Recursive technique for evaluation of Feynman diagrams

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Abstract

A method is presented in which matrix elements for some processes are calculated recursively. This recursive calculational technique is based on the method of basis spinors.

1 Introduction

The possibility of investigation higher and higher energies at present and future colliders, entails the necessity of predicting and calculating with high precision more and more complicated processes. When the number of final particles is high it becomes hard, even to calculate the corresponding tree level Feynman diagrams and the final expression of cross section is often an intricated function of several variables, inadequate for practical use. This has lead to that it is necessary abandon the standard methods for perturbative calculations and to use instead the new effective ones.

The standard method to obtain a cross section with the fermions in perturbative quantum field theories is to reduce the squared amplitude to a trace from products of $\gamma$-matrices. An alternative approach is to calculate the Feynman amplitudes directly. The idea of calculating amplitudes has a long enough history. In 1949 it was suggested in Ref. [1] to calculate a matrix element by means of explicit form of $\gamma$-matrices and Dirac spinors (more detailed bibliography on the problem can be found in Refs. [2, 3]).

Various methods of calculating the reaction amplitudes with fermions have been proposed and successfully applied in recent years. In general the methods of matrix element calculation can be classified into two basic types. The first type includes methods of direct numerical calculation of the Feynman diagrams. The second type includes methods of analytical calculations of amplitudes with the subsequent numerical calculations of cross sections. Notice that there are methods of calculating cross sections without the Feynman diagrams [4, 5].

Analytical methods of calculating the Feynman amplitudes can be divided into two basic groups. The first group involves the analytical methods that reduce the calculation of $S$-matrix element to a trace calculation. The reduction of a matrix element to trace calculation from products of $\gamma$-matrices underlies a large number of methods (see, Refs. [2, 3] and Refs. [6]-[13] etc.). In this method the matrix element is expressed as the algebraic function in terms of scalar products of four-vectors and their contractions with the Levi-Civita tensor.

The second group involves the analytical methods that practically do not use the operations with traces from products of $\gamma$-matrices. The method of the CALCUL group which was used for the calculations of the reactions with massless fermions is the most famous among [14]-[18]. The basic idea behind the CALCUL method is to replace $S$-matrix element by spinor products of bispinors and to use the fact that expressions $\bar{u}_\lambda(p)\,u_{-\lambda}(k)$ are simple scalar functions of the momenta $p,k$ and the helicity $\lambda$. However, the operation of matrix element reduction is not so simple as the calculation of traces. It requires the use of Chisholm spinor identities

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(see [16]). Also it takes the representation of contraction $\not{p} = p^\mu \gamma_\mu$ with four-momenta $p^\mu$ and polarization vectors of external photons through bispinors. For gauge massive bosons the additional mathematical constructions are needed [16].

There are generalizations of the CALCUL method for massive Dirac particles both for special choices of the fermion polarization ([16], [19]-[22]) and for an arbitrary fermion polarization [23]. We call the polarization states of fermions in Ref. [19] [16] as Berends-Daverveldt-Kleiss-Stirling or $BDKS$-states.

Notice that for massless fermions we can obtain amplitude in terms of the scalar products of four-momentum vectors and current-like constructions of the type $J^\mu \sim \bar{u}_\lambda (p) \gamma^\mu u_\lambda (k)$. The components of $J^\mu$ are calculated by means of momentum components $p, k$ (so called $E$-vector formalism; see Ref. [25]).

For $BDKS$-states Ref. [26] presents the iterative scheme of calculation that reduces expression for the fermion chain $\bar{u}_\lambda (p) Q u_\lambda (k)$ to the combination of spinor products $\bar{u}_\lambda (p) u_\lambda (k)$ and (or) $\bar{u}_\lambda (p) \gamma^\mu (g_V + g_A \gamma_5) u_\lambda (k)$ by means of inserting the complete set of non-physical states of bispinors (with $p^2 < 0$) into the fermion chain.

In all above-mentioned methods the spinor products and current constructions were calculated by means of traces and then used as scalar functions of the momenta and of the helicities (similar to scalar product of four-vectors).

Due to their easy implementation methods of matrix element calculation have become a basis for modern programs dealing with evaluation cross-sections for various processes. Examples for such programs are generators HELAS [27], GRACE [28], MadGraph [29], O’MEGA [30], FeynArts/FormCalc4 [31], [32] and CompHEP [33] (planned the calculation of the matrix element [34]). There is large number of more specialized programs as AMEGIC++ [35], ALPGEN [36], WPHACT [37], LUSIFER [38] and et al. The detailed list of such programs can be found in Ref. [39].

In the paper we describe an approach to Feynman diagrams which is based on the using of an isotropic tetrad in Minkowski space and basis spinors connected with it (see [40], [41]). Here we don’t use an explicit form of Dirac spinors and $\gamma$-matrices and the operation of trace calculations. The method is based on the active using of the massless basis spinors connected with isotropic tetrad vectors and we will call it as Method of Basis Spinors (MBS).

In this method as well as in the trace methods the matrix element of Feynman amplitudes is reduced to the combination of scalar products of momenta and polarization vectors. Unlike spinor technique in different variants [14]-[20], this method doesn’t use either Chisholm identities, or the presentation of the contraction $\not{p}$ with four vector $p$ and of the polarization vector of bosons through the bispinors. Unlike $WvD$ spinor technique [18], [21] MBS doesn’t use special Feynman rules for calculating of the matrix elements.

We propose to use recursion relations as a technique to evaluate the Feynman amplitudes of processes. The advantage of recursive technique is that for calculation of a $n + 1$ matrix element of some process we can use the calculation of $n$ process. Both for analytic and numerical evaluation this is asset.
2 Method of Basis Spinors

When evaluating a Feynman amplitude involving the fermions, the amplitude is expressed as the sum of terms that have the form

\[ M_{\lambda_p, \lambda_k} (p, s_p, k, s_k ; Q) = \]
\[ = M_{\lambda_p, \lambda_k} ([p], [k] ; Q) = \bar{u}_{\lambda_p} (p, s_p) Q u_{\lambda_k} (k, s_k) , \]  

(1)

where \( \lambda_p \) and \( \lambda_k \) are the polarizations of the external particles with four-momentum \( p, k \) and arbitrary polarization vectors \( s_p, s_k \). The operator \( Q \) is the sum of products of Dirac \( \gamma \)-matrices.

