TAME COVERS AND COHOMOLOGY OF RELATIVE CURVES
OVER COMPLETE DISCRETE VALUATION RINGS WITH
APPLICATIONS TO THE BRAUER GROUP

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Abstract. We prove the existence of noncrossed product division algebras
and indecomposable division algebras of unequal period and index over the
function field of any $p$-adic curve, extending the results and methods of [10].

1. Introduction

We study the cohomology and the Brauer group of a field $F$ that is finitely
generated and of transcendence degree one over the $p$-adic field $\mathbb{Q}_p$. Such a field is
always the function field of a regular (projective, flat) relative curve $X/\mathbb{Z}_p$. In [10]
it was shown that if $F$ admits a smooth model $X/\mathbb{Z}_p$ then there exist noncrossed
product $F$-division algebras, and indecomposable $F$-division algebras of unequal
prime-power period and index. These were constructed from objects defined over
the generic point $(p)$ of the closed fiber $X_0 = X \otimes_{\mathbb{Z}_p} \mathbb{F}_p$ using a homomorphism
$\lambda : \text{Br}(F_p) \to \text{Br}(F)$ that splits the restriction map $\text{res} : \text{Br}(F) \to \text{Br}(F_p)$, where
the subscript $p$ denotes completion with respect to the valuation defined by $(p)$, and
" $'$ " denotes "prime-to-$p$ part". The fields $F$ that have a smooth $X/\mathbb{Z}_p$ include
fields such as $\mathbb{Q}_p(t)$ but do not include the function fields of all $p$-adic curves.

In this paper we generalize the machinery and results of [10] to arbitrary $p$-adic
curves. We prove that if $F$ is the function field of a $p$-adic curve then there exist
noncrossed product $F$-division algebras, and indecomposable $F$-division algebras of
unequal prime-power period and index. The machinery we develop here is used
in [11] to prove that every $F$-division algebra of prime period $\ell$ and index $\ell^2$
decomposes into two cyclic $F$-tensor factors, hence is a crossed product, generalizing
Suresh’s result [36], which assumes roots of unity. In the terminology of [5, Sections
3,4], this shows the $\mathbb{Z}/\ell$-length in $H^2(F,\mu_{\ell})$ equals the $\ell$-Brauer dimension, which
is two by a theorem of Saltman ([33, Theorem 3.4]). In general our work is motivated
by the work of Saltman over these fields in [33] and [35] (see also [9]).

We summarize the technical results. Let $R$ be a complete discrete valuation
ring with fraction field $K$, and let $F$ be a finitely generated field extension of $K$
of transcendence degree one. Let $X/R$ be a regular (projective, flat) model for $F$
whose closed fiber $X_0$ has normal crossings on $X$. Let $C = X_{0,\text{red}}$, let $\{C_i\}$ denote
the set of irreducible components of $C$, let $S$ be the set of singular points of $C$, and
set $F_C = \prod_i F_{C_i}$, the product of the completions with respect to the valuations

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on $F$ defined by the $C_i$. We construct for any prime-to-char($\kappa(C)$) number $n$, any integer $r$, and any $q \geq 0$, a homomorphism

$$\lambda : H^q(O_{C,S}, \mathbb{Z}/n(r)) \to H^q(O_{X,S}, \mathbb{Z}/n(r)) \to H^q(F, \mathbb{Z}/n(r))$$

that splits the restriction map $H^q(O_{X,S}, \mathbb{Z}/n(r)) \to H^q(O_{C,S}, \mathbb{Z}/n(r))$. We use $\lambda$ to construct another map $\lambda$ that splits the subgroup of the image of $H^q(F, \mathbb{Z}/n(r)) \to H^q(F_c, \mathbb{Z}/n(r))$ consisting of tuples of classes that are unramified at $S$ and glue along $S$. When $q = 2$, $\mathbb{Z}/n(r) = \mu_n$, and $R = \mathbb{Z}_p$, we show that $\lambda$ preserves index. This allows us to construct indecomposable $F$-division algebras and noncrossed product $F$-division algebras as mentioned above, in the same manner as [10]. When the dual graph of the closed fiber $X_0$ has nontrivial topology, i.e., nonzero (first) Betti number, we construct cyclic covers of $X$ that are (defined and) trivial at every point of $X$ except the generic point of $X$. These arise as cyclic covers of the closed fiber $X_0$ that are trivial at every point, and transported to $X$ via $\lambda$. When $R = \mathbb{Z}_p$ they are the completely split cyclic covers of Saito ([30]). We thank Colliot-Thélène for drawing our attention to these interesting specimens.

2. Background and Conventions

We use [26] Chapter 8.9, [23] Section 2], and [22] Chapter XIII] for many of the following definitions.

2.1. General Conventions. Let $S$ be an excellent scheme, $n$ a number that is invertible on $S$, and $\Lambda = (\mathbb{Z}/n)(i)$ the twisted étale sheaf. We write $H^q(S, \Lambda)$ for the étale cohomology group, and if $\Lambda$ is understood (or doesn’t matter) we write $H^q(S, r)$ instead of $H^q(S, \Lambda(r))$, and $H^q(S)$ in place of $H^q(S, 0)$. If $S = \text{Spec } A$ for a ring $A$ then we write $H^q(A, r)$.

If $T$ is an integral scheme contained in $S$ we write $k(T)$ for its function field. If $T \to S$ is a morphism of schemes then the restriction $\text{res}_{T|S} : H^q(T) \to H^q(S)$ is defined, and we write $\beta_S = \text{res}_{T|S}(\beta)$. If $Z \subset S$ is a subscheme, we write $Z_T$ for the preimage $Z \times_S T$.

2.2. Valuation Theory. If $v$ is a valuation on a field $F$ we write $\kappa(v)$ for the residue field of the valuation ring $O_v$, and $F_v$ for the completion of $F$ at $v$. If $S$ is a connected normal scheme with function field $F$ and $v$ arises from a prime divisor $D$ on $S$, we write $v_D$ for $v$, $\kappa(D)$ for $\kappa(v)$, and $F_D$ for $F_v$. If $D$ is a sum of prime divisors $D_i$ we write $F_D = \prod_i F_{D_i}$. Each $f \in F^*$ defines a divisor $\text{div}(f) = \sum v_D(f)D$, where the (finite) sum is over prime divisors on $S$.

Recall that if $F = (F, v)$ is a discretely valued field and $\alpha \in H^q(F, \Lambda)$ then $\alpha$ has a residue $\partial_v(\alpha)$ in $H^{q-1}(\kappa(v), \Lambda(-1))$. More generally, suppose $T$ is a noetherian scheme, $\xi$ is a generic point of $T$, and $\alpha \in H^q(T, \Lambda)$. Then for each discrete valuation $v$ on the field $F = \kappa(\xi_{red})$ $\alpha$ has a residue

$$\partial_v(\alpha) \overset{\text{df}}{=} \partial_v(\alpha_F) \in H^{q-1}(\kappa(v), \Lambda(-1))$$

We say $\alpha$ is unramified at $v$ if $\partial_v(\alpha) = 0$, ramified at $v$ if $\partial_v(\alpha) \neq 0$, and tamely ramified at $v$ if $\partial_v(\alpha)$ is contained in the prime-to-char($\kappa(v)$) part of $H^{q-1}(\kappa(v), \Lambda(-1))$.

If $\alpha$ is unramified at $v$ the value of $\alpha$ at $v$ is the element

$$\alpha(v) = \text{res}_{F|F_v}(\alpha_F) \in H^q(\kappa(v), \Lambda) \leq H^q(F_v, \Lambda)$$

(see [16] 7.13, p.19]). Suppose $T \to S$ is a birational morphism of noetherian schemes (see [17] 1.2.2.9]). The ramification locus of $\alpha$ on $S_{\text{red}}$ is the sum of the
prime divisors on \( S_{\text{red}} \) that determine valuations at which \( \alpha \) is ramified, over all generic points of \( S_{\text{red}} \).

Let \( D \) be a divisor on a noetherian normal scheme \( S \), set \( U = S - D \), and for each generic point \( \xi \) of \( \text{Supp} \, D \), let \( K_{\xi} \) denote the fraction field of the discrete valuation ring \( O_{S,\xi} \). We say a morphism \( \rho : T \rightarrow S \) is \textit{tamely ramified along} \( D \) if \( T \) is normal, \( \rho_U : V = T \times_S U \rightarrow U \) is étale, and for each generic point \( \xi \) of \( \text{Supp} \, D \), the étale \( K_{\xi} \)-algebra \( L \) defined by \( \text{Spec} \, L = V \times_U \text{Spec} \, K_{\xi} \) is tamely ramified with respect to \( O_{S,\xi} \). Since \( L/K_{\xi} \) is étale it is a finite product of separable field extensions of \( K_{\xi} \), and \( L \) is tamely ramified if each field extension is tamely ramified (with respect to \( O_{S,\xi} \)) in the usual sense. If \( S \) is a noetherian scheme whose irreducible components are normal, we’ll say a morphism \( \rho : T \rightarrow S \) is tamely ramified along \( D \) if again \( V \rightarrow U = S - D \) is étale, and the restriction \( \rho_{S_i} \) to each irreducible component \( S_i \) of \( S \) is tamely ramified along \( D_{S_i} \). If \( S = \text{Spec} \, A \) and \( T = \text{Spec} \, B \), we will also say \( B \) is a \textit{tamely ramified} \( A \)-algebra. We say a map \( \rho : T \rightarrow S \) of noetherian schemes is a \textit{cover} if it is finite, generically étale, and each connected component of \( T \) dominates a connected component of \( S \).

2.3. Relative Curves. In this paper, \( R \) will be a complete discrete valuation ring with residue field \( k \) and fraction field \( K \), \( F \) will be a field finitely generated of transcendence degree one over \( K \), and \( X/R \) will be a regular 2-dimensional scheme \( X \) that is flat and projective over \( \text{Spec} \, R \) and has function field \( K(X) = F \). We call \( X/R \) a \textit{regular relative curve}, write \( X_0 = X \otimes_R k \) for the closed fiber, \( C = X_0,\text{red} \) for the reduced subscheme underlying the closed fiber, and \( C_1, \ldots, C_m \) for the irreducible components of \( C \). We assume each \( C_i \) is regular, and at most two of them meet at any closed point of \( X \), a situation that can always be obtained by blowing up using Lipman’s embedded resolution theorem (see \cite[9.2.4]{20}). For all closed points \( z \in X \), we have \( \dim O_{X,z} = 2 \) by \cite[8.3.4(c)]{26}, and since \( X \) is regular, \( O_{X,z} \) is factorial by Auslander-Buchsbaum’s theorem.

