Hamiltonian Floer theory on surfaces

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Abstract

We develop connections between the qualitative dynamics of Hamiltonian isotopies on a surface \( \Sigma \) and their chain-level Floer theory using ideas drawn from Hofer-Wysocki-Zehnder’s theory of finite energy foliations. We associate to every collection of capped 1-periodic orbits which is ‘maximally unlinked relative the Morse range’ a singular foliation on \( S^1 \times \Sigma \) which is positively transverse to the vector field \( \partial_t \oplus X^H \) and which is assembled in a straight-forward way from the relevant Floer moduli spaces. As a consequence, we derive a purely topological and Turing-computable characterization of the spectral invariant \( c(H; [\Sigma]) \) for generic Hamiltonians on arbitrary closed surfaces. This completes, for generic Hamiltonians, a project initiated by Humilière-Le Roux-Seyfaddini, in addition to fulfilling a desideratum expressed by Gambaudo-Ghys seeking a topological characterization of the Entov-Polterovich quasi-morphism on \( \text{Ham}(S^2) \).

1 Introduction

The goal of this paper is to introduce a set of tools for tying together the qualitative dynamics of Hamiltonian isotopies on surfaces and the chain-level structure of their Floer complexes. Since its introduction by Floer in his approach to the famed Arnol’d conjecture, Hamiltonian Floer theory has become a standard tool in the modern symplectic geometer’s toolkit. However, in spite of the numerous uses to which it has been put, the relationship between the Floer complex \( CF_\ast(H,J) \) of a generic Floer pair \((H,J)\) and features of the dynamics of the Hamiltonian isotopy \( \phi^H_t \) induced by \( H \) remains rather unclear, essentially due to the difficulty in understanding the relationship between solutions to Floer’s equations and the qualitative properties of the (generically non-autonomous) vector field \( X^H \) used to define them.

To give a better sense of what we mean by this, and to give some sense of the analogy that our work seeks to develop, consider the case when \( H \) is a \( C^2 \)-small Morse function on a surface and \( J \) is constant; because of the tight relationship between negative gradient-flow lines and the defining Hamiltonian vector field — away from singular points of \( X^H \), the negative gradient trajectories are \textit{positively transverse} to the trajectories of the Hamiltonian flow — it’s easy to construct the filtered Floer complex simply from knowledge of the qualitative dynamics of \( X^H \) and the critical values of \( H \), and contrariwise, one can readily extract a significant amount of information about the qualitative dynamics of \( X^H \) from knowledge of the Floer complex. The situation becomes much murkier if we allow \( H \) or \( J \) to vary with time or even if we omit the \( C^2 \)-smallness condition. In fact, the strongest result that exists in this direction has been given by Humilière-Le Roux-Seyfaddini in \([12]\), in which the authors give a characterization of the Hamiltonian spectral invariant \( c(H; [\Sigma]) \) (see Section 4.1 for the definition) in terms of the topology of certain braids of 1-periodic orbits of \( X^H \) in the case when \( H \) is \textit{any} autonomous Hamiltonian and \( \Sigma \neq S^2 \). However, even the approach therein hints at the limitations of our collective understanding of the situation; in order to circumvent the difficulties posed by the relationship between solutions to Floer’s equations and the qualitative properties of the (generically non-autonomous) vector field \( X^H \) used to define them.

The view developed in this paper attacks the question of the relationship of Floer cylinders to dynamics in low dimensions head-on, and we develop a theory which essentially reduces the study of this

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relationship for a generic Hamiltonian to the case of understanding $C^2$-small Morse functions, related to the underlying dynamics in a straight-forward way. This gives a transparent relationship between the homologically non-trivial part of the (filtered) Floer complex and the dynamics of the Hamiltonian isotopy. As a corollary, we obtain an effective topological characterization of the spectral invariants associated to the point and fundamental classes on arbitrary closed symplectic surfaces.

1.1 Main results

Unless otherwise noted, we always work on a closed symplectic surface $(\Sigma, \omega)$. In order to treat the case of $C^2$-small Morse functions on even footing with generic Hamiltonians, we consider the flow of the lifted vector field $\hat{X}^H := \partial_t \oplus X^H$ on $S^1 \times \Sigma$. Hamiltonian Floer theory from our point of view then becomes a matter of studying the relationship between the dynamics of the flow of $\hat{X}^H$ and cylinders of the form $(s, t) \mapsto (t, u(s, t)) \in S^1 \times \Sigma$, for $u$ a Floer cylinder running between strands of the braid given by graphs $t \mapsto (t, x(t))$ for $x$ an element of $\text{Per}_0(H)$, the collection of a contractible 1-periodic orbits of $H$.

In order to treat all surfaces at once (including the case of $\Sigma = S^2$), it becomes useful to introduce the notion of a capped braid. Essentially, a capped braid $\hat{X}$ is a finite collection of capped loops $\{\hat{x}_1, \ldots, \hat{x}_k\}$ called its strands such that the graphs of the underlying loops are all disjoint in $S^1 \times \Sigma$. We say that $\hat{X}$ is unlinked if the capping disks may be chosen to have disjoint graphs in $D^2 \times \Sigma$ (see Section 3 for formal definitions). For a Hamiltonian $H$ and a capped braid $\hat{X} \subseteq \text{Per}_0(H)$ of capped 1-periodic orbits of $\phi^H$, we say that $\hat{X}$ is maximally unlinked relative the Morse range if $\hat{X}$ is unlinked, each capped loop in $\hat{X}$ has Conley-Zehnder index lying in the set $\{1, 0, 1\}$, and such that $\hat{X}$ is maximal (as a subset of $\text{Per}_0(H)$) with respect to these two properties. We denote the collection of such capped braids by $\text{murm}(H)$. Our main result is the following

**Theorem A.** Let $(H, J)$ be a Floer non-degenerate pair on a closed symplectic surface $(\Sigma, \omega)$, then to each $\hat{X} \in \text{murm}(H)$ we may associate a singular foliation $\mathcal{F}^X$ of $S^1 \times \Sigma$ with singular leaves given by the graphs of the strands of $\hat{X}$ and regular leaves parametrized by maps of the form $(s, t) \mapsto (t, u(s, t))$ for $u$ a Floer cylinder for $(H, J)$ connecting $\hat{x}, \hat{y} \in \hat{X}$. Moreover, the vector field $\partial_t \oplus X^H$ is positively transverse to $\mathcal{F}^X$.

Similar foliations played a crucial role in Bramham’s construction of periodic approximations for irrational pseudo-rotations of the disk in [2]. Our approach gives both the existence of such foliations for generic Hamiltonians, and moreover ties their behaviour directly to the structure of the Floer complex and the dynamics of the Hamiltonian isotopy.

The structure of the foliations $\mathcal{F}^X$ could be a priori rather complicated, however, with Theorem A in hand, we can define the $\hat{X}$-restricted action functional $A^\hat{X} \in C^\infty(S^1 \times \Sigma)$ by $A^\hat{X}(t, u(s, t)) = A_H(\hat{u}_s)$, for $A_H : \mathcal{C}_0(\Sigma) \rightarrow \mathbb{R}$ the Hamiltonian action functional on the space of capped loops (the capping of $u_s$ is naturally induced by the cappings of the limiting orbits). $A^\hat{X}$ turns out to be a Morse-Bott function, and if we define $A^\hat{X}_t \in C^\infty(\Sigma)$, $t \in S^1$, to be its restriction to the fiber $\{t\} \times \Sigma$, we obtain an $S^1$-family of Morse functions, such that the negative gradient flow of $A^\hat{X}_t$ provides a singular foliation which coincides with the foliation $\mathcal{F}^X$ given by intersecting $\mathcal{F}^\hat{X}$ with the fiber over $t \in S^1$. Sliding the fiber $\{0\} \times \Sigma$ along the circles $t \mapsto (t, u_s(t))$ provides a loop $(\psi^X_t)_{t \in S^1}$, and we prove

**Theorem B.** For every $t \in S^1$, $\mathcal{F}^X_t$ is a singular foliation of Morse type. Moreover, the loop $(\psi^X_t)_{t \in S^1}$ is a contractible Hamiltonian loop such that the orbits of $(\psi^X_t)^{-1} \circ \phi^H$ are positively transverse to the foliation $\mathcal{F}^X_0$.

We thereby reduce the study of the qualitative dynamics of the isotopy $\phi^H$ to the much better understood situation of dynamics which are positively transverse to a Morse-type foliation. Note that similar foliations (with a weaker notion of positive transversality) have been constructed by Le Calvez in [17] for Hamiltonian homeomorphisms and play a central role in the forcing theory developed in [18]. The above result can be viewed as giving a Floer-theoretic construction of certain of Le Calvez’s foliations, along with additional insight into their structure in the smooth case. We also obtain as a corollary
the following somewhat surprising structural result about the topology of the braid generated by the 1-periodic orbits of $H$ (a capped braid is said to be linked if it is not unlinked):

**Theorem C.** Let $H$ be non-degenerate $\hat{X} \in \text{murm}(H)$, and let $\hat{y} \notin \hat{X}$ be a capped time-$k$ orbit of $\phi^H$ for $k \in \mathbb{Z}$, then $\hat{X} \cup \hat{y}$ is linked. In particular every capped braid which is maximally unlinked relative the Morse range is maximally unlinked as a subset of $\text{Per}_0(H)$.

Finally, as the foregoing discussion suggests, our approach gives a transparent relationship between the filtered Floer complex in non-trivial degrees and the dynamics of a generic Hamiltonian isotopy. As an example of the sort of chain-level information which can be extracted by these techniques we give the following characterization of the spectral invariants (see Section 4.1 for definitions) associated to the point and the fundamental class:

**Theorem D.** Let $H$ be a non-degenerate Hamiltonian on a closed symplectic surface $(\Sigma, \omega)$, then

$$c(H; [\Sigma]) = \min_{\hat{x} \in \text{murm}(H)} \max_{\hat{x} \in \hat{X}} A_H(\hat{x}),$$

$$c(H; [pt]) = \max_{\hat{x} \in \text{murm}(H)} \min_{\hat{x} \in \hat{X}} A_H(\hat{x}).$$

This seems to be the first such result giving an effectively computable formula (modulo the task of first calculating the 1-periodic orbits) for spectral invariants of generic Hamiltonians. The closest precedent appears to be [12], which computes these spectral invariants for autonomous Hamiltonians when $\Sigma \neq S^2$ through similar minimax formulas over slightly different collections of orbits. There are also characterizations of $c(H; [\Sigma])$ for higher dimensional manifolds with $H$ non-autonomous but with significant restrictions on $H$ (see [13] for the aspherical case and [21] for an extension to general symplectic manifolds). Additionally, this result is easily seen to give a (non-effective) topological characterization of the Entov-Polterovich quasi-morphism $\mu : \text{Ham}(S^2, \omega) \to \mathbb{R}$ defined in [4] by

$$\mu(\phi) := \text{Vol}(S^2, \omega) \cdot \lim_{k \to \infty} \frac{c(H^k; [S^2])}{k},$$

for $H$ any Hamiltonian with $\phi^H_1 = \phi$, in terms of the long term structure of the collections of capped braids $\text{murm}(H^k)$. The question of the existence of such a topological characterization was first raised explicitly by Gambaudo-Ghys [5] but little positive progress has been made since (although see [14] for some impossibility results, which in particular imply that the ‘braid quasi-morphism averaging’ procedure developed in [5] cannot produce $\mu$).

### 1.2 Main ideas

The ideas in this paper evolve reasonably naturally from adopting the position — implied by the work of Humilière-Le Roux-Seyfaddini — that the topology of the capped braids which make up $\text{Per}_0(H)$ ought to be in some way related to the structure of the filtered Floer complex of $H$, and proceeding to study why this should be the case.

The homological linking number, presented in Section 3, produces a homotopy invariant for pairs of capped braids $(\hat{X}, \hat{Y})$ by counting the signed number of intersections made by the strands throughout a generic deformation which takes $\hat{X}$ to $\hat{Y}$, subject to a certain homotopy condition. The graph of such a deformation gives rise to a collection of cylinders in $[0,1] \times S^1 \times \Sigma$, which we call a braid cobordism, whose signed intersections are precisely what is counted by the homological linking number. The proof of Theorem A exploits the interaction of this invariant with the positivity of intersections of pseudoholomorphic curves in dimension 4 by viewing, via the Gromov trick, the Floer boundary and continuation operators as giving rise to collections of braid cobordisms between capped braids made up of capped orbits lying over the corresponding ends (we adopt, therefore, the field-theoretic perspective on Floer theory presented in [24]). Positivity of intersections implies that the homological linking number cannot decrease along such cobordisms. In order to deal with cylinders which emerge from or converge to the same orbit, we make use of the analysis of the relative asymptotic behaviour of pseudo-holomorphic curves developed by Siefring in [29], which pairs with work of Hofer-Wysocki-Zehnder in [8] to connect the Conley-Zehnder index of an orbit to bounds on the winding behaviour of pairs of cylinders which emerge from or converge to that orbit.
In the contact setting, these sorts of bounds (in the non-relative case), along with the insight that under appropriate index conditions on the asymptotic orbits families of pseudoholomorphic curves automatically form local foliations in the symplectization of a contact manifold, go back to the pioneering work of Hofer-Wysocki-Zehnder in [9], [8], [10] and [11]. Siefring has also more recently put this circle of ideas to use in [30] to define an intersection number for arbitrary pseudoholomorphic curves in 4-dimensional symplectic cobordisms which is invariant under homotopy.

In comparison with [30] and [11] which put many of these same ideas to work, the main novelties in this paper are our use of the homological intersection number to provide a means by which to probe the global topology of the collection of $\overline{\operatorname{Per}}_0(H)$, and to relate this to the behaviour of Floer moduli spaces, along with the use of Floer-theoretic tools — continuation maps and the cap product — to provide existence principles for the required families of Floer cylinders against which we may pit our a priori controls. In addition, the fact that we place ourselves in a Floer-theoretic (rather than an SFT-type setting, as in the previously mentioned works) considerably simplifies the analytic prerequisites; hopefully this work may serve to initiate non-experts to this productive circle of ideas while minimizing the technical requirements. This circle of ideas is developed in the the form needed for our purposes in Section 4.2 (linking and asymptotic behaviour of Floer cylinders) and Section 4.3 (producing local foliations from Floer moduli spaces).

The idea of relating the global topology of the braids formed by periodic orbits of a Hamiltonian to Floer theory via positivity of intersections arguments also has some precedent on the disk in [33], in which the authors use Floer homology to define braid invariants and obtain a forcing theory for periodic orbits of Hamiltonians, and it is likely that the results in this paper could be combined with these to productive ends, but we make no attempt at this here.

A final novelty of our approach is that we take the topology of capped braids seriously (rather than assigning primacy to the trivializations which they induce). In addition to allowing us to treat all surfaces in a uniform manner, the value of this perspective in the Floer-theoretic context is supported by the identification provided by Proposition 3.23 between the area of a capping disk for a capped orbit $\hat{x}$ and the average homological linking of $\hat{x}$ with each point of the surface (see Proposition 3.23). This provides crucial tools for relating the actions of various capped orbits to their dynamics relative other orbits in $\overline{\operatorname{Per}}_0(H)$ (see Lemma 8.1 and Proposition 8.2), which prove essential in characterizing the spectral invariants.

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2 Stefan-Sussmann foliations

It will be convenient in what follows to have some explicit language with which to speak about singular foliations. To that end, we will make use of some elementary notions from the theory of Stefan-Sussmann foliations (cf. [3] and the references therein, in particular [31]) which will be suitable to our purposes.

Let $G^k(M) \to M$ denote the $k$-Grassmannian of $M$, having fiber $\operatorname{Gr}(k,T_xM)$ over $x \in M$, and let $G^*(M) := \sqcup_{k=0}^n G^k(M)$, where $n = \dim M$ denote the total Grassmannian of $M$.

**Definition 2.1.** A (generalized) distribution on a manifold $M$ is a section $D : M \to G^*(M)$. A local section $X : M \to TM$ is said to belong to $D$ if $X(x) \in D(x)$ for all $x \in \operatorname{dom}(X)$. The set of all smooth local sections $X \in \mathcal{X}_{\text{loc}}(M)$ belonging to $D$ is denoted by $\Delta_D$.

**Definition 2.2.** A generalized distribution $D$ is said to be smooth if, $D(x) = \text{span} \langle X(x) \rangle_{X \in \Delta_D}$, for every $x \in M$.

**Definition 2.3.** A (smooth) $k$-leaf of $M$ is a subset $L \subseteq M$ equipped with a differentiable structure $\sigma$ such that $(L, \sigma)$ is a connected $k$-dimensional immersed submanifold of $M$ and such that for any
continuous map \( f : N \to M \) with \( f(N) \subseteq L \) and \( N \) a locally connected topological space, we have that \( f : N \to (L, \sigma) \) is continuous.

**Definition 2.4.** A \((C^\infty)\)-singular (Stefan-Sussmann) foliation of \( M \) is a partition \( \mathcal{F} \) of \( M \) into smooth leaves of \( M \) such that for every \( x \in M \), there exists a local smooth chart \( \varphi : U \to \mathcal{O}(x) \subseteq M \), from \( U \subseteq \mathbb{R}^n \) an open neighbourhood of \( 0 \in \mathbb{R}^n \), with \( \varphi(0) = x \) and

1. \( U = V \times W \) for \( V \) an open neighbourhood of \( 0 \) in \( \mathbb{R}^k \) and \( W \) an open neighbourhood of \( 0 \) in \( \mathbb{R}^{n-k} \), where \( k \) is the dimension of the smooth leaf \( L_x \in \mathcal{F} \) containing \( x \).
2. \( L \cap \varphi(U \times W) = \varphi(U \times l) \), for every leaf \( L \in \mathcal{F} \), where \( l := \{ w \in W : \varphi(0, w) \in L \} \).

**Definition 2.5.** A smooth generalized distribution \( D \) is said to be integrable if for every \( x \in M \) there exists an immersed manifold \( L \subseteq M \), such that \( x \in L \), and \( T_y L \subseteq D(y) \) for all \( y \in L \). Such an immersed submanifold is called an integral submanifold of \( D \).

**Theorem 2.6** ([31]). If \( D \) is a smooth integrable generalized distribution and \( \mathcal{F}_D \) is the partition of \( M \) formed by taking the collection maximal connected integral submanifolds of \( D \), then \( \mathcal{F}_D \) is a smooth singular Stefan-Sussmann foliation.

For a singular foliation \( \mathcal{F} \), we let \( d(-, \mathcal{F}) \) denote the function which sends \( x \in M \) to the dimension of the leaf of \( \mathcal{F} \) passing through \( x \in M \).

**Definition 2.7.** A smooth singular foliation is said to have codimension \( k \) if \( n - k = \max_{x \in M} d(x, \mathcal{F}) \). For such a foliation, we define the domain to be \( \text{dom}(\mathcal{F}) := \{ x \in M : d(x, \mathcal{F}) = n - k \} \), while we define the singular set of \( \mathcal{F} \) to be \( \text{sing}(\mathcal{F}) := M \setminus \text{dom}(\mathcal{F}) \).

A leaf of \( \mathcal{F} \) is said to be regular if it is of maximal dimension, otherwise it is said to be singular. \( \mathcal{F} \) is said to be oriented if every regular leaf of \( \mathcal{F} \) is in addition equipped with an orientation, and the local charts about points on the regular leaves may be taken to be orientation-preserving.

## 3 Capped braids and homological linking

Throughout this section and the rest of the paper, \( \Sigma \) will always denote a smooth symplectic surface \((\Sigma, \omega)\), \( L(M) \) the (smooth) loop space of the manifold \( M \), \( L_0(M) \) its space of contractible loops and \( \tilde{L}_0(M) \) its Novikov covering space (see [19], Section 12.1). For \( x \in L(M) \), we write \( \bar{x}(t) := (t, x(t)) \) for its graph. For \( u : I \times S^1 \to M \), where \( I \subseteq \mathbb{R} \), we write \( \bar{u}(s, t) := (s, t, u(s, t)) \) for its graph and \( \bar{u}(s, t) := (t, u(s, t)) \) for the projection of its graph onto \( S^1 \times M \).

**Definition 3.1.** For any \( k \in \mathbb{N} \), we define the \( k \)-configuration space

\[
C_k(\Sigma) := \{ (z_1, \ldots, z_k) \in \Sigma^k : (i \neq j) \Rightarrow z_i \neq z_j \}
\]

**Definition 3.2.** An (ordered) \( k \)-braid is an element \( X = (x_1, \ldots, x_k) \in L(C_k(\Sigma)) \). Denote by \( B^k(\Sigma) \) the space of ordered \( k \)-braids. The loop \( x_i \) is called the \( i \)-th strand of \( X \), for \( i = 1, \ldots, k \).

**Definition 3.3.** An unordered \( k \)-braid is an element \( [X] \in L(C_k(\Sigma))/S_k \), where \( S_k \) acts by permutation of coordinates. Such unordered braids may be identified with certain finite subsets of \( L(\Sigma) \).

**Remark.** We raise the distinction between ordered and unordered braids here mainly to flag for the reader that we will make no real effort outside of this section to maintain the distinction between these two concepts. In particular, we will routinely treat ordered braids as finite subsets of \( L(\Sigma) \) and perform set-wise operations on them, when properly speaking we should be speaking of the unordered braids which they represent. We will moreover speak simply of ‘braids’ relying on the context to make clear whether these braids are ordered or unordered. For the remainder of this section, we will make a clear distinction between ordered and unordered braids, mainly to convince the suspicious reader that nothing essential is lost in making this elision.

**Definition 3.4.** The graph \( \bar{X} \) of an (ordered) \( k \)-braid is the set-valued map \( \bar{X}(t) = \bigsqcup_{i=1}^k \bar{x}(t) \subseteq S^1 \times \Sigma \), \( t \in S^1 \). The graph of an unordered braid \([X]\) is the graph of some (hence every) representative \( X \) of \([X]\).
Definition 3.5. An ordered \( k \)-braid \( Y \in B^k(\Sigma) \) is an ordered sub-braid of \( X \in B^k(\Sigma) \) if \( Y \subseteq X \), as an ordered set. An unordered braid \([Y]\) is a sub-braid of \([X]\) if \([Y] \subseteq [X]\) as sets. There is an obvious partial ordering on the collection of sub-braids of \( X \) (resp. of \([X]\)).

Definition 3.6. \( X \in B^k(\Sigma) \) is contractible if each strand of \( X \) is a contractible loop. We write \( B^k(\Sigma) \) for the space of contractible ordered \( k \)-braids. \([X] \in B^k(\Sigma)/S_k \) is contractible if some (hence every) representative \( X \) of \([X]\) is contractible.

Definition 3.7. A continuous map \( h : [0, 1] \to B^k(\Sigma) \) with \( h(0) = X, h(1) = Y \) is a braid homotopy from \( X \) to \( Y \). When such a map exists, we shall say that \( X \) and \( Y \) are braid homotopic, denoted \( X \simeq Y \). The map \((s, t) \mapsto h_\ast(s, t)\) is called the \( i \)-th strand of \( h \). To any braid homotopy, we associate its graph \( h(s, t) = \bigcup_{t=0}^{1} h_i(s, t) \subseteq [0, 1] \times S^1 \times \Sigma \), \((s, t) \in [0, 1] \times S^1\).

Definition 3.8. An ordered braid \( X \) will be said to be trivial if all of its strands are constant maps. We will sometimes write \( 0 \in B^k(\Sigma) \) to stand for some fixed but arbitrary trivial braid, when the particular choices of the constant maps are unimportant. An unordered braid \([X]\) is trivial if some (hence every) ordered representative is trivial.

Definition 3.9. \( X \in B^k(\Sigma) \) is unlinked if \( X \simeq 0 \). An ordered braid is linked if it is not unlinked. An unordered braid is unlinked (resp. linked) if some (hence every) ordered representative is unlinked (resp. linked).

Definition 3.10. Two unordered braids \([X], [Y]\) are said to be unlinked if \([X] \bigcup [Y]\) is unlinked, and are said to be linked otherwise. Two ordered braids \( X, Y \in B^k(\Sigma) \) are said to be unlinked (resp. linked) if \([X]\) and \([Y]\) are unlinked (resp. linked).

Definition 3.11. A continuous map \( h : [0, 1] \to L(\Sigma)^k \) with \( h(0) = X \in B^k(\Sigma), h(1) = Y \in B^k(\Sigma) \) will be called a braid cobordism if there exists some \( \delta > 0 \) such that \( h(s) \in B^k(\Sigma), \forall s \in [0, \delta) \cup (1-\delta, 1) \).

