Stability of partially locked states in the Kuramoto model through Landau damping with Sobolev regularity

Helge Dietert

July 13, 2017

The Kuramoto model is a mean-field model for the synchronisation behaviour of oscillators, which exhibits Landau damping. In a recent work, the nonlinear stability of a class of spatially inhomogeneous stationary states was shown under the assumption of analytic regularity. This paper proves the nonlinear Landau damping under the assumption of Sobolev regularity. The weaker regularity required the construction of a different more robust bootstrap argument, which focuses on the nonlinear Volterra equation of the order parameter.

1 Introduction

The Kuramoto model [10, 11] is a mean-field model for the interaction of oscillators, which shows the Landau damping behaviour [1, 3, 4, 5, 7, 20]. On the particle level the model consists of oscillators $i = 1, \ldots, N$, which are modelled by their position $\theta_i$, the phase angle on the torus $\mathbb{T} = \mathbb{R} / (2\pi \mathbb{Z})$, and their velocity $\omega_i \in \mathbb{R}$, the intrinsic frequency. The evolution is determined by the system of ODEs

$$\begin{cases}
\frac{d}{dt} \theta_i = \omega_i + \frac{K}{N} \sum_{j=1}^{N} \sin(\theta_j - \theta_i), \\
\frac{d}{dt} \omega_i = 0
\end{cases}$$

for $i = 1, \ldots, N$, where $K$ is the coupling constant. The intuition is that each oscillator $i$ evolves according to its own intrinsic frequency $\omega_i$ and according to a global coupling, which tries to synchronise the oscillators.

With the order parameter

$$r = \frac{1}{N} \sum_{j=1}^{N} e^{i \theta_j},$$

the coupling can be written in the mean-field form

$$\frac{d}{dt} \theta_i = \omega_i + \frac{K}{2i} (r e^{-i \theta_i} - r e^{i \theta_i}).$$

Moreover, the order parameter is used to measure the synchronisation of oscillators.
For the study of a large number of oscillators \((N \to \infty)\), the mean-field limit can be used. In the mean-field limit the system is described by a measure \(f(t, \theta, \omega) \, d\theta \, d\omega\) for the distribution of oscillators and the evolution is described by the PDE
\[
\begin{cases}
\partial_t f(t, \theta, \omega) + \partial_\theta \left( \left( \omega + \frac{K}{2i} \left( r(t) e^{-i\theta} - \overline{r(t)} e^{i\theta} \right) \right) f(t, \theta, \omega) \right) = 0, \\
\end{cases}
\]
\[
r(t) = \int_{\mathbb{R}} \int_{\mathbb{T}} e^{i\theta} f(t, \theta, \omega) \, d\theta \, d\omega.
\]
Due to the Lipschitz regular interaction, this limit can be justified in the mean-field limit framework by Braun and Hepp [2], Dobrushin [6], and Neunzert [14], see [12, 16, 19]. In particular, this shows the well-posedness of the PDE (1).

The evolution is invariant under the rotation symmetry \(R_\Theta\) changing \(\theta \to \theta + \Theta\), i.e.
\[\left( R_\Theta f \right)(\theta, \omega) = f(\theta + \Theta, \omega).\]

The velocity distribution
\[g(\omega) = \int_{\mathbb{T}} f(\theta, \omega) \, d\theta\]
is constant over time. Fixing the velocity distribution \(g\), we look for stationary solutions with a fixed order parameter \(r\). By the rotation symmetry \(R_\Theta\), we can assume that the order parameter is \(r_{\text{st}} \in [0, 1]\) and find the stationary state
\[
f_{\text{st}}(\theta, \omega) = \begin{cases} 
\delta_{\arcsin(\omega/(Kr_{\text{st}}))}(\theta) \, g(\omega) & \text{if } |\omega| \leq Kr_{\text{st}} \\
\frac{\sqrt{\omega^2 - (Kr_{\text{st}})^2}}{2\pi |\omega - Kr_{\text{st}} \sin \theta|} \, g(\omega) & \text{if } |\omega| > Kr_{\text{st}} 
\end{cases}
\]
if the self-consistency equation
\[r_{\text{st}} = \int_{\mathbb{R}} \int_{\mathbb{T}} e^{i\theta} f_{\text{st}}(\theta, \omega) \, d\theta \, d\omega\]
is satisfied [13, 15, 18]. The existence of such states can be assured [5] and we choose the fixed-point \(\arcsin(\omega/(Kr_{\text{st}}))\) \(|\arcsin(\omega/(Kr_{\text{st}}))| < \pi/2\) for the locked oscillators \(|\omega| < Kr_{\text{st}}\) as the other fixed-point \(\pi - \arcsin(\omega/(Kr_{\text{st}}))\) is unstable.

Factoring out the rotation symmetry, the author identified with Fernandez and Gérard-Varet in [5] the linear stability criterion and showed nonlinear stability, under the assumption of analytic regularity in \(\omega\).

In this work, we extend the analysis to the case of Sobolev regularity of the stationary state and the perturbation. In the analytic setting, the linear evolution was regularising enough to control the nonlinearity as a forcing bounded by its norm. In the case of Sobolev regularity, this is no longer possible and we need to devise a bootstrap argument taking into account the structure of the nonlinearity.

We control the evolution by considering the effect of the perturbation on the stationary state as forcing. As the perturbation acts on the stationary state through the order parameter, this implies that the order parameter satisfies a Volterra integral equation. On the remaining homogeneous problem, we can formulate energy estimates, which quantify the damping. This then allows us to study the nonlinear behaviour on the level of the Volterra equation, where no regularity issues remain. The obtained control of the order parameter can then be injected in the full behaviour, which allows a bootstrap argument controlling the evolution.

The rotation symmetry is handled by taking out a possible rotation of the stationary state. Controlling this projection determines in our proof the minimal needed regularity and the rotation symmetry limits the obtained decay rate of the order parameter.
2 Overview

The aim of the work is the stability study of a stationary state \( f_{\text{st}} \) of the form given in (2) and as noted we may choose the order parameter \( r_{\text{st}} \in [0, 1] \) by the rotation symmetry. Throughout this work, we assume the existence of the stationary state and keep it fixed.

We understand the stability in Fourier variables. For the Fourier transform \( \hat{f} \) of \( f \), which takes \( \theta \) to \( \ell \) and \( \omega \) to \( \xi \), we use the convention

\[
(\hat{f})_\ell(\xi) = \int_R \int_T e^{-i\ell\theta - i\xi \omega} f(\theta, \omega) \, d\theta \, d\omega
\]

and accordingly

\[
\hat{g}(\xi) = \int_R e^{-i\xi \omega} f(\theta, \omega) \, d\omega.
\]

The evolution PDE (1) then becomes

\[
\begin{align*}
\partial_t \hat{f}_{\ell}(t, \xi) &= \ell \partial_\xi \hat{f}_{\ell}(t, \xi) + \frac{K\ell}{2} \left( r_{st}(\hat{f}_{\ell-1}(t, \xi) - r_{st}(\hat{f}_{\ell+1}(t, \xi)) \right), \\
\frac{r(t)}{r(0)} &= \hat{f}_1(0).
\end{align*}
\]

The stability of the stationary state \( f_{\text{st}} \) is studied by considering a solution \( f = f_{\text{st}} + f_{\text{pt}} \) where \( f_{\text{pt}} \) is the perturbation. From (3) the evolution is given by

\[
\partial_t \hat{f}_{\ell}(t, \xi) = L \hat{f}_{\ell}(t, \xi) + Q(\hat{f}_{\ell}(t, \xi))
\]

where \( L = L_1 + L_2 \) with

\[
\begin{align*}
(L_1 \hat{f}_{\ell})(t, \xi) &= \ell \partial_\xi (\hat{f}_{\ell}(t, \xi)) + \frac{K\ell}{2} \left( r_{st}(\hat{f}_{\ell-1}(t, \xi) - r_{st}(\hat{f}_{\ell+1}(t, \xi)) \right), \\
(L_2 \hat{f}_{\ell})(t, \xi) &= \frac{K\ell}{2} \left( (\hat{f}_{\ell+1}(0)) - (\hat{f}_{\ell-1}(0)) \right), \\
(Q(\hat{f}_{\ell}))(t, \xi) &= \frac{K\ell}{2} \left( (\hat{f}_{\ell+1}(0)) - (\hat{f}_{\ell-1}(0)) \right).
\end{align*}
\]

In Fourier space the rotation symmetry \( R_\Theta \) acts as

\[
(R_\Theta \hat{f})(\xi) = e^{i\Theta} \hat{f}(\xi).
\]

The symmetry means that for any \( \Theta \) and solution \( f \) also \( R_\Theta f \) is a solution. In particular, \( R_\Theta f_{\text{st}} \) is also a stationary solution with the same behaviour. Therefore, along the rotation symmetry, a perturbation does not decay and we need to study orbital stability, i.e. if a solution \( f \) converges to the set \( \{R_\Theta f_{\text{st}}\}_{\Theta \in \mathbb{T}} \).

In order to separate the rotation behaviour, we introduce polar type coordinates for states close to the circle \( \{R_\Theta f_{\text{st}}\}_{\Theta \in \mathbb{T}} \). In these coordinates, the solution is written as

\[
\hat{f}(t) = \hat{R}_{\Theta(t)}(\hat{f}_{\text{st}} + \hat{u}(t)),
\]

where \( \Theta(t) \) is a suitable chosen angle and \( u \) is the remaining perturbation. The time evolution of \( u \) is then given by

\[
\partial_t u = Lu + Q(u) - \frac{d\Theta}{dt} \left( D\hat{R} f_{\text{st}} + D\hat{R} u \right),
\]

where \( D\hat{R} \) denotes the differential of \( \Phi \mapsto \hat{R}_\Phi \) at \( \Phi = 0 \). We then obtain the stability, if we can show that the remaining perturbation \( u \) is damped. For this, we want to project \( \hat{f}_{\text{pt}} \) onto the circle \( \{R_\Theta f_{\text{st}}\}_{\Theta \in \mathbb{T}} \) such that the remaining difference \( u \) is in the stable subspace of \( L \).
From the linear stability theory, we characterise the projection to the rotation eigenmode by a linear functional $\alpha$. The kernel of $\alpha$ is then the stable subspace under $L$ and a suitable condition for the choice of $\Theta(t)$ is therefore the condition

$$\alpha(u) = 0.$$  

As long as the perturbation is small enough, this condition is propagated if

$$\frac{d\Theta}{dt} = \dot{\Theta},$$  

where

$$\dot{\Theta} := \frac{\alpha(Qu)}{\alpha(D\hat{R}_f) + \alpha(D\hat{R}u)}.$$  

Here $\dot{\Theta}$ is a function of $u$. Hence we find a closed equation for the evolution of $u$. Explicitly, it takes the form

$$\partial_t u = B^q u + L^2 u$$

where

$$B^q = B^q_1 + B^q_2,$$

$$B^q_1 = L_1 + B^q_{1n},$$

$$(B^q_{1n}u)_\ell = \frac{K_\ell}{2} (u_{\ell-1} - u_{\ell+1}) - \dot{\Theta} (\hat{D}\hat{R}u)_\ell,$$

$$(B^q_2 u)_\ell = -\Theta (D\hat{R}f_{st}).$$

By (4), we find that

$$(D\hat{R}u)_\ell = i\ell u_\ell.$$  

Therefore, $B^q_{1n}$ has the same divergence structure as $L_1$. By an energy estimate, we therefore have a quantified damping under the evolution of $B^q_1$.

The operator $B^q_2$ is like $L_2$ a finite-rank operator whose image is derived from the stationary state. It is the effect of $\dot{\Theta}$ on the stationary state, which ensures that $u$ stays in the kernel of $\alpha$. Explicitly,

$$B^q_2 u = -\alpha (B^q_{1n} u) r_{\Theta},$$

where

$$r_{\Theta} := \frac{D\hat{R}f_{st}}{\alpha(D\hat{R}f_{st})}.$$  

For the remaining perturbation $u$ we let $\eta(t) = u_1(t, 0)$. Considering $\eta(t)$ and $\dot{\Theta}(t)$ from (5) as known coefficients, we can define the corresponding linear operators $B_1, B_2$ and $B$ with time-varying coefficients by

$$(B_1 u)_\ell = \frac{K_\ell}{2} (\eta_{\ell-1} - \eta_{\ell+1}) - \dot{\Theta} i\ell u_\ell,$$

$$(B_2 u)_\ell = -\alpha (B_1 u) r_{\Theta},$$

$$Bu = B_1 u + B_2 u.$$  

Over a time range, where the coefficients are continuous, the evolution under $B$ has a unique weak solution and we let $S^B_0 \to t$ be the corresponding solution operator from time $s$ to time $t$. The solution $u$ can then be expressed by Duhamel’s principle as

$$u(t) = S^B_0 \to t u_0 + \int_0^t S^B_\tau \to t L^2 u(s) \, ds$$  

(6)
with the initial data $u_{in}$ at time $t = 0$.

The rotation symmetry is a one-dimensional real symmetry. Hence the corresponding functional $\alpha$ maps into $\mathbb{R}$ and is only linear over $\mathbb{R}$, but not over $\mathbb{C}$. This implies that $B_2$ and $B$ are only real linear. We therefore rewrite $L_2$ as

$$L_2 u = r_r \Re\eta(t) + r_i \Im\eta(t)$$

with

$$(r_r)_{\ell}(\xi) = \frac{K \ell}{2} (\hat{f}_{a\ell-1}(\xi) - (\hat{f}_{a\ell+1})(\xi)) \quad \text{and} \quad (r_i)_{\ell}(\xi) = \frac{K \ell}{2} (\hat{f}_{a\ell-1}(\xi) + (\hat{f}_{a\ell+1})(\xi)).$$

Then the linearity over $\mathbb{R}$ implies that

$$S_{s=t}^B(L_2u(s)) = (S_{s=t}^B, (\Re\eta(t)) + (S_{s=t}^B, (\Im\eta(t)).$$

Computing $\eta(t) = u_1(t, 0)$ over (6), we find the Volterra equation

$$\left( \begin{array}{c} \Re\eta(t) \\ \Im\eta(t) \end{array} \right) = F(t) \left( \begin{array}{c} k \star (\Re\eta(t)) \\ k \star (\Im\eta(t)) \end{array} \right)$$

with the Volterra kernel

$$k(t, s) = -\left( \begin{array}{c} \Re (S_{s=t}^B, (\Re\eta_1)(0)) \\ \Re (S_{s=t}^B, (\Im\eta_1)(0)) \end{array} \right) \left( \begin{array}{c} \Re (S_{s=t}^B, (\Re\eta_1)(0)) \\ \Re (S_{s=t}^B, (\Im\eta_1)(0)) \end{array} \right)$$

using the product notation

$$k \star (\Re\eta(t)) = \int_0^t k(t, s) (\Re\eta(s)) ds.$$

and the forcing

$$F(t) = \left( \begin{array}{c} \Re F(t) \\ \Im F(t) \end{array} \right) \quad \text{with} \quad F(t) = (S_{s=t}^B, u_{in})(0).$$

When we treat $\Re\eta$ and $\Im\eta$ as independent complex variables, Eq. (7) is complex linear and its linearised behaviour can be understood by a spectral analysis. In [5], an equivalent complexification is done on the linearised evolution of $u$ and the operators $L_1$ and $L_2$. In the spectral analysis, we find the same stability condition. As in [5], this does not create any spurious eigenmodes.