Matrix element (1) with Dirac spinors is a scalar function. Thus, it should be expressible in terms of scalar functions formed from the spin and momentum four-vectors of the fermions, including \( p, s_p, k, s_k \) and the operator \( Q \).

We will now consider that in the our approach this matrix element (1) can be represented as linear combinations of the products of the lower-order matrix elements.

2.1 Isotropic tetrad

We use the metric and matrix convention as in the book by Bjorken and Drell [42], i.e. the Levi-Civita tensor is determined as \( \epsilon_{0123} = 1 \) and the matrix \( \gamma_5 = i\gamma^0\gamma^1\gamma^2\gamma^3 \). Let us introduce the orthonormal four-vector basis in Minkowski space which satisfies the relations:

\[ l_\mu^0 \cdot l_\nu^0 - l_\mu^1 \cdot l_\nu^1 - l_\mu^2 \cdot l_\nu^2 - l_\mu^3 \cdot l_\nu^3 = g_{\mu\nu} , \]
\[ (l_A \cdot l_B) = g_{AB} , \]  

(2)

where \( g \) is the Lorentz metric tensor.

With the help of the basis vectors \( l_A (A = 0, 1, 2, 3) \) we can define lightlike vectors, which form the isotropic tetrad in Minkowski space (see, [43])

\[ b_\rho = \frac{l_0 + \rho l_3}{2} , \quad n_\lambda = \frac{\lambda l_1 + i l_2}{2} , \quad (\rho, \lambda = \pm 1) . \]  

(3)

From Eqs. (2), (3) it follows that

\[ (b_\rho \cdot b_{-\lambda}) = \frac{\delta_{\lambda, \rho}}{2} , \quad (n_\lambda \cdot n_{-\rho}) = \frac{\delta_{\lambda, \rho}}{2} , \quad (b_\rho \cdot n_\lambda) = 0 , \]  

(4)

\[ g^{\mu\nu} = 2 \sum_{\lambda = -1}^1 \left[ b_\mu^{\lambda} \cdot b_\nu_{-\lambda} + n_\mu^{\lambda} \cdot n_\nu_{-\lambda} \right] . \]  

(5)

It is always possible to construct the basis of an isotropic tetrad [43] as numerical four-vectors

\[ (b_{\pm1})_\mu = (1/2) \{1, 0, 0, \pm1\} , \quad (n_{\pm1})_\mu = (1/2) \{0, \pm1, i, 0\} \]  

(6)

or by means of the physical vectors for reaction.

For practical applications it is convenient to introduce additional four-vectors

\[ \tilde{b}_\rho = 2 b_\rho , \quad \tilde{n}_\lambda = 2 n_\lambda \]  

(7)
By means of the isotropic tetrad vectors we can determine the polarization vectors of massless (also and massive) vector bosons. For photons with momentum \( k^\mu \) and helicity \( \lambda = \pm 1 \) we use the following definition of polarizations in the axial gauge

\[
\varepsilon_\lambda (k) = \frac{(k \cdot \tilde{n}_{-\lambda}) \tilde{b}_{-1}}{\sqrt{2} (k \cdot b_{-1})} - \frac{\tilde{n}_{-\lambda}}{\sqrt{2}} \tag{8}
\]

provided that, the four-vectors \( k, b_1, b_{-1} \) are linearly independent.

### 2.2 Massless basis spinors

By means of the isotropic tetrad vectors (3) we define massless basis spinors \( u_\lambda (b_{-1}) \) and \( u_\lambda (b_1) \)

\[
\gamma_{-1} u_\lambda (b_{-1}) = 0 \ , \quad u_\lambda (b_1) \equiv \gamma_1 u_{-\lambda} (b_{-1}) \ , \tag{9}
\]

\[
\omega_\lambda u_\lambda (b_{\pm 1}) = u_\lambda (b_{\pm 1}) \tag{10}
\]

with the matrix \( \omega_\lambda = 1/2 (1 + \lambda \gamma_5) \) and the normalization condition

\[
u\lambda (b_{\pm 1}) \bar{u}_\lambda (b_{\pm 1}) = \omega_\lambda \gamma_{\pm 1} . \tag{11}\]

The relative phase between basis spinors with different helicity is given by

\[
\not\psi u_{-\nu} (b_{-1}) = \delta_{\lambda,\nu} u_{\lambda} (b_{-1}) \tag{12}.\]

The important property of basis spinors (9) is the completeness relation

\[
\sum_{\lambda, A = -1}^1 u_\lambda (b_A) \bar{u}_{-\lambda} (b_{-A}) = I \ , \tag{13}\]

which follows from Eqs.(9), (12). Thus, the arbitrary Dirac spinor can be decomposed in terms of basis spinors \( u_\lambda (b_A) \).

### 2.3 Dirac spinors and basis spinors

Arbitrary Dirac spinor can be determined through the basis spinor (9) with the help of projection operators \( u_{\lambda p} (p, s_p) \bar{u}_{\lambda p} (p, s_p) \).

The Dirac spinors \( w_\lambda^A (p, s_p) \) for massive fermion and antifermion with four-momentum \( p \) \( (p^2 = m_p^2) \), arbitrary polarization vector \( s_p \) and spin number \( \lambda = \pm 1 \) can be obtained with the help of basis spinors by means of equation:

\[
w_\lambda^A (p, s_p) = \frac{(\not{p} + A m_p) (1 + \lambda \gamma_5 \not{s}_p)}{2 \sqrt{(b_{-1} \cdot (p + m_p s_p))}} u_{-A \times \lambda} (b_{-1})
\]

\[
= \left[ \frac{g^p}{s^p_{-1}} + A \frac{g^p}{s^p_{1}} \frac{g^p/m_p}{s^p_{1}} \right] u_{-A \times \lambda p} (b_{-1})
\]

\[
= T_\lambda (p, s_p) u_{-A \times \lambda} (b_{-1}) \ . \tag{14}\]
The notation \( w^A_{\lambda p} (p, s_p) \) stands for either \( u_{\lambda p} (p, s_p) \) (bispinor of fermion; \( A = +1 \)) or \( v_{\lambda p} (p, s_p) \) (bispinor of antifermion; \( A = -1 \)). Here we have introduced the abbreviations

\[
\xi_{\pm 1} = \frac{p \pm m_p s_p}{2}.
\]  

