Hugenholtz - Pines relations and the critical temperature of a
Rabi coupled symmetric Bose mixture

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Abstract

Using a theoretical field Gaussian approximation we have studied Rabi coupled two component Bose mixtures at low temperatures. We have derived extended Hugenholtz - Pines relations taking into account one body interaction (e.g. Rabi coupling) and studied the critical temperature $T_c$ of BEC transition. We have shown that, the shift of $T_c$ due to this interaction can not exceed $\sim 60\%$ and goes to a plateau with increasing the parameter $\Omega_R/T_0^c$, where $\Omega_R$ is the intensity of the coupling and $T_0^c$ is the critical temperature of the system with $\Omega_R = 0$. Moreover, the shift is always positive and does not depend on the sign of the one body interaction.

I. INTRODUCTION

The experimental possibility of achieving quantum degeneracy with mixtures of atomic gases occupying different hyperfine states or with mixtures of different atomic species has opened rich opportunities for novel experimental and theoretical studies. The mixtures of atomic gases are much more flexible, due to the large variety of available atomic species, characterized by different hyperfine states, the possibility of generating coherently coupled configurations, and tuning the interaction between the different components for the mixtures [1]. For the mixtures made of atoms occupying different hyperfine states, it is possible experimentally to generate coherently coupled configurations via radio frequency transitions, giving rise to typical Rabi oscillations. Such experiments were performed with atoms evaporatively cooled in the $|F = 2, m_F = 2\rangle$ and $|F = 1, m_F = -1\rangle$ spin states of $^{87}\text{Rb}$ [2-4] and very recently, with $^{39}\text{K}$ atoms in $|F = 1, m_F = -1\rangle$ and $|F = 1, m_F = 0\rangle$ states [5]. Although, in the most of experimental and theoretical [5-13] studies the existence of two component Bose-Einstein condensate (BEC) has been predicted, possible finite temperature effects on the properties of the condensates were not considered.

It is well known that a system of bosons, where number of particles is conserved, experiences normal to BEC phase transition with a certain temperature $T_c$. This critical temperature essentially depends on internal properties of the system as well as on the geometry of the trapping potential. In the simplest case of a free homogeneous gas of bosonic atoms the critical temperature is given by $T_c^0 = (2\pi/m)(\rho/\zeta(3/2))^{2/3}$ where $\rho$ is the average

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particle density of the Boson gas of atoms with a mass $m$, and $\zeta(n)$ is the Reiman zeta function, $\zeta(3/2) \approx 2.61$. The question, attracting for long time attention, is how this expression varies under switching on one or two body interactions. This problem turned on to be highly non-trivial [14] even for a one component Bose system.

The modification of the critical temperature $T_c$ is usually expressed in terms of the relative temperature shift defined as

$$\frac{\Delta T_c}{T_c^0} = \frac{T_c - T_c^0}{T_c^0}$$

whose determination has a long history. For example, there are several articles, where the authors made an attempt to obtain a proper analitycal expression for the shift due to repulsive contact interaction [15–21], disorder [22, 23], anisotropic effect in the BEC of triplons [24–26] and trap geometry [14]. However, in our knowledge, the shift of critical temperature of a two component mixture due to the inter-component interaction (with a coupling constant $g_{ab}$ between atoms of $a$ and $b$ components) or especially due to the Rabi coupling has never been studied.

The goals of the present work are derivation Hugenholtz - Pines (HP) relations and determination of the behavior of the transition temperature of two component Bose mixture in the presence of a one body interaction implemented by optical Rabi coupling. For this purpose we use Gaussian (one loop ) approximation [12, 27, 28] with Rabi coupling in the framework of the mean filed theory and derive an analytical expression for the Rabi induced shift of the critical tempearture. We will show that the shift is positive and increases toward an asymptotic value with increasing the strength of Rabi coupling. During the calculations we shall derive also Hugenholtz -Pines relations for this system, which is essential for the BEC with a gapless spectrum.

This work is organised as follows. In Sec. II starting from the hamiltonian with one and two body interactions, we derive explicite expressions for the Green functions and self energies in order to find HP relations in Gaussian approximation. In Sec. III we discuss extremums of the free energy and study conditions for the existence of a pure BEC in an equilibrum state of a Rabi coupled two component Bose mixture. In Sec. IV we concentrate on a symmetric mixture to study BEC and normal phases separately. The critical temperature and its shift will be studied in Sec. V , where we present our numerical results also. In the last section we summarize our findings.
II. HUGENHOLTZ-PINES RELATIONS

We start with the grand canonical Hamiltonian for homogenous two component Bose mixture with Rabi coupling:

\[ \hat{H} = \hat{H}_a + \hat{H}_b + \hat{H}_{ab} \quad (2a) \]

\[ \hat{H}_a = \int dr \{-\psi^\dagger \vec{\nabla}^2 \psi - \mu_a \psi^\dagger \psi + \frac{g_a}{2} (\psi^\dagger \psi)^2 \} \quad (2b) \]

\[ \hat{H}_b = \int dr \{-\phi^\dagger \vec{\nabla}^2 \phi - \mu_b \phi^\dagger \phi + \frac{g_b}{2} (\phi^\dagger \phi)^2 \} \quad (2c) \]

\[ \hat{H}_{ab} = \int dr \{g_{ab} (\psi^\dagger \psi)(\phi^\dagger \phi) + \frac{\Omega_R}{2} (\omega_R \psi^\dagger \phi + \omega_R^* \phi^\dagger \psi) \} \quad (2d) \]

where the associated chemical potentials are represented by \( \mu_{a,b} \) while \( m_{a,b} \) represent the masses. In terms of the corresponding s-wave scattering lengths \( a_s \), the coupling constants can be written as \( g_{a,b} = \frac{4\pi a_{a,b}}{m_{a,b}} \), while the cross coupling is \( g_{ab} = \frac{2\pi a_{ab}}{m_{ab}} \), where \( m_{ab} = \frac{m_a m_b}{m_a + m_b} \) represents reduced mass. Here and below we set \( \hbar = 1, k_B = 1 \).

