The Gravitational Sine-Gordon Model

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We use matrix model results to investigate the Sine-Gordon model coupled to two dimensional gravity. For relevant (in the RG sense) potentials, we show that the $c = 1$ string, which appears in the ultraviolet limit of this model, flows to a set of decoupled $c = 0$ (pure gravity) models in the infrared. The torus partition sum, which was argued previously to count the number of string degrees of freedom and hence satisfy a new $c$ – theorem, is shown to be a monotonically decreasing function of the scale (given by the quantum area of the world-sheet). The model discussed describes an interesting time dependent solution of two dimensional string theory.

December 1992

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1. Introduction

The Lagrangian

\[ \mathcal{L} = (\partial X)^2 + \sum_{i=1}^{N} \lambda_i \cos p_i X \]  

for an interacting scalar field in two dimensions generalizes the Sine-Gordon (SG) model (the case with \( N = 1 \)). In general not much is known about the Quantum Field Theory (QFT) to which it gives rise, but a qualitative picture of what one may expect arises from the work of Zamolodchikov [1], according to which the infrared limit is governed by the minima of the effective potential for \( X, V(X) \). According to this picture, at large scales the \( X \) field freezes at one of these minima and is described by a minimal model; precisely which model depends on the behavior of \( V(X) \) around the minimum. If \( V(X) \) has more than one minimum, the theory presumably splits into a decoupled set of minimal models.

While this picture is plausible and has received a certain amount of verification over the years, exact calculations in (1.1) are difficult because of the large fluctuations of the \( X \) field in two dimensions. For example, the effective potential \( V(X) \) is renormalized and may differ significantly from \( \sum_{i=1}^{N} \lambda_i \cos p_i X \) in (1.1).

From this point of view, it is interesting to consider the theory (1.1) coupled to gravity. Two dimensional gravity coupled to a scalar field, and the generalization (1.1), is quite well understood [2][3]; thus one may hope to develop a good understanding of the space of \( \{\lambda_i\} \), at least in the presence of gravity. But then, it is generally expected that the structure of the space of couplings is the same with or without gravity; e.g., if (1.1) exhibits a renormalization group (RG) trajectory leading to a certain CFT in the infrared limit in the presence of gravity, one would expect this trajectory to exist also when gravity is turned off (the converse is obviously true). Hence, one may study the space of QFT’s (1.1) using the exact results of matrix models. An example of what we have in mind is the space of minimal models. In flat space QFT only very partial results regarding this space have been established; after coupling to gravity the problem is exactly solvable in terms of certain integrable hierarchies [2]. In particular the flow structure (at least genus by genus) is trivial to obtain and it of course agrees with the known results in the absence of gravity.

An additional motivation to study the model (1.1) coupled to gravity stems from the interpretation of this system as a string theory in two dimensional space-time. There are several applications for (1.1) in that context. One is to the study of time dependent solutions of string theory. Equation (1.1) describes a solution in which a non-trivial “tachyon”
profile is propagating through the two dimensional space-time. There have been many discussions of gravitational back-reaction to the “tachyon” stress tensor, but the issue is still not completely resolved. Perhaps a better understanding of \((1.1)\) will contribute to the solution of that problem. In addition, a good understanding of non-trivial time dependent vacua is bound to be useful.

A second application of \((1.1)\) is to the study of the role of non-singlet states in the \(c = 1\) matrix model. According to \([4]\) non-singlets correspond in the continuum to vortices (winding modes). Thus one imagines having in the action terms like \((1.1)\) (which in the dual picture correspond to winding states), with \(p_i = n_i R\), \(R\) being the radius of \(X\). At large enough radius all winding terms are irrelevant (in the RG sense). As the radius decreases, first one \((n = 1)\), then two \((n = 1, 2)\), etc. winding modes become relevant and \((1.1)\) describes the resulting theory. One expects \([4]\) a dynamical transition to a theory where \(X\) is discrete, but the detailed picture should follow from a solution of \((1.1)\).

