Analytical Representations of Divisors of Integers

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Abstract

Certain analytical expressions which ”feel” the divisors of natural numbers are investigated. We show that these expressions encode to some extent the well-known algorithm of the sieve of Eratosthenes.

Most part of the text is written in pedagogical style, however some formulas are new.

MSC: Primary 11A51; Secondary 26A06

1 Notation and Conventions

Throughout this paper we shall adopt the following notation and conventions: \( n \) is a given natural number and \( k \) is a possible divisor of \( n \). If \( k \) actually divides \( n \) then \( j = n/k \). Let \( f(x) \) denotes any real analytic function defined in the neighborhood of the origin by a power series

\[
f(x) = \sum_{j=0}^{\infty} c_j x^j
\]

with all \( c_j \neq 0 \) (\( i = 1, 2, 3... \)). It will be shown that \( j \) is also the exponent of \( x \) in the expansion (1) around zero and \( j \) labels half-lines or rays of divisors (see below).
2 Motivation

The theory of divisors of integers is the cornerstone of elementary number theory. It is convenient to introduce the characteristic function for divisors:

**Definition.** For any \( n, k \in \mathbb{N} \)

\[
\alpha_{nk} := \begin{cases} 
1 & \text{if } k \mid n \\
0 & \text{if } k \nmid n
\end{cases}
\]  
(2)

Another pretty obvious (and rather useless in numerical calculations) representation of (2) is:

\[
\alpha_{nk} = \frac{1}{\Gamma(1 - \text{mod}(n, k))}
\]  
(3)

where \( \Gamma(s) \) denotes the Euler gamma function and \( \text{mod}(n, k) \) gives the remainder on division of \( n \) by \( k \). In fact (3) is more general than (2) since it may be calculated also for non-integer or even complex values of \( n \) and \( k \) but this leads to some interpretation difficulties which we shall not discuss here.

Consider the following expression for some natural numbers \( n \) and \( k \):

\[
\alpha_{nk} = \frac{d^n}{dx^n}e^{x^k} \bigg|_{x=0}
\]  
(4)

We will prove the following

**Theorem.** Apart from a trivial normalization factor, \( \alpha_{nk} \) defined in formula (2) is equal to \( \alpha_{nk} \) defined in (4).

**Proof.** Expanding the exponential function in (4) in power series and performing term-by-term differentiation we get:

\[
\alpha_{nk} = \frac{d^n}{dx^n} \sum_{j=0}^{\infty} \frac{(x^k)^j}{j!} \bigg|_{x=0} = \sum_{j=0}^{\infty} \frac{1}{j!} \frac{d^n}{dx^n} x^{jk} \bigg|_{x=0}
\]  
(5)

Recall the general formulas for the \( n \)-th derivative of \( x^p \) with respect to \( x \)

\[
\frac{d^n}{dx^n} x^p = \frac{\Gamma(p+1)}{\Gamma(p+1-n)} x^{p-n} = n! \binom{p}{n} x^{p-n}
\]  
(6)

\[
\frac{d^n}{dx^n} x^p = (-1)^n \frac{\Gamma(n-p)}{\Gamma(-p)} x^{p-n}
\]  
(7)

where the second formula stems from properties of the gamma function and is suitable for integer negative \( p \) (see e.g. [4]). Note that the order of derivative \( n \) does not have to be integer but for integer \( n \) both (6) and (7) reduce to the well-known elementary differentiation rule. Using (6) we get:

\[
\alpha_{nk} = \sum_{j=0}^{\infty} \frac{1}{j!} \frac{\Gamma(jk+1)}{\Gamma(jk+1-n)} x^{jk-n} \bigg|_{x=0} = n! \sum_{j=0}^{\infty} \frac{1}{j!} \binom{jk}{n} x^{jk-n} \bigg|_{x=0}
\]  
(8)
By simple inspection of (8) we see why this expression "feels" the divisors of the integer \( n \). Indeed, when taking the limit \( x \to 0 \) the only non-zero term in the series appears when \( jk = n \) for some integer \( j \), and this occurs if and only if \( k \) divides \( n \). All terms with \( jk > n \) disappear in the limit \( x \to 0 \) whereas those with \( jk < n \), although singular in \( x = 0 \), vanish since the binomial coefficient term is zero. Therefore, in the summation (8) at most only one term can survive in the limit process.