The coherent (Rabi) coupling is given by the last term of (2d) with the intensity \( \Omega_R \geq 0 \) and the phase \( \omega_R = \exp(i\theta_R) \). Depending on the physical system, this term can have its origin on a two-photon (Raman) process also. It is clear that, when \( \Omega_R = 0 \), the phase of each component is independent and \( \hat{H} \) is invariant under the transformations \{ \( \psi \rightarrow \exp(i\theta_1)\psi \), \( \phi \rightarrow \exp(i\theta_2)\phi \) \}. The spontaneous breaking of this invariance leads to the emergence of Goldstone (gapless) modes for both components. However, when \( \Omega_R \neq 0 \) the Hamiltonian (2a) is invariant under the unique gauge transformation \{ \( \psi \rightarrow \exp(i\theta_0)\psi \), \( \phi \rightarrow \exp(i\theta_0)\phi \) \} with the same phase angle \( \theta_1 = \theta_2 = \theta_0 \). In this case the spontaneous breaking of this symmetry makes at least one of branches of excitation spectrum as gapless.

In fact, the particle spectrum under the spontaneously broken gauge symmetry has to be gapless. This is, actually, one of the main conditions for the existence of a stable BEC. Since if there would be a gap in the spectrum, there could be no macroscopic occupation of a single ground state level. Hugenholtz and Pines [29], (and later Bogolyubov [30]) showed that for one component Bose system the chemical potential is expressed through the normal \( \Sigma_n(\vec{k}, \omega) \) and anomalous \( \Sigma_{an}(\vec{k}, \omega) \) self energies as \( \mu = \Sigma_n(0,0) - \Sigma_{an}(0,0) \). Further, this relation has been extended [31, 32] for a two component Bose system with a two body coupling. The question arises, how the presence of a one body coupling, (e.g. Rabi coupling) in a two component Bose mixture will modify Hugenholtz-Pines relations?
In this section we make an attempt to find an answer to this question, at least on the level of the Gaussian (bilinear) approximation \[27, 33\], which is a particular case of a more accurate approach as Hartree - Fock - Bogoliubov \[34, 35\]. For this purpose we derive explicite expressions for Green functions, as well as for the particle spectrum and find relations between the self energies. Note that, particularly, for a one component Bose system HP relation can be directly obtained by the condition of the existence of the Goldstone mode.

A. The Green functions

The finite temperature Euclidean ($\tau = it$) space time action, corresponding to the Hamiltonian \((2a)\) is given by

\[
S = \int_0^\beta d\tau \int d\vec{r} \left[ \psi^\dagger \hat{K}_a \psi + \phi^\dagger \hat{K}_b \phi + \frac{g_a}{2} (\psi^\dagger \psi^\dagger) + \frac{g_b}{2} (\phi^\dagger \phi^\dagger) + g_{ab} (\psi^\dagger \phi^\dagger) + \frac{\Omega}{2} (\omega_R \hat{\psi}^\dagger \hat{\phi} + \omega_R^* \hat{\phi}^\dagger \hat{\psi}) \right] (3)
\]

where $K_{a,b} = \partial / \partial \tau - \hat{O}_{a,b}$, $\hat{O}_{a,b} = \vec{\nabla}^2 / 2m_{a,b} + \mu_{a,b}$. In Eq. \((3)\) the fields $\psi(\vec{r}, \tau)$ and $\phi(\vec{r}, \tau)$ are periodic in $\tau$ with period $\beta = 1/T$. Now we introduce Bogoliubov shift

\[
\psi = \sqrt{\rho_{0a}} + \tilde{\psi}, \quad \phi = \xi \sqrt{\rho_{0b}} + \tilde{\phi} \tag{4}
\]

where $\rho_{0a,b}$ is the condensate fraction of the component $a$, $(b)$, $\xi = e^{i\theta}$ with the relative phase angle $\theta$ between two Bose-Einstein Condensates and $\tilde{\psi}$ and $\tilde{\phi}$ are fluctuating fields which will be integrated out. Due to the \((2d)\) term in the Hamiltonian, both components are coherently coupled, and hence the relative phase $\xi$ should be determined from the minimum condition of the thermodynamic potential. On the other hand it has been proven \[25, 26\] that, the phase angle of a pure BEC should be equal to $\pi n$ with an integer $n$. Consequently, the relative phase should be real, $\xi = \pm 1$.

Note that the Bogoliubov shift is an exact canonical transformation \[36\], and not an approximation, as sometimes it is stated. For a uniform system at equilibrium, $\rho_{0a}$ and $\rho_{0b}$ are real variational constants, which are fixed by the minimum of the free energy $\Omega$ as $\partial \Omega / \partial \rho_{0,a,b} = 0$, $\partial^2 \Omega / \partial^2 \rho_{0,a,b} \geq 0$ \[25\]. As to the numbers of uncondensed particles $N_{1a}$ and $N_{1b}$, they are related to the fields $\tilde{\psi}$ and $\tilde{\phi}$:

\[
N_{1a} = V \rho_{1a} = \int d\vec{r} \langle \tilde{\psi}^\dagger (r) \tilde{\psi}(r) \rangle, \tag{5}
\]

\[
N_{1b} = V \rho_{1b} = \int d\vec{r} \langle \tilde{\phi}^\dagger (r) \tilde{\phi}(r) \rangle, \tag{6}
\]
so that
\[ N_a = \int d\vec{r} \langle \psi^\dagger(r) \psi(r) \rangle, \quad (7) \]
\[ N_b = \int d\vec{r} \langle \phi^\dagger(r) \phi(r) \rangle \quad (8) \]

with the normalization conditions \( N = N_a + N_b, \) \( N_a = V\rho_a = V(\rho_{0a} + \rho_{1a}) \), and \( N_b = V\rho_b = V(\rho_{0b} + \rho_{1b}) \), where \( N_{a(b)} \) is the number of particles in the component \( a, (b) \) and \( N \) is the particle number in the whole two-component system and \( V \) is the total volume of the system. Due to the flipping term only the total number of particles (total density in the uniform system) is conserved. Thus the chemical potential should be the same for both components: \( \mu_a = \mu_b = \mu \). Since we are considering a homogeneous system, the densities \( \rho_a \) and \( \rho_b \) are uniform.

