Neutron stars in $f(T)$ gravity with realistic models of matter

Rui-Hui Lin,1,* Xiao-Ning Chen,1 and Xiang-Hua Zhai1,†

1Division of Mathematics and Theoretical Physics, Shanghai Normal University, 100 Guilin Road, Shanghai 200234, China

Abstract

We investigate the nonrotating neutron stars in $f(T)$ gravity with $f(T) = T + \alpha T^2$, where $T$ is the torsion scalar in the teleparallel formalism of gravity. In particular, we utilize the SLy and BSk family of equations of state for perfect fluid to describe the neutron stellar matter and search for the effects of the $f(T)$ modification on the models of neutron stars. For positive $\alpha$, the modification results in a stronger gravitation exerted on the stellar matter, leading to a smaller stellar mass in comparison to general relativity. Moreover, there seems to be an upper limit for the central density of the neutron stars with $\alpha > 0$, beyond which the effective $f(T)$ fluid would have a steplike phase transition in density and pressure profiles, collapsing the numerical system. For negative $\alpha$, the $f(T)$ modification provides additional support for neutron stars to contain larger amount of matter.

We obtain the mass-radius relations of the realistic models of neutron stars and subject them to the joint constraints from the observed massive pulsars PSR J0030+0451, PSR J0740+6620, and PSR J2215+5135, and gravitational wave events GW170817 and GW190814. For BSk19 equation of state, the neutron star model in $f(T)$ gravity can accommodate all the mentioned data when $\alpha \leq 3.5G^2M^2_\odot/c^4$. For BSk20, BSk21 and SLy equations of state, the observational data constrain the model parameter $\alpha$ to be negative. If one considers the unknown compact object in the event GW190814 not to be a neutron star and hence excludes this dataset, the constraints for BSk20 and BSk21 models can be loosened to $\alpha \leq 0.4G^2M^2_\odot/c^4$ and $\alpha \leq 1.9G^2M^2_\odot/c^4$, respectively.

* linrh@shnu.edu.cn
† zhaixh@shnu.edu.cn
I. INTRODUCTION

General relativity (GR) seems to work perfectly well against the local weak field tests of gravity. However, the long known challenge of it when applied to the entire Universe, including the dark contents of the Universe and the singularities of the spacetime, still lacks a consensus solution. It is believed that a quantum theory of gravitation may possibly help understand or resolve these problems. Nonetheless, since a generally accepted theory of quantum gravity is still missing, modifications of GR which may hint the quantum corrections are widely considered. In this sense, the Hilbert-Einstein action of GR may be seen as a classical approximation at low energy scale. In the Riemannian formulation of GR, gravitation manifests itself in the curvature of the spacetime manifold. One would expect that higher order curvature term(s) may become relevant as the energy goes higher. Therefore, alternative theories of gravity involving nonlinear terms of curvature appear to be of particular interests, among which \( f(R) \) model with an arbitrary function \( f \) of the Ricci scalar \( R \) is one of the most renowned schemes (see, e.g., Refs. [1–4] for extensive reviews).

On the other hand, GR can also be formulated in various forms by choosing different affine connections, which may constitute different but equivalent descriptions of gravity [5, 6] and provide different aspects of insight. As one of the variants of GR that can be dated back to the time of Einstein, the teleparallel equivalent of GR (TEGR) can be formulated where the tetrad field is used as the dynamical variable instead of the metric and the torsion scalar \( T \) is constructed to be the underlying Lagrangian while the curvature \( R \) vanishes [7, 8]. Following this line, the correction corresponding to higher energy scale may appear as terms of higher orders of torsion. Similar to \( f(R) \) gravity, this may be achieved by \( f(T) \) gravity with a nonlinear function of \( T \) (see, e.g., Refs.[9–12]). Although TEGR is equivalent to GR, the extended theories, \( f(R) \) and \( f(T) \), are generally different in that the difference between their Lagrangians is no longer a total derivative term when the function \( f \) is nonlinear, and hence cannot be discarded via integration by parts. It follows that \( f(T) \) gravity may possess new features and provide new angle to investigate the geometry of the spacetime. One particular advantage of \( f(T) \) gravity is that the field equations under this framework are of second order of derivatives as in GR instead of fourth order ones in \( f(R) \) gravity. This makes \( f(T) \) gravity somewhat a simpler and more natural way to modify (TE)GR.

However, in some early formulations of \( f(T) \) gravity, it was found that the Lorentz
invariance is not respected by the theory in the sense that the torsion tensor, and hence the field equations, are in general not invariant with respect to the tetrad [13–15]. The frame-dependent nature of this formalism complicates the studies of various subjects in $f(T)$ gravity and hinders the development of the theory, which has then led to efforts in finding the “good” tetrad in certain situations [16, 17]. But the head-on solution to this problem should be restoring the spin connection and constructing the covariant formulation of teleparallel gravities [18–21]. This covariant formulation then allows one to pursue the relatively more difficult problems beyond Minkowski spacetime or spatial flat cosmology in $f(T)$ gravity, which, for example, include the spherically symmetric configurations [22–24]. In particular, by considering the regularity of the matter, the relativistic stars in $f(T)$ gravity are proved existing for a priori assumed metrics[25]. After that, various sorts of star solutions in $f(T)$ gravity have been found and studied in this approach [26–29].

Another route to study relativistic stars is to start from the physical properties of the matter that may form the stars. Following this scheme, stars consisting of Yang-Mills field [30] and boson field [31] are investigated. For stellar matter that can be approximated by perfect fluid, if energy is conserved, a correlation between the spacetime metric and the stellar structure can be found. In GR, this is the well-known Tolman-Oppenheimer-Volkoff (TOV) equation. Moreover, the equation of state (EOS) of matter is also essential to the interior structure of the star. Mathematically speaking, this relation between matter density and pressure is required to close the differential equation system. Physically, EOS represents the model one employs to describe the stellar matter. For a compact star like a neutron star (NS) in which the matter is so dense that a fully relativistic treatment is needed, the interior structure is mostly supported by the degeneracy pressure of nuclear particles. If one assumes a neutron stellar matter that involves pure nucleon-nucleon interaction, a polytropic approximation of the EOS,

$$p = \kappa \rho^\gamma,$$

(1)
can be utilized, where $p$ and $\rho$ are the pressure and density of the matter, respectively, and $\kappa$ and $\gamma$ are parameters related to the matter model. The compact stars in $f(T)$ gravity utilizing the polytropic EOS has been studied in Ref. [32]. In reality, however, NSs may consist of several layers that have different physical properties. A unified EOS that is valid in all of these crusts and core segments is then needed to construct realistic models of stars. Various unified EOSs have been employed to study the stellar structures in $f(R)$ gravity.
A comprehensive review on the relativistic stars in modified gravities can be found in Ref. [37].

In addition to the theoretical development of NS models, recent observations suggest that massive NSs with the mass close or beyond the prediction of GR may exist [38]. In particular, from the astrophysical data released by various projects including Neutron Star Interior Composition Explorer (NICER), North American Nanohertz Observatory for Gravitational Waves (NANOGrav) and Green Bank Telescope (GBT), it can be derived that there exist massive pulsars with masses in the range $1.44 \sim 2.27 M_\odot$ [39–41]. Moreover, the gravitational wave (GW) events of binary mergers observed by LIGO/Virgo collaboration also provide independent measurement of the properties of the event participants, among which the GW signal GW190814 indicates that a compact object, possibly an NS, with a mass around $2.6 M_\odot$ may exist [42]. The existence of these super-massive NSs may be explained by the modification of gravitational theories such as $f(T)$ gravity studied in this work. And hence, it is necessary to compare the theoretical predictions with the observational data and put constraints to the theory so that it can accommodate the observed NSs.