In contrast to [5], we perform the linear stability analysis on the level of the Volterra equation, where we can handle the nonlinearities. Under the linearised evolution the kernel $k_L$ takes the form

$$k_L(t, s) = k_{Lc}(t - s) := -\left( \begin{array}{c} \Re (e^{(t-s)Lc}, Lc_{T})(0) \\ \Re (e^{(t-s)Lc}, Lc_{T})(0) \end{array} \right) \left( \begin{array}{c} \Re (e^{(t-s)Lc}, Lc_{T})(0) \\ \Re (e^{(t-s)Lc}, Lc_{T})(0) \end{array} \right).$$

The Volterra equation then takes the convolution form

$$\left( \begin{array}{c} \Re\eta(t) \\ \Im\eta(t) \end{array} \right) = F_L(t),$$

where the product simplifies to the convolution with $k_{Lc}$ as

$$k \star (\Re\eta(t)) = \int_0^t k_{Lc}(t - s) (\Re\eta(s)) ds := k_{Lc} \star (\Re\eta(t)).$$
and the linear forcing becomes

\[ F_L(t) = \left( \Re F_L(t) \right) \quad \text{with} \quad F_L(t) = (e^{tL_1}u_{in})_1(0). \]

In order to quantify the decay we introduce the submultiplicative weight function \( p_{A,b}(t) = (A + t)^b \) with \( A \geq 1 \) and \( b \geq 0 \) by

\[ p_{A,b}(t) = (A + t)^b \quad \text{with} \quad p_b = p_{1,b} \]

and the weighted norms for a function \( h \) of time in \( \mathbb{R}^+ \) by

\[
\| h \|_{L^1(J,\phi)} = \int_{t \in J} \| h(t) \| \phi(t) \, dt, \\
\| h \|_{L^\infty(J,\phi)} = \text{ess sup}_{t \in J} \| h(t) \| \phi(t),
\]

where \( J = [0,T] \) or \( J = \mathbb{R}^+ \) is the considered time range.

We solve the Volterra equation by introducing the resolvent \( r_{Lc} \) satisfying

\[ r_{Lc} + k_{Lc} * r_{Lc} = r_{Lc} + r_{Lc} * k_{Lc} = k_{Lc}, \]

which has a unique solution. The elements of the kernel \( k_{Lc} \) come from propagating \( r_r \) and \( r_i \) by \( L_1 \). As the operator \( L_1 \) is damping regular states, the kernel \( k_{Lc} \) is decaying, if \( f_{st} \) is regular enough. Taking the Laplace transformation, we can formulate a precise stability condition imposing that the rotation symmetry is the only non-decaying eigenmode. In this case the resolvent takes the form

\[ r_{Lc} = K_{\Theta} + r_{Lcs}, \]

where \( K_{\Theta} \) is a constant matrix corresponding to the rotation eigenmode and \( r_{Lcs} \) is the decaying remainder. The decay is quantified by

\[ \| r_{Lcs} \|_{L^1(\mathbb{R}^+,p_b)} < \infty \]

for the parameter \( b \geq 0 \).

Looking at rotation eigenmode, we find that \( K_{\Theta} \) takes the form

\[ K_{\Theta} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \otimes \left( c_{\Theta,r} \ c_{\Theta,i} \right). \quad (9) \]

Hence the order parameter is decaying if and only if the forcing \( F \) satisfies

\[ K_{\Theta} \int_0^\infty F(t) \, dt = 0 \quad \text{or} \quad \int_0^\infty \left( c_{\Theta,r} \Re F(t) + c_{\Theta,i} \Im F(t) \right) \, dt = 0. \]

Therefore, the appropriate functional for the stable part is

\[ \alpha(u) = \int_0^\infty \left( c_{\Theta,r} \Re (e^{tL_1}u)_{1,0} + c_{\Theta,i} \Im (e^{tL_1}u)_{1,0} \right) \, dt. \quad (10) \]

For a forcing of the form \( F(t) = (e^{tL_1}u_{in})_1(0) \) with \( \alpha(u_{in}) = 0 \), the order parameter is controlled under the linear evolution as

\[ |\eta(t)| \leq |\alpha(e^{tL_1}u_{in})| + (1 + t)^{-b} \| r_{Lcs} \|_{L^1(\mathbb{R}^+,p_b)} \| F \|_{L^\infty([0,t],p_b)}. \]
As $L_1$ induces the decay for a regular initial datum $u_{in}$, this shows under the linearised evolution that
\[ |\eta(t)| \leq (1 + t)^{-b} \|u_{in}\| \]
for a suitable norm of $u_{in}$.

In order to control the nonlinearity, we introduce the bootstrap control
\[ R_d(t) = \sup_{s \in [0,t]} (1 + t)^{b_d} M_d(s) \quad \text{with} \quad M_d(t) = K|\eta(t)| + |\dot{\Theta}(t)| \]
for a decay rate $b_d \geq 0$.

In the kernel of the Volterra equation, we propagate initial data derived from the stationary state. Assuming enough regularity on the stationary state, we can still control the appropriate nonconvolution resolvent of the nonlinear evolution in a similar fashion. Also the bootstrap assumption $R_d$ allows us to control the forcing $(\mathcal{S}_{\mathbb{R}^+} u_{in})_1(0)$ so that we can conclude in the nonlinear case that
\[ |\eta(t)| \leq 2C(1 + t)^{-b} \|u_{in}\| \]
if $R_d(t)$ is sufficiently small.

Knowing the decay of $\eta(t)$, we can go back to (6) and estimate the behaviour of $u$. As the remaining growing terms are quadratic, we can close the bootstrap argument for small enough initial data.

## 3 Results

We now give the precise result and describe the main steps.

In order to find suitable norms, note that the velocity distribution, i.e. the spatial mode $\ell = 0$, is constant. We therefore assume that the perturbation $f_{pt}$ does not change the velocity distribution so that $(\hat{f}_{pt})_0 \equiv 0$ and so $u_0 \equiv 0$. Moreover, the evolution (3) only couples neighbouring modes, so that the positive modes $\ell \geq 1$ are separated from the negative modes $\ell \leq 1$ by the constant mode $\ell = 0$. We therefore restrict our attention to $\ell \geq 1$. In this restriction the transport operator in (3) has the same sign for all modes $\ell \geq 1$ and the nonlinear coupling is controlled by the special value $(\hat{f}_{1})_1(0)$. Hence the region $\ell \geq 1$ and $\xi \geq 0$ is its own domain of dependency and we can further restrict the attention to this domain.

Similar to the used norms in [5], we want to use a Hilbert space norm in order to take advantage of the divergence structure. Moreover, we need a pointwise control for the order parameter in the nonlinearity. Hence we introduce the weighted Sobolev spaces
\[ \mathcal{X}_{\phi,k} = \{ u : N \times \mathbb{R} \rightarrow \mathbb{C} \text{ with } \|u\|_{\phi,k} < \infty \} \]
with the norm
\[ \|u\|_{\phi,k} = \sum_{\ell \geq 1} \int_0^\infty \left( |u_\ell(\xi)|^2 + |\partial_\xi u_\ell(\xi)|^2 \right) |\phi(\xi)|^2 \ell^2k \, d\xi \]
for the weight $\phi$ and degree $k$ and use the shorthands
\[ \mathcal{X}_\phi = \mathcal{X}_{\phi,-\frac{1}{2}} \quad \text{and} \quad \|u\|_\phi = \|u\|_{\phi,-\frac{1}{2}}. \]

For the Fourier transform of the velocity profile, which has no spatial modes, we introduce accordingly
\[ \|\hat{g}\|_\phi = \int_0^\infty \left( |\hat{g}(\xi)|^2 + |\partial_\xi \hat{g}(\xi)|^2 \right) |\phi(\xi)|^2 \, d\xi. \]

The norms are well-adapted to the stationary states $\hat{f}_{st}$, as defined in (2).
Proposition 1. Let $b \geq 0$ and $k = 0, 1/2, 1, 3/2, \ldots$. Then there exists a constant $C_{\text{st}}$ only depending on $k$, $b$ and $K_{\text{st}}$ such that the stationary state satisfies
\[
\|\hat{f}_{\text{st}}\|_{p_0,k} \leq C_{\text{st}}\|\hat{g}\|_{p_{0+k+1}},
\]
where $\hat{f}_{\text{st}}$ is restricted to $\ell \geq 1$ and $\xi \geq 0$.

In particular, $C_{\text{st}}$ can be chosen such that
\[
\|r_{\phi}\|_{p_0,k} \leq C_{\text{st}}\|\hat{g}\|_{p_{0+k+2}} \quad \text{and} \quad \|r_{\psi}\|_{p_0,k} \leq C_{\text{st}}\|\hat{g}\|_{p_{0+k+2}} \quad \text{and} \quad \|r_{\eta}\|_{p_0,k} \leq C_{\text{st}}\|\hat{g}\|_{p_{0+k+2}}.
\]

A crucial ingredient for the control is that we have chosen our stationary state $\hat{f}_{\text{st}}$ in (2) such that all locked oscillators are at the stable fixed-point, cf. [5].

The linear stability is then determined by the linear stability of the Volterra equation and we will later find a precise stability condition in Definition 8. Postponing the stability condition to Definition 8, we can state our main result.

Theorem 2. Let $b > 3/2$ and $b_{\text{r}} > b + 3/2$. Let $\hat{f}_{\text{st}}$ be a stationary state which is linearly stable in the sense of Definition 8 and is regular enough such that
\[
\|r_{i}\|_{p_0} < \infty, \quad \|r_{\psi,0}\|_{p_0} < \infty, \quad \|r_{\eta}\|_{p_0} < \infty,
\]
and
\[
\|r_{\eta}\|_{p_{0} - \frac{3}{2},0} < \infty, \quad \|r_{i}\|_{p_{0} - \frac{3}{2},0} < \infty, \quad \|r_{\psi,0}\|_{p_{0} - \frac{3}{2},0} < \infty.
\]
Furthermore, assume that one of the following conditions holds:

- $b > 3$,
- $b_{\text{r}} > b + 7/2 + \max\{0, 3/2 - b\}$.

Then there exist constants $C$ and $\delta$ such that for initial data $\hat{f}_{\text{in}}$ with the same velocity distribution $\hat{g} = (\hat{f}_{\text{in}})_0 = (\hat{f}_{\text{st}})_0$ and
\[
\|\hat{f}_{\text{in}} - \hat{f}_{\text{st}}\|_{p_0} \leq \delta,
\]
there exists a unique global weak solution $\hat{f}$ of (3) and $\Theta : \mathbb{R}^+ \rightarrow \mathbb{R}$ with
\[
\frac{d}{dt}\Theta(t) \leq C (1 + t)^{2 - b}\|\hat{f}_{\text{in}} - \hat{f}_{\text{st}}\|_{p_0}^2
\]
such that the order parameter $\pi(t) = u_1(t,0)$ of the remaining perturbation
\[
u = R_{-\Theta(t)}\hat{f} - \hat{f}_{\text{st}}
\]
is controlled for all times $t$ by
\[
|\eta(t)| \leq C (1 + t)^{b - 1}||u_{\text{in}}||_{p_0}.
\]

Measuring the decay through the order parameter $\eta$ of the remaining perturbation $u$, this shows the decay of a small initial perturbation. Furthermore, the bound on $\Theta$ shows that the system will converge to a nearby partially locked state, because $\Theta(t)$ converges to some $\Theta_\infty$ and $|\Theta_\infty - \Theta(0)|$ is controlled by $||u_{\text{in}}||_{p_0}$.

The minimal needed regularity for the perturbation comes from the requirement to control $\alpha(Q(u))$ and $\alpha(B_{1,u}u)$, where we use Lemma 5, which imposes $||u||_{p_0} < \infty$ for $b > 3/2$. This
control is crucially needed to make the projection to handle the rotation symmetry and appears in the estimates for the control of $B_2$ and $\dot{\Theta}$.

The achieved decay $(1 + t)^{\frac{1}{2} - b}$ is also limited by the rotation eigenmode. Under the linearised evolution, the rotation eigenmode creates a contribution $K_{\Theta} \int_0^t F(s) \, ds$ at time $t$ and this is controlled by $\beta_\alpha(e^{tL_1}u_m)$. This is bounded in Lemma 5, which gives the achieved decay. We suspect that it cannot be improved as the estimate is sharp in the limiting case $r_{st}$, where the evolution under $L_1$ can be solved explicitly.

The plan of this paper is to first study the evolution operator, for which we can take $K_\Theta$ as given matrix with finite real coefficients and use the corresponding definition of $\alpha$ in (10). Afterwards, we study the Volterra equation, where we find from the linearised evolution the explicit form $K_\Theta$, which only depends on the stationary state. Finally, we conclude the result by a local well-posedness result and a bootstrap argument.

For the evolution operators, we always assume that, for the considered time range, the coefficients of the operator $B_1n$ are continuous. Moreover, we define all the operators over the restriction $\ell \geq 1$, where they explicitly take the form

\[
(L_1 u)_1 = \partial_\xi u_1 - \frac{K}{2} r_{st} u_2,
\]
\[
(B_1n u)_1 = -\frac{K}{2} \eta(t) u_2 - \dot{\Theta}(t) i u_1
\]

and for $\ell \geq 2$

\[
(L_1 u)_\ell = \ell \partial_\xi u_\ell + \frac{K \ell}{2} (r_{st} u_{\ell-1} - r_{st} u_{\ell+1}),
\]
\[
(B_1n u)_\ell = \frac{K \ell}{2} \left( \frac{\eta(t) u_{\ell-1} - \eta(t) u_{\ell+1}}{\ell} \right) - \dot{\Theta}(t) i u_\ell.
\]

For notational convenience we also use sometimes the convention $u_0 \equiv 0$.

Under this setup, the evolution under $L_1$ and $B_1$ is well-defined.

**Lemma 3.** Let $E = L_1$, or let $E = B_1$ and assume that the coefficients of $B_1$ are continuous for the considered time range $J = [0, T]$. Fix $b \geq 0$. Then the evolution equation

\[
\begin{align*}
\partial_t w &= E w, \\
w(s) &= v
\end{align*}
\]

for $v \in X_p$ has a unique weak solution $w \in C_w([s, T], X_p)$, i.e. $w \in L^\infty([s, T], X_p)$ and is weakly continuous.

This shows that the solution operators $S^E_{s \to t}$ are well-defined. For $L_1$ the coefficients are constant and thus $L_1$ generates a semigroup $e^{tL_1}$ with $S^L_{s \to t} = e^{(t-s)L_1}$.

