Magnetic field effects on the optimal fidelity of standard teleportation via the two qubits Heisenberg XX chain in thermal equilibrium

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Abstract

We study the two qubits Heisenberg XX chain with magnetic impurity in the presence of the external magnetic field and calculate the optimal fidelity of standard teleportation via the thermal equilibrium state. It is found that the combined influence of magnetic impurity and external magnetic field can increase the critical temperatures of entanglement and quantum teleportation without limit. The relation between two kinds of critical temperatures is revealed.

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In recent years, considerable attention has been devoted to the quantum entanglement [1-4] and teleportation [5,6,7] in Heisenberg spin chains, which is closely associated with the study of many practical systems and can be used to realize universal gate operations in solid state quantum computation [8]. The thermal entanglement of spins in a Heisenberg chain with an external magnetic field in thermal equilibrium with a nonzero temperature has been studied [1]. It is shown that the entanglement of a two qubit isotropic Heisenberg system decreases with increasing the temperature $T$ and vanishes beyond a critical value $T_c$ [1]. The teleportation via the thermal states of the two-qubit Heisenberg XX chain has also been investigated [7]. There is a critical temperature above which the thermal state of two-qubit Heisenberg XX chain is useless for quantum standard teleportation. In this Letter, we study the two qubits Heisenberg XX chain with magnetic impurity in the presence of the external magnetic field and calculate the entanglement and the optimal fidelity of standard teleportation via the thermal equilibrium state. It is found that the combined influence of magnetic impurity and external magnetic field can increase the critical temperatures of entanglement and standard teleportation. We construct the envelope of the critical temperature of teleportation fidelity, which is just as the same as the critical temperature of entanglement.

The Hamiltonian for the two qubit Heisenberg XX chain in an external magnetic field $B$ (along the $z$ axis) with a magnetic impurity $B_1$ in the first qubit can be expressed as [9]

$$H = (B + B_1)S_z^1 + BS_z^2 + J(S_x^1 S_x^2 + S_y^1 S_y^2),$$  

(1)

where $S_z^i = S_i^x \pm iS_i^y$, $S_i^\alpha = \sigma_i^\alpha/2$ ($\alpha = x,y,z$ and $i = 1,2$) denotes the local spin $1/2$ operator of the $i$th qubit and $\sigma_i^\alpha$ are the Pauli operators. The chain is said to be antiferromagnetic for the coupling constant $J > 0$ and ferromagnetic for $J < 0$. The eigenvectors and eigenvalues of the Hamiltonian $H$ are

$$|\Psi_1\rangle = |00\rangle, \quad E_1 = -B - \frac{1}{2}B_1,$$

$$|\Psi_2\rangle = |11\rangle, \quad E_2 = B + \frac{1}{2}B_1,$$

$$|\Psi_3\rangle = \frac{J}{\sqrt{2\eta^2 - B_1^2}}|10\rangle + \frac{\eta - B_1/2}{\sqrt{2\eta^2 - B_1^2}}|01\rangle, \quad E_3 = \eta,$$

$$|\Psi_4\rangle = \frac{J}{\sqrt{2\eta^2 + B_1^2}}|10\rangle - \frac{\eta + B_1/2}{\sqrt{2\eta^2 + B_1^2}}|01\rangle, \quad E_4 = -\eta,$$

(2)

where $\eta = \sqrt{J^2 + B_1^2}/4$. From Eq.(2), we can see that the ground state of $H$ is just the Bell singlet state if $B_1 = 0$ and $B^2 < J^2$. For the system in thermal equilibrium at temperature $T$, the density operator can be derived as

$$\rho_{AB} = \frac{1}{Z}[u|11\rangle\langle 11| + v|00\rangle\langle 00| + w_1|10\rangle\langle 10| + w_2|01\rangle\langle 01| + y(|10\rangle\langle 01| + |01\rangle\langle 10|)],$$  

(3)

where

$$u = \exp[-\beta(B + \frac{1}{2}B_1)], \quad v = \exp[\beta(B + \frac{1}{2}B_1)],$$

$$w_1 = e^{-\frac{B}{2}B_1/(\sqrt{2\eta^2 - B_1^2})}, \quad w_2 = e^{-\frac{B}{2}B_1/(\sqrt{2\eta^2 + B_1^2})},$$

