GROWTH OF QUADRATIC FORMS UNDER ANOSOV SUBGROUPS

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Abstract. Let \( \rho : \Gamma \to \text{PSL}_d(\mathbb{K}) \) be a Zariski dense Borel-Anosov representation, for \( \mathbb{K} \) equal to \( \mathbb{R} \) or \( \mathbb{C} \). Let \( o \) be a form of signature \( (p, d-p) \) on \( \mathbb{K}^d \) (where \( 0 < p < d \)). Let \( S^o \) be the corresponding geodesic copy of the Riemannian symmetric space of \( \text{PSO}(o) \), inside the Riemannian symmetric space of \( \text{PSL}_d(\mathbb{K}) \). For certain choices of \( o \) and every \( t \) large enough, we show exponential bounds for the number of \( \gamma \in \Gamma \) for which the distance between \( S^o \) and \( \rho\gamma \cdot S^o \) is smaller than \( t \). Under an extra assumption, satisfied for instance when the boundary of \( \Gamma \) is connected, we show an asymptotic as \( t \to \infty \) for the counting function relative to a functional in the interior of the dual limit cone.

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1. Introduction

Let \( d = p+q \geq 3 \), where \( p \) and \( q \) are positive integers and \( V \) be a \( \mathbb{K} \)-vector space of dimension \( d \), where \( \mathbb{K} = \mathbb{R} \) or \( \mathbb{K} = \mathbb{C} \). Let \( G := \text{PSL}(V) \) and \( X_G \) be the Riemannian symmetric space of \( G \). A form of signature \( (p,q) \) on \( V \) is a quadratic or Hermitian form on \( V \) of that signature, depending respectively on whether \( \mathbb{K} = \mathbb{R} \) or \( \mathbb{K} = \mathbb{C} \). The space of homothety classes of forms of signature \( (p,q) \) on \( V \) is denoted by \( Q_{p,q} \). The group of projectivized linear isometries of a basepoint \( o \in Q_{p,q} \) is denoted by \( H^o \). It is isomorphic to \( \text{PSO}(p,q) \) (resp. \( \text{PSU}(p,q) \)) if \( \mathbb{K} = \mathbb{R} \) (resp. \( \mathbb{K} = \mathbb{C} \)). We let \( S^o \subset X_G \) be the corresponding totally geodesic copy of the Riemannian symmetric space of \( H^o \). In this paper we study the following problem.

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Main Problem. Let \( \Xi \) be a discrete subgroup of \( G \). Describe the set of points \( o \in \mathbb{Q}_{p,q} \) for which the counting function
\[
(1.1) \quad t \mapsto \#\{g \in \Xi : d_{\mathcal{X}_G}(S^o, g \cdot S^o) \leq t\}
\]
is finite\(^1\) for every positive \( t \) and study its asymptotic behaviour as \( t \to \infty \).

In the previous formulation \( d_{\mathcal{X}_G}(\cdot, \cdot) \) denotes the distance on \( \mathcal{X}_G \) coming from a \( G \)-invariant Riemannian structure and, for closed subsets \( A \) and \( B \) of \( \mathcal{X}_G \), we let
\[
d_{\mathcal{X}_G}(A, B) := \inf\{d_{\mathcal{X}_G}(a, b) : a \in A, b \in B\}.
\]

In this paper we contribute to the study of the previous problem when \( \Xi \) is the image of a word hyperbolic group \( \Gamma \) under a \( \Delta \)-Anosov representation \( \rho : \Gamma \to G \), where \( \Delta \) denotes the set of simple roots of some Weyl chamber. Anosov representations were introduced by Labourie [32] and further extended by Guichard-Wienhard [23]. They provide a (stable) class of faithful and discrete representations from word hyperbolic groups into semisimple Lie groups that share many features with holonomies of convex co-compact hyperbolic manifolds. They have been object of intensive research in recent years (see e.g. Kassel [29], Pozzetti [45] or Wienhard [58]). Reminders on this notion are given in Subsection 7.1, but we recall here that these representations come equipped with a continuous equivariant limit map
\[
\xi_\rho : \partial_\infty \Gamma \to F(V).
\]
Here \( \partial_\infty \Gamma \) denotes the Gromov boundary of \( \Gamma \) and \( F(V) \) denotes the full flag manifold of \( G \), that is, the space of \( d \)-uples of the form
\[
\xi = (\xi^1 \subset \cdots \subset \xi^d)
\]
where \( \xi^j \) is a \( j \)-dimensional subspace of \( V \) for each \( j = 1, \ldots, d \). A central feature about the limit map \( \xi_\rho \) is that it is transverse, i.e. for every \( x \neq y \) in \( \partial_\infty \Gamma \) and every \( j = 1, \ldots, d-1 \), the subspace \( \xi^j_\rho(x) \) is linearly disjoint from \( \xi^{d-j}_\rho(y) \) (see Subsection 7.1 for further precisions and references).

Let \( \rho : \Gamma \to G \) be a \( \Delta \)-Anosov representation and define \( \Omega_\rho \subset \mathbb{Q}_{p,q} \) to be the set consisting of forms for which the subspaces \( \xi^j_\rho(x) \) are non degenerate for every \( x \in \partial_\infty \Gamma \) and every \( j = 1, \ldots, d \). In Subsection 7.2 we discuss examples of Anosov representations for which the set \( \Omega_\rho \) is non empty.

Towards the understanding of the Main Problem above we prove the following result.

**Corollary A** (Corollaries 7.4 & 7.5). Assume that \( \rho \) is Zariski dense and let \( o \in \Omega_\rho \). Then there exist positive constants \( \delta_\rho, C_1 \) and \( C_2 \) such that for every \( t \) large enough one has
\[
C_1 e^{\delta_\rho t} \leq \#\{\gamma \in \Gamma : d_{\mathcal{X}_G}(S^o, \rho \gamma \cdot S^o) \leq t\} \leq C_2 e^{\delta_\rho t}.
\]

The constant \( \delta_\rho \) in Corollary A is independent of the choice of the basepoint \( o \) and coincides with the critical exponent
\[
\limsup_{t \to \infty} \frac{\log \#\{\gamma \in \Gamma : d_{\mathcal{X}_G}(\tau, \rho \gamma \cdot \tau) \leq t\}}{t}
\]
of \( \rho \) (for any point \( \tau \in \mathcal{X}_G \)). Corollary A is a consequence of a uniform estimate of the distance \( d_{\mathcal{X}_G}(S^o, \rho \gamma \cdot S^o) \) in terms of the Cartan projection of \( \rho \gamma \), and a corresponding counting theorem for this projection due to Sambarino [54]. In [54]

\(^1\)Observe that if this is the case then the intersection \( \Xi \cap H^o \) must be finite.
Sambarino proves his theorem when $\Gamma$ is the fundamental group of a closed negatively curved manifold. In Appendix A we explain how his proof adapts to our more general setting.

### 1.1. Main result.

The main result of this paper (Theorem B below) provides a precise asymptotic, as $t \to \infty$, for a counting function similar to (1.1) but with the distance $d_{\mathcal{L}_\rho} (\cdot, \cdot)$ replaced by the choice of a linear functional in the interior of the dual limit cone of $\rho$. We now state this result in a proper way.

A (maximal) flat of $X_G$ is a totally geodesic copy of $\mathbb{R}^{d-1}$ inside $X_G$. The space of flats identifies naturally with the space of Cartan subspaces $\mathfrak{a}$ of the Lie algebra $\mathfrak{g}$ of $G$. Cartan subspaces corresponding to flats orthogonal to $S^o$ will be denoted with the symbol $\mathfrak{b}$.

Let $o$ be a point in $Q_{p,q}$ and define $\mathcal{R}_{o,G}$ to be the set consisting of elements $g \in G$ for which there exists a flat of $X_G$ orthogonal both to $S^o$ and $g \cdot S^o$. The Lie theoretic description of $\mathcal{R}_{o,G}$ goes as follows. Fix a Cartan subspace $\mathfrak{b}$ whose corresponding flat is orthogonal to $S^o$ and let $\tau$ be the intersection point between this flat and $S^o$. We think here the point $\tau \in S^o$ as a Cartan involution of $\mathfrak{g}$ that commutes with the derivative $d\sigma^o$ of the involution $\sigma^o : G \to G$,

whose fixed point set is $H^o = \text{PSO}(o)$ (see Subsections 2.1 and 2.2).

Let $\mathfrak{g}^\tau$ be the subalgebra of $\mathfrak{g}$ consisting of fixed points of the involution $\tau(d\sigma^o)$ and $\mathfrak{b}^+$ be a (closed) Weyl chamber of the set of restricted roots $\Sigma(\mathfrak{g}^\tau, \mathfrak{b})$ (c.f. Subsection 3.2). Consider the Weyl group $W := N_{K^+}(\mathfrak{b})/Z_{K^+}(\mathfrak{b})$, where $K^+$ is the maximal compact subgroup of $G$ associated to $\tau$ and $N_{K^+}(\mathfrak{b})$ (resp. $Z_{K^+}(\mathfrak{b})$) is the normalizer (resp. centralizer) of $\mathfrak{b}$ in $K^+$.

**Proposition** (Propositions 4.2 and 4.4). One has a decomposition

$$\mathcal{R}_{o,G} = H^o W \exp(\mathfrak{b}^+) H^o.$$  

Furthermore, the $\mathfrak{b}^+$-coordinate in this decomposition is uniquely determined.

The previous decomposition of $\mathcal{R}_{o,G}$ will be called $(p,q)$-Cartan decomposition. We then introduce a $(p,q)$-Cartan projection

$$b^o : \mathcal{R}_{o,G} \to \mathfrak{b}^+$$

which is characterized by the equality

$$g = hw \exp(b^o(g)) h'$$

for every $g \in \mathcal{R}_{o,G}$, where $h, h' \in H^o$ and $w \in W$. In Lemma 4.5 we show the equality

$$\|b^o(g)\|_b = d_{\mathcal{L}_\rho}(S^o, g \cdot S^o)$$

for every $g \in \mathcal{R}_{o,G}$, where $\| \cdot \|_b$ is the Euclidean norm on $\mathfrak{b}$ induced by the $G$-invariant Riemannian structure of $X_G$.

In Corollary 7.4 we prove that for every $\Delta$-Anosov representation $\rho : \Gamma \to G$ and every basepoint $o \in \mathcal{O}_\rho$, then apart from possibly finitely many exceptions $\gamma \in \Gamma$ one has $\rho \gamma \in \mathcal{R}_{o,G}$.

We can now state our main result. Recall that since $\mathfrak{b}$ is a Cartan subspace of $\mathfrak{g}$ we can use the notation $\mathfrak{a} = \mathfrak{b}$. A closed Weyl chamber of the system $\Sigma(\mathfrak{g}, \mathfrak{b}) = \Sigma(\mathfrak{g}, \mathfrak{a})$ will be denoted by $\mathfrak{a}^+$. It will be said to be $\mathfrak{b}^+$-compatible if the inclusion $\mathfrak{a}^+ \subset \mathfrak{b}^+$ holds. The corresponding asymptotic cone, as introduced by Benoist in [3], will be denoted by $\mathcal{L}_\rho^*$ be the dual cone.
Theorem B (Proposition 8.10). Let \( \rho : \Gamma \to G \) be a Zariski dense \( \Delta \)-Anosov representation and \( o \) be a basepoint in \( \Omega \), such that for each \( j = 1, \ldots, d \) the signature of the form \( o \) restricted to \( \xi_j(x) \) is independent on \( x \in \partial_\infty \Gamma \). Then there exists a \( b^+ \)-compatible Weyl chamber \( a^+ \) such that for every linear functional \( \varphi \) in the interior of \( \mathcal{L}_o^+ \) there exist constants \( h^o_\rho > 0 \) and \( m = m_{o,\varphi} > 0 \) satisfying
\[
me^{-h^o_\rho t} \# \{ \gamma \in \Gamma : \rho \gamma \in \mathcal{B}_{o,G} \text{ and } \varphi(b^o(\rho \gamma)) \leq t \} \to 1
\]
as \( t \to \infty \).

The constant \( h^o_\rho \) in Theorem B coincides with the \( \varphi \)-entropy of \( \rho \), defined by
\[
h^o_\rho := \limsup_{t \to \infty} \frac{\log \# \{ [\gamma] \in [\Gamma] : \varphi(\lambda(\rho \gamma)) \leq t \}}{t}.
\]
Here \( \lambda(\cdot) \) denotes the Jordan projection of \( G \) and \( [\gamma] \) denotes the conjugacy class of \( \gamma \in \Gamma \). In other words, the choice of \( \varphi \) induces a H"older reparametrization \( \phi_t^\rho \) of the geodesic flow of \( \rho \), and \( h^o_\rho \) is the topological entropy of \( \phi_t^\rho \). Recall that the geodesic flow of \( \rho \) was introduced by Bridgeman-Canary-Labourie-Sambarino [8].

On the other hand, the constant \( m = m_{o,\varphi} \) is related to the total mass of a specific measure in the Bowen-Margulis measure class of this reparametrization (i.e. the homothety class of measures maximizing the entropy of the flow \( \phi_t^\rho \)).

1.2. Method and outline of the proof. After defining the \((p,q)\)-Cartan projection we begin the study of its asymptotic properties. Given a \( b^+ \)-compatible Weyl chamber \( a^+ \), we let \( \Delta \subset \Sigma(g,a) \) be the set of simple roots and
\[
G = K^\gamma \exp(a^+)K^\gamma
\]
be the associated Cartan decomposition of \( G \). We denote by
\[
a^\tau : G \to a^+
\]
the corresponding Cartan projection. Recall that an element \( g \in G \) is said to have a gap of index \( \Delta \) if \( a(a^\tau(g)) \) is positive for every \( a \in \Delta \). In this case we have well defined full flags \( U^\tau(g) \) and \( S^\tau(g) \) which are called respectively the Cartan attractor and Cartan repellor of \( g \) (see Subsection 5.1).

A full flag \( \xi \in F(V) \) is said to be \( o \)-generic if for every \( j = 1, \ldots, d \) the restriction of the form \( o \) to the subspace \( \xi_j \) is non degenerate. The space of \( o \)-generic flags is denoted by \( F(V)^o \) and coincides with the union of open orbits of the action \( H^o \curvearrowright F(V) \) (see Subsection 2.3.2).

Under the assumption that \( g \in G \) has a “sufficiently strong” gap of index \( \Delta \) and \( o \)-generic Cartan attractor and repellor, we compute in Subsection 5.3 the \( b^+ \)-compatible Weyl chamber that contains \( b^o(g) \). This makes the study of the \((p,q)\)-Cartan projection tractable. Indeed for a given \( b^+ \)-compatible Weyl chamber \( a^+ \), in Subsection 5.4 we show the inequality
\[
\|b^o(g) - w_g \cdot a^\tau(g)\|_b \leq D
\]
for some \( D > 0 \) and an element \( w_g \) of the Weyl group that we can precisely describe. Further, we can describe \( b^o(g) \) using the Jordan projection \( \lambda \) of \( G \):
\[
b^o(g) = \frac{1}{2} w_g \cdot \lambda(\sigma^o(g^{-1})g).
\]
Note that the estimate (1.2) and the equality (1.3) are not canonical: they strongly depend on the choice of a \( b^+ \)-compatible Weyl chamber \( a^+ \). Because of this, we emphasize that we do not fix the \( b^+ \)-compatible Weyl chamber \( a^+ \), only the Weyl chamber \( b^+ \) is fixed beforehand.
If a Zariski dense $\Delta$-Anosov representation $\rho : \Gamma \to G$ and a basepoint $o$ in $\Omega_\rho$ are given, the estimate (1.2) combined with the work of Sambarino [54] allows us to prove Corollary A. Furthermore, let $\mathcal{L}_\rho^{p,q}$ be the $(p,q)$-asymptotic cone of $\rho$. By definition, it is the subset of $b^+$ consisting on all possible limits of the form

$$\frac{b^o(\rho \gamma_n)}{t_n}$$

where $t_n \to \infty$. Recall that for a given Weyl chamber $a^+$ of the system $\Sigma(g,a)$, the asymptotic cone of $\rho$ (in the sense of Benoist [3]) is denoted by $\mathcal{L}_\rho$. We show the following.

**Proposition (Proposition 7.8).** Let $a^+$ be a $b^+$-compatible Weyl chamber and $\mathcal{L}_\rho \subset \text{int}(a^+)$ be the associated asymptotic cone. Then there exists a subset $W_{\rho,a^+}$ of the Weyl group $W$ for which one has

$$\mathcal{L}_\rho^{p,q} = \bigcup_{w \in W_{\rho,a^+}} w \cdot \mathcal{L}_\rho.$$

The subset $W_{\rho,a^+}$ is in one to one correspondence with the set open orbits of the action of $H^o \curvearrowright F(V)$ that intersect the limit set $\xi_\rho(\partial_\infty \Gamma)$. In contrast with Benoist’s asymptotic cone [3], the $(p,q)$-asymptotic cone is not necessarily convex (c.f. Remark 7.6).

We then begin the study of finer asymptotic properties of the $(p,q)$-Cartan projection. In the classical setting (i.e. for the Cartan projection $a^+(\cdot)$), the key object that appears is the (vector valued) Busemann cocycle $^2\text{Busemann cocycle}$ of $G$, introduced by Quint [48]. In Section 6 we introduce an analogue of the Busemann cocycle, that we call the $o$-Busemann cocycle, and that plays the role of the classical Busemann cocycle in our setting. We remark here that its definition depends on the choice of a $b^+$-compatible Weyl chamber.

The assumption over the limit set in Theorem B is equivalent to the fact that $\xi_\rho(\partial_\infty \Gamma)$ is contained in a single open orbit of the action $H^o \curvearrowright F(V)$. As we shall see (c.f. Subsection 3.4) this assumption canonically selects a $b^+$-compatible Weyl chamber $a^+$ and for this Weyl chamber we have the equality

$$\mathcal{L}_\rho^{p,q} = \mathcal{L}_\rho.$$

Furthermore, for every $\gamma$ large enough the equality

$$b^o(\rho \gamma) = \frac{1}{2} \lambda(\sigma^o(\rho \gamma^{-1})\rho \gamma)$$

can be assumed to hold (see Corollary 8.1) and we have also a well defined vector valued $o$-Busemann cocycle for $\rho$ (Subsection 8.1). Benoist’s framework [4] allows us to obtain a precise comparison between $\frac{1}{2} \lambda(\sigma^o(\rho \gamma^{-1})\rho \gamma)$ and $\lambda(\rho \gamma)$ in terms of some appropriate vector valued cross-ratio, which turns out to be closely related with the $o$-Busemann cocycle of $\rho$ (c.f. Corollaries 6.9 and 8.5). Equipped with this precise estimate, if a functional $\varphi$ in the interior of the dual cone $\mathcal{L}_\rho^*$ is given we are in position of applying Sambarino’s adaptation [53] of Roblin’s method [50] to obtain Theorem B.

1.3. Final remarks. To finish this introduction we discuss some work related to the present paper.

2Sometimes also called the Iwasawa cocycle of $G$. 

1.3.1. **Domains of discontinuity.** In joint work with F. Stecker, which is still in progress, we prove that $\Omega_\rho$ is a *domain of discontinuity* for $\rho$, i.e. the action of $\Gamma$ on $\Omega_\rho$ induced by $\rho$ is properly discontinuous. In fact, our construction provides examples of domains of discontinuity for Anosov representations in a large class of $G$-homogeneous spaces (for a connected semisimple Lie group $G$ with no compact factors) that include *symmetric spaces* of $G$. This construction generalizes that of Kapovich-Leeb-Porti [27] (see Stecker [56] or C. [12] for further details). We mention here that in the present paper we do not use the fact that the action of $\Gamma$ on $\Omega_\rho$ is properly discontinuous.

1.3.2. **The classical counting problem.** Classically, the *orbital counting problem* concerns the study of the asymptotic behaviour of the function 

$$t \mapsto \# \{ g \in \Xi : d_X(o, g \cdot o) \leq t \}$$

as $t \to \infty$, where $\Xi$ is a discrete group of isometries of a given proper non compact metric space $X$, and $o$ is a basepoint in $X$. This problem has been studied in many situations, by authors among who we find notably Gauss, Huber, Patterson and Margulis. Of course, this list is highly incomplete (we refer the reader to Babillot’s survey [1] for a more complete picture). Let us mention here that the asymptotic behaviour of the function (1.4) has been studied when $X$ coincides with the Riemannian symmetric space of a semisimple Lie group $G$ with no compact factors. Indeed, when $\Xi < G$ is a lattice one finds the work of Duke-Rudnick-Sarnak [16] and more generally that of Eskin-McMullen [18]. In the non lattice case one also finds the work of Quint [49] and Sambarino [54] (who deal with $\Delta$-Anosov subgroups of $G$), and the work of Thirion [57] (who deals with Ping-Pong subgroups of $\text{SL}_d(\mathbb{R})$). The approach by Sambarino and Thirion is inspired by Roblin’s method [50].

1.3.3. **Relation with the work of Parkkonen-Paulin.** When $d = 2$ the Main Problem of this paper has been studied by Parkkonen-Paulin [42]. The results of [42] are valid also in some situations of variable curvature bounded above by a negative constant, and in some of these situations the authors obtain estimates on the error terms for their counting results (see [42] for precisions). Parkkonen and Paulin’s work generalizes (to the variable curvature setting) previous work of Eskin-McMullen [18], Oh-Shah [38, 39, 40, 41] and Mohammadi-Oh [37], which include counting problems associated to symmetric spaces of $\text{SO}_0(n, 1)$.

1.3.4. **Relation with the work of Edwards-Lee-Oh.** In a recent preprint [17], Edwards, Lee and Oh treat a counting problem for Zariski dense $\Delta$-Anosov subgroups which is related to ours. They look at a space of the form $G/H$ (where $G$ is a semisimple Lie group and $H \subset G$ consists of fixed points of an involution of $G$), and prove a counting theorem for the *polar projection* (see e.g. [56, Section 7]) of discrete $\Gamma$-orbits$^3$ in $G/H$. The norm of the polar projection of an element $g \in G$ can be interpreted as the distance between a basepoint in $S^H$ and the submanifold $g \cdot S^H$, where $S^H \subset X_G$ is a totally geodesic copy of the Riemannian symmetric space of $H$ (see C. [12, Proposition 1.4.6]). Here by “norm” we mean the one induced by a $G$-invariant Riemannian structure on $X_G$ and part of the results of [17] include, in some specific situations, counting theorems with respect to this norm (see [17, Theorem 1.11] for precisions).

$^3$The authors do not assume that the intersection $\rho(\Gamma) \cap H$ is finite.
We mention here that counting problems for the polar projection and also the Main Problem of the present paper have been studied as well in C. [11] for $G = \text{PSO}(p, q)$, $H = \text{PSO}(p, q - 1)$ and $\rho : \Gamma \to G$ a projective Anosov representation.