In this paper we are going to analyze in some detail the physics of \((1.1)\) for \(N = 1\) (the gravitational SG model). The action is described after coupling to gravity by:

\[
S = \int d^2 z \sqrt{\hat{g}} \left[ \frac{1}{8\pi} (\hat{\nabla} \phi)^2 + \frac{\mu}{16\pi} e^{\sqrt{2}\phi} + \frac{\sqrt{2}}{4\pi} \phi \hat{R}(\hat{g}) \right] + \int d^2 z \sqrt{\hat{g}} \left[ \frac{1}{4\pi} (\hat{\nabla} X)^2 + \lambda e^{\beta_p \phi} \cos(pX) \right] \tag{1.2}
\]

where \(\hat{g}\) is some fiducial metric, \(\hat{R}\) is its curvature, \(\mu\) is the cosmological constant, \(\phi\) is the Liouville field, \(\beta_p = (\sqrt{2} - |p|/\sqrt{2})\), and \(X\) is a free scalar field, which we will take to be compact, \(X \simeq X + 2\pi R\), so that the momenta \(p\) are quantized \(Rp \in \mathbb{Z}\). The non-compact case is a trivial generalization. This model was first considered by Moore \([3]\) where the spherical partition sum was calculated as a function of \(\lambda\) at fixed cosmological constant \(\mu\). A physical picture for \((1.2)\) and in general for all models in \((1.1)\) was suggested in \([3]\). According to that picture, for \(p < 2\) in \((1.2)\) (so that the cosine is relevant\([\text{1}]\)), the infrared limit should correspond to a set of \(n\) decoupled \(c = 0\), pure gravity models, where \(n\) is the number of minima of the potential in \((1.2)\) related to \(p\) through \(p = \frac{n}{\pi}\). In this paper we will verify that picture.

The plan is as follows. In section 2 we use the results of \([3]\) for the spherical partition sum and show that in the infrared the theory \((1.2)\) with \(p < 2\) has \(\gamma_{st} = -\frac{1}{2}\), the right

\[\text{1} \quad X \text{ is normalized such that the dimension of } e^{ikX} \text{ is } \frac{1}{4} k^2.\]
value for pure gravity. We also discuss a finite renormalization of the vacuum energy which arises and clarify the relation to \[5\]. In section 3 we examine the partition sum of the model on the torus and find that as expected the infrared partition sum is \( \frac{n}{48} \) (for \( p = \frac{n}{R} \) in (1.2)), in agreement with \[6\]. We also show that the torus partition sum is a monotonically decreasing function of the scale, confirming its interpretation as counting the number of degrees of freedom \[7\], and satisfying a new \( c \)-theorem \[6\]. In section 4 we briefly describe the fate of the operators in the ultraviolet \( c = 1 \) model as one flows to the infrared and the more difficult case of an irrelevant cosine perturbation (\( p > 2 \) in (1.2)). Section 5 includes some remarks about our results. Some of the details appear in the appendices.

2. The Gravitational SG Model on the Sphere

Our goal is to calculate the partition sum

\[
Z_0 = \langle e^{-S(\lambda, \mu)} \rangle
\]

where \( \langle \cdots \rangle \) denotes a path integral over \( \phi \) and \( X \) with the action (1.2). Since the theory with \( \lambda = 0 \) (a scalar field coupled to gravity) is exactly solvable \[2\], it is natural to try to calculate \( Z_0(\mu, \lambda) \) by perturbing in \( \lambda \):

\[
Z_0(\mu, \lambda) = \sum_{n=0}^{\infty} \frac{\lambda^n}{n!} \langle (\cos pX \ e^{\beta_p \phi})^n \rangle_{\lambda=0} = \sum_{n=0}^{\infty} \frac{(\frac{1}{2} \lambda^2)^n}{(n!)^2} \langle (e^{ipX+\beta_p \phi})^n (e^{-ipX+\beta_p \phi})^n \rangle_{\lambda=0}
\]

(2.2)

