CONJUGACY CLASSES OF POLYSPINAL GROUPS

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Abstract. Spinal groups and multi-GGS groups are both generalisations of the well-known Grigorchuk-Gupta-Sidki (GGS-)groups. Here we give a necessary condition for spinal groups to be conjugate, and we establish a necessary and sufficient condition for multi-GGS groups to be conjugate. We also introduce a natural common generalisation of both classes, which we call polyspinal groups. Our results enable us to give a negative answer to a question of Bartholdi, Grigorchuk and Šunič, on whether every finitely generated branch group is isomorphic to a weakly branch spinal group.

1. Introduction

Let $m \in \mathbb{N}_{\geq 2}$ and let $T = T_m$ be the $m$-adic tree. Groups acting on $m$-adic trees have received quite a bit of attention, especially in the case when $m$ is a prime. The interest in these groups is largely due to their nice structure, their importance in the theory of just infinite groups, and the fact that many such groups have exotic algebraic properties; we refer the reader to [2] for a good introduction.

The (first) Grigorchuk group [5] was the first notable group acting on an $m$-adic tree that was constructed, and it continues to play a central role in the subject. It is a 3-generated infinite periodic group acting on the binary rooted tree $T_2$ with many interesting properties. Its three generators include the rooted automorphism $a$ which swaps the two maximal subtrees of $T_2$, and two directed automorphisms $\beta$ and $\gamma$, both of which stabilise the rightmost infinite ray of the tree. They are defined recursively as follows:

$$\beta = (a, \gamma), \quad \gamma = (a, \delta),$$

where for $x$ and $y$ automorphisms of $T_2$, the notation $(x, y)$ indicates the independent actions on the respective maximal subtrees, and

$$\delta = (1, \beta).$$

Soon after the Grigorchuk group was defined, other similar constructions followed, including the well-studied classes of Grigorchuk-Gupta-Sidki (GGS-)groups and Šunič groups (also called siblings of the Grigorchuk group).

These constructions were generalised to a natural class of so-called spinal groups, which are generated by a group $R$ of rooted automorphisms and a group $D$ of directed automorphisms. Both are defined by restricting the area of the tree where the automorphisms act non-trivially; rooted automorphisms act only at the root of the tree, while directed automorphisms only act on a 1-sphere around a constant path. We refer the reader to Section 2.2 for the precise definition. To any group $D$ of directed automorphisms, there is an associated sequence of homomorphisms $\omega = (\omega_n)_{n \in \mathbb{N}}$ from $D$ to $\text{Sym}(X)$, prescribing the action of $D$
on the different levels of the tree. This sequence fully determines the directed automorphisms. We write $\sigma^n R$ for the group generated by the images of the associated rooted automorphisms at level $n$. Note also that we can identify the vertices of the $m$-adic tree $T$ with the elements of the free monoid $X^*$, for the alphabet $X = \{0, 1, \ldots, m - 1\}$; see Section 2 for precise details.

Recently, Petschick [12] identified the conjugacy classes of GGS-groups within the automorphism group of their respective trees. In this paper, we give a condition for spinal groups to be conjugate in the same sense.

**Theorem 1.1** (A necessary condition for spinal groups to be conjugate). Let $G$ and $\tilde{G}$ be spinal groups with defining data $R, \omega$, respectively $\tilde{R}, \tilde{\omega}$, and let $f \in \text{Aut} T$ be such that $\tilde{G}^f = G$. Then there is an integer $N$ and an isomorphism $\iota : D \to \tilde{D}$, such that for all $n \geq N$ there is:

(i) an inner automorphism $\phi_n$ of $\text{St}_{\text{Sym}(X)}(0)$ such that $\phi_n(\sigma^n \tilde{R}) = \sigma^n R$,

(ii) a tuple of inner automorphisms $\rho_n \in (\text{Inn}(\sigma^n \tilde{R}))^{X \backslash \{0\}}$,

(iii) an automorphism $\alpha_n$ of $\text{Sym}(X)^{X \backslash \{0\}}$ permuting the direct factors by an element $\alpha' \in \text{St}_{\text{Sym}(X)}(0)$,

such that

$$\omega_n = \phi_n^{X \backslash \{0\}} \circ \rho_n \circ \alpha_n \circ \tilde{\omega} \circ \iota.$$

A subclass of spinal groups, resembling the GGS-groups more closely, have received increased attention in recent times. These are called multi-GGS groups, which are generated by a rooted automorphism permuting the maximal subtrees cyclically, and the sequences defining their directed generators are constant, but – in contrast to the GGS-groups – they have possibly more than one directed generator. Up to relabelling the vertices of the tree, all multi-GGS groups on a fixed tree have the same rooted group $A_m = \langle (0 1 \cdots m - 1) \rangle \cong C_m$. These groups were first defined in [1], where they were originally called multi-edge spinal groups. We prefer the term multi-GGS groups, as the original name can be easily confused with the term multispiral group, which represents a more generalised family of groups generated by rooted and tree-like automorphisms, as defined in [13]. We also consider the class of multi-EGS groups, which allows for directed generators associated to different constant paths, and includes branch groups that do not have the congruence subgroup property. To deal with both the classes of multi-EGS and spinal groups simultaneously, we introduce a common natural generalisation, which we call polyspiral groups; see Section 2.2 for details.

For multi-GGS groups, we are able to provide a necessary and sufficient condition for the groups to be conjugate.

