PARTITIONS WEIGHTED BY THE PARITY OF THE CRANK

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Abstract. A partition statistic ‘crank’ gives combinatorial interpretations for Ramanujan’s famous partition congruences. In this paper, we establish an asymptotic formula, Ramanujan type congruences, and q-series identities that the number of partitions with even crank $M_e(n)$ minus the number of partitions with odd crank $M_o(n)$ satisfies. For example, we show that $M_e(5n + 4) - M_o(5n + 4) \equiv 0 \pmod{5}$. We also determine the exact values of $M_e(n) - M_o(n)$ in case of partitions into distinct parts, which are at most two and zero for infinitely many $n$.

1. Introduction

A partition $\lambda$ of a positive integer $n$ is a weakly decreasing sequence of positive integers $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k$ such that $\lambda_1 + \lambda_2 + \cdots + \lambda_k = n$. Then $\lambda_1$ is the largest part and $k$ is the number of parts of $\lambda$. In 1944, Dyson [15] defined the rank of a partition $\lambda$ by
\[
\text{rank}(\lambda) := \lambda_1 - k
\]
and observed that this partition statistic appeared to give combinatorial interpretations for Ramanujan’s famous congruences $p(5n + 4) \equiv 0 \pmod{5}$ and $p(7n + 5) \equiv 0 \pmod{7}$, where $p(n)$ is the number of partitions of $n$. Dyson also conjectured the existence of another partition statistic, named the ‘crank’, that would explain the last Ramanujan’s partition congruence $p(11n + 6) \equiv 0 \pmod{11}$ combinatorially. His observations on the rank were first proved by Atkin and Swinnerton-Dyer [8] in 1954 and the crank was found by Andrews and Garvan [5] in 1988. This Andrews-Garvan crank is defined by
\[
\text{crank}(\lambda) = \begin{cases} 
\lambda_1, & \text{if } \mu(\lambda) = 0; \\
\nu(\lambda) - \mu(\lambda), & \text{if } \mu(\lambda) > 0,
\end{cases}
\]
where $\mu(\lambda)$ denotes the number of ones in $\lambda$ and $\nu(\lambda)$ denotes the number of parts of $\lambda$ that are strictly larger than $\mu(\lambda)$.

Studying the number of partitions with even rank minus the number with odd rank has led to some rather intriguing mathematics. For example, if we let $N_e(n)$ (resp. $N_o(n)$) denote the number of partitions of $n$ with even (resp. odd) rank, then we have
\[
\sum_{n=0}^{\infty} (N_e(n) - N_o(n)) q^n = \sum_{n=0}^{\infty} \frac{q^{n^2}}{(-q; q)_n^2} =: f(q),
\]

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Theorem 1.1. Let \( I_s(x) \) be the usual \( I \)-Bessel function of order \( s \). If \( n \) is a positive integer, then

\[
N_e(n) - N_o(n) = \pi (24n - 1)^{-1/4} \sum_{k=1}^{\infty} \frac{(-1)^{(k+1)/2} A_{2k}(n - \frac{k(1+(-1)^k)}{4})}{k} I_{1/2} \left( \frac{\pi \sqrt{24n - 1}}{12k} \right),
\]

where

\[
A_k(n) := \frac{1}{2} \sqrt{\frac{k}{12}} \sum_{x \equiv \left\lfloor \frac{k}{24} \right\rfloor \pmod{24k}} \chi_{12}(x) e^{2\pi i x/12k},
\]

with the sum running over the residue classes modulo \( 24k \) and \( \chi_{12}(x) := \left( \frac{12}{x} \right) \).

For another example, let \( N_e(D, n) \) (resp. \( N_o(D, n) \)) denote the number of partitions into distinct parts with even (resp. odd) rank. Then

\[
\sum_{n=0}^{\infty} (N_e(D, n) - N_o(D, n))q^n = \sum_{n=0}^{\infty} \frac{q^{n(n+1)/2}}{(-q; q)_n} =: R(q).
\]

In \cite{AndrewsDysonHickerson}, Andrews, Dyson, and Hickerson showed that the coefficients of \( R(q) \) have multiplicative properties determined by a certain Hecke character associated to the ring of integers of the real quadratic field \( \mathbb{Q}(\sqrt{6}) \), and Cohen \cite{Cohen} subsequently showed that \( R(q) \) belongs to the theory of Maass waveforms. To state the main theorem of \cite{AndrewsDysonHickerson}, we must note that every number which is 1 modulo 6 and greater than 1 has a factorization of the form

\[
6m + 1 = p_1^{e_1} p_2^{e_2} \cdots p_r^{e_r},
\]

where \( r \geq 1 \), each \( p_i \) is either a prime \( \equiv 1 \pmod{6} \) or the negative of a prime \( \equiv 5 \pmod{6} \), the \( p_i \)'s are distinct, and \( e_i \)'s are positive integers.

Theorem 1.2. For all positive integers \( n \) we have \( N_e(D, n) - N_o(D, n) = T(24n + 1) \), where \( 24n + 1 \) has the factorization \( \prod p_i^{e_i} \) and \( T(m) = T(p_1^{e_1}) \cdots T(p_r^{e_r}) \) is the multiplicative function defined on powers of primes by:

\[
T(p^e) = \begin{cases} 
0, & \text{if } p \not\equiv 1 \pmod{24} \text{ and } e \text{ is odd}, \\
1, & \text{if } p \equiv 13 \text{ or } 19 \pmod{24} \text{ and } e \text{ is even}, \\
(-1)^{e/2}, & \text{if } p \equiv 7 \pmod{24} \text{ and } e \text{ is even}, \\
e + 1, & \text{if } p \equiv 1 \pmod{24} \text{ and } T(p) = 2, \\
(-1)^{e}(e + 1), & \text{if } p \equiv 1 \pmod{24} \text{ and } T(p) = -2.
\end{cases}
\]
This “exact” formula has numerous interesting consequences, such as the fact that $N_e(D, n) - N_o(D, n)$ is almost always zero and assumes every integer infinitely often.

In this paper we pass from the rank to the crank, studying the number of partitions with even crank minus the number with odd crank. In doing so, we leave the world of weak Maass forms and Maass waveforms and enter the world of classical modular forms. Specifically, if $M(m, n)$ denotes the number of partitions of $n$ with crank $m$, then the generating function for $M(m, n)$ is given in [5] by

\[ \sum_{m=-\infty}^{\infty} \sum_{n=0}^{\infty} M(m, n) a^m q^n = \frac{(q; q)_{\infty}}{(aq; q)_{\infty}(q/a; q)_{\infty}}, \]

where

\[(z; q)_{\infty} = \lim_{n \to \infty} (z; q)_n.\]

Letting $M_e(n)$ (resp. $M_o(n)$) denote the number of partitions of $n$ with even (resp. odd) crank, then by setting $a = -1$ in (1.3) we have

\[ \sum_{n=0}^{\infty} (M_e(n) - M_o(n)) q^n = \frac{(q; q)_{\infty}}{(-q; q)_{\infty}^2} := g(q). \]

Apparentely the only study of $g(q)$ was done by Andrews and Lewis [6], who proved that $M_e(n) > M_o(n)$ if $n$ is even and $M_e(n) < M_o(n)$ if $n$ is odd (by showing all the coefficients of $g(-q)$ are positive). Here we shall examine several other aspects of $g(q)$, such as congruences, asymptotics, and $q$-series identities.