Remark. We will frequently find ourselves concerned with maps \( h : I \to L(\Sigma)^k \), where \( I = \mathbb{R} \) or \( I = [a, b] \) for some \( a, b \in \mathbb{R} \), and in the case that \( I = \mathbb{R} \), it will always be the case that \( h \) extends continuously to a map \( \mathbb{R} \to L(\Sigma)^k \) such that on some neighbourhood of \( \pm \infty \), the graphs of the strands of \( h \) do not intersect. In such a case, we will speak freely of ‘the’ braid cobordism induced by \( h \), which is simply any braid cobordism \( h \circ \varphi \), where \( \varphi : I \to [0, 1] \) is any orientation-preserving diffeomorphism.

Definition 3.12. An (ordered) capped \( k \)-braid \( \hat{X} \) is an equivalence class \([X, \hat{w}]\) where \( X \in B^k_0(\Sigma) \) and \( \hat{w} = (w_1, \cdots, w_k) \) with \( w_i : D^2 \to \Sigma \) a capping disk for the \( i \)-th strand of \( X \), subject to the equivalent relation \([X, \hat{w}] \sim [X', \hat{w}']\) if and only if \( X = X' \) and \( \{w_i\} \# \{\neg[w_i]\} = 0 \in \pi_2(\Sigma) \) for each \( i = 1, \cdots, k \). The space of ordered capped \( k \)-braids is denoted by \( \hat{B}^k_0(\Sigma) \). The capped loop \( \hat{x} = (x_i, w_i) \in L_0(\Sigma) \) is called the \( i \)-th strand of \( \hat{X} \). The notion of capped sub-braids \( \hat{Y} \subseteq \hat{X} \) is defined in the obvious way.

The distinction between ordered capped braids and unordered capped braids obtains here as well, and we adopt parallel conventions as those discussed in the case of braids in Remark 3.

\( \pi_2(\Sigma)^k \) acts on \( \hat{B}^k(\Sigma) \) by the obvious ‘gluing of spheres’:

\[
(A_1, \ldots, A_k) \cdot ([x_1, w_1], \ldots, [x_k, w_k]) = ([x_1, A_1 \# w_1], \ldots, [x_k, A_k \# w_k]),
\]

where here we abuse notation slightly by thinking of \( A_i \in \pi_2(\Sigma, x_i(0)) \) as being both a homotopy class of maps, as well as a particular choice of a representative from that class. This action does not descend to an action on \( \hat{B}^k_0(\Sigma)/S_k \). However, if we denote by \( Fix_{S_k}(\pi_2(\Sigma)^k) \simeq \pi_2(\Sigma) \) the set of fixed points of the action of the symmetric group on \( \pi_2(\Sigma)^k \) by permutation of coordinates, we obtain a well-defined induced action on unordered braids given by \((A, \ldots, A) \cdot [\hat{X}] = [(A, \ldots, A) \cdot \hat{X}], \) for \( A \in \pi_2(\Sigma) \).

Definition 3.13. A trivial braid \( 0 \in B^k(\Sigma) \) has a naturally associated capping \( \hat{0} \in \hat{B}^k_0(\Sigma) \) given by capping each strand of \( 0 \) with the constant capping. We call any such braid a trivial capped braid.

When the particular components of a trivial capped braid are unimportant, we denote some fixed but arbitrary capped braid by the symbol \( \hat{0} \). An unordered capped braid is said to be trivial if some (hence every) ordered representative is trivial.
Definition 3.14. For \( A = (A_1, \ldots, A_k) \in \pi_2(\Sigma)^k \), an ordered braid cobordism \( h \) from \( X \) to \( Y \) will be called an \( A \)-cobordism from \([X, w] \) to \([Y, v] \) if \([w_i] \# [h_i] \# (-v_i) \) = \( A_i \), for all \( i = 1, \ldots, k \). Whenever such a map exists, \([X, w] \) and \([Y, v] \) will be said to be \( A \)-cobordant. This notion descends to unordered capped braids provided that \( A \in Fix_{S_\Sigma}(\pi_2(\Sigma)^k) \).

Definition 3.15. If \( u : [0, 1] \to L_0(\Sigma) \) is homotopy from \( x \) to \( y \), then for any choice of cappings \( \hat{x} = [x, w_x] \) and \( \hat{y} = [y, w_y] \), \( u \) is an \( A \)-cobordism from \( \hat{x} \) to \( \hat{y} \), for \( A = [w_x] \# [u] \# (-[w_y]) \in \pi_2(\Sigma) \).

Moreover, for any \( s \in [0, 1] \), there are two natural choices of capping for the loop \( u_s \in L_0(\Sigma) \). Namely, if we write \( \alpha^s(\tau) := u(s \cdot \tau, t) \) and \( \beta(\tau) := u(1 - (1 - s) \cdot \tau, t) \) for \( s \in [0, 1] \), then we may associate to \( u_s \) either of the cappings \([u_s, w_x \# \alpha^s]\) or \([u_s, w_y \# \beta^s]\), and these two cappings are obviously related by \( A \cdot [u_s, w_x \# \alpha^s] = [u_s, w_y \# \beta^s] \). Consequently, if \( u \) is a 0-homotopy between \( \hat{x} \) and \( \hat{y} \), these two cappings agree and we may associate a unique capping

\[
\hat{u}_s := [u_s, w_x \# \alpha^s] = [u_s, w_y \# \beta^s]
\]

to each \( u_s \) in this case. We will call such a capping the natural capping of \( u_s \) whenever \( u \) is such a 0-homotopy.

Definition 3.16. An \( A \)-cobordism \( h \) from \( \tilde{X} = [X, w] \) to \( \tilde{Y} = [Y, v] \) is called an \( A \)-homotopy if \( h \) is in addition a braid homotopy from \( X \) to \( Y \). In such a situation, we will say that \( \tilde{X} \) and \( \tilde{Y} \) are \( A \)-homotopic, and we will define the notation by \( \tilde{X} \simeq_A \tilde{Y} \). This notion descends to unordered capped braids, provided that \( A \in Fix_{S_\Sigma}(\pi_2(\Sigma)) \).

Definition 3.17. \( \tilde{X} \in \tilde{B}_0^k(\Sigma) \) is unlinked if \( \tilde{X} \simeq_0 \tilde{0} \). An unordered capped braid \([\tilde{X}] \) is unlinked if some (hence every) ordered representative is unlinked. The notion of linkedness for a capped braid or for a pair of capped braids is defined as in the case of braids.

3.1 Winding of capped loops with close strands

To any \( x \in L_0(\Sigma) \), we may associate the set

\[
S_x := \{ y \in L_0(\Sigma) : \exists t \in S^1 \text{ such that } x(t) = y(t) \},
\]

with the property that \( L_0(\Sigma) \setminus S_x \) consists of precisely those loops \( y \) such that \( (x, y) \in L_0(\Sigma)^2 \) is a braid.

We fix some family \( J = (J_t)_{t \in S^1} \) of \( \omega \)-compatible almost complex structures, and let \( g_J = (g_{J_t})_{t \in S^1} \) denote the associated family of compatible metrics. This data provides us with an exponential neighbourhoood \( \mathcal{O} \subseteq L_0(M) \) of \( x \), along with a diffeomorphism \( Exp : U \to \mathcal{O} \), defined by \( (Exp(\xi))(t) := \exp_{J(t)}^x(\xi(t)) \), for \( U \subseteq \Gamma^\infty(x^*T\Sigma) \) a neighbourhood of the zero section.

Remark that any choice of a lift \( \hat{x} = [x, \alpha] \in \tilde{L}_0(\Sigma) \) of \( x \) gives rise to a lift \( \tilde{\alpha} \) of \( \alpha \), and \( T_x \tilde{L}_0(\Sigma) \simeq (x^*T\Sigma, J, \omega) \) comes equipped with a homotopically unique unitary trivialization

\[
T_x : S^1 \times (\mathbb{R}^2, J_0, \omega_0) \to (x^*T\Sigma, J, \omega),
\]

provided by any trivialization which extends the capping. For any \( y \in \mathcal{O} \setminus S_x \) and any capping \( \hat{x}_\alpha := [x, \alpha] \) of \( x \), let \( \hat{y}_\alpha \) denote the unique lift of \( y \) lying in \( \tilde{\alpha} \). We define the winding number of \( \hat{x}_\alpha \) and \( \hat{y}_\alpha \) by

\[
\ell(\hat{x}_\alpha, \hat{y}_\alpha) := wind((T_x^{-1} \circ \tilde{Exp}^{-1})(\hat{y}_\alpha)),
\]

where \( wind(\xi) \) denotes the classical winding number of a non-vanishing family of vectors \( t \mapsto \xi(t) \) for \( t \in S^1 \) in \( \mathbb{R}^2 \). Note that for \( A \in \pi_2(\Sigma) \), we clearly have \( \ell(A \cdot \hat{x}_\alpha, \hat{y}_\alpha) = \ell(\hat{x}_\alpha, \hat{y}_\alpha) + c_1(A) \), and \( \ell \) is symmetric in its arguments. In order to extend this definition to arbitrary cappings of \( x \) and \( y \), let \( A, B \in \pi_2(\Sigma) \) and we define

\[
\ell(A \cdot \hat{x}_\alpha, B \cdot \hat{y}_\alpha) := \ell(\hat{x}_\alpha, \hat{y}_\alpha) + \frac{1}{2}(c_1(A) + c_1(B)).
\]

It is easy to check that this definition does not depend on the choice of \( \alpha \), nor the choice of compatible almost complex structure \( J = (J_t)_{t \in S^1} \), and agrees with the previous definition in the case that \( \hat{x} \) and \( \hat{y} \) are close in \( \tilde{L}_0(\Sigma) \). A more geometric view of this formula will be provided in the following subsection.
3.2 The homological linking number for capped braids

Definition 3.18. Let $\hat{X}, \hat{Y} \in \hat{B}_0^k(\Sigma)$ and $A \in \pi_2(\Sigma)^k$. We define the homological $(A)$-linking number of $\hat{Y}$ relative to $\hat{X}$

\[
L_A(\hat{X}; \hat{Y}) := \sum_{1 \leq i < j \leq k} \#(\tilde{h}_i \cap \tilde{h}_j),
\]

where $h = (h_1, \cdots, h_k)$ is any $A$-cobordism from $\hat{X}$ to $\hat{Y}$ such that the graphs of the strands of $h$ in $[0,1] \times S^1 \times \Sigma$ are all pairwise transversal, and $\#(\tilde{h}_i \cap \tilde{h}_j)$ denotes the signed count of the intersections of the graphs $\tilde{h}_i$ and $\tilde{h}_j$ (recall that in our setting $\Sigma$ carries the orientation induced by $\omega$).

Remark. The above definition may be generalized straightforwardly by replacing the cylinder $[0,1] \times S^1$ with a surface $S_{g,k-1,k^+}$ of genus $g$, having $k^-$ negatively oriented boundary components and $k^+$ positively oriented boundary components. This provides a family of homotopy invariants for collections of $\Sigma$ and oriented boundary components. These invariants likely provide interesting insight into the structure of the field-theoretic operations in Floer theory described in [24] at the chain level, but we will not pursue this here.

The following proposition summarizes the main properties of the homological linking number which we will need in our investigations.

Proposition 3.19. For any $\hat{X}, \hat{Y}, \hat{Z} \in \hat{B}_0^k(\Sigma)$ and $A, B \in \pi_2(\Sigma)^k$ we have that:

1. $L_A(\hat{X}, \hat{Y})$ is well-defined.
2. For any $\sigma \in S_k$, $L_{\sigma A}(\sigma \cdot \hat{X}; \sigma \cdot \hat{Y}) = L_A(\hat{X}; \hat{Y}).$
3. $L_A(\hat{X}, \hat{Y}) + L_B(\hat{Y}, \hat{Z}) = L_{A+B}(\hat{X}, \hat{Z}).$
4. If $\hat{X}$ and $\hat{Y}$ are $A$-homotopic, then $L_A(\hat{X}, \hat{Y}) = 0.$
5. $L_A(\hat{X}, \hat{Y}) = -L_{-A}(\hat{Y}, \hat{X}).$
6. $L_A(\hat{X}, B \cdot \hat{Y}) = L_{A+B}(\hat{X}, \hat{Y}).$
7. $L_0(\hat{X}, A \cdot \hat{X}) = (k-1) \sum_{i=1}^k \frac{c_i(A)}{2}.$

Proof. Items 1-6 are straightforward consequences of the definition of $L_A(\hat{X}, \hat{Y})$. To prove item 7, we note first that item 3 implies that

\[
L_0(\hat{0}, \hat{X}) + L_0(\hat{X}, A \cdot \hat{X}) + L_0(A \cdot \hat{X}, A \cdot \hat{0}) = L_0(\hat{0}, A \cdot \hat{0}).
\]

Next, items 5 and 6 imply that

\[
L_0(A \cdot \hat{X}, A \cdot \hat{0}) = L_A(A \cdot \hat{X}, \hat{0}) = -L_{-A}(\hat{0}, A \cdot \hat{X}) = -L_0(\hat{0}, \hat{X}),
\]

whence we need only show that the desired formula holds when $\hat{X} = \hat{0}$. To reduce to an even simpler case, let us write $A$ as

\[
(A_1, \cdots, A_k) = (A_1, 0, \cdots, 0) + (0, A_2, 0, \cdots, 0) + \cdots + (0, 0, \cdots, A_k) =: A'_1 + \cdots + A'_k.
\]

By items 3 and 6, demonstrating the desired equality is therefore equivalent to showing that $L_0(\hat{0}, A'_i \cdot \hat{0}) = (k-1) \cdot \frac{c_i(A)}{2}$ for any $i = 1, \cdots, k$. In what follows, let $(\hat{p}_1, \cdots, \hat{p}_k) = \hat{0}$ represent the trivial capped braid. Since the statement is trivial when $\Sigma \neq S^2$, as then $\pi_2(\Sigma) = \{0\}$ and every capped braid is 0-homotopic to itself, we now suppose $\Sigma = S^2$. For $m \in \mathbb{Z}$, if $u_i : (S^2, *) \to (\Sigma, p_i)$ represents $A_i = m[S^2] \in \pi_2(\Sigma, p_i)$, we may pull $u_i$ back along the quotient $[0,1] \times S^1 \to S^2$, given by collapsing the boundary circles to points, to a map which we will denote $h_i : [0,1] \times S^1 \to \Sigma$. If we take $h$ to be the 0-cobordism from $\hat{0}$ to $A'_i \cdot \hat{0}$ given by $h_i$ as the $i$-th strand and the constant strand $h_j(s,t) \equiv p_j$ for all other strands $j \neq i$, then the important point is that $PD(c_1) = 2[S^2]$ and hence the intersection of the graph of $h_i$ with the constant cylinder $h_j(s,t) \equiv p_j$ for $j \neq i$ contributes precisely $(u_i)[S^2] \cap [p_j] = \frac{c_i(A)}{2}$ to the sum defining $L_0(\hat{0}, A'_i \cdot \hat{0})$, and such intersections are the only ones that occur, since all other strands are constant and disjoint. The desired equality follows.
Proposition 3.20. For $A \in \pi_2(\Sigma)^k$, and $[\hat{X}], [\hat{Y}] \in \hat{B}_0^k(\Sigma)/S_k$, the function $L_A([\hat{X}];[\hat{Y}]) := L_A(\hat{X};\hat{Y})$, is well-defined.

Proof. The previous proposition implies that for any $\sigma, \tau \in S_k$, we have

$$L_A(\sigma \cdot \hat{X}; \tau \cdot \hat{Y}) = L_A(\hat{0}; 0) - L_0(\hat{0}; \sigma \cdot \hat{X}) + L_0(\hat{0}; \tau \cdot \hat{Y}),$$

so it suffices to show that the expression $L_0(\hat{0}; \sigma \cdot \hat{X})$ is independent of $\sigma \in S_k$. To see this, note that item 2 of the previous proposition, together with the fact that $0 \in \text{Fix}_S(\pi_2(\Sigma)^k)$ implies that $L_0(\hat{0}; \sigma \cdot \hat{X}) = L_0(\sigma^{-1} \cdot \hat{0}; \hat{X})$. Moreover, it is easy to see that $\hat{0} = (\hat{p}_1, \ldots, \hat{p}_k)$ is 0-homotopic to $\sigma \cdot \hat{0}$ for any $\sigma \in S_k$, and consequently, $L_0(\sigma^{-1} \cdot \hat{0}; \hat{X}) = L_0(\hat{0}; \hat{X})$, which is independent of $\sigma \in S_k$.

Proposition 3.21. Let $\hat{X} = (\hat{x}_1, \hat{x}_2) \in \hat{B}_0^2(\Sigma)$ with $x_2$ lying in some exponential of neighbourhood of $x_1$ in $\mathcal{L}_0(\Sigma)$, then $L_0(\hat{0}; \hat{X}) = \ell(\hat{x}_1, \hat{x}_2)$.

Proof. As $L_0(\hat{0}; \hat{X})$ depends on $\hat{X}$ only up to a 0-homotopy, we may assume without loss of generality that $x_1$ is a constant loop. Moreover, noting that if $A, B \in \pi_2(\Sigma)$, then

$$L_0(\hat{0}; (A, B) \cdot \hat{X}) - L_0(\hat{0}; \hat{X}) = \frac{1}{2}(c_1(A) + c_1(B)) = \ell(A \cdot \hat{x}_1, B \cdot \hat{x}_2) - \ell(\hat{x}_1, \hat{x}_2),$$

and so it suffices to prove the statement in the case in which $\hat{x}_1$ is a trivially capped constant loop and $\hat{x}_2$ lies inside an exponential neighbourhood of $\hat{x}_1$ in $\hat{L}_0(\Sigma)$.

As discussed in Section 3.1, $T_2\hat{L}_0(\Sigma)$ is naturally identified (up to a homotopy of trivializations) with $\Gamma^\infty(S^1 \times \mathbb{R}^2)$ and so (in the notation of that section) we may write $\hat{x}_2$ in local coordinates as $v(t) := (T_{\hat{x}_1}^r \circ \exp^{-1})(\hat{x}_2)$, and we have that $\ell(\hat{x}_1, \hat{x}_2) = \text{wind}(v_2)$ by definition. By the capped braid homotopy-invariance of the homological linking number, and the homotopy invariance of the winding number in $\mathbb{R}^2 \setminus \{0\}$, we may assume that $v(t) = r_0e^{2\pi it} \in S$, $t \in [0, 1]$, for some small $r_0 > 0$ and $l = \ell(\hat{x}_1; \hat{x}_2)$. The claim then immediately follows from computing the transverse intersections of the strands of the braid homotopy given by $h(s, t) = (0, (1 - s)e^{2\pi it} + s\frac{\hat{x}_2}{2})$.

The previous proposition justifies the following extension of the winding of capped loops with close strands in Section 3.1.

Definition 3.22. For $\hat{x}, \hat{y} \in \hat{L}_0(\Sigma)$ such that $\hat{X} = (\hat{x}, \hat{y}) \in \hat{B}_0^2(\Sigma)$, we define the linking of $\hat{x}$ and $\hat{y}$ as $\ell(\hat{x}, \hat{y}) := L_0(\hat{0}; \hat{X})$. As a matter of convention, we also define $\ell(\hat{x}, A \cdot \hat{x}) := \frac{c_1(A)}{2}$ for $A \in \pi_2(\Sigma)$, and for $A \neq 0 \in \pi_2(\Sigma)$. We will additionally declare, simply as a matter of convention, that $\hat{x}$ and $A \cdot \hat{x}$ are linked.

The following proposition hints at the role that linking plays in understanding the relation of the action functional to dynamics and will prove crucial in Section 8.

Proposition 3.23. Suppose that $\hat{\gamma} = [\gamma, w]$ is a capped loop such that $\gamma$ is smooth, then

$$\int_{D^2} w^*\omega = \int_{\Sigma \setminus \text{im } \gamma} \ell(\hat{\gamma}, \hat{x})\omega,$$

where for $x \in \Sigma$, $\hat{x} = [x, x]$ denotes the trivially capped constant loop based at $x$.

Proof. Suppose without loss of generality that $w : D^2 \to \Sigma$ is a smooth map with $w(0) = \gamma(0)$. The main point is to notice that for any $x \in \Sigma \setminus \text{im } \gamma$ such that $w$ is transversal to $x$, we have that $\ell(\hat{\gamma}, \hat{x}) = \deg(w)_x$, where $\deg(w)_x$ denotes the local degree of $w$ at $x$. Indeed, transversality of $w$ to $x$ implies the transversality of the maps into $[0, 1] \times S^1 \times \Sigma$ defined by $\tilde{w}(s, t) = (s, t, w(se^{2\pi it}))$ and $\hat{x}(s, t) = (s, t, x)$, for $(s, t) \in [0, 1] \times S^1$, and obviously the algebraic count of the intersection number between these two graphs is identical with $\deg(w)_x$. One immediate consequence is that $\ell(\hat{\gamma}, \hat{x}) = 0$ for all $x \not\in \text{im } w$, so it suffices to show that

$$\int_{D^2} w^*\omega = \int_{\text{im } w \setminus \text{im } \gamma} \ell(\hat{\gamma}, \hat{x})\omega.$$
To this end, denote by $S(w) \subseteq D^2$ the set of points such that $Dw$ is not of full rank and define $G(w) \subseteq D^2$ to be $G(w) = w^{-1}(\text{im } \gamma) \setminus S(w)$. Note that $G(w)$ is a set of measure 0, since we may realize $G(w)$ as the projection onto $D^2 \setminus S(w)$ of $(w|_{D^2} \times \gamma)^{-1}(\Delta)$, which is a submanifold of $D^2 \setminus S(w)$ of codimension 2. Next note that we must have $\overline{S(w)}w \omega = 0$, since $w \omega$ vanishes on $S(w)$. Consequently, writing $N := S(w) \cup G(w)$, we note that $\text{im } \gamma \subseteq w(N)$ and so it suffices to establish that

$$\int_{D^2 \setminus N} w^* \omega = \int_{\overline{w(D^2 \setminus N)}} \ell(\gamma, \dot{x}) \omega = \int_{\overline{w(D^2 \setminus N)}} \deg(w) \omega \omega.$$

The local degree is a locally constant function of $x$, and so we obtain

$$\int_{w(D^2 \setminus N)} \deg(w) \omega \omega = \sum_{C \in \pi_0(w(D^2 \setminus N))} \deg(w)|_C \int_{w^{-1}(C)} \omega \omega = \sum_{C \in \pi_0(w(D^2 \setminus N))} \int_{w^{-1}(C)} w^* \omega = \int_{D^2 \setminus N} w^* \omega$$

as claimed. 

\[ \square \]

4 Elements of Floer theory and linking

In this section, we give a rapid overview of the elements of Floer theory of which we will have need, mainly to fix notation and conventions. For a more detailed treatment, see [1], [25] for standard accounts of Hamiltonian Floer theory (see also [7] for its adaptation to the weakly monotone case) and [27], [24] or [15] for a more detailed treatment of how Floer theory fits into a field theory over surfaces. Throughout, we assume that $(M, \omega)$ is a weakly monotone compact symplectic manifold of dimension $2n$, while $J(M, \omega)$ denotes the space of all smooth $\omega$-compatible almost complex structures. For convenience, we work only with $\mathbb{Z}_2$-coefficients, but this restriction is inessential.

4.1 Floer theory

A smooth Hamiltonian function $H : S^1 \times M \to \mathbb{R}$ induces a time-dependent vector field $(X_H(t))_{t \in [0, 1]}$ on $M$ defined by the relation $\omega(X_H(t), -) = −dH_t$. The Hamiltonian isotopy obtained as the flow by this vector field is denoted $\phi_H := (\phi_H(t))_{t \in [0, 1]}$.

The Hamiltonian $H$ defines a corresponding action functional on the space of capped loops $\tilde{\mathcal{L}}_0(M)$, $A_H([\gamma, v]) := \int_0^1 H_t(\gamma(t)) dt − \int_{D^2} v^* \omega$. We write $\text{Per}_0(H) := \text{Crit } A_H$, and $\text{Per}_0(H) := \pi(\text{Per}_0(H)) \subseteq \mathcal{L}_0(M)$ which consists precisely of the contractible 1-periodic orbits of $\phi_H$.

$H$ is said to be non-degenerate if for all $x \in \text{Per}_0(H)$, $(D\phi_H^1)_{x(0)}$ has no eigenvalues equal to 1. When $H$ is non-degenerate, there exists a well-defined Conley-Zehnder index $\mu([x, v], \mathbb{Z}) \in \mathbb{Z}$ for $[x, v] \in \text{Per}_0(H)$. We shall normalize the Conley-Zehnder index by insisting that if $H$ is a $C^2$-small Morse function and $x$ a critical point of $H$, then $\mu(\dot{x}) = \mu_{\text{Morse}}(x) − n$, where $\mu_{\text{Morse}}$ is the Morse index of $x$, and $\dot{x}$ denotes the trivial capping of the constant orbit $x$. For $k \in \mathbb{Z}$, and any $P \subseteq \text{Per}_0(H)$ we define $P(k)$ to be the collection of capped orbits in $P$ with Conley-Zehnder index $k$.