The evolution under $L_1$ and $B_1$ is damping by a mixture of phase mixing for the unlocked oscillators and convergence to a fixed-point for the locked oscillators.

**Lemma 4.** Let $E = L_1$ or $E = B_1$. Then for initial data $v$ at time $s$ it holds that

\[
\|S^E_{s \to t} v\|^2_{\ell_{A, s}^{A \pm s}} \leq \|v\|^2_{\ell_{A, s}^{A \pm s}} + \int_s^t (\text{Re} S^E_{t \to r} v) (0)^2 (A + \tau - s)^{2b} \, dr
\]

for $A \geq 1$ and $b \geq 0$. 

Introduce the seminorms $\beta_\eta$, $\beta_\alpha$ and $\beta_d$ by
\[
\beta_\eta(u) = |u_1(0)|,
\]
\[
\beta_\alpha(u) = \|K_\Theta\| \int_0^\infty |(e^{tL_1}u)_1(0)| \, dt,
\]
\[
\beta_d(u) = \max \left\{ \beta_\alpha \left( (\ell u_{t-1})_\epsilon \right), \beta_\alpha \left( (\ell u_t)_\epsilon \right), \beta_\alpha \left( (\ell u_{t+1})_\epsilon \right) \right\}.
\]
The seminorms $\beta_\eta$ and $\beta_\alpha$ control the order parameter $\eta$ and the functional $\alpha$. The seminorm $\beta_d$ is used for the nonlinearity in $B_2$ as
\[
|\alpha(B_{1w}u)| \leq M_d(t)\beta_d(u).
\]
These seminorms can be controlled by the weighted Sobolev norms as follow.

**Lemma 5.** There exists a numerical constant $C_S$ such that for $A \geq 1$ and $b \geq 0$
\[
\beta_\eta(u) \leq C_S A^{-b} \|u\|_{p_A,b}.
\]
If $b > 1/2$ it holds that
\[
\beta_\alpha(u) \leq \frac{A^{1-b} \|K_\Theta\|}{\sqrt{2b-1}} \|u\|_{p_A,b}.
\]
For $b > 3/2$ there exists a constant $C_{3d}$ only depending on $b$, $\|K_\Theta\|$ and $Kr_{\alpha}$ such that for $A \geq 1$
\[
\beta_d(u) \leq C_{3d} A^{\frac{3}{2}-b} \|u\|_{p_A,b}.
\]
This allows us to control the effect of $B_2$.

**Lemma 6.** Assume that the coefficients of $B_1$ are continuous for the time range $J = [0, T]$ and that
\[
\|r_\Theta\|_{p_{bc}} < \infty
\]
for some $b_c > 3/2$. Then for $b_c \geq b > 3/2$ the evolution under $B$ starting at $s \in J$ has a unique weak solution in $C_w([s, T], X_{p_s})$.

With the control (11) of the coefficients, it holds for $v \in X_{p_s}$ and $0 \leq s \leq t \leq T$ that
\[
\beta_d(S_{b_{s \to t}}^B v) \leq C_{3d}(1 + t - s)^{\frac{5}{2} - b}\|v\|_{p_s}
\]
\[+ C_{3d}\|r_\Theta\|_{p_{bc}}, R_d(t) \int_s^t (1 + t - \tau)^{\frac{5}{2} - b - (1 + \tau) - b_d} \beta_d(S_{b_{s \to \tau}}^B, v) \, d\tau,
\]
\[
\beta_\alpha(S_{b_{s \to t}}^B v) \leq \frac{(1 + t - s)^{\frac{5}{2} - b} \|K_\Theta\|}{\sqrt{2b-1}} \|v\|_{p_s}
\]
\[+ \|r_\Theta\|_{p_{bc}}, \|K_\Theta\| R_d(t) \int_s^t (1 + t - \tau)^{\frac{5}{2} - b - (1 + \tau) - b_d} \beta_d(S_{b_{s \to \tau}}^B, v) \, d\tau,
\]
\[
\beta_\eta(S_{b_{s \to t}}^B v) \leq C_S (1 + t - s)^{-b} \|v\|_{p_s}
\]
\[+ C_S \|r_\Theta\|_{p_{bc}}, R_d(t) \int_s^t (1 + t - \tau)^{-b - (1 + \tau) - b_d} \beta_d(S_{b_{s \to \tau}}^B, v) \, d\tau.
\]

If additionally $b_r > b + 1$ or $bd > 1$, then there exists a constant $\delta_R$ such that
\[
\beta_d(S_{b_{s \to t}}^B v) \leq 2C_{3d}(1 + t - s)^{\frac{3}{2} - b} \|v\|_{p_s}.
\]
if $R_d(t) \leq \delta_R$. 

10
Applying the decay to \( r_r \) and \( r_i \), we can estimate the decay of the kernel of the Volterra equation (8) of the linearised evolution.

**Lemma 7.** Let \( 0 \leq b \leq b_r - 1/2 \). Then there exists a numerical constant \( C \) such that

\[
\int_0^\infty \|k_{Lc}(t)\| (1 + t)^b \, dt \leq C \left( \|r_r\|_{p_{u_r}} + \|r_i\|_{p_{u_r}} \right).
\]

If \( \|r_r\|_{p_{u_r}} < \infty \) and \( \|r_i\|_{p_{u_r}} < \infty \) for \( b_r > 1/2 \), this shows that the Laplace transform

\[
(\mathcal{L}k_{Lc})(z) = \int_0^\infty k_{Lc}(t) e^{-zt} \, dt
\]

is defined for \( \Re z \geq 0 \) by an absolutely converging integral. Moreover, if \( b_r > 3/2 \), then the Laplace transform \( \mathcal{L}k_{Lc} \) is continuous differentiable in the whole region \( \{ z \in \mathbb{C} : \Re z \geq 0 \} \), in particular, including the critical line \( \Re z = 0 \).

Therefore, we can discuss the linear stability through the characteristic equation

\[
\det \left( \mathrm{Id} + (\mathcal{L}k_{Lc})(z) \right) = 0.
\]

(12)

The rotation invariance always implies an eigenmode with \( z = 0 \). Imposing that this is the only non-decreasing eigenmode, we arrive at the definition of linear stability for states satisfying

\[
\|r_r\|_{p_{u_r}} + \|r_i\|_{p_{u_r}} < \infty
\]

for \( b_r > 3/2 \).

**Definition 8.** The stationary state \( f_{st} \) is linearly stable up to the rotation invariance if \( z = 0 \) is the only solution of the characteristic equation (12) in \( \Re z \geq 0 \) and

\[
\frac{d}{dz} \det \left( \mathrm{Id} + (\mathcal{L}k_{Lc})(z) \right) \bigg|_{z = 0} \neq 0.
\]

Even though \( \Re \eta \) and \( \Im \eta \) are treated as separate complex variables in this spectral analysis, we can show that the condition is sharp, see Section 5.1.

By studying the Volterra equation, we arrive at the following nonlinear control of the order parameter.

**Lemma 9.** Let \( b_\eta \geq 0 \), \( b_d \geq 0 \) and \( b_r > \max\{5/2, b_\eta + 2\} \) with

\[
\|r_r\|_{p_{u_r}} < \infty, \quad \|r_i\|_{p_{u_r}} < \infty, \quad \|r_{\Theta r}\|_{p_{u_r}} < \infty,
\]

and

\[
\|r_r\|_{p_{u_r} - \frac{1}{2}, 0} < \infty, \quad \|r_i\|_{p_{u_r} - \frac{1}{2}, 0} < \infty, \quad \|r_{\Theta r}\|_{p_{u_r} - \frac{1}{2}, 0} < \infty.
\]

Assume that one of the following conditions holds:

- \( b_d > 5/2 \),

- \( b_r > b_\eta + 4 + \max\{0, 1 - b_d\} \) and \( b_r > b_\eta + \frac{17}{4} + \max\{0, 1 - b_d\} + \frac{\max\{0, \frac{1}{2}, b_r\}}{2} - \frac{b_r}{2} \).
Furthermore, assume that the stationary state \( f_{st} \) is linearly stable in the sense of Definition 8. Then there exist constants \( C_\eta \) and \( \delta_R \) such that, for a forcing

\[
F(t) = \left( \frac{\partial F(t)}{\partial t} \right) \quad \text{with} \quad F(t) = (S_{0 \to t} u_{in})_1(0)
\]

with \( \| u_{in} \|_{p_\eta} < \infty \) for \( b > 3/2 \) and \( \alpha(u_{in}) = 0 \), the solution \( \mathcal{N}_\eta(t) \) and \( \mathcal{S}_\eta(t) \) of the Volterra equation (6) is controlled by

\[
|\eta(t)| \leq C_\eta (1 + t)^{-b_\eta} \sup_{s \in [0,t]} \left( |F(s)| + \beta_\alpha \left( S_{0 \to s} u_{in} \right) \right) (1 + s)^{b_\eta}
\]

if \( R_d(t) \leq \delta_R \).

Combining Lemma 9 with the control of the forcing, we can control the order parameter under the bootstrap hypothesis.

**Lemma 10.** Let \( b > 3/2 \), \( b_d = b - 1/2 \) and \( b_r > b + 3/2 \). Let \( f_{st} \) be a linearly stable stationary state in the sense of Definition 8 such that

\[
\| r_{\eta} \|_{p_\eta} < \infty, \quad \| r_{i} \|_{p_{\eta_{\infty}}} < \infty, \quad \| r_{\phi} \|_{p_{\eta_{\infty}}} < \infty,
\]

and

\[
\| r_{\eta} \|_{p_{\eta_{\infty}} - \frac{1}{2}} < \infty, \quad \| r_{i} \|_{p_{\eta_{\infty}} - \frac{1}{2}} < \infty, \quad \| r_{\phi} \|_{p_{\eta_{\infty}} - \frac{1}{2}} < \infty.
\]

Furthermore, assume that one of the following conditions holds:

- \( b_d > 5/2 \),
- \( b_r > b + 7/2 + \max\{0,3/2 - b_\eta\} \).

Then there exist constants \( \delta_R \) and \( C \) such that

\[
|\eta(t)| \leq C (1 + t)^{-b_r} \| u_{in} \|_{p_\eta}\n\]

if \( R_d(t) \leq \delta_R \).

With the control of the order parameter, we can use (6) to control \( \beta_d(u(t)) \).

**Lemma 11.** Let \( b > 3/2 \), \( b_d = b - 1/2 \) and \( b_r > b + 1 \) and assume that

\[
\| r_{\phi} \|_{p_{\eta_{\infty}}} < \infty \quad \text{and} \quad \| r_{\eta} \|_{p_{\eta_{\infty}}} < \infty \quad \text{and} \quad \| r_{i} \|_{p_{\eta_{\infty}}} < \infty.
\]

Then there exist constants \( \delta_R \) and \( C \) such that

\[
|\beta_d(u(t))| \leq C (1 + t)^{\frac{7}{2} - b} \left( \| u_{in} \|_{p_\eta} + \sup_{s \in [0,t]} (1 + s)^{b_d} |\eta(s)| \right)
\]

if \( R_d(t) \leq \delta_R \).

By a well-posedness result of the nonlinear evolution, we can prove that \( \eta \) and \( \Theta \) vary continuously as long as

\[
|\beta_d(u(t))| < \frac{1}{2} |\alpha(D\hat{R} f_{st})|.
\]

In this case, we can also control \( \Theta \) by

\[
|\hat{\Theta}(t)| \leq C |\eta(t)| |\beta_d(u(t))|
\]

for a constant \( C \). Combining the previous estimates we can therefore prove the result by a bootstrap argument.
4 Norms and time-evolution under the transport operators

The bound on the stationary state comes from an energy estimate with an appropriate approximation scheme for this class of partially locked states.

Proof of Proposition 1. As $\hat{f}_{st}$ is a stationary state, by (3) it satisfies

$$0 = \ell \partial_{\xi}(\hat{f}_{st})_{e}(\xi) + \frac{K\ell}{2} \left( r_{st}(\hat{f}_{st})_{e-1} - r_{st}(\hat{f}_{st})_{e+1} \right)$$

and $(\hat{f}_{st})_{0} = \hat{g}$. For an a priori estimate, let $b \geq 0$ and take the inner product in $\mathcal{X}_{p_{b}-\frac{1}{2}}$. This shows

$$0 \leq -2b\|\hat{f}_{st}\|_{p_{b}-\frac{1}{2},0}^{2} + \frac{Kr_{st}}{2}\|\hat{g}\|_{p_{b},\frac{1}{2}}\|\hat{f}_{st}\|_{p_{b},\frac{1}{2},0},$$

which shows the result for $k = 0$.

Fixing $k = -1, -3/2, \ldots$, we find the following a priori estimate by taking the inner product in $\mathcal{X}_{p_{b},k}$

$$0 \leq -2b\|\hat{f}_{st}\|_{p_{b},\frac{1}{2}+\frac{k}{2}+\frac{1}{2}}^{2} + C\|\hat{f}_{st}\|_{p_{b},k}^{2} + C\|\hat{f}_{st}\|_{p_{b},k}\|\hat{g}\|_{p_{b}},$$

which shows the result by induction over $k$.

The a priori estimates are justified for states with all locked oscillators at the stable fixed points. For this construct approximate states $\hat{f}_{st}^{n}$ as Fourier transform of $\hat{f}_{st}$ given by

$$\hat{f}_{st}^{n}(\theta, \omega) = \begin{cases} \delta_{\arcsin(\omega/(Kr_{st}))}(\theta) g^{n}(\omega) & \text{if } |\omega| \leq Kr_{st} \\ \frac{\sqrt{\omega^{2} - (Kr_{st})^{2}}}{2\pi|\omega - Kr_{st}\sin\theta|} g^{n}(\omega) & \text{if } |\omega| > Kr_{st} \end{cases}$$

where $g^{n}$ is an approximation of $g$ such that $g^{n}$ has analytic regularity and $\|\hat{g}^{n} - \hat{g}\|_{p_{b}} \to 0$, e.g. $g^{n}$ is obtained by convolution of $g$ with a Gaussian. By [5], we then control

$$\int_{0}^{\infty} \left( |\hat{f}_{st}^{n}|^{2} + |\partial_{\xi} \hat{f}_{st}^{n}|^{2} \right) e^{2\alpha \xi} d\xi \leq C \delta^{\ell}$$

with $\alpha > 0$, $\ell \in \mathbb{N}$ for constants $C$ and $\delta$. Since $\left( f_{st}^{n}\right)_{e}(\xi) \to (\hat{f}_{st})_{e}(\xi)$ as $n \to \infty$, this shows the claimed bound.

The control of $r_{\theta}$ and $r_{r}$ and $r_{t}$ follows directly from their definition. \qed

The results on the evolution operators are based on energy estimates. The derivatives $\partial_{\xi}u_{\ell}$ can always be handled in the same way, because they satisfy the same evolution equation.

