The relativistic causal Newton gravity law vs. general relativity

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The equations of the relativistic causal Newton gravity law for the planets of the solar system are studied in the approximation when the Sun rests at the coordinates origin and the planets do not interact between each other. The planet orbits of general relativity are also studied in the same approximation.

1 I. INTRODUCTION

Poincaré\textsuperscript{1} tried to find a modification of Newton gravity law: ”In the paper cited Lorentz\textsuperscript{2} found it necessary to supplement his hypothesis in such a way that the relativity postulate could be valid for other forces in addition to the electromagnetic ones. According to his idea, because of Lorentz transformation (and therefore because of translational movement) all forces behave like electromagnetic (despite their origin).

”It turned out to be necessary to consider this hypothesis more attentively and to study the changes it makes in the gravity laws in particular. First, it obviously enables us to suppose that the gravity forces propagate not instantly, but at the speed of light. One could think that this is a sufficient for rejecting such a hypothesis, because Laplace has shown that this cannot occur. But, in fact, the effect of this propagation is largely balanced by some other circumstance, hence, there is no any contradiction between the law proposed and the astronomical observations.

”Is it possible to find a law satisfying the condition stated by Lorentz and at the same time reducing to Newton law in all the cases where the velocities of the celestial bodies are small to neglect their squares (and also the products of the accelerations and the distance) compared with the square of the speed of light?”

The relativistic Newton gravity law was proposed in Ref. 3

\[
\frac{d}{dt} \left( \left( 1 - \frac{1}{c^2} \left| \frac{dx_k}{dt} \right|^2 \right) \right)^{-1/2} \frac{dx_k}{dt} = -\eta_{\mu\nu} \sum_{\nu=0}^{3} \frac{1}{c} \left( \frac{dx_\nu}{dt} \right) \sum_{j=1,2,j \neq k} F_{j;\mu\nu}(x_k, x_j), \tag{1.1}
\]

\(k = 1, 2, \mu = 0, ..., 3\). The world line \(x_\mu(t)\) satisfies the condition \(x_0(t) = ct\); \(c\) is the speed of light; the diagonal \(4 \times 4\) matrix \(\eta_{\mu\nu} = \eta_{\nu\mu}\), \(\eta_{00} = -\eta_{11} = -\eta_{22} = -\eta_{33} = 1\); the strength \(F_{j;\mu\nu}(x_k, x_j)\) is expressed through the vector potential

\[
F_{j;\mu\nu}(x_k, x_j) = \frac{\partial A_{j;\nu}(x_k, x_j)}{\partial x_k^\mu} - \frac{\partial A_{j;\mu}(x_k, x_j)}{\partial x_k^\nu}, \tag{1.2}
\]

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\[
A_{j;\mu}(x_k, x_j) = 4\pi m_j G \eta_{\mu\nu} \int dt e_0(x_k - x_j(t)) \frac{dx_j^\mu(t)}{dt} = \\
\eta_{\mu\nu} m_j G \left( \frac{d}{dt} x_j^\mu(t') \right) \left( c |x_k - x_j(t')| - \sum_{i=1}^{3} (x_k^i - x_j^i(t')) \frac{d}{dt} x_j^i(t') \right)^{-1}, \tag{1.3}
\]

the gravitation constant \( G = (6.673 \pm 0.003) \cdot 10^{-11} m^3 kg^{-1} s^{-2} \) and \( m_j \) is the \( j \) body mass. The distribution \( e_0(x) \in S'(\mathbb{R}^4) \) with support in the closed upper light cone is the fundamental solution of the wave equation. It is unique. The distribution \( e_0(x) \) is not a regular function.

The relativistic Newton law is based on the fundamental solution \( e_0(x) \) of the wave equation similar as Newton law is based on the fundamental solution \( -(4\pi)^{-1} |x|^{-1} \) of Laplace equation. The vector potential (1.3) was proposed by Liénard (1898) and Wiechert (1900). The vector potential (1.3) is the relativistic version of Coulomb potential.

Newton gravity law requires the instant propagation of the force action. The special relativity requires that the propagation speed does not exceed the speed of light. If the propagation speed is independent of gravitating body speed, then it is equal to that of light. The vector potential (1.3) depends not on its simultaneous positions and speeds but on the positions and the speeds at the time moments \( t \) and \( t' \) which differ from each other in the time interval \( c^{-1} |x_k(t) - x_j(t')| \) needed for light covering the distance between the physical points \( x_k(t) \) and \( x_j(t') \). The equations (1.1) - (1.3) satisfy the causality condition: some event in the system can influence the evolution of the system in the future only and can not influence the behavior of the system in the past, in the time preceding the given event.

The equations (1.1) - (1.3) are the relativistic causal version of Newton gravity law equations. Sommerfeld (Ref. 4, Sec. 38): "The question may arise: what is the relativistic form of Newton gravity law? If the law is supposed to have a vector form, this question is wrong. The gravitational field is not a vector field. It has the incomparably complicated tensor structure." It seems the reason why the relativistic Newton gravity law\(^1\) was not studied.

For the resting body world line \( x_0^j(t) = ct, x_j(t) = \text{const} \) the vector potential (1.3) is

\[
A_{j,0}(x_k, x_j) = m_j G |x_k - x_j(c^{-1} x_0^j)|^{-1}, \quad A_{j,i}(x_k, x_j) = 0, \quad i = 1, 2, 3. \tag{1.4}
\]

99.87\% of the solar system total mass belongs to the Sun. We consider the Sun resting at the coordinates origin. The substitution of the vector potential (1.4) for the Sun world line \( x_j^0(t) = ct, x_j^i(t) = 0, i = 1, 2, 3 \) into the equation (1.1) yields

\[
\frac{d}{dt} \left( \left( 1 - \frac{1}{c^2} \frac{|dx|}{dt} \right)^{1/2} \frac{dx^i}{dt} \right) = -\frac{m_{10} G x^i}{|x|^3}, \quad i = 1, 2, 3. \tag{1.5}
\]

\( m_{10} \) is the Sun mass. The right-hand side of the equation (1.5) coincides with the right-hand side of the Newton gravity law equation for a planet. We neglect the interaction between the planets. The equation (1.5) is solved in Ref. 3. We choose the third axis to be orthogonal to the orbit plain

\[
x^1(t) = r(t) \cos \phi(t), \quad x^2(t) = r(t) \sin \phi(t), \quad x^3(t) = 0. \tag{1.6}
\]
The orbit radius \( r(t) \) is given by
\[
\frac{a(1 - e^2)}{r(t)} = 1 + e \cos \gamma (\phi(t) - \phi_0).
\] (1.7)

\( \phi(t) \) is the orbit angle, \( \phi_0 \) is the perihelion orbit angle, \( e \) is the planet orbit eccentricity and \( a \) is the ”ellipse” (1.7) major ”semi - axis”. In the Section II the time dependence of the orbit radius \( r(t) \) is determined. We get the precession coefficient
\[
\gamma \approx 1 - \frac{\omega^2 a^2}{2(1 - e^2)c^2}.
\] (1.8)

\( \omega = 2\pi T^{-1} \) is the mean ”angular frequency” and \( T \) is the planet ”period”. According to (Ref. 5, Chap. 25, Sec. 25.1, Appendix 25.1) for Mercury \( \omega^2 a^2 c^{-2} = 1477 m \), \( a = 0.5791 \cdot 10^{11} m \), \( e = 0.21 \) and \( 2^{-1}\omega^2 a^2 c^{-2}(1 - e^2)^{-1} \approx 1.3341 \cdot 10^{-8} \). The advance of Mercury’s perihelion, observed from the Sun, is \((\gamma^{-1} -1)\cdot360^\circ \) per ”period” of Mercury. The advance of Mercury’s perihelion, observed from the Sun, is \((\gamma^{-1} -1)\cdot360 \cdot 415 \cdot 3600^\circ \approx 2^{-1}\omega^2 a^2 c^{-2}(1 - e^2)^{-1} \cdot 360 \cdot 415 \cdot 3600^\circ \approx 7^\circ.175 \) per century (415 ”periods” of Mercury). \( 1^\circ = 60^\circ = 3600^\circ \). The advance of Mercury’s perihelion, observed by the astronomers from the Earth, is 5599”.74 ± 0”.41 per century (Ref. 5, Chap. 40, Sec. 40.5, Appendix 40.3). By using Newton gravity law it is possible to calculate the advance of Mercury’s perihelion caused by the non-inertial system connected with the Earth. It turns out to be 5025”.645 ± 0”.50 per century (Ref. 5, Chap. 40, Sec. 40.5, Appendix 40.5). By using Newton gravity law it is possible to calculate the advance of Mercury’s perihelion caused by the gravity of other planets. It turns out to be 531”.54 ± 0”.68 per century (Ref. 5, Chap. 40, Sec. 40.5, Appendix 40.5). The rest advance of Mercury’s perihelion 5599”.74 − 5025”.645 − 531”.54 ≈ 42”.56 per century can not be explained by the disturbing forces. It is not obvious that we can add the advance of Mercury’s perihelion obtained for the orbits in Newton gravity theory and the advance of Mercury’s perihelion 7”.175, observed from the Sun and obtained for the orbits (1.6) - (1.8). In our opinion for the experimental verification of the relativistic causal Newton gravity law (1.1) - (1.3) it is necessary to obtain the advance of Mercury’s perihelion 5599”.74, observed from the Earth, by making use of the relativistic causal Newton gravity law (1.1) - (1.3) without Newton gravity theory.