Given $\dot{x}^\pm = [x^\pm, w^\pm] \in \overline{\text{Per}_0(H)}$, we write $C^\infty_{\dot{x}^-, \dot{x}^+}(\mathbb{R} \times S^1; M)$ for the subspace of $C^\infty(\mathbb{R} \times S^1; M)$ consisting of cylinders which induce a 0-homotopy from $\dot{x}^−$ to $\dot{x}^+$. Letting $\mathcal{E} \to C^\infty(\mathbb{R} \times S^1; M)_{\dot{x}^-, \dot{x}^+}$ be the infinite dimensional vector bundle with fiber $\mathcal{E}_u = \Gamma^\infty(u^*TM)$ at $u$, any smooth $S^1$-family $J = (J_t)_{t \in S^1} \subseteq \mathcal{J}(M, \omega)$, permits the definition of the Floer operator $\mathcal{F}_{H,J}(u) := \partial_s u + J(\partial_t u - X_H^t) \in \mathcal{E}_u$, for $u \in C^\infty_{\dot{x}^-, \dot{x}^+}(\mathbb{R} \times S^1; M)$. After passing to appropriate Banach space completions, $\mathcal{F}_{H,J}$ defines a Fredholm operator with index $\mu(\dot{x}^−) − \mu(\dot{x}^+)$. The intersection of $\mathcal{F}_{H,J}$ with the 0-section gives rise to Floer’s equation

$$\partial_s u + J_i(\partial_t u - X_H^t) = 0 \quad (1)$$

for smooth maps $u : \mathbb{R} \times S^1 \to M$. If we define the energy of $u \in C^\infty(\mathbb{R} \times S^1; M)$ by $E(u) := \int_{\mathbb{R} \times S^1} \|\partial_s u\|^2_{\mathcal{E}_u} ds$, then the finite energy solutions of Floer’s equation may be thought of as the projections to $M$ of negative gradient flow lines of $A_H$ with respect to the $L^2$-metric on $\tilde{\mathcal{L}}_0(M)$ induced
by \( J \). It follows easily from this that if \( u \in C_{\dot{a}_-}^\infty(\mathbb{R} \times S^1; M) \) is such a finite energy solution, then \( E(u) = A_H(\dot{x}) - A_H(\dot{x}^+) \).

For any \( \dot{x}^\pm \in \tilde{Per}_0(H) \), we define define \( \tilde{M}(\dot{x}^-, \dot{x}^+; H, J) \) to be the zero set of \( F_{H,J} \) on \( C_{\dot{a}_-}^\infty(\mathbb{R} \times S^1; M) \). It carries an obvious \( \mathbb{R} \)-action given by translation in the \( s \)-coordinate. The reduced moduli space is defined by \( \tilde{M}(\dot{x}, \tilde{y}; H, J) := \tilde{M}(\dot{x}, \tilde{y}; H, J)/\mathbb{R} \).

**Definition 4.1.** A pair \((H, J)\) with \( H \) and \( J \) as above, \( H \) non-degenerate, are said to be Floer non-degenerate if the intersection of the Floer operator with the 0-section is transverse (after taking appropriate Banach space completions) for all \( \dot{x}, \dot{y} \in \tilde{Per}_0(H) \) with \( \mu(\dot{x}) - \mu(\dot{y}) \leq 2 \).

For \( H \) non-degenerate, let \( \mathcal{J}^\text{nd}(H) \subseteq C^\infty(S^1; \mathcal{J}(M, \omega)) \) denote the space of \( S^1 \)-families of complex structures such that \((H, J)\) is Floer non-degenerate. \( \mathcal{J}^\text{nd}(H) \) is residual in \( C^\infty(S^1; \mathcal{J}(M, \omega)) \).

If \((H, J)\) is Floer non-degenerate, then \( \tilde{M}(\dot{x}, \tilde{y}; H, J) \) is a compact manifold of dimension 0 whenever \( \mu(\dot{x}) - \mu(\dot{y}) = 1 \), and in this case we may define the Floer chain complex \( CF_*(H, J) := \mathbb{Z}_{\tilde{2}}(\tilde{x})_{\tilde{\dot{x}}}^{\tilde{\dot{x}} \in \tilde{Per}_0(H)} \), which is graded by \( \mu \) and which has differential defined on generators by \( \partial_{H,J} \tilde{x} := \sum_{\mu(\dot{x}) - \mu(\dot{y}) = 1} n(\dot{x}, \dot{y}) \tilde{y} \), with \( n(\dot{x}, \dot{y}) \) being the mod 2 count of elements in \( \tilde{M}(\dot{x}, \tilde{y}; H, J) \). The homology of this complex \( FH_*(H) \) is the Floer homology of \( H \) and is independent of the choice of \( J \).

The Floer complex has the structure of a filtered complex, with the filtration coming from the action functional. Explicitly, for \( \sigma = \sum_{\dot{x} \in \tilde{Per}_0(H)} a_{\dot{x}} \dot{x} \in CF_*(H, J) \), we define \( \text{supp} \; \sigma := \{ \dot{x} \in \tilde{Per}_0(H) : a_{\dot{x}} \neq 0 \} \), and we define the level of \( \sigma \) to be

\[
\lambda_H(\sigma) := \sup_{\dot{x} \in \text{supp} \; \sigma} A_H(\dot{x}).
\]

For a Floer homology class \( \alpha \in HF_*(H) \) the spectral invariant associated to \( \alpha \) is defined by

\[
c(\; H; \alpha) := \inf \{ \lambda_H(\sigma) : [\sigma] = \alpha \}.
\]

A cycle \( \sigma \) such that \( [\sigma] = \alpha \) and \( \lambda_H(\sigma) = c(\; H; \alpha) \) is called tight (for \( \alpha \) ). It is a non-trivial fact that such cycles always exist (see [32] or [22]).

Let \( H^S(M) \) denote the image in \( H_2(M; \mathbb{Z}) \) of the Hurewicz morphism, and let \( \Gamma_\omega := H^S(M)/\ker c_1 \cap \ker [\omega] \). We define the Novikov ring

\[
\Lambda_\omega := \left\{ \sum_{A \in \Gamma_\omega} \lambda_A e^A : \lambda_A \in \mathbb{Z}_2, \#\{\lambda_A \neq 0, \omega(A) \leq c\} < \infty, \text{ for all } c \in \mathbb{R} \right\}.
\]

This is a graded commutative ring with grading given by declaring \( \text{deg}(A) := 2c_1(A) \). \( CF_*(H,J) \) is a \( \Lambda_\omega \)-module where the action of \( e^A \) in \( \Lambda_\omega \) is defined on generators \( \dot{x} = [x, v] \) of \( CF_*(H,J) \) by \( e^A \cdot \dot{x} := [x, A \# v] \), and extended linearly. Note that we have the relations

\[
\mu(e^A \cdot \dot{x}) = \mu(\dot{x}) - 2c_1(A),
\]

\[
A_H(e^A \cdot \dot{x}) = A_H(\dot{x}) - \omega(A).
\]

It is a standard fact in Floer theory that if \( f \in C^\infty(M) \) is a sufficiently \( C^2 \)-small Morse function and \( J \in \mathcal{J}(M, \omega) \) is such that \( (f, g_J) \) is Morse-Smale, then \( CF_*(f, J) = C_{+n}^{\text{Mor}-}(f, g_J) \circ \Lambda_\omega \). Taking homology then gives a natural identification with the quantum homology of \((M, \omega)\): \( HF_n(f) = H_{+n}(M; \Lambda_\omega) = QH_{+n}(M; \omega) \).

**4.1.1 Continuation maps**

For \( X \) a smooth manifold, a function \( F \in C^\infty(\mathbb{R} \times X) \) is said to be \( T \)-adapted for \( T \in (0, \infty) \) if \( (\partial F)_s \equiv 0 \) for all \( |s| \geq T \). \( F \) is said to be adapted if it is \( T \)-adapted for some \( T \). For \( X = C^\infty(S^1 \times M) \), and \( H^\pm \in C^\infty(S^1 \times M) \), we denote by \( \mathcal{H}(H^-, H^+) \) the space of adapted homotopies \( \mathcal{H} \) having \( \lim_{s \to \pm \infty} \mathcal{H}(s) = H^\pm \). We make a similar definition for \( \mathcal{J}(J^-, J^+) \) in the case where
A pair \((\mathcal{H}, J)\) is an adapted homotopy of Floer data from \((H^-, J^-)\) to \((H^+, J^+)\) if \(\mathcal{H} \in \mathcal{HF}(H^-, H^+)\) and \(J \in \mathcal{J}(J^-, J^+)\). We will write \(\mathcal{HF}(H^-, J^-; H^+, J^+)\) for the collection of all such adapted homotopies, often omitting the dependence on \((H^\pm, J^\pm)\) if it is clear from context. Just as in the \(s\)-independent case, for any adapted homotopy of Floer data \((\mathcal{H}, J)\), we obtain a corresponding Floer operator \(\mathcal{F}_{\mathcal{H}, J}\). For any pair \(\hat{x}^\pm \in \overline{\text{Per}}_0(H^\pm)\), consideration of the zeros of \(\mathcal{F}_{\mathcal{H}, J}\) along \(C^\infty(\mathbb{R} \times S^1; M)\) gives rise to the \(s\)-dependent Floer equation

\[ \partial_s u + J^+_s(\partial_t u - X_{\mathcal{H}_s}) = 0, \]

and everything proceeds as before, with the proviso that now, if \(u \in C^\infty_{\hat{x}^-, \hat{x}^+}(\mathbb{R} \times S^1; M)\) solves Equation 2, then its energy is given by \(E(u) = A_{H^-}((\hat{x}^-) - A_{H^+_s}(\hat{x}^+) + \int_{-\infty}^{\infty} \int_0^1 (\partial_s \mathcal{H})(s, t, u(s, t)) \, dt \, ds\). The moduli space \(\mathcal{M}(\hat{x}^-, \hat{x}^+; \mathcal{H}, J)\) is defined to be the zero set of \(\mathcal{F}_{\mathcal{H}, J}\) on \(C^\infty_{\hat{x}^-, \hat{x}^+}(\mathbb{R} \times S^1; M)\).

**Remark.** When \((H^-, J^-) = (H^+, J^+)\), then the \(s\)-independent homotopy \((\mathcal{H}, J) = (H^-, J^-) = (H^+, J^+)\) is a special case of an adapted homotopy. In this case, \(\mathcal{M}(\hat{x}, \hat{y}; \mathcal{H}, J) = \mathcal{M}(\hat{x}, \hat{y}; H^\pm, J^\pm)\). In the sequel, when we speak of adapted homotopies of Floer data, this case is included.

**Definition 4.2.** Given \((H^\pm, J^\pm)\) Floer non-degenerate, \(\hat{x}^\pm \in \overline{\text{Per}}_0(H^\pm)\), and \((\mathcal{H}, J) \in \mathcal{HF}\), we will say that \((\mathcal{H}, J)\) is \((\hat{x}^-, \hat{x}^+)-\text{regular}\) if \(\mathcal{F}_{\mathcal{H}, J}\) is transverse to the zero section along \(C^\infty_{\hat{x}^-, \hat{x}^+}(\mathbb{R} \times S^1; M)\). We denote the collection of all such adapted homotopies by \(\mathcal{HF}_{\hat{x}^-, \hat{x}^+}(\mathbb{R} \times S^1; M)\). \((\mathcal{H}, J)\) will be said to be \(\text{Floer-regular}\) if it is \((\hat{x}^-, \hat{x}^+)-\text{regular}\) whenever \(\mu(\hat{x}^-) - \mu(\hat{x}^+) \leq 1\). We denote the space of Floer-regular adapted homotopies \((H^-, J^-)\) to \((H^+, J^+)\) by \(\mathcal{HF}_{\hat{x}^-, \hat{x}^+}(H^-, J^-; H^+, J^+)\), suppressing the dependence on \((H^\pm, J^\pm)\) when no confusion will arise.

For any fixed \(J \in \mathcal{J}(J^-, J^+)\), the set \(\mathcal{HF}_{\hat{x}^-, \hat{x}^+}(J; H^-, H^+) \subseteq \mathcal{HF}(H^-, H^+)\) of adapted homotopies \(\mathcal{H}\) such that \((\mathcal{H}, J)\) is Floer regular is residual.

For \((\mathcal{H}, J) \in \mathcal{HF}_{\hat{x}^-, \hat{x}^+}\), the spaces \(\mathcal{M}(\hat{x}^-, \hat{x}^+; \mathcal{H}, J)\) are all compact manifolds of dimension 0 whenever \(\mu(\hat{x}^-) = \mu(\hat{x}^+)\), and so we may define the continuation morphism

\[ h_{\mathcal{H}, J} : CF_s(H^-, J^-) \to CF_s(H^+, J^+) \]

on generators by setting \(h_{\mathcal{H}, J}(x^-) := \sum_{\mu(\hat{x}^-) - \mu(\hat{x}^+) = 0} n(\hat{x}^-, \hat{x}^+) \hat{x}^+\), where \(n(\hat{x}^-, \hat{x}^+)\) is the mod 2 count of elements in the moduli space \(\mathcal{M}(\hat{x}^-, \hat{x}^+; \mathcal{H}, J)\). The continuation morphism is a morphism of complexes, which descends to an isomorphism at the level of homology. Moreover any two continuation maps between \((H^-, J^-)\) and \((H^+, J^+)\) define the same map at the level of homology, and further these isomorphism satisfy the obvious composition law \(h_{21} \circ h_{10} = h_{20}\), where \(h_{ij} : HF(H_i) \to HF(H_j)\), Consequently, for any quantum homology class \(\alpha \in QH_{*-n}(M) \simeq H_{\text{Mor}}^s(f; \mathcal{A}_\omega) \simeq HF_s(f)\), with \(f\) some \(C^2\)-small Morse function, letting \(h : HF_s(f) \to HF_s(H)\) be such a continuation morphism, \(h(\alpha) \in HF_s(H)\) is a well-defined homology class, independent of the Morse function \(f\) and the continuation morphism \(h\).

**Definition 4.3.** For \(H\) Floer non-degenerate and \(\alpha \in QH_{*-n}(M)\) the spectral invariant of \(H\) associated to \(\alpha\) is defined by \(c(H; \alpha) := c(H; h(\alpha))\).

We conclude this by recalling the so-called ‘Gromov trick’, which forms the basis of much of this paper by establishing that we may use pseudo-holomorphic techniques to analyze the graphs of Floer-type cylinders.

**Theorem 4.4** (1.4.C’ in [6]). Let \((\mathcal{H}, J)\) be an adapted homotopy of Floer data, then there exists a unique almost complex structure \(\tilde{J}\) on \(\mathbb{R} \times S^1 \times M\) with the property that a section

\[ \tilde{u} : \mathbb{R} \times S^1 \to \mathbb{R} \times S^1 \times M \]

\((s, t) \mapsto (s, t, u(s, t))\)

is \((\tilde{J}_0, \tilde{J})\)-holomorphic if and only if \(u\) satisfies Equation 2, where \(\tilde{J}_0\) denotes the standard complex structure on the cylinder.
4.2 Asymptotic analysis for pseudoholomorphic cylinders

The main analytic fact that gives us control over the asymptotic winding behaviour of Floer cylinders, as well as that of vector fields lying in the the kernel of the Floer differential, is the following theorem which describes the asymptotic behaviour of solutions to an appropriately perturbed Cauchy–Riemann equation. This result is originally due to [20], although the version we reproduce here for the convenience of the reader is from the appendix of [29].

**Theorem 4.5.** Let $w : [0, \infty) \times S^1 \to \mathbb{R}^{2n}$ satisfy the equation

$$\partial_s w + J_0 \partial_t w + (S(t) - \Delta(s,t))w = 0,$$

where $S : S^1 \to \text{End}(\mathbb{R}^{2n})$ is a smooth family of symmetric matrices and $\Delta : [0, \infty) \times S^1 \to \text{End}(\mathbb{R}^{2n})$ is smooth. Suppose that for $\beta \in \mathbb{N}^2$, there exist constants $M_\beta, d > 0$ such that $|\langle \partial^2 \Delta(s,t) \rangle| \leq M_\beta e^{-ds}$, and $|\langle \partial^2 w(s,t) \rangle| \leq M_\beta e^{-ds}$. Then either $w \equiv 0$ or $w(s,t) = e^{\lambda s}(\xi(t) + r(s,t))$, where $\lambda$ is a negative eigenvalue of the self-adjoint operator $A : H^1(S^1; \mathbb{R}^{2n}) \subset L^2(S^1; \mathbb{R}^{2n}) \to L^2(S^1; \mathbb{R}^{2n})$

$$(\partial_t - J_0 S)h,$$

which the remainder term satisfies the decay estimates $|\nabla^i_\xi \nabla^j_\xi r(s,t)| \leq M_{ij} e^{-ds}$ for all $(i, j) \in \mathbb{N}^2$.

This theorem is useful in the following setting. Let $(H,J)$ be Floer non-degenerate. To any $x \in \text{Per}_{0}(H)$, we may assign the asymptotic operator $A_{x,J} : \Gamma(x^*TM) \to \Gamma(x^*TM)$ as follows. Viewing $\xi \in \Gamma(x^*TM)$ as a section of the vertical tangent bundle $\mathcal{V}|_x \leq T(S^1 \times M)|_x$ along the graph $\hat{x}$ of $x$, we let $X_H := \partial_t \oplus X_H \in \mathcal{X}(S^1 \times M)$, and we view $J = (J_t)_{t \in \mathbb{S}^1}$ as an endomorphism of the vertical tangent bundle by setting $J_{t,x} := J_t(x)$. $A_{x,J}$ is then defined by setting $A_{x,J}(\xi) := -J L_{X_H} \xi$, where $L_X Y$ denotes the Lie derivative of $Y$ along $X$. $A_{x,J}$ extends to an unbounded self-adjoint operator with discrete spectrum (still denoted $A_{x,J}$) from $W^{1,2}(x^*TM)$ to $L^2(x^*TM)$.

By taking an exponential chart as in Section 3.1 on a neighbourhood $\hat{O}$ of $\hat{x} \in \text{Per}_{0}(H)$, Floer’s equation may be written in the local coordinates provided by this chart in the form of Equation 3, with $A_{x,J}$ being sent via these coordinates to $A$. Following [29], we define

**Definition 4.6.** Let $x \in L_0(M)$ and suppose that $\lim_{s \to \infty} u_s \equiv x$ for a map $u : \mathbb{R} \times S^1 \to M$. For any $R > 0$, a map $U^+ : [R, \infty) \to \Gamma(x^*TM)$ will be said to be a positive asymptotic representative of $u$ if $u(s,t) = \text{Exp}(U^+(s))(t)$ for all $(s,t) \in [R, \infty) \times S^1$, where $\text{Exp}$ is as in Section 3.1. The notion of a negative asymptotic representative of $u$, $U^- : (-\infty, -R] \to \Gamma(x^*TM)$ is defined in the obvious analogous manner.

Every Floer-type cylinder considered in this paper admits, due to exponential convergence at the ends, essentially unique positive and negative asymptotic representatives, determined up to a restriction of the domains of $U^\pm$ to larger values of $|R|$.

The main result that we will need from [29] (paraphrased for our setting) is the following

**Theorem 4.7.** Let $(H,J)$ be Floer non-degenerate, $x \in \text{Per}_{0}(H)$ and let $u$ and $v$ solve Equation 2 for $s >> 0$ (resp. for $s << 0$), where the adapted homotopy used in defining Equation 2 satisfies $\langle H^+, J^+ \rangle = (H,J)$ (resp. $\langle H^-, J^- \rangle = (H,J)$). Suppose moreover that $u_s$ and $v_s$ both converge to $x$ as $s \to \infty$ (resp. $s \to -\infty$). Let $U$ and $V$ be positive (resp. negative) asymptotic representatives of $u$ and $v$ respectively. Then either $U \equiv V$ or there exists a strictly negative (resp. strictly positive) eigenvalue $\lambda \in \sigma(A_{x,J})$ and an eigenvector $\xi$ with eigenvalue $\lambda$ such that

$$(U - V)(s,t) = e^{\lambda s}(\xi(t) + r(s,t)),$$

where the remainder term satisfies the decay estimates $|\nabla^i_\xi \nabla^j_\xi r(s,t)| \leq M_{ij} e^{-ds}$ for all $(i,j) \in \mathbb{N}^2$ and $M_{ij}, d > 0$ (resp. $d < 0$).
Whenever \( u, v \) and \( x \) are as above, we will write \( \xi_{u,v}^+ \) (resp. \( \xi_{u,v}^- \)) for the eigenvectors of \( A_{x,J} \) whose existence is guaranteed by the above theorem. We will call \( \xi_{u,v}^+ \) the **positive** (resp. **negative** asymptotic eigenvector of \( v \) relative to \( u \)). Note that the above result only requires that \( u \) and \( v \) solve Equation 2 on some neighbourhood of \( s = \infty \) (resp. \( s = -\infty \)), and that, for \((H,J) \in \mathcal{H}^J\), the trivial cylinder \( v(s,t) = x(t) \) is always a solution to Equation 2 outside some compact set. We will write \( \xi_u^\pm := \xi_{u,x} \) and call these the **positive and negative** asymptotic eigenvectors of \( u \).

This asymptotic information becomes especially useful when combined with the following fact (see [8] p. 285 or [29] p.1637).

**Proposition 4.8.** If \( \xi \in \Gamma(x^*TM) \) is an eigenvector of \( A_{x,J} \), then \( \xi(t) \neq 0 \) for all \( t \in S^1 \).

**Corollary 4.9.** Let \( u, v : \mathbb{R} \times S^1 \rightarrow M \) be distinct finite energy solutions of Equation 2, then there is a compact subset \( K \subseteq \mathbb{R} \times S^1 \) such that \( u(s,t) = v(s,t) \) only if \((s,t) \in K \).

These results become even stronger in the case when \( \dim M = 2 \), as in this case Proposition 4.8 implies that eigenvectors of the asymptotic operator have a well-defined winding number, once we fix a trivialization of \( x^*TM \) via a choice of capping disk. More precisely, when \( M = \Sigma \), if \( x \in \mathcal{L}_0(\Sigma) \), and \( T_x : S^1 \times (\mathbb{R}^2, \omega_0) \rightarrow (x^*T\Sigma, \omega) \) is a symplectic trivialization as in Section 3, then for any \( \xi \in \mathcal{A}_{x,J} \), the map \( t \mapsto T_x(t)^{-1}\xi(t) \) has a well-defined winding number \( \text{wind}(\xi(t); x) \), by Proposition 4.8. Proposition 3.21 then implies

**Corollary 4.10.** Let \( u, v \) be distinct finite energy solutions of Equation 2 with \( \lim_{s \rightarrow -\infty} u_s = \lim_{s \rightarrow -\infty} v_s = x \). Then there exists \( R > 0 \) such that for all \( s < -R \) and any capping \( \tilde{x} = [x, \alpha] \), we have \( \ell(\tilde{u}^*_s, \tilde{u}^*_s) = \text{wind}(\xi_{u,v}^+; \tilde{x}) \), where \( \tilde{u}^*_s \) (resp. \( \tilde{u}^*_s \)) denotes the capping of \( u_s \) (resp. \( v_s \)) such that \([x, \alpha]\) and \( u_s \) (resp. \( v_s \)) are 0-homotopic. The analogous statement when \( \lim_{s \rightarrow \infty} u_s = \lim_{s \rightarrow \infty} v_s = x \) also holds.

If we combine the positivity of intersection of holomorphic curves in dimension 4 with the foregoing discussion, we arrive at the principal point of this section

**Lemma 4.11.** Let \((H^\pm, J^\pm)\) be Floer non-degenerate, \((H,J) \in \mathcal{H}^J\) and let \( u, v \in C^\infty(\mathbb{R} \times S^1; \Sigma) \) be distinct finite energy solutions to Equation 2 for \((H,J)\). Then for any lifts \( \tilde{u}, \tilde{v} \) of \( u, v : \mathbb{R} \rightarrow \mathcal{L}_0(\Sigma) \), the function \( \ell_{\tilde{u},\tilde{v}}(s) := \ell(\tilde{u}_s, \tilde{v}_s) \) is non-decreasing, locally constant, and well-defined for all but finitely many values \( s \in \mathbb{R} \). Moreover, for \( s, s' \in \text{dom}(\ell_{\tilde{u},\tilde{v}}) \), with \( s < s' \), \( \ell_{\tilde{u},\tilde{v}}(s) \neq \ell_{\tilde{u},\tilde{v}}(s') \) if and only if there exists \( s_0 \in (s, s') \) and some \( t_0 \in S^1 \) such that \( u(s_0, t_0) = v(s_0, t_0) \).