The above reasoning might appear far too excessive. However, it guarantees that among divisors none have been omitted. It should also be stressed that it may be used as a starting point for various generalizations since \( n \) need not to be integer.

It is easy to guess the normalizing factor:

\[
\alpha_{nk} = \frac{1}{n!} \left( \frac{n}{k} \right)! \frac{d^n}{dx^n} e^{x k} \bigg|_{x=0}
\]  

(9)

Using the same reasoning we can derive similar expression for \( \alpha_{nk} \):

\[
\alpha_{nk} = \frac{(k)!^{n/k}}{n!} \left( \frac{n}{k} \right)! \frac{d^n}{dx^n} e^{x k} \bigg|_{x=0}
\]  

(10)

3 Simple example

In a natural way coefficients \( \alpha_{nk} \) may be regarded as a square matrix of arbitrarily large dimension where the running integer \( n \) labels rows and the potential divisor \( k \) labels columns. The entries of this matrix are either one or zero depending on whether \( k \) divides \( n \) or not. This matrix is always triangular, since of course no divisor can exceed a given number, and its determinant (for any dimension) is 1.

\[
\begin{pmatrix}
\begin{array}{cccccccccc}
 n \backslash k & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & \ldots \\
 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \ldots \\
 2 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \ldots \\
 3 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \ldots \\
 4 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & \ldots \\
 5 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & \ldots \\
 6 & 1 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & \ldots \\
 7 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & \ldots \\
 8 & 1 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & \ldots \\
 9 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & \ldots \\
 10 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & \ldots \\
 \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\
\end{array}
\end{pmatrix}
\]  

(11)

(Matrix (11) is closely related to the Redheffer matrix, see e.g. [5], [9].) Introducing

\[
\sigma_0(n) := \sum_{k=1}^{n} \alpha_{nk}
\]

(12)
we see that \( \sigma_0(n) \) just counts the number of all divisors of a given \( n \) including both unity and \( n \) itself.

It is known (see e.g. [1]) that the inverse of matrix \((11)\) is:

\[
\beta_{nk} = \begin{cases} 
\mu \left( \frac{n}{k} \right) & \text{if } k \mid n \\
0 & \text{if } k \nmid n
\end{cases}
\]

(13)

where \( \mu \) denotes the Möbius function:

\[
\mu(n) = \begin{cases} 
0 & \text{if } n \text{ has squared prime factor} \\
+1 & \text{if } n \text{ is a square-free positive integer with an even number of prime factors} \\
-1 & \text{if } n \text{ is a square-free positive integer with an odd number of prime factors}
\end{cases}
\]

(14)

Note that the numbers in \((15)\) when summed in rows give zero except for the first row which stems from the following identity:

\[
\sum_{d \mid n} \mu(d) = \delta_{n,1}
\]

(16)

Matrices \((11)\) and \((15)\) are visualized in Figure 1.

Somewhat similar but purely qualitative results have been published in [3].
Figure 1. Graphic distribution of divisors for \( n = 1, 2, ..., 50 \) as a square matrix (left panel). Each blue square denotes +1. In the inverse matrix (right panel) blue square denotes +1 and red square denotes −1.

4 General case

The particular choice of the exponential function in (4) is not crucial to our reasoning. Indeed, instead of this function we can take any regular function \( f(x) \) provided that it has all non-zero coefficients in its power series expansion

\[
f(x) = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + ... \quad c_i \neq 0 \text{ for } i = 1, 2, 3, ...
\]  

Thus in general we have (up to appropriate normalizing factor)

\[
\alpha_{nk} = \frac{d^n}{dx^n} f(x^k) \bigg|_{x=0}
\]

For example, taking

\[
f(x) = \frac{x}{1-x} = x + x^2 + x^3 + ...
\]

we get:

\[
\alpha_{nk} = \sum_{j=1}^{\infty} \binom{j}{k} x^{j-k-n} \bigg|_{x=0}
\]
or simply

\[ \alpha_{nk} = \sum_{j=1}^{\infty} \frac{x^{jk-n}}{(jk-n)!} \bigg|_{x=0} \]  \hspace{1cm} (21)

Taking

\[ f(x) = \log(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \ldots \]  \hspace{1cm} (22)

we get:

\[ \alpha_{nk} = (-1)^{n/k} \frac{n}{k} \sum_{j=1}^{\infty} \frac{(-1)^j}{j} \binom{jk}{n} x^{jk-n} \bigg|_{x=0} \]  \hspace{1cm} (23)

