Quasiparticle Lifetimes in Superconductors: Relationship with the Conductivity Scattering Rate

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Abstract

We compute the single particle inverse lifetime, evaluated in the superconducting state. Within the BCS framework, the calculation can be done non-perturbatively, i.e. poles can be found well away from the real axis. We find that perturbative calculations are in good agreement with these results, even for non-zero impurity scattering. With electron-phonon scattering added to the problem, we use the Eliashberg equations, with impurity scattering, to calculate the quasiparticle inverse lifetime perturbatively. In all cases we find that the inverse lifetime is significantly suppressed in the superconducting state, particularly in the presence of impurity scattering. We also compute the far-infrared and microwave conductivity, and describe procedures for extracting scattering rates from conductivity measurements. While these procedures lead to scattering rates in qualitative agreement with the inverse lifetime, we find that quantitative agreement is lacking, in general.
I. INTRODUCTION

The quasiparticle lifetime is a concept which is useful for clarifying the nature of the system of interest, i.e. is it a Fermi vs. marginal Fermi vs. a Luttinger liquid, etc [1]. Various techniques are available for measuring lifetimes [2]; perhaps the most direct is through tunnel junction detection [3]. In this paper we analyze electron scattering rates, as measured by microwave and far-infrared conductivity measurements, and examine their relationship to quasiparticle lifetimes. The experimental results for YBaCu$_3$O$_{7-x}$ [4,5] and their theoretical implications [5–8] have been discussed extensively in the literature. We include a treatment of inelastic scattering (through the electron-phonon interaction) and use Eliashberg theory to further analyze the problem. To some extent this has been done in the normal state already [9,10]. Certain aspects of quasiparticle lifetimes are clarified and the analysis is extended to the superconducting state, following our recent work on the optical conductivity [10,11].

We will proceed by defining what we mean by a single particle lifetime, bearing in mind that sometimes such a definition is not very meaningful, or useful. We review two ways in which conductivity data can be used to extract a quasiparticle scattering rate, one based on the two-fluid model, and the other based on a straightforward Drude fit to the low frequency conductivity. The instances in which these procedures yield a qualitative or quantitative facsimile of the quasiparticle inverse lifetime will be clarified. In this way, one can evaluate the usefulness of the two-fluid hypothesis, for example, in systems with both elastic and inelastic scattering channels.

This paper has been divided into two sections. The first reviews quasiparticle lifetimes, which are calculated from the single particle Green function. Previous work is extended to include both electron-impurity scattering (in the Born approximation) and electron-phonon scattering. Perturbative methods within the Eliashberg framework are used. The second section elucidates two methods recently used to extract scattering rates from optical and microwave conductivity measurements, and how these relate to the single particle inverse lifetime. We should make it clear at the start that the calculation we use for the optical conductivity omits vertex corrections. The main result of this is that the “1 − cos θ”
factor that occurs in a Boltzmann formulation of transport properties is absent \[12\], so in fact, when theoretical comparisons between the quasiparticle lifetime and the conductivity-derived scattering rate are made, the agreement will in general be better than it would be, given an exact calculation. This is made obvious in the next section, which treats elastic impurity scattering. Conversely, good agreement between these two quantities in the theory does not imply good agreement between the two measured quantities.

II. QUASIPARTICLE LIFETIMES

II.A Normal State

The fact that the quasiparticle lifetime is related to the conductivity is most apparent in the Drude model, which incorporates electron-impurity scattering into the problem. In the Born approximation, the single-electron self-energy is given by

\[
\Sigma(\omega + i\delta) = -\frac{i}{2\tau}
\]

where \(1/\tau\) is the impurity scattering rate and \(\tau\) is the single particle lifetime. We will utilize an expression for the optical conductivity which omits vertex corrections, and so it is given simply as the convolution of two single particle Green functions \[13–16\]. The result reduces to the Drude conductivity \[17\],

\[
\sigma(\nu) = \frac{ne^2}{m} \frac{1}{1/\tau - i\nu}
\]

whose real part is given by the usual Lorentzian form with half-width \(1/\tau\). Note that vertex corrections would actually alter the \(1/\tau\) that appears in eq. \(2\); however, if only s-wave phase shifts are included, the two inverse lifetimes would be identical. Here \(n\), \(e\), and \(m\) are the free electron density, charge magnitude, and mass. Thus the single particle lifetime appears explicitly (and can easily be measured) in the optical conductivity.

The next step is to include phonons, and thus an additional scattering mechanism, via the electron-phonon interaction \[18,9\]. Then the electron self-energy is given by \[19\]

\[
\Sigma(\omega + i\delta) = \int_0^{\infty} d\nu \alpha^2 F(\nu) \left\{-2\pi i [N(\nu) + \frac{1}{2}] + \psi\left(\frac{1}{2} + \frac{i\nu - \omega}{2\pi T}\right) - \psi\left(\frac{1}{2} - i\frac{\nu + \omega}{2\pi T}\right)\right\} - \frac{i}{2\tau}
\]
where $\alpha^2 F(\nu)$ is the electron-phonon spectral function, $N(\nu)$ is the bose function, and $\psi(x)$ is the digamma function. The quasiparticle has energy and lifetime defined by the pole of the single particle retarded Green function, i.e. the zero of

$$G^{-1}(k, \omega + i\delta) = \omega - \epsilon_k - \Sigma(\omega + i\delta), \quad (4)$$

where now the functions are analytically continued to the lower half-plane. Defining the energy and inverse lifetime by

$$\omega \equiv E_k - i\Gamma_k \quad (5)$$

we obtain the two equations,

$$E_k = \epsilon_k + \int_0^\infty d\nu \alpha^2 F(\nu) \Re \left\{ \psi\left(\frac{1}{2} + i\frac{\nu - E_k + i\Gamma_k}{2\pi T}\right) - \psi\left(\frac{1}{2} - i\frac{\nu + E_k - i\Gamma_k}{2\pi T}\right) \right\} \quad (6)$$

$$\Gamma_k = \frac{1}{2\tau} + \int_0^\infty d\nu \alpha^2 F(\nu) \left\{ 2\pi [N(\nu) + \frac{1}{2}] - \Im \left\{ \psi\left(\frac{1}{2} + i\frac{\nu - E_k + i\Gamma_k}{2\pi T}\right) - \psi\left(\frac{1}{2} - i\frac{\nu + E_k - i\Gamma_k}{2\pi T}\right) \right\} \right\} \quad (7)$$

