A REMARK ON THE GROUP-COMPLETION THEOREM

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Abstract. Suppose that \( M \) is a topological monoid satisfying \( \pi_0 M = \mathbb{N} \) to which the McDuff-Segal group-completion theorem applies. This implies that a certain map \( f : M_\infty \to \Omega BM \) defined on an infinite mapping telescope is a homology equivalence with integer coefficients. In this short note we give an elementary proof of the result that if left- and right-stabilisation commute on \( H_1(M) \), then the “McDuff-Segal comparison map” \( f \) is acyclic. For example, this always holds if \( \pi_0 M \) lies in the centre of the Pontryagin ring \( H_\ast(M) \). As an application we describe conditions on a commutative I-monoid \( X \) under which \( \text{hocolim}_I X \) can be identified with a Quillen plus-construction.

1. Introduction and result

Let \( M \) be a topological monoid and let \( BM \) be its classifying space. The group-completion theorem of McDuff and Segal [6] relates the homology of \( \Omega BM \) to the localization of the Pontryagin ring \( H_\ast(M) \) at its multiplicative subset \( \pi_0 M \).

**Theorem 1.1** ([6, Prop. 1]). Suppose the localization \( H_\ast(M)[(\pi_0 M)^{-1}] \) can be constructed by right-fractions. Then the natural map \( M \to \Omega BM \) induces an isomorphism \( H_\ast(M)[(\pi_0 M)^{-1}] \cong H_\ast(\Omega BM) \).

In this paper all homology groups are understood to be singular homology with integer coefficients. For what is meant by “can be constructed by right-fractions” we refer the reader to [6, Rem. 1], where this is explained. Let us now assume, for simplicity, that \( \pi_0 M \) is finitely generated. Let \( x_1, \ldots, x_k \in M \) be a set of generators and define \( x := x_1 \cdots x_k \in M \). Then the infinite mapping telescope

\[ M_\infty := \text{tel}(M \xrightarrow{x} M \xrightarrow{x} M \xrightarrow{x} \ldots) \]

has \( \pi_0 M \)-local homology \( H_\ast(M_\infty) = H_\ast(M)[(\pi_0 M)^{-1}] \). There is not directly a map \( M_\infty \to \Omega BM \) inducing the isomorphism of Theorem 1.1. However, a natural map into a space weakly equivalent to \( \Omega BM \) can be constructed, see [6,8] for details. We shall not take this too precisely, but simply speak of a comparison map

\[ f : M_\infty \to \Omega BM . \]

It is desirable to know if this map induces an isomorphism on homology for all choices of local coefficients on the target space. Such a map is **acyclic**, i.e. its homotopy fibre is an acyclic space. If the comparison map \( f \) is acyclic, it can be converted into a weak homotopy equivalence by means of the Quillen plus-construction. Randal-Williams [8] has proved the following strengthening of Theorem 1.1 under the hypothesis that \( M \) is homotopy commutative (see also [7]).

**Theorem 1.2** ([8]). Suppose \( M \) is homotopy commutative. Then the comparison map \( f \) is acyclic. As a consequence, the fundamental group of \( M_\infty \) with any choice of basepoint has a perfect commutator subgroup.

The objective of this note is to study the acyclicity of the comparison map under a weaker condition than homotopy commutativity. This is condition (†) below. We
restrict ourselves to a simplified setting by assuming that the monoid of components is the natural numbers \( \pi_0 M = \mathbb{N} \). Our main result is:

**Theorem 1.3.** Let \( M \) be a topological monoid satisfying \( \pi_0 M = \mathbb{N} \). Suppose that the localization \( H_\ast(M)[[\pi_0 M]^{-1}] \) can be constructed by right-fractions and that \((\dagger)\) holds. Then the fundamental group of \( M_\infty \) with any choice of basepoint has a perfect commutator subgroup and the comparison map \( f : M_\infty \to \Omega BM \) is acyclic.

As a direct consequence of acyclicity we obtain

**Corollary 1.4.** Under the assumptions of Theorem 1.3 the induced map \( f^+ : M_\infty^+ \to \Omega BM \) is a weak homotopy equivalence.

Here the plus-construction is applied to each path-component separately and with respect to the maximal perfect subgroup of the fundamental group.

