Bogdanov–Takens Bifurcation in a Shape Memory Alloy Oscillator with Delayed Feedback

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1. Introduction

In recent years, smart materials have been widely used in many fields such as aircraft manufacturing [1, 2], control field [3], energy [4, 5], and medical [6] due to their special properties. The discovery and application of shape memory alloys [7–9] is an important part of smart materials. The so-called shape memory alloy (SMA) [10] is a new type of smart material with special shape memory effect and pseudo-elasticity, which can restore the previously defined shape when subjected to an appropriate thermomechanical loading process.

SMA spring oscillators can exhibit rich dynamic behaviors based on their pseudo-elasticity, thus promoting the study of nonlinear dynamics and bifurcation of shape memory oscillators [11–15]. Savi et al. [16] studied the nonlinear dynamics of shape memory alloy systems and established the constitutive model of the SMA. Fu and Lu [17] investigated the nonlinear dynamics and vibration damping of dry friction oscillators with SMA restraints. Costa et al. [18] applied the extended time-delayed feedback approach to investigate the chaos control of an SMA two-bar truss. de Paula et al. [19] controlled a shape memory alloy two-bar truss by the delayed feedback method.

The governing equation of motion of a shape memory oscillator [20, 21] is given by

\[ m\ddot{x} + c\dot{x} + K(x, T) = F \cos(\omega t), \]

(1)

where \( \bar{q} = (qA/L), \bar{b} = (bA/L^3), \) and \( \bar{e} = (eA/L^5). \) \( m \) is the mass of the oscillator. \( F \cos(\omega t) \) is a periodic external force, and \( K(x, T) = \bar{q}(T - T_m)x - \bar{b}x^3 + \bar{c}x^5 \) is the restoring force of the spring. \( L \) and \( A, \) respectively, denote a shape memory element of length and cross-section area. \( b, c, \) and \( q \) are constants of the material. \( T_M \) corresponds to the temperature where the martensitic phase is stable.

In 2016, Yu et al. [22] considered a typical dimensionless system of the SMA oscillator based on equation (1) as follows:

\[ \ddot{x} + \alpha_1 \dot{x} + \alpha_2 x - \alpha_3 x^3 + \alpha_4 x^5 = k \cos(\theta t), \]

(2)

and they added a time-delayed feedback to control equation (2), and equation (2) can be rewritten as

\[ \ddot{x} + \alpha_1 \dot{x} + \alpha_2 x - \alpha_3 x^3 + \alpha_4 x^5 \]

(3)

\[ \ddot{x} + \alpha_1 \dot{x} + \alpha_2 x - \alpha_3 x^3 + \alpha_4 x^5 = k \cos(\theta t + \tau). \]

(4)

where \( \tau \) represents time delay.
\[ \ddot{x} + \alpha_1 x + \alpha_2 x - \alpha_3 x^3 + \alpha_4 x^5 = k \cos (\theta t) + A_1 x_\tau + A_2 x_\tau, \]
\[ \] (3)
where \( x_\tau = x(t - \tau) \), \( \tau \) is denoted as delay, and \( A_{1,2} \) is the delay position feedback parameter.

If \( k \cos (\theta t) \) is considered as the control parameter \( \delta \), equation (3) can be rewritten as
\[ \ddot{x} + \alpha_1 x + \alpha_2 x - \alpha_3 x^3 + \alpha_4 x^5 = \delta + A_1 x_\tau + A_2 x_\tau, \]
\[ \] (4)

They used the normal form theory (NFT) and center manifold theorem (CMT) to calculate the conditions of the Hopf bifurcation and stability of equation (4).

The deep insight of the system dynamics is helpful to understand the nonlinear dynamics of shape memory alloy systems. However, many studies on time-delay systems have focused on analyzing the bifurcations of codimension-1, such as Hopf bifurcation [23]. Actually, the time-delay system may have more complicated dynamics when two separate parameters or many parameters are changed simultaneously. B-T bifurcation, which is a typical codimension-2 bifurcation, is studied in [24–28].

Motivated by the above works, we consider system (4) and investigate the B-T bifurcation under some critical conditions. The main contributions of this paper are as follows:

1. The feedback parameters \( A_{1,2} \) and time delay \( \tau \) are selected to analyze their impact on codimension-2 bifurcations of system (4).
2. The bifurcation diagram and topological classification of the trajectory of a universal unfolding are given.
3. The second-order terms of the normal form on a center manifold of the SMA system are obtained.

