First successful renormalization of a QCD-inspired Hamiltonian

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Abstract
The long standing problem of non perturbative renormalization of a gauge field theoretical Hamiltonian is addressed and explicitly carried out within an (effective) light-cone Hamiltonian approach to QCD. The procedure is in line with the conventional ideas: The Hamiltonian is first regulated by suitable cut-off functions, and subsequently renormalized by suitable counter terms to make it cut-off independent. The formalism is applied to physical mesons with a different flavor of quark and anti-quark. The excitation spectrum of the $\rho$-meson with its excellent agreement between theory and experiment is discussed as a pedagogical example.

1. Introduction
When starting in 1984 with Discretized Light-Cone Quantization (DLCQ) and with a revival of Dirac’s Hamiltonian front form dynamics, all challenges of a gauge field Hamiltonian theory were essentially open questions, particularly the non perturbative bound state problem, the many-body aspects, regularization, renormalization, confinement, chirality, vacuum structure and condensates, just to name a few. The step from the gauge field QCD Lagrangian down to a nonrelativistic Schrödinger equation was completely mysterious. Now we know better. We have understood, for example, that the chiral phase transition, in which the quarks are supposed to get their mass, is not the major challenge. The challenge is to understand what happens after the phase transition, at zero temperature. The challenge is to understand the spectrum of physical hadrons and to get the corresponding eigenfunctions, the light-cone wave functions. The light-cone wave functions $\Psi$ for a hadron with mass $M$ encode all possible quark and gluon momentum, helicity and flavor correlations and, in principle, are obtained by diagonalizing the QCD light-cone Hamiltonian

$$H_{LC}|\Psi_i\rangle = M_i^2|\Psi_i\rangle,$$

in a complete basis of Fock states with increasing complexity. For example, the positive pion has the Fock expansion:

$$|\Psi_{\pi^+}\rangle = \sum_n \langle n|\pi^+\rangle|n\rangle = \Psi_{ud/\pi}(x_i, k_{\perp i})|ud\rangle + \Psi_{udg/\pi}(x_i, k_{\perp i})|udg\rangle + \ldots,$$

representing the expansion of the exact QCD eigenstate at scale $\Lambda$ in terms of non-interacting quarks and gluons. The particles in a Fock state have longitudinal light-cone momentum fractions $x_i = k_{\perp i}^+/P^+$ and relative transverse momenta $k_{\perp i}$. The form of $\Psi_{n/H}(x_i, k_{\perp i})$ is invariant under longitudinal and transverse boosts; i.e., the light-cone wave functions expressed in the relative coordinates $x_i$ and $k_{\perp i}$ are independent of the total momentum $(P^+, P_\perp)$ of the hadron. The first term in the expansion is referred to as the valence Fock state, as it relates to the hadronic description in the constituent quark model. The higher terms are related to the sea components of the hadronic structure. It has been shown that the rest of the light-cone wave function is determined once the valence Fock state is known [4][5], with explicit expressions given in [5].

The key issue is to overcome the problem of any gauge theory, that the unregulated theory exposes logarithmic singularities. The problem of regularization and renormalization has been solved in the perturbative context of scattering theory, but not in the non perturbative context of a Hamiltonian. It is addressed to in the first two sections and applied in the remainder of this paper.
It took several years to realize that it is the Feynman four-momentum transfer across a vertex, $Q^2 = -(k_1 - k_2)^2$, which governs any effective interaction. The minimal requirement for such a form factor is $\lim_{Q \to 0} F(Q; \Lambda) = 1$ and $\lim_{Q \to \infty} F(Q; \Lambda) = 0$. The job would be done by a step function, $F(Q) = \Theta(Q^2 - \Lambda^2)$. The limit $\Lambda \to 0$ suppresses the interaction all together, the limit $\Lambda \to \infty$ restores

$\langle k_1, h_1|V|k_2, h_2; k_3, h_3 \rangle \Rightarrow \langle k_1, h_1|V|k_2, h_2; k_3, h_3 \rangle F(Q)$ .

(1)

**2. Regularization**

Canonical field theory with the conventional QCD Lagrangian allows to derive the components of the total canonical four-momentum $P^\mu$. Its front form version \[8\] rests on two assumptions, the light cone gauge $A^+ = 0$ \[6\] and the suppression of all zero modes \[3, 7\]. The front form vacuum is then trivial. I find it helpful to discuss the problem in terms of DLCQ \[1, 3\]. In the back of my mind I visualize an explicit finite dimensional matrix representation of the Light-Cone Hamiltonian as it occurs for finite harmonic resolution. Such one is schematically displayed in Fig. 2 of \[3\]. All of its matrix elements are finite for any finite $x$ and $k_\perp$.

