Non-Abelian String Conductivity

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Abstract

We examine current-carrying configurations of cosmic strings in non-Abelian gauge theories. We study the solutions numerically and point out that the currents will be at best dynamically stable and not subject to any topological quantisation or conservation, as in conventional models of string superconduction. We suggest that non-Abelian string loops may be unable to support persistent currents in the absence of external fields. This will have relevance to vorton stability.

1 Introduction

It has been known for several years that many cosmic string models appear to exhibit superconductivity [1]. The currents may be very large for strings formed during the Grand-Unified phase transition, and may have significant cosmological implications. For instance, it has been argued that decaying superconducting loops can provide an explosive mechanism for galaxy formation [2] (although it is not clear if this scenario is compatible with the observed anisotropies in the microwave background [3]). Observational bounds on cosmic rays from superconducting strings networks [4] and the synchrotron radiation signatures of these topological defects have been also discussed [5]. String loops stabilised against collapse by the effect of a current flowing on them – vortons [6] – are a potential dark matter candidate, and indeed if they exist from the time of the GUT scale they would present an overdensity problem that could rule out string scenarios [7].

In his original paper [1], Witten proposed two mechanisms for cosmic string superconductivity. The first is based on a charged scalar boson field which condenses in the core of the string, breaking electromagnetic gauge invariance there and supporting supercurrents in a manner directly analogous to conventional BCS theory. The second involves charged fermionic zero modes trapped in the core of the string and travelling along it at the speed of light. Both mechanisms are shown to operate in a simple U(1)×U(1) model, although for the scalar-bosonic case the presence of superconductivity requires a rather intricate fine-tuning of parameters – indicating that this is unlikely to be a generic feature of string-forming models. Despite this, as the model is sufficiently simple it has been studied thoroughly and the astrophysical consequences of superconducting strings have been explored by many authors. We refer the reader to the reviews [8] for more details.
A distinct and more natural mechanism for string conductivity arises in non-Abelian string models, that is free of the coupling dependence that restricts the mechanisms above. Here the charge carriers are (super-heavy) vector bosons which, if absent in the model mentioned above, are present in any Grand-Unified model. This possibility was first mentioned by Preskill and it was subsequently considered to some extent by Everett and Alford et al. By vector-boson conductivity we are not referring to the phenomenon of W-condensation near an ordinary GUT cosmic string, which could be the source of electromagnetic currents at the electroweak scale (see, for instance, references [2]-[10]), but rather the case in which the string itself is constructed from electromagnetically charged vector bosonic fields.

In this paper we will attempt a more systematic study of the conduction properties of this type of string. In section 2 we will briefly review the basics of scalar boson superconductivity in the U(1) × U(1) model, analysing the cases of a straight infinite string and a string loop. This provides the motivation for some distinctions made later in the paper. In section 3 we introduce a toy model (SU(2) × U(1) with a complex Higgs in the adjoint representation) which is the simplest theory that supports conducting non-Abelian strings, and allows us to study their electromagnetic properties without going through the technical complications of examining a realistic Grand-Unified group. We show in section 4, however, that this model can be embedded directly in SO(10) with the Higgs in the 126 representation, which is the simplest Grand-Unified theory exhibiting topologically stable strings. Finally, in section 5 we explain why the electrodynamic properties of loops of non-Abelian conducting string appear to be very different to those of their Abelian counterparts, speculate about the cosmological implications and summarize our results.

2 Scalar boson superconductivity

Superconductivity can be understood generally as broken electromagnetic gauge invariance. In general this appears when the expectation value of a charged, bosonic condensate or order parameter is non-zero. Witten examined a U(1) × U(1) model in which a charged scalar φ condensed around the core of an uncharged Abelian string, breaking electromagnetic symmetry there and allowing currents to flow along the string. We review it briefly here in order to introduce some concepts that will be useful later.

We begin with a Lagrangian with U(1) × U(1) symmetry:

$$
\mathcal{L} = -(D_\mu \Phi)^* D^\mu \Phi - (D_\mu \phi)^* D^\mu \phi - \frac{1}{4} G_{\mu\nu} G^{\mu\nu} - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} - V(\Phi, \phi),
$$

where we use a polar coordinate metric $g_{00} = -1, \ g_{zz} = g_{rr} = 1, \ g_{\theta\theta} = r^2$ and

$$
D_\mu \Phi = (\partial_\mu - ig R_\mu) \Phi \nonumber \\
D_\mu \phi = (\partial_\mu - ie A_\mu) \phi \\
G_{\mu\nu} = \partial_\mu R_\nu - \partial_\nu R_\mu \\
F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \\
V(\Phi, \phi) = \frac{1}{2} \alpha (|\Phi|^2 - \sigma^2)^2 + \frac{1}{2} \chi (|\phi|^2 - \eta^2)^2 + \frac{1}{2} \gamma |\Phi|^2 |\phi|^2.
$$

The U(1) symmetry gauged by $R_\mu$ is broken by the complex scalar $\Phi$ if $\chi \eta^4 < \alpha \sigma^4$ while the U(1) gauged by $A_\mu$ remains unbroken if $\gamma \sigma^2 / 2 > \chi \eta^2$. Nevertheless, in the presence of a string it may be favorable for $\phi$ to acquire a vacuum expectation value in the core of the string (where $\Phi$ is zero), effectively breaking the U(1) of electromagnetism there but not outside the string. This is indeed the case when, for instance,

$$
\gamma \sigma^2 / 2 \gg \chi \eta^2 \quad \left( \frac{\sigma}{\eta} \right)^3 \gg \left( \frac{\chi}{\alpha} \right)^2 \quad \frac{\gamma}{\chi} \ll \frac{\chi \eta^4}{\alpha \sigma^4}.
$$

Throughout the paper we will use convention in which indices $\mu, \nu, ..$ run from 0 to 3; $\alpha, \beta, ..$ denote $z, t$ and $i, j, ..$ denote $r, \theta$. 2
The ansatz for the bare string, with no current, is
\[ R_\theta = b(r) \quad \Phi(r, \theta) = f(r)e^{i\theta} \quad \phi(r, \theta) = \phi(r), \quad (2) \]
where \( b(r) \) and \( f(r) \) have the usual Nielsen-Olesen profiles and
\[ \phi(0) \neq 0, \quad \phi(\infty) = 0. \quad (3) \]

