Near Lightcone Thermal Conformal Correlators and Holography

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Heavy-heavy-light-light (HHL) correlators of pairwise identical scalars in CFTs with a large central charge in any number of dimensions admit a double scaling limit where the ratio of the heavy conformal dimension to the central charge becomes large as the separation between the light operators becomes null. In this limit the stress tensor sector of a generic HHL correlator receives contributions from the multi stress tensor operators with any number of stress tensors, as long as their twist is not increased by index contractions. We show how one can compute this leading twist stress tensor sector when the conformal dimension of the light operators is large and the stress tensor sector approximates the thermal CFT correlator. In this regime the value of the correlator is related to the length of the spacelike geodesic which approaches the boundary of the dual asymptotically AdS spacetime at the points of light operator insertions. We provide a detailed description of the infinite volume limit. In two spacetime dimensions the HHLL Virasoro vacuum block is reproduced, while in four spacetime dimensions the result is written in terms of elliptic integrals.

1. Introduction: In $d$-dimensional CFTs with a large central charge $C_T$ there are multi-stress tensor operators composed out of stress-tensors and derivatives. The contribution of such composite operators to a four-point function forms the stress-tensor sector of this correlator. The case where the correlator includes two pairwise identical operators $O_L$ with conformal dimension $\Delta_L$ of order unity and two identical light operators $O_H$ with conformal dimension $\Delta_H \sim C_T$ (with the ratio $\mu \sim \Delta_H/C_T$ fixed) has recently been studied in various ways [1]-[11].

Such heavy-heavy-light-light (HHLL) correlators can be used to determine the OPE coefficients of the two light operators with the multi-stress tensors; with this data one can compute the stress tensor sector of any correlator. In CFTs with holographic duals [12]-[14] we expect generic heavy states created by $O_H$ to thermalize; thermal holographic CFTs are described by asymptotically AdS black holes [15]. (For recent work on thermalization in CFTs see e.g. [16]-[18].) Hence, the HHLL correlators provide a window into thermal correlators in CFTs.

It will be convenient to place the CFT on a lorentzian cylinder whose base is a $(d-1)$-dimensional sphere of radius $R$. Consider a kinematical setup where the heavy operators create a heavy state in the infinite past (and annihilate it in the future) while the light operators approach each other’s lightcone from the spatial direction. In the language of conformal cross-ratios $z, \bar{z}$, this corresponds to taking $(1-\bar{z})$ to zero, while keeping $z$ fixed. In this case the main contribution to the $O(\mu^k)$ term in the stress tensor sector comes from the multi stress tensor operators with $k$ stress tensors with minimal twist $\tau = k(d-2)$. An interesting double-scaling limit involves taking $\mu$ large, keeping $\Delta x^- \equiv i\mu^{\frac{1}{d-2}}(1-\bar{z})$ fixed [19]. In this limit all minimal twist multi stress tensor operators contribute.

In [2] it was argued, in a holographic setting, that the OPE coefficients of such operators with two light scalars are universal, i.e. do not depend on the gravitational lagrangian in the bulk. A procedure for computing these OPE coefficients using lightcone conformal bootstrap [20, 21] was proposed in [7]. At each twist there are infinitely many multi stress tensors labeled by spin, but one can perform the summation and obtain a simple expression [5]. As argued in [7], the leading twist part of the stress tensor sector exponentiates:

$$\langle O_H(x_L)O_L(1)O_L(z, \bar{z})O_H(0) \rangle \sim e^{\Delta L F},$$

where $F = F(\Delta x^-, z, \Delta_L)$ is a universal function which can be computed order by order in $(\Delta x^-)^{\frac{1}{d-2}}$ and has a finite limit as $\Delta_L \to \infty$, which we denote by $F_{\infty}$. Hence, the situation is very similar to that in two spacetime dimensions, where the HHLL Virasoro vacuum block also has the form [1]. The HHLL Virasoro block was computed exactly in a variety of ways (see e.g. [22, 28]).