**Proof.** Since the map \( f^+ \) is acyclic, the induced map on universal coverings is a homology isomorphism and therefore a weak equivalence by the Whitehead theorem. The map \( f^+ \) is in addition a \( \pi_1 \)-isomorphism, hence \( f^+ \) is a weak equivalence. \( \square \)

**Notation.** For the rest of the paper we assume that \( \pi_0 M = \mathbb{N} \). Thus we study a sequence of path-connected spaces \( (M_n)_{n \geq 0} \) with basepoints \( m_n \in M_n \) and associative, basepoint preserving product maps

\[
\mu_{m,n} : M_m \times M_n \to M_{m+n}
\]

for all \( m, n \geq 0 \). The basepoint \( m_0 \in M_0 \) serves as a two-sided unit. We write

\[
G_n := \pi_1(M_n, m_n)
\]

and \( G_n' := [G_n, G_n] \) for its derived subgroup. We avoid the use of a designated symbol for the concatenation product on fundamental groups. For \( a, b \in G_n \) we employ the convention where \( ab \) means that \( a \) is traversed first, followed by \( b \). We let \( e_n \) denote the neutral element of \( G_n \), that is the class of the constant loop based at \( m_n \in M_n \). Via pointwise multiplication of loops the maps \( \mu_{m,n} \) induce homomorphisms

\[
\oplus : G_m \times G_n \to G_{m+n}
\]

which we commonly denote by the symbol \( \oplus \) to keep the notation simple. Multiplication by \( e_n \) from the right is distributive over the concatenation product in \( G_m \) and therefore describes a homomorphism \( - \oplus e_n : G_m \to G_{m+n} \). Similarly, multiplication from the left \( e_m \oplus - \) is a homomorphism. We clearly have \( e_n = e_1 \oplus \cdots \oplus e_1 \) (\( n \) times).

**The commutativity condition.** The following condition on the Pontryagin ring \( H_\ast(M) \) expresses commutativity of left- and right-stabilisation in \( H_1(M) \). For each \( n \geq 0 \) let us regard the basepoint \( m_n \in M_n \) as a class in \( H_0(M_n) \) and write a single dot \( - \) for the Pontryagin product.

\( (\dagger) \) There is a cofinal sequence \( n_0, n_1, n_2, \ldots \) in \( \mathbb{N} \) so that for all \( k \in \mathbb{N} \) and all \( a \in H_1(M_{n_k}) \) the equality

\[
a \cdot m_{n_k} = m_{n_k} \cdot a
\]

holds in \( H_1(M) \subseteq H_\ast(M) \).

Equivalently,

\( (\dagger') \) there is a cofinal sequence \( n_0, n_1, n_2, \ldots \) in \( \mathbb{N} \) so that for all \( k \in \mathbb{N} \) and all \( a \in G_{n_k} \) there exists \( c \in G_{2n_k} \) so that the equality

\[
a \oplus e_{n_k} = (e_{n_k} \oplus a)c
\]

holds in \( G_{2n_k} \).
Indeed, the definition of the Pontryagin product implies that for all \( n \geq 0 \) the diagram

\[
\begin{array}{ccc}
G_n & \xrightarrow{e_1 \oplus -} & G_{n+1} \\
\downarrow & & \downarrow \\
H_1(M_n) & \xrightarrow{m_1 \cdot -} & H_1(M_{n+1})
\end{array}
\]

commutes, where the vertical maps are abelianization. The diagram shows (†) ⇔ (†').

**Example 1.5.** If we assume that \( \pi_0 M \) is central in \( H_* (M) \), then (†) holds. In particular, (†) holds for homotopy commutative \( M \).

**Organisation of this paper.** The proof of the main result, Theorem 1.3, occupies Section 2. The proof is a standard spectral sequence argument which rests on two preparatory lemmas. We will first prove the theorem and then prove the necessary lemmas. In Section 3 we begin by recalling the notion of an \( \mathbb{I} \)-space and of a (commutative) \( \mathbb{I} \)-monoid. We then apply Theorem 1.3 to show that under suitable hypotheses the infinite loop space associated to a commutative \( \mathbb{I} \)-monoid can be identified with a Quillen plus-construction. Finally, in Section 4 we show that \( \mathbb{G}^\infty = \text{colim}_n G_n \) has a perfect commutator subgroup whenever \( (G_n)_{n \geq 0} \) is a direct system of groups with multiplication maps \( \oplus \) as described in the preceding paragraphs and which satisfies condition (†'). This is merely a replication of an argument of Randal-Williams [8, Prop. 3.1] and is not relevant for the body of the paper, but it provides a direct proof of the first statement in Theorem 1.3.

