Unfoldings for regular $F$-manifolds

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Abstract: A regular $F$-manifold is an $F$-manifold (with Euler field) $(M, o, e, E)$, such that the endomorphism $U(X) := E \circ X$ of $TM$ is regular at any $p \in M$. We prove that the germ $((M, p), o, e, E)$ is uniquely determined (up to isomorphism) by the conjugacy class of $U_p : T_p M \to T_p M$. This leads to various (equivalent) local descriptions for regular $F$-manifolds. As an application, we show that any regular $F$-manifold is weak Frobenius.

1 Introduction

The motivation for this note comes from various universal deformations and unfoldings for meromorphic connections and Frobenius manifolds, which exist in the literature. By a result of Malgrange [9, 10], a meromorphic connection on the trivial rank $n$ bundle ($n \geq 1$) over a small disc $D$ around the origin in $\mathbb{C}^1$, with pole of Poincaré rank one at 0, and whose residue is a regular endomorphism, admits a universal deformation. Similar type of results appear in the setting of Frobenius manifolds. Unfoldings for Frobenius manifolds at semisimple points were constructed in [1], and later on, generalizations where the point was replaced by an entire submanifold, were developed in [6].

The notion of $F$-manifold, defined by Hertling and Manin [5], is weaker than the notion of Frobenius manifold. An $F$-manifold is a manifold $(M, o, e, E)$ together with an associative, commutative, with unit field multiplication $\circ$ on $TM$, which satisfies a certain integrability condition, and a vector field $E$ (called the Euler field), which satisfies $L_E(\circ) = \circ$. Any Frobenius manifold stripped by its metric is an $F$-manifold. Conversely, any semisimple $F$-manifold supports, around any tamed point, a Frobenius metric (in the language of [5], is weak Frobenius on the open dense subset of such points). This general construction of Frobenius metrics on semisimple $F$-manifolds reduces to solutions of Schlesinger’s equations (see e.g. [11], Chapter II, Section 3). The question of existence of Frobenius metrics on a given $F$-manifold
is, in its full generality, still open. Examples of $F$-manifolds which do not support Frobenius metrics are known \[4, 7\]. We shall come to this point in more detail and we shall give two sources of obstructions in Remark 16. The notion of $F$-manifold is met also in quantum cohomology \[12\] and integrable systems \[8, 15\].

Owing to their relation with meromorphic connections and Frobenius manifolds, it is natural to ask if unfoldings exist for $F$-manifolds. In this paper we address this question for a large class of $F$-manifolds, namely the regular ones. Our main result is as follows:

**Theorem 1.** There is a unique (up to isomorphism) germ $((M,p),\circ,e,E)$ of (regular) $F$-manifolds, for which the endomorphism $U_p(X) := X \circ E_p$ of $T_pM$ is regular and belongs to a given conjugacy class.

(An endomorphism of a (complex) vector space is regular if it has, for each eigenvalue, only one Jordan block. The conjugacy class of an endomorphism is determined by its Jordan normal form; two endomorphisms, defined on not necessarily the same vector space, belong to the same conjugacy class if they can be reduced to the same Jordan normal form).

As a main application of Theorem 1, we obtain that any regular $F$-manifold $(M,\circ,e,E)$ admits a preferred system of local coordinates, in which the multiplication and Euler field take a particularly simple form (see Proposition 13). Using this coordinate system, we easily show that $(M,\circ,e,E)$ can be locally enriched to a (rather special and simple) Frobenius manifold (see Corollary 14).

**Outline of the paper.** The paper is structured as follows. In Section 2 we recall basic definitions and results we need about $F$-manifolds, Euler fields and Frobenius manifolds. Section 3 represents a first step in the proof of Theorem 1. Here we prove that a regular $F$-manifold $(M,\circ,e,E)$, for which $U = C_E$ has exactly one eigenvalue at a point, is globally nilpotent around that point (see Proposition 6). The proof of this fact uses the field of frames $\{X_0, \cdots, X_{n-1}\}$, obtained by successive multiplications of the unit field $e$ by the Euler field $E$, and a general formula (see Theorem 6 of \[5\]) for the Lie brackets of such vector fields. As a consequence, we obtain that on a regular globally nilpotent $F$-manifold $(M,\circ,e,E)$, the coordinates of $[X_i, X_j]$, in the frame $\{X_0, \cdots, X_{n-1}\}$, are polynomials (with constant coefficients) in the eigenfunction $a$ of $U = C_E$, and $X_i(a) = a^i$, for any $i$ (see Corollary 7). This fact will play a key role in the proof of our main result. In Section 4 we prove Theorem 1. The main issue is the uniqueness (the existence is easy). Owing to Hertling’s decomposition for $F$-manifolds \[4\],
there is no loss of generality to assume in Theorem 1 that the given conjugacy class is determined by a single Jordan block. Then the endomorphism $U_p$ of any germ of $F$-manifolds $((M, p), \circ, e, E)$, as in Theorem 1, has exactly one eigenvalue and from Proposition 5 mentioned above, $((M, p), \circ, e, E)$ is globally nilpotent. The problem reduces to showing that any two systems $\{X_0, \ldots, X_{n-1}, a\}$ and $\{Y_0, \ldots, Y_{n-1}, b\}$ on the germ $(\mathbb{C}^n, 0)$, formed by a field of frames and a holomorphic function, which satisfy the relations from Corollary 7 (see also Remark 8) and coincide at $p = 0$, are isomorphic via a biholomorphic transformation of $(\mathbb{C}^n, 0)$. We prove this statement in Proposition 11. In the proof of Proposition 11 we use the (uniqueness part) of the Cauchy-Kovalevskaia theorem, together with a well-known characterization of the differentials of biholomorphic transformations, between all tangent bundle automorphisms (see Lemma 10). Corollary 12 concludes the proof of Theorem 1. Section 5 is devoted to applications of Theorem 1. We prove that any regular $F$-manifold $((M, \circ, e, E)$ admits a preferred system of local coordinates (see Proposition 13) and we use this system of local coordinates to show that $(M, \circ, e, E)$ is weak Frobenius (see Corollary 14). We also remark, as a consequence of Theorem 1 that $(M, \circ, e, E)$ is locally isomorphic to the parameter space (with the canonical $F$-manifold structure) of a certain universal meromorphic connection (see Remark 15). This leads to another natural local description for regular $F$-manifolds. At the end of the paper we give examples of (non-regular) $F$-manifolds which are not weak Frobenius (see Remark 16).

