Abstract. We introduce three non-compact moduli stacks parametrizing noncommutative deformations of Hirzebruch surfaces; the first is the moduli stack of locally free sheaf bimodules of rank 2, which appears in the definition of noncommutative $\mathbb{P}^1$-bundle in the sense of Van den Bergh [VdB12], the second is the moduli stack of relations of a quiver in the sense of [AOU], and the third is the moduli stack of quadruples consisting of an elliptic curve and three line bundles on it. The main result of this paper shows that they are naturally birational to each other. We also give an Orlov-type semiorthogonal decomposition for noncommutative $\mathbb{P}^1$-bundles, an explicit classification of locally free sheaf bimodules of rank 2, and a noncommutative generalization of the (special) McKay correspondence as a derived equivalence for the cyclic group $\langle 1, \frac{1}{7}(1, 1) \rangle$.

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1. Introduction

A noncommutative $\mathbb{P}^1$-bundle over a smooth scheme $X$ is a Grothendieck category of the form $\text{Qgr}\mathcal{S}(\mathcal{E})$, where $\mathcal{S}(\mathcal{E})$ is the noncommutative symmetric algebra over a locally free sheaf bimodule $\mathcal{E}$ of rank 2 on $X$ [VdB12] (see Section 2 for a recap). One motivation to study such objects comes from Artin’s conjecture [Art97, Conjecture 4.1], which states that any noncommutative surface is birational to either

1. a noncommutative projective plane,
2. a noncommutative $\mathbb{P}^1$-bundle over a commutative curve, or
3. a noncommutative surface which is finite over its center.

If $X$ is a smooth projective curve of genus $g \geq 2$ and $\mathcal{E}$ is a stable vector bundle of rank 2 on $X$, then one can easily show $H^0(\mathbb{P}(\mathcal{E}), \mathcal{E} \otimes \mathcal{O}(1)) = 0$ and $H^2(\mathbb{P}(\mathcal{E}), \mathcal{O}(2)) = 0$, so that $\text{HH}^2(\mathbb{P}(\mathcal{E})) \approx H^1(\mathbb{P}(\mathcal{E}), \mathcal{O}(1))$ which means that every formal noncommutative deformation of $\mathbb{P}_X(\mathcal{E})$ is commutative.

In contrast, any ‘continuous’ noncommutative deformation of a Hirzebruch surface $\Sigma_d := \mathbb{P}_p(\mathcal{O}_p \oplus \mathcal{O}_p(d))$ is strictly noncommutative, since isomorphism classes of commutative Hirzebruch surfaces are parametrized by natural numbers, and hence discrete. It is shown in [VdB12, Theorem 1.3] that any noncommutative deformation, i.e., flat deformation of the abelian category of quasi-coherent sheaves in the sense of [LVdB06], of a commutative Hirzebruch surface over a complete Noetherian local ring is a noncommutative $\mathbb{P}^1$-bundle over $\mathbb{P}^1$.

For $d \in \mathbb{Z}_{\geq 0}$, the dimension of the Hochschild cohomology of $\Sigma_d$ is given by

\[
\dim \text{HH}^1(\Sigma_d) = h^0(\Theta_{\Sigma_d}) = \max\{d - 1, 0\} + 6, \quad (1.1) \\
\dim \text{HH}^2(\Sigma_d) = h^0(\mathcal{E} \otimes \mathcal{O}_{\Sigma_d}) + h^1(\Theta_{\Sigma_d}) = (\max\{d - 3, 0\} + 9) + \max\{d - 1, 0\}, \quad (1.2) \\
\dim \text{HH}^3(\Sigma_d) = h^2(\Theta_{\Sigma_d}) + h^1(\mathcal{E} \otimes \mathcal{O}_{\Sigma_d}) + \max\{d - 3, 0\}, \quad (1.3)
\]

so that the expected dimension of the moduli space of noncommutative Hirzebruch surfaces is

\[
- \dim \text{HH}^1(\Sigma_d) + \dim \text{HH}^2(\Sigma_d) - \dim \text{HH}^3(\Sigma_d) = 3. \quad (1.4)
\]

In contrast with the case of del Pezzo surfaces, noncommutative deformation of $\Sigma_d$ is obstructed when $d > 3$ by [Got16, Remark 5 and Proposition 9]. The non-vanishing of $h^1(\mathcal{E} \otimes \mathcal{O}_{\Sigma_d}) = h^1(\mathcal{O}_{\Sigma_d}(-K_{\Sigma_d}))$ also leads to the absence of a geometric helix (in the sense of [BS10]) of vector bundles.

The first result of this paper, given in Section 3, is a structure theorem and an explicit and complete classification of locally free sheaf bimodules of rank 2 on $\mathbb{P}^1$. We also study their Gieseker stability with respect to the anti-canonical bundle in Section 4. It turns out that many of them are semi-stable, which implies the unobstructedness of their deformations as explained in Proposition 1.7.

The second result of this paper, given in Section 5.1, is an Orlov-type semi-orthogonal decomposition of the bounded derived category $\text{D}^b\text{qgr}\mathcal{S}(\mathcal{E})$; the derived category decomposes into two copies of $\text{D}^b\text{coh} X$, and the sheaf bimodule $\mathcal{E}$ gives the integral kernel of the dual gluing functor. In Section 5.2, we show that a certain exceptional collection (5.9) on $\text{D}^b\text{qgr}\mathcal{S}(\mathcal{E})$ is strong, for an appropriate choice of a parameter $m$. This follows from the computation in Section 3 of the action of the dual gluing functor $\phi^!$ on the line bundles $\mathcal{O}_{\mathbb{P}_1}$ and $\mathcal{O}_{\mathbb{P}_1}(-1)$. In Corollary 5.3, we give the list of sheaf bimodules of degree either 2 or 1 for which the exceptional collection (5.9) with $m = 1$ is strong. The total morphism algebra of the collection is isomorphic to the quotient of the path algebra of
the quiver $Q^0$ or $Q^1$, shown in Figure 1.1 or Figure 1.2 respectively, by a two-sided ideal $I$ called relations of the quiver.

The relations $I$ depend on the isomorphism class of $E$, and it is natural to ask to which extent $E$ can be reconstructed from $I$. To study this kind of problems, we introduce three moduli stacks parametrizing noncommutative Hirzebruch surfaces in suitable senses.

The first moduli stack, introduced in Section 4.3, is the quotient

$$M_{\text{sh}} := \left[ \tilde{M}_{\text{sh}} / (\text{PGL}_2)^2 \right]$$

of the stack $\tilde{M}_{\text{sh}}$ of locally free sheaf bimodules of rank 2 on $\mathbb{P}^1$ by the natural action of the group $(\text{PGL}_2)^2 \simeq \text{Aut}(\mathbb{P}^1) \times \text{Aut}(\mathbb{P}^1)$. It is natural to take the quotient by this action, because it preserves the equivalence classes of the associated category $\text{Qgr} \mathcal{S}(E)$ as we recall in Section 2.5. The stack $\tilde{M}_{\text{sh}}$ is divided into connected components by the degree, but only the parity of the degree matters, since components whose degrees are different by a multiple of 2 are isomorphic. We write the connected component parametrizing sheaf bimodules of degree 2 (resp. 1) as $M_{\text{sh},0}$ (resp. $M_{\text{sh},1}$).

The second moduli stack, introduced in Section 5.3, is the moduli stack of relations of the quivers $Q^0$ and $Q^1$ in the sense of [AOU]. They will be denoted by $M_{\text{rel}},0$ and $M_{\text{rel}},1$, respectively.

The third moduli stack, introduced in Section 6.1, is the moduli stack of non-singular admissible quadruples, which parametrizes isomorphism classes of collections $(E, L_0, L_1, L_2)$ of an elliptic curve $E$ and three line bundles $(L_0, L_1, L_2)$ such that $L_0 \not\cong L_2$. We consider two connected components $M_{\text{ell},0}$ and $M_{\text{ell},1}$ such that $\deg(L_0, L_1, L_2) = (2, 2, 2)$ and $(2, 1, 2)$ respectively.

The third result of this paper is the comparison of these three moduli stacks:

**Theorem 1.1.** For each $i \in \{0, 1\}$, three moduli stacks $M_{\text{sh},i}$, $M_{\text{rel},i}$ and $M_{\text{ell},i}$ are naturally birational to each other. When $i = 0$, for a generic triple

$$(E, I \subset \mathbb{k}Q^0, (E, L_0, L_1, L_2))$$

which correspond to each other by the birational maps, there are derived equivalences

$$D^b \text{qgr} \mathcal{S}(E) \simeq D^b \text{mod} \mathbb{k}Q^0 / I \simeq D^b \text{qgr} A(E, L_0, L_1, L_2),$$

where $A(E, L_0, L_1, L_2)$ is the 3-dimensional cubic AS-regular $\mathbb{Z}$-algebra associated to the quadruple $(E, L_0, L_1, L_2)$ by the correspondence of [VdB11].
It is an interesting problem to find a complete moduli scheme (or a proper Deligne–Mumford stack) birational to the above moduli stacks.

Recall that \( \Sigma_d \) for \( d \geq 2 \) admits a birational morphism to the weighted projective plane \( \mathbb{P}(1,1,d) \) which contracts the negative section to a \( \frac{1}{d}(1,1) \)-singularity. As is well-known, there is a fully faithful functor \( \mathcal{D} \) which is a derived equivalence when \( d = 2 \). This phenomenon is a particular case of the (special) McKay correspondence as a derived equivalence. In [Ste00], Stephenson studied AS-regular algebras \( S \) generated by 3 elements of degree 1, \( d \). By taking the associated category \( \mathcal{Q} \), we obtain noncommutative \( \mathbb{P}(1,1,d) \). In Section 7, we consider one family of such algebras \( S \), and show the existence of sheaf bimodules \( \mathcal{E} \) and fully faithful functors \( \mathcal{D} : \mathcal{Q} \rightarrow \mathcal{D} \), which are equivalences if \( d = 2 \).

### Notation and Conventions

Throughout this paper, we fix an algebraically closed base field \( \mathbb{k} \) of characteristic zero. Schemes are defined over \( \mathbb{k} \), and categories and functors are linear over \( \mathbb{k} \). Triangulated categories and exact functors are enhanced in the sense of [BK90]. For a pair \((X,Y)\) of objects in an enhanced triangulated category \( \mathbb{D} \), the complex of \( \mathbb{k} \)-vector spaces underlying \( \text{Hom}_\mathbb{D}(X,Y) \) will be denoted by \( \text{hom}_\mathbb{D}(X,Y) \).

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### 2. Recapitulation on noncommutative \( \mathbb{P}^1 \)-bundle

#### 2.1. Sheaf bimodules

Let \( X \) and \( Y \) be schemes. A coherent sheaf \( \mathcal{E} \) on \( X \times Y \) is said to be a sheaf bimodule if the scheme-theoretic support

\[
W := \text{Spec}_{X \times Y} \text{Im} \left( \mathcal{O}_{X \times Y} \rightarrow \mathcal{E} \right)
\]

is finite over both \( X \) and \( Y \) [VdB96, Pat97b, Pat00]. The full subcategory of \( \text{coh}(X \times Y) \) consisting of sheaf bimodules is denoted by \( \text{shbimod}(X,Y) \). A sheaf bimodule \( \mathcal{E} \) defines a right exact functor

\[
(-) \otimes_{\mathcal{O}_X} \mathcal{E} : \text{coh} X \rightarrow \text{coh} Y, \quad \mathcal{F} \mapsto p_Y^* p_X^* \mathcal{F} \otimes_{\mathcal{O}_{X \times Y}} \mathcal{E},
\]

where \( p_X : X \times Y \rightarrow X \) and \( p_Y : X \times Y \rightarrow Y \) are the projections. Let \( u \) and \( v \) be the restrictions of \( p_X \) and \( p_Y \) to \( W \) respectively, so that the inclusion is given by \( \iota = u \times v : W \rightarrow X \times Y; \)

\[
\begin{array}{ccc}
W & \xrightarrow{\iota} & X \times Y \\
\uparrow & & \downarrow p_X \quad p_Y \\
X & & Y
\end{array}
\]
Let $\mathcal{U}$ be the unique coherent $\mathcal{O}_W$-module such that $\iota_*\mathcal{U} = \mathcal{E}$. Then the functor (2.2) can be rewritten as

$$(-) \otimes_{\mathcal{O}_X} \mathcal{E} \simeq \iota_* (u^*(-) \otimes_{\mathcal{O}_W} \mathcal{U}) : \text{coh} X \to \text{coh} Y. \quad (2.4)$$

The convolution of sheaf bimodules $\mathcal{E} \in \text{shbimod}(X, Y)$ and $\mathcal{F} \in \text{shbimod}(Y, Z)$ is defined by

$$\mathcal{E} \otimes_{\mathcal{O}_Y} \mathcal{F} := p_{XZ*} (p_{XY}^* \mathcal{E} \otimes_{\mathcal{O}_{X \times Y \times Z}} p_{YZ}^* \mathcal{F}) \in \text{shbimod}(X, Z), \quad (2.5)$$

where $p_{\ast}$ denotes the projection from $X \times Y \times Z$ to $\bullet \times \circ$. It is clear that there exists an isomorphism

$$( (-) \otimes_{\mathcal{O}_X} \mathcal{E} ) \otimes_{\mathcal{O}_Y} \mathcal{F} \simeq (-) \otimes_{\mathcal{O}_X} (\mathcal{E} \otimes_{\mathcal{O}_Y} \mathcal{F}) \quad (2.6)$$

of functors from $\text{coh} X$ to $\text{coh} Z$.

A sheaf bimodule $\mathcal{E}$ is said to be locally free of rank $r$ if both $p_X^* \mathcal{E}$ and $p_Y^* \mathcal{E}$ are locally free of rank $r$ [VdB12, Definition 3.1.3]. The structure sheaf $\mathcal{O}_{\Delta_X}$ of the diagonal $\Delta_X \subset X \times X$ is called the diagonal sheaf bimodule. It is locally free of rank one, and the corresponding functor $(-) \otimes_{\mathcal{O}_X} \mathcal{O}_{\Delta_X} : \text{coh} X \to \text{coh} X$ is isomorphic to the identity functor.

The locally-freeness of a sheaf bimodule implies that the integral transformation is defined on the underived level.

**Lemma 2.1.** Suppose that a sheaf bimodule $\mathcal{E}$ is locally free. Then the functor (2.4) is exact.

The functor (2.4) extends to an exact functor of triangulated categories from $D^b \text{coh} X$ to $D^b \text{coh} Y$, which we write $(-) \otimes_{\mathcal{O}_X} \mathcal{E}$ again by an abuse of notation.

It is known [VdB12, Section 3] that if $\mathcal{E}$ is locally free, then the functor (2.2) admits the left and the right adjoint functors. They are given by $(-) \otimes_{\mathcal{O}_Y} \ast \mathcal{E} : \text{coh} Y \to \text{coh} X$ and $(-) \otimes_{\mathcal{O}_Y} \mathcal{E}^* : \text{coh} Y \to \text{coh} X$ where

$$\mathcal{E}^* := p_Y^* \omega_Y^{-1} \otimes_{\mathcal{O}_{Y \times X}} \mathcal{E}^D, \quad (2.7)$$

$$\ast \mathcal{E} := \mathcal{E}^D \otimes_{\mathcal{O}_{Y \times X}} p_X^* \omega_X^{-1}, \quad (2.8)$$

and

$$\mathcal{E}^D := \mathcal{H} \text{om}^{\text{codim}}(\mathcal{E}, \omega_{X \times Y}) = \mathcal{H} \text{om}^{\text{op}}_{\mathcal{O}_{X \times Y}}(\mathcal{E}, \omega_{X \times Y}) \quad (2.9)$$

is the Cohen-Macaulay dual of $\mathcal{E}$ [HL10, Definition 1.1.7].

### 2.2 Sheaf $\mathbb{Z}$-algebras

A sheaf $\mathbb{Z}$-algebra over a scheme $X$ is a category $\mathcal{C}$ enriched over the monoidal category $(\text{shbimod}(X, X), - \otimes_{\mathcal{O}_X} -)$ equipped with a bijection $\mathbb{Z} \to \text{Obj} \mathcal{C}$. Concretely, a sheaf $\mathbb{Z}$-algebra

$$\mathcal{A} = \text{Alg}(\mathcal{C}) = \left( (\mathcal{A}_{ij})_{i,j \in \mathbb{Z}}, (\eta_i)_{i \in \mathbb{Z}}, (\mu_{ijk})_{i,j,k \in \mathbb{Z}} \right) \quad (2.10)$$

on a scheme $X$ consists of

- sheaf bimodules $\mathcal{A}_{ij} \in \text{shbimod}(X, X)$,
- morphisms $\eta_i : \mathcal{O}_{\Delta_X} \to \mathcal{A}_{ii}$ of sheaf bimodules called the units, and
- morphisms $\mu_{ijk} : \mathcal{A}_{ij} \otimes_{\mathcal{O}_X} \mathcal{A}_{jk} \to \mathcal{A}_{ik}$ called multiplication maps

such that

- the compositions

$$\mathcal{A}_{ij} \cong \mathcal{A}_{ij} \otimes_{\mathcal{O}_X} \mathcal{O}_{\Delta_X} \xrightarrow{id \otimes \eta_i} \mathcal{A}_{ij} \otimes_{\mathcal{O}_X} \mathcal{A}_{jj} \xrightarrow{\mu_{ijj}} \mathcal{A}_{ij} \quad (2.11)$$


are the identity morphisms, and
• the diagrams
\[
\begin{align*}
A_{ij} \otimes_{\mathcal{O}_X} A_{jk} \otimes_{\mathcal{O}_X} A_{kl} & \xrightarrow{\mu_{ijk} \otimes \text{id}} A_{ik} \otimes_{\mathcal{O}_X} A_{kl} \\
\text{id} \otimes \mu_{jkl} & \
A_{ij} \otimes_{\mathcal{O}_X} A_{jl} & \xrightarrow{\mu_{ijl}} A_{il}
\end{align*}
\]
are commutative.