The situation in $d>2$ is much more intricate and the function $F$ is not known. In this paper we use holography to compute $F_{\infty}$. Note that in the $\Delta_L \gg 1$ limit the multi-trace operators which include one or more insertion of $O_L$ decouple and the stress tensor sector approximates the full thermal correlator, at least for holographic theories. The value of $F_{\infty}$ equals minus the (regularized) length of the spacelike geodesic which propagates in the $d+1$-dimensional AdS-Schwarzschild background and approaches the positions of the light operator insertions at the boundary. This has recently been used to partially compute the multi stress tensor OPE coefficients [2, 5] (see also [22]). We pay particular attention to the large volume limit, where $R \to \infty$ and $(1-z) \to 0$. In this case the function $F_{\infty}$ simplifies and we provide direct comparison (and exact matching) for the first few terms in the expansion.

The rest of the paper is organized as follows. In the next Section we provide a brief review and set up notation along the way. In Section 3 we compute the effective metric where the geodesic, whose length computes the near lightcone correlator, propagates. We also write down expressions which allow the computation of $F_{\infty}$.

In Section 4 we consider the large volume limit and re-
produce the Virasoro vacuum block in $d = 2$. We also compute $\mathcal{F}_\infty$ in the four-dimensional case and match the first few terms in the expansion in $\Delta x^-$ with the known results. We discuss the results in Section 5. Appendix A contains some details related to the $d = 2$ case, while Appendix B describes the $d = 4$ case.

2. Review: We will consider the stress-tensor sector of the correlator, where the heavy operators $O_H$ are inserted at $t = \pm \infty$, and the relative coordinates of the light operators differ by $\Delta t$ in the time direction and by $\Delta \varphi$ on the $(d-1)$-sphere. The contribution of the multi stress-tensors (denoted by the subscript "MST") can be written as

$$\mathcal{G}(z, \bar{z}) = \lim_{x \to \infty} x^2 \Delta_H (O_H(x_4)O_L(1)O_L(z, \bar{z})O_H(0))_{\text{MST}}.$$  

(2)

The conformal dimension of the heavy operators, $O_H$, is proportional to the central charge with

$$\mu \equiv \frac{4\Gamma(d+2)}{(d-1)\Gamma(\frac{d}{2})^2} \frac{\Delta H}{C_T}$$

(3)

fixed. The cross-ratios $z$ and $\bar{z}$ are related to the relative spacetime positions of the light operators as (see e.g. [1])

$$z = \exp(i \Delta x^+), \quad \bar{z} = \exp(i \Delta \bar{x}^-),$$

(4)

where

$$\Delta x^+ \equiv (\Delta t + \Delta \varphi), \quad \Delta \bar{x}^- \equiv (\Delta t - \Delta \varphi).$$

(5)

In the equations above, and in most of what follows we set $R = 1$. To retain the contributions of all leading twist multi stress tensors we take the double-scaling limit $\mu \to \infty$, $(1 - z) \to 0$ with

$$\Delta x^- \equiv \mu^{\frac{2}{d-2}} \Delta \bar{x}^- \approx i \mu^{\frac{2}{d-2}} (1 - \bar{z})$$

(6)

fixed. The double scaling limit of the stress-tensor sector of the HHLL correlator, which we denote by $\tilde{\mathcal{G}}$, exponentiates,

$$\tilde{\mathcal{G}} = \exp(\Delta_L \mathcal{F}), \quad \mathcal{F} = \sum_{k=1}^{\infty} \mu^k (1 - z) \frac{d-2k}{d} \mathcal{F}^{(k)}.$$  

(7)

In [7], $\mathcal{F}^{(k)} = \mathcal{F}^{(k)}(z, \Delta_L)$ receives contributions from the minimal twist multi stress-tensor operators with $k$ stress tensors. In $d = 2$ the function $\mathcal{F}$ is $\Delta_L$-independent,

$$\mathcal{F}_{|d=2} = -\log \sinh \left( \frac{\sqrt{\mu - 1}}{2} \Delta x^+ \right).$$

