A new approach to $q$-zeta function

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Abstract We construct the new $q$-extension of Bernoulli numbers and polynomials in this paper. Finally we consider the $q$-zeta functions which interpolate the new $q$-extension of Bernoulli numbers and polynomials.

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1. Introduction

Throughout this paper $\mathbb{Z}, \mathbb{Z}_p, \mathbb{Q}_p$ and $\mathbb{C}_p$ will be denoted by the ring of rational integers, the ring of $p$-adic integers, the field of $p$-adic rational numbers and the completion of algebraic closure of $\mathbb{Q}_p$, respectively; cf. [7, 8, 9, 10]. Let $\nu_p$ be the normalized exponential valuation of $\mathbb{C}_p$ with $|p|_p = p^{-\nu_p(p)} = p^{-1}$. When one talks of $q$-extension, $q$ is variously considered as an indeterminate, a complex number $q \in \mathbb{C}$, or $p$-adic number $q \in \mathbb{C}_p$. If $q \in \mathbb{C}_p$, then we normally assume $|q - 1|_p < p^{-\nu_p(1)}$, so that $q^x = \exp(x \log q)$ for $|x|_p \leq 1$. If $q \in \mathbb{C}$, then we normally assume that $|q| < 1$. For $f \in UD(\mathbb{Z}_p, \mathbb{C}_p) = \{f : \mathbb{Z}_p \to \mathbb{C}_p \text{ is uniformly differentiable function}\}$, the $p$-adic $q$-integral (or $q$-Volkenborn integration) was defined as

$$I_q(f) = \int_{\mathbb{Z}_p} f(x)d\mu_q(x) = \lim_{N \to \infty} \frac{1}{[p^N]_q} \sum_{x=0}^{p^N-1} f(x)q^x, \quad (1)$$

where $[x]_q = \frac{1 - q^x}{1 - q}$, cf. [1, 2, 3, 4, 11]. Thus we note that

$$I_1(f) = \lim_{q \to 1} I_q(f) = \int_{\mathbb{Z}_p} f(x)d\mu_1(x) = \lim_{N \to \infty} \frac{1}{p^N} \sum_{0 \leq x < p^N} f(x), \quad (2)$$

where $\mu_1$ is the Haar measure.
By (2), we easily see that
\[ I_1(f_1) = I_1(f) + f'(0), \text{ cf. [5, 6, 7]}, \]
where \( f_1(x) = f(x + 1), f'(0) = \frac{d}{dx} f(x) \bigg|_{x=0}. \)

In [8], the \( q \)-Bernoulli polynomials are defined by
\[ \beta_n^{(h)}(x, q) = \int_{\mathbb{Z}_p} [x + x_1]^n q^{x_1(h-1)} d\mu_q(x_1), \text{ for } h \in \mathbb{Z}. \]  
(4)

In this paper we consider the new \( q \)-extension of Bernoulli numbers and polynomials. The main purpose of this paper is to construct the new \( q \)-extension of zeta function and \( L \)-function which interpolate the above new \( q \)-extension of Bernoulli numbers at negative integer.

2. On the New \( q \)-Extensions of Bernoulli numbers and polynomials

In (3), if we take \( f(x) = e^{hx}e^t \), then we have
\[ \int_{\mathbb{Z}_p} q^{hx} e^t d\mu_1(x) = \frac{h \log q + t}{q^h e^t - 1}. \]  
(5)

for \( |t| \leq p^{-\frac{1}{p-1}}, h \in \mathbb{Z}. \)

Let us define the \((h, q)\)-extension of Bernoulli numbers and polynomials as follows:
\[ F_q^{(h)}(t) = \frac{h \log q + t}{q^h e^t - 1} = \sum_{n=0}^{\infty} B_{n,q}^{(h)} \frac{t^n}{n!}, \]
\[ F_q^{(h)}(t, x) = \frac{h \log q + t}{q^h e^t - 1} e^t = \sum_{n=0}^{\infty} B_{n,q}^{(h)}(x) \frac{t^n}{n!}. \]  
(6)

Note that \( B_{n,q}^{(h)}(0) = B_{n,q}^{(h)}, \lim_{q \to 1} B_{n,q}^{(h)} = B_n \), where \( B_n \) are the \( n \)-th Bernoulli numbers.

By (5) and (6), we obtain the following Witt’s formula.