In this paper, we intend to study the realistic models of NSs in $f(T)$ gravity. We aim at obtaining the stellar structure numerically and searching for possible signatures of the modification of $f(T)$. We therefore use the simple model $f(T) = T + \alpha T^2$ that can be seen as an analog to the Starobinsky model in curvature-based gravity [43]. We employ several unified EOSs (SLy and BSk family) to describe the neutron stellar matter. Then the NSs in $f(T)$ gravity are subjected to comparison with the observations of massive pulsars and GW events. The paper is organized as follows. In Sec. II, we briefly review the basis of the covariant teleparallel gravities and set the equations for spherically symmetric stellar structure. Concrete models and the internal structures of NSs in $f(T)$ gravity are presented numerically in Sec. III. Based on the numerical results, we present the mass-radius relations of the NSs and subject them to the constraints of observations in Sec. IV. Section V contains our concise summary and discussions. Throughout the paper, we use the units with $c = 8\pi G = 1$. 
II. EQUATIONS FOR STELLAR STRUCTURE IN COVARIANT $f(T)$ GRAVITY

A. The covariant $f(T)$ gravity

As the spacetime manifold $\mathcal{M}$ is assumed to be a parallelizable metric space, one can generally find a trivialization $e_a = e^\mu_a \partial_\mu$ of the tangent bundle of the manifold. The dual vector basis 1-form to $e_a$, i.e. the tetrad, is given by $h_a = h^\mu_a dx^\mu$, so that $h_a(e_b) = \delta^a_b$. The spacetime metric $g$ is related to the tangent space metric $\eta$ by

$$g = g_{\alpha\beta} dx^\alpha \otimes dx^\beta = \eta_{ab} h^a \otimes h^b,$$

or, in terms of components,

$$g_{\alpha\beta} = \eta_{ab} h^a_{\alpha} h^b_{\beta}, \quad \eta_{ab} = g_{\alpha\beta} e^a_{\alpha} e^b_{\beta}.$$  \hspace{1cm} (3)

The torsion 2-form is given by \cite{7, 19}

$$T^a = \mathcal{D}h^a = dh^a + \omega^a_b \wedge h^b,$$  \hspace{1cm} (4)

where the covariant exterior derivative $\mathcal{D}$ and the spin connection $\omega^a_b$ are introduced such that for any vector $V^a$ in the tangent space at a given point, $\mathcal{D}_\mu V^a$ is covariant under Lorentz rotation. Generally, one can always find a specific tetrad, called proper tetrad, in which all components of the spin connection vanish. Hence, in terms of components, the torsion tensor with proper tetrad is

$$T^a_{\beta\gamma} = e^a_{\alpha} \left( \partial_\beta h^a_{\gamma} - \partial_\gamma h^a_{\beta} \right).$$  \hspace{1cm} (5)

Then, the torsion scalar is given by

$$T = T^a \wedge \star \left( T_a - h^a \wedge (e_b \cdot T^b) - \frac{1}{2} e^a \cdot (h^b \wedge T_b) \right)$$

where $\star$ denotes the Hodge dual and $\cdot$ indicates the interior product. In terms of contraction of tensors, it is

$$T = T^a_{\beta\gamma} S_{\alpha}^{\beta\gamma},$$  \hspace{1cm} (7)

with the super potential

$$S_{\alpha}^{\beta\gamma} = \frac{1}{4} \left( T^a_{\alpha} T_{\beta\gamma} + T^\gamma_{\alpha} T_{\beta} - T^\gamma_{\beta} T_{\alpha} \right) + \frac{1}{2} \left( \delta^\beta_{\alpha} T_{\lambda\gamma} - \delta^\gamma_{\alpha} T_{\lambda\beta} \right).$$  \hspace{1cm} (8)
TEGR takes $T$ as its Lagrangian, and the $f(T)$ gravity considers an arbitrary function of $T$ instead, i.e.,

$$ S = -\frac{1}{2} \int |h| f(T) d^4 x + \int |h| L_M d^4 x, \quad (9) $$

where $|h| = \det(h^a_{\alpha}) = \sqrt{-g}$ is the determinant of the tetrad $h^a_{\alpha}$, and $L_M$ is the Lagrangian of matter. Variation of Eq. (9) with respect to the tetrad gives the field equations

$$ \frac{2}{|h|} \partial_\beta \left( |h| S^\alpha_\sigma e^\sigma_a f_T \right) + \frac{f}{2} e^\alpha_a = T^\alpha_\beta e^\beta_a, \quad (10) $$

where $f_T$ denotes $df/dT$, and the energy-momentum tensor $T^\alpha_\beta$ of matter is given by

$$ \frac{\delta (|h| L_M)}{\delta h^a_{\alpha}} = |h| T^\alpha_\beta e^\beta_a. \quad (11) $$

**B. Spherically symmetric stellar equations**

For spherically symmetric stars, we consider a static metric

$$ ds^2 = e^{A(r)} dt^2 - e^{B(r)} dr^2 - r^2 d\theta^2 - r^2 \sin^2 \theta d\phi^2, \quad (12) $$

where the unknown metric functions $A(r)$ and $B(r)$ depend only on the radial coordinate $r$. The proper tetrad for this metric is chosen as

$$ h^a_{\alpha} = \begin{pmatrix}
  e^{\frac{A}{2}} & 0 & 0 & 0 \\
  0 & e^{\frac{B}{2}} \sin \theta \cos \phi & e^{\frac{B}{2}} \sin \theta \sin \phi & e^{\frac{B}{2}} \cos \theta \\
  0 & -r \cos \theta \cos \phi & -r \sin \theta \sin \phi & r \sin \theta \\
  0 & r \sin \theta \sin \phi & -r \sin \theta \cos \phi & 0
\end{pmatrix}. \quad (13) $$

Then, the torsion scalar is given by

$$ T(r) = \frac{2}{r^2} e^{-B(r)} \left( e^{\frac{B}{2}} - 1 \right) \left( e^{\frac{B}{2}} - 1 - rA'(r) \right). \quad (14) $$

The stellar matter, if considered as a perfect fluid, can be described by

$$ T_{\mu\nu} = (p + \rho) u_\mu u_\nu - pg_{\mu\nu} \quad (15) $$

with the 4-velocity $u^\mu$. The $tt$ and $rr$ components of Eq. (11) read

$$ \rho = -\frac{e^{-B}}{2r^2} \left\{ 4rf_T T' \left( e^{\frac{B}{2}} - 1 \right) + 2f_T \left[ rB' + \left( e^{\frac{B}{2}} - 1 \right) \left( 2 + rA' \right) \right] + fr^2 e^B \right\}, $$

$$ p = \frac{f}{2} + \frac{2f_T e^{-\frac{B}{2}}}{r^2} \left( 1 - e^{-\frac{B}{2}} \right) - \frac{f_T e^{-\frac{B}{2}}}{r} \left( 2e^{-\frac{B}{2}} - 1 \right) A', \quad (16) $$
where the prime indicates a derivative with respect to \( r \) and \( f_{TT} = d^2 f/dT^2 \). For a third equation, one can take either the angular component of Eq. (11) or the conservation of matter \( \nabla \mu T^{\mu \nu} = 0 \), which gives the TOV equation

\[
p' = -\frac{1}{2} (\rho + p) A'.
\]  

