PERIODIC POINTS ON SHIFTS OF FINITE TYPE AND COMMENSURABILITY INVARIANTS OF GROUPS

DAVID CARROLL AND ANDREW PENLAND

ABSTRACT. We explore the relationship between subgroups and the possible shifts of finite type (SFTs) that can be defined on the group. In particular, we investigate two group invariants, weak periodicity and strong periodicity, defined via symbolic dynamics on the group. We show that these properties are invariants of commensurability. Thus, many known results about periodic points in SFTs defined over groups are actually results about entire commensurability classes. Additionally, we show that the property of being not strongly periodic (also called weakly aperiodic) is preserved under extensions with finitely generated kernels. We conclude by raising questions and conjectures about the relationship of these invariants to the geometric notions of quasi-isometry and growth.

1. INTRODUCTION

Given a group $G$ and a finite alphabet $A$, a shift of finite type (SFT) on $G$ is a compact, $G$-equivariant subset of the full shift $A^G$. It is natural to ask how properties of the group are related to the dynamical properties of its SFTs. Since it is possible to define essentially equivalent SFTs on a group using different alphabets and different sets of forbidden patterns, we should seek properties which are defined in terms of all possible SFTs defined on the group.

One of the most natural properties of a dynamical system is the existence and type of its periodic points. We say that $G$ is weakly periodic if every SFT defined on $G$ has a weakly periodic point (a point whose stabilizer subgroup in $G$ is nontrivial), and we say $G$ is strongly periodic if every SFT defined on $G$ has a strongly periodic point (a point with finite orbit under the action of $G$). It should be noted that weakly periodic is precisely the opposite of what many authors call strongly aperiodic, i.e. a group $G$ is strongly aperiodic if there exists a SFT on $G$ which has no $G$-periodic points.

Our objective is to show that these dynamical properties are influenced by the finite-index subgroups of $G$. We prove the following results.

**Theorem 1.** Let $G_1$ and $G_2$ be finitely generated groups such that $G_1$ and $G_2$ are commensurable. If $G_1$ is weakly periodic, then $G_2$ is weakly periodic.

**Theorem 2.** Let $G_1$ and $G_2$ be finitely generated groups such that $G_1$ and $G_2$ are commensurable. If $G_1$ is strongly periodic, then $G_2$ is strongly periodic.

**Theorem 3.** Suppose $1 \to N \to G \to Q \to 1$ is a short exact sequence of groups and $N$ is finitely generated. If $G$ is strongly periodic, then $Q$ is strongly periodic.

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Corollary 4. If $G$ is a finitely generated group of polynomial growth $\gamma(n) \sim n^d$, where $d \geq 2$, then $G$ is not strongly periodic.

The study of periodic points in SFTs over arbitrary groups is well-established and arises from two major sources. The first is a generalization of classical symbolic dynamics, which considers shift spaces defined on $\mathbb{Z}$ or $\mathbb{Z}^2$, to arbitrary semigroups $\mathbb{S}$ and groups $\mathbb{G}$. For instance, the group $\mathbb{Z}$ is a strongly periodic group, which follows from the fact (well-known in symbolic dynamics) that every two-sided SFT contains a periodic point and the fact (well-known in group theory) that every subgroup of $\mathbb{Z}$ has finite index. The proof is a straightforward exercise in the use of the well-known higher block shift, a notion which we will generalize to arbitrary groups in Section 3.1. It is worth noting that the higher block shift for arbitrary groups is implicitly suggested in [15] and [6]. A higher block shift is also explicitly defined in recent work by Aubrun, Barbieri, and Sablik on effective subshifts [1], though the details of their construction are somewhat different from ours. Piantadosi ([15], [16]) showed that a free group on $n > 1$ generators is weakly periodic but not strongly periodic.