In the Section II we study the orbits (1.6) - (1.8) of Mercury and of the Earth and show that the value of the Mercury’s perihelion advance, observed from the Earth, depends on the perihelion angle \( \phi_0 \) of the Mercury orbit and on the perihelion angle \( \phi_0 \) of the Earth orbit.

Kepler (Astronomia nova seu physica coelestis, tradita commentariis de motibus stellae Martis ex observationibus Tychonis Brahe. MDCXIX) found that the planet orbits are elliptic in the coordinate system where the Sun rests. Kepler used Tycho Brahe’s astronomical observations (1580-1597). Due to Brahe, the Mars orbit deviation from the circular orbit was 8’. Ptolemaeus and Copernicus had the instrument precision 10’. Brahe had the instrument precision 2’. The intensive astronomical observations from the middle of the XIX century and the radio-location after 1966 discovered the advances of orbit perihelion for different planets. Is the orbit (1.6) - (1.8) consistent with the observable Mercury’s orbit? Clemence⁶: “Observations of Mercury are among the most difficult in positional astronomy. They have to be made in the daytime, near noon, under unfavorable conditions of the atmosphere; and they are subject to large systematic and accidental errors arising both from this cause and from the shape of the visible disk of the planet. The planet’s path in Newtonian space is
not an ellipse but an exceedingly complicated space-curve due to the disturbing effects of all of the other planets. The calculation of this curve is a difficult and laborious task, and significantly different results have been obtained by different computers.”

By making use of Hamilton-Jacobi equation Boguslavskii (Ref. 7, P. 233 - 403. Boguslavskii used the German transcription: Boguslawski.) solved the equation (1.5) and obtained the orbit formula (1.7). Boguslavskii did not calculate all integrals needed for Hamilton-Jacobi equation and did not obtain the time dependence of the orbit (1.6), (1.7) radius. Boguslavskii (Ref. 7, P. 386): “Since any material point mass changes in the special relativity according to the same law as the electron mass does, Einstein tried to apply the theory described above for the explanation of the part of Mercury’s movement which can not be explained by the disturbing forces. However, the movement calculated due to the formula (1.7) turned out to be six times less than the observable movement. Einstein\(^8\) obtained the correct explanation by means of his general relativity principle containing the new gravity theory.” By making use of the formulas (1.7), (1.8) Einstein obtained probably the estimate of advance of Mercury’s perihelion, observed from the Sun, \(42'\, 0.56 : 6 \approx 7'\, 0.09\) per century. Could Boguslavskii have learned Einstein’s calculation of Mercury’s movement by means of the formulas (1.7), (1.8)? In 1913 - 1914 he worked in Göttingen University together with Max Born (Ref. 7, P. 9 - 17).

The metric (Ref. 9, Chap. 38, relation (38.8))

\[
(ds)^2 = \left(1 - 2\frac{m_{10}G}{rc^2}\right) c^2 (dt)^2 - \left(1 - 2\frac{m_{10}G}{rc^2}\right)^{-1} (dr)^2 - r^2 (d\theta)^2 - r^2 \sin^2 \theta (d\phi)^2, \tag{1.9}
\]

\(r = ((x^1)^2 + (x^2)^2 + (x^3)^2)^{1/2}, \theta = \arctan\left[x^3/((x^1)^2 + (x^2)^2)^{1/2}\right], \phi = \arctan(x^2/x^1),\)

is a solution of Einstein’s equations: \(R_{\mu\nu}(x) - (1/2)g_{\mu\nu}(x)R(x) = 0, x \neq 0\). The solution (1.9) was obtained by Schwarzschild (1916). Eddington\(^9\) considers the Sun resting at the coordinates origin and neglects the interaction between the planets. The Sun gravitational field is described by the metric (1.9). A planet orbit is a solution of the geodesic equation (Ref. 9, Chap. 39, equation (39.1)) for the metric (1.9). The constant angle \(\theta = \pi/2\) satisfies the geodesic equation (39.2) from Ref. 9, Chap. 39. Choose the coordinates in such a way that a planet moves in the plane \(\theta = \pi/2\). The geodesic equations (39.1) from Ref. 9, Chap. 39 for the metric (1.9) imply the following equations (Ref. 9, Chap. 39, equations (39.61), (39.62))

\[
\frac{d^2}{d\phi^2} \left[\frac{1}{r} + \frac{1}{r} - \frac{m_{10}G}{c^2 h^2} - \frac{3m_{10}G}{c^2 r^2}\right] = 0, \quad r \frac{d\phi}{ds} = h = \text{const.} \tag{1.10}
\]

For Venus the orbit eccentricity \(e = 0.007\). Venus moves along the approximately circular orbit with approximately constant radius \(a\) and with approximately constant angular frequency \(\omega\). The substitution of the relation (1.6) for \(r(t) = a, \phi(t) = \omega t - \phi_0\) into the Newton gravity law for Venus and the Sun yields the third Kepler law

\[
m_{10}G = \omega^2 a^3. \tag{1.11}
\]

According to (Ref. 5, Chap. 25, Sec. 25.1, Appendix 25.1) for Mercury, Venus, the Earth, Mars and Saturn \(\omega^2 a^3 c^{-2} = 1477\, m\), for Jupiter and Neptune \(\omega^2 a^3 c^{-2} = 1478\, m\), for Uranus \(\omega^2 a^3 c^{-2} = 1476\, m\), for Pluto \(\omega^2 a^3 c^{-2} = 1469\, m\). The value \(m_{10}G a^{-1} c^{-2} = \omega^2 a^2 c^{-2}\) is negligible for any planet. For the nearest planet to the Sun, Mercury \(a = 0.5791 \cdot 10^{11}\, m\).
We substitute the function (1.7) into the first equation (1.10) and multiply the obtained equality by \(a(1 - e^2)\)

\[
1 - \frac{m_{10}Ga(1 - e^2)}{h^2c^2} + (1 - \gamma^2) e \cos \gamma(\phi - \phi_0) - \frac{3m_{10}G}{a(1 - e^2)c^2}(1 + e \cos \gamma(\phi - \phi_0))^2 = 0,
\]

(1.12)

If we neglect the last term \(-3m_{10}Ga^{-1}c^{-2}(1 - e^2)^{-1}(1 + e \cos \gamma(\phi - \phi_0))^2\) in the left-hand side of the equality (1.12), we get two equalities \(m_{10}Ga(1 - e^2)h^{-2}c^{-2} = 1, \gamma = 1\) (see Ref. 9, Chap. 40, relation (40.2)). The advance of Mercury’s perihelion, observed from the Sun, is \((\gamma^{-1} - 1) \cdot 360 \cdot 415 \cdot 3600'' = 0''\) per century (415 ”periods” of Mercury). By making use of the identity \(2\cos^2\gamma(\phi - \phi_0) = 1 + \cos 2\gamma(\phi - \phi_0)\) we rewrite the equality (1.12)

\[
1 - \frac{m_{10}Ga(1 - e^2)}{h^2c^2} - \frac{3m_{10}G(2 + e^2)}{2a(1 - e^2)c^2}
+ (1 - \frac{6m_{10}G}{a(1 - e^2)c^2} - \gamma^2) e \cos \gamma(\phi - \phi_0) - \frac{3m_{10}Ge^2}{2a(1 - e^2)c^2} \cos 2\gamma(\phi - \phi_0) = 0.
\]

(1.13)

If we neglect the last term \(-3m_{10}Ga^{-1}c^{-2}2^{-1}e^2(1 - e^2)^{-1}\cos 2\gamma(\phi - \phi_0)\) in the left-hand side of the equality (1.13), we get two equalities:

\[
\frac{m_{10}Ga(1 - e^2)}{h^2c^2} + \frac{3m_{10}G(2 + e^2)}{2a(1 - e^2)c^2} = 1,
\]

\[
\gamma = \left(1 - \frac{6m_{10}G}{a(1 - e^2)c^2}\right)^{1/2} \approx 1 - \frac{3m_{10}G}{a(1 - e^2)c^2}.
\]

(1.14)

The equality \(m_{10}Ga(1 - e^2)h^{-2}c^{-2} = 1\) and the second equality (1.14) are used in Ref. 9, Chap. 40, relations (40.5), (40.6). For the Mercury orbit the ellipse major ”semi - axis” \(a = 0.5791 \cdot 10^{11}m\) and the ellipse eccentricity \(e = 0.21\). The relations (1.7), (1.11), (1.14) imply that the advance of Mercury’s perihelion observed from the Sun is \((\gamma^{-1} - 1) \cdot 360 \cdot 415 \cdot 3600'' \approx 3m_{10}Ga^{-1}c^{-2}(1 - e^2)^{-1} \cdot 360 \cdot 415 \cdot 3600'' \approx 3\omega^2a^2c^{-2}(1 - e^2)^{-1} \cdot 360 \cdot 415 \cdot 3600'' \approx 43''.05 = 6.7''.175\) (see the relations (1.8) and (1.14)) per century (415 ”periods” of Mercury).

