Algorithmic market making in foreign exchange cash markets with hedging and market impact

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Abstract

In OTC markets, one of the main tasks of dealers / market makers consists in providing prices at which they agree to buy and sell the assets and securities they have in their scope. With ever increasing trading volume, this quoting task has to be done algorithmically. Over the last ten years, many market making models have been designed that can be the basis of quoting algorithms in OTC markets. Nevertheless, in most (if not all) OTC market making models, the market maker is a pure internalizer, setting quotes and waiting for clients. However, on many markets such as foreign exchange cash markets, market makers have access to liquidity pools where they can hedge part of their inventory.

In this paper, we propose a model taking this possibility into account, therefore allowing market makers to externalize part of their risk by trading in a liquidity pool. The model displays an important feature well known to practitioners that within a certain inventory range the market maker internalizes the flow by appropriately adjusting the quotes and externalize outside of that range. The larger the market making franchise, the wider is the inventory range suitable for internalization. The model is illustrated numerically with realistic parameters for USDCNH spot market.

Key words: Market making, Algorithmic trading, Stochastic optimal control, Viscosity solutions

1 Introduction

In financial markets, liquidity has traditionally been provided by a specific category of agents who, on a continuous and regular basis, set prices at which they agree to buy or sell assets and securities. These agents, called market makers or dealers, play a key role in the price formation process in all markets but their exact role and behavior depend on the considered asset class.

In most order-driven markets, such as stock markets, traditional exchanges have converted from open outcry communications between human traders to electronic platforms organized around all-to-all limit order books, and computers now handle almost all market activity. Official market makers and traditional market making companies still make money by providing liquidity to the market but they are now, somehow, in competition with all market participants who can post liquidity-providing orders. In quote-driven markets, electronification has also been one of the major upheavals of the last decade, with important consequences for market makers / dealers. In foreign exchange (FX) cash markets for instance, dealers set up their own private electronic platforms enabling clients to directly send them requests for quotes (RFQ) and to be connected to their stream (RFS). Many dealer-to-dealer and all-to-all platforms have also emerged, therefore blurring the frontier between OTC and organized markets (see [31, 32] for a recent analysis of FX markets).

Alongside the multifaceted electronification of trading means, most human market makers have been replaced by market making algorithms. This evolution has naturally gone along with the development of many market making models in the academic literature. In 2008, largely inspired by a paper from the 1980s

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by Ho and Stoll [27], Avellaneda and Stoikov [6] proposed a stochastic optimal control model to determine the optimal bid and ask quotes that a single-asset risk-averse market maker should set. The authors paved the way to a new literature on market making that complements the contributions of the economic literature on the topic. The resulting new models can be divided into two groups: those adapted to the problem of a market maker in a limit order book and those adapted to OTC markets.

To build a relevant market making model for order-driven markets, and especially stock markets, it is important to take microstructure into account. For instance, Guilbaud and Pham [25] modeled the market bid-ask spread with a discrete Markov chain and studied the performance of a market maker submitting limit buy/sell order at the best bid/ask quotes and/or at the best bid plus one tick and best ask minus one tick. Guilbaud and Pham [20] studied a similar problem of optimal market making in a pro-rata limit order book, where the dealer may post limit orders but also market orders represented by impulse controls. Fodra and Pham [17, 18] considered a model in which market orders arrive in the limit order book according to a point process correlated with the stock price itself. They modeled the market maker as an agent placing limit orders of constant size at the best bid and at the best ask, and solved the problem faced by a risk-averse market maker. More recently, in Abergel et al. [2], the authors proposed a different model for the limit order book (first introduced in Abergel et al. [1]), in which the limit orders, market orders, and cancel orders arrive according to Markov jump processes with intensities depending only on the state of the limit order book. They considered the case of a market maker trading in this limit order book, and proposed a quantization-based algorithm to numerically solve the resulting high-dimensional problem. Finally, Capponi et al. [11] studied a discrete-time problem, assuming that the market maker can place bid and ask limit orders simultaneously on both sides at prespecified dates. In their framework, the number of filled orders during each period depends linearly on the distance between the fundamental price and the price of the market maker’s limit order, with random slope and intercept coefficients. Using discrete-time optimal control theory, the authors managed to get an explicit characterization of the optimal strategy.

In the case of OTC markets, or for order-driven markets in which the spread-to-tick ratio is large, most market making models derive from the seminal work of Avellaneda and Stoikov [6]. For instance, Guéant et al. [21] provided a rigorous analysis of the stochastic optimal control problem introduced in Avellaneda and Stoikov [6] and showed that, by adding risk limits to the inventory of the market maker, the problem boils down to a finite system of ordinary differential equations (ODEs) – in particular, those ODEs are linear in the case of the exponential intensity functions proposed in [6]. Cartea, Jaimungal, and Ricci considered in [14] a different kind of objective function: instead of the Von Neumann-Morgenstern expected utility of [6], they optimized a risk-adjusted expectation of the PnL. Cartea and Jaimungal, along with diverse coauthors, then proposed several extensions. For instance, Cartea, Donnelly, and Jaimungal studied in [12] the impact of uncertainty on the parameters of the model. Multi-asset market making models have been proposed by Guéant in [22, 23] for both kinds of objective function, and the author showed again that the problem boils down to a system of ODEs. In all these models the trade size is assumed constant: the same number of assets is bought/sold at each trade. In [9], Bergault and Guéant introduced a distribution of trade sizes in the model along with the possibility for the market maker to answer different quotes for different sizes. They characterized the value function of the problem with an integro-differential equation of the Hamilton-Jacobi (HJ) type that can be seen as an ODE in an infinite-dimensional space. They also proposed a dimensionality reduction technique in order to approximate numerically the optimal bid and ask quotes of the market maker in high dimension, by projecting the market risk on a low-dimensional space of factors. This problem of approximating the solution across a large universe of assets has been addressed using different approaches. In Bergault et al. [8], the authors regarded the Hamilton-Jacobi equation of the problem as a perturbation of another simpler equation whose solution can be computed in closed form. In Guéant and Manziuk [21], the authors used neural networks instead of grids to compute an approximate

1 Economists had studied for a long time the behaviour of market makers / dealers with the aim of understanding market liquidity and the magnitude of bid-ask spreads. Models where one or several risk-averse market makers optimize their pricing policy for managing their inventory risk models include Amihud and Mendelson [13], Ho and Stoll [27, 28], and O’Hara and Oldfield [29]. Models focused on information asymmetries where bid-ask spreads derive from adverse selection include Copeland and Galai [10] and Glosten and Milgrom [19]. Other classic economic references on market making include Grossman and Miller [20] and the review paper of Stoll [25].

2 See the books of Cartea et al. [13] and Guéant [22] for a detailed discussion.
solution – a method inspired by approximate dynamic programming and reinforcement learning techniques.

In most academic models adapted to OTC markets, market makers are pure liquidity providers: they buy assets at the bid price they quote and sell them at the ask price they quote – ideally earning the difference between these two prices. Of course, market makers seldom buy and sell simultaneously: they carry inventory and bear price risk. The problem faced by market makers in these models is already a subtle dynamic optimization problem in which market makers must mitigate the risk associated with price changes by skewing their quotes as a function of their inventory. In practice, however, market makers in FX cash markets have an additional way to manage their inventory risk since they can partially or completely hedge it by trading in liquidity pools on the Dealer-to-Dealer (D2D) segment of the market and in a variety of all-to-all platforms.

The co-existence of requests for quotes / requests for stream and liquidity pools has seldom been studied in the academic literature on optimal market making in OTC markets (the only instance we found beyond our paper is the very recent paper [7] that proposes a reinforcement learning approach). The trade-off faced by dealers in FX markets between internalization and externalization is nevertheless discussed in the literature. It is discussed by Butz and Oomen in [10] on the basis of queuing theory, though not with optimized quotes. It is also abundantly discussed on empirical grounds in the recent BIS Triennial Survey that concludes on the growing prevalence of internalization (see [31]). A wide spectrum of behaviors is documented, from pure externalization to large ratios of internalization. It is noteworthy that even though internalization ratios for top trading centers exceed 80%, hedging through externalization remains an essential component of risk management for any dealer. In particular, this feature should be included in FX optimal market making models.

The main goal of our paper is to build an optimal market making strategy that includes the possibility for the market maker to hedge by buying and selling (in continuous time) in a liquidity pool, in order to better mitigate inventory risk. By trading in a liquidity pool, the market maker adds certainty to inventory risk management but that comes with execution costs and market impact, in part due to revealing of trading intentions to a wider audience. Our setup is inspired by Almgren-Chriss-like models of optimal execution (see Almgren et al. [4, 3], and Guéant [22] for a general presentation). More precisely, compared to existing optimal OTC market making models, ours includes a new form of control – in addition to the bid and ask quotes – that represents the trading rate of the market maker in a liquidity pool and features (i) execution costs to proxy transaction costs and nonlinear liquidity costs, and (ii) permanent market impact (assumed to be linear in the trading rate).

In Section 2 we present our market making model involving a currency pair for which the market maker has a classical market making quoting activity together with the possibility to hedge risk by trading in a liquidity pool. We then introduce the stochastic optimal control problem of the market maker. In Section 3 we characterize the associated value function as the unique continuous viscosity solution of a Partial Integro-Differential Equation (PIDE) of the HJ type. We illustrate our model numerically in Section 4 and discuss the results. In particular, we highlight the existence of a threshold of inventory under which it is not optimal for the market maker to trade in the liquidity pool.

2 The model

We consider a market maker in charge of a single currency pair. This currency pair can be traded by the market maker in two ways: (i) via requests she receives from clients and (ii) via market orders sent to a liquidity pool (for instance an all-to-all trading platform, but we can also regard the liquidity pool of our model as an aggregation of numerous liquidity pools). In the former case, the market maker may receive a RFQ from a client wishing to buy or sell the currency pair, and then proposes a price to the client who finally decides whether she accepts to trade at that price or not – another possibility is that a client connected
to the stream of quotes proposed by the market maker decides to trade (this is the RFS channel). In the latter case, the market maker buys or sells at a market price that depends on the traded volume.

In Section 2.1 we present an optimization problem with hard constraints on the trading strategy in the liquidity pool. In order to carry out a rigorous mathematical analysis, we shall slightly modify (regularize) the problem in Section 2.2.

2.1 Modeling framework and notations

We consider a filtered probability space \((\Omega, F, \mathbb{F}, \mathbb{F}) = (\mathcal{F}_t)_{t \geq 0}\) satisfying the usual conditions. We assume this probability space is large enough to support all the processes we introduce.

Let us start with the modelling of the price. We model the reference price (for instance the mid-price of the liquidity pool) of the currency pair by a process \((S_t)_{t \geq 0}\). We consider that the market maker can trade continuously in the liquidity pool and we denote by \((w_t)_{t \geq 0}\) the trading rate of the market maker (she buys when \(w_t \geq 0\) and sells otherwise). Taking into account the permanent market impact of trades within the liquidity pool, the price process \((S_t)_{t \geq 0}\) is modeled as follows:

\[
dS_t = \sigma dW_t + kw_t dt,
\]

with \(S_0\) given, where \(k\) and \(\sigma\) are positive constant and the process \((W_t)_{t \geq 0}\) is a standard Brownian motion adapted to the filtration \(\mathbb{F}\).

Regarding requests, the market maker proposes bid and ask quotes depending on the size \(z \in \mathbb{R}^*_+\) of each request. These quotes are modeled by maps \(S^b, S^a : \Omega \times [0, T] \times \mathbb{R}^*_+ \to \mathbb{R}^*_+\) which are \(\mathcal{P} \otimes \mathcal{B}(\mathbb{R}^*_+)\)-measurable, where \(\mathcal{P}\) denotes the \(\sigma\)-algebra of \(\mathbb{F}\)-predictable subsets of \(\Omega \times [0, T]\) and \(\mathcal{B}(\mathbb{R}^*_+)\) denotes the Borelian sets of \(\mathbb{R}^*_+\).

We introduce \((J^b(dt, dz))\) and \((J^a(dt, dz))\) two \(\mathbb{C}^{adl\acute{a}g} \mathbb{R}^*_+\)-marked point processes corresponding to the number of transactions resulting from requests at the bid and at the ask, respectively. We denote respectively by \(\tilde{J}^b(dt, dz)\) and \(\tilde{J}^a(dt, dz)\) the associated compensated processes.

The inventory of the market maker, modeled by the process \((q_t)_{t \geq 0}\), has the following dynamics:

\[
dq_t = \int_{\mathbb{R}^*_+} zJ^b(dt, dz) - \int_{\mathbb{R}^*_+} zJ^a(dt, dz) + w_t dt,
\]

with \(q_0\) given. We assume that the processes \((J^b(dt, dz))\) and \((J^a(dt, dz))\) do not have simultaneous jumps almost surely. Moreover, we denote by \((\nu^b_t(dz))_{t \geq 0}\) and \((\nu^a_t(dz))_{t \geq 0}\) the intensity kernels of \((J^b(dt, dz))\) and \((J^a(dt, dz))\), respectively.

