Factorization of Non-Commutative Polynomials

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Abstract

We describe an algorithm for the factorization of non-commutative polynomials over a field. The first sketch of this algorithm appeared in an unpublished manuscript (literally hand written notes) by James H. Davenport more than 20 years ago. This version of the algorithm contains some improvements with respect to the original sketch. An improved version of the algorithm has been fully implemented in the Axiom computer algebra system.

1 Introduction

We describe an algorithm for the factorization of non-commutative polynomials over a field, the first version of which was described but never published by James H. Davenport more than 20 years ago. He wrote these notes [Dav] on the occasion of a visit to Pisa for a series of lectures and later forgot them. These (hand-written) notes have been kept by Teo Mora (Univ. of Genoa) who passed them to Carlo Traverso (Univ. of Pisa) who passed them to me.

The main ideas of the original algorithm and an application to cryptanalysis have been treated in [CCT08], where we have shown how to construct an attack on the non-commutative Polly-Crackers [FK94] proposed in [Rai04].

We are considering a free $K$-algebra over a field $K$ for which effective polynomial system solving is possible. We do not treat the cases where some algebraic relations on the elements are imposed, such as for example the case of linear differential operators ([Bro94], [vdPS03]). The problem of factorizing a non-commutative polynomial in this setting is clearly solvable by a brute-force approach. Our algorithm if applied to univariate polynomials coincides with the brute-force approach, but it performs much better in the multivariate case.

We give the full details of the algorithm, two improvements of the original algorithm and we present its complete implementation in the Axiom computer algebra system. The implementation will be submitted to the Axiom maintainers for inclusion in the next version of the system.

In order to simplify the description we will only consider the problem of factorizing a polynomial in two factors of given total degrees.
2 The Homogeneous Case

The homogeneous case is a very special case. The algorithm is simple and its correctness is self-evident: we construct the two factors of a given polynomial \( F \) by selecting a monomial \( m = m_1m_2 \) and taking the sum of the monomials left-divisible by \( m_1 \) and those right-divisible by \( m_2 \).

**Algorithm 1** Homogeneous Non-Commutative Factorization

**Require:** A homogeneous polynomial \( F \) of degree \( n \);
the desired degrees \( h \) and \( k \) for the factorization

**Ensure:** Either a negative answer or the two factors \( G \) and \( H \) of desired degree

1: “Select” any monomial in \( F \) and factor it in two parts \( \hat{G} \) of degree \( h \) and \( \hat{H} \) of degree \( k \).
2: for all monomials \( M \) in \( F \) do
3: if \( \hat{G} \) left-divides \( M \) then
4: \( H := H + R, \) with \( R \) the left-quotient of \( M \) by \( \hat{G} \)
5: end if
6: if \( \hat{H} \) right-divides \( M \) then
7: \( G := G + L, \) with \( L \) the right-quotient of \( M \) by \( \hat{H} \)
8: end if
9: end for
10: if \( F = G \cdot H \) then
11: Return \( G \) and \( H \)
12: else
13: Return “Irreducible”
14: end if

**Remark 1.** The correctness of Algorithm 1 implies that for given degrees there is a unique factorization. One can prove more: the factorization is essentially unique, i.e. if a polynomial \( F \) is factored as \( F = G_1H_1 = G_2H_2 \) with \( \deg G_1 = i < j = \deg G_2 \) then the two factorizations must have a common refinement: \( F = G_1JH_2 \).

This is proved as follows: we can always assume

\[
G_1 = x_1 \cdots x_i + R_1, \quad G_2 = x_1 \cdots x_j + R_2. \tag{1}
\]

after dividing by a suitable element of the field.

We can now consider the polynomial \( J_0 \) given by the sum of the elements in

\[
\{ \text{monomial } m \text{ in } G_2 \text{ left-divisible by } x_1 \cdots x_i, \text{ i.e. of the form } cx_1 \cdots x_iy_1 \cdots y_{j-i} \}. \tag{2}
\]
We have that the set of monomials \( m \) in \( F \) that are left-divisible by \( x_1 \cdots x_i \) can be obtained as \( J_0 H_2 \) as well as \( x_1 \cdots x_i H_1 \).

Therefore we can take the polynomial \( J \) given by the sum of the elements in

\[
\{ \text{left-quotient of an element of } J_0 \text{ by } x_1 \cdots x_i, \]
\[
\text{i.e. a monomial of the form } cy_1 \cdots y_{j-i} \}
\]

from which follows

\[
x_1 \cdots x_i H_1 = x_1 \cdots x_i J H_2
\]

that implies \( H_1 = J H_2 \) and \( G_2 = G_1 J \).

