Supplementary Materials for

*Anyonic-parity-time symmetry in complex-coupled lasers*

Geva Arwas *et al.*

Corresponding author: Geva Arwas, arwasgeva@gmail.com

*Sci. Adv.* 8, eabm7454 (2022)
DOI: 10.1126/sciadv.abm7454

This PDF file includes:

- Supplementary Text
- Figs. S1 to S8
- Tables S1 and S2
- References
The detailed experimental setup is presented in Fig. S1. The detailed experimental arrangement of the digital degenerate ring cavity laser (DDRCL) is schematically presented in Fig. S1(a). The DDRCL includes a gain medium, two 4f telescopes with one common lens, a reflective phase only spatial light modulator (SLM), two retroreflectors and pentaprism-like 90 degree reflector (all from high reflectivity mirrors), two polarizing beam splitters (PBS), two half-wave plates ($\lambda/2$) and a Faraday rotator.

The laser gain medium was a 1.1% doped Nd-YAG rod of 10-mm diameter and 11-cm length. For quasi-CW operation, the gain medium was pumped above threshold by a 200$\mu$s pulsed xenon flash lamp operating at 1800-1950V and a repetition rate of 1 Hz to avoid thermal lensing. Each 4f telescope consists of two plano-convex lenses, with diameters of 50.8mm and focal lengths of $f_1 = 750mm$ and $f_1 = 500mm$ at the lasing wavelength of 1064nm. The SLM was Meadowlark (liquid crystal on silicon (LCOS)) with a zero order diffraction efficiency of 88%, an area of 17.6mm by 10.7mm, 1920 by 1152 resolution, 9.2$\mu$m pixel size, and a high damage threshold (200 $W/cm^2$).

In the DDRCL, each of the two 4f telescopes has one lens $f_1$ and a common lens $f_2$. The first telescope images the field distribution at the center of the gain medium onto the SLM where the reflectivity of each effective pixel is controlled (52). The second telescope, images the field distribution at the SLM back onto the gain medium. The coupling is done by moving RR1 to the left and distorting the imaging condition of the second telescope. Since the SLM operates on axis and by reflection on horizontal polarized light, half of the ring degenerate cavity was designed as a twisted-mode (53) linear degenerate cavity (54) and the other half as regular ring cavity laser. The two halves are connected by PBS1, which separates the two counter-propagating beams into two different cross-polarized paths. A large aperture Faraday rotator together with a half-wave plate (HWP) at 22.5° and another PBS2 (which also serves as ~ 5% output coupler) enforce unidirectional operation of the DDRCL. A 90° reflector flips left and right areas of the beam. A second HWP at 45° rotates the polarization from vertical to horizontal to pass through PBS1.

The detection arrangement is shown in Fig. S1(b). It includes a CMOS camera, lenses, beam splitters and a pinhole to form a two channels interferometer. The first interferometer channel images the near field distribution from the DDRCL onto the camera with an 8-f telescope. In the second interferometer channel, one of the lasers is selected by the pinhole and expanded by an additional lens to serve as a reference beam. The light distributions from the two channels are then combined by a beam splitter, to interfere on the camera with a small relative angle.

The local reflectivity magnitude of the SLM is determined by local phase differences between adjacent pixels and affects the amount of light diffracted out of the cavity. The local reflectivity phase is determined by the local average phase of the adjacent pixels (52). For example, adjacent pixels with phases of [0,0] will result in high reflectivity and 0 phase, whereas adjacent pixels with phases of [0, $\pi$] will result in no reflectivity and $\pi/2$ phase. The reflectivity pattern can form arbitrary loss and phase distribution, and it is used to form the lasers, and to add relative loss and detuning between them.
**Diffractive coupling**

The coupling between the lasers was achieved by translating RR1 (see Fig. S1) to the left by 5cm such that the distance between the two $f_1$ lenses was increased by $z = 10cm$. As a result, the perfect imaging condition of the 8f cavity was distorted such that a portion of the light from each laser leaked to the neighboring laser. The complex-valued coupling coefficient of lasers $i$ and $j$ is calculated by the following normalized overlap integral

$$\kappa_{ij} = \frac{\langle E_i^F | E_i^I \rangle}{\langle E_i^F | E_i^I \rangle},$$

where $E_i^I$ is the initial field of laser $i$, and $E_i^F$ is the field of the same laser after propagating a distance $z$. For two Gaussian lasers with waist $w_0$, separated by distance $r_{ij}$, the result is (55)

$$\kappa_{ij}(r_{ij}, z) = \frac{1}{1 + \frac{2iz\lambda}{\pi w_0^2}} \exp \left( - \frac{2r_{ij}^2}{w_0^2 - \frac{2iz\lambda}{\pi}} \right).$$

