LAGUERRE BV SPACES, LAGUERRE PERIMETER AND THEIR APPLICATIONS

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Abstract. In this paper, we introduce the Laguerre bounded variation space and the Laguerre perimeter, thereby investigating their properties. Moreover, we prove the isoperimetric inequality and the Sobolev inequality in the Laguerre setting. As applications, we derive the mean curvature for the Laguerre perimeter.

1. Introduction

The spaces BV of functions of bounded variation in Euclidean spaces have been a classical setting now where several problems, mainly (but not exclusively) of variational nature, find their nature framework. For instance, when working with minimization problems, reflexivity or the weak compactness property of the function space $W^{1,p}(\mathbb{R}^d)$ for $p > 1$, the space BV usually plays an important role. For the case of the space $W^{1,1}(\mathbb{R}^d)$, one possible way to deal with this lack of reflexivity is to consider the space $\text{BV}(\mathbb{R}^d)$. However, the importance of generalizing the classical notion of variation has been pointed out in several occasions by E. De Giorgi in [4]. Recently, Huang, Li and Liu in [11] investigate the capacity and perimeters derived from $\alpha$-Hermite bounded variation. In a general framework of strictly local Dirichlet spaces with doubling measure, Alonso-Ruiz, Baudoin and Chen et al. in [1] introduce the class of bounded variation functions and proved the Sobolev inequality under the Bakry-Émery curvature type condition. For further information on this topic, we refer the reader to [6, 12, 13] and the references therein.

One of the aims of this paper is intended to discuss several basic questions of geometric measure theory related to the Laguerre operator in Laguerre BV spaces. At first, we will present a very short introduction to the Laguerre operator.

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Given a multiindex $\alpha = (\alpha_1, \cdots, \alpha_d)$, $\alpha \in (-1, \infty)^d$, the Laguerre differential operator is defined by:
\[ L^\alpha = - \sum_{i=1}^{d} \left[ x_i \frac{\partial^2}{\partial x_i^2} + (\alpha_i + 1 - x_i) \frac{\partial}{\partial x_i} \right]. \]

Consider the probabilistic gamma measure $\mu_\alpha$ in $\mathbb{R}^d_+ = (0, \infty)^d$ given by
\[ d\mu_\alpha(x) = \prod_{i=1}^{d} x_i^{\alpha_i} e^{-x_i} \frac{dx}{\Gamma(\alpha_i + 1)} = \omega(x) dx. \]

It is well-known that $L^\alpha$ is positive and symmetric in $L^2(\mathbb{R}^d_+, d\mu_\alpha)$. Moreover, $L^\alpha$ has a closure which is selfadjoint in $L^2(\mathbb{R}^d_+, d\mu_\alpha)$ and which also will be denoted by $L^\alpha$. We define the $i$-th partial derivative associated with $L^\alpha$ by
\[ \delta_i = \sqrt{x_i} \frac{\partial}{\partial x_i}, \]
see [9] or [10]. One of the motivations of such definition is that $L^\alpha = \sum_{i=1}^{d} \delta_i^* \delta_i$.

Naturally, we use $BV_{L^\alpha}(\Omega)$ to represent the class of all functions with the Laguerre bounded variation ($L^\alpha$-BV in short) on $\Omega$, as a continuation of [11], the goal of this paper is to consider some related topics for the Laguerre setting, and the plan of the notes is as follows. Section 2.1 contains some basic facts and notations needed in the sequel, the lower semicontinuity (Lemma 2.1), the completeness (Lemma 2.2), the structure theorem (Theorem 2.3) and approximation via $C^\infty$-functions (Theorem 2.4). It should be noted that in contrast with Theorem 2 in [7, Section 5.2.2], we need to use the mean value theorem of multivariate functions and the intrinsic nature of the Laguerre variation.
Section 2.2 is devoted to the perimeter \( P_{\mathcal{L}^\alpha}(-,\Omega) \) induced by \( \mathcal{BV}_{\mathcal{L}^\alpha}(\Omega) \), see (10) below.

Recall that the classical perimeter of \( E \subseteq \mathbb{R}^d \) is defined by

\[
P(E) = \sup_{\varphi \in \mathcal{F}(\mathbb{R}^d)} \left\{ \int_E \text{div}\varphi(x)dx \right\},
\]

where \( \mathcal{F}(\mathbb{R}^d) \) denotes the class of all functions

\[
\varphi = (\varphi_1, \cdots, \varphi_d) \in C^1_c(\mathbb{R}^d, \mathbb{R}^d)
\]
satisfying

\[
\|\varphi\|_\infty = \sup_{x \in E} \left\{ (|\varphi_1(x)|^2 + \cdots + |\varphi_d(x)|^2)^{\frac{1}{2}} \right\} \leq 1.
\]

An elementary property of \( P(E) \) is

\[
(1) \quad P(E) = P(E^c), \quad \forall E \subset \mathbb{R}^d.
\]

In Lemma 2.10, we proved that (1) is valid for the Laguerre perimeter \( P_{\mathcal{L}^\alpha}(-) \). In section 2.3, we obtain a coarea formula for \( \mathcal{L}^\alpha\text{-BV} \) functions. As an application, we deduce that the Sobolev type inequality

\[
\|f\|_{L^{d/d-1}(\Omega_1, d\mu_\alpha)} \lesssim |\nabla_{\mathcal{L}^\alpha}f|_{\Omega_1}
\]

is equivalent to the following isoperimetric inequality

\[
\mu_\alpha(E) \lesssim P_{\mathcal{L}^\alpha}(E, \Omega_1),
\]

see Theorem 2.12. We point out that, in the proof of (2), the inequality

\[
|\nabla f(x)|_{\Omega_1} \lesssim |\nabla_{\mathcal{L}^\alpha}f(x)|_{\Omega_1}
\]

holds true. With this in mind, we consider the subset

\[
(3) \quad \Omega_1 = \Omega \setminus \{x \in \mathbb{R}^d_+ : \exists i \in 1, \cdots, d \text{ such that } \sqrt{x_i} < 1\}
\]
of \( \Omega \) which is a reasonable substitute of \( \Omega \) and whose figure is given as follows:

Our motivation comes not only from the fact that these objects are interesting on their own, but also from the possibility of their potential applications in further research concerning the Laguerre operator. Consequently, in Section 3, we want to investigate the Laguerre mean curvature of a set with finite Laguerre perimeter. For the special case, i.e., the Laplace operator \( \Delta \), sets of finite perimeter were introduced by E. De Giorgi in the 1950s, and were applied to the research on some classical problems of the calculus of variations, such as the Plateau problem and the isoperimetric problem, see \([5, 8]\) and \([15]\). Barozzi-Gonzalez-Tamanini \([3]\) proved that every set \( E \) of finite classical perimeter \( P(E) \) in \( \mathbb{R}^d \) has mean curvature in \( L^1(\mathbb{R}^d) \). A natural question is that if the result of \([3]\) holds for \( P_{\mathcal{L}^\alpha}(E, \Omega), \alpha \in (-1, \infty)^d \). We point out that, in the proof of main theorem of \([3]\), the identity (1) is required. In Theorem 3.1, we generalize the result of \([3]\) to \( P_{\mathcal{L}^\alpha}(-, \Omega_1) \) and prove that every set \( E \) with \( P_{\mathcal{L}^\alpha}(E, \Omega_1) < \infty \) in \( \Omega_1 \) has mean curvature in \( L^1(\Omega_1, d\mu_\alpha) \).
Throughout this article, we will use $c$ and $C$ to denote positive constants, which are independent of main parameters and may be different at each occurrence. $U \approx V$ indicates that there is a constant $c > 0$ such that $c^{-1}V \leq U \leq cV$, whose right inequality is also written as $U \lesssim V$. Similarly, one writes $V \gtrsim U$ for $V \geq cU$. For convenience, the positive constant $C$ may change from one line to another and this usually depends on the spatial dimension $d$ and other fixed parameters.

2. $L^\infty$-BV functions

2.1. Fundamentals of $L^\infty$-BV Space

In this section, we introduce the $L^\infty$-BV space, i.e. the class of all functions with the Laguerre bounded variation and investigate its properties. The Laguerre variation ($L^\infty$-variation in short) of $f \in L^1(\Omega, d\mu_\alpha)$ is defined by

$$|\nabla_{L^\infty} f|(\Omega) = \sup_{\varphi \in \mathcal{F}(\Omega)} \left\{ \int_{\Omega} f(x) \text{div}_{L^\infty} \varphi(x) d\mu_\alpha(x) \right\},$$

where $\mathcal{F}(\Omega)$ denotes the class of all functions $\varphi = (\varphi_1, \varphi_2, \ldots, \varphi_d) \in C^1_c(\Omega, \mathbb{R}^d)$ satisfying

$$\|\varphi\|_{L^\infty} = \sup_{x \in \Omega} \left\{ (|\varphi_1(x)|^2 + \ldots + |\varphi_d(x)|^2)^{1/2} \right\} \leq 1.$$

An function $f \in L^1(\Omega, d\mu_\alpha)$ is said to have the $L^\infty$-bounded variation on $\Omega$ if

$$|\nabla_{L^\infty} f|(\Omega) < \infty,$$

and the collection of all such functions is denoted by $\mathcal{B}V_{L^\infty}(\Omega)$, which is a Banach spaces with the norm

$$\|f\|_{\mathcal{B}V_{L^\infty}(\Omega)} = \|f\|_{L^1(\Omega, d\mu_\alpha)} + |\nabla_{L^\infty} f|(\Omega).$$

**Definition.** Suppose $\Omega$ is an open set in $\mathbb{R}^d_+$. Let $1 \leq p \leq \infty$. The Sobolev space $W^{k,p}_{L^\infty}(\Omega)$ associated with $L^\infty$ is defined as the set of all functions $f \in L^p(\Omega, d\mu_\alpha)$ such that

$$\delta_j_1 \ldots \delta_j_m f \in L^p(\Omega, d\mu_\alpha), \quad 1 \leq j_1, \ldots, j_m \leq d, \quad 1 \leq m \leq k.$$

The norm of $f \in W^{k,p}_{L^\infty}(\Omega)$ is given by

$$\|f\|_{W^{k,p}_{L^\infty}(\Omega)} := \sum_{1 \leq j_1 \ldots j_m \leq d, \ 1 \leq m \leq k} \|\delta_j_1 \ldots \delta_j_m f\|_{L^p(\Omega, d\mu_\alpha)} + \|f\|_{L^p(\Omega, d\mu_\alpha)}.$$

In what follows, we will collect some properties of the space $\mathcal{B}V_{L^\infty}(\Omega)$.

**Lemma 2.1.**
Suppose \( f \in W_{L^\alpha}^{1,1}(\Omega) \), then
\[
|\nabla_{L^\alpha} f|(\Omega) = \int_\Omega |\nabla_{L^\alpha} f(x)|d\mu_\alpha(x),
\]
which implies \( W_{L^\alpha}^{1,1}(\Omega) \subseteq BV_{L^\alpha}(\Omega) \).