3 The General Case

The general case is more complicated because the factorization is not unique anymore even if we fix the degrees of the factors (see Subsection 3.3 for a simple example).

3.1 Exponential Growth of the Number of Factorizations

Teo Mora (Univ. of Genoa) has noticed that for any univariate polynomial \( f(t) \) that is factored as \( f_1(t) \cdots f_k(t) \) (with all distinct factors), if we consider \( f(XY) \) we have the following non-commutative factorizations for the polynomial \( Y f(XY) \):

\[
Y f(XY) = Y f_1(XY) \cdots f_k(XY) =
\]
\[
f_1(YX) Y \cdots f_k(XY) = f_1(XY) \cdots Y f_k(XY).
\]

In such a way Teo Mora proves an exponential lower-bound on the number of factorizations with respect to the degree. However if we homogenize an inhomogeneous polynomial we are left with just one possible factorization. This is explained by the fact that there are different ways to homogenize.

Example 1. Clearly \( x^2 - 1 \) is factored as \( (x-1)(x+1) \) but it can be homogenized as \( x^2 - y^2 \), which is irreducible, or as \( x^2 - xy + xy - y^2 \), which is factored as \( (x-y)(x+y) \), which corresponds to the factorization \( (x-1)(x+1) \).

Remark 2. The finiteness of the factorization is still unproved. A formal proof could be achieved by proving the 0-dimensionality of the system produced by the algorithm.
3.2 The General Algorithm

Let us consider the problem of factorizing $F$ of degree $n$ as $F = GH$, with $G$ of degree $h$ and $H$ of degree $k$. The main idea of the algorithm is to use the relations between the homogeneous parts $F_{n-j}$ (of degree $n-j$) of $F$ and the homogeneous parts $G_{h-j}$ (of degree $h-j$) of $G$, $H_{k-j}$ (of degree $h-j$) of $H$:

$$
F_n = G_h H_k
$$

$$
F_{n-1} = G_h H_{k-1} + G_{h-1} H_k
$$

$$
F_{n-2} - G_{h-1} H_{k-1} = G_h H_{k-2} + G_{h-2} H_k
$$

$$
F_{n-3} - G_{h-1} H_{k-2} - G_{h-2} H_{k-1} = G_h H_{k-3} + G_{h-3} H_k
$$

(6)

It is possible to determine $G_{h-j}$ and $H_{k-j}$ in the right hand side by “inspection” of the left hand side (similarly to the homogeneous case, by searching for monomials that have certain “substrings”). The main difference from the homogeneous case is that we must take into account possible cancellations of terms in the right-hand side, which corresponds to possible partial overlaps of monomials in $G_h$ and $H_k$. For each possible cancellation between $G_h H_{k-j}$ and $G_{h-j} H_k$ we introduce new “symbols”, i.e. an extension of our ground field. The subsequent relations will determine algebraic relations on the new elements of the field that will produce a system of polynomial equation, which we can solve by a Gröbner basis computation.

Remark 3. The description of the possible values of the new symbols is in general provided by a system of polynomial equations on the new symbols. Our implementations allow to choose whether the system should be normalized in the form of the reduced lexicographic Gröbner basis.

3.3 One Interesting Simple Example

Let us consider $K = \mathbb{F}_p$, with $p > 2$ and $F := yxyxy - y$. We can use our algorithm to factorize $F$ in two factors of degree 2 and 3.

Our procedure for this example could be summarized as follows

1. The head $F_n$ of $F$ is just the monomial $yxyxy$ which is factored in $G_2 := yx$ and $H_3 := yxy$.
2. From $F_4 = 0 = yx H_2 + G_1 yxy$ we get $H_2 = G_1 = 0$.
3. From $F_3 - G_1 H_2 = 0 = yx H_1 + G_0 yxy$ we detect a possible cancellation in the right hand side, which forces us to introduce a new symbol $\alpha$ as possible coefficient, which implies $H_1 := \alpha y$, $G_0 = -\alpha$.
4. From $F_2 - G_1 H_1 - G_0 H_2 = 0 = yx H_0$ we get $H_0 = 0$.
5. By using the relation $F = yxyxy - y = (\sum_{i=0}^{2} G_i)(\sum_{j=0}^{3} H_j) = (yx - \alpha)(yxy + \alpha y)$ we get $\alpha^2 = 1$ we gives the two different solutions to the factorization $(yx \pm 1)(yxy \mp y)$. 


