Generation, concentration and purification for ionic entangled states

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Abstract. In cavity QED, the atoms would be sent through the sequential arrays of cavities for the generation of multi-cavity entanglement, or several atoms would be sent into the same cavity mode one by one for the generation of multi-atom entanglement. The complexity of these processes will impose limitations on the experimental feasibility of it. So, following our previous publication [International Journal Of Quantum Information 2, 231 (2004)] we will propose an alternative scheme for the preparation of multi-cavity W state via cavity QED, which uses the geometrical method to do what other authors have proposed previously using sequential arrays of cavities. Due to the impossibility that one quantum system can be isolated from the environment absolutely, the entanglement of the entangled objects will decrease exponentially with the propagating distance of the objects, and the practically available quantum entangled states are all non-maximally entangled states or the more general case—mixed states. Following our previous publications [Phys. Rev. A 72, 042307 (2005), ibid. 71, 012308 (2005)], we will propose an entanglement generation, concentration and purification scheme for atomic or ionic system, which is mainly based on Cavity QED and linear optical elements. This purification process avoids the controlled-NOT (C-NOT) operations needed in the original purification protocol, which simplifies the whole purification process.

1. Introduction

Entanglement is a fundamental resource in quantum information processing, so the preparation of entanglement is a central task in quantum information theory (QIT). Many theoretical and experimental schemes for the generation of entangled states have been proposed in Cavity QED [1], ion trap [2], and NMR [3].

W state is a special kind of entangled state in the tripartite system. There is a more robust entanglement in it than in the GHZ state when one of the three particles was traced out [4]. Because of the special property, when W state is used in the quantum communication, there will be some novel results [5]. So it is of practical significance to generate the state. The generation schemes for W state have been proposed recently both in optical domain [6, 7] and in cavity QED [8–11]. But in most of the previous generation schemes for multiple cavity fields’ states, the atoms will be sent through the sequential arrays of cavities, the complexity of which will impose limitations on the experimental feasibility of it. Here, we will propose an alternative scheme for the preparation of three cavities W state via cavity QED, which uses the geometry to do what other authors have proposed previously using sequential arrays of cavities [12]. We need to modulate the interaction time between atom and cavities.
only once, which is more efficient than the previous proposals. As a straight extension, we will discuss the generation of N-cavity entangled W state using this scheme [12]. Alternatively, the entangled states also can be generated via atomic interference [13–22]. The schemes of this type can entangle the spatially separated atoms, but there is still a serious difficulty. These schemes require the two photons reach the beam splitter analyzer (BSA) or polarization beam splitter (PBS) simultaneously. To solve these problems, we will propose a novel entanglement generation scheme, which is motivated by Xing-Xiang Zhou’s proposal on non-distortion quantum interrogation [23]. In our scheme, the photon wave function of one incident circular polarized photon will be split into two parts, transmitted part and the reflected part, by an ordinary nonpolarizing 50/50 Beam Splitter (BS). Two multi-level ions will be pre-placed on the two possible paths of the photon. After interacting with the ions, the two parts of the photon wave function will be re-combined by the second ordinary nonpolarizing 50/50 BS. Through detecting the photon after the second BS, we can decide whether the entangled ionic pairs has been created or not. The main part of the setup can be regarded as a Mach-Zehnder interferometer (MZI). The BS is an ordinary one, and the relative phase problem inherent in the previous schemes has been avoided in our scheme. The photon detected in this scheme is a circular polarized one, and can be detected easier than the scattered one used in the previous schemes. It is not easy to make the two photons from two different ions interfere in the previous schemes. But in our scheme, the photon wave function has been split into two parts in MZI, and the coherent condition is satisfied naturally [24].

Due to the impossibility that one quantum system can be isolated from the environment absolutely, the entanglement of the entangled objects will decrease exponentially with the propagating distance of the objects, and the practically available quantum entangled states are all non-maximally entangled states or the more general case-mixed states. So, if nothing has been done on the distributed states before used in Quantum Communication, the long distance Quantum Communication is impossible. To overcome the dissipation and decoherence, various schemes of entanglement distillation [25, 26], entanglement concentration [27, 28] and entanglement purification [29–41] have been proposed. In the original entanglement purification scheme [29], C-NOT operations construct the main step of the purification process. But, in experiment, there is no implementation of C-NOT operations can meet the error rate level, which is needed for the logic gates in long distance Quantum Communication. So more and more attention are focused on finding the realizable schemes for entanglement purification. In optical domain, J.-W. Pan et al use the Polarization Beam Splitter (PBS) [30] to replace the C-NOT gate needed in the original scheme [29] to get the newly-obtained polarization-entangled photon pairs with a larger fraction of fidelity [31].

Most of the previous entanglement purification schemes only apply to the polarization-entangled photon pairs. There is few schemes for distillation and purification of atomic and ionic entangled states in the literature [42]. Although, photons are the attractive carriers of information for the implementation of Quantum Communication, ions are also the preferred carrier for quantum information, because the realization of Quantum Computer and Quantum Computation relies on the optimal quantum carriers, which should can be integrated. So the purification of ionic entangled states is of practical significance not only in Quantum Communication but also in Quantum Computation.