The general problem requires a numerical solution of these equations and is beyond the scope of this paper. Here, we assume that $E_k$ and $\Gamma_k$ are small so that one can obtain closed expressions. They are \[13,20\]

$$E_k = \frac{\epsilon_k}{1 + \lambda^*(T)} \quad (8)$$

$$\Gamma_k = \frac{1}{2\tau} + 2\pi \int_0^\infty d\nu \alpha^2 F(\nu) [N(\nu) + f(\nu)] \frac{1}{1 + \lambda^*(T)} \quad (9)$$

where $f(\nu)$ is the Fermi function and

$$\lambda^*(T) = -\frac{1}{\pi T} \int_0^\infty d\nu \alpha^2 F(\nu) Im \psi\left(\frac{1}{2} + i\frac{\nu}{2\pi T}\right). \quad (10)$$

As $T \to 0$, $\lambda^*(T) \to \lambda \equiv 2 \int_0^\infty \frac{d\nu}{\pi} \alpha^2 F(\nu)$, while at high temperature, $\lambda^*(T) \propto 1/T^2$. Typical results for various electron-phonon spectral functions are shown in Fig. 1.
II.B Superconducting State

Quasiparticle lifetimes in electron-phonon superconductors were discussed long ago by Kaplan et al. [2]. Before proceeding to their relationship with the optical conductivity we give a brief review of the formalism [2,21]. As in the normal state the quasiparticle scattering rate is defined by (twice) the imaginary part of the pole in the single particle Green function.

In the superconducting state, the diagonal component of the single particle Green function is [22]

\[ G_{11}(k, \omega) = \frac{\omega Z(\omega) + \epsilon_k}{\omega^2 Z^2(\omega) - \epsilon_k^2 - \phi^2(\omega)}, \tag{11} \]

where \( Z(\omega) \) and \( \phi(\omega) \) are the renormalization and pairing functions given by solutions to the Eliashberg equations [23,22,24] which are repeated here for convenience:

\[
\phi(\omega) = \pi T \sum_{m=-\infty}^{\infty} \frac{|\lambda(\omega - i\omega_m) - \mu^*(\omega_c)\theta(\omega_c - |\omega_m|)|}{\omega_m^2 Z^2(i\omega_m) + \phi_m^2} \phi_m \sqrt{\omega^2 Z^2(\omega) - \epsilon_k^2 - \phi^2(\omega)} \\
+ i\pi \int_0^\infty d\nu \alpha^2 F(\nu) \left\{ [N(\nu) + f(\nu - \omega)] \frac{\phi(\omega - \nu)}{\sqrt{\omega^2 Z^2(\omega - \nu) - \phi^2(\omega - \nu)}} \\
+ [N(\nu) + f(\nu + \omega)] \frac{\phi(\omega + \nu)}{\sqrt{\omega^2 Z^2(\omega + \nu) - \phi^2(\omega + \nu)}} \right\}, \tag{12}
\]

and

\[
\tilde{\omega}(\omega) = \omega + i\pi T \sum_{m=-\infty}^{\infty} \lambda(\omega - i\omega_m) \frac{\omega Z(i\omega_m)}{\omega_m^2 Z^2(i\omega_m) + \phi_m^2} \\
+ i\pi \int_0^\infty d\nu \alpha^2 F(\nu) \left\{ [N(\nu) + f(\nu - \omega)] \frac{\tilde{\omega}(\omega - \nu)}{\sqrt{\tilde{\omega}^2(\omega - \nu) - \phi^2(\omega - \nu)}} \\
+ [N(\nu) + f(\nu + \omega)] \frac{\tilde{\omega}(\omega + \nu)}{\sqrt{\tilde{\omega}^2(\omega + \nu) - \phi^2(\omega + \nu)}} \right\}, \tag{13}
\]

where \( \tilde{\omega}(\omega) \equiv \omega Z(\omega) \). Here, \( N(\nu) \) and \( f(\nu) \) are the Bose and Fermi distribution functions, respectively. The electron-phonon spectral function is given by \( \alpha^2 F(\nu) \), its Hilbert transform is \( \lambda(z) \). The Coulomb repulsion parameter is \( \mu^*(\omega_c) \) with cutoff \( \omega_c \). A negative value for this parameter can be used to model some BCS attraction of unspecified origin. The renormalization and pairing functions are first obtained on the imaginary axis at the Matsubara frequencies, i.e. \( \omega = i\omega_n \equiv i\pi T(2n - 1) \), with \( \phi_m \equiv \phi(i\omega_m) \) by setting the complex...
variable $\omega$ in these equations to the Matsubara frequencies $[25,26]$. Then the equations are iterated as written, with $\omega$ set to a frequency on the real axis. Note that the square roots with complex arguments are defined to have a positive imaginary part.