**Proof.** That \( \ell_{\tilde{u},\tilde{v}} \) has only finitely many points at which it is ill-defined follows the fact that, by definition, \( \ell(\tilde{u}_{s_0}, \tilde{v}_{s_0}) \) is undefined only when there exists \( \ell_0 \in S^1 \) such that \( u(s_0, \ell_0) = v(s_0, \ell_0) \). By Corollary 4.9, the set of all such \( (s_0, \ell_0) \) in \( \mathbb{R} \times S^1 \) must lie inside some compact set, and we may then apply Theorem 4.4 to choose an almost complex structure on \( \mathbb{R} \times S^1 \times \Sigma \) such that the graphs \( \tilde{u} \) and \( \tilde{v} \) are pseudoholomorphic, whence all such intersections must be isolated, and so finite in number. That \( \ell_{\tilde{u},\tilde{v}} \) is non-decreasing follows by applying the positivity of intersections for holomorphic curves in dimension 4 at these intersections, which implies moreover that any such intersection contributes strictly positively to the change in \( \ell_{\tilde{u},\tilde{v}}(s) \) as \( s \) increases and passes from one connected component of \( \text{dom}(\ell_{\tilde{u},\tilde{v}}) \) to another.

**Definition 4.12.** For \((H^\pm, J^\pm)\) Floer non-degenerate, \((H,J) \in \mathcal{H}^J\) and \( u, v \in M(\tilde{x}, \tilde{y}; H,J) \), \( u \neq v \), we define \( \ell_{\pm,\infty}(u, v) := \lim_{s \rightarrow \pm,\infty} \ell(\tilde{u}_s, \tilde{v}_s) \), where \( \tilde{u}_s \) and \( \tilde{v}_s \) are the natural cappings of \( u_s \) and \( v_s \) (cf. Definition 3.15).

Note that the previous lemma implies that these quantities exist and are finite. Indeed, if \( \tilde{u}_s \) and \( \tilde{v}_s \) tend to \( \tilde{x} \) as \( s \rightarrow \pm,\infty \), then \( \ell_{\pm,\infty}(u, v) = \text{wind}(\xi_{u,v}^+; \tilde{x}) \) by Corollary 4.10, while if \( \lim_{s \rightarrow \pm,\infty} u_s = \tilde{x} \) and \( \lim_{s \rightarrow \pm,\infty} v_s = \tilde{y} \) with \( x \neq y \), then \( \ell_{\pm,\infty}(u, v) = \ell(\tilde{x}, \tilde{y}) \).

### 4.2.1 Winding of eigenvectors of \( A_{x,J} \)

We summarize here some necessary facts from [8] on the winding numbers of eigenvectors of \( A_{x,J} \) which appeared in the previous subsection (while [8] works in the aspherical case, our previous discussion makes clear how this winding number depends on the choice of cappings of \( x \) and \( y \) and this is all that is needed to extend the results to capped orbits). For a loop \( x \in \mathcal{L}_0(\Sigma) \), let \( \pi_2(\Sigma; x) \) denote the set of homotopy classes of capping disks for \( x \).
Proposition 4.13. Let \( x \in \text{Per}_0(H) \) with \( H \) non-degenerate and \( J : S^1 \to \mathcal{J}(\Sigma, \omega) \) arbitrary. There is a well-defined function

\[
W = W_{x,J} : \pi_2(\Sigma; x) \times \sigma(A_{x,J}) \to \mathbb{Z}
\]

\[
(\alpha, \lambda) \mapsto \text{wind}(T^{-1}_{[x,\alpha]} \circ \xi),
\]

where \( \xi \in \Gamma(x^*T\Sigma) \) is any eigenvector with eigenvalue \( \lambda \). Moreover, \( W \) satisfies the following properties

1. For any \( \alpha \in \pi_2(\Sigma; x) \), \( \lambda < \lambda' \Rightarrow W(\alpha, \lambda) \leq W(\alpha, \lambda') \).

2. For any \( \alpha \in \pi_2(\Sigma; x) \), and any \( k \in \mathbb{Z} \), \( \sum_{\lambda \in W_{x,J}^{-1}([k])} \text{dim} E_\lambda = 2 \), where \( W_\alpha(\lambda) = W(\alpha, \lambda) \), and \( E_\lambda \) is the eigenspace associated to the eigenvalue \( \lambda \).

3. For any \( A \in \pi_2(\Sigma) \), \( W(A \cdot \alpha, \lambda) = W(\alpha, \lambda) + c_1(A) \).

In view of the control over the sign of the eigenvector provided by Theorem 4.7, combined with the monotonicity of the winding number provided by item (1) of the above proposition, we make the following

Definition 4.14. For \( (H, J) \) and \( \hat{x} = [x, \alpha] \in \text{Per}_0(H) \) as above, define

\[
a(\hat{x}) = a(\hat{x}; H) := \sup_{\lambda \in \sigma(A_{x,J}) \cap (-\infty,0)} W(\alpha, \lambda),
\]

\[
b(\hat{x}) = b(\hat{x}; H) := \inf_{\lambda \in \sigma(A_{x,J}) \cap (0,\infty)} W(\alpha, \lambda).
\]

Remark that we have, by the monotonicity of \( W \) and by Theorem 4.7, that

Corollary 4.15. Let \( (H^\pm, J^\pm) \) be Floer non-degenerate, \( (H, J) \in \mathcal{H} \), \( x^\pm \in \text{Per}_0(H^\pm) \) for \( i = 0, 1 \). If \( u_i \in \mathcal{M}(\hat{x}_0^-, \hat{x}_1^+; H, J), i = 0, 1 \), then \( b(\hat{x}_0^-) \leq \ell_\infty(a_0, u_1) \). If \( v_i \in \mathcal{M}(\hat{x}_1^+, \hat{x}_0^-; H, J), i = 0, 1 \), then \( \ell_\infty(v_0, v_1) \leq a(\hat{x}_0^-) \).

The result which relates this discussion to the behaviour of the Floer complex is the following

Theorem 4.16 ([8] Theorem 3.10),

\[-\mu(\hat{x}) = a(\hat{x}) + b(\hat{x})\]

Remark. Strictly speaking, it is shown in [8] that \(-\mu(x) = 2a(x) + p(x)\), where \( p(x) \) denotes the parity of the index, but it is straightforward to show, using Proposition 4.13, that \( b(x) = a(x) + p(x) \). Note that our sign convention for the Conley-Zehnder index is the negative of that used in [8].

Recall from Section 4.1 that to any \((H, J) \in \mathcal{H}\) \((H^-, J^-; H^+, J^+)\), we associate the operator \( \mathcal{F}_{H,J} : C^\infty(\mathbb{R} \times S^1; \Sigma) \to \mathcal{E} \). Whenever \( \mathcal{F}_{H,J}(u) = 0 \), for \( u \in C^\infty(\mathbb{R} \times S^1; \Sigma) \), \( \hat{x}^{-, \hat{x}^+} \), \( T_u \mathcal{E} \) splits canonically as \( T^u \mathcal{E} \mid_{\hat{x}^-} \oplus \hat{x}^+ \). In such a case, we denote by \( D\mathcal{F}_{H,J} \) the projection of the differential of \( \mathcal{F}_{H,J} \) onto \( \hat{x}^+ \), and we call \( D\mathcal{F}_{H,J} \) the linearized Floer operator. The transversality of \( \mathcal{F}_{H,J} \) to the 0-section of \( \hat{x}^+ \) is equivalent to the surjectivity of \( (D\mathcal{F}_{H,J})_u \), which is in turn related to the behaviour of its kernel by the Fredholm property. The following result is essentially proved in [8] as Proposition 5.6 and serves to give significant control over elements in the kernel of the linearized Floer operator. We give a simple proof here in the Floer-theoretic setting for the convenience of the reader.

Proposition 4.17. Let \((H^\pm, J^\pm) \) be Floer non-degenerate, let \((H, J) \in \mathcal{H}, u \in \mathcal{M}(\hat{x}^-, \hat{x}^+; H, J)\), and let \( \xi \in \ker(D\mathcal{F}_{H,J})_u \). Suppose that \( \xi \neq 0 \) and denote by \( Z(\xi) \) the algebraic count of the number of zeros of \( \xi \), then \( Z(\xi) \) is finite and satisfies the inequality \( 0 \leq Z(\xi) \leq a(\hat{x}^+) - b(\hat{x}^-) \).

Proof. It is a standard result in Floer theory (see for instance [25], Section 2.2) that for any \( u \in \mathcal{M}(\hat{x}^-; H, J) \), any element \( \xi \in \ker(D\mathcal{F}_{H,J})_u \) may be expressed (with respect to the unitary trivialization \( \Phi : \mathbb{R} \times S^1 \times (\mathbb{R}^2, J_0) \to u^*(T\Sigma, J) \) along \( u \) induced by the cappings of \( \hat{x}^- \) and \( \hat{x}^+ \)) as solving an equation of the form

\[
\partial_t \xi + J_0 \partial_s \xi + S \xi = 0,
\]

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where we may write $S$ on the positive and negative ends as $S^\pm(s,t) = \Phi^{-1}A_{x, \pm} - \Delta^\pm(s,t)$, with $\Delta^\pm$ satisfying the decay estimates of Theorem 4.5. Consequently, any $\xi \in \ker(DF)_u$ must be non-vanishing outside of some compact neighbourhood of $\mathbb{R} \times S^1$, and the Carlemann similarity principle, combined with positivity of intersections of holomorphic curves in dimension 4, implies that $Z(\xi)$ is finite and non-negative.

To see that $Z(\xi) \leq a(\hat{x}^+) - b(\hat{x}^-)$, we take $R > 0$ sufficiently large so that $\xi$ is non-vanishing outside of $(-R, R) \times S^1$ and consider the homotopy of 2-braids in $\mathbb{R}^2$ induced by $h(s) = (0, \Phi^{-1}(\xi_s)) \in \mathcal{L}_0(\mathbb{R}^2)$, $s \in [-R, R]$. Theorem 4.5 implies that for $R > 0$ sufficiently large,

$$\ell(0, \xi_R) = \text{wind}(\Phi^{-1}(\xi_R)) \leq b(\hat{x}^-), \quad \text{and} \quad \ell(0, \xi_{-R}) = \text{wind}(\Phi^{-1}(\xi_{-R})) \leq a(\hat{x}^+),$$

(since $\mathbb{R}^2$ is aspherical, we omit any mention of cappings), and the algebraic count zeros of $\xi$ correspond to the algebraic count of the intersections of the graphs of the strands of $h$ from which the proposition follows.

$\square$

### 4.3 Foliated sectors and Floer moduli spaces as leaf spaces

We proceed to an investigation of the topology of the capped braid $(\hat{x}, \hat{y})$ for $\hat{x}, \hat{y} \in \text{Per}_0(H)$, and collective behaviour in $S^1 \times \Sigma$ of maps $(s,t) \mapsto (t, u(s,t))$, as $u$ varies in $\mathcal{M}(\hat{x}, \hat{y}; H, J)$.

**Lemma 4.18.** Let $(H, J)$ be an adapted homotopy of Floer data with $(H^\pm, J^\pm)$ Floer non-degenerate. Suppose that $\hat{x}^\pm \in \text{Per}_0(H^-) \cap \text{Per}_0(H^+)$, with $x^\pm \neq x^\pm$, and that $\mathcal{M}(\hat{x}^-, \hat{x}^+; H, J), \mathcal{M}(\hat{x}^-, \hat{x}^-; H, J)$, and $\mathcal{M}(\hat{x}^+, \hat{x}^+; H, J)$ are all non-empty, then $b(\hat{x}^-; H^-) \leq \ell(\hat{x}^-; H^-) \leq b(\hat{x}^+; H^+)$. 

**Proof.** That $b(\hat{x}^-; H^-) \leq \ell(\hat{x}^-, \hat{x}^+)$ follows from applying Corollary 4.15 with $u_0 \in \mathcal{M}(\hat{x}^-, \hat{x}^+; H, J)$ and $u_1 \in \mathcal{M}(\hat{x}^-, \hat{x}^-; H, J)$. The second inequality uses $v \in \mathcal{M}(\hat{x}^+, \hat{x}^+; H, J)$. 

Applying the preceding lemma in the case where $(H, J)$ is $s$-independent yields

**Corollary 4.19.** Let $(H, J)$ be Floer non-degenerate and suppose that $\mathcal{M}(\hat{x}, \hat{y}; H, J) \neq \emptyset$, $x \neq y$, then

$$b(\hat{x}) \leq \ell(\hat{x}, \hat{y}) \leq a(\hat{y}).$$

Applying Lemma 4.11 to the constant cylinder $u \in \mathcal{M}(\hat{x}', \hat{x}^+; H, J)$ and $v \in \mathcal{M}(\hat{x}, \hat{y}; H, J)$ shows

**Proposition 4.20.** Let $(H, J)$ be Floer non-degenerate and let $\hat{x}, \hat{y} \in \text{Per}_0(H)$ with $\hat{y} \in \text{supp} \partial_{H,J} \hat{x}$, then for all $x' \in \text{Per}_0(H)$, $x' \notin \{x, y\}$, we have $\ell(\hat{x}, x') \leq \ell(\hat{y}, x')$. 

A straight-forward computation using Theorem 4.16 gives

**Lemma 4.21.** Suppose that $\mu_{CZ}^H(\hat{x}) \in \{2k - 1, 2k\}$ for some $k \in \mathbb{Z}$, then $a(\hat{x}; H) = -k$. If $\mu_{CZ}^H(\hat{x}) \in \{2k, 2k + 1\}$ for some $k \in \mathbb{Z}$, then $b(\hat{x}; H) = -k$.

And the preceding lemma combines with Corollary 4.19 to give

**Corollary 4.22.** If $\mu(\hat{x}), \mu(\hat{y}) \in \{2k - 1, 2k, 2k + 1\}$ for some $k \in \mathbb{Z}$ and $\mathcal{M}(\hat{x}, \hat{y}; H, J) \neq \emptyset$, then $\ell(\hat{x}, \hat{y}) = -k$.

The main geometric input for this section is the following (cf. Theorem 5.6 in [10])

**Proposition 4.23.** Let $(H, J)$ be Floer non-degenerate, $\hat{x}, \hat{y} \in \text{Per}_0(H)$, $2k - 1 \leq \mu(\hat{x}), \mu(\hat{y}) \leq 2k + 1$, for some $k \in \mathbb{Z}$. Then the map $\hat{Ev} : \mathbb{R} \times S^1 \times \mathcal{M}(\hat{x}, \hat{y}; H, J) \to \mathbb{R} \times S^1 \times \Sigma$ defined by $\hat{Ev}(s,t,u) := \hat{u}(s,t) = (s, t, u(s,t))$ is a diffeomorphism onto its image.
Proof. We may suppose that $\tilde{\mathcal{M}}(\hat{x}, \hat{y}; H, J) \neq \emptyset$, or else the lemma is vacuously true. Moreover, if $\hat{x} = \hat{y}$, then the lemma is obvious. Thus, we may suppose that $\mu(\hat{x}) - \mu(\hat{y}) \in \{1, 2\}$. We will show that $\tilde{E}v$ is a proper injective immersion. That $\tilde{E}v$ is one-to-one follows from the fact that by Lemma 4.18, we have that for any $u, v \in \tilde{\mathcal{M}}(\hat{x}, \hat{y}; H, J)$, $u \neq v$, and all $s \in \mathbb{R}$, we have

$$0 = b(\hat{x}) \leq \ell(\hat{u}_s, \hat{v}_s) \leq a(\hat{y}) = 0,$$

and so by Lemma 4.11, the graphs $\hat{u}$ and $\hat{v}$ cannot intersect for any $u \neq v$. That $\tilde{E}v$ is proper and compact follows from compactness results in Floer theory; if $\{(s_n, t_n, u_n)\}_{n \in \mathbb{N}} \subseteq \mathbb{R} \times S^1 \times \tilde{\mathcal{M}}(\hat{x}, \hat{y}; H, J)$ is some sequence which eventually leaves any compact set, then either $s_n \rightarrow \pm\infty$ and $(s_n, \tau, u_n)$ converges to a point on either the graph $\tilde{x}(s, t) = (s, t, x(t))$ or on the graph $\tilde{y}(s, t) = (s, t, y(t))$, or $(s_n)_{n \in \mathbb{N}}$ remains bounded, in which case $(s_n, t_n, u_n)$ must converge to a point on the graph of some broken Floer cylinder between $x$ and $y$. In either case, the sequence $(s_n, t_n, u_n)$ eventually leaves every compact subset of $\text{im} \tilde{E}v$.

It remains to show that $\tilde{E}v$ is an immersion when $\mu(\hat{x}) - \mu(\hat{y}) \in \{1, 2\}$. We note that

$$T(\mathbb{R} \times S^1 \times \tilde{\mathcal{M}}(\hat{x}, \hat{y}; H, J)) = T(\mathbb{R} \times S^1) \oplus (\ker(dF))|_{\tilde{\mathcal{M}}(\hat{x}, \hat{y}; H, J)},$$

and since $d\tilde{E}v$ is obviously non-vanishing on $T(\mathbb{R} \times S^1)$, the problem reduces to showing that $\xi \in \ker(dF)$ is a nowhere-vanishing vector field along $u$, which follows by combining Proposition 4.17 with Corollary 4.22 to deduce that $\tilde{Z}(\xi) = 0$ whenever $\xi$ is not identically zero.

Note that as a consequence of the previous lemma, whenever $\mu(\hat{x}), \mu(\hat{y}) \in \{2k - 1, 2k, 2k + 1\}$ for $k \in \mathbb{Z}$, then $\tilde{E}v(\mathbb{R} \times S^1 \times \tilde{\mathcal{M}}(\hat{x}, \hat{y}; H, J))$ carries a smooth 2-dimensional foliation $\tilde{F}^{\hat{x}, \hat{y}}$, the leaves of which are nothing but the graphs $\tilde{u}$ of $u \in \tilde{\mathcal{M}}(\hat{x}, \hat{y}; H, J)$.

**Definition 4.24.** For $\hat{x}, \hat{y} \in \tilde{\text{Per}_0}(H)$, the connecting subspace of $\hat{x}$ and $\hat{y}$ will denote the subspace $W(\hat{x}, \hat{y}) := \{(t, u(s, t)) \in S^1 \times \Sigma : s \in \mathbb{R}, u \in \tilde{\mathcal{M}}(\hat{x}, \hat{y}; H, J)\}$.

Remark that if we write $\tilde{W}(\hat{x}, \hat{y}) := \tilde{E}v(\mathbb{R} \times S^1 \times \tilde{\mathcal{M}}(\hat{x}, \hat{y}; H, J))$, then the map $\tilde{\pi} : \mathbb{R} \times S^1 \times \Sigma \rightarrow S^1 \times \Sigma$ restricts to a projection $\tilde{\pi} : \tilde{W}(\hat{x}, \hat{y}) \rightarrow W(\hat{x}, \hat{y})$, with fiber $\tilde{\pi}^{-1}(t, p) = \tilde{E}v(s, t, u) : u(s, t) = p)$, which under the hypotheses of Proposition 4.23 may be identified via $\tilde{E}v$ with the orbit of any $(s_0, t, u_0) \in \mathbb{R} \times S^1 \times \tilde{\mathcal{M}}(\hat{x}, \hat{y}; H, J)$ such that $u_0(s_0, t) = p$ under the $\mathbb{R}$-action $\tau \cdot (s, t, u) = (s - t, u^\tau)$, for $\tau \in \mathbb{R}$, where $u^\tau(s, t) = u(s + \tau, t)$. Consequently, $\tilde{E}v$ descends to a well-defined map $\tilde{E}v([s, t, u]) = (t, u(s(t)) \in W(\hat{x}, \hat{y}), [s, t, u] \in (\mathbb{R} \times S^1 \times \tilde{\mathcal{M}}(\hat{x}, \hat{y}; H, J))/\mathbb{R})$.

Hence, under the hypotheses of Proposition 4.23, $\tilde{\pi}$ restricts to a submersion on $\tilde{W}(\hat{x}, \hat{y})$ with fiber diffeomorphic to $\mathbb{R}$. Moreover, if we choose a section $\sigma : \tilde{\mathcal{M}}(\hat{x}, \hat{y}; H, J) \rightarrow \tilde{\mathcal{M}}(\hat{x}, \hat{y}; H, J)$, then we may thereby (non-canonically) identify

$$\phi_{\sigma} : \mathbb{R} \times S^1 \times \tilde{\mathcal{M}}(\hat{x}, \hat{y}; H, J) \xrightarrow{\sim} (\mathbb{R} \times S^1 \times \tilde{\mathcal{M}}(\hat{x}, \hat{y}; H, J))/\mathbb{R}$$

$$\{ (s, t, [u]) \rightarrow [s, t, \sigma([u])].$$

Finally, to understand the behaviour of the foliation $\tilde{F}^{\hat{x}, \hat{y}}$ under this projection, note that $\ker d\tilde{\pi} = \langle \partial_s \rangle$, and that since the tangent space of any leaf of $\tilde{F}$ is given by $\langle \partial_s + (\partial_s u)_{u(s, t)}, \partial_t + (\partial_t u)_{u(s, t)} \rangle$, where $\partial_s u \in \ker(dF)_{u}$, and is therefore nowhere-vanishing whenever $u$ is not an orbit cylinder by our index constraint. So the leaves of the foliation are nowhere tangent to the fibers of the projection map whenever $\hat{x} \neq \hat{y}$. As a consequence, we deduce

**Corollary 4.25.** Let $\hat{x}, \hat{y} \in \tilde{\text{Per}_0}(H)$ satisfy $\mu(\hat{x}), \mu(\hat{y}) \in \{2k - 1, 2k, 2k + 1\}$, for some $k \in \mathbb{Z}$ then for any section $\sigma : \tilde{\mathcal{M}}(\hat{x}, \hat{y}; H, J) \rightarrow \tilde{\mathcal{M}}(\hat{x}, \hat{y}; H, J)$, as above,

$$\tilde{E}v \circ \phi_{\sigma} : \mathbb{R} \times S^1 \times \tilde{\mathcal{M}}(\hat{x}, \hat{y}; H, J) \rightarrow W(\hat{x}, \hat{y})$$

$$(s, t, [u]) \rightarrow (t, \sigma([u])(s, t))$$

is a smooth embedding. Moreover, writing $\sigma([u]) = u_s$, the partition $F^{\hat{x}, \hat{y}} := \{ (\text{im} \hat{u}_s)_{u \in \tilde{\mathcal{M}}(\hat{x}, \hat{y}; H, J) \text{ is a smooth 2-dimensional foliation of } W(\hat{x}, \hat{y}) \text{ whenever } \hat{x} \neq \hat{y}.}$$


5 Contracting Seidel morphisms

In order to build the foliation described by Theorem A, our goal is to find appropriate conditions on capped braids \( \tilde{X} \subseteq \text{Per}_0(H) \) such that the foliated sectors \( W(\hat{x}, \hat{y}) \) from the preceding section considered for pairs \( \hat{x}, \hat{y} \in \tilde{X} \) will glue together to produce the desired foliation. Minimally, any such braid will have to be unlinked. In this section, we introduce a useful tool for reducing the study of unlinked capped braids to that of trivial capped braids without changing the qualitative dynamics or the Floer-theoretic properties of the situation.

We briefly recall the definition of the Seidel morphism on Floer chain complexes introduced in [28]. Let \( \mathcal{G} = C^\infty(S^1, 0; (\text{Ham}(M, \omega), id)) \) be the space of Hamiltonian loops, based at the identity. Any \( g \in \mathcal{G} \) induces a diffeomorphism \( g_* : \mathcal{L}_0(M) \to \mathcal{L}_0(M) \), defined by \( (g_*x)(t) := g(x(t)), t \in S^1 \). We denote by \( \mathcal{G} \) the covering of \( \mathcal{G} \) given by pairs \( (g, \tilde{g}) \) such that \( g \in \mathcal{G} \) and \( \tilde{g} : \mathcal{L}_0(M) \to \mathcal{L}_0(M) \) is a \( \Gamma_\omega \)-equivariant lift of the map \( g_* \). For any Floer non-degenerate pair \( (H, J) \), an element \((g, \tilde{g}) \in \mathcal{G} \) with \( g = \phi_G \), where \( G \) is a 1-periodic Hamiltonian, gives rise to an isomorphism of Floer complexes (modulo a grading shift)

\[
S(g, \tilde{g}) : CF_*(H, J) \to CF_{*-1}(G\# H, g_*J) \quad \hat{x} \to \tilde{g}(\hat{x}),
\]

where \( I = I(g, \tilde{g}) \in \mathbb{Z} \) is the Maslov index of the loop of symplectic linear maps \( T_{\tilde{g}}(s)(t) \circ Dg(t)(x(t)) \circ T_{\tilde{g}}^{-1}(t) \). In addition, \( S(g, \tilde{g}) \) shifts the action upwards by a constant \( a(g, \tilde{g}) \) which, when \( (\mathcal{M}, \omega) \) is monotone, is a scalar multiple of \( I(g, \tilde{g}) \). \( S(g, \tilde{g}) \) is called the Seidel morphism associated to \( (g, \tilde{g}) \in \mathcal{G} \).