The mean-field plus Gaussian approximation is obtained by expanding \( S \) up to the second order in fluctuating fields [12]. So, inserting (11) into (3), we represent the effective action as
\[ S \approx S_0 + S_2 \]
\[ S_0 = \int_0^\beta d\tau \int d\vec{r} \{ \hat{K}_a \rho_{0a} + \hat{K}_b \rho_{0b} + \frac{g_a}{2} \rho_{0a}^2 + \frac{g_b}{2} \rho_{0b}^2 + g_{ab} \rho_{0a} \rho_{0b} + \frac{\Omega_R \sqrt{\rho_{0a} \rho_{0b}}}{2} \xi (\omega_R + \omega_R^*) \} \quad (9a) \]
\[ S_2 = -\frac{1}{2} \int d\tau d\tau' d\vec{r} d\vec{r}' [\tilde{\psi}^\dagger(\tau, \vec{r}), \tilde{\psi}(\tau, \vec{r}), \tilde{\phi}^\dagger(\tau, \vec{r}), \tilde{\phi}(\tau, \vec{r})] D^{-1}(\tau, \tau', \vec{r}, \vec{r}') \quad (9b) \]
where the inverse Green function in momentum space is given by
\[ D^{-1}(k, \omega_n) = \begin{pmatrix}
  i\omega_n - \varepsilon_k^a + \Lambda_a & -g_a \rho_{0a} & -g_{ab} \sqrt{\rho_{0a} \rho_{0b}} \xi - \frac{\Omega_R}{2} \omega_R & -g_{ab} \sqrt{\rho_{0a} \rho_{0b}} \xi \\
  -g_a \rho_{0a} & i\omega_n - \varepsilon_k^a + \Lambda_a & -g_{ab} \sqrt{\rho_{0a} \rho_{0b}} \xi & -g_{ab} \sqrt{\rho_{0a} \rho_{0b}} \xi - \frac{\Omega_R}{2} \omega_R \omega_{tar} \\
  -g_{ab} \sqrt{\rho_{0a} \rho_{0b}} \xi - \frac{\Omega_R}{2} \omega_R^* \omega_{tar} & -g_{ab} \sqrt{\rho_{0a} \rho_{0b}} \xi & i\omega_n - \varepsilon_k^b + \Lambda_b & -g_b \rho_{0b} \\
  -g_{ab} \sqrt{\rho_{0a} \rho_{0b}} \xi & -g_{ab} \sqrt{\rho_{0a} \rho_{0b}} \xi - \frac{\Omega_R}{2} \omega_R & -g_b \rho_{0b} & i\omega_n - \varepsilon_k^b + \Lambda_b
\end{pmatrix} \quad (10) \]
where \( \varepsilon_{a,b}^k = -\frac{\vec{k}^2}{2m_{a,b}} \), \( \Lambda_a = \mu - 2g_a\rho_{0a} - g_{ab}\rho_{0b} \); \( \Lambda_b = \mu - 2g_b\rho_{0b} - g_{ab}\rho_{0a} \). Now we define self-energies as \[ \Sigma_{ij} = (D^{-1})_{ij} - D_{ij}^{-1} \] (11)

where \( \hat{D}_0 \) corresponds to the "ideal gas" with Rabi coupling:

\[
\hat{D}_0^{-1}(k,\omega_n) = D^{-1}(k;\omega_n)|_{g_{a,b}=0, g_{ab}=0} = \\
\begin{pmatrix}
    i\omega_n - \varepsilon_k^a + \mu & 0 & -\frac{\Omega_R}{2}\omega_R & 0 \\
    0 & -i\omega_n - \varepsilon_k^b + \mu & 0 & -\frac{\Omega_R}{2}\omega_R^b \\
    -\frac{\Omega_R}{2}\omega_R^* & 0 & i\omega_n - \varepsilon_k^b + \mu & 0 \\
    0 & -\frac{\Omega_R}{2}\omega_R & 0 & -i\omega_n - \varepsilon_k^b + \mu
\end{pmatrix}
\] (12)

Note that, the Green function \( \hat{D}_0 \) may be used in organizing perturbative scheme in terms of coupling constants. From (10), (11) and (12) we immediately obtain

\[
\hat{\Sigma}(k,\omega_n) = \\
\begin{pmatrix}
    2g_a\rho_{0a} + g_{ab}\rho_{0b} & g_a\rho_{0a} & g_{ab}\sqrt{\rho_{0a}\rho_{0b}}\xi & g_{ab}\sqrt{\rho_{0a}\rho_{0b}}\xi \\
    g_a\rho_{0a} & g_a\rho_{0a} + 2g_{ab}\rho_{0b} & g_{ab}\sqrt{\rho_{0a}\rho_{0b}}\xi & g_{ab}\sqrt{\rho_{0a}\rho_{0b}}\xi \\
    g_{ab}\sqrt{\rho_{0a}\rho_{0b}}\xi & g_{ab}\sqrt{\rho_{0a}\rho_{0b}}\xi & 2g_b\rho_{0b} + g_{ab}\rho_{0a} & g_b\rho_{0b} \\
    g_{ab}\sqrt{\rho_{0a}\rho_{0b}}\xi & g_{ab}\sqrt{\rho_{0a}\rho_{0b}}\xi & g_b\rho_{0b} & 2g_b\rho_{0b} + g_{ab}\rho_{0a}
\end{pmatrix}
\] (13)

Thus normal and anomalous self-energies are

\[
\Sigma^a_n = 2g_a\rho_{0a} + g_{ab}\rho_{0b}, \quad \Sigma^b_n = 2g_b\rho_{0b} + g_{ab}\rho_{0a} \\
\Sigma^a_{an} = g_a\rho_{0a}, \quad \Sigma^b_{an} = g_b\rho_{0b} \\
\Sigma^{ab}_{n} = \Sigma^{ab}_{an} = g_{ab}\sqrt{\rho_{0a}\rho_{0b}}\xi,
\] (14) (15) (16)