**Theorem 1.2** (A condition for multi-GGS groups to be conjugate). Let $G$ and $\tilde{G}$ be multi-GGS groups with defining data $\omega$ and $\tilde{\omega}$ respectively. There is an element $f \in \text{Aut} T$ such that $\tilde{G}^f = G$ if and only if there exists

(i) an automorphism $\alpha$ of $A_{m - 1}^n$ permuting the direct factors by an element of $\text{N}_{\text{Sym}(X)}(A_m) \cap \text{St}_{\text{Sym}(X)}(0)$ and

(ii) an isomorphism $\iota : D \to \tilde{D}$ such that

$$\omega = \alpha \circ \tilde{\omega} \circ \iota.$$

It was asked by Bartholdi, Grigorchuk and Šunič [2, Question 4] whether every finitely generated branch group is isomorphic to a spinal group. Using the above results, we give a negative answer to this question within the class of weakly branch
groups. We refer the reader to Section 2 for the definitions of weakly branch and branch groups.

**Theorem 1.3.** There exists a finitely generated branch group \( G \leq \text{Aut} \ T_3 \) such that \( G \) is not isomorphic to any spinal group \( S \leq \text{Aut} \ T_3 \). Also, if \( G \) is isomorphic to a spinal group \( S \leq \text{Aut} \ \tilde{T} \) on a different tree \( \tilde{T} \), then \( S \) is not weakly branch with respect to its embedding into \( \text{Aut} \ \tilde{T} \).

**Organisation.** Section 2 consists of background material on groups acting on the \( m \)-adic tree and the definitions of weakly branch, branch, spinal, and polyspinal groups. In Section 3 we prove Theorems 1.1 and 1.2, and in Section 4 we prove Theorem 1.3.

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## 2. Preliminaries

For two integers \( l, u \in \mathbb{Z} \), we denote by \([l, u]\) and \((l, u)\) the set of integers within the respective intervals.

Let \( m \in \mathbb{N}_{\geq 2} \) and let \( T = T_m \) be the \( m \)-adic tree, that is, a rooted tree where all vertices have \( m \) children. Using the alphabet \( X = [0, m) \), we identify \( T \) with the Cayley graph of the free monoid \( X^* \) with respect to \( X \), by identifying the root of the tree with the empty word. Thereby we establish a natural length function on \( T \). We will routinely refer to vertices of the tree as words, using the said identification. The words \( u \) of length \(|u| = n\), (i.e. vertices of distance \( n \) from the root) are called the \( n \)th level vertices and constitute the \( n \)th layer of the tree.

By \( T_u \) we denote the full rooted subtree of \( T \) that has its root at a vertex \( u \) and includes all vertices having \( u \) as a prefix. For any two vertices \( u \) and \( v \), the map induced by replacing the prefix \( u \) by \( v \), yields an isomorphism between the subtrees \( T_u \) and \( T_v \).

Every \( f \in \text{Aut} \ T \) fixes the root, and the orbits of \( \text{Aut} \ T \) on the vertices of the tree \( T \) are precisely its layers. For \( u \in X^* \) and \( x \in X \) we have \( f(u x) = f(u) x' \), for \( x' \in X \) uniquely determined by \( u \) and \( f \). This induces a permutation \( f|^u \) of \( X \) which satisfies

\[
f(u x) = f(u) f|^u(x).
\]

The permutation \( f|^u \in \text{Sym}(X) \) is called the label of \( f \) at \( u \), and the collection of all labels of \( f \) constitutes the portrait of \( f \). There is a one-to-one correspondence between automorphisms of \( T \) and portraits. We say that an automorphism \( f \in \text{Aut} \ T \) has constant portrait induced by a permutation \( \sigma \) of \( X \) if all labels of \( f \) equal \( \sigma \); this automorphism is denoted by \( \kappa(\sigma) \).

The automorphism \( f \) is rooted if \( f|^\omega = 1 \) for \( \omega \) unequal to the root. Rooted automorphisms can be thought of as both elements of \( \text{Aut} \ T \) and \( \text{Sym}(X) \).

Let \( x \in X \) be a letter. We write \( \bar{x} \) for the infinite simple rooted path \((x^n)_{n \in \mathbb{N}_0} \). The automorphism \( f \) is directed, with directed path \( \bar{x} \) for some \( x \in X \), if the support \( \{ \omega \mid f|^\omega \neq 1 \} \) of its labelling is infinite and marks only vertices at distance 1 from the set of vertices corresponding to the path \( \bar{x} \).
More generally, for \( f \) an automorphism of \( T \), since the layers are invariant under \( f \), for \( u, v \in X^* \), the equation
\[
f(uv) = f(u)f|_u(v)
\]
defines a unique automorphism \( f|_u \) of \( T \) called the section of \( f \) at \( u \). This automorphism can be viewed as the automorphism of \( T \) induced by \( f \) upon identifying the rooted subtrees of \( T \) at the vertices \( u \) and \( f(u) \) with the tree \( T \). For \( G \) a subgroup of \( \text{Aut} \ T \), we will denote the set of all sections of group elements at \( u \) by \( G|_u \).

The action of \( \text{Aut} \ T \) on \( T \) will be on the left. We observe that, for any \( u, v \in X^* \) and automorphisms \( f, g \in \text{Aut} \ T \), we have
\[
(fg)|_u = f|_{g(\mu)}g|_u, \\
(f^{-1})|_u = (f|_{f^{-1}(u)})^{-1}.
\]
The corresponding equations also hold for the labels \( f|_u \).

