The canonical geometry of a Lie group

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Abstract

An abstract Lie group $G$ admits many left-invariant metrics and it is well known that these metrics possess drastically different curvature properties. However, $G$ admists a canonical metric if we view $G$ as a flat and globalizable absolute parallelism as in [O1]. We study some surprising consequences of this shift in perspective.

1 Basic concepts

Our main object is a pair $(M, w)$ where $M$ is a smooth manifold and $w = (w^i_j(x))$ is a geometric object on $M$, called the structure object in [O1] and [O2]. To understand the meaning of $w$, let $F(M) \to M$ be the principal frame bundle of $M$ whose fiber over $p \in M$ is the set of 1-jets (which we call 1-arrows) $j_1(f)^o$ of local diffeomorphisms $f$ with the source at the origin $o \in \mathbb{R}^n$ and the target at $f(o) = p$. The 1-arrows with the source and target at $o$ is the Lie group $GL(n, \mathbb{R})$, $n = \dim M$, and acts freely on the fiber over $p$ by composition at the source. Therefore $F(M) \to M$ is a right principal bundle with the structure group $GL(n, \mathbb{R})$. Now suppose that $F(M) \to M$ is trivial and we fix one trivialization $w$ once and for all. For a coordinate neighborhood $(U, x) \subset M$, the unique 1-arrow $j_1(f)^o$ with target at $f(o) = x \in (U, x)$ is of the form $w^i_j(x) = \left[ \frac{\partial f^i(z)}{\partial z^j} \right]_{z = o}$ where $(z^i)$ are the standard coordinates in $\mathbb{R}^n$. By the chain rule, a coordinate change $(U, x) \to (U, y)$ at the target transforms the components $(w^i_j(x))$ according to

$$w^i_j(y) = \frac{\partial y^i}{\partial x^a} w^a_j(x)$$

(1)

Therefore, a trivialization determines a geometric $w$ on $M$ with components $(w^i_j(x))$ on $(U, x)$ subject to (1). Conversely, the geometric object $w = (w^i_j(x))$ subject to (1) defines uniquely a trivialization in the obvious way. Therefore, trivializations and the geometric objects $w$ on $M$ satisfying (1) can be identified and henceforth we will adhere to this identification but remark here that the geometric object viewpoint generalizes in a natural way to geometric structures other than trivializations. Note that if $\tilde{w} = w^{-1}$, i.e., $\tilde{w}^a_i(x)w^i_j(x) = w^a_i(x)\tilde{w}^a_j(x) = \delta^a_j$, then $\tilde{w} = (\tilde{w}^i_j(x))$ is another trivialization subject to
\[ \bar{w}^i_j(y) = \bar{w}^i_a(x) \frac{\partial x^a}{\partial y^j} (2) \]

and (2) is obtained by inverting (1). Clearly, \( \bar{w} \) can be interpreted as the corresponding trivialization of the coframe bundle \( \bar{F}(M) \to M \) whose fiber over \( p \) is the set of 1-arrows with source at \( o \in \mathbb{R}^n \) and \( \bar{F}(M) \to M \) is a left \( GL(n, \mathbb{R}) \)-principal bundle.

Given \( (M, w) \), there are three objects canonically associated with \( w \) which will play a fundamental role below. The first is a very special groupoid \( \Upsilon \) on \( M \). For \( p, q \in M \), we define a 1-arrow \( \varepsilon(p, q) \) from \( p \) to \( q \) by composing the inverse of the 1-arrow from \( o \) to \( p \) with the 1-arrow from \( o \) to \( q \). This amounts to defining \( \varepsilon(p, q) \) in coordinates as

\[ \varepsilon_{ij}(x, y) = w^i_a(y) \bar{w}^a_j(x) (3) \]

We easily check that the 1-arrows defined by \( \varepsilon \) are closed under composition and inversion of arrows and we obtain a groupoid \( \Upsilon \) as a subgroupoid of the universal jet groupoid \( U_1 \). By construction, \( w \) is left invariant by the arrows of \( \Upsilon \), i.e., \( \Upsilon \subset U_1 \) is the invariance subgroupoid of \( w \). Part 1 of [O1] and the second chapter of [O2] are devoted to a detailed study of \( \Upsilon \) which we will assume henceforth. It is crucial to observe that different \( w \) may define the same groupoid \( \Upsilon \).

