already derived above MC.

Comparing the definitions of GMC and WGMC, it is clear that GMC is stronger than WGMC, that is, GMC implies WGMC. We also have a counter example that shows GMC is strictly stronger than WGMC. By Proposition 23 GMC is strictly stronger than MC. The advantage of using GMC is that it is easier to check GMC than to check MC, because it uses only computationally cheap inverse projections, while MC uses both projections and inverse projections.

5 Distributed Computation of Supremal Normal Sublanguages

In this section we show how structural conditions proposed in Section 3 can be used for distributed computation of supremal normal sublanguages. We assume that at least one local supervisor does not have complete observations.
The local alphabets admit a partition into locally observable and locally unobservable event sets as specified in Section II. We recall that observability of events is consistent over local supervisors, which can also be stated as $A_{1,o} \cap A_2 = A_1 \cap A_{2,o}$.

Recall $K$ is normal with respect to $L$ and $A_i$ if $O^{-1}O(K) \cap L \subseteq K$. For local languages, $K_i$ is normal with respect to $L_i$ and $A_{i,o}$ if $O_{i}^{-1}O_{i}(K_i) \cap L_i \subseteq K_i$. The supremal normal sublanguages are denoted by $(K,L)^{\uparrow n}$ and $(K_i,L_i)^{\uparrow n}$ respectively. By Corollary 12, $(\cdot)^{\uparrow n}$ is monotonically increasing. We show that $(P_i^{-1}(K_i))^{\uparrow n}$ is exchangeable in the following proposition.

**Proposition 25** (Exchangeability of $(\cdot)^{\uparrow n}$) Given $K_i \subseteq L_i \subseteq A_i^*$, $(P_i^{-1}(K_i))^{\uparrow n}$ is exchangeable with respect to $P_i$, that is

$$(P_i^{-1}(K_i))^{\uparrow n} = P_i^{-1}(K_i^{\uparrow n}).$$

**Proof 26** Let the automata generating $K_i$ and $L_i$ be $H_i = (Q_H,A_i,\delta_H,q_0)$ and $G_i = (Q,A_i,\delta,q_0)$ respectively, with $H_i$ being a subautomaton of $G_i$. It can be proven from results in [1] that the subautomaton

$$H_i^{\uparrow n} = (Q_H^{\uparrow n},A_i,\delta_H^{\uparrow n},q_0)$$

generating $K_i^{\uparrow n}$ can be obtained from $H_i$ by removing the states from $Q_H$ that are indistinguishable from states in $Q \setminus Q_H$, that is,

$$Q_H^{\uparrow n} = Q_H \setminus \{ q \in Q_H : (\exists s,s' \in L_i) O_i(s) = Q_i(s') \land \delta(q_0,s) = q \land \delta(q_0,s') \in Q \setminus Q_H \}$$

and $\delta_H^{\uparrow n} = \delta_H|_{Q_H^{\uparrow n}}$.

The automata for $P_i^{-1}(K_i)$ and $P_i^{-1}(L_i)$ can be obtained from $H_i$ and $G_i$ by adding self-loops of $A \setminus A_i$ to all states. Denote the resulting automata by $\tilde{H}_i = (Q_H,A,\tilde{\delta}_H,q_0)$ and $\tilde{G}_i = (Q,A,\tilde{\delta},q_0)$ respectively. Since the only difference between $H_i$ and $\tilde{H}_i$ (and $G_i$ and $\tilde{G}_i$) is the self-loops of $A \setminus A_i$, the states that are are indistinguishable from states in $Q \setminus Q_H$ in $H_i$ and $\tilde{H}_i$ are same, because $(\forall s,s' \in P_i^{-1}(L_i))O(s) = Q(s') \Leftrightarrow O_i(P_i(s)) = O_i(P_i(s'))$. Hence,

$$\tilde{H}_i^{\uparrow n} = (Q_H^{\uparrow n},A,\tilde{\delta}_H^{\uparrow n},q_0),$$

where $\tilde{\delta}_H^{\uparrow n} = \tilde{\delta}_H|_{Q_H^{\uparrow n}}$. Therefore,

$$L(\tilde{H}_i^{\uparrow n}) = P_i^{-1}(L(\tilde{H}_i^{\uparrow n}))$$

$$\Rightarrow (P_i^{-1}(K_i))^{\uparrow n} = P_i^{-1}(K_i^{\uparrow n}).$$

The following proposition gives a sufficient condition for distributivity of $(\cdot)^{\uparrow n}$. 

