A Design Framework for Epsilon-Private Data Disclosure

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Abstract

In this paper, we study a stochastic disclosure control problem using information-theoretic methods. The useful data to be disclosed depend on private data that should be protected. Thus, we design a privacy mechanism to produce new data which maximizes the disclosed information about the useful data under a strong $\chi^2$-privacy criterion. For sufficiently small leakage, the privacy mechanism design problem can be geometrically studied in the space of probability distributions by a local approximation of the mutual information. By using methods from Euclidean information geometry, the original highly challenging optimization problem can be reduced to a problem of finding the principal right-singular vector of a matrix, which characterizes the optimal privacy mechanism. In two extensions we first consider a noisy disclosure channel and then we look for a mechanism which finds $U$ based on observing $X$, maximizing the mutual information between $U$ and $Y$ while satisfying the privacy criterion on $U$ and $Z$ under the Markov chain $(Z,Y) - X - U$.

I. INTRODUCTION

The amount of data created by humans, robots, advanced cyber-physical and software systems and billions of interconnected sensors is growing rapidly. Unwanted inference possibilities from this data cause privacy threats. Thus, privacy mechanisms are required before data can be disclosed.

Accordingly, the information theoretic approach to privacy is receiving increased attention and related works can be found in [1]–[18]. One of the earliest works is [1], where a source coding problem with secrecy is studied. In both [1] and [2], the privacy-utility trade-off is considered using expected distortion and equivocation as measure of utility and privacy. In [3], the concept of a privacy funnel is introduced, where the privacy-utility trade-off under log-loss distortion is considered. In [4], the concept of differential privacy is introduced, which aims to minimize the chance of identifying the membership in a database. In [5], the hypothesis test performance of an adversary is used to measure the privacy leakage. The concept of maximal leakage is introduced in [6] and some bounds on privacy utility trade-off are provided. In [7], fundamental limits of privacy utility trade-off are studied measuring the leakage using estimation-theoretic guarantees. Properties of rate-privacy functions are studied in [8], where either maximal correlation or mutual information are used for measuring privacy. Biometric identification systems with no privacy leakage are studied in [9].

Our problem formulation is closest related to [10], where the problem of maximizing mutual information $I(U;Y)$ given the leakage constraint $I(U;X) \leq \epsilon$ and Markov chain $X - Y - U$ is studied. Under the assumption of perfect privacy, i.e., $\epsilon = 0$, it is shown that the privacy mechanism design problem can be reduced to a standard linear program. In [11], the work has been extended considering the privacy utility trade-off with a rate constraint for the disclosed data.

In this paper, we consider a similar problem as in [10] depicted in Fig. 1, where an agent wants to disclose some useful data to a user. The useful data is denoted by the random variable (RV) $Y$. Furthermore, $Y$ is dependent on the private data denoted by RV $X$, which is not accessible to the agent. Due to privacy considerations, the agent cannot release the useful data directly. So, the agent uses a privacy mechanism to produce data $U$ that can be disclosed. $U$ should disclose as much information about $Y$ as possible and at the same time satisfy the privacy criterion. In this work, the perfect privacy condition considered in [10] is relaxed considering an element-wise $\chi^2$ privacy criterion which we call “Strong $\chi^2$-privacy criterion”. A $\chi^2$-privacy criterion has been also considered in [7], studying a related privacy-utility trade-off problem. Since the optimization problem is difficult, only upper and lower bounds on the optimal privacy-utility trade-off have been derived. Furthermore, a convex program for designing the privacy mechanism is introduced, where additional constraints are added to the main privacy problem. In contrast, we in this paper focus on finding an explicit design for the privacy mechanism problem for small leakage considering our strong $\chi^2$-privacy criterion. As a side result we show that the upper bound in [7] is achievable in the small leakage regime.

We use methods from Euclidean information theory [19], [20] to study the design optimization problem. There exist many problems in information theory, where one main difficulty is not having a geometric structure on the space of probability distributions. If we assume that the distributions of interest are close to each other, then KL divergence can be well approximated by weighted squared Euclidean distance. This results in a framework where a mutual information term has been approximated in order to simplify the optimization problem. This framework has been used in [19], [20], specifically, in [19], where it was employed for point-to-point channels and some specific broadcast channels. In this paper, due to the strong $\chi^2$-privacy criterion, we can exploit the information geometry approach and approximate the KL divergence and mutual information in case of a
small leakage $\epsilon$. This allows us to transfer the main problem into an analytically simple largest singular value problem, which also provides deep intuitive understanding of the mechanism.

In more detail we can summarize our contribution as follows

(i) We present an information-theoretic disclosure control problem using a strong $\chi^2$-privacy criterion in Section II.
(ii) We introduce and utilize concepts from Euclidean information theory to linearize the problem and derive a simple approximate solution for small leakage in Section III. In particular our result shows that the upper bound found in [7] is actually achievable for small leakage.
(iii) We provide a geometrical interpretation of the privacy mechanism design problem and two examples are given. Significantly that enhance the intuitive understanding of the privacy mechanism.
(iv) We transfer our methods to two extended problems which demonstrates the value of our approach as a design framework.

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**II. SYSTEM MODEL AND PROBLEM FORMULATION**

Let $P_{XY}$ denote the joint distribution of discrete random variables $X$ and $Y$ defined on finite alphabet $\mathcal{X}$ and $\mathcal{Y}$ with equal cardinality, i.e., $|\mathcal{X}| = |\mathcal{Y}| = K$. We represent $P_{XY}$ by a matrix defined on $\mathbb{R}^{K \times K}$ and marginal distributions of $X$ and $Y$ by vectors $P_X$ and $P_Y$ defined on $\mathbb{R}^K$. We assume that each element in vectors $P_X$ and $P_Y$ is non-zero. Furthermore, we represent the leakage matrix $P_{X|Y}$ by a matrix defined on $\mathbb{R}^{K \times K}$ which is assumed to be invertible. In the related privacy problem with perfect privacy [10], it has been shown that information can be only revealed if $P_{X|Y}$ is not invertible. This result was also proved in [21] in a source coding setup. RVs $X$ and $Y$ denote the private data and the useful data. In this work, privacy is measured by the strong $\chi^2$-privacy criterion which we introduce next.

**Definition 1.** Given two random variables $X \in \mathcal{X}$ and $U \in \mathcal{U}$ with joint pmf $P_{XU}$ where $X$ describes the private data and $U$ denotes the disclosed data, for $\epsilon > 0$, the strong $\chi^2$-privacy criterion is defined as follows

$$
\chi^2(P_{X|U=u}|P_X) = \sum_{x \in \mathcal{X}} \left( \frac{P_{X|U=u}(x) - P_X(x)}{P_X(x)} \right)^2 = \left\| [\sqrt{P_X}]^{-1}(P_{X|U=u} - P_X) \right\|^2 \leq \epsilon^2, \forall u \in \mathcal{U},
$$

where $[\sqrt{P_X}]^{-1}$ is a diagonal matrix with diagonal entries $\{\sqrt{P_X(x)}^{-1}, x \in \mathcal{X}\}$. The norm is the Euclidean norm.

The strong $\chi^2$-privacy criterion means that the all distributions (vectors) $P_{X|U=u}$ for all $u \in \mathcal{U}$ are close to $P_X$ in the Euclidean sense. The closeness of $P_{X|U=u}$ and $P_X$ allows us to use the concepts of information geometry so that we can locally approximate the KL divergence and mutual information between $U$ and $Y$ for small $\epsilon > 0$. In [7], the concept of $\chi^2$-information between $U$ and $X$ is employed as privacy criterion. The relation between these two criteria is as follows

$$
\chi^2_{\text{information}}(X;U) = \mathbb{E}_U \left[ \chi^2(P_{X|U=u}|P_X) \right].
$$

Our goal is to design the privacy mechanism that produces the disclosed data $U$, which maximizes $I(U;Y)$ and satisfies the strong $\chi^2$-privacy criterion. The relation between $U$ and $Y$ is described by the kernel $P_{U|Y}$ defined on $\mathbb{R}^{U \times K}$. Thus, the privacy problem can be stated as follows

$$
\max_{P_{U|Y}} I(U;Y),
$$

subject to:

$$
X - Y - U,
$$

$$
\left\| [\sqrt{P_X}]^{-1}(P_{X|U=u} - P_X) \right\|^2 \leq \epsilon^2, \forall u \in \mathcal{U}.
$$

The strong $\chi^2$-privacy criterion for small $\epsilon$ results in closeness of $P_{Y|U=u}$ and $P_Y$ in the output distributions space, which allows us to transfer the main problem into a linear algebra problem, specifically, finding the largest singular value of a matrix. We refer to problem (1a) as $\epsilon$-private data disclosure.
Remark 1. Although we are interested in small $\epsilon$, we do not allow $\epsilon = 0$ since this conflicts with our assumption that $P_{X|Y}$ is invertible [21, Th. 4].

Remark 2. By using an inequality between KL divergence and the strong $\chi^2$-privacy criterion [22, Page 130], we have
\[
D(P_{X|U=u}||P_X) \leq \chi^2(P_{X|U=u}||P_X) \leq \epsilon^2, \forall u,
\]
where $D(P_{X|U=u}||P_X)$ denotes KL divergence between distributions $P_{X|U=u}$ and $P_X$. Thus, we have
\[
I(U;X) = \sum_{u \in U} P_U(u) D(P_{X|U=u}||P_X) \leq \epsilon^2. \tag{2}
\]
Consequently, information leakage using the strong $\chi^2$-privacy criterion implies also a bound on the mutual information $I(U;X)$. In the following we show that by using the Euclidean information theory method, we can strengthen [2] and show that (1c) implies $I(U;X) \leq \frac{1}{2} \epsilon^2 + o(\epsilon^2)$ for small $\epsilon$.