Plan of the paper. Sections 2 and 3 are mainly intended to fix terminology and notations. Proposition 3.4 is the most important result of those sections, as it will be helpful in the study of the $(p, q)$-Cartan projection. The $(p, q)$-Cartan decomposition is introduced in Section 4 and in Section 5 we begin the study of the associated projection. Notably, the contents of Subsection 5.4 will be of central importance for the rest of the paper (we prove the estimate (1.2) and the equality (1.3)). In Section 6 we introduce the vector valued $o$-Busemann cocycle and study some basic properties. Corollary A is proved in Subsection 7.3 and Theorem B is proved in Section 8. In Subsection 7.1 we recall the definition and main properties of Anosov representations, and in Appendix A we explain how Sambarino’s results [53, 54] still hold in our setting.

Dependence between sections is as follows:

![Diagram]

2. Forms of fixed signature

From now on we fix a real or complex vector space $V$ of dimension $d \geq 3$ and denote by $G$ the group $\text{PSL}(V)$ of projectivized elements of $\text{SL}(V)$. We let $\mathfrak{g}$ be the Lie algebra of $G$ and $\kappa : \mathfrak{g} \times \mathfrak{g} \to \mathbb{R}$ be the Killing form of $\mathfrak{g}$.

2.1. Notations and terminology. Consider two non negative integers $p$ and $q$ such that $p + q = d$. A form of signature $(p, q)$ on $V$ is a quadratic or Hermitian form of that signature, depending respectively on whether $K = \mathbb{R}$ or $K = \mathbb{C}$. We denote by $Q_{p,q}$ the space of homothety classes of forms of signature $(p, q)$ on $V$, where two forms are said to be homothetic if they differ by multiplication by a positive real number. Note that $G$ acts on $Q_{p,q}$ in a transitive way.

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4When $K = \mathbb{C}$, we compute the Killing form using the trace form associated to the underlying real structure of $\mathfrak{g}$. 
2.1.1. Structure of symmetric space. Let \( o \) be a point in \( Q_{p,q} \) and \( \langle \cdot, \cdot \rangle_o \) be the form associated to a representative of \( o \). Since \( o \) is non degenerate, we can define the \( o \)-adjoint operator
\[
*o : \mathfrak{gl}(V) \rightarrow \mathfrak{gl}(V)
\]
where, for \( T \in \mathfrak{gl}(V) \), \( *T \) is the unique linear transformation of \( V \) that satisfies the equality
\[
\langle T \cdot u, v \rangle_o = \langle u, *T \cdot v \rangle_o
\]
for all \( u \) and \( v \) in \( V \). Note that the \( o \)-adjoint \( *T \) does not depend on the choice of the representative \( \langle \cdot, \cdot \rangle_o \) of \( o \). Moreover, \( * \cdot o \) preserves \( \text{SL}(V) \) and descends to a map \( G \rightarrow G \), that we still denote by \( * \cdot o \). Define \( \sigma^o \) to be the involutive automorphism of \( G \) given by
\[
\sigma^o(g) := *^o g^{-1}.
\]
Then \( Q_{p,q} \) identifies with \( G/H^o \), where \( H^o \) is the subgroup of \( G \) consisting of fixed points of the involution \( \sigma^o \). The space \( Q_{p,q} \) is then a symmetric space of \( G \).

Let \( ds^o \) be the derivative of \( \sigma^o \) at the identity element of \( G \). The map
\[
X \mapsto \left. \frac{d}{dt} \right|_{t=0} \exp(tX) \cdot o
\]
gives a \( G \)-equivariant identification between
\[
q^o := \{ ds^o = -1 \}
\]
and the tangent space to \( Q_{o,q} \) at the point \( o \). If \( h^o \) denotes the subalgebra of \( g \) consisting of fixed points of \( ds^o \), one has the following decomposition
\[
g = h^o \oplus q^o.
\]
This decomposition is orthogonal with respect to the Killing form \( \kappa \).

**Remark 2.1.** For \( j = 1, \ldots, d \) consider the group \( S_j \) of permutations of \( \{1, \ldots, j\} \). Let \( \varepsilon(\omega) \) denote the sign of \( \omega \in S_j \). Let \( \Lambda_j \) be the \( j^{th} \)-exterior power representation of \( G \). Then the form on \( \Lambda_j V \) defined by
\[
\langle v_1 \wedge \cdots \wedge v_j, v'_1 \wedge \cdots \wedge v'_j \rangle_j := \frac{1}{j!} \sum_{\omega \in S_j} \varepsilon(\omega) \prod_{i=1}^{j} \langle v_i, v'_i(\omega) \rangle_o
\]
is non degenerate and invariant under the action of \( \Lambda_j H^o \). Denote by \( o_j \) the ray containing the form associated to \( \langle \cdot, \cdot \rangle_o \) and observe that for every \( g \in G \) one has
\[
\sigma^o(\Lambda_j g) = \Lambda_j \sigma^o(g).
\]

2.1.2. The Riemannian symmetric space. A Cartan involution of \( g \) is an \( \mathbb{R} \)-bilinear involutive automorphism \( \tau : g \rightarrow g \) for which the form on \( g \) given by
\[
(X, Y) \mapsto -\kappa(X, \tau \cdot Y)
\]
is positive definite. The Lie group \( G \) acts on the space \( X_G \) of Cartan involutions of \( g \) in a transitive way and the stabilizer \( K^\tau \) of a point \( \tau \) in \( X_G \) is a maximal compact subgroup of \( G \) (see Knapp [31, Corollary 6.19 and Theorem 6.31]). In particular, the space \( X_G \) can be endowed with a \( G \)-invariant Riemannian metric which is necessarily non positively curved (see Helgason [24, Theorem 4.2 of Ch. IV]). We fix once and for all such a Riemannian metric and call the space \( X_G \) the Riemannian symmetric space of \( G \). We let \( d_{X_G}(\cdot, \cdot) \) denote the \( (G \text{-invariant}) \) distance on \( X_G \) induced by the Riemannian structure. For a point \( \tau \in X_G \) we define
\[
p^\tau := \{ \tau = -1 \} \text{ and } \mathfrak{t}^\tau := \{ \tau = 1 \}.
\]
The Riemannian structure on \( X_G \) induces a Euclidean norm on \( p^\tau \).
Remark 2.2. Note that one has natural identifications
\[ Q_{0,d} \cong Q_{d,0} \cong X_G. \]
Indeed, for \( o \in Q_{d,0} \) we have that \( d\sigma^o \in X_G \). For this reason (and to avoid confusions), from now on the notation \( Q_{p,q} \) will always mean that \( p \) and \( q \) are positive. However, we will identify points in \( X_G \) with homothety classes of inner products on \( V \) whenever convenient.

2.1.3. Cartan subspaces and flats. Fix a basepoint \( o \in Q_{p,q} \). There exist Cartan involutions \( \tau \) of \( g \) that commute with \( d\sigma^o \) and two of them differ by the action of \( H^0_o \), the connected component of \( H^0 \) containing the identity element (see Matsuki [35, Lemmas 3 and 4]).

Take a point \( \tau \in X_G \) for which \( \tau \) and \( d\sigma^o \) commute and let \( b \subset p^\tau \cap q^o \) be a (necessarily abelian) maximal subalgebra. The norm on \( b \) induced by the Riemannian structure on \( X_G \) is denoted by \( \| \cdot \|_b \). For all \( X \in b \) one has
\[
\|X\|_b = \|X\| = d_{X_G}(\tau, \exp(X) \cdot \tau) = \|X\|_b.
\]

Let us now describe a maximal subalgebra \( b \subset p^\tau \cap q^o \) in a more concrete way. In the process we fix some terminology that will remain valid for the rest of the paper.

Fix a representative \( \langle \cdot, \cdot \rangle_o \) of \( o \). For a subspace \( \pi \) of \( V \) denote
\[
\pi^{+o} := \{v \in V : \langle v, u \rangle_o = 0 \text{ for all } u \in \pi\}.
\]
A set is said to be \( o \)-orthogonal if its elements are pairwise orthogonal with respect to the form \( \langle \cdot, \cdot \rangle_o \). The \( o \)-sign of a vector \( v \) in \( V \) is defined by
\[
sg_o(v) := \begin{cases} 
1 & \text{if } \langle v, v \rangle_o > 0 \\
-1 & \text{if } \langle v, v \rangle_o < 0 \\
0 & \text{if } \langle v, v \rangle_o = 0
\end{cases}.
\]
This vector is said to be \( \langle \cdot, \cdot \rangle_o \)-unitary if
\[
\langle v, v \rangle_o \in \{-1, 1\}.
\]
A basis \( B \) of \( V \) is said to be \( \langle \cdot, \cdot \rangle_o \)-orthonormal if it is \( o \)-orthogonal and its elements are \( \langle \cdot, \cdot \rangle_o \)-unitary. Finally, the \( o \)-sign of a line \( \ell \) in \( V \) is defined in an analogous way and a basis of lines of \( V \) is a set of \( d \) lines in \( V \) that span \( V \).

Example 2.3. Let \( B \) be a \( \langle \cdot, \cdot \rangle_o \)-orthonormal basis of \( V \) and \( \langle \cdot, \cdot \rangle \) be the inner product of \( V \) for which this basis is orthonormal. The associated point in \( X_G \) will be denoted by \( \tau \) and we emphasize the link between the inner product \( \langle \cdot, \cdot \rangle \) and the point \( \tau \) by denoting \( \langle \cdot, \cdot \rangle_{\tau} := \langle \cdot, \cdot \rangle \). It can be seen that \( d\sigma^o \) and \( \tau \) commute.

Pick \( b \) to be the subset of \( g \) consisting of elements which are diagonal in the basis \( B \), with real eigenvalues. Since \( B \) is both \( \langle \cdot, \cdot \rangle_o \)-orthonormal and \( \langle \cdot, \cdot \rangle_{\tau} \)-orthonormal, one has
\[
b \subset p^\tau \cap q^o
\]
and one can see that \( b \) is a maximal subalgebra in \( p^\tau \cap q^o \).

From Example 2.3 we conclude that maximal subalgebras \( b \subset p^\tau \cap q^o \) are in fact maximal in \( p^\tau \), that is, they are Cartan subspaces of \( g \). It is standard to denote Cartan subspaces of \( g \) with the symbol \( a \) instead of \( b \). We will use the notation \( b \) if we want to emphasize that this is a maximal subalgebra in \( p^\tau \cap q^o \) and use the notation \( a := b \) if we want to emphasize that this subalgebra is maximal in \( p^\tau \) and apply the classical theory to it.
As outlined in Example 2.3, the space of Cartan subspaces of $\mathfrak{g}$ is in natural bijection with the space of bases of lines of $V$. A Cartan subspace is contained in $\mathfrak{p}^\tau$ (resp. $\mathfrak{q}^o$) if and only if the corresponding basis of lines is $\tau$-orthogonal (resp. $\sigma$-orthogonal).

A flat in $X_G$ is a maximal dimensional totally geodesic submanifold of $X_G$ on which sectional curvature vanishes. The space of flats in $X_G$ (through the basepoint $\tau$) is in one to one correspondence with the space of Cartan subspaces of $\mathfrak{g}$ (contained in $\mathfrak{p}^\tau$). Concretely, if $\mathfrak{a}$ is a Cartan subspace of $\mathfrak{g}$ (contained in $\mathfrak{p}^\tau$) and $\mathcal{C}$ is the associated $(\tau$-orthogonal) basis of lines of $V$, then

$$\{\tau' \in X_G : \mathcal{C} \text{ is } \tau'$-orthogonal\}$$

is a flat in $X_G$. It coincides with $\exp(\mathfrak{a}) \cdot \tau$.

2.2. The submanifold $S^o$. Let

$$S^o := \{\tau \in X_G : \tau(d\sigma^o) = (d\sigma^o)\tau\}.$$

Concretely, an homothety class of inner products $\tau$ belongs to $S^o$ if and only if there exist representatives $\langle \cdot, \cdot \rangle_\mathfrak{o}$ of $\sigma$ and $\langle \cdot, \cdot \rangle_\mathfrak{r}$ of $\tau$ and a basis $\mathcal{B}$ of $V$ which is both $\langle \cdot, \cdot \rangle_\mathfrak{o}$-orthonormal and $\langle \cdot, \cdot \rangle_\mathfrak{r}$-orthonormal.

The Riemannian symmetric space $X_{H^o}$ of $H^o$ can be identified with the space of $q$-dimensional subspaces of $V$ on which the form $\sigma$ is negative definite. We then find an isometry

$$S^o \to X_{H^o}$$

that sends each $\tau \in S^o$ to the subspace of $V$ spanned by the vectors of $\mathcal{B}$ which are negative for the form $\sigma$, where $\mathcal{B}$ is a basis of $V$ as in the above paragraph. It follows also that $S^o$ is $H^o$-invariant and $S^o = H^o_0 \cdot \tau$ for any $\tau \in S^o$. In particular, the tangent space to $S^o$ at $\tau$ identifies with

$$\mathfrak{p}^\tau \cap \mathfrak{h}^o$$

and $S^o$ is totally geodesic (see e.g. Helgason [24, Theorem 7.2 of Ch. IV]). We deduce the following.

**Corollary 2.4.** Let $\tau$ be a point in $S^o$ and $\mathfrak{a} \subset \mathfrak{p}^\tau$ be a maximal subalgebra. Then the following are equivalent:

1. The flat $\exp(\mathfrak{a}) \cdot \tau$ is orthogonal to $S^o$ at $\tau$.
2. The inclusion $\mathfrak{a} \subset \mathfrak{p}^\tau \cap \mathfrak{q}^o$ holds.
3. The basis of lines $\mathcal{C}$ associated to $\mathfrak{a}$ is $\sigma$-orthogonal (and $\tau$-orthogonal).

2.3. Generic flags. For $j = 1, \ldots, d$ we denote by $\text{Gr}_j(V)$ the Grassmannian of $j$-dimensional subspaces of $V$. Denote by $F(V)$ the space of complete (or full) flags of $V$, that is,

$$F(V) := \{\xi = (\xi^1 \subset \cdots \subset \xi^d) : \xi^j \in \text{Gr}_j(V) \text{ for every } j = 1, \ldots, d\}.$$

Two complete flags $\xi_1$ and $\xi_2$ are said to be transverse if for every $j = 1, \ldots, d$ the subspace $\xi_1^j$ is linearly disjoint from $\xi_2^{d-j}$. Equivalently, recall that for every $j = 1, \ldots, d$ there are equivariant maps

$$\Lambda^j : F(V) \to P(\Lambda^j V) \text{ and } \Lambda^j_\xi : F(V) \to P(\Lambda^j V^\tau)$$

into the projective spaces of $\Lambda^j V$ and $\Lambda^j V^\tau$ respectively. Then $\xi_1$ is transverse to $\xi_2$ if and only if for every $j = 1, \ldots, d-1$ the line $\Lambda^j(\xi_1)$ is linearly disjoint from
the hyperplane $\Lambda_{\xi}(\xi)$. The space of ordered pairs of transverse full flags of $V$ is denoted by $F(V)^{(2)}$.

2.3.1. **Genericity of flags with respect to a basepoint.** In this subsection we introduce the notion of $o$-generic flag and discuss some characterizations. This notion is introduced by means of an involution

$$\cdot^o : F(V) \to F(V)$$

which is defined in the following way. Given a full flag $\xi = (\xi^1, \ldots, \xi^d)$ in $F(V)$ we denote by $\xi^o$ the complete flag of $V$ defined by the equalities

$$(\xi^o)^j := (\xi^{d-j})^o$$

for $j = 1, \ldots, d$.

The following lemma is direct.

**Lemma 2.5.** For every $g \in G$ and every $\xi$ in $F(V)$ one has

$$\sigma^o(g) \cdot (\xi^o) = (g \cdot \xi)^o = g \cdot (\xi^{o^{-1}} o).$$

In particular, the following holds:

$$(g \cdot \xi)^o = g \cdot (\xi^o).$$

A full flag $\xi \in F(V)$ is said to be $o$-generic if it is transverse to $\xi^o$. We have the following useful equivalences, which hold by definitions.

**Lemma 2.6.** Let $\xi = (\xi^1, \ldots, \xi^d)$ be an element of $F(V)$. Then the following are equivalent:

1. The flag $\xi$ is $o$-generic.
2. For every $j = 1, \ldots, d$, the restriction of the form $o$ to $\xi^j$ is non degenerate.
3. There exists a unique $o$-orthogonal ordered basis of lines

$$\{\ell^j_1(\xi), \ldots, \ell^j_d(\xi)\}$$

of $V$ such that the equality

$$\xi^j = \ell^j_1(\xi) \oplus \cdots \oplus \ell^j_d(\xi)$$

holds for every $j = 1, \ldots, d$.
4. For every $j = 1, \ldots, d$, the line $\Lambda^j \xi$ is transverse to the hyperplane $(\Lambda^j \xi)^o$.

The following remark is useful.

**Remark 2.7.** Fix $j = 1, \ldots, d$ and let $\xi$ be an $o$-generic flag. Then the following equality holds

$$\Lambda^j(\xi^o) = (\Lambda^j \xi)^o.$$ 

In particular, if $\xi' \in F(V)$ is a flag transverse to $\xi$ then the hyperplane $(\Lambda^j(\xi^o))^{o}$ is transverse to the line $\Lambda^j \xi'$.

2.3.2. **Open orbits of point-stabilizers.** Denote by $F(V)^o$ the subset of $F(V)$ consisting of $o$-generic flags. One can see that $F(V)^o$ coincides with the union of open orbits of the action of $H^o$ on $F(V)$.

The proof of the following proposition is direct.

**Proposition 2.8.** Let $\xi$ and $\xi'$ be two $o$-generic flags. Then the following are equivalent:

\[^5\text{As usual, for a finite dimensional vector space } W \text{ one identifies } P(W^*) \text{ with the Grassmannian of codimension-one subspaces of } W \text{ by the map } \vartheta \mapsto \ker \vartheta.\]
1. The flags $\xi$ and $\xi'$ belong to the same orbit of the action $H^o \lhd F(V)$.

2. For every $j = 1, \ldots, d$, the signature of $o$ restricted to $\xi_j$ coincides with the signature of $o$ restricted to $\xi'_j$.

3. For every $j = 1, \ldots, d$, one has $sg_o(\ell_o(\xi)) = sg_o(\ell_o(\xi'))$.

4. For every $j = 1, \ldots, d$, one has $sg_o(\Lambda^j \xi) = sg_o(\Lambda^j \xi')$.

3. Weyl chambers and generic flags

Fix a point $\tau \in S^o$ and a maximal subalgebra $b \subset p^\tau \cap q^o$. Let $C$ be the $o$-orthogonal and $\tau$-orthogonal basis of lines of $V$ determined by this choice.

3.1. Weyl group. Recall that $b$ is a maximal subalgebra in $p^\tau$ and that we use the notation $a := b$ whenever we want to emphasize this. Let $W := Weyl_g := N_K(\tau) \cap b$ be the Weyl group of the pair $(g, a)$. If $w$ belongs to $W$, we denote by $\tilde{w}$ its class in $W$. Conversely, for a given $w \in W$ we denote by $\tilde{w} \in W$ any representative of $w$.

Fix a representative $\langle \cdot, \cdot \rangle_o$ of $o$. Since the involutions $d \sigma^o$ and $\tau$ commute we can find a representative $\langle \cdot, \cdot \rangle_\tau$ of $\tau$ for which the following holds: for each line $\ell \in C$ and each vector $v \in \ell$ one has

$$\langle \tilde{w} \cdot v, \tilde{w} \cdot v \rangle_o = \langle v, v \rangle_\tau.$$

From this observation the following technical lemma can be easily deduced.

**Lemma 3.1.** Let $\tilde{w}$ be an element of $G$ that preserves the set $C$. Then the following holds:

1. If $\tilde{w}$ belongs to $H^o$, then it belongs to $K^\tau$.

2. Suppose that $\tilde{w}$ belongs to $K^\tau$. Then for every line $\ell \in C$ and every vector $v \in \ell$ one has

$$\langle \tilde{w} \cdot v, \tilde{w} \cdot v \rangle_o = \langle v, v \rangle_\tau.$$

In particular, if $\tilde{w}$ preserves the $o$-sign of each line of $C$ then $\tilde{w}$ belongs to $H^o$.

By Lemma 3.1 we have that $M$ coincides with the centralizer of $b$ in $H^o$ and in particular it is contained in $H^o$.

3.2. Restricted roots and Weyl chambers. Let $\mathfrak{g}$ be a subalgebra of $g^\tau$ invariant under the (adjoint) action of $b$. A non zero functional $\alpha \in b^*$ is called a restricted root of $b$ in $\mathfrak{g}$ if the subspace

$$\mathfrak{g}_\alpha := \{ Y \in \mathfrak{g} : [X, Y] = \alpha(X) Y \ \text{for all} \ X \in b \}$$

is non zero. In that case, $\mathfrak{g}_\alpha$ is called the associated root space. The set of restricted roots of $b$ in $\mathfrak{g}$ is denoted by $\Sigma(\mathfrak{g}, b)$. A (closed) Weyl chamber of this set is the closure of a connected component of

$$b \setminus \bigcup_{\alpha \in \Sigma(\mathfrak{g}, b)} \ker(\alpha).$$

Let $\Sigma^+(\mathfrak{g}, b)$ be the corresponding positive system, that is, the set of restricted roots in $\Sigma(\mathfrak{g}, b)$ which are non negative on this Weyl chamber. In this paper we look at Weyl chambers of two different sets of restricted roots.
3.2.1. **The case** $\Sigma(g, a)$. In this case we think $b$ as a maximal subalgebra in $p^\tau$ and therefore we denote $b = a$. Weyl chambers of the system $\Sigma(g, a)$ will be denoted with the symbol $a^+$.

Given a line $\ell \in C$, let $\varepsilon_\ell \in a^*$ be the functional that assigns to each element $X \in a$ its eigenvalue in the line $\ell$. For different $\ell$ and $\ell'$ in $C$ let

$$\alpha_{\ell\ell'}(X) := \varepsilon_\ell(X) - \varepsilon_{\ell'}(X).$$

Then one has the equality

$$\Sigma(g, a) = \{\alpha_{\ell\ell'} : \ell \neq \ell' \in C\}$$

and the choice of a closed Weyl chamber corresponds to the choice of a total order $C$ on $C$. If this choice is given we set

$$\varepsilon_j := \varepsilon_{\ell_j} \text{ and } \alpha_{ji} := \varepsilon_j - \varepsilon_i$$

for each $j$ different from $i$ in $\{1, \ldots, d\}$, and the corresponding positive system is

$$\Sigma^+(g, a) := \left\{\alpha_{\ell_j\ell_i}^+ : 1 \leq j < i \leq d\right\}.$$ 

3.2.2. **The case** $\Sigma(g^\tau_o, b)$. Let

$$g^\tau_o := (p^\tau \cap q^\tau) \oplus (t^\tau \cap b^\tau)$$

be the ($b$-invariant) Lie algebra consisting of fixed points of the involution $\tau(da^\tau)$. As suggested in Schlichtkrull [55, p. 117], Weyl chambers of the set $\Sigma(g^\tau_o, b)$ will be denoted with the symbol $b^+$.

