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Abstract. We prove two results about degree of polynomial mappings of \( \mathbb{C}^2 \) to \( \mathbb{C}^2 \).

§0. Let \( X_1, X_2 \) be the canonical coordinates in \( \mathbb{C}^2 \), \( \mathbb{C}[X_1, X_2]_{\leq m} \) be the linear space of polynomials of degree \( \leq m \) in the variable \( X = (X_1, X_2) \). Fix \( n = (n_1, n_2) \in \mathbb{N}^2 \) and put

\[
\tilde{V}_n = \mathbb{C}[X_1, X_2]_{\leq n_1} \times \mathbb{C}[X_1, X_2]_{\leq n_2}.
\]

For \( F = (F_1, F_2) \in \tilde{V}_n \) consider the polynomial mapping (denote it by the same letter)

\[
F : \mathbb{C}^2 \rightarrow \mathbb{C} \times \mathbb{C},
\]

\[
X = (X_1, X_2) \mapsto (F_1(X_1, X_2), F_2(X_1, X_2)) = F(X)
\]

and let \( J(F) \) be jacobian of the mapping \( F \). In the linear space \( \tilde{V}_n \) consider the following closed subvarieties

\[
D_i = \{ F \in \tilde{V}_n \mid deg J(F) \leq i \}, \quad 0 \leq i \leq n_1 + n_2 - 2,
\]

\[
W_i = \{ F \in \tilde{V}_n \mid dim F^{-1}(0) > 0 \text{ or } |F^{-1}(0)| \leq i \text{ or } \deg F_1 < n_1 \text{ and } \deg F_2 < n_2 \}, \quad 0 \leq i \leq n_1 n_2.
\]

The results of the article are around the following problem.

Problem 0.1. For that \( k, l \) the inclusion \( D_k \subset W_l \) holds?

Note that the inclusion \( D_0 \subset W_1 \) for all \( n \) is equivalent to the 2-dimensional Jacobian Conjecture [3].

Theorem 0.2. For all \( k \geq 0 \) the inclusion

\[
D_k \subset W_{\min\{n_1, n_2\}(k+1)}
\]

holds. In particular if jacobian of a polynomial mapping \( F \) is identically equal to 1, then degree of \( F \) does not exceed \( \min\{n_1, n_2\} \).

By \( C[X_1, X_2]_m \) denote the linear space of homogeneous polynomials of degree \( m \) in the variable \( X = (X_1, X_2) \). Pair of elements \( F \in \tilde{V}_n, H' \in \mathbb{C}[X_1, X_2]_1 \) are called general iff for \( i = 1 \) or for \( i = 2 \) the restriction of the polynomial \( F_i(X) \) on the line \( H'(X) = 0 \) is a polynomial of degree \( n_i \).
Theorem 0.3. Let \( F \in \tilde{V}_n, \ H' \in \mathbb{C}[X_1, X_2]_1 \) be general elements, \(|F^{-1}(0)| < \infty\). Define the linear subspaces

\[ K = K(F, H'), K_i = K_i(F, H') \subset \mathbb{C}[X_1, X_2]_{\leq n_1 + n_2 - 1} \]

in the following way

\[ K = \{ F_1 Q_2 + F_2 Q_1 \mid Q_i \in \mathbb{C}[X_1, X_2]_{n_i - 1} \}, \]
\[ K_0 = 0, \]
\[ K_{i+1} = (K + H' K_i) \cap \mathbb{C}[X_1, X_2]_{\leq n_1 + n_2 - 2}. \]

Then

\[ K_0 \subset \cdots \subset K_i \subset \cdots \subset \mathbb{C}[X_1, X_2]_{n_1 + n_2 - 2}, \]
\[ 2 \text{dim} K_i \geq \text{dim} K_{i-1} + \text{dim} K_{i+1}, \quad i \geq 1, \]
\[ |F^{-1}(0)| = n_1 n_2 - \dim K_\infty. \]

We prove Theorem 0.2 in §1 and Theorem 0.3 in §2 - §4.