2.1. **Subgroups of \( \text{Aut} \ T \).** Let \( G \) be a subgroup of \( \text{Aut} \ T \) acting spherically transitively, that is, transitively on every layer of \( T \). The vertex stabiliser \( \text{st}_G(u) \) is the subgroup consisting of elements in \( G \) that fix the vertex \( u \). For \( n \in \mathbb{N} \), the \( n^{\text{th}} \) level stabiliser \( \text{St}_G(n) = \bigcap_{|u| = n} \text{st}_G(u) \) is the subgroup consisting of automorphisms that fix all vertices at level \( n \).

We also write \( \text{st}_G(u)|_u \) for the restriction of the vertex stabiliser \( \text{st}_G(u) \) to the subtree \( T_u \) rooted at a vertex \( u \). Since \( G \) acts spherically transitively, the vertex stabilisers at every level are conjugate under \( G \). The group \( G \) is fractal if \( \text{st}_G(u)|_u \) coincides with \( G|_u \) for all vertices \( u \).

Recall the cyclic subgroup \( A_m \) of \( \text{Sym}(X) \) generated by \((0 \ 1 \ \cdots \ m - 1)\). We denote by \( \Gamma \) the subgroup of all automorphisms, whose labels are all contained in \( A_m \). In other words, the group \( \Gamma \) is the inverse limit of \( n \)-fold iterated wreath products of \( A_m \):
\[
\Gamma = \lim_{\longrightarrow} \prod_{n \in \mathbb{N}} A_m \triangleleft \cdots \triangleleft A_m.
\]

A group \( G \leq \text{Aut} \ T \) is called reducing with respect to \((N_n)_{n \in \mathbb{N}}\), if there is a sequence of finite sets \( N_n \subset \text{Aut} \ T \) such that for all \( g \in G \) there is a positive integer \( N \) such that \( g|_u \in N_m \) for all \( u \in X^m \) with \( m \geq N \). The sequence \((N_n)_{n \in \mathbb{N}}\) is called the nuclear sequence of \( G \). If there is some \( k \in \mathbb{N} \) such that the sequence \((N_n)_{n \geq k}\) is constant, then \( G \) is called contracting and the set \( N = N_k \) is called the nucleus of \( G \).

Let \( u \in T \) be a vertex. The rigid vertex stabiliser of \( u \) is the subgroup \( \text{rst}(u) \leq \text{Aut} \ T \) consisting of a automorphisms whose portrait is trivial outside of \( T_u \), and the \( n^{\text{th}} \) rigid level stabiliser \( \text{Rst}(n) \) is the product of all rigid vertex stabilisers of vertices at the \( n^{\text{th}} \) level. A group \( G \leq \text{Aut} \ T \) is called weakly branch if it acts spherically transitive and \( G \cap \text{Rst}(n) \) is non-trivial for all \( n \in \mathbb{N} \). It is called branch if it is weakly branch and \( G \cap \text{Rst}(n) \) is of finite index in \( G \) for all \( n \in \mathbb{N} \).

2.2. **Polyspinal groups.** Spinal groups were first introduced by Bartholdi and Šunič in [3] as a common generalisation of the Grigorchuk group and the class of GGS-groups. A more general definition was formulated by Bartholdi, Grigorchuk and Šunič in [2]. Below we define a natural generalisation of spinal groups acting on the \( m \)-adic tree \( T \).
Let \((a_n)_{n \in \mathbb{N}}\) be any sequence. We denote the shift operator \((a_n)_{n \in \mathbb{N}} \mapsto (a_{n+1})_{n \in \mathbb{N}}\) by \(\sigma\).

A directed automorphism \(f\) is described by a sequence of \((m - 1)\)-tuples of permutations of \(X\) and a path \(\overline{\mathbf{x}}\). More generally, given a group \(D\), a directed path \(\overline{\mathbf{x}}\) and a sequence \(\omega = (\omega_n)_{n \in \mathbb{N}}\) of homomorphisms \(\omega_n : D \to \text{Sym}(X^{\mathbb{N}})\), we (recursively) define a tree automorphism for every \(d \in D\) by

\[
d_{\omega,x}|y = \begin{cases} d_{\sigma \omega,x} & \text{if } y = x, \\
\pi_y \omega_1(d) & \text{otherwise.}
\end{cases}
\]

Here \(\pi_y\) denotes the projection to the \(y\)-th component, for \(y \in X\). Write \(D_{\omega,x} = \{d_{\omega,x} \mid d \in D\}\) for the set of all such automorphisms. Since \(d_{\omega,x}\) fixes \(\overline{\mathbf{x}}\), we have \(d'_{\omega,x} d_{\omega,x} = (d' d'')_{\omega,x}\) for all \(d', d'' \in D\), and hence a homomorphism \(D \to D_{\omega,x}\) with kernel \(\bigcap_{n \in \mathbb{N}} \ker(\omega_n)\).

All sections \(D_{\omega,x}|_{X^n}\) are isomorphic to \(D\) if and only if \(\bigcap_{n \geq k} \ker(\omega_n) = 1\) for all \(k \in \mathbb{N}\). In this case we call \(D_{\omega,x}\) a directed group defined by \(\overline{\mathbf{x}}\) and \(\omega\) and drop the indices, identifying it with \(D\).