Misner, Thorne and Wheeler discuss and modify (Ref. 5, Chap. 40, Sec. 40.1, relation (40.1)) the Schwarzschild’s metric (1.9)

\[
\sum_{\mu, \nu=0}^3 g_{\mu\nu}(x)dx^\mu dx^\nu = \left(1 - 2\frac{m_{10}G}{rc^2} + 2\left(\frac{m_{10}G}{rc^2}\right)^2\right)(dx^0)^2
- \left(1 + 2\frac{m_{10}G}{rc^2}\right)((dx^1)^2 + (dx^2)^2 + (dx^3)^2), \quad r = ((x^1)^2 + (x^2)^2 + (x^3)^2)^{1/2}.
\]

(1.15)

A planet orbit is a solution of the geodesic equation (Ref. 5, Chap. 13, Sec. 13.4, equation (13.36)) for the metric (1.15). However, the approximation of Hamilton-Jacobi equation is used for a planet orbit in (Ref. 5, Chap. 40, Sec. 40.5). A planet orbit (Ref. 5, Chap. 40, Sec. 40.5, equations (40.17), (40.18)) is given by the equation (1.7) for the precession coefficient (1.14). The sketch of the equation (1.7), (1.14) proof is given in Exercise 40.4 from (Ref. 5, Chap. 40, Sec. 40.5). There is no a time dependence of the orbit radius \(r(t)\) and of the orbit angle \(\phi(t)\) in (Ref. 5, Chap. 40, Sec. 40.5). It needs the complete orbits of Mercury and the Earth to get the advance of Mercury’s perihelion, observed from the Earth.
In the Section III we solve the Kepler problem with the proper time $\tau$

$$\frac{d^2 x^i}{d\tau^2} = -\frac{m_{10} G x^i}{r^3}, \quad \frac{dt}{d\tau} = \frac{d t}{d t}, \quad i = 1, 2, 3,$$ \hfill (1.16)

$$\frac{dt}{d\tau} = c \left( \sum_{\mu, \nu = 0}^{3} g_{\mu\nu}(x(t)) \left( \frac{d x^\mu}{d t} \right) \left( \frac{d x^\nu}{d t} \right) \right)^{-1/2}.$$

The second relation (1.16) is the relation (13.35) from (Ref. 5, Chap. 13, Sec. 13.4) for the metric (1.15). The exact solution of the Kepler problem (1.16) is the planet orbit (1.6), (1.7) with the precession coefficient $\gamma = 1$. It is the approximate solution of the geodesic equations (Ref. 5, Chap. 13, Sec. 13.4, equations (13.36)) for the metric (1.15). It is possible to prove that the solution (1.6) - (1.8) of the equation (1.5) is an approximate solution of the equation (1.16) for the world line $x^\mu(t)$, $x^0(t) = ct$.

2 II. RELATIVISTIC PLANET ORBITS

Let us consider the relativistic Newton second law

$$mc \frac{dt}{ds} \left( \frac{d t}{d s} \frac{d x^\mu}{d t} \right) + q c^{-1} \sum_{k=0}^{N} \sum_{\alpha_1, ..., \alpha_k = 0}^{3} \eta^{\mu \alpha_1 \cdots \alpha_k} F_{\mu \alpha_1 \cdots \alpha_k}(x) \frac{dt}{ds} \frac{d x^{\alpha_1}}{d t} \cdots \frac{dt}{ds} \frac{d x^{\alpha_k}}{d t} = 0,$$ \hfill (2.1)

$$\frac{dt}{ds} = c^{-1} \left( 1 - c^{-2} |v|^2 \right)^{-1/2}, \quad v^i = \frac{d x^i}{d t}, \quad i = 1, 2, 3.$$

where $\mu = 0, ..., 3$ and the world line $x^\mu(t)$ satisfies the condition: $x^0(t) = ct$. The force is the polynomial of the speed in the equation (2.1). For an infinite series of the speed we need to define the series convergence. The second relation (2.1) implies the identities

$$\sum_{\alpha = 0}^{3} \eta^{\alpha \alpha} \left( \frac{dt}{ds} \frac{d x^\alpha}{d t} \right)^2 = 1, \quad \sum_{\alpha = 0}^{3} \eta^{\alpha \alpha} \frac{dt}{ds} \frac{d x^\alpha}{d t} \frac{d t}{ds} \frac{d x^\alpha}{d t} = 0.$$ \hfill (2.2)

The equation (2.1) and the second identity (2.2) imply

$$\sum_{k=0}^{N} \sum_{\alpha_1, ..., \alpha_{k+1} = 1}^{3} F_{\alpha_1 \cdots \alpha_{k+1}}(x) \frac{dt}{ds} \frac{d x^{\alpha_1}}{d t} \cdots \frac{dt}{ds} \frac{d x^{\alpha_{k+1}}}{d t} = 0.$$ \hfill (2.3)

Let the functions $F_{\alpha_1 \cdots \alpha_{k+1}}(x)$ satisfy the equation (2.3). Then three equations (2.1) for $\mu = 1, 2, 3$ are independent

$$m \frac{d}{d t} \left( (1 - c^{-2} |v|^2)^{-1/2} v^i \right) = \sum_{k=0}^{N} \sum_{\alpha_1, ..., \alpha_k = 0}^{3} F_{\alpha_1 \cdots \alpha_k}(x) \frac{d x^{\alpha_1}}{d t} \cdots \frac{d x^{\alpha_k}}{d t}, \quad i = 1, 2, 3.$$ \hfill (2.4)

**Lemma** (Ref. 3): Let there exist Lagrange function $L(x, v, t)$ such that for any world line $x^\mu(t)$, $x^0(t) = ct$, the relation

$$\frac{d}{d t} \frac{\partial L}{\partial v^i} - \frac{\partial L}{\partial x^i} = m \frac{d}{d t} \left( (1 - c^{-2} |v|^2)^{-1/2} v^i \right) - \sum_{k=0}^{N} \sum_{\alpha_1, ..., \alpha_k = 0}^{3} F_{\alpha_1 \cdots \alpha_k}(x) \frac{d x^{\alpha_1}}{d t} \cdots \frac{d x^{\alpha_k}}{d t}, \quad i = 1, 2, 3.$$ \hfill (2.5)
We define the coefficients $q$. The interaction is defined by the product of the charge and the external vector potential $A_0(x,t)$.

The relativistic causal Coulomb law is given by the equations of the type (2.11)

$$m c^2 \frac{d^2 x}{dt^2} \left( \frac{1 - q^2 |v|^2}{c^2} \right)^{1/2} + q \sum_{i=1}^3 A_i(x,t) c^{-1} v + q A_0(x,t) = 0$$

and the coefficients $F_{i\alpha_1\cdots \alpha_k}(x)$ in the equations (2.4) are

$$F_{i\alpha_1\cdots \alpha_k}(x) = 0, \ k \neq 1, \ i = 1, 2, 3, \ \alpha_1, \ldots, \alpha_k = 0, \ldots, 3,$$

$$F_{ij}(x) = \frac{\partial A_j(x,t)}{\partial x^i} - \frac{\partial A_i(x,t)}{\partial x^j}, \ i, j = 1, 2, 3,$$

$$F_{i0}(x) = \frac{\partial A_0(x,t)}{\partial x^i} - \frac{1}{c} \frac{\partial A_i(x,t)}{\partial t}, \ i = 1, 2, 3. \quad (2.8)$$

We define the coefficients

$$F_{00}(x) = 0, \ F_{0i}(x) = -F_{i0}(x), \ i = 1, 2, 3. \quad (2.9)$$

Then the tensor $F_{\alpha\beta}(x)$ is antisymmetric and the identity

$$\sum_{\alpha,\beta=0}^3 F_{\alpha\beta}(x) \frac{dt}{ds} \frac{dx^\alpha}{dt} \frac{dt}{ds} \frac{dx^\beta}{dt} = 0 \quad (2.10)$$

of the type (2.3) holds. By making use of the second identity (2.2) and the relations (2.8) - (2.10) we can rewrite the equation (2.4) with the coefficients (2.7), (2.8) as the relativistic Newton second law with Lorentz force

$$m c^2 \frac{d^2 x}{dt^2} \left( \frac{1 - q^2 |v|^2}{c^2} \right)^{1/2} + \sum_{\alpha,\beta=0}^3 F_{\alpha\beta}(x) c^{-1} \frac{dt}{ds} \frac{dx^\alpha}{ds} = 0.$$

The interaction is defined by the product of the charge $q$ and the external vector potential $A_\mu(x,t)$.

