RECOVERING A LARGE NUMBER OF DIFFUSION CONSTANTS IN A PARABOLIC EQUATION FROM ENERGY MEASUREMENTS

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Abstract. Let \((H, \langle \cdot , \cdot \rangle)\) be a separable Hilbert space and \(A_i : D(A_i) \rightarrow H\) \((i = 1, \cdots , n)\) be a family of nonnegative and self-adjoint operators mutually commuting. We study the inverse problem consisting in the identification of a function \(u : [0, T] \rightarrow H\) and \(n\) constants \(\alpha_1, \cdots , \alpha_n > 0\) (diffusion coefficients) that fulfill the initial-value problem
\[
\frac{du}{dt} + \alpha_1 A_1 u(t) + \cdots + \alpha_n A_n u(t) = 0, \quad t \in (0, T), \quad u(0) = x,
\]
and the additional conditions
\[
\langle A_1 u(T), u(T) \rangle = \varphi_1, \quad \cdots , \langle A_n u(T), u(T) \rangle = \varphi_n,
\]
where \(\varphi_i\) are given positive constants. Under suitable assumptions on the operators \(A_i\) and on the initial data \(x \in H\), we shall prove that the solution of such a problem is unique and depends continuously on the data. We apply the abstract result to the identification of diffusion constants in a heat equation and of the Lamé parameters in an elasticity problem on a plate.

1. Introduction. We consider the abstract parabolic Cauchy problem
\[
\frac{du}{dt} + \alpha_1 A_1 u(t) + \cdots + \alpha_n A_n u(t) = 0, \quad t \in (0, T), \quad u(0) = x,
\]
where \(A_i : D(A_i) \rightarrow H\) \((i = 1, \cdots , n)\) are nonnegative and self-adjoint operators in a separable Hilbert space \(H\). The positive diffusion coefficients \(\alpha_1, \cdots , \alpha_n\) will be always considered as purely scalar quantities, independent of time. In the case when \(\alpha_i\) are assumed as given data, then problem (1) admits a unique solution \(u\) for all the initial datum \(x \in H\), which can be recovered through the semigroup theory (cf. [3]) and perturbation theory (cf. [7]). This will be called the direct problem.

The main task of the paper is the one to investigate the well-posedness for the inverse problem consisting in the identification, along with the \(H\)-valued function \(u\), also of parameters \(\alpha_i\), supposed to be unknown, under the overdeterminating conditions of the “final-time” measurements
\[
\langle A_1 u(T), u(T) \rangle = \varphi_1, \quad \cdots , \langle A_n u(T), u(T) \rangle = \varphi_n,
\]
where \(\varphi_i\) are given positive constants, to be regarded from now on as data of our problem, together with the initial value \(x \in H\).

The inverse problem we presented is in fact the continuation of a long-standing work developed in the last years. First, in the basic case when \(n = 1\), existence

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and uniqueness have been achieved in a Hilbert setting in [13], and subsequently extended in the framework of Banach spaces in [15].

Moving forward to the more difficult case $n = 2$, the first results have been provided in [10], in the special case when operator $A_2$ is the identity on $H$. Therein existence and uniqueness of a weak solution $(u, \alpha_1, \alpha_2)$ was analyzed, as long as its continuous dependence on the data $(x, \varphi_1, \varphi_2)$, by adapting a finite-dimensional Faedo-Galerkin approximation scheme to the inverse problem. More recently, in [14], the same results have been extended for general operators $A_1$ and $A_2$ and rely on the abstract semigroup theory. In this case, a classic solution has been constructed. However, the commutation of operators $A_1$ and $A_2$ is a necessary assumption, as it is shown via an explicit counterexample (cf. Remark 2).

The task of the present paper is the one to generalize the uniqueness and the Lipschitz dependence in [14] to an arbitrary number of diffusion constants, under the overdetermination delivered by the same number of energy measurements performed at the time-instant $T$. As, for a larger number of unknown constants, the approach proposed in [14] - involving implicit functions - seems difficult to be performed, in order to recover $\alpha_1, \cdots, \alpha_n$ from the additional measurements we have to use a global injectivity theorem based on the positivity of the Jacobian matrix (see Subsection 3.1), which is the key tool of our investigation. Besides, this allows to prove only a uniqueness result, whereas understanding suitable conditions on the data which allow to recover also the existence of solutions is still an open issue, although less interesting for the sake of inverse problems.

Moreover, we shall also provide such a statement for a larger class of initial data, which do not necessarily need to belong to the intersection of the domains of operators $A_i$ as in [14]. As a consequence, here we deal with weak solutions.

Applications of our abstract theory have been displayed. As expected, the problem of recover diffusion constants in a parabolic equation for a thermal diffusion phenomena is straightforward. More interesting is the identification of the Lamé parameters of a plate in an elasticity equation under the structural assumption that the inertial forces are neglected. The latter problem has already been analyzed in [1] and [16] in the steady case when the Lamé coefficients depend on space.

Finally, it is also noteworthy mentioning that, to the best of our knowledge, there are only a few papers (see [2, 4, 8, 9, 11, 12, 19]) where a constant rather than a function is identified. The main novelty in our formulation of the inverse problem consists in imposing the final conditions of energy type, which, although natural by the viewpoint of measurements, leads to nonlinear (quadratic) conditions, which appear to be new in literature.

**Further developments.** We conclude this introductory section by exploiting the further directions of our investigation. First, we aim at analyzing the determination of time-dependent coefficients $\alpha_i(t)$ from energy measurements of the type $\|A_i^{1/2}u(t)\| = \varphi_i(t), t \in (0, T) (i = 1, \cdots, n)$. A suitable extension of the abstract theory presented in this paper in order to cover the above problem seems to be affordable and will be the subject of a communication in preparation.

On the other hand, a more rich and challenging task is the one to achieve some deeper acquaintance in the non-commutative case, possibly changing the overdeterminating conditions. This problem, in fact, seems to have some connection with the identification of diffusion coefficients which may depend on space. Also, the non-commutative case would allow to analyze the second-order in time equation,
including the wave propagation, i.e.

$$u''(t) + \alpha_1 A_1 u(t) + \cdots + \alpha_n A_n u(t) = 0, \quad t \in (0, T).$$

In fact, if we consider the usual split of such an equation into a system of two first-order in time equations, we see that

$$\frac{d}{dt} \begin{bmatrix} u(t) \\ v(t) \end{bmatrix} + \alpha_1 \begin{bmatrix} 0 & 0 \\ A_1 & 0 \end{bmatrix} \begin{bmatrix} u(t) \\ v(t) \end{bmatrix} + \cdots + \alpha_n \begin{bmatrix} 0 & 0 \\ A_n & 0 \end{bmatrix} \begin{bmatrix} u(t) \\ v(t) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad t \in (0, T).$$

where it is immediate to check that the extended operators

$$\begin{bmatrix} 0 & 0 \\ A_1 & 0 \end{bmatrix}, \quad \cdots, \quad \begin{bmatrix} 0 & 0 \\ A_n & 0 \end{bmatrix}$$

are mutually commuting, but do not commute with

$$\begin{bmatrix} 0 & -I \\ 0 & 0 \end{bmatrix}.$$

All of these problems, at the moment, are far away from being included in the theory so far developed, and would possibly require a meaningful step forward in order to be understood.