The next proposition gives a topological interpretation of \( I(g, \tilde{g}) \) in the case that \( g \) is a contractible Hamiltonian loop.

Proposition 5.1. Let \( g \in \mathcal{G} \) be contractible and \( \tilde{g} \in \mathcal{G} \) any lift of \( g \), then \( I(g, \tilde{g}) = k \) if and only if for some (hence any) \( \hat{x} \in \tilde{L}_0(M) \), there exists some homotopy \( s \to g^s \in \mathcal{G}, s \in [0, 1] \), from \( g^0 \equiv id \) to \( g^1 = g \), such that the induced homotopy of loops given by \( h(s) := (g^s)_*x \in \mathcal{L}_0(M), s \in [0, 1] \), is an \( A \)-homotopy from \( \hat{x} \) to \( \tilde{g}(\hat{x}) \) for \( A \in \pi_2(M) \) with \( c_1(A) = k \).

Proof. Let \( (g, \tilde{g}) \) be as above and fix an arbitrary \( \hat{x} = [x, \alpha] \in \tilde{L}_0(M) \) and an arbitrary homotopy \((g^s)_x \in [0, 1]\) from \( id \) to \( g \). Let us write \( c(s, t) := (g^s)_x(t), (s, t) \in [0, 1] \times S^1 \) be the cylinder in \( M \) traced out by \((g^s)_x \) throughout the homotopy. Note first that \( h \) is obviously a 0-homotopy between \( \hat{x} \) and \([g, x, \alpha\# c]\), and the map

\[
(s, t) \mapsto T_{(g^s)_x, x, \alpha\# c(0, \alpha)}^{-1}(t) \circ (Dg^s(t))_{x(t)} \circ T_{x}^{-1}(t), \quad (s, t) \in [0, 1] \times S^1
\]

provides a homotopy from the constant loop to the loop \( T_{(g^s)_x, x, \alpha\# c(0, \alpha)}^{-1}(t) \circ \tilde{T}_{x}(t), (s, t) \in [0, 1] \times S^1 \). So, in the event that \( \tilde{g}(\hat{x}) = [g, x, \alpha] \) (and so \( h \) is a 0-homotopy), we have that \( I(g, \tilde{g}) = 0 \). Next note that we may always write \( \tilde{g}(\hat{x}) = [g, x, A\# \alpha\# c] \) for some \( A \in \pi_2(M) \), and in such a case, we know both that \( h \) is a \((-A)\)-homotopy from \( \hat{x} \) to \( \tilde{g}(\hat{x}) \) and also that \( T_{(g^s)_x, x, A\# \alpha\# c(0, \alpha)}^{-1}(t) \circ \tilde{T}_{x}(t) \circ T_{(g^s)_x, x, \alpha\# c(0, \alpha)}^{-1}(t) \circ L_{x} \) for \( (L_x)_{x \in S^1} \) a loop of symplectic matrices with Maslov index equal to \( I(g, \tilde{g}_x) = c_1(A) \), where \( \tilde{g}_x \) is the lift of \( g \) sending \( \hat{x} \) to \([g, x, \alpha\# c]\). The statement follows readily.

Corollary 5.2. Let \( M = \Sigma \) be a surface, and let \((g, \tilde{g}) \in \mathcal{G} \) be such that \( I(g, \tilde{g}) = 0 \). Then, for any capped k-braid \( \tilde{X} = (\hat{x}_1, \ldots, \hat{x}_k), L_0(0, \tilde{X}) = L_0(0, \tilde{g}(\hat{X})), \) where \( \tilde{g}(\hat{X}) = (\tilde{g}(\hat{x}_1), \ldots, \tilde{g}(\hat{x}_k)) \).

Proof. This follows immediately from Proposition 3.19 and consideration of the braid homotopy \( h(s) := (g^s_{x_1}, \ldots, g^s_{x_k}) \) for \( g \), a path of contractible loops from \( id \) to \( g \).

The following maybe be viewed as an analogue of Lemma 9.2 in [26] for contractible loops with additional controls over the index shift induced by a well-chosen lift.

Lemma 5.3. For any \( \hat{x} \in \tilde{L}_0(M) \), there exists a contractible loop \( g \in \mathcal{G} \) and a lift \( \tilde{g} \in \mathcal{G} \) such that \( I(g, \tilde{g}) = 0 \) and \( \tilde{g}(\hat{x}) = \hat{x}_0 \), where \( \hat{x}_0 = [x_0, x_0] \) denotes the trivially capped constant loop based at \( x_0 = x(0) \).

Proof. Consider the map \( E_{\hat{x}_0} : \text{Ham}(M, \omega) \to M \). It is easy to see that this a locally trivial fibration with model fiber \( S_{x_0} = \{ \hat{\phi} \in \text{Ham}(M, \omega) : \phi(x_0) = x_0 \} \) over \( x_0 \). Let \( w : (D^2, -1) \to (M, x_0) \) represent
the capping of \( \hat{x} \) (here we view \( D^2 \) as the unit disk in \( \mathbb{C} \)), because \((D^2, -1)\) is homeomorphic as a pair to \(([0,1]^2, [0,1] \times \{0\} \cup \{0\} \times [0,1])\,\)
we may lift \( x \) to a map \( \tilde{w} : [0,1]^2 \to \text{Ham}(M, \omega) \) which covers \( w \) via \( Ev_x \), and which moreover satisfies \( w(s,0) = w(0,t) = w(1, t) = id \), for all \((s,t) \in [0,1]^2\). Let us write \( k^1(t) := \tilde{w}(s,t) \in \text{Ham}(M, \omega)_\ast \) gives a homotopy from the constant loop to \( k^1 \), with \( k^1_0 = x \). Moreover, since \( \tilde{w} \) covers \( w \), it is immediate that \( h(s) := k^1_0 \in \mathbb{L}_0(M) \) provides a homotopy from \( x_0 \) to \( \hat{x} \), and so the lift of \( k^1 \) defined by insisting that \( \tilde{k}^1(\hat{x}_0) = \hat{x} \) satisfies \( I(k^1, \tilde{k}^1) = 0 \) by Proposition 5.1. Setting \((g, \tilde{g}) = ((k^1)^{-1}, (\tilde{k}^1)^{-1}) \) proves the statement. \( \Box \)

If \( \hat{X} = (\hat{x}_1, \ldots, \hat{x}_k) \) is an unlinked capped braid, then reasoning in an analogous manner to the previous subsection — with the fibration \( Ev_{\tilde{x}_0} : \text{Ham}(M, \omega) \to M \) replaced with the fibration \( Ev_{\tilde{x}_0} : \text{Ham}(M, \omega)_\ast \to \mathbb{L}_0 \) given by \( Ev_{\tilde{x}_0}(\phi) := (\phi(x_1(0)), \ldots, \phi(x_k(0))) \) — allows us to prove the following

Lemma 5.4. For any unlinked capped braid \( \hat{X} \), there exists a contractible loop \( \tilde{g} \in \tilde{G} \) such that \( I(g, \tilde{g}) = 0 \) and \( \tilde{g}(\hat{x}) = [x(0), x(0)] \) for every \( \hat{x} \in \hat{X} \).

We will call any Seidel morphism \( S(g, \tilde{g}) \) satisfying the conclusions of the above lemma for a given \( \hat{X} \) a contracting Seidel morphism associated to \( \hat{X} \). Since the precise choice of \((g, \tilde{g}) \) will be irrelevant for our purposes, we will write \( S(\hat{X}) := S(g, \tilde{g}) \) for any contracting Seidel morphism associated to the capped orbit \( \hat{X} \).

6 The restricted complex associated to a capped braid
Let \((H, J)\) be a non-degenerate Floer pair. To any capped braid \( \hat{X} \subset \bar{\text{Per}_0}(H) \), we may associate the submodule \( C_\ast(\hat{X}) := \Lambda_\omega(\hat{x})_{\hat{x} \in \hat{X}} \), which comes with the projection \( \pi^\hat{X} : CF_\ast(\hat{H}, J) \to C_\ast(\hat{X}) \) associated to the splitting \( CF_\ast(\hat{H}, J) = C_\ast(\hat{X}) \oplus C_\ast(\hat{Y}) \), for \( \hat{Y} \subset \bar{\text{Per}_0}(H) \) any capped braid such that \( \text{Per}_0(H) = X \sqcup Y \). \( C_\ast(\hat{X}) \) is not generally a subcomplex of \( CF_\ast(\hat{H}, J) \), since there is no reason that Floer cylinders should only run between strands of \( X \). However, we will see that if we define the restricted differential
\[ \partial^\hat{X} := \pi^\hat{X} \circ \partial_{H,J} \]
then under suitable conditions on \( \hat{X} \),
\[ CF_\ast(\hat{X}; H, J) := (C_\ast(\hat{X}), \partial^\hat{X}) \]
is a chain complex whose homology is isomorphic to the homology of the full complex \( CF_\ast(\hat{H}, J) \).

6.1 Maximal unlinkedness relative the Morse range
Definition 6.1. For any capped braid \( \hat{X} \subset \bar{\text{Per}_0}(H) \), we define
\[ \text{Pos}(\hat{X}) := \{ \sigma \in CF_\ast(\hat{H}, J) : \forall \hat{\gamma} \subset \hat{X}, \ell(\hat{x}, \hat{\gamma}) \geq 0 \}, \]
\[ \text{Pos}^\ast(\hat{X}) := \{ \sigma \in \text{Pos}(\hat{X}) : \forall \hat{\gamma} \subset \hat{X}, \exists \hat{x} \in \hat{X}, \text{such that } \ell(\hat{x}, \hat{\gamma}) > 0 \}. \]
We define \( \text{Neg}(\hat{X}) \) and \( \text{Neg}^\ast(\hat{X}) \) in the obvious manner simply by reversing the inequalities in the above.

Definition 6.2. Let \( \hat{X} \subset \bar{\text{Per}_0}(H) \) be a capped braid for some Hamiltonian \( H \). \( \hat{X} \) will be said to be maximally unlinked if it is unlinked and if for any \( \tilde{y} \in \tilde{\text{Per}_0}(H) \) either \( \tilde{y} \in \tilde{X} \) or \( \tilde{y} \) and \( \hat{X} \) are linked. We write \( \text{mu}(H) \) for the collection of all such capped braids.

Definition 6.3. Let \( \hat{X} \subset \bar{\text{Per}_0}(H) \) be a capped braid for some Hamiltonian \( H \). \( \hat{X} \) of \( X \) will be said to be maximally unlinked relative the Morse range \( \hat{X} \) is unlinked, \( \mu(x) \in \{-1, 0, 1\} \) for all \( x \in \hat{X} \), and moreover if for any \( y \in \tilde{\text{Per}_0}(H) \) such that \( \mu(y) \in \{-1, 0, 1\} \), either \( y \in \tilde{X} \) or \( y \) and \( \hat{X} \) are linked.
We write \( \text{mu}^\ast(H) \) for the collection of all capped braids \( \hat{X} \subset \bar{\text{Per}_0}(H) \) which are maximally unlinked relative the Morse range.

The next lemma is a direct consequence of the definitions.

Lemma 6.4. Let \( \hat{X} \subset \bar{\text{Per}_0}(H) \) be an unlinked braid, then \( \text{Pos}^\ast(\hat{X}), \text{Neg}^\ast(\hat{X}) \subset \ker \pi^\hat{X} \)

The following situation will occur frequently enough that it will be useful to isolate it as a
Lemma 6.5. Let $(H^\pm, J^\pm)$ be Floer non-degenerate, $(\mathcal{H}, J) \in \mathcal{HF}$, and $\hat{x}_i \in \hat{\text{Per}}_0(H^-) \cap \hat{\text{Per}}_0(H^+)$, $i = 1, \ldots, k$. Suppose that $\mu(\hat{x}_i) \in \{0, 1\}$, $X = (\hat{x}_1, \ldots, \hat{x}_k)$ is unlinked, and $\nu' \in M(\hat{x}_i, \hat{x}_i; \mathcal{H}, J)$ for $i = 1, \ldots, k$. Let $v \in M(\hat{x}_1, \hat{x}_1; \mathcal{H}, J)$, with $\hat{x}^+ \in \hat{\text{Per}}_0(H^+)$ such that $X \cup \{\hat{x}^+\}$ is linked, then $\hat{x}^+ \in \text{Pos}^*(\hat{X})$.

Proof. Note that we have $0 = \ell(\hat{x}_1, \hat{x}_i) \leq \ell(\hat{x}^+, \hat{x}_i)$ for $i = 2, \ldots, k$ by Lemma 4.11 and $0 = b(\hat{x}_1) \leq \ell(\hat{x}_1, \hat{x}_1)$ by Lemma 4.18, so we need only show that $\ell(\hat{x}^+, \hat{x}_i) > 0$ for some $i = 1, \ldots, k$. We write $h(s) = (u_1^i, \ldots, u_k^i, v_0)$, $s \in \mathbb{R}$. $h$ does not induce a braid cobordism, because $u_1$ and $v$ degenerate to the same orbit as $s \to -\infty$, however Lemma 4.11 implies that for $R > 0$ sufficiently large, $h|_{(-R, \infty)}$ induces a braid cobordism from $\hat{X} \cup \{\hat{v}_R\}$ to $\hat{X} \cup \{\hat{y}\}$. Since $b(\hat{x}_1) \geq 0$, there are two possibilities: either $0 < \ell_{-\infty}(u^1, v)$, or $\ell_{-\infty}(u^1, v) = 0$. In the former case, Lemma 4.11 immediately implies that $0 < \ell_{-\infty}(u^1, v) = \ell(\hat{x}_1, \hat{x}^+)$, and we are done.

We may therefore assume that $\ell_{-\infty}(u^1, v) = 0$. In this case, $\hat{X} \cup \{\hat{v}_R\}$ is unlinked. Indeed by Corollary 4.9, $R > 0$ may be chosen such that $\hat{v}$ has no intersections with $\hat{u}_i$, $i = 1, \ldots, k$ for $s < -R$, and the property of being unlinked is invariant under 0-homotopies, so $\hat{X} \cup \{\hat{v}_R\}$ is unlinked only if $\{\hat{x}_1, \hat{v}_R\}$ is unlinked. But we may take $R > 0$ sufficiently large such that $\hat{v}_R \in \hat{L}_0(S)$ lies in an exponential neighbourhood of $\hat{x}_1$, and in this neighbourhood the homological linking number reduces to the classical winding number by Proposition 3.21, and so that $\{\hat{x}_1, \hat{v}_R\}$ is unlinked follows directly from the fact that the winding number classifies homotopy classes of loops into $\mathbb{R}^2 \setminus 0$.

Thus, $\hat{X} \cup \{\hat{v}_R\}$ is unlinked, while $\hat{X} \cup \{\hat{x}^+\}$ is linked, whence the graphs of some of the strands of $h$ must intersect. Since $\ell_{-\infty}(u^i, u^j) = \ell_{\infty}(u^i, u^j) = \ell(\hat{x}_i, \hat{x}_j) = 0$ for $i \neq j$, it follows from Lemma 4.11 that the graphs of $u^i$ and $u^j$ are disjoint for $i \neq j$. Thus, there exists some $i = 1, \ldots, k$ such that the graphs of $u^i$ and $v$ intersect, so $0 < \ell_{-\infty}(u^i, v) = \ell(\hat{x}_i, \hat{x}^+)$, as claimed.

Lemma 6.6. Let $\hat{X} \in \text{murm}(H)$, then for all $\hat{x} \in \hat{X}$, $\partial H,J \hat{x} \in \mathbb{Z}_2[\hat{x}]_{\hat{x} \in \hat{X}} \oplus \text{Pos}^*(\hat{X})$.

Proof. Let $\hat{x} \in \hat{X}$, and $\hat{y} \in \text{supp} \partial H,J \hat{x}$. Either $\mu(\hat{x}) = -1$ or $\mu(\hat{x}) = \{0, 1\}$. If $\mu(\hat{x}) = -1$, then $b(\hat{x}) = 1$ and so Corollary 4.19 implies that $1 \leq \ell(\hat{x}, \hat{y})$ while Proposition 4.20 implies that $0 \leq \ell(\hat{x}, \hat{y})$ for all $\hat{x} \in \hat{X}$, $\hat{x} \neq \hat{x}^+$, and so $\hat{y} \in \text{Pos}^*(\hat{X})$. If $\mu(\hat{x}) = \{0, 1\}$, then Corollary 4.19 and Proposition 4.20 imply that $0 \leq \ell(\hat{x}, \hat{y})$ for all $\hat{x} \in \hat{X}$. To see that $\hat{y} \in \hat{X} \cup \text{Pos}^*(\hat{X})$, note that either $\hat{X} \cup \{\hat{y}\}$ is unlinked, in which case $\hat{y} \in \hat{X}$ by the maximality of $\hat{X}$, or $\hat{X} \cup \{\hat{y}\}$ is linked, in which case Lemma 6.5 directly implies that $\hat{y} \in \text{Pos}^*(\hat{X})$.

Lemma 6.7. Let $\hat{X} \subseteq \hat{\text{Per}}_0(H)$ be any capped braid, then $\partial H,J \text{Pos}^*(\hat{X}) \subseteq \text{Pos}^*(\hat{X})$.

Theorem 6.8. Let $\hat{X} \in \text{murm}(H)$, then $CF_*(\hat{X}; H, J)$ is a chain complex. That is, $\partial \hat{X} \circ \partial \hat{X} = 0$.

Proof. First, consider that $\Sigma$ is either an aspherical surface, in which case $\Lambda_\omega = \mathbb{Z}_2$, or else a sphere, in which case $\Sigma$ has minimal Chern number 2, and in the case that $\Sigma = S^2$, $CF_*(\hat{X}; H, J)$ vanishes in any degree congruent to 2 mod 4, whence by the $\Lambda_\omega$-equivariance of the Floer boundary map, it suffices in all cases to prove that $(\partial \hat{X})^2$ vanishes in the Morse range. To wit, by the previous two lemmas, we see that for any $\hat{x} \in \hat{X}$,

$$\partial H,J \hat{x} = \partial \hat{X} \hat{x} + (\pi_{\text{Pos}^*(\hat{X})} \circ \partial H,J)(\hat{x}),$$

where $\pi_{\text{Pos}^*(\hat{X})}$ denotes projection onto $\text{Pos}^*(\hat{X})$. Thus, since $\partial^2 H,J = 0$, $(\partial \hat{X})^2 \hat{x} + \sigma = 0$, where $\sigma \in \text{Pos}^*(\hat{X})$. It follows that $(\partial \hat{X})^2 \hat{x} = 0$.

We will write $HF_*(\hat{X}; H)$ for the homology of the complex $CF_*(\hat{X}; H, J)$ when $\hat{X} \in \text{murm}(H)$.

6.2 Moduli spaces for continuation morphisms and pseudolinear homotopies

We obtain results giving us a measure of control over the moduli spaces which contribute to continuation morphisms. In addition, we introduce the notion of pseudolinear adapted homotopies which enable us to avoid regularity issues which arise in considering linear homotopies while retaining sufficient control over the moduli spaces for our linking arguments to go through.
Corollary 6.11. Let \((H^+, J^+)\) be Floer non-degenerate, then for any \(\hat{x}^\pm \in \overline{\text{Per}_0}(H^\pm)\) such that \(\mu(\hat{x}^-) = 2k + 1\), \(k \in \mathbb{Z}\), and any \((H, J) \in \mathcal{H}^f\), we have \(|\mathcal{M}(\hat{x}^-, \hat{x}^+; H, J)| \in (0, 1]\) and this quantity is independent of the pair \((H, J)\).

Corollary 6.12. Let \((H^+, J^+)\) be Floer non-degenerate, \((H, J) \in \mathcal{H}^f\) and \(\hat{x} \in \overline{\text{Per}_0}(H^-) \cap \text{Per}_0(H^+)\) such that \(\mu(\hat{x}; H^+^) = 2k + 1\) for some \(k \in \mathbb{Z}\). Let \(\hat{x}^\pm \in \text{Per}_0(H^\pm)\), \(\hat{x}^\pm \neq \hat{x}\), also satisfy \(\mu(\hat{x}^\pm) = 2k + 1\). Then \(\mathcal{M}(\hat{x}^-, \hat{x}; H, J) \neq \emptyset\) only if \(\ell(\hat{x}; \hat{x}^-) \leq a(\hat{x})\) and \(\mathcal{M}(\hat{x}, \hat{x}^+; H, J) \neq \emptyset\) only if \(b(\hat{x}) \leq \ell(\hat{x}; \hat{x}^+)\).

We fix, once and for all, some smooth, surjective non-decreasing function \(\beta : \mathbb{R} \to [0, 1]\) which is constant outside some compact set. Given two Hamiltonians \(H^\pm \in C^\infty(S^1 \times \Sigma)\), we will call the adapted homotopy \(H^\in\!(s,t) := \beta(s)H^-(t) + (1 - \beta(s))H^+(t)\) the linear homotopy from \(H^-\) to \(H^+\).

If \(x \in \text{Per}_0(H^-) \cap \text{Per}_0(H^+)\), then the trivial cylinder \(u(s,t) = x(t)\) solves Equation 2 for \((H^\in\!, J)\) with \(J \in \mathcal{J}(J^-, J^+)\) arbitrary.

We denote by \(\mathcal{H}^{\text{reg}}(H^\in\!)\) the collection of adapted homotopies \(H \in \mathcal{H}(H^-^, H^+^)\) which agree with \(H^\in\!\) up to second order along the graphs \((s,t,x(t)) \in \mathbb{R} \times S^1 \times M\) for all \(x \in \text{Per}_0(H^-^) \cap \text{Per}_0(H^+)\) such that \(\mu(\hat{x}; H^+^) = \mu(\hat{x}; H^-^)\) for some (hence every) capping \(\hat{x}\) of \(x\) (we thus admit arbitrary perturbations along the graphs of constant cylinders which are 0-homotopies between capped orbits with index differing on each end). We may use the fact that any finite energy solution to Equation 2 has only finitely many intersections with any given constant cylinder solving the same equation to adapt the usual proof (see the proof of Lemma 11.1.9 in [1], for example) that generic perturbations of \(H\) suffice to achieve regularity of the Floer operator to establish the following

Proposition 6.13. Let \((H^+, J^+)\) be Floer non-degenerate and \(J \in \mathcal{J}(J^-, J^+)\). There exists a residual set \(\mathcal{H}^{\text{reg}}(H^\in\!, J)\) inside \(\mathcal{H}^{\text{reg}}(H^\in\!, J)\) such that \((J, J^+)\) is \((\hat{x}^-, \hat{x}^+)\)-regular for all \(\hat{x}^\pm \in \text{Per}_0(H^\pm)\) with either \(\hat{x}^- \neq \hat{x}^+\) and \(\mu(\hat{x}^-) = \mu(\hat{x}^+)\) or with \(\hat{x}^- = \hat{x}^+\) and \(\mu(\hat{x}^-; H^-) \neq \mu(\hat{x}^+; H^+)\).

The essential point is that if \(u\) is a non-constant cylinder, then we may always find some \((s_0, t_0) \in \mathbb{R} \times S^1\) and a neighbourhood \(U\) of \((s_0, t_0, u(s_0,t_0)) \in \mathbb{R} \times S^1 \times \Sigma\) which is disjoint from the graph of every constant solution and construct a perturbation with support contained in \(U\).