Let \(D\) be a directed group. For every \(n \in \mathbb{N}\) we define the \(n\)-th rooted companion group

\[
\sigma^n R(D) = \langle d^v \mid |v| = n \rangle = \langle d|x^{n-1}y \mid y \in X \rangle.
\]

To shorten the notation, we write

\[
\sigma^n d = d|x^n \quad \text{for } d \in D, n \in \mathbb{N},
\]

\[
\sigma^n D = D_{\sigma^n \omega,x} \quad \text{for } n \in \mathbb{N}.
\]

**Definition 2.1.** Let \(R\) be a group of rooted automorphisms acting transitively on \(X\), and for some \(r \in [1, m]\), let \(x^{(0)}, \ldots, x^{(r-1)}\) be \(r\) distinct elements in \(X\). Let \(D^{(0)}, \ldots, D^{(r-1)}\) be directed groups defined by the constant paths given by \(x^{(0)}, \ldots, x^{(r-1)}\) and \(\omega^{(0)}, \ldots, \omega^{(r-1)}\) respectively, where the latter are sequences of homomorphisms \(\omega_n^{(i)} : D^{(i)} \to \text{Sym}(X^{\mathbb{N}})\) such that

(i) the groups \(\sigma^n R(D^{(i)})\) for \(i \in [0, r)\) act transitively on \(X\) for all \(n \in \mathbb{N}\), and

(ii) for all \(i \in [0, r)\) and \(k \in \mathbb{N}\)

\[
\bigcap_{n \geq k} \ker(\omega_n^{(i)}) = 1.
\]

Then

\[
G = \langle R, D^{(i)} \mid i \in [0, r) \rangle
\]

is called the polyspinal group with data \(R, \omega^{(0)}, \ldots, \omega^{(r-1)}, x^{(0)}, \ldots, x^{(r-1)}\). If \(r = 1\) we drop the superscript \((0)\), and call \(G\) the spinal group with data \(R, \omega\).

**Remark 2.2.** The choice of the path \(\overline{\mathbf{x}}\) does not matter in case of spinal groups, which is why it is omitted from the defining data. More generally, one easily defines directed elements along \(\ell\), an arbitrary infinite simple rooted path in \(T\), and, with this in mind, one can more generally define a spinal group to be equipped with an arbitrary directed path. However, by conjugating by an appropriate element \(f \in \text{Aut} T\), we can always assume that \(\ell = \overline{0}\). The same cannot be said about polyspinal groups with more than one arbitrary, non-constant, directed path; compare [4, Lemma 2.3]. We will not consider this more general case here.
For any \( n \in \mathbb{N} \), the \( n \)th shifted companion of \( G \),
\[
\sigma^n G = \langle \sigma^n R, \sigma^n D^{(i)} \mid i \in [0, r) \rangle,
\]
where \( \sigma^n R := \langle \sigma^n R(D^{(i)}) \mid i \in [0, r) \rangle \), is again a polyspinal group.

**Definition 2.3.** We record some previously studied special cases:

- If \( \sigma^n R \) is equal to the group \( A_n = \langle \{0 1 \cdots m - 1\} \rangle \) for all \( n \), all \( D^{(i)} \) are necessarily direct products of cyclic groups of order \( m \). In this case, we drop the rooted group from the defining data. Assume further that the sequences \( \omega^{(i)} \) are all constant. In this case, one calls \( G \) a multi-EGS group; compare [7]. Clearly, all multi-EGS groups are subgroups of \( \Gamma \).
- A group that is both spinal and a multi-EGS group is called a multi-GGS group; compare [1].
- A multi-GGS group such that the unique non-trivial \( D^{(0)} \) is cyclic is called a GGS-group.

**Lemma 2.4.** Let \( G \) be a polyspinal group with defining data \( R, \omega^{(0)}, \ldots, \omega^{(r-1)} \), and \( x^{(0)}, \ldots, x^{(r-1)} \). Then \( G \) is reducing with respect to
\[
\left( \sigma^n R \bigcup \bigcup_{i=0}^{r-1} \sigma^n D^{(i)} \right)_{n \in \mathbb{N}}.
\]

**Proof.** Every element \( g \in G \) can be represented by a word of the form \( \left( \prod_{j=0}^{l-1} d_j^{r_j} \right) r_l \) for some \( l \in \mathbb{N} \), with \( r_j \in R \) and \( d_j \in D^{(i_j)} \) with \( i_j \in [0, r) \), for \( j \in [0, l] \). We further assume that \( i_j \neq i_{j+1} \) if \( r_j = r_{j+1} \), for some \( j \in [0, l] \). An element of the form \( d_j^{r_j} \) is called a syllable, and consequently \( l = \text{syl}(g) \) is called the syllable length of \( g \). It is enough to prove \( \text{syl}(g|_{xy}) < \text{syl}(g) \) for all \( g \in G \) with \( \text{syl}(g) > 1 \) and \( xy \in X^2 \).

Let \( d_j^{r_j} d_{j+1}^{r_{j+1}} \) be two neighbouring syllables. Then
\[
(d_j^{r_j} d_{j+1}^{r_{j+1}})|_x = d_j|x^{r_j}, d_{j+1}|x^{r_{j+1}}
\]
has syllable length one or zero if \( r_j \neq r_{j+1} \). Otherwise, for \( x = i_j^{r_j} = i_{j+1}^{r_{j+1}} \) it is \( (d_j^{r_j} d_{j+1}^{r_{j+1}})|_x = d_j d_{j+1} \). If \( i_j = i_{j+1} \), this is again of syllable length 1. Hence we may assume \( i_j \neq i_{j+1} \). But then
\[
(d_j^{r_j} d_{j+1}^{r_{j+1}})|_{xy} = (d_j d_{j+1})|_y = d_j|y d_{j+1}|y
\]
has syllable length at most 1.

We have proven that upon taking sections at vertices of level 2 at most every second syllable may contribute a syllable to the section. Hence \( \text{syl}(g|_{xy}) < \text{syl}(g) \) if \( \text{syl}(g) > 1 \).

