A Single Particle Interpretation
of Relativistic Quantum Mechanics
in 1+1 Dimensions

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Abstract

The relativistic free particle system in 1+1 dimensions is formulated as a
"bi-Hamiltonian system". One Hamiltonian generates ordinary time transla-
tions, and another generates (essentially) boosts. Any observer, accelerated
or not, sees evolution as the continuous unfolding of a canonical transforma-
tion which may be described using the two Hamiltonians. When the system
is quantized both Hamiltonians become Hermitian operators in the standard
positive definite inner product. Hence, each observer sees the evolution of
the wave function as the continuous unfolding of a unitary transformation in
the standard positive definite inner product. The result appears to be a con-
sistent single particle interpretation of relativistic quantum mechanics. This
interpretation has the feature that the wave function is observer dependent,
and observables have non-local character, similar to what one might expect
in quantum gravity.
Traditionally it has not been possible to extend the single particle interpretation of non-relativistic quantum mechanics to relativistic quantum mechanics. The main problem has been the lack of a positive definite inner product associated with a locally conserved current. The necessity that the probability density transform as the zeroth component of a 4-vector seems to preclude such inner product.

It is shown in this paper that the Heisenberg picture formalism developed in [1] (motivated by Rovelli’s “evolving constants of the motion” [2]) leads to a relativistic quantum mechanics in 1+1 dimensions which is a natural extension of non-relativistic quantum mechanics, and which includes a new notion of conserved current. To make contact with the problems of ordinary relativistic quantum mechanics it is necessary to switch to the Schrödinger picture. It is found that doing this requires a paradigm shift; the wavefunction must be associated with an observer-system pair, instead of with a system alone.

As has been suggested by Rovelli [3], perhaps the notion of an observer-independent state of a system is flawed in a way analogous to the flaw in the notion of observer-independent simultaneity. I implement this concept in the simple example of the quantum mechanics of the relativistic free particle by associating different observers (accelerated or not) with different sequences of flat spacelike submanifolds of Minkowski space, the submanifolds perpendicular to their worldlines. The notion of conserved current becomes the notion of consistent unitary evolution for different observers. This may be summarized by saying that we regard the important object of the formalism to be not a wave function which is a mapping of spacetime to the complex numbers, but rather a mapping of the space of flat spacelike submanifolds of Minkowski space to wavefunctions on space.

We will find that the physical consequence of adopting the above point of view is that observables have non-local character; it is important for the interpretation that detectors not be infinitesimal. This makes the result interesting from the point of view of quantum gravity, which is expected to have this feature. Whether or not this theory can be sensibly “second quantized”, and whether or not the results would agree with experiment is not discussed. It is not clear if the type of non-locality predicted would be excluded by existing experimental data. However, our results appear to be equivalent to Fleming’s hyperplane dependent relativistic quantum mechanics [4], so that his discussion [5, 6, 4, 7] is relevant.

1. Bi-Hamiltonian System

In general relativity, asking what is the “time” should be replaced by asking what is the “spacelike submanifold”. This presents a problem in the interpretation of quantum mechanics; the standard interpretations of quantum mechanics use the assumption that the “time” is a real number. Because the technical problems of general relativity are so immense, it is useful to note that there is an analogous interpretation of time in special relativity. In special relativity asking what is the “time” should be replaced by asking what is the “flat spacelike submanifold in Minkowski space”. These planes are parametrized by Lorentz transformations and ordinary time translations. If we restrict to the (1+1)-dimensional case and fix a reference coordinate system, then the straight spacelike lines (flat spacelike submanifolds) can be parametrized by the ordinary times where the lines cross the time axis.
t, and a boost parameter, \( u \), which parametrizes the angles that the lines make with the horizontal. Specifically let \( \alpha \) be the angle that a line makes with the horizontal and take
\[
u = \frac{\tan \alpha}{\sqrt{1 - \tan^2 \alpha}} = \frac{\beta}{\sqrt{1 - \beta^2}}.
\]
which ranges \((-\infty, \infty)\).