We say an effective divisor \( D \) on a relative curve \( X/R \) is \textit{horizontal} if each of its irreducible components maps surjectively (hence finitely) to \( \text{Spec} \, R \), and \textit{vertical} if its support is contained in the support of the closed fiber. If \( D \) is a reduced and irreducible horizontal divisor then it is flat over \( \text{Spec} \, R \), since \( R \) is a discrete valuation ring. Every effective divisor on a relative curve \( X/R \) is a sum of horizontal and vertical divisors, and the horizontal prime divisors are exactly the closures of the closed points of the generic fiber (\cite[8.3.4(b)]{26}). Since \( R \) is henselian, each irreducible horizontal divisor has a single closed point.

2.4. Distinguished Divisors. In general there will be many horizontal divisors on a relative curve \( X \) that restrict to a given divisor on \( C \). In order to construct our lifts of covers and cohomology classes from \( C \) to \( X \) we select a single regular horizontal divisor for each closed point, as follows.

Proposition 2.5. Assume the setup of \cite{20}. Then for each closed \( z \in X \) there exists a regular irreducible horizontal divisor \( D \subset X \) that intersects each irreducible component of \( C \) passing through \( z \) transversally at \( z \).

Proof. Transversal intersection with a single component is by \cite[8.3.35(g)]{26} and its proof (see also \cite[21.9.12]{21}). Thus if \( z \in C_i \cap C_j \) \((i \neq j)\) and \( t_i \) and \( t_j \) are local equations for \( C_i \) and \( C_j \), then we have local equations \( f_i \) and \( f_j \) for effective
regular horizontal divisors such that \((f_i, t_i) = (f_j, t_j) = m_z \subset O_{X,z}\). If \((f_j, t_i) = m_z\) or \((f_i, t_j) = m_z\) then a suitable \(D\) is defined locally by \(f_j\) or \(f_i\). Otherwise \((f_i + f_j, t_i) = (f_i + f_j, t_j) = m_z\), and we define \(D\) locally by \(f_i + f_j\). The rest of the proof proceeds as in [20, 3.3.35].

We fix a set of these (prime) divisors, and let \(\mathcal{D}\) denote the set of supports of the semigroup they generate in \(\text{Div} \ X\). We will say a divisor \(D\) is distinguished and write \(D \in \mathcal{D}\) whenever \(D\) is reduced and supported in \(\mathcal{D}\). Though \(\mathcal{D}\) is fixed in principle, we reserve the right to declare any divisor satisfying the definition to be a member of this set retroactively. Let \(\mathcal{D}_S\) denote the subset that avoids \(S\). Note that each \(D \in \mathcal{D}\) is a disjoint union of its irreducible components, each of which meets each irreducible component of \(C\) transversally.

### 3. Structure of Tame Covers

**Lemma 3.1 (Structure).** Assume the setup of [23]. Suppose \(\rho : Y \to (X, D)\) is a tamely ramified cover, where \(D \in \mathcal{D}\). Then

a) The structure map \(\rho : Y \to X\) is flat.

b) \(Y/R\) is a regular relative curve, \(Y_{0, \text{red}} = C_Y\), each irreducible component of \(C_Y\) is regular, \(\mathcal{S}_Y\) is the set of singular closed points of \(C_Y\), and exactly two irreducible components of \(C_Y\) meet at each point of \(\mathcal{S}_Y\).

c) The support of the irreducible components of \(D'_Y\) for \(D' \in \mathcal{D}\) generate a set \(\mathcal{D}_Y\) of distinguished divisors on \(Y\).

**Proof.** Since \(Y \to X\) is finite, \(\dim(X) = \dim(Y) = 2\) by [19, 5.4.2], and \(Y \to \text{Spec} \ R\) is projective as the composition of projective morphisms ([20, 3.3.32]). Let \(y \in Y\) be a closed point and set \(x = f(y)\), \(A = O_{X,x}\), \(B' = O_{Y,x}\), and \(B = O_{Y,y}\). Choose a geometric point over \(x\) that lifts to each point of \(Y\) lying over \(x\), and use this in the following to define the strict henselizations with respect to the maximal ideals of these points.

Since the statements involving \(D\) are local and \(D\) is a disjoint union of its irreducible components we may assume \(D\) is irreducible. Let \(C_i \subset C\) be a (regular) irreducible component going through \(x\), and let \(\{f, t\} \subset A\) be the regular system of parameters formed by local equations for the distinguished prime divisor passing through \(x\), and for \(C_i\), respectively. Then the strict henselization \(A_{\text{sh}}\) of \(A\) with respect to the maximal ideal of \(A\) is a two-dimensional regular local ring, faithfully flat over \(A\), with regular system of parameters \(\{f, t\}\) (see [21, 18.8]).

If \(x \notin D\) then \(B' \otimes_A A_{\text{sh}}\) is a finite étale \(A_{\text{sh}}\)-algebra by base change, since \(\rho|_{X - D}\) is finite-étale. If \(x \in D\) then \(B' \otimes_A A_{\text{sh}}\) is a finite tamely ramified \(A_{\text{sh}}\)-algebra by [23 Lemma 2.2.8]. By [21, 18.8.10], \(B_{\text{sh}}\) is a factor of the direct product decomposition of \(B' \otimes_A A_{\text{sh}}\), hence \(B_{\text{sh}}\) is a finite tamely ramified local \(A_{\text{sh}}\)-algebra, in particular it is a normal local ring, hence it is a normal domain. It follows that \(B_{\text{sh}}\) is the integral closure of \(A_{\text{sh}}\) in the field \(\bar{L} = \text{Frac} \ B_{\text{sh}}\). Since the tame fundamental group of the strictly henselian regular local ring \(A_{\text{sh}}\) is abelian ([22, XIII.5.3]) the field extension \(\bar{L}/\text{Frac} \ A_{\text{sh}}\) is Galois, and by Abhyankar’s Lemma ([13, A.I.11], see also [23 Corollary 2.3.4])

\[
B_{\text{sh}} = A_{\text{sh}}[T]/(T^e - f) \quad (\text{some } e \geq 1)
\]
By [23] Lemma 1.8.6| $B^\text{sh}$ is a regular (2-dimensional) local ring with system of parameters $\{\sqrt{T}, t\}$. Since $B \rightarrow B^\text{sh}$ is faithfully flat and $B^\text{sh}$ is regular, $B$ is regular by flat descent ([13] 6.5.1) or [24] 23.7(i)), and since $B$ is the local ring of an arbitrary closed point, we conclude $Y$ is regular. It follows that $\rho : Y \rightarrow X$ is flat by [24] 23.1), proving (a), and since $Y$ is regular and $Y \rightarrow \text{Spec } R$ is flat and projective, $Y/R$ is a regular relative curve.

We derive a system of parameters for $B$. The prime ideal $(\sqrt{T}) \subset B^\text{sh}$ is the only one lying over $(f, A^\text{sh})$ since, for $\kappa(f) = \text{Frac } A^\text{sh}/(f)A^\text{sh}$, the ring $B^\text{sh} \otimes_{A^\text{sh}} \kappa(f) = \kappa(f)[T]/(T^e)$ of the fiber over $\text{Spec } \kappa(f)$ consists of a single prime ideal. The image $(\sqrt{T})$ in $\text{Spec } B$ is therefore a unique prime $(g) \subset B$ lying over $(f) \subset A$, and $(\sqrt{T})$ is the unique prime lying over $(g)$. Therefore, since $B \rightarrow B^\text{sh}$ is unramified, $(g)B^\text{sh} = (\sqrt{T})$. Since $B \rightarrow B^\text{sh}$ is faithfully flat, $IB^\text{sh} \cap B = I$ for all ideals $I$ of $B$ (by e.g. [4] Exercise 3.16)), so since $(g, t)B^\text{sh} = (\sqrt{T}, t)$ is maximal, $(g, t)B^\text{sh} \cap B = (g, t)$ is the maximal ideal of $B$. Thus $(g, t)$ is a regular system for $B$.

Since $t$ is a local equation for $\rho^{-1}C_i$, $\rho^{-1}C_i$ is regular and irreducible at $y$ for each $C_i$ passing through $x$. In particular $C_Y = \bigcup_i \rho^{-1}C_i$ is reduced, and so equals $Y_0, \text{red}$. Since at most two irreducible components of $C$ meet at $x$, the same holds for $C_Y$ at $y$, and $y$ is a singular point on $C_Y$ if and only if $x = f(y) \in S$. This completes the proof of (b).

If $D' \in D$ is the distinguished (horizontal) prime divisor running through $x$ then there is a single irreducible component of $D'_Y$ passing through $y$, whose support $D'_Y, \text{red}$ has local equation $g$ at $y$. Thus each irreducible component of $D'_Y$ covers $D'$, hence $\text{Spec } R$, hence $D'_Y$ is horizontal. Since $g$ is part of the regular system $\{g, t\}$ at the arbitrary closed point $y$ we see that $D'_Y, \text{red}$ is regular, and since $t$ is a local equation for an arbitrary irreducible component of $C_Y$ passing through $y$, $D'_Y, \text{red}$ intersects each component of $C_Y$ transversally. Thus the support of the irreducible components of $D'_Y$ generate a set of distinguished divisors $D_Y$ for $Y$. This proves (c).

\begin{lemma}
Suppose $X$ is a regular noetherian scheme and $L$ is an étale $K(X)$-algebra that is tamely ramified along a divisor $D$. Then the normalization $Y$ of $X$ in $L$ defines a tamely ramified cover $\rho : Y \rightarrow (X, D)$.
\end{lemma}

\begin{proof}
Since $X$ is regular, its connected components are integral regular schemes, hence we may assume $X$ is integral. Since $L/K(X)$ is étale, $L$ is a product of finite separable field extensions of $K(X)$, hence we may assume $L/K(X)$ is itself a finite separable field extension. Then the normalization $Y$ exists, $Y$ is normal by definition, and $\rho : Y \rightarrow X$ is finite by [20] 4.1.25). Since $Y$ is normal and connected it is irreducible, so $Y$ dominates $X$. Let $U = X - D$, and set $V = Y \times_X U$. Since $X$ is normal, $Y$ is connected, and $\rho|_Y$ is unramified, $\rho|_Y$ is étale by [22] I.9.11 (see also [28] I.3.20)). Therefore $Y \rightarrow (X, D)$ is a tamely ramified cover.
\end{proof}

The next lemma shows how distinguished divisors split in tamely ramified covers.

\begin{lemma}
Assume the setup of (a). Suppose $\rho : Y \rightarrow (X, D)$ is a tamely ramified cover, where $D \in D$, and $D' \in D_S$ is irreducible. Suppose $E \subset D'_Y, \text{red}$ is a distinguished prime divisor lying over $D'$ as in Lemma 3.1(c), $y = E \times_Y C_Y$, and
\end{lemma}
$x = D' \times_X C$. Then $y$ and $x$ are regular closed points, and the ramification (resp. inertia) degree of $v_E$ over $v_{D'}$ equals the ramification (resp. inertia) degree of $v_y$ over $v_x$.