**Proposition 1.** It suffices to consider $U$ such that $|U| \leq |Y|$. Furthermore, the supremum in (1a) is achieved so that we used maximum instead of supremum.

**Proof.** The proof is provided in Appendix A. \qed

### III. Privacy Mechanism Design

In this section we follow the method used in [19], [20] and show that input and output spaces can be reduced to linear spaces where the kernel describes a linear mapping between these two spaces. Thus, the privacy problem can be reduced to a linear algebra problem. The solution to the linear problem is provided and elucidates the optimal mechanism for producing $U$. We consider the resulting design framework to be the main contribution of this work.

By using (1c), we can rewrite the conditional distribution $P_{X|U=u}$ as a perturbation of $P_X$. Thus, for any $u \in U$, we can write $P_{X|U=u} = P_X + \epsilon \cdot J_u$, where $J_u \in \mathbb{R}^K$ is a perturbation vector that has the following three properties:
\[
\sum_{x \in X} J_u(x) = 0, \forall u, \tag{3}
\]
\[
\sum_{u \in U} P_U(u) J_u(x) = 0, \forall x, \tag{4}
\]
\[
\sum_{x \in X} \frac{J_u^2(x)}{P_X(x)} \leq 1, \forall u. \tag{5}
\]
The first two properties ensure that $P_{X|U=u}$ is a valid probability distribution and the third property follows from (1c). The next proposition shows that $I(U;X)$ can be locally approximated by a squared Euclidean metric. In the following we use the Bachmann-Landau notation where $o(\epsilon)$ describes the asymptotic behaviour of a function $f : \mathbb{R}^+ \rightarrow \mathbb{R}$ which satisfies that $\frac{f(\epsilon)}{\epsilon} \rightarrow 0$ as $\epsilon \rightarrow 0$.

**Proposition 2.** For all $\epsilon < \frac{\min_{x \in X} P_X(x)}{\sqrt{\max_{x \in X} P_X(x)}}$ (1c) results in a leakage constraint as follows
\[
I(X;U) \leq \frac{1}{2} \epsilon^2 + o(\epsilon^2), \tag{6}
\]

**Proof.** The proof is provided in Appendix B. \qed

Now we show that the distribution $P_{Y|U=u}$ can be written as a linear perturbation of $P_Y$. Since we have the Markov chain $X - Y - U$, we can write
\[
P_{X|U=u} - P_X = P_{X|Y}[P_{Y|U=u} - P_Y] = \epsilon \cdot J_u.
\]
Due to the assumed non-singularity of the leakage matrix we obtain
\[
P_{Y|U=u} - P_Y = P_X^{-1}[P_X|U=u] - P_Y = \epsilon \cdot P_X^{-1} J_u. \tag{7}
\]
The next proposition shows that $I(U;Y)$ can be locally approximated by a squared Euclidean metric [19], [20]. In the following we use the notation $\cong$ which is defined as
\[
f(x) = g(x) + o(x^2) \rightarrow f(x) \cong g(x),
\]
where $g(x)$ is the second Taylor expansion of $f(x)$ at point $x$. 

Proposition 3. For all \( \epsilon < \frac{\sigma_{\text{min}}(P_{X|Y}) \min_{y \in Y} P_Y(y)}{\sqrt{\max_{x \in X} P_X(x)}} \), \( I(U; Y) \) can be approximated as follows

\[
I(Y; U) \approx \frac{1}{2} \epsilon^2 \sum_u P_U(u) \| \sqrt{P_Y^{-1}} P_{X|Y}^{-1} \| \sqrt{P_X} \| L_u \|^2,
\]

where \( \sqrt{P_Y^{-1}} \) and \( \sqrt{P_X} \) are diagonal matrices with diagonal entries \( \{ \sqrt{P_Y^{-1}}, \forall y \in Y \} \) and \( \{ \sqrt{P_X}, \forall x \in X \} \). Furthermore, for every \( u \in U \) we have \( L_u = \sqrt{P_X} \) \( J_u \in \mathbb{R}^K \).

Proof. For the local approximation of the KL-divergence we follow similar arguments as in [19, 20]:

\[
I(Y; U) = \sum_u P_U(u) D(P_{Y|U=u} || P_Y) = \sum_u P_U(u) \sum_y P_{Y|U=u}(y) \log \left( \frac{P_{Y|U=u}(y)}{P_Y(y)} \right)
= \sum_u P_U(u) \sum_y P_{Y|U=u}(y) \log \left( \frac{1 + \epsilon \frac{P_{X|Y}^{-1} J_y}{P_Y(y)} }{P_Y(y)} \right)
\approx \frac{1}{2} \epsilon^2 \sum_u P_U(u) \sum_y \left( \frac{P_{X|Y}^{-1} J_y}{P_Y} \right)^2 + o(\epsilon^2)
= \frac{1}{2} \epsilon^2 \sum_u P_U(u) \| \sqrt{P_Y^{-1}} P_{X|Y}^{-1} J_u \|^2 + o(\epsilon^2)
\approx \frac{1}{2} \epsilon^2 \sum_u P_U(u) \| \sqrt{P_Y^{-1}} P_{X|Y}^{-1} \sqrt{P_X} \| L_u \|^2,
\]

where (a) comes from second order Taylor expansion of \( \log(1 + x) \) which is equal to \( x - \frac{x^2}{2} + o(x^2) \) and using the fact that we have \( \sum_y P_{X|Y}^{-1} J_y(y) = 0 \). The latter follows from (5) and the property of the leakage matrix \( 1^T \cdot P_{X|Y} = 1^T \), we have

\[ 0 = 1^T \cdot J_u = 1^T \cdot P_{X|Y}^{-1} J_u, \]

where \( 1 \in \mathbb{R}^K \) denotes a vector with all entries equal to 1. For approximating \( I(U; Y) \), we use the second Taylor expansion of \( \log(1 + x) \). Therefore we must have \( \frac{e}{P_{X|Y}^{-1} J_u(y) / P_Y(y)} < 1 \) for all \( u \) and \( y \). One sufficient condition for \( \epsilon \) to satisfy this inequality is to have \( \epsilon < \frac{\sigma_{\text{min}}(P_{X|Y}) \min_{y \in Y} P_Y(y)}{\sqrt{\max_{x \in X} P_X(x)}} \), since in this case we have

\[
\epsilon^2 \| P_{X|Y}^{-1} J_u(y) \|^2 \leq \epsilon^2 \| P_{X|Y}^{-1} J_u \|^2 \leq \epsilon^2 \sigma_{\text{max}}^2 \left( P_{X|Y}^{-1} \right) \| J_u \|^2
\leq \frac{\epsilon^2 \max_{x \in X} P_X(x)}{\sigma_{\text{min}}^2(P_{X|Y})} \leq \min_{y \in Y} P_Y^2(y),
\]

which implies \( \frac{e}{P_{X|Y}^{-1} J_u(y) / P_Y(y)} < 1 \). The step (a) follows from \( \sigma_{\text{max}}^2 \left( P_{X|Y}^{-1} \right) = \frac{1}{\sigma_{\text{min}}^2(P_{X|Y})} \) and \( \| J_u \|^2 \leq \max_{x \in X} P_X(x) \). The latter inequality follows from (5) since we have

\[
\frac{\| J_u \|^2}{\max_{x \in X} P_X(x)} \leq \sum_{x \in X} \frac{J_u^2(x)}{P_X(x)} \leq 1.
\]

The following result shows that by using local approximation in (8), the privacy problem defined in (1a) can be reduced to a linear algebra problem. In more detail, by substituting \( L_u \) in (5), (4) and (5) we obtain next corollary.

Corollary 1. For all \( \epsilon < \frac{\sigma_{\text{min}}(P_{X|Y}) \min_{y \in Y} P_Y(y)}{\sqrt{\max_{x \in X} P_X(x)}} \), the privacy mechanism design problem in (1a) can be approximately solved by the following linear problem

\[
\max_{\{L_u, P_U\}} \sum_u P_U(u) \| W \cdot L_u \|^2,
\]

subject to: \( \| L_u \|^2 \leq 1, \forall u \in U \),

\[
\sum_u \sqrt{P_X(x)} L_u(x) = 0, \forall u,
\]

\[
\sum_u P_U(u) \sqrt{P_X(x)} L_u(x) = 0, \forall x,
\]
where \( W = [\sqrt{P_X^{-1}}]P_{X|Y}^{-1}[\sqrt{P_X}] \) and the \( o(e^2) \)-term is ignored.

Condition (11) can be interpreted as an inner product between vectors \( L_u \) and \( \sqrt{P_X} \), where \( \sqrt{P_X} \in \mathbb{R}^K \) is a vector with entries \( \{ \sqrt{P_X(x)} \, x \in \mathcal{X} \} \). Thus, condition (11) states an orthogonality condition. Furthermore, (12) can be rewritten in vector form as \( \sum_u P_U(u)L_u = 0 \in \mathbb{R}^K \) using the assumption that \( P_X(x) > 0 \) for all \( x \in \mathcal{X} \). Therewith, the problem in Corollary 1 can be rewritten as

\[
\max_{L_u, P_U: \|L_u\|_2^2 \leq 1, \sum_u P_U(u)L_u = 0} \sum_u P_U(u) ||W \cdot L_u||^2.
\] (13)

The next proposition shows how to simplify (13).