If \(\text{Per}_0(H^-) \cap \text{Per}_0(H^+) \neq \emptyset\), we cannot expect \((H, J) \in \mathcal{H}^f\) with \(H \in \mathcal{H}^{\text{reg}}(H^\in\!, J)\) to be Floer-regular, owing to the fact that constant solutions are not in general regular points of the Floer operator. The following lemma provides the existence of regular pairs \((H', J') \in \mathcal{H}^{\text{reg}}\) such that, for the purposes of controlling the linking behaviour of orbits contributing to the continuation map \(b_{H', J}\), we may reason ‘as if’ these constant solutions exist.
Lemma 6.14. Let \((H^\pm, J^\pm)\) be Floer non-degenerate and \(J \in \mathcal{J}(J^-, J^+)\). There exists an open neighbourhood \(U\) of \(\mathcal{H}_{\text{reg}}(H^{\text{lin}}) \subseteq \mathcal{H}(H^-, H^+)\) such that for every \(\mathcal{H} \in \mathcal{H}_{\text{reg}}(J) \cap U\) there exists some \(\mathcal{H}' \in \mathcal{H}_{\text{reg}}(H^{\text{lin}}, J)\) with the property that for all \(\hat{x}^\pm \in \text{Per}_0(H^\pm), \hat{x}^\neq \hat{x}^+\), with \(\mu(\hat{x}^-) = \mu(\hat{x}^+)\),

\[\mathcal{M}(\hat{x}^-, \hat{x}^+; H, J) \neq \emptyset \Rightarrow \mathcal{M}(\hat{x}^-, \hat{x}^+; H', J) \neq \emptyset.\]

(4)

Proof. It will suffice to show that if we are given a pair \(\hat{x}^\pm \in \text{Per}_0(H^\pm)\) such that \(\mu(\hat{x}^-) = \mu(\hat{x}^+)\), then around each \(\mathcal{H}' \in \mathcal{H}_{\text{reg}}(H^{\text{lin}}, J)\), there exists an open neighbourhood \(V = \mathcal{V}_{\mathcal{H}'}^{\hat{x}^-, \hat{x}^+} \subseteq \mathcal{H}(H^-, H^+)\) such that (4) holds for every \(\mathcal{H} \in \mathcal{H}_{\text{reg}}(J) \cap V\). Indeed, if this holds, then since the sets \(\text{Per}_0(H^\pm)\) are finite and the moduli spaces \(\mathcal{M}(\hat{x}^-, \hat{x}^+; H, J)\) are invariant under the \(\Gamma^-\) action \(A \cdot (\hat{x}^-, \hat{x}^+) = (A \cdot \hat{x}^-, A \cdot \hat{x}^+)\), and in our setting any \(A\) with \(c_1(A) = 0\) represents \(0 \in \Gamma_\omega\), the set

\[V_{\mathcal{H}'} := \bigcup_{\hat{x}^\pm \in \text{Per}_0(H^\pm)} \mathcal{V}_{\mathcal{H}'}^{\hat{x}^-, \hat{x}^+} \cap \mathcal{V}_{\mathcal{H}'}^{\hat{x}^-, \hat{x}^+} \neq \emptyset \cap \mathcal{V}_{\mathcal{H}'}^{\hat{x}^-, \hat{x}^+} \neq \emptyset\]

is an intersection of finitely many open sets, and hence open (the bounds \(-2 \leq \mu(\hat{x}^-) = \mu(\hat{x}^+) \leq 2\) appear in the last intersection due to 2 being a lower bound for the minimal Chern number of a surface).

By the density of \(\mathcal{H}_{\text{reg}}(H^{\text{lin}}, J)\) in \(\mathcal{H}(H^{\text{lin}})\), taking \(U := \bigcup V_{\mathcal{H}'}\), where the union is over all \(\mathcal{H}' \in \mathcal{H}_{\text{reg}}(H^{\text{lin}}, J)\), we obtain the desired open set.

We therefore fix some \(\mathcal{H}' \in \mathcal{H}_{\text{reg}}(H^{\text{lin}})\) and some \(\hat{x}^\pm \in \text{Per}_0(H^\pm)\) with \(\mu(\hat{x}^-) = \mu(\hat{x}^+)\), and suppose that there exists no such neighbourhood \(V_{\mathcal{H}'}^{\hat{x}^-, \hat{x}^+}\). Then it must be the case both that \(\mathcal{M}(\hat{x}^-, \hat{x}^+; H', J) = \emptyset\) and that there is some sequence \((\mathcal{H}^\nu)'\) in \(\mathcal{H}_{\text{reg}}(J)\) tending to \(\mathcal{H}'\) in the \(C^\infty\)-topology such that \((\mathcal{H}^\nu, J) \in \mathcal{H}^{\text{reg}}\) and \(\mathcal{M}(\hat{x}^-, \hat{x}^+; H^\nu, J) \neq \emptyset\), for each \(\nu \in \mathbb{N}\). We thereby obtain a sequence of maps \(u_\nu \in \mathcal{M}(\hat{x}^-, \hat{x}^+; H^\nu, J)\), \(\nu \in \mathbb{N}\), which have uniformly bounded energy, owing to the fact that \(\mathcal{H}^\nu \to \mathcal{H}'\) in the \(C^\infty\)-topology. Consequently, up to passing to a subsequence, \(u_\nu\) converges modulo bubbling in the sense of Floer to a collection of broken cylinders \(v_1^1 \# \ldots v_k^- \# w v_i^+ \ldots v_k^+ \), \(k \in \mathbb{Z}_{\geq 0}\), with \(v_i^\pm \in \mathcal{M}(\hat{y}_i^\pm, \hat{y}_{i+1}^\pm; H^\pm, J^\pm)\), and \(w \in \mathcal{M}(\hat{y}_k^- + 1, \hat{y}_1^-; H', J)\), where \(\hat{y}_1^- = \hat{x}^-\) and \(\hat{y}_{k+1}^+ = \hat{x}^+\).

Because the \(H^\pm, J^\pm\) are Floer non-degenerate, we must have for each \(i = 1, \ldots, k^\pm\) that either \(v_i^\pm\) is constant, or that \(\mu(\hat{y}_{i}^\pm) < \mu(\hat{y}_{i+1}^\pm)\), and similarly since \((\mathcal{H}', J) \in \mathcal{H}^{\text{reg}}(H^{\text{lin}}, J)\), either \(\mu(\hat{y}_{i}^-) = \mu(\hat{y}_{i+1}^-)\) and \(\hat{y}_{k_i^-}^- = \hat{y}_{i+1}^-\), or \(\mu(\hat{y}_{i}^-) < \mu(\hat{y}_{i+1}^-)\); in either case, \(\mu(\hat{y}_{i}^-) \leq \mu(\hat{y}_{i+1}^-)\). Since there are no holomorphic spheres with negative Chern number in our setting, we have that

\[\sum_{i=1}^{k^-} (\mu(\hat{y}_{i+1}^-) - \mu(\hat{y}_{i}^-)) + (\mu(\hat{y}_{k_i^-}^-) - \mu(\hat{y}_{i+1}^-)) + \sum_{i=1}^{k^+} (\mu(\hat{y}_{i+1}^+) - \mu(\hat{y}_{i}^+)) \leq 0,
\]

and consequently the Conley-Zehnder index of all the orbits involved must be equal. In particular we must have that \(\mu(\hat{y}_{i}^-) = \mu(\hat{y}_{i+1}^-)\) for each \(i = 1, \ldots, k^\pm\), whence each \(v_i^\pm\) is constant and thus \(\hat{y}_i^- = \hat{y}_{i+1}^-\), for each \(i = 1, \ldots, k^\pm\). It follows that \(w \in \mathcal{M}(\hat{x}^-, \hat{x}^+; H', J)\), which is a contradiction to our assumption of the emptiness of \(\mathcal{M}(\hat{x}^-, \hat{x}^+; H', J)\).

\[\square\]

Definition 6.15. Let \((H^\pm, J^\pm)\) be Floer non-degenerate. We denote by \(\mathcal{H}^{\text{bl}} \subseteq \mathcal{H}^{\text{reg}}\) the collection of regular pairs \((\mathcal{H}', J)\) such that \(\mathcal{H}' \in \mathcal{H}^{\text{reg}}(J) \cap U\), where \(U\) is the neighbourhood given by the above lemma. We will call any \((\mathcal{H}, J) \in \mathcal{H}^{\text{bl}}\) pseudolinear, and any \(\mathcal{H}' \in \mathcal{H}^{\text{reg}}(H^{\text{lin}}, J)\) such that (4) holds for all \(\hat{x}^\pm \in \text{Per}_0(H^\pm), \hat{x}^- \neq \hat{x}^+\), with \(\mu(\hat{x}^-) = \mu(\hat{x}^+)\), will be called an essentially linear approximation for \(\mathcal{H}\).

6.3 Pseudolinear continuation maps from dominating Morse functions

If \(\hat{X} \in \text{murm}(H)\), then by Lemma 5.4 we may choose a contracting Seidel morphism for \(\hat{X}, \text{S}((g, \tilde{g})) : CF_*(H, J) \to CF_*(G#H, g, J)\), such that \(\tilde{g}(\hat{X})\) is a trivial capped braid which lies in \text{murm}(G#H) by Corollary 5.2. Consequently, modulo composing the Hamiltonian flow generated by \(H\) with a contractible loop of Hamiltonian diffeomorphisms, there is no loss in generality in assuming that \(\hat{X}\) is a trivial capped braid.
Definition 6.16. Let $\hat{X} \subseteq \overline{\text{Per}}_0(H)$ be a trivial capped braid. We will say that a Morse function $f \in C^\infty(\Sigma)$ is $\hat{X}$-dominating if $X_{(k)} \subseteq \overline{\text{Per}}_0(f)_{(k)}$ for all $k \in \mathbb{Z}$.

For the remainder of this section, we fix the following setting: $(H, J^+)$ is Floer non-degenerate, $\hat{X} \in \text{murm}(H)$ is a trivial capped braid, $(f, J^-)$ is Floer-regular, with $f$ a $C^2$-small $\hat{X}$-dominating Morse function, and $(\mathcal{H}, J) \in \mathcal{HF}^+$ is such that $(\mathcal{H}, J) \in \mathcal{HF}^+(H, J^+; f, J^-)$, where $\mathcal{H}(s, t, x) = \mathcal{H}(-s, t, x)$, $J(s, t) = J(-s, t)$. With this setting understood, we have

**Proposition 6.17.** The continuation map $h_{\mathcal{H}} : CF^*(f, J^-) \to CF^*(H, J^+)$ satisfies the following.

1. For all $\tilde{x} \in \hat{X}_{(1)} \cup \hat{X}_{(-1)}$, $h_{\mathcal{H}}(\tilde{x}) = \tilde{x} + \sigma$, where $\text{supp} \sigma \subseteq \text{Pos}^*(\hat{X})$.

2. For all $\tilde{p} \in \overline{\text{Per}}_0(f) \setminus \hat{X}$ with $\mu(\tilde{p}) \in \{\pm 1\}$, $h_{\mathcal{H}}(\tilde{p}) \in \text{Pos}^*(\hat{X})$.

3. $\text{Pos}^*(\hat{X}) \subseteq \ker h_{\mathcal{H}}$.

4. for all $\tilde{p} \in \overline{\text{Per}}_0(f)$ with $\mu(\tilde{p}) \in \{-1, 0, 1\}$, $h_{\mathcal{H}}(\tilde{p}) \in \mathbb{Z}_2(\tilde{x} \in \hat{X} \cup \text{Pos}^*(\hat{X}) \oplus \text{Pos}^*(\hat{X})$.

**Proof.** Item (1) follows immediately from Corollary 6.11, Corollary 6.12 and Lemma 4.21. Item (2) follows from Corollary 6.12 and Lemma 4.21. For items (3) and (4) we let $H' \in \mathcal{HF}^+(H^\text{lin}; J)$ be such that $M(\tilde{x}^-, \tilde{x}^+; H', J) \neq 0$ whenever $M(\tilde{x}^-, \tilde{x}^+; H, J) \neq 0$, for $\tilde{x}^- \neq \tilde{x}^+$, and $\mu(\tilde{x}) = \mu(\tilde{x})$.

Item (3) then follows by remarking that by our choice of $(H, J)$ we may choose an essentially linear approximation $H' \in \mathcal{HF}^+(H^\text{lin}; J)$ for $(H, J)$. Therefore if $t(\tilde{x}^+, \tilde{x}) > 0$ for some $\tilde{x} \in \hat{X}$, then the existence of some $u \in M(\tilde{x}^+, \tilde{x}^-; H', J)$ would imply, by taking $v \in M(\tilde{x}, \tilde{x}^-; H', J)$, that $t(\tilde{x}^+, \tilde{x}) \leq t_{\infty}(u, v)$ and this latter is either equal to 0 (if $\tilde{x}^+ \neq \tilde{x}$, since all capped orbits of a $C^2$-small Morse function in the Morse range are unlinked), or bounded above by $a(\tilde{x})$, which is itself bounded above by 0 when $\tilde{x}$ lies in the Morse range. In either case, we obtain a contradiction.

To prove item (4), we note first that items (1) and (2) suffice to establish (4) in the event that $\mu(\tilde{p}) = \pm 1$, so we can, and do, assume that $\mu(\tilde{p}) = 0$. In this case, we let $(H', J)$ be an essentially linear approximation for $(H, J)$ and take $\tilde{x} \in \overline{\text{Per}}_0(H) \setminus \hat{X}$ with $\mu(\tilde{x}) = 0$, and suppose that $M(\tilde{p}, \tilde{x}; H, J) \neq 0$. Therefore either $\tilde{p} = \tilde{x}$, or $M(\tilde{p}, \tilde{x}; H', J) \neq 0$. The former case is absurd, since this would imply that $\tilde{x}$ is a trivially capped constant orbit, and therefore unlinked with $\hat{X}$, which contradicts the maximality of $\hat{X}$. Thus, $M(\tilde{p}, \tilde{x}; H', J) \neq 0$, and $\hat{X} \cup \{\tilde{x}\}$ is linked. There are two possibilities: either $\tilde{p} \in \tilde{X}$ or $\tilde{p} \notin \tilde{X}$. In the former case, Lemma 6.5 immediately gives that $\tilde{x} \in \text{Pos}^*(\hat{X})$. If $\tilde{p} \notin \tilde{X}$, then consideration of the braid cobordism between the unlinked capped braid $\tilde{X} \cup \{\tilde{p}\}$ and the linked capped braid $\tilde{X} \cup \{\tilde{x}\}$ given by the taking the constant cylinders between strands of $\tilde{X}$ and by $u \in M(\tilde{p}, \tilde{x}; H, J)$ between $\tilde{p}$ and $\tilde{x}$ yields that the graph of $u$ must intersect the graph of some constant cylinder at least once, and any crossings may only contribute positively to the change in linking number as all the maps solve Equation 2 for $(H', J)$. Consequently, $\tilde{x} \in \text{Pos}^*(\hat{X})$.

**Proposition 6.18.** The maps $\pi^\hat{X} \circ h_{\mathcal{H}}$ and $h_{\mathcal{H}|_{CF_*(\hat{X}, H, J)}}$ are morphisms of chain complexes.

**Proof.** As in the proof of Theorem 6.8, it suffices to prove that the maps are chain maps in the Morse range. Thus, we may assume that $\tilde{p} \in CF_k(f, J^-)$ for $k \in \{-1, 0, 1\}$. We note that $h_{\mathcal{H}}(\tilde{p}) \in \mathbb{Z}_2(\tilde{x} \in \hat{X} \cup \text{Pos}^*(\hat{X})$ by Proposition 6.17 and $\partial_{H, J} \text{Pos}^*(\hat{X}) \subseteq \text{Pos}^*(\hat{X}) \subseteq \ker \pi^\hat{X}$ by Lemmas 6.4 and 6.7. Consequently, we see that

$$\partial_{\mathcal{H}}(\tilde{p}) = (\partial_{\mathcal{H}} \circ \pi^\hat{X} \circ h_{\mathcal{H}})(\tilde{p}) + (\partial_{\mathcal{H}} \circ \pi^\text{Pos}^*(\hat{X}) \circ h_{\mathcal{H}})(\tilde{p}) = (\partial_{\mathcal{H}} \circ \pi^\hat{X} \circ h_{\mathcal{H}})(\tilde{p}) + (\pi^\hat{X} \circ \partial_{H, J=+} \circ \pi^\text{Pos}^*(\hat{X}) \circ h_{\mathcal{H}})(\tilde{p}) = (\partial_{\mathcal{H}} \circ \pi^\hat{X} \circ h_{\mathcal{H}})(\tilde{p})$$

Thus, since $h_{\mathcal{H}}$ is a chain map with respect to the full Floer differential, we compute

$$(\pi^\hat{X} \circ h_{\mathcal{H}})(\partial_{f,J^-} \tilde{p}) = (\pi^\hat{X} \circ \partial_{H, J=+} h_{\mathcal{H}})(\tilde{p}) = (\partial_{\mathcal{H}} \circ h_{\mathcal{H}})(\tilde{p}) = \partial_{\mathcal{H}}((\pi^\hat{X} \circ h_{\mathcal{H}})(\tilde{p}))$$

which shows that $\pi^\hat{X} \circ h_{\mathcal{H}}$ is a chain map. Parallel reasoning shows that $h_{\mathcal{H}} \circ \pi^\hat{X}$ is a chain map, from which it immediately follows that $h_{\mathcal{H}|_{CF_*(\hat{X}, H, J)}}$ is a chain map.
Corollary 6.19. \( H_*(CF(\hat{X}; H, J)) \simeq QH_*(\Sigma) \).

Proof. Once more, it suffices to show that \( \pi^X \circ h_H \) induces an isomorphism on homology in degrees lying in the Morse range. Since, for any \( \hat{p} \in CF_k(f, J^-) \), with \( k \in \{-1, 0, 1\} \), we have that \( h_H(\hat{p}) \in \mathbb{Z}_2(\hat{x}) \in X \oplus \text{Pos}^*(\hat{X}) \), and \( \text{Pos}^*(\hat{X}) \subseteq \ker(\pi^X \cap \ker h_H) \), so we compute
\[
(h_H \circ h_H)(\hat{p}) = h_H((\pi^X \circ h_H)(\hat{p}) + (\pi^\text{Pos}^*(\hat{X}) \circ h_H)(\hat{p}))
\]
\[
= (h_H \circ \pi^X \circ h_H)(\hat{p})
\]
\[
= (h_H \circ \pi^X \circ \pi^X \circ h_H)(\hat{p})
\]
\[
= h_H|_{CF_k(X; H, J)} \circ (\pi^X \circ h_H).
\]

But it is a standard fact in Floer theory that \( h_H \circ h_H \) induces the identity map on homology, and so it must be that case that the composition
\[
HF_k(f) \xrightarrow{(\pi^X \circ h_H)} HF_k(\hat{X}; H) \xrightarrow{(h_H)} HF_k(f)
\]
is the identity map for \( k \in \{-1, 0, 1\} \) (and analogous reasoning shows that the same holds for the above diagram with the arrows reversed), consequently, \( (\pi^X \circ h_H)_* \) is an isomorphism on homology, and the claim follows. \( \square \)

Corollary 6.20. For any \( \alpha \in QH_*(\Sigma) \), there exists \( \sigma \in \Lambda_\cup(\hat{x}) \in X \oplus \text{Pos}^*(\hat{X}) \subseteq CF_*(H, J) \) such that \([\sigma] = \alpha \in HF_*(H) \simeq QH_*(\Sigma) \) and \([\pi^X(\sigma)] = \alpha \in HF_*(\hat{X}; H) \simeq QH_*(\Sigma) \), where these identifications with the quantum homology is understood to be those naturally induced by Floer continuation maps.

7 Construction and properties of \( \mathcal{F}^\hat{X} \)

Our construction of the singular foliation in Theorem A proceeds by establishing that a generic point in \( S^1 \times \Sigma \) lies inside the foliated sector \( W(\hat{x}, y) \) for some \( \hat{x} \in \hat{X}_{(1)}, \hat{y} \in \hat{X}_{(-1)} \). The remaining points lie in the closure of these sectors, and so lie either on leaves parametrized by broken cylinders, or the graphs of orbits in \( X \). To establish existence of the requisite leaves, we make use of the cap action of a point on the Floer complex, first introduced in detail in [16] (see also [24]).

Given a singular homology class \( \alpha \in H_k(M; \mathbb{Z}_2) \), we represent \( \alpha \) by a smooth chain \( \alpha^# : \sqcup \Delta^k \to M \), and for any \((H, J)\) and any \( t \in S^1 \), we may consider, for any \( \hat{x}, \hat{y} \in \text{Per}_0(H) \), the moduli space
\[
\mathcal{M}^\alpha^#, t(\hat{x}, \hat{y}; H, J) := \{ u \in \tilde{M}(\hat{x}, \hat{y}; H, J) : u(0, t) \in \text{im} \alpha^# \}.
\]

For \( t \in S^1 \), we will say that the smooth chain \( \alpha^# \) is \((H, J, t)\)-generic when the evaluation map \( ev_t(u, q) := (u(0, t), \alpha^#(q)) \in M \times M \), \( (u, q) \in \tilde{M}(\hat{x}, \hat{y}; H, J) \times \sqcup \Delta^k \), is transversal to the diagonal whenever \( \mu(\hat{x}) - \mu(\hat{y}) \leq (2n-k) + 1 \). Such chains form a residual set for fixed \( H \) if we permit generic perturbations of \( J \), and in such a case we define the cap product of \( \alpha \) on \( HF_*(H) \) (at \( t \) time) at the chain level by defining, for \( \hat{x} \in \text{Per}_0(H) \),
\[
\alpha^# \cap_t \hat{x} := \sum_{\hat{y} \in \text{Per}_0(H) : \mu(\hat{x}) - \mu(\hat{y}) = 2n-k} n^{\alpha^#, t}(\hat{x}, \hat{y}) \hat{y},
\]
where \( n^{\alpha^#, t}(\hat{x}, \hat{y}) \) is the mod 2 count of the number of elements in \( \mathcal{M}^\alpha^#, t(\hat{x}, \hat{y}; H, J) \). The cap action descends to homology, and is independent at the homology level of all choices. Moreover, for generic adapted homotopies of Floer data, the cap action commutes with continuation maps at the chain level. That is, for generic \((H, J)\) we have
\[
\hat{h}_H(\alpha^# \cap_t \hat{x}) = \alpha^# \cap_t h_H(\hat{x}),
\]
whenever the Floer pairs \((H^\pm, J^\pm)\) at the ends of the homotopy are such that the relevant moduli spaces are transversal. It follows from the above that, under the identification of \(HF_*(H)\) with \(QH_{*+n}(M)\), the cap action on \(HF_*(H)\) is identified with the standard cap action of the homology of \(M\) on its quantum homology.

A rather important point for us will be that the cap action interacts nicely with the respect to the chain maps \(\pi^X \circ h_H\) and \(h_R \circ \pi^X\) introduced in the previous section. We take up again the setting and notation of Section 6.3, with the added stipulation that \((H, J) \in \mathcal{H}^{2m}\) is generic in the sense that relation \(5\) holds.

**Proposition 7.1.** Suppose that \(\alpha^#\) represents \(\alpha\) as above and \(\alpha^#\) is both \((H, J^+; t)\)-generic and \((f, J^-; t)\)-generic for some \(t \in S^1\), then

\[
(\pi^X \circ h_H)(\alpha^# \cap_t \hat{p}) = \pi^X(\alpha^# \cap_t (\pi^X \circ h_H)(\hat{p})), \forall \hat{p} \in \overline{\text{Per}_0}(f) \quad \text{and} \quad (h_R \circ \pi^X)(\alpha^# \cap_t \pi^X(\hat{y})) = \alpha^# \cap_t (h_R \circ \pi^X)((\hat{y})), \forall \hat{y} \in \overline{\text{Per}_0}(H)
\]

**Proof.** Since \(f\) is \(C^2\)-small, we may reason similarly as in the proof of Theorem 6.8, and reduce to the case where \(\mu(\hat{p}) \in \{-1, 0, 1\}\). Note that Proposition 6.17 implies that we may write \(h_H(\hat{p}) = \sigma + \beta\) for \(\sigma \in CF_*(\mathcal{X}; H, J)\) and \(\beta \in \text{Pos}^*(\hat{X})\), so that

\[
\pi^X(\alpha^# \cap_t (\pi^X \circ h_H)(\hat{p})) = \pi^X(\alpha^# \cap_t \sigma).
\]

The central point is that capping with \(\alpha^#\) preserves \(\text{Pos}^*(\hat{X})\). Indeed, if \(\hat{y} \in \text{Pos}^*(\hat{X})\) and there exists \(u \in \mathcal{M}^{\alpha^#}(\hat{y}, \hat{y}'; H, J)\) for some \(\hat{y}' \in \overline{\text{Per}_0}(H)\), then for each \(\hat{x} \in \hat{X}\), Lemma 4.11 with \(v(s, t) = x(t)\) implies that \(\ell(\hat{y}, \hat{x}) \leq \ell(\hat{y}', \hat{x})\). Thus \(\alpha^# \cap_t \beta \in \text{Pos}^*(\hat{X})\) and we have

\[
(\pi^X \circ h_H)(\alpha^# \cap_t \hat{p}) = \pi^X(\alpha^# \cap_t \sigma + \alpha^# \cap_t \beta) = \pi^X(\alpha^# \cap_t \sigma),
\]

where we use Lemma 6.4 in the last equality. The second equality is proved similarly, needing only the additional remark that for \(\hat{x} \in \hat{X}\), \(\alpha^# \cap_t \hat{x} \in \mathbb{Z}_2(\hat{x}, \hat{x}, H, J) \oplus \text{Pos}^*(\hat{X})\), which follows by the same reasoning as above, using that \(b(\hat{x}) \geq 0\) when \(\mu(\hat{x})\) lies in the Morse range.