**Proposition 27** (Distributivity of $\ast^n$) Given three languages $L$, $M$, and $K \subseteq L$, if $M$ is normal with respect to $L$, then $(K, L)^n$ is distributable with respect to $M$, that is,

$$(K \cap M)^n = K^n \cap M.$$ 

Or,

$$(K \cap M, L \cap M)^n = (K, L)^n \cap M.$$ 

**Proof 28** ($\subseteq$) Clearly,

$$(K \cap M)^n \subseteq K \cap M \subseteq M.$$ 

To prove $(K \cap M)^n \subseteq K \cap M \subseteq M$, we need to show the following. (1) $(K \cap M)^n \subseteq K \cap M$, which is clearly true. (2) $(K \cap M)^n$ is normal with respect to $L \cap M$. Let us show that $(K \cap M)^n$ is normal with respect to $L \cap M$ as follows.

$$
O^{-1}O(L \cap M) \cap L \\
\subseteq O^{-1}O(L) \cap O^{-1}O(M) \cap L \\
= O^{-1}O(M) \cap L \subseteq M \cap L
$$

(because $M$ is normal with respect to $L$)

By the chain property of normality[^1], $(K \cap M)^n$ is normal with respect to $L$. ($\subseteq$) To prove $K^n \cap M \subseteq (K \cap M)^n$, we need to show the following. (1) $K^n \cap M \subseteq (K \cap M)^n$, which is clearly true. (2) $K^n \cap M$ is normal with respect to $L \cap M$. Indeed, this is true because

$$
O^{-1}O(K \cap M) \cap L \cap M \\
\subseteq O^{-1}O(K^n) \cap O^{-1}O(M) \cap L \cap M \\
= O^{-1}O(K^n) \cap L \cap M \\
\subseteq K^n \cap M
$$

(because $K^n$ is normal with respect to $L$)

From Proposition 27, we conclude that in order to use Theorem 8 for the supremal normal sublanguage $(\ast)^n$, it is required that (1) $P_1^{-1}(L_1)$ is normal with respect to $P_2^{-1}(L_2)$ and (2) $P_2^{-1}(L_2)$ is normal with respect to $P_1^{-1}(L_1)$. We call this requirement globally mutual normality (GMN), which is defined as follows.

**Definition 29** The modular plant languages $L_1$ and $L_2$ are globally mutually normal if for all $i, j \in \{1, 2\}$, $i \neq j$,

$$O^{-1}O(P_i^{-1}(L_i)) \cap P_j^{-1}(L_j) \subseteq P_i^{-1}(L_i).$$

We then have the following result for the supremal normal sublanguage $(\ast)^n$.

[^1]: If $M_1$ is normal with respect to $M_2$ and $M_2$ is normal with respect to $M_3$, then $M_1$ is normal with respect to $M_3$. [^2]
Theorem 30 (Comparison for $(\cdot)^n$). Given $K_i \subseteq L_i \subseteq A^*$, $i = 1, 2$, and $K = K_1 \parallel K_2$, $L = L_1 \parallel L_2$. If $L_1$ and $L_2$ are globally mutually normal, then

$$K^\parallel L \subseteq K_1^\parallel L_2^\parallel n.$$  

That is,

$$(K, L)^\parallel n \subseteq (K_1, L_1)^\parallel n \parallel (K_2, L_2)^\parallel n.$$  

Proof 31 By Corollary 12 $(\cdot)^n$ is monotonically increasing. By Proposition 25, $(P_1^{-1}(K_i))^\parallel n$ is exchangeable with respect to $P_i$. Since $L_1$ and $L_2$ are globally mutually normal, by Propositions 27, $(P_1^{-1}(K_1), P_1^{-1}(L_1))^\parallel n$ is distributable with respect to $P_2^{-1}(L_2)$ and $(P_2^{-1}(K_2), P_2^{-1}(L_2))^\parallel n$ is distributable with respect to $P_1^{-1}(L_1)$. Thus, by Theorem 8

$$K^\parallel L \subseteq K_1^\parallel L_2^\parallel n.$$  

To compare the above result with existing results in the literature, we show that GMN is equivalent to mutual normality (MN) introduced in [6]. Let us recall the definition of MN and compare it with GMN.