The problem arises for ever increasing harmonic resolution, on the way to the continuum limit: The numerical eigenvalues are numerically unstable and diverge logarithmically \[16\], contrary to the calculations in 1+1 dimension \[11\]. One must regulate the theory.

As usual, regularization is not unique, and can be done in many ways. But not all regularization techniques of the past are applicable in an Hamiltonian approach. Dimensional regularization, for example, is not applicable in a matrix approach which is stuck with the precisely 3+1 dimensions of the physical world. Perturbative regularization \[8\] is not applicable in the non-perturbative context. The Fock space regularization of Lepage and Brodsky \[6\], see also \[3\] has blocked renormalization for many years. It was impossible to find suitable counter terms. In recent work, the invariant mass squared regularization has been abandoned in favor of Pauli-Villars regularization \[9, 10\]. But thus far it is unclear how the approach \[10\] is applicable to the spectra of physical mesons. After applauding the light-cone approach \[11\], Wilson and collaborators \[12\] have attempted to base their considerations almost entirely on a renormalization group analysis, but no concrete technology has emerged thus far.

In recent years I have favored an other regularization, which allows for renormalization and which is technically sufficiently simple to be carried out explicitly. All divergences can be traced back to Dirac’s relativistic vertex interaction $\langle k_1, h_1|V|k_2, h_2; k_3, h_3 \rangle$, in which some particle ‘1’ is scattered into two particles ‘2’ and ‘3’ with their respective four-momenta $k$ and helicities $h$, see Fig. I. The matrix element for bremsstrahlung, for example, is proportional to $k_\perp$, $\langle k_1, \uparrow|V|k_2, \uparrow; k_1, \uparrow \rangle \propto |x\rangle$ \[8\]. The vertex interaction is regulated by multiplying each (typically off-diagonal) matrix element with a regulating form factor $F$:

$$\langle k_1, h_1|V|k_2, h_2; k_3, h_3 \rangle \Rightarrow \langle k_1, h_1|V|k_2, h_2; k_3, h_3 \rangle F(Q) .$$

(1)
the interaction and its problems. Any finite value of $\Lambda^2$ restricts $Q^2$ to be finite and eliminates the singularities. But the sharp cut-off generates problems in an other corner of the theory and $F(Q)$ must be an analytic function of $Q$, as to be seen below.

**Vertex regularization** takes thus care of the ultraviolet divergences. The (light-cone) infrared singularities are taken care of as usual by a kinematical gluon mass.

### 3. Renormalization

The non perturbative renormalization of the Hamiltonian was stuck for many years by the fact that the coupling constant $g$ and the regulator function $F(Q)$ multiply each other in Eq. (1). It was always clear that one may add *non local counter terms* [1], but is was not clear how they could be constructed. Progress has come from recent work on a particular model [19], which did allow to formulate a paradigmatic example in modern renormalization theory.

Here is the general but abstract procedure.

Suppose to have solved $H_{\Lambda C}\langle \Psi_i \rangle = M_i^2 \langle \Psi_i \rangle$ for a fixed value of the 7 ‘bare’ parameters in the Lagrangian, for the coupling constant $g = g_0$ and the 6 flavor quark masses $m_f = (m_f)_0$, and for a fixed value of exterior cut-off scale $\Lambda = \Lambda_0$. Suppose further that these 7+1 parameters are chosen such, that the calculated $M_i^2$ agree with the corresponding experimental values. Next, suppose to change the cut-off by a small amount $\delta \Lambda$. Every calculated eigenvalue will then change by $\delta M_i^2$. Renormalization theory is then the attempt to reformulate the Hamiltonian, such, that all changes $\delta M_i^2$ vanish identically.

The fundamental renormalization group equation is therefore *for all eigenstates* $i$:

$$dM_i^2 \bigg|_0 = dM_i^2 \bigg|_{g=g_0,m_f=(m_f)_0,\Lambda=\Lambda_0} = 0 .$$

Equivalently one requires that the Hamiltonian is stationary, $\delta H_{\Lambda C} \bigg|_0 = 0$, with respect to small $\delta \Lambda$. Hence forward reference to the renormalization point $(g_0, m_f, \Lambda_0)$ will be suppressed.