§1. In this section we prove Theorem 0.2.

Let us remember some facts on Puiseux serieses.

A Puiseux series (at the infinity) is a convergent for \(|t| > R\) series

\[ \alpha(t) = \sum_{i \leq i_0} a_i (t^\frac{1}{d})^i, \]

where \( R > 0 \). Degree \( \text{deg}_t \alpha(t) \) of the series \( \alpha(t) \) is the greatest \( \frac{i}{d} \) such that \( a_i \neq 0 \). We write \( \alpha \sim \beta \) iff

\[ \beta(t) = \sum_{i \leq i_0} a_i \theta^i (t^\frac{1}{d})^i, \]

where \( \theta \in \mathbb{C} \) is a \( d \)-th root of 1. The series \( \alpha(t) \) is called reduced iff g.c.d. of \( \{ i \mid a_i \neq 0 \} \) is equal to 1. Suppose \( \alpha(t) \) is a reduced Puiseux series. The series \( \alpha(t) \) defines \( d \)-valued analytical function

\[ \tilde{\alpha} : \{ t \in \mathbb{C} \mid |t| > R \} \rightarrow \mathbb{C}, \]
\[ t \mapsto \alpha(t). \]

The number \( d \) is called the denominator of \( \alpha(t) \) (denote it by \( \text{den}(\alpha) \)).

By \( \mathbb{C}\{t\} \) denote the ring of Puiseux serieses. We have the canonical differential operator

\[ \frac{d}{dt} : \mathbb{C}\{t\} \rightarrow \mathbb{C}\{t\}. \]

By \( \mathbb{C}[X_2]\{X_1\} \) denote the ring of polynomials in \( X_2 \) that coefficients are Puiseux serieses in \( X_1 \). We have the canonical differential operators

\[ \frac{\partial}{\partial x_i} : \mathbb{C}[X_2]\{X_1\} \rightarrow \mathbb{C}[X_2]\{X_1\}, \quad i = 1, 2. \]
An element $G \in \mathbb{C}[X_1, X_2]$ is called proper with respect to $X_2$ iff

$$G(X_1, X_2) = G_0 X_2^p + G_1(X_1)X_2^{p-1} + \cdots + G_p(X_1),$$

where $0 \neq G_0 \in \mathbb{C}$.

Let $G$ be proper with respect to $X_2$ polynomial in the variables $X_1, X_2$. One can decompose the polynomial $G$ in the ring $\mathbb{C}[X_2]\{X_1\}$. Namely there exist Puiseux serieses $\alpha_1, \ldots, \alpha_m$ such that

1. $\text{den}(\alpha_1) + \cdots + \text{den}(\alpha_m) = \text{deg}_{X_2}G,$
2. $G(X_1, X_2) = G_0 \prod_{1 \leq i \leq m, \alpha_i \sim \alpha} (X_2 - \alpha(X_1)).$

The Puiseux serieses $\alpha_1, \ldots, \alpha_m$ are called roots of $G$ with respect to $X_2$.

Let $G$ be proper with respect to $X_2$ polynomial in the variables $X_1, X_2$. There is well known procedure to construct roots of $G$ with respect to $X_2$ (by means of Newton’s polygons). If we replace the polynomial $G$ by $G + c$, where $c \in \mathbb{C}$, then the roots (and sometimes the number of the roots) are changed. The following fact is a corollary of the procedure of the construction of roots.

