Quasi-projections in Teichmüller space

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Many parallels have been drawn between the geometric properties of the Teichmüller space of a Riemann surface, and those of complete, negatively curved spaces (see for example [B, K2, W]). This paper investigates one such parallel – the contracting properties of certain projections to geodesics.

We will use the Teichmüller metric throughout the paper (the other famous metric on Teichmüller space, the Weil-Petersson metric, is negatively curved although it is not complete. The Teichmüller metric is complete, but not negatively curved – see [Ma].)

Every closed subset $C$ of a complete metric space $X$ determines a “closest-points” projection, defined as a map $\pi_C : X \to \mathcal{P}(C)$, where $\mathcal{P}(C)$ is the set of closed subsets of $C$. Namely,

$$\pi_C(x) = \{ y \in C : d(x, y) = d(x, C) \}$$

where $d(x, C) = \inf_{y \in C} d(x, y)$. Suppose now that $X$ is a simply-connected Riemannian manifold with non-positive sectional curvatures, and that $C$ is a geodesic segment, ray or line. In this case $\pi_C(x)$ always contains a single point. If the sectional curvatures are bounded above by a negative constant, then $|d\pi_C| \to 0$ as $d(x, C) \to \infty$. In other words, $\pi$ is contracting at large distances.

A standard consequence of this is that “quasi-geodesics” are always in bounded neighborhoods of actual geodesics in such a space (see section 3).

If $X = T(S)$, the Teichmüller space of a surface $S$ of finite type, we must be satisfied with coarser results. We show that, for a certain class of geodesics, the images of the projection do not deviate too much from being single points (see section 4 for definitions): 

**Theorem 2.1** (Quasi-map) Given $\epsilon$ there is a constant $b_0(\epsilon, \chi(S))$ such that for any
\( \varepsilon \)-thick geodesic \( L \) the projection \( \pi_L \) satisfies:
\[
\text{diam}(\pi_L(\sigma)) \leq b_0
\]
for all \( \sigma \in \mathcal{T}(S) \).

This theorem is a consequence of a characterization of \( \pi_L \) in terms of an extremal-length ratio problem. Minimizing the distance of \( \sigma \in \mathcal{T}(S) \) to a geodesic \( L \) is equivalent to solving the problem of finding
\[
\inf_{\tau \in L} \sup_{\lambda} \frac{E_{\tau}(\lambda)}{E_{\sigma}(\lambda)}
\]
where the supremum is over \( \lambda \) in \( PF(S) \), the set of the set of projectivized measured foliations of compact support in \( S \). This is because the supremum of extremal length ratios is just \( \exp 2d(\sigma, \tau) \). Our second theorem, which we state informally, is:

**Theorem 2.2** (Characterize projection) If \( \sigma \in \mathcal{T}(S) \) and \( L \) is \( \varepsilon \)-thick, the solutions \( \tau \) of the inverted problem
\[
\sup_{\lambda} \inf_{\tau \in L} \frac{E_{\tau}(\lambda)}{E_{\sigma}(\lambda)}
\]
are a bounded distance apart from \( \pi_L(\sigma) \).

This turns out to be of great assistance in controlling the projection. In particular we obtain the following average contraction theorem:

**Theorem 2.6** (Contraction at a distance) There exist constants \( b_1, b_2 \) depending on \( \varepsilon, \chi(S) \) such that if \( L \) is an \( \varepsilon \)-thick geodesic and \( d(\sigma, L) > b_1 \) then, setting \( r = d(\sigma, L) - b_1 \),
\[
\text{diam}(\pi_L(\mathcal{N}_r(\sigma))) \leq b_2.
\]

Here \( \mathcal{N}_r \) denotes an \( r \)-neighborhood in the Teichmüller metric, and we have taken the liberty of denoting by \( \pi_L(A) \) the union \( \cup_{a \in A} \pi_L(a) \).

In section 3 we give some fairly easy consequences of these results, which are directly analogous to well-known properties of hyperbolic space. The first, which was the original motivation for the results, is the aforementioned “stability” of certain quasi-geodesics: a quasi-geodesic whose endpoints are connected by an \( \varepsilon \)-thick geodesic must remain in a bounded neighborhood of the geodesic. The second concerns the projected images of regions in Teichmüller space which are analogous to horoballs in hyperbolic space.
1 Preliminaries

We begin with a brief summary of results and notation. Let \( S \) be a surface of finite genus with finitely many punctures, and let \( \mathcal{T}(S) \) be the Teichmüller space of conformal structures of finite type on \( S \), where two structures are considered equivalent if there is a conformal isomorphism of one to the other, which is isotopic to the identity on \( S \). By “finite type” we mean that neighborhoods of the punctures are conformally equivalent to the plane minus a point, rather than a disk (hyperbolically, these are cusps rather than funnels).

\( \mathcal{T}(S) \) has a natural topology, and is homeomorphic to a finite-dimensional Euclidean space. The Teichmüller distance between two points \( \sigma, \tau \in \mathcal{T}(S) \) is defined as

\[
d(\sigma, \tau) = \frac{1}{2} \log K(\sigma, \tau),
\]

where \( K(\sigma, \tau) = K(\tau, \sigma) \) is the smallest possible quasi-conformal dilatation of a homeomorphism from \((S, \sigma)\) to \((S, \tau)\) isotopic to the identity.

Let \( \alpha \) be a homotopy class of simple closed curves in \( S \). Given \( \sigma \in \mathcal{T}(S) \), one may define the extremal length of \( \alpha \) in \( \sigma \), as the inverse of the conformal modulus of the thickest embedded annulus homotopic to \( \alpha \) in \( S \):

\[
E_\sigma(\alpha) = \inf_{A \sim \alpha} \frac{1}{\text{Mod}(A)}.
\]

This number is positive provided \( \alpha \) is non-peripheral – that is, not deformable into a puncture. We will make heavy use of the fact (see [K1]) that

\[
K(\sigma, \tau) = \sup_{\alpha} \frac{E_\tau(\alpha)}{E_\sigma(\alpha)},
\]

(1.1)

In order to obtain this supremum as a maximum, we may enlarge the space of homotopy classes of non-peripheral simple closed curves to obtain the space of “measured foliations with compact support” (see [FLP, L]), \( \mathcal{MF}(S) \). Elements of \( \mathcal{MF}(S) \) are equivalence classes of foliations of \( S \) with saddle singularities (negative index) in the interior, and index 1/2 singularities at the poles, equipped with transverse measures. The equivalence is via isotopy, and saddle-collapsing (Whitehead) moves. \( \mathcal{MF}(S) \) carries a natural topology, and is homeomorphic to a Euclidean space of the same dimension as \( \mathcal{T}(S) \). Multiplication of the measure by positive scalars gives a ray structure to \( \mathcal{MF}(S) \), and the quotient of \( \mathcal{MF}(S) \) (minus the empty foliation) by this multiplication is the projectivized space \( \mathcal{PMF}(S) \), which is a sphere.

A homotopy class of simple closed curves, equipped with a real number giving the measure, is represented in \( \mathcal{MF}(S) \) as a foliation whose non-singular leaves are
§1. Preliminaries

in the homotopy class and fit together in a cylinder whose height is the measure. These special foliations are dense in \( \mathcal{MF}(S) \), and the extremal length function extends [K1] to a function \( E_\sigma(\lambda) \) which is continuous in both \( \sigma \) and \( \lambda \), and scales quadratically:

\[
E_\sigma(a\lambda) = a^2 E_\sigma(\lambda).
\]

Moreover, the supremum in (1.1) is realized by a unique projective class in \( \mathcal{PMF}(S) \).

Fixing \( \sigma \in \mathcal{T}(S) \), let \( QD(S,\sigma) \) denote the space of integrable holomorphic quadratic differentials on \( (S,\sigma) \) (see [Ga]). This space may be identified with \( \mathcal{MF}(S) \) by associating to a quadratic differential \( \Phi \) its horizontal foliation, which we call \( \Phi_h \). This correspondence is a homeomorphism [HM]. We also have the equality [K1]

\[
E_\sigma(\Phi_h) = ||\Phi||.
\]

We note also that the vertical foliation \( \Phi_v \), which is the orthogonal foliation to \( \Phi_h \), is just \( (-\Phi)_h \).

Geodesics in \( \mathcal{T}(S) \) are determined by holomorphic quadratic differentials (and hence measured foliations) as follows: Fix \( \sigma \in \mathcal{T}(S) \) and \( \Phi \in QD(S,\sigma) \). The family \( \{\sigma_t\}_{t\geq 0} \) of conformal structures obtained by contracting along the leaves of \( \Phi_h \) by \( e^t \) and expanding along \( \Phi_v \) by \( e^{-t} \), forms a geodesic ray in the Teichmüller metric. Geodesics exist and are unique between any two points in \( \mathcal{T}(S) \). We also have

\[
d(\sigma,\sigma_t) = t
\]

and

\[
E_{\sigma_t}(\Phi_h) = e^{-2t} E_\sigma(\Phi_h),
\]

\[
E_{\sigma_t}(\Phi_v) = e^{2t} E_\sigma(\Phi_v).
\]

Our last group of definitions has to do with intersection number. The standard geometric (unoriented) intersection number between two simple closed homotopy classes may be extended to a continuous function on \( \mathcal{MF}(S) \times \mathcal{MF}(S) \), homogeneous in both arguments. One may best visualize this in the case when one of the arguments is a simple closed curve and the other is any measured foliation. The intersection number is the smallest possible transverse measure deposited on a homotopic representative of the simple closed curve.

Intersection numbers may be related to extremal lengths by the following easily proven lemma which appears for example in [Mi2]:

**Lemma 1.1 (Extremal length and intersection number)** If \( \alpha, \beta \) are any two measured foliations on any Riemann surface \( X \), then

\[
E_X(\alpha)E_X(\beta) \geq i(\alpha,\beta)^2.
\]
§2. Projecting onto thick geodesics

We say a foliation $\lambda$ is complete if for any other foliation $\mu$, $i(\lambda, \mu) = 0$ implies that the underlying leaf structures of $\lambda$ and $\mu$ are the same (up to isotopy and Whitehead moves).

Let $L$ denote a geodesic segment, ray or line in $T(S)$. We say that $L$ is $\epsilon$-thick, for $\epsilon > 0$, if for all $\tau \in L$ and all non-peripheral simple closed curves $\gamma$ in $S$, $E_\tau(\gamma) \geq \epsilon$.