2. Main result.

2.1. Preliminaries. Let \(H\) be a (complex) separable Hilbert space endowed with inner product \(\langle \cdot, \cdot \rangle\) and related norm \(\| \cdot \|\). Let also \(A\) be a nonnegative self-adjoint operator with domain \(D(A)\) and range \(R(A)\) in \(H\), i.e., \(A = A^*\) and \(\langle Ax, x \rangle \geq 0\) \((x \in D(A))\). Notice also that \(D(A)\) is dense in \(H\).

In this case (cf. [7]) we recall that the square root of \(A\) is given by

$$A^{1/2}x = \frac{1}{\pi} \int_0^\infty \xi^{-1/2}(A + \xi)^{-1}Ax \, d\xi, \quad x \in D(A),$$

and displays the following properties:

\[(\mu A)^{1/2}x = \mu^{1/2}A^{1/2}x, \quad x \in D(A), \quad \mu > 0;\]

\[\|A^{1/2}x\| \leq \|x\| \cdot \|Ax\|, \quad x \in D(A).\]

Thus, we define

\[D(A^{1/2}) = \{ x \in H; \exists \{ x_n \} \in H \subset D(A) \text{ s.t. } x_n \to x \text{ in } H, \langle A(x_n - x), x_n - x \rangle \to 0 \}.\]

In other words, \(D(A)\) is a core for \(A^{1/2}\), and on \(D(A^{1/2})\) the operator \(A^{1/2}\) is also nonnegative and self-adjoint.

Then, under these assumptions, the operator \(-A\) generates an analytic semigroup of contractions on \(H\) denoted by \(\{e^{-tA}\}_{t \geq 0}\). This is well-known as the self-adjoint case of the Hille-Yosida Theorem (see [3], [7]).

To this purpose, we recall the definition of Yosida approximation of \(A\) with a continuous parameter \(\varepsilon > 0\)

$$A_\varepsilon := \varepsilon^{-1}[1 - (1 + \varepsilon A)^{-1}] = A(1 + \varepsilon A)^{-1}.$$ 

Throughout the rest of the paper, we shall consider the family of operators

$$A_i : D(A_i) \to H \quad (i = 1, \cdots, n),$$

which will be always assumed to fulfill the same requirements as the ones above described for \(A\). Accordingly, we shall denote by \(A_{i,\varepsilon}\) its Yosida approximation. In
this case, as in [14, Lemma 2.1] (cf. [17, Theorem 5.4]), it is possible to prove that under the conditions

\(\text{Re} \langle A_i x, A_j \varepsilon x \rangle \geq 0, \quad \forall x \in D(A_i), \forall \varepsilon > 0 \quad (i, j = 1, \cdots, n),\)

the operator

\[\alpha_1 A_1 + \cdots + \alpha_n A_n\]
on the domain \(D(\alpha_1 A_1 + \cdots + \alpha_n A_n) = \bigcap_{i=1}^{n} D(A_i)\)
is nonnegative and self-adjoint in \(H\) for all \((\alpha_1, \cdots, \alpha_n) \in \mathbb{R}_+^n\) and, consequently, its opposite generates an analytic contraction semigroup \(\{e^{-t(\alpha_1 A_1 + \cdots + \alpha_n A_n)}\}_{t \geq 0}\)
on \(H\).

**Commutation.** From now on, we assume further that operators \(A_i\) are mutually commuting, i.e.

\[A_i A_j = A_j A_i \quad \text{on} \quad D(A_i A_j) = D(A_j A_i) \quad \forall i, j = 1, \cdots, n.\]

In this case, many more properties can be achieved. First, notice that the identity

\[(3) \quad \langle A_i x, A_j x \rangle = \left\| A_i^{1/2} A_j^{1/2} x \right\|^2 \quad \forall x \in D(A_i) \cap D(A_j), \quad (i, j = 1, \cdots, n),\]

implies condition (2), so that the semigroup generation is always achieved. Moreover, by straightforward computations, it is then immediate to prove that the following factorization holds

\[e^{-t(\alpha_1 A_1 + \cdots + \alpha_n A_n)} x = e^{-t \alpha_1 A_1} \cdots e^{-t \alpha_n A_n} x \quad \forall x \in H, \quad \forall t > 0,\]

and, as a consequence

\[(4) \quad \partial_t e^{-t(\alpha_1 A_1 + \cdots + \alpha_n A_n)} x = -t A_i e^{-t(\alpha_1 A_1 + \cdots + \alpha_n A_n)} x.\]

We now recall the next important result, subsuming Theorem 4.3 and Corollary 4.4 in [18], which says that commuting self-adjoint operators can be simultaneously diagonalized.

**Lemma 2.1.** Let \(A_1, \cdots, A_n\) mutually commuting self-adjoint operators. Then there exist two sequences

\[\{(\lambda_{1,k}, \cdots, \lambda_{n,k})\}_{k \in \mathbb{N}} \subset [0, \infty)^n\] (eigenthuples)

and

\[\{v_k\}_{k \in \mathbb{N}} \subset \bigcap_{i=1}^{n} D(A_i)\] (eigenfunctions),

such that

\[A_i v_k = \lambda_{i,k} v_k, \quad i = 1, \cdots, n, \quad k \in \mathbb{N},\]

where \(0 \leq \lambda_{i,k} \to \infty\) as \(k \to \infty\) and the set of \(v_k\) forms an orthonormal complete system in \(H\), i.e.

\[\bigcup_{k=1}^{\infty} \text{span}\{v_1, \ldots, v_k\}\]
is dense in \(H\) and \(\langle v_k, v_h \rangle = \delta_{k,h}.\)
From the above lemma, we learn that every $x \in H$ can be represented as

$$x = \sum_{k=1}^{\infty} \langle x, v_k \rangle v_k,$$

where $\langle x, v_k \rangle$ is the $k$-th Fourier coefficient of $x$ with respect to the common set of eigenfunctions of $A_1, \cdots, A_n$. As a consequence, we introduce the following decomposition formulæ

$$A_i x = \sum_{k=1}^{\infty} \lambda_{i,k} \langle x, v_k \rangle v_k, \quad \forall x \in D(A_i), \quad i = 1, \cdots, n,$$

and

$$e^{-t(\alpha_1 A_1 + \cdots + \alpha_n A_n)} x = \sum_{k=1}^{\infty} e^{-t(\alpha_1 \lambda_{i,k} + \cdots + \alpha_n \lambda_{i,k})} \langle x, v_k \rangle v_k, \quad \forall x \in H.$$

Along with the spectral decomposition above introduced, it is possible to characterize the intermediate operator $A_i^{1/2}$ $(i = 1, \cdots, n)$ as

$$A_i^{1/2} x = \sum_{k=1}^{\infty} \lambda_{i,k}^{1/2} \langle x, v_k \rangle v_k$$

and its domain

$$D(A_i^{1/2}) = \left\{ x \in H; \| A_i^{1/2} x \|^2 = \sum_{k=1}^{\infty} \lambda_{i,k}^{1/2} | \langle x, v_k \rangle |^2 < \infty \right\}.$$

The inverse problem. On account to the functional setting above introduced, we can state the rigorous abstract formulation of the inverse problem we aim at investigating.

**Problem P.** Find an $H$-valued function $u$ and $n$ real positive constants $\alpha_1, \cdots, \alpha_n$ fulfilling the equation

$$u'(t) + \alpha_1 A_1 u(t) + \cdots + \alpha_n A_n u(t) = 0, \quad t \in (0, T),$$

the initial datum

$$u(0) = x,$$

and the additional constraints

$$\| A_1^{1/2} u(T) \|^2 = \varphi_1, \cdots, \| A_n^{1/2} u(T) \|^2 = \varphi_n,$$

where $x \in H$, and $\varphi_1, \cdots, \varphi_n > 0$ are given.