The layout of this work is organized as follows: in Section 2, we, respectively, give conditions for the occurrence of the B-T bifurcation and mainly discuss the normal forms for the B-T bifurcation. In Section 3, some numerical simulations are implemented to validate the above analysis. We give some conclusions in Section 4, respectively.

### 2. Stability and B-T Bifurcation

In this section, we mainly establish the existence of the B-T bifurcation under some critical conditions.

Firstly, let \( \ddot{x} = y \); then, system (4) can be equivalent to
\[ \begin{cases} \dot{x} = y, \\ \dot{y} = -\alpha_1 y - \alpha_2 x + \alpha_3 x^3 - \alpha_4 x^5 + \delta + A_1 x_\tau + A_2 x_\tau. \end{cases} \]
\[ \] (5)

Denoting the equilibrium of system (4) as \( E_0 = (x_0, 0) \), \( x_0 \) satisfies an algebraic equation as follows:
\[ g(x) = e_1 x^5 + e_2 x^3 + e_3 x + e_4 = 0, \]
\[ \] (6)
where \( e_1 = \alpha_4, e_2 = -(\alpha_3 + A_2), e_3 = \alpha_2 - A_1, \) and \( e_4 = -\delta \).

Next, we discuss the existence conditions of the root of equation (6).

#### Lemma 1

For the roots of equation (6), the following results hold:

1. If \( e_1 < 0, e_3 > 0, \) and \( e_4 = 0 \), then equation (6) has three roots, and they are \( x_1 = 0 \) and \( x_{2,3} = \pm \sqrt{((-e_2 - \sqrt{e_2^2 - 4e_1 e_3})/2e_3)}, \)
2. If \( e_1 > 0, e_3 < 0, \) and \( e_4 = 0 \), then equation (6) has three roots, and they are \( x_1 = 0 \) and \( x_{2,3} = \pm \sqrt{((-e_2 - \sqrt{e_2^2 - 4e_1 e_3})/2e_3)}, \)
3. If \( (e_2/e_3) < 0, \) (\( e_3/e_1 \)) > 0, \( \Delta = e_2^2 - 4e_1 e_3 > 0, \) and \( e_4 = 0 \), then equation (6) has five roots, and they are
\[ x_1 = 0, \quad x_{2,3,4} = \pm \sqrt{((-e_2 - \sqrt{e_2^2 - 4e_1 e_3})/2e_3)}, \]
\[ \text{and} \]
\[ x_{4,5} = \pm \sqrt{((-e_2 - \sqrt{e_2^2 - 4e_1 e_3})/2e_3)}, \]
4. If \( e_1 > 0 (i = 1, 2, 3, 4), \) then equation (6) has no real root.
5. If \( e_1 > 0 \) and \( e_4 < 0 \), then equation (6) has at least one positive real root.
6. If \( e_1 > 0 \) and \( e_4 > 0 \) and \( \exists \xi > 0 \) such that \( g(\xi) < 0, \) then equation (6) has at least two positive real roots.
7. If \( x > 0, e_3 > 0, e_1 > 0, e_4 > 0, \) and \( e_4 > 0 \) and there exists \( \xi > 0 \) such that \( g(\xi) < 0, \) then equation (6) has two positive real roots.

**Proof**

(i), (ii), and (iii) are easy to prove, and we do not show the process of proof.

iv. If \( e_i > 0 (i = 1, 2, 3, 4), \) then \( g(0) = e_4 > 0 \) and \( g'(x) > 0. \) Thus, we can obtain that equation (6) has no real root.

v. If \( e_1 > 0 \) and \( e_4 > 0 \), then \( g(0) = e_4 < 0 \) and \( \lim_{x \to +\infty} g(x) = +\infty, \) and we can obtain that equation (6) has at least one positive real root.

vi. If \( e_1 > 0 \) and \( e_4 > 0 \) and there exists \( \xi > 0 \) such that \( g(\xi) = 0, \) we can obtain that \( g(0) = e_4 > 0 \) and \( \lim_{x \to +\infty} g'(x) = +\infty, \) and \( g(x) = 0 \) have only one positive real root. Furthermore, from \( g(0) = e_4 > 0 \) and \( \lim_{x \to +\infty} g'(x) = +\infty, \) we get that equation (6) has two positive real roots. This completes the proof.

