An Ariki-Koike like extension of the 
Birman-Murakami-Wenzl Algebra

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Abstract

We introduce an Ariki-Koike like extension of the Birman-Murakami-Wenzl Algebra and show it to be semi-simple. This algebra supports a faithful Markov trace that gives rise to link invariants of closures of Coxeter type B braids.

1 Introduction

The theory of quantum invariants of links nowaday rests on a broad theory that includes quantum groups, their centraliser algebras and tensor categories. It is the ultimate goal of the 'Knot Theory and Root Systems' programme initiated in [3] to carry over this theory to the braid groups associated to the other root systems. The greatest progress sofar has been taken for the braid group of Coxeter type B where the notions of quasi triangular Hopf algebra and monoidal categories have been defined and nontrivial examples have been found [6], [7], [9]. Furthermore, Temperley-Lieb algebras and Hecke algebras have been studied intensively for this root system. In the present paper we continue the study of generalisations of the Birman-Murakami-Wenzl algebra [17], [11].

Every Coxeter diagram defines a braid group that is an infinite covering of its Coxeter group. The braid group ZB_n of Coxeter type B has generators τ_i, i = 0, 1, . . ., n − 1. Generators τ_i, i ≥ 1 satisfy the relations of Artin’s braid group (which is the braid group of Coxeter type A): τ_iτ_j = τ_jτ_i if |i − j| > 1, and τ_iτ_jτ_i = τ_jτ_iτ_j if |i − j| = 1. The generator τ_0 has relations

\[ τ_0τ_iτ_0 = τ_iτ_0τ_i \quad (1) \]
\[ τ_0τ_i = τ_iτ_0 \quad i \geq 2 \quad (2) \]
The braid group $\mathbb{Z}B_n$ may be graphically interpreted (cf. figure 1) as symmetric braids or cylinder braids: The symmetric picture shows it as the group of braids with $2n$ strands (numbered $-n, \ldots, -1, 1, \ldots, n$) which are fixed under a 180 degree rotation about the middle axis. In the cylinder picture one adds a single fixed line (indexed 0) on the left and obtains $\mathbb{Z}B_n$ as the group of braids with $n$ strands that may surround this fixed line. The generators $\tau_i, i \geq 0$ are mapped to the diagrams $X_i^{(G)}$ given in figure 1.

More generally, tangles of B-type may be defined. The special case of tangles without crossings is the B-type Temperley-Lieb algebra $TB_n$ that has been introduced by tom Dieck in [3].

The Ariki-Koike Algebra is the quotient of the group algebra of $\mathbb{Z}B_n$ where the images $X_i$ of the generators $\tau_i$ for $i \geq 1$ fulfil quadratic relations while $X_0$ satisfies a polynomial of arbitrary degree. The Hecke Algebra of B type is a special case where $X_0$ satisfies also a quadratic relation.

The standard Birman-Murakami-Wenzl algebra $BA_n$ of type A imposes cubic relations on its generators in a way that enables its interpretation as an algebra of tangles with a skein relation that comes from the Kauffman polynomial.

Thus it is natural to define an Ariki-Koike like extension of the BMW algebra $BB^k_n$ that contains a generator $Y$ as image of $\tau_0$ that satisfies $\prod_{i=0}^{k-1}(Y - p_i) = 0$. The special case $k = 2$ has been called restricted B-type BMW algebra and has been studied in [11].

The current interest in the study of B type braid groups has several origins. Closing B type braids yields links that can be interpreted as links in a solid torus [12] and Markov traces on group algebras of $\mathbb{Z}B_n$ hence allow the calculations of invariants of such links (cf. end of section 8). Braid groups of all finite root systems further act as symmetries on the corresponding quantum groups [13]. The B braid group occurs furthermore in several physical situations [8], [10]. The general idea is that the B type braids allow to treat with knot theoretic methods also physical models with a boundary. The $\tau_0$ generator is interpreted as a reflection at the boundary.

We now outline the structure of the paper and point out the main results. After a short review of the Birman-Wenzl algebra of A-type we go on to define the Ariki-Koike-Birman-Murakami-Wenzl algebra of B-type $BB^k_n$ in section 2 and list a number of fundamental relations. They are used extensively in section 5 to determine a partial normal form of words in $BB^k_n$. Section 3 shows how to obtain the Ariki-Koike algebra as a quotient of $BB^k_n$. Furthermore, it investigates the B type Temperley Lieb sub-algebra.

Section 6 introduces the graphical interpretation of our algebra and studies its classical limit. The construction of a Markov trace fills section 7.

The main theorem of this paper is contained in section 8. We prove that $BB^k_n$ is semi-simple in the generic case and show how its simple components can be enumerated in terms of Young diagrams. The Bratteli diagram is given and we show that the Markov trace is faithful. As an application a generalisation of the Kauffman polynomial to links in the solid torus is discussed.

**Algebraic preliminaries:**

We collect some simple results from algebra that will be needed later on. Our first topic is the specialisation of the ground ring of an algebra. Let $R$ and $R'$
be rings and let $\pi : R \rightarrow R'$ denote an epimorphism. Let $S$ be a set that is considered as a set of generators. The set of words over this alphabet is denoted by $S^*$. The free $R$ module with basis $S^*$ is denoted by $\langle S \rangle_R$. We then have an induced mapping

$$\pi_* : \langle S \rangle_R \rightarrow \langle S \rangle_{R'}, \quad \sum_i r_i w_i \mapsto \sum_i \pi(r_i) w_i, \quad r_i \in R, w_i \in S^*.$$  

An alternative description of $\pi_*$ is possible in terms of tensor products by viewing $R'$ to be a $R$ module (with the action given by $\pi$). We then have $\langle S \rangle_R \otimes_R R' = \langle S \rangle_{R'}$. Now, let $M = \{ \sum_j m_{ij} w_j \mid m_{ij} \in R, w_j \in S^* \}$ be a set of relations and $\overline{M}$ the ideal generated by these elements in $\langle S \rangle_R$. The projection is denoted by $p : \langle S \rangle_R \rightarrow \langle S \rangle_R/\overline{M}, a \mapsto a + \overline{M}$. Furthermore, let $\overline{\pi_*} = \langle \pi_* \rangle_{R'/\overline{M}}$ be the ideal generated by $\pi_*(M)$ and denote by $p'$ the corresponding projection. It follows that

$$\pi_* p = p' \pi_*.$$  

We now turn to results about the construction of integral domains.

**Proposition 1** Let $K$ be an infinite field. An ideal $I \subset K[x_1, \ldots, x_n]$ is prime if, and only if its affine variety $V(I)$ is irreducible.
$V(I)$ is irreducible if it can be rationally parametrised

$$x_i = \frac{f_i(t_1, \ldots, t_m)}{g_i(t_1, \ldots, t_m)}, \quad i = 1, \ldots, n$$

Here $f_i, g_i$ denote polynomials.

**Corollary 2** Let $K$ be an infinite field. Assume the ideal $I \subset K[x_1, \ldots, x_n]$ is generated by $m$ polynomials $h_i$ which may in the field of fractions $K(x_1, \ldots, x_n)$ be solved uniquely for $x_1, \ldots, x_m$:

$$x_i = \frac{f_i(x_{m+1}, \ldots, x_n)}{g_i(x_{m+1}, \ldots, x_n)}, \quad i = 1, \ldots, m$$

Then $I$ is prime.

**Proposition 3** Let $K$ denote an algebraically closed field and let $\sigma_i, i = 1, \ldots, n$ be the elementary symmetric polynomials in $K[x_1, \ldots, x_n]$. Then the system of equations $\sigma_i = y_i$ over $K[x_1, \ldots, x_n, y_1, \ldots, y_n]$ can be solved for the $x_i$.

## 2 The Definition of the Ariki-Koike-Birman-Murakami-Wenzl-Algebra

This section introduces a generalisation of the Birman-Murakami-Wenzl that is related to the B-type braid group. Because the algebras of Ariki and Koike appear as quotients we call our algebra an Ariki-Koike-BMW algebra. We set off by recalling the definition of the ordinary BMW algebra.

**Definition 4** Let $R$ denote an integral domain with units $x, \lambda \in R$ such that with a further element $\delta \in R$ the relation $(1 - x)\delta = \lambda - \lambda^{-1}$ holds. The Birman-Murakami-Wenzl (BMW) algebra $BA_n(R)$ is generated by $X_1, \ldots, X_{n-1}, e_1, \ldots, e_{n-1}$ and relations:

\[
\begin{align*}
X_i X_j &= X_j X_i, \quad |i - j| > 1 & (3) \\
X_i X_j X_i &= X_j X_i X_j, \quad |i - j| = 1 & (4) \\
x_i e_i &= e_i X_i = \lambda e_i & (5) \\
e_i X_j^\pm e_i &= \lambda^\mp e_i, \quad |i - j| = 1 & (6) \\
e_i^2 &= x e_i & (7) \\
X_i^{-1} &= X_i - \delta + \delta e_i & (8) \\
e_i e_j &= e_j e_i, \quad |i - j| > 1 & (9) \\
e_i X_j X_i &= X_j^\pm X_i^\pm e_j, \quad |i - j| = 1 & (10) \\
e_i e_j e_i &= e_i, \quad |i - j| = 1 & (11)
\end{align*}
\]
Lemma 5

\[ X_i^2 = 1 + \delta X_i - \delta \lambda e_i \] (12)
\[ X_i^3 = X_i^2(\lambda + \delta) + X_i(1 - \lambda \delta) - \lambda \] (13)
\[ X_i^{-2} = 1 + \delta^2 - \delta X_i + \delta(\lambda - 1 - \delta) e_i = 1 - \delta X_i^{-1} + \delta \lambda^{-1} e_i \] (14)
\[ X_i X_i^{\pm 1} X_i = X_j X_i^{\pm 1} X_j^{-1} \quad |i - j| = 1 \] (15)
\[ X_i^{\pm 1} e_j e_i = X_j^{\pm 1} e_i \quad |i - j| = 1 \] (16)
\[ e_i e_j X_i^{\pm 1} = e_i X_j^{\pm 1} \quad |i - j| = 1 \] (17)
\[ e_i X_j^\pm e_i = e_i e_j \quad |i - j| = 1 \] (18)
\[ X_i^{\pm} X_j e_i = e_j e_i \quad |i - j| = 1 \] (19)
\[ X_i e_j X_i^{-1} = X_j^{-1} e_i X_j \quad |i - j| = 1 \] (20)
\[ X_i e_j X_i = X_j^{-1} e_i X_j^{-1} \quad |i - j| = 1 \] (21)

Proof: (12)-(14) are simple restatements of (8).

(15): \[ X_i X_j X_i = X_j X_i X_j \Rightarrow X_j X_i X_j^{-1} = X_i^{-1} X_j X_i \Rightarrow X_i X_j X_i^{-1} = X_j^{-1} X_i^{-1} X_j \]
(16): \[ X_i^{\pm} e_j e_i = X_j^{\pm} X_i^{\pm} e_j e_i \quad |i - j| = 1 \]
(17): \[ e_i e_j X_i^{\pm} = e_j e_i X_j^{\pm} X_i^{\pm} = e_i X_j^{\pm} e_j e_i X_j^{\pm} = e_j e_i X_j^{\pm} \]
(18): Using (11), (13) and (14) we have

\[ e_i X_j^{\pm} = e_i e_j e_i X_j^{\pm} X_i^{\pm} = e_i X_j^{\pm} X_i^{\pm} e_j e_i X_j^{\pm} = e_i X_j^{\pm} X_i^{\pm} X_j^{\pm} X_i^{\pm} = e_i e_j \]

(19) is shown similarly to (18). (20)(21) follow from (14).