**Lemma 2.5.** Polyspinal groups are fractal.

**Proof.** Since \( \sigma G = G|_u \), where \( u \) is any first-level vertex, is again a polyspinal group, it suffices to show that \( \text{st}_G(u)|_u \) equals \( \sigma G \). The result follows from the definition of \( \sigma G \) and upon considering \( \langle D^{(0)}, \ldots, D^{(r-1)} \rangle G|_u \).

3. **Conditions to be conjugate**

3.1. **Necessary conditions for spinal groups to be conjugate.** Here, let \( G \), respectively \( \bar{G} \), denote polyspinal groups with defining data \( R, \omega^{(0)}, \ldots, \omega^{(r-1)}, \)
\[ x^{(0)}, \ldots, x^{(r-1)}, \] respectively \( \tilde{R}, \tilde{\omega}^{(0)}, \ldots, \tilde{\omega}^{(r-1)} \). To be consistent, we write \( \sigma^n \tilde{R} \) for the rooted generators of \( \sigma^n \tilde{G} \).

**Lemma 3.1.** Let \( G \) and \( \tilde{G} \) be polyspinal groups that are conjugate via \( f \in \text{Aut} T \); that is, \( G^f = \tilde{G} \). Then \( f|_u \equiv f|_v \pmod{\sigma^n G} \) for all \( n \in \mathbb{N} \) and \( u, v \in X^n \). In particular, \( f|_u \equiv f|_v \pmod{\sigma^n R} \).

**Proof.** Let \( n \in \mathbb{N} \) and \( u, v \in X^n \). Since \( \tilde{G} \) acts spherically transitively, there is an element \( g' \in G \) such that \( u^{g'} = v \). Now \( g'|_u \in \sigma^n G \), and since \( \text{st}_G(u)|_u = \sigma^n G \) there is an element \( g'' \in \text{st}_{\tilde{G}}(u) \) such that \( g''|_u = (g'|_u)^{-1} \). Thus \( g = g'g'' \) maps \( u \) to \( v \) and \( g|_u = 1 \). Let \( h \in G \) be such that \( h^f = g \). Then, recalling that the action on the tree is on the left, we have \( h(f(u)) = f(f^{-1}h)f(u) = f(v) \). Thus,

\[
1 = g|_u = (h^f)|_u = f^{-1}|_{h \circ f}| f|_u = f^{-1}|_{f(v)} h|_u \circ f|_u = f|_v^{-1} h|_u \circ f|_u.
\]

Restricting to the label at the vertex \( u \) yields the second statement. \( \square \)

**Lemma 3.2.** Let \( G = \langle R, D \rangle \) be a spinal group directed along \( \overline{t} \) and \( H \leq \text{Aut} T \) be reducing with respect to \( (N_n)_{n \in \mathbb{N}} \). If there is some \( f \in \text{Aut} T \) such that \( H^f = G \), then there is a positive integer \( N \) such that for all \( n \geq N \),

\[
\sigma^n D \subseteq N_n^{f|_n}.
\]

**Proof.** For every \( d \in D \) there is some element \( h(d) \in H \) such that \( h(d)^f = d \). Since \( H \) is reducing, there is a positive integer \( N \) such that for all \( d \in D \), we have \( h(d)|_u \in N_n \) for all \( u \in X^n \) with \( n \geq N \). Since \( d \) fixes \( \overline{t} \), the conjugate \( h(d) \) must fix \( f(\overline{t}) \). Thus,

\[
\sigma^n d = d|_{f|_n} = h(d)^f|_{f|_n} = (h(d)|_{f(\overline{t})})|_{f(\overline{t})} \in N_n^{f|_n}.
\]

**Lemma 3.3.** Let \( D \) and \( \tilde{D} \) be two directed groups defined by \( \pi, \omega \) and \( \overline{\pi}, \overline{\omega} \) respectively. If there exists an automorphism \( f \in \text{Aut} T \) such that \( \tilde{D}^f = D \), then there exists an isomorphism \( \iota : D \to \tilde{D} \) such that for all \( n \in \mathbb{N} \),

\[
\omega_n = (c(f|_{X^{n-1}})) \circ p(f|_{X^{n-1}}) \circ \omega_n \circ \iota,
\]

where \( c(\alpha) \) is the inner automorphism induced by \( \alpha \) and \( p(\alpha) \) is the relabelling of a direct product \( \text{Sym}(X)^{X\{y\}} \) by \( \alpha \) for any \( \alpha \in \text{Sym}(X) \), \( i.e. \) the \( k \)-th component of \( (\text{Sym}(X)^{X\{y\}})^{\alpha|_{X\{\alpha^{-1}(y)\}}} = \text{Sym}(X)^{X\{\alpha^{-1}(y)\}} \) is the \( \alpha(k) \)-th component of \( \text{Sym}(X)^{X\{y\}} \).

**Proof.** Denote by \( \iota \) the isomorphism induced by \( D = \tilde{D}^f \). Since all elements of \( D \) have labels only at distance 1 from \( \overline{t} \), respectively all elements of \( \tilde{D} \) have labels only at distance 1 from \( \overline{\theta} \), we have \( f(\overline{\pi}) = \overline{\theta} \). For any \( d \in D, z \in X \) and \( n \in \mathbb{N} \), it follows that

\[
\begin{align*}
\pi_n \omega_n(d) & \quad \text{for } z \neq x \\
\sigma^n d & \quad \text{for } z = x \\
\end{align*}
\]

\[
\begin{align*}
= \sigma^{n-1} d|_z \\
= d|_{x^{n-1}} z \\
= \iota(d)^f|_{x^{n-1}} z \\
= (\sigma^{n-1} \iota(d)^f)|_{x^{n-1}} z \\
= ((\pi f)^{t(x)}(\overline{\omega}_n(\iota(d))]|_{x^{n-1}} z & \quad \text{for } f|_{x^{n-1}}(z) \neq y \Leftrightarrow z \neq x, \\
= (\sigma^n \iota(d)^f)|_{x^{n-1}} z & \quad \text{for } f|_{x^{n-1}}(z) = y \Leftrightarrow z = x.
\end{align*}
\]