Proof. Since we assume (2.3) and $E$ is reduced and $E \subset D'_{\text{red}}$ is distinguished. Note that either $D' \cap D = \emptyset$ or $D' \subset D$. Since $D'$ and $E$ are distinguished and avoid the singular points of $X$ and $Y$, they intersect the reduced closed fibers $C$ and $Cy$ transversally, hence $x = D' \times_X C$ and $y = E \times_Y C_y$ are regular closed points. We must show that $[\kappa(E) : \kappa(D')] = [\kappa(y) : \kappa(x)]$ and that $v_E(f) = v_y(f_0)$, where $f \in O_{X,D'}$ is a local equation for $D'$ on $X$ and $f_0 \in O_{C,x}$ is a local equation for $x$ on $C$.

Since $D'$ is horizontal and irreducible, $D' = \text{Spec } S$ for a finite local $R$-algebra by 1.4.2, and $S$ is a discrete valuation ring since $D'$ is regular. The map $E \to \rho^{-1}D' \to D'$ is finite as a composition of finite morphisms, hence $E = \text{Spec } T$ for $T$ a finite local $S$-algebra, again a discrete valuation ring since $E$ is regular. Since $S$ is a discrete valuation ring and $S \to T$ is finite, $T$ is a free $S$-module of finite rank, and so $[T : S]$ is well defined. Since the generic point of $E$ lies over that of $D'$, we have $\text{Frac } T = T \otimes_S \text{Frac } S$, hence $[\kappa(E) : \kappa(D')] = [T : S]$.

Let $A = O_{X,x}$, $B = O_{Y,y}$, let $t$ be a local equation for $C$ at $x$, and set $A_0 = A/(t)$ and $B_0 = B/(t)B$, the (reduced) local rings of the fibers through $x$ and $y$, as in the proof of Lemma 3.1. Already $\kappa(x) = S \otimes_A A_0$ and $\kappa(y) = T \otimes_B B_0$ by the transversality of the intersections. Since $B_0 = B \otimes_A A_0$ we have $\kappa(y) = T \otimes_A A_0$, hence $[\kappa(y) : \kappa(x)] = [T : S] = [\kappa(E) : \kappa(D')]$ by base change.

Let $g \in A$ be defined as above. To compute the ramification degree, note that since $B \to B^{\text{sh}}$ is faithfully flat, $(gf')B = (gf')B^{\text{sh}} \cap B = (f)B^{\text{sh}} \cap B = (f)B$, hence $g' = fu$ for some $u \in B^*$. Since $f$ and $g$ are uniformizers for $v_{D'}$ and $v_E$, respectively, it follows that $e(v_E/v_{D'}) = e(v_E(f)) = e$. On the other hand, let $f_0$ be the image of $f$ in $A_0$, and let $g_0$ be the image of $g$ in $B_0$. Then $f_0$ cuts out the closed point $x$ on $C$ and $g_0$ cuts out $y$ on $C_y$ by transversality. Thus $f_0$ and $g_0$ are uniformizers for $v_x$ and $v_y$, and since $g_0^x = f_0^x g_0$, where $u_0$ is the image of $u$ in $B_0^*$, we have $e(v_y/v_x) = v_y(f_0) = e$, as desired. This completes the proof.

4. LIFTING COHOMOLOGY CLASSES

4.1. Let $k$ be a field, and let $C/k$ be a reduced connected projective curve with regular irreducible components $C_1, \ldots, C_m$, at most two of which meet at any closed point. Denote the singular points of $C$ by $S$ and write $O_{C,S}$ for the semilocal ring $\lim_{U} O_{C}(U)$, where $U$ varies over (dense) open subsets of $C$ containing $S$. Then $O_{C,S}$ is a subring of the rational function ring $\kappa(C) = \prod_i \kappa(C_i)$. For each $z \in S \cap C_i$, let $K_{i,z} = \text{Frac } O_{C_{i,z}}^h$, a field since $z$ is a normal point of $C_i$, and if $\alpha_i \in H^q(\kappa(C_i))$, let $\alpha_{i,z}$ denote the image in $H^q(K_{i,z})$.

Lemma 4.2 (Gluing). Assume the setup of 4.1. There exists an element $\alpha \in H^q(O_{C,S}, \Lambda)$ that restricts to $\alpha_C = (\alpha_1, \ldots, \alpha_m) \in \bigoplus_i H^q(\kappa(C_i), \Lambda)$ if and only if $\alpha_i$ is unramified at each $z \in S \cap C_i$, and $\alpha_{i,z} = \alpha_{j,z}$ whenever $z \in C_i \cap C_j$. 

Proof. There is an exact sequence ([28, III.1.25])

\[ 0 \to H^0_\mathcal{S}(O_{C,S}) \to H^0(O_{C,S}) \to H^0(\kappa(C)) \to H^1_\mathcal{S}(O_{C,S}) \to \]
\[ \to H^1(O_{C,S}) \to H^1(\kappa(C)) \to H^2_\mathcal{S}(O_{C,S}) \to \]
\[ \to H^2(O_{C,S}) \to H^2(\kappa(C)) \to H^3_\mathcal{S}(O_{C,S}) \]

(*)

where the maps into the direct sum are restrictions. Since \( \mathcal{S} \) is a disjoint union of closed points, \( H^q_\mathcal{S}(O_{C,S}) = \bigoplus_{x \in \mathcal{S}} H^q_x(O_{C,x}) \) by excision ([28, III.1.28, p.93]). Since \( A \) is a smooth group scheme, \( H^q(O_{C,x}^h) = H^q(\kappa(x)) \), by the cohomological Hensel’s lemma ([28, III.3.11(a), p.116]). Since the C map given by inflation from \( \kappa \) (4.3) 0

Thus the long exact sequence breaks up into short exact sequences

\[ 0 \to H^q(\kappa(z)) \to H^q(K_{i,z} \times K_{j,z}) \to H^q_z(O_{C,S}) \to \]
\[ \to H^q(K_{i,z}) \to H^q(K_{i,z} \times K_{j,z}) \to H^q_z(O_{C,S}) \to \]
\[ \to H^q(K_{j,z}) \to H^q(K_{i,z} \times K_{j,z}) \to H^q_z(O_{C,S}) \to \cdots \]

where the map \( H^q(\kappa(z)) \to H^q(K_{i,z} \times K_{j,z}) = H^q(K_{i,z}) \oplus H^q(K_{j,z}) \) is the diagonal map given by inflation from \( \kappa(z) \) to the “local fields” \( K_{i,z} \) and \( K_{j,z} \). Since \( n \) is prime-to-\( p \), the map \( H^q(\kappa(z)) \to H^q(K_{i,z}) \) is an isomorphism, so \( H^q_z(O_{C,S}) = 0 \), and for \( q \geq 1 \) we have short exact Witt-type sequences

\[ 0 \to H^q(\kappa(z)) \to H^q(K_{i,z}) \xrightarrow{\partial_z} H^{q-1}(\kappa(z), -1) \to 0 \]

Thus the long exact sequence breaks up into short exact sequences

\[ (4.3) \quad 0 \to H^q(\kappa(z)) \to H^q(K_{i,z} \times K_{j,z}) \to H^{q+1}_z(O_{C,S}) \to 0 \quad (q \geq 0) \]

By the compatibility of the localization sequence with the excised sequence, the map \( H^q(\kappa(C_i)) \to H^{q+1}_z(O_{C,S}) \) of (\#) factors through \( \text{res}_{\kappa(C_i)}K_{i,z} \). Therefore an element \( \alpha_C = (\alpha_1, \ldots, \alpha_m) \in H^q(\kappa(C)) \) maps to zero in \( H^{q+1}_z(O_{C,S}) \) if and only if each couple \( (\alpha_{i,z}, \alpha_{j,z}) \) is in the image of some \( \tilde{\alpha} \in H^q(\kappa(z)) \); i.e., \( \alpha_{i,z} = \alpha_{j,z} \), and both are unramified. Thus by the exactness of (\#), \( \alpha_C \) comes from \( H^q(O_{C,S}) \) if and only if each \( \alpha_i \) is unramified at each \( z \in S \cap C_i \), and \( \alpha_{i,z} = \alpha_{j,z} \) whenever \( z \in C_i \cap C_j \).

\[ \square \]

Suppose \( C \) is as in (4.1). Since exactly two irreducible components meet at any \( z \in S \) the dual graph \( G_C \) is defined, and consists of a vertex for each irreducible component of \( C \) and an edge for each singular point, such that an edge and a vertex are incident when the corresponding singular point lies on the corresponding irreducible component ([32, 2.23], see also [26, 10.1.48]). The (first) Betti number for \( G_C \) is \( \beta_C \overset{\text{df}}{=} \text{rk}(H_1(G_C, Z)) = N + E - V \), where \( V, E \), and \( N \) are the numbers of vertices, edges, and connected components of \( G_C \), respectively.