The Hamiltonian can be made stationary by making $g$ and the $m_f$ functions of $\Lambda$, by introducing physical coupling constants and masses, $\overline{g}$ and $m_f$, respectively, which themselves are functions of the bare $g$ and $m_f$, and which are functionals of the regulator $F = F$. The variation of $H_{\Lambda C}$ reads then

$$\delta H_{\Lambda C} = \frac{\partial H_{\Lambda C}}{\partial \overline{g}} + \sum_f \frac{\partial H_{\Lambda C}}{\partial m_f} F + \frac{\partial H_{\Lambda C}}{\partial F} = 0 ,$$

with the familiar variational derivatives. However, since $\overline{g}$ and $m_f$ are themselves functionals of $F$, this reduces to

$$\delta H_{\Lambda C} = \frac{\partial H_{\Lambda C}}{\partial F} = 0 .$$

The fundamental equation of renormalization theory (2) is then replaced by

$$\delta F = \delta \Lambda \frac{\partial F}{\partial \Lambda} = 0 .$$

It can be solved by counter term technology, as follows. A counter term is added to the Hamiltonian, whose interaction has exactly the same structure except that the regulator $F(Q)$ is replaced by $C(Q)$. This defines

$$\overline{F}(Q, \Lambda) = F(Q, \Lambda) + C(Q, \Lambda) ,$$

subject to the constraint that the counter term vanishes at the renormalization point, $C(Q, \Lambda)|_{\Lambda=\Lambda_0} = 0$. The fundamental equation (3) defines then a differential equation

$$dC(Q; \Lambda)/d\Lambda = -dF(Q; \Lambda)/d\Lambda ,$$

which, in its integral form, includes the initial condition

$$C(Q, \Lambda) = -\int_{\Lambda_0}^{\Lambda} ds \frac{dF(Q, s)}{ds} = F(Q, \Lambda_0) - F(Q, \Lambda) .$$

The renormalized regulator function, $\overline{F} = F + C$, is *manifestly independent of $\Lambda$*:

$$\overline{F}(Q, \Lambda) = F(Q, \Lambda_0) .$$
By construction, the value of \( \Lambda_0 \) is determined by experiment. One should emphasize an important point: In deriving Eq. (6), use was made of assuming the regulator function has well defined derivatives with respect to \( \Lambda \). The theta function of the sharp cut-off, however, is a distribution with only ill defined derivatives. This raises an other important point: If \( F(Q, \Lambda) \) is a function of \( Q/\Lambda \) other than a theta function, one must specify how the function approaches the limiting values of 1 and of 0. The case of the ‘soft’ regulator \( F(Q, \Lambda) = \Lambda^2/(\Lambda^2 + Q^2) \) is only a very special example. In a more general approach the soft regulator plays the role of a generating function

\[
F(Q, \Lambda) = \left[ 1 + \sum_{n=1}^{N} (-1)^n s_n \Lambda^n \frac{\partial^n}{\partial \Lambda^n} \right] \frac{\Lambda^2}{\Lambda^2 + Q^2} .
\]

The partials \( \Lambda^n \frac{\partial^n}{\partial \Lambda^n} \) are dimensionless and independent of a change in \( \Lambda \). The arbitrarily many coefficients \( s_n \) are renormalization group invariants and, as such, subject to be determined by experiment.

4. The effective (light-cone) Hamiltonian

In a field theory, one is confronted with a many-body problem of the worst kind: Not even the particle number is conserved. For to formulate effective Hamiltonians more systematically, a novel many-body technique had to be developed, the method of iterated resolvents \([5, 14]\), whose details are not important here. Important is that the effective light-cone Hamiltonian \( H_{\text{eLC}} \) has the same eigenvalue as the full light-cone Hamiltonian \( H_{\text{LC}} \) and that it generates the bound state wave function of valence quarks by an one-body integral equation in \( (x, \vec{k}_1) \):

\[
M^2 \psi_{h_1 h_2}(x, \vec{k}_1) = \left[ \frac{\vec{m}_1^2 + \vec{k}_1^2}{x} + \frac{\vec{m}_2^2 + \vec{k}_2^2}{1-x} \right] \psi_{h_1 h_2}(x, \vec{k}_1) - \frac{1}{3\pi^2} \sum_{h'_1, h'_2} \int \frac{dx' d^2 \vec{k}'_1}{\sqrt{x'(1-x')x(1-x)}} F(Q) F(Q) \left( \frac{\vec{m}(Q) q_{\ell_2}}{2Q^2} + \frac{\vec{m}(Q) q_{\ell_1}}{2Q^2} \right) \left[ \overline{\psi}(k_1, h_1) \gamma^\mu u(k_1', h_1') \right] \left[ \overline{\psi}(k_2, h_2) \gamma^\mu v(k_2, h_2) \right] .
\]