We begin with congruence properties satisfied by $M_e(n) - M_o(n)$. From work of Treneer [23], we know immediately that $M_e(n) - M_o(n)$ has infinitely many congruences in arithmetic progressions modulo any prime coprime to 6. The obvious question, then, is whether any of these congruences are as simple and elegant as those of Ramanujan for the partition function. We prove in fact that the crank difference $M_e(n) - M_o(n)$ satisfies a family of congruences modulo powers of 5.

**Theorem 1.3.** For all $\alpha \geq 0$, we have

\[ M_e(n) - M_o(n) \equiv 0 \pmod{5^{\alpha+1}} \quad \text{if} \quad 24n \equiv 1 \pmod{5^{2\alpha+1}}. \]

Following from the proof of Theorem 1.3 will be a generating function for $M_e(5n + 4) - M_o(5n + 4)$:

**Theorem 1.4.**

\[ \sum_{n=0}^{\infty} (M_e(5n + 4) - M_o(5n + 4)) q^n = \frac{5(q; q^2)_{\infty}^2 (q^5; q^5)_{\infty} (q^{10}; q^{10})_{\infty}^2}{(q^2; q^2)_{\infty}^2}. \]

Next we apply the Hardy-Ramanujan circle method to the generating function $g(q)$, obtaining an asymptotic formula for $M_e(n) - M_o(n)$:
Theorem 1.5. If \( n \) is a positive integer, then
\[
M_e(n) - M_o(n) = \frac{1}{\sqrt{n - 1/24}} \sum_{0 < k < 5\sqrt{n}/2} \frac{B_k(n)}{\sqrt{k}} \cosh \left( \frac{\pi}{k} \sqrt{\frac{n - 1/24}{6}} \right) + E_n,
\]
where
\[
B_k(n) = \sum_{(h,2k)=1} e^{\pi i (2s(h,k) - 3s(h,2k))} e^{-\frac{2\pi i nh}{2k}},
\]
for the Dedekind sum \( s(h,k) \), and where \(|E_n| < 194n^{1/4}\).

To finish our study of the function \( M_e(n) - M_o(n) \), we discuss consequences of some \( q \)-series expansions for \( g(q) \). As an example, we give a weighted partition identity involving \( M_e(n) - M_o(n) \). To state this we use the notion of the “initial run” of a partition, by which we mean the largest increasing sequence of part sizes starting with 1. For example, the partition \((7,7,5,3,3,3,2,1,1)\) has initial run \((1,2,3)\), while the partition \((6,6,5,2,2,2)\) has no initial run at all.

Theorem 1.6. For a nonempty partition \( \lambda \), define the weight \( \omega(\lambda) \) to be
\[
\omega(\lambda) = 1 + 4 \sum_j (-1)^j,
\]
where the sum is over those \( j \) in the initial run which occur an odd number of times in \( \lambda \). Then
\[
M_e(n) - M_o(n) = \sum_\lambda \omega(\lambda),
\]
where the sum is over all partitions \( \lambda \) of \( n \).

For example, take \( n = 4 \). The partition 4 has weight 1, \((3,1)\) has weight 1 - 4 = 3, \((2,2)\) has weight 1, \((2,1,1)\) has weight 1 + 4 = 5, and \((1,1,1,1)\) has weight 1. Summing these weights gives 1 - 3 + 1 + 5 + 1 = 5, which is, as expected, \( M_e(4) - M_o(4) \).

Finally, inspired by the work of Andrews, Dyson and Hickerson in Theorem 1.2, we look at what happens if we restrict our crank difference to partitions into distinct parts. In this case, the definition of the crank simplifies considerably, and we are able to use basic manipulations of \( q \)-series to prove an exact formula.

Let \( M_e(D,n) \) (resp. \( M_o(D,n) \)) denote the number of partitions into distinct parts with even (resp. odd) crank. For partitions into distinct parts the crank is either the largest part, if there is no one appearing, or the number of parts minus 2 if there is a one. Let \( P \) denote the set of pentagonal numbers, i.e., numbers of the form \( m(3m+1)/2 \) for \( m \) an integer. If \( n = m(3m+1)/2 \), we write \( R(n) = m \). Finally, we use the notations \( \lfloor n \rfloor_p \) and \( \lceil n \rceil_p \) to denote the pentagonal floor and ceiling of \( n \), i.e., the largest (resp. smallest) pentagonal number \leq (resp. \geq) \( n \).
Theorem 1.7. For positive integers $n$ we have

$$M_e(D, n) - M_o(D, n) = \begin{cases} 1, & \text{if } n \in \mathcal{P} \text{ and } R(n) \text{ is odd and positive}, \\ -1, & \text{if } n \in \mathcal{P} \text{ and } R(n) \text{ is not as above}, \\ 2, & \text{if } n \not\in \mathcal{P}, R([n]_p) \text{ is odd and positive, and } n \equiv [n]_p \pmod{2}, \\ -2, & \text{if } n \not\in \mathcal{P}, R([n]_p) \text{ is even and positive, and } n \equiv [n]_p \pmod{2}, \\ -2(-1)^{n-[n]_p}, & \text{if } n \not\in \mathcal{P} \text{ and } R([n]_p) \text{ is even and negative}, \\ 0, & \text{otherwise}. \end{cases}$$

Corollary 1.8. The quantities $M_e(D, n)$ and $M_o(D, n)$ differ by at most 2 and are equal for infinitely many $n$.

To illustrate the above theorem, take $n = 6$. Then $n$ is not a pentagonal number and $R([6]_p) = R(5) = -2$. Hence by the penultimate case we expect $M_e(D, 6) - M_o(D, 6)$ to be $-2(-1)^{6-5} = 2$. Indeed, the four partitions of 6 into distinct parts are (6), (5, 1), (4, 2), and (3, 2, 1), each of which has even crank except for (3, 2, 1).

The paper is organized as follows. In the next section, we prove the family of congruences in Theorem 1.3 and the generating function in Theorem 1.4. In Section 3, we establish the asymptotic formula in Theorem 1.5. In Section 4, we prove the weighted identity in Theorem 1.6 and discuss a similar identity. Finally, In Section 5 we treat the exact formula in Theorem 1.7.

