On Structural Parameterizations of the Offensive Alliance Problem

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Abstract. The Offensive Alliance problem has been studied extensively during the last twenty years. A set $S \subseteq V$ of vertices is an offensive alliance in an undirected graph $G = (V, E)$ if each $v \in N(S)$ has at least as many neighbours in $S$ as it has neighbours (including itself) not in $S$. We study the parameterized complexity of the Offensive Alliance problem, where the aim is to find a minimum size offensive alliance. Our focus here lies on parameters that measure the structural properties of the input instance. We enhance our understanding of the problem from the viewpoint of parameterized complexity by showing that the problem is W[1]-hard parameterized by a wide range of fairly restrictive structural parameters such as the feedback vertex set number, treewidth, pathwidth, and treedepth of the input graph.

Keywords: Defensive and Offensive alliance · Parameterized Complexity · FPT · W[1]