Proof of Lemma 3. By Morray’s inequality, a function $w \in C_{w}([s, T], \mathcal{X}_{p_{b}})$ is uniformly continuous in $\xi$ over compact regions. Moreover, by the weak continuity $w_{\ell}(\cdot, \xi)$ is continuous in time for all $\ell \in \mathbb{N}$ and $\xi \in \mathbb{R}^{+}$ and so $w$ is a continuous function. By standard arguments on the scalar transport equation, this shows the uniqueness of solutions.

For constructing a solution we use the following a priori estimate for $B_{1}$

$$\partial_{t}\|w\|_{p_{b}}^{2} = \sum_{\ell \geq 1} \int_{0}^{\infty} \partial_{\xi} \left( |w_{\ell}(t, \xi)|^{2} + |\partial_{\xi} w_{\ell}(t, \xi)|^{2} \right) p_{A,b}(\xi) |X_{\ell}| d\xi$$

$$+ Kr_{st} \sum_{\ell \geq 1} \int_{0}^{\infty} \Re \left[ \frac{w_{\ell}(t, \xi) w_{\ell-1}(t, \xi) - w_{\ell-1}(t, \xi) w_{\ell}(t, \xi)}{w_{\ell}(t, \xi) w_{\ell}(t, \xi)} \right] p_{A,b}(\xi) |X_{\ell}| d\xi$$

$$+ Kr_{st} \sum_{\ell \geq 1} \int_{0}^{\infty} \Re \left[ \frac{\partial_{\xi} w_{\ell}(t, \xi) \partial_{\xi} w_{\ell}(t, \xi)}{\partial_{\xi} w_{\ell}(t, \xi) w_{\ell}(t, \xi)} - \partial_{\xi} w_{\ell-1}(t, \xi) \partial_{\xi} w_{\ell}(t, \xi) \right] p_{A,b}(\xi) |X_{\ell}| d\xi.$$
By the weak compactness, we extract a weak solution \( w \) where \( \langle \xi \rangle \) satisfies
\[
\|w^n\|_{p_0} \leq 0
\]
and the same estimate holds for \( L_1 \).

The result now follows from an approximation scheme and a standard compactness argument. We construct approximate solutions \( w^n \) by restricting the evolution to \( t \in [1, n] \) and smooth initial data with compact support \( v^n \) with \( v^n \to v \) in \( X_{p_0} \). By the a priori estimate, these solutions satisfy \( \|w^n\|_{p_0} \leq \|v^n\|_{p_0} \) for \( t \in [s, T] \). Hence \( \{w^n : n \in \mathbb{N}\} \) is a bounded set in \( L^\infty([s, T], X_{p_0}) \). By the weak compactness, we extract a weak solution \( w \). This shows the existence of a solution.

For the weak continuity use that
\[
\partial_t w \in L^\infty([s, T], \mathcal{Y}_{p_0, 0}),
\]
where \( \mathcal{Y}_{p_0, 0} \) is the Hilbert space defined by the norm
\[
\|u\|^2_{\mathcal{Y}_{p_0, 0}} = \sum_{\ell \geq 1} \int_0^\infty |v_\ell(t, \xi)|^2 |p_\ell(\xi)|^2 \, d\xi.
\]
The space \( X_{p_0} \) is dense in \( \mathcal{Y}_{p_0, 0} \) so that (13) implies that \( w \) is weakly continuous by standard functional analysis, see e.g. Theorem 2.1 in [17].

The remaining controls follow from refined energy estimates. For these, we will only present the a priori estimates, which can be justified in the same way.

**Proof of Lemma 4.** Let \( v(t, \xi) = (S^E_{t-1} v)(\xi) \) and \( v(t) = r_{st} \) in the case of \( E = L_1 \) and \( v(t) = r_{st} + \eta(t) \) in the case of \( E = B_1 \). Then with the weight \( \phi = p_{A,b} \) it holds with the convention \( v_0 \equiv 0 \) that
\[
\frac{d}{dt} \sum_{\ell \geq 1} \int_0^\infty |v_\ell(t, \xi)|^2 |\phi(\xi + t - s)|^2 \ell^{-1} \, d\xi
\]
\[
= \sum_{\ell \geq 1} \int_0^\infty \delta_t \left(|v_\ell(t, \xi)|^2 \right) |\phi(\xi + t - s)|^2 \ell^{-1} \, d\xi
\]
\[
+ K \sum_{\ell \geq 1} \int_0^\infty \Re \left[ v(t) v_{\ell-1}(t, \xi) v_\ell(t, \xi) - v(t) v_{\ell+1}(t, \xi) v_\ell(t, \xi) \right] |\phi(\xi + t - s)|^2 \ell^{-1} \, d\xi
\]
\[
= - \sum_{\ell \geq 1} |v_\ell(t, 0)|^2 |\phi(t - s)|^2 - \sum_{\ell \geq 1} \int_0^\infty |v_\ell(t, \xi)|^2 \delta_t \left(|\phi(\xi + t - s)|^2 \right) (1 - \ell^{-1}) \, d\xi
\]
\[
\leq - |v_1(t, 0)|^2 |\phi(t - s)|^2,
\]
where we used that \( \phi \) has non-negative derivative as \( b \geq 0 \) and
\[
\sum_{\ell \geq 1} \Re \left[ v(t) v_{\ell-1}(t, \xi) v_\ell(t, \xi) \right] = \sum_{\ell \geq 1} \Re \left[ v(t) v_{\ell}(t, \xi) v_{\ell+1}(t, \xi) \right].
\]
Likewise we control \( \partial_\xi u \) and as \( \phi(\xi + t - s) = p_{A+t-s,b}(\xi) \) the claimed result follows.}

In preparation of Lemma 5, we first prove the following lemma.
Lemma 12. Let $k = -1/2, -1, -3/2, \ldots$ and $b > 1/2 + |k|$. Then there exists a constant $C_k$ only depending on $k$, $b$, $\|K\|$ and $K_{\text{rat}}$ such that
\[
\beta_\alpha(v) \leq C_k A^{1-b} \|v\|_{p_{A,b,k}}
\]
and
\[
C_{-1/2} = \frac{\|K\|}{\sqrt{2b - 1}}.
\]

Proof. We prove it by induction over $k$ starting at $k = -1/2$ and going downwards. The base case is a simple application of Lemma 4 as by the Cauchy-Schwarz inequality
\[
\left( \int_0^\infty \left| (e^{tL_1}v)_1(0) \right| \, dt \right)^2 \leq \left( \int_0^\infty \left| (e^{tL_1}v)_1(0) \right|^2 (A + t)^{2b} \, dt \right) \left( \int_0^\infty (A + t)^{-2b} \, dt \right)
\leq \frac{A^{1-2b}}{2b-1} \|v\|_{p_{A,b}}^2.
\]

For the induction step we use that the transport evolution is regularising in the spatial modes $\ell$ at the expense of a power in $\xi$. Assuming it is true for $k + 1/2$, we look at $k$ and find with the notation $v(t) = e^{tL_1}v$
\[
d\frac{d}{dt} \|v(t)\|_{p_{A,b},k}^2 = \sum_{\ell \geq 1} \int_0^\infty \partial_\xi \left( |v_\ell(t,\xi)|^2 + |\partial_\xi v_\ell(t,\xi)|^2 \right) |p_{A,b}(\xi)|^2 \xi^{2k+1} \, d\xi
+ K_{\text{rat}} \sum_{\ell \geq 1} \int_0^\infty \Re \left[ v_{\ell-1}(t,\xi)v_{\ell-1}(t,\xi) - v_{\ell+1}(t,\xi)v_{\ell+1}(t,\xi) \right] |p_{A,b}(\xi)|^2 \xi^{2k+1} \, d\xi
+ K_{\text{rat}} \sum_{\ell \geq 1} \int_0^\infty \Re \left[ (\partial_\xi v_{\ell-1}(t,\xi))(\partial_\xi v_{\ell-1}(t,\xi)) - (\partial_\xi v_{\ell+1}(t,\xi))(\partial_\xi v_{\ell+1}(t,\xi)) \right] |p_{A,b}(\xi)|^2 \xi^{2k+1} \, d\xi
\leq -A^{2b} |v_1(t,\xi)|^2 - 2b \sum_{\ell \geq 1} \int_0^\infty \left( |v_\ell(t,\xi)|^2 + |\partial_\xi v_\ell(t,\xi)|^2 \right) |p_{A,b,-\frac{1}{2}}(\xi)|^2 \xi^{2k+1} \, d\xi
+ K_{\text{rat}} \sum_{\ell \geq 1} \int_0^\infty \Re \left[ v_\ell(t,\xi)v_{\ell+1}(t,\xi) + (\partial_\xi v_\ell(t,\xi))(\partial_\xi v_{\ell+1}(t,\xi)) \right] ((\ell+1)^{2k+1} - (\ell+1)^{2k+1}) |p_{A,b}(\xi)|^2 \xi \, d\xi.
\]
As $|(\ell+1)^{2k+1} - (\ell+1)^{2k+1}| \leq (2k+1)^{\ell+1}$, this means that there exists a constant $C$ such that
\[
d\frac{d}{dt} \|v(t)\|_{p_{A,b},k}^2 \leq -A^{2b} |v_1(t,0)|^2 - 2b \|v(t)\|_{p_{A,b,-\frac{1}{2},k+\frac{1}{2}}}^2 + C \|v(t)\|_{p_{A,b},k}^2.
\]

Therefore,
\[
\|v\|_{p_{A,b},k}^2 \geq \int_0^\infty e^{-Ct} \left[ A^{2b} |v_1(t,0)|^2 + 2b \|v(t)\|_{p_{A,b,-\frac{1}{2},k+\frac{1}{2}}}^2 \right] \, dt.
\]
Hence there exists a time $t^* \in [0, 1]$ such that
\[
\|e^{t^*L_1}v\|_{p_{A,b,-\frac{1}{2},k+\frac{1}{2}}}^2 \leq \frac{e^C}{2b} \|v\|_{p_{A,b,k}}^2.
\]
Using the semigroup property, we find with $v^* = e^{t^*L_1}v$
\[
\beta_\alpha(v) = \|K\| \int_0^{t^*} |v_1(t,0)| \, dt + \|K\| \int_0^\infty |(e^{tL_1}v^*)_1(0)| \, dt.
\]

15
The first term can be controlled by (14) and the second term by the induction hypothesis. This proves the induction step.

The bounds on the seminorms are now an easy consequence.

**Proof of Lemma 5.** The bound on $\beta_0$ is already proved in the previous lemma. For $\beta_d$ we use the previous lemma with $k = -3/2$ as

$$
\|(\ell u_{\ell-1})\ell\|_{p_0} \leq \|u\|_{p_0}, \quad \|(\ell u_\ell)\ell\|_{p_0} \leq \|u\|_{p_0}, \quad \|(\ell u_{\ell+1})\ell\|_{p_0} \leq 2\|u\|_{p_0}.
$$

Finally, the control of $\beta_0$ is a consequence of the Sobolev embedding theorem, which implies a constant $C_S$ such that a function $u_1$ of $\xi$ satisfies

$$
u_1(0) \leq C_S \int_0^1 (|u_1(\xi)|^2 + |\partial_\xi u_1(\xi)|^2) \, d\xi.
$$

The controls of $B$ are a direct consequence of Duhamel’s principle.

**Proof of Lemma 6.** With the bound of $\beta_d$ from Lemma 5, the operator $B_2$ is a bounded operator. Hence the evolution has a unique solution given by Duhamel’s principle as

$$S^B_{s \to t} v = \mathcal{S}^{B_1}_{s \to t} v + \int_0^t \mathcal{S}^{B_1}_{r \to t} B_2(S^B_{s \to r} v) \, d\tau.
$$

For the quantified estimates, recall that

$$B_2(S^B_{s \to \tau} v) = -\alpha(B_1 v S^B_{s \to \tau} v) \, r_\Theta.
$$

By the bootstrap assumption we control

$$|\alpha(B_1 v S^B_{s \to \tau} v)| \leq M_d(t) \beta_d(S^B_{s \to \tau} v),$$

so that the result follows from the propagation of $B_1$, see Lemmas 4 and 5.

Finally, under the assumption $b_r > b + 1$ or $b_d > 1$ we find that

$$(1+t-s)^{-\frac{b}{2}+b} \int_s^t (1+t-\tau)^{-b_r} (1+\tau)^{-b_d} (1+\tau-s)^{-b_d} \, d\tau \leq \int_s^t (1+t-\tau)^{-b_r} (1+\tau)^{-b_d} \, d\tau$$

is uniformly bounded by a constant. Moreover, $\beta_d$ varies continuously by the weak-continuity. Hence by a bootstrap argument, we can find $\delta_R$ such that the claimed control holds.

5 **Volterra equation**

For controlling the order parameter, we use the theory of the Volterra equation and follow the setup of the book by Gripenberg, Londen, and Staffans [8].

We denote the entries of the kernel as follows

$$k(t, s) = \begin{pmatrix}
\Re k_r(t, s) & \Re k_i(t, s) \\
\Im k_r(t, s) & \Im k_i(t, s)
\end{pmatrix}
$$

and accordingly for $k_Lc$

$$k_Lc(t) = \begin{pmatrix}
\Re k_{Lc}(t) & \Re k_{Lc}(t) \\
\Im k_{Lc}(t) & \Im k_{Lc}(t)
\end{pmatrix}.
5.1 Linearised evolution

Using the decay under $L_1$, we find the bound on the kernel.

**Proof of Lemma 7.** By the Cauchy Schwarz inequality it holds that

$$
\left( \int_0^\infty |k_{L,r}(t)(1+t)^b\,dt \right)^2 \leq \left( \int_0^\infty |k_{L,r}(t)|^2(1+t)^{2b}\,dt \right) \left( \int_0^\infty (1+t)^{2(b-b_r)}\,dt \right)
$$

and likewise for $k_{L,i}$. As $b_r > b + 1/2$, the last integral is finite and as $k_{L,r}(t) = -(e^{tL_1}r_r)_1(0)$ and $k_{L,i}(t) = -(e^{tL_1}r_i)_1(0)$, Lemma 4 implies the result.

The rotation symmetry implies that $\hat{D}\hat{f}_{st}$ is a zero eigenmode of the linearised evolution and its corresponding order parameter is $ir_{st}$, i.e. purely imaginary.

The stability condition in Definition 8 is necessary to exclude other non-decaying eigenmodes, similar to the results in [5]. We prove the results directly, because we need to handle the case of poles at $\Re z = 0$, which can be the boundary of the resolvent.

**Proposition 13.** Let $b_r \geq 0$ and assume the stationary state $\hat{f}_{st}$ is regular enough such that $\|r_r\|_{p_r} < \infty$ and $\|r_i\|_{p_i} < \infty$.

If $\lambda$ is a root of the characteristic equation with $\Re \lambda > 0$ and $\Im \lambda = 0$, then there exists an eigenmode $v_\lambda$ with $\|v_\lambda\|_{p_\lambda} < \infty$, i.e. satisfying $L v_\lambda = \lambda v_\lambda$.