Let us then examine how currents arise in this model. Clearly the configuration above must be extended to introduce some \( z, t \) dependence of the fields, as well as a non-zero electromagnetic field. The most natural way to do this is with
\[ \phi(r) \rightarrow \phi(r)e^{ic_0\beta(z,t)} \]
\[ A_\alpha \rightarrow -\frac{1}{e}s(r)\partial_\alpha\beta(z,t) \quad \alpha \in \{z,t\} \quad (4) \]
and keeping the other fields as before. The factor \( c_0 \) could, of course, be absorbed into the function \( \beta(z,t) \) but it is useful to retain it, as we shall see below. It appears initially that we could simply remove the \( z, t \) dependence of \( \phi \) with a gauge transformation. This is obviously true for the case of an infinite string, but as we will see it is essential to consider the boundary conditions, and the situation is more delicate when we extend the analysis to the case of string loops.

In addition, we will demand that \( \eta(z,t) \) satisfies the transverse field equations,
\[ \partial_\alpha \partial^\alpha \beta = 0. \quad (5) \]
Introducing this ansatz into the Euler-Lagrange equations, one finds that for consistency one also has to demand
\[ \partial_\alpha \beta \partial^\alpha \beta = \epsilon, \quad (6) \]
where \( \epsilon \) is a constant. This last constraint arises as a result of our attempt to use separation of variables in a non linear equation. Although it might appear too restrictive, notice that it includes three relevant physical cases,
\[ \beta = k z \quad \beta = \omega t \quad \beta = k(z-t), \quad (7) \]
corresponding to a static current, a static charge density and a charged current pulse travelling at the speed of light respectively. This last case, (\( \epsilon = 0 \), or the so-called ‘chiral’ state) is special, since the backreaction of the electromagnetic fields on string vanishes. We will return to this point in studying the non-Abelian model.

The equations of motion for the \( z, t \) components of the gauge fields reduce to
\[ \nabla^2 s(r) = 2e^2|\phi(r)|^2(s(r) - c_0). \quad (8) \]
As we have indicated, in the case of a straight infinite string the constant \( c_0 \) is irrelevant since the phase of the condensate can be removed with a gauge transformation. In this case, then, we have a homogeneous equation for \( s(r) \)
\[ \nabla^2 s(r) = 2e^2|\phi(r)|^2s(r). \quad (9) \]
This equation certainly admits the trivial solution \( s(r) = 0 \), unless of course we impose a non trivial boundary condition at infinity – specifically,
\[ s'(r) \rightarrow \frac{Ie}{2\pi r}. \quad (10) \]
representing the presence of a current (or charge, or both) on the string. The key to the presence of supercurrents is the condensate \( |\phi(r)| \) which is non-vanishing at the origin (the core of the string), and without which we would not have a regular solution to equation (8).
The case of a string loop is significantly different, however. We restrict ourselves for the moment to the uncharged case $\eta = k z$, where now we designate with $z$ the coordinate along the loop (which we take of a large radius $R$ in order to avoid curvature effects). Naturally, in order for the condensate field to be single-valued, we require that $c_0 2\pi R k = N$ where $N$ is an integer. Now the phase of $\phi$ has a clear physical meaning – it cannot be removed by a gauge transformation and is something that the loop can acquire on becoming superconducting. In another words, if one assumes (following Davis and Shellard [17]) that an ansatz of the type given by equation (4) will still be valid on a loop, the constant $c_0$ in equation (8) being nozero, means that the gauge fields are sourced by the phase winding number. Notice that the role played for the straight string by the boundary condition at infinity translates on a loop into the specification of $N$, which is related to the total current.

In the following section we will confine ourselves to the infinite straight string case, and study the generalisation of equation (5) for both a simple non-Abelian model and the well-known $\text{SO}(10)$ string. Note that a crucial distinction arises when we discuss the case of loops, however, which we mention here. The extension of the comments above regarding the topologically conserved phase of the scalar condensate does not appear to be possible for a general non-Abelian theory – in fact, it appears to be impossible to construct field configurations with any such topological invariants – and it is more appropriate to view the currents as carried purely by gauge field components. This is a distinction analogous to the difference between superconductors, which carry currents non-dissipatively and, in addition, support conserved currents in multiply-connected samples in the absence of external fields, and perfect conductors, which display similar bulk features (the Meissner effect, for instance) but whose currents appear and disappear reversibly in the presence or absence of an external field, with no possibility of ‘topological’ preservation. This may have important cosmological consequences – particularly for studies of vorton stability. We return to this point in the conclusion.

3 A non-Abelian string model: $\text{SU}(2) \times \text{U}(1)$

3.1 The model

To investigate non-Abelian string conductivity, we consider a model which will bring out the relevant features of the mechanism. In the next section we look at its embedding in more realistic GUT model.

The simplest non-Abelian model that displays string solutions is given by the breaking of $\text{SU}(2)$ to $\mathbb{Z}_2$ by two real Higgs in the adjoint representation. In this case $\text{SU}(2)$ is effectively $\text{SO}(3)$. The homotopy group $\pi_1(\text{SO}(3)/1) = \mathbb{Z}_2$ and so topologically stable strings are formed. Different ansätze leading to string solutions have been analysed in the past (see for instance [18], [19]).

Since all the symmetries are broken in this case, to provide for the possibility of a long-range massless field playing the role of electromagnetism we extend the theory to $\text{SU}(2) \times \text{U}(1)$. We do this by arranging the two real Higgs in a complex multiplet and gauging the corresponding $\text{U}(1)$ symmetry.

