Search for primes of the form \( m^2 + 1 \)

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Abstract

The results of the computer hunt for the primes of the form \( q = m^2 + 1 \) up to \( 10^{20} \) are reported. The number of sign changes of the difference \( \pi_q(x) - \frac{C_q}{2} \int_x^2 \frac{du}{\sqrt{u} \log(u)} \) and the error term for this difference is investigated. The analogs of the Brun’s constant and the Skewes number are calculated. An analog of the B conjecture of Hardy–Littlewood is formulated. It is argued that there is no Chebyshev bias for primes of the form \( q = m^2 + 1 \). All encountered integrals we were able to express by the logarithmic integral.

Mathematics Subject Classification: 11L20, 11N13, 11N32, 11A41, 11N05, 11Y35, 11Y60

1 Introduction

This paper is devoted to investigation of the set of prime numbers

\[ Q = \{2, 5, 17, 37, 101, 197, 257, 401, 577, \ldots \} \]

given by the quadratic polynomial \( m^2 + 1 \) and let \( q_n \) denote the \( n \)-th prime of this form. By the conjecture E of Hardy and Littlewood \[19\] the number of primes \( q < x \) of the form \( q = m^2 + 1 \) is given by

\[ \pi_q(x) \sim C_q \frac{\sqrt{x}}{\log(x)}, \] (2)

where

\[ C_q = \prod_{p \geq 3} \left( 1 - \frac{(-1)^{(p-1)/2}}{p-1} \right) = 1.37281346281824609112192696727\ldots \] (3)

The primes \( q_n \) were investigated in the past both theoretically and numerically. One of the strongest theoretical results is the theorem of H. Iwaniec \[22\], who proved that there exist infinitely many integers \( m^2 + 1 \) which are 2-almost-primes. In 1998 H. we adopt here the convention that all functions describing usual prime numbers will have the subscript \( q \) when applied to primes of the form \( q = m^2 + 1 \).
Iwaniec and J. Friedlander [15] have proved that there is an infinity of primes given by the polynomial of two variables \( m, n \) of the form \( m^2 + n^4 \), thus the case of the polynomial of one variable \( m^2 + 1 \) is not covered by their theorem. Quite recently there appeared two papers by S. Baier and L. Zhao [2, 3], treating the more general problem of the primes of the form \( m^2 + k \) in average over square-free parameter \( k \) in appropriate intervals. In the computational part we should cite the papers by Shanks [38, 36] and Wunderlich [42]; in the last paper the table of \( \pi_q(x) \) for \( x < 1.96 \times 10^{14} \) is given.

By analogy with the case of all primes, where substitution of the logarithmic integral \( \text{Li}(x) = \int_{2}^{x} \frac{du}{\log(u)} \) instead of \( x/\log(x) \) gives better approximation for \( \pi(x) \), we recast the original Hardy and Littlewood conjecture E in the form:

\[
\pi_q(x) \sim \frac{1}{2} C_q \int_{2}^{x} \frac{du}{\sqrt{u} \log u} \tag{4}
\]

\[= C_q \left( \frac{\sqrt{x}}{\log x} + \frac{2\sqrt{x}}{\log^2 x} + \frac{8\sqrt{x}}{\log^3 x} + \frac{48\sqrt{x}}{\log^4 x} + \ldots + \frac{2^{n-1}(n-1)!\sqrt{x}}{\log^n x} + \ldots \right).\]

The series in parenthesis above is asymptotic one: the terms initially decrease, but for sufficiently large \( n \) they become to increase. Thus this series has to be cut at such \( n_0 \), which depends on \( x \), that the \( n_0 \)-th term (and consecutive terms) is larger than previous one with \( n_0(x) - 1 \) what gives for the threshold at which the series should be cut the inequality:

\[n_0(x) > \frac{1}{2} \log(x) + 1.\tag{5}\]

Besides this asymptotic representation the integral (4) can be linked to the logarithmic integral by the change of variables \( u = t^2 \):

\[
\int_{a}^{b} \frac{du}{\sqrt{u} \log u} = \int_{\sqrt{a}}^{\sqrt{b}} \frac{dt}{\log t} = \text{li}(\sqrt{b}) - \text{li}(\sqrt{a}). \tag{6}
\]

Here we use the following convention for the lower limit of integration:

\[
\text{li}(x) = \text{v.p.} \int_{0}^{x} \frac{du}{\log(u)} = \lim_{\epsilon \to 0} \left( \int_{0}^{1-\epsilon} \frac{du}{\log(u)} + \int_{1+\epsilon}^{x} \frac{du}{\log(u)} \right). \tag{7}
\]

Integration by parts gives the asymptotic expansion:

\[
\text{li}(x) \sim \frac{x}{\log(x)} + \frac{x}{\log^2(x)} + \frac{2x}{\log^3(x)} + \frac{6x}{\log^4(x)} + \ldots + \frac{n!x}{\log^{n+1}(x)}. \tag{8}
\]

which should be cut at \( n_0 = \log(x) \). There is a series giving \( \text{li}(x) \) for all \( x \) and quickly convergent which has \( n! \) in denominator and \( \log^n(x) \) in nominator instead of opposite order in (8) (see [1] Sect. 5.1])

\[
\text{li}(x) = \gamma + \log \log(x) + \sum_{n=1}^{\infty} \frac{\log^n(x)}{n \cdot n!} \quad \text{for } x > 1, \tag{9}
\]
Here $\gamma = 0.57721566490153286\ldots$ is the Euler-Mascheroni constant. Even faster converging series was discovered by Ramanujan [6, p.123]:

$$\text{li}(x) = \gamma + \log(\log(x)) + \sqrt{x} \sum_{n=1}^{\infty} \frac{(-1)^{n-1}(\log(x))^n}{n! 2^{n-1}} \sum_{k=0}^{[(n-1)/2]} \frac{1}{2k+1} \quad \text{for } x > 1. \quad (10)$$

Skipping the number $2 = 1^2 + 1$ all odd primes $q_n$ can be expressed by the form $m^2 + 1$ only for $m$ even, thus we put $m = 2k$, i.e. we are looking for primes of the form $4k^2 + 1$. We have collected in one file of the size roughly 3.5 GB values of $k$ (thus the prime $2 = 1^2 + 1$ is absent here!) up to $k = 499999978$ what corresponds to all primes of the form $m^2 + 1 < 10^{20}$. The compressed data occupies 760 MB and is available for downloading from [http://www.ift.uni.wroc.pl/~mwolf/4k-2+1-data.zip](http://www.ift.uni.wroc.pl/~mwolf/4k-2+1-data.zip). Initially up to $6.65 \times 10^{16}$ I have used my own program written in Fortran for Alpha DEC workstation, but to reach $10^{20}$ we have used the free package PARI/GP [39] developed especially for number theoretical purposes. To scan the interval $(6.65 \times 10^{16}, 10^{20})$ it took about 10 days of CPU time on the PC with the clock 2.5 GHz using the built-in PARI very fast function `isprime(p)`. The Table I gives comparison of $\pi_q(x)$ with conjectures (2) and (4). Among those 312,357,934 values of $k$ there were 11,864,645 such $k$ that they in turn were primes, i.e. there were 11,864,645 pairs of numbers $(p, 4p^2 + 1)$ both being prime. We have checked separately that up to $10^{22}$ there are 96,817,209 such pairs $(p, 4p^2 + 1)$ [40].

As it is seen from the fourth column, the actual number $\pi_q(x)$ is always larger than prediction (2). Contrary to this the ratio of $\pi_q(x)$ to the integral (4) sometimes is larger than 1 and sometimes is smaller than 1, see the last column. It means, that the difference $\pi_q(x) - \frac{1}{2}C_q \int_2^x \frac{du}{\sqrt{u \log u}}$ changes the sign, see the Sect.3. In the next Section 2 we discuss the problem of the error term, in Sect.4 the analog of the Brun’s constant is calculated. Sections 5 and 6 contains some heuristics about the distribution of $k$ in $4k^2 + 1$ giving the prime. We formulate an analog of the B conjecture of Hardy and Littlewood for the case of primes $q = m^2 + 1$. In Sect. 6 we discuss the distribution of the gaps between consecutive $k$’s giving the prime $4k^2 + 1$. Heuristic arguments allow us to make conjecture about the growth of the difference $q_{n+1} - q_n = O(\sqrt{q_n} \log^2(q_n))$. The last Section 7 contains the discussion of the analog of the Chebyshevs Bias for the case of primes in $Q$.

