EMBEDDING OPEN RIEMANN SURFACES IN
4-DIMENSIONAL RIEMANNIAN MANIFOLDS

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ABSTRACT. Any open Riemann surface has a conformal model in any orientable Riemannian manifold of dimension 4. Precisely, we will prove that, given any open Riemann surface, there is a conformally equivalent model in a prespecified orientable 4-dimensional Riemannian manifold. This result along with [5] now shows that an open Riemann surface admits conformal models in any Riemannian manifold of dimension ≥ 3.

1. Introduction

In 1999, the author [5] used Teichmüller theory to prove that, for given open Riemann surface \( S_0 \), there exists a conformally equivalent model surface \( S \) in a prespecified orientable Riemannian manifold \( \mathcal{M} \) of dimension \( \mathcal{M} \geq 3 \) except the partial proof for the embedding into 4-dimensional Riemannian manifold. In [5], the case of a 4-dimensional \( \mathcal{M} \) required the extra technical assumption that the normal bundle have a nowhere vanishing cross-section. In the present paper we remove this assumption and thus conclude, with [5], that an open Riemann surface now admits conformal embedding into any Riemannian manifold of dimension ≥ 3.

2. The main results

We will see in this paper that the methods used in the Ko’s Embedding Theorems ([3, 4, 5, 6, 7, 8]) are even strong enough to prove this theorem for non-compact Riemann surfaces in 4-dimensional Riemannian manifold too.

\( C^\infty \)-embedded surfaces are called classical surfaces if they are viewed as Riemann surfaces whose conformal structure is given in the following natural way: the local coordinates are those which preserve angles and orientation.

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In this paper, we follow carefully all the notations and arguments of [5], but the detailed expressions and computations may differ because we define a new deformation function \( h \), to prove:

**Theorem 2.1 (Embedding Theorem).** Assume that \( S \) is any Riemann surface, \( C^\infty \)-embedded in the orientable Riemannian manifold \( \mathcal{M} \) of \( \dim \mathcal{M} = 4 \). Let \( S_0 \) be any Riemann surface structure on \( S \). Then \( S_0 \) is conformally equivalent to a complete classical surface in \( \mathcal{M} \). A model can be constructed by deforming a given topologically equivalent complete Riemann surface \( S \) on each element in compact exhaustion of \( S \) (via the map (4.2)).

We know that, in the above case, there are sections of the normal bundle of \( S \) in \( \mathcal{M} \) with isolated zeroes. (See Ko [6, Section 2.2] and the next paragraph for details on the sections of the normal bundle.) We consider this case only, otherwise the theorem is true [5].

**Remark.** It can be shown that if \( \dim \mathcal{M} \neq 4 \), then there always exists a nowhere vanishing section of the normal bundle \( NS \) of \( S \) in \( \mathcal{M} \) if \( S \) is compact. When \( \dim \mathcal{M} = 4 \), the nowhere vanishing section of the normal bundle \( NS \) exists if there are no obstructions. In this case the obstruction lies in the Euler class \( e(\text{NS}) \) of the normal bundle \( NS \). That is, if \( e(\text{NS}) = 0 \), then there is always such a section. For the proof see Ko [3, 6].

The argument now continues as in [5], Section 2. We need several supporting lemmas, especially Garsia’s Continuity Lemma (Lemma 5.2 in [5]) and (revised) Brouwer’s Fixed Point Lemma (Lemma 6.5 in [5]).

For the theory and the coordinate systems of Teichmüller space of a Riemann surface, we refer to [5], Section 4.

3. **Outline of the proof**

Let \( S \) be any Riemann surface \( C^\infty \)-embedded in the orientable Riemannian manifold \( \mathcal{M} \) of \( \dim \mathcal{M} = 4 \) and \( S_0 \) be any Riemann surface structure on \( S \). We may assume that \( S \) and \( S_0 \) are non-compact because the Embedding Theorem is known to be true, otherwise ([7, 8]). Then there exists a topological mapping \( f : S_0 \to S \) by a consequence of the choice of \( S \) and \( S_0 \). In terms of exhaustions, we do the following constructions. Since every Riemann surface admits a countable compact exhaustion by a subsurface, we may choose a regular exhaustion, that is, a sequence \( \{S_i\} \) on \( S_0 \), of relatively compact regular subregions, such that \( S_i \subset S_{i+1} \subseteq \cup S_0 = S_0 \) and \( \partial S_i \) consists of analytic arcs.