**Lemma 4.4.** Assume the setup of (4.7). Then:

a) For any integer \( r \), \( H^1(C, Z/n(r)) \to H^1(O_{C,S}, Z/n(r)) \) is injective.
b) The map $H^q(\Omega_{C,S},\mathbb{Z}/n(q - 1)) \to H^q(\kappa(C),\mathbb{Z}/n(q - 1))$ is injective for $q = 0, 2$, and for $q = 1$ we have

$$H^1(\Omega_{C,S},\mathbb{Z}/n) \simeq (\mathbb{Z}/n)^{\beta C} \oplus \Gamma$$

where $(\mathbb{Z}/n)^{\beta C}$ is the kernel of $H^1(\Omega_{C,S},\mathbb{Z}/n) \to H^1(\kappa(C),\mathbb{Z}/n)$, and $\Gamma \leq H^1(\kappa(C),\mathbb{Z}/n)$ is the group of tuples that glue as in Lemma 4.2.

Proof. We suppress the notation for $\Lambda = \mathbb{Z}/n(r)$. Let $z \in C - S$ be a (regular) closed point, and set $U = C - \{z\}$, a dense open subset containing $S$. The localization exact sequence is

$$0 \to H^0_z(C) \to H^0(C) \to H^0(U) \to \cdots$$

$$\cdots \to H^2_z(C) \to H^2(C) \to H^2(U) \to H^2_z+1(C) \to \cdots$$

By excision we have an exact sequence

$$0 \to H^0_z(C) \to H^0_{O_{C,z}^h} \to H^0(K_z) \to \cdots$$

$$\cdots \to H^2_z(C) \to H^2_{O_{C,z}^h} \to H^2(K_z) \to H^2_z+1(C) \to \cdots$$

where $K_z = \text{Frac} O_{C,z}^h$. Since $z$ is a regular point $O_{C,z}^h$ is a discrete valuation ring, and by [12] Section 3.6] we may replace $H^2_{O_{C,z}^h}$ with $H^2_{O_{C,z}^h}$, and the map from $H^2(U)$, which factors through $H^2(K_z)$, is then the residue map $\partial_z$. We conclude $H^0(C) = H^0(U)$, and we have a long exact sequence

$$0 \to H^1(C) \to H^1(U) \xrightarrow{\partial_z} H^0(\kappa(z), -1) \to \cdots$$

$$(**) \to H^0(C) \to H^0(U) \xrightarrow{\partial_z} H^0(\kappa(z), -1) \to \cdots$$

As $H^1(\Omega_{C,S}) = \lim_{\to} H^1(U)$, where the limit is over dense open subsets of $C$ containing $S$, $H^1(C) \to H^1(\Omega_{C,S})$ is injective by the exactness of the injective limit functor, proving (a).

For (b) we go back to $\Lambda = \mathbb{Z}/n$. By (*) we have an exact sequence

$$0 \to H^0(\Omega_{C,S},\mathbb{Z}/n) \xrightarrow{\phi_1} H^0(\kappa(C),\mathbb{Z}/n) \xrightarrow{\phi_2} H^1_{O_{C,S},\mathbb{Z}/n} \xrightarrow{\phi_3}$$

$$\xrightarrow{\phi_4} H^1(\Omega_{C,S},\mathbb{Z}/n) \xrightarrow{\phi_4} H^1(\kappa(C),\mathbb{Z}/n)$$

The groups $H^0(\Omega_{C,S},\mathbb{Z}/n)$ and $H^0(\kappa(C),\mathbb{Z}/n)$ are free finite $\mathbb{Z}/n$-modules whose ranks are the number of $C$'s connected components $N$ and irreducible components $m$, respectively. We claim $H^0_{O_{C,S},\mathbb{Z}/n}$ is a finite free $\mathbb{Z}/n$-module. For by [13] for each $z \in S$ we have an exact sequence

$$0 \to H^0(\kappa(z),\mathbb{Z}/n) \to H^0(K_{i,z},\mathbb{Z}/n) \oplus H^0(K_{j,z},\mathbb{Z}/n) \to H^1_{O_{C,z}^h,\mathbb{Z}/n} \to 0$$

This shows $H^1_{O_{C,z}^h,\mathbb{Z}/n} \simeq \mathbb{Z}/n$, and since $H^1_{O_{C,S},\mathbb{Z}/n}$ is a finite direct sum of these groups, it is a finite free $\mathbb{Z}/n$-module, of rank $|S|$.

The result [13] 27.1] implies that a free $\mathbb{Z}/n$-submodule of a $\mathbb{Z}/n$-module is a direct summand. Therefore we have a decomposition

$$H^0(\kappa(C),\mathbb{Z}/n) \simeq \text{im}(\phi_1) \oplus \text{im}(\phi_2)$$

and since $H^0(\kappa(C),\mathbb{Z}/n)$ is a finite free $\mathbb{Z}/n$-module, $\text{im}(\phi_2)$ is a finite free $\mathbb{Z}/n$-module by the structure theorem for finitely generated abelian groups. Similarly,
We will show that $H^1_\kappa(C,z)$ is a finite free $\mathbb{Z}/n$-module, 

$$H^2_\kappa(O_{C,S},Z/n) \simeq \text{im}(\phi_2) \oplus \text{cok}(\phi_2)$$

and since $\text{im}(\phi_2)$ and $H^1_\kappa(O_{C,S},Z/n)$ are finite free $\mathbb{Z}/n$-modules, so is $\text{cok}(\phi_2)$. Since $H^1(O_{C,S},Z/n)$ is a $\mathbb{Z}/n$-module, $\text{cok}(\phi_2)$ is a direct summand of $H^1(O_{C,S},Z/n)$, again by [14, 27.1]. Thus we have a decomposition 

$$H^1(O_{C,S},Z/n) \simeq \text{cok}(\phi_2) \oplus \text{im}(\phi_4)$$

Now we set $\Gamma = \text{im}(\phi_4)$, and compute $\text{rk}(\text{cok}(\phi_2)) = N + |S| - m = \beta C$. This proves the case $q = 1$ part of (b).

The $q = 0$ case of (b) is in the proof of Lemma 4.2. Suppose $q = 2$. To show $H^2(O_{C,S},\mu_n) \to H^2(\kappa(C),\mu_n)$ is injective, we will show $H^1(\kappa(C),\mu_n) \to H^2_\kappa(O_{C,S},\mu_n)$ is onto and apply the exactness of $(*)$.

For each closed point $z \in C_1 \cap C_2 \subset S$, we have a diagram

$$H^1(\kappa(C_1),\mu_n) \oplus H^1(\kappa(C_2),\mu_n) \to H^2(O_{C,S},\mu_n)$$

$$0 \to H^1(\kappa(z),\mu_n) \to H^1(\kappa(C_1),\mu_n) \oplus H^1(\kappa(C_2),\mu_n) \to H^2_\kappa(O_{C,S},\mu_n) \to 0$$

We will show that $H^1(\kappa(C_1),\mu_n) \oplus H^1(\kappa(C_2),\mu_n) \to H^2_\kappa(O_{C,S},\mu_n)$ is onto, by showing the downarrow is onto. Since $z$ is a regular point of each $C_i$, each $O_{C_i,z}$ is a discrete valuation ring with residue field $\kappa(z)$ and fraction field $\kappa(C_i)$, and we have a diagram of split short exact sequences

$$0 \to H^1(O_{C_i,z},\mu_n) \to H^1(\kappa(C_i),\mu_n) \to H^0(\kappa(z),\mathbb{Z}/n) \to 0$$

$$0 \to H^1(O_{C_i,z},\mu_n) \to H^1(\kappa(C_i),\mu_n) \to H^0(\kappa(z),\mathbb{Z}/n) \to 0$$

To show the middle downarrow is onto it suffices (by a standard diagram chase) to prove that the left downarrow is onto. Since $O_{C_i,z}$ is henselian $H^1(O_{C_i,z},\mu_n) = H^1(\kappa(z),\mu_n)$, and by Kummer theory and Hilbert 90 we have $H^1(O_{C_i,z},\mu_n) = O_{C_i,z}/\mu_n$ and $H^1(\kappa(z),\mu_n) = \kappa(z)^*/\mu_n$. Since $O_{C_i,z} \to \kappa(z)$ is onto and $O_{C_i,z}$ is local, the induced map $O_{C_i,z} \to \kappa(z)^*$ is onto, hence $H^1(O_{C_i,z},\mu_n)$ maps onto $H^1(\kappa(z),\mu_n)$. We conclude $H^1(\kappa(C_i),\mu_n) \to H^1(\kappa(C_i),\mu_n)$ is onto. Now each map $H^1(\kappa(C_1),\mu_n) \oplus H^1(\kappa(C_2),\mu_n) \to H^2_\kappa(O_{C,S},\mu_n)$ is onto.

Suppose $(b_z) \in H^2_\kappa(O_{C,S},\mu_n) = \bigoplus \kappa H^2_\kappa(O_{C,z},\mu_n)$. We have just seen that for each closed point $z \in C_i \cap C_j$ there exists a pair $([a_{i,z},t_{i,z}^e],[a_{j,z},t_{j,z}^f]) \in \kappa(C_i)^*/\mu_n \oplus \kappa(C_j)^*/\mu_n$ mapping to $b_z$, for $z$-units $a_{\ell,z} \in O_{C_{\ell,z}}$, $z$-uniformizers $t_{k,z} \in \kappa(C_{k,z})$, and integers $\epsilon_k$ for $k = i,j$. Let $v_{k,z}$ be the discrete valuation on $\kappa(C_k)$ determined by $z$. By standard approximation (e.g. [25, XII.1.2]) there exist elements $a_{k,z},t_{k,z} \in \kappa(C_k)$ such that

$$v_{k,z}(a_{k,z} - a_{k,z}) > 0 \quad \text{and} \quad v_{k,z}(t_{k,z} - t_{k,z}) > 1 \quad \text{for all z}.$$ 

The image of $a_{k,z}t_{k,z}^e$ in $\kappa(C_k)^*/\mu_n$ is $[a_{k,z},t_{k,z}^e]$. Therefore the $m$-tuple

$$([a_{k,z}t_{k,z}^e]) \in H^1(\kappa(C),\mu_n)$$

is
maps to \((b_z)\). This proves the induced map \(H^1(\kappa(C), \mu_n) \to H^2_S(O_{C,S}, \mu_n)\) is onto, and completes the proof.