**Lemma 1.1.** There exists an nonempty open in $\mathbb{C}$ subset $C = C(G)$ such that for $c \in C$, roots $\alpha_1, \ldots, \alpha_m$ of $G + c$ with respect to $X_2$, and

$$G(X_1, X_2 + \alpha_l(X_1)) + c = X_2 G_{l1}(X_1) + X_2^2 G_{l2}(X_1) + \cdots, \quad 1 \leq l \leq m$$

we have

$$\text{deg}_{X_1} G_{l1}(X_1) \geq 0, \quad 1 \leq l \leq m.$$

Let $G_1, G_2$ be polynomials in $X_1, X_2$ such that $G_1$ is proper with respect to $X_2$ and system of equations

(1.1)

$$\begin{cases} G_1(X_1, X_2) = 0 \\ G_2(X_1, X_2) = 0 \end{cases}$$

has finitely many solutions. Suppose $\alpha_1, \ldots, \alpha_m$ is roots of the polynomial $G_1$ with respect to $X_2$; then the number of solutions (with multiplicities) of the system (1.1) is equal to

$$\sum_{1 \leq l \leq m} \text{den}(\alpha_l) \text{deg}_{X_1} G_2(X_1, \alpha_l(X_1))$$

(the Zeuthen formula [4]).

**Proof of Theorem 0.2.** Suppose $F = (F_1, F_2) \in D_k$ and let $J = J(F)$ be jacobian of $F$. We may assume that $\text{deg} F_2 = n_2 \geq \text{deg} F_1 = n_1 = \text{deg}_{X_2} F_1$. We have to prove that

$$\text{deg} F \leq n_1(k + 1) \quad \text{or} \quad \text{dim} F(\mathbb{C}^2) < 2.$$
Set

\[ F_{1l}(X_1, X_2) = F_1(X_1, X_2 + \alpha_l(X_1)) \in \mathbb{C}[X_2][X_1], \]
\[ F_{1l}(X_1, X_2) = X_2 F_{1l1}(X_1) + X_2^2 F_{1l2}(X_1) + \ldots, \quad 1 \leq l \leq m. \]

Suppose \( \dim F(\mathbb{C}^2) = 2 \). We may assume that

1. the number of the solutions (with multiplicities) of the system of equations

\[ \begin{cases} F_1(X_1, X_2) = 0 \\ F_2(X_1, X_2) = 0 \end{cases} \tag{1.3} \]

is equal to degree of the mapping \( F \),

2. \( \deg_{X_1} F_{1l1}(X_1) \geq 0, \quad 1 \leq l \leq m \) (see Lemma 1.1).

Fix \( 1 \leq l \leq m \) and consider

\[ F_{2l}(X_1, X_2) = F_2(X_1, X_2 + \alpha_l(X_1)) \in \mathbb{C}[X_2][X_1], \]
\[ F_{2l}(X_1, X_2) = F_{2l0}(X_1) + X_2 F_{2l1}(X_1) + \ldots, \]
\[ J_{l}(X_1, X_2) = J(X_1, X_2 + \alpha_l(X_1)) \in \mathbb{C}[X_2][X_1], \]
\[ J_{l}(X_1, X_2) = J_{l0}(X_1) + X_2 J_{l1}(X_1) + \ldots. \]

From (1.2) it follows that

\[ \deg_{X_1} J_{l0}(X_1) \leq k. \tag{1.4} \]

We have

\[ \det \left( \frac{\partial F_{il}}{\partial X_j} \right)_{1 \leq i, j \leq 2} (X_1, X_2) = J_l(X_1, X_2) \]

whence

\[ \deg_{X_1} \left( \frac{d}{dX_1} F_{2l0}(X_1) \right) + \deg_{X_1} F_{1l1}(X_1) = \deg_{X_1} J_{l0}(X_1). \]

Using (2) and (1.4), we get

\[ \deg_{X_1} F_{2l0}(X_1) \leq \deg_{X_1} J_{l0}(X_1) + 1 \leq k + 1. \tag{1.5} \]

The number of the solutions (with multiplicities) of the system of equations (1.3) is equal to

\[ \sum_{1 \leq l \leq m} \text{den}(\alpha_l) \deg_{X_1} F_2(X_1, \alpha_l(X_1)) = \sum_{1 \leq l \leq m} \text{den}(\alpha_l) \deg_{X_1} F_{2l0}(X_1) \leq \]
\[ \leq \sum_{1 \leq l \leq m} \text{den}(\alpha_l)(k + 1) = n_1(k + 1) \]

(see (1.5)). Therefore, degree of the mapping \( F \) does not exceed \( \min\{n_1, n_2\}(k+1) \).
For a linear space $V$ and dual space $V^*$ by
\[
\langle \cdot, \cdot \rangle : V \times V^* \to \mathbb{C},
\quad (v, v^*) \mapsto \langle v, v^* \rangle
\]
denote the canonical bilinear mapping.