Lemma 6 If \( \delta \) is a unit in \( R \) the algebra \( BA(R) \) is isomorphic to the algebra generated by invertible \( X_1, \ldots, X_{n-1} \) and relations (3)-(4). The element \( e_i \) is now defined by

\[ e_i := 1 - \frac{X_i - X_i^{-1}}{\delta} \quad i = 1, \ldots, n - 1 \] (22)

Proof: (7): \[ e_i^2 = (1 - \delta^{-1}(X_i - X_i^{-1}))e_i = e_i - \delta^{-1}(\lambda e_i - \lambda^{-1} e_i) = xe_i \]
(8): follows from (3) using (22)
(10): from (22) and (13)
(11): The middle \( e_j \) is replaced by (22):

\[ e_i e_j e_i = xe_i - \delta^{-1}(e_i X_j e_i - e_i X_j^{-1} e_i) = (1 - \delta^{-1}(\lambda - \lambda^{-1}))(e_i - \delta^{-1}(\lambda^{-1} e_i - \lambda e_i) = e_i \]

We now define our generalised algebra.
Definition 7 Fix $k \in \mathbb{N}$ and let $x, \lambda, \kappa, p_0, \ldots, p_{k-1} \in R$ be units and let $\delta, A_1, \ldots, A_{k-1} \in R$ be some further elements. Assume that the relation $(1-x)\delta = \lambda - \lambda^{-1}$ holds. The Ariki-Koike-BWM-Algebra on $n$ strands $BB^k_n(R)$ is defined as $R$ algebra generated by $Y, X_1, \ldots, X_n, e_1, \ldots, e_{n-1}$ and the relations of the Birman-Murakami-Wenzl-Algebra $BA_n$ and

\begin{align*}
X_1 Y_1 Y &= Y X_1 Y X_1 \\
Y X_i &= X_i Y & i > 1 \\
Y X_1 Y e_1 &= \kappa e_1 \\
0 &= \prod_{i=0}^{k-1} (Y - p_i) \\
e_1 Y^i e_1 &= A_i e_1 & i = 1, \ldots, k - 1
\end{align*}

Relation (27) suggests to define $A_0 := x$.

These relations are motivated by our intended graphical interpretation. Section 3 will give precise definitions of the graphical version of the algebra. Here we only shed some light on the interpretation of the relations. (23) is the four braid relation (1) which is visualised in figure 2. Relation (24) stems from the braid group as well. Relation (25) is visualised in figure 3. The graphical calculus suggests to take either $\kappa = 1$ or $\kappa = \lambda$ (depending on the precise ribbon graph which $Y$ should represent). Disconnected components of a graph may be eliminated using (27). Finally, (26) is motivated by algebraic considerations.

\begin{figure}
\centering
\includegraphics[width=0.5\textwidth]{figure3.png}
\caption{Relation (23) (on the left) and relation (27) (on the right) in the cylinder picture}
\end{figure}

The generic ground ring for our algebra is a quotient of a Laurent polynomial ring. We denote by $R[x]$ the polynomial ring and by $R\{x\}$ the Laurent ring in $x$ over $R$.

$$R_0 := C\{\delta, A_1, \ldots, A_{k-1} \} \{p_0, \ldots, p_{k-1}, x, \kappa, \lambda\}/(x\delta - \delta - \lambda^{-1} + \lambda)$$

The ring’s dependence on $k$ is not written explicitly.

Remark 1 There is an involution of $BB^k_n(R_0)$ as a $C$ algebra such that

$$X_i^* := X_i^{-1}, e_i^* := e_i, Y^* := Y^{-1}, \delta^* := -\delta, x^* := x, \lambda^* := \lambda^{-1}, p_i^* := p_i^{-1}, \kappa^* := \kappa^{-1}$$

$A_i^*$ has to be calculated from $e_1 Y^{-i} e_1 = A_i^* e_1$ using the following formulas.
Definition 8 Let \( q_{k-1}, \ldots, q_0 \) be the signed elementary symmetric polynomials in \( p_0, \ldots, p_{k-1} \) such that:

\[
Y^k = \sum_{i=0}^{k-1} q_i Y^i \quad (30)
\]

Note that \( q_0 = (-1)^{k-1} \prod_i p_i \) is invertible. We calculate \( Y^{-1} \):

\[
Y^{-1} = \sum_{i=0}^{k-1} \bar{q}_i Y^i \quad \text{with} \quad \bar{q}_{k-1} = q_0^{-1} \quad \bar{q}_{i-1} = -q_i q_0^{-1} \quad (31)
\]

The coefficients are determined uniquely if the \( Y^i \) are linearly independent.

Iterating one obtains expressions \( \bar{Q}_{i,j} \) such that:

\[
Y^{-i} = \sum_{j=0}^{k-1} \bar{Q}_{i,j} Y^j \quad (32)
\]

Acting with the involution * one obtains

\[
Y^i = \sum_{j=0}^{k-1} Q_{i,j} Y^{-j} \quad \text{with} \quad Q_{i,j} = \bar{Q}^*_{i,j} \quad (33)
\]

The following definitions will prove useful later on.

\[
Y_i := X_{i-1}X_{i-2} \cdots X_1 \cdot Y \cdot X_1^{-1} \cdots X_{i-2}X_{i-1}^{-1} \quad (34)
\]

\[
Y_i^{(m)} := X_{i-1}X_{i-2} \cdots X_1 \cdot Y^m \cdot X_1 \cdots X_{i-2}X_{i-1} \quad (35)
\]

\[
Y_i' := Y_i^{(1)} = X_{i-1}X_{i-2} \cdots X_1 \cdot Y \cdot X_1 \cdots X_{i-2}X_{i-1} \quad (36)
\]

The next lemma collects a stock of relations that show among other things that the most important properties of \( Y \) can be shifted to other strands.

**Lemma 9**

\[
Y_i^k = \sum_{j=0}^{k-1} q_j Y_i^j \quad (37)
\]

\[
Y_i^{-1} = \sum_{j=0}^{k-1} \bar{q}_j Y_i^j \quad (38)
\]

\[
0 = [X_1 Y \cdot X_1, \{ Y, e_1, X_1 \}] \quad (39)
\]

\[
0 = [Y_i^j, X_j] = [Y_i, e_j] \quad j \neq i, i-1 \quad (40)
\]

\[
0 = [Y_i^{(m)}, X_j] = [Y_i^{(m)}, e_j] \quad j \neq i, i-1 \quad (41)
\]

\[
Y_i^{(m)} X_{i+1} = X_i Y_i^{(m)} \quad (42)
\]

\[
Y_{i+1} X_i = X_i Y_i \quad (43)
\]

\[
X_i Y_i X_i = Y_i X_i Y_i X_i \quad (44)
\]
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\[ X_i Y_i' X_i Y'_i = Y_i' X_i Y_i' X_i \] (45)

\[ \kappa e_i = e_i Y_i X_i = Y_i X_i e_i \] (46)

\[ \kappa e_i = e_i Y_i' X_i Y'_i = Y_i' X_i Y'_i e_i \] (47)

\[ e_i Y_m^j e_i = A_m e_i \] (48)

\[ Y_i Y_{j-1}^{-1} = Y_{i-1}^{-1} X_{i-1}^{-1} X_i Y_i^{-1} \] (50)

\[ Y_i e_{i-1} = \kappa \lambda^{-1} Y_i^{-1} e_{i-1} \] (51)

\[ e_{i-1} Y_i = \kappa \lambda e_{i-1} Y_i^{-1} - \lambda \delta e_{i-1} Y_{i-1} + \delta \lambda A_i e_1 \] (52)

\[ e_{i-1} Y_i' = \kappa \lambda e_{i-1} Y_i'^{-1} \] (53)

\[ Y'_i e_{i-1} = \kappa \lambda Y_i'^{-1} e_{i-1} \] (54)

\[ X_i Y_{i+1} = X_i Y_i - \delta Y_i + \delta Y_i e_i + \delta Y_{i+1} - \kappa \delta e_i Y_i^{-1} + \delta^2 \lambda e_i Y_i - \delta^2 \lambda A_i e_i \] (55)

\[ Y_i X_i' = X_i Y_i' \] (56)

\[ e_i Y_i^j X_i = \kappa \lambda e_i Y_i'^{-1} X_i Y_i'^{-1} - \kappa \delta e_i Y_i'^{-2} + \kappa \delta \lambda A_{i-1} e_i Y_i'^{-1} \] (57)

\[ X_i Y_i'^j e_i = \kappa \lambda Y_i'^{-1} X_i Y_i'^{-1} e_i - \delta \kappa Y_i'^{-2} e_i + \delta \lambda A_{i-1} Y_i'^{-1} e_i \] (58)

Proof:

(39): Using (23), one has \(X_1 X_1 Y X_1 Y = X_1 Y X_1 Y X_1\) and hence \(X_1 Y X_1 Y\) commutes with \(X_1\). Thus it also commutes with \(X_1^{-1}\) and \(e_1\).

(40)-(44): For \(j \geq i + 1\) commutativity follows from (32)-(44) and for \(j \leq i - 1\) it is an application of (4).

(42) and (43) are trivial. (44), (45) are shown by induction. The induction step for (44) reads:

\[ Y_i X_i Y_i X_i = X_{i-1} Y_{i-1} X_{i-1}^{-1} X_i X_{i-1}^{-1} Y_{i-1}^{-1} X_i Y_i \]
\[ = X_{i-1} Y_{i-1} X_i X_{i-1}^{-1} Y_{i-1}^{-1} Y_{i-1} X_i \]
\[ = X_{i-1} X_i Y_{i-1} X_i X_{i-1}^{-1} Y_{i-1} X_i \]
\[ = X_{i-1} X_i Y_{i-1} X_{i-1}^{-1} X_i X_{i-1}^{-1} X_i \]
\[ = X_{i-1} X_i Y_{i-1} X_{i-1}^{-1} Y_{i-1} X_i \]
\[ = X_{i-1} X_i Y_{i-1} X_{i-1}^{-1} Y_{i-1} X_i \]
\[ = X_{i-1} Y_{i-1} X_{i-1}^{-1} X_i X_{i-1}^{-1} X_i \]
\[ = X_{i-1} Y_{i-1} X_{i-1}^{-1} X_i X_{i-1}^{-1} X_i \]
\[ = X_i Y_i X_i Y_i \]

The induction step for (45) is almost identical.

(46)-(47): The inductive proofs start from (25) and its mirror version:

\[ \lambda e_1 Y X_i Y = e_1 X_1 Y X_i Y \] (48)  \[ X_i Y X_1 Y e_1 = \lambda Y X_1 Y e_1 = \kappa \lambda e_1 \]
In the induction step for (46) equation (10) is used to eliminate $e_{i+1}$ in terms of $e_i$:

\[
Y^{(1)}_{i+1}X_{i+1}Y^{(1)}_{i+1}e_{i+1} = X_iY^{(1)}_iX_{i+1}X_iY^{(1)}_iX_i^{-1}X_{i+1}^{-1}e_iX_{i+1}X_i
\]
\[
= X_iY^{(1)}_iX_{i+1}X_iX_{i+1}Y^{(1)}_iX_i^{-1}e_iX_{i+1}X_i
\]
\[
= X_iX_{i+1}Y^{(1)}_iX_{i+1}X_{i+1}^{-1}Y^{(1)}_iX_{i+1}X_i
\]
\[
= X_iX_{i+1}Y^{(1)}_iX_{i+1}Y^{(1)}_iX_{i+1}X_i = \kappa X_iX_{i+1}e_iX_{i+1}X_i = \kappa e_{i+1}
\]

The induction step for (17) is:

\[
e_{i+1}Y_{i+1}X_{i+1}X_{i+1}Y_{i+1} = e_{i+1}X_iY_iX_i^{-1}X_{i+1}Y_iX_i^{-1} = e_{i+1}X_iY_iX_{i+1}X_iX_{i+1}^{-1}Y_iX_i^{-1}
\]
\[
= e_{i+1}X_iX_{i+1}Y_iY_iX_i^{-1}X_i^{-1} = X_iX_{i+1}e_iY_iY_iX_{i+1}X_i^{-1}
\]
\[
= \kappa X_iX_{i+1}e_iX_{i+1}^{-1}X_i^{-1} = \kappa e_{i+1}
\]

(48): The proof is by induction.

\[
e_iY^m e_i = e_iX_{i-1}Y^m_{i-1}X_{i-1}^{-1}e_i = e_iX_{i-1}Y^m_{i-1}X_{i-1}^{-1}e_i
\]
\[
= e_iX_{i-1}Y^m_{i-1}X_{i-1}^{-1}X_{i-1}^{-1}e_i = e_iX_{i-1}Y^m_{i-1}e_{i-1}X_{i-1}^{-1}X_{i-1}^{-1}
\]
\[
= A_m e_iX_{i-1}X_{i-1}^{-1}X_{i-1}^{-1}
\]

(49): $[Y, Y^{(1)}_1] = [Y, Y^{(1)}_2] = 0$ is trivial. For $i > 1$ the claim follows by induction: $[Y, Y^{(1)}_i] = 0 \Rightarrow [Y, Y^{(1)}_{i+1}] = [Y, X_iY^{(1)}_iX_i] = 0$. In the general case $[Y^{(1)}_j, Y^{(1)}_i]$ we may assume $j < i$. Using (49) the induction step is: $[Y^{(1)}_j, Y^{(1)}_i] = [X_{j-1}Y^{(1)}_{j-1}X_{j-1}, Y^{(1)}_i] = 0$.