Let a distribution $e_0(x) \in S'(\mathbb{R}^4)$ with support in the closed upper light cone be a fundamental solution of the wave equation

$$-(\partial_\mu, \partial_\mu) e_0(x) = \delta(x), \ (\partial_\mu, \partial_\nu) = \left( \frac{\partial}{\partial x^\mu} \right)^2 - \sum_{i=1}^3 \left( \frac{\partial}{\partial x^i} \right)^2. \quad (2.12)$$

The equation (2.12) solution is unique in the class of distributions with supports in the closed upper light cone (Ref. 10, relation (2.42)). Due to (Ref. 11, Sect. 30) this unique causal distribution is

$$e_0(x) = -(2\pi)^{-1} \theta(x^0) \delta((x,x)), \quad (2.13)$$

$$(x,y) = x^0 y^0 - \sum_{k=1}^3 x^k y^k, \ \theta(x) = \begin{cases} 1, & x \geq 0, \\ 0, & x < 0. \end{cases}$$

The relativistic causal Coulomb law is given by the equations of the type (2.11)

$$m_k \frac{d}{dt} \left( \left( 1 - \frac{1}{c^2} \frac{dx_k}{dt} \right)^{1/2} \frac{dx_k^\mu}{dt} \right) = -q_k \eta^{\mu\nu} \sum_{\nu=0}^3 \frac{c^{-1}}{dt} \sum_{j=1, j \neq k}^2 F_{j;\mu\nu}(x_k, x_j). \quad (2.14)$$
where the strength $F_{j\mu}(x_k, x_j)$ is given by the relation (1.2) with Liénard - Wiechert vector potential of the type (1.3)

$$A_{j\mu}(x_k, x_j) = -4\pi q_jK \sum_{\nu=0}^{3} \eta_{\mu\nu} \int dt e_0(x_k - x_j(t)) \frac{dx^\nu_j(t)}{dt} =$$

$$-q_jK \frac{d}{dt} x^\mu_j(t) \left( c|x_k - x_j(t)| - \sum_{i=1}^{3} (x^i_k - x^i_j(t)) \frac{dx^i_j(t)}{dt} \right)^{-1} \bigg|_{t=t(0)}, \quad (2.15)$$

$$x^0_k - ct(0) = |x_k - x_j(t(0))|. \quad (2.16)$$

Here $K$ is the constant of the causal electromagnetic interaction for two particles with the charges $q_j$. The support of the distribution (2.13) lies in the upper light cone boundary. The interaction speed is equal to that of light. It is easy to prove the second relation (2.15) by making change of the integration variable

$$x^0_k - ct(s) = (|x_k - x_j(t(s))|^2 + s)^{1/2}. \quad (2.17)$$

For $s = 0$ the relation (2.17) coincides with the relation (2.16).

The equations (2.14), (1.2), (2.15) are the relativistic causal version of the Coulomb law. The Lorentz invariant distribution (2.13) defines the delay. The Lorentz invariant solutions of the equation (2.12) are described in Ref. 3. By making use of these solutions it is possible to describe the Lorentz covariant equations of the type (2.14), (1.2), (2.15). The equations (2.14), (1.2), (2.15) are Lorentz covariant and causal due to the distribution (2.13). The quantum version of the equations (2.14), (1.2), (2.15) is defined in Ref. 10. The solutions of these causal equations do not contain the diverging integrals similar to the diverging integrals of the quantum electrodynamics.

For a world line $x^\mu_j(t)$ we define the vector

$$J^\mu(x, x_j) = -(\partial_x, \partial_{x_j}) \int dt e_0(x - x_j(t)) \frac{dx^\mu_j(t)}{dt} = \int dt \delta(x - x_j(t)) \frac{dx^\mu_j(t)}{dt} =$$

$$\left( \frac{d}{dx^0} x^\mu_j \left( c^{-1}x^0 \right) \right) \delta \left( x - x_j \left( c^{-1}x^0 \right) \right), \mu = 0, ..., 3. \quad (2.18)$$

The condition $x^0_j(t) = ct$ implies the equalities

$$\frac{\partial}{\partial x^0} J^0(x, x_j) = -\sum_{i=1}^{3} \left( \frac{d}{dx^0} x^i_j \left( c^{-1}x^0 \right) \right) \frac{\partial}{\partial x^i} \delta \left( x - x_j \left( c^{-1}x^0 \right) \right), \quad (2.19)$$

$$\frac{\partial}{\partial x^i} J^i(x, x_j) = \left( \frac{d}{dx^0} x^i_j \left( c^{-1}x^0 \right) \right) \frac{\partial}{\partial x^i} \delta \left( x - x_j \left( c^{-1}x^0 \right) \right), \quad i = 1, 2, 3. \quad (2.20)$$

The equalities (2.19), (2.20) imply the continuity equation

$$\sum_{\mu = 0}^{3} \frac{\partial}{\partial x^\mu} J^\mu(x, x_j) = 0. \quad (2.21)$$

The integration of the relation

$$e_0(x - x_j(t)) \frac{dx^\mu_j(t)}{dt} = \int d^4y e_0(x - y) \delta(y - x_j(t)) \frac{dx^\mu_j(t)}{dt} \quad (2.22)$$
yields
\[ \int dt e_0(x - x_j(t)) \frac{dx'^\mu(t)}{dt} = \int d^4y e_0(x - y) J^\mu(y, x_j). \] (2.23)

The relations (2.21), (2.23) imply the gauge condition for the vector potential (2.15)
\[ \sum_{\mu=0}^{3} \eta^{\mu\nu} \frac{\partial}{\partial x^\nu} A_{j,\mu}(x, x_j) = 0. \] (2.24)

Due to the gauge condition (2.24) the tensor (1.2), (2.15) satisfies Maxwell equations with the current proportional to the current (2.18).

The substitution \( K = G \) and two positive or two negative gravitational masses \( q_1 = \pm m_1, q_2 = \pm m_2 \) into the equations (2.14), (1.2), (2.15) yields the relativistic causal Newton gravity law (1.1) – (1.3). By changing the constants \( K = G, q_1 = \pm m_1, q_2 = \pm m_2 \) in the equations from Ref. 10 we have the quantum version of the equations (1.1) – (1.3). The substitution \( K = G \) and also one positive and one negative gravitational masses \( q_1 = \pm m_1, q_2 = \mp m_2 \) into the equations (2.14), (1.2), (2.15) yields the galaxies scattering with an acceleration. For the negative constant \( K \) of the causal electromagnetic interaction the protons and electons really couldn’t exist together.

The idea of the electromagnetic and gravitational interactions similarity is not new. Einstein\(^{12}\): “The theoretical physicists studying the problems of general relativity can hardly doubt now that the gravitational and electromagnetic fields should have the same nature.”

The relativistic Newton gravity law for the solar system was proposed in Ref. 3
\[ \frac{d}{dt} \left( (1 - \frac{1}{c^2} \left| \frac{dx_k}{dt} \right|^2 \right)^{-1/2} \frac{dx'^\mu}{dt} = -m_{10} \eta^{\mu\nu} \sum_{\nu=0}^{3} \frac{1}{c} \frac{dx'_k}{dt} \sum_{j=1,\ldots,10, j\neq k} \frac{1}{m_{10}} F_{j, \mu\nu}(x_k, x_j), \] (2.25)

\[ k = 1, \ldots, 10, \mu = 0, \ldots, 3. \] We give the number \( k = 1 \) for Mercury, the number \( k = 2 \) for Venus, the number \( k = 3 \) for the Earth, the number \( k = 4 \) for Mars, the number \( k = 5 \) for Jupiter, the number \( k = 6 \) for Saturn, the number \( k = 7 \) for Uranus, the number \( k = 8 \) for Neptune, the number \( k = 9 \) for Pluto and the number \( k = 10 \) for the Sun.

The calculation of the equation (2.25) orbits is a difficult and laborious task. We consider the Sun resting at the coordinates origin. Substituting the Sun world line \( x^0_{10}(t) = ct, x^i_{10}(t) = 0, i = 1, 2, 3, \) into the equalities (1.2), (1.3) we have
\[ F_{10;i}(x; x_{10}) = 0, i, j = 1, 2, 3, F_{10;00}(x; x_{10}) = -m_{10} G |x|^{-3} x^i, i = 1, 2, 3. \] (2.26)

Due to the relations (1.2), (1.3) the value \( m_{10}^{-1} F_{j, \mu\nu}(x_k, x_j) \) is proportional to \( m_j m_{10}^{-1}, j = 1, \ldots, 9. \) According to (Ref. 5, Chap. 25, Sec. 25.1, Appendix 25.1) for Jupiter the ratio \( m_5 m_{10}^{-1} \approx 0.95 \cdot 10^{-3} \) is maximal. For the Earth the ratio \( m_3 m_{10}^{-1} \approx 3.01 \cdot 10^{-6}. \) We neglect the action of any planet on all of the other planets. Substituting the Sun world line \( x^0_{10}(t) = ct, x^i_{10}(t) = 0, i = 1, 2, 3, \) and the relations (2.26) into the equations (2.25) we get
\[ \frac{d}{dt} \left( (1 - \frac{1}{c^2} \left| \frac{dx_k}{dt} \right|^2 \right)^{-1/2} \frac{dx_k}{dt} = -m_{10} G \frac{x_k}{|x_k|^3}, \] (2.27)
\( k = 1, ..., 9 \). Due to the equations (2.27) the angular momentum and the energy

\[
M_l(x_k) = \sum_{i,j=1}^{3} \epsilon_{ijl} \left( x^i_k \frac{dx^j_k}{dt} - x^j_k \frac{dx^i_k}{dt} \right) \left( 1 - \frac{1}{c^2} \left| \frac{dx_k}{dt} \right|^2 \right)^{-1/2}, \tag{2.28}
\]

\[
E(x_k) = c^2 \left( 1 - \frac{1}{c^2} \left| \frac{dx_k}{dt} \right|^2 \right)^{-1/2} - \frac{m_{10} G}{|x_k|}, \tag{2.29}
\]

\( l = 1, 2, 3, k = 1, ..., 9 \), are time independent. The antisymmetric in all indices tensor \( \epsilon_{ijl} \) has the normalization \( \epsilon_{123} = 1 \). Let the third axis coincide with the constant vector (2.28). The vector \( x_k \) is orthogonal to the constant vector (2.28). We introduce the polar coordinates in the plane orthogonal to the vector (2.28)

\[
x_k^1(t) = r_k(t) \cos \phi_k(t), \quad x_k^2(t) = r_k(t) \sin \phi_k(t), \quad x_k^3(t) = 0, \tag{2.30}
\]