(15)

The bispinors \( u_{\lambda p} (p, s_p) \) and \( v_{\lambda p} (p, s_p) \) satisfy Dirac equations and spin conditions for massive fermion and antifermion

\[
\not{\!} u_{\lambda p} (p, s_p) = m_p u_{\lambda p} (p, s_p) , \quad \not{\!} v_{\lambda p} (p, s_p) = -m_p v_{\lambda p} (p, s_p) , \\
\gamma_5 \not{\!} s_p u_{\lambda p} (p, s_p) = \lambda u_{\lambda p} (p, s_p) , \quad \gamma_5 \not{\!} s_p v_{\lambda p} (p, s_p) = \lambda v_{\lambda p} (p, s_p) .
\]  

(16)

We also found, that the Dirac spinors of fermions and antifermions are related by

\[
u_{\lambda p} (p, s_p) = \lambda \gamma_5 \bar{u}_{\lambda p} (p, s_p) , \quad \bar{\nu}_{\lambda p} (p, s_p) = \bar{\lambda} \gamma_5 \bar{u}_{\lambda p} (p, s_p).
\]  

(17)

Let us consider the particular case of Eq.(14)–the \( BDKS \) polarization states of fermions, as they are the most-used in calculations of matrix elements. The polarization vector of \( BDKS \)-states is defined as follows [16, 19, 24, 26]:

\[
s_{KS} \equiv s_p = \frac{p}{m_p} - \frac{m_p b_{1-1}}{(p \cdot b_{1-1})}.
\]  

(18)

Performing a calculation for Eqs.(9), (14) and Eq.(18), we find the simple result for massive Dirac spinor [16, 26, 24]:

\[
w^\lambda_{\lambda p} (p, s_{KS}) = \left( \not{\!} + A m_p \right) u_{-A \times \lambda} (b_{1-1}) \sqrt{(p \cdot b_{1-1})}.
\]  

(19)

Notice that, in Ref.\[26\] relation between the Dirac spinor of fermion and the Dirac spinor of antifermion differs from the Eq.(17).

The Dirac spinor \( u_{\lambda p} (p) \) of massless fermion with momentum \( p \) \((p^2 = 0, (p \cdot b_{1-1}) \neq 0)\) and helicity \( \lambda \) is defined by (for example, see Ref.[16])

\[
u_{\lambda p} (p) = \frac{(\not{\!} + A m_p) u_{-A \times \lambda} (b_{1-1})}{\sqrt{(p \cdot b_{1-1})}}.
\]  

(20)

2.4 Main equations of MBS

The spinor products of massless basis spinors \([9]\) are determined by

\[
\bar{u}_{\lambda p} (b_C) u_{\rho p} (b_A) = \delta_{\lambda,-\rho} \delta_{C,-A} , \quad C, A = \pm 1, \quad \lambda, \rho = \pm 1.
\]  

(21)

With the help of Eq.(5) Dirac matrix \( \gamma^\mu \) can be rewritten as

\[
\gamma^\mu = \sum_{\lambda=-1}^{1} \left[ \gamma_{-\lambda p} \tilde{b}_A^\mu + \gamma_{-\lambda p} \tilde{n}_A^\mu \right].
\]  

(22)
Using the Eqs. (9), (10) and (12) we can obtain that
\[ \gamma^\mu u_\lambda (b_A) = \tilde{b}_A^\mu u_{-\lambda} (b_{-A}) - A \tilde{n}_{-A \times \lambda}^\mu u_{-\lambda} (b_A) \] (23)
and
\[ \gamma_5 u_\rho (b_A) = \rho u_\rho (b_A) . \] (24)
Eqs. (23) - (24) and Eq. (21) underlie the method of basis spinors (MBS) [40, 41].
By means of the Eq. (23) we can determine that product of two \( \gamma \)-matrices can be represented as
\[ \gamma^\mu \gamma^\nu u_\lambda (b_A) = Y^{\mu, \nu}_{A, \lambda} u_{-\lambda} (b_A) , \] (25)
where \( X^{\mu, \nu}_{A, \lambda} \) and \( Y^{\mu, \nu}_{A, \lambda} \) are the Lorentz tensors:
\[ X^{\mu, \nu}_{A, \lambda} = \tilde{b}_A^\mu \cdot \tilde{n}_{-A \times \lambda}^\nu - \tilde{n}^{\mu}_{-A \times \lambda} \cdot \tilde{n}_A^\nu , \] (26)
\[ Y^{\mu, \nu}_{A, \lambda} = \tilde{b}_A^\mu \cdot \tilde{b}_A^\nu + \tilde{n}^{\mu}_{A \times \lambda} \cdot \tilde{n}^\nu_{-A \times \lambda} . \] (27)
The product \( S^n = \gamma^\mu_1 \gamma^\mu_2 \ldots \gamma^\mu_n \) can be written as
\[ S^n u_\lambda (b_A) = \mathcal{B}^{\{ \mu_1, \ldots, \mu_n \}}_{A, \lambda} u_{\lambda_n} (b_{A_n}) - A \mathcal{N}^{\{ \mu_1, \ldots, \mu_n \}}_{A, \lambda} u_{\lambda_n} (b_{-A_n}) , \] (28)
where
\[ \lambda_n' = (-1)^n \lambda , \quad A_n' = (-1)^n A \] (29)
and \( \mathcal{B}^{\{ \mu_1, \ldots, \mu_n \}}_{A, \lambda} \), \( \mathcal{N}^{\{ \mu_1, \ldots, \mu_n \}}_{A, \lambda} \) are some Lorentz tensors, which are related with isotropic tetrad vectors [7].
As follows from Eqs. (23), (25) we have that in particular cases:
\[ \mathcal{B}^{\{ \mu_1 \}}_{A, \lambda} = \tilde{b}_A^{\mu_1} , \quad \mathcal{N}^{\{ \mu_1 \}}_{A, \lambda} = \tilde{n}^{\mu_1}_{-A \times \lambda} , \] (30)
\[ \mathcal{B}^{\{ \mu_1, \mu_2 \}}_{A, \lambda} = Y^{\mu_1, \mu_2}_{A, \lambda} , \quad \mathcal{N}^{\{ \mu_1, \mu_2 \}}_{A, \lambda} = X^{\mu_1, \mu_2}_{A, \lambda} . \] (31)