We will soon need the following technical lemma in order to replace \(X_0\) with \(C\).

**Lemma 4.5.** Suppose \(A\) is a noetherian ring. Then \((\text{Frac } A)_{\text{red}} = \text{Frac } (A_{\text{red}})\) if and only if \(A\) has no embedded primes.

**Proof.** By definition \(\text{Frac } A = S^{-1}A\), where \(S = A - \bigcup \text{Ass } A p\), and \(S^{-1}N_A = N_{S^{-1}A}\) by [4, 3.12], hence \(S^{-1}(A_{\text{red}}) = (S^{-1}A)_{\text{red}}\). It remains to show that \(S^{-1}(A_{\text{red}}) = \text{Frac } (A_{\text{red}})\) if and only if \(A\) has no embedded primes. By \(S^{-1}(A_{\text{red}})\) of course we mean \(f(S)^{-1}(A_{\text{red}})\), where \(f : A \to A_{\text{red}}\). This localization equals the localization with respect to the multiplicative set \(T\), where \(T\) is the saturation of \(f(S)\) in \(A_{\text{red}}\), and this is the complement of the union of prime ideals of \(A_{\text{red}}\) that don’t meet \(f(S)\) by [4, Exercise 3.7]. Thus \(S^{-1}(A_{\text{red}}) = \text{Frac } (A_{\text{red}})\) if and only if the union of the primes of \(A_{\text{red}}\) that don’t meet \(f(S)\) equals the union of the associated primes of \(A_{\text{red}}\), which are just the minimal primes of \(A_{\text{red}}\). But \(A\) and \(A_{\text{red}}\) have identical underlying topological spaces, and the primes of \(A_{\text{red}}\) that don’t meet \(f(S)\) correspond to the primes of \(A\) that don’t meet \(S\), i.e., the associated primes. These correspond to the minimal primes of \(A_{\text{red}}\) if and only if the associated primes of \(A\) are the minimal primes of \(A\), i.e., \(A\) has no embedded primes.

**Theorem 4.6.** Assume the setup of (2.3), with \(X\) connected. Then for \(q \geq 0\) there is a map

\[ \lambda : H^q(O_{C,S}, \Lambda) \to H^q(K(X), \Lambda) \]

and a commutative diagram

(4.7) \[
\begin{array}{ccc}
H^q(O_{C,S}, \Lambda) & \xrightarrow{\lambda} & H^q(K(X), \Lambda) \\
\oplus_{i \text{res}_i} & & \oplus_{i \text{res}_i} \\
\downarrow \text{res}_i & & \downarrow \text{res}_i \\
\oplus_{i \text{inf}} H^q(\kappa(C), \Lambda) & \xrightarrow{\oplus \text{ res}_i} & \oplus_{i \text{res}_i} H^q(K(X)_C, \Lambda)
\end{array}
\]

such that if \(\alpha_0 \in H^q(O_{C,S}, \Lambda)\) and \(\alpha = \lambda(\alpha_0)\) then:

a) \(\alpha\) is defined at the generic points of \(C_i\), and \(\alpha(C_i) = \text{res}_i(\alpha_0)\).
b) The ramification locus of \(\alpha\) (on \(X\)) is contained in \(\mathcal{D}_S\).
c) If \(D \in \mathcal{D}_S\) is prime and \(z = D \cap C\), then \(\partial_D \cdot \lambda = \inf_{\kappa(z)}(\kappa(D)) \cdot \partial_z\).
d) If \(\alpha_0\) is unramified at a closed point \(z\), and \(D\) is any (horizontal) prime lying over \(z\), then \(\alpha\) is unramified at \(D\), and has value \(\alpha(D) = \inf_{\kappa(z)}(\kappa(D)) (\alpha_0(z))\).

**Proof.** Let \(D_0\) be an effective divisor on \(C\) that avoids \(S\), let \(D \in \mathcal{D}_S\) be the distinguished lift of \(D_0\), set \(U = X - D\), and set \(U_0 = C - D_0\). Since \(X\) and \(D\) are regular and \(D\) has pure codimension 1, we have \(H^0(X) \simeq H^0(U)\), and an exact Gysin sequence

\[ 0 \to H^1(X) \to H^1(U) \xrightarrow{\partial_0} H^0(D, -1) \to H^2(X) \to \cdots \]

by Gabber’s absolute purity theorem ([15, Theorem 2.1.1]) and the standard construction of the Gysin sequence ([12, Section 3.2]). (Note that the result in [15] is stated for the \(\Lambda = \mathbb{Z}/n\) case only, but the result holds in general since the sheaves \(\mathcal{H}^q_D(X)\) and \(\mathcal{H}^q_D(X, \mathbb{Z}/n)\) are locally isomorphic, and the morphism \(i^* \Lambda(-1) \to \)
Thus we have a commutative ladder
\[ \mathcal{H}_2^0(X) \] is canonical. We use the notation \( \partial_D \) since this map is compatible with the one defined above on \( H^0(K(X)) \) when \( D \) is prime.

We may replace \( X_0 \) by \( C = X_{0,\text{red}} \) in the cohomological computations below since \( \Lambda \) is finite and \( n \) is prime-to-\( p \), by \text{[28] V.2.4(c)} (see also \text{[28] II.3.11}). To substitute \( O_{C,S} \) and \( \kappa(C) \) for \( O_{X_0,S} \) and \( \kappa(X_0) \) we must check that the former are the canonical reduced quotients of the latter. But the ring \( O_{X_0,S} \) can be obtained by localizing some affine open subset \( \text{Spec} A_0 \) containing \( S \) (which exists since \( X_0/k \) is projective) with respect to the multiplicative set \( T = A_0 - \bigcup S \mathfrak{m}_z \). Since \( O_{C,S} \) is obtained by localizing \( A_{0,\text{red}} \) with respect to the image of \( T \) in \( A_{0,\text{red}} \), we have \( O_{C,S} = (O_{X_0,S})_{\text{red}} \) since the formation of the nilradical commutes with localization (see e.g. \text{[H] 3.12}).

To show \( \kappa(C) = \kappa(X_0)_{\text{red}} \) it suffices to show \( X_0 \) has no embedded points by Lemma 4.5. But if \( z \) is any closed point of \( X \) then \( O_{X,z} \) is a regular local ring, and a local equation for the closed fiber \( O_{X,z} \otimes_R k \) passing through \( z \) is given by the uniformizer \( p \) in \( R \). Since \( O_{X,z} \) is factorial and at most two components of \( X_0 \) pass through \( z \) we have \( p = \pi_{x_1}^{a_1} \pi_{x_2}^{a_2} \) for primes \( \pi_i \) and numbers \( a_i \geq 0 \). The associated primes of \( O_{X,z}/(\pi_{x_1}^{a_1} \pi_{x_2}^{a_2}) \) are evidently just the \( (\pi_i) \), which shows \( X_0 \) has no embedded point at \( z \).

Since \( D_0 \) is a disjoint union of regular closed points, by (**) and the work that immediately precedes it we have \( H^0(C) \simeq H^0(U_0) \) and an exact sequence
\[
0 \to H^1(C) \to H^1(U_0) \to H^0(D_0, -1) \to H^2(C) \to \cdots
\]
Thus we have a commutative ladder
\[
\begin{array}{ccccccc}
0 & \to & H^1(X) & \to & H^1(U_0) & \xrightarrow{\partial_D} & H^0(D_0, -1) & \to & H^2(X) & \to & \cdots \\
0 & \to & H^1(C) & \to & H^1(U_0) & \xrightarrow{\partial_{D_0}} & H^0(D_0, -1) & \to & H^2(C) & \to & \cdots \\
\end{array}
\]
Since \( R \) is complete, \( H^q(X) \to H^q(C) \) and \( H^q(D, -1) \to H^q(D_0, -1) \) are isomorphisms for \( q \geq 0 \) by proper base change (\text{[28] VI.Corollary 2.7}). Therefore, in light of the isomorphisms in degree zero and the 5-lemma in degree \( q \geq 1 \), we obtain isomorphisms
\[
H^q(U) \xrightarrow{\sim} H^q(U_0)
\]
for \( q \geq 0 \). Let \( \tilde{U} \) denote the inverse limit over all these open sets \( U \) (this is a scheme by \text{[20] 8.2.3}). Then \( H^q(\tilde{U}) \) is the direct limit of the \( H^q(U) \) by \text{[28] III.1.16}, and since the direct limit functor is exact we have an isomorphism \( H^q(\tilde{U}) \xrightarrow{\sim} H^q(O_{C,S}) \). Composing the inverse with \( H^q(\tilde{U}) \to H^q(K(X)) \) yields our lift
\[
\lambda : H^q(O_{C,S}) \to H^q(K(X))
\]
The commutative diagram \text{[A:1]} follows by applying cohomology to the diagram
\[
\begin{array}{ccccccc}
\text{Spec} O_{C,S} & \to & \tilde{U} & \to & \text{Spec} K(X) \\
\text{Spec} \kappa(C_i) & \to & \text{Spec} O_{K(X)C_i} & \to & \text{Spec} K(X)_{C_i}
\end{array}
\]
incorporating the isomorphisms induced by the upper and lower left horizontal arrows. If \( \alpha_0 \in H^1(O_{C,S}) \) and \( \alpha = \lambda(\alpha) \), then since \( \bar{U} \) contains \( S \) and the generic points of the \( C_i \), \( \alpha \) is defined at these points, and the formula \( \alpha(C_i) = \text{res}_{O_{C,S}|K(C_i)}(\alpha_0) \) follows immediately from \([13]\), proving (a).

If \( D \) is a horizontal prime divisor not in \( \mathcal{D}_S \), then the generic point \( \text{Spec } \kappa(D) \) is contained in \( \bar{U} \), hence the map \( H^q(\bar{U}) \rightarrow H^q(K(X)_D) \) factors through \( H^q(O_{K(X)}_{D}) \), which shows \( \partial_D \cdot \lambda = 0 \). Thus the ramification locus of any element in the image of \( \lambda \) must be contained in \( \mathcal{D}_S \), proving (b). Now if \( D \in \mathcal{D}_S \) is prime and \( z = D \cap C \) then \( D \) is the prime spectrum of a complete local ring with residue field \( \kappa(z) \), and the isomorphism

\[
H^{q-1}(D, -1) \xrightarrow{\sim} H^{q-1}(z, -1) = H^{q-1}(\kappa(z), -1)
\]

is the standard identification. Thus the formula \( \partial_D \cdot \lambda = \inf_{\kappa(z)/K(D)} \partial_z \) is immediate by the commutative ladder of Gysin sequences above, proving (c).