A typical intensity distribution of one laser is shown in Fig. S2(a). A corresponding cross-section of the intensity distribution and a Gaussian fit are shown in Fig. S2(b). As evident, while the Gaussian fit is good in the central part of the beam, there are significant deviations at the tails. Since the coupling is obtained by the overlap of the tail of one laser with the mode of the other laser, we cannot use the coupling coefficient described by equation above, which is the overlap integral of two Gaussian beams. Instead, we find the coupling coefficient experimentally.

**Relative detuning control**

As noted above, the diameter of the lasers and the distance between them is controlled by the SLM. The phase of each pixel on the SLM can take any value between 0 and $2\pi$ with 8 bit resolution. For identical lasers (not detuned, and without relative loss), the reflective spots are defined by circular areas on the SLM with uniform $\pi$ phase. The circular spots are confined by non-reflective area that is obtained by pixels stripes with alternating phases of $\pi/2$ and $3\pi/2$, as shown in Fig. S3 (the width of each stripe is two pixels).

The frequency detuning between the lasers is achieved by changing the phase of the reflective spots in a symmetric way. The relation between the phase difference of the reflective spots $\Delta\phi$ and relative detuning between the lasers $\Delta\Omega$ is given by

$$\Delta\Omega = -\frac{\Delta\phi}{2\pi} \tau^{-1}_{RT},$$

where $\tau^{-1}_{RT} = 60MHz$ is the FSR of the DDRCL. Hence, a relative detuning of $\Delta\Omega$ is obtained by imposing a uniform phase of $\pi + \Delta\phi$ for one laser and $\pi - \Delta\phi$ for the other laser.

**Relative gain/loss control**

The addition of loss to the reflective spots is done by introducing a phase grating on top of the uniform phase. The contrast of the grating determines the amount of loss, as is elaborated in (52). The addition of linear gain (instead of changing the pumping rate) is done as follows. If the relative loss between the lasers is scanned from $-\Delta\alpha_{max}$ to $\Delta\alpha_{max}$, then a global loss of $\Delta\alpha_{max}$ is added to the two lasers throughout the scan. Hence, if the relative loss between the lasers is
\( \Delta \alpha \), then a loss of \( \Delta \alpha_{\text{max}} + \Delta \alpha \) will be added to the “lossy” laser, and loss of \( \Delta \alpha_{\text{max}} - \Delta \alpha \) will be added to the laser that is said to experience gain. For example, when there is no relative loss between the lasers, their reflectivity is equal to \( r_1 = r_2 = e^{-\Delta \alpha_{\text{max}}} \). When the relative loss is maximal, one laser has a reflectivity of \( r_{1/2} = e^{-2\Delta \alpha_{\text{max}}} \) and the second has a reflectivity of \( r_{2/1} = 1 \).

**Dispersive coupling sensitivity**

When two identical lasers are coupled with purely dispersive coupling, they are expected to phase-lock randomly in either in-phase (IP) or out-of-phase (OOP) modes, with equal probability. If the coupling is not purely dispersive, and it contains a small dissipative component, the degeneracy between the modes is lifted. In Fig. S4 we show diagrams of the relative phase between the two lasers as a function of \( \Delta \Omega \) and \( \Delta \alpha \) for different values of lasers distance: 453\( \mu \)m, 455\( \mu \)m and 459\( \mu \)m. Each pixel in the diagram is an average over 10 shots. The color hue represents the relative phase \( \phi \) and the brightness represents the lasers coherence, defined as \( C = \langle e^{i \phi} \rangle_{\text{shots}} \). When the lasers are separated by 455\( \mu \)m, the coupling is purely dispersive, as evidenced by the degeneracy of the IP and OOP modes at low values of \( \Delta \Omega \) and \( \Delta \alpha \) (and is manifested by the low coherence). A change of few microns in the lasers distance, adds a small dissipative part to the coupling, and breaks the degeneracy. It is demonstrated in panel (a) where the dissipative component is positive so the minimal loss mode is the IP, and in panel (c) where the dissipative component is negative so the minimal loss mode is the OOP.