Actually, these functions are obtained at frequencies just above the real axis, and then, in principle, the pole is given by the zero of the denominator continued to the lower half-plane. However, since the solutions are readily known only along certain lines in the complex plane (e.g. the imaginary axis or just above the real axis) we follow previous authors and look for the pole perturbatively. That is, we write $\omega \equiv E - i \Gamma$, and linearize the imaginary part so that $[23,24]$

$$\Gamma(E) = \frac{E Z_2(E)}{Z_1(E)} - \frac{\phi_1(E) \phi_2(E)}{E Z_1^2(E)}$$

(14)

Note that $E$ is determined by equating the real parts (again linearizing in imaginary components), which yields

$$E = \sqrt{\epsilon_k^2 + \Delta^2}$$

(15)

At the Fermi surface, $\epsilon_k \equiv 0$ so $E = \phi_1(E)/Z_1(E) \equiv \Delta_1(E)$ becomes the definition for the lowest energy excitation, i.e. the gap in the excitation spectrum. It has become common practice to consider $E$ as an independent variable, and then to study $\Gamma(E)$ as a function of $E$. In Fig. 2 we show $\Gamma(E)$ vs. $E$ for various temperatures, using a Debye model spectrum. It is clear that the scattering rate near the gap edge (shown by the arrow) decreases very quickly as the temperature decreases. Note that $\Gamma(E)$ actually becomes negative at intermediate temperatures near $E \approx 10$ meV. This is real (i.e. not a numerical artifact) and is simply a property of the function $\Gamma(E)$ given by eq. (14) through the linearization procedure. The true pole in eq. (11) will always have a negative imaginary part (i.e. $\Gamma$ is positive).

What happens in the BCS limit? Then $Z_2(E) \to 0$ and $\phi_2(E) \to 0$, i.e. the functions involved in the solution are pure real, with $Z_1(E) \to 1$ and $\phi_1(E) \to \Delta(T)$, where $\Delta(T)$ is obtained self-consistently from the BCS equation. Clearly then the quasiparticle energy is $E = \sqrt{\epsilon_k^2 + \Delta^2}$ and the scattering rate, $\Gamma = 0$. Thus, within the BCS approximation the
quasiparticle states are infinitely long-lived, as there is no means by which a quasiparticle can decay. In the electron-phonon theory, the phonons provide the means. In this way the BCS approximation is pathological.

Before including the electron-phonon interaction as a scattering mechanism, one can first introduce impurities to provide elastic electron scattering. Then

\[
\phi(\omega) = \phi_{cl}(\omega) + \frac{i}{2\tau} \frac{\phi(\omega)}{\sqrt{\omega^2 Z^2(\omega) - \phi^2(\omega)}}
\]

\[
Z(\omega) = Z_{cl}(\omega) + \frac{i}{2\tau} \frac{Z(\omega)}{\sqrt{\omega^2 Z^2(\omega) - \phi^2(\omega)}}
\] (16)

where the subscript ‘cl’ refers to the clean limit and \(1/\tau\) is the (normal) impurity scattering rate. The gap function is defined

\[
\Delta(\omega) = \phi(\omega)/Z(\omega)
\] (17)

and is independent of impurities. In the BCS limit,

\[
\phi(\omega) = \Delta + \frac{i}{2\tau} \frac{\Delta \text{sgn}\omega}{\sqrt{\omega^2 - \Delta^2}}
\]

\[
Z(\omega) = 1 + \frac{i}{2\tau} \frac{\text{sgn}\omega}{\sqrt{\omega^2 - \Delta^2}}
\] (18)

As before, the quasiparticle energy is given by \(E = \sqrt{\epsilon_k^2 + \Delta^2}\), but the scattering rate is now

\[
\Gamma = \left|E\right| \frac{1}{2\tau} \frac{\sqrt{E^2 - \Delta^2}}{\left|E\right|} = \left|\epsilon_k\right| \frac{1}{2\tau} \frac{1}{\sqrt{\epsilon_k^2 + \Delta^2}}
\] (19)

Eq. (19) shows that at the Fermi surface impurities are ineffectual for electron scattering, i.e. this is another manifestation of Anderson’s theorem [27]. This happens abruptly at \(T_c\); as soon as a gap develops the scattering rate becomes zero. Away from the Fermi surface the scattering rate is reduced from its value in the normal state. In particular, well away from the Fermi surface the scattering rate in the superconducting state approaches the normal state rate. This behaviour is illustrated in Fig. 3.

The calculations in Fig. 3 rely on a perturbative search for the pole in the lower half-plane, as given by eq. (14). A more rigorous calculation reveals poles “within the gap”, but
these do not appear in the spectral function because of coherence factors. This can be seen by rewriting eq. (11) as

\[
G_{11}(k, \omega) = \frac{1}{2} \left( 1 + \frac{\omega}{\sqrt{\omega^2 - \Delta^2}} \right) \frac{1}{\sqrt{\omega^2 - \Delta^2} - \epsilon_k + \frac{i}{2\tau} \text{sgn}\omega} \\
- \frac{1}{2} \left( 1 - \frac{\omega}{\sqrt{\omega^2 - \Delta^2}} \right) \frac{1}{\sqrt{\omega^2 - \Delta^2} + \epsilon_k + \frac{i}{2\tau} \text{sgn}\omega}
\]  

(20)

The coherence factors, \(1 \pm \frac{\omega}{\sqrt{\omega^2 - \Delta^2}}\), are such that the spectral function, \(A(k, \omega) \equiv -\frac{1}{\pi} \text{Im}G(k, \omega)\), is zero for \(|\omega| < \Delta\), independent of the impurity scattering rate, as can be readily verified explicitly from eq. (20). On the other hand the pole of the Green function is given by solving for \(\omega = E - i\Gamma\) in the two coupled equations

\[
E^2 - \Gamma^2 = \Delta^2 + \epsilon_k^2 - \left(\frac{1}{2\tau}\right)^2
\]

(21)

\[
\Gamma = \frac{|\epsilon_k|}{|E|} \frac{1}{2\tau}
\]

(22)