Let \( \bar{x} = x - x_0 \) and \( \bar{y} = y. \) Omitting the tilde, then system (5) can be rewritten as
\[ \begin{cases} \dot{x} = y, \\ \dot{y} = -\alpha_1 y + c_1 x + c_2 x^2 + c_3 x^3 + c_4 x^4 + c_5 x^5 + d_1 x_\tau + d_2 x_\tau^2 + d_3 x_\tau^3. \end{cases} \]
\[ \] (7)
where
\[\begin{align*}
  c_1 &= -\alpha_2 + 3\alpha_3 x_0 + 5\alpha_4 x_0^2, \\
  c_2 &= 3\alpha_3 x_0 - 10\alpha_4 x_0^3, \\
  c_3 &= \alpha_2 + 10\alpha_4 x_0^2, \\
  c_4 &= -5\alpha_4 x_0, \\
  c_5 &= -\alpha_4, \\
  d_1 &= A_1 + 3x_0^2A_2, \\
  d_2 &= 3A_2x_0, \\
  d_3 &= A_2.
\end{align*}\] (8)

The characteristic equation of system (7) at the zero equilibrium \( E_0 = (0,0)^T \) is
\[F(\lambda, \tau) = \lambda^2 + \alpha_1 \lambda - c_1 - d_0 e^{-\lambda \tau} = 0.\] (9)

Next, we give the conditions for the existence of the B-T bifurcation and investigate the dynamical classification near the B-T bifurcation point.

**Lemma 2.** If \( d_0 < 0 \), then the following is obtained:

(i) If \( d_1 = d_0 \) and \( \tau \neq \tau_0, \lambda = 0 \) is a single root of equation (9)

(ii) If \( \lambda = 0, d_1 = d_0, \) and \( \tau = \tau_0, \lambda = 0 \) is a double root of equation (9)

(iii) If \( d_1 = d_0, \) all the roots of equation (9) have negative real parts except for the zero roots

Here, \( \tau_0 = -\frac{\alpha_1}{d_0} \) and \( d_0 = -c_1. \)

**Proof.** Clearly, \( F(0, \tau) = d_1 + c_1 = 0. \) By calculating, we can obtain the following result:
\[\frac{\partial F(\lambda, \tau)}{\partial \lambda} = 2\lambda + \alpha_1 + d_0 \tau e^{-\lambda \tau},
\[\frac{\partial^2 F(\lambda, \tau)}{\partial \lambda^2} = 2 - d_0 \tau^2 e^{-\lambda \tau}.\] (10)

It is easy to obtain if \( \lambda = 0 \) and \( \tau \neq \tau_0, \) then \((\partial F(\lambda, \tau)/\partial \lambda)|_{\lambda=0, \tau \neq \tau_0} \neq 0. \) Thus, (i) holds.

If \( \lambda = 0 \) and \( \tau = \tau_0, \) then \((\partial^2 F(\lambda, \tau)/\partial \lambda^2)|_{\lambda=0, \tau = \tau_0} = 2 - (\alpha_1^2/d_0) > 0. \) Thus, (ii) holds.

If \( \tau \neq 0, \alpha_1 > 0, \) and \( d_1 = d_0, \) equation (9) has roots \( \lambda_1 = 0 \) and \( \lambda_2 = -\alpha_1 < 0. \) When \( \tau \neq 0, \) let \( \lambda = \omega e^{i\theta} \) be a root of equation (9); then, we have
\[\begin{align*}
  -\omega^2 + d_0 - d_0 \cos \omega \tau &= 0, \\
  \alpha_1 \omega + d_0 \sin \omega \tau &= 0, \\
  \omega^4 + (\alpha_1^2 - 2d_0) \omega^2 &= 0. \tag{11}
\end{align*}\]

Let \( \omega^2 = t > 0; \) then, equation (12) can be rewritten as
\[t^2 + pt = 0, \tag{13}\]
where \( p = \alpha_1^2 - 2d_0. \) If \( d_0 < 0, \) it results in \( p > 0. \) Clearly, equation (13) has no positive roots. Thus, (iii) holds. This completes the proof.

