Semiparametric Efficient Fusion of Individual Data and Summary Statistics

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Abstract

Suppose we have individual data from an internal study and various summary statistics from relevant external studies. External summary statistics have the potential to improve statistical inference for the internal population; however, it may lead to efficiency loss or bias if not used properly. We study the fusion of individual data and summary statistics in a semiparametric framework to investigate the efficient use of external summary statistics. Under a weak transportability assumption, we establish the semiparametric efficiency bound for estimating a general functional of the internal data distribution, which is no larger than that using only internal data and underpins the potential efficiency gain of integrating individual data and summary statistics. We propose a data-fused efficient estimator that achieves this efficiency bound. In addition, an adaptive fusion estimator is proposed to eliminate the bias of the original data-fused estimator when the transportability assumption fails. We establish the asymptotic oracle property of the adaptive fusion estimator. Simulations and application to a Helicobacter pylori infection dataset demonstrate the promising numerical performance of the proposed method.

Keywords: Causal inference; Data fusion; Integrative data analysis; Semiparametric efficiency bound.

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1 Introduction

Suppose individual data from an internal study is available to investigate a particular scientific purpose. It is appealing to fuse external datasets from different sources with the internal data to improve statistical inference. Methods for data fusion with external individual data have grown in popularity in recent years (e.g., Yang and Ding, 2020; Li et al., 2023b; Li and Luedtke, 2023; Sun and Miao, 2022; Chen et al., 2021). However, sometimes it is impossible to access the individual data due to ethics and privacy concerns and one can only have certain summary statistics from external studies. Meta-analysis has been widely applied to integrate summary statistics on a common parameter from multiple studies (e.g., Singh et al., 2005; Lin and Zeng, 2010; Li et al., 2020), but it becomes challenging if participating studies are analyzed with different statistical models and are concerned with different parameters. In order to assimilate various types of summary statistics from multiple sources, previous authors have developed a suite of methods which essentially view external summary statistics as certain constraints on the internal data distribution. For example, Bickel et al. (1993, Section 3.2 Example 3) established the semiparametric theory for using external summary statistics as moment equation constraints; Qin (2000) proposed an empirical likelihood method and Chatterjee et al. (2016) proposed a constrained maximum likelihood approach, which leverage summary statistics obtained from a large external dataset to improve estimation efficiency for a parametric model of the internal dataset. In situations where the uncertainty of external summary statistics is negligible, these methods can achieve higher efficiency than the maximum likelihood estimator based solely on the internal individual data. Nonetheless, Zhang et al. (2020) cautioned that if the uncertainty of external summary statistics is not negligible, the efficiency gain of the constrained maximum likelihood estimator is not guaranteed, and paradoxically, it can be even less efficient than the maximum likelihood estimator based on internal individual data solely.

The external summary statistics provide additional information, and when incorporated appropriately, they are expected to improve the efficiency of statistical inference. The efficiency paradox arises mainly because the use of the external summary statistics is not optimized. Therefore, it is
of both practical and theoretical interest to investigate the efficient fusion of individual data and summary statistics. Semiparametric theory is a widely used framework to study the optimality of estimators, providing guidance for constructing efficient estimators under minimal assumptions on the data distribution. However, the classic semiparametric theory, which is designed for independent and identically distributed (i.i.d.) individual data (Bickel et al., 1993; Li and Luedtke, 2023; Li et al., 2023a), is not applicable to the current setting involving both individual data and summary statistics. The literature on semiparametric theory for non-i.i.d. data is relatively sparse (Strasser, 1989; McNeney and Wellner, 2000), and, to our knowledge, none of these works address the incorporation of summary statistics. Zhang et al. (2020) developed a generalized data integration method (GIM) under a parametric conditional density model, which avoids the efficiency paradox and is optimal in the sense that it achieves the smallest asymptotic variance over a class of maximum log-pseudolikelihood estimators. Recently, Chen et al. (2024) extended the method of Zhang et al. (2020) to account for prior probability shifts between internal and external populations, thereby improving robustness against population heterogeneity. However, the general semiparametric theory for fusing individual data and summary statistics is not available yet. In particular, it is not clear how to efficiently estimate a general functional in a semi/non-parametric model when given both individual data and summary statistics.

In addition to potential efficiency loss, integrating external summary statistics can also introduce estimation bias if the summary statistics are not transportable. This bias frequently occurs in the presence of population heterogeneity, biased sampling, or model misspecification. It has been noted in conventional meta-analysis, and various robust methods have been proposed to address this problem (Singh et al., 2005; Shen et al., 2020; Wang et al., 2023), ensuring consistent and asymptotically normal estimation despite untransportable summary statistics. For the fusion of individual data, Chen et al. (2021), Kallus et al. (2018), Yang et al. (2023), Li et al. (2023b) and Yang et al. (2020) considered combining randomized trial data and observational data to remove unmeasured confounding bias. However, these methods require the availability of individual data.
from multiple data sources. For integration of individual data and summary statistics, Zhai and Han (2022); Huang et al. (2023) proposed penalized constrained maximum likelihood methods that extend the empirical likelihood framework of Qin (2000), Chatterjee et al. (2016) and Zhang et al. (2020) to accommodating possibly untransportable external summary statistics. However, these methods focus on parameters in a parametric model and may not be suitable for general semi/nonparametric inference problems, such as estimation of the average treatment effect estimation in causal inference or the outcome mean in missing data analysis.

In this paper, we develop a semiparametric framework for the integration of internal individual data and external summary statistics. In contrast to previous works that primarily focus on parametric models, our approach accommodates general semiparametric and nonparametric models. We derive the semiparametric efficiency bound for inference on a general functional of the internal data distribution in the presence of external summary statistics, which is shown to be no larger than that obtained using only the internal data. We construct a data-fused efficient estimator that achieves the efficiency bound under data fusion. By adopting the efficient estimator which incorporates the external summary statistics in an optimal way, the efficiency paradox is resolved. The data-fused efficient estimator has a closed form, which greatly reduces the computational complexity compared to the empirical likelihood methods proposed by Zhang et al. (2020, 2022). To address potential issues with untransportable summary statistics, we further propose an adaptive fusion estimator by leveraging some carefully designed weighting matrices. The adaptive fusion estimator is continuous with respect to the observed data, which is a desirable property that improves numerical stability (Fan and Li, 2001). In addition, it is easy to compute due to its closed-form expression. The adaptive fusion estimator is consistent and asymptotically normal even if some external summary statistics are untransportable. It is also asymptotically equivalent to the oracle estimator that uses only transportable summary statistics. The asymptotic results hinge on the fact that one can consistently assess whether the summary statistics are transportable in large sample size. However, distinguishing between transportable and untransportable summary statistics in fi-
nite samples can be challenging, particularly when the internal and external populations are close but not identical. This problem may lead to undercoverage of confidence intervals that are based on the adaptive fusion estimator and its asymptotic distribution. We thus propose a re-bootstrap procedure to mitigate the undercoverage issue in finite samples. This procedure maintains robust coverage rates even under the challenging setting where the heterogeneity between populations is comparable to the magnitude of the estimation error. We discuss theoretical results for the scenario where the transportability of certain summary statistics holds asymptotically at certain rates in the Supplementary Material. We evaluate the performance of the proposed estimation methods via simulations and apply them to test the causal effect of a combined therapy on Helicobacter pylori infection.

The rest of this paper is organized as follows. In Section 2, we briefly review the classic semiparametric theory and discuss the gap between the classic theory and the problem under consideration. In Section 3, we establish the semiparametric efficiency theory for fusing individual data and summary statistics. In Section 4, we propose an adaptive fusion estimator to integrate the summary statistics in the presence of untransportable components and a re-bootstrap procedure to make inference. We report extensive simulation results and a real data analysis in Section 5, followed by discussions. Additional simulations, discussion on the scenario when transportability holds asymptotically and all proofs are provided in the Supplementary Material.

2 The Classic Semiparametric Theory

Suppose we have \( n \) i.i.d. individual-level observations \((Z_1, \ldots, Z_n)\) from the internal distribution/study \( P_0 \in \mathcal{P}_0 \) and a \( q \)-dimensional vector of summary statistics \( \tilde{\beta} = (\tilde{\beta}_1, \ldots, \tilde{\beta}_q)^T \) based on individual observations \((W_1, \ldots, W_m)\) from the external distribution/study \( P_1 \in \mathcal{P}_1 \), where \( \mathcal{P}_0 \) and \( \mathcal{P}_1 \) denote collections of models for internal data and external data, respectively. The external sample size \( m \) is known, the external individual data are unavailable, and \( \tilde{\beta} \) is assumed to be an estimator of some functional \( \beta(P_1) \). The parameter of interest is a \( p \)-dimensional functional of the
internal data distribution, \( \tau = \tau(P_0) \), which may differ from \( \beta \). Throughout the paper, we let \( E(\cdot) \) denote the expectation with respect to \( P_0 \) and \( \hat{E}(\cdot) \) the empirical mean in the internal data, unless otherwise specified.

We briefly review the classical semiparametric theory when only the internal i.i.d. individual data are used for estimating \( \tau \). Most reasonable estimators in statistical inference problems are regular and asymptotically linear (RAL). Following Tsiatis (2006, chapter 3), RAL estimators in a parametric model indexed by a finite-dimensional parameter \( \theta \), say \( \{P_0(Z; \theta); \theta \in \mathbb{R}^k\} \), are described as follows.

**Definition 1.** An estimator \( T_n = T_n(Z_1, \ldots, Z_n) \) of \( \tau \) is regular if \( n^{1/2}\{T_n(Z_1^{(n)}, \ldots, Z_n^{(n)}) - \tau(\theta_n)\} \) has a limiting distribution that does not depend on the local data generating process where for each \( n \) the data \( \{Z_1^{(n)}, \ldots, Z_n^{(n)}\} \) are i.i.d. distributed according to \( P_0(Z; \theta_n) \) with \( n^{1/2}(\theta_n - \theta) \) converging to a constant; an estimator \( T_n \) is asymptotically linear if \( T_n = \tau + n^{-1}\sum_{i=1}^n \phi(Z_i) + o_P(n^{-1/2}) \) for some vector function \( \phi \) with \( E\{\phi(Z)\} = 0 \) and \( E\{\phi(Z)\phi^T(Z)\} \) finite and nonsingular; and an estimator \( T_n \) is RAL if it is both regular and asymptotically linear.

