ON THE DIOPHANTINE EQUATION \((5pn^2 - 1)x + (p(p - 5)n^2 + 1)y = (pn)^z\)

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Abstract. Let \(p\) be a prime number with \(p > 3\), \(p \equiv 3 \pmod{4}\) and let \(n\) be a positive integer. In this paper, we prove that the Diophantine equation \((5pn^2 - 1)x + (p(p - 5)n^2 + 1)y = (pn)^z\) has only the positive integer solution \((x, y, z) = (1, 1, 2)\) where \(pn \equiv \pm 1 \pmod{5}\). As another result, we show that the Diophantine equation \((35n^2 - 1)x + (14n^2 + 1)y = (7n)^z\) has only the positive integer solution \((x, y, z) = (1, 1, 2)\) where \(n \equiv \pm 3 \pmod{5}\) or \(5 | n\). On the proofs, we use the properties of Jacobi symbol and Baker’s method.

1. Introduction

Let \(N\) be the set of positive integers, \(Z\) the set of integers and \(Q\) the field of rational numbers. Suppose that \(a, b, c\) are relatively prime positive integers with \(\min\{a, b, c\} > 1\). The equation
\[
a^x + b^y = c^z, \quad x, y, z \in N,
\]
(1.1)
is an important and general exponential Diophantine equation which attracts the attention of many authors.

In 1933, the first work was recorded by Mahler [18]. He proved finiteness of solutions of equation (1.1). His method is a \(p\)-adic analogue of that of Thue-Siegel, so it is ineffective in the sense that it gives no indication on the number of possible solutions. In 1940, an effective result for solutions of (1.1) was given by Gel’fond [10], who used Baker’s method, which is on lower bounds for linear form in the logarithms of algebraic numbers. If \(x = y = p\), then the equation (1.1) becomes the Catalan-Fermat equation \(x^p + y^q = z^q\) (see [7], [8]). The case \(p = 2\) is an important ingredient to the topic. Essentially this case has been solved in a quite general form by Ellenberg [8].

Some authors determined complete solutions of equation (1.1) for small values of \(a, b, c\) by using elementary congruences, the quadratic reciprocity law and the arithmetic of quadratic (or cubic) fields (see [11], [22] and [31]).

Equation (1.1) has been considered in detail for Pythagorean numbers \(a, b, c\) as well. In 1956, Jeśmanowicz [12] set a conjecture that if \(a, b, c\) are Pythagorean triples i.e. positive integers satisfying \(a^2 + b^2 = c^2\), then (1.1) has only the positive integer solution \((x, y, z) = (2, 2, 2)\). Other conjectures related to equation (1.1) were set and discussed. One is the extension of Jeśmanowicz’ conjecture due to Terai. In 1994, Terai [26], conjectured that if equation (1.1) has a solution \((x, y, z) = (p, q, r)\) with \(\min(p, q, r) > 1\), then the equation (1.1) has only the solution \((x, y, z) = (p, q, r)\). But, the years between 1999 and 2011, some authors determined the exceptional cases of Terai’s conjecture on the equation (1.1) (see [4], [5], [16], [19], [20], [27]). On the other hand, heuristics show that the equation (1.1) has at most one solution \((x, y, z)\) with \(\min(x, y, z) > 1\) (see [6], [17]). The conjecture was proved for some special cases. But, in general, the problem is still unsolved.

For the details, see the survey paper on Jeśmanowicz and Terai conjectures, which is written by Soydan et al [24].

Now we consider the Diophantine equation
\[
(am^2 + 1)^x + (bn^2 - 1)^y = (cm)^z,
\]
(1.2)
where \(a, b, c, m\) are positive integers such that \(a + b = c^2\). Recently, there are many papers discussing the solutions \((x, y, z)\) of the equation (1.2) (see [1], [9], [14], [21], [23], [24], [28]).
In this paper, we are interested in the positive integer solutions of the Diophantine equation

\[(5pn^2 - 1)^x + (p(p - 5)n^2 + 1)^y = (pm)^z, \quad (1.3)\]

where \(p \equiv 3 \pmod{4}\) and \(p > 3\) is prime. Our main theorem is the following.

**Theorem 1.1** (Main theorem). Suppose that \(n\) is a positive integer, \(p > 3\) is a prime number with \(p \equiv 3 \pmod{4}\) and \(pM \equiv \pm 1 \pmod{5}\). Then the equation \((1.3)\) has only the positive integer solution \((x, y, z) = (1, 1, 2)\).