Hence the result. \( \square \)
Proof of Theorem 1.1. By Lemma 2.4 and Lemma 3.2 there is $N \in \mathbb{N}$ such that 

$$\sigma^n D \subseteq (\sigma^n \tilde{D})^{f|_{0^n}} \cup (\sigma^n \tilde{R})^{f|_{0^n}},$$

for all $n \geq N$. But since $\sigma^n D \leq \text{St}(1)$ for all such $n$, one obtains $\sigma^n D \leq (\sigma^n \tilde{D})^{f|_{0^n}}$. By symmetry (possibly increasing $N$) we have equality for all $n \geq N$. By Lemma 3.3 we have 

$$\omega_n = (\epsilon(f|_{0^n-1}^n))_{x \in X \setminus \{0\}} \circ p(f|_{0^n-1}) \circ \tilde{\omega}_n \circ \iota$$

for some isomorphism $\iota : D \to \tilde{D}$ and all $n \geq N$. By Lemma 3.1 we may write $f|_{0^n-1}^n = r_x f|_{0^n}$ for some $r_x \in \sigma^n \tilde{R}$. For all $x \in X \setminus \{0\}$, it follows that 

$$\pi_x \omega_n(d) = (\pi_f|_{0^n-1}(x) \tilde{\omega}_n(\iota(d))) f|_{0^n-1}^n = (\pi_f|_{0^n-1}(x) \tilde{\omega}_n(\iota(d))) r_x f|_{0^n},$$

implying $(\sigma^n \tilde{R})^{f|_{0^n}} = \sigma^n R$. Thus for every $n \geq N$ we choose 

- the inner automorphisms of $\text{St}_{\text{Sym}(X)}(0)$ induced by $f|_{0^n}$ as $\phi_n$, 
- the inner automorphisms of $\sigma^n \tilde{R}$ induced by the $r_x$ as $\rho_n$, and 
- the automorphism $p(f|_{0^n-1})$ induced by the permutation $f|_{0^n-1}$ as $\alpha_n$.

Then (3.1) implies the statement. \qed

3.2. Multi-EGS and multi-GGS groups. Recall the group $\Gamma$, which is the inverse limit of $n$-fold iterated wreath products of $A_m = \langle (0 1 \cdots m - 1) \rangle$.

Lemma 3.4. Let $g, h \in \Gamma$ be directed elements along $\overline{\eta}$, and $f \in \text{Aut} T$ such that $g^f \in \Gamma$ is directed along $\overline{\eta}$, for some $x, y \in X$. Then $h^f$ is directed along $\overline{\eta}$.

Proof. Without loss of generality one can consider the case $x = y = 0$. Since $g^f|_0$ is directed, the element $f$ stabilises 0. For any $x \in X \setminus \{0\}$ there is $n \in \mathbb{Z} / m \mathbb{Z}$ such that 

$$g^f|_x = (g|_{f(x)})^{f|_x} = ((0 1 \cdots m - 1)^n)^{f|_{0^n}}.$$

Since $g^f$ is directed and a member of $\Gamma$, 

$$g^f|_x \in A_m,$$

hence $f|_x$ normalises $A_m$. Thus 

$$h^f|_x = \begin{cases} (h|_0)^{f|_0} & \text{if } x = 0, \\ (h|_{f(x)})^{f|_x} \in A_m & \text{otherwise.} \end{cases}$$

Repeating this argument for $g|_0, h|_0$ and $f|_0$ shows that $h^f$ fixes $\overline{0}$ and has non-trivial labels only at vertices of distance 1 to this ray. Thus $h^f$ is directed along $\overline{0}$, as required. \qed

Recall that for a multi-EGS group $G$, for $i \in X$ the directed groups $D^{(i)}$ are direct products of cyclic groups of order $m$ and the rooted group is equal to $A_m$.

Proposition 3.5. Let $G$ and $\tilde{G}$ be multi-EGS groups defined by $\omega$ and $\tilde{\omega}$, respectively, such that $\tilde{G}^f = G$ for an element $f \in \text{Aut} T$. Then for $i \in [0, r)$, there exist 

- automorphisms $\alpha^{(i)}$ of $A_m^{X \setminus \{i^{(i)}\}}$ permuting the direct factors by an element of $\text{N}_{\text{Sym}(X)}(A_m) \cap \text{St}_{\text{Sym}(X)}(\overline{x}^{(i)})$, 
- a map $\theta : [0, r) \to [0, \tilde{r})$ such that $\text{rk} D^{(i)} = \text{rk} \tilde{D}^{(\theta(i))}$, and 
- isomorphisms $\iota^{(\theta(i))} : D^{(i)} \to \tilde{D}^{(\theta(i))}$ such that
Lemma 3.2

there is a positive integer $x$. It follows that since there are elements

Lemma 3.4

and Lemma 2.4

Lemma 3.3

as required.