If $\lambda$ is a root of the characteristic equation with $\Re \lambda > 0$ and $\Im \lambda \neq 0$, then also $\overline{\lambda}$ is a root of the characteristic equation and there exist modes $v_{\lambda,c}$ and $v_{\lambda,s}$ with $\|v_{\lambda,c}\|_{p_{\lambda}} < \infty$ and $\|v_{\lambda,s}\|_{p_{\lambda}} < \infty$ satisfying

$$
L v_{\lambda,c} = (\Re \lambda) v_{\lambda,c} - (\Im \lambda) v_{\lambda,s},
$$

$$
L v_{\lambda,s} = (\Im \lambda) v_{\lambda,c} + (\Re \lambda) v_{\lambda,s}.
$$

If $\lambda$ is a root of the characteristic equation with $\Re \lambda = 0$ and $b_r > 1$, then the above modes exist with the bound $\|v_{\lambda,c}\|_{p_{\lambda}} < \infty$ and $\|v_{\lambda,s}\|_{p_{\lambda}} < \infty$, $\|v_{\lambda,c}\|_{p_{\lambda}} < \infty$ for $0 \leq b < b_r - 1$.

If $b_r > 2$ and $\lambda = 0$ is not a simple root, i.e.

$$
\frac{d}{dz} \det \left( \Id + (\mathcal{L}k_{L,c})(z) \right) \bigg|_{z=\lambda} = 0
$$

then at least one of the following possibilities holds:

- There exist two eigenmodes $v_{0,r}$ and $v_{0,i}$ with $\|v_{0,r}\|_{p_r} < \infty$ and $\|v_{0,r}\|_{p_i} < \infty$ for $0 \leq b < b_r - 1$.

- There exist two modes $v_{0,0}$ and $v_{0,1}$ with $\|v_{0,0}\|_{p_0} < \infty$ and $\|v_{0,1}\|_{p_{b-1}} < \infty$ for $1 \leq b < b_r - 2$ satisfying

$$
L v_{0,0} = 0 \text{ and } L v_{0,1} = v_{0,0}.
$$

**Proof.** If $\lambda$ is satisfying $\Im \lambda = 0$ and $\Re \lambda \geq 0$, then $\mathcal{L}k_{L,c}$ is a real matrix. Hence if $\lambda$ is a root, there exists $w_r, w_i \in \mathbb{R}$ such that

$$
\begin{pmatrix} w_r \\ w_i \end{pmatrix} \in \ker [1 + \mathcal{L}k_{L,c}(\lambda)].
$$

Then define the mode $v_\lambda$ by

$$
v_\lambda = \int_0^\infty e^{tL_1}(w_r r_r + w_i r_i)e^{-\lambda t}\,dt.
$$
which is a converging Bochner integral with the claimed bounds by Lemma 4.

Moreover, we find
\[
\begin{pmatrix}
\Re(v_\lambda)_0(1) \\
\Im(v_\lambda)_0(1)
\end{pmatrix}
= -L\mathcal{K}_Lc(\lambda)
\begin{pmatrix}
\Re(w_r) \\
\Im(w_i)
\end{pmatrix}
\] so that
\[
L_2 v_\lambda = w_r r_r + w_i r_i.
\]
On the other hand
\[
L_1 v_\lambda = \lambda v_\lambda - (w_r r_r + w_i r_i),
\]
which shows that \( v_\lambda \) is the claimed eigenmode.

In the case that \( \lambda \) is not a simple root with \( \Re \lambda \geq 0 \) and \( \Im \lambda = 0 \), one possibility is that \( 1 + L\mathcal{K}_Lc(\lambda) = 0 \). In this case, we have the following two eigenmodes
\[
v_{\lambda, r} = \int_0^\infty e^{tL_1} r_r e^{-\lambda t} \, dt \quad \text{and} \quad v_{\lambda, i} = \int_0^\infty e^{tL_1} r_i e^{-\lambda t} \, dt.
\]
Otherwise, find \( a_r \) and \( a_i \) such that
\[
\begin{pmatrix}
a_r \\
a_i
\end{pmatrix}
= \begin{pmatrix}
\Re(w_r) \\
\Im(w_i)
\end{pmatrix}
\]
with the adjoint
\[
\text{adj}(1 + L\mathcal{K}_Lc(\lambda)) = \begin{pmatrix}
1 + \Im k_{L, i}(\lambda) & -\Re k_{L, r}(\lambda) \\
-L\Re k_{L, i}(\lambda) & 1 + L\Re k_{L, r}(\lambda)
\end{pmatrix}.
\]
Let
\[
\begin{pmatrix}
w_r' \\
w_i'
\end{pmatrix}
= \frac{d}{dz} \begin{pmatrix}
a_r \\
a_i
\end{pmatrix}
\]
and define the modes
\[
v_{\lambda, 0} = \int_0^\infty e^{tL_1} (w_r r_r + w_i r_i) e^{-\lambda t} \, dt
\]
\[
v_{\lambda, 1} = \int_0^\infty e^{tL_1} \left( (-t w_r + w_r') r_r + (-t w_i + w_i') r_i \right) e^{-\lambda t} \, dt.
\]
As before, we have that \( L v_{\lambda, 0} = \lambda v_{\lambda, 0} \) and for the mode \( v_{\lambda, 1} \) we find that
\[
\begin{pmatrix}
\Re(v_{\lambda, 1})_1(0) \\
\Im(v_{\lambda, 1})_1(0)
\end{pmatrix}
= -\frac{d}{dz} (1 + L\mathcal{K}_Lc(z)) \begin{pmatrix}
w_r \\
w_i
\end{pmatrix} - L\mathcal{K}_Lc(z) \begin{pmatrix}
w_r' \\
w_i'
\end{pmatrix}.
\]
With \( M(z) = 1 + L\mathcal{K}_Lc(z) \), this can be written as
\[
\begin{pmatrix}
\Re(v_{\lambda, 1})_1(0) \\
\Im(v_{\lambda, 1})_1(0)
\end{pmatrix}
= -\frac{d}{dz} M(z) \begin{pmatrix}
\Re(M(\lambda)) + M(\lambda) \frac{d}{dz} \Re M(z) \bigg|_{z=\lambda} \\
\Im(M(\lambda)) + M(\lambda) \frac{d}{dz} \Im M(z) \bigg|_{z=\lambda}
\end{pmatrix}
\begin{pmatrix}
a_r \\
a_i
\end{pmatrix}
+ \begin{pmatrix}
w_r' \\
w_i'
\end{pmatrix}
\]
\[
= -\frac{d}{dz} \det M(z) \bigg|_{z=\lambda} \begin{pmatrix}
a_r \\
a_i
\end{pmatrix}
+ \begin{pmatrix}
w_r' \\
w_i'
\end{pmatrix}.
\]
As we assume that \( \lambda \) is not a simple root of \( 1 + L\mathcal{K}_Lc \), the first term vanishes and we find that
\[
\begin{pmatrix}
\Re(v_{\lambda, 1})_1(0) \\
\Im(v_{\lambda, 1})_1(0)
\end{pmatrix}
= \begin{pmatrix}
w_r' \\
w_i'
\end{pmatrix}.
\]
Then we can directly verify as before that
\[ L_{v\lambda,1} = \lambda v_{\lambda,1} + v_{\lambda,0}, \]
which is the claimed relation.

In the case of a root \( \lambda \) with \( \Re \lambda \geq 0 \) and \( \Im \lambda \neq 0 \), let
\[ \left( \begin{array}{c} w_1 \\ w_2 \end{array} \right) \in \ker(1 + LkLc(\lambda)). \]
Taking the conjugate shows that
\[ \left( \begin{array}{c} w_1 \\ w_2 \end{array} \right) \in \ker(1 + LkLc(\bar{\lambda})), \]
so that \( \bar{\lambda} \) is also a root. Then define the modes as
\[
v_{\lambda,c} = \int_{0}^{\infty} e^{tL_1} \left[ (w_1 e^{-\lambda t} + \overline{w_1} e^{-\overline{\lambda} t}) r_r + (w_2 e^{-\lambda t} + \overline{w_2} e^{-\overline{\lambda} t}) r_i \right] dt,
\]
\[
v_{\lambda,s} = \int_{0}^{\infty} e^{tL_1} \left[ -i(w_1 e^{-\lambda t} + \overline{w_1} e^{-\overline{\lambda} t}) r_r - i(w_2 e^{-\lambda t} + \overline{w_2} e^{-\overline{\lambda} t}) r_i \right] dt,
\]
which satisfy the claimed bounds. Moreover, as before
\[
\left( \begin{array}{c} \Re(v_{\lambda,c})_1(0) \\ \Im(v_{\lambda,c})_1(0) \end{array} \right) = \left( \begin{array}{c} w_1 + \overline{w_1} \\ w_2 + \overline{w_2} \end{array} \right) \quad \text{and} \quad \left( \begin{array}{c} \Re(v_{\lambda,s})_1(0) \\ \Im(v_{\lambda,s})_1(0) \end{array} \right) = \left( \begin{array}{c} -i(w_1 - \overline{w_1}) \\ -i(w_2 - \overline{w_2}) \end{array} \right),
\]
Therefore, we find directly
\[
Lv_{\lambda,c} = (\Re \lambda)v_{\lambda,c} - (\Im \lambda)v_{\lambda,s},
\]
\[
Lv_{\lambda,s} = (\Im \lambda)v_{\lambda,c} + (\Re \lambda)v_{\lambda,s},
\]
which is the claimed relation.

The convolution Volterra equation (8) can be solved through the resolvent \( r_{Lc} \) satisfying
\[
r_{Lc} + k_{Lc} \ast r_{Lc} = r_{Lc} + r_{Lc} \ast k_{Lc} = k_{Lc}, \tag{15}
\]
which has a unique locally integrable solution, cf. [8, Theorem 3.1 of Chapter 2]. The Volterra equation then has a unique solution given by
\[
\left( \begin{array}{c} \Re \eta \\ \Im \eta \end{array} \right) = F_L - r_{Lc} \ast F_L,
\]
see [8, Theorem 3.5 of Chapter 2].

The weights \( p_{A,b} \) are submultiplicative meaning
\[
p_{A,b}(s + t) \leq p_{A,b}(s) p_{A,b}(t) \quad \text{for} \ s, t \in \mathbb{R}^+,
\]
as we assume \( A \geq 1 \) and \( b \geq 0 \). This allows to control the convolution with a Young inequality

**Lemma 14.** Let \( \alpha \in L^1(\mathbb{R}^+, p_{A,b}) \) and \( \beta \in L^\infty(\mathbb{R}^+, p_{A,b}) \) with \( A \geq 1 \) and \( b \geq 1 \), then
\[
\| \alpha \ast \beta \|_{L^\infty(\mathbb{R}^+, p_b)} \leq \| \alpha \|_{L^1(\mathbb{R}^+, p_b)} \| \beta \|_{L^\infty(\mathbb{R}^+, p_b)}.
\]

19
Proof. The result follows directly from the submultiplicativity with Fubini’s theorem, see [4, Lemma 19].

If the kernel is sufficiently decaying, then the single root of the characteristic equation at $z = 0$ must behave like a pole and can be separated.

**Proposition 15.** Let $b \geq 0$ and $b_r > b + 5/2$. Assume that $\hat{f}_{st}$ is such that

$$\|r_r\|_{p_b} < \infty \quad \text{and} \quad \|r_i\|_{p_b} < \infty$$

and $k_{LC}$ satisfying the stability condition from Definition 8. Then the resolvent $r_{LC}$ takes the form

$$r_{LC} = K_\Theta + r_{LC,s},$$

where $\|r_{LC,s}\|_{L^1(\mathbb{R}^+, p_b)} < \infty$ and $K_\Theta$ is a constant matrix.

Proof. We use Section 3 of Chapter 7 of [8], which applies as our weight is submultiplicative.

By the assumed regularity on $\hat{f}_{st}$, we have that

$$\int_0^\infty \|k_{LC}(t)\|(1 + t)^{b+2} dt < \infty$$

so that $k_{LC}$ is smooth of order at least 2 in $L^1(\mathbb{R}^+, p_b)$ with the Definitions 3.1 and 3.5 of Chapter 7 of [8], see also Lemma 4.3 of [9]. By the condition on the derivative, $1 + Lk_{LC}$ has a zero of order 1 (see Definition 3.6 of Chapter 6 of [8]). Hence by the corresponding version of Theorem 3.7 of Chapter 7 of [8] or Theorem 3.6 of [9] the result follows.

We can identify $K_\Theta$ more precisely.

**Lemma 16.** Assume the setup of Proposition 15 and

$$\|r_\Theta\|_{p_b} < \infty.$$

Then $K_\Theta$ can be written as

$$K_\Theta = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \otimes \begin{pmatrix} c_{\Theta, r} \\ c_{\Theta, i} \end{pmatrix}$$

for constants $c_{\Theta, r}$ and $c_{\Theta, i}$. Moreover, its kernel is determined by

$$K_\Theta(1 + Lk_{LC}(0)) = 0$$

and

$$\alpha(D\hat{R}f_{st}) \neq 0.$$

Proof. Consider the rotation eigenmode and its forcing

$$F_{L, \Theta}(t) = (e^{L_1} D\hat{R}f_{st})(0),$$

which is decaying as $(1 + t)^{-b_r}$ by Lemma 4 and Lemma 5, because $D\hat{R}f_{st}$ is proportional to $r_{\Theta}$. As it is an eigenmode, the order parameter is the constant $(D\hat{R}f_{st})(0) = ir_{st}$, so that

$$\begin{pmatrix} 0 \\ 1 \end{pmatrix} = F_{L, \Theta} - r_{LC} \ast F_{L, \Theta} = F_{L, \Theta} - K_\Theta \ast F_{L, \Theta} - r_{LC,s} \ast F_{L, \Theta}.$$
By Lemma 14, the term $r_{Lcs} * F_{L,\Theta}$ is also vanishing as $t \to \infty$. Therefore we find from the limit $t \to \infty$ that
\[
\begin{pmatrix} 0 \\ 1 \end{pmatrix} = - \int_0^\infty K_\Theta F_{L,\Theta}(t) dt.
\] (16)
This shows that
\[
\begin{pmatrix} 0 \\ 1 \end{pmatrix} \in \text{ran } K_\Theta.
\]
and $\alpha(D \hat{R}_g) \neq 0$.

Taking the Laplace transform of (15) shows that
\[
(1 + \mathcal{L}k_{Lc}(0))K_\Theta = 0 \quad \text{and} \quad K_\Theta(1 + \mathcal{L}k_{Lc}(0)) = 0.
\]
The stability condition implies that $1 + \mathcal{L}k_{Lc}(0) \neq 0$, because otherwise $\det(1 + \mathcal{L}k_{Lc}(z))$ would have a root of order at least two. Therefore, the range of $K_\Theta$ must be one-dimensional and $K_\Theta$ takes the given form.