We will have need of another simple observation, which appears in [Ma] and again in [Mi1]:

**Lemma 1.2** (Thick implies complete) If $\Phi_\epsilon \in \mathcal{MF}(S)$ determines an $\epsilon$-thick Teichmüller ray, then $\Phi_\epsilon$ is complete.

## 2 Projecting onto thick geodesics

Our first result is that the projection to a thick geodesic is a “quasi-map”, in the following sense:

**Theorem 2.1** (Quasi-map) Given $\epsilon$ there is a constant $b_0(\epsilon, \chi(S))$ such that for any $\epsilon$-thick geodesic $L$ the projection $\pi_L$ satisfies:

$$\text{diam}(\pi_L(\sigma)) \leq b_0$$

for all $\sigma \in T(S)$.

**Proof of 2.1:** Let $\overline{L}$ denote the complete geodesic containing $L$. Choose an arbitrary origin and orientation on $\overline{L}$ and let the parameter $t$ denote twice the signed distance from the origin (the factor of 2 is for later convenience). Then $L$ corresponds to an interval $[a, b] \cap \mathbb{R}$, where $-\infty \leq a < b \leq \infty$. (We will henceforth abbreviate this to just $[a, b]$). Denote by $L(t)$ the conformal structure on $S$ given by the point parametrized by $t$; or by $L(t)$ if $t \in [a, b]$.

For each $\alpha \in \mathcal{MF}(S)$ and $t \in \mathbb{R}$, let $E_t(\alpha)$ be short for the extremal length $E_{\overline{L}(t)}(\alpha)$. Fixing a point $\sigma \in T(S)$, define also

$$R_{\sigma,t}(\alpha) = \frac{E_t(\alpha)}{E_\sigma(\alpha)},$$

which we will tend to abbreviate as $R_t(\alpha)$. Note that $R_t$ is invariant under scaling, making it a function defined on projective classes in $\mathcal{PMF}(S)$.

For each $t$ we have $d(\sigma, L(t)) = \frac{1}{2} \log \sup_\alpha R_t(\alpha)$, so finding $\pi_L(\sigma)$ reduces to minimizing this quantity over $[a, b]$. 

Accordingly, define
\[
\text{Minmax}(\sigma, L) = \{ (\alpha, t) \in \mathcal{MF}(S) \times [a, b] : R_t(\alpha) = \inf_s \sup_{\beta} R_s(\beta) \\
= \sup_{\beta} R_t(\beta) \}
\]
(where infima and suprema are over \([a, b]\) and \(\mathcal{MF}(S)\), respectively). Let \(T_1(\sigma, L) \subset [a, b]\) denote the set of values of \(t\) which appear in \(\text{Minmax}(\sigma, L)\), and let \(A_1(\sigma, L) \subset \mathcal{MF}(S)\) denote the set of values of \(\alpha\) which appear. \(\pi_L(\sigma)\) is exactly the set \(\{L(t) : t \in T_1(\sigma, L)\}\).

Now consider the dual problem,
\[
\text{Maxmin}(\sigma, L) = \{ (\alpha, t) \in \mathcal{MF}(S) \times [a, b] : R_t(\alpha) = \sup_{\beta} \inf_s R_s(\beta) \\
= \inf_s R_t(\beta) \}
\]
and similarly define \(T_2\) and \(A_2\). The main idea of this paper is to replace the first problem with the second. In particular theorem 2.1 will follow from

**Theorem 2.2** (Characterize projection) Let \(L\) be an \(\epsilon\)-thick geodesic in \(\mathcal{T}(S)\). Then for any \(\sigma \in \mathcal{T}(S)\), \(T_1(\sigma, L)\) and \(T_2(\sigma, L)\) have bounded diameters, and are a bounded distance apart. These bounds depend only on \(\epsilon\) and \(\chi(S)\).

**Proof of 2.2:** The first step is to show that \(E_t(\alpha)\), as a function of \(t\), is coarsely approximated by a strictly convex function. This in particular will show that the minima of \(E_t(\alpha)\) occur on sets of bounded diameter.

Let \(\Phi\) be the holomorphic quadratic differential on \(\mathcal{T}(0)\) that determines the geodesic. The extremal lengths of the horizontal and vertical foliations of \(\Phi\) along the geodesic are
\[
E_t(\Phi_h) = e^{-t}E_0(\Phi_h)
\]
and
\[
E_t(\Phi_v) = e^tE_0(\Phi_v).
\]
We also have \(E_0(\Phi_h) = E_0(\Phi_v) = ||\Phi||\), and we may normalize so that \(||\Phi|| = 1\).

We will need to know (see \(\text{[GM]}\)) that \(\Phi_h\) and \(\Phi_v\) bind, in the sense that every non-zero element of \(\mathcal{MF}(S)\) has positive intersection with at least one of them. Thus we may define the strictly positive function
\[
e_t(\alpha) = \frac{1}{2} \left( \frac{i(\alpha, \Phi_h)^2}{E_t(\Phi_h)} + \frac{i(\alpha, \Phi_v)^2}{E_t(\Phi_v)} \right).
\]
We then have
Lemma 2.3 (Extremal length estimate) There is a fixed $c_0 = c_0(\epsilon, \chi(S))$ such that if $L$ is $\epsilon$-thick then for any $\alpha \in \mathcal{MF}(S)$ and $t \in [a, b]$,
\[ e_t(\alpha) \leq E_t(\alpha) \leq c_0 e_t(\alpha). \]

Proof of 2.3: The left side actually does not depend on the $\epsilon$-thickness – it follows directly from lemma [1.1], applied to $\alpha$ and each of $\Phi_h$ and $\Phi_v$.

The right side follows from a compactness argument. Let $\Phi^t$ denote the image of $\Phi$ in $L(t)$ under the Teichmüller map from $L(0)$, scaled so that $||\Phi^t|| = 1$. Then as measured foliations we have $\Phi^t_h = e^{-t/2}\Phi_h$ and $\Phi^t_v = e^{t/2}\Phi_v$. Thus, 
\[ e_t(\alpha) = \frac{1}{2} \left( i(\alpha, \Phi^t_h)^2 + i(\alpha, \Phi^t_v)^2 \right). \]

The ratio $E_t(\alpha)/e_t(\alpha)$ is therefore a continuous positive function of the projectivized measured foliation $[\alpha] \in \mathcal{PF}(S)$ and the pair $(L(t), \Phi^t)$ in the total space of Riemann surfaces equipped with holomorphic quadratic differentials. Fixing the Riemann surface $L(t)$, the space $\mathcal{PMF}(S) \times \mathcal{PQD}(L(t))$ is compact; and the function is invariant under the action of the mapping class group. Further, the image of $L(t)$ in the moduli space is restricted to a compact set by the assumption of $\epsilon$-thickness. It follows that $E_t(\alpha)/e_t(\alpha)$ is bounded.

Fix a foliation $\alpha \in \mathcal{MF}(S)$, and let $s_\alpha$ denote the point in $[a, b]$ where $e_t(\alpha)$ takes its minimum. Since $e_t(\alpha)$ is a strictly positive sum of exponentials, a brief calculation shows that, excluding the case $s_\alpha = \pm\infty$,
\[ \frac{1}{2} e_{s_\alpha}(\alpha) \exp |t - s_\alpha| \leq e_t(\alpha) \leq 2 e_{s_\alpha}(\alpha) \exp |t - s_\alpha| \tag{2.1} \]
for $t \in [a, b]$.

Let $m(\alpha)$ denote the set of minima of $E_t(\alpha)$. It follows immediately from (2.1) and lemma 2.3 that $m(\alpha)$ is constrained by
\[ m(\alpha) \subseteq [s_\alpha - 2 \log 2 c_0, s_\alpha + 2 \log 2 c_0]. \tag{2.2} \]

Since $T_2(\sigma, L) = \bigcup_{\alpha \in A_2(\sigma, L)} m(\alpha)$, our next task will be to control the points $s_\alpha$.

Define
\[ I_t(\alpha, \beta) = \frac{i(\alpha, \beta)^2}{E_t(\alpha)E_t(\beta)}. \]

Lemma 2.4 (Distance implies intersection) There are constants $D$ and $c_1$, depending only on $\epsilon$ and $\chi(S)$, such that for any $\alpha, \beta \in \mathcal{MF}(S)$,
\[ |s_\alpha - s_\beta| \geq D \implies I_{s_\alpha}(\alpha, \beta) \geq c_1. \]
Remark. This lemma is similar to (and was motivated by) lemma 3.2 from [Mi2], in which the role of $L$ is played by a hyperbolic 3-manifold $N$ homeomorphic to $S \times \mathbb{R}$, and the role of the minimum point $s_\alpha$ is played by the geodesic representative of $\alpha$ in $N$.

Proof of 2.4: Suppose the theorem is false; then there is a sequence of $\epsilon$-thick geodesics $L_i$ parametrized by $[a_i, b_i]$, with foliations $\alpha_i, \beta_i$ such that $|s_{\alpha_i} - s_{\beta_i}| \to \infty$ while $I_{s_\alpha}(\alpha, \beta) \to 0$.

Adjusting if necessary by automorphisms of $S$, we may assume that the points $L_i(s_{\alpha_i})$ lie in a fixed compact subset of a fundamental domain of the action of the mapping class group on $\mathcal{T}(S)$. Further, the quantity $I_t$ is invariant under scaling of the measures, and so is defined on the compact space $\mathcal{PMF}(S) \times \mathcal{PMF}(S)$. Thus we may take a convergent subsequence and obtain a limit example $(L, \alpha, \beta)$ in which $I_{s_\alpha}(\alpha, \beta) = 0$, and $|s_\beta - s_\alpha| = \infty$. Without loss of generality we assume $s_\beta = +\infty$, so that $L$ must contain an $\epsilon$-thick ray $[s_\alpha, \infty)$.

However, by lemma 1.2, this implies that the foliation $\Phi_h$ is complete. Since $s_\beta = +\infty$, we have $i(\beta, \Phi_h) = 0$, so that $\beta$ must be topologically equivalent to $\Phi_h$. On the other hand, the fact that $s_\alpha \neq +\infty$ implies that $i(\alpha, \Phi_h) > 0$, and therefore that $i(\alpha, \beta) > 0$, a contradiction.