Definition 32 The modular plant languages $L_1$ and $L_2$ are mutually normal if for all $i, j \in \{1, 2\}$, $i \neq j$,

$$O^{-1}_i O_i(L_i) \cap P_i(P_j^{-1}(L_j)) \subseteq L_i.$$  

Proposition 33 The modular plant languages $L_1$ and $L_2$ are globally mutually normal if and only if they are mutually normal.

Proof 34 We first prove

$$O^{-1} O_i(P_i^{-1}(L_i)) = P_i^{-1}(O^{-1}_i O_i(L_i)).$$  

(2)

Let $G_i = (Q, A_i, \delta, q_0)$ be the automata generating $L_i$ and $G_{i, \text{obs}} = (X, A_{i, o}, \xi, x_0)$ be the observer generating $O_i(L_i)$. By adding self-loops of $A \setminus A_{i, o}$ to $G_{i, \text{obs}}$, we obtain the automaton $\hat{G}_{i, \text{obs}} = (X, A, \xi, x_0)$. Clearly $L(\hat{G}_{i, \text{obs}}) = P_i^{-1}(O^{-1}_i O_i(L_i))$. On the other hand, adding self-loops of $A \setminus A_i$ and then building observer for $O$ is same as building observer and then adding self-loops. Hence, $L(\hat{G}_{i, \text{obs}}) = O^{-1} O_i(P_i^{-1}(L_i))$ as well.

Let us now prove that MN implies GMN as follows.

$$O^{-1}_i O_i(L_i) \cap P_i(P_j^{-1}(L_j)) \subseteq L_i$$  

$\Rightarrow P_i^{-1}(O^{-1}_i O_i(L_i) \cap P_i(P_j^{-1}(L_j))) \subseteq P_i^{-1}(L_i)$  

$\Leftrightarrow P_i^{-1}(O^{-1}_i O_i(L_i)) \cap P_i^{-1}(P_i(P_j^{-1}(L_j))) \subseteq P_i^{-1}(L_i)$  

(because $P_i^{-1}$ preserves intersections)  

$\Leftrightarrow P_i^{-1}(O^{-1}_i O_i(L_i)) \cap P_j^{-1}(L_j) \subseteq P_i^{-1}(L_i)$  

$\Rightarrow O^{-1} O_i(P_i^{-1}(L_i)) \cap P_j^{-1}(L_j) \subseteq P_i^{-1}(L_i)$  

(by Equation 2)
To prove that GMN implies MN, we do the following.

\[
O^{-1}O(P_i^{-1}(L_i)) \cap P_j^{-1}(L_j) \subseteq P_i^{-1}(L_i)
\]
\[
\iff P_i^{-1}(O^{-1}O(P_i^{-1}(L_i))) \cap P_j^{-1}(L_j) \subseteq P_i^{-1}(L_i)
\]
(by Equation (3))
\[
\iff O_i^{-1}O_i(L_i) \parallel L_j \subseteq P_i^{-1}(L_i)
\]
\[
\implies P_i(O^{-1}O_i(L_i) \parallel P_i^{-1}(L_i)) \subseteq P_i(P_i^{-1}(L_i)) = L_i
\]

It is proved in [5] that, if the event set which \( P_i \) projects to \( A_i \) (in the current case) contains the shared events of the shuffle \( \parallel \) \( (A_i \cap A_j) \) (in the current case), then \( P_i \) can be distributed over \( \parallel \). Hence

\[
P_i(O_i^{-1}O_i(L_i) \parallel P_i(L_i)) \subseteq L_i
\]
\[
\iff P_i(O_i^{-1}O_i(L_i)) \parallel P_i(L_i) \subseteq L_i
\]
\[
\iff O_i^{-1}O_i(L_i) \parallel P_i(L_i) \subseteq L_i
\]
\[
\iff O_i^{-1}O_i(L_i) \cap P_i(P_i^{-1}(L_i)) \subseteq L_i.
\]

We conclude that mutual normality and global mutual normality are equivalent.