The function \( \phi \) is called an influence function for \( \tau \), describing the influence of each observation on the estimation of \( \tau \). Regularity is often desirable, which rules out pathological estimators such as the superefficient estimator of Hodges and estimators that invoke more information than is contained in the model. Moreover, it can be shown that the most efficient regular estimator is asymptotically linear (Hájek, 1970); hence, it is reasonable to restrict attention to RAL estimators.

For a differentiable parameter \( \tau(\theta) \) in the parametric model \( P_0(Z; \theta) \), letting \( S_\theta \) denote the score for \( \theta \), then the Cramer–Rao bound, \( V_\theta = \{\partial\tau(\theta)/\partial\theta\} \{E(S_\theta S^T_\theta)\}^{-1}\{\partial\tau(\theta)/\partial\theta^T\} \), characterizes the smallest possible asymptotic variance for RAL estimators of \( \tau \). However, lack of flexibility and thus potential misspecification of parametric models incur untransportable inferences, and in many situations, one is only interested in a finite-dimensional parameter rather than the full data distribution. This leads to the adoption of semiparametric or nonparametric models that admit infinite-dimensional parameters embodying less restrictive assumptions beyond the parameter of...
interest. Bickel et al. (1993) described the efficiency theory for inference in semiparametric and nonparametric models. One can view a semiparametric model as the collection of many parametric submodels that satisfy the semiparametric assumptions and contain the true data generating process but impose no additional restrictions. A (pathwise) differentiable functional $\tau$ on a semiparametric model needs to be differentiable on all parametric submodels and satisfy $\partial \tau(\theta)/\partial \theta = E\{\phi S_\theta\}$ for some squared integrable function $\phi$ and score function $S_\theta$ of an arbitrary parametric submodel. An estimator is said to be regular on a semiparametric model if it is regular on all parametric submodels. A key concept in the semiparametric theory is the semiparametric efficiency bound, which is the supremum of the Cramer-Rao bounds for all parametric submodels. The semiparametric efficiency bound is the lower bound for the asymptotic variance of any RAL estimator. The influence function attaining the semiparametric efficiency bound is called the efficient influence function, and the corresponding estimator is the efficient estimator.

In the rest of this paper, we let $\phi_{\text{eff}}$ denote the efficient influence function, $E(\phi_{\text{eff}}\phi_{\text{eff}}^T)$ the efficiency bound, $\hat{\tau}_{\text{eff}}^{\text{int}}$ the efficient estimator of $\tau$ based on the internal data in the class of semiparametric or nonparametric models under consideration. We illustrate these concepts with an influential causal inference example; see Bang and Robins (2005) and Hahn (1998) for details.

**Example 1.** Suppose we have internal individual data on $Z = (T, X, Y)$ from an observational study $P_0$ about the effect of a binary treatment $T$ on the outcome $Y$ with covariates $X$. Let $Y_t$ denote the potential outcome if the treatment were set to $T = t$ for $t = 0, 1$ and $\tau = E(Y_1 - Y_0)$ the average treatment effect. Let $p(X) = pr(T = 1 \mid X)$ be the treatment propensity score and $\mu_t(X) = E(Y \mid T = t, X)$ ($t = 0, 1$) the outcome regression function. Under the ignorability assumption ($Y_t \perp \perp T \mid X$ and $0 < p(X) < 1$), $\tau$ is identified from the observed data with $\tau = E\{\mu_1(X) - \mu_0(X)\}$. The efficient influence function for $\tau$ in the nonparametric model that imposes no other restrictions than the ignorability is

$$
\phi_{\text{eff}}(Z; \tau) = \frac{T}{p(X)}\{Y - \mu_1(X)\} - \frac{1-T}{1-p(X)}\{Y - \mu_0(X)\} + \mu_1(X) - \mu_0(X) - \tau.
$$

The efficiency bound for $\tau$ is $E(\phi_{\text{eff}}^2)$. The efficient estimator $\hat{\tau}_{\text{eff}}^{\text{int}}$ can be obtained by firstly estima-
ing \{p(X), \mu_t(X)\} and then solving \( \hat{E}\{\phi_{\text{eff}}(Z; \tau)\} = 0 \) with these nuisance estimators plugged in.

For instance, one can specify and fit parametric working models \( p(X; \tilde{\zeta}) \) and \( \mu_t(X; \hat{\psi}_t) \), then

\[
\hat{\tau}_{\text{eff}} = \frac{1}{n} \sum_{i=1}^{n} \left[ \frac{T_i}{p(X_i; \tilde{\zeta})} \{Y_i - \mu_1(X_i; \hat{\psi}_1)\} - \frac{1 - T_i}{1 - p(X_i; \tilde{\zeta})} \{Y_i - \mu_0(X_i; \hat{\psi}_0)\} + \mu_1(X_i; \hat{\psi}_1) - \mu_0(X_i; \hat{\psi}_0) \right].
\]

We aim to combine both the internal individual data and the external summary statistics \( \hat{\beta} \) to improve the estimation of \( \tau \). Suppose \( \beta \) can also be identified in the internal data, and let \( \eta_{\text{eff}} \) be the efficient influence function for \( \beta \) based on internal individual data, which depends on \( \beta \) in general. For simplicity, We omit the dependency in the notation \( \eta_{\text{eff}} \) when evaluated at the true value \( \beta \), and use \( \eta_{\text{eff}}(\beta^\dagger) \) to denote the corresponding influence function evaluated at some \( \beta^\dagger \) that may not be equal to \( \beta \). Applying the classical semiparametric theory, Bickel et al. (1993, Section 3.2 Example 3) established a well-known result that the efficient influence function for \( \tau \) is \( \phi_{\text{eff}} - E(\phi_{\text{eff}}\eta_{\text{eff}}^T) \{E(\eta_{\text{eff}}\eta_{\text{eff}}^T)\}^{-1} \eta_{\text{eff}} \) when \( \beta \) is known. This influence function has a variance no larger than that of \( \phi_{\text{eff}} \) and motivates the estimator \( T_n(\beta) = \hat{\tau}_{\text{eff}} - E(\phi_{\text{eff}}\eta_{\text{eff}}^T) \{E(\eta_{\text{eff}}\eta_{\text{eff}}^T)\}^{-1} \hat{E}(\eta_{\text{eff}}) \) whose asymptotic variance is no larger than \( \hat{\tau}_{\text{eff}} \). However, the estimator \( T_n(\beta) \) is infeasible if \( \beta \) is unknown. In practice, the true value \( \beta \) is typically unknown but is instead estimated by the external summary statistics \( \hat{\beta} \). Thus, a primitive data-fused estimator of \( \tau \) is the plug-in estimator \( T_n(\hat{\beta}) = \hat{\tau}_{\text{eff}} - E(\phi_{\text{eff}}\eta_{\text{eff}}^T) \{E(\eta_{\text{eff}}\eta_{\text{eff}}^T)\}^{-1} \hat{E}(\eta_{\text{eff}}(\hat{\beta})) \). In the definition of \( T_n(\hat{\beta}) \), we retain the unknown term \( E(\phi_{\text{eff}}\eta_{\text{eff}}^T) \{E(\eta_{\text{eff}}\eta_{\text{eff}}^T)\}^{-1} \) for clarity in our illustrations. Replacing this term with a consistent estimator does not alter the asymptotic distribution. The estimator \( T_n(\hat{\beta}) \) is expected to still deliver better efficiency than \( \hat{\tau}_{\text{eff}} \). Nonetheless, somewhat paradoxically, the estimator \( T_n(\hat{\beta}) \) that incorporates external information may be less efficient than the internal data-based estimator \( \hat{\tau}_{\text{eff}} \), which we will show later. The phenomenon that incorporating external information may lead to efficiency loss has been previously pointed out and resolved under a parametric conditional density model by Zhang et al. (2020). The constrained likelihood estimator studied in Zhang et al. (2020) is asymptotically equivalent to the estimator \( T_n(\hat{\beta}) \) when estimating the parameter in a parametric conditional density model according to the first-order equivalence of empirical
likelihood method and generalized method of moments and the efficiency theory of generalized method of moments (Imbens and Lancaster, 1994; Newey and Smith, 2004).

Roughly speaking, the above efficiency paradox arises because the estimator $T_{n}^{\text{eff}}(\tilde{\beta})$ ignores the uncertainty of $\tilde{\beta}$ and thus fails to incorporate the external summary statistics in an efficient way. This motivates us to investigate the efficiency of the data-fused estimators and derive the most efficient one in a semiparametric framework where the parameter of interest is a generic functional of internal data distribution. However, the classical semiparametric theory for i.i.d. individual data does not apply to the setting here, which involves the external summary statistics $\tilde{\beta}$ that are distributed differently from the internal data. Although there exists a sparse literature on the semiparametric theory for non-i.i.d data (Strasser, 1989; McNeney and Wellner, 2000), their theory is not applicable here. In the next section, we extend the classical semiparametric theory to the data fusion setting where both individual data and summary statistics are available.

3 Semiparametric Theory for Fusion of Individual data and Summary Statistics

3.1 Assumptions and Data-fused RAL Estimators

In order to make use of the external summary statistic, we make the following assumptions.

Assumption 1. The external summary statistic $\tilde{\beta}$ is a RAL estimator of a $q$-dimensional functional $\beta(\cdot)$ of the external data distribution and $m^{1/2}\{\tilde{\beta} - \beta(P_1)\} \rightarrow N(0, \Sigma_1)$; a consistent covariance estimator $\hat{\Sigma}_1$ is also available; and $m/n \rightarrow \rho \in (0, +\infty)$.

Assumption 2 (Transportability). $\beta(P_0) = \beta(P_1)$.

Assumption 1 is met with standard estimation methods under mild regularity conditions and has been widely adopted in meta-analysis (Singh et al., 2005; Xie et al., 2011; Kundu et al., 2019; Zhang et al., 2020). Note that the functional $\beta(\cdot)$ is not necessarily the same as $\tau$, the parameter of
interest. Assumption 2 states that the values of \( \beta(\cdot) \) are the same across the internal and external studies. This transportability assumption establishes the connection between the internal data distribution \( P_0 \) and external data distribution \( P_1 \), which is essential for efficiency improvement with external summary statistics. Analogous assumptions such as mean/distribution exchangeability have been used in previous work (e.g., Dahabreh et al., 2019; Li et al., 2023b). In Section 4, we discuss scenarios where Assumption 2 fails and bias arises. Furthermore, we consider the scenario where the transportability of a subset of the summary statistics holds only asymptotically at a suitable rate in Supplementary Material. In the rest of the paper, we denote \( \beta = \beta(P_0) \), and we assume that \( \tau \) is pathwise differentiable on \( P_0 \) at \( P_0 \) and \( \beta \) is pathwise differentiable on \( P_0 \) and \( P_1 \) at \( P_0 \) and \( P_1 \), respectively. We let \( \eta_{\text{eff}} \) denote the efficient influence function for \( \beta \) when only internal individual data are available.

