SEIFERT FIBERED FOUR-MANIFOLDS WITH NONZERO
SEIBERG-WITTEN INVARIANT

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Abstract. The main result of this paper asserts that if a Seifert fibered 4-manifold
has nonzero Seiberg-Witten invariant, the homotopy class of regular fibers has infi-
tinite order. This is a nontrivial obstruction to smooth circle actions; as applications,
we show how to destroy smooth circle actions on a 4-manifold by knot surgery, with-
out changing the integral homology, intersection form, and even the Seiberg-Witten
invariant. Results concerning classification of Seifert fibered complex surfaces or
symplectic 4-manifolds are included. We also show that every smooth circle action
on the 4-torus is smoothly conjugate to a linear action.

1. Introduction

A 4-manifold is called Seifert fibered if it admits a smooth, fixed-point free circle
action. As such, the 4-manifold is given as the total space of a circle bundle over a 3-
dimensional orbifold, whose singular set consists of at most disjoint embedded circles.
An oriented Seifert fibered 4-manifold must have zero Euler number and signature,
which seems to be the only known obstructions. For basic results concerning smooth
circle actions on 4-manifolds, we refer the reader to [15, 16].

In this paper we derive new obstructions for Seifert fibrations on 4-manifolds which
have nontrivial Seiberg-Witten invariant. Recall that for an oriented 4-manifold $M
with $b^+_2 > 1$, the Seiberg-Witten invariant of $M$ is a map $SW_M$ that assigns to each
$Spin^c$-structure $L$ of $M$ an integer $SW_M(L)$ which depends only on the diffeomorphism
class of $M$ (cf. [31]). In the case of $b^+_2 = 1$, the definition of $SW_M(L)$ requires an
additional choice of orientation of the 1-dimensional space $H^{2,+}(M, \mathbb{R})$, as discussed in
Taubes [36]. A convention throughout this paper is that when we say $M$ has nonzero
Seiberg-Witten invariant in the case of $b^+_2 = 1$, it is meant that the map $SW_M$ has
nonzero image in $\mathbb{Z}$ for some choice of orientation of $H^{2,+}(M, \mathbb{R})$. Note that under
this convention, every symplectic 4-manifold has nonzero Seiberg-Witten invariant by
the work of Taubes [34].

Baldridge in [3] showed how to compute the Seiberg-Witten invariant of a Seifert
fibered 4-manifold in terms of the base 3-orbifold. We observe that, through a de-
singularization formula of Seiberg-Witten invariants of 3-orbifolds whose details are
given in a separate paper [10], the non-vanishing of Seiberg-Witten invariant yields
topologically interesting informations about the base 3-orbifold. For the purpose here,
we paraphrase the said de-singularization formula as follows:

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Let $Y$ be a closed, oriented $3$-orbifold with $b_1 > 1$, and let $Y_0$ be the oriented $3$-orbifold obtained from $Y$ by removing an embedded circle from its singular set. Then $Y$ has nonzero Seiberg-Witten invariant if and only if $Y_0$ does.

The center of a group $G$ will be denoted by $z(G)$.

1.1. **Topological constraints.** Among the new constraints of Seifert fibered $4$-manifolds discussed in this paper, the central result, as stated in the following theorem, asserts that the fundamental group of the $4$-manifold has infinite center. The class of Seifert fibered $4$-manifolds whose $\pi_1$ has infinite center is further studied in [11], where the main tools are from $3$-dimensional topology centered around the recently proved Thurston’s Geometrization Conjecture (cf. [7, 32]).

**Theorem 1.1.** Let $X$ be an oriented $4$-manifold with $b_2^+ \geq 1$ and nonzero Seiberg-Witten invariant, and let $\pi : X \to Y$ be any Seifert fibration. Then (1) the homotopy class of a regular fiber of $\pi$, which lies in $z(\pi_1(X))$, has infinite order, and (2) if $b_2^+ > 1$, the Hurwitz homomorphism $\pi_2(X) \to H_2(X)$ has finite image.

In the course of the proof, we will show that $Y$ has a regular, finite manifold cover, $Y = \tilde{Y}/G$. The pull-back of $\pi$ to $\tilde{Y}$, denoted by $\tilde{X}$, is a circle bundle over a $3$-manifold, which finitely covers $X$.

It can be shown that the $4$-manifold $X$ in Theorem 1.1 is minimal, and moreover, $z(\pi_1(X))$ is finitely generated and torsion-free (details are given in [11]).

**Remarks:** (1) A special case of Theorem 1.1(1), where $X$ is symplectic and the circle action is free, was due to Kotschick [24] (see also [8]) using a different argument. As already hinted in [24], the assumption of non-vanishing Seiberg-Witten invariant is absolutely necessary. For instance, take $X$ to be the fiber-sum of $S^1 \times N$ and $S^1 \times S^3$ where $N$ is a closed oriented $3$-manifold with $b_1(N) > 1$. Then $X$ is a circle bundle over $N \# S^1 \times S^2$, and $\pi_1(X) = \pi_1(N \# S^1 \times S^2) = \pi_1(N) \ast \mathbb{Z}$ which is centerless. Since $N \# S^1 \times S^2$ has zero Seiberg-Witten invariant and $b_2^+(X) = b_1(N) > 1$, the Seiberg-Witten invariant of $X$ is also zero (cf. [2]).