(50): is a consequence of (44).

\[
Y_iX_{i-1} = X_{i-1}Y_iX_{i-1}^{-1}e_{i-1} = \lambda^{-1}X_{i-1}Y_{i-1}X_{i-1}^{-1} = \kappa \lambda^{-1}Y_{i-1}^{-1}e_{i-1}
\]

(52): \[
e_{i-1}Y_i = e_{i-1}X_{i-1}Y_{i-1}X_{i-1}^{-1} = \lambda e_{i-1}X_{i-1}Y_{i-1}X_{i-1}^{-1}
\]
\[
= \lambda e_{i-1}X_{i-1}X_{i-1}^{-1} - \delta \lambda e_{i-1}X_{i-1}^{-1}e_{i-1} + \delta \lambda e_{i-1}Y_{i-1}Y_{i-1}^{-1}
\]
\[
= \kappa \lambda e_{i-1}X_{i-1}^{-1} - \lambda \delta e_{i-1}Y_{i-1}^{-1} + \delta \lambda A_1 e_i
\]

(53,54) are shown in the following way:

\[
e_{i-1}Y^{(1)}_i = e_{i-1}X_{i-1}Y^{(1)}_{i-1}X_{i-1} = \lambda e_{i-1}Y^{(1)}_{i-1}X_{i-1}Y^{(1)}_{i-1}X_{i-1}^{-1} = \kappa \lambda e_{i-1}Y^{(1)}_{i-1}
\]

(55): \[
X_iY_{i+1} = X_i^2Y_iX_i^{-1} = Y_iX_i^{-1} + \delta Y_{i+1} - \delta \lambda e_{i}Y_{i}X_{i}^{-1} = X_{i}Y_{i} - \delta Y_{i} + \delta Y_{i+1} - \kappa \delta \lambda e_{i}Y_{i}^{-1} + \delta^2 \lambda e_{i}Y_{i} - \delta^2 \lambda A_1 e_i
\]
\( R/erates \) the same ideal and the quotient \( A \) and \( Y \) generates the ideal generated by this sub-algebra is isomorphic to the Ariki-Koike algebra. For specific parameter values one may also obtain the A-type BMW algebra as a quotient.

The \( e_1 \) together with a projector \( e_0 \) on the \( p_0 \) eigenvalue of \( Y \) generate a sub-algebra that is a homomorphic image of a Type-B-Temperley-Lieb algebra. The quotient by this ideal generated by this sub-algebra is isomorphic to the Ariki-Koike algebra. For specific parameter values one may also obtain the A-type BMW algebra as a quotient.

**Lemma 10** Let \( J_n \) be the ideal generated by \( Y_n - p_0 \). Every other \( Y_i - p_0, i = 1, \ldots, n \) generates the same ideal and the quotient \( R/(\kappa - \lambda p_0^2, xp_0^2 - A_i) \otimes_R BB^k_n(R)/J_n \) is isomorphic to the A-type BMW algebra \( BA_n(R/(\kappa - \lambda p_0^2, xp_0^2 - A_i)) \).

Proof: The first claim is a consequence of the definition of the \( Y_i \). The specialisation of \( \kappa \) and \( A_i \) is necessary since in the quotient one obtains 0 = \( e_1 Y_i X_i Y_i - \kappa e_1 \) = \( e_1 p_0 X_i p_0 - \kappa e_1 \) = \( e_1 (p_0^2 \lambda - \kappa) \) and \( A_i e_1 = e_1 Y_i^i e_1 = p_0^2 e_1 \). The remaining relations present no further restrictions.

\( \Box \)
\textbf{Definition 11} \(I_n\) denotes the ideal generated by \(e_{n-1}\) in \(B^k_n\).

As we shall see, the quotient by this ideal is an Ariki-Koike algebra.

\textbf{Definition 12} \(AK^k_n\) denotes the Ariki-Koike algebra \([3]\) with generators \(X_0, X_1, \ldots, X_{n-1}\) and parameters \(\delta, p_i, i = 0, \ldots k - 1\) and relations:

\[
\begin{align*}
X_0X_1X_0X_1 &= X_1X_0X_1X_0 \\
X_iX_j &= X_jX_i \quad |i - j| > 1 \\
X_iX_jX_i &= X_jX_iX_j \quad |i - j| = 1 \\
X_i^2 &= \delta X_i + 1 \quad i \geq 0 \\
0 &= \prod_{i=0}^{k-1} (X_0 - p_i)
\end{align*}
\]

We use a slightly different normalisation of the parameters than Ariki and Koike did. From their work we need the result that \(AK^k_n\) is semi-simple. The proof of the following lemma is now trivial.

\textbf{Lemma 13} \(I_n\) is generated by any of the \(e_i\) and the quotient by it is isomorphic to \(AK^k_n\).

Of some interest in knot theoretical applications is the projector on the eigenvalue \(p_0\) of \(Y\). Such a projector is given by \(\prod_{i=1}^{k-1} (Y - p_i)\), but we prefer to work with sums of \(Y^i\).

\textbf{Definition 14} Let \(e_0 = \sum_{i=0}^{k-1} \alpha_i Y^i\) be a projector on the eigenvalue \(p_0\) of \(Y\), i.e. it satisfies \(e_0Y = Ye_0 = p_0e_0\).

\textbf{Lemma 15}

\[
\begin{align*}
\alpha_{k-1} &= \alpha_0 p_0 q_0^{-1} \\
\alpha_{j-1} &= p_0 \alpha_j - \alpha_{k-1} q_j \quad (59) \\
e_0^2 &= x_0 e_0 \quad x_0 := \sum_{i=0}^{k-1} \alpha_i p_0^i \quad (60) \\
e_1 e_0 e_1 &= x_0' e_1 \quad x_0' := \sum_{i=0}^{k-1} \alpha_i A_i \quad (61)
\end{align*}
\]

The proofs are simple.

The modified B-Temperley-Lieb Algebra (\([3], [10]\)) \(TB'_n\) is defined by generators \(e_0, e_1, \ldots, e_{n-1}\), parameters \(c, c', c\) and relations \(e_0^2 = ce_0, e_i^2 = de_i, e_j e_l = e_l e_j, e_i e_j e_i = e_i, e_j e_0 e_1 = c' e_1, 1 \leq i, j \leq n - 1, 0 \leq l \leq n - 1, |i - j| = 1, |j - l| > 1\). Obviously, we have a morphism \(TB'_n \to B^k_n\).
4 Two strands $n = 2$ and ground rings

The algebra $\mathbb{B}^k_n(R)$ is in general not semisimple. This section studies conditions that suffice to make $B(R) := \mathbb{B}^k_2(R)$ semisimple. For the sake of notational convenience we omit the index 1 of $e_1$ and $X_1$.

The parameters of the algebra cannot be chosen independently. Note for example that both $e = \kappa eYXY$ and $Y^k = \sum_i q_i Y^i$ fix the length of $Y$.

**Definition 16** Define a ring $R_1$ as a quotient $R_1 := R_0/c$ of $R_0$. The ideal $c \subset R_0$ is generated by $k$ Laurent polynomials that are obtained by the following procedure: Expand $Ye - \kappa X_1^{-1}Y^{-1}e$ using (27), (8), (30) and (58) into a linear combination $\sum_{i=0}^{k-1} h_i Y^i e$. The coefficient $h_i$ of this sum are the generators of $c = (h_0, \ldots, h_{k-1})$.

**Lemma 17** The expansion of the expression in the definition of $c$ in terms of $Y^{-i}e$ defines polynomials $h'_i$ such that $Ye - \kappa X_1^{-1}Y^{-1}e = \sum_{i=0}^{k-1} h'_i Y^{-i}e$. They generate the same ideal: $c = (h'_0, \ldots, h'_{k-1})$. Furthermore, $c$ is closed under the involution $c^* = c$.

Proof: We have $h_j = \sum_i q_{i,j} h'_i$ and $h'_i = \sum_i Q_{i,j} h_i$. This implies equality of both the ideals generated by these sets of polynomials. The second claim follows from $h'_i = h_i$. $\square$

To shed some light on the ideal $c$ we first note that (58) implies:

$$X_1 Y^m e = \lambda^{m-1} \kappa^m Y^{-m} e + \sum_{s=1}^{m-1} \kappa^s \lambda^s \delta(A_{m-s} Y^{-s} e - Y^{m-2s} e)$$

This renders the defining equation into the form

$$Ye + \delta \kappa Y^{-1} e - \kappa \delta \sum_{m} \lambda_m A_m e_1$$

(62)

We now introduce a ring that will become relevant later on as the ring of the classical limit of the algebra. At this stage we need it purely as a tool.

**Definition 18** The ideal $J_c \subset R_1$ is given by $J_c := (\kappa - 1, \lambda - 1, q - 1, q_0 - 1, q_1, \ldots, q_{k-1})$. Set $R_c := R_1/J_c$.

According to proposition 3 the equation for the $q_i$ are solvable. Hence the ring $R_c$ is nontrivial. The same polynomials $(\kappa - 1, \lambda - 1, q - 1, q_0 - 1, q_1, \ldots, q_{k-1})$ define an ideal in $R_0$. It contains $c$ since after dividing by $J_c$ we have $Y^{-1} = Y^{k-1}, \lambda_{k-1} = 1, \lambda_i = 0$ and hence (62) becomes trivial. It follows that $R_c$ is the quotient of $R_0$ by $J_c$.

The ring $R_1$ plays a special role in the contruction of a $B$-module. Using (27), (30) and (58) we see that the ideal $I_2$ is spanned $Y^i eY^j, i, j = 0, 1, \ldots, k - 1$. 

Definition 19 Let $R$ be as in definition 7. Let $V := V(R)$ be the free $R$-module of dimension $k$. The basis is denoted by $b_i, 0 \leq i < k$. $V$ is turned into a module of the free algebra generated by $e, X, Y$ by the following definitions:

$$e \cdot b_i := A_i b_0 \quad \text{with } A_0 = x$$
$$Y \cdot b_{k-1} := \sum_{j=0}^{k-1} q_j b_j \quad Y \cdot b_i := b_{i+1}$$

$$X \cdot b_0 := \lambda b_0 \quad X \cdot b_1 := \kappa Y^{-1} b_0$$
$$X \cdot b_i := \kappa \lambda (Y^{-1}.X.b_{i-1} - \delta b_{i-2} + \delta A_{i-1} Y^{-1} b_0) \quad i \geq 2$$

The definition of this action is guided by the desire that it should factor over $B(R)$. $Y^{-1}$ and $X^{-1}$ shall act by their expansions in terms of $Y^i$ (implying $(Y^{-1})_i = (Y^{-1})_i$), resp. $X, e, 1$. It turns out, however, that $V$ is not in general a $B$-module. Most relations are easy to check but two of them may not hold: (a) $XYXY = YXYX$ and (b) $X^2 = 1 + \delta X - \delta \lambda e$. Relation (b) is equivalent to $(X^{-1})_1 = (X^{-1})_1$.

Lemma 20 $V(R_1)$ is a $B(R_1)$-module.