6 Distributed Computation of Supremal Relatively Observable Sublanguages

In this section, we investigate relative observability. We first take \( C = K \). Clearly, \( K \) is observable if and only if \( K \) is \( K \)-observable. However, we emphasize that relative observability is preserved by language unions. Hence, for any two sublanguages \( K_1 \) and \( K_2 \) of \( K \), if both \( K_1 \) and \( K_2 \) are \( K \)-observable, then \( K_1 \cup K_2 \) is also \( K \)-observable. Therefore, the supremal \( K \)-observable sublanguage of \( K \) with respect to \( L \) and \( A_o \) exists and is denoted by \( (K, L)^{\uparrow r} \). For local languages \( K_i \) and \( L_i \), the supremal \( K_i \)-observable sublanguage of \( K_i \) with respect to \( L_i \) and \( A_{i,o} \) is denoted by \( (K_i, L_i)^{\uparrow r} \). In the literature, there exist several different algorithms for computation of supremal relatively observable sublanguages, see [3], for example.

By Corollary 12, \((.)^{\uparrow r}\) is monotonically increasing. We show that \((P_i^{-1}(K_i))^{\uparrow r}\) is exchangeable.

**Proposition 35** (Exchangeability of \((.)^{\uparrow r}\)) Given \( K_i \subseteq L_i \subseteq A_i^* \), we have

\[
(P_i^{-1}(K_i))^{\uparrow r} = P_i^{-1}(K_i^{\uparrow r}).
\]

Or,

\[
(P_i^{-1}(K_i), P_i^{-1}(L_i))^{\uparrow r} = P_i^{-1}((K_i, L_i)^{\uparrow r}).
\]
Proof 36 Let the automata generating $K_i$ and $L_i$ be $H_i = (Q_H, A_i, \delta_H, q_0)$ and $G_i = (Q, A_i, \delta, q_0)$ respectively, with $H_i$ being a subautomaton of $G_i$. Based on these two automata, an algorithm is given in [3] to construct

$$H_i^{tr} = (Q_H^{tr}, A_i, \delta_H^{tr}, q_0)$$

that generates $K_i^{tr}$. The algorithm can be viewed as a refinement of the algorithm that constructs an automaton $H_i^{trn}$ generating $K_i^{trn}$ outlined in the proof of Proposition 28. It is clear from the algorithm that the addition of self-loops of events not in $A_i$ to all states, does not change the result.

The automata for $P_i^{-1}(K_i)$ and $P_i^{-1}(L_i)$ can be obtained from $H_i$ and $G_i$ by adding self-loops of $A \setminus A_i$ to all states. Denote the resulting automata by $\tilde{H}_i = (Q_H, A, \delta_H, q_0)$ and $\tilde{G}_i = (Q, A, \delta, q_0)$ respectively. Since the only difference between $H_i$ and $\tilde{H}_i$ ($G_i$ and $\tilde{G}_i$) is the self-loops of $A \setminus A_i$, the automaton generating $(P_i^{-1}(K_i))^{tr}$ can be obtained from $H_i^{tr}$ by adding the self-loops of $A \setminus A_i$. Denote the resulting automaton by

$$\tilde{H}_i^{tr} = (Q_H^{tr}, A, \delta_H^{tr}, q_0).$$

Then

$$L(\tilde{H}_i^{tr}) = P_i^{-1}(L(H_i^{tr}))$$

$$\Rightarrow (P_i^{-1}(K_i))^{tr} = P_i^{-1}(K_i^{tr}).$$

We now prove the following proposition.

Proposition 37 (Distributivity of $(\cdot)^{tr}$) Given three languages $L$, $M$, and $K \subseteq L$, if $M$ is $K$-observable with respect to $L$, then $(K, L)^{tr}$ is distributable with respect to $M$, i.e.

$$(K \cap M)^{tr} = K^{tr} \cap M.$$ Or,

$$(K \cap M, L \cap M)^{tr} = (K, L)^{tr} \cap M.$$ Proof 38 It is not difficult to see that $M$ is $K$-observable with respect to $L$ if and only if

$$\forall w \in A^* \forall w' \in O^{-1}O(w) \forall a \in A \wedge w \in M \Rightarrow w'a \in L \Rightarrow w'a \in M.$$ (3)