In the absence of gravity, this expansion is expected to have a finite radius of convergence, at least in the super-renormalizable case (\( p < 2 \)). The qualitative argument for that is that the potential in (1.1) is a conformal primary (or a sum thereof), and does not significantly alter the large field \( (X) \) behavior of the action. Turning on gravity should not spoil this. Indeed, consider the action (1.2). For \( 0 < p < 2, 0 < \beta_p < \sqrt{2} \) so that the \( \lambda \) perturbation is soft: in the ultraviolet, \( \phi \to -\infty \), the coefficient of the cosine \( e^{\beta_p \phi} \to 0 \), whereas in the infrared, \( \phi \to \infty \) that coefficient is large but it is still a small perturbation compared to the dominant cosmological term proportional to \( \mu \). Thus, as in flat space, one expects the expansion (2.2) to be good (convergent) for \( p < 2 \) (and any \( \lambda \)). For \( p > 2 \) we see that the \( \lambda \) perturbation in (1.2) qualitatively changes the behavior of the potential in the ultraviolet region. Instead of dying off, it now fluctuates wildly with an amplitude which diverges as
φ → −∞. Thus a perturbative expansion in λ is expected to be asymptotic in this case. All of the above discussion is of course independent of topology and we expect this picture to be valid for all genera. We will see that these expectations are indeed realized in the exact solution.

One additional point should be made before we turn to the calculations. In (1.2) we have presented the theory at fixed cosmological constant \( \mu \). Although this is not always emphasized in the literature, the more “fundamental” quantity in two dimensional gravity is the fixed area partition sum, in terms of which

\[
Z(\mu, \lambda) = \int_0^\infty dA \ Z(A, \lambda) \ e^{-\mu A}
\]

(2.3)

For example, it is in the fixed area representation that KPZ scaling holds; the Laplace transform (2.3) sometimes introduces divergences at small area which lead to logarithmic scaling violations in \( \mu \). Also, at fixed area the sum over genera is always absolutely convergent while at fixed \( \mu \) is it always asymptotic. Below we will mention other aspects of this issue, but a more complete discussion is beyond the scope of this paper.

After these initial remarks, we are ready to present the calculation of \( Z_0(A, \lambda) \) (2.2), (2.3). Utilizing the results of [5] one finds that

\[
A^3 Z_0(A, \lambda) = \sum_{n=0}^{\infty} \frac{\lambda^{2n} (1-p)^n A^{n(2-p)}}{n! \Gamma(n(1-p) + 1)}
\]

(2.4)

In (2.4) we are using a certain normalization of the tachyon operators \( T_p = e^{ipX+\beta_p \phi} \) which can be controlled by rescaling \( \lambda \). Note that (2.4) has the structure discussed above: for \( p < 2 \) it is an absolutely convergent series for all \( \lambda \) and \( A \), while for \( p > 2 \) it is an asymptotic series, in agreement with the heuristic picture above.

In the ultraviolet (2.4) trivially leads to \( Z_0(A, \lambda) \sim A^{-3} \), as appropriate for the \( c = 1 \) model coupled to gravity. It is easy to sum the series (2.4) explicitly when it is absolutely convergent (\( p < 2 \)). We do that in Appendix A with the result (for large \( A \)):

\[
Z_0(A) \sim A^{-3-\frac{1}{2}} e^{\mu_c(p) A} ; \ (A \to \infty)
\]

(2.5)

with \footnote{\( \mu_c \) has also a trivial dependence on \( \lambda \) following from KPZ scaling, \( \mu_c(p) \sim \lambda^{\frac{2}{2-p}} \). Below we set \( \lambda = 1 \) with no loss of generality.}

\[
\mu_c(p) = \frac{(2-p)}{(1-p)} |1-p|^{\frac{2}{2-p}}.
\]

(2.6)
The meaning of the results (2.5) and (2.6) is quite transparent given our introductory remarks. $\mu_c(p)$ of (2.5), (2.6) corresponds to a finite renormalization of the vacuum energy of the infrared theory compared to that of the ultraviolet one. This is of course expected on general grounds since gravity is sensitive to the vacuum energy of matter (in this case $X$). The actual value of $\mu_c$ (2.6) is not universal; e.g. if we perturb the potential in (1.2) (as in (1.1)), $\mu_c$ will change. We could tune it to zero by an appropriate choice of $\mu$ in (1.2).

On the other hand the power behavior $Z_0(A) \sim A^{-3-\frac{1}{2}}$ of (2.5) is universal (stable under small deformations (1.1)) and it implies that in the infrared the theory has $\gamma_{st} = -\frac{1}{2}$. Since the potential in (1.2) is real we know that the theory we get is a unitary matter theory coupled to Liouville; the only candidate is $c = 0$ — pure gravity, as expected.