**Theorem 1.** For \( h \in \mathbb{Z}, q \in \mathbb{C}_p \) with \( |1 - q|_p \leq p^{-\frac{1}{p-1}}, \) we have
\[ \int_{\mathbb{Z}_p} q^{hx}x^n d\mu_1(x) = B_{n,q}^{(h)}; \]
\[ \int_{\mathbb{Z}_p} q^{hy}(x + y)^n d\mu_1(y) = B_{n,q}^{(h)}(x). \]  
(7)

By above theorem, we easily see that
\[ B_{n,q}^{(h)}(x) = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} B_{k,q}^{(h)}. \] (8)

Let \( d \) be any fixed positive integer with \((p, d) = 1\). Then we set

\begin{align*}
X &= X_d = \lim_{N} (\mathbb{Z}/dp^N\mathbb{Z}), X_1 = \mathbb{Z}_p, \\
X^* &= \bigcup_{0 < a < dp} a + dp\mathbb{Z}_p, \\
a + dp^N\mathbb{Z}_p &= \{ x \in X \mid x \equiv a \pmod{dp^N} \},
\end{align*}

where \( a \in \mathbb{Z} \) with \( 0 \leq a < dp^N \). Note that

\[ \int_{\mathbb{Z}_p} f(x) d\mu_1(x) = \int_{X} f(x) d\mu_1(x), \]

for \( f \in UD(\mathbb{Z}_p, \mathbb{C}_p) \), cf. [1, 2, 3, 4, 10].

In Eq. (7), it is easy to see that

\[ B_{k,q}^{(h)}(x) = \int_{X} (x + t)^k q^{ht} d\mu_1(t) = \lim_{l \to \infty} \frac{1}{mp^l} \sum_{n=0}^{mp^{l-1}} q^{hn}(x + n)^k, \]

\[ = \frac{1}{m} \lim_{l \to \infty} \frac{1}{p^l} \sum_{i=0}^{m-1} \sum_{n=0}^{p^{l-1}} q^{h(i + mn)}(x + i + mn)^k = m^{k-1} \sum_{i=0}^{m-1} q^{hi} B_{k,q}^{(h)} \left( \frac{x + i}{m} \right). \]

Therefore we have the below theorem.

**Theorem 2.** For any positive integer \( m \), we have

\[ B_{k,q}^{(h)}(x) = m^{k-1} \sum_{i=0}^{m-1} q^{hi} B_{k,q}^{(h)} \left( \frac{x + i}{m} \right), \text{ for } k \geq 0. \]

Let \( \chi \) be the Dirichlet character with conductor \( d \in \mathbb{Z}_+ \). Then we define the \((h, q)\)-extension of generalized Bernoulli numbers attached to \( \chi \). For \( n \geq 0 \), define

\[ B_{n,q,\chi}^{(h)} = \int_{X} \chi(x) q^{hx} x^n d\mu_1(x). \] (9)

By (9), we easily see that

\[ B_{n,q,\chi}^{(h)} = \lim_{l \to \infty} \frac{1}{dp^l} \sum_{x=0}^{dp^{l-1}} \chi(x) q^{hx} x^n = \frac{1}{d} \lim_{l \to \infty} \frac{1}{p^l} \sum_{i=0}^{d-1} \sum_{x=0}^{p^{l-1}} \chi(i + dx) q^{h(i + dx)} (i + dx)^n = \frac{1}{d} \sum_{i=0}^{d-1} \chi(i) q^{hi} \int_{\mathbb{Z}_p} q^{hdx} (i + dx)^n d\mu_1(x) = d^{n-1} \sum_{i=0}^{d-1} \chi(i) q^{hi} B_{n,q}^{(h)} \left( \frac{i}{d} \right). \]
Therefore we obtain the below lemma.

**Lemma 3.** For \( d \in \mathbb{Z}_+ \), we have

\[
B_{k,q,\chi}^{(h)} = d^{k-1} \sum_{i=0}^{d-1} \chi(i)q^{hi} B_{k,q^d}^{(h)} \left( \frac{i}{d} \right), \text{ for } n \geq 0. \tag{10}
\]

By induction in Eq.(3), we easily see that

\[
I_1(f_b) = I_1(f) + \sum_{i=0}^{b-1} f'(i), \text{ where } f_b(x) = f(x+b), b \in \mathbb{Z}_+. \tag{11}
\]

In Eq.(11), if we take \( f(x) = q^{hx}e^{tx} \chi(x) \), then we have

\[
I_1(e^{tx}q^{hx} \chi(x)) = \sum_{d=0}^{d-1} \left( te^{it} \chi(i)q^{hi} + e^{ti}(h \log q)q^{hi} \chi(i) \right) \frac{q^{hd}e^{dt}}{q^{hd}e^{dt} - 1}. \tag{12}
\]