Proof: For the ring $R_1$ one has by its construction:

$$b_1 = Y \cdot b_0 = \kappa X^{-1} Y^{-1} b_0 \quad (63)$$

On $b_0$ relation (b) holds trivially. We check (a):

$$X \cdot Y \cdot X \cdot Y \cdot b_0 = X \cdot Y \cdot X \cdot b_1 = \kappa Y \cdot X \cdot Y^{-1} b_0 = \kappa Y \cdot b_0 = \kappa \lambda b_0$$

$$= \lambda \kappa Y^{-1} b_0 = \lambda Y \cdot X \cdot b_1 = Y \cdot X \cdot Y \cdot b_0$$

Furthermore, we check the inverse of (a):

$$X^{-1} \cdot Y^{-1} \cdot X^{-1} \cdot Y^{-1} \cdot b_0 \quad \boxed{63} \quad \kappa^{-1} X^{-1} \cdot Y^{-1} \cdot Y \cdot b_0 = \kappa^{-1} X^{-1} \cdot b_0 = \kappa^{-1} \lambda^{-1} b_0$$

$$Y^{-1} \cdot X^{-1} \cdot Y^{-1} \cdot X^{-1} \cdot b_0 \quad = \lambda^{-1} Y^{-1} \cdot X^{-1} \cdot Y^{-1} \cdot b_0 \quad \boxed{63} \quad \kappa^{-1} \lambda^{-1} Y^{-1} \cdot Y \cdot b_0 = \lambda^{-1} \kappa^{-1} b_0$$

$(63)$ enables us to write for all $i = 0, \ldots, k - 1$:

$$X \cdot Y \cdot b_{i-1} = X \cdot b_i = \kappa \lambda Y^{-1} \cdot X^{-1} \cdot b_{i-1} \quad (64)$$

Here we used the convention that $b_{-1} = Y^{-1} b_0$. The case $i = 0$ follows from $(63)$, the case $i = 1$ is trivial, and the cases $i > 1$ are simple rewritings of the action of $X$.

Now we can start the inductive proof that (a) and (b) hold on all basis vectors. The induction assumption $H_i$ is: Relations (a) and (b) hold on $b_{i-1}$. We show that the inverse of relation (a) holds on $b_{i-1}$.

$$Y^{-1} \cdot X^{-1} \cdot Y^{-1} \cdot X^{-1} \cdot b_{i-1} \quad \boxed{63} \quad \kappa^{-1} \lambda^{-2} X \cdot Y \cdot b_{i-1} \quad = \quad \kappa^{-1} \lambda^{-1} X \cdot Y \cdot b_{i-1} \quad H_i \quad \kappa^{-1} \lambda^{-1} b_{i-1}$$

$$X^{-1} \cdot Y^{-1} \cdot X^{-1} \cdot b_{i-1} \quad = \quad X^{-1} \cdot Y^{-1} \cdot X^{-1} \cdot b_{i-2} \quad \boxed{63} \quad \kappa^{-1} \lambda^{-1} X^{-1} \cdot X \cdot b_{i-1}$$

$$\quad \quad \quad H_i \quad \kappa^{-1} \lambda^{-1} b_{i-1}$$
We now check (b):

\[ X^{-1}.X.b_i = \kappa\lambda Y^{-1}.X^{-1}.b_{i-1} = \kappa\lambda Y^{-1}.X^{-1}.b_{i-1} \]
\[ = \kappa\lambda Y^{-1}.X^{-1}.Y^{-1}.Y^{-1}.X^{-1}.b_{i-1} \]
\[ \overset{\overset{\text{64}}{\text{[4]}}}{} = Y^{-1}.X^{-1}.b_{i-1} = b_i \]

Finally, we look at (a):

\[ Y.X.Y.X.b_i = \kappa\lambda Y.X.Y.Y^{-1}.X^{-1}.b_{i-1} = \kappa\lambda Y.X^{-1}.b_{i-1} = \kappa\lambda Y.b_i \]
\[ X.Y.X.Y.b_i = \kappa\lambda Y.Y^{-1}.X^{-1}.b_i = \kappa\lambda X^{-1}.b_i = \kappa\lambda b_i \]
\[ \square \]

**Definition 21** \( U_m := \operatorname{span}_{R_1}\{Y^i e^m \mid i = 0, \ldots, k - 1\} \)

**Lemma 22** Each \( U_m \) is a \( B(R_1) \)-module isomorphic to \( V = V(R_1) \). They have pairwise trivial intersections.

Proof: We show that the map \( \varrho : V(R_1) \to B(R_1), b_i \mapsto Y^i e \) defines a module isomorphism of \( V \) and \( U_0 \). It is a surjection of \( R_1 \)-modules, and, by the above lemma, a morphism of \( B(R_1) \)-modules. It remains to check injectivity. Suppose we had \( 0 = \sum_i \alpha_i Y^i e, \alpha_i \in R_1 \). Applying this to \( b_0 \) we obtain \( 0 = x \sum_i \alpha_i b_i \). Now, \( x \) is invertible, and hence all the \( \alpha_i \) have to vanish. Thus we have shown that \( \operatorname{span}_{R_1}\{Y^i e_1\} \) is a free \( R_1 \) module. The same is true for the isomorphic \( B(R_1) \) modules \( U_m \). Now, we are going to show that the \( e_1 \) ideal \( \operatorname{span}\{Y^i e_1 Y^j\} \) as a whole is a free \( R_1 \) module. It suffices to show that the \( U_m \) form a direct sum decomposition, i.e. that \( m \neq r \Rightarrow U_m \cap U_r = \{0\} \). Since \( Y \) is invertible, it suffices to show for \( m \geq 1 \) that \( U_m \cap U_0 = \{0\} \). Assume there is a non zero element \( a \) in the intersection of \( U_0 \) and \( U_m \):

\[ a = \sum_i a_i Y^i e_1 = \sum_i b_i Y^i e_1 Y^m \]

Multiplying from the right with \( e_1 \) the righthand side is mapped to \( U_0 \) and we may compare the coefficients in its basis: \( a_i = x^{-1} A_m b_i \). Thus

\[ a = x^{-1} A_m \sum_i b_i Y^i e_1 = \sum_i b_i Y^i e_1 Y^m \]

Hence \( x^{-1} A_m a Y^{-m} = a Y^{-m} Y^m, \) that is \( a Y^{-m} \) is an eigenvector of \( Y^m \) to the eigenvalue \( x^{-1} A_m \). Now, \( Y^m \) and \( x \) are invertible and hence \( A_m \) is a unit in \( R_1 \) and furthermore in any quotient of \( R_1 \). However, we have already defined the quotient \( R_c \) in which \( A_m \) is obviously not invertible. \[ \square \]

**Lemma 23** \( R_1 \) is an integral domain.
Proof: We have to show that the ideal $c$ is prime. Due to proposition 2 it suffices to show that the defining equations may be solved uniquely in the field of fractions.

Using lemma 17 we consider the coefficients of $Y^{-e_1}, i = 1, \ldots, k - 2$. They form a triangular (and hence a soluble) system of equations in the variables $A_i$. To solve it one first determines $A_1$ from the coefficient of $Y^{-(k-2)}e_1$. Secondly, $A_2$ is calculated from the coefficient of $Y^{-(k-3)}e_1$. We end with $A_{k-2}$ and $Y^{-1}e_1$. Thereafter the coefficient of $e_1$ can be used to isolate $A_{k-1}$ which appears just once in this expression. It remains to investigate the coefficient of $Y^{-(k-1)}e_1$. It is $q_0 = \kappa q_0^{-1}\lambda^{k-2} \kappa^{-1} - \kappa \sum_{m=0}^{k-1} \sum_{s=1}^{m-1} q_m \kappa \lambda^s (\lambda - \lambda^{-1})(1 - x)^{-1}\text{Coeff}(Y^{m-2s}e_1, Y^{-(k-1)}e_1)$. This can be solved for $x$.

Note that the proof of this lemma breaks down if one chooses to specify $\kappa = \lambda^{-1}$ since then we can’t be sure that the coefficient of $(1 - x)^{-1}$ is non-zero.

**Definition 24** Let $K_1$ denote the field of fractions of $R_1$.

The quotient of $B(R_1)$ by the ideal $I_2$ is isomorphic to the Ariki-Koike algebra $AK^k_2(R_1)$ which is a free module over any integral domain. We summarise:

**Definition and Lemma 25** $B(R_1)$ is a free $R_1$ module and the subset $Y^i e_1 Y^j$ is linearly independent.

## 5 The word problem in $BB^k_n$

In this section we single out a set of words in standard form that linearly generate $BB^k_n$. However, this does not lead to a linear basis of $BB^k_n$ but it is fundamental to the following analysis.

**Proposition 26** Every element in $BB^k_n$ is a linear combination of words of the form $w_1 \gamma w_2$ where $w_i \in BB^k_{n-i}$ and $\gamma \in \Gamma_n := \{1, e_{n-1}, X_{n-1}, Y^j, j = 1, \ldots, k - 1\}$. The same is true if in $\Gamma_n$ the generators $X_{n-1}$ or $Y_n$ or both are replaced by their inverses.

Proof: We prove the proposition by induction. The case $n = 1$ is trivial and $n = 2$ can also be verified easily.

Let $w_0 \gamma_0 w_1 \gamma_1 \cdots w_m \gamma_m w_{m+1} \in BB^k_n, w_i \in BB^k_{n-i}$ be an arbitrary word. It suffices to show that any two neighbouring $\gamma_i$ can be combined together. Hence the situation we have to investigate is $w = \gamma_1 w_1 \gamma_2, w_1 \in BB^k_{n-1}, \gamma_1, \gamma_2 \in \Gamma_n$. By induction hypothesis we have $w_1 = u_1 \alpha u_2, u_i \in BB^k_{n-2}, \alpha \in \Gamma_{n-1}$ and hence $w = \gamma_1 u_1 \alpha u_2 \gamma_2 = u_1 \gamma_1 \alpha \gamma_2 u_2$. Thus it suffices to investigate $w' = \gamma_1 \alpha \gamma_2$. The cases $\gamma_1 = 1$ or $\gamma_2 = 1$ are trivial. We now investigate in turn the four possible values of $\alpha$.

1. Case $\alpha = 1$: The following table gives the relation that allows to reduce the product $\gamma_1 \gamma_2$ to the standard form of the proposition.

| $\gamma_1 \gamma_2$ | $Y_n^j$ | $e_{n-1}$ | $X_{n-1}$ |
|---------------------|---------|-----------|-----------|
| $Y_n^j$             | (37)    | (53)      | (56)      |
| $e_{n-1}$           | (38)    | (3)       | (12)      |
| $X_{n-1}$           | (53)    | (3)       | (12)      |

Proof: We have to show that the ideal $c$ is prime. Due to proposition 2 it suffices to show that the defining equations may be solved uniquely in the field of fractions.

Using lemma 17 we consider the coefficients of $Y^{-e_1}, i = 1, \ldots, k - 2$. They form a triangular (and hence a soluble) system of equations in the variables $A_i$. To solve it one first determines $A_1$ from the coefficient of $Y^{-(k-2)}e_1$. Secondly, $A_2$ is calculated from the coefficient of $Y^{-(k-3)}e_1$. We end with $A_{k-2}$ and $Y^{-1}e_1$. Thereafter the coefficient of $e_1$ can be used to isolate $A_{k-1}$ which appears just once in this expression. It remains to investigate the coefficient of $Y^{-(k-1)}e_1$. It is $q_0 = \kappa q_0^{-1}\lambda^{k-2} \kappa^{-1} - \kappa \sum_{m=0}^{k-1} \sum_{s=1}^{m-1} q_m \kappa \lambda^s (\lambda - \lambda^{-1})(1 - x)^{-1}\text{Coeff}(Y^{m-2s}e_1, Y^{-(k-1)}e_1)$. This can be solved for $x$.

Note that the proof of this lemma breaks down if one chooses to specify $\kappa = \lambda^{-1}$ since then we can’t be sure that the coefficient of $(1 - x)^{-1}$ is non-zero.

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The quotient of $B(R_1)$ by the ideal $I_2$ is isomorphic to the Ariki-Koike algebra $AK^k_2(R_1)$ which is a free module over any integral domain. We summarise:

**Definition and Lemma 25** $B(R_1)$ is a free $R_1$ module and the subset $Y^i e_1 Y^j$ is linearly independent.

## 5 The word problem in $BB^k_n$

In this section we single out a set of words in standard form that linearly generate $BB^k_n$. However, this does not lead to a linear basis of $BB^k_n$ but it is fundamental to the following analysis.