$$y = e^{-\beta(B + \frac{1}{2}B_1)},$$
\[ w_1 = J^2 \left[ \frac{\exp(-\eta \beta)}{2 \eta^2 - \eta B_1} + \frac{\exp(\eta \beta)}{2 \eta^2 + \eta B_1} \right] , \]
\[ w_2 = \frac{\exp(-\eta \beta)(\eta - B_1/2)^2}{2 \eta^2 - \eta B_1} + \frac{\exp(\eta \beta)(\eta + B_1/2)^2}{2 \eta^2 + \eta B_1} , \]
\[ y = -\frac{J}{\eta} \sinh(\eta \beta) , \quad Z = 2 \cosh[(B + \frac{1}{2} B_1)\beta] + 2 \cosh(\eta \beta) , \]
\[ w_2 = \frac{\exp(-\eta \beta)(\eta - B_1/2)^2}{2 \eta^2 - \eta B_1} + \frac{\exp(\eta \beta)(\eta + B_1/2)^2}{2 \eta^2 + \eta B_1} , \]
\[ y = -\frac{J}{\eta} \sinh(\eta \beta) , \quad Z = 2 \cosh[(B + \frac{1}{2} B_1)\beta] + 2 \cosh(\eta \beta) , \]
\[ and \, \beta = \frac{1}{k_B T} \text{ with } k_B \text{ the Boltzmann’s constant.} \]
\[ \text{Next, we briefly investigate the entanglement characterized by concurrence } [10] \text{ of the two-qubit Heisenberg XX chain with magnetic impurity in an external magnetic field in the thermal equilibrium. The concurrence } C \text{ of density operator } \rho_{AB} \text{ can be calculated,} \]
\[ C = 2 \max[0, (|y| - 1)/Z] , \]
\[ \text{where } |x| \text{ gives the absolute value of } x. \text{ From Eq.(5), we can see that the thermal entanglement is invariant under the substitution } B \rightarrow -B \text{ and } B_1 \rightarrow -B_1 \text{ or } J \rightarrow -J. \]
\[ \text{The latter indicates that the entanglement is the same for the antiferromagnetic and ferromagnetic cases. For } T = 0, \text{ at certain critical value } B_c^\pm \text{ of the magnetic field } B, \text{ the entanglement becomes a nonanalytical function of the magnetic field and a quantum phase transition occurs } [1,11]. \text{ It is easy to see that the critical magnetic field } B_c^\pm \text{ is dependent of the magnetic impurity } B_1 \text{ and can be expressed as } B_c^\pm = \eta \pm \frac{1}{2} B_1. \]
\[ \text{In Fig.1(a), the concurrence } C \text{ is plotted as a function of } k_B T \text{ and } B \text{ with } J = 1 \text{ and } B_1 = 0. \text{ In Fig.1(b), the concurrence } C \text{ is plotted as a function of } k_B T \text{ and } B_1 \text{ with } J = 1 \text{ and } B = 0. \text{ From Fig.1(a), we can see that there exists a critical temperature } k_B T_c \approx 1.13459 J, \text{ beyond which the thermal entanglement is zero. However, in Fig.1(b), it is shown that the critical temperature is increased due to the presence of the magnetic impurity. In fact, the critical temperature can be explicitly expressed as} \]
\[ k_B T_c = \frac{\eta}{\ln(\eta + \sqrt{J^2 + \eta^2}) - \ln J} . \]
\[ \text{In the following, we consider the standard teleportation protocol by making use of the above two-qubit thermal state } \rho_{AB} \text{ as resource. Horodecki et al. have defined a optimal fidelity } f(\rho) \text{ of the standard teleportation scheme } [12] \text{ which quantifying the quality of the teleportation that can be achieved with the given state } \rho. \text{ The optimal fidelity of standard teleportation is related with the maximal singlet fraction via the equation } f(\rho) = \frac{2 F(\rho) + 1}{3} , \text{ in which the maximal singlet fraction } F(\rho) \text{ is defined as the maximal overlap of the state } \rho \text{ with a maximally entangled (ME) state } [12] \]
\[ F(\rho) = \max_{|\psi\rangle=\text{ME}} \langle \psi | \rho | \psi \rangle . \]
\[ \text{An explicit value for the maximal singlet fraction has been derived } [13]. \text{ If one considers the real } 3 \times 3 \text{ matrix } \hat{R} = \text{Tr}(\rho \sigma_i \otimes \sigma_j) \text{ with } \{ \sigma_i, \ i = 1,2,3 \} \text{ the Pauli matrices, then} \]
\[ F(\rho) = \frac{1 + \lambda_1 + \lambda_2 - \text{sgn}(|\text{det}(\hat{R})|) \lambda_3}{4} \]
Figure 1: (a) The concurrence $C$ is plotted as a function of $k_B T$ and $B$ with $J = 1$ and $B_1 = 0$; (b) the concurrence $C$ is plotted as a function of $k_B T$ and $B_1$ with $J = 1$ and $B = 0$. 
Figure 2: The optimal fidelity $f$ of the thermal state $\rho_{AB}$ is plotted as a function of $k_BT$ with $J = 1$ for four different cases: (Solid Line) $B = -1$ and $B_1 = 2$; (Dot Line) $B = 0$ and $B_1 = 2$; (Dash Dot Line) $B = -0.5$ and $B_1 = 0$; (Dash Dot Dot Line) $B = 0$ and $B_1 = 0$.