For \(1/\tau = 0\) the solution is as before, \(E = E_k \equiv \sqrt{\epsilon_k^2 + \Delta^2}\), and \(\Gamma = 0\). For finite impurity scattering, however, the solution depends on \(\frac{1}{2\tau\Delta}\) at the Fermi surface (\(\epsilon_k = 0\)):

\[
E = \sqrt{\Delta^2 - \left(\frac{1}{2\tau}\right)^2}, \quad \Gamma = 0 \quad \text{for} \quad \frac{1}{2\tau\Delta} < 1
\]

(23)

\[
E = 0, \quad \Gamma = \sqrt{\left(\frac{1}{2\tau}\right)^2 - \Delta^2} \quad \text{for} \quad \frac{1}{2\tau\Delta} > 1
\]

(24)

In either case the solution is irrelevant, because the real part lies within the gap (i.e. the quasiparticle residue is zero). The relevant energies are \(|E| > \Delta\), and then the scattering rate given by Eq. (22) agrees with the linearized solution, Eq. (14), although the quantity \(\Gamma\) no longer truly represents a quasiparticle inverse lifetime. Away from the Fermi surface the quasiparticle energy requires the solution of a quadratic equation and \(E\) may lie below or above the gap, depending on \(\epsilon_k\) and \(1/\tau\). Once more solutions with \(|E| < \Delta\) are irrelevant because the residue is zero. In Fig. 4a (b) we show the real (imaginary) parts of the pole for some typical parameters. As can be seen, some atypical behaviour can occur as a function of temperature, for example when \(\epsilon_k = 0\). As the temperature is lowered the pole moves.
from a point on the negative imaginary axis to the origin (near $T/T_c = 0.8$) and then moves along the real axis towards $E = \sqrt{\Delta^2 - (\frac{1}{2\tau})^2}$. Nonetheless, comparison of Fig. 4(b) with Fig. 3 shows that the perturbative is in qualitative agreement with the non-perturbative solution.

Coherence factors have always been thought to play an important role for two-particle response functions (such as the NMR relaxation rate or the microwave conductivity). In particular they lead to the Hebel-Slichter \[28\] singularity in the NMR relaxation rate. However they are considered not to play a big role in single particle functions. Eq. (20) shows, however, that the coherence factors play a very important role in the single particle spectral function, in that they maintain a gap equal to $\Delta$, even when impurities cause the single particle pole to have a real part whose value lies in the gap.

The frequency dependence of the spectral function is shown in Fig. 5. For energies close to the Fermi surface the spectral function is dominated by the square root singularity at the gap edge, which comes from the coherence factors rather than the single particle pole. For larger energies the peak present is due to the pole and differs from the normal state spectral function only at low frequencies.

We return now to the strong coupling case. In the superconducting state, we do not have access to the analytic continuation of the Green function to the lower half-plane \[29\]. We thus follow Karakozov et al. \[30\], and use an expansion near $\omega = 0$. We also follow Kaplan et al. \[2\] and henceforth use Eq. (14) for the scattering rate, noting that while the energy $E$ at which $\Gamma(E)$ is evaluated ought, in principle, to be computed self-consistently as was done in BCS, i.e. eqs. (21,22), in practice we will choose the relevant energy, and so treat energy as an independent variable.

Karakozov et al. \[30\] pointed out that at any finite temperature the solutions to the Eliashberg equations have the following low frequency behaviour:

\begin{align}
Z(\omega) & \approx Z_1 + \frac{i\gamma_2}{\omega} \quad (25) \\
\Delta(\omega) & \approx \delta_1 \omega^2 - i\delta_2 \omega \quad (26)
\end{align}
where $Z_1, \gamma_2, \delta_1$ and $\delta_2$ are real (positive) constants. This implies that a quasiparticle pole exists with

$$ E \approx \frac{\epsilon_k}{Z_1 \sqrt{1 + \delta_2^2}} $$

(27)

$$ \Gamma \approx \frac{1}{Z_1} \left( 2\pi \int_0^\infty d\nu \alpha^2 F(\nu) g(\nu) \left[ N(\nu) + f(\nu) \right] + \frac{g(0)}{2\tau} \right) $$

(28)

where we have used Eq. (14) and assumed $E \to 0$ (i.e. we are near the Fermi surface). In Eq. (28) $g(\nu)$ is the single electron density of states in the superconducting state,

$$ g(\nu) = \text{Re} \left( \frac{\nu}{\sqrt{\nu^2 - \Delta^2}} \right), $$

(29)

which is non-zero for $\nu = 0$ at any finite temperature [30–32]. In Fig. 6 we show the quasiparticle scattering rate, $2\Gamma(T, E = 0)$ vs. $T/T_c$ for various impurity scattering rates, $1/\tau$, in the superconducting state. The normal state result is also shown for reference. Note that in the clean limit there is an enhancement just below $T_c$ in the superconducting state, but at low temperatures there is an exponential suppression, compared to the power law behaviour observed in the normal state. When impurity scattering is present there is an immediate suppression below $T_c$ (a knee is still present, however, in the superconducting state). It is clear that at low temperatures the superconducting state is impervious to impurity scattering, as one would expect.