**Proposition 26** Every element in $BB^k_n$ is a linear combination of words of the form $w_1 \gamma w_2$ where $w_i \in BB^k_{n-i}$ and $\gamma \in \Gamma_n := \{1, e_{n-1}, X_{n-1}, Y^j, j = 1, \ldots, k - 1\}$. The same is true if in $\Gamma_n$ the generators $X_{n-1}$ or $Y_n$ or both are replaced by their inverses.

Proof: We prove the proposition by induction. The case $n = 1$ is trivial and $n = 2$ can also be verified easily.

Let $w_0 \gamma_0 w_1 \gamma_1 \cdots w_m \gamma_m w_{m+1} \in BB^k_n, w_i \in BB^k_{n-i}$ be an arbitrary word. It suffices to show that any two neighbouring $\gamma_i$ can be combined together. Hence the situation we have to investigate is $w = \gamma_1 w_1 \gamma_2, w_1 \in BB^k_{n-1}, \gamma_1, \gamma_2 \in \Gamma_n$. By induction hypothesis we have $w_1 = u_1 \alpha u_2, u_i \in BB^k_{n-2}, \alpha \in \Gamma_{n-1}$ and hence $w = \gamma_1 u_1 \alpha u_2 \gamma_2 = u_1 \gamma_1 \alpha \gamma_2 u_2$. Thus it suffices to investigate $w' = \gamma_1 \alpha \gamma_2$. The cases $\gamma_1 = 1$ or $\gamma_2 = 1$ are trivial. We now investigate in turn the four possible values of $\alpha$.

1. Case $\alpha = 1$: The following table gives the relation that allows to reduce the product $\gamma_1 \gamma_2$ to the standard form of the proposition.

| $\gamma_1 \gamma_2$ | $Y_n^j$ | $e_{n-1}$ | $X_{n-1}$ |
|---------------------|---------|-----------|-----------|
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| $e_{n-1}$           | (38)    | (3)       | (12)      |
| $X_{n-1}$           | (53)    | (3)       | (12)      |
5 THE WORD PROBLEM IN $BB^K_N$

\[ Y_n^j e_{n-1} = X_{n-1} Y_n^j X_{n-1}^{-1} e_{n-1} = \lambda^{-1} X_{n-1} Y_n^j e_{n-1} \text{ apply (58) recursively} \quad (65) \]

\[ e_{n-1} Y_n^j = \lambda e_{n-1} Y_n^j X_{n-1}^{-1} \]
\[ = \lambda e_{n-1} Y_n^j X_{n-1} - \delta \lambda e_{n-1} Y_n^j + \delta \lambda A_n e_{n-1} \quad (66) \]

The first term is reduced by applying (57) recursively.

\[ X_{n-1} Y_n^j = X_{n-1} Y_n^j X_{n-1}^{-1} \]
\[ = Y_{n-1} X_{n-1}^{-1} - \delta \lambda e_{n-1} Y_n^j + \delta X_{n-1}^{-1} X_{n-1} \]
\[ = Y_{n-1} X_{n-1}^{-1} - \delta Y_{n-1} + \delta Y_n^j \]
\[ - \delta \lambda (e_{n-1} Y_n^j X_{n-1} - e_{n-1} Y_n^j + \delta A_n e_{n-1}) \]
\[ + \delta Y_n^j \quad (68) \]

Again, one needs (57) for recursive reduction.

2. Case $\alpha = X_{n-2}$:

\[ \gamma_1 \setminus \gamma_2 \quad || \quad Y_n^j \quad || \quad e_{n-1} \quad || \quad X_{n-1} \]
\[ Y_n^j = X_{n-2} Y_n^j e_{n-1} \quad (67) \]
\[ e_{n-1} = e_{n-1} Y_n^j X_{n-2} \quad (66) \]
\[ X_{n-1} = X_{n-1} Y_n^j e_{n-2} \quad (67) \]

3. Case $\alpha = e_{n-2}$:

\[ \gamma_1 \setminus \gamma_2 \quad || \quad Y_n^j \quad || \quad e_{n-1} \quad || \quad X_{n-1} \]
\[ Y_n^j = e_{n-2} Y_n^j e_{n-2} \quad (67) \]
\[ e_{n-1} = e_{n-1} Y_n^j e_{n-2} \quad (66) \]
\[ X_{n-1} = X_{n-1} Y_n^j e_{n-2} \quad (67) \]

4. Case $\alpha = Y_{n-1}^m$:

\[ \gamma_1 \setminus \gamma_2 \quad || \quad Y_n^j \quad || \quad e_{n-1} \quad || \quad X_{n-1} \]
\[ Y_n^j \quad * \quad \text{like (70)} \quad * \]
\[ e_{n-1} \quad (71) \quad (48) \quad (57) \]
\[ X_{n-1} \quad * \quad (48) \quad (72) \]

\[ e_{n-1} Y_{n-1}^m Y_n^j = e_{n-1} Y_{n-1}^m X_{n-1} Y_n^j X_{n-1}^{-1} \quad (70) \]
\[ \subseteq \text{span}\{e_{n-1} Y_{n-1}^s \mid 0 \leq s < k\} Y_n^j X_{n-1}^{-1} \quad (57) \]
\[ \subseteq \text{span}\{e_{n-1} Y_{n-1}^s \mid 0 \leq s < k\} \quad (71) \]
The remaining cases (marked by * in the table) are

\[
\begin{align*}
Y_n^m Y_{n-1}^j &= X_{n-1} Y_{n-1} X_{n-1}^{-1} Y_{n-1}^m Y_{n-1}^j X_{n-1}^{-1}
\end{align*}
\]

The last term can be reduced using (65)

We note that we are dealing with sequences of generators where all indices are equal. Hence we will suppress the index in further calculations. Equations (57) and (58) imply that every such sequence containing \( e \) is reducible to \( Y^t e Y^s \) and thus is of the standard form. This motivates the following notation: We write

\[
\begin{align*}
XY X^{-1} &= \sum_{i,j} \alpha_{i,j} X Y^i X Y^j + \sum_{i,j} \beta_{i,j} Y^i X Y^j \quad \text{(73)}
\end{align*}
\]

We prove (73) by induction on \( s \). For \( s = 1 \) we have \( XY X Y^t X = Y^t X X Y Y^t \sim Y X Y + \delta Y X Y X Y = Y^t X Y Y^t \). Assume that (73) holds for \( s \). We show it for \( s + 1 \):

\[
\begin{align*}
XY^{s+1} X Y^t &= XY X^{-1} X Y^s X Y^t
\sim& \sum_{i,j} \alpha_{i,j} X Y^i X Y^t + \sum_{i,j} \beta_{i,j} X Y^i X Y^t
\sim& \sum_{i,j} \alpha_{i,j} X Y Y^i X Y^t + \sum_{i,j} \beta_{i,j} X Y Y^i X Y^t - \delta \sum_{i,j} \beta_{i,j} X Y Y^i X Y^t
\end{align*}
\]

The first and third summand are already in a form in which their contribution to \( \alpha_{i,j}^{s+1} \) can be read off. In the second summand we apply the induction hypothesis once again

\[
\begin{align*}
\sum_{i,j} \beta_{i,j} X Y Y^i X Y^j
\sim& \sum_{i,j} \beta_{i,j} \left( \alpha_{p,q}^{1,i} X Y P XY^{q+j} + \beta_{p,q}^{1,i} Y^p X Y^{q+j} \right)
\end{align*}
\]
Lemma 27 There exist elements $R_{i,m} \in \text{BB}^{k}_{i-1}$ such that $e_i Y''_i e_i = R_{i,m} e_i$.

Proof: To prove the first statement one writes

$$e_i Y''_i X_i = e_i Y''_i X_i X_i^{-1} Y''_i = e_i X_i Y''_i X_i X_i^{-1} Y''_i$$

$$= \kappa \lambda e_i Y''_i X_i^{-1} Y''_i - \kappa \delta \lambda e_i Y''_i + \kappa \delta \lambda R_{i,j-1} e_i Y''_i$$

The proposition implies that $\text{BB}^{k}_{i}$ is finite dimensional.

Proposition 28 In proposition 26 one may replace $\Gamma_n$ by $\Gamma'_n := \{1, e_{n-1}, X_{n-1}, Y''_{n,j}, j = 1, \ldots, k-1\}$.

Proof: We express an arbitrary element $a$ in $\text{BB}^{k}_{n}$ as $a = \sum_j f_j h_j g_j$ with $f_j, g_j \in \text{BB}^{k}_{n-1}$, $h_j \in \Gamma_n$. We are finished if we can show that $Y''_n = \sum_j f_j^{(n)} g_j^{(n)} h_j^{(n)}$ with $f_j^{(n)} \in \Gamma'_n$, $g_j^{(n)} \in \text{BB}^{k}_{n-1}$ since in this case we can simply substitute this expressions for the $Y''_n$ which appear among the $f_j$. We show $Y''_n = \sum_j f_j^{(n)} g_j^{(n)} h_j^{(n)}$ by induction. The case $n = 1$ is trivial. Now assume that the formula holds for $n - 1$.

$$Y''_n = X_{n-1} Y''_{n-1} X_{n-1}^{-1} = \sum_s X_{n-1} f_s^{(n-1)} g_s^{(n-1)} h_s^{(n-1)} X_{n-1}^{-1}$$

$$= \sum_s f_s^{(n-1)} X_{n-1} Y''_{n-1} X_{n-1}^{-1}$$

The cases $\gamma_s^{(n-1)} \in \{1, e_{n-2}, X_{n-2}\}$ are easily reduced using lemma 9. It remains to investigate the case $\gamma_s^{(n-1)} = Y''_{n-1}$. 

$$X_{n-1} Y''_{n-1} X_{n-1}^{-1} = X_{n-1} Y''_{n-1} X_{n-1}^{-1} Y''_{n-1} X_{n-1}$$

$$= Y''_{n-1} (X_{n-1} - \delta + \delta e_{n-1}) Y''_{n-1} X_{n-1}^{-1}$$

$$= Y''_{n-1} X_{n-1} Y''_{n-1} X_{n-1}^{-1} - \delta Y''_{n-1} Y''_{n-1} X_{n-1}^{-1}$$

$$+ \delta Y''_{n-1} e_{n-1} Y''_{n-1} X_{n-1}^{-1}$$

(76)
The second summand is \(-\delta Y'_{n-1}X_{n-1}Y'_{n-1}\) which is already of the standard form. The third summand is
\[
\delta Y'_{n-1}X_{n-1}Y'_{n-1} = \delta \lambda X_{n-1}Y'_{n-1}e_{n-1}Y'_{n-1}(X_{n-1} - \delta + \delta e_{n-1}) = \delta \lambda \kappa Y'_{n-1}e_{n-1}Y'_{n-1}X_{n-1} - \delta^2 \lambda \kappa Y'_{n-1}e_{n-1}Y'_{n-1} + \delta^2 \lambda \kappa Y'_{n-1}e_{n-1}Y'_{n-1} e_{n-1}
\]

Here the last summand is reduced using the formula for \(e_iY'_{m}e_i\) from Lemma 27 while the first summand needs \((\ref{74})\). The middle summand is already of the standard form.

The first summand of \((\ref{76})\) is reduced by iteration. \(\square\)

We continue our study of words in \(BB_k^n\) by cutting down the size of sets that linearly generate the algebra.

**Lemma 29** \(BB_k^n\) is linearly spanned by the set \(S_n\) which is recursively defined:

\[
S_1 := \{Y^i \mid i = 0, \ldots, k - 1\}
\]

\[
S_n := \Gamma_1 \cdots \Gamma_n S_{n-1}
\]

It suffices to take out of \(\Gamma_1 \cdots \Gamma_n\) those elements that are of the following form:

\[
Y_{i_1}^{m_1} \cdots Y_{i_s}^{m_s} X_i \cdots X_j e_{j+1} \cdots e_n, \quad m_t \in \{0, \ldots, k - 1\}, m_s = i, l_1 < \cdots < l_s
\]

Here we have \(1 \leq i \leq n\) and \(i - 1 \leq j \leq n\) so that the chains of \(X\) and \(e\) may be empty.