2. **Proof of Theorem 1.3**

Let \( \mathbb{G}^\infty \) be the colimit of the direct system of groups \( - \oplus e_1 : G_n \rightarrow G_{n+1} \). It is isomorphic to the fundamental group of the infinite mapping telescope

\[
\mathbb{M}^\infty := \text{tel}(M_0 \xrightarrow{m_0} M_1 \xrightarrow{m_1} M_2 \xrightarrow{m_1} \ldots),
\]

which we give the basepoint \( m_0 \in M_0 \subset \mathbb{M}^\infty \). Note that \( \mathbb{M}^\infty \simeq \mathbb{Z} \times \mathbb{M}^\infty \). We will be working with covering spaces, so we shall assume that all our spaces be locally path-connected and semi-locally simply connected. For example, this includes all CW-complexes. For our purposes this is not a restriction, since all of our results remain valid by replacing the spaces by the realization of their singular complex. Let \( Y^\infty \) denote the covering space of \( \mathbb{M}^\infty \) corresponding to the derived subgroup \( \mathbb{G}^\infty \). This is a regular covering and the action via deck translations of the fundamental group \( \mathbb{G}^\infty \) on \( Y^\infty \) factors through the abelianisation \( \mathbb{G}^\infty / \mathbb{G}^\infty \). The proof of Theorem 1.3 only depends upon the following lemma, which we will prove below.

**Lemma 2.1.** Suppose \( M \) satisfies (†). Then the action of \( \mathbb{G}^\infty / \mathbb{G}^\infty \) on \( H_* (Y^\infty) \) through deck translations of \( Y^\infty \) is trivial.

Conceptually our proof of Theorem 1.3 is similar to Wagoner [10, Prop. 1.2].

**Proof of Theorem 1.3.** Our assumptions allow us to apply Theorem 1.1 which asserts that \( f : \mathbb{M}^\infty \rightarrow \Omega BM \) is a homology equivalence for integer coefficients. In particular, \( f \) is bijective on path-components and induces an isomorphism

\[
H_1(M^\infty) \xrightarrow{\cong} \mathbb{G}^\infty / \mathbb{G}^\infty \xrightarrow{\cong} \pi_1(\Omega_0 BM),
\]

where \( \Omega_0 BM \subset \Omega BM \) is the component of the basepoint. Consider the map of fibration sequences.
where the bottom row is the universal covering sequence for $\Omega_0BM$, i.e. $W$ is the universal covering space of $\Omega_0BM$. The map $f_0 : M_\infty \to \Omega_0BM$ is the restriction of $f$ to the basepoint components, and the left hand square is a pullback with the horizontal arrows fibrations. The space $\Omega_0BM$ is a connected $H$-space and therefore weakly simple, that is its fundamental group acts trivially on the integral homology of its universal covering. For the latter fact we refer the reader to the proof of [3 Lem. 6.2]. Thus $G_\infty / G'_\infty$ acts trivially on $H_\ast(W)$. Moreover, by Lemma 2.1, the action of $G_\infty / G'_\infty$ on $H_\ast(Y_\infty)$ is trivial. This shows that both fibrations in (2.2) have a simple system of local coefficients.

Consider now the map of Serre spectral sequences associated to these fibrations. It follows from Zeeman’s comparison theorem [5 Thm. 3.26] that the map of coverings $f'_0 : Y_\infty \to W$ is an integer homology equivalence. Since $W$ is a simply connected space, this implies that $H_1(Y_\infty) = 0$, i.e. that $\pi_1Y_\infty = G'_\infty$ is perfect. It is known that a map into a simply connected space which is an integer homology equivalence is in fact acyclic. Thus $f'_0$ is acyclic and, because the left hand square in (2.2) is a homotopy pullback, so is $f_0$ (we call a map acyclic, if each of its homotopy fibres has the integral homology of a point, and this property is clearly preserved under homotopy pullbacks). This proves the theorem for the basepoint components of $M_\infty$ and $\Omega BM$.

However, the exact same argument applies to any other component of $M_\infty$. Namely, $f$ is a bijection on path-components, each path-component of $M_\infty$ has the homotopy type of $M_\infty$ (and thus Lemma 2.1 applies to it), and $\Omega BM$ is an $H$-group, hence all its path-components are homotopy equivalent to the component of the basepoint, which is a weakly simple space. \hfill $\square$

It remains to show Lemma 2.1. Consider one component $M_n$ of the monoid $M$ and let $I = [0,1]$ denote the unit interval. As a model for the universal covering space $\tilde{M}_n \to M_n$ we may take

$$M_n = \{ \text{homotopy classes of paths } \gamma : (I, 0) \to (M_n, m_n) \text{ rel } \partial I \},$$

suitably topologised [1 §3.8]. As the notation suggests, paths originate from the basepoint $m_n \in M_n$ and the homotopies are required to fix the endpoints of a path.