3 Recursion relations for the matrix elements

The basic idea of Method of Basis Spinors is to replace Dirac spinors in Eq. (1) by massless basis spinors (9), and to use the Eq. (21) and Eqs. (23) - (24) for calculation of matrix element (1) in terms of scalar functions \( \mathcal{B}, \mathcal{N} \). With the help of the Eq. (14) the matrix element (1) transforms to fermion “string” with massless basis spinors \( u_\lambda (b_A) \) i.e.
\[ \mathcal{M}_{\lambda p, \lambda k} (p, s_p, k, s_k ; Q) = \]
\[ = \bar{u}_{-\lambda_p} (b_{-1}) T_{\lambda_p} (p, s_p) Q T_{\lambda_k} (k, s_k) u_{-\lambda_k} (b_{-1}) = \]
\[ = \mathcal{M}_{-\lambda_p, -\lambda_k} (b_{-1}, b_{-1}; T_{\lambda_p} (p, s_p) Q T_{\lambda_k} (k, s_k)) , \] (32)
where operator \( T_\lambda \) is determined by Eq. (14). Let us consider a special variants of a matrix element.
3.1 Basic matrix element

Let us consider an important type of a matrix element (1), when \( p = b_C \) and \( k = b_A \), i.e.

\[
\mathcal{M}_{\sigma, \rho} (b_C ; b_A ; Q) = \Gamma_{\sigma, \rho}^{C, A} [Q]
\]

\[= \bar{u}_{\sigma} (b_C) Q u_{\rho} (b_A) . \quad (33)\]

We call this type of matrix element as **basic matrix element**.

Note that, the matrix element (1) is a particular case of basic matrix element i.e.

\[
\mathcal{M}_{\lambda_p, \lambda_k} (p, s_p, k, s_k ; Q) = \Gamma_{\lambda_p, \lambda_k}^{1,-1} [T_{\lambda_p} (p, s_p) Q T_{\lambda_k} (k, s_k)] . \quad (34)
\]

With the help of the completeness relation (13) we can obtain the recursion formula for \( \Gamma_{\sigma, \rho}^{C, A} [Q_1 Q_2] \)

\[
\Gamma_{\sigma, \rho}^{C, A} [Q_1 Q_2] = \sum_{D, \lambda = -1}^{1} \Gamma_{\sigma, \rho}^{C, D} [Q_1] \Gamma_{\lambda, \rho}^{D, A} [Q_2] . \quad (35)
\]

By means of the relations (23), (25), (28) and Eq. (21) it is easy to calculate \( \Gamma_{\sigma, \rho}^{C, A} \) in terms of the isotropic tetrad vectors. For instance,

\[
\Gamma_{\sigma, \rho}^{C, A} [\gamma^\mu] = \delta_{\sigma, -\rho} \left( \delta_{C, -A} \tilde{b}_A^\mu - A \delta_{C, A} \tilde{\eta}^\mu_{-A \times \rho} \right) , \quad (36)
\]

\[
\Gamma_{\sigma, \rho}^{C, A} [\gamma^\mu \gamma^\nu] = \delta_{\sigma, \rho} \left( \delta_{C, A} \gamma_{A, \rho}^\nu - A \delta_{C, -A} \gamma^\mu_{A, \rho} \right) \quad (37)
\]

and

\[
\Gamma_{\sigma, \rho}^{C, A} [\gamma^{\mu_1} \gamma^{\mu_2} \ldots \gamma^{\mu_n}] = \Gamma_{\sigma, \rho}^{C, A} [S^n]
\]

\[
= \delta_{\sigma, \rho} \left( \delta_{C, A, \rho} B_{A, \rho}^{(\mu_1, \ldots, \mu_n)} - A \delta_{C, -A, \rho} N_{A, \rho}^{(\mu_1, \ldots, \mu_n)} \right) . \quad (38)
\]

With the help of the Eqs. (35) and (38) we obtain recursion relations for \( B_{A, \lambda}^{(\mu_1, \ldots, \mu_n)} \) and \( N_{A, \lambda}^{(\mu_1, \ldots, \mu_n)} \)

\[
B_{A, \lambda}^{(\mu_1, \ldots, \mu_n)} = B_{A, \lambda}^{(\mu_1, \ldots, \mu_k, \mu_{k+1}, \ldots, \mu_n)} + (-1)^{n-k+1} N_{A, \lambda}^{(\mu_1, \ldots, \mu_{k-1}, \lambda_{n-k})} N_{A, \lambda}^{(\mu_{k+1}, \ldots, \mu_n)} , \quad (39)
\]

\[
N_{A, \lambda}^{(\mu_1, \ldots, \mu_n)} = B_{A, \lambda}^{(\mu_1, \ldots, \mu_k, \lambda_{n-k})} N_{A, \lambda}^{(\mu_{k+1}, \ldots, \mu_n)} + (-1)^{n-k} N_{A, \lambda}^{(\mu_1, \ldots, \mu_{k-1}, \lambda_{n-k})} B_{A, \lambda}^{(\mu_{k+1}, \ldots, \mu_n)} . \quad (40)
\]