**Proof:** Proposition \(26\) yields the following representation of \(BB_k^n:\)

\[
BB_k^n = \text{span}BB_{n-1} \Gamma_n BB_k^{n-1} = \text{span}BB_{n-2} \Gamma_n BB_k^{n-2} \Gamma_n BB_k^{n-1} = \text{span} \Gamma_1 \cdots \Gamma_n BB_k^{n-1}
\]

To establish the second statement we consider the \(Y'_{m}\) that appears at the leftmost position in a chain \(Z_i \cdots Z_{j-1} Y'_j Z_{j+1} \cdots Z_n\) of generators \(\hat{Z}_s \in \Gamma_s\). Then \(Z_i \cdots Z_{j-1}\) consists only of \(e\) and \(X\) and hence it can be commuted to the right and be absorbed in \(BB_k^{n-1}\). Similarly \(e\) and \(X\) that appear between two \(Y\) can be commuted to the right. Iterating this argument we obtain only chains of the form \(Y_{i_1}^{m_1} \cdots Y_{i_s}^{m_s} Z_{j+1} \cdots Z_n, i_1 < \cdots < i_s\).

If \(e_i X_{i+1}\) appears in such a chain it may be converted to \(e_i X_{i+1} = e_i e_{i+1} X_{i-1}^{-1}\). The \(X_{i-1}\) can be absorbed in \(BB_k^{n-1}\). Hence all \(X\) have to appear to the left of all \(e\). \(\square\)

A similar proof establishes a related lemma using the \(Y'_{i}\):

**Lemma 30** \(BB_k^n\) is linearly spanned by \(S'_n:\)

\[
S'_1 := \{Y'^i \mid i = 0, \ldots, k - 1\}
\]

\[
S'_n := \Gamma'_1 \cdots \Gamma'_n S'_{n-1}
\]

From \(\Gamma'_1 \cdots \Gamma'_n\) only elements of the following form are needed:

\[
Y_{i_1}^{m_1} X_i \cdots X_j e_{j+1} \cdots e_n, \quad m = 0, \ldots, k - 1
\]

The chains of \(x\) and \(e\) may be empty.
6 Graphical interpretation and classical limit

The very definition of \( \text{BB}^k_n \) is motivated by knot theory as was vaguely explained in section 2. Here we fill in the details.

Consider the free \( R \)-algebra (\( R \) may denote any commutative ring) of isotopy classes of ribbons in \( (\mathbb{R}^2 - \{0\}) \times [0,1] \) where \( n \) ribbons end at the upper and lower plane each. The ribbons touch these planes in small intervals which have as their lower starting point one of the set \( \{1, 2, \ldots, n\} \times 0 \times \{0, 1\} \). Closed components are allowed. Multiplications is given by putting graphs on top of each other. This forms the algebra of cylinder tangles.

The pictures on the right hand side of figure 1 may now be easily interpreted as regular diagrams of such cylinder tangles. We need to specify the total number of strands in these pictures. Thus, we write \( X^{(G)}_{i,n}, e^{(G)}_{i,n}, Y^{(G)}_{1,n} \) for the generators that act at the \( i \)-th of \( n \) strands. Let \( G_{\text{BB}}^k_n'(R) \) be the sub-algebra of the algebra of cylinder tangles that is generated by \( X^{(G)}_{i,n}, e^{(G)}_{i,n}, Y^{(G)}_{1,n}, 1 \leq i \leq n - 1 \). Each isotopy class thus has a representative that is a product in these generators. We define \( G_{\text{BB}}^k_n(R) \) (where \( R \) is now as in definition 7) to be the quotient of this algebra by skein relations that result from (5), (6) and (26) by replacing \( X_i, e_i, Y \) by \( X_{i,j}, e_{i,j}, Y_{1,j} \). Here, we don’t restrict \( j \) so that it may be greater than \( n \). This is necessary to account for the fact that by introducing maxima and minima the number of strands that intersect some horizontal plane may be arbitrary.

The remaining relations of \( G_{\text{BB}}^k_n(R) \) have obvious topological content so that we have a surjective morphism \( \Psi_n : G_{\text{BB}}^k_n(R) \to G_{\text{BB}}^k_n(R) \). We remark that this graphical algebra is not defined in terms of a basis but in terms of generators and relations. However, some of the relations are not stated explicitly. The existence of \( \Psi_n \), however, shows that the statements of section 5 carry over. However, we have to keep in mind the possibility that \( \Psi_n \) could fail to be injective.

The graphical interpretation suggests special settings for \( \kappa \). Recall that \( \lambda \) amounts to a twist of the ribbon. If we interpret \( Y^{(G)} \) as a ribbon band that lies flat in the projection plane then we should have \( \lambda = \kappa \). On the other hand, if the transversal vector field of the ribbon is always oriented towards the cylinder axes we should have \( \kappa = 1 \). However, we can (and will) decide to keep \( \kappa \) free by renormalising \( Y \).

The classical limit of tangle algebra is a specialisation in which braidings degenerate to permutations. We define \( \text{BP}^k_n(R) \) in its own right as algebra of Brauer graphs where each arc carries an element of \( \mathbb{Z}_k \). We visualise this as dotted Brauer graphs, i.e. \( \text{BP}_n^k(R) \) is the free \( R \)-module of dimension \( k^n(2n-1)!! \) that has as basis the set of Brauer graphs where each arc carries at most \( k - 1 \) points. We require that vertical arcs have no extrema with respect to the height function and that horizontal arcs have exactly one extremum. Furthermore, we demand that the dots of vertical arcs are concentrated at the left endpoint.

Multiplication is given as for graphs. Dots may flow along an arc and may cross another arc. If a dot traverses an extremum it gets replaced by \( k - 1 \) dots. Dot numbers are reduced modulo \( k \). Using this we may isolate cycles and concentrate dots on their leftmost position. Such a cycle with \( i \) dots on it may be deleted at the expense of a factor \( A_i \). Dots on vertical arcs may be brought to the lower endpoint and thereafter the arc may
be straightened. Similarly, dots on horizontal arcs may be concentrated according to our convention. Just as in the case of ordinary Brauer graphs we see that $$BP^k_n(R)$$ is generated by $$X^{(G)}_{i,n}, e^{(G)}_{i,n}, Y^{(G)}_{i,n}$$ (where $$X^{(G)}_{i,n}$$ is to be understood as a permutation two-cycle).

Let compare $$BP^k_n$$ with the classical limit of $$BB^k_n(R_1)$$.

**Definition 31** The classical limit of $$BB^k_n(R_1)$$ is defined to be the algebra

$$CBB^k_n := BB^k_n(R_1) \otimes_{R_1} (R_1/J_c)$$

$$J_c := (\kappa - 1, \lambda - 1, \delta, q_0 - 1, q_1, \ldots, q_{k-1}) \subset R_1$$

The new ground ring $$R_1/J_c$$ is denoted by $$R_c$$.

Note that $$(\kappa - 1, \lambda - 1, q_0 - 1, q_1, \ldots, q_{k-1})$$ viewed as ideal in $$R_0$$ contains the consistency ideal $$c$$ because in the limit $$Y^{-1} = Y^{k-1}, \bar{q}_{k-1} = 1, \bar{q}_i = 0$$ and hence $$[64]$$ becomes trivial. Thus, $$R_c$$ is the quotient of $$R_0$$ by this ideal.

In $$CBB^k_n$$ we have $$X_i = X_i^{-1}$$ and hence $$Y_i^j = Y_{i}^{(j)} = Y^{ij}_i$$. An important consequence is that $$Y'$$ behaves naturally with respect to the braiding $$X_i$$. In the system $$S'_n$$ from lemma $[30]$ we may read $$Y'$$ as $$Y$$. Using this we are going to prove that $$BB^k_n$$ is linearly spanned by a set of elements of the form $$\alpha\beta\gamma$$, where $$\alpha$$ is a product of $$Y_i$$, $$\gamma$$ is a product of $$Y^{-1}$$ and $$\beta$$ is an element of a basis of the A-type BMW algebra $$BA_n$$. The proof is by induction on $$n$$, so assume the claim is already shown for $$n - 1$$. It suffices to show that all $$Y_i$$ which appear on the left of the generating system $$S_{n-1}$$ of $$BB^k_{n-1}$$ can be moved to the left through the left chain or that it can (in negated form) be moved to right of $$BB^k_{n-1}$$. We investigate the various arising cases. In the first case $$e_{n-1}Y_{n-1}$$ appears. We rewrite it according to

$$e_{n-1}Y_{n-1} = e_{n-1}Y_{n-1}X_{n-1}Y_{n-1}^{-1}Y_{n-1}^{-1}X_{n-1}^{-1} = e_{n-1}Y_{n-1}^{-1}X_{n-1}^{-1}$$

$$= e_{n-1}X_{n-1}^{-1}Y_{n-1}^{-1}X_{n-1}^{-1} = e_{n-1}Y_{n-1}^{-1}$$

The $$Y_{n-1}^{-1}$$ may then be moved to the right. If $$e_ie_{i+1}Y_i = e_iY_iY_{i+1}$$ occurs a twofold application of this result shows that $$Y_{i+2}$$ may be moved to the left. The only remaining situation $$X_iY_i = Y_{i+1}X_i$$ is trivial.

In each step of the recursive construction of $$S'_n$$ only one additional $$Y_i^m$$ can occur and these occurrences stick together in the above process. The dimension of $$CBB^k_n$$ is therefore at most $$k^n$$ times the dimension of the ordinary Brauer algebra: $$\dim CBB^k_n \leq k^n(2n-1)!$$.

**Lemma 32** The algebras $$CBB^k_n$$ and $$BP^k_n(R_c)$$ are isomorphic.

Proof: We define the morphism $$\chi_n : CBB^k_n \rightarrow BP^k_n(R_c)$$ that maps $$e_i \mapsto e_i^{(G)}, X_i \mapsto Y_i^{(G)}$$ and $$Y$$ to a dot on the first strand. It is easy to see that this is a morphism (It is relation $[25]$ that requires the somewhat strange minimum/maximum rule.). It is surjective. Injectivity may be seen by looking at the dimension of these algebras. \[ \square \]

**Lemma 33** The quotient of $$GBB^k_n$$ by the ideal $$I^{(G)}$$ generated by $$e_{1,n}^{(G)}$$ is isomorphic to the Ariki-Koike Algebra $$AK^k_n$$. 

Proof: A graph is of the form $ae_{1,n}^{(G)} b$ if and only if it contains horizontal arcs. The quotient consists hence of those graphs that have only vertical arcs. It is therefore the group of ribbon braids in the cylinder. The relations of this group are known to be a subset of the relations of the Ariki-Koike algebra. The remaining relations follow from the imposed skein relations.

At this point the importance of the index $n$ (total number of strands) of the generator $e_{i,n}^{(G)}$ becomes obvious. Without fixing the total number of strings the ideal would be the whole algebra because minima and maxima can be introduced within the isotopy class of any diagram (cf. figure 4 right). On the other hand we have avoided to restrict the number of strands when defining the skein relations. Using this we obtain the following lemma.

**Lemma 34** The map $\BB_k^n(R) \to \BB_k^{n+2}(R), a \mapsto x^{-1}ae_{n+1}$ is injective for $n \geq 1$.

Proof: By deforming the $n$-th strand of a graph $a$ we may generate maxima and minima as shown in figure 4 (on the left). Thus, locally, we obtain $ae_{n+1}$. If $a$ is in the kernel then this vanishes and hence $a = 0$. \hfill \Box

![Figure 4](image)

**7 Conditional expectation and Markov trace**

The graphical calculus as well as the relationship with the A-type BMW algebra suggest that there should exist a Markov trace on $\BB_k^n$. We follow Wenzl’s original approach \cite{17} as close as possible.

The constructions of this section can equally well be carried out for $\BB_k^n$ and for its graphical counterpart $\BBB_k^n$. Notationally, however, we’ll stick to the former case.

The fundamental hypothesis for the following construction is:

**Hypothesis 35** The map $\BB_k^n \to \BB_k^{n+2}, a \mapsto x^{-1}ae_{n+1}$ is injective.

Lemma 34 has shown that this hypothesis is valid for the graphical algebra.