We have the equality

$$\Sigma(g^\tau_o, b) = \{\alpha_{\ell\ell'} \in \Sigma(g, a) : \ell \neq \ell' \in C \text{ and } sg_o(\ell) = sg_o(\ell')\}.$$

Indeed, one has $\Sigma(g^\tau_o, b) \subset \Sigma(g, a)$ and therefore a non zero linear functional $\alpha \in b^*$ belongs to $\Sigma(g^\tau_o, b)$ if and only if it belongs to $\Sigma(g, a)$ and $g^\tau_o \cap \mathfrak{a} \neq \{0\}$.

Now for $\alpha = \alpha_{\ell\ell'} \in \Sigma(g, a)$ the associated root space $\mathfrak{g}_\alpha$ intersects the subalgebra $g^\tau_o$ if and only if $sg_o(\ell)$ coincides with $sg_o(\ell')$ and the claim follows.

Note then that $\Sigma(g^\tau_o, b)$ is not a root system, because it does not generate $b^*$. However, it is in one to one correspondence with

$$\Sigma(\mathfrak{sl}_p, a_{st_p}) \sqcup \Sigma(\mathfrak{sl}_q, a_{st_q}).$$

An explicit way of prescribing a Weyl chamber $b^+$ is the following. Write $C = C^+ \sqcup C^-$ where $C^+$ (resp. $C^-$) is the subset of $C$ consisting of lines which are positive (resp. negative) for the form $o$. Fix total orders

$$C^+ = \{\ell_1^+, \ldots, \ell_p^+\} \text{ and } C^- = \{\ell_1^-, \ldots, \ell_q^-\}$$

on $C^+$ and $C^-$ respectively. Then a positive system in $\Sigma(g^\tau_o, b)$ is given by

$$\Sigma^+(g^\tau_o, b) := \left\{\alpha_{\ell_j\ell_i}^+ : 1 \leq j < i \leq p\right\} \sqcup \left\{\alpha_{\ell_j\ell_i}^- : 1 \leq j < i \leq q\right\}.$$

Conversely, the choice of a Weyl chamber $b^+$ induces total orders on $C^+$ and $C^-$. Note that a Weyl chamber $a^+$ is contained in $b^+$ if and only if the total order on $C$ induced by $a^+$ restricts to the total orders on $C^+$ and $C^-$ determined by $b^+$. In particular, each $b^+$ contains $\left(\frac{d!}{p!q!}\right)\text{ Weyl chambers of } \Sigma(g, a)$ (c.f. Figure 1).
Figure 1. Weyl chambers for $Q_{2,1}$. In light grey, a Weyl chamber $b^+$ of the set $\Sigma(g^{\tau_0}, b)$. This Weyl chamber is a union of three Weyl chambers $a_1^+, a_2^+$ and $a_3^+$ of the system $\Sigma(g, a)$.

Remark 3.2. Even though it will not be used in the future, we mention here that Weyl chambers $b^+$ of the set $\Sigma(g^{\tau_0}, b)$ admit the following geometric interpretation: a vector $X$ belongs to the interior $\text{int}(b^+)$ of $b^+$ if and only if the flat $\exp(b) \cdot \tau$ is the unique flat of $X_G$ that contains the geodesic $\exp(R \cdot \tau)$ and that is orthogonal to $S^o$ at $\tau$.

3.3. Opposition involution of $b^+$. For the rest of the section we fix a closed Weyl chamber $b^+$ of the set $\Sigma(g^{\tau_0}, b)$. Let $w_{b^+} \in W$ be the unique element that preserves $C^+$ and $C^-$ and acts on these sets by reversing the total order induced by $b^+$. By Lemma 3.1 we have that $\hat{w}_{b^+}$ belongs to $H^o$ and, by definition, it satisfies

$$w_{b^+} \cdot (-b^+) = b^+.$$

The opposition involution of $b^+$ is defined by

$$\iota_{b^+} : b \to b : \quad \iota_{b^+}(X) := -w_{b^+} \cdot X.$$

Note that $\iota_{b^+}$ preserves $b^+$.

3.4. Compatible Weyl chambers and generic flags. A $b^+$-compatible Weyl chamber is a Weyl chamber of the system $\Sigma(g, a)$ that is contained in $b^+$.

Lemma 3.3. Let $a^+ \subset b$ be any Weyl chamber of the system $\Sigma(g, a)$. Then there exists an element $\hat{w} \in \hat{W} \cap H^o$ such that

$$w \cdot a^+ \subset b^+.$$

Furthermore, if $\hat{w}' \in \hat{W} \cap H^o$ satisfies $w' \cdot a^+ \subset b^+$ then $\hat{w}$ and $\hat{w}'$ define the same element in $W$.

Proof. Let $\{\ell_1, \ldots, \ell_d\}$ be the total order on $C$ induced by the choice of $a^+$ and

$$C^+ = \{\ell_1^+, \ldots, \ell_p^+\} \quad \text{and} \quad C^- = \{\ell_1^-, \ldots, \ell_q^-\}$$

be the total orders on $C^+$ and $C^-$ determined by the choice of $b^+$.

We define the element $\hat{w}$ inductively. If $sg_o(\ell_1) = 1$ we define $\hat{w} \cdot \ell_1$ to be $\ell_1^+$, and if $sg_o(\ell_1) = -1$ we define $\hat{w} \cdot \ell_1$ to be $\ell_1^-$. Say that $sg_o(\ell_1) = 1$ (the other case being analogous). Now define $\hat{w} \cdot \ell_2$ to be $\ell_2^+$ if $sg_o(\ell_2) = 1$, or to be $\ell_2^-$ if $sg_o(\ell_2) = -1$. 


Procede inductively to obtain an element \( \hat{w} \in \hat{W} \) such that the restrictions of the total order
\[
\{ \hat{w} \cdot \ell_1, \ldots, \hat{w} \cdot \ell_d \}
\]
to both \( C^+ \) and \( C^- \) coincide with the total orders of equation (3.1). That is, the Weyl chamber \( w \cdot a^+ \) is \( b^+ \)-compatible. Furthermore, by construction we have \( sg_o(\hat{w} \cdot \ell_j) = sg_o(\ell_j) \) for every \( j = 1, \ldots, d \). Lemma 3.1 implies \( \hat{w} \in H^o \).

On the other hand, suppose that \( \hat{w}' \in \hat{W} \cap H^o \) satisfies \( \hat{w}' \cdot a^+ \subset b^+ \). Say \( sg_o(\ell_1) = 1 \) (the other case being analogous). Then since \( \hat{w}' \in H^o \) we have \( sg_o(\hat{w}' \cdot \ell_1) = 1 \). Furthermore, since \( \hat{w}' \cdot a^+ \subset b^+ \) the total order
\[
\{ \hat{w}' \cdot \ell_1, \ldots, \hat{w}' \cdot \ell_d \}
\]
on \( C \) must restrict to the total order of \( C^+ \) given by equation (3.1). We see then that \( \hat{w}' \cdot \ell_1 \) must be equal to \( \ell_1^+ = \hat{w} \cdot \ell_1 \). Proceeding inductively we conclude that \( \hat{w} \cdot \ell_j = \hat{w}' \cdot \ell_j \) for all \( j = 1, \ldots, d \), and this completes the proof. \( \square \)

Recall that there exists a \( W \)-equivariant identification between the set of Weyl chambers of the system \( \Sigma(\mathfrak{g}, \mathfrak{a}) \) and the set of \( (\mathfrak{o}\text{-generic}) \) flags determined by the lines of \( C \). If \( a^+ \subset \mathfrak{b} \) is such a Weyl chamber the corresponding flag is denoted by \( \xi_{a^+} \). Conversely, the Weyl chamber of the system \( \Sigma(\mathfrak{g}, \mathfrak{a}) \) determined by a flag \( \xi \) spanned by \( C \) will be denoted by \( a_\xi^+ \). A flag \( \xi \in F(V) \) is said to be \( b^+\text{-compatible} \) if it is spanned by the elements of \( C \) and the Weyl chamber \( a_\xi^+ \) is \( b^+ \)-compatible.

The following is a concrete instance of a result due to Matsuki [35, Section 3] and Rossmann [51, Theorem 13 and Corollaries 15 to 17].

**Proposition 3.4.** The map
\[
\xi \mapsto F(V)_\xi^o := H^o \cdot \xi
\]
defines a one to one correspondence between the set of \( b^+ \)-compatible flags and the set consisting on open orbits of the action \( H^o \circlearrowleft F(V) \).

For a \( b^+ \)-compatible Weyl chamber \( a^+ \) we use the notation
\[
F(V)_{a^+}^o := H^o \cdot \xi_{a^+}.
\]

**Proof of Proposition 3.4.** Let \( \xi \) and \( \xi' \) be two \( b^+ \)-compatible flags in the same \( H^o \)-orbit. By Proposition 2.8 we have \( sg_o(\ell_j^o(\xi)) = sg_o(\ell_j^o(\xi')) \) for every \( j = 1, \ldots, d \) and, as unordered sets, one has
\[
\{\ell_j^o(\xi), \ldots, \ell_d^o(\xi)\} = C = \{\ell_j^o(\xi'), \ldots, \ell_d^o(\xi')\}.
\]

By Lemma 3.1 there exists an element \( \hat{w} \in \hat{W} \cap H^o \) such that \( \hat{w} \cdot \ell_j^o(\xi) = \ell_j^o(\xi') \) for all \( j = 1, \ldots, d \). That is, \( w \cdot a^+_\xi = a^+_\xi' \) and Lemma 3.3 implies \( w = 1 \). Hence \( \xi = \xi' \).

For surjectivity, let \( \xi' \) be any \( \mathfrak{o}\text{-generic} \) flag. By Lemma 2.6 we can find an element \( h \in H^o \) such that \( h \cdot \ell_j^o(\xi') \in C \) for every \( j = 1, \ldots, d \). That is, \( h \cdot \xi' \) determines a Weyl chamber of the system \( \Sigma(\mathfrak{g}, \mathfrak{a}) \). By Lemma 3.3 the proof is complete. \( \square \)
4. \((p, q)\)-Cartan decomposition

The choice of a point \(\tau \in X_G\), a Cartan subspace \(a \subset \mathfrak{p}^\tau\) and a Weyl chamber \(a^+\) of \(\Sigma(g, a)\) induces a Cartan decomposition \(G = K^\tau \exp(a^+)K^\tau\) of \(G\) and a Cartan projection \(a^+ : G \to a^+\) (see e.g. Knapp [31, Chapter VI]). Geometrically, for all \(g \in G\) we have

\[d_{K^\tau}(\tau, g \cdot \tau) = \|a^+(g)\|,\]

where \(\|\cdot\|\) is the norm on \(\mathfrak{p}^\tau\) induced by the \(G\)-invariant Riemannian structure on \(X_G\). In this section we describe a way to generalize this picture to our context.

Fix a basepoint \(o \in \mathcal{Q}_{p,q}\) and let \(\mathcal{B}_o\) be the set consisting of \(o\)-orthogonal bases of lines of \(V\). Define

\[\mathcal{B}_{o,G} := \{g \in G : \exists \mathcal{C} \in \mathcal{B}_o \text{ such that } g \cdot \mathcal{C} \in \mathcal{B}_o\},\]

that is, \(\mathcal{B}_{o,G}\) is the set of elements in \(G\) that take some \(o\)-orthogonal basis of lines of \(V\) into an \(o\)-orthogonal basis of lines of \(V\). We remark here that \(\mathcal{B}_{o,G}\) is different from \(G\) and that it is not open, but has non empty interior (see Remark 4.3 below).

Fix a point \(\tau \in S_o\), a maximal subalgebra \(\mathfrak{b} \subset \mathfrak{p}^\tau \cap \mathfrak{q}^o\) and a Weyl chamber \(\mathfrak{b}^+ \subset \mathfrak{b}\) of the set \(\Sigma(g_{\mathfrak{q}^o}, \mathfrak{b})\). Let \(\mathcal{C} \in \mathcal{B}_o\) be the element determined by \(\mathfrak{b}\).

**Remark 4.1.** For every \(\hat{\omega} \in \hat{\mathcal{W}}\) the element \(m := \sigma^o(\hat{\omega}^{-1})\hat{\omega}\) belongs to \(M\). Indeed, as \(\tau\) and \(da^\circ\) commute we have that \(m\) belongs to \(K^\tau\). Furthermore, since \(da^\circ(X) = -X\) for all \(X \in \mathfrak{b}\) we have

\[\sigma^o(\hat{\omega}) \exp(X)\sigma^o(\hat{\omega}^{-1}) = \sigma^o(\hat{\omega} \exp(X)^{-1}\hat{\omega}^{-1}) = \hat{\omega} \exp(X)\hat{\omega}^{-1}\]

and this proves the claim.

Note moreover that when \(\mathbb{K} = \mathbb{C}\), the eigenvalues of \(m\) are real (hence equal to \(\pm 1\)).

The analogue of the Cartan decomposition is the following.

**Proposition 4.2.** Let \(g\) be an element of \(G\). Then the following are equivalent:

1. The element \(g\) belongs to \(\mathcal{B}_{o,G}\).
2. The element \(g\) belongs to \(H^p\hat{\mathcal{W}}\exp(\mathfrak{b}^+\mathfrak{H}^o)\).
3. The element \(\sigma^o(g^{-1})g\) is diagonalizable with real eigenvalues\(^6\).

In this case, let \(\tilde{\mathcal{C}}\) be an element of \(\mathcal{B}_o\). Then \(g \cdot \tilde{\mathcal{C}}\) belongs to \(\mathcal{B}_o\) if and only if \(\tilde{\mathcal{C}}\) diagonalizes \(\sigma^o(g^{-1})g\).

A decomposition \(g = h\hat{\omega}\exp(X)\tilde{h}\) of an element \(g \in \mathcal{B}_{o,G}\), where \(h, \tilde{h} \in \mathbb{H}^o\), \(\hat{\omega} \in \hat{\mathcal{W}}\) and \(X \in \mathfrak{b}^+\) will be called a \((p, q)\)-Cartan decomposition of \(g\).

**Proof of Proposition 4.2.** Suppose first that \(g\) belongs to \(\mathcal{B}_{o,G}\) and let \(\tilde{\mathcal{C}} \in \mathcal{B}_o\) be such that \(g \cdot \tilde{\mathcal{C}} \in \mathcal{B}_o\). Write \(\mathcal{C} = \mathcal{C}^+ \cup \mathcal{C}^-\) the decomposition of \(\tilde{\mathcal{C}}\) into positive and negative lines for the form \(o\). There exists \(h \in \mathbb{H}^o\) such that \(h^{-1}g \cdot \tilde{\mathcal{C}} = \mathcal{C}\). Further, we can take \(\hat{\omega}\) in \(\hat{\mathcal{W}}\) such that \(\hat{\omega}^{-1}h^{-1}g \cdot \tilde{\mathcal{C}}^\pm = \mathcal{C}^\pm\). Now let \(\tilde{h} \in \mathbb{H}^o\) be such that \(\tilde{h}^{-1} \cdot \mathcal{C} = \mathcal{C}\). We can assume that \(\hat{\omega}^{-1}h^{-1}g\hat{\omega}^{-1}\) fixes each line of \(\mathcal{C}\) and therefore one has

\[\hat{\omega}^{-1}h^{-1}g\hat{\omega}^{-1} = m \exp(X)\]

\(^6\)Formally, elements of \(G\) are not linear transformations of \(V\) but rather projective classes of linear transformations. Nevertheless, we can say that \(g \in G\) is diagonalizable if it preserves the elements of some basis of lines of \(V\). We say that this basis of lines diagonalizes the projective class \(g\).
for some $X \in \mathfrak{b}$ and $m \in M$. Since $M$ is contained in $\hat{W}$ and $X$ is conjugate to an element in $\mathfrak{b}^+$ by an element in $\hat{W} \cap \mathfrak{h}^o$, we conclude that $g$ belongs to $\mathfrak{h}^o \hat{W} \exp(\mathfrak{b}^+) \mathfrak{h}^o$.

Now suppose that $g$ admits a $(p,q)$-Cartan decomposition $g = h \hat{w} \exp(X) \hat{h}$. Then

$$\sigma^o(g^{-1}) = \hat{h}^{-1} \exp(X) \sigma^o(\hat{w}^{-1}) \exp(X) \hat{h}.$$  \hspace{1cm} (4.1)

By Remark 4.1 we conclude that $\sigma^o(g^{-1})$ is diagonalizable with real eigenvalues.

Finally, suppose that $\sigma^o(g^{-1})$ is diagonalizable with real eigenvalues. It is not hard to show that in this case $\sigma^o(g^{-1})$ must be diagonalizable in an $o$-orthogonal basis of lines $\hat{C}$ of $V$. Given $\ell \neq \ell'$ in $\hat{C}$ we have

$$\langle g \cdot \ell, g \cdot \ell' \rangle_o = \langle \sigma^o(g^{-1}) g \cdot \ell, g \cdot \ell' \rangle_o = \langle \ell, \ell' \rangle_o = 0,$$

where the above equalities hold up to scalar multiples. Hence $g \cdot \hat{C} \subset \mathcal{B}_o$.

\hspace{1cm} □

Remark 4.3. Take an element $k \in \exp(\mathfrak{t} \cap \mathfrak{q})$. Then $\sigma^o(k^{-1}) = k^2$ which, in general, is not diagonalizable with real eigenvalues. This shows that $\mathcal{B}_{o,G}$ is strictly contained in $G$ and that it is not open: one can approximate $1 \in \mathcal{B}_{o,G}$ by a sequence $\{k_n\}_n \subset \exp(\mathfrak{t} \cap \mathfrak{q})$. However, we will see that $\mathcal{B}_{o,G}$ has non empty interior (c.f. Remark 5.3 below).

4.1. $(p,q)$-Cartan projection. Define the $(p,q)$-Cartan projection $b^o : \mathcal{B}_{o,G} \rightarrow \mathfrak{b}^+$, where $g = h \hat{w} \exp(b^o(g)) \hat{h}$ is a $(p,q)$-Cartan decomposition of $g \in \mathcal{B}_{o,G}$. We now prove that this map is well defined.

Proposition 4.4. Let $g$ be an element of $\mathcal{B}_{o,G}$ and write

$$h_1 \hat{w}_1 \exp(X_1) \hat{h}_1 = g = h_2 \hat{w}_2 \exp(X_2) \hat{h}_2$$

two $(p,q)$-Cartan decompositions of $g$. Then $X_1 = X_2$.

Proof. Let $\mu$ be an eigenvalue of $\sigma^o(g^{-1})$. By Remark 4.1 and equation (4.1) we have that $|\mu|$ is an eigenvalue of $\exp(2X_i)$ for $i = 1, 2$. Moreover if $V_{|\mu|}^i$ is the corresponding eigenspace, the signature of $o$ restricted to $V_{|\mu|}^i$ is independent on $i = 1, 2$. We will show by induction that $V_{|\mu|}^i$ itself does not depend on $i = 1, 2$.

Recall that the choice of $\mathfrak{b}^+$ induces total orders on $C^+$ and $C^-$. An element $X \in \mathfrak{b}$ belongs to $\mathfrak{b}^+$ if and only if its eigenvalues on the lines of $C^+$ (resp. $C^-$) are ordered decreasingly, according to the total order of $C^+$ (resp. $C^-$).

Take an eigenvalue $\mu$ of $\sigma^o(g^{-1})$ such that $|\mu|$ is maximal and let $(p_{|\mu|}, q_{|\mu|})$ be the signature of $o$ restricted to $V_{|\mu|}^i$ (for $i = 1, 2$). It follows that $V_{|\mu|}^i$ is the sum of the first $p_{|\mu|}$ lines of $C^+$ and the first $q_{|\mu|}$ lines of $C^-$. Since this description is independent on $i = 1, 2$, we conclude that $V_{|\mu|}^1 = V_{|\mu|}^2$. Proceeding inductively over all eigenvalues of $\sigma^o(g^{-1})$, the result follows.

\hspace{1cm} □

4.2. Geometric interpretation.

Lemma 4.5. Let $g = h \hat{w} \exp(b^o(g)) \hat{h}$ be a $(p,q)$-Cartan decomposition of $g \in \mathcal{B}_{o,G}$. Then
\[ h \hat{w} \exp(b) \cdot \tau = h \exp(b) \cdot \tau \]
is a flat of \(X_G\) orthogonal to \(S^o\) at \(h \cdot \tau\) and to \(g \cdot S^o\) at \(h \hat{w} \exp(b^o(g)) \cdot \tau\). In particular,
\[ d_{X_G}(S^o, g \cdot S^o) = \|b^o(g)\|_b. \]

**Proof.** Indeed, this follows from the fact that \(\exp(b) \cdot \tau\) is orthogonal to \(S^o\) (resp. \(\exp(b^o(g)) \cdot S^o\)) at \(\tau\) (resp. \(\exp(b^o(g)) \cdot \tau\)) and the fact that \(\hat{W}\) fixes \(\tau\) and preserves \(\exp(b) \cdot \tau\) (see Figure 2).

\[ \tau \\
\hat{w}b \cdot \tau \\
g \cdot S^o \\
S^o \\
\exp(b) \cdot \tau \]

**Figure 2.** The proof of Lemma 4.5 for \(g = \hat{w}b\hat{h}\) (and \(b = \exp(b^o(g))\)). The red curves represent the submanifolds \(S^o, b \cdot S^o\) and \(g \cdot S^o\), that intersect orthogonally the flat \(\exp(b) \cdot \tau\).

5. \((p,q)\)-Cartan decomposition for elements with gaps

We now begin a more precise study of the \((p,q)\)-Cartan decomposition for elements that have "strong enough dynamics" on \(F(V)\) and \(\sigma\)-generic attractor and repellor. Subsection 5.1 is intended to fix notations and terminology. Subsection 5.2 is preparatory to the contents of Subsection 5.3, where we compute the \(b^+\)-compatible Weyl chamber that contains \(b^\sigma(g)\) for \(g\) as above. The contents of Subsection 5.4 will be crucial for our understanding of the new projection: we interpret the vector \(b^\sigma(g)\) using the Jordan projection of \(\sigma^\alpha(g^{-1})g\) and estimate \(b^\sigma(g)\) in terms of the Cartan projection \(a^\tau(g)\).

### 5.1. Reminders on Cartan and Jordan projections.

Given a Weyl chamber \(a^+ \subset b = a\) of the system \(\Sigma(g, a)\), let \(\Delta \subset \Sigma^+(g, a)\) be the set of simple roots. For each \(j = 1, \ldots, d\) we set \(a^+_j := \varepsilon_j \circ a^+\) (recall the notation introduced in Subsection 3.2.1).

An element \(g \in G\) is said to have a **gap of index** \(\Delta\) if for every \(\alpha \in \Delta\) one has \(\alpha(a^\tau(g)) > 0\). In this case, the **Cartan attractor** of \(g\) is the full flag \[
U^\tau(g) := k \cdot \xi_{a^+},
\]
where $\xi_{n+} \in F(V)$ is the flag determined by $a^+ \in a^+$ and $g = k \exp(a^+(g))l$ is a Cartan decomposition of $g$. Since $g$ has a gap of index $\Delta$, the Cartan attractor does not depend on the particular choice of the Cartan decomposition of $g$ (c.f. [31, Chapter VII]). Note that $g^{-1}$ also has a gap of index $\Delta$ and we denote

$$S^\tau(g) := U^\tau(g^{-1}).$$

This flag is called the Cartan repellor of $g$.