The recursion Eqs. (39)-(40) allow to convert scalar functions \( B, N \) into Lorentz tensors in terms of isotropic tetrad vector with the help of Eq. (30) or Eq. (31). For example, the constructions \( q_1^\mu q_2^\nu q_3^\rho B_{A, \lambda}^{(\mu_1, \mu_2, \mu_3)} = B_{A, \lambda}^{(q_1, q_2, q_3)} \) and \( q_1^\mu q_2^\nu q_3^\rho N_{A, \lambda}^{(\mu_1, \mu_2, \mu_3)} = N_{A, \lambda}^{(q_1, q_2, q_3)} \) can be represented as a combination of the scalar functions \( X, Y \) and scalar products of the isotropic tetrad vectors:

\[
q_1^\mu q_2^\nu q_3^\rho B_{A, \lambda}^{(\mu_1, \mu_2, \mu_3)} = B_{A, \lambda}^{(q_1, q_2, q_3)}
= (q_3 \cdot \tilde{b}_A) Y_{A, \lambda}^{q_1, q_2} + (q_3 \cdot \tilde{n}_{A \times \lambda}) X_{A, \lambda}^{q_1, q_2} , \quad (41)
\]

\[
q_1^\mu q_2^\nu q_3^\rho N_{A, \lambda}^{(\mu_1, \mu_2, \mu_3)} = N_{A, \lambda}^{(q_1, q_2, q_3)}
= (q_3 \cdot \tilde{n}_{A \times \lambda}) Y_{A, \lambda}^{q_1, q_2} - (q_3 \cdot \tilde{b}_A) X_{A, \lambda}^{q_1, q_2} . \quad (42)
\]
3.2 Decomposition coefficients

The next type of lower-order matrix element (11) is
\[
\mathcal{M}_{\rho, \lambda_{p}} (b_A, [p]; I) \equiv \mathcal{M}_{\rho, \lambda_{p}} (b_A, [p]) = \bar{u}_{\rho} (b_A) u_{\lambda_{p}} (p, s_{p}).
\]  

The matrix element (43) is determined by the decomposition coefficients of an arbitrary Dirac spinor \( u_{\lambda_{p}} (p, s_{p}) \) on basis spinors (9).

With the help of the Eqs. (14), (20) the matrix element (43) transforms to
\[
\mathcal{M}_{\rho, \lambda_{p}} (b_A, [p]) = \bar{u}_{\rho} (b_A) \left( \xi_{p}^{\mu} + \xi_{p}^{\mu} / m_p \right) \bar{u}_{\lambda_{p}} (b_{-1}) \sqrt{\left( b_{-1} \cdot \xi_{p}^{\mu} \right)}
\]  

for the massive fermions with arbitrary vector of polarization and transforms to
\[
\mathcal{M}_{\rho, \lambda_{p}} (b_A, p) = \bar{u}_{\rho} (b_A) \left( \xi_{p}^{\mu} + \xi_{p}^{\mu} / m_p \right) \bar{u}_{\lambda_{p}} (b_{-1}) \sqrt{\left( b_{-1} \cdot \xi_{p}^{\mu} \right)}
\]  

for massless fermions.

Using the Eqs. (23)-(24) and (25) the matrix element (43) is reduced to an algebraic expression in terms of scalar products of isotropic tetrad vectors and physical vectors or in terms of components of four-vectors.

Let us consider massless fermions. Using Eqs. (21), (23) we obtain, that
\[
\mathcal{M}_{\rho, \lambda_{p}} (b_A, p) = \delta_{\lambda_{p}, -\rho} \left( \delta_{A, -1} \sqrt{\left( p \cdot b_{-1} \right)} + \delta_{A, 1} \frac{p \cdot \bar{n}_{-\lambda}}{\sqrt{\left( p \cdot b_{-1} \right)}} \right).
\]  

For numerical calculations, as well as in the case of spinor techniques, it is convenient to determine the (46) through the momentum components \( p = (p^0, \ p^x = p^0 \sin \theta_p \sin \varphi_p, \ p^y = p^0 \sin \theta_p \cos \varphi_p, \ p^z = p^0 \cos \theta_p) \)

\[
\mathcal{M}_{\rho, \lambda} (b_A, p) = \delta_{\lambda_{p}, -\rho} \left[ \delta_{A, -1} \sqrt{p} - \delta_{A, 1} \lambda \exp (-i \lambda \varphi_p) \sqrt{p} \right] =
\]  

\[
= \delta_{\lambda_{p}, -\rho} \sqrt{2p_0} \left[ \delta_{A, -1} \sin \frac{\theta_p}{2} - \delta_{A, 1} \lambda \cos \frac{\theta_p}{2} \exp (-i \lambda \varphi_p) \right],
\]  

where
\[
p^\pm = p^0 \pm p^x, \ p^x + i \lambda p^y = \sqrt{(p^x)^2 + (p^y)^2} \exp (i \lambda \varphi_p).
\]

Let us consider massive Dirac particles with arbitrary polarization vector \( s_{p} \). After evaluations we obtain, that the decomposition coefficients for a massive fermion with momentum \( p \),
an arbitrary polarization vector \( s_p \) and mass \( m_p \) can be written as scalar products of tetrad and physical vectors

\[
\mathcal{M}_{\rho, \lambda_p} (b_A, p, s_p) = \frac{1}{\sqrt{(b_{-1} \cdot \xi_1^p)}} \left\{ \delta_{\lambda_p, -\rho} \left\{ \delta_{A, -1} \left( \tilde{n}_{-1} \cdot \xi_1^p \right) + \delta_{A, 1} \left( \tilde{n}_{-1} \cdot \xi_1^p \right) \right\} + \delta_{\lambda_p, \rho} \left\{ \frac{m_p}{(p \cdot \tilde{n}_{-1})} \left( \frac{p \cdot \tilde{n}_{-1}}{m_p} \right) + \frac{m_p}{(p \cdot \tilde{n}_{-1})} \right\} \right\},
\]

(48)

where the scalar functions \( Y^{p, q}, X^{p, q} \) are determined by Eqs. (26)-(27).