Next, we will investigate the B-T bifurcation of system (7) near \( (d_0, \tau_0) \) by choosing \( d_1 \) and \( \tau \) as bifurcation parameters.

Taking \( t = (t/\tau), \) \( d_1 = d_0 + \mu_1, \) and \( \tau = \tau_0 + \mu_2, \) system (5) can be rewritten as
\[\begin{align*}
  \dot{x} &= (\tau_0 + \mu_2) y, \\
  \dot{y} &= (\tau_0 + \mu_2)(-\alpha_1 y + c_1 x + c_2 x^2 + c_3 x^3 + c_4 x^4 + c_5 x^5 \\
  &+ (d_0 + \mu_1) x(t-1) + d_2 x^2 (t-1) + d_3 x^3 (t-1)),
\end{align*}\] (14)
where \( \mu_1 \) and \( \mu_2 \) are perturbation parameters.

The phase space \( C = C([-1,0]; R^2) \) is chosen as the Banach space of the continuous mappings from \([-1,0]\) to \( R^2. \) For \( \forall \phi \in C, \) we define \( x_1(t + \theta) \) and \( \|x\| = \sup_{-1 < \theta < 0}|x(\theta)|. \) System (14) becomes
\[\dot{x}(t) = EX_x + FX_x, \tag{15}\]
where \( E \) and \( F \) are operators, given by
\[\begin{align*}
  E\psi &= \hat{\phi} D(E) \psi \in C([-1,0]; R^2), \\
  \hat{\phi}(0) &= \int_1^0 d\eta(\theta) \psi(-\theta), \\
  d\eta(\theta) &= (A_0 \delta(\theta) + B_0 \delta(\theta + 1)) d\theta, \\
  F\psi &= \begin{cases}
    0, & \theta \in [-1,0), \\
    L_1(\mu)x_1 + G(x_1, \mu), & \theta = 0,
  \end{cases}
\end{align*}\] (16)

where \( \delta(\theta) \) is the Dirac delta function, \( A_0 = \begin{pmatrix} 0 & \tau_0 \\ c_1 \tau_0 & -\alpha_1 \tau_0 \end{pmatrix}, \) and \( B_0 = \begin{pmatrix} 0 & 0 \\ d_0 \tau_0 & 0 \end{pmatrix}. \)

\[L_1(\mu)x_1 = \begin{pmatrix} 0 & \mu_2 y(0) \\ -\mu_2 \alpha_1 y(0) + c_1 \mu_2 x(0) + d_2 \mu_2 x(-1) + \tau_0 \mu_4 x(-1) \end{pmatrix},
\[G(x_1, \mu) = \begin{pmatrix} c_2 \tau_0 x^2(0) + c_1 \tau_0 x^2(0) + c_2 \tau_0 x^2(0) + c_2 \tau_0 x^2(0) \end{pmatrix}
\[+ \tau_0 d_2 x^2(-1) + \tau_0 d_3 x^3(-1).\] (17)

From Lemma 1, equation (6) has a double-zero root, and all other eigenvalues have negative real parts. Let \( \Lambda \) be the set of eigenvalues with zero real part; \( C \) can be decomposed as \( C = A \oplus B, \) where \( A \) is the generalized eigenspace associated with \( \Lambda \) which has two zero eigenvalues and \( A^* \) is the space adjoint with \( A, \) and \( B = \{ \phi \in C : \langle \phi, \psi \rangle = 0, \forall \phi \in A^* \}. \) Next, we define
\[E^* \psi = -\hat{\psi}, D(E^*) \psi = \psi \in C([-1,0]; R^2*): \psi(0) = -\int_1^0 d\eta(\theta) \psi(-\theta),\] (18)
and the bilinear form on \( C \times C^* \) is
\[ \langle \psi, \varphi \rangle = \psi(0)\varphi(0) - \int_{-1}^{0} \psi(\xi - \theta)d\eta(\theta)\varphi(\xi)d\xi. \quad (19) \]