A (right) module \( \mathcal{M} = ((\mathcal{M}_i)_{i \in \mathbb{Z}}, (h_{ij})_{i,j \in \mathbb{Z}}) \) over a sheaf \( \mathbb{Z} \)-algebra \( \mathcal{A} \) on a scheme \( X \) consists of
• \( \mathcal{O}_X \)-modules \( \mathcal{M}_i \), and
• morphisms \( h_{ij} : \mathcal{M}_i \otimes_{\mathcal{O}_X} \mathcal{A}_{ij} \to \mathcal{M}_j \) of \( \mathcal{O}_X \)-modules called the action
such that
• the composition
\[
\mathcal{M}_i \xrightarrow{\mathcal{M}_i} \mathcal{M}_i \otimes_{\mathcal{O}_X} \mathcal{O}_X \xrightarrow{\text{id} \otimes \eta_i} \mathcal{M}_i \otimes_{\mathcal{O}_X} \mathcal{A}_{ii} \xrightarrow{h_{ii}} \mathcal{M}_i
\]
(2.14)
is the identity morphism, and
• the diagrams
\[
\begin{align*}
\mathcal{M}_i \otimes_{\mathcal{O}_X} \mathcal{A}_{ij} \otimes_{\mathcal{O}_X} \mathcal{A}_{jk} & \xrightarrow{h_{ij} \otimes \text{id}} \mathcal{M}_j \otimes_{\mathcal{O}_X} \mathcal{A}_{jk} \\
\text{id} \otimes \mu_{ijk} & \\
\mathcal{M}_i \otimes_{\mathcal{O}_X} \mathcal{A}_{ik} & \xrightarrow{h_{ik}} \mathcal{M}_k
\end{align*}
\]
(2.15)
are commutative.

A morphism \( f = (f_i)_{i \in \mathbb{Z}} : \mathcal{M} \to \mathcal{N} \) of \( \mathcal{A} \)-modules consists of morphisms \( f_i : \mathcal{M}_i \to \mathcal{N}_i \) of \( \mathcal{O}_X \)-modules such that the diagrams
\[
\begin{align*}
\mathcal{M}_i \otimes_{\mathcal{O}_X} \mathcal{A}_{ij} & \xrightarrow{f_i \otimes \text{id}} \mathcal{N}_i \otimes_{\mathcal{O}_X} \mathcal{A}_{ij} \\
h_{ij}^M & \\
\mathcal{M}_j & \xrightarrow{f_j} \mathcal{N}_j
\end{align*}
\]
(2.16)
are commutative. The category of \( \mathcal{A} \)-modules is denoted by \( \text{Gr} \mathcal{A} \).

An \( \mathcal{A} \)-module is right-bounded if \( \mathcal{M}_i \simeq 0 \) for \( i \gg 0 \). An \( \mathcal{A} \)-module is torsion if it is a direct limit of right-bounded objects. The full subcategory of \( \text{Gr} \mathcal{A} \) consisting of torsion modules is denoted by \( \text{Tors} \mathcal{A} \). A Grothendieck category is locally Noetherian if it has a small generating family of Noetherian objects. In the rest of the paper we will assume that \( \mathcal{A} \) is right Noetherian and positively graded in the sense that \( \mathcal{A}_{ij} = 0 \) for \( i > j \), so that the category \( \text{Gr} \mathcal{A} \) is locally Noetherian and \( \text{Tors} \mathcal{A} \subset \text{Gr} \mathcal{A} \) is a localizing subcategory. The quotient abelian category is denoted by \( \text{Qgr} \mathcal{A} := \text{Gr} \mathcal{A} / \text{Tors} \mathcal{A} \). The torsion functor \( \tau : \text{Gr} \mathcal{A} \to \text{Tors} \mathcal{A} \), the quotient functor \( \pi : \text{Gr} \mathcal{A} \to \text{Qgr} \mathcal{A} \), and its right adjoint \( \omega : \text{Qgr} \mathcal{A} \to \text{Gr} \mathcal{A} \) are defined as in [AZ94]. In this case, the image of the unit \( \eta_i \) will be denoted by \( e_i \), which is an \( \mathcal{O}_{X \times X} \)-submodule of \( \mathcal{A}_{ii} \). It defines the functor
\[
(-) \otimes_{\mathcal{O}_X} e_i \mathcal{A} : \text{Qcoh} X \to \text{Gr} \mathcal{A}.
\]
(2.17)
2.3. Noncommutative symmetric algebras. Let $\mathcal{E}$ be a locally free sheaf bimodule of rank 2 on a smooth scheme $X$. We define a sequence $(\mathcal{E}^i)_{i \in \mathbb{Z}}$ of sheaf bimodules of rank 2 on $X$ inductively by

$$
\mathcal{E}^i := \begin{cases} 
(\mathcal{E}^{(i-1)*}) & i \geq 1, \\
\mathcal{E} & i = 0, \\
\mathcal{E}^{(i+1)*} & i \leq -1.
\end{cases}
$$

Let

$$i_n : \mathcal{O}_X \to \mathcal{E}^{i_n} \otimes \mathcal{O}_X \mathcal{E}^{(n+1)}$$

be the canonical morphism coming from the adjunction

$$\text{Hom}(\mathcal{E}^{*n}, \mathcal{E}^{*n}) \simeq \text{Hom}(\mathcal{O}_X, \mathcal{E}^{*n} \otimes \mathcal{O}_X \mathcal{E}^{(n+1)})$$

in shbimod($X, X$), and $Q_n \subset \mathcal{E}^{*n} \otimes \mathcal{O}_X \mathcal{E}^{(n+1)}$ be the image of $i_n$. The sheaf bimodule $Q_n$ is invertible since $i_n$ is injective by [VdB12, Proposition 3.1.10]. The noncommutative symmetric algebra is the sheaf $\mathbb{Z}$-algebra $S(\mathcal{E})$ on $X$ generated by $\mathcal{E}^i$ subject to the relations $Q_i$. To be more explicit, it is a sheaf $\mathbb{Z}$-algebra with

$$S(\mathcal{E})_{ij} = \begin{cases} 
0 & i > j, \\
\mathcal{O}_X & i = j, \\
\mathcal{E}^{i_j} \otimes \mathcal{O}_X \cdots \otimes \mathcal{O}_X \mathcal{E}^{(j-1)} / \mathcal{R}_{ij} & j > i + 1,
\end{cases}
$$

where

$$\mathcal{R}_{ij} := \sum_{k=i}^{j-2} \mathcal{E}^{i_k} \otimes \mathcal{O}_X \cdots \otimes \mathcal{O}_X \mathcal{E}^{(k-1)} \otimes \mathcal{O}_X Q_k \otimes \mathcal{O}_X \mathcal{E}^{(k+2)} \otimes \mathcal{O}_X \cdots \otimes \mathcal{O}_X \mathcal{E}^{(j-1)}.
$$

2.4. Noncommutative $\mathbb{P}^1$-bundles over commutative schemes. Recall from [VdB01, Section 3.5] that a quasi-scheme $X$ is a symbol with which one associates a Grothendieck category $\text{Mod} X$, and an enriched quasi-scheme is a pair $(X, \mathcal{O}_X)$ of a quasi-scheme $X$ and an object $\mathcal{O}_X$ of $\text{Mod} X$.

Let $\mathcal{E}$ be a locally free sheaf bimodule of rank 2 on a smooth scheme $X$. The noncommutative $\mathbb{P}^1$-bundle $\mathbb{P}(\mathcal{E})$ in the sense of [VdB12] is the quasi-scheme with

$$\text{Mod} \mathbb{P}(\mathcal{E}) := \text{Qgr} S(\mathcal{E}).
$$

It is naturally enriched by

$$\mathcal{O}_{\mathbb{P}(\mathcal{E})} := \pi (\mathcal{O}_X \otimes \mathcal{O}_X e_0 S(\mathcal{E})).
$$

The category $\text{Gr} S(\mathcal{E})$ and hence $\text{Qgr} S(\mathcal{E})$ is locally Noetherian by [VdB12, Theorem 1.2]. The full subcategory of $\text{Qgr} A$ consisting of Noetherian objects will be denoted by $\text{qgr} A$. The exact functor

$$f_n^* := \pi (-) \otimes \mathcal{O}_X e_n A : \text{Qcoh} X \to \text{Qgr} A
$$

induces a functor from $\text{coh} X$ to $\text{qgr} A$ by [Nym05, Proposition 2.16], which will be denoted by $f_n^*$ again by an abuse of notation. The images of $\text{coh} X$ by $f_n^*$ for all $n \in \mathbb{Z}$ together generates $\text{Qgr} S(\mathcal{E})$ by [Nym05, Proposition 2.19]. The functor

$$f_n := \omega (-)_n : \text{Qgr} A \to \text{Qcoh} X.
$$
is right adjoint to $f_n^*$ by [Nym05, Lemma 2.15]. It is left exact since it has a left adjoint. Its derived functor, which exists since $Qgr\mathcal{A}$ is a Grothendieck category, is denoted by $Rf_n^*$. It satisfies

$$R\text{Hom}(f_n^*\mathcal{F}, \mathcal{M}) \simeq R\text{Hom}(\mathcal{F}, Rf_n^*\mathcal{M})$$

(2.27)

by [Mor07, Lemma 4.2]. For any locally free $\mathcal{O}_X$-module $\mathcal{F}$, one has

$$Rf_n^* (f_n^*\mathcal{F}) \simeq \begin{cases} \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{A}_{n,m} & n \leq m, \\ 0 & n = m + 1, \\ \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{Q}^*_{n-2} \otimes_{\mathcal{O}_X} (\mathcal{A}_{m,n-2})^*[-1] & n \geq m + 2 \end{cases}$$

(2.28)

by [Mor07, Lemma 4.4]. It follows that

$$Rf_n^* \circ f_n^* \simeq (-) \otimes_{\mathcal{O}_X} \mathcal{O}_{\Delta X} \simeq \text{id}: D^b\text{coh} X \to D^b\text{coh} X$$

(2.29)

for any $n \in \mathbb{Z}$, so that $f_n^*$ is full and faithful.

Recall from [BK89] that a Serre functor in a Hom-finite $k$-linear category $D$ is an additive equivalence $S: D \to D$ with bi-functorial isomorphisms $\phi_{A,B}: \text{Hom}(A,B) \xrightarrow{\sim} \text{Hom}(B,S(A))^*$. The category $D^b\text{qgr} S(E)$ has a Serre functor by [Nym05, CN13].

2.5. Invariance of the module categories under the action of invertible sheaf bimodules. Let $\mathcal{A} = (A_{ij})_{i,j \in \mathbb{Z}}$ be a sheaf $\mathbb{Z}$-algebra on $X$, and $\mathcal{T} = (T_i)_{i \in \mathbb{Z}}$ be a sequence of invertible objects in $\text{shbimod}(X,X)$. The twist of $\mathcal{A}$ by $\mathcal{T}$ is defined in [VdB12, Section 3.2] as the sheaf $\mathbb{Z}$-algebra $B := A\mathcal{T}$ given by

$$B_{ij} := T_i^{-1} \otimes_{\mathcal{O}_X} A_{ij} \otimes_{\mathcal{O}_X} T_j$$

(2.30)

with the obvious multiplications. Then the functor

$$(\mathcal{M}_i)_{i \in \mathbb{Z}} \mapsto (\mathcal{M}_i \otimes_{\mathcal{O}_X} T_i)_{i \in \mathbb{Z}}$$

(2.31)

defines an equivalence $\text{Gr}(\mathcal{A}) \xrightarrow{\sim} \text{Gr}(\mathcal{B})$, which descends to an equivalence $\text{Qgr}(\mathcal{A}) \xrightarrow{\sim} \text{Qgr}(\mathcal{B})$.

Let $\mathcal{E}$ be a locally free sheaf bimodule. The definition

$$\text{Hom}(\mathcal{M} \otimes_{\mathcal{O}_X} \mathcal{E}, \mathcal{N}) \simeq \text{Hom}(\mathcal{M}, \mathcal{N} \otimes_{\mathcal{O}_X} \mathcal{E}^*)$$

(2.32)

of the adjunction implies

$$(T_0^{-1} \otimes_{\mathcal{O}_X} \mathcal{E} \otimes_{\mathcal{O}_X} T_1)^* \simeq T_1^{-1} \otimes_{\mathcal{O}_X} \mathcal{E}^* \otimes_{\mathcal{O}_X} T_0$$

(2.33)

for any pair $(T_0, T_1)$ of invertible sheaf bimodules. It follows that

$$(T_0^{-1} \otimes_{\mathcal{O}_X} \mathcal{E} \otimes_{\mathcal{O}_X} T_1)^{sm} \simeq \begin{cases} T_0^{-1} \otimes_{\mathcal{O}_X} \mathcal{E}^{sm} \otimes_{\mathcal{O}_X} T_1 & m: \text{even}, \\ T_1^{-1} \otimes_{\mathcal{O}_X} \mathcal{E}^{sm} \otimes_{\mathcal{O}_X} T_0 & m: \text{odd} \end{cases}$$

(2.34)

so that

$$S(T_0^{-1} \otimes_{\mathcal{O}_X} \mathcal{E} \otimes_{\mathcal{O}_X} T_1) \simeq S(\mathcal{E})_{\mathcal{T}},$$

(2.35)

where $\mathcal{T} = (T_i)_{i \in \mathbb{Z}}$ is defined by

$$T_i := \begin{cases} T_0 & i: \text{even}, \\ T_1 & i: \text{odd} \end{cases}$$

(2.36)

(see also [Mor09, p. 136]).
Example 2.2. (1) An example of an invertible sheaf bimodule is given by the structure sheaf $\mathcal{O}_Y$ of the graph
\[ \Gamma_g := \{(x, gx) \in X \times X \mid x \in X\} \] (2.37)
of $g \in \text{Aut} X$. One can easily check $(-) \otimes_{\mathcal{O}_X} \iota_* \mathcal{O}_{\Gamma_g}(-) \simeq g_*$. For any pair $(g, h)$ of automorphisms on $X$, consider the pair of invertible sheaf bimodules $(\mathcal{T}_0, \mathcal{T}_1) = (\mathcal{O}_Y, \mathcal{O}_Y)$. One can easily check
\[ \mathcal{T}_0^{-1} \otimes_{\mathcal{O}_X} \mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{T}_1 \simeq (g \times h)_* \mathcal{E}. \] (2.38)
By the arguments above, this sheaf bimodule gives rise to the equivalent category as $\mathcal{E}$ does. In short, $\text{Aut} X \times \text{Aut} X$ acts on the space of sheaf bimodules preserving the equivalence classes of the resulting abelian categories $\text{qgr} \mathcal{S}(\mathcal{E})$.

(2) Another example is given by the line bundle $\Delta_* L$ on the diagonal of $X \times X$. Consider the pair of invertible sheaf bimodules $(\mathcal{T}_0, \mathcal{T}_1) = (\Delta_* L, \Delta_* M)$. By standard arguments one can verify
\[ \mathcal{T}_0^{-1} \otimes_{\mathcal{O}_X} \mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{T}_1 \simeq \mathcal{E} \otimes_{\mathcal{O}_{X \times X}} \left( L^{-1} \boxtimes M \right). \] (2.39)
In particular, when $X = \mathbb{P}^1$ and $(L, M) = (\mathcal{O}_{\mathbb{P}^1}(a), \mathcal{O}_{\mathbb{P}^1}(b))$, then
\[ \mathcal{T}_0^{-1} \otimes_{\mathcal{O}_1} \mathcal{E} \otimes_{\mathcal{O}_1} \mathcal{T}_1 \simeq \mathcal{E} \otimes_{\mathcal{O}_{1 \times 1}} \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(a \otimes b). \] (2.40)
Note that the line bundle $\mathcal{L} := u^* \mathcal{O}_{\mathbb{P}^1}(-1) \otimes v^* \mathcal{O}_{\mathbb{P}^1}(1)$ is non-trivial of degree zero on $W$, unless $W_{\text{red}}$ is a divisor of bidegree $(1, 1)$. This implies that there exists the action $\mathcal{E} = \iota_* \mathcal{U} \mapsto \iota_* (\mathcal{U} \otimes \mathcal{L})$ on the space of sheaf bimodules of fixed degree. This is an effective action of $\mathbb{Z}$ if $\mathcal{L} \in \text{Pic}^0(W)$ is not a torsion point.

Combining these examples with the isomorphism $\text{Auteq} (\text{coh} X) \simeq \text{Pic} (X) \rtimes \text{Aut} X$, it follows that the natural action of the group $\text{Auteq} (\text{coh} X) \times \text{Auteq} (\text{coh} X)$ on sheaf bimodules preserves the equivalence classes of the associated categories $\text{Gr} \mathcal{S}(-)$ and $\text{qgr} \mathcal{S}(-)$.

3. Explicit classification of sheaf bimodules on $\mathbb{P}^1$

In this section, we give an explicit classification of locally free sheaf bimodules of rank 2 on $\mathbb{P}^1$. The first part of this section has overlap with [Pat97a, Chapter 4], especially with [Pat97a, Theorem 4.5]. A difference is in that our argument is more abstract and based on the Serre’s conditions.