Proof. Let $i \in [0, r]$. Then the group generated by $A_m$ and $D^{(i)}$ is a multi-GGS group. Write $x$ for $x^{(i)}$, for $x \in X$. By Lemma 3.2 and Lemma 3.4, recalling that $\sigma D^{(i)} = D^{(i)}$ and $\sigma R = R = A_m$ for a multi-EGS group, there is some $n \in \mathbb{N}$ such that

$$D^{(i)} \subseteq \left( A_m \cup \bigcup_{j=0}^{\bar{r}} \tilde{D}^{(j)} \right) f^{x^n}_1.$$

Since $D^{(i)}$ stabilises the first layer, in fact

$$D^{(i)} \subseteq \left( \bigcup_{j=0}^{\bar{r}} \tilde{D}^{(j)} \right) f^{x^n}_1.$$

Let $d \in D^{(i)}$ and $e \in \tilde{D}^{(j)}$, for some $j \in [0, \bar{r})$, be such that $e f^{x^n}_1 = d$. Then

$$d = d f^{x^n}_1 = (e f^{x^n}_1) f^{x^n}_1 = (e | f^{x^n}_1(x)) f^{x^n}_1.$$

Now write $y$ for $x^{(j)}$. Since there is only one non-trivial first layer section of $e$ stabilising the first layer, this implies $y = f^{x^n}_1(x)$. Defining $\theta(i) = j$ we have

$$D^{(i)} \leq (\tilde{D}^{(\theta(i))}) f^{x^n}_1.$$

Now let $e \in \tilde{D}^{(\theta(i))}$. By [12, Lemma 3.3] and the fact that multi-EGS groups are fractal, we obtain $e f^{x^n}_1 \in G$. However by Lemma 3.4 it follows that since there are elements $e' \in \tilde{D}^{(\theta(i))}$ such that $e' f^{x^n}_1 \in D^{(j)}$ is $\pi$-spinal, the element $e f^{x^n}_1$ is $\pi$-spinal. Hence by Lemma 3.4 there is a positive integer $k(e)$ such that $(e f^{x^n}_1)_{x^{k(e)}} \in D^{(i)}$. Thus

$$(e f^{x^n}_1)_{x^{k(e)}} = (e | f^{x^n}_1(x^{k(e)})) f^{x^n}_1 \in St_G(1) \setminus \{1\},$$

hence $f^{x^n}_1(x^{k(e)}) = y^{k(e)}$ and $(e f^{x^n}_1)_{x^{k(e)}} = e f^{x^n}_1 \in D^{(i)}$.

Set $k_{\max} = \max\{k(e) \mid e \in \tilde{D}^{(\theta(i))}\}$ to obtain

$$(\tilde{D}^{(\theta(i))}) f^{x^n}_1 \leq D^{(i)}.$$

Hence $D^{(i)} = (\tilde{D}^{(\theta(i))}) f^{x^n}_1 k_{\max} + D^{(i)}$. Taking further sections it is clear that

$$D^{(i)} = (\tilde{D}^{(\theta(i))}) f^{x^k}_1$$

for all $k \geq n + k_{\max}$.

Since all directed groups involved are abelian and both rooted groups are equal and cyclic, we have by Lemma 3.3 that for all $i \in [0, r)$,

$$\omega^{(i)} = (\phi^{(i)}) X \setminus \{x\} \circ \alpha^{(i)} \circ \tilde{\omega}^{(i)} \circ \tilde{\nu}^{(i)}$$

with $\alpha^{(i)}$ as desired, and $\phi^{(i)} \in N_{Sym(X)}(A_m) \cap St(x) \cong (\mathbb{Z} / m \mathbb{Z})^\times$. Hence, writing $c = (0 \ 1 \ \cdots \ m - 1)$, we have $\phi^{(i)}(c) = c^{k^{(i)}}$ for some $k^{(i)}$ coprime to $m$. Therefore, replacing $\nu^{(i)}$ with

$$\tilde{\nu}^{(i)} = \mu_{k^{(i)}} \circ \tilde{\nu}^{(i)} : D^{(i)} \to \tilde{D}^{(\theta(i))}$$

where $\mu_{k^{(i)}}(\tilde{\nu}^{(i)}(d)) = (\tilde{\nu}^{(i)}(d))^{k^{(i)}}$, we obtain

$$\omega^{(i)} = \alpha^{(i)} \circ \tilde{\omega}^{(i)} \circ \tilde{\nu}^{(i)};$$

as required. □
Lemma 3.6. Let $G$ be a multi-GGS group defined by $\omega$, and let $\iota \in \text{Aut}(D)$, and $\alpha \in N_{\text{Sym}(X)}(A_m) \cap \text{St}_{\text{Sym}(X)}(0)$. Then the multi-GGS group $\tilde{G}$ defined by $p(\alpha) \circ \omega \circ \iota$ is conjugate to $G$.

Proof. The multi-GGS group defined by $\omega \circ \iota$ is equal to $G$, as for any $d \in D$,

$$\iota(d) = d_{\omega \iota}$$

and vice versa.

Let $\kappa = \kappa(\alpha)$ be the automorphism with constant portrait $\alpha$. Then

$$d^\kappa = (d^\kappa, [d]_1^\kappa, \ldots, [d]_{(m-1)^\kappa})$$

for all $d \in D$, hence $d^\kappa = d_{p(\alpha) \omega \iota}$, using the notation $\iota$ from the previous lemma. Since $\kappa$ centralises rooted elements, we have $G^\kappa = \tilde{G}$. $\square$

Proof of Theorem 1.2. The necessity of the condition is a direct consequence of Proposition 3.5. The sufficiency follows from Lemma 3.6. $\square$

4. Finitely generated non-spinal branch groups

We now prove that one of the (finitely generated branch) Extended Gupta–Sidki groups defined by Pervova [11] is not isomorphic to a weakly branch spinal group. The attentive reader will notice that this by far is not the only example for a group with this property within the class of polyspinal groups.