**Corollary 1.1.1.** Suppose that \(n \equiv \pm 3 \pmod{5}\) or \(5 \mid n\). Then the exponential Diophantine equation

\[(35n^2 - 1)^x + (14n^2 + 1)^y = (7n)^z \quad (1.4)\]

has only the positive integer solution \((x, y, z) = (1, 1, 2)\).

2. **Auxiliary results**

In this section, we recall some results that will be important for the proofs of the main theorem and the corollary. We first need a result on lower bounds for linear forms in the logarithms of two algebraic numbers to obtain an upper bound for the solution \(y\) of Pillai’s equation \(u^x - v^y = u\) under some conditions. (We will give the details about Pillai’s equation in Section 3.3). Here, we first present some notations. Suppose that \(\beta_1\) and \(\beta_2\) are real algebraic numbers with \(|\beta_1| \geq 1\) and \(|\beta_2| \geq 1\). Consider the linear form

\[\Omega = b_2 \log \beta_2 - b_1 \log \beta_1,\]

where \(b_1\) and \(b_2\) are positive integers. For any non-zero algebraic number \(\gamma\) of the degree \(d\) over \(\mathbb{Q}\), whose minimal polynomial over \(\mathbb{Z}\) is \(a_0 \prod_{j=1}^{d}(X - \gamma^{(j)})\), we denote by

\[h(\gamma) = \frac{1}{d} \left[\log |a_0| + \sum_{j=1}^{d} \log \max(1, |\gamma^{(j)}|)\right]\]

its absolute logarithmic height, where \((\gamma^{(j)})_{1 \leq j \leq d}\) are conjugates of \(\gamma\). Suppose that \(B_1\) and \(B_2\) are real numbers greater than 1 with

\[\log B_1 \geq \max\{h(\gamma_i), \frac{\log \gamma_i}{D}, \frac{1}{D}\},\]

for \(i \in \{1, 2\}\), where \(D\) is the degree of the number field \(\mathbb{Q}(\gamma_1, \gamma_2)\) over \(\mathbb{Q}\). Define

\[b' = \frac{b_1}{D \log B_2} + \frac{b_2}{D \log B_1} \quad (1.5)\]

To use Laurent’s result in [15, Corollary 2], we take \(m = 10\) and \(C_2 = 25.2\).

**Proposition 2.1.** (Laurent [15]) Suppose that \(\Omega\) is given as above, with \(\gamma_1 > 1\) and \(\gamma_2 > 1\). Let \(\beta_1\) and \(\beta_2\) be multiplicatively independent. Then

\[\log |\Omega| \geq -25.2D^4 (\max\{\log b' + 0.38, \frac{10}{D}\})^2 \log B_1 \log B_2.\]

Now, we need a result of Bugeaud [3], which is based on linear forms in \(p\)-adic logarithms. Here, we take the case where \(y_1 = y_2 = 1\) in the notation of [3, p.375].

Suppose that \(p\) is a prime and let \(v_p\) denote the standard \(p\)-adic valuation normalized by \(v_p(p) = 1\). Suppose that \(\delta_1\) and \(\delta_2\) are non-zero integers prime to \(p\) and let \(h\) denote the smallest positive integer such that

\[v_p(\delta_1^h - 1) > 0, \quad v_p(\delta_2^h - 1) > 0.\]

Suppose that there exists a real number \(F\) such that \(v_p(\delta_1^h - 1) \geq F > \frac{1}{p^h}\). Bugeaud [3] obtained explicit upper bounds for the \(p\)-adic valuation of

\[\Omega = \delta_1^{b_1} - \delta_2^{b_2},\]

where \(b_1\) and \(b_2\) are positive integers. We have
Proposition 2.2. (Bugeaud [3]) Let $B_1 > 1, B_2 > 1$ be real numbers such that
$$\log B_i \geq \max \{\log |\delta_i|, F \log p\}, \ i = 1, 2,$$
and we put
$$b' = \frac{b_1}{\log B_2} + \frac{b_2}{\log B_1}.$$
If $\delta_1$ and $\delta_2$ are multiplicatively independent, then we have the upper estimates
$$v_p(\Omega) \leq \frac{36.1 h}{F^3(\log p)^4} (\max \{\log b' + \log(F \log p) + 0.4, 6F \log p, 5\})^2 \log B_1 \log B_2.$$

3. Proof of Theorem 1.1

In this section, we prove Theorem 1.1.

Suppose that $(x, y, z)$ is a solution of (1.3). Considering (1.3) modulo $p$ implies that $(-1)^x + 1 \equiv 0 \pmod{p}$. Therefore $x$ is odd.