Also we explicitly compute the values of $a, b, a', b'$ which are defined by $\mathcal{O}_{\mathbb{P}^1} \otimes \mathcal{E} = v_* \mathcal{U} = \mathcal{O}_{\mathbb{P}^1}(a) \oplus \mathcal{O}_{\mathbb{P}^1}(b)$ and $\mathcal{O}_{\mathbb{P}^1}(-1) \otimes \mathcal{E} = \mathcal{O}_{\mathbb{P}^1}(a') \oplus \mathcal{O}_{\mathbb{P}^1}(b')$. This will later be used to check the strongness of certain exceptional collections on $\mathbb{D}^b \text{qgr} \mathcal{S}(\mathcal{E})$ and compute its endomorphism algebra.

Let $\mathcal{E}$ be a locally free sheaf bimodule on smooth projective schemes $X$ and $Y$ of the same dimension $n$. Since $u_* \mathcal{U}$ is locally free and $X$ is smooth over a field, $u_* \mathcal{U}$ is maximally Cohen-Macaulay. Since $u$ is finite, this implies that $\mathcal{U}$ is also maximally Cohen-Macaulay over $W$ (this argument is quoted from the proof of [VdB12, Proposition 3.1.6]) and $\mathcal{E} = \iota_* \mathcal{U}$ is a (non-maximal) Cohen-Macaulay module over $X \times Y$. Hence by [HL10, Proposition 1.1.10], the sheaf $\mathcal{E}$ is pure, i.e., all associated points of $\mathcal{E}$ have the same dimension; equivalently, non-trivial subsheaves of $\mathcal{E}$ have the same dimension as $\mathcal{E}$ (see [HL10, p. 3]). It follows that $W$ is an equi-$n$-dimensional scheme and the restriction of $u: W \to X$ to any irreducible component of $W$ is dominant and finite. Note that here and below, any claim on $u$ also holds for $v$ by symmetry.

Lemma 3.1 below shows that the scheme $W$ satisfies the $S_1$ condition of Serre.
Lemma 3.1. Let $A$ be a Noetherian ring and $M$ a finitely generated $A$-module which satisfies the $S_1$ condition and $\text{ann}_A(M) = 0$. Then $A$ also satisfies the $S_1$ condition. In particular, $A$ is pure as a module over $A$.

Proof. All the properties discussed here are local, so we may assume without loss of generality that $A = (A, \mathfrak{m})$ is a local ring. Since $M$ satisfies the $S_1$ condition, there exists a non-zero element $a \in \mathfrak{m}$ such that $M \to M$ is injective. Then the assumptions imply that $A \to A$ is also injective. Thus we obtain the first claim.

The second claim simply follows from the fact that the condition $S_1$ is equivalent to the purity; this is essentially stated in [HL10, Proposition 1.1.10] modulo the simple observation that a coherent sheaf on a smooth projective scheme whose support is of codimension $c$ satisfies the $S_{k,c}$ condition precisely when it satisfies $S_k = S_{k,0}$ on its support. □

Since the reducedness of a scheme is equivalent to $S_1 + R_0$ (= regular in codimension 0), we see that $W$ is reduced if and only if it is reduced at the generic point of any irreducible component.

Proposition 3.2. If the rank of $E$ is 2, then one of the following cases occur.

(1) $W$ is an irreducible and reduced ($\iff$ integral) scheme. Then either
(a) $u$ is birational. In this case, since $X$ is normal, $u$ is an isomorphism. In particular, $U$ is a locally free sheaf of rank 2 on $X \cong W$, or
(b) $u$ is of degree 2 and $U$ is a pure sheaf of rank 1 on $W$. Then by the assumption $\text{rank}(v_*U) = 2$, the degree of $v$ has to be 2 as well.

(2) $W$ is irreducible and not reduced. In this case, by the arguments above, $W$ is not reduced at the generic point. Therefore $W_{\text{red}} \to X$ and $W_{\text{red}} \to Y$ are both birational and finite, and hence are isomorphisms. The sheaf $U$ is pure, and isomorphic to $\mathcal{O}_W$ on a non-empty Zariski open subset of $W$.

(3) $W$ is not irreducible. In this case, $W$ is reduced and admits exactly 2 irreducible components. $u$ sends each component of $W$ onto $X$ (so does $v$ onto $Y$). The sheaf $U$ is pure and of rank 1 on each irreducible component.

Let us assume that $X$ and $Y$ are smooth projective curves. As shown in Lemma 3.1 there exists no embedded point in $W$. We also checked that any irreducible component of $W$ dominates the generic point of $X$. Therefore [Har77, Chapter III, Proposition 9.7] implies:

Corollary 3.3. $u: W \to X$ and $v: W \to Y$ are flat.

One can also show:

Corollary 3.4. $W$ is isomorphic to a Cartier divisor of $X \times Y$.

Proof. Consider the standard short exact sequence
$$0 \to I_W \to \mathcal{O}_{X \times Y} \to \iota_* \mathcal{O}_W \to 0.$$ (3.1)

For any coherent sheaf $E$ on $X \times Y$, there exists the canonical map to the double dual $\theta_E: E \to E^{DD}$, (3.2)

where $(-)^D$ is the Cohen-Macaulay dual as defined in (2.9). Then $E$ is said to be reflexive if $\theta_E$ is an isomorphism. Note that $W$ is a Cartier divisor if and only if $I_W$ is an invertible sheaf. This is, in turn, equivalent to the condition that $I_W$ is reflexive in the above sense.
Apply the functor $\mathcal{E}xt_{X \times Y}^1(-, \omega_{X \times Y})$ to (3.1) to obtain the following long exact sequence:

$$
\mathcal{E}xt_{X \times Y}^1(\iota_* \mathcal{O}_W, \omega_{X \times Y}) \to \mathcal{E}xt_{X \times Y}^1(\mathcal{O}_X \otimes \mathcal{O}_{X \times Y}, \omega_{X \times Y}) = 0 \to \mathcal{E}xt_{X \times Y}^1(I_W, \omega_{X \times Y})
$$

$$
\to \mathcal{E}xt_{X \times Y}^2(\iota_* \mathcal{O}_W, \omega_{X \times Y}) \to \mathcal{E}xt_{X \times Y}^2(\mathcal{O}_X \otimes \mathcal{O}_{X \times Y}, \omega_{X \times Y}) = 0 \to \mathcal{E}xt_{X \times Y}^2(I_W, \omega_{X \times Y})
$$

(3.3)

$$\to 0.$$  

Since $\iota_* \mathcal{O}_W$ satisfies $S_{1,1}$, $\mathcal{E}xt_{X \times Y}^2(\iota_* \mathcal{O}_W, \omega_{X \times Y}) = 0$ by [HL10 Proposition 1.1.10]. Hence $\mathcal{E}xt_{X \times Y}^q(I_W, \omega_{X \times Y}) = 0$ for $q = 1, 2$, which in turn implies $I_W$ is reflexive again by [HL10 Proposition 1.1.10].

From now on, we will restrict ourselves to the case when $X = Y = \mathbb{P}^1$ and $\mathcal{E}$ is a locally free sheaf bimodule of rank 2. As we proved in Proposition 3.2, such a sheaf bimodule is either of the following two types (we borrow the labels from [VdB12 Definition 6.2.1]):

(I) $W$ is a divisor of bidegree $(1,1)$, and $\mathcal{U}$ is a locally free sheaf on $W$ of rank 2.

(II) $W$ is a divisor of bidegree $(2,2)$, and $\mathcal{U}$ is a pure sheaf on $W$ of rank 1.

Since

- a divisor of bidegree $(1,1)$ in $\mathbb{P}^1 \times \mathbb{P}^1$ is isomorphic to $\mathbb{P}^1$ and transferred to the diagonal by an automorphism of $\mathbb{P}^1 \times \mathbb{P}^1$.
- a pure sheaf on a smooth curve is locally free, and
- a locally free sheaf of rank 2 on $\mathbb{P}^1$ is isomorphic to $\mathcal{O}_{\mathbb{P}^1}(a) \oplus \mathcal{O}_{\mathbb{P}^1}(b)$ for some $a, b \in \mathbb{Z}$ by Birkhoff–Grothendieck theorem,

a noncommutative Hirzebruch surface associated to a sheaf bimodule of type (II) is equivalent to a commutative Hirzebruch surface by Example 2.2 (1). Hence in the rest of this section we may and will consider sheaf bimodules of type (II). Since $\mathcal{O}_{\mathbb{P}^1} \otimes \mathcal{O}_{\mathbb{P}^1} \mathcal{E} \simeq v_* \mathcal{U}$ is a locally free sheaf of rank 2 on $\mathbb{P}^1$, there exist integers $a \leq b \in \mathbb{Z}$ such that

$$v_* \mathcal{U} \simeq \mathcal{O}_{\mathbb{P}^1}(a) \oplus \mathcal{O}_{\mathbb{P}^1}(b).$$  

(3.4)

Since $W$ is a divisor of bidegree $(2,2)$ in $\mathbb{P}^1 \times \mathbb{P}^1$, the dualizing sheaf of $W$ is trivial, so that the Serre duality theorem implies

$$h^1(\mathcal{U}) = h^0(\mathcal{U}^\vee)$$  

(3.5)

for any locally free $\mathcal{O}_W$-module $\mathcal{U}$. Also, it follows from the Riemann-Roch theorem [BHPVdV04, Chapter II, Theorem (3.1)] that

$$\chi(\mathcal{U}) = h^0(\mathcal{U}) - h^1(\mathcal{U}) = \text{rank} \mathcal{U} \cdot \deg \mathcal{U},$$  

(3.6)

where $\deg \mathcal{U} := m \cdot \deg (\mathcal{U}|_{W_{\text{red}}})$ when $W = mW_{\text{red}}$ as Weil divisors on $\mathbb{P}^1 \times \mathbb{P}^1$. As an immediate corollary, we easily obtain the formula

$$\deg \mathcal{U} = \chi(\mathcal{U}) = \chi(v_* \mathcal{U}) = a + b + 2$$  

(3.7)

3.1. Non-reduced $W$. Suppose that $W$ is not reduced. Then the reduced subscheme $W_{\text{red}}$ is a divisor of bidegree $(1,1)$, which we may and will assume $W_{\text{red}} = \Delta_{\mathbb{P}^1}$ without changing the abelian category by Example 2.2 (1).

**Theorem 3.5.** The sheaf $\mathcal{U}$ on $W$ sits in an exact sequence of the following form

$$0 \to \mathcal{U} \to \mathcal{L} \to \mathcal{L} \otimes \mathcal{O}_D \simeq \mathcal{O}_D \to 0,$$  

(3.8)

where

- $\mathcal{L}$ is an invertible sheaf on $W$,
- $D = \sum_{i=1}^N n_i x_i \subset W_{\text{red}} \subset W$ is a 0-dimensional closed subscheme of $W$, where $N \in \mathbb{Z}_{\geq 0}$, $x_1, \ldots, x_N \in W_{\text{red}}$ are distinct closed points, and $n_1, \ldots, n_N \in \mathbb{Z}_{> 0}$. 


• $c$ is the restriction morphism.

Conversely, any such sheaf $\mathcal{U}$ on $W$, regarded as a sheaf on $\mathbb{P}^1 \times \mathbb{P}^1$, is a locally free sheaf bimodule of rank 2 whose support coincides with $W$.

By an abuse of notation we let $\mathcal{O}_D$ denote both the structure sheaf of $D$ and its push-forwards to ambient schemes, depending on the context. Theorem 3.5 is an immediate corollary of Lemma 3.6 below on the local structure of $\mathcal{U}$.

**Lemma 3.6 (=[BGSS87] Proposition 4.1).** Consider the local $k$-algebra $A = k[t, \varepsilon]/(\varepsilon^2)(t)$ and let $M$ be an MCM $A$-module such that $\text{ann}_A(M) = 0$. Then there exists a uniquely determined non-negative integer $n$ such that $M$ is isomorphic to the ideal $(t^n, \varepsilon)$ as $A$-modules.

Let $x \in W$ be a closed point at which $\mathcal{U}$ is not locally free. By taking the module $M$ of Lemma 3.6 to be the stalk $\mathcal{U}_x$ and $A$ to be $\mathcal{O}_{W,x}$, one can find an embedding $\mathcal{U}_x \hookrightarrow \mathcal{O}_{W,x}$ as an ideal. The invertible sheaf $\mathcal{L}$ is obtained from $\mathcal{U}$ by locally replacing $\mathcal{U}_x$ with the over module $\mathcal{O}_{W,x}$ at each such point $x$.

Tensoring (3.8) with $\mathcal{L}^{-1}$, we obtain

$$0 \rightarrow \mathcal{U} \otimes \mathcal{L}^{-1} \rightarrow \mathcal{O}_W \rightarrow \mathcal{O}_D \rightarrow 0. \quad (3.9)$$

On the other hand, since $I_{W_{\text{red}}/W} \simeq i^*\mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(-1, -1) \simeq \mathcal{O}_{W_{\text{red}}}(-2)$, where $i : W_{\text{red}} \rightarrow W$ is the canonical inclusion, we obtain a morphism of exact sequences as follows. The rightmost vertical map corresponds to the closed immersion $D \hookrightarrow W_{\text{red}}$.

$$0 \longrightarrow i_*\mathcal{O}_{W_{\text{red}}}(-2) \longrightarrow \mathcal{O}_W \longrightarrow i_*\mathcal{O}_{W_{\text{red}}} \longrightarrow 0 \quad (3.10)$$

Applying the snake lemma, we obtain the short exact sequence

$$0 \rightarrow i_*\mathcal{O}_{W_{\text{red}}}(-2) \rightarrow \mathcal{U} \otimes \mathcal{L}^{-1} \rightarrow i_*\mathcal{O}_{W_{\text{red}}}(-D) \rightarrow 0. \quad (3.11)$$

Applying $- \otimes \mathcal{L}$ we obtain the exact sequence

$$0 \rightarrow \mathcal{L} \otimes_W i_*\mathcal{O}_{W_{\text{red}}}(-2) \rightarrow \mathcal{U} \rightarrow \mathcal{L} \otimes_W i_*\mathcal{O}_{W_{\text{red}}}(-D) \rightarrow 0, \quad (3.12)$$

which locally around the point $x_i$ is isomorphic to the exact sequence

$$0 \rightarrow (\varepsilon) \rightarrow (t^n, \varepsilon) \rightarrow (t^n, \varepsilon)/(\varepsilon) \rightarrow 0 \quad (3.13)$$

of $A$-modules. From this local description, one can conclude that the support of the divisor $D$ in Theorem 3.5 coincides with the non-locally-free locus of $\mathcal{U}$.

Applying $v_*$ to (3.12), we obtain the following exact sequence on $\mathbb{P}^1$.

$$0 \rightarrow i^*\mathcal{L}(-2) \rightarrow v_*\mathcal{U} \rightarrow i^*\mathcal{L}(-D) \rightarrow 0. \quad (3.14)$$

If $\deg D > 0$, then the sequence (3.14) splits, and one has $v_*\mathcal{U} \simeq i^*\mathcal{L}(-2) \oplus i^*\mathcal{L}(-\deg D)$. Therefore when $\deg D > 0$, the deformations of $\mathcal{U}$ correspond to the deformations of the pair $(\mathcal{L}, D)$, so that we obtain a $\dim \text{Pic}^0 W + \deg D =$ $(\deg D + 1)$-dimensional family of sheaf bimodules $\mathcal{U}$. Taking into account the action of $\text{Aut}(\mathbb{P}^1 \times \mathbb{P}^1)$ on the space of sheaf bimodules, which preserves the equivalence classes of the associated categories $\text{Qgr}\text{S}(\mathcal{E})$, when $\deg D \geq 2$, we obtain a $((\deg D + 1) - 3 =) (\deg D - 2)$-dimensional family of noncommutative Hirzebruch surfaces; note that the anti-diagonal subgroup of $\text{Aut}(\mathbb{P}^1 \times \mathbb{P}^1)$ has been already used to translate $W_{\text{red}}$ to the diagonal.
To describe invertible sheaves on $W$, consider the defining ideal $\mathcal{I} \subset \mathcal{O}_W$ of $W_{\text{red}}$. It satisfied $\mathcal{I}^2 = 0$, and since $W_{\text{red}}$ is a divisor of bidegree $(1, 1)$, it follows that $\mathcal{I} \simeq i_*\mathcal{O}_{W_{\text{red}}}(-2)$. Note that $N_{\mathcal{O}_{W_{\text{red}}}/W} \simeq (\mathcal{I}/\mathcal{I}^2)^\vee \simeq \mathcal{O}_{W_{\text{red}}}(2)$. It then follows that

$$W \simeq V(\mathcal{J}^2) \subset N_{\mathcal{O}_{W_{\text{red}}}/W} = \text{Sym}^\bullet_{\mathcal{O}_{W_{\text{red}}}} \mathcal{O}_{W_{\text{red}}}(-2),$$

(3.15)

where $\mathcal{J} = \text{Sym}_{\mathcal{O}_{W_{\text{red}}}}^0 \mathcal{O}_{W_{\text{red}}}(-2)$ is the ideal sheaf of the 0-section of the normal bundle $N_{\mathcal{O}_{W_{\text{red}}}/W}$. From this isomorphism, we see that the scheme $W$ is obtained by gluing $U_1 := \text{Spec} k[z, u]/(u^2)$ and $U_2 := \text{Spec} k[w, v]/(v^2)$ along $U_{12} := \text{Spec} k[z, w, u, v]/(u^2, v^2, zw - 1, u - z^2v)$.

Consider the short exact sequence

$$1 \to \mathcal{I} \to \mathcal{O}_W \to \mathcal{O}_{W_{\text{red}}} \to 1$$

(3.16)

of sheaves of abelian groups on the topological space underlying $W$, where $e$ is the map defined by $x \mapsto x + 1$. By taking the long exact sequence, we obtain the exact sequence

$$1 \to H^1(W_{\text{red}}, \mathcal{O}_{W_{\text{red}}}(-2)) \xrightarrow{H^1(e)} \text{Pic}(W) \xrightarrow{\text{deg}(\mathcal{O}_{W_{\text{red}}})} \text{Pic}(W_{\text{red}}) \simeq \mathbb{Z} \to 1$$

(3.17)

of abelian groups. In particular, $\text{deg}(\mathcal{U}) = 2 \text{deg}(\mathcal{U}|_W_{\text{red}})$ is always an even integer.