A second motivation for studying SFTs over arbitrary groups is connected to geometric tilings. The literature on tilings is vast and varied, so we mention only here results which are relevant to finitely generated groups. Many authors have investigated sets of tiles which can tile (tesselate) a Riemannian manifold without admitting any translation (i.e. fixed-point-free) symmetries from the manifold’s isometry group. In certain cases, such tilings naturally lead to SFTs on a group. The question of the existence of a set of tiles which can tesselate the plane $\mathbb{R}^2$ without any translation symmetries was first asked by Wang [17], who was motivated by connections to decidability problems. Berger [3] constructed such a set of tiles, now known as Wang tiles. From this work, it follows that $\mathbb{Z}^2$ is neither weakly periodic nor strongly periodic. Culik and Kari [12] showed how to produce aperiodic tilings analogous to the Wang tiles for $\mathbb{Z}^n$ whenever $n \geq 3$. Block and Weinberger [4] used a homology theory connected to coarse geometry to construct aperiodic tiling systems for a large class of metric spaces. Similar homological techniques were recently used by Marcinkowski and Nowak [13] to produce aperiodic tilings for a large class of manifolds which admit actions by amenable groups, including the Grigorchuk group and other groups of intermediate growth. Mozes [14] constructed aperiodic tiling systems for a class of Lie groups which contain uniform lattices satisfying certain conditions. Mozes also associated these tiling systems to labelings of vertices of the Cayley graph of a lattice. Aubrun and Kari [2] constructed weakly aperiodic tilings (having no strongly periodic points) on the Baumslag-Solitar groups. Sahin, Schraudner, and Ugarcovici have given an example of a strongly aperiodic SFT on the discrete Heisenberg group, as well as proving that every ascending HNN extension of $\mathbb{Z}^2$ has a weakly aperiodic SFT [9].

Recently, Cohen [7] has independently obtained results connecting strong aperiodicity to the coarse geometry of groups. In particular, he has shown that in the case of finitely presented torsion-free groups, strong aperiodicity is a quasi-isometry invariant, and that no group with at least two ends is strongly aperiodic. It should be noted that Cohen conjectures that strong aperiodicity is a quasi-isometric invariant for all finitely generated groups. Since commensurability implies quasi-isometry (but not vice versa), a proof of his conjecture would subsume Theorem 1. The techniques we use in the paper are not explicitly geometric. However, Cohen’s results,
as well as the connection to isometry groups of Riemannian manifolds, suggest that a deeper geometric investigation of SFTs on groups is warranted.

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2. Background

2.1. Symbolic dynamics on groups. Throughout this section, we assume \( A \) is a finite discrete set, called the alphabet, and \( G \) is an arbitrary group. The set of all functions \( G \to A \) is called the full shift of \( A \) over \( G \) and is denoted \( A^G \). We endow \( A^G \) with the product topology and the (continuous) left \( G \)-action defined by \((gx)(h) = x(g^{-1}h)\) for \( g, h \in G \), \( x \in A^G \). Elements of \( A^G \) are called configurations.

Often we just call \( A^G \) the full shift if \( A \) and \( G \) are understood.

A subset \( X \subset A^G \) is shift-invariant if \( gx \in X \) for all \( g \in G \), \( x \in X \). The stabilizer group of a configuration \( x \) will be denoted \( \text{Stab}^G(x) = \{ g \in G : gx = x \} \), and we call \( x \in A^G \) weakly periodic, or just periodic, if \( \text{Stab}(x) \) is nontrivial. If \( \text{Stab}(x) \) is of finite index in \( G \), so that \( x \) has finite \( G \)-orbit, we call \( x \) strongly periodic.

Definition. \( S \subset A^G \) is a shift space (or shift) of \( A \) over \( G \) if \( S \) is closed and shift-invariant.

An equivalent definition can be formulated as follows. Call a function \( p : \Omega \to A \) a pattern if \( \Omega \) is a finite subset of \( G \) and write \( \text{supp}(p) = \Omega \). We say \( p \) appears in a configuration \( x \in A^G \) if \( (gx)|_{\Omega} = p \) for some \( g \in G \). Then we have:

Proposition 5. \( S \subset A^G \) is a shift if and only if there exists a set of patterns \( P \subset \bigcup_{\Omega \subset G, \Omega \text{ finite}} A^\Omega \) such that \( S = \{ x \in A^G : \text{no } p \in P \text{ appears in } x \} \).