3.1. The case where $n$ is odd with $n \not\equiv 0 \pmod{5}$.

Lemma 3.1. Let $n$ be odd and $pn \equiv \pm 1 \pmod{5}$. If the equation (1.3) has a positive integer solution, then $x = 1$.

Proof. Assume that $pn \equiv \pm 1 \pmod{5}$. If $x > 2$, then we desire to get a contradiction. We consider the proof in two cases: Case 1: $n \equiv 3 \pmod{4}$, Case 2: $n \equiv 1 \pmod{4}$.

Case 1: $n \equiv 3 \pmod{4}$. Then considering (1.3) modulo 4 implies that $3^y \equiv 1 \pmod{4}$, then $y$ is even.

Case 2: $n \equiv 1 \pmod{4}$. Then $$(5pn^2 - 1)(p(p - 5)n^2 + 1) = 1$$ and considering (1.3) modulo 5, one obtains
$$(-1) + (\pm 1) \equiv 1 \pmod{5}$$
which is a contradiction. Therefore we get $x = 1$. □
3.2. The case \( n \) is even.

**Lemma 3.2.** Let \( n \) be even. Then the equation (1.3) has only the positive integer solution \((x, y, z) = (1, 1, 2)\).

**Proof.** If \( z \leq 2 \), we obtain \((x, y, z) = (1, 1, 2)\) by (1.3). Therefore we may assume that \( z \geq 3 \). Considering (1.3) modulo \( n^3 \) implies that

\[
5pn^2 x - 1 + p(p - 5)n^2 y + 1 \equiv 0 \pmod{n^3},
\]

so

\[
5px + p(p - 5)y \equiv 0 \pmod{n^3},
\]

which is impossible, as \( x \) is odd and \( n \) is even. Hence, we get that since \( n \) is even, the equation (1.3) has only the positive integer solution \((x, y, z) = (1, 1, 2)\). This completes the proof of Lemma 3.2. \( \square \)

3.3. **Pillai’s equation** \( w^z - v^y = u \). By Lemma 3.1, we get \( x = 1 \) from (1.3), satisfying \( n \) is odd with \( pn \equiv \pm 1 \pmod{5} \). If \( y \leq 2 \), then we obtain \( y = 1 \) and \( z = 2 \) from (1.3). So, we may assume that \( y \geq 3 \).

Thus, our theorem is reduced to solving Pillai’s equation

\[
w^z - v^y = u \tag{3.2}
\]

with \( y \geq 3 \), where \( u = 5pn^2 - 1, v = p(p - 5)n^2 + 1 \) and \( w = pn \).

We first want to determine an upper bound for \( y \).

**Lemma 3.3.** \( y < 2521 \log w \).

**Proof.** By (3.2), we now consider the following linear form in two logarithms:

\[
\Omega = z \log w - y \log v (> 0).
\]

By the inequality \( \log(1 + k) < k \) for \( k > 0 \), we obtain

\[
0 < \Omega = \log \left( \frac{w^z}{v^y} \right) = \log(1 + \frac{u}{v^y}) < \frac{u}{v^y}.
\]

So we get

\[
\log \Omega < \log u - y \log v. \tag{3.4}
\]

On the other hand, using Proposition 2.1, we want to find a lower bound for \( \Omega \). We obtain the following inequality

\[
\log \Omega \geq -25.2(\max\{\log b' + 0.38, 10\})^2(\log v)(\log w), \tag{3.5}
\]

where \( b' = \frac{y}{\log w} + \frac{z}{\log v} \).

Note that \( v^{y+1} > w^z \). Indeed,
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\[
v^y + 1 - w^z = v(w^z - u) - w^z = (v - 1)w^z - uv \geq p(p - 5)n^2p^2n^2
\]
\[-(5pn^2 - 1)(p(p - 5)n^2 + 1) > 0.
\]

Thus \(b' < \frac{2y + 1}{\log w}\)

Set \(N = \frac{y}{\log w}\). Using the inequalities \(3.4\) and \(3.5\), we obtain

\[
y \log v < \log u + 25.2(\max\{\log(2N + 1) + \log w, 0.38, 10\})^2(\log v)(\log w).
\]

Since \(\log w = \log(pm) \geq \log 21 > 1\), we get

\[
y \log v < \log u + 25.2(\max\{\log(2N + 1) + \log w, 0.38, 10\})^2(\log v)(\log w),
\]

so

\[
N < 1 + 25.2(\max\{\log(2N + 1) + \log w, 0.38, 10\})^2
\]

since \(\frac{\log u}{(\log v)(\log w)} < 1\). Therefore we have \(N < 2521\). Thus, the proof Lemma 3.3 is completed.

