A NOTE ON LINEAR HIGHER CHOW GROUPS

MUXI LI

ABSTRACT. We give a counterexample to the proof in [dJ02] of the existence of linear representatives of higher Chow groups of number fields.

1. Introduction

Higher Chow groups were introduced by Bloch [Bl86] three decades ago to geometrize Quillen’s higher algebraic K-theory. Let X be a quasi-projective variety over an infinite field k. Writing \( \Delta^m_k := \mathbb{P}^m_k \setminus H \cong \mathbb{A}^m_k \), where \( H := \{ x_0 + \cdots + x_m = 0 \} \cong \mathbb{P}^{m-1} \), denote by \( Z^p(X, m) \) the free abelian group on closed irreducible subvarieties of \( X \times \Delta^m_k \) of codimension \( p \), properly intersecting each face \( X \times \partial_i \Delta^m_k \) (given by \( x_i = 0 \forall i \in I \)). One then defines \( \text{CH}^p(X, m) \) to be the \( m \)th homology of the complex

\[ (Z^p(X, \bullet), \partial := \Sigma_{i=0}^m (-1)^i \rho_i^*) \]

where \( \rho_i : \Delta^{\bullet-1} \hookrightarrow \Delta^\bullet \) is the inclusion of \( \partial_i \Delta^\bullet \). If \( X \) is smooth and \( k \) is a subfield of \( \mathbb{C} \), one has Bloch’s Abel-Jacobi maps

\[ \text{AJ} : \text{CH}^p(X, m) \to H^{2p-m}_n(X^\text{an}_{\mathbb{C}}, \mathbb{Z}(p)) \]

into absolute Hodge cohomology, which may be described \((\otimes \mathbb{Q})\) in terms of explicit maps of complexes \( \tilde{\text{AJ}} \) [BKLL18]. The homology of the subcomplex \( LZ^p(X, \bullet) \) given by equations linear in the \( \{ x_i \} \) defines the linear higher Chow groups \( L\text{CH}^p(X, m) \), which map naturally to \( \text{CH}^p(X, m) \).

This note concerns the case \( \text{CH}^p(k, m) \) of a point over a number field, where \( X = \text{Spec}(k) \). Working \( \otimes \mathbb{Q} \), this is zero unless \( (p, m) = (n, 2n-1) \), in which case \( \text{CH}^n(k, 2n-1)_\mathbb{Q} \cong K_{2n-1}(k)_\mathbb{Q} \cong K_{2n-1}(\mathcal{O}_k)_\mathbb{Q} \). The linear group \( L\text{CH}^n(k, 2n-1)_\mathbb{Q} \) is (for each \( n \geq 1 \)) the image of a canonical homomorphism

\[ \psi_n : H_{2n-1}^n(\text{GL}_n(k), \mathbb{Q}) \to \text{CH}^n(k, 2n-1)_\mathbb{Q}, \]

induced by the morphism of complexes

\[ \tilde{\psi}_n : C^{\text{grp}}_n(n) \to Z^n(k, \bullet)_\mathbb{Q} \]

given (for \( \bullet = m \)) by

\[ (g_0, \ldots, g_m) \mapsto \left\{ \sum_{i=0}^m x_i \cdot g_i \cdot \mathbf{x} = 0 \right\} \subset \Delta^m \]
for some choice of $v \in k^n \setminus \{0\}$. (Here we consider $C^\Grp_i$ resp. $Z^n(k,i)$ to be in degree $-i$.) Now given an embedding $\sigma : k \hookrightarrow \mathbb{C}$, the Bloch-Beilinson regulator map (i.e., $AJ$ composed with projection $\mathbb{C}/\mathbb{Q}(n) \to \mathbb{R}$) sends $\text{CH}^n(\sigma(k), 2n-1)_{\mathbb{Q}} \to \mathbb{R}$, so that composing with all $r = [k : \mathbb{Q}] = r_1 + 2r_2$ embeddings maps $\text{CH}^n(k, 2n-1) \to \mathbb{R}^r$. This factors through the invariants $\mathbb{R}^{d_n} [d_n := r_2 (n \text{ even}) \text{ resp. } r_1 + r_2 (n \text{ odd})]$ under de Rham conjugation, and is known to be equivalent to $\frac{1}{2}$ the Borel regulator $r_{\text{Bo}} : K_{2n-1}(\mathcal{O}_k)_{\mathbb{Q}} \to \mathbb{R}^{d_n}$ [Bu02].