The second object naturally associated with \( w \) is the canonical metric \( g \) on \( M \) whose value \( g(p) \) at \( p \in M \) is obtained by mapping the standard Euclidean metric of \( \mathbb{R}^n \) to \( p \in M \) using the 1-arrow of the trivialization \( w \) from \( o \) to \( p \). In coordinates, we have

\[ g^{ij}(x) \overset{\text{def}}{=} \sum_{1 \leq a \leq n} w^i_a(x) w^a_j(x) \quad g_{ij}(x) \overset{\text{def}}{=} \sum_{1 \leq a \leq n} \bar{w}^a_i(x) \bar{w}^a_j(x) \quad \bar{w} = w^{-1} (4) \]

Clearly \( g = (g_{ij}(x)) \) is symmetric, positive definite and also \( \Upsilon \)-invariant. Now if we fix the trivialization \( w \), then it is easily checked that the metrics of \( Aw \) range over all \( \Upsilon \)-invariant metrics on \( M \) as \( A \) ranges over \( GL(n, \mathbb{R}) \). If \( \mathcal{R} = 0 \) and the groupoid \( \Upsilon \) integrates to a globalizable \( \mathcal{G} \), i.e., if the pseudogroup \( \mathcal{G} \) of local solutions globalizes to a transitive transformation group \( \mathcal{G} \) of \( M \) acting simply transitively on \( M \), then these metrics are of course also \( \mathcal{G} \)-invariant.

Using \( \mathcal{G} \), we can now define a Lie group structure on \( M \) in such a way that \( \mathcal{G} \) becomes left or right translations according to our choice and therefore all these metrics become left or right invariant. As a surprising fact, it turns out that these metrics have drastically different curvature properties (see the excellent survey article [M] and the recent book [AB]). However, if we shift our focus from metrics to transformations, i.e., from Riemannian geometry to Lie theory...
as proposed in [O1] where an abstract Lie group is by definition a flat and globalizable $w$ modulo the choices of left/right and a unit, then the primary object for us is $w$ which has the canonical metric (4)! Though it does not interest here, a trivialization $(M, w)$ has also a canonical symplectic form, almost complex structure...etc...any first order structure which has a canonical meaning in $\mathbb{R}^n$ carries over $M$ by $w$.

Finally, the third object naturally associated with $w$ is the integrability object $I = (I^i_{jk}(x))$ defined by

$$I^i_{jk}(x) \overset{\text{def}}{=} \left[ \frac{\partial w^i_a(x)}{\partial x^j} \tilde{w}_k^a(x) \right]_{[jk]} = \frac{\partial w^i_a(x)}{\partial x^j} \tilde{w}_k^a(x) - \frac{\partial w^i_a(x)}{\partial x^k} \tilde{w}_j^a(x)$$  

Both the linear and nonlinear curvatures $\mathfrak{R}$ and $\mathcal{R}$ in [O1] are determined by $I$. Now $w$ is $\Upsilon$-invariant by definition but $I$ is not necessarily $\Upsilon$-invariant. It turns out that $\mathfrak{R} = 0 \iff \mathcal{R} = 0 \iff I$ is $\Upsilon$-invariant ([O1], [O2]).

In Part 1 of [O1] we introduced two linear connections $\tilde{\nabla}, \nabla$ both determined by $w$ but in [O2] we showed that these connections are consequences of more fundamental concepts: $\tilde{\nabla}$ is formal Lie derivative and $\nabla$ is the Spencer operator. Therefore, it is possible to develop the whole theory without even mentioning the word “linear connection” and this approach applies to all geometric structures ([O2]). However, since geometers are much familiar with linear connections, we will continue to use this interpretation below.