(2) There is a $4$-manifold which satisfies all the known constraints yet is not Seifert fibered: Let $X$ be the projective bundle $\mathbb{P}(E \times \mathbb{C})$ where $E$ is a holomorphic line bundle of odd degree over an elliptic curve. Then $X$ is minimal with nonzero Seiberg-Witten invariant, and $z(\pi_1(X)) = \pi_1(X) = \mathbb{Z}^2$. The claim that $X$ is not Seifert fibered follows from a classification theorem in [11], which says that a Seifert fibered $4$-manifold with $z(\pi_1) = \mathbb{Z}^2$ and $\pi_2 \neq 0$ must be diffeomorphic to $T^2 \times S^2$. (Note that although $X$ is not Seifert fibered, a double cover of $X$, which is diffeomorphic to $T^2 \times S^2$, is Seifert fibered.)

As a corollary of Theorem 1.1 and the aforementioned classification result in [11], the class of complex surfaces which may possess a Seifert fibration is determined. Except for the class $VII$ surfaces, a complex surface in all other cases clearly admits a Seifert fibration.

**Theorem 1.2.** A complex surface is not Seifert fibered unless it is either a class $VII$ surface with $b_2 = 0$, or has an elliptic fibration with trivial monodromy representation, or is a ruled surface diffeomorphic to $T^2 \times S^2$. 
Examples of class VII surfaces with $b_2 = 0$ are Hopf surfaces or Inoue surfaces. Many Hopf surfaces admit a smooth fixed-point free circle action, hence are naturally Seifert fibered. However, there are no naturally defined circle actions on an Inoue surface (cf. [5]). Note that since these 4-manifolds have $b_2 = 0$, Seiberg-Witten invariants are not well-defined. Consequently, for a given class VII surface with $b_2 = 0$ which is not naturally Seifert fibered, the question as whether it is indeed not Seifert fibered is currently beyond reach.

In the following theorem, we give a classification of smooth circle actions on $T^4$.

**Theorem 1.3.** Every smooth circle action on the 4-torus is smoothly conjugate to a linear action.

1.2. **Destruction of circle actions.** Theorem 1.1 gives new obstructions to smooth circle actions on 4-manifolds. For convenience we shall collect in the next theorem all the known obstructions, with the last item being a corollary of Theorem 1.1 and a theorem of Baldridge (cf. [4], Theorem 1.1). The latter says that a 4-manifold with $b_2^+ > 1$ and admitting a smooth circle action with nonempty fixed-point set must have vanishing Seiberg-Witten invariant.

**Theorem 1.4.** (Obstructions for smooth circle actions on 4-manifolds)

1. (Atiyah-Hirzebruch [1], Herrera [23]) If a 4-manifold with even intersection form admits a smooth circle action, then its signature must vanish.

2. (Fintushel [16]) If a simply connected 4-manifold admits a smooth circle action, then it must be diffeomorphic to a connected sum of $S^4$, $\pm \mathbb{CP}^2$, or $S^2 \times S^2$.

3. (Baldridge [4]) Let $X$ be a 4-manifold with either nonzero Euler characteristic or nonzero signature, and suppose $X$ admits a smooth circle action. (a) If $X$ is symplectic, then $X$ must be diffeomorphic to a rational or ruled surface. (b) If $b_2^+ > 1$, then $X$ must have vanishing Seiberg-Witten invariant.

4. Let $X$ be a 4-manifold with $b_2^+ > 1$ admitting a smooth circle action. If $X$ has nonzero Seiberg-Witten invariant, then (i) $z(\pi_1(X))$ must be infinite, and (ii) the Hurwitz homomorphism $\pi_2(X) \rightarrow H_2(X)$ must have finite image.

We fell upon these new obstructions (i.e. Theorem 1.4(4)) when investigating methods for destroying smooth circle actions on 4-manifolds using the Fintushel-Stern knot surgery [19], as motivated by considerations in [9]. In order to explain this, suppose we are given a Seifert fibered 4-manifold $X$ with $b_2^+ > 1$. We denote by $\pi : X \rightarrow Y$ the Seifert fibration, and furthermore, we assume that there is an embedded loop $l \subset X$ satisfying the following conditions:

(i) $\pi(l)$ is an embedded loop lying in the complement of the singular set of $Y$, and $l$ is a section of $\pi$ over $\pi(l)$,

(ii) no nontrivial powers of the homotopy class of $l$ are contained in $z(\pi_1(X))$,

(iii) the 2-torus $T \equiv \pi^{-1}(\pi(l))$ is non-torsion in $H_2(X)$.

Now for any (nontrivial) knot $K \subset S^3$, we let $X_K$ be the 4-manifold obtained from knot surgery on $X$ along $T$ using knot $K$, i.e.,

$$X_K \equiv X \setminus Nd(T) \cup_{\phi} (S^3 \setminus Nd(K)) \times S^1.$$  

1This was proved modulo the 3-dimensional Poincaré Conjecture, which is now resolved, cf. [32].
Here in the knot surgery we require that the diffeomorphism \( \phi \) identifies the meridian of \( T \) with the longitude of \( K \), a fiber of \( \pi \) with the meridian of \( K \), and a push-off of \( l \) with \( \{ pt \} \times S^1 \). It follows easily from the Mayer-Vietoris sequence that \( X_K \) has the same integral homology and the same intersection pairing of \( X \), and moreover, the Seiberg-Witten invariant of \( X_K \) can be calculated from that of \( X \) and the Alexander polynomial of \( K \) (cf. [19, 17]).

With the preceding understood, we have the following theorem.

**Theorem 1.5.** Let \( X \) be a Seifert fibered 4-manifold with \( b_2^+ > 1 \) such that

\[
SW_X(z) \equiv \sum_{c_1(\mathcal{L})=z} SW_X(\mathcal{L}) \neq 0
\]

for some \( z \in H^2(X) \). Let there be an embedded loop \( l \) in \( X \) satisfying (i) – (iii). Suppose \( K \) is not a torus knot. Then the 4-manifold \( X_K \) constructed above does not support any smooth circle actions.