\( P \) as above is called a set of forbidden patterns for \( S \). If \( P \) can be taken to be finite, we call \( S \) a shift of finite type.

Remark. Let \( S \) be a SFT and \( P \) be a finite set of forbidden patterns defining \( S \). We may assume that \( P \subset A^\Omega \) for a fixed finite subset \( \Omega \subset G \) by taking \( \Omega \supset \bigcup_{p \in P} \text{supp}(p) \) and extending forbidden patterns in all possible ways. In fact, we may assume that \( \Omega \) is a ball of radius \( n \) with respect to some finite generating set of \( G \).

Henceforth, we use the acronym SFT as shorthand for a nonempty shift of finite type of some finite alphabet over a group.

Different shift spaces over a group \( G \) may use use differing alphabets. Our primary concern is for properties common to all shift spaces over a fixed group \( G \), independent of the choice of finite alphabet. We begin with two such invariants.

Definition. \( G \) is weakly periodic if every SFT over \( G \) contains a (weakly) periodic configuration.
Definition. $G$ is strongly periodic if every SFT over $G$ contains a strongly periodic configuration.

In other words, $G$ is strongly periodic if every SFT over $G$ contains a point with finite orbit under the $G$-shift action. Currently the only strongly periodic groups of which the authors are aware are virtually cyclic groups.

As mentioned in the introduction, the properties which we call weak periodicity and strong periodicity have been established for several well-known classes of groups. It is not hard to see that any finite group is strongly periodic, but not weakly periodic. For infinite groups, strongly periodic implies weakly periodic.

2.2. Quasi-Isometry and Commensurability of Groups. Here we give a very brief introduction to certain geometric notions in group theory, nearly all of which can be found in the textbook [10].

For the remainder of this subsection, let $G$ be a group with a finite generating set $A = \{a_1, a_2, \ldots, a_m\}$, so that any group element in $G$ can be represented by at least one word in $A^\pm$.

We define the length of $g \in G$ to be the length of the shortest word in $A$ which represents $g$. The length of $g$ is denoted by $|g|_A$, or $|g|$ if $A$ is understood. This makes $G$ as a metric space with left-invariant distance function $d^A_G$ given by $d^A_G(g_1, g_2) = |g_1^{-1}g_2|_A$ for $g_1, g_2 \in G$. This distance is equivalent to the combinatorial distance between $g_1$ and $g_2$ in the right Cayley graph of $G$ with respect to $A$.

The growth function of $G$ (with respect to $A$) is a map $\gamma^A_G : \mathbb{N} \to \mathbb{N}$, which counts the number of elements in a balls of radius $n$ with respect to $d^A_G$, i.e.

$$\gamma^A_G(n) = |\{g \in G \mid |g| \leq n\}|.$$

The growth function $\gamma^A_G$ depends on $A$, but this dependence is superficial. Take a partial order $\prec$ on growth functions such that $f \prec g$ if and only if there exists a constant $C$ so that $f(n) \leq g(Cn)$ for all $n \in \mathbb{N}$, and define an equivalence relation $\simeq$ on this set of functions so that $f \simeq g$ if and only if $f \prec g$ and $g \prec f$. We call an equivalence class under $\simeq$ a growth type. The growth type of a finitely generated group is an invariant of the group, independent of the choice of finite generating set.

Gromov [11] established a striking connection between the growth of a group and algebraic properties of its finite index subgroups. Recall that if $P$ is a property of groups, a group $G$ is virtually $P$ if $G$ has a finite index subgroup which has the property $P$. If $G$ and $H$ are groups, we say that $G$ is virtually $H$ if $G$ has a finite index subgroup isomorphic to $H$.

Theorem 6 (Gromov). A finitely generated group has polynomial growth if and only if it is virtually nilpotent.

Growth is related to the coarse geometry of a group, as expressed by the notion of quasi-isometry.