A comment on the relation to the work of [5] is in order. In [5] the fixed cosmological constant situation was considered. The large order behavior of $Z_0(A)$, (2.5), means that the Laplace transform (2.3) converges for $\mu > \mu_c(p)$, with a singularity $Z_0(\mu) \sim (\mu - \mu_c)^{\frac{3}{2}}$ as one approaches $\mu_c$. Note that $\mu_c(p) > 0$ for $p < 1$ and $\mu_c(p) < 0$ for $1 < p < 2$, (2.6). For $p < 1$ this means that the infrared fixed point appears along a line $\mu = \mu_c(p, \lambda)$ while for $1 < p < 2$, where $\mu_c$ is negative, this occurs at negative $\mu$ or complex $\lambda$. We would describe the situation by saying that the infrared theory is always pure gravity, and that the only difference between $p < 1$ and $1 < p < 2$ is the sign of the (non-universal) shift in the vacuum energy, whose physical significance is unclear to us.

3. The Gravitational SG Model on the Torus

Further information on the physics of the gravitational SG model can be obtained by analyzing the partition sum (1.2) on the torus. Our interest in this problem is twofold. First we would like to verify the intuitive picture that in the infrared, the model consists of $n (= pR)$ copies of pure gravity ($c = 0$) models. Since the partition sum of a single pure gravity model is [8]

$$\Omega(A) = AZ_1(A) = \frac{1}{48}$$

we expect to find in the infrared limit of (1.2),

$$\Omega(A) \sim \frac{n}{48} e^{\mu_c(p)A}$$

with the same $\mu_c$ as on the sphere (2.6). $\mu_c$ is expected to be independent of genus because it corresponds to a bulk effect, therefore it is insensitive to the topology of the world sheet.
As on the sphere, we expand the partition sum in powers of the interaction (2.2), and the problem reduces to the calculation of \( \langle T^n T_{-p}^n \rangle \). Using the results of [3], [5], [9] one finds

\[
\langle T^n T_{-p}^n \rangle_{h=1} = -R \mu^n (p - 2)(p - 1)^n \frac{n!}{24} \left( f_n(p) + \frac{g_n(p)}{R^2} \right)
\] (3.3)

where \( f_n \) is a calculable polynomial of degree \( n + 1 \) and,

\[
g_n(p) = (-1)^n \frac{\Gamma(n(2 - p))}{\Gamma(n(1 - p) + 1)}
\] (3.4)

are related to the spherical results. The polynomials \( f_n(p) \) can be in principle obtained from [3], and our main technical problem is to find a compact expression for them for all \( n \). To this end we have calculated the first few \( f_n(p) \) (the first nine are reproduced in Appendix B). On the sphere Moore was able to guess the general form by studying the roots of analogous polynomials. Here, one finds that \( f_n(p) \) always have \( n \) roots with \( 1 \leq p \leq 2 \) and one with \( -1 \leq p \leq 0 \). Unlike the sphere, these roots do not occur at simple \( p \)'s in general, but for large \( n \) the pattern is much simpler. Up to small corrections we find that (see Appendix B for more details):

\[
f_n(p) = \frac{(-1)^n}{\sqrt{2n}} (\sqrt{\pi n p} + 1) \frac{\Gamma(n(2 - p) + \frac{1}{2})}{\Gamma(n(1 - p) + \frac{1}{2})}; \quad n \to \infty
\] (3.5)

In fact, as we show in Appendix B, (3.5) is a good approximation for \( f_n(p) \) even for low \( n \). The overall normalization in (3.5) is determined from \( f_n(p = 0) \) which is known. As in [3] we should emphasize that we have only checked this for low values of \( n \), but expect it to be valid in general.

Knowledge of the large \( n \) behavior of \( f_n(p) \) allows us to estimate the large area limit of \( Z_1(A) \). Plugging in (3.3), (3.4) into (2.2) we find\(^3\) as \( A \to \infty \) (see Appendix B):

\[
Z_1(A) \sim \frac{pR}{48A} e^{\mu_c(p)A}
\] (3.6)

with \( \mu_c(p) \) given by (2.6). This is in agreement with the anticipated result [3], according to which when \( p = n/R \) the infrared theory consists of a decoupled set of \( n c = 0 \) models. Apart from the by now familiar vacuum renormalization \( \exp(\mu_c A) \), we have \( \Omega(IR) \equiv A Z_1(A \to \infty) = n/48 \).