To simplify the notation, we shall denote a path and its homotopy class rel $\partial I$ by the same letter. The covering projection $\tilde{M}_n \to M_n$ is evaluation at the endpoint of a path. A loop $a \in G_n$ acts on a path $\gamma \in \tilde{M}_n$ via deck translation. In our chosen model this action corresponds to precomposition of paths $\gamma \mapsto a\gamma$. We define

$$Y_n := G'_n \backslash \tilde{M}_n$$

to be the quotient by the action by commutators, equipped with the quotient topology. The induced projection $Y_n \to M_n$ is the connected regular covering space of $M_n$ corresponding to the commutator subgroup $G'_n = [G_n, G_n]$. In the same way we define the covering space $Y_\infty \to M_\infty$ corresponding to the subgroup $G'_\infty \subset G_\infty$. Note that, despite the notation, $Y_\infty$ is not a telescope built from the covering spaces $\{Y_n\}_{n \geq 0}$, though it is weakly equivalent to such, as we shall see shortly.

Since the homomorphism $\oplus : G_m \times G_n \to G_{m+n}$ restricts to a homomorphism of commutator subgroups $G'_m \times G'_n \to G'_{m+n}$, the lifting property of a covering space
allows us to fill in the dashed arrow in the diagram

\[
\begin{array}{c}
Y_m \times Y_n \xrightarrow{\oplus} Y_{m+n} \\
\mu_{m,n} \downarrow \downarrow \downarrow \downarrow \downarrow \\
M_m \times M_n \xrightarrow{\mu_{m,n}} M_{m+n}.
\end{array}
\]

The requirement that \(e_m \oplus e_n\) be \(e_{m+n}\) in \(Y_{m+n}\) specifies a unique continuous pairing

\[
\oplus : Y_m \times Y_n \rightarrow Y_{m+n}.
\tag{2.3}
\]

In fact, this map is just induced by pointwise multiplication of paths, i.e. for \(\gamma \in Y_m\) and \(\eta \in Y_n\) the homotopy class \(\gamma \oplus \eta\) is represented by the path \((\gamma \oplus \eta)(t) := \mu_{m,n}(\gamma(t), \eta(t))\) for \(t \in I\). In particular, we have the diagram of spaces

\[
\begin{array}{c}
Y_0 \xrightarrow{-\oplus e_1} \cdots \xrightarrow{-\oplus e_1} Y_n \xrightarrow{-\oplus e_1} Y_{n+1} \xrightarrow{-\oplus e_1} \cdots
\end{array}
\]

and we can consider the group colim, \(H_*(Y_n)\) (where \(* \geq 0\) is any fixed degree).

Then \(G_\infty = \text{colim}_n G_n\) acts upon the colimit in the following way. Let \([a] \in G_\infty\) be represented by some \(a \in G_m\) and let \([z] \in \text{colim}_n H_*(Y_n)\) be represented by some \(z \in H_*(Y_t)\). Then choose \(k \geq \text{max}\{m, t\}\) and define an action by

\[
[a][z] := [(a \oplus e_{k-m})(z \oplus e_{k-t})],
\tag{2.4}
\]

where on the right hand side we use the action of \(G_k\) on \(H_*(Y_k)\) through deck translations of \(Y_k\). One may verify that the action (2.4) is well-defined.

**Lemma 2.2.** There is an isomorphism \(\text{colim}_n H_*(Y_n) \cong H_*(Y_\infty)\) which respects the action of \(G_\infty\).

**Proof.** We begin by describing a map \(j : \text{tel}_n Y_n \rightarrow Y_\infty\). It suffices to give maps \(j_n : Y_n \times [n, n+1] \rightarrow Y_\infty\) for all \(n \geq 0\) which are compatible at the endpoints of the intervals where they are glued together in the telescope. A point in the telescope \(M_\infty\) is specified by a pair \((x, t)\) with \(t \in \mathbb{R}\) and \(x \in M_{[t]}\), where \([t] \in \mathbb{N}\) denotes the integral part of \(t\). For \(t \in \mathbb{R}\) define a path

\[
\alpha_t : I \rightarrow M_\infty
\]

\[
s \mapsto (m_{[st]}, st).
\]

This is the “straight” path from the basepoint \(m_0 \in M_\infty\) to the basepoint \(m_{[t]} \in M_{[t]}\) of the “slice” at coordinate \(t\) in the telescope.