One has achieved \( H_{\text{eLC}} | \Psi_{q_1 q_2} \rangle = M^2 | \Psi_{q_1 q_2} \rangle \). Here, \( M^2 \) is the eigenvalue of the invariant-mass squared. The associated eigenfunction \( \psi_{h_1 h_2}(x, \vec{k}_1) \) is the probability amplitude \( \langle x, \vec{k}_1; h_1, 1-x, -\vec{k}_1; h_2 | \Psi_{q_1 q_2} \rangle \) for finding the quark with momentum fraction \( x \), transversal momentum \( \vec{k}_1 \) and helicity \( h_1 \), and correspondingly the anti-quark. Expressions for the (effective) quark masses \( \vec{m}_1 \) and \( \vec{m}_2 \) and the (effective) coupling function \( \vec{m}(Q) \) are given in \([14]\). \( Q_q \) and \( Q_\bar{q} \) are the Feynman momentum transfers of quark and anti-quark, respectively, and \( u(k_1, h_1) \) and \( v(k_2, h_2) \) are their Dirac spinors in Leigie Brodsky convention \([6]\), given explicitly in \([4]\). Finally, the form factors \( F(Q) \) restrict the range of integration and regulate the interaction. Note that the equation is fully relativistic and covariant. Note also that Eq. (8) is valid only for quark and anti-quark having different flavors \([5, 14]\). The additional annihilation term for identical flavors is omitted here, and presently investigated by Krahl \([15]\). The ‘mean momentum transfer’, \( Q^2 = \frac{1}{2} (Q_q^2 + Q_{\bar{q}}^2) \), allows to replace Eq. (8) by

\[
M^2 \psi_{h_1 h_2}(x, \vec{k}_1) = \left[ \frac{\vec{m}_1^2 + \vec{k}_1^2}{x} + \frac{\vec{m}_2^2 + \vec{k}_2^2}{1-x} \right] \psi_{h_1 h_2}(x, \vec{k}_1) - \frac{1}{3\pi^2} \sum_{h'_1, h'_2} \int \frac{dx' d^2 \vec{k}'_1}{\sqrt{x'(1-x')x(1-x)}} F(Q) F(Q) \left( \frac{\vec{m}(Q) q_{\ell_2}}{2Q^2} + \frac{\vec{m}(Q) q_{\ell_1}}{2Q^2} \right) \left[ \overline{\psi}(k_1, h_1) \gamma^\mu u(k_1', h_1') \right] \left[ \overline{\psi}(k_2, h_2) \gamma^\mu v(k_2, h_2) \right] .
\]

The form factors \( F(Q) \) have made their way into the regulator function \( \overline{R}(Q) = F^2(Q) \). Krautgärtner and Trittman \([16]\) have shown how to solve numerically such an equation with a high precision. But since the numerical effort is so considerable, it is reasonable to work first with (over-)simplified models, as specified next.
The Singlet-Triplet model. Quarks are at relative rest when \( \vec{k}_\perp = 0 \) and \( x = \bar{m}_1/(\bar{m}_1 + \bar{m}_2) \). An inspection of Eq.(33) in [13] reveals that for very small deviations from the equilibrium values, the spinor matrix \( \langle h_1, h_2 | S | h_1', h_2' \rangle = [\bar{\tau}(k_1, h_1) \gamma^\mu u(k_1', h_1')] [\bar{\tau}(k_2, h_2) \gamma^\nu v(k_2, h_2)] \) is proportional to the unit matrix, \( \langle h_1, h_2 | S | h_1', h_2' \rangle \approx 4m_1m_2 \delta_{h_1, h_1'} \delta_{h_2, h_2'} \). For very large deviations, particularly for \( k_1^{\perp 2} \gg k_2^{\perp 2} \), holds \( Q^2 \approx k_2^{\perp 2} \) and \( \langle \uparrow \downarrow | S | \uparrow \downarrow \rangle \approx 2k_2^{\perp 2} \).

The Singlet-Triplet (ST) model combines these aspects:

\[
\frac{\langle h_1, h_2 | S | h_1, h_2 \rangle}{Q^2} = \begin{cases} \frac{4m_1m_2}{Q^2} + 2, & \text{for } h_1 = -h_2, \\ \frac{4m_1m_2}{Q^2} & \text{for } h_1 = h_2. \end{cases}
\]

For anti parallel helicities \( h_1 = -h_2 \) (singlets) the model interpolates between two extremes: For small momentum transfer \( Q \), the ‘2’ in Eq.(10) is unimportant and the Coulomb aspects of the first term prevail. For large \( Q \), the Coulomb aspects are unimportant and the hyperfine interaction is dominant. For parallel helicities \( h_1 = h_2 \) (triplets) the model reduces to the Coulomb kernel. The model over emphasizes many aspects but its simplicity has proven useful for fast and analytical calculations. Most importantly, the model allows to drop the helicity summations which technically simplifies the problem enormously.