**Proposition 4.** Let \( L^* \) be the maximizer of (14), then (13) and (14) achieve the same maximum value while \( U \) as a uniform binary RV with \( L_0 = -L_1 = L^* \) maximizes (15).

\[
\max_{L: \|L\|_2^2 \leq 1} ||W \cdot L||^2.
\] (14)

**Proof.** Let \( \{L_u^*, P_U^*\} \) be the maximizer of (13). Furthermore, let \( u' \) be the index that maximizes \( ||W \cdot L_u^*||^2 \), i.e., \( u' = \arg\max_{u \in \mathcal{U}} ||W \cdot L_u^*||^2 \). Then we have

\[
\sum_u P_U(u) ||W \cdot L_u^*||^2 \leq ||W \cdot L_{u'}^*||^2 \leq ||W \cdot L^*||^2,
\]

where the right inequality comes from the fact that \( L^* \) has to satisfy one less constraint than \( L_{u'}^* \). However, by choosing \( U \) as a uniform binary RV and \( L_0 = -L_1 = L^* \) the constraints in (13) are satisfied and the maximum in (14) is achieved. Thus, without loss of optimality we can choose \( U \) as a uniformly distributed binary RV and (13) reduces to (14).

After finding the solution of (14), the conditional distributions \( P_{X|U=u} \) and \( P_{Y|U=u} \) are given by

\[
P_{X|U=0} = P_X + \epsilon \sqrt{P_X} L^*,
\] (15)

\[
P_{X|U=1} = P_X - \epsilon \sqrt{P_X} L^*,
\] (16)

\[
P_{Y|U=0} = P_Y + \epsilon P_{X|Y}^{-1} \sqrt{P_X} L^*,
\] (17)

\[
P_{Y|U=1} = P_Y - \epsilon P_{X|Y}^{-1} \sqrt{P_X} L^*.
\] (18)

In next theorem we derive the solution of (14).

**Theorem 1.** \( L^* \), which maximizes (14), is the right singular vector corresponding to the largest singular value of \( W \).

**Proof.** The proof is provided in Appendix C.

By using Theorem 1 the solution to the problem in Corollary 1 can be summarized as \( \{P_{U^*}, L_u^*\} = \{U \text{ uniform binary RV}, L_0 = -L_1 = L^*\} \), where \( L^* \) is the solution of (14). Thus, we have the following result.

**Corollary 2.** The maximum value in (13) can be approximated by \( \frac{1}{2} \epsilon^2 \sigma_{\text{max}}^2 \) for small \( \epsilon \) and can be achieved by a privacy mechanism characterized by the conditional distributions found in (17) and (18), where \( \sigma_{\text{max}} \) is the largest singular value of \( W \) corresponding to the right singular vector \( L^* \).
IV. GEOMETRIC INTERPRETATION AND DISCUSSIONS

In Figure 2, four spaces are illustrated. Space B and space C are probability spaces of the input and output distributions, where the points are inside a simplex. Multiplying input distributions by $P_{X|Y}^{-1}$ results in output distributions. Space A illustrates vectors $L_u$ with norm smaller than 1, which corresponds to the strong $\chi^2$-privacy criterion. The red region in this space includes all vectors that are orthogonal to $\sqrt{P_X}$. For the optimal solution with $U$ chosen to be equiprobable binary RV, it is shown that it remains to find the vector $L_u$ in the red region that results in a vector that has the largest norm in space D. This is achieved by the principal right-singular vector of $W$. The mapping between space A and B is given by $\sqrt{P_X}$ and also the mapping between space C and D is given by $\sqrt{P_Y^{-1}}$. Thus $W$ is given by $\sqrt{P_Y^{-1}}P_{X|Y}^{-1}\sqrt{P_X}$.

In following, we provide an example where the procedure of finding the mechanism to produce $U$ is illustrated.

Example 1. Consider the leakage matrix $P_{X|Y} = \begin{bmatrix} \frac{1}{4} & \frac{2}{5} \\ \alpha & 1 - \alpha \end{bmatrix}$ and $P_Y$ is given as $\begin{bmatrix} \frac{1}{2} & \frac{3}{4} \end{bmatrix}^T$. Thus, we can calculate $W$ and $P_X$ as

$$P_X = P_{X|Y}P_Y = [0.3625, 0.6375]^T,$$

$$W = [\sqrt{P_Y^{-1}}]P_{X|Y}^{-1}\sqrt{P_X} = \begin{bmatrix} -4.8166 & 4.2583 \\ 3.4761 & -1.5366 \end{bmatrix}.$$ 

The singular values of $W$ are $7.4012$ and $1$ with corresponding right singular vectors $[0.7984, -0.6021]^T$ and $[0.6021, 0.7954]^T$, respectively. Thus the maximum of $I(U; Y)$ is achieved by the following conditional distributions:

$P_{Y|U=0} = P_Y + \epsilon P_{X|Y}^{-1}\sqrt{P_X}L^* = [0.25 - 3.2048 \epsilon, 0.75 + 3.2048 \epsilon]^T,$

$P_{Y|U=1} = P_Y - \epsilon P_{X|Y}^{-1}\sqrt{P_X}L^* = [0.25 + 3.2048 \epsilon, 0.75 - 3.2048 \epsilon]^T.$

Note that the approximation is valid if $\epsilon P_{X|Y}^{-1}\sqrt{P_X}L^* \ll 1$ holds for all $y$ and $u$. For the example above we have $\epsilon \cdot P_{X|Y}^{-1}\sqrt{P_X}L^* \ll 1$ holds for all $y$ and $u$. Thus, we can calculate $W$ and $P_X$ with corresponding right singular vectors $[0.7984, -0.6021]^T$ and $[0.6021, 0.7954]^T$.

In next example we consider a BSC(\(\alpha\)) channel as leakage matrix. We provide an example with a constant upper bound on the approximated mutual information.

Example 2. Let $P_{X|Y} = \begin{bmatrix} 1 - \alpha & \alpha \\ \alpha & 1 - \alpha \end{bmatrix}$ and $P_Y$ is given as $\begin{bmatrix} \frac{1}{2} & \frac{3}{4} \end{bmatrix}^T$. By following the same procedure we have

$$P_X = P_{X|Y}P_Y = \begin{bmatrix} \frac{2\alpha + 1}{4} & \frac{3 - 2\alpha}{4} \end{bmatrix},$$

$$W = [\sqrt{P_Y^{-1}}]P_{X|Y}^{-1}\sqrt{P_X} = \begin{bmatrix} \frac{\sqrt{2\alpha + 1}(\alpha - 1)}{(2\alpha - 1)^2} \\ \frac{\alpha \sqrt{2\alpha + 1}}{\sqrt{3(2\alpha - 1)}} \end{bmatrix}.$$ 

Singular values of $W$ are $\sqrt{\frac{(2\alpha + 1)(3 - 2\alpha)}{4(2\alpha - 1)^2}} \geq 1$ for $\alpha \in [0, \frac{1}{2})$ and $1$ with corresponding right singular vectors $[-\sqrt{\frac{3 - 2\alpha}{4}}, \sqrt{\frac{2\alpha + 1}{4}}]^T$ and $[\sqrt{\frac{2\alpha + 1}{4}}, \sqrt{\frac{3 - 2\alpha}{4}}]^T$, respectively. Thus, we have $L^* = [-\sqrt{\frac{3 - 2\alpha}{4}}, \sqrt{\frac{2\alpha + 1}{4}}]^T$ and $\max I(U; Y) \approx \epsilon^2 \frac{(2\alpha + 1)(3 - 2\alpha)}{6(2\alpha - 1)^2}$ with the following conditional distributions

$P_{Y|U=0} = P_Y + \epsilon \cdot P_{X|Y}^{-1}\sqrt{P_X}L^* = \begin{bmatrix} \frac{1}{4} + \epsilon \frac{\sqrt{3 - 2\alpha}(2\alpha + 1)}{4(2\alpha - 1)} & \frac{3}{4} - \epsilon \frac{\sqrt{3 - 2\alpha}(2\alpha + 1)}{4(2\alpha - 1)} \\ \frac{3}{4} + \epsilon \frac{\sqrt{3 - 2\alpha}(2\alpha + 1)}{4(2\alpha - 1)} & \frac{1}{4} - \epsilon \frac{\sqrt{3 - 2\alpha}(2\alpha + 1)}{4(2\alpha - 1)} \end{bmatrix},$

$P_{Y|U=1} = P_Y - \epsilon \cdot P_{X|Y}^{-1}\sqrt{P_X}L^* = \begin{bmatrix} \frac{1}{4} - \epsilon \frac{\sqrt{3 - 2\alpha}(2\alpha + 1)}{4(2\alpha - 1)} & \frac{3}{4} + \epsilon \frac{\sqrt{3 - 2\alpha}(2\alpha + 1)}{4(2\alpha - 1)} \\ \frac{3}{4} - \epsilon \frac{\sqrt{3 - 2\alpha}(2\alpha + 1)}{4(2\alpha - 1)} & \frac{1}{4} + \epsilon \frac{\sqrt{3 - 2\alpha}(2\alpha + 1)}{4(2\alpha - 1)} \end{bmatrix}.$