Fix $\sigma \in \mathcal{T}(S)$, and for $\alpha \in \mathcal{MF}(S)$ and $t \in \mathbb{R}$ define

$$r_t(\alpha) = \frac{e_j(\alpha)}{E_{\sigma}(\alpha)}.$$ 

This function is invariant under scaling, so it may be defined for $[\alpha] \in \mathcal{PMF}(S)$. Since intersection number and extremal length are continuous functions, it follows that the values $r_{s_\alpha}(\alpha)$ depend continously on $[\alpha]$. Thus a maximum is realized, and we can make these definitions:

$$\mathcal{L}(\sigma, L) = \{[\lambda] \in \mathcal{PMF}(S) : r_{s_{\lambda}}(\lambda) = \max_{\alpha \in \mathcal{MF}(S)} r_{s_{\alpha}}(\alpha)\},$$

$$\mathcal{S}(\sigma, L) = \{s_{\lambda} : [\lambda] \in \mathcal{L}(\sigma, L)\}.$$ 

The ratios $R_t(\alpha)$ are approximated by $r_t(\alpha)$, by virtue of lemma 2.3 (Extremal length estimate).

We claim now that, for $[\lambda] \in \mathcal{L}(\sigma, L)$, the ratio $R_{s_{\lambda}}(\lambda)$ (or $r_{s_{\lambda}}(\lambda)$) provides a good estimate for $d(\sigma, L(s_{\lambda}))$. More precisely,
Lemma 2.5 (Ratio estimates distance) There exists a constant \( c_3(\epsilon, \chi(S)) \) such that

\[
R_{s\lambda}(\lambda) \leq \exp 2d(\sigma, L(s\lambda)) \leq c_3 R_{s\lambda}(\lambda)
\]

for \([\lambda] \in \mathcal{L}(\sigma, L)\).

**Proof of 2.5:** The left side of the inequality is immediate from (1.1).

Let \([\mu] \in \mathcal{PMF}(S)\) denote the unique projective class of measured foliations, such that

\[
d(\sigma, L(s\lambda)) = \frac{1}{2} \log R_{s\lambda}(\mu),
\]

and, in particular, \([\mu]\) maximizes \( R_{s\lambda} \) over \( \mathcal{PMF}(S) \). Let \( A = R_{s\lambda}(\mu)/R_{s\lambda}(\lambda) \). By virtue of (2.1), we have

\[
r_{s\lambda}(\mu) \leq 2e|s\lambda - s\mu|r_{s\mu}(\mu).
\]

Since \( R_{s\lambda}(\mu) \leq c_0 r_{s\lambda}(\mu) \), and by choice of \( \lambda \), \( r_{s\mu}(\mu) \leq r_{s\lambda}(\lambda) \leq R_{s\lambda}(\lambda) \), we conclude

\[
A \leq \frac{c_0}{2e}|s\lambda - s\mu|,
\]

or

\[
|s\lambda - s\mu| \geq \log A/2c_0.
\]

Now suppose \( A \geq 2c_0 \exp D \), with \( D \) the constant from lemma 2.4 (Distance implies intersection). It follows that

\[
i(\lambda, \mu)^2 \geq c_1 E_{s\lambda}(\lambda)E_{s\lambda}(\mu).
\]

On the other hand, lemma 1.1 assures us that

\[
E_{s\lambda}(\lambda)E_{s\lambda}(\mu) \geq i(\lambda, \mu)^2.
\]

Therefore we have

\[
R_{s\lambda}(\lambda)R_{s\lambda}(\mu) \leq \frac{1}{c_1} \tag{2.3}
\]

or

\[
A \leq (c_1 R_{s\lambda}^2(\lambda))^{-1}.
\]

To bound \( A \) we need only note that there is a lower bound for \( R_{s\lambda}(\lambda) \): There is a constant \( \ell_0 \) depending on \( \chi(S) \) such there is always a simple closed curve \( \alpha \) on \( S \) with \( E_{s\lambda}(\alpha) \leq \ell_0 \). Since \( L \) is \( \epsilon \)-thick, \( R_t(\alpha) \geq \epsilon/\ell_0 \) for all \( t \in [a, b] \). Tracing through our definitions, it follows that \( r_{s\alpha}(\alpha) \geq \epsilon/\ell_0 c_0 \), and hence that \( R_{s\lambda}(\lambda) \geq \epsilon/\ell_0 c_0 \) as well. This gives the desired bound on \( A \).
Next we will bound the diameter of $S(\sigma, L)$. Suppose $\lambda, \mu \in L(\sigma, L)$. If $|s_\lambda - s_\mu| \geq D$ then, as above, inequality (2.3) holds. On the other hand, $R_{s_\lambda}(\lambda) \geq r_{s_\lambda}(\lambda)$, and $R_{s_\lambda}(\mu) \geq \frac{1}{2}r_{s_\mu}(\mu) \exp |s_\lambda - s_\mu|$, since, as in the previous paragraph, $r_{s_\lambda}(\lambda) = r_{s_\mu}(\mu) \geq r_0$ for a fixed $r_0$, we get

$$\frac{1}{2}r_0^2 e^{|s_\lambda - s_\mu|} \leq \frac{1}{c_1}$$

which bounds $|s_\lambda - s_\mu|$ by some $c_4$.

Similarly, if $\lambda \in L(\sigma, L)$ and $\mu \in A_2(\sigma, L)$ then $\inf_t R_t(\mu)$ is maximal in $\mathcal{MF}(S)$ and in particular is at least $\epsilon/\ell_0$. Thus we may repeat the argument of the previous paragraph to obtain a bound on $|s_\lambda - s_\mu|$ in this case as well.

Using (2.2), we conclude that $T_2(\sigma, L)$ lies in a bounded neighborhood of $S(\sigma, L)$, and thus its diameter is bounded, and half the theorem is proven.

To complete the proof, we will show that every point $t \in T_1(\sigma, L)$ is at a bounded distance from $T_2(\sigma, L)$, or what is now equivalent, from $S(\sigma, L)$.

Fix $t \in T_1(\sigma, L)$, and let $[\mu] \in \mathcal{PML}(S)$ maximize $R_t$—so that $d(\sigma, L) = d(\sigma, L(t)) = \frac{1}{2} \log R_t(\mu)$.

Since $d(\sigma, L(s_\lambda)) \geq d(\sigma, L)$, and using lemma 2.5, we obtain

$$c_3 R_{s_\lambda}(\lambda) \geq R_t(\mu)$$

for any $\lambda \in L(\sigma, L)$. But by choice of $\mu$, $R_t(\mu) \geq R_t(\lambda)$. Finally, $R_t(\lambda) \geq r_t(\lambda) \geq \frac{1}{2} \exp |t - s_\lambda| r_{s_\lambda}(\lambda)$, and $R_{s_\lambda}(\lambda) \leq c_0 r_{s_\lambda}(\lambda)$. We conclude

$$c_0 c_3 \geq \frac{1}{2} \exp |t - s_\lambda|.$$ 

This bounds $|t - s_\lambda|$ by some $c_5$, and the proof is complete.

2.2

Theorem 2.1 follows too, recalling that the parameter $t$ measures half of Teichmüller arclength along $L$.

2.1

We proceed to the contraction theorem:
§2. Projecting onto thick geodesics

Theorem 2.6 (Contraction at a distance) There exist constants \( b_1, b_2 \) depending on \( \epsilon, \chi(S) \) such that if \( L \) is an \( \epsilon \)-thick geodesic and \( d(\sigma, L) > b_1 \) then, setting \( r = d(\sigma, L) - b_1 \),

\[
\text{diam}(\pi_L(N_r(\sigma))) \leq b_2.
\]

Proof of 2.6: Let \( \tau \) be such that \( d(\sigma, \tau) \leq r \). Suppose \( \lambda \in \mathcal{L}(\sigma, L) \) and \( \mu \in \mathcal{L}(\tau, L) \). By the proof of theorem 2.2, it is sufficient to find a value for \( b_1 \) that allows us to bound \( |s_\lambda - s_\mu| \).

Suppose \( |s_\lambda - s_\mu| > D \), where \( D \) is the constant in lemma 2.4. Then inequality (2.3) holds again, or in other words

\[
R_{\sigma,s_\lambda}(\lambda) R_{\sigma,s_\lambda}(\mu) \leq \frac{1}{c_1}.
\]

By choice of \( \lambda \), and lemma 2.5,

\[
R_{\sigma,s_\lambda}(\lambda) \geq c_0 \exp 2d(\sigma, L),
\]

for an appropriate constant \( c_0 \), and similarly with \( \sigma \) and \( \lambda \) replaced by \( \tau \) and \( \mu \). We note also that

\[
E_\sigma(\mu) \leq e^{2r} E_\tau(\mu)
\]

by choice of \( \tau \) and by (1.1), and therefore

\[
R_{\tau,s_\lambda}(\mu) \leq e^{2r} R_{\sigma,s_\lambda}(\mu).
\]

By definition of \( s_\mu \) and by lemma 2.3

\[
R_{\tau,s_\mu}(\mu) \leq c_0 R_{\tau,s_\lambda}(\mu).
\]

Putting all this together, we obtain

\[
d(\sigma, L) + d(\tau, L) \leq r + C
\]

for some new constant \( C \). By the definition of \( r \), we obtain \( d(\tau, L) \leq C - b_1 \). But, by the choice of \( \tau \) and the triangle inequality, \( d(\tau, L) \geq b_1 \), so if we choose \( b_1 > C/2 \) we obtain a contradiction here, implying that \( |s_\lambda - s_\mu| \leq D \).
§3. Applications

We mention some easily stated consequences of the projection theorems. Some more substantive applications to the area of hyperbolic three-manifolds will hopefully be presented in a future paper.

Define a $K$-quasi-geodesic in $\mathcal{T}(S)$ as a path $\Gamma : I \to \mathcal{T}(S)$, say parametrized by arclength, such that for any $a, b \in I$,

$$|a - b| \leq Kd(\Gamma(a), \Gamma(b)).$$

**Theorem 3.1** (Stability of quasi-geodesics) Let $\Gamma$ be a $K$-quasi-geodesic path in $\mathcal{T}(S)$, whose endpoints are connected by an $\varepsilon$-thick Teichmüller geodesic $L$. Then $\Gamma$ remains in a $B(K, \varepsilon)$-neighborhood of $L$.

The proof is standard – see [Mi2] for a proof in a very similar situation. The idea is that any segment of $\Gamma$ which is outside a sufficiently large neighborhood of $L$ is, because of the contractive properties of $\pi_L$, much less efficient than the path obtained by going back to $L$ and moving along the projection.