2.2. Statement of the main result. We now describe the main result in the paper. We first need to exploit the set of conditions we shall require.

Assumptions on the operators. Following the general results discussed in Subsection 2.1, we shall require operators $A_1, \cdots, A_n$ to be nonnegative, self-adjoint and mutually commuting.
Assumptions on the data. An initial datum \( x \in H \) is admissible for the sake of our investigation if, roughly speaking, it is such that the vectors
\[
A_1 u(T), \ldots, A_n u(T)
\]
are linearly independent in \( H \). As a matter of fact, like in the case \( n = 2 \) (cf. [14]), it is not difficult to realize that such a condition is strictly necessary for proving our result. In fact, without losing of generality, suppose
\[
A_n u(T) = \sum_{i=1}^{n-1} c_i A_i u(T)
\]
for some \( c_1, \ldots, c_{n-1} \in \mathbb{C} \). Then back to (8) we get
\[
\varphi_n = \langle A_n u(T), u(T) \rangle = \sum_{i=1}^{n-1} c_i \langle A_i u(T), u(T) \rangle = \sum_{i=1}^{n-1} c_i \varphi_i.
\]
As a consequence of the above decomposition, in the case when \( \varphi_n = \sum_{i=1}^{n-1} c_i \varphi_i \), the system is underdetermined as one overdetermining condition is redundant, and therefore solution of such an inverse problem fails to be unique (cf. Section 3). On the other hand, if \( \varphi_n \neq \sum_{i=1}^{n-1} c_i \varphi_i \), one condition is not compatible with the others, so that no solution can exist.

In order to state a rigorous condition, notice first that, by means of formulae (5), we learn that for all \( i = 1, \ldots, n \) and for all \( t > 0 \) there holds
\[
A_i u(t) = \sum_{k=1}^{\infty} \lambda_{i,k} e^{-t(\alpha_1 \lambda_{1,k} + \cdots + \alpha_n \lambda_{n,k})} \langle x, v_k \rangle v_k.
\]
We can therefore identify each vector \( A_i u(t) \) with the sequence of its Fourier coefficients
\[
\left\{ \lambda_{i,k} e^{-t(\alpha_1 \lambda_{1,k} + \cdots + \alpha_n \lambda_{n,k})} \langle x, v_k \rangle \right\}_{k \in \mathbb{N}} \quad i = 1, \ldots, n.
\]
Then, the linearly independence condition is equivalent to the existence of a \( n \times n \) nonsingular minor \( M \) of the infinite-dimensional matrix
\[
\begin{bmatrix}
\lambda_{1,1} e^{-t(\alpha_1 \lambda_{1,1} + \cdots + \alpha_n \lambda_{n,1})} \langle x, v_1 \rangle & \cdots & \lambda_{n,1} e^{-t(\alpha_1 \lambda_{1,1} + \cdots + \alpha_n \lambda_{n,1})} \langle x, v_1 \rangle \\
\vdots & \ddots & \vdots \\
\lambda_{1,k} e^{-t(\alpha_1 \lambda_{1,k} + \cdots + \alpha_n \lambda_{n,k})} \langle x, v_k \rangle & \cdots & \lambda_{n,k} e^{-t(\alpha_1 \lambda_{1,k} + \cdots + \alpha_n \lambda_{n,k})} \langle x, v_k \rangle \\
\vdots & \ddots & \vdots
\end{bmatrix}.
\]
If we label by \( w_1, \ldots, w_n \) the simultaneous eigenfunctions of \( A_1, \ldots, A_n \) arising from \( M \), and by \( (\mu_{1,1}, \ldots, \mu_{1,n}), \ldots, (\mu_{n,1}, \ldots, \mu_{n,n}) \) the respective eigenvalues, we see that
\[
M = \begin{bmatrix}
\mu_{1,1} e^{-t(\alpha_1 \mu_{1,1} + \cdots + \alpha_n \mu_{n,1})} \langle x, w_1 \rangle & \cdots & \mu_{n,1} e^{-t(\alpha_1 \mu_{1,1} + \cdots + \alpha_n \mu_{n,1})} \langle x, w_1 \rangle \\
\vdots & \ddots & \vdots \\
\mu_{1,n} e^{-t(\alpha_1 \mu_{1,n} + \cdots + \alpha_n \mu_{n,n})} \langle x, w_n \rangle & \cdots & \mu_{n,n} e^{-t(\alpha_1 \mu_{1,n} + \cdots + \alpha_n \mu_{n,n})} \langle x, w_n \rangle
\end{bmatrix}
\]
which, by means of the standard matrix product, may be decomposed in the left product of the diagonal matrix
\[
\begin{bmatrix}
e^{-t(\alpha_1 \mu_{1,1} + \cdots + \alpha_n \mu_{n,1})} \langle x, w_1 \rangle & \cdots & 0 \\
\vdots & \ddots & \vdots \\
0 & \cdots & e^{-t(\alpha_1 \mu_{1,n} + \cdots + \alpha_n \mu_{n,n})} \langle x, w_n \rangle
\end{bmatrix}
\]
by

\[
\begin{bmatrix}
  \mu_{1,1} & \cdots & \mu_{n,1} \\
  \vdots & \ddots & \vdots \\
  \mu_{1,n} & \cdots & \mu_{n,n}
\end{bmatrix}.
\]

Thus, condition \( \det M \neq 0 \) is equivalent to requiring that

\[
\det\begin{bmatrix}
  e^{-t(\alpha_1 \mu_{1,1} + \cdots + \alpha_n \mu_{n,1})} \langle x, w_1 \rangle & \cdots & 0 \\
  \vdots & \ddots & \vdots \\
  0 & \cdots & e^{-t(\alpha_1 \mu_{1,n} + \cdots + \alpha_n \mu_{n,n})} \langle x, w_n \rangle
\end{bmatrix}
= \prod_{i=1}^{n} e^{-t(\alpha_1 \mu_{1,i} + \cdots + \alpha_n \mu_{n,i})} \langle x, w_i \rangle \neq 0
\]

and

\[
\det\begin{bmatrix}
  \mu_{1,1} & \cdots & \mu_{n,1} \\
  \vdots & \ddots & \vdots \\
  \mu_{1,n} & \cdots & \mu_{n,n}
\end{bmatrix} \neq 0.
\]

Therefore, we define an initial datum \( x \in H \) to be \textit{admissible} if, and only if, there exist \( w_1, \cdots, w_n \) simultaneous eigenfunctions of \( A_1, \cdots, A_n \) with respect to the eigenthuples \( (\mu_{1,1}, \cdots, \mu_{1,n}), \cdots, (\mu_{n,1}, \cdots, \mu_{n,n}) \), such that the following assumptions are fulfilled:

\[\begin{align*}
(i) & \quad \text{the eigenthuples (square) matrix} \\
& \quad \begin{bmatrix}
  \mu_{1,1} & \cdots & \mu_{n,1} \\
  \vdots & \ddots & \vdots \\
  \mu_{1,n} & \cdots & \mu_{n,n}
\end{bmatrix} \text{ is nonsingular;} \\
(ii) & \quad \langle x, w_i \rangle \neq 0, \quad \forall i = 1, \cdots, n.
\end{align*}\]

Notice that we have to require, as we shall do from now on, that the dimension of \( H \) is equal or larger than \( n \).