(8)

In higher dimensions it is possible to compute $\mathcal{F}^{(k)}$ using conformal bootstrap, order by order in $k$. The $k = 1$ terms is just the stress tensor contribution, $\mathcal{F}^{(1)} \sim f^2 \pi^2 (z)$ where $f(a) \equiv (1 - z)^a F_1(a, a; 2a, 1 - z)$. The $k = 2$ term is a result of the summation of all twist-2($d-2$) double stress tensors with varying spin (one can find the explicit expression for $d = 4$ in [3]). It will be useful to consider the $\Delta_L \to \infty$ limit of the $k = 2$ term,

$$\mathcal{F}^{(2)}_{|d=4} = -\frac{5 f_3(z)^2 + 18 f_2(z) f_4(z) + 40 f_1(z) f_5(z)}{28800}.$$  

(9)

The next, $k = 3$ term in the expansion, was computed using bootstrap in [7]. To take the large volume limit one needs to set the hypergeometric functions in $f_a(z)$ to unity. (This corresponds to keeping only multi stress tensors without derivatives). The result is

$$\mathcal{F}_{|d=4} \approx -\log(\Delta x^+ \Delta x^-) + \frac{\Delta x^- (\Delta x^+)^3}{120} + \frac{\Delta x^- (\Delta x^+)^6}{10080} + \frac{1583 (\Delta x^-)^3 (\Delta x^+)^9}{648648000} + \ldots$$

(10)

where we used [1] to write the result in terms of $\Delta x^+, \Delta x^-$. The first term in the right hand side of (10) corresponds to the vacuum two-point function.

3. Effective metric and spacelike geodesics: We would like to analyze spacelike geodesics in $(d+1)$-dimensional AdS-Schwarzschild spacetime,

$$ds^2 = -f_B H dt^2 + f_B H dr^2 + r^2(d\varphi^2 + \sin^2 \varphi d\Omega_{d-2})^2,$$

(11)

where

$$f_B H = 1 + r^2 - \frac{\mu}{r^{d-2}}.$$  

(12)

As discussed above, we are interested in the limit $\mu \to \infty$. We will consider spacelike geodesics which approach the AdS boundary at points separated in the lightlike direction $\bar{x}^- = t - \varphi$ by $\Delta \bar{x}^- \to 0$ with $\Delta x^- = \mu^{\frac{2}{d-2}} \Delta \bar{x}^-$ fixed [30]. The geodesics can be taken to live in the $t, \varphi, r$ part of the spacetime. It is natural to write the metric (11) in coordinates $x^+, x^- = \bar{x}^- \mu^{\frac{2}{d-2}}$ and $y = r \mu^{\frac{1}{d-2}}$. In the $\mu \to \infty$ limit the metric of the three-dimensional spacetime where the geodesics propagate becomes

$$ds^2 = -\frac{1}{4} \left( 1 - \frac{1}{y^{d-2}} \right) dx^+ dy^2 dx^- + \frac{dy^2}{y^2}.$$  

(13)

The two Killing vectors give rise to two conserved quantities,

$$K_+ = -\frac{1}{4} \left( 1 - \frac{1}{y^{d-2}} \right) \dot{x}^+ - \frac{y^2}{2} \dot{x}^-, \quad K_- = -K = -\frac{y^2}{2} \dot{x}^+.$$  

(14)

The geodesic equation becomes

$$y^2 + 4K K_+ + (y^{-2} - y^{-d}) K^2 - y^2 = 0.$$  

(15)

Eq. (15) describes the one-dimensional motion of a particle in an effective potential which can be inferred from
An important quantity is the largest (real) solution of the equation

\[ 4KK_+ + (y_0^{-2} - y_0^{-d})K^2 - y_0^2 = 0. \]  \hfill (16)