It is easy to show that \( S_0 \) can be mapped by \( f_i \) topologically on a classical surface \( S_i \) such that \( \partial S_i \) consists of circles contained in \( \partial B_i \), where, for \( n = \dim \mathcal{M} \),

\[
B_i = \left\{ (x_1, x_2, \ldots, x_n) \left| \sum_{j=1}^{n} x_j^2 \leq i^2 \right. \right\}, \quad S_i \subset B_i, \quad S_{i+1} \cap B_i = \overline{S_i}
\]
with \( \dim B_i = 2 = \dim S = \dim S_0 \), and \( f_{i+1}|S_0^i = f_i \), \( f = \lim f_i \), and \( S = \cup S_i \) satisfy the above conditions ([11]).

We may assume that \( S_0^1 \) is a disk. Let \( p \in S_0^1 \) and \( q \in \partial S_0^1 \) be distinguished points and put \( p' = f(p) \), \( q' = f(q) \) and \( f(S_0^1) = S' \). If \( S_0^1 \) is simply connected, we introduce 4 distinguished points.

We will deform \( S \) in successive steps such that the \( i \)-th deformation \( (i \geq 2) \) takes place on \( S_i - S_{i-1} \) only, and we will denote the resulting surface by \( S' \).

Let \( S'_i \) be the part of \( S' \) corresponding to \( S_i \). We will show that \( S_0 \) can be mapped conformally onto \( S'_i \) by a mapping \( f_i \) with the additional properties \( f_i(p) = p', \ f_i(q) = q' \), \( i \geq 1 \).

If this is accomplished, then we can derive our theorem that \( S_0 \) and \( S \) are equivalent by [5, Lemma 3.1].

4. The existence of the functions \( f_i \)

Suppose \( S \) is any open Riemann surface, \( \mathcal{C}^{\infty} \)-embedded in the orientable Riemannian manifold \( \mathbb{M} \) of \( \dim \mathbb{M} = 4 \). Let \( S_0 \) be any Riemann surface structure on \( S \). Let \( \{S_0^i\} \) and \( \{S_i\} \) be exhaustions of \( S_0 \) and \( S \) respectively.

Assume that \( S_{i-1} \) is deformed into a surface \( S'_{i-1} \) such that conformal map \( f_{i-1} : S_{i-1}^0 \rightarrow S'_{i-1} \) with \( f_{i-1}(p) = p' \) and \( f_{i-1}(q) = q' \) exists. We are going to construct \( S'_i \) and \( f_i \) which is different from the function \( f_i \) in [5]. The existence of \( f_i \) follows by Riemann’s mapping theorem, the existence of \( f_i \), \( i \geq 2 \), will be proved by induction.

Let \( S''_i = (S_i - S_{i-1}) \cup S'_{i-1} \). Let \( \Gamma : S''_i \hookrightarrow NS''_i \setminus \Gamma_0 \) be a smooth section of the normal bundle \( NS''_i \) of \( S''_i \) in \( \mathbb{M} \) which vanishes at some exceptional points, where \( \Gamma_0 \) is the zero section of \( NS''_i \). \( \Gamma \) has a maximum length 1.

By Nash’s Embedding Theorem ([10]), there is a \( \mathcal{C}^{\infty} \)-isometric embedding \( j : \mathbb{M} \hookrightarrow \mathbb{R}^m \) for some sufficiently large \( m \). This allows us to consider \( S \) and \( \mathbb{M} \) as subsets of \( \mathbb{R}^m \).

We know that there are certain number of exceptional points \( z_j \) on \( S''_i \) where a section of the normal bundle \( NS''_i \) vanishes, as we indicated in the Section 2. Fix a global coordinate \( z \in S''_i \) such that \( X : S''_i \rightarrow \mathbb{R}^m \) is a conformal parametrization of \( S''_i \). Then the Riemann surface structure of \( S''_i \) may be viewed as induced by the metric \( (dX)^2 = \lambda^2(z)|dz|^2 \), where \( \lambda(z) \) is a smooth real valued \( (1,1) \)-form.