The previous discussion showed in fact that assumptions (i) and (ii) imply that the vectors \( \{A_1u(t), \cdots, A_nu(t)\} \) are linearly independent for all \( t > 0 \), and in particular for \( t = T \). We subsume this result in the next

**Lemma 2.2.** For all admissible \( x \in H \) and all \( (\alpha_1, \cdots, \alpha_n) \in \mathbb{R}_+^n \) the solution \( u \in C^1([0, \infty) ; H) \cap C ((0, \infty); \bigcap_{i=1}^{n} D(A_i)) \) to abstract Cauchy problem (6)-(7) is such that \( \{A_1u(t), \cdots, A_nu(t)\} \) are linearly independent vectors for all \( t > 0 \).

**Remark 1.** If we consider the further regularity assumption \( x \in \bigcap_{i=1}^{n} D(A_i) \), then (i) and (ii) may be replaced by the equivalent condition

\[\{A_1x, \cdots, A_nx\} \text{ are linearly independent vectors.}\]

Such a condition is in fact implying the same conclusion of Lemma 2.2, and this can be proven exactly like in [14, Lemma 2.4]. However, in that case the solutions are forced to be classical, meanwhile the admissibility assumptions we are now proposing allow the initial datum to range in a larger set, so that here the solutions may also be of weak type.

We end this subsection by establishing an important topological property of the set of the admissible data. Although the “size” of the set is highly depending on the operators \( A_1, \cdots, A_n \), the next result holds in general.
Proposition 1. The set of admissible data, i.e. fulfilling assumptions (i) and (ii), is path-connected.

Proof. Let \( x, \varphi \in H \) be admissible data, and accordingly with assumptions (i) and (ii) we denote the respective eigenvalues by
\[
w_1, \cdots, w_n \quad \text{and} \quad \overline{w}_1, \cdots, \overline{w}_n\]
as well as the respective eigenthuples
\[(\mu_1, \cdots, \mu_{1,n}), \cdots, (\mu_{n,1}, \cdots, \mu_{n,n}) \quad \text{and} \quad (\overline{\mu}_1, \cdots, \overline{\mu}_{1,n}), \cdots, (\overline{\mu}_{n,1}, \cdots, \overline{\mu}_{n,n}).\]

We now construct a smooth path originating in \( x \) and arriving in \( \varphi \). For every \( i = 1, \cdots, n \), we choose \( \gamma_i, \overline{\gamma}_i : [0, 1) \to \mathbb{C} \) to be smooth paths such that
\[
\gamma_i(0) = (x, w_i), \quad \gamma_i(1) = (\varphi, w_i) \quad \text{and} \quad \gamma_i(\tau) \neq 0 \quad \forall \tau \in (0, 1)
\]
and, analogously
\[
\overline{\gamma}_i(0) = (x, \overline{w}_i), \quad \overline{\gamma}_i(1) = (\varphi, \overline{w}_i) \quad \text{and} \quad \overline{\gamma}_i(\tau) \neq 0 \quad \forall \tau \in (0, 1).
\]

Then, any smooth map \( \Gamma : [0, 1) \to H \) with \( \Gamma(0) = x \) and \( \Gamma(1) = \varphi \) and
\[
(\Gamma(\tau), w_i) = \gamma_i(\tau) \quad \text{and} \quad (\Gamma(\tau), \overline{w}_i) = \overline{\gamma}_i(\tau), \quad \forall \tau \in [0, 1), \quad \forall i = 1, \cdots, n
\]
is, by construction, a path connecting \( x \) and \( \varphi \) consisting of admissible data. In fact, assumption (i) is implying that the matrices
\[
\begin{bmatrix}
\mu_{1,1} & \cdots & \mu_{1,n} \\
\vdots & \ddots & \vdots \\
\mu_{n,1} & \cdots & \mu_{n,n}
\end{bmatrix} \quad \text{and} \quad
\begin{bmatrix}
\overline{\mu}_{1,1} & \cdots & \overline{\mu}_{1,n} \\
\vdots & \ddots & \vdots \\
\overline{\mu}_{n,1} & \cdots & \overline{\mu}_{n,n}
\end{bmatrix}
\]
are nonsingular. Moreover, as a consequence of (ii), for every \( \tau \in (0, 1) \) the point \( \Gamma(\tau) \) is such that its projections on both subspaces of \( H \) generated by \( w_1, \cdots, w_n \) and \( \overline{w}_1, \cdots, \overline{w}_n \) are non-vanishing. Therefore, \( \Gamma(\tau) \) is admissible. \( \square \)

Main result. We are now in a position to state our main result. From now on we shall assume that, for a set of data \((x, \varphi_1, \cdots, \varphi_n) \in H \times \mathbb{R}^n_+\), \( x \) being admissible in \( H \), there exists a solution
\[
(u, \alpha_1, \cdots, \alpha_n) \in (C([0, T]; H) \cap C^1((0, T]; H)) \cap \bigcap_{i=1}^n D(A_i)) \times \mathbb{R}^n_+
\]
to Problem \( P \). We point out that such a regularity for \( u \) is the same as the solution to the direct problem, once that the positive coefficients \((\alpha_1, \cdots, \alpha_n)\) are known quantities.

Then, our main result concerns uniqueness of such a solution \((u, \alpha_1, \cdots, \alpha_n)\), which is a consequence of the Lipschitz dependence estimate delivered by the next

Theorem 2.3. Let \((x, \varphi_1, \cdots, \varphi_n)\) and \((\varphi, \overline{\varphi}_1, \cdots, \overline{\varphi}_n)\) be two sets of data, and denote by \((u, \alpha_1, \cdots, \alpha_n)\) and \((\overline{u}, \overline{\alpha}_1, \cdots, \overline{\alpha}_n)\) the respective solutions to Problem \( P \). Then the continuous dependence estimate holds:
\[
\|u(t) - \overline{u}(t)\|^2 + \sum_{i=1}^n \left[ \int_0^t \|A_i^{1/2} [u(\tau) - \overline{u}(\tau)]\|^2 d\tau + |\alpha_i - \overline{\alpha}_i| \right] \\
\leq C \left( \|x - \varphi\|^2 + \sum_{i=1}^n |\varphi_i - \overline{\varphi}_i| \right), \forall t > 0,
\]
where the Lipschitz constant \( C > 0 \) depends continuously on \((x, \varphi_1, \cdots, \varphi_n)\) and \((\varphi, \overline{\varphi}_1, \cdots, \overline{\varphi}_n)\).
Although commutativity condition may appear severe, its necessity - for the sake of uniqueness Theorem 2.3 - has already been discussed in [14] via an explicit counterexample in the case \( n = 2 \) (see Remark 2).