(17)

By extracting the Einstein tensor \( G_{\mu \nu} \), Eq. (10) can be written as

\[
G_{\mu \nu} = T_{\mu \nu} + \tilde{T}_{\mu \nu},
\]  

(18)

where the modification from the nonlinear term(s) of \( f(T) \) is absorbed in the effective energy-momentum tensor \( \tilde{T}_{\mu \nu} \) of "\( f(T) \) fluid" given by

\[
\tilde{T}_{\mu \nu} = \left( f_T T - f \right) \frac{1}{2} g_{\mu \nu} - 2 S_{\mu \sigma} \partial_{\sigma} f_T + (1 - f_T) G_{\mu \nu}.
\]  

(19)

The components of Eq. (19) can be denoted as the effective density, radial and transverse pressures as follows

\[
\tilde{\rho} = \frac{f}{2} + \frac{f_T}{r^2 e^B} \left[ (e^\frac{B}{2} - 1) (2 + r A') + r B' \right] + \frac{2f_{TT}}{r e^B} \left( e^\frac{B}{2} - 1 \right) T' + \frac{1}{r^2 e^B} \left( e^B + r B' - 1 \right),
\]

\[
\tilde{p}_r = -\frac{f}{2} - \frac{1}{r^2 e^B} \left( 1 - e^B + r A' \right) - \frac{f_T}{r^2 e^B} \left[ 2 \left( e^\frac{B}{2} - 1 \right) + \left( e^\frac{B}{2} - 2 \right) r A' \right],
\]

\[
\tilde{p}_t = -\frac{f}{2} + \frac{f_T}{4 r^2 e^B} \left[ 4 - 8 e^\frac{B}{2} + 4 e^B - 2 \left( 2 e^\frac{B}{2} - 3 \right) r A' + r^2 A'^2 - r B' (2 + r A') + 2 r A'' \right]
\]

\[
- \frac{f_{TT}}{2 r e^B} T' \left( 2 e^\frac{B}{2} - 2 - r A' \right) - \frac{1}{4 r e^B} \left[ 2 A' + r A'^2 - B' (2 + r A') + 2 r A'' \right],
\]  

(20)

so that \( \tilde{T}_{\mu \nu} \) can be written in the form of an anisotropic perfect fluid

\[
\tilde{T}_{\mu \nu} = (\tilde{\rho} + \tilde{p}_t) u_\mu u_\nu - \tilde{p}_t g_{\mu \nu} + (\tilde{p}_r - \tilde{p}_t) \chi_\mu \chi_\nu,
\]  

(21)

where \( \chi^\mu \) is the radial space-like unit vector.

III. INTERNAL STRUCTURE

In this section, we study numerically the internal structure of the NSs in \( f(T) \) gravity. As a simple model, we consider the concrete form of \( f(T) = T + \alpha T^2 \), which can be viewed as an analog of the Starobinsky model in the curvature framework of gravity[43].
A. Boundary conditions and EOSs

Since $A(r)$ does not appear explicitly in Eq. (16) or (17), we therefore solve the system for the functions $A'(r), B(r) \text{ and } \rho(r)$. The boundary conditions appropriate for the system can be set at the center of the star. The regularity condition then requires

$$A'(0) = 0, \quad B(0) = 0, \quad \rho(0) = \rho_c$$

for some central density $\rho_c$ of the NS. The radius $R$ of the star is defined by

$$\rho(R) = 0.$$  \hspace{1cm} (23)

For the last piece to close the system given by Eqs. (16) and (17), one needs the EOS of the stellar matter, i.e., the algebra relation between $p$ and $\rho$. Since the interaction of matter under the extreme environment within the NS is not yet very well understood, various EOSs for the neutron stellar matter have been proposed to describe the different crust and core segments in a unifying way. The main features of the representative EOSs that we use in the present paper are reported as follows:

- From the Skyrme Lyon (SLy) effective nucleon-nucleon interaction, the SLy EOS for nonrotating neutron stellar matter is obtained by many-body methods [44]. Nuclei in the neutron star crust are described by the compressible liquid drop model. The calculation is also continued to the characteristic densities of the expected liquid core of NSs. The analytic representation of SLy EOS can be written as

$$\zeta = \frac{a_1 + a_2 \xi + a_3 \xi^2}{1 + a_4 \xi} k(a_5(a_6 - \xi)) + (a_7 + a_8 \xi) k(a_9(a_{10} - \xi))$$

$$+ (a_{11} + a_{12} \xi) k(a_{13}(a_{14} - \xi)) + (a_{15} + a_{16} \xi) k(a_{17}(a_{18} - \xi)),$$  \hspace{1cm} (24)

where $\xi = \log(\rho/(g \text{ cm}^{-3}))$ and $\zeta = \log(p/(\text{dyn cm}^{-2}))$, and the function $k(x)$ is defined as $k(x) = 1/(e^x + 1)$.

- The BSk family[45] of EOSs bases on the generalized Skyrme interaction, supplemented by several interaction corrections, which is developed for cold catalyzed nuclear matter. Each BSk EOS corresponds to a specific numerical fit. At the typical density of neutron matter, BSk19, BSk20, and BSk21 describe approximately the soft, moderate, and stiff
matter EOS, respectively. The analytic representation of BSk EOSs can be written as

$$\zeta = \frac{a_1 + a_2 \xi + a_3 \xi^2}{1 + a_4 \xi} k (a_5 (\xi - a_6)) + (a_7 + a_8 \xi) k (a_9 (a_6 - \xi))$$

$$+ (a_{10} + a_{11} \xi) k (a_{12} (a_{13}-\xi)) + (a_{14} + a_{15} \xi) k (a_{16} (a_{17} - \xi))$$

$$+ \frac{a_{18}}{1 + (a_{19} (\xi - a_{20}))^2} + \frac{a_{21}}{1 + (a_{22} (\xi - a_{23}))^2}. \tag{25}$$

The parameters $a_i$ in Eqs. (24) and (25) are listed in Appendix A.

Note that for electromagnetic field and non-interacting matter, the trace $T$ of Eq. (15)
is positively determined, i.e., $T = \rho - 3p \geq 0$. This condition is then sometimes assumed

Note that for electromagnetic field and non-interacting matter, the trace $T$ of Eq. (15)
is positively determined, i.e., $T = \rho - 3p \geq 0$. This condition is then sometimes assumed
to hold for any perfect fluid [35]. If this is true, there will be an extra constraint on the
density $\rho$ for the EOSs mentioned above. However, it is pointed out that $T$ may be negative
for relativistic invariant, causal theories describing strong interacting systems [46]. The
correlation between the possibility of a negative $T$ and the macroscopic properties of NSs
is studied in Ref. [47]. In the present work, we assume that $T$ is allowed to be negative.
Moreover, other conditions for $\rho$ and $p$ such as the energy conditions do not lead to any
constraint in regard of the EOSs discussed here. Therefore, in the following discussions, we
do not put any extra constraint on $\rho$ or $p$.

B. Energy density and pressure profiles

For a given central density $\rho_c$, we integrate the system of Eqs. (16) and (17) using the
EOSs described in the previous subsection. When $\alpha > 0$, there seems to be an upper limit for
$\rho_c$ beyond which the numerical system cannot converge stably. In Figs. 1 and 2, we show the
density and pressure profiles, as well as the profiles of the effective density $\tilde{\rho}$ and pressures
$\tilde{p}_r, \tilde{p}_t$ of the $f(T)$ fluid given in Eq. (20), for the aforementioned EOSs with $\alpha = 10r_g^2$ and
some representative values of $\rho_c$, where $r_g = GM_{\odot}/c^2 \simeq 1.48 \times 10^5$ cm.