On the other hand, the Green functions and excitations spectrum can be presented in a more compact form in Cartesian representation (real field formalism) as

\[
\tilde{\psi} = \frac{1}{\sqrt{2}}(\psi_1 + i\psi_2), \quad \tilde{\phi} = \frac{1}{\sqrt{2}}(\phi_3 + i\phi_4)
\] (17)

with real functions \([\psi_1, \psi_2, \psi_3, \psi_4]\), such that

\[
S_2 = \frac{1}{2} \int d\tau d\mathbf{r} d\tau' d\mathbf{r}' \sum_{i,j=1}^{4} \psi_i(\tau, \mathbf{r}) G_{ij}^{-1}(\tau, \mathbf{r}, \tau', \mathbf{r}') \psi_j(\tau', \mathbf{r}')
\] (18)
Now from (3), (16), (17) and (18) one finds

\[
G^{-1}(\vec{k}, \omega_n) = \begin{pmatrix}
\varepsilon_a^k + X_1 & \omega_n & X_5 & 0 \\
-\omega_n & \varepsilon_b^k + X_2 & 0 & X_6 \\
X_5 & 0 & \varepsilon^b_k + X_3 & \omega_n \\
0 & X_6 & -\omega_n & \varepsilon^b_k + X_4
\end{pmatrix}
\]  
(19)

where

\[
X_1 = -\mu + 3\rho_0 \alpha a + \rho_0 \beta a = \Sigma^a_n + \Sigma^a_{an} - \mu \\
X_2 = -\mu + \rho_0 \alpha a + \rho_0 \beta a = \Sigma^a_n - \Sigma^a_{an} - \mu \\
X_3 = -\mu + 3\rho_0 \beta b + \rho_0 \alpha a = \Sigma^b_n + \Sigma^b_{an} - \mu \\
X_4 = -\mu + \rho_0 \beta b + \rho_0 \alpha a = \Sigma^b_n - \Sigma^b_{an} - \mu \\
X_5 = 2\xi g_{ab} \sqrt{\rho_0 \alpha \rho_0 b} + \frac{\Omega_R \omega_R}{2}, \quad X_6 = \frac{\Omega_R \omega_R}{2}
\]
(20)

so that \(\Sigma^a_{ab} = \Sigma^a_{an} = g_{ab} \sqrt{\rho_0 \alpha \rho_0 b} = (X_5 - X_6)/2\). From the condition \(\det[G^{-1}(\vec{k}, \omega_n)] = 0\) one may find the excitation spectrum:

\[
\omega_1^2 = \frac{E_a^2 + E_b^2}{2} + X_5X_6 + \frac{\sqrt{D_s}}{2}, \quad \omega_2^2 = \frac{E_a^2 + E_b^2}{2} + X_5X_6 - \frac{\sqrt{D_s}}{2}
\]  
(21)

where

\[
D_s = (E_a^2 - E_b^2)^2 + 4X_5X_6(E_a^2 + E_b^2) + 4X_6^2 E_{13}^2 + 4X_5^2 E_{24}^2,
\]

\[
E_a^2 = W_1 W_2, \quad E_b^2 = W_3 W_4, \quad E_{13}^2 = W_1 W_3, \quad E_{24}^2 = W_2 W_4
\]  
(22)

\[
W_1 = \varepsilon_k^a + X_1, \quad W_2 = \varepsilon_k^a + X_2, \quad W_3 = \varepsilon_k^b + X_3, \quad W_4 = \varepsilon_k^b + X_4
\]

Now we come back to the question on HP relations. For simplicity we limit ourselves to the symmetrical case with \(m_a = m_b = m, g_a = g_b = g, \rho_{0a} = \rho_{0b} = \rho_0/2\), \(\Sigma^a_n = \Sigma^b_n = \Sigma_n\) and \(\Sigma^a_{an} = \Sigma^b_{an} = \Sigma_{an}\), to rewrite the dispersions in (21) as

\[
\omega_1^2 = (\varepsilon_k + X_2 + X_6)(\varepsilon_k + X_1 + X_5)
\]

\[
\omega_2^2 = (\varepsilon_k + X_2 - X_6)(\varepsilon_k + X_1 - X_5)
\]  
(23)

Both from experimental and theoretical studies it is well known [1] that, spontaneous symmetry breaking of gauge invariance in the Hamiltonian (2a) leads to the appearance of two modes, \(\omega_d\) and \(\omega_s\), at that the density mode \(\omega_d\) is gapless, while the spin mode \(\omega_s\) has a finite
gap. From Eq. (23) it is seen that this may be achieved by imposing a simple condition as $X_2 = X_6$, which in terms of the self energies will be equivalent to the relation

$$\mu = \Sigma_n - \Sigma_{an} - \frac{\Omega R}{2}$$

(24)

where we used (20). Note that, in the absence of the Rabi coupling one comes back to the familiar HP relation $\Sigma_n - \Sigma_{an} = \mu$. In the next section we show that the chemical potential determined by the relation (24) indeed will correspond to the minimum of the thermodynamic potential in an equilibrium.

III. THERMODYNAMIC POTENTIAL AND THE CONDITION FOR THE EXISTENCE OF A PURE BEC

The grand canonical potential can be easily found as

$$\Omega = -T \ln Z, \quad Z = \int D\psi_1 D\psi_2 D\psi_3 D\psi_4 e^{-S[\psi_1,\psi_2,\psi_3,\psi_4]}$$

(25)

where $S = S_0 + S_2$ with $S_0$ and $S_2$ are given in Eqs. (9b) and (18), respectively. The path integral in (25) is Gaussian and can be evaluated exactly. As a result one obtains:

$$\Omega = \Omega_0 + \Omega_{ln}$$

(26)

$$\Omega_0 = V[-\mu \rho_{0a} - \mu \rho_{0b} + \frac{g_a \rho_{0a}^2}{2} + \frac{g_b \rho_{0b}^2}{2} + g_{ab} \rho_{0a} \rho_{0b} + \frac{\Omega R \sqrt{\rho_{0a} \rho_{0b}} (\omega_R + \omega_R^*)}{2}]$$

(27)

$$\Omega_{ln} = \frac{1}{2} \sum_k [\omega_1 + \omega_2 - \varepsilon_k^a - \varepsilon_k^b + (\text{counterterms})] +$$

$$T \sum_k \ln(1 - e^{-\beta \omega_1}) + T \sum_k \ln(1 - e^{-\beta \omega_2})$$

(28)

(29)

where $\omega_{1,2}$ are given in Eqs. (21) and (22). In stable equilibrium, the variational parameters $\rho_{0a}$ and $\rho_{0b}$ should satisfy the saddle-point equations (12, 27, 28)

$$\frac{\partial \Omega_0}{\partial \rho_{0a}} = 0, \quad \frac{\partial \Omega_0}{\partial \rho_{0b}} = 0, \quad \left(\frac{\partial^2 \Omega_0}{\partial \rho_{0a}^2}\right) - \left(\frac{\partial^2 \Omega_0}{\partial \rho_{0a} \partial \rho_{0b}}\right) > 0$$