Is $2\Gamma(T, E = 0)$ the relevant inverse lifetime by which properties of the superconducting state can be understood? The answer is no, at least not at low temperatures, where the spectral function essentially develops a gap for low energies. In Fig. 7 we plot the spectral function at the Fermi surface, $A(k_F, \omega)$ for various temperatures [31]. While no true gap exists at any finite temperature, it is clear that for low temperatures the spectral weight at low frequency is exponentially suppressed. Thus we have a situation similar to that in BCS theory, where the imaginary part of the pole is irrelevant; rather it is the scattering rate (as defined by Eq. (14)) evaluated above the ”gap-edge” where a large spectral weight is present, that is most relevant. When impurities are added the spectral peak is broadened,
even at low temperatures, as shown in Fig. 8 for an intermediate temperature, $T/T_c = 0.5$. However the lineshape is very asymmetric as a gap remains at low frequencies (particularly prominent at low temperatures). Fig. 7 demonstrates that the relevant energy is a function of temperature. In Fig. 9 we plot $2\Gamma(T, E = \Delta(T))$ vs. $T/T_c$, where $\Delta(T)$ is determined from the relation (at the Fermi surface)

$$E = \text{Re}\Delta(E, T).$$

By Eq. (30) we understand that the $E = 0$ solution is excluded (similarly the very low energy solution is also excluded). We are interested in the conventional solution which gives rise to the peak in the spectral function illustrated in Fig. 7 or 8. If no non-zero solution is present (as is the case near $T_c$) then we utilize the $E = 0$ solution. Note that in this case care must be taken when obtaining the $E \to 0$ limit of Eq. (14). Also there is present in this definition a discontinuity at some temperature near $T_c$, which is where a nonzero solution first appears. In this way we hope to show the scattering rate at an energy where the spectral weight is large, and therefore of most relevance to observables. At any rate, it is clear by comparing Fig. 9 to Fig. 6 (see also Fig. 2) that there is very little difference in the scattering rate in the gap region of energy. However, as the energy increases beyond the gap the scattering rate increases, resulting in short “lifetimes”, even at zero temperature [2].

**III. THE OPTICAL CONDUCTIVITY**

Using the Kubo formalism [12], the optical conductivity can be related to a current-current correlation function. The final result for the frequency dependence of the conductivity in the long wavelength limit is [13–16]

$$\sigma(\nu) = i\frac{\nu}{\nu}(\Pi(\nu + i\delta) + \frac{ne^2}{m}),$$

(31)

where the paramagnetic response function, $\Pi(\nu + i\delta)$, is given by

$$\Pi(\nu + i\delta) = \frac{ne^2}{m} \left\{ -1 + \int_0^\infty d\omega \tanh\left(\frac{\beta\omega}{2}\right) \left( h_1(\omega, \omega + \nu) - h_2(\omega, \omega + \nu) \right) + \int_{-\nu}^\nu d\omega \tanh\left(\frac{\beta(\omega + \nu)}{2}\right) \left( h_1^*(\omega, \omega + \nu) + h_2^*(\omega, \omega + \nu) \right) \right\}$$

(32)
with

\[ h_1(\omega_1,\omega_2) = \frac{1 - N(\omega_1)N(\omega_2) - P(\omega_1)P(\omega_2)}{2(\epsilon(\omega_1) + \epsilon(\omega_2))} \]

\[ h_2(\omega_1,\omega_2) = \frac{1 + N^*(\omega_1)N(\omega_2) + P^*(\omega_1)P(\omega_2)}{2(\epsilon(\omega_2) - \epsilon^*(\omega_1))} \]

\[ N(\omega) = \frac{\tilde{\epsilon}(\omega + i\delta)}{\epsilon(\omega + i\delta)} \]

\[ P(\omega) = \frac{\phi(\omega + i\delta)}{\epsilon(\omega + i\delta)} \]

\[ \epsilon(\omega) = \sqrt{\tilde{\epsilon}^2(\omega + i\delta) - \phi^2(\omega + i\delta)}. \] (33)

In eq. (32) \( D \) is a large cutoff of order the electronic bandwidth.

To summarize the previous set of equations: for a given model for the spectral density, \( \alpha^2 F(\nu) \) and a choice of impurity scattering rate \( 1/\tau \), we can compute the conductivity \( \sigma(\nu) \) at any frequency and temperature, including effects due to both elastic and inelastic scattering mechanisms, the latter being determined by the choice of electron-phonon spectral density. As explained in the introduction, however, vertex corrections are omitted.

In the analysis of experimental data it is possible to use several methods to extract a single temperature dependent scattering rate. One such method utilized the microwave conductivity in YBaCu3O7-x \[4,5\] and in Nb \[7\], and adopted a two-fluid model description of the superconducting state. It was assumed that the absorptive component of the conductivity (real part of \( \sigma \) at finite frequency, denoted by \( \sigma_1(\nu) \)) was due only to the normal component of the fluid. Thus an expression of the form

\[ \sigma_1(\nu,T) = \frac{ne^2}{m} \frac{m}{m^*(T)} \left[ 1 - \frac{\lambda^2(0)}{\lambda^2(T)} \right] \frac{\tau(T)}{1 + (\nu\tau(T))^2} \] (34)

is assumed to hold approximately. In eq. (34) \( \tau(T) \) has units of a scattering time and \( \lambda(T) \) is the penetration depth at temperature \( T \). Here we will calculate \( \sigma_1(\nu,T) \) for a model \( \alpha^2 F(\nu) \) and \( 1/\tau \) using the full expression (31). At the same time we can calculate the penetration depth, either from a zero frequency limit of eq. (31), or directly from the imaginary axis [13,38]:

\[ \frac{\lambda^2(0)}{\lambda^2(T)} = \pi T \sum_{m=\infty}^{\infty} \frac{\phi_m^2}{\omega_m^2 Z^2(i\omega_m) + \phi_m^2} \frac{1}{3/2}. \] (35)
The idea is to examine the zero frequency limit of eq. (34), and thus define a scattering rate relative to the rate at $T_c$:

$$\frac{\tau(T_c)}{\tau(T)} \equiv \left(1 - \frac{\lambda^2(0)}{\lambda^2(T)}\right) \frac{\sigma_N(T_c)}{\sigma_1(T)},$$

(36)

where it has been assumed that the mass enhancement factor in eq. (34) does not change with temperature.