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2 F-manifolds and Frobenius manifolds

This section is intended to fix notation. We work in the holomorphic category: the manifolds are complex and the vector bundles, sections, connections etc. are holomorphic. We denote by $T_M$ the sheaf of holomorphic vector fields on a complex manifold $M$, by $\mathcal{O}_M$ the sheaf of holomorphic functions on $M$ and by $\Omega^1(M, V)$ the sheaf of 1-forms with values in a vector bundle $V$.

Despite the usual terminology, we assume that $F$-manifolds and Frobenius manifolds always come with an Euler field (of weight one).

Definition 2. i) An $F$-manifold is a manifold $M$ together with a commutative, associative multiplication $\circ$ on $T_M$, with unit field $e$, and an additional
field E (called the Euler field), such that the following conditions hold:

\[ L_{X \circ Y}(\circ) = X \circ L_Y(\circ) + Y \circ L_X(\circ) \]  

and

\[ L_E(\circ)(X,Y) = X \circ Y, \]

for any vector fields \( X,Y \in T_M \).

ii) An \( F \)-manifold \((M, \circ, e, E)\) is called globally nilpotent if, for any \( p \in M \) and \( X \in T_pM \), the endomorphism \( C_X : T_pM \to T_pM \), \( C_X(Y) := X \circ Y \), has exactly one eigenvalue. Equivalently, if it is of the form

\[ C_X = \mu(X)\text{Id} + N_X, \]

with \( \mu(X) \in \mathbb{C} \) and \( N_X \in \text{End}(T_pM) \) nilpotent.

iii) An \( F \)-manifold \((M, \circ, e, E)\) is called regular if the endomorphism \( U : TM \to TM \), \( U := C_E \), is regular at any \( p \in M \).

We make several comments on the above definition.

Remark 3. i) The Lie derivative \( L_X(\circ) \) of the multiplication \( \circ \) along a vector field \( X \) is defined by

\[ L_X(\circ)(Y,Z) := [X,Y \circ Z] - [X,Y] \circ Z - X \circ [Y,Z], \quad Y,Z \in T_M. \]

ii) An endomorphism \( A : V \to V \) of a complex vector space \( V \) is regular, if any two distinct Jordan blocks from its Jordan normal form have distinct eigenvalues. Alternatively, if any of the following three (equivalent) conditions holds: 1) the characteristic and minimal polynomials of \( A \) coincide; 2) the vector space of endomorphisms of \( V \) commuting with \( A \) has dimension \( n = \dim(V) \) (and a basis \( \{\text{Id}, A, \cdots, A^{n-1}\} \)); 3) there is a cyclic vector for \( A \), i.e. a vector \( v \in V \) such that \( \{v, A(v), \cdots, A^{n-1}(v)\} \) is a basis of \( V \).

Let \((M, \circ, e, E)\) be an \( F \)-manifold. For any \( k \geq 0 \), we denote \( X_k := E \circ \cdots \circ E \) (\( k \)-times), with \( X_0 := e \). As proved in [5],

\[ [X_i, X_j] = (j - i)X_{i+j-1}, \quad i,j \geq 0. \]  

(2)

Suppose now that for a point \( p \in M \), the endomorphism \( \mathcal{U}_p(X) = X \circ E_p \) of \( T_pM \) is regular. Let \( V \) be a small neighborhood of \( p \), such that \( \mathcal{U}_q : T_qM \to T_qM \) is regular, for any \( q \in V \). The vector fields \( \{X_0, \cdots, X_{n-1}\} \) form a field of frames on \( V \). The following simple lemma holds.

Lemma 4. In the above setting, assume that \( \mathcal{U}_q : T_qV \to T_qV \) has exactly one eigenvalue, for any \( q \in V \). Then \((V, \circ, e)\) is globally nilpotent.
Proof. By hypothesis, $U = \mu \text{Id} + N$ on $TV$, where $\mu \in \mathcal{O}_V$ and $N : TV \to TV$ is nilpotent (at any point). Let $X := f_0 X_0 + \cdots + f_{n-1} X_{n-1}$ be an arbitrary vector field on $V$, with $f_i \in \mathcal{O}_V$. We obtain

$$C_X = (\sum_{k=0}^{n-1} f_k \mu^k)\text{Id} + \sum_{k=1}^{n-1} f_k \sum_{p=1}^{k} C_{k}^{p} \mu^{k-p} N^{p}.$$ 

The second term in the right hand side of the above relation is a nilpotent endomorphism. Our claim follows. \hfill \Box

Rather than the usual definition of Frobenius manifolds [1], we prefer the alternative one (see e.g. [4]) where Frobenius manifolds are viewed as an enrichment of $F$-manifolds.

**Definition 5.**

i) A Frobenius manifold is an $F$-manifold $(M, \circ, e, E)$ together with a (non-degenerate) flat, multiplication invariant metric $g$ (i.e. $g(X \circ Y, Z) = g(X, Y \circ Z)$, for any $X, Y, Z \in TM$), such that $L_E(g) = Dg$ (with $D \in \mathbb{C}$) and $\nabla(e) = 0$ (where $\nabla$ is the Levi-Civita connection of $g$).

ii) A Frobenius metric on an $F$-manifold $(M, \circ, e, E)$ is a metric $g$ which makes $(M, \circ, e, E, g)$ a Frobenius manifold.

iii) A weak Frobenius manifold is an $F$-manifold which can be locally enriched to a Frobenius manifold.

3 Globally nilpotent regular $F$-manifolds

Our aim in this section is to prove the following result.

