A Model of Weighted Network: the Student Relationships in a Class

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A simple model is proposed to simulate the evolution of interpersonal relationships in a class. The small social network is simply assumed as an undirected and weighted graph, in which students are represented by vertices, and the extent of favor or disfavor between two of them are denoted by the weight of corresponding edge. Various weight distributions have been found by choosing different initial configurations. Analysis and experimental results reveal that the effect of first impressions has a crucial influence on the final weight distribution. The system also exhibits a phase transition in the final hostility (negative weights) proportion depending on the initial amity (positive weights) proportion.

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I. INTRODUCTION

Recently, physicists have displayed much interests in social phenomena that exhibit complex behaviors and nonlinear dynamics. A social network is a set of people with some pattern of contacts or interactions between them [1]. The patterns of friendships between individuals [2], business relationships between corporations [3, 4], and intermarriages between families [5] are all examples of networks that have been studied in the past. Hidden behind such complex phenomena, however, are many factors hard to control, including human nature, social environment, social distance and opportunity. Fortunately, human beings have accumulated precious experiences of themselves. Sociologists and psychologists have long noticed that the first impression between two individuals is often the seed of their future relationship, not only seen in romantic stories. Good seed may promise a good harvest, while ill seed may be supposed to portend illness. This effect seems to be more distinct in campus life, where the influence of social distances is not so remarkable. Pupils of a class serve as a typical example that exhibits relatively simple friendships. See the studies of friendship networks of school children by Rapoport [6]. According to empirical observation, students are more likely to get along with their friends in daily activities, such as dinner, discussion, entertainment, etc. Often, their ties are strengthened through frequent contacts. Assumably, people with common friends or common “enemies” are prone to unite; likewise, the “enemy” of Jack’s friends or the friend of Jack’s “enemies” may be very difficult to associate with Jack. Similar human relations and social environments are an effective catalyzer for friendships. As a result of restricted social scope and in the light of psychology, the encounters between “en-

II. THE MODEL

The model system consists of \( N \) individuals (students of a class). Since the size of a class is not too large, it is suitable to assume that each student has chances to contact all his classmates. For clarity, we introduce a generalization of the \( N \times N \) adjacency matrix to describe the interpersonal relationships of the small social network. The matrix elements \( \omega_{ij} \) represent the weight of edge \( e_{ij} \), where \( i, j = 1, 2, \ldots, N \). Postulate that the value of \( \omega_{ij} \) is discrete and can be negative for the case of disfavor relationships. If most elements of the matrix are positive, the system can be called harmonious; otherwise, it contains considerable hostility. As an original model,
we add an assumption that each contact can only alter weight by $\pm 1$ at most. In other words, love or hatred is not formed in a day (or individuals will not fall in “love” at first sight). This condition makes the contacts moderate and can be interpreted by the fact that true friends (or enemies) are selected by time.

Now, take $N=5$ for instance, the adjacency matrix is as below:

$$
\begin{pmatrix}
0 & \omega_{12} & \omega_{13} & \omega_{14} & \omega_{15} \\
\omega_{21} & 0 & \omega_{23} & \omega_{24} & \omega_{25} \\
\omega_{31} & \omega_{32} & 0 & \omega_{34} & \omega_{35} \\
\omega_{41} & \omega_{42} & \omega_{43} & 0 & \omega_{45} \\
\omega_{51} & \omega_{52} & \omega_{53} & \omega_{54} & 0
\end{pmatrix}
$$

For undirected and weighted graphs, $\omega_{ij}=0$ and $\omega_{ji}=\omega_{ij}$. In the following, we will only discuss the case of symmetric weights. Since the $i$th row of the matrix records the information of interpersonal relationships of student $i$, we will use it to judge the interpersonal similarity. This point is a basic assumption of our model and will be reviewed later. At the beginning of evolution, it is reasonable that some initial weights have non-zero values, due to first impressions among individuals. For convenience, we assign value 1 with probability $p$, and $-1$ with probability $1-p$ to the elements of matrix, i.e. the seed of the model is given. Here, $p$ is called the initial amity possibility. The symmetry requirement of the matrix must be satisfied. The initial configuration and the evolution of the system are moderate in degree. The definition of the model is based on the weights’ dynamics:

(i) First, suppose student $i$ has been randomly selected from the class. Then, he takes the initial to contact student $j$ with a certain probability. A natural idea is to set this possibility as:

$$P_{i\rightarrow j} = \frac{|\omega_{ij}|}{\sum_{j=1}^{N} |\omega_{ij}|},$$

recalling the point that “in the deepest of one’s heart are always those he hates or he loves, while people without intensified contact are easy to fade from memory” (see Introduction). However, this method may lead to the absurd case that individual $j$ with $\omega_{ij}=0$ would not be chosen by $i$, nor would $i$ by $j$. Therefore, the edges with 0 weight keep invariant, that is, unfamiliar ones are always unfamiliar. To avoid this unrealistic scenario, let $i$ choose $j$ with possibility:

$$W_{i\rightarrow j} = \frac{|\omega_{ij}| + 1}{\sum_{j=1}^{N} (|\omega_{ij}| + 1)},$$

obviously,

$$W_{i\rightarrow j} = W_{j\rightarrow i}.$$  

The adaption is based on the moderate evolution mechanism: the minimum unit of weight is 1, and each contact can only alter weight by $\pm 1$ at most. A non-zero value of $W_{i\rightarrow j}$ is needed but we expect a least deflection to $P_{i\rightarrow j}$. When $i$ selects $j$ with possibility $W_{i\rightarrow j}$, for small $|\omega_{ij}|$ this perturbation is quite significant and reasonable. In a fresh environment, people will try to get familiar with others and we call it unfamiliar-familiar period. The differences of weights are not significant; thus, the contacts between them behave no obvious preferences. Once some weight becomes 0 during the period, it is still possible to be altered in later contacts. Initially, the interpersonal relations of the class are unsteady, and the first impressions rising in the period will play an important role in future weight polarization. When friends and enemies (large $|\omega_{ij}|$) form in the system, situation is quite different and we call it friend-enemy period. The contacts between friends and encounters between enemies now become more frequent, and the interpersonal relationships tend to be steady. However, the emergence of new friends and enemies is not forbidden.

(ii) Now, $i$ and $j$ have been chosen for interaction, then $\omega_{ij}$ will be altered with a certain possibility:

$$\omega_{ij} \rightarrow \omega_{ij} \pm 1.$$  

The crux of the problem now is how to determine the possibility. Recall that similar human relations and social environments are more likely to promote friendships. We could define $\gamma_{ij}$ as below to describe the interpersonal relation similarity:

$$\gamma_{ij} = C^{-1} \sum_{\alpha} \omega_{i\alpha} \cdot \omega_{\alpha j},$$

where

$$C = \sqrt{\sum_{\alpha} \omega_{i\alpha}^2} \cdot \sqrt{\sum_{\beta} \omega_{j\beta}^2}.$$  

It is manifest that $\gamma_{ij}=\gamma_{ji}$ and $-1 \leq \gamma_{ij} \leq 1$. One can see that the definition of $\gamma_{ij}$ is equivalent to the inner product of two normalized vectors. So $\gamma_{ij}$ could be regarded as a signed possibility. Then in detail, the rules are: when $\gamma_{ij} \geq 0$

$$\omega_{ij} \rightarrow \omega_{ij} + 1, \omega_{ji} \rightarrow \omega_{ji} + 1$$  

with possibility $\gamma_{ij}$, and nothing is altered with possibility $1 - \gamma_{ij}$;

when $\gamma_{ij} < 0$

$$\omega_{ij} \rightarrow \omega_{ij} - 1, \omega_{ji} \rightarrow \omega_{ji} - 1$$  

with possibility $|\gamma_{ij}|$, and nothing is altered with possibility $1 - |\gamma_{ij}|$.