An observer determines a continuous ordered sequence of planes, the planes perpendicular to the 4-velocity of its timelike path. Hence, an observer defines a path in \( \mathbb{R} \times \mathbb{R} \) by the evolution of the parameters \( t \) and \( u \) associated with it. We will refer to \( \mathbb{R} \times \mathbb{R} \) as the space of global time. Clearly the concept of global time can be extended to higher dimensional special relativity; the space of global time will be higher dimensional. In reference [1] the concept of a space of global time is discussed in a more general way, with reference to quantum gravity.

Let \( q(t,u) \) and \( p(t,u) \) be the position and momentum in the coordinate system boosted relative to the reference coordinate system as determined by \( u \), at the time determined by \( t \). Then we have the following:

**Result 1** Any observer, that is any path in the space of global time:
\[
\phi : \mathbb{R} \rightarrow \mathbb{R} \times \mathbb{R} = \text{space of global time},
\]
which together determine the evolution seen by any observer according to
\[
dq = dt \{ q, H_t \}_{pb} + du \{ q, H_u \}_{pb},
\]
\[
dp = dt \{ p, H_t \}_{pb} + dp \{ p, H_u \}_{pb}.
\]
It may be verified by direct substitution that the evolution determined by (1.6) satisfies equation (1.5). (Again the calculation is not illuminating and will be omitted.) Hence, the evolution is by canonical transformation and Result 1 is demonstrated. Of course it may also be verified that \[ \left\{ q(t, u), p(t, u) \right\}_{p_b} = 1. \]

2. Quantization After Evolution

For an ordinary Hamiltonian system, evolution is given by the Hamiltonian equations of motion:
\[
\frac{dq}{dt} = \{ q, H \}, \quad \frac{dp}{dt} = \{ p, H \}. \tag{2.1}
\]

If at some fixed initial time \( t_0 \) we have \( q(t_0) = q_0 \) and \( p(t_0) = p_0 \), then at each later time, \( t \), \( q(t) \) and \( p(t) \) can be expressed as functions of \( q_0 \) and \( p_0 \):
\[
q(t) = Q(t, q_0, p_0),
\]
\[
p(t) = P(t, q_0, p_0). \tag{2.2}
\]

The evolution is by canonical transformation so that
\[
\left\{ q(t), p(t) \right\} = \{ q_0, p_0 \} \overset{def}{=} 1. \tag{2.3}
\]

Ordinarily in quantizing a Hamiltonian system we make \( q_0, p_0 \) and \( H \) into Hermitian operators \( \hat{q}_0, \hat{p}_0 \) and \( \hat{H} \), and evolve by means of the Heisenberg equations of motion:
\[
\frac{d\hat{q}}{dt} = \frac{1}{i\hbar} [\hat{q}, \hat{H}], \quad \frac{d\hat{p}}{dt} = \frac{1}{i\hbar} [\hat{p}, \hat{H}]. \tag{2.4}
\]

If at some fixed time \( t_0 \) we have \( \hat{q}(t_0) = \hat{q}_0 \) and \( \hat{p}(t_0) = \hat{p}_0 \), then at each later time, \( t \), \( \hat{q}(t) \) and \( \hat{p}(t) \) can be expressed as functions of \( \hat{q}_0 \) and \( \hat{p}_0 \):
\[
\hat{q}(t) = \hat{Q}(t, \hat{q}_0, \hat{p}_0),
\]
\[
\hat{p}(t) = \hat{P}(t, \hat{q}_0, \hat{p}_0). \tag{2.5}
\]

The evolution is by unitary transformation so that
\[
\frac{1}{i\hbar} [\hat{q}(t), \hat{p}(t)] = \frac{1}{i\hbar} [\hat{q}_0, \hat{p}_0] \overset{def}{=} 1. \tag{2.6}
\]