The general explicit formula for \( \alpha_{nk} \) using arbitrary function \( f \) satisfying (17) is:

\[ \alpha_{nk} = \left( \frac{n}{k} \right)! \frac{1}{f(n/k)(0)} \sum_{j=0}^{\infty} \frac{f^{(j)}(0)}{j!} \binom{jk}{n} x^{jk-n} \bigg|_{x=0} \]  \hspace{1cm} (24)

where \( f^{(j)}(0) \) denotes the \( j \)-th derivative of \( f \) with respect to \( x \) taken at \( x = 0 \). (If \( n/k \) in (24) is non-integer then the value of fractional derivative \( f^{(n/k)}(0) \) is unimportant since in this case the sum vanishes.)

The table below contains normalizing factors for \( \alpha_{nk} \), for several different choices of function \( f(x) \), obtained using (24).

\[
\begin{array}{ll}
\hline
f(x) & \alpha_{nk} \\
\hline
\log(1-x) & (-1)^{n/k} \frac{n}{k} \sum_{j=1}^{\infty} \frac{(-1)^j}{j} \binom{jk}{n} x^{jk-n} \\
\frac{x}{1-x} & \frac{1}{n!} \frac{d^n}{dx^n} f(x) \bigg|_{x=0} \\
\sqrt{1+x} & \frac{1}{(2^n - 3)!} \frac{1}{k!} \frac{d^n}{dx^n} f(x) \bigg|_{x=0} \\
\frac{1}{\sqrt{1+x}} & \frac{(-1)^{n/k}}{(2^n - 1)!} \frac{1}{k!} \frac{d^n}{dx^n} f(x) \bigg|_{x=0} \\
(1 + x)^{-3/2} & \frac{(-2)^{n/k}}{(2^n + 1)!} \frac{1}{k!} \frac{d^n}{dx^n} f(x) \bigg|_{x=0} \\
W(x) & (-1)^{n/k-1} \frac{1}{(k + \frac{1}{2})^n} \frac{1}{n!} \frac{d^n}{dx^n} f(x) \bigg|_{x=0} \\
\sqrt{1-x-x^2} & \frac{1}{(2^n + 1)!} \frac{1}{n!} \frac{d^n}{dx^n} f(x) \bigg|_{x=0} \\
\hline
\end{array}
\]

\( (W(x) \) is the Lambert \( W \)-function and \( F_n \) in the last row denotes the \( n \)-th Fibonacci number.)
5 Interpretation

Let us now explain in more details how it all works. The thing is that all formulas for $\alpha_{nk}$ presented so far encode, at least to some extent, the ancient algorithm known as the sieve of Eratosthenes.

Indeed, consider as $f(x)$ the function $f(x) = x/(1-x)$ and let us temporarily restrict ourselves to the linear case: $f(x) \approx x$. According to the general formula (18) we have

$$\alpha_{nk} = \frac{1}{n!} \frac{d^n}{dx^n} f(x^k) \bigg|_{x=0} = \frac{1}{n!} \frac{d^n}{dx^n} x^k \bigg|_{x=0} = \binom{k}{n} x^{k-n} \bigg|_{x=0} = \delta_{k,n} \quad (25)$$

and this produces a single line of ones on the diagonal $n = k$ in the divisor matrix (11) – cf. Figure 2 below. This is equivalent to the trivial statement that all integers are divisible both by one and by themselves. Let us further consider more precise approximation $f(x) \approx x + x^2$. We get from (18) another sequence of ones on the line $n = 2k$. This is equivalent to selecting all even integers $n$ and adding to the divisor matrix their divisors $n/2$. Taking into account higher powers of $x$ we select all numbers $n$ which are multiplies of 3, 4, 5, ... and this adds to the matrix further lines of divisors: $n/3$, $n/4$, $n/5$, respectively.