**Lemma 3** (see [24, 28, 29]). The bases \( \Phi \) and \( \Psi \) for \( A \) and \( A^\star \) can be chosen such that \( \langle \Psi, \Phi \rangle = I, \Phi = \Phi \left( \begin{array}{cc} 0 & 1 \\ 0 & 0 \end{array} \right), \) and

\[
\Psi = \begin{pmatrix}
0 & 1 \\
0 & 0
\end{pmatrix} \Psi, \text{ where } \Phi(\theta) = (\psi_1(\theta), \psi_2(\theta))(\theta \in [-1,0]),
\]

\[
\psi(s) = \text{col}(\psi_1(s), \psi_2(s)) (s \in [0,1]), \quad \psi_1(\theta) = \phi_1^0 \in \mathbb{R}^\star \setminus \{0\}, \quad \psi_2(\theta) = \phi_2^0 + \phi_1^0 B, \phi_1^0 \in \mathbb{R}^\star, \text{ and } \psi_2(s) = \psi_2^0 \in \mathbb{R}^\star \setminus \{0\}, \quad \psi_1(s) = \psi_1^0 - s\psi_2^0, \psi_2^0 \in \mathbb{R}^\star, \text{ which satisfy}
\]

(i) \( (A_0 + B_0)\psi_1^0 = 0. \)

(ii) \( (A_0 + B_0)\psi_2^0 = (B_0 + I)\psi_1^0. \)

(iii) \( \psi_1^0(A_0 + B_0) = 0. \)

(iv) \( \psi_1^0(A_0 + B_0) = \psi_1^0(B_0 + I). \)

(v) \( \psi_1^0 \phi_2^0 - (1/2)\psi_2^0 B_0 \psi_1^0 + \psi_1^0 B_0 \phi_1^0 = 1. \)

(vi) \( \psi_1^0 \phi_2^0 - (1/2)\psi_2^0 B_0 \phi_1^0 + \psi_1^0 B_0 \phi_2^0 + (1/6) \psi_1^0 B_0 \phi_2^0 \psi_1^0 - (1/2) \psi_2^0 B_0 \phi_2^0 = 0. \)

By calculating, we can obtain

\[
\Phi(\theta) = \left( \begin{array}{cc} 1 & b^* + \theta \\ 0 & 1 \end{array} \right), \quad \Psi(s) = \begin{pmatrix}
-s\alpha_1 d^* + s d^* - \frac{d^*}{\alpha_1} t_0 \\
\alpha_1 d^* & d^*
\end{pmatrix}, \quad (20)
\]

where \( b^* = (\alpha_1^2 t_0^2 + 6 + 3\alpha_1 t_0 + 6\alpha_1 t_0 + 3\alpha_1^2 t_0^2) \) and \( d^* = (2t_0/2 + \alpha_1 t_0). \)

Let \( X = \Phi Z + W, \) where \( Z \in \mathbb{R}^2 \) and \( W \in B, \) namely,

\[
f^{1}_{2(2,0,\theta)} = \begin{pmatrix} h_1 Z_1^2 + h_2 Z_2 Z_1 + h_3 Z_1^3 + h_4 Z_1 Z_2 + h_5 Z_1^2 + h_6 Z_2^2 + h_7 Z_1 Z_2 + h_8 Z_2 + h_9 Z_1 + h_{10} Z_2 + h_{11} Z_1 + h_{12} Z_2 + h_{13} Z_1 + h_{14} Z_2 + h_{15} Z_1 + h_{16} Z_2 & f_1 Z_1 + f_2 Z_2 + f_3 Z_1^2 + f_4 Z_2 + f_5 Z_1 Z_2 + f_6 Z_1 + f_7 Z_2 \end{pmatrix} \quad (24)
\]

where

\[
\begin{align*}
h_1 &= \frac{d^*}{\alpha_1} (c_2 + d_2), h_2 = \frac{d^*}{\alpha_1} (2c_2 b^* + 2d_2 (b^* - 1)), \\
h_3 &= \frac{d^*}{\alpha_1} (c_2 b^* + d_2 (b^* - 1)), h_4 = \frac{d^*}{\alpha_1}, \\
h_5 &= 0, h_6 = -\frac{d^*}{\alpha_1} (b^* - 1), h_7 = -\frac{d^*}{\alpha_1} \left( \frac{\alpha_1}{t_0} - d_0 \right), \\
f_1 &= d^* t_0 (c_2 + d_2), f_2 = d^* t_0 (2c_2 b^* + 2d_2 (b^* - 1)), \\
f_3 &= d^* t_0 (c_2 b^* + d_2 (b^* - 1)), \\
f_4 &= d^* t_0, f_5 = 0, f_6 = d^* t_0 (b^* - 1), f_7 = -d_0 d^*.
\end{align*}
\]