### 2 The problem of error term

Nothing is known about the error term for the formula (4) (see however [2, 3]), thus the only way to gain some information and intuition is to appeal to the available computer data. The Figure 1 presents the plot of the difference

$$|\Delta_q(x)| = \left| \pi_q(x) - \frac{1}{2}C_q \int_2^x \frac{du}{\sqrt{u \log u}} \right|. \quad (11)$$

for $x \in (10, 10^{20})$. As it is seen from this plot the graph of the error term is very erratic, thus the plot of the maximal value of the absolute difference:

$$\omega(x) = \max_{2 < t < x} |\Delta_q(t)|, \quad (12)$$
what is a kind of envelope for $|\Delta_q(x)|$, is also plotted in green in the Figure 1.

The error term present in the ordinary Prime Number Theorem (PNT) under the Riemann Hypothesis is $\sqrt{x} \log(x)$:

$$\pi(x) = \text{Li}(x) + \mathcal{O}(\sqrt{x} \log(x)).$$

which can be written in the slightly weaker form

$$\pi(x) = \text{Li}(x) + \mathcal{O}(x^{1/2+\epsilon}).$$

TABLE I

| $x$  | $\pi_q(x)$ | $C_q \sqrt{x}/\log(x)$ | $\pi_q(x)/\text{eq. (2)}$ | formula (4) | $\pi_q(x)/\text{eq. (4)}$ |
|------|------------|-------------------------|-----------------------------|-------------|-----------------------------|
| $10^6$ | 112        | 99                      | 1.12713                     | 122         | 0.91869                     |
| $10^7$ | 316        | 269                     | 1.17325                     | 318         | 0.99440                     |
| $10^8$ | 841        | 745                     | 1.12847                     | 855         | 0.98321                     |
| $10^9$ | 2378       | 2095                    | 1.13516                     | 2357        | 1.00888                     |
| $10^{10}$ | 6656    | 5962                    | 1.11639                     | 6610        | 1.00696                     |
| $10^{11}$ | 18822    | 17140                   | 1.09815                     | 18787       | 1.00184                     |
| $10^{12}$ | 54110    | 49684                   | 1.08909                     | 53971       | 1.00258                     |
| $10^{13}$ | 156081   | 145028                  | 1.07621                     | 156386      | 0.99805                     |
| $10^{14}$ | 456362   | 425861                  | 1.07162                     | 456405      | 0.99999                     |
| $10^{15}$ | 1339875  | 1256912                 | 1.06601                     | 1340089     | 0.99984                     |
| $10^{16}$ | 3954181  | 3726285                 | 1.06116                     | 3955222     | 0.99974                     |
| $10^{17}$ | 11726896 | 11090399                | 1.05739                     | 11726340    | 1.00005                     |
| $10^{18}$ | 34900213 | 33122538                | 1.05367                     | 34903278   | 0.99991                     |
| $10^{19}$ | 10428948 | 99229889                | 1.05058                     | 104251624  | 0.99997                     |
| $10^{20}$ | 312357934 | 298102838               | 1.04782                    | 312353427  | 1.00001                     |

The $\sqrt{x}$ behavior is confirmed by the computer data, see e.g. [32, Table 14, p. 175] or [17, Table 5 and 6], where the difference $\text{Li}(x) - \pi(x)$ has roughly half digits of the value $\pi(x)$. Because there are roughly $\sqrt{x}$ candidates for primes of the form $m^2 + 1$ up to $x$ it is natural to expect that the error term for (4) will be square root of the error term for PNT. This heuristic seems to be confirmed by the fact, that $\omega(x)$ is well approximated by the power--like error term:

$$\alpha_1 x^{\beta_1}, \quad \alpha_1 = 0.38\ldots, \quad \beta_1 = 0.22\ldots$$

and indeed here $\beta_1 \approx 1/4$. This function was obtained by fitting the straight line to the points $\log(\omega(x))$ vs $\log(x)$ for $x > 10^9$ by the least-square method and to bound the difference $\omega(x)$ from above it is sufficient to shift the above curve (15) parallel up to leave the plot of $|\Delta_q(x)|$ below. In the Figure 1 the function $5x_1^{\beta_1}$ was chosen, but at least for $x < 10^{20}$ the smaller choice for the constant hidden in the big-$\mathcal{O}$ in $\omega(x) = \mathcal{O}(x^{\beta_1})$ will also do. These heuristic arguments and computer data lead us to guess the following
Conjecture 1:

\[ \pi_q(x) = \frac{1}{2} C_q \int_2^x \frac{du}{\sqrt{u} \log u} + O(x^{1/4 + \epsilon}). \] (16)

More stringent error term \( O(x^{1/4} \sqrt{\log(x)}) \) is also a possibility which cannot be ruled out by available data. Let us notice, that (16) is heuristically supported by the relation (6) and \( \pi(\sqrt{x}) = \text{li}(\sqrt{x}) + O(x^{1/4}) \). Another support in favor of (16) will be given in Sect.7 (see Fig.8).

The difference \( \Delta_q(x) \) fluctuates roughly symmetrically around zero. As the computer check of possible future oscillations theorems we present in the Fig.2 the plot of \( \Delta_q(x)/x^{1/4} \). The amplitude of this function practically does not change in the interval \((10^2, 10^{20})\) giving support in favor of the error term \( O(x^{1/4}) \).

3 An analog of the Skewes number for primes of the form \( m^2 + 1 \)

It turns out that the there is a lot of sign changes of the difference

\[ \Delta_q(x) = \pi_q(x) - \frac{1}{2} C_q \int_2^x \frac{du}{\sqrt{u} \log u} \] (17)

in the investigated interval \( x \in (2, 10^{20}) \). In the generic problem of all prime numbers it was shown by J.E. Littlewood in the 1914 [28] (see also [13]) that the difference between the number of primes smaller than \( x \) and the logarithmic integral \( \text{li}(x) \) infinitely often changes the sign. The smallest value \( x_S \) such that for the first time the difference \( \Delta(x) = \pi(x) - \text{li}(x) \) changes the sign is called Skewes number and the lowest present day known estimate of the Skewes number is around \( 10^{316} \), see [5] and [11]. However in the case of the primes given by the quadratic polynomial \( m^2 + 1 \) the first sign change of the difference \( \Delta_q(x) \) occurs already at the prime \( q_{13} = 2917 = 54^2 + 1 \) and there are 20634 such sign changes up to \( 10^{20} \). Let \( \nu_q(T) \) denotes the number of sign changes of the function \( \Delta_q(x) \) for \( x \in (2, T) \). The Fig.3 presents the plot of the function \( \nu_q(T) \) for \( T < 10^{20} \). The fitting of the power–like dependence of \( \nu_q(T) \) on \( T \) gives parameters which depend on the number of discarded initial points. For example fitting \( \log(\nu_q(T)) \) vs \( \log(T) \) for \( T \in (10^7, 10^{20}) \) gives \( \nu_q(T) \sim T^{0.23308} \) while for \( T \in (10^{12}, 10^{20}) \) we obtained \( \nu_q(T) \sim T^{0.23834} \), and we have checked for other intervals of \( T \) that the first digits 0.23 persists, thus we write:

\[ \nu_q(T) \approx \alpha_2 T^{\beta_2} \quad \alpha_2 = 0.38 \ldots, \quad \beta_2 = 0.23 \ldots \] (18)

Let us mention that for the case of all primes Knapowski [25] proved that the number of sign changes of \( \Delta(x) = \) in the interval \((1, T)\)

\[ \nu(T) \geq e^{-35} \log \log \log T \] (19)

provided \( T \geq \exp \exp \exp \exp(35) \). There is a remarkable coincidence in the values of the parameters \( \alpha \) and \( \beta \) present in the fits (15) and (18) and when the functions \( \omega(x) \) and \( \nu_q(T) \) are plotted on the same graph they appear very close to each other, despite the fact that \( \omega(x) \) and \( \nu_q(T) \) represent quantities at first sight unrelated.
4 The analog of the Brun constant

Because \( \sum_{n=1}^{\infty} \frac{1}{n^2} = \pi^2/6 < \infty \) thus the sum of reciprocals of all primes of the form \( q = m^2 + 1 \) is trivially convergent:

\[
\sum_{q \in \mathbb{Q}} \frac{1}{q} = \frac{1}{2} + \frac{1}{5} + \frac{1}{17} + \frac{1}{37} + \frac{1}{101} + \ldots < \infty
\]  

but the actual numerical value of this sum is unknown \[31\]. In 1919 Brun \[9\] has shown that the sum of reciprocals of all twin primes is finite:

\[
B_2 = \left( \frac{1}{3} + \frac{1}{5} \right) + \left( \frac{1}{5} + \frac{1}{7} \right) + \left( \frac{1}{11} + \frac{1}{13} \right) + \ldots < \infty.
\]  

Numerically \( B_2 = 1.9021605823 \ldots \), see \[29\], \[8\]. It is natural to call the above sum \( \sum_{x>q \in \mathbb{Q}} \frac{1}{q} \). From the computer data we can calculate the finite size approximations:

\[
B_q(x) = \sum_{x>q \in \mathbb{Q}} \frac{1}{q}.
\]