In Fig. 10 we show results for $\tau(T_c)/\tau(T)$ defined by eq. (36) in the superconducting state (solid curve) with which we compare the inverse lifetime for both the normal (short-dashed curve, eq. (9), and the superconducting (long-dashed curve, eq. (14)) states. The results are based on a Debye model spectrum for $\alpha^2F(\nu)$ used in the previous section, with mass enhancement parameter $\lambda = 1$ and $T_c = 100$ K. (A negative $\mu^*$ is required.) Note that the normal state scattering rate (short-dashed curve) is only sensitive to the low frequency part of $\alpha^2F(\nu)$ at low temperatures: a $\nu^2$ dependence in $\alpha^2F(\nu)$ implies a $T^3$ dependence in $1/\tau(T)$ (Again, vertex corrections would alter this to the familiar $T^5$ law[17]). The agreement between the inverse lifetime (long-dashed curve) and the scattering rate defined by the two-fluid model (solid curve) in the superconducting state is remarkable. This indicates that the two-fluid description makes sense[6,8], and the quantities shown in Fig. 10 apply to the normal component of the superfluid. Fig. 10 illustrates the comparison in the clean limit, where the two-fluid description is expected to be most accurate[8]. Before investigating impurity dependence, we turn to a second possible procedure for extracting a scattering rate from conductivity data[34,35], which is simply a generalization of that used by Shulga et al. [9] to the superconducting state. One simply fits the low frequency absorptive part of the conductivity to a Drude form:

$$\sigma_1(\nu, T) = \frac{ne^2}{m} \frac{1}{m^*/m} \frac{\tau^*(T)}{1 + (\nu\tau^*)^2}.$$  

(37)

As described in Refs. [10], it is possible to fit a Drude form to the low frequency part of the optical conductivity in the normal state. Such a fit is also possible in the superconducting state [34]. Theoretically, the fit is problematic in a BCS approach because there is a Hebel-Slichter logarithmic singularity at low frequency at all temperatures in the superconducting...
However, with the Eliashberg approach, the Hebel-Slichter singularity is smeared, and one can fit a Drude form over a limited range of frequency. Such a fit for a Debye spectrum is also included in Fig. 10 (dotted curve). While the fit is in qualitative agreement with the inverse lifetime, this method of characterizing the inverse lifetime in the superconducting state is clearly not as accurate as the two-fluid model. The fits themselves are shown in Fig. 11. It is clear that the fits fail at sufficiently high frequency, as one would expect, but that they characterize well the low frequency response in the superconducting state.

Fig. 10 clearly shows that there is considerable freedom and hence ambiguity in extracting a temperature dependent scattering rate from conductivity data. Nevertheless, in the clean limit it is evident that the two-fluid model is a useful device for extracting the low energy quasiparticle inverse lifetime in the superconducting state.

With the addition of impurities the situation changes considerably. This is illustrated in Fig. 12, where the same calculations as in Fig. 10 are shown, but with an additional impurity scattering, $1/\tau = 25$ meV included. The use of formula (34), inspired by the two-fluid model, gives a scattering rate (solid curve) that falls much less rapidly around $T = T_c$ then does the inverse lifetime in the superconducting state (long-dashed curve). The latter curve drops almost vertically as the temperature drops below $T_c$, as has already been discussed. The result based on the Drude fit (dotted curve) shows a peak which is reduced in size from that in the clean limit (Fig. 10). Indeed, for increased impurity scattering, the peak just below $T_c$ disappears. In both cases a rapid suppression is expected just below $T_c$: in the case of the inverse lifetime, this suppression is a consequence of Anderson’s theorem [27], as already discussed. In the case of the Drude fit, it is easy to see that this is the case in the dirty limit. In the dirty limit the conductivity is almost flat as a function of frequency on the scale of $1/\tau$, at $T_c$. Just below $T_c$, however, a gap in the spectrum begins to develop, so that weight is shifted from roughly the gap region to the delta function at the origin. Any low energy fit will then use a Lorentzian width which senses this depression in the conductivity, which is on an energy scale of the gap. This represents a significant suppression from the normal state scattering rate (infinite in the dirty limit). Here we are in an intermediate regime,
with $1/\tau = 25$ meV (note: $\Delta(T = 0) = 20.2$ meV). The corresponding fits for Fig. 12 are shown in Fig. 13.

Finally, we examine the low frequency conductivity vs. reduced temperature, a quantity measured in the high $T_c$ oxides by microwave techniques. In Fig. 14 we show the real part of the conductivity, $\sigma_1(\nu)$ vs. reduced temperature, for several low frequencies. Note the relative insensitivity to frequency, a feature of strong coupling pointed out in Ref. [36]. Also note the lack of a coherence peak just below $T_c$. Nonetheless, a broad peak exists at lower temperatures, somewhat reminiscent of that observed in YBaCu$_3$O$_{7-x}$ [37, 4, 8]. This peak exists because of a competition between an increasing scattering time (making $\sigma$ increase) and a decreasing normal component (making $\sigma$ decrease, particularly as $T \to 0$). We should note that this peak is most prominent in the clean limit. As Fig. 13 indicates (see values at the intercept), the peak is absent for a sufficiently large impurity scattering rate.

It is of interest to examine what dependence these results have on the electron-phonon spectral function. As an extreme we utilize a spectrum which is sharply peaked at some high frequency, and, in contrast to the Debye model employed above, coupling to low frequency modes is absent; such a spectrum models a strong coupling to an optic mode. We choose a triangular shape for convenience, starting at $\omega_0 = 34.8$ meV with a cutoff at $\omega_E = 35.5$ meV. The coupling constant is chosen so that the mass enhancement value is $\lambda = 1$, in agreement with that chosen for the Debye spectrum. As was the case there, a negative $\mu^*$ is used to give $T_c = 100$ K, and the zero temperature gap was found to be close to the Debye value.