Algorithm 2 Non-Commutative Factorization

Require: A polynomial $F$ of degree $n$; the desired degrees $h$ and $k$ for the factorization

Ensure: The list of possible factorization of $F$ in $G_h$ and $H_k$ in two parts of degree $h$ and $k$

1: Use Algorithm 1 to factorize the homogeneous part of $F$ of highest degree in $G_h$ and $H_k$
2: “Select” monomials $\hat{G} = x_1 \ldots x_h$ of $G_h$, and $\hat{H} = y_1 \ldots y_k$ of $H_k$
3: $\hat{F}_{n-j} := F_{n-j}$ for all $j = 1 \ldots n$.
4: for $j$ in $1 \ldots n$ do
5: $\hat{F}_{n-j} := \hat{F}_{n-j} - \sum_{i=1}^{j-1} G_{h-i}H_{k-j+i}$
6: if the last $j$ variables $x_{h-j+1} \ldots x_h$ of $\hat{G}$ are equal to the first $j$ variables $y_1 \ldots y_j$ of $\hat{H}$ then
7: Consider the coefficient $c$ of $x_1 \ldots x_{h-j}y_{j+1} \ldots y_k$ in $\hat{F}_{n-j}$
8: $\hat{F}_{n-j} := \hat{F}_{n-j} - cx_1 \ldots x_{h-j}y_{j+1} \ldots y_k$
9: $\tilde{K} = \tilde{K}(\alpha)$ for a new symbol $\alpha$
10: $G_{h-j} := G_{h-j} + \alpha x_1 \ldots x_{h-j}y_{j+1} \ldots y_k$
11: $H_{k-j} := H_{k-j} + (c - \alpha)x_1 \ldots x_{h-j}y_{j+1} \ldots y_k$
12: end if
13: for all monomials $M$ in $\hat{F}_{n-j}$ do
14: if $\hat{G}$ left-divides $M$ then
15: $H_{h-j} := H_{h-j} + dR$, with $R$ the left-quotient of $M$ by $\hat{G}$
16: end if
17: if $\hat{H}$ right-divides $M$ then
18: $G_{k-j} := G_{k-j} + dL$, with $L$ the right-quotient of $M$ by $\hat{H}$
19: end if
20: end for
21: end for
22: Consider $G := \sum_{i=0}^{h} G_i$ and $H := \sum_{i=0}^{k} H_i$
23: Find the possible values of the new symbols such that $F = GH$
24: Return $G$, $H$ and a description of all possible values of the new symbols
4 Improvements

The main problem of non-commutative factorization is the exponential number of cases to be considered. Our improvements reduce the number of possible cases to be considered by

1. reducing the number of algebraic extensions;
2. reducing the number of possible factorizations in two factors.

4.1 Reducing the Extensions

It is possible to avoid the introduction of extensions in the coefficient field by carefully choosing $\hat{G}$ and $\hat{H}$ in Algorithm 2. In particular we must choose them in a way that reduces the number of overlaps between them, since for each overlap between the last part of $G$ and the first of $H$ of length $j$ we are forced to consider a possible cancellation which produces a new extension.

4.2 Commutative Images

It is possible to immediately detect some impossible factorization by considering the commutative version of the given polynomial and its commutative factors. If the commutative polynomial has the same degree as the original one, then the commutative factors are in general a refinement of the possible non-commutative factorization and therefore we can greatly limit the number of cases to be considered. A very simplified version of this idea could be used as follows: if the commutative version of a given polynomial $F$ can be factored in irreducible factors of degree $a_1, \ldots, a_k$, then we need only consider as possible non-commutative factors those of degrees $b_1$ and $b_2$ where $b_2 = \deg F - b_1$ (i.e. $b_1$ is a sum of some $a_i$).

This simple approach can be extended to the other cases except when the commutative image is 0, by considering the commutative homogeneous parts of the original polynomial and use them to reduce the cases. Another improvement may come from considering other quotients of the algebra.

5 The Axiom Implementation

We have implemented Algorithm 2 and its improvements in the Axiom computer algebra system. The choice of the Axiom computer algebra system is due to the flexibility of this system and to the fact that Axiom already provides constructors for non-commutative algebraic structures. In particular MonoidRing provides a constructor for polynomials over any monoid and any coefficient ring, and FreeMonoid provides one for free monoids, which is exactly what is needed in our case.
This package is going to be part of standard Axiom but before this happens you will need to load the code by typing

\texttt{)r daven.input}

The main command is \texttt{NCFactor} which is used as follows

\texttt{NCFactor(y*x*y*x*y-y*x*y)};

which outputs the list of its factorizations for all possible degrees.

6 Future Work

We plan to fully integrate the non-commutative factorization in the next version of the Axiom computer algebra system. There are two open questions we will be working on: a formal proof (see Remark 2) of the finiteness of factorizations and the question whether other quotients of the algebra can be used.

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