2 The curvature of a local Lie group

We start with the first Bianchi identity (Proposition 6.4 in [O1])

$$R^i_{kj,r} + R^i_{jr,k} + R^i_{rk,j} = I^i_{kj} I^r_{ar} + I^i_{jr} I^r_{ak} + I^i_{rk} I^r_{aj}$$  

We recall that $\mathfrak{R}$ is the linear curvature of $(M, w)$ defined by $\tilde{\nabla}_r I^i_{kj} = \mathfrak{R}^i_{kj,r}$ in [O1] (a more conceptual definition is given in [O2]) and it turns out that $I$ and $\mathfrak{R}$ are the torsion and curvature of the linear connection $\tilde{\nabla}$. We define

$$S^i_{kj,r} \overset{\text{def}}{=} I^a_{kj} I^i_{ar}$$  

We observe that we differentiate $w$ twice to define $\mathfrak{R}$ whereas we differentiate $w$ only once to define $S$. This suggests that $S$ is a more fundamental object than $\mathfrak{R}$.

**Definition 1** $S$ is the primary curvature of $(M, w)$.

Obviously $S$ satisfies

$$S^i_{kj,r} = -S^i_{jk,r}$$  

Now suppose $(M, w)$ is a local Lie group (LLG), i.e., $\mathfrak{R} = 0$. In this case we denote $(M, w)$ by $(M, w, \mathcal{G})$ where $\mathcal{G}$ is the pseudogroup obtained by locally
integrating the 1-arrows of the groupoid \( \Upsilon \). Now the left hand side (LHS) of (6) vanishes and \( S \) satisfies also

\[
S_{k,j,r} + S_{j,r,k} + S_{r,k,j} = 0 \tag{9}
\]

We define

\[
S_{k,j,r} \overset{\text{def}}{=} S_{k,j,r} g_{ai} \tag{10}
\]

The main idea of this note is based on the following elementary observation.

**Lemma 2** On a LLG \((M, w, \mathcal{G})\), we also have

\[
S_{k,j,r} = -S_{k,j,r} \tag{11}
\]

Since the proof we will make use \( \nabla, \tilde{\nabla} \), it is useful to use a notation coherent with these operators. So we make the definitions

\[
\Gamma^i_{jk} \overset{\text{def}}{=} \frac{\partial w^a_i}{\partial x^j} \tilde{w}^a_k(x) \quad T^i_{jk} \overset{\text{def}}{=} \Gamma^i_{jk} - \Gamma^i_{kj} = I^i_{jk} \tag{12}
\]

Note that \( \Gamma = (\Gamma^i_{jk}) \) are the components of \( \nabla \) and not the Christoffel symbols of the canonical metric \( g \). In particular \( \Gamma^i_{jk} \) is not necessarily symmetric in \( j, k \).

As we remarked above, \( I = T = (T^i_{jk}) \) and \( R = (R^i_{jk,r}) \) are the torsion and curvature of \( \nabla \). Now a straightforward computation using the definitions gives the formula (called the Ricci formula in the old books)

\[
\nabla_s \nabla_r g_{ij} - \nabla_r \nabla_s g_{ij} = -R^t_{sr,ij} - R^t_{sr,ji} - T^t_{sr} \nabla_a g_{ij} \tag{13}
\]

We also have

\[
\nabla_r g_{ij} = \tilde{\nabla}_r g_{ij} + T^a_{ir} g_{aj} + T^a_{jr} g_{ai} \tag{14}
\]

since \( g \) is \( \Upsilon \)-invariant and therefore \( \tilde{\nabla} \)-parallel. Substituting (14) into (13) gives

\[
\nabla_s \nabla_r g_{ij} - \nabla_r \nabla_s g_{ij} = -T^a_{sr} T^b_{ia} g_{bj} - T^a_{sr} T^b_{ja} g_{ia} \tag{15}
\]