(ii) (Lower semicontinuity). Suppose \( f_k \in BV_{L^\alpha}(\Omega), \ k \in \mathbb{N} \text{ and } f_k \to f \text{ in } L_{L^\alpha}^\infty(\Omega,d\mu_\alpha) \), then
\[
|\nabla_{L^\alpha} f|(\Omega) \leq \liminf_{k \to \infty} |\nabla_{L^\alpha} f_k|(\Omega).
\]

Proof. (i) For every \( \varphi \in C^1_c(\Omega,\mathbb{R}^d) \) with \( \|\varphi\|_{L^\infty(\Omega)} \leq 1 \), we have
\[
\left| \int_\Omega f(x) \text{div}_{L^\alpha} \varphi(x)d\mu_\alpha(x) \right| = \left| \int_\Omega \nabla_{L^\alpha} f(x) \cdot \varphi(x)d\mu_\alpha(x) \right| \leq \int_\Omega |\nabla_{L^\alpha} f(x)|d\mu_\alpha(x).
\]
By taking the supremum over \( \varphi \), it is obvious that
\[
|\nabla_{L^\alpha} f|(\Omega) \leq \int_\Omega |\nabla_{L^\alpha} f(x)|d\mu_\alpha(x).
\]

Define \( \varphi \in L^\infty(\Omega,\mathbb{R}^d) \) as follows:
\[
\varphi(x) := \begin{cases} \frac{\nabla_{L^\alpha} f(x)}{|\nabla_{L^\alpha} f(x)|}, & \text{if } x \in \Omega \text{ and } \nabla_{L^\alpha} f(x) \neq 0, \\ 0, & \text{otherwise.} \end{cases}
\]

It is easy to see that \( \|\varphi\|_{L^\infty(\Omega)} \leq 1 \). Moreover, we can obtain the approximating smooth fields \( \varphi_n := (\varphi_{n,1}, \ldots, \varphi_{n,d}) \) such that \( \varphi_n \to \varphi \) pointwise as \( n \to \infty \), with \( \|\varphi_n\|_{L^\infty(\Omega)} \leq 1 \) for all \( n \in \mathbb{N} \). Combining the definition of \( |\nabla_{L^\alpha} f|(\Omega) \) with integration by parts derives that for every \( n \geq 1 \),
\[
|\nabla_{L^\alpha} f|(\Omega) \geq \int_\Omega f(x) \text{div}_{L^\alpha} \varphi_n(x)d\mu_\alpha(x)
= \int_\Omega f(x)(\delta_{L^\alpha}^1 \varphi_{n,1}(x) + \cdots + \delta_{L^\alpha}^d \varphi_{n,d}(x))d\mu_\alpha(x)
= \int_\Omega \nabla_{L^\alpha} f(x) \cdot \varphi_n(x)d\mu_\alpha(x).
\]
Using the dominated convergence theorem and the definition of \( \varphi \) in (4), we have
\[
|\nabla_{L^\alpha} f|(\Omega) \geq \int_\Omega |\nabla_{L^\alpha} f(x)|d\mu_\alpha(x)
\]
by letting \( n \to \infty \).

(ii) Fix \( \varphi \in C^1_c(\Omega,\mathbb{R}^d) \) with \( \|\varphi\|_{L^\infty(\Omega)} \leq 1 \). We use the definition of \( |\nabla_{L^\alpha} f_k|(\Omega) \) to obtain
\[
|\nabla_{L^\alpha} f_k|(\Omega) \geq \int_\Omega f_k(x) \text{div}_{L^\alpha} \varphi(x)d\mu_\alpha(x).
\]
The convergence of \( \{f_k\}_{k \in \mathbb{N}} \) in \( L^1_{\text{loc}}(\Omega, d\mu_\alpha) \) to \( f \) implies that
\[
\liminf_{k \to \infty} |\nabla L^\alpha f_k|(\Omega) \geq \int_\Omega f(x) \text{div} L^\alpha \varphi(x) d\mu_\alpha(x).
\]
Therefore, (ii) can be proved by the definition of \(|\nabla L^\alpha f|(\Omega)\) and the arbitrariness of such functions \( \varphi \). \( \square \)

**Lemma 2.2.** The space \((\mathcal{BV}_{L^\alpha}(\Omega), \|\cdot\|_{\mathcal{BV}_{L^\alpha}(\Omega)})\) is a Banach space.

**Proof.** It is easy to check that \( \|\cdot\|_{\mathcal{BV}}(\Omega) \) is a norm and we omit the details. In what follows, we prove the completeness of \( \mathcal{BV}_{L^\alpha}(\Omega) \). Let \( \{f_n\}_{n \in \mathbb{N}} \subset \mathcal{BV}_{L^\alpha}(\Omega) \) be a Cauchy sequence, namely, for every \( \varepsilon > 0 \), there exists \( n_0 \in \mathbb{N} \) such that
\[
\forall n, m \geq n_0, \quad |\nabla L^\alpha (f_m - f_n)|(\Omega) < \varepsilon.
\]
Especially, \( \{f_n\}_{n \in \mathbb{N}} \) is a Cauchy sequence in the Banach space \((L^1(\Omega, d\mu_\alpha), \|\cdot\|_{L^1(\Omega, d\mu_\alpha)})\), which implies that there exists \( f \in L^1(\Omega, d\mu_\alpha) \) with \( \|f_n - f\|_{L^1(\Omega, d\mu_\alpha)} \to 0 \) as \( n \to \infty \). Hence, via Lemma 2.1(ii), we have
\[
|\nabla L^\alpha (f_m - f_n)|(\Omega) \leq \liminf_n |\nabla L^\alpha (f_m - f_n)|(\Omega) \leq \varepsilon, \quad (\forall m \geq n_0)
\]
which implies that \( |\nabla L^\alpha (f_m - f)|(\Omega) \to 0 \) as \( m \to \infty \). This completes the proof. \( \square \)

The following lemma gives the structure theorem for \( L^\alpha \)-BV functions and it can be proved by the Hahn-Banach theorem and the Riesz representation theorem.

**Lemma 2.3.** *(Structure Theorem for \( \mathcal{BV}_{L^\alpha} \) functions).* Let \( f \in \mathcal{BV}_{L^\alpha}(\Omega) \). Then there exists a Radon measure \( \mu_{L^\alpha} \) on \( \Omega \) such that
\[
\int_\Omega f(x) \text{div} L^\alpha \varphi(x) d\mu_\alpha(x) = \int_\Omega \varphi(x) \cdot d\mu_{L^\alpha}(x)
\]
for every \( \varphi \in C_c^\infty(\Omega, \mathbb{R}^d) \) and
\[
|\nabla L^\alpha f|(\Omega) = |\mu_{L^\alpha}|(\Omega),
\]
where \( |\mu_{L^\alpha}| \) is the total variation of the measure \( \mu_{L^\alpha} \).

**Proof.** It is easy to see that
\[
|\int_\Omega f(x) \text{div} L^\alpha \varphi(x) d\mu_\alpha(x)| \leq |\nabla L^\alpha f|(\Omega) \|\varphi\|_{L^\infty(\Omega)}, \quad \forall \varphi \in C_c^\infty(\Omega, \mathbb{R}^d).
\]
Denote by the functional \( \Phi \) with
\[
\Phi : C_c^\infty(\Omega, \mathbb{R}^d) \to \mathbb{R},
\]
where
\[
\langle \Phi, \varphi \rangle := \int_\Omega f(x) \text{div} L^\alpha \varphi(x) d\mu_\alpha(x).
\]
Then the Hahn-Banach theorem derives that there exists a linear and continuous extension $L$ of $\Phi$ to the normed space $(C_c(\Omega, \mathbb{R}^d), \|\cdot\|_{L^\infty(\Omega)})$ such that 

$$
\|L\| = \|\Phi\| = |\nabla L_\alpha f|_{\Omega}.
$$

By the Riesz representation Theorem (cf. [2, Corollary 1.55]), there exists a unique $\mathbb{R}^d$-valued finite Radon measure $\mu_L$ with 

$$
L(\varphi) = \int_{\Omega} \varphi(x) \cdot d\mu_L(x), \quad \forall \varphi \in C_c(\Omega, \mathbb{R}^d)
$$

and so that $|\mu_L|_{\Omega} = \|L\|$. Thus we have $|\mu_L|_{\Omega} = |\nabla L_\alpha f|_{\Omega}$, which completes the proof.

In the following theorem, we can obtain the approximation result for the $L_\alpha$-variation.

**Theorem 2.4.** Let $\Omega_1$ be an open set defined in (3). Assume that $u \in BV_{L_\alpha}(\Omega_1)$, then there exists a sequence $\{u_h\}_{h \in \mathbb{N}} \in BV_{L_\alpha}(\Omega_1) \cap C_\infty_c(\Omega_1)$ such that 

$$
\lim_{h \to \infty} \|u_h - u\|_{L^1(\Omega_1, d\mu_{\alpha})} = 0
$$

and 

$$
\lim_{h \to \infty} \int_{\Omega_1} |\nabla L_\alpha u_h(x)| d\mu_{\alpha}(x) = |\nabla L_\alpha u|_{\Omega_1}.
$$

**Proof.** We adapt the method of the proof in [7, Section 5.2.2, Theorem 2], but different from its proof, we need to use the mean value theorem of multivariate functions and the intrinsic nature of the $L_\alpha$-variation. Via the lower semicontinuity of $L_\alpha$-BV functions, we only need to show that for $\varepsilon > 0$, there exists a function $u_\varepsilon \in C_\infty(\Omega_1)$ such that 

$$
\int_{\Omega_1} |u_\varepsilon(x) - u(x)| d\mu_{\alpha}(x) < \varepsilon
$$

and 

$$
|\nabla L_\alpha u_\varepsilon|_{\Omega_1} \leq |\nabla L_\alpha u|_{\Omega_1} + \varepsilon.
$$