**SLM Calibration: Compensation for detuning-loss coupling**

The SLM has a fill-factor of 95\%, and the remaining 5\% have some unknown constant phase. As a result, the SLM pixels and the spacing between them lead to a parasitic grating that diffracts light outside the cavity. The parasitic grating depends on the phase of the SLM pixels. Since the detuning between the lasers is achieved by applying a constant phase on the reflective spots of the SLM, then when the detuning is changed, the parasitic grating function also changes, resulting in a detuning-induced loss (DIL). The addition of loss to the lasers is accompanied by a small change in the effective detuning of the laser, probably due to edge effects. We refer to this kind of coupling as the loss-induced detuning (LID).

The DIL and LID compensation is done according to the phase diagram of the dispersively coupled lasers. When two lasers are dispersively coupled (with coupling strength \( \kappa \)) and have the same frequency, they will fail to synchronize along the symmetry line for \( |\Delta \alpha| < \kappa \). In a diagram of the standard deviation of the phase as a function of \( \Delta \Omega \) and \( \Delta \alpha \) of two dispersively coupled lasers, this will be manifested by a horizontal line from \( -\Delta \alpha_{\text{max}} \) to \( \Delta \alpha_{\text{max}} \) of high standard deviation values, indicating lack of coherence. We use this property of the dispersively coupled lasers to determine the LID, and then compensate for it in all of the measurements. Another “symmetry line” in the case of dispersive coupling, is the vertical line of \( \Delta \alpha = 0 \), in which the two eigenmodes are degenerate in their loss. We use this line to compensate for the DIL.

The calibration procedure is demonstrated in Fig. S5. The dashed red line marks the zero-relative-loss line, and the dashed black line marks the no-detuning line. The transformation to the data is done by the following linear functions:
\[ \Delta \alpha_c(\Delta \alpha_e, \Delta \Omega_e) = \Delta \alpha_e + 0.64 \cdot \Delta \Omega_e \]
\[ \Delta \Omega_c(\Delta \alpha_e, \Delta \Omega_e) = -0.12 \cdot \Delta \alpha_e + \Delta \Omega_e \]

where \( \Delta \alpha_e \) and \( \Delta \Omega_e \) are the experimental relative loss and detuning before compensation, and \( \Delta \alpha_c \) and \( \Delta \Omega_c \) the relative loss and detuning after the DIL and LID compensation. We noticed that the DIL depends on the global loss \( \Delta \alpha_{\text{max}} \). Hence, we repeated the calibration measurement for different global loss values. The detuning-loss coupling slightly depends on the laser’s location on the SLM, so the calibration is not exact for the anyonic and dissipative cases, resulting in an asymmetric stretch of the corresponding phase-diagrams shown in figure 1(d).

**Figure parameters and raw data**

In figure 1 (main text) we display 2D diagrams of the relative intensity and the phase difference of the two lasers as a function of their relative detuning and loss (both experimental and theoretical), for three different types of coupling: Dispersive, Anyonic and Dissipative. Each point in the phase diagram is averaged over 10 repetitions. In the experimental 2D phase diagram, we assigned to each pixel an average phase \( \langle \phi \rangle \) and coherence \( C(\Delta \alpha, \Delta \Omega) \), defined as the

\[ C(\Delta \alpha, \Delta \Omega) = \left| \langle e^{i\phi} \rangle \right| \]

where the angle brackets stand for an average over the 10 repetitions. Note that low coherence can arise from fundamentally different laser states: the lasers can be bi-stable, unstable (not phase locked) and even turned off.

In figure 4 (main text) we display the allowed synchronization regimes for the same 3 experiments (dispersive, anyonic and dissipative coupling). In order to draw the theoretical lines, we had to find the values of the coupling strength \( \kappa \) and coupling phase \( \beta \) of the experiment. The coupling strength (in MHz) was calculated according to

\[ \kappa = \frac{\sqrt{(\Delta \alpha_{\text{EP1}} - \Delta \alpha_{\text{EP2}})^2 + (\Delta \Omega_{\text{EP1}} - \Delta \Omega_{\text{EP2}})^2}}{2} \]

where \( \Delta \alpha_{\text{EP1,2}} \) and \( \Delta \Omega_{\text{EP1,2}} \) are the locations of the exceptional points, in terms of the relative loss and detuning, in the DIL and LID compensated phase diagrams. The coupling phase \( \beta \) was obtained from the angle of the symmetry line (equal laser amplitudes), which connects the two EPs. In Table S1 we summarize the experimental parameters. In Fig. S6 we display the EP locations on top of the relative intensity and phase diagrams.