**Proposition 6.** Let $(M, \circ, e, E)$ be an $F$-manifold of dimension $n \geq 2$, such that at a point $p_0 \in M$, the endomorphism $\mathcal{U}_{p_0} = (C_E)_{p_0} : T_{p_0}M \to T_{p_0}M$ is regular, with exactly one eigenvalue. Then there is a neighborhood $V$ of $p_0$, such that $(V, \circ, e, E)$ is globally nilpotent (and regular).

Proof. Let $V$ be a small neighborhood of $p_0$, such that, for any $p \in V$, the endomorphism $\mathcal{U}_p = (C_E)_p : T_p M \to T_p M$ is regular, and let $P(p, z) = z^n + \sum_{k=0}^{n-1} \lambda_k(p) z^k$ be the characteristic (or minimal) polynomial of $\mathcal{U}_p$. As before, let $X_k := E \circ \cdots \circ E$ ($k$-times, $k \geq 0$), with the convention $X_0 := e$. The system $\{X_0, X_1, \cdots, X_{n-1}\}$ is a frame field on $V$ and, since $P(p, \mathcal{U}_p) = 0$,

$$X_n + \sum_{k=0}^{n-1} \lambda_k X_k = 0. \quad (3)$$
Define the functions

\[ f_k := \lambda_k - \frac{C^n_k}{n^{n-k}} \lambda_{n-k}, \quad 0 \leq k \leq n - 2. \]

(Remark that \( f_k(p) = 0 \) for \( p \in V \) and all \( 0 \leq k \leq n - 2 \), if and only if \( P(p, z) = (z + \frac{\lambda_{n-1}(p)}{n})^n \), if and only if \( \mathcal{U}_p \) has exactly one eigenvalue). By hypothesis, \( f_k(p_0) = 0 \), for any \( 0 \leq k \leq n - 2 \). Our aim is to compute the derivatives \( X_s(f_k) \) and to show, using the Cauchy-Kovalevskaya theorem, that \( f_k = 0 \) on \( V \), for any \( 0 \leq k \leq n - 2 \). This implies that \( \mathcal{U} \) has exactly one eigenvalue at any point of \( V \) and we conclude from Lemma 4 that \((V, \circ, e, E)\) is globally nilpotent, as required. Details are as follows.

We take the Lie derivative of \( (3) \) with respect to \( X_s \) \((s \geq 0)\) and we use \( (2) \). We obtain

\[ (n - s)X_{s+n-1} + \sum_{k=0}^{n-1} (X_s(\lambda_k))X_k + (k - s)\lambda_k X_{s+k} - 1) = 0, \quad (4) \]

which is equivalent, by taking \( \lambda_n := 1 \), to

\[ \sum_{k=0}^{n-1} X_s(\lambda_k)X_k + \sum_{k=0}^{n} (k - s)\lambda_k X_{s+k} - 1 = 0. \quad (5) \]

Relation \((5)\), with \( s = 0 \), gives

\[ X_0(\lambda_k) = -(k + 1)\lambda_{k+1}, \quad 0 \leq k \leq n - 1. \quad (6) \]

Relation \((5)\), with \( s = 1 \), gives

\[ \sum_{k=0}^{n-1} X_1(\lambda_k)X_k + \sum_{k=0}^{n} (k - 1)\lambda_k X_{k+1} = 0. \]

From \((3)\), \( X_n = -\sum_{k=0}^{n-1} \lambda_k X_k \). Replacing this expression of \( X_n \) into the above relation we obtain

\[ X_1(\lambda_k) = (n - k)\lambda_k, \quad 0 \leq k \leq n - 1. \quad (7) \]

The computation of \( X_2(\lambda_k) \) is done in the same way, but is a bit more complicated. Taking in \((5)\) \( s = 2 \) we obtain

\[ \sum_{k=0}^{n-1} X_2(\lambda_k)X_k + \sum_{k=0}^{n} (k - 2)\lambda_k X_{k+1} = 0 \]
or, equivalently,

$$
\sum_{k=0}^{n-1} X_2(\lambda_k)U^k + \sum_{k=0}^{n} (k - 2)\lambda_kU^{k+1} = 0.
$$

Since $P(p, \cdot)$ is the minimal polynomial of $U_p$ (for any $p \in V$) there are holomorphic functions $b_0, b_1 \in \mathcal{O}_V$ such that

$$
\sum_{k=0}^{n-1} X_2(\lambda_k)z^k + \sum_{k=0}^{n} (k - 2)\lambda_kz^{k+1} = (b_0 + b_1z)\sum_{k=0}^{n} \lambda_kz^k.
$$

Identifying in (8) the coefficients of $z^{n+1}$ we obtain $b_1 = n - 2$. Relation (8) becomes

$$
\sum_{k=0}^{n} (X_2(\lambda_k) - b_0\lambda_k)z^k - \sum_{k=1}^{n} (n - k + 1)\lambda_{k-1}z^k = 0.
$$

Then (from the coefficient of $z^0$), $X_2(\lambda_0) = b_0\lambda_0$, and the remaining terms in (9) give

$$
\sum_{k=1}^{n} (X_2(\lambda_k) - b_0\lambda_k - (n - k + 1)\lambda_{k-1})z^k = 0.
$$

Using $\lambda_i = 1$, we obtain $b_0 = -\lambda_{n-1}$ (from the coefficient of $z^n$) and relation (10) becomes

$$
X_2(\lambda_k) = -\lambda_{n-1}\lambda_k + (n - k + 1)\lambda_{k-1}, \quad 0 \leq k \leq n - 1.
$$

(We use the convention $\lambda_i = 0$ for $i < 0$; similarly, below $f_i = 0$ whenever $i < 0$).