Suppose \( \alpha = \lambda(\alpha_0) \) has ramification locus \( D_\alpha \), then \( D_\alpha \in \mathcal{D}_S \). Set \( U = X - D_\alpha \). If \( \alpha_0 \) is unramified at a point \( z \), then \( \alpha \) is unramified at every prime divisor \( D \) lying over \( z \). For if \( D \in \mathcal{D}_S \) then \( \partial_D(\alpha) = \inf(\partial_z(\alpha_0)) \) by the formula just proved, and if \( D \notin \mathcal{D}_S \) then \( \partial_D(\alpha) = 0 \) since \( D_\alpha \in \mathcal{D}_S \). Thus if \( \alpha_0 \) is unramified at \( z \), and \( D \) is a prime divisor lying over \( z \), then \( U \) contains \( D \). The maps \( z = \text{Spec } \kappa(z) \rightarrow U_0 \) and \( D \rightarrow U \) then induce a commutative diagram

\[
\begin{array}{ccc}
H^q(U) & \xrightarrow{\text{res}} & H^q(D) \\
\downarrow{\text{res}} & & \downarrow{\text{res}} \\
H^q(U_0) & \xrightarrow{\text{res}} & H^q(z)
\end{array}
\]

Both vertical down-arrows are isomorphisms by proper base change. The inverse of the left one is \( \lambda \) by definition, and the composition of the inverse of the right one and the restriction \( H^q(D) \rightarrow H^q(K(D)) \) is inflation, as shown. Since \( \kappa(D) \) is complete, the top composition of horizontal restrictions factors through the restriction \( H^q(U) \rightarrow H^q(O_{X,D}) \) and the bottom factors through the restriction \( H^q(U_0) \rightarrow H^q(O_{C,z}) \). Since these are restriction maps, the images of \( \alpha \) and \( \alpha_0 \) are the values \( \alpha(D) \) and \( \alpha_0(z) \). We conclude \( \inf_{\kappa(z)/K(D)}(\alpha_0(z)) = \alpha(D) \), as in (d).

\[\blacksquare\]

4.8. By weak approximation \([34]\) Lemma] there exists a \( \pi \in K(X) \) such that

\[
\text{div } \pi = C + E
\]

where \( E \) contains no components of \( C \), and avoids any finite set of points \( \mathcal{N} \). We fix such a \( \pi \) for \( \mathcal{N} \) containing \( S \). For each \( i \), the choice of \( \pi \) determines a noncanonical “Witt” isomorphism

\[
H^q(\kappa(C_i)) \oplus H^{q-1}(\kappa(C_i), -1) \xrightarrow{\sim} H^q(K(X)_{C_i})
\]

Taking \( (\alpha, \theta) \) to \( \alpha + (\pi) \cdot \theta \), where \( \alpha \) and \( \theta \) are inflated from \( \kappa(C_i) \) to \( K(X)_{C_i} \), \( (\pi) \) is the image of \( \pi \) in \( H^1(K(X)_{C_i}, \mu_n) \), and \( (\pi) \cdot \theta \) is the cup product. Although we cannot in general lift all of \( \bigoplus_i H^q(K(X)_{C_i}) \) to \( H^q(K(X)) \), we can now prove the following.
Corollary 4.9. Let $(\pi)$ denote the image of $\pi$ in $H^1(K(X), \mu_n)$. The choice of $\mathcal{D}_S$ and $\pi$ determines a homomorphism for $q \geq 1$,
$$
\lambda : H^q(O_{C,S}, \Lambda) \oplus H^{q-1}(O_{C,S}, \Lambda(-1)) \longrightarrow H^q(K(X), \Lambda)
$$
\((\alpha_0, \theta_0) \longmapsto \lambda(\alpha_0) + (\pi) \cdot \lambda(\theta_0)\)
such that $$(\bigoplus_i \text{res}_{K(X)}(K(X)_{C_i}) \cdot \lambda = \bigoplus_i (\inf_{\kappa(C_i)}(K(X)_{C_i} \cdot \text{res}_{O_{C,S}}(\kappa(C_i))))$$.

Proof. This is an immediate consequence of Theorem 4.10. \hfill \blacksquare

Remark 4.10. a) Theorem 4.6 and Corollary 4.9 apply with obvious amendments to the case where $X$ is not connected. For if $X = \coprod_k X_k$ is a decomposition into connected components, then $K(X) = \coprod_k K(X_k)$, $X_0 = \coprod_k (X_k)_0$, $O_{X_0, S} = \coprod_k O_{(X_k)_0, S_k}$ (where $S_k = S \cap X_k$), all of the cohomology groups break up into direct sums, and we define the map $\lambda$ to be the direct sum of the maps on the summands. This will come up in the next section.

b) If $X$ is smooth, then $S$ is empty, and $O_{C,S} = \kappa(C)$. By Witt’s theorem we have $H^q(K(X)_{C}, \Lambda) \simeq H^q(\kappa(C), \Lambda) \oplus H^{q-1}(\kappa(C), \Lambda(-1))$, and we obtain a map
$$
\lambda : H^q(K(X)_{C}, \Lambda) \longrightarrow H^q(K(X), \Lambda)
$$
that splits the restriction map. This is the map of [10].

4.11. Completely split characters. In [30, 2.1] Saito defines a completely split covering of a noetherian scheme $X$ to be a finite étale cover $Y \to X$ such that $Y \times_X \text{Spec} \kappa(x) = \coprod \text{Spec} \kappa(x)$, for all closed points $x \in X$. We abuse Saito’s terminology (see Remark 4.13 below) and in the setup of (2.3) denote by $H^1_{cs}(C, \mathbb{Z}/n)$ the kernel of the map $H^1(O_{C,S}, \mathbb{Z}/n) \to H^1(\kappa(C), \mathbb{Z}/n)$ in Lemma 4.3. If $\beta \in H^1_{cs}(C, \mathbb{Z}/n)$ then $\partial_z(\beta) = 0$ for all closed points $z \in C - S$ since $\partial_z$ factors through $\kappa(C_i)_z$. Therefore $\beta$ is defined on $C$, hence $H^1_{cs}(C, \mathbb{Z}/n) \leq H^1(C, \mathbb{Z}/n)$. Let $H^1_{cs}(X, \mathbb{Z}/n)$ denote the preimage of $H^1_{cs}(C, \mathbb{Z}/n)$ under the proper base change isomorphism.

Proposition 4.12. Assume the setup of (2.3). Then elements of $H^1_{cs}(C, \mathbb{Z}/n)$ are trivial at all points of $C$, and the nontrivial elements of $H^1_{cs}(X, \mathbb{Z}/n)$ are trivial at all points of $X$ except for the generic point $\text{Spec} K(X)$, where they are nontrivial.

Proof. Suppose $\beta_0 \in H^1_{cs}(C)$. Then $\beta_0$ is trivial at each generic point of $C$ by definition of $H^1_{cs}(C)$. If $z \in C$ is a closed point lying on the irreducible component $C_i$ then the map $H^1(C_i) \to H^1(\kappa(z))$ factors through $H^1(C_i)$. Since $C_i$ is regular the map $H^1(C_i) \to H^1(\kappa(C_i))$ is injective by purity, and consequently $\beta_0(z) = 0$ by definition. Thus the elements of $H^1_{cs}(C)$ are trivial at all points of $C$.

Suppose $\beta = \lambda(\beta_0) \in H^1_{cs}(X)$. If $x \in X$ is a generic point of some irreducible component $C_i$ of $C$ then the image of $\beta$ in $H^1(\kappa(C_i))$ is zero since the map $H^1_{cs}(X, \mathbb{Z}/n) \to H^1(\kappa(C_i))$ factors through $H^1_{cs}(C)$. If $x$ is the generic point of a horizontal divisor $D$ with closed point $z$ then $\beta(D) = \inf_{\kappa(z)}(\kappa(D))(\beta_0(z))$ by Theorem 1.1 (d), and this is zero since $\beta_0(z) = 0$. If $z$ is a closed point of $X$ then $z$ is on $C$, and the map $H^1_{cs}(X) \to H^1(\kappa(z))$ factors through $H^1_{cs}(C)$, hence $\beta$ is trivial at $z$. Finally, since $X$ is regular the map $H^1(X) \to H^1(K(X))$ is injective by purity, hence $\beta$ is nontrivial at the generic point of $X$. \hfill \blacksquare
Remark 4.13. Proposition [11,2] shows the elements of $H^1_{et}(C, \mathbb{Z}/n)$ are completely split in the sense of [30]. However, in the general case our $H^1_{et}(C, \mathbb{Z}/n)$ does not account for elements that are split at every closed point but nontrivial at generic points of $C$. This is not an issue if $k$ is finite as shown by Saito in [30, Theorem 2.4], since then the $C_1$ have no nontrivial completely split covers, essentially by Cebotarev’s density theorem (see [29, Lemma 1.7]).

5. Index Calculation in the Brauer Group

5.1. Cyclic Covers. If $U$ is any scheme, and $\bar{u}$ is a geometric point, the fiber functor defines a category equivalence between (finite) étale covers of $U$ and finite continuous $\pi_1(U, \bar{u})$-sets, yielding a canonical isomorphism

$$H^1(U, \mathbb{Z}/n) \simeq H^1(\pi_1(U, \bar{u}), \mathbb{Z}/n) = \text{Hom}_{cont}(\pi_1(U, \bar{u}), \mathbb{Z}/n)$$

(see [13, I.2.11]). If $\theta \in H^1(U, \mathbb{Z}/n)$, we will write $U[\theta]$ for the finite cyclic étale cover determined by $\theta$. If $U = \text{Spec} \, A$ is affine, we will write $A[\theta]$ for the corresponding ring, or $A[\theta]$ if $A$ is a field. If $U$ is a connected normal scheme, and $\theta \in H^1(U, \mathbb{Z}/n)$ has order $m$, then $U[\theta]$ is a disjoint sum of $n/m$ connected $\mathbb{Z}/m$-Galois covers of $U$.