We remark that when \( K \) has trivial Alexander polynomial and \( H^2(X) \) has no 2-torsions, \( X_K \) and \( X \) have the same Seiberg-Witten invariant.

Consider the following example. Let \( X = S^1 \times N^3 \), where \( N^3 = [0,1] \times T^2 / \sim \) with \((0, x, y) \sim (1, x + y, y)\). Then \( X \) is naturally a \( T^2 \)-bundle over \( T^2 \), with \( x, y \) being coordinates on the fiber. Note further that the translations along the \( x \)-direction define naturally a free smooth circle action on \( X \), so in this sense \( X \) is Seifert fibered. The integral homology of \( X \) is given as follows:

\[
H_1(X) = H_3(X) = \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}, \quad \text{and} \quad H_2(X) = \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}.
\]

In particular, \( b_2^+ = 2 \). Finally, as a \( T^2 \)-bundle over \( T^2 \), \( X \) has a symplectic structure with \( c_1(K_X) = 0 \).

In order to perform the construction of manifold \( X_K \) described in Theorem 1.5, we consider the following embedded loop \( l \) in \( X \) which clearly satisfies the conditions (i)-(iii) required in the construction: we pick any \( T^2 \)-fiber in \( X \) and take in that fiber an embedded loop parametrized by the \( y \)-coordinate. Notice that in the present case the 2-torus \( T \equiv \pi^{-1}(\pi(l)) \) is simply the \( T^2 \)-fiber we picked.

Now for any fixed, nontrivial knot \( K \subset S^3 \) which is not a torus knot, we obtain a smooth 4-manifold \( X_K \) which has the same integral homology of \( X \) and does not support any smooth circle actions. Furthermore, note that \( X_K \) is symplectic if \( K \) is chosen to be a fibered knot, and \( X_K \) has the same Seiberg-Witten invariant of \( X \) if the Alexander polynomial of \( K \) is trivial.

We shall also consider an equivariant version of the above construction. Note that the \( S^1 \)-factor in \( X \) defines a natural circle action, so that for any integer \( p \geq 2 \) there is an induced free smooth \( \mathbb{Z}_p \)-action on \( X \), which preserves the \( T^2 \)-bundle structure. For a fixed, nontrivial knot \( K \), we let \( X_{K,p} \) denote the 4-manifold resulted from a repeated application of the above construction which is equivariant with respect to the free \( \mathbb{Z}_p \)-action. Using the fact that repeated knot surgery along parallel copies is equivalent to a single knot surgery using the connected sum of the knot (cf. Example 1.3 in [9]), we obtain the following corollary.
Corollary 1.6. Let \( p \geq 2 \) be an integer, and let \( K \subset S^3 \) be a nontrivial knot. There are smooth 4-manifolds \( X_{K,p} \) such that the following statements are true.

1. \( H_1(X_{K,p}) = H_3(X_{K,p}) = \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z} \), \( H_2(X_{K,p}) = \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z} \).
2. \( X_{K,p} \) does not admit any smooth circle actions, but has a smooth free \( \mathbb{Z}_p \)-action.

Moreover,

3. If \( K \) is a fibered knot, then \( X_{K,p} \) has a symplectic structure \( \omega \) where \( [\omega] \in H^2_{dR}(X_{K,p}) \) is integral, with respect to which the free \( \mathbb{Z}_p \)-action on \( X_{K,p} \) is symplectic and \( c_1(K_{X_{K,p}}) \cdot [\omega] \) grows linearly with respect to \( p \).
4. If \( K \) has trivial Alexander polynomial, then the Seiberg-Witten invariants of the 4-manifolds \( X_{K,p} \) are all the same, i.e., independent of \( p \) and \( K \).

Remarks: In [9] the author considered an extension of the classical Hurwitz theorem on automorphisms of surfaces to 4 dimensions, where bounds for the order of periodic diffeomorphisms of 4-manifolds admitting no smooth circle actions were investigated. It was shown that the order of a holomorphic \( \mathbb{Z}_p \)-action is bounded by a constant depending only on the integral homology of the manifold, however, in the symplectic case the bound was shown to depend in addition on \( c_1(K_X) \cdot [\omega] \), a term reflecting the smooth and symplectic structures of the 4-manifold. Examples were given in [9] which show that the involvement of \( c_1(K_X) \cdot [\omega] \) is indeed necessary, at least in the case of \( b_2^+ = 1 \). With this understood, note that Corollary 1.6(3) gives further examples for the case of \( b_2^+ > 1 \). Furthermore, Corollary 1.6(4) shows that the integral homology and the Seiberg-Witten invariants alone are insufficient in bounding the order of a smooth \( \mathbb{Z}_p \)-action on a 4-manifold, and that certain measurement for the complexity of fundamental group must also be involved.

1.3. Symplectic Seifert fibered 4-manifolds. Concerning smooth classification of Seifert fibered 4-manifolds which admit a symplectic structure, we have the following conjecture.

Conjecture 1.7. Let \( X \) be a symplectic 4-manifold and \( \pi : X \to Y \) be a Seifert fibration. Then \( Y = \hat{Y}/G \), where \( \hat{Y} \) is a fibered 3-manifold, and \( G \) is a finite group acting on \( \hat{Y} \) which preserves a fibration \( \hat{Y} \to \mathbb{S}^1 \) such that the induced action on \( \mathbb{S}^1 \) is orientation-preserving.

Remarks: (1) Seminal works of Taubes (concerning near symplectic structure [35]) and Kronheimer (on minimal genus [25]) led to the following conjecture: If \( S^1 \times M^3 \) is symplectic, \( M^3 \) must be fibered over \( S^1 \). First progress was made in [12] where the conjecture was confirmed under a stronger assumption, i.e., \( S^1 \times M^3 \) admits a symplectic Lefschetz fibration. After a series of partial results, the conjecture was finally resolved by Friedl and Vidussi in [20]. The extension of the conjecture to circle bundles over 3-manifold was considered in [2] (the question was raised even earlier in [14]), and partial results were obtained in [8] and [21].