The following normal form with versal unfolding on the center manifold can be obtained by some calculations:

\[
\begin{align*}
\dot{Z}_1 &= Z_2, \\
\dot{Z}_2 &= \lambda_1 Z_1 + \lambda_2 Z_2 + \eta_1 Z_1^2 + \eta_2 Z_1 Z_2,
\end{align*}
\]

where \( \lambda_1 = d^* t_0 \mu_1, \lambda_2 = t_0 (b^* - 1) d^* \mu_1 - d_0 d^* \mu_2, \eta_1 = d^* t_0 (c_2 + d_2), \) and \( \eta_2 = -2d^*/\alpha_1 (c_2 + d_2) + d^* t_0 (2c_2 b^* + 2d_2 (b^* - 1)). \)

The detailed calculations can be found in Appendix.

By calculating, we can get

\[
\left( \frac{\partial (\lambda_1, \lambda_2)}{\partial (\mu_1, \mu_2)} \right)_{\mu=0} = -d_0 (d^*)^2 t_0 \neq 0. \quad (27)
\]

Thus, the map \( (\mu_1, \mu_2) \rightarrow (\lambda_1, \lambda_2) \) is regular, and system (14) is equivalent to the normal form (26), where \( \eta_1 \cdot \eta_2 > 0. \)
Let $Z_1 \longrightarrow (\eta_1/\eta_2^2)(Z_1 - \lambda_1 (\eta_2^2/2\eta_1^2))$, $Z_2 \longrightarrow - (\eta_1^2/\eta_2^3)Z_2$, and $t \longrightarrow - (\eta_1/\eta_2) \tau$; system (26) can be rewritten as

$$\begin{align*}
\dot{Z}_1 &= Z_2, \\
\dot{Z}_2 &= v_1 + v_2Z_2 + Z_1 - Z_1Z_2,
\end{align*}$$

(28)

where $v_1 = - (\lambda_1^2\eta_1^2/4\eta_1^3)$ and $v_2 = - \lambda_1 (\eta_1/\eta_2) - \lambda_1 (\eta_2^2/2\eta_1^2)$.

System (28) as a universal unfolding [30, 31] with codimension-2 has been well studied. We can acquire the complete bifurcation diagram and topological classification of the trajectory of system (28), and the bifurcation diagram of system (28) on the perturbation parameter $v_1$ and $v_2$ planes is shown in Figure 1 [17, 21]. Furthermore, we give a concise form to list the conclusion as follows:

(1) System (28) undergoes a saddle-node bifurcation on the set

$$SN = \{(v_1, v_2) : v_1 = 0, v_2 \neq 0\}$$

$$= \left\{ (\mu_1, \mu_2) : \mu_1 = 0, -\left[ \tau_0 (b^* - 1)d^* \mu_1 - d_0d^* \mu_2 \eta_2^2/\eta_1^2 \eta_1 - d^* \tau_0 \mu_1 \eta_2^2/2\eta_1^3 \right] \neq 0 \right\}.$$  \hspace{1cm} (29)

(2) System (28) undergoes a stable Hopf bifurcation on the set

$$H = \{(v_1, v_2) : v_1 = -v_2^2, v_2 > 0\}$$

$$= \left\{ (\mu_1, \mu_2) : \frac{(d^* \tau_0 \mu_1)^2}{4\eta_1^3} \eta_2^4 \left( \tau_0 (b^* - 1)d^* \mu_1 - d_0d^* \mu_2 \eta_2^2/\eta_1^2 \eta_1 + (d^* \tau_0 \mu_1) \eta_2^2/2\eta_1^3 \right)^2 \right\}.$$  \hspace{1cm} (30)

(3) System (28) undergoes a saddle homoclinic bifurcation on the set

$$T = \{(v_1, v_2) : v_1 = \frac{-49}{25}v_2^2\}$$

$$= \left\{ (\mu_1, \mu_2) : \frac{(d^* \tau_0 \mu_1)^2}{4\eta_1^3} \eta_2^4 \left( \tau_0 (b^* - 1)d^* \mu_1 - d_0d^* \mu_2 \eta_2^2/\eta_1^2 \eta_1 + (d^* \tau_0 \mu_1) \eta_2^2/2\eta_1^3 \right)^2 \right\}.$$  \hspace{1cm} (31)

### 3. Numerical Simulation

In this section, we use the dde23 method in MATLAB and show some numerical simulations to illustrate the analysis results given in the previous sections.

