Using a recently proposed new renormalization group method (tensor renormalization group), we analyze the Ising model on the 2-dimensional square lattice. For the lowest order approximation with two domain wall states, it realizes the idea of coarse graining of domain walls. We write down explicit analytic renormalization transformation and prove that the picture of the coarse graining of the physical domain walls does hold for all physical renormalization group flows. We solve it to get the fixed point structure and obtain the critical exponents and the critical temperature. These results are very near to the exact values. We also briefly report the improvement using four domain wall states.

1 Introduction

Recently a new type of renormalization group method called the tensor renormalization group (TRG) is introduced by Levin and Nave [2] and has been applied to various models, particularly classical and quantum spin models in 2-dimension [3, 4]. In this article we concentrate on the square lattice Ising model. The tensor renormalization group method for spin systems can be interpreted as the domain wall renormalization group. We hope it might give deep insight for the idea of the coarse graining of such topological objects like conserved walls.

Our aim here is to clarify in detail the properties of this domain wall renormalization group transformation in the lowest order approximation with only two domain wall states on the coarse grained links. We will be able to write down the explicit analytic form of the renormalization group transformation (RGT) with the help of the physical region condition. This analytic formula will help us to prove that the picture of the coarse grained domain walls does hold for all renormalization group flows starting from the physical points.

Then we analytically prove the existence of a single non-trivial fixed point controlling the phase transition and evaluate the correlation length critical exponent which is actually unexpectedly good value, that is, within 2% error of the exact value.

We also obtain the critical temperature within 10% error of the exact value. Using this lowest order RGT we also numerically calculate the partition function for finite
size systems as a function of the temperature. In the infinite size limit, we observe the logarithmic divergence of the specific heat in the neighborhood of the criticality.

We finally mention quickly that enlarging the coarse grained state space we set up 4-state version of the domain wall renormalization group and it gives an yet better result for the critical temperature within 2% error.

We would like to stress here about new arguments and results in our study. We obtain explicit analytic forms of the renormalization group transformation and solve its fixed point condition to have a single non-trivial fixed point solution. Although we work in the lowest order approximation, this is a good example of partially solvable renormalization group equation to clarify total phase structure. Analytic forms give us much greater inspiration than numerical calculations in the black box, and we hope it will contribute to physical formulation of renormalization group equation for topological objects in the higher order approximation. In this course of calculation it is important that we define the physical region condition of parameters, because the condition assures that the notion of the conserved domain walls hold also for coarse grained variables. Due to this fact, the coarse grained variables can be regarded as to describe the coarse grained domain wall configurations, and therefore we can call the renormalization group as the domain wall renormalization group. Also the critical behaviors we evaluated for this system of the square lattice Ising model has not been reported explicitly elsewhere.

2 Domain wall representation and RGT

It is well-known that in the 2-dimensional Ising model, any spin configuration is equivalentally represented by a domain wall configuration defined on the dual link of the dual lattice. As for the square lattice, the dual lattice is also square. Domain walls are said to exist on a dual link when two spins separated by the dual link are different (Fig.1).

Note that a domain wall configuration represents two spin configurations of $Z_2$ pair. Thus summing up all domain wall configuration gives just a half of the partition function and this factor 1/2 does not affect physical quantities discussed below.

We define the partition function of the Ising spin system by

$$Z = \sum_{\sigma} \exp \left( \beta J \sum_{\text{n.n.}} \sigma_i \sigma_j \right).$$

(1)

According to the recently proposed general method of the tensor renormalization group method[2], we express the partition function in terms of a product of vertices (sometimes called tensors, the origin of the name, tensor renormalization group).

$$Z = \sum_{a b c d e f g \ldots} T_{a b c d e f g} \ldots$$

(2)
The vertex $T$ is defined on each dual site and has 4 indices representing domain wall state on 4 dual links. We assign 0 for no-domain wall and 1 for existence of domain wall. Then the components of $T$ are given by

$$T_{0000} = \exp(2\beta J),$$
$$T_{0101} = T_{0110} = 1,$$
$$T_{1111} = \exp(-2\beta J),$$

where the index cyclic symmetry is assumed.