\(^3\) For technical reasons explained in Appendix B the analysis below is valid for \( p < 1 \) only. While the physics is expected to be similar for \( 1 < p < 2 \), a more accurate analysis is needed to reveal it.
A second application of (3.5) is to calculate the function \( \Omega(A) \equiv AZ_1(A) \) as a function of \( A \) to see whether it is monotonically decreasing to its infrared value, as argued in [6]. One can use the \( f_n(p) \) given in (3.3) to calculate \( \Omega(A) \); one might worry that since for small \( n \) the difference between (3.3) and the exact result (Appendix B) is not necessarily small, the small area behavior of \( \Omega(A) \) thus obtained may differ from the correct one. In practice, however, the difference turns out to be insignificant. We have checked that by calculating \( \Omega \) in two different ways – using (3.3) for all \( n \), and using the exact \( f_n \) for low \( n \) and the approximation (3.3) for large \( n \)’s. The two functions agree to within a few percent. In fig. 1 we show as an example \( \Omega(A) \), of course multiplied by \( \exp(-\mu c A) \) to offset the trivial vacuum energy effects, and also by 48 for convenience, versus \( \log A \), for \( R = 10 \) and \( n = 8 \) (\( p = 0.8 \)). The resulting function is monotonically decreasing to the appropriate infrared value, in agreement with the ideas of [6]. It is natural to define \( \Omega \) as a “\( c \) – function” in two dimensional gravity. In fig. 2 we exhibit more examples of the behavior of this \( c \) – function for given \( R \) and different values of \( n \), for which the infrared theory contains different numbers of copies of pure gravity models.

4. Other Issues

The simple picture for the space of theories (1.1) coupled to gravity which emerges from the results of [6] and this paper leads of course to many additional predictions which can in principle be checked against the matrix model results [8]. As an example, one may consider the flow of the operators in the \( c = 1 \) string as one approaches the IR pure gravity fixed point. There are three kinds of operators in the UV \( c = 1 \) theory:

1) Tachyon modes \( \mathcal{T}_p \).

2) Oscillator states (also known as discrete states of ghost number 2 [10].)

3) “Ground ring” or discrete states of ghost number zero [10] [11].

Consider first the tachyon operators \( \mathcal{T}_p \). On general grounds one expects them to flow to the physical operators of the IR minimal models, which form Kac tables. In particular, when the IR theory consists of \( n \) copies of pure gravity, we expect all \( \mathcal{T}_p \) to flow to the \( n \) identity operators in the \( n \) decoupled vacua. This is quite plausible because of the following argument. We have observed before that when the gravitational SG model (1.2) is slightly perturbed by other cosines, as in (1.1), one should find (as \( A \to \infty \))

\[
Z(\lambda_i, A) \approx A^{-3-\frac{1}{2} \mu c (\lambda_i, p_i) A} \tag{4.1}
\]
That means that the correlation function of $T_p$, given by
\[
\langle T_{p_1} T_{p_2} \cdots T_{p_n} \rangle \approx \frac{1}{Z(\lambda_i, A)} \frac{\partial}{\partial \lambda_1} \frac{\partial}{\partial \lambda_2} \cdots \frac{\partial}{\partial \lambda_n} Z(\lambda_i, A) \approx A^n \tag{4.2}
\]
so that all $T_p$ behave in the IR as area operators. Of course the conclusion assumes (4.1), but that can in principle be checked by using the general tachyon correlation functions of [8]. It is also clear that there are precisely $n (= pR)$ distinct area operators, since from (1.2) we see that
\[
\langle T_{l_1/R} T_{-l_2/R} \rangle = \delta_{l_1,l_2} \quad l_1, l_2 = 0, 1, 2, \cdots, n - 1 \tag{4.3}
\]
so that $T_{l/R}, l = 0, 1, \cdots, n - 1$ are independent whereas $T_{(l+n)/R} \approx T_{l/R}$ due to the interaction in (1.2). Hence there are precisely $n$ independent area operators in the IR theory, in agreement with the expected picture. Of course, if we now turn on $\lambda_i$ with finite strength, the above picture will be correct for generic $\lambda_i$, but at multicritical points, higher minimal models will appear and the spectrum will change appropriately to accommodate the larger Kac table [6].