From the integral test for convergence of the series in the form:

\[
\sum_{n=N}^{\infty} f(n) \leq f(N) + \int_{N}^{\infty} f(u) \, du.
\]  

we have:

\[
B_q = \sum_{x>q \in \mathbb{Q}} \frac{1}{q} + \sum_{x<q \in \mathbb{Q}} \frac{1}{q} < B_q(x) + \sum_{n^2>x} \frac{1}{n^2} < B_q(x) + \frac{1}{x} + \int_{\sqrt{x}}^{\infty} \frac{du}{u^2} = B_q(x) + \frac{1}{x} + \frac{1}{\sqrt{x}}
\]

Thus from this trivial inequality we can expect for \( x = 10^{20} \) the accuracy of 10 digits for \( B_q \). Indeed, from the second column of the Table II we see that the number of stabilizing digits of \( B_q(x) \) is roughly half of the digits of the exponent of \( x \) in the first column. However, using some heuristics it is possible to obtain from \( B_q(10^{20}) \) 15 digits of \( B_q \). Namely, we can obtain the analytical formula for dependence of \( B_q(x) \) on \( x \). From the equation (4) it follows that the chance to find a prime of the form \( m^2 + 1 \) around \( x \) is \( C_q \frac{1}{\sqrt{x} \log(x)} \), thus we can write:

\[
B_q(x) = \sum_{x>q \in \mathbb{Q}} \frac{1}{q} \approx B_q(\infty) - \frac{C_q}{2} \int_{x}^{\infty} \frac{du}{u^{3/2} \log(u)}
\]

Integrating by parts gives:

\[
\int_{x}^{\infty} \frac{du}{u^{3/2} \log(u)} = -\frac{2}{\sqrt{x} \log x} + \frac{4}{\sqrt{x} \log^2 x} \ldots + (-1)^n \frac{2^n (n-1)!}{\sqrt{x} \log^n x} + \ldots
\]

\(^2\) Let us remark, that in the case of all primes \( \pi(x) \sim \int_{x}^{\infty} \frac{du}{\log(u)} = \frac{x}{\log(x)} + \ldots \) as well as of twin primes \( \pi_2(x) \sim C_2 \int_{x}^{\infty} \frac{du}{\log^2(u)} = C_2 \frac{x}{\log^2(x)} + \ldots \) the chance to find a prime around \( x \) following from dividing the first terms on r.h.s. by \( x \) coincide with integrands on l.h.s, what is not true in the case of (4), where the factor \( \frac{1}{2} \) makes the difference.
This series is asymptotic one and the condition for the dropped terms is \( n > \frac{1}{2} \log(x) \) — the same as threshold \([5]\). But by the change of the variable \( u = 1/\sqrt{t} \) it is possible to express the above integral by the logarithmic integral:

\[
\int_{a}^{b} \frac{du}{u^{3/2} \log(u)} = \operatorname{li} \left( \frac{1}{\sqrt{b}} \right) - \operatorname{li} \left( \frac{1}{\sqrt{a}} \right).
\]  \(\text{(27)}\)

This formula is useless for our purposes, because \([9] \text{ or } [10]\) is valid for \( x > 1 \). The analog of \([25]\) for usual Brun’s constant is given by

\[
\mathcal{B}_2(\infty) = \mathcal{B}_2(x) + \frac{2C_2}{\log(x)},
\]  \(\text{(28)}\)

where \( C_2 = \prod_{p>2} \left( 1 - \frac{1}{(p-1)^2} \right) = 0.66016 \ldots \) is the Twins constant.

The third column in Table II gives the sample of values

\[
\mathcal{B}_q^*(x) = \mathcal{B}_q(x) + \frac{C_q}{2} \int_{x}^{\infty} \frac{du}{u^{3/2} \log(u)}
\]  \(\text{(29)}\)

which are supposed to be constant and equal to \( \mathcal{B}_q(\infty) \). As it seen from the Table II indeed with increasing \( x \) growing number of digits of the sum \([29]\) is stabilizing.

To produce the data for this Table we calculated in PARI the value of \( C_q \) with over 30 digits accuracy using the formula (10) from \([36]\). We calculated the finite approximations \( \mathcal{B}_q(x) \) in DEC Fortran using the quadruple precision (REAL*16) with 33 decimal digits and the Mathematica v.7 was used to calculate integrals \([26]\) with over 30 digits of accuracy.

| \( x \) | \( \mathcal{B}_q(x) \) | \( \mathcal{B}_q^*(x) \) |
|---|---|---|
| \(10^2\) | 0.79575154653780163 | 0.81798411614326626 |
| \(10^3\) | 0.81119372199008314 | 0.8162594881529443 |
| \(10^4\) | 0.8135296609757082 | 0.81460891435244917 |
| \(10^5\) | 0.81432340308206016 | 0.81465046053349854 |
| \(10^6\) | 0.81450766696668914 | 0.81459563341678427 |
| \(10^7\) | 0.81457232824055968 | 0.8145963781164721 |
| \(10^8\) | 0.81458971836435488 | 0.81459649673657666 |
| \(\ldots\) | \(\ldots\) | \(\ldots\) |
| \(10^{13}\) | 0.81459655805079256 | 0.81459657169340710 |
| \(10^{14}\) | 0.81459656768254395 | 0.81459657104790996 |
| \(10^{15}\) | 0.81459657051185242 | 0.8145965710321129 |
| \(10^{16}\) | 0.81459657134862317 | 0.8145965710292293 |
| \(10^{17}\) | 0.81459657159724819 | 0.8145965710299049 |
| \(10^{18}\) | 0.81459657167131276 | 0.8145965710297237 |
| \(10^{19}\) | 0.81459657169347024 | 0.8145965710297623 |
| \(10^{20}\) | 0.81459657170012661 | 0.8145965710298816 |

It is seen from the last column that starting with \( x = 10^{16} \) all first 14 digits remain the same — the change appears at the 14-th place after the dot. In the
paper \cite{30} Nicely has performed complicated statistical analysis to get the 95% confidence interval for the value of $B_2$. In our case it is possible to estimate the error appearing in \eqref{29} by using the form of the function $\omega(x)$ given by Conjecture 1. Namely, the “density” of the error for the chance $C_q \sqrt{2x \log(x)}$ to find the prime of the form $m^2 + 1$ around $x$ is less than $O(x^{-3/4})$, thus we have:

$$|B_q(\infty) - B_q^*(x)| = O(x^{-3/4}).$$

(30)

From this we see, that for $x = 10^{20}$ the value of $B_q(\infty)$ lies in the interval of approximate length $10^{-15}$ around $B_q^*(10^{20})$ and we can claim that with 15 digits accuracy

$$B_q(\infty) \equiv B_q = 0.81459657170299\ldots.$$  

(31)

S. Plouffe has checked using his Symbolic Inverse Calculator (http://pi.lacim.uqam.ca/eng/), that this constant can not be expressed by other mathematical constants \cite{31}, thus the value of $B_q$ could be treated as a new mathematical constant. The comparison of numbers in the second and third column reveals that addition of the term $C_q \int_x^\infty \frac{du}{u^{3/2} \log(u)}$ causes that about 3-4 digits more than in the values of $B_q(x)$ alone settle down, thus the rate of convergence of $B_q^*(x)$ is a few orders faster than that of $B_q(x)$.

The correctness of the choice of $\frac{1}{2}$ in front of the integral in \eqref{25} can be checked by comparing the values of the equation

$$\mathcal{B}_q(x_2) - \mathcal{B}_q(x_1) = C_q \int_{x_1}^{x_2} \frac{du}{u^{3/2} \log(u)}$$

(32)

following from \eqref{25} with the actual computer data. For example for $x_1 = 10^6$ and $x_2 = 10^{20}$ for the l.h.s. of \eqref{32} from the computer data we get 0.000177\ldots while the r.h.s. is equal to 0.000176\ldots.

Let us mention that two first primes from $Q$ give contribution $\frac{1}{2} + \frac{1}{5} = 0.7$ to $B_q$, i.e. 86% of the total value 0.8145966\ldots!

We can define the Brun’s measure of the set of numbers $S = \{a_1, a_2, \ldots\}$ as the sum

$$\mathcal{M}_S(S) = \sum_{i} \frac{1}{a_i}$$

(33)

provided it is finite. Thus we can say that the Brun’s measure of the set of twin primes is $1.9021605823\ldots$, while the Brun’s measure of the set $Q$ is $\mathcal{M}_S(Q) = 0.8145965717\ldots$.