2. Proof of Theorems 1.3 and 1.4

Here we follow the exposition in Gordon and Hughes’ rediscovery of some congruences of Rødseth [22]. The reader may should have some familiarity with the preliminaries in [16]. Define

$$F(\tau) := \eta(\tau)^3 \frac{\eta(50\tau)^2}{\eta(2\tau)^2 \eta(25\tau)^3}.$$

Here $\eta(\tau) := q^{1/24} \prod_{n \geq 1} (1 - q^n)$ and $q := e^{2\pi i \tau}$. Applying [16] Theorem 2, Theorem 3, we have $F(\tau) \in M_0(\Gamma_0(50))$, the space of weight 0 modular functions on $\Gamma_0(50)$. Then $F(\tau)$ is holomorphic on the upper half plane $\mathbb{H}$ and its orders at the cusps $\nu/\delta$ are as given below by [16] Theorem 3.

| $\delta$ | 1 | 2 | 5 | 10 | 25 | 50 |
|---|---|---|---|---|---|---|
| $\text{ord}_D F$ | 4 | -1 | 0 | 0 | -4 | 1 |

Next, recall the $U_d$-operator, which acts on power series by

$$\sum_{n=0}^{\infty} a(n)q^n|U_d := \sum_{n=0}^{\infty} a(dn)q^n.$$

Note that if $f$ and $g$ are power series in $q$, we have

$$\left(f(q^d)g(q)\right)|U_d = f(q)(g(q)|U_d).$$

By [16] Theorem 5 we have that $F(\tau)|U_5 \in M_0(\Gamma_0(10))$ with the following lower bounds for the orders of $F|U_5$ at the cusps.
Now consider the function
\[
G(\tau) := \frac{\eta(\tau)^2 \eta(10\tau)^4}{\eta(2\tau)^4 \eta(5\tau)^2}.
\]
Applying [16, Theorem 2, Theorem 3], we find that \( G(\tau) \in M_0(\Gamma_0(10)) \) and that its orders at the cusps are as follows.

| \( \delta \) | 1 | 2 | 5 | 10 |
|------------|---|---|---|----|
| \( \text{ord}_{\nu/\delta} G \) | 0 | -1 | 0 | 1 |

Since the only holomorphic modular functions of weight 0 are the constant functions, comparing the last two tables and the Fourier series expansions of \( F|U_5 \) and \( G \) gives
\[
F|U_5 = 5G.
\]
Now, if we consider \( G(\tau) \) as a function in \( M_0(\Gamma_0(50)) \), rather than the subfield \( M_0(\Gamma_0(10)) \), we find from [16, Theorem 3] that its orders at the cusps are as follows.

| \( \delta \) | 1 | 2 | 5 | 10 | 25 | 50 |
|------------|---|---|---|----|----|----|
| \( \text{ord}_{\nu/\delta} G \) | 0 | -5 | 0 | 1 | 0 | 1 |

Hence by [16, Theorem 5] applied to \( G^i \) and \( FG^i \), these functions are on \( \Gamma_0(10) \) and we have the following lower bounds for the orders of \( G^i|U_5 \) and \( FG^i|U_5 \) at the cusps.

| \( \delta \) | 1 | 2 | 5 | 10 |
|------------|---|---|---|----|
| \( \text{ord}_{\nu/\delta} G^i|U_5 \geq \) | 0 | -5i | 0 | i/5 |
| \( \text{ord}_{\nu/\delta} FG^i|U_5 \geq \) | 0 | -5i - 1 | 0 | (i + 1)/5 |

If \( i \geq 0 \), this implies that \( G^i|U_5 \) and \( FG^i|U_5 \) are polynomials in \( G \) of degrees at most \( 5i \) and \( 5i + 1 \) respectively. Hence
\[
G^i|U_5 = \sum_{j \geq 0} a_{ij} G^j \quad \text{and} \quad FG^i|U_5 = \sum_{j \geq 0} b_{ij} G^j
\]
for complex coefficients \( a_{ij} \) and \( b_{ij} \).

Let \( S \) be the vector space of all polynomials \( P = \sum_{j \geq 0} c_j G^j \) and \( T \) be the subspace of such polynomials with 0 as constant terms. By considering our lower bounds for the orders of \( G^i|U_5 \) and \( FG^i|U_5 \) and (2.1), we see that \( U_5 \) maps \( S \) to itself as well as \( T \) to itself. In addition, the linear transformation \( V : P \to (FP)|U_5 \) maps \( S \) into \( T \). With respect to the basis \( G, G^2, G^3, \ldots \) of \( T \) the matrices of \( U_5 \) and \( V \) restricted to \( T \) are respectively
\[
U_5 = (A := (a_{ij})) \quad \text{and} \quad V = (B := (b_{ij})),
\]
for \( 1 \leq i, j < \infty \).

If we define a sequence of functions \( L_\nu \) (\( \nu \geq 0 \)) inductively by putting for \( \alpha \geq 0 \),
\[
L_0 = 1, \quad L_{2\alpha + 1} = FL_{2\alpha}|U_5, \quad \text{and} \quad L_{2\alpha + 2} = L_{2\alpha + 1}|U_5,
\]
then
\[
L_1 = 5G = (5, 0, 0, \ldots),
\]
\[
L_{2\alpha + 1} = (5, 0, 0, \ldots)(AB)\alpha,
\]
and
\[
L_{2\alpha + 2} = (5, 0, 0, \ldots)(AB)\alpha A.
\]
On the other hand, it follows from induction on $\alpha$ that
\[
(2.3) \quad L_{2\alpha+1} = \left(\frac{q^{10}; q^{10}}{q^3; q^5}\right) \sum_{n=1}^{\infty} (M_e(m) - M_o(m))q^n,
\]
where $m = 5^{2\alpha+1} - 1 - 5^2 - \cdots - 5^{2\alpha}$. Theorem 1.3 will then follow from the following theorem.

**Theorem 2.1.** If we set $L_{2\alpha+1} = (l_1(2\alpha+1), l_2(2\alpha+1), \ldots)$, then $l_i(2\alpha+1)$ are integers divisible by $5^{\alpha+1}$.

We prove Theorem 2.1 in three steps.

**Lemma 2.2.** For all $i, j$, we have $a_{ij} \in \mathbb{Z}$ and
\[
\pi(a_{ij}) \geq \left\lfloor \frac{5j - i - 1}{6} \right\rfloor,
\]
where $\pi(n)$ denotes the $5$-adic order of $n$.