One can see that for all the cases the density $\rho$ and pressure $p$ decrease outwardly and
reach zero at some finite radius, defining the surface radius of the star. Hence the stellar
matter is indeed confined in a finite region independently of EOS. The effective density $\tilde{\rho}$, radial and transverse pressures $\tilde{p}_r, \tilde{p}_t$ of the $f(T)$ fluid vanish at the center of the star and increase outwardly for the inner part of the stellar interior. They reach their maxima at some point inside the star and start to decrease and change sign within the radius of the
FIG. 1: Density profiles, as well as the profiles of the effective density of the $f(T)$ fluid, for SLy, BSk19, BSk20, and BSk21 EOSs with $\alpha = 10r_s^2$ and several representative values of $\rho_c$.

star. At the surface of the star, while the effective radial pressure $\tilde{p}_r$ returns to zero, the effective density $\tilde{\rho}$ and transverse pressure $\tilde{p}_t$ remain negative, indicating a likely different external solution for nonlinear $f(T)$ gravity than the Schwarzschild vacuum in GR.
FIG. 2: Pressure profiles, as well as the profiles of the effective radial and transverse pressures of the $f(T)$ fluid, for SLy, BSk19, BSk20, and BSk21 EOSs with $\alpha = 10r_g^2$ and several representative values of $\rho_c$.

The last panel of each graph in Figs. 1 and 2 corresponds to the value of $\rho_c$ close to the stably integrable limit of the numerical system, which, for $\alpha = 10r_g^2$, are 1.678, 1.859, 1.563,
and 1.523 (in the unit of $10^{15}$ g/cm$^3$) for SLy, BSk19, BSk20, and BSk21 EOSs, respectively. The profiles of the effective $f(T)$ fluid in these limit cases are qualitatively similar in that $\tilde{\rho}, \tilde{p}_t$ change abruptly at some point, indicating rigid systems that lead to the breakdowns of the numerical procedures. The core region of the NS defined by the steplike phase transition of the effective $f(T)$ fluid is then formed by a total effective fluid that has a slightly increasing density $\rho + \tilde{\rho}$. This behavior of the mixture material of neutron matter and the effective $f(T)$ fluid is similar to that in the polytropic model where it is referred to as mimicking of an incompressible matter [32].

From Fig. 2 or more explicitly, from calculation, one can easily find that the central pressures for different EOSs are similar, i.e., $\sim 4.8 \times 10^{35}$ dyn/cm$^2$ for the limit cases. This can be interpreted as follows. It is the pressure that balances the gravitation which, in the current cases, is given by the same $f(T)$ model. If the numerical system describing this gravitation has any sharp transition, its occurrence will be most likely dominated by the initial value of pressure rather than density. This correlation between pressure and the behavior of the numerical system may be extended to the noncritical cases. However, it does not mean similarity among the NS structures with different EOSs. On the contrary, softer stellar matter will be compressed more tightly by the same level of gravitation and hence the NS will have smaller radius. From Figs. 1 and 2, one can see that for BSk19, SLy, BSk20, and BSk21 EOSs, in turn, describing stellar matter from soft to stiff, the radii of NSs generally vary from relatively smaller to larger.

Although all models with positive $\alpha$’s have qualitatively similar behaviors and we have chosen a representative value $\alpha = 10r_{g}^2$ to depict in Figs. 1 and 2, quantitatively, different values of $\alpha$ will no doubt affect the behaviors of the numerical systems. For example, when $\alpha = 5r_{g}^2$, the limits of $\rho_c$ will be raised to 2.474, 2.670, 2.277, and 2.329 (in the unit of $10^{15}$ g/cm$^3$) for SLy, BSk19, BSk20, and BSk21 EOSs, respectively. And when $\alpha = r_{g}^2$, the corresponding limit values are 8.628, 8.384, 7.935, and 8.239 (in the unit of $10^{15}$ g/cm$^3$), respectively. It seems that the limit of $\rho_c$ will be higher for a smaller modification. We also check $f(T) = T e^{\alpha T}$ model for confirmation of this observation. Since the torsion scalar is generally negative for the gravitational system under consideration, the factor $e^{\alpha T}$ leads to a smaller modification than the term $\alpha T^2$ for the same value of $\alpha$. The limits of $\rho_c$ for the steplike behavior of the $f(T)$ fluid are raised in $T e^{\alpha T}$ model as expected.

In Fig. 3, we present the density and pressure profiles, as well as the profiles of the
The effective density and pressures of the $f(T)$ fluid, for the EOSs of SLy and BSk family with $\alpha = -10r_g^2$ and $\rho_c = 5.0 \times 10^{15}\text{g/cm}^3$ as a representative case of negative $\alpha$. The matter density and pressure also decrease outwardly and vanish at a finite surface radius of the star. The profiles for different EOSs with a negative $\alpha$ are qualitatively similar to each other, and also to the polytropic model [32]. The effective density and pressures of the $f(T)$ fluid, in contrast with the cases of positive $\alpha$, are negative for most part of the interior region of the star. They change sign near the surface and remain positive at the radius of the star, which, like the cases of positive $\alpha$, indicates a possibly different external solution of $f(T)$ gravity than the Schwarzschild one. No steplike changes can be seen in profiles of the cases.

FIG. 3: Density and pressure profiles for SLy, BSk19, BSk20, and BSk21 EOSs with $\alpha = -10r_g^2$ and $\rho_c = 5.0 \times 10^{15}\text{g/cm}^3$. The radii of the NSs are 10.86km, 10.24km, 11.15km, and 11.53km, respectively.
of negative $\alpha$. More precisely, the sharp transition of $f(T)$ fluid only happens when the modification term is positive. This can be checked in the $f(T) = T + \alpha T^3$ model, where the steplike behavior occurs when $\alpha < 0$.

IV. MASSES AND RADII

A. Mass-radius relation

The term stellar mass generally means the total mass (or energy) of the stellar gravitation system that may be measured at a distance. Due to the non-linearity of the gravitation field, this total (or effective, active) mass is usually a combination of the material (or passive, rest) mass and the energy of the corresponding gravitation field. In spherical symmetry with a metric written in Eq.(12), the material mass within a hypersurface $\Sigma$ of radius $r$ is given by

$$m(r) = \int_{\Sigma} \star 1 = 4\pi \int_0^r \rho(x)e^{B(x)}x^2dx,$$

where $\star 1$ is the volume form and $\rho$ is the energy density. The definition of the active mass, however, relies on the gravitation theory. In GR, the active mass of an asymptotically flat spacetime can be defined at spatial infinity by the Arnowitt-Deser-Misner (ADM) mass [48] and at null infinity by the Bondi-Sachs (BS) mass [49, 50]. Within a finite hypersurface $\Sigma$, the Misner-Sharp (MS) mass [51] gives a definition of the active mass that can reduce to the ADM or BS mass asymptotically at the corresponding infinity [52]. For a spherically symmetric spacetime either dynamic or static, the metric can be written in the following coordinates,

$$ds^2 = I_{ab}dx^adx^b + r^2d\Omega_2^2,$$

where $d\Omega_2^2$ represents a unit 2-sphere, $I_{ab}$ is the induced metric that does not depend on the inner coordinates of the unit 2-sphere, and $a, b, \cdots$ run from 0 to 1. The MS mass in GR then can be written as

$$M_{MS}^{(GR)} = \frac{r}{2G} \left(1 - I^{ab} \partial_a r \partial_b r \right).$$

One can see that Eq.(28) does not involve explicitly any term of material source, e.g., energy density $\rho$ or pressure $p$. In fact, the MS mass, as well as the ADM mass and BS mass, has utilized the gravitational field equations to transform the combination of the material mass and the gravitational energy into the geometric terms of the spacetime. A known solution
of the metric and the material source can, of course, reproduce the active mass written in terms of $\rho$ and $p$. But they depend closely on the gravitation theory, or more precisely, the field equations since the active mass is in fact the solution to the field equation in a different, integral form.