In order to easily verify the obtained results, we choose parameters $a_1 = 1$, $a_2 = 0.1$, $a_3 = 1$, $a_4 = 1$, $\delta = 0.8$, $A_1 = -1$, and $A_2 = 1$. Furthermore, based on results of Lemmas 1 and 2, we can calculate $x_0 = 0.634$, $\tau_0 = 0.344$, $d_0 = -2.902$, $\delta = -0.646$, $c_1 = 5.02$, $c_2 = -3.17$, $c_3 = -1$, $d_1 = -1.902$, and $d_2 = -1$. In Figure 1, the bifurcation diagrams of system (16) are composed of codimension-2 bifurcation point $(v_1, v_2) = (0, 0)$ and three codimension-1 curves (saddle-node bifurcation curve, Hopf bifurcation curve, and saddle homoclinic bifurcation curve). When the parameters $v_1$ and $v_2$ change in different regions, system (16) will produce different dynamic properties.

To easily analyze the dynamics of system (7), we fix $\mu_1 = -0.001$ and only choose the value of $\mu_2$ to change. From Lemma 2, the $B-T$ bifurcation point $(d_0, \tau_0) = (-2.902, 0.344)$ is obtained. Thus, we give some numerical examples as follows:

(i) If setting $(\mu_1, \mu_2) = (-0.001, 0.006)$, there exist a saddle and a stable focus in region $A_3$ (see Figure 1), as shown in Figures 2 and 3

(ii) Fix $(\mu_1, \mu_2) = (-0.001, 0.026)$; Figures 4 and 5 show that a stable periodic solution occurs when bifurcation parameters pass through the Hopf bifurcation line $H$ in region $A_3$ (see Figure 1)

(iii) Fix $(\mu_1, \mu_2) = (-0.001, 0.1664)$; Figures 6–8 show that a closed orbit exists through the homoclinic bifurcation line $T \in \Omega$ (see Figure 1)
\[ v_1 = -v^2_2 \]
\[ v_1 = -(49/25)v^2_2 \]

**Figure 1:** The bifurcation diagram (a) and phase portraits (b) of system (15) on the perturbation parameter \( v_1 \) and \( v_2 \) planes.

**Figure 2:** Waveform plot of system (7) near \((d_0, \tau_0) = (-2.902, 0.344)\) with parameters \((\mu_1, \mu_2) = (-0.001, 0.006)\), \( \tau = 0.35 \), \( \text{tspan} = [0, 500] \), and \((x_0, y_0) = (0.1, 0.2)\).

**Figure 3:** State trajectory of system (7) near \((d_0, \tau_0) = (-2.902, 0.344)\) with parameters \((\mu_1, \mu_2) = (-0.001, 0.006)\), \( \tau = 0.35 \), \( \text{tspan} = [0, 500] \), and \((x_0, y_0) = (0.1, 0.2)\).
In this work, a shape memory alloy oscillator with delayed feedback has been analyzed. We mainly choose the two parameters $d_1 = A_1 + 3x_0^2A_2$ and $\tau$ to investigate the B-T bifurcation of system (6). It is demonstrated that the feedback parameters $A_{1,2}$ and time delay $\tau$ have an important influence on the shape memory alloy oscillator. As the two parameters of the SMA oscillator change, the conditions for the occurrence of B-T bifurcation and some phase portraits and bifurcation diagrams are given. By using the CMT and NFT of functional differential equations, we investigate some typical codimension-1 bifurcations such as saddle-node bifurcation, Hopf bifurcation, and saddle homoclinic bifurcation. Some numerical simulations further verify the obtained analytic results.

In our paper, second-order terms of the normal form on a center manifold are given, but the higher order is not investigated. System (2) or (3) is only discussed by considering $k\cos(\delta t)$ as the control parameter $\delta$ (see [22]).
However, the periodic force $k \cos(\theta t)$ has an important
effect on the vibration and memory characteristics of the SMA system. Therefore, further discussion and analysis of
the SMA system will be our future work.

**Appendix**

The following calculations of the norm forms of the equation
are based on [25, 26, 32, 33].