**Proof.** Define
\[
\phi(\tau) := \frac{\eta(\tau) \eta(50\tau)^2}{\eta(2\tau)^2 \eta(25\tau)} \in M_0(\Gamma_0(50)).
\]
Again, by [18, Theorem 3, Theorem 5], we obtain the orders of $\phi$ and the lower bounds for the orders of $\phi^\mu|U_5$ for any nonnegative number $\mu$ at the cusps as follows.

| $\delta$ | 1 | 2 | 5 | 10 | 25 | 50 |
|----------|---|---|---|----|----|----|
| $\text{ord}_{\nu/5}\phi$ | -3 | 0 | 0 | 0 | 3 |

and

| $\delta$ | 1 | 2 | 5 | 10 |
|----------|---|---|---|----|
| $\text{ord}_{\nu/5}\phi^\mu|U_5 \geq$ | 0 | -3$\mu$ | 0 | $3\mu/5$ |

Hence for any $\mu \geq 0$,
\[
\phi(\tau)^\mu|U_5 = \frac{1}{5} \sum_{\lambda=0}^{4} (\frac{\tau + \lambda}{5})^\mu \in M_0(\Gamma_0(10))
\]
is a polynomial in $G$ of degree at most $3\mu$. Then by Newton’s identities relating power sums of the roots to the coefficients of a polynomial (see [21] for example), $\phi(\frac{\tau + \lambda}{5})$ ($0 \leq \lambda \leq 4$) are the roots of an equation
\[
(2.4) \quad t^5 - \sigma_1 t^4 + \sigma_2 t^3 - \sigma_3 t^2 + \sigma_4 t - \sigma_5 = 0,
\]
where $\sigma_i$s are elementary symmetric functions in $\mathbb{C}[G]$. As a matter of convenience, we consider the reciprocal equation
\[
u^5 - \frac{\sigma_4}{\sigma_5} u^4 + \frac{\sigma_3}{\sigma_5} u^3 - \frac{\sigma_2}{\sigma_5} u^2 + \frac{\sigma_1}{\sigma_5} u - \frac{1}{\sigma_5} = 0.
\]
The roots of this equation are the functions $\phi(\frac{\tau + \lambda}{5})^{-1}$ ($0 \leq \lambda \leq 4$).

Note that

| $\delta$ | 1 | 2 | 5 | 10 | 25 | 50 |
|----------|---|---|---|----|----|----|
| $\text{ord}_{\nu/5}\phi^{-1}$ | 0 | 3 | 0 | 0 | 0 | -3 |
Since both $\phi^{-\mu}|U_5$ and $G^{-1}$ belong to $M_0(\Gamma_0(10))$ and are holomorphic except for a pole at infinity, it follows that $\phi^{-\mu}|U_5$ is a polynomial in $G^{-1}$ of degree at most $[3u/5]$. From

$$G^{-1}(\tau) = q^{-1} + 2 + q + 2q^2 + \cdots$$

and

$$G^{-2}(\tau) = q^{-2} + 4q^{-1} + 6 \cdots,$$

we can find that

$$\phi^{-1}|U_5 = 1,$$

$$\phi^{-2}|U_5 = 2q^{-1} + 3 + O(q) = 2G^{-1} - 1,$$

$$\phi^{-3}|U_5 = 6q^{-1} + 7 + O(q) = 6G^{-1} - 5,$$

$$\phi^{-4}|U_5 = 6q^{-2} + 24q^{-1} + 31 + O(q) = 6G^{-2} - 5,$$

which happen to give exactly the same polynomials in equations (17) in [16]. Now the rest of the proof of Theorem 2.1 is almost identical to [16, pp. 341-346], and so we shall only sketch the remaining details.

Using Newton’s identities and the polynomials above, we obtain the same $\sigma_i$’s as in [16, Eq. (18)], which shows that $\sigma_i \in \mathbb{Z}[G]$ for $1 \leq i \leq 5$. Hence, from

\[(5.5) \quad \phi^\mu|U_5 = \sigma_1 \phi^{\mu-1}|U_5 - \sigma_2 \phi^{\mu-2}|U_5 + \sigma_3 \phi^{\mu-3}|U_5 - \sigma_4 \phi^{\mu-4}|U_5 + \sigma_5 \phi^{\mu-5}|U_5,\]

for all $\mu \in \mathbb{Z}$, we obtain $\phi^\mu|U_5 = \sum_{\nu=-\infty}^{\infty} c_{\mu\nu}G^\nu$ with integral coefficients $c_{\mu\nu}$. And arguing as in [16, Lemma 6, Lemma 7] gives

$$\pi(c_{\mu\nu}) \geq \left[ \frac{5\nu - 3\mu - 1}{6} \right]$$

from which we deduce Lemma 2.2 by using $G^i|U_5 = (\phi^{2i}|U_3)G^{-i}$. \hfill \Box

**Lemma 2.3.** For all $i, j \in \mathbb{Z}$, we have $b_{ij} \in \mathbb{Z}$ and

$$\pi(b_{ij}) \geq \left[ \frac{5j - i - 1}{6} \right] \quad \text{and} \quad \pi(b_{ij}) \geq 1 \quad \text{if} \quad i \equiv 1 \pmod{5}.$$

**Proof.** We note that $F \phi^\mu|U_5$ satisfies the same Newton recurrence (2.5) as $\phi^\mu|U_5$, and

$$F|U_5 = 5G,$$

$$F \phi^{-1}|U_5 = -1,$$

$$F \phi^{-2}|U_5 = q^{-1} + 2 + O(q) = G^{-1},$$

$$F \phi^{-3}|U_5 = -5 + O(q) = -5,$$

$$F \phi^{-4}|U_5 = q^{-2} + 9q^{-1} - 9 + O(q) = G^{-2} + 5G^{-1} - 25.$$ 

It hence follows that for all $\mu \in \mathbb{Z}$,

$$F \phi^\mu|U_5 = \sum_{\nu} d_{\mu\nu}G^\nu,$$
where $d_{\mu\nu}$ are integers. Arguing as in [16 Lemma 8, Lemma 9], we have that for all $\mu, \nu \in \mathbb{Z}$,
\[
\pi(d_{\mu\nu}) \geq \left\lfloor \frac{5\nu - 3\mu - 1}{6} \right\rfloor \quad \text{and} \quad \pi(d_{\mu\nu}) \geq 1 \quad \text{if} \quad \mu \equiv 2 \pmod{5}.
\]
Now, this lower bound for $\pi(d_{\mu\nu})$ along with the fact $FG^i|U_5 = (F\phi^2|U_5)G^{-i}$ gives Lemma 2.3.

Using the fact that $F|U_5 = 5G$ together with the lower bounds for $\pi(a_{ij})$ and $\pi(b_{ij})$ in the lemmas above as in the proof of [16 Theorem 10], we obtain Lemma 2.4. For all $\alpha \geq 0$ and $j \geq 1$, we have
\[
\pi(l_j(2\alpha + 1)) \geq \alpha + 1 + \left\lfloor \frac{j-1}{2} \right\rfloor,
\]
\[
\pi(l_j(2\alpha + 2)) \geq \alpha + 1 + \left\lfloor \frac{j}{2} \right\rfloor.
\]
This implies that $l_j(2\alpha + 1) \equiv 0 \pmod{5^{\alpha+1}}$, which is Theorem 2.1. This then completes the proof of Theorem 1.3.