The ground ring states of the $c = 1$ model presumably flow to the ground ring states of the $c = 0$ models. It is not completely clear what is the IR limit of the oscillator states. The only states in the IR fixed point which are not accounted for are the negative ghost number states of [10] and in principle the oscillator states could flow to those, but the details remain to be worked out. In any case, these issues are more difficult to study using matrix models since both the discrete oscillator states and the ground ring operators are far less understood in that framework than are the tachyon modes.

Another issue we have ignored so far is the physics of (1.2) when the cosine is irrelevant, $p > 2$. In that case we have seen that the fixed area partition sums are given by asymptotic series both on the sphere (2.4), and on the torus (3.3), (3.5). Consider for example the spherical result (2.4). Purely on dimensional grounds, the irrelevant coupling $\lambda_p$ goes to zero in the IR. Hence naively the IR theory is the $c = 1$ string. But this raises a problem. Since we know [8] that the number of degrees of freedom decreases as we flow from the UV to the IR, the UV fixed point must have more d.o.f. than the $c = 1$ string, but we know that theories with more d.o.f. than $c = 1$ are tachyonic [7] and are not likely to be produced by the flow (1.2) with $p > 2$. Then what is the UV behavior of this model? To answer that question one must sum the series (2.4) in the region of large effective coupling. Since the series is asymptotic one must use resummation techniques. One possibility is to
Laplace transform \( Z(\lambda, A) \to Z(\lambda, \mu) \) as in (2.3); one finds then as observed in [3] a series with a finite radius of convergence (in \( \mu \), for \( \lambda = 1 \)), converging for \( \mu < \mu_c(p) \). In fact, one can take \( Z(\lambda, \mu) \) as fundamental as in [5]; there is nothing wrong with that, but from our point of view it corresponds to choosing one of many possible resumptions of (2.4). In any case, \( Z(\lambda, \mu) \) can be easily summed and in the UV, \( \mu \to \infty \), one finds [5]

\[
Z(\lambda, \mu) \sim \mu^2 \log \mu
\]

indicative of a \( c = 1 \) behavior at short distances. This is a general feature of all resum- mations of (2.4). This however raises a new problem: we know that a \( c = 1 \) model in the UV does not flow to a \( c = 1 \) model in the IR under the perturbation one finds in the UV analysis alluded to above. Different resumptions of (2.4) give rise in the IR either to the \( c = 1 \) model or – via a nonperturbative contribution – to a collection of \( c = 0 \) ones. This is in qualitative agreement with the phase diagram of the flat space Sine- Gordon model, but a few puzzles remain; a complete understanding of this region in Sine- Gordon coupling space must await further work.

5. Conclusion

In this paper we have shown that the matrix model results for the partition sum of the gravitational Sine-Gordon model lead to a description of the corresponding RG flow consistent with what one expects from heuristic considerations. In particular, the “c – theorem” of [3] for the torus partition sum is satisfied and the number of d.o.f. in the IR corresponds to a number of decoupled pure gravity systems. There are still many interesting open questions. In particular, the understanding of the Sine-Gordon model with an irrelevant cosine is unsatisfactory. A better understanding is certainly needed for the application of (1.2) to two dimensional string theory.

We have also argued that the KdV and generalized KdV hierarchies of [2] are hidden in the generating functional of the \( c = 1 \) matrix model. There should be a simpler and more elegant way of deriving this result. The form of the \( c = 1 \) generating functional obtained in [12] seems closest to the \( c < 1 \) results and perhaps it can be used to prove our results in much greater generality and with much less work.