5 Analog of the conjecture B of Hardy–Littlewood

The conjecture B of Hardy–Littlewood \cite{19} says that the number

$$\pi(d; x) = \sum_{\substack{p < x \\ p - p' = d}} 1 = |\{p < x : p \text{ and } p - d \text{ are primes}\}|$$

(34)

of prime pairs $p - d, p < x$ separated by $d$ ($d = 2, 4, 6, \ldots$) is given by

$$\pi(d; x) \sim C_2 \prod_{p \mid d} \left(\frac{p - 1}{p - 2}\right) \int_2^x \frac{dt}{\log^2(t)},$$

(35)
where
\[ C_2 = 2 \prod_{p > 2} \left( 1 - \frac{1}{(p - 1)^2} \right) = 1.320323631693739 \ldots \] (36)
is the Twin constant. The integral in (35) again can be expressed by the logarithmic integral and thus calculated quickly from the series (9) or (10):
\[
\int_a^b \frac{dt}{\log^2(t)} = \int_a^b \frac{dt}{\log(t)} + \frac{a}{\log(a)} - \frac{b}{\log(b)} = \text{li}(b) - \text{li}(a) + \frac{a}{\log(a)} - \frac{b}{\log(b)} \quad (37)
\]
The values of the gaps between primes \( m^2 + 1 \) and \( (m - d)^2 + 1 \) grow linearly with \( m \), but we can formulate an analog of the conjecture B of Hardy–Littlewood when we will focus on the gaps between \( k \)'s appearing in \( 4k^2 + 1 \). Thus let us define:
\[
\pi_q(d; x) = | \{ 2k < \sqrt{x} : 4k^2 + 1 \text{ and } 4(k - d)^2 + 1 \text{ are both primes} \} | \quad (38)
\]
In contrast to all primes here \( d = 1, 2, 3, \ldots \). The Fig. 4 presents the plots of \( \pi_q(d; x) \) obtained from our computer data for \( x = 10^{10}, x = 10^{12}, \ldots x = 10^{20} \) and \( d \leq 300 \).

There is a heuristic procedure of Bateman and Horn [4] (see also [33, chap. 3]) allowing to guess the formula for the number of constellations of primes of different types. Let \( f = \{ f_1(x), f_2(x), \ldots, f_l(x) \} \) be the set of distinctive irreducible polynomials with integral coefficients and positive leading coefficient such that \( f(x) = f_1(x)f_2(x) \ldots f_l(x) \) has no fixed divisor \( > 1 \). Let \( \pi(f; x) \) denote the number of positive integers \( n < x \) such that all \( f_1(n), f_2(n), \ldots, f_l(n) \) are simultaneously primes. Then the Bateman–Horn conjecture reads:
\[
\pi(f; x) \sim \prod_{p} \frac{1 - \frac{w(p)}{p}}{(1 - \frac{1}{p})^l} \int_2^x \frac{du}{\prod_{i=1}^{l} \log(f_i(u))}, \quad (39)
\]
where \( w(p) \) is the number of distinct solutions to \( f_1(u)f_2(u) \cdots f_l(u) \equiv 0 \pmod{p} \) with \( u \in \{ 0, 1, \ldots, p - 1 \} \). In our case \( f_1(u) = 4u^2 + 1, f_2(u) = 4(u - d)^2 + 1 \). It is well known [20, chap. VII] that equation \( 4u^2 + 1 \equiv 0 \pmod{p} \) has two solutions for primes of the form \( p = 4n + 1 \) and does not have solutions for primes of the form \( p = 4n + 3 \). For the case \( p = 4n + 1 \) if \( p \nmid d \) then equations \( f_1(u) \equiv 0 \pmod{p} \) and \( f_2(u) \equiv 0 \pmod{p} \) have the same solutions, thus \( w(p) = 2 \). When \( p \mid d \) then there are two possibilities: \( w(p) = 3 \) when \( f_1(u) \equiv 0 \pmod{p} \) and \( f_2(u) \equiv 0 \pmod{p} \) have one common solution, or \( w(p) = 4 \) when \( f_1(u) \equiv 0 \pmod{p} \) and \( f_2(u) \equiv 0 \pmod{p} \) have distinct solutions. The equations \( f_1(u) \equiv 0 \pmod{p} \) and \( f_2(u) \equiv 0 \pmod{p} \) can have one common solution only when \( p \mid \pm 2u^* + d \), where \( u^* \) is the solutions of
\[
4u^2 + 1 \equiv 0 \pmod{p}.
\]
The solutions of this equation are of the form (see [20, p.88])
\[
2u_{1,2} = \pm u^*, \quad u^* = \left( \frac{p - 1}{2} \right)!
\]
Because the conditions \( p \mid \pm 2u^* + d \) for the case \( w(p) = 3 \) can be written as the one condition \( p \mid d^2 - 4u^* \) and there is an identity \( 4u^*^2 + 1 \equiv 0 \pmod{p} \) we can write the condition for \( w(p) = 3 \) simply as \( p \mid d^2 + 1 \). Thus we have for \( p = 4n + 1 \)
Analog of the conjecture B of Hardy–Littlewood

\[ w(p) = \begin{cases} 
4 & \text{if } p \nmid d, \ p \nmid d^2 + 1 \\
3 & \text{if } p \mid d^2 + 1 \\
2 & \text{if } p \mid d 
\end{cases} \tag{40} \]

For \( p = 2 \) and for \( p = 4n + 3 \) there are no solutions, i.e. \( w(p) = 0 \), hence finally the product appearing in the Bateman–Horn conjecture takes the form:

\[ \prod_p \frac{1 - w(p)/p}{(1 - 1/p)^2} = \tag{41} \]

\[ \prod_{p \equiv 3 \pmod{4}} 1 - \frac{w(p)/p}{(1 - 1/p)^2} \prod_{p \equiv 1 \pmod{4}} \frac{1 - 4/p}{(1 - 1/p)^2} \prod_{p \mid d} \frac{1 - 2/p}{(1 - 1/p)^2} \prod_{p \mid d^2 + 1} \frac{1 - 3/p}{(1 - 1/p)^2}. \]

We can get rid of the product over \( p \nmid d \) by extending it to the product over all \( p \equiv 1 \pmod{4} \) and simultaneously by dividing by an appropriate term. Because the conditions \( p \mid d \) and \( p \mid d^2 + 1 \) cannot be satisfied simultaneously these additional factors can be incorporated into the last two products above. Finally the factor describing oscillations takes the form (we separated 4 to cancel it later with 4 coming from the degrees of \( f_1(u) \) and \( f_2(u) \)):

\[ \prod_p \frac{1 - w(p)/p}{(1 - 1/p)^4} = 4C_1 P(d), \tag{42} \]

where the constant \( C_1 \):

\[ C_1 = \prod_{p \equiv 3 \pmod{4}} \frac{p^2}{(p - 1)^2} \prod_{p \equiv 1 \pmod{4}} \frac{p(p - 4)}{(p - 1)^2} = 0.975245556223143537223292783\ldots \tag{43} \]

and \( P(d) \) denotes the product:

\[ P(d) = \prod_{p \equiv 1 \pmod{4}} \frac{p - 2}{p - 4} \prod_{p \equiv 1 \pmod{4}} \frac{p - 3}{p - 4}. \tag{44} \]

In above expressions the condition \( p \equiv 1 \pmod{4} \) means that products are over primes \( p \geq 5 \), thus all these products are positive. In \( \text{(41)} \) the conditions \( p \mid d \) and \( p \mid d^2 + 1 \) are fulfilled only by finite number of \( p \)'s, hence it is obvious that these products are convergent.

Finally we obtain the number of such \( k < x \) that both \( f_1(k) = (2k)^2 + 1 \) and \( f_2(k) = (2(k - d))^2 + 1 \) are prime:

\[ \pi(f_1, f_2; x) = C_1 \prod_{p \equiv 1 \pmod{4}} \frac{p - 2}{p - 4} \prod_{p \equiv 1 \pmod{4}} \frac{p - 3}{p - 4} \int_1^x \frac{du}{\log^2(2u)} \tag{45} \]

We put here for a while 1 as the lower limit of integration, since \( k = 1 \) gives the prime \( 4 \cdot 1^2 + 1 = 5 \) (let us notice, that Ramanujan often did not specify the lower
limit of integration, see [6, p.123]). Because we skip 1 in \( m^2 + 1 \) in manipulations of integrals below, alternatively we can say that the lower limit of integration is \( \sqrt{5}/4 = 1.118033989 \ldots \).