We assume that the market maker wants her inventory to remain in an interval \(Q = [-\bar{q}, \bar{q}]\), with \(\bar{q} > 0\). The intensities \((\nu^b_t(dz))_{t \geq 0}\) and \((\nu^a_t(dz))_{t \geq 0}\) verify:

\[
\nu^b_t(dz) = 1_{\{q_- + z \in Q\}} \Lambda^b(\delta^b(t, z)) \mu^b(dz),
\]

\[
\nu^a_t(dz) = 1_{\{q_- - z \in Q\}} \Lambda^a(\delta^a(t, z)) \mu^a(dz),
\]

\[^{3}\text{The mathematical modelling of RFQs and RFSs is the same because market makers do not answer a fixed price in the case of RFQs but rather stream prices for a given period of times.}\)

\[^{4}\text{The model can easily be generalized to multiple currency pairs (see [23] for the way to go multi-asset in market making models à la Avellaneda-Stoikov). In that case, each currency pair may be traded through requests only, market orders only, or both ways.}\)

\[^{5}\text{In practice, in FX we take the mid-price on platforms like EBS or Refinitiv.}\)

\[^{6}\text{This is similar to what was done in Almgren [3].}\)
where (i) $\mu^b$ and $\mu^a$ are probability measures on $\mathbb{R}_+^*$, absolutely continuous with respect to the Lebesgue measure and such that $\int_{\mathbb{R}_+^*} z \mu^b(dz) =: \Delta^b < +\infty$ and $\int_{\mathbb{R}_+^*} z \mu^a(dz) =: \Delta^a < +\infty$, (ii) $\delta^b(t, z)$ and $\delta^a(t, z)$ are defined by $\delta^b(t, z) = S_t - S^b(t, z)$ and $\delta^a(t, z) = S^a(t, z) - S_t$, and (iii) $\Lambda^b$, $\Lambda^a$ are two functions satisfying the following hypotheses $(H_\Lambda)$:

- $\Lambda^b$ and $\Lambda^a$ are twice continuously differentiable,
- $\Lambda^b$ and $\Lambda^a$ are decreasing, with $\forall \delta \in \mathbb{R}, \Lambda^b'(\delta) < 0$ and $\Lambda^a'(\delta) < 0$,
- $\lim_{\delta \to +\infty} \Lambda^b(\delta) = 0$ and $\lim_{\delta \to +\infty} \Lambda^a(\delta) = 0$,
- $\sup_{\delta \in \mathbb{R}} \frac{\Lambda^b(\delta)\Lambda^{b''}(\delta)}{(\Lambda^a(\delta))'^2} < 2$ and $\sup_{\delta \in \mathbb{R}} \frac{\Lambda^a(\delta)\Lambda^{a''}(\delta)}{(\Lambda^a(\delta))'^2} < 2$.

In the first version of the model the process $(X_t)_{t \geq 0}$ modeling the market maker’s cash account has the dynamics:

$$
\frac{dX_t}{X_t} = \int_{\mathbb{R}_+^*} z \left( S_t + \delta^a(t, z) \right) J^a(dt, dz) - \int_{\mathbb{R}_+^*} z \left( S_t - \delta^b(t, z) \right) J^b(dt, dz) - w_t S_t dt - L(u, q_{-}) dt
= \int_{\mathbb{R}_+^*} z \delta^b(t, z) J^b(dt, dz) + \int_{\mathbb{R}_+^*} z \delta^a(t, z) J^a(dt, dz) - L(u, q_{-}) dt - S_t dq_t,
$$

with $L(w, q) = L(w) + \chi(w, q)$ $\forall (w, q) \in \mathbb{R}^2$, where the penalty function $L : \mathbb{R} \to \mathbb{R}_+$ (that results from the temporary price impact of the market maker when she chooses to externalize trading in the liquidity pool) satisfies the following hypotheses $(H_L)$:

- $L(0) = 0$,
- $L$ is strictly convex, increasing on $\mathbb{R}_+$ and decreasing on $\mathbb{R}_-$,
- $L$ is asymptotically superlinear, i.e., $\lim_{|q| \to +\infty} \frac{L(q)}{|q|} = +\infty$,

and the function $\chi : \mathbb{R}^2 \to \{0, +\infty\}$ verifies $\chi(w, q) = \begin{cases} 0 & \text{if } q = w = 0 \text{ or } \text{sign}(q) \neq \text{sign}(w) \\ +\infty & \text{otherwise.} \end{cases}$

The latter penalty function prevents the market maker from selling in the liquidity pool when her inventory is already short, and to buy in the liquidity pool when her inventory is already long.\footnote{This is the hard constraint that we shall relax in an astute manner in Section 2.2 to be able to use viscosity solutions.}

We define the following set $A_M$ of admissible controls for the quotes:

$$
A_M = \left\{ \delta = (\delta^b, \delta^a) : \Omega \times [0, T] \times \mathbb{R}_+^* \to \mathbb{R}^2 \big| \delta \text{ is } \mathcal{P} \otimes \mathcal{B}(\mathbb{R}_+^*)\text{-measurable and } \delta^b(t, z) \wedge \delta^a(t, z) \geq -\delta_\infty \mathbb{P} \otimes dt \otimes dz \text{ a.e.} \right\}
$$

where $\delta_\infty > 0$ is a prespecified constant.
We define the set $\mathcal{A}_T$ of admissible continuous trading strategies:

$$\mathcal{A}_T = \left\{ w : \Omega \times [0,T] \to \mathbb{R} \mid w \text{ is } \mathcal{P} \text{-measurable and } |w_t| \leq v_\infty \mathbb{P} \otimes dt \text{ a.e.} \right\}.$$  

for a given $v_\infty > 0$.

For two given continuous penalty functions $\psi : \mathbb{R} \to \mathbb{R}_+$ and $\ell : \mathbb{R} \to \mathbb{R}_+$, modeling the risk aversion of the market maker and the cost of liquidity, we aim at maximizing

$$\mathbb{E} \left[ X_T + q_T S_T - \ell(q_T) - \int_0^T \psi(q_t) dt \right]$$

over the set $\mathcal{A}_M \times \mathcal{A}_T$ of admissible controls $(\delta, w)$.

**Remark 1.** For instance, for a constant $\gamma > 0$, we can choose $\psi(q) = \frac{1}{2} \sigma^2 q^2$ or $\psi(q) = \gamma |q|$ and $\ell(q) = 0$, $\ell(q) = \frac{1}{2} \sigma^2 q^2$ or $\ell(q) = l \sigma |q|$ (for $l > 0$), as done in [13], [14], [23], and [21].

### 2.2 A slightly modified version

Carrying out a mathematical analysis of the above optimal control problem is complicated because the associated Hamiltonian function is discontinuous (because of the hard trading constraint introduced through the penalty function $\chi$). In order to carry out a rigorous mathematical analysis we do not consider the above problem but a slight modification of it. We fix $\bar{c} \in (0, \bar{q})$ and introduce a function $\zeta : \mathbb{R} \to [0, 1]$ which is a Lipschitz approximation of the indicator of the penalty function $\chi$.

$$\zeta(q) = 1 \quad \forall q \in \mathbb{R}_+ \quad \text{and} \quad \zeta(q) = 0 \quad \forall q \leq -\bar{c}.$$ 

Then, instead of considering the trading rate process as a control variable, we use a new control process $(v_t)_{t \geq 0} \in A_T$ and choose a trading rate of the form

$$w_t = -(v_t)_- \zeta(q_t-) + (v_t)_+ \zeta(q_t+).$$

We also introduce a new cost function $\tilde{L}(v, q) = L(-v_-) \zeta(q) + L(v_+) \zeta(-q)$ for $(v, q) \in \mathbb{R}^2$.

**Remark 2.** The model is exactly the same as the initial one whenever $|q| \geq \bar{c}$. However, for small values of the inventory ($|q| < \epsilon$), the market maker is allowed to trade in both directions in the liquidity pool but the cost to buy (resp. sell) increases when the inventory becomes negative (resp. positive).

The price process $(S_t)_{t \geq 0}$ has now the following dynamics:

$$dS_t = \sigma dW_t + k \left( -(v_t)_- \zeta(q_t-) + (v_t)_+ \zeta(q_t+) \right) dt,$$

with

$$L \left( \frac{w_-}{\zeta(q)} \right) \zeta(q) = \begin{cases} +\infty & \text{if } \zeta(q) = 0 \text{ and } w_- \neq 0 \\ 0 & \text{if } \zeta(q) = 0 \text{ and } w_- = 0, \end{cases}$$

and similarly

$$L \left( \frac{w_+}{\zeta(-q)} \right) \zeta(-q) = \begin{cases} +\infty & \text{if } \zeta(-q) = 0 \text{ and } w_+ \neq 0 \\ 0 & \text{if } \zeta(-q) = 0 \text{ and } w_+ = 0. \end{cases}$$

These conventions are natural as $L$ is assumed to be asymptotically superlinear.
and the inventory \((q_t)_{t \geq 0}\) has dynamics:
\[
dq_t = \int_{\mathbb{R}_+} zJ^b(dt, dz) - \int_{\mathbb{R}_+} zJ^a(dt, dz) + ((v_t)_{-} - \zeta(q_t -) + (v_t)_{+} + \zeta(-q_t -))dt.
\]  
(5)

Finally, the process \((X_t)_{t \geq 0}\) modeling the market maker’s cash account has the dynamics:
\[
\begin{align*}
&dX_t = \int_{\mathbb{R}_+} z(S_t + \delta^a(t, z))J^a(dt, dz) - \int_{\mathbb{R}_+} z(S_t - \delta^b(t, z))J^b(dt, dz) \\
&\quad - ((v_t)_{-} - \zeta(q_t -) + (v_t)_{+} + \zeta(-q_t -))S_t dt - \hat{L}(v_t, q_t -) dt \\
&= \int_{\mathbb{R}_+} z\delta^b(t, z)J^b(dt, dz) + \int_{\mathbb{R}_+} z\delta^a(t, z)J^a(dt, dz) \\
&\quad - \hat{L}(v_t, q_t -) dt - S_t dq_t.
\end{align*}
\]
(6)

The resulting optimization problem is that of maximizing
\[
\mathbb{E}\left[ X_T + q_T S_T - \ell(q_T) - \int_0^T \psi(q_t)dt \right] = \mathbb{E}\left[ X_{T^-} + q_{T^-} S_T - \ell(q_{T^-}) - \int_0^T \psi(q_{t^-})dt \right]
\]
over the set \(A := \mathcal{A}_M \times \mathcal{A}_T\) of admissible controls \((\delta, v)\).

After applying Itô’s formula to \((X_t + q_t S_t)_{t \geq 0}\) between 0 and \(T^-\), it is easy to see that the problem is equivalent to maximizing:
\[
\mathbb{E}\left[ \int_0^T \left\{ \int_{\mathbb{R}_+^+} (z\delta^b(t, z)1_{\{q_{t^-} + z \in \mathcal{Q}\}}\Lambda^b(\delta^b(t, z))\mu^b(dz) + z\delta^a(t, z)1_{\{q_{t^-} - z \in \mathcal{Q}\}}\Lambda^a(\delta^a(t, z))\mu^a(dz) \\
\quad + kq_t ((v_t)_{-} - \zeta(q_t -) + (v_t)_{+} + \zeta(-q_t -)) - \hat{L}(v_t, q_t -) - \psi(q_t -) \right\} dt - \ell(q_{T^-}) \right],
\]
over the set of admissible controls \(A\).

We therefore introduce the function \(J : [0, T] \times \mathcal{Q} \times \mathcal{A}_M \times \mathcal{A}_T \to \mathbb{R}\) such that, for all \(t \in [0, T]\), for all \(q \in \mathcal{Q}\) and for all \((\tilde{\delta}, \tilde{v}) \in \mathcal{A}\), if we denote by \((q_{s_{t^-}}^{t, \tilde{\delta}, \tilde{v}})_{s \geq t}\) the inventory process starting in state \(q\) at time \(t\) and controlled by \((\tilde{\delta}, \tilde{v})\):
\[
J(t, q, \tilde{\delta}, \tilde{v}) = \mathbb{E}\left[ \int_t^T \left\{ \int_{\mathbb{R}_+^+} (z\delta^b(s, z)1_{\{q_{s_{t^-}} \tilde{\delta}, \tilde{v} + z \in \mathcal{Q}\}}\Lambda^b(\delta^b(s, z))\mu^b(dz) \\
\quad + z\delta^a(s, z)1_{\{q_{s_{t^-}} \tilde{\delta}, \tilde{v} - z \in \mathcal{Q}\}}\Lambda^a(\delta^a(s, z))\mu^a(dz) \\
\quad + kq_{s_{t^-}} \tilde{\delta}, \tilde{v} ((v_s)_{-} - \zeta(q_{s_{t^-}}^{t, \tilde{\delta}, \tilde{v}}) + (v_s)_{+} + \zeta(-q_{s_{t^-}}^{t, \tilde{\delta}, \tilde{v}})) \\
\quad - \hat{L}(v_s, q_{s_{t^-}}^{t, \tilde{\delta}, \tilde{v}}) - \psi(q_{s_{t^-}}^{t, \tilde{\delta}, \tilde{v}}) \right\} ds - \ell(q_{T^-}^{t, \tilde{\delta}, \tilde{v}}) \right].
\]

The value function \(\theta : [0, T] \times \mathcal{Q} \to \mathbb{R}\) of the problem is then defined as follows:
\[
\theta(t, q) = \sup_{(\tilde{\delta}, \tilde{v}) \in \mathcal{A}} J(t, q, \tilde{\delta}, \tilde{v}), \quad \forall (t, q) \in [0, T] \times \mathcal{Q}.
\]
We will show that $\theta$ is the unique continuous viscosity solution on $[0, T] \times Q$ to the following Hamilton-Jacobi partial integro-differential equation:

$$
\begin{cases}
0 = -\partial_t \theta(t, q) + \psi(q) - \int_{\mathbb{R}^*_+} \mathbb{1}_{\{q + z \in Q\}} z H^b \left( \frac{\theta(t, q) - \theta(t, q + z)}{z} \right) \mu^b(dz) \\
- \int_{\mathbb{R}^*_+} \mathbb{1}_{\{q - z \in Q\}} z H^a \left( \frac{\theta(t, q) - \theta(t, q - z)}{z} \right) \mu^a(dz) - \mathcal{H}(\partial_q \theta(t, q), q) \quad \forall t \in [0, T)
\end{cases}
$$

(\text{HJ})

where

$$H^b : p \in \mathbb{R} \mapsto \sup_{\delta \geq -\delta_{\infty}} \Lambda^b(\delta)(\delta - p)$$

and

$$H^a : p \in \mathbb{R} \mapsto \sup_{\delta \geq -\delta_{\infty}} \Lambda^a(\delta)(\delta - p),$$

and

$$\mathcal{H} : (p, q) \in \mathbb{R} \times Q \mapsto \sup_{|v| \leq v_{\infty}} \left( -v_- \zeta(q) + v_+ \zeta(-q) \right)(p + kq) - \tilde{L}(v, q).$$

3 Mathematical analysis

3.1 Preliminary results

We first recall a result (Lemma 1) which is proved in [9].