Therefore, in modified gravities with different field equations than the Einstein equation, the definition of the active mass, or, the MS mass in a finite spacetime with spherical symmetry, needs to be reconsidered, which is generally thought to be much easier and more definitive when an explicit solution is known [32]. In $f(R)$ gravity, this issue is studied in Refs. [53, 54]. Following the idea of these works, in $f(T)$ gravity, one starts from the physical meaning of the active mass, i.e., the unified first law [55],

$$dM_{MS} = A \psi_a dx^a + W dV,$$  \hspace{2cm} (29)

where $A$ and $V$ are the surface area and volume of the space region being considered, respectively. $W$ is the work term and $\psi_a$ is the energy supply term, which are given by

$$W = -\frac{1}{2} I^{ab}T_{ab}, \quad \psi_a = T^b_a \partial_b r + W \partial_a r.$$  \hspace{2cm} (30)

Note that although Eq.(30) does not involve any explicit feature of the gravitation theory, they are consistent with the Einstein equation. If Eq.(30) is assumed to still hold in modified gravities [53, 54], one then immediately obtains,

$$dM_{MS} = A I^{ab} (T_{ab} \partial_c r - T_{ac} \partial_b r) dx^c.$$  \hspace{2cm} (31)

In the static case with the usual spherical coordinates, Eq.(31) gives the intuitive definition of the active mass

$$dM_{MS} = A T^0_0 dr = 4\pi r^2 \rho dr.$$  \hspace{2cm} (32)

Further derivation of the MS mass involves substitution of $T^0_0$ in Eq.(32) with the field equation, i.e., Eq.(10) in the current case. This, however, is effectively the procedure of solving the field equation for external solutions and is beyond the scope of this work. Nonetheless, since we have already obtained the energy density profile $\rho(r)$ numerically, Eq.(32) is sufficient to define the active mass of an NS in $f(T)$ gravity. Due to the fact that $\rho(r)$ vanishes at the stellar surface $R$, the active stellar mass is given by

$$M = 4\pi \int_0^R \rho(r) r^2 dr.$$  \hspace{2cm} (33)
Then, for a given central density $\rho_c$ and hence a numerically determined profile $\rho(r)$, a pair \( \{ M, R \} \) can be found. Thus, one obtains an $M – R$ curve on which every point corresponds to a different $\rho_c$. In Fig. 4 we present the $M – R$ curves for some representative values of $\alpha$ and all the EOSs described in the previous subsection. At relatively low central density

around or below $\rho_c \sim 10^{13} \text{g/cm}^3$, the radius and mass only show subtle differences for various values of $\alpha$ with a given EOS. This may indicate that a unified treatment, possibly

FIG. 4: Mass-radius curves for SLy, BSk19, BSk20, and BSk21 EOSs, considering various representative values of $\alpha$. The curve corresponding to GR is recovered when $\alpha = 0$. 
a nonrelativistic one like Newtonian star, is sufficient for the models at this order of density. At neutron stellar density around or above $\rho_c \sim 10^{15} \text{g/cm}^3$, changes of the active mass and radius of the star can be observed.

For positive $\alpha$’s, the numerical procedure continues till the system reaches its limit as discussed in the previous subsection and cannot converge stably. Relatively small $\alpha$ (see the $\alpha = 0.2r_g^2$ lines in each panel) results in $M - R$ curves that are quite close to GR, somewhat validating the stability of the system and the reliability of the numerical procedure. As $\alpha$ grows and hence the model moves away from GR, significantly less stellar matter can be contained in a given radius. In these cases, there exist such critical configurations that the stellar masses reach their maxima. If the ratio between pressure and energy density goes beyond this critical point, the structure of the star may become unstable. The critical configuration of different values of $\alpha$ for each EOS, if exists, are marked in Fig. 4 with circle symbols. The detail values of the critical configurations can be found in Appendix B.

For negative $\alpha$’s, the numerical system allows considerably high central densities. Relatively small $|\alpha|$ (see the $\alpha = -0.2r_g^2$ lines in each panel) may still present a peak in $M - R$ curve as GR. But as the central densities are set to be higher, all curves with negative $\alpha$ tilt up and the stellar masses increase with $\rho_c$. For $\rho_c = 10^{19} \text{g/cm}^3$ and $\alpha = -10r_g^2$, the stellar mass can reach 11.43$M_\odot$, 23.23$M_\odot$, 23.35$M_\odot$, and 24.15$M_\odot$ (beyond the plots in Fig. 4) for SLy, BSk19, BSk20, and BSk21 EOS, respectively. Even larger neutron stellar masses may be reached for higher central densities, which, however, may indicate a different matter state beyond the description of the EOSs used in this work. As $\alpha$ decreases, more stellar matter can be contained in a given radius, resulting in a more compact star. A qualitatively analogous pattern is reported for the material mass of the polytropic model of stars in $f(T)$ gravity [32].

B. Observation constraints

In this subsection, the mass-radius curves of the NSs in $f(T)$ gravity will be subjected to the joint constraint from the observed massive pulsars and the gravitational wave events. The detection of the first GW signal from a binary NS merger GW170817 by LIGO/Virgo collaboration [56], together with its electromagnetic counterpart, GRB170817, provides information of the masses and radii of the two NSs in this event. Another GW event, GW190814 [42],
indicates that a compact object, possibly an NS, with a mass of $2.59 \pm 0.08 M_\odot$ may exist. Besides this compact object, an NS with a mass of $2.27^{+0.17}_{-0.15} M_\odot$, hosted by PSR J2215+5135 [39], is reported as one of the most massive NSs known to date. Moreover, joint observation of the mass and radius of the pulsar PSR J0030+0451 by NICER [40] also provides independent constraint on NS properties. By combining the data from NANOGrav 12.5-year dataset with the orbital-phase-specific observations using the GBT, another massive NS PSR J0740+6620 with a mass of $2.14^{+0.20}_{-0.18} M_\odot$ has been reported [41]. With the NICER and X-ray Multi-Mirror (XMM) X-ray observation, the radius of PSR J0740+6620 has also been measured [57]. The constraints on the mass-radius relation from the aforementioned observations are presented in Fig. 5. The inner solid contour and outer dashed contour for each set of observational data correspond to the $1\sigma$ (68.3%) and $2\sigma$ (95.4%) confidential levels, respectively. The GW170817 dataset produces two sets of contours, corresponding to two components of the binary. Observation of PSR J2215+5135 and the GW event GW190814 only give constraints of the mass, which are depicted as the blue and purple bands in the graphs, respectively. We compare these constraints with the $M-R$ curves of NSs in $f(T)$ gravity in Fig. 5, and obtain the constraints for the model parameter $\alpha$ for different EOSs.

