OPTIMAL MAXIMUM NORM ESTIMATES FOR VIRTUAL
ELEMENT METHODS

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Abstract. The maximum norm error estimations for virtual element methods are studied. To
establish the error estimations, we prove higher local regularity based on delicate analysis of Green’s
functions and high-order local error estimations for the partition of the virtual element solutions.
The maximum norm of the exact gradient and the gradient of the projection of the virtual element
solutions are proved to achieve optimal convergence results. For high-order virtual element methods,
we establish the optimal convergence results in $L^{\infty}$ norm. Our theoretical discoveries are validated
by a numerical example on general polygonal meshes.

Key words. virtual element method, maximal error estimate, Green’s function

AMS subject classifications. 65N30, 65N25, 65N15.

1. Introduction. In recent years, there has been a surge of interest in developing
numerical methods for numerically solving partial differential equation using general
polygonal/polyhedral meshes. The construct of finite element shape functions on
convex polygons was first articulated by Wachspress (1971) [40] and popularized in
his book: A Rational Finite Element Basis [41]. Since that, considerable literature
has grown up around the theme of developing finite element/difference methods using
general polygons/polyhedra. Famous examples include the polygonal finite element
methods [38, 39], mimetic finite difference methods [10, 28, 30, 35, 36], hybrid high-
order methods [22, 23, 31], polygonal discontinuous Galerkin methods [33], etc. The
interesting readers are referred to [32] for the recent review.

Virtual element methods were originated from developing higher-order mimetic
finite difference methods [9, 14] using the framework of finite element methods. The
key idea is to use non-polynomial basis functions which are similar to the polygonal
finite element methods [39] or the extended/generalized finite element methods [3, 25].
Different from other numerical methods using general polygons/polyhedra, virtual
element methods preserve the form of classical finite element methods on simplexes
while use general polygons/polyhedra. The beauty of virtual element methods is
that the non-polynomial basis functions are never explicitly constructed (or needed).
The local stiffness (or mass) matrix is assembled only using the polynomial basis
functions and suitable projections [1]. This capability allows virtual element methods
to handle more general continuity requirements like $C^r$ continuity with $r \geq 1$. The
usage of polygonal/polyhedral meshes makes virtual element methods handle hanging
nodes naturally and simplifies the procedure of adaptive mesh refinement.

Virtual element method was firstly proposed by Beirão da Veiga etc. In their
seminal study [5], Beirão da Veiga etc. used the Poisson equation to illustrate the
abstract framework of constructing and analyzing arbitrary order virtual element
methods using a local $H^1$-projector. Within the abstract framework, the virtual
element method is proven to be convergent optimally in both the $H^1$ norm and $L^2$

norm. In a study conducted by Ahmad etc. [1], it was shown that a local $L^2$-orthogonal
projector can be easily computed using the local $H^1$-projector by slightly changing the

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local virtual element space (or non-polynomial basis functions), which may facilitate the treatment of lower order terms and $L^2$ error analysis. In [8], Beirão Da Veiga etc. investigated the virtual element methods for general second-order elliptic equations with variable coefficients. Thereafter, there has been a lot of study of virtual element for other equations, just to name a few, see [2, 6, 8, 15–17, 43]. In the most recent work [26], Guo etc. examined the superconvergence property of the linear virtual element method and its corresponding post-processing technique.

The greater part of the literature on virtual element methods seems to have been based on the optimal error estimate in the energy norm. Up to now, there are only limited studies focusing on the analysis of the maximal norm error for virtual element methods in the literature. The first attempts on maximal norm error estimation for virtual element methods was presented by Brenner and Sung in [13], where a suboptimal $L^\infty$ error estimate was obtained. To our best knowledge, there is no optimal maximal norm error estimation for virtual element methods. In contrast, the maximal norm error estimates [27, 34] for classical numerical methods like finite element methods [12, 20] and finite volume methods [42] have been in the maturity stage.

The main purpose of this paper is to establish the optimal maximum norm estimates for virtual element methods under the general setting. For the maximum norm estimates for the classical numerical methods, the main ingredient is the inverse error estimate [12, 20]. However, for the virtual element solutions, they are no longer piecewise polynomials even though we don’t explicitly construct their non-polynomial parts. To overcome the difficulty for maximum error for gradients, we consider the gradient of projection of the virtual element solutions that are actually what we can compute and use the inverse estimate for polynomials on general polygonal domain [21]. For the maximum norm error for functions values, we bypass those difficulties by establishing an inverse error estimate using only the maximum principle for $H^1$ functions [11]. Our error estimations use the regularity results based on the delicate estimation of Green’s functions. By using a special partition of unity, we can prove the high-order local error estimates. It is worth pointing out that the high-order local $L^2$ error works except linear virtual element method because of the $2k$ conjecture [18]. The established high-order local error estimates pave the way for our optimal maximum norm error estimates.

The rest of the paper is organized as follows. The model problem and its related Sobolev spaces are introduced in Section 2. The construct of virtual element methods is given in Section 3. Then, the presentation and proof our main results are given in Section 4. Two numerical examples are provided to verify our theoretical results in Section 6. Finally, some conclusion is drawn in Section 6.

2. Model problem. In this paper, the standard notation for Sobolev spaces as in [12, 20, 24] will be adopted. Let $\Omega \subset \mathbb{R}^2$ be a bounded polygonal domain with Lipschitz boundary $\partial \Omega$. For any subdomain $D$ of $\Omega$ and $1 \leq p \leq \infty$, let $W^{k,p}(D)$ denote the Sobolev space with norm $\|\cdot\|_{W^{k,p}(D)}$ and seminorm $|\cdot|_{W^{k,p}(D)}$. When $p = 2$, $W^{k,2}(D)$ is abbreviated as $H^k(D)$ and $p$ is omitted from its norms. Similarly, $W^{0,p}(D)$ is abbreviated as $L^p(D)$. $(\cdot, \cdot)_D$ denotes the standard $L^2$ inner product in $D$ and the subscript is ignored when $D = \Omega$. Let $P_k(D)$ denote the space of polynomials of degree less than or equal to $k$ over $D$. In this paper, the letter $C$ or $c$ denotes a generic positive constant which may be different at different occurrences. We will use $x \lesssim y$ to denote $x \leq Cy$ for some constant $C$ independent of the mesh size.
We take the following second order elliptic equation as our model equation

\[
\begin{cases}
-\nabla \cdot (\alpha \nabla u) = f, & \text{in } \Omega, \\
u = 0, & \text{on } \partial \Omega,
\end{cases}
\]  

(2.1)

where the diffusion coefficient matrix \( \alpha \) is a constant coefficient matrix satisfying that

\[\alpha_0 \|x\|^2 \leq x^T \alpha x \leq \alpha_1 \|x\|^2\]

for some positive constant \( \alpha_0 \) and \( \alpha_1 \). To simplify the notation, we denote the differential operator as \( L := -\nabla \cdot (\alpha \nabla) \).

The variational formulation of (2.1) is to find \( u \in H^1_0(\Omega) \) such that

\[
(a(u, v) = (f, v), \quad \forall v \in H^1_0(\Omega),
\]

where the bilinear form \( a(\cdot, \cdot) \) is defined to be

\[
a(w, v) = (\alpha \nabla w, \nabla v).
\]

(2.3)

The boundedness of the coefficient matrix \( \alpha \) and the Lax-Milgram theorem imply the variational problem (2.2) has a unique solution \( u \).

3. Virtual element methods. One of the important merits of virtual element methods is that they allow for general polygonal partitions of the computational domain \( \Omega \). Suppose \( T_h \), consisting of non-overlapping polygonal elements, is a partition of \( \Omega \). For any element \( E \in T_h \), its diameter is denoted by \( h_E \).

The largest element diameter of \( T_h \) is denoted by \( h \). Without loss of generality, we assume that there exists \( \rho \in (0, 1) \) such that the mesh \( T_h \) satisfies the following two assumptions [8,11]:

(i). every element \( E \) is star-shaped with respect to every point of a disk \( D \) of radius \( \rho h_E \);

(ii). every edge \( e \) of \( E \) has length \( |e| \geq \rho h_E \).

For a polygonal element \( E \), we denote the barycenter of \( E \) by \( x_E \). For any nonnegative integer \( r \in \mathbb{N} \), let \( \mathcal{M}_r(E) \) be the set of polynomials defined as

\[
\mathcal{M}_r(E) := \left\{ m \middle| m = \left( \frac{x - x_E}{h_E} \right)^s \text{ for } |s| \leq r \right\},
\]

(3.1)

with \( s = (s_1, s_2) \in \mathbb{N}^2 \) being the multi-index and \( |s| = s_1 + s_2 \). We can check that \( \mathcal{M}_r(E) \) is a basis for the space of polynomials of degree \( \leq r \) on \( E \). Furthermore, we define the subspace \( \mathcal{M}_r^+(E) \) of \( \mathcal{M}_r(E) \) as

\[
\mathcal{M}_r^+(E) := \left\{ m \middle| m = \left( \frac{x - x_E}{h_E} \right)^s \text{ for } |s| = r \right\}.
\]

(3.2)

To define the virtual element space, we begin with defining the local virtual element spaces on each element. For any positive integer \( k \), let

\[
\mathcal{B}_k(\partial E) := \left\{ v \in C^0(\partial E) : v|_e \in \mathbb{P}_k(e), \quad \forall e \in \partial E \right\}.
\]

(3.3)

Then, the enlarged local virtual element space \( V_k(E) \) [1] on the element \( E \) can be defined as

\[
V_k(E) = \left\{ v \in H^1(E) : v|_{\partial E} \in \mathcal{B}_k(\partial E), \quad \Delta v|_{E} \in \mathbb{P}_{k-2}(E) \right\},
\]

(3.4)

with \( \mathbb{P}_{-1}(E) = \{0\} \).
The soul of virtual element methods is that the non-polynomial basis functions are never explicitly constructed and needed. This is made possible by introducing the projection operator \( \Pi_{\nabla}^k \). For any function \( v^h \in V(E) \), its projection \( \Pi_{\nabla}^k v^h \) is defined to satisfy the following orthogonality:

\[
(\nabla p, \nabla (\Pi_{\nabla}^k v^h - v^h))_E = 0, \quad \forall p \in \mathbb{P}_k(E),
\]

plus (to take care of the constant part of \( \Pi_{\nabla}^k \)):

\[
\int_{\partial E} (\Pi_{\nabla}^k v^h - v^h) ds = 0, \quad \text{for } k = 1,
\]

or

\[
\int_E (\Pi_{\nabla}^k v^h - v^h) dx = 0, \quad \text{for } k \geq 2.
\]

The modified local virtual element space \([1]\) is defined as

\[
W_k(E) = \{ v^h \in V_k(E) : (v^h - \Pi_{\nabla}^k v^h, q^*) = 0, \quad \forall q^* \in M_{k-1}(E) \cup M_k(E) \}.
\]

Then, the virtual element space \([1,7]\) is

\[
V_h = \{ v \in H^1(\Omega) : v|_E \in W_k(E), \quad \forall E \in \mathcal{T}_h \}.
\]

Furthermore, let \( V_{h,0} = V_h \cap H^1_0(\Omega) \) be the subspace of \( V_h \) with homogeneous boundary condition.

Similarly, we can define the \( L^2 \) projection \( \Pi_0^k \) as

\[
(p, \Pi_0^k v^h - v^h)_E = 0, \quad \forall p \in \mathbb{P}_k(E).
\]

For the linear virtual element method \([1,7]\), these two projections are equivalent, i.e. \( \Pi_{\nabla}^k = \Pi_0^k \).

On each element \( E \in \mathcal{T}_h \), we can define the following discrete bilinear form

\[
a_h^E(u^h, v^h) = (\alpha \nabla \Pi_{\nabla}^k u^h, \nabla \Pi_{\nabla}^k v^h)_E + S^E(u^h - \Pi_{\nabla}^k u^h, v^h - \Pi_{\nabla}^k v^h)
\]

for any \( u^h, v^h \in V(E) \). The discrete bilinear form \( S^E \) is symmetric, positive definite, and continuous, which is also fully computable using only the degrees of freedom of \( u^h \). The readers are referred to \([1,5,7]\) for the detail definition of \( S^E \), which is selected to make \( a_h^E(\cdot, \cdot) \) satisfy

- \( k \)-Consistency: For all \( p \in \mathbb{P}_k(E) \) and all \( v_h \in W_k(E) \),

\[
a_h^E(p, v_h) = a^E(p, v_h),
\]

where \( a^E(u, v) = \int_E \alpha \nabla u \cdot \nabla v dx \).