The model is then described by the following Lagrangian density:

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{1}{4} G_{\mu\nu} \cdot G^{\mu\nu} - D_\mu \phi \cdot D^\mu \phi - V(\phi),$$

(11)

where

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu,$$

$$G_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu + g B_\mu \times B_\nu,$$

$$D_\mu \phi = \partial_\mu \phi - ie A_\mu \phi + g B_\mu \times \phi,$$

$$V(\phi) = \frac{1}{2} \lambda (\phi^\dagger \cdot \phi - \frac{1}{2} \eta^2)^2 + \frac{1}{2} \kappa |\phi| \cdot |\phi|^2.$$  (12)

Here, $B_\mu$ and $\phi$ are three-dimensional vectors in the $\text{SU}(2)$ Lie Algebra and “$\cdot$” and “$\times$” denote the standard scalar and cross product in internal space.
If we choose $\kappa > 0$, then the vacuum is characterised by
\[ \phi \cdot \phi = 0, \quad \phi^\dagger \cdot \phi = \frac{1}{2} \eta^2. \] (13)

We then define the vacuum to be
\[ \Phi_0 = \frac{\eta}{2} \begin{pmatrix} 1 \\ 0 \\ i \end{pmatrix}. \] (14)

Let the SU(2) generators in the adjoint representation be $T_i$ (for $i = 1, 2, 3$) and the U(1) generator be $T_0$,
\[ (-iT_i \phi)_j = -\epsilon_{ijk} \phi_k, \quad (-iT_0 \phi)_j = -i \phi_j. \] (15)

The generator that annihilates $\phi_0$, and which we identify with electromagnetic charge, is
\[ Q = T_2 + T_0. \] (16)

### 3.2 Bare string solutions

This theory supports topologically stable strings. Imagine that a string is formed, by $\phi$ acquiring its non-zero expectation value above in the usual way, and take the string generator to be
\[ T_S = T_3. \] (17)

At large distances from the string core,
\[ \phi(\theta) = e^{-i\theta T_3} \Phi_0 = \frac{\eta}{2} \begin{pmatrix} \cos \theta \\ \sin \theta \end{pmatrix}. \] (18)

A distinctive feature of our ansatz is that the string generator does not commute with the charge generator (given by equation (16)):
\[ [T_S, Q] = [T_3, T_2] = -iT_1. \] (19)

As a result, in the presence of this string the generator $Q$ which annihilates the vacuum acquires an angular dependence:
\[ Q(\theta) = e^{-i\theta T_3} Q e^{i\theta T_3} = T_0 + T_2 \cos \theta - T_1 \sin \theta. \] (20)

We define the following unit vectors in internal space:
\[ e_1 = \begin{pmatrix} \cos \theta \\ \sin \theta \\ 0 \end{pmatrix}, \quad e_2 = \begin{pmatrix} -\sin \theta \\ \cos \theta \\ 0 \end{pmatrix}, \quad e_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}. \] (21)

The ansatz for the scalar field is then
\[ \phi(r, \theta) = \frac{\eta}{2} (e_1 f(r) + i e_3 h(r)). \] (22)

The real functions $f$ and $h$ satisfy
\[ f(0) = 0, \quad f(\infty) = h(\infty) = 1, \] (23)

and there is no topological restriction on $h(0)$. We shall see below that it can be energetically favourable (in the absence of a current) for this component to persist to the core of the string, to reduce both gradient and potential energy.
Although the basis given by equation (24) is the most convenient for the calculation, we will introduce two other bases which will be useful for the physical interpretation of the solutions. One of them is given by the eigenvectors of the string generator,

\[ u_{+1} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ i \\ 0 \end{pmatrix}, \quad u_{-1} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -i \\ 0 \end{pmatrix}, \quad u_0 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \]

and the other by the eigenvectors of \( Q(\theta) \);

\[ v_0 = \frac{1}{\sqrt{2}}(e_1 + ie_3), \quad v_+ = e_2, \quad v_+ = \frac{1}{\sqrt{2}}(e_1 - ie_3). \]

In these bases, the scalar field is expressed as

\[ \phi(r, \theta) = \frac{\eta}{2} \left( \frac{f(r)}{\sqrt{2}} (e^{i\theta} u_{+1} + e^{-i\theta} u_{-1}) + ih(r)u_0 \right), \]

\[ = \frac{\eta}{2} \left( \frac{f(r) + h(r)}{\sqrt{2}} v_0 + \frac{f(r) - h(r)}{\sqrt{2}} v_+ \right). \]

It is interesting to point out that instead of the form in equation (22), we could have chosen another ansatz for the scalar field:

\[ \phi_{NO}(\theta) = \frac{\eta}{2} f_{NO}(r)e^{-i\theta} \Phi_0. \]

Here the string generator is proportional to

\[ T'_S = T_2 - T_0, \]

and so

\[ [T'_S, Q] = 0. \]

Notice also that in contrast to the form (22), there is only one function \( f(r) \) involved, which is constrained to vanish at the origin since every component of the Higgs is wound by the string generator \( T'_S \). In this case, the string is just an embedding of the U(1) Nielsen-Olesen vortex in a larger group. Which of the two ansätze is realized in a given model is a matter of energetics and, as we shall discuss below, depends largely on the parameters of the model. Indeed, in the out-of-equilibrium conditions of a phase transition it seems likely that all possible string configurations will arise. However, only those in which the string fields are charged – \([T_s, Q] \neq 0 \) – support currents in the sense we discuss in this paper.