5. The potential energy

It is possible to subtract a c-number from \( H_{eLC} \) and to define an effective Hamiltonian \( H_{\text{eff}} \) implicitly by

\[
H_{eLC} \equiv (m_1 + m_2)^2 + 2 (m_1 + m_2) H_{\text{eff}}, \quad H_{\text{eff}} | \varphi \rangle = E | \varphi \rangle.
\]

Its eigenvalues have the dimension of an energy. Note that mass and energy in the front form, on the light cone, are related by

\[
M^2 = (m_1 + m_2)^2 + 2 (m_1 + m_2) E,
\]

and not by \( M^2 = (m_1 + m_2)^2 + 2 (m_1 + m_2) E + E^2 \), as usual. Only if the energy is negligible as compared to the quark masses, i.e. only if \( (E/(m_1 + m_2))^2 \ll 1 \), the two relations coincide.

A rather drastic technical simplification is achieved by a transformation of the integration variable. One can substitute the integration variable \( x \) by the integration variable \( k_z \), which, for all practical purposes, can be interpreted [3] as the \( z \)-component of a 3-momentum vector \( \vec{p} = (k_z, \vec{k}_\perp) \). For equal masses \( m_1 = m_2 = m \), the transformation is, together with its inverse,

\[
x(k_z) = \frac{1}{2} \left[ 1 + \frac{k_z}{\sqrt{m^2 + \vec{k}_\perp^2 + k_z^2}} \right], \quad k_z(x) = \left( m^2 + \vec{k}_\perp^2 \right) \frac{x - \frac{1}{2}}{x(1 - x)}. \tag{13}
\]

Inserting these substitutions into Eq.(9) and defining the reduced wave function \( \varphi \) by

\[
\psi_{h_1h_2}(x, \vec{k}_\perp) = \sqrt{\frac{A(k_z, \vec{k}_\perp)}{x(1 - x)}} \varphi_{h_1h_2}(k_z, \vec{k}_\perp), \quad A(\vec{p}) = \sqrt{1 + \frac{\vec{p}^2}{m^2}}, \tag{14}
\]

leads to an integral equation in the components of \( \vec{p} \), in which all reference to light-cone variables has disappeared. Using in addition the ST-model of Eq.(10), Eq.(9) translates for singlets identically into

\[
M^2 \varphi(\vec{p}) = 4 \left[ m^2 + \vec{p}^2 \right] \varphi(\vec{p}) - \frac{\alpha_e}{2\pi^2} \int \frac{d^3p'}{\sqrt{A(p)A(p')}} \left( \frac{4m^2}{Q^2} + 2 \right) \frac{R(Q)}{m} \varphi(\vec{p}'), \tag{15}
\]
with \( \alpha_c = \frac{4}{3} \alpha_s \). The equation for the triplets is obtained by dropping the ‘2’. In the ST-model, the helicity arguments in the wave functions can be suppressed. Applying the relation between mass and energy, as given in Eq.\((12)\), the equation is converted to
\[
E \varphi(\vec{p}) = \frac{\vec{p}^2}{2m_r} \varphi(\vec{p}) - \frac{\alpha_c}{2\pi^2} \int \frac{d^3 p'}{\sqrt{A(p)A(p')}} \left( \frac{4m^2}{Q^2} + 2 \right) \frac{R(Q)}{4m^2} \varphi(\vec{p}') ,
\]
since the reduced mass for \( m_1 = m_2 = m \) is \( m_r = m/2 \). The first term in this equation, \( \vec{p}^2/2m_r \), coincides with the kinetic energy in a conventional non-relativistic Hamiltonian. This is remarkable in view of the fact that no approximation to this extent has been made. The fully relativistic and covariant light-cone approach has no relativistic corrections in the kinetic energy!