The choice of $\tau$ induces a continuously distance in each exterior power of $V$ and in $F(V)$. By abuse of notations, $d(\cdot, \cdot)$ will denote any of these distances. Let $j = 1, \ldots, d-1$ and $\varepsilon$ be a positive number. For a line $\ell \in P(\Lambda^j V)$ and a hyperplane $\vartheta \in P(\Lambda^j V^*)$ we let

$$b_\varepsilon(\ell) := \{\ell' \in P(\Lambda^j V) : \ d(\ell, \ell') \leq \varepsilon\}$$

and

$$B_\varepsilon(\vartheta) := \{\ell' \in P(\Lambda^j V) : \ d(\ell', \vartheta) \geq \varepsilon\},$$

where $d(\ell', \vartheta)$ denotes the minimal distance between $\ell'$ and lines contained in $\vartheta$.

The following remark is classical (see e.g. Horn-Johnson [25, Section 7.3 of Chapter 7]).

**Remark 5.1.** Fix a positive $\varepsilon$. There exists $L > 0$ such that for every $j = 1, \ldots, d-1$ and every $g \in G$ with

$$\min_{\alpha \in \Delta} \alpha(a^+(g)) > L$$

one has

$$\Lambda^j g : B_\varepsilon(\Lambda^j S^\tau(g)) \subset b_\varepsilon(\Lambda^j U^\tau(g)).$$

Recall that the Jordan projection $\lambda : G \to a^+$ can be defined by the formula

$$\lambda(g) := \lim_{n \to \infty} \frac{a^+(g^n)}{n}$$

for every $g \in G$. For each $j = 1, \ldots, d$ we set $\lambda_j := \varepsilon_j \circ \lambda$.

An element $g$ in $G$ is said to be proximal (on $P(V)$) if $\lambda_1(g) > \lambda_2(g)$ and is said to be loxodromic if $\Lambda^j g$ is proximal for every $j = 1, \ldots, d-1$. If $g$ is loxodromic, we let $g_+ \in F(V)$ (resp. $g_- \in F(V)$) denote the unique attracting (resp. repelling) fixed point of $g$ acting on $P(V)$. For every $j = 1, \ldots, d-1$, the attracting (resp. repelling) fixed line (resp. hyperplane) of $\Lambda^j g$ acting on $P(\Lambda^j V)$ is $\Lambda^j(g_+)$ (resp. $\Lambda^j(g_-)$).

In order to study products of loxodromic elements we will need the following quantified version of proximality. Given real numbers $0 < \varepsilon \leq r$, we say that a loxodromic element $g$ is $(r, \varepsilon)$-loxodromic if for every $j = 1, \ldots, d-1$ one has

$$d(\Lambda^j(g_+), \Lambda^j(g_-)) \geq 2r$$

and

$$g \cdot B_\varepsilon(\Lambda^j(g_-)) \subset b_\varepsilon(\Lambda^j(g_+)).$$

Since $\sigma^o$ preserves $K^*$ and acts as $-\text{id}$ on $b = a$, Lemma 2.5 implies the following.

**Corollary 5.2.** For every $g \in G$ one has

$$a^+(\sigma^o(g^{-1})) = a^+(g) \text{ and } \lambda(\sigma^o(g^{-1})) = \lambda(g).$$

Moreover, if $g$ has a gap of index $\Delta$ (resp. if $g$ is loxodromic) then

$$U^\tau(\sigma^o(g^{-1})) = S^\tau(g)^{\Delta_+} \text{ (resp. } \sigma^o(g^{-1})_+ = (g_-)^{\Delta_+}).$$
5.2. \((p, g)\)-Cartan attractors. We now focus on elements \(g \in G\) such that \(\sigma^o(g^{-1})g\) is loxodromic.

**Remark 5.3.** It is not hard to see that if \(\sigma^o(g^{-1})g\) is loxodromic then its eigenvalues must be real numbers. In particular, for all such \(g\) we have \(g \in \mathcal{B}_oG\). Moreover, since being loxodromic is an open condition in \(G\) we conclude that \(\mathcal{B}_oG\) has non empty interior.

By Proposition 4.2 and Remark 4.1, the element \(\sigma^o(g^{-1})g\) is loxodromic if and only if there exists a \(b^+\)-compatible Weyl chamber \(a^+\) for which one has
\[
g \in H^o W \exp(\text{int}(a^+)) H^o.
\]
In this case we set
\[
U^o(g) := (g \sigma^o(g^{-1}))_+ \in F(V).
\]
By similar reasons the element \(\sigma^o(g)g^{-1}\) is loxodromic as well and we denote
\[
S^o(g) := U^o(g^{-1}).
\]
Note that
\[
S^o(g) = (\sigma^o(g^{-1})g)_-.
\]

**Remark 5.4.** Suppose that \(g = h \hat{w} \exp(X) \hat{h}\) is a \((p, g)\)-Cartan decomposition of \(g\) such that \(X \in \text{int}(a^+)\) for some Weyl chamber \(a^+ \subset b^+\). Then
\[
U^o(g) = h \hat{w} \cdot \xi_{a^+} \quad \text{and} \quad S^o(g) = \hat{h}^{-1} \cdot \xi_{-a^+} = \hat{h}^{-1} \cdot (\xi_{a^+}^\perp).
\]
In particular, both \(U^o(g)\) and \(S^o(g)\) belong to the set \(F(V)^o\) of \(o\)-generic flags of \(V\).

**Lemma 5.5.** Let \(C \subset F(V)^o\) be a compact set. Then there exist \(0 < \varepsilon_0 \leq r_0\) such that for every \(0 < \varepsilon \leq r\) with \(\varepsilon \leq \varepsilon_0\) and \(r \leq r_0\) there exists a positive \(L\) with the following property: fix a \(b^+\)-compatible Weyl chamber \(a^+\). For every \(g \in G\) for which
\[
\min_{\alpha \in \Sigma^+_{\sigma^o(g, a)} \alpha(a^\perp(g))} > L
\]
holds, and such that \(U^\sigma(g) \in C\) and \(S^\sigma(g) \in C\), then \(\sigma^o(g^{-1})g\) is \((2r, 2\varepsilon)\)-loxodromic.
Furthermore one has
\[
d(U^o(g), U^\sigma(g)) \leq \varepsilon \quad \text{and} \quad d(S^o(g), S^\sigma(g)) \leq \varepsilon.
\]

**Proof.** We apply a “ping-pong” argument (see Figure 3 below). Namely, because of Lemma 2.6 there exists a positive \(r_0\) for which for every \(j = 1, \ldots, d - 1\) and every \(\xi \in C\) one has
\[
d(\Lambda^j \xi, (\Lambda^j \xi)^{\perp_{a^+}}) \geq 6r_0 \quad \text{and} \quad d(\Lambda^j (\xi^{\perp_{a^+}}), (\Lambda^j (\xi^{\perp_{a^+}}))^{\perp_{a^+}}) \geq 6r_0.
\]
By Remark 2.7 this implies
\[
d(\Lambda^j \xi, \Lambda^j (\xi^{\perp_{a^+}})) \geq 6r_0 \quad \text{and} \quad d(\Lambda^j (\xi^{\perp_{a^+}}), \Lambda^j (\xi^{\perp_{a^+}})) \geq 6r_0
\]
for all \(\xi \in C\). We now find \(\varepsilon_0 \leq r_0\) for which for every \(j = 1, \ldots, d - 1\) and every \(\xi \in C\) one has
\[
b_{\varepsilon_0}(\Lambda^j \xi) \subset B_{r_0}(\Lambda^j (\xi^{\perp_{a^+}})) \quad \text{and} \quad b_{\varepsilon_0}(\Lambda^j (\xi^{\perp_{a^+}})) \subset B_{r_0}(\Lambda^j (\xi^{\perp_{a^+}})).
\]
Fix \(0 < \varepsilon \leq r\) with \(\varepsilon \leq \varepsilon_0\) and \(r \leq r_0\). By Remark 5.1 we can find a positive \(L\) for which for every \(g\) such that
\[
\min_{\alpha \in \Sigma^+_{\sigma^o(g, a)} \alpha(a^\perp(g))} > L
\]
one has
\[ (5.3) \quad \Lambda^j g \cdot B_{\varepsilon} (\Lambda_j^i S^r(g)) \subset b_{\varepsilon}(\Lambda^j U^r(g)). \]
for every \( j = 1, \ldots, d - 1 \). Further, by Corollary 5.2 we have as well
\[ (5.4) \quad \Lambda^j \sigma^\alpha(g^{-1}) \cdot B_{\varepsilon} (\Lambda_j^i S^r(\sigma^\alpha(g^{-1}))) \subset b_{\varepsilon}(\Lambda^j U^r(\sigma^\alpha(g^{-1}))). \]
Now suppose moreover that \( U^r(g) \in C \) and \( S^r(g) \in C \). By equation (5.1) and Corollary 5.2 we have
\[ (5.5) \quad d \left( \Lambda^j U^r(\sigma^\alpha(g^{-1})), \Lambda_j^i S^r(g) \right) \geq 6r. \]
Now by equation (5.3) we have
\[ \Lambda^j (\sigma^\alpha(g^{-1})g) \cdot B_{\varepsilon}(\Lambda_j^i S^r(g)) \subset \Lambda^j \sigma^\alpha(g^{-1}) \cdot b_{\varepsilon}(\Lambda^j U^r(g)) \]
and by Corollary 5.2 and equation (5.2) this is contained in
\[ \Lambda^j \sigma^\alpha(g^{-1}) \cdot B_{\varepsilon}(\Lambda_j^i S^r(\sigma^\alpha(g^{-1}))). \]
By equation (5.4) we have therefore
\[ \Lambda^j (\sigma^\alpha(g^{-1})g) \cdot B_{\varepsilon}(\Lambda_j^i S^r(g)) \subset b_{\varepsilon}(\Lambda^j U^r(\sigma^\alpha(g^{-1}))). \]
By Benoist [4, Lemme 1.2], this inclusion together with equation (5.5) gives that
\[ \sigma^\alpha(g^{-1})g \text{ is } (2r, 2\varepsilon)\text{-loxodromic and} \]
\[ d((\sigma^\alpha(g^{-1})g)_-, S^r(g)) \leq \varepsilon. \]
Working with \( g\sigma^\alpha(g^{-1}) \) (instead of \( \sigma^\alpha(g^{-1})g \)) we can also assume
\[ d(U^r(g), U^r(g)) \leq \varepsilon. \]
\[ \Box \]

5.3. Computation of the Weyl chamber and the \( \hat{W} \)-coordinate. Recall that \( \mathcal{C} \) is the basis of lines determined by \( \mathfrak{b} \). Given two flags \( \xi \) and \( \xi' \) spanned by \( \mathcal{C} \), we denote by \( w_{\xi \xi'} \) the unique element of \( \mathbf{W} \) for which
\[ w_{\xi \xi'} \cdot \xi' = \xi. \]
We also denote by \( \iota_{\mathfrak{b}^+}(\xi) \) the flag determined by the \( \mathfrak{b}^+ \)-compatible Weyl chamber \( \iota_{\mathfrak{b}^+}(a_\xi^+) \).
Recall that \( F(V)_\xi^o \) denotes the (open) \( \mathbf{H}^o \)-orbit of \( \xi \).

**Proposition 5.6.** Let \( \xi_s \) and \( \xi_u \) be two \( \mathfrak{b}^+ \)-compatible flags. Fix two compact sets \( C_s \subset F(V)_{\xi_s}^o \) and \( C_u \subset F(V)_{\xi_u}^o \). Then there exists a positive \( L \) with the following property: let \( \mathfrak{a}^+ \) be a \( \mathfrak{b}^+ \)-compatible Weyl chamber. For every \( g \in G \) for which
\[ \min_{\alpha \in \Sigma^{\mathfrak{b}^+}(g, \mathfrak{a})} \alpha(\mathfrak{a}^+(g)) > L \]
holds, and such that \( S^r(g) \in C_s \) and \( U^r(g) \in C_u \), one has
\[ g \in \mathbf{H}^o w_{\xi_u \iota_{\mathfrak{b}^+}(\xi_u) \exp(\int \iota_{\mathfrak{b}^+}(a_\xi^+))} \mathbf{H}^o. \]
**Proof.** The proof is illustrated in Figure 4 below.
By Lemma 5.5 we can take a positive \( L \) such that for every \( g \) as in the statement one has
\[ U^\alpha(g) \in F(V)_{\xi_u}^o \text{ and } S^\alpha(g) \in F(V)_{\xi_u}^o. \]
If \( h\hat{w}\exp(b^o(g))\hat{h} \) is a \((p,q)\)-Cartan decomposition of \( g \) we then have

\[
U^o(g) = h\hat{w} \cdot b^o(g)_+ \in H^o \cdot \xi_u \quad \text{and} \quad S^o(g) = \hat{h}^{-1} \cdot b^o(g)_- \in H^o \cdot \xi_s.
\]

In particular,

\[
\hat{w} \cdot b^o(g)_+ \in H^o \cdot \xi_u \quad \text{and} \quad b^o(g)_- \in H^o \cdot \xi_s.
\]

We conclude that

\[
b^o(g)_+ = b^o(g)^{1-o}_+ \in H^o \cdot (\xi_s)^{1-o} = H^o \cdot \iota_{b^+}^\circ (\xi_s),
\]

because \( w_{b^+} \) belongs to \( H^o \) (c.f. Subsection 3.3). Since both \( b^o(g)_+ \) and \( \iota_{b^+}(\xi_s) \) are \( b^+ \)-compatible, Proposition 3.4 implies \( b^o(g)_+ = \iota_{b^+}(\xi_s) \) and therefore \( b^o(g) \) belongs to \( \text{int}(\iota_{b^+}(a_\xi^\circ)) \). Further, we have

\[
\hat{w} \cdot \iota_{b^+}(\xi_s) \in H^o \cdot \xi_u
\]

and Lemma 3.1 implies the existence of an element \( \hat{w}' \in \hat{W} \cap H^o \) such that

\[
\hat{w}'\hat{w} \cdot \iota_{b^+}(\xi_s) = \xi_u.
\]

Then \( w'w = w_{\xi_u \iota_{b^+}(\xi_s)} \) and the proof is complete. \( \square \)
Corollary 5.7. Let $\xi$ be a $b^+\!$-compatible flag and $C \subset F(V)_\xi^0$ be a compact set. Then there exists a positive $L$ with the following property: let $a^+\!$ be a $b^+\!$-compatible Weyl chamber. For every $g \in G$ for which

$$\min_{\alpha \in \Sigma^+(g,a)} \alpha(a^+(g)) > L$$

holds, and such that $S^+(g) \in C$ and $U^+(g) \in C$, one has

$$g \in H^o w_{\xi_b^+}(\xi) \exp(\text{int}(t_{b^+}(a^+_c)))H^o.$$

5.4. **Linear algebraic interpretation and first estimates.** Fix a $b^+\!$-compatible Weyl chamber $a^+\!$. Remark 4.1 and equation (4.1) imply the following: for every element $g$ of $\mathcal{B}_{a,G}$ there exists an element $w_g \in W$ such that

$$b^o(g) = \frac{1}{2} w_g \cdot \lambda(\sigma^o(g^{-1})g).$$

In particular, one has:

$$\|b^o(g)\|_b = \frac{1}{2} \|\lambda(\sigma^o(g^{-1})g)\|_b.$$
Whenever $g$ has a (sufficiently large) gap of index $\Delta$ and $o$-generic Cartan attractor and repellor we have a more precise result.

**Proposition 5.8.** Let $\xi_s$ be a $b^+$-compatible flag and fix compact sets $C_s \subset F(V)_{\xi_s}^o$ and $C \subset F(V)^o$. Then there exists a positive $L$ with the following property: for every $g \in G$ for which

$$\min_{\alpha \in \Sigma^{\pm}(g, a)} \alpha(a^\tau(g)) > L$$

holds, and such that $S^\tau(g) \in C_s$ and $U^\tau(g) \in C$, one has

$$b^\alpha(g) = \frac{1}{2} w_{i_b+}(\xi_s) \cdot \lambda(\alpha^\alpha(g^{-1})g).$$

In particular, if $i_{b^+}(\xi_s) = \xi_{a^+}$ we have

$$b^\alpha(g) = \frac{1}{2} \lambda(\alpha^\alpha(g^{-1})g).$$

**Proof.** Apply Proposition 5.6 to each intersection of $C$ with each open orbit of the action $H^o \cap F(V)$. For every $g$ as in the statement we know that there exists some $X \in \text{int}(a^\tau)$ such that

$$b^\alpha(g) = w_{i_b+}(\xi_s) \cdot X.$$

Hence

$$\frac{1}{2} \lambda(\alpha^\alpha(g^{-1})g) = \lambda(\exp(b^\alpha(g))) = (w_{i_b+}(\xi_s) \cdot a^\tau) \cdot b^\alpha(g).$$

\[\square\]

**Proposition 5.9.** Let $\xi_s$ be a $b^+$-compatible flag and fix compact sets $C_s \subset F(V)_{\xi_s}^o$ and $C \subset F(V)^o$. Then there exist positive numbers $L$ and $D$ with the following property: for every $g \in G$ for which

$$\min_{\alpha \in \Sigma^{\pm}(g, a)} \alpha(a^\tau(g)) > L$$

holds, and such that $S^\tau(g) \in C_s$ and $U^\tau(g) \in C$, one has

$$\|b^\alpha(g) - w_{i_b+}(\xi_s) \cdot a^\tau(g)\| \leq D.$$

**Proof.** As we saw in the proof of Lemma 5.5, there exists a positive constant $r_0$ such that for every $j = 1, \ldots, d-1$ and every $g$ as in the statement one has

$$d\left(\Lambda_j U^\tau(g), \Lambda_j S^\tau(\alpha^\alpha(g^{-1}))\right) \geq r_0.$$

It is not hard to show (see e.g. [6, Lemma A.7]) that this implies the existence of a constant $D$ such that, for all $j = 1, \ldots, d-1$,

$$|a^\tau_j(\alpha^\alpha(g^{-1})g) - a^\tau_j(\alpha^\alpha(g^{-1})) - a^\tau_j(g)| \leq D.$$

By Corollary 5.2 we have

$$\frac{1}{2} |a^\tau_j(\alpha^\alpha(g^{-1})g) - a^\tau_j(g)| \leq D/2$$

and we conclude that, up to changing $D$ by a larger constant if necessary, one has

$$\|\frac{1}{2} a^\tau(\alpha^\alpha(g^{-1})g) - a^\tau(g)\| \leq D.$$

Fix $r$, $\varepsilon$ and $L$ as in Lemma 5.5 and Proposition 5.8 (for each intersection of $C$ with $F(V)^o$). For every $g$ as in the statement we know that $\alpha^\alpha(g^{-1})g$ is $(2r, 2\varepsilon)$-loxodromic and therefore we can enlarge $D$ if necessary in order to have

$$\|\frac{1}{2} a^\tau(\alpha^\alpha(g^{-1})g) - \frac{1}{2} \lambda(\alpha^\alpha(g^{-1})g)\| \leq D.$$
6. BUSEMANN COCYCLES

6.1. τ-Busemann cocycle. For future reference we begin by recalling the definition of the Busemann cocycle of G.

Let N (resp. P\(_{a^+}\)) be the unipotent radical (resp. minimal parabolic) associated to \(a^+\). Quint [48] introduces the Busemann cocycle \(\beta^\tau\) of G

\[ \beta^\tau : G \times F(V) \to \mathfrak{b} \]

which is defined by means of the Iwasawa decomposition of G. Indeed, for \(g \in G\) and \(\xi \in F(V)\) the Busemann cocycle is characterized by the equality

\[ gk = l \exp(\beta^\tau(g, \xi))n \]

where \(k, l \in K^\tau, n \in N\) and \(k \cdot \xi_{a^+} = \xi\). Note that for every \(g_1, g_2 \in G\), and every \(\xi \in F(V)\) one has

\[ \beta^\tau(g_1g_2, \xi) = \beta^\tau(g_1, g_2 \cdot \xi) + \beta^\tau(g_2, \xi). \]

In this paper we call this cocycle the τ-Busemann cocycle of G.

6.2. \(\omega\)-Busemann cocycle. For \(j = 1, \ldots, d-1\) we denote by \(\chi_j \in \mathfrak{b}\) the highest weight of the exterior power representation \(\Lambda^j\). Let

\[ (G \times F(V))^\omega := \{(g, \xi) \in G \times F(V)^\omega : g \cdot \xi \in F(V)^\omega\}. \]

Define the \(\omega\)-Busemann cocycle of G

\[ \beta^\omega : (G \times F(V))^\omega \to \mathfrak{b} \]

by the equations for \(j = 1, \ldots, d-1\):

\[ \chi_j(\beta^\omega(g, \xi)) := \frac{1}{2} \log \left| \frac{\langle \Lambda^j \cdot v, \Lambda^j \cdot v \rangle_{o_j}}{\langle v, v \rangle_{o_j}} \right|, \]

where \(\langle \cdot, \cdot \rangle_{o_j}\) is any form representing \(o_j\) and \(v\) is any non zero vector in the line \(\Lambda^j \cdot \xi\). Thanks to Lemma 2.6 the map \(\beta^\omega\) is well defined.

The proof of the following is straightforward.

Lemma 6.1. Let \(g_1\) and \(g_2\) be two elements of G, \(\xi\) be a flag in V and suppose that \((g_1g_2, \xi)\) and \((g_2, \xi)\) belong to \((G \times F(V))^\omega\). Then

\[ \beta^\omega(g_1g_2, \xi) = \beta^\omega(g_1, g_2 \cdot \xi) + \beta^\omega(g_2, \xi). \]

Remark 6.2. The \(\omega\)-Busemann cocycle generalizes the τ-Busemann cocycle of G, in the sense that whenever \(pq = 0\) and \(o = \tau\) one has

\[ (G \times F(V))^\omega = G \times F(V) \]

and \(\beta^\omega\) coincides with \(\beta^\tau\) (c.f. Quint [48, Lemma 6.4]). Since we are assuming \(pq \neq 0\), the set \((G \times F(V))^\omega\) does not coincide with \(G \times F(V)\), but we still have a relation between \(\beta^\omega\) and \(\beta^\tau\): there exists a smooth function

\(\beta^\omega\). Sometimes also called the Iwasawa cocycle of G.
for which one has
\[ V_{\sigma r}(g \cdot \xi) - V_{\sigma r}(\xi) = \beta^o(g, \xi) - \beta^r(g, \xi) \]
for every pair \((g, \xi) \in (G \times F(V))^o\). Indeed, it suffices to take \(V_{\sigma r}\) defined by the formulas
\[
\chi_j(V_{\sigma r}(\xi)) := \frac{1}{2} \log \left| \frac{\langle \Lambda^j g \cdot v, \Lambda^j g \cdot v \rangle_{a_j}}{\langle v, v \rangle_{a_j}} \right|
\]
for \(j = 1, \ldots, d - 1\) and \(0 \neq v \in \Lambda^j \xi_j\), and for the inner product \(\tau_j\) on \(\Lambda^j V\) induced by \(\tau\).