The mechanism (ii) have simple physical and realistic interpretations. Take Jack and Mike for instance. If they have common friends or common enemies, they are
more likely to strengthen their friendship ($\gamma > 0$). Suppose, however, they are good fellows at first and Jack’s pals are all Mike’s foes. If Jack goes on associating with Mike, he may be excluded by some of his friends and have to confront his foes under certain circumstances. Thus, their social relations have a potential to separate them ($\gamma < 0$), just like an electron-positron system under external electro-magnetic field. In this case, we can equally say that Jack and Mike have distinct social tastes and in the long run, their friendship is on test.

After the weights have been updated, the process is iterated by randomly selecting a new individual for the next contact, i.e. going back to step (i) until the class disbands.

To better understand the micro dynamics, it is beneficial to analyze the case for $N=3$. Suppose Jack(A), Mike(B) and John(C) interact with each other according to above mechanism. The possible states of this triangle-relation evolution are shown in Fig. 1. Triangle (b) represents the friend-friend-enemy relation, that is, two edges of the triangle have positive (+) weights and another is negative (−) represents hostility.

FIG. 1: the possible states of triangle relationship. Positive (+) edge means friendly relation and negative (−) represents hostility.

III. EXPERIMENTAL RESULTS

We choose different initial amity possibility $p$ to perform simulations. In order to obtain the weight distribution, the range of weight $\omega$ is equally divided by $M$. Then, the range $[\omega_{\min}, \omega_{\max}]$ becomes $[\omega_1, \omega_2, \omega_3, \ldots, [\omega_M, \omega_{M+1}]$, where $\omega_1=\omega_{\min}, \omega_{M+1}=\omega_{\max}$. Define $n_\omega$, as the number of weights in $[\omega_l, \omega_{l+1})$, $l=1, 2, \ldots, M$; when $l=M$ the interval is $[\omega_M, \omega_{M+1}]$. Plotted in Fig. 2-7 are typical weight distributions ($n_\omega \sim \omega$) which behave a pinnacle for $p=0$ or 0.50, power-law with a heavy tail for $p=0.59$ or 0.60, an exponential decay for $p=0.70$, and a peak structure for $p=1.00$.

When $p=0$, i.e. the initial non-diagonal elements are all −1, the final weight distribution exhibits a symmetric pinnacle near $\omega=0$. The peak value is 2690, and on both sides $n_\omega$ is quite low (but many non-zero), see Fig. 2. The weight distribution for $p \leq 0.50$ exhibits a similar behavior, as experiments can test. For $p=0.50$, the peak value is 1693, see Fig. 3. When the initial hostility proportion $1-p$ is significant, the model mechanics has a tendency to push the peak towards the right. However, no matter how many time steps are run, the peak cannot move further beyond zero. We can conclude from these results that when the initial amity is insufficient, the harmony of the class is out of the question. The most majority are indifferent to others, and true friends and foes can rarely “survive” under such environments.

Presented in Fig. 4 and Fig. 5 are positive weight distributions for $p=0.60$ and $p=0.59$. The negative weights are discarded in the log-log plot. Apparently, each distribution obeys power law with a heavy tail; this power-law property is independent from iterated steps. Here, negative weights in the matrix are quite sparse, compared with the positive, see Fig. 8 and related discussion.

Exponential distributions are found near $p=0.70$ and
independent from time steps, as shown in semi-log plot (see Fig. 6). The negative weights have disappeared under such circumstances. It is clear that for large $\omega$, $n_\omega$ increases with the passage of time.

When $p=1.00$, i.e. the initial non-diagonal elements are all 1, we find a peak structure in $n_\omega \sim \omega$ diagram(Fig. 7). Different from the above, the maximum of $n_\omega$ is reached somewhat far from $\omega=0$. By increasing the iterated times, the peak and the upper limit of $\omega$ are both pushed to the right. It means the harmony of the system is boosted up. One can check that for $p \geq 0.8$, the weight distribution displays similar behaviors.