The following proposition is essentially tailor-made.

**Proposition 7.2.** Let \((H, J)\) be Floer non-degenerate, \(t \in S^1\), and suppose that \(p \in \Sigma\) is \((H, J; t)\)-generic (for the point class in homology). Then \(p \in W(\hat{x}, \hat{y})\) implies that \(\mu(\hat{x}) - \mu(\hat{y}) \geq 0\), and \(\hat{y} \in \text{supp}(p \cap t \hat{x})\) if and only if \((t, p) \in W(\hat{x}, \hat{y})\).

Combining the above with Corollary 4.25, allows us to conclude

**Corollary 7.3.** Suppose that \(\mu(\hat{x}) = 2k + 1\) for \(k \in \mathbb{Z}\) and \(p\) is \((H, J; t)\)-generic, then \(\hat{y} \in \text{supp}(p \cap t \hat{x})\) if and only if there exists an open neighbourhood \((t, p) \in S^1 \times \Sigma\) which is foliated by leaves of \(\mathcal{F}^{2\cdot \hat{y}}\).

Recall from section 4.3 that \(W(\hat{x}, \hat{y}) = \{(t, u(s, t)) \in S^1 \times \Sigma : u \in \mathcal{M}(\hat{x}, \hat{y}; H, J)\}\), and write \(W(\hat{X})\) for the union of all \(W(\hat{x}, \hat{y})\) where \(\hat{x} \in \hat{X}_{(1)}\), \(\hat{y} \in \hat{X}_{(-1)}\).

**Lemma 7.4.** Let \((H, J)\) be Floer non-degenerate and \(\hat{X} \in \text{mumn}(H)\), then \(W(\hat{X})\) is open and dense in \(S^1 \times \Sigma\).

**Proof.** By Lemma 5.4, we may suppose without loss of generality that \(\hat{X}\) is trivial. Let \((H, J^+), (f, J^-)\) and \((H, J)\) be as in the setting of Proposition 7.1. We fix \(t \in S^1\) arbitrarily and let \(p \in \Sigma\) be both \((f, J^-; t)\)-generic and \((H, J; t)\)-generic. We let \(\sigma \in CF_1(f, J^-)\) represent the fundamental class \([\Sigma] \in QH_2(\Sigma) \cong HF_1(f)\), and we note that we must have \([p \cap t \sigma] = \{pt\} \in QH_2(\Sigma)\), which is in particular not 0. But \((\pi^X \circ h_H)\), is an isomorphism of homology groups, and so

\[
0 \neq (\pi^X \circ h_H)(p \cap t \sigma) = \pi^X(p \cap t (\pi^X \circ h_H)(\sigma)),
\]

and this implies that for every such generic \(p\), there must exist some \(\hat{x} \in \text{supp}(\pi^X \circ h_H)(\sigma) \subseteq \hat{X}_{(1)}\) and some \(\hat{y} \in \hat{X}_{(-1)}\) such that \(\hat{y} \in \text{supp} p \cap t \hat{x}\), and hence every \(p\) which is both \((H, J; t)\)-generic and \((f, J^-; t)\)-generic lies inside the open 3-dimensional connecting submanifold \(W(\hat{x}, \hat{y})\) for some such \(\hat{x}, \hat{y} \in \hat{X}\), which proves the lemma.
Lemma 7.5. Let \((H, J)\) be Floer non-degenerate, and \(\tilde{X} \subseteq \mathcal{P}er_0(H)\) any unlinked capped braid. Then for any \(\tilde{x} \in \mathcal{P}er_0(H)\) such that \(\mathcal{M}(\tilde{x}^+, \tilde{x}; H, J) \neq \emptyset\) for some \(\tilde{x}^+ \in \tilde{X}(1), \tilde{x}^- \in \tilde{X}(-1)\), \(\tilde{X}\) and \(\tilde{x}\) are unlinked.

Proof. Suppose that \(\tilde{X} \cup \{\tilde{x}\}\) is linked, then Lemma 6.5 implies that \(\tilde{\gamma} \in Pos^*(\tilde{X})\), and so \(\ell(\tilde{\gamma}, \tilde{x}) > 0\) for some \(\tilde{x} \in \tilde{X}\), but then Proposition 4.20 implies that \(\ell(\tilde{x}^-, \tilde{x}) > 0\), which contradicts the assumption that \(\tilde{X}\) is unlinked.

Inductively applying Lemma 7.5 yields

Corollary 7.6. Suppose that \((H, J)\) is Floer non-degenerate, and let \(\tilde{X} \subseteq \mathcal{P}er_0(H)\) be such that \(\tilde{X}\) is unlinked and \(\tilde{X} = \tilde{X}(1) \cup \tilde{X}(-1)\), then \(\tilde{X}\) and \(\tilde{Y}\) are unlinked, where \(\tilde{Y}\) is the capped braid consisting of all \(\tilde{\gamma} \in \mathcal{P}er_0(H)\) with \(\mu(\tilde{\gamma}) = 0\) such that \(\mathcal{M}(\tilde{x}^+, \tilde{\gamma}; H, J) \neq \emptyset\) for some \(\tilde{x}^+ \in \tilde{X}\).

Finally, we are ready to prove the existence of the advertised foliation.

Theorem 7.7 (Existence part of Theorem A). Let \((H, J)\) be a non-degenerate Floer pair, and \(\tilde{X} \in \text{murn}(H)\), then the collection of submanifolds \(\mathcal{F}^X := \bigcup_{\tilde{u} \in \mathcal{M}^{H,J}(\tilde{X})} \{\text{im} \tilde{u}\}\) forms a Stefan-Sussmann foliation of \(S^1 \times \Sigma\).

Proof. We adopt a strategy used in [11] that shows that the foliation \(\tilde{\mathcal{F}}^X\) with leaves given by the graphs \(\tilde{u}\) of all the \(u \in \tilde{\mathcal{M}}(\tilde{x}, \tilde{y}; H, J), \tilde{x}, \tilde{y} \in \tilde{X}\), is a smooth 2-dimensional foliation of \(\mathbb{R} \times S^1 \times \Sigma\), from which it follows immediately that \(\mathcal{F}^X\) is a Stefan-Sussmann foliation. Indeed, in this event, \(\mathcal{F}^X\) integrates the distribution \(\mathcal{D}^X = \pi_* \bar{\mathcal{D}}^X\), where \(\bar{\mathcal{D}}^X\) is the distribution integrated by \(\tilde{\mathcal{F}}^X\), and this realizes \(\bar{\mathcal{D}}^X\) in a way that is manifestly smooth in the sense of generalized distributions (see Definition 2.2). To see that \(\tilde{\mathcal{F}}^X\) is a smooth foliation, note that by Lemma 7.4, the set \(W(\tilde{X})\) is open and dense in \(S^1 \times \Sigma\). This implies, by the \(\mathbb{R}\)-invariance of solutions to Equation 1, that the set of points \(\tilde{W}(\tilde{X})\) lying on the graph \(\tilde{u}\) of some \(u \in \tilde{\mathcal{M}}(\tilde{x}, \tilde{y}; H, J), \tilde{x} \in \tilde{X}(1), \tilde{y} \in \tilde{X}(-1)\) is open and dense in \(\mathbb{R} \times S^1 \times \Sigma\). Consequently, the partition \(\tilde{\mathcal{F}}^X\), where the union runs over all \(\tilde{x} \in \tilde{X}(1), \tilde{y} \in \tilde{X}(-1)\), gives a smooth foliation of an open, dense set of \(\mathbb{R} \times S^1 \times \Sigma\). Consequently we may argue just as in [11] in the paragraphs following the proof of Lemma 6.10 (p. 231-232); all of the remaining leaves in \(\tilde{\mathcal{F}}^X\) are graphs of constant orbits or of cylinders \(u\) which connect orbits of index difference equal to 1. In either case, by standard compactness theorems of Floer theory those graphs which form the leaves of the foliation of \(W(\tilde{X})\) converge modulo reparametrization in the \(C_{loc}^\infty\)-topology either to the graphs \((s, t) \mapsto (s, t, x(t))\) of the orbits \(x\) for \(\tilde{x} \in \tilde{X}\), or to graphs of cylinders connecting orbits of index difference 1, which come in pairs \((u, v) \in \tilde{\mathcal{M}}(\tilde{x}, \tilde{y}; H, J) \times \tilde{\mathcal{M}}(\tilde{\gamma}, \tilde{\gamma}; H, J), \tilde{x} \in \tilde{X}(1), \tilde{\gamma} \in \tilde{X}(0), \tilde{y} \in \tilde{X}(-1)\). By Corollary 7.6 and Lemma 7.5, the capped braid formed by the collection of all the \(\tilde{\gamma} \in \mathcal{P}er_0(H)\) on which such pairs break are unlinked with \(\tilde{X}\), and so lie in \(\tilde{X}\) by maximality. Consequently, the graphs of such broken trajectories cannot intersect, nor can they intersect any leaf of \(\tilde{\mathcal{F}}^X\) in the dense set \(\tilde{W}(\tilde{X})\). Since every point in \(\mathbb{R} \times S^1 \times \Sigma\) lies in the closure of \(\tilde{W}(\tilde{X})\), every such point much lie on the graph of an Orbit in \(\tilde{X}\) or the graph of such a broken cylinder. It follows that the union of all the leaves in \(\tilde{\mathcal{F}}^X\) thus fits together into a smooth foliation on all of \(\mathbb{R} \times S^1 \times \Sigma\), and so the theorem follows.

7.1 \(\mathcal{F}^\tilde{X}\) as negative gradient flow-lines of the restricted action functional

For \(\tilde{X} \in \text{murn}(H)\), denote

\[\mathfrak{M}_\tilde{X} = \mathfrak{M}_{\tilde{X}; H,J} := \{\tilde{\alpha} \in \tilde{\mathcal{L}}_0(\Sigma) : \exists \tilde{x}, \tilde{y} \in \tilde{X}, \exists u \in \tilde{\mathcal{M}}(\tilde{x}, \tilde{y}; H, J), \text{ such that } \tilde{u}_s = \tilde{\alpha}, \text{for some } s \in \mathbb{R}\}\]

Proposition 7.8. The map \(Ev : S^1 \times \mathfrak{M}_\tilde{X} \rightarrow S^1 \times \Sigma\), given by \(Ev(t, \tilde{\alpha}) = (t, \alpha(t))\) is a diffeomorphism.

Proof. The generalized distribution \(\mathcal{D}^\tilde{X}_{(t,u(s,t))}\) contains the one-dimensional distribution \(\mathcal{D}^\mathfrak{M}_{(t,u(s,t))} = (\partial_t + \partial_u)\), which is easily seen to be smooth near the singular fibers by employing the local model for leaves of a Stefan-Sussmann foliation of section 2. Consequently, \(\mathcal{D}^\mathfrak{M}\) is a smooth foliation which integrates precisely to the graphs of the maps \(\alpha : S^1 \rightarrow \Sigma\) for \(\tilde{\alpha} \in \mathfrak{M}_\tilde{X}\). This is obviously equivalent to the proposition.
Definition 7.9. For \( \tilde{X} \in \text{murm}(H) \), define the (\( \tilde{X} \))-restricted action functional \( A^\tilde{X} \in C^\infty(S^1 \times \Sigma) \) by \( A^\tilde{X} := A_H \circ E\nu^{-1} \). Additionally, for each \( t \in S^1 \), we define \( A^\tilde{X}_t := \nu_t^* A^\tilde{X} \), where \( \nu_t : \Sigma \to S^1 \times \Sigma \) is the inclusion of the fiber over \( t \in S^1 \).

Note that each \( A^\tilde{X}_t \) is automatically Morse, since the Hessian of \( A^\tilde{X}_t \) at \( x(t) \) for \( \hat{x} \in \tilde{X} \) obviously inherits the non-degeneracy of the Hessian of \( A_H \) at \( \hat{x} \). In fact, our construction clearly identifies Floer trajectories connecting orbits in \( \tilde{X} \) with negative gradient flow lines of the \( A^\tilde{X}_t \), giving us Morse models for the foliation \( \mathcal{F}^\tilde{X} \).

Proposition 7.10. If \( (H, J) \) is Floer non-degenerate, \( \tilde{X} \in \text{murm}(H) \) and \( \epsilon > 0 \) is sufficiently small, then for every \( t \in S^1 \), and every \( \hat{x}, \hat{y} \in \tilde{X} \), there is a natural identification \( \overline{M}(\hat{x}, \hat{y}; H, J) \cong \overline{M}(\hat{x}, \hat{y}; \epsilon A^\tilde{X}_t, J_t) \) given by \( u(s, t) \mapsto u(\epsilon s, t) \).

Corollary 7.11. Let \( (H, J) \) be Floer non-degenerate and \( \tilde{X} \in \text{murm}(H) \). Then for every \( t \in S^1 \), and any \( \epsilon > 0 \) sufficiently small, \( CF_*(\tilde{X}; H, J) \cong C_{+1}^\text{Morse}(A^\tilde{X}_t, g_{J_t}) \oplus \Lambda_\omega \cong CF_*(\epsilon A^\tilde{X}_t, J_t) \).

7.2 \( \mathcal{F}^\tilde{X} \) as a positively transverse foliation

Each regular leaf of the foliation \( \mathcal{F}^\tilde{X} \) arises naturally as the image of an embedding \( \hat{u} : \mathbb{R} \times S^1 \to S^1 \times \Sigma \) for \( u \), the standard orientation on the cylinder induces the orientation \( \partial_s u \wedge \partial_u u \) on each regular leaf, so we may view \( \mathcal{F}^\tilde{X} \) in a natural way as an oriented singular foliation.

Definition 7.12. Let \( \mathcal{F} \) be an oriented codimension 1 Steffan-Sussmann foliation of an oriented \( d \)-dimensional manifold \( (M^d, \partial_M) \). We will say that a smooth path \( \alpha : [0, 1] \to M \) is positively transverse to \( \mathcal{F} \) if the following dichotomy holds, either

1. \( \alpha \) is contained in a singular leaf of \( \mathcal{F} \), or
2. for every \( t \in [0, 1] \), \( \{\partial_t \alpha)_t, v_1, \ldots, v_{d-1}\} \) is an oriented basis for \( (T_{\alpha(t)} M, \partial_M) \), where \( \{v_1, \ldots, v_{d-1}\} \) is an oriented basis for the tangent space of the regular leaf of \( \mathcal{F} \) passing through \( \alpha(t) \).

Definition 7.13. Let \( \mathcal{F} \) be an oriented codimension 1 Steffan-Sussmann foliation on an oriented \( d \)-dimensional manifold \( (M^d, \partial_M) \) and let \( X \in \mathcal{X}(M) \) be a vector field generating an isotopy \( (\phi^X_t)_{t \in \mathbb{R}} \). We say that \( X \) (or \( (\phi^X_t)_{t \in \mathbb{R}} \)) is positively transverse to \( \mathcal{F} \) if every integral curve of \( X \) is positively transverse to \( \mathcal{F} \).

Proposition 7.14 (Positive transversality part of Theorem A). Let \( (H, J) \) be Floer non-degenerate, \( \tilde{X} \in \text{murm}(H) \), and \( \tilde{X}_H := \partial_t \oplus X_H \in \mathcal{X}(S^1 \times \Sigma) \), then \( \mathcal{F}^\tilde{X} \) is positively transverse to \( \tilde{X}_H \).

Proof. As the singular leaves of \( \mathcal{F}^\tilde{X} \) are orbits of \( \tilde{X}_H \), it suffices to consider points \( (t, p) \in S^1 \times \Sigma \) lying on regular leaves. In such a case, since \( u \) solves Equation 1, the basis formed by \( \{\chi_t, \partial_u u, \partial_t u\} \) is easily seen to be orientation-equivalent to the basis \( \{\chi_t, \partial_t u, J_t \partial_u u\} \), which is a positively oriented basis, as \( J_t \in \mathcal{J}(\Sigma, \omega) \) for all \( t \in S^1 \).

The previous proposition tells us that to any non-degenerate Hamiltonian \( H \) and each \( \tilde{X} \in \text{murm}(H) \), we may associate a foliation on \( S^1 \times \Sigma \) with respect to which the graph of the isotopy is well-behaved in a certain sense. However, if we’re willing to modify the isotopy by a contractible loop, then we can in fact do better and obtain a positively transverse singular foliation on \( \Sigma \) itself.

To see this, consider the distribution \( D_{t \cdot u(s, t)}^\text{murm} \) introduced in the proof of Proposition 7.8. As noted therein, \( D_{t \cdot u(s, t)}^\text{murm} \) integrates to a smooth 1-dimensional foliation by the graphs of the loops \( t \mapsto u_s(t) \) for \( u_s \in \mathfrak{M}_X \). This induces a natural loop of diffeomorphisms \( (\psi^X_t)_{t \in S^1} \), given by sliding the fiber \( \{0\} \times \Sigma \) along the foliation which integrates \( D_{t \cdot u}^\text{murm} \). In other words, we have the isotopy \( \psi^X_t(p) = u_p(s, t), t \in S^1 \), where \( u_p \in \mathcal{M}(\hat{x}, \hat{y}; H, J) \), \( \hat{x}, \hat{y} \in \tilde{X} \), is any Floer cylinder such that \( u_p(s, 0) = p \). It follows from Corollary 4.25 and the fact that if \( \hat{x} = \hat{y} \) then \( u(s, t) = x(t) \) that \( \psi^X \) is well-defined. Let us establish some elementary properties of this isotopy.

Proposition 7.15. \( \psi := (\psi^X_t)_{t \in S^1} \) is a contractible Hamiltonian loop. Moreover, if \( \hat{\psi} : \mathcal{L}_0(\Sigma) \to \mathcal{L}_0(\Sigma) \) is the lift of \( \psi \) which sends \( [x(0), x(0)] \) to \( \hat{x} \in \tilde{X} \), then \( \mathcal{S}(\psi^{-1}, \hat{\psi}^{-1}) \) is a contracting Seidel morphism for \( \tilde{X} \).
Proof. \( \psi \) defines a loop of diffeomorphisms based at the identity by construction, and it follows easily from the fact that \( \partial_t u \) verifies \( -\nabla_A X + J(\partial_t u - X_H) = 0 \) that the generating Hamiltonian function is given by \( H_t - A X_t \). To see that \( \psi \) is contractible, we may as well-suppose that \( \Sigma = S^1 \), or else contractibility is immediate, and up to composing with a contracting Seidel morphism, we may suppose that \( \hat{X} \) is a trivial capping braid, then clearly \( \psi \) fixes \( x_0 = x(0) \) for all \( t \in S^1 \), for each \( \hat{x} \in \hat{X} \) and we may simply compute the Maslov index of the linearization of \( \psi \) about some such \( x_0 \), relative the trivial capping, but this Maslov index turns out to vanish and so \( \psi \) is necessarily contractable. Indeed, suppose without loss of generality that \( \mu([x_0,x_0]) = 1 \) and let \( \xi_1, \xi_2 \in \mu_x \) be a basis of eigenvectors of \( A_{x_0} \) winding number 0 relative the trivial capping, then \( (D\psi_t)_{\xi}(0) = \xi_t(0) \) for \( t \in S^1 \) by the asymptotic estimates of Theorem 4.7, and so \( (D\psi_t)_{\xi} \) is homotopic to \( (Id)_{\xi} \). Thus \( \psi^{-1} \) is contractible as well and \( S(\psi^{-1}, \psi^{-1}) \) is clearly a contracting Seidel morphism for \( \hat{X} \) by Proposition 5.1.

Note that \( \mathcal{F}^X \) is everywhere transverse to the fibers \( \{t\} \times \Sigma \) of \( S^1 \times \Sigma \), and so may be viewed as an \( S^1 \)-family of (singular) foliations on \( \Sigma \). Let us write \( \mathcal{F}^X_t \) for the foliation obtained on \( \Sigma \) by intersecting \( \mathcal{F}^X \) with \( \{t\} \times \Sigma \).

**Theorem 7.16.** Let \((H,J)\) be a Floer non-degenerate pair, \( \hat{X} \in \text{murm}(H) \), then the orbits of the Hamiltonian isotopy \((\psi^X_t)^{-1} \circ \phi^H_t \) are positively transverse to the foliation \( \mathcal{F}^X_0 \).

Proof. Writing \( \psi = \psi^X \), observe that the vector field \((Z_t)_{t \in [0,1]} \) which generates the isotopy \( \psi^t \circ \phi^H_t \) is easily computed via the chain rule as \((Z_t)_{u(s,t)} = (\psi^t \circ \phi^H_t) u(s,t) \). Consequently, because \( \psi_t \) is a Hamiltonian diffeomorphism for all \( t \in S^1 \), we see that

\[
\omega_{\mu(s,t)} (Z_t, \partial_s u) = (\psi_t \omega)(Z_t, \partial_s u) = \omega_{\mu(s,t)} (X_t - \partial_t u, \partial_s u) = \omega_{\mu(s,t)} (-J_t \partial_t u, \partial_s u)
\]

from which the claim follows.

Theorem B is an immediate consequence of Proposition 7.10 and the preceding Theorem.

### 7.2.1 Consequences for the structure of Hamiltonian isotopies

**Definition 7.17.** For \((H,J)\) non-degenerate and \( \hat{X} \in \text{murm}(H) \), we define the **Piexoto graph** of \( \mathcal{F}^X \) to be the directed graph \( \Gamma(\mathcal{F}^X) \) whose vertex set is \( \hat{X} \) and such that there is a directed edge from \( \hat{x} \) to \( \hat{y} \) only if \( \mu(\hat{x}) - \mu(\hat{y}) = 1 \), and in this case there is an edge from \( \hat{x} \) to \( \hat{y} \) for each element in \( \mathcal{M}(\hat{x}, \hat{y}; H,J) \).

**Remark.** Note that since \( \mathcal{F}^X_t \) may be realized as the singular foliation obtained by the negative gradient flow of \((A^X_t, g_{J_t}) \), \( \Gamma(\mathcal{F}^X) \) may be naturally identified with the Piexoto graph (see [23]) of \((A^X_t, g_{J_t}) \).

**Definition 7.18.** Let \((H,J)\) be non-degenerate and \( \hat{X} \in \text{murm}(H) \). To any capped loop \( \gamma \in \tilde{L}_0(\Sigma) \) such that \( (\gamma, x) \) is a braid for all \( \hat{x} \in \hat{X} \), we may define the **linking cochain** \( \ell_{\gamma} : \tilde{L}_0(\Gamma(\mathcal{F}^X)) \to \hat{X} \) as well as the **intersection cochain** \( I_{\gamma} : E(\Gamma(\mathcal{F}^X)) \to \mathbb{Z} \), where \( I_{\gamma}(u) \) counts the signed intersection number of (some transverse perturbation of) the maps \( u(s,t) = (t, u(s,t)) \) and \( \gamma(t) = (t, \gamma(t)) \).

The following relation between these two quantities is immediate from the definition of the homological linking number.

**Proposition 7.19.** Let \((H,J)\) be non-degenerate and \( \hat{X} \in \text{murm}(H) \). For any capped loop \( \gamma \in \tilde{L}_0(\Sigma) \) such that \( (\gamma, x) \) is a braid for all \( \hat{x} \in \hat{X} \), we have \( I_{\gamma} = \delta \ell_{\gamma} \).

For any Hamiltonian \( H \), we write \( H^k := H \# \ldots \# H \) for the \( k \)-fold concatenated Hamiltonian which generates \( (\phi^H_t)^k \) as its time-1 map. Positive transversality of \( \hat{X}^H \) to \( \mathcal{F}^X \) implies positive transversality of \( \hat{X}^{H^k} \) for every \( k \in \mathbb{Z}_{>0} \) and so we obtain

**Corollary 7.20.** Let \((H,J)\) be non-degenerate, \( \hat{X} \in \text{murm}(H) \) and let \( \hat{\gamma} \in \tilde{Per}_0(H^k) \) for \( k \in \mathbb{Z}_{>0} \), then \( \ell_{\hat{\gamma}} \) is non-decreasing along edges of \( \Gamma(\mathcal{F}^X) \). Moreover, for every \( \hat{x}, \hat{z} \in \hat{X} \), \( \ell(\hat{x}, \hat{\gamma}) < \ell(\hat{z}, \hat{\gamma}) \) if and only if \( \hat{\gamma}(t) \in W(\hat{x}, \hat{z}) \) for some \( t \in S^1 \).
Definition 7.21. For a Hamiltonian \( H \), we will say that a capped braid \( \hat{X} \subseteq \overline{Per}_0(H) \) is strongly linking if for any \( \hat{\gamma} \in \overline{Per}_0(H) \), \( \ell(\hat{\gamma}, \hat{x}) = 0 \) for all \( \hat{x} \in \hat{X} \) implies that \( \hat{\gamma} \in \hat{X} \). Denote by \( usl(H) \) the collection of all \( X \subseteq \overline{Per}_0(H) \) such that \( X \) is both unlinked and strongly linking.