We focus on the estimation of \( \tau \) in the semiparametric model

\[
P_{\text{trans}} = \{P_0 \times P_1 \in P_0 \times P_1 : P_0, P_1 \text{ satisfy Assumption 2}\}.
\]

We consider the following class of estimators that incorporate both internal individual data and the external summary statistics.

**Definition 2 (Data-fused RAL estimator).** Let \( T_n(Z_1, \ldots, Z_n, \bar{\beta}) \) denote a data-fused estimator of \( \tau \) and we write \( T_n(\bar{\beta}) \) for shorthand.

(i) \( T_n(\bar{\beta}) \) is regular if for every parametric submodel \( P_0(Z; \theta) \times P_1(W; \theta) \in P_{\text{trans}} \), the quantity \( n^{1/2}\left\{ T_n(Z_1^{(n)}, \ldots, Z_n^{(n)}, \bar{\beta}^{(m)}) - \tau(P_0(Z; \theta_n)) \right\} \) has a limiting distribution that does not depend on the local data generating process, where the data \( \{Z_1^{(n)}, \ldots, Z_n^{(n)}\} \) is an i.i.d. sample from \( P_0(Z; \theta_n) \), and \( \bar{\beta}^{(m)} \) is obtained from an i.i.d. sample \( \{W_1^{(m)}, \ldots, W_m^{(m)}\} \) from \( P_1(W; \theta_n) \), with \( m/n \to \rho \in (0, \infty) \) and \( n^{1/2}(\theta_n - \theta) \) converging to a constant.

(ii) \( T_n(\bar{\beta}) \) is asymptotically linear if \( T_n(\bar{\beta}) = \tau + n^{-1} \sum_{i=1}^n \psi(Z_i) + \gamma(\bar{\beta}) + o_P(n^{-1/2}) \) with \( E\{\psi(Z)\} = 0, E\{\psi(Z)\psi(Z)\} \) finite and nonsingular, \( \gamma(\bar{\beta}) \) continuously differentiable in \( \bar{\beta} \) and \( \gamma(\beta) = 0 \).
(iii) \( T_n(\tilde{\beta}) \) is regular and asymptotically linear (RAL) if it satisfies both (i) and (ii).

Analogous to the classical semiparametric theory, Definition 2 (i) characterizes the regularity with respect to both the internal data distribution and external data distribution. This class of data-fused regular estimators contains all the regular estimators based only on the internal individual data. Following the spirit of classical asymptotic linearization, Definition 2 (ii) treats \( \tilde{\beta} \) as an additional sample to the internal data and uses \( \psi(Z_i) \) as well as \( \gamma(\tilde{\beta}) \) to depict the influence of \((Z_1, \ldots, Z_n, \tilde{\beta})\) on the estimation of \( \tau \). The restrictions on \( \gamma(\tilde{\beta}) \) ensure that \( T_n(\tilde{\beta}) \) satisfying Definition 2 (ii) is consistent and asymptotically normal. The class of data-fused RAL estimators in Definition 2 (iii) includes all the RAL estimators that use only the internal data.

**Proposition 1.** Under Assumptions 1, 2 and a regularity condition (Condition ??) in Supplementary Material, a data-fused RAL estimator \( T_n(\tilde{\beta}) \) has the following representation,

\[
T_n(\tilde{\beta}) = \tau + \frac{1}{n} \sum_{i=1}^{n} \{ \phi(Z_i) - \xi_{\text{eff}}(Z_i) \} + \xi(\tilde{\beta} - \beta) + o_P(n^{-1/2}),
\]

and its asymptotic variance is

\[
E(\phi\phi^T) + \xi E(\eta_{\text{eff}}^T \eta_{\text{eff}}) \xi^T - 2\xi E(\eta_{\text{eff}} \phi^T) + \rho^{-1} \xi \Sigma_1 \xi^T,
\]

where \( \phi \) is an influence function for \( \tau \) based only on the internal data, \( \xi = \xi(P_0) \) is a \( p \times q \) matrix, and the forms of \( \phi \) and \( \xi \) depend on the estimator \( T_n(\tilde{\beta}) \).

Condition ?? in Supplementary Material is a regularity condition concerning the continuity and boundness of \( \tilde{\beta} \)'s density, invoked primarily for technical purposes. Proposition 1 reveals how the estimation of \( \beta \) in the external data affects the efficiency of a data-fused RAL estimator \( T_n(\tilde{\beta}) \). If the estimation \( \tilde{\beta} \) in the external study is very precise, or if the true value \( \beta \) is known, then \( T_n(\tilde{\beta}) \) reduces to \( T_n(\beta) \), which is a RAL estimator with influence function \( \psi = \phi - \xi_{\text{eff}} \). Recall that \( T_n^{\text{eff}}(\beta) \) is the efficient estimator when \( \beta \) is known and \( T_n^{\text{eff}}(\tilde{\beta}) \) is the plug-in estimator obtained by replacing \( \beta \) by its estimate \( \tilde{\beta} \) in the definition of \( T_n^{\text{eff}}(\beta) \). Then, \( T_n^{\text{eff}}(\tilde{\beta}) = \tau + n^{-1} \sum_{i=1}^{n} \{ \phi_{\text{eff}} -
\( A_{\text{eff}} \} + A(\bar{\beta} - \beta) + o_P(n^{-1/2}) \) with \( A = E(\phi_{\text{eff}} \eta_{\text{eff}}^T) \{ E(\eta_{\text{eff}} \eta_{\text{eff}}^T) \}^{-1} \), and its asymptotic variance is

\[
E(\phi_{\text{eff}} \phi_{\text{eff}}^T) + A \{ \Sigma_1 / \rho - E(\eta_{\text{eff}} \eta_{\text{eff}}^T) \} A^T. \tag{2}
\]

This result has three meaningful implications: First, when the external data sample size is much larger than the internal data, i.e., \( \rho \to +\infty \), the asymptotic variance of the plug-in estimator \( T_n^\text{eff}(\bar{\beta}) \) approximates the semiparametric efficiency bound when \( \beta \) is known; this has been noted by Qin (2000) and Chatterjee et al. (2016) for the estimation of parameters in a parametric model for the internal data; here we extend this result to semiparametric models where the parameter of interest is a functional of data distribution. Second, when the external data sample size is much smaller than the internal data, i.e., \( \rho \to 0 \), the asymptotic variance of \( T_n^\text{eff}(\bar{\beta}) \) diverges, suggesting that it has a slower convergence rate than \( n^{-1/2} \). In this case, the large variability in external summary statistics will severely damage the estimation efficiency of the plug-in estimator. Third, for sufficiently small \( \rho \) such that \( \Sigma_1 / \rho - E(\eta_{\text{eff}} \eta_{\text{eff}}^T) > 0 \) (positive definite), the asymptotic variance of \( T_n^\text{eff}(\bar{\beta}) \) is larger than that of the efficient estimator using only internal data. This explains why the efficiency paradox arises: the external summary statistics are used as the true values while their uncertainty is not negligible. The following example illustrates a situation frequently encountered in practice where the efficiency paradox occurs.

**Example 2.** Suppose \( Z = (X, Y) \sim P_0 \) in the internal study and \( W = X \sim P_1 \) in the external study with \( P_1 \) equal to the marginal distribution of \( P_0 \). This is a common setting in semi-supervised learning or missing data analysis where \( Z \) is the labeled/complete data and \( W \) is the unlabeled/incomplete data. Consider the estimation of \( \tau = E(Y) \). Suppose individual samples in the internal study and \( \bar{\beta} \), the sample mean of \( X \) in external data, are available. Using only the internal data, the efficiency bound for \( \tau \) is \( \text{var}(Y) \). Treating \( \bar{\beta} \) as the true underlying value \( \beta \), then the plug-in estimator is \( T_n^\text{eff}(\bar{\beta}) = n^{-1} \sum_{i=1}^n \{ Y_i - \hat{\zeta}(X_i - \bar{\beta}) \} \), where \( \hat{\zeta} \) is the least squares coefficient of \( X \) in the linear regression of \( Y \) on \( X \). The asymptotic variance of \( T_n^\text{eff}(\bar{\beta}) \) is \( \text{var}(Y) - (1 - 1 / \rho) \text{cov}^2(X, Y) \{ \text{var}(X) \}^{-1} \). If \( \rho < 1 \) and \( \text{cov}(X, Y) \neq 0 \), the efficiency paradox
emerges.

3.2 The Efficiency Bound

To assess how external summary statistics can improve the efficiency for estimating $\tau$, we establish the semiparametric efficiency bound for the data-fused RAL estimators. Theorem 1 characterizes their asymptotic distribution.

**Theorem 1 (Convolution theorem).** Under Assumptions 1, 2 and a regularity condition (Condition ??) in Supplementary Material, for any data-fused RAL estimator $T_n(\tilde{\beta})$ we have

$$n^{1/2} \left[ T_n(\tilde{\beta}) - \tau - n^{-1} \sum_{i=1}^{n} (\phi_{\text{eff}} - M \eta_{\text{eff}}) - M(\tilde{\beta} - \beta) \right] \rightarrow \left( \Delta_0 \right) \left( S_0 \right),$$

where $M = E(\phi_{\text{eff}} \eta_{\text{eff}}^T) \{ \Sigma_1 / \rho + E(\eta_{\text{eff}} \eta_{\text{eff}}^T) \}^{-1}$, $S_0$ and $\Delta_0$ are independent, and $S_0 \sim N(0, B)$ with $B = E(\phi_{\text{eff}} \phi_{\text{eff}}^T) - E(\phi_{\text{eff}} \eta_{\text{eff}}^T) \{ \Sigma_1 / \rho + E(\eta_{\text{eff}} \eta_{\text{eff}}^T) \}^{-1} E(\phi_{\text{eff}} \eta_{\text{eff}}^T)^T$.

In classical semiparametric theory, convolution theorem is a key venue for establishing the asymptotic bound for the limiting distribution of RAL estimators. Here we extend it to the data fusion setting with both individual data and summary statistics. The proof follows from Bickel et al. (1993), the innovation here is that we incorporate the external summary statistics, which is different from the i.i.d case considered in classical semiparametric theory. Theorem 1 asserts that the asymptotic distribution of any data-fused RAL estimator $T_n(\tilde{\beta})$ can be decomposed into two independent parts $\Delta_0$ and $S_0$, where $S_0$ follows a normal distribution. We have $n^{1/2}\{T_n(\tilde{\beta}) - \tau\} \rightarrow \Delta_0 + S_0$ and $\text{var}(\Delta_0 + S_0) = \text{var}(\Delta_0) + \text{var}(S_0) \geq \text{var}(S_0) = B$, which is a lower bound for the asymptotic variance of any data-fused RAL estimator $T_n(\tilde{\beta})$.