Our purpose is to show that the RHS of (15) vanishes. Now we will compute the LHS of (15) in a different way. Applying \( \nabla_s \) to (14) gives

\[
\nabla_s \nabla_r g_{ij} = (\nabla_s T^a_{ir}) g_{aj} + T^a_{ir} \nabla_s g_{aj} + (\nabla_s T^a_{jr}) g_{ai} + T^a_{jr} \nabla_s g_{ai} \tag{16}
\]

We have

4
Now we substitute (17) and (14) into (16), alternate $s, r$ in (16) and equate it to (15). Simplifying the resulting equality using the Bianchi identity (6) and
\[
\tilde{\nabla}_r T_{jk}^i = \tilde{\nabla}_r T_{jk}^i = R^i_{jk,r} = 0,
\]
after a straightforward computation we obtain the desired result
\[
T_{sr}^a T_{aj}^b g_{bi} = -T_{sr}^a T_{ai}^b g_{bj} \tag{18}
\]

To summarize what we have done so far, we state

**Proposition 3** On a LLG $(M, w, \mathcal{G})$, the primary curvature $S$ satisfies the identities
\[
S_{ij,kr} = -S_{ij,kr} \tag{19}
\]
\[
S_{ij,kr} + S_{jk,ir} + S_{ki,jr} = 0
\]
\[
S_{ij,kr} = -S_{ij,rk}
\]

Therefore $S$ must satisfy also
\[
S_{ij,kr} = S_{kr,ij} \tag{20}
\]
(see [KN] for a coordinate free derivation of this fact).

From (10) and (11) we deduce
\[
S_{ij,kr} = -S_{ij,r} \tag{21}
\]
and therefore
\[
S_{ij,kr}^a = 0 \tag{22}
\]

Now summing over $i, r$ in (6) and using (22), we get
\[
S_{ak,j}^a = S_{aj,k}^a \tag{23}
\]

We define
\[
Ric(S)_{kj}^a \overset{df}{=} S_{ak,j}^a \tag{24}
\]
\[
Ric(S)_{a} = Ric(S)_{ba}g^{ab} \overset{df}{=} K \tag{25}
\]
and note that $Ric(S)_{kj}$ is symmetric by (23).

**Definition 4** $Ric(S)$ is the Ricci curvature and $K$ is the scalar curvature of the LLG $(M, w, \mathcal{G})$. 


At this point, it is natural to suspect that $S$ is the Riemann curvature tensor $R$ of the canonical metric $g$. Clearly, without the assumption $R = 0$, there is no reason to believe this and indeed this belief is not justified. From our standpoint of Lie theory, however, it is more natural to ask first the geometric meaning of $S$ in terms of the group structure of the LLG $(M, w, G)$. For this purpose, we first take a closer look at (14). In addition to $\tilde{\nabla} g = 0$, suppose we also have $\nabla g = 0$. This gives

$$T^a_{rj}g_{ai} = -T^a_{rj}g_{ai}$$

(26)

Multiplying (26) with $T^a_{sm}$ and summing over $r$ gives

$$T^b_{sm}T^a_{aj} = -T^b_{sm}T^a_{bj}$$

(27)

which is (11). Therefore (26) is a stronger condition than (11). If the pseudogroup $G$ globalizes, then the conditions $\tilde{\nabla} g = \nabla g = 0$ are equivalent to the bi-invariance of the metric $g$ with respect to $G$ and its centralizer $C(G)$. Thus we see that Lemma 2 drops the hypothesis of globalizability and bi-invariance, but makes the weaker conclusion (27) rather than (26).