For Theorem 1.4, recall that we have shown that $F|U_5 = 5G$. Hence we have
\[
\frac{5\eta^3(\tau)\eta(5\tau)\eta^2(10\tau)}{\eta^2(2\tau)} = \frac{\eta^3(\tau)\eta(5\tau)\eta^2(10\tau)}{\eta^2(2\tau)} |U_5|
\]
\[
= (q^5; q^5)_\infty (q^{10}; q^{10})^2 \sum_{n=0}^{\infty} (M_e(n) - M_o(n)) q^{n+1} |U_5|
\]
\[
= (q)_\infty (q^2; q^2)^2 \sum_{n=0}^{\infty} (M_e(5n+4) - M_o(5n+4)) q^{n+1}.
\]
Multiplying the first and last terms in the above string of equations by
\[
\frac{1}{q \prod_{n=1}^{\infty} (1 - q^n)(1 - q^{2n})^2}
\]
gives Theorem 1.4.

3. Proof of Theorem 1.5

This proof is very similar to Kane’s treatment of the circle method in [17] where he proved a conjecture by Andrews and Lewis [6] on cranks of partitions modulo 3. By Cauchy’s integral formula, for a circle $C$ centered on the origin and inside the unit circle, we have
\[
M_e(n) - M_o(n) = \frac{1}{2\pi i} \int_C g(q)q^{-n-1} dq
\]
\[
= \int_0^1 g(\exp(-\rho + 2\pi i\phi)) e^{n\rho - 2\pi in\phi} d\phi
\]
\[
= \sum_{0<k<N, \substack{0<h<k \\ (h,k)=1}} \int_{\theta'_{h,k}}^{\theta''_{h,k}} g \left( \exp \left\{ \frac{2\pi ih}{k} - (\rho - 2\pi i\phi) \right\} \right) e^{n(\rho - 2\pi i\phi) - \frac{2\pi ih\phi}{k}} d\phi,
\]
where the last equality follows from Andrews’ dissection of the circle of integration in [3, Ch. 5] and \(\rho\) is a positive real number. As each of \(\theta'_{h,k}\) and \(\theta''_{h,k}\) is the mediant of the Farey number \(h/k\) and the adjacent Farey numbers, it satisfies \(1/(2kN) \leq \theta \leq 1/(kN)\). If we substitute \(y = \rho - 2\pi i\phi\) in (3.1), we obtain

\[
M_e(n) - M_o(n) = -\frac{1}{2\pi i} \sum_{\substack{0 < h < k < N \\ (h,k) = 1}} \int_{\rho = 2\pi i \theta'_{h,k}}^{\rho = 2\pi i \theta''_{h,k}} g \left( \exp \left( \frac{2\pi i h}{k} - y \right) \right) e^{ny - \frac{2\pi i n h}{k}} dy.
\]

Set

\[
I := -\frac{1}{2\pi i} \int_{\rho = 2\pi i \theta'_{h,k}}^{\rho = 2\pi i \theta''_{h,k}} g \left( \exp \left( \frac{2\pi i h}{k} - y \right) \right) e^{ny - \frac{2\pi i n h}{k}} dy.
\]

Then we can write

\[
M_e(n) - M_o(n) = \sum_{2k, k < N \\ 0 < h < k \\ (h,k) = 1} I + \sum_{2k, k < N \\ 0 < h < k \\ (h,k) = 1} I := \Sigma_1 + \Sigma_2.
\]

Recall from [4, Theorem 5.1] that if \(F(q) = 1/(q; q)_{\infty}\), then

\[
F \left( \exp \left( \frac{2\pi i h}{k} - \frac{2\pi z}{k^2} \right) \right) = e^{\pi i s(h,k)} \left( \frac{z}{k} \right)^{1/2} \exp \left( \frac{\pi}{12z} - \frac{\pi z}{12k^2} \right) F \left( \exp \left( \frac{2\pi i H}{k} - \frac{2\pi}{z} \right) \right),
\]

where \(\Re(z) > 0, k > 0, (h,k) = 1\), and \(hH \equiv -1 \pmod{2}\). We apply (3.5) to the integrand of \(I\) and evaluate a main term and estimate an error term. We first find a bound for \(\Sigma_1\):

**Lemma 3.1.** If \(\Sigma_1\) is as defined in (3.4), then

\[|\Sigma_1| < 53n^{1/4}.\]

**Proof.** Since

\[
g(q) = \langle q; q \rangle_{\infty} \langle q; q^2 \rangle_{\infty} = \frac{(q; q^3)_{\infty}}{(q^2; q^2)_{\infty}} = \frac{F(q^2)^2}{F(q)^3},
\]

if \(k > 0\) is not divisible by 2, \(h, H\) are integers defined as in (3.5), \(h' \equiv 2h \pmod{k}\), and \(k'H' \equiv -1 \pmod{k}\), then

\[
g \left( \exp \left( \frac{2\pi i h}{k} - y \right) \right) = 2e^{\pi i (2s(h',k) - 3s(h,k))} \left( \frac{2\pi}{yk} \right)^{1/2} \exp \left( -\frac{\pi^2}{3k^2 y} - \frac{y}{24} \right) F^2 \left( \exp \left( \frac{2\pi i H'}{k} - \frac{4\pi^2}{k^2 y} \right) \right) / F^3 \left( \exp \left( \frac{2\pi i H}{k} - \frac{4\pi^2}{k^2 y} \right) \right).
\]

Note that

\[
F^2 \left( \exp \left( \frac{2\pi i H'}{k} - \frac{4\pi^2}{k^2 y} \right) \right) / F^3 \left( \exp \left( \frac{2\pi i H}{k} - \frac{4\pi^2}{k^2 y} \right) \right) \leq F^5 \left( \exp \left( -\frac{2\pi^2}{k^2 y} \Re \left( \frac{1}{y} \right) \right) \right) \leq F^5 \left( \exp \left( -\frac{2\pi^2 a}{a^2 + 4\pi^2} \right) \right),
\]

because

\[
\Re \left( \frac{1}{y} \right) = \frac{\rho}{\rho^2 + 4\pi^2 y^2} \geq \frac{k^2 N^2 \rho}{k^2 N^2 \rho^2 + 4\pi^2} \geq \frac{k^2 a}{a^2 + 4\pi^2} \quad \text{if} \quad a = N^2 \rho.
\]
We will choose \(a\) in the interval \(6.25 \leq a \leq 6.5\) so that
\[
F \left( \exp \left\{ -\frac{2\pi^2 a}{a^2 + 4\pi^2} \right\} \right) \leq 1.35.
\]
Also note that
\[
\exp \left( -\frac{\pi^2}{3k^2} \Re \left( \frac{1}{y} \right) \right) \left| y \right|^{-1/2} \leq \Re (y)^{-1/2} = \rho^{-1/2}
\]
and the length of the integral is at most \(4\pi/(kN)\). Then,
\[
|\Sigma_1| \leq \sum_{2k, k<N \atop 0<h<k \atop (h,k)=1} \frac{1}{2\pi} \int_{\rho+2\pi i\theta_{h,k}}^{\rho-2\pi i\theta'_{h,k}} \left( \frac{8\pi}{k} \right)^{1/2} \left| e^{(n-\frac{1}{2})y} \right| \rho^{-1/2} F^5 \left( \exp \left\{ -\frac{2\pi^2 a}{a^2 + 4\pi^2} \right\} \right) |dy|
\]
\[
\leq 4.4 \sum_{2k, k<N \atop 0<h<k \atop (h,k)=1} \frac{2\pi}{kN} \left( \frac{8\pi}{k} \right)^{1/2} e^{n\rho} \rho^{-1/2} \leq 45 \sum_{2k, k<N} \frac{e^{n\rho} \rho^{-1/2}}{k^{1/2}/N} \leq 29 e^{n\rho} \rho^{-1/4}.
\]
Setting \(\rho = \frac{1}{4n}\), (that is, when \(25n \leq N^2 \leq 26n\)) yields Lemma 3.1 \(\Box\)