**Theorem 2.** Under Assumptions 1, 2 and a regularity condition (Condition ??) in Supplementary Material, the efficiency bound for data-fused RAL estimators given in Definition 2 is

$$B = E(\phi_{\text{eff}} \phi_{\text{eff}}^T) - E(\phi_{\text{eff}} \eta_{\text{eff}}^T) \{ \Sigma_1 / \rho + E(\eta_{\text{eff}} \eta_{\text{eff}}^T) \}^{-1} E(\phi_{\text{eff}} \eta_{\text{eff}}^T)^T.$$

(3)
Note that when only internal data are available, the efficiency bound is $E(\phi_{eff}\phi_{eff}^T)$. Theorem 2 suggests that the efficiency bound does not increase with the inclusion of external summary statistics. However, if $B$ is larger than $E(\phi_{eff}\phi_{eff}^T) - E(\phi_{eff}\eta_{eff}^T)\{E(\eta_{eff}\eta_{eff}^T)\}^{-1}E(\phi_{eff}\eta_{eff}^T)^T$, which is the efficiency bound when $\beta$ is known. This indicates that $\hat{\beta}$ provides no more information than the true value of $\beta$ for estimating $\tau$, and $B$ converges to the latter as $\rho \rightarrow +\infty$. The efficiency bound $B$ also depends on the efficiency of $\hat{\beta}$, captured by $\Sigma_1$; specifically, $B$ increases as $\Sigma_1$ increases.

**Proposition 2.** Under Assumptions 1, 2 and a regularity condition (Condition ??) in Supplementary Material, if $P_0 = P_1$ and $\hat{\beta}$ is an efficient estimator of $\beta$ in the external study, i.e., $\Sigma_1 = E(\eta_{eff}\eta_{eff}^T)$, the efficiency bound for $\tau$ is $E(\phi_{eff}\phi_{eff}^T) - \rho(1+\rho)^{-1}E(\phi_{eff}\eta_{eff}^T)\{E(\eta_{eff}\eta_{eff}^T)\}^{-1}E(\phi_{eff}\eta_{eff}^T)^T$.

Theorem 2 also shows that external summary statistics do not bring efficiency gain if $E(\phi_{eff}\eta_{eff}^T) = 0$; that is, knowing $\beta$ has no influence on the estimation of $\tau$. This happens if $P_0$ factorizes as $P_0(Z) = f_1(Z)f_2(Z)$ and $\tau(P_0) = \tau(f_1), \beta(P_0) = \beta(f_2)$, i.e., $\tau$ and $\beta$ are functionals of variationally independent components of the internal data distribution. Proposition 2 is a special case of Theorem 2 when we have $\Sigma_1 = E(\eta_{eff}\eta_{eff}^T)$. The following is an example concerning marginal and joint regressions in genome-wide association studies, etc.

**Example 3.** Suppose the internal data are random samples of $(X_1, X_2, Y) \sim P_0$ and the external individual data are random samples of $(X_1, X_2, Y) \sim P_1$, with $P_0 = P_1$, $Y = X_1\tau_1 + X_2\tau_2 + \varepsilon$, $X_1, X_2$ mean zero, $E(\varepsilon \mid X_1, X_2) = 0$ and $\text{var}(\varepsilon \mid X_1, X_2) = \sigma^2$. The external summary statistics are $\hat{\beta} = (\hat{\beta}_1, \hat{\beta}_2)^T$, where $\hat{\beta}_1$ and $\hat{\beta}_2$ are the ordinary least squares coefficients obtained by regressing $Y$ on $X_1$ and $X_2$ separately with the external data. This happens in genome-wide association studies where researchers provide summary statistics of separate univariate regression coefficients of a quantitative trait ($Y$) on each centered genotype ($X$) (Zhu and Stephens, 2017).

The efficient influence function for $\beta$ is $\eta_{eff} = (\eta_{eff,1}, \eta_{eff,2})^T$, where $\eta_{eff,1} = \{E(X_1^2)\}^{-1}X_1(Y - X_1\beta_1)$ and $\eta_{eff,2} = \{E(X_2^2)\}^{-1}X_2(Y - X_2\beta_2)$. Denoting $X = (X_1, X_2)^T$, the data-fused efficiency bound for $\tau = (\tau_1, \tau_2)^T$ is

$$
\sigma^2\{E(XX^T)\}^{-1} - \frac{\rho\sigma^4}{1+\rho}\text{diag}\{1/E(X_1^2), 1/E(X_2^2)\}\{E(\eta_{eff}\eta_{eff}^T)\}^{-1}\text{diag}\{1/E(X_1^2), 1/E(X_2^2)\}.
$$
If only $\tilde{\beta}_1$ is available and without loss of generality we assume $X_1, X_2$ have unity variance and correlation coefficient $\kappa$, the data-fused efficiency bound for $\tau$ is

$$\sigma^2 \begin{pmatrix} \frac{1}{1-\kappa^2} & -\kappa \\ -\kappa & \frac{1}{1-\kappa^2} \end{pmatrix} - \frac{\rho}{1 + \rho} \begin{pmatrix} \frac{\sigma^4}{\text{var}(\eta_{\text{eff},1})} & 0 \\ 0 & 0 \end{pmatrix}.$$ 

There is no efficiency gain for estimating $\tau_2$ from external marginal regression estimate $\tilde{\beta}_1$. Zhang et al. (2020) has obtained this result under a special case where the distribution of $\varepsilon$ is $N(0, 1)$. Nevertheless, we note that the efficiency gain for estimating $\tau_2$ emerges when $\varepsilon$ is heteroscedastic, that is, $\text{var}(\varepsilon \mid X_1, X_2)$ is not a constant. A simulation is provided in Supplementary Material for illustration.

Applying Theorem 2 to estimation of the generalized linear model, we have the following proposition that provides a formal justification of the result conjectured by Zhang et al. (2020).

**Proposition 3.** Suppose $P_0 = P_1$ and $E(Y \mid X_1, X_2) = g^{-1}(X_1^T \tau + X_2^T \zeta)$ with $g$ being the canonical link function. Suppose in the external study $g^{-1}(X_2^T \beta)$ is used as a working model for $E(Y \mid X_2)$ and $\beta$ is estimated by solving estimating equation $\hat{E}[X_2 \{ Y - g^{-1}(X_2^T \beta) \}] = 0$. Here $\hat{E}$ means the empirical mean operator in the external study. Then the resultant estimator $\tilde{\beta}$ does not bring efficiency gain for estimating $\tau$.

### 3.3 An Efficient Data-fused Estimator

Let $\hat{\beta}_{\text{eff}}^{\text{int}}$ denote the efficient estimator of $\beta$ and $\hat{\tau}_{\text{eff}}^{\text{int}}$ the efficient estimator of $\tau$ based only on internal individual data. Motivated by Theorem 1, we propose the following data-fused estimator:

$$\hat{\tau}_{\text{eff}} = \hat{\tau}_{\text{eff}}^{\text{int}} - \hat{\Sigma}_{\phi \eta}^{\text{eff}} \left( \hat{\Sigma}_1 / \rho + \hat{\Sigma}_{\eta \eta}^{\text{eff}} \right)^{-1} (\hat{\beta}_{\text{eff}}^{\text{int}} - \tilde{\beta}),$$

where $\hat{\Sigma}_{\phi \eta}^{\text{eff}}, \hat{\Sigma}_{\eta \eta}^{\text{eff}}$ are consistent estimators of $E(\phi_{\text{eff}} \eta_{\text{eff}}^T), E(\eta_{\text{eff}} \eta_{\text{eff}}^T)$ based on the internal data respectively, and $\hat{\Sigma}_1$ is a consistent estimator of $\Sigma_1$. 

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Theorem 3. Under Assumptions 1–2, suppose that \( n^{1/2}(\hat{\tau}_{\text{eff}} - \tau) \rightarrow N\{0, E(\phi_{\text{eff}}\phi_{\text{eff}}^T)\} \) and 
\( n^{1/2}(\hat{\beta}_{\text{eff}} - \beta) \rightarrow N\{0, E(\eta_{\text{eff}}\eta_{\text{eff}}^T)\}, \)
then \( n^{1/2}(\hat{\tau}_{\text{eff}} - \tau) \) is asymptotically normal with asymptotic variance equal to \( B \) in (3).

Theorem 3 shows that \( \hat{\tau}_{\text{eff}} \) attains the efficiency bound for estimating a general functional in semiparametric or nonparametric models when both internal individual data and external summary statistics are available. This generalizes previous efficiency results (Zhang et al., 2020) on parameter estimation in parametric models. We refer to \( \hat{\tau}_{\text{eff}} \) as the data-fused efficient estimator. This estimator is more efficient than any RAL estimator using only internal data, and thus, resolves the efficiency paradox. The estimator \( \hat{\tau}_{\text{eff}} \) has a closed form. This fact benefits the computation of the estimator in general, although the calculation of efficient influence functions \( \phi_{\text{eff}}, \eta_{\text{eff}} \) may not be straightforward in complicated semiparametric models. For example, for estimating parameters in parametric models, \( \hat{\tau}_{\text{eff}} \) can greatly reduce the computational burden compared to the constrained maximum likelihood or empirical likelihood methods (Chatterjee et al., 2016; Zhang et al., 2020, 2022).