Since the first term in Eq.\((16)\) is a kinetic energy, the second must be a potential energy — in a momentum representation. In principle, it could be Fourier transformed with \( \sqrt{A(p)A(p')} \) to a configuration space with the variable \( \vec{r} \). But due to the factor \( A(p)A(p') \) in the kernel, the resulting potential energy would be non-local. The non-locality of the potential is certainly mathematically exact. But I do not expect this to generate aspects of leading importance, and avoid it by the simplification \( A(p) = 1 \), both in Eqs.\((14)\) and \((16)\). With \( A(p) = 1 \), the mean four momentum transfer \( Q^2 \) reduces to the three momentum transfer \( q^2 = (\vec{p} - \vec{p}')^2 \). In consequence, the kernel of Eq.\((16)\) depends only on \( \vec{q} = \vec{p} - \vec{p}' \),
\[
U(\vec{q}) = -\frac{\alpha}{2\pi^2} \left( \frac{4m^2}{q^2} + 2 \right) \frac{R(q)}{4m^2} ,
\]
Its Fourier transform is a local function, which plays the role of a conventional potential energy \( V(r) \) in the Fourier transform of Eq.\((16)\), i.e. in
\[
E \psi(\vec{r}) = \left[ \frac{\vec{p}^2}{2m_r} + V(\vec{r}) \right] \psi(\vec{r}) .
\]
Here is the simple Schrödinger equation! Despite its conventional structure it is a front form equation, designed to calculate the light-cone wave function \( \psi(\vec{r}) \rightarrow \varphi(\vec{p}) \rightarrow \psi_{\vec{q}}(x, \vec{k}_\perp) \). I conclude this section with a subtitle point, which needs clarification in the future. The simplification \( A(p) = 1 \) is different from a non-relativistic approximation. The approach is certainly valid also for relativistic momenta \( p^2 \gg m^2 \), particularly Eqs.\((16)\) and \((17)\). The reason is that \( A(p) \) occurs only under the integral. There, the large momenta are suppressed by the regulator, anyway.

6. The renormalized Coulomb potential

Hence forward, I restrict consideration to the triplet case, i.e. to Coulomb kernels like \( U(\vec{q}) \sim R(q)/q^2 \). The point is that he renormalized Coulomb potential is always finite at the origin, as opposed to the conventional Coulomb potential with its \( \frac{1}{r} \)-singularity. It is instructive to verify this explicitly for the sharp cut-off as a regulator, that is for \( U(q) = -\frac{\alpha_c}{2\pi^2 q^2} \Theta(q^2 - \lambda^2) \). The Fourier transform according to Eq.\((17)\) gives \( V(r) = -\frac{\alpha_c}{r} \frac{2}{\pi} \text{Si}(\lambda r) \). Using the well known asymptotic expansions of the Integral Sine \( \text{Si}(\lambda r) \) gives \( \text{lim}_{r \to \infty} V(r) = -\frac{2}{\pi} \) and \( \text{lim}_{r \to 0} V(r) = \alpha_c \lambda (-\frac{2}{\pi} + \frac{\lambda r^2}{9\pi}) \). The regulated Coulomb potential is finite at the origin. The cut-off dependence near the origin is one of the most important insights of the present work and has a deep physical reason to be discussed below. In analogy to Eq.\((7)\) the regulator is chosen as
\[
R(q) = \left[ 1 + \sum_{n=1}^{N} (-1)^n s_n \lambda^n \frac{\partial^n}{\partial \lambda^n} \right] \frac{\lambda^2}{\lambda^2 + q^2} ,
\]
which gives straightforwardly the generalized Coulomb potential
\[
V(r) = -\frac{\alpha_c}{r} \left[ 1 + \sum_{n=1}^{N} (-1)^n s_n \lambda^n \frac{\partial^n}{\partial \lambda^n} \right] \left( 1 - e^{-\lambda r} \right) = \frac{\alpha_c}{r} \left[ -1 + e^{-\lambda r} \sum_{n=0}^{N} s_n (\lambda r)^n \right] ,
\]
potential $V(r)$ for the radius $r$ under weak coupling.

Coulomb potential $\sim \frac{1}{r}$, oscillator $\sim r^2$ (arbitrary units).

Fig. 2: Schematic behavior of the renormalized Coulomb potential, see also the discussion in the text.

The physical picture which develops is illustrated in Fig. 2. In the far zone, for sufficiently large $r$, the potential energy coincides with the conventional Coulomb potential $-\frac{\alpha_c}{r}$. Since the potential is attractive, it can host bound states which are probably those realized in weak binding. In the near zone, for sufficiently small $r$, the potential behaves like a power series $c_0 + c_1 r + c_2 r^2$ which potentially can host the bound states of strong coupling. Since Eq. (20) is an analytic function of $r$, the actual potential must interpolate between these two extremes in an intermediate zone, for example by developing a barrier of finite height. The onset of the near and intermediate regimes must occur for relative distances of the quarks, which are comparable to the Compton wave length associated with their reduced mass. If the distance is smaller, one expects deviations from the classical regime by elementary considerations on quantum mechanics, indeed.