It specifies the turning point of the particle. Now one can compute the length of the geodesic \( \ell \), as well as \( \Delta x^+ \) and \( \Delta x^- \), in terms of \( K, K_+ \):

\[ \Delta x^+ = 4K \int_{y_0}^{\infty} \frac{dy}{y^2((y^{-d} - y^{-2})K^2 - 4KK_+ + y^2)^{1/2}}, \]  \hfill (17)

\[ \Delta x^- = 2 \int_{y_0}^{\infty} \frac{dy}{y^2((y^{-d} - y^{-2})K^2 - 4KK_+ + y^2)^{1/2}}, \]  \hfill (18)

\[ \ell = 2 \int_{y_0}^{\infty} \frac{dy}{(y^{-d} - y^{-2})K^2 - 4KK_+ + y^2)^{1/2}}. \]  \hfill (19)

Rewriting \(-\ell\) in terms of \( \Delta x^+ \) and \( \Delta x^- \) yields the value of \( \mathcal{F}_\infty \).

Let us consider the large \( K \) behavior. The substitution \( y = \sqrt{K} \tilde{y} \) can be used to argue that in this limit the \( y^{-d}K^2 \) term under the square root in the denominators of (17)–(19) can be dropped. The subsequent integration yields \( \Delta x^+ = 2 \cot^{-1}(2K) \) and \( \Delta x^- = -K^{-1}(K_+ + \frac{1}{2})\Delta x^+ \). Now we see that the large \( K \) limit corresponds to the \( \Delta x^- \rightarrow 0 \) limit. We can now compute the length and recover \( \mathcal{F}_\infty \), but the immediate technical difficulty is that the length is divergent. This means we need to regularize it – this corresponds to introducing a UV cutoff in the dual CFT. It will be easier to do this in the large volume limit.

### 4. Large volume limit

In the following we simplify the setup further and take the large volume limit (CFT on \( \mathbb{R}^{d-1,1} \), as opposed to \( S^{d-1} \times \mathbb{R} \)). This can be achieved in two equivalent ways: either taking the limit \( \Delta x^- \sim R^{-\frac{2d-2}{d-2}} \rightarrow \infty \), \( \Delta x^+ \sim R^{-1} \rightarrow 0 \) (which corresponds to taking \( K_+ \sim R \rightarrow \infty \), \( K \sim R^{-\frac{2d-2}{d-2}} \rightarrow 0 \)) or dropping the unity in the \( \text{d}t \) metric component in (13), which describes the asymptotically AdS black hole with a planar horizon.

Either way, in this limit the integrals become

\[ \Delta x^+ \simeq 4K \int_{y_0}^{\infty} \frac{dy}{y^2((y^{-d}K^2 - 4KK_+ + y^2)^{1/2}}. \]  \hfill (20)

\[ \Delta x^- \simeq 2 \int_{y_0}^{\infty} \frac{dy}{y^2((y^{-d}K^2 - 4KK_+ + y^2)^{1/2}}. \]  \hfill (21)

\[ \ell_{\Lambda} \simeq 2 \Lambda \int_{y_0}^{\Lambda} \frac{dy}{y^2((y^{-d}K^2 - 4KK_+ + y^2)^{1/2}}. \]  \hfill (22)

where the subscript \( \Lambda \) has been introduced to signify the cutoff dependence. In the integrals above the lower limit of integration corresponds to the largest root of the expression inside the square root and “\( \simeq \)” means equality up to terms subleading in \( 1/R \). It is useful to rewrite the expressions above as

\[ \Delta x^+ \simeq \frac{1}{K}I_+(\alpha), \quad I_+(\alpha) \equiv 4 \int_{u_0}^{\infty} \frac{du}{u^{2d} + 2u^d + x^2}, \]  \hfill (23)

\[ \Delta x^- \simeq \frac{1}{K}I_-(\alpha), \quad I_-(\alpha) \equiv 4 \int_{u_0}^{\infty} \frac{(1 - \frac{x^2}{2u^2}) du}{u^{2d} + 2u^d + x^2}, \]  \hfill (24)