Via the explicit description of $W$ given above, the Čech complex for $\mathcal{O}_{W_{\text{red}}}(-2)$ with respect to the affine cover $W_{\text{red}} = U_{1,\text{red}} \cup U_{2,\text{red}}$ is described as follows:

$$k[z] \oplus k[w] \xrightarrow{d} k[z, w]/(zw - 1); \quad (f(z), g(w)) \mapsto f(z) - \frac{1}{z^2}g\left(\frac{1}{z}\right)$$

(3.18)

Thus we see that $\tilde{H}^1(W_{\text{red}}, \mathcal{O}_{W_{\text{red}}}(-2)) = k[w]$.

Under the morphism $\tilde{H}^1(W_{\text{red}}, \mathcal{O}_{W_{\text{red}}}(-2)) \xrightarrow{H^1(e)} \tilde{H}^1(W, \mathcal{O}_W(-2))$, the element $a[w]$ is mapped to the class represented by the cocycle $1 + auw + 1 + avz \in \mathcal{O}_W^\times(U_{12})$. Therefore it follows that any line bundle on $W$ of degree zero is given as the line bundle $\mathcal{L}_a$ obtained by gluing $\mathcal{O}_{U_1}$ and $\mathcal{O}_{U_2}$ by $1 + azv \in \mathcal{O}_W^\times(U_{12})$ for $a \in k$.

The Čech complex for $\mathcal{L}_a$ is given by

$$k[z, u]/(u^2) \oplus k[w, v]/(v^2) \to k[z, w, u, v]/(u^2, v^2, zw - 1, u - z^2v),$$

(3.19)

$$ (f(z, u), g(w, v)) \mapsto (1 + azv)f(z, u) - g(w, v),$$

(3.20)

which is acyclic if and only if $a \neq 0$. More precisely, it follows that

$$\dim H^i(\mathcal{L}_a) = \begin{cases} 1 & a = 0, \\ 0 & a \neq 0, \end{cases}$$

(3.21)

for $i = 0, 1$. Since $v$ is an affine morphism, we have

$$H^i(\mathcal{L}_a) \simeq H^i(v_*(\mathcal{L}_a)).$$

(3.22)

Combining this with (3.21), we see that

$$v_*\mathcal{L}_a \simeq \begin{cases} \mathcal{O}_{p_1} \oplus \mathcal{O}_{p_1}(-2) & a = 0, \\ \mathcal{O}_{p_1}(-1) \oplus \mathcal{O}_{p_1}(-1) & a \neq 0. \end{cases}$$

(3.23)

Since the morphisms $u$ and $v$ are isomorphic, one also has

$$\mathcal{O}_{p_1}(-1) \otimes \mathcal{E} = v_*(v^*(\mathcal{O}_{p_1}(-1)) \otimes \mathcal{L}_a) = \mathcal{O}_{p_1}(-1) \otimes v_*\mathcal{L}_a.$$  

(3.24)

Consider in general $\mathcal{U}$ with $\mathcal{L}|_{W_{\text{red}}} \simeq \mathcal{O}_{W_{\text{red}}}(q)$ and $\text{deg} D > 0$. Then $v_*\mathcal{U} \simeq \mathcal{O}_{p_1}(q - 2) \oplus \mathcal{O}_{p_1}(q - \text{deg} D)$ as we observed after (3.13).

Summing up, we obtain the following conclusion.
Lemma 3.7. Suppose that $W$ is not reduced.

(1) When $\mathcal{U}$ is invertible, then

$$
(a, b), b - a = \begin{cases}
\left(\frac{\deg U}{2} - 2, \frac{\deg U}{2}\right), & \mathcal{U} \simeq v^*\mathcal{O}_{\mathbb{P}^1}(k) \ (\exists k \in \mathbb{Z}), \\
\left(\frac{\deg U}{2} - 1, \frac{\deg U}{2} - 1\right), & \mathcal{U} \not\simeq v^*\mathcal{O}_{\mathbb{P}^1}(k) \ (\forall k \in \mathbb{Z}).
\end{cases}
$$

(3.25)

(2) When $\mathcal{U}$ is not invertible, so that $\deg D > 0$,

$$
(a, b), b - a = \begin{cases}
\left(\frac{x(U)-3}{2}, \frac{x(U)-1}{2}\right), & \deg D = 1 \\
\left(\frac{x(U)-\deg D}{2}, \frac{x(U)+\deg D}{2} - 2\right), & \deg D - 2 \quad \deg D \geq 2.
\end{cases}
$$

(3.26)

3.2. Reduced $W$. In this section we assume that $W$ is a reduced divisor of bidegree $(2,2)$. Either by studying the branched double covers $u: \mathbb{Z} \to \mathbb{P}^1$ or by sending $W$ by a birational map $\mathbb{P}^1 \times \mathbb{P}^1 \dashrightarrow \mathbb{P}^2$ and using the classification of cubic curves in $\mathbb{P}^2$, one can show that there are five possibilities for $W$ as follows. In the rest of this paper, for convenience, we will use the Kodaira’s symbol for singular elliptic fibers to describe the type of $W$. Note that in the list below, except the case $I_0$, the type uniquely determines the isomorphism class of $W$.

(i) $W$ is an elliptic curve ($I_0$).

(ii) $W$ is an irreducible nodal rational curve ($I_1$).

(iii) $W$ is the union $W_1 \cup W_2$ of two smooth rational curves intersecting at two points ($I_2$).

(iv) $W$ is a cuspidal rational curve ($II$).

(v) $W$ is the union $W_1 \cup W_2$ of two smooth rational curves intersecting at one point with multiplicity two ($III$).

Lemma 3.8. Suppose that $W$ is irreducible and $\mathcal{U}$ is invertible. Then one has

$$
(a, b), b - a = \begin{cases}
\left(\frac{\deg U}{2} - 2, \frac{\deg U}{2}\right), & \mathcal{U} \simeq v^*\mathcal{O}_{\mathbb{P}^1}(k) \ (\exists k \in \mathbb{Z}), \\
\left(\frac{\deg U}{2} - 1, \frac{\deg U}{2} - 1\right), & \mathcal{U} \not\simeq v^*\mathcal{O}_{\mathbb{P}^1}(k) \ (\forall k \in \mathbb{Z}) \text{ and } \deg \mathcal{U} \equiv 0 \mod 2, \\
\left(\frac{\deg U}{2} - 1, \frac{\deg U}{2} - 1\right), & \deg \mathcal{U} \equiv 1 \mod 2,
\end{cases}
$$

(3.27)

where the integers $a, b$ are those in (3.4).

Proof. The first claim follows from the following computations

$$
\chi_W(\mathcal{U}) = \chi_{\mathbb{P}^1}(v_\mathcal{U}) = \chi_{\mathbb{P}^1}(\mathcal{O}_{\mathbb{P}^1}(a) \otimes \mathcal{O}_{\mathbb{P}^1}(b)) = (a + 1) + (b + 1).
$$

(3.28)

For the first equality we used that $v$ is an affine morphism.

Let us show the second claim. By replacing $\mathcal{U}$ with $\mathcal{U} \otimes v^*\mathcal{O}_{\mathbb{P}^1}(k)$ for some integer $k$, we may and will assume $\deg \mathcal{U} = 0$ or 1, without changing the value $b - a$.

If $\deg \mathcal{U} = 0$, (3.7) implies

$$
(a + b + 2) = \chi(\mathcal{U}) = 0
$$

(3.29)

On the other hand we have the following dichotomy

$$
h^0(v_\mathcal{U}) = h^0(\mathcal{U}) = \begin{cases}
1 & \mathcal{U} \simeq \mathcal{O}_W, \\
0 & \text{otherwise}.
\end{cases}
$$

(3.30)

In the first case it follows that $a < b = 0$, and in the second case $a \leq b \leq -1$. Combining this with (3.29), in each of the two cases we conclude $(a, b) = (-2, 0)$ and $(-1, -1)$, respectively.
If \( \deg \mathcal{U} = 1 \), then one has

\[
h^1(\mathcal{U}) = h^0(\mathcal{U}^\vee) = 0, \quad (3.31)
\]

which together with (3.6) implies

\[
h^0(\mathcal{U}) = \chi(\mathcal{U}) = 1. \quad (3.32)
\]

Combining this with (3.7) as before, we conclude that \((a, b) = (-1, 0)\). Thus we finish the proof of Lemma 3.8.

\[\square\]

**Lemma 3.9.** Suppose that \( W \) is not irreducible (i.e., \( I_2 \) or \( III \)), so that it is a sum of two distinct effective divisors of bidegree \((1, 1)\) and \( \mathcal{U} \) is an invertible sheaf of bidegree \((p, q)\) on \( W \) with \( p \leq q \). Then

\[
(a, b), b - a = \begin{cases} 
(p - 2, p), 2 & \text{if } q - p = 0 \text{ and } \mathcal{U} \simeq \nu^* \mathcal{O}_{	ext{P}^1}(p), \\
(p - 1, p - 1), 0 & \text{if } q - p = 0 \text{ and } \mathcal{U} \not\simeq \nu^* \mathcal{O}_{	ext{P}^1}(p), \\
(p - 1, p), 1 & \text{if } q - p = 1, \\
(p, q - 2), q - p - 2 & \text{if } q - p \geq 2.
\end{cases} \quad (3.33)
\]

**Proof.** Replacing \( \mathcal{U} \) with \( \mathcal{U} \otimes v^* \mathcal{O}_{	ext{P}^1}(-p) \), we may and will assume \( p = 0 \) throughout the proof. Note first the constraint

\[
q = \chi(\mathcal{U}) = \chi(v_\ast \mathcal{U}) = a + b + 2, \quad (3.34)
\]

where the first equality follows from the Riemann-Roch formula (3.6).

Take the normalization morphism \( \nu: W' \simeq \text{P}^1 \coprod \text{P}^1 \to W \), so that \( \nu^* \mathcal{U} \simeq \mathcal{O}_{	ext{P}^1} \coprod \mathcal{O}_{	ext{P}^1}(q) \).

Note that there exists the exact sequence

\[
0 \to \mathcal{U} \to \nu_\ast \nu^* \mathcal{U} \to \nu_\ast \mathcal{O}_{W'} / \mathcal{O}_W \simeq \mathcal{O}_W / \mathcal{I} := C \to 0, \quad (3.35)
\]

where \( \mathcal{I} := \text{Hom}_W (\nu_\ast \mathcal{O}_{W'}, \mathcal{O}_W) \subset \mathcal{O}_W \) is the conductor ideal. Depending on whether \( W = I_2 \) or \( III \), we have \( \mathcal{O}_W / \mathcal{I} \simeq k \times k \) or \( k[t]/(t^2) \), respectively.

Applying \( \nu_\ast \) to (3.35), we obtain the exact sequence

\[
0 \to \nu_\ast \mathcal{U} \simeq \mathcal{O}_{	ext{P}^1}(a) \oplus \mathcal{O}_{	ext{P}^1}(b) \xrightarrow{\iota} \mathcal{O}_{	ext{P}^1} \oplus \mathcal{O}_{	ext{P}^1}(q) \to \nu_\ast C \simeq C \to 0. \quad (3.36)
\]

Now suppose that \( q = 0 \), so that \( \mathcal{U} \in \text{Pic}^0(W) \). Since the map \( \iota \) of (3.36) has trivial kernel, it follows that \( b \leq 0 \). Combining this with the assumption \( a \leq b \) and (3.34), we immediately see \((a, b)\) is either \((-2, 0)\) or \((-1, -1)\). It is then easy to observe that \( \mathcal{U} \simeq \mathcal{O}_W \iff h^0(W, \mathcal{U}) = 1 \iff h^0(\text{P}^1, \nu_\ast \mathcal{U}) = 1 \iff (a, b) = (-2, 0) \).

Next assume that \( q > 0 \). Apply \( \mathbb{R} \Gamma (\text{P}^1, -) \) to (3.36) and take the associated long exact sequence, to obtain:

\[
0 \to H^0 (\mathcal{O}_{	ext{P}^1}(a)) \oplus H^0 (\mathcal{O}_{	ext{P}^1}(b)) \to H^0 (\mathcal{O}_{	ext{P}^1}) \oplus H^0 (\mathcal{O}_{	ext{P}^1}(q)) \xrightarrow{\iota} H^0 (C) = C \to 0 \quad (3.37)
\]

with the restriction map

\[
H^0 (W, \nu_\ast v^* \mathcal{U}) \to C, \quad (3.38)
\]

which is surjective when \( q > 0 \). This then implies \( H^1 (\mathcal{O}_{	ext{P}^1}(a)) \oplus H^1 (\mathcal{O}_{	ext{P}^1}(b)) = 0 \), so that \(-1 \leq a \leq b \). When \( q = 1 \), combining this again with (3.34), we immediately see that \((a, b) = (-1, 0)\).

Consider the remaining case \( q \geq 2 \). We see that \( a \leq 0 \), since otherwise the composition of the map \( \iota \) of (3.36) with the projection to the component \( \mathcal{O}_{	ext{P}^1} \) is trivial, contradicting that \( \text{coker } \iota \) is a torsion sheaf. Finally, suppose for a contradiction that \( a = -1 \). Then by
applying $- \otimes \mathcal{O}_p(-b)$ to (3.36) and taking the associated long exact sequence, we obtain the following exact sequence.

$$0 \rightarrow H^0(\mathcal{O}_p(-b - 1) \oplus \mathcal{O}_p) \rightarrow H^0(\mathcal{O}_p(-b) \oplus \mathcal{O}_p(1)) \xrightarrow{r'} C.$$  \hspace{1cm} (3.39)

As before, one can check that the map $r'$ is an isomorphism by identifying it with the restriction map

$$H^0(W, \nu_0 \nu' \mathcal{U} \otimes v^* \mathcal{O}(-b)) \rightarrow C.$$  \hspace{1cm} (3.40)

Thus we obtain a contradiction, concluding the proof. □

We next consider the case when $\mathcal{U}$ is not an invertible sheaf on $W$. Since $\mathcal{U}$ is torsion free of rank 1, it is invertible on the smooth locus of $W$. Moreover, as we observed in the beginning of Section III, the sheaf $\mathcal{U}$ is a maximally Cohen-Macaulay (MCM) module over $W$. Since the singularity of $W$ is either of type $A_1$, $A_2$, or $A_3$, we can use the (finite!) classification of the indecomposable MCM modules on those singularities to classify the local structure of $\mathcal{U}$ around the singularity of $W$, which is described for example in [Yos90, (5.12)] (for $A_{\text{odd}}$) and [Yos90, (9.9)] (for $A_{\text{even}}$). Taking into account that the support of $\mathcal{U}$ coincides with $W$, the local structures of $\mathcal{U}$ around the singularities of $W$ are classified as follows.

| $W$ | Singularity of $W$ | Stalk of $\mathcal{U}$ at the singularity up to isomorphism |
|-----|-------------------|-----------------------------------------------------------|
| $I_1$ or $I_2$ | $R = \mathbf{k}[x,y]_{(x,y)}/(y^2 - x^2)$ (A1) | $R$ or $(x,y)$ |
| II  | $R = \mathbf{k}[x,y]_{(x,y)}/(y^2 - x^2)$ (A2) | $R$ or $(x,y)$ |
| III (A1) | $R = \mathbf{k}[x,y]_{(x,y)}/(y^2 - x^4)$ (A3) | $R$ or $(x,y)$ or $(x^2,y)$ |

(3.41)

**Remark 3.10.** The MCM modules on $A_n$ singularities have an interpretation as the conductor ideals of the partial resolutions of the singularity. We illustrate this in the case when $n = 3$. Consider the following sequence of commutative $\mathbf{k}$-algebras.

$$S_2 = \mathbf{k}[u] \times \mathbf{k}[v],$$  \hspace{1cm} (3.42)

$$S_1 = \mathbf{k}[u + v, u - v] \subset S_2,$$  \hspace{1cm} (3.43)

$$S_0 = \mathbf{k}[u + v, u^2 - v^2] \subset S_1.$$  \hspace{1cm} (3.44)

By an abuse of notation we will denote the localization of these rings by the multiplicative set $S_0 \setminus (u + v, u^2 - v^2)$ $S_0$ by the same symbols. If we let $\mathfrak{m}_S$ denote the maximal ideal of a local ring $S$, then we have $S_1 \simeq \text{End}_{S_0} \mathfrak{m}_{S_0}$ and $S_2 \simeq \text{End}_{S_1} \mathfrak{m}_{S_1}$. Then under the isomorphism $R_{III} \xrightarrow{\sim} S_0$; $x, y \mapsto u + v, u^2 - v^2$, the MCM modules $R_{III}, (x,y), (x^2,y)$ of $R_{III}$ are identified with the ideals

$$\text{Hom}_{S_0}(S_0, S_0) (= S_0),$$  \hspace{1cm} (3.45)

$$\text{Hom}_{S_0}(S_1, S_0),$$  \hspace{1cm} (3.46)

$$\text{Hom}_{S_0}(S_2, S_0),$$  \hspace{1cm} (3.47)

respectively.