For BDKS polarization states with the polarization vector (18) the matrix element (48) has a compact form

\[
\mathcal{M}_{\rho, \lambda} (b_A, p, s_{KS}) = \delta_{\lambda_p, -\rho} \left\{ \delta_{A, -1} \sqrt{(p \cdot \tilde{n}_{-1})} + \delta_{A, 1} \frac{(p \cdot \tilde{n}_{-1})}{\sqrt{(p \cdot \tilde{n}_{-1})}} \right\} + \delta_{\lambda_p, \rho} \frac{m_p}{\sqrt{(p \cdot \tilde{n}_{-1})}}.
\]

(49)

The matrix element (43) with the antifermion can be easily obtained with the help of Eq. (17):

\[
\widetilde{\mathcal{M}}_{\rho, \lambda_p} (b_A, [p]) = \bar{u}_p (b_A) u_{\lambda_p} (p, s_p) = \rho \lambda_p \mathcal{M}_{\rho, -\lambda_p} (b_A, [p]).
\]

(50)

### 3.3 Recursion relation

With the help of completeness relation (13) the amplitude (1) with \( Q = Q_2 Q_1 \) is expressed as combinations of the lower-order matrix element

\[
\mathcal{M}_{\lambda_p, \lambda_k} ([p], [k] ; Q_2 Q_1) =
\]

\[
= \sum_{\sigma, A = -1} \mathcal{M}_{\lambda_p, \sigma} ([p] ; b_A ; Q_2) \mathcal{M}_{\lambda_k, \lambda_k} (b_{-A}, [k] ; Q_1).
\]

(51)

This insertion allows us to “cut” fermion chain into pieces of fermion chains with basis spinors \( u_{\lambda} (b_A) \). Hence our formalism enables to calculate the blocks of the Feynman diagrams and then to use them in the calculation as scalar functions. All possible Feynman amplitudes can be built up from a set of “building” blocks.

Let us consider the matrix element (1) with an operator

\[
Z^{(n)} = Q_n Q_{n-1} \cdots Q_1 Q_0
\]

(52)
with $Q_0 = I$. In the Eq. (52) all operators $Q_j$ have an identical mathematical expressions. Using Eq. (51) we find that

$$M_{\lambda_p, \lambda_k} ([p], [k]; Z^{(n)}) \equiv M^{(n)}_{\lambda_p, \lambda_k} ([p], [k]) =$$

$$= \sum_{\sigma, A=1}^1 M_{\lambda_p, \sigma} ([p], b_A) M^{(n)}_{-\sigma, \lambda_k} (b_{-A}, [k]), \quad (53)$$

where matrix element $M^{(n)}_{-\sigma, \lambda_k} (b_{-A}, [k])$ can be calculated with the help of recursion relation

$$M^{(n)}_{-\sigma, \lambda_k} (b_{-A}, [k]) =$$

$$= \sum_{\rho, C=-1}^1 \Gamma^{A,C}_{\sigma, \rho} [Q_n] M^{(n-1)}_{-\rho, \lambda_k} (b_{-C}, [k]). \quad (54)$$

Once scalar functions $M_{\rho, \lambda_k} (b_A, [k])$ and $\Gamma^{A,C}_{\sigma, \rho} [Q_j]$ are known (see Eqs. (46), (48)-(49) and Eqs. (36)-(37)), it is possible to evaluate the higher order of $M^{(n)}_{-\sigma, \lambda_k} (b_{-A}, [k])$ with the help of recursion relation (54).

4 Examples

Consider the “toy” example

$$M^{(n)}_{\lambda_p, \lambda_k} (p, s_p, k, s_k) =$$

$$= \bar{u}_{\lambda_p} (p, s_p) \, g_n \, g_{n-1} \ldots g_1 \, u_{\lambda_k} (k, s_k), \quad (55)$$

where $q_j$ are some arbitrary four-vectors.

Therefore, we have that

$$Z^{(n)} = g_n \, g_{n-1} \ldots g_1. \quad (56)$$

Using the Eqs. (36 - 37), (54) we find that

$$M^{(j)}_{-\rho, \lambda_k} (b_{-C}, [k]) = \left( q_j \cdot \bar{b}_{-C} \right) M^{(j-1)}_{\rho, \lambda_k} (b_C, [k]) -$$

$$- C \left( q_j \cdot \bar{n}_{C\rho} \right) M^{(j-1)}_{\rho, \lambda_k} (b_{-C}, [k]), \quad (57)$$

and

$$M^{(j)}_{-\rho, \lambda_k} (b_{-C}, [k]) = B_{C'}_{j-k} \rho_{j-k} \, [q_j, \ldots q_{j-k}] \, M^{(j-k)}_{-\rho_{j-k}, \lambda_k} (b_{-C'_{j-k}}, [k]) +$$

$$+ C'_{j-k} N_{-C'_{j-k}, \rho_{j-k}} \, [q_j, \ldots q_{j-k}] \, M^{(j-k)}_{-\rho_{j-k}, \lambda_k} (b_{C'_{j-k}}, [k]), \quad (58)$$

where $C'_{j-k} = (-1)^{j-k} C$, $\rho'_{j-k} = (-1)^{j-k} \rho$, $k < j$. 

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With the help of Eq. (53) we can obtain the recursion formulas for calculation matrix element (59):

\[ M_{\lambda_p, \lambda_k}^{(j)} ([p], [k]) = \]

\[ = \sum_{\rho, \bar{C} = -1} M_{\lambda_p, \rho} ([p], b_C) \left( q_j \cdot \bar{b}_{-C} \right) M_{\rho, \lambda_k}^{(j-1)} (b_C, [k]) - \]

\[ - C' \left( q_j \cdot \bar{n}_{C_p} \right) M_{\rho, \lambda_k}^{(j-1)} (b_C, [k]) \]

and

\[ M_{\lambda_p, \lambda_k}^{(j)} ([p], [k]) = \]

\[ = \sum_{\rho, \bar{C} = -1} M_{\lambda_p, \rho} ([p], b_C) \left[ B_{C_{j-k}'} \left( q_j, \ldots q_j, k \right) M_{\rho_{j-k}, \lambda_k}^{(j-k)} \left( b_{C_{j-k}'}, [k] \right) + \right. \]

\[ + C'_{j-k} N_{C_{j-k}'} \left( q_j, \ldots q_j, k \right) M_{\rho_{j-k}, \lambda_k}^{(j-k)} \left( b_{C_{j-k}'}, [k] \right) \] .

We have obtained that the matrix element (55) can be represented as a combination of the scalar functions \( B, N \), decomposition coefficients \( M_{\lambda_p, \rho} ([p], b_C) \) and lower order matrix elements.