\( k = 1, ..., 9 \). The relations (2.28), (2.29) imply

\[
\begin{align*}
&\quad \frac{r_k^2(t)}{E(x_k) + \frac{m_{10} G}{r_k(t)}} \frac{d\phi_k}{dt} = c^2 |M(x_k)|, \\
&\quad \frac{\left( E(x_k) + \frac{m_{10} G}{r_k(t)} \right)^2}{r_k(t)} \frac{dr_k}{dt}^2 = \\
&\quad c^2((E(x_k))^2 - c^4) + \frac{2m_{10} Gc^2E(x_k)}{r_k(t)} + \frac{m_{10}^2 G^2 c^2 - |M(x_k)|^2 c^4}{r_k^2(t)}, \tag{2.31}
\end{align*}
\]

\( k = 1, ..., 9 \). Let the constants (2.28), (2.29) satisfy the inequalities

\[
(E(x_k))^2 - c^4 < 0, \tag{2.32}
\]

\[
c^2 |M(x_k)|^2 - m_{10}^2 G^2 > 0, \tag{2.33}
\]

\[
c^2 |M(x_k)|^2 ((E(x_k))^2 - c^4) + m_{10}^2 G^2 c^4 > 0, \tag{2.34}
\]

\( k = 1, ..., 9 \). Due to Ref. 3 the equations (2.31) have the solutions

\[
a_k(1 - e_k^2) \frac{r_k(t)}{r_k(t)} = 1 + e_k \cos \gamma_k(\phi_k(t) - \phi_{k;0}). \tag{2.35}
\]

The orbit radius \( r_k(t) \) is given by

\[
r_k(t(\xi_k)) = m_{10} G E(x_k)(c^4 - (E(x_k))^2)^{-1}(1 + e_k \sin \xi_k),
\]

\[
t(\xi_k) = m_{10} G c^{-1}(c^4 - (E(x_k))^2)^{-3/2}(c^4(\xi_k - \xi_{k;0}) - e_k (E(x_k))^2 \cos \xi_k). \tag{2.36}
\]

The perihelion angle \( \phi_{k;0} \), the parameter \( \xi_{k;0} \) and the values

\[
a_k(1 - e_k^2) = (c^2 |M(x_k)|^2 - m_{10}^2 G^2)(m_{10} G E(x_k))^{-1},
\]

\[
e_k = (c^2 |M(x_k)|^2 ((E(x_k))^2 - c^4) + m_{10}^2 G^2 c^4)^{1/2}(m_{10} G E(x_k))^{-1},
\]

\[
\gamma_k = \left( 1 - \frac{m_{10}^2 G^2}{c^2 |M(x_k)|^2} \right)^{1/2}, \quad k = 1, ..., 9. \tag{2.37}
\]
are constant. The orbit eccentricities: \( e_1 = 0.21, e_2 = 0.007, e_3 = 0.017, e_4 = 0.093, e_5 = 0.048, e_6 = 0.056, e_7 = 0.047, e_8 = 0.009, e_9 = 0.249. \) Therefore \( 0 < e_k < 1, k = 1, \ldots, 9. \) Let us suppose \( E(x_k) > 0, k = 1, \ldots, 9. \) The inequalities (2.33) and \( e_k^2 < 1 \) imply the inequality (2.32). For \( e_k^2 < 1 \) the equation (2.35) defines an ellipse with a precession given by the coefficient \( \gamma_k. \) The focus of this ellipse is the coordinates origin. It is a relativistic analogue of the first Kepler law. The time independence of the vector (2.28) is the relativistic second Kepler law. The equations (2.36) define the time dependence of the radius \( r_k. \) For \( c \to \infty \) the equations (2.27) solutions tend to the Kepler problem solutions. (For the Kepler problem equations the multiplier \( (1 - c^2(d^2x_k/dt^2)^{-1/2}} \) is absent in the equations (2.27)).

Let us express the constants in the equations (2.35), (2.36) trough the astronomical data. The ellipse (2.35) major "semi-axis" is equal to

\[
a_k = m_{10}GE(x_k)(c^2 - (E(x_k))^2)^{-1}, \quad k = 1, \ldots, 9. \tag{2.38}
\]

For the parameters \( \pm \pi/2 \) we have the extremal radii

\[
r_k(t(\pm \pi/2)) = a_k(1 \pm e_k), \quad k = 1, \ldots, 9. \tag{2.39}
\]

Hence, the "period" of the motion along the ellipse (2.35) is equal to

\[
T_k = 2(t_k(\pi/2) - t_k(-\pi/2)) = 2\pi m_{10}Ge^3(c^2 - (E(x_k))^2)^{-3/2}, \quad k = 1, \ldots, 9. \tag{2.40}
\]

Let us define the mean "angular frequency" \( \omega_k = 2\pi T_k^{-1}. \) The relation (2.40) implies

\[
\omega_k = (c^2 - (E(x_k))^2)^{3/2}(m_{10}Ge^3)^{-1},
\]

\[
(E(x_k))^2 = c^2(\omega_k m_{10}G)^{2/3}, \quad k = 1, \ldots, 9. \tag{2.41}
\]

The substitution of the expression (2.41) into the equality (2.38) yields

\[
m_{10}G = \omega_k^2a_k^3(2^{-1}(1 + \sigma_k(1 - (2ak\omega_k c^{-1})^{1/2}))^{3/2}, \quad \sigma_k = \pm 1, \quad k = 1, \ldots, 9. \tag{2.42}
\]

According to (Ref. 5, Chap. 25, Sec. 25.1, Appendix 25.1) the values \( \omega_k^2a_k^3c^{-2} = 1477m \) for \( k = 1, 2, 3, 4, 6 \) (Mercury, Venus, the Earth, Mars and Saturn), the values \( \omega_k^2a_k^3c^{-2} = 1478m \) for \( k = 5, 8, \) (Jupiter and Neptune), the value \( \omega_k^2a_k^3c^{-2} = 1476m \) for Uranus, the value \( \omega_k^2a_k^3c^{-2} = 1469m \) for Pluto; the major semi-axes \( a_1 = 0.5791 \cdot 10^{11}m, \quad a_2 = 1.0821 \cdot 10^{11}m, \quad a_3 = 1.4960 \cdot 10^{11}m, \quad a_4 = 2.2794 \cdot 10^{11}m, \quad a_5 = 7.783 \cdot 10^{11}m, \quad a_6 = 14.27 \cdot 10^{11}m, \quad a_7 = 28.69 \cdot 10^{11}m, \quad a_8 = 44.98 \cdot 10^{11}m, \quad a_9 = 59.00 \cdot 10^{11}m. \) The values \( \omega_k^2a_k^3c^{-2} = a_k^{-1} \cdot \omega_k^2a_k^3c^{-2}, \quad k = 1, \ldots, 9, \) are negligible and therefore \( m_{10}G \approx \omega_k^2a_k^3(2^{-1}(1 + \sigma_k))^{3/2}. \) For \( \sigma_k = 1 \) this expression agrees with the third Kepler law (1.11): \( m_{10}G = \omega_k^2a_k^3. \) Choosing \( \sigma_k = 1 \) in the relation (2.42) we get the third Kepler law

\[
m_{10}G = \omega_k^2a_k^3\left(2^{-1}(1 + (1 - 4\omega_k^2a_k^3c^{-2})^{1/2})\right)^{-3/2} \approx \omega_k^2a_k^3\left(1 + \frac{3}{2}\omega_k^2a_k^3c^{-2}\right) \tag{2.43}
\]

for the orbits (2.35), (2.36). The Sun mass values (2.43) obtained in the relativistic Kepler problem agrees perfectly with values \( \omega_k^2a_k^3 \) obtained in Kepler problem. The substitution of the vector (2.30), \( r_k(t) = a_k, \phi_k(t) = \omega_k(t - t_0), \) into the equation (2.27) yields the third Kepler law

\[
m_{10}G = \omega_k^2a_k^3(1 - \omega_k^2a_k^3c^{-2})^{-1/2} \approx \omega_k^2a_k^3\left(1 + \frac{1}{2}\omega_k^2a_k^3c^{-2}\right), \quad k = 1, \ldots, 9, \tag{2.44}
\]
for the equation (2.27) periodic circular orbits. The substitution of the expression (2.43) into the equality (2.41) yields

\[ c^{-4}(E(x_k))^2 = 1 - 2\omega_k^2 a_k^2 c^{-2} \left(1 + (1 - 4\omega_k^2 a_k^2 c^{-2})^{1/2}\right)^{-1} \approx 1 - \omega_k^2 a_k^2 c^{-2}, \quad k = 1, \ldots, 9. \]  

(2.45)

By making use of the relations (2.37), (2.38), (2.43), (2.45) we have

\[ \gamma_k = \left(1 + 4\omega_k^2 a_k^2 c^{-2}(1 - e_k^2)^{-1} \right) \left(1 + (1 - 4\omega_k^2 a_k^2 c^{-2})^{1/2}\right)^{-2/3} \approx 1 - \frac{\omega_k^2 a_k^2 c^{-2}}{2(1 - e_k^2)}, \]  

(2.46)