The approximation of $I(U; Y)$ holds when we have $|\epsilon \cdot P_{X|Y}^{-1}\sqrt{P_X}L^*| \ll 1$ for all $y$ and $u$, which leads to $\epsilon \ll \frac{2|\alpha - 1|}{\sqrt{(3 - 2\alpha)(2\alpha + 1)}}.$ If $\epsilon < \frac{2|\alpha - 1|}{\sqrt{(3 - 2\alpha)(2\alpha + 1)}}$, then the approximation of the mutual information $I(U; Y) \approx \frac{1}{2} \epsilon^2 \sigma_{\text{max}}^2$ is upper bounded by $\frac{1}{6}$ for all $0 \leq \alpha < \frac{1}{2}.$

Our next result discusses the relation to [7]. While the focus in the present paper is on introducing the proposed design framework, our result also shows that an upper bound in [7] is actually achievable since we can achieve it considering even a strong privacy criterion. In order to compare the results one needs to substitute $S$, $X$, $Y$ and $\epsilon$ in [7], by $X$, $Y$, $U$ and $\epsilon^2$, respectively.
Proposition 5. For all $\epsilon < \frac{[\sigma_{\text{min}}(P_{X|Y})][\min_{y \in Y} P_Y(y)]}{\sqrt{\max_{x \in X} P_X(x)}}$, the upper bound on the privacy-utility trade-off derived in [7, Th.2], is tight.

Proof. First we show that the approximation of $I(U;Y)$ found in (8), is equal to half of the $\chi^2$-information between $U$ and $Y$. By using Proposition 2, we have

$$\frac{1}{2} \epsilon^2 \sum_u P_U \| [\sqrt{P_Y^{-1}} P_{X|Y} \sqrt{P_X}] L_u \|^2 = \frac{1}{2} \epsilon^2 \sum_u P_U \sum_y \frac{(P_{X|Y}^{-1} J_u)^2}{P_Y} = \frac{1}{2} \sum_u P_U \sum_y \frac{(P_{Y|U=u} - P_Y)^2}{P_Y}.$$

Thus, the problem found in (9), is equivalent to the following problem

$$\max_{P_{U|Y}} \lambda^2_{\text{information}}(Y;U), \quad \text{(19a)}$$

subject to: $X - Y - U,$

$$\left[ [\sqrt{P_X^{-1}}]_{(P_{X|U=u} - P_X)} \right]^2 \leq \epsilon^2, \forall u \in U. \quad \text{(19b)}$$

Since the strong privacy criterion in (19c) implies the privacy criterion in [7, Definition 4] and the objective functions are the same, we conclude that the problem defined in [7, Definition 4] is an upper bound to (19a). Furthermore, the upper bound in (7) is equal to $\frac{1}{\lambda_{\text{min}}(X,Y)} \epsilon^2$ for $\epsilon^2 \leq \lambda_{\text{min}}(X,Y)$, where $\lambda_{\text{min}}(X,Y)$ is the minimum singular value of $Q_{X,Y}$, which is defined in [7, Definition 2]. Next, we show that $\frac{1}{\lambda_{\text{min}}(X,Y)} = \sigma^2_{\text{max}}(W)$. The relation between $W$ and $Q_{X,Y}$ is as follows

$$Q_{X,Y} = [\sqrt{P_X^{-1}}]_{P_{X|Y} \sqrt{P_Y^{-1}}} = [\sqrt{P_X^{-1}}]_{P_{X|Y} \sqrt{P_Y}},$$

Thus, we have $\frac{1}{\lambda_{\text{min}}(X,Y)} = \sigma^2_{\text{max}}(W)$. Also, $\epsilon < \frac{[\sigma_{\text{min}}(P_{X|Y})][\min_{y \in Y} P_Y(y)]}{\sqrt{\max_{x \in X} P_X(x)}}$ leads to the first region ($\epsilon^2 \leq \lambda_{\text{min}}(X,Y)$) of the upper bound, since we have

$$||W|| \leq \frac{1}{\min P_Y} [||P_{X|Y}^{-1}||(\max P_X),$$

which implies

$$\frac{(\sigma_{\text{min}}(P_{X|Y}))^2 (\min P_Y)^2}{\max P_X} \leq \frac{(\sigma_{\text{min}}(P_{X|Y}))^2 \min P_Y}{\max P_X} \leq \frac{1}{\sigma^2_{\text{max}}(W)} \leq \lambda_{\text{min}}(X,Y),$$

where we used spectral norm defined as $||A|| = \max_{|x|=1} \|Ax\|_2$, also $\max P_X = \max_{x \in X} P_X(x)$ and $\min P_Y = \min_{y \in Y} P_Y(y)$. The privacy mechanism found in this paper achieves $\sigma^2_{\text{max}}(W) \epsilon^2$ for (19a), and since (19a) is a lower bound to the problem defined in [7, Definition 4] and achieves the upper bound in [7, Th.2] for small $\epsilon$, we can conclude the upper bound in [7, Th.2] is tight for all $\epsilon < \frac{[\sigma_{\text{min}}(P_{X|Y})][\min_{y \in Y} P_Y(y)]}{\sqrt{\max_{x \in X} P_X(x)}}$.

In next section, we study two extensions, where the idea of information geometry approximation is used.

V. EXTENSIONS

In this section, two problems are introduced. First, a fixed binary channel between the agent and the user is considered and the agent is trying to find a mechanism to produce binary random variable $U$, which maximizes $I(U;Y)$ under the Markov chain $X - Y - U - U'$ and privacy criterion on $X$ and $U'$. In second extension, the agent looks for a mechanism which finds $U$ based on observing $Y$, maximizing the mutual information between $U$ and $Z$ while satisfying the privacy criterion on $U$ and $X$ under the Markov chain $(X,Z) - Y - U$. In these extensions, small enough $\epsilon$ stands for all $\epsilon$ such that the second Taylor expansion can be used.
A. Privacy problem with noisy disclosure channel

Similar to the previous problem let $P_{XY}$ denote the joint distribution of discrete random variables $(X, Y)$ and the leakage matrix defined by $P_{X|Y}$ be invertible. Similarly, let $X$ and $Y$ denote the private and the useful data with equal cardinality, i.e, $|X| = |Y| = K$. Other considerations on $(X, Y)$ mentioned in section II are assumed in this problem. Here, we add an invertible fixed binary channel between the agent and user denoted by $P_{U|U'}$ on $\mathbb{R}^{2 \times 2}$, where we assume $|U| = |U'| = 2$. $U'$ is the message sent by the user and $U$ is the message sent by the agent. The agent tries to find a mechanism to produce $U$ such that maximizes $I(U; Y)$ while satisfying privacy criterion on $X$ and $U'$ under the Markov chain $X - Y - U - U'$. The privacy criterion employed in this problem is as follows

$$\left\| \left( \sqrt{P_X} X_{U'=u'} - P_X \right) \right\|^2 \leq \epsilon^2, \ u' \in \{u_0', u_1'\}.$$  

The information theoretic privacy problem can be characterized as follows

$$\begin{aligned}
&\max_{P_{U|Y}} I(U; Y), \\
&\text{subject to: } X - Y - U - U', \\
&\left\| \left( \sqrt{P_X} X_{U'=u'} - P_X \right) \right\|^2 \leq \frac{1}{2} \epsilon^2, \ u' \in \{u_0', u_1'\},
\end{aligned}$$  

(20a) (20b) (20c)

Same as before, we assume that $\epsilon$ is a small quantity. We define the matrix $P_{U|U'} \in \mathbb{R}^{2 \times 2}$ by $\begin{bmatrix} x & y \\ z & t \end{bmatrix}$, where $x + z = 1, y + t = 1$, and all $x, y, z$ and $t$ are non-negative. Furthermore, we show $P_{U|U'}^{-1}$ by $\begin{bmatrix} a & c \\ b & d \end{bmatrix}$, where $a = \frac{t}{zt - zy}, b = \frac{z}{zt - zy}, c = \frac{y}{zt - zy}$ and $d = \frac{x}{zt - zy}$.

Proposition 6. The tuple $(a, b, c, d)$ belongs to one of the following sets

$$\begin{aligned}
A_1 &= \{(a, b, c, d) | a \leq 0, d \leq 0, b \geq 1, c \geq 1, a + b = 1, c + d = 1\}, \\
A_2 &= \{(a, b, c, d) | a \geq 1, d \geq 1, b \leq 0, c \leq 0, a + b = 1, c + d = 1\}.
\end{aligned}$$

Proof. Since $x + z = 1$ and $y + t = 1$, we have

$$a + b = \frac{t - z}{zt - zy} = \frac{t - z}{(1 - z)(1 - t)} = 1.$$  

Since $t \geq 0$ and $z \geq 1$, one of $a$ and $b$ is non-negative and the other one is non-positive. Furthermore, since $a + b = 1$, we have $a \leq 0, b \geq 1$ or $a \geq 1, b \leq 0$. Same proof can be used for $c$ and $d$. 

By using (20c), we can write $P_{X|U=u} = P_X + \epsilon \cdot J_{u'}$, where $J_{u'} \in \mathbb{R}^K$ is the perturbation vector that has three properties as follows

$$\begin{aligned}
\sum_{x \in X} J_{u'}(x) &= 0, \ u' \in \{u_0', u_1'\}, \\
P_{U'}(u' = 0)J_0 + P_{U'}(u' = 1)J_1 &= 0, \\
\sum_{x \in X} J_{u'}^2(x) &\leq 1, \ u' \in \{u_0', u_1'\},
\end{aligned}$$  

(21) (22) (23)

where $0 \in \mathbb{R}^K$.