In order to compute \(\Sigma_2\), consider
\[
\frac{1}{2\pi i} \int_{\rho+2\pi i\theta'_{h,k}}^{\rho-2\pi i\theta'_{h,k}} g \left( \exp \left\{ \frac{2\pi ih}{2k} - y \right\} \right) e^{ny} dy.
\]
Then by the functional equation of the Dedekind-eta function in (3.3), it is equal to
\[
\frac{1}{2\pi i} \int_{\rho+2\pi i\theta'_{h,k}}^{\rho-2\pi i\theta'_{h,k}} \omega_{h,k} \sqrt{\frac{\pi}{yk}} e^{(n-1/4)y} \exp \left( \frac{\pi^2}{24k^2 y} \right) g \left( \exp \left\{ \frac{2\pi i H}{2k} - \frac{4\pi^2}{4k^2 y} \right\} \right) dy,
\]
where \(\omega_{h,k} = e^{\pi i (2s(h,k) - 3s(h,2k))}\). Now we approximate \(g\) by 1. Let \(\Psi(x) = g(x) - 1\). Then
\[
\Sigma_2 = \sum_{2k<N \atop 0<h<2k \atop (h,2k)=1} \frac{-1}{2\pi i} \exp \left( -\frac{2\pi ih}{2k} \right) \omega_{h,k} \sqrt{\frac{\pi}{k}} \int_{\rho+2\pi i\theta'_{h,k}}^{\rho-2\pi i\theta'_{h,k}} \sqrt{\frac{1}{y}} e^{(n-1/4)y} \exp \left( \frac{\pi^2}{24k^2 y} \right) dy
\]
\[
\times \left\{ \Psi \left( \exp \left\{ \frac{2\pi i H}{2k} - \frac{4\pi^2}{4k^2 y} \right\} \right) \right\} dy + 1dy \right\} := \Sigma_3 + \Sigma_4.
\]

**Lemma 3.2.** If \(\Sigma_3\) is as defined in (3.9), then
\[
|\Sigma_3| < 22n^{1/4}.
\]

**Proof.** Since \(|M_e(n) - M_o(n)| \leq p(n), |g(x) - 1| \leq |F(x) - 1|, and
\[
\frac{|\Psi(x)|}{x^{1/24}} \leq \frac{F(|x|)}{|x^{1/24}|}.
\]
Thus
\[
\left| \Psi \left( \exp \left( \frac{2\pi i H}{2k} - \frac{4\pi^2}{4k^2y} \right) \right) \exp \left( -\Re(1/y) \frac{\pi^2}{24k^2} \right) \right| \leq \frac{F \left( \exp \left( -\Re(1/y) \frac{\pi^2}{24k^2} \right) \right) - 1}{\exp \left( -\Re(1/y) \frac{\pi^2}{24k^2} \right)} < 1.844,
\]
by \ref{eq:3.7} and the choice of \( a \). Hence
\[
|\Sigma_3| < 1.844 \sum_{2k<N \atop 0<h<2k} \sqrt{\frac{1}{4\pi k}} \int_{\rho+2\pi i \theta_{h,k}'} \frac{1}{\Re(y)^{1/2}} \exp \left( \Re(y)(n - 1/24) \right) |dy|
\]
\[
\leq 1.844 \sum_{2k<N \atop 0<h<2k} \sqrt{\frac{1}{4\pi k}} \frac{4\pi}{kN} \rho^{-1/2} e^{\rho(1/24)} \leq 1.844 \sum_{2k<N} \frac{4\sqrt{\pi}}{k^{1/2}N} \rho^{-1/2} e^{\rho}
\]
\[
\leq 7.376 \sqrt{2\pi} \rho^{-1/2} e^{\rho} N^{-1/2} \leq 18.489 \rho^{-1/4} e^{\rho} a^{-1/4}.
\]
Since \( \rho = \frac{1}{4n} \) and \( 6.25 \leq a \leq 6.5 \), we have Lemma \ref{lem:3.2} \( \square \)

Now, it remains to approximate \( \Sigma_4 \):

**Lemma 3.3.** If \( B_k(n) \) is as defined in Theorem \ref{thm:1.2} then
\[
\Sigma_4 = \frac{1}{\sqrt{n-1/24}} \sum_{0<h<5\sqrt{n}/2} \frac{B_k(n)}{\sqrt{k}} \cosh \left( \frac{\pi}{24k} \sqrt{\frac{n-1/24}{6}} \right) + E',
\]
where \( |E'| \leq 119n^{1/4} \).

**Proof.** Consider
\[
\frac{1}{2\pi i} \int_{\rho+2\pi i \theta_{h,k}'} \sqrt{\frac{1}{y}} e^{(n-1/24)y} \exp \left( \frac{\pi^2}{24k^2y} \right) dy.
\]
This is
\[
\frac{1}{2\pi i} \left( \int_{-\infty}^{(0+)} - \int_{-\infty}^{(0-)} \int_{-\infty}^{(0+)} + \int_{-\infty}^{(0-)} \right) \sqrt{\frac{1}{y}} e^{(n-1/24)y} \exp \left( \frac{\pi^2}{24k^2y} \right) dy := J_0 + J_1 + J_2,
\]
where \( \int_{-\infty}^{(0+)} \) denotes integration over the contour leading from one branch of \( -\infty \) around 0 to the other branch. To compute an error contributed by \( J_1 \) and \( J_2 \), note that on the lines \( y = x + 2\pi i \theta \), \( (-\infty < x \leq \rho, \theta = \pm \theta_{h,k} \) we have
\[
\Re \left( \frac{\pi^2}{24k^2y} \right) = \frac{x\pi^2}{24k^2(x^2 + 4\pi^2\theta^2)} \leq \frac{\rho}{96k^2\theta^2} \leq \frac{a}{24} \leq 0.271
\]
and
\[
|y|^{-1/2} = \left( \frac{1}{\rho^2 + 4\pi^2\theta^2} \right)^{1/4} \leq \left( \frac{1}{\theta} \right)^{1/2} \leq \sqrt{2kN}.
\]
Hence
\[
|J_1 + J_2| \leq 2e^{0.271} \sqrt{2kN} \frac{e^{\rho(1/24)}}{n - 1/24} \leq \frac{e^{0.271}}{n} \sqrt{2kN} \frac{3e^{1/4}}{2n} \leq 1.137 \sqrt{2kN/n}.
\]
Thus the total error made by $J_1$ and $J_2$ is at most
\[
\sum_{2k<N \atop 0<h<2k} \frac{1.137 \sqrt{kN}}{n} \sqrt{\frac{\pi}{k}} \leq 1.137 \frac{\sqrt{\pi N}}{n} \frac{N(N+1)}{2} \leq 2.016 N^{5/2} n^{-1} \leq 119n^{1/4}.
\]