Fix $\varepsilon > 0$. Given a positive integer $m$, define a sequence of open sets, 

$$
\Omega_{1,j} := \left\{x \in \Omega_1 : \text{dist}(x, \partial \Omega_1) > \frac{1}{m+j}\right\} \cap B(0, m+j), \quad j \in \mathbb{N},
$$

where $\text{dist}(x, \partial \Omega_1) = \inf\{|x-y| : y \in \partial \Omega_1\}$. Note that $\Omega_{1,j} \subset \Omega_{1,j+1} \subset \Omega_1$, $j \in \mathbb{N}$ and $\bigcup_{j=0}^{\infty} \Omega_{1,j} = \Omega_1$. Since $|\nabla L_\alpha u(\cdot)|$ is a measure, then choose a $m \in \mathbb{N}$ so large such that 

$$
|\nabla L_\alpha u|_{\Omega_1 \setminus \Omega_{1,0})} < \varepsilon.
$$

Set $U_0 := \Omega_{1,0}$ and $U_j := \Omega_{1,j+1} \setminus \overline{\Omega_{1,j}}$ for $j \geq 1$. By standard results from [7, Section 5.2.2, Theorem 2], we conclude that there is a partition of unity associated to the covering $\{U_j\}_{j \in \mathbb{N}}$. Namely, there exist functions $\{f_j\}_{j \in \mathbb{N}} \in$
$C_c^\infty(U_j)$ such that $0 \leq f_j \leq 1$, $j \geq 0$ and $\sum_{j=0}^{\infty} f_j = 1$ on $\Omega_1$. Thus we have the fact that
\begin{equation}
\sum_{j=0}^{\infty} \nabla_{\mathcal{L}^o} f_j = \left( \sqrt{x_1} \frac{\partial}{\partial x_1} \left( \sum_{j=0}^{\infty} f_j \right), \sqrt{x_2} \frac{\partial}{\partial x_2} \left( \sum_{j=0}^{\infty} f_j \right), \ldots, \sqrt{x_d} \frac{\partial}{\partial x_d} \left( \sum_{j=0}^{\infty} f_j \right) \right) = 0 \tag{6}
\end{equation}
on $\Omega_1$. Given $\varepsilon > 0$ and $u \in L^1(\Omega_1, \mathbb{R})$, extended to zero out of $\Omega_1$, we define the following regularization
\[ u_\varepsilon(x) := \frac{1}{\varepsilon^d} \int_{B(x, \varepsilon)} \eta \left( \frac{x-y}{\varepsilon} \right) u(y) d\mu_\alpha(y), \]
where $\eta \in C_c^\infty(\mathbb{R}^d)$ is a nonnegative radial function satisfying $\frac{1}{\varepsilon^d} \int_{\mathbb{R}^d} \eta \left( \frac{x-y}{\varepsilon} \right) d\mu_\alpha(x) = 1$, $\forall \ j \in \mathbb{N}$, and $\text{supp} \ \eta \subseteq B(0, 1) \cap \mathbb{R}^d$. Then for each $j$, there exists $0 < \varepsilon_j < \varepsilon$ so small such that
\begin{equation}
supp((f_j u)_\varepsilon) \subseteq U_j, \tag{7}
\end{equation}
\begin{align*}
\int_{\Omega_1} |(f_j u)_\varepsilon(x) - f_j u(x)| d\mu_\alpha(x) < \varepsilon 2^{-(j+1)}, \\
\int_{\Omega_1} |(u \nabla_{\mathcal{L}^o} f_j)_\varepsilon(x) - u \nabla_{\mathcal{L}^o} f_j(x)| d\mu_\alpha(x) < \varepsilon 2^{-(j+1)}.
\end{align*}
Construct
\[ v_\varepsilon(x) := \sum_{j=0}^{\infty} (uf_j)_\varepsilon(x). \]
In some neighborhood of each point $x \in \Omega_1$, there are only finitely many nonzero terms in this sum, hence $v_\varepsilon \in C^\infty(\Omega_1)$ and $u = \sum_{j=0}^{\infty} u f_j$. Therefore, by a simple computation, we obtain
\[ \|v_\varepsilon - u\|_{L^1(\Omega_1, d\mu_\alpha)} \leq \sum_{j=0}^{\infty} \int_{\Omega_1} |(f_j u)_\varepsilon(x) - f_j u(x)| d\mu_\alpha(x) < \varepsilon. \]
Consequently,
\[ v_\varepsilon \to u \ \text{in} \ L^1(\Omega_1, d\mu_\alpha) \ \text{as} \ \varepsilon \to 0. \]
Now, assume $\varphi \in C_c^1(\Omega_1, \mathbb{R}^d)$ and $|\varphi| \leq 1$. We decompose the integral as follows:
\begin{align*}
\int_{\Omega_1} v_\varepsilon(x) \text{div}_{\mathcal{L}^o} \varphi(x) d\mu_\alpha(x) \\
= \int_{\Omega_1} \left( \sum_{j=0}^{\infty} (uf_j)_\varepsilon(x) \right) \text{div}_{\mathcal{L}^o} \varphi(x) d\mu_\alpha(x)
\end{align*}
\[ \begin{aligned}
&= \sum_{j=0}^{\infty} \int_{\Omega_1} (u f_j)_{e_j}(x) \left( \delta_1 \varphi_1(x) + \delta_2 \varphi_2(x) + \cdots + \delta_d \varphi_d(x) \right) d\mu_\alpha(x) \\
&:= I + II,
\end{aligned} \]

where
\[
\begin{aligned}
I &:= -\sum_{j=0}^{\infty} \int_{\Omega_1} (u f_j)_{e_j}(x) \left( \sqrt{x_1} \frac{\partial}{\partial x_1} \varphi_1(x) + \cdots + \sqrt{x_d} \frac{\partial}{\partial x_d} \varphi_d(x) \right) d\mu_\alpha(x), \\
II &:= -\sum_{j=0}^{\infty} \int_{\Omega_1} (u f_j)_{e_j}(x) \left( \frac{\alpha_1 + \frac{1}{2} - x_1}{\sqrt{x_1}} \varphi_1(x) + \cdots + \frac{\alpha_d + \frac{1}{2} - x_d}{\sqrt{x_d}} \varphi_d(x) \right) d\mu_\alpha(x).
\end{aligned}
\]

For the sake of research, let
\[
\overline{\operatorname{div}}_{\mathcal{L}_\alpha} \varphi = \delta_1 \varphi_1 + \delta_2 \varphi_2 + \cdots + \delta_d \varphi_d.
\]

As for \( I \), we obtain
\[
\begin{aligned}
I &= -\sum_{j=0}^{\infty} \int_{\Omega_1} (u f_j)_{e_j}(x) \overline{\operatorname{div}}_{\mathcal{L}_\alpha} \varphi(x) d\mu_\alpha(x) \\
&= -\sum_{j=0}^{\infty} \int_{\Omega_1} (u f_j)(y) \overline{\operatorname{div}}_{\mathcal{L}_\alpha} (\eta_{e_j} \ast \varphi(y)) d\mu_\alpha(y) \\
&= -\sum_{j=0}^{\infty} \int_{\Omega_1} u(y) \overline{\operatorname{div}}_{\mathcal{L}_\alpha} (f_j(\eta_{e_j} \ast \varphi))(y) d\mu_\alpha(y) \\
&\quad + \sum_{j=0}^{\infty} \int_{\Omega_1} u(y) \nabla_{\mathcal{L}_\alpha} f_j \cdot (\eta_{e_j} \ast \varphi)(y) d\mu_\alpha(y) \\
&= -\sum_{j=0}^{\infty} \int_{\Omega_1} u(y) \overline{\operatorname{div}}_{\mathcal{L}_\alpha} (f_j(\eta_{e_j} \ast \varphi))(y) d\mu_\alpha(y) \\
&\quad + \sum_{j=0}^{\infty} \int_{\Omega_1} \varphi(y) \left( \eta_{e_j} \ast (u \nabla_{\mathcal{L}_\alpha} f_j)(y) - u \nabla_{\mathcal{L}_\alpha} f_j(y) \right) d\mu_\alpha(y) \\
&:= I_1 + I_2,
\end{aligned}
\]

where in the last equality we have used the fact (6). In fact, when \( \| \varphi \|_{L^\infty} \leq 1 \), it holds that \( |f_j(\eta_{e_j} \ast \varphi)(x)| \leq 1, \ j \in \mathbb{N}, \) and each point in \( \Omega \) belongs to at most three of the sets \( \{ U_j \}_{j=0}^{\infty} \). Furthermore, (7) implies that \( |I_2| < \varepsilon \).

On the other hand, we change the order of integration to get
\[
\begin{aligned}
II &= -\sum_{j=0}^{\infty} \int_{\Omega_1} (u f_j)_{e_j}(x) \left( \frac{\alpha_1 + \frac{1}{2} - x_1}{\sqrt{x_1}} \varphi_1(x) + \cdots + \frac{\alpha_d + \frac{1}{2} - x_d}{\sqrt{x_d}} \varphi_d(x) \right) d\mu_\alpha(x) \\
&= -\sum_{j=0}^{\infty} \int_{\Omega_1} \int_{\Omega_1} \frac{1}{\varepsilon_j} (x - y) u(y) f_j(y) \left( \sum_{k=1}^{d} \frac{\alpha_k + \frac{1}{2} - y_k}{\sqrt{y_k}} \varphi_k(x) \right) d\mu_\alpha(y) d\mu_\alpha(x)
\end{aligned}
\]
\[-\sum_{j=0}^{\infty} \int_{\Omega_1} \int_{\Omega_1^j} \frac{1}{\varepsilon_j} |y \ominus \eta_j| u(y) f_j(y) \times \left( \sum_{k=1}^{d} \frac{\alpha_k + \frac{1}{2} - x_k}{\sqrt{x_k}} - \frac{\alpha_k + \frac{1}{2} - y_k}{\sqrt{y_k}} \right) \varphi_k(x) \, d\mu_\alpha(y) d\mu_\alpha(x) \]

\[= -\sum_{j=0}^{\infty} \int_{\Omega_1} \int_{\Omega_1} \frac{1}{\varepsilon_j} |y \ominus \eta_j| u(y) f_j(y) \times \left( \sum_{k=1}^{d} \frac{\alpha_k + \frac{1}{2} - x_k}{\sqrt{x_k}} - \frac{\alpha_k + \frac{1}{2} - y_k}{\sqrt{y_k}} \right) \varphi_k(x) \, d\mu_\alpha(y) d\mu_\alpha(x). \]

Thus, the above estimate of the term $I_2$ shows that

\[\left| \int_{\Omega_1} v_\varepsilon(x) \text{div}_{L^\infty} \varphi(x) d\mu_\alpha(x) \right| = |I_1 + I_2 + II| \leq J_1 + J_2 + \varepsilon, \]

where

\[J_1 := \left| -\sum_{j=0}^{\infty} \int_{\Omega_1} u(y) \text{div}(f_j(\eta_j \ast \varphi))(y) d\mu_\alpha(y) \right| \]

and

\[J_2 := \left| -\sum_{j=0}^{\infty} \int_{\Omega_1} \int_{\Omega_1} \frac{1}{\varepsilon_j} |y \ominus \eta_j| u(y) f_j(y) \times \left( \sum_{k=1}^{d} \frac{\alpha_k + \frac{1}{2} - x_k}{\sqrt{x_k}} - \frac{\alpha_k + \frac{1}{2} - y_k}{\sqrt{y_k}} \right) \varphi_k(x) \, d\mu_\alpha(y) d\mu_\alpha(x) \right|. \]

Furthermore,

\[J_1 = \left| -\sum_{j=0}^{\infty} \int_{\Omega_1} u(y) \tilde{\text{div}}_{L^\infty} (f_j(\eta_j \ast \varphi))(y) d\mu_\alpha(y) \right| \]

\[\leq -\sum_{j=0}^{\infty} \int_{\Omega_1} u(y) f_j(y) \left( \sum_{k=1}^{d} \frac{\alpha_k + \frac{1}{2} - y_k}{\sqrt{y_k}} \varphi_k(\eta_j)(y) \right) d\mu_\alpha(y) \]

and

\[J_2 := \left| -\sum_{j=0}^{\infty} \int_{\Omega_1} \int_{\Omega_1} \frac{1}{\varepsilon_j} |y \ominus \eta_j| u(y) f_j(y) \times \left( \sum_{k=1}^{d} \frac{\alpha_k + \frac{1}{2} - x_k}{\sqrt{x_k}} - \frac{\alpha_k + \frac{1}{2} - y_k}{\sqrt{y_k}} \right) \varphi_k(x) \, d\mu_\alpha(y) d\mu_\alpha(x) \right|. \]
by the mean value theorem of multivariate functions, there exists \( \theta \) such that

\[
J \leq \int_{\Omega_1} u(y) f_j(y) \left( \sum_{k=1}^d \alpha_k + \frac{1}{\sqrt{y_k}} \phi_k \ast \eta \right) \, d\mu_\alpha(y)
\]