Here we will use the MZI developed in the preceeding generation scheme to realize the purification of the unknown mixed ionic entangled states. This purification process avoids the controlled-NOT (C-NOT) operations needed in the original purification protocol, which simplifies the whole purification process [42].

This article is organized as follows: In Section 2 we discuss the generation scheme for multi-cavity-mode W state. And the generation of two-ion maximally entangled state is discussed in Section 3. Section 4 is the purification process for the unknown mixed ionic entangled states. Finally, in section 5, the entanglement concentration for unknown non-maximally entangled ionic. We will discuss the feasibility of the proposed schemes in Section 6.
2. Generation of multi-cavity-mode W state in cavity QED

We first discuss the three-cavity-mode case. An atom initially prepared in excited state and three identical cavities initially prepared in vacuum state are required. The three cavities have been placed in a common plane, and the focus area of the three fields constructs a cylinder area, which is vertical to the common field plane. This has been depicted in Fig. 1. Then the atom will be sent through the cylinder area from one side (top or bottom) of it. In the focus area, the atom will interact simultaneously with three cavity fields. The interaction can be described by Hamiltonian [12]:

$$\hat{H} = \omega_0 s_z + \sum_{i=1}^{3} \omega_i a_i^+ a_i + \sum_{i=1}^{3} \varepsilon_i \left( a_i s_+ + a_i^+ s_- \right),$$

(1)

where \(s_z, s_+\), and \(s_-\) are atomic operators, and \(s_z = \frac{1}{2} (|e\rangle_a \langle e| - |g\rangle_a \langle g|)\), \(s_+ = |e\rangle_a \langle g|\), \(s_- = |g\rangle_a \langle e|\) with \(|e\rangle_a\) and \(|g\rangle_a\) being the excited and ground states of the atom. \(a_i^+\) and \(a_i\) denote the creation and annihilation operators of the ith cavity mode. \(\omega_0\) is the transition frequency of the atom and \(\omega_i\) is the ith cavity frequency. \(\varepsilon_i\) is the coupling constant between the atom and the ith cavity mode. Because the cavities are all identical, if we suppose that the interaction is a resonant one we get: \(\varepsilon_1 = \varepsilon_2 = \varepsilon_3 = \varepsilon\) and \(\omega_0 = \omega_1 = \omega_2 = \omega_3\).

The evolution of the total system reads:

$$\begin{align*}
|0\rangle_1 |0\rangle_2 |0\rangle_3 |e\rangle_a &\rightarrow \cos \left( \sqrt{3} \varepsilon t \right) |0\rangle_1 |0\rangle_2 |0\rangle_3 |e\rangle_a \\
- i \frac{\sin \left( \sqrt{3} \varepsilon t \right)}{\sqrt{3}} \left( |0\rangle_1 |0\rangle_2 |1\rangle_3 + |0\rangle_1 |1\rangle_2 |0\rangle_3 + |1\rangle_1 |0\rangle_2 |0\rangle_3 \right) |g\rangle_a .
\end{align*}$$

(2)

Then if we select the interaction time \(\frac{\pi}{2\sqrt{3}\varepsilon}\), we can get the maximally entangled W state for three cavities without detection on the atom, and the successful probability is 100%, where we have discarded the common phase factor.
Figure 2. The level configuration of the ions. The ions in $|m_+\rangle$ (or $|m_-\rangle$) can be excited into the excited state $|e\rangle$ by absorbing one $\sigma^+\sigma^-$ polarized photon, and then it will decay to the stable ground state $|g\rangle$ and scatter a photon rapidly. Here, we assume that the decay process is so rapid that the probability of excited emission can be neglected.

As a generalization, we can generate N-cavity-mode W state using the same method. In this case, the Hamiltonian becomes:

$$\hat{H}_N = \omega_0 s_z + \sum_{i=1}^{N} \omega_i a_i^+ a_i + \sum_{i=1}^{N} \varepsilon_i (a_i s_+ + a_i^+ s_-).$$

(3)

Then we can use the same analysis and conclude that, If the interaction time is $\frac{\pi}{2\sqrt{N\varepsilon}}$, the N cavity will be left in a maximally entangled W state with probability 100% without detection on the atom, and the atom is left in ground state.

Compared with the proposals in [9], [11], our scheme uses the geometry to do what other authors have proposed previously using sequential arrays of cavities and needs fewer operations, which makes it more easily to be realized. In addition, no detection on atom is needed in our scheme, which is another unique advantage. But in experiment, there will be a practical problem about how to measure the length of the focus area, which will affect the accuracy of the interaction time. Nevertheless, the simplicity of the scheme makes it more easily to be realized in experiment.

3. Generation of ionic entangled states via linear optical elements

Here, we will consider two identical ions, and they are all multi-level systems [24]. The level configuration of the ions has been depicted in Fig. 2.