Extend the function \( f_{i-1} \) to \( S_0^1 \) such that the extended map \( \xi : S_0^1 \rightarrow \mathbb{R}^m \subset \mathbb{R}^m \) is \( K \)-quasiconformal for a suitable \( K \), \( \mathcal{C}^{\infty} \) except perhaps on \( \partial S_0^{i-1} \) and such that, for the complex dilatation \( \mu(\xi) = \frac{\partial \xi}{\partial z} \) of \( \xi \),

\[
\mu(\xi) = 0 \quad \text{on} \quad S_0^{i-1}, \quad \mu(\xi) = \frac{i}{2} \quad \text{on a disk} \quad D \quad \text{in} \quad S_0^i - S_0^{i-1}.
\]

Such an extension is certainly possible (see Lehto and Virtanen [9, p. 92ff]).

By the previous constructions, \( S_0^{i-1} \rightarrow S'_{i-1} \) is conformal. By the previous paragraph and the properties of the quasiconformal mappings, \( \xi : S_0^1 \rightarrow S''_i \) is a
homeomorphism and so it has an inverse $g_0 : S''_i \rightarrow S'_0$ which is quasiconformal. Let $T^\#(S''_i)$ be a Teichmüller space of $S''_i$, then it can be identified with the open unit ball $Q_1(S''_i)$ in a normed linear vector space of symmetric holomorphic quadratic differentials on $S''_i$.

Assume that $\omega \in T^\#(S''_i)$ is a local coordinate for a neighborhood of $[ds_{S''_i}]$ in $T^\#(S''_i)$ provided $\|\omega\| = \int_{S''_i} |\omega| \leq 2r < 1$.

For later use, we define a surface $(S''_i)_\omega$ as follows: For any $\omega = \phi_\omega(z)dz^2 \in T^\#(S''_i) \setminus \{0\}$, define a metric $ds^2_{\omega}$ by
\begin{equation}
(4.1) \quad ds^2_{\omega} := \lambda^2(z)dz + \Psi_\omega(z)dz^2,
\end{equation}
where $\lambda$ is a smooth real-valued $(1,1)$-form and
\[
\Psi_\omega(z) = \begin{cases} 
\|\omega\| \frac{\overline{\omega(z)}}{\overline{|\omega(z)|}} & \text{on } \xi(D) \\
0 & \text{outside } \xi(D).
\end{cases}
\]

The metric $(4.1)$ defines a new conformal structure on $S''_i$ which will be denoted by $(S''_i, ds^2_{\omega}) := (S''_i)_\omega$.

If $g_\omega : S''_i \rightarrow (S''_i)_\omega$ is a quasiconformal map and $[g_\omega] \in T^\#(S''_i)$, then we write $[g_\omega] = \omega$. Let $g_0 : S''_i \rightarrow S'_0$ be a homeomorphism so that $[g_0] \in T^\#(S''_i)$. Assume that $[g_0] = \omega_0 = \omega_0 \in T^\#(S''_i)$ and denote by $B_\epsilon(\omega_0) \subset T^\#(S''_i)$ the set of elements in $T^\#(S''_i)$ with $\|\omega - \omega_0\| < \epsilon$.

Then García’s Continuity Lemma ([5], Lemma 5.2) and (revised) Brouwer’s Fixed Point Lemma ([5], Lemma 6.5) follow.

Now we fix a map
\begin{equation}
(4.2) \quad h : S''_i \times \overline{B}_\epsilon(\omega_0) \rightarrow (-\epsilon, \epsilon)
\end{equation}
so that $h$ is a $C^\infty$-function with support on $S_i - S_{i-1}$ for each fixed $\omega$. This $h$ (which is different from the function $h$ given in [5]) will be defined explicitly in Section 5.