Next, we compute the derivatives $X_3(\lambda_k)$. Relation (5), with $s = 3$, gives, by a similar argument as for $s = 2$,

$$
X_3(\lambda_k) = (\lambda^2_{n-2} - 2\lambda_{n-2})\lambda_k + (n - k + 2)\lambda_{k-2} - \lambda_{n-1}\lambda_{k-1}, \quad 0 \leq k \leq n - 1.
$$

From relations (6), (7), (11) and (12), we obtain, from long but straightforward computations, the expressions for the derivatives $X_i(f_k)$ (for any $0 \leq i \leq 3$ and $0 \leq k \leq n - 2$):

$$
\begin{align*}
X_0(f_k) &= -(k+1)f_{k+1} \quad (k \leq n-3), \quad X_0(f_{n-2}) = 0, \quad X_1(f_k) = (n-k)f_k \quad (13)
\end{align*}
$$

and

$$
\begin{align*}
X_2(f_k) &= -\lambda_{n-1}f_k + (n - k + 1)f_{k-1} - \frac{2C^k_n(n - k)}{n^{n-k}}\lambda^{n-k-1}_{n-1}f_{n-2}, \\
X_3(f_k) &= (\lambda^2_{n-2} - 2\lambda_{n-2})f_k + \frac{C^k_n(3n - 3k - 2)}{n^{n-k}}\lambda^{n-k}_{n-1}f_{n-2} \\
&\quad + (n - k + 2)f_{k-2} - \lambda_{n-1}f_{k-1} - \frac{3C^k_n(n - k)}{n^{n-k}}\lambda^{n-k-1}_{n-1}f_{n-3}.
\end{align*}
$$

(14)
In particular, both (13) and (14) are of the form
\[ X_i(f_k) = \sum_{s=0}^{n-2} a_{ks}^{(i)} f_s, \quad 0 \leq i \leq 3, \quad 0 \leq k \leq n - 2, \]  
(15)
for some \( a_{ks}^{(i)} \in \mathcal{O}_V \). Since \([X_i, X_j] = (j - i) X_{i+j-1}\) (see relation (2)), we obtain that the derivatives \( X_i(f_k) \), for any \( 0 \leq i \leq n - 1 \) (and \( 0 \leq k \leq n - 2 \)), are of the form (15). Considering now a local chart \( \chi = (y^0, \cdots, y^{n-1}) : V \to \mathbb{C}^n \), with \( \chi(p_0) = 0 \), we obtain
\[
\frac{\partial(f_k \circ \chi^{-1})}{\partial y^i} = \sum_{s=0}^{n-2} b_{ks}^{(i)} (f_s \circ \chi^{-1}), \quad 0 \leq i \leq n - 1, \quad 0 \leq k \leq n - 2,
\]
for some \( b_{ks}^{(i)} \in \mathcal{O}_V \). Also, \((f_k \circ \chi^{-1})(0) = 0\) for any \( 0 \leq k \leq n - 2 \). Applying successively the uniqueness statement of the Cauchy-Kovalevskaya theorem (in the form stated e.g. in [6], relations (2.42) and (2.43), with no \((t^i)\)-parameters in the notations of this reference), we obtain that \( f_k = 0 \) on \( V \), as required.

The computations from the above proof imply the following corollary.

**Corollary 7.** Let \((M, \circ, e, E)\) be a globally nilpotent regular \( F \)-manifold of dimension \( n \geq 2 \). Let \( a \in \mathcal{O}_M \) be the eigenfunction of \( \mathcal{U} = \mathcal{C}_E \) and let, for any \( i \geq 0 \), \( X_i := E^i \) (with \( X_0 = e \)). Then
\[
[X_i, X_j] = \begin{cases} (j - i) X_{i+j-1}, & i + j \leq n \\ (i - j) \sum_{k=0}^{n-1} c_k^{(i+j-1-n)} a^{i+j-1-k} X_k, & i + j > n, \end{cases}
\]  
(16)
where \( c_k^{(p)} \) (\( p \geq 0 \) and \( 0 \leq k \leq n - 1 \)) are constants, defined inductively by \( c_k^{(0)} = (-1)^{n-k} C_n^k \) and for any \( s \geq 0 \), \( c_k^{(s+1)} = c_k^{(s)} - c_k^{(0)} c_{n-1}^{(s)} \) (when \( k \geq 1 \)) and \( c_0^{(s+1)} = -c_0^{(0)} c_{n-1}^{(s)} \). Moreover,
\[
X_i(a) = a^i, \quad i \geq 0.
\]  
(17)

**Proof.** Relation (16) for \( i + j \leq n \) is just (2). We now prove (16) for \( i + j > n \). Since \((\mathcal{U} - a\text{Id})^n = 0\),
\[
\mathcal{U}^n + \sum_{k=0}^{n-1} c_k^{(0)} a^{n-k} \mathcal{U}^k = 0.
\]  
(18)
Multiplying the above relation with \( U, U^2, \) etc, and using an induction argument, we obtain

\[
U^{n+s} + \sum_{k=0}^{n-1} c_k^{(s)} a^{n-k+s} U^k = 0, \quad s \geq 0,
\]

or, equivalently,

\[
X_{n+s} = -\sum_{k=0}^{n-1} c_k^{(s)} a^{n-k+s} X_k, \quad s \geq 0. \tag{19}
\]

Relations (2) and (19) imply (16) for \( i + j > n \), as required.

It remains to prove (17). With the notation from the proof of Proposition 6, \( a = -\frac{\lambda_{n-1}}{n} \) and \( \lambda_k = C_n^k (-a)^{n-k} \), for any \( 0 \leq k \leq n - 1 \) (because \( f_k = 0 \), \((M, \circ, e, E)\) being globally nilpotent). On the other hand, in the proof of Proposition 6 we computed the following derivatives:

\[
X_0(\lambda_{n-1}) = -n, \quad X_1(\lambda_{n-1}) = \lambda_{n-1}
\]
\[
X_2(\lambda_{n-1}) = -\lambda_{n-1}^2 + 2\lambda_{n-2}
\]
\[
X_3(\lambda_{n-1}) = \lambda_{n-1}^3 + 3\lambda_{n-3} - 3\lambda_{n-1}\lambda_{n-2}.
\]

These expressions, written in terms of \( a \), give (17), for \( 0 \leq i \leq 3 \). Using (2) we obtain (17), for any \( i \geq 0 \).