Fix a non-peripheral simple closed curve $\alpha$ in $S$. The set

$$\text{Thin}(\alpha, \delta) = \{\sigma \in \mathcal{T}(S) : E_\sigma(\alpha) \leq \delta\}$$

is somewhat analogous to a horoball in hyperbolic space. We have the following:

**Theorem 3.2** (Cusp projections) Given $\varepsilon, \delta$ there is a constant $B$ such that

$$\text{diam}(\pi_L(\text{Thin}(\alpha, \delta))) \leq B$$

for any simple closed curve $\alpha \subset S$ and $\varepsilon$-thick geodesic $L$.

**Proof of 3.2:** We show first that there is some $\delta_0$ and $B$ for which the bound $\text{diam}(\pi_L(\text{Thin}(\alpha, \delta_0))) \leq B$ holds. In fact we will show that $\pi_L(\text{Thin}(\alpha, \delta_0))$ lies in a bounded neighborhood of $L(s_\alpha)$.

By the results of the previous section, it suffices to bound $|s_\lambda - s_\alpha|$ for $\lambda \in \mathcal{L}(\sigma, L)$, when $\sigma \in \text{Thin}(\alpha, \delta_0)$. Suppose $|s_\lambda - s_\alpha| \geq D$. Then, as in (2.3) we get

$$R_{s_\alpha}(\alpha)R_{s_\lambda}(\lambda) \leq \frac{1}{c_1}.$$ 

Now, $R_{s_\alpha}(\alpha) \geq \varepsilon/\delta_0$, so $R_{s_\alpha}(\lambda) \leq \delta_0/c_1$. On the other hand, $R_{s_\alpha}(\lambda) \geq r_{s_\lambda}(\lambda) \geq r_0$ as in the previous section, so if we choose $\delta_0 < \varepsilon c_1 r_0$ we obtain a contradiction, and conclude $|s_\lambda - s_\alpha| < D$. 
To prove the theorem for general $\delta > \delta_0$, we observe that $Thin(\alpha, \delta) \in \text{neighborhood of } Thin(\alpha, \delta_0)$ of radius $\frac{1}{2} \log \delta / \delta_0$ (since the extremal length of $\alpha$ can always be reduced at a constant rate along an appropriate Teichmüller geodesic), and apply theorem 2.6.

Let $\tau$ be a point at a distance $a$ from $Thin(\alpha, \delta_0)$, and let $\sigma \in Thin(\alpha, \delta_0)$ be such that $d(\sigma, \tau) = a$. We may assume $a > b_1$. If $d(\sigma, L) > a + b_1$ then by theorem 2.6 we have $d(\pi_L(\sigma), \pi_L(\tau)) \leq b_2$ (here distance between sets is the maximal distance between pairs of elements), and we are done.

If $d(\sigma, L) \leq a + b_1$ then $d(\tau, \pi_L(\sigma)) \leq 2a + b_1$. This implies $d(\tau, \pi_L(\tau)) \leq 2a + b_1$, so $d(\pi_L(\sigma), \pi_L(\tau)) \leq 4a + 2b_1$, and again we are done.

3.2

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QUASI-PROJECTIONS IN TEICHMÜLLER SPACE

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Many parallels have been drawn between geometric properties of the Teichmüller space of a Riemann surface, and those of complete, negatively curved spaces (see for example Bers [2], Kerckhoff [8], Masur [12], Wolpert [17]). This paper investigates one such parallel – the contracting properties of certain projections to geodesics.

We will use the Teichmüller metric throughout the paper (the other famous metric on Teichmüller space, the Weil-Petersson metric, is negatively curved although it is not complete – see Wolpert [16]. The Teichmüller metric is complete, but not negatively curved – see Masur [10].)

Every closed subset $C$ of a metric space $X$ determines a “closest-points” projection, defined as a map $\pi_C : X \to \mathcal{P}(C)$, where $\mathcal{P}(C)$ is the set of closed subsets of $C$. Namely,$$\pi_C(x) = \{ y \in C : d(x, y) = d(x, C) \}$$where $d(x, C) = \inf_{y \in C} d(x, y)$. (Provided $X$ is proper, i.e. $R$-balls are compact, $\pi_C(x)$ is never empty.) Suppose now that $X$ is a simply-connected Riemannian manifold with non-positive sectional curvatures, and that $C$ is a geodesic segment, ray or line. In this case $\pi_C(x)$ always consists of a single point. If the sectional curvatures are bounded above by a negative constant, then $||d\pi_C|| \to 0$ as $d(x, C)$ goes to $\infty$ (where $d\pi_C$ is the differential of $\pi_C$). In other words, $\pi$ is contracting at large distances.

If $X = T(S)$, the Teichmüller space of a surface $S$ of finite type, this contraction property fails in general, but a coarse form of it can be shown to hold for a special class of geodesics. Call a geodesic (segment, ray, or line) $L$ in $T(S)$ precompact if its projection to the moduli space $M(S)$ lies in a compact set. We say $L$ is $\epsilon$-precompact if $\epsilon > 0$ is a lower bound on the extremal length of any non-peripheral simple closed curve in $S$, for any conformal structure in $L$ (see section 1 for definitions). Our main theorem is the following:

**Theorem 3.1** Given $\epsilon$ there are constants $b_1, b_2$ depending on $\epsilon, \chi(S)$ such that for any $\epsilon$-precompact geodesic $L$, if $\sigma \in T(S)$ is such that $d(\sigma, L) > b_1$ then, setting $r = d(\sigma, L) - b_1$, $$\text{diam}(\pi_L(N_r(\sigma))) \leq b_2.$$In particular, for some $b_0 = b_0(\epsilon, \chi(S))$ and all $\sigma \in T(S)$ we have $$\text{diam}(\pi_L(\sigma)) \leq b_0.$$
Here \( \mathcal{N}_r \) denotes an \( r \)-neighborhood in the Teichmüller metric, and we have taken the liberty of denoting by \( \pi_L(A) \) the union \( \bigcup_{a \in A} \pi_L(a) \).

Precompact geodesics are actually rare: by Masur’s ergodicity theorem [12], the infinite ones have zero measure in the geodesic flow. In Theorem 5.2 we show that Theorem 3.1 is sharp, in the sense that it fails for all non-precompact geodesics.

Theorem 3.1 is a consequence of a characterization of \( \pi_L \) in terms of an extremal-length ratio problem. Minimizing the distance of \( \sigma \in \mathcal{T}(S) \) to a geodesic \( L \) is equivalent to solving the problem of finding

\[
\inf_{\tau \in L} \sup_{\lambda} \frac{E_{\tau}(\lambda)}{E_{\sigma}(\lambda)}
\]

where the supremum is over \( \lambda \) in \( \mathcal{PMF}(S) \), the set of projectivized measured foliations of compact support in \( S \). This is because the supremum of extremal length ratios is just \( \exp 2d(\sigma, \tau) \). In section 2 we prove the following (stated a bit differently):

**Theorem 2.1** If \( \sigma \in \mathcal{T}(S) \) and \( L \) is \( \epsilon \)-precompact, the points \( \tau \in L \) which appear in the solutions of the dual problem

\[
\sup_{\lambda} \inf_{\tau \in L} \frac{E_{\tau}(\lambda)}{E_{\sigma}(\lambda)}
\]

are within a bounded distance of \( \pi_L(\sigma) \).

In section 4 we give some consequences of the contraction theorem, which are directly analogous to well-known properties of hyperbolic space. Theorem 4.2 is a “stability” property for a class of quasi-geodesics: a quasi-geodesic whose endpoints are connected by an \( \epsilon \)-precompact geodesic must remain in a bounded neighborhood of the geodesic. Theorem 4.3 bounds the projected images of regions in Teichmüller space which are analogous to horoballs in hyperbolic space. Theorem 4.4 is a lower bound for the translation distance of a pseudo-Anosov automorphism of \( \mathcal{T}(S) \), in terms of distance from its axis; a similar estimate holds for arbitrary loxodromic isometries in \( \mathbb{H}^n \).

**§1. Preliminaries**

We begin with a brief summary of results and notation. Let \( S \) be a surface of finite genus with finitely many punctures, and let \( \mathcal{T}(S) \) be the Teichmüller space of conformal structures of finite type on \( S \), where two structures are considered equivalent if there is a conformal isomorphism of one to the other, which is isotopic to the identity on \( S \). By “finite type” we mean that each puncture has a neighborhood conformally equivalent to a punctured disk.

\( \mathcal{T}(S) \) has a natural topology, and is homeomorphic to a finite-dimensional Euclidean space. The *Teichmüller distance* between two points \( \sigma, \tau \in \mathcal{T}(S) \) is defined as

\[
d(\sigma, \tau) = \frac{1}{2} \log K(\sigma, \tau),
\]
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where $K(\sigma, \tau) = K(\tau, \sigma)$ is the smallest possible quasi-conformal dilatation of a homeomorphism from $(S, \sigma)$ to $(S, \tau)$ isotopic to the identity.

Let $\alpha$ be a homotopy class of simple closed curves in $S$. Given $\sigma \in \mathcal{T}(S)$, one may define the extremal length of $\alpha$ in $\sigma$, as the inverse of the conformal modulus of the thickest embedded annulus homotopic to $\alpha$ in $S$:

$$E_\sigma(\alpha) = \inf_{A \sim \alpha} \frac{1}{\text{Mod}(A)}.$$

This number is positive provided $\alpha$ is non-peripheral – that is, not deformable into a puncture. We will make heavy use of Kerckhoff’s theorem (see [7]), which states that

$$K(\sigma, \tau) = \sup_{\alpha} \frac{E_\tau(\alpha)}{E_\sigma(\alpha)}. \tag{1.1}$$

Kerckhoff in fact obtains this supremum as a maximum, by using a completion of the space of homotopy classes of non-peripheral simple closed curves known as the space of “measured foliations with compact support”, $\mathcal{MF}(S)$ (see [3, 9]). Elements of $\mathcal{MF}(S)$ are equivalence classes of foliations of $S$ with saddle singularities (negative index) in the interior, and index 1/2 singularities at the poles, equipped with transverse measures. The equivalence is via isotopy, and saddle-collapsing (Whitehead) moves. $\mathcal{MF}(S)$ carries a natural topology, and is homeomorphic to a Euclidean space of the same dimension as $\mathcal{T}(S)$. Multiplication of the measure by positive scalars gives a ray structure to $\mathcal{MF}(S)$, and the quotient of $\mathcal{MF}(S)$ (minus the empty foliation) by this multiplication is the projectivized space $\mathcal{PMF}(S)$, which is a sphere.