Returning our attention to the lower-energy solution, the ansatz for the string gauge field is

\[ B_\theta = b(r)e_3, \]

where \( b(0) = 0 \) and \( b(\infty) = -1/g \). This, together with equation (22), completes the ansatz for the non-conducting string fields. The equations of motion

\[ \frac{1}{\sqrt{g}} \partial_{\mu} \sqrt{g} F^{\mu\nu} = j^\alpha = ie[\phi^i \cdot D^\nu \phi - \phi \cdot (D^\nu \phi)^\dagger], \]

\[ \frac{1}{\sqrt{g}} D_{\mu} \sqrt{g} G^{\mu\nu} = J^\alpha = g[\phi^i \times D^\nu \phi + \phi \times (D^\nu \phi)^\dagger], \]

\[ \frac{1}{\sqrt{g}} D_{\mu} \sqrt{g} D^\mu \phi = \frac{\delta V}{\delta \phi^i}, \]

then become

\[ \left( \frac{1}{r} f' \right)' = \frac{\eta^2}{2r} g(1 + gb)f^2, \]

\[ \frac{1}{r^2} \frac{d}{dr} (1 + gb)^2 f + \frac{1}{4} \lambda \eta^2 f(f^2 + h^2 - 2) + \frac{1}{4} \kappa \eta^2 f(f^2 - h^2), \]

\[ \frac{1}{r} (r h')' = \frac{1}{4} \lambda \eta^2 h(f^2 + h^2 - 2) + \frac{1}{4} \kappa \eta^2 h(h^2 - f^2). \]
The energy per unit length of this configuration is given by

\[ H_{na} = \int r \, dr \, d\theta \left\{ \frac{1}{2r^2} b'^2 + \frac{\eta^2}{4} [f'^2 + h'^2] + \frac{\eta^2}{4r^2} (1 + gb)^2 f^2 + \frac{\lambda \eta^4}{32} \left( f^2 + h^2 - 2 \right)^2 + \frac{\kappa \eta^4}{32} \left( f^2 - h^2 \right)^2 \right\}. \]  

(32)

We note that for \( \kappa = \lambda \), the Higgs component \( h \) decouples from the equations and can be set to unity everywhere. In this limit, the equations reduce to those of the Abelian Higgs model, whose solutions are well-known. Further, if we now impose the additional constraint \( g^2 = 2\lambda \), the second order differential equations can be shown to be equivalent to the first order Bogomol’nyi equations [20],

\[ b' = \frac{g \eta^2}{4r} (f^2 - 1), \]

\[ f' = \frac{1}{r} \left( 1 + gb \right). \]

For the Abelian \textit{ansatz} of Eq.(27) we take instead

\[ \alpha Z_\theta (r) = eA_\theta (r) - gB_\theta^2 (r), \]

(33)

where \( \alpha = \sqrt{e^2 + g^2} \). In this case, the equations become

\[ \left( \frac{1}{r} Z' \right)' = \frac{\eta^2}{r} \alpha (1 + \alpha Z) f^2, \]

\[ \frac{1}{r} r (rf')' = \frac{1}{r^2} (1 + \alpha Z)^2 f + \frac{1}{2} \lambda \eta^2 f (f^2 - 1), \]

with a resulting energy per unit length

\[ H_{ab} = \int r \, dr \, d\theta \left\{ \frac{1}{2r^2} Z'^2 + \frac{\eta^2}{2} f'^2 + \frac{\eta^2}{2r^2} (1 + \alpha Z)^2 f^2 + \frac{\lambda \eta^4}{8} (f^2 - 1)^2 \right\}. \]

(34)

Similar to the non-Abelian ansatz, the equations of motion also reduce to a Bogomol’nyi form when \( \lambda = \alpha^2 / 2 \).

Notice that for the non-Abelian ansatz, \( h(r) \) does not vanish at the origin. We can estimate its the value by equating to zero the right hand side of the last equation, giving

\[ h(0)^2 \approx \frac{2\lambda + \mathcal{O}(h(0)^\prime / h(0))}{\lambda + \kappa}, \]

which means that in the general case (that is, unless \( \kappa \) is infinitely large) the full symmetry is not restored inside the string.

The equations (33) can be solved numerically, either with a fifth-order Runge-Kutta shooting method – giving initial estimates for \( h(0), f'(0) \) and \( b''(0) \) – or by a relaxation method, giving initial estimates for \( h(r), f(r) \) and \( b(r) \). The resulting fields for a representative set of values for the couplings are shown in figure 1 (produced with the shooting method).

As we mentioned in the introduction, we wish to embed this solution in the SO(10) model and this will be possible provided that a certain relation between the gauge couplings holds. As will become clear in section 4 or just by a direct comparison with the equations of motion in [21], the embedding of the Abelian ansatz for the SU(2)×U(1) model in SO(10) requires \( g^2 = \frac{2}{9} e^2 \). In this case, it was shown by Aryal and Everett [22] and Ma [21] that the non-Abelian string form has a lower energy than the Abelian one for all parameters in the potential. This result is somewhat surprising. Notice, for instance, that the
Figure 1: Fields for a string formed in the breaking $\text{SU}(2) \times \text{U}(1) \rightarrow \text{U}(1)$. Here $\eta = 1$. 
magnetic energy – represented by the first term in each of the equations (32) and (34) – is proportional to $1/g^2$ and $2/5g^2$ for the non-Abelian and Abelian string respectively. Neither does the difference between the two arise from the potential energy of the Higgs, since for instance in the limit of very large $\kappa$, $h \to f$ in the non-Abelian ansatz, and both forms have the same potential energy. The reason the non-Abelian ansatz is energetically preferred to the Abelian one seems to arise instead from the Higgs-gauge field interaction; while in the first case only one component of the Higgs field ($f$) interacts with the gauge fields, in the second case both components (which are identical) do. According to the analyses performed in references [21, 22], then, the non-Abelian string is energetically favoured in SO(10). We speculate that in the out-of-equilibrium conditions of a phase transition, however, all possible string solutions will arise – nevertheless, the measure of Abelian solutions is outweighed by that of the non-Abelian form. We will show in the next section that this latter type of string supports currents.

We mention here that a less direct mechanism by which Abelian strings can become superconducting, through electroweak gauge boson condensation, has been analysed by several authors [12, 13, 16].