Lemma 1. $H^b$ and $H^a$ are two continuously differentiable decreasing functions and the supremum in the definition of $H^b(p)$ (respectively $H^a(p)$) is reached at a unique $\delta^b_*(p)$ (respectively $\delta^a_*(p)$). Furthermore, $\delta^b_*$ and $\delta^a_*$ are continuous and nondecreasing in $p$.

We then state another useful lemma:

Lemma 2. Let $\varphi : [0, T] \times Q \mapsto \mathbb{R}$ be a bounded function. The functions

$$(t, q, z) \in [0, T] \times Q \times \mathbb{R}^*_+ \mapsto \mathbb{1}_{\{q + z \in Q\}} z H^b \left( \frac{\varphi(t, q) - \varphi(t, q + z)}{z} \right)$$

and

$$(t, q, z) \in [0, T] \times Q \times \mathbb{R}^*_+ \mapsto \mathbb{1}_{\{q - z \in Q\}} z H^a \left( \frac{\varphi(t, q) - \varphi(t, q - z)}{z} \right)$$

are bounded.

Proof. We only prove it for the ask side (the proof for the bid side is similar).

Let $t \in [0, T]$, $q \in Q$ and $z \in \mathbb{R}^*_+$ such that $q - z \in Q$.

We have

$$z H^a \left( \frac{\varphi(t, q) - \varphi(t, q - z)}{z} \right) = z \sup_{\delta \geq -\delta_{\infty}} \Lambda^a(\delta) \left( \delta - \frac{\varphi(t, q) - \varphi(t, q - z)}{z} \right)$$

$$\leq z \sup_{\delta \geq -\delta_{\infty}} \Lambda^a(\delta) \delta + \sup_{\delta \geq -\delta_{\infty}} -\Lambda^a(\delta) \left( \varphi(t, q) - \varphi(t, q - z) \right)$$

$$\leq 2q H^a(0) + 2\Lambda^a(-\delta_{\infty}) \sup_{[0, T] \times Q} |\varphi|. $$
For the other bound, we have
\[
zh^a \left( \frac{\varphi(t, q) - \varphi(t, q - z)}{z} \right) = \sup_{\delta \geq -\delta_\infty} \left\{ z\Lambda^a(\delta) - \Lambda^a(\delta) \left( \varphi(t, q) - \varphi(t, q - z) \right) \right\}.
\]
\[
\geq -\Lambda^a(0) \left( \varphi(t, q) - \varphi(t, q - z) \right)
\]
\[
\geq -2\Lambda^a(0) \sup_{[0, T] \times Q} |\varphi|.
\]

We can now state a first simple result about the value function \( \theta \):

**Proposition 1.** The value function \( \theta \) is bounded on \([0, T] \times Q\).

**Proof.** \( \forall (t, q, \tilde{d}, \tilde{v}) \in [0, T] \times Q \times A_M \times A_T \), we have

\[
J(t, q, \tilde{d}, \tilde{v}) = \mathbb{E} \left[ \int_t^T \left\{ \int_{\mathbb{R}^+} \left( z\delta^a(s, z) \mathbb{1}_{\{q_{t, q, \delta, \tilde{v}} + z \in Q\}} \Lambda^b(\delta^a(s, z)) \mu^b(dz) 
+ z\delta^a(s, z) \mathbb{1}_{\{q_{t, q, \delta, \tilde{v}} - z \in Q\}} \Lambda^a(\delta^a(s, z)) \mu^a(dz) \right) 
+ kq_{t, q, \delta, \tilde{v}} \left( (v_s - \zeta(q_{t, q, \delta, \tilde{v}})) + (v_s + \zeta(q_{t, q, \delta, \tilde{v}}) \right) 
- \tilde{L}(v_s, q_{t, q, \delta, \tilde{v}}) - \psi(q_{t, q, \delta, \tilde{v}}) \right) ds - \ell(q_{T, q, \delta, \tilde{v}}) \right] .
\]

As \( \ell, \psi, \) and \( \tilde{L} \) are nonnegative, we get

\[
J(t, q, \tilde{d}, \tilde{v}) \leq \mathbb{E} \left[ \int_t^T \left\{ \int_{\mathbb{R}^+} \left( z\delta^a(s, z) \mathbb{1}_{\{q_{t, q, \delta, \tilde{v}} + z \in Q\}} \Lambda^b(\delta^a(s, z)) \mu^b(dz) 
+ z\delta^a(s, z) \mathbb{1}_{\{q_{t, q, \delta, \tilde{v}} - z \in Q\}} \Lambda^a(\delta^a(s, z)) \mu^a(dz) \right) 
+ kq_{t, q, \delta, \tilde{v}} \left( (v_s - \zeta(q_{t, q, \delta, \tilde{v}})) + (v_s + \zeta(q_{t, q, \delta, \tilde{v}}) \right) \right] ds \]
\[
\leq T \left( \Delta^b \sup_{\delta \geq -\delta_\infty} \delta \Lambda^b(\delta) + \Delta^a \sup_{\delta \geq -\delta_\infty} \delta \Lambda^a(\delta) + 2kv_\infty \tilde{q} \right)
\]
\[
\leq T \left( \Delta^b H^b(0) + \Delta^a H^a(0) + 2kv_\infty \tilde{q} \right).
\]

The right-hand side is independent from \( t, q, \tilde{d} \) and \( \tilde{v} \), so it is clear that \( J \) and \( \theta \) are bounded from above. By considering the control process corresponding to \( \delta^b(t, z) = \delta^a(t, z) = v_t = 0 \), we obtain

\[
\theta(t, q) \geq -T\psi(q) - \ell(q). \tag{8}
\]

As \( \psi \) and \( \ell \) are continuous and \( Q \) is compact, we get that \( \theta \) is bounded from below. \( \square \)

Turning to the Hamiltonian function associated with the possibility to trade in a liquidity pool, we will need the following lemma:
**Lemma 3.** \( \mathcal{H} \) is a continuous function that satisfies

\[ \exists C_\mathcal{H} > 0, \forall p \in \mathbb{R}, \forall q, y \in \mathcal{Q}, |\mathcal{H}(p, q) - \mathcal{H}(p, y)| \leq C_\mathcal{H} (1 + |p|)|q - y|. \]

Furthermore, the supremum in the definition of \( \mathcal{H}(p, q) \) is reached for \( v^*(p, q) = \bar{\mathcal{H}}(p + kq) \), where

\[ \mathcal{H}(v) = \sup_{|v| \leq v_\infty} (rv - L(v)). \]

**Proof.** Let us first recall from classical results of convex analysis that, given the hypotheses \((H_L)\), \( \bar{\mathcal{H}} \) is a continuously differentiable function that satisfies \( \bar{\mathcal{H}}(0) = 0 \). Moreover, the supremum in the definition of \( \bar{\mathcal{H}}(v) \) is reached uniquely at \( \bar{\mathcal{H}}(v) \). In particular, \( \bar{\mathcal{H}} \) is a Lipschitz function with Lipschitz constant equal to \( v_\infty \).

For all \( p \in \mathbb{R} \) and \( q \in \mathcal{Q} \), we have

\[ \mathcal{H}(p, q) = \sup_{|v| \leq v_\infty} \left( (-v_-\zeta(q) + v_+\zeta(-q)) (p + kq) - \bar{L}(v, q) \right). \]

Let us consider first the case where \( p + kq \geq 0 \). If \( v < 0 \), then \((-v_-\zeta(q) + v_+\zeta(-q)) (p + kq) - \bar{L}(v, q) \leq 0 \). Therefore, when \( p + kq \geq 0 \), we have

\[ \mathcal{H}(p, q) = \zeta(-q) \sup_{0 \leq v \leq v_\infty} ((p + kq)v - L(v)) = \zeta(-q) \sup_{|v| \leq v_\infty} ((p + kq)v - L(v)) = \zeta(-q)\bar{\mathcal{H}}(p + kq) \]

and the supremum is reached for \( v^*(p, q) = \bar{\mathcal{H}}(p + kq) \) (though not uniquely because of the term \( \zeta(-q) \) that can be equal to nought).

Let us now come to the case where \( p + kq \leq 0 \). If \( v > 0 \), then \((-v_-\zeta(q) + v_+\zeta(-q)) (p + kq) - \bar{L}(v, q) \leq 0 \). Therefore, when \( p + kq \leq 0 \), we have

\[ \mathcal{H}(p, q) = \zeta(q) \sup_{-v_\infty \leq v \leq 0} ((p + kq)v - L(v)) = \zeta(q) \sup_{|v| \leq v_\infty} ((p + kq)v - L(v)) = \zeta(q)\bar{\mathcal{H}}(p + kq) \]

and the supremum is reached for \( v^*(p, q) = \bar{\mathcal{H}}(p + kq) \) (though not uniquely because of the term \( \zeta(q) \) that can be equal to nought).

Overall, because \( \bar{\mathcal{H}}(0) = 0 \), we can write \( \mathcal{H}(p, q) = \zeta(-q)\bar{\mathcal{H}}((p + kq)_+) + \zeta(q)\bar{\mathcal{H}}(-(p + kq)_-) \) and consequently \( \mathcal{H} \) is continuous.

Now, let us take \( p \in \mathbb{R} \) and \( q, y \in \mathcal{Q} \). We have

\[
\begin{align*}
|\mathcal{H}(p, q) - \mathcal{H}(p, y)| & \leq |\zeta(-q)\bar{\mathcal{H}}((p + kq)_+) - \zeta(-y)\bar{\mathcal{H}}((p + ky)_+)| \\
& + |\zeta(q)\bar{\mathcal{H}}(-(p + kq)_-) - \zeta(y)\bar{\mathcal{H}}(-(p + ky)_-)| \\
& \leq |(\zeta(-q) - \zeta(-y))\bar{\mathcal{H}}((p + kq)_+) + \bar{\mathcal{H}}((p + kq)_+) - \bar{\mathcal{H}}((p +ky)_+)| \\
& + |(\zeta(q) - \zeta(y))\bar{\mathcal{H}}(-(p + kq)_-) + \bar{\mathcal{H}}(-(p + kq)_-) - \bar{\mathcal{H}}(-(p + ky)_-)| \\
& \leq L_\zeta|q - y| [(\bar{\mathcal{H}}((p + kq)_+) + \bar{\mathcal{H}}(-(p + kq)_-)) \\
& + |\bar{\mathcal{H}}((p + kq)_+) - \bar{\mathcal{H}}((p + ky)_+)| + |\bar{\mathcal{H}}(-(p + ky)_-) - \bar{\mathcal{H}}(-(p + kq)_-)|]
\end{align*}
\]

As \( \bar{\mathcal{H}} \) is Lipschitz with Lipschitz constant \( v_\infty \) and \( \bar{\mathcal{H}}(0) = 0 \), we get

\[
|\mathcal{H}(p, q) - \mathcal{H}(p, y)| \leq L_\zeta|q - y|v_\infty(|p| + k|q|) + 2v_\infty k|q - y| \\
\leq C_\mathcal{H}(1 + |p|)|q - y|,
\]

for \( C_\mathcal{H} = \max(L_\zeta v_\infty, L_\zeta v_\infty k\bar{q} + 2v_\infty k) \).
3.2 Existence result

We denote by $C^1 := C^1([0,T) \times \mathbb{R})$ the class of functions $\varphi : [0,T) \times \mathbb{R} \to \mathbb{R}$ that are continuously differentiable on $[0,T) \times \mathbb{R}$.