**Corollary 17.** Assume the setup of Lemma 16, then for a forcing $F_L$ created by $F_L$ through
\[
F_L = \left( \begin{array}{c} \Re F_L \\ \Im F_L \end{array} \right)
\]
the formula
\[
K_\Theta \ast F_L(t) = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \int_0^t \left( c_{\Theta, r} \Re F_L(s) + c_{\Theta, i} \Im F_L(s) \right) ds.
\]

By the solution formula, the order parameter can therefore only decay if
\[
\int_0^\infty \left( c_{\Theta, r} \Re F_L(t) + c_{\Theta, i} \Im F_L(t) \right) dt = 0,
\]
which motivates the definition of $E$ in (10). Precisely, we find:

**Lemma 18.** Let $E = L_1$ and $\|u_\infty\|_{p_0} < \infty \text{ for } b > 1/2$ or let $E = B$ and $\|u_\infty\|_{p_0} < \infty \text{ and } \|r_\Theta\|_{p_0} < \infty \text{ for } b > 3/2$. Then for $t \in \mathbb{R}^+$ it holds that
\[
\alpha(S_{0,t}^{E} u_\infty) + \int_0^t \left( c_{\Theta, r} \Re (S_{0,t}^{E} u_\infty)(s) + c_{\Theta, i} \Im (S_{0,t}^{E} u_\infty)(s) \right) ds = \alpha(u_\infty).
\]

**Proof.** Note that
\[
\frac{d}{dt}\alpha(S_{0,t}^{E} u_\infty) = \alpha[L_1(S_{0,t}^{E} u_\infty)],
\]
from where the result follows. \(\square\)

For this kind of forcing we can therefore formulate the following corollary.

**Corollary 19.** Let $E = L_1$ and $\|u_\infty\|_{p_0} < \infty \text{ for } b > 1/2$ or let $E = B$ and $\|u_\infty\|_{p_0} < \infty \text{ for } b > 3/2$. Furthermore, assume the setup of Lemma 16. If $\alpha(u_\infty) = 0$, then the forcing
\[
F(t) = \begin{pmatrix} \Re F(t) \\ \Im F(t) \end{pmatrix}
\]
with $F(t) = (S_{0,t}^{E} u_\infty)(t)$ satisfies for $t \geq 0$ that
\[
|K_\Theta \ast F(t)| \leq \beta_\alpha(S_{0,t}^{E} u_\infty).
\]

**Proof.** By the previous lemma, we can estimate
\[
\left| \int_0^t \left( c_{\Theta, r} \Re (S_{0,t}^{E} u_\infty)(s) + c_{\Theta, i} \Im (S_{0,t}^{E} u_\infty)(s) \right) ds \right| = |\alpha(S_{0,t}^{E} u_\infty)| \leq \beta_\alpha(S_{0,t}^{E} u_\infty). \square
\]

The contribution of the stable part $r_{Lcs}$ can easily be controlled by Lemma 14.
5.2 Estimates on the nonlinear deviation to the Volterra kernel

Over a time range \( J = [0, T] \) the Volterra kernel \( k \) is a function on \( J \times J \) and satisfies \( k(t, s) = 0 \) for \( t < s \). Following Section 2 of Chapter 9 of [8], we introduce a suitable norm and convolution-like product.

For Volterra kernels we define the norm by

\[
\| k \|_{L^\infty(J, \phi)} := \sup_{t \in J} \int_{J} \| k(t, s) \| \phi(s) \|^{-1} ds,
\]

where \( \phi \) is a submultiplicative weight (i.e. \( \phi(t+s) \leq \phi(t) \phi(s) \)) and we let \( V(J, \phi) \) be the class of such functions. As in the formulation of (6), we generalise the convolution product between \( k \in V(J, \phi) \) and a function \( F \) on \( J \) as

\[
(k \ast F)(t) = \int_{0}^{t} k(t, s) F(s) ds
\]

and for \( \beta, \gamma \in V(J, \phi) \) we define the product

\[
(\beta \ast \gamma)(t, s) = \int_{\tau=s}^{t} \beta(t, \tau) \gamma(\tau, s) d\tau.
\]

The following lemma collects the basic properties.

**Lemma 20.** Let \( \beta, \gamma \in V(J, \phi) \). Then

\[
\| \beta \ast \gamma \|_{L^\infty(J, \phi)} \leq \| \beta \|_{L^\infty(J, \phi)} \| \gamma \|_{L^\infty(J, \phi)}.
\]

If \( \beta(t, s) = \beta_c(t-s) \) then

\[
\| \beta \|_{L^1(J, \phi)} \leq \| \beta_c \|_{L^1(J, \phi)}
\]

and for a function \( F \) on \( J \)

\[
\| (\beta \ast F)(t) \|_{L^\infty(J, \phi)} \leq \| \beta \|_{L^\infty(J, \phi)} \| F \|_{L^\infty(J, \phi)}.
\]

**Proof.** The inequalities follow directly from the submultiplicativity, see [8, Section 2 of Chapter 9].

In particular, this shows that the product defines a Banach algebra and we can hope that a small deviation can be handled by a series expansion, which will be done in Section 5.3.

In the remainder of this subsection, we prove the needed control.

**Lemma 21.** Let \( b_\eta \geq 0, b_d \geq 0, \) and \( b_r > b_\eta + 2 \) with

\[
\| r_r \|_{p_{\eta \sigma}} < \infty, \quad \| r_i \|_{p_{\eta \sigma}} < \infty, \quad \| r_\Theta \|_{p_{\eta \sigma}} < \infty,
\]

and

\[
\| r_r \|_{p_{\eta \sigma - \frac{1}{2}, 0}} < \infty, \quad \| r_i \|_{p_{\eta \sigma - \frac{1}{2}, 0}} < \infty, \quad \| r_\Theta \|_{p_{\eta \sigma - \frac{1}{2}, 0}} < \infty.
\]

Additionally assume one of the following conditions

- \( b_d > 5/2, \)
- \( b_r > b_\eta + 4 + \max\{0, 1 - b_d\} \) and \( b_r > b_\eta + 1/4 + \max\{0, 1 - b_d\} + \frac{\max\{0, 1 - b_d\}}{2} = \frac{b_\eta}{2}. \)
Then there exist constant $\delta_R$ and $C_Q$ such that for $J = [0, t]$ the deviation $k_Q = k - k_L$ is controlled as

$$\|k_Q\|_{L^\infty(J,p_{b_0})} \leq C_Q \sqrt{R_d(t)}$$

and

$$\|K\star k_Q\|_{L^\infty(J,p_{b_0})} \leq C_Q \sqrt{R_d(t)}$$

if $R_d(t) \leq \delta_R$.

**Remark 22.** The square root dependency of the bound on $R_d(t)$ can be improved. However, for the purpose of controlling the deviation it is sufficient.

We control the difference from the quadratic term by obtaining higher regularity estimates. The second part crucially depends on the fact, that $k_Q$ creates elements in the stable subspace due to the $B_2$ term in the evolution in $B$.

We start with a simple lemma controlling the norm.

**Lemma 23.** Let $b > 3/2$, $b_d \geq 0$ and $b_r \geq b$ with $\|r_\Theta\|_{p_{b_r}} < \infty$. Assume that $b_r > b + 1$ or $b_d > 5/2$, then there exist constants $\delta_R$ and $C$ such that

$$\|S_{s,t}^E v\|_{p_{1+\epsilon,b}} \leq C(1 + t - s)^{b_r} \|v\|_{p_0} \quad \text{with} \quad b_d = \max\left\{0, \frac{3}{2} - b_d\right\}$$

for $E = L_1$ or $E = B$ if $R_d(t) \leq \delta_R$.

**Proof.** The case $E = L_1$ follows directly from Lemma 4.

In the case $E = B$, we can chose $\delta_R$ small enough to apply Lemma 6 in order to control $\beta_d(S_{s,t}^E v)$. With this we find by Duhamel’s principle using Lemma 4 that

$$\|S_{s,t}^B v\|_{p_{1+\epsilon,b}} \leq \|v\|_{p_0} + 2C_d R_d(t) \int_s^t (1 + \tau)^{-b_d} (1 + \tau - s)^{\frac{3}{2} - b}\|v\|_{p_0} \|S_{\tau,t}^B r_\Theta\|_{p_{1+\epsilon,b}} d\tau.$$

We can bound

$$\|S_{\tau,t}^B r_\Theta\|_{p_{1+\epsilon,b}} \leq (1 + \tau - \tau)^{b_r} (1 + \tau - s)^b \|S_{\tau,t}^B r_\Theta\|_{p_{1+\epsilon,b}} \leq (1 + t - \tau)^{b_r} (1 + \tau - s)^b \|r_\Theta\|_{p_{b_r}},$$

where we used Lemma 4 again. Plugging in this bound gives the claimed result.

Using the transport part, we control the evolution with higher regularity.

**Lemma 24.** Let $b_0 \geq 0$, $b_d \geq 0$, $b \geq b_0 + 1/2$ with $b > 3/2$ and $b_r \geq b$ with $$\|r_\Theta\|_{p_{b_r}} < \infty \quad \text{and} \quad \|r_\Theta\|_{p_{b_r} - \frac{1}{4}} < \infty.$$

Assume that one of the following conditions holds

- $b_d > 5/2$,
- $b_r > b + 1$ and

$$b > b_0 + \frac{7}{4} + \frac{b_d - b_d}{2}$$

with $\tilde{b}_d$ from Lemma 23.

Then there exist constants $\delta_R$ and $C$ such that

$$\|S_{s,t}^E v\|_{p_{1+\epsilon,b_0},0} \leq C(\|v\|_{p_{b_0},0} + \|v\|_{p_b})$$

for $E = L_1$ and $E = B$ if $R_d(t) \leq \delta_R$.  

---

23
Proof. We present the proof for $E = B$ which is the more difficult part. The proof for $E = L_1$ follows from dropping the additional terms.

We start with showing a $L^2$ in time control. For this note that

$$\frac{d}{dt}\|S_{s \rightarrow t}^{B}v\|^2_{p_1 + \frac{1}{2}b_0 + \frac{1}{2}} \leq -\left(\frac{b+1}{2}\right)\|S_{s \rightarrow t}^{B}v\|^2_{p_1 + \frac{1}{2}b_0,0} + 2R_d(t)(1+t)^{-b_d}\beta_d(S_{s \rightarrow t}^{B}v)||r_\Theta||_{p_1 + \frac{1}{2}b_0 + \frac{1}{2}}\|S_{s \rightarrow t}^{B}v\|_{p_1 + \frac{1}{2}b_0 + \frac{1}{2}}. $$

By choosing $\delta_R$ small enough to apply Lemma 23, we control the growth term as

$$2R_d(t)(1+t)^{-b_d}\beta_d(S_{s \rightarrow t}^{B}v)||r_\Theta||_{p_1 + \frac{1}{2}b_0 + \frac{1}{2}}\|S_{s \rightarrow t}^{B}v\|_{p_1 + \frac{1}{2}b_0 + \frac{1}{2}} \leq C R_d(t)(1+t)^{-b_d}(1+t-s)^{-b_d+1}(1+t-s)^{b_0+1} \|r_\Theta\|_{p_0}^2$$

for a constant $C$ with $\delta_d$ from Lemma 23 if $R_d(t) \leq \delta_R$. By the assumptions, the term is integrable, so that there exists a constant $C$ such that

$$\int_s^t\|S_{s \rightarrow \tau}^{B}v\|^2_{p_1 + \frac{1}{2}b_0,0} d\tau \leq 2^{2b_0+1} \int_s^t\|S_{s \rightarrow \tau}^{B}v\|^2_{p_1 + \frac{1}{2}b_0,0} d\tau \leq C\|v\|_{p_0}^2$$

if $R_d(t) \leq \delta_R$.

Adapting the estimate from the proof of Lemma 12, we can convert the $L^2$ control to a pointwise control. From the estimate, we find that there exists a constant $C$ such that

$$\frac{d}{dt}\|S_{s \rightarrow t}^{B}v\|^2_{p_1 + \frac{1}{2}b_0,0} \leq C(r_{st} + |\phi(t)|)\|S_{s \rightarrow t}^{B}v\|^2_{p_1 + \frac{1}{2}b_0,0} + R_d(t)(1+t)^{-b_d}\beta_d(S_{s \rightarrow t}^{B}v)||r_\Theta||_{p_1 + \frac{1}{2}b_0,0}\|S_{s \rightarrow t}^{B}v\|_{p_1 + \frac{1}{2}b_0,0}. $$

Here $\beta_d(S_{s \rightarrow t}^{B}v)$ can again be controlled by $2C\beta_d(1+t)^{-b_d}\|v\|_{p_0}$ using Lemma 6. With this the second term can be controlled with a constant $C$ as

$$R_d(t)(1+t)^{-b_d}\beta_d(S_{s \rightarrow t}^{B}v)||r_\Theta||_{p_1 + \frac{1}{2}b_0,0}\|S_{s \rightarrow t}^{B}v\|_{p_1 + \frac{1}{2}b_0,0} \leq R_d(t)\|S_{s \rightarrow t}^{B}v\|^2_{p_1 + \frac{1}{2}b_0,0} + C R_d(t)(1+t)^{-2b_d}(1+t-s)^{2b_0}\|v\|_{p_0}^2(1+t-s)^{2b_0}\|r_\Theta\|_{p_0}^2.$$ 

The second term is in both cases integrable. This follows form the exponents as, in the case $b_d > 5/2$, we find

$$-2b_d + 3 - 2b + 2b_0 \leq -2b + 2 \leq -3$$

and in the other case

$$-2b_d + 3 - 2b + 2b_0 < -\frac{1}{2} - b_d - b_d \leq -2.$$

Therefore, there exists a constant constant $C$ such that

$$\|S_{s \rightarrow t}^{B}v\|^2_{p_1 + \frac{1}{2}b_0,0} \leq C\|v\|_{p_0}^2 + C \int_s^t\|S_{s \rightarrow \tau}^{B}v\|^2_{p_1 + \frac{1}{2}b_0,0} d\tau.$$ 

The first control then shows the claimed result. $\square$

The deviation kernel can be expressed as

$$k_Q = \begin{pmatrix} \Re k_{Q,r}(t,s) & \Re k_{Q,t}(t,s) \\ \Im k_{Q,r}(t,s) & \Im k_{Q,t}(t,s) \end{pmatrix}$$

with

$$k_{Q,r}(t,s) = (S_{s \rightarrow t}^{B}r_r - S_{s \rightarrow t}^{L_1}r_r)(0) \quad \text{and} \quad k_{Q,t}(t,s) = (S_{s \rightarrow t}^{B}r_t - S_{s \rightarrow t}^{L_1}r_t)(0).$$

As a last preparation step, we adapt Lemma 18.
Lemma 25. Let \( b > 3/2 \) and \( \|r_\Theta\|_{p_0} < \infty \) and \( \|v\|_{p_0} < \infty \). Then for \( t \geq s \) it holds that

\[
\alpha(S_{s \rightarrow t}^Bv - S_{s \rightarrow t}^L_1) + \int_0^t \left( c_{\Theta,j} \Re(S_{s \rightarrow t}^Bv - S_{s \rightarrow t}^L_1) + c_{\Theta,j} \Im(S_{s \rightarrow t}^Bv - S_{s \rightarrow t}^L_1) \right) ds = 0
\]

Proof. As in Lemma 18, note that

\[
\frac{d}{dt} \alpha(S_{s \rightarrow t}^Bv - S_{s \rightarrow t}^L_1) = \alpha \left[ L_1(S_{s \rightarrow t}^Bv - S_{s \rightarrow t}^L_1) \right]
\]

from where the result follows again. \( \square \)

Finally, we can prove the needed control.