- Stability: There exist two positive constants \( \alpha_* \) and \( \alpha^* \), independent of \( h \) and \( E \), such that

\[
\alpha_* a^E(v_h, v_h) \leq a_h^E(v_h, v_h) \leq \alpha^* a^E(v_h, v_h), \quad \forall v_h \in W_k(E).
\]

Then, we can define the discrete bilinear form \( a_h(\cdot, \cdot) \):

\[
a_h(u^h, v^h) = \sum_{E \in \mathcal{T}_h} a_h^E(u^h, v^h),
\]
for any $u^h, v^h \in V_h$. The virtual element method for the model problem (2.1) is to find $u^h \in V_{h,0}$ such that

$$a_h(u^h, v^h) = (f, \Pi_k^0 v^h), \quad \forall v^h \in V_{h,0}. \tag{3.15}$$

Assume $\chi_1, \cdots, \chi_{NE}$ are the basis functions of the dual space of $W_k(E)$. Define the local interpolation $I_{h,E} w \in W_k(E)$ of a smooth enough function $w$ as a

$$\chi_i(w - I_{h,E} w) = 0. \tag{3.16}$$

The global interpolation operator $I_h : C^0(\Omega) \rightarrow V_h$ is defined as

$$I_h w|_E := I_{h,E} w, \quad \forall w \in C^0(\Omega). \tag{3.17}$$

For the virtual element method (3.15), [1, 5, 8] established the following convergence results in $L^2$ and $H^1$ norms.

**Theorem 3.1.** Let $u$ be the solution to the problem (2.1), and let $u_h \in V_{h,0}$ be the solution of the discretized problem. Assume further that $\Omega$ is convex, and that the exact solution $u(x)$ belongs to $H^{k+1}(\Omega)$. Then the following estimate holds:

$$h \|u - u^h\|_{H^1(\Omega)} + \|u - u^h\|_{L^2(\Omega)} \lesssim h^{k+1} \|u\|_{H^{k+1}(\Omega)}. \tag{3.18}$$

In the proof of maximum norm error estimate, we shall introduce second-order elliptic equations with inhomogeneous boundary condition. For the inhomogeneous boundary value problem, we have the following error estimate.

**Theorem 3.2.** Assume $\phi \in H^1(\Omega)$ is the solution for the following second-order elliptic equation with inhomogeneous boundary condition:

$$\begin{cases}
L \phi = -\nabla \cdot (a \nabla \phi(x)) = f(x), & \text{in } \Omega, \\
\phi(x) = g(x), & \text{on } \partial \Omega.
\end{cases} \tag{3.19}$$

We further assume that $\partial \Omega$ is smooth enough such that $\phi \in H^{k+1}(\Omega)$ and $g \in H^{k+\frac{1}{2}}(\partial \Omega)$. Let $\phi^h$ be the $k$th order virtual element approximation of $\phi$, there holds

$$h \|\phi - \phi^h\|_{H^1(\Omega)} + \|\phi - \phi^h\|_{L^2(\Omega)} \lesssim h^{k+1} (\|\phi\|_{H^{k+1}(\Omega)} + \|g\|_{H^{k+\frac{1}{2}}(\partial \Omega)}). \tag{3.20}$$

**Proof.** The error estimate (3.20) can be established by the optimal error estimate for homogeneous problems (3.18) and the standard lift argument [12, 20, 37]. \qed

**4. Main results.** Suppose the domain $\Omega$ is convex. The main result of this paper is the following theorem.

**Theorem 4.1.** Let $u$ be the solution of (2.2) and $u^h$ be its virtual element solution. If $u \in W^{k+1, \infty}(\Omega)$ and $f \in H^k(\Omega)$, then the following result holds

$$\|\nabla u - \nabla \Pi_k^u u_h\|_{L^\infty(\Omega)} \lesssim h^k |\ln h|^2 (\|u\|_{W^{k+1, \infty}(\Omega)} + \|f\|_{H^k(\Omega)}). \tag{4.1}$$

Furthermore, if we assume $k \geq 2$, then we have

$$\|u - u^h\|_{L^\infty(\Omega)} \lesssim h^{k+1} |\ln h|^4 (\|u\|_{W^{k+1, \infty}(\Omega)} + \|f\|_{H^k(\Omega)}). \tag{4.2}$$
Remark 4.2. For the classical finite element method, Schatz and Wahlbin [34] proved the following maximum norm error estimates

\begin{align}
\|u - u_h\|_{W^{1, \infty}(\Omega)} & \lesssim h^k \|u\|_{W^{k+1, \infty}(\Omega)}, \\
\|u - u_h\|_{L^{\infty}(\Omega)} & \lesssim h^{k+1} |\ln h| \|u\|_{W^{k+1, \infty}(\Omega)},
\end{align}

where \( u_h \) is the classical continuous finite element solution of degree \( k \) and \( k = 1 \) if \( k = 1 \) and \( k = 0 \) if \( k \geq 1 \). In contrast the maximal norm error estimates of the classical finite element methods, the maximal norm error estimates for virtual element methods have a high-order power of \( |\ln h| \).

In the rest of this section, we shall present a proof of our main result.

4.1. Local regularity result. For any point \( x \) in \( \Omega \), let \( G_x \in W^{1,1}(\Omega) \) be the standard Green's function for (2.1) which is defined as

\begin{equation}
a(v, G_x) = v(x), \quad \forall v \in H^1_0(\Omega).
\end{equation}

Let \( M \) be the set of corner points(or vertices) of the domain \( \Omega \). Let

\begin{equation}
B(M, r) = \Omega \cap \bigcup_{y \in M} B(y, r),
\end{equation}

where \( B(y, r) \) is the disk with radius \( r \) centered at \( y \). For any \( E_0 \in T_h \) and \( r > 0 \), define

\begin{equation}
B_{E_0, r} = \{ y \in \Omega \mid \rho(y, E_0) \leq r \} \cup B(M, r),
\end{equation}

where the distance function \( \rho(\cdot, \cdot) \) is defined as

\begin{equation}
\rho(x, E_0) = \inf_{y \in E_0} |x - y|.
\end{equation}

Similarly, for any \( x \in \Omega \), let

\begin{equation}
B_{x, r} = (\Omega \cap B(x, r)) \cup B(M, r).
\end{equation}

In this subsection, we shall establish some local regularity result. Under the assumption that \( \Omega \) is smooth enough such as ball, [29] showed, for any \( r > 0 \), there holds

\begin{equation}
\|G_x\|_{W^{s, \infty}(\Omega \setminus B(x, r))} \lesssim r^{-s}.
\end{equation}

In this paper, we prove a more generalized results. The estimate of Green’s function is one the key ingredient in the establishment local regularity result (4.37) which will be used to prove (4.103).

We start with the following lemma.

**Lemma 4.3.** Assume that \( s \geq 3 \) is a positive integer. Then there holds

\begin{equation}
\|G_x\|_{H^s(\Omega \setminus B_{x, r})} \lesssim r^{1-s} |\ln r|,
\end{equation}

and for any \( s \geq 2 \), there holds

\begin{equation}
\|G_x\|_{W^{s, \infty}(\Omega \setminus B_{x, r})} \lesssim r^{-s} |\ln r|.
\end{equation}
Next, we estimate $\|G_x\|_{H^2(\Omega)}$. To do this, we observe that, for any $x \in \Omega \setminus B_{x,\frac{r}{2}}$, there holds

$$
G_x(z) = \nu_1(z)G_x(z).
$$

Furthermore, $L\mathbf{G}_x(z) = 0$ if $z \in B_{x,\frac{r}{2}} \cup (\Omega \setminus B_{x,r})$. From (4.14), we can deduce that

$$
\|G_x\|_{H^2(\Omega)} \lesssim \|L\mathbf{G}_x\|_{L^2(\Omega)} = \|L\mathbf{G}_x\|_{L^2(B_{x,\frac{r}{2}} \setminus B_{x,\frac{r}{2}})}
$$

where we have used the following estimate (see [27, Lemma 2.1])

$$
\|G_x\|_{H^2(B_{x,\frac{r}{2}} \setminus B_{x,\frac{r}{2}})} + r^{-1}\|G_x\|_{H^1(B_{x,\frac{r}{2}} \setminus B_{x,\frac{r}{2}})} + r^{-2}\|G_x\|_{L^2(B_{x,\frac{r}{2}} \setminus B_{x,\frac{r}{2}})} \lesssim r^{-1} - |\ln r|.
$$

In the following, we shall use (4.15) and (4.16) to estimate $\|G_x\|_{H^((\Omega \setminus B_{x,r})}$ for $s \geq 3$. Assume that $\theta(x) = \rho(x, M)$ where $M$ is the set of corner points for the domain $\Omega$. We defined the weighted norm $\| \cdot \|_{\kappa^2_{(\frac{r}{2})}}$ as

$$
\|\omega\|_{\kappa^2_{(\frac{r}{2})}} = \sum_{|\beta| \leq \alpha_2} \|\theta^{\beta}|^{\alpha_1} \theta^2\omega\|_{L^2(\Omega)}.
$$

We start with the estimation of $\|\mathbf{G}_x\|_{\kappa^2_{(\frac{r}{2})}}$ and $\|L\mathbf{G}_x\|_{H^1(B_{x,\frac{r}{2}} \setminus B_{x,\frac{r}{2}})}$. Let $r_0 = \frac{r}{2}$. Note that $\mathbf{G}_x(z) = 0$ for all $z \in B_{x,\frac{r}{2}}$. Assume that $s \geq 3$. In (4.16), we have

$$
\|\mathbf{G}_x\|_{\kappa^2_{(\frac{r}{2})}}^2 = \|\mathbf{G}_x\|_{\kappa^2_{(\frac{r}{2})}}^2 + \|G_x\|_{\kappa^2_{(\frac{r}{2})}}^2 = \int_{\Omega \setminus B(M,r_0)} \theta^{-2s}\mathbf{G}_x^2(z)dz
$$

where

$$
\|\theta^{-2s}\|_{L^1(\Omega \setminus B(M,r_0))} \leq \|\theta^{-2s}\|_{L^\infty(\Omega \setminus B_{x,\frac{r}{2}})} \leq r_0^{-2s}\|G_x^2\|_{L^\infty(\Omega \setminus B_{x,\frac{r}{2}})} \quad \text{and} \quad \theta^{-2s} \lesssim \|\theta^{-2s}\|_{L^\infty(\Omega \setminus B_{x,\frac{r}{2}})} \leq r_0^{-2s}\|G_x^2\|_{L^\infty(\Omega \setminus B_{x,\frac{r}{2}})} \leq r^{-2s} \ln r^2.
$$

Next, we estimate $\|L\mathbf{G}_x\|_{H^1(\Omega)}$. We observes that, for any $z \in \Omega \setminus B_{x,\frac{r}{2}}$, there holds

$$
L\mathbf{G}_x(z) = - \nabla \cdot (\nu_1(z)\alpha(z)\nabla G_x(z)) - \nabla \cdot (G_x(z)\alpha(z)\nabla \nu_1(z))
$$

where

$$
= - \nabla \nu_1(z) \cdot \nabla G_x(z) - \nu_1(z) \nabla \cdot (\alpha(z)\nabla G_x(z)) - \nabla \cdot (G_x(z)\alpha(z)\nabla \nu_1(z))
$$

and

$$
= - \nabla \nu_1(z) \cdot \nabla G_x(z) - G_x(z) \nabla \cdot (\alpha(z)\nabla \nu_1(z)) - \nabla G_x(z) \cdot (\alpha(z)\nabla \nu_1(z)).
$$