**Definition 1.** (i) If $u$ is an upper semicontinuous (USC) function on $[0,T) \times \mathbb{Q}$, we say that $u$ is a viscosity subsolution to $\mathcal{H}$ on $[0,T) \times \mathbb{Q}$ if $\forall (\bar{t}, \bar{q}) \in [0,T) \times \mathbb{Q}$, $\forall \varphi \in C^1$ such that $(u - \varphi)(\bar{t}, \bar{q}) = \max_{(t,q) \in [0,T) \times \mathbb{Q}} (u - \varphi)(t,q)$, we have:

$$
- \frac{\partial \varphi}{\partial t}(\bar{t}, \bar{q}) + \psi(\bar{q}) - \int_{\mathbb{R}^*_+} \mathbb{1}_{\{q+z \in \mathbb{Q}\}} zH^b\left( \frac{\varphi(\bar{t}, \bar{q}) - \varphi(\bar{t}, \bar{q} + z)}{z} \right) \mu^b(dz)
- \int_{\mathbb{R}^*_+} \mathbb{1}_{\{q-z \in \mathbb{Q}\}} zH^a\left( \frac{\varphi(\bar{t}, \bar{q}) - \varphi(\bar{t}, \bar{q} - z)}{z} \right) \mu^a(dz) - \mathcal{H}(\partial_\varphi(\bar{t}, \bar{q}), \bar{q}) \leq 0.
$$

(ii) If $v$ is a lower semicontinuous (LSC) function on $[0,T) \times \mathbb{Q}$, we say that $v$ is a viscosity supersolution to $\mathcal{H}$ on $[0,T) \times \mathbb{Q}$ if $\forall (\bar{t}, \bar{q}) \in [0,T) \times \mathbb{Q}$, $\forall \varphi \in C^1$ such that $(v - \varphi)(\bar{t}, \bar{q}) = \min_{(t,q) \in [0,T) \times \mathbb{Q}} (v - \varphi)(t,q)$, we have:

$$
- \frac{\partial \varphi}{\partial t}(\bar{t}, \bar{q}) + \psi(\bar{q}) - \int_{\mathbb{R}^*_+} \mathbb{1}_{\{q+z \in \mathbb{Q}\}} zH^b\left( \frac{\varphi(\bar{t}, \bar{q}) - \varphi(\bar{t}, \bar{q} + z)}{z} \right) \mu^b(dz)
- \int_{\mathbb{R}^*_+} \mathbb{1}_{\{q-z \in \mathbb{Q}\}} zH^a\left( \frac{\varphi(\bar{t}, \bar{q}) - \varphi(\bar{t}, \bar{q} - z)}{z} \right) \mu^a(dz) - \mathcal{H}(\partial_\varphi(\bar{t}, \bar{q}), \bar{q}) \geq 0.
$$

(iii) If $\theta$ is locally bounded on $[0,T) \times \mathbb{Q}$, we say that $\theta$ is a viscosity solution to $\mathcal{H}$ on $[0,T) \times \mathbb{Q}$ if its upper semicontinuous envelope $\theta^*$ and its lower semicontinuous envelope $\theta_*$ are respectively subsolution on $[0,T) \times \mathbb{Q}$ and supersolution on $[0,T) \times \mathbb{Q}$ to $\mathcal{H}$.

The following result is proved in the appendix:

**Proposition 2.** (i) Let $u$ be a USC function on $[0,T) \times \mathbb{Q}$, $u$ is a viscosity subsolution to $\mathcal{H}$ on $[0,T) \times \mathbb{Q}$ if and only if $\forall (\bar{t}, \bar{q}) \in [0,T) \times \mathbb{Q}$, $\forall \varphi \in C^1$ such that $\max_{(t,q) \in [0,T) \times \mathbb{Q}} (u - \varphi)(t,q) = (u - \varphi)(\bar{t}, \bar{q})$, we have

$$
- \frac{\partial \varphi}{\partial t}(\bar{t}, \bar{q}) + \psi(\bar{q}) - \int_{\mathbb{R}^*_+} \mathbb{1}_{\{q+z \in \mathbb{Q}\}} zH^b\left( \frac{u(\bar{t}, \bar{q}) - u(\bar{t}, \bar{q} + z)}{z} \right) \mu^b(dz)
- \int_{\mathbb{R}^*_+} \mathbb{1}_{\{q-z \in \mathbb{Q}\}} zH^a\left( \frac{u(\bar{t}, \bar{q}) - u(\bar{t}, \bar{q} - z)}{z} \right) \mu^a(dz) - \mathcal{H}(\partial_\varphi(\bar{t}, \bar{q}), \bar{q}) \leq 0.
$$

(ii) Let $v$ be a LSC function on $[0,T) \times \mathbb{Q}$, $v$ is a viscosity supersolution to $\mathcal{H}$ on $[0,T) \times \mathbb{Q}$ if and only if $\forall (\bar{t}, \bar{q}) \in [0,T) \times \mathbb{Q}$, $\forall \varphi \in C^1$ such that $\min_{(t,q) \in [0,T) \times \mathbb{Q}} (v - \varphi)(t,q) = (v - \varphi)(\bar{t}, \bar{q})$, we have

$$
- \frac{\partial \varphi}{\partial t}(\bar{t}, \bar{q}) + \psi(\bar{q}) - \int_{\mathbb{R}^*_+} \mathbb{1}_{\{q+z \in \mathbb{Q}\}} zH^b\left( \frac{v(\bar{t}, \bar{q}) - v(\bar{t}, \bar{q} + z)}{z} \right) \mu^b(dz)
- \int_{\mathbb{R}^*_+} \mathbb{1}_{\{q-z \in \mathbb{Q}\}} zH^a\left( \frac{v(\bar{t}, \bar{q}) - v(\bar{t}, \bar{q} - z)}{z} \right) \mu^a(dz) - \mathcal{H}(\partial_\varphi(\bar{t}, \bar{q}), \bar{q}) \geq 0.
$$

We can now prove that $\theta$ is a viscosity solution to $\mathcal{H}$.

**Proposition 3.** $\theta$ is a viscosity subsolution to $\mathcal{H}$ on $[0,T) \times \mathbb{Q}$. 

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Proof. $\theta$ is bounded on $[0, T] \times Q$ so we can define $\theta^*$ its upper semicontinuous envelope.

Let $(\bar{t}, \bar{q}) \in [0, T) \times Q$ and $\varphi \in C^1$ such that

$$0 = (\theta^* - \varphi)(\bar{t}, \bar{q}) = \max_{(t,q) \in [0, T) \times Q} (\theta^* - \varphi)(t,q).$$

We can classically assume this maximum to be strict. By definition of $\theta^*(\bar{t}, \bar{q})$, their exists $(t_m, q_m)_m$ a sequence of $[0, T) \times Q$ such that

$$(t_m, q_m) \underset{m \to +\infty}{\to} (\bar{t}, \bar{q}),$$

$$\theta(t_m, q_m) \underset{m \to +\infty}{\to} \theta^*(\bar{t}, \bar{q}).$$

We prove the result by contradiction. Assume there exists $\eta > 0$ such that

$$-\frac{\partial \varphi}{\partial t}(\bar{t}, \bar{q}) + \psi(\bar{q}) - \int_{R^+} 1_{\{\bar{q}+z \in Q\}} z H^b \left( \frac{\varphi(\bar{t}, \bar{q}) - \varphi(\bar{t}, \bar{q}+z)}{z} \right) \mu^b(dz)$$

$$\quad - \int_{R^+} 1_{\{\bar{q}-z \in Q\}} z H^a \left( \frac{\varphi(\bar{t}, \bar{q}) - \varphi(\bar{t}, \bar{q}-z)}{z} \right) \mu^a(dz) - \mathcal{H}(\partial_q \varphi(\bar{t}, \bar{q}), \bar{q}) > 2\eta. $$

Then, as $\varphi$ is continuously differentiable and $\mu^b$ and $\mu^a$ absolutely continuous with respect to the Lebesgue measure, we must have

$$-\frac{\partial \varphi}{\partial t}(t, q) + \psi(q) - \int_{R^+} 1_{\{q+z \in Q\}} z H^b \left( \frac{\varphi(t, q) - \varphi(t, q+z)}{z} \right) \mu^b(dz)$$

$$\quad - \int_{R^+} 1_{\{q-z \in Q\}} z H^a \left( \frac{\varphi(t, q) - \varphi(t, q-z)}{z} \right) \mu^a(dz) - \mathcal{H}(\partial_q \varphi(t, q), q) \geq 0$$

on $B := ((\bar{t} - r, \bar{t} + r) \cap [0, T)) \times ((\bar{q} - r, \bar{q} + r) \cap Q)$ for a sufficiently small $r \in (0, T - \bar{t})$. Without loss of generality, we can assume that $B$ contains the sequence $(t_m, q_m)_m$.

Then, by potentially reducing the value of $\eta$, we have

$$\theta \leq \theta^* \leq \varphi - \eta $$

on the parabolic boundary $\partial_p B$ of $B$, i.e.

$$\partial_p B = \left( ((\bar{t} - r, \bar{t} + r) \cap [0, T)) \times ((\bar{q} - r, \bar{q} + r) \cap Q) \right) \cup \left( \{\bar{t} + r\} \times (\bar{q} - r, \bar{q} + r) \cap Q \right).$$

Without loss of generality we can assume that the above inequality holds on

$$\tilde{B} := \{(t, q+z) \mid (t, q, z) \in B \times \mathbb{R}, \ q + z \in (\bar{q} - r, \bar{q} + r) \cap Q \},$$

which is also bounded.

We introduce an arbitrary control $\delta = (\delta^b, \delta^a) \in A_M$. We also introduce an arbitrary control $v \in A_T$. We denote by $\pi_m$ the first exit time of $(t, q^m_t)_{t \geq t_m}$ from $B$ (where $q^m_t := q^m_{t_m, q^m, \delta^b, v}$):

$$\pi_m = \inf \{t \geq t_m \mid (t, q^m_t) \notin B \}.$$
By Itô’s formula,

\[
\varphi(\pi_m, q_{\pi_m}^m) = \varphi(t_m, q_m) + \int_{t_m}^{\pi_m} \frac{\partial \varphi}{\partial t}(s, q_{s-}^m) ds
\]

\[
+ \int_{t_m}^{\pi_m} \int_{\mathbb{R}^+} 1_{\{q_{s-}^m + z \in \mathcal{Q}\}} \Lambda^b(\delta^b(s, z)) \left( \varphi(s, q_{s-}^m + z) - \varphi(s, q_{s-}^m) \right) \mu^b(dz) ds
\]

\[
+ \int_{t_m}^{\pi_m} \int_{\mathbb{R}^+} 1_{\{q_{s-}^m - z \in \mathcal{Q}\}} \Lambda^a(\delta^a(s, z)) \left( \varphi(s, q_{s-}^m - z) - \varphi(s, q_{s-}^m) \right) \mu^a(dz) ds
\]

\[
+ \int_{t_m}^{\pi_m} (\varphi(s, q_{s-}^m + z) - \varphi(s, q_{s-}^m)) \tilde{J}^b(ds, dz)
\]

\[
+ \int_{t_m}^{\pi_m} (\varphi(s, q_{s-}^m - z) - \varphi(s, q_{s-}^m)) \tilde{J}^a(ds, dz)
\]

\[
+ \int_{t_m}^{\pi_m} \partial_q \varphi(s, q_{s-}^m) \left( -(v_s - \zeta(q_{s-}^m)) + (v_s) + \zeta(-q_{s-}^\mu) \right) ds
\]

which we can write

\[
\varphi(\pi_m, q_{\pi_m}^m) = \varphi(t_m, q_m) + \int_{t_m}^{\pi_m} \left\{ \frac{\partial \varphi}{\partial t}(s, q_{s-}^m)ight.
\]

\[
+ \int_{\mathbb{R}^+} 1_{\{q_{s-}^m + z \in \mathcal{Q}\}} \Lambda^b(\delta^b(s, z)) \left( \delta^b(s, z) - \frac{\varphi(s, q_{s-}^m) - \varphi(s, q_{s-}^m + z)}{z} \right) \mu^b(dz)
\]

\[
+ \int_{\mathbb{R}^+} 1_{\{q_{s-}^m - z \in \mathcal{Q}\}} \Lambda^a(\delta^a(s, z)) \left( \delta^a(s, z) - \frac{\varphi(s, q_{s-}^m) - \varphi(s, q_{s-}^m + z)}{z} \right) \mu^a(dz)
\]

\[
+ (v_s - \zeta(q_{s-}^m)) + (v_s) + \zeta(-q_{s-}^m) \left( \partial_q \varphi(s, q_{s-}^m) + kq_{s-}^m \right) - \tilde{\mathcal{L}}(v_s, q_{s-}^m) - \psi(q_{s-}^m)
\left\} ds
\]

\[
+ \int_{t_m}^{\pi_m} \left( \psi(q_{s-}^m) - k \left( -(v_s - \zeta(q_{s-}^m) + (v_s) + \zeta(-q_{s-}^m)) q_{s-}^m - \int_{\mathbb{R}^+} 1_{\{q_{s-}^m + z \in \mathcal{Q}\}} \Lambda^b(\delta^b(s, z)) \delta^b(s, z) \mu^b(dz)
\right.
\]

\[
- \int_{\mathbb{R}^+} 1_{\{q_{s-}^m - z \in \mathcal{Q}\}} \Lambda^a(\delta^a(s, z)) \delta^a(s, z) \mu^a(dz) + \tilde{\mathcal{L}}(v_s, q_{s-}^m)
\left\} ds
\]

\[
+ \int_{t_m}^{\pi_m} (\varphi(s, q_{s-}^m + z) - \varphi(s, q_{s-}^m)) \tilde{J}^b(ds, dz)
\]

\[
+ \int_{t_m}^{\pi_m} (\varphi(s, q_{s-}^m - z) - \varphi(s, q_{s-}^m)) \tilde{J}^a(ds, dz).
\]
From (9), and by definition of $H^b$, $H^a$, and $\mathcal{H}$, we then get

$$\varphi(\pi_m, q_{\pi_m}^m) \leq \varphi(t_m, q_{t_m}^m) + \int_{t_m}^{\pi_m} \left\{ \psi(q_{s_r}^m) - k (-(v_s) - \zeta(q_{s_r}^m) + (v_s) + \zeta(-q_{s_r}^m)) q_{s_r}^m \right. \right.$$

$$- \int_{\mathbb{R}^+} \mathbb{1}_{(q_{s_r}^m + z \in \mathcal{Q})} \bar{z} \Lambda^b(\delta^b(s, z)) \delta^b(s, z) \mu^b(dz)$$

$$- \int_{\mathbb{R}^+} \mathbb{1}_{(q_{s_r}^m - z \in \mathcal{Q})} \bar{z} \Lambda^a(\delta^a(s, z)) \delta^a(s, z) \mu^a(dz)$$

$$+ \int_{\mathbb{R}^+} \left. \varphi(s, q_{s_r}^m - z) - \varphi(s, q_{s_r}^m) \right\} \bar{J}^b(ds, dz)$$

$$+ \int_{\mathbb{R}^+} \left. \varphi(s, q_{s_r}^m + z) - \varphi(s, q_{s_r}^m) \right\} \bar{J}^a(ds, dz).$$