Suppose that $I \subset W$ is an ideal sheaf which is isomorphic to $\mathcal{U}$ around the singularity and cosupported in $\text{Sing} W$, which always exists by Remark 3.10. In order to resolve the
non-invertibility of $\mathcal{U}$, consider the blowup

$$f : \tilde{W} := \text{Bl}_f W = \text{Proj}_W \bigoplus_{d \geq 0} I^d \to W.$$  \hspace{1cm} (3.48)

It follows from the definition that $f$ is projective and an isomorphism over the smooth locus of $W$, which hence is surjective. Also we have an invertible sheaf $\tilde{U}$ on $\tilde{W}$ such that $f_* \tilde{U} \simeq \mathcal{U}$. An explicit computation will tell us that $f$ locally normalizes $W$ at the points where $\mathcal{U}$ is not invertible, except when $W$ is the curve $\text{III}$ and the ideal $I$ is locally isomorphic to $(x, y)$ around the singularity, in which case $f$ is locally isomorphic to $\text{Spec} \, S_1 \to \text{Spec} \, S_0$ and $\tilde{W}$ is isomorphic to the union of two copies of $\mathbb{P}^1$ meeting transversally in a point.

**Lemma 3.11.** Suppose that $W$ is irreducible (i.e., $I_1$ or $\text{II}$) and $\mathcal{U}$ is not invertible, so that there exists an invertible sheaf $\tilde{U} = O(i)$ on the normalization $f : W^\nu \simeq \mathbb{P}^1 \to W$ such that $f_* O(i) = \mathcal{U}$. Then

$$(a, b), b - a = \begin{cases} \left( \frac{i}{2} - 1, \frac{i}{2} \right), 1 & \text{if } i \equiv 0 \mod 2 \\ \left( \frac{i-1}{2}, \frac{i+1}{2} \right), 0 & \text{if } i \equiv 1 \mod 2. \end{cases} \hspace{1cm} (3.49)$$

**Proof.** We can again assume that $i = 0$ or $1$. As before, we have $\chi(\mathcal{U}) = \chi(O(i)) = i + 1 = a + b + 2$. We also have $0 = h^0(\mathcal{O}_{W^\nu}(i - 2)) = h^0(\mathcal{O}_{\mathbb{P}^1}(a - 1) \oplus \mathcal{O}_{\mathbb{P}^1}(b - 1))$, so that $a \leq 0$ and $b \leq 0$. Therefore $(a, b) = (-1, 0)$ (if $i = 0$) and $(0, 0)$ (if $i = 1$). \hfill \Box

When $W$ is not irreducible (i.e., either $I_2$ or $\text{III}$), it is easy to classify $\mathcal{U}$ which is not invertible, since in this case $\tilde{W}$ of (3.48) has trivial $H^1(O)$, so that $\mathcal{U} = f_* \tilde{U}$ is uniquely determined by the (multi-)degree of $\tilde{U}$. We give a more explicit description as follows.

**Lemma 3.12.** The classifications of non-invertible $\mathcal{U}$ on a non-irreducible (and reduced) $W$ is given by the following table.

| $W$  | Type of $\mathcal{U}$                  | $\tilde{W}$ | $\tilde{U}$ $(p \leq q)$ | $(a, b)$ and $b - a$ |
|------|----------------------------------------|-------------|--------------------------|----------------------|
| $I_2$ | Not invertible at a point              | nodal conic | $O(p, q)$                | $(p - 1, p), 1$      |
|      |                                        |             |                          | $(p, q - 1), q - p - 1$ (if $p < q$) |
| $\text{III}$ | $\simeq (x, y)$ around Sing $W$       | nodal conic | $O(p, q)$                | $(p - 1, p), 1$      |
|      |                                        |             |                          | $(p, q - 1), q - p - 1$ (if $p < q$) |
| $I_2$ | Not invertible at two points           | $\mathbb{P}^1$ | $\mathbb{P}^1$         | $O(p), q - p$        |
| $\text{III}$ | $\simeq (x^2, y)$ around Sing $W$     | $\mathbb{P}^1$ | $\mathbb{P}^1$         | $O(p), q - p$        |

**Remark 3.13.** The last two cases are the so-called "decomposable case".

From the classification of locally free sheaf bimodules which we have just completed, we obtain the following corollary.

**Corollary 3.14.** If a locally free sheaf bimodule $\mathcal{E}$ satisfies $b - a \geq 3$, then $W$ is not irreducible.

The following corollary will be used to check the strongness of certain full exceptional collections of the derived category of $\text{qgr} \, S(\mathcal{E})$.

**Corollary 3.15.** Set $\mathcal{O}(-1) \otimes \mathcal{E} \simeq \mathcal{O}_{\mathbb{P}^1}(a') \oplus \mathcal{O}_{\mathbb{P}^1}(b')$ with $a' \leq b', \text{ so that } a' + b' = \chi(\mathcal{U}) - 4$.

(1) If $W$ is not reduced and
(i) $U$ is invertible, then

$$(a', b'), b' - a' = \begin{cases} (\deg U - 3, \deg U, 2) & U \simeq v^*O_{\mathbb{P}^1}(k) (\exists k \in \mathbb{Z}), \\ (\deg U - 2, \deg U, 0) & U \not\simeq v^*O_{\mathbb{P}^1}(k) (\forall k \in \mathbb{Z}) \end{cases} \quad (3.51)$$

(ii) $U$ is not invertible, so that $\deg D > 0$, then

$$(a', b'), b' - a' = \begin{cases} \left(\frac{\chi(U) - 5}{2}, \frac{\chi(U)}{2} - 1\right), 1 & \deg D = 1 \\ \left(\frac{\deg U}{2}, \frac{\deg D}{2} - 1\right), 1 & \deg D - 2 \quad \deg D \geq 2. \end{cases} \quad (3.52)$$

(2) If $W$ is integral (i.e., $I_0, I_1$ or $III$) and $U$ is invertible, then

$$(a', b'), b' - a' = \begin{cases} (\deg U - 3, \deg U, 2) & u^*O_{\mathbb{P}^1}(-1) \otimes U \simeq v^*O_{\mathbb{P}^1}(k) (\exists k \in \mathbb{Z}) \\ (\deg U - 2, \deg U, 0) & u^*O_{\mathbb{P}^1}(-1) \otimes U \not\simeq v^*O_{\mathbb{P}^1}(k) (\exists k \in \mathbb{Z}) \\ (\deg U - 1, \deg U - 1, 1) & \deg U \equiv 1 \mod 2 \end{cases} \quad (3.53)$$

(3) If $W$ is integral and $U$ is not invertible, then

$$(a', b'), b' - a' = \begin{cases} (\frac{i}{2} - 2, \frac{i}{2} - 1, 1) & i \equiv 0 \mod 2, \\ (\frac{i - 1}{2} - 1, \frac{i - 1}{2} - 1, 0) & i \equiv 1 \mod 2. \end{cases} \quad (3.54)$$

(4) If $W$ is not irreducible (i.e., $I_2$ or $III$) and $U$ is invertible, then

$$(a', b'), b' - a' = \begin{cases} (p - 3, p - 1, 2) & q - p = 0 \text{ and } U \otimes u^*O_{\mathbb{P}^1}(-1) \simeq v^*O_{\mathbb{P}^1}(p - 1), \\ (p - 2, p - 2, 0) & q - p = 0 \text{ and } U \otimes u^*O_{\mathbb{P}^1}(-1) \not\simeq v^*O_{\mathbb{P}^1}(p - 1), \\ (p - 2, p - 1, 1) & q - p = 1, \\ (p - 1, q - 3, q - p - 2) & q - p \geq 2. \end{cases} \quad (3.55)$$

(5) If $W$ is not irreducible (i.e., $I_2$ or $III$) and $U$ is not invertible, then

(i) When $\tilde{W} \simeq X$, then

$$(a', b'), b' - a' = \begin{cases} (p - 2, p - 1, 1) & (if \ p = q) \\ (p - 1, q - 2, q - p - 1) & (if \ p < q) \end{cases} \quad (3.56)$$

(ii) When $\tilde{W} \simeq W^\nu \simeq \mathbb{P}^1 \coprod \mathbb{P}^1$, then

$$(a', b'), b' - a' = (p - 1, q - 1), q - p \quad (3.57)$$

Proof. Set $\mathcal{M} := u^*\mathcal{O}(-1) \otimes v^*\mathcal{O}(1) \in \text{Pic}^0 W$. Then one observes

$$\mathcal{O}(-1) \otimes \mathcal{E} = v_* (u^*\mathcal{O}(-1) \otimes U) = v_* (v^*\mathcal{O}(-1) \otimes \mathcal{U} \otimes \mathcal{M}) \simeq v_* (U \otimes \mathcal{M}) \otimes \mathcal{O}(-1),$$

where the last isomorphism is the projection formula. Then one immediately obtains the conclusions just by applying our results above to $\mathcal{U} \otimes \mathcal{M}$.

\end{quote}

4. Moduli stack of sheaf bimodules

4.1. Gieseker stability. Let $\mathcal{E} = \iota_* \mathcal{U}$ be a locally free sheaf bimodule on $\mathbb{P}^1$. In this section we completely determine the Gieseker stability of sheaf bimodules with respect to the polarization $-K_{\mathbb{P}^1 \times \mathbb{P}^1} = \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(2, 2)$. We recall the definition of Gieseker stability from [HL10].
**Definition 4.1.** Let \((X, H)\) be a polarized projective scheme over a field \(k\). For a coherent sheaf \(E \in \text{coh} \ X\) on \(X\), the Hilbert polynomial \(P_E(t)\) of \(E\) with respect to \(H\) is the polynomial in \(\mathbb{Q}[t]\) which satisfies \(P_E(m) = \chi(E \otimes \mathcal{O}_X(mH))\) for all \(m \in \mathbb{Z}\). When \(\text{dim Supp} \ E = d\), there are rational numbers \(a_i \in \mathbb{Q}\) for \(i = 0, 1, \ldots, d\) such that \(a_d > 0\) and \(P_E(t) = \sum_{i=0}^{d} a_i t^i\). The reduced Hilbert polynomial of \(E\) is defined to be \(p_E(t) = \frac{1}{a_d} P(t)\).

A pure sheaf \(E\) is said to be Gieseker (semi-)stable with respect to the polarization \(H\) if the inequality

\[
p_F(t) < (\leq) p_E(t)
\]

holds for any subsheaf \(0 \neq F \subseteq E\), where (4.1) should be interpreted that the leading coefficient of the polynomial \(p_E(t) - p_F(t)\) is strictly positive (non-negative, respectively).

Note that if \(\iota: Y \hookrightarrow X\) is a closed immersion, then \(E \in \text{coh} \ Y\) is \(\iota^*H\) (semi-)stable if and only if so is \(\iota_*E\) with respect to \(H\).

The following lemma immediately follows from the definition.

**Lemma 4.2.** Let \(\mathcal{E}\) be an indecomposable locally free sheaf bimodule over \(\mathbb{P}^1\) of type (I). Then \(\mathcal{U}\) (hence \(\mathcal{E}\)) is unstable if \(a < b\), and is semi-stable but not stable if \(a = b\).

In the rest of this subsection, we consider the case (II).

**Proposition 4.3.** Let \(\mathcal{E}\) be a locally free sheaf bimodule over \(\mathbb{P}^1\) whose scheme theoretic support \(W\) is not reduced. With the notation of Theorem 3.5, the stability of \(\mathcal{U}\) with respect to the polarization \(\mathcal{O}_W(-K_{\mathbb{P}^1 \times \mathbb{P}^1})\) is classified as follows, depending on \(\deg D\).

| \(\deg D\) | Stability of \(\mathcal{U}\) |
|----------------|-----------------------------|
| \(\geq 3\)  | unstable                    |
| 2            | semi-stable but not stable  |
| 0 or 1       | stable                      |

**Proof.** Since \(\mathcal{L}\) is an invertible sheaf on \(W\), the stability of \(\mathcal{U}\) coincides with that of \(\mathcal{U} \otimes \mathcal{L}^{-1}\). Hence we may and will assume \(\mathcal{L} = \mathcal{O}_W\), without loss of generality. Then we have the short exact sequence (3.11), with the middle term = \(\mathcal{U}\).

By the direct computation, we obtain

\[
p_{\mathcal{U}}(m) = m - \frac{1}{8} \deg D
\]

and

\[
p_{\iota_*\mathcal{O}_{W\text{red}}(-2)}(m) = m - \frac{1}{4}
\]

Therefore the subsheaf \(\iota_*\mathcal{O}_{W\text{red}}(-2) \subset \mathcal{U}\) ensures that \(\mathcal{U}\) is not stable if \(\deg D \geq 2\), and is unstable if \(\deg D \geq 3\). From now on, let us assume \(\deg D \leq 2\). Take an arbitrary subsheaf \(0 \neq F \subsetneq \mathcal{U}\), and consider the exact sequence

\[
0 \to F \xrightarrow{i} \mathcal{U} \xrightarrow{\mathcal{P}} \mathcal{U}/F \to 0.
\]

If \(\text{dim Supp} \mathcal{U}/F \leq 0\), then \(P_{\mathcal{U}}(t) = P_F(t) + C\), where \(C = \dim_k \mathcal{U}/F\) is a positive constant. In this case, clearly \(P_F < p_{\mathcal{U}}\). Suppose that \(\text{dim Supp} \mathcal{U}/F = 1\).

Consider the case when \(F \cap \iota_*\mathcal{O}_{W\text{red}}(-2) \neq 0\). Then the assumption implies \(\text{dim Supp} \ p(F) \leq 0\), so that \(p(F) = 0\) and hence \(F \subseteq \iota_*\mathcal{O}_{W\text{red}}(-2)\). Therefore in this case \(p_F \leq p_{\mathcal{U}}\) when \(\deg D = 2\), and \(p_F < p_{\mathcal{U}}\) when \(\deg D \leq 1\).
Consider then the other case when \( F \cap i_* O_{W_{\text{red}}}(-2) = 0 \), so that \( p|_F : F \to p(F) \subset i_* O_{W_{\text{red}}}(-D) \). At this point we can already conclude that \( U \) is semi-stable when \( \deg D = 2 \).

Note that \( p(F) \subset i_* O_{W_{\text{red}}}(-D) \), since otherwise \( p \) splits so as to contradict the assumption that \( W \) is the scheme theoretic support of \( U \). Finally, by the standard long exact sequence argument we can check that \( \text{Hom}_W (i_* O_{W_{\text{red}}}(-1), U) = 0 \), to conclude that \( U \) is stable when \( \deg D \leq 1 \).

**Proposition 4.4.** Let \( E \) be an indecomposable locally free sheaf bimodule over \( \mathbb{P}^1 \) whose scheme theoretic support \( W \) is reduced and irreducible (i.e., either \( I_0, I_1 \), or \( III \)). Then \( E \) is stable with respect to any polarization.

**Proof.** Let \( H \) be a polarization of \( W \). We show the stability of \( U \) with respect to \( H \).

Take any subsheaf \( 0 \neq F \subset U \), and consider the short exact sequence (4.5). The assumption implies that \( W \) is an integral scheme, and since \( U \) has rank 1, it follows that the stalk of the map \( F \to U \) at the generic point of \( W \) is an isomorphism, so that \( \dim \text{Supp} U/F \leq 0 \). As is already discussed in the proof of the previous proposition, from this we immediately obtain the conclusion.

**Proposition 4.5.** Let \( E \) be a locally free sheaf bimodule over \( \mathbb{P}^1 \) whose scheme theoretic support \( W \) is not irreducible (i.e., either \( I_2 \) or \( III \)), and \( U \) is an invertible sheaf of bidegree \( (p,q) \) on \( W \) with \( p \leq q \). Then the stability of \( U \) with respect to the polarization \( O_W(-K_{\mathbb{P}^1 \times \mathbb{P}^1}) \) is classified as follows, depending on \( q - p \).

| \( q - p \) | Stability of \( U \) |
|----------|------------------|
| \( \geq 3 \) | unstable         |
| 2        | semi-stable but not stable |
| 0 or 1   | stable           |

**Proof.** Let \( i_p, i_q \) be the closed immersions into \( W \) of its irreducible components on which the restriction of \( U \) is isomorphic to \( O(p) \) and \( O(q) \), respectively. On \( W \), there exists a short exact sequence

\[
0 \to i_{q*} O(q-2) \to U \to i_{p*} O(p) \to 0.
\]

This immediately implies that \( U \) is unstable when \( q - p \geq 3 \) and is not stable when \( q - p = 2 \).

By the similar arguments as in the proof of Proposition 4.3, the proof of the assertion amounts to showing \( \text{Hom}_W (i_p O(p-1), U) = 0 \). The rest of the proof will be devoted to this step.

Applying \( \text{Hom}_W (i_p O(p-1), -) \) to the sequence (4.7), as part of the long exact sequence we obtain

\[
0 \to \text{Hom}_W (i_p O(p-1), U) \to \text{Hom}_W (i_p O(p-1), i_{p*} O(p)) \to \text{Ext}_W^1 (i_p O(p-1), i_{q*} O(q-2)) \to \text{Ext}_W^1 (i_p O(p-1), U) .
\]

(4.8)

Note first that \( \text{Hom}_W (i_p O(p-1), i_{p*} O(p)) \sim \mathbb{k}^2 \). To compute the next term, note that

\[
\text{Ext}_W^1 (i_p O(p-1), i_{q*} O(q-2)) \sim H^0 \left( W, \text{Ext}_W^1 (i_p O(p-1), i_{q*} O(q-2)) \right),
\]

(4.9)

since \( \text{Ext}_W^1 (i_p O(p-1), i_{q*} O(q-2)) \) is supported in the intersection of the two irreducible components and hence the local-to-global spectral sequence degenerates at sheet.
2. A local computation ensures that the RHS is isomorphic to $k \times k$ or $k[t]/(t^2)$, depending on whether $W$ is $I_2$ or $III$. Hence in any case $\dim_k \text{Ext}^1_W \left( i_p \mathcal{O}(p-1), i_q \mathcal{O}(q-2) \right) = \dim_k k^2 = 2$.