Lemma 5.2. Assume the setup of (2.4). Let $\theta_0 \in H^1(O_{C, S}, \mathbb{Z}/n)$ be a (tamely ramified) character with ramification divisor $D_0$ on $C$. Then the (tamely) ramification divisor of $\theta = \lambda(\theta_0)$ is the distinguished lift $D \in \mathcal{D}_S$ of $D_0$ on $X$, and $\theta$ defines a tamely ramified cover $\rho : Y \to (X, D)$ as in Lemma 3.2. Restriction to $C$ yields a tamely ramified cover $\rho_0 : C_Y \to (C, D_0)$ such that $O_{C_Y, S_Y} = O_{C, S}[\theta_0]$, and the reduced closed fiber $C_Y$ of $Y$ is the normalization of $C$ in $\kappa(C)(\theta_0)$.

Proof. The lift $\theta$ is tamely ramified with respect to $D$ by Theorem 1.6. Let $Y$ be the normalization of $X$ in $L = K(X)(\theta)$. Since $D$ is in $\mathcal{D}_S$ and $X/R$ satisfies the setup of (2.3), by Lemma 3.2, $\rho : Y \to (X, D)$ is a tamely ramified cover, $Y/R$ is a regular relative curve with reduced closed fiber $C_Y$, the irreducible components of $C_Y$ are regular with singular points $S_Y$, and $D_Y$ is in $\mathcal{D}_{Y, S_Y}$.

Let $U = X - D$, $V = U \times_X Y$, $U_0 = U \times_X X_0$ and $V_0 = V \times_Y Y_0$. Then $\theta_0 \in H^1(U_0)$. We have $H^1(U) \simeq \text{Hom}(\pi_1(U), \mathbb{Z}/n)$ and $H^1(U_0) \simeq \text{Hom}(\pi_1(U_0), \mathbb{Z}/n)$, and the restriction map $\text{res} : H^1(U) \to H^1(U_0)$, which sends $\theta$ to $\theta_0$, is induced by the natural map $\pi_1(U_0) \to \pi_1(U)$, which is induced on covers by $W \mapsto W \times_U U_0$. Therefore $V_0 = U_0[\theta_0]$.

We show $O_{C_Y, S_Y} = O_{C, S}[\theta_0]$. If $U_0 = \text{Spec} \, A_0 \subset C - D_0$ is a dense affine open subset of $C$ containing $S$, then its preimage in $C_Y$ is a dense affine open subset $V_0 = \text{Spec} \, B_0$ containing $S_Y$. By base change we have $B_0 = A_0[\theta_0]$, and $S^{-1}B_0 = O_{C, S}[\theta_0]$, where $S = A_0 - \left( \bigcup_{x \in S} m_x \right)$ is the multiplicative set defining $O_{C, S}$. Since $S_Y = \rho^{-1}S$, the saturation $T$ of $S$ in $B_0$ is $T = B_0 - \left( \bigcup_{y \in S_Y} m_y \right)$, which shows $S^{-1}B_0 = O_{C_Y, S_Y}$. Therefore $O_{C_Y, S_Y} = O_{C, S}[\theta_0]$, and it follows immediately that $O_{C_Y, S_Y} = O_{C, S}[\theta_0]$.

The map $C_Y \to C$ is finite by base change, and since each irreducible component of $C_Y$ is regular by Lemma 3.1, each irreducible component of $C_Y$ is the normalization of a component of $C$ in a field extension which is a direct factor of $\kappa(C_Y)$. Equivalently, $C_Y$ is the normalization of $C$ in $\kappa(C)(\theta_0)$, by [13, 6.3.7]. This completes the proof.
5.3. Index. Assume the setup of (2.3) with $R = \mathbb{Z}_p$ and $\Lambda = \mu_n$. Fix $\alpha_C \in \mathbb{H}^2(O_{C,S})$ and $\theta_C \in \Gamma \leq \mathbb{H}^1(O_{C,S}, -1)$, and write

$$\alpha_C = (\alpha_{C_1}, \ldots, \alpha_{C_m}) \in \mathbb{H}^2(\kappa(C))$$

$$\theta_C = (\theta_{C_1}, \ldots, \theta_{C_m}) \in \mathbb{H}^1(\kappa(C), -1)$$

as per Lemma 4.4(b). Suppressing the inflation maps, we form the element

$$\gamma_C = \alpha_C + (\pi) \cdot \theta_C = (\alpha_{C_1} + (\pi) \cdot \theta_{C_1}, \ldots, \alpha_{C_m} + (\pi) \cdot \theta_{C_m}) \in \bigoplus_{i=1}^m \mathbb{H}^2(K(X)_{C_i})$$

where $\pi$ is as in (4.8). Let $\theta = \lambda(\theta_C)$, and let $Y$ be the normalization of $X$ in $K(X)(\theta)$, as in Lemma 5.2. Then $C_Y$ is the reduced closed fiber of $Y$, and we write $C_{i,Y}$ for the preimage of $C_i$, so that $\kappa(C_{i,Y}) = \kappa(C_i)(\theta_{C_i})$. Thus

$$\alpha_{C_Y} = (\alpha_{C_{1,Y}}, \ldots, \alpha_{C_{m,Y}}) \in \bigoplus_{i=1}^m \mathbb{H}^2(\kappa(C_{i,Y}))$$

Note $\kappa(C_{i,Y})$ is a product of the function fields of the irreducible components of $C_{i,Y}$. By the (well-known) Nakayama-Witt index formula,

$$\text{ind}(\alpha_{C_Y} + (\pi) \cdot \theta_{C_Y}) = |\theta_{C_Y}| \cdot \text{ind}(\alpha_{C_Y})$$

We have $|\theta_{C_Y}| = \text{lcm}_i\{|\theta_{C_i}|\}$ by Lemma 4.4(b). We now define

$$\text{ind}(\gamma_C) \overset{\text{df}}{=} \text{lcm}_i\{\text{ind}(\alpha_{C_i})\}$$

$$\text{ind}(\gamma_C) \overset{\text{df}}{=} |\theta_{C_Y}| \cdot \text{ind}(\alpha_{C_Y})$$

**Theorem 5.4.** Assume the setup of (2.3) with $R = \mathbb{Z}_p$. Let $\Gamma \leq \mathbb{H}^1(O_{C,S})$ be as in Lemma 4.4(b). Then the map $\lambda : \mathbb{H}^1(O_{C,S}) \oplus \Gamma \to \mathbb{H}^2(K(X))$ preserves index.

**Proof.** We may assume $X$ is connected. We identify $\Gamma$ with the image of $\mathbb{H}^1(O_{C,S}, -1)$ in $\mathbb{H}^1(\kappa(C), -1)$, as in Lemma 4.4(b), and adopt the notation of (5.3). Set $\gamma = \lambda(\gamma_C)$, $\alpha = \lambda(\alpha_C)$, and $\theta = \lambda(\theta_C)$, so that $\gamma = \alpha + (\pi) \cdot \theta$ as in Corollary 4.9. By restricting to connected components if necessary we may assume that $Y$ is connected, hence that $K(Y)$ is a field. Even so, the cyclic-Galois étale $\kappa(C_i)$-algebra $\kappa(C_{i,Y})$ may not be a field. Since $\kappa(C_i)$ is in $\mathbb{H}^1(O_{C,S}, -1)$ the ramification divisor $D_0$ of $\theta_C$ avoids $S$, and the distinguished lift $D \in \mathcal{D}_S$ of $D_0$ is the ramification divisor of $\theta$ by Theorem 4.6. By Lemma 5.2 $\rho : Y \to (X, D)$ is a cyclic tamely ramified cover, and by Lemma 3.1 $Y/R$ satisfies the properies of (2.3), with reduced closed fiber $C_Y$, $S_Y = \rho^{-1} S$ the singular points of $C_Y$, and $\mathcal{D}_Y$ generated by $D_{Y,\text{red}}$ and the preimages of the other distinguished divisors of $X$.

The index of $\gamma$ cannot exceed $|\theta_{\text{ind}}(\alpha_{C_Y})$. For if $M/K(Y)$ is a separable maximal subfield of the division algebra associated with $\alpha_Y$, then $M/K(X)$ splits $\gamma$, and has degree $|\theta_{\text{ind}}(\alpha_Y)|$. Since in our case $|\theta| = [K(Y) : K(X)] = [\kappa(C_Y) : \kappa(C)] = |\theta_C|$, to prove the theorem it is enough to prove $\text{ind}(\alpha_Y) = \text{ind}(\alpha_{C_Y})$.

By Lemma 5.2 $O_{C,S}[\theta_C] = O_{C_Y,S_Y}$. Each $\kappa(C_{i,Y}) = \kappa(C_i)(\theta_{C_i})$ is a product of global fields, and by class field theory the division algebra associated with the restriction of $\alpha_{C_{i,Y}}$ to each field component is cyclic. Since $\alpha_{C_{i,Y}}$ is in $\mathbb{H}^2(O_{C_{i,Y},S_Y})$ (by Lemma 5.2 and Lemma 4.2), $\alpha_{C_{i,Y}}$ is unramified at $S_Y$. By Grunwald-Wang’s
there exists a tuple \( \psi_{C_Y} = (\psi_{C_{1,Y}}, \ldots, \psi_{C_{n,Y}}) \in \bigoplus_i H^1(\kappa(C_{i,Y}), -1) \) such that \( |\psi_{C_{i,Y}}| = \text{ind}(\alpha_{C_{i,Y}}), \kappa(\psi_{C_{i,Y}}) \). In particular, \( \psi_{C_Y} \) is unramified and equal at the local fields defined by the singular points \( \lambda \).

By Theorem 4.6 (and Remark 4.10a) if \( Y \) is not connected we have a map \( \lambda_Y : H^1(O_{C_Y}, S_Y, -1) \to H^1(K(Y), -1) \). Since the distinguished divisors on \( Y \) are the (reduced) preimages of those on \( X \), \( \lambda_Y \) is compatible with \( \lambda \) and the residue maps. Set \( \psi = \lambda_Y(\psi_{C_Y}) \), and let \( D_\psi \) denote the distinguished lift of the ramification divisor \( D_{\psi_C} \) of \( \psi_{C_Y} \) on \( C_Y \).