Let
\[
\Sigma = \begin{pmatrix} E(\phi_{\text{eff}}\phi_{\text{eff}}^T) & E(\phi_{\text{eff}}\eta_{\text{eff}}^T) \\ E(\eta_{\text{eff}}\phi_{\text{eff}}^T) & E(\eta_{\text{eff}}\eta_{\text{eff}}^T) \end{pmatrix}
\]
denote the asymptotic covariance of \((\hat{\tau}_{\text{eff}}^{\text{int}}, \hat{\beta}_{\text{eff}}^{\text{int}})\). The estimator \( \hat{\tau}_{\text{eff}} \) and its asymptotic variance in (3) are determined once we obtain \((\hat{\tau}_{\text{eff}}^{\text{int}}, \hat{\beta}_{\text{eff}}^{\text{int}}, \hat{\Sigma})\) and \((\tilde{\beta}, \hat{\Sigma}_1)\) where \( \hat{\Sigma} \) is a consistent estimator of \( \Sigma \). Specifically, let \( \hat{\Sigma}_{\phi\phi} \) be a consistent estimator of \( E(\phi_{\text{eff}}\phi_{\text{eff}}^T) \). The asymptotic variance of \( \hat{\tau}_{\text{eff}} \) can be estimated by \( \hat{\Sigma}_{\phi\phi} - \hat{\Sigma}_{\phi\eta} (\hat{\Sigma}_1 / \rho + \hat{\Sigma}_{\eta\eta})^{-1} \hat{\Sigma}_{\phi\eta}^T \). The estimators \((\hat{\tau}_{\text{eff}}^{\text{int}}, \hat{\beta}_{\text{eff}}^{\text{int}}, \hat{\Sigma})\) can be obtained with internal individual data, \( \tilde{\beta} \) is available from external data, and \( \hat{\Sigma}_1 \) is routinely reported as summary statistics together with \( \tilde{\beta} \). One can also consistently estimate \( \Sigma_1 \) using internal data if \( P_1 \) is a marginal distribution of \( P_0 \) and the method for estimating \( \tilde{\beta} \) is known. Otherwise, one can use a positive-definite working matrix \( \Omega \) for \( \Sigma_1 \) without compromising consistency of \( \hat{\tau}_{\text{eff}} \), but in this case, there is no guarantee of efficiency gain from external summary statistics. Additional
discussion on the choice of the working covariance matrix and the efficiency is provided in the proof of Proposition 4 in Supplementary Material.

If \( \beta \) is the same functional as \( \tau \), then \( \hat{\tau}_{\text{eff}} \) reduces to

\[
\frac{\hat{\tau}_{\text{eff}}^{\text{int}} / \text{var}(\hat{\tau}_{\text{eff}}^{\text{int}}) + \hat{\beta} / \text{var}(\hat{\beta})}{1 / \text{var}(\hat{\tau}_{\text{eff}}^{\text{int}}) + 1 / \text{var}(\hat{\beta})},
\]

which is the well-known inverse variance weighted estimator in meta-analysis (Lin and Zeng, 2010). The estimator \( \hat{\tau}_{\text{eff}} \) can be viewed as a calibration estimator where the external summary statistics \( \tilde{\beta} \) are used to calibrate the internal data-based efficient estimator \( \hat{\tau}_{\text{eff}}^{\text{int}} \). Calibration is a standard technique used in survey sampling for efficiency improvement with auxiliary information. Chen and Chen (2000), Wang and Wang (2015) and Yang and Ding (2020) considered calibration with validation data in the contexts of measurement error and confounding adjustment, where the validation dataset contains individual random samples from the big internal data. In contrast, here we consider the situation where only summary statistics are available from the external study and the external data are not necessarily random samples from the internal population.

The estimator \( \hat{\tau}_{\text{eff}} \) can also be viewed as a generalization of the confidence density estimator (Liu et al., 2015) which only employs summary statistics \( (\hat{\tau}_{\text{eff}}^{\text{int}}, \hat{\beta}_{\text{eff}}^{\text{int}}, \hat{\Sigma}) \) and \( (\tilde{\beta}, \hat{\Sigma}_1) \) to estimate \( \tau \).

**Proposition 4.** Given \( (\hat{\tau}_{\text{eff}}^{\text{int}}, \hat{\beta}_{\text{eff}}^{\text{int}}, \hat{\Sigma}) \) and \( (\tilde{\beta}, \hat{\Sigma}_1) \), then for some \( \hat{\beta} \),

\[
(\hat{\tau}_{\text{eff}}, \hat{\beta}) = \arg \min_{\tau, \beta} \left\{ \left( \begin{array}{c} \hat{\tau}_{\text{eff}}^{\text{int}} - \tau \\ \hat{\beta}_{\text{eff}}^{\text{int}} - \beta \end{array} \right)^T \hat{\Sigma}^{-1} \left( \begin{array}{c} \hat{\tau}_{\text{eff}}^{\text{int}} - \tau \\ \hat{\beta}_{\text{eff}}^{\text{int}} - \beta \end{array} \right) + (\tilde{\beta} - \beta)^T \left( \begin{array}{c} \Sigma_1 \\ \rho \end{array} \right)^{-1} (\tilde{\beta} - \beta) \right\}. \tag{5}
\]

The confidence density approach of Liu et al. (2015) concerns the estimation of parameters in a parametric model with summary statistics whose probability limit has a completely known functional relationship to the parameters of interest. Here however, we consider the estimation of a general functional in semiparametric models and the availability of internal individual data of the internal population obviates the need to know the explicit functional relationship between the probability limit of the summary statistics and the parameter of interest.

For illustration, we apply our estimation method to causal inference and semi-supervised learning examples.
Example 4 (Continuation of Example 1). Suppose in addition to individual samples on \( Z = (T, X, Y) \sim P_0 \) in the internal study, we also have the ordinary least squares estimate \( \tilde{\beta} \) of \( \beta = \{E(VV^T)\}^{-1}E(VY) \) obtained from the linear regression of \( Y \) on \( V = (1, X^T, T)^T \) in the external study. Suppose \( P_1 = P_0 \). The efficient influence function for \( \beta \) using only internal data is

\[
\eta_{\text{eff}} = \{E(VV^T)\}^{-1}V(Y - V^T\beta) .
\]

Given \((\hat{\tau}_{\text{eff}}^{\text{int}}, \phi_{\text{eff}})\) described in Example 1 and \( \hat{\beta}_{\text{eff}}^{\text{int}} \) by regressing \( Y \) on \( V \) in the internal data, \( \hat{\tau}_{\text{eff}} \) is equal to

\[
\hat{\tau}_{\text{eff}}^{\text{int}} = \frac{m}{m + n} \left[ \hat{E}\{\phi_{\text{eff}}(Y - V^T\hat{\beta}_{\text{eff}}^{\text{int}})V\} \right]^T \left[ \hat{E}\{(Y - V^T\hat{\beta}_{\text{eff}}^{\text{int}})^2V^TV\} \right]^{-1} \hat{E}(VV^T)(\hat{\beta}_{\text{eff}}^{\text{int}} - \beta). \]

Example 5 (Continuation of Example 2). We have \( \hat{\tau}_{\text{eff}}^{\text{int}} = \bar{Y} \) and \( \hat{\beta}_{\text{eff}}^{\text{int}} = \bar{X} \), the sample mean of \( Y \) and \( X \), respectively, in the internal data. The data-fused efficient estimator is \( \hat{\tau}_{\text{eff}} = \bar{Y} - \rho(1 + \rho)^{-1}\hat{\zeta}(\bar{X} - \tilde{\beta}) \), where \( \hat{\zeta} \) is the least squares coefficient of \( X \) in the linear regression of \( Y \) on \( X \). This recovers the semi-supervised least squares estimator given by Zhang et al. (2019).

4 Adaptive Fusion in the Presence of Population Heterogeneity

4.1 An Adaptive Fusion Estimator

The transportability assumption automatically holds if the internal and external populations are the same. However, there may well be heterogeneity between populations in different studies. In the presence of population heterogeneity, the external summary statistics may be partially transportable or untransportable, i.e., Assumption 2 may not hold for some or all the components of \( \beta(\cdot) \). In this case, the integration of summary statistics as in (4) will introduce bias. Specifically, we have

\[
\hat{\tau}_{\text{eff}} - \tau \rightarrow E(\phi_{\text{eff}}\eta_{\text{eff}}^Y)\{\Sigma_1/\rho + E(\eta_{\text{eff}}\eta_{\text{eff}}^Y)\}^{-1}h ,
\]

in probability, where \( h = \beta(P_1) - \beta(P_0) \) is the heterogeneity parameter. This implies that \( \hat{\tau}_{\text{eff}} \) may have a non-negligible asymptotic bias. To mitigate this problem, we construct a robust estimator that can effectively use the transportable components of the external summary statistics to improve the efficiency while keeping invulnerable to untransportable components.
For any integer \( j \), index set \( \mathcal{I} \), vector \( v \) and matrix \( V \), let \( v_j \) be the \( j \)-th component of \( v \), \( V_{jj} \) the \( j \)-th diagonal element of \( V \), \( v_{\mathcal{I}} \) the vector consisting of the components of \( v \) in \( \mathcal{I} \) and \( V_{\mathcal{I}} \) the matrix consisting of elements with indices in \( \mathcal{I} \times \mathcal{I} \). Let \( \beta = \{ j : \beta_j(P_0) = \beta_j(P_1), j = 1, \ldots, q \} \) denote the set of transportable external summary statistics. If such a set is known a priori, an oracle estimator could be obtained by incorporating only this subset of external summary statistics, \( \tilde{\beta}_\mathcal{A} \), utilizing the efficient data-fusion method proposed in (4). Let \( \hat{\tau}_{orc} = \hat{\tau}_{\mathcal{A}}^{eff} \) denote such an oracle estimator and
\[
\hat{B}_\mathcal{A} = \mathbb{E}(\phi_{\mathcal{A}}^{eff} \eta_{\mathcal{A}}^{eff})^{-1} \mathbb{E}(\phi_{\mathcal{A}}^{eff} \eta_{\mathcal{A}}^{eff})^T
\]
denote its asymptotic variance. In practice, \( \mathcal{A} \) is unknown. One can first select the transportable components based on \( \tilde{\beta} - \hat{\beta}_{int}^{eff} \) and then construct the data-fused estimator using the selected component. However, such a select-and-fuse procedure leads to an estimator that is discontinuous with respect to the observed data. The discontinuity is undesirable and can diminish the numerical stability of the procedure in practice (Fan and Li, 2001). To resolve the problem, we propose an adaptive fusion estimator that is continuous with respect to the observed data and shares the same asymptotic distribution as the oracle estimator. Specifically, define
\[
\hat{a}_j^2 = \max \left\{ 0, 1 - \lambda |\tilde{\beta}_j - \hat{\beta}_{eff,j}^{int}|^\alpha \right\}
\]
for \( j = 1, \ldots, q \) where \( \lambda, \alpha > 0 \) are tuning parameters. Let \( \hat{A} = \text{diag}\{\hat{a}_1^2, \ldots, \hat{a}_q^2\} \) and \( \hat{a} = (\hat{a}_1, \ldots, \hat{a}_q) \). The adaptive fusion estimator is defined as
\[
\hat{\tau}_{adf} = \hat{\tau}_{eff} - \hat{\Sigma}_{\phi \eta} \hat{A} \left\{ (I - \hat{A} + \hat{a}\hat{a}^T) \odot \left( \hat{\Sigma}_1 / \rho + \hat{\Sigma}_{\eta \eta} \right) \right\}^{-1} (\hat{\beta}_{eff}^{int} - \tilde{\beta}),
\]
where \( \odot \) denotes the element-wise product and \( I \) is the identity matrix.