The large number of parameters in Eq. (20) can be controlled by expressing all coefficients $\{s_n\}$ in terms of three parameters $a$, $b$, and $c$:

$$s_n = \frac{1}{n!} + \frac{a}{(n-1)!} + \frac{b}{(n-2)!} + \frac{c}{(n-3)!}. \quad (21)$$

The first 3 coefficients are then explicitly $s_0 = 1$, $s_1 = 1 + a$, and $s_2 = \frac{1}{2} + a + b$. The dimensionless Coulomb potential depends then only on three parameters: $W_N(y; a, b, c)$. In the near zone, it is at most a quadratic function of $y$, $W_N(y; a, b, c) = a + by + cy^2$, independent of $N$. The remainder starts at most with power $y^{N+1}$. A value of $a = c = 0$ and $b = 1$ should therefore yield a linear set of functions $W_N(y; a, b, c) = y$ in the near zone. As shown in Fig. 3, this happens to be true for surprisingly large values of $y$. The value of $N$ essentially controls the height of the barrier. Similarly, $W_N(y; 0, 0, 1) = y^2$ generates a set of functions which are strictly quadratic in the near zone. Again, $N$ controls the height of the barrier, as seen in Fig. 3.

7. Determining the parameters by experiment

The QCD-inspired model developed thus far has a considerable number of renormalization group invariant parameters, which must be determined once and for all by experiment. In doing this, we
Fig. 4: The invariant mass-squares of all available $\pi^+$- and $\rho^+$-states are plotted versus a counting index $n$. The straight lines correspond to $M_n^2 = M_0^2 + n\chi$, with the value $\chi = 1.39$ GeV$^2$, taken from Anisovich et al. \[22\]. The filled circles correspond to states which have been seen empirically \[21\], the empty ones correspond to the predictions \[22\]. — Plot courtesy of Shan-Gui Zhou.

Fig. 5: The continuous lines display the generalized Coulomb potential
$$V(r) = \alpha c \lambda W_N(\lambda r; a, 0, c)$$
in physical units as function of $r$, for the values $N = 4, 5, 6$ from bottom to top. The dashed line displays the harmonic approximation. The horizontal lines on the left indicate the oscillator states. — The circles indicate the experimental eigenvalues $E_n$ for the $\rho^+$. They agree with the calculated eigenvalues for $N = 6$, shown by the horizontal lines. See the discussion in the text.

Fig. 4: The invariant mass-squares of all available $\pi^+$- and $\rho^+$-states are plotted versus a counting index $n$. The straight lines correspond to $M_n^2 = M_0^2 + n\chi$, with the value $\chi = 1.39$ GeV$^2$, taken from Anisovich et al. \[22\]. The filled circles correspond to states which have been seen empirically \[21\], the empty ones correspond to the predictions \[22\]. — Plot courtesy of Shan-Gui Zhou.

have been inspired by the work of Anisovich et al. \[22\]. Enumerating the excited states of a hadron by a counting index $n = 0, 1, 2, \ldots$, these authors have found the linear relation $M_n^2 = M_0^2 + n\chi$ for practically all hadrons. As an example, I present in Fig. 4 the spectrum of the $\pi$- and the $\rho$-meson. The linear relation between mass–squared and energy on the light cone, Eq.(12), allows then to conclude that the potential energy in the near zone must be a pure oscillator,

$$V(r) = -c_t + \frac{1}{2} f_t r^2,$$

at least to first approximation, thus $b = 0$. If one addresses to reproduce the spectra of all flavor off-diagonal triplet mesons (pseudo-vector mesons), except the topped ones, one has to determine 6 parameters: The 2 constants from the oscillator model, $c_t$ and $f_t$, and the 4 effective flavor quark masses $m_u = m_d$, $m_s$, $m_c$, and $m_b$. To determine them, one needs 6 experimental triplet masses. I take from \[21\]: The ground and first excited states of the $u\bar{d}$ and $u\bar{s}$ mesons, and the ground states of $u\bar{c}$ and $u\bar{b}$. The so obtained parameter values are:

$$m_u = 0.218 \text{ GeV}, \quad m_s = 0.438 \text{ GeV}, \quad m_c = 1.749 \text{ GeV}, \quad m_b = 5.068 \text{ GeV},$$

$$c_t = 0.880 \text{ GeV}, \quad f_t = 0.0869 \text{ GeV}^3.$$

The numbers differ slightly from those in \[20\], due to choosing a different set of empirical data, but they yield about the same overall agreement with all available experimental states of pseudo-vector mesons.

Reverting the argument, one concludes as in \[20\] that the oscillator model in Eq.(22) explains quite naturally the systematics found by Anisovich et al. \[22\].