The coarse graining procedure consists of two steps. The first step is to break $T$-vertex into a product of two $S$-vertex. From the Feynman diagram view, this step can be seen as introducing an intermediating boson for the 4-fermi weak interactions.

The intermediating line will become a coarse grained dual link and coarse grained domain wall states are defined on it. If we take 4 states for that, we do not lose any information. However we like to set up a finite dimensional system of the renormalization group transformation and we have to discard some degrees of freedom here. Then we take only 2 states on the intermediate states. It means this decomposition of $T$-vertex loses information, and this is the approximation of this type of renormalization group transformation.

The 2-state approximation must be the lowest approximation and we will analyze its RGT in detail. Which two states among totally $2 \times 2 = 4$ states should be picked up? There are two ways to determine it. One is a physical condition that coarse grained states should obey notions of the coarse grained domain walls. The other is a practical condition that we should minimize the information loss of this procedure. Fortunately, we prove below that these two conditions are consistent for some region of parameters which we call physical region, and it is respected by the physical finite temperature system.

The second step of RGT is to integrate out all $S$-vertices and will make the system described only by new variables, the intermediate states. The new variables are nothing but the coarse grained domain walls on the newly made coarse grained dual links (Fig[2]). These two steps define RGT with scale factor of $\sqrt{2}$. 

Figure 2: Integrate out $S$-vertex to define renormalized $T$-vertex
3 Analytic expression of RGT

We express \( T \)-vertex (with four legs) by a \( 4 \times 4 \) matrix. We combine two legs each into row or column index of the matrix. The correspondence rules are

\[
\{00\} \Rightarrow \{1\} , \quad \{11\} \Rightarrow \{2\} , \quad \{01\} \Rightarrow \{3\} , \quad \{10\} \Rightarrow \{4\} ,
\]

Then we assume the texture of \( T \) matrix as follows:

\[
T = \begin{pmatrix}
00 & 11 & 01 & 10 \\
00 & 1 & b & 0 & 0 \\
11 & b & c & 0 & 0 \\
01 & 0 & 0 & a & b \\
10 & 0 & 0 & b & a
\end{pmatrix}.
\]

(6)

where \( a,b,c \) are real positive parameters obeying the conditions defined by

\[
0 < a \leq b < 1 , \\
0 < c < b , \\
2b < c + 1 ,
\]

(7)

which we call the physical region condition. In general, renormalization group transformation modifies \( T_{11} \) component and we always normalize the matrix by dividing all elements by a constant so that \( T_{11} \) should be 1.

These two properties, the texture and physical region condition, are satisfied by the initial \( T \) matrix given by

\[
a = b = \alpha , \\
c = \alpha^2 , \\
\alpha = \exp(-2\beta J) < 1.
\]

(8)

Now will prove that our RGT does not break these two properties and therefore all flows starting from the physical region have the same texture and belongs to the physical region.

First we calculate eigenvalues of \( T \).

\[
\lambda_{1,2} = \frac{1}{2} \left( 1 \mp \sqrt{(1-c)^2 + 4b^2} \right) , \\
\lambda_{3,4} = a \pm b.
\]

(9)

We examine the absolute values of these 4 eigenvalues and select two larger components according to the policy of TRG. Using the physical region condition, we can prove the following inequalities

\[
\lambda_1 > b , \quad \lambda_3 > b , \\
|\lambda_4| < b , \quad |\lambda_2| < b.
\]

(10)