Similarly, we can show that by using the concept of Euclidean Information theory, (20c) results in a leakage constraint.

Proposition 7. For a small enough $\epsilon$, (1c) results in a leakage constraint as follows

$$I(U'; X) \leq \frac{1}{2} \epsilon^2 + o(\epsilon^2).$$

Proof. The proof is similar to Proposition 3.  

We show that $P_{Y|U=u}$ can be written as a linear perturbation of $P_Y$. Since the Markov chain $X - Y - U - U'$ holds, we can write

$$P_{X|u_0} = P_{X|U}P_{U|u_0}, \ P_{X|u_1} = P_{X|U}P_{U|u_1}.$$  

Thus, $P_{X|U'} = P_{X|U}P_{U|U'}$ and since $P_{U|U'}$ is invertible, we obtain

$$[P_{X|u_0} \ P_{X|u_1}] = P_{X|U'} \begin{bmatrix} a & c \\ b & d \end{bmatrix}.$$
Furthermore, by using the Markov chain we have $P_{Y|U=u} = P_{X|Y}^{-1} P_{X|U=u}$, which results in

\[ P_{Y|u_0} = P_{X|Y}^{-1}[a P_{X|u_0} + b P_{X|u_1}], \]
\[ P_{Y|u_1} = P_{X|Y}^{-1}[c P_{X|u_0} + d P_{X|u_1}]. \]

Considering $P_{Y|U=u_0}$, we have

\[ P_{Y|u_0} - P_Y = P_{X|Y}^{-1}[a P_{X|u_0} + b P_{X|u_1} - P_X] = P_{X|Y}^{-1}[a(P_{X|u_0} - P_X) + b(P_{X|u_1} - P_X)] \]
\[ 
\rightarrow P_{Y|u_0} = P_Y + aP_{X|Y}^{-1}J_{u_0} + bP_{X|Y}^{-1}J_{u_1}. \]

Similarly, $P_{Y|U=u_1}$ is found as follows

\[ P_{Y|u_1} = P_Y + cP_{X|Y}^{-1}J_{u_0} + dP_{X|Y}^{-1}J_{u_1}. \]

Now we can approximate $I(U;Y)$ by a squared Euclidean metric.

**Proposition 8.** For a small enough $\epsilon$, $I(U;Y)$ can be approximated as follows

\[ I(U;Y) \approx \frac{1}{2} \epsilon^2 \left( \sum_u P_{U}(u) D(P_{Y|U=u} \| P_Y) \right) \]
\[ = \frac{1}{2} \epsilon^2 \left( \sum_u P_{U}(u) \log \left( \frac{P_{Y|u_0}}{P_Y} + \sum_y P_{Y|u_1} \log \left( \frac{P_{Y|u_1}}{P_Y} \right) \right) \right) \]
\[ = \frac{1}{2} \epsilon^2 \left( \sum_u P_{U}(u) \log \left( \frac{P_{Y|u_0}}{P_Y} \right) + \sum_y P_{Y|u_1} \log \left( \frac{P_{Y|u_1}}{P_Y} \right) \right) \]
\[ = \frac{1}{2} \epsilon^2 \sum_u P_{U}(u) \sum_y \left( \frac{P_{X|Y}(a J_{u_0} + b J_{u_1})}{P_Y} \right)^2 + \frac{1}{2} \epsilon^2 \sum_y P_{Y|u_1} \sum_u \left( \frac{P_{X|Y}(c J_{u_0} + d J_{u_1})}{P_Y} \right)^2 \]
\[ = \frac{1}{2} \epsilon^2 \sum_u P_{U}(u) \sum_y \left( \frac{P_{X|Y}(a J_{u_0} + b J_{u_1})}{P_Y} \right)^2 + \frac{1}{2} \epsilon^2 \sum_y P_{Y|u_1} \sum_u \left( \frac{P_{X|Y}(c J_{u_0} + d J_{u_1})}{P_Y} \right)^2 \]
\[ = \frac{1}{2} \epsilon^2 \sum_u P_{U}(u) \left\| \sqrt{P_Y^{-1}} P_{X|Y}(a J_{u_0} + b J_{u_1}) \right\|^2 + \frac{1}{2} \epsilon^2 \sum_u P_{U}(u) \left\| \sqrt{P_Y^{-1}} P_{X|Y}(c J_{u_0} + d J_{u_1}) \right\|^2 \]
\[ \leq \frac{1}{2} \epsilon^2 \sum_u P_{U}(u) \left\| W(a L_{u_0} + b L_{u_1}) \right\|^2 + \frac{1}{2} \epsilon^2 \sum_u P_{U}(u) \left\| W(c L_{u_0} + d L_{u_1}) \right\|^2 \]

**Proof.** By using (24) and (25) we have

\[ I(Y;U) = \sum_u P_{U}(u) D(P_{Y|U=u} \| P_Y) \]

By locally approximating $I(U;Y)$, the main privacy problem in (20a) can be reduced to a simple quadratic problem. Substituting $L_u$ in (21), (27) and (28) leads to the following result.

**Corollary 3.** For a small enough $\epsilon$, the privacy mechanism design problem in (20a) can be approximately solved by the following linear problem

\[ \max_{\{L_u, P_u\}} \sum_u P_{U}(u) \left\| W(a L_{u_0} + b L_{u_1}) \right\|^2 + \sum_u P_{U}(u) \left\| W(c L_{u_0} + d L_{u_1}) \right\|^2 \]
\[ \text{subject to:} \quad \sum_u L_u = 1, \quad u \in \{u_0, u_1\}, \]
\[ \sum_u \sqrt{P_X(x)} L_u'(x) = 0, \quad u \in \{u_0, u_1\}, \]
\[ P_{u_0} L_{u_0} + P_{u_1} L_{u_1} = 0, \]

where $0 \in \mathbb{R}^K$. 
Remark 3. Condition (28) can be rewritten as $L_{u_0} \perp \sqrt{P_X}$ and $L_{u_1} \perp \sqrt{P_X}$. Also the maximization is over $\{L_{u_0}, L_{u_1}, P_{u_0}, P_{u_1}\}$. $P_{u_0}$ and $P_{u_1}$ are replaced by $aP_{u_0} + cP_{u_1}$ and $bP_{u_0} + dP_{u_1}$, since we have $\begin{bmatrix} P_{u_0} \\ P_{u_1} \end{bmatrix} = P^{-1}_{U|U'} \begin{bmatrix} P_{u_0} \\ P_{u_1} \end{bmatrix}$.

In next proposition we derive the solution of (26).

**Proposition 9.** The solution of (26) is as follows

$$L_{u_0} = -L_{u_1} = \psi, \quad P_{u_0} = \frac{c - \frac{1}{2}}{c - a}, \quad P_{u_1} = \frac{\frac{1}{2} - a}{c - a}, \quad P_{u_0}' = P_{u_1}' = \frac{1}{2}$$

**Maximum value** = $4(c - \frac{1}{2})(\frac{1}{2} - a)\sigma^2$,

where $\sigma^2$ is the largest singular value of $W$ with corresponding singular vector $\psi$.

**Proof.** The proof is provided in Appendix C.

**Corollary 4.** The maximum value in (20a) can be approximated by $2a^2\sigma^2(c - \frac{1}{2})(\frac{1}{2} - a)$ for small $\epsilon$ and can be achieved by conditional distributions as follows

$$P_{Y|u_0} = P_Y + \epsilon(a - b)P^{-1}_{X|Y}[\sqrt{P_X}]\psi,$$

$$P_{Y|u_1} = P_Y + \epsilon(c - d)P^{-1}_{X|Y}[\sqrt{P_X}]\psi,$$

where $\sigma^2$ is the largest singular value of $W$ with corresponding singular vector $\sqrt{P_X}$. Furthermore, the distribution of $U$ is as follows

$$P_{u_0} = \frac{c - \frac{1}{2}}{c - a}, \quad P_{u_1} = \frac{\frac{1}{2} - a}{c - a}.$$

**B. Privacy problem with utility provider**

In this part, we consider a similar framework as in [1], where we have an agent and a utility provider. The agent observes useful data denoted by RV $X$ and the utility provider is interested in target data denoted by RV $Y$ which is not directly accessible by the agent but correlated with RV $X$. The agent receives utility by disclosing information about $Y$. Furthermore, we assume $X$ is dependent on the private data denoted by RV $Z$, and tried to keep it private and not disclose much information about $Z$. Thus, the agent uses a privacy mechanism to produce $U$ and tries to maximize the utility measured by $I(U;Y)$ and at the same time satisfies the privacy criterion. RV $U$ denotes the disclosed data. Here we assume that all random variables are discrete and have finite support, i.e., $|X|, |Y|, |Z| < \infty$. Since the disclosed data is produced by observing $X$ and the variables $X$, $Y$ and $Z$ are correlated, we have the Markov chain $(Z,Y) - X - U$. We assume that $|X| = |Z| = K$ and the leakage matrix $P_{Z|X} \in \mathbb{R}^{K \times K}$ is invertible. Furthermore, the marginal vectors $P_X$, $P_Z$ and $P_Y$ contain non-zero elements. Here, privacy is measured as follows

$$\left\| [\sqrt{P_X^{-1}}](P_{Z|U = u} - P_Z) \right\|^2 \leq \epsilon^2, \ \forall u \in U.$$

The privacy problem is characterized as follows

$$\max_{P_{U|X}} I(U;Y),$$

subject to: $(Z,Y) - X - U$, \hspace{1cm} (30a)

$$\left\| [\sqrt{P_X^{-1}}](P_{Z|U = u} - P_Z) \right\|^2 \leq \frac{1}{2} \epsilon^2, \ \forall u \in U.$$

**Remark 4.** By using Fenchel-Eggleston-Carathéodory’s Theorem [23], it can be shown that it suffices to consider $U$ such that $|U| \leq |X| + 1$. Furthermore, the maximum in (30a) is achieved so we used maximum instead of supremum.