We will refer to the function \( \mathcal{O} \) on phase space (for example \( q_0 \)) which corresponds to the operator \( \hat{\mathcal{O}} \) (for example \( \hat{\mathcal{O}} \)) as the classical limit of \( \mathcal{O} \), or \( \mathcal{O} = \lim_{\hbar \to 0} \hat{\mathcal{O}} \). In Heisenberg picture quantization we define the classical limit of \( \hat{q}_0 \) to be \( q_0 \) and the classical limit of \( \hat{p}_0 \) to be \( p_0 \). That is, the quantization is carried out at some fixed time \( t_0 \). The use of the Heisenberg equations of motion to find \( \hat{q}(t) \) and \( \hat{p}(t) \) for \( t > t_0 \) is supposed to ensure that \( \lim_{\hbar \to 0} \hat{q}(t) = q(t) \) and \( \lim_{\hbar \to 0} \hat{p}(t) = p(t) \). This means that the functions \( \hat{Q}(t, *, *) \) and \( \hat{P}(t, *, *) \) of \( \hat{q}_0 \) and \( \hat{p}_0 \) are obtained from the functions \( Q(t, *, *) \) and \( P(t, *, *) \) of \( q_0 \) and \( p_0 \).
by substituting \( \hat{q}_0 \) and \( \hat{p}_0 \) for \( q_0 \) and \( p_0 \) and choosing “the correct operator ordering”. (Of course the choice of operator ordering does not effect the classical limit.) Hence, one might attempt to quantize the system directly by making this substitution, without reference to the Heisenberg equations of motion. This operation will be called “quantization after evolution”, because the evolution problem is solved classically and the whole “already evolved classical system” is quantized at once. It is closely related to Rovelli’s “evolving constants” picture of dynamics [3]. Examples of this construction have been worked out by Carlip [8] in the context of comparing different quantizations.

The obvious approach to quantizing our bi-Hamiltonian system is to turn \( q_0, p_0, H_t(t, u) \) and \( H_u(t, u) \) into hermitian operators \( \hat{q}_0, \hat{p}_0, \hat{H}_t(t, u) \) and \( \hat{H}_u(t, u) \), and evolve to arbitrary global time \((t, u)\) using the pair of Heisenberg equations of motion. There is, however, a potential problem here. Due to operator ordering ambiguities, evolving from the global time \((t_0, u_0)\) to the global time \((t, u)\) by different paths in the space of global time may yield different results. That is to say, an observer who moves from global time \((t_0, u_0)\) to global time \((t, u)\) by accelerating from speed \(v_0\) and then decelerating back to speed \(v_0\) may find different operators \(\hat{q}(t, u)\) and \(\hat{p}(t, u)\) then an observer who makes the trip (through the space of global time) by staying at the constant velocity \(v_0\). If the name “space of global time” is worth its salt, then all observers should agree on \(\hat{q}(t, u)\) and \(\hat{p}(t, u)\).

Note now that the operation of quantization after evolution, applied to the bi-Hamiltonian system, is inherently path independent. If we fix some “initial” global time \((t_0, u_0)\), then for each \((t, u)\) we find that \(q(t, u)\) and \(p(t, u)\) are expressed as functions of \(q(t_0, u_0) = q_0\) and \(p(t_0, u_0) = p_0\):

\[
q(t, u) = Q(t, u, q_0, p_0) \\
p(t, u) = P(t, u, q_0, p_0).
\]

The functions \(Q\) and \(P\) are given explicitly in equation (1.6). Quantization after evolution involves substituting \(\hat{q}_0\) and \(\hat{p}_0\) for \(q_0\) and \(p_0\) in the functions \(Q\) and \(P\) and choosing the operator orderings so that \(\hat{q}(t, u)\) and \(\hat{p}(t, u)\) are hermitian and related to \(\hat{q}_0\) and \(\hat{p}_0\) by a unitary transformation for all \((t, u)\). The important point, however, is that no matter what operator orderings are chosen, the above substitution yields unique values for \(\hat{q}(t, u)\) and \(\hat{p}(t, u)\) for each \((t, u)\), and therefore avoids the issue of path dependence mentioned above.