Let $w = w_1\gamma w_2 \in \BB_k^{n+1}$ where $w_i \in \BB_k^n, \gamma \in \Gamma_{n+1}$. Then we have $e_{n+1}we_{n+1} = w_1e_{n+1}\gamma e_{n+1}w_2 = sw_1w_2e_{n+1}$ with a factor $s$ that assumes the values $s = x, 1, \lambda^{-1}, A_m$ if $\gamma = 1, e_n, X_n, Y_{n+1}^m$. Hypothesis 35 guarantees that the following map is well defined.
Definition 36 Let $e_n : BB^k_{n+1} \rightarrow BB^k_n$ be defined by $e_{n+1}a_{n+1} = x e_n(a) e_{n+1}$.

Obviously, we have $e_n(w_1 a w_2) = w_1 e_n(a) w_2$ if $w_i \in BB^k_n$. Moreover, (33) implies $e_{n+1} = e_{n+1}e_ne_{n+1} = x e_n(e_n) e_{n+1}$ and thus $e_n(e_n) = x^{-1}$. Similarly, (31) implies $e_{n+1} = \lambda^x e_{n+1} X_{n}^x e_{n+1} = \lambda^x e_n(X_n) e_{n+1}$ and thus $e_n(X_n^x) = x^{-1}\lambda^x$. From relation (T3) we deduce $e_{n+1} = A_m^1 e_{n+1} X_{n+1}^m e_{n+1} = A_m^1 x e_n(Y_{n+1}) e_{n+1}$ and hence $e_n(Y_{n+1}) = A_m x^{-1}$.

Iterating the conditional expectation yields a map which will turn out to be a Markov trace.

Definition and Lemma 37 The iterated application of the conditional expectation is denoted by $\text{tr}(a) := \text{tr}(e_{n-1}(a)), \text{tr}(1) := 1$ and fulfils $\text{tr}(e_n) = e_n(e_n) = x^{-1}$, $\text{tr}(X_n^x) = e_n(X_n^x) = x^{-1}\lambda^x$, $\text{tr}(Y_{n+1}^m) = e_n(Y_{n+1}^m) = A_m x^{-1}$.

Lemma 38 For any $w_1, w_2 \in BB^k_n, \gamma \in \Gamma_{n+1}$ we have $\text{tr}(w_1 \gamma w_2) = \text{tr}(\gamma)\text{tr}(w_1 w_2)$ and $e_n(w_1 \gamma w_2) = \text{tr}(\gamma) e_n w_1 w_2$.

Proof: The first statement follows from the second which is shown by the following calculation. $x e_n(w_1 \gamma w_2) e_{n+1} = e_{n+1} w_1 e_n w_2 e_{n+1} = w_1 e_n \gamma e_{n+1} w_2 = w_1 x e_n(\gamma) e_{n+1} w_2 = w_1 w_2 x e_n(\gamma) e_{n+1}$.

Lemma 39 $\forall a \in BB^k_n \ e_n(X_n^{-1} a X_n) = e_n(X_n a X_n^{-1}) = e_n(e_n a e_n) = e_n(a)$

Proof: Let $a = w_1 \gamma w_2 \in BB^k_n, w_i \in BB^k_{n-1}, \gamma \in \Gamma_n$. Multiplying by $x e_{n+1}$ we obtain:

$x e_n(X_n^{-1} w_1 \gamma w_2 X_n)e_{n+1} = x e_n(X_n w_1 \gamma w_2 X_n^{-1}) e_{n+1} = x e_n w_1 \gamma w_2 e_{n+1} = x e_n w_1 \gamma w_2 e_{n+1}$

Omitting the arbitrary factors $w_1, w_2$ yields:

$e_{n+1}(X_n^{-1} \gamma X_n)e_{n+1} = e_{n+1}(X_n \gamma X_n^{-1})e_{n+1} = e_{n+1}(e_n \gamma e_n) e_{n+1} = x \text{tr}(\gamma) e_{n+1}$.

This is checked by analysing the cases for the various values of $\gamma$ successively. For $\gamma = 1$ nothing is to be shown. For $\gamma = e_{n-1}$ we have

$e_{n+1}(X_n^{-1} e_{n-1} X_n)e_{n+1} = e_{n+1}(X_n e_{n-1} X_n^{-1})e_{n+1} = e_{n+1}(e_n e_{n-1} e_n) e_{n+1} = x e_{n-1} e_{n+1}$

The case $\gamma = Y_{n}^m$ yields

$e_{n+1}(X_n^{-1} Y_{n}^m X_n)e_{n+1} = e_{n+1}(X_n Y_{n}^m X_n^{-1}) e_{n+1} = e_{n+1}(e_n Y_{n}^m e_n) e_{n+1} = x \text{tr}(Y_{n}^m) e_{n+1}$

This is checked by analysing the cases for the various values of $\gamma$ successively. For $\gamma = 1$ nothing is to be shown. For $\gamma = e_{n-1}$ we have

$e_{n+1}(X_n^{-1} Y_{n}^m X_n)e_{n+1} = e_{n+1}(X_n Y_{n}^m X_n^{-1}) e_{n+1} = e_{n+1}(e_n Y_{n}^m e_n) e_{n+1} = x e_{n-1} e_{n+1}$

We rewrite the first expression to obtain:

$e_{n+1} Y_{n}^m X_n e_{n+1} = e_{n+1} Y_{n}^m X_n e_{n+1} = e_{n+1} Y_{n}^m X_n e_{n+1} = e_{n+1} Y_{n}^m X_n e_{n+1} = A_m e_{n+1}$
The last case is \( \gamma = X_{n-1} \).

\[
e_{n+1}(X^{-1}_nX_{n-1}X_n)e_{n+1} = e_{n+1}(X_nX_{n-1}X^{-1}_n)e_{n+1} = \\
e_{n+1}(e_nX_{n-1}e_n)e_{n+1} = xtr(X_{n-1})e_{n+1} \\
\Leftrightarrow e_{n+1}(X^{-1}_nX_nX^{-1}_n)e_{n+1} = e_{n+1}(X^{-1}_nX_nX^{-1}_n)e_{n+1} = \\
e_{n+1}(\lambda^{-1}e_n)e_{n+1} = \lambda^{-1}e_{n+1} \\
\Leftrightarrow X_{n-1}e_{n+1}X_n e_{n+1}X^{-1}_n = X_{n-1}e_{n+1}X_n e_{n+1}X^{-1}_n = \lambda^{-1}e_{n+1} = \lambda^{-1}e_{n+1} \\
\Leftrightarrow X_{n-1}\lambda^{-1}e_{n+1}X^{-1}_n = X_{n-1}\lambda^{-1}e_{n+1}X^{-1}_n = \lambda^{-1}e_{n+1}
\]

\( \Box \)

Just as in [17] we have the trace property in the semi-simple case.

**Lemma 40** If \( I_{n+1} \) is semi-simple and \( \text{tr} \) is a trace on \( \text{BB}_n^k \) then \( \text{tr} \) is a trace on \( \text{BB}_n^k \).

Proof: It suffices to show that \( \text{tr}(uv) = \text{tr}(vu) \forall u, v \in \text{BB}_n^k \). If one of the factors (say \( u \)) is contained in \( \text{BB}_n^k \) this is easily seen: \( \text{tr}(uv) = \text{tr}(e_n(uv)) = \text{tr}(u e_n(v)) = \text{tr}(e_n(v)u) = \text{tr}(e_n(vu)) = \text{tr}(vu) \).

According to proposition 26 we may write \( u, v \in \text{BB}_n^k \) in the form

\[
u = u_1 + u_2 Y_{n+1} + u_3 e_n u_4 + u_5 X_n u_6 \tag{79}
\]

\[
v = v_1 + v_2 Y_{n+1} + v_3 e_n v_4 + v_5 X_n^{-1} v_6 \tag{80}
\]

Since \( \text{tr} \) is linear it suffices to investigate all possible combinations of summands. The calculations are similar to those in [17] and [11]. Thus we only give calculations for the cases that involve \( Y \). Our first case is: \( a = a_1 Y_{n+1}^t, b = b_1 Y_{n+1}^t \).

\[
\text{tr}(ab) = \text{tr}(a_1 Y_{n+1}^t b_1 Y_{n+1}^t) = \text{tr}(a_1 Y_{n+1}^t b_1)
\]

\[
= \text{tr}(Y_{n+1}^t b_1 a_1) = \text{tr}(b_1 Y_{n+1}^t a_1 Y_{n+1}^t) = \text{tr}(ba)
\]

Next we look at \( a = a_1 e_n a_2, b = a_3 Y_{n+1}^t \).

\[
\text{tr}(ab) = \text{tr}(a_1 e_n (e_n a_2 a_3 Y_{n+1}^t)) = \text{tr}(a_1 e_n (e_n Y_{n+1}^t) a_2 a_3)
\]

\[
= \lambda x^{-1} \text{tr}(a_1 e_n (e_n Y_{n+1}^t) a_2 a_3) = \lambda x^{-1} \text{tr}(a_2 a_3 a_1 e_n Y_{n+1}^{-1})
\]

\[
= \lambda x^{-1} \text{tr}(a_2 a_3 a_1 e_n (e_n Y_{n+1}^{-1})) = \text{tr}(a_2 a_3 a_1 e_n Y_{n+1}^{-1})
\]

\[
= \lambda x^{-1} \text{tr}(a_2 a_3 a_1 e_n Y_{n+1}^{-1}) = \text{tr}(a_2 a_3 a_1 e_n Y_{n+1}^{-1})
\]

\[
= \text{tr}(e_n (a_3 Y_{n+1}^t a_1 e_n) a_2) = \text{tr}(ba)
\]

The case \( b = a_1 e_n a_2, a = a_3 Y_{n+1}^t \) is treated similarly.

Next case: \( a = a_1 X_n a_2, b = b_1 Y_{n+1}^t \).

\[
\text{tr}(ab) = \text{tr}(a_1 e_n (X_n Y_{n+1}^t) a_2 b_1) = \text{tr}(a_1 e_n (X_n Y_{n+1}^t X_n) a_2 b_1)
\]

\[
= \text{tr}(a_1 e_n (X_n Y_{n+1}^t a_2 b_1) + \delta \text{tr}(a_1 e_n (Y_{n+1}^t X_n) a_2 b_1) - \delta \text{tr}(a_1 e_n (e_n Y_{n+1}^t X_n) a_2 b_1)
\]
The trace is nondegenerate on $\text{CBB}^k_n = \text{BP}_n^k$.

Proof: Let $\{v_i \mid i = 1, \ldots, k^n(2n-1)!!\}$ be a linear basis of dotted Brauer graphs. It suffices to show $\det(\text{tr}(v_iv_j^*)_{i,j}) \neq 0$.

The involution $a \mapsto a^*$ maps graphs to their top-down mirror image and replaces each dot by $k - 1$ dots. Hence the closure of $aa^*$ is free of dots. Now assume that $a$ has $s$ upper (and hence $s$ lower) horizontal arcs. Then there are $s$ cycles in $aa^*$. Upon closing another $s$ cycles are produced from the the remaining horizontal arcs. The vertical arcs form a permutation and $a^*$ contains the inverse permutation. Upon closing these $n - 2s$ vertical arcs yield $n - 2s$ cycles. The closure of $aa^*$ has therefore a total of $n$ cycles and $\text{tr}(aa^*) = 1$.

We now specialise the ground ring: $A_1 := \ldots := A_{k-1} := x^{-1}$. The trace is then a Laurent polynomial in $x$. The choice for the $A_i$ implies that additional dots on an arc decrease the degree (in $x$) of the trace. If $\beta$ is an arc of $a$ and $b$ is any other graph which does not contain an arc which is the mirror image of $\beta$. By considering the cases that $\beta$ is vertical and horizontal individually one easily sees that the cycle in the closure of $ab$ which contains $\beta$ consists of more than two arcs from $a$ and $b$. The closure of $ab$ has therefore less cycles than the closure of $aa^*$. We conclude that $b = a^*$ is the unique graph with highest $x$ degree of $\text{tr}(ab)$. We now consider the determinant of the trace.

$$\det(\text{tr}(v_iv_j^*)_{i,j}) = x^{-nk^n(2n-1)!!}\det((x^{n_0(v_iv_j)} x^{n_1(v_iv_j)/2})_{i,j})$$

In each row the element at the diagonal is the unique element with highest degree in $x$. Calculating the determinant thus yields a sum with a unique term of highest degree. Thus the determinant does not vanish.
8 The structure theorem

In this section we determine the structure of $\text{BB}_n^k(K_1)$. It will turn out to be semi-simple over this generic ground field. We only need a few definitions on Young diagrams before we can state the structure theorem.