3.3 Incorporating a current

Now we add fields to our solution representing a current along the string. We allow the time and $z$-components of the gauge fields to be non-zero:

$$A_\alpha = a(r, \theta) \partial_\alpha \beta(z, t)$$
$$B_\alpha = k(r, \theta) \partial_\alpha \beta(z, t),$$

where $\alpha = t, z$ and $\beta$ satisfies the transverse wave equation

$$\partial_\alpha \partial^{\alpha} \beta = 0, \quad (35)$$

The equations of motion are now (where $i, j = r, \theta$)

$$\frac{1}{\sqrt{g}} \partial_i \sqrt{g} \partial^i A^\alpha = j^\alpha$$
$$\frac{1}{\sqrt{g}} D_i \sqrt{g} D^i B^\alpha = J^\alpha \quad (36)$$

While the equation of motion for the transverse components of $A$ is not modified,

$$\frac{1}{\sqrt{g}} \partial_i \sqrt{g} F^{ij} = j^i,$$

there are new terms in the equations for $B_i$ and $\phi$ which represent the backreaction of the currents on the string fields:

$$\frac{1}{\sqrt{g}} D_i \sqrt{g} G^{ij} = J^j + J^{jBR} \quad (37)$$
$$\frac{1}{\sqrt{g}} D_i \sqrt{g} D^i \phi = \frac{\delta V}{\delta \phi} + F^{BR} \quad (38)$$

where

$$J^{jBR}_i = g k \times D^j k \partial_\alpha \beta \partial^{\alpha} \beta,$$  \quad (39)

and

$$F^{BR} = \left[-(e^2 a^2 \phi - 2iega k \times \phi + g^2 k \times (k \times \phi)) \partial_\alpha \beta \partial^{\alpha} \beta \right]. \quad (40)$$

We see here that the backreaction vanishes in the so-called ‘chiral’ state – that is, when $\beta$ is function of light-cone coordinates $\beta = \beta(z \pm t)$.

Taking the form for the string background $\phi$ in equation (22), and

$$k = k(r)e_2 \quad (41)$$
(all other components of $k$ decouple and can be set to zero), the current field equations are

$$
(ra')' = \frac{1}{2} \eta^2 r [e^2 (f^2 + h^2) a - 2 egf k h k] \\
(rk')' = \frac{1}{r} (1 + gb)^2 k + \frac{1}{2} \eta^2 r (h^2 + f^2) k - 2 egf k a
$$

(42)

We see that for consistency in the transverse field equations, we have to demand

$$
\partial_{\alpha} \partial^{\alpha} = \epsilon.
$$

(43)

where $\epsilon$ is an arbitrary constant, so that all transverse field components are independent of $z$ and $t$. We see then that the string equations of motion are modified in the following way:

$$
\begin{align*}
\left( \frac{1}{r} b' \right)' &= \frac{\eta^2}{2r} g (1 + gb) f^2 + \epsilon \frac{1}{r} g (1 + gb) k^2, \\
\frac{1}{r} (rf')' &= \frac{1}{r^2} (1 + gb)^2 f + \frac{1}{4} \lambda \eta^2 f (f^2 + h^2 - 2) + \frac{1}{4} \kappa \eta^2 f (f^2 - h^2) + \\
&\qquad \epsilon (e^2 a^2 f - 2 eg ak h + g^2 k^2 f), \\
\frac{1}{r} (rh')' &= \frac{1}{4} \lambda \eta^2 h (f^2 + h^2 - 2) + \frac{1}{4} \kappa \eta^2 h (h^2 - f^2) + \\
&\qquad \epsilon (e^2 a^2 h - 2 eg af k + g^2 k^2 h).
\end{align*}
$$

(44)

Before entering into a full numerical study of the equations, we can see the pertinent features of the solutions by making some approximations. We can analyse the asymptotic behavior of the electromagnetic field by using an approximate form for the string background to solve equations (42) – namely,

1. $r < R, \ f = b = 0, \ h = h_0 = \text{constant}$
2. $r > R, \ b = -1/g, \ f = h = 1$.

We have in region (1)

$$
\begin{align*}
\frac{1}{r} (ra')' &= \frac{\eta^2 e^2 h_0^2}{2} a, \\
\frac{1}{r} (rk')' &= \left( \frac{1}{r^2} + \frac{\eta^2 g^2 h_0^2}{2} \right) k,
\end{align*}
$$

and in region (2)

$$
\begin{align*}
\frac{1}{r} (ra')' &= e \eta^2 (ea - gk), \\
\frac{1}{r} (rk')' &= g \eta^2 (ek - ea).
\end{align*}
$$

If we transform to our new variables

$$
\gamma = \frac{ga + ek}{\sqrt{e^2 + g^2}}, \quad Z = \frac{ea - gk}{\sqrt{e^2 + g^2}},
$$

(45)

we see that

$$
\begin{align*}
\frac{1}{r} (rZ')' &= (e^2 + g^2) \eta^2 Z, \\
\frac{1}{r} (r\gamma')' &= 0.
\end{align*}
$$
Now we solve these equations, satisfying the boundary conditions we require. In region (1),

\[
a = C_1 I_0 \left( \frac{\eta h_0 r}{\sqrt{2}} \right),
\]

\[
k = C_2 I_1 \left( \frac{\eta h_0 r}{\sqrt{2}} \right),
\]

where \( I_0 \) and \( I_1 \) are modified Bessel functions that are regular at the origin, so that \( a \sim \text{constant} \) and \( k \sim r \) at \( r = 0 \), and in region (2)

\[
Z(r) = C_3 K_0 \left( \sqrt{e^2 + g^2} \eta r \right),
\]

\[
\gamma(r) = C_4 \ln \frac{r}{R} + C_5,
\]

so that \( Z \) decays exponentially at large \( r \). The five arbitrary constants are related by the four continuity conditions at \( r = R \), and so as expected there remains one arbitrary constant that sets the magnitude of the current. The effective electromagnetic current on the string is obviously related to the asymptotic magnetic field \( B \) by the relation

\[
B(r) = \partial_r \gamma = \frac{I}{2\pi r},
\]

and so

\[
I = 2\pi C_4.
\]

We can now study the back reaction of this current on the string by studying the linearised version of equations (44) in the presence of an electromagnetic gauge field of the form (47). Defining

\[
b(r) = -\frac{1}{g} + \sqrt{r} X_1(r) \quad f(r) - h(r) = \frac{1}{\sqrt{r}} X_2(r),
\]

noting that clearly only the charged linear combination \( f - h \) couples to the current, one obtains the following at large distances from the core:

\[
X_1'' - \lambda_1(r) X_1 = 0,
\]

\[
X_2'' - \lambda_2(r) X_2 = 0,
\]

where

\[
\lambda_1(r) = \frac{g^2}{2} \left( 1 + \epsilon \frac{2e^2}{e^2 + g^2} \gamma^2(r) \right)
\]

\[
\lambda_2(r) = \kappa + \epsilon \frac{4e^2 g^2}{e^2 + g^2} \gamma^2(r)
\]