Now let us quantize the bi-Hamiltonian system using quantization after evolution. We substitute the operators \(\hat{q}_0\) and \(\hat{p}_0\) for \(q_0\) and \(p_0\) in the functions \(Q\) and \(P\) (equation (1.6)), and choose the most obvious operator orderings which make \(\hat{q}(t, u)\) and \(\hat{p}(t, u)\) Hermitian in the inner product

\[
\int \psi^*(q, t, u) \psi(q, t, u) dq
\]

(2.8)

(where \(\hat{q}_0\) is represented by multiplication by \(q_0\) and \(\hat{p}_0\) is represented by \(\frac{\hbar}{\imath \partial_{q_0}}\)). The result is

\[
\hat{q}(t, u) = \frac{1}{2} \hat{q}_0 \frac{\sqrt{m^2 + \hat{p}_0^2}}{\sqrt{1 + u^2 \sqrt{m^2 + \hat{p}_0^2 - u\hat{p}_0}}}
\]

(2.9)
\[ \hat{q}(t, u) = \sqrt{1 + u^2} \hat{p}_0 - u \sqrt{m^2 + \hat{p}_0^2}. \] (2.9)

The operator \( \sqrt{m^2 + \hat{p}_0^2} \) may be defined by means of spectral decomposition. To get a “single particle interpretation” only the positive (or negative) root of the eigenvalues of the operator \( m^2 + \hat{p}_0^2 \) should be retained. This is in some sense justified by the fact that the classical Hamiltonian is the positive branch of the square root of \( m^2 + \hat{p}_0^2 \). The definition of the square root operator in this way has been discussed in [9].

We do not yet know if \( \hat{q}(t, u) \) and \( \hat{p}(t, u) \) are related to \( \hat{q}_0 \) and \( \hat{p}_0 \) by a unitary transformation for all \( (t, u) \). To verify this, we make \( \hat{H}_t \) and \( \hat{H}_u \) into Hermitian operators \( \hat{H}_t^\dagger = \hat{H}_t \) and \( \hat{H}_u^\dagger = \hat{H}_u \) by choosing the most obvious operator orderings which make them Hermitian:

\[ \hat{H}_t = \sqrt{1 + u^2} \sqrt{m^2 + \hat{p}_0^2}, \quad \text{and} \]
\[ \hat{H}_u = \frac{tu}{\sqrt{1 + u^2}} \sqrt{m^2 + \hat{p}_0^2} + \frac{1}{2} \sqrt{1 + u^2} \hat{q} \sqrt{m^2 + \hat{p}_0^2} + \frac{1}{2} \sqrt{1 + u^2} \sqrt{m^2 + \hat{p}_0^2} \hat{q} - \hat{t}. \] (2.10)

It is straightforward to verify that the expressions for \( \hat{q}(t, u) \) and \( \hat{p}(t, u) \) (2.9) satisfy the Heisenberg equations of motion

\[ \frac{\partial \hat{q}}{\partial t} = \frac{1}{i\hbar} [\hat{q}, \hat{H}_t], \quad \frac{\partial \hat{p}}{\partial t} = \frac{1}{i\hbar} [\hat{p}, \hat{H}_t], \]
\[ \frac{\partial \hat{q}}{\partial u} = \frac{1}{i\hbar} [\hat{q}, \hat{H}_u], \quad \frac{\partial \hat{p}}{\partial u} = \frac{1}{i\hbar} [\hat{p}, \hat{H}_u], \] (2.11)

so that operator evolution is by unitary transformation.

We have now demonstrated the following:

**Result 2** Any observer, that is any path in the space of global time:

\[ \phi : R \to \mathbb{R} \times \mathbb{R} = \text{space of global time}, \] (2.12)
\[ \tau \mapsto (t(\tau), u(\tau)), \] (2.13)

sees evolution of the free particle operators \( \hat{q}(t(\tau), u(\tau)) \) and \( \hat{p}(t(\tau), u(\tau)) \) as the continuous unfolding of a unitary transformation in the standard positive definite inner product,

\[ \int \psi^\dagger(q, t, u) \psi(q, t, u) dq \] (2.14)