Finally, by the similar reasoning one obtains

$$\text{Ext}^1_W \left( i_p \mathcal{O}(p-1), \mathcal{U} \right) \simeq H^0 \left( W, \text{Ext}^1_W \left( i_p \mathcal{O}(p-1), \mathcal{U} \right) \right).$$

(4.10)

However, from local computations we always obtain $\text{Ext}^1_W \left( i_p \mathcal{O}(p-1), \mathcal{U} \right) = 0$. Thus we conclude the proof. □

**Proposition 4.6.** Let $\mathcal{E}$ be a locally free sheaf bimodule over $\mathbb{P}^1$ whose scheme theoretic support $W$ is not irreducible (i.e., either $I_2$ or $III$), and $\mathcal{U}$ is a non-invertible sheaf of rank 1 of bidegree $(p, q)$ on $W$ with $p \leq q$. Then the stability of $\mathcal{U}$ with respect to the polarization $\mathcal{O}_W(-K_{\mathbb{P}^1 \times \mathbb{P}^1})$ is classified as follows, depending on $q - p$.

| $\widetilde{\mathcal{W}}$ | Stability of $\mathcal{U}$ |
|---------------------------|--------------------------|
| nodal conic               | unstable                 |
| $\geq 2$                  |                          |
| 1                         | semi-stable but not stable |
| 0                         | stable                   |
| $\mathbb{P}^1 \coprod \mathbb{P}^1$ | unstable         |
| $\geq 1$                  |                          |
| 0                         | semi-stable but not stable |

(4.11)

**Proof.** The assertion for the case $\widetilde{\mathcal{W}} \simeq$ nodal conic (two copies of $\mathbb{P}^1$ glued transversally together at a point) can be proved by the same arguments as in the proof of the previous proposition. The assertion for the case $\widetilde{\mathcal{W}} \simeq \mathbb{P}^1 \coprod \mathbb{P}^1$ is rather obvious. □

4.2. **Deformation theory.** The deformation theory of the sheaf bimodule $\mathcal{E}$ as a coherent sheaf on $\mathbb{P}^1 \times \mathbb{P}^1$ is controlled by the differential graded Lie algebra $\text{Hom}^*_{\mathbb{P}^1 \times \mathbb{P}^1} (\mathcal{E}, \mathcal{E})$. As a corollary, the first order infinitesimal automorphisms and the first order infinitesimal deformations of $\mathcal{E}$ are classified by $\text{Ext}^0_{\mathbb{P}^1 \times \mathbb{P}^1} (\mathcal{E}, \mathcal{E})$ and $\text{Ext}^1_{\mathbb{P}^1 \times \mathbb{P}^1} (\mathcal{E}, \mathcal{E})$, respectively, and $\text{Ext}^2_{\mathbb{P}^1 \times \mathbb{P}^1} (\mathcal{E}, \mathcal{E})$ serves as an obstruction space. Based on our stability analysis above, we can partially compute these spaces as follows.

**Proposition 4.7.** Let $\mathcal{E}$ be a locally free sheaf bimodule on $\mathbb{P}^1$.

1. It holds that

$$\chi (\mathcal{E}, \mathcal{E}) = \sum_{i=0}^2 (-1)^i \dim_k \text{Ext}^i_{\mathbb{P}^1 \times \mathbb{P}^1} (\mathcal{E}, \mathcal{E}) = -8.$$  

(4.12)

2. When $\mathcal{E}$ is semi-stable with respect to the polarization $\mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(-K_{\mathbb{P}^1 \times \mathbb{P}^1})$, then $\text{Ext}^2_{\mathbb{P}^1 \times \mathbb{P}^1} (\mathcal{E}, \mathcal{E}) = 0$. In particular, the deformation of $\mathcal{E}$ is unobstructed.

3. When $\mathcal{E}$ is stable with respect to the polarization $\mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(-K_{\mathbb{P}^1 \times \mathbb{P}^1})$ or the sheaf $\mathcal{U}$ on $W$ is invertible, then

$$\left( \dim_k \text{Ext}^i_{\mathbb{P}^1 \times \mathbb{P}^1} (\mathcal{E}, \mathcal{E}) \right)_{i=0,1,2} = (1, 9, 0).$$

(4.13)

**Proof.** (1) Since $\dim \mathbb{P}^1 \times \mathbb{P}^1 = 2$, the extension groups between coherent sheaves are concentrated in degrees 0, 1, 2.

It follows from the classification that any locally free sheaf bimodule on $\mathbb{P}^1$ is deformation equivalent to the standard one $\mathcal{O}_\Delta (a') \oplus \mathcal{O}_\Delta (b')$ for some $(a', b')$. Using the deformation invariance of $\chi (\mathcal{E}, \mathcal{E})$ one can reduce the computation to that for the standard one, to obtain the result.
(2) By the Serre duality, \( \dim_k \text{Ext}^2_{\mathbb{P}^1 \times \mathbb{P}^1}(\mathcal{E}, \mathcal{E}) = \dim_k \text{Ext}^0_{\mathbb{P}^1 \times \mathbb{P}^1}(\mathcal{E}, \mathcal{E}(K_{\mathbb{P}^1 \times \mathbb{P}^1})) \). Since \( p_\mathcal{E}(K_{\mathbb{P}^1 \times \mathbb{P}^1})(t) = p_\mathcal{E}(t - 1) \), it always follows that \( p_\mathcal{E} > p_\mathcal{E}(K_{\mathbb{P}^1 \times \mathbb{P}^1}) \). Hence if \( \mathcal{E} \) is semi-stable, there is no non-trivial homomorphism from \( \mathcal{E} \) to \( \mathcal{E}(K_{\mathbb{P}^1 \times \mathbb{P}^1}) \) by [HL10 Proposition 1.2.7].

(3) When \( \mathcal{E} \) is stable, then \( \text{Ext}^0_{\mathbb{P}^1 \times \mathbb{P}^1}(\mathcal{E}, \mathcal{E}) = k \text{id}_\mathcal{E} \) by [HL10 Corollary 1.2.8]. When \( \mathcal{U} \) is invertible we have \( \text{Hom}_W(\mathcal{U}, \mathcal{U}) \simeq H^0(W, \mathcal{O}_W) \) and the latter is isomorphic to \( k \), since \( W \) is an effective Cartier divisor of \( \mathbb{P}^1 \times \mathbb{P}^1 \). Combining these with the first two items, we obtain the conclusion.

\[ \square \]

4.3. Moduli stack. Let \( \widetilde{\mathcal{M}}'_{\text{sh}} \) be the stack of locally free sheaf bimodules on \( \mathbb{P}^1 \), which is an open substack of the stack of coherent sheaves on \( \mathbb{P}^1 \times \mathbb{P}^1 \). The automorphism group of every object of \( \mathcal{M}'_{\text{sh}} \) contains the multiplicative group \( G_m \), and we write the rigidification as

\[ \widetilde{\mathcal{M}}_{\text{sh}} := \widetilde{\mathcal{M}}'_{\text{sh}}/BG_m, \]

which is written as \( (\widetilde{\mathcal{M}}'_{\text{sh}})^{G_m} \) and \( \widetilde{\mathcal{M}}'_{\text{sh}} \sslash G_m \) in [ACV03] and [Rom05] respectively.

The stack \( \widetilde{\mathcal{M}}_{\text{sh}} \) is decomposed into open and closed substacks by \( \text{deg} \mathcal{U} \). Note that the map \( \mathcal{U} \mapsto \mathcal{U} \otimes v^* \mathcal{O}_{\mathbb{P}^1}(k) \) induces an isomorphism between components whose degrees differ by an even integer, preserving the equivalence classes of the associated categories \( \text{Qgr} \). We write the connected component parametrizing sheaf bimodules of degree 2 (resp. 3) as \( \widetilde{\mathcal{M}}_{\text{sh},0} \) (resp. \( \widetilde{\mathcal{M}}_{\text{sh},1} \)).

The discussion in the previous subsections shows that \( \mathcal{E} \) is simple and has 9-dimensional unobstructed deformation space if \( W \) is non-singular. Hence the dimension of the open substack of \( \widetilde{\mathcal{M}}_{\text{sh}} \) consisting of \( \mathcal{E} \) with non-singular \( W \) has dimension 9. Roughly speaking, 8 out of 9 comes from the linear system

\[ |W| = | - K_{\mathbb{P}^1 \times \mathbb{P}^1} | = \mathbb{P} \ast H^0(\mathcal{O}_{\mathbb{P}^1}(2) \boxtimes \mathcal{O}_{\mathbb{P}^1}(2)) \simeq \mathbb{P}^8, \]

and the remaining 1 comes from \( \text{Pic}^0(W) \). The group \( \text{Aut}(\mathbb{P}^1) \times \text{Aut}(\mathbb{P}^1) \simeq (\text{PGL}_2)^2 \) acts naturally on the stack \( \widetilde{\mathcal{M}}_{\text{sh}} \) by pull-back of coherent sheaves. The dimension of the quotient stack \( \mathcal{M}_{\text{sh}} := [\widetilde{\mathcal{M}}_{\text{sh}}/(\text{PGL}_2)^2] \) is the same as the expected dimension 3 of the moduli space of noncommutative Hirzebruch surfaces. We let \( \mathcal{M}_{\text{sh},0} \) and \( \mathcal{M}_{\text{sh},1} \) denote the quotients of \( \widetilde{\mathcal{M}}_{\text{sh},0} \) and \( \widetilde{\mathcal{M}}_{\text{sh},1} \) respectively.

5. Quivers with relations from noncommutative Hirzebruch surfaces

5.1. Sheaf bimodule as the kernel of the dual gluing functor. Let \( \mathcal{D} \) be a triangulated category. A full triangulated subcategory \( \mathcal{N} \subset \mathcal{D} \) is right (resp. left) admissible if the inclusion functor \( i: \mathcal{N} \hookrightarrow \mathcal{D} \) has a right (resp. left) adjoint functor \( i^! \) (resp. \( i^* \)). It is admissible if it is both right and left admissible. The right (resp. left) orthogonal \( \mathcal{N}^\perp \) (resp. \( {}^\perp \mathcal{N} \)) is the full subcategory of \( \mathcal{D} \) consisting of objects \( X \in \mathcal{D} \) satisfying \( \text{Hom}(N, X) = 0 \) (resp. \( \text{Hom}(X, N) = 0 \)) for any \( N \in \mathcal{N} \). The subcategory \( \mathcal{N} \) is right (resp. left) admissible if and only if for any \( X \in \mathcal{D} \), there exists \( N \in \mathcal{N} \) and \( M \in \mathcal{N}^\perp \) (resp. \( N' \in \mathcal{N} \) and \( M' \in {}^\perp \mathcal{N} \)) forming a distinguished triangle \( N \to X \to M \to N[1] \) (resp. \( M' \to X \to N' \to M'[1] \)). Let \( (\mathcal{N}_1, \ldots, \mathcal{N}_n) \) be a sequence of subcategories of \( \mathcal{D} \). For \( i = 1, \ldots, n \), we let \( \mathcal{D}_i \) denote the smallest full triangulated subcategory of \( \mathcal{D} \) containing \( \text{objects of } \mathcal{N}_1, \ldots, \mathcal{N}_i \). The sequence \( (\mathcal{N}_1, \ldots, \mathcal{N}_n) \) is a semiorthogonal decomposition of \( \mathcal{D} \) if
\textbullet \ \mathcal{D}_{i-1} \text{ is right admissible in } \mathcal{D}_i \text{ for } i = 1, \ldots, n-1,
\textbullet \ \text{the left orthogonal of } \mathcal{D}_{i-1} \text{ in } \mathcal{D}_i \text{ is equivalent to } \mathcal{N}_i, \text{ and}
\textbullet \ \mathcal{D}_n \text{ is equivalent to } \mathcal{D}.

We write \( \mathcal{D} = \langle \mathcal{N}_1, \ldots, \mathcal{N}_n \rangle \) if \( \langle \mathcal{N}_1, \ldots, \mathcal{N}_n \rangle \) is a semiorthogonal decomposition of \( \mathcal{D} \).

Let \( \mathcal{D} = \langle \mathcal{N}_1, \mathcal{N}_2 \rangle \) be a semiorthogonal decomposition, so that the inclusion functor \( i_1 : \mathcal{N}_1 \to \mathcal{D} \) has a right adjoint \( i_1^! \) and the inclusion functor \( i_2 : \mathcal{N}_2 \to \mathcal{D} \) has a left adjoint \( i_2^! \). The \textit{gluing bimodule} is given by

\[
\text{hom}_\mathcal{D}(i_1(-), i_2(-)) \cong \text{hom}_{\mathcal{N}_2}(\phi(-), -) \cong \text{hom}_{\mathcal{N}_1}(-, \phi^!(-)) : \mathcal{D} \to D^b(\mathbb{k}),
\]

where

\[
\phi = i_2^! \circ i_1 : \mathcal{N}_1 \to \mathcal{N}_2
\]

is the \textit{gluing functor} and

\[
\phi^! = i_1^! \circ i_2 : \mathcal{N}_2 \to \mathcal{N}_1
\]

is the \textit{dual gluing functor}. The category \( \mathcal{D} \) can be recovered from the categories \( \mathcal{N}_1, \mathcal{N}_2 \) and the gluing bimodule by the upper-triangular matrix construction \([\text{Orl}]\).

The following is a generalization of \([\text{Orl92}]\).

**Theorem 5.1.** Let \( X \) be a smooth projective scheme and \( \mathcal{E} \) be a locally free sheaf bimodule of rank 2 on \( X \). Then one has a semiorthogonal decomposition

\[
D^b \text{qgr } \mathcal{S}(\mathcal{E}) = \langle f_1^* D^b \text{coh } X, f_0^! D^b \text{coh } X \rangle,
\]

and the dual gluing functor is given by

\[
f_0^! D^b \text{coh } X \xrightarrow{(f_0^!)^{-1}} D^b \text{coh } X \cong \mathcal{E} \xrightarrow{\om_{\mathcal{O}_X}} D^b \text{coh } X \xrightarrow{f_1^!} f_1^* D^b \text{coh } X.
\]

**Proof.** The subcategory \( f_1^* D^b \text{coh } X \) is right admissible since \( f_1^* \) has a right adjoint functor \( \mathbb{R}f_1^* \). The right admissibility also implies the left admissibility since \( D^b \text{qgr } \mathcal{S}(\mathcal{E}) \) has a Serre functor. It follows from (2.28) that the full subcategory \( f_1^* D^b \text{coh } X \) is contained in the right orthogonal of \( f_0^! D^b \text{coh } X \). For any \( m \in \mathbb{Z} \) and any locally-free \( \mathcal{O}_X \)-module \( \mathcal{F} \), one has an exact sequence

\[
0 \to \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{Q}_m \otimes_{\mathcal{O}_X} \mathcal{E}_{e_m+2} \to \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{E}^{sm} \otimes_{\mathcal{O}_X} \mathcal{E}_{e_{m+1}} \to \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{E}_{e_m} \to \mathcal{F} \to 0
\]

in \( \text{Gr } \mathcal{S}(\mathcal{E}) \) (see \([\text{Nym05}]\) Theorem 1.4), which induces an exact sequence

\[
0 \to f^*_{m+2}(\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{Q}_{m}) \to f^*_{m+1}(\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{E}^{sm}) \to f^*_m \mathcal{F} \to 0
\]

in \( \text{qgr } \mathcal{S}(\mathcal{E}) \). It follows from (5.7) that if a full triangulated subcategory \( \mathcal{T} \) of \( D^b \text{qgr } \mathcal{S}(\mathcal{E}) \) contains \( f^*_{m+2}D^b \text{coh } X \) and \( f^*_{m+1}D^b \text{coh } X \) for some \( m \in \mathbb{Z} \), then it also contains \( f^*_mD^b \text{coh } X \). Similarly, if a full triangulated subcategory \( \mathcal{T} \) of \( D^b \text{qgr } \mathcal{S}(\mathcal{E}) \) contains \( f^*_{m+1}D^b \text{coh } X \) and \( f^*_mD^b \text{coh } X \), then it also contains \( f^*_{m+2}D^b \text{coh } X \), since the sheaf bimodule \( \mathcal{Q}_m \) is invertible. It follows that if a full triangulated subcategory \( \mathcal{T} \) of \( D^b \text{qgr } \mathcal{S}(\mathcal{E}) \) contains \( f^*_{m+2}D^b \text{coh } X \) and \( f^*_{m+1}D^b \text{coh } X \) for some \( m \in \mathbb{Z} \), then it also contains \( f^*_nD^b \text{coh } X \) for all \( n \in \mathbb{Z} \). It follows from \([\text{Nym05}]\) Proposition 2.19] that the subcategory of \( D^b \text{qgr } \mathcal{S}(\mathcal{E}) \) right orthogonal to \( f^*_nD^b \text{coh } X \) for all \( n \in \mathbb{Z} \) is zero, and \( (5.4) \) is proved. The dual gluing functor is given on locally-free \( \mathcal{O}_X \)-modules by

\[
\mathbb{R}f_1^* f_0^! \cong (-) \otimes_{\mathcal{O}_X} \mathcal{A}_{01} \cong (-) \otimes_{\mathcal{O}_X} \mathcal{E}
\]

by (2.28). \( \square \)
5.2. Full strong exceptional collections on noncommutative Hirzebruch surfaces. Let $\mathcal{E}$ be a locally free sheaf bimodule of rank 2 on $\mathbb{P}^1$. By Theorem 5.1, the derived category $D^b \text{qgr} \mathbb{S}(\mathcal{E})$ is obtained by gluing two copies of $D^b \text{coh} \mathbb{P}^1$ by the dual gluing functor $\phi : (-) \otimes_{\mathcal{O}_{\mathbb{P}^1}} \mathcal{E} : D^b \text{coh} \mathbb{P}^1 \to D^b \text{coh} \mathbb{P}^1$. If one sets

$$
(E_1, E_2, E_3, E_4) := (f_1^* \mathcal{O}_{\mathbb{P}^1}(-m - 1), f_1^* \mathcal{O}_{\mathbb{P}^1}(-m), f_0^* \mathcal{O}_{\mathbb{P}^1}(-1), f_0^* \mathcal{O}_{\mathbb{P}^1}),
$$

then this is a full exceptional collection for any $m \in \mathbb{Z}$.