(2) Conjecture 1.7 is true if the fixed-point free circle action on \( X \) preserves the symplectic structure. This follows easily from a generalized moment map argument, see [28, 14].
(3) Conjecture 1.7 also holds true if rank $z(\pi_1(X)) > 1$. It is shown in [11] that any Seifert fibration on such a 4-manifold may be extended to a principal $T^2$-bundle over a 2-orbifold, which can be given a compatible complex structure. If the complex structure is non-Kähler, then $X$ can not be symplectic unless it is a primary Kodaira surface (cf. [6], also [10]), and in this case the circle action is free and $Y$ is fibered. If $X$ is Kähler, then $X = (T^2 \times \Sigma)/G$ for some free finite group action of $G$ preserving the product structure on $T^2 \times \Sigma$, cf. [22], Theorem 7.7 in p. 200, in which case $\tilde{Y} = S^1 \times \Sigma$ with the $G$-invariant fibration $pr_1 : S^1 \times \Sigma \to S^1$.

Generalizing the work of Friedl-Vidussi [21], we obtain the following result.

**Proposition 1.8.** Let $X$ be a symplectic 4-manifold and $\pi : X \to Y$ be a Seifert fibration. Then there exists a 3-orbifold $\tilde{Y}$ and a finite group action of $G$ such that $Y = \tilde{Y}/G$. Furthermore, if the canonical class $K_X$ is torsion, then $\tilde{Y}$ is fibered, and there is a $G$-invariant fibration $\tilde{Y} \to S^1$ such that the induced $G$-action on the base $S^1$ is orientation-preserving.

**Additional Remarks:** It was shown in [4] that if a symplectic 4-manifold admits a smooth circle action with nonempty fixed-points, it must be diffeomorphic to a rational or ruled surface. Thus the smooth classification of symplectic 4-manifolds with smooth circular symmetry hinges on the resolution of Conjecture 1.7, which remains open in the following case: $X$ has Kodaira dimension 1 (cf. [26]), and rank $z(\pi_1(X)) = 1$ (or equivalently, $X$ supports no complex structures).

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2. New constraints of Seifert fibered 4-manifolds

**Lemma 2.1.** Suppose $X$ has nonzero Seiberg-Witten invariant. Let $\pi : X \to Y$ be any Seifert fibration where $b_1(Y) > 1$. Then the 3-orbifold $Y$ has nonzero Seiberg-Witten invariant.

**Proof.** Let $\mathcal{L}$ be a $Spin^c$-structure such that $SW_X(\mathcal{L}) \neq 0$. If $b_2^+ > 1$, then $\mathcal{L} = \pi^* \mathcal{L}_0$ for some $Spin^c$-structure $\mathcal{L}_0$ on $Y$, and moreover

$$SW_X(\mathcal{L}) = \sum_{\mathcal{L} \equiv \mathcal{L}_0 \mod \chi} SW_Y(\mathcal{L}),$$

where $\chi$ stands for the Euler class of $\pi$, cf. Baldridge [3], Theorem C. It follows readily that $Y$ has nonzero Seiberg-Witten invariant in this case.

When $b_2^+ = 1$ and $b_1(Y) > 1$, the above formula continues to hold (cf. [3], Corollary 25) as long as $\mathcal{L} = \pi^* \mathcal{L}_0$. Thus it remains to show that $\mathcal{L} = \pi^* \mathcal{L}_0$ for some $Spin^c$-structure $\mathcal{L}_0$ on $Y$. 
In order to see this, we observe first that $H^2(X; \mathbb{Z})/\text{Tor}$ has rank 2, and any torsion element of $H^2(X; \mathbb{Z})$ is the $c_1$ of the pull-back of an orbifold complex line bundle on $Y$. Moreover, there exists an embedded loop $\gamma$ lying in the complement of the singular set of $Y$, such that an element of $H^2(X; \mathbb{Z})$ is the $c_1$ of the pull-back of an orbifold complex line bundle on $Y$ if and only if it is a multiple of the Poincaré dual of the 2-torus $T = \pi^{-1}(\gamma) \subset X$ in $H^2(X; \mathbb{Z})/\text{Tor}$, see Baldridge [3], Theorem 9. With this understood, $L = \pi^*L_0$ for some $\text{Spin}^c$-structure $L_0$ on $Y$ if and only if $c_1(L)$ is Poincaré dual to a multiple of $T$ over $\mathbb{Q}$.

We shall prove this using the product formula of Seiberg-Witten invariants in Taubes [36]. To this end, we let $N$ be the boundary of a regular neighborhood of $T$ in $X$. Then $N$ is a 3-torus, essential in the sense of [36], which splits $X$ into two pieces $X_+$, $X_-$. With this understood, Theorem 2.7 of [36] asserts that $SW_X(L)$ can be expressed as a sum of products of the Seiberg-Witten invariants of $X_+$ and $X_-$. In particular, $c_1(L)$ is expressed as a sum of $x$, $y$, where $x$, $y$ are in the images of $H^2(X_+, N; \mathbb{Z})$ and $H^2(X_-, N; \mathbb{Z})$ in $H^2(X; \mathbb{Z})$ respectively. It is easily seen that mod torsion elements both are generated by the Poincaré dual of $T$. Hence the claim $L = \pi^*L_0$.

Corollary 2.2. Let $\pi : X \to Y$ be a Seifert fibration where $b_1(Y) > 1$. If the underlying 3-manifold $|Y|$ contains a non-separating 2-sphere, then $X$ has vanishing Seiberg-Witten invariant.