**Remark 8.** For the proof of Theorem 1 it is convenient to express the Lie brackets \([X_i, X_j]\) computed in Corollary 7 in a unified form (not as in (16), where the cases \( i + j \leq n \) and \( i + j > n \) are separated). This can be done as follows. Consider the constants \( c_k^{(p)} \) from Corollary 7. They were defined for \( p \geq 0 \) and \( 0 \leq k \leq n - 1 \). For \( p < 0 \) (and \( 0 \leq k \leq n - 1 \)), let \( c_k^{(p)} := 0 \), unless \( p = k - n \), in which case \( c_k^{(k-n)} := -1 \). With this notation, the two relations (16) reduce to the single one

\[
[X_i, X_j] = (i - j) \sum_{k=0}^{n-1} c_k^{(i+j-1-n)} a^{i+j-1-k} X_k, \quad i, j \geq 0.
\]

### 4 Proof of Theorem 1

Using the material from the previous section, we now prove Theorem 1. The existence part is easy: the \( F \)-manifold with coordinates as in Proposition 13 from the next section proves the existence. Therefore, we only need to deal
with the uniqueness. Let \(((M, p), \circ_M, e_M, E_M)\) and \(((N, q), \circ_N, e_N, E_N)\) be
two germs of \(F\)-manifolds of dimension \(n \geq 2\), such that \((\mathcal{U}_M)_p : T_pM \to T_qM\)
and \((\mathcal{U}_N)_q : T_qN \to T_mN\) are regular and belong to the same conjugacy class.
Our aim is to show that the two germs are isomorphic (as \(F\)-manifolds). A
first simplification follows from the next lemma.

**Lemma 9.** There is a linear isomorphism

\[
 f : (T_pM, (\circ_M)_p, (e_M)_p, (E_M)_p) \to (T_qN, (\circ_N)_q, (e_N)_q, (E_N)_q).
\]

**Proof.** Define \(f : T_pM \to T_qN\) by \(f((E_M)_p^i) = (E_N)_q^i\), for any \(0 \leq i \leq n-1\).
For \(i \geq n\), the coordinates of \((E_M)_p^i\) in the basis \((e_M)_p, (E_M)_p, \cdots, (E_M)_p^{n-1}\)
coincide with the coordinates of \((E_N)_q^i\) in the basis \((e_N)_q, (E_N)_q, \cdots, (E_N)_q^{n-1}\)
(because the characteristic polynomials of \((\mathcal{U}_M)_p\) and \((\mathcal{U}_N)_q\) coincide). We
deduce that \(f((E_M)_p^i) = (E_N)_q^i\), for any \(i \geq 0\), i.e. \(f\) preserves multiplications. \(\square\)

From the above lemma, we can assume, without loss of generality, that
\((M, p) = (N, q) = (\mathbb{C}^n, 0)\), \((\circ_M)_0 = (\circ_N)_0\) and \((E_M)_0 = (E_N)_0\). Owing to
Hertling’s decomposition of \(F\)-manifolds (which states that the germ of an
\(F\)-manifold at a point decomposes as a product of germs of \(F\)-manifolds, according to the (distinct)
eigenvalues of the multiplication endomorphism by the Euler field at the given point; see [4], Theorem 2.11, page 16) we can make
a further simplification, namely that the (regular) endomorphism \(\mathcal{U}_0(X) =
X \circ_M (E_M)_0 = X \circ_N (E_N)_0\) of \(T_0\mathbb{C}^n\) has precisely one eigenvalue. From
Proposition 6 the germs \(((\mathbb{C}^n, 0), \circ_M, e_M, E_M)\) and \(((\mathbb{C}^n, 0), \circ_N, e_N, E_N)\) are
globally nilpotent. Let \(a\) and \(b\) be the eigenfunctions of \(\mathcal{U}_M\) and \(\mathcal{U}_N\) respectively.
Remark that \(a(0) = b(0)\).

Let \(\{X_i = (E_M)_i, 0 \leq i \leq n - 1\}\) and
\(\{Y_i = (E_N)_i, 0 \leq i \leq n - 1\}\) be the field of frames generated by \(E_M\) and \(E_N\)
(with \(X_0 = e_M\) and \(Y_0 = e_N\), as usual).

The Lie brackets \([X_i, X_j]\) and similarly \([Y_i, Y_j]\) were computed in Corollary 7 (see also Remark 8).
Using these fields of frames, we prove that the two germs of \(F\)-manifolds are isomorphic,
as needed (see Proposition 11 and its Corollary 12 below). Before, we need to
recall the following characterization of those tangent bundle automorphisms,
which are differentials of biholomorphic transformations of the base manifold
(see e.g. Proposition 3.22 of [2]; it is stated in the smooth setting, but it
holds, with the same proof, in the holomorphic setting as well). It will play
an essential role in our argument.

**Lemma 10.** Let \(\psi : M \to M\) be a biholomorphic transformation of a
complex manifold. Let \(F : TM \to TM\) be a holomorphic map which covers
\(\psi\) (i.e. \(F(T_xM) \subset T_{\psi(x)}(M)\), for any \(x \in M\)) and such that \(F : T_xM \to
$T_{\psi(x)}M$ is a linear isomorphism, for any $x \in M$. Define $F : T_M \to T_M$, by $F(X)_{\psi(x)} = F(X_x)$, for any $X \in T_M$. Suppose that

$$F([Z,V]_x) = [F(Z),F(V)]_{\psi(x)}, \quad x \in M, \quad Z,V \in T_M.$$ 

Then $F = \psi_*$. 

Below we use the notation $\phi_s^X$ for the flow of a vector field $X$.