Consequently, we obtain

\[
J = \left| \int_{\Omega_1} u(y) f_j(y) \left( \sum_{k=1}^d \frac{\alpha_k + \frac{1}{\sqrt{y_k}}}{\sqrt{y_k}} \phi_k \ast \eta \right) \, d\mu_\alpha(y) \right|
\]

where we have used the fact (5) in the last inequality. Note that \( \psi(x_k) = \frac{\alpha_k + \frac{1}{\sqrt{y_k}}}{\sqrt{y_k}}, \| \varphi \|_{L^\infty} \leq 1 \) and \( \operatorname{supp} \eta \subseteq B(0, 1) \cap \mathbb{R}^d_+ \). When \( \|x_k - y_k\| < \varepsilon_j < \|y_k\|/2 \), by the mean value theorem of multivariate functions, there exists \( \theta \in (0, 1) \) such that

\[
|\psi(x_k) - \psi(y_k)| = \left| \frac{\alpha_k + \frac{1}{\sqrt{y_k}}}{\sqrt{y_k}} (y_k + \theta(x_k - y_k)) - \frac{\alpha_k + \frac{1}{\sqrt{y_k}}}{\sqrt{y_k}} (y_k + \theta(y_k - x_k)) \right| \leq \left( \frac{\alpha_k + \frac{1}{\sqrt{y_k}}}{\sqrt{y_k}} \right) |\theta(x_k - y_k)|
\]

Consequently, we obtain

\[
J_2 = \left| \int_{\Omega_1} u(y) f_j(y) \left( \sum_{k=1}^d \frac{\alpha_k + \frac{1}{\sqrt{y_k}}}{\sqrt{y_k}} \phi_k(x) \right) \right|
\]

\[
\leq \frac{\varepsilon_j}{2} \sum_{j=0}^\infty \int_{\Omega_1} \int_{\Omega_1} \left| \frac{1}{\varepsilon_j} \eta \left( \frac{x - y}{\varepsilon_j} \right) u(y) f_j(y) \right| \left| \sum_{k=1}^d \frac{\alpha_k + \frac{1}{\sqrt{y_k}}}{\sqrt{y_k}} \phi_k(x) \right| \, d\mu_\alpha(y) \, d\mu_\alpha(x)
\]

\[
\leq C \varepsilon_j \sum_{j=0}^\infty \int_{\Omega_1} \int_{\Omega_1} \left| \frac{1}{\varepsilon_j} \eta \left( \frac{x - y}{\varepsilon_j} \right) u(y) f_j(y) \right| \left| \sum_{k=1}^d \frac{\alpha_k + \frac{1}{\sqrt{y_k}}}{\sqrt{y_k}} \phi_k(x) \right| \, d\mu_\alpha(y) \, d\mu_\alpha(x)
\]
\[
+ C \varepsilon_j \sum_{j=0}^{\infty} \int_{\Omega_1} \int_{\Omega_1} \left| \frac{1}{\varepsilon_j} \eta \left( \frac{x-y}{\varepsilon_j} \right) u(y) f_j(y) \right| \frac{d}{L} \sum_{k=1}^{d} \left| y_k \right|^{-\frac{2}{}\varepsilon_j} d\mu_{\alpha}(y) d\mu_{\alpha}(x) \\
\leq C \varepsilon_j \sum_{j=0}^{\infty} \int_{\Omega_1} \int_{\Omega_1} \left| \frac{1}{\varepsilon_j} \eta \left( \frac{x-y}{\varepsilon_j} \right) \right| d\mu_{\alpha}(x) \sum_{k=1}^{d} \alpha_k + \frac{1}{2} \left| u(y) \right| \left| f_j(y) \right| \left| y_k \right|^{-\frac{2}{}\varepsilon_j} d\mu_{\alpha}(y) \\
+ C \varepsilon_j \sum_{j=0}^{\infty} \int_{\Omega_1} \int_{\Omega_1} \left| \frac{1}{\varepsilon_j} \eta \left( \frac{x-y}{\varepsilon_j} \right) \right| d\mu_{\alpha}(x) \sum_{k=1}^{d} \left| u(y) \right| \left| f_j(y) \right| \left| y_k \right|^{-\frac{2}{}\varepsilon_j} d\mu_{\alpha}(y) \\
\leq C \varepsilon_j \int_{\Omega_1} \left| u(y) \right| \sum_{k=1}^{d} \alpha_k + \frac{1}{2} \left| y_k \right|^{-\frac{2}{}\varepsilon_j} d\mu_{\alpha}(y) + C \varepsilon_j \int_{\Omega_1} \left| u(y) \right| \sum_{k=1}^{d} \left| y_k \right|^{-\frac{2}{}\varepsilon_j} d\mu_{\alpha}(y) \\
\lesssim \varepsilon,
\]

where we have used the facts that

\[
\begin{cases}
\int_{\Omega_1} \left| u(y) \right| \sum_{k=1}^{d} \left| y_k \right|^{-\frac{2}{}\varepsilon_j} d\mu_{\alpha}(y) < \infty,

\int_{\Omega_1} \left| u(y) \right| \sum_{k=1}^{d} \alpha_k + \frac{1}{2} \left| y_k \right|^{-\frac{2}{}\varepsilon_j} d\mu_{\alpha}(y) < \infty,
\end{cases}
\]

and in the third inequality we have used the fact that

\[|y_k + \theta (x_k - y_k)| \geq \left| y_k \right| - \theta (x_k - y_k) = \left( 1 - \frac{\theta}{2} \right) \left| y_k \right| \]

By taking the supremum over \( \varphi \) and the arbitrariness of \( \varepsilon > 0 \), the theorem can be proved. \( \square \)

**Remark 2.5.** By computation, we conclude that the function \( u \in BV_{L^\infty}(\Omega) \) satisfies (9) in Theorem 2.4 when \( d \geq 3 \), at this time, Theorem 2.4 is valid for any open set \( \Omega \subseteq \mathbb{R}^d \).

Moreover, by Lemma 2.1 and Theorem 2.4, we have the following max-min property of the \( L^\alpha \)-variation.

**Theorem 2.6.** Let \( \Omega_1 \) be an open set defined in (3). Suppose \( u, v \in L^1(\Omega_1, d\mu_{\alpha}) \), then

\[|\nabla_{L^\infty} \max \{ u, v \}|(\Omega_1) + |\nabla_{L^\infty} \min \{ u, v \}|(\Omega_1) \leq |\nabla_{L^\infty} u|(\Omega_1) + |\nabla_{L^\infty} v|(\Omega_1).\]
Proof. Without loss of generality, we may assume
\[ |\nabla_{L^\alpha} u| (\Omega_1) + |\nabla_{L^\alpha} v| (\Omega_1) < \infty. \]

By Theorem 2.4, we take two functions
\[ u_h, v_h \in BV_{L^\alpha}(\Omega_1) \cap C^\infty_c(\Omega_1), \quad h = 1, 2, \ldots, \]
such that
\[ \begin{align*}
  u_h & \to u, v_h \to v \quad \text{in} \quad L^1(\Omega_1, d\mu_\alpha), \\
  \int_{\Omega_1} |\nabla_{L^\alpha} u_h(x)| d\mu_\alpha (x) & \to |\nabla_{L^\alpha} u| (\Omega_1), \\
  \int_{\Omega_1} |\nabla_{L^\alpha} v_h(x)| d\mu_\alpha (x) & \to |\nabla_{L^\alpha} v| (\Omega_1).
\end{align*} \]

Since
\[ \max\{u_h, v_h\} \to \max\{u, v\} \quad \text{&} \quad \min\{u_h, v_h\} \to \min\{u, v\} \quad \text{in} \quad L^1(\Omega_1, d\mu_\alpha). \]
Via Lemma 2.1, it follows that
\[ |\nabla_{L^\alpha} \max\{u, v\}| (\Omega_1) + |\nabla_{L^\alpha} \min\{u, v\}| (\Omega_1) \]
\[ \leq \liminf_{h \to \infty} \int_{\Omega_1} |\nabla_{L^\alpha} \max\{u_h, v_h\}| d\mu_\alpha (x) + \liminf_{h \to \infty} \int_{\Omega_1} |\nabla_{L^\alpha} \min\{u_h, v_h\}| d\mu_\alpha (x) \]
\[ \leq \liminf_{h \to \infty} \left( \int_{\Omega_1} |\nabla_{L^\alpha} \max\{u_h, v_h\}| d\mu_\alpha (x) + \int_{\Omega_1} |\nabla_{L^\alpha} \min\{u_h, v_h\}| d\mu_\alpha (x) \right) \]
\[ \leq \liminf_{h \to \infty} \left( \int_{\{x \in \Omega_1: u_h \leq v_h\}} |\nabla_{L^\alpha} u_h| d\mu_\alpha (x) + \int_{\{x \in \Omega_1: u_h > v_h\}} |\nabla_{L^\alpha} v_h| d\mu_\alpha (x) + \int_{\{x \in \Omega_1: u_h \leq v_h\}} |\nabla_{L^\alpha} u_h| d\mu_\alpha (x) + \int_{\{x \in \Omega_1: u_h > v_h\}} |\nabla_{L^\alpha} v_h| d\mu_\alpha (x) \right) \]
\[ = \liminf_{h \to \infty} \int_{\Omega_1} |\nabla_{L^\alpha} u_h(x)| d\mu_\alpha (x) + \liminf_{h \to \infty} \int_{\Omega_1} |\nabla_{L^\alpha} v_h(x)| d\mu_\alpha (x) \]
\[ \leq \lim_{h \to \infty} \int_{\Omega_1} |\nabla_{L^\alpha} u_h(x)| d\mu_\alpha (x) + \lim_{h \to \infty} \int_{\Omega_1} |\nabla_{L^\alpha} v_h(x)| d\mu_\alpha (x) \]
\[ = |\nabla_{L^\alpha} u| (\Omega_1) + |\nabla_{L^\alpha} v| (\Omega_1). \]

\[ \square \]

2.2. Basic properties of Laguerre perimeter

In this subsection, we introduce a kind of new perimeter: the Laguerre perimeter ($L^\alpha$-perimeter in short). Moreover, we establish the related results for it.

The $L^\alpha$-perimeter of $E \subset \Omega$ can be defined as follows:
\[ P_{L^\alpha}(E, \Omega) = |\nabla_{L^\alpha} 1_E| (\Omega) = \sup_{\phi \in \mathcal{F}(\Omega)} \left\{ \int_E \div_{L^\alpha} \phi(x) d\mu_\alpha (x) \right\}, \]
where $\mathcal{F}(\Omega)$ is defined in Section 2.1. In particular, we shall also write
\[ P_{L^\alpha}(E, \mathbb{R}^d_+) = P_{L^\alpha}(E) \]
The following conclusion is a direct corollary of Lemma 2.1 to replace \( f \) with \( 1_E \).