Similarly, (30c) results in $P_{Z|U = u} = P_Z + \epsilon J_u$, where $J_u \in \mathbb{R}^K$ is the perturbation vector that has the following three properties

$$\sum_{z \in Z} J_u(z) = 0, \forall u \in U,$$

$$\sum_{u \in U} P_T(u) J_u = 0,$$

$$\sum_{z \in Z} (\frac{J^2_u}{P_Z(z)}) \leq 1, \forall u \in U,$$
where \(0 \in \mathbb{R}^K\). By following the same procedure in Proposition 1 and using the Euclidean information concept, it can be shown that for a small enough \(\epsilon\), (30c) results in the following leakage constraint
\[
I(Z;U) \leq \frac{1}{2} \epsilon^2 + o(\epsilon^2).
\]

Now we show that \(P_{Y|U=u}\) can be written as a linear perturbation of \(P_Y\). Since the Markov chain \((Z, Y) - X - U\) holds, we can write \(P_{X|U=u} = P_{Z|X}^{-1}P_{Z,U}^{-1}\). Thus,
\[
P_{Y|U=u} = P_{Y|X}P_{X|U=u} = P_{Y|X}P_{Z|X}^{-1}P_{Z,U}^{-1}.
\]

By using \(P_{Z|U=u} = P_Z + \epsilon J_U\), \(P_{Y|U=u}\) can be written as follows
\[
P_{Y|U=u} = P_{Y|X}P_{Z|X}^{-1}(P_Z + \epsilon J_u) = P_Y + \epsilon P_{Y|X}P_{Z|X}^{-1}J_u.
\]

The next proposition shows that \(I(U; Y)\) can be locally approximated by a squared Euclidean metric.

**Proposition 10.** For a small enough \(\epsilon\), \(I(U; Y)\) can be approximated as follows
\[
I(U; Y) \approx \frac{1}{2} \epsilon^2 \sum_u P_U(u) \left\| \sqrt{P_Y^{-1}}J_u \| \right\|, \quad (34)
\]
where \(L_u = \sqrt{P_Z^{-1}}J_u\).

**Proof.** By using the local approximation of the KL-divergence we have
\[
I(Y; U) = \sum_u P_Y(u)D(P_{Y|U=u}\|P_Y)
\]
\[
= \sum_u P_Y(u) \sum_y P_{Y|U=u}(y) \log \left( \frac{P_{Y|U=u}(y)}{P_Y(y)} \right)
\]
\[
= \sum_u P_Y(u) \sum_y P_{Y|U=u}(y) \log \left(1 + \epsilon \frac{P_{Y|X}P_{Z|X}^{-1}J_u(y)}{P_Y(y)} \right)
\]
\[
= \frac{1}{2} \epsilon^2 \sum_u P_Y(u) \sum_y \frac{(P_{Y|X}P_{Z|X}^{-1}J_u(y))^2}{P_Y(y)} + o(\epsilon^2)
\]
\[
= \frac{1}{2} \epsilon^2 \sum_u P_Y(u) \left\| \sqrt{P_Y^{-1}}J_u \| \right\|^2 + o(\epsilon^2)
\]
\[
\approx \frac{1}{2} \epsilon^2 \sum_u P_Y(u) \left\| \sqrt{P_Y^{-1}}J_u \| \right\|^2.
\]

By substituting \(L_u\) in (31), (32) and (33), and using the local approximation in (34) we obtain the following result.

**Corollary 5.** For a small enough \(\epsilon\), the privacy mechanism design problem in (30a) can be approximately solved by the following linear problem
\[
\max_{\{L_u, P_U\}} \sum_u P_U(u) \left\| W_1 W_2 L_u \right\|^2, \quad (35)
\]
subject to: \(\|L_u\|^2 \leq 1, \forall u \in U, \quad \sqrt{P_Z} \perp L_u, \forall u, \quad \sum_u P_U(u) L_u = 0, \quad (36)
\]

where \(W_1 = \sqrt{P_Y^{-1}}P_{Y|X}\sqrt{P_X}\) and \(W_2 = \sqrt{P_X^{-1}}P_{Z|X}\sqrt{P_Z}\).

Similar to the Proposition 3, without loss of optimality we can choose \(U\) as a uniform binary RV. Thus, (35) reduces to the following problem
\[
\max_{L: L \perp \sqrt{P_Z}, \|L\|^2 \leq 1} \left\| W_1 W_2 \cdot L \right\|^2. \quad (39)
\]
Let $L^*$ maximizes (39), thus, the conditional distributions $P_{Y|U=u}$ which maximizes (35) are given by
\[
P_{Y|U=0} = P_Y + \epsilon P_{Y|X} P_{Z|X}^{\perp} [\sqrt{P_Z}] L^*,
\]
(40)
\[
P_{Y|U=1} = P_Y - \epsilon P_{Y|X} P_{Z|X}^{\perp} [\sqrt{P_Z}] L^*.
\]
(41)
In the next theorem, the solution of (39) is derived.

**Theorem 2.** Let $\sigma_{\max}$ be the largest singular value of $W_1 W_2$ corresponding to the singular vector $\psi$. Furthermore, let $\phi$ be the singular vector of $W_1 W_2$ corresponding the second largest singular value. If $\sigma_{\max} > 1$, $\psi$ maximizes (39), and if $\sigma_{\max} = 1$, $\phi$ is the maximizer of (39).

**Proof.** The largest singular value of $W_1$ is 1 corresponding to singular vector $\sqrt{P_Z}$ and the smallest singular value of $W_2$ is 1 corresponding to singular vector $\sqrt{P_Z}$. Furthermore, we show that 1 is one of the singular values of $W_1 W_2$ corresponding to singular vector $\sqrt{P_Z}$. We have
\[
W_2^T W_1^T W_1 W_2 [\sqrt{P_Z}] = [\sqrt{P_Z}]^T \left( P_{Z|X}^{\perp} \right)^T \left[ \sqrt{P_X} \right]^T P_{Y|X} \left[ \sqrt{P_Y} \right]^T P_{Z|X} \left[ \sqrt{P_Z} \right]
\]
\[
= [\sqrt{P_Z}]^T \left( P_{Z|X}^{\perp} \right)^T \left[ \sqrt{P_X} \right]^T P_{Y|X} \left[ \sqrt{P_Z} \right] = [\sqrt{P_Z}]^T [\sqrt{P_Z}]
\]
Thus, we have two cases as $\sigma_{\max} > 1$ and $\sigma_{\max} = 1$. In first case, $\psi$ is orthogonal to $\sqrt{P_Z}$ and so maximizes (39). In second case, $\psi = \sqrt{P_Z}$ and $\phi$ is orthogonal to $\sqrt{P_Z}$. Thus, $\phi$ maximizes (39). There are no other cases since 1 is one of the singular values.

**Corollary 6.** Let $\sigma_{\max}$ and $\sigma_2$ be the first and second largest singular values of $W_1 W_2$. If $\sigma_{\max} > 1$, the maximum value in (30a) can be approximated by $\frac{1}{2} \epsilon^2 \sigma_2^2$ and can be achieved by a privacy mechanism characterized by conditional distributions found in (40) and (41) where $L^* = \psi$. Otherwise, the maximum value can be approximated by $\frac{1}{2} \epsilon^2 \sigma_2^2$ and can be achieved by (40) and (41) where $L^* = \phi$.

**Remark 5.** One simple example for the second case where $\sigma_{\max} = 1$ is letting $P_{Z|X} = P_{Y|X}$. In this case, $W_1 = W_2^{-1}$ and so all singular values of $W_1 W_2$ are equal to one. The maximum value in (39) is 1 and can be achieved by any vector orthogonal to $\sqrt{P_Z}$.

**Remark 6.** One sufficient condition for the first case where $\sigma_{\max} > 1$, is to have $\sigma_{\max}(W_2) = \frac{1}{\sigma_{\min}(W_1)}$ and not all singular values are equal to 1. Since in this case we have
\[
|||W_1 W_2||| \geq |||W_2||| \cdot \frac{\sigma_{\max}(W_2)}{\sigma_{\min}(W_1)} = \sigma_{\max}(W_2) \sigma_{\min}(W_1) = 1,
\]
where we used the spectral norm.

**VI. CONCLUSION**

We have shown that Euclidean information theory can be used to linearize an information-theoretic disclosure control problem. When a small $\epsilon$ privacy leakage is allowed, a simple approximate solution is derived. A geometrical interpretation of the privacy mechanism design is provided. Four linear spaces are introduced to further interpret the structure of the optimization problem. In particular, we look for a vector satisfying the constraint of having the largest Euclidean norm in other space, leading to finding the largest principle singular value of a matrix. The proposed approach establishes a useful and general design framework, which has been demonstrated in two problem extensions that included a noisy disclosure channel and privacy design with utility provider.