Next, we want to determine a lower bound for \(y\).

**Lemma 3.4.** \(y > n^2 - 2\).

**Proof.** Using equation \(3.2\), we obtain the following inequality:

\[
(pm)^2 \geq 5pn^2 - 1 + (p(p - 5)n^2 + 1)^3 > (pn)^3,
\]

since \(y \geq 3\). Thus \(z \geq 4\). Considering \(3.2\) modulo \(p^2n^4\) implies that

\[
5pn^2 - 1 + p(p - 5)n^2y + 1 \equiv 0 \pmod{p^2n^4}.
\]

This shows that \(5 + (p - 5)y \equiv 0 \pmod{p^2n^4}\). Thus one gets

\[
y \geq \frac{pm^2 - 5}{p - 5} = \frac{p}{p - 5}n^2 - \frac{5}{p - 5} > n^2 - 2,
\]

which completes the proof.

Now we are ready to prove Theorem 1.1. By Lemmas 3.3 and 3.4, we obtain

\[
n^2 - 2 < 2521\log(pn).
\]

We want to determine an upper bound for \(p\) and then one for \(n\). Firstly, we will show that if \(n \equiv 1 \pmod{4}\), then \(p < 6307\). We proved that \(z\) is even when \(n \equiv 1 \pmod{4}\). So set \(z = 2t\) with \(t\) positive integer. Then, we determine the equation \(3.2\) as follows:

\[
(u^2)^t - v^y = w^z - v.
\]

Thus \(y \geq t\). If \(y = t\), then we get \(y = t = 1\). If \(y > t\), then we consider a “gap” between the trivial solution \((y, t) = (1, 1)\) and (possibly) another solution \((y, t)\).

Using \(u + v = w^z\) and \(u + v^y = w^{2t}\), we consider the following two linear forms in two logarithms:

\[
\Omega_0 = 2\log w - \log v(> 0), \quad \Omega = 2t \log w - y \log v(> 0).
\]

Hence

\[
y\Omega_0 - \Omega = 2(y - t) \log w \geq 2\log w,
\]

which implies that

\[
y > \frac{2}{\Omega_0} \log w.
\]
Using Lemma 3.3, we obtain \( \frac{2}{\Omega_0} \log w < 2521 \log w \). So,

\[
\frac{2}{2521} < \Omega_0 = \log \left( \frac{w^2}{v} \right) = \log \left( 1 + \frac{u}{v} \right) < \frac{u}{v} = \frac{5pn^2 - 1}{p(p - 5)n^2 + 1} < \frac{5pn^2}{p(p - 5)n^2}
\]

\[
= \frac{5}{p - 5}.
\]

From here, one gets \( p < 6307 \). And hence by (3.6), we obtain \( n \leq 187 \). Secondly, we will show that if \( n \equiv 3 \pmod{4} \), then \( p < 12610 \). We proved that \( y \) is even when \( n \equiv 3 \pmod{4} \) and we know that \( y > 1 \). If \( z \geq 2y \), then by (3.2) we get

\[
5pn^2 - 1 = w^z - v^y \geq w^{2y} - v^y > w^2 - v = p^2n^2 - (p^2n^2 - 5pn^2 + 1) = 5pn^2 - 1,
\]

which is a contradiction. So, \( z > 2y \).

Using \( u + v = w^2 \) and \( u + v^y = w^z \), we consider the following two linear forms in two logarithms:

\[
\lambda_0 = 2 \log w - \log v > 0, \quad \lambda = z \log w - y \log v > 0.
\]

Therefore,

\[
y \lambda_0 - \lambda = (2y - z) \log w \geq \log w,
\]

which implies that

\[
y > \frac{1}{\lambda_0} \log w.
\]

By Lemma 3.3, we get \( \frac{1}{\lambda_0} \log w < 2521 \log w \). Thus,

\[
\frac{1}{2521} < \lambda_0 = \log \left( \frac{w^2}{v} \right) = \log \left( 1 + \frac{u}{v} \right) < \frac{u}{v} = \frac{5pn^2 - 1}{p(p - 5)n^2 + 1} < \frac{5pn^2}{p(p - 5)n^2}
\]

\[
= \frac{5}{p - 5}.
\]

From here, one obtains \( p < 12610 \). And hence by (3.6), we have \( n \leq 192 \).