In this section, we specify the pairs $(\mathcal{E}, m)$ for which (5.9) is a strong exceptional collection.

If we set

$$
(F_1, F_2, F_3, F_4) := (\mathcal{O}_{\mathbb{P}^1}(-m - 1), \mathcal{O}_{\mathbb{P}^1}(-m), \phi_1^! \mathcal{O}_{\mathbb{P}^1}(-1), \phi_1^! \mathcal{O}_{\mathbb{P}^1}),
$$

then one has

$$
\mathbb{R}\text{Hom}_{D}(E_i, E_j) \simeq \begin{cases} 
\mathbb{R}\text{Hom}_{\mathcal{O}_{\mathbb{P}^1}}(F_i, F_j) & 3 \neq i < j, \\
\mathbb{R}\text{Hom}_{\mathcal{O}_{\mathbb{P}^1}}(F_i, F_j) & (i, j) = (3, 4), \\
\mathbb{R}\text{Hom}_{\mathcal{O}_{\mathbb{P}^1}}(F_i, F_j) & i = j, \\
0 & \text{otherwise}.
\end{cases}
$$

Hence we obtain the following explicit characterization of strongness, which immediately follows from Corollary 3.15.

**Theorem 5.2.** (5.9) is strong if and only if

$$
\text{Ext}^1_{\mathcal{O}_{\mathbb{P}^1}}(F_i, F_j) = 0
$$

for $(i, j) = (1, 3), (1, 4), (2, 3), (2, 4)$, which is the case if and only if

$$
a' \geq -m - 1.
$$

**Corollary 5.3.** Let $\mathcal{E}$ be a sheaf bimodule on $\mathbb{P}^1$ such that $\chi(\mathcal{U}) = 2$. Then the exceptional collection (5.9) for $m = 1$ is strong if and only if

- $W$ is non-reduced and $\deg D = 0, 2$ or $4$,
- $W$ is integral, or
- $W$ is not irreducible and $p \geq -1$.

Similarly, when $\chi(\mathcal{U}) = 1$, the exceptional collection (5.9) for $m = 1$ is strong if and only if

- $W$ is non-reduced and $\deg D = 1$ or $3$,
- $W$ is integral or
- $W$ is not irreducible and $p \geq -1$.

**Proof.** Immediately follows from Corollary 5.15. □

5.3. Quivers with relations and the moduli stack of relations. If we set

$$
(G_1, G_2, G_3, G_4) := (v^* \mathcal{O}_{\mathbb{P}^1}(-m - 1), v^* \mathcal{O}_{\mathbb{P}^1}(-m), u^* \mathcal{O}_{\mathbb{P}^1}(-1) \otimes \mathcal{U}, \mathcal{U}),
$$

then

$$
\mathbb{R}\text{Hom}_{\mathcal{O}_{\mathbb{P}^1}}(F_2, F_3) \simeq \mathbb{R}\text{Hom}_{\mathcal{O}_{\mathbb{P}^1}}(O_{\mathbb{P}^1}(-m), \phi_1^! \mathcal{O}_{\mathbb{P}^1}(-1)) \\
\simeq \mathbb{R}\text{Hom}_{\mathcal{O}_{\mathbb{P}^1}}(O_{\mathbb{P}^1}(-m), \mathcal{O}_{\mathbb{P}^1}(-1) \otimes_{\mathcal{O}_{\mathbb{P}^1}} \mathcal{E}) \\
\simeq \mathbb{R}\text{Hom}_{\mathcal{O}_{\mathbb{P}^1}}(O_{\mathbb{P}^1}(-m), v_*(u^* \mathcal{O}_{\mathbb{P}^1}(-1) \otimes \mathcal{U}))
$$

(5.15)

$$
\simeq \mathbb{R}\text{Hom}_{\mathcal{O}_{\mathbb{P}^1}}(O_{\mathbb{P}^1}(-m), u^* \mathcal{O}_{\mathbb{P}^1}(-1) \otimes \mathcal{U}) \\
\simeq \mathbb{R}\text{Hom}_{\mathcal{O}_{\mathbb{P}^1}}(G_2, G_3),
$$

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and similarly
\[ \mathbb{R} \text{Hom}_{p_1}(F_i, F_j) \simeq \mathbb{R} \text{Hom}_{p_1}(G_i, G_j) \]  
for all \( 1 \leq i < j \leq 4 \) except \((i, j) = (3, 4)\).

Consider the case when \( \mathcal{U} \) is invertible, \( \deg \mathcal{U} = 2 \), the collection (5.9) for \( m = 1 \) is strong, and \( b - a = 0 = b' - a' \). Set
\[ (L_0, L_1, L_2) := (v^* \mathcal{O}_{p_1}(1), v^* \mathcal{O}_{p_2}(1) \otimes u^* \mathcal{O}_{p_1}(-1) \otimes \mathcal{U}, u^* \mathcal{O}_{p_1}(1)), \]  
so that
\[ \text{Hom}(E_1, E_2) \simeq H^0(L_0), \quad \text{Hom}(E_2, E_3) \simeq H^0(L_1), \quad \text{Hom}(E_3, E_4) \simeq H^0(L_2). \]  
(5.18)

Consider the quiver \( Q^0 \) as in Figure 5.1. For \( i = 0, 1, 2 \), fix an isomorphism of vector spaces
\[ \text{span} \{a_i, b_i\} \simeq H^0(L_i). \]  
(5.19)

Under these isomorphisms, the following subspace is identified with a 2-dimensional linear subspace \( I \subset e_4 \mathbb{k} Q^0 e_1 \), which automatically is a 2-sided ideal of \( \mathbb{k} Q^0 \).

\[ \ker \left( H^0(L_2) \otimes H^0(L_1) \otimes H^0(L_0) \to H^0(L_0 \otimes L_1 \otimes L_2) \right) \]  
(5.20)

Then the endomorphism algebra of the full strong exceptional collection is isomorphic to the path algebra of \( Q^0 \) by the relations (5.20).

On the other hand, consider the 3-dimensional AS-regular cubic \( \mathbb{Z} \)-algebra \( A = A(W, L_0, L_1, L_2) \) associated to the admissible quadruple
\[ (W, L_0, L_1, L_2) \]  
(5.21)

studied in [vdB11]. Then the relation (5.20) is identified with the cubic relations
\[ \ker (A_{23} \otimes A_{12} \otimes A_{01} \to A_{03}), \]  
(5.22)

which implies the derived equivalences
\[ D^b \text{qgr } \mathcal{S}(\mathcal{E}) \simeq D^b \text{ mod } \mathbb{k} Q^0/I \simeq D^b \text{ qgr } A(W, L_0, L_1, L_2). \]  
(5.23)

Combined with the arguments below, this shows the 2nd assertion of Theorem 1.1.

The moduli stack of relations of the quiver \( Q^0 \) is defined as
\[ \widetilde{\mathcal{M}}_{rel,0} := [V_1 \otimes V_2 \otimes V_3 \otimes V_4 / \text{GL}(V_1) \times \text{GL}(V_2) \times \text{GL}(V_3) \times \text{GL}(V_4)], \]  
(5.24)

where \( V_i = \text{span} \{a_i, b_i\} \) for \( i = 1, 2, 3 \) and \( V_4 \) is a 2-dimensional vector space. It is studied in detail in [OUa]. The generic stabilizer of \( \mathcal{M}_{rel,0} \) is the kernel \( K_0 \cong (\mathbb{G}_m)^3 \) of the map
\[ Z \left( \prod_{i=1}^{4} \text{GL}(V_i) \right) \cong (\mathbb{G}_m)^4 \to \mathbb{G}_m, \quad (\lambda_i)_{i=1}^{4} \mapsto \lambda_1 \lambda_2 \lambda_3 \lambda_4. \]  
(5.25)

We write the rigidified stack as \( \mathcal{M}_{rel,0} := \widetilde{\mathcal{M}}_{rel,0}/\mathbb{K} K_0 \). The corresponding GIT quotient
\[ \text{Proj } \mathbb{k}[V_1 \otimes V_2 \otimes V_3 \otimes V_4]_{\text{SL}(V_1) \times \text{SL}(V_2) \times \text{SL}(V_3) \times \text{SL}(V_4)} \cong \mathbb{P}(2, 4, 4, 6) \]  
(5.26)

can also be interpreted as the SLOCC moduli space of 4 qubits (cf. [OUb] and references therein).
Consider next the case when $\mathcal{U}$ is invertible, $\deg \mathcal{U} = 1$, the collection \((5.9)\) for \(m = 1\) is strong, and \(b - a = 1 = b' - a'\). Define the line bundles \((L_0, L_1, L_2)\) as in \((5.17)\), so that
\[
\text{Hom}(E_1, E_2) \simeq H^0(L_0), \quad \text{Hom}(E_2, E_3) \simeq H^0(L_1), \quad \text{Hom}(E_3, E_4) \simeq H^0(L_2).
\]
(5.27)
Consider the quiver $Q^1$ defined as in Figure 5.2. Fix isomorphisms
\[
\text{span} \{a_1, a_2\} \simeq H^0(L_0), \quad \text{span} \{a_7\} \simeq H^0(L_1), \quad \text{span} \{a_4, a_5\} \simeq H^0(L_2),
\]
(5.28)
and lifts
\[
\text{span} \{a_3\} \hookrightarrow \text{Hom}(E_0, E_2), \quad \text{span} \{a_6\} \hookrightarrow \text{Hom}(E_1, E_3)
\]
(5.29)
of isomorphisms
\[
\text{span} \{a_3\} \xrightarrow{\sim} \text{coker} (\text{Hom}(E_0, E_1) \otimes \text{Hom}(E_1, E_2) \to \text{Hom}(E_0, E_2))
\]
(5.30)
and
\[
\text{span} \{a_6\} \xrightarrow{\sim} \text{coker} (\text{Hom}(E_1, E_2) \otimes \text{Hom}(E_2, E_3) \to \text{Hom}(E_1, E_3)),
\]
(5.31)
respectively. From these choices one obtains a surjective homomorphism of $k^4$-algebras from $kQ^1$ to the endomorphism algebra of the full strong exceptional collection. The kernel of the homomorphism is a 3-dimensional linear subspace of $e_4 k Q^1 e_1$, which automatically is a 2-sided ideal of the path algebra.

The moduli stack of relations of the quiver \((5.2)\) is defined as follows. Let
- $V_1 = \text{span} \{a_3\}$
- $V_2 = \text{span} \{a_1, a_2\}$
- $V_3 = \text{span} \{a_7\}$
- $V_4 = \text{span} \{a_4, a_5\}$
- $V_5 = \text{span} \{a_6\}$

and $V_6$ be a 3-dimensional vector space. Consider the group
\[
G = (\text{Hom}(V_1, V_2 \otimes V_3) \times \text{Hom}(V_5, V_3 \otimes V_4)) \ltimes \prod_{1 \leq i \leq 6} \text{GL}(V_i),
\]
(5.32)
where the semi-direct product is defined by the obvious left action
\[
(g_i)_{i=1,...,6} : (\varphi, \psi) \mapsto (g_2 \otimes g_3 \circ \varphi \circ g_1^{-1}, g_3 \otimes g_4 \circ \psi \circ g_5^{-1}).
\]
(5.33)
Then
\[
\tilde{\mathcal{M}}_{\text{rel,1}} := [(V_1 \otimes V_4 \oplus V_2 \otimes V_3 \oplus V_4 \oplus V_2 \otimes V_5) \otimes V_6/G].
\]
(5.34)
The generic stabilizer of $\tilde{\mathcal{M}}_{\text{rel,1}}$ is the direct product $K_1$ of the kernels of the maps
\[
Z(\text{GL}(V_4) \times \text{GL}(V_3)) \to \mathbb{G}_m, \quad (\lambda_1, \lambda_4) \mapsto \lambda_1 \lambda_4,
\]
(5.35)
\[
Z(\text{GL}(V_2) \times \text{GL}(V_5)) \to \mathbb{G}_m, \quad (\lambda_2, \lambda_5) \mapsto \lambda_2 \lambda_5,
\]
(5.36)
and
\[ Z(GL(V_2) \times GL(V_3) \times GL(V_4)) \to \mathbb{G}_m, \quad (\lambda_2, \lambda_3, \lambda_5) \mapsto \lambda_2 \lambda_3 \lambda_5. \] (5.37)

The rigidified stack will be denoted by
\[ \mathcal{M}_{\text{rel}, i} := \widetilde{\mathcal{M}}_{\text{rel}, i}/BK_i. \] (5.38)

6. MODULI STACK OF NONSINGULAR ADMISSIBLE QUADRUPLES

6.1. The moduli stack \( \mathcal{M}_{\text{ell}} \).
A nonsingular admissible quadruple \((E, L_0, L_1, L_2)\) consists of a smooth proper curve \(E\) of genus 1 and three line bundles \((L_0, L_1, L_2)\) on \(E\) such that \(L_i \not\cong L_j\) for \(i \neq j\). An isomorphism of nonsingular admissible quadruples \((E, L_0, L_1, L_2)\) and \((E', L'_0, L'_1, L'_2)\) consists of an isomorphism \(\varphi : E \to E'\) of \(k\)-schemes and isomorphisms \(\varphi_i : L_i \to \varphi^* L'_i\) of line bundles on \(E\) for \(i = 0, 1, 2\).

Let \(\mathcal{M}_{\text{ell}}\) be the category fibered in groupoids over the category of \(k\)-schemes, where an object of its fiber category over a scheme \(S \to \text{Spec} k\) is a collection \((C, L_0, L_1, L_2)\) of a family \(C \to S\) of elliptic curves over \(S\) and three line bundles \((L_0, L_1, L_2)\) on \(C\), and a morphism from \((C, L_0, L_1, L_2)\) to \((C', L'_0, L'_1, L'_2)\) consists of an isomorphism \(\varphi : C \to C'\) of elliptic curves over \(S\) and isomorphisms \(\varphi_i : L_i \to \varphi^* L'_i\) of line bundles for \(i = 0, 1, 2\). It is an algebraic stack, which is a gerbe banded by \((\mathbb{G}_m)^3\), and we write its rigidification as \(\mathcal{M}_{\text{ell}} := \widetilde{\mathcal{M}}_{\text{ell}}/B(\mathbb{G}_m)^3\).

\(\mathcal{M}_{\text{ell}}\) is decomposed into connected components by the degrees of line bundles. The connected components of \(\mathcal{M}_{\text{ell}}\) parametrizing the quadruples \((E, L_0, L_1, L_2)\) with
\[ \deg(L_0, L_1, L_2) = (2, 2, 2) \text{ and } (2, 1, 2) \] (6.1)
will be denoted by \(\mathcal{M}_{\text{ell}, 0}\) and \(\mathcal{M}_{\text{ell}, 1}\) respectively.

6.2. Sheaf bimodules and nonsingular admissible quadruples. For \(i = 0, 1\), let \(\mathcal{M}_{\text{sh}, i} \subset \mathcal{M}_{\text{ell}, i}\) be the open substack of sheaf bimodules whose support \(W\) is a smooth divisor of bidegree \((2, 2)\). There is a natural morphism \(\Phi_i : \mathcal{M}_{\text{sh}, i} \to \mathcal{M}_{\text{ell}, i}\) which sends \(E\) to \((W, L_0, L_1, L_2)\), where \((L_0, L_1, L_2)\) are the line bundles defined in \((5.17)\). Conversely, given a nonsingular admissible quadruple \((E, L_0, L_1, L_2)\), since the line bundles \(L_2\) and \(L_0\) are assumed to be non-isomorphic, they together define a closed immersion \(\iota : E \hookrightarrow \mathbb{P}^1 \times \mathbb{P}^1\), and the push-forward \(E := \iota_* (L_0^{-1} \otimes L_1 \otimes L_2)\) gives a sheaf bimodule on \(\mathbb{P}^1\). This gives the inverse morphism \(\mathcal{M}_{\text{ell}, i} \to \mathcal{M}_{\text{sh}, i}^0\), so that \(\mathcal{M}_{\text{sh}, i}^0\) and \(\mathcal{M}_{\text{ell}, i}\) are isomorphic to each other.

6.3. From relations to nonsingular admissible quadruples. For \(i = 0, 1\), there is a natural morphism \(\Psi_i : \mathcal{M}_{\text{ell}, i} \to \mathcal{M}_{\text{rel}, i}\) sending a nonsingular admissible quadruple \((E, L_0, L_1, L_2)\) to the relation coming from the full strong exceptional collection \((5.9)\) for \(m = 1\) of the derived category \(D^b \text{qgr } S\left(\Phi_i^{-1}(E, L_0, L_1, L_2)\right)\). The inverse birational map from \(\mathcal{M}_{\text{rel}, i}\) to \(\mathcal{M}_{\text{ell}, i}\) is given by considering the moduli space of representations of the quiver with relations.

For a 2-sided ideal \(I\) of the quiver \(Q^0\), let \(\text{rep}(Q^0, I)\) be the stack of finite dimensional right \(kQ^0/I\)-modules. Note that there are isomorphisms of abelian groups as follows.
\[ K_0 \left( \text{rep}(Q^0, I) \right) \simeq \mathbb{Z}^4; \quad [M] \mapsto (\dim_k M e_i)_{i=1}^4 \] (6.2)

For \(M \in \text{rep}(Q^0, I)\), the corresponding element \((\dim_k M e_i)_{i=1}^4 \in \mathbb{Z}^4\) is called the dimension vector of \(M\).
Consider the map \( \theta : K_0(\text{rep}(Q^0, I)) \to \mathbb{Z} \) which is identified with \( \mathbb{Z}^4 \xrightarrow{(-3,1,1,1)} \mathbb{Z} \) under the isomorphism \((6.2)\). A module \( M \in \text{rep}(Q^0, I) \) is said to be \( \theta \)-semistable if the inequality \( \theta(N) \leq (\leq) \theta(M) = 0 \) holds for any submodule \( 0 \neq N \subset M \).