Definition. Let $X$ and $Y$ be metric spaces. A map $\phi : X \to Y$ is a quasi-isometry if it satisfies the following properties

(i.) there exist constants $B \geq 1$, $C \geq 0$ such that for all $x_1, x_2 \in X$,

$$\frac{1}{B} d_X(x_1, x_2) - C \leq d_Y(\phi(x_1), \phi(x_2)) \leq Bd_X(x_1, x_2) + C.$$
are such that $ht$.

We conclude

$x$.

Hence

$r / Z$.

That all bounded metric spaces are quasi-isometric, and that $\mathbb{Z}^n$ is quasi-isometric to $\mathbb{R}^n$ for any $n \geq 1$.

Two groups $G_1, G_2$ are said to be commensurable if there exists a group $H$ such that $G_1$ and $G_2$ are both virtually $H$. Commensurability is an equivalence relation on the class of all groups. Commensurability implies quasi-isometry, but the converse is not true (see [10] IV.44, IV.47, and IV.48).

3. Special SFTs on Groups

In this section we construct some useful shifts of finite type and apply them to the question of shift periodicity of groups.

3.1. Higher block shift. Let $A$ be a finite set. Let $G$ be a group, $H$ a subgroup of finite index, and $T$ a finite set of elements of $G$ such that $HT = G$. Set $B = A^T$. The higher block map $\psi_{H,T} : A^G \to B^H$ is defined by $\psi_{H,T}(x)(h) = (h^{-1}x)|_T$. In other words, $\psi_{H,T}(x)$ is the configuration $z \in B^H$ such that

$$z(h)(t) = x(ht) \text{ for any } h \in H, t \in T.$$  

Notice that $[1]$ guarantees $\psi_{H,T}$ is injective, as $HT = G$. However, not all configurations $z$ in $B^H$ are of the form $[1]$ for some $x \in A^G$—one needs to restrict to those which overlap correctly.

We define a set of non-overlapping patterns $P$ for $B^H$ as follows. Let $E = H \cap TT^{-1}$ and

$$P = \{ p \in B^E : \text{there exist } h \in E \text{ and } t \in T \text{ such that } h^{-1}t \in T \text{ and } p(1)(t) \neq p(h)(h^{-1}t) \}.$$  

The higher block shift (over $G$, relative to $H,T$) is the shift of finite type $I \subset B^H$ defined by the forbidden patterns $P$.

Proposition 7. $\psi_{H,T}$ maps $A^G$ onto $I$ bijectively.

Proof. First we show $z = \psi_{H,T}(x) \in I$ for any $x \in A^G$, which amounts to showing that for any $k \in H$, the restriction $r = (kz)|_E$ is not in $P$. For $h \in E$ and $t \in T$ such that $h^{-1}t \in T$,

$$r(1)(t) = (kz(1))(t) = z(k^{-1})(t) = x(k^{-1}ht) = z(k^{-1}h)(h^{-1}t) = r(h)(h^{-1}t).$$  

Hence $r \notin P$.

It remains to show $\psi_{H,T}$ is surjective. If $z \in I$, define $x \in A^G$ by setting $x(ht) = z(h)(t)$ for $h \in H, t \in T$. To see $x$ is well-defined, suppose $h_2 \in H, t_2 \in T$ are such that $ht = h_2t_2$. Then $h^{-1}h_2 = t_2t_2^{-1} \in E$ and $h_2^{-1}ht = t_2 \in T$. Thus the non-overlapping restrictions in $I$ imply that

$$z(h)(t) = (h^{-1}z)(1)(t) = ((h^{-1}z)(h^{-1}h_2))(h_2^{-1}ht) = z(h_2)(t_2).$$  

We conclude $x$ is well-defined, and it is evident that $\psi_{H,T}(x) = z$. \qed
Remark. Notice that if \( z \in I \) and \( x = \psi_{H,T}^{-1}(z) \), \( \text{Stab}_H(z) \) is a subgroup of \( \text{Stab}_G(x) \).

Remark. Recall that given an SFT, we may assume that the forbidden patterns are supported on the ball of radius \( n \) with respect to some generating set. By applying the higher block shift construction with \( G = H \) and \( T \) equal to the ball of radius \( n \) around the identity \( e_G \), we may further assume that the forbidden patterns are supported on the ball of radius 1 with respect to the same generating set.