\( k = 1, \ldots, 9. \) The value \( 2^{-1}\omega_k^2 a_k^2 c^{-2}(1 - e_k^2)^{-1} \) is maximal for Mercury: \( 2^{-1}\omega_1^2 a_1^2 c^{-2}(1 - e_1^2)^{-1} \approx 1.3341 \cdot 10^{-8}. \) The precession coefficients (2.46) of the orbits (2.35) are practically equal to one for all planets. It agrees with Tycho Brahe’s astronomical observations used by Kepler. The relations (2.35), (2.46) imply the perihelion angle

\[ \phi_{k,l} \approx \phi_{k,0} + 2\pi l(1 + 2^{-1}\omega_k^2 a_k^2 c^{-2}(1 - e_k^2)^{-1}), \quad l = 0, \pm 1, \pm 2, \ldots \]  

(2.47)

The substitution of the relations (2.38), (2.43), (2.45), (2.46) into the equalities (2.35), (2.36) yields

\[ e_k \cos \left(\left(1 - 2^{-1}\omega_k^2 a_k^2 c^{-2}(1 - e_k^2)^{-1}\right) \left(\phi_k(t) - \phi_{k,0}\right)\right) \approx a_k \left(1 - e_k^2\right) r_k^{-1}(t) - 1, \]  

(2.48)

\[ r_k(t_k(\xi_k)) \approx a_k(1 + e_k \sin \xi_k), \quad \omega_k t_k(\xi_k) \approx \xi_k - \xi_{k,0} - e_k(1 - \omega_k^2 a_k^2 c^{-2}) \cos \xi_k, \]  

(2.49)

\( k = 1, \ldots, 9. \) Let us define the constant \( \xi_{k,0} \) in the second equality (2.49) by choosing the initial time moment \( t_k(0) = 0. \) Then the equalities (2.49) have the form

\[ r_k(t_k(\xi_k)) \approx a_k(1 + e_k \sin \xi_k), \quad \omega_k t_k(\xi_k) \approx \xi_k + e_k(1 - \omega_k^2 a_k^2 c^{-2}) (1 - \cos \xi_k), \]  

(2.50)

\( k = 1, \ldots, 9. \) Let the direction of the first axis be orthogonal to the vector \( \mathbf{M}(\mathbf{x}_1). \) Let the direction of the third axis coincide with the direction of vector \( \mathbf{M}(\mathbf{x}_3). \) Then the second axis lies in the plane stretched on the vectors \( \mathbf{M}(\mathbf{x}_1) \) and \( \mathbf{M}(\mathbf{x}_3). \) Due to the relations (2.30) the Mercury and Earth orbits

\[ x_1^1(t) = r_1(t) \cos \phi_1(t), \quad x_1^2(t) = -r_1(t) \cos \theta_1 \sin \phi_1(t), \quad x_1^3(t) = r_1(t) \sin \theta_1 \sin \phi_1(t), \]  

\[ x_3^1(t) = r_3(t) \cos \phi_3(t), \quad x_3^2(t) = r_3(t) \sin \phi_3(t), \quad x_3^3 = 0 \]  

(2.51)

where the inclination of Mercury orbit plane \( \theta_1 = 7^\circ \) and the values \( r_k(t), \phi_k(t), k = 1, 3, \) satisfy the equations (2.48), (2.50). For the definition of Mercury and the Earth trajectories it is necessary to define the perihelion angles \( \phi_{1,0}, \phi_{3,0} \) in the equations (2.48).

"Observations of Mercury do not give the absolute position of the planet in space but only the direction of a line from the planet to the observer." (Ref. 6, p. 363.) The advance of Mercury’s perihelion is given by the angle

\[ \cos \alpha = \frac{(\mathbf{x}_1(t_1(\xi_{1,1})) - \mathbf{x}_3(t_3(\xi_{3,1})), \mathbf{x}_1(t_1(\xi_{1,2})) - \mathbf{x}_3(t_3(\xi_{3,2})))}{|\mathbf{x}_1(t_1(\xi_{1,1})) - \mathbf{x}_3(t_3(\xi_{3,1}))||\mathbf{x}_1(t_1(\xi_{1,2})) - \mathbf{x}_3(t_3(\xi_{3,2}))|}, \]  

\[ c(t_3(\xi_{3,k}) - t_1(\xi_{1,k})) = |\mathbf{x}_1(t_1(\xi_{1,k})) - \mathbf{x}_3(t_3(\xi_{3,k}))|, \quad k = 1, 2, \]  

\[ t_1(\xi_{1,2}) - t_1(\xi_{1,1}) \leq 100T_3 \leq t_1(\xi_{1,2}) - t_1(\xi_{1,1}) + T_1 \]  

(2.52)
where the parameters $\xi_{1,1}, \xi_{1,2}$ are defined by Mercury’s perihelion points, the parameters $\xi_{3,1}, \xi_{3,2}$ are the solutions of the second equation (2.52), the numbers $T_1, T_3$ are the orbit "periods" of Mercury and the Earth. The quotient $T_3/T_1$ of the Earth and Mercury orbit "periods" is approximately equal to 4.15.

By making use of the equations (2.50) we obtain the parameters corresponding to Mercury’s perihelion points:

$$a_i^{-1}r_1(t_1(\xi_{1,k})) \approx 1 - e_i, \quad \omega_1t_1(\xi_{1,k}) \approx \pi(2l_k + 3/2) + e_i(1 - \omega_1^2a_i^2c^{-2}),$$

$$\xi_{1,k} \approx \pi(2l_k + 3/2), \quad k = 1, 2, \quad (2.53)$$

where $l_k$ are the integers. The first relation (2.53) coincides with the equality (2.39).

According to (Ref. 5, Chap. 25, Sec. 25.1, Appendix 25.1) $c^{-1}\omega_1 = 275.8 \cdot 10^{-17} m^{-1}$, $c^{-1}\omega_3 = 66.41 \cdot 10^{-17} m^{-1}, a_1 = 0.5791 \cdot 10^{11} m, a_3 = 1.4960 \cdot 10^{11} m.$ The substitution of the second equality (2.53) into the third relation (2.52) yields $l_2 - l_1 = 415.$

Due to the second relation (2.52)

$$x_3(t_3(\xi_{3,k})) = x_3(t_1(\xi_{1,k})) + c^{-1}|x_1(t_1(\xi_{1,k})) - x_3(t_3(\xi_{3,k}))/v_3(t_{3,k}), \quad k = 1, 2. \quad (2.54)$$

The Earth speed is small compared with the speed of light: $c^{-1}|v_3| \approx c^{-1}\omega_3a_3 \approx 0.9935 \cdot 10^{-4}.$ We neglect this value (arcsin $10^{-4} \approx 0.00057$). Then the relations (2.52), (2.54) imply

$$\cos \alpha \approx \frac{(x_1(t_1(\xi_{1,1})) - x_3(t_1(\xi_{1,1})), x_1(t_1(\xi_{1,2})) - x_3(t_1(\xi_{1,2})))}{|x_1(t_1(\xi_{1,1})) - x_3(t_1(\xi_{1,1}))||x_1(t_1(\xi_{1,2})) - x_3(t_1(\xi_{1,2}))|} \quad (2.55)$$

where the parameters $\xi_{1,k}, \quad k = 1, 2,$ are given by the third relation (2.53) and the relation $l_2 = l_1 + 415.$

Let us consider Mercury’s perihelion points corresponding to the integers $l_1 = 0$ and $l_2 = 415.$ The substitution of the values corresponding to the Mercury’s perihelion, defined by the first equation (2.53), into the equation (2.48) yields

$$r_1(t_1(\pi(2l + 3/2))) \approx a_1(1 - e_1),$$

$$\phi_1(t_1(\pi(2l + 3/2))) \approx \phi_{1,0} + 2\pi l \left(1 + 2^{-1}\omega_1^2a_1^2c^{-2}(1 - e_1^2)^{-1}\right) \quad (2.56)$$

since the value $\omega_1^2a_1^2c^{-2} \approx 2.5509 \cdot 10^{-8}$ is negligible. We substitute the time, defined by the second relation (2.53), into the second relation (2.50) for the Earth

$$r_3(t_1(\xi_{3}(l))) \approx a_3(1 + e_3) \sin \xi_{3}(l),$$

$$\omega_3t_1(\pi(2l + 3/2)) \approx \omega_3\omega_1^{-1}(\pi(2l + 3/2) + e_1(1 - \omega_1^2a_1^2c^{-2})) \approx$$

$$\xi_{3}(l) - e_3(1 - \omega_3^2a_3^2c^{-2}) (\cos \xi_{3}(l) - 1). \quad (2.57)$$

Solving the second equation (2.57) we get $\xi_{3}(0) \approx 1.1748, \xi_{3}(415) \approx 629.09.$ Substituting these values in the first equation (2.57) we have $a_3^{-1}r_3(\xi_{3}(0)) \approx 1.0157, a_3^{-1}r_3(\xi_{3}(415)) \approx 1.0118.$ We substitute the first equation (2.57) in the equation (2.48) for the Earth

$$\cos \left(1 - \frac{\omega_3^2a_3^2c^{-2}}{2(1 - e_3^2)}\right) \left(\phi_3(t_1(\pi(2l + 3/2))) - \phi_{3,0}\right) \approx \frac{e_3 + \sin \xi_{3}(l)}{1 + e_3 \sin \xi_{3}(l)}. \quad (2.58)$$
The function in the right-hand side of the equation (2.58) is monotonic with respect to the variable $e_3$ on the interval $0 \leq e_3 \leq 1$. Calculating the values of this function at the points $e_3 = 0, 1$ we get the estimation for the module of this function which implies that the equation (2.58) has a solution. Substituting the solutions $e_3(l), l = 0, 415$, of the second equation (2.57) in the equation (2.58) we get the angles in radians

$$\phi_3(t_1(\pi(2 \cdot 0 + 3/2))) \approx \phi_{3,0} + 2.7521,$$

$$\phi_3(t_1(\pi(2 \cdot 415 + 3/2))) \approx \phi_{3,0} + 2.3544 + 2\pi \cdot 99 \left(1 + 2^{-1/3}a_0^2c^{-2}(1 - e_3^2)^{-1}\right)$$