Figure 3 (main text) displays the amplitude ratio and relative phase between the lasers along the symmetry line. The symmetry line was obtained by connecting the two EPs, and the values were taken from data points along the line. The maximal allowed distance from the line was chosen to be 0.05 MHz (hence the “missing segments”). The points that were used for figure 3 (main text) are marked by orange dots on the 2D diagrams in Fig. S7. The experimental parameters are summarized in Table S2.

**Nonlinear Simulations**

Here we present examples of regime diagrams obtained by numerically simulating the full laser rate equations. As we argued in the main text, the lack of phase coherence and de-
synchronization results from the nonlinearity of the laser system. In Fig. S8 we show numerical regime diagrams of $\beta = -0.7\pi, -0.9\pi$, for two different pump strengths. The regions of scrambled phase indicate where the lasers fail to synchronize, and the relative phase is not well-defined. In panel (a) the pump strength is slightly above the lasing threshold. Here the lasers synchronize and the phase coherence is almost perfect, apart for a narrow region at the vicinity of the pseudo-Hermiticity symmetry lines. When the pump strength is increased in panel (b), we see that the low coherence region develops around the symmetry lines, where the two underlying linear eigenmodes are degenerate in loss.
Fig. S1. Experimental Arrangement. Schematic drawing of the experimental setup. (a) Folded digital degenerate ring cavity laser, supporting the two lasers. (b) Interferometer for intensity and phase measurements.
Fig. S2. Typical laser’s intensity distribution. (a) Typical image of one laser’s near-field. (b) Intensity cross-section with a Gaussian fit.

Fig. S3. Typical SLM pattern. SLM pixels configuration for two identical lasers with diameter of 300μm, separated by 490μm.
**Fig. S4. Dispersive coupling sensitivity.** The relative phase and coherence of two coupled lasers, as a function of their relative detuning and loss. (a) Lasers distance $453\mu m$, resulting in a small positive dissipative part. (b) Lasers distance $455\mu m$. (c) Lasers distance $459\mu m$.

**Fig. S5. Loss-detuning coupling calibration.** Standard deviation of the relative phase of two dispersively coupled lasers. (a) The phase diagram before correction, dashed line marks the line of high standard deviation. (b) The phase diagram after DIL and LID corrections.
Fig. S6. Exceptional points locations. Intensity ratio and Panels (a), (c) and (e) display the intensity ratio diagrams obtained for the dispersive, anyonic and dissipative coupling measurements correspondingly. The color-map was modified to emphasize the symmetry line. The white dots mark the location of the EPs. Panels (b), (d) and (f) display the EP locations on top of the corresponding phase diagrams.
Fig. S7. Raw data of figure 2. 2D diagrams of the data presented in figure 3(a) (main text). White points mark the location of the EPs. The blue line show the line that connects the EPs. The data that was used is marked by the orange dots. (a) Laser distance $d = 459\mu m$ and coupling phase $\beta = 181^\circ$. (b) Laser distance $d = 470\mu m$ and coupling phase $\beta = 193^\circ$. (c) Laser distance $d = 500\mu m$ and coupling phase $\beta = 203^\circ$. (d) Laser distance $d = 540\mu m$ and coupling phase $\beta = 226^\circ$. (e) Laser distance $d = 570\mu m$ and coupling phase $\beta = 235^\circ$. 
Fig. S8. The effect of nonlinearity on the phase diagrams. Numerical simulations of the (steady-state) amplitude ratio and relative phase of the lasers for $\beta = -0.7\pi$ (top row) and $\beta = -0.9\pi$ (bottom row). In (a) the pump strength is $P = \alpha_0$ and in (b) $P = 1.2\alpha_0$, where $\alpha_0$ is the mean loss. In both panels the coupling strength is $\kappa = 0.25\alpha_0$. 
| Coupling type                  | Dispersive | Anyonic  | Dissipative |
|-------------------------------|------------|----------|-------------|
| Laser distance                | 455\(\mu m\) | 500\(\mu m\) | 625\(\mu m\) |
| Laser diameter                | 300\(\mu m\) | 300\(\mu m\) | 300\(\mu m\) |
| Coupling strength \(\kappa\)  (MHz) | 10.9\(MHz\) | 6.7\(MHz\) | 2.2\(MHz\) |
| Coupling phase \(\beta\) (degrees) | 180° | 200° | 256° |
| Normalization loss (before DIL correction) | 22\(MHz\) | 16.5\(MHz\) | 22\(MHz\) |
| DIL coefficient               | 0.64       | 0.52     | 0.64        |
| LID coefficient               | 0.1        | 0.1      | 0.1         |