For \( \epsilon > 0 \), the current backreaction completely dominates the asymptotic behavior of the fields;

\[
X_i = \exp \left( -|c_i|r \ln r \right),
\]

where the constants \( c_i \) are

\[
c_1^2 = \frac{e^2 g^2}{e^2 + g^2} \frac{I^2}{4\pi^2}, \quad c_2^2 = 4c_1^2.
\]

For the case \( \epsilon < 0 \) (a string carrying a pure charge) we cannot go to arbitrarily large distances from the core, since \( \lambda_i(r) \) would turn negative and we would obtain oscillatory solutions which would produce a linearly divergent energy per unit length. Solutions for this case may have a physical meaning if there is a natural large-distance cut-off in the problem, as can be for the case of loops (a situation analogous to this occurs for Abelian strings – see, for instance, the study by Davis and Shellard [17]).
We have performed a full numerical study of the equations of motion. A relaxation method was employed to solve the five coupled second-order field equations (42) and (44), and the results are shown for $\epsilon = +1$ – an uncharged string – in figures 2 and 3. In figure 3 we can see that as the current is increased, the string width (given by the characteristic scales of the fields $f$ and $b$) initially decreases. At higher values of the current, the gauge field width of the string continues to decrease and the fields $f$ and $h$ are forced to be approximately equal, as in the Abelian configuration, and vanish out to larger and larger distances, increasing the energy stored in the string core and restoring the symmetry completely near $r = 0$. The case of a charged string with no space-like current, $\epsilon = -1$, is more difficult to solve numerically due to the oscillatory behaviour of the fields at large $r$, but the qualitative features for small charge densities are that the field $h$ is increased in value near the origin and the gauge field width of the string increases.

The studies above refer to straight strings. The case of a loop is a more difficult to interpret. The chiral case $\epsilon = 0$, in which the electromagnetic 4-current vanishes, is free of backreaction, as we have shown above. This feature is common to all models of string conductivity. In the conclusion we discuss the possibility that the dynamically stable chiral currents may be the only stable persistent currents on non-Abelian string loops.
Figure 3: The backreaction of a non-chiral current ($\epsilon = +1$) on the string for progressively larger space-like currents. The couplings are as in the previous figure.
4 Strings in SO(10)

Having explored the support of a current in a simple model of a non-Abelian string, we now consider strings formed in a realistic Grand-Unified symmetry breaking phase transition. We hope to establish that the features we have seen above can be embedded in such a theory. We take the model GUT of Spin(10), the simply-connected covering group of SO(10) and the simplest Grand-Unified group that supports topologically stable strings. The group can break to the Standard Model in many ways and these have been studied and categorised according to the types of defect they produce \[23\]. However, the pattern of symmetry breaking of most interest to us, as it is similar to that in the SU(2) model discussed above, is \[24\]

\[
\begin{align*}
\text{SO(10)} & \rightarrow \text{SU(5) } \times \text{Z}_2 \\
& \rightarrow \text{SU(3)}_c \times \text{SU(2)}_L \times \text{U(1)}_Y \times \text{Z}_2 \\
& \rightarrow \text{SU(3)}_c \times \text{U(1)}_{em} \times \text{Z}_2.
\end{align*}
\]

(Strings are formed at the first phase transition, with a Higgs in the complex 126-dimensional representation of SO(10). We embed SU(5) in SO(10) by treating 5-dimensional complex vectors in the fundamental representation of SU(5) as 10-dimensional real vectors in SO(10). The discrete \text{Z}_2 symmetry corresponds to the element \{-1\} that is not an element of SU(5). The homotopy group \(\pi_1(\text{Spin}(10)/\text{SU}(5) \times \text{Z}_2) = \text{Z}_2\) and so stable strings are formed. The unbroken subgroup is not simply connected, and so a closed path in configuration space enclosing the string corresponds to movement between the disconnected pieces of the unbroken subgroup \(H\) on the group manifold of \(G\), which cannot be continuously deformed to a point.

\(\text{SO(10)}\) has 45 Hermitian generators that are defined by the relation

\[
M_{ab} = -\frac{i}{4}[\Gamma_a, \Gamma_b], \quad a, b = 1...10,
\]

where the \(\Gamma_a\) are as usual constructed iteratively from the Pauli matrices and are 32 \times 32 matrices in this representation:

\[
\Gamma_{2k+1} = \times_{i=1}^5 a_i, \quad \Gamma_{2k} = \times_{i=1}^5 b_i,
\]

where

\[
a_i = \begin{cases} 
I & i < k \\
\sigma_1 & i = k \\
\sigma_3 & i > k
\end{cases}, \quad b_i = \begin{cases} 
I & i < k \\
\sigma_2 & i = k \\
\sigma_3 & i > k
\end{cases}
\]

We can decompose the group space of \(G (\text{Spin}(10))\) into five 2 \times 2 subspaces. Specifically, the five diagonal generators of \(\text{SO(10)}\) are

\[
\begin{align*}
M_{12} &= \frac{1}{2} \sigma_3 \times I \times I \times I \times I, \\
M_{34} &= \frac{1}{2} I \times \sigma_3 \times I \times I \times I, \\
& \vdots \\
M_{910} &= \frac{1}{2} I \times I \times I \times I \times \sigma_3.
\end{align*}
\]

The basis we employ is conveniently expressed in terms of linear combinations of the \(M_{ab}\). We can now write down the explicit forms of the unbroken generators — for example, we have the 8 generators of the SU(3) colour symmetry, of which

\[
T_3^{\text{SU(3)}} = \frac{1}{2}(M_{12} - M_{34}).
\]
with the corresponding $\sigma_1$ and $\sigma_2$ equivalents

\[
T_{12}^L = \frac{1}{2}(M_{13} + M_{24}),
\]

\[
T_{12}^R = \frac{1}{2}(M_{14} - M_{23})
\]

respectively; we have

\[
T_{3}^{SU(2)L} = \frac{1}{2}(M_{78} - M_{910}) \equiv L_3
\]

generating the third component of left-handed weak isospin (coupling to the electron and antineutrino), and

\[
Y = \frac{2}{3}(M_{12} + M_{34} + M_{56}) - (M_{78} + M_{910})
\]

\[
Q = \frac{1}{3}(M_{12} + M_{34} + M_{56}) - M_{910}.
\]

being the hypercharge and the electric charge generators.