Let the surface $(S''_i)_\omega$ be the $\epsilon$-normal deformation of $S''_i$ defined by the map
\begin{equation}
(4.2) \quad (S''_i)_\omega : S''_i \rightarrow \mathcal{F} \subset \mathbb{R}^m
\end{equation}
where $X$ is a local coordinate for $S''_i$ and $\Gamma(X(z))$ is a unit tangent vector in $\mathbb{R}^m$ to the curve $\exp th(X(z))\Gamma(X(z))$ at the point $X(z)$. (For more details, see the Section 2 of [5].) Denote by $[(S''_i)_\omega]$ the conformal equivalence class of the surface $(S''_i)_\omega$ as a marked surface $((S''_i)_\omega, (S''_i)_\omega)$. We then define a map $\Xi : \overline{B}_\epsilon(\omega_0) \rightarrow T^\#(S''_i)$ by
\[
\Xi : \overline{B}_\epsilon(\omega_0) \rightarrow T^\#(S''_i)
\]
where
\[
\omega \mapsto [(S''_i)_\omega].
\]
In addition, the function $h$ will be so small that all the surfaces $(S''_i)_\omega$ are embedded surfaces. Then as a consequence of (revised) Brouwer Fixed Point Lemma, we will have proved the existence of the conformal model if we can prove that, given $[g_0] = \omega_0$ and $\epsilon > 0$, for $\omega$ in the closed ball $\overline{B}_\epsilon(\omega_0) \subset Q_1(S''_i)$
there is a family of deformations \((S''_i)^{\omega}\) of \(S''_i\) depending on \(\omega\) so that Lemma 6.3 (with the newly defined present deformation function \(h\)) of [5] is true.

By Lemma 6.3 of [5], \(\Xi\) satisfies the hypothesis of the (revised) Brouwer Fixed Point Lemma. Therefore there is a point \(\omega_1 \in \overline{B}_r(\omega_0)\) so that

\[
\Xi(\omega_1) = [(S''_i)^{\omega_1}] = \omega_0 = [g_0], \quad \text{where} \quad g_0 : S''_i \to S''_0.
\]

This means that \(S'_i\) can be mapped conformally onto the deformed surface \((S''_i)^{\omega_1} = S'_i\) by a conformal map, call it \(f_i\), homotopic to \((S''_i)^{\omega_1} \circ \xi\) and satisfies the condition \(f_i(p) = p', f_i(q) = q'.

### 5. The construction of the family \((S''_i)^{\omega}\)

As we said previously, we will deform \(S''_i\). Since the whole space we are considering here is \(S''_i\) (not \(S\)), we take the fundamental domain \(P\) for \(S''_i = (S_i - S_{i-1}) \cup (S'_{i-1})\) in \(S''_i\). Let \(P_i\) (respectively \(P_{i-1}\)) be the fundamental domain for \(S_i\) (respectively \(S_{i-1}\)). Then the fundamental domain for \(S_i - S_{i-1}\) will be the domain \(P_i - P_{i-1}\). We may assume that \(\partial P_i\) (respectively \(\partial P_{i-1}\)) has measure zero and hence \(\partial (P_i - P_{i-1})\) has measure zero. We will construct a \(\mathcal{C}^\infty\) deformation function \(h\) non-vanishing only on \(P_i - P_{i-1}\). In fact, we are assumed the deformation \(S'_{i-1}\) for \(S_{i-1}\) is already done, so the deformation function \(h\) for this part of \(S''_i\) needs to be zero. Therefore the deformation will actually take place only on \(S_i - S_{i-1}\).

#### 5.1. The metric \(ds^2_2\)

The metric of the \(\epsilon\)-normal deformation \((S''_i)^{\omega}\), defined by the map \((S''_i)^{\omega}\) in equation (4.2), of \(S''_i\) satisfies the equation

\[
(d(S''_i)^{\omega})^2 = (dX)^2 + (dh)^2 + o(h)|dz|^2
\]

\[
\quad = \lambda^2(z)|dz|^2 + \left(\frac{\partial h}{\partial x} dx + \frac{\partial h}{\partial y} dy\right)^2 + o(h)|dz|^2.
\]

Let \(\chi : (S''_i)_\omega \to (S''_i)^{\omega}\) be a mapping of \((S''_i)_\omega\) onto \((S''_i)^{\omega}\). Let \(ds^2_2\), given by (4.1), and \((d(S''_i)^{\omega})^2\), be metrics for \((S''_i)_\omega\) and \((S''_i)^{\omega}\) respectively. We want to show that the dilatation \(K_\chi\) (will be given in Lemma 5.1) of \(\chi\) satisfies the hypotheses of Garsia’s Continuity Lemma ([5], Lemma 5.2). It will be helpful to split \(ds^2_2\) into the form given in (5.1). For fixed choice of global coordinate \(z\) in \(S''_i, \omega\) uniquely defines \(\phi_\omega(z)\). Let

\[
\Pi(\omega) = \left\{ z \in S''_i \mid \exists \phi_\omega(z) \neq 0 \right\}.
\]