Usually we are interested directly in the number of primes \( 4k^2 + 1 \) and after the change of the integration variable \( u = \sqrt{t}/2 \) we have

\[
\int_{\sqrt{a}/2}^{\sqrt{b}/2} \frac{du}{\log^2(2u)} = \int_{\sqrt{a}/2}^{\sqrt{b}/2} \frac{dt}{\sqrt{t}\log^2(t)}
\]

and finally for the quantity \( \pi_q(d; x) \) defined in (38) we obtain **Conjecture 2:**

\[
\pi_q(d; x) = C_1 \prod_{p \equiv 1 \pmod{4}} \frac{p-2}{p-4} \prod_{p \equiv 1 \pmod{4}} \frac{p-3}{p-4} \int_{\sqrt{x}}^{x} \frac{dt}{\sqrt{t}\log^2(t)} + \text{error term} \quad (47)
\]

We cheated a little here replacing 4 by 5 as the lower limit of integration; better possibility as the lower limit of integration is perhaps the Soldner-Ramanujan constant \( \mu = 1.4513692348838105 \ldots \) defined by \( \lim(\mu) = 0 \), see [6, p.123]. Appearing here integral by the change of the variable \( t = u^2 \) can be expressed by the logarithmic integral:

\[
\int_{a}^{b} \frac{dt}{\sqrt{t}\log^2(t)} = \frac{1}{2} (\text{li}(\sqrt{b}) - \text{li}(\sqrt{a})) + \frac{\sqrt{a}}{\log(a)} - \frac{\sqrt{b}}{\log(b)}. \quad (48)
\]

The product \( P(d) \) from (44) is responsible for characteristic oscillations seen in the Fig. 4. The conjecture 2 agrees with the computer data quite well. Instead of producing some table to corroborate this statement, we give “visual argument”: in the Fig.4 by the red boxes are plotted values of the quotient \( \pi_q(d; x)/P(d) \) for \( x = 10^{12}, \ldots x = 10^{20} \) and \( d \leq 300 \). In green are plotted values of the integral \( \int_{a}^{b} du/\sqrt{t}\log^2(t) \) for the same values of \( x \).

6 Heuristics on the gaps between adjacent \( k \)

It is interesting to restrict the analysis from the previous section to the case of consecutive values of \( k \) giving the prime \( 4k^2 + 1 \). Let us define the quantity:

\[
h(d; x) = \{ \text{number of pairs } k < k' = k + d, \text{ such that } 4k^2 + 1 \text{ and } 4k'^2 + 1 < x \text{ are consecutive primes of the form } m^2 + 1 \} = \sum_{q_n < x \atop q_n - q_{n-1} = 4d(\sqrt{q_n - 1} - d)} 1. \quad (49)
\]

The Fig. 5 presents the plot of \( h(d; x) \) obtained from our computer data for \( x = 10^{10}, x = 10^{12}, \ldots x = 10^{20} \). On the semi-logarithmic scale the points display characteristic oscillations around straight lines representing the fits to exponential decrease obtained by the least-square method. To describe the oscillations we take the product \( P(d) \) from (44). The Fig.5 presents the corroboration of this mechanism of oscillations: dividing the values of \( h(d, x) \) by the product \( P(d) \) leaves pure exponential decrease — small deviations could be attributed to the fluctuations of \( h(d, x) \) and incorporated into the (unknown) error term.
The Figure 5 suggests the following

**Ansatz:**

\[ h(d, x) = C_1 P(d) B(x) e^{-dA(x)}. \]  

(50)

The functions \( A(x) \) and \( B(x) \), giving the slopes and the intercepts of straight lines seen in the Fig. 5, can be determined by exploiting two selfconsistency conditions that \( h(d, x) \) has to obey just from the definition. First of all, the number of all gaps between \( k' \)'s is by one smaller than the number of primes of the form \( 4k^2 + 1 \) smaller than \( x \):

\[ K(x) = \sum_{d=1}^{x} h(d, x) = \pi_q(x) - 1, \]  

(51)

where \( K(x) \) denotes the largest gap between two consecutive \( k, k' < x \). The second selfconsistency condition comes from the observation, that the sum of distances between adjacent \( k \) is equal to the \( k \) producing the largest prime \( q = 4k^2 + 1 \leq x \). For large \( x \) we can write:

\[ K(x) = \sum_{d=1}^{x} h(d, x) = \frac{\sqrt{x}}{2}. \]  

(52)

The erratic behavior of the product \( P(d) \) is an obstacle in calculation of the above sums (51) and (52). Thus we will replace \( P(d) \) by the mean value:

\[ \sum_{d=1}^{n} P(d) = ns + E(n), \]  

(53)

where we assume that the unknown error term \( E(n) \) is an increasing function of \( n \) which grows slower than \( n \):

\[ \lim_{n \to \infty} \frac{E(n)}{n} = 0, \]  

(54)

what means that:

\[ s = \lim_{n \to \infty} \frac{1}{n} \sum_{d=1}^{n} \left( \prod_{p \equiv 1 \pmod{4}}^{p \mid d} \frac{p-2}{p-4} \prod_{p \equiv 1 \pmod{4}}^{p \mid d^2+1} \frac{p-3}{p-4} \right). \]  

(55)

There is known at least one example of the error term which grows slower than \( n \) for the similar problem. Namely, E. Bombieri and H. Davenport [7] have proved that the number \( 1/\prod_{p>2}(1 - \frac{1}{(p-1)^2}) \) is the arithmetical average for the product \( \prod_{p \mid d} \frac{p-1}{p-2} \):

\[ \sum_{d=1}^{n} \prod_{p \mid d, p > 2} \frac{p-1}{p-2} = \frac{n}{\prod_{p>2}(1 - \frac{1}{(p-1)^2})} + \mathcal{O}(\log^2(n)). \]  

(56)

To get rid of \( P(d) \) in (51) and (52) the Abel summation formula can be used in the form:

\[ \sum_{i=1}^{n} a_i b_i = - \sum_{i=1}^{n-1} S(i)c_i + A(n)b_n, \]
where \( S(i) = a_1 + \ldots a_i \) and \( c_i = b_{i+1} - b_i \). Putting here \( a_i = P(i) \), \( b_i = f(i) \) and replacing \( E(1) < E(2) < \ldots < E(n - 1) \) by larger \( E(n) \) we obtain:

\[
\sum_{l=1}^{n} P(l)f(l) = s \sum_{l=1}^{n-1} f(l) + \mathcal{O}(f(n)E(n)). \tag{57}
\]

In our case \( f(l) = C_1 B(x)e^{-A(x)l} \) for equation \( (51) \) and \( f(l) = C_1 B(x)e^{-A(x)l} \) for equation \( (52) \). From the Fig.5 we see, that \( h(d, x) \) decreases exponentially with \( d \) and to solve \( (51) \) and \( (52) \) for \( e^{-A(x)} \) and \( B(x) \) we have to drop the term \( \mathcal{O}(f(n)E(n)) \).

The sums \( (51) \) and \( (52) \) are the geometrical and differentiated geometrical series respectively; because \( h(d, x) \) decreases exponentially with \( d \) we have replaced \( K(x) \) in \( (51) \) and \( (52) \) by \( \infty \) and \( (51), (52) \) turn into the equations:

\[
\frac{sC_1 B(x)e^{-A(x)}}{1 - e^{-A(x)}} = \pi_q(x), \tag{58}
\]

\[
\frac{sC_1 B(x)e^{-A(x)}}{(1 - e^{-A(x)})^2} = \sqrt{x}. \tag{59}
\]

The solutions for \( e^{-A(x)} \) and \( B(x) \) of the above equations are:

\[
e^{-A(x)} = 1 - \frac{2\pi_q(x)}{\sqrt{x}}, m \tag{60}
\]

\[
B(x) = \frac{2\pi_q^2(x)}{sC_1 \sqrt{x}(1 - \frac{2\pi_q(x)}{\sqrt{x}})}. \tag{61}
\]

Hence we have finally:

\[
h(d; x) = \frac{2P(d)\pi_q^2(x)}{s\sqrt{x}} \left( 1 - \frac{2\pi_q(x)}{\sqrt{x}} \right)^{d-1} + \text{error term}. \tag{62}
\]

The Table III gives a comparison of the formulas \( (60) \) and \( (61) \) with the slopes and intercepts of \( \log(h(d; x)/C_1 P(d)) \) vs \( d \) obtained from the computer data by means of the least square method (roughly 1/4 values of \( h(d; x) \) for largest \( d \) were discarded to avoid the fluctuations of \( h(d; x) \) in the region of large \( d \)).