6.2.1. \((p, q)\)-Iwasawa decomposition. The Lie theoretic description of the \(\alpha\)-Busemann cocycle is as follows. A \((p, q)\)-Iwasawa decomposition of an element \(g \in G\) is a decomposition of the form
\[
g = h \hat{w} \exp(X)n
\]
where \(h \in H^o\), \(\hat{w} \in \hat{W}\), \(X \in b\) and \(n \in N\). Note that if this decomposition holds, then the element \(X\) is uniquely determined: for every \(j = 1, \ldots, d - 1\) one has
\[
\chi_j(X) = \frac{1}{2} \log \left| \frac{\langle \Lambda^j \hat{w} \cdot v, \Lambda^j \hat{w} \cdot v \rangle_{a_j}}{\langle v, v \rangle_{a_j}} \right|
\]
where \(\langle \cdot, \cdot \rangle_{a_j}\) is any representative of \(a_j\) and \(v\) is any non zero vector in the line \(\Lambda^j \xi_{a^+}\). Indeed, this follows from the fact that \(\Lambda^j H^o\) preserves the form \(a_j\) and the fact that one has the equality
\[
|\langle \Lambda^j \hat{w} \cdot v, \Lambda^j \hat{w} \cdot v \rangle_{a_j}| = |\langle v, v \rangle_{a_j}|
\]
for every \(j = 1, \ldots, d - 1\) (c.f. Lemma 3.1).

On the other hand, if an element \(g \in G\) admits a \((p, q)\)-Iwasawa decomposition then one has
\[
(g, \xi_{a^+}) \in (G \times F(V))^o.
\]
Conversely, we have the following.

**Lemma 6.3.** Suppose that the pair \((g, \xi_{a^+})\) belongs to \((G \times F(V))^o\). Consider an element \(\hat{w} \in \hat{W}\) for which the Weyl chamber \(\hat{w} \cdot a^+\) is \(b^+\)-compatible and such that \(g \cdot \xi_{a^+} \in F(V)^o_{\hat{w} \cdot a^+}\). Then \(g\) admits a \((p, q)\)-Iwasawa decomposition of the form
\[
g = h \hat{w} \exp(X)n.
\]

**Proof.** There exists \(\hat{h} \in H^o\) such that \(g \cdot \xi_{a^+} = \hat{h} \hat{w} \cdot \xi_{a^+}\) and therefore \(\hat{w}^{-1} \hat{h}^{-1} g\) belongs to \(P_{a^+}\). Since \(P_{a^+} = M \exp(b) N\) we conclude that
\[
\hat{w}^{-1} h = m \exp(X)n
\]
for some \(m \in M\), \(X \in b\) and \(n \in N\). Now \(\hat{w}m = m' \hat{w}\) for some \(m' \in M \subset H^o\) and the lemma follows.

**Corollary 6.4.** Let \((g, \xi)\) be an element in \((G \times F(V))^o\) and take \(h' \in H^o\) and \(\hat{w}' \in \hat{W}\) such that \(h' \hat{w}' \cdot \xi_{a^+} = \xi\). Then \(gh' \hat{w}'\) admits a \((p, q)\)-Iwasawa decomposition of the form
\[
gh' \hat{w}' = h \hat{w} \exp(\beta^o(g, \xi))n.
\]
Proof. Since \( gh'\hat{w} \cdot \xi_{a^+} \) is \( o \)-generic, there exists \( \hat{w} \in \hat{W} \) such that \( \hat{w} \cdot \xi_{a^+} \) is \( b^+ \)-compatible and \( gh'\hat{w}' \cdot \xi_{a^+} \in F(V)_{a^+}^\circ \). By Lemma 6.3 we find a \((p,q)\)-Iwasawa decomposition of \( gh'\hat{w}' \) of the form
\[
gh'\hat{w}' = h\hat{w} \exp(X)n.
\]

Now the element \( X \) in this decomposition is characterized by the equalities
\[
\chi_j(X) = \frac{1}{2} \log \left| \frac{\langle \Lambda^j(gh'\hat{w}') \cdot v, \Lambda^j(gh'\hat{w}') \cdot v \rangle_{o_j}}{\langle v, v \rangle_{o_j}} \right|,
\]
where \( v \) is any non-zero vector in the line \( \Lambda^j\xi_{a^+} \). Since \( \Lambda^jH^o \) preserves the form \( o_j \), equality (6.1) finishes the proof.

□

6.2.2. Dual cocycle. Let \( \iota_{a^+: b} \) be the opposition involution associated to the choice of \( a^+: b \):
\[
\iota_{a^+: b} : X \mapsto -w_{a^+} \cdot X,
\]
where \( w_{a^+} \in W \) is the unique element that takes the Weyl chamber \( -a^+ \) to \( a^+ \).

Remark 6.5. Recall that the choice of \( a^+ \) determines a total order \( C = \{\ell_1, \ldots, \ell_d\} \) in the basis of lines associated to \( b \). Even though we will not use it in the future, we mention that it is possible to show that the equality \( \iota_b = \iota_{a^+} \) holds if and only if
\[
\sg_{o_j}(\ell_j) = \sg_{o_j}(\ell_{d-j+1})
\]
holds for every \( j = 1, \ldots, d \). In particular, such a choice of \( a^+ \) is not always possible.

In higher rank, cocycles come usually in pairs (c.f. [53, 54]). The following corollary gives an explicit description of the cocycle “dual” to \( \beta^o \).

Corollary 6.6. Let \((g, \xi)\) be an element in \((G \times F(V))^o\). Then \((\sigma^o(g), \xi^\perp_o)\) belongs to \((G \times F(V))^o\) and one has
\[
\beta^o(\sigma^o(g), \xi^\perp_o) = \iota_{a^+} \circ \beta^o(g, \xi).
\]

Proof. Lemma 2.5 implies that \((\sigma^o(g), \xi^\perp_o)\) belongs to \((G \times F(V))^o\). Further, let \( h' \in H^o \) and \( \hat{w}' \in \hat{W} \) be two elements such that \( h'\hat{w}' \cdot \xi_{a^+} = \xi \). By Corollary 6.4 we can write
\[
gh'\hat{w}' = h\hat{w} \exp(\beta^o(g, \xi))n
\]
and therefore
\[
\sigma^o(g) = h\sigma^o(\hat{w}) \exp(-\beta^o(g, \xi))\sigma^o(n)\sigma^o(\hat{w}')^{-1}(h')^{-1}.
\]
Now by Lemma 2.5 we have
\[
\sigma^o(\hat{w}')^{-1}(h')^{-1} \cdot (\xi^\perp_o) = \xi^\perp_o = w_{a^+} \cdot \xi_{a^+}
\]
and since \( \sigma^o(n)w_{a^+} = w_{a^+}n' \) for some \( n' \in N \), the result follows.

□
6.2.3. Geometric interpretation. We now discuss a geometric interpretation for the $o$-Busemann cocycle. This is not formally needed for the reminder of the paper, and the reader not interested in this discussion may go directly to Subsection 6.3.

Define a map

$$
\Pi^o : F(V)^o \to S^o
$$

in the following way: given $\xi \in F(V)^o$, consider the $o$-orthogonal basis of lines of $V$

$$
\{\ell_1^o(\xi), \ldots, \ell_d^o(\xi)\}
$$

given by Proposition 2.6. Define $\Pi^o(\xi)$ to be the unique element of $S^o$ for which this basis of lines is orthogonal. Geometrically, the projection $\Pi^o(\xi)$ is the intersection between $S^o$ and the unique flat of $X_G$ which is orthogonal to $S^o$ and that contains a Weyl chamber “asymptotic” to $\xi$.

For every $(g,\xi) \in (G \times F(V))^o$ one has

$$
\Pi^{g^{-1}o}(\xi) = g^{-1} \cdot \Pi^o(g \cdot \xi).
$$

On the other hand, the Busemann function (in $X_G$) is the map

$$
\beta : X_G \times X_G \times F(V) \to b
$$

given by

$$
(\tau_1, \tau_2, \xi) \mapsto \beta_\xi(\tau_1, \tau_2) := \beta^\tau(\gamma_1^{-1}, \xi) - \beta^\tau(\gamma_2^{-1}, \xi),
$$

where $g_i \cdot \tau = \tau_i$ for $i = 1, 2$. A geometric interpretation of the $o$-Busemann cocycle is given by the following proposition (c.f. Figure 5 below).

Proposition 6.7. For every $(g,\xi) \in (G \times F(V))^o$ one has

$$
\beta^o(g, \xi) = \beta_\xi(\Pi^{g^{-1}o}(\xi), \Pi^o(\xi)).
$$

Proof. Let $h' \in H^o$ and $\hat{w}' \in \hat{W}$ be such that $h' \hat{w}' \cdot \xi_{a+} = \xi$ and write

$$
gh'\hat{w}' = h\hat{w} \exp(\beta^o(g,\xi))n.
$$

Equivariance property (6.2) gives the following:

$$
\Pi^{g^{-1}o}(\xi) = g^{-1}h\hat{w} \cdot \tau \text{ and } \Pi^o(\xi) = h'\hat{w}' \cdot \tau.
$$

We then have

$$
\beta_\xi(\Pi^{g^{-1}o}(\xi), \Pi^o(\xi)) = \beta^\tau(\exp(\beta^o(g,\xi))n, \xi_{a+}) = \beta^o(g,\xi).
$$

\[ \square \]

6.3. $o$-Gromov product. We now introduce the “Gromov product” associated to the pair $(\tau_{a+} \circ \beta^o, \beta^o)$. Its definition goes as follows. Let

$$
(F(V)^o)^{(2)} := \{(\xi, \xi') \in F(V)^o \times F(V)^o : \xi \text{ is transverse to } \xi'\}.
$$

For an element $(\xi, \xi')$ in $(F(V)^o)^{(2)}$ and $j = 1, \ldots, d-1$ we know by Remark 2.7 that the hyperplane $(\Lambda^j(\xi_{a+}))^\perp$ is transverse to the line $\Lambda^j\xi'$. Further, by Lemma 2.6 the lines $\Lambda^j(\xi_{a+})$ and $\Lambda^j\xi'$ are not isotropic for the form $\sigma_j$. Therefore the following $o$-Gromov product, that naturally generalizes the one introduced by Sambarino in [53], is well defined: let

$$
G_\sigma : (F(V)^o)^{(2)} \to b
$$

be defined by the equalities

$$
\chi_j(G_\sigma(\xi, \xi')) := \frac{1}{2} \log \left| \frac{(v, v')_{\sigma_j}(v, v')_{\sigma_j}}{(v, v)_{\sigma_j}(v', v')_{\sigma_j}} \right|
$$
for every \( j = 1, \ldots, d - 1 \), where \( v \) (resp. \( v' \)) is any non zero vector in the line \( \Lambda_j(\xi_{-o}) \) (resp. \( \Lambda_j\xi' \)).

The classical relation between Busemann functions and Gromov products is still satisfied in our framework.

Lemma 6.8. Let \((\xi,\xi') \in (F(V)^o)^{(2)} \) and \( g \in G \) be an element such that \((g,\xi)\) and \((g,\xi')\) belong to \((G \times F(V))^o\). Then the following equality holds:

\[
g_o(g \cdot \xi, g \cdot \xi') - g_o(\xi, \xi') = -(\iota a^+ o \beta^o(g,\xi) + \beta^o(g,\xi')).
\]

Proof. Fix \( j = 1, \ldots, d - 1 \) and note that, by equation (2.1) and Lemma 2.5, if \( v \in \Lambda_j(\xi_{-o}) \) then

\[
\sigma^o(\Lambda_j g) \cdot v \in \Lambda_j((g \cdot \xi)_{-o}).
\]

Let \( v' \in \Lambda_j \xi' \) be a non zero vector. We have

\[
\chi_j(g_o(g \cdot \xi, g \cdot \xi')) = \frac{1}{2} \log \left| \frac{\langle \sigma^o(\Lambda_j g) \cdot v, \Lambda_j g \cdot v' \rangle_o \langle \sigma^o(\Lambda_j g) \cdot v, \Lambda_j g \cdot v' \rangle_o}{\langle \sigma^o(\Lambda_j g) \cdot v, \sigma^o(\Lambda_j g) \cdot v \rangle_o \langle \Lambda_j g \cdot v', \Lambda_j g \cdot v' \rangle_o} \right|
\]

\[
= \frac{1}{2} \log \left| \frac{\langle v, v' \rangle_o \langle v, v' \rangle_o}{\langle \sigma^o(\Lambda_j g) \cdot v, \sigma^o(\Lambda_j g) \cdot v \rangle_o \langle \Lambda_j g \cdot v', \Lambda_j g \cdot v' \rangle_o} \right|
\]

where the last equality holds by definition of \( \sigma^o \). If we subtract to the previous equality the number

\[
\chi_j(g_o(\xi, \xi')) = \frac{1}{2} \log \left| \frac{\langle v, v' \rangle_o \langle v, v' \rangle_o}{\langle v, v \rangle_o \langle v', v' \rangle_o} \right|
\]

we obtain that \( \chi_j(g_o(g \cdot \xi, g \cdot \xi')) - \chi_j(g_o(\xi, \xi')) \) equals

\[
\frac{1}{2} \log \left| \frac{\langle v, v \rangle_o \langle v', v' \rangle_o}{\langle \sigma^o(\Lambda_j g) \cdot v, \sigma^o(\Lambda_j g) \cdot v \rangle_o \langle \Lambda_j g \cdot v', \Lambda_j g \cdot v' \rangle_o} \right|
\]
and by Corollary 6.6 the result follows. □

We now discuss a geometric interpretation for the $o$-Gromov product, that will be of central importance in Section 8 (c.f. Corollary 8.5). Let $B$ be the vector valued cross-ratio, defined by Benoist [4, p. 6] in the following way. Given a 4-tuple $(\xi_1, \xi_2, \xi_3, \xi_4) \in F(V)^4$ such that $(\xi_i, \xi_k)$ belongs to $F(V)^{(2)}$ for all $(i, k) \in \{(1, 2), (1, 4), (2, 3), (3, 4)\}$, the vector

$$B(\xi_1, \xi_2, \xi_3, \xi_4) \in b$$

is defined by the following equalities for $j = 1, \ldots, d - 1$:

$$\chi_j(B(\xi_1, \xi_2, \xi_3, \xi_4)) := \log \left| \frac{\varphi_1(v_4) \varphi_3(v_2)}{\varphi_1(v_2) \varphi_3(v_4)} \right|,$$

where for $i = 2, 4$, $v_i$ is any non zero vector in the line $\Lambda^j \xi_i \in P(\Lambda^j V)$ and for $k = 1, 3$, $\varphi_k$ is any non zero linear functional in the line $\Lambda^j \xi_k \in P(\Lambda^j V^*)$.

Corollary 6.9. For every $(\xi, \xi') \in (F(V)^o)^{(2)}$ the following equality holds:

$$G_o(\xi, \xi') = -\frac{1}{2} B((\xi')^{\perp_o}, \xi, \xi').$$

Proof. Let $j = 1, \ldots, d - 1$ and $v \in \Lambda^j (\xi^{\perp_o})$ and $v' \in \Lambda^j (\xi')$ be non zero vectors. Then by Remark 2.7 we have

$$\langle v, \cdot \rangle_{o_j} \in \Lambda^j \xi$$ and $$\langle v', \cdot \rangle_{o_j} \in \Lambda^j ((\xi')^{\perp_o}).$$

It follows that

$$\chi_j(B((\xi')^{\perp_o}, \xi, \xi')) = \log \left| \frac{\langle v', v \rangle_{o_j} \langle v, v \rangle_{o_j}}{\langle v', v \rangle_{o_j} \langle v, v \rangle_{o_j}} \right|$$

and the result is proven. □

7. $(p, q)$-Cartan projection for Anosov representations

In this section we begin the study of asymptotic properties of the $(p, q)$-Cartan projection for elements in the image of a $\Delta$-Anosov representation. In Subsection 7.1 we briefly recall this notion and some of its main features. In Subsection 7.2 we introduce the subset $\Omega\rho$ and discuss some examples. In Subsection 7.3 we prove Corollary A and in Subsection 7.4 we describe the $(p, q)$-asymptotic cone.

7.1. Reminders on Anosov representations. A lot of work has been done in order to simplify Labourie’s original definition of Anosov representations. Here we follow mainly the work of Bochi-Potrie-Sambarino [6], Guichard-Guérinod-Kassel-Wienhard [22] and Kapovich-Leeb-Porti [26].

Let $\Gamma$ be a finitely generated group. Consider a finite symmetric generating set $\mathcal{S}$ of $\Gamma$ and take $|\cdot|_\Gamma$ to be the associated word length: for $\gamma$ in $\Gamma$, it is the minimum number required to write $\gamma$ as a product of elements of $\mathcal{S}$. This number depends on the choice of $\mathcal{S}$. However, the set $\mathcal{S}$ will be fixed from now on hence we do not emphasize the dependence on this choice in the notation.

8This number depends on the choice of $\mathcal{S}$. However, the set $\mathcal{S}$ will be fixed from now on hence we do not emphasize the dependence on this choice in the notation.
Fix a Cartan decomposition $G = K\exp(a^+)K^+$ and let $\Delta$ be the set of simple roots. A representation $\rho : \Gamma \to G$ is said to be $\Delta$-Anosov if there exist positive constants $c$ and $c'$ such that for all $\gamma \in \Gamma$ and all $\alpha \in \Delta$ one has
\begin{equation}
\alpha(a^+(\rho\gamma)) \geq c|\gamma|_\Gamma - c'.
\end{equation}

By Kapovich-Leeb-Porti [28, Theorem 1.4] (see also [6, Section 3]), condition (7.1) implies that $\Gamma$ is word hyperbolic. In this paper we assume that $\Gamma$ is non elementary. Let $\partial_\infty \Gamma$ be the Gromov boundary of $\Gamma$ and $\Gamma_H$ be the set of infinite order elements in $\Gamma$. Every $\gamma$ in $\Gamma_H$ has exactly two fixed points in $\partial_\infty \Gamma$: the attractive one denoted by $\gamma_+$ and the repelling one denoted by $\gamma_-$. The dynamics of $\gamma$ on $\partial_\infty \Gamma$ is of type “north-south”.

As shown in [6, 22, 26], a central feature about $\Delta$-Anosov representations is that they admit a continuous equivariant map
\[ \xi_\rho : \partial_\infty \Gamma \to F(V) \]
which is transverse, i.e. for every $x \neq y$ in $\partial_\infty \Gamma$ one has
\begin{equation}
(\xi_\rho(x), \xi_\rho(y)) \in F(V)^2.
\end{equation}
Moreover, this map is dynamics-preserving, i.e. for every $\gamma$ in $\Gamma_H$ the element $\xi_\rho(\gamma_+)$ (resp. $\xi_\rho(\gamma_-)$) is an attractive (resp. repelling) fixed point of $\rho\gamma$ acting on $F(V)$. It follows that $\rho\gamma$ is loxodromic with
\[ \xi_\rho(\gamma_\pm) = (\rho\gamma)_\pm. \]
In particular, $\xi_\rho$ is injective and uniquely determined by $\rho$: it is called the limit map of $\rho$. This map varies in a continuous way with the representation and is Hölder continuous (see Guichard-Wienhard [23, Theorem 5.13] and Bridgeman-Canary-Labourie-Sambarino [8, Theorem 6.1]).

The limit set of $\rho$ is the image of $\xi_\rho$ and admits the following useful characterization (see [22, Theorem 5.3] or [6, Subsection 3.4] for a proof).

**Proposition 7.1.** Let $\rho : \Gamma \to G$ be a $\Delta$-Anosov representation. Then the limit set of $\rho$ coincides with the set of accumulation points of sequences of the form $\{ U^*(\rho\gamma_n) \}_n$, where $\gamma_n \to \infty$. Moreover, given a (continuous) distance $d(\cdot, \cdot)$ on $F(V)$ and a positive $\varepsilon$, one has
\[ d(U^*(\rho\gamma), (\rho\gamma)_+) < \varepsilon \]
and
\[ d(S^*(\rho\gamma), (\rho\gamma)_-) < \varepsilon \]
for every $\gamma \in \Gamma_H$ with $|\gamma|_\Gamma$ large enough.

Anosov representations have strong proximality properties (c.f. [23, Subsection 5.2]). We now establish one of them that will be useful in Section 8: it will provide the correct framework to estimate the geometric quantity involved in our counting functions.

**Lemma 7.2** (c.f. Sambarino [53, Lemma 5.7]). Let $\rho : \Gamma \to G$ be a $\Delta$-Anosov representation and fix real numbers $0 < \varepsilon \leq r$. Then there exists a positive $L$ with the following property: for every $\gamma \in \Gamma_H$ satisfying $|\gamma|_\Gamma > L$ and such that
\[ d(\Lambda_j^1(\rho\gamma)_+, \Lambda_j^1(\rho\gamma)_-) \geq 2r \]
holds for every $j = 1, \ldots, d - 1$, one has that $\rho\gamma$ is $(r, \varepsilon)$-loxodromic.

**Proof.** Consider a sequence $\gamma_n \to \infty$ in $\Gamma_H$ such that
\[ |\gamma_i|_\Gamma \to \infty \text{ as } n \to \infty. \]
for all $n$ and $j$. By Proposition 7.1, for every $n$ large enough the following holds

$$B_\varepsilon(\Lambda_j^+(\rho\gamma_n)) \subset B_\varepsilon(\Lambda_j^+(\rho\gamma_n)) + B_\varepsilon(\Lambda_j^+(\rho\gamma_n)) \subset B_\varepsilon(\Lambda_j^+(\rho\gamma_n))$$

By Remark 5.1 and equation (7.1) the condition

$$\rho\gamma_n \cdot B_\varepsilon(\Lambda_j^+(\rho\gamma_n)) \subset B_\varepsilon(\Lambda_j^+(\rho\gamma_n))$$

is satisfied for sufficiently large $n$.

\[\square\]

7.2. The set $\Omega_\rho$. Given a $\Delta$-Anosov representation $\rho$ define the (open) set

$$\Omega_\rho := \{o \in \mathbb{Q}_{p,q} : \xi_\rho(\partial_\infty \Gamma) \subset F(V)^o\}.$$

Let us discuss some examples of $\Delta$-Anosov representations into $G$ for which the set $\Omega_\rho$ is non-empty. Observe that since the limit map of an Anosov representation varies in a continuous way, if $\Omega_\rho$ is non empty for some specific $\rho$ then the same will hold for every small enough deformation of $\rho$.