Proof of Lemma 21. Fix the initial time \( s \) and denote the two evolutions as

\[
v(t) = S_{s \rightarrow t}^Bfr_r, \quad w(t) = S_{s \rightarrow t}^Lfr_r.
\]

By the assumptions, we can choose \( b_0 > b_\eta + 3/2 + \max\{0, 3/2 - b_d\} \) and \( b \geq b_0 + 1/2 \) and \( b_r \geq b \) satisfying the conditions of Lemma 24 and \( b \geq b_0 + 3/2 \) in the case \( b_d \leq 5/2 \). Hence there exist constants \( \delta_R \) and \( C \) such that

\[
\begin{align*}
\|v(t)\|_{p_1 + t - s, b_\eta, 0} & \leq C, \\
\|w(t)\|_{p_1 + t - s, b_\eta, 0} & \leq C, \\
\beta_d(v(t)) & \leq (1 + t - s)^{3/2 - b}C
\end{align*}
\]

if \( R_d(t) \leq \delta_R \).

From the definition

\[
\partial_t(v(t) - w(t)) = L_1(v(t) - w(t)) + B_{1n}(v) + B_2(v).
\]

Therefore, we find

\[
\begin{align*}
\frac{d}{dt}\|v(t) - w(t)\|^2_{p_1 + t - s, b_\eta} & \leq 2R_d(t)(1 + t)^{-b_d}\|v(t)\|_{p_1 + t - s, b_\eta, 0}\|v(t) - w(t)\|_{p_1 + t - s, b_\eta, 0} \\
& \quad + 2\beta_d(v(t))R_d(t)(1 + t)^{-b_d}\|r_\Theta\|_{p_1 + t - s, b_\eta, 0}\|v(t) - w(t)\|_{p_1 + t - s, b_\eta, 0}.
\end{align*}
\]

Using the control of \( v \) and \( w \), we find a constant \( C \) such that

\[
\frac{d}{dt}\|v(t) - w(t)\|^2_{p_1 + t - s, b_\eta} \leq CR_d(t)(1 + t)^{-b_d}\left[1 + (1 + t - s)^{3/2 - b}(1 + t - s)^{b_\eta}\right]
\]

If \( b_d > 5/2 \), then the RHS is integrable and we find

\[
\|v(t) - w(t)\|_{p_1 + t - s, b_\eta} \leq C\sqrt{R_d(t)}
\]

for a constant \( C \) if \( R_d(t) \leq \delta_R(t) \).

If \( b_d \leq 5/2 \), then by the choice of \( b \) we have \((1 + t - s)^{3/2 - b}(1 + t - s)^{b_\eta} \leq 1\), so that

\[
\|v(t) - w(t)\|_{p_1 + t - s, b_\eta} \leq C(1 + t - s)^{\max\{1 - b_d, 0\} \sqrt{R_d(t)}}
\]

for a constant \( C \) if \( R_d(t) \leq \delta_R(t) \).
By Lemma 5, this gives us directly the pointwise control
\[ |k_Q,r(t,s)| \leq \beta_0|v(t) - w(t)| \leq C(1 + t - s)^{-b_0 + \max\{1-b_d,0\}} \sqrt{R_d(t)} \]
and
\[ \beta_0(\tilde{v}(t) - w(t)) \leq C(1 + t - s)^{1-b_0 + \max\{1-b_d,0\}} \sqrt{R_d(t)}. \] (17)

Exactly, in the same way the same bounds hold for \( k_{Q,i} \).

The first part of the lemma follows by integration for \( t \geq s \) as
\[ (1 + t)^{b_0} \int_0^t \|k_Q(t,s)\|(1 + s)^{-b_0} \, ds \leq C_0 \sqrt{R_d(t)} \int_0^t (1 + t - s)^{b_0 - b_0 + \max\{1-b_d,0\}} \, ds, \]
which is uniformly bounded.

For the effect on the rotation eigenmode, we find explicitly
\[ (K_{\Theta} \ast k_Q)(t,s) = -\left( \frac{0}{1} \right) \int_s^t \left( c_{\Theta,r} \quad c_{\Theta,i} \right) \begin{pmatrix} \Re(S_{s_{\rightarrow t}^+}^{L_1} r_r - S_{s_{\rightarrow t}^-}^{L_1} r_r) \beta_{11}(0) & \Re(S_{s_{\rightarrow t}^+}^{L_1} r_r - S_{s_{\rightarrow t}^-}^{L_1} r_r) \beta_{12}(0) & \Re(S_{s_{\rightarrow t}^+}^{L_2} r_r - S_{s_{\rightarrow t}^-}^{L_2} r_r) \beta_{11}(0) & \Re(S_{s_{\rightarrow t}^+}^{L_2} r_r - S_{s_{\rightarrow t}^-}^{L_2} r_r) \beta_{12}(0) \end{pmatrix} d\tau. \]

The integral can be identified with Lemma 25 and controlled by (17) as
\[ \|(K_{\Theta} \ast k_Q)(t,s)\| \leq C(1 + t - s)^{\frac{1}{2} - b_0 + \max\{1-b_d,0\}} \sqrt{R_d(t)}. \]

By the choice of \( b_0 \), this bound gives the claimed control on \( K_{\Theta} \ast k_Q \).

5.3 Nonconvolution bound

We control the nonconvolution Volterra equation (7) for small nonlinear contributions as perturbation of the linearised evolution. In this case we can solve the Volterra equation again using a resolvent \( r \) satisfying
\[ r + k \ast r = r + r \ast k = k. \] (18)

The resolvent \( r_{Lc} \) is not in \( L^1(\mathbb{R}, p_{b_0}) \) due to the rotation eigenmode contribution \( K_{\Theta} \). However, we can circumvent the problem by only using \( r_L \ast k_Q \), which is better behaved, because the kernel \( k_Q \) only creates contributions in the stable subspace. Adapting Lemma 3.7 of Chapter 9 of [8] to the case of an eigenmode, we find the resolvent.

**Lemma 26.** Let \( b_0 \geq 0 \) and let \( k_{Lc} \) be a convolution kernel with \( k_{Lc} \in L^1(\mathbb{R}^+, p_{b_0}) \) and resolvent
\[ r_{Lc} = K_{\Theta} + r_{Lcs} \]
where \( K_{\Theta} \) is a constant matrix and \( r_{Lcs} \in L^1(\mathbb{R}^+, p_{b_0}) \). Let \( k_L \) and \( k_{Ls} \) be the Volterra kernels corresponding to the convolution kernels \( k_{Lc} \) and \( k_{Ls} \), respectively.

Assume a Volterra kernel \( k_Q \) for a time range \( J = [0, T] \) satisfying
\[ \|k_Q - r_L \ast k_Q\|_{L^\infty(J,p_{b_0})} < 1. \]

Then \( k = k_L + k_Q \) has a resolvent \( r \) for the time range \( J \) of the form
\[ r = K_{\Theta} + r_Q \ast K_{\Theta} + r_s, \]
where
\[ \|r_Q\|_{L^\infty(J,p_{b_0})} \leq \frac{\|k_Q - r_L \ast k_Q\|_{L^\infty(J,p_{b_0})}}{1 - \|k_Q - r_L \ast k_Q\|_{L^\infty(J,p_{b_0})}} \]
and
\[ \|r_s\|_{L^\infty(J,p_{b_0})} \leq \frac{\|k_Q - r_L \ast k_Q + r_{Ls}\|_{L^\infty(J,p_{b_0})}}{1 - \|k_Q - r_L \ast k_Q\|_{L^\infty(J,p_{b_0})}}. \]
Proof. We define \( r \) by
\[
r = (k - r_L * k) + \left( \sum_{n=1}^{\infty} (-1)^n (k_Q - r_L * k_Q)^n \right) * (k - r_L * k),
\]
which is an absolutely converging sum by the assumed bound. Moreover, we note that
\[
k - r_L * k = K_\Theta + (k_Q - r_L * k_Q + r_{Ls}),
\]
so that \( r \) has the claimed form with the bounds of \( r_Q \) and \( r_s \).

In order to show that \( r \) satisfies (18), we first note that multiplying by \( k_Q - r_L * k_Q \) from the left shows that
\[
(k_Q - r_L * k_Q) * r = -r + k - r_L * k.
\]
(19)

Multiplying by \( k_L \) from the left shows together for the resolvent equation for the kernel \( k_L \) that
\[
(k_L + r_L * k_Q) * r = r_L * k.
\]
(20)

Combining (19) and (20) then gives
\[
r + k \star r = k.
\]

For the other part, note that from the definition of \( r \) it follows that
\[
r + r \star k_L = k_L - \left( \sum_{n=1}^{\infty} (-1)^n (k_Q - r_L * k_Q)^n \right).
\]
(21)

Multiplying with \( k_Q - r_L * k_Q \) then shows that
\[
(r + r \star k_L) * (k_Q - r_L * k_Q) = k_L * (k_Q - r_L * k_Q) + \left( \sum_{n=1}^{\infty} (-1)^n (k_Q - r_L * k_Q)^n \right) + (k_Q - r_L * k_Q).
\]
Replacing the sum by (21) and rearranging then shows the required equality.
\[
r + r \star k = k.
\]

Combining this result with the bound on \( k_Q \), we can prove the result for the Volterra equation.

Proof of Lemma 9. By Proposition 15, the linear convolution kernel \( k_{Lc} \) has a resolvent
\[
r_{Lc} = K_\Theta + r_{Lcs},
\]
with \( \| r_{Lcs} \|_{L^1(\mathbb{R}^+, p_{\eta})} < \infty \).

For the nonlinear behaviour, we fix the time range \( J = [0, t] \) and assume that \( R_d(t) \leq \delta_R \), where \( \delta_R \) is chosen small enough so that Lemma 21 implies
\[
\| k_Q \|_{L^\infty(J, p_{\eta})} \leq \frac{2 + \| r_{Lcs} \|_{L^1(\mathbb{R}^+, p_{\eta})}}{4},
\]
\[
\| K_\Theta * k_Q \|_{L^\infty(J, p_{\eta})} \leq \frac{2 + \| r_{Lcs} \|_{L^1(\mathbb{R}^+, p_{\eta})}}{4}.
\]

Then Lemma 20 implies that
\[
\| k_Q - r_L * k_Q \|_{L^\infty(J, p_{\eta})} \leq \left( 1 + \| r_{Ls} \|_{L^\infty(J, p_{\eta})} \right) \| k_Q \|_{L^\infty(J, p_{\eta})} + \| K_\Theta * k_Q \|_{L^\infty(J, p_{\eta})} \leq \frac{1}{2}.
\]

27
Therefore, Lemma 26 shows that the kernel $k$ has the resolvent
\[
 r = K_\Theta + r_Q \star K_\Theta + r_s,
\]
where
\[
 \|r_Q\|_{L^\infty(J,p_{\eta_0})} \leq 1 \quad \text{and} \quad \|r_s\|_{L^\infty(J,p_{\eta_0})} \leq 1 + 2\|r_{L^\infty_c}(\mathbb{R}^+p_{\eta_0})\|.
\]
By Theorem 3.6 of Chapter 9 of [8], the Volterra equation (7) then has over the time range $J$ the unique solution
\begin{equation}
 \begin{pmatrix} \Re \eta \\ \Im \eta \end{pmatrix} = F - r \star F.
\end{equation}
This follows by elementary calculations from (18), which we repeat here. Indeed (22) defines a solution, because
\[
 F - r \star F + k \star (F - r \star F) = F + (k - r - k \star r) \star F = F.
\]
On the other hand for a solution
\[
 \begin{pmatrix} \Re \eta \\ \Im \eta \end{pmatrix} + k \star \begin{pmatrix} \Re \eta \\ \Im \eta \end{pmatrix} = F,
\]
we find by multiplying from the left with $r$ that
\[
 k \star \begin{pmatrix} \Re \eta \\ \Im \eta \end{pmatrix} = r \star F,
\]
which shows that the solution has the claimed form (22).

By Corollary 19, we find
\[
 \|K_\Theta \star F\|_{L^\infty(J,p_\eta)} \leq \sup_{s \in J} \beta_{\alpha}(S_{0,t}^B u_{in})(1 + s)^{b_\eta}.
\]
Hence by Lemma 20,
\[
 \|r \star F\|_{L^\infty(J,p_\eta)} \leq \left(1 + \|r_Q\|_{L^\infty(J,p_{\eta_0})}\right)\|K_\Theta \star F\|_{L^\infty(J,p_\eta)} + \|r_s\|_{L^\infty(J,p_{\eta_0})}\|K_\Theta \star F\|_{L^\infty(J,p_\eta)},
\]
which is the claimed control. □

6 Bootstrap argument

The obtained estimates allow us to control $\eta(t)$.

Proof of Lemma 10. First we chose $\delta_R$ small enough to apply Lemma 6 to conclude under $R_d(t) \leq \delta_R$ that
\[
 \beta_d(S_{0,t}^B u_{in})(1 + t)^{-\frac{d}{2} + b} \leq 2C_{\beta_d} \|u_{in}\|_{p_0}.
\]

With this control we find by Lemma 6
\[
 \beta_\alpha(S_{0,t}^B u_{in})(1 + t)^{-\frac{d}{2} + b} \leq \left(2b - 1\right)^{-1/2}\|K_\Theta\|\|u_{in}\|_{p_0}
 + \frac{2C_{\beta_d}}{\sqrt{2b - 1}}\|r_\Theta\|_{p_0}\|K_\Theta\|\|R_d(t)(1 + t)^{-\frac{d}{2} + b}\| \int_0^t (1 + t - s)^{-b_\alpha}(1 + s)^{-b_d+\frac{d}{2}} ds.
\]
Here we find
\[
(1+t)^{-\frac{1}{2}+b} \int_0^t (1+t-s)^{-\frac{1}{2}-br} (1+s)^{-bd+\frac{1}{2}-b} ds \leq \int_0^t (1+t-s)^{b-br} (1+s)^{1-bd} ds,
\]
which is bounded uniformly over \( t \), as \( b_d > 1 \) and \( b_r > b + 1 \).
Likewise, it holds that
\[
\beta_\eta (S_{0\to t}^B u_{in}) (1+t)^{-\frac{1}{2}+b} \leq C_S \|u_{in}\|_{p_b}
\]
\[
+ 2C_\beta d C_S \|r_{\eta \delta}\|_{p_b_r} R_d(t)(1+t)^{-\frac{1}{2}+b} \int_0^t (1+t-s)^{-br} (1+s)^{-bd+\frac{1}{2}-b} ds,
\]
which can be controlled in the same way. Hence there exists a constant \( C \) such that \( F(t) = (S_{0\to t}^B u_{in})^1(0) \) satisfies
\[
\sup_{s \in [0,t]} \left( |F(s)| + \beta_\alpha (S_{0\to s}^B u_{in}) (1+s)^{b_d} \right) \leq C \|u_{in}\|_{p_b}
\]
if \( R_d(t) \leq \delta_R \).
Decreasing \( \delta_R \) if needed, we apply Lemma 9 for the Volterra equation with \( b_\eta = b - 1/2 \) and obtain the result.