**Corollary 2.7.** *(Lower semicontinuity of \( P_{E^*} \)).* Suppose \( 1_{E_k} \to 1_E \) in \( L_{\text{loc}}^1(\Omega, d\mu_\alpha) \), where \( E \) and \( E_k \), \( k \in \mathbb{N} \), are subsets of \( \Omega \), then

\[
P_{E^*}(E, \Omega) \leq \liminf_{k \to \infty} P_{E_k^*}(E_k, \Omega).
\]

Moreover, by Theorem 2.6, via choosing \( u = 1_E \) and \( v = 1_F \) for any compact subsets \( E, F \) in \( \Omega_1 \), we immediately obtain the following corollary. According to Xiao and Zhang’s result in [16, Section 1.1 (iii)], we also give the equality condition of (11).

**Corollary 2.8.** For any compact subsets \( E, F \) in \( \Omega_1 \), we have

\[
P_{E^*}(E \cap F, \Omega_1) + P_{E^*}(E \cup F, \Omega_1) \leq P_{E^*}(E, \Omega_1) + P_{F^*}(F, \Omega_1),
\]

where \( \Omega_1 \) is an open set defined in (3). Especially, if \( P_{E^*}(E \setminus (E \cap F), \Omega_1) \) \( P_{\Omega^*}(F \setminus (E \cap F), \Omega_1) = 0 \), \( P_{E^*}(E \setminus (E \cap F), \Omega_1) = 0 \) or \( P_{\Omega^*}(F \setminus (E \cap F), \Omega_1) = 0 \). Suppose \( P_{E^*}(E \setminus (E \cap F), \Omega_1) = 0 \). Via (11), we have

\[
P_{E^*}(E, \Omega_1) = P_{E^*}((E \setminus (E \cap F)) \cup (E \cap F), \Omega_1)
\]

\[
\leq P_{E^*}(E \setminus (E \cap F), \Omega_1) + P_{\Omega^*}(E \cap F, \Omega_1)
\]

\[
= P_{E^*}(E \cap F, \Omega_1).
\]

Using (10) and \( E \cup F = F \cup (E \setminus (E \cap F)) \), we obtain

\[
P_{\Omega^*}(F, \Omega_1) = \sup_{\varphi \in \mathcal{F}(\Omega_1)} \left\{ \int_F \div \cdot \varphi(x) d\mu_\alpha(x) \right\}
\]

\[
= \sup_{\varphi \in \mathcal{C}(\Omega_1)} \left\{ \int_{E \cup F} \div \cdot \varphi(x) d\mu_\alpha(x) - \int_{E \setminus (E \cap F)} \div \cdot \varphi(x) d\mu_\alpha(x) \right\}
\]

\[
\leq \sup_{\varphi \in \mathcal{C}(\Omega_1)} \left\{ \int_{E \cup F} \div \cdot \varphi(x) d\mu_\alpha(x) \right\}
\]

\[
+ \sup_{\varphi \in \mathcal{C}(\Omega_1)} \left\{ \int_{E \setminus (E \cap F)} \div \cdot \varphi(x) d\mu_\alpha(x) \right\}
\]

\[
= P_{\Omega^*}(E \cup F, \Omega_1) + P_{\Omega^*}(E \setminus (E \cap F), \Omega_1)
\]

\[
= P_{\Omega^*}(E \cap F, \Omega_1).
\]

Combining (12) with (13) deduces that

\[
P_{E^*}(E, \Omega_1) + P_{\Omega^*}(F, \Omega_1) \leq P_{E^*}(E \cup F, \Omega_1) + P_{\Omega^*}(E \cap F, \Omega_1),
\]
which derives the desired result. Another case can be similarly proved, we omit
the details. □

Next we show that the Gauss-Green formula is valid on sets of finite $L^\alpha$-perimeter.

**Theorem 2.9. (Gauss-Green formula).** Let $E \subseteq \Omega$ be subset with finite $L^\alpha$-
perimeter. Then we have

\[
\int_E \tilde{\text{div}}_{L^\alpha} \varphi (x) \, d\mu_\alpha(x) \\
= - \int_{\partial E^c} (\sqrt{x_1} \varphi_1(x), \ldots, \sqrt{x_d} \varphi_d(x)) \cdot \tilde{n}\omega(x) d\mathcal{H}^{d-1}(x) \\
- \int_{E} \sum_{i=1}^d \alpha_i + \frac{d-1}{d-1} \frac{x_i}{\sqrt{x_i}} \varphi_i(x) \omega(x) dx,
\]

where the unit vector $\tilde{n}(x)$ is the outward normal to $E$ and $\tilde{\text{div}}_{L^\alpha}(\cdot)$ is defined
in (8).

**Proof.** By calculating, we have

\[
\int_E \tilde{\text{div}}_{L^\alpha} \varphi (x) \, d\mu_\alpha(x) \\
= \int_E \left( \sum_{i=1}^d \frac{1}{\sqrt{x_i}} \frac{\partial}{\partial x_i} \varphi_i(x) \right) \omega(x) dx \\
= \int_E \text{div}(\sqrt{x_1} \varphi_1(x) \omega(x), \ldots, \sqrt{x_d} \varphi_d(x) \omega(x)) dx - \int_{E} \sum_{i=1}^d \sqrt{x_i} \varphi_i(x) \frac{\partial}{\partial x_i} \omega(x) dx \\
- \frac{1}{2} \int_{E} \sum_{i=1}^d \frac{1}{\sqrt{x_i}} \varphi_i(x) \omega(x) dx \\
= - \int_{\partial E^c} (\sqrt{x_1} \varphi_1(x), \ldots, \sqrt{x_d} \varphi_d(x)) \cdot \tilde{n}\omega(x) d\mathcal{H}^{d-1}(x) \\
- \int_{E} \sum_{i=1}^d \sqrt{x_i} \varphi_i(x) \frac{\partial}{\partial x_i} \omega(x) dx + \frac{1}{2} \int_{E} \sum_{i=1}^d \frac{1}{\sqrt{x_i}} \varphi_i(x) \omega(x) dx \\
= - \int_{\partial E^c} (\sqrt{x_1} \varphi_1(x), \ldots, \sqrt{x_d} \varphi_d(x)) \cdot \tilde{n}\omega(x) d\mathcal{H}^{d-1}(x) \\
- \int_{E} \sum_{i=1}^d \alpha_i + \frac{d-1}{d-1} \frac{x_i}{\sqrt{x_i}} \varphi_i(x) \omega(x) dx,
\]
where we have used the classical Gauss-Green formula and the following facts for the derivatives of $\omega(x)$:

$$
\frac{\partial}{\partial x_i} \left( \prod_{j=1}^d x_j^{\alpha_j} e^{-x_j} \right) = \prod_{j=1, j \neq i}^d \frac{x_j^{\alpha_j} e^{-x_j}}{\Gamma(\alpha_j + 1)} \frac{1}{\Gamma(\alpha_i + 1)} (-e^{-x_i} x_i^{\alpha_i} + \alpha_i e^{-x_i} x_i^{\alpha_i - 1})
$$

$$
= \left( -1 + \frac{\alpha_i}{x_i} \right) \omega(x)
$$

for $1 \leq i \leq d$. This completes the proof. □

**Lemma 2.10.** For any set $E$ in $\Omega$ with finite $\mathcal{L}^\alpha$-perimeter, then

$$
P_{\mathcal{L}^\alpha}(E, \Omega) = P_{\mathcal{L}^\alpha}(\Omega \setminus E, \Omega).
$$

**Proof.** For any $\varphi \in \mathcal{F}(\mathbb{R}^d)$, since $P_{\mathcal{L}^\alpha}(E, \Omega) < \infty$, then

$$
\sup_{\varphi \in \mathcal{F}(\mathbb{R}^d)} \int_E \text{div}_{\mathcal{L}^\alpha} \varphi(x) d\mu_{\alpha}(x) < \infty.
$$

Via the extended Gauss-Green formula (Theorem 2.9) and noting the compact support of $\varphi$, we have

$$
\int_E \text{div}_{\mathcal{L}^\alpha} \varphi(x) d\mu_{\alpha}(x)
$$

$$
= - \int_E \tilde{\text{div}}_{\mathcal{L}^\alpha} (\varphi_1(x), \ldots, \varphi_d(x)) d\mu_{\alpha}(x) - \int_E \sum_{k=1}^d \frac{\alpha_i + \frac{1}{2} - x_i}{\sqrt{x_i}} \varphi_i(x) d\mu_{\alpha}(x)
$$

$$
= \int_{\partial E^c} (\sqrt{x_1} \varphi_1(x), \ldots, \sqrt{x_d} \varphi_d(x)) \cdot \tilde{n}(x) d\mathcal{H}^{d-1}(x)
$$

$$
+ \int_E \sum_{k=1}^d \frac{\alpha_i + \frac{1}{2} - x_i}{\sqrt{x_i}} \varphi_i(x) d\mu_{\alpha}(x) - \int_E \sum_{k=1}^d \frac{\alpha_i + \frac{1}{2} - x_i}{\sqrt{x_i}} \varphi_i(x) d\mu_{\alpha}(x)
$$

$$
= \int_{\partial E^c} (\sqrt{x_1} \varphi_1(x), \ldots, \sqrt{x_d} \varphi_d(x)) \cdot \tilde{n}(x) d\mathcal{H}^{d-1}(x)
$$

$$
= \int_{E^c} \tilde{\text{div}}_{\mathcal{L}^\alpha} \varphi(x) d\mu_{\alpha}(x) - \int_{E^c} \sum_{k=1}^d \frac{\alpha_i + \frac{1}{2} - x_i}{\sqrt{x_i}} \varphi_i(x) d\mu_{\alpha}(x)
$$

$$
= \int_{E^c} \text{div}_{\mathcal{L}^\alpha} \varphi(x) d\mu_{\alpha}(x),
$$

where $\tilde{n}(x)$ is the unit exterior normal to $E$ at $x$. Due to the arbitrariness of $\varphi$, taking the supremum reaches

$$
P_{\mathcal{L}^\alpha}(E, \Omega) = P_{\mathcal{L}^\alpha}(\Omega \setminus E, \Omega).
$$

□
2.3. Coarea formula of $\mathcal{L}^\alpha$-BV functions and the Sobolev inequality

Below we prove the coarea formula and the Sobolev inequality for $\mathcal{L}^\alpha$-perimeter.

**Theorem 2.11.** Let $\Omega_1$ be an open set defined in (3). If $f \in \mathcal{BV}_{\mathcal{L}^\alpha}(\Omega_1)$, then

\begin{equation}
|\nabla_{\mathcal{L}^\alpha} f|(\Omega_1) \approx \int_{-\infty}^{+\infty} P_{\mathcal{L}^\alpha}(E_t, \Omega_1) dt,
\end{equation}

where $E_t = \{x \in \Omega_1 : f(x) > t\}$ for $t \in \mathbb{R}$.

**Proof.** Firstly, assume

$\varphi = (\varphi_1, \varphi_2, \ldots, \varphi_d) \in C^1_c(\Omega_1, \mathbb{R}^d)$.