Next, we explain the rationale behind the adaptive fusion estimator. Without loss of generality, assume \( \mathcal{A} = \{1, \ldots, q_1\} \) for some positive integer \( q_1 \). Suppose \( \lambda \to \infty \) and \( \lambda n^{-\alpha/2} \to 0 \). Then, \( \hat{a}_j \to 1 \) in probability if \( j \in \mathcal{A} \) and \( \hat{a}_j \to 0 \) if \( j \notin \mathcal{A} \). We have
\[
\hat{\Sigma}_{\phi \eta} \hat{A} \to \begin{pmatrix} E(\phi_{\mathcal{A}}^{eff} \eta_{\mathcal{A}}^{eff})^T & 0 \end{pmatrix}
\]
and
\[
(I - \tilde{A} + \hat{a}a^T) \odot \left( \tilde{\Sigma}_1 / \rho + \hat{\Sigma}^\text{eff} \right) \to \begin{pmatrix}
\Sigma_{1,A} / \rho + E(\eta_{\text{eff},A}\eta_{\text{eff},A}^T) & 0 \\
0 & D_{\eta,A^c}
\end{pmatrix}
\]
in probability, where \( D_{\eta,A^c} \) is the diagonal matrix consisting of the diagonal elements of \( \Sigma_{1,A} / \rho + E(\eta_{\text{eff},A}\eta_{\text{eff},A}^T) \) and \( A^c = \{ j : j \notin A \} \). This implies that
\[
\hat{\tau}_{\text{adf}} = \hat{\tau}_{\text{int}} - E(\phi_{\text{eff}}\eta_{\text{eff},A}^T) \left( \Sigma_{1,A} / \rho + E(\eta_{\text{eff},A}\eta_{\text{eff},A}^T) \right)^{-1} \left( \hat{\beta}^\text{int} - \tilde{\beta} \right) + o_P(n^{-1/2})
\]
\[
= \hat{\tau}_{\text{ort}} + o_P(n^{-1/2}).
\]

The formal result is summarized in the following theorem.

**Theorem 4.** Under Assumption 1, if \( \lambda \to \infty, \lambda n^{-\alpha/2} \to 0 \), \( n^{1/2}(\hat{\tau}_{\text{int}} - \tau) \to N\{0, E(\phi_{\text{eff}}\phi_{\text{eff}}^T)\} \) and \( n^{1/2}(\hat{\beta}_{\text{eff}} - \beta) \to N\{0, E(\eta_{\text{eff}}\eta_{\text{eff}}^T)\} \), then \( \hat{\tau}_{\text{adf}} \) has the same asymptotic distribution as \( \hat{\tau}_{\text{ort}} \) and \( n^{1/2}(\hat{\tau}_{\text{adf}} - \tau) \to N(0, B^A) \).

Theorem 4 shows that the adaptive fusion estimator \( \hat{\tau}_{\text{adf}} \) retains consistency and asymptotic normality even if \( \tilde{\beta} \) contains untransportable components, and it is asymptotically as efficient as the oracle estimator \( \hat{\tau}_{\text{ort}} \). The asymptotic variance \( B^A \) can be consistently estimated by
\[
\hat{\Sigma}_{\phi \phi}^\text{eff} - \hat{\Sigma}_{\phi \eta}^\text{eff} \hat{A} \left( I - \hat{A} + \hat{a}a^T \right) \odot \left( \hat{\Sigma}_1 / \rho + \hat{\Sigma}_\eta^\text{eff} \right)^{-1} \hat{A} \hat{\Sigma}_{\phi \eta}^\text{eff}^T.
\]

Zhai and Han (2022) have previously proposed a penalized constrained maximum likelihood method for the fusion of internal individual data and possibly untransportable summary statistics; however, they have focused on inference about parameters in a parametric conditional density model and ignored the uncertainty of external summary statistics, which may lead to efficiency loss as discussed at the end of Section 2. In contrast, our method applies to a general functional in semiparametric models and accounts for the uncertainty of external summary statistics, which can achieve the semiparametric efficiency bound of combining individual data and summary statistics.
In addition, our estimator enjoys a closed form, which bypasses sophisticated programming and computation and can be easily implemented.

For the choice of the tuning parameters, we set \( \alpha = 4 \) and \( \lambda = cn^{1/2} \), which meet the requirement of Theorem 4. As a thumb of rule, we adopt the cross-validation method to choose the constant term \( c \). Specifically, we split the full internal data into \( K \geq 2 \) balanced subsets. Given a candidate set \( C \) for \( c \), we propose to select \( c \) using the cross-validation procedure summarized in Algorithm 1. We set \( K = 3 \) and \( C = \{1, \ldots, 10\} \) in the simulation and real data analysis.

**Algorithm 1:** Cross validation method for selecting \( c \).

\[
\text{for } c \in C \text{ do}
\]

\[
\text{for } k = 1, \ldots, K \text{ do}
\]

\[
\text{Compute the internal data-based efficient estimator } \hat{\tau}_{\text{int eff},(k)} \text{ using the observations in the } k\text{-th subset ;}
\]

\[
\text{Compute the adaptive fusion estimator } \hat{\tau}_{\text{adf},(k)}^{(c)} \text{ based on the external summary statistics } \tilde{\beta} \text{ and all internal data except for the } k\text{-th subset using the tuning parameter } \lambda = cn^{1/2};
\]

\[
\text{end}
\]

\[
\text{end}
\]

Select

\[
c^* = \arg \min_{c \in C} \frac{1}{K} \sum_{k=1}^{K} \left( \hat{\tau}_{\text{eff},(k)}^{\text{int}} - \hat{\tau}_{\text{adf},(k)}^{(c)} \right)^2.
\]

### 4.2 A Re-bootstrap Procedure to Improve Finite Sample Coverage

Theorem 4 establishes the asymptotic distribution of \( \hat{\tau}_{\text{adf}} \). The result is obtained by treating the difference \( h_j = \beta_j(P_1) - \beta_j(P_0) \) as fixed for \( j = 1, \ldots, p \) when the sample size \( n \) is large. Thus, the difference \( h_j \), if non-zero, asymptotically dominates the estimation error of \( \tilde{\beta}_j - \hat{\beta}_{\text{int eff},j} \) as \( n \to \infty \).
This dominance ensures that one can consistently determine whether the summary statistics are transportable or not, which is essential for the oracle result in Theorem 4. However, the asymptotic distribution may not accurately approximate the actual distribution of \( \hat{\tau}_{adf,j} \) when \( h_j \) is of a similar magnitude to the estimation error of \( \tilde{\beta}_j - \hat{\beta}_{int,j} \) for some \( j = \{1, \ldots, q\} \), i.e., \( |h_j| \asymp n^{-1/2} \). Here, for any positive sequences \( \{a_{1n}\} \) and \( \{a_{2n}\} \), \( a_{1n} \asymp a_{2n} \) means \( C^{-1} a_{1n} \leq a_{2n} \leq C a_{1n} \) for some \( C > 1 \). The above scenario, referred to as moderate heterogeneity, may lead to undercoverage of the confidence intervals based on the asymptotic distribution in Theorem 4. Please refer to Section 5.3 for a numerical illustration of this issue. We provide further theoretical analyses of this issue in Section ?? of the Supplementary Material.

In this section, we propose a re-bootstrap procedure to mitigate the undercoverage issue in finite samples. We aim to construct the confidence interval for \( \tau_j \) for some \( j = 1, \ldots, p \) based on the estimator \( \hat{\tau}_{adf,j} \). For \( j = 1, \ldots, p \) and \( 0 < z < 1 \), let \( q_j(z; h) \) denote the \( z \)-quantile of the distribution of \( \hat{\tau}_{adf,j} - \tau_j \), which may depend on the heterogeneity parameter \( h \). Consequently, \([\hat{\tau}_{adf,j} - q_j(1 - z/2; h), \hat{\tau}_{adf,j} - q_j(z/2; h)]\) forms a valid \( 1 - z \) confidence interval for \( \tau_j \). Since \( \hat{\tau}_{adf,j} - \tau_j \) is a function of \( \hat{\tau}_{eff} - \tau \) and \( \hat{\beta}_{eff} - \tilde{\beta} \), the quantile \( q_j(z; h) \) is thus a functional of the distribution of these two quantities. The moderate heterogeneity does not invalidate the normal approximation for \( \hat{\tau}_{eff} - \tau \) and \( \hat{\beta}_{eff} - \tilde{\beta} \), but it may affect the mean of \( \hat{\beta}_{eff} - \tilde{\beta} \), which is approximately equal to \( h \). Therefore, the quantile \( q_j(z; h) \) can be estimated by the bootstrap quantile \( \hat{q}_j(z; h) \), which is obtained based on the bootstrap counterparts of \( \hat{\tau}_{eff} - \tau \) and \( \hat{\beta}_{eff} - \tilde{\beta} \) generated from a joint normal distribution that depends on \( h \). See Section ?? of the Supplementary Material for more details. The remaining challenge for inference is the unknown heterogeneity parameter \( h \). Although \( \tilde{\beta} - \hat{\beta}_{eff} \) is a \( \sqrt{n} \)-consistent estimator of \( h \), the estimation error of \( \tilde{\beta} - \hat{\beta}_{eff} \) has a comparable scale to the quantile \( q_j(z, h) \) and hence non-negligible. To improve the robustness against this estimation error, we propose to calculate bootstrap quantiles under multiple reasonable candidate values for the heterogeneity parameter and use the most conservative quantile to construct the confidence interval. To generate these candidate values, we sample from the
asymptotic confidence distribution $N(\tilde{\beta} - \hat{\beta}_{\text{int}}^{\text{eff}}, \tilde{\Sigma}_1/m + \hat{\Sigma}_{\eta_{\eta}}/n)$ of $h$, which is induced by the asymptotic likelihood function of $h$ as suggested by Xie and Singh (2013), and make some calibration to the sampled heterogeneity parameters. Further details are deferred to Section ?? of the Supplementary Material. The resulting candidate heterogeneity parameters are denoted as $\hat{h}_{\text{cal}}^{(r)}$ for $r = 1, \ldots, \bar{r}$, where $\bar{r}$ is a user-specified integer. The confidence interval for $\tau_j$ is then constructed as

$$\left[\hat{\tau}_{\text{adf},j} - \max_{r=1,\ldots,\bar{r}} \hat{q}_j(1 - z/2; \hat{h}_{\text{cal}}^{(r)}), \hat{\tau}_{\text{adf},j} - \min_{r=1,\ldots,\bar{r}} \hat{q}_j(z/2; \hat{h}_{\text{cal}}^{(r)})\right].$$

The above procedure constructs the confidence interval using the most conservative quantile derived from multiple candidate values. The quantiles $q_j(z/2; h)$ and $q_j(1 - z/2; h)$ are continuous functions of $h$ due to the continuity of the adaptive fusion estimator with respect to the observed data. If the number of candidate values $\bar{r}$ is large, then with high probability, there exists some candidate value that is extremely close to the true heterogeneity parameter $h$ (Guo, 2023), making the resulting quantiles very close to the true quantiles. Thus, the coverage rate can be ensured by adopting the most conservative quantile among those derived from $\bar{r}$ different candidate values. On the other hand, the candidate values are designed to lie within a small neighborhood of $h$, ensuring that the quantiles calculated from these values are likely to be similar. This prevents the confidence interval from being overly conservative. The desired properties of the re-bootstrap procedure are confirmed by our numerical results in Section 5.3.