For SLy, one can see that the GR curve cannot reach the mass range given by the dataset of PSR J2215+5135. It follows that the model parameter $\alpha$ needs to be negative so that the curve can tilt up and pass both mass ranges given by PSR J2215+5135 and GW190814. On the other hand, when $\alpha$ is small enough ($\leq 1.8 r_g^2$), the curve can pass through all the contours. Therefore, for the NS model to accommodate the observation data, the interception of constraints leads to the requirement that $\alpha < 0$.

For BSk19, one can see that most of the curves can pass through the contours of GW170817 and PSR J0030+0451. And, as long as $\alpha < 0$, the curve will tilt up for large $\rho_c$ and reach the mass ranges given by PSR J2215+5135 and GW190814. However, only the curves with $\alpha \leq -3.5 r_g^2$ (see the solid line) can pass through the $2\sigma$ contour given by the dataset of PSR J0740+6620 and accommodate all the considered observations. For BSk20, the GR curve can already accommodate all the data except GW190814. The models with $\alpha > 0.4 r_g^2$ predict that the NS mass (see the solid line) cannot reach the mass range given by the observation of PSR J2215+5135, and hence are ruled out. Moreover, if one considers the compact object in the GW event GW190814 to be an NS and wishes the mass range given by this dataset to be reached, the model parameter $\alpha$ has to be negative.
so that the curve would tilt up and increase as $\rho_c$ is set high enough.

For BSk21, once again the GR curve can already accommodate all the data except GW190814. The requirement for the models to accommodate the observation of PSR J2215+5135 is $\alpha \leq 1.9r_g^2$ (see the solid line). And a negative $\alpha$ can also enable the model to accommodate the GW190814 data for a possible NS.
In this paper, we employ the realistic EOSs to investigate the NSs in \( f(T) \) gravity. In particular, we study the static and spherically symmetric configuration with neutron stellar matter described by SLy and BSk family of EOSs in a simple nonlinear model \( f(T) = T + \alpha T^2 \). For both positive and negative values of \( \alpha \), we show that the model indeed provides a compact star solution. As depicted in Fig. 4, significant changes of stellar mass-radius relation can be seen as \( \alpha \) moves away from zero and hence the model departs from GR. Moreover, regardless of the EOS, models with a negative \( \alpha \) may support more matter for the compact star than in GR, while less matter can be contained with a positive \( \alpha \). This may be understood qualitatively as follows. In the interior region of an NS, \( f(T) \) model with a positive modification exerts a stronger gravity than GR for a given amount of matter. Thus, at the same pressure level, e.g., degeneracy pressure of nuclear particles, less matter can be supported in such a case. On the other hand, a negative modification may act as diminishment of the gravity, or, if considered as an \( f(T) \) fluid, provide a pressure to resist the gravity; hence, more matter can be supported. This can also be seen in Figs. 2 and 3. The effective pressures of the \( f(T) \) fluid are positive for most part of the interior of the star when \( \alpha \) is positive, while a negative value of \( \alpha \) may lead to almost always negative pressures of the \( f(T) \) fluid inside the star.

In GR, due to the limit of nuclear degeneracy pressure, there exists a configuration that corresponds to a maximum stellar mass [38]. In \( f(T) \) gravity, this upper bound of stellar mass may be passed with the help of positive radial and transverse pressures of the \( f(T) \) fluid. For a negative \( \alpha \), there may still be a peak of the \( M - \mathcal{R} \) curve for relatively small \(|\alpha|\). But the peak no longer corresponds to the maximum NS mass in that for \( \alpha < 0 \) the mass can always go up as the central density \( \rho_c \) is set high enough. This makes \( f(T) \) gravity be able to accommodate some observations of NS that are beyond GR’s prediction. Therefore, we compare the NS models in \( f(T) \) gravity with the astrophysical observations of the NS PSR J0030+0451 [40], PSR J0740+6620 [41, 57], PSR J2215+5135 [39], and the GW event GW170817 [56] and GW190814 [42]. For SLy EOS, a negative \( \alpha \) is required to account for all the considered NS observation datasets. For BSk19 EOS, the models with \( \alpha > -3.5r_g^2 \) are ruled out by the NS observations. For BSk20 and BSk21, the observation datasets excluding GW190814 constrain that \( \alpha \leq 0.4r_g^2 \) and \( \alpha \leq 1.9r_g^2 \), respectively. If the compact object in
the GW190814 event is to be considered as an NS and needs to be accounted for by the $f(T)$ modification, then $\alpha$'s in these two EOS models are constrained to be negative.

These constraints, however, may still be flawed in that an NS model for all of the four EOSs with a negative $\alpha$ can produce large enough stellar mass only if the central density $\rho_c$ is set to be high enough. Since the EOSs all have their effective ranges and the matter may not be in the same state and phase when the density and pressure are beyond these ranges, a joint constraint for both the EOS and the $f(T)$ theory may be more physical. Another issue may come from the GW. The constraints of NS parameters from GW events are adapted to the waveform in GR. A GW theory in $f(T)$ gravity may result in shifts of these constraints. Finally, the NS models considered in the present work are restricted to static spherically symmetric cases with an isotropic perfect fluid and a simple form of $f(T) = T + \alpha T^2$. It is known that the anisotropy of the fluids and the rotation of a star may raise the stellar mass and reduce the central density [37]. These issues are worth further studying in the future.

**ACKNOWLEDGEMENT**

This work is supported by the National Science Foundation of China under Grant No. 12105179.

**Appendix A: Parameters $a_i$ in the analytic forms of the EOSs**

For SLy EOS, the parameters $a_i$ are listed in Table I.

For BSk family of EOSs, the parameters $a_i$ in Eq. (25) are listed in Table II.

**Appendix B: The critical configurations of NS models**

For positive $\alpha$ or for negative $\alpha$ with relatively small $|\alpha|$, the $M - R$ curves of the NS models will have a peak, corresponding to the critical configurations, which are listed in Table III and are already marked with circle symbols in Fig. 4.
TABLE I: $a_i$ (SLy) parameters for Eq. (24).

| $i$ | $a_i$  | $i$ | $a_i$ |
|-----|--------|-----|--------|
| 1   | 6.22   | 10  | 11.4950|
| 2   | 6.121  | 11  | -22.775|
| 3   | 0.005925 | 12  | 1.5707 |
| 4   | 0.16326 | 13  | 4.3    |
| 5   | 6.48   | 14  | 14.08  |
| 6   | 11.4971| 15  | 27.80  |
| 7   | 19.105 | 16  | -1.653 |
| 8   | 0.8938 | 17  | 1.50   |
| 9   | 6.54   | 18  | 14.67  |

[1] Antonio De Felice and Shinji Tsujikawa, “f(R) theories,” Living Rev. Rel. 13, 3 (2010), arXiv:1002.4928 [gr-qc].
[2] Thomas P. Sotiriou and Valerio Faraoni, “f(R) Theories Of Gravity,” Rev. Mod. Phys. 82, 451–497 (2010), arXiv:0805.1726 [gr-qc].
[3] Salvatore Capozziello and Mariafelicia De Laurentis, “Extended Theories of Gravity,” Phys. Rept. 509, 167–321 (2011), arXiv:1108.6266 [gr-qc].
[4] Shin’ichi Nojiri and Sergei D. Odintsov, “Unified cosmic history in modified gravity: from F(R) theory to Lorentz non-invariant models,” Phys. Rept. 505, 59–144 (2011), arXiv:1011.0544 [gr-qc].
[5] Jose Beltrán Jiménez, Lavinia Heisenberg, and Tomi S. Koivisto, “The Geometrical Trinity of Gravity,” Universe 5, 173 (2019), arXiv:1903.06830 [hep-th].
[6] Junpei Harada, “Connection independent formulation of general relativity,” Phys. Rev. D 101, 024053 (2020).
[7] Ruben Aldrovandi and José Geraldo Pereira, Teleparallel Gravity: An Introduction, Vol. 173 (Springer, 2013).
TABLE II: $a_i$ parameters for BSk family of EOSs (25).