Similarly, since $(K \cap M)^{tr}$ is $(K \cap M)$-observable with respect to $(L \cap M)$,

$$\forall w \in A^* \forall w' \in O^{-1}O(w) \forall a \in A \wedge w \in (K \cap M)^{tr} \Rightarrow w'a \in (K \cap M)^{tr}.$$ (4)

Since $K^{tr}$ is $K$-observable with respect to $L$,  

\footnote{In the definition of relative observability [3], it is assumed that $M \subseteq K \subseteq L$. This is needed in order to show $M$ is $K$-observable with respect to $L$ implies $M$ is observable with respect to $L$. This implication is not needed for this proposition. Therefore, we relax the assumption of $M \subseteq K \subseteq L$.}
We now prove the result.

(\subseteq) Clearly, 

\((K \cap M)^{\uparrow_r} \subseteq K \cap M \subseteq M.\)

To prove \((K \cap M)^{\uparrow_r} \subseteq K^{\uparrow_r},\) we need to show that 

(1) \((K \cap M)^{\uparrow_r} \subseteq K\) (obvious) and 

(2) \((K \cap M)^{\uparrow_r}\) is \(K\)-observable with respect to \(L,\) that is, 

\((\forall w \in A^*)(\forall w' \in O^{-1}O(w))(\forall a \in A)\)

\(wa \in (K \cap M)^{\uparrow_r} \land w' \in K \land w'a \in L \Rightarrow w'a \in (K \cap M)^{\uparrow_r}.\)

This is true because

\(wa \in (K \cap M)^{\uparrow_r} \land w' \in K \land w'a \in L\)

\(\Rightarrow wa \in (K \cap M)^{\uparrow_r} \land w' \in K \land w'a \in L \land wa \in M\)

\(\Rightarrow wa \in (K \cap M)^{\uparrow_r} \land w' \in K \land w'a \in L \land w'a \in M\)

(by Equation (3))

\(\Rightarrow wa \in (K \cap M)^{\uparrow_r} \land w' \in (K \cap M) \land w'a \in (L \cap M)\)

\(\Rightarrow w'a \in (K \cap M)^{\uparrow_r}\) (by Equation (4)).

(\supseteq) To prove \(K^{\uparrow_r} \cap M \subseteq (K \cap M)^{\uparrow_r},\) we need to show the following. (1) \(K^{\uparrow_r} \cap M \subseteq K \cap M,\) which is clearly true. (2) \(K^{\uparrow_r} \cap M\) is \((K \cap M)\)-observable with respect to \((L \cap M),\) that is, 

\((\forall w \in A^*)(\forall w' \in O^{-1}O(w))(\forall a \in A)\)

\(wa \in (K^{\uparrow_r} \cap M) \land w' \in (K \cap M)\)

\(\land w'a \in (L \cap M) \Rightarrow w'a \in (K^{\uparrow_r} \cap M).\)

This is true because

\(wa \in (K^{\uparrow_r} \cap M) \land w' \in (K \cap M) \land w'a \in (L \cap M)\)

\(\Rightarrow (wa \in K^{\uparrow_r} \land w' \in K \land w'a \in L) \land w'a \in M\)

\(\Rightarrow wa' \in K^{\uparrow_r} \land w'a \in M\) (by Equation (4))

\(\Rightarrow wa' \in (K^{\uparrow_r} \cap M).\)

From Proposition 37 we conclude that in order to use Theorem 8 for the suprema-l relatively observable sublanguage \((\cdot)^{\uparrow_r},\) it is required that (1) \(P_{\uparrow_r}^{-1}(L_1)\) is \(P_{\uparrow_r}^{-1}(K_2)\)-observable with respect to \(P_{\uparrow_r}^{-1}(L_2)\) and (2) \(P_{\uparrow_r}^{-1}(L_2)\) is \(P_{\uparrow_r}^{-1}(K_1)\)-observable with respect to \(P_{\uparrow_r}^{-1}(L_1)\).
**Definition 39** The modular plant languages $L_1$ and $L_2$ are globally mutually $K$-observable if for all $i, j \in \{1, 2\}$, $i \neq j$,

$$(\forall w \in A^*) (\forall w' \in O^{-1}O(w)) (\forall a \in A)$$

$$wa \in P_i^{-1}(L_i) \land w' \in P_j^{-1}(L_j) \land w'a \in P_j^{-1}(L_j)$$

$$\Rightarrow w'a \in P_i^{-1}(L_i).$$

We then have the following result.