The metric \(ds^2_2\) is smooth on the set \(\Pi(\omega)\). Let

\[
\gamma_\omega := (1 - \|\Psi_\omega\|_\infty)^{-2} = (1 - \|\omega\|)^{-2}.
\]
Then define the following real valued functions on $\Pi(\omega)$.

$$
\alpha_\omega^2 := 2\gamma(\|\Psi_{\omega}\|_\infty + \Re\Psi_{\omega}(z)) = 2\gamma(\|\omega\| + \Re\Psi_{\omega}(z)), \\
\beta_\omega^2 := 2\gamma(\|\Psi_{\omega}\|_\infty - \Re\Psi_{\omega}(z)) = 2\gamma(\|\omega\| - \Re\Psi_{\omega}(z)).
$$

On each connected component of the set $\Pi(\omega)$, choose continuous (real) branches of $\alpha_\omega, \beta_\omega$ so that

$$
\text{sign}(\alpha_\omega\beta_\omega) = \text{sign}(\Im\Psi_{\omega}(z)) \quad \text{and} \quad \beta_\omega > 0.
$$

Since $dz^2 = dx^2 - dy^2 + 2idxdy$ and $ds^2 = dx^2 - dy^2 - 2idxdy$, we get

$$
\gamma_\omega ds^2 = \lambda_\omega^2(z)\left(|dz|^2 + (\alpha_\omega dx + \beta_\omega dy)^2\right).
$$

5.2. The deformation function $h$

To complete Theorem 2.1, we need to describe a function $h$ on $S''_P$ satisfying the following properties:

1. $h$ is $C^\infty$.
2. $\|h\|_\infty < \epsilon$.
3. $(dh)^2 \approx (\alpha_\omega dx + \beta_\omega dy)^2$ in view of equations (5.1) and (5.5).

We would like to define a function $h$ satisfying condition 3 except on a sufficiently small set. But $dh$ remains bounded on this set. Condition 3 suggests that we express $(dh)^2$ in terms of $\alpha_\omega$ and $\beta_\omega$. On general Riemann surfaces, $\alpha_\omega$ and $\beta_\omega$ must be non-constant functions of $z$. The definition of $h$ will come as a solution of a differential equation in which $\alpha_\omega$, $\beta_\omega$ and their derivatives appear as coefficients. In order to get a $C^\infty$ solution, we need $\alpha_\omega$, $\beta_\omega$ to be smooth on all of $P$, that is on $S''_P$. Also they, together with their derivatives, must change as little as possible. For $h$ to be well-defined on $S''_P$, it is convenient that it be zero on $P_{-1}$ and in a neighborhood of the edges of $P_1 - P_{-1}$ (and neighborhoods of exceptional points) but remains smooth.

In this section, we will eventually construct the deformation function $h$ in terms of $\lambda(z)$, $\alpha_\omega(z)$, $\beta_\omega(z)$ and some large number $N$.

For this purpose, we need to extend the functions $\alpha_\omega$ and $\beta_\omega$ on whole of $P$ since they are not defined on $P \setminus \Pi(\omega)$. This work has been done in [5], Section 3.2. And in [5], Sections 7.2.1 to 7.2.3, we set them as $\bar{\alpha}_\omega$ and $\bar{\beta}_\omega$, and, for a compact subset $F$ in $Q_1(S''_P)$ not containing 0, constructed several other auxiliary functions on $\Delta \times F$ such as the real-valued continuous functions $\mu(z,\omega)$ and $u(z,\omega)$, maximum value $u_0$ of $|u(z,\omega)|$, $\gamma(z,\omega) = e^{u_0 - u(z,\omega)}$, an exact function $g(z,\omega) = \gamma(z,\omega)(\bar{\alpha}_\omega dx + \bar{\beta}_\omega dy)$ and a differentiable function $k(z,\omega)$ with $|k| \leq k_0$ satisfying $g = dk$.