| $i$ | BSk19 | BSk20 | BSk21 |
|-----|-------|-------|-------|
| 1   | 3.916 | 4.078 | 4.857 |
| 2   | 7.701 | 7.587 | 6.981 |
| 3   | 0.00858 | 0.00839 | 0.00706 |
| 4   | 0.22114 | 0.21695 | 0.19351 |
| 5   | 3.269 | 3.614 | 4.085 |
| 6   | 11.964 | 11.942 | 12.065 |
| 7   | 13.349 | 13.751 | 10.521 |
| 8   | 1.3683 | 1.3373 | 1.5905 |
| 9   | 3.254 | 3.606 | 4.104 |
| 10  | -12.953 | -22.996 | -28.726 |
| 11  | 0.9237 | 1.6229 | 2.0845 |
| 12  | 6.20 | 4.88 | 4.89 |
| 13  | 14.383 | 14.274 | 14.302 |
| 14  | 16.693 | 23.560 | 22.881 |
| 15  | -1.0514 | -1.5564 | -1.7690 |
| 16  | 2.486 | 2.095 | 0.989 |
| 17  | 15.362 | 15.294 | 15.313 |
| 18  | 0.085 | 0.084 | 0.091 |
| 19  | 6.23 | 6.36 | 4.68 |
| 20  | 11.68 | 11.67 | 11.65 |
| 21  | -0.029 | -0.042 | -0.086 |
| 22  | 20.1 | 14.8 | 10.0 |
| 23  | 14.19 | 14.18 | 14.15 |

[8] José W. Maluf, “The teleparallel equivalent of general relativity,” Annalen der Physik 525, 339–357 (2013), https://onlinelibrary.wiley.com/doi/pdf/10.1002/andp.201200272.
TABLE III: Some critical configurations of NS models with BSk family and SLy EOSs.

| EOS  | $\alpha/r_g^2$ | $\rho_c/(10^{15}\text{g/cm}^3)$ | $R$/km | $M/M_\odot$ |
|------|----------------|---------------------------------|--------|-------------|
| BSk19 | 0(GR)         | 3.48                            | 9.11   | 1.86        |
|      | 0.2           | 3.27                            | 9.18   | 1.84        |
|      | 1             | 2.78                            | 9.37   | 1.75        |
|      | 5             | 1.91                            | 9.75   | 1.48        |
|      | 10            | 1.54                            | 9.91   | 1.29        |
| BSk20 | 0(GR)         | 2.69                            | 10.17  | 2.17        |
|      | 0.2           | 2.57                            | 10.22  | 2.14        |
|      | 1             | 2.23                            | 10.37  | 2.05        |
|      | 5             | 1.58                            | 10.66  | 1.76        |
|      | 10            | 1.29                            | 10.77  | 1.53        |
| BSk21 | 0(GR)         | 2.30                            | 11.03  | 2.28        |
|      | 0.2           | 2.19                            | 11.10  | 2.26        |
|      | 1             | 1.97                            | 11.21  | 2.19        |
|      | 5             | 1.43                            | 11.49  | 1.93        |
|      | 10            | 1.17                            | 11.59  | 1.71        |
| SLy  | 0(GR)         | 2.85                            | 9.98   | 2.05        |
|      | 0.2           | 2.76                            | 10.01  | 2.03        |
|      | 1             | 2.38                            | 10.18  | 1.95        |
|      | 5             | 1.67                            | 10.53  | 1.68        |
|      | 10            | 1.35                            | 10.68  | 1.47        |

[9] Gabriel R. Bengochea and Rafael Ferraro, “Dark torsion as the cosmic speed-up,” Phys. Rev. D 79, 124019 (2009).
Eric V. Linder, “Einstein’s other gravity and the acceleration of the universe,” Phys. Rev. D 81, 127301 (2010).

Yi-Fu Cai, Salvatore Capozziello, Mariafelicia De Laurentis, and Emmanuel N. Saridakis, “f(T) teleparallel gravity and cosmology,” Rept. Prog. Phys. 79, 106901 (2016), arXiv:1511.07586 [gr-qc].

S. Nojiri, S.D. Odintsov, and V.K. Oikonomou, “Modified Gravity Theories on a Nutshell: Inflation, Bounce and Late-time Evolution,” Phys. Rept. 692, 1–104 (2017), arXiv:1705.11098 [gr-qc].

Thomas P. Sotiriou, Baojiu Li, and John D. Barrow, “Generalizations of teleparallel gravity and local lorentz symmetry,” Phys. Rev. D 83, 104030 (2011).

Baojiu Li, Thomas P. Sotiriou, and John D. Barrow, “f ( t ) gravity and local lorentz invariance,” Phys. Rev. D 83, 064035 (2011).

Rafael Ferraro and Franco Fiorini, “Remnant group of local lorentz transformations in $f(t)$ theories,” Phys. Rev. D 91, 064019 (2015).

Rafael Ferraro and Franco Fiorini, “Non-trivial frames for f(t) theories of gravity and beyond,” Phys. Lett. B 702, 75 – 80 (2011).

Nicola Tamanini and Christian G. Böhmer, “Good and bad tetrads in f ( t ) gravity,” Phys. Rev. D 86, 044009 (2012).

Yuri N. Obukhov and Guillermo F. Rubilar, “Covariance properties and regularization of conserved currents in tetrad gravity,” Phys. Rev. D 73, 124017 (2006).

Martin Krššák and Emmanuel N. Saridakis, “The covariant formulation of f(T) gravity,” Class. Quant. Grav. 33, 115009 (2016), arXiv:1510.08432 [gr-qc].

Alexey Golovnev, Tomi Koivisto, and Marit Sandstad, “On the covariance of teleparallel gravity theories,” Class. Quantum Grav. 34, 145013 (2017).

Rui-Hui Lin and Xiang-Hua Zhai, “New proper tetrad for teleparallel gravity in curved space-times,” Phys. Rev. D 99, 024022 (2019), arXiv:1911.02240 [gr-qc].

Andrew DeBenedictis and Saša Ilijić, “Spherically symmetric vacuum in covariant $f(t) = t + \frac{\alpha}{2} T^2 + O(T^\gamma)$ gravity theory,” Phys. Rev. D 94, 124025 (2016).

Alexey Golovnev and María-José Guzmán, “Approaches to spherically symmetric solutions in f(T) gravity,” Universe 7, 121 (2021), arXiv:2103.16970 [gr-qc].
[24] Christian Pfeifer and Sebastian Schuster, “Static spherically symmetric black holes in weak f(T)-gravity,” Universe 7, 153 (2021), arXiv:2104.00116 [gr-qc].

[25] Christian G. Boehmer, Atifah Mussa, and Nicola Tamanini, “Existence of relativistic stars in f(T) gravity,” Class. Quant. Grav. 28, 245020 (2011), arXiv:1107.4455 [gr-qc].