Where $|m_+\rangle$ and $|m_-\rangle$ are two degenerate metastable states which are used to store quantum information. $|e\rangle$ is a excited state of ions and $|g\rangle$ is the stable ground state. Ions in states $|m_+\rangle$ (or $|m_-\rangle$) can be excited into the $|e\rangle$ state by absorbing one $\sigma^+\sigma^-$ circular polarized photon with unit efficiency, then it will decay to ground state $|g\rangle$ rapidly and scatter a photon. This process can be expressed as:

$$\hat{a}_\pm |0\rangle |m_\pm\rangle \longrightarrow |S\rangle |g\rangle.$$  

(4)

where $|S\rangle$ denotes the scattered photons which we assume will not be reabsorbed by the ions and can be filtered away from the detectors. Although this process does not always occur despite the photon impinging on the ions, we still consider the ideal case to demonstrate the process, and then we will consider how to enhance the scattering rate by adding an optical cavity. The setup for generation of maximally entangled ionic states is depicted in Fig. 3.
One MZI with two BSs is the main part of the generation setup. One $\sigma^+$ polarized photon enters the MZI from the left lower port. The MZI is initially adjusted (without ions) such that the upper detector($D_u$) registers photons with certainty. In the case of the existence of two ions in arbitrary superposition states of $|m_+\rangle$ and $|m_-\rangle$ at each arm of the MZI (the two ions can be placed on the two arms of the MZI by using the trapping techniques [43]), the upper and the lower detectors($D_u$ and $D_l$) all have the probability of fire. If we select the superposition coefficients of the initial states of the two ions appropriately, we can get the maximally entangled ionic states conditioned on the fire at $D_l$. Suppose that the two ions ($U$, $L$) are initially prepared in the following states:

$$
|\Psi\rangle_U = \alpha|m_+\rangle_U + \beta|m_-\rangle_U, \quad (5a)
$$

$$
|\Psi\rangle_L = a|m_+\rangle_L + b|m_-\rangle_L, \quad (5b)
$$

where the coefficients $\alpha, \beta, a, b$ satisfy $|\alpha|^2 + |\beta|^2 = 1$ and $|a|^2 + |b|^2 = 1$. These states can be prepared by a laser pulse focused on the ion. The effect of the BS on the input photon can be expressed as:

$$
\hat{a}_{\pm,u}^+|0\rangle \longrightarrow \frac{1}{\sqrt{2}}(\hat{a}_{\pm,u}^+ \pm i\hat{a}_{\mp,l}^+)|0\rangle, \quad (6a)
$$

$$
\hat{a}_{\pm,l}^+|0\rangle \longrightarrow \frac{1}{\sqrt{2}}(\hat{a}_{\pm,u}^+ \pm i\hat{a}_{\mp,l}^+)|0\rangle. \quad (6b)
$$

That is to say, the BS takes no effect on the polarization of the input photon, and reflects the wave function with a $\pm \pi/2$ phase shift corresponding to the propagation direction of the photon [23].

Next, we will trace the input photon and give the evolution of the total system. After one $\sigma^+$ polarized photon entering the left lower port of the MZI, its wave function will be split into two parts (the upper arm and the lower arm) by $BS_1$. Because the two ions are placed on the two arms, they will interact with the different parts of the wave function. Then the two parts of the wave function will be combined by $BS_2$. To analyze the evolution of the total system, we will consider the evolution of the following four product states of two ions:

$$
a_{m_+}^+|0\rangle|m_+\rangle_U|m_+\rangle_L \longrightarrow \frac{1}{\sqrt{2}}(|S\rangle_U|g\rangle_U|m_+\rangle_L + i|m_+\rangle_U|S\rangle_L|g\rangle_L), \quad (7a)
$$
any entanglement between the ions being created. To enhance the scattering rate, an optical cavity will so a resonant cavity must be introduced for each ion to achieve directional emission of the photons from the ions. In the free space, the efficiency of entanglement creation will be essentially incoherent and very few of the photons will be in the correct direction in the free space, where the probability states:

\[ a_{l+}^+|0\rangle|m_+\rangle_L \rightarrow \frac{1}{\sqrt{2}}|S\rangle_L|g\rangle_U|m_-\rangle_L + \frac{i}{2}(a_{u+}^+ + ia_{i+}^+)|0\rangle|m_+\rangle_L \quad (7b) \]

\[ a_{l+}^+|0\rangle|m_-\rangle_U|m_+\rangle_L \rightarrow \frac{i}{\sqrt{2}}|m_-\rangle_U|S\rangle_L|g\rangle_L + \frac{1}{2}(a_{l+}^+ + ia_{d+}^+)|0\rangle|m_-\rangle_U|m_+\rangle_L, \quad (7c) \]

\[ a_{l+}^+|0\rangle|m_-\rangle_U|m_+\rangle_L \rightarrow ia_{u+}^+|0\rangle|m_-\rangle_U|m_+\rangle_L. \quad (7d) \]