**Example 7.3.**

- The simplest way of constructing a $\Delta$-Anosov representation $\rho$ is to use a “Schottky construction” (see Benoist [2]). If one fixes beforehand a base-point $o \in \mathbb{Q}_{p,q}$, it is easy to construct this representation in such a way that $o \in \Omega_\rho$.
- Suppose that $p$ and $q$ are different modulo 2 and consider a splitting $V = \pi^+ \oplus \pi^-$ where $\pi^+$ (resp. $\pi^-$) is a $p$-dimensional (resp. $q$-dimensional) subspace of $V$. Consider a representation

$$\Lambda : SL_2(\mathbb{K}) \to SL(V) : \Lambda := \Lambda^+ \oplus \Lambda^-$$

where $\Lambda^\pm : SL_2(\mathbb{K}) \to SL(\pi^\pm)$ are irreducible. Let $\rho_0$ be given by

$$\Gamma \to SL_2(\mathbb{K}) \to G,$$

where the first arrow is an Anosov representation into $SL_2(\mathbb{K})$ and the second arrow is induced by $\Lambda$. Then $\rho_0$ is $\Delta$-Anosov. Pick a form $o \in \mathbb{Q}_{p,q}$ for which the splitting $V = \pi^+ \oplus \pi^-$ is orthogonal and such that $\pi^+$ (resp. $\pi^-$) is positive definite (resp. negative definite). Then the point $o$ belongs to $\Omega_{\rho_0}$.
- Let $p = 2$ and $q = 1$ and consider this time a Hitchin representation $\rho : \Gamma_\rho \to PSL_3(\mathbb{R})$, where $\Gamma_\rho$ is the fundamental group of a closed orientable surface of genus $g \geq 2$ (see Labourie [32]). Suppose that $o$ is a point in $\mathbb{Q}_{2,1}$ such that for every $x \in \partial_\infty \Gamma_\rho$ either

  (i) the line $\xi_1^o(x)$ is negative definite for $o$,
  (ii) the hyperplane $\xi_2^o(x)$ is positive definite for $o$.

In any of these situations one has $o \in \Omega_\rho$. Since the *projective limit set* $\xi_3^o(\partial_\infty \Gamma_\rho)$ of $\rho$ is contained in an affine chart of $P(V)$ (see Choi-Goldman [13]), it is not hard to construct forms of signature $(2,1)$ on $V$ for which either (i) or (ii) above is satisfied.
The previous example generalizes to all odd dimensions as follows. Let \( \rho : \Gamma \rightarrow \text{PSL}_{2k+1}(\mathbb{R}) \) be a Hitchin representation. As shown by Danciger-Gueritaud-Kassel [15, Proposition 1.7] (see also Zimmer [59, Corollary 1.33]), there exists a \( \Gamma \)-invariant non empty open subset \( C \subset P(\mathbb{R}^{2k+1}) \) disjoint from \( \xi^k(\partial_\infty \Gamma_g) \). Then any element \( \sigma \in Q_{2k,1} \) whose projectivized isotropic cone is contained in \( C \) belongs to \( \Omega_\rho \).

7.3. **Proof of Corollary A.** For the rest of the paper we fix a \( \Delta \)-Anosov representation \( \rho : \Gamma \rightarrow G \) and a \((p,q)\)-Cartan decomposition of \( B_{o,G} \), for a basepoint \( o \in \Omega_\rho \). The following is a consequence of Lemma 5.5 and Proposition 7.1.

**Corollary 7.4.** There exist \( 0 < \varepsilon_0 \leq r_0 \) with the following property: for every \( 0 < \varepsilon \leq r \) such that \( \varepsilon \leq \varepsilon_0 \) and \( r \leq r_0 \), there exists a positive \( L \) such that if \( \gamma \in \Gamma \) satisfies \( |\gamma|_\Gamma > L \) then one has that \( \sigma^o(\rho_\gamma^{-1}) \rho_\gamma \) is \((2r,2\varepsilon)\)-loxodromic. In particular \( \rho_\gamma \) belongs to \( B_{o,G} \). Furthermore, given a positive \( \delta \) the number \( L > 0 \) can be chosen in such a way that \( d(U^o(\rho_\gamma), U^r(\rho_\gamma)) < \delta \) and \( d(S^o(\rho_\gamma), S^r(\rho_\gamma)) < \delta \).

The entropy of \( \rho \), introduced by Bridgeman-Canary-Labourie-Sambarino in [8], is defined by

\[
\h^1_\rho := \limsup_{t \to \infty} \frac{\log \# \{ [\gamma] \in [\Gamma] : \lambda_1(\rho_\gamma) \leq t \}}{t}.
\]

Here \([\gamma]\) denotes the conjugacy class of the element \( \gamma \in \Gamma \) and \( \lambda_1(\cdot) \) is defined as in Subsection 5.1 (for any given Weyl chamber of the system \( \Sigma(g,a) \)). The entropy of \( \rho \) is positive and finite (see [8, Sections 4 & 5]).

On the other hand, the **projective critical exponent** of \( \rho \) is defined by

\[
\delta^1_\rho := \limsup_{t \to \infty} \frac{\log \# \{ \gamma \in \Gamma : a_\gamma^o(\rho_\gamma) \leq t \}}{t}.
\]

A consequence of Sambarino’s work [53] is that the entropy of \( \rho \) coincides with the projective critical exponent\(^{10}\). We conclude that \( \delta^1_\rho \) is positive and finite, and therefore the **critical exponent**

\[
\delta_\rho := \limsup_{t \to \infty} \frac{\log \# \{ \gamma \in \Gamma : \|a_\gamma^o(\rho_\gamma)\|_b \leq t \}}{t}
\]

of \( \rho \) must also be positive and finite.

We now prove Corollary A.

**Corollary 7.5.** The following holds:

1. The function

\[
t \mapsto \# \{ \gamma \in \Gamma : \rho_\gamma \in B_{o,G} \text{ and } \|b^o(\rho_\gamma)\|_b \leq t \}
\]

is finite for every positive \( t \). Moreover, the following equality is satisfied

\[
\delta_\rho = \limsup_{t \to \infty} \frac{\log \# \{ \gamma \in \Gamma : \rho_\gamma \in B_{o,G} \text{ and } \|b^o(\rho_\gamma)\|_b \leq t \}}{t}.
\]

In particular, the right hand side of the last equality is finite, positive and independent on the choice of the basepoint \( o \in \Omega_\rho \).

\(^{10}\)In [53] the author treats the case in which \( \Gamma \) is the fundamental group of a closed negatively curved manifold. His results remain valid when \( \Gamma \) is a word hyperbolic group admitting an Anosov representation, as proven by Glorieux-Tholozan-Monclair [20, Theorem 2.31] (see also Appendix A).
Suppose that \( \rho \) is Zariski dense. Then there exist positive constants \( C_1 \) and \( C_2 \) such that for every \( t \) large enough one has
\[
C_1 e^{\delta \rho t} \leq \# \{ \gamma \in \Gamma : \rho \gamma \in \mathcal{B}_{o,G} \text{ and } \| b^\rho(\rho \gamma) \|_b \leq t \} \leq C_2 e^{\delta \rho t}.
\]

Proof. Let \( a^+ \) be a \( b^+ \)-compatible Weyl chamber.

(1) Follows from Propositions 5.9 and 7.1, and W-invariance of \( \| \cdot \|_b \).

(2) By the work of Sambarino [54] (see Theorem A.5 for a proof in our setting) there exists a positive constant \( C \) such that
\[
C e^{-\delta \rho t} \# \{ \gamma \in \Gamma : \| a^\tau(\rho \gamma) \|_b \leq t \} \to 1
\]
as \( t \to \infty \). Propositions 5.9 and 7.1 and W-invariance of \( \| \cdot \|_b \) finish the proof.

\[\square\]

7.4. \((p,q)\)-asymptotic cone. Fix a \( b^+ \)-compatible Weyl chamber \( \hat{a}^+ \) for which the intersection
\[
\xi_{\rho}(\partial_{\infty}\Gamma) \cap F(V)^o_{\hat{a}^+}
\]
is non empty and set
\[
a^+ := \iota_{b^+}(\hat{a}^+).
\]
Let \( a^\tau : G \to a^+ \) be the associated Cartan projection and denote by \( \mathcal{L}_\rho \) the asymptotic cone of \( \rho(\Gamma) \), introduced by Benoist in [3]. By definition, it is the subset of \( a^+ \) consisting on all possible limits of sequences of the form
\[
\frac{a^\tau(\rho \gamma_n)}{t_n}
\]
where \( t_n \to \infty \). Benoist [3, 4] showed that, for Zariski dense subgroups of \( G \), the set \( \mathcal{L}_\rho \) coincides with the smallest closed cone containing \( \lambda(\rho(\Gamma)) \) (which is also showed to be convex and with non empty interior).

We define a new asymptotic cone, that we denote by \( \mathcal{L}_\rho^{p,q} \), and that consists on all possible limits of sequences of the form (7.3) but with \( a^\tau(\rho \gamma_n) \) replaced by \( b^\rho(\rho \gamma_n) \). The main goal of this subsection is to give an explicit description of this new cone by means of Benoist’s asymptotic cone: see Proposition 7.8 below. As a consequence of this description (c.f. Remark 7.9), we note that the topology of \( \mathcal{L}_\rho^{p,q} \) may depend on the specific choice of the basepoint \( o \). However, since \( o \in \Omega_\rho \) is fixed for the rest of the paper, we prefer not to stress this dependence in the notation.

Define
\[
W_{\rho,a^+} := \{ w \in W : w \cdot a^+ \subset b^+ \text{ and } \xi_{\rho}(\partial_{\infty}\Gamma) \cap F(V)^o_{b^+(w \cdot a^+)} \neq \emptyset \}.
\]

Remark 7.6. By definition the identity element of \( W \) belongs to \( W_{\rho,a^+} \). Moreover, the equality \( W_{\rho,a^+} = \{ 1 \} \) is equivalent to the inclusion
\[
\xi_{\rho}(\partial_{\infty}\Gamma) \subset F(V)^o_{b^+(w \cdot a^+)}.
\]
In particular, if \( \partial_{\infty}\Gamma \) is connected we always have \( W_{\rho,a^+} = \{ 1 \} \). However, one can do a Schottky construction as in Example 7.3 to find \( \Delta \)-Anosov representations for which \( W_{\rho,a^+} \) does not consists on a single element: it suffices to play “ping-pong” with matrices whose attractors and repellors belong to different open orbits of the action of \( H^o \) on \( F(V) \) (c.f. Figure 6 below).

Remark 7.7. Let \( \{ \gamma_n \} \) be a sequence in \( \Gamma \) diverging to infinity and suppose that there exists a \( b^+ \)-compatible Weyl chamber \( \hat{a}^+ \) such that
for every \( n \) large enough. Let \( w \in W_{\rho,a^+} \) be the element defined by the equality \( w \cdot a^+ = t_{b^+}(\hat{a}^+) \).

Then Proposition 5.9, the definition of a \( \Delta \)-Anosov representation and Proposition 7.1 imply that the sequence

\[
\|b^\tau(\rho\gamma_n) - w \cdot a^\tau(\rho\gamma_n)\|_b
\]

is bounded.

**Proposition 7.8.** The following equality holds:

\[
L_{\rho,q} = \bigcup_{w \in W_{\rho,a^+}} w \cdot L_{\rho}.
\]

**Proof.** We first prove the inclusion

\[
L_{\rho,q} \subset \bigcup_{w \in W_{\rho,a^+}} w \cdot L_{\rho}.
\]

Let

\[
X = \lim_{n \to \infty} \frac{b^\nu(\rho\gamma_n)}{t_n}
\]

be a point in \( L_{\rho,q} \). By taking a subsequence if necessary we may assume

\[
S^\tau(\rho\gamma_n) \in F(V)_{\hat{a}^+}^o,
\]

for all \( n \) and some Weyl chamber \( \hat{a}^+ \subset b^+ \). Take \( w \in W_{\rho,a^+} \) as in Remark 7.7. Then the sequence

\[
\frac{1}{t_n}\|b^\nu(\rho\gamma_n) - w \cdot a^\tau(\rho\gamma_n)\|_b
\]

converges to zero and we conclude that \( X \) belongs to \( w \cdot L_{\rho} \).

Conversely, let \( w \in W_{\rho,a^+} \) and

\[
X = \lim_{n \to \infty} \frac{a^\tau(\rho\gamma_n)}{t_n}
\]

be a point in \( L_{\rho} \). Define \( \hat{a}^+ := t_{b^+}(w \cdot a^+) \), which is a \( b^+ \)-compatible Weyl chamber and, by definition of \( W_{\rho,a^+} \), the intersection

\[
\xi(\partial_{\infty}\Gamma) \cap F(V)_{\hat{a}^+}^o
\]

is non empty. By taking a subsequence if necessary we may suppose

\[
S^\tau(\rho\gamma_n) \to \xi_{\rho}(y)
\]

as \( n \to \infty \), for some \( y \in \partial_{\infty}\Gamma \). Since the intersection \( \xi_{\rho}(\partial_{\infty}\Gamma) \cap F(V)_{\hat{a}^+}^o \) is non empty we can fix an element \( \gamma^0 \) in \( \Gamma \) such that

\[
(\rho\gamma^0)^{-1} \cdot \xi_{\rho}(y) \in F(V)_{\hat{a}^+}^o.
\]

Further, by Proposition 7.1 we may choose the element \( \gamma^0 \) in such a way that the flag \( U^\tau(\rho\gamma^0) \) is transverse to \( S^\tau(\rho\gamma_n) \) for all \( n \) large enough. Hence we find a constant \( D > 0 \) such that for every \( j = 1, \ldots, d-1 \) one has

\[
d \left( N_j U^\tau(\rho\gamma^0), \Lambda_j S^\tau(\rho\gamma_n) \right) \geq D
\]
for every \( n \) large enough. Therefore the sequence

\[
\|a^\tau(\rho_{n\gamma}0)) - a^\tau(\rho\gamma_n)\|_b
\]

is bounded (see e.g. [6, Lemma A.7]) and we conclude that

\[
X = \lim_{n \to \infty} a^\tau(\rho\gamma_n) = \lim_{n \to \infty} a^\tau(\rho(\gamma_n\gamma^0)).
\]

Thanks to Remark 7.7 in order to finish it suffices to show that \( S^\tau(\rho(\gamma_n\gamma^0)) \) belongs to \( F(V)_{a+}^o \) for every \( n \) large enough. But applying [6, Lemma A.5] we have

\[
\lim_{n \to \infty} S^\tau(\rho(\gamma_n\gamma^0)) = (\rho\gamma^0)^{-1} \cdot \xi_\rho(y)
\]

which by construction belongs to \( F(V)_{a^+}^o \).

\[ \square \]

**Remark 7.9.** The topology of \( \mathcal{L}^{p,q}_{\rho} \) may depend on the choice of the basepoint \( a \). Indeed, one can fix beforehand two points \( o_1 \) and \( o_2 \) in \( \mathbb{Q}_{p,q} \) and find a Schottky representation \( \rho \) such that \( \xi_\rho(\partial_\infty \Gamma) \) is contained in a single open orbit of the action \( H^\rho \rtimes F(V) \), but in more than one open orbit for the action of \( H^a \) (see Figure 6). Note however that, by Proposition 7.8, the asymptotic cone \( \mathcal{L}^{p,q}_{\rho} \) is independent on the choice of basepoints \( a \) for which the limit set is contained in a single open orbit of the action \( H^a \rtimes F(V) \).

![Figure 6](image_url)

**Figure 6.** An example of a free Schottky subgroup of \( G \) generated by (large enough) powers of \( g_1, g_2 \in G \). The red conics represent the projectivized isotropic cones of points \( o_1, o_2 \in \mathbb{Q}_{2,1} \). The \((p,q)\)-asymptotic cones for \( o_1 \) and \( o_2 \) have different topologies.
8. Counting on a given direction

Through this section we assume further that there exists a single open orbit of the action \( H^o \curvearrowright F(V) \) that contains the limit set \( \xi_\rho(\partial_\infty \Gamma) \). The goal of this section is to prove Theorem B (see Proposition 8.10 below). The method we use is that of Roblin [50] and Sambarino [53], therefore in Subsection 8.1 we introduce a pair of dual Hölder cocycles over \( \partial_\infty \Gamma \) and a corresponding Gromov product to study our problem. In Subsection 8.2 we introduce the corresponding Patterson-Sullivan measures and in Subsection 8.3 we establish an equidistribution result for fixed points of elements in \( \Gamma_H \), due to Sambarino [53, Proposition 4.3]. From this result we deduce our main theorem.

To introduce our pair of dual cocycles over \( \partial_\infty \Gamma \) we use the \( o \)-Busemann cocycle of Section 6. Recall that to define this cocycle we need to pick a \( b^+ \)-compatible Weyl chamber. Our assumption on the limit set of \( \rho \) provide us with a \( b^+ \)-compatible Weyl chamber \( \tilde{a}^+ \) such that

\[
\xi_\rho(\partial_\infty \Gamma) \subset F(V)_{\tilde{a}^+}^o.
\]

We set \( a^+ := \iota_{b^+}(\tilde{a}^+) \) and with this Weyl chamber we will construct our Hölder cocycles over \( \partial_\infty \Gamma \). This is the reason why we need assumption (8.1) to hold.

By Remark 7.6 we have \( W_{\rho, a^+} = \{1\} \) and Proposition 7.8 gives us the equality \( L_{p,q}^o(\rho) = L_{\rho}^o \).

Let \( w \in W \) be the unique element of the Weyl group such that \( w \cdot \tilde{a}^+ = \tilde{a}^+ \).

We have the following equalities:

\[
w_{\tilde{a}^+} \iota_{b^+}(\tilde{a}^+) = w \quad \text{and} \quad w \iota_{b^+}(\tilde{a}^+) a^+ = 1.
\]

Therefore the following is a consequence of Corollary 5.7, Proposition 5.8 and Remark 7.7.

**Corollary 8.1.** There exist positive constants \( L \) and \( D \) such that for every \( \gamma \) in \( \Gamma \) with \( |\gamma|_\Gamma > L \) one has

\[
\rho \gamma \in H^o w \exp \left( \frac{1}{2} \lambda(\sigma^o(\rho \gamma^{-1}) \rho \gamma) \right) H^o
\]

and

\[
\|b^o(\rho \gamma) - a^+(\rho \gamma)\|_b \leq D.
\]

8.1. \( o \)-Busemann cocycle and \( o \)-Gromov product for \( \rho \). Fix a linear functional \( \varphi \in b^* \) in the interior of the dual cone \( \mathcal{L}_\rho^o \). The \( \varphi \)-entropy of \( \rho \) is defined by

\[
h^o_\varphi := \limsup_{t \to \infty} \frac{\log \# \{ [\gamma] \in [\Gamma] : \varphi(\lambda(\rho \gamma)) \leq t \}}{t}.
\]

Since the entropy \( h^1_\varphi = h^o_\varphi \) of \( \rho \) is positive and finite and \( \varphi \) is positive on \( \mathcal{L}_\rho \), we conclude that \( h^o_\varphi \) must also be positive and finite.

Let

\[
ce_\varphi^o : \Gamma \times \partial_\infty \Gamma \to \mathbb{R} : \ne_\varphi^o(\gamma, x) := \varphi(\beta^o(\rho \gamma, \xi_\rho(x)))
\]

and

\[
ne_\varphi^o : \Gamma \times \partial_\infty \Gamma \to \mathbb{R} : ne_\varphi^o(\gamma, x) := \varphi((\iota_{a^+} \circ \beta^o(\rho \gamma, \xi_\rho(x))))
\]}
where $\beta^\circ$ is the $o$-Busemann cocycle (associated to $a^+$). These are called the $(\varphi, o)$-Busemann cocycles of $\rho$.

Recall that a Hölder cocycle over $\partial_\infty \Gamma$ is a function
c$$c : \Gamma \times \partial_\infty \Gamma \to \mathbb{R}$$satisfying that for every $\gamma_0, \gamma_1$ in $\Gamma$ and $x$ in $\partial_\infty \Gamma$ one has
c$$c(\gamma_0 \gamma_1, x) = c(\gamma_0, \gamma_1 \cdot x) + c(\gamma_1, x),$$and such that the map $c(\gamma_0, \cdot)$ is Hölder continuous (with the same exponent for every $\gamma_0$). The period of an element $\gamma \in \Gamma_H$ for such a cocycle is defined by 
$p_c(\gamma) := c(\gamma, \gamma_+).$

This is an invariant of the conjugacy class $[\gamma]$ of $\gamma$. If $c$ has positive periods, we let 
h$$h_c := \limsup_{t \to \infty} \log \# \{ [\gamma] \in [\Gamma] : p_c(\gamma) \leq t \} \in [0, \infty]$$be the entropy of $c$. A Hölder cocycle $\tau$ over $\partial_\infty \Gamma$ is said to be dual to $c$ if the equality
$$p_\tau(\gamma) = p_c(\gamma^{-1})$$holds for every $\gamma \in \Gamma_H$. Note that dual cocycles have the same entropy.

The following lemma holds by direct computations (c.f. Lemma 6.1).

**Lemma 8.2.** The pair $(\tau_\circ^\circ, c_\circ^\circ)$ is a pair of dual Hölder cocycles. The periods of $c_\circ^\circ$ are given by
$$p_{c_\circ^\circ}(\gamma) = \varphi(\lambda(\rho\gamma)) > 0$$for every $\gamma \in \Gamma_H$. In particular, one has the equalities 
h$$h_{c_\circ^\circ} = h_\tau^\circ = h_{\tau_\circ^\circ}.$$**Remark 8.3.** The definition of the $(\varphi, o)$-Busemann cocycles of $\rho$ takes inspiration from Sambarino’s work [53]. The author defines
c$$c_\tau^\circ : \Gamma \times \partial_\infty \Gamma \to \mathbb{R} : c_\tau^\circ(\gamma, x) := \varphi(\beta^\tau(\rho\gamma, \xi_\rho(x)))$$and
c$$\tau_\circ^\circ : \Gamma \times \partial_\infty \Gamma \to \mathbb{R} : \tau_\circ^\circ(\gamma, x) := \varphi(\iota_{a^\circ} \circ \beta^\tau(\rho\gamma, \xi_\rho(x))),$$where $\beta^\tau$ is the $\tau$-Busemann cocycle of $G$ (see [53, Section 7]). The cocycles $c_\tau^\circ$ and $c_\circ^\circ$ (resp. $\tau_\circ^\circ$ and $\tau_\circ^\circ$) are cohomologous in the sense of Livšic [34]. By definition this means that there exist Hölder continuous functions 
v : $\partial_\infty \Gamma \to \mathbb{R}$ and $\varpi : \partial_\infty \Gamma \to \mathbb{R}$such that for every $\gamma$ in $\Gamma$ and $x$ in $\partial_\infty \Gamma$ one has 
c$$c_\circ^\circ(\gamma, x) - c_\tau^\circ(\gamma, x) = v(\gamma \cdot x) - v(x)$$and
$$\tau_\circ^\circ(\gamma, x) - \tau_\circ^\circ(\gamma, x) = \varpi(\gamma \cdot x) - \varpi(x).$$Indeed, this follows directly from Remark 6.2.

Define
$$\lbrack \cdot, \cdot \rbrack_\circ^\circ : \partial_\infty^2 \Gamma \to \mathbb{R} : \lbrack x, y \rbrack_\circ^\circ := \varphi(\mathcal{G}_\circ(\xi_\rho(x), \xi_\rho(y))).$$The following is a consequence of Lemma 6.8.