Let \( A \) and \( B \) be finite alphabets. We set \( C = A \times B \). If \( c = (a, b) \in A \times B \), we use the notation \( c_1 = a \), \( c_2 = b \).

Suppose now that \( X \) is an arbitrary set. Notice that \( C^X \) can be identified with \( A^X \times B^X \): if \( f: X \to C \) is a function, we write \( f_1: X \to C \to A \) and \( f_2: X \to C \to B \) for \( f \) composed with the corresponding projections, and make the identification \( f = (f_1, f_2) \).

Some easy consequences of this identification are as follows. If \( G \) is a group and \( x \in C^G \), \( (gx)_i = g(x_i) \) for \( i = 1, 2 \). If \( \Omega \subset G \), then \( (x|_{\Omega})_i = (x_i)|_{\Omega} \) for \( i = 1, 2 \).

If \( S_1 \subset A^G \), \( S_2 \subset B^G \) are shifts on \( A \) and \( B \), respectively, then \( S_1 \times S_2 \) is a shift on \( C \). Moreover, if \( S_1 \) is a SFT defined by forbidden patterns \( P \subset A^\Omega \), then the SFT on \( C \) defined by forbidden patterns \( P \times B^\Omega \) equals \( S_1 \times S_2 \). It follows that if \( S_1 \) and \( S_2 \) are SFTs, then \( S_1 \times S_2 \) is a SFT: assuming \( S_1 \) is defined by forbidden patterns \( P \subset A^\Omega \) and \( S_2 \) by forbidden patterns \( Q \subset B^\Omega \), \( S_1 \times S_2 \) is the SFT defined by forbidden patterns

\[
P \times B^\Omega \cup A^\Omega \times Q.
\]

3.2. Locked shift. Suppose \( A \) is a finite alphabet, \( G \) is a group, and \( N \) is a finitely generated normal subgroup of \( G \). We define \( \text{Fix}_{\Lambda \circ G}(N) = \text{Fix}(N) = \{ x \in A^G : nx = x \text{ for all } n \in N \} \). We claim that \( \text{Fix}(N) \) is an SFT over \( G \). To see this, let \( \Lambda = \{ a_1, \ldots, a_m \} \) be a symmetric generating set for \( N \). Then \( \text{Fix}(N) \) is the SFT determined by the forbidden patterns

\[
\{ p : \{1, a_i \} \to A : a_i \in \Lambda, p(a_i) \neq p(1) \}.
\]

Now suppose \( N \) is also of finite index in \( G \). Let \( T = \{ t_1, \ldots, t_n \} \) be a complete set of distinct left coset representatives for \( N \) in \( G \) with \( t_1 = 1 \). We use \( T \) as alphabet and define the \( N \)-locked shift (over \( G \), on \( T \)) to be \( L = \text{Fix}_{\Lambda \circ G}(N) \cap S \), where \( S \) is the SFT defined by forbidden patterns

\[
\{ p : \{1, t \} \to T : t \in T \setminus \{1\}, p(1) = p(t) \}.
\]

Proposition 8. \( L \) is a nonempty shift of finite type. Moreover, for any \( x \in L \), \( gx = x \) if and only if \( g \in N \).

Proof. First, we show \( L \) is nonempty. Let \( y \in T^G \) be the configuration sending each \( g \in G \) to its coset representative: \( y(tn) = t \) whenever \( t \in T \), \( n \in N \). Then if \( n, n' \in N \), \( t \in T \),

\[
ny(tn') = y(n^{-1}tn') = y(t(t^{-1}n^{-1}t)n') = t = y(tn'),
\]

so \( ny = y \) and we conclude \( y \in \text{Fix}(N) \). Moreover, \( y \in S \), for if \( g = tn \in G \),

\[
gy(1) = ty(1) = y(t^{-1}) \neq y(t^{-1}t_2) = gy(t_2)
\]

whenever \( t_2 \in T \setminus \{1\} \).
For the second assertion, suppose \( x \in L \) and \( g = tn \in G \) satisfies \( gx = x \). Then \( tx = x \), so \( x(1) = tx(t) = g(t) = x(t) \).