**Proposition 11.** Let $\{X_0, \cdots, X_{n-1}\}$ and $\{Y_0, \cdots, Y_{n-1}\}$ be two frames of vector fields on the germ $(\mathbb{C}^n,0)$, such that $(X_i)_0 = (Y_i)_0$ for any $0 \leq i \leq n-1$ and $a, b : (\mathbb{C}^n,0) \to \mathbb{C}$ holomorphic functions, such that $a(0) = b(0)$. Assume that the system $\{X_0, \cdots, X_{n-1}, a\}$ satisfies

$$[X_i, X_j] = (i-j) \sum_{k=0}^{n-1} c_k^{(i+j-1-n)} a^{i+j-1-k} X_k, \quad X_i(a) = a^i, \quad 0 \leq i, j \leq n-1, \quad (20)$$

where $c_k^{(i+j-1-n)}$ are the constants from Remark 8 and that the same relations hold for the system $\{Y_0, \cdots, Y_{n-1}, b\}$. Then

$$\psi : (\mathbb{C}^n,0) \to (\mathbb{C}^n,0), \quad \psi((\phi_{s_0}^{X_0} \circ \cdots \circ \phi_{s_{n-1}}^{X_{n-1}})(0)) = (\phi_{s_0}^{Y_0} \circ \cdots \circ \phi_{s_{n-1}}^{Y_{n-1}})(0)$$

is a well-defined isomorphism, which satisfies $\psi_*(X_i) = Y_i \ (0 \leq i \leq n-1)$ and $b \circ \psi = a$.

**Proof.** Consider the map

$$f^X : (\mathbb{C}^n,0) \to (\mathbb{C}^n,0), \quad f^X(s_0, \cdots, s_{n-1}) = (\phi_{s_0}^{X_0} \circ \cdots \circ \phi_{s_{n-1}}^{X_{n-1}})(0).$$

Its differential is given by: for any $0 \leq i \leq n-1$,

$$f^X_* \left( \frac{\partial}{\partial s_i} \right) = (\phi_{s_0}^{X_0} \circ \cdots \circ \phi_{s_{n-1}}^{X_{n-1}})_*((X_i)_{(\phi_{s_0}^{X_0} \circ \cdots \circ \phi_{s_{n-1}}^{X_{n-1}})(0)}).$$

In particular,

$$(f^X)_0(0) = (X_i)_0, \quad 0 \leq i \leq n-1.$$ 

Since $X_0, \cdots, X_{n-1}$ are linearly independent at 0, $(f^X)_0 : T_0\mathbb{C}^n \to T_0\mathbb{C}^n$ is a linear isomorphism and $f^X : (\mathbb{C}^n,0) \to (\mathbb{C}^n,0)$ is a germ isomorphism. Defining $f^Y : (\mathbb{C}^n,0) \to (\mathbb{C}^n,0)$ in the same way as $f^X$, we obtain that $\psi = f^Y \circ (f^X)^{-1}$ is a (well-defined) germ isomorphism.

Next, we prove that $b \circ \psi = a$. For this, we notice that

$$\frac{d}{ds}(a \circ \phi_s^{X_i}(0)) = a_*((X_i)_{\phi_s^{X_i}(0)}) = X_i(a)|_{\phi_s^{X_i}(0)} = (a \circ \phi_s^{X_i}(0))^i$$

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The following proposition provides a preferred system of local coordinates on regular \( F \)-manifolds and similarly for \( b \circ \phi^Y_i(0) \). Thus,
\[
\frac{d}{ds}(a \circ \phi^X_i(0)) = (a \circ \phi^X_i(0))^i, \quad \frac{d}{ds}(b \circ \phi^Y_i(0)) = (b \circ \phi^Y_i(0))^i.
\]
Since \( a(0) = b(0) \) we deduce that
\[
(a \circ \phi^X_i)(0) = (b \circ \phi^Y_i)(0), \quad 0 \leq i \leq n - 1.
\]
More generally, we claim that
\[
(a \circ \phi^X_i \circ \phi^X_j)(0) = (b \circ \phi^Y_i \circ \phi^Y_j)(0), \quad 0 \leq i, j \leq n - 1.
\]
This follows by a similar argument, by noticing that both functions \( s \to (a \circ \phi^X_i \circ \phi^X_j)(0) \) and \( s \to (b \circ \phi^Y_i \circ \phi^Y_j)(0) \) satisfy the same system of differential equations and coincide at \( s = 0 \) (from (21)). By induction, for any \( (s_0, \ldots, s_{n-1}) \in (\mathbb{C}^n, 0) \) and \( 0 \leq i \leq n - 1 \),
\[
(a \circ \phi^{X_{s_0}} \circ \cdots \circ \phi^{X_{s_{n-1}}})(0) = (b \circ \phi^{Y_{s_0}} \circ \cdots \circ \phi^{Y_{s_{n-1}}})(0),
\]
i.e. \( b \circ \psi = a \), as required. It remains to show that \( \psi_*(X_i) = Y_i \), for any \( 0 \leq i \leq n - 1 \). For this, we apply Lemma 10 to \( \psi : (\mathbb{C}^n, 0) \to (\mathbb{C}^n, 0) \) and
\[
F : T(\mathbb{C}^n, 0) \to T(\mathbb{C}^n, 0)
\]
defined by \( F((x_i)_x) := (Y_i)_{\psi(x)} \), for any \( x \in (\mathbb{C}^n, 0) \) and \( 0 \leq i \leq n - 1 \). In the notation from Lemma 10, \( F(X_i) = Y_i \). Using (21) (together with the similar relation for \( [Y_i, Y_j] \)) and \( b \circ \psi = a \), we obtain
\[
F([X_i, X_j]) = [Y_i, Y_j]_{\psi(x)}.
\]
Therefore, \( F = \psi_* \), i.e. \( \psi_*(X_i) = Y_i \) \( 0 \leq i \leq n - 1 \) as needed.

**Corollary 12.** The map \( \psi \) from Proposition 11 defines an isomorphism between the germs of \( F \)-manifolds \( ((\mathbb{C}^n, 0), o_M, e_M, E_M) \) and \( ((\mathbb{C}^n, 0), o_N, e_N, E_N) \).

**Proof.** We need to prove that \( \psi_* \) preserves multiplications. This follows from \( \psi_*(X_i) = Y_i \) for any \( 0 \leq i \leq n - 1 \) (see Proposition 11) and an argument as in the proof of Lemma 9. (We remark that for any \( x \in (\mathbb{C}^n, 0) \), the characteristic polynomials of \( (U_M)_x \) and \( (U_N)_x \) coincide, being equal to \( P(x, z) = (z - a(x))^n = (z - (b \circ \psi)(x))^n \).)