Table S1. Experimental parameters for figures 1 and 4.

| Coupling type                  | Anyonic  | Anyonic  | Anyonic  | Anyonic  | Anyonic  |
|-------------------------------|----------|----------|----------|----------|----------|
| Laser distance                | 459\(\mu m\) | 470\(\mu m\) | 500\(\mu m\) | 540\(\mu m\) | 570\(\mu m\) |
| Laser diameter                | 300\(\mu m\) | 300\(\mu m\) | 300\(\mu m\) | 300\(\mu m\) | 300\(\mu m\) |
| Coupling strength \(\kappa\)  (MHz) | 12\(MHz\) | 5.5\(MHz\) | 5.4\(MHz\) | 4.4\(MHz\) | 4.1\(MHz\) |
| Coupling phase \(\beta\) (degrees) | 181° | 193° | 203° | 226° | 235° |
| Normalization loss (before DIL correction) | 22\(MHz\) | 16.5\(MHz\) | 9.5\(MHz\) | 8.5\(MHz\) | 5.7\(MHz\) |
| DIL coefficient               | 0.64     | 0.52     | 0.33     | 0.29     | 0.29     |
| LID coefficient               | 0.1      | 0.1      | 0.1      | 0.1      | 0.1      |

Table S2. Experimental parameters for figure 3.
REFERENCES AND NOTES

1. N. Moiseyev, *Non-Hermitian Quantum Mechanics* (Cambridge Univ. Press, 2011).

2. C. M. Bender, Making sense of non-hermitian hamiltonians. *Rep. Prog. Phys.* **70**, 947–1018 (2007).

3. M. V. Berry, Physics of nonhermitian degeneracies. *Czechoslovak J. Phys.* **54**, 1039–1047 (2004).

4. W. Heiss, The physics of exceptional points. *J. Phys. A Math. Theor.* **45**, 444016 (2012).

5. C. M. Bender, S. Boettcher, Real spectra in non-hermitian hamiltonians having PT symmetry. *Phys. Rev. Lett.* **80**, 5243–5246 (1998).

6. A. Mostafazadeh, Pseudo-hermiticity versus PT symmetry: The necessary condition for the reality of the spectrum of a non-hermitian hamiltonian. *J. Math. Phys.* **43**, 205–214 (2002).

7. C. M. Bender, D. C. Brody, H. F. Jones, Complex extension of quantum mechanics. *Phys. Rev. Lett.* **89**, 270401 (2002).

8. I. Rotter, A non-hermitian hamilton operator and the physics of open quantum systems. *J. Phys. A Math. Theor.* **42**, 153001 (2009).

9. J. Schindler, A. Li, M. C. Zheng, F. M. Ellis, T. Kottos, Experimental study of active LRC circuits with PT symmetries. *Phys. Rev. A* **84**, 040101 (2011).

10. X. Zhu, H. Ramezani, C. Shi, J. Zhu, X. Zhang, PT-symmetric acoustics. *Phys. Rev. X* **4**, 031042 (2014).

11. R. Fleury, D. Sounas, A. Alu, An invisible acoustic sensor based on parity-time symmetry. *Nat. Commun.* **6**, 5905 (2015).

12. L. Feng, R. El-Ganainy, L. Ge, Non-hermitian photonics based on parity–time symmetry. *Nat. Photonics* **11**, 752–762 (2017).

13. Ş. Özdemir, S. Rotter, F. Nori, L. Yang, Parity–time symmetry and exceptional points in photonics. *Nat. Mater.* **18**, 783–798 (2019).
14. M.-A. Miri, A. Alù, Exceptional points in optics and photonics. Science 363, eaar7709 (2019).

15. S. Klaiman, U. Günther, N. Moiseyev, Visualization of branch points in PT-symmetric waveguides. Phys. Rev. Lett. 101, 080402 (2008).