The last two terms have expectations equal to zero and we obtain

$$\varphi(t_m, q_{t_m}) \geq \mathbb{E}\left[ \varphi(\pi_m, q_{\pi_m}^m) + \int_{t_m}^{\pi_m} \left\{ \int_{\mathbb{R}^+} \mathbb{1}_{(q_{s_r}^m + z \in \mathcal{Q})} \bar{z} \Lambda^b(\delta^b(s, z)) \delta^b(s, z) \mu^b(dz) \right. \right.$$

$$+ \int_{\mathbb{R}^+} \mathbb{1}_{(q_{s_r}^m - z \in \mathcal{Q})} \bar{z} \Lambda^a(\delta^a(s, z)) \delta^a(s, z) \mu^a(dz)$$

$$+ kq_{s_r}^m (-(v_s) - \zeta(q_{s_r}^m) + (v_s) + \zeta(-q_{s_r}^m)) - \bar{L}(v_s, q_{s_r}^m) - \psi(q_{s_r}^m) \right\} ds \right].$$

Therefore

$$\varphi(t_m, q_{t_m}) \geq \eta \mathbb{E}\left[ \theta(\pi_m, q_{\pi_m}^m) + \int_{t_m}^{\pi_m} \left\{ \int_{\mathbb{R}^+} \mathbb{1}_{(q_{s_r}^m + z \in \mathcal{Q})} \bar{z} \Lambda^b(\delta^b(s, z)) \delta^b(s, z) \mu^b(dz) \right. \right.$$

$$+ \int_{\mathbb{R}^+} \mathbb{1}_{(q_{s_r}^m - z \in \mathcal{Q})} \bar{z} \Lambda^a(\delta^a(s, z)) \delta^a(s, z) \mu^a(dz)$$

$$+ kq_{s_r}^m (-(v_s) - \zeta(q_{s_r}^m) + (v_s) + \zeta(-q_{s_r}^m)) - \bar{L}(v_s, q_{s_r}^m) - \psi(q_{s_r}^m) \right\} ds \right].$$

As $\varphi(t_m, q_{t_m}) \xrightarrow{m\to+\infty} \varphi(\bar{t}, \bar{q}) = \theta^*(\bar{t}, \bar{q})$ and $\theta(t_m, q_{t_m}) \xrightarrow{m\to+\infty} \theta^*(\bar{t}, \bar{q})$, we have for $m$ large enough the inequality $\theta(t_m, q_{t_m}) + \frac{\pi}{2} \geq \varphi(t_m, q_{t_m})$, from which we deduce

$$\theta(t_m, q_{t_m}) \geq \frac{\pi}{2} \mathbb{E}\left[ \theta(\pi_m, q_{\pi_m}^m) + \int_{t_m}^{\pi_m} \left\{ \int_{\mathbb{R}^+} \mathbb{1}_{(q_{s_r}^m + z \in \mathcal{Q})} \bar{z} \Lambda^b(\delta^b(s, z)) \delta^b(s, z) \mu^b(dz) \right. \right.$$

$$+ \int_{\mathbb{R}^+} \mathbb{1}_{(q_{s_r}^m - z \in \mathcal{Q})} \bar{z} \Lambda^a(\delta^a(s, z)) \delta^a(s, z) \mu^a(dz)$$

$$+ kq_{s_r}^m (-(v_s) - \zeta(q_{s_r}^m) + (v_s) + \zeta(-q_{s_r}^m)) - \bar{L}(v_s, q_{s_r}^m) - \psi(q_{s_r}^m) \right\} ds \right].$$

By taking the supremum over all the controls in $\mathcal{A}$ on the right-hand side, we contradict the dynamic programming principle.
Necessarily, we deduce:

$$-rac{\partial \varphi}{\partial t}(\bar{t}, \bar{q}) + \psi(\bar{q}) - \int_{\mathbb{R}_+^*} 1_{\{q + z \in \mathcal{Q}\}} z H^b \left( \frac{\varphi(\bar{t}, \bar{q}) - \varphi(\bar{t}, \bar{q} + z)}{z} \right) \mu^b(dz)$$

$$- \int_{\mathbb{R}_+^*} 1_{\{q - z \in \mathcal{Q}\}} z H^a \left( \frac{\varphi(\bar{t}, \bar{q}) - \varphi(\bar{t}, \bar{q} - z)}{z} \right) \mu^a(dz) - \mathcal{H}(\partial_q \varphi(\bar{t}, \bar{q}), \bar{q}) \leq 0,$$

and $\theta$ is a viscosity subsolution to $[HJ]$ on $[0, T) \times \mathcal{Q}$.

**Proposition 4.** $\theta$ is a viscosity supersolution to $[HJ]$ on $[0, T) \times \mathcal{Q}$.

**Proof.** $\theta$ is bounded on $[0, T) \times \mathcal{Q}$, so we can define $\theta_*$ its lower semicontinuous envelope.

Let $(\bar{t}, \bar{q}) \in [0, T) \times \mathcal{Q}$ and $\varphi \in C^1$ such that

$$0 = (\theta_* - \varphi)(\bar{t}, \bar{q}) = \min_{(t,q) \in [0,T) \times \mathcal{Q}} (\theta_* - \varphi)(t, q).$$

We can assume this minimum to be strict. By definition of $\theta_*(\bar{t}, \bar{q})$, there exists $(m, q_m)_m$ a sequence of $[0, T) \times \mathcal{Q}$ such that

$$(m, q_m) \xrightarrow{m \to +\infty} (\bar{t}, \bar{q}),$$

$$\theta(m, q_m) \xrightarrow{m \to +\infty} \theta_*(\bar{t}, \bar{q}).$$

Let us prove the proposition by contradiction. Assume there is $\eta > 0$ such that

$$-rac{\partial \varphi}{\partial t}(\bar{t}, \bar{q}) + \psi(\bar{q}) - \int_{\mathbb{R}_+^*} 1_{\{q + z \in \mathcal{Q}\}} z H^b \left( \frac{\varphi(\bar{t}, \bar{q}) - \varphi(\bar{t}, \bar{q} + z)}{z} \right) \mu^b(dz)$$

$$- \int_{\mathbb{R}_+^*} 1_{\{q - z \in \mathcal{Q}\}} z H^a \left( \frac{\varphi(\bar{t}, \bar{q}) - \varphi(\bar{t}, \bar{q} - z)}{z} \right) \mu^a(dz) - \mathcal{H}(\partial_q \varphi(\bar{t}, \bar{q}), \bar{q}) < -2\eta.$$

Then, as $\varphi$ is continuously differentiable and $\mu^b$ and $\mu^a$ absolutely continuous with respect to the Lebesgue measure, we must have

$$-rac{\partial \varphi}{\partial t}(t, q) + \psi(q) - \int_{\mathbb{R}_+^*} 1_{\{q + z \in \mathcal{Q}\}} z H^b \left( \frac{\varphi(t, q) - \varphi(t, q + z)}{z} \right) \mu^b(dz)$$

$$- \int_{\mathbb{R}_+^*} 1_{\{q - z \in \mathcal{Q}\}} z H^a \left( \frac{\varphi(t, q) - \varphi(t, q - z)}{z} \right) \mu^a(dz) - \mathcal{H}(\partial_q \varphi(t, q), q) \leq 0 \quad (10)$$

on $B := ((\bar{t} - r, \bar{t} + r) \cap [0, T)) \times ((\bar{q} - r, \bar{q} + r) \cap \mathcal{Q})$ for a sufficiently small $r \in (0, T - \bar{t})$. Without loss of generality, we can assume that $B$ contains the sequence $(m, q_m)_m$.

Then, by potentially reducing $\eta$, we have

$$\theta \geq \theta_* \geq \varphi + \eta$$

on $\partial_p B$. We can also without loss of generality assume that this inequality is true on

$$\hat{B} := \{(t, q + z) \mid (t, q, z) \in B \times \mathbb{R} \}, \quad q + z \in (\bar{q} - r, \bar{q} + r) \cap \mathcal{Q},$$

which is also bounded.

We introduce the control $\delta = (\delta^b, \delta^a) \in \mathcal{A}_M$ such that $\forall t \geq t_m, \forall z \in \mathbb{R}_+^*$,

$$\delta^b(t, z) = \delta^{b*} \left( \frac{\varphi(t, q^m_t) - \varphi(t, q^m_t + z)}{z} \right) \quad \text{and} \quad \delta^a(t, z) = \delta^{a*} \left( \frac{\varphi(t, q^m_t) - \varphi(t, q^m_t - z)}{z} \right),$$

where

$$\delta^{b*} \left( \frac{\varphi(t, q^m_t) - \varphi(t, q^m_t + z)}{z} \right) \quad \text{and} \quad \delta^{a*} \left( \frac{\varphi(t, q^m_t) - \varphi(t, q^m_t - z)}{z} \right).$$
where $\delta^b$ and $\delta^a$ are defined in Lemma 1. Similarly, we introduce the control $v \in A_T$ such that $\forall t \geq t_m$, 

$$v_t = v^*(\partial_q \varphi(t, q^m_t, q^m_t),$$

where $v^*$ is defined in Lemma 3. As before, we denote by $\pi_m$ the first exit time of $(t, q^m_t)_{t \geq t_m}$ from $B$ (where $q^m_t := q^m_{t_m, q^m_{t_m}, q^m_t}$). By Itô’s lemma, we obtain

$$\varphi(\pi_m, q^m_{\pi_m}) = \varphi(t_m, q_m) + \int_{t_m}^{\pi_m} \frac{\partial \varphi}{\partial t}(s, q^m_s) \, ds +$$

$$+ \int_{t_m}^{\pi_m} \int_{\mathbb{R}^*_+} \mathbb{1}_{\{q^m_z + \pi \in \mathcal{Q}\}} \Lambda^b(\delta^b(s, z)) \left( \varphi(s, q^m_{s-} + z) - \varphi(s, q^m_{s-}) \right) \mu^b(dz) \, ds$$

$$+ \int_{t_m}^{\pi_m} \int_{\mathbb{R}^*_+} \mathbb{1}_{\{q^m_z - \pi \in \mathcal{Q}\}} \Lambda^a(\delta^a(s, z)) \left( \varphi(s, q^m_{s-} - z) - \varphi(s, q^m_{s-}) \right) \mu^a(dz) \, ds$$

$$+ \int_{t_m}^{\pi_m} \left( \varphi(s, q^m_{s-} + z) - \varphi(s, q^m_{s-}) \right) \tilde{J}^b(ds, dz)$$

$$+ \int_{t_m}^{\pi_m} \sum_{i \in I_m} \int_{\mathbb{R}^*_+} \left( \varphi(s, q^m_{s-} - z) - \varphi(s, q^m_{s-}) \right) \tilde{J}^a(ds, dz)$$

$$+ \int_{t_m}^{\pi_m} \partial_q \varphi(s, q^m_{s-}) \left( -(v_s) - \zeta(q^m_{s-}) + (v_s) + \zeta(-q^m_{s-}) \right) \, ds,$$

which we can write

$$\varphi(\pi_m, q^m_{\pi_m}) = \varphi(t_m, q_m) + \int_{t_m}^{\pi_m} \left\{ \frac{\partial \varphi}{\partial t}(s, q^m_s) +$$

$$+ \int_{\mathbb{R}^*_+} \mathbb{1}_{\{q^m_z + \pi \in \mathcal{Q}\}} \Lambda^b(\delta^b(s, z)) \left( \delta^b(s, z) - \frac{\varphi(s, q^m_{s-} + z) - \varphi(s, q^m_{s-})}{z} \right) \mu^b(dz)$$

$$+ \int_{\mathbb{R}^*_+} \mathbb{1}_{\{q^m_z - \pi \in \mathcal{Q}\}} \Lambda^a(\delta^a(s, z)) \left( \delta^a(s, z) - \frac{\varphi(s, q^m_{s-} - z) - \varphi(s, q^m_{s-})}{z} \right) \mu^a(dz)$$

$$+ \left( -(v_s) - \zeta(q^m_{s-}) + (v_s) + \zeta(-q^m_{s-}) \right) \left( \partial_q \varphi(s, q^m_{s-}) + kq_{s-} \right) - \tilde{L}(v_s, q^m_{s-}) - \psi(q^m_{s-}) \right\} \, ds$$

$$+ \int_{t_m}^{\pi_m} \left\{ \psi(q^m_{s-}) - kq_{s-} \left( -(v_s) - \zeta(q^m_{s-}) + (v_s) + \zeta(-q^m_{s-}) \right) \right\} ds$$

$$- \int_{\mathbb{R}^*_+} \mathbb{1}_{\{q^m_z + \pi \in \mathcal{Q}\}} \Lambda^b(\delta^b(s, z)) \delta^b(s, z) \mu^b(dz)$$

$$- \int_{\mathbb{R}^*_+} \mathbb{1}_{\{q^m_z - \pi \in \mathcal{Q}\}} \Lambda^a(\delta^a(s, z)) \delta^a(s, z) \mu^a(dz) + \tilde{L}(v_s, q^m_{s-}) \right\} \, ds$$

$$+ \int_{t_m}^{\pi_m} \int_{\mathbb{R}^*_+} \left( \varphi(s, q^m_{s-} + z) - \varphi(s, q^m_{s-}) \right) \tilde{J}^b(ds, dz)$$

$$+ \int_{t_m}^{\pi_m} \int_{\mathbb{R}^*_+} \left( \varphi(s, q^m_{s-} - z) - \varphi(s, q^m_{s-}) \right) \tilde{J}^a(ds, dz).$$
By (10), we then get
\[ \varphi(\pi, q^m_{\pi}) \geq \varphi(t_m, q_m) + \int_{t_m}^{\pi_m} \left\{ \psi(q^m_{\pi}) - kq^m_{s} \left( -(v_s) - \zeta(q^m_{s}) + (v_s) + \zeta(-q^m_{s}) \right) \right\} \]
\[ - \left( \int_{\mathbb{R}_+^m} 1_{(q^m_{s} + z \in \mathcal{Q})} z\Lambda^{b}(\delta^{b}(s, z))\delta^b(s, z)\mu^b(dz) \right) \]
\[ - \left( \int_{\mathbb{R}_+^m} 1_{(q^m_{s} - z \in \mathcal{Q})} z\Lambda^{a}(\delta^{a}(s, z))\delta^a(s, z)\mu^a(dz) \right) + \tilde{\mathcal{L}}(v_s, q^m_{s}) \right\} ds \]
\[ + \int_{t_m}^{\pi_m} \left( \varphi(s, q^m_{s} + z) - \varphi(s, q^m_{s}) \right) J^b(ds, dz) \]
\[ + \int_{t_m}^{\pi_m} \left( \varphi(s, q^m_{s} - z) - \varphi(s, q^m_{s}) \right) J^a(ds, dz) . \]