(30)

Besides, the relative phase angle $\theta$, should also correspond to the minimum of $\Omega_0$:

$$\frac{\partial \Omega_0}{\partial \theta} = 0, \quad \frac{\partial^2 \Omega_0}{\partial \theta^2} > 0$$

(31)

Now bearing in mind $\omega_R = e^{-i \theta_R}$, and using (26), (27), (31) we obtain

$$\frac{\partial \Omega_0}{\partial \theta} = -V \Omega R \rho_{0a} \rho_{0b} \sin(\theta - \theta_R) = 0,$$

$$\frac{\partial^2 \Omega_0}{\partial \theta^2} = -V \Omega R \rho_{0a} \rho_{0b} \cos(\theta - \theta_R) > 0$$

(32)
which lead to the well known [1] [9] [10] result: $\theta - \theta_R = \pi (2n + 1)$ i.e. $\omega_R \xi = -1$. This physically means that, when the system goes into its equilibrium state, it will choose the relative phase between the two condensates by itself, preferring the case with $\xi = -\omega_R$, where $\omega_R$ is in fact the sign of the one body interaction. For example, if the latter is negative, $\omega_R = -1$, then the condensates will coexist with the same phase $\xi = +1$, and vice versa. It is intuitively clear that, for a homogenous Bose mixture physical observable should not depend on the relative phase $\xi$, or more precisely on its sign: $\xi = +1$ or $\xi = -1$.\[1\] Moreover, particularly, in both cases, with $\xi = \pm 1$ there always exist in phase and out of phase excitations. The former correspond to the density branch ($\omega_d$), while the latter to the spin branch ($\omega_s$), as it has been clarified in Refs. [37, 38].

So, using the condition $\omega_R \xi = -1$ one can rewrite a physical $\Omega_0$ as

$$\Omega_0 = V \left\{ -\mu \rho_{0a} - \mu \rho_{0b} + \frac{g_a \rho_{0a}^2}{2} + \frac{g_b \rho_{0b}^2}{2} + g_{ab} \rho_{0a} \rho_{0b} - \Omega_R \sqrt{\rho_{0a} \rho_{0b}} \right\} \tag{33}$$

and hence Eq.s (30) will have following explicite form:

$$\frac{\partial \Omega_0}{\partial \rho_{0a}} = -\frac{V}{2} \left[ 2\mu - 2g_a \rho_{0a} - 2g_{ab} \rho_{0b} + \frac{\Omega_R \rho_{0b}}{\sqrt{\rho_{0a} \rho_{0b}}} \right] = 0$$

$$\frac{\partial \Omega_0}{\partial \rho_{0b}} = -\frac{V}{2} \left[ 2\mu - 2g_b \rho_{0b} - 2g_{ab} \rho_{0a} + \frac{\Omega_R \rho_{0a}}{\sqrt{\rho_{0a} \rho_{0b}}} \right] = 0 \tag{34}$$

which give

$$\mu = g_a \rho_{0a} + g_{ab} \rho_{0b} - \frac{\Omega_R \rho_{0b}}{2 \sqrt{\rho_{0a} \rho_{0b}}}$$

$$\mu = g_b \rho_{0b} + g_{ab} \rho_{0a} - \frac{\Omega_R \rho_{0a}}{2 \sqrt{\rho_{0a} \rho_{0b}}} \tag{35}$$

From these equations one may come to the following important conclusion: If $\rho_{0a}(T) \neq \rho_{0b}(T)$, then there would be no pure BEC with a certain critical temperature above which $\rho_{0a}(T > T_c)$ or $\rho_{0b}(T > T_c)$ absolutely vanish. Instead, since the chemical potential should be finite, ($\mu \neq \infty$), one has to deal with a crossover from BEC to the normal phase transition. The situation may be cured by assumption $\rho_{0a}(T) = \rho_{0b}(T)$ in the whole ranges of the temperatures. Clearly this can take place in symmetric case with equal masses and coupling constants. For this reason in the rest of this work we shall consider only symmetric Bose mixtures.

\[1\] This rule may be referred as a phase invariance.
IV. SYMMETRIC BOSE MIXTURES WITH ONE BODY COUPLING

Let \( \rho_a = \rho_b = \rho / 2, \rho_{0a} = \rho_{0b} = \rho_0 / 2, g_a = g_b = g, m_a = m_b = m, \varepsilon_a^a = \varepsilon_a^b = \varepsilon_k \) and hence \( X_3 = X_1, X_4 = X_2 \). Below we first derive explicit expressions for the density of the uncondensed fraction \( \rho_{1a} = \rho_{1b} = \rho_1 / 2 \) and then discuss its features in the normal and BEC phases.

A. Uncondensed fraction \( \rho_1 = \rho - \rho_0 \)

It is well known that quantum and temperature fluctuations tend to destroy BEC leading to an emergence of a depletion defined as

\[
\rho_{1a} = \frac{1}{\mathcal{V}} \int d\vec{r} \overline{\psi}^{(1)}(\vec{r}) \psi(\vec{r}) = \frac{1}{2\mathcal{V}} \int d\vec{r} [G_{11}(\vec{r}, \tau; \vec{r}', \tau') + G_{22}(\vec{r}, \tau; \vec{r}', \tau')]_{|\vec{r} = -\vec{r}', \tau = -\tau'}
\]  
(36)

where

\[
G_{ij}(r, \tau; r', \tau') = \frac{1}{V} \sum_{\omega, \vec{k}} e^{i\vec{k}(r-r')} e^{i\omega(\tau-\tau')} G_{ij}(\omega, \vec{k}),
\]
(37)