The total evolution of the system can be expressed as follow [24]:

\[ \hat{a}_{l+}^+|0\rangle(\alpha|m_+\rangle_U + \beta|m_-\rangle_U)(a|m_+\rangle_L + b|m_-\rangle_L) \rightarrow \frac{1}{\sqrt{2}}a|S\rangle_U|g\rangle_U(a|m_+\rangle_L + b|m_-\rangle_L) + \frac{i}{\sqrt{2}}\beta a|m_-\rangle_U|m_+\rangle_L \]

\[ + \frac{1}{2}a_{u+}^+|0\rangle(\beta a|m_-\rangle_U|m_+\rangle_L + \alpha b|m_+\rangle_U|m_-\rangle_L) \]

\[ + \frac{1}{2}a_{d+}^+|0\rangle(\beta a|m_-\rangle_U|m_+\rangle_L - \alpha b|m_+\rangle_U|m_-\rangle_L). \quad (8) \]

From the above result, we can get that the two ions will be left in three possible states corresponding to three measurement results on the two output ports respectively. If the \( D_i \) fires, we get two-ion entangled states: \( \beta a|m_-\rangle_U|m_+\rangle_L - \alpha b|m_+\rangle_U|m_-\rangle_L \). If we modulate the coefficients of the initial states to make \( \alpha, \alpha, \beta, \beta \) satisfy \( |\alpha| = |a| \) and \( |\beta| = |b| \), the two ions can be left in maximally entangled state \( \Psi_{UL} = \frac{1}{\sqrt{2}}(|m_-\rangle_U|m_+\rangle_L + |m_+\rangle_U|m_-\rangle_L) \) with probability \( P = \frac{1}{2}|a|^2(1 - |a|^2) \). From this analysis, we conclude that the two ions must be prepared in the same superposition state initially, then we can get the two-ion maximally entangled state. The successful probability is a function of the modulus of the initial state.

Here we only discussed the ideal case where the ion decays are coherent. In fact, the ion decays will be essentially incoherent and very few of the photons will be in the correct direction in the free space, so a resonant cavity must be introduced for each ion to achieve directional emission of the photons from each ion [22]. To simplify the figures, we do not draw these cavities in the generation setups. In addition, we have only discussed the ideal case where we suppose a photon impinging on an ion always leads to the process described by Eq.(4). But in most cases the photon will not be scattered by the ions. If the ions are placed inside theMZI, it would mean that detector \( D_u \) will most likely fire as before but without any entanglement between the ions being created. To enhance the scattering rate, an optical cavity will be added to the MZI. The added optical cavity enclosure cavity enhance the scattering rate and the efficiency of entanglement creation by a factor 4/3 [24].

Here, we only discussed the product initial states case. After analysis, we found that, if the initial state is a mixed state we also can generate the pure maximally entangled states. Suppose that the initial mixed state is in the following form [30, 31]:

\[ \rho_{UL} = F|\Psi^+\rangle_{UL}\langle \Psi^+| + (1 - F)|\Phi^+\rangle_{UL}\langle \Phi^+|. \quad (9) \]

where \( |\Psi^+\rangle_{UL} = \frac{1}{\sqrt{2}}(|m_+\rangle_U|m_-\rangle_L + |m_-\rangle_U|m_+\rangle_L) \) and \( |\Phi^+\rangle_{UL} = \frac{1}{\sqrt{2}}(|m_+\rangle_U|m_+\rangle_L + |m_-\rangle_U|m_-\rangle_L) \) are two Bell states of the two ions. To express the evolution clearly, we will consider the mixed state as the probabilistic mixture of pure two-ion entangled states, i.e. the state \( |\Psi^+\rangle_{UL} \) with probability \( F \) and the state \( |\Phi^+\rangle_{UL} \) with probability \( 1 - F \). In the \( |\Psi^+\rangle_{UL} \) case, if \( D_u \) fires the two ions will be left in \( \frac{1}{\sqrt{2}}(|m_+\rangle_U|m_-\rangle_L + |m_-\rangle_U|m_+\rangle_L) \) state. If \( D_l \) fires the two ions will collapse into \( \frac{1}{\sqrt{2}}(|m_-\rangle_U|m_+\rangle_L - |m_+\rangle_U|m_-\rangle_L) \) state. On the contrary, the \( |\Phi^+\rangle_{UL} \) case only leads to fire at \( D_u \) with the two ions in \( |m_-\rangle_U|m_-\rangle_L \) state.
If we detect a photon at $D_A$, the two ions will be left in a mixed state whose fidelity (with respect to $|\Psi^+\rangle_{UL}$) is lower than the initial one. So we consider this result as a garboge. If $D_I$ registers one photon, the two ions are left in a pure maximally entangled state $|\Psi\rangle_{UL} = \frac{1}{\sqrt{2}}(|m_−\rangle_U|m_+\rangle_L - |m_+\rangle_U|m_−\rangle_L)$ with probability $P' = \frac{F}{4}$. So long as the initial fidelity satisfies $F > \frac{1}{2}$, the successful probability of the mixed states case will be larger than the pure product states case. This point can be understood easily. The pure states case starts from a product state, but the mixed states case from a partially entangled state. Naturally, the probability of the later case is larger than the former one.