By the restrictions on \( S \), \( t = 1 \) and \( g \in N \).

4. Commensurability and periodicity

Using the basic constructions above, we show the properties of weak and strong periodicity are preserved under finite index extensions and, in the case when the group is finitely generated, preserved in finite index subgroups. As a corollary, we find that weak and strong periodicity are commensurability invariants.

Proposition 9. Let \( G \) be a group.

1. If \( G \) is virtually weakly periodic, then \( G \) is weakly periodic.
2. If \( G \) is virtually strongly periodic, then \( G \) is strongly periodic.

Proof. Suppose \( H \) is a weakly periodic subgroup of \( G \) with \([G:H]=n\) and \( S \subset A^G \) is a nonempty shift of finite type over \( G \). Let \( P' \) be a finite set of forbidden patterns for \( S \); by extending patterns if necessary, we may assume \( P' \subset A^H \) for some fixed finite subset \( \Omega \subset G \) with \( 1 \in \Omega \). Let \( T' = \{a_1, \ldots, a_n\} \) be a set of distinct right coset representatives for \( H \) in \( G \), where \( a_1 = 1 \), and set \( T = T' \Omega \). Define the higher block map \( \psi_{H,T} : A^G \to I \subset B^H \) as in Section 3, and let \( J \subset B^H \) be the shift of finite type defined by the following set of forbidden patterns:

\[
\{ p : \{1_H\} \to B : \text{there is } p' \in P', t \in T' \text{ such that } (p(1_H))(t \omega) = p'(\omega) \text{ for all } \omega \in \Omega \}.
\]

Then \( I \cap J \) is a shift of finite type and \( S = \psi_{H,T}^{-1}(I \cap J) \). Since \( S \) is nonempty, \( I \cap J \) is nonempty and contains a periodic configuration \( z \in B^H \) by hypothesis; i.e., \( \text{Stab}_H(z) \) is nontrivial. Let \( x = \psi_{H,T}^{-1}(z) \in S \). By Remark 3, \( \text{Stab}_G(x) \) is nontrivial, proving assertion 1. If in addition \( z \) can be chosen so that \( \text{Stab}_H(z) \) is of finite index in \( H \), then \( \text{Stab}_G(x) \) is of finite index in \( G \), giving assertion 2. \( \square \)

Proposition 10. Let \( G \) be a finitely generated group and \( H \) a finite index subgroup of \( G \).

1. If \( G \) is weakly periodic, then \( H \) is weakly periodic.
2. If \( G \) is strongly periodic, then \( H \) is strongly periodic.

Proof. Since \( H \) contains a finite index subgroup that is normal in \( G \), by Proposition 9 we may assume without loss of generality that \( H \) is normal in \( G \). Let \( T \) be a complete set of left coset representatives for \( H \) in \( G \) with \( 1 \in T \).

Suppose \( A \) is a finite alphabet and \( S \subset A^H \) is a SFT over \( H \) defined by forbidden patterns \( P \subset A^\Omega \), where \( \Omega \subset H \) is finite. By regarding \( \Omega \) as a subset of \( G \), we can consider the SFT \( S' \subset A^G \) with the same forbidden pattern set \( P \). Notice that \( S' \) is nonempty, for we can choose \( x \in S \) and define \( x' \in S' \) by letting \( x'(th) = x(h) \) for \( t \in T, h \in H \). Notice also that if \( y \in S' \), \( y|_H \in S \).

Define \( L \) to be the \( H \)-locked shift as in section 3.2. Then \( S' \times L \) is a SFT over \( G \). Moreover, whenever \( y \in S' \times L \), \( \text{Stab}_G(y) \subset H \), for \( gy = y \) implies in particular that \( gy_2 = y_2 \). Regarding \( x = (y_1)|_H \) as a configuration in \( S \subset A^H \), it follows that \( \text{Stab}_H(x) \) contains \( \text{Stab}_G(y) \). In conclusion, \( H \) is weakly (strongly) periodic whenever \( G \) is. \( \square \)
Theorem 11. Let $G_1$ and $G_2$ be finitely generated commensurable groups. If $G_1$ is weakly (strongly) periodic, then $G_2$ is weakly (strongly) periodic.