\[ \ell_f \simeq \log(K_+ K) + I_f(\alpha), \]  \hfill (25)

where \( \alpha = K^{-\frac{d-2}{2}}K_+^{\frac{d-2}{2}} \) and the subscript “\( f \)” in (25) stands for the regularized value. The appearance of the \( \log(K_+ K) \) term in (25) comes from the regularization and the change of variables \( u = (KK_+)^{-1/2}y \) – this implies that the original cutoff \( \Lambda \) is related to \( \Lambda_u \) via \( \Lambda = (KK_+)^{1/2}\Lambda_u \). In (25) the limit \( \Lambda_u \rightarrow \infty \) is implied; in this limit the expression for \( I_f(\alpha) \) is cutoff-independent.

Here and in what follows we set the regularization-dependent constant term to zero (this corresponds to the canonical normalization of the conformal \( \mathcal{O}_L\mathcal{O}_L \) two point function). Combining everything,

\[ \ell_f \simeq \log(\Delta x^+ \Delta x^-) + \delta I, \]  \hfill (26)

where

\[ \delta I = -\log[I_+(\alpha)I_-(\alpha)] + I_f(\alpha) \]  \hfill (27)

determines the correlator with the vacuum part subtracted and \( \alpha \) is a solution of

\[ (-\Delta x^-)^{\frac{d-2}{2}}(\Delta x^+)^{\frac{d-2}{2}} = \alpha I_+^{\frac{d-2}{2}}(\alpha) I_+^{\frac{d-2}{2}}(\alpha). \]  \hfill (28)

The expressions above provide a complete solution in the large volume limit, although one still needs to compute (and invert) a few functions which are determined by simple one-dimensional integrals.

To extract the \( d = 2 \) result, we need to take the \( d \rightarrow 2 \) limit of (28), keeping \( \Delta x^- \) fixed. This yields \( \sqrt{\mu} \Delta x^+ = \alpha I_+^{(d-2)}(\alpha) \). One can further show \( I_+^{(d-2)}(\alpha) = 1 \) and (see Appendix for details)

\[ -\ell_f|_{d=2} \simeq -\log \sinh \frac{\sqrt{\mu} \Delta x^+}{2}, \]  \hfill (29)

which agrees with (8) in the large volume limit, \( \mu \gg 1 \).
the two scalars on the right-hand side of (10) determines the OPE coefficient of \( e^{F_{\infty}} \), the exponential of \( F \) that one can read off the OPE coefficients from the correlator in the large \( R \) limit. The result provides a close analog of the Virasoro vacuum block in large-\( \Delta \) limit. Note that this is the lightcone limit of the small temperature expansion of the correlator (in the large volume limit \( \Delta_x \equiv \Delta x^-(\Delta x^+)^3 \)):

\[
-\ell I_{d=4} \simeq -\log(\Delta x^- \Delta x^+) + \frac{\Delta x^-}{120} + \frac{\Delta x^2}{10080} + \frac{1583 \Delta x^3}{648648000} + \frac{3975313 \Delta x^4}{4940103168000} + \cdots.
\]

Terms up to \( \mathcal{O}(\Delta x^3) \) agree with the previously known result \([10]\). Note that this is the lightcone limit of the small temperature expansion of the correlator (in the large volume limit \( \Delta_x = c T^4 \Delta x^- (\Delta x^+)^3 \) where \( T \) is the temperature, \( c \) is a theory-dependent numerical coefficient, and factors of \( R \) have explicitly canceled out).

5. Discussion: We have computed the near lightcone behavior of HHLL correlators in CFTs with a large central charge in the large \( \Delta_L \) limit. The result provides a close analog of the Virasoro vacuum block in large-\( C_T \) two-dimensional CFTs. It would be nice to see if the \( d = 4 \) result could be simplified further (while we do have an expression in terms of the Appell functions/elliptic integrals, it is still a bit involved). Perhaps this would allow understanding of the algebraic structure behind the near-lightcone correlators (see \([81, 82]\) for recent work in this direction).