3. Proof of Theorem 2.3. We recall that, if \( e^{-t(\alpha_1 A_1 + \cdots + \alpha_n A_n)} \) is the semigroup devised in Subsection 2.1, then the \( H \)-valued function

\[
(9) \quad u(t) = e^{-t(\alpha_1 A_1 + \cdots + \alpha_n A_n)}x, \quad t \geq 0
\]

belongs to the space \( C([0, T]; H) \cap C^4((0, T]; H) \cap C((0, T]; \cap_{i=1}^n D(A_i)) \) and fulfills both equation (6) and initial condition (7). Consequently, by replacing the above expression at the time-instant \( t = T \) in the constraints (8), we have

\[
\|A_1^{1/2} e^{-T(\alpha_1 A_1 + \cdots + \alpha_n A_n)} x\|^2 = \varphi_i, \quad i = 1, \ldots, n.
\]

Therefore, we are lead to the system of nonlinear equations

\[
(10) \quad \Phi(\alpha_1, \ldots, \alpha_n) = \begin{bmatrix}
\Phi_1(\alpha_1, \cdots, \alpha_n) \\
\vdots \\
\Phi_n(\alpha_1, \cdots, \alpha_n)
\end{bmatrix} = \begin{bmatrix}
\varphi_1 \\
\vdots \\
\varphi_n
\end{bmatrix}
\]

for any fixed \((\varphi_1, \ldots, \varphi_n) \in \mathbb{R}_+^n\), where the map

\[
\Phi = (\Phi_1, \cdots, \Phi_n): \mathbb{R}_+^n \to \mathbb{R}_+^n
\]

(depending on \( x \)) is component-wise defined as

\[
\Phi_i(\alpha_1, \cdots, \alpha_n) = \|A_1^{1/2} e^{-T(\alpha_1 A_1 + \cdots + \alpha_n A_n)} x\|^2, \quad i = 1, \ldots, n.
\]

The immediate properties of \( \Phi \) are stated in the following proposition, which can be easily proven using identities (4).

**Proposition 2.** Let \( \Phi = (\Phi_1, \cdots, \Phi_n) \) be as in (10). Then \( \Phi \in C^1(\mathbb{R}_+^n; \mathbb{R}_+^n) \), with

\[
\partial_{\alpha_i} \Phi_j(\alpha_1, \cdots, \alpha_n) = -2T \left\langle A_i e^{-T(\alpha_1 A_1 + \cdots + \alpha_n A_n)} x, A_j e^{-T(\alpha_1 A_1 + \cdots + \alpha_n A_n)} x \right\rangle
\]

for all \((\alpha_1, \cdots, \alpha_n) \in \mathbb{R}_+^n \) and all \( i,j = 1, \cdots, n \).

As it is clear, the main features of the identification problem we are investigating are related to the map \( \Phi \). In the sequel we shall first prove that it is injective, and subsequently that it is locally Lipschitz. As a byproduct, this will imply that the solution to nonlinear system (10) is unique and depends continuously on the data.

3.1. Injectivity. As an immediate consequence of Proposition 2, we deduce that the \((n \times n)\)-Jacobian matrix of the map \( \Phi \) is defined by

\[
\mathcal{J}_\Phi = -2T \left[ \left\langle A_i e^{-T(\alpha_1 A_1 + \cdots + \alpha_n A_n)} x, A_j e^{-T(\alpha_1 A_1 + \cdots + \alpha_n A_n)} x \right\rangle \right]_{i,j=1,\ldots,n}.
\]

Then, by identity (3), we learn that \( \mathcal{J}_\Phi \) is real and symmetric for all the \( n \)-tuples \((\alpha_1, \cdots, \alpha_n) \in \mathbb{R}_+^n \). More is true, in fact \( \mathcal{J}_\Phi \) is, up to the factor \(-2T\), the Gram matrix associated to the family of vectors

\[
\{A_1 u(T), \cdots, A_n u(T)\}.
\]

Here below we subsume the basic algebraic facts related to Gramian matrices.
Proposition 3. Let $y_1, \ldots, y_n \in H$ and consider the related $(n \times n)$-Gram matrix

$$G = [\langle y_i, y_j \rangle]_{i,j=1,\ldots,n}. $$

Then the following conclusions hold:

(a) $G$ is non-singular if, and only if, the vectors $\{y_1, \ldots, y_n\}$ are linearly independent;

(b) $G$ is positive semi-defined, i.e. $x^T G x \geq 0$ for all $x \in \mathbb{R}^n$, where the inequality is strict if, and only if, $\{y_1, \ldots, y_n\}$ are linearly independent.

Therefore, thanks to Proposition 3, we deduce the following statement, which follows from Lemma 2.2 applied at the time-instant $T$.

Proposition 4. For all $(\alpha_1, \ldots, \alpha_n) \in \mathbb{R}_+^n$ the Jacobian matrix $J_\Phi(\alpha_1, \ldots, \alpha_n)$ is real, symmetric and negative defined, and therefore nonsingular. As a byproduct, $\Phi$ is a local homeomorphism of $\mathbb{R}_+^n$ into itself.

However, as it is well-known, this fact alone does not imply that the map $\Phi$ is globally injective. To reach the task, we shall use the next general result, of which we shall give a direct proof (cf. also [6, Theorem 6]).

Theorem 3.1. Let $\Omega \subseteq \mathbb{R}^n$ to be a convex nonempty set, and let $F \in C^1(\Omega; \mathbb{R}^n)$ to be such that $J_F(x)$ is positive (negative) defined at every point $x \in \Omega$. Then $F$ is injective on $\Omega$.

Proof. Given any couple $x, y \in \Omega$ with $x \neq y$, first notice that by convexity the segment

$$y + \tau(x - y), \quad \tau \in [0,1]$$

lies in $\Omega$. Then we define the real function

$$f(\tau) = (x - y) \cdot [F(y + \tau(x - y)) - F(y)], \quad \tau \in [0,1]$$

where “$\cdot$” denotes the usual scalar product in $\mathbb{R}^n$. As it is clear, $f \in C^1([0,1])$ with

$$f'(\tau) = (x - y)J_F(y - \tau(x - y))(x - y)^T, \quad \tau \in [0,1].$$

Using the definiteness of $J_F$ at every point of $\Omega$, we deduce that $f'(\tau)$ is strictly positive (negative) on $[0,1]$. Then, since $f(0) = 0$, a fortiori we have

$$f(1) = (x - y) \cdot [F(x) - F(y)] \neq 0,$$

which is implying $F(x) \neq F(y)$. \hfill \Box

Finally, summing up Proposition 2 and Theorem 3.1, we have

Lemma 3.2. The map $\Phi$ defined above is injective. Therefore, the nonlinear system (10) admits at most one solution.

As a consequence of Lemma 3.2, we deduce that $\Phi$ is a global homeomorphism of $\mathbb{R}_+^n$ into its image $\Phi(\mathbb{R}_+^n)$, and we denote by $\Phi^{-1}$ its inverse function. Moreover, by the standard Inverse Function Theorem, we deduce that

$$\Phi^{-1} \in C^1(\Phi(\mathbb{R}_+^n); \mathbb{R}_+^n) \quad \text{and} \quad J_{\Phi^{-1}}(\varphi_1, \ldots, \varphi_n) = (J_\Phi(\alpha_1, \ldots, \alpha_n))^{-1}. $$
Thus, we may write
\[
\begin{bmatrix}
\alpha_1 \\
\vdots \\
\alpha_n
\end{bmatrix} = \Phi^{-1} \begin{bmatrix}
\varphi_1 \\
\vdots \\
\varphi_n
\end{bmatrix},
\]
so that \((J_\Phi(\alpha_1, \ldots, \alpha_n))^{-1}\) can be identified with the Jacobian matrix of the coefficients \((\alpha_1, \ldots, \alpha_n)\) with respect to variables \((\varphi_1, \ldots, \varphi_n)\), which we denote by \(J_{(\alpha_1, \ldots, \alpha_n)}(\varphi_1, \ldots, \varphi_n)\). Summing up, we have established the identities
\[
J_{(\alpha_1, \ldots, \alpha_n)}(\varphi_1, \ldots, \varphi_n) = J_{\Phi^{-1}}(\varphi_1, \ldots, \varphi_n) = (J_\Phi(\alpha_1, \ldots, \alpha_n))^{-1}.
\]