Proof. If $X$ has nonzero Seiberg-Witten invariant, so does $Y$ by Lemma 2.1. By the de-singularization formula in [10], the underlying 3-manifold $|Y|$ also has nonzero Seiberg-Witten invariant. But if $|Y|$ contains a non-separating 2-sphere, we have a decomposition $|Y| = Y_1 \# S^1 \times S^2$ where $b_1(Y_1) = b_1(Y) - 1 > 0$. This implies that $|Y|$ has vanishing Seiberg-Witten invariant, which is a contradiction.

Recall that a 3-orbifold is called pseudo-good if it contains no bad 2-suborbifold.

Lemma 2.3. Let $X$ be a Seifert fibered 4-manifold with Seifert fibration $\pi : X \to Y$. Then $Y$ is pseudo-good if either the Euler class of $\pi$ is torsion, or $X$ has nontrivial Seiberg-Witten invariant.

Proof. We shall show that if $Y$ has a bad 2-suborbifold $\Sigma$, then the Euler class of $\pi$ must be non-torsion and $X$ has vanishing Seiberg-Witten invariant. By definition, $\Sigma$ is an embedded 2-sphere in the underlying 3-manifold $|Y|$, such that either $\Sigma$ contains exactly one singular point of $Y$ or $\Sigma$ contains two singular points of different multiplicities. For simplicity, we let $p_1, p_2$ be the singular points on $\Sigma$, with $p_1, p_2$ contained in the components $\gamma_1, \gamma_2$ of the singular set of $Y$, with multiplicities $\alpha_1, \alpha_2$ respectively. We assume that $\alpha_1 < \alpha_2$, with $\alpha_1 = 1$ representing the case where $\Sigma$ contains only one singular point $p_2$. Note that since $\alpha_1 \neq \alpha_2$, $\gamma_1, \gamma_2$ are distinct.

We first show that the Euler class of $\pi$ must be non-torsion. Let $e(\pi)$ denote the image of the Euler class of $\pi$ in $H^2(|Y|; \mathbb{Q})$. Then

$$\int_{|\Sigma|} e(\pi) = b + \beta_1/\alpha_1 + \beta_2/\alpha_2,$$
where \( b \in \mathbb{Z}, 1 \leq \beta_1 < \alpha_1 \) or \( \alpha_1 = \beta_1 = 1 \), and \( 1 \leq \beta_2 < \alpha_2 \) with \( \gcd (\alpha_2, \beta_2) = 1 \). It follows from the fact that \( \alpha_1 < \alpha_2 \) and \( \alpha_2, \beta_2 \) are co-prime that \( \int_{|\Sigma|} e(\pi) \neq 0 \). Hence the Euler class of \( \pi \) is non-torsion.

Now note that \( b_1(Y) = b_1^+ + 1 > 1 \) since the Euler class of \( \pi \) is non-torsion (cf. [3]). On the other hand, clearly \( |\Sigma| \) is a non-separating 2-sphere in \( |Y| \). By Corollary 2.2, \( X \) has vanishing Seiberg-Witten invariant.

\[ \square \]

Proof of Theorem 1.1

We first introduce some notations. Let \( y_0 \in Y \) be a regular point and \( F \) be the fiber of the Seifert fibration over \( y_0 \), and fix a point \( x_0 \in F \). Denote by \( i : F \hookrightarrow X \) the inclusion. We shall prove that \( i_* : \pi_1(F, x_0) \rightarrow \pi_1(X, x_0) \) is injective first.

By Lemma 2.3, \( Y \) is pseudo-good. As a corollary of the proven Thurston’s Geometrization Conjecture for 3-orbifolds (cf. [7, 27]), \( Y \) is very good, i.e., there is a 3-manifold \( \tilde{Y} \) with a finite group action \( G \) such that \( Y = \tilde{Y}/G \). Let \( pr : \tilde{Y} \rightarrow Y \) be the quotient map, and let \( \tilde{X} \) be the 4-manifold which is the total space of the pull-back circle bundle of \( \pi : X \rightarrow Y \) via \( pr \). Then \( \tilde{X} \) has a natural free \( G \)-action such that \( X = \tilde{X}/G \). We fix a point \( \tilde{y}_0 \in \tilde{Y} \) in the pre-image of \( y_0 \), and let \( \tilde{F} \subset \tilde{X} \) be the fiber over \( \tilde{y}_0 \). We fix a \( \tilde{x}_0 \in \tilde{F} \) which is sent to \( x_0 \) under \( \tilde{X} \rightarrow X \). Then consider the following commutative diagram

\[
\begin{array}{ccc}
\pi_2(\tilde{Y}, \tilde{y}_0) & \xrightarrow{\delta} & \pi_1(\tilde{F}, \tilde{x}_0) \\
\downarrow & & \downarrow \\
\pi_2^{orb}(Y, y_0) & \xrightarrow{\delta} & \pi_1(F, x_0)
\end{array}
\]

Since \( \pi_1(\tilde{X}, \tilde{x}_0) \rightarrow \pi_1(X, x_0) \) is injective, it follows that \( i_* \) is injective if \( \delta \) has zero image.

With the preceding understood, by the Equivariant Sphere Theorem of Meeks and Yau (cf. [30], p. 480), there are disjoint, embedded 2-spheres \( \Sigma_i \) in \( \tilde{Y} \) such that the union of \( \Sigma_i \) is invariant under the \( G \)-action and the classes of \( \Sigma_i \) generate \( \pi_2(\tilde{Y}) \) as a \( \pi_1(\tilde{Y}) \)-module. Suppose the image of \( \delta \) is non-zero. Then there is a \( \Sigma \in \{\Sigma_i\} \) which is not in the kernel of \( \delta \). It follows that the Euler class of \( \tilde{\pi} : \tilde{X} \rightarrow \tilde{Y} \) is nonzero on \( \Sigma \), and consequently, \( \Sigma \) is a non-separating 2-sphere in \( \tilde{Y} \). It also follows that the Euler class of \( \pi \) is non-torsion, and in this case, we have \( b_1(Y) = b_1^+ + 1 > 1 \).