(2.59)

since the value $\omega_3^2a_0^2c^{-2} \approx 0.9870 \cdot 10^{-8}$ is negligible. Substituting the radii and the angles (2.56), the radii (2.57) and the angles (2.59) in the equations (2.51), (2.55) we get the equation

$$\cos \alpha(0, 415) \approx (((1 - e_1)\cos \phi_{1,0} - 1.0157a_3a_1^{-1}\cos(\phi_{3,0} + 2.7521))$$

$$\times((1 - e_1)\cos(\phi_{1,0} + 415\pi\omega_0^2a_1^2c^{-2}(1 - e_1^2)^{-1})$$

$$-1.0118a_3a_1^{-1}\cos(\phi_{3,0} + 2.3544 + 99\pi\omega_0^2a_0^2c^{-2}(1 - e_3^2)^{-1}))$$

$$+(0.99255(1 - e_1)\sin\phi_{1,0} + 1.0157a_3a_1^{-1}\sin(\phi_{3,0} + 2.7521))$$

$$\times(0.99255(1 - e_1)\sin(\phi_{1,0} + 415\pi\omega_0^2a_1^2c^{-2}(1 - e_1^2)^{-1})$$

$$+(1.0118a_3a_1^{-1}\sin(\phi_{3,0} + 2.3544 + 99\pi\omega_0^2a_0^2c^{-2}(1 - e_3^2)^{-1}))$$

$$+0.01485(1 - e_1^2)\sin(\phi_{1,0} + 415\pi\omega_0^2a_1^2c^{-2}(1 - e_1^2)^{-1}))$$

$$\times(((1 - e_1)\cos(\phi_{1,0} - 1.0157a_3a_1^{-1}\cos(\phi_{3,0} + 2.7521))^2$$

$$+(0.99255(1 - e_1)\sin\phi_{1,0} + 1.0157a_3a_1^{-1}\sin(\phi_{3,0} + 2.7521))^2$$

$$+0.01485(1 - e_1^2)\sin^2\phi_{1,0})^{-1/2}$$

$$\times(((1 - e_1)\cos(\phi_{1,0} + 415\pi\omega_0^2a_1^2c^{-2}(1 - e_1^2)^{-1})$$

$$-1.0118a_3a_1^{-1}\cos(\phi_{3,0} + 2.3544 + 99\pi\omega_0^2a_0^2c^{-2}(1 - e_3^2)^{-1}))^2$$

$$+(0.99255(1 - e_1)\sin(\phi_{1,0} + 415\pi\omega_0^2a_1^2c^{-2}(1 - e_1^2)^{-1})$$

$$+1.0118a_3a_1^{-1}\sin(\phi_{3,0} + 2.3544 + 99\pi + \omega_0^2a_0^2c^{-2}(1 - e_3^2)^{-1}))^2$$

$$+0.01485(1 - e_1^2)\sin^2(\phi_{1,0} + 415\pi\omega_0^2a_1^2c^{-2}(1 - e_1^2)^{-1}))^{-1/2}. \quad (2.60)$$

The perihelion angles $\phi_{1,0}, \phi_{3,0}$ are needed. Let the perihelion angles $\phi_{1,0}, \phi_{3,0}$ in the equation (2.60) be equal to zero. Then $\alpha(0, 415) = 17^\circ.889$. According to (Ref. 5, Chap. 40, Sec. 40.5, Appendix 40.3), the advance of Mercury’s perihelion, observed by the astronomers from the Earth, is $1^\circ.55548 \pm 0^\circ.00011$ per century.

### 3 III. GENERAL RELATIVITY PLANET ORBITS

The equations (1.16) are the Kepler problem equations for the proper time $\tau$. Due to the equations (1.16) the angular momentum

$$M_l(x) = \sum_{i,j=1}^{3} \epsilon_{ijl}(x^i dx^j d\tau - x^j dx^i d\tau), \quad l = 1, 2, 3, \quad (3.1)$$

and the energy

$$E(x) = \frac{1}{2} \left(\frac{dx}{d\tau}\right)^2 - \frac{m_0G}{|x|} \quad (3.2)$$

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are independent of the proper time $\tau$. The antisymmetric in all indices tensor $\epsilon_{ijl}$ has the normalization $\epsilon_{123} = 1$. Let the third axis coincide with the constant vector (3.1). The vector $\mathbf{x}(\tau)$ is orthogonal to the constant vector (3.1): $x^3(\tau) = 0$. Inserting the vector
\[ x^1(\tau) = r(\tau) \cos \phi(\tau), \quad x^2(\tau) = r(\tau) \sin \phi(\tau), \quad x^3(\tau) = 0 \] into the relations (3.1), (3.2) we get for the angular momentum
\[ |\mathbf{M}(x)| = r^2(\tau) \frac{d\phi}{d\tau} \] and for the energy
\[ E(x) = \frac{1}{2} \left( \left( \frac{dr}{d\tau} \right)^2 + r^2(\tau) \left( \frac{d\phi}{d\tau} \right)^2 \right) - \frac{m_{10}G}{r(\tau)}. \] It follows from the equations (3.4), (3.5) that
\[ \left( \frac{dr}{d\tau} \right)^2 = 2E(x) + 2\frac{m_{10}G}{r} - \frac{|\mathbf{M}(x)|^2}{r^2}. \] Dividing the equation (3.6) by the square of the equation (3.4) we get
\[ \left( \frac{d}{d\phi} \frac{1}{r} \right)^2 = \frac{2E(x)}{|\mathbf{M}(x)|^2} + \frac{2m_{10}G}{|\mathbf{M}(x)|^2r} - \frac{1}{r^2}. \] The function
\[ a(1 - e^2)r^{-1} = 1 + e \cos (\phi - \phi_0), \] \[ a(1 - e^2) = m_{10}^{-1}G^{-1}|\mathbf{M}(x)|^2, \] \[ e^2 = 1 + 2m_{10}^2G^{-2}E(x)|\mathbf{M}(x)|^2 \] is the solution of the equation (3.7). $\phi_0$ is the perihelion angle. We assume that the constants (3.1), (3.2) satisfy the inequality
\[ 0 < -2E(x)|\mathbf{M}(x)|^2 \leq m_{10}^2G^2. \] The third equality (3.8) and the inequality (3.9) imply the inequality $0 \leq e^2 < 1$ for the orbit eccentricity. The relations (3.4), (3.8) imply the angle $\phi(\tau)$ dependence on $\tau$
\[ \tau = \tau_0 + \int_{\phi_0}^{\phi} d\psi \frac{a^2(1 - e^2)^2|\mathbf{M}(x)|^{-1}}{(1 + e \cos (\psi - \phi_0))^2}. \] Let us consider the metric (1.15). The world line $x^\mu(t)$ is called geodesic, if it satisfies the geodesic equations (Ref. 5, Chap. 13, Sec. 13.4, equations (13.36)) with the proper time $\tau$ (the second relation (1.16))
\[ \sum_{\nu=0}^{3} g_{\nu\nu}(x) \frac{d^2 x^\nu}{d\tau^2} = -\frac{1}{2} \sum_{\mu,\nu=0}^{3} \left( \frac{\partial g_{\nu\nu}(x)}{\partial x^\mu} + \frac{\partial g_{\sigma\nu}(x)}{\partial x^\mu} - \frac{\partial g_{\nu\sigma}(x)}{\partial x^\mu} \right) \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau}. \]
\[ \sum_{\mu, \nu = 0}^{3} g_{\mu\nu}(x) \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau} = c^2 \] (3.12)

similar to the first identity (2.2). The differentiation of the identity (3.12) yields the identity

\[ \sum_{\sigma, \nu = 0}^{3} g_{\sigma\nu}(x) \frac{dx^\sigma}{d\tau} \left( \frac{dx^\nu}{d\tau} \right)^2 = \]

\[ -\frac{1}{2} \sum_{\sigma, \mu, \nu = 0}^{3} \left( \frac{\partial g_{\sigma\nu}(x)}{\partial x^\mu} + \frac{\partial g_{\sigma\mu}(x)}{\partial x^\nu} - \frac{\partial g_{\mu\nu}(x)}{\partial x^\sigma} \right) \frac{dx^\sigma}{d\tau} \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau}. \] (3.13)