16. C. E. Rüter, K. G. Makris, R. El-Ganainy, D. N. Christodoulides, M. Segev, D. Kip, Observation of parity–time symmetry in optics. Nat. Phys. 6, 192–195 (2010).

17. A. Guo, G. Salamo, D. Duchesne, R. Morandotti, M. Volatier-Ravat, V. Aimez, G. Siviloglou, D. Christodoulides, Observation of $\mathcal{P}\mathcal{T}$-symmetry breaking in complex optical potentials. Phys. Rev. Lett. 103, 093902 (2009).

18. R. El-Ganainy, K. Makris, D. Christodoulides, Z. H. Musslimani, Theory of coupled optical $\mathcal{P}\mathcal{T}$-symmetric structures. Opt. Lett. 32, 2632–2634 (2007).

19. Y. D. Chong, L. Ge, A. D. Stone, $\mathcal{P}\mathcal{T}$-symmetry breaking and laser-absorber modes in optical scattering systems. Phys. Rev. Lett. 106, 093902 (2011).

20. Z. Lin, H. Ramezani, T. Eichelkraut, T. Kottos, H. Cao, D. N. Christodoulides, Unidirectional invisibility induced by PT-symmetric periodic structures. Phys. Rev. Lett. 106, 213901 (2011).

21. K. G. Makris, R. El-Ganainy, D. Christodoulides, Z. H. Musslimani, Beam dynamics in $\mathcal{P}\mathcal{T}$ symmetric optical lattices. Phys. Rev. Lett. 100, 103904 (2008).

22. S. Longhi, PT-symmetric laser absorber. Phys. Rev. A 82, 031801 (2010).

23. B. Peng, Ş. K. Özdemir, F. Lei, F. Monifi, M. Gianfreda, G. L. Long, S. Fan, F. Nori, C. M. Bender, L. Yang, Parity–time-symmetric whispering-gallery microcavities. Nat. Phys. 10, 394–398 (2014).

24. J. Wiersig, Enhancing the sensitivity of frequency and energy splitting detection by using exceptional points: Application to microcavity sensors for single-particle detection. Phys. Rev. Lett. 112, 203901 (2014).

25. W. Chen, Ş. K. Özdemir, G. Zhao, J. Wiersig, L. Yang, Exceptional points enhance sensing in an optical microcavity. Nature 548, 192–196 (2017).
26. M. P. Hokmabadi, A. Schumer, D. N. Christodoulides, M. Khajavikhan, Non-hermitian ring laser gyroscopes with enhanced sagnac sensitivity. *Nature* **576**, 70–74 (2019).

27. L. Feng, Z. J. Wong, R.-M. Ma, Y. Wang, X. Zhang, Single-mode laser by parity-time symmetry breaking. *Science* **346**, 972–975 (2014).

28. H. Hodaei, M.-A. Miri, M. Heinrich, D. N. Christodoulides, M. Khajavikhan, Parity-time–symmetric microring lasers. *Science* **346**, 975–978 (2014).

29. M. Brandstetter, M. Liertzer, C. Deutsch, P. Klang, J. Schöberl, H. E. Türeci, G. Strasser, K. Unterrainer, S. Rotter, Reversing the pump dependence of a laser at an exceptional point. *Nat. Commun.* **5**, 4034 (2014).

30. B. Peng, Ş. Özdemir, S. Rotter, H. Yilmaz, M. Liertzer, F. Monifi, C. Bender, F. Nori, L. Yang, Loss-induced suppression and revival of lasing. *Science* **346**, 328–332 (2014).

31. P. Peng, W. Cao, C. Shen, W. Qu, J. Wen, L. Jiang, Y. Xiao, Anti-parity–time symmetry with flying atoms. *Nat. Phys.* **12**, 1139–1145 (2016).

32. Y. Choi, C. Hahn, J. W. Yoon, S. H. Song, Observation of an anti-PT-symmetric exceptional point and energy-difference conserving dynamics in electrical circuit resonators. *Nat. Commun.* **9**, 2182 (2018).

33. Y. Li, Y.-G. Peng, L. Han, M.-A. Miri, W. Li, M. Xiao, X.-F. Zhu, J. Zhao, A. Alù, S. Fan, C.-W. Qiu, Anti-parity-time symmetry in diffusive systems. *Science* **364**, 170–173 (2019).

34. H. Cao, R. Chriki, S. Bittner, A. A. Friesem, N. Davidson, Complex lasers with controllable coherence. *Nat. Rev. Phys.* **1**, 156–168 (2019).