Note that in general, strong aperiodicity is not a hereditary property of groups. A group which is strongly aperiodic (such as $\mathbb{Z}^2$) can contain a subgroup which is strongly periodic (such as $\mathbb{Z}$).

4.1. SFTs and quotient groups. Suppose $G, Q$ are groups and $f : G \to Q$ is a surjective homomorphism. Let $N$ be the kernel of $f$ so that $Q = G/N$. Notice that $f$ induces a bijection $F : A^Q \to \text{Fix}_{A^G}(N)$ defined by $F(x) = x \circ f$. Moreover, $F$ is $G/N$-equivariant in the following sense: if $q = gN \in Q$ and $x \in A^Q$, then $F(qx) = gF(x)$.

Now, if $\overline{S} \subset A^Q$ is a SFT defined by forbidden patterns $\overline{P} \subset A^\Omega$, $\Omega \subset Q$ finite, we can choose a (finite) set $\Omega \subset G$ such that $f$ maps $\Omega$ onto $\overline{\Omega}$ bijectively. $f|_\Omega$ induces a bijection $g : A^\overline{\Omega} \to A^\Omega$. We let $P = g(\overline{P})$ and $S \subset A^G$ be the SFT with forbidden patterns $P$.

Proposition 12. $F(\overline{S}) = S \cap \text{Fix}(N)$.

Proof. Suppose $x \in A^Q$. If $\overline{p} \in \overline{P}$ and $p = g(\overline{p}) \in P$, then $\overline{p}$ appears in $x$ if and only if $p$ appears in $F(x)$. Since $F$ is a bijection from $A^Q$ to $\text{Fix}(N)$, the result follows. \qed

It follows from Section 3.2 that $F(\overline{S})$ is a SFT in $A^G$ whenever $N$ is finitely generated. This leads to a useful general result.

Theorem 13. Suppose $1 \to N \to G \to Q \to 1$ is a short exact sequence of groups and $N$ is finitely generated. If $G$ is strongly periodic, then $Q$ is strongly periodic.

Proof. Suppose $\overline{S} \subset A^Q$ is a SFT over $Q$. Defining $F$ as above, by hypothesis there is a configuration $F(x) \in F(\overline{S}) \subset A^G$ such that $H = \text{Stab}_G(F(x))$ is of finite index in $G$. Of course, $N \subset H$, so $H/N$ is of finite index in $G/N = Q$. Moreover, by equivariance of $F$, $qx = x$ for every $q \in H/N$. In conclusion, $x$ has finite index stabilizer in $Q$. \qed

As an example application, this result yields information about the SFTs which can be defined on groups of polynomial growth.

Corollary 14. If $G$ is a finitely generated group of polynomial growth $\gamma(n) \sim n^d$, where $d \geq 2$, then $G$ is not strongly periodic. In other words, $G$ is weakly aperiodic, i.e. there exists a SFT on $G$ without strongly periodic points.

Proof. By Gromov’s theorem, $G$ is virtually nilpotent, so by Proposition 10 we may assume $G$ is nilpotent. Since $G$ is not virtually cyclic, there exists a surjective homomorphism $f : G \to \mathbb{Z}^2$ ([5 Lemma 13]) and as every subgroup of a finitely generated nilpotent group is itself finitely generated, $N = \ker f$ is finitely generated. Since $\mathbb{Z}^2$ is not strongly periodic, $G$ is not strongly periodic. \qed

Proposition 14 does not rule out the existence of periodic points for SFTs defined on groups of polynomial growth. However, the examples we know of strongly aperiodic groups(such as $\mathbb{Z}^2$ and the discrete Heiseinberg group) are all groups of polynomial growth. This leads to the following question.

Question. Is there a group of non-linear polynomial growth on which every shift of finite type has a periodic point?
We also make the following conjecture (which is very similar to Cohen’s).

**Conjecture.** *Strong periodicity is a quasi-isometric invariant.*

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