Next we define a function to take care of some (as noted in the Section 2) fixed exceptional points on $S''_P$ (actually, we work on exceptional points on $S_j - S_{j-1}$ since we already deformed a surface $S_{j-1}$) where the section $\Gamma$ of the normal bundle $NS''_P$ vanishes. Let, for $j = 1, \ldots, r$, $z_j$ be fixed exceptional points on $P_j - P_{j-1}$ and $E_j := E_{\delta_j}(z_j)$ be a small neighborhood of $z_j$ so that the area $E \cap (P_j - P_{j-1}) < \ell_E \cdot \eta$, where $E = \cup_{j=1}^r E_j$ and $\ell_E$ is a small constant($< \frac{1}{4}$).
depending on $E$. We define a real-valued $C^\infty$-function (Beltrami differential on $S_j''$) $v(x, y)$ $(\|v\|_\infty < 1)$ on $P$ so that its support lies in the complement of the set $E$. (This can be done, using a theorem of Bers, by defining a $C^\infty$-function (Beltrami differential $v_j(x, y)$ $(\|v_j\|_\infty < 1)$) having a support on a complement of each $E_j$ and multiplying them all. See [1] for more information.) Let $I_j$ be a small neighborhood containing $E_j$ (closure of $E_j$), $j = 1, \ldots, r$, with the area $I \cap (P_i - P_{i-1}) < \lambda \cdot \eta$, where $I = \bigcup_{j=1}^r I_j$ and $\lambda \cdot \eta$ is a small constant($< \frac{1}{4}$) depending on $I \supset E$. We define a function $\mathcal{U}(x, y)$ on $P$ using $v(x, y)$ as follow so that it is $C^\infty$ on $P$:

$$
\mathcal{U}(x, y) = \begin{cases} 
0 & \text{for } (x, y) \in E \cup [S_j'' - (P_i - P_{i-1})] \\
1 & \text{for } (x, y) \in P_i - P_{i-1} - I \quad \text{(where } I \supset E). \end{cases}
$$

On $I - E$, we may define any $C^\infty$-function (since it does not really matter which form we use as you may see in Lemma 5.2 as long as its sup norm is less than 1 and it is expressed) in terms of $v(x, y)$ so that $\mathcal{U}$ is $C^\infty$ and $\|\mathcal{U}\|_\infty \leq 1$ on whole $P$, that is, on whole $S_j''$.

As a final auxiliary function, we define a real-valued function $\nu_\eta(x)$ for $\eta < \frac{1}{16}$ as follow.

1. $|\nu_\eta(x)| \leq 1$,
2. $\nu_\eta(x) = \begin{cases} 
x & \text{for } \eta - 1 \leq x \leq 1 - \eta \\
2 - x & \text{for } 1 + \eta \leq x \leq 3 - \eta, \end{cases}$
3. $\nu_\eta(x + 4) = \nu_\eta(x)$.

Let $F$ be a compact subset in $Q_1(S_j'')$ which does not contain 0. For $(x, y) \in S_j''$ (i.e., $P$) and $\omega \in F$, define $h$ by

$$
h(x, y, \omega, N) = \frac{1}{N} \lambda(x, y) \mathcal{U}(x, y) \mu_\eta(x, y, \omega) \frac{1}{\gamma(z, \omega)} \cdot \nu_\eta(N \cdot k(x, y, \omega)),
$$

where $N$ (which will be determined at the end of this section) is a sufficiently large natural number depending on $F$ and $\epsilon$ (which will be determined in the Section 5.4 and it will guarantee the existence of $\epsilon$-normal deformation surface $(S_j'')^\omega$. Refer to Theorem 2.1 of [6], Section 2). Then for each $N$, $h$ is a $C^\infty$-function on $S_j''$ having support on $P_i - P_{i-1}$ and continuous on $S_j'' \times F$ and we have

$$
dh^2 = \lambda^2 \cdot \mathcal{U}^2 \cdot \mu_\eta^2 \cdot \nu_\eta^2(Nk) \frac{1}{\gamma^2(z, \omega)}(dk)^2 + o\left(\frac{1}{N}\right)|dz|^2.
$$