We have explicitly matched the expansion of the length of the near-null geodesic to the near lightcone behavior of the correlator in the large \( R \) and large \( \Delta_L \) limit. Note that one can read off the OPE coefficients from the exponential of \( F_{\infty} \). For example, the second term in the right-hand side of (10) determines the OPE coefficient of the two scalars \( O_L \) with the stress-tensor and the leading \( \mathcal{O}(\Delta_L, k) \) behavior of the OPE coefficients of \( O_L \) with the leading twist \( k \)-stress tensor operator \( T_{\mu\nu} \).

One may ask whether the full correlator is well approximated by its stress-tensor sector. Generally we expect the multi trace operators of the form \( \mathcal{O}_L^k T_{\mu\nu} \) to contribute. In the \( d = 2 \) case, such contributions vanish in the large \( R \) limit and the HHLL Virasoro vacuum block agrees with the full thermal correlator in the high temperature limit (in this limit the correlator can be computed by using a conformal transformation from \( \mathbb{R}^2 \) to \( \mathbb{R} \times S^1 \)). The situation in higher dimensions is more intricate (see e.g. \([2]\) for a recent discussion). Nevertheless, in the large \( \Delta_L \) limit, multi-trace operators involving \( O_L \) become heavy and decouple even for finite \( R \). Hence, in the large \( \Delta_L \) limit the stress tensor sector is a good approximation to the thermal correlator.

The results of Section 4 make it evident that as \( \alpha \) approaches a certain critical value, the spacelike geodesic gets closer to the horizon (this corresponds to the \( \Delta x^+ \to \infty \) limit; similar behavior of geodesics was recently studied in \([33]\). It would be interesting to investigate this limit from the CFT point of view. It would also be interesting to relate our results to the behavior of quasinormal modes in the UV region \([33, 35, 36]\) which is relevant for the conformal collider bounds \([37]\).

Other natural directions include explicit computations for finite \( R \) (this would involve keeping the hypergeometric functions in \([10]\) and generalization for finite values of \( \Delta_L \). Also note that the universality of the multi stress tensor OPE coefficients may have a wider region of applicability than just the set of holographic theories. In this case the holographic calculation of this paper will have a wider regime of applicability as well.

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As explained in Section 2, $\Delta z^-$ is the separation between the insertions of the light operators in the rescaled light-like direction.

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In the following we mostly consider $d > 2$, but the $d = 2$ limit can also be recovered.

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Supplemental Material

A. Details of the $d = 2$ case

In two spacetime dimensions some simplifications occur. Eq. (28) can be written as

$$\sqrt{\mu} \Delta x^+ = 4 \tanh \sqrt{x_1}, \quad \bar{x}_1 \equiv \frac{x_1}{x_0} = \frac{\alpha}{x_0},$$

(31)

where $x_{1,2}$ are the roots of the polynomial $P_2(x) = x^2 - 4x + \alpha$ in the decreasing order. Note that $x_0 = 4/(1 + \bar{x}_1)$. Combining everything yields

$$I_{\ell_d=2} = \log \frac{\sqrt{x_1}}{1 - \bar{x}_1},$$

(32)

which, upon the substitution of (31) results in eq. (29).

B. Details of the $d = 4$ case

The integrals can be rewritten in terms of special functions. For example,

$$I^{(d=4)}_+ = \frac{4}{x_0} F_1(1, 1 \frac{1}{2} \frac{3}{2}, \bar{x}_1, \bar{x}_2),$$

(33)

where $\bar{x}_{1,2} = x_{1,2}/x_0$, $i = 0, 1, 2$ are the ($\alpha$-dependent) roots of the polynomial $P_4(x) = x^3 - 4x^2 + \alpha$ in the decreasing order and $F_1$ is the Appel function. Likewise,

$$I^{(d=4)}_\ell = \frac{4}{x_0} F_1(1, 1 \frac{1}{2} \frac{3}{2}, \bar{x}_1, \bar{x}_2) - \frac{16\alpha}{15x_0} F_1(3, 1 \frac{1}{2} \frac{3}{2}, \bar{x}_1, \bar{x}_2).$$