The group $SL_3$ acts canonically in the spaces $\mathbb{C}^3, \mathbb{C}^3^*, S^m\mathbb{C}^3^*, \ldots$. Let $e_1, e_2, e_3$ be the standard basis of $\mathbb{C}^3$, $x_1, x_2, x_3$ be the standard basis of $\mathbb{C}^3^*$. Set
\[
\Delta = \sum \frac{\partial}{\partial e_i} \otimes \frac{\partial}{\partial x_i} : \mathbb{C}[\mathbb{C}^3^*] \otimes \mathbb{C}[\mathbb{C}^3] \to \mathbb{C}[\mathbb{C}^3^*] \otimes \mathbb{C}[\mathbb{C}^3].
\]
Recall that
\[
dim S^m\mathbb{C}^3^* = \frac{1}{2} (m+1)(m+2).
\]
For $g \in S^m\mathbb{C}^3^*$ put
\[
V(g) = \{ \overline{a} \in P\mathbb{C}^3 | g(\overline{a}) = 0 \} \subset P\mathbb{C}^3.
\]
Let $n_1, n_2$ be natural numbers. Set $n = (n_1, n_2)$, $N = n_1 n_2$, and
\[
V_n = S^{n_1}\mathbb{C}^3^* \times S^{n_2}\mathbb{C}^3^*.
\]
In this section we make the following.

(1) We define the covariant
\[
Q : V_n \to \mathbb{C}
\]
(eliminant) and prove some of its properties.

(2) For $h \in \mathbb{C}^3^*, h \neq 0, f \in V_n$ we define affine space $A(h) \subset P\mathbb{C}^3$ and the polynomial mapping
\[
I_h(f) : A(h) \to \mathbb{C} \times \mathbb{C}.
\]
We have $Q(f) = 0$ iff $\dim(I_h(f)^{-1}(0)) > 0$ or polynomial degree of $I_h(f)$ is less $(n_1; n_2)$.

(3) For $f \in V_n, h, h' \in \mathbb{C}^3^*$ such that $Q(f) \neq 0, h \neq 0, h' \neq 0$, and $V(h) \cap V(h') \cap V(f_1) \cap V(f_2) = \emptyset$ we prove that
\[
|I_h(f)^{-1}(0)| = \deg R(f, h' + th),
\]
where
\[
R : V_n \times \mathbb{C}^3^* \to \mathbb{C}
\]
is the resultant.

1. Consider the resultant
\[
R : V_n \times \mathbb{C}^3^* \to \mathbb{C}
\]
The resultant $R$ is a polyhomogeneous (of polydegree $(n_1, n_2, n_3 n_2)$) invariant. The resultant $R$ defines canonically the polyhomogeneous (of polydegree $(n_2, n_1)$) covariant

$$Q : V_n \rightarrow (S^N \mathbb{C}^3^*)^\ast = S^N \mathbb{C}^3.$$  

We have

$$R(f, h) = \langle Q(f), h^N \rangle$$

for $(f, h) \in V_n \times \mathbb{C}^3^*$. It follows from this formula that

1. if $f \in V_n$, then $Q(f) = 0$ iff $\dim(V(f_1) \cap V(f_2)) > 0$,

2. if $f \in V_n$, $\dim(V(f_1) \cap V(f_2)) = 0$, then $Q(f) = l_1 \ldots l_N$, where $l_1, \ldots, l_N \in \mathbb{C}^3$ and $V(f_1) \cap V(f_2) = \{ \overline{t_1}, \ldots, \overline{t_N} \}$.