Thus we take two eigenvalues \( \lambda_1, \lambda_3 \) to define \( S \) vertex. Note that these two eigenvalues are positive.
Now we can split $T$-vertex into a product of two $S$-vertices. First we introduce two orthogonal $2 \times 2$ matrices $R_{1,2}$ which diagonalize $T$ as follows:

\[
T = \begin{pmatrix}
1 & b & 0 & 0 \\
0 & b & c & 0 \\
0 & 0 & a & b \\
0 & 0 & b & a
\end{pmatrix}
= \begin{pmatrix}
R_1 & 0 \\
0 & R_2
\end{pmatrix}
\begin{pmatrix}
\lambda_1 & 0 & 0 & 0 \\
0 & \lambda_2 & 0 & 0 \\
0 & 0 & \lambda_3 & 0 \\
0 & 0 & 0 & \lambda_4
\end{pmatrix}
\begin{pmatrix}
R_1^t & 0 \\
0 & R_2^t
\end{pmatrix},
\] (11)

where $R_{1,2}$ are defined by angles $\theta_{1,2}$

\[
R_1 = \begin{pmatrix}
\cos \theta_1 & -\sin \theta_1 \\
\sin \theta_1 & \cos \theta_1
\end{pmatrix},
\]

\[
R_2 = \begin{pmatrix}
\cos \theta_2 & -\sin \theta_2 \\
\sin \theta_2 & \cos \theta_2
\end{pmatrix}.
\] (12)

Therefore we have the following decomposition:

\[
T = S^A \cdot S^B,
\] (13)

where $S^A$ is $4 \times 2$ matrix and $S^B$ is $2 \times 4$ matrix given by

\[
S^A = \begin{pmatrix}
\sqrt{\lambda_1} c_1 & 0 \\
\sqrt{\lambda_1} s_1 & 0 \\
0 & \sqrt{\lambda_3} c_2 \\
0 & \sqrt{\lambda_3} s_2
\end{pmatrix},
\]

\[
S^B = \begin{pmatrix}
\sqrt{\lambda_1} c_1 & \sqrt{\lambda_1} s_1 & 0 & 0 \\
0 & 0 & \sqrt{\lambda_3} c_2 & \sqrt{\lambda_3} s_2
\end{pmatrix}.
\] (14)

It should be noted here that these $S$-vertices satisfy $S^A = (S^B)^t$ and they are essentially the same. This decomposition is nothing but the singular value decomposition which is frequently used in order to pick up important degrees of freedom. Hereafter we use simple notations as follows:

\[
s_1 = \sin \theta_1, \quad c_1 = \cos \theta_1, \quad s_2 = \sin \theta_2, \quad c_2 = \cos \theta_2.
\] (15)

### 4 Feynman rules to calculate RGT

The $T$-vertex decomposition is expressed as

\[
T = S^A \cdot S^B
\] (16)

where the inner thick line represents two possible states. We assign names to these two states to be 0 ($\lambda_1$ component) and 1 ($\lambda_3$ component) respectively. This assignment is essential for interpreting the coarse grained domain wall variables, which will be clear soon. The $S$-vertex gives the following Feynman rules:
where we draw single line for 0-state and double line for 1-state. These rule applies to all legs. Now we see $S$-vertex conserves the number of legs of double line (1-state), thus conserves the domain wall.

The renormalization group transformation is defined by one-loop diagrams and they are calculated as follows:

\[
\begin{align*}
\lambda_1 c_1 & \rightarrow \lambda_1^2 (c_1^4 + s_1^4), \\
\lambda_1 s_1 & \rightarrow 2\lambda_1 \lambda_3 c_1 s_1 c_2 s_2, \\
\lambda_3 c_2 & \rightarrow \lambda_1 \lambda_3 c_2 s_2 (c_1^2 + s_1^2), \\
\lambda_3 s_2 & \rightarrow 2\lambda_3^2 c_2^2 + s_2^2.
\end{align*}
\]