Except on a small set $A$ of $P_i - P_{i-1}$ (in fact, on the set $S_j'' - (P_i - P_{i-1})$, we have $h = 0$, so that $dh^2 = 0$), this reduces to

$$
dh^2 = \lambda^2(x, y)(\bar{\alpha} \cdot dx + \bar{\beta} \cdot dy)^2 + o\left(\frac{1}{N}\right)|dz|^2,
$$

where $A$ is given by

$$
A = \{(x, y) \in P_i - P_{i-1} \mid \mu_\eta^2 \cdot \mathcal{U}^2 \cdot \nu_\eta^2(N \cdot k(x, y, \omega)) \neq 1\}.
$$
Let $A_1$ be the set $A_1 = \{(x, y) \in P_1 - P_{1-1} \mid \mu^2(x, y, \omega) \neq 1\} \equiv \{(x, y) \in P_1 - P_{1-1} \mid \mu(x, y, \omega) \neq 1\}$ (since $0 \leq \mu(x, y, \omega) \leq 1$), then $A_1$ has an area ([5], Section 7.2.2)

\[
\text{area } A_1 < \frac{\eta}{2}.
\]

Let

\[
A_2 = \{(x, y) \in P_1 - P_{1-1} \mid \tilde{\Omega}^2(x, y) \neq 1\}
\]

and

\[
A_3 = \{(x, y) \in P_1 - P_{1-1} \mid \tilde{\nu}^2(Nk(x, y, \omega)) \neq 1\},
\]

then it becomes $A = A_1 \cup A_2 \cup A_3$. Since we know that

\[
\text{area } A_2 = (P_1 - P_{1-1}) \cap I < l_I \cdot \eta,
\]

we only need to compute the area of the set $A_3$. But $A_3$ becomes

\[
A_3 = \left\{ (x, y) \in P_1 - P_{1-1} \mid |k(x, y, \omega) - \frac{1}{N}| \leq \frac{\eta}{N} \left( \text{mod } \frac{2}{N} \right) \right\}.
\]

$A_3$ has an area ([5], Section 7.2.3)

\[
\text{area } (A_3) = \int_{\Phi_\omega(A_3)} |\det(D\Phi_\omega^{-1})| \, dxdy \leq \frac{1}{\sigma} \text{area}(\Phi_\omega(A_3)) < l_F \cdot \eta
\]

if $l_F > \frac{4\omega}{\omega^2}$, where $e^{\omega_0^0} \cdot \beta_0 \geq \sigma$ for all $(x, y, \omega) \in (P_1 - P_{1-1}) \times F$.

So finally we obtain

\[
\text{area } A = \text{area } (A_1 \cup A_2 \cup A_3) < \left( \frac{1}{2} + l_I + l_F \right) \eta.
\]

Here we take $N > 4(l_F \sigma - 4k_0) + \frac{1}{2} \max_{z \in F} |\lambda(z)|$, so that for this $N$ the inequality (5.10) is true.

5.3. Comparison of the metrics $(d(S''_\omega)^2)$ and $ds_\omega^2$

Recall that the deformed surface $(S''_\omega)$ is defined by (4.2). Then for $K^2$, we will get:

**Lemma 5.1.** Assume that $h(x, y, \omega, N)$ is given by the formula (5.7) and that the supremum and the infimum are taken over all directions at a point $z$. Then the metric of the deformed surface $(S''_\omega) := (S''_\omega)^{(h(\omega, \omega))}$, defined by the map $(S''_\omega)(x, y)$ as given in the equation (4.2), satisfies the relations:

1. $\lim_{\omega \to \infty} ((d(S''_\omega)^{p_1})^2 / (d(S''_\omega)^{p_2})) = 1$
2. $K^2 \leq \left( \frac{\sup ((d(S''_\omega)^{p_1})^2 / (d(S''_\omega)^{p_2}))}{\inf ((d(S''_\omega)^{p_1})^2 / (d(S''_\omega)^{p_2}))} \right)$

where $\omega \in F$, the constant $c_1$ can be made arbitrarily small for each fixed $\eta$ and for sufficiently large $N$, $c_2$ is some constant which is not necessarily small. The area of $A$ is given in (5.11).
Remark. On \( P_i - 1 \), since \( h = 0 \) and \( ds^2_\omega = \lambda^2(z)|dz|^2 \), \((d(\mathcal{S}_\omega^\eta))^\omega = \lambda^2(z)|dz|^2 \) so that (1) and (2) of Lemma 5.1 follows immediately. Therefore we need to consider all computations on \( P_i - P_i - 1 \) only.