Let us consider the moduli stack \( \mathcal{N} = \mathcal{N}_{Q^0}(1, \theta) \) of \( \theta \)-stable representations of the quiver \( Q^0 \) (without relations) of dimension vector \( \mathbf{1} = (1,1,1,1) \). It follows from [Kin94] that \( \mathcal{N} \) admits a projective fine moduli scheme which is described as the following GIT quotient, as we explain next.

For \( j = 0, 1, 2, 3 \), set \( U_j := \mathbf{k} \) and \( H' := \prod_j \text{GL}(U_j) = \mathbb{G}_m^4 \). Consider the affine space

\[
\prod_{a \in (Q^0)_1} \text{Hom} \left( U_{(a)}, U_{(s(a))} \right) = V_0 \times V_1 \times V_2,
\]

where

\[
V_i := \prod_{a \in (Q^0)_1} \text{Hom} \left( U_{i+1}, U_i \right) \simeq \mathbf{k}^2.
\]

The group \( H' \) naturally acts on it, but the small diagonal \( \mathbb{G}_m \subset H' \) acts trivially. Hence we consider the induced action of the quotient group \( H := H'/\mathbb{G}_m \). In what follows, the symbol \( \chi(-) \) denotes the group of characters of a group \( - \).

**Lemma 6.1.** There exists a canonical isomorphism \( \chi(H') \sim \xrightarrow{\sim} K_0(Q^0, I)^{\vee} \) which extends to an isomorphism of short exact sequences of abelian groups as follows, where the right-most vertical map sends \( \text{id}_{\mathbb{G}_m} \) to 1.

\[
\begin{array}{ccc}
0 & \xrightarrow{\chi(H)} & \chi(H') \xrightarrow{\chi(\mathbb{G}_m)} \mathbb{Z} \text{id}_{\mathbb{G}_m} \xrightarrow{\sim} 0 \\
\downarrow{\simeq} & & \downarrow{\simeq} \\
0 & \xrightarrow{\text{Ker } \varphi} & K_0(Q^0, I)^{\vee} \xrightarrow{\varphi:=(-,1)} \mathbb{Z} \xrightarrow{\sim} 0
\end{array}
\]

**Proof.** We only describe the canonical isomorphism. The rest of the proof is rather straightforward.

Take \( \theta \in \chi(H') \) and \( M \in \text{rep}(Q^0, I) \). Consider the map \( t : \mathbb{G}_m \to \prod_i \text{GL}(M_{e_i}) \) which sends \( \lambda \in \mathbb{G}_m \) to \( \left(M_{e_i} \xrightarrow{\lambda} M_{e_i} \right)_i \). Then we let the integer \( n \) which is associated to the class \([M] \in K_0(Q^0, I)\) to be the one defined by the equality \( \theta(t(\lambda)) = \lambda^n \).

Since \( \langle \theta, \mathbf{1} \rangle = 0 \), we thus obtain a character of the group \( H \), which will also be denoted by \( \theta \) by an abuse of notation. It, in turn, defines a linearization of the space \( V_1 \times V_2 \times V_3 \).

By the results of [Kin94], the moduli stack \( \mathcal{N} \) is isomorphic to the GIT quotient

\[
(V_1 \times V_2 \times V_3) \sslash_{\theta} \left( \mathbb{G}_m^4 \sslash \mathbb{G}_m \right) \simeq \mathbb{P}(V_1) \times \mathbb{P}(V_2) \times \mathbb{P}(V_3).
\]

A relation \([I] \in \mathcal{M}_{\text{rel,0}}\) determines the moduli space \( \mathcal{N}_I = \mathcal{N}_{(Q^0, I)}(1, \theta) \subset \mathcal{N} \) of representations of the quiver with relations \((Q^0, I)\), which is a complete intersection of two divisors of multidegree \((1,1,1,1)\) in \( \mathbb{P}(V_1) \times \mathbb{P}(V_2) \times \mathbb{P}(V_3) \). It is an elliptic curve if \( I \) is sufficiently general.

Since \( \mathcal{N}_I \) is a fine moduli scheme, it comes with the universal representation on it. In particular, there are tautological line bundles \( M_1, \ldots, M_4 \) corresponding to the 4 vertices of the quiver \( Q^0 \), which are unique up to simultaneous tensoring by a line bundle on \( \mathcal{N}_I \). Now

\[
(E, L_0, L_1, L_2) := (\mathcal{N}_I, M_2, M_2^{\vee} \otimes M_3, M_3^{\vee} \otimes M_4)
\]

(6.7)
ideal

I

[72x-2796]ideal

When

Remark 6.2. This induces a morphism $\Phi_0$ to $\mathcal{M}_{\text{rel}, 0}$ from the open substack of $\mathcal{M}_{\text{rel}, 0}$ consisting of points $[I]$ for which $\mathcal{N}_I$ is nonsingular and $L_i \neq L_j$ for $i \neq j$, which is a birational inverse to the morphism $\Psi_0$. This is shown by checking $\Psi_0 \Phi_0 = \text{id}$, since both $\mathcal{M}_{\text{rel}, 0}$ and $\mathcal{M}_{\text{rel}, 0}$ are irreducible smooth stacks of the same dimension 3 with trivial generic stabilizers.

To see $\Psi_0 \Phi_0 (E, L_0, L_1, L_2) = (E, L_0, L_1, L_2)$, note that for $[I] = \Phi_0 (E, L_0, L_1, L_2)$ there exists the canonical embedding $E \hookrightarrow \mathcal{N}_I$. In fact, if one chooses bases for $H^0 (E, L_i)$ for $i = 1, 2, 3$, then one obtains it as the classifying morphism by regarding the collection $(\mathcal{O}_E, L_0, L_0 \otimes L_1, L_0 \otimes L_1 \otimes L_2)$ and the set of bases as a family of stable representations of $(Q^0, I)$ parametrized by $E$. From the description of $\mathcal{N}_I$ as a complete intersection, one concludes that the closed immersion is actually an isomorphism. The rest follows from the definition of the notion of classifying morphism.

The parallel arguments as above work for the moduli space $\mathcal{N}$ of stable representations of the quiver $Q^1$ of dimension vector $1$ (we use the same symbols as above for the counterparts). For each choice of a generic stability condition, $\mathcal{N}$ is the toric variety obtained as the quotient of the stable locus of $\mathbb{A}^7$ by a 3-dimensional torus $\mathbb{G}_m^3$. The weight matrix of the torus action is given by

$$a_1 \ a_2 \ a_3 \ a_4 \ a_5 \ a_6 \ a_7$$
$$t_2 \begin{pmatrix} 1 & 1 & 1 & 0 & 0 & -1 & -1 \end{pmatrix}$$
$$t_3 \begin{pmatrix} 0 & 0 & 1 & -1 & -1 & 0 & 1 \end{pmatrix}$$
$$t_4 \begin{pmatrix} 0 & 0 & 0 & 1 & 1 & 1 & 0 \end{pmatrix}.

(6.8)

whose kernel is given by the image of

$$\begin{pmatrix} a_1 \ & -1 & -1 & 0 & 0 \\ a_2 \ & 1 & 0 & 0 & 0 \\ a_3 \ & 0 & 1 & 0 & 0 \\ a_4 \ & 0 & 0 & 1 & 0 \\ a_5 \ & 0 & 0 & -1 & -1 \\ a_6 \ & 0 & 0 & 0 & 1 \\ a_7 \ & 0 & -1 & 0 & -1 \end{pmatrix}.

(6.9)

One-dimensional cones of the corresponding fan are generated by the rows of this matrix. The choice of the stability condition determines which cones of higher dimensions are included in the fan, and it turns out that for an appropriate choice of a generic stability condition, the toric variety is isomorphic to $\mathbb{P}^2 \times \mathbb{P}^2$ blown-up at one point. Although the anti-canonical bundle is not ample, it is big and globally generated, and yields a flopping contraction. For a general relations $I$ of $Q^1$, the moduli space $\mathcal{N}_I$ is the intersection of three divisors of multidegree $(-1, 0, 0, 1)$ in $\text{Hom}(\mathbb{G}_m^{(Q^1)_0}, \mathbb{G}_m)$. Since a divisor of multidegree $(-1, 0, 0, 1)$ is the cubic root of the anti-canonical divisor, it is base point free and big. Taking into account that it does not contract a divisor, by the Bertini’s theorem we can conclude that the moduli space $\mathcal{N}_I$ is a connected elliptic curve for a general relation $[I]$.

The contraction $\mathcal{N} \to \mathbb{P}^2 \times \mathbb{P}^2$ corresponds to the inversion of the arrow $a_7$, which in turn corresponds to passage to the Beilinson quiver for $\mathbb{P}^2$.

Remark 6.2. When $\mathcal{U}$ is not invertible, it is expected that the moduli space $\mathcal{N}_I$ for the ideal $I$ corresponding to $\mathcal{U}$ is isomorphic to the fine moduli space of point modules, which
is isomorphic to the scheme $\mathbb{P}_W U$ by [VdB12, Theorem 4.5.1]. At least there should be the classifying morphism

$$P := \mathbb{P}_W U \to \mathcal{N}_i; \quad p \mapsto \text{Hom}(\mathcal{T}, \mathcal{O}_p),$$

where the object $\mathcal{T}$ should be the direct sum of four line bundles, whose successive differences in turn should give rise to admissible quadruples in the sense of [VdB11].

7. Noncommutative derived (special) McKay correspondence

The McKay correspondence is a name given to the relation between various invariants of quotient stacks by finite groups and those of crepant resolutions (if any) of their coarse moduli. In characteristic 0, a version of McKay correspondence as an equivalence of derived categories of coherent sheaves has been established in various cases, starting with [KV00] and then by [BKR01] and many more. This, in turn, is a particular case of the DK-hypothesis by Bondal, Orlov, Bridgeland, and Kawamata (see, e.g., [Kaw18] and references therein).

The simplest case of the derived McKay correspondence appears in the surface $A_1$-singularity. It globalizes to the derived equivalence between the Hirzebruch surface $\Sigma_2$ and the stack-theoretic weighted projective plane $\mathbb{P}(1, 1, 2) := [(\mathbb{A}^3 \setminus \{0\})/\mathbb{G}_m]$, both of which are crepant ‘resolutions’ of the weighted projective plane $\mathbb{P}(1, 1, 2)$:

$$\begin{array}{ccc}
\Sigma_2 & \xrightarrow{\text{crepant resolution}} & \mathbb{P}(1, 1, 2) \\
\downarrow & & \downarrow \\
\mathbb{P}(1, 1, 2) & & \text{étale in codimension 1}
\end{array}$$

On the one hand, a sheaf bimodule $\mathcal{E}$ on $\mathbb{P}^1$ gives an abelian category $\text{qgr } \mathbb{S}(\mathcal{E})$, which is a noncommutative deformation of $\text{coh } \Sigma_2$. On the other hand, a 3-dimensional AS-regular algebra $S$ generated by three elements of degree 1, 1, 2, classified by Stephenson [Ste96], gives another abelian category $\text{qgr } S$, which is a noncommutative deformation of $\text{qgr } k[x, y, z] \simeq \text{coh } \mathbb{P}(1, 1, 2)$. It is natural to ask if there is a derived equivalence

$$D^b \text{qgr } \mathbb{S}(\mathcal{E}) \simeq D^b \text{qgr } S.$$  

In the rest of this section, we will discuss an example of a pair of $\mathcal{E}$ and $S$ satisfying (7.2).

**Remark 7.1.** Although there are essentially 3-dimensional moduli of deformations for the abelian category $\text{coh } \Sigma_2$ as we have seen so far, families of algebras on the list [IP02, Theorem 2.11] apparently can not cover all of them. This suggests one to study 3-dimensional AS-regular $\mathbb{Z}$-algebras generated by three elements of degree 1, 1, 2, instead of AS-regular algebras generated by three elements of degree 1, 1, 2. Note that a generic 3-dimensional AS-regular $\mathbb{Z}$-algebra generated by three elements of degree 1, 1, 2 is isomorphic to a 3-dimensional cubic AS-regular $\mathbb{Z}$-algebra; the relation in degree 2 turns the degree 2 generator into a linear combination of quadratic monomials of degree 1 generators, and one is left with cubic relations between degree 1 generators.

For a 3-dimensional AS-regular algebra $S$ generated by three elements of degree 1, 1, 2, we let $\pi: \text{grmod } S \to \text{qgr } S$ be the quotient functor, and write $\mathcal{O}(i) := \pi(S(i))$ and $\mathcal{O} := \mathcal{O}(0)$. The sequence

$$(\mathcal{O}, \mathcal{O}(1), \mathcal{O}(2), \mathcal{O}(3))$$

(7.3)
of objects of $\text{qgr} S$ is a full strong exceptional collection in $D^b \text{qgr} S$ (see, e.g., [Orl09, Corollary 18] and references therein). We consider the algebra

$$S = k \langle x, y, z \rangle / (yx - xy, zx - \lambda xz, zy - yz)$$

(7.4)

appearing as the first item in [IP02, Theorem 2.11]. The total morphism algebra of the collection (7.3) in this case is isomorphic to the quotient of the path algebra of the quiver in Figure 7.1 by the relations

$$I = (b_2a_4 - a_2b_1, b_3a_2 - a_3b_2, c_2a_1 - \lambda a_3c_1, c_2b_1 - b_3c_1).$$

(7.5)

Consider the sheaf bimodule

$$\mathcal{E} = \mathcal{O}_\Delta(2) \oplus \mathcal{O}_{\Gamma_\lambda},$$

(7.6)

where $\Gamma_\lambda$ is the graph of the automorphism $\lambda: \mathbb{P}^1 \to \mathbb{P}^1$ given by $(x : y) \mapsto (\lambda x : y)$. As explained in (5.10), the quiver with relations describing the total morphism algebra of the collection (5.9) can be computed using

$$(F_1, F_2, F_3, F_4) = (\mathcal{O}_{\mathbb{P}^1}(-1), \mathcal{O}_{\mathbb{P}^1}, \mathcal{O}_{\mathbb{P}^1}(1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1), \mathcal{O}_{\mathbb{P}^1}(2) \oplus \mathcal{O}_{\mathbb{P}^1}).$$

(7.7)

For example, the arrows $c_1$ and $c_2$ correspond to the unique (up to scalar) morphisms $\text{Hom}_{\mathbb{P}^1} (\mathcal{O}_{\mathbb{P}^1}(-1), \lambda_* \mathcal{O}_{\mathbb{P}^1}(1))$ and $\text{Hom}_{\mathbb{P}^1} (\mathcal{O}_{\mathbb{P}^1}, \lambda_* \mathcal{O}_{\mathbb{P}^1})$, which will be denoted by $z_1$ and $z_2$ respectively.

Choose a homogeneous coordinate $x, y$ of $\mathbb{P}^1$, so that $\text{Hom}_{\mathbb{P}^1} (\mathcal{O}_{\mathbb{P}^1}(-1), \mathcal{O}_{\mathbb{P}^1}) = \text{span} \{x, y\}$. A moment’s reflection will convinces one that the dual gluing functor $\phi^!$ sends $x, y$ to the morphisms

$$\begin{pmatrix} x & 0 \\ 0 & \lambda^{-1} x \end{pmatrix}, \begin{pmatrix} y & 0 \\ 0 & y \end{pmatrix}: \mathcal{O}_{\mathbb{P}^1}(1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1) \to \mathcal{O}_{\mathbb{P}^1}(2) \oplus \mathcal{O}_{\mathbb{P}^1},$$

(7.8)

respectively.

Putting together, the endomorphism algebra of the full strong exceptional collection can be described as in Figure 7.2.
The relations (7.5) can be readily seen from this. For example, the relation
\[
\begin{pmatrix} 0 \\ 1 \end{pmatrix} \circ x = \lambda \begin{pmatrix} x & 0 \\ 0 & \lambda^{-1} \end{pmatrix} \circ \begin{pmatrix} 0 \\ 1 \end{pmatrix}
\]
(7.9)
comes from the equality
\[
\langle \mathcal{O}, \mathcal{O}(1), \mathcal{O}(d), \mathcal{O}(d + 1) \rangle \subset D^b \text{coh } \mathbb{P}(1, 1, d) = \langle \mathcal{O}, \mathcal{O}(1), \mathcal{O}(2), \ldots, \mathcal{O}(d + 1) \rangle.
\] (7.11)

This is a global version of the derived special McKay correspondence for the cyclic quotient singularity \(\frac{1}{d}(1, 1)\). Similarly, the category \(D^b \text{qgr } S\) for any 3-dimensional AS-regular algebra \(S\) generated by elements of degrees \(1, 1, d\) has an admissible subcategory \(\langle \mathcal{O}, \mathcal{O}(1), \mathcal{O}(d), \mathcal{O}(d + 1) \rangle\) admitting a semiorthogonal decomposition into two copies of \(D^b \text{coh } \mathbb{P}^1\). It is an interesting problem to see if the integral kernel for the dual gluing functor is given by a locally free sheaf bimodule \(E\), so that one obtains a fully faithful functor
\[
D^b \text{qgr } S(E) \hookrightarrow D^b \text{qgr } S,
\] (7.12)
which is a non-commutative generalization of (7.10). A calculation parallel to the one given above shows that this is the case for (7.4).

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