We can easily prove that for $i = 1, 2, \ldots, d$,

\[
\int_{\Omega_1} f(x) \frac{\partial}{\partial x_i} \varphi_i(x) d\mu_\alpha(x) = \int_{-\infty}^{+\infty} \left( \int_{E_t} \frac{\partial}{\partial x_i} \varphi_i(x) d\mu_\alpha(x) \right) dt,
\]

and

\[
\int_{\Omega_1} f(x) \frac{\alpha_i + \frac{d}{\alpha} - x_i}{\sqrt{x_i}} \varphi_i(x) d\mu_\alpha(x) = \int_{-\infty}^{+\infty} \left( \int_{E_t} \frac{\alpha_i + \frac{d}{\alpha} - x_i}{\sqrt{x_i}} \varphi_i(x) d\mu_\alpha(x) \right) dt,
\]

where the latter can be seen in the proof of [7, Section 5.5, Theorem 1]. It follows that

\[
\int_{\Omega_1} f(x) \text{div}_{\mathcal{L}^\alpha} \varphi(x) d\mu_\alpha(x) = \int_{-\infty}^{+\infty} \left( \int_{E_t} \text{div}_{\mathcal{L}^\alpha} \varphi(x) d\mu_\alpha(x) \right) dt.
\]

Therefore, we conclude that for all $\varphi \in \mathcal{F}(\Omega_1)$,

\[
\int_{\Omega_1} f(x) \text{div}_{\mathcal{L}^\alpha} \varphi(x) d\mu_\alpha(x) \leq \int_{-\infty}^{+\infty} P_{\mathcal{L}^\alpha}(E_t, \Omega_1) dt.
\]

Furthermore,

\[
|\nabla_{\mathcal{L}^\alpha} f|(\Omega_1) \leq \int_{-\infty}^{+\infty} P_{\mathcal{L}^\alpha}(E_t, \Omega_1) dt.
\]

Secondly, without loss of generality, we only need to verify that

\[
|\nabla_{\mathcal{L}^\alpha} f|(\Omega_1) \geq \int_{-\infty}^{+\infty} P_{\mathcal{L}^\alpha}(E_t, \Omega_1) dt
\]

holds for $f \in \mathcal{BV}_{\mathcal{L}^\alpha}(\Omega_1) \cap C^\infty(\Omega_1)$. This proof can refer to the idea of [14, Proposition 4.2]. Let

\[
m(t) = \int_{\{x \in \Omega_1 : f(x) \leq t\}} \left| \sum_{i=1}^{d} \sqrt{x_i} \frac{\partial}{\partial x_i} f(x) \right| d\mu_\alpha(x).
\]

It is obvious that

\[
\int_{-\infty}^{+\infty} m'(t) dt = \int_{\Omega_1} \left| \sum_{i=1}^{d} \sqrt{x_i} \frac{\partial}{\partial x_i} f(x) \right| d\mu_\alpha(x).
\]
Define the following function \( g_h \) as

\[
  g_h(s) := \begin{cases} 
    0, & \text{if } s \leq t, \\
    h(s - t), & \text{if } t \leq s \leq t + 1/h, \\
    1, & \text{if } s \geq t + 1/h,
  \end{cases}
\]

where \( t \in \mathbb{R} \). Set the sequence \( v_h(x) := g_h(f(x)) \). At this time, \( v_h \to 1_{E_t} \) in \( L^1(\Omega_1, d\mu_\alpha) \). In fact,

\[
  \int_{\Omega_1} |v_h(x) - 1_{E_t}| d\mu_\alpha(x) = \int_{\{x \in \Omega_1 : t < f(x) \leq t + 1/h\}} |g_h(f(x)) - 1| d\mu_\alpha(x) \\
  \leq \int_{\{x \in \Omega_1 : t < f(x) \leq t + 1/h\}} d\mu_\alpha(x) \to 0.
\]

Since \( \{x \in \Omega_1 : t < f(x) \leq t + 1/h\} \to \emptyset \) as \( h \to \infty \), we obtain

\[
  \int_{\Omega_1} |\nabla_{\mathcal{L}^\alpha} v_h(x)| d\mu_\alpha(x) \\
  = \int_{\{x \in \Omega_1 : t < f(x) \leq t + 1/h\}} |\nabla_{\mathcal{L}^\alpha} (h(f(x) - t))| d\mu_\alpha(x) + \int_{\{x \in \Omega_1 : f(x) \geq t + 1/h\}} |\nabla_{\mathcal{L}^\alpha} 1| d\mu_\alpha(x) \\
  = h \int_{\{x \in \Omega_1 : t < f(x) \leq t + 1/h\}} \sum_{i=1}^d \sqrt{x_i} \frac{\partial}{\partial x_i} f(x) | d\mu_\alpha(x).
\]

Taking the limit with \( h \to \infty \) and using Theorem 2.4, we obtain

\[
  |\nabla_{\mathcal{L}^\alpha} 1_{E_t}|(\Omega_1) \leq \limsup_{h \to \infty} \int_{\Omega_1} |\nabla_{\mathcal{L}^\alpha} v_h(x)| d\mu_\alpha(x) \\
  = h \limsup_{h \to \infty} \int_{\{x \in \Omega_1 : t < f(x) \leq t + 1/h\}} \left| \sum_{i=1}^d \sqrt{x_i} \frac{\partial}{\partial x_i} f(x) \right| d\mu_\alpha(x) \\
  = m'(t).
\]

Integrating (16) reaches

\[
  \int_{-\infty}^{+\infty} P_{\mathcal{L}^\alpha}(E_t, \Omega_1) dt \leq \int_{-\infty}^{+\infty} m'(t) dt \\
  = \int_{\Omega_1} \left| \sum_{i=1}^d \sqrt{x_i} \frac{\partial}{\partial x_i} f(x) \right| d\mu_\alpha(x) \\
  \leq \int_{\Omega_1} |\nabla_{\mathcal{L}^\alpha} f(x)| d\mu_\alpha(x).
\]

Finally, by approximation and using the lower semicontinuity of the \( \mathcal{L}^\alpha \)-perimeter, we conclude that (15) holds true for all \( f \in \mathcal{BV}_{\mathcal{L}^\alpha}(\Omega_1) \). \qed
Finally, we can develop the Sobolev inequality and the isoperimetric inequality for $L^\alpha$-BV functions. Since the domain $\Omega_1$ is a reasonable substitute of $\Omega$, we can obtain the isoperimetric inequality and the Sobolev inequality for $f \in BV_{L^\alpha}(\Omega_1)$, where $\Omega_1$ is given in (3).

**Theorem 2.12.**

(i) *(Sobolev inequality).* Let $\Omega_1$ be an open set defined in (3). Then for all $f \in BV_{L^\alpha}(\Omega_1)$, we have

$$\|f\|_{L^{d+1}(\Omega_1,d\mu_\alpha)} \lesssim |\nabla_{L^\alpha} f|(\Omega_1).$$

(ii) *(Isoperimetric inequality).* Let $E$ be a bounded set of finite $L^\alpha$-perimeter in $\Omega_1$. Then

$$\mu_\alpha(E)^{\frac{d+1}{d-1}} \lesssim P_{L^\alpha}(E,\Omega_1).$$

(iii) The above two statements are equivalent.

**Proof.**

(i) Choose $f_k \in C^\infty_c(\Omega_1) \cap BV_{L^\alpha}(\Omega_1)$, $k = 1, 2, \ldots$, such that

$$\int_{\Omega_1} f_k \to f \text{ in } L^1(\Omega_1,d\mu_\alpha),$$

$$\int_{\Omega_1} |\nabla_{L^\alpha} f_k(x)|d\mu_\alpha(x) \to \| \nabla_{L^\alpha} f \|(\Omega_1).$$

Since $\Omega_1 = \Omega \setminus \{ x \in \mathbb{R}^d : \exists i \in 1, \ldots, d \text{ such that } \sqrt{x_i} < 1 \}$, then for any $i = 1, \ldots, d$, we obtain $\sqrt{x_i} \geq 1$. It is easy to see that

$$|\nabla f(x)| \leq |\nabla_{L^\alpha} f(x)| = \left( \sum_{i=1}^d (\sqrt{x_i} \frac{\partial}{\partial x_i} f(x))^2 \right)^{\frac{1}{2}}.$$

Then by Fatou’s lemma and the weighted Gagliardo-Nirenberg-Sobolev inequality, we have

$$\|f\|_{L^{d+1}(\Omega_1,d\mu_\alpha)} \leq \liminf_{k \to \infty} \|f_k\|_{L^{d+1}(\Omega_1,d\mu_\alpha)} \lesssim \lim_{k \to \infty} \|f\|_{L^1(\Omega_1,d\mu_\alpha)} \lesssim \lim_{k \to \infty} \|\nabla_{L^\alpha} f\|_{L^1(\Omega_1,d\mu_\alpha)} = |\nabla_{L^\alpha} f|(\Omega_1),$$

where we have used the relation between the gradient $\nabla$ and the Laguerre gradient $\nabla_{L^\alpha}$ in (19).

(ii) We can show that (18) is valid via letting $f = 1_E$ in (17).

(iii) Apparently, (i)$\Rightarrow$(ii) has been proved. In what follows, we prove (ii)$\Rightarrow$(i). Assume that $0 \leq f \in C^\infty_c(\Omega_1)$. By the coarea formula in Theorem 2.11 and (ii), we have

$$\int_{\Omega_1} |\nabla_{L^\alpha} f(x)|d\mu_\alpha(x) = \int_0^{+\infty} \int_{\Omega_1} 1_{E_t}(\Omega_1) \, dt \lesssim \int_0^{+\infty} |E_t|^{\frac{d-1}{d+1}} \, dt,$$
where \( E_t = \{ x \in \Omega_1 : f(x) > t \} \). Let

\[
f_t = \min\{ t, f \} \quad \text{and} \quad \chi(t) = \left( \int_{\Omega_1} f_t^\frac{d}{d-p}(x) \, d\mu_\alpha(x) \right)^{\frac{d-p}{d}} , \quad \forall \ t \in \mathbb{R}.
\]

It is easy to see that

\[
\lim_{t \to \infty} \chi(t) = \left( \int_{\Omega_1} |f(x)|^\frac{d}{d-p} \, d\mu_\alpha(x) \right)^{\frac{d-p}{d}}.
\]

In addition, we can check that \( \chi(t) \) is nondecreasing on \((0, \infty)\) and for \( h > 0 \),

\[
0 \leq \chi(t + h) - \chi(t) \leq \left( \int_{\Omega_1} |f_{t+h}(x) - f_t(x)|^\frac{d}{d-p} \, d\mu_\alpha(x) \right)^{\frac{d-p}{d}} \leq h |E_t|^\frac{d-p}{d}.
\]

Then \( \chi(t) \) is locally a Lipschitz function and \( \chi'(t) \leq \frac{d-p}{d} |E_t|^\frac{d-p}{d} \) for a.e. \( t \in (0, \infty) \).