States in this 32 representation can be labelled according to the eigenvalues $\frac{1}{2}\epsilon_i$ of the diagonal operators $M_{k,k+1}$ (see Eq.(56)), where $\epsilon_i = \pm 1$ -

\[
\phi_{32} \equiv |\epsilon_1 \epsilon_2 \epsilon_3 \epsilon_4 \epsilon_5\rangle.
\]

(57)

As it is well known, this representation is reducible to two 16 representations, which are characterised by the value of the chirality operator $\chi$:

\[
\chi = -i\Gamma_1...\Gamma_{10}.
\]

(58)

The sixteen left-handed fermions are then assigned to the 16-dimensional representation of positive chirality.

As we mentioned above, strings are formed when the Higgs field associated with the GUT phase transition is in the 126 representation, which is included in the the symmetric product of two 16 representations

\[
16 \times 16 = 10_S + 120_A + 126_S.
\]

(59)

After the GUT phase transition, the Higgs can be defined such that it acquires an expectation value in the (16,16) position,

\[
|---|--\rangle|---|--\rangle \equiv |16\rangle|16\rangle,
\]

(60)

which is a singlet under SU(5) gauge transformations.

The criterion for the strings formed to be topologically stable is that the string generator $T_s$ satisfies the relation $e^{2\pi iT_s} = -U$, where $U \in SU(5)$. The simplest string solution is generated by the diagonal generator

\[
T^U = \frac{1}{5}(M_{12} + M_{34} + M_{56} + M_{78} + M_{910})
\]

(61)

that commutes with all the elements of SU(5), including of course the electromagnetic generator. This is the most symmetric way of generating a U(1) subgroup within SO(10) that contains the $Z_2$ symmetry, and the Higgs and gauge fields for this type of string are of the usual Nielsen-Olesen form. However, Aryal and Everett [22] and Ma [21] examined this solution and found it to be of a higher energy per unit length than that generated by

\[
T_s \equiv \frac{1}{2}(M_{59} - M_{610}) = \frac{1}{4}(I \times I \times \sigma_1 \times \sigma_3 \times \sigma_2 + I \times I \times \sigma_2 \times \sigma_3 \times \sigma_1),
\]

(62)

a linear combination of the generators of right-handed weak isospin. At the level of this first symmetry breaking, there is a degeneracy of possible choices for $T_s$, but this degeneracy is progressively lifted at subsequent breakings by a combination of energetic considerations and demanding that the generators corresponding to physical observables be single-valued [25]. This generator does not commute with charge.
Together with $T_s$, the two generators
\[
A = \frac{1}{2}(M_{56} + M_{9\,10}) = \frac{1}{4}(I \times I \times I \times I \times \sigma_3 + I \times I \times \sigma_3 \times I \times I),
\]
\[
B = \frac{1}{2}(M_{5\,10} + M_{6\,9}) = \frac{1}{4}(I \times I \times \sigma_2 \times I \times \sigma_2 - I \times I \times \sigma_1 \times I \times \sigma_1),
\]
generate an SU(2) algebra,
\[
[T_s, A] = -iB, \quad [T_s, B] = iA, \quad [B, A] = iT_s.
\]
We introduce the generator
\[
T_0 = -\frac{1}{4}(M_{1\,2} + M_{3\,4}) - M_{56} + M_{9\,10}
\]
\[
= \frac{1}{4}(\sigma_3 \times I \times I \times I \times I + I \times \sigma_3 \times I \times I \times I
+ 2I \times I \times \sigma_3 \times I \times I - 2I \times I \times I \times I \times \sigma_3),
\]
which commutes with $T_s$, and so the charge generator breaks into the components
\[
Q = -\frac{2}{3}(A + T_0).
\]
The subspace of the $126$ representation in which these generators act is generated by the states
\[
\left\{ |14\rangle |14\rangle, \frac{1}{\sqrt{2}} \left( |14\rangle |16\rangle + |16\rangle |14\rangle \right), \quad |16\rangle |16\rangle \right\}, \quad (63)
\]
where
\[
|14\rangle = | - - + + \rangle. \quad (64)
\]
The form of the generators in this subspace is then
\[
T_s = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad B = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix},
\]
while $T_0$ is the identity matrix. It is straightforward to check that these generators are related to the ones we discussed in section $\mathbb{H}$ by the transformations
\[
A = C^{-1}T_2C, \quad B = C^{-1}T_1C, \quad T_s = C^{-1}T_3C, \quad (65)
\]
where
\[
C = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & i \sqrt{2} & 0 \\ -i & 0 & i \end{pmatrix}.
\]
After this change of base then, the model becomes identical to the SU(2)×U(1) model above. For instance, the covariant derivative
\[
D_\mu \Phi = (\partial_\mu + ig(A_\mu^A A + A_\mu^B B + A_\mu^T T + A_\mu^0 T_0))\Phi
\]
is equivalent to the SU(2)×U(1) definition in Eqs.(12) after the obvious identification of gauge fields as indicated in Eq.(13) and setting the two gauge couplings $e$ and $g$ to be equal.
For the Abelian ansatz, notice that in the subspace defined in equation $\mathbb{K}$ the generator $T^U$ is given by
\[
T^U = \frac{1}{3}(2A - 3I), \quad (66)
\]
and after the change of basis (equation 65),

$$T^U = \frac{1}{5}(2T_2 - 3I).$$  

(67)

In this ansatz, then, the covariant derivative of the Higgs field reads

$$D_\mu \Phi = (\partial_\mu + igA_\mu)\Phi.$$  

(68)

In the SU(2)×U(1) model, we had

$$D_\mu \Phi = (\partial_\mu - i(-eA_\mu + gB_\mu^2 T_2))\Phi.$$  

We define

$$A_\mu = \frac{e}{\alpha} Z_\mu, \quad B_\mu^2 = -\frac{g}{\alpha} Z_\mu,$$

where $\alpha = \sqrt{e^2 + g^2}$. We can see that in this case, $e^2 = (3/2)g^2$.