### 4.1 Photon emission

Let us consider the matrix element where photon with momentum \( k \) and helicity \( \sigma = \pm 1 \) emission occurs from the incoming electron

\[ M_{\lambda_2, \lambda_1} ([p_2], [p_1]; Q Z_k) = \bar{u}_{\lambda_2} (p_2, s_{p_2}) Q Z_k u_{\lambda_1} (p_1, s_{p_1}) \quad \text{with} \quad (61) \]

\[ Z_k = e \left( \frac{p_1 - k + m}{(p_1 - k)^2 - m^2} \right) . \]

Using the algebra of \( \gamma \)-matrix and Dirac equation the operator \( Z_k \) can be rewritten as

\[ Z_k = -e \left\{ \frac{(p_1 \varepsilon_{\sigma} (k))}{(p_1 k)} - \frac{k \varepsilon_{\sigma} (k)}{2 (p_1 k)} \right\} . \]

(63)

Now we get

\[ M_{\lambda_2, \lambda_1} ([p_2], [p_1]; Q Z_k) = -e \frac{(p_1 \varepsilon_{\sigma} (k))}{(p_1 k)} M_{\lambda_2, \lambda_1} ([p_2], [p_1]; Q) + \]

\[ + \frac{e}{2 (p_1 k)} M_{\lambda_2, \lambda_1} ([p_2], [p_1]; Q \ k \varepsilon_{\sigma} (k)) . \]

(64)

The recursion technique (see Eq. (51)) imply

\[ M_{\lambda_2, \lambda_1} ([p_2], [p_1]; Q \ k \varepsilon_{\sigma} (k)) = \]

\[ = \sum_{\sigma, \bar{C} = -1} M_{\lambda_2, \sigma} ([p_2], b_C; Q) M_{-\sigma, \lambda_1} (b_{-C}, [p_1]; k \varepsilon_{\sigma} (k)) = \]

\[ = \sum_{\sigma, \bar{C} = -1} M_{\lambda_2, \sigma} ([p_2], b_C; Q) \Gamma_{\sigma, \lambda_1}^{C_{\bar{C}} - 1} [k, \varepsilon_{\sigma} (k), T_{\lambda_1} (p_1, s_{p_1})] , \]

(65)
where

\[ T_\lambda (p, s_p) = \frac{\sqrt{p_1^0 + \frac{\theta}{\xi_1^0} - \frac{\theta}{\xi_1^0}} / m_p}{\sqrt{b_{-1} \cdot \xi_1^p}} \]  

(66)

for fermion with arbitrary polarization vector (see Eq. (14)),

\[ T_\lambda (p, s_p) = \frac{p^0 + m_p}{\sqrt{(p \cdot \bar{b}_{-1})}} \]  

(67)

for fermion with BDKS polarization vector (19),

\[ T_\lambda (p, s_p) = \frac{\sqrt{p^0}}{\sqrt{(p \cdot \bar{b}_{-1})}} \]  

(68)

for massless fermion (see Eq. (20)).

Let us consider initial massive fermion with BDKS polarization vector in expression (65). After calculations, for the matrix element (61) with the elements with operator \( Q \) fermion and helicity \( \sigma \) for massless fermion (see Eq. (20)).

\[ M_{\lambda_2, \lambda_1} ([p_2], [p_1]; Q, Z_k) = -e \frac{(p_1 \varepsilon_{\sigma(k)})}{(p_1^0)} M_{\lambda_2, \lambda_1} ([p_2], [p_1]; Q) + \]

\[ + \frac{e}{2 (p_1 \cdot k)} \sqrt{(p_1 \cdot \bar{b}_{-1})} \left[ M_{\lambda_2, \lambda_1} ([p_2], b_{-1}; Q) N_{-1,-\lambda_1}^{k, \varepsilon_{\sigma(k)}, p_1} + \right. \]

\[ + M_{\lambda_2, \lambda_1} ([p_2], b_1; Q) B_{-1,-\lambda_1}^{k, \varepsilon_{\sigma(k)}, p_1} + m_{p_1} \left\{ M_{\lambda_2, -\lambda_1} ([p_2], b_{-1}; Q) Y_{-1,-\lambda_1}^{k, \varepsilon_{\sigma(k)}} + M_{\lambda_2, -\lambda_1} ([p_2], b_1; Q) X_{-1,-\lambda_1}^{k, \varepsilon_{\sigma(k)}} \right\} \]  

(69)

Scalar functions \( B_{A, \lambda}^{k, \varepsilon_{\sigma(k)}, p_1}, N_{A, \lambda}^{k, \varepsilon_{\sigma(k)}, p_1} \) are determined in terms of scalar product with the help of the Eqs. (14)-(15) and scalar functions \( X_{-1,-\lambda_1}^{k, \varepsilon_{\sigma(k)}}, Y_{-1,-\lambda_1}^{k, \varepsilon_{\sigma(k)}} \) are determined by Eqs. (26)-(27).

Scalar functions can be easily calculated in terms of physical vector components. Using a vector of polarization of a photon as Eq. (8) we obtain simple result:

\[ B_{-1,-\lambda_1}^{k, \varepsilon_{\sigma(k)}, p} = \sqrt{2} \sigma (p^- k_{T, \lambda_1} [\delta_{\lambda_1, \sigma} - 1] + k^- p_{T, -\sigma}) \],

\[ N_{-1,-\lambda_1}^{k, \varepsilon_{\sigma(k)}, p} = \frac{\sqrt{2}}{k^-} \left[p^- k_{T, -\lambda_1} k_{T, -\sigma} + k^- \left(\delta_{\lambda_1, -\sigma} k^+ p^- - \delta_{\lambda_1, \sigma} p_{T, -\sigma} k_{T, -\sigma}\right)\right] \],

\[ Y_{-1,-\lambda_1}^{k, \varepsilon_{\sigma(k)}} = -\sqrt{2} \lambda_1 \delta_{\lambda_1, -\sigma} k_{T, -\sigma}, \quad X_{-1,-\lambda_1}^{k, \varepsilon_{\sigma(k)}} = -\sqrt{2} \delta_{\lambda_1, -\sigma} k^- \],

(70)

where

\[ p^\pm = p^0 \pm p^x, \quad p_{T, \lambda} = p^x + i \lambda p^y \].

(71)
4.2 The process $e^+ e^- \rightarrow n\gamma$

Consider the process

$$e^+ (p_2, \sigma_2) + e^- (p_1, \sigma_1) \rightarrow \gamma (k_1, \lambda_1) + \gamma (k_2, \lambda_2) + \cdots + \gamma (k_n, \lambda_n),$$

(72)

where the momenta of the particles and spin numbers are given between parentheses.