Let $x = \Phi z + w$. Then, system $\dot{x}(t) = E x_t + F x_t$ can be
decomposed into the following form:

$$
\begin{cases}
\dot{z} = Bz + \psi(0)F(\Phi z + w, \mu), \\
\dot{w} = A_{\psi}w + (I - \pi)x_0F(\Phi z + w, \mu),
\end{cases}
$$

(A.1)

where $A_{\psi}$ is the restriction of $A_0$ as an operator from $Q^1$ to
the Banach space $\text{ker} \pi$. Employing Taylor expansion, system
(A.1) becomes

$$
\begin{cases}
\dot{z} = Bz + \frac{1}{2!} f_j(z, w, \mu), \\
\dot{w} = A_{\psi}w + \frac{1}{2!} f_j^2(z, w, \mu),
\end{cases}
$$

(A.2)

where $f_j(z, w, \mu) (j = 1, 2)$ denote the homogeneous poly-
nomials of degree $j$ in $(z, w, \mu)$. Then, we can obtain the
following form:

$$
\dot{z} = Bz + \frac{1}{2} g^1_z(z, 0, \mu) + h.o.t.
$$

(A.3)

Denote $V_2^1(z)$ as the linear space of homogeneous polynomials and $M_1^1$ as the operator on $V_2^1(R^2)$ with

$$
M_1^1 \begin{pmatrix} p_1 \\ p_2 \end{pmatrix} = \begin{pmatrix} \frac{\partial p_1}{\partial z_1} z_2 - p_2 \\ \frac{\partial p_1}{\partial z_1} z_2 - p_2 \end{pmatrix},
$$

where $\begin{pmatrix} p_1 \\ p_2 \end{pmatrix} \in V_2^1(R^2)$. $V_2^1(R^2)$ can be decomposed as $V_2^1(R^2) = \text{Im}(M_1^1) \oplus \text{Im}(M_1^1)^c$. Then, $(1/2) g^1_z(z, 0, \mu)$ can be written as

$$
g^1_z(z, 0, \mu) = \text{Project}_{\text{Im}(M_1^1)^c} f^1_z(z, 0, \mu).
$$

(A.4)

The base of $V_2^1(R^2 \times \text{ker} \pi)$ can be chosen as follows:
where

\[
e_1 = \frac{d^*}{\alpha_1^2}(c_2 + d_2), e_2 = \frac{d^*}{\alpha_1(2c_2b^* + 2d_2(b^* - 1))}, e_3 = \frac{d^*}{\alpha_1}(c_2b^* + 2d_2(b^* - 1)^2),
\]

\[
e_4 = \frac{d^*}{\alpha_1}, e_5 = 0, e_6 = \frac{d^*}{\alpha_1(2c_2b^* + 2d_2(b^* - 1))}, e_7 = -\frac{d^*}{\tau_0}(\frac{\alpha_1}{\tau_0 - d_0}),
\]

\[
f_1 = d^*\tau_0\left(1 + d_2\right), f_2 = d^*\tau_0\left(2c_2b^* + 2d_2(b^* - 1)\right), f_3 = d^*\tau_0\left(c_2b^* + 2d_2(b^* - 1)^2\right),
\]

\[
f_4 = d^*\tau_0, f_5 = 0, f_6 = d^*\tau_0(b^* - 1), f_7 = -d_0\alpha.
\]

Thus, the following normal form with versal unfolding on the center manifold can be obtained by some calculations:

\[\begin{cases}
Z_1 = Z_2, \\
Z_2 = \lambda_1 Z_1 + \lambda_2 Z_2 + \eta_1 Z_1^2 + \eta_2 Z_1 Z_2,
\end{cases}\]

(A.10)

where \(\lambda_1 = d^*\tau_0\mu_1\), \(\lambda_2 = \tau_0(b^* - 1)d_0\mu_1 - d_0\alpha\), \(\eta_1 = d^*\tau_0\left(1 + d_2\right)\), \(\eta_2 = -\left(2d^*/\alpha_1\right)(c_2 + d_2) + d^*\tau_0\left(2c_2b^* + 2d_2(b^* - 1)\right)\).

Data Availability

The data used to support the findings of this study are included within the article.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Authors’ Contributions

J. B. Wang carried out the study. L. F. Ma supervised the work and provided the support of funds. All authors read and approved the final manuscript.

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