(where \( \hat{q}_0 \) is represented by multiplication with \( q_0 \) and \( \hat{p}_0 \) is represented by \( \frac{\hbar}{i} \frac{\partial}{\partial q_0} \)). There are two Hamiltonians (2.10) which together determine this evolution for any observer according to

\[ dq = dt \frac{1}{i\hbar} [\hat{q}, \hat{H}_t] + du \frac{1}{i\hbar} [\hat{q}, \hat{H}_u], \]
\[ dp = dt \frac{1}{i\hbar} [\hat{p}, \hat{H}_t] + du \frac{1}{i\hbar} [\hat{p}, \hat{H}_u]. \] (2.15)
The evolution is consistent in the sense that all observers agree on the operators \( \hat{q}(t,u) \) and \( \hat{p}(t,u) \) for each global time \((t,u)\).

The first Hamiltonian, \( \hat{H}_t \), describes the operator evolution as seen by an observer who does not accelerate (i.e. \( t \) changes while \( u \) remains fixed). The second, \( \hat{H}_u \), describes the evolution seen by a “highly” accelerated observer (i.e. \( u \) changes while \( t \) remains fixed). For an unaccelerated observer there is a wave function on spacetime which satisfies the square root Schrödinger equation. This equation has been discussed in [9] and in [10]. In [9] the positive root is taken so that there are no negative frequency modes. Strictly speaking, it is only in this way that a “single particle interpretation” is obtained, since the negative frequency modes are generally considered to represent different particles. This gives rise to many interesting issues concerning the localization of particles and position operators [11, 12, 13, 14, 5]. As mentioned below, these issues lead to an alternative motivation for the present work. It is also possible to relax the restriction to a single particle and allow negative energy modes; the eigenvalues of the square root operator acting on the negative energy modes are simply taken to be negative. However, as discussed in [10], there is some question as to the correct way to introduce interactions between the positive and negative energy modes in the setting of the square root Schrödinger equation.

3. Interpretation of Results

We have defined operators \( \hat{q}(t,u) \) and \( \hat{p}(t,u) \) on the space of global time, \( \mathbb{R} \times \mathbb{R} \), which “evolve” by unitary transformation along any path in the space of global time, with the standard positive definite inner product. (The reader is reminded that the space of global time in the example of this paper is the space of flat spacelike submanifolds of Minkowski space.) The relevant path in the space of global time is singled out by the choice of observer. It is the set of submanifolds perpendicular to the observer’s worldline. This allows us to make physical predictions of the following form: an observer may measure an observable at global time \((t_0,u_0)\), for example \( \hat{q}(t_0,u_0) \), thus producing a state \( |\psi_0\rangle \) which is an eigenstate of \( \hat{q}(t_0,u_0) \),

\[
\hat{q}(t_0,u_0)|\psi_0\rangle = q(t_0,u_0)|\psi_0\rangle;
\]

the observer may then evolve along any path in the space of global time to \((t,u)\) (if \( u \neq u_0 \) this will involve acceleration), and predict the outcome of its measurement of, for example, \( \hat{q}(t,u) \). The expectation value for the observer’s second measurement is \( \langle \psi_0|\hat{q}(t,u)|\psi_0\rangle \).

The above implies that different global times are regarded as different “times”. To understand the physical implication of this, consider the case in which the observer is located on the time axis of the reference coordinate system when it makes its second measurement. Then the different global times \((t,u)\) and \((t,u')\) find the observer at the same point in spacetime. Hence the fact that \( \hat{q}(t,u) \neq \hat{q}(t,u') \) when \( u \neq u' \) implies that the probability amplitude for the outcome of a position measurement depends on the observer’s velocity, not just on the observer’s position in spacetime. This reflects the fact that different velocities correspond to different definitions of “simultaneous”, that is to different spatial slices through the observer’s position in spacetime. It is physically sensible as long as the observer’s detector
is of finite size (not infinitesimal), because a detector of finite size at global time \((t, u)\) occupies a different region of spacetime (different spatial slice) than when it is at global time \((t, u')\). It follows that measurements of the position operators \(\hat{q}(t, u)\) and \(\hat{q}(t, u')\) are genuinely different physical measurements. Although it would seem to be impossible to determine a particle’s position with an infinitesimal detector, we should note that these statements break down for an infinitesimal detector; the position in spacetime of an infinitesimal detector located on the time axis is independent of \(u\). In this sense, non-locality of physical observables plays an important part in this formulation of relativistic quantum mechanics. It is not immediately clear if this kind of non-locality is incompatible with experimental results. A careful study of the measurement process is necessary.