Hence,

\[
\left( \int_{\Omega_1} |f(x)|^\frac{d}{d-p} \, d\mu_\alpha(x) \right)^{\frac{d-p}{d}} = \int_0^\infty \chi'(t) \, dt \leq \int_0^\infty \frac{d-p}{d} |E_t|^\frac{d-p}{d} \, dt \lesssim \int_{\Omega_1} |\nabla_{L^p} f(x)| \, d\mu_\alpha(x).
\]

For all \( f \in BV_{L^p}(\Omega_1) \), we conclude that (17) is valid by Theorem 2.4. \( \square \)

As a direct result of the proof of (i) in Theorem 2.12, we can get the following corollary.

**Corollary 2.13.** Let \( 1 < p < d \) and let \( \Omega_1 \) be an open set defined in (3). For any \( f \in W^{1,1}_{L^p}(\Omega_1) \) one has

\[
\| f \|_{L^p(\Omega_1, d\mu_\alpha)} \lesssim \| \nabla_{L^p} f \|_{L^p(\Omega_1, d\mu_\alpha)}.
\]

**Proof.** For some \( \gamma > 1 \) to be fixed later, via the Lemma 2.1 (i) and Hölder inequality we obtain

\[
\left( \int_{\Omega_1} |f(x)|^\frac{d}{d-p} \, d\mu_\alpha(x) \right)^{\frac{d-p}{d}} \lesssim \int_{\Omega_1} |f(x)|^{-\gamma} |\nabla_{L^p} f(x)| \, d\mu_\alpha(x)
\]

\[
\lesssim \left( \int_{\Omega_1} |f(x)|^{\frac{d-p}{d-1}} \, d\mu_\alpha(x) \right)^{1 - \frac{1}{\gamma}} \left( \int_{\Omega_1} |\nabla_{L^p} f(x)|^p \, d\mu_\alpha(x) \right)^{\frac{1}{p}}.
\]

Choosing

\[
\gamma = \frac{p(d-1)}{d-p}
\]
and noting
\[ \gamma - 1 = \frac{d(p - 1)}{d - p}, \]
then we conclude that (20) holds true.

\[ \square \]

3. Laguerre mean curvature

In this section we focus on the question whether every set of finite \( \mathcal{L}^\alpha \)-perimeter in \( \Omega_1 \subseteq \mathbb{R}^d_+ \) has mean curvature in \( L^1(\Omega_1, d\mu_\alpha) \). For the classical case, please refer to [3] for details. During the proof of Theorem 3.1, we need to use the important result for the Laguerre perimeter in Corollary 2.8. Therefore, we assume that the dimension \( d \geq 3 \) via Remark 2.5.

For a given \( u \in L^1(\Omega_1, d\mu_\alpha) \), the Massari type functional corresponding to the \( \mathcal{L}^\alpha \)-perimeter is defined as
\[ \mathcal{F}_{u, \mathcal{L}^\alpha}(E) := P_{\mathcal{L}^\alpha}(E, \Omega_1) + \int_E u(x)d\mu_\alpha(x), \]
where \( E \) is an arbitrary set of finite \( \mathcal{L}^\alpha \)-perimeter in \( \mathbb{R}^d_+ \).

**Theorem 3.1.** For every set \( E \) of finite \( \mathcal{L}^\alpha \)-perimeter in \( \mathbb{R}^d_+ \), there exists a function \( u \in L^1(\Omega_1, d\mu_\alpha) \) such that
\[ \mathcal{F}_{u, \mathcal{L}^\alpha}(E) \leq \mathcal{F}_{u, \mathcal{L}^\alpha}(F) \]
holds for every set \( F \) of finite \( \mathcal{L}^\alpha \)-perimeter in \( \Omega_1 \).

**Proof.** At first, for a given set \( E \), we need to find a function \( u \in L^1(\Omega_1, d\mu_\alpha) \) such that
\[ (21) \quad \mathcal{F}_{u, \mathcal{L}^\alpha}(E) \leq \mathcal{F}_{u, \mathcal{L}^\alpha}(F) \]
holds for every \( F \) with either \( F \subseteq E \) or \( E \subseteq F \), then Theorem 3.1 is proved, i.e. (21) holds for every \( F \subseteq \Omega_1 \). In fact, by adding the inequality (21) corresponding to the test sets \( E \cap F \) and \( E \cup F \),
\[
\begin{align*}
P_{\mathcal{L}^\alpha}(E \cap F, \Omega_1) + \int_{E \cap F} u(x)d\mu_\alpha(x) & \leq P_{\mathcal{L}^\alpha}(E \cap F, \Omega_1) + \int_{E \cap F} u(x)d\mu_\alpha(x), \\
P_{\mathcal{L}^\alpha}(E, \Omega_1) + \int_E u(x)d\mu_\alpha(x) & \leq P_{\mathcal{L}^\alpha}(E \cup F, \Omega_1) + \int_{E \cup F} u(x)d\mu_\alpha(x).
\end{align*}
\]
Then noting that
\[ (22) \quad P_{\mathcal{L}^\alpha}(E \cap F, \Omega_1) + P_{\mathcal{L}^\alpha}(E \cup F, \Omega_1) \leq P_{\mathcal{L}^\alpha}(E, \Omega_1) + P_{\mathcal{L}^\alpha}(F, \Omega_1), \]
we can get
\[
\begin{align*}
2P_{\mathcal{L}^\alpha}(E, \Omega_1) + 2 \int_E u(x)d\mu_\alpha(x) & \\
\leq P_{\mathcal{L}^\alpha}(E \cap F, \Omega_1) + P_{\mathcal{L}^\alpha}(E \cup F, \Omega_1) + \int_{E \cap F} u(x)d\mu_\alpha(x) + \int_{E \cup F} u(x)d\mu_\alpha(x)
\end{align*}
\]
\[
\leq P_{\mathcal{L}^\alpha}(E, \Omega_1) + P_{\mathcal{L}^\alpha}(F, \Omega_1) + \int_E u(x)d\mu_\alpha(x) + \int_F u(x)d\mu_\alpha(x),
\]
that is, (21) holds for arbitrary $F$. Also, if (21) holds for $F \subset E$, then for the set $F$ such that $E \subset F$, i.e. $\Omega_1 \setminus F \subset \Omega_1 \setminus E$,

$$P_{L^\infty}(E, \Omega_1) + \int_E u(x)d\mu_\alpha(x)$$

$$= P_{L^\infty}(\Omega_1 \setminus E, \Omega_1) + \int_{\Omega_1 \setminus E} u(x)d\mu_\alpha(x) - \int_{\Omega_1 \setminus E} u(x)d\mu_\alpha(x) + \int_E u(x)d\mu_\alpha(x)$$

$$\leq P_{L^\infty}(\Omega_1 \setminus F, \Omega_1) + \int_{\Omega_1 \setminus F} u(x)d\mu_\alpha(x) - \int_{\Omega_1 \setminus F} u(x)d\mu_\alpha(x) + \int_E u(x)d\mu_\alpha(x)$$

$$= P_{L^\infty}(F, \Omega_1) + \int_{\Omega_1 \setminus F} u(x)d\mu_\alpha(x) - \int_{\Omega_1 \setminus F} u(x)d\mu_\alpha(x) + \int_E u(x)d\mu_\alpha(x)$$

$$= \mathcal{F}_{u, L^\infty}(F) - \int_F u(x)d\mu_\alpha(x) + \int_{\Omega_1 \setminus F} u(x)d\mu_\alpha(x) - \int_{\Omega_1 \setminus F} u(x)d\mu_\alpha(x)$$

$$\geq \mathcal{F}_{u, L^\infty}(F),$$

where we have used the fact that $u$ vanishes outside the set $E$ and Lemma 2.10. Hence, we only need to prove that $u$ defined on $E$ is integrable and (21) holds for any $F \subset E$.

**Step I.** Denote by $h(\cdot)$ a measurable function satisfying $h > 0$ on $E$ and $\int_E h(x)d\mu_\alpha(x) < \infty$, and denote by $\Lambda$ the (positive and totally finite) measure:

$$\Lambda(F) = \int_F h(x)d\mu_\alpha(x), \quad F \subset E.$$ 

It is obvious that $\Lambda(F) = 0$ if and only if $\mu_\alpha(F) = 0$. For $\lambda > 0$ and $F \subset E$, consider the functional

$$\mathcal{F}_\lambda(F) := P_{L^\infty}(F, \Omega_1) + \lambda \Lambda(E \setminus F).$$

It is well known that every minimizing sequence is compact in $L^1_{loc}(\Omega_1, d\mu_\alpha)$ and the functional is lower semi-continuous with respect to the same convergence. Hence, we conclude that, for every $\lambda > 0$, there is a solution $E_\lambda$ to the problem:

$$\mathcal{F}_\lambda(F) \rightarrow \min, \quad F \subset E.$$ 

Choose a sequence $\{\lambda_i\}$ of positive numbers which is strictly increasing to $\infty$ and denote the corresponding solutions by $E_i \equiv E_{\lambda_i}$, so that $\forall i \geq 1$,

$$\mathcal{F}_{\lambda_i}(E_i) \leq \mathcal{F}_{\lambda_j}(F), \quad \forall F \subset E.$$ 