It implies

$$
\|L\mathbf{G}_x\|_{H^1(\Omega)} \lesssim r^{-1-s} |\ln r|.
$$
By (4.20), for any $z \in B_{x,r} \setminus B_{x,\frac{r}{2}}$, there holds
\[
\frac{\partial L \mathcal{G}_x(z)}{\partial z_l} = - \frac{\partial \nabla \nu_1(z) \cdot \alpha(z) \nabla G_x(z) - \nabla \nu_1(z) \cdot \partial (\alpha(z) \nabla G_x(z))}{\partial z_l} - \frac{\partial G_x(z) \cdot \nabla \cdot (\alpha(z) \nabla \nu_1(z)) - G_x(z) \partial \nabla \cdot (\alpha(z) \nabla \nu_1(z))}{\partial z_l} - \frac{\partial \nabla G_x(z) \cdot \alpha(z) \nabla \nu_1(z) - \nabla G_x(z) \cdot \partial (\alpha(z) \nabla \nu_1(z))}{\partial z_l}.
\]
Combining the estimates (4.16) and (4.21), we can deduce that
\[
\| L \mathcal{G}_x \|_{H^1(\Omega)} = \| L \mathcal{G}_x \|_{H^1(B_{x,r} \setminus B_{x,\frac{r}{2}})} \\
\lesssim \| \nu_1 \|_{W^3,\infty(B_{x,r} \setminus B_{x,\frac{r}{2}})} \| G_x \|_{L^2(B_{x,r} \setminus B_{x,\frac{r}{2}})} + \\
\| \nu_1 \|_{W^2,\infty(B_{x,r} \setminus B_{x,\frac{r}{2}})} \| G_x \|_{H^1(B_{x,r} \setminus B_{x,\frac{r}{2}})} + \\
\| \nu_1 \|_{W^{1,\infty}(B_{x,r} \setminus B_{x,\frac{r}{2}})} \| G_x \|_{H^2(B_{x,r} \setminus B_{x,\frac{r}{2}})} \\
\lesssim r^{-3} |\ln r| + r^{-2} |\ln r| + r^{-1} r^{-1} |\ln r| \\
\lesssim r^{-2} |\ln r|.
\]
Note that $\mathcal{G}_x(z) = 0$ for all $z \in \partial \Omega$. By Theorem 3.7 in [4], (4.19), and (4.22), we can derive that
\[
\| G_x \|_{H^s(\Omega)} \lesssim (\| G_x \|_{\kappa_2^s(\Omega)} + \| L \mathcal{G}_x \|_{H^1(\Omega)}) \lesssim r^{-2} |\ln r|,
\]
which implies
\[
\| G_x \|_{H^1(\Omega \setminus B_{x,r})} = \| \mathcal{G}_x \|_{H^1(\Omega \setminus B_{x,r})} \lesssim \| \mathcal{G}_x \|_{H^2(\Omega)} \lesssim r^{-2} |\ln r|.
\]
When $s \geq 4$, we shall prove it by induction on $s$. Assume that, for any $1 \leq l \leq s$, there holds
\[
\| G_x \|_{H^{s-l}(\Omega \setminus B_{x,r})} \lesssim r^{1-l-s} |\ln r|.
\]
Similarly to (4.21) and (4.22), we have
\[
\| L \mathcal{G}_x \|_{H^{s-2}(\Omega)} = \| L \mathcal{G}_x \|_{H^{s-2}(B_{x,r} \setminus B_{x,\frac{r}{2}})} \\
\lesssim \sum_{l=1}^{s} \| \nu_1 \|_{W^{1,\infty}(B_{x,r} \setminus B_{x,\frac{r}{2}})} \| G_x \|_{H^{s-l-1}(B_{x,r} \setminus B_{x,\frac{r}{2}})} \\
\lesssim \sum_{l=1}^{s} r^{-l} cr^{1+l-s} |\ln r| \\
\lesssim r^{-1} |\ln r|.
\]
Again, Theorem 3.7 in [4], (4.19), and (4.22) give
\[
\| \mathcal{G}_x \|_{H^s(\Omega)} \lesssim (\| G_x \|_{\kappa_2^s} + \| L \mathcal{G}_x \|_{H^{s-2}(\Omega)}) \lesssim r^{1-s} |\ln r| + cr^{1-s} |\ln r| \lesssim r^{1-s} |\ln r|,
\]
which implies

\[(4.27) \quad \|G_x\|_{H^s(\Omega\setminus B_{x,r})} = \|G_x\|_{H^s(\Omega\setminus B_{x,r})} \leq \|G_x\|_{H^s(\Omega)} \lesssim r^{1-s} |\ln r| \]

Now, we turn to prove (4.12). Let \( \Phi = \{y-x \mid y \in \Omega \} \) and \( \Phi_1 = \{y-x \mid y \in B_{x,r} \} \).

We introduce \( \psi(t) \) by letting

\[(4.28) \quad \psi(z) = G_x(rz), \quad \forall z \in \Phi.\]

For any \( s \geq 3 \), (4.11) and (4.28) give

\[\|\psi\|_{H^s(\Phi \setminus \Phi_1)} = r^{s-1} \|G_x\|_{H^s(\Omega \setminus B_{x,r})} \lesssim |\ln r|.\]

For any \( s_1 \geq 2 \), it implies that

\[(4.29) \quad \|\psi\|_{W^{s_1,\infty}(\Phi \setminus \Phi_1)} \leq c|\ln r|.\]

From (4.29), we can derive that

\[\|G_x\|_{W^{s_1,\infty}(\Omega \setminus B_{x,r})} \leq cr^{-s_1} \|\psi\|_{W^{s_1,\infty}(\Phi \setminus \Phi_1)} \leq cr^{-s_1} |\ln r|,\]

which concludes the proof of (4.12).

Using the above Lemma, we shall prove the following point-wise estimate.

**Lemma 4.4.** Let \( \gamma > 2 \), \( \gamma = \gamma - 1 \), and \( B_{E_0,r} \) be defined as in (4.7). Assume that \( y \in \Omega \setminus B_{E_0,\gamma r} \) and \( |\beta| \geq 2 \) is a positive integer. Then

\[(4.30) \quad \left| \frac{\partial^\beta G_x(z)}{\partial z_1^{\beta_1} \partial z_2^{\beta_2}} \right|_{z=y} \lesssim \gamma^{-|\beta|} r^{-|\beta|} |\ln r|.\]

**Proof.** Assume that \( x \in \Omega \). Notice that

\[(4.31) \quad G_y(x) = G_x(y) \quad \text{and} \quad G_{y+\Delta y}(x) = G_x(y + \Delta y).\]

By setting \( |\Delta y| \to 0 \) in (4.31), we can deduce that \( \frac{\partial G_x(x)}{\partial z} \bigg|_{z=y} = \frac{\partial G_x(y)}{\partial y} \). Furthermore, we have

\[(4.32) \quad \frac{\partial^\beta G_x(x)}{\partial z_1^{\beta_1} \partial z_2^{\beta_2}} \bigg|_{z=y} = \frac{\partial^\beta G_x(y)}{\partial y_1^{\beta_1} \partial y_2^{\beta_2}}.\]

Since \( y \in \Omega \setminus B_{E_0,\gamma r} \), it follows that for any \( x \in B_{E_0,r} \), there holds

\[(4.33) \quad |x - y| \geq \rho(y, E_0) - \rho(x, E_0) = \gamma r - r = \gamma r,\]

and

\[(4.34) \quad \rho(y, M) \geq \gamma r > \gamma r.\]

From (4.33) and (4.34), we can show

\[(4.35) \quad y \in \Omega \setminus B_{x,\gamma r}.\]
It implies that
\[
\left\| \frac{\partial^3 G_x}{\partial y_1^2 \partial y_2^2} \right\|_{L^\infty(B_{E_0,r})} \leq \max_{x \in B_{E_0,r}} |\nabla \chi| L^\infty(\Omega \setminus B_{E_0,r}) \leq \gamma - |\beta| r^{-1} |\ln r|,
\]
where we have used (4.12).

Based on Lemma 4.4, we establish the following local regularity result.

**Theorem 4.5.** Let \( r \geq d > 0 \) and \( \gamma > 2 \). Assume that \( \nu \in L^2(\Omega) \) satisfies \( \nu(x) = 0 \) if \( x \in \Omega \setminus B_{E_0,d} \) and \( \chi(x) \in H^3_0(\Omega) \) is the solution to \( L\chi(x) := -\nabla \cdot (\alpha \nabla \chi(x)) = \nu(x) \). Then there holds
\[
\|\chi\|_{H^3(\Omega \setminus B_{E_0,\gamma d})} \lesssim \gamma^{-1} d^{-1} |\ln d| \|\nu\|_{L^2(\Omega)}.
\]

**Proof.** Let \( r \geq d \). Assume that \( y \in \Omega \setminus B_{E_0,\gamma r} \). Note that \( \nu(x) = 0 \) if \( x \in \Omega \setminus B_{E_0,r} \). Then, Lemma 4.4 implies
\[
\frac{\partial^3 \chi(y)}{\partial y_1^2 \partial y_2^2} \leq \int_\Omega \left| \frac{\partial^3 G_x}{\partial y_1^2 \partial y_2^2} \right| |\nu(x)| \, dx = \int_{B_{E_0,r}} \left| \frac{\partial^3 G_x}{\partial y_1^2 \partial y_2^2} \right| |\nu(x)| \, dx
\]
\[
\leq \left\| \frac{\partial^3 G_x}{\partial y_1^2 \partial y_2^2} \right\|_{L^2(\Omega)} \|\nu\|_{L^2(\Omega)} \lesssim (\gamma - 1)^{-1} r^{-1} |\ln r| \|\nu\|_{L^2(\Omega)}.
\]
Let \( |\beta| = 3 \). By (4.38), we have
\[
\|\chi\|_{H^3(\Omega \setminus B_{E_0,\gamma d})} \lesssim (\gamma - 1)^{-2} r^{-2} |\ln r| \|\nu\|_{L^2(\Omega)}.
\]

Let \( d_i = 2^{-i} d \) for any positive integer \( i \). We also assume that \( i_0 \) is the minimal positive integer satisfying \( \Omega \subset B_{E_0,\gamma d_{i_0}+1} \). Note that \( \gamma \geq 2 \). By (4.39), we have
\[
\|\chi\|_{H^3(\Omega \setminus B_{E_0,\gamma d})} \leq \sum_{i=1}^{i_0} \|\chi\|_{H^3(B_{E_0,\gamma d_{i+1}} \setminus B_{E_0,\gamma d_i})} \leq \sum_{i=1}^{i_0} c \gamma^2 d_i^2 \|\chi\|_{W^{3,\infty}(B_{E_0,\gamma d_{i+1}} \setminus B_{E_0,\gamma d_i})} \leq (\gamma - 1)^{-2} r^{-2} |\ln r| \|\nu\|_{L^2(\Omega)}
\]
\[
\leq (\gamma - 1)^{-2} r^{-2} |\ln r| \|\nu\|_{L^2(\Omega)} \sum_{i=1}^{i_0} d_i^{-2} |\ln d_i|^2 \leq (\gamma - 1)^{-2} r^{-2} |\ln r| \|\nu\|_{L^2(\Omega)} \leq (\gamma - 1)^{-2} d^{-1} |\ln d| \|\nu\|_{L^2(\Omega)} \leq (\gamma - 1)^{-2} d^{-1} |\ln d| \|\nu\|_{L^2(\Omega)},
\]
which completes the proof of (4.37). 

\[\Box\]
4.2. Cutoff functions. Let $E_0$ be an arbitrary element in $T_h$. Assume that $l_0$ is a positive integer satisfying

\[(l_0 - 1)2^{l_0 - 1} \leq \max_{x \in \Omega} \rho(x, E_0) \leq l_0 2^{l_0}.\]

From (4.41) and the regularity of the mesh, it not hard to deduce that

\[(l_0 - 1)2^{l_0 - 1} \leq \ln h.|\]

Let $\hat{B}_{E_0,r}$ be defined by

\[\hat{B}_{E_0,r} = \{x \mid \rho(x, E_0) \leq r\} \cup \{x \mid \rho(x, M) \leq r\},\]

with $M$ being the set of corner points (or vertices) of the domain $\Omega$.