To understand how this interpretation relates to the standard problems of relativistic quantum mechanics it is necessary to switch to the Schrödinger picture. Corresponding to each global time is an instantaneous wave function defined on the spacelike submanifold associated with the global time. However, spacelike submanifolds which correspond to different global times may intersect in spacetime. There appears to be no reason to believe that the instantaneous wave functions defined on the submanifolds will agree at the intersection points. In the Appendix we show that in general they do not. Hence, it is not possible to define the wave function as a mapping of spacetime to the complex numbers. Instead, the important object of the formalism is a mapping of the space of global time (flat spacelike planes in Minkowski space) to wave functions on space:

\[ \Phi : \{ \text{embeddings } \Sigma \hookrightarrow M \mid \text{spacelike and flat} \} \rightarrow \Sigma^*, \] (3.2)

where \(\Sigma^*\) is the complex dual of \(\Sigma\), and in \(n+1\) dimensions \(\Sigma\) is \(\mathbb{R}^n\). An observer is then associated with a sequence of instantaneous wave functions, different sequences for different observers. Only if the observer is unaccelerated is it clear that this gives rise to a wave function on spacetime. (As mentioned earlier, this wave function will satisfy the square root Schrödinger equation.)

In essence, the observer has been “relativized” (as Smolin puts it \cite{Smolin}). The state is not considered as an object associated with the physical system, but rather as an object associated with an observer observing a physical system. In Rovelli’s 1993 preprint “On Quantum Mechanics” \cite{Rovelli}, he suggests that the notion of observer-independent state of a system may be flawed in a way analogous to the flaw in the notion of observer-independent simultaneity. In the construction of the present paper, if \(q(t, u)\) and \(q(t, u')\) have different values when \(u \neq u'\), then the operators associated with them, \(\hat{q}(t, u)\) and \(\hat{q}(t, u')\), are different. This seems to be sensible and indeed necessary if the theory is to have the correct classical limit. Yet it means that, in the Heisenberg picture, observers at the same point in spacetime moving at different velocities naturally measure different operator observables. Switching to the Schrödinger picture, it means that these observers are naturally associated with different wavefunctions. This is a consequence of the fact that their notions of simultaneity are different. Hence we are led to accept the suggestion of Rovelli, and closely related suggestions

*Stated another way, it is an object associated with a set of measurements that have been made together with a set of measurements that can be made. This seems to tie in with consistent histories quantum mechanics.
of Smolin [13], and Crane [14], and to propose this somewhat unconventional relativistic quantum mechanics.

To evaluate the self-consistency of a theory of this form we must discuss the issue of multiple observers. Consider the free particle system with two observers, Observer Number One and Observer Number Two. In its calculations, Observer Number One can always, if necessary, consider the larger system of Observer Number Two interacting with the free particle. As long as this is a sensible physical system, then Observer Number One will not find any contradictions. Hence, Observer Number One will not find a contradiction as long as it doesn’t ignore part of the physical system, namely Observer Number Two, in its calculations. Of course if Observer Number Two never interacts with Observer Number One, and never interacts with the free particle (except, possibly, to duplicate some of the measurements made by Observer Number One), then Observer Number One need only consider the free particle system in its calculations. Rovelli has discussed interpretational issues of this type in [3]. The important point is that we avoid internal inconsistencies by studying observer-system pairs, instead of systems in the abstract. Another way to say this is that we study sequences of measurements. Hence, this approach is related to consistent histories quantum mechanics. (The relativistic quantum mechanics of the free particle appears to be sensible in the consistent histories formalism [14,15].)