In Fig. 15 we plot the real part of the low frequency conductivity, $\sigma_1(\nu)$ vs. reduced temperature, now using the triangular spectrum for $\alpha^2 F(\nu)$. In contrast to the result for the Debye model, the low frequency conductivity is strongly frequency dependent at low temperatures. In fact, for $\nu = 0$, the conductivity appears to diverge. We believe that at sufficiently low temperature, this curve will actually achieve a maximum and approach zero at zero temperature, but we have been unable to obtain this result numerically. Once again there is a competition between an increasing scattering time and a decreasing normal density component as the temperature is lowered from $T_c$. Here, however, the increasing
scattering time appears to be overwhelming the decrease in normal fluid density. The key difference with the Debye spectrum is that here the spectrum has a big gap, so that the lifetime is increasing exponentially with decreasing temperature, already in the normal state. Recall that in the Debye case the increase followed a power law behaviour with decreasing temperature. Since the decrease in normal fluid density is always exponential, this term dominates in the Debye case, whereas in the case of the gapped spectrum the competition is subtle, and will depend strongly on the details of the spectrum (an electron-phonon spectrum with a much smaller gap will yield a zero frequency conductivity which approaches zero at zero temperature, for example).

The physics of this conductivity peak is different from what has been already proposed for YBaCu$_3$O$_{7-x}$. It has been suggested that the excitation spectrum becomes gapped due to the superconductivity, i.e. a feedback mechanism exists which creates a low frequency gap in the excitation spectrum as the superconducting order parameter opens up below $T_c$. Such a scenario has been explored within a marginal Fermi liquid scheme \cite{37,38,39}, and is also consistent with thermal conductivity experiments \cite{40}. Here the conductivity peak arises because the spectrum is already gapped, and the scattering rate is sufficiently high at $T_c$ because $T_c$ itself is fairly high. We should warn the reader that this mechanism requires a “fine tuning” of the spectrum, i.e. a large gap is required.

Note that for any non-zero frequency the conductivity has a visible maximum, and quickly approaches zero at sufficiently low temperature. Nonetheless this turn around occurs for yet another reason: as one lowers the temperature the Drude-like peak at low frequencies gets narrower while at the same time the magnitude of the zero temperature intercept increases. For any given finite frequency, then, a temperature is eventually reached below which this frequency is now on the tail of the Drude-like peak. This means that while the zero frequency conductivity increases, that at any finite frequency will eventually decrease as the width becomes smaller than the frequency. So in this case the increase in scattering lifetime still dominates the decrease in normal fluid density, but, because we are fixed at a finite frequency, the conductivity decreases. Note that frequencies of order 0.01 meV ($\approx 2.4$
GHz) are within the range of microwave frequencies that are used in experiments.

To show this more explicitly we illustrate in Fig. 16 the various scattering rates obtained with the gapped spectrum. These are to be compared with those shown for the Debye model in Fig. 10. Clearly the quantitative agreement between the scattering rate inspired by the two-fluid model and the inverse lifetime that was seen in Fig. 10 with the Debye spectrum was fortuitous. While a qualitative correspondence between these two entities continues to exist with the gapped spectrum, they are no longer in quantitative agreement. We have verified that this is generically true, by investigating other spectra, not shown here.

IV. SUMMARY

We have investigated the quasiparticle lifetime in an Eliashberg s-wave superconductor, generalizing earlier work [2] to include impurity scattering as well. We find that the quasiparticle lifetime becomes infinite at low temperatures, independent of the impurity scattering rate, which we understand as simply a manifestation of Anderson’s theorem [27]. Thus, on general grounds, within a BCS framework, the scattering rate should collapse to zero in the superconducting state.

We have also investigated two methods for extracting the scattering rate from the low frequency conductivity. One relies on a two-fluid model picture, and the other simply utilizes a low frequency Drude fit. Neither should necessarily correspond very closely to the quasiparticle inverse lifetime, and we find that in general they do not, quantitatively. Qualitatively, however, the scattering rate defined by either procedure gives the correct temperature dependence for the inverse lifetime. In the presence of impurities, the two-fluid prescription appears to be less accurate, presumably because such a prescription takes into account only the lower normal fluid density as the temperature is lowered, and not the fact that impurity scattering is less effective in the superconducting state. Thus we caution that interpreting a conductivity-derived scattering rate as a quasiparticle inverse lifetime can lead to inaccuracies.

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• Fig. 1. (a) Mass enhancement parameter, as defined by Eq. (10) in the text, vs. reduced temperature $T/T_c$, for various electron-phonon spectral functions. In all cases $T_c = 100$ K. The 4 spectra used are a Debye spectrum (solid line), linear spectrum (dotted line), a spectrum proportional to $\sqrt{\nu}$ (dashed line), and a triangular spectrum (dot-dashed line). In all cases the strength is such that $\lambda = 1$. In the first three cases a cutoff frequency equal to 30 meV was used. In the last case, the spectrum starts at 34.8 meV and is cut off at 35.5 meV.

(b) Inverse lifetime, $2\Gamma(T)$ (in meV) vs. reduced temperature for the same spectra as in (a). Note the different low temperature behaviour, depending on the low frequency characteristics of the electron-phonon spectrum used.

• Fig. 2. Scattering rate, $2\Gamma(T, E)$ (in meV) vs. $E$ (in meV) for various temperatures in the superconducting state. The zero temperature gap at the Fermi surface is indicated by the arrow. Note that for $T/T_c = 0.5$ the function plotted actually becomes negative (near 10 meV). The physical pole occurs at an energy, $E$, where $2\Gamma(T, E)$ is always positive, however.

• Fig. 3. Scattering rate normalized to the zero temperature gap, $2\Gamma(T, E)/\Delta$ vs. reduced temperature $T/T_c$, for various quasiparticle energies, $\epsilon$. These are computed using the perturbative expansion, Eq. (19). The impurity scattering rate is $1/(\tau)\Delta = 1$. Note that on the Fermi surface (solid curve), the scattering rate is zero immediately below $T_c$.