A Young diagram $\lambda$ of size $n$ is a partition of the natural number $n$. $\lambda = (\lambda_1, \ldots, \lambda_k), \sum_i \lambda_i = n, \lambda_i \geq \lambda_{i+1}$. In the following we use ordered tuples of Young diagrams $\lambda = (\lambda^1, \ldots, \lambda^k)$ (cf. \[1\]). The size of a tuple of Young diagrams is the sum of sizes of its components. Let $\hat{\Gamma}_n^k$ be the set of all $k$ tuples of Young diagrams of sizes $n, n-2, \ldots$.

**Proposition 42**

1. $\text{BB}_n^k(K_1)$ is a semi-simple algebra isomorphic to $\text{GBB}_n^k(K_1)$.

   The simple components are indexed by $\hat{\Gamma}_n^k$.

   $\text{BB}^k_n = \bigoplus_{\lambda \in \hat{\Gamma}_n^k} \text{BB}_{n,\lambda}^k$ \hspace{1cm} (82)

2. The Bratteli rule for restrictions of modules: A simple $\text{BB}^k_n, \nu$ module $V_{\nu, \lambda} \in \hat{\Gamma}_n^k$ decomposes into $\text{BB}^k_{n-1}$ modules such that the $\text{BB}^k_{n-1}$ module $\lambda \in \hat{\Gamma}_{n-1}^k$ occurs iff $\lambda$ may be obtained from $\nu$ by adding or removing a box.

3. $\text{tr}$ is a faithful trace. To every tuple of Young diagrams $\lambda \in \hat{\Gamma}_n^k$ there is an idempotent $p_\lambda$ and a non vanishing, rational function $Q_\lambda$ which does not depend on $n$ and satisfies $\text{tr}(p_\lambda) = Q_\lambda/n^\infty$.

For the proof of the structure theorem we need some facts from Jones-Wenzl theory of inclusions of finite dimensional semi-simple algebras.

Let $A \subset B \subset C$ be a unital embedding of finite dimensional semi-simple algebras and let $\text{tr}$ be a trace on $A, B$ that is compatible with the inclusion. The associated conditional expectation is denoted by $\epsilon_A : B \rightarrow A, \text{tr}(ab) = \text{tr}(a\epsilon_A(b))$. It is assumed that there is an idempotent $e \in C$ such that $e^2 = e, ebc = e\epsilon_A(b)\forall b \in B$ and $\varphi : A \rightarrow C, a \mapsto ae$ is injective.

Such a situation can be realized starting from an inclusion pair $A \subset B$ with a common faithful trace $\text{tr}$ and conditional expectation $\epsilon_A$. We set $\hat{C} := \{\alpha : B \rightarrow B \mid \text{linear}, \alpha(ba) = \alpha(b)\alpha(a) \in A, b \in B\}$. The inclusion $B \subset \hat{C}$ is given by $b \mapsto \alpha_b, \alpha_b(b_1) := bb_1$. Here $e$ is given by $e_A = \epsilon_A : B \rightarrow B$. The sub-algebra of $\hat{C}$ generated by $B$ and $e_A$ is denoted by $\langle B, e_A \rangle$. For this setup Wenzl has obtained the following results \[7\] Theorem 1.1

1. $\langle B, e_A \rangle \cong \text{End}_A(B)$

2. The simple components of $A$ and $\langle B, e_A \rangle$ are in 1-1 correspondence. The inclusion matrices of $A \subset B \subset \langle B, e_A \rangle$ are relatively transposed. If $p$ is a minimal idempotent in $A$ then $pe_A$ is a minimal idempotent in $\langle B, e_A \rangle$

3. $\langle B, e_A \rangle \cong B e_A B$
4. \( < B, e > \cong < B, e_A > \oplus \bar{B} \) where \( \bar{B} \) is a sub-algebra of \( B \).

5. \( \bar{e} \) implies that the ideal generated by \( e \) in \( C \) is isomorphic to \( < B, e_A > \).

We now prove the main theorem.

Proof: \( BB^k_0 \) is simply the ground ring. Thus the proposition is true with \( \text{tr}(p_\langle \ldots \rangle) = \text{tr}(1) = Q_{\langle \ldots \rangle}/x^0, Q_{\langle \ldots \rangle} = 1 \). The algebra \( BB^k_1 \) is of dimension \( k \) and has a basis \( \{1,Y, \ldots Y^{k-1}\} \). It is commutative and semi-simple. The simple blocks are given by the eigen spaces of \( Y \). Existence of idempotents is clear. The graphical version is isomorphic as a simple consequence of Turaev’s result on the skein module of the solid torus [1].

We have to establish that the trace on \( BB^k_1 \) is nondegenerate. Now, suppose it were degenerate. Choose \( 0 \neq a = \sum m_am^iY^m \in BB^k_1 \) such that \( \forall b \in BB^k_1 : \text{tr}(ab) = 0 \). It is \( \text{tr}(ab)e_1 = x^{-1}e_1ab \). We have already noted in the discussion following definition [1] that the ideal generated by \( e_1 \) is isomorphic and that it is spanned by \( Y^i e_1 Y^j \). The trace does not vanish on this module and hence it is faithful. Furthermore the embedding \( BB^k_1 \subset BB^k_2 \) is faithful and hence there must be an element \( b' \) in the ideal \( I_2 \) such that \( \text{tr}(ab') \neq 0 \). Suppose \( b' = \sum_{i,j} c_{i,j}^Y e_1 Y^j \). We then have \( xe_1(ab')e_2 = \sum_{i,j,m} a_m c_{i,j}^Y e_2 Y^i e_1 Y^j e_2 = \sum_{i,j,m} a_m c_{i,j}^Y e_2 e_1 Y^j = \sum_{i,j,m} a_m c_{i,j}^Y e_2 Y^j \) and hence \( \text{tr}(ab')e_1 = x^{-2} e_1 \sum_{i,j,m} a_m c_{i,j}^Y Y^{m+i} e_1 = x^{-2} \sum_{m,i,j} a_m c_{i,j}^Y e_1 Y^{m+i} e_1 \neq 0 \). This is the required contradiction.

Assume the proposition is shown by induction for \( BB^k_n \).

By induction assumption \( BB^k_n = GBB^k_n \). We investigate the kernel of the inclusion \( i : BB^k_n \to BB^k_{n+2} \) introduced in section [7]. Assume \( i(a) = 0 \), then we have \( 0 = \Psi_{n+2}(i(a)) = i(G)(a) \). Since \( i(G) \) is injective according to lemma [3] it follows that \( a = 0 \) and hence that \( i \) is injective. Thus we can use the results from section [4].

We apply Jones-Wenzl theory to the following situation: \( A = BB^k_{n-1}, B = BB^k_n, C = BB^k_{n+1}, e = x^{-1}e_n, e_A = e_{n-1} \). This is possible because \( A, B \) are semi-simple algebras with a faithful trace by induction assumption. All properties needed for \( e \) have already been established. Statement [1] of Jones-Wenzl theory asserts the semi-simplicity of \( \text{End}_A(B) \cong < B, e_A > \) which is by [3] the ideal generated by \( e \). Thus \( I_{n+1} \) is semi-simple. The quotient algebra \( BB^k_{n+1}/I_{n+1} \) is the Ariki-Koike algebra \( Ak^{(k)}_{n+1} \) and is semi-simple according to [1]. Since we work over a field we can conclude (by looking at the radicals) that \( BB^k_{n+1} \) is semi-simple and that it is isomorphic to the direct sum \( BB^k_{n+1} = I_{n+1} \oplus BB^k_{n+1}/I_{n+1} \). Statement [2] asserts that the simple components of \( I_{n+1} \) are indexed by \( \hat{\Gamma}^k_{n-1} \). The simple components of \( Ak^{(k)}_{n+1} \) are indexed by tuples of Young diagrams of size \( n+1 \) (see [1]).

Consider the situation for the graphical algebra \( GBB^k_{n+1} \). By Jones-Wenzl theory we know that the ideals \( I_{n+1} \) and \( I_{n+1}^{(G)} \) are isomorphic. The quotient \( GBB^k_{n+1}/I_{n+1}^{(G)} \) is by lemma [33] isomorphic to the Ariki-Koike algebra. Hence, we have \( GBB^k_{n+1} = BB^k_{n+1} \). This completes the proof of point [1] of the theorem.

The inclusion matrix for the part \( I_{n+1} \) is the transpose of the inclusion matrix of \( BB^k_{n-1} \subset BB^k_n \). For the part \( Ak^{(k)}_{n+1} \) the Bratteli rule follow from [1].

We have to show that \( \mathfrak{tr} \) is faithful, i.e. that the \( Q \) functions don’t vanish. If \( p_\Delta \in BB^k_{n-1} \) is a minimal idempotent in \( BB^k_{n-1} \) then \( x^{-1} p_\mu \epsilon_n \) is a minimal idempotent in
The trace of this idempotent is $\text{tr}(x^{-1}p_\Delta e_n) = x^{-2}\text{tr}(p_\Delta) = Q_\Delta/x^{n-1+2}$. Obviously, this is non vanishing (using the induction assumption). The idempotents of this kind are those of $I_{n+1}$. For the other idempotents (which are those of $BB^k_{n+1}/I_{n+1}$) the function $Q$ is defined by $\text{tr}(p_\Delta) = Q_\Delta/x^n$.

To establish faithfulness of the trace we use the classical limit. A minimal idempotent $p_\Delta$ of $BB^k_n$ yields an idempotent in the classical limit described in section 8. We know already that on the classical limit algebra the trace is nondegenerate. Hence the function $Q_\Delta$ does have a non vanishing classical limit and hence can’t be zero itself.

The Ariki-Koike-Birman-Wenzl algebra with $Y$ satisfying a quadratic relation is of special interest and has been studied in [11].

Naturally, one would like to study the algebra not only over the generic field but also with complex parameters. The classical limit is then a point in parameter space and we know that at this point the algebra is semi-simple. The functions $Q$ are apart from a finite set of poles continuous and hence there is a neighbourhood of the classical point where the algebra is semi-simple. While some necessary conditions for semi-simplicity may be derived easily from the knowledge of the Ariki-Koike algebra the determination of sufficient conditions has to await further studies.

The Markov trace can be used to define a link invariant for links of B-type which are links in a solid torus. There is an analog of Markov’s theorem for type B links found by S. Lambrodopoulou in [12]. It takes the same form as the usual Markov theorem, i.e. two B-braids $\beta_1, \beta_2$ have isotopic closures $\hat{\beta}_1, \hat{\beta}_2$ if $\beta_1, \beta_2$ may transformed in one another by a finite sequence of moves of the following two kinds: I Conjugation $\beta \sim \alpha\beta\alpha^{-1}$ and II $\alpha \sim \alpha\tau_n$ for $\alpha \in \mathbb{Z}_B$.

This theorem implies that there exists an extension of the Kauffman polynomial to braids of B-type. Denote by $\pi : \mathbb{Z}_B \rightarrow BB^k_n$ the morphism $\tau_i \mapsto X_i$. Then we obtain without any further proof an invariant of the B-type link $\hat{\beta}$ that is the closure of a B-braid $\beta \in \mathbb{Z}_B$ by the following definition:

**Definition 43** The B-type Kauffman polynomial of a B-link $\hat{\beta}$ is defined to be

$$L(\hat{\beta}, n) := x^{n-1}e^{(\beta)}\text{tr}(\beta) \quad \beta \in \mathbb{Z}_B$$

(83)

e : \mathbb{Z}_B \rightarrow \mathbb{Z} is the exponential sum with $e(X_i) = 1, e(Y) = 0$. 

Figure 5: The first three lines of the Bratteli digram of $BB^2_n$
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