2. Suppose $h \in \mathbb{C}^3^*$, $h \neq 0$. Define the 2-dimensional affine space

$$A(h) = PC^3 \setminus V(h).$$

Let $\mathbb{C}[A(h)]_{\leq m}$ be the linear space of polynomial mappings (of polynomial degree $\leq m$) of the affine space $A(h)$ to $\mathbb{C}$. The linear space $\mathbb{C}[A(h)]_{\leq n_1} \times \mathbb{C}[A(h)]_{\leq n_2}$ is the space of polynomial mappings (of polynomial degree $\leq (n_1; n_2)$) of the affine space $A(h)$ to $\mathbb{C} \times \mathbb{C}$. Fix the isomorphisms of the liner spaces

$$i_h : S^m \mathbb{C}^3^* \rightarrow \mathbb{C}[A(h)]_{\leq m},$$

$$i_h(g)(a) = \frac{g(a)}{h(a)^m},$$

$$I_h : V_n \rightarrow \mathbb{C}[A(h)]_{\leq n_1} \times \mathbb{C}[A(h)]_{\leq n_2},$$

$$I_h(f)(a) = \left( \frac{f_1(a)}{h(a)^{n_1}}, \frac{f_2(a)}{h(a)^{n_2}} \right).$$

From (1) of item 1 it follows that $Q(f) = 0$ iff $\dim(I_h(f)^{-1}(0)) > 0$ or polynomial degree of $I_h(f)$ is less $(n_1; n_2)$.

3. \textbf{Lemma 2.1.} Suppose $f \in V_n$, $h, h' \in \mathbb{C}^3^*$, $Q(f) \neq 0$, $h \neq 0$, $h' \neq 0$, and $V(h) \cap V(h') \cap V(f_1) \cap V(f_2) = \emptyset$; then

$$|I_h(f)^{-1}(0)| = \deg R(f, h' + th).$$

\textbf{Proof.} We have

$$Q(f) = l_1 \ldots l_N,$$

where $l_i \in \mathbb{C}^3$. We may assume that $h(l_i) \neq 0$ for $1 \leq i \leq d$ and $h(l_i) = 0$ for $d + 1 \leq i \leq N$. From the suppositions of the Lemma it follows that $h'(l_i) \neq 0$ for $d + 1 \leq i \leq N$. We have

$$I_h(f)^{-1}(0) = \{ \overline{t_1}, \ldots, \overline{t_d} \},$$

$$R(f, h' + th) = \langle Q(f), (h' + th)^N \rangle =$$

$$= \langle l_1 \ldots l_N, h'^N + tN h'^{(N-1)}h + \ldots + t^N h^N \rangle,$$

$$\langle l_1 \ldots l_N, h'^{(N-d)}h^d \rangle \neq 0,$$

$$\langle l_1 \ldots l_N, h'^{(N-i)}h^i \rangle = 0, \quad d + 1 \leq i \leq N.$$
From these formulas it follows the Lemma.

§3. We use the notations of §2.
Suppose \( n = (n_1, n_2) \in \mathbb{N}^2 \).
Recall a procedure of a calculation of the resultant

\[
\mathcal{R} : V_n \times \mathbb{C}^{3*} \rightarrow \mathbb{C}.
\]

Fix \((f, s) \in V_n \times \mathbb{C}^{3*}\) and consider the complex of vector spaces

\[
M(f, s) : 0 \rightarrow M \xrightarrow{\beta(f,s)} M' \xrightarrow{\beta'(f,s)} M'' \rightarrow 0,
\]

where

\[
M = S^{n_1-2} \mathbb{C}^{3*} \times S^{n_2-2} \mathbb{C}^{3*},
M' = S^{n_1-1} \mathbb{C}^{3*} \times S^{n_2-1} \mathbb{C}^{3*} \times S^{n_1+n_2-2} \mathbb{C}^{3*},
M'' = S^{n_1+n_2-1} \mathbb{C}^{3*},
\]

\[
\beta(f,s)(r_1, r_2) = (sr_1, sr_2, f_1 r_2 + f_2 r_1),
\]

\[
\beta'(f,s)(q_1, q_2, g) = (f_1 q_2 + f_2 q_1 - sg).
\]