The key element that allows this single particle interpretation to succeed is the replacement of the notion of locally conserved current with the notion of consistent unitary evolution for different observers. For a classical particle flux the answer to the question “What is the probability for finding a particle in a certain region of spacetime?” is observer independent. This is the essence of the concept of locally conserved current. However, an actual measurement determining if a particle is in a region of spacetime consists of performing a measurement to determine if a particle is in a region of space, and then continuing this measurement for some period of time. If different spatial slices are used to define simultaneity, then a different experimental procedure is required. Although classical special relativity predicts that these two experimental procedures will yield the same result, it is not clear that this equivalence should be regarded as an essential element of a relativistic theory. The version of quantum mechanics presented here is proclaimed to be relativistic because the quantization map \((q(t,u) \mapsto \hat{q}(t,u), p(t,u) \mapsto \hat{p}(t,u))\) induces an isomorphism between two representations of the group generated by Lorentz transformations and time translations, one with representation space \(\{(q(t,u), p(t,u)) \mid t, u \in \mathbb{R}\}\) and the other with representation space \(\{\langle \hat{q}(t,u), \hat{p}(t,u) \rangle \mid t, u \in \mathbb{R}\}\). It is a matter of taste whether or not this is enough to call the theory relativistic. The present theory does not predict that the two experimental procedures described above will yield the same result. In this version of relativistic quantum mechanics the answer to the question “What is the probability for finding a particle in a certain region of spacetime?” is observer dependent. The closest concept to that of locally conserved current that is allowed is that of consistent unitary evolution for different observers: stated in the Heisenberg picture, all observers see unitary evolution of

\[1\]Certainly, the result obtained here does not capture the full spirit of the work of Rovelli, Smolin, and Crane. The free particle system is not sufficiently complex; it is not possible to consider a variety of observer-system splits. We have merely considered a variety of abstract observers observing the free particle system.
the operator observables, and all observers present at a particular global time agree on the
operator observables; stated in the Schrödinger picture, all observers see unitary evolution
of their wavefunctions, and all observers present at a particular global time have the same
wavefunction.

In the late stages of this work I was introduced to the work of Fleming.‡ It appears that
the theory developed here is, in essence, an evolving constants approach to Fleming’s hyperplane (flat spacelike submanifold) dependent relativistic quantum mechanics. In a series of papers Fleming has developed the theory of hyperplane dependent operator observables. He has even begun study of hyperplane dependent quantum field theory. His work is motivated by the study of position operators and the fact that the Newton-Wigner position operator for a single particle (constructed out of only positive frequency modes) is not relativistically invariant; if a particle is localized on one hyperplane, then it is not localized on another that intersects the first at the location of the particle. In the present work, the emphasis is on quantization after evolution as a way to understand quantization of systems with space of global time not equal to \( \mathbb{R} \). We hope that our approach will help to clarify the interpretation of Fleming’s work. However, our main purpose is to build up ideas that may be applicable to interpreting quantum gravity.

Smolin’s discussion of quantum cosmology in [15], and Crane’s discussion of quantum
gravity in [16], have much in common with the above discussion. The relativistic quantum
mechanics presented here may be an example of the type of situation we are faced with in quantum gravity. There may be a space of global time which is something like the space of spacelike submanifolds. (However, this particular space of global time doesn’t quite make sense in the context of quantum gravity. For a more technical discussion of the possibility of viewing gravity in this way see [1,13,20].) An observer may define a path in this (infinite dimensional) space of global time. If we could find a natural way to relate an experimental apparatus to such a path, then the understanding described here may provide a conceptually acceptable view of quantum gravity, without structural modification of general relativity or quantum mechanics.

4. Appendix: Demonstration of Non-Locality

To see that it is not possible to define the wave function as a mapping of spacetime to the complex numbers, we will show that the wave functions on two different spacelike submanifolds (i.e. at two different global times) through the origin of the reference coordinate system will not always agree at the origin. The restriction to positive energy states will be assumed. Let \( |p(t, u)\rangle \) be the basis eigenket for the operator \( \hat{p}(t, u) \). From (2.9) we find

\[
\hat{p}(t, u)|p(0, 0)\rangle = (\sqrt{1 + u^2}p - u\sqrt{m^2 + p^2})|p(0, 0)\rangle.
\]

(4.1)