Note that the conservation of domain wall prohibits other diagrams. These amplitudes give the renormalized $T$-vertex. Taking account of the discrete rotational symmetry,

\[
\begin{align*}
\lambda_1 c_1 & = \lambda_1 c_1, \\
\lambda_1 s_1 & = \lambda_1 s_1, \\
\lambda_3 c_2 & = \lambda_3 c_2, \\
\lambda_3 s_2 & = \lambda_3 s_2,
\end{align*}
\]

we have the renormalized $T$-vertex ($T'$)

\[
T' = \begin{pmatrix}
1 & b' & 0 & 0 \\
b' & c' & 0 & 0 \\
0 & 0 & a' & b' \\
0 & 0 & b' & a'
\end{pmatrix},
\]

(23)
where we made additional total renormalization of the matrix elements so that $T'_{11}$ should be equal to 1.

Thus we have proved that RGT conserves the texture defined in Eq.(6). This comes from the property that the $S$-vertex conserves the domain wall with our assignment of coarse grained domain walls. Also the discrete rotational symmetry is necessary. In fact the physical region condition (7) is not necessary for this texture conservation. Strictly speaking the necessary and sufficient condition is that each of the two coarse grained components should come from the upper block and the lower block respectively. The physical region condition corresponds to another property that each of the two largest values among 4 eigenvalues are in the upper block and the lower block respectively. Thus the physical region condition assures that both of the best approximate coarse graining and the domain wall conservation holds simultaneously.

We should discuss here the notion of coarse grained domain walls. The $S$-vertex defines the relation of the micro domain walls and macro (coarse grained) domain wall. The micro domain wall does not always generates macro domain wall. The small loop of the micro domain wall does not correspond to the macro domain wall, whereas the domain walls connecting two coarse grained vertices (domain walls residing on the coarse grained links) are defined as macro domain walls. These interpretation naturally defines the coarse graining of the domain walls and renormalization of the $T$-vertex.

5 Renormalization Transformation

The renormalization transformation for parameters $a, b, c$ are written down as follows:

$$a' = \frac{2\lambda_3 c_1 s_1 c_2 s_2}{\lambda_1 c_1^4 + s_1^4},$$

$$b' = \frac{\lambda_3 c_2 s_2}{\lambda_1 c_1^4 + s_1^4},$$

$$c' = \frac{2\lambda_3^2 c_2^2 s_2^2}{\lambda_1 c_1^4 + s_1^4}.$$ (24)

Angles are obtained as a function of parameters $a, b, c$

$$\sin 2\theta_1 = \frac{2b}{\lambda_1 - \lambda_2} = \frac{2b}{\sqrt{(1-c)^2 + 4b^2}},$$

$$\sin 2\theta_2 = 1.$$ (25)

Using these angles we rewrite the transformation

$$a' = \left(\frac{\lambda_3}{\lambda_1}\right) \frac{s}{2 - s^2},$$

$$b' = \left(\frac{\lambda_3}{\lambda_1}\right) \frac{1}{2 - s^2},$$

$$c' = \left(\frac{\lambda_3}{\lambda_1}\right)^2 \frac{1}{2 - s^2}.$$ (26)
where eigenvalues and diagonalization angles are
\[
\begin{align*}
\lambda_1 &= \frac{1}{2} \left( 1 + c + \sqrt{(1 - c)^2 + 4b^2} \right), \\
\lambda_3 &= a + b, \quad s = \frac{2b}{\sqrt{(1 - c)^2 + 4b^2}}.
\end{align*}
\] (27)

This completes the explicit analytic formula of the domain wall renormalization group transformation.

Finally we check the physical region condition for renormalized parameters. Due to the inequality \(0 < a \leq b\), we have
\[
\lambda_3 < 2b . \quad (28)
\]

On the other hand, we define a function \(f(\lambda)\)
\[
f(\lambda) = \lambda^2 - (1 + c)\lambda + c - b^2 = 0 \quad (29)
\]
where \(\lambda_1\) is a larger root of this equation. We have the following inequality
\[
f(2b) = (1 - b)(c - b) + b(2b - c - 1) < 0 \quad \text{(30)}
\]
and it proves \(\lambda_1 > 2b\). Then we have
\[
\frac{\lambda_3}{\lambda_1} < 1 . \quad \text{(31)}
\]

Also using \(0 \leq s \leq 1\), we can finally prove
\[
\begin{align*}
0 &< a' \leq b' < 1 , \\
0 &< c' < b , \\
2b' &< c' + 1 .
\end{align*} \quad (32)
\]

Therefore we have proved that the physical region condition is conserved by the renormalization transformation. Then the above analytic forms of the transformation and physical region condition always hold for any flows starting from a point in the physical region.