• Fig. 4. (a) Real and (b) imaginary parts of the quasiparticle pole vs. reduced temperature, $T/T_c$, with $1/(\tau)\Delta = 1$ and various quasiparticle energies. These are computed non-perturbatively from Eqs. (21,22). Note that at the Fermi surface the quasiparticle energy has an abrupt onset at a temperature somewhat below $T_c$. The results in (b) are
similar to those in Fig. 3, except that at the Fermi surface, the decrease in scattering rate below $T_c$ is not as abrupt.

- Fig. 5. The single particle spectral function, $A(k, \omega)$, vs. normalized frequency, $\omega/\Delta$, with $1/(\tau \Delta) = 1$, for various quasiparticle energies, $\epsilon/\Delta$. The Fermi surface result (solid curve) has particle-hole symmetry. The result for $\epsilon/\Delta = 0.5$ would have a peak within the gap (between -1 and 1) except that the coherence factors in Eq. (20) give zero residue for the gap region. As the quasiparticle energy increases, the spectral function begins to resemble the normal state spectral function. Small square-root singularities still exist, nonetheless at the particle and hole gap edges.

- Fig. 6. The quasiparticle scattering rate, $2\Gamma(T, E = 0)$ vs. $T/T_c$ for various impurity scattering rates, $1/\tau$, in both the superconducting and normal (long-dashed curves) states. In the clean limit (solid curve) there is an enhancement immediately below $T_c$. When impurity scattering is present, this scattering is immediately suppressed in the superconducting state, as shown by the dotted and dashed curves. Below a temperature of about $0.8T_c$ the scattering rate is independent of the amount of impurity scattering present in the normal state. A Debye electron-phonon spectrum was used with $\lambda = 1$ and cutoff frequency, $\omega_D = 30$ meV.

- Fig. 7. The single particle spectral function, $A(k_F, \omega)$ at the Fermi surface, vs. frequency, in the clean limit, for various reduced temperatures. The spectral function is considerably broadened near $T_c$, due to temperature alone. At the two lowest temperatures shown, while there is no true gap in the excitation spectrum, this plot makes it clear that, practically speaking, an effective gap in the excitation spectrum is present. A Debye electron-phonon spectrum was used with $\lambda = 1$ and cutoff frequency, $\omega_D = 30$ meV.

- Fig. 8. The single particle spectral function, $A(k_F, \omega)$ at the Fermi surface, vs. frequency, for various impurity scattering rates, as indicated. Note the broadening which
occurs with increasing impurity scattering. However, spectral weight remains absent in the “gap region”. Results are for a temperature $T/T_c = 0.5$, and with a Debye electron-phonon spectrum with $\lambda = 1$ and cutoff frequency, $\omega_D = 30$ meV.

- **Fig. 9.** The quasiparticle scattering rate, $2\Gamma(T, E = \Delta(E))$, evaluated at the quasiparticle energy, given on the Fermi surface by $E = \Delta(E)$, vs. $T/T_c$ for various impurity scattering rates, $1/\tau$, in both the superconducting and normal (long-dashed curves) states. These results are in quantitative agreement with those in Fig. 6, since the energy scale, $\Delta$, is still small compared to other (phonon) energy scales in the problem. In particular, below about $0.8T_c$, the scattering rate is independent of the amount of impurity scattering present in the normal state. A Debye electron-phonon spectrum was used with $\lambda = 1$ and cutoff frequency, $\omega_D = 30$ meV.

- **Fig. 10.** Various normalized scattering rates vs. reduced temperature. The normal and superconducting scattering rates come from the quasiparticle inverse lifetime, given by eq. (14), at zero energy. The two-fluid result comes from eq. (36) while the Drude fit is obtained by fitting eq. (37) to the low frequency conductivity in the superconducting state. Note the agreement of the scattering rate as extracted from the two-fluid analysis with the inverse lifetime in the superconducting state. A Debye electron-phonon spectrum was used with $\lambda = 1$ and cutoff frequency, $\omega_D = 30$ meV.

- **Fig. 11.** The low frequency conductivity in the superconducting state (solid curves) along with their fits based on eq. (37) (dashed curves). These fits were used in Fig. 10 (dotted curves).

- **Fig. 12.** Same as Fig. 10, except now with an impurity scattering rate of 25 meV. Note that the two-fluid analysis agrees poorly with the quasiparticle inverse lifetime.

- **Fig. 13.** The low frequency conductivity fits used in Fig. 12.

- **Fig. 14.** The very low frequency conductivity as a function of reduced temperature,
in the clean limit. A Debye electron-phonon spectrum was used with $\lambda = 1$ and cutoff frequency, $\omega_D = 30$ meV. Note that the results are relatively insensitive to frequency (the $\nu = 0.01$ meV result, given by the dotted curve, is essentially hidden by the zero frequency result). A microwave experiment yields essentially zero frequency results.

- Fig. 15. Same as for Fig. 14, but now with the triangular spectrum as described in the text. Note that the zero frequency conductivity appears to diverge as $T \to 0$. A microwave experiment will yield a large low temperature peak as a function of reduced temperature, whose magnitude will depend strongly on the frequency. These results are for the clean limit. The peak is also reduced as impurity scattering is added (not shown).

- Fig. 16. Comparison of scattering rates (as in Fig. 10) calculated for the triangular spectrum, in the clean limit. Note that the normal state result approaches $T = 0$ exponentially due to the gap in the $\alpha^2 F(\nu)$ spectrum (in Fig. 10 the corresponding curve approached zero with a power law behaviour). While the results in the superconducting state are qualitatively similar, they no longer agree quantitatively with one another, as in Fig. 10.