For $0 \leq l \leq l_0$, let

\[r_l = 2^l \ln h.\]

Let $\hat{B}_{E_0,r_{l-1}} = \emptyset$. Assume that $\phi_l(x)$ ($0 \leq l \leq l_0$) is a $C^\infty(\mathbb{R}^2)$ satisfies, $0 \leq \phi_l(x) \leq 1$ for all $x \in \mathbb{R}^2$, and $\phi_l(x) = 1$ if $x \in \hat{B}_{E_0,r_l}$, and $\phi_l(x) = 0$ if $x \in \mathbb{R}^2 \setminus \hat{B}_{E_0,r_{l+1}}$. We also assume that there exists a constant $c$ independent of $x$ and $l$ such that

\[|\nabla^m \phi_l(x)| \leq cr_l^{-m}, \quad \forall 0 \leq l \leq l_0.\]

Set

\[\hat{\phi}_l(x) = \begin{cases} \phi_l(x), & \text{if } l = 0, \\ \phi_l(x) - \phi_{l-1}(x), & \text{if } 1 \leq l \leq l_0 - 1. \end{cases}\]

Notice that (4.41) implies $\Omega \subset \hat{B}_{E_0,r_{l_0}}$. By (4.45) and (4.46), we have, if $x \in \Omega$, then

\[\sum_{l=0}^{l_0} \hat{\phi}_l(x) = \phi_{l_0}(x) = 1,\]

and

\[|\nabla^m \hat{\phi}_l(x)| \leq |\nabla^m \phi_l(x)| + |\nabla^m \phi_{l-1}(x)| \lesssim r_l^{-m} + r_{l-1}^{-m} \lesssim r_l^{-m}.\]

Using the above cutoff functions, we define a partition of the exact solution $u$. Let $x_0 \in E_0$ satisfy $|\nabla^m u(x_0)| \leq \|u\|_{W^{m, \infty}(\Omega)}$ ($0 \leq m \leq k + 1$). Assume that $\hat{u}(x)$ is a global polynomial of degree $k$ on $\Omega$ satisfying

\[\nabla^m \hat{u}(x_0) = \nabla^m u(x_0), \quad \forall 0 \leq m \leq k.\]

The existence of $\hat{u}(x)$ can be guaranteed using the averaged Taylor polynomial in [12]. By (4.49), we have, for any $r > 0$, there holds [12]

\[\|u - \hat{u}\|_{W^{m, \infty}(\Omega \cap \hat{B}_{E_0,r})} \lesssim r^{k+1-m}\|u\|_{W^{k+1, \infty}(\Omega)}, \quad \forall 0 \leq m \leq k.\]

Then, we can define the following partition of the exact solution

\[u(x) = \hat{u}(x) + \sum_{l=0}^{l_0} u_l(x),\]
where
\[(4.52)\quad \dot{u}(x) = \hat{\phi}(x)[u(x) - \hat{u}(x)].\]
We can define a similar partition for the right hand side function as
\[(4.53)\quad f(x) = \hat{f}(x) + \sum_{l=0}^{l_0} f_l(x),\]
such that
\[(4.54)\quad a(\hat{u}, v) = (\hat{f}, v), \quad \forall v \in H_0^1(\Omega),\]
and
\[(4.55)\quad a(u_l, v) = (f_l, v), \quad \forall v \in H_0^1(\Omega),\]
for \(l = 0, \ldots, l_0\).

Once we obtain the decomposition of \(f\), we define the virtual element solution \(\hat{u}^h \in V_h\) of \(\hat{u}\) as
\[(4.56)\quad \begin{cases} 
  a_h(\hat{u}^h, v^h) = (\hat{f}, \Pi_{k-1}^0 v^h), & \forall v^h \in V_{h,0}, \\
  \hat{u}^h(x) = \hat{u}(x), & \forall x \in \partial \Omega,
\end{cases}\]
and the virtual element solution \(u_l^h \in V_h\) of \(u_l\) as
\[(4.57)\quad \begin{cases} 
  a_h(u_l^h, v^h) = (f_l, \Pi_{k-1}^0 v^h), & \forall v^h \in V_{h,0}, \\
  \hat{u}_l^h(x) = I_h \hat{u}_l(x), & \forall x \in \partial \Omega,
\end{cases}\]
where \(I_h\) is the interpolation operator of the virtual element space \(V_h\) defined in (3.17).

For the global polynomial, we can show the following equivalence relationship:

**Lemma 4.6.** Let \(\hat{u}(x)\) be the polynomial defined in (4.50) and \(\hat{u}^h(x)\) be its virtual element solution defined in (4.56). Then, we have
\[(4.58)\quad \hat{u}^h(x) = \hat{u}(x).\]

**Proof.** Since \(\hat{u}\) is a polynomial of degree \(k\), it follows that
\[(4.59)\quad a_h(\dot{u}, v^h) = a(\hat{u}, v^h), \quad \forall v^h \in V_{h,0}.
\]
The assumption that \(\alpha\) is a constant coefficient matrix and \(\hat{u}\) is a polynomial of degree \(k\) means \(\dot{f}\) is also a polynomial of degree \(k - 2\). Then, the definition of \(L^2\)-projection in (3.10) implies that
\[(4.60)\quad (\dot{f}, \Pi_{k-1}^0 v^h - v^h) = 0, \quad \forall v \in V_{h,0}.
\]
Using the definition of the variational formulation (2.2) and the VEM variational formulation (3.15), it is easy to see that
\[(4.61)\quad a_h(\hat{u}^h, v^h) = a(\hat{u}, v^h) = (\dot{f}, \Pi_{k-1}^0 v^h - v^h), \quad \forall v^h \in V_h.
\]
Then, the orthogonal relationship (4.60) implies
\[(4.62)\quad a_h(\hat{u}^h, v^h) = a(\hat{u}, v^h), \quad v \in V_{h,0}.
\]
Note that the fact that \(\hat{u}\) is a polynomial with degree \(k\) and \(\hat{u}^h(x) = \hat{u}(x), \forall x \in \partial \Omega.
\]
Using (4.59), (4.62), we have (4.58). \(\square\)
4.3. Approximation property of the partition of the exact solution.

For the sake of simplifying the notation, we use the shorthand \( B_{r_l} := B(x_0, r_l) \). Let \( u_l^h \) be the VEM solution of the problem (4.55) defined in (4.57). Assume that \( g_1(x) = u_l(x), \ \forall x \in \partial \Omega \). To estimate \( u_l - u_l^h \), we first estimate \( \| g_l \|_{H^{k+\frac{1}{2}}(\partial \Omega)} \).

**Lemma 4.7.** Assume that \( u \in W^{k+1, \infty}(\Omega) \) and \( 1 \leq l \leq l_0 \). Then, we have

\[
\| g_l \|_{H^{k+\frac{1}{2}}(\partial \Omega)} \lesssim r_l \| u \|_{W^{k+1, \infty}(\Omega)}.
\]

**Proof.** Assume that \( \hat{B}_{E_0, r} \) is defined as in (4.43). Let \( \Omega_l = \hat{B}_{E_0, r_{l+1}} \cap \Omega \). Note that \( \phi_l(x) = 0, \ \forall x \in \mathbb{R}^2 \setminus \hat{B}_{E_0, r_{l+1}} \). We observes that

\[
\| g_l \|_{H^{k+\frac{1}{2}}(\partial \Omega)} \lesssim r_l \| u \|_{W^{k+1, \infty}(\Omega)}.
\]

Note that (4.50) implies

\[
\| \nabla^m \psi_0 \|_{L^\infty(\Omega)} = r_l^m \| \nabla^m (u - \hat{u}) \|_{L^\infty(\Omega)} \lesssim r_l^m r_l^{k+1-m} \| u \|_{W^{k+1, \infty}(\Omega)} \leq r_l^{k+1} \| u \|_{W^{k+1, \infty}(\Omega)}.
\]

Combining (4.66), (4.70) and (4.71) gives

\[
\| g_l \|_{H^{k+\frac{1}{2}}(\partial \Omega)} \lesssim r_l \| u \|_{W^{k+1, \infty}(\Omega)}.
\]
Now, we are in a perfect position to present the estimate for \( u_l - u_l^h \).

**Lemma 4.8.** Assume that \( u \in W^{k+1, \infty}(\Omega) \) and \( 0 \leq l \leq l_0 \). Then, we have

\[
\| u_l - u_l^h \|_{H^1(\Omega)} \lesssim h^{k+1} r_l \| u \|_{W^{k+1, \infty}(\Omega)},
\]

and

\[
\| u_l - u_l^h \|_{L^2(\Omega)} \lesssim h^{k+1} r_l \| u \|_{W^{k+1, \infty}(\Omega)}.
\]

**Proof.** Let \( \hat{B}_{E_0, r_{l-2}} = \hat{B}_{E_0, r_{l-1}} = \emptyset \). Note that \( u_l(x) = 0, \forall x \in \Omega \setminus \hat{B}_{E_0, r_{l+1}} \).

Combining Theorem 3.2, (4.48), (4.50) and Lemma 4.7, we have

\[
\| u_l - u_l^h \|_{H^1(\Omega)} \leq h^k (\| u_l \|_{H^{k+1}(\Omega)} + \| \nabla u \|_{H^{k+\frac{1}{2}}(\partial \Omega)})
\]

\[
= h^k (\| u_l \|_{H^{k+1}(\hat{B}_{E_0, r_{l+1}} \cap \Omega)} + r_l \| u \|_{W^{k+1, \infty}(\Omega)})
\]

\[
= h^k (\sum_{m=0}^{k+1} \| \tilde{u}_m \|_{W^{k+1-m}(\hat{B}_{E_0, r_{l+1}} \cap \Omega)} + r_l \| u \|_{W^{k+1, \infty}(\Omega)})
\]

\[
\leq h^k (\sum_{m=0}^{k+1} r_l^{m-m} \| u - \tilde{u} \|_{H^{k+1-m}(\hat{B}_{E_0, r_{l+1}} \cap \Omega)} + r_l \| u \|_{W^{k+1, \infty}(\Omega)})
\]

\[
\leq h^k (\sum_{m=0}^{k+1} r_l^{m-m} r_l^{m} + r_l) \| u \|_{W^{k+1, \infty}(\Omega)}
\]

\[
\leq h^k r_l \| u \|_{W^{k+1, \infty}(\Omega)}.
\]

Using a similar argument, we can prove (4.74). \( \square \)

**4.4. A technical lemma.** To prepare for the proof of the main result, we firstly establish a technical lemma on the local error estimates for the partition of the exact solution.

**Lemma 4.9.** Assume that \( u \in W^{k+1, \infty}(\Omega) \). Under the assumption that \( 3 \leq l \leq l_0 \), there holds

\[
\| u_l - u_l^h \|_{H^1(\hat{B}_{E_0, r_{l-2}})} \lesssim h^{k+1} |\ln h| \| u \|_{W^{k+1, \infty}(\Omega)}.
\]

**Proof.** Firstly, we notice that Lemma 4.8 implies that there exists a constant \( \gamma_0 \) such that

\[
\| u_l - u_l^h \|_{H^1(\hat{B}_{E_0, r_{l-2}})} \leq \gamma_0 h^k r_l \| u \|_{W^{k+1, \infty}(\Omega)}.
\]

Without loss of generality, we assume that

\[
\| u_l - u_l^h \|_{H^1(\hat{B}_{E_0, r_{l-2}})} \geq \gamma_0 h^{k+1} \| u \|_{W^{k+1, \infty}(\Omega)},
\]

since otherwise we already have (4.75). Combining the above two estimates gives

\[
\frac{\| u_l - u_l^h \|_{H^1(\hat{B}_{E_0, r_{l-2}})}}{\| u_l - u_l^h \|_{H^1(\hat{B}_{E_0, r_{l-2}})}} \geq \frac{h}{r_l} = 2^{-l-1}.
\]
Let $D_0 = B_{E_0, r_{l-2}}$ and $D_m = \bigcup_{\rho(E_0, D_{m-1}) \leq h, E \in T_h} E$ for $m \geq 1$ where $\rho(E_0, D_{m-1})$ is the distance between $E_0$ and $D_{m-1}$. Also, let $D_m = B_{E_0, r_{l-1}}$. We can observe that there exists a constant $\gamma_1$ independent of $l$ such that

$$m_0 = \gamma_1 2^{l-1} (l-1).$$

Let $s_m = 0$ if $m = 1$, and $s_m$ be the minimal positive integer satisfying $s_m \geq (m-1)\gamma_1 2^{l-1}$ if $m \geq 2$. Assume that $1 \leq \hat{m} \leq l-1$ is the index such that

$$m_0 = \gamma_1 2^{l-1} (l-1).$$

By noticing the fact that $\gamma_1 = 0$ if $m \geq 1$, and $\gamma_1 = 1$ if $m = 0$, we can deduce the following inequality using a simple calculation

$$\left( \frac{\|u^h_l\|_{H^1(E)}}{\|u^h_l\|_{H^1(B_{E_0, r_{l-1}})}} \right) \geq \left( \frac{\|u^h_l\|_{H^1(B_{E_0, r_{l-2}})}}{\|u^h_l\|_{H^1(B_{E_0, r_{l-1}})}} \right) \geq \left( \frac{\|u^h_l\|_{H^1(B_{E_0, r_{l-2}})}}{\|u^h_l\|_{H^1(B_{E_0, r_{l-1}})}} \right)^{1/2} \geq \gamma_1 \geq c,$$

where we have used (4.77), (4.79) and (4.80).