Now we inspect the first formula of (6) more closely. The index $a$ in $w^a_i(x)w^a_j(x)$ represents $\left[ \frac{\partial}{\partial y^a} \right]_{y=x}$ where $y = (y^i)$ are the standard coordinates in $\mathbb{R}^n$. According to (1), $w^a_i(x)$ does not transform in the index $a$ and transforms like a vector field in the index $i$. Identifying the variables $y^1, ..., y^n$ with $1, 2, ..., n$, we now define $n$ global vector fields $v(k)$ on $M$ by

$$v(k) = (w^i_j(x)) = w^a_i(k) \frac{\partial}{\partial x^a} \quad 1 \leq k \leq n$$

(28)

According to (4) and (2), we have

$$g_{ab}w^a_i(k)w^b_j(l) = \left( \sum_{1 \leq c \leq n} \tilde{\nabla}_{a}^c(x)w^c_i(x) \right)w^a_i(k)w^b_j(l)$$

(29)

Therefore the vector fields $v(k)$ are orthonormal. Since $\tilde{\nabla} g = 0$, they are also $\Upsilon$-invariant and form $n$ orthonormal global vector fields on $M$. Note that the assumption $R = 0$ localizes any object $A$ with $\nabla A = 0$ at some arbitrary point and therefore reduces all computations to pure algebra. Clearly, if $R = 0$, then $\nabla S = 0$ and therefore also $\nabla \text{Ric}(S) = 0$ and $\nabla K = 0$. Henceforth we will denote $v(k)$ by $w_k$.

Now we define

$$S_{kl} \overset{\text{def}}{=} -S_{ab,cd}w^a_kw^b_lw^c_kw^d_l$$

(30)
The minus sign will be clear shortly. The RHS of (30) has a coordinatefree meaning since we contract tensors. We observe that $S_{kl}$ are \textit{not the components of a tensor} but are numbers depending on $w_k, w_l$ which are defined canonically on $M$. Since $\nabla w_k = 0$, $1 \leq k \leq n$, and also $\nabla S = 0$ if $R = 0$, on a LLG $(M, w, G)$ we have $S_{kl}(p) = S_{kl}(q)$ for all $p, q \in M$, i.e., $S_{kl}$ is constant on $M$ for all $1 \leq k, l \leq n$. However, note that $S_{kl}$ and $S_{jm}$ need not be the same constants. Clearly $K$ is also constant on $(M, w, G)$.

**Definition 5** $S_{kl}$ is the sectional curvature determined by $w_k$ and $w_l$.

For the moment, $S$ is a function which assigns numbers to the pairs $w_k, w_l$ but it is easy to extend its definition to \textit{all planes}. A miracle is hidden in (30): Summing over $l$ and using (4), we get

$$\sum_{1 \leq l \leq n} S_{kl} = -S_{ab,c}w^c_kw^d_kg^{bd}$$

(31)

$$= -S_{ab,c}w^c_kw^d_k$$

$$= S_{bac}w^c_kw^d_k$$

$$= Ric(S)_{ab}w^a_kw^b_k$$

Summing over $k$ in (31) gives

$$\sum_{1 \leq l, k \leq n} S_{kl} = \sum_{1 \leq k \leq n} Ric(S)_{ab}w^a_kw^b_k = Ric(S)_{ab}g^{ab} = Ric(S)^a_a = K$$

(32)

Turning back to the question of the group theoretic meaning of $S$, we fix a point $p \in M$ and consider all paths $c(t)$ with $c(0) = p$ satisfying the following condition: The translation of the tangent vector $c(0)$ to $c(t)$ by the 1-arrow of $\Upsilon$ from $c(0)$ to $c(t)$ gives the tangent vector $c(t)$ (see [O1], pg.51). These paths are defined for all $t$ and we call them pre-1-parameter subgroups. If $R = 0$ and $G$ globalizes to a Lie group, they become 1-parameter subgroups. Differentiation of the pre-1-parameter condition gives the geodesics of the linear connections $\nabla$ and $\tilde{\nabla}$. Therefore, from our standpoint, pre-1-parameter condition is more fundamental than the geodesic condition. Now we fix a 2-dimensional subspace of $T_pM$. As the tangent vectors range over this 2-dimensional subspace, the pre-1-parameter subgroups spreading out from $p$ with those tangents sweep a surface. Now suppose $R = 0$ and assume for simplicity that $G$ globalizes to a Lie group. Do the transformations of $G$ permute these surfaces? Equivalently, for $p, q \in M$, does the unique transformation of $G$ that maps $p$ to $q$ restrict to the slices passing through $p, q$? If so, we get a 2-dimensional foliation on $M$ whose slices are subgroups of $G$. However, even though a Lie group always has 1-dimensional subgroups, it does not always admit 2-dimensional subgroups, i.e., it is not always possible to put together 1-parameter subgroups and form higher dimensional subgroups. However, this can be done if $G$ is solvable. Recalling the
Cartan-Killing form and its interpretation in terms of solvability, it is natural to suspect a relation between vanishing of $S$ and solvability. So we now take a closer at (7). We recall that the algebraic bracket of two vector fields $\xi, \eta$ on $M$ is defined by

