Non-existence of solutions for non-autonomous elliptic systems

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Abstract

We extend the classical Pohozaev’s identity to semilinear elliptic systems of Hamiltonian type, providing a simpler approach, and a generalization, of the results of E. Mitidieri [6], R.C.A.M. Van der Vorst [14], and Y. Bozhkov and E. Mitidieri [1].

Key words: Pohozaev’s identity, non-existence of solutions.

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1 Introduction

Any solution $u(x)$ of semilinear Dirichlet problem on a bounded domain $\Omega \subset \mathbb{R}^n$

\begin{equation}
\Delta u + f(x, u) = 0 \text{ in } \Omega, \quad u = 0 \text{ on } \partial \Omega
\end{equation}

satisfies the well known Pohozaev’s identity

\begin{equation}
\int_{\Omega} [2nF(x, u) + (2 - n)uf(x, u) + 2\sum_{i=1}^{n} x_i F_{x_i}(x, u)] \, dx = \int_{\partial \Omega} (x \cdot \nu)|\nabla u|^2 \, dS.
\end{equation}

Here $F(x, u) = \int_{0}^{u} f(x, t) \, dt$, and $\nu$ is the unit normal vector on $\partial \Omega$, pointing outside. (From the equation (1.1), $\int_{\Omega} uf(x, u) \, dx = \int_{\Omega} |\nabla u|^2 \, dx$, which gives an alternative form of the Pohozaev’s identity.) Pohozaev’s identity is usually written for the case $f = f(u)$, but the present version is also known, see e.g., K. Schmitt [13]. A standard use of this identity is to conclude that if $\Omega$ is a star-shaped domain with respect to the origin, i.e., $x \cdot \nu \geq 0$ for all $x \in \partial \Omega$, and $f(u) = u|u|^{p-1}$, for some constant $p$, then the problem (1.1) has no non-trivial solutions in the super-critical case, when $p > \frac{n+2}{n-2}$. In this note we present a proof of Pohozaev’s identity, which appears a little more straightforward than the usual one, see e.g., L. Evans [2], and then use a similar idea for systems, generalizing the well-known results of E. Mitidieri [6], see also R.C.A.M. Van der Vorst [14], and of Y. Bozhkov and E. Mitidieri [1], by allowing explicit dependence on $x$ in the Hamiltonian function.

Let $z = x \cdot \nabla u = \sum_{i=1}^{n} x_i u_{x_i}$. It is straightforward to verify that $z$ satisfies

\begin{equation}
\Delta z + f_u(x, u)z = -2f(x, u) - \sum_{i=1}^{n} x_i f_{x_i}(x, u).
\end{equation}

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We multiply the equation (1.1) by \( z \), and subtract from that the equation (1.3) multiplied by \( u \), obtaining

\[
\begin{align*}
(1.4) & \quad \Sigma_{i=1}^{n} (zu_{x_{i}} - uz_{x_{i}})_{x_{i}} + \Sigma_{i=1}^{n} (f(x, u) - uf(x, u)) x_{i}u_{x_{i}} = 2f(x, u)u + \Sigma_{i=1}^{n} x_{i}f_{x_{i}}(x, u)u.
\end{align*}
\]

We have

\[
\begin{align*}
(1.4) & \quad \Sigma_{i=1}^{n} (f(x, u) - uf(x, u)) x_{i}u_{x_{i}} = \Sigma_{i=1}^{n} x_{i}i\frac{\partial}{\partial x_{i}}(2F - uf) - 2\Sigma_{i=1}^{n} x_{i}F_{x_{i}} + \Sigma_{i=1}^{n} x_{i}f_{x_{i}}(x, u)u = \\
& \quad \Sigma_{i=1}^{n} \frac{\partial}{\partial x_{i}} [x_{i}(2F - uf)] - n(2F - uf) - 2\Sigma_{i=1}^{n} x_{i}F_{x_{i}} + \Sigma_{i=1}^{n} x_{i}f_{x_{i}}(x, u)u.
\end{align*}
\]

We then rewrite (1.3)

\[
(1.5) & \quad \Sigma_{i=1}^{n} [(zu_{x_{i}} - uz_{x_{i}}) + x_{i}(2F(x, u) - uf(x, u))]_{x_{i}} = 2nF(x, u) + (2 - n)uf(x, u) + 2\Sigma_{i=1}^{n} x_{i}F_{x_{i}}.
\]