A homotopy class of simple closed curves, equipped with a real number giving the measure, is represented in $\mathcal{MF}(S)$ as a foliation whose non-singular leaves are in the homotopy class and fit together in a cylinder whose height is the measure. These special foliations are dense in $\mathcal{MF}(S)$, and Kerckhoff shows that the extremal length function extends to a function $E_\sigma(\lambda)$ which is continuous in both $\sigma$ and $\lambda$, and scales quadratically:

$$E_\sigma(a\lambda) = a^2 E_\sigma(\lambda).$$

Moreover, the supremum in (1.1) is realized by a unique projective class in $\mathcal{PMF}(S)$.

Fixing $\sigma \in \mathcal{T}(S)$, let $QD(S, \sigma)$ denote the space of integrable holomorphic quadratic differentials on $(S, \sigma)$ (see [4]). This space may be identified with $\mathcal{MF}(S)$ by associating to a quadratic differential $\Phi$ its horizontal foliation, which we call $\Phi_h$. Hubbard-Masur [6] showed that this correspondence is a homeomorphism. From [7] we also have the equality

$$E_\sigma(\Phi_h) = ||\Phi|| \tag{1.2}$$

where $||\Phi|| = \int_S |\Phi|$. We note also that the vertical foliation $\Phi_v$, which is the orthogonal foliation to $\Phi_h$, is just $(-\Phi)_h$. Geodesics in $\mathcal{T}(S)$ are determined by holomorphic quadratic differentials (and hence measured foliations) as follows: Fix $\sigma \in \mathcal{T}(S)$ and $\Phi \in QD(S, \sigma)$. The family
{σ_t}_{t \geq 0} of conformal structures obtained by contracting along the leaves of Φ_h by e^t and expanding along Φ_v by e^{-t}, forms a geodesic ray in the Teichmüller metric. Geodesics exist and are unique between any two points in T(S). We also have

\[ d(σ, σ_t) = t \]

and

\[ E_{σ_t}(Φ_h) = e^{-2t} E_σ(Φ_h), \]
\[ E_{σ_t}(Φ_v) = e^{2t} E_σ(Φ_v). \]  

(1.3)

(1.4)

Our last group of definitions has to do with intersection number. The standard geometric (unoriented) intersection number between two simple closed homotopy classes may be extended to a continuous function on \( MF(S) \times MF(S) \), homogeneous in both arguments. One may best visualize this in the case when one of the arguments is a simple closed curve and the other is any measured foliation. The intersection number is the smallest possible transverse measure deposited on a homotopic representative of the simple closed curve.

Intersection numbers may be related to extremal lengths by the following basic lemma which appears for example in [15]:

**Lemma 1.1.** If \( α, β \) are any two measured foliations on any Riemann surface \( X \), then

\[ E_X(α)E_X(β) \geq i(α, β)^2. \]

We say a foliation \( λ \) is *complete* if for any other foliation \( µ \), \( i(λ, µ) = 0 \) implies that the underlying leaf structures of \( λ \) and \( µ \) are the same (up to isotopy and Whitehead moves).

Let \( L \) denote a geodesic segment, ray or line in \( T(S) \). We say that \( L \) is \( \varepsilon \)-precompact, for \( \varepsilon > 0 \), if for all \( τ \in L \) and all non-peripheral simple closed curves \( γ \) in \( S \), \( E_τ(γ) \geq \varepsilon \). Equivalently, let \( M(S) \) denote the moduli space of \( S \), which is the quotient of \( T(S) \) by the induced action of the group of homeomorphisms of \( S \). A geodesic is precompact exactly when its image in \( M(S) \) is contained in a compact set.

We will have need of another observation, which appears in [10] and again in [14]:

**Lemma 1.2.** If \( Φ_h \in MF(S) \) determines an \( \varepsilon \)-precompact Teichmüller ray, then \( Φ_h \) is complete.

§2. Two optimization problems

Let \( L \) be an \( \varepsilon \)-precompact geodesic segment, ray, or line and let \( \overline{L} \) denote the complete geodesic line containing \( L \). Choose an arbitrary origin and orientation on \( \overline{L} \) and let the parameter \( t \) denote signed distance from the origin. Then \( L \) corresponds to an interval \([a, b]\), where \(-∞ \leq a < b \leq ∞\) (if \( a = -∞ \) we really mean \((-∞, b]\), and similarly for \( b = ∞\)). Denote by \( \overline{L}(t) \) the conformal structure on \( S \) given by the point parametrized by \( t \); or by \( L(t) \) if \( t \in [a, b] \).
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For each \( \alpha \in \mathcal{MF}(S) \) and \( t \in \mathbb{R} \), let \( E_t(\alpha) \) be short for the extremal length \( E_{\pi_t}(\alpha) \). Fixing a point \( \sigma \in \mathcal{T}(S) \), define also

\[
R_{\sigma,t}(\alpha) = \frac{E_t(\alpha)}{E_\sigma(\alpha)}
\]

which we will tend to abbreviate as \( R_t(\alpha) \). Note that \( R_t \) is invariant under scaling, making it a function defined on projective classes in \( \mathcal{PMF}(S) \).

For each \( t \) we have \( d(\sigma, L(t)) = \frac{1}{2} \log \sup_\alpha R_t(\alpha) \), so finding \( \pi_L(\sigma) \) reduces to minimizing this quantity over \([a,b]\).

Accordingly, define

\[
\text{Minmax}(\sigma, L) = \{(\alpha, t) \in \mathcal{MF}(S) \times [a,b] : R_t(\alpha) = \inf_{\beta} \sup_{s} R_s(\beta) = \sup_{\beta} \inf_{s} R_s(\beta)\}
\]

(where infima and suprema are over \([a,b]\) and \( \mathcal{MF}(S) \), respectively). Denote by \( A_{mm}(\sigma, L) \subset \mathcal{MF}(S) \) the projection of \( \text{Minmax}(\sigma, L) \) to its first factor, and let \( T_{mm}(\sigma, L) \subset [a,b] \) denote the projection of \( \text{Minmax}(\sigma, L) \) to its second factor. The projection \( \pi_L(\sigma) \) is exactly the set \( \{L(t) : t \in T_{mm}(\sigma, L)\} \).

Now consider the dual problem,

\[
\text{Maxmin}(\sigma, L) = \{(\alpha, t) \in \mathcal{MF}(S) \times [a,b] : R_t(\alpha) = \sup_{\beta} \inf_{s} R_s(\beta) = \inf_{s} \sup_{\beta} R_s(\alpha)\}
\]

and similarly define \( T_{Mm} \) and \( A_{Mm} \). Note that, in this problem, the first optimization, \( \inf_{s} R_s(\beta) \), is independent of \( \sigma \). The main idea of this paper is to replace the Minmax problem with the Maxmin problem. In particular,

**Theorem 2.1.** Let \( L \) be an \( \epsilon \)-precompact geodesic in \( \mathcal{T}(S) \). Then for any \( \sigma \in \mathcal{T}(S) \), \( T_{mm}(\sigma, L) \) and \( T_{Mm}(\sigma, L) \) have bounded diameters, and are a bounded distance apart. These bounds depend only on \( \epsilon \) and \( \chi(S) \).

**Proof.** The first step is to show that \( E_t(\alpha) \), as a function of \( t \), is coarsely approximated by a strictly convex function. This in particular will show that the minima of \( E_t(\alpha) \) occur on sets of bounded diameter.

Let \( \Phi \) be the holomorphic quadratic differential on \( \mathcal{T}(0) \) that determines the geodesic. The extremal lengths of the horizontal and vertical foliations of \( \Phi \) vary exponentially with \( t \) according to (1.3) and (1.4). We also have \( E_0(\Phi_h) = E_0(\Phi_v) = ||\Phi|| \) by (1.2), and we may normalize so that \( ||\Phi|| = 1 \).

We will need to know (see e.g. Gardiner-Masur [5]) that \( \Phi_h \) and \( \Phi_v \) fill up \( S \), in the sense that every non-zero element of \( \mathcal{MF}(S) \) has positive intersection with at least one of them. Thus we may define the strictly positive function

\[
e_t(\alpha) = \frac{1}{2} \left( \frac{i(\alpha, \Phi_h)^2}{E_t(\Phi_h)} + \frac{i(\alpha, \Phi_v)^2}{E_t(\Phi_v)} \right)
\]

which will serve as an approximation for \( E_t(\alpha) \) (see lemma 2.2).
We can then introduce the following quantities, after fixing $\sigma \in \mathcal{T}(S)$.

- Let $s_\alpha$ denote the unique value of $t$ where $e_t(\alpha)$ is minimal (possibly $s_\alpha = \infty$ or $-\infty$).
- Let $m(\alpha)$ be the set of $t$ values where $E_t(\alpha)$ is minimal.
- Let $r_s(\alpha) = \frac{e_s(\alpha)}{E_s(\alpha)}$. This function is an approximation for $R_s(\alpha)$, and takes its minimum at $t = s_\alpha$.
- Let $\bar{A}_{Mm}(\sigma, L) = \{[\alpha] \in \mathcal{PMF}(S) : r_{s_\alpha}(\alpha) = \max_{\beta \in \mathcal{MF}(S)} r_{s_\beta}(\beta)\}$.
- Let $\bar{T}_{Mm}(\sigma, L) = \{s_\alpha : [\alpha] \in \bar{A}_{Mm}(\sigma, L)\}$.

Note that $\bar{A}_{Mm}$ and $\bar{T}_{Mm}$ are defined similarly to $A_{Mm}$ and $T_{Mm}$, respectively, with $R_t$ replaced with $r_t$. These definitions can be made for the following reason: The function $r_t(\alpha)$ is invariant under scaling of $\alpha$, so it may be defined for $[\alpha] \in \mathcal{PMF}(S)$. Since intersection number and extremal length are continuous functions, it follows that the minimal values $r_{s_\alpha}(\alpha)$ depend continuously on $[\alpha]$. Therefore a maximum is realized, since $\mathcal{PMF}(S)$ is compact.

The proof will divide into the following basic steps:

1. Show that $e_t(\alpha)$ and $r_t(\alpha)$ are approximations for $E_t(\alpha)$ and $R_t(\alpha)$, respectively. As a consequence, the set of minima $m(\alpha)$ is contained in a bounded neighborhood of $s_\alpha$.

2. Show that for a solution $\alpha \in A_{Mm}(\sigma, L)$ of the Maxmin problem, $R_{s_\alpha}(\alpha)$ and $r_{s_\alpha}(\alpha)$ are also good approximations to $\exp 2d(\sigma, L(s_\alpha))$.