Given the close relation between homology of $\text{GL}_n$ and the original context of Borel’s theorem, it is natural to consider the composite morphism of complexes $\tilde{AJ} \circ \tilde{\psi}_n$. Replacing $k$ by $\mathbb{C}$, these should yield explicit cocycles in $H^{2n-1}_{\text{meas}}(\text{GL}_n(\mathbb{C}), \mathbb{C}/\mathbb{Z}(n))$ lifting the Borel classes in $H^{2n-1}_{\text{cont}}(\text{GL}_n(\mathbb{C}), \mathbb{R})$ [BKLL18]. This would also deepen our understanding of the equivalence of the Beilinson and Borel regulators. The first test of this proposal is to check its simplest implication:

**Conjecture 1.** For a number field $k$, the linear higher Chow cycles surject (rationally) onto the simplicial higher Chow groups. Equivalently, $\psi_n$ is surjective for every $n \geq 1$.

### 2. A STRATEGY FOR SURJECTIVITY?

In fact, Conjecture 1 is claimed as Proposition 16 in R. de Jeu’s paper [dJ02]. His approach is to fit (for each $n \geq 1$) $\tilde{\psi}_n$ into a commuting triangle

\[
\begin{align*}
C^\Grp_i(n) & \xrightarrow{\tilde{\psi}_n} Z^n(k, \bullet) \\
\downarrow & \downarrow \\
\mathbb{R}^{2n-1} & = \mathbb{R}^{2n-1}.
\end{align*}
\]

Taking homology yields the diagram

\[
\begin{align*}
H_{2n-1}(\text{GL}_n(k), \mathbb{Q}) & \xrightarrow{\psi_n} \text{CH}^n(k, 2n-1)_{\mathbb{Q}} \\
\downarrow & \downarrow \\
\mathbb{R} & \xrightarrow{r_{\text{Be}}} \mathbb{R}^{d_n},
\end{align*}
\]

in which $r_{\text{Bo}}$ [resp. $r_{\text{Be}}$] is the Borel [resp. Beilinson] regulator, composed with a choice of embedding $k \hookrightarrow \mathbb{C}$. By composing with all embeddings (and using Borel’s theorem), we get a diagram of the form

\[
\begin{align*}
H_{2n-1}(\text{GL}_n(k), \mathbb{R}) & \xrightarrow{\psi_n} \text{CH}^n(k, 2n-1)_{\mathbb{R}} \\
\downarrow & \downarrow \\
\mathbb{R} & \xrightarrow{r_{\text{Be}}} \mathbb{R}^{d_n},
\end{align*}
\]

proving Conjecture 1.
The problem here is with de Jeu’s choice of Goncharov’s simplicial regulator $r_{\text{Gon}}$ for $\tilde{r}_{\text{Be}}$. While this appears to make (1) commute, by the calculation on pp. 228-230 of [dJ02], it is now known [BKLL18] that $r_{\text{Gon}}$ is not a map of complexes. Specifically, in

$$
\cdots \to Z^2(k, 2n) \xrightarrow{\partial} Z^n(k, 2n - 1) \xrightarrow{r_{\text{Gon}}} Z^n(k, 2n - 2) \to \cdots
$$

we do not have $r_{\text{Gon}}(\partial C_{2n}) = 0$. So we must replace $r_{\text{Gon}}$ by the “corrected” version in [BKLL18], which we will denote by $\text{reg}_G$. It is given on

$$
\text{reg}_G(Y) := \int_{Y(C)} r_{2n-1} \left( \frac{x_1 + \cdots + x_{2n-1}}{-x_0}, \frac{x_2 + \cdots + x_{2n-1}}{-x_1}, \ldots, \frac{x_{2n-2}}{x_{2n-2}} \right),
$$

which is known to induce $\tilde{r}_{\text{Be}}$.

On the group homology side, de Jeu [dJ02] also uses a formula of Goncharov for $\tilde{r}_{\text{Bor}}$; we denote this by $\text{reg}_B$. Given $(g_0, \ldots, g_{2n-1}) \in C^{\text{grp}}_{2n-1}(n)$, let $\{f_i\}_{i=1}^{2n-1}$ denote nonzero rational functions on $P^{n-1}_C$ with divisors

$$
D_i = \{ [X] \in P^{n-1} | (X_0, \ldots, X_{n-1}) \cdot g_i = 0 \} - \{ [X] \in P^{n-1} | (X_0, \ldots, X_{n-1}) \cdot g_0 = 0 \}.
$$

Then according to [Go04, Thm. 5.12],

$$
\text{reg}_B(g_0, \ldots, g_{2n-1}) := \int_{P^{n-1}_C} r_{2n-1}(f_1, \ldots, f_{2n-1})
$$

induces $r_{\text{Bor}}$. At least in the $n = 2$ case we treat below, this formula is correct. (See the calculation in §3 below.) Moreover, it is well-defined for any $n$, in the sense that the RHS of (5) is invariant when we rescale any $f_i$ by a constant.