By (12) and (9), we can give the generation function of \( B_{n,q,\chi}^{(h)} \) as follows:

\[
F_{q,\chi}^{(h)}(t) = \sum_{n=0}^{\infty} B_{n,q,\chi}^{(h)} t^n = \frac{\sum_{n=0}^{\infty} \left( te^{it} \chi(i)q^{hi} + e^{ti}(h \log q)q^{hi} \chi(i) \right)}{q^{hd}e^{dt} - 1}. \tag{13}
\]

3. The analogue of zeta function

In this section we assume that \( q \in \mathbb{C} \) with \(|q| < 1\). Let \( \Gamma(s) \) be the gamma function. By (6), we can readily see that

\[
\frac{1}{\Gamma(s)} \int_0^\infty t^{s-2}e^{-t} F_{q}^{(h)}(-t) dt = \frac{1}{\Gamma(s)} \int_0^\infty t^{s-2}e^{-t} \left\{ \frac{-t}{q^h e^{-t} - 1} + \frac{h \log q}{q^h e^{-t} - 1} \right\} dt
\]

\[
= \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1}e^{-t} \frac{1}{1 - q^h e^{-t}} dt - \frac{h \log q}{\Gamma(s)} \int_0^\infty \frac{t^{s-2}e^{-t}}{1 - q^h e^{-t}} dt
\]

\[
= \sum_{n=0}^{\infty} q^{nh} \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1}e^{-(n+1)t} dt - h \log q \sum_{n=0}^{\infty} q^{nh} \int_0^\infty t^{s-1}e^{-(n+1)t} dt
\]

\[
= \sum_{n=1}^{\infty} q^{(n-1)h} \frac{1}{n^s} - \log q^h \frac{\infty}{s-1} \sum_{n=1}^{\infty} q^{(n-1)h} n^{s-1}. \tag{14}
\]

Using (14), we define the new \( q \)-extensions of zeta functions as follows:
Definition 4. For $s \in \mathbb{C}, x \in \mathbb{R}^+$, we define

$$
\zeta_q^{(h)}(s) = \sum_{n=1}^{\infty} \frac{q^{(n-1)h}}{n^s} - \frac{h \log q}{s-1} \sum_{n=1}^{\infty} \frac{q^{(n-1)h}}{n^{s-1}},
$$

(14 - a)

$$
\zeta_q^{(h)}(s, x) = \sum_{n=0}^{\infty} \frac{q^{nh}}{(n+x)^s} - \frac{h \log q}{s-1} \sum_{n=0}^{\infty} \frac{q^{nh}}{(n+x)^{s-1}},
$$

(14 - b)

Remark. By (14-a) and (14-b), we easily see that $\zeta_q^{(h)}(s) = \zeta_q^{(h)}(s, 1)$. Also, we note that $\zeta_q^{(h)}(s)$ is analytic continuation for $R(s) > 1$.

Using Mellin transforms in Eq.(6), we obtain

$$
\frac{1}{\Gamma(s)} \int_0^{\infty} t^{s-2} F_q^{(h)}(-t, x)dt = \zeta_q^{(h)}(s, x).
$$

(15)

By (15) and (6), we readily see that

$$
\zeta_q^{(h)}(s, x) = \sum_{n=0}^{\infty} \frac{(-1)^n B_q^{(h)}(x)}{n!} \frac{1}{\Gamma(s)} \int_0^{\infty} t^{s-2+n} dt.
$$

Therefore we obtain the following:

**Theorem 5.** For $n \in \mathbb{N}$, we have

$$
\zeta_q^{(h)}(1-n, x) = -\frac{B_q^{(h)}(x)}{n}.
$$

By Mellin transforms and (13), we note that

$$
\frac{1}{\Gamma(s)} \int_0^{\infty} F_q^{(h)}(-t) t^{s-2} dt = \sum_{n=1}^{\infty} \frac{q^{nh}\chi(n)}{n^s} - \frac{h \log q}{s-1} \sum_{n=1}^{\infty} \frac{q^{nh}\chi(n)}{n^{s-1}}.
$$

(16)

Thus we can define the new $q$-extension of Dirichlet $L$-function as follows:

**Definition 6.** For $s \in \mathbb{C}$, we define

$$
L_q^{(h)}(s, \chi) = \sum_{n=1}^{\infty} \frac{q^{nh}\chi(n)}{n^s} - \frac{h \log q}{s-1} \sum_{n=1}^{\infty} \frac{q^{nh}\chi(n)}{n^{s-1}}.
$$

By (16) and (13), we have the following

**Theorem 7.** For $n \in \mathbb{N}$, we obtain

$$
L_q^{(h)}(1-n, \chi) = -\frac{B_q^{(h)}(\chi)}{n}.
$$
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