## 5 Local models for regular \( F \)-manifolds and Frobenius metrics

The following proposition provides a preferred system of local coordinates on regular \( F \)-manifolds and shows their existence.
**Proposition 13.** Any regular $F$-manifold is locally isomorphic to a product of $F$-manifolds of the form $(\mathbb{C}^m, \circ, e, E)$, where, in the canonical frame field $\{\partial_i := \frac{\partial}{\partial t^i}, 0 \leq i \leq m - 1\}$, determined by coordinates $(t^0, \cdots, t^{m-1}) \in \mathbb{C}^m$, the multiplication $\circ$ is given by

$$\partial_i \circ \partial_j = \begin{cases} 
\partial_{i+j}, & i + j \leq m - 1 \\
0, & i + j \geq m,
\end{cases}$$

and the unit and Euler fields by

$$e = \partial_0, \quad E = (t^0 + a)\partial_0 + (t^1 + 1)\partial_1 + \cdots + t^{m-1}\partial_{m-1},$$

where $a \in \mathbb{C}$.

*Proof.* One checks directly that $(\mathbb{C}^m, \circ, e, E)$ is an $F$-manifold. At $t = 0$, $\mathcal{U} := \mathcal{C}_E$ is a Jordan block in the basis $\{\partial_i\}$, with eigenvalue $a$. The claim follows from Theorem 1 and Hertling’s local decomposition of $F$-manifolds already mentioned before. \qed

It is easy to construct explicitly Frobenius metrics in the coordinate system from Proposition 13.

**Corollary 14.** i) The globally nilpotent $F$-manifold $(\mathbb{C}^m, \circ, e, E)$ from Proposition 13 can be enriched with the following metric $g$ to a Frobenius manifold:

$$g(\partial_i, \partial_j) = \delta_{i+j,m-1}.$$  

The coordinates $t^i$ and the coordinate vector fields $\partial_i$ are flat (with respect to the Levi-Civita connection of $g$) and $L_E(g) = 2g$.

ii) Any regular $F$-manifold is weak Frobenius.

*Proof.* The metric $g$ is obviously flat and multiplication invariant and $L_E(g) = 2g$ can be checked immediately. Claim i) follows. For claim ii), we notice that any product of globally nilpotent Frobenius manifolds as in i) is also a Frobenius manifold (here it is essential that in any factor of such a product, the Euler field rescales the metric by the same constant 2). We conclude from Proposition 13. \qed

We end the paper with various other comments on Theorem 1 and Corollary 14. To keep the text short, we omit the details.

**Remark 15.** Using Theorem 1 one can show that any regular $F$-manifold is locally isomorphic to the parameter space (with its canonical $F$-manifold structure), of the universal deformation $\nabla^{\text{can}}$ of a meromorphic connection.
\( \nabla^0 \), defined on a trivial bundle \( V^0 = D \times \mathbb{C}^n \rightarrow D \) (\( D \) a small disc around the origin \( 0 \in \mathbb{C}^1 \)), with pole of Poincaré rank one at 0, and whose residue at 0 is a regular endomorphism (see \([9, 10]\)). This leads to another local description for regular \( F \)-manifolds. We present here, without proofs, this alternative description. We first recall, following \([13, \text{ch.7}]\), the parameter space mentioned above. Let \( B_0^0, B_\infty \in M_n(\mathbb{C}^n) \) be two complex matrices. Assume that \( B_0^0 \) is regular. Define \( \mathcal{D}_\Gamma := \text{Span}_\mathbb{C}\{\text{Id}, (B_0)_{\Gamma}, \cdots, (B_0)^{-1}_{\Gamma}\} \subset T_\Gamma M_n(\mathbb{C}) = M_n(\mathbb{C}) \) (22)

where \((B_0)_{\Gamma} := B_0^0 - \Gamma + [B_\infty, \Gamma]\). Because \( B_0^0 \) is regular, so is \((B_0)_{\Gamma}\), for any \( \Gamma \in W \), where \( W \) is a small open neighborhood of 0 in \( M_n(\mathbb{C}) \). For any \( \Gamma \in W \), \( \mathcal{D}_\Gamma \) is the \((n\text{-dimensional})\) vector space of polynomials in \((B_0)_{\Gamma}\). The distribution \( \mathcal{D} \rightarrow W \) is integrable (see \([13\text{, Theorem 3.2, page 209}]\)). Let \( M_{\text{can}} = M_{\text{can}}(B_0^0, B_\infty) \) be the maximal integral submanifold of \( \mathcal{D}_\Gamma \), passing through 0. It is the parameter space of the universal deformation \( \nabla_{\text{can}} \) of the meromorphic connection \( \nabla^0 \) on \( V^0 \), with connection form, in the standard trivialization of \( V^0 \), given by \( \Omega^0 = (\frac{\partial}{\partial t} + B_\infty)\tau^0 \) (we denote by \( \tau \) the canonical coordinate on \( \mathbb{C}^1 \)). The connection \( \nabla_{\text{can}} \) is defined on the trivial bundle \( V_{\text{can}} = (M_{\text{can}} \times D) \times \mathbb{C}^n \rightarrow (M_{\text{can}} \times D) \), has poles of Poincaré rank one along \( M_{\text{can}} \times \{0\} \) and is in the Birkhoff normal form. We do not recall its definition (it can be found in \([13\text{, ch.7}]\), Section 3.a). The connection \( \nabla_{\text{can}} \) induces a Saito structure on \( V_{\text{can}}|_{M_{\text{can}} \times \{0\}} = M_{\text{can}} \times \mathbb{C}^n \rightarrow M_{\text{can}} \), which determines, by means of a choice of a primitive section of \( V_{\text{can}}|_{M_{\text{can}} \times \{0\}} \) (e.g. the section \( s(\Gamma) = (\Gamma, v) \), where \( v \) is a cyclic vector of \( B_0^0 \)), a multiplication \( \circ_{\text{can}} \) on \( TM_{\text{can}} \), with unit field \( \text{Id}_{\text{can}} \), and an additional vector field \( E_{\text{can}} \), which make \((M_{\text{can}}, \circ_{\text{can}}, \text{Id}_{\text{can}}, E_{\text{can}})\) a (regular) \( F \)-manifold. (A Saito structure on a vector bundle consists of a system \((D, \Phi, R_0, R_\infty)\) formed by a flat connection \( D \), a Higgs field \( \Phi \), and two bundle endomorphisms \( R_0, R_\infty \), satisfying the compatibility conditions 2.17 from \([13]\), page 207. For the relation between meromorphic connections and Saito structures, see e.g. \([13\text{, ch.7}]\), Section 2.a. For the relation between Saito structures and \( F \)-manifolds, see Lemmas 4.1 and 4.3 of \([3]\)).