This leaves us to show that the main term of $\Sigma_4$ is from $J_0$ which is equal to
\[
\sum_{2k<N \atop 0<h<2k} \frac{1.137 \sqrt{kN}}{n} \sqrt{\frac{\pi}{k}} \leq 1.137 \frac{\sqrt{\pi N}}{n} \leq 2.016 N^{5/2} n^{-1} \leq 119n^{1/4}.
\]

By Hankel’s loop integral formula, we have
\[
J_0 = (n - \frac{1}{24})^{-1/2} \sum_{s=0}^{\infty} \frac{1}{2\pi i} \frac{1}{s!} \phi(n - \frac{1}{24})^s \int_{-\infty}^{(0+)} z^{-s-1/2} e^{(n - \frac{1}{24})y} dy.
\]

Then by the dominated convergence theorem and making a change of variable $z = (n - 1/24)y$, we find that
\[
J_0 = (n - \frac{1}{24})^{-1/2} \sum_{s=0}^{\infty} \frac{1}{2\pi i} \frac{1}{s!} \phi(n - \frac{1}{24})^s \int_{-\infty}^{(0+)} z^{-s-1/2} e^{(n - \frac{1}{24})y} dy.
\]

By Hankel’s loop integral formula, we have
\[
J_0 = (n - \frac{1}{24})^{-1/2} \sum_{s=0}^{\infty} \frac{1}{2\pi i} \frac{1}{s!} \phi(n - \frac{1}{24})^s \int_{-\infty}^{(0+)} z^{-s-1/2} e^{(n - \frac{1}{24})y} dy.
\]

Therefore, we obtain the main term of $\Sigma_4$ as
\[
\sum_{2k<N \atop 0<h<2k} \frac{1.137 \sqrt{kN}}{n} \sqrt{\frac{\pi}{k}} \leq 1.137 \frac{\sqrt{\pi N}}{n} \leq 2.016 N^{5/2} n^{-1} \leq 119n^{1/4}.
\]

Recalling $25n \leq N^2 \leq 26n$ completes the proof of Lemma 3.3.

Theorem 1.5 now follows from the three lemmas above.

4. Weighted identities

In this section we prove Theorem 1.6 and discuss another weighted identity of similar type.

We begin with a $q$-series of expansion of the crank generating function [9, Theorem 2.1]:
\[
(q; q)_\infty (xq; q)_\infty = \frac{1}{(q; q)_\infty} \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{n+1/2} (1 - x)}{1 - xq^n}.
\]
Setting $x = -1$, this may be rewritten in the following way:

$$
\sum_{n \geq 0} (M_e(n) - M_o(n))q^n = \frac{1}{(q; q)_\infty} + 4 \sum_{n \geq 1} (-1)^n q^{n(n+1)/2} \frac{(q; q)_{n-1}(1 - q^{2n})(q^{n+1}; q)_\infty}{(q; q)_n(1 + q^{n+1})(q^{n+2}; q)_\infty}.
$$

We shall interpret the right hand side of the identity as the weighted count of partitions in Theorem 1.6. How does this go? The term $1/(q; q)_\infty$ initializes the weight of each partition $\lambda$ to 1. If there are no ones in $\lambda$, then it is not counted at all by the sum on the right hand side, and so the weight just remains 1.

Otherwise, we look at the “initial run” of $\lambda$. A partition $\lambda$ will be counted by the sum on the right hand side for each $j$ in the initial run that occurs an odd number of times. For each such $j$, we add $4(-1)^j$ to the weight. (Those that occur an even number of times contribute nothing.) Then, summing over all partitions of $n$, each counted according to its weight, gives $M_e(n) - M_o(n)$.

As an application, we give a combinatorial proof of the following $q$-series identity:

**Corollary 4.1.**

$$
\frac{1}{(q; q)_\infty} + 4 \sum_{n \geq 1} (-1)^n q^{n(n+1)/2} \frac{(q; q)_{n-1}(1 - q^{2n})(q^{n+1}; q)_\infty}{(q; q)_n(1 + q^{n+1})(q^{n+2}; q)_\infty} = \sum_{n=0}^\infty \frac{(-1)^n q^{n(n+1)/2}(1 - q^{n+1})}{(q; q)_n(1 + q^{n+1})(q^{n+2}; q)_\infty}.
$$

Of course, this identity may also be established analytically. For example, take $a = b = 0, c = d = q$, and $z = -1$ in the following identity of S.H. Chan [12, Eq. (3.1)], valid for $|a/c| < 1$ and $|bq/d| < 1$,

$$
\frac{(az, b/z, q)_{\infty}}{(cz, d/z)_{\infty}} = \frac{(a/c, bc)_{\infty}}{(cd)_{\infty}} \sum_{n=0}^\infty \frac{(cq/a, cd)_n(a/c)_n}{(q, bc)_n(1 - czq^n)} + \frac{(d/ad, b/d)_{\infty}}{(cd)_{\infty}} \sum_{n=0}^\infty \frac{(dq/b, cd)_n(bq/d)_n}{(q, ad)_n(z - dq^n)},
$$

where $(z_1, z_2, \ldots, z_k)_n = \prod_{i=1}^k (z_i; q)_n$. The result shows that the right hand side of (4.3) is indeed the generating function for $M_e(n) - M_o(n)$. This with (4.2) implies Corollary 4.1.

**Proof.** We define a second weight, $\omega_1$ by

$$
\omega_1(\lambda) = (-1)^{\text{length of initial run}} - 2 \sum_j (-1)^j(-1)^\# \text{ occurrences of } j,
$$

where the sum is over those $j$ in the initial run of $\lambda$.

We shall argue by induction on the length of the initial run that for any partition $\lambda$, we have $\omega(\lambda) = \omega_1(\lambda)$. First, if the initial run is empty, then $\omega(\lambda) = \omega_1(\lambda)$. Now, suppose $\lambda$ has an initial run of length $n + 1$. Let \( \lambda - \overline{n+1} \) denote the partition $\lambda$ with all of the parts of size $n + 1$ removed. Then the induction hypothesis says that $\omega(\lambda - \overline{n+1}) = \omega_1(\lambda - \overline{n+1})$.