We tried to emulate the approach in [dJ02] to see if the new diagram (1) (with $\tilde{r}_{\text{Bor}} = \text{reg}_B$ unchanged and $\tilde{r}_{\text{Be}}$ corrected to $\text{reg}_G$) commutes, with no success. At this point, we decided to attempt the first nontrivial case by hand, and arrived at a negative result:

**Proposition 2.** For $n = 2$, the amended triangle (1) does not commute.

### 3. Proof of Proposition 2

In [Go04], Goncharov mentions the formula

$$
\int_{P^1} r_3(f_1, f_2, f_3) = \sum_{(x_1, x_2, x_3) \in C^3} \nu_{x_1}(f_1)\nu_{x_2}(f_2)\nu_{x_3}(f_3)D_2(CR(x_1, x_2, x_3, \infty))
$$

where $\nu_x(f)$ is the order of $f$ at $x$. One easily verifies that this is correct; it will be required below.
Now take \( \mathbf{v} = \left( \begin{array}{c} 1 \\ 0 \end{array} \right) \in \mathbb{C}^2 \), and \((g_0, g_1, g_2, g_3) \in C_3(2)\). We can do a change of coordinate to let \( g_0 = \left( \begin{array}{cc} 1 & * \\ 0 & 1 \end{array} \right) \), \( g_1 = \left( \begin{array}{cc} 0 & * \\ 1 & 1 \end{array} \right) \), \( g_2 = \left( \begin{array}{cc} a & * \\ c & 1 \end{array} \right) \), \( g_3 = \left( \begin{array}{cc} b & * \\ d & * \end{array} \right) \). For convenience, we set \( \Delta := ad - bc \).

Write \( z := \frac{x_1}{x_0} \) and \( f_1(z) = z \), \( f_2(z) = cz + a \), and \( f_3(z) = dz + b \). According to (6) and (7), we have

\[
\text{reg}_B(g_0, g_1, g_2, g_3) = \int_{\mathbb{P}^1} r_3(z, cz + a, dz + b) = D_2 \left( \frac{\Delta}{ad} \right).
\]

This is consistent with evaluating the cocycle \( \varepsilon_2 \in H^3_{\text{cont}}(GL_2(\mathbb{C}), \mathbb{R}) \) (cf. Intro. to [BKLL18]) on the “group homology chain” \((g_0, g_1, g_2, g_3)\).

For the other side, applying \( \psi \) to this chain produces the linear higher Chow chain \( Y \subset \Delta^3 \) cut out by

\[
x_0 + ax_2 + bx_3 = 0 \quad \text{and} \quad x_1 + cx_2 + dx_3 = 0.
\]

Parametrizing \( Y \cong \mathbb{P}^1 \) by \( t \mapsto (\Delta, \Delta t, bt - d, c - at) \), (5), (7) and the rescaling property yield \( \text{reg}_G(\tilde{\psi}(g_0, g_1, g_2, g_3)) = \)

\[
\text{reg}_G(Y) = \int_{\mathbb{P}^1} r_3 \left( \frac{(d-c) + (a-b-\Delta)t}{\Delta}, \frac{(c-d) + (a-b)t}{\Delta t}, \frac{at-c}{bt-d} \right)
= \int_{\mathbb{P}^1} r_3 \left( (c-d) + (\Delta + b - a)t, \frac{(c-d) + (b-a)t}{t}, \frac{at-c}{bt-d} \right)
= D_2 \left( \frac{(d-1)\Delta}{b(c-d)} \right) - D_2 \left( \frac{(c-1)\Delta}{a(c-d)} \right) - D_2 \left( \frac{(b-a)(d-1)}{b(d-c)} \right) + D_2 \left( \frac{(b-a)(c-1)}{a(d-c)} \right).
\]

To check that these two results disagree, put \( a = 1 \), \( b = -1 \), \( c = 1 - i \), \( d = 1 + i \), so that \( \Delta = 2 \) and \( \frac{ad}{bc} = -i \). Of course, \( D_2(-i) \neq 0 \). On the other hand,

\[
\frac{(d-1)\Delta}{b(c-d)}, \frac{(c-1)\Delta}{a(c-d)}, \frac{(b-a)(d-1)}{b(d-c)}, \frac{(b-a)(c-1)}{a(d-c)}
\]

are all 1, \( D_2 \) of which is 0.

4. Concluding remarks

Naturally, it is still possible that (2) commutes, since there we restrict to closed chains. In fact, even if we don’t accept the proof in [dJ02], there is the earlier result of Gerdes [Ge91] which gives surjectivity of \( \psi_n \) for \( n = 2 \). Moreover, there is the agreement between the Beilinson and Borel regulators in [Bu02], though this does not involve \( \psi_n \) in any way. To sum up, we conclude with the

**Question 3.** Are there any techniques to prove that (2) commutes even though the amended diagram (1) does not, for \( n = 2 \) and more generally? Or is it more likely that \( \psi_n \) has to be somehow modified?
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