Now suppose \((\gamma, t) \in Y_n \times [n, n+1]\). Then \(\gamma\) represents a homotopy class of a path in \(M_{[t]}\) based at \(m_{[t]}\). To this pair we assign the class in \(Y_\infty\) of the path

\[
j_n(\gamma, t) = \alpha_t \gamma : I \rightarrow M_\infty.
\]

This map is well-defined and continuous, as one may easily check. In fact, upon passage to quotient spaces, the map \(j_n\) arises as a lift of the composite map \(M_n \times [n, n+1] \rightarrow M_n \times [n, n+1] \rightarrow M_\infty\) under the universal covering map \(\tilde{M}_\infty \rightarrow M_\infty\). Moreover, the maps \(j_n\) for all \(n \geq 0\) fit nicely together whenever \(t\) approaches an integer, and we obtain the desired map \(j\) from the telescope to \(Y_\infty\).

We now consider the following diagram

\[
\begin{array}{ccc}
tel_n Y_n & \xrightarrow{ev_1} & tel_n M_n \\
\downarrow & & \downarrow \\
y_\infty & \xrightarrow{ev_1} & M_\infty
\end{array}
\]

The map denoted \(ev_1\) is evaluation at the endpoint of a path. By construction of \(j\) the left hand square commutes. The right most vertical arrow is obtained by
factoring the family of maps $K(G_n/G'_n, 1) \to K(G_\infty/G'_\infty, 1)$ induced by inclusion through the telescope. It is a weak homotopy equivalence, as one can see by commuting homotopy groups with the telescope. The map $M_\infty \to K(G_\infty/G'_\infty, 1)$ is the classifying map for the covering space $Y_\infty$. The top row of the diagram is the telescope over the natural homotopy fibre sequences $Y_n \to M_n \to K(G_n/G'_n, 1)$ and therefore again a homotopy fibre sequence. The right hand square commutes up to homotopy: It is easily verified that the square commutes on the level of $H_1(-; \mathbb{Z})$ and hence, by the universal coefficient theorem, also on $H^1(-; G_\infty/G'_\infty)$, but homotopy classes of maps into $K(G_\infty/G'_\infty, 1)$ are uniquely determined by their effect on $H^1(-; G_\infty/G'_\infty)$. It follows that the map $j$ is a weak homotopy equivalence and therefore induces an isomorphism $H(j) : \text{colim}_n H_*(Y_n) \xrightarrow{\cong} H_*(Y_\infty)$.

Finally, we must check that $H(j)$ is compatible with the action of $G_\infty$. Consider the following two diagrams

$$Y_k \times \{k\} \xrightarrow{j} Y_\infty \quad \xrightarrow{\text{incl.}} \quad H_*(Y_k) \xrightarrow{H(- \oplus e_i)} H_*(Y_{k+1})$$

(2.5)

The left hand triangle defines the map $i_k : Y_k \to Y_\infty$. With this definition it is readily verified that the right hand triangle commutes. Now suppose we are given equivalence classes $[a] \in G_\infty$ and $[z] \in \text{colim}_n H_*(Y_n)$ represented respectively by $a \in G_m$ and $z \in H_*(Y_n)$. Then, recalling (2.4) we find

$$H(j)([a][z]) = H(j)([(a \oplus e_{k-m})(z \oplus e_{k-n})]) = H(i_k)((a \oplus e_{k-m})(z \oplus e_{k-n})) = \alpha_k(a \oplus e_{k-m})(z \oplus e_{k-n}),$$

for some $k \geq \max\{m, n\}$. On the other hand, we have the action of $G_\infty$ on $H_*(Y_\infty)$ through deck translations. If $[a] \in G_\infty$ is represented by $a \in G_m$, then the isomorphism $G_\infty \cong \pi_1(M_\infty, m_0)$ takes $[a] \mapsto \alpha_m a \alpha_m$. Here we denote by $\alpha_m$ the inverse path of $\alpha_m$. Thus $[a]$ acts on $\gamma \in Y_\infty$ as $\gamma \mapsto (\alpha_m a \alpha_m \gamma)$. Therefore, using commutativity of the right hand diagram in (2.5) we find

$$[a]H(j)([z]) = [a]H(i_n)(z) = [a \oplus e_{k-m}]H(i_k)(z \oplus e_{k-n}) = \alpha_k(a \oplus e_{k-m})\alpha_k\alpha_k(z \oplus e_{k-n}),$$

which equals (2.6). So $H(j)$ commutes with the action of $G_\infty$ and the assertion of the lemma follows.