Acknowledgements

We would like to thank G. Moore, D. Petrich and S. Kachru for useful discussions. This work was partially supported by NSF grant PHY90-21984.
Appendix A. Large Area Partition Function on the Sphere

In this appendix we calculate the large area behavior of \( (2.4) \) for relevant momenta \( p < 2 \). Defining \( F(A) = A^3 Z_0(A) \) and \( A^{(2-p)} = x^{(1-p)} \), we have

\[
F(x) = \sum_{n=0}^{\infty} \frac{(1-p)^n x^{n(1-p)}}{n! \Gamma(n(1-p) + 1)} \tag{A.1}
\]

Laplace transforming (A.1) gives

\[
\bar{F}(t) = \int_0^\infty F(x) e^{-xt} dx = \sum_{n=0}^{\infty} \frac{t^{-1-n(1-p)}}{n!} (1-p)^n = \frac{1}{t} e^{(1-p) \frac{t^{p-1}}{2}} \tag{A.2}
\]

Consider first \( 0 < p < 1 \), where large \( A \) corresponds to large \( x \). We assume an asymptotic form for \( F(x) \) at large \( x \), \( F(x) \sim x^{a(p)} e^{b(p)x^{c(p)}} \). Substituting this into (A.2) and using the method of steepest descent to evaluate the integral, we obtain \( \bar{F}(t) \) for small \( t \) in terms of \( a(p) \), \( b(p) \), and \( c(p) \). Equating this with the last expression in (A.2), we find

\[
a(p) = -\frac{1}{2} \frac{(1-p)}{(2-p)}, \quad b(p) = \frac{(2-p)}{(1-p)} |1-p|^{\frac{2}{2-p}}, \quad c(p) = \frac{(1-p)}{(2-p)} \tag{A.3}
\]

Transforming variables back from \( x \) to \( A \), we thus have the desired large area behavior

\[
F(A) \sim A^{-\frac{1}{2}} e^{\mu_c(p)A} \tag{A.4}
\]

where \( \mu_c(p) = b(p) \).

For \( 1 < p < 2 \), large \( A \) corresponds to small \( x \), but in this case, we may simply assume \( F(x) \sim x^{a(p)} e^{b(p)x^{c(p)}} \) for small \( x \). Performing the same analysis as above, we again find (A.4).

Appendix B. Calculations on the Torus

We use the formula derived in [5] for the calculation of the torus correlation functions and find:

\[
\langle T_p^n T_{-p}^n \rangle_{h=1} = -\mu^{n(p-2)} (p-1)^n \frac{n!}{24} f_n(p) \tag{B.1}
\]
where the \( f_n \)'s are polynomials of degree \( n + 1 \). The first nine \( f_n \)'s are:

\[
\begin{align*}
f_1(p) &= p^2 - p - 1 \\
f_2(p) &= 3p^3 - 8p^2 + 3p + 3 \\
f_3(p) &= 17p^4 - 72p^3 + 90p^2 - 17p - 20 \\
f_4(p) &= 2(71p^5 - 410p^4 + 842p^3 - 670p^2 + 65p + 105) \\
f_5(p) &= 1569p^6 - 11455p^5 + 32460p^4 - 43475p^3 + 24915p^2 - 1014p - 3024 \\
f_6(p) &= 12(1798p^7 - 15858p^6 + 57137p^5 - 106455p^4 + 105027p^3 - 46322p^2 + 63p + 4620) \\
f_7(p) &= 355081p^8 - 3669526p^7 + 16023411p^6 - 38107279p^5 + 52657584p^4 - 40905739p^3 + 14464814p^2 + 416424p - 1235520 \\
f_8(p) &= 16(425331p^9 - 5038436p^8 + 25830780p^7 - 74616206p^6 + 132018894p^5 - 144550364p^4 + 92211780p^3 - 26867634p^2 - 1440855p + 207025) \\
f_9(p) &= 9(1654101p^{10} - 220912245p^9 + 1299983709p^8 - 4415181390p^7 + 9504928503p^6 - 13361715045p^5 + 12088018191p^4 - 6518013480p^3 + 1595066820p^2 + 120188240p - 108908800)
\end{align*}
\]