The last two terms have expectations equal to zero and we obtain
\[ \varphi(t_m, q_m) \leq \mathbb{E} \left[ \varphi(\pi, q^m_{\pi}) + \int_{t_m}^{\pi_m} \left\{ \int_{\mathbb{R}_+^m} 1_{(q^m_{s} + z \in \mathcal{Q})} z\Lambda^{b}(\delta^{b}(s, z))\delta^b(s, z)\mu^b(dz) \right\} \right] \]
\[ + \int_{\mathbb{R}_+^m} 1_{(q^m_{s} - z \in \mathcal{Q})} z\Lambda^{a}(\delta^{a}(s, z))\delta^a(s, z)\mu^a(dz) \]
\[ + kq^m_{s} \left( -(v_s) - \zeta(q^m_{s}) + (v_s) + \zeta(-q^m_{s}) \right) - \tilde{\mathcal{L}}(v_s, q^m_{s}) - \psi(q^m_{s}) \right\} ds \].

Therefore,
\[ \varphi(\pi, q^m_{\pi}) \leq -\eta + \mathbb{E} \left[ \varphi(\pi, q^m_{\pi}) + \int_{t_m}^{\pi_m} \left\{ \int_{\mathbb{R}_+^m} 1_{(q^m_{s} + z \in \mathcal{Q})} z\Lambda^{b}(\delta^{b}(s, z))\delta^b(s, z)\mu^b(dz) \right\} \right] \]
\[ + \int_{\mathbb{R}_+^m} 1_{(q^m_{s} - z \in \mathcal{Q})} z\Lambda^{a}(\delta^{a}(s, z))\delta^a(s, z)\mu^a(dz) \]
\[ + kq^m_{s} \left( -(v_s) - \zeta(q^m_{s}) + (v_s) + \zeta(-q^m_{s}) \right) - \tilde{\mathcal{L}}(v_s, q^m_{s}) - \psi(q^m_{s}) \right\} ds \].

As \( \varphi(t_m, q_m) \xrightarrow{m \to +\infty} \varphi(\bar{t}, \bar{q}) = \theta_*(\bar{t}, \bar{q}) \) and moreover \( \theta(t_m, q_m) \xrightarrow{m \to +\infty} \theta_*(\bar{t}, \bar{q}) \), we have that for \( m \) sufficiently large, \( \theta(t_m, q_m) - \frac{\eta}{2} \leq \varphi(t_m, q_m) \) and we deduce:
\[ \theta(t_m, q_m) < \mathbb{E} \left[ \theta(\pi, q^m_{\pi}) + \int_{t_m}^{\pi_m} \left\{ \int_{\mathbb{R}_+^m} 1_{(q^m_{s} + z \in \mathcal{Q})} z\Lambda^{b}(\delta^{b}(s, z))\delta^b(s, z)\mu^b(dz) \right\} \right] \]
\[ + \int_{\mathbb{R}_+^m} 1_{(q^m_{s} - z \in \mathcal{Q})} z\Lambda^{a}(\delta^{a}(s, z))\delta^a(s, z)\mu^a(dz) \]
\[ + kq^m_{s} \left( -(v_s) - \zeta(q^m_{s}) + (v_s) + \zeta(-q^m_{s}) \right) - \tilde{\mathcal{L}}(v_s, q^m_{s}) - \psi(q^m_{s}) \right\} ds , \]

which contradicts the dynamic programming principle.
In conclusion, we necessarily have
\[
- \frac{\partial \varphi}{\partial t}(t, \tilde{q}) + \psi(\tilde{q}) - \int_{\mathbb{R}_+^n} 1_{(\tilde{q} + z \in \mathcal{Q})} z H^b \left( \varphi(t, \tilde{q}) - \varphi(t, \tilde{q} + z) \right) \mu^b(dz) \\
- \int_{\mathbb{R}_+^n} 1_{(\tilde{q} - z \in \mathcal{Q})} z H^a \left( \varphi(t, \tilde{q}) - \varphi(t, \tilde{q} - z) \right) \mu^a(dz) - \mathcal{H}(\partial_q \varphi(t, \tilde{q}), \tilde{q}) \geq 0,
\]
and $\theta$ is a viscosity supersolution to $[HJ]$ on $[0, T) \times \mathcal{Q}$.

\[\square\]

**Proposition 5.** $\forall q \in \mathcal{Q}$, we have $\theta_*(T, q) = \theta^*(T, q) = -\ell(q)$.

**Proof.** Let $q \in \mathcal{Q}$ and let us take $(t_m, q_m)_{m \in \mathbb{N}}$ a sequence of $[0, T] \times \mathcal{Q}$ such that

\[
(t_m, q_m) \underset{m \to +\infty}{\longrightarrow} (T, q) \quad \text{and} \quad \theta(t_m, q_m) \underset{m \to +\infty}{\longrightarrow} \theta_*(T, q).
\]

We introduce arbitrary controls $\delta = (\delta^b, \delta^a) \in \mathcal{A}_M$. We also introduce an arbitrary control $v \in \mathcal{A}_T$. Then we have for all $m \in \mathbb{N}$, by denoting $q^m_{t_m} = q^{t_m, q_m, \delta, v}$ for all $t \in [t_m, T]$:

\[
\theta(t_m, q_m) \geq \mathbb{E} \left[ \int_{t_m}^T \left\{ \int_{\mathbb{R}_+^n} \left( z \delta^b(s, z) 1_{\{q^m_{t_m} + z \in \mathcal{Q}\}} \Lambda^b(\delta^b(s, z)) \mu^b(dz) \\
+ z \delta^a(s, z) 1_{\{q^m_{t_m} - z \in \mathcal{Q}\}} \Lambda^a(\delta^a(s, z)) \mu^a(dz) \right) \right. \\
+ k q^m_{t_m} (-v_s + \zeta(q^m_{t_m}) + (v_s + \zeta(-q^m_{t_m}))) - \hat{\mathcal{L}}(v_s, q^m_{t_m}) - \psi(q^m_{t_m}) \right\} ds - \ell(q^m_{t_m}) \right]
\]

But,

\[
\left| \int_{t_m}^T \left\{ \int_{\mathbb{R}_+^n} \left( z \delta^b(s, z) 1_{\{q^m_{t_m} + z \in \mathcal{Q}\}} \Lambda^b(\delta^b(s, z)) \mu^b(dz) + z \delta^a(s, z) 1_{\{q^m_{t_m} - z \in \mathcal{Q}\}} \Lambda^a(\delta^a(s, z)) \mu^a(dz) \right) \right\} ds \right|
\leq (T - t_m) \left( \Delta^b H^b(0) + \Delta^a H^a(0) \right),
\]

\[
\left| \int_{t_m}^T k q^m_{t_m} (-v_s + \zeta(q^m_{t_m}) + (v_s + \zeta(-q^m_{t_m}))) ds \right| \leq 2(T - t_m) kv_{\infty} \tilde{q},
\]

\[
\left| \int_{t_m}^T \hat{\mathcal{L}}(v_s, q^m_{t_m}) ds \right| \leq (T - t_m) \left( L(-v_{\infty}) + L(v_{\infty}) \right)
\]

and

\[
\left| \int_{t_m}^T \psi(q^m_{t_m}) ds \right| \leq (T - t_m) \sup_{q \in \mathcal{Q}} \psi(q).
\]

By dominated convergence (because $(q^m_{t_m})_m$ converges in probability towards $q$ and $\ell$ is continuous on the compact set $\mathcal{Q}$), we have $\mathbb{E}[\ell(q^m_{t_m})] \underset{m \to +\infty}{\longrightarrow} \ell(q)$, and therefore $\theta_*(T, q) \geq -\ell(q)$. But as $\theta_*(T, q) \leq \theta(T, q) = -\ell(q)$, we get $\theta_*(T, q) = -\ell(q)$.

The proof for $\theta^*$ is similar, by taking $\varepsilon$-optimal controls and showing that $\theta^*(T, q) - \varepsilon \leq -\ell(q)$ for all $\varepsilon > 0$. \[\square\]
3.3 Uniqueness result

**Theorem 1.** Let \( u \) be a bounded USC subsolution and \( v \) be a bounded LSC supersolution to \( (H) \) on \([0,T) \times \mathcal{Q}\) such that \( u \leq v \) on \( \{T\} \times \mathcal{Q} \). Then \( u \leq v \) on \([0,T) \times \mathcal{Q}\).

**Proof.** We prove it by contradiction. Let us assume \( \sup_{[0,T] \times \mathcal{Q}} (u - v) > 0 \). Then this supremum cannot be reached on \( \{T\} \times \mathcal{Q} \). For \( n \geq 0 \) and \( \varepsilon > 0 \), we introduce:

\[
\phi_{n,\varepsilon}(t, s, q, y) = u(t, q) - v(s, y) - n(q - y)^2 - n(t - s)^2 - \varepsilon(2T - t - s).
\]

We also introduce \((t_{n,\varepsilon}, s_{n,\varepsilon}, q_{n,\varepsilon}, y_{n,\varepsilon})\) such that:

\[
\phi_{n,\varepsilon}(t_{n,\varepsilon}, s_{n,\varepsilon}, q_{n,\varepsilon}, y_{n,\varepsilon}) = \max_{[0,T]^2 \times \mathcal{Q}^2} \phi_{n,\varepsilon}(t, s, q, y).
\]

Then for all \( n \geq 0, \varepsilon > 0 \) and for all \((t, q) \in [0,T] \times \mathcal{Q}\), we have

\[
\phi_{n,\varepsilon}(t_{n,\varepsilon}, s_{n,\varepsilon}, q_{n,\varepsilon}, y_{n,\varepsilon}) \geq \phi_{n,\varepsilon}(t, t, q, q) = u(t, q) - v(t, q) - 2\varepsilon(T - t).
\]

In particular,

\[
\phi_{n,\varepsilon}(t_{n,\varepsilon}, s_{n,\varepsilon}, q_{n,\varepsilon}, y_{n,\varepsilon}) \geq \sup_{[0,T] \times \mathcal{Q}} (u - v) - 2\varepsilon T. \tag{11}
\]

We can now fix \( \varepsilon \) such that

\[
0 < \varepsilon < \frac{\sup_{[0,T] \times \mathcal{Q}} (u - v)}{4T}
\]

to ensure that the right-hand side of \( (11) \) is larger than \( \frac{1}{2} \sup_{[0,T] \times \mathcal{Q}} (u - v) \), which is positive by assumption.

\( \varepsilon \) will remain fixed throughout the rest of the proof and, for ease of notation, we now write \( \phi_n = \phi_{n,\varepsilon}, \ t_n = t_{n,\varepsilon}, \ s_n = s_{n,\varepsilon}, \ q_n = q_{n,\varepsilon}, \) and \( y_n = y_{n,\varepsilon}. \)

From what precedes, we know that the sequence \( (n(q_n - y_n)^2 + n(t_n - s_n)^2) \) is bounded, so necessarily, \(|t_n - s_n| \xrightarrow{n \to +\infty} 0\) and \(|q_n - y_n| \xrightarrow{n \to +\infty} 0\). Then, up to a subsequence, there exist \((\bar{t}, \bar{q}) \in [0,T] \times \mathcal{Q}\) such that

\[
s_n, t_n \xrightarrow{n \to +\infty} \bar{t} \text{ and } q_n, y_n \xrightarrow{n \to +\infty} \bar{q}.
\]

Moreover, we know that

\[
\phi_n(t_n, s_n, q_n, y_n) \geq \phi_n(\bar{t}, \bar{t}, \bar{q}, \bar{q}),
\]

which implies

\[
u(t_n, q_n) - v(s_n, y_n) - n(q_n - y_n)^2 - n(t_n - s_n)^2 - \varepsilon(2T - t_n - s_n) \geq u(\bar{t}, \bar{q}) - v(\bar{t}, \bar{q}) - 2\varepsilon(T - \bar{t}).
\]

Hence we have

\[
n(q_n - y_n)^2 + n(t_n - s_n)^2 \leq u(t_n, q_n) - u(\bar{t}, \bar{q}) + v(\bar{t}, \bar{q}) - v(s_n, y_n) + 2\varepsilon(T - \bar{t}) - \varepsilon(2T - t_n - s_n).
\]

As \( u \) is USC and \( v \) is LSC, the \( \limsup \) when \( n \to +\infty \) of the left-hand side is nonpositive, which implies \( n(q_n - y_n)^2 + n(t_n - s_n)^2 \xrightarrow{n \to +\infty} 0 \).