Integrating over \( \Omega \), we conclude the Pohozaev’s identity (1.2). (The only non-zero boundary term is \( \Sigma_{i=1}^{n} \int_{\partial \Omega} z u_{x_{i}} \nu_{i} dS \). Since \( \partial \Omega \) is a level set of \( u \), \( \nu = \pm \frac{\nabla u}{|\nabla u|} \), i.e., \( u_{x_{i}} = \pm |\nabla u| \nu_{i} \). Then \( z = \pm (x \cdot \nu) \nabla u \), and \( \Sigma_{i=1}^{n} u_{x_{i}} \nu_{i} = \pm |\nabla u| \).

We refer to (1.5) as a differential form of Pohozaev’s identity. For radial solutions on a ball, the corresponding version of (1.5) played a crucial role in the study of exact multiplicity of solutions, see T. Ouyang and J. Shi [7], and also P. Korman [5], which shows the potential usefulness of this identity.

## 2 Non-existence of solutions for a class of systems

The following class of systems has attracted considerable attention recently

\[
(2.1) \quad \Delta u + H_{u}(u, v) = 0 \text{ in } \Omega, \quad u = 0 \text{ on } \partial \Omega,
\]

\[
\Delta v + H_{u}(u, v) = 0 \text{ in } \Omega, \quad v = 0 \text{ on } \partial \Omega,
\]

where \( H(u, v) \) is a given differentiable function, see e.g., the following surveys: D.G. de Figueiredo [3], P. Quittner and P. Souplet [11], B. Ruf [12], see also P. Korman [4]. This system is of Hamiltonian type, so that it has some of the properties of scalar equations.

More generally, let \( H = H(x, u_{1}, u_{2}, \ldots, u_{m}, v_{1}, v_{2}, \ldots, v_{m}) \), with integer \( m \geq 1 \), and consider the Hamiltonian system of \( 2m \) equations

\[
(2.2) \quad \Delta u_{k} + H_{v_{k}} = 0 \quad \text{in } \Omega, \quad u_{k} = 0 \text{ on } \partial \Omega, \quad k = 1, 2, \ldots, m,
\]

\[
\Delta v_{k} + H_{u_{k}} = 0 \quad \text{in } \Omega, \quad v_{k} = 0 \text{ on } \partial \Omega, \quad k = 1, 2, \ldots, m.
\]

We call solution of (2.2) to be positive, if \( u_{k}(x) > 0 \) and \( v_{k}(x) > 0 \) for all \( x \in \Omega \), and all \( k \). We consider only the classical solutions, with \( u_{k} \) and \( v_{k} \) of class \( C^{2}(\Omega) \cap C^{1}(\Omega) \). We have the following generalization of the results of [1] and [6].

**Theorem 2.1** Assume that \( H(x, u_{1}, u_{2}, \ldots, u_{m}, v_{1}, v_{2}, \ldots, v_{m}) \in C^{2}(\Omega \times R^{n}_{+} \times R^{n}_{+}) \cap C(\Omega \times R^{n}_{+} \times R^{m}_{+}) \) satisfies

\[
(2.3) \quad H(x, 0, \ldots, 0, 0, \ldots, 0) = 0 \quad \text{for all } x \in \partial \Omega.
\]

Then for any positive solution of (2.2), and any real numbers \( a_{1}, \ldots, a_{m} \), one has

\[
(2.4) \quad \int_{\Omega} [2nH + (2 - n)\Sigma_{k=1}^{m} (a_{k}u_{k}H_{v_{k}} + (2 - a_{k})v_{k}H_{u_{k}}) + 2\Sigma_{l=1}^{n} x_{l}H_{x_{l}}] \ dx = 2\Sigma_{k=1}^{m} \int_{\partial \Omega} (x \cdot \nu)|\nabla u_{k}| \nabla v_{k}| \ dS.
\]
Proof: Define $p_k = x \cdot \nabla u_k = \sum_{i=1}^{n} x_i u_{kx_i}$, and $q_k = x \cdot \nabla v = \sum_{i=1}^{n} x_i v_{kx_i}$, $k = 1, 2, \ldots, m$. These functions satisfy the system

\begin{align}
\Delta p_k + 2 \sum_{j=1}^{m} H_{vk} u_j p_j + \sum_{j=1}^{m} H_{vk} v_j q_j &= -2 H_{uk} - \sum_{i=1}^{n} \xi_i H_{vx_i}, \quad k = 1, 2, \ldots, m
\end{align}