### TABLE III

| \( x \) | \( e^{-A(x)} \) | \( 1 - \frac{2\pi_q(x)}{\sqrt{x}} \) | \( e^{-2\pi_q^2(x)} \) | \( B(x) \) | \( \frac{2\pi_q^2(x)}{sC_1 \sqrt{x}-2\pi_q(x)} \) | \( \frac{2\pi_q^2(x)}{sC_1 \sqrt{x}} \) |
|---|---|---|---|---|---|---|
| 10^{15} | 0.91014186 | 0.91525886 | 0.91875009 | 75421.77 | 65826.02 | 60247.85 |
| 10^{16} | 0.91611160 | 0.92091638 | 0.92396266 | 206903.63 | 180179.71 | 165930.45 |
| 10^{17} | 0.92141730 | 0.92583260 | 0.92851624 | 570171.11 | 498478.78 | 461507.91 |
| 10^{18} | 0.92632759 | 0.93019957 | 0.93257992 | 1573509.35 | 1389610.28 | 1292614.89 |
| 10^{19} | 0.93060739 | 0.93406718 | 0.93619375 | 4388166.12 | 3904615.30 | 3647172.98 |
| 10^{20} | 0.93444100 | 0.93752841 | 0.93943975 | 12318241.09 | 11044184.29 | 10352363.57 |
The expression for the mean value $s$ of the product $P(d)$ can be obtained in the following way: Because the pairs of primes of the form $(2k)^2 + 1$ and $(2k + 2)^2 + 1$ (Shanks calls in \cite{Shanks} such pairs Gaussian Twins) correspond to $d = 1$ thus they are necessarily consecutive $(q_n, q_{n+1})$ and the formula for the number $h(1, x)$ of such pairs smaller than $x$ obtained from \eqref{eq:heuristics_pairs} has to be equal to $\pi_q(1; x)$ from \eqref{eq:pi_q}:

$$h(1; x) = \frac{2\pi_q^2(x)}{s\sqrt{x}} = C_1 \int_5^x \frac{dt}{\sqrt{t} \log^2(t)} \tag{63}$$

For large $x$ we have $\pi_q(x) = C_q \sqrt{x} / \log(x)$ and

$$\int_5^x \frac{dt}{\sqrt{t} \log^2(t)} = \frac{1}{2} \text{li}(\sqrt{x}) - \frac{\sqrt{x}}{\log(x)} = \frac{2\sqrt{x}}{\log^2(x)} + \ldots$$

where we have used two first terms of the asymptotic expansion \eqref{eq:asymptotic_expansion} and fortunately the term $\sqrt{x}/\log(x)$ cancels out leaving on both sides of \eqref{eq:heuristics_pairs} the same dependence on $x$ and thus we obtain $s = C^2_q / C_1$:

$$s = \lim_{n \to \infty} \frac{1}{n} \sum_{d=1}^{n} \left( \prod_{p \equiv 1 \pmod{4}} \frac{p-2}{p-4} \prod_{p \equiv 1 \pmod{4}} \frac{p-3}{p-4} \prod_{p \equiv 1 \pmod{4}} \frac{p-4}{p-4} \right) = \frac{C^2_q}{C_1}. \tag{64}$$

In the full (mysterious) form it reads:

$$\lim_{n \to \infty} \frac{1}{n} \sum_{d=1}^{n} \left( \prod_{p \equiv 1 \pmod{4}} \frac{p-2}{p-4} \prod_{p \equiv 1 \pmod{4}} \frac{p-3}{p-4} \prod_{p \equiv 1 \pmod{4}} \frac{p-4}{p-4} \right) = \frac{\left( \prod_{p \geq 3} \left( 1 - \frac{(-1)(p-1)/2}{p-1} \right) \right)^2}{\prod_{p \equiv 3 \pmod{4}} \frac{p^2}{(p-1)^2} \prod_{p \equiv 1 \pmod{4}} \frac{p(p-4)}{(p-1)^2}}. \tag{65}$$

We were not able to prove this identity analytically — usual methods of calculating the sums of arithmetic functions, see e.g. \cite{Knuth} Chap.1, are not applicable here because $P(d)$ is not a multiplicative function. The computer checking of \eqref{eq:heuristics_pairs} for large number of terms on the l.h.s. is also difficult because the calculation of the average has almost cubic complexity in $n$ (i.e. finding the value of l.h.s. of \eqref{eq:heuristics_pairs} involves $O(n^3/\log^2(n))$ operations). We have calculated the sum of $P(d)$ for $d$ up to 150000 and we obtained $\sum_{d=1}^{150000} P(d)/150000 = 1.93242674 \ldots$, while the r.h.s. of \eqref{eq:heuristics_pairs} is $(1.3728134628)^2/0.97524552 = 1.93245368$, thus the first 5 digits are the same.

In \cite{Shanks}, \cite[p. 90]{Guy}, \cite{Guy} heuristically the formula for $h(1; x)$ was obtained in the form:

$$h(1, x) = F \frac{\sqrt{x}}{\log^2(x)} + \text{error term} \tag{66}$$

where:

$$F = \frac{\pi^2}{2} \prod_{p \equiv 1 \pmod{4}} \left( 1 - \frac{4}{p} \right) \left( \frac{p+1}{p-1} \right)^2 = 1.9504911124462870744465855658 \ldots. \tag{67}$$
In [35] the 50 digits of this constant are given. Therefore we have \( C_1 = F/2 \), \( s = 2C_q/F \) and the combination on the r.h.s of (65) can be transformed to the form:

\[
s = \frac{2C_q^2}{F} = \frac{1}{4} \prod_{p \equiv 1 \pmod{4}} (p-2)^2(p-1)^2 \prod_{p \equiv 3 \pmod{4}} \left( \frac{p+1}{p-1} \right)^2
\]

(68)

Finally we state the

**Conjecture 3:**

\[
h(d, x) = \frac{FP(d)p_q^2(x)}{C_q^2 \sqrt{x}} \left( 1 - \frac{2\pi_q(x)}{\sqrt{x}} \right)^{d-1} + \text{error term.}
\]

(69)

For large \( x \) we can simplify considerably the above formulas by writing \( e^{-A(x)} = 1 - \frac{2\pi_q(x)}{\sqrt{x}} \) as \( e^{-A(x)} = e^{-\frac{2\pi_q(x)}{\sqrt{x}}} \) and \( B(x) = \frac{2\pi_q^2(x)}{sC_q^2 \sqrt{x}} \), therefore in the limit of large \( x \) we have:

\[
h(d, x) = \frac{FP(d)}{C_q^2 \sqrt{x}} \pi_q^2(x) e^{-\frac{2\pi_q(x)}{\sqrt{x}}} + \text{error term.}
\]

(70)

The Table III gives the comparison of the quantities \( e^{-A(x)} \) and \( B(x) \) obtained from least-square method applied to \( \log(h(d; x)/C_1 P(d)) \) vs \( d \) and analytical expressions for them.

As a corroboration of the above conjectures we will obtain the formula for the maximal gap \( K(x) \) between two consecutive values of \( k < \sqrt{x}/2 \) giving the prime \( 4k^2 + 1 \). Assuming, that the maximal gap \( K(x) \) appears only once we have the equation \( h(K(x), x) = 1 \) and putting the Hardy–Littlewood formula for \( \pi_q(x) \) in (70) and replacing \( P(d) \) by \( s \) we obtain:

\[
K(x) \sim \frac{\log(x)}{2C_q} \left( \frac{1}{2} \log(x) + \log(2C_q^2) - 2 \log(\log(x)) \right)
\]

(71)

what for large \( x \) goes to the

\[
K(x) \sim \frac{1}{4C_q} \log^2(x).
\]

(72)

Because for all primes it is widely believed that \( G(x) \equiv \max_{p_n < x} (p_{n+1} - p_n) \sim \log^2(x) \), we see that \( K(x) \) differs from \( G(x) \) just by a constant (but here only a fraction of \( k' \)'s are primes!). The largest gap between adjacent \( k \) giving \( 4k^2 + 1 < 10^{20} \) was 290. The comparison of this formula with computer data is shown in the Figure 6. From (72) we deduce the following

**Conjecture 4:**

\[
q_{n+1} - q_n = \mathcal{O}(\sqrt{q_n} \log^2(q_n))
\]

(73)

Let us recall that for all primes the Riemann Hypothesis gives \( p_{n+1} - p_n = \mathcal{O}(p_n^{1+\epsilon}) \) for any \( \epsilon > 0 \), but in reality gaps between consecutive primes are smaller and the Cramer conjecture [10] states that \( p_{n+1} - p_n = \mathcal{O}(\log^2(p_n)) \), see however [16]. Because our Conjecture 4 is obtained from the guessed formula for maximal gap between \( k' \)'s we expect (73) to be close to the optimal bound for \( q_{n+1} - q_n \).
7 Analog of the Chebyshev’s bias