**Theorem 40** (Comparison for $(\cdot)^{tr}$) Let $K_i \subseteq L_i \subseteq A_i^*$, $i = 1, 2$, $K = K_1 \| K_2$, and $L = L_1 \| L_2$. If $L_1$ and $L_2$ are globally mutually $K$-observable then

$$K^{tr} \subseteq K_1^{tr} \| K_2^{tr}.$$  

That is,

$$(K, L)^{tr} \subseteq (K_1, L_1)^{tr} \| (K_2, L_2)^{tr}.$$  

**Proof 41** By Corollary 12 $(\cdot)^{tr}$ is monotonically increasing. By Proposition 37 $(P_i^{-1}(K_i))^{tr}$ is exchangeable with respect to $P_i$. Since $L_1$ and $L_2$ are globally mutually $K$-observable, by Proposition 37 $(P_i^{-1}(K_i), P_i^{-1}(L_i))^{tr}$ is distributable with respect to $P_i^{tr}(L_2)$ and $(P_2^{-1}(K_2), P_2^{-1}(L_2))^{tr}$ is distributable with respect to $P_1^{-1}(L_1)$. Therefore, by Theorem 3

$$K^{tr} \subseteq K_1^{tr} \| K_2^{tr}.$$  

We observe that unlike mutual normality for computation of supremal normal sublanguages, mutual $K$-observability depends on the specification $K$. We might want to replace $K$ by a larger language $C$ such that $M$ becomes $C$-observable with respect to $L$. If we insists on having structural conditions that depend on the the plant only, we need to consider a stronger version of relative ($C$-)observability, namely with $C = L$ instead of $C = K$. We recall that $L$-observability of $K$ is still weaker than normality (because in case of normality the requirement for unobservable $a$-steps is much stronger). We denote by $(K, L)^{tr}$ the supremal sublanguage of $K$ that is $L$-observable with respect to $L$. Since $L$-observability is weaker than normality, $(K, L)^{tr} \supseteq (K, L)^{tr}.$

Exchangeability of $(\cdot)^{tr}$ can be established using the same arguments as Proposition 37. Distributivity of $(\cdot)^{tr}$ is proved below.

**Proposition 42** (Distributivity of $(\cdot)^{tr}$) Let $K, L, M \subseteq A^*$ and $K \subseteq L$. If $M$ is $L$-observable with respect to $L$, then $(K, L)^{tr}$ is distributable with respect to $M$, that is,