Numerically, we find the \( f_n \)'s have \( n \) roots between \( 1 < p < 2 \) of the approximate form \( C_i = 1 + \frac{i - 1}{n} \) where \( 1 \leq i \leq n \), and one negative root at \( C_- = -\frac{1}{\sqrt{\pi n}} \). To determine how well these values approximate the exact zeros of the \( f_n \)'s, \( \tilde{C}_i \) and \( \tilde{C}_- \), we look at the relative error, \( (\tilde{C}_i - C_i)/\tilde{C}_i \). As shown in fig. 3 and fig. 4, where we have plotted the relative errors of the positive roots and the negative root, respectively, versus \( n \), these approximations get better at larger \( n \). We see from these plots that the relative errors decrease steadily as \( n \) increases, and although this does not constitute a proof, we believe this behavior will continue at higher values of \( n \). In fact, one may show that these errors obey some power law, \( (\tilde{C}_i - C_i)/\tilde{C}_i \simeq \alpha_i n^{-\beta_i} \), where \( \alpha_i < 0.1 \), and \( \beta_i > 0 \). We may therefore write

\[
f_n(p) = c_n(\sqrt{\pi np} + 1)\frac{\Gamma(n(2 - p) + \frac{1}{2})}{\Gamma(n(1 - p) + \frac{1}{2})}
\]

where the \( c_n \)'s are some \( n \)-dependent normalization factors. To fix the \( c_n \)'s, we use the fact that \( f_n(0) = g_n(0) \) given by (3.4) with \( p = 0 \). Using Stirling’s formula \( (\Gamma(x) \sim \sqrt{2\pi x} x^{x - \frac{1}{2}} e^{-x}) \), we find the \( c_n \)'s to be \( \frac{(-1)^n}{\sqrt{2n}} \). Thus,

\[
f_n(p) = \frac{(-1)^n}{\sqrt{2n}}(\sqrt{\pi np} + 1)\frac{\Gamma(n(2 - p) + \frac{1}{2})}{\Gamma(n(1 - p) + \frac{1}{2})}; \quad n \to \infty
\]
Although (B.4) strictly holds only as $n \to \infty$, it is a very good approximation for low $n$, and the difference between (B.4) and the exact $f_n$’s are only a few percent.

The one-loop partition sum for fixed $\mu$ is

$$Z_1 = \frac{R}{24} (1 + \frac{1}{R^2}) \log \mu + \frac{R}{24} \sum_{n \geq 1} \frac{(p-1)^n}{n!} \mu^{n(p-2)} \left( f_n(p) + \frac{g_n(p)}{R^2} \right) \quad (B.5)$$

We Laplace transform (B.5) to fixed area, and neglect the contribution from the $g_n$’s, since as follows from Appendix A, it is suppressed by $A^{-\frac{1}{2}}$ relative to the contribution of $f_n(p)$ at large area. Using (B.4), we find that the large area partition sum is

$$Z_1(A) \sim \frac{R}{24A} \sum_{n \geq 1} \frac{(1-p)^n}{\sqrt{2n} n! \Gamma(n(2-p))} A^{n(2-p)-1} \Gamma(n(2-p)+\frac{1}{2}) \quad (B.6)$$

Using Stirling’s formula to expand the gamma functions and approximating the sum by an integral, we get

$$Z_1(A) \sim \frac{p R}{48A} e^{\mu_c(p) A} \quad (B.7)$$

Finally, we note that the above analysis for the large area behavior of the torus partition sum only hold for $p < 1$. For $1 < p < 2$, the series in (B.6) alternates in sign, and we can no longer neglect the error in $Z_1$ due to the error in the location of the zeros of the $f_n$’s. The point is that $Z_1(A)$ is then much smaller as $A \to \infty$ than the individual terms in the sum defining it, so that a small mistake in each term can lead to a large mistake in estimating $Z_1(A)$.  

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Figure Captions

Fig. 1. $48\Omega(A)e^{-\mu_c A}$ as a function of log $A$ for $R = 10$ and $n = 8$ ($p = 0.8$). The dashed line shows the proper infrared limit for $n = 8$.

Fig. 2. $48\Omega(A)e^{-\mu_c A}$ as a function of log $A$ for $R = 5$ and $n = 1, 2, 3, 4$ ($p = 0.2, 0.4, 0.6, 0.8$). The dashed lines indicate the different infrared limits, showing the different copies of pure gravity models.

Fig. 3. The relative error of the approximate positive roots of the polynomials $f_n$ for the different values of $n$.

Fig. 4. The relative error of the approximate negative root of the polynomials $f_n$ for the different values of $n$. 