We multiply the first equation in (2.2) by $q_k$, and subtract from that the first equation in (2.5) multiplied by $v_k$. The result can be written as

\begin{align}
\sum_{k=1}^{m} \left[ \left( u_{kx_i} q_k - p_{kx_i} v_k \right) x_i + \left( -u_{kx_i} q_k x_i + v_{kx_i} p_k x_i \right) \right] + H_{vk} q_k - \sum_{j=1}^{m} H_{vk} u_j p_j v_k - \sum_{j=1}^{m} H_{vk} v_j q_j v_k = 2 v_k H_{vk} + v_k \sum_{i=1}^{n} x_i H_{vx_i}.
\end{align}

Similarly, we multiply the second equation in (2.2) by $p_k$, and subtract from that the second equation in (2.5) multiplied by $u_k$, and write the result as

\begin{align}
\sum_{k=1}^{m} \left[ \left( v_{kx_i} p_k - q_{kx_i} u_k \right) x_i + \left( -v_{kx_i} p_k x_i + u_{kx_i} q_k x_i \right) \right] + H_{uk} p_k - \sum_{j=1}^{m} H_{uk} u_j p_j u_k - \sum_{j=1}^{m} H_{uk} v_j q_j u_k = 2 u_k H_{uk} + u_k \sum_{i=1}^{n} x_i H_{vx_i}.
\end{align}

Adding the equations (2.6) and (2.7), we get

\begin{align}
\sum_{k=1}^{m} \left[ u_{kx_i} q_k - p_{kx_i} v_k + v_{kx_i} p_k - q_{kx_i} u_k \right] x_i + H_{uk} p_k + H_{vk} q_k - \sum_{j=1}^{m} H_{uk} u_j p_j u_k - \sum_{j=1}^{m} H_{vk} v_j q_j v_k = 2 u_k H_{uk} + 2 v_k H_{vk} + \sum_{i=1}^{n} x_i H_{vx_i}.
\end{align}

We now sum in $k$, putting the result into the form

\begin{align}
\sum_{k=1}^{m} \sum_{i=1}^{n} [u_{kx_i} q_k - p_{kx_i} v_k + v_{kx_i} p_k - q_{kx_i} u_k x_i] + \sum_{i=1}^{n} x_i (2H - \sum_{k=1}^{m} u_k H_{uk} - \sum_{k=1}^{m} v_k H_{vk}) x_i = 2 \sum_{k=1}^{m} u_k H_{uk} + 2 \sum_{k=1}^{m} v_k H_{vk} + 2 \sum_{i=1}^{n} x_i H_{vx_i}.
\end{align}

Writing,

\begin{align}
\sum_{i=1}^{n} x_i \frac{\partial}{\partial x_i} \left( 2H - \sum_{k=1}^{m} u_k H_{uk} - \sum_{k=1}^{m} v_k H_{vk} \right) = \sum_{i=1}^{n} \frac{\partial}{\partial x_i} \left[ x_i \left( 2H - \sum_{k=1}^{m} u_k H_{uk} - \sum_{k=1}^{m} v_k H_{vk} \right) \right],
\end{align}

we obtain the differential form of Pohozaev’s identity

\begin{align}
\sum_{k=1}^{m} \sum_{i=1}^{n} [u_{kx_i} q_k - p_{kx_i} v_k + v_{kx_i} p_k - q_{kx_i} u_k x_i] + \sum_{i=1}^{n} x_i \left( 2H - \sum_{k=1}^{m} u_k H_{uk} - \sum_{k=1}^{m} v_k H_{vk} \right) x_i = 2nH + (2 - n) \left( \sum_{k=1}^{m} u_k H_{uk} + \sum_{k=1}^{m} v_k H_{vk} \right) + 2n \sum_{i=1}^{n} x_i H_{vx_i}.
\end{align}