Compared to the previous generation schemes, our scheme has the following advantages:
1. The relative phase problem has been avoided successfully in our scheme by using MZI, and the relative phase in our scheme is adjusted to zero and will not change in the process. The common phase of the state takes no effect on the entanglement of the generated entangled states.
2. The photon we want to detect is the input circular polarized photon, which makes it easier to be registered than the scattered ones, because the input photon has a better directionality than the scattered one.
3. In the previous schemes, the BSA is a necessity, but in our scheme, only two ordinary BS are needed. So the current scheme is simpler than the previous ones.
4. The simultaneity of the two scattered photons is a main difficulty of the preceding schemes. But in our scheme, the simultaneity will be satisfied naturally because of the MZI.

4. Purification for unknown mixed ionic entangled states using linear optical elements

In this section, we will discuss the Purification for unknown mixed ionic entangled states. Here, the ionic system and the setup are all same to those developed in the preceding section.

Suppose that the mixed state to be purified is in the form:

$$\rho_{AB} = F|\Phi^\pm\rangle_{AB}\langle \Phi^\pm| + (1 - F)|\Psi^+\rangle_{AB}\langle \Psi^+|.$$ (10)

Because a general mixed state can be rotated into the form in equation (10), the discussion on the state in equation (10) applies to general mixed cases [30]. Further, to complete the purification scheme, we suppose that Alice and Bob have shared an ionic ensemble, each pair of which can be described by the state in equation (10). Here, $F = \langle \Phi^+ | \rho_{AB} | \Phi^+ \rangle$ is the fidelity of the pairs with respect to $|\Phi^+\rangle$.

Next, we will discuss the purification procedure in details [42]. To complete the purification process, we must carry out operations on two pairs of the ensemble. We denote the four ions of the two pairs as $1, 2$ and $3, 4$, and the total state of the two pairs before purification can be regarded as a probabilistic mixture of four pure states: $|\Phi^+\rangle_{12}|\Phi^+\rangle_{34}$ with probability $F^2$, $|\Phi^+\rangle_{12}|\Psi^+\rangle_{34}$ with probability $F(1 - F)$, $|\Psi^+\rangle_{12}|\Phi^+\rangle_{34}$ with probability $(1 - F)F$, and $|\Psi^+\rangle_{12}|\Psi^+\rangle_{34}$ with probability $(1 - F)^2$.

The main setup, depicted in Fig.4, are two Mach-Zehnder interferometers ($M_A$ and $M_B$) located at Alice and Bob’s side respectively. We suppose the input photon at Alice side is $\sigma^+$ polarized, and it is superimposed on $BS_{A1}$ at the left lower input port of $M_A$, and analogously for the description of Bob’s side. Then we can give the evolution of the four probabilistic pure states:

$$F^2 :$$

$$a_{i+}^+(0)Aa_{i+}^+(0)B|\Phi^+\rangle_{12}|\Phi^+\rangle_{34} \rightarrow -\frac{1}{8}(a_{i-}^{a+} + ia_{i+}^{a+})|0\rangle_A(a_{i+}^{a+} + ia_{i+}^{a+})|0\rangle_B \times |m_+\rangle_1|m_+\rangle_2|m_-\rangle_3|m_-\rangle_4$$

$$+ \frac{1}{8}(a_{i+}^{a+} + ia_{i+}^{a+})|0\rangle_A(a_{i-}^{a+} + ia_{i+}^{a+})|0\rangle_B \times |m_-\rangle_1|m_-\rangle_2|m_+\rangle_3|m_+\rangle_4$$

$$- \frac{1}{2}a_{i+}^{a+}|0\rangle_Aa_{i+}^{a+}|0\rangle_B|m_-\rangle_1|m_-\rangle_2|m_-\rangle_3|m_+\rangle_4 + \frac{\sqrt{10}}{4}|\text{Scatter}\rangle.$$ (11a)

$$F(1 - F) :$$

$$a_{i+}^+(0)Aa_{i-}^+(0)B|\Phi^+\rangle_{12}|\Psi^+\rangle_{34} \rightarrow \frac{i}{4}(a_{i+}^{a+} + ia_{i+}^{a+})|0\rangle_Aa_{i+}^{a+}|0\rangle_B|m_-\rangle_1|m_-\rangle_2|m_+\rangle_3|m_-\rangle_4$$