The \( F \)-manifold \((M_{\text{can}}, \circ_{\text{can}}, \text{Id}_{\text{can}}, E_{\text{can}})\) has the following simple description: at any \( \Gamma \in M_{\text{can}} \), \((\circ_{\text{can}})_\Gamma \), acting on \( T_\Gamma (M_{\text{can}}) = \mathcal{D}_\Gamma \) is the multiplication of matrices. The unit field \( \text{Id}_{\text{can}} \) at \( \Gamma \) is the identity matrix \((\text{Id}_{\text{can}})_{\Gamma} = \text{Id} \) and the Euler field is \((E_{\text{can}})_{\Gamma} := -(B_0)_{\Gamma} \). Now, we claim that any regular \( F \)-manifold arises locally in this way. More precisely, let \((M, \circ, e, E)\) be a regular \( F \)-manifold of dimension \( n \), \( p \in M \), and \(-B_0^p\) the representation of \( \mathcal{U}_p : T_p M \rightarrow T_p M \), \( \mathcal{U}_p(X) = X \circ E_p \), in a basis of \( T_p M \). Let \( B_\infty \in M_n(\mathbb{C}) \) be any matrix and let \((M_{\text{can}} = M_{\text{can}}(B_0^p, B_\infty), \circ_{\text{can}}, \text{Id}_{\text{can}}, E_{\text{can}})\) be the \( F \)-manifold constructed as above. The endomorphisms \((\mathcal{U}_{\text{can}})_0 : T_0 M_{\text{can}} \rightarrow T_0 M_{\text{can}} \) are conjugated. From
Theorem 1. The germs \(((M, p), \circ, e, E)\) and \(((M^{\text{can}}, 0), \circ_{\text{can}}, \text{Id}_{\text{can}}, E_{\text{can}})\) are isomorphic. (In particular, the latter is independent of \(B_\infty\)).

The regularity assumption in Corollary 14 ii) is essential. The following remark discusses this issue.

Remark 16. The question whether any given \(F\)-manifold is weak Frobenius was raised in \([5, \text{ch. 3}]\). For certain \(F\)-manifolds (e.g. some generically semisimple \(F\)-manifolds near points where they are not semisimple) the answer is still unknown. But there are \(F\)-manifolds which are not weak Frobenius. Below two sources of such \(F\)-manifolds are described.

a) Proposition 5.32 and Remark 5.33 in \([4]\) provide examples of germs \((M, 0)\) of generically semisimple \(F\)-manifolds such that \(T_0M\) is a local algebra, but not a Frobenius algebra, so it does not allow a nondegenerate multiplication invariant metric. In Proposition 5.32 the \(F\)-manifolds are 3 dimensional, and \(T_0M\) is as an algebra isomorphic to \(\mathbb{C}\{x,y\}/(x^2, xy, y^2)\).

b) There are also examples of globally nilpotent \(F\)-manifolds which do not support any Frobenius metric. Such \(F\)-manifolds are described in \([7]\), 2.5.2 and 2.5.3. Recall that an associative, commutative, with unit multiplication \(\circ\) on the tangent bundle \(TM\) of a manifold \(M\) defines a (possible non-reduced) subvariety \(Y\) of \(T^*M\), the spectral cover, by the ideal \(I = (y^0 - 1, y^i y^j - \sum_k a^k_{ij}(x) y^k) \subset \mathcal{O}_{T^*M}\) (where \((x^i)\) are coordinates on \(M\), with \(\partial_0 = e\) the unit field, and \((x^i, y^j)\) are the induced coordinates on \(T^*M\)). The integrability condition \(\{\cdot, \cdot\}\) from the definition of \(F\)-manifolds is equivalent to \(\{I, I\} \subset I\), where \(\{\cdot, \cdot\}\) is the canonical Poisson bracket of \(T^*M\) (see Theorem 2.5 of \([7]\)). The reduced variety \(Y_{\text{red}}\), defined by \(\sqrt{I}\), is the support of the Higgs bundle \((TM, \mathcal{C}_X(Y) = X \circ Y)\):

\[Y_{\text{red}} = \bigcup_{x \in M} \{ \lambda \in T^*_x M, \forall X \in T_x M, \ker(\mathcal{C}_X - \lambda(X) \text{id} : T_x M \to T_x M) \neq 0 \}.\]

If the \(F\)-manifold can be enriched to a Frobenius manifold (even without Euler field), then on the pull-back of \(T^*M\) to \(\mathbb{C}^1 \times M\) there is an induced \((T)\)-structure (in the notation of \([3]\)) respectively a holonomic \(\mathcal{R}_X\) module (in the notation of \([14]\), where \(X = M\)). This is essentially the construction of the Saito bundle from the Frobenius manifold, but without the data from the Euler field. A result of Sabbah (\([14]\), Proposition 1.2.5) on holonomic \(\mathcal{R}_X\)-modules says that the reduced variety \(Y_{\text{red}}\) is Lagrangian, or, equivalently, \(\{\sqrt{T}, \sqrt{I}\} \subset \sqrt{I}\). The ideals defining the spectral covers in the examples of \(F\)-manifolds from \([7]\), mentioned above, do not satisfy this last condition. Thus, these \(F\)-manifolds do not support any Frobenius metric.
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