By (3.3), we obtain the inequality

\[
\left| \frac{\log v - \frac{z}{y}}{\log w} \right| < \frac{u}{yv^y \log w},
\]

which implies that \( \left| \frac{\log v}{\log w} - \frac{z}{y} \right| < \frac{1}{2y^y} \) since \( y \geq 3 \). So \( \frac{z}{y} \) is a convergent of the continued fraction expansion of \( \frac{\log v}{\log w} \).

On the other hand, if \( \frac{p_r}{q_r} \) is the \( r \)-th such convergent, then

\[
\left| \frac{\log v}{\log w} - \frac{p_r}{q_r} \right| > \frac{1}{(u_{r+1} + 2)q_r^2},
\]

where \( u_{r+1} \) is the \( (r + 1) \)-st partial quotient to \( \frac{\log v}{\log w} \) (see e.g. Khinchin [13]). Set \( \hat{y} = \frac{p_r}{q_r} \). Note that \( q_r \leq y \). Then it follows that

\[
u_{r+1} > \frac{v^y \log w}{uy} - 2 \geq \frac{v^y \log w}{uq_r} - 2.
\]

Finally, running MAGMA [2] for each \( p < 12610 \) with \( p \equiv 3 \pmod{4} \), it is seen that the above inequality is not satisfied for any \( r \) with \( q_r < 2521 \log(pm) \) in the range \( 1 \leq n \leq 192 \). Thus, the proof of Theorem 1.1 is completed.
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4. Proof of Corollary 1.1.1

Suppose that $(x, y, z)$ is a solution of (1.4). By Theorem 1.1, we may assume that $n \equiv 0 \pmod{5}$. We know that $x$ is odd. Here, we use Proposition 2.2. So, put $p := 5, \delta_1 := 14n^2+1, \delta_2 := 1-35n^2, b_1 := y, b_2 := x$ and

$$\Omega := (14n^2+1)y - (1-35n^2)x.$$ 

Then we may take $h = 1, F = 2, B_1 = 14n^2+1, B_2 = 35n^2-1$. Therefore, we obtain

$$z \leq \frac{36.1}{8(\log 5)^4}(\max\{\log b' + \log(2 \log 5) + 0.4, 12 \log 5, 5\})^2$$

$$\times \log(14n^2+1) \log(35n^2-1),$$

where $b' := \frac{y}{\log(35n^2-1)} + \frac{x}{\log(14n^2+1)}$. Assume that $z \geq 4$. We show that this will lead to a contradiction. Considering (1.4) modulo $n^4$, we get

$$35x + 14y \equiv 0 \pmod{n^2}.$$ 

Particularly, we obtain $N := \max\{x, y\} \geq \frac{n^2}{36}$. Hence, since $z \geq N$ and $b' \leq \frac{N}{\log n}$, we find

$$N \leq \frac{36.1}{8(\log 5)^4}(\max\{\log(\frac{N}{\log n}) + \log(2 \log 5) + 0.4, 12 \log 5, 5\})^2$$

$$\times \log(14n^2+1) \log(35n^2-1).$$

If $n \geq 173356$, then

$$N \leq 0.68(\log(\frac{N}{\log n}) + \log(2 \log 5) + 0.4)^2 \log(14n^2+1) \log(35n^2-1).$$

When $n^2 \leq 49N$, from the above inequality, we have

$$N \leq 0.68(\log N - \log(\log(173356))) + 1.57)^2 \log(686N + 1) \log(1715N - 1).$$

Therefore, we get $N \leq 13732$, which contradicts the fact that $N \geq n^2/49 \geq 6.133123007 \times 10^8$.

If $n < 173356$, then using inequality (4.1), we obtain

$$\frac{n^2}{49} \leq \frac{36.1}{8(\log 5)^4}(12 \log 5)^2 \log(14n^2+1) \log(35n^2-1),$$

namely,

$$\frac{n^2}{49} \leq 250.8 \log(14n^2+1) \log(35n^2-1).$$

From the above inequality, we find $n \leq 2031$. Therefore all $x, y$ and $z$ are also bounded. Using MAGMA [2], we see that the eq. (1.4) has no solution $(n, x, y, z)$ with $5 | n$. So, we conclude $z \leq 3$. In this case, we can easily show that $(x, y, z) = (1, 1, 2)$. Thus, the proof of Corollary 1.1.1 is completed.

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ON THE DIOPHANTINE EQUATION \((5pn^2 - 1)x + (p(p - 5)n^2 + 1)y = (pn)^2\)

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