$$+ \frac{i}{4}a_{i-}^{a+}|0\rangle_A(a_{i+}^{a+} + ia_{i+}^{a+})|0\rangle_B|m_-\rangle_1|m_-\rangle_2|m_-\rangle_3|m_+\rangle_4 + \frac{\sqrt{3}}{2}|\text{Scatter}\rangle.$$ (11b)
Figure 4. The setup for purification scheme. Alice places two ions, 1,3 which are at her side on the two arms of $M_A$, ion 1 on upper path and ion 3 on lower one by using the trapping technology [43], and analogously for the two ions 2,4 at Bob’s side. One $\sigma^+$-polarized photon at each side will be superimposed on the first two BS ($BS_{A1}$ and $BS_{B1}$) of $M_A$ and $M_B$ respectively. After the first BS the photon will take two possible paths ($u$ denotes the upper path and $l$ denotes the lower one). Reflected by two mirrors, the two possible paths will re-combined at the second BS ($BS_{A2}$ and $BS_{B2}$). Because the two ions at one side are initially placed on the two optical pathes, the ions and the photon will interact. This interaction will generate a shift of the interference after the second BS ($BS_{A2}$ and $BS_{B2}$). Then through single photon measurement after the two second BS ($BS_{A2}$, $BS_{B2}$), Alice and Bob can compare their measurement results via classical communication. If the two lower output ports ($D_{A1}, D_{B1}$) all fire, the purification succeeds.

\[(1 - F)^2 :\]

\[a_{l,+}^+|0\rangle_A a_{l,+}^+|0\rangle_B |\Psi^+\rangle_{12} |\Phi^+\rangle_{34} \rightarrow \frac{i}{8}(a_{u,+}^+ + i a_{l,+}^+)|0\rangle_A (a_{u,+}^+ + i a_{u,+}^+)|0\rangle_B |m_+\rangle_1 |m_-\rangle_2 |m_-\rangle_3 |m_-\rangle_4 + \sqrt{\frac{3}{4}} |\text{Scatter}\rangle,\]  

\[(1 - F)^2 :\]

\[a_{l,+}^+|0\rangle_A a_{l,+}^+|0\rangle_B |\Psi^+\rangle_{12} |\Phi^+\rangle_{34} \rightarrow \frac{i}{8}(a_{u,+}^+ + i a_{l,+}^+)|0\rangle_A (a_{u,+}^+ + i a_{u,+}^+)|0\rangle_B |m_-\rangle_1 |m_-\rangle_2 |m_-\rangle_3 |m_-\rangle_4 + \sqrt{\frac{3}{4}} |\text{Scatter}\rangle.\]  

Where $|\text{Scatter}\rangle$ denotes the normalized vectors describing the state of the scattered photons, which can be filtered out from the detector. After evolution, Alice and Bob will operate single photon measurements at the lower and upper output ports of $M_A, M_B$ respectively. In this purification scheme, the first ($|\Phi^+\rangle_{12} |\Phi^+\rangle_{34}$) and the fourth ($|\Psi^+\rangle_{12} |\Psi^+\rangle_{34}$) cases will lead to the measurement result that the two lower output ports ($D_{A1}$ and $D_{B1}$) fire simultaneously, but the second ($|\Phi^+\rangle_{12} |\Psi^+\rangle_{34}$) and the third ($|\Psi^+\rangle_{12} |\Phi^+\rangle_{34}$) cases never lead to. From the evolution result, we get that if the two lower output ports ($D_{A1}$ and $D_{B1}$) fire simultaneously, Alice and Bob will get the four-ion maximally entangled state $\frac{1}{\sqrt{2}}(|m_+\rangle_1 |m_+\rangle_2 |m_-\rangle_3 |m_-\rangle_4 + |m_-\rangle_1 |m_-\rangle_2 |m_+\rangle_3 |m_+\rangle_4)$ with probability $\frac{F^2}{32}$, and get another four-ion maximally entangled state $\frac{1}{\sqrt{2}}(|m_+\rangle_1 |m_-\rangle_2 |m_-\rangle_3 |m_-\rangle_4 + |m_-\rangle_1 |m_+\rangle_2 |m_+\rangle_3 |m_-\rangle_4)$ with probability $\frac{(1-F)^2}{32}$. If Alice and Bob measure the ions 3 and 4 in the $|\pm\rangle$ basis, then
probability for obtaining two-ion maximally entangled state is still measurement in the basis 
\(|±⟩\). So the remaining ions will be left in the new states expressed by the new density operator:

\[ \rho_{12} = F′|Ψ^+⟩_{12}⟨Ψ^+| + (1 - F′)|Ψ^+⟩_{12}⟨Ψ^+. (12) \]

where, \( F′ = \frac{F^2}{F^2 + (1 - F^2)} \), is the new fidelity. If the fidelity of the initial shared entangled ensemble satisfies \( F > \frac{1}{2} \), \( F′ > F \), the initial entangled state is purified [30, 31]. Because \( F \) can be an arbitrary number between 0.5 and 1.0, the iteration of our scheme can extract a near-perfect maximally entangled state from the ensemble shared by Alice and Bob. Here concludes the discussion of the entanglement purification for mixed ion states.