Therefore the geodesic equation (3.11), \( \sigma = 0 \) is a linear combination of the geodesic equations (3.11), \( \sigma = 1, 2, 3 \). We consider these equations for the world line \( x^\mu(t), x^0(t) = ct \),

\[ (1 + 2 \frac{m_{10} G}{rc^2}) \left( 1 - 2 \frac{m_{10} G}{rc^2} + 2 \frac{m_{10} G^2}{r^2c^4} - \left( 1 + 2 \frac{m_{10} G}{rc^2} \right) \frac{1}{c^2} \left( \frac{dx}{dt} \right)^2 \right) \left( \frac{dx^i}{dt} \right)^2 = \]

\[ -\frac{m_{10} G}{r^2} \left( \frac{x^i}{r} \left( 1 - 2 \frac{m_{10} G}{rc^2} + \frac{1}{c^2} \left( \frac{dx}{dt} \right)^2 \right) - \frac{4}{c^2} \frac{dx^i}{dt} \frac{dr}{dt} \right), \quad i = 1, 2, 3. \] (3.14)

The geodesic equations (3.14) without the terms

\[ \pm 2 \frac{m_{10} G}{rc^2}, \quad 2 \frac{m_{10} G^2}{r^2c^4}, \quad \pm \frac{1}{c^2} \left( \frac{dx}{dt} \right)^2, \quad -\frac{4}{c^2} \frac{dx^i}{dt} \frac{dr}{dt}, \quad i = 1, 2, 3, \] (3.15)

coincide with the Kepler problem equations (1.16). The equations (1.16) are solved exactly. Let us insert the solution (3.3), (3.8) of the equations (1.16) into the geodesic equations (3.14) and estimate the terms (3.15)

\[ \frac{m_{10} G}{r_k c^2} = \omega_k^2 a_k^2 c^{-2} \frac{1 + e_k \cos(\phi_k - \phi_{k;0})}{1 - e_k^2} \ll 1, \] (3.16)

\[ \frac{m_{10} G^2}{r_k^2 c^4} = \omega_k^4 a_k^4 c^{-4} \frac{(1 + e_k \cos(\phi_k - \phi_{k;0}))^2}{(1 - e_k^2)^2} \ll 1, \] (3.17)

\[ \frac{1}{c^2} \left( \frac{dx_k}{dt} \right)^2 \approx \frac{a_k^2 \omega_k^2 c^{-2}(1 - e_k^2)^2}{(1 + e_k \cos(\phi_k - \phi_{k;0}))^2} \left( \frac{e_k^2 \sin^2(\phi_k - \phi_{k;0})}{(1 + e_k \cos(\phi_k - \phi_{k;0}))^2} + 1 \right) \ll 1, \] (3.18)

\[ \frac{4}{c^2} \left( \sum_{i=1}^{3} \left( \frac{dx^i}{dt} \right)^2 \right)^{1/2} \left( \frac{dr_k}{dt} \right) \approx 4 \omega_k^2 a_k^2 c^{-2} e_k (1 - e_k^2)^2 \sin(\phi_k - \phi_{k;0}) \right) \times \]

\[ \left( \frac{e_k^2 \sin^2(\phi_k - \phi_{k;0})}{(1 + e_k \cos(\phi_k - \phi_{k;0}))^2} + 1 \right)^{1/2} \ll \left( \sum_{i=1}^{3} \left( \frac{x^i_k}{r_k} \right)^2 \right)^{1/2} = 1, \] (3.19)

For \( k = 1, ..., 9 \). We assumed that \( m_{10} G c^{-2} \approx \omega_k^2 a_k^2 c^{-2} \) (the third Kepler law (1.11)) and \( |d\phi_k/dt| \approx \omega_k = 2\pi T_k^{-1} \). According to (Ref. 5, Chap. 25, Sec. 25.1, Appendix 25.1) the
values $ω_k^2 a_k^2 c^{-2} = 1477m$ for $k = 1, 2, 3, 4, 6$ (Mercury, Venus, the Earth, Mars and Saturn), the values $ω_l^2 a_l^2 c^{-2} = 1478m$ for $l = 5, 8$ (Jupiter and Neptune), the value $ω_5^2 a_5^2 c^{-2} = 1476m$ for Uranus, the value $ω_6^2 a_6^2 c^{-2} = 1469m$ for Pluto. The major semi-axes: $a_1 = 0.5791 \cdot 10^{11}m$, $a_2 = 1.0821 \cdot 10^{11}m$, $a_3 = 1.4960 \cdot 10^{11}m$, $a_4 = 2.2794 \cdot 10^{11}m$, $a_5 = 7.783 \cdot 10^{11}m$, $a_6 = 14.27 \cdot 10^{11}m$, $a_7 = 28.69 \cdot 10^{11}m$, $a_8 = 44.98 \cdot 10^{11}m$, $a_9 = 59.00 \cdot 10^{11}m$. For Mercury $ω_1^2 a_1^2 c^{-2} \approx 2.6 \cdot 10^{-8}$, for Venus $ω_2^2 a_2^2 c^{-2} \approx 1.4 \cdot 10^{-8}$, for the Earth $ω_3^2 a_3^2 c^{-2} \approx 9.9 \cdot 10^{-9}$, for Mars $ω_4^2 a_4^2 c^{-2} \approx 6.5 \cdot 10^{-9}$, for Jupiter $ω_5^2 a_5^2 c^{-2} \approx 1.9 \cdot 10^{-9}$, for Saturn $ω_6^2 a_6^2 c^{-2} \approx 1.04 \cdot 10^{-9}$, for Uranus $ω_7^2 a_7^2 c^{-2} \approx 5.1 \cdot 10^{-10}$, for Neptune $ω_8^2 a_8^2 c^{-2} \approx 3.3 \cdot 10^{-10}$, for Pluto $ω_9^2 a_9^2 c^{-2} \approx 2.5 \cdot 10^{-10}$. The orbit eccentricities: $e_1 = 0.21$, $e_2 = 0.007$, $e_3 = 0.017$, $e_4 = 0.093$, $e_5 = 0.048$, $e_6 = 0.056$, $e_7 = 0.047$, $e_8 = 0.009$, $e_9 = 0.249$.

For the world line $x^μ(t)$, $x^0(t) = ct$, the second relation (1.16) is

$$\frac{dt}{dτ} = \left(1 - 2 \frac{m_{10}G}{rc^2} + 2 \frac{m_{10}G^2}{r^2c^4} - \left(1 + 2 \frac{m_{10}G}{rc^2}\right) \frac{1}{c^2} \left|\frac{dx}{dt}\right|^2\right)^{-1/2}. \quad (3.20)$$

The vector $x(τ)$ is the solution of the equations (1.16). It depends on the proper time $τ$. Then the equality (3.20) implies

$$\left(1 - 2 \frac{m_{10}G}{rc^2} + 2 \frac{m_{10}G^2}{r^2c^4}\right) \left(\frac{dt}{dτ}\right)^2 = 1 + \left(1 + 2 \frac{m_{10}G}{rc^2}\right) \frac{1}{c^2} \frac{dt}{dτ} \left|\frac{dx}{dt}\right|^2. \quad (3.21)$$

The equation (3.21) has the solution

$$t(τ) = t(0) + \int_0^τ dτ' \left(1 + \left(1 + 2 \frac{m_{10}G}{r(τ')c^2}\right) \frac{1}{c^2} \frac{dt}{dτ'} \left|\frac{dx}{dt'}\right|^2\right)^{1/2} \times \left(1 - 2 \frac{m_{10}G}{r(τ')c^2} + 2 \frac{m_{10}G^2}{r^2(τ')c^4}\right)^{-1/2}, \quad (3.22)$$

$$g_{00}(x) = 1 - 2 \frac{m_{10}G}{rc^2} + 2 \frac{m_{10}G^2}{r^2c^4} = \left(1 - \frac{m_{10}G}{rc^2}\right)^2 + \frac{m_{10}G^2}{r^2c^4} \geq \frac{1}{2}. \quad (3.23)$$

The relations (3.3), (3.8), (3.10), (3.22) are the complete description of planet orbit. It is the orbit (1.6), (1.7) with the precession coefficient $γ = 1$.

By making use of the relation (3.20) and of the estimates similar to (3.16) - (3.18) it is possible to prove for the solution (1.6) - (1.8) of the equation (1.5)

$$\frac{dt}{dτ} \frac{d}{dt} \left(\frac{dt}{dτ} \left|\frac{dx}{dt}\right|^2\right)^{-1/2} = -\frac{m_{10}G}{|x|^3} x. \quad (3.23)$$

Due to the relation (3.23) the solution (1.6) - (1.8) of the equation (1.5) is an approximate solution of the equation (1.16) for the world line $x^μ(t)$, $x^0(t) = ct$.

Let us compare the function (1.7), (1.8) with the function (1.7), (1.11), (1.14) from (Ref. 5, Chap. 40, Sec. 40.5)

$$1 + e \cos \left(1 - \frac{3ω^2 a^2}{(1 - e^2)c^2}\right) (φ(t) - φ_0) = 1 + e \cos \left(1 - \frac{ω^2 a^2}{2(1 - e^2)c^2}\right) (φ(t) - φ_0)$$

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\[ + \frac{\omega^2 a^2 (\phi(t) - \phi_0)}{(1 - e^2)c^2} \int_{1/2}^{3} ds \sin \left( \left( 1 - \frac{s\omega^2 a^2}{(1 - e^2)c^2} \right)(\phi(t) - \phi_0) \right). \]  

(3.24)

The value \( \omega^2 a^2 c^{-2} \) is negligible for any planet.

ACKNOWLEDGMENTS

This work was supported in part by the Program for Supporting Leading Scientific Schools (Grant No. 4612.2012.1) and by the RAS Program "Fundamental Problems of Nonlinear Mechanics."

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