3. Conclude that $\text{diam } \bar{T}_{Mm}$ is bounded, and that $T_{Mm}$ and $T_{mM}$ lie in a bounded neighborhood of $\bar{T}_{Mm}$.

Of these, step 2 contains the essential geometric idea of the paper, and brings us most of the way to connecting the Maxmin and the Minmax problems.

Begin with this estimate:

**Lemma 2.2.** There is a fixed $c_0 = c_0(\epsilon, \chi(S))$ such that if $L$ is $\epsilon$-precompact then for any $\alpha \in \mathcal{MF}(S)$ and $t \in [a, b]$, $$ e_t(\alpha) \leq E_t(\alpha) \leq c_0 e_t(\alpha). $$

**Proof.** The left side actually does not depend on the precompactness of $L$ – it follows directly from lemma 1.1, applied to $\alpha$ and each of $\Phi_h$ and $\Phi_v$.

The right side follows from a compactness argument. Let $\Phi'$ denote the image of $\Phi$ in $L(t)$ under the Teichmüller map from $L(0)$, scaled so that $||\Phi'|| = 1$. Then as measured foliations we have $\Phi_h' = e^{-t}\Phi_h$ and $\Phi_v' = e^{t}\Phi_v$. Thus,

$$ e_t(\alpha) = \frac{1}{2} (i(\alpha, \Phi_h')^2 + i(\alpha, \Phi_v')^2). $$

The ratio $E_t(\alpha)/e_t(\alpha)$ is therefore a continuous positive function of the projectivized measured foliation $[\alpha] \in \mathcal{PMF}(S)$ and the pair $(L(t), \Phi')$ in the total space of Riemann surfaces equipped with holomorphic quadratic differentials. Fixing the Riemann surface $L(t)$, the space $\mathcal{PMF}(S) \times PQD(L(t))$ is compact; and the function is invariant under the action of the mapping class group. Further, the image
of $L(t)$ in the moduli space is restricted to a compact set by our definition of pre-
compactness. It follows that $E_t(\alpha)/e_t(\alpha)$ is bounded. 

Now since $e_t(\alpha)$ is a strictly positive sum of exponentials, a brief calculation using
(1.3,1.4) shows that, excluding the case $s_\alpha = \pm \infty$,
\[
\frac{1}{2} e_{s_\alpha}(\alpha) \exp 2|t - s_\alpha| \leq e_t(\alpha) \leq 2 e_{s_\alpha}(\alpha) \exp 2|t - s_\alpha|
\] (2.1)
for $t \in [a, b]$. It follows immediately from (2.1) and lemma 2.2 that $m(\alpha)$ is con-
strained by
\[
m(\alpha) \subseteq [s_\alpha - \frac{1}{2} \log 2c_0, s_\alpha + \frac{1}{2} \log 2c_0].
\] (2.2)
This concludes step 1 of the proof.

Define
\[
I_t(\alpha, \beta) = \frac{i(\alpha, \beta)^2}{E_t(\alpha)E_t(\beta)}.
\]

We begin the proof of step 2 with the following crucial lemma. It states that the
minima $s_\alpha$ and $s_\beta$ for two measured foliations $\alpha, \beta$ can occur far apart only if $\alpha$ and
$\beta$ are sufficiently different to have a definite intersection number, as measured by
$I_t$:

**Lemma 2.3.** There are constants $D$ and $c_1$, depending only on $\epsilon$ and $\chi(S)$, such
that for any $\alpha, \beta \in MF(S)$,
\[
|s_\alpha - s_\beta| \geq D \quad \implies \quad I_{s_\alpha}(\alpha, \beta) \geq c_1.
\]

**Remark.** This lemma is similar to (and was motivated by) lemma 3.2 from [15], in
which the role of $L$ is played by a hyperbolic 3-manifold $N$ homeomorphic to $S \times \mathbb{R}$,
and the role of the minimum point $s_\alpha$ is played by the geodesic representative of $\alpha$
in $N$.

**Proof.** Suppose the lemma is false; then there exists a sequence of $\epsilon$-precompact
godesics $L_i$ parametrized by $[a_i, b_i]$, with foliations $\alpha_i, \beta_i$ such that $|s_{\alpha_i} - s_{\beta_i}| \to \infty$
while $I_{s_{\alpha_i}}(\alpha, \beta) \to 0$.

Adjusting if necessary by automorphisms of $S$, we may assume that the points
$L_i(s_{\alpha_i})$ lie in a fixed compact subset of a fundamental domain of the action of the
mapping class group on $\mathcal{T}(S)$. Further, the quantity $I_t$ is invariant under scaling
of the measures, and so is defined on the compact space $PMF(S) \times PMF(S)$.
Thus we may take a convergent subsequence and obtain a limit example $(L, \alpha, \beta)$
in which $I_{s_\alpha}(\alpha, \beta) = 0$, and $|s_\beta - s_\alpha| = \infty$. Without loss of generality we assume
$s_\beta = +\infty$, so that $L$ must contain an $\epsilon$-precompact ray $[s_\alpha, \infty)$.

However, by lemma 1.2, this implies that the foliation $\Phi_h$ is complete. Since
$s_\beta = +\infty$, we have $i(\beta, \Phi_h) = 0$, so that $\beta$ must be topologically equivalent to $\Phi_h$.
On the other hand, the fact that $s_\alpha \neq +\infty$ implies that $i(\alpha, \Phi_h) > 0$, and therefore
that $i(\alpha, \beta) > 0$, a contradiction. 

As a consequence of this lemma we can show that two foliations with sufficiently
distant minima cannot both give a large minimal value for $R_t$. 

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Now since $e_t(\alpha)$ is a strictly positive sum of exponentials, a brief calculation using
(1.3,1.4) shows that, excluding the case $s_\alpha = \pm \infty$,
Corollary 2.4. For any $\alpha, \beta \in \mathcal{MF}(S)$, if $|s_\alpha - s_\beta| > D$ then
\[ R_{s_\alpha} (\alpha) R_{s_\beta} (\beta) \leq 1/c_1 \]
where $D, c_1$ are the constants of lemma 2.3.

Proof. Lemma 2.3 implies that $i(\alpha, \beta)^2 \geq c_1 E_{s_\alpha}(\alpha) E_{s_\beta}(\beta)$. On the other hand, lemma 1.1 assures us that $E_{s_\alpha}(\alpha) E_{s_\beta}(\beta) \geq i(\alpha, \beta)^2$. Combining these inequalities yields the desired conclusion. \[
\square
\]

We can now complete step 2 of the proof:

Lemma 2.5. There exists a constant $c_3(\epsilon, \chi(S))$ such that
\[ R_{s_\lambda}(\lambda) \leq \exp 2d(\sigma, L(s_\lambda)) \leq c_3 R_{s_\lambda}(\lambda) \]
for $[\lambda] \in \tilde{A}_{Mm}(\sigma, L)$.

Proof. The left side of the inequality is immediate from (1.1).

Let $[\mu] \in \mathcal{PMF}(S)$ denote the unique projective class of measured foliations, such that
\[ d(\sigma, L(s_\lambda)) = \frac{1}{2} \log R_{s_\lambda}(\mu), \]
and, in particular, $[\mu]$ maximizes $R_{s_\lambda}$ over $\mathcal{PMF}(S)$. Thus our job is to find an upper bound for the ratio
\[ Q = R_{s_\lambda}(\mu) / R_{s_\lambda}(\lambda). \]

The idea will be that, if $Q$ is large, then by the exponential growth of $e_t$ and the fact that the minimum of $r_t(\mu)$ is no larger than the minimum of $r_t(\lambda)$, we may conclude that $|s_\lambda - s_\mu|$ is large. But then corollary 2.4 implies that $R_{s_\lambda}(\lambda)$ and $R_{s_\lambda}(\mu)$ cannot both be large, which will yield a contradiction.

By virtue of the estimate (2.1) on $e_t$, we have
\[ r_{s_\lambda}(\mu) \leq 2e^{2|s_\lambda - s_\mu|} r_{s_\mu}(\mu). \]
Lemma 2.2 implies that $R_{s_\lambda}(\mu) \leq c_0 r_{s_\lambda}(\mu)$, and by choice of $\lambda$ we have $r_{s_\mu}(\mu) \leq r_{s_\lambda}(\lambda) \leq R_{s_\lambda}(\lambda)$. Therefore we conclude
\[ Q \leq 2c_0 e^{2|s_\lambda - s_\mu|}, \]
or
\[ |s_\lambda - s_\mu| \geq \frac{1}{2} \log Q / 2c_0. \]

Now suppose $Q \geq 2c_0 \exp 2D$, with $D$ the constant from lemma 2.3. Corollary 2.4 implies that
\[ R_{s_\lambda}(\lambda) R_{s_\lambda}(\mu) \leq \frac{1}{c_1} \]
and hence
\[ Q \leq (c_1 R_{s_\lambda}(\lambda))^{-1}. \]

To bound $Q$ we need only note that there is a lower bound for $R_{s_\lambda}(\lambda)$: There is a constant $\ell_0$ depending on $\chi(S)$ such that the shortest simple closed curve $\alpha$ on $S$ has $E_{s_\alpha}(\alpha) \leq \ell_0$. Since $L$ is $\epsilon$-precompact, $R_t(\alpha) \geq \epsilon / \ell_0$ for all $t \in [a, b]$. Tracing through
our definitions, it follows that \( r_\alpha(s) \geq r_0 = \epsilon/\ell_0 c_0 \), and hence that \( R_\alpha(\lambda) \geq \epsilon/\ell_0 c_0 \) as well. This gives the desired bound on \( Q \).

We proceed with the proof of step 3, and bound the diameter of \( \tilde{T}_{Mm}(\sigma, L) \).

Suppose \( \lambda, \mu \in \tilde{A}_{Mm}(\sigma, L) \). If \( |s_\lambda - s_\mu| \geq D \) then corollary 2.4 bounds \( R_\alpha(\lambda) R_\alpha(\mu) \).

On the other hand, \( R_\alpha(\lambda) \geq r_\alpha(\lambda) \), and by (2.1) \( R_\alpha(\mu) \geq \frac{1}{e} r_\alpha(\mu) \exp |s_\lambda - s_\mu| \).

Since, as in the previous proof, \( r_\alpha(\lambda) = r_\mu(\mu) \geq r_0 \) for a fixed \( r_0 \), we get

\[
\frac{1}{2} r_0^2 e^{2|s_\lambda - s_\mu|} \leq \frac{1}{c_1}
\]

which bounds \( |s_\lambda - s_\mu| \) by some \( c_4 \).