8. Relating the oscillator model to QCD

The oscillator model in Eq.(22) is only the harmonic approximation to the QCD–inspired, generalized Coulomb potential in Eq.(20). Their parameters are related obviously by $c_t = -\alpha c \lambda a$, $b = 0$, and
\[ f_L = 2\alpha_c \lambda^3 c. \] One needs more experimental information to pin down the value of \( a, c \) and \( N \). Choosing \( \lambda \) as the QCD scale, \( i.e. \lambda = 200 \text{ MeV} \), one can use the expressions for \( \overline{\alpha}(Q) \) in \([14]\) to calculate \( \alpha_s \equiv \overline{\alpha}(0) \) from the measured value of the coupling constant at the \( Z \)-mass \( M_Z = 91.2 \text{ GeV} \), \( \overline{\alpha}(M_Z) = 0.118 \), thus \( \alpha_s \equiv \overline{\alpha}(0) = 0.1695 \). Having fixed \( \alpha_c = \frac{4}{3} \alpha_s \) and \( \lambda \) allows to calculate \( a \) and \( c \) from \( c_t \) and \( f_L \), \( i.e. \) \( a = -19.5 \) and \( c = 24.0 \), one can draw the generalized Coulomb potential \( V(r) = \alpha_c \lambda W_N(\lambda r; a, 0, c) \) for different \( N \) as done in Fig. 5. The ‘experimental’ eigenvalues \( E_0, E_1, E_2 \), and \( E_3 \) for the \( \rho \)-meson are obtained from \( E_n = (M_n^2 - 4m_0^2)/(4m_0) \) and also inserted together with the empirical limits of error. The experimental error \( \delta E_{\rho,3} \sim \pm 0.5 \text{ GeV} \) (thus \( \delta M_{\rho,3} \sim \pm 0.1 \text{ GeV} \)) is hypothetical, since \( M_{\rho,3} \) is not confirmed. Taking it for granted, the lowest possible value for \( N \) is thus \( \tilde{N} = 6 \). This completes the determination of all parameters. They are universal within the model. I thank Omer\[23\] for the exact eigenvalues prior to publication.

9. Summary and Conclusions

This work is an important mile stone on the long way from the canonical Lagrangian for quantum chromodynamics down to the composition of physical hadrons in terms of their constituting quarks and gluons, by the eigenfunctions of a Hamiltonian. As part of a on-going effort \([24]\), a denumerable number of simplifying assumptions had to be phrased for getting a manageable formalism. Among them is the formulation of an effective interaction by the method of iterated resolvents. As long as the assumption can be relaxed the easiest in upcoming work. But its simplicity has been an advantage to unreveal the physical content of gauge theory by analytical relations.

The biggest progress of the present work is related to a consistent regularization and renormalization of a gauge theory. The ultraviolet divergences in gauge theory are caused less by the possibly large momenta \( \text{transfers} \) in the interaction. In a Hamiltonian approach, such as the present, one has not much choice else than to chop them off by a regulating form factor in the elementary vertex interaction. The form factor makes its way into a regulator function which suppresses the large momentum transfers in the Fourier transform of the Coulomb interaction.

The arbitrariness in chopping off the \( \text{large momentum transfers} \) is reflected in the arbitrariness of the potential at \( \text{small relative distances} \). It is this arbitrariness which allows for a potential pocket which binds the quarks in a hadron. The problem how to fix this arbitrariness by experiment, in practice, is less difficult than anti-cipated. It suffices to determine only three parameters: \( a, c, \) and \( N \).

The potential energy of the present work vanishes at an infinite separation of the quarks. This seems be be in conflict with the potential energies of phenomenological models which rise forever. It also seems to be in conflict with lattice gauge calculations. Is a finite ionization limit in conflict also with ‘confinement’, \( i.e. \) with the empirical fact that free quarks have not been observed? — The present model prohibits free quarks as a stable solution, since the sum of the constituent quark masses is always larger than the mass of the corresponding hadron and a pion. Free constituent quarks would hadronize very quickly into bound states. This is different from atomic physics with its free constituents, where the binding energy is always much smaller than the mass of positronium proper.

The most disturbing aspect of the present work is its obvious conflict with lattice gauge calculations. I have not checked to which extent a possible linear term in the potential spoils the present excellent agreement between theory and experiment. The calculation of the potential energy on the lattice rests on the assumptions of static quarks, of quarks with an infinitely large mass. Even with present day computers lattice gauge calculations have a hard time in extrapolating them down to such light systems as the \( \pi \) or the \( \rho \).

The present work opens a broad avenue of further applications, among them also the baryons. But thus far, it is only a first step.
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