Let us assume \( \bar{t} = T \). Then we have, as \( u \) is USC and \( v \) is LSC:

\[
\limsup_{n \to +\infty} \phi_n(t_n, s_n, q_n, y_n) \leq \limsup_{n \to +\infty} u(t_n, q_n) - \liminf_{n \to +\infty} v(s_n, y_n) \leq u(T, q) - v(T, q) \leq 0,
\]

which constitutes a contradiction. Necessarily, \( \bar{t} < T \).

Hence, for \( n \) large enough we must have \( t_n, s_n < T \), and we know that \((t_n, q_n)\) is a maximum point of \( u - \varphi_n \) where

\[
\varphi_n(t, q) = v(s_n, y_n) + n(q - y_n)^2 + n(t - s_n)^2 + \varepsilon(2T - t - s_n).
\]
By Proposition 2, we have:
\[
\varepsilon - 2n(t_n - s_n) + \psi(q_n) - \int_{\mathbb{R}_+^n} \mathbf{1}_{(q_n \in \mathbb{Q})} \varepsilon H^b \left( \frac{u(t_n, q_n) - u(t_n, q_n + z)}{z} \right) \mu^b(dz) - \int_{\mathbb{R}_+^n} \mathbf{1}_{(q_n \in \mathbb{Q})} \varepsilon H^a \left( \frac{u(t_n, q_n) - u(t_n, q_n - z)}{z} \right) \mu^a(dz) - H(2n(q_n - y_n), q_n) \leq 0.
\]
Furthermore, \((s_n, y_n)\) is a minimum point of \(v - \xi_n\) where
\[
\xi_n(s, y) = u(t_n, y_n) - n(q_n - y)^2 - n(t_n - s)^2 - \varepsilon(2T - t_n - s),
\]
and by the same argument
\[
-\varepsilon - 2n(t_n - s_n) + \psi(y_n) - \int_{\mathbb{R}_+^n} \mathbf{1}_{(y_n \in \mathbb{Q})} \varepsilon H^b \left( \frac{v(s_n, y_n) - v(s_n, y_n + z)}{z} \right) \mu^b(dz) - \int_{\mathbb{R}_+^n} \mathbf{1}_{(y_n \in \mathbb{Q})} \varepsilon H^a \left( \frac{v(s_n, y_n) - v(s_n, y_n - z)}{z} \right) \mu^a(dz) - H(2n(q_n - y_n), y_n) \geq 0.
\]
Therefore by combining the two inequalities we get
\[
\int_{\mathbb{R}_+^n} \mathbf{1}_{(y_n \in \mathbb{Q})} \varepsilon H^b \left( \frac{v(s_n, y_n) - v(s_n, y_n + z)}{z} \right) - \mathbf{1}_{(q_n \in \mathbb{Q})} \varepsilon H^b \left( \frac{u(t_n, q_n) - u(t_n, q_n + z)}{z} \right) \mu^b(dz) + \int_{\mathbb{R}_+^n} \mathbf{1}_{(y_n \in \mathbb{Q})} \varepsilon H^a \left( \frac{v(s_n, y_n) - v(s_n, y_n - z)}{z} \right) - \mathbf{1}_{(q_n \in \mathbb{Q})} \varepsilon H^a \left( \frac{u(t_n, q_n) - u(t_n, q_n - z)}{z} \right) \mu^a(dz) \leq -2\varepsilon + (\psi(q_n) - \psi(y_n)) - (H(2n(q_n - y_n), q_n) - H(2n(q_n - y_n), y_n)).
\]
By rearranging the terms, we get:
\[
\int_{\mathbb{R}_+^n} \mathbf{1}_{(y_n \in \mathbb{Q})\cap(q_n \in \mathbb{Q})} \varepsilon H^b \left( \frac{v(s_n, y_n) - v(s_n, y_n + z)}{z} \right) - H^b \left( \frac{u(t_n, q_n) - u(t_n, q_n + z)}{z} \right) \mu^b(dz) + \int_{\mathbb{R}_+^n} \mathbf{1}_{(y_n \in \mathbb{Q})\cap(q_n \in \mathbb{Q})} \varepsilon H^a \left( \frac{v(s_n, y_n) - v(s_n, y_n - z)}{z} \right) - H^a \left( \frac{u(t_n, q_n) - u(t_n, q_n - z)}{z} \right) \mu^a(dz) \leq -2\varepsilon + (\psi(q_n) - \psi(y_n)) - (H(2n(q_n - y_n), q_n) - H(2n(q_n - y_n), y_n)).
\]
We know that \( q_n, y_n \xrightarrow{n \to +\infty} \tilde{q} \). Therefore, \( (\psi(q_n) - \psi(y_n)) \xrightarrow{n \to +\infty} 0 \).

Moreover, by Lemma 3 there exists a constant \( C_\mathcal{H} > 0 \) such that for all \( n \),
\[
|\mathcal{H}(2n(q_n - y_n), q_n) - \mathcal{H}(2n(q_n - y_n), y_n)| \leq C_\mathcal{H}(1 + 2n|q_n - y_n|)|q_n - y_n| \xrightarrow{n \to +\infty} 0.
\]

We also have for almost every \( z > 0 \) that \( \mathbb{1}_{\{y_n + z \in \mathcal{Q} \} \cap \{q_n + z \notin \mathcal{Q} \}} \xrightarrow{n \to +\infty} 0 \).

By Lemma 2, the term \( zH^b\left(\frac{v(s_n, y_n) - v(s_n, y_n + z)}{z}\right) \) is bounded uniformly in \( n \) and \( z \), and by the absolute continuity of \( \mu^b \), the dominated convergence theorem enables us to conclude that:
\[
\int_{\mathbb{R}_+^*} \mathbb{1}_{\{y_n + z \in \mathcal{Q} \} \cap \{q_n + z \notin \mathcal{Q} \}} zH^b\left(\frac{v(s_n, y_n) - v(s_n, y_n + z)}{z}\right) \mu^b(dz) \xrightarrow{n \to +\infty} 0.
\]

By the same reasoning:
\[
\int_{\mathbb{R}_+^*} \mathbb{1}_{\{y_n - z \in \mathcal{Q} \} \cap \{q_n - z \notin \mathcal{Q} \}} zH^a\left(\frac{v(s_n, y_n) - v(s_n, y_n - z)}{z}\right) \mu^a(dz) \xrightarrow{n \to +\infty} 0,
\]
\[
\int_{\mathbb{R}_+^*} \mathbb{1}_{\{y_n - z \in \mathcal{Q} \} \cap \{q_n - z \notin \mathcal{Q} \}} zH^a\left(\frac{u(t_n, q_n) - u(t_n, q_n - z)}{z}\right) \mu^a(dz) \xrightarrow{n \to +\infty} 0,
\]
\[
\int_{\mathbb{R}_+^*} \mathbb{1}_{\{y_n - z \in \mathcal{Q} \} \cap \{q_n - z \notin \mathcal{Q} \}} zH^a\left(\frac{u(t_n, q_n) - u(t_n, q_n - z)}{z}\right) \mu^a(dz) \xrightarrow{n \to +\infty} 0.
\]

We can thus choose \( n \) large enough so that the right-hand side of (12) is negative.

However, on the left-hand side of (12), all the integrals are always nonnegative; indeed, we have
\[
u(t_n, q_n - z) - v(s_n, y_n - z) - n(q_n - y_n)^2 - n(t_n - s_n)^2 - \varepsilon(2T - t_n - s_n)
\leq u(t_n, q_n) - v(s_n, y_n) - n(q_n - y_n)^2 - n(t_n - s_n)^2 - \varepsilon(2T - t_n - s_n), \tag{13}
\]
therefore \( v(s_n, y_n) - v(s_n, y_n - z) \leq u(t_n, q_n) - u(t_n, q_n - z) \) and as \( H^a \) is nonincreasing, we get the result (the proof is identical for the integrals with \( H^b \)).

Therefore, the left-hand side is nonnegative for every \( n \). But, as we said before, for \( n \) large enough, the right-hand side of (12) is negative, which yields a contradiction.

In conclusion, we necessarily have \( \sup_{[0,T] \times \mathcal{Q}} (u - v) \leq 0. \)

\[ \square \]

**Theorem 2.** \( \theta \) is the only continuous viscosity solution to (\( HJ \)).

**Proof.** We know that \( \theta \) is a bounded viscosity solution of (\( HJ \)), and in particular, \( \theta_\ast \) is a bounded subsolution of (\( HJ \)), \( \theta^\ast \) is a bounded subsolution of (\( HJ \)), and \( \theta_\ast(T, \cdot) = \theta^\ast(T, \cdot) = -\ell \).

Hence \( \theta_\ast \) and \( \theta^\ast \) verify the assumptions of Theorem 1 and we get that \( \theta_\ast \geq \theta^\ast \) on \( [0,T] \times \mathcal{Q} \). But by definition of \( \theta_\ast \) and \( \theta^\ast \), we have \( \theta_\ast \leq \theta \leq \theta^\ast \). Thus we have \( \theta_\ast = \theta = \theta^\ast \), and \( \theta \) is continuous.

Let us now assume that we have another continuous viscosity solution \( \tilde{\theta} \). In particular, \( \tilde{\theta} \) is a subsolution to (\( HJ \)) and \( \theta \) is a supersolution to (\( HJ \)), and as \( \theta(T, q) = \tilde{\theta}(T, q) = -\ell(q) \quad \forall q \in \mathcal{Q} \), we know by the comparison principle that \( \tilde{\theta} \leq \theta \) on \( [0,T] \times \mathcal{Q} \). But we also have that \( \theta \) is a supersolution and \( \tilde{\theta} \) is a subsolution, so by the same argument we have \( \tilde{\theta} \geq \theta \) and finally \( \tilde{\theta} = \theta \) on \( [0,T] \times \mathcal{Q} \). Hence the uniqueness. 

\[ \square \]
4 Numerical results

4.1 Context and parameters

In this section, we apply our model to the foreign exchange spot market where market makers have access to a variety of trading venues in a number of geographical locations with overall depth of liquidity often exceeding their own resources.

In order to derive realistic parameters we consider a set of HSBC FX streaming clients trading the US Dollar against offshore Chinese Renminbi, USDCNH. The set is sufficiently diverse to provide realistic results but by no means complete to fully represent HSBC FX market making franchise. In particular, in this work we mainly consider connections sensitive to pricing and do not take into account any cross currency trading which may significantly contribute to risk management of the chosen currency pair.

Many FX market participants submit quotes to electronic communication networks and their prices may not necessarily indicate commitment (the so-called “Last Look” practice, see Oomen [30] and Cartea et al. [15]). Therefore we take the firm primary mid price as the reference price at any point in time. Typically, Refinitiv and/or Electronic Broking Services (EBS) are considered as primary FX spot sources, depending on currency pair. USDCNH trades on both.

Since market practitioners are used to reason in basis points (bps) as far as quotes are concerned, we decided to slightly adapt our model by factoring out $S_0$ from $\sigma$, $k$, $\delta$, $L$, $\psi$, and $\ell$ (with of course an adjustment of parameters in accordance for intensities). This boils down to factoring out $S_0$ from the value function $\theta$ and we can therefore reason in bps throughout (up to a little approximation because the base price at any time $t$ is $S_0$, not $S_t$, but this makes very little difference given the time frame of the problem we shall consider).

In what follows we consider therefore the following parameters (rescaled as above):

- Volatility parameter: $\sigma = 50$ bps · day$^{-\frac{1}{2}}$.
- Using a standard log-likelihood optimisation technique, we fit the following logistic intensity functions:

$$\Lambda^b(\delta) = \Lambda^a(\delta) = \lambda_R \frac{1}{1 + e^{\alpha_A + \beta_A \delta}},$$

and obtained $\lambda_R = 1000$ day$^{-1}$, $\alpha_A = -1$, and $\beta_A = 10$ bps$^{-1}$. In terms of RFQs, this would correspond to 1000 requests per day, a probability of $\frac{1}{1 + e^{-1}} \approx 63\%$ to trade when the answered quote is the reference price and a probability of $\frac{1}{1 + e^{-1}} \approx 27\%$ to trade when the answered quote is the reference price worsen by 0.2 bps.
- In practice, a pricing ladder for only a few characteristic sizes is quoted. We therefore discretize the distribution of request sizes, with 4 possible sizes: $z^1 = 1$ M$, z^2 = 5$ M$, z^3 = 10$ M$, and $z^4 = 20$ M$, with respective probability $p^1 = 0.76$, $p^2 = 0.15$, $p^3 = 0.075$ and $p^4 = 0.015$.