The matrix elements of the Green function \( G_{ij}(x, x) \) can be obtained by inverting \( \hat{G}^{-1} \) in Eq. (19) and performing Matsubara summations in (37). The result is given by

\[
G_{11}(x, x) = G_{33}(x, x) = A_1(\vec{k}, \omega, X_1, X_2, X_3, X_6) + A_1(\vec{k}, \omega, X_1, X_2, -X_5, -X_6)
\]

\[
G_{22}(x, x) = G_{44}(x, x) = A_2(\vec{k}, \omega, X_1, X_2, X_5, X_6) + A_2(\vec{k}, \omega, X_1, X_2, -X_5, -X_6)
\]

\[
G_{13} = A_1(\vec{k}, \omega, X_1, X_2, X_5, X_6) - A_1(\vec{k}, \omega, X_1, X_2, -X_5, -X_6)
\]

\[
G_{24} = A_2(\vec{k}, \omega, X_1, X_2, X_5, X_6) - A_2(\vec{k}, \omega, X_1, X_2, -X_5, -X_6)
\]

(38)

where \( x \equiv (\vec{r}, \tau) \) and the rest of matrix elements equals to zero, e.g. \( G_{12}(x, x) = 0 \). Here we have introduced following notations

\[
A_1(\vec{k}, \omega, X_1, X_2, X_5, X_6) = \frac{1}{2} \sum_{\vec{k}} \frac{X_3^2 W_1 + X_5 W_2^2 + W_2 X_5 X_6 + W_1 W_2 X_6 W(\omega)}{\omega_1(W_2 X_5 + W_1 X_6)}
\]

\[
A_2(\vec{k}, \omega, X_1, X_2, X_5, X_6) = \frac{1}{2} \sum_{\vec{k}} \frac{X_3^2 W_2 + X_6 W_2 + W_1 X_5 X_6 + W_1 W_2 X_5 W(\omega)}{\omega_1(W_2 X_5 + W_1 X_6)}
\]

(39)

where \( W_1 = X_1 + \varepsilon_k, W_2 = X_2 + \varepsilon_k, W(\omega) = 1/2 + 1/(\exp(\omega \beta) - 1) \) and \( \omega_{1,2} \) are defined in the Eqs. (23). Now from Eqs. (36)-(39) one can immediately find

\[
\rho_{1a} = \rho_{1b} = \frac{1}{2\mathcal{V}} \sum_{\vec{k}} \left\{ \frac{X_3^2 W_1 + X_5 W_2 + (W_1 + W_2)(W_1 X_6 + W_2 X_5 + X_5 X_6) W(\omega)}{2\omega_1(W_2 X_5 + W_1 X_6)} + \omega_1 \to \omega_2, X_5 \to -X_5, X_6 \to -X_6 \right\} \]  
(40)
B. Phase invariance

In the symmetric case for the self energies we have

\[ X_1 = X_3 = -\mu + 3\rho_{0a}g + \rho_{0a}g_{ab} = \Sigma_n + \Sigma_{an} - \mu \]
\[ X_2 = X_4 = -\mu + \rho_{0a}g + \rho_{0a}g_{ab} = \Sigma_n - \Sigma_{an} - \mu \]
\[ X_5 = 2\xi_{g_{ab}}\rho_{0a} + \Omega_R \omega_R \]
\[ X_6 = \Omega_R \omega_R \]

and for the dispersions:

\[ \omega_1 = \sqrt{(\varepsilon_k + X_2 + X_6)(\varepsilon_k + X_1 + X_5)}, \quad \omega_2 = \sqrt{(\varepsilon_k + X_2 - X_6)(\varepsilon_k + X_1 - X_5)} \]

From Eqs. (40) - (42) it is seen that \( \rho_{1a} \) is phase invariant, that is \( \rho_{1a}(\xi, \omega_R) = \rho_{1a}(-\xi, -\omega_R) \), since the transformation \( (\xi, \omega_R) \leftrightarrow (-\xi, -\omega_R) \) is equalent to the folowing replacements: \( X_5 \leftrightarrow -X_5, \ X_6 \leftrightarrow -X_6, \ \omega_1 \leftrightarrow \omega_2 \). Note that, this statement holds true both for the condensed as well as normal states. For this reason in the rest of the paper we set \( \xi = 1, \ \omega_R = -1 \) for convenience, and discuss BEC and normal states separately.

C. Normal phase \( (T > T_c) \).

Here we have \( \rho_{0a} = \rho_{0b} = 0 \) and Eqs. (41) give

\[ X_1(T > T_c) = X_2(T > T_c) = -\mu, \quad X_5(T > T_c) = X_6(T > T_c) = -\Omega_R/2, \]

and hence

\[ \omega_1^2(T > T_c) = (\varepsilon_k + X_1 + X_5)^2 = (\varepsilon_k - \mu - \Omega_R/2)^2 \]
\[ \omega_2^2(T > T_c) = (\varepsilon_k + X_1 - X_5)^2 = (\varepsilon_k - \mu + \Omega_R/2)^2 \]

The density of uncondensed particles, say of the component \( a \), given by the general expression (40) will be exactly equal to the density of the particles of the sort \( a \) with the following explicite expression

\[ \rho_{1a}(T > T_c) = \rho_a = \frac{\rho}{2} = \frac{1}{2V} \sum_k \left\{ \frac{1}{e^{\beta \omega_1} - 1} + \frac{1}{e^{\beta \omega_2} - 1} \right\} \]

where \( \omega_{1,2} \) are defined in Eqs. (44). Note that in the normal state HP relations do not make sense, so the chemical potential \( \mu(T > T_c) \) should be evaluated as the solution to the equation (45) with the given input parameters such as \( \rho, \ T \) and \( \Omega_R \).
D. Condensed phase.

On the other hand, at enough low temperatures, \( T < T_c \), HP relations are appropriate. Moreover, it can be easily shown that, the chemical potential \( \mu \) determined by the HP relation in Eq. (24) corresponds to the minimum of \( \Omega \). In fact, \( \mu \) in Eq. (35) can be rewritten as

\[
\mu = g\rho_0 a + g_{ab}\rho_0 a - \frac{\Omega_R}{2}
\]

in the symmetric case. Now inverting the explicit expressions for the self energies (46), one finds

\[
g\rho_0 a = \Sigma_n, \quad g_{ab}\rho_0 a = \Sigma_n - 2\Sigma_{an}
\]

and inserting these into (46) comes back exactly to the HP relations in Eq. (24). Thus we conclude that HP relations for a symmetric Bose mixture with Rabi coupling, introduced by the Hamiltonian (2d) are

\[
\Sigma_n - \Sigma_{an} = \mu + \frac{\Omega_R}{2}, \quad \Sigma_{ab}^n = \Sigma_{ab}^{an}
\]