Similarly, if \( \lambda \in \tilde{A}_{Mm}(\sigma, L) \) and \( \mu \in A_{Mm}(\sigma, L) \) then \( \inf_i R_i(\mu) \) is maximal in \( \mathcal{MF}(S) \) and in particular is at least \( r_0 \). Thus we may repeat the argument of the previous paragraph to obtain a bound on \( |s_\lambda - s_\mu| \) in this case as well.

Using the bound (2.2) on \( m(\alpha) \) in terms of \( s_\alpha \), and the fact that \( T_{Mm}(\sigma, L) = \bigcup_{\alpha \in A_{Mm}(\sigma, L)} m(\alpha) \), we conclude that \( T_{Mm}(\sigma, L) \) lies in a bounded neighborhood of \( \tilde{T}_{Mm}(\sigma, L) \), and thus its diameter is bounded, and half the theorem is proven.

To complete the proof, we must show that \( T_{mM}(\sigma, L) \) is in a bounded neighborhood of \( \tilde{T}_{Mm}(\sigma, L) \).

Fix \( t \in T_{mM}(\sigma, L) \), and let \( [\mu] \in PML(S) \) maximize \( R_i - \) so that

\[
d(\sigma, L) = d(\sigma, L(t)) = \frac{1}{2} \log R_i(\mu).
\]

Since \( d(\sigma, L(s_\lambda)) \geq d(\sigma, L) \), and using lemma 2.5, we obtain

\[
c_3 R_\alpha(\lambda) \geq R_i(\mu)
\]

for any \( \lambda \in \tilde{A}_{Mm}(\sigma, L) \). But by choice of \( \mu \), \( R_i(\mu) \geq R_i(\lambda) \). Finally, \( R_i(\lambda) \geq r_i(\lambda) \geq \frac{1}{e} r_\alpha(\lambda) \exp 2|t - s_\lambda| \), and \( R_\alpha(\lambda) \leq c_0 r_\alpha(\lambda) \). We conclude

\[
c_0 c_3 \geq \frac{1}{2} \exp 2|t - s_\lambda|.
\]

This bounds \( |t - s_\lambda| \) by some \( c_5 \), and the proof of theorem 2.1 is complete.

\section{The contraction theorem}

Our main theorem follows almost directly from the results of the previous section.

\textbf{Theorem 3.1.} Given \( \epsilon \) there are constants \( b_1, b_2 \) depending on \( \epsilon, \chi(S) \) such that for any \( \epsilon \)-precompact geodesic \( L \), if \( \sigma \in \mathcal{T}(S) \) is such that \( d(\sigma, L) > b_1 \) then, setting \( r = d(\sigma, L) - b_1 \),

\[
diam(\pi_L(\mathcal{N}_0(\sigma))) \leq b_2.
\]

In particular, for some \( b_0 = b_0(\epsilon, \chi(S)) \) and all \( \sigma \in \mathcal{T}(S) \) we have

\[
diam(\pi_L(\sigma)) \leq b_0.
\]
Proof. Since \( \pi_L(\sigma) = \{ L(t) : t \in T_{mM}(\sigma, L) \} \), and \( t \) measures Teichmüller arclength along \( L \), the bound on \( \text{diam}(\pi_L(\sigma)) \) follows directly from theorem 2.1. (Note also that it is a logical consequence of the first diameter bound, with \( b_0 = \max(b_2, 2b_1) \)).

Let \( \tau \) be such that \( d(\sigma, \tau) \leq r \). Suppose \( \lambda \in \tilde{A}_{\text{mm}}(\sigma, L) \) and \( \mu \in \tilde{A}_{\text{mm}}(\tau, L) \). By the proof of theorem 2.1, it is sufficient to find a value for \( b_1 \) that allows us to bound \( |s_\lambda - s_\mu| \) (note that \( s_\lambda \) and \( s_\mu \) are independent of \( \sigma \) and \( \tau \)).

Suppose \( |s_\lambda - s_\mu| > D \), where \( D \) is the constant in lemma 2.3. Then corollary 2.4 again gives

\[
R_{\sigma, s_\lambda}(\lambda) R_{\sigma, s_\lambda}(\mu) \leq \frac{1}{c_1}.
\]

By choice of \( \lambda \), and lemma 2.5,

\[
R_{\sigma, s_\lambda}(\lambda) \geq c_3^{-1} \exp 2d(\sigma, L),
\]

and similarly with \( \sigma \) and \( \lambda \) replaced by \( \tau \) and \( \mu \). We note also that

\[
E_\sigma(\mu) \leq e^{2r} E_\tau(\mu)
\]

by choice of \( \tau \) and by (1.1), and therefore

\[
R_{\tau, s_\lambda}(\mu) \leq e^{2r} R_{\sigma, s_\lambda}(\mu).
\]

By definition of \( s_\mu \) and by lemma 2.2,

\[
R_{\tau, s_\lambda}(\mu) \leq c_0 R_{\tau, s_\lambda}(\mu).
\]

Putting all this together, we obtain

\[
d(\sigma, L) + d(\tau, L) \leq \frac{1}{2} \log(c_3^2 R_{\sigma, s_\lambda}(\lambda) R_{\tau, s_\mu}(\mu))
\]

\[
\leq \frac{1}{2} \log(c_3^2 c_0 e^{2r} R_{\sigma, s_\lambda}(\lambda) R_{\sigma, s_\lambda}(\mu))
\]

\[
\leq r + C
\]

where \( C = \frac{1}{2} \log(c_3^2 c_0 / c_1) \). Now if we choose \( b_1 > C/2 \) and assume \( d(\sigma, L) > b_1 \), then since \( r = d(\sigma, L) - b_1 \) we have \( d(\tau, L) \leq C - b_1 < b_1 \). But, by the choice of \( \tau \) and the triangle inequality, \( d(\tau, L) \geq b_1 \). This contradiction implies that \( |s_\lambda - s_\mu| \leq D \). \( \square \)

§4. Applications

We shall need the following immediate consequence of the contraction theorem:

**Corollary 4.1.** Let \( L \) be an \( \epsilon \)-precompact geodesic in \( \mathcal{T}(S) \) and let \( b_1, b_2 \) be the constants given in theorem 3.1. If \( R > b_1 \) and points \( x, y \in \mathcal{T}(S) \) are connected by a path of length \( T \) which remains outside an \( R \)-neighborhood of \( L \), then

\[
\text{diam}(\pi_L(x) \cup \pi_L(y)) \leq b_2 \left( \frac{T}{R - b_1} + 1 \right).
\]

Furthermore for any \( x, y \in \mathcal{T}(S) \),

\[
\text{diam}(\pi_L(x) \cup \pi_L(y)) \leq d(x, y) + B(\epsilon).
\]
§4. Applications

Proof. Dividing the path from $x$ to $y$ into pieces of size at most $R - b_1$ and applying theorem 3.1 yields inequality (4.1). To obtain (4.2), let $R = 2b_1$ and consider the geodesic $G$ connecting $x$ to $y$. If $G$ stays outside of an $R$-neighborhood of $L$ then apply (4.1). If not, let $x', y'$ be the closest points in $G$ to $x, y$ (respectively) such that $d(x', L) = d(y', L) = R$. The triangle inequality together with (4.1) yields

$$\text{diam}(\pi_L(x) \cup \pi_L(y)) \leq d(x, x') + 2R + d(x', y') + d(y, y') + 2b_2$$

which suffices. □

Define a $(K, \delta)$-quasi-geodesic in $\mathcal{T}(S)$ as a path $\Gamma: [a, b] \to \mathcal{T}(S)$, parametrized by arclength, such that for any $s, t \in [a, b]$,

$$|s - t| \leq Kd(\Gamma(s), \Gamma(t)) + \delta.$$ 

Theorem 4.2. Let $\Gamma$ be a $(K, \delta)$-quasi-geodesic path in $\mathcal{T}(S)$, whose endpoints are connected by an $\epsilon$-precompact Teichmüller geodesic $L$. Then $\Gamma$ remains in a $B(K, \delta, \epsilon)$-neighborhood of $L$.

Proof. The idea of the argument is standard: any segment of $\Gamma$ which is outside a sufficiently large neighborhood of $L$ is, because of the contractive properties of $\pi_L$, much less efficient than the path obtained by going back to $L$ and moving along the projection. Thus large excursions from $L$ violate the quasi-geodesic property.

Fix $R = \max(Kb_2, 2b_1)$. Let $[s, t]$ be a maximal interval for which $\Gamma((s, t))$ is outside an $R$-neighborhood of $L$. Applying corollary 4.1, we have

$$\text{diam}(\pi_L(\gamma(s)) \cup \pi_L(\gamma(t))) \leq \frac{b_2}{2R}|s - t| + b_2.$$ 

The maximality of $[s, t]$ implies that $d(\Gamma(s), \pi_L(\Gamma(s))) = R$, and similarly for $t$. Thus

$$d(\Gamma(s), \Gamma(t)) \leq 2R + \frac{b_2}{2R}|s - t| + b_2.$$ 

On the other hand, the quasi-geodesic property of $\Gamma$ gives a lower bound on the distance $d(\Gamma(s), \Gamma(t))$, and together we obtain (using the definition of $R$), a bound $|s - t| \leq (4K + 2)R + 2\delta$. It follows that $\Gamma$ cannot exit a $(2K + 2)R + \delta$ neighborhood of $L$. □

Let $\alpha$ be a measured foliation in $\mathcal{M}(S)$. The set

$$\text{Thin}(\alpha, \delta) = \{\sigma \in \mathcal{T}(S) : E_\alpha(\sigma) \leq \delta\}$$

is somewhat analogous to a horoball in hyperbolic space. As in hyperbolic space, it has infinite diameter, but its projection to a precompact geodesic is nevertheless bounded, bar one exceptional case:

Theorem 4.3. Given $\epsilon, \delta$ there is a constant $B$ such that, if $L$ is a complete $\epsilon$-precompact geodesic and $\alpha \in \mathcal{M}(S)$, then

$$\text{diam}(\pi_L(\text{Thin}(\alpha, \delta))) \leq B + \text{diam}(L \cap \text{Thin}(\alpha, \delta)).$$
Remarks: (1) A special case of this is when $\alpha$ is a non-peripheral simple closed curve, with weight 1. Then $L \cap \text{Thin}(\alpha, \delta)$ is uniformly bounded for all $\epsilon$-precompact $L$, so we have a uniform bound on $\text{diam}(\pi_L(\text{Thin}(\alpha, \delta)))$. (2) If a complete geodesic $L$ is determined by a quadratic differential $\Phi$, let us call $[\Phi_h]$ and $[\Phi_v]$ in $\mathcal{PMF}(S)$ the endpoints at infinity for $L$. Another special case occurs when $[\alpha]$ is equal in $\mathcal{PMF}(S)$ to an endpoint, and then both sides of the inequality are infinite.