**Remark 2.** As discussed in Section 2, the commutativity of operators \(A_1, \ldots, A_n\) is necessary. In fact, in the Appendix at the end of [14] we have showed that for a large class of admissible data \(x = (x, y)^T\) the \(2 \times 2\) symmetric nonnegative matrices
\[
A = \begin{bmatrix}
2 & 0 \\
0 & 0
\end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix}
2 & 1 \\
1 & 2
\end{bmatrix},
\]
which do not commute, are such that the map \(\beta \mapsto \|A^{1/2}e^{-\alpha A - \beta B}x\|^2\) is not monotone in a neighborhood of \(\beta = 0\) for every fixed \(\alpha\) small enough (here \(e^M\) denotes the exponential matrix of \(M\)). This implies that the level set \(\|A^{1/2}e^{-\alpha A - \beta B}x\|^2 = \varphi\) is not necessarily a unique function and, therefore, the solution \((\alpha, \beta) \in \mathbb{R}_+^2\) to the system of nonlinear equations
\[
\begin{cases}
\|A^{1/2}e^{-\alpha A - \beta B}x\|^2 = \varphi \\
\|B^{1/2}e^{-\alpha A - \beta B}x\|^2 = \psi
\end{cases}
\]
which is the analogous of (10) in the two-dimensional case, may be multiple. As a byproduct, following the argument introduced in the above subsection, this corrupts the uniqueness result on the inverse problem.

**3.2. Lipschitz continuity.** We now focus on the continuous dependence on \(x\) and \((\varphi_1, \ldots, \varphi_n)\) of \((\alpha_1, \ldots, \alpha_n)\) which, from now on, will be chosen to be the solution to system (10), and regarded as a functional of the data. Throughout this section, we shall denote by \((L(H), \| \cdot \|_{L(H)})\) the space of continuous linear operators from \(H\) into itself, and by \((H^*, \| \cdot \|_*)\) the dual space of \(H\).

First, we need to study the dependence of \((\alpha_1, \ldots, \alpha_n)\) on \((\varphi_1, \ldots, \varphi_n)\). From (11) we learn
\[
\|J_{(\alpha_1, \ldots, \alpha_n)}(\varphi_1, \ldots, \varphi_n)\| = \|J_{\Phi^{-1}}(\varphi_1, \ldots, \varphi_n)\| \leq \| (J_\Phi(\alpha_1, \ldots, \alpha_n))^{-1} \|,
\]
having denoted by \(\| \cdot \|\) any matrix-norm.

Next, we point out the behavior of \((\alpha_1, \ldots, \alpha_n)\) with respect to \(x \in H\). We need the next abstract results (cf. [14, Proposition 4.1]).

**Proposition 5.** Let \(L \in L(H)\), and consider the nonlinear map \(F : H \to \mathbb{R}\) defined as \(F(x) = \|Lx\|^2\). Then \(F \in C^1(H)\), and the Riesz representation of its Fréchet derivative \(\nabla_x F\) is given by the formula
\[
\nabla_x F(x) = 2L^*Lx, \quad \forall x \in H,
\]
being $L^* \in \mathcal{L}(H)$ the adjoint operator of $L$. Moreover, for every $x \in H$, $d\mathcal{F}(x) \in H^*$, where $d\mathcal{F}(x)$ is defined as $d\mathcal{F}(x)(y) = 2(Lx, Ly)$.

From the above proposition, we deduce that, for all fixed $(\alpha_1, \cdots, \alpha_n) \in \mathbb{R}^n_+$ and for all $i = 1, \cdots, n$, the maps

$$x \mapsto \Phi_i(\alpha_1, \cdots, \alpha_n) = \|A_1^{1/2}e^{-T(\alpha_1A_1+\cdots+\alpha_nA_n)x}\|^2,$$

belong to $C^1(H)$, and their Fréchet gradients are computed by the formulæ

$$\nabla_x \Phi_i(\alpha_1, \cdots, \alpha_n) = \left(A_1^{1/2}e^{-T(\alpha_1A_1+\cdots+\alpha_nA_n)}\right)^* A_1^{1/2}e^{-T(\alpha_1A_1+\cdots+\alpha_nA_n)x}.$$

Consequently, by using again (10), the following limitations hold

$$\|\nabla_x \Phi_i(\alpha_1, \cdots, \alpha_n)\|_* \leq \left\|\left(A_1^{1/2}e^{-T(\alpha_1A_1+\cdots+\alpha_nA_n)}\right)^* \right\|_{\mathcal{L}(H)} \|A_1^{1/2}e^{-T(\alpha_1A_1+\cdots+\alpha_nA_n)}\|$$

$$= \|A_1^{1/2}e^{-T(\alpha_1A_1+\cdots+\alpha_nA_n)}\|_{\mathcal{L}(H)} \varphi_i^{1/2}.$$

Thus, invoking *Implicit Function Theorem* (cf. [5] for its infinite dimension statement), we deduce that the Fréchet gradient of the map $x \mapsto \Phi(\alpha_1, \cdots, \alpha_n)$ is given by

$$\begin{bmatrix}
\nabla_x \alpha_1 \\
\vdots \\
\nabla_x \alpha_n \\
\end{bmatrix} = (J_\Phi(\alpha_1, \cdots, \alpha_n))^{-1} \begin{bmatrix}
\nabla_x \Phi_1(\alpha_1, \cdots, \alpha_n) \\
\vdots \\
\nabla_x \Phi_n(\alpha_1, \cdots, \alpha_n) \\
\end{bmatrix}.$$  

Collecting the above estimates, we deduce the upper bound

(13)

$$\sum_{i=1}^{n} \|\nabla_x \alpha_i\|_* \leq \left\|\left(J_\Phi(\alpha_1, \cdots, \alpha_n)\right)^{-1}\right\| \sum_{i=1}^{n} \|\nabla_x \Phi_i(\alpha_1, \cdots, \alpha_n)\|_*$$

$$\leq \left\|\left(J_\Phi(\alpha_1, \cdots, \alpha_n)\right)^{-1}\right\| \sum_{i=1}^{n} \|A_1^{1/2}e^{-T(\alpha_1A_1+\cdots+\alpha_nA_n)}\|_{\mathcal{L}(H)} \varphi_i^{1/2}.$$

We take into account of limitations (12) and (13) in the following

**Lemma 3.3.** There exists a positive constant $c$, continuously depending on the data $(x, \varphi_1, \cdots, \varphi_n)$, and on the solution $(\alpha_1, \cdots, \alpha_n)$ to system (10), such that

$$\|J_{(\alpha_1, \cdots, \alpha_n)}(\varphi_1, \cdots, \varphi_n)\| + \sum_{i=1}^{n} \|\nabla_x \alpha_i\|_* \leq c.$$

Now, consider $(x, \varphi_1, \cdots, \varphi_n)$ and $(\bar{x}, \bar{\varphi}_1, \cdots, \bar{\varphi}_n)$, and denote by $(\alpha_1, \cdots, \alpha_n)$ and $(\bar{\alpha}_1, \cdots, \bar{\alpha}_n)$ be the respective solutions to (10). Then, recalling that both $x$ and $\bar{x}$ are admissible data, by Proposition 1, the limitation in Lemma 3.3 and the mean value theorem, we get the estimate

$$\sum_{i=1}^{n} |\alpha_i - \bar{\alpha}_i| \leq C \left(\|x - \bar{x}\| + \sum_{i=1}^{n} |\varphi_i - \bar{\varphi}_i|\right).$$
Using the above estimate, it is easy to recover the one on \( u - \bar{u} \) in \( H \) and in the higher-order space \( L^2 \left( 0, T; \cap_{i=1}^n D\left( A_i^{1/2} \right) \right) \) by means of a standard multiplication argument.