The determinant of the complex \(M(f, s)\) is equal to \(R(f, s)\).
One can calculate the determinant of the complex \(M(f, s)\) in the following way [2]. Fix \(a \in \mathbb{C}^3, s(a) \neq 0\) and consider the linear mappings

\[
\alpha(a) : M' \rightarrow M,
(q_1, q_2, g) \mapsto (\Delta(q_1 a), \Delta(q_2 a)),
\]

\[
t\alpha(a) \circ \beta(f,s) : M \rightarrow M,
(t\alpha(a), \beta'(f,s)) : M' \rightarrow M \times M''.
\]

We have

\[
R(f,s) = \det(t\alpha(a), \beta'(f,s))(\det(t\alpha(a) \circ \beta(f,s)))^{-1}
\]

(we calculate \(\det\) with respect to the canonical \(SL_3\)-invariant forms of maximal degree in the spaces \(S^m \mathbb{C}^{3*}\)).

**Lemma 3.1.**

\[
\det(t\alpha(a) \circ \beta(f,s)) = c(t(s(a)))^{\dim M},
\]

where \(0 \neq c \in \mathbb{C}\).

**Proof.** The function \(\det(\alpha(a) \circ \beta(f,s))\) is a nonzero polyhomogeneous (of polydegree \((\dim M, \dim M)\) \(SL_3\)-invariant function in \((a, s) \in \mathbb{C}^3 \times \mathbb{C}^{3*}\). As is known, there exists only one (up to a nonzero factor) that function that is \((s(a))^{\dim M}\).

Therefore,

\[
(3.1) \quad R(f,s) = \det(t\alpha(a), \beta'(f,s))c^{-1}(ts(a))^{-\dim M}.
\]

We need the following fact of linear algebra.
Lemma 3.2. Let \( \eta, \eta' : V \longrightarrow V' \)  
be linear mappings, \( \dim V = \dim V' \), and \( \det(\eta' + t\eta) \neq 0 \). Define  
\[ L_i = L_i(\eta, \eta') \subset V', \quad i \geq 0 \]
in the following way  
\[ L_0 = 0, \]
\[ L_{i+1} = \eta'(\eta^{-1}(L_i)) \cap \text{Im}\eta. \]

Then  
\[ L_0 \subset \cdots \subset L_i \subset \cdots \subset \text{Im}\eta, \]
\[ 2\dim L_i \geq \dim L_{i-1} + \dim L_{i+1}, \quad i \geq 1, \]
\[ \deg_t \det(\eta' + t\eta) = \dim V - \dim \eta^{-1}(0) - \dim L_{\infty}. \]

Proof. Let \( b_1, \ldots, b_n \) be a basis of \( V \), \( m \geq 0 \), and  
\[ \epsilon : V' \longrightarrow V \]
be an isomorphism such that  
\[ (\epsilon \circ \eta')(b_i) = \begin{cases} b_i & \text{for } 0 \leq i \leq m, \\ 0 & \text{for } m + 1 \leq i \leq n, \end{cases} \]
\[ (\epsilon \circ \eta)(\langle b_1, \ldots, b_m \rangle) \subset \langle b_1, \ldots, b_m \rangle, \]
\[ (\epsilon \circ \eta)(\langle b_{m+1}, \ldots, b_n \rangle) \subset \langle b_{m+1}, \ldots, b_n \rangle, \]
the matrix of \( \epsilon \circ \eta \) in the basis \( b_1, \ldots, b_n \) is a Jordan matrix. It can easily be checked that the Lemma holds for \( \epsilon \circ \eta, \epsilon \circ \eta' \). Therefore, the Lemma holds for \( \eta, \eta' \).