5. Concentration for unknown pure non-maximally entangled ionic states using linear optical elements

We find that the above scheme can also be used to concentrate the non-maximally entangled pure states. We can suppose the non-maximally entangled pure state is in the form:

\[ |Ψ⟩_{AB} = a|m+_A⟩_A|m−_B⟩_B + b|m−_A⟩_A|m+_B⟩_B. (13) \]

where \(|a|^2 + |b|^2 = 1 \). Just like the mixed state case, two pairs of ions(1, 2 and 3, 4) will be placed on \( M_A, M_B. \) The evolution of the total state of the system can be expressed as:

\[ a_{i_A}^+(0)a_{i_B}^+(0)|Ψ⟩_{12}|Ψ⟩_{34} \rightarrow \frac{iab}{4}(a_{u_A}^+ + ia_{i_A}^+)|0⟩_A(a_{u_B}^+ + ia_{i_B}^+)|0⟩_B \times |m+_1⟩_1|m−_2⟩_2|m−_3⟩_3|m+_4⟩_4 \]

\[ + \frac{iab}{4}(a_{i_A}^+ + ia_{u_A}^+)|0⟩_A(a_{u_B}^+ + ia_{i_B}^+)|0⟩_B \times |m−_1⟩_1|m+_2⟩_2|m+_3⟩_3|m−_4⟩_4 + \sqrt{\frac{2 - |a|^2|b|^2}{2}}|Scatter⟩ (14) \]

After evolution, if the detectors \( D_{AI}D_{BI} \) fire, the four ions are left in maximally entangled state:

\[ \frac{1}{\sqrt{2}}(|m+_1⟩_1|m−_2⟩_2|m−_3⟩_3|m+_4⟩_4 + |m−_1⟩_1|m+_2⟩_2|m+_3⟩_3|m−_4⟩_4) \] with probability \( \frac{|a|^2|b|^2}{8} \). Although we probably can get the four-ion maximally entangled states corresponding to the measurement results: \( D_{Au}D_{Bu}, D_{Ai}D_{Bi}, D_{Au}D_{Bi}, D_{Ai}D_{Bu} \), we should omit these results for the reason that the fire at the upper output\( (D_{Au}, D_{Bu}) \) probably means the ions are not precisely placed on the optical pathes.

If the initial non-maximally entangled state is in following form: \( a|m+_A⟩_A|m+_B⟩_B + b|m−_A⟩_A|m−_B⟩_B, \) the concentration will similarly succeed, provided that the \( D_{Ai}D_{Bi} \) fire, and the successful probability is also \( \frac{|a|^2|b|^2}{8} \). After obtaining the four-ion maximally entangled states, Alice and Bob can make single ion measurement on ions 3, 4 in the basis \( |±⟩ \) just like in the mixed states case. Then the remaining ions 1, 2 will be left in two-ion maximally entangled state. From analysis, the successful probability for obtaining two-ion maximally entangled state is still \( \frac{|a|^2|b|^2}{8} \).

If we want to get four-ion maximally entangled states, there is no need for us to operate the ion measurement in the basis \( |±⟩ \). So our purification and concentration scheme can not only generate two-ion maximally entangled states but also can generate four-ion maximally entangled states. In this sense, the present scheme is more efficient than the previous scheme [30]. In Pan’s scheme, the four-photon entangled states can not be extracted, because the measurement on one pair of photons are needed to
complete the purification procedure, otherwise we cannot get to know whether the purification succeeds or not. While, in our scheme, after the single photon measurement, the purification process can concludes if we need four-ion maximally entangled states. Then the four-ion maximally entangled states can be used as a robust entanglement resource in Quantum Communication. That is to say, our scheme is a purification scheme without postselection measurement [44].

6. Discussions
After discussion on the generation and purification schemes, we will consider the feasibility of the current schemes. Singly positively charged alkaline ions, which have only one electron outside a closed shell, are commonly used in the quantum information experiments using trapped ions [45,46]. Here we discuss a possible implementation of our schemes using \textsuperscript{40}Ca\textsuperscript{+} as example. The relevant levels of \textsuperscript{40}Ca\textsuperscript{+} has been depicted in Fig. 5 [22].

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{levels.png}
\caption{Relevant levels of \textsuperscript{40}Ca ions [22].}
\end{figure}