**Corollary 8.4.** The map $\lbrack \cdot, \cdot \rbrack_\circ^\circ$ is a Gromov product for the pair $(\tau_\circ^\circ, c_\circ^\circ)$, that is, for every $\gamma \in \Gamma$ and every $(x, y) \in \partial_\infty^2 \Gamma$ one has 
$$\lbrack \gamma \cdot x, \gamma \cdot y \rbrack_\circ^\circ - \lbrack x, y \rbrack_\circ^\circ = -(\tau_\circ^\circ(\gamma, x) + c_\circ^\circ(\gamma, y)).$$
We now state a crucial result that allows us to compare $\varphi(b^\circ(\rho \gamma))$ with the period $\varphi(\lambda(\rho \gamma))$ by means of the Gromov product defined above. This is the analogue of C. [11, Lemma 6.6(4)] in the present framework.

**Corollary 8.5.** Fix a positive $\delta$ and $A$ and $B$ two disjoint compact subsets of $\partial_{\infty}\Gamma$. Then there exists a positive $L$ such that for every $\gamma \in \Gamma_H$ satisfying $|\gamma|_\Gamma > L$ and $(\gamma_-, \gamma_+) \in A \times B$ one has

$$|\varphi(b^\circ(\rho \gamma)) - \varphi(\lambda(\rho \gamma)) + [\gamma_-, \gamma_+]_\varphi| < \delta.$$  

**Proof.** From Corollaries 5.2 and 6.9 we know that

$$[\gamma_-, \gamma_+]_\varphi = -\frac{1}{2}\varphi(\mathbb{B}(\sigma^a(\rho \gamma^{-1})_+, \sigma^a(\rho \gamma^{-1})_+, (\rho \gamma)_-, (\rho \gamma)_+))$$

holds for every $\gamma \in \Gamma_H$. Because of Corollary 8.1 we can suppose that

$$b^\circ(\rho \gamma) = \frac{1}{2}\lambda(\sigma^a(\rho \gamma^{-1})\rho \gamma)$$

holds as well. To finish the proof we apply Benoist [4, Lemme 3.4]. Indeed, by transversality condition (7.2) we can find a positive $r$ for which for every $\gamma$ as in the statement and every $j = 1, \ldots, d - 1$ one has:

$$d(\Lambda^j(\rho \gamma)_+, \Lambda^j(\rho \gamma)_-) \geq 2r.$$  

Given a positive $\varepsilon \leq r$, Lemma 7.2 states that we can assume further that $\rho \gamma$ is $(r, \varepsilon)$-loxodromic and, because of Corollary 5.2, we can suppose that analogue assertions hold for $\sigma^a(\rho \gamma^{-1})$. Moreover, by changing $r$ by a smaller constant if necessary we have

$$d(\Lambda^j(\rho \gamma)_+, \Lambda^j(\rho \gamma)_-) \geq 6r \text{ and } d(\Lambda(\rho \gamma)_+, \Lambda(\rho \gamma)^{-1}_{-}) \geq 6r$$

for every $j = 1, \ldots, d - 1$. By [4, Lemme 3.4] the proof is finished. \qed

**8.2. Patterson-Sullivan measures.** Let $c$ be a Hölder cocycle over $\partial_{\infty}\Gamma$ and $\delta$ be a positive number. A probability measure $\mu_c$ on $\partial_{\infty}\Gamma$ is called a **Patterson-Sullivan measure of dimension $\delta$** for $c$ if the equality$^{11}$

$$\frac{d\gamma_*\mu_c}{d\mu_c}(x) = e^{-\delta c(\gamma^{-1}, x)}$$  

is satisfied for every $\gamma \in \Gamma$.

The **$\varphi$-critical exponent of** $\rho$ is defined by the equality

$$\delta^c_\rho := \limsup_{t \to \infty} \frac{\log \# \{ \gamma \in \Gamma : \varphi(\sigma^a(\rho \gamma)) \leq t \}}{t}.$$  

It is positive and finite (because $\delta^1_\rho$ is).

Assume from now on that $\rho$ is Zariski dense. Quint [48, Théorème 8.4] shows the existence of Patterson-Sullivan measures $\mu^x_\varphi$ and $\mu^{x_0}_\varphi$ of dimension $\delta^x_\rho$ for the cocycles $c^x_\varphi$ and $c^{x_0}_\varphi$ respectively$^{12}$. Because of Remark 8.3, we find Patterson-Sullivan measures $\mu^x_\varphi$ and $\mu^{x_0}_\varphi$ of the same dimension for the cocycles $c^x_\varphi$ and $c^{x_0}_\varphi$ respectively.

We mention here that, in the case of fundamental groups of negatively curved closed manifolds, the existence (and uniqueness) of these probability measures is also shown by Ledrappier [33].

$^{11}$Recall that if $f : X \to Y$ is a map and $m$ is a measure on $X$ then $f_*m$ denotes the measure on $Y$ defined by $A \mapsto m(f^{-1}(A))$.

$^{12}$Indeed, since $\varphi$ is positive in the limit cone $\mathcal{L}_\rho$, the linear form $\delta^x_\rho \varphi$ is tangent to the growth indicator of $\rho$ in a direction contained in $\mathcal{L}_\rho \subset \text{int}(a^\circ)$. 

The following lemma is well-known and will be used in Subsection 8.4. We include a proof for completeness.

**Lemma 8.6.** The probability measures $\mu_\phi^\circ$ and $\pi_\phi^\circ$ have no atoms.

**Proof.** It suffices to show that $\mu_\phi^\tau$ and $\pi_\phi^\tau$ have no atoms. We only do it for $\mu_\phi^\tau$ (the other case being analogous). Let $\nu := \xi_\rho \mu_\tau^\circ \cdot \phi$ and $\varepsilon > 0$. We will find a covering of $\xi_\rho(\partial_\infty \Gamma)$ by open sets of $\nu$-measure less than or equal to $\varepsilon$.

By Pozzetti-Sambarino-Wienhard [46, Lemma 5.15], there exists a positive $\delta_0$ such that for every $0 < \delta < \delta_0$ there exists $L = L(\delta) > 0$ satisfying that for every $\gamma \in \Gamma$ with $|\gamma|_\Gamma > L$ one has

$$\nu(\rho \gamma \cdot B_\delta(S^\tau(\rho \gamma))) \leq \varepsilon.$$ 

Here we denote

$$B_\delta(S^\tau(\rho \gamma)) := \{ \xi \in F(V) : \Lambda^j \xi \in B_\delta(\Lambda^j S^\tau(\rho \gamma)) \text{ for all } j = 1, \ldots, d - 1 \}.$$ 

We now follow the outline of [46, Proposition 3.5] to show that $\delta$ can be chosen in such a way that

$$\{ \rho \gamma \cdot B_\delta(S^\tau(\rho \gamma)) : |\gamma|_\Gamma > L \}$$

is a covering of $\xi_\rho(\partial_\infty \Gamma)$.

By Bochi-Potrie-Sambarino [6, Lemma 2.5], there exist positive constants $\eta_\rho$ and $L_\rho$ such that for every geodesic segment $(\gamma_i)_{i=0}^k$ in $\Gamma$ passing through the identity element of $\Gamma$ and such that $|\gamma_0|_\Gamma > L_\rho$ and $|\gamma_k|_\Gamma > L_\rho$ one has

$$d \left( \Lambda^j U^\tau(\rho \gamma_k), \Lambda^j U^\tau(\rho \gamma_0) \right) \geq \eta_\rho$$

for every $j = 1, \ldots, d - 1$.

Let $\delta < \eta_\rho$, we may take $L = L(\delta)$ to be larger than $L_\rho$. Fix $y \in \partial_\infty \Gamma$ and a geodesic ray $(\gamma_i)_{i=0}^\infty$ in $\Gamma$ starting at the identity element of $\Gamma$ and converging to $y$. Let also $i_0 \geq 0$ be chosen in such a way that if $\gamma := \gamma_{i_0}$ then one has

$$|\gamma^{-1}|_\Gamma > L.$$ 

Applying [6, Lemma 2.5] to the geodesic ray $\gamma^{-1} \cdot (\gamma_i)_{i=0}^\infty$ we find that for every $k$ large enough one has

$$d \left( \Lambda^j U^\tau(\rho \gamma^{-1} \gamma_k), \Lambda^j U^\tau(\rho \gamma^{-1}) \right) \geq \delta$$

for every $j = 1, \ldots, d - 1$. By [6, Lemma 4.7] we have

$$U^\tau(\rho \gamma^{-1} \gamma_k) \to \rho \gamma^{-1} \cdot \xi_\rho(y)$$

as $k \to \infty$. Therefore up to changing $\delta$ by a smaller constant if necessary we may assume that

$$d \left( \Lambda^j (\rho \gamma^{-1} \cdot \xi_\rho(y)), \Lambda^j S^\tau(\rho \gamma) \right) \geq \delta$$

holds for every $j = 1, \ldots, d - 1$. Hence $\rho \gamma^{-1} \cdot \xi_\rho(y) \in B_\delta(S^\tau(\rho \gamma))$. 

$\square$
8.3. Distribution of fixed points. For a metric space $X$, we denote by $C^*_c(X)$ the dual of the space of compactly supported continuous functions on $X$ equipped with the weak-star topology. For a point $x \in X$, we let $\delta_x \in C^*_c(X)$ be the Dirac mass at $x$.

Denote by $\partial^2_\infty \Gamma$ the space of ordered pairs of distinct points in $\partial_\infty \Gamma$. When $\Gamma$ is the fundamental group of a closed negatively curved manifold, Sambarino [53] shows the following (for a proof in our setting see Appendix A).

Proposition 8.7 (c.f. [53, Proposition 4.3 & Theorem C]). The number $\delta^\varphi_\rho$ coincides with the $\varphi$-entropy $h^\varphi_\rho$ of $\rho$ and there exists a positive constant $m = m_{\rho,o,\varphi}$ such that

$$m e^{-h^\varphi_\rho t} \sum_{\gamma \in \Gamma \varphi(b^\varphi(\rho \gamma)) \leq t} \delta_{\gamma_-} \otimes \delta_{\gamma_+} \rightarrow e^{-h^\varphi_\rho t} [\cdot]_\varphi^\rho \otimes \mu^\varphi_\rho$$

as $t \to \infty$ on $C^*_c(\partial^2_\infty \Gamma)$.

8.4. Proof of Theorem B. As noticed by Roblin [50], equidistribution statements as that of Proposition 8.7 can be used to study counting problems. In [53] Sambarino applies Roblin’s method to obtain a counting theorem for the operator norm of elements in the image of a (projective) Anosov representation and in C. [11] we applied this method to study counting problems in pseudo-Riemannian hyperbolic spaces.

The following proposition is an intermediate step towards the proof of Theorem B. It implies

$$m e^{-h^\varphi_\rho t} \#\{\gamma \in \Gamma_H : \rho \gamma \in \mathcal{B}_{o,G} \text{ and } \varphi(b^\varphi(\rho \gamma)) \leq t\} \rightarrow 1$$

as $t \to \infty$. In order to obtain Theorem B we must also count the amount of torsion elements $\gamma$ for which $\varphi(b^\varphi(\rho \gamma)) \leq t$. This will be a consequence of Proposition 8.8.

Proposition 8.8. There exists a positive constant $m = m_{\rho,o,\varphi}$ such that

$$m e^{-h^\varphi_\rho t} \sum_{\gamma \in \Gamma \varphi(b^\varphi(\rho \gamma)) \leq t} \delta_{\gamma_-} \otimes \delta_{\gamma_+} \rightarrow [\cdot]_\varphi^\rho \otimes \mu^\varphi_\rho$$

as $t \to \infty$ on $C^*_c(\partial_\infty \Gamma \times \partial_\infty \Gamma)$.

The sum in Proposition 8.8 is taken over all elements $\gamma \in \Gamma_H$ for which $\rho \gamma \in \mathcal{B}_{o,G}$ and $\varphi(b^\varphi(\rho \gamma)) \leq t$. To make the formula more readable we do not emphasize the fact that $\rho \gamma$ must belong to $\mathcal{B}_{o,G}$ (recall from Corollary 7.4 that this holds with only finitely many exceptions $\gamma \in \Gamma$).

Proof of Proposition 8.8. Let

$$\theta_t := m e^{-h^\varphi_\rho t} \sum_{\gamma \in \Gamma \varphi(b^\varphi(\rho \gamma)) \leq t} \delta_{\gamma_-} \otimes \delta_{\gamma_+}.$$

The proof follows line by line the proof of [53, Theorem 6.5] or [11, Proposition 7.11]. Namely, for open subsets $A$ and $B$ of $\partial_\infty \Gamma$ with disjoint closure and negligible boundary, the convergence

$$\theta_t(A \times B) \rightarrow \mathcal{P}^\varphi_\rho(A) \mu^\varphi_\rho(B)$$

is implied by Proposition 8.7 and Corollary 8.5. On the other hand, since $\mathcal{P}^\varphi_\rho$ and $\mu^\varphi_\rho$ have no atoms (Lemma 8.6), one has

$$\mathcal{P}^\varphi_\rho \otimes \mu^\varphi_\rho(\{(x,x) : x \in \partial_\infty \Gamma\}) = 0.$$
In order to finish the proof it suffices to show the following: for every positive \( \varepsilon_0 \) there exists an open covering \( \mathcal{U} \) of \( \partial_\infty \Gamma \) such that

\[
\limsup_{t \to \infty} \theta_t \left( \bigcup_{U \in \mathcal{U}} U \times U \right) \leq \varepsilon_0.
\]

Provided Lemma 8.9 below, the proof of this fact follows exactly as the proof of the analogue fact in [53, Theorem 6.5] or in [11, Proposition 7.11].

\[
\square
\]

**Lemma 8.9** (c.f. [11, Proposition 6.10]). Fix an element \( \gamma_0 \in \Gamma \). Then there exist positive constants \( L \) and \( D_{\gamma_0} \) such that for every \( \gamma \) in \( \Gamma \) satisfying \( |\gamma|_\Gamma > L \) one has

\[
\|b^o(\rho(\gamma_0 \gamma)) - b^o(\rho \gamma)\|_b \leq D_{\gamma_0}.
\]

**Proof.** By Corollary 8.1 there exist positive constants \( L \) and \( D \) such that for every \( \gamma \) with \( |\gamma|_\Gamma > L \) one has

\[
\|b^o(\rho(\gamma_0 \gamma)) - b^o(\rho \gamma)\|_b \leq ||a^\tau(\rho(\gamma_0 \gamma)) - a^\tau(\rho \gamma)||_b + D.
\]

To finish observe that there exists a positive constant \( d_{\gamma_0} \) for which the inequality

\[
\|a^\tau(\rho(\gamma_0 \gamma)) - a^\tau(\rho \gamma)||_b \leq d_{\gamma_0}
\]

is satisfied for every \( \gamma \in \Gamma \).

\[
\square
\]

The following proposition finishes the proof of Theorem B.

**Proposition 8.10.** There exists a positive constant \( m = m_{\rho, o, \varphi} \) such that

\[
me^{-h^\rho_{\varphi} t} \sum_{\gamma \in \Gamma, o(\rho(\gamma)) \leq t} \delta_{S^o(\rho \gamma)} \otimes \delta_{U^o(\rho \gamma)} \rightarrow \xi_{\rho, o}{\mathcal{P}}_0 \otimes \xi_{\rho, o}{\mathcal{M}}_0
\]

as \( t \to \infty \) on \( C^*(F(V) \times F(V)) \).

**Proof.** The proof is analogous to the proof of [11, Proposition 7.13], the main steps being:

- Let

\[
\nu^H_t := me^{-h^\rho_{\varphi} t} \sum_{\gamma \in \Gamma, o(\rho(\gamma)) \leq t} \delta_{S^o(\rho \gamma)} \otimes \delta_{U^o(\rho \gamma)}.
\]

Provided Corollary 7.4 and Proposition 7.1, the convergence

\[
\nu^H_t \rightarrow \xi_{\rho, o}{\mathcal{P}}_0 \otimes \xi_{\rho, o}{\mathcal{M}}_0
\]

follows from Proposition 8.8.

- Let

\[
\nu_t := me^{-h^\rho_{\varphi} t} \sum_{\gamma \in \Gamma, o(\rho(\gamma)) \leq t} \delta_{S^o(\rho \gamma)} \otimes \delta_{U^o(\rho \gamma)},
\]

which differs from \( \nu^H_t \) only from the fact that we allow torsion elements in the defining sum. Fix a continuous function \( f \) on \( F(V) \times F(V) \) whose support \( \text{supp}(f) \) is contained in \( F(V)^{(2)} \). An application of Corollary 7.4 and Benoist [4, Lemme 6.2] yields

\[
\# \{ \gamma \in \Gamma : (S^o(\rho \gamma), U^o(\rho \gamma)) \in \text{supp}(f) \text{ and } \gamma \notin \Gamma_H \} < \infty.
\]

This implies the convergence \( \nu^H_t(f) - \nu_t(f) \to 0 \).
To finish it remains to analyze the asymptotic behaviour of the measure \( \nu_t \) over the set \( \mathcal{D} := F(V)^2 \setminus F(V)^{(2)} \). As in [11, Proposition 7.13], for every positive \( \varepsilon_0 \) one can find an open covering \( \mathcal{U} \) of \( \mathcal{D} \) such that
\[
\limsup_{t \to \infty} \nu_t \left( \bigcup_{U \in \mathcal{U}} U \right) \leq \varepsilon_0.
\]
Since the number \( \xi_{\rho*} \mu \otimes \xi_{\rho*} \mu(\mathcal{D}) \) equals zero (Lemma 8.6), the proof is complete.

Appendix A. Distribution of fixed points, mixing and counting

We now explain why the results of Sambarino [52, 53, 54] hold when we replace the fundamental group of a closed negatively curved manifold by a general word hyperbolic group \( \Gamma \) admitting an Anosov representation. The central dynamical tool used by the author along the above works is the thermodynamical formalism for the geodesic flow of the manifold. Provided the work of Bridgeman-Canary-Labourie-Sambarino [8], the thermodynamical formalism also applies to the geodesic flow of \( \Gamma \).

A.1. The geodesic flow. Let \( \Gamma \) be a word hyperbolic group. Gromov [21] introduced a compact space \( \tilde{U} \Gamma \) endowed with a transitive Hölder flow, called the geodesic flow of \( \Gamma \), which is well defined up to Hölder reparametrization (see Mineyev [36] for details). If \( \Gamma \) admits an Anosov representation, the geodesic flow of \( \Gamma \) is metric Anosov [8, Section 5]. By work of Pollicott [43], the geodesic flow of \( \Gamma \) admits then a Markov coding and therefore the techniques coming from the thermodynamical formalism of subshifts of finite type apply (see [8, 14]).

The following way of constructing parametrizations of the geodesic flow is useful. Let \( L \) be a one dimensional Hölder real vector bundle over \( \partial^2 \Gamma \). Let \( \hat{L} \) be the \( \mathbb{R} \)-principal bundle over \( \partial^2 \Gamma \) whose fibers are
\[
\hat{L}_{(x,y)} := (L_{(x,y)} \setminus \{0\})/\sim,
\]
where \( v \sim -v \). Here the action of \( t \in \mathbb{R} \) on \( \hat{L} \) is given by \( t : [v] \mapsto [e^t v] \). Suppose furthermore that \( L \) is endowed with an action of \( \Gamma \) by bundle automorphisms. For every \( \gamma \in \Gamma_H \), let \( p_L(\gamma) \) be the real number such that for every \( v \in L_{(\gamma \tau, \gamma \tau)} \) one has
\[
\gamma \cdot v = \pm e^{p_L(\gamma)} v.
\]
The following proposition is essentially proven in [8, Proposition 4.2] (see also [9, Proposition 2.4]).

**Proposition A.1.** Let \( \rho : \Gamma \to G \) be a \( \Delta \)-Anosov representation. Assume that there exists a positive constant \( \alpha \) such that the inequality
\[
p_L(\gamma) \geq \alpha \lambda_1(\rho' \gamma)
\]
holds for every \( \gamma \in \Gamma_H \). Then the action of \( \Gamma \) on \( \hat{L} \) is properly discontinuous and the quotient space \( U_L \) is Hölder homeomorphic to \( \mathbb{R} \). Furthermore, the flow on \( U_L \) induced by the action of \( \mathbb{R} \) on \( \hat{L} \) is Hölder conjugate to a Hölder reparametrization of the geodesic flow of \( \Gamma \).

**Proof.** Let
\[
\phi_t : U_{\rho} \Gamma \to U_{\rho} \Gamma
\]
be the \textit{geodesic flow} of $\rho$, which is Hölder conjugate to a Hölder reparametrization of the geodesic flow of $\Gamma$ (see [8, Section 4]). Let $\xi = (\xi_1, \ldots, \xi_d)$ be the limit map of $\rho$. We recall that $U_\rho \Gamma$ is the quotient, under the natural action of $\Gamma$, of the fiber bundle over $\partial \infty \Gamma$ whose fiber over $(x, y)$ is 
\begin{align*}
\{(\vartheta, v) : \vartheta \in \xi_{d-1}(x), v \in \xi_1(y), \vartheta(v) = 1\} / \sim,
\end{align*}
where $(\vartheta, v) \sim (-\vartheta, -v)$. In particular, for every periodic orbit $a$ of $\phi$ we may find an element $\gamma_a \in \Gamma_H$ such that the period of $a$ coincides with $\lambda_1(\rho \gamma_a)$ when $K = \mathbb{R}$, or twice this number when $K = \mathbb{C}$ (see [8, Section 4] and [10, Corollary 2.10]).

Let $E$ be the vector bundle over $U_\rho \Gamma$ whose fiber over $[x, y, \vartheta, v]$ is $L(x, y)$. The geodesic flow on $U_\rho \Gamma$ naturally lifts to a flow $\tilde{\phi}_t$ on $E$.

\textbf{Claim A.2.} Let $\| \cdot \|$ be a Hölder norm on $E$. There exists $t_0 > 0$ such that for all $t > t_0$, all $z \in U_\rho \Gamma$ and all $u \in E_z \setminus \{0\}$ one has 
\begin{align*}
\|\tilde{\phi}_t(u)\| > \left\| u \right\| < 1/4.
\end{align*}

Provided Claim A.2, the same proof of [8, Proposition 4.2] applies to conclude the proof of Proposition A.1.