We first consider the case where there are no elements of \( G \) which leaves the 2-sphere \( \Sigma \) invariant. In this case the image of \( \Sigma \) in \( Y \) under the quotient \( \tilde{Y} \rightarrow Y \) is an embedded non-separating 2-sphere which contains no singular points of \( Y \). By Corollary 2.2, \( X \) has vanishing Seiberg-Witten invariant, a contradiction.

Now suppose that there are nontrivial elements of \( G \) which leave \( \Sigma \) invariant. We denote by \( G_0 \) the maximal subgroup of \( G \) which leaves \( \Sigma \) invariant. We shall first argue that the action of \( G_0 \) on \( \Sigma \) is orientation-preserving. Suppose not, and let \( \tau \in G_0 \) be an involution which acts on \( \Sigma \) reversing the orientation. Then since \( \tau \) preserves the Euler class of \( \tilde{\pi} \) and reverses the orientation of \( \Sigma \), the Euler class of \( \tilde{\pi} \) evaluates to 0 on \( \Sigma \), which is a contradiction. Hence the action of \( G_0 \) on \( \Sigma \) is orientation-preserving.
Let $\Sigma$ be the image of $\tilde{\Sigma}$ in $Y$. Then $\Sigma$ is a non-separating, spherical 2-suborbifold of $Y$. Consequently $|Y|$ contains a non-separating 2-sphere, which, by Corollary 2.2, implies that $X$ has vanishing Seiberg-Witten invariant, a contradiction.

For part (2) of the theorem, we claim that there exist embedded 2-spheres $C_1, \cdots, C_N$ of self-intersection 0 which generate the image of $\pi_2(X) \to H_2(X)$. Assume the claim momentarily. Since $X$ has $b^+_2 > 1$ and nonzero Seiberg-Witten invariant, a theorem of Fintushel and Stern (cf. [18]) asserts that each $C_i$ must be torsion in $H_2(X)$, from which part (2) follows.

To see the existence of $C_1, \cdots, C_N$, note that the restriction of $\tilde{\pi}$ over each $\tilde{\Sigma}_i$ must be trivial from the proof of part (1). We claim that there are sections $\Sigma_i$ of $\tilde{\pi}$ over $\tilde{\Sigma}_i$ or a push-off of it, such that no nontrivial element of $G$ will leave $\Sigma_i$ invariant. The image of such a $\Sigma_i$ under $\tilde{X} \to X$ is an embedded 2-sphere of self-intersection 0, which will be our $C_i$. Since $\tilde{\pi}_*: \pi_2(\tilde{X}) \to \pi_2(\tilde{Y})$ is isomorphic and $\pi_2(\tilde{X}) = \pi_2(X)$, the 2-spheres $C_i$ generate the image of $\pi_2(X) \to H_2(X)$.

To construct these $\Sigma_i$’s, note that there are three possibilities: (i) no nontrivial element of $G$ leaves $\Sigma_i$ invariant, (ii) there is a nontrivial $g \in G$ (and no other elements) which acts on $\Sigma_i$ preserving the orientation, and (iii) there is an involution $\tau \in G$ (and no other elements) which acts on $\Sigma_i$ reversing the orientation. The existence of $\Sigma_i$ is trivial in case (i). In case (ii), any section of $\tilde{\pi}$ will do because $g$ acts as translations on the fiber of $\tilde{\pi}$. In case (iii), we take $\Sigma_i$ to be a section of $\tilde{\pi}$ over a push-off of $\tilde{\Sigma}_i$. This shows the existence of $\Sigma_i$’s, and the proof of Theorem 1.1 is completed.

**Proof of Theorem 1.2**

According to the Enriques-Kodaira classification, $X$ falls into one of the following classes of surfaces: class $VII$, rational or ruled, elliptic, $K3$, or general type.

First consider the case where $X$ is a class $VII$ surface. Since $b_1 = 1$, the Euler number of $X$ equals $b_2$. This shows that if $X$ is Seifert fibered, $b_2$ must be zero.

Suppose $X$ is of general type or $K3$. Since $c_2(X) > 0$ (cf. [5], Theorem 1.1 in Chapter VII for the case of general type), $X$ can not be Seifert fibered.

Suppose $X$ is elliptic. Then the Euler number of $X$ is non-negative. Hence if $X$ is Seifert fibered, $X$ must be a minimal elliptic surface with Euler number 0. Suppose the monodromy representation of the elliptic fibration is nontrivial. Then $X$ is Kähler (cf. [22], Theorem 7.8, p.201), and therefore $z(\pi_1(X))$ is infinite by Theorem 1.1. On the other hand, since the monodromy is nontrivial, the image of the $\pi_1$ of a generic fiber of the elliptic fibration does not lie in the center $z(\pi_1(X))$, which implies that the base of the elliptic fibration must be a nonsingular torus. In this case $X$ is a bi-elliptic surface, which admits another elliptic fibration with trivial monodromy representation.

Finally, let $X$ be rational or ruled. If $X$ is Seifert fibered, then the vanishing of Euler number and signature implies that $X$ must be a ruled surface over an elliptic curve. In particular, $\pi_1(X) = \mathbb{Z} \times \mathbb{Z}$ and $\pi_2(X) \neq 0$. By the classification result concerning such Seifert fibered 4-manifolds in [11], $X$ is diffeomorphic to $T^2 \times S^2$.