In Fig.2, the optimal fidelity $f$ of the thermal state $\rho_{AB}$ is plotted as a function of $k_BT$ with $J = 1$ for four different cases. In the case of $B = 0$ and $B_1 = 0$ (Dash Dot Dot line in Fig.2), we can see that there exists a temperature $T_c^{(f)}(0, 0)$, beyond which the optimal fidelity is smaller than the classical limit $2/3$. This critical temperature $T_c^{(f)}(0, 0)$ is given by $T_c^{(f)}(0, 0) \simeq 1.13459J/k_B$. The critical temperature decreases with the external magnetic field (See dash dot line in Fig.2) or the magnetic impurity (See dot line in Fig.2). However, the combined influence of the external magnetic field and the magnetic impurity can increase the critical temperature $T_c^{(f)}$ above $1.13459J/k_B$ (See solid line in Fig.2). It is easy to verify that the critical temperature $T_c^{(f)}$ is the same as $T_c$ when $B_1 = -2B$.

In Fig.3, the critical temperature $T_c$ of entanglement is plotted as a function of the
Figure 3: The critical temperature $k_B T_c$ (Solid Line) of entanglement is plotted as a function of the magnetic impurity $B_1$ with $J = 1$. The critical temperature $k_B T_c^{(f)}$ of the teleportation fidelity is plotted as a function of the magnetic impurity $B_1$ with $J = 1$ for five different values of external magnetic field: (Dash Line) $B = 0$; (Dot Line) $B = -1$; (Dash Dot Line) $B = -2$; (Dash Dot Dot Line) $B = -3$; (Short Dash Line) $B = -4$.

magnetic impurity $B_1$ with $J = 1$ and the critical temperature $T_c^{(f)}$ of the teleportation fidelity is plotted as a function of the magnetic impurity $B_1$ with $J = 1$ for five different values of external magnetic field. It is shown that the critical value $T_c$ increases with the magnetic impurity $B_1$ and is independent of $B$. From Fig.3, it is easy to see that the curve of $T_c$ constructs a envelope of the curves \{T_c^{(f)}(J,B_1,B)\}$_B$. To prove it, we recall the classical theory of envelopes [14]: Suppose one has a set of functions $g(x; \alpha)$ parametrized by $\alpha$. Then the envelope of \{g(x; \alpha)\}$_\alpha$ can be obtained by solving $\frac{\partial g(x; \alpha)}{\partial \alpha} = 0$ and substituting the solution $\alpha(x)$ into the original function, i.e., $g(x; \alpha(x))$. In our case, $g = T_c^{(f)}$ and $\alpha = B$. Thus we reach a condition

$$\frac{\partial T_c^{(f)}(J,B_1,B)}{\partial B} = 0. \quad (10)$$

From Eq.(9), we know that $T_c^{(f)}$ satisfies the following equation

$$\sinh(\eta/k_B T_c^{(f)}) = \frac{\eta}{J} \cosh[(\frac{1}{2} B_1 + B)/k_B T_c^{(f)}]. \quad (11)$$

Combining Eq.(10) and Eq.(11), we can derive that

$$B = -\frac{1}{2} B_1 \quad (12)$$

is the unique solution. Substituting Eq.(12) into Eq.(11), it is easy to verify that $T_c^{(f)}(J,B_1,-\frac{1}{2} B_1)$ is equal to $T_c(J,B_1)$.
In conclusion, we studied the two-qubit Heisenberg XX chain with magnetic impurity. Firstly, we investigated the entanglement in the two-qubit Heisenberg XX chain with magnetic impurity, and it is shown that the critical temperature $T_c$ of entanglement, which is independent of the external magnetic field, can be improved with the increasing intensity of magnetic impurity. Then, We show that the combined influence of magnetic impurity and the external magnetic field can improve the critical temperature (beyond which the optimal fidelity is smaller than the classical limit $2/3$) of thermal state of the two-qubit Heisenberg XX chain as a useful resource for standard quantum teleportation.

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