(34)

It will be convenient to parameterize the solutions in terms of a new variable, $\tilde{\alpha} = \alpha/x_0^\ell$. One can derive the following expressions:

$$\bar{x}_1(\tilde{\alpha}) = \frac{\tilde{\alpha} + \sqrt{(16 - 3\tilde{\alpha})\tilde{\alpha}}}{8 - 2\tilde{\alpha}}, \quad \bar{x}_2(\tilde{\alpha}) = \frac{\tilde{\alpha} - \sqrt{(16 - 3\tilde{\alpha})\tilde{\alpha}}}{8 - 2\tilde{\alpha}}, \quad x_0(\tilde{\alpha}) = 4 - \tilde{\alpha}. \quad (35)$$

Hence, eq. (28) becomes

$$\Delta x^-(\Delta x^+) \tilde{\alpha} = -\frac{16\tilde{\alpha}}{(1 - \frac{1}{2})} F_1(1, 1 \frac{1}{2} \frac{3}{2}, \bar{x}_1, \bar{x}_2) \left[ F_1(1, 1 \frac{1}{2} \frac{3}{2}, \bar{x}_1, \bar{x}_2) - \frac{4\tilde{\alpha}}{15} F_1(3, 1 \frac{1}{2} \frac{3}{2}, \bar{x}_1, \bar{x}_2) \right],$$

(36)

where $x_{1,2} = x_{1,2}(\tilde{\alpha})$. We generally need to invert this, which is a simple exercise perturbatively,

$$\tilde{\alpha} = -\frac{\Delta x}{16} \left[ 1 + \frac{5\Delta x}{96} + \frac{6299\Delta x^2}{2150400} + \frac{76228319\Delta x^3}{442810368000} + \cdots \right], \quad \Delta x = \Delta x^-(\Delta x^+) \tilde{\alpha}. \quad (37)$$

To deal with the log divergence, one can differentiate $I_\ell$ in (28) with respect to $\tilde{\alpha}$. This makes the integral convergent and kills the log $\Lambda$ term, but one would need to integrate back with respect to $\tilde{\alpha}$ to recover the result:

$$I_{\ell_d=4} = \log(\Delta x^+\Delta x^-) - \log(I^{(d=4)}_+ I^{(d=4)}_\ell) + I^{(d=4)}_+(\tilde{\alpha}), \quad (38)$$

where $I^{(d=4)}_+$ and $I^{(d=4)}_\ell$ are given by (33), (34) and $\bar{x}_{1,2}$ and $x_0$ are related to $\tilde{\alpha}$ by (35), while $I^{(d=4)}(\tilde{\alpha})$ is given by

$$I^{(d=4)}_+(\tilde{\alpha}) = \frac{2}{x_0} \int_0^x \frac{dt}{(t - 4)^2} \left[ (8 - t + \sqrt{(16 - 3t)^2}) F_1(1, 3 \frac{1}{2} \frac{3}{2}, \bar{x}_1(t), \bar{x}_2(t)) + (t - 8 + \sqrt{(16 - 3t)^2}) F_1(1, 3 \frac{1}{2} \frac{3}{2}, \bar{x}_1(t), \bar{x}_2(t)) \right] - \log(4 - \tilde{\alpha}). \quad (39)$$

Eq. (39) can be expanded perturbatively in $\tilde{\alpha}$,

$$I^{(d=4)}_+(\tilde{\alpha}) = \frac{2\tilde{\alpha}}{3} + \frac{5\tilde{\alpha}^2}{21} + \frac{647\tilde{\alpha}^3}{5544} + \frac{391\tilde{\alpha}^4}{6006} + \cdots, \quad (40)$$

where we omit a constant term. Substituting eq. (40) into eq. (38) yields eq. (30).