As it has been pointed out in Sect.II, due to these relations the excitation spectrum has two branches: one is gapless with

\[
\omega_1^2 = \varepsilon_k (\varepsilon_k + X_1 + X_5) = \varepsilon_k [\varepsilon_k + 2\rho_0 a (g + g_{ab})] \approx c k^2 + O(k^4)
\]

and the other one with a gap:

\[
\omega_2^2 = [\varepsilon_k + \Omega_R] [\varepsilon_k + \Omega_R + 2\rho_0 a (g - g_{ab})], \quad \omega_2(k = 0) \neq 0
\]

In above equations we have used following explicit expressions for the self energies:

\[
X_1 = 2g\rho_0 a + \frac{\Omega_R}{2}, \quad X_2 = \frac{\Omega_R}{2},
\]

\[
X_5 = 2g_{ab}\rho_0 a - \frac{\Omega_R}{2}, \quad X_6 = -\frac{\Omega_R}{2}
\]

The Gaussian approximation possesses a nice feature: The same chemical potential, corresponding to the minimum of the free energy is also relevant both for the spectrum and HP theorem. However, an extension of this approach e.g. by taking into account anomalous density, \( \sigma \sim < \tilde{\psi}\tilde{\psi} > \), faces a problem, referred as Hohenberg - Martin dilemma [39, 40], which declines the existence of a universal chemical potential in a BEC phase. The solution of this problem has been found in Refs. [37, 41], by introducing two chemical potentials. We shall discuss such extension in our forthcoming paper. Here we note that, the dispersions
being in a complete agreement with results by Cappelaro et al. [12], coincide with the results by other theoretical works e.g. [9–11] in the Bogoliubov approximation, \( \rho_{0a} \approx \rho_a = \rho/2 \).

In reality, especially at finite temperatures, the Bogoliubov approximation becomes irrelevant. So, in practical calculations within present approximation one may evaluate the condensate fraction from the normalization condition

\[
\rho_{0a} = \rho/2 - \rho_{1a}
\]  

(52)

which is, in fact, a nonlinear algebraic equation with respect to \( \rho_{0a} \) with a fixed \( \rho \) and \( \rho_{1a} \) given by Eqs. (40),(49) - (51). It is naturally expected that, the solutions should diminish by increasing the temperature to reach zero at a critical one.

V. CRITICAL TEMPERATURE AND ITS SHIFT

It is well known that MFT is not enough good to describe critical properties of even one component Bose gases near BEC transition [28]. One of the reasons of such failure is that, due to the Bogoliubov shift (4) formulas for an observable parameter include a factor like \( g\rho_0 \), which goes to zero near \( T_c \) due to \( \rho_0(T = T_c) = 0 \). As a result, a system of particles "forgets" about the presence of interparticle interaction and behaves like an ideal gas with \( T_c = T_c^0 \) and \( \mu(T = T_c) = 0 \). Particularly, the present approximation also fails to predict e.g. the shift of the critical temperature due interparticle interaction. However, this is not the case for a one body interaction. Below we derive explicite expressions for \( T_c \) as well as for its shift due to this interaction, which can be realized e.g. by Rabi coupling.

A. Equation for \( T_c \)

It is understood that this coupling modifies an effective chemical potential of the system. In previous section we have shown that, to evaluate \( \mu \) one has to solve the algebraic equation (45) or (46), (52) for the normal \( T > T_c \) or BEC \( T < T_c \) phases, respectively. But how about \( \mu(T = T_c) \)? Actually, it can be found, for example, from continuity condition of the self energies as \( X_1(T = T_c - 0) = X_1(T = T_c + 0) \) and \( X_2(T = T_c - 0) = X_2(T = T_c + 0) \). So,
by using Eqs. (43) and (51) we immediately find

\[ X_1(T = T_c) = X_2(T = T_c) = -\mu(T = T_c) = -\frac{\Omega_R}{2} \]  
\[ X_5(T = T_c) = X_6(T = T_c) = -\frac{\Omega_R^2}{2} \]  

and inserting these into (42) recognise that there are still two different branches of the spectrum:

\[ \omega_1(T = T_c) = \varepsilon_k, \quad \omega_2(T = T_c) = \varepsilon_k + \Omega_R \]  

Now using Eqs. (53) and (54) in (40) we obtain following equations for the critical temperature \( T_c \equiv 1/\beta_c \):

\[ \rho_{\text{in}}(T = T_c) = \rho_a = \rho/2 = \frac{1}{2V} \sum_k \left\{ \frac{1}{e^{\varepsilon_k/T_c} - 1} + \frac{1}{z^{-1}e^{\varepsilon_k/T_c} - 1} \right\} \]  

where \( z = \exp(-\Omega_R/T_c) \) is the fugacity due to the Rabi coupling, \( 0 \leq z \leq 1 \). As to the \( \rho \) in this equation, it is clear that, when the Rabi coupling is switched on or off the density of the whole uniform system remains unchanged: \( \rho = \rho(\Omega_R = 0) = \rho(\Omega_R \neq 0) \), being as an fixed input parameter. So, bearing in mind this trivial condition one can define \( T_c^0 \) as the solution to following equation:

\[ \rho_a = \rho/2 = \frac{1}{V} \sum_k \left\{ \frac{1}{e^{\varepsilon_k/T_c^0} - 1} \right\} \]  

which is just the particular case of (55) for \( \Omega_R = 0 \). Thus we obtain following relation between \( T_c \) and \( T_c^0 \)

\[ \frac{1}{2} \sum_k \left\{ \frac{1}{e^{\varepsilon_k/T_c} - 1} + \frac{1}{e^{\varepsilon_k/T_c} z^{-1} - 1} \right\} - \sum_k \frac{1}{e^{\varepsilon_k/T_c^0} - 1} = 0 \]  

with \( \sum_k \equiv V \int d^3k/(2\pi)^3 \). This relation can be presented in a rather compact form as

\[ T_c = T_c^0 \left\{ \frac{2\zeta(3/2)}{\left[ \zeta(3/2) + Li(3/2, z) \right]} \right\}^{2/3} \]  

by using following well known integrals

\[ \int \frac{d^3k}{(2\pi)^3} \left[ e^{\varepsilon_k/T} - 1 \right] = \bar{c}(mT)^{3/2} \]  
\[ \int \frac{d^3k}{(2\pi)^3} \left[ e^{\varepsilon_k/T} - 1 \right] = \frac{\bar{c}(mT)^{3/2}}{\zeta(3/2)} Li(3/2, z) \]  

where \( \bar{c} = \zeta(3/2)/2\sqrt{2\pi}^{3/2} \), \( \zeta(3/2) = Li(3/2, 1) \) and \( Li(3/2, z) \) is the polylogarithm function. The equation (58) is a nonlinear algebraic equation with respect to \( T_c \) with given input parameters \( T_c^0 \) and \( \Omega_R \). In the next subsection we will discuss its exact as well as approximative solutions.
B. The shift vs $\Omega_R$