$$(K \cap M)^{tr} = K^{tr} \cap M.$$  

Or,

$$(K \cap M, L \cap M)^{tr} = (K, L)^{tr} \cap M.$$
Proof 43 Since $L$ is closed, $w' \in L \wedge w'a \in L \iff w'a \in L$. Hence, $M$ is $L$-observable with respect to $L$ if and only if
\[(\forall w \in A^*)(\forall w' \in O^{-1}O(w))(\forall a \in A)\]
wa \in M \wedge w'a \in L \Rightarrow w'a \in M. \tag{6}
For the same reason, $(K \cap M)^{\uparrow R}$ is $(L \cap M)$-observable with respect to $(L \cap M)$ if and only if
\[(\forall w \in A^*)(\forall w' \in O^{-1}O(w))(\forall a \in A)\]
wa \in (K \cap M)^{\uparrow R} \wedge w'a \in (L \cap M) \Rightarrow w'a \in (K \cap M)^{\uparrow R}. \tag{7}
\]
$K \uparrow R$ is $L$-observable with respect to $L$ if and only if
\[(\forall w \in A^*)(\forall w' \in O^{-1}O(w))(\forall a \in A)\]
wa \in K \uparrow R \wedge w'a \in L \Rightarrow w'a \in K \uparrow R. \tag{8}
\]
We now prove the result.
\[\subseteq: \] Clearly,
\[(K \cap M)^{\uparrow R} \subseteq K \cap M \subseteq M.\]
To prove $(K \cap M)^{\uparrow R} \subseteq K \uparrow R$, we need to show the following. (1) $(K \cap M)^{\uparrow R} \subseteq K$, which is clearly true. (2) $(K \cap M)^{\uparrow R}$ is $L$-observable with respect to $L$, that is,
\[(\forall w \in A^*)(\forall w' \in O^{-1}O(w))(\forall a \in A)\]
wa \in (K \cap M)^{\uparrow R} \wedge w'a \in L \Rightarrow w'a \in (K \cap M)^{\uparrow R}.
This holds because
wa \in (K \cap M)^{\uparrow R} \wedge w'a \in L \Rightarrow wa \in (K \cap M)^{\uparrow R} \wedge w'a \in L \wedge wa \in M \Rightarrow wa \in (K \cap M)^{\uparrow R} \wedge w'a \in L \wedge w'a \in M \tag{by Equation 6} \Rightarrow w'a \in (K \cap M)^{\uparrow R} \tag{by Equation 7}.
\[\supseteq: \] To prove $K \uparrow R \cap M \subseteq (K \cap M)^{\uparrow R}$, we need to show that (1) $K \uparrow R \cap M \subseteq K \cap M$ (clearly true) and (2) $K \uparrow R \cap M$ is $(L \cap M)$-observable with respect to $(L \cap M)$, i.e.,
\[(\forall w \in A^*)(\forall w' \in O^{-1}O(w))(\forall a \in A)\]
wa \in (K \uparrow R \cap M) \wedge w'a \in (L \cap M) \Rightarrow w'a \in (K \uparrow R \cap M).
This is true because
wa \in (K \uparrow R \cap M) \wedge w'a \in (L \cap M) \Rightarrow wa \in K \uparrow R \wedge w'a \in L \wedge w'a \in M \Rightarrow wa' \in K \uparrow R \wedge w'a \in M \tag{by Equation 8} \Rightarrow wa' \in (K \uparrow R \cap M).
Theorem 44 follows from Theorem 8 and Proposition 42.

**Theorem 44 (Comparison for $(\cdot)^R$)** Given $K_i \subseteq L_i \subseteq A_i^*$, $i = 1, 2$, and $K = K_1 || K_2$, $L = L_1 || L_2$. If $P_{1}^{-1}(L_1)$ is $P_{2}^{-1}(L_2)$-observable with respect to $P_{2}^{-1}(L_2)$ and $P_{2}^{-1}(L_2)$ is $P_{1}^{-1}(L_1)$-observable with respect to $P_{1}^{-1}(L_1)$, then

$$K^R \subseteq K_1^R || K_2^R.$$ 

That is,

$$(K, L)^R \subseteq (K_1, L_1)^R || (K_2, L_2)^R.$$ 

Finally we show an example of local plants that satisfy the condition of Theorem 44 but are not mutually normal.

**Example 45** Let $A_1 = \{a, b_1, \tau\}$, $A_2 = \{a, b_2, \tau\}$, $A_o = \{a, b_1, b_2\}$, and $A_{uo} = A \setminus A_o$. Consider local plant languages $L_1$ and $L_2$ on figure 2.

Then $L_1$ is not normal with respect to $P_{1}P_{2}^{-1}(L_2)$ because $a\tau \in O_{1}^{-1}O_{1}(L_1) \cap P_{1}P_{2}^{-1}(L_2)$, but $a\tau \notin L_1$. However, $P_{1}^{-1}(L_1)$ is $P_{2}^{-1}(L_2)$-observable with respect to $P_{2}^{-1}(L_2)$ and $P_{2}^{-1}(L_2)$ is $P_{1}^{-1}(L_1)$-observable with respect to $P_{1}^{-1}(L_1)$. This shows that not only supervisors based on $(\cdot)^R$ are more permissive than those based on $(\cdot)^n$, but sufficient condition for locally computing the former is strictly weaker than mutual normality.

7 Concluding remarks

We have studied distributed computation of supremal sublanguages in modular (and more generally coordination) control of large DES. We have presented a new approach based on three algebraic conditions that are key for modular control to be as permissive as monolithic control. Sufficient conditions for maximal permissiveness of modular/coordination control are formulated in terms of global mutual normality and relative observability.

In a future research we plan to extend these results to distributed computation of supremal languages with respect to other properties closed under unions such as opacity.
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