Proof. In view of remark (2), let us assume that $[\alpha] \neq [\Phi_h], [\Phi_v]$ in $\mathcal{PMF}(S)$. Masur proved in [11] that, if $L$ is $\epsilon$-precompact, then its endpoints at infinity are uniquely ergodic – they support a unique projective class of transverse measures. Thus the assumption that $[\alpha]$ is not an endpoint implies that $\alpha$ is topologically distinct from both foliations, and therefore (by lemma 1.2) it intersects both of them non-trivially. It follows that $s_\alpha \neq \pm \infty$ and that $E_t(\alpha)$ has a positive infimum, $E_0$.

Fix $\delta_0 < E_0c_1r_0$, where $c_1$ and $r_0$ are the constants that appear in §2. We will show that $\pi_L(\text{Thin}(\alpha, \delta_0))$ lies in a neighborhood of $L(s_\alpha)$.

By the results of section 2, it suffices to bound $|s_\lambda - s_\alpha|$ for $\lambda \in \hat{A}_{M_m}(\sigma, L)$, when $\sigma \in \text{Thin}(\alpha, \delta_0)$. Suppose $|s_\lambda - s_\alpha| \geq D$. Then by corollary 2.4 we get

$$R_{s_\alpha}(\alpha)R_{s_\alpha}(\lambda) \leq \frac{1}{c_1}.$$

On the other hand, $R_{s_\alpha}(\alpha) \geq E_0/\delta_0$ by assumption, and $R_{s_\alpha}(\lambda) \geq r_{s_\alpha}(\lambda) \geq r_0$ as in lemma 2.5, so we obtain a contradiction. It follows that $|s_\lambda - s_\alpha| < D$.

To prove the theorem for general $\delta > \delta_0$, we first observe that $\text{Thin}(\alpha, \delta)$ lies in a neighborhood of $\text{Thin}(\alpha, \delta_0)$ of radius $\frac{1}{2} \log \delta/\delta_0$ (since one can move along a Teichmüller geodesic defined by $\Phi$ such that $\Phi_h = \alpha$, and apply (1.3)).

Applying corollary 4.1 (inequality 4.2) we conclude immediately that the projection $\pi_L(\text{Thin}(\alpha, \delta))$ lies in a $(C + \frac{1}{2} \log \delta/\delta_0)$-neighborhood of $L(s_\alpha)$, for an appropriate $C(\epsilon)$. On the other hand, applying lemma 2.2 and estimate (2.1), we have $E_t(\alpha) \geq \frac{1}{2} \frac{E_0}{c_0} \exp 2|t - s_\alpha|$. Using the definition of $E_0$, we have

$$\text{diam}(\text{Thin}(\alpha, \delta) \cap L) \leq \log \frac{\delta}{\delta_0} + \log \frac{E_0}{c_0c_1r_0}.$$  

This completes the proof of the theorem. □

Our last application is a geometric property of pseudo-Anosov homeomorphisms. Let $f : S \to S$ be pseudo-Anosov, and let $f_*$ denote the induced action on $\mathcal{T}(S)$ (see [3] or [1]). Bers showed in [2] that the infimum $\inf_{x \in \mathcal{T}(S)} d(\sigma, f_\sigma)$ of translation distance is achieved in $\mathcal{T}(S)$ on a geodesic, namely the axis of $f_*$, on which $f_*$ acts by translation. The contraction theorem can be used to give lower bounds on how fast the translation distance grows as we consider points far away from the axis (this question was suggested by Feng Luo).

**Theorem 4.4.** Let $f : S \to S$ be a pseudo-Anosov homeomorphism, and $L \subset \mathcal{T}(S)$ the axis of $f_*$. There are $c_0, c_1 > 0$ such that, for all $x \in \mathcal{T}(S)$,

$$d(x, f_*x) \geq c_0 d(x, L) - c_1.$$
Remark: it is easy to see that the rate of increase of $d(x, f_* x)$ is at most linear with $d(x, L)$, and so in an asymptotic sense the theorem gives maximal growth. However the constants $c_0$ and $c_1$ are completely non-constructive.

Proof. Since the axis $L$ is invariant by $f_*$, its image in the moduli space is a closed curve, so that $L$ is precompact. Let $b_1, b_2$ be the constants given by theorem 3.1 for $L$.

For $x \in \mathcal{T}(S)$, let $R = d(x, L)$ and let $t = d(x, f_*(x))$. Let $t_0$ denote the translation distance of $f_*$ on $L$, so that $t \geq t_0$, and $t = t_0$ only for $x \in L$.

For any $n > 0$ we can connect $x$ to $f^n_* x$ by a chain of geodesics of length $nt$, which remains outside an $R - t/2$ neighborhood of $L$. By corollary 4.1 we have

$$\text{diam}(\pi_L(x) \cup \pi_L(f^n_* x)) \leq b_2 \left( \frac{nt}{R - b_1 - t/2} + 1 \right).$$

(Note that we may assume that $R - b_1 - t/2 > 0$, because otherwise we would have $t \geq 2R - b_1$, and we would be done.) It follows that

$$d(x, f^n_* x) \leq 2R + b_2 \left( \frac{nt}{R - b_1 - t/2} + 1 \right).$$

On the other hand $L$ is also the axis of $f^n_*$, so that $d(x, f^n_* x) \geq nt_0$. Letting $n$ go to infinity, we conclude that

$$t_0 \leq b_2 \frac{t}{R - b_1 - t/2}.$$

Solving for $t$, we obtain the desired inequality.

§5. The case of non-precompact geodesics

In this section we consider the sharpness of theorem 3.1. Let $L$ be a geodesic which is not precompact. Then $L$ has segments which enter, arbitrarily deeply, into the “thin parts” $\text{Thin}(\alpha, \delta)$ defined in the previous section.

For small $\delta$, the geometry of $\text{Thin}(\alpha, \delta)$ is quite far from negatively curved. In [13] we showed that it is approximated by a product space in the following sense.

Let $X_\alpha$ denote $\mathcal{H}^2 \times \mathcal{T}(S \setminus \alpha)$, endowed with the sup metric $d_X = \max(d_{\mathcal{H}^2}, d_{\mathcal{T}(S \setminus \alpha)})$ of the metrics on the factors. There is a natural homeomorphism $\Pi : \mathcal{T}(S) \to X_\alpha$, defined using Fenchel-Nielsen coordinates, so that the coordinates in the hyperbolic plane $\mathcal{H}^2$ encode the length and twist parameters associated to $\alpha$. In particular, the $y$ coordinate in $\mathcal{H}^2$ (in the upper half-plane model) for $\Pi(\sigma)$ is $1/\ell_\sigma(\alpha)$ where $\ell_\sigma$ is hyperbolic length, and it follows easily that $\Pi(\text{Thin}(\alpha, \delta))$ is contained in $\{ y > \ell_1 \} \times \mathcal{T}(S \setminus \alpha)$ and contains $\{ y > \ell_2 \} \times \mathcal{T}(S \setminus \alpha)$, where $\ell_1, \ell_2$ are approximately inversely proportional to $\delta$. The main theorem of [13] says that

**Theorem 5.1.** (Theorem 6.1 of [13]) For sufficiently small $\delta$, the map $\Pi$ restricted to $\text{Thin}(\alpha, \delta)$ has bounded additive distortion; that is,

$$|d_{\mathcal{T}(S)}(\sigma, \tau) - d_X(\Pi(\sigma), \Pi(\tau))| \leq c$$

for $\sigma, \tau \in \text{Thin}(\alpha, \delta)$, where $c$ depends only on $\delta$ and the topological type of $S$. 
This product geometry suffices to give a converse to theorem 3.1:

**Theorem 5.2.** If a Teichmüller geodesic $L$ is not precompact, then there is no choice of constants $b_1, b_2$ for which the contraction property of theorem 3.1 holds for $\pi_L$.

Remark: This theorem does not rule out the diameter bound for images of single points which is stated in theorem 3.1. In fact it is not ruled out that $\pi_L(\sigma)$ is in all cases a single point; resolving this question seems to require finer techniques than we use here.

**Proof.** Using the product structure we can show that large segments of $L$ can be replaced by quasi-geodesics that stray arbitrarily far from $L$. This will contradict theorem 4.2.

Since $L$ is not precompact, for arbitrarily small $\delta > 0$ and large $T > 0$ there exists a segment in $L$ of length $T$ contained in $\text{Thin}(\alpha, \delta)$ for some $\alpha$. (We are using here the fact that the Teichmüller metric on the moduli space is complete, or equivalently that the distance from the boundary of $\text{Thin}(\alpha, \delta)$ to $\text{Thin}(\alpha, \delta')$ goes to infinity as $\delta/\delta' \to \infty$ – in fact it is $\frac{1}{2} \log(\delta/\delta')$.)

Let $\sigma_1, \sigma_2$ be the endpoints of such a segment, and let $\Pi(\sigma_i) = q_i$ be their images in the product $X_\alpha$, so that $|d(q_1, q_2) - T| < c$. Using the sup metric on $X_\alpha$ it is easy to construct two $(2, 0)$-quasi-geodesics $m, m'$ connecting $q_1$ to $q_2$, so that neither quasi-geodesic lies in a $(T - c)$-neighborhood of the other. (restricting to an $\mathbb{R}^2$ slice, construct an appropriate quadrilateral whose diagonal is the line $[q_1, q_2]$, and let $m, m'$ be the two paths around the perimeter). Furthermore $m, m'$ can be made to remain in $\Pi(\text{Thin}(\alpha, \delta'))$ for $\delta'$ slightly bigger than $\delta$. Then applying $\Pi^{-1}$, we obtain two $(2, c)$-quasi-geodesics connecting $\sigma_1$ to $\sigma_2$, neither of which lies in a $(T - 2c)$-neighborhood of the other. It follows that at least one of the quasi-geodesics cannot be in a bounded neighborhood of $L$, as $T$ is chosen arbitrarily large. This violates the stability of quasi-geodesics proved in Theorem 4.2, and therefore $\pi_L$ cannot have the contraction property.

Remark: For a product space with the sup metric the stability of quasi-geodesics is actually violated in a stronger way than used above. See the last section of [13] for a brief discussion of this.

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