With the order parameter, we can control the full solution.

**Proof of Lemma 11.** By (6) we have
\[
\beta_d(u(t)) \leq \beta_d(S_{0\to t}^B u_{in}) + \int_0^t \beta_d(S_{s\to t}^B L_2u(s)) ds.
\]
Recall that
\[
L_2u(s) = 3\eta(s) r_r + 3\eta(s) r_s
\]
By choosing \( \delta_R \) small enough, Lemma 6 implies under \( R_d(t) \leq \delta_R \) that
\[
\beta_d(u(t)) \leq 2C_\beta d (1+t)^{\frac{1}{2}-b} \|u_{in}\|_{p_b} + 2C_\beta d C_r \int_0^t (1+t-s)^{\frac{1}{2}-br} (1+s)^{-bd} ds \sup_{s \in [0,t]} (1+s)^{b_d} |\eta(s)|,
\]
where
\[
C_r = \|r_r\|_{p_{r_r}} + \|r_s\|_{p_{r_s}}.
\]
As \( b - b_r < -1 \) and \(-3/2 + b - b_d = -1\), there exists a constant \( C \) such that for all time \( t \) it holds that
\[
(1+t)^{-\frac{1}{2}+b} \int_0^t (1+t-s)^{\frac{1}{2}-b_r} (1+s)^{b_d} ds \leq \int_0^t (1+t-s)^{b-br} (1+s)^{-\frac{1}{2}+b_d} \leq C,
\]
which shows the claimed control.

In preparation of the bootstrap argument, we prove a well-posedness result.

**Lemma 27.** Let \( b \geq 0 \) and \( f_{in} \) be initial data. Assume that the velocity marginal \( \hat{g} = (f_{in})_0 \) satisfies \( \|\hat{g}\|_{p_b} < \infty \) and that the restriction to \( t \geq 1 \) is \( f_{in} \in \mathcal{X}_{p_b} \). Then for any time \( T > 0 \), there exists a global unique solution \( f \in C_w([0,T], \mathcal{X}_{p_b}) \) to (3) with initial data \( f_{in} \) and the constant velocity marginal \( \hat{g} \).
Proof. Given a solution $\hat{f} \in C_w([0, T], X_p)$, we find that $\hat{f}$ is a continuous function of time $t$ and frequency $\xi$ by Morray’s inequality. Hence Theorem 15 in [4] applies to show uniqueness.

The existence can be proven similar to Lemma 3.2 of [5]. The key point is the a priori estimate

$$\partial_t \|\hat{f}\|_{p_b}^2 \leq K\|\hat{f}(0)\|_p \|\hat{f}\|_{p_b},$$

similar to the estimate in the proof of Lemma 3. By the Sobolev embedding used in Lemma 5, this shows that there exists a constant $C$ such that

$$\partial_t \|\hat{f}\|_{p_b}^2 \leq C\|\hat{f}\|_{p_b}^2.$$ 

As in [5], we can build approximate solutions $\hat{f}_n$ satisfying this bound by restricting to the spatial modes $\ell \in [1, \ldots, n]$ and compact smooth initial data.

Looking at $\ell = 1$ and $\xi \in [0, 1]$, the estimate shows that $\hat{f}_n \in L^\infty([0, T], H^1([0, 1]))$ and $\partial_t \hat{f}_n \in L^\infty([0, T], L^2([0, 1]))$. By the Aubin-Lions Lemma, we extract a subsequence for which $\hat{f}_n(\cdot, 0)$ converges strongly in $L^\infty([0, T])$. By the weak compactness, we can extract a further subsequence converging to a weak solution.

For the initial data $\hat{f}_i$, we show that close to $\hat{f}_{st}$ it can be written in the required polar coordinate form.

Lemma 28. Given a stationary state $\hat{f}_{st}$ and $b > 3/2$. Then there exist $\delta_{in}$ and $\delta_{\Theta}$ such that for $\hat{f}_i$ with $\|\hat{f}_i - \hat{f}_{st}\|_{p_b} < \delta_{in}$, there exists a unique $\Theta \in (-\delta_{\Theta}, \delta_{\Theta})$ such that $u = R_{-\Theta} \hat{f}_i - \hat{f}_{st}$ satisfies

$$\alpha(u) = 0.$$ 

Moreover, there exists a constant $C$ such that

$$|\Theta| \leq C\|\hat{f}_i - \hat{f}_{st}\|_{p_b}$$

and

$$\|u\|_{p_b} \leq \|\hat{f}_i - \hat{f}_{st}\|_{p_b} + |\Theta| \|\hat{f}_{st}\|_{p_b,0}.$$ 

Proof. Define the function $F : \mathbb{T} \times X_{p_b} \mapsto \mathbb{R}$ by

$$F(\Theta, \hat{f}_i) = \alpha(R_{-\Theta} \hat{f}_i - \hat{f}_{st}).$$

By Lemma 5 the function $F$ is continuously differentiable and

$$\partial_{\Theta} F(0, \hat{f}_{st}) = -\alpha(D\hat{R}_{\hat{f}_{st}}) \neq 0.$$ 

Hence by the implicit function theorem, there exists a unique inverse in the neighbourhood of $\hat{f}_{st}$ with the given control.

For the control on $u$ note that

$$\|u\|_{p_b} = \|\hat{R}_{\Theta} u\|_{p_b} \leq \|\hat{f}_i - \hat{f}_{st}\|_{p_b} + \|\hat{R}_{\Theta} \hat{f}_{st} - \hat{f}_{st}\|_{p_b}.$$ 

As $|e^{i\ell \Theta} - 1| \leq \ell|\Theta|$, the second term can be bounded as

$$\|\hat{R}_{\Theta} \hat{f}_{st} - \hat{f}_{st}\|_{p_b} \leq |\Theta| \|\hat{f}_{st}\|_{p_b,0},$$

which is the claimed result.

With this we can assemble the proof of the main theorem.
Proof of Theorem 2. The conditions on $r_r$ and $r_i$ imply that $\|\hat{g}\|_{p_b} < \infty$. By Lemma 27, there exists a global weak solution $f$, which is locally bounded in $X_{p_b}$ and weakly continuous.

By Lemma 28, we can choose $\delta$ small enough so that there exists an initial angle $\Theta(0)$ such that

$$|\Theta(0)| \leq C \| f_{in} - \hat{f}_{st} \|_{p_b}$$

and for a constant $C_f$

$$\| u_{in} \|_{p_b} \leq C_f \| f_{in} - f_{st} \|_{p_b},$$

as the bound of $\|r\|_{p_b}$ implies the control of $\|f_{st}\|_{p_b,0}$.

By setting $u = R_{-\Theta} \hat{f} - f_{st}$ and evolving $\Theta$ by

$$\frac{d}{dt} \Theta = \dot{\Theta}$$

given by (5), we find an evolution for $u$ and $\Theta$ as long as

$$\beta_d(u(t)) \leq \frac{1}{2} |\alpha(D \hat{R} \hat{f}_{st})|,$$

(23)

because $|\alpha(D \hat{R} u)| \leq \beta_d(u(t))$. Under this assumption, $u$ is locally bounded and weakly continuous, and $\dot{\Theta}$ is continuous. Moreover, if (23) holds up to a time $t$, the solution can be extended by a positive amount of time.

Then $u$ is given by (6), because the evolution under $B$ has a unique solution and $L_2$ is a bounded operator. With the control $R_d$ on the coefficients, the estimates from Lemmas 10 and 11 show that there exist constants $\delta_R$ and $C$ such that

$$|\eta(t)| \leq C (1 + t)^{-b_d} \| u_{in} \|_{p_b},$$

$$|\beta_d(u(t))| \leq C (1 + t)^{\frac{a}{2} - b} \| u_{in} \|_{p_b}$$

if $R_d(t) \leq \delta_R$. This in particular implies the existence of a constant $C_Q$ with

$$|\alpha(Q(u(t)))| \leq K |\eta(t)| \beta_d(u(t)) \leq C_Q (1 + t)^{\frac{a}{2} - b - b_d} \| u_{in} \|_{p_b}^2$$

and the existence of $\delta_\Theta$ such that

$$\beta_d(u(t)) \leq \frac{1}{4} |\alpha(D \hat{R} \hat{f}_{st})|$$

if $R(t) \leq \delta_R$ and $\| u_{in} \| \leq \delta_\Theta$.

Therefore, we can find a small enough $\delta$ such that for $\| u_{in} \| \leq \delta$, the bootstrap assumption $R_d(t) \leq \delta_R$ implies

$$\beta_d(u(t)) \leq \frac{1}{4} |\alpha(D \hat{R} \hat{f}_{st})|$$

and

$$R_d(t) \leq \frac{1}{2} \delta_R.$$

By the continuity of the solution and the local well-posedness, this implies the existence of a global solution $u$ and $\Theta$ satisfying $R_d(t) \leq \delta_R$ for all times $t$. Then, in particular, the stated bounds on $\eta(t)$ and $\dot{\Theta}$ hold.

\[ \square \]
Acknowledgements

The author would like to thank David Gérard-Varet and Bastien Fernandez for the helpful discussions during the work.

The author thankfully acknowledges support by the ANR Chaire d’Excellence ANR-11-IDEX-005 and the People Programme (Marie Curie Actions) of the European Union’s Seventh Framework Programme (FP7/2007-2013) under REA grant agreement n. POCOFUND-GA-2013-609102, through the PRESTIGE programme coordinated by Campus France.

References

[1] Dario Benedetto, Emanuele Caglioti, and Umberto Montemagno. “Exponential dephasing of oscillators in the kinetic Kuramoto model”. In: J. Stat. Phys. 162.4 (2016), pp. 813–823. DOI: 10.1007/s10955-015-1426-3.

[2] W. Braun and K. Hepp. “The Vlasov dynamics and its fluctuations in the 1/N limit of interacting classical particles”. In: Communications in Mathematical Physics 56.2 (June 1977), pp. 101–113. DOI: 10.1007/bf01611497.

[3] Hayato Chiba. “A proof of the Kuramoto conjecture for a bifurcation structure of the infinite-dimensional Kuramoto model”. In: Ergodic Theory Dynam. Systems 35.3 (2015), pp. 762–834. DOI: 10.1017/etds.2013.68.

[4] Helge Dietert. “Stability and bifurcation for the Kuramoto model”. In: J. Math. Pures Appl. (9) 105.4 (2016), pp. 451–489. DOI: 10.1016/j.matpur.2015.11.001.

[5] Helge Dietert, Bastien Fernandez, and David Gérard-Varet. Landau damping to partially locked states in the Kuramoto model. 2016. arXiv: 1606.04470v1 [math.AP].

[6] R. L. Dobrushin. “Vlasov equations”. In: Functional Analysis and Its Applications 13.2 (1979), pp. 115–123. DOI: 10.1007/bf01077243.

[7] Bastien Fernandez, David Gérard-Varet, and Giambattista Giacomin. “Landau Damping in the Kuramoto Model”. In: Ann. Henri Poincaré 17.7 (2016), pp. 1793–1823. DOI: 10.1007/s00023-015-0450-9.

[8] G. Gripenberg, S.-O. Londen, and O. Staffans. Volterra integral and functional equations. Vol. 34. Encyclopedia of Mathematics and its Applications. Cambridge University Press, Cambridge, 1990, pp. xxii+701. DOI: 10.1017/CBO9780511662805.

[9] G. S. Jordan, Olof J. Staffans, and Robert L. Wheeler. “Local analyticity in weighted $L^1$-spaces and applications to stability problems for Volterra equations”. In: Transactions of the American Mathematical Society 274.2 (Feb. 1982), pp. 749–749. DOI: 10.1090/s0002-9947-1982-0675078-8.

[10] Y. Kuramoto. Chemical oscillations, waves, and turbulence. Vol. 19. Springer Series in Synergetics. Springer-Verlag, Berlin, 1984, pp. viii+156. DOI: 10.1007/978-3-642-69689-3.

[11] Yoshiki Kuramoto. “Self-entrainment of a population of coupled non-linear oscillators”. In: International Symposium on Mathematical Problems in Theoretical Physics (Kyoto Univ., Kyoto, 1975). Springer, Berlin, 1975, 420–422. Lecture Notes in Phys., 39.

[12] Carlo Lancellotti. “On the Vlasov limit for systems of nonlinearly coupled oscillators without noise”. In: Transport Theory Statist. Phys. 34.7 (2005), pp. 523–535. DOI: 10.1080/00411450508951152.

[13] R. Mirollo and S.H. Strogatz. “The Spectrum of the Partially Locked State for the Kuramoto Model”. In: Journal of Nonlinear Science 17.4 (June 2007), pp. 309–347. DOI: 10.1007/s00332-006-0806-x.
[14] H. Neunzert. “An introduction to the nonlinear Boltzmann-Vlasov equation”. In: Lecture Notes in Math. 1048 (1984), pp. 60–110. doi: 10.1007/BFb0071878.

[15] Oleh E. Omel’chenko and Matthias Wolfrum. “Bifurcations in the Sakaguchi-Kuramoto model”. In: Phys. D 263 (2013), pp. 74–85. doi: 10.1016/j.physd.2013.08.004.

[16] Hidetsugu Sakaguchi. “Cooperative phenomena in coupled oscillator systems under external fields”. In: Progr. Theoret. Phys. 79.1 (1988), pp. 39–46. doi: 10.1143/PTP.79.39.

[17] Walter Strauss. “On continuity of functions with values in various Banach spaces”. In: Pacific Journal of Mathematics 19.3 (Dec. 1966), pp. 543–551. doi: 10.2140/pjm.1966.19.543.

[18] Steven H. Strogatz. “From Kuramoto to Crawford: exploring the onset of synchronization in populations of coupled oscillators”. In: Phys. D 143.1-4 (2000). Bifurcations, patterns and symmetry, pp. 1–20. doi: 10.1016/S0167-2789(00)00994-4.

[19] Steven H. Strogatz and Renato E. Mirollo. “Stability of incoherence in a population of coupled oscillators”. In: J. Statist. Phys. 63.3-4 (1991), pp. 613–635. doi: 10.1007/BF01029202.

[20] Steven H. Strogatz, Renato E. Mirollo, and Paul C. Matthews. “Coupled nonlinear oscillators below the synchronization threshold: relaxation by generalized Landau damping”. In: Phys. Rev. Lett. 68.18 (1992), pp. 2730–2733. doi: 10.1103/PhysRevLett.68.2730.