This completes the proof of Theorem 1.2.

**Proof of Theorem 1.3**

Since $T^4$ has $b^+_2 > 1$ and nonzero Seiberg-Witten invariant, any smooth circle action on $T^4$ must be fixed-point free by Baldridge [4]. Let $\pi : T^4 \to Y$ be the corresponding
Seifert fibration. By Lemma 2.3, $Y$ is pseudo-good, hence there is a 3-manifold $\tilde{Y}$ and a finite group action of $G$ such that $Y = \tilde{Y}/G$. Since $\pi_1(T^4) = \mathbb{Z}^4$, we see that the center of $\pi_1(\tilde{Y})$ must have rank 3, and consequently, $\tilde{Y} = T^3$. Finally, observe that in this case the $\pi_1^{orb}$ of $Y = \tilde{Y}/G = T^3/G$ must have a torsionless center, and $Y = T^3$ must be true. It follows that $\pi: T^4 \to Y = T^3$ is a trivial $S^1$-bundle, and the original circle action is smoothly conjugate to a linear action. This proves Theorem 1.3.

**Proof of Proposition 1.8**

First, every symplectic 4-manifold has nonzero Seiberg-Witten invariant by work of Taubes [34]. More precisely, the Seiberg-Witten invariant associated to the canonical $Spin^c$-structure $\mathbb{I} \oplus K_X^{-1}$ equals $\pm 1$, where in the case of $b_2^+ = 1$, the convention is to orient $H^{2,+}(X; \mathbb{R})$ by the symplectic form. Hence by Lemma 2.3, $Y$ is pseudo-good, and there exists a 3-manifold $\tilde{Y}$ and a finite group action of $G$ such that $Y = \tilde{Y}/G$ (cf. [7, 27]). We denote by $\tilde{\pi}: \tilde{X} \to \tilde{Y}$ the pull-back of $\pi$ via $\tilde{Y} \to Y$. Then $pr: \tilde{X} \to X$ is a finite cover of $X$, hence $\tilde{X}$ is also symplectic.

Now suppose $K_X$ is torsion. Then $K_\tilde{X} = pr^*K_X$ is also torsion, and by [21], $\tilde{Y}$ must be a $T^2$-bundle over $S^1$. We will show that there exists a $G$-invariant fibration $\tilde{Y} \to S^1$ such that the induced action of $G$ on the base $S^1$ is orientation-preserving. The discussion will be according to the following three cases.

**Case 1:** $b_1(\tilde{Y}) = 1$. In this case, our claim follows from Edmonds-Livingston [13], Theorem 5.2, together with Meeks-Scott [29], Theorem 8.1, since $b_1(D)$, which is nonzero, must also be equal to 1, and the condition $H_1(\tilde{Y}; \mathbb{Q})^G = \mathbb{Q}$ in [13] is verified.

**Case 2:** $b_1(\tilde{Y}) = 2$. In this case, $\tilde{Y}$ is a nontrivial $S^1$-bundle over $T^2$. Note that $z(\pi_1(\tilde{Y}))$ has rank 1, so it must be preserved under $G$. By Meeks-Scott [29], the $G$-action is equivalent to a fiber-preserving $G$-action. If the induced action on the base $T^2$ is homologically nontrivial, $Y$ would have $b_1 = 0$, which is a contradiction. Hence the induced action on $T^2$ is homologically trivial, which must be given by translations. In this case, it is easily seen that the $T^2$-bundle over $S^1$ structure on $\tilde{Y}$ is $G$-invariant, with the induced action preserving the orientation of the base $S^1$.

**Case 3:** $b_1(\tilde{Y}) = 3$. In this case, $\tilde{Y} = T^3$. By Meeks-Scott [29], the $G$-action is smoothly conjugate to a linear action. Furthermore, since $Y = \tilde{Y}/G$ has only 1-dimensional singular set, the $G$-action must preserve a product structure $T^3 = S^1 \times T^2$. This case follows easily. (Note that there is a finite group action on $T^3$ which does not preserve any product structure $T^3 = S^1 \times T^2$, cf. Meeks-Scott [29], p. 291. Of course, such an action has an isolated fixed point.)

3. **Destroying smooth circle actions via knot surgery**

We begin by reviewing the knot surgery and the knot surgery formula for Seiberg-Witten invariants, following Fintushel-Stern [19]. To this end, let $M$ be a smooth 4-manifold with $b_2^+ > 1$, which possesses an essential embedded torus $T$ of self-intersection 0. Given any knot $K \subset S^3$, the knot surgery of $M$ along $T$ with knot $K$ is the smooth 4-manifold, denoted by $M_K$, which is constructed as follows:

$$M_K \equiv M \setminus Nd(T) \cup_{\phi} (S^3 \setminus Nd(K)) \times S^1,$$
where the only requirement on the identification $\phi$ is that it sends the meridian of $T$ to the longitude of $K$. It follows easily from the Mayer-Vietoris sequence that the integral homology of $M_K$ is naturally identified with the integral homology of $M$, under which the intersection pairings on $M$ and $M_K$ agree. (In [19], it is assumed that $M$ is simply connected, $T$ is $c$-embedded, and $\pi_1(M \setminus T)$ is trivial. These assumptions are irrelevant to the discussions here.)

An important aspect of knot surgery is that the Seiberg-Witten invariant of $M_K$ can be computed from that of $M$ and the Alexander polynomial of $K$, through the so-called knot surgery formula. In order to state the formula, let

$$SW_M = \sum (\sum_{t \in H^2(M;\mathbb{Z})} t^z),$$

where $t_z = \exp(z)$ is a formal variable, regarded as an element of the group ring $\mathbb{Z}[H^2(M;\mathbb{Z})]$.