35. S. Longhi, E. Pinotti, Anyonic $\mathcal{PT}$ symmetry, drifting potentials and non-hermitian delocalization. *EPL* **125**, 10006 (2019).

36. Y.-P. Gao, Y. Sun, X.-F. Liu, T.-J. Wang, C. Wang, Parity-time-anyonic coupled resonators system with tunable exceptional points. *IEEE Access* **7**, 107874–107878 (2019).
37. A. Pikovsky, J. Kurths, M. Rosenblum, J. Kurths, *Synchronization: A Universal Concept in Nonlinear Sciences* (Cambridge Univ. Press, 2003), vol. 12.

38. J. Ding, I. Belykh, A. Marandi, M.-A. Miri, Dispersive versus dissipative coupling for frequency synchronization in lasers. *Phys. Rev. Appl.* 12, 054039 (2019).

39. L. Bello, M. C. Strinati, E. G. Dalla Torre, A. Pe’er, Persistent coherent beating in coupled parametric oscillators. *Phys. Rev. Lett.* 123, 083901 (2019).

40. F. Rogister, K. S. Thornburg Jr, L. Fabiny, M. Möller, R. Roy, Power-law spatial correlations in arrays of locally coupled lasers. *Phys. Rev. Lett.* 92, 093905 (2004).

41. A. Siegman, *Lasers* (University Science Books, 1986).

42. C. Tradonsky, I. Gershenson, V. Pal, R. Chriki, A. Friesem, O. Raz, N. Davidson, Rapid laser solver for the phase retrieval problem. *Sci. Adv.* 5, eaax4530 (2019).

43. I. Gershenson, G. Arwas, S. Gadasi, C. Tradonsky, A. Friesem, O. Raz, N. Davidson, Exact mapping between a laser network loss rate and the classical XY hamiltonian by laser loss control. *Nanophotonics* 9, 4117–4126 (2020).

44. Y. D. Chong, A. D. Stone, General linewidth formula for steady-state multimode lasing in arbitrary cavities. *Phys. Rev. Lett.* 109, 063902 (2012).

45. J. Zhang, B. Peng, Ş. K. Özdemir, K. Pichler, D. O. Krimer, G. Zhao, F. Nori, Y.-x. Liu, S. Rotter, L. Yang, A phonon laser operating at an exceptional point. *Nat. Photonics* 12, 479–484 (2018).

46. M. Fridman, V. Eckhouse, N. Davidson, A. A. Friesem, Effect of quantum noise on coupled laser oscillators. *Phys. Rev. A* 77, 061803 (2008).

47. L. Ge, R. El-Ganainy, Nonlinear modal interactions in parity-time (PT) symmetric lasers. *Sci. Rep.* 6, 24889 (2016).

48. M. H. Teimourpour, M. Khajavikhan, D. N. Christodoulides, R. El-Ganainy, Robustness and mode selectivity in parity-time (PT) symmetric lasers. *Sci. Rep.* 7, 10756 (2017).
49. J. Doppler, A. A. Mailybaev, J. Böhm, U. Kuhl, A. Girschik, F. Libisch, T. J. Milburn, P. Rabl, N. Moiseyev, S. Rotter, Dynamically encircling an exceptional point for asymmetric mode switching. Nature 537, 76–79 (2016).

50. M. Nixon, E. Ronen, A. A. Friesem, N. Davidson, Observing geometric frustration with thousands of coupled lasers. Phys. Rev. Lett. 110, 184102 (2013).

51. D. Leykam, A. Andreanov, S. Flach, Artificial flat band systems: From lattice models to experiments. Adv. Phys. X 3, 1473052 (2018).

52. S. Ngcobo, I. Litvin, L. Burger, A. Forbes, A digital laser for on-demand laser modes. Nat. Commun. 4, 2289 (2013).

53. V. Evtuhov, A. E. Siegman, A “twisted-mode” technique for obtaining axially uniform energy density in a laser cavity. Appl. Optics 4, 142–143 (1965).

54. J. A. Arnaud, Degenerate optical cavities. Appl. Optics 8, 189–195 (1969).

55. C. Tradonsky, V. Pal, R. Chriki, N. Davidson, A. A. Friesem, Talbot diffraction and fourier filtering for phase locking an array of lasers. Appl. Optics 56, A126–A132 (2017).