$$\{\xi, \eta\} = T_{ab}^i \xi^a \eta^b$$

Note that $\{,\}$ is an algebraic operation and does not satisfy the Jacobi identity in general. However, if $\xi, \eta$ are $\Upsilon$-invariant and $R = 0$, then $\{\xi, \eta\}$ coincides with their true bracket $[\xi, \eta]$. From (7) and (33), we deduce

$$S(\xi, \eta; \gamma) = S_{bc,d}^i \xi^b \eta^c \gamma^d = I_{ab}^i \xi^a \eta^b \gamma^d = \{\{\xi, \eta\}, \gamma\}$$

Now we assume $R = 0$ so that

$$S(\xi, \eta; \gamma) = [[\xi, \eta], \gamma]$$

where $\xi, \eta, \gamma$ are $\Upsilon$-invariant, i.e., they belong to the "Lie algebra" of $G$. Denoting the Killing form by $\kappa$, we have

$$\kappa(\xi, \eta) = Tr(ad(\xi) \circ ad(\eta)) = Tr(x \rightarrow [\xi, [\eta, x]]) = Tr(x \rightarrow S(x; \eta, \xi)) = Tr(x \rightarrow S_{ab,c}^i x^a \xi^b \eta^c) = S_{ab,c}^i \xi^b \eta^c$$

We call the LLG $(M, w, G)$ semisimple (solvable) if the Lie algebra $\mathfrak{g}$ of the $\Upsilon$-invariant vector fields is semisimple (solvable). From (35), we see that $S = 0$ if and only if $\mathfrak{g}^{(2)} = [[\mathfrak{g}, \mathfrak{g}], \mathfrak{g}] = 0$, i.e., $\mathfrak{g}$ is 2-step nilpotent. Combining this fact with the Cartan-Killing criterion, we obtain

**Proposition 6** Let $(M, w, G)$ be a LLG. Then

1) $S = 0$ if and only if $(M, w, G)$ is 2-step nilpotent
2) If $(M, w, G)$ is nilpotent, then $\text{Ric}(S) = 0$
3) $(M, w, G)$ is semisimple if and only if $\text{Ric}(S)$ is nondegenerate

It is known that a compact Lie group admits a bi-invariant metric which is not necessarily unique. The existence and uniqueness of a bi-invariant metric on an abstract Lie group is a subtle issue (see [M], [AB]). On the other hand, we observe that Proposition 6 assumes neither compactness, nor globalizability nor the existence of any extra structure like a bi-invariant metric and it is worthwhile to compare its conceptual simplicity to [M], [AB]. Some other interesting results follow from the present framework. For instance, using the linear Spencer sequence with representations we can define closed forms in the Lie algebra cohomology using $S$, we can refine, extend and reinterpret the results in [YB], [G] about the Betti numbers in terms of the group theory of $(M, w, G)$...We will relegate these problems to some mythical future work.