In view of the above Remarks, to prove Lemma 5.1, we have to consider the following lemmas.

**Lemma 5.2.** Given \( h \) as in equation (5.7), and \( \gamma_\omega ds^2_\omega \) as in (5.5),

\[
|dh^2 + \lambda^2|dz|^2 - \gamma_\omega ds^2_\omega| \leq \begin{cases} \frac{R(\eta; N)}{\delta}ds^2_\omega & \text{on } P_1 - P_i - 1 - A, \omega \in F \\
\frac{\hat{R}(\eta; N)}{\delta}ds^2_\omega & \text{on } A, \omega \in F, \end{cases}
\]

where area \( A \) is given in (5.11). The inequalities are valid for \( N > N_F + \frac{1}{\max_{x \in F}|\lambda(x)|} \), where \( N_F > 4(l_F - 4k_0) \) with \( l_F \) (given in (5.10)) a constant depending on the compact set \( F \). For each fixed \( \eta \), \( R(\eta; N) \) can be made small as \( N \to \infty \) and \( \hat{R}(\eta; N) \) is some constant which is bounded as a function of \( N \).

**Proof.** Use the equations (5.8) and (5.5) to obtain

\[
|dh^2 + \lambda^2|dz|^2 - \gamma_\omega ds^2_\omega| = \left| \lambda^2 \cdot (\tilde{\alpha}_\omega dx + \tilde{\beta}_\omega dy) \cdot (\mathcal{U}^2 \mu_\eta^2 - 1) + o\left(\frac{1}{N}\right)|dz|^2 \right|.
\]

On \( P_1 - P_i - 1 - A \), we have \( \mu_\eta^2 = \tilde{\nu}_\eta^2 = 1 \) and \( \mathcal{U}^2(x, y) = 1 \) (see the above equation (5.9) and property (2) of the function \( \mu_\eta \) in [5], Section 7.2.2), so the right hand side of the equation (5.12) becomes

\[
\left| o\left(\frac{1}{N}\right)|dz|^2 \right| \leq R(\eta; N)ds^2_\omega
\]

for some small constant \( R(\eta; N) \).

On \( A \), since \( \mu_\eta^2 \cdot \tilde{\nu}_\eta^2 \neq 1 \) or \( \mathcal{U}^2(x, y) \neq 1 \), the right hand side (RHS') of the equation (5.12) becomes

\[
\text{RHS'} \leq \left| \gamma_\omega \left[ (\mathcal{U}^2 \cdot \mu_\eta^2 \cdot \tilde{\nu}_\eta^2 - 1) + o\left(\frac{1}{N}\right) \right] \right| ds^2_\omega \leq \hat{R}(\eta; N)ds^2_\omega
\]

for some constant \( \hat{R}(\eta; N) \) which is not necessarily very small. \( \square \)

**Lemma 5.3.** Given \( h(x, y, \omega, N) \) as in the equation (5.7), the metric of the deformed surface \( (\mathcal{S}_\omega^\eta)^\omega \) defined by the equation (4.2) satisfies the inequality

\[
|d(\mathcal{S}_\omega^\eta)^\omega| = dh_0^2 - dX^2 \leq c(\eta; N)|ds^2_\omega|
\]

for each fixed \( \eta \) and \( \omega \in F \), where \( c(\eta; N) \to 0 \) as \( N \to \infty \).

**Proof.** Apply the proof of Lemma 7.43 of [5] to the \( h \) given in equation (5.7). \( \square \)

To prove Lemma 5.1, we now apply Lemmas 5.2 and 5.3 using the same arguments, with \( h \) given in (5.7), in the proof of Lemma 7.38 of [5].
5.4. Final words

Thus far we have checked every condition we need in the hypotheses of Garsia’s Continuity lemma for some compact set $F$ in $T^#(S''_i)$. Therefore if we take $\epsilon = \frac{1}{2}\min\{1 - \|\omega_0\|, \|\omega_0\|\}$ and $F = B_{\epsilon}(\omega_0) \subset T^#(S''_i) \setminus \{0\}$, then we may now complete the process in the Section 4.

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