Integrating, we obtain, in view of (2.3),

\begin{align}
\int_{\Omega} \left( 2nH(u, v) + (2 - n) \left( \sum_{k=1}^{m} u_k H_{uk} + \sum_{k=1}^{m} v_k H_{vk} \right) + 2n \sum_{i=1}^{n} x_i H_{vx_i} \right) dx
\end{align}

(Since we consider positive solutions, and $\partial \Omega$ is a level set for both $u_k$ and $v_k$, we have $\nu = \frac{\sum u_k}{\left| \nabla u_k \right|}$, i.e., $u_k = -\left| \nabla u_k \right| \nu_i$ and $v_k = -\left| \nabla v_k \right| \nu_i$ on the boundary $\partial \Omega$.) From the first equation in (2.2), $\int_{\Omega} v_k H_{vk} dx = \int_{\Omega} \nabla v_k \cdot \nabla v_k dx$, while from the second equation $\int_{\Omega} u_k H_{uk} dx = \int_{\Omega} \nabla u_k \cdot \nabla v_k dx$, i.e., for each $k$

\begin{align}
\int_{\Omega} v_k H_{vk} dx = \int_{\Omega} u_k H_{uk} dx.
\end{align}

Using this in (2.8), we conclude the proof.

Remarks

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1. We consider only the classical solutions. Observe that by our conditions and elliptic regularity, classical solutions are in fact of class $C^3(\Omega)$, so that all quantities in the above proof are well defined.

2. In case $H$ is independent of $x$, the condition (2.3) can be assumed without loss of generality.

As a consequence, we have the following non-existence result.

**Proposition 1** Assume that $\Omega$ is a star-shaped domain with respect to the origin, and for some real constants $\alpha_1, \ldots, \alpha_m$, all $u_k > 0$, $v_k > 0$, and all $x \in \Omega$, we have

$$nH + (2 - n)\sum_{k=1}^{m} \left( \alpha_k u_k H u_k + (1 - \alpha_k) v_k H v_k \right) + \sum_{i=1}^{n} x_i H x_i < 0. \quad (2.9)$$

Then the problem (2.2) has no positive solutions.

**Proof:** We use the identity (2.4), with $a_k/2 = \alpha_k$. Then, assuming existence of positive solution, the left hand side of (2.4) is negative, while the right hand side is non-negative, a contradiction. $\diamond$

Observe, that it suffices to assume that $\Omega$ is star-shaped with respect to any one of its points (which we then take to be the origin).

In case $m = 1$, and $H = H(u, v)$, we recover the following condition of E. Mitidieri [6].

**Proposition 2** Assume that $\Omega$ is a star-shaped domain with respect to the origin, and for some real constant $\alpha$, and all $u > 0$, $v > 0$ we have

$$\alpha u H_u(u, v) + (1 - \alpha) v H_v(u, v) > \frac{n}{n - 2} H(u, v). \quad (2.10)$$

Then the problem (2.1) has no positive solutions.

Comparing this result to E. Mitidieri [6], observe that we do not require that $H_u(0,0) = H_v(0,0) = 0$.

An important subclass of (2.1) is

$$\Delta u + f(v) = 0 \text{ in } \Omega, \quad u = 0 \text{ on } \partial \Omega$$
$$\Delta v + g(u) = 0 \text{ in } \Omega, \quad v = 0 \text{ on } \partial \Omega,$$

which corresponds to $H(u, v) = F(v) + G(u)$, where $F(v) = \int_{0}^{v} f(t) \, dt$, $G(u) = \int_{0}^{u} g(t) \, dt$. Unlike [6], we do not require that $f(0) = g(0) = 0$. The Theorem 2.1 now reads as follows.

**Theorem 2.2** Let $f, g \in C(\bar{R}_+)$, For any positive solution of (2.11), and any real number $a$, one has

$$\int_{\Omega} \left[ 2n(F(v) + G(u)) + (2 - n)(af(v) + (2 - a)ug(u)) \right] \, dx$$
$$= 2 \int_{\partial \Omega} (x \cdot \nu) |\nabla u||\nabla v| \, dS. \quad (2.12)$$

More generally, we consider

$$\Delta u + f(x, v) = 0 \text{ in } \Omega, \quad u = 0 \text{ on } \partial \Omega$$
$$\Delta v + g(x, u) = 0 \text{ in } \Omega, \quad v = 0 \text{ on } \partial \Omega,$$

with $H(x, u, v) = F(x, v) + G(x, u)$, where $F(x, v) = \int_{0}^{v} f(x, t) \, dt$, $G(x, u) = \int_{0}^{u} g(x, t) \, dt$. 