Now the following proposition should not come as a surprise.
Proposition 7 \(-S = R = \text{the Riemann curvature tensor of the canonical metric } g.\)

The proof is not difficult and we will leave it to the interested reader (The minus sign suggests to modify (7) by sign, see (40) below). Therefore, by a classical result (see [C]), the above sectional curvatures are the Gaussian curvatures of the sweeping surfaces, indicating remarkable relations between metric and Lie theoretic properties of \((M, w, G)\).

Now given the trivialization \((M, w)\), we define

\[
R^i_{jk,r} \overset{\text{def}}{=} \mathfrak{R}_{k,j,r}^i - R^a_{k,j} I^i_{ar} \tag{37}
\]

Clearly we have

\[
R^i_{jk,r} = -R^i_{kj,r} \tag{38}
\]

\[
R^i_{jk,r} + R^i_{kr,j} + R^i_{rj,k} = 0 \tag{39}
\]

We will write (37) in the form

\[
R = \mathfrak{R} - S \tag{40}
\]

To close the scene, we will state here the following Decomposition Theorem for absolute parallelism which we hope will attract the attention of young researchers and add more suspense to this adventure.

Theorem 8 \(R\) is the Riemann curvature tensor \(R\) of the canonical metric of the trivialization \((M, w)\) and decomposes as (40).

Note that (40) decomposes a metrical object into two pre-Lie theoretic objects. If \(\mathfrak{R} = 0\), then \(R\) becomes \(-S.\) If \(S\) vanishes, then the RHS of (6) vanishes, i.e., \(\{,\}\) satisfies the Jacobi identity. This condition does not imply \(\mathfrak{R} = 0\) but makes the trivialization \((M, w)\) in some vague sense "close to a 2-step nilpotent LLG", i.e., almost 2-step nilpotent by Proposition 6. With this assumption, we observe the intriguing fact that the homogeneous tensor \(\mathcal{H} = (\mathcal{H}^i_j)\) defined in Chapter 13 of [O1] becomes

\[
\mathcal{H}^i_j = -\text{Ric}(R)_{(ij)}^i \tag{41}
\]

with the passive index \((j)\) as explained in [O1].

Theorem 8 has many consequences and here is an immediate one which is already nontrivial: Can two Lie groups \(G_1, G_2\) with their left invariant metrics \(g_1, g_2\) be globally isometric but nonisomorphic as Lie groups? The answer is affirmative and such examples abound everywhere. For instance, let \(G\) be a simply connected 2-step nilpotent Lie group which is nonabelian. By (40) it is globally isometric to \(\mathbb{R}^n\), \(\dim G = n\), but \(\mathbb{R}^n\) is also abelian with its left invariant Euclidean metric. This example shows that a "Lie structure" is much more refined than a "metric structure". Understanding this relation, we believe,
is equivalent to understanding the meaning of (40), i.e., the meaning of the first Bianchi Identity (6) (see Proposition 12.2, [O1] for the second BI).

A final remark: A trivial principle bundle is a very boring object for a topologist. It is a great wonder that a trivialization of the principal frame bundle (a first order jet bundle) is such an immensely rich geometric structure.

References

[AB] Alexandrino, M.M., Bettiol, R.G., Lie Groups and Geometric Aspects of Isometric Actions, Springer, 2015
[C] do Carmo, M.P., Riemannian Geometry, Birkhauser, 1992
[G] Goldberg, S.I., Curvature and Homology, Academic Press, New York, 1962
[KN] Kobayashi, S., Nomizu, K., Foundations of Differential Geometry, Interscience Publishers, John Wiley & Sons, Vol1, 1963
[M] Milnor, J.W., Curvatures of left invariant metrics on Lie groups, Adv. in Math., 21, 293-329, 1976
[O1] Orta¸cgil, E.H., An Alternative Approach to Lie Groups and Geometric Structures, Oxford University Press, 2018
[O2] Orta¸cgil, E.H., Curvature without connection, arXiv:2003.06593 2020
[YB] Yano, K., Bochner, S., Curvature and Betti Numbers, Annals of Math. Studies, No. 32, Princeton Univ. Press, 1953

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