From (1.11) it follows that the basis eigenkets \( |p(0, 0)\rangle \) and \( |(\sqrt{1 + u^2}p - u\sqrt{m^2 + p^2})(t, u)\rangle \) agree up to phase:

\[
|p(0, 0)\rangle = |(\sqrt{1 + u^2}p - u\sqrt{m^2 + p^2})(t, u)\rangle e^{i\theta(p, t, u)}.
\]

(4.2)

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‡I have Steve Carlip to thank for this.
From (4.2) we can relate the momentum space wave functions \( \psi_p(p,0,0) \) and \( \psi_p(p,t,u) \) as follows:

\[
\psi_p(p,0,0) = \langle p(0,0)|\psi \rangle \\
= \langle (\sqrt{1 + u^2}p - u\sqrt{m^2 + p^2})(t,u)|\psi \rangle e^{if(p,t,u)} \\
= \psi_p(\sqrt{1 + u^2}p - u\sqrt{m^2 + p^2},t,u)e^{if(p,t,u)}.
\]

(4.3)

Finally, we write the position space wave functions \( \psi_q(q,0,0) \) and \( \psi_q(q',t,u) \) in terms of the momentum space wave functions and use (4.3) to obtain:

\[
\psi_q(q,0,0) = \langle q(0,0)|\psi \rangle \\
= \int dp \langle q(0,0)|p(0,0)\rangle \langle p(0,0)|\psi \rangle \\
= \int dp \frac{1}{\sqrt{2\pi}} e^{ipq} \psi_p(p,0,0) \\
= \int dp \frac{1}{\sqrt{2\pi}} e^{ipq} \psi_p(\sqrt{1 + u^2}p - u\sqrt{m^2 + p^2},t,u)e^{if(p,t,u)} \text{ and,}
\]

\[
\psi_q(q',t,u) = \langle q'(t,u)|\psi \rangle \\
= \int dp \langle q'(t,u)|p(t,u)\rangle \langle p(t,u)|\psi \rangle \\
= \int dp \frac{1}{\sqrt{2\pi}} e^{ipq'} \psi_p(p,t,u).
\]

(4.4)

Note that while we could certainly introduce invariant measures in (4.4) without changing the below conclusion, this would not be in the spirit of the interpretation of relativity adopted in the present work. The wave function \( \psi_p(p,t,u) \) is associated with a particular global time, and a particular global time is associated with a particular boost parameter. Changing global times is viewed as evolution. (This is the “clean separation of the dynamical evolution problem from the kinematical transformation problem” that Fleming speaks of. [4] )

The wave function at the origin of the reference coordinate system on the spacelike submanifold \( (t = 0, u = 0) \) is \( \psi_q(0,0,0) \), for which from (4.4) we have

\[
\psi_q(0,0,0) = \int \frac{dp}{\sqrt{2\pi}} \psi_p(\sqrt{1 + u^2}p - u\sqrt{m^2 + p^2},0,u)e^{if(p,0,u)}.
\]

(4.5)

(Here we have used the fact that the left hand side is independent of \( t \) to choose \( t = 0 \) on the right hand side.) The wave function at the origin of the reference coordinate system on the spacelike submanifold \( (t = 0, u) \) is \( \psi_q(0,0,u) \), for which from (4.4) we have

\[
\psi_q(0,0,u) = \int \frac{dp}{\sqrt{2\pi}} \psi_p(p,0,u).
\]

(4.6)
This demonstrates that our formalism allows $\psi_q(0,0,0)$ and $\psi_q(0,0,u)$ to be unequal. For example, consider the special case $\psi_p(0,0,u) = \delta(p)$. Direct calculation gives

$$\psi_q(0,0,0) = \frac{1}{\sqrt{2\pi(1 + u^2)}} e^{i f(u,0,0)}, \text{ and}$$

$$\psi_q(0,0,u) = \frac{1}{\sqrt{2\pi}}.$$  \hspace{1cm} (4.7)

Thus it is demonstrated that the formalism developed here is non-local in the sense that it possesses what Fleming calls “hyperplane dependence”.

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