**APPENDIX A**

As shown in (7), $P_{Y|U=u}$ must belong to $\Psi$ for every $u \in \mathcal{U}$ which is defined as follows
\[
\Psi = \{ y \in \mathbb{R}^K | y = P_Y + \epsilon P_{X|Y}^{-1} J, \| J \|_{P_X}^2 \leq 1, J^T \cdot J = 0 \},
\]
where $\| J \|_{P_X}^2 = \sum_x J(x)^2 P_X(x)$ is the weighted Euclidean norm. For all $\epsilon < \frac{\sigma_{\max}(P_{X|Y}) \min_{x \in X} P_Y(x)}{\sqrt{\max_{x \in X} P_X(x)}}$, any point in $\Psi$ is a probability distribution and hence $\Psi$ is a subset of the standard $K-1$ dimension simplex. Thus, $\Psi$ is bounded. Let $\mathcal{J}_1 = \{ J \in \mathbb{R}^K | J^T \cdot J = 0 \}$ and $\mathcal{J}_2 = \{ J \in \mathbb{R}^K | \| J \|_{P_X}^2 \leq 1 \}$. $\mathcal{J}_1$ and $\mathcal{J}_2$ correspond a hyperplane and an ellipsoid, respectively. The set $\mathcal{J} = \mathcal{J}_1 \cap \mathcal{J}_2$ is closed since each $\mathcal{J}_1$ and $\mathcal{J}_2$ is closed. Considering the sequence $\{ y_0, y_1, \ldots \}$ where each $y_i$ is inside the set $\Psi$, we have $\lim_{i \to \infty} y_i = \lim_{i \to \infty} P_Y + \epsilon P_{X|Y}^{-1} J_i = P_Y + \epsilon P_{X|Y}^{-1} \lim_{i \to \infty} J_i$. 


Since \( J \) is a closed set \( \lim_{i \to \infty} J_i \in J \) and hence \( \lim_{i \to \infty} y_i \in \Psi \). Thus, \( \Psi \) is a compact set. We define a vector mapping \( \theta : \Psi \rightarrow \mathbb{R}^K \) as follows

\[
\theta_i(p_{Y|U(\cdot|U')}) = p_{Y|U}(y_i|u), \quad i \in [1 \colon K - 1],
\]

\[
\theta_K = H(Y|U = u).
\]

Since the mapping \( \theta \) is continuous and the set \( \Psi \) is compact, by using Fenchel-Eggleton-Carathéodory’s Theorem [23] for every \( U \) with p.m.f \( F(u) \) there exists a random variable \( U' \) with p.m.f \( F(u') \) such that \( |U'| \leq K \) and collection of conditional p.m.f.s \( P_{Y|U'}(\cdot|u') \in \Psi \) where

\[
\int_u \theta_i(p(y|u))dF(u) = \sum_{u'\in U'} \theta_i(p(y|u'))p(u').
\]

It ensures that by replacing \( U \) by \( U' \), \( I(U;Y) \) and the distribution \( P_Y \) are preserved. Furthermore, the condition \( \sum_{u'} P_{U'}(u')J_{u'} = 0 \) is satisfied since we have

\[
P_Y = \sum_{u'} P_{U'}P_{Y|U'=u'} \rightarrow P_X = \sum_{u'} P_{U'}P_{X|U'=u'} \sum_{u'} P_{U'}(P_{X|U'=u'} - P_X) = 0 \rightarrow \sum_{u'} P_{U'}(u')J_{u'} = 0.
\]

Note that any point in \( \Psi \) satisfies the strong privacy criterion, i.e., the equivalent \( U' \) satisfies the per-letter privacy criterion as well. Thus, without loss of optimality we can assume \( |U| \leq K \).

Let \( \mathcal{A} = \{ P_{U|Y}(\cdot|U) \mid U \in \mathcal{U}, Y \in \mathcal{Y}, |U| \leq K \} \) and \( \mathcal{A}_y = \{ P_{U|Y}(\cdot|y) \mid U \in \mathcal{U}, |U| \leq K \}, \forall y \in \mathcal{Y} \). \( \mathcal{A}_y \) is a standard \( |U| - 1 \) simplex and since \( |U| \leq |Y| < \infty \) it is compact. Thus \( \mathcal{A} = \bigcup_{y \in \mathcal{Y}} \mathcal{A}_y \) is compact. And the set \( \mathcal{A}' = \{ P_{U|Y}(\cdot|U) \mid U \in \mathcal{A}|X - Y - U, \|P_X|_{U=U'} - P_X\|_2^2 \leq \epsilon^2, \forall u \} \) is a closed subset of \( \mathcal{A} \) since \( \chi^2 \) information is closed of the interval \([0, \epsilon^2]\). Therefore, \( \mathcal{A}' \) is compact. Since \( I(U;Y) \) is a continuous mapping over \( \mathcal{A}' \), the supremum is achieved. Thus, we use maximum instead of supremum.

**APPENDIX B**

The KL divergence is denoted by \( D(\cdot||\cdot) \).

\[
I(X;U) = \sum_{u \in U} P_U(u)D(P_{X|U=u}||P_X)
\]

\[
= \sum_{u} P_U(u) \sum_{x} P_{X|U=u} \log \left( \frac{P_{X|U=u}}{P_X} \right)
\]

\[= (a) \sum_{u} P_U(u) \sum_{x} (P_X + \epsilon \cdot J_u) \log(1 + \epsilon \frac{J_u}{P_X})
\]

\[= \sum_{u} P_U(u) \sum_{x} (\epsilon J_u + \frac{1}{2} \epsilon^2 \frac{J_u^2}{P_X}) + o(\epsilon^2)
\]

\[= \frac{1}{2} \epsilon^2 \sum_{u \in U} P_U(u) \left[ \|P_X^{-1} \|_2^2 J_u \|_2^2 + o(\epsilon^2) \right]
\]

\[\leq (b) \frac{1}{2} \epsilon^2 + o(\epsilon^2),
\]

where (a) follows from \( P_{X|U=u} = P_X + \epsilon \cdot J_u \) and (b) follows from the third property of \( J_u \) stated in [5]. Furthermore, for approximating \( I(U;X) \) we should have \( |\epsilon J_u(x) - 1| < 1 \) for all \( x \) and \( u \). One sufficient condition is to have \( \epsilon < \frac{\text{min}_{x \in X} P_X(x)}{\sqrt{\text{max}_{x \in X} P_X(x)}} \).

Thus the privacy criterion implies a bounded mutual information leakage.

**APPENDIX C**

We first show that the smallest singular value of \( W \) is 1 with \( \sqrt{P_X} \) as corresponding right singular vector. We have

\[
W^TW \sqrt{P_X} = [\sqrt{P_X}]^T[\sqrt{P_Y}^{-1}]^T[\sqrt{P_Y}]^{-1}P_X^{-1} \sqrt{P_Y}^{-1} \sqrt{P_Y} \sqrt{P_X} \sqrt{P_X}
\]

\[= [\sqrt{P_X}]^T[\sqrt{P_Y}]^{-1} = [\sqrt{P_X}]^T \sqrt{P_Y} \sqrt{P_X}
\]

\[= [\sqrt{P_X}]^T \mathbf{1} = \sqrt{P_X} \mathbf{1} = \sqrt{P_X}.
\]
Now we show that all other singular values are greater than or equal to 1. Equivalently, we show that all singular values of $W^{-1} = [\sqrt{P_X}]^{-1}P_X[Y][\sqrt{P_Y}]$ are smaller than or equal to 1, i.e., we need to prove that for any vector $\alpha \in \mathbb{R}^K$ we have

$$||W^{-1}\alpha||^2 \leq ||\alpha||^2.$$  \hfill (42)

In the following, we use $P_{Yj} = P_Y(y_j)$, $P_{Xi} = P_X(x_i)$ and $P_{Xi}|Y_j = P_X|Y(x_i|y_j)$ for simplicity. More explicitly, we claim to have

$$\alpha^T(W^{-1})^TW^{-1}\alpha = \sum_{j=1}^K \alpha_j^2 \sum_{i=1}^K \frac{P_{Xi}|Yj P_{Yj}}{P_{Xi}} + \sum_{m,n=1}^K \alpha_m \alpha_n \sum_{i=1}^K \frac{P_{Xi}|Ym P_{Xm} P_{Ym} \sqrt{P_{Ym} P_{Ym}}}{P_{Xi}} \leq \sum_{i=1}^K \alpha_i^2.$$

By using $\frac{P_{Xi}|Yj P_{Yj}}{P_{Xi}} = P_{Xi}|Yj P_{Ym}|X_i$, we can rewrite the last inequality as follows

$$\sum_{j=1}^K \alpha_j^2 \sum_{i=1}^K P_{Xi}|Yj P_{Yj}|X_i + \sum_{m,n=1}^K \alpha_m \alpha_n \sum_{i=1}^K \frac{P_{Xi}|Ym P_{Xm} P_{Ym} \sqrt{P_{Ym} P_{Ym}}}{P_{Xi}} \leq \sum_{i=1}^K \alpha_i^2,$$