5 Numerical Experiments

5.1 Simulation with Transportable Summary Statistics

We conduct simulation studies to evaluate the performance of the proposed estimators. We consider different scenarios with transportable and untransportable external summary statistics, respectively. In Scenario I, a triplet of treatment $T$, outcome $Y$ and covariate $X$ are generated as
follows in both the internal and external data:

$$X \sim N(0, 1), \quad \text{pr}(T = 1 \mid X) = \expit(1 - X),$$

$$Y = 1 + X + TX^2 + T\varepsilon_1 + (1 - T)\varepsilon_0, \quad (\varepsilon_0, \varepsilon_1) \perp \perp (X, T),$$

where $$\varepsilon_1 \sim N(0, 4), \varepsilon_0 \sim N(0, 1)$$ and $$\expit(x) = 1/\{1 + \exp(-x)\}$$. The internal sample size $$n$$ is 1000 throughout the simulations and the external sample size $$m$$ increases from 200 to 2000.

The functional of interest $$\tau$$ is the treatment effect of $$T$$ on $$Y$$. Following Example 4 we use the internal individual data and the ordinary least squares estimate $$\tilde{\beta}$$ obtained by regressing $$Y$$ on $$(1, X^T, T)^T$$ in the external study for estimating $$\tau$$. We implement four estimators, (i) INT: $$\hat{\tau}_{\text{eff}}^{\text{int}}$$ using only internal data; (ii) PRM: the primitive estimator ignoring uncertainty of $$\tilde{\beta}$$; (iii) EFF: the data-fused efficient estimator $$\hat{\tau}_{\text{eff}}$$; (iv) KNW: the efficient estimator knowing the true value of $$\beta$$.

The GIM estimator by Zhang et al. (2020) does not apply to the causal effect functional, and we do not implement it.

We replicate 1000 simulations. Figure 1 and Table 1 show root mean squared error (RMSE), average standard error and coverage probability of the four estimators under different external sample sizes. The crude estimator improves efficiency against $$\hat{\tau}_{\text{eff}}^{\text{int}}$$ only if the external sample size is sufficiently large ($$m = 2000$$) relative to that of internal data, otherwise ($$m \leq 1000$$) efficiency loss emerges. The EFF estimator outperforms the INT and PRM estimators and ensures efficiency gain under all sample sizes. The EFF estimator is less efficient than the KNW estimator, but the KNW estimator is not feasible in practice because one does not know $$\beta$$. For all methods, coverage probabilities of the 95% confidence intervals are close to the nominal level.

![Figure 1: Boxplots of various estimators in Scenario I. The horizontal line marks the true value.](image-url)
Table 1: RMSE, average standard error (ASE) and coverage probability of 95% confidence interval (CP) for Scenario I. All numbers are multiplied by 100.

|      | m = 200   | m = 500   | m = 1000  | m = 2000  |
|------|-----------|-----------|-----------|-----------|
|      | RMSE      | ASE       | CP        | RMSE      | ASE       | CP        | RMSE      | ASE       | CP        | RMSE      | ASE       | CP        |
| INT  | 11.62     | 11.78     | 95.4      | 11.62     | 11.78     | 95.4      | 11.62     | 11.78     | 95.4      | 11.62     | 11.78     | 95.4      |
| PRM  | 20.86     | 22.64     | 96.2      | 15.30     | 15.26     | 95.5      | 11.79     | 11.79     | 94.2      | 9.44      | 9.59      | 95.1      |
| EFF  | 10.97     | 11.10     | 94.8      | 10.25     | 10.37     | 95.5      | 9.56      | 9.58      | 94.4      | 8.65      | 8.72      | 94.8      |
| KNW  | 6.94      | 6.69      | 93.1      | 6.94      | 6.69      | 93.1      | 6.94      | 6.69      | 93.1      | 6.94      | 6.69      | 93.1      |

5.2 Simulation with Partially Transportable Summary Statistics

In Scenario II, we consider fusion with partially transportable external summary statistics. We generate \((Y, X_1, X_2)\) in the internal data and \((Y, X_1, \tilde{X}_2)\) in the external data as follows:

\[
Y = X_1 \tau_1 + X_2 \tau_2 + \varepsilon_1, \quad (X_1, X_2)^T \sim N \left\{ \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 & 0.6 \\ 0.6 & 1 \end{pmatrix} \right\},
\]

\[
\tilde{X}_2 = X_2 + \varepsilon_2, \quad \varepsilon_1 \sim N(0, 4), \quad \varepsilon_2 \sim N(0, 1).
\]

where \(\tilde{X}_2\) in the external data is viewed as a surrogate of \(X_2\) with measurement error. The internal sample size is \(n = 1000\) and the external sample size is \(m = 4000\). Let \(\hat{\beta} = (\hat{\beta}_1, \hat{\beta}_2)\) be the ordinary least squares estimate obtained from the external data by regressing \(Y\) on \(X_1\) ad \(\tilde{X}_2\) separately. We use the internal individual data and \(\hat{\beta}\) to estimate \(\tau = (\tau_1, \tau_2)^T\). We implement five estimation methods: (i) INT: \(\hat{\tau}^{\text{int}}\) using only internal data; (ii) ORC: the oracle estimator using internal data and only \(\hat{\beta}_1\); (iii) ADF: the adaptive fusion estimator \(\hat{\tau}_{\text{adf}}\); (iv) EFF: the efficient estimator \(\hat{\tau}_{\text{eff}}\) in (4) using both \(\hat{\beta}_1\) and \(\hat{\beta}_2\); (v) GIM: the estimator of Zhang et al. (2020).

We replicate 1000 simulations. Figure 2 and Table 2 show the RMSE, average standard error and coverage probability of different estimators. The GIM and EFF estimators exhibit large RMSE due to the inclusion of the untransportable component of the summary statistics. In contrast, the
ADF estimator can adaptively select and use the transportable component of the external summary statistics; as a result, it performs similarly to the oracle estimator, both showing negligible bias. Besides, the ADF estimator of $\tau_1$ has smaller variance than the INT estimator while the ADF estimator of $\tau_2$ does not enjoy the efficiency gain, which is consistent with our analysis in Example 3. Coverage probabilities of the 95% confidence intervals are close to the nominal level for INT, ORC and ADF estimators; while for EFF and GIM estimators, the coverage probabilities are close to zero due to the bias introduced by the untransportable component.

We also evaluate the performance of these estimators when the external data has accurate measurement of $X_2$, in which case, $\tilde{\beta}_2$ is also transportable. Figure 3 and Table 3 show the RMSE, average standard error and coverage probability of the five estimators. In this setting, the ORC, ADF, EFF, and GIM estimators exhibit similar performance, indicating that the ADF estimator has minimal efficiency loss when all the components are transportable. Overall, we recommend the ADF estimator because it is statistically efficient, computationally convenient and empirically robust against untransportable external summary statistics.

| $\tau_1$ | $\tau_2$ | RMSE | ASE | CP | RMSE | ASE | CP |
|---------|---------|------|-----|----|------|-----|----|
| INT     | 7.75    | 7.90 | 95.8| 7.79| 7.90 | 95.2|
| ORC     | 5.85    | 5.90 | 94.8| 7.79| 7.87 | 94.8|
| ADF     | 5.89    | 5.97 | 95.0| 7.80| 7.87 | 95.0|
| EFF     | 42.63   | 3.57 | 0.0 | 82.82| 3.55 | 0.0 |
| GIM     | 47.21   | 3.73 | 0.0 | 73.16| 4.59 | 0.0 |

Table 2: RMSE, ASE and CP for Scenario II with partially transportable summary statistics. All numbers are multiplied by 100.

| $\tau_1$ | $\tau_2$ | RMSE | ASE | CP | RMSE | ASE | CP |
|---------|---------|------|-----|----|------|-----|----|
| INT     | 7.84    | 7.91 | 95.4| 7.72| 7.91 | 95.2|
| ORC     | 4.66    | 4.75 | 94.7| 4.83| 4.75 | 94.1|
| ADF     | 4.80    | 4.90 | 95.1| 5.03| 4.91 | 95.0|
| EFF     | 4.66    | 4.75 | 94.7| 4.83| 4.75 | 94.1|
| GIM     | 4.64    | 4.76 | 95.1| 4.81| 4.76 | 94.4|

Table 3: RMSE, ASE and CP for Scenario II with transportable summary statistics. All numbers are multiplied by 100.
5.3 Simulation with Moderate Heterogeneity

In the presence of moderate heterogeneity, the confidence interval based on the adaptive fusion estimator may have coverage probability below the nominal coverage rate. In this section, we run simulations where $\beta_1(P_1) = \beta_1(P_0)$ and $|\beta_2(P_1) - \beta_2(P_0)| \approx n^{-1/2}$ in the setting of Scenario II to evaluate the performance of different methods in the presence of moderate heterogeneity. Specifically, we let $\tilde{X}_2 = X_2 + \varepsilon_2$, where $\varepsilon_2 \sim N(0, \sigma^2)$ and $\sigma^2 = Cn^{-1/2}$. Other settings are the same as in Scenario II in the previous section. The results are shown in Table 4.