Three points must be stressed here. First, we have observed four typical kinds of distributions from $p=0$ to $p=1.00$, and each kind appears in a certain range. However, the distributions in some unmentioned ranges are not so typical and may be influenced by increasing time steps. Second, by comparing the weight distributions of different $p$ from 0 to 1, one can see that this system exhibits a potential to become harmonious. Third, the properties of the above weight distributions also suggest a critical transition. Define the hostility proportion $h$ as

$$h = \frac{\sum_{i,j,\omega_{ij}<0} \omega_{ij}}{\sum_{i,j} |\omega_{ij}|} \tag{9}$$

which can describe the harmony degree of the class from an opposite sight. The dependence of $h$ on $p$ is shown in Fig. 8, and a phase transition is found near $p_c=0.6$, where the weight distribution exhibits power-law (Fig. 4
FIG. 7: weight distribution for N=100, p=1.00 after 0.5 × 10^6, 1.0 × 10^6 and 2.0 × 10^6 time steps.

FIG. 8: the dependence of hostility proportion h on the initial amity proportion p, hostility-amity phase transition for N=100 after 1.0 × 10^6 time steps.

and Fig. 5). Below the critical value, the hostility proportion h is non-trivial, that is, there exists considerable hostility in the interpersonal atmosphere; while above the critical value, the final hostility proportion h is close to 0, i.e. the interpersonal atmosphere of the class is quite harmonious. Conflicts and grievances may melt gradually under such harmonious environment.

IV. REVIEW AND OUTLOOK

Of the academic disciplines the social sciences have the longest history of the substantial quantitative study of real-world networks [8, 9]. Of particular note among the early works on the subject are: Jacob Moreno’s work in the 1920s and 30s on friendship patterns within small groups [3]; the so-called “southern women study” of Davis et al. [10], which focused on the social circles of women in an unnamed city in the American south in 1936; the study by Elton Mayo and colleagues of social networks of factory workers in the late 1930s in Chicago [11]; the studies of friendship networks of school children by Rapoport and others [4, 12]; and the mathematical models of Anatol Rapoport [13], who was one of the most theorists to stress the significance of the degree distribution in networks of all kinds, not just social networks. In more recent years, studies of business communities [4, 17, 18] and of patterns of sexual contacts [16, 17, 18, 19, 20] have attracted particular attention. However, traditional social network studies often suffer from problems of inaccuracy, subjectivity, and small sample size. Because of these problems many researchers have turned to other methods for probing social networks. One source of copious and relatively reliable data is collaboration networks [21, 22]; another source of reliable data about personal connections between people is communication records of certain kinds [23, 24]. It is quite possible for researchers to investigate the (weighted) relationships in a certain class or club, for its finite size and simple patterns. Previous researches had stressed the significance of the degree distribution in social networks, while more practical studies on the weighted social networks are required.

Recently, Alain Barrat, et al. have proposed a general model for the growth of weighted networks [25], considering the effect of the coupling between topology and weights’ dynamics. It appears that there is a need for a modelling approach to complex networks that goes beyond the purely topological point.

The simple model here is self-generated and allows various further modifications. Some acute interacting ingredients could be taken into this system. For instance, Jack and Mike might be intimate friends long before, so the initial weight between them must be larger. In the present model, moreover, good friendship will not collapse instantly, nor will old grievance; thus it seems more reasonable to take some acute interaction into account. In the framework of this model, it is possible and interesting to study the adaptive process of a new student joining the class midway [26]. Meanwhile, this model could be easily extended to directed graph more close to real world where human relations are often asymmetric. Finally, generalizing it to complex social, economic and political networks is also an interesting and challenging task. The relationships between individuals, economic entities or nations are amazingly similar in many aspects. The basic assumptions and concepts here are expected to have well applications in related fields.
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