**Proof of Lemma 2.3.** Let $[a] \in G_\infty$ and let $z$ be a class in $H_*(Y_\infty)$. Let $n_0, n_1, n_2, \ldots$ be the sequence in $\mathbb{N}$ which appears in the commutativity condition (1). By Lemma 2.2 and cofinality of $n_0, n_1, n_2, \ldots$ we may assume that there is $n \in \{n_k\}_{k \in \mathbb{N}}$ so that $z$ is a class in $H_*(Y_n)$ and that $[a]$ is represented by an element $a \in G_n$. Recall that the colimit system of homology groups is induced by the maps $- \oplus e_1 : Y_n \to Y_{n+1}$. Therefore $z \in H_*(Y_n)$ and $z \oplus e_n \in H_*(Y_{2n})$ represent the same classes in the direct limit. Similarly, $a \in G_n$ and $a \oplus e_n \in G_{2n}$ coincide in $G_\infty$. The commutativity relation (1) implies that there exists $c \in G_{2n}$ so that $a \oplus e_n = (e_n \oplus a)c$. Since the action of $G_{2n}$ on $Y_{2n}$ factors through the abelianization, the action of $a \oplus e_n$ on $z \oplus e_n$ can be written

$$(a \oplus e_n)(z \oplus e_n) = (e_n \oplus a)(z \oplus e_n) = (e_n z) \oplus (ae_n) = z \oplus a.$$ (2.7)

Note that $G_n/G'_n$ is a discrete subspace of $Y_n$, so we can consider the loop $a \in G_n$ as a point in $Y_n$. Therefore, using (2.7), the action of $[a] \in G_\infty$ on $[z] \in H_*(Y_\infty)$ can
be described by choosing representatives \( a \in Y_n \) and \( z \in H_*(Y_n) \) and computing the image of \( z \) under the map
\[
H_*(Y_n) \xrightarrow{H(- \oplus a)} H_*(Y_{2n}) \to \text{colim}_n H_*(Y_n) \cong H_*(Y_\infty). \tag{2.8}
\]
Here \( H(- \oplus a) \) is the map induced on homology by the map of spaces \(- \oplus a : Y_n \to Y_{2n}\), see \((2.3)\). Since \( Y_n \) is path-connected, there is a path from \( a \) to \( e_n \) which induces a homotopy from \(- \oplus a\) to \(- \oplus e_n\) as maps \( Y_n \to Y_{2n}\). As a consequence, the first map in \((2.8)\) is the stabilisation map \( H(- \oplus e_n) \) for the colimit \( \text{colim}_n H_*(Y_n) \), which shows that the action of \( a \) on \( z \) is trivial. \( \square \)

3. An application to commutative \( \mathbb{I} \)-monoids

Let \( \mathbb{I} \) denote the skeletal category of finite sets \( n = \{1, \ldots, n\} \) (including the empty set \( 0 := \emptyset \)) and injective maps between them. It is a permutative category under the disjoint union of sets, i.e.
\[
(m, n) \mapsto m \sqcup n := \{1, \ldots, m, m + 1, \ldots, m + n\}.
\]
The monoidal unit is given by the initial object \( 0 \in \mathbb{I} \) and the commutativity isomorphism \( m \sqcup n \cong n \sqcup m \) is the evident block permutation.

Let \( \mathcal{T} \) be the category of based spaces. A functor \( \mathbb{I} \to \mathcal{T} \) is called an \( \mathbb{I} \)-space. By the usual construction, the category of \( \mathbb{I} \)-spaces and natural transformations inherits a symmetric monoidal structure from \( \mathbb{I} \). The following definition is standard in the literature.

**Definition 3.1** (E.g. [9, §2.2]). A commutative \( \mathbb{I} \)-monoid \( X : \mathbb{I} \to \mathcal{T} \) is a commutative monoid object in the symmetric monoidal category of \( \mathbb{I} \)-spaces.