Regarding the objective function, we consider the following:

- Time horizon given by $T = 0.05$ days. This horizon (although it seems short) ensures convergence towards stationary quotes at time $t = 0$.
- $L : v \in \mathbb{R} \mapsto \eta v^2 + \phi |v|$ with $\eta = 10^{-5}$ bps · day · M$^{-1}$ and $\phi = 0.2$ bps.
- Permanent market impact: $k = 0.005$ bps · M$^{-1}$.
- $\psi : q \in \mathbb{R} \mapsto \frac{\gamma}{2} q^2$ with $\gamma = 0.0005$ bps$^{-1}$ · M$^{-1}$.
- $\ell = 0$.

\footnotesize{A basis point is one hundredth of a percent.}
We impose risk limits in the sense that no trade that would result in an inventory \(|q| \geq \tilde{q}\) is admitted, where \(\tilde{q} = 100 \text{ M$}\). We then approximate the solution \(\theta\) to (HJ) using a monotone implicit Euler scheme on a grid with 201 points for the inventory.

### 4.2 Results

The value function (at time \(t = 0\)) is plotted in Figure 1.

![Value function](image)

Figure 1: Value function of the problem for different values of the inventory.

We plot in Figure 2 the optimal trading rate of the market maker as a function of her inventory. Of course that trading rate is nonincreasing but we observe an interesting effect: a plateau around zero, thereafter referred to as the pure flow internalization area. This is due to the proportional transaction cost (given by the parameter \(\phi\)), that discourages the trader to buy or sell externally in the liquidity pool when her inventory is small enough (she prefers to bear this small risk than to pay the execution costs). Permanent impact also discourages external execution. We note that permanent impact has no influence on classical Almgren-Chriss optimal schedules (or marginal influence in the discrete formulation). The reason is that it is proportional to the overall quantity and thus independent of the way the order is executed. The situation is quite different here as no market impact is expected when the flow is internalized. Therefore, external trading brings additional relative cost by pushing the expected price for all the subsequent fills.

To observe the impact of the request size on the optimal quotes, we plot in Figure 3 the four functions

\[
q \mapsto -\tilde{\delta}^b(0, q, z^k), \quad k \in \{1, \ldots, 4\},
\]

and the four functions

\[
q \mapsto -\tilde{\delta}^a(0, q, z^k), \quad k \in \{1, \ldots, 4\},
\]

where \(\tilde{\delta}^b\) and \(\tilde{\delta}^a\) represent the optimal quotes as a function of inventory and size of request, at the bid and at the ask, respectively. We see that accounting for the size of requests impacts the optimal bid and ask quotes. The monotonicity of the quotes is of course unsurprising. It is noteworthy that no market spread was parametrically introduced into the model during the estimation of the logistic parameters. Therefore, it is interesting to compare the bid-ask spread we obtain from the pure flow consideration against the actual market spread. Our current estimation produces 0.32 bps for $1M size. The average composite interbank spread of USDCNH at London open as of this writing (early June 2021) is 0.38 bps.
Throughout this section, the optimal quotes are those derived from Lemma 1 and the optimal execution rates are those derived from Lemma 3. To confirm empirically that these controls are in line with the value function obtained with our numerical scheme, we performed Monte-Carlo simulations using those controls. The comparison between the value function approximated numerically and the proceed of the Monte-Carlo simulations is plotted in Figure 4. We see that the values coincide in our case.
4.3 Comparative statics regarding the pure flow internalization area

We now study the influence of the parameters on the width of the pure flow internalization area.

We plot in Figure 5 the optimal trading rate of the market maker as a function of her inventory, when execution cost parameter $\phi$ is set to 0.4 bps. We see that increasing $\phi$ leads to a wider pure flow internalization area: the market maker is less inclined to pay for immediate hedging and waits for her inventory to reach a higher level of risk to start trading externally in the liquidity pool.

We plot in Figure 6 the optimal trading rate of the market maker as a function of her inventory, when the permanent market impact parameter $k$ is set to $0.01 \text{ bps} \cdot \text{M$}^{-1}$. We see that increasing $k$ leads to a wider pure flow internalization area: the market maker is less inclined to impact the market, and waits for her inventory to reach a higher level of risk to start trading externally in the liquidity pool.

We plot in Figure 7 the optimal trading rate of the market maker as a function of her inventory, when the risk aversion parameter $\gamma$ is set to $0.005 \text{ bps}^{-1} \cdot \text{M$}^{-1}$. We see that increasing $\gamma$ leads to a narrower pure flow internalization area: the market maker is more risk averse, and therefore starts externalizing sooner to bring her inventory closer to 0.

Finally, we plot in Figure 8 the optimal trading rate of the market maker as a function of her inventory when the intensity parameter $\lambda_R$ is set to $3000 \text{ day}^{-1}$. We see that increasing $\lambda_R$ leads to a wider pure flow internalization area because the market maker has more frequent opportunities to trade and therefore less reasons to pay the costs of externalization.
Figure 5: Optimal execution rate as a function of the inventory when $\phi$ increases.

Figure 6: Optimal execution rate as a function of the inventory when $k$ increases.
Figure 7: Optimal execution rate as a function of the inventory when $\gamma$ increases.

Figure 8: Optimal execution rate as a function of the inventory when $\lambda_R$ increases.
Conclusion

In this paper, we generalized existing OTC market making models to introduce the possibility for the market maker to trade in some liquidity pools for hedging purpose. This extension led to a partial integro-differential equation of the Hamilton-Jacobi (HJ) type and we proved that the value function of the problem was its unique continuous viscosity solution. We illustrated our results numerically by solving the equation on a grid using an implicit Euler scheme and computing the optimal quotes and trading rates. We highlighted the existence of a pure flow internalization area. This area depicts a subtle balance between uncertainty, execution cost, and market impact. It is wider for a less risk averse market maker with a larger franchise, exposed to higher transaction costs and market impact.

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Appendix

In this appendix, we prove Proposition 2. More exactly, we only prove the subsolution part (the proof for the supersolution part is identical).

Let us first assume that the following inequality holds:

\[-\frac{\partial \varphi}{\partial \bar{t}}(\bar{t}, \bar{q}) + \psi(\bar{q}) - \int_{\mathbb{R}_+^*} \mathbf{1}_{\{\bar{q} + z \in \mathcal{Q}\}} z H^b \left( \frac{u(\bar{t}, \bar{q}) - u(\bar{t}, \bar{q} + z)}{z} \right) \mu^b(dz) - \int_{\mathbb{R}_+^*} \mathbf{1}_{\{\bar{q} - z \in \mathcal{Q}\}} z H^a \left( \frac{u(\bar{t}, \bar{q}) - u(\bar{t}, \bar{q} - z)}{z} \right) \mu^a(dz) - \mathcal{H}(\partial_{\bar{q}} \varphi(\bar{t}, \bar{q}), \bar{q}) \leq 0.\]

We know that \(\forall z > 0:\)

\[u(\bar{t}, \bar{q} - z) - \varphi(\bar{t}, \bar{q} - z) \leq u(\bar{t}, \bar{q}) - \varphi(\bar{t}, \bar{q}).\]

Thus:

\[H^a \left( \frac{u(\bar{t}, \bar{q}) - u(\bar{t}, \bar{q} - z)}{z} \right) \leq H^a \left( \frac{\varphi(\bar{t}, \bar{q}) - \varphi(\bar{t}, \bar{q} - z)}{z} \right),\]

and the same holds for \(H^b:\)

\[H^b \left( \frac{u(\bar{t}, \bar{q}) - u(\bar{t}, \bar{q} + z)}{z} \right) \leq H^b \left( \frac{\varphi(\bar{t}, \bar{q}) - \varphi(\bar{t}, \bar{q} + z)}{z} \right).\]

So we get:

\[-\frac{\partial \varphi}{\partial \bar{t}}(\bar{t}, \bar{q}) + \psi(\bar{q}) - \int_{\mathbb{R}_+^*} \mathbf{1}_{\{\bar{q} + z \in \mathcal{Q}\}} z H^b \left( \frac{\varphi(\bar{t}, \bar{q}) - \varphi(\bar{t}, \bar{q} + z)}{z} \right) \mu^b(dz) - \int_{\mathbb{R}_+^*} \mathbf{1}_{\{\bar{q} - z \in \mathcal{Q}\}} z H^a \left( \frac{\varphi(\bar{t}, \bar{q}) - \varphi(\bar{t}, \bar{q} - z)}{z} \right) \mu^a(dz) - \mathcal{H}(\partial_{\bar{q}} \varphi(\bar{t}, \bar{q}), \bar{q}) \leq 0,\]

and \(u\) is a viscosity subsolution.
Let us now assume that \( u \) is a viscosity subsolution. Without loss of generality, we can assume that 
\[
\varphi(t, \bar{q}) = u(t, \bar{q}).
\]

Let \( B_\eta \) be the open ball of center \((\bar{t}, \bar{q})\) and radius \( \eta > 0 \). Let \((u_n)\) be a sequence of smooth functions uniformly (in \( n \)) bounded such that \( u_n \geq u \) \( \forall n \) and \( u_n \xrightarrow{n \to +\infty} u \) pointwise. Let \( \xi \) be a smooth nondecreasing function such that \( \xi(x) = 1 \) if \( x > \eta/4 \) and \( \xi(x) = 0 \) if \( x < -\eta/4 \). Let \( d_{\eta/2} \) be the algebraic distance to \( \partial B_{\eta/2} \) (with \( d_{\eta/2} > 0 \) on \( B_{\eta/2} \) and \( d_{\eta/2} \leq 0 \) on \( B_{\eta/2}^c \)); this function is continuously differentiable. We introduce:
\[
\varphi_n^a = \varphi \times (\xi \circ d_{\eta/2}) + u_n \times (1 - \xi \circ d_{\eta/2}).
\]

Then \((\bar{t}, \bar{q})\) is still a max of \( -\varphi_n^a \) and \((u - \varphi_n^a)(\bar{t}, \bar{q}) = 0 \). Furthermore we have \( \partial_{\bar{t}} \varphi_n^a(\bar{t}, \bar{q}) = \partial_{\bar{t}} \varphi(\bar{t}, \bar{q}) \) and \( \partial_q \varphi_n^a(\bar{t}, \bar{q}) = \partial_q \varphi(\bar{t}, \bar{q}) \) for all \( i \in I_T \). Thus:
\[
- \frac{\partial \varphi}{\partial \bar{t}}(\bar{t}, \bar{q}) + \psi(\bar{q}) - \int_{\mathbb{R}_+^1} \mathbf{1}_{\{q+z \in Q\}} z H^b \left( \frac{\varphi_n^a(\bar{t}, \bar{q}) - \varphi_n^a(\bar{t}, \bar{q} + z)}{z} \right) \mu^b(dz) = 0,
\]

\[
- \int_{\mathbb{R}_+^1} \mathbf{1}_{\{q-z \in Q\}} z H^a \left( \frac{\varphi_n^a(\bar{t}, \bar{q}) - \varphi_n^a(\bar{t}, \bar{q} - z)}{z} \right) \mu^a(dz) - \mathcal{H}(\partial_q \varphi(\bar{t}, \bar{q}), \bar{q}) \leq 0.
\]

Plus we have \( \varphi_n^a \xrightarrow{n \to +\infty} \varphi_\eta \) pointwise with \( \varphi_\eta = \varphi \times (\xi \circ d_{\eta/2}) + u \times (1 - \xi \circ d_{\eta/2}) \) which is smooth on \( B_{\eta/4} \) and such that \( \varphi_\eta = u \) on \( B_{\eta/2}^c \) and \( \varphi_\eta(\bar{t}, \bar{q}) = u(\bar{t}, \bar{q}) \).

By continuity of \( H^a \) and \( H^b \), absolute continuity of \( \mu^b \) and \( \mu^a \) and by dominated convergence (using the same argument than in Lemma 2 and the fact that the \( \varphi_n^a \) are bounded uniformly in \( n \)) we get:
\[
- \frac{\partial \varphi}{\partial \bar{t}}(\bar{t}, \bar{q}) + \psi(\bar{q}) - \int_{\mathbb{R}_+^1} \mathbf{1}_{\{q+z \in Q\}} z H^b \left( \frac{\varphi(\bar{t}, \bar{q}) - \varphi(\bar{t}, \bar{q} + z)}{z} \right) \mu^b(dz) = 0,
\]

\[
- \int_{\mathbb{R}_+^1} \mathbf{1}_{\{q-z \in Q\}} z H^a \left( \frac{\varphi(\bar{t}, \bar{q}) - \varphi(\bar{t}, \bar{q} - z)}{z} \right) \mu^a(dz) - \mathcal{H}(\partial_q \varphi(\bar{t}, \bar{q}), \bar{q}) \leq 0.
\]

By then sending \( \eta \) to 0 and using again dominated convergence, we get the result:
\[
- \frac{\partial \varphi}{\partial \bar{t}}(\bar{t}, \bar{q}) + \psi(\bar{q}) - \int_{\mathbb{R}_+^1} \mathbf{1}_{\{q+z \in Q\}} z H^b \left( \frac{u(\bar{t}, \bar{q}) - u(\bar{t}, \bar{q} + z)}{z} \right) \mu^b(dz) = 0,
\]

\[
- \int_{\mathbb{R}_+^1} \mathbf{1}_{\{q-z \in Q\}} z H^a \left( \frac{u(\bar{t}, \bar{q}) - u(\bar{t}, \bar{q} - z)}{z} \right) \mu^a(dz) - \mathcal{H}(\partial_q \varphi(\bar{t}, \bar{q}), \bar{q}) \leq 0.
\]

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