5 Conclusion

We have seen that the essential feature of non-Abelian string conductivity models is that the string generator does not commute with electromagnetic charge. If it did, the Higgs would be annihilated by the charge generator at all distances from the string core, and so would be uncharged. It is precisely because $T_s$ and $Q$ do not share a common set of eigenvectors, and that different eigenvectors of $T_s$ have different profiles (in the discussion in section 3, $f$ and $h$) near the string core, that a charged component of the Higgs exists there.

Charge is, however, generically ill-defined at the origin. The charged component of the Higgs (see equation 26) cannot really be considered to be a well-defined 'condensate' in the sense of the Witten model. Moreover, we cannot universally give this charged component a varying phase with an electromagnetic gauge transformation around a loop of such string, as it does not vanish at $r = 0$ where $Q$ itself is ill-defined. If we were to try to construct such a solution we would have to give the Higgs a phase with a generator that commuted with $T_s$ at $r = 0$ and deformed into $Q$ outside the string. That implies there must be some radial dependence of the Lie algebra element that acts on the condensate, and we cannot construct a non-trivial $r$ and $z$-dependent condensate that is periodic in $z$, as is required for a loop. This can be seen as follows. Following the infinite-string ansatz of Alford et al. [11] for a zero-mode excitation, we require a Higgs of the form

$$\Phi(r, \theta, z, t) = e^{i\beta(z, t)S(r, \theta)}\Phi(r, \theta),$$

where $\Phi$ is the unperturbed string solution, and $S$ lies in the Lie algebra of the unbroken group $H$ and tends to charge $Q(\theta)$ away from the string core. We wish to implement this on the topology of a loop of length $L$. Now the coordinate $z$ measures the distance along the string. The requirement

$$\Phi(z + L) = \Phi(z)$$

implies that

$$\beta(z + L, t)S(r, \theta) \equiv \beta(z, t)S(r, \theta) + 2\pi N_1.$$  

(69)

This cannot be satisfied everywhere unless $\beta$ is trivial, or $S$ is independent of $r$ and $\theta$ (which is not possible for the case $[T_s, Q] \neq 0$ we require). We see, then, that we must view current excitations on such non-Abelian strings as being supported by the string gauge field, which is also charged, and sources excitations of the form

$$A_\alpha = -S(r, \theta)\beta(z, t).$$  

(70)

This viewpoint is consistent with the topology of a string loop.
The boundary conditions on the gauge field equations of motion (which, in the absence of a Higgs phase, are homogeneous in $S$) now demand that the $A_\alpha$ vanish both at the center of the string loop and at spatial infinity [26]. The lowest energy solution in the absence of an external field is simply $S = 0$ and hence we suggest that an uncharged non-Abelian string loop ($\epsilon > 0$) will be incapable of supporting a persistent current in this case, making it in effect a perfect conductor. We aim to explore the issue of perfect conductor electrodynamics in a subsequent paper[6]. Similarly, we would not expect the charged $\epsilon < 0$ case to persist either. It is conceivable that the chiral state, in which the current is identified with the charge density, could be dynamically stable, however. The solution of the equations of motion on a string loop need to be solved fully to confirm this, however, to take into account curvature effects.

Studies of current-carrying loop stability have to date worked with the Witten model of a superconducting string, in which a loop (or vorton) has two associated conserved quantities $N$ and $Z$; these correspond to the condensate winding and the net charge carried by the loop. In terms of the function $\beta(z, t) = k z \pm \omega t$ defined on the string worldsheet, these come from the first term (a topological invariant) and second term (Noether charge) respectively. The fact that there is no condensate in non-Abelian models seems to preclude the existence of the quantum number $N$. However, it may be that the chiral current-carrying states may be dynamically stable on a loop, with a net charge $Q$. These may be vulnerable to the loss of charge carriers at cusps [28], however, and so these currents can leak from the string when the loop has a contorted small-scale structure.

Despite the questions raised about the stability of currents, it seems that there are so many competing perfect-conduction mechanisms clamouring for a string’s attention that it seems almost certain that if strings exist at all they will be current-carrying at some scale. It seems that if GUT strings are formed they are likely to be capable of carrying enormous currents – we see that the Euler-Lagrange equations provide no natural upper limit, and so quantum effects must intervene through particle production in the vicinity of the string. Chiral current magnitudes may be limited by the amount of charge a loop can randomly acquire, either at formation or after intercommutation of strings carrying different currents. If the bare GUT string does not support currents, whether fermionic or vector bosonic, then subsequent dressing at the electroweak scale appears to provide them. This provides a further degree of freedom for string-based models of structure formation and dark matter.

Note that the induced vector-boson conductivity models [13]-[16] are not included in this. They are complementary to the models listed and will presumably display similar topological currents and/or zero modes according to whether charge commutes with the string generator or not.

In this paper we have explored non-Abelian string conductivity in some detail. We have emphasised the role of topology in true superconductivity, and its absence in non-Abelian string models. We have analysed the current solution (with and without backreaction) in an illustrative non-Abelian string model. We have also shown that the standard, lowest-energy SO(10) string solution is a simple embedding of this model and hence also supports currents. We suggest that only charged non-Abelian strings will be able to carry persistent currents in the absence of an external field, due to dynamical stability of the chiral zero-modes. It seems that vorton stability is less likely in realistic GUT string models, then, which avoids the overdensity problem and reinforces structure formation scenarios involving defects at such a scale.

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