Let $a = (0 \ 1 \ 2)$ be rooted and define $b = (b, a, a^2)$, $c = (a^2, c, a)$. The group $G = \langle a, b, c \rangle$ is a polyspinal group with defining data $R = (a) \cong A_3$ and $\omega_n(0) = (1 : x \mapsto x, \ 2 : x \mapsto x^{-1})$, $\omega_n(1) = (0 : x \mapsto x^{-1}, \ 2 : x \mapsto x)$ for all $n \in \mathbb{N}$, where the letter in front of a colon signifies the component of the homomorphism after it. Both $\langle a, b \rangle$ and $\langle a, c \rangle$ are isomorphic to the Gupta–Sidki 3-group. We record some of the results of [11] in a lemma.

Lemma 4.1. The group $G$ is a just infinite torsion branch group.

We now show the following.

Lemma 4.2. The group $G$ is not conjugate to any spinal group $S \leq \text{Aut} T_3$.

Proof. Assume for contradiction that $G^f = \tilde{G} = \langle R, D \rangle$ is a spinal group, for some $f \in \text{Aut} T_3$. Clearly, the rooted group of $\tilde{G}$ must be cyclic of order 3. Following our usual strategy, by Lemma 3.2 and Lemma 2.4 we find $n \in \mathbb{N}$ such that

$$D_{\sigma^n \omega} \subseteq (\langle a \rangle \cup \langle b \rangle \cup \langle c \rangle)^{f|_{\omega^n}}.$$ 

Since $D_{\sigma^n \omega}$ stabilises the first layer, we have $D_{\sigma^n \omega} \cap \langle a \rangle^{f|_{\omega^n}} = 1$. Let $d_{\sigma^n \omega} \in D_{\sigma^n \omega}$ equal $(c^i)^{f|_{\omega^n}}$ for some $i \in \mathbb{F}_3$. Then, recalling that $c \in \text{St}(1)$, we obtain

$$d_{\sigma^n+1 \omega} = d_{\sigma^n \omega}|_0 = (c^i)^{f|_{\omega^n}}|_0 = (f|_{\omega^n})^{-1}|_{f|_{\omega^n}(0)} c^i |_{f|_{\omega^n}(0)} f|_{\omega^{n+1}} = (c^i |_{f|_{\omega^n}(0)})^{f|_{\omega^{n+1}}}.$$ 

As $d_{\sigma^n+1 \omega} \in \text{St}(1)$, it follows that $f|_{\omega^n}(0) = 1$. Repeating the argument with some $e_{\sigma^n \omega} \in D_{\sigma^n \omega}$, which equals $(b^j)^{f|_{\omega^n}}$ for some $j \in \mathbb{F}_3$, we see that

$$e_{\sigma^n+1 \omega} = e_{\sigma^n \omega}|_0 = (b^j)^{f|_{\omega^n}}|_0 = (b^j |_{\omega^n})^{f|_{\omega^{n+1}}} = (a^i)^{f|_{\omega^{n+1}}} \in \text{St}(1),$$

hence $j = 0$ and $e_{\sigma^n \omega} = 1$. It follows $D_{\sigma^n \omega} \subseteq \langle c \rangle^{f|_{\omega^n}}$. Thus $D \cong C_3$ and $\tilde{G}$ is generated by two elements. But since $G/\text{St}_G(1)' \cong C_3 \times C_3^2$ the group $G$ cannot be two-generated. This is a contradiction. $\square$
Proof of Theorem 1.3. By Lemma 4.1, the group $G$ is branch, torsion, and just infinite. If $S \leq \text{Aut} \tilde{T}_3$ is isomorphic to $G$, it is conjugate to $G$ by [6, Corollary 1(a)] and [7, Proof of Corollary 3.8]. However by Lemma 4.2, this is impossible.

If $S \leq \text{Aut} \tilde{T}$ is weakly branch (with respect to its embedding into $\text{Aut} \tilde{T}$) and isomorphic to $G$, it is just infinite and hence branch. Thus by [6, Theorem 2] we have $\tilde{T} \cong T_3$ and the first assertion implies the second. □

We remark that all known spinal groups that are not weakly branch act on the binary tree such that the sequence of companion groups stabilises as the infinite dihedral group; compare [10, Proposition 3.4]. Therefore we ask: “Is every involution-free spinal group weakly branch with respect to its natural embedding?”

It is natural to update [2, Question 4] to “Does every finitely generated branch group admit an embedding into some $\text{Aut} T$ as a branch polyspinal group?”. To conclude this paper, we will make some remarks concerning the candidate presented in [2], which is the perfect regular branch group defined by Peter Neumann [9]. For a short description see [2, Section 1.6.6].

**Proposition 4.3.** Neumann’s group $G \leq \text{Aut} T$ is not conjugate to any spinal group $S \leq \text{Aut} T$.

**Proof.** Assume for contradiction that $G^f$ is a spinal group with data $R, \omega$. Then by Lemma 3.2 there is a number $n \in \mathbb{N}$ such that

$$\sigma^n D \subseteq N^{f|0}.$$

As the elements of $\sigma^n D$ stabilise the first layer, we arrive at a contradiction since $N$ consists of elements that do not belong to $\text{St}(1)$. □

By a rigidity result of Lavreniuk and Nekrashevych [8, Section 8] the automorphism group of Neumann’s group $G$ coincides with its normaliser in $\text{Aut} T$, but further results allowing us to reduce any isomorphism to a subgroup of $\text{Aut} T$ (or even $\text{Aut} \tilde{T}$) would be necessary to negatively answer the updated question.

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