\textbf{Proof of Claim A.2.} Let $\| \cdot \|$ be a Hölder norm on $E$. There exists $t_0 > 0$ such that for all $t > t_0$, all $z \in U_\rho \Gamma$ and all $u \in E_z \setminus \{0\}$ one has 
\begin{align*}
\|\tilde{\phi}_t(u)\| > \left\| u \right\| < 1/4.
\end{align*}

Let also $f : U_\rho \Gamma \to \mathbb{R}$ be given by 
\begin{align*}
f(z) := \frac{d}{dt} \bigg|_{t=0} j_c(z, t).
\end{align*}

Finally, let $\kappa_c : U_\Gamma \to \mathbb{R}$ be defined by 
\begin{align*}
\kappa_c(z, t) := \int_0^t f(\phi_s(z)) \, ds.
\end{align*}

Then $\kappa_c$ is a Hölder cocycle and its periods coincide with those of $\kappa$ (recall that the \textit{period} according to $\kappa$ of a periodic orbit $a$ of $\phi$ of period $p_\phi(a)$ is defined by $\kappa(z, p_\phi(a))$, for $z \in a$). Livsic’s Theorem [34] guarantees that $\kappa$ and $\kappa_c$ are then cohomologous. This means that we may find a Hölder continuous function $U : U_\rho \Gamma \to \mathbb{R}$ such that for all $z \in U_\rho \Gamma$ and $t \in \mathbb{R}$ one has 
\begin{align*}
\kappa(z, t) - \kappa_c(z, t) = U(\phi_t(z)) - U(z).
\end{align*}

Furthermore, let $a$ be a periodic orbit of $U_\rho \Gamma$ and $\gamma_a \in \Gamma_H$ be such that $p_\phi(a) = \lambda_1(\rho \gamma_a)$. Then
\[ \int_{a} f = \kappa(z, p_{\phi}(a)) = p_{L}(\gamma_{a}) \geq \alpha \lambda_{1}(\rho \gamma_{a}) > 0, \]

for \( z \in a \). Since \( \phi : U_{\rho} \Gamma \to U_{\rho} \Gamma \) is a transitive metric Anosov flow we find a positive Hölder continuous function \( h : U_{\rho} \Gamma \to \mathbb{R} \) cohomologous to \( f \) (see e.g. [44, Lemma 2.5]). Combining this fact with (A.1) we find a Hölder continuous function \( V : U_{\rho} \Gamma \to \mathbb{R} \) such that for all \( z \in U_{\rho} \Gamma \), all \( t \in \mathbb{R} \) and all \( u \in E_{z} \setminus \{0\} \) one has

\[
\log \frac{\|u\|}{\|\phi_{t}(u)\|} = \int_{0}^{t} h(\phi_{s}(z)) \, ds = V(\phi_{t}(z)) - V(z).
\]

Since \( h > 0 \) the proof is complete.

\[ \square \]

A.2. The flow associated to the \((\varphi, \tau)\)-Busemann cocycle. Fix a \( \Delta \)-Anosov representation \( \rho : \Gamma \to G \). Let \( \tau \in X_{G} \) be a basepoint and \( a^{\tau} \) be a closed Weyl chamber of the system \( \Sigma(g, a) \), for some Cartan subspace \( a \subset p^{\tau} \). Let \( \varphi \) be a functional in the interior \( \text{int}(Z^{\ast}_{\rho}) \) of the dual limit cone \( Z^{\ast}_{\rho} \). Let \( c_{\varphi}^{\tau} \) and \( c_{\varphi}^{\tau} \) be the \((\varphi, \tau)\)-Busemann cocycles of \( \rho \) (c.f. Remark 8.3) and consider the action of \( \Gamma \) on \( \partial_{\infty}^{\rho} \Gamma \times \mathbb{R} \) given by

\[ \gamma \cdot (x, y, s) := (\gamma \cdot x, \gamma \cdot y, s - c_{\varphi}^{\tau}(\gamma, y)). \]

The translation flow is the flow \( \psi_{t} = \psi^{\rho,\tau,\varphi} \) on the quotient space of this action induced by the action of \( \mathbb{R} \) on \( \partial_{\infty}^{\rho} \Gamma \times \mathbb{R} \) given by

\[ t : (x, y, s) \mapsto (x, y, s - t). \]

We assume from now on that \( \rho \) is Zariski dense. Let

\[ [\cdot, \cdot]^{\varphi}_{\rho} : \partial_{\infty}^{\rho} \Gamma \to \mathbb{R} \]

be the Gromov product associated to the pair \((c_{\varphi}^{\tau}, c_{\varphi}^{\tau})\) (see [53, Lemma 7.10]) and recall that \( \pi_{\varphi}^{\tau} \) and \( \mu_{\varphi}^{\tau} \) are Patterson-Sullivan probability measures for the cocycles \( c_{\varphi}^{\tau} \) and \( c_{\varphi}^{\tau} \) respectively, of dimension \( \delta_{\varphi}^{\rho} \).

Applying Ledrappier’s framework [33], the following theorem is proved in [53, Theorem 3.2 & Theorem C] for fundamental groups of closed negatively curved manifolds. We briefly explain the main lines of the proof in our setting.

**Theorem A.3.** Let \( \rho : \Gamma \to G \) be a Zariski dense \( \Delta \)-Anosov representation and \( \varphi \in \text{int}(Z^{\ast}_{\rho}) \). Then the following holds:

1. The action of \( \Gamma \) on \( \partial_{\infty}^{\rho} \Gamma \times \mathbb{R} \) induced by \( c_{\varphi}^{\tau} \) is properly discontinuous and the translation flow \( \psi_{t} \) is conjugate, by a Hölder homeomorphism, to a Hölder reparametrization of the geodesic flow of \( \Gamma \).
2. For every periodic orbit of \( \psi \) one may find an element \( \gamma \in \Gamma_{H} \) such that the period of this periodic orbit coincides with \( \varphi(\lambda(\rho \gamma)) \).
3. Let \( h_{\text{top}}(\psi) \) be the topological entropy of \( \psi \). One has the equalities \( h_{\text{top}}(\psi) = \delta_{\varphi}^{\rho} = h_{\varphi}^{\rho} \) and the measure

\[ e^{-h_{\text{top}}(\psi) \cdot \cdot \cdot \cdot} \pi_{\varphi}^{\tau} \otimes \mu_{\varphi}^{\tau} \otimes dt \]

on \( \partial_{\infty}^{\rho} \Gamma \times \mathbb{R} \) descends to a measure on the quotient space that maximizes entropy for \( \psi_{t} \).

**Proof.** (1) Endow \( L := \partial_{\infty}^{\rho} \Gamma \times \mathbb{R} \) with the action of \( \Gamma \) by bundle automorphisms given by

- \( \square \)
\( \gamma : (x, y, s) \mapsto (\gamma \cdot x, \gamma \cdot y, e^{c_\epsilon}(\gamma \cdot y) \cdot s) \).

For every \( \gamma \in \Gamma_H \) one has
\[
 p_L(\gamma) = \varphi(\lambda(\rho \gamma))
\]
and since \( \varphi \) is positive in the asymptotic cone \( \mathcal{L}_\rho \), we find a positive constant \( \alpha \) for which
\[
 p_L(\gamma) \geq \alpha \lambda(\rho \gamma)
\]
holds. To finish apply Proposition A.1.

(2) Direct computation.

(3) By [43, Subsection 3.5] the flow \( \psi_t \) admits a unique probability measure maximizing entropy (called the Bowen-Margulis measure). This measure is ergodic (see Bowen-Ruelle [7]).

It can be seen that the quotient measure of
\[
e^{-\delta_p^\varphi [\cdot] \mathcal{L}^\varphi \otimes \mu^\varphi \otimes dt}
\]
is absolutely continuous with respect to a measure \( \nu \) that can locally be written as
\[
\nu = \nu^ss_{\text{loc}} \otimes \nu^cu_{\text{loc}},
\]
where the families \( \{ \nu^cu_{\text{loc}} \} \) and \( \{ \nu^ss_{\text{loc}} \} \) satisfy the following: each measure \( \nu^cu_{\text{loc}} \) (resp. \( \nu^ss_{\text{loc}} \)) is a finite Borel measure on local leaves \( \mathcal{U}^cu_{\text{loc}} \) (resp. \( \mathcal{U}^ss_{\text{loc}} \)) of the central unstable (resp. strong stable) lamination of \( \psi_t \), and for every \( t \in \mathbb{R} \) the following equalities are satisfied
\[
\phi_{t*} \nu^cu_{\text{loc}} = e^{-\delta_p^\varphi [\cdot] \mathcal{L}^\varphi \otimes \mu^\varphi \otimes dt} \text{ and } \phi_{t*} \nu^ss_{\text{loc}} = e^{\delta_p^\varphi [\cdot] \mathcal{L}^\varphi \otimes \mu^\varphi \otimes dt}.
\]
Standard techniques of the theory of (metric) Anosov flows (see e.g. Katok-Hasselblatt [30, Section 5 of Chapter 20]) imply that \( \delta_p^\varphi \) must coincide with the topological entropy \( h_{\text{top}}(\psi) \) of \( \psi_t \) and that \( \nu \) is proportional to the Bowen-Margulis measure of this flow. By ergodicity, the measure
\[
e^{-\delta_p^\varphi [\cdot] \mathcal{L}^\varphi \otimes \mu^\varphi \otimes dt}
\]
descends to a measure proportional to the Bowen-Margulis measure of \( \psi \).

Finally, the inequalities
\[
h_{\text{top}}(\psi) \leq h_p^\varphi \leq \delta_p^\varphi
\]
are easy to check, and this completes the proof.

(\[ \square \]

A.3. Distribution of fixed points. We now finish the proof of Proposition 8.7.

Proof of Proposition 8.7. The Zariski density assumption over \( \rho \) implies that the set of periods of periodic orbits of \( \psi_t \) span a dense subgroup of \( \mathbb{R} \) (see Benoist [4, Main Proposition]). On the other hand, by Remark 8.3 one has that the equality
\[
e^{-h_p^\varphi [\cdot] \mathcal{L}^\varphi \otimes \mu^\varphi \otimes dt} = e^{-h_p^\varphi [\cdot] \mathcal{L}^\varphi \otimes \mu^\varphi \otimes dt}
\]
holds up to scaling. Using the thermodynamical formalism the proof of [53, Proposition 4.3] can now be adapted to our setting. Indeed, the only difference is that in our general setting the group \( \Gamma \) may have torsion elements. However, the proof adapts as follows (c.f. Blayac [5, Section 8.3.3]).

Let \( \Gamma_{SP} \subset \Gamma_H \) be the set of strongly primitive elements, i.e. elements \( \gamma \in \Gamma_H \) for which the period of the corresponding periodic orbit of \( \psi \) is precisely \( \varphi(\lambda(\rho \gamma)) \). By Pollicott [43, Subsection 3.5] there exists a positive \( m \) such that
\[ \sigma_t := m e^{-h_{\varphi}^\gamma t} \sum_{\gamma \in \Gamma^0 \cdot \rho(\lambda(\rho)) \leq t} \frac{1}{\varphi(\lambda(\rho))} \delta_{\gamma^-} \otimes \delta_{\gamma^+} \to e^{-h_{\varphi}^\gamma \cdot [\cdot]} \mu_{\varphi}^\gamma \otimes \mu_{\varphi}. \]

Pick a subset \( \Gamma^0 \subset \Gamma_{\text{SP}} \) in one to one correspondence with the set of axes \( \{(\gamma^-, \gamma^+) \times \mathbb{R}\}_{\gamma \in \Gamma_{\text{H}}}. \)

For \( \gamma \in \Gamma^0 \) let \( S_\gamma \subset \Gamma \) be the set of elements that act trivially on the corresponding axis \( (\gamma^-, \gamma^+) \times \mathbb{R} \). We have
\[ \sigma_t = m e^{-h_{\varphi}^\gamma t} \sum_{\gamma \in \Gamma^0 \cdot \rho(\lambda(\rho)) \leq t} \frac{\# S_\gamma}{\varphi(\lambda(\rho))} \delta_{\gamma^-} \otimes \delta_{\gamma^+}. \]

On the other hand, let
\[ \nu_t := m e^{-h_{\varphi}^\gamma t} \sum_{\gamma \in \Gamma_{\text{H}} \cdot \rho(\lambda(\rho)) \leq t} \delta_{\gamma^-} \otimes \delta_{\gamma^+}. \]

It follows easily that
\[ \nu_t = m e^{-h_{\varphi}^\gamma t} \sum_{\gamma \in \Gamma^0 \cdot \rho(\lambda(\rho)) \leq t} \frac{\# S_\gamma}{\varphi(\lambda(\rho))} \delta_{\gamma^-} \otimes \delta_{\gamma^+}. \]

The convergence \( \sigma_t - \nu_t \to 0 \) can now be proven exactly as in \([53, \text{Proposition 4.3}]\). \( \square \)

A.4. **Mixing and counting.** Let \( \mathbb{R}^+ X_\varphi \subset \mathcal{L}_\rho \) be the direction dual to \( \mathbb{R}^+ \varphi \subset \text{int}(\mathcal{L}_\varphi) \). By definition, it is the unique direction in \( \mathcal{L}_\rho \) in which the form \( h_{\varphi}^\gamma \varphi \) is tangent to Quint’s growth indicator \( \psi_\rho \) (c.f. Quint \([47]\) and \([54, \text{Theorem 4.20}]\)). Here we pick the vector \( X_\varphi \) in such a way that \( \varphi(X_\varphi) = 1 \). Given a Euclidean norm \( |\cdot| \) on a let
\[ I(X) := \frac{|X|^2 |X_\varphi|^2 - (X, X_\varphi)^2}{|X_\varphi|^2} \]
for \( X \in \ker \varphi \).

On the other hand, the Weyl chamber flow is the action of \( a \) on the space \( \rho(\Gamma)G/M \) given by
\[ X : \rho(\Gamma)gM \mapsto \rho(\Gamma)g \exp(X)M \]
for every \( X \in a \) and \( g \in G \). The \( \tau \)-Busemann cocycle of \( G \) induces a \( G \)-equivariant identification \( G/M \cong \mathbb{F}(V)^{(2)} \times a \) and, for a given a Lebesgue measure \( \text{Leb}_a \) on \( a \), the pushforward of
\[ e^{-h_{\varphi}^\gamma [\cdot]} \mu_{\varphi}^\gamma \otimes \mu_{\varphi}^\gamma \otimes \text{Leb}_a \]
under the map \( \xi_\varphi \times \xi_\varphi \times \text{id}_a \) descends to a measure \( \chi_\varphi \) on \( \rho(\Gamma)G/M \) invariant under the Weyl chamber flow. This measure is called the \( \varphi \)-Bowen-Margulis measure of \( \rho(\Gamma) \).

Given two functions \( f_0, f_1 : \rho(\Gamma)G/M \to \mathbb{R} \) and an element \( X \in a \) denote by \( (X \cdot f_0)f_1 \) the function \( \rho(\Gamma)G/M \to \mathbb{R} \) given by
\[ z \mapsto f_0(X \cdot z)f_1(z). \]

The proof of the following theorem follows line by line \([54, \text{Theorem 4.23}]\). Indeed, the central tools involved in the proof of that result are the Reparametrizing Theorem \([53, \text{Theorem 3.2}]\) (here replaced by Theorem A.3) and the thermodynamical formalism for the flow \( \psi_1 \).
Theorem A.4. There exists a positive constant \(c\) and a Euclidean norm \(\lVert \cdot \rVert\) on \(a\) such that for every pair of compactly supported continuous functions \(f_0, f_1 : \rho(\Gamma) \setminus G/M \to \mathbb{R}\) and every \(X_0 \in \text{ker} \, \varphi\) one has
\[
(2\pi t)^{(\dim a-1)/2} \chi_\varphi((tX_\varphi + \sqrt{t}X_0) \cdot f_0) f_1) = c e^{-\frac{I(X_0)}{2}} \chi_\varphi(f_0) \chi_\varphi(f_1)
\]
as \(t \to \infty\).

Equipped with Theorem A.4, the contents of [54, Section 5] remain valid in our setting and give the following counting theorem (recall that \(\delta_\rho\) is the critical exponent of \(\rho\)).

Theorem A.5. There exists a positive constant \(C\) such that
\[
Ce^{-\delta_\rho t} \# \{\gamma \in \Gamma : \lVert a^\tau(\rho\gamma) \rVert_a \leq t\} \to 1
\]
as \(t \to \infty\), where \(\lVert \cdot \rVert_a\) is the Euclidean norm on \(a\) induced by the Riemannian structure on \(X_G\).

References

[1] Martine Babillot. Points entiers et groupes discrets: de l’analyse aux systèmes dynamiques. Panor. Synthèses, 13:1–119, 2002.
[2] Yves Benoist. Actions propres sur les espaces homogènes réductifs. Ann. of Math., 144:315–347, 1996.
[3] Yves Benoist. Propriétés asymptotiques des groupes linéaires. Geom. Funct. Anal., 7:1–47, 1997.
[4] Yves Benoist. Propriétés asymptotiques des groupes linéaires II. Adv. Stud. Pure Math., 26:33–48, 2000.
[5] Pierre-Louis Blayac. Aspects dynamiques des structures projectives convexes. PhD thesis, Université Paris-Saclay, 2021.
[6] Jairo Bochi, Rafael Potrie, and Andrés Sambarino. Anosov representations and dominated splittings. J. Eur. Math. Soc. (JEMS), 11:3343–3414, 2019.
[7] Rufus Bowen and David Ruelle. The ergodic theory of axiom A flows. Invent. Math., 29:181–202, 1975.
[8] Martin Bridgeman, Richard Canary, François Labourie, and Andrés Sambarino. The pressure metric for Anosov representations. Geom. Funct. Anal., 25:1089–1179, 2015.
[9] Martin Bridgeman, Richard Canary, François Labourie, and Andrés Sambarino. Simple root flows for Hitchin representations. Geom. Dedicata, 192:57–86, 2018.
[10] Martin Bridgeman, Beatrice Pozzetti, Andrés Sambarino, and Anna Wienhard. Hessian of Hausdorff dimension on purely imaginary directions. Preprint, arXiv:2010.16308 [math.DG].
[11] León Carvajales. Convex real projective structures on closed surfaces are closed. Proc. Amer. Math. Soc., 118(2):657–661, 1993.
[12] Dave Constantine, Jean-François Lafont, and Dan Thompson. Strong symbolic dynamics for geodesic flow on CAT(-1) spaces and other metric Anosov flows. J. Éc. polytech. Math, 7:201–231, 2020.
[13] Jeffrey Danciger, François Guérıtaud, and Fanny Kassel. Convex cocompact actions in real projective geometry. Preprint, arXiv:1704.08711 [math.GT].
[14] William Duke, Zeev Rudnick, and Peter Sarnak. Density of integer points on affine homogeneous varieties. Duke Math. J., 71(1):143–179, 1993.
[15] Sam Edwards, Minju Lee, and Hee Oh. Anosov groups: local mixing, counting, and equidistribution. Preprint, arXiv:2003.14277 [math.DS].
[16] Étienne Ghys and Pierre de la Harpe. Sur les groupes hyperboliques d’après Mikhael Gromov. Number 83 in Progr. Math. Springer Science+Business Media, LLC, 1990.
[20] Olivier Glorieux, Nicolas Tholozan, and Daniel Monclair. Hausdorff dimension of limit sets for projective Anosov representations. Preprint, arXiv:1902.01844 [math.DG].
[21] Mikhael Gromov. Hyperbolic groups. Essays in group theory. Springer-Verlag, 1987.
[22] Olivier Guichard, François Guéritaud, Fanny Kassel, and Anna Wienhard. Anosov representations and proper actions. Geom. Topol., 21:485–584, 2017.
[23] Olivier Guichard and Anna Wienhard. Anosov representations: domains of discontinuity and applications. Invent. Math., 190:357–438, 2012.
[24] Sigurdur Helgason. Differential geometry, Lie groups, and symmetric spaces. Academic Press, 1978.
[25] Roger Horn and Charles Johnson. Matrix analysis. Cambridge Univ. Press, 1985.
[26] Michael Kapovich, Bernhard Leeb, and Joan Porti. Anosov subgroups: Dynamical and geometric characterizations. Eur. J. Math., 3:808–898, 2017.
[27] Michael Kapovich, Bernhard Leeb, and Joan Porti. Dynamics on flag manifolds: domains of proper discontinuity and cocompactness. Geom. Topol., 22(1):157–234, 2018.
[28] Michael Kapovich, Bernhard Leeb, and Joan Porti. A Morse Lemma for quasigeodesics in symmetric spaces and euclidean buildings. Geom. Topol., 22(7):3827–3923, 2018.
[29] Fanny Kassel. Geometric structures and representations of discrete groups. Proc. Int. Cong. of Math., 1:1113–1150, 2018.
[30] Anatole Katok and Boris Hasselblatt. Introduction to the modern theory of dynamical systems. Cambridge Univ. Press, 1995.
[31] Anthony Knapp. Lie groups beyond an introduction. Number 140 in Progr. Math. Springer Science+Business Media, LLC, 1996.
[32] François Labourie. Structure au bord des variétés à courbure négative. Séminaire de Théorie spectrale et géométrie de Grenoble, 13:97–122, 1994.
[33] A N Livšic. Cohomology of dynamical systems. Math. USSR Izvestija, 6:1278–1301, 1972.
[34] Toshihiko Matsuki. The orbits of affine symmetric spaces under the action of minimal parabolic subgroups. J. Math. Soc. Japan, 31(2):331–357, 1979.
[35] Igor Mineyev. Flows and joins of metric spaces. Geom. Topol., 9:403–482, 2005.
[36] Amir Mohammadi and Hee Oh. Matrix coefficients, counting and primes for orbits of geometrically finite groups. J. Eur. Math. Soc. (JEMS), 17:837–897, 2015.
[37] Hee Oh and Nimish Shah. The asymptotic distribution of circles in the orbits of Kleinian groups. Invent. Math., 187:1–35, 2012.
[38] Hee Oh and Nimish Shah. Equidistribution and counting for orbits of geometrically finite hyperbolic groups. J. Amer. Math. Soc., 26:511–562, 2013.
[39] Hee Oh and Nimish Shah. Limits of translates of divergent geodesics and integral points on one-sheeted hyperboloids. Israel J. Math., 199:915–931, 2014.
[40] Hee Oh and Nimish Shah. Counting visible circles on the sphere and Kleinian groups. Proceedings of the conference on “Geometry, Topology and Dynamics in negative curvature”, LMS series, 425:272–288, 2016.
[41] Thomas Roblin. Ergodicité et équidistribution en courbure négative. Mém. Soc. Math. Fr. (N.S.), 95, 2003.
[51] W. Rossmann. The structure of semisimple symmetric spaces. *Canad. J. Math.*, 31(1):157–180, 1979.

[52] Andrés Sambarino. Hyperconvex representations and exponential growth. *Erg. Theo. Dyn. Sys.*, 34(3):986–1010, 2014.

[53] Andrés Sambarino. Quantitative properties of convex representations. *Comment. Math. Helv.*, 89:443–488, 2014.

[54] Andrés Sambarino. The orbital counting problem for hyperconvex representations. *Ann. Inst. Fourier (Grenoble)*, 65(4):1755–1797, 2015.

[55] Henrik Schlichtkrull. *Hyperfunctions and harmonic analysis on symmetric spaces*. Number 49 in *Progr. Math.*, Birkhäuser-Verlag, 1984.

[56] Florian Stecker. *Domains of discontinuity of Anosov representations in flag manifolds and oriented flag manifolds*. PhD thesis, Ruprecht-Karls-Universität Heidelberg, 2019.

[57] Xavier Thirion. *Sous-groupes discrets de SL(d, R) et équidistribution dans les espaces symétriques*. PhD thesis, Université de Tours, 2007.

[58] Anna Wienhard. An invitation to higher Teichmüller theory. *Proc. Int. Cong. of Math.*, 1:1007–1034, 2018.

[59] Andrew Zimmer. Projective Anosov representations, convex cocompact actions, and rigidity. *Preprint, arXiv:1704.08582 [math.DG]*, 2018.

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