It is well known that for a commutative \( \mathbb{I} \)-monoid \( X \) the space \( \text{hocolim}_\mathbb{I} X \) is an \( E_\infty \)-space structured by an action of the Barratt-Eccles operad. For details we refer the reader to [9], or to [1] and the references therein, where the basic definitions and results are summarized. Let us write \( X_n := X(n) \) for short. Let \( \Sigma_n \) denote the symmetric group on \( n \) letters. Unravelling the definition, a commutative \( \mathbb{I} \)-monoid \( X \) consists of a sequence of \( \Sigma_n \)-spaces \( X_n \) with \( \Sigma_n \)-fixed basepoints and basepoint preserving, equivariant structure maps
\[
\oplus : X_m \times X_n \to X_{m+n}
\]
for all \( m, n \geq 0 \) satisfying suitable associativity and unit axioms. Moreover, commutativity of \( X \) implies that for all \( m, n \geq 0 \) the diagram
\[
\begin{array}{ccc}
X_m \times X_n & \xrightarrow{\oplus} & X_{m+n} \\
\downarrow t & & \downarrow \tau_{m,n} \\
X_n \times X_m & \xrightarrow{\oplus} & X_{n+m}
\end{array}
\tag{3.1}
\]
commutes, where \( t \) is the transposition and \( \tau_{m,n} \in \Sigma_{m+n} \) is the block permutation \( m \sqcup n \to n \sqcup m \).

Let us write \( X_\infty = \text{tel}_n X_n \) for the infinite mapping telescope which is formed using the maps \( X_n \to X_{n+1} \) induced by the standard maps \( n = n \sqcup 0 \to n \sqcup 1 \).

Combining our Theorem 1.3 with results of [1] we obtain conditions under which the homotopy colimit over \( \mathbb{I} \) is equivalent to the Quillen plus-construction on \( X_\infty \).

**Corollary 3.2.** Let \( N \geq 0 \) be a fixed integer and let \( X \) be a commutative \( \mathbb{I} \)-monoid such that for all \( n \geq N \) the space \( X_n \) is path-connected and the induced \( \Sigma_n \)-action on \( H_*(X_n) \) is trivial. Then the fundamental group of \( X_\infty \) has a perfect commutator subgroup and \( X_\infty^+ \) is an infinite loop space.
Proof. Without loss of generality we may assume \( N = 0 \), since we can always replace \( X \) by the commutative \( \mathbb{I} \)-monoid \( X' \) with \( X'_n = pt \) for all \( n < N \) and \( X'_n = X_n \) for all \( n \geq N \). Then \( X' \) satisfies the assumptions of the corollary for \( N = 0 \) and \( X'_\infty \simeq X_\infty \).

Now consider the topological monoid \( X := \coprod_{n \geq 0} X_n \). The commutative diagrams of \([5,1]\) and the assumption that \( \Sigma_n \) acts trivially on \( H_*(X_n) \) for all \( n \geq 0 \) imply that the Pontryagin ring \( H_*(X) \) is abelian. Thus we can apply Theorem \([1,3]\) (cf. Example \([1,5]\)). It follows that the fundamental group of \( X_\infty \) has a perfect commutator subgroup and that \( X_\infty^+ \simeq \Omega_0 BX \). Hence \( X_\infty^+ \) is a simple space, i.e. its fundamental group is abelian and acts trivially on all higher homotopy groups. On passage to colimits as \( n \to \infty \) we also have that \( \Sigma_\infty \) acts trivially on \( H_*(X_\infty) \cong \text{colim}_n H_*(X_n) \). The assertion of the corollary follows now from \([1, \text{Thm. } 3.1]\). The argument is short and we shall spell it out, because it explains how the condition that \( X_\infty^+ \) be simple enters into the proof.

The axioms of a commutative \( \mathbb{I} \)-monoid imply that \( \coprod_{n \geq 0} E\Sigma_n \times \Sigma_n X_n \) is an \( E_\infty \)-space and thus, using Theorem \([2,2]\) its group completion is the infinite loop space \( \mathbb{Z} \times (E\Sigma_\infty \times \Sigma_\infty X_\infty)^+ \). The projection to \( \mathbb{Z} \times B\Sigma_\infty^+ \), which is induced by collapsing \( X_\infty \) to a point, is a map of infinite loop spaces, so its homotopy fibre is an infinite loop space too. It remains to show that this homotopy fibre is precisely \( X_\infty^+ \). This last fact follows from Berrick’s fibration theorem \([2, \text{Thm. } 1.1(b)]\) applied to the fibration

\[
X_\infty \to \mathbb{Z} \times E\Sigma_\infty \times \Sigma_\infty X_\infty \to \mathbb{Z} \times B\Sigma_\infty ,
\]

using the fact that the fundamental group of the base, that is, \( \Sigma_\infty \) acts trivially on \( H_*(X_\infty) \) and that the fibre after plus-construction, that is, \( X_\infty^+ \) is simple, hence nilpotent. \( \square \)