Proceeding in the same way we finally arrive at the full expansion of $f(x)$:

$$f(x) = \frac{x}{1-x} = \sum_{j=1}^{\infty} x^j \quad (26)$$

which produces the entire sequence of lines $n = jk$ labelled by parameter $j = 1, 2, 3, ....$. In this way we have selected and visualized all divisors for all integers. It is clear that there are certain well-defined numbers $n$ (marked in bold in Figure 2) which have exactly two divisors: unity and themselves, i.e. prime numbers: 2, 3, 5, 7, 11, 13, ... At the same time we see the importance of condition $c_i \neq 0$ in (17) since even a single coefficient $c_i = 0$ would cause a skipping of certain divisors. In view of this the characteristic function for divisors may also be written in a very natural form as a sum over Kronecker deltas:

$$\alpha_{nk} = \sum_{j=1}^{n} \delta_{jk,n} \quad (27)$$

Note that combining (12), (18) and (26) gives:

$$\sigma_0(n) := \sum_{k=1}^{n} \alpha_{nk} = \frac{1}{n!} \frac{d^n}{dx^n} \sum_{k=1}^{\infty} \frac{x^k}{1-x^k} \bigg|_{x=0} \quad (28)$$

Hence

$$\sum_{k=1}^{\infty} \frac{x^k}{1-x^k} = \sum_{n=1}^{\infty} \sigma_0(n) x^n \quad (29)$$
which is consistent with the theory of Lambert series (see e.g. [2]) which is the generating function for the sequence $\sigma_0(n)$ where $\sigma_0(n)$ is the total number of divisors for a given integer $n$.

Figure 2. Distribution of divisors of integers computed from $\alpha_{nk}$. This figure illustrates how various terms in the sum (27) contribute to the whole pattern of divisors. Each term corresponds to a ray of divisors. Rows are labelled by consecutive integers $n$ and columns are labelled by potential divisors $k$. Each colored disc means that given $k$ actually divides $n$, otherwise there is small black circle. To better visualize the whole pattern discs are in 3 different colors and lines connecting them are drawn. Of course, above the diagonal ($k > n$)
6 Concluding remarks

A few elementary comments at the end of this note. As we have seen, all divisors $k$ of integers $n$ lie on rays passing through the origin of the coordinate system on the $(n,k)$ plane and are labelled by an integer parameter $j = 1, 2, 3...$

$$n = jk$$

(30)

We have also seen that this simple condition has a natural interpretation since $j$ may be identified with the exponent in $x^j$ in the expansion \[17\]. The key thing is that these rays must pass through certain points of an integer lattice and only then a potential divisor can be an actual divisor. For large $n$ these rays typically get closer and closer to one another. Therefore we qualitatively see why it is so difficult to factorize large integers.

Moreover, numerical experiments suggest that all divisors lie on countable families of parabolas passing through the origin (see Figures 3, 4 and 5 below). These parabolas are "quantized" in the sense that each family is characterized by two discrete parameters $\mu = 1, 2, 3...$ and $\nu = 1, 2, 3...$ and inside any family parabolas are labelled by another integer parameter $i$:

$$g_i^{(\mu\nu)}(k) = -\frac{\mu}{\nu}k^2 + \frac{i}{\nu}k$$

(31)

Careful simulations using Mathematica revealed that parameter $i$ assumes equidistant values with integer constant step:

$$\delta = \gcd(\mu, \nu)$$

(32)

starting from $i = \mu + \nu$ where gcd denotes greatest common divisor, i.e. $i = \mu + \nu$, $\mu + \nu + \delta$, $\mu + \nu + 2\delta$, ...
Figure 3. Various families of parabolas for $\mu = 1, 2, 3$ and $\nu = 1, 2, 3$. Step $\delta$ (32) described in the main text is also indicated.
Figure 4. Family of parabolas for $\mu = 1$ and $\nu = 1$ (red), 2 (orange), 3 (green), and 4 (cyan) for $n < 100$. For clarity of the plot parameter $i$ assumes only 50 consecutive values. Prime numbers among $n$s are indicated by vertical lines.
Figure 5. Family of parabolas (31) for $\mu = 1$ and $\nu = 1$ (red), 2 (yellow) and 3 (green) around $n = 740$. For clarity of the plot parameter $i$ assumes only 5 consecutive values. Prime numbers among $n$s are indicated by vertical lines.

As far as I am aware the unexpected parabolas in the distribution of divisors have been independently noticed by Jeffrey Ventrella (see his popular book [7], page 33) but with no quantitative considerations.

Finally, it should be stressed that, unfortunately, expressions presented in this note do not tell us much about distribution of primes. They are even not very suitable for numerical calculations for large $n$ therefore may be treated merely as a curiosity. Nevertheless, we have shown some unexpected relationship between number theory and calculus.

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The results presented in this paper were inspired by experimenting with Wolfram Mathematica. Also all calculations were checked using this powerful soft-
ware.

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