The Feynman diagrams of the processes [72] contain the matrix element

$$M^{(n)}_{\sigma_2, \sigma_1} ([p_2], [p_1]) = \sum_{\rho, \sigma} M^{(n)}_{\sigma_2, \rho} ([p_2], b_C; \lambda_n, (k_n)) M^{-\rho, \sigma_1} (b_C, [p_1]; Z^{(n-1)}).$$

(73)

where

$$Q_j = p_1 - \sum_{i=1}^j k_i.$$  

(74)

Hence, we have that

$$Z^{(n)} = \frac{Q_n + m}{Q_n^2 - m^2} \tilde{\lambda}_n (k_n) \cdots \frac{Q_2 + m}{Q_2^2 - m^2} \tilde{\lambda}_2 (k_2)$$

and

$$M^{(n)}_{\sigma_2, \sigma_1} ([p_2], [p_1])$$

$$= \sum_{\rho, \sigma = -1}^1 \tilde{M}_{\sigma_2, \rho} ([p_2], b_C; \lambda_n, (k_n)) M^{-\rho, \sigma_1} (b_C, [p_1]; Z^{(n-1)}).$$

(76)

Here

$$\tilde{M}_{\sigma_2, \rho} ([p_2], b_C; \lambda_n, (k_n)) = \tilde{v}_{\sigma_2} (p_2, s_{p_2}) \mathcal{M}_{\lambda_n} (u_\rho (b_C))$$

(77)

and

$$M^{(n-1)}_{\rho, \sigma_1} (b_C, [p_1]) = M^{-\rho, \sigma_1} (b_C, [p_1]; Z^{(n-1)}).$$

(78)

Using the expressions [37] and [8] we obtain, that [77] for BDKS massive Dirac spinor [19] is determined by

$$\tilde{M}_{\sigma_2, \rho} ([p_2], b_C; \lambda_n, (k_n)) =$$

$$= 1/ \sqrt{(p_2 \cdot b_{-1})} \left[ \delta_{\sigma_2, \rho} (\delta_{\sigma_1, 1} X_{\sigma_2, \sigma_2}^{p_2, \varepsilon_n} + \delta_{\sigma_1, 1} X_{\sigma_2, \sigma_2}^{p_2, \varepsilon_n})$$

$$- m \delta_{\sigma_2, -\rho} \left( \delta_{\sigma_1, 1} (\bar{b}_{-1} \cdot \varepsilon_n) - \delta_{\sigma_1, 1} (\bar{b}_{-1} \cdot \varepsilon_n) \right) \right]$$

(79)

with $\varepsilon_n = \varepsilon_{\lambda_n} (k_n)$. 

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The final recursion relation of the process (72) with arbitrary helicities of the photons and BDKS polarization states of positron is written as

\[
M_{\sigma_2,\sigma_1}^{(n)} ([p_2], [p_1]) = \frac{1}{\sqrt{(p_2 \cdot b_{-1})}} \left[ X_{-1,\sigma_2}^{p_2, \varepsilon_\mu} M_{\sigma_2,\sigma_1}^{(n-1)} (b_1, [p_1]) + Y_{-1,\sigma_2}^{p_2, \varepsilon_\mu} M_{\sigma_2,\sigma_1}^{(n-1)} (b_{-1}, [p_1]) + m \left( (\tilde{\alpha}_{\sigma_2} \cdot \varepsilon_n) M_{\sigma_2,\sigma_1}^{(n-1)} (b_{-1}, [p_1]) + \left( \tilde{\beta}_{-1} \cdot \varepsilon_n \right) M_{\sigma_2,\sigma_1}^{(n-1)} (b_1, [p_1]) \right) \right],
\]

where (see Eq.(59) and Eq.(60)) the matrix element \( M_{-\rho,\sigma}^{(j)} (b_{-C}, [p_1]) \) with arbitrary polarization vector of electron is calculated by means of the recursion formula

\[
M_{-\rho,\sigma}^{(j)} (b_{-C}, [p_1]) = \frac{1}{Q_j - m^2} \left\{ m \left[ (\varepsilon_j \cdot b_{-C}) M_{-\rho,\sigma}^{(j-1)} (b_{-C}, [p_1]) - C \left( \varepsilon_j \cdot n_{C\rho} \right) M_{-\rho,\sigma}^{(j-1)} (b_C, [k]) \right] + Y_{C,\rho} Q_j^{\varepsilon_j} M_{\rho,\sigma}^{(j-1)} (b_C, [p_1]) + C X_{-C,\rho}^{Q_j, \varepsilon_j} M_{\rho,\sigma}^{(j-1)} (b_{-C}, [p_1]) \right\}.
\]

5 Summary and Acknowledgements

In present paper we have formulated a new effective method to calculate the Feynman amplitudes for various processes with fermions of arbitrary polarizations. In our method it is much easier to keep track of partial results and to set up recursive schemes of evaluation which compute and store for later use subdiagrams of increasing size and complexity.

In our approach of the matrix element calculation:

1 we don’t use an explicit form of Dirac spinors and \( \gamma \)-matrices
2 we don’t use the calculation of traces
3 as well as in the trace methods the matrix element of Feynman amplitudes is reduced to the combination of scalar products of momenta and polarization vectors.
4 Unlike spinor technique in different variants \[14\]-\[17\] in this method we don’t use either Chisholm identities, or the presentation of the contraction \( \hat{\rho} \) with four vector \( \rho \) and of the polarization vector of bosons through the Dirac spinors.
5 Unlike WvD technique \[18\],[21] in this method we don’t use special Feynman rules for calculating of the matrix elements.
6 Expression for matrix element calculates \( M_{\lambda_p,\lambda_k} (p, s_p, k, s_k ; Q) \) \[\square\] for all values \( \lambda_p, \lambda_k \) simultaneously.

The recursive algorithms can be easily realized in the various systems of symbolic calculation (Mathematica, Maple, Reduce, Form) and in such packages as FeynArts \[31\], FeynCalc \[44\], HIP \[45\] and so on.

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