\textit{D}_{5/2} and \textit{D}_{3/2} are two metastable levels of \textsuperscript{40}Ca\textsuperscript{+} with lifetimes of the order of 1s. \textit{s}_1 and \textit{s}_2 are two sublevels of \textit{D}_{5/2} with \textit{m} = -5/2 and \textit{m} = -1/2, and this two sublevels are coupled to |\textit{e}\rangle by \sigma_- and \sigma_+ light at 854nm. Here \textit{e}, \textit{s}_1, \textit{s}_2, \textit{s}_{1/2} correspond to \textit{e}, \textit{m} = -5/2, \textit{m} = -1/2, \textit{g} in Fig. 2, respectively, i.e. we use the \textit{S}_{1/2} as stable ground state, \textit{S}_1, \textit{S}_2 as two degenerate metastable state and \textit{P}_{3/2} as excited state. Arbitrary superposition state of this two degenerate metastable states can be prepared by applying a laser pulse of appropriate length, and this process can be realized in a few microsecond [47]. The \textsuperscript{40}Ca\textsuperscript{+} in state \textit{S}_1 or \textit{S}_2 can be excited into the excited state \textit{P}_{3/2} by applying one \sigma_- or \sigma_+ light at 854nm. Then decay from |\textit{e}\rangle to \textit{S}_1, \textit{S}_2, to \textit{D}_{3/2} and to \textit{S}_{1/2} are all possible. But the branching ratio for \textit{P}_{3/2} \rightarrow \textit{D}_{5/2}(854nm) versus \textit{P}_{3/2} \rightarrow \textit{S}_{1/2}(393nm) can be estimated as 1:30, giving 0.5 \times 10^7/s for the transition probability [22,46]. So in most case, the \textsuperscript{40}Ca\textsuperscript{+} in the excited state will decay into the stable ground state \textit{S}_{1/2}. The detection of the internal states of \textsuperscript{40}Ca\textsuperscript{+} can be realized by using a cycling transition between \textit{S}_{1/2} and \textit{P}_{3/2}(397nm) [45,46].

To achieve directional emission of the photons from the ions, we introduced an optical resonant cavity to surround each ion. We will introduce a cavity on the \textit{S}_{1/2} to \textit{P}_{3/2} transition to enhance the emission of the photons from atom transition \textit{P}_{3/2} to \textit{S}_{1/2}. Then the following two items will affect the emission efficiency of the photon from the ions: (1) The coupling between cavity mode and the \textit{P}_{3/2} \rightarrow \textit{S}_{1/2}(393nm) transition; (2) Decay from \textit{P}_{3/2} to \textit{D}_{5/2}; (3) Cavity decay. The
probability \( p_{\text{cav}} \) for a photon to be emitted into the cavity mode after excitation to \( |e\rangle \) can be expressed as
\[
p_{\text{cav}} = \frac{4\gamma \Omega^2}{(\gamma + 1)(\Omega^2 + 4\Omega^2)},
\]
where \( \gamma = 4\pi c / F_{\text{cav}} L \) is the decay rate of the cavity, \( F_{\text{cav}} \) its finesse, \( L \) its length, \( \Omega = D \sqrt{\frac{\hbar c^2 \epsilon_0}{\lambda V}} \) is the coupling constant between the transition and the cavity mode, \( D \) the dipole element, \( \lambda \) the wavelength of the transition, \( V \) the mode volume (which can be made as small as \( L^2 \lambda / 4 \) for a confocal cavity with waist \( \sqrt{L \lambda / \pi} \)), and \( \Gamma \) is the non-cavity related loss rate \([22, 48]\). From the discussion of Ref. [22], the photon package is about 100ns, and such a long coherence time makes it easy to achieve good overlap for the wave function of the photon on the beam splitter.

For example, when calculating the total efficiency of the generation scheme, we must consider the following items:

- The emission efficiency of photon: \( p_{\text{cav}} \), which has included the cavity decay; To maximize the \( p_{\text{cav}} \), we have chosen \( F_{\text{cav}} = 19000 \), \( L = 3 \text{mm} \). Then \( \gamma = 9.9 \times 10^6 / \text{s} \), \( p_{\text{cav}} = 0.01 \) \([22]\);
- The efficiency of the photon detectors is expressed as \( \eta \). Here we let the detection efficiency \( \eta = 0.7 \), which is a level that can be reached within the current technology.
- Coupling the photon out of the cavities will introduce another error \( \xi \), which can be modulated to be close to unit.

In addition, because the two ions have been placed on the MZI symmetrically, the different transition times for the ions and the consequent pulse broadening will affect the efficiency of the scheme slightly. To complete the generation scheme, we suppose that the state maker has held two ionic ensembles. After considering the above factors, the total success probability can be expressed as follow (considering the modified schemes as example):

- \( P = \frac{F}{3} \times p_{\text{cav}} \times \eta \) for mixed state, that is to say, if we input photon with the rate of 5000/s, we can get eight pairs of pure maximally entangled \(^{40}\text{Ca}^+\) ions per second for \( F = 0.7 \).
- \( P = \frac{2a^2(1-a^2)}{3} \times p_{\text{cav}} \times \eta \) for product initial states, that is to say, if we input photon with the rate of 5000/s, we can get five pairs of pure maximally entangled \(^{40}\text{Ca}^+\) ions per second for \( a^2 = 0.7 \).

And the same analysis applies to the purification scheme. From the experimental point of view, because the efficiency of the current schemes would be greatly enhanced if there were enough photons in the resonant system to induce stimulated emission from the ions, we will input enough photons into the MZI simultaneously. That is to say, the current scheme becomes more realizable.

In conclusion, we presented a simple generation scheme for the multi-cavity-mode W states in cavity QED. No detection on atom is needed in our scheme, and all the operations needed in this scheme are to modulate the interaction time only once. Then, we designed a setup using the combination of cavity QED and linear optical elements. This setup can realize the generation, concentration and purification of ionic entangled states. The feasibility of the setup is also discussed.

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