## 6 Fixed points

According to the standard procedure, we look for fixed points of the renormalization transformation. We can see there are two trivial fixed points.

Low temperature fixed point: \(\{a = 0 , b = 0 , c = 0\}\)

High temperature fixed point: \(\{a = 1 , b = 1 , c = 1\}\)

These fixed points are not in the physical region but on the boundary of the region. They are both infrared fixed points which almost all flows (except for those on the critical surface) in the physical region approaches in the infrared limit. We call them low or high temperature fixed point respectively because they are equal to the physical initial \(T\)-vertex with \(a = 0(\beta \to \infty)\) or \(a = 1(\beta \to 0)\).
We check the eigenvalues of the renormalization transformation in the neighborhood of these trivial fixed points and find that all eigenvalues are less than unity for both fixed points. Thus, we have two infrared fixed points at the border of the physical region. Then there must necessarily exist a critical surface which divides the space into two subspace (different phases). Phases are characterized by these infrared fixed point. On the critical surface there must exist at least one (non-trivial) fixed point.

The non-trivial fixed point satisfies the following set of equations

\[
\begin{align*}
    a &= \left( \frac{\lambda_3}{\lambda_1} \right) \frac{s}{2 - s^2}, \\
    b &= \left( \frac{\lambda_3}{\lambda_1} \right) \frac{1}{2 - s^2}, \\
    c &= \left( \frac{\lambda_3}{\lambda_1} \right)^2 \frac{1}{2 - s^2},
\end{align*}
\]

(33)

where \( \lambda_1, \lambda_3, s \) are defined in Eq. (27). Noticing that parameters \( s \) and \( \lambda_1 \) are functions of \( b, c \) only, we first solve \( a \). Dividing the first equation by the second equation in the above, we have

\[
a = bs = \frac{2b^2}{\sqrt{(1 - c)^2 + 4b^2}}. \tag{34}
\]

Also dividing the second equation by the third equation in Eq. (33), we have

\[
b^2(2 - s^2) = c. \tag{35}
\]

and we solve it with respect to \( b \),

\[
b = \frac{1}{2} \sqrt{2c - (1 - c)^2 + \sqrt{4c^2 + (1 - c)^4}}. \tag{36}
\]

Next adding the first and second equations in Eq. (33) we have

\[
\lambda_1(2 - s^2) = 1 + s. \tag{37}
\]

Substituting Eq. (36) into the above equation, we have an equation for \( c \).

\[
136c^5 - 145c^4 + 116c^3 - 54c^2 + 12c - 1 = 0. \tag{38}
\]

This 5th order polynomial equation is easily found to have only one real root. We finally get the unique non-trivial fixed point

\[
\begin{align*}
    c^* &= 0.238902743, \\
    b^* &= 0.402938077, \\
    a^* &= 0.292942734.
\end{align*}
\]

(39)

Numerical results for the critical surface and the renormalized trajectory are drawn in Fig. 3. Note that we plotted here whole region of \( 0 < a, b, c < 1 \). There is no singular behavior at the boundary of the physical region condition once the renormalization transformation is given. Also the physical initial points and a sample renormalization group
flow are plotted there. The cross point of physical initial points and the critical surface determines the critical temperature. We have the criticality in terms of $\alpha$

$$\alpha_c = 0.37036 ,$$  \hspace{1cm} (40)

which should be compared with the exact value $\alpha_c(\text{Exact}) = \sqrt{2} - 1 = 0.41421$. Our result suffers about 10% error.