Let us now determine the effect on the weights of adding back in the parts of size $n + 1$. First, consider the effect on $\omega$. If $n + 1$ occurs an even number of times, the weight is unchanged. If it occurs an odd number of times, then 4 is added to the weight if $n$ is odd and $-4$ is added if $n$ is even.

Now, consider the effect on $\omega_1$. If $n$ is even, the length of the initial run changes from even to odd, giving us a $-2$. If $n + 1$ occurs an even number of times, then we get a $+2$ and the net change is 0. If $n + 1$ occurs an odd number of times, we get a $-2$ and the net change is $-4$. This matches the change to $\omega$ in these cases. A similar argument in the case where $n$ is odd.
shows that the changes to ω and ω₁ are always the same. Hence, we have ω(λ) = ω₁(λ) for all partitions λ.

To complete the proof, it suffices to argue as in the proof of Theorem 1.6 that the right hand side of (4.3) is the generating function for partitions λ counted with weight ω₁(λ). The details are very similar to the case of (4.2), so we omit them. □

Before continuing, we note that similar weighted identities can be found for the rank difference f(q), for example by using the Watson’s equation [24]

\[
\frac{1}{(q;q)\infty} \left( 1 + 4 \sum_{k=1}^{\infty} (-1)^k \frac{q^{k(3k+1)/2}}{1 + q^k} \right).
\]

5. PROOF OF THEOREM 1.7

In this section we prove Theorem 1.7. We begin by deducing a key generating function for \( M_e(\mathcal{D}, n) - M_o(\mathcal{D}, n) \). It is easily seen using the definition of the crank that

\[
\sum_{n=1}^{\infty} (M_e(\mathcal{D}, n) - M_o(\mathcal{D}, n)) q^n = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} q^{n(n+3)/2}}{(-q; q)_n} + \sum_{n=1}^{\infty} \frac{(-1)^n q^{n(n+1)/2}}{(q; q)_{n-1}}.
\]

But this generating function does not tell us much about \( M_e(\mathcal{D}, n) - M_o(\mathcal{D}, n) \). However, we can prove the more useful:

**Theorem 5.1.**

\[
\sum_{n=1}^{\infty} (M_e(\mathcal{D}, n) - M_o(\mathcal{D}, n)) q^n = \frac{1}{1 + q} \sum_{n=1}^{\infty} q^{n(3n+1)/2} (1 - q^{2n+1}) - q(q^2; q)\infty.
\]

**Proof.** We shall demonstrate that

\[
\sum_{n=1}^{\infty} \frac{(-1)^{n+1} q^{n(n+3)/2}}{(-q; q)_n} = \frac{1}{1 + q} \sum_{n=1}^{\infty} q^{n(3n+1)/2} (1 - q^{2n+1})
\]

and

\[
\sum_{n=1}^{\infty} \frac{(-1)^n q^{n(n+1)/2}}{(q; q)_{n-1}} = -q(q^2; q)\infty.
\]

Shifting the summation variable by 1 on both sides of (5.2), we obtain the equivalent identity

\[
\sum_{n=0}^{\infty} \frac{(-1)^n q^{n(n+5)/2}}{(-q^2; q)_n} = \sum_{n=0}^{\infty} q^{(3n^2+7n)/2} (1 - q^{2n+3}).
\]

But this is precisely the specialization \((a, b, c, d, e) \to (q^3, -q^2, \infty, \infty, q)\) of the following limiting case of the Watson-Whipple transformation:

\[
\sum_{n=0}^{\infty} \frac{(aq/bc, d, e)_n (aq)^n}{(q, aq/b, aq/c)_n} = (aq/d, aq/e)_\infty \sum_{n=0}^{\infty} \frac{(a, \sqrt{a}q, -\sqrt{a}q, b, c, d, e)_n (aq)^{2n} (-1)^n q^{n(n-1)/2}}{(q, \sqrt{a}, -\sqrt{a}, aq/b, aq/c, aq/d, aq/e)_n (bcde)^n}.
\]
For (5.3), shifting the summation variable as before leaves us with the task of proving that

\[
\sum_{n=0}^{\infty} \frac{(-1)^n q^{n(n+3)/2}}{(q;q)_n} = (q^2;q)_\infty.
\]

This follows immediately from the case \( z = -q \) of

\[
\sum_{n=0}^{\infty} \frac{z^n q^{n(n+1)/2}}{(q;q)_n} = (-zq;q)_\infty.
\]

This completes the proof Theorem 5.1. \( \square \)

We may now use this theorem to deduce the formula for \( M_e(D, n) - M_o(D, n) \). First we record a formula for each of the two series in (5.1).

**Lemma 5.2.** If \( a(n) \) is defined by

\[
\sum_{n=1}^{\infty} a(n)q^n = \frac{1}{1+q} \sum_{n=1}^{\infty} q^{n(3n+1)/2}(1-q^{2n+1}),
\]

then

\[
a(n) = \begin{cases} 
-(-1)^{n-[n]_p}, & \text{if } R([n]_p) \text{ is even and positive}, \\
0, & \text{if } R([n]_p) \text{ is odd and negative}, \\
(-1)^{n-[n]_p}, & \text{if } R([n]_p) \text{ is odd and positive}, \\
-2(-1)^{n-[n]_p}, & \text{if } R([n]_p) \text{ is even and negative}.
\end{cases}
\]

**Proof.** The proof is elementary. One simply expands \( 1/(1+q) \) as \( \sum (-1)^n q^n \) in (5.8), multiplies the series together, and verifies that the result is (5.9). \( \square \)

**Lemma 5.3.** If \( b(n) \) is defined by

\[
\sum_{n=1}^{\infty} b(n)q^n = -q(q^2;q)_\infty,
\]

then

\[
b(n) = \begin{cases} 
0, & \text{if } R([n]_p) \text{ is positive}, \\
-1, & \text{if } R([n]_p) \text{ is odd and negative}, \\
1, & \text{if } R([n]_p) \text{ is even and negative}.
\end{cases}
\]

**Proof.** This is another elementary calculation, using the fact that

\[
(q^2;q)_\infty = \frac{1}{1-q} \sum_{n \in \mathbb{Z}} (-1)^n q^{n(3n+1)/2}.
\]

\( \square \)

Theorem 1.7 now follows by combining the above two lemmas and checking all of the different cases. \( \square \)

In closing this section, we might mention that it is also possible to adapt Franklin’s involution \([3, pp.10-11]\) to prove Theorem 1.7.
6. Concluding Remarks

We wish to end by offering two suggestions for future research. First, there do not seem to be any simple congruences of the form $M_e(pn + a) - M_o(pn + a) \equiv 0 \pmod{p}$ for $p$ prime except when $p = 5$. Can the work of Ahlgren-Boylan [1] and Kiming-Olsson [18] be adapted to prove that this is indeed the case? Second, can the ideas of [20] be applied to extend the congruences modulo $5^a$ to congruences modulo $5^{a+1}$ within certain arithmetic progressions?

References

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