Equivalently, by using $\sum_{i=1}^K \sum_{m=1}^K P_{Xi}|Yj P_{Ym}|X_i = 1$, we claim to have

$$\sum_{m,n=1}^K \alpha_m \alpha_n \sum_{i=1}^K \frac{P_{Xi}|Ym P_{Xm} P_{Ym} \sqrt{P_{Ym} P_{Ym}}}{P_{Xi}} \leq \sum_{j=1}^K \alpha_j^2 \sum_{i=1}^K \sum_{m \neq j} P_{Xi}|Yj P_{Ym}|X_i.$$

Finally, we can see that the last inequality holds, since for any $i$ by using the inequality of arithmetic and geometric means and $P_{Xi}|Ym P_{Ym}|X_i, P_{Xi}|Yn P_{Yn}|X_i = \frac{P_{Xi}|Ym P_{Xm} P_{Ym} P_{Xn} P_{Ym}}{P_{Xi}}$, we have

$$2\alpha_m \alpha_n \frac{P_{Xi}|Ym P_{Xm} P_{Ym} \sqrt{P_{Ym} P_{Ym}}}{P_{Xi}} \leq \alpha_m^2 P_{Xi}|Ym P_{Ym}|X_i + \alpha_n^2 P_{Xi}|Yn P_{Yn}|X_i,$$

where we use

$$P_{Xi}|Ym P_{Ym}|X_i, P_{Xi}|Yn P_{Yn}|X_i = \frac{P_{Xi}|Ym P_{Xm} P_{Xn} P_{Ym}}{P_{Xi}^2} = \left(\frac{P_{Xi}|Ym P_{Xm} P_{Ym} \sqrt{P_{Ym} P_{Ym}}}{P_{Xi}} \right)^2.$$

Therefore, one is the smallest singular value of $W$ with $\sqrt{P_X}$ as corresponding right singular vector. Furthermore, we have that the right singular vector of the largest singular value is orthogonal to $\sqrt{P_X}$. Thus, the principal right-singular vector is the solution of $[14]$.

**APPENDIX D**

First, assume that the maximum occurs in non-zero $P_{u0}$ and $P_{u1}$. For simplicity we show $P_{u0}$ and $P_{u1}$ by $P_0$ and $P_1$, also we show $P_{u0}$ and $P_{u1}$ by $P'_0$ and $P'_1$. By using (29), we have

$$L_{u1} = \frac{P'_1}{P_0} L_{u1} = -\frac{bP_0 + dP_1}{aP_0 + cP_1} L_{u1}.$$  \hfill (43)

Since $||L_{u0}||^2 \leq 1$, thus, $||\frac{P'_1}{P_0} L_{u1}||^2 \leq 1$, which results in $||L_{u1}|| \leq \min(1, (\frac{P'_1}{P_0})^2) \leq 1$. With the same argument $||L_{u0}|| \leq \min(1, (\frac{P'_0}{P_1})^2) \leq 1$. Now we consider two cases:

1. Case 1: $|P'_0| \geq |P'_1|$. 2. Case 2: $|P'_1| \geq |P'_0|$.

**Case 1**: In this case we have $|aP_0 + cP_1| \geq |bP_0 + dP_1|$, which results in

$$(a-c)P_0 + c \geq \frac{1}{2},$$  \hfill (43)
since \( b = 1 - a \) and \( d = 1 - c \). Then, we substitute \( L_{u_0}' \) by \(-\frac{bP_0 + dP_1}{aP_0 + cP_1}L_{u_1}'\) in the objective function, which results in

\[
P_0 \left\| W(aL_{u_0}' + bL_{u_1}') \right\|^2 + P_1 \left\| W(cL_{u_0}' + dL_{u_1}') \right\|^2
\]

\[
= \left\| WL_{u_1}' \right\| P_0 \left( b - \frac{abP_0 + adP_1}{aP_0 + cP_1} \right)^2 + \left\| WL_{u_1}' \right\| P_1 \left( b - \frac{bcP_0 + cdP_1}{aP_0 + cP_1} \right)^2
\]

\[
= \left\| WL_{u_1}' \right\| P_0 \left( \frac{(bc - ad)^2 P_0^2}{(aP_0 + cP_1)^2} \right) + \left\| WL_{u_1}' \right\| P_1 \left( \frac{(ad - bc)^2 P_0^2}{(aP_0 + cP_1)^2} \right)
\]

\[
= \left\| WL_{u_1}' \right\| (bc - ad)^2 \left( \frac{P_0(1 - P_0)}{((a - c)P_0 + c)^2} \right) .
\]

Now we show that the maximum of \( f(P_0) = \frac{P_0(1 - P_0)}{(a - c)P_0 + c} \) occurs in \( P_0^* = \frac{c - \frac{1}{2}}{c - a} \). The derivative of \( f \) with respect to \( P_0 \) is as follows

\[
\frac{d}{dP_0} f = \frac{c - (a - c)P_0}{((a - c)P_0 + c)^2} .
\]

By using Proposition [6] we have two cases for \( a \) and \( c, a \geq 1, c \leq 0 \) or \( a \leq 0, c \geq 1 \). For \( a \geq 1, c \leq 0 \) we have \( a - c \leq 0 \), which implies \( P_0 \geq \frac{b - c}{a - c} \) by using (43). We show that \( f(P_0) \) is a decreasing function in this case. If \( a + c \geq 0 \), then \( c - (a + c)P_0 \leq 0 \) and if \( a + c 
leq 0 \), then \(-(a + c)P_0 \leq (a + c)P_0 \leq -a \leq 1 \). Thus, for \( a \geq 1, c \leq 0 \), \( f(P_0) \) is decreasing and its maximum happens in \( P_0^* = \frac{c - \frac{1}{2}}{a - c} \). Now consider \( a \leq 0, c \geq 1 \). In this case we have \( P_0 \leq \frac{1 - c}{a - c} \). We show that \( f(P_0) \) is an increasing function. If \( a + c \leq 0 \), then \( P_0(a + c) \leq 0 \) which results in \( c \geq 1 > 0 \geq (a + c)P_0 \). And if \( a + c \geq 0 \), then \( (a + c)P_0 \leq \frac{(c - \frac{1}{2})(a + c)}{c - a} \leq c \), since \( 2ac \leq 0 \leq \frac{4(c - \frac{1}{2})^2}{(c - a)^2} \). Thus, \( f(P_0) \) is an increasing function and its maximum occurs in \( P_0^* = \frac{c - \frac{1}{2}}{a - c} \). The maximum value of \( f(P_0) \) is \( \frac{(c - \frac{1}{2})(a + c)}{4(c - a)^2} \).

**Case 2:** In this case we have \( |aP_0 + cP_1| \leq |bP_0 + dP_1| \), which results in

\[
(a - c)P_0 + c \geq 1 - \frac{1}{2} .
\]

We substitute \( L_{u_1}' \) by \(-\frac{aP_0 + cP_1}{bP_0 + dP_1}L_{u_0}' \) in the objective function, which results in

\[
P_0 \left\| W(aL_{u_0}' + bL_{u_1}') \right\|^2 + P_1 \left\| W(cL_{u_0}' + dL_{u_1}') \right\|^2
\]

\[
= \left\| WL_{u_1}' \right\| \left( \frac{bc - ad}{(b - d)P_0 + d} \right)^2 \left( \frac{P_0(1 - P_0)}{(b - d)P_0 + d} \right) .
\]

By the same arguments it can be shown that maximum of \( f(P_0) = \frac{P_0(1 - P_0)}{(b - d)P_0 + d} \) occurs in \( P_0^* = \frac{d - \frac{1}{2}}{b - d} = \frac{b - c}{a - c} \). Thus for both cases we have

\[
P_0^* = \frac{\frac{1}{2} - c}{a - c}, \quad P_1^* = \frac{a - \frac{1}{2}}{a - c}, \quad P_0'^* = P_1'^* = \frac{1}{2} .
\]

So the maximum of (26) occurs in \( L_{u_1}' = -L_{u_1}' = \psi \), where \( \psi \) is the singular vector corresponding to largest singular value of \( W \), if both \( P_0' \) and \( P_1' \) are non-zero, and the maximum value is \( 4(c - \frac{1}{2})(\frac{1}{2} - a)^2 \).

Now we assume that \( P_0' \) or \( P_1' \) for instance \( P_0' \) is zero, which implies \( L_{u_1}' = 0 \) and \( P_1' = 1 \). Thus, the objective function reduces to

\[
\| WL_{u_0}' \|^2 \left( a^2 P_0 + c^2 P_1 \right) .
\]

Since \( P_0' = aP_0 + cP_1 = 0 \), we have

\[
P_0 = -\frac{c}{a} P_1 \rightarrow P_0 = -\frac{c}{a - c}, \quad P_1 = \frac{a}{a - c} ,
\]

Thus, the objective function is \( \| WL_{u_0}' \|^2 (ac) \), where the maximum value is \( -ac \sigma^2 \). We show that \( 4(c - \frac{1}{2})(\frac{1}{2} - a) \geq -ac \). This is true since we have \( 2(a + c) - 3ac \geq 1 \) due to \( a \geq 1, c \leq 0 \) or \( a \leq 0, c \geq 1 \). Thus, the maximization of (26) occurs in \( P_0' = P_1' = \frac{1}{2} \). Furthermore, \( L_{u_0}' = -L_{u_1}' = \psi \) satisfies the conditions (28) and (29), since \( \psi \) is orthogonal to \( \sqrt{P_X} \).

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