Clearly, \( usl(H) \subseteq mu(H) \).

Theorem 7.22 (Theorem C). Let \( H \) be a non-degenerate Hamiltonian, then \( murm(H) \subseteq \cap_{k=1}^\infty usl(H^{2k}) \).

Proof. Let \( \hat{X} \in murm(H) \), fix some \( J \) such that \((H,J)\) is Floer non-degenerate, and suppose for a contradiction that \( \hat{\gamma} \in \overline{Per}_0(H^{2k}) \setminus \hat{X} \) but \( \ell(\hat{x}, \hat{\gamma}) = 0 \) for all \( \hat{x} \in \hat{X} \). We may in fact suppose that \( \hat{\gamma} \notin \pi_2(\Sigma) \cdot \hat{X} \), since if \( \hat{\gamma} = A \cdot \hat{x} \) for some \( A \in \pi_2(\Sigma) \), then for any \( \hat{y} \in \hat{X} \), \( \hat{x} \neq \hat{y} \), Proposition 3.19 implies \( \ell(\hat{\gamma}, \hat{y}) = \ell(\hat{x}, \hat{y}) + c_{\hat{\gamma}}(A) = c_{\hat{\gamma}}(A) \), since \( \hat{X} \) is unlinked. So we may as well assume that \( \gamma \neq x \) for any \( \hat{x} \in \hat{X} \). Since \( \mathcal{F}^X \) foliates \( S^1 \times \Sigma \), it is necessary that \( \hat{\gamma}(0) \in W(\hat{x}, \hat{y}) \) for some \( \hat{x}, \hat{y} \in \hat{X} \) and so by Corollary 7.20 implies that \( \ell(\hat{x}, \hat{\gamma}) < \ell(\hat{y}, \hat{\gamma}) \), a contradiction.

8 Dynamical characterization of the spectral invariants

We apply the theory developed in the preceding sections to prove Theorem D. Fix some Floer non-degenerate \((H, J)\) throughout this section. For \( X \in murm(H) \), let us write

\[
\widetilde{M}(X) := \bigcup_{\hat{x}, \hat{y} \in \hat{X}} \widetilde{M}(\hat{x}, \hat{y}; H, J)
\]

and also \( \psi := \psi^X \), where the latter is the loop constructed in the paragraph preceding Proposition 7.15.

Lemma 8.1. Let \( \hat{X} \in murm(H) \), \( \hat{\gamma} = [\gamma, u] \in \overline{Per}_0(H) \), and write \( w_{\psi^{-1}} \) for the capping disk of \( (\psi^{-1})\hat{\gamma} \), where \( \psi \) sends \( \hat{X} \) to a trivial capped braid.

1. if \( \hat{\gamma} \in Pos(\hat{X}) \), then \( \int_{D^2} (w_{\psi^{-1}})^* \omega > 0 \), and

2. if \( \hat{\gamma} \in Neg(\hat{X}) \), then \( \int_{D^2} (w_{\psi^{-1}})^* \omega < 0 \).

Proof. We prove the first statement, with the proof of the second being entirely dual. Note that by Corollary 5.2, \( \hat{\gamma} \in Pos(\hat{X}) \) if and only if \( \psi^{-1} \hat{\gamma} \in Pos(\psi^{-1} \hat{X}) \). For convenience of notation, let us write \( \hat{\gamma}' = [\gamma', u'] = \psi^{-1} \hat{\gamma} \) and remark that Proposition 3.23 implies

\[
\int_{D^2} (u')^* \omega = \int_{\Sigma} \ell(\gamma', \bar{p}) \omega,
\]

where \( \bar{p} \) denotes the trivially capped disk assigned to a point \( p \in \Sigma \). We will therefore be done if we can show that \( \ell(\gamma', \bar{p}) \geq 0 \), \( \forall \bar{p} \in \Sigma \). To see that this is so, consider the leaf \( F_p \) of \( \mathcal{F}^X_0 \) which passes through \( p \); it is parametrized by a map of the form \( s \mapsto u(s,0), s \in \mathbb{R} \), for some \( u \in \widetilde{M}(\hat{X}) \). Without loss of generality, we may assume that \( u(0,0) = 0 \), and so we may consider the 0-homotopy induced by \( h(s) = (u_s, \gamma') \in L_0(\Sigma)^2 \), \( s \in (\infty, 0] \), between the capped braids \((\psi^{-1} \hat{x}, \gamma') = ([x(0), x(0)], \gamma') \) and \((\bar{p}, \gamma') \). Since \( \psi^{-1} \circ \phi^h \) is positively transverse to \( \mathcal{F}^X_0 \), any intersections between the graphs of the strands of \( h \) must contribute positively, whence \( \ell(\bar{p}, \gamma') \geq \ell(\psi^{-1} \hat{x}, \gamma') = \ell(\hat{x}, \gamma) \geq 0 \).

The following result gives us some control over the action of periodic orbits in terms of their winding.

Proposition 8.2. Let \( (H, J) \) be Floer non-degenerate and \( \hat{X} \in murm(H) \). For any \( \hat{\gamma} \in \overline{Per}_0(H) \), if \( \hat{\gamma} \in Pos(\hat{X}) \), then \( \mathcal{A}_H(\hat{\gamma}) < \max_{\hat{x} \in \hat{X}(1)} \mathcal{A}_H(\hat{x}) \).

Proof. Without loss of generality, we assume that \( \hat{X} \) is a trivial braid and moreover, we may replace \( (H, J) \) with \((\psi^{-1}_X)^* H, (\psi^{-1}_X)^* J \) without affecting the action of any of the capped orbits or their grading, and hence we may suppose that the flow of \( H \) is positively transverse to \( \mathcal{F}^X_0 \), which is the singular foliation produced by the negative gradient flow-lines of \( A_0^X \). Equivalently, for each \( t \in S^1 \), we have that
\[ dH(-\nabla J_0 A^X) \leq 0, \text{ with equality only at critical points of } A^X. \] 

Because and the critical points of \( A^X \) are precisely the points \( x(0) = x \) for \( x \in \Sigma \), this implies that for all \( t \in S^1 \), we have that

\[ H_t(\gamma(t)) < H_t(x_{\text{max}}) \quad (6) \]

for \( x_{\text{max}} \) the maximum of \( A^X \), which is, by construction, the orbit such that \( \int_0^1 H_t(x_{\text{max}}) \, dt = \max_{x \in X_{(1)}} A_H(\hat{x}) \). Integrating \( (6) \) gives

\[ \int_0^1 H_t(\gamma(t)) \, dt < \int_0^1 H_t(x_{\text{max}}) \, dt = \max_{\hat{x} \in X_{(1)}} A_H(\hat{x}), \]

and since \( \gamma \in \text{Pos}(\Sigma) \), the previous lemma implies that \( \int_{D^2} w^* \omega > 0 \), from which the claim follows. \( \square \)

For \( \sigma \in CF_* (H, J) \), write \( \text{mur}m(\sigma) := \{ \hat{Y} \leq \text{supp } \sigma : \hat{Y} \) is maximally unlinked relative \( \text{supp } \sigma \} \).

**Proposition 8.3.** Let \( \sigma \in CF_* (H, J) \) be a cycle representing the fundamental class, then for every \( \hat{Y} \in \text{mur}m(\sigma) \), and every \( \hat{X} \in \text{mur}m(H) \) such that \( \hat{Y} \leq \hat{X} \), we have \( \hat{Y} = \hat{X}_{(1)} \).

**Proof.** Let \( \sigma \) be generic and take any \( \hat{Y} \in \text{mur}m(\sigma) \), we will write \( \text{mur}m(H; \hat{Y}) := \{ \hat{X} \in \text{mur}m(H) : \hat{Y} \subseteq \hat{X} \} \). Note that \( \text{mur}m(H; \hat{Y}) \) is non-empty, because \( \hat{Y} \) is unlinked and has all strands lying in the Morse range. In view of a contradiction, we take an arbitrary \( \hat{X} \in \text{mur}m(H; \hat{Y}) \) and suppose that \( \hat{X} := \hat{X}_{(1)} \setminus \hat{Y} \) is non-empty. Without loss of generality, up to applying a contracting Seidel morphism, we henceforth assume that \( \hat{X} \) (and also \( \hat{Y} \) and \( \hat{X}' \)) is a trivial capped braid. We adopt once more the setting and notation of Section 6.3 with \( (f, \gamma^-) = (\epsilon A^X_0, J_0) \), for some small \( \epsilon > 0 \) and \( J^+ = J \).

That \( h_R : CF_*(H, J) \to CF_* (f, J_0) \) induces an isomorphism on homology, implies that \( h_R(\sigma) = \sum_{\hat{x} \in \hat{X}} \hat{x} \), since \( \sum_{\hat{x} \in \hat{X}} \hat{x} \in CF_*(f, J_0) \) is the unique cycle representing the fundamental class as \( f \) is small and Morse. Corollaries 6.11 and 6.12, together with the observation that \( CF_*(f, J_0) = CF_*(\hat{X}; H, J) \), imply that \( h_R(\hat{z}) = \hat{y} \) for all \( \hat{y} \in \hat{Y} \). Thus, if we write \( \text{supp } \sigma = \hat{Y} \cup \hat{Z} \), we see that \( h_R(\sum_{\hat{z} \in \hat{Z}} \hat{z}) = \sum_{\hat{z} \in \hat{Z}} \hat{z}' \). Since continuation maps are chain morphisms we see

\[ h_R(\partial_{H,J} \sum_{\hat{z} \in \hat{Z}} \hat{z}) = \partial_{f,J_0} \sum_{\hat{z} \in \hat{Z}} \hat{z}', \]

and this last quantity must be non-vanishing, because \( Z_2 \simeq HF_1(f) = \ker \partial_{f,J_0} \) is generated by \( \sum_{\hat{z} \in \hat{Z}} \hat{z} \), and \( \hat{Z}' \subseteq \hat{X}_{(1)} \). We will derive a contradiction by showing that, in fact, \( h_R(\partial_{H,J} \hat{z}) = 0 \) for each \( \hat{z} \in \hat{Z} \).

To see this, let \( \hat{z} \in \hat{Z} \) be arbitrary, fix some \( t \in S^1 \), and let \( \beta : D^2 \to \Sigma \) be an embedding of a closed disk into \( \Sigma \) such that \( \alpha := \partial \beta : S^1 \to \Sigma \) is \( (H, J; t) \)-regular (and so \( \text{a fortiori} (f, J_0) \)-regular), and such that \( \beta(D^2) \) is an isolating neighbourhood for \( z(t) \) in the sense that for any \( \gamma \in \text{Per}_{\Sigma}(H) \), \( \gamma(t) \in \text{im } \beta \) implies \( \gamma = z \). Because \( \beta \) is isolating for \( z(t) \), we have immediately that \( \partial_{H,J} \hat{z} = \alpha \cup_t \hat{z} \), and so, up to perturbing \( (H, J) \) slightly to achieve regularity of the evaluation maps relative to \( \alpha \), we compute

\[ h_R(\partial_{H,J} \hat{z}) = h_R(\alpha \cap_t \hat{z}) = \alpha \cap_t h_R(\hat{z}) = \partial_{f,J_0}(\beta \cap_t h_R(\hat{z})) + \beta \cap_t \partial_{f,J_0} h_R(\hat{z}), \]

where in the last equality, we have used that if \( \alpha \cap \beta \) (as smooth cycles), then \( \alpha \cap \beta = \partial (\beta \cap \sigma) + \beta \cap \partial \sigma \), since the cap product is independent of the representing smooth cycle at the level of homology. However, since \( \beta(D^2) \) is isolating for \( z(t) \), in particular \( \text{Crit}(A^X_0) \cap \text{im } \beta = \emptyset \), and so both \( \beta \cap \partial h_R(\hat{z}) = 0 \), and \( \beta \cap_t \partial_{f,J_0} h_R(\hat{z}) = 0 \), from which we readily conclude that \( h_R(\partial_{H,J} \hat{z}) = 0 \), which gives the desired contradiction. \( \square \)

**Proof of Theorem D.** By the behaviour of the filtered Floer complexes under Poincaré duality (see [4], Lemma 2.2), we have that for surfaces, \( c(H; [pt]) = -c(H; [\Sigma]) \), where \( H_t(x) := -H_t(\phi_t(x)) \), and it is easy to see that \( \text{mur}m(H) = \text{mur}m(H) \). It therefore suffices to prove that for any non-degenerate Hamiltonian \( H, c(H; [\Sigma]) = \text{mil} \max_{\hat{X} \in \text{mur}m(H)} \max_{\hat{x} \in \hat{X}} A_H(\hat{x}) \). To bound the spectral invariant from above, we may apply Corollary 6.20, so as to see that for any \( \hat{X} \in \text{mur}m(H) \), we may represent the fundamental class of \( CF_* (H, J) \) by a cycle of the form \( \sum_{\hat{x} \in \hat{X}_{(1)}} \hat{x} + \beta \), where \( \beta \in \text{Pos}^* (\hat{X}) \), consequently Proposition
8.2 implies that \( \lambda_H(\beta) < \max_{\hat{x} \in \hat{X}(1)} A_H(\hat{x}) = \max_{\hat{x} \in \hat{X}} A_H(\hat{x}) \). Since \( \hat{X} \in \text{murm}(H) \) was arbitrary, we deduce that

\[
 c(H; [\Sigma]) \leq \min_{\hat{X} \in \text{murm}(H)} \max_{\hat{x} \in \hat{X}} A_H(\hat{x}).
\]

To obtain the opposite inequality, we let \( \sigma \in CF_*(H, J) \) be a tight cycle for \([\Sigma]\). Applying Proposition 8.3 to \( \sigma \), we may take any \( \hat{Y} \in \text{murm}(\sigma) \) and extend it to some \( \hat{X} \in \text{murm}(H) \) with \( \hat{Y} \subseteq \hat{X} \) and \( \hat{X}(1) = \hat{Y} \), and thereby immediately obtain

\[
 \min_{\hat{X} \in \text{murm}(H)} \max_{\hat{x} \in \hat{X}} A_H(\hat{x}) \leq c(H; [\Sigma]).
\]

\[\square\]

9 Remarks on higher dimensional generalizations

It is clear that the methods used in this paper have no chance of establishing the existence of singular foliations on \( S^1 \times M \) of the type described here when \((M, \omega)\) is a symplectic manifold of dimension greater than 2; linking becomes a vacuous notion and one loses the positivity of intersections which is essential for controlling the behaviour of Floer trajectories. However, it is at the same time evident that at least some pairs \((H, J)\) and some collections of their orbits do admit such foliations: namely, if \( H \) is any Morse function (even if not \( C^2 \)-small) and \( J \) is also autonomous, then restricting our attention to Floer cylinders which are actually negative gradient trajectories of \( H \) with respect to \( g_t \) between the critical points of \( H \) provides such a \((S^1\text{-invariant})\) foliation (in this case, \( A_X^1 \) simply recovers \( H \) for every \( t \in S^1 \)).

Let us define, therefore, for a Hamiltonian \( H \) on an arbitrary symplectic manifold \( M^{2n} \) the set

\[
 \text{murm}(H) := \left\{ \hat{X} \subseteq \text{Per}_0(H) \mid \mu(\hat{x}) \in \{-n, \ldots, n\}, \forall \hat{x} \in \hat{X}, Ev : S^1 \times \mathcal{M}_\hat{X} \to S^1 \times M \text{ is a diffeomorphism for some } J \right\},
\]

where \( \mathcal{M}_\hat{X} \) is defined as in Section 7.1, with the possible caveat that one may need to take only certain connected components of the moduli spaces \( \mathcal{M}(\hat{x}, \hat{y}; H, J), \hat{x}, \hat{y} \in \hat{X} \). The non-emptiness of \( \text{murm}(H) \) may then be viewed as containing interesting information about the degree to which \( H \) behaves ‘roughly like an autonomous Morse Hamiltonian’ with the non-triviality of the associated Hamiltonian loop \( \psi^X \) measuring an associated obstruction to something like \( H \) actually being an autonomous Morse function with critical points given by \( \hat{X} \) (more specifically, the non-triviality of \( \psi^X \) is equivalent to the failure of the foliating Floer cylinders to be \( S^1\text{-invariant} \)). In any case, according to this definition, on any symplectic manifold the set of Hamiltonians with non-empty \( \text{murm}(H) \) is itself non-empty, and so it may be an interesting question to investigate under what conditions one can guarantee that \( \text{murm}(H) \) is non-empty for a given \( H \), and what the structure of \( \text{murm}(H) \) is for interesting classes of Hamiltonians.

One of the main obstructions to attacking this problem lies in finding good conditions on orbits \( \hat{x}, \hat{y} \in \text{Per}_0(H) \) and selection principles for connected components \( C \subset \mathcal{M}(\hat{x}, \hat{y}; H, J) \), such that, for any Floer cylinder \( u \in C \), one can guarantee that every element in \( \ker(dF)_u \) is everywhere non-vanishing. It’s clearly necessary that any pair of orbits \( \hat{x}, \hat{y} \in \hat{X} \) and certain components of their Floer moduli space must have such a property if \( \hat{X} \in \text{murm}(H) \). Let us provisionally call this the \textbf{linearized non-vanishing property}. One would need moreover a condition on collections of capped orbits \( \hat{Y} \subseteq \text{Per}_0(H) \) which guarantees that the graphs \( \hat{u} : \mathbb{R} \times S^1 \to \mathbb{R} \times S^1 \times M \) of Floer cylinders running between orbits of \( \hat{Y} \) do not intersect. Let us unimaginatively call such a property the \textbf{non-intersecting Floer cylinders property}. We may then hope to equivalently characterize \( \text{murm}(H) \) as the set of collections of orbits \( \hat{X} \) with each \( \hat{x} \in \hat{X} \) having Conley-Zehnder index in the Morse range, such that each pair of orbits has the linearized non-vanishing property, while the entire collection has the non-intersecting Floer cylinders property, where the gap between this characterization and the above definition of \( \text{murm}(H) \) is precisely the \textbf{existence of Floer cylinders in the moduli spaces connecting orbits of} \( \hat{X} \) \( \text{pass through each point of} \ M \). Presumably, this gap could be filled by the same type of argument as employed in Section 7, which guarantees the existence of Floer cylinders passing through generic points of \( M \) via the capping
action with the point — assuming that such collections could be shown to be part of the support of a Floer cycle which represents the fundamental class of $M$.

We hasten to point out that this sketch is purely speculative, as it’s unclear how one might find interesting and practical conditions which guarantee the linearized non-vanishing condition or the non-intersecting Floer cylinders property, but, in some sense, any approach which seeks to build these sorts of foliations in higher dimensions will have to address these issues in one form or another.

References

[1] M. Audin and M. Damian. *Morse theory and Floer homology*. Springer, 2014.

[2] B. Bramham. Periodic approximations of irrational pseudo-rotations using pseudoholomorphic curves. *Annals of Mathematics*, pages 1033–1086, 2015.

[3] C. Debord. Holonomy groupoids of singular foliations. *Journal of Differential Geometry*, 58(3):467–500, 2001.

[4] M. Entov and L. Polterovich. Calabi quasimorphism and quantum homology. *International Mathematics Research Notices*, 2003(30):1635–1676, 2003.

[5] J.-M. Gambaudo and É. Ghys. Commutators and diffeomorphisms of surfaces. *Ergodic Theory and Dynamical Systems*, 24(5):1591–1617, 2004.

[6] M. Gromov. Pseudo holomorphic curves in symplectic manifolds. *Inventiones mathematicae*, 82(2):307–347, 1985.

[7] H. Hofer and D. A. Salamon. Floer homology and novikov rings. In *The Floer memorial volume*, pages 483–524. Springer, 1995.

[8] H. Hofer, K. Wysocki, and E. Zehnder. Properties of pseudo-holomorphic curves in symplectisations II: Embedding controls and algebraic invariants. In *Geometries in Interaction*, pages 270–328. Springer, 1995.

[9] H. Hofer, K. Wysocki, and E. Zehnder. Properties of pseudoholomorphic curves in symplectisations I: Asymptotics. *Annales de l’Institut Henri Poincare (C) Non Linear Analysis*, 13(3):337 – 379, 1996.

[10] H. Hofer, K. Wysocki, and E. Zehnder. Properties of pseudoholomorphic curves in symplectizations III: Fredholm theory. In *Topics in nonlinear analysis*, pages 381–475. Springer, 1999.

[11] H. Hofer, K. Wysocki, and E. Zehnder. Finite energy foliations of tight three-spheres and Hamiltonian dynamics. *Annals of Mathematics*, pages 125–255, 2003.

[12] V. Humilière, F. Le Roux, and S. Seyfaddini. Towards a dynamical interpretation of Hamiltonian spectral invariants on surfaces. *Geometry & Topology*, 20(4):2253–2334, 2016.

[13] E. Kerman and F. Lalonde. Length minimizing Hamiltonian paths for symplectically aspherical manifolds. *Annales de l’institut Fourier*, 53(5):1503–1526, 2003.

[14] M. Khanevsky. Quasimorphisms on surfaces and continuity in the Hofer norm. arXiv:1906.08429, 2019.

[15] F. Lalonde. A field theory for symplectic fibrations over surfaces. *Geometry & Topology*, 8(3):1189–1226, 2004.

[16] H. V. Lê and K. Ono. Cup-length estimate for symplectic fixed points. *Contact and symplectic geometry (Cambridge, 1994)*, 8:268–295, 1996.

[17] P. Le Calvez. Une version feuilletée équivariante du théoreme de translation de Brouwer. *Publications Mathématiques de l’IHÉS*, 102:1–98, 2005.
[18] P. Le Calvez and F. A. Tal. Forcing theory for transverse trajectories of surface homeomorphisms. Inventiones mathematicae, 212(2):619–729, 2018.

[19] D. McDuff and D. Salamon. J-holomorphic curves and symplectic topology, volume 52. American Mathematical Soc., 2012.

[20] E. Mora. Pseudoholomorphic Cylinders in Symplectisations. PhD thesis, New York University, 2003.

[21] Y.-G. Oh. Spectral invariants and the length minimizing property of Hamiltonian paths. Asian J. Math., 9(1):001–018, 03 2005.

[22] Y.-G. Oh. Floer mini-max theory, the Cerf diagram, and the spectral invariants. J. Korean Math. Soc., 46(2):363–447, 2009.

[23] M. Peixoto. On the classification of flows on 2-manifolds. In Dynamical systems, pages 389–419. Elsevier, 1973.

[24] S. Piunikhin, D. Salamon, and M. Schwarz. Symplectic Floer-Donaldson theory and quantum cohomology. Contact and symplectic geometry (Cambridge, 1994), 8:171–200, 1996.

[25] D. Salamon. Lectures on Floer homology. In L. T. Yakov Eliashberg, editor, Symplectic Geometry and Topology, pages 144–227. AMS/IAS, 1997.

[26] D. Salamon and E. Zehnder. Morse theory for periodic solutions of Hamiltonian systems and the Maslov index. Communications on pure and applied mathematics, 45(10):1303–1360, 1992.

[27] M. Schwarz. Cohomology Operations from $S^1$-cobordisms in Floer homology. PhD thesis, ETH Zurich, 1995.

[28] P. Seidel. $\pi_1$ of symplectic automorphism groups and invertibles in quantum homology rings. Geometric & Functional Analysis GAFA, 7(6):1046–1096, 1997.

[29] R. Siefring. Relative asymptotic behavior of pseudoholomorphic half-cylinders. Communications on Pure and Applied Mathematics: A Journal Issued by the Courant Institute of Mathematical Sciences, 61(12):1631–1684, 2008.

[30] R. Siefring. Intersection theory of punctured pseudoholomorphic curves. Geom. Topol., 15(4):2351–2457, 2011.

[31] P. Stefan. Accessible sets, orbits, and foliations with singularities. Proceedings of the London Mathematical Society, 3(4):699–713, 1974.

[32] M. Usher. Spectral numbers in Floer theories. Compositio Mathematica, 144(6):1581–1592, 2008.

[33] J. van den Berg, R. Ghrist, R. Vandervorst, and W. Wójcik. Braid Floer homology. Journal of Differential Equations, 259(5):1663–1721, 2015.