Theorem 3.3. Suppose \( f \in V_n, a \in \mathbb{C}^3, h, h' \in \mathbb{C}^3^*, \ Q(f) \neq 0, h \neq 0, h' \neq 0, \ (h' + th)(a) \neq 0, \) and \( V(h) \cap V(h') \cap V(f_1) \cap V(f_2) = \emptyset \) and define  
\[ L_i = L_i(f, a, h, h') \subset M \times M'', \quad i \geq 0 \]
in the following way  
\[ L_0 = 0, \]
\[ L_{i+1} = (0, \beta'(f, h'))((\alpha(a), \beta'(0, h))^{-1}(L_i)) \cap \text{Im}(\alpha(a), \beta'(0, h)); \]
then  
\[ L_0 \subset \cdots \subset L_i \subset \cdots \subset \{0\} \times M'', \]
\[ 2\dim L_i \geq \dim L_{i-1} + \dim L_{i+1}, \quad i \geq 1, \]
\[ \deg P(f, h' + th) = n, n - \dim L_{\infty}. \]
Proof. The Theorem is a corollary of (3.1), Lemma 3.2, the formula for dimension of \( S^m \mathbb{C}^3 \) (see \( \S 2 \)), and the following evident fact

\[
\dim(\alpha(a), \beta'(0, h))^{-1}(0) = n_1 + n_2.
\]

\( \S 4. \) In this section we prove Theorem 0.3. We use the notations of \( \S 2 \) and \( \S 3. \) Fix the isomorphisms of the linear spaces (denote them by one letter)

\[
\theta : \mathbb{C}[X_1, X_2]_{\leq m} \rightarrow S^m \mathbb{C}^3, \quad m \geq 0,
\]

\[
G(X_1, X_2) \mapsto x_3^m G\left(\frac{x_1}{x_3}, \frac{x_2}{x_3}\right).
\]

We see that

\[
\theta = i_{x_3}^{-1} \circ j_3^*,
\]

where

\[
j_3 : \mathbb{A}(x_3) \rightarrow \mathbb{C}^2, \quad (a_1 : a_2 : 1) \mapsto (a_1, a_2)
\]

is the isomorphism of the affine varieties.

Set

\[
\tilde{M} = \mathbb{C}[X_1, X_2]_{\leq n_1} \times \mathbb{C}[X_1, X_2]_{n_2 - 2},
\]

\[
\tilde{M}' = \mathbb{C}[X_1, X_2]_{\leq n_1 - 1} \times \mathbb{C}[X_1, X_2]_{n_2 - 1} \times \mathbb{C}[x_1, X_2]_{n_1 + n_2 - 2},
\]

\[
\tilde{M}'' = \mathbb{C}[X_1, X_2]_{n_1 + n_2 - 1}.
\]

The isomorphisms \( \theta \) define canonically the following isomorphisms (denote them by the same letter)

\[
\theta : \tilde{V}_n \rightarrow V_n, \quad \tilde{M} \rightarrow M, \quad \tilde{M}' \rightarrow M', \quad \tilde{M}'' \rightarrow M''.
\]

Proof of Theorem 0.3. Set \( f = \theta(F), \ h' = \theta(H'). \) From the suppositions of the Theorem it follows that \( Q(f) \neq 0, (h' + tx_3)(e_3) \equiv t, \) and \( V(x_3) \cap V(h') \cap V(f_1) \cap V(f_2) = \emptyset. \) Using Lemma 2.1 and Theorem 3.3, we get

\[
|F^{-1}(0)| = |I_{x_3}(f)^{-1}(0)| = \deg_t R(f, h' + tx_3) = n_1 n_2 - \dim L_\infty(f, e_3, x_3, h').
\]