By Lemma 5.2 \( \psi \) determines a cyclic tamely ramified cover \( \sigma : Z \to (Y, D_\psi) \) with reduced closed fiber \( C_Z \), such that \( C_Z \) is cyclic and tamely ramified over \((C_Y, D_{\psi_C})\), inducing \( \psi_{C_Y} \). Since \( \kappa(C_{i,Z}) \) splits \( \alpha_{C_{i,Y}}, \alpha_{C_Z} = 0 \).

Again we may assume \( Z \) is connected. By construction, \( |K(Z) : K(Y)| = |\psi| = |\psi_{C_Y}| = |\kappa(C_Z) : \kappa(C_Y)| = \text{ind}(\alpha_{C_Y}) \). By (1.7) we have \( \text{ind}(\alpha_{C_Y}) \leq \text{ind}(\alpha_Y) \), and it remains to show \( \alpha_Z = 0 \). It is then enough to show \( \alpha_Z \) is unramified with respect to all Weil divisors on \( Z \), by [10] Lemma 3.5.

Let \( D' = D \cup \rho(D_\psi) \). Since \( D_\psi \in S_{Y,S_Y} \), \( \rho(D_\psi) \in S_\Sigma \), and the composition \( \rho' : Z \to (X, D') \) is a tamely ramified cover. Since \( X \) is regular, \( \rho' \) is (finite and) flat by Lemma 3.1, and so the image of any prime divisor \( J \) of \( Z \) is a prime divisor \( \rho'(J) = I \) of \( X \). By the functoriality of the residue maps, \( \alpha_Z \) can only be ramified at prime divisors lying over irreducible components of \( D_\alpha \). By Theorem 4.6(a) \( D_\alpha \) is in \( S_\Sigma \), and the divisors of \( Z \) lying over \( D_\alpha \) are distinguished by Lemma 3.3. Thus it is enough to show that \( \alpha_Z \) is unramified at these distinguished divisors. Clearly we may assume \( D_\alpha \) is irreducible.

Let \( E \subset Z \) be a (distinguished) prime divisor lying over \( D_\alpha \in S_\Sigma \). Since \( D_\alpha \cap D' = \emptyset \) or \( D_\alpha \subset D' \), we have \( e(v_E/v_{D_\alpha}) = e(v_E/v_{D_\alpha}) = e \) for some \( e \geq 1 \), by Lemma 4.6. By Lemma 4.6 and the functorial behavior of the residue and restriction maps, we have a commutative diagram

\[
\begin{array}{cccccc}
H^2(O_{C,S}) & \xrightarrow{\lambda} & H^2(K(X)) & \xrightarrow{\text{res}} & H^2(K(Z)) \\
\downarrow{\partial_{D_{\alpha C}}} & & \downarrow{1} & & \downarrow{\partial_E} \\
H^1(\kappa(D_{\alpha C}), -1) & \xrightarrow{\text{res}} & H^1(\kappa(D_{\alpha}), -1) & \xrightarrow{\text{inf}} & H^1(\kappa(E), -1) \\
\downarrow{\inf} & & \downarrow{\text{inf}} & & \downarrow{\partial_E} \\
H^2(O_{C,S}) & \xrightarrow{\text{res}} & H^2(O_{C_Z,S_Z})
\end{array}
\]

Since \( \alpha = \lambda(\alpha_C), \partial_E(\alpha_Z) = e \cdot (\partial_{D_{\alpha C}}(\alpha_C))_{\kappa(E)} \) by squares (1) and (2), and by square (4), \( \partial_{E_0}(\alpha_{C_Z}) = e \cdot (\partial_{D_{\alpha C}}(\alpha_C))_{\kappa(E_0)} \). Therefore \( \partial_E(\alpha_Z) = \inf_{\kappa(E_0)}(\partial_E)^{\kappa(E)}(\partial_{E_0}(\alpha_{C_Z})) \) by square (3). Since \( \alpha_{C_Z} = 0 \), we conclude \( \partial_E(\alpha_Z) = 0 \), as desired. This proves the theorem.
6. **Noncrossed Products and Indecomposable Division Algebras**

A (finite-dimensional) division algebra $D$ central over a field $F$ is called a *noncrossed product* if it has no Galois maximal subfield. Its algebra structure then cannot be given by a Galois 2-cocycle, counter to almost all known division algebra constructions (see [24] for a construction of a noncrossed product). Noncrossed product division algebras were long conjectured to be fictional, until they were shown to exist by Amitsur in [1].

We say $D$ is *indecomposable* if it does not contain a subalgebra that is also central over $F$, or equivalently if it is not an $F$-tensor product of two nontrivial $F$-division algebras. It is not hard to show that all division algebras of composite period are decomposable, and that all division algebras of equal prime-power period and index are indecomposable, but it is nontrivial to construct indecomposable division algebras of unequal prime-power period and index. The first examples appeared in [3] and in [31]. For additional discussion of either of these topics, see almost any survey treating open problems on division algebras, such as [5], [2], or [32].

We can use Theorem 5.4 to prove the existence of noncrossed product and indecomposable division algebras over the function field $F$ of any $p$-adic curve $X_{\mathbb{Q}_p}$. Noncrossed products over $K(t)$ for $K$ a local field were first constructed in [8], and then constructed more systematically over the function field of a smooth relative $\mathbb{Z}_p$-curve in [10]. Indecomposable division algebras of unequal period and index were also constructed in [10], over the same types of fields. Modulo gluing, the method we use below is the same as the one used in [10] Theorem 4.3, Corollary 4.8.

**Theorem 6.1.** Let $F/\mathbb{Q}_p$ be a finitely generated field extension of transcendence degree one. Let $X/\mathbb{Z}_p$ be a regular relative curve with function field $F$, let $C_i$ be a reduced irreducible component of the closed fiber, let $\ell \neq p$ be a prime, and let $r$ and $s$ be numbers that are maximal such that $\mu_{\ell^r} \subset \kappa(C_i)$ and $\mu_{\ell^s} \subset \kappa(C_i)(\mu_{\ell^{r+1}})$. Then there exist noncrossed product $F$-division algebras of period and index as low as $\ell^{s+1}$ if $r = 0$, and $\ell^{2r+1}$ if $r \neq 0$.

**Proof.** We may assume (without changing $r$ and $s$) that $C$ has regular irreducible components, at most two of which meet at any closed point of $X$. The idea is to use the (known) existence of such algebras over the fields $F_{C_i}$, modify the construction to produce a class in $H^2(O_{C_i,S}) \oplus \Gamma$, and then apply Theorem 4.9 and Theorem 5.4 to prove existence over $F$.

By [10] Theorem 4.7, if $F$ admits a smooth $X$ then there exist noncrossed product division algebras over $F_C$ of period and index as low as $\ell^{s+1}$ if $r = 0$, and $\ell^{2r+1}$ if $r > 0$. The resulting Brauer class has the form $\alpha_C + (\pi) \cdot \theta_C \in H^2(F_C)$, where $\alpha_C \in H^2(\kappa(C))$ and $\theta_C \in H^1(\kappa(C), -1)$. A look at the construction, which proceeds exactly as in [10] Theorem 1], shows we may pre-assign values at any finite set of points of $C$. Thus we may thus produce a noncrossed product $F_{C_i}$-division algebra with class $\gamma_{C_i} = \alpha_{C_i} + (\pi) \cdot \theta_{C_i}$ of the desired period and index, and elements $\gamma_C = (\alpha_{C_1} + (\pi) \cdot \theta_{C_1}, \ldots, \alpha_{C_m} + (\pi) \cdot \theta_{C_m}) \in H^2(O_{C_i,S}) \oplus (\pi) \cdot \Gamma \leq \bigoplus_i H^2(F_{C_i})$

Then $\gamma_C$ lifts to $\gamma = \lambda(\gamma_C) \in H^2(F)$ by Theorem 4.10. By Theorem 5.4, $\text{ind}(\gamma) = \text{ind}(\gamma_C) = \ell^{s+1}$ if $r = 0$, and $\ell^{2r+1}$ if $r \neq 0$. It is clear that the $F$-division algebra...
$D$ associated to $\gamma$ is a noncrossed product, since any Galois maximal subfield of $D$ over $F$ could be constructed over $F_{C_1}$, contradicting the fact that $D \otimes_F F_{C_1}$ is a noncrossed product $F_{C_1}$-division algebra.

Theorem 6.2. Let $F/\mathbb{Q}_p$ be a finitely generated field extension of transcendence degree one, and let $\ell \neq p$ be a prime. Then there exist indecomposable $F$-division algebras of $(\text{period}, \text{index}) = (a^a, b^b)$, for any numbers $a$ and $b$ satisfying $1 \leq a \leq b \leq 2a - 1$.

Proof. Let $X$, $C$, $C_i$, and $S$ be as in Theorem 6.1. The construction over $F_{C_i}$ is exactly as in [10, Proposition 4.2] and [7], and we merely have to observe that we may assume all of the data in the constructed class $\gamma_{C_i} = \alpha_{C_i} + (\pi) \cdot \theta_{C_i} \in H^2(F_{C_i})$ is trivial at the singular points $S \cap C_i$, so that by Lemma 4.12 we may construct a class $\gamma_{C} = \alpha_{C} + (\pi) \cdot \theta_{C} \in H^2(O_{C,S}) \oplus (\pi) \cdot \Gamma \leq \bigoplus_i H^2(F_{C_i})$ whose $i$-th component is $\alpha_{C_i} + (\pi) \cdot \theta_{C_i}$. This class lifts to a class $\gamma = \lambda(\gamma_{C})$ by Theorem 4.6 and $\text{ind}(\gamma) = \text{ind}(\gamma_{C})$ by Theorem 5.3. Since the indexes are the same, the division algebra $D$ associated to $\gamma$ is indecomposable, since any decomposition would extend to $D_{C_i} = D \otimes_F F_{C_i}$, contradicting the construction of $\gamma_{C_i}$.

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