**Final remarks.** We conclude this section by pointing out some limitations of the abstract theory so far developed, which are intended to be improved in the future.

As first, notice that the surjectivity of map \( \Phi \), which is related to the existence of solutions to our inverse problem, has been achieved in [14] in the case \( n = 2 \) under suitable conditions involving also \( (\varphi_1, \varphi_2) \), along with \( x \). This corresponds to give a more precise description of \( \Phi(\mathbb{R}^n_+) \) with respect to the data. However, how to extend the validity of those conditions for larger values of \( n \) is still an open problem.

Secondly, the estimate of \( C \) delivered in (3.5) is in fact not fully satisfactory, as it is invoking a control from above of some matrix norm of \( J_\Phi \) with respect to data \( x \) and \( (\varphi_1, \ldots, \varphi_n) \), and of the unknown parameters \( (\alpha_1, \ldots, \alpha_n) \). This is left as a further step in the theory, to be the subject of further investigations.

4. **Applications.** In this last section we shall display concrete applications of our abstract result to a family of initial-boundary value problems. In particular, in Subsection 4.1 we deal with an inverse diffusion problem for a parabolic equation on a bounded set. On the other hand, in Subsection 4.2 we study the problem of the identification of Lamé coefficients in a plane elasticity problem (cf. [1] and [16] for a similar problem in the steady case).

4.1. **Realization on bounded sets.** Let \( D_i \subseteq \mathbb{R}^{d_i}, i = 1, \ldots, n \), be bounded domains with regular boundary \( \partial D_i \), and set \( \Omega = D_1 \times \cdots \times D_n \). Denote by \( \Delta_{x_i} \) the realizations of the Dirichlet Laplacian in the space of the complex-valued square-summable functions \( L^2(\Omega) \), and define \( A_i = -\Delta_{x_i} \), on the domain \( D(A_i) = \{ u \in L^2(\Omega); \Delta_{x_i} u \in L^2(\Omega) \} \), with respect to the variables \( x_i = (x_{i1}, \ldots, x_{id_i}) \in D_i \).

Thus the operators \( A_i \) are self-adjoint and nonnegative, with discrete spectrum

\[
\{ \lambda_{i,k_i} \}_{k_i \in \mathbb{N}} \subseteq \mathbb{R}_+
\]

consisting in monotone increasing sequences of nonnegative numbers

\[
0 \leq \lambda_{i,1} \leq \cdots \leq \lambda_{i,k_i-1} \leq \lambda_{i,k_i} \to \infty \text{ as } k_i \to \infty,
\]

to be associated to the smooth product eigenfunctions \( v_{k_1}(x_1) \cdots v_{k_n}(x_n) \), respectively. For example, in the special case when

\[
d_1 = \cdots = d_n = 1 \quad \text{and} \quad D_1 = \cdots = D_n = (0, \pi),
\]

as it is well known, one has

\[
\lambda_{k_i} = k_i^2, \quad v_{k_1}(x_1) \cdots v_{k_n}(x_n) = \frac{2^n}{\pi^n} \sin(k_1 x_1) \cdots \sin(k_n x_n),
\]

for \( (x_1, \cdots, x_n) \in (0, \pi)^n \). Here, we stress that

\[
D(A_i^{1/2}) = \{ u \in L^2(\Omega); \nabla_{x_i} u \in L^2(\Omega)^{d_i}, u(\cdot, x_i, \cdot) \equiv 0 \text{ on } \partial D_i \},
\]

with \( \| A_i^{1/2} u \|_{L^2(\Omega)} = \| \nabla_{x_i} u \|_{L^2(\Omega)^{d_i}} \), having denoted by \( \nabla_{x_i} = (\partial_{x_{i1}}, \cdots, \partial_{x_{idi}}) \) the gradient operators with respect to \( x_i \). By the topological equality

\[
\partial(D_1 \times \cdots \times D_n) = (\partial D_1 \times \cdots \times D_n) \cup \cdots \cup (D_1 \times \cdots \times \partial D_n)
\]
it is easy to check that
\[ \bigcap_{i=1}^{n} D(A_i^{1/2}) = H^1_0(\Omega) \quad \text{and} \quad \bigcap_{i=1}^{n} D(A_i) = H^2(\Omega) \cap H^1_0(\Omega). \]
Furthermore, they are commuting (in the sense of distributions).

On account to the functional setting above introduced, the inverse problem we want to study reads

**Inverse Diffusion Problem.** Find the function \( u : \Omega \times [0, T] \to \mathbb{C} \) and the positive constants \( \alpha_1, \ldots, \alpha_n \) such that the following parabolic initial-boundary value problem is satisfied:

\[
\begin{aligned}
\partial_t u - \alpha_1 \Delta_{x_1} u - \cdots - \alpha_n \Delta_{x_n} u &= 0 \quad \text{in} \quad \Omega \times (0, T), \\
u(x_1, \ldots, x_n, 0) &= u_0(x_1, \ldots, x_n) \quad \text{in} \quad \Omega, \\
u(x_1, \ldots, x_n, t) &= 0 \quad \text{on} \quad \partial \Omega \times (0, T), \\
\|\nabla_{x_i} u(t, \cdot)\|_{L^2(\Omega)^{d_x}}^2 &= \varphi_1, \ldots, \|\nabla_{x_n} u(t, \cdot)\|_{L^2(\Omega)^{d_x}}^2 = \varphi_n,
\end{aligned}
\]

where \( u_0 \) and \( (\varphi_1, \ldots, \varphi_n) \) are given.

Then, abstract Theorem 2.3 yields

**Theorem 4.1.** Let \( u_0 \in L^2(\Omega) \), to be admissible. Then the solution to the Inverse Diffusion Problem \((u, \alpha_1, \ldots, \alpha_n)\), with

\[
\begin{aligned}
u \in C([0, T]; L^2(\Omega)) \cap C^1((0, T]; H^2(\Omega) \cap H^1_0(\Omega)) \cap L^2(0, T; H^1_0(\Omega)) \quad \text{is unique and depends continuously on} \ u_0 \ \text{and} \ (\varphi_1, \ldots, \varphi_n). \\
(\alpha_1, \ldots, \alpha_n) \in \mathbb{R}^n,
\end{aligned}
\]

4.2. **Identification of Lamé coefficients in a plane elasticity problem.** Consider a bounded domain \( \Omega \subset \mathbb{R}^2 \) with smooth boundary \( \partial \Omega \) which is occupied by an elastic homogeneous plate. We denote by \( \mathbf{u} = (u_1, u_2)^T \) the displacement vector, by \( \nabla = (\partial_x, \partial_y)^T \) the transposed gradient operator and by \( \Delta = (\partial_x^2 + \partial_y^2, \partial_x^2 + \partial_y^2)^T \) the 2-dimensional Laplace operator. Accordingly, we set the vectorial notations

\[
\mathbf{L}^2(\Omega) = L^2(\Omega) \times L^2(\Omega), \quad \mathbf{H}^1(\Omega) = H^1(\Omega) \times H^1(\Omega) \quad \text{and} \quad \mathbf{H}^2(\Omega) = H^2(\Omega) \times H^2(\Omega).
\]

We shall suppose that the plate is clamped on \( \Gamma_D \subset \partial \Omega \) i.e.

\[
\mathbf{u} \equiv 0 \quad \text{on} \quad \Gamma_D \quad (\text{Dirichlet homogeneous conditions}),
\]

and is stress-free on \( \Gamma_N \subset \partial \Omega \), i.e.

\[
\partial_n \mathbf{u} \equiv 0 \quad \text{on} \quad \Gamma_N \quad (\text{Neumann homogeneous conditions}),
\]

where \( \partial_n \) is the outer normal derivative on \( \Gamma_N \). Here we assume \( \partial \Omega = \Gamma_D \cup \Gamma_N \) and \( \Gamma_D \cap \Gamma_N = \emptyset \).