Corollary 3.3. Let \( X \) be as in Corollary \([5,2]\). If in addition all maps \( X_m \to X_n \) induced by injections \( m \to n \) are injective, and for all \( x \in X_m \) and \( y \in X_n \) the element \( x \oplus y \in X_{m+n} \) is in the image of a map induced by a non-identity order preserving injection if and only if \( x \) or \( y \) is, then the inclusion \( X_\infty^+ \to \text{hocolim}_n X_n \) induces a weak homotopy equivalence of infinite loop spaces \( X_\infty^+ \simeq \text{hocolim}_n X_n \).

Proof. This follows directly from Corollary \([5,2]\) and \([1, \text{Thm. } 3.3]\). \( \square \)

4. Perfection of the commutator group

Let \( (G_n)_{n \geq 0} \) be a sequence of groups with homomorphisms \( \oplus : G_m \times G_n \to G_{m+n} \) for all \( m, n \geq 0 \) and suppose that \((\dagger')\) holds. We now repeat an argument of Randall-Williams \([8, \text{Prop. } 3.1]\) to show that in this situation \( G_\infty = \text{colim}_n G_n \) has a perfect commutator subgroup.

Lemma 4.1. The commutator subgroup of \( G_\infty \) is perfect.

Proof. For simplicity let us assume that the sequence \( n_0, n_1, n_2, \ldots \) in \((\dagger')\) is all of \( \mathbb{N} \). The proof is essentially the same in the more general case of only a cofinal subsequence. It suffices to show that every commutator \([a, b]\) in \( G_\infty \) can be written as a commutator \([c, d]\) with \( c, d \in G_{2n} \). We may assume that \( a, b \in G_\infty \) be represented by \( a, b \in G_n \). Using \((\dagger')\) we find

\[
[a \oplus e_n, b \oplus e_n] = [a \oplus e_n, (e_n \oplus b)d]
\]

for some \( d \in G_{2n} \). Since the product \( \oplus \) is a homomorphism, we have that

\[
(a \oplus e_n)(e_n \oplus b) = (ae_n) \oplus (e_nb) = a \oplus b = (e_na) \oplus (be_n) = (e_n \oplus b)(a \oplus e_n),
\]

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that is $a \oplus e_n$ and $e_n \oplus b$ commute with respect to the product in $G_{2n}$. Thus the commutator can be written as
\[ [a \oplus e_n, b \oplus e_n] = (e_n \oplus b)[a \oplus e_n, d](e_n \oplus b)^{-1}, \]
where we used Hall's identity $[x, yz] = [x, y][x, z]^y$. Multiplication by $e_{2n}$ from the right defines a homomorphism $G_{2n} \to G_{4n}$. Applied to the previous line it gives
\[ [a \oplus e_{3n}, b \oplus e_{3n}] = (e_n \oplus b \oplus e_{2n})[a \oplus e_{3n}, d \oplus e_{2n}](e_n \oplus b \oplus e_{2n})^{-1}. \]
Again by (\dagger′) there exists $c \in G'_{4n}$ such that
\[ a \oplus e_{3n} = (a \oplus e_n) \oplus e_{2n} = (e_{2n} \oplus a \oplus e_n)c. \]
Now $e_{2n} \oplus a \oplus e_n$ commutes with $d \oplus e_{2n}$ in $G_{4n}$, and using Hall's identity we can write
\[ [a \oplus e_{3n}, b \oplus e_{3n}] = (e_n \oplus b \oplus e_{2n})(e_{2n} \oplus a \oplus e_n)[c, d \oplus e_{2n}](e_{2n} \oplus a \oplus e_n)^{-1}. \]
Let $v$ be the element which is represented by $(e_n \oplus b \oplus e_{2n})(e_{2n} \oplus a \oplus e_n) \in G_{4n}$ in the direct limit $G_{\infty}$. Then in $G_{\infty}$ the previous equation reads
\[ [a, b] = v[c, d]v^{-1}, \]
where $c, d \in G'_{\infty}$. Since conjugation by $v$ is a homomorphism, we conclude $[a, b] \in [G'_{\infty}, G'_{\infty}]$. \qed

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