**Theorem 3.1.** (Knot Surgery Formula [19]) With the integral homology of $M$ and $M_K$ naturally identified, one has

$$SW_{M_K} = SW_M \cdot \Delta_K(t),$$

where $t = \exp(2[T])$. Here $\Delta_K(t)$ is the Alexander polynomial of $K$, and $[T]$ stands for the Poincaré dual of the 2-torus $T$.

**Remarks:**

1. The knot surgery formula was originally proved in [19] under the assumption that $T$ is $c$-embedded. However, the assumption of $c$-embeddedness of $T$ is not essential in the argument and it may be removed (cf. Fintushel [17]).

2. We note that in Theorem 1.5, the condition $SW_X(z) = \sum_{t \in H^2(X;\mathbb{Z})} t^z$ is equivalent to $SW_X \neq 0$, which implies, by Theorem 3.1, that $SW_{X_K} \neq 0$ for any knot $K$. Consequently, $X_K$ has nonzero Seiberg-Witten invariant. Moreover, if $H^2(X;\mathbb{Z})$ has no 2-torsions, so does $H^2(X_K;\mathbb{Z})$, and in this case, $X$ and $X_K$ have the same Seiberg-Witten invariant provided that the Alexander polynomial of $K$ is trivial.

Now we give a proof for Theorem 1.5. To this end, we introduce the following notations. Let $Y_1 \equiv Y \setminus Nd(\pi(l))$ and $X_1 \equiv X \setminus Nd(T)$, where $T$ is the 2-torus $T = \pi^{-1}(\pi(l))$, such that $X_1$ is the restriction of $\pi$ to the 3-orbifold $Y_1$. Note that the boundary $\partial X_1$ is a 3-torus $T^3$. We fix three embedded loops in $\partial X_1$: $m$, a meridian of $T$, $h$, a fiber of $\pi$, and $l'$, a push-off of $l$, which all together generate $\pi_1(\partial X_1)$.

We assume $m$, $l'$ are sections over $\pi(m)$, $\pi(l')$ respectively. Finally, recall that the underlying manifolds of $Y_1$, $Y$ are denoted by $|Y_1|$, $|Y|$ respectively.

**Lemma 3.2.** The map $i_* : \pi_1(\partial X_1) \to \pi_1(X_1)$ induced by the inclusion is injective.

**Proof.** Suppose to the contrary that $i_* : \pi_1(\partial X_1) \to \pi_1(X_1)$ has a nontrivial kernel. Then $\pi_1(\partial Y_1) \to \pi_1(|Y_1|)$ must also have a nontrivial kernel. To see this, suppose $\gamma \neq 0$ lies in the kernel of $i_*$. Then $\pi_1(\partial Y_1)$ lies in the kernel of $\pi_1(\partial Y_1) \to \pi_1^{orb}(Y_1)$. On the other hand, if $\pi_1(\partial Y_1) \to \pi_1^{orb}(Y_1) \to \pi_1(|Y_1|)$ has a nontrivial kernel.
Now by the Loop Theorem, $\partial Y_1$ is compressible in $|Y_1|$. This means that there is an embedded disc $D \subset |Y_1|$ such that (1) $D \cap \partial Y_1 = \partial D$, (2) $\partial D$ is a homotopically nontrivial simple closed loop in $\partial Y_1$. We claim that $\partial D$ must be a copy of the meridian $\pi(m)$ of $\pi(l)$ in $Y$.

To see this, write $[\partial D] = s \cdot [\pi(m)] + t \cdot [\pi(l')]$ in $\pi_1(\partial Y_1)$. If $t \neq 0$, then $[\pi(l')]$ has finite order in $\pi_1^{orb}(Y)$. This implies that a nontrivial power of $[l]$ lies in the subgroup generated by $[h]$, which lies in $z(\pi_1(X))$, a contradiction to (ii). Hence $t = 0$, and $s = 1$ with $[\partial D] = [\pi(m)]$, so that $\partial D$ is isotopic to $\pi(m)$.

With the preceding understood, the median $\pi(m)$ bounds an embedded disk $D \subset |Y_1|$. It follows that there is a non-separating 2-sphere in $|Y|$ intersecting with $\pi(l)$ in exactly one point. By Corollary 2.2, $X$ has vanishing Seiberg-Witten invariant, a contradiction.

\hspace{1cm}  \Box

**Proof of Theorem 1.5**

Since $K$ is a nontrivial knot, $\pi_1(\partial((S^3 \setminus Nd(K)) \times S^1)) \to \pi_1((S^3 \setminus Nd(K)) \times S^1)$ is injective. With Lemma 3.2, this implies that $\pi_1(X_K)$ is an amalgamated free product

$$
\pi_1(X_K) = \pi_1(X_1) \ast_{\pi_1(T^3)} \pi_1((S^3 \setminus Nd(K)) \times S^1),
$$

where $\pi_1(T^3) = \pi_1(\partial X_1) = \pi_1(\partial((S^3 \setminus Nd(K)) \times S^1))$.

Since $K$ is not a torus knot, the knot group of $K$ has trivial center (cf. [33]). This implies that the center of $\pi_1((S^3 \setminus Nd(K)) \times S^1)$ is generated by the class of $\{pt\} \times S^1$. On the other hand, by the construction of $X_K$, $\{pt\} \times S^1$ is identified with a push-off of $l$, and furthermore, by condition (ii) on $l$, no nontrivial powers of the homotopy class of $l$ is contained in the center $z(\pi_1(X))$. Consequently, $z(\pi_1(X_K))$ is trivial. By Theorem 1.1 (together with Balridge [4], Theorem 1.1), $X_K$ does not support any smooth circle actions.

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