Following the Hooke’s law, the planar stress tensor is given by

\[
\mathbf{\tau}_{\lambda, \mu}(\mathbf{u}) = \lambda \text{Tr} (\mathbf{\varepsilon}(\mathbf{u})) \mathbf{I} + 2\mu \mathbf{\varepsilon}(\mathbf{u})
\]

where we use the standard notations

\[
\mathbf{\varepsilon}(\mathbf{u}) = \frac{1}{2} (\nabla \mathbf{u} + \nabla \mathbf{u}^T) = \frac{1}{2} \begin{bmatrix} 2\partial_x u_1 & \partial_y u_1 + \partial_z u_2 \\ \partial_y u_1 + \partial_z u_2 & 2\partial_y u_2 \end{bmatrix} \quad \text{(strain tensor)}
\]
and
\[
\text{Tr}(\varepsilon(u)) I = \begin{bmatrix}
\partial_x u_1 + \partial_y u_2 & 0 \\
0 & \partial_x u_1 + \partial_y u_2
\end{bmatrix} \quad (\text{trace tensor}).
\]

The parameters \(\lambda\) and \(\mu\) are the Lamé coefficients, which, from now on, are to be considered independent on time and space. Since
\[
\nabla \cdot \text{Tr}(\varepsilon(u)) I = \begin{bmatrix}
\partial_x^2 u_1 + \partial_y^2 u_2 \\
\partial_x \partial_y u_1 + \partial_y^2 u_2
\end{bmatrix} \quad \text{and} \quad 2\nabla \cdot \varepsilon(u) = \begin{bmatrix}
2\partial_x^2 u_1 + \partial_y^2 u_1 + \partial_x \partial_y u_2 \\
\partial_x \partial_y u_1 + \partial_x^2 u_2 + 2\partial_y^2 u_2
\end{bmatrix},
\]
then the second-order operator \(-\nabla \cdot \tau_{\lambda,\mu}(u)\) can be split as
\[
-\nabla \cdot \tau_{\lambda,\mu}(u) = (\lambda + \mu)Au + \mu Bu
\]
where we define
\[
Au = -\nabla (\nabla \cdot u) = -\begin{bmatrix}
\partial_x^2 u_1 + \partial_y \partial_y u_2 \\
\partial_x \partial_y u_1 + \partial_y^2 u_2
\end{bmatrix}
\]
on the domain
\[
D(A) = \{ u \in L^2(\Omega); Au \in L^2(\Omega) \text{ and } u \equiv 0 \text{ on } \Gamma_D \}
\]
and
\[
Bu = -\Delta u = -\begin{bmatrix}
\partial_x^2 u_1 + \partial_y^2 u_1 \\
\partial_x^2 u_2 + \partial_y^2 u_2
\end{bmatrix}
\]
on the domain
\[
D(B) = \{ u \in L^2(\Omega); Bu \in L^2(\Omega) \text{ and } u \equiv 0 \text{ on } \Gamma_D \}.
\]

By straightforward computations, it is not difficult to see that both operators \(A\) and \(B\) are nonnegative and self-adjoint, and mutually commute. Then, following the general theory, we can define \(A^{1/2}\) and \(B^{1/2}\) on the respective domains \(D(A^{1/2})\) and \(D(B^{1/2})\), with
\[
\int_\Omega |A^{1/2}u|^2 dxdy = \int_\Omega Au \cdot \nabla u dxdy = \int_\Omega |\nabla \cdot u|^2 dxdy
\]
and
\[
\int_\Omega |B^{1/2}u|^2 dxdy = \int_\Omega Bu \cdot \nabla u dxdy = \int_\Omega |\nabla u|^2 dxdy.
\]
Here, since \(D(B^{1/2}) \subseteq D(A^{1/2})\) and \(D(B) \subseteq D(A)\), there holds
\[
D(B^{1/2}) \cap D(A^{1/2}) = D(B^{1/2}) = H^1_D(\Omega) = \{ u \in H^1(\Omega); u \equiv 0 \text{ on } \Gamma_D \}
\]
and
\[
D(B) \cap D(A) = D(B) = H^2(\Omega) \cap H^1_D(\Omega).
\]

Then, we can state our inverse problem consisting in the identification of the Lamé coefficients, along with the displacement function \(u\), as in the following...
Inverse Lamé Problem. Find the function \( u : \Omega \times [0, T] \to \mathbb{C}^2 \) and the constants \( \lambda \) and \( \mu \) that fulfill the parabolic initial-boundary value problem:

\[
\begin{cases}
\partial_t u - \nabla \cdot \tau_{\lambda, \mu}(u) = 0 & \text{in } \Omega \times (0, T), \\
 u(x, y, 0) = u_0(x, y) & \text{in } \Omega, \\
 u(x, y, t) = 0 & \text{on } \Gamma_D \times (0, T), \\
 \partial_{\nu} u(x, y, t) = 0 & \text{on } \Gamma_N \times (0, T), \\
 \int_{\Omega} |\nabla \cdot u(x, y, T)|^2 dxdy = \varphi & \text{and } \int_{\Omega} |\nabla u(x, y, T)|^2 dxdy = \psi,
\end{cases}
\]

where \( u_0, \varphi \) and \( \psi \) are given.

In order to deduce our uniqueness result, we now apply Theorem 2.3 where, according to decomposition (14), we set the positive unknown constants as

\[
\alpha = \lambda + \mu \quad \text{and} \quad \beta = \mu.
\]

Notice that \( \alpha = \lambda + \mu > 0 \) is in fact the strong convexity condition in dimension two (cf. [1, 16]). Along with this choices, we have

**Theorem 4.2.** Let \( u_0 \in L^2(\Omega) \), to be admissible. Then the solution to the Inverse Lamé Problem \( (u, \lambda, \mu) \), with

\[
\begin{cases}
 u \in C([0, T]; L^2(\Omega)) \cap C^1((0, T]; H^2(\Omega) \cap H_D^1(\Omega)) \cap L^2(0, T; H_D^1(\Omega)) \\
 \lambda \in \mathbb{R}, \quad \mu \in \mathbb{R}_+ \quad \text{and} \quad \lambda + \mu > 0,
\end{cases}
\]

is unique and depends continuously on \( u_0, \varphi \) and \( \psi \).

**Remark 3.** We stress that the physical meaning of the Inverse Lamé Problem, as it is here stated, represents the in-plane motion of an elastic flat plate of negligible mass under the action of a local damping force characterized by a unitary damping coefficient. That is, the term \( \partial_{tt}^2 u \) in the dynamic equilibrium equation has been neglected, as the parabolic abstract theory we developed cannot manage directly a second-order equation in time. As a matter of fact, the problem of identification of Lamé constants \( \lambda \) and \( \mu \) in the more realistic equation

\[
\partial_{tt}^2 u - \nabla \cdot \tau_{\lambda, \mu}(u) = 0
\]

is an interesting direction for further investigations. This corresponds at removing the plate negligible mass assumption and tackle the more meaningful problem representing the flat plate dynamics under the action of local elastic and inertial forces.

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