We linearize the renormalization transformation around the non-trivial fixed point and find the eigenvalues

$$\{1.4224236, -0.29079698, 0\}.$$  \hspace{1cm} (41)

There is only one eigenvalue $\lambda = 1.4224236$ which is larger than unity. Thus the fixed point has only one relevant operator as is reduced by the general argument. According to the standard method\cite{1} to evaluate the critical exponent $\nu$ of the correlation length divergence, we have

$$\nu = \frac{\log \sqrt{2}}{\log \lambda} = 0.98357.$$  \hspace{1cm} (42)

Here we have used that the renormalization transformation changes the lattice scale by factor $\sqrt{2}$. This result seems extremely good, less than 2% error compared to the exact value $\nu = 1$, considering that our domain wall renormalization group has only two states on the coarse grained link and only 3-dimensional interaction space has been used.

We proceed to investigate the behavior of the mean energy and the specific heat. The partition function is calculated by tracing out the $T$-vertex. Note that we have to take account of the additional total normalization factor which is introduced at each
Figure 4: Mean energy as a function of $\beta J$

Figure 5: Specific heat $C$ as a function of $\beta J$
renormalization step to make $T_{11}$ to be unity. We denote the factor by $C_n$ which is used to divide $T$ at the $n$-th renormalization step. After $n$-times renormalization transformation, the $T$-vertex is renormalized to become $T^{(n)}$ and we define

$$Z^{(n)}(\beta) = C_{\text{tot}} T^{(n)}_{\text{abab}} = C_0 2^n C_1 2^{n-1} C_2 2^{n-2} \cdots C_n T^{(n)}_{\text{abab}},$$

which is the partition function of the system with size $2^n$ (the total number of sites).

We calculate the mean energy and the specific heat per site. The results are shown in Figs. 4 and 5. In the mean energy plots, increasing $n$, there start to appear a would-be singular point near $\beta J \simeq 0.5$. The specific heat would diverge at the point due to non-analyticity at $n \to \infty$. The divergence behavior of the specific heat is almost completely logarithmic as is derived by the scaling relation.

We can increase the number of states defined on the coarse grained link. As a next approximation we set up 4-state version of the domain wall renormalization group. The corresponding $T$-vertex has $4^4 = 256$ components. The texture analysis shows that only 25 components are non-vanishing in the procedure of the renormalization group transformation. The details of our study will be reported in a full paper [5]. Here we only mention some results. The critical temperature is obtained as

$$\alpha_c[4\text{-state}] = 0.42205,$$

and the error is now less than 2%. This is a great improvement compared to the 2-state result. As for the correlation length critical exponent, we have

$$\nu[4\text{-state}] = 0.98359,$$

and this is almost equal to the 2-state result.

In the 4-state version RGT, there are some subtle issues to be discussed in detail. For example, notion of the coarse grained domain wall is not trivial like 2-state version. To evaluate the magnetic quantities like the magnetization and the susceptibility, we need to introduce external source field. This will break the symmetry property we have used in this article and it needs additional consideration.

We would like to thank fruitful discussion with H. Suzuki who was collaborating with us in the initial stage of this work. This research was partially supported by the Ministry of Education, Science, Sports and Culture, Grant-in-Aid for Scientific Research (B), 17340070, 2007.

References

[1] K. G. Wilson, Rev. Mod. Phys. 47 (1975) 773.

[2] M. Levin and C. P. Nave, Phys. Rev. Lett. 99 (2007) 120601.

[3] M. Hinczewski and N. Berker, Phys. Rev. E 77 (2008) 011104.

[4] Z.-C. Gu, M. Levin and X.G. Wen, Phys. Rev. B 78 (2008) 205116 and [arXiv:0806.3509 [cond-mat.str-el]].

[5] K-I. Aoki, T. Kobayashi and H. Tomita in preparation.