[26] M. Zubair and G. Abbas, “Analytic models of Anisotropic Strange Stars in f(T) Gravity with Off-diagonal tetrad,” Astrophys. Space Sci. 361, 27 (2016), arXiv:1507.00247 [physics.gen-ph].

[27] Ksh. Newton Singh, Farook Rahaman, and Ayan Banerjee, “Einstein’s cluster mimicking compact star in the teleparallel equivalent of general relativity,” Phys. Rev. D 100, 084023 (2019), arXiv:1909.10882 [gr-qc].

[28] Pameli Saha and Ujjal Debnath, “Study of anisotropic compact stars with quintessence field and modified chaplygin gas in f(T) gravity,” Eur. Phys. J. C 79, 919 (2019), arXiv:1911.10908 [physics.gen-ph].

[29] Gamal G. L. Nashed and Salvatore Capozziello, “Stable and self-consistent compact star models in teleparallel gravity,” Eur. Phys. J. C 80, 969 (2020), arXiv:2010.06355 [gr-qc].

[30] Andrew DeBenedictis and Saša Ilijić, “Regular solutions in f(T)-Yang-Mills theory,” Phys. Rev. D 98, 064056 (2018), arXiv:1806.11445 [gr-qc].

[31] Saša Ilijić and Marko Sossich, “Boson stars in f(T) extended theory of gravity,” Phys. Rev. D 102, 084019 (2020), arXiv:2007.12451 [gr-qc].

[32] Saša Ilijić and Marko Sossich, “Compact stars in f(t) extended theory of gravity,” Phys. Rev. D 98, 064047 (2018).

[33] A. Savas Arapoglu, Cemsinan Deliduman, and K. Yavuz Eksi, “Constraints on Perturbative f(R) Gravity via Neutron Stars,” JCAP 07, 020 (2011), arXiv:1003.3179 [gr-qc].

[34] Artyom V. Astashenok, Salvatore Capozziello, and Sergei D. Odintsov, “Further stable neutron star models from f(R) gravity,” JCAP 12, 040 (2013), arXiv:1309.1978 [gr-qc].

[35] Salvatore Capozziello, Mariafelicia De Laurentis, Ruben Farinelli, and Sergei D. Odintsov, “Mass-radius relation for neutron stars in f(R) gravity,” Phys. Rev. D 93, 023501 (2016), arXiv:1509.04163 [gr-qc].

[36] Artyom V. Astashenok, Sergei D. Odintsov, and Alvaro de la Cruz-Dombriz, “The realistic models of relativistic stars in f(R) = R + αR^2 gravity,” Class. Quant. Grav. 34, 205008 (2017), arXiv:1704.08311 [gr-qc].
[37] Gonzalo J. Olmo, Diego Rubiera-Garcia, and Aneta Wojnar, “Stellar structure models in modified theories of gravity: Lessons and challenges,” Phys. Rept. 876, 1–75 (2020), arXiv:1912.05202 [gr-qc].

[38] Clifford E. Rhoades, Jr. and Remo Ruffini, “Maximum mass of a neutron star,” Phys. Rev. Lett. 32, 324–327 (1974).

[39] Manuel Linares, Tariq Shahbaz, and Jorge Casares, “Peering into the dark side: Magnesium lines establish a massive neutron star in PSR J2215+5135,” Astrophys. J. 859, 54 (2018), arXiv:1805.08799 [astro-ph.HE].

[40] M. C. Miller et al., “PSR J0030+0451 Mass and Radius from NICER Data and Implications for the Properties of Neutron Star Matter,” Astrophys. J. Lett. 887, L24 (2019), arXiv:1912.05705 [astro-ph.HE].

[41] H. T. Cromartie et al. (NANOGrav), “Relativistic Shapiro delay measurements of an extremely massive millisecond pulsar,” Nature Astron. 4, 72–76 (2019), arXiv:1904.06759 [astro-ph.HE].

[42] R. Abbott et al. (LIGO Scientific, Virgo), “GW190814: Gravitational Waves from the Coalescence of a 23 Solar Mass Black Hole with a 2.6 Solar Mass Compact Object,” Astrophys. J. Lett. 896, L44 (2020), arXiv:2006.12611 [astro-ph.HE].

[43] Alexei A. Starobinsky, “A New Type of Isotropic Cosmological Models Without Singularity,” Adv. Ser. Astrophys. Cosmol. 3, 130–133 (1987).

[44] F. Douchin and P. Haensel, “A unified equation of state of dense matter and neutron star structure,” Astron. Astrophys. 380, 151 (2001), arXiv:astro-ph/0111092.

[45] A. Y. Potekhin, A. F. Fantina, N. Chamel, J. M. Pearson, and S. Goriely, “Analytical representations of unified equations of state for neutron-star matter,” Astron. Astrophys. 560, A48 (2013), arXiv:1310.0049 [astro-ph.SR].

[46] Ya. B. Zel’ dovich, “The equation of state at ultrahigh densities and its relativistic limitations,” Zh. Eksp. Teor. Fiz. 41, 1609–1615 (1961).

[47] Dylan M. Podkowka, Raissa F. P. Mendes, and Eric Poisson, “Trace of the energy-momentum tensor and macroscopic properties of neutron stars,” Phys. Rev. D 98, 064057 (2018), arXiv:1807.01565 [gr-qc].

[48] Richard L. Arnowitt, Stanley Deser, and Charles W. Misner, “Dynamical Structure and Definition of Energy in General Relativity,” Phys. Rev. 116, 1322–1330 (1959).
[49] H. Bondi, M. G. J. van der Burg, and A. W. K. Metzner, “Gravitational waves in general relativity. 7. Waves from axisymmetric isolated systems,” Proc. Roy. Soc. Lond. A 269, 21–52 (1962).

[50] R. K. Sachs, “Gravitational waves in general relativity. 8. Waves in asymptotically flat space-times,” Proc. Roy. Soc. Lond. A 270, 103–126 (1962).

[51] Charles W. Misner and David H. Sharp, “Relativistic equations for adiabatic, spherically symmetric gravitational collapse,” Phys. Rev. 136, B571–B576 (1964).

[52] Sean A. Hayward, “Gravitational energy in spherical symmetry,” Phys. Rev. D 53, 1938–1949 (1996), arXiv:gr-qc/9408002.

[53] Rong-Gen Cai, Li-Ming Cao, Ya-Peng Hu, and Nobuyoshi Ohta, “Generalized Misner-Sharp Energy in f(R) Gravity,” Phys. Rev. D 80, 104016 (2009), arXiv:0910.2387 [hep-th].

[54] Hongsheng Zhang, Yapeng Hu, and Xin-Zhou Li, “Misner-Sharp Mass in N-dimensional f(R) Gravity,” Phys. Rev. D 90, 024062 (2014), arXiv:1406.0577 [gr-qc].

[55] Sean A. Hayward, “Unified first law of black hole dynamics and relativistic thermodynamics,” Class. Quant. Grav. 15, 3147–3162 (1998), arXiv:gr-qc/9710089.

[56] B. P. Abbott et al. (LIGO Scientific, Virgo), “GW170817: Measurements of neutron star radii and equation of state,” Phys. Rev. Lett. 121, 161101 (2018), arXiv:1805.11581 [gr-qc].

[57] M. C. Miller et al., “The Radius of PSR J0740+6620 from NICER and XMM-Newton Data,” arxiv (2021), arXiv:2105.06979 [astro-ph.HE].