Let $\mathfrak{s} \in [s_{\hat{m}+1}, s_m+1]$ be the index such that

$$\min_{s_{\hat{m}+1} \leq s \leq s_{\hat{m}+1}} \|u^h_l\|_{H^1(D_s \setminus D_{s-1})} \|u^h_l\|_{L^2(D_s \setminus D_{s-1})}.$$

Then, we can show that

$$\sum_{s' = s_{\hat{m}+1}}^{s_{\hat{m}+1}} \|u^h_l\|_{H^1(D_{s-1} \setminus D_{s-2})} \|u^h_l\|_{L^2(D_{s-1} \setminus D_{s-2})} \leq \left( \sum_{s' = s_{\hat{m}+1}}^{s_{\hat{m}+1}} \|u^h_l\|_{L^2(D_{s-1} \setminus D_{s-2})} \right)^{1/2} \left( \sum_{s' = s_{\hat{m}+1}}^{s_{\hat{m}+1}} \|u^h_l\|_{L^2(D_{s-1} \setminus D_{s-2})} \right)^{1/2}.$$

Also, using (4.78), we can establish the following relationship

$s_{\hat{m}+1} - s_{\hat{m}} = m_1 \geq m_0 = \gamma_1 2^{l-1} (l-1).$

By (4.83), (4.82) and (4.84), we have

$$\|u^h_l\|_{H^1(D_s \setminus D_{s-1})} \|u^h_l\|_{L^2(D_s \setminus D_{s-1})} \leq \frac{1}{s_{\hat{m}+1} - s_{\hat{m}}} \sum_{s' = s_{\hat{m}+1}}^{s_{\hat{m}+1}} \|u^h_l\|_{H^1(D_{s-1} \setminus D_{s-2})} \|u^h_l\|_{L^2(D_{s-1} \setminus D_{s-2})} \leq \frac{1}{2^{l-1}} \left( \sum_{s' = s_{\hat{m}+1}}^{s_{\hat{m}+1}} \|u^h_l\|_{L^2(D_{s-1} \setminus D_{s-2})} \right)^{1/2} \left( \sum_{s' = s_{\hat{m}+1}}^{s_{\hat{m}+1}} \|u^h_l\|_{L^2(D_{s-1} \setminus D_{s-2})} \right)^{1/2} \leq \frac{1}{2^{l-1}} \|u^h_l\|_{H^1(D_{s_{\hat{m}+1}})} \|u^h_l\|_{L^2(D_{s_{\hat{m}+1}})}.$
Let $\psi(x)$ be a cutoff function such that $\psi(x) = 1$ if $x \in D_\sigma$ satisfies $\rho(x, \partial D_\sigma) \leq c_1 h$ for some constant $0 < c_1 < 1$, and $\psi(x) = 0$ if $x \in D_{\sigma-1}$. It is easy to deduce that $\|\psi\|_{W^{s,\infty}(\Omega)} \leq c h^{-p}$ for any positive integer $p$. We decompose $u_h^l(x)$ as

\begin{equation}
(4.86) \quad u_h^l(x) = \bar{u}_l(x) + \hat{u}_l(x),
\end{equation}

where

\begin{equation}
(4.87) \quad \bar{u}_l(x) = I_h(\psi u_h^l)(x),
\end{equation}

with $I_h$ being the interpolation operator of $V_h$ defined in (3.17). Then, we have

\begin{equation}
(4.88) \quad a_h^{D_\sigma}(u_h^l, u_h^l) = a_h^{D_\sigma}(u_h^l, \bar{u}_l) + a_h^{D_\sigma}(u_h^l, \hat{u}_l).
\end{equation}

We first estimate $a_h^{D_\sigma}(u_h^l, \bar{u}_l)$. Using the property of $\psi$, we can show that

\begin{equation}
a_h^{D_\sigma}(u_h^l, \bar{u}_l) \leq a_h^{D_\sigma}(u_h^l, \bar{u}_l) + c h^{-1}\|u_h^l\|_{H^1(D_\sigma \setminus D_{\sigma-1})}\|u_h^l\|_{L^2(D_\sigma \setminus D_{\sigma-1})}.
\end{equation}

Rearranging the above equation implies

\begin{equation}
(4.89) \quad a_h^{D_\sigma}(u_h^l, \bar{u}_l) - a_h^{D_\sigma}(u_h^l, u_h^l) \leq c h^{-1}\|u_h^l\|_{H^1(D_\sigma \setminus D_{\sigma-1})}\|u_h^l\|_{L^2(D_\sigma \setminus D_{\sigma-1})}.
\end{equation}

Moving on now to estimate $a_h^{D_\sigma}(u_h^l, \hat{u}_l)$. Notice that $\hat{u}_l(x) = 0$ for all $x \in \partial D_\sigma$. We can introduce a function $\omega(x) \in V_h$ by $\omega(x) = \hat{u}_l(x)$ if $x \in D_\sigma$, and $\omega(x) = 0$ if $x \in \Omega \setminus D_\sigma$. It follows that

\begin{equation}
(4.90) \quad \Pi_0^h \omega(x) = 0, \quad \forall x \in \Omega \setminus D_\sigma.
\end{equation}

Notice that $D_\sigma \subset B_{E_0, r_{\sigma-1}} \subset \tilde{B}_{E_0, r-1}$, so

\begin{equation}
(4.91) \quad u_l(x) = 0, \quad \forall x \in D_\sigma,
\end{equation}

which implies

\begin{equation}
(4.92) \quad f_l(x) = L u_l(x) = 0, \quad \forall x \in D_\sigma.
\end{equation}

Then, equation (4.90) and (4.92) tell us that

\begin{equation}
(4.93) \quad a_h^{D_\sigma}(u_h^l, \omega) = (f_l^h, \Pi_0^h \omega) = (f_l^h, \Pi_0^h \omega)_{D_\sigma} + (f_l^h, \Pi_0^h \omega)_{\Omega \setminus D_\sigma} = (0, \omega)_{D_\sigma} + (f_l^h, 0)_{\Omega \setminus D_\sigma} = 0 + 0 = 0.
\end{equation}

Combining the results (4.73), (4.85), (4.89) and (4.93), we have

\begin{equation}
(4.94) \quad \begin{aligned}
& a_h^{D_\sigma}(u_h^l, u_h^l) \\
& \leq h^{-1}\|u_h^l\|_{H^1(D_\sigma \setminus D_{\sigma-1})}\|u_h^l\|_{L^2(D_\sigma \setminus D_{\sigma-1})} \\
& \leq h^{-1}\|u_h^l\|_{H^1(D_{\sigma_{m+1}})}\|u_h^l\|_{L^2(D_{\sigma_{m+1}})} \\
& \leq h^{-1}\|u_h^l\|_{H^1(D_{\sigma_{m+1}})}\|u_h^l\|_{W^{k+1,\infty}(\Omega)} \\
& \leq h^{k+1}\|u_h^l\|_{H^1(D_{\sigma_{m+1}})}\|u||_{W^{k+1,\infty}(\Omega)}.
\end{aligned}
\end{equation}
To show the local error estimate (4.75), we use (4.94) and (4.81), which gives us that
\[
\|u^h_l\|_{H^1(D_{\epsilon,l})} \leq a_h^{D_{\epsilon,l}}(u^h_l, u^h_l) \\
\leq h^{k+1}\|u^h\|_{H^1(D_{\epsilon,l+1})}\|u\|_{W^{k+1,\infty}(\Omega)} \\
\leq h^{k+1}\|u^h\|_{H^1(D_{\epsilon,l})}\|u\|_{W^{k+1,\infty}(\Omega)}.
\] (4.95)

By the definition of the partition of the exact solution in (4.52), we have
\[ u_l(\mathbf{x}) = 0, \quad \forall \mathbf{x} \in B_{E_0,l+1}. \] (4.96)

From (4.95) and (4.96), it follows that
\[
\|u_l - u^h\|_{H^1(E_{0,l+1})} = \|u^h\|_{H^1(E_{0,l+1})} \\
\leq h^{k+1}\|u\|_{W^{k+1,\infty}(\Omega)} \\
\leq h^{k+1}\|u\|_{W^{k+1,\infty}(\Omega)},
\] (4.97)

which completes the proof.

**4.5. Local error estimates.** Using the results in previous subsections, we shall establish a local \( H^1 \) error estimate for virtual element method.

**Lemma 4.10.** Let \( 1 \leq l \leq l_0 \). Assume that \( E_0 \) is a given element in \( T_h \) and \( B_{E_0,l} \) is defined as in (4.7). Under the same assumptions of Theorem 4.1, there holds
\[
\|u - u^h\|_{H^1(E_{0,l})} \lesssim h^k r_l \|u\|_{W^{k+1,\infty}(\Omega)}.
\] (4.98)

**Proof.** Assume that \( \bar{u}(\mathbf{x}), u_0(\mathbf{x}) \) and \( u^h_l(\mathbf{x}) (1 \leq l \leq l_0) \) are defined as in subsection 4.2. By (4.51) and Lemma 4.6, we have
\[
(u - u^h)(\mathbf{x}) = \sum_{l=0}^{l+1} (u_l - u^h_l)(\mathbf{x}) + \sum_{l=l+2}^{l_0} (u_l - u_l^h)(\mathbf{x}).
\] (4.99)

We first estimate \( \left\| \sum_{l=0}^{l+1} (u_l - u^h_l) \right\|_{H^1(E_{0,l})} \). By (4.44) and Lemma 4.8, we have, if
\[ 0 \leq l \leq l + 1, \]
\[
\|u_l - u^h_l\|_{H^1(E_{0,l})} \leq \|u_l - u^h_l\|_{H^1(\Omega)} \lesssim h^k r_l \|u\|_{W^{k+1,\infty}(\Omega)} \lesssim 2^{l-\tilde{l}} h^k r_l \|u\|_{W^{k+1,\infty}(\Omega)},
\]
which implies
\[
\left\| \sum_{l=0}^{l+1} (u_l - u^h_l) \right\|_{H^1(E_{0,l})} \lesssim \sum_{l=0}^{l+1} 2^{l-\tilde{l}} h^k r_l \|u\|_{W^{k+1,\infty}(\Omega)} \lesssim h^k r_l \|u\|_{W^{k+1,\infty}(\Omega)}.
\] (4.100)
Next we estimate \( \left\| \sum_{l=1+2}^{l=0} (u_l - u_l^h) \right\|_{H^1(B_{E_0}, r_2^*)} \). By Lemma 4.9, we have

\[
\sum_{l=1}^{l=0} (u_l - u_l^h) \right\|_{H^1(B_{E_0}, r_2^*)} \leq \sum_{l=1}^{l=0} \| u_l - u_l^h \|_{H^1(B_{E_0}, r_2^*)}
\]

(4.101)

\[
\leq \sum_{l=1}^{l=0} \| u_l - u_l^h \|_{H^1(B_{E_0}, r_1^2 - 2)} \leq \sum_{l=1}^{l=0} ch^{k+1} \| \ln h \|_{W^{k+1, \infty}(\Omega)}
\]

\[\lesssim h^{k+1} \| \ln h \|_{W^{k+1, \infty}(\Omega)}\cdot\]

Inserting (4.100) and (4.101) into (4.99), we get the desired result (4.98).