The following linear mappings correspond to \( (0, \beta'(f, h')) \) and \( (\alpha(e_3), \beta'(0, x_3)) \) under the isomorphisms \( \theta: \)

\[
\gamma'(F, H') = \theta^{-1} \circ (0, \beta'(f, h')) \circ \theta : \tilde{M}' \rightarrow \tilde{M} \times \tilde{M}'', \quad (Q_1, Q_2, G) \mapsto (0, F_1 Q_2 + F_2 Q_1 - H' G),
\]

\[
\gamma(F, H') = \theta^{-1} \circ (\alpha(e_3), \beta'(0, x_3)) \circ \theta : \tilde{M}' \rightarrow \tilde{M} \times \tilde{M}'', \quad (O, Q_1, C) \mapsto (O, \Delta_{x_3}(O) - \Delta_{x_3}(O) - C).
\]
where
\[ \Delta_m : \mathbb{C}[X_1, X_2] \longrightarrow \mathbb{C}[X_1, X_2], \]
\[ X_1^{m_1} X_2^{m_2} \mapsto (m - m_1 - m_2)X_1^{m_1} X_2^{m_2}. \]

Define the linear subspaces
\[ \tilde{L}_i = \tilde{L}_i(F, H') \subset \tilde{M} \times \tilde{M}'', \]
in the following way
\[ \tilde{L}_0 = 0, \]
\[ \tilde{L}_{i+1} = \gamma'((F, H') (\gamma(F, H')^{-1}(\tilde{L}_i)) \cap Im \gamma(F, H'). \]

We have
\[ \tilde{L}_i(F, H') = \theta(L_i(f, e_3, x_3, h')), \quad i \geq 0 \]
and therefore,
\[ \tilde{L}_0 \subset \cdots \subset \tilde{L}_i \subset \cdots \subset \tilde{M} \times \tilde{M}'', \]
\[ 2\dim \tilde{L}_i \geq \dim \tilde{L}_{i-1} + \tilde{L}_{i-1}, \quad i \geq 1, \]
\[ |F^{-1}(0)| = n_1 n_2 - \dim \tilde{L}_\infty. \]

It follows from the definitions that
\[ \tilde{L}_i \subset \{0\} \times \tilde{M}'', \quad \gamma(F, H')^{-1}(0, Z) = \]
\[ \begin{cases} 0 & \text{if } \text{deg}Z = n_1 + n_2 - 1, \\ \mathbb{C}[X_1, X_2]_{n_1-1} \times \mathbb{C}[X_1, X_2]_{n_2-1} \times \{-Z\} & \text{if } \text{deg}Z \leq n_1 + n_2 - 2. \end{cases} \]

It is easy to prove by induction that
\[ \tilde{L}_i(F, H') = \{0\} \times K_i(F, H') \]
and therefore,
\[ K_0 \subset \cdots \subset K_i \subset \cdots \subset \mathbb{C}[X_1, X_2]_{n_1+n_2-2}, \]
\[ 2\dim K_i \geq \dim K_{i-1} + \dim K_{i+1}, \quad i \geq 0, \]
\[ \dim \tilde{L}_\infty = \dim K_\infty, \]
\[ |F^{-1}(0)| = n_1 n_2 - \dim K_\infty. \]

(see (4.1)).

REFERENCES

1. W.Fulton and J.Harris, *Representation Theory*, Springer Graduate Text in Math., v. 129, Springer-Verlag, Berlin, 1991.
2. I.M.Gelfand, M.M.Kapranov, and A.V.Zelevinsky, *Discriminants, Resultants and Multidimensional Determinants*, Birkhauser, Boston, 1994.
3. H.Bass, E.H.Connell, and D.Wright, *The Jacobian Conjecture: reduction of degree and formal expansion of the inverse*, Bull. Amer. Math. Soc. **1982**, no. 7, 287 - 330.
4. W.Fulton, *Intersection Theory*, Springer-Verlag Berlin, Heidelberg, 1984.