Theorem 2.3 Let \( f, g \in C(\Omega \times \mathbb{R}_+) \). For any positive solution of (2.13), and any real number \( a \), one has
\[
(2.14) \int_{\Omega} \left[ 2n(F(x,v) + G(x,u)) + (2 - n)(avf(x,v) + (2 - a)ug(x,u)) + 2\sum_{i=1}^{n} x_i (F_{x_i} + G_{x_i}) \right] \, dx = 2 \int_{\partial\Omega} (x \cdot \nu) |\nabla u||\nabla v| \, dS .
\]
We now consider a particular system
\[
(2.15) \quad \Delta u + v^p = 0 \quad \text{in} \quad \Omega, \quad u = 0 \quad \text{on} \quad \partial\Omega \quad \Delta v + g(x,u) = 0 \quad \text{in} \quad \Omega, \quad v = 0 \quad \text{on} \quad \partial\Omega ,
\]
with \( g(x,u) \in C(\Omega \times \mathbb{R}_+) \), and a constant \( p > 0 \).

Theorem 2.4 Assume that \( \Omega \) is a star-shaped domain with respect to the origin, and
\[
(2.16) \quad nG(x,u) + (2 - n) \left( 1 - \frac{n}{(n-2)(p+1)} \right) ug(x,u) + \sum_{i=1}^{n} x_i G_{x_i} < 0, \quad \text{for} \quad x \in \Omega, \quad \text{and} \quad u > 0 .
\]
Then the problem (2.15) has no positive solutions.

Proof: We use the identity (2.14), with \( f(v) = v^p \). We select the constant \( a \), so that
\[
2nF(v) + (2 - n)avf(v) = 0 ,
\]
i.e., \( a = \frac{2n}{(n-2)(p+1)} \). Then, assuming existence of a positive solution, the left hand side of (2.14) is negative, while the right hand side is non-negative, a contradiction. \( \square \)

Observe that in case \( p = 1 \), the Theorem 2.4 provides a non-existence result for a biharmonic problem with Navier boundary conditions
\[
(2.17) \quad \Delta^2 u = g(x,u) \quad \text{in} \quad \Omega, \quad u = \Delta u = 0 \quad \text{on} \quad \partial\Omega .
\]

Proposition 3 Assume that \( \Omega \) is a star-shaped domain with respect to the origin, and the condition (2.16), with \( p = 1 \), holds. Then the problem (2.17) has no positive solutions.

Finally, we consider the system
\[
(2.18) \quad \Delta u + v^p = 0 \quad \text{in} \quad \Omega, \quad u = 0 \quad \text{on} \quad \partial\Omega \quad \Delta v + u^q = 0 \quad \text{in} \quad \Omega, \quad v = 0 \quad \text{on} \quad \partial\Omega .
\]
The curve \( \frac{1}{p+1} + \frac{1}{q+1} = \frac{n-2}{n} \) is called a critical hyperbola. We recover the following well known result of E. Mitidieri [6], see also R.C.A.M. Van der Vorst [14]. (Observe that we relax the restriction \( p, q > 1 \) from [6].)

Proposition 4 Assume that \( p, q > 0 \), and
\[
(2.19) \quad \frac{1}{p+1} + \frac{1}{q+1} < \frac{n-2}{n} .
\]
Then the problem (2.18) has no positive solutions.

Proof: Condition (2.19) implies (2.16), and then the Theorem 2.4 applies. \( \square \)

In case \( p = 1 \), we recover the following known result, see E. Mitidieri [6].

Proposition 5 Assume that \( \Omega \) is a star-shaped domain with respect to the origin, and \( q > \frac{n+4}{n-4} \). Then the problem
\[
(2.20) \quad \Delta^2 u = u^q \quad \text{in} \quad \Omega, \quad u = \Delta u = 0 \quad \text{on} \quad \partial\Omega
\]
has no positive solutions.
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