Now, we shall estimate the local error \( \| u - u^h \|_{H^1(E_0)} \) and \( \| u - u^h \|_{L^2(E_0)} \).

**Lemma 4.11.** Assume that \( E_0 \) is a given element in \( T_h \) and \( u \in W^{k+1, \infty}(\Omega) \).

Then

\[
\| u - u^h \|_{H^1(E_0)} \lesssim h^{k+1} \| \ln h \|_{W^{k+1, \infty}(\Omega)}.
\]

Furthermore, under the assumption that \( k \geq 2 \), there holds

\[
\| u - u^h \|_{L^2(E_0)} \lesssim h^{k+2} \| \ln h \|_{W^{k+1, \infty}(\Omega)} + \| f \|_{H^k(\Omega)}.
\]

**Proof.** Let \( \tilde{l} = 1 \). By (4.98), we have

\[
\| u - u^h \|_{H^1(E_0)} \leq \| u - u^h \|_{H^1(B_{E_0}, r_2^*)}
\]

(4.104)

\[\lesssim h^{k+1} \| \ln h \|_{W^{k+1, \infty}(\Omega)}\lesssim h^{k+1} \| \ln h \|_{W^{k+1, \infty}(\Omega)},\]

which complete the proof of (4.102).

To show (4.103), let \( \theta_0(\mathbf{x}) \) be the cutoff function satisfying \( \theta_0(\mathbf{x}) = 1 \) if \( \mathbf{x} \in E_0 \), and \( \theta_0(\mathbf{x}) = 0 \) if \( \mathbf{x} \in \Omega \setminus B_{E_0, r_1} \). Assume that \( \theta_1(\mathbf{x}) \) satisfies the following problem

\[
\begin{cases}
-\nabla \cdot (\alpha \nabla \theta_1(\mathbf{x})) = \theta_0(\mathbf{x})(u - u^h)(\mathbf{x}), & \text{in } \Omega, \\
\theta_1(\mathbf{x}) = 0, & \text{on } \partial \Omega.
\end{cases}
\]

(4.105)

Assume also that \( f^h = \Pi_{k-1}^0 f \). By the definition of the cutoff function \( \theta_0(x) \), it is easy to deduce that

\[
\| u - u^h \|_{L^2(E_0)}^2 \leq (u - u^h, \theta_0(u - u^h))
\]

(4.106)

\[= (u - u^h, -\nabla \cdot (\alpha \nabla \theta_1))
\]

\[= a(u - u^h, \theta_1 - l_0 \theta_1) + a(u - u^h, I_0 \theta_1)
\]

\[= a(u - u^h, \theta_1 - l_0 \theta_1) + (f, I_0 \theta_1) - a(u^h, I_0 \theta_1)
\]

\[= a(u - u^h, \theta_1 - l_0 \theta_1) + (f - f^h, I_0 \theta_1) + [a_h(u^h, I_0 \theta_1) - a(u^h, I_0 \theta_1)]
\]

\[:= I_1 + I_2 + I_3.
\]
We first concentrate on the estimate of $I_1$. For such purpose, we make an observation that we can split $I_1$ as

$$I_1 = a_{E_{0}, r_2}(u - u^h, \theta_1 - I_h \theta_1) + \sum_{l=2}^{l_0} a_{E_{0}, r_{l+1} \setminus B_{E_{0}, r_l}}(u - u^h, \theta_1 - I_h \theta_1)$$  \hspace{1cm} (4.107)

$$:= \sum_{l=1}^{l_0} I_{1,l}.$$  

It is sufficient to estimate $I_{1,l}$ for $1 \leq l \leq l_0$.

Lemma 4.10 implies that

$$|I_{1,l}| \lesssim \|u^h - u\|_{H^1(B_{E_{0}, r_l})} \|\theta_1 - I_h \theta_1\|_{H^1(B_{E_{0}, r_l})} \lesssim h^k r_l \ln h^2 \|u\|_{W^{k+1, \infty}(\Omega)} h^2 \|\theta_1\|_{H^2(\Omega)}$$

$$\lesssim h^{k+1} \ln h^2 \|u\|_{W^{k+1, \infty}(\Omega)} h^2 \|\theta_0(u - u^h)\|_{L^2(\Omega)}$$

$$\lesssim h^{k+2} \ln h^2 \|u\|_{W^{k+1, \infty}(\Omega)} \|\theta_0(u - u^h)\|_{L^2(\Omega)}.$$  \hspace{1cm} (4.108)

To estimate $I_{1,l}$ for $l \geq 2$, notice that $\theta_0(x) = 0$ if $x \in \Omega \setminus B_{E_{0}, r_1}$. Using the Theorem 4.5, we have the following regularity result for $\theta_1$

$$\|\theta_1\|_{H^1(\Omega \setminus B_{E_{0}, r_l})} \lesssim r^{-1}_l \ln r_l \|\theta_0(u - u^h)\|_{L^2(\Omega)},$$  \hspace{1cm} (4.109)

for $l \geq 2$. From (4.98) and (4.109), we can derive that

$$|I_{1,l}| \lesssim \|u^h - u\|_{H^1(B_{E_{0}, r_{l+1} \setminus B_{E_{0}, r_l}})} \|\theta_1 - I_h \theta_1\|_{H^1(B_{E_{0}, r_{l+1} \setminus B_{E_{0}, r_l}})} \lesssim h^{k+1} r_l \ln h^2 \|u\|_{W^{k+1, \infty}(\Omega)} h^2 \|\theta_1\|_{H^1(\Omega \setminus B_{E_{0}, r_l})}$$

$$\lesssim h^{k+2} \ln h^2 \|u\|_{W^{k+1, \infty}(\Omega)} \|\theta_0(u - u^h)\|_{L^2(\Omega)}.$$  \hspace{1cm} (4.110)

Combining the estimates of (4.108) and (4.110), we get

$$|I_1| \lesssim h^{k+2} \ln h^4 \|u\|_{W^{k+1, \infty}(\Omega)} \|\theta_0(u - u^h)\|_{L^2(\Omega)}.$$  \hspace{1cm} (4.111)

Now, we turn to estimate $I_2$. By the definition of the projection operator $\Pi^0_{k-1}$, there holds

$$(f - f^h, q)_K = 0, \hspace{0.5cm} \forall q \in P_{k-1}(K)$$

for any $K \in T_h$. We can deduce that

$$I_2 = (f - f^h, I_h \theta_1) = (f - \Pi^0_{k-1} f, I_h \theta_1 - \Pi^0_{k-1} \theta_1)$$
$$= (f - \Pi^0_{k-1} f, I_h \theta_1 - \theta_1) + (f - \Pi^0_{k-1} f, \theta_1 - \Pi^0_{k-1} \theta_1)$$

$$:= I_{2,1} + I_{2,2}.$$  \hspace{1cm} (4.112)

We only need to estimate $I_{2,1}$ and $I_{2,2}$. We start with the estimation of $I_{2,1}$. For $I_{2,1}$, we have

$$|I_{2,1}| \lesssim \|f - \Pi^0_{k-1} f\|_{L^2(\Omega)} \|I_h \theta_1 - \theta_1\|_{L^2(\Omega)} \lesssim h^k \|f\|_{H^k(\Omega)} h^2 \|\theta_1\|_{H^2(\Omega)}$$

$$\lesssim h^{k+2} \|f\|_{H^k(\Omega)} \|\theta_0(u - u^h)\|_{L^2(\Omega)}.$$
Similarly, for $I_{2,2}$, we have

$$
|I_{2,2}| \lesssim \|f - \Pi_{h-1}^0 f\|_{L^2(\Omega)} \|\Pi_{h-1}^0 \theta_1 - \theta_1\|_{L^2(\Omega)} \lesssim h^k \|f\|_{H^k(\Omega)} h^2 \|\theta_1\|_{H^2(\Omega)} \\
\lesssim h^{k+2} \|f\|_{H^k(\Omega)} \|\theta_0(u - u^h)\|_{L^2(\Omega)}.
$$

Plugging the estimates of $I_{2,1}$ and $I_{2,2}$ into (4.112), we have

$$
|I_2| \lesssim h^{k+2} \|f\|_{H^k(\Omega)} \|\theta_0(u - u^h)\|_{L^2(\Omega)}.
$$

Then, we move to consider the estimation of $I_3$. We proceed as

$$
I_3 = a_h(u^h, I_h \theta_1) - a(u^h, I_h \theta_1) = \sum_{E \in T_h} [a_h^E(u^h, I_h \theta_1) - a^E(u^h, I_h \theta_1)]
$$

$$
= \sum_{E \in T_h} [a_h^E(u^h - \Pi^0_h u, I_h \theta_1) - a^E(u^h - \Pi^0_h u, I_h \theta_1)]
$$

$$
= \sum_{E \in T_h} [a_h^E(u^h - \Pi^0_h u, I_h \theta_1 - \Pi^0_2 \theta_1) - a^E(u^h - \Pi^0_h u, I_h \theta_1 - \Pi^0_2 \theta_1)]
$$

$$
= \sum_{E \in B_{E_0, r_2}} \sum_{l=1}^{l_0} [a_h^E(u^h - \Pi^0_h u, I_h \theta_1 - \Pi^0_2 \theta_1) - a^E(u^h - \Pi^0_h u, I_h \theta_1 - \Pi^0_2 \theta_1)]
$$

$$
:= \sum_{l=1}^{l_0} I_{3,l}.
$$

We first estimate $I_{3,1}$. Lemma 4.10 implies

$$
\|u^h - \Pi^0_h u\|_{H^1(B_{E_0, r_2})} \lesssim \|u^h - u\|_{H^1(B_{E_0, r_2})} + \|u - \Pi^0_h u\|_{H^1(B_{E_0, r_2})} \\
\lesssim h^{k_r} \ln h^2 \|u\|_{W^{k+1, \infty}(\Omega)} + h^{k_r} \|u\|_{W^{k+1, \infty}(\Omega)} \\
\lesssim h^{k+1} \ln h^2 \|u\|_{W^{k+1, \infty}(\Omega)},
$$

and

$$
\|I_h \theta_1 - \Pi^0_2 \theta_1\|_{H^1(B_{E_0, r_2})} \lesssim \|I_h \theta_1 - \theta_1\|_{H^1(B_{E_0, r_2})} + \|\theta_1 - \Pi^0_2 \theta_1\|_{H^1(B_{E_0, r_2})} \\
\lesssim h \|\theta_1\|_{H^2(\Omega)} + h \|\theta_1\|_{H^2(\Omega)} \lesssim h \|\theta_1\|_{H^2(\Omega)} \\
\lesssim h \|\theta_0(u - u^h)\|_{L^2(\Omega)}.
$$

Combining those two estimates, we have

$$
|I_{3,l}| \lesssim \|u^h - \Pi^0_h u\|_{H^1(B_{E_0, r_2})} \|I_h \theta_1 - \Pi^0_2 \theta_1\|_{H^1(B_{E_0, r_2})} \\
\lesssim h^{k+1} \ln h^2 \|u\|_{W^{k+1, \infty}(\Omega)} \|\theta_1\|_{H^2(\Omega)} \\
\lesssim h^{k+2} \ln h^2 \|u\|_{W^{k+1, \infty}(\Omega)} \|\theta_0(u - u^h)\|_{L^2(\Omega)}.
$$
Similarly, we have
\[
\left| \sum_{E \subset \mathcal{B}_{0, r_l}} a_h^E (u^h - \Pi_h^0 u, I_h \theta_1 - \Pi_h^0 \theta_1) \right| \lesssim h^{k+2} | \ln h |^2 \| u \|_{W^{k+1, \infty}(\Omega)} \| \theta_0 (u - u^h) \|_{L^2(\Omega)}.
\]

Using those two estimates, we can deduce that
\[
|I_{3,1}| \lesssim h^{k+2} | \ln h |^2 \| u \|_{W^{k+1, \infty}(\Omega)} \| \theta_0 (u - u^h) \|_{L^2(\Omega)}.
\]