The simulation results show that the ADF estimator has similar or better performance than the INT and ORC estimators in terms of RMSE and average standard error across all settings. While the RMSEs of the EFF and GIM estimators are smaller than that of the ADF estimator when $C = 0.05$, they become significantly larger when $C = 20$. Confidence intervals based on the INT and ORC estimators can always achieve the nominal level. However, the confidence interval based on the adaptive fusion estimator may have coverage probability below the nominal level, especially when the bias of the external summary statistics is moderate ($C = 1$). In addition, confidence intervals based on the EFF and GIM estimators tend to fall below the nominal level when $C = 1$ and may even be zero when $C = 20$.

Next, we compare the average width and coverage probability of the confidence intervals based
Table 4: RMSE, average standard error (ASE) and coverage probability (CP) for Scenario II with moderate heterogeneity. All numbers are multiplied by 100.

| Method | C = 0.05 |   |   | C = 1 |   |   | C = 20 |   |   |
|--------|----------|---|---|-------|---|---|--------|---|---|
|        | RMSE     | ASE| CP | RMSE  | ASE| CP | RMSE   | ASE| CP |
| INT    | 8.01     | 7.92| 95.4| 8.01  | 7.92| 95.4| 8.01   | 7.92| 95.4|
| ORC    | 5.94     | 5.92| 95.0| 5.94  | 5.92| 95.0| 5.94   | 5.92| 95.0|
| ADF    | 4.84     | 4.92| 95.2| 5.39  | 4.94| 90.5| 6.02   | 5.98| 94.9|
| EFF    | 4.67     | 4.76| 95.6| 5.33  | 4.76| 91.4| 33.64  | 4.76| 0.0 |
| GIM    | 4.68     | 4.77| 95.3| 5.37  | 4.70| 90.9| 37.41  | 3.91| 0.0 |
| INT    | 7.97     | 7.92| 95.1| 7.97  | 7.92| 95.1| 7.97   | 7.92| 95.1|
| ORC    | 7.97     | 7.91| 94.7| 7.97  | 7.91| 94.7| 7.97   | 7.91| 94.7|
| ADF    | 4.99     | 4.92| 95.2| 6.75  | 5.01| 84.9| 7.99   | 7.91| 94.8|
| EFF    | 4.80     | 4.76| 94.8| 6.80  | 4.76| 83.6| 63.76  | 4.76| 0.0 |
| GIM    | 4.80     | 4.77| 95.2| 6.76  | 4.75| 84.3| 58.51  | 4.65| 0.0 |

on the INT, ORC, ADF, EFF and GIM estimators with that provided by the re-bootstrap (ReBoot) procedure in Section 4.2. We set $\bar{r} = 10$ in this simulation. The results in Table 5 show that the ReBoot procedure mitigates the undercoverage issue of the confidence intervals based on the ADF estimator at the cost of some increase in average width. Although the ReBoot procedure is conservative in some cases, it still produces narrower confidence intervals than those based solely on the internal data in most settings.

### 5.4 Real Data Analysis

We apply the proposed methods to analyze a Helicobacter pylori infection dataset described by Li et al. (2023b). Helicobacter pylori infection is a leading worldwide infectious disease. The triple therapy (clarithromycin, amoxicillin, and omeprazole) is a standard treatment for Helicobacter pylori infection. The internal study is a two-arm randomized clinical trial conducted at a traditional
Table 5: Average width (AW) and coverage probability (CP) for Scenario II with moderate heterogeneity. All numbers are multiplied by 100.

| Method | $C = 0.05$ |  |  | $C = 1$ |  |  | $C = 20$ |  |  |
|--------|-----------|---|---|---------|---|---|---------|---|---|
|        | AW        | CP|    | AW      | CP|    | AW      | CP|    |
| INT    | 30.96     | 95.4|100| 30.96    | 95.4|100| 30.96    | 95.4|100|
| ORC    | 23.11     | 94.8|100| 23.11    | 94.8|100| 23.11    | 94.8|100|
| $\tau_1$ ADF | 19.17 | 95.3|100| 19.27   | 91.5|100| 23.36 | 95.2|100|
|        | EFF       | 18.60   | 95.1|100| 18.60 | 91.4|100| 18.60 | 0.0|100|
|        | GIM       | 18.65   | 95.0|100| 18.40 | 90.4|100| 15.27 | 0.0|100|
|        | ReBoot    | 25.05   | 98.0|100| 25.52 | 96.5|100| 27.83 | 97.5|100|
| INT    | 30.96     | 94.5|100| 30.96    | 94.5|100| 30.96    | 94.5|100|
| ORC    | 30.86     | 94.5|100| 30.86    | 94.5|100| 30.86    | 94.5|100|
| $\tau_2$ ADF | 19.21 | 95.9|100| 19.50   | 85.7|100| 30.86 | 94.5|100|
|        | EFF       | 18.60   | 95.4|100| 18.60 | 82.9|100| 18.60 | 0.0|100|
|        | GIM       | 18.65   | 95.7|100| 18.58 | 83.2|100| 18.21 | 0.0|100|
|        | ReBoot    | 24.93   | 97.6|100| 26.33 | 93.1|100| 33.89 | 96.3|100|

Chinese medicine hospital. This trial aims to investigate whether the additional taking of traditional Chinese medicine ($T = 1$) has better efficacy than the standard triple therapy treatment ($T = 0$) on Helicobacter pylori infection. It contains 362 observations, of which 180 patients are assigned to the triple therapy and the rest are assigned to a combination treatment including both triple therapy and traditional Chinese medicine. The external study is a single-arm study conducted at a Western-style hospital, where 110 patients are all assigned to the triple therapy. The outcome $Y$ is the post-treatment infection status assessed with the C–14 urea breath test and baseline covariates $X$ include age, gender, height, BMI, occupation, education level, marital status, and information on patients’ symptoms. The internal and external studies adopt the same inclusion and exclusion criteria and the same treatment protocols. The parameter of interest is the average causal effect of the combination treatment against the standard triple therapy treatment, i.e., $\tau = E(Y_1 - Y_0)$. We
illustrate how to use the individual data from the internal study and the outcome sample mean \((\tilde{\beta})\) from the external study to make inference about \(\tau\), where \(\tilde{\beta}\) is a consistent estimator of \(\beta = E(Y_0)\) in the external study.

We implement four methods INT, PRM, EFF and ADF to estimate \(\tau\). Our goal is to test the null hypothesis \(H_0: \tau \leq 0\) against \(H_1: \tau > 0\) to investigate whether the combination treatment can improve the efficacy. We calculate the estimate of \(\tau\) and its standard error and then use z-test to calculate one-sided p-values. Table 6 presents the analysis results. The four point estimates are close to each other, all showing a potentially beneficial effect of the additional use of traditional Chinese medicine. The EFF estimate and the ADF estimate are identical, suggesting ‘transportability of the external summary statistic; this is because the same inclusion criterion and the same treatment protocols are adopted in both the internal and external studies, and it is reasonable to assume that they are from the same population. The INT estimate based solely on the internal data does not reject the null hypothesis \(H_0\) at level 0.1. Test based on the PRM estimate rejects \(H_0\) at level 0.1 but the PRM estimate has an even larger standard error than the INT estimate. In contrast, by appropriate integration of the external summary statistic, the EFF and ADF estimates achieve smaller standard errors and reject \(H_0\) at level 0.1. This may serve as evidence in favor of the beneficial effect of the additional use of traditional Chinese medicine in the treatment for Helicobacter pylori infection.

Table 6: Point estimates of \(\tau\), standard errors, and \(p\)-values

|     | point estimate | standard error | \(p\)-value |
|-----|----------------|----------------|-------------|
| INT | 0.0543         | 0.0442         | 0.1100      |
| PRM | 0.0773         | 0.0520         | 0.0684      |
| EFF | 0.0628         | 0.0394         | 0.0553      |
| ADF | 0.0628         | 0.0394         | 0.0554      |
6 Discussion

The proposed methods can be extended to integrate summary statistics from multiple external studies. Consider $S$ independent external studies, each study $s$ ($1 \leq s \leq S$) with sample size $m_s$ from population $P_s$ providing summary statistics $\tilde{\beta}(s)$ on a functional $\beta(s)$. Define $\tilde{\beta} = (\tilde{\beta}(1)^T, \ldots, \tilde{\beta}(S)^T)^T$ with some abuse of notation.

Theorem 5. Suppose $m_s^{1/2}\{\tilde{\beta}(s) - \beta(s)(P_s)\} \to N(0, \Sigma_s)$ for $s = 1, \ldots, S$ and

$$\{\beta^T(1)(P_1), \ldots, \beta^T(S)(P_S)\}^T = \{\beta^T(1)(P_0), \ldots, \beta^T(S)(P_0)\}^T =: \beta(P_0).$$

Denote the efficient influence function for $\beta(P_0)$ by $\eta_{\text{eff}} = (\eta_{\text{eff},(1)}, \ldots, \eta_{\text{eff},(S)})^T$. Suppose $m_s/n \to \rho_s \in (0, \infty)$ for each $s$. Then the data-fused efficiency bound for $\tau$ is

$$E(\phi_{\text{eff}}^T\eta_{\text{eff}}) - E(\phi_{\text{eff}}\eta_{\text{eff}}) \{\Sigma_{\text{ext}} + E(\eta_{\text{eff}}\eta_{\text{eff}}^T)\}^{-1} E(\phi_{\text{eff}}\eta_{\text{eff}}^T)^T,$$

where $\Sigma_{\text{ext}} = \text{diag}(\Sigma_1/\rho_1, \ldots, \Sigma_S/\rho_S)$.

Theorem 5 establishes the efficiency bound for estimating $\tau$ when multiple external summary statistics are available; the proof is analogous to that of Theorem 2 and is omitted. The data-fused efficient estimator with transportable summary statistics is $\hat{\tau}_{\text{eff}} = \hat{\tau}_{\text{int}} - \hat{\Sigma}_{\text{ext}}^{-1}(\hat{\beta}_{\text{eff}} - \tilde{\beta})$. In the presence of untransportable summary statistics, the adaptive fusion estimator can be obtained analogously, with $\hat{\Sigma}_1/\rho$ replaced by $\hat{\Sigma}_{\text{ext}}$ in formula (6).

We have focused on the integration of finite-dimensional summary statistics. It is of both theoretical and practical interest to study how to integrate infinite-dimensional external summary curves, such as estimates of a density function, regression curve, or conditional mean, and a trained neural network model. We plan to pursue this extension in the future.

Supplementary Material

Supplementary Material available online includes proofs of Theorems 1–4 and Propositions 1–4, details of Examples 2 and 3, additional results about the re-bootstrap procedure, and discussions.
on the scenario when Assumption 2 holds only asymptotically.

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