For ordinary primes the Dirichlet’s Theorem on the primes in arithmetical progressions asserts that the number \( \pi(x; 4, 1) \) of primes \( < x \) giving 1 as the remainder when divided by 4 should be equal to the number \( \pi(x; 4, 3) \) of primes \( < x \) giving 3 as the remainder when divided by 4, see e.g. [34]. But the direct inspection shows that for small \( x \) there are more primes \( p \equiv 3 \pmod{4} \) than \( p \equiv 1 \pmod{4} \): \( \pi(x; 4, 3) > \pi(x; 4, 1) \), what is called Chebyshev’s bias [24]. For the first time 1’s takes the lead at \( p = 26861 \) — up to this prime 3’s win or there is a tie: \( \pi(x; 4, 3) \geq \pi(x; 4, 1) \) for \( x < 26861 \). The next time \( \pi(x; 4, 3) < \pi(x; 4, 1) \) at 616841 and in general there is preponderance of primes in the progression \( 4n + 3 \). The same phenomenon was observed in other arithmetical progressions [17]. However for the case of number of twins \( \pi(2; x) \) (primes pairs \( p, p + 2 \)) and number of cousins \( \pi(4; x) \) (primes pairs \( p, p + 4 \)) which by the B conjecture of Hardy and Littlewood (35) should be the same \( \pi(2; x) \approx \pi(4; x) \), there is no Chebyshev bias at least up to \( 2^{12} \approx 1.4 \times 10^{12} \): sometimes twins and sometimes cousins take the lead, [41]. In this paper it was shown numerically that \( \pi(2; x) - \pi(4; x) \) behaves as the uncorrelated random walk; also the number of returns of this random walk to the origin (the number of such \( x \) that \( \pi(2; x) = \pi(4; x) \)) follows the usual square root law \( \sqrt{x} \) [41]. It is in agreement with last sentences of the paper [17] suggesting that there is no Chebyshev bias for pairs of primes \( p, p + 2k \) in general.

TABLE IV

| \( x \) | \( \pi_q(x; 3, 2) \) | \( \pi_q(x; 3, 2) / \pi_q(x; 3, 1) \) | \( w_2(x) / \pi_q(x) \) | \( \delta_1(x) \) | \( \delta_2(x) \) | \( \delta_0(x) \) |
|---|---|---|---|---|---|---|
| \( 10^4 \) | 8 | 4.000000 | 0.818182 | 0.000000 | 0.850210 | 0.000000 |
| \( 10^4 \) | 12 | 1.714286 | 0.700000 | 0.023927 | 0.873320 | 0.024337 |
| \( 10^5 \) | 32 | 1.684211 | 0.269231 | 0.217321 | 0.709371 | 0.030297 |
| \( 10^6 \) | 71 | 1.731707 | 0.132743 | 0.340751 | 0.597523 | 0.028523 |
| \( 10^7 \) | 215 | 2.128713 | 0.470032 | 0.341211 | 0.592423 | 0.039598 |
| \( 10^8 \) | 560 | 1.992883 | 0.769596 | 0.301877 | 0.640202 | 0.034797 |
| \( 10^9 \) | 1565 | 1.924969 | 0.322825 | 0.363780 | 0.582380 | 0.033401 |
| \( 10^{10} \) | 4410 | 1.963491 | 0.140454 | 0.424147 | 0.526904 | 0.030546 |
| \( 10^{11} \) | 12503 | 1.978636 | 0.049673 | 0.476509 | 0.479021 | 0.027770 |
| \( 10^{12} \) | 36069 | 1.999279 | 0.073571 | 0.514925 | 0.444198 | 0.025570 |
| \( 10^{13} \) | 104214 | 2.009254 | 0.671500 | 0.476702 | 0.485462 | 0.025570 |
| \( 10^{14} \) | 304482 | 2.004754 | 0.887649 | 0.442651 | 0.52215 | 0.023710 |
| \( 10^{15} \) | 893184 | 1.999557 | 0.610524 | 0.442717 | 0.524391 | 0.022016 |
| \( 10^{16} \) | 2636218 | 2.000222 | 0.709507 | 0.433234 | 0.535911 | 0.020650 |
| \( 10^{17} \) | 7817678 | 1.999806 | 0.474900 | 0.442119 | 0.528811 | 0.019378 |
| \( 10^{18} \) | 23265155 | 1.999574 | 0.169032 | 0.471831 | 0.500710 | 0.018267 |
| \( 10^{19} \) | 69497579 | 1.999852 | 0.141396 | 0.491985 | 0.481998 | 0.017256 |
| \( 10^{20} \) | 208240005 | 2.000040 | 0.365513 | 0.499533 | 0.475749 | 0.016352 |

All the primes of the form \( q = 4k^2 + 1 \) are necessarily disposed somewhere in the arithmetical progression \( 4l + 1: q \equiv 1 \pmod{4} \). But the situation changes when we consider residues modulo 3. Let \( \pi_q(x; 3, 1) \) denote the number of primes
\[ q = m^2 + 1 < x \] such that \( q \equiv 1 \pmod{3} \) and let \( \pi_q(x; 3, 2) \) denote the number of primes \( q = m^2 + 1 < x \) such that \( q \equiv 2 \pmod{3} \). The direct inspection of all possibilities shows that the residue 2 should appear twice as often as the residue 1:

\[ \pi_q(x; 3, 2) \approx 2\pi_q(x; 3, 1). \quad (74) \]

It translates into the human 10-base system as the observation, that except for the first two cases 2 and 5, the last digit of the primes of the form \( m^2 + 1 \) can be only 1 or 7, see [1]. The direct inspection of all 10 possibilities under the assumption of normality of digits of \( k \) in the base 10 shows, that there are two times more ways of obtaining 7 than 1.

The ratio of the number of those primes \( p = 4k^2 + 1 \) congruent to 2 modulo 3 to those congruent to 1 modulo 3 can give some information about irregularities in the distribution of primes of the form \( 4k^2 + 1 \). The Table IV shows the values of the ratio \( \pi_q(x; 3, 2)/\pi_q(x; 3, 1) \) for \( x = 10^5, \ldots 10^{20} \). As is seen from this Table after the initial transient interval below \( 10^7 \) this ratio begins to oscillate around the predicted value 2. Initially \( \pi_q(x; 3, 2) > 2\pi_q(x; 3, 1) \) and for the first time \( \pi_q(x; 3, 2) < 2\pi_q(x; 3, 1) \) at \( q_{17} = 4 \cdot 42^2 + 1 = 7057 \), i.e. at the 17-th prime of the form \( q = m^2 + 1 \). Recently A. Granville and G. Martin [17] have discussed several examples of ”prime races”. Primes of the form \( q = 4k^2 + 1 \) provide another example of such a race. Namely we will say that two wins at a given \( x \) if \( \pi_q(x; 3, 2) > 2\pi_q(x; 3, 1) \) and let \( w_2(x) \) denote the number of those primes \( q = 4k^2 + 1 < x \) that residue two wins over residue one. In the third column of the Table IV the ratio \( w_2(x)/\pi_q(x) \) is shown. As it is seen from this sample of numbers there are large fluctuations of the ratio \( w_2(x)/\pi_q(x) \). The Fig.7 shows the plot of \( \pi_q(x; 3, 2) - 2\pi_q(x; 3, 1) \) up to \( x = 10^{20} \). This plot can be interpreted as a kind of one dimensional random walk: let \( y(x) \) denote the displacement of the walker at the “time” \( x \), which plays the role of the time. If for a given \( q \in Q \) we find that \( q \equiv 1 \pmod{3} \) the random walker performs step down of length 2 and if \( q \equiv 2 \pmod{3} \) the random walker performs step up of length 1 at the moment \( x = q \). In other moments of time \( x \) the walker simply does not move. Thus we have \( y(x) = \pi_q(x; 3, 2) - 2\pi_q(x; 3, 1) \). This plot resembles usual random walk, there were 21349 returns to the origin of this random walk up to \( 10^{20} \). There are large regions that \( y(x) > 0 \) as well as \( y(x) < 0 \) suggesting that there is no Chebyshev bias in the distribution of primes \( q_n \). In the Fig.8 the plot of \( y(x)/x^{1/4} = (\pi_q(x; 3, 2) - 2\pi_q(x; 3, 1))/x^{1/4} \) is shown. The amplitude of oscillations in this plot seems to be constant over the interval \((10^3, 10^{20})\) and contained in the very short interval \((-\tfrac{1}{2}, \frac{1}{2})\). Dividing \((\pi_q(x; 3, 2) - 2\pi_q(x; 3, 1))\) by \( x^\alpha \) results in amplitudes going to zero when \( \alpha > \tfrac{1}{4} \) and increasing when \( \alpha < \tfrac{1}{4} \). It is another argument in favor of the error term conjectured in the Sect. 2. There are probably logarithmic factors present, like for the usual Chebyshev bias, see Fig. 6 in [17], but we are not able to separate it. The amplitude of oscillations of \( y(x)/x^{1/4} \) is very small, less than 0.5 and roughly half of the plot in Fig.8 is greater than zero and roughly half is below line zero. In [33] it was proposed to use the logarithmic density to measure the Chebyshev bias. Here we will define these densities for primes from \( Q \) as follows:

\[
\delta_1 = \lim_{x \to \infty} \frac{1}{\log(x)} \sum_{\substack{2 \leq n < x \\ 2\pi_q(n; 3, 1) > \pi_q(n; 3, 2)}} \frac{1}{n};
\quad (75)
\]
\[ \delta_2 = \lim_{{x \to \infty}} \frac{1}{{\log(x)}} \sum_{{2 \leq n < x \atop 2\pi q(n;3,1) < \pi q(n;3,2)}} \frac{1}{n} \] (76)

\[ \delta_0 = \lim_{{x \to \infty}} \frac{1}{{\log(x)}} \sum_{{2 \leq n < x \atop 2\pi q(n;3,1) = \pi q(n;3,2)}} \frac{1}{n} \] (77)