From Eq. (58) one determines the shift of the critical temperature due to Rabi coupling as:

$$\Delta_R = \frac{\Delta T_c}{T_c^0} = \frac{T_c - T_c^0}{T_c^0} = \left\{ \frac{2\zeta(3/2)}{\zeta(3/2) + Li(3/2, z)} \right\}^{2/3} - 1 \quad (60)$$

In Figs. 1 we presented $\Delta_R$ vs $\Omega_R/T_c^0$. As it is seen from Figs. 1 the shift is always positive and tends to a plateau at large $\Omega_R$ for a finite $T_c^0$. This asymptotic value can be easily evaluated from the Eq. (58). In fact, since $z(\Omega_R \to \infty) = 0$, and $Li(3/2, 0) = 0$ we obtained that $\Delta_R(\Omega_0 \to \infty) = 2^{2/3} - 1 \approx 0.5874$. The dashed and dotted curves in Figs. 1 are approximative solutions of Eq. (58).

In the literature one can find two kinds of approximations for $Li(3/2, z)$ (see e.g. [42]):

$$Li(3/2, z) = z + \frac{\sqrt{2}z^2}{4} + \frac{\sqrt{3}z^3}{9} + O(z^4) \quad (61a)$$

$$Li(3/2, z) = -2\sqrt{\pi}\sqrt{\ln(1/z)} + \zeta(3/2) + \zeta(1/2)\ln(z) + \frac{\zeta(-1/2)}{2}\ln^2(z) + O(\ln^3(z)) \quad (61b)$$

The former (61a) is good for small $z < 0.3$ (that is for large $\Omega_R$) and the latter (61b) for rather large $0.3 \leq z \leq 1$ (for small $\Omega_R$). Thus using these expansions in (58) and solving the equation iteratively we find following approximations

$$\Delta_R(\Omega_R/T_c^0 \leq 1) = \frac{432^{1/3}/18\zeta(3/2)^{2/3}}{\sqrt{x} + \frac{\pi + \zeta(1/2)\zeta(3/2)}{3(\zeta(3/2)^2)}}x + O(x^{3/2}) \approx 0.45\sqrt{\frac{\Omega_R}{T_c^0}} - 0.033\frac{\Omega_R}{T_c^0} \quad (62)$$
for small $\Omega_R/T_c^0$ and

$$\Delta_R(\Omega_R/T_c^0 \gg 1) \approx 2^{2/3} - 1 - \frac{25/2 \exp(-\frac{\Omega_R}{T_c^0 2^{2/3}})}{3\zeta(3/2)} \approx 0.5874 - 0.4 \exp(-\frac{0.63\Omega_R}{T_c^0})$$  \(63\)

for large values of the dimensionless parameter $\Omega_R/T_c^0$. This is illustrated in Fig.1b, where dashed and dotted curves are appropriate for the Eqs. \(63\) and \(62\) respectively.

Note that, existing experiments are performed with $\Omega_R \sim 1000 H \approx 0.48 \cdot 10^{-7} K$ at temperatures $T \sim 2 \cdot 10^{-7} K$, which corresponds to rather small values of $\Omega_R/T_c^0 \sim 0.25$. From Fig.1b we conclude that, in reality one may use the approximation given by Eq. \(62\) which gives about 20% near this region for the shift of the critical temperature due to Rabi coupling.

VI. CONCLUSION

In present work we have studied Rabi coupled two component Bose mixtures within the Gaussian (bilinear) approximation in the framework of the mean field theory. We have shown that for this system at finite temperatures BEC can exist only for a symmetric case $(m_a = m_b, g_a = g_b)$ with an equal population, otherwise only a crossover transition may take place. For the first time we have derived Hugenholtz - Pines relations and an equation for the shift of the critical temperature in the presence of one body interaction. Although our extended HP relations: $\Sigma_n - \Sigma_{an} = \mu + \Omega_R/2$, and $\Sigma_{ab} = \Sigma_{an}^{ab}$ are proved within the field theoretical Gaussian approximation, we expect that they will remain true beyond this approximation also and can be proved more accurately, say in the spirit of works \[30, 32\].

As to the critical temperature $T_c$ of a BEC transition in Rabi coupled two component system, it has never been studied before. So, we obtained rather simple equation to determine $T_c$ as well as its shift due to Rabi coupling. The solutions of this equation may be apparently presented analytically for small (or large) values of the input parameter $\Omega_R/T_c^0$. The shift, as we predicted, is positive and should be about $\sim 20\%$ in existing experimental measurements with the moderate value of parameter $\Omega_R/T_c^0 \sim 0.2$. Principally, one may assume possibility of making measurements with rather large values of $\Omega_R/T_c^0$ also. For such experiments, we predict that the shift should go to its asymptotic value $\Delta T_c/T_c^0 \leq 2^{2/3} - 1 \approx 0.58$, displaying a plateau. We hope that, this prediction can be verified in future experiments performed at finite temperatures.
Moreover, we have proved that, physical observables, particularly $\Delta T_c/T_c^0$ do not depend on the sign of one body interaction i.e. on the phase of Rabi coupling $\omega_R$. Possible effect of changing the sign of this interaction could be reflected only in the relative phase of the two condensates $\xi$ under the condition $\xi\omega_R = -1$.

Note that, other problems concerning these systems, such as the temperature dependence of thermodynamic parameters, as well as stability conditions [12] are left beyond the scope of the present work. We shall study these problems in our forthcoming paper, where we use a more accurate approximation than the Gaussian one.

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