Given $i < j$. Let $F = E_i \cap E_j$. It follows from (23) that

$$\mathcal{F}_{\lambda_i}(E_i) \leq \mathcal{F}_{\lambda_i}(E_i \cap E_j),$$
that is, 
\[ P_{\mathcal{L}^n}(E_i, \Omega_1) + \lambda_i \Lambda(E \setminus E_i) \leq P_{\mathcal{L}^n}(E_i \cap E_j, \Omega_1) + \lambda_i \Lambda(E \setminus (E_i \cap E_j)), \]
which implies
\[ P_{\mathcal{L}^n}(E_i, \Omega_1) + \lambda_i \int_{E \setminus E_i} h(x) d\mu_\alpha(x) \leq P_{\mathcal{L}^n}(E_i \cap E_j, \Omega_1) + \lambda_i \int_{E \setminus (E_i \cap E_j)} h(x) d\mu_\alpha(x). \]
A direct computation gives
\[ P_{\mathcal{L}^n}(E_i, \Omega_1) \leq \lambda_i \int_{E \setminus E_j} h(x) d\mu_\alpha(x) + P_{\mathcal{L}^n}(E_i \cap E_j, \Omega_1). \]
On the other hand, taking \( F = E_i \cup E_j \subset E \) in (23), we can get \( \mathcal{F}_{\lambda_j}(E_j) \leq \mathcal{F}_{\lambda_j}(E_i \cup E_j) \). Hence,
\[ P_{\mathcal{L}^n}(E_j, \Omega_1) + \lambda_j \int_{E \setminus E_j} h(x) d\mu_\alpha(x) \leq P_{\mathcal{L}^n}(E_j \cup E_i, \Omega_1) + \lambda_j \int_{E \setminus (E_j \cup E_i)} h(x) d\mu_\alpha(x), \]
equivalently,
\[ P_{\mathcal{L}^n}(E_j, \Omega_1) + \lambda_j \int_{E \setminus E_j} h(x) d\mu_\alpha(x) \leq P_{\mathcal{L}^n}(E_i \cup E_j, \Omega_1) \]
which implies that
\[ P_{\mathcal{L}^n}(E_i, \Omega_1) + P_{\mathcal{L}^n}(E_j, \Omega_1) + \lambda_j \int_{E \setminus E_j} h(x) d\mu_\alpha(x) \leq P_{\mathcal{L}^n}(E_i \cup E_j, \Omega_1) + \lambda_i \int_{E \setminus E_i} h(x) d\mu_\alpha(x) + P_{\mathcal{L}^n}(E_i \cap E_j, \Omega_1). \]
Recall that \( h > 0 \). The above estimate, together with (22) and the fact that \( \lambda_i < \lambda_j \), indicates that
\[ (\lambda_j - \lambda_i) \Lambda(E_i \setminus E_j) = (\lambda_j - \lambda_i) \int_{E_i \setminus E_j} h(x) d\mu_\alpha(x) = 0, \]
that is, \( E_i \subset E_j \) and the sequence of minimizers \( \{E_i\} \) is increasing. On the other hand, letting \( F = E \), we get
\[ P_{\mathcal{L}^n}(E_i, \Omega_1) + \lambda_i \Lambda(E \setminus E_i) \leq P_{\mathcal{L}^n}(E, \Omega_1) + \lambda_i \Lambda(E \setminus E) = P_{\mathcal{L}^n}(E, \Omega_1) \quad \forall \ i \geq 1, \]
which deduces that \( E_i \) converges monotonically and in \( L^1_{\text{loc}}(\mathbb{R}^d_+, d\mu_\alpha) \) to \( E \). Via Lemma 2.1 (ii), we get
\[ \begin{aligned}
P_{\mathcal{L}^n}(E, \Omega_1) &\leq \liminf_{i \to \infty} P_{\mathcal{L}^n}(E_i, \Omega_1) \leq P_{\mathcal{L}^n}(E, \Omega_1), \\
P_{\mathcal{L}^n}(E, \Omega_1) &\leq \liminf_{i \to \infty} P_{\mathcal{L}^n}(E_i, \Omega_1) \leq \limsup_{i \to \infty} P_{\mathcal{L}^n}(E_i, \Omega_1) \leq P_{\mathcal{L}^n}(E),
\end{aligned} \]
which means
\[ P_{\mathcal{L}^n}(E, \Omega_1) = \lim_{i \to \infty} P_{\mathcal{L}^n}(E_i, \Omega_1). \]
Step II. Let \( \lambda_0 = 0 \) and \( E_0 = \emptyset \), and define
\[
u(x) = \begin{cases} -\lambda_i h(x), & x \in E_i \setminus E_{i-1}, \ i \geq 1, \\ 0, & \text{otherwise}. \end{cases}
\]
Clearly, \( \nu \) is negative almost everywhere on \( E \), and
\[
\int_{\mathbb{R}^d_+} |\nu(x)| d\mu_\alpha(x) = \int_{\bigcup_{i=0}^{\infty} E_{i+1} \setminus E_i} |\nu(x)| d\mu_\alpha(x) \\
= \sum_{i=0}^{\infty} \int_{E_{i+1} \setminus E_i} \lambda_{i+1} h(x) d\mu_\alpha(x) \\
= \sum_{i=0}^{\infty} \lambda_{i+1} \Lambda(E_{i+1} \setminus E_i).
\]
In (23), taking \( F = E_{i+1} \), we have
\[
P_{L^\alpha}(E_i, \Omega_1) + \lambda_i \Lambda(E \setminus E_i) \leq P_{L^\alpha}(E_{i+1}, \Omega_1) + \lambda_i \Lambda(E \setminus E_{i+1}),
\]
that is, for every \( i \geq 0 \),
\[
\lambda_i \Lambda(E_{i+1} \setminus E_i) \leq P_{L^\alpha}(E_{i+1}, \Omega_1) - P_{L^\alpha}(E_i, \Omega_1).
\]
Then for sufficiently large \( N \),
\[
\sum_{i=0}^{N} \lambda_i \Lambda(E_{i+1} \setminus E_i) \leq \sum_{i=0}^{N} \left[ P_{L^\alpha}(E_{i+1}, \Omega_1) - P_{L^\alpha}(E_i, \Omega_1) \right] \\
= P_{L^\alpha}(E_N, \Omega_1) - P_{L^\alpha}(E_0, \Omega_1) \\
= P_{L^\alpha}(E_N, \Omega_1).
\]
Letting \( N \to \infty \), (24) indicates that
\[
\sum_{i=0}^{\infty} \lambda_i \Lambda(E_{i+1} \setminus E_i) \leq P_{L^\alpha}(E, \Omega_1).
\]
We make the additional assumption that \( 0 < \lambda_{i+1} - \lambda_i \leq c, \ i \geq 0 \), where \( c \) is a constant independent of \( i \). Then for any \( N > 0 \),
\[
\sum_{i=0}^{N} (\lambda_{i+1} - \lambda_i) \Lambda(E_{i+1} \setminus E_i) \leq c \sum_{i=0}^{N} \Lambda(E_{i+1} \setminus E_i) \\
= c \sum_{i=0}^{N} \int_{E_{i+1} \setminus E_i} h(x) d\mu_\alpha(x) \\
= c \int_{\bigcup_{i=0}^{N} (E_{i+1} \setminus E_i)} h(x) d\mu_\alpha(x),
\]
which gives
\[
\sum_{i=0}^{\infty} (\lambda_{i+1} - \lambda_i) \Lambda(E_{i+1} \setminus E_i) \leq c \Lambda(E).
\]
Therefore, we obtain that
\[ \int_{\mathbb{R}^d_+} |u(x)| d\mu_\alpha(x) = \sum_{i=0}^{\infty} \lambda_{i+1} \Lambda(E_{i+1} \setminus E_i) \]
\[ = \sum_{i=0}^{\infty} (\lambda_{i+1} - \lambda_i) \Lambda(E_{i+1} \setminus E_i) + \sum_{i=0}^{\infty} \lambda_i \Lambda(E_{i+1} \setminus E_i) \]
\[ \leq c \Lambda(E) + P_{\mathcal{L}^\alpha}(E, \Omega_1) < \infty. \]

In conclusion, \( u \in L^1(\mathbb{R}^d_+, d\mu_\alpha) \).

**Step III.** We claim that for every \( i \geq 1 \) the inequality
\begin{equation}
P_{\mathcal{L}^\alpha}(E_i, \Omega_1) \leq P_{\mathcal{L}^\alpha}(F, \Omega_1) + \sum_{j=1}^{i} \lambda_j \Lambda((E_j \setminus E_{j-1}) \setminus F)
\end{equation}
holds for any \( F \subset E \).

For \( i = 1 \), \( E_{i-1} = E_0 = \emptyset \). Then (25) becomes
\[ P_{\mathcal{L}^\alpha}(E_i, \Omega_1) \leq P_{\mathcal{L}^\alpha}(F, \Omega_1) + \lambda_1 \Lambda(E_1 \setminus F), \]
which coincides with (23) for \( i = 1 \).

Now we assume that (25) holds for a fixed \( i \geq 1 \) and every \( F \subset E \). Take \( F \cap E_i \) as a test set. Note that \( \{E_j\} \) is increasing. It is easy to see that
\[ (E_j \setminus E_{j-1}) \setminus (F \cap E_i) = (E_j \setminus E_{j-1}) \setminus F. \]

Then
\[ P_{\mathcal{L}^\alpha}(E_i, \Omega_1) \leq P_{\mathcal{L}^\alpha}(F \cap E_i, \Omega_1) + \sum_{j=1}^{i} \lambda_j \Lambda((E_j \setminus E_{j-1}) \setminus (F \cap E_i)) \]
\[ = P_{\mathcal{L}^\alpha}(F \cap E_i, \Omega_1) + \sum_{j=1}^{i} \lambda_j \Lambda((E_j \setminus E_{j-1}) \setminus F). \]

On the other hand, \( E_{i+1} \) is a minimizer of \( \mathcal{F}_{\lambda_{i+1}} \). Hence,
\[ \mathcal{F}_{\lambda_{i+1}}(E_{i+1}) \leq \mathcal{F}_{\lambda_{i+1}}(F \cup E_i), \]
and noticing that
\[ E \setminus E_i = (E \setminus E_{i+1}) \cup (E_{i+1} \setminus E_i), \]
we can get
\[ E \setminus (F \cup E_i) = ((E \setminus E_{i+1}) \setminus F) \cup ((E_{i+1} \setminus E_i) \setminus F). \]

This gives
\[ P_{\mathcal{L}^\alpha}(E_{i+1}, \Omega_1) + \lambda_{i+1} \Lambda(E \setminus E_{i+1}) \leq P_{\mathcal{L}^\alpha}(F \cup E_i, \Omega_1) + \lambda_{i+1} \Lambda((E \setminus E_i) \setminus F) \]
\[ \leq P_{\mathcal{L}^\alpha}(F \cup E_i, \Omega_1) + \lambda_{i+1} \Lambda((E \setminus E_{i+1}) \setminus F) + \lambda_{i+1} \Lambda((E_{i+1} \setminus E_i) \setminus F). \]

Therefore, we obtain that
\[ P_{\mathcal{L}^\alpha}(E_i, \Omega_1) + P_{\mathcal{L}^\alpha}(E_{i+1}, \Omega_1) + \lambda_{i+1} \Lambda(E \setminus E_{i+1}) \]
\[ \leq P_{\mathcal{L}}(F \cap E_i, \Omega_1) + \sum_{j=1}^{i} \lambda_j \Lambda((E_j \setminus E_{j-1}) \setminus F) \\
+ P_{\mathcal{L}}(F \cup E_i, \Omega_1) + \lambda_{i+1} \Lambda((E \setminus E_{i+1}) \setminus F) + \lambda_{i+1} \Lambda((E \setminus E_{i+1}) \setminus F) \\
\leq P_{\mathcal{L}}(E_i, \Omega_1) + P_{\mathcal{L}}(F, \Omega_1) + \sum_{j=1}^{i+1} \lambda_j \Lambda((E_j \setminus E_{j-1}) \setminus F) + \lambda_{i+1} \Lambda((E \setminus E_{i+1}) \setminus F) \\
\leq P_{\mathcal{L}}(E_i, \Omega_1) + P_{\mathcal{L}}(F, \Omega_1) + \sum_{j=1}^{i+1} \lambda_j \Lambda((E_j \setminus E_{j-1}) \setminus F) + \lambda_{i+1} \Lambda(E \setminus E_{i+1}), \]

that is, (25) holds for \( i + 1 \). Finally,

\[ P_{\mathcal{L}}(E, \Omega_1) = \lim_{i \to \infty} P_{\mathcal{L}}(E_i, \Omega_1) \]

\[ \leq P_{\mathcal{L}}(F, \Omega_1) + \lim_{i \to \infty} \sum_{j=1}^{i} \lambda_j \Lambda((E_j \setminus E_{j-1}) \setminus F) \]

\[ = P_{\mathcal{L}}(F, \Omega_1) - \int \bigcup_{j=0}^{\infty} (E \setminus E_{j-1}) \setminus F \ u(x) \, d\mu_\alpha(x) \]

\[ = P_{\mathcal{L}}(F, \Omega_1) - \int_{E \setminus F} u(x) \, d\mu_\alpha(x), \]

which gives (23). \qed

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