We do not have at our disposal any formulas like those in [34] and we have to turn to the brute force numerical calculation of finite size approximations \( \delta_1(x) \), \( \delta_2(x) \) and \( \delta_0(x) \) given by expressions (75) — (77) without limit operation \( \lim_{{x \to \infty}} \). The results are presented in the 5-th, 6-th and 7-th column of the Table IV and in Figure 9. Up to \( x = 2^{31} \) the data for the Table IV and Figure 9 was obtained by direct summing of the harmonic sums, for \( x > 2^{31} \approx 2.15 \times 10^9 \) the incredible accurate approximation [12, [21, pp. 76-78]

\[ \sum_{{k=n}}^{m} \frac{1}{k} = \log \left( m + \frac{1}{2} \right) - \log \left( n - \frac{1}{2} \right) + \mathcal{O} \left( \frac{1}{n^2} \right) \] (78)

was used, thus the error of each summand was smaller than \( 10^{-19} \), and as there were \( \mathcal{O}(10^8) \) terms, in the worst case of adding up all roundoffs we expect the total error to be smaller than \( 10^{-10} \). To calculate the harmonic series up to \( x = 10^{20} \) directly by adding all numbers \( 1/n \) would take a few thousands years of CPU time, but it is in general impossible using standard programming languages as the loop can have only integer counter and on 64 bit processors the largest integer is \( 2^{63} \approx 9.22 \times 10^{18} \).

For all primes Chebyshev conjectured that

\[ \lim_{{x \to \infty}} \sum_{{p > 2}} (-1)^{{p-1}} e^{-p/x} = -\infty \] (79)

It was proved by Hardy and Littlewood [18] and Landau [27, [26] that (79) is true if and only if the L-function

\[ L(s, \chi_1) = \sum_{{n=0}}^{\infty} \frac{(-1)^n}{(2n + 1)^s} \] (80)

does not have nontrivial zeros outside the critical line \( \Re(s) = \frac{1}{2} \). Here we formulate the analogous function for the case of primes from \( Q \):

\[ F(x) = \sum_{{q \in Q}} c_q e^{-q/x} \] (81)

where

\[ c_q = \begin{cases} 2 & \text{when } q \mod 3 = 1 \\ -1 & \text{when } q \mod 3 = 2 \end{cases} \] (82)

We made the plot of \( F(x) \) and in contrast to (79) this function seems to not possess a limit when \( x \to \infty \). Because values of \( F(x) \) sometimes are close to zero calculating the sum (81) on the computer we have stopped summation at such \( q' \) that

\[ \left| \frac{c_{q'} e^{-q'/x}}{\sum_{{q=2}}^{q'} c_q e^{-q/x}} \right| < 10^{-8} \] (83)
thus we have used relative error. Using such a condition (83) for terminating the
sum in (81) is necessary as $F(x)$ sometimes crosses zero and using absolute error can
be misleading at small values of $F(x)$. The last 43 points in the Fig. 10 were not
fulfilling the requirement (83) as all primes $q = m^2 + 1$ generated were exhausted,
but as it is seen in the plot of $F(x)$ in Fig. 10 values of the function $F(x)$ are well
above 1000 for $x > 10^{17}$.

The numbers presented in the Table IV and plots in the Figures 8, 9 and 10
allow us to formulate the

**Conjecture 5:** There is no Chebyshev bias for primes of the form $m^2 + 1$:
$\delta_1 = \delta_2 = \frac{1}{2}, \quad \delta_0 = 0$

It is the last conjecture formulated in this paper.

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Fig. 1 The plot of the error term for the conjecture \(|\pi_q(x) - C_q \int_2^{x} \frac{du}{u^{\frac{1}{2}} \log(u)}|\) up to \(x = 10^{20}\) on the double logarithmic axes. The plot of the difference \(|\pi_q(x) - \frac{1}{2} C_q \int_2^{x} \frac{du}{u^{\frac{1}{2}} \log(u)}|\) was made from the 101566 points. Up to \(10^{12}\) values of \(|\Delta_q(x)|\) at all 54110 primes of the form \(m^2 + 1 < 10^{12}\) are included. For larger values of \(x > 10^{12}\) the following decimation procedure was used: changes of local maxima larger than 1, all local minima and all sign changes were recorded with the provision that values corresponding to the sign changes of the \(|\Delta_q(x)|\) and smaller than \(10^{-3}\) were artificially set to \(10^{-3}\). Additionally the plot contains 18430 points recorded at the geometrical progression \(x = 10^{12} \times (1.001)^n\). In blue the power-like fit \(0.51877 \times x^{0.2139}\) to \(\omega(x)\) obtained by the least-square method is shown and in red the possible choice for the bound of \(\omega(x)\) from above is plotted.
Fig. 2 The plot of the oscillations of the \( \frac{\pi q(x) - \frac{1}{2} C_q \int_2^x \frac{du}{\sqrt{u \log u}}}{x^{1/4}} \). It consists of 106027 points and the procedure for recording values of this difference was similar to the previous one.
Fig. 3 The plot of the number of sign changes of the difference
\[ \pi_q(x) - \frac{1}{2} C \int_2^x \frac{du}{\sqrt{\log(u)}} \] up to \( x = 10^{20} \). There are 20456 data points and power-like fit was performed with respect to all points. In red the power fit obtained by the least-square method is shown.
Fig. 4 The plot of the function $\pi_q(d; x)$ for $x = 10^{10}, 10^{12}, \ldots, 10^{20}$ and $d \leq 300$. By red boxes the values of $\pi_q(d; x)/P(d)$ are shown; for clarity only every 4-th value of $d$ is shown — discarding small fluctuations they fit the green lines representing the integrals $C_1 \int_5^x \frac{dt}{\sqrt{t \log^2(t)}}$. 
Fig. 5 The plot of the histogram $h(d; x)$ for $x = 10^{10}, 10^{12}, \ldots, 10^{20}$. By red boxes the values of $h(x, d)/P(d)$ are shown; for clarity only every 4-th red box is shown.

They are supposed to lie along the green lines representing the plots of the

$$\frac{\pi_2(x)}{s\sqrt{x}} \left(1 - \frac{2\pi_2(x)}{\sqrt{x}}\right)^{d-1}.$$
Fig. 6 The plot of the function $K(x)$
Fig. 7 The plot of the difference ($\pi_q(x; 3, 2) - 2\pi_q(x; 3, 1)$). There are 102714 points: up to $10^{12}$ all primes $q_n$ are plotted, for $x > 10^{12}$ the following decimations procedure was used: all local maxima, minima and sign changes of this difference were recorded plus 18430 values at the geometrical progression $10^{12} \times (1.001)^n$. Because the amplitude grows with $x$ we have presented the interval $(10^{10}, 10^{20})$ in the inset with smaller range on the $y$-axis.
Fig. 8 The plot of the ratio $y(x)/x^{1/4} = (\pi_q(x; 3, 2) - 2\pi_q(x; 3, 1))/x^{1/4}$. This plot is made of 102714 points, exactly as previous plot.
Fig. 9 The plots of the logarithmic densities $\delta_1(x), \delta_2(x)$ and $\delta_0(x)$ defined in the text. Each plot consists of 72542 points: up to $10^{12}$ all all primes $q_n$ are plotted, for $x > 10^{12}$ the values of $\delta(x)$'s were recorded at the progression $10^{12} \times (1.001)^n$. 
Fig. 10 The plot of the function $F(x)$ defined by equation (79). There are 2021 points on these figures: the arguments $x$ were chosen as the geometrical progression $100.0 \times 1.02^n$, $n = 0, 1, \ldots, 2020$ and the largest value of $x$ is here $2.35693149 \ldots \times 10^{19}$, for which the last term in the sum (81) was roughly $e^{-10^{20}/2.35693149 \times 10^{19}} = 0.01436723 \ldots$. 