We are moving to estimate \( I_{3,l} \) for \( 2 \leq l \leq l_0 \). Using the regularity (4.109), we have
\[
\| I_h \theta_1 - \Pi_h^0 \theta_1 \|_{H^1(\mathcal{B}_{0, r_{l+1}} \setminus \mathcal{B}_{0, r_l})} \lesssim h^2 \| \theta_1 \|_{H^2(\mathcal{B}_{0, r_{l+1}} \setminus \mathcal{B}_{0, r_l})} \lesssim h^2 r_l^{-1} | \ln h | \| \theta_0 (u - u^h) \|_{L^2(\Omega)},
\]

and
\[
\| I_h \theta_1 - \theta_1 \|_{H^1(\mathcal{B}_{0, r_{l+1}} \setminus \mathcal{B}_{0, r_l})} \lesssim h^2 r_l^{-1} | \ln h | \| \theta_0 (u - u^h) \|_{L^2(\Omega)}.
\]

Also, (4.98) implies
\[
\| u^h - \Pi_h^0 u \|_{H^1(\mathcal{B}_{0, r_{l+1}} \setminus \mathcal{B}_{0, r_l})} \lesssim |u^h - u|_{H^1(\mathcal{B}_{0, r_{l+1}} \setminus \mathcal{B}_{0, r_l})} + \| u - \Pi_h^0 u \|_{H^1(\mathcal{B}_{0, r_{l+1}} \setminus \mathcal{B}_{0, r_l})} \lesssim h^2 r_l | \ln h |^2 \| u \|_{W^{k+1, \infty}(\Omega)} + h^2 r_l \| u \|_{W^{k+1, \infty}(\Omega)} \lesssim h^2 r_l | \ln h |^2 \| u \|_{W^{k+1, \infty}(\Omega)}.
\]

Using the above three estimates and the definition of \( I_3 \), for \( 2 \leq l \leq l_0 \), we can show
\[
\left| \sum_{E \subset \mathcal{B}_{0, r_{l+1}} \setminus \mathcal{B}_{0, r_l}} a_h^E (u^h - \Pi_h^0 u, I_h \theta_1 - \Pi_h^0 \theta_1) \right| \lesssim \| u^h - \Pi_h^0 u \|_{H^1(\mathcal{B}_{0, r_{l+1}} \setminus \mathcal{B}_{0, r_l})} \| I_h \theta_1 - \Pi_h^0 \theta_1 \|_{H^1(\mathcal{B}_{0, r_{l+1}} \setminus \mathcal{B}_{0, r_l})} \lesssim \| u^h - \Pi_h^0 u \|_{H^1(\mathcal{B}_{0, r_{l+1}} \setminus \mathcal{B}_{0, r_l})} \| I_h \theta_1 - \Pi_h^0 \theta_1 \|_{H^1(\mathcal{B}_{0, r_{l+1}} \setminus \mathcal{B}_{0, r_l})} \lesssim h^2 r_l | \ln h |^2 \| u \|_{W^{k+1, \infty}(\Omega)} h^2 r_l^{-1} | \ln h | \| \theta_0 (u - u^h) \|_{L^2(\Omega)} + h^2 r_l \| u \|_{W^{k+1, \infty}(\Omega)} h^2 r_l^{-1} | \ln h | \| \theta_0 (u - u^h) \|_{L^2(\Omega)} \lesssim h^{k+2} | \ln h |^2 \| u \|_{W^{k+1, \infty}(\Omega)} \| \theta_0 (u - u^h) \|_{L^2(\Omega)}.
\]

Similarly, we have
\[
\left| \sum_{E \subset \mathcal{B}_{0, r_{l+1}} \setminus \mathcal{B}_{0, r_l}} a_h^E (u^h - \Pi_h^0 u, I_h \theta_1 - \Pi_h^0 \theta_1) \right| \lesssim h^{k+2} | \ln h |^3 \| u \|_{W^{k+1, \infty}(\Omega)} \| \theta_0 (u - u^h) \|_{L^2(\Omega)}.
\]
Substituting (4.123) and (4.124) into (4.115), we have

\[ |I_{3,l}| \lesssim h^{k+2} |\ln h|^3 \|u\|_{W^{k+1,\infty}(\Omega)} \|\theta_0(u-u^h)\|_{L^2(\Omega)}. \]  

Combining (4.119) and (4.125) gives

\[ |I_3| \lesssim h^{k+2} |\ln h|^4 \|u\|_{W^{k+1,\infty}(\Omega)} \|\theta_0(u-u^h)\|_{L^2(\Omega)}. \]  

By (4.106), (4.111), (4.114) and (4.126), we have

\[ \|\theta_0(u-u^h)\|_{L^2(\Omega)} \lesssim h^{k+1} |\ln h|^4 (\|u\|_{W^{k+1,\infty}(\Omega)} + \|f\|_{H^k(\Omega)}) \|\theta_0(u-u^h)\|_{L^2(\Omega)}. \]  

This implies

\[ \|u-u^h\|_{L^2(E_0)} \leq \|\theta_0(u-u^h)\|_{L^2(\Omega)} \lesssim h^{k+1} |\ln h|^4 (\|u\|_{W^{k+1,\infty}(\Omega)} + \|f\|_{H^k(\Omega)}). \]  

This concludes the proof of the local $L^2$ error estimate.

### 4.6. An inverse estimate for VEM functions.

It is well known that the classical polynomial inverse estimates [12,20] are no longer valid for VEM functions. In this subsection, we establish an inverse estimate using the maximum principle of harmonic function [13].

**Lemma 4.12.** Assume that $v \in V_h$. Under the same assumptions of Theorem 4.1, there holds

\[ \|v\|_{L^\infty(E_0)} \lesssim h^{-1}\|v\|_{L^2(E_0)}. \]

**Proof.** Brenner and Sung (See [13, Lemma 3.3]) showed the following maximum principle

\[ \|v\|_{L^\infty(E_0)} \lesssim (\|v\|_{L^\infty(\partial E_0)} + \|v\|_{H^1(\Omega)}), \quad \forall v \in H^1(\Omega). \]

The key observation is that $v|_{\partial E_0}$ is a polynomial and the standard polynomial inverse estimates [12,20] are applicable, which implies

\[ \|v\|_{L^\infty(\partial E_0)} \lesssim h^{-1}_{E_0} \|v\|_{L^2(\partial E_0)}. \]

Using the scaled trace inequality for $H^1$ functions [11], we obtain

\[ \|v\|_{L^2(\partial E_0)} \lesssim h^{-\frac{1}{2}}_{E_0} \|v\|_{L^2(E_0)} + h^{\frac{1}{2}}_{E_0} \|\nabla v\|_{L^2(E_0)}. \]

Combining the above two estimates, we have

\[ \|v\|_{L^\infty(\partial E_0)} \lesssim h^{-1}_{E_0} \|v\|_{L^2(E_0)} + \|\nabla v\|_{L^2(E_0)}. \]

Also, the inverse estimate in [19, Theorem 3.6] implies

\[ \|\nabla v\|_{L^2(E_0)} \lesssim h^{-1}_{E_0} \|v\|_{L^2(E_0)}. \]

Using the above two estimates, it is relatively easy to deduce (4.127). }
4.7. Proof of the main result. With the above preparation, we are now in a perfect position to present the proof of our main numerical results. Before we start, we recall the approximation property of the $L^2$-projection operator $\Pi_k^0$ as defined in (3.10). For $\Pi_k^0$, the following approximation result holds [21, Theorem 1.45]

\begin{equation}
|v - \Pi_k^0 v|_{W^{m,p}(E_0)} \leq h^{s-m} |v|_{W^{s,m}(E_0)},
\end{equation}

for any $v \in W^{s,p}(E_0)$, $m \in \{0, 1, \ldots, s\}$, and $p \in [1, \infty]$

We begin with the maximum error estimate of the difference between the gradient of exact solution and the gradient of the projection of the virtual element method solution. For such purpose, let $E_0 \in \mathcal{T}_h$ such that

\begin{equation}
\|\nabla u - \nabla \Pi_k^0 u_h\|_{L^\infty(\Omega)} = \|\nabla v\|_{L^2(E_0)}, \quad \forall v \in H^1(E_0).
\end{equation}

Then, (4.102), (4.130), and (4.132) imply

\begin{equation}
\|\nabla u - \nabla \Pi_k^0 u_h\|_{L^\infty(\Omega)}
= \|\nabla u - \nabla \Pi_k^0 u_h\|_{L^\infty(E_0)}
\leq \|\nabla u - \nabla \Pi_k^0 u\|_{L^\infty(E_0)} + \|\nabla \Pi_k^0 u - \nabla \Pi_k^0 u_h\|_{L^\infty(E_0)} + \|\nabla \Pi_k^0 u - \nabla \Pi_k^0 u_h\|_{L^\infty(E_0)}
\leq \|\nabla u - \nabla \Pi_k^0 u\|_{L^\infty(E_0)} + h^{-1}\|\nabla \Pi_k^0 u - \nabla \Pi_k^0 u_h\|_{L^2(E_0)} + h^{-1}\|\nabla \Pi_k^0 u - \nabla \Pi_k^0 u_h\|_{L^2(E_0)}
\leq \|\nabla u - \nabla \Pi_k^0 u\|_{W^{k+\frac{1}{2}, \infty}(\Omega)} + h^{-1}\|\nabla u - \nabla \Pi_k^0 u\|_{L^2(E_0)} + h^{-1}\|\nabla u - \nabla \Pi_k^0 u_h\|_{L^2(E_0)}
\leq h^k\|u\|_{W^{k+1, \infty}(\Omega)} + h^{-1}\|\nabla u\|_{H^{k+1}(E_0)} + h^{-1}\|\nabla u - \nabla u_h\|_{L^2(E_0)}
\leq h^k\|u\|_{W^{k+1, \infty}(\Omega)} + h^{-1}\|\nabla u\|_{H^{k+1}(E_0)} + h^{-1}\|\nabla u - \nabla u_h\|_{L^2(E_0)}
\leq h^k\ln h\|u\|_{W^{k+1, \infty}(\Omega)} + h^{-1}\|\nabla u\|_{H^{k+1}(E_0)} + h^{-1}\|\nabla u - \nabla u_h\|_{L^2(E_0)}
\leq h^k\ln h\|u\|_{W^{k+1, \infty}(\Omega)} + h^{-1}\|\nabla u\|_{H^{k+1}(E_0)} + h^{-1}\|\nabla u - \nabla u_h\|_{L^2(E_0)}
\leq h^k\ln h^2\|u\|_{W^{k+1, \infty}(\Omega)},
\end{equation}

where we have used the $L^\infty$ to $L^2$ inverse estimate for polynomial on polygons in obtaining the second inequality. This completes the proof of (4.1).

Then, we are proving the $L^\infty$ error estimate. Assume $k \geq 2$. In that case, we can analogously let $E_0 \in \mathcal{T}_h$ satisfy

\begin{equation}
\|u - u_h\|_{L^\infty(\Omega)} = \|u - u_h\|_{L^\infty(E_0)},
\end{equation}
Then, Lemma 4.11, Lemma 4.12, and the approximation property (4.130) give that
\[
\|u - u_h\|_{L^\infty(\Omega)}
\leq \|u - \Pi_k^0 u\|_{L^\infty(E_0)} + \|\Pi_k^0 u - u^h\|_{L^\infty(E_0)}
\lesssim h^{k+1} \|u\|_{W^{k+1,\infty}(\Omega)} + h^{-1} \|\Pi_k^0 u - u^h\|_{L^2(E_0)}
\lesssim h^{k+1} \|u\|_{W^{k+1,\infty}(\Omega)} + h^{-1} \left[\|\Pi_k^0 u - u\|_{L^2(E_0)} + \|u - u^h\|_{L^2(E_0)}\right]
\lesssim h^{k+1} \|u\|_{W^{k+1,\infty}(\Omega)} + h^{-1} \left[h^{k+2} \|u\|_{W^{k+1,\infty}(E_0)} + h^{k+2} \|f\|_{H^k(\Omega)}\right]
\lesssim h^{k+1} \|u\|_{W^{k+1,\infty}(\Omega)} + h^{-1} \left[h^{k+2} \|u\|_{W^{k+1,\infty}(E_0)} + h^{k+2} \|f\|_{H^k(\Omega)}\right].
\]

(4.135)

It concludes the proof of (4.2).

5. Numerical Examples. In this section, we present a numerical example to validate our theoretical results. In all the following numerical, the stabilizing bilinear form \(S^E(\cdot, \cdot)\) is chosen as \[5\]
\[
S^E(\phi_i - \Pi_k^g \phi_i, \phi_j - \Pi_k^g \phi_j) = \sum_{r=1}^{N_E} \chi_r(\phi_i - \Pi_k^g \phi_i) \chi_r(\phi_j - \Pi_k^g \phi_j),
\]
where \(\chi_1, \cdot \cdot \cdot, \chi_{N_E}\) are the basis functions of the dual space of \(W_k(E)\).

The optimal convergence of the maximal norm errors shall be measured using discrete maximal norm error at vertices of mesh. Let \(N_h\) denote the set of all vertices of \(T_h\). We shall consider the discrete \(L^\infty\) norm of \(u - u_h\) as \(\|u - u_h\|_{L^\infty} = \max_{p \in N_h} |u(p) - u_h(p)|\). In the virtual element method, the piecewise gradient of virtual element solution \(u_h\) at vertices is not directly available. In practice, we use the piecewise gradient of the projection \(\Pi_k^g u_h\) to approximate \(\nabla u_h\). We take simple averaging to obtain the gradient of \(\Pi_k^g u_h\) at a mesh vertex \(p \in N_h\), which is denoted by \(\nabla \Pi_k^g u_h(p)\).

The discrete \(W^{1,\infty}\) norm of \(u - u_h\) is defined as \(\|u - u_h\|_{W^{1,\infty}} = \max_{p \in N_h} |\nabla u(p) - \nabla \Pi_k^g u_h(p)|\). Let \(N\) be the number of vertices of \(T_h\).

Fig. 1: Sample meshes for numerical tests: (a) structured hexagonal mesh; (b) transformed hexagonal mesh; (c) mesh with non-convex elements.
One of the merits of VEM is its ability to use arbitrary polygonal meshes. To illustrate the generality of our theoretical results in terms of the flexibility of VEM, we test our numerical example using three different types of polygonal meshes. The first level of each type of meshes are plotted in Figure 1. The first type of mesh $T_{h,1}$ is uniform hexagonal mesh. The second type of mesh $T_{h,2}$ is generated by applying the following coordinate transform

$$
\begin{align*}
    x_1 &= \hat{x}_1 + \frac{1}{10} \sin(2\pi \hat{x}_1) \sin(2\pi \hat{x}_2), \\
    x_2 &= \hat{x}_2 + \frac{1}{10} \sin(2\pi \hat{x}_1) \sin(2\pi \hat{x}_2),
\end{align*}
$$

to the uniform hexagonal mesh $T_{h,1}$. The third type of mesh $T_{h,3}$ is the uniform non-convex mesh.

### 5.1. Test case I: Smooth solution

In this test, we consider the following exemplary equation with homogeneous Dirichlet boundary condition:

$$(5.2) \quad -\Delta u = 2\pi^{2} \sin(\pi x_1) \sin(\pi x_2), \quad \text{in } \Omega = (0,1) \times (0,1).$$

The exact solution is $u(x) = \sin(\pi x_1) \sin(\pi x_2)$.

Table 1: Numerical errors of test case I on structured hexagonal meshes

| Degree | $N$ | $\|u - u^h\|_{L^\infty}$ | Order | $\|u - u^h\|_{W^{1,\infty}}$ | Order |
|--------|-----|-------------------------|-------|-------------------------|-------|
| $k=1$  | 514 | 2.39e-03                | –     | 4.04e-01                | –     |
|        | 2050| 5.75e-04               | 2.06  | 2.05e-01               | 0.98  |
|        | 8194| 1.41e-04               | 2.03  | 1.03e-01               | 1.00  |
|        | 32770| 3.49e-05              | 2.02  | 5.13e-02               | 1.00  |
|        | 131074| 8.67e-06             | 2.01  | 2.57e-02               | 1.00  |
| $k=2$  | 514 | 7.72e-05                | –     | 2.38e-02                | –     |
|        | 2050| 9.93e-06               | 2.96  | 6.02e-03               | 1.99  |
|        | 8194| 1.26e-06               | 2.98  | 1.51e-03               | 2.00  |
|        | 32770| 1.58e-07              | 2.99  | 3.77e-04               | 2.00  |
|        | 131074| 1.98e-08             | 3.00  | 9.44e-05               | 2.00  |
| $k=3$  | 514 | 1.02e-05                | –     | 1.52e-03                | –     |
|        | 2050| 5.93e-07               | 4.11  | 1.94e-04               | 2.98  |
|        | 8194| 3.56e-08               | 4.06  | 2.44e-05               | 2.99  |
|        | 32770| 2.18e-09              | 4.03  | 3.05e-06               | 3.00  |
|        | 131074| 1.36e-10             | 4.00  | 3.82e-07               | 3.00  |

In the numerical test, we consider virtual element methods of degrees from 1 to 3. The numerical errors are documented in Table 1 for the structured hexagonal meshes. What is striking in this table is the $O(h^{k+1})$ optimal convergence rate for $L^\infty$ error and $O(h^{k})$ optimal convergence rate for $W^{1,\infty}$. The observed convergence rates are consistent with the theoretical results predicted by Theorem 4.1. Even though our theoretical results for $L^\infty$ error work for virtual element methods of degree $k \geq 2$, we can observe the optimal convergence results for the linear virtual element method.

Let us now turn to the numerical results for the transformed hexagonal meshes, which is displayed in Table 2. Despite the unstructured nature of the mesh $T_{h,2}$, we can still observe the optimal maximal error, which demonstrates the Theorem 4.1.
For the non-convex meshes $\mathcal{T}_{h,3}$, we show the convergence history in Table 3. Note that in the case the element is not always convex, which is not allowed in classical finite element methods. Similar to the previous two tests, the same optimal convergence rates are observed as anticipated by the Theorem 4.1.

Table 2: Numerical errors of test case I on transformed hexagonal meshes

| Degree | N     | $\|u - u_h\|_{L^\infty}$ | Order | $\|u - u_h\|_{W^{1,\infty}}$ | Order |
|--------|-------|--------------------------|-------|-----------------------------|-------|
| k=1    | 514   | 6.64e-03                 | –     | 4.27e-01                    | –     |
|        | 2050  | 1.64e-03                 | 2.02  | 2.12e-01                    | 1.02  |
|        | 8194  | 4.01e-04                 | 2.03  | 1.05e-01                    | 1.02  |
|        | 32770 | 9.91e-05                 | 2.02  | 5.18e-02                    | 1.01  |
|        | 131074| 2.46e-05                 | 2.01  | 2.58e-02                    | 1.01  |
| k=2    | 514   | 3.12e-04                 | –     | 3.98e-02                    | –     |
|        | 2050  | 4.12e-05                 | 2.93  | 1.07e-02                    | 1.89  |
|        | 8194  | 5.09e-06                 | 3.02  | 2.09e-03                    | 2.00  |
|        | 32770 | 6.22e-07                 | 3.03  | 6.70e-04                    | 2.00  |
|        | 131074| 7.66e-08                 | 3.02  | 1.67e-04                    | 2.00  |
| k=3    | 514   | 3.26e-05                 | –     | 2.07e-03                    | –     |
|        | 2050  | 2.01e-06                 | 4.03  | 2.89e-04                    | 2.85  |
|        | 8194  | 1.25e-07                 | 4.01  | 3.59e-05                    | 3.01  |
|        | 32770 | 7.76e-09                 | 4.01  | 4.48e-06                    | 3.00  |
|        | 131074| 4.83e-10                 | 4.01  | 5.57e-07                    | 3.01  |

Table 3: Numerical errors of test case I on non-convex meshes

| Degree | N     | $\|u - u_h\|_{L^\infty}$ | Order | $\|u - u_h\|_{W^{1,\infty}}$ | Order |
|--------|-------|--------------------------|-------|-----------------------------|-------|
| k=1    | 833   | 7.47e-03                 | –     | 3.53e-01                    | –     |
|        | 3201  | 1.89e-03                 | 2.04  | 1.79e-01                    | 1.01  |
|        | 12545 | 4.75e-04                 | 2.02  | 8.98e-02                    | 1.01  |
|        | 49665 | 1.19e-04                 | 2.01  | 4.50e-02                    | 1.01  |
|        | 197633| 2.98e-05                 | 2.01  | 2.25e-02                    | 1.00  |
| k=2    | 833   | 1.85e-04                 | –     | 2.22e-02                    | –     |
|        | 3201  | 2.36e-05                 | 3.06  | 5.63e-03                    | 2.04  |
|        | 12545 | 2.97e-06                 | 3.03  | 1.41e-03                    | 2.02  |
|        | 49665 | 3.72e-07                 | 3.02  | 3.54e-04                    | 2.01  |
|        | 197633| 4.65e-08                 | 3.01  | 8.85e-05                    | 2.01  |
| k=3    | 833   | 1.35e-05                 | –     | 1.36e-03                    | –     |
|        | 3201  | 8.04e-07                 | 4.19  | 1.74e-04                    | 3.05  |
|        | 12545 | 4.92e-08                 | 4.09  | 2.19e-05                    | 3.04  |
|        | 49665 | 3.08e-09                 | 4.03  | 2.74e-06                    | 3.02  |
|        | 197633| 1.94e-10                 | 4.00  | 3.43e-07                    | 3.01  |

5.2. Test case II: Problem with a Gaussian surface. In this test, we consider the following Poisson equation

\[
- \Delta u = f, \quad \text{in } \Omega = (0,1) \times (0,1),
\]
with non-homogenous boundary condition \( u|_{\partial \Omega} = g \). The right hand side function \( f(x) \) and the boundary condition \( g(x) \) can be calculated from the exact solution \( u(x) = \exp(-\ell((x_1 - 0.5)^2 + (x_2 - 0.5)^2)) \). When \( \ell \) is large, the function \( u \) has a rapidly varying gradient.

In this test, we select \( \ell = 25 \). The numerical errors on non-convex meshes are displayed in Table 4. Similar to the previous test case, we can observe the optimal convergence rates in maximal norm errors. But it requires a little bit finer meshes to observe the perfect optimal convergence rates since the rapidly varying gradient. We also test it on other two polygonal meshes, which also gives us the same results.

| Degree | \( N \) | \( ||u - u^h||_{L^\infty} \) | Order | \( ||u - u^h||_{W^{1,\infty}} \) | Order |
|-------|-------|------------------|-------|------------------|-------|
| \( k=1 \) | 833 | 2.80e-02 | - | 7.06e-01 | - |
|       | 3201 | 7.54e-03 | 1.95 | 2.68e-01 | 1.44 |
|       | 12545 | 1.92e-03 | 2.00 | 1.10e-01 | 1.31 |
|       | 49665 | 4.83e-04 | 2.01 | 5.06e-02 | 1.13 |
|       | 197633 | 1.21e-04 | 2.01 | 2.46e-02 | 1.04 |
| \( k=2 \) | 833 | 1.08e-03 | - | 1.94e-01 | - |
|       | 3201 | 1.41e-04 | 3.02 | 5.51e-02 | 1.87 |
|       | 12545 | 1.78e-05 | 3.03 | 1.42e-02 | 1.98 |
|       | 49665 | 2.23e-06 | 3.02 | 3.59e-03 | 2.00 |
|       | 197633 | 2.79e-07 | 3.01 | 8.99e-04 | 2.00 |
| \( k=3 \) | 833 | 5.77e-04 | - | 2.27e-02 | - |
|       | 3201 | 3.86e-05 | 4.02 | 3.34e-03 | 2.85 |
|       | 12545 | 2.45e-06 | 4.03 | 4.37e-04 | 2.98 |
|       | 49665 | 1.54e-07 | 4.02 | 5.52e-05 | 3.01 |
|       | 197633 | 9.64e-09 | 4.01 | 6.92e-06 | 3.01 |

6. Conclusion. In this paper, we consider the error estimations in the maximum norm for virtual element methods. We establish the optimal maximum norm error estimations as to the classical numerical methods. In special, we show \( O(h^k) \) order convergence between the exact gradient and the gradient of the projection of virtual element solution for \( k \)th order virtual element methods. When \( k \geq 2 \), we prove the optimal \( O(h^{k+1}) \) order convergence for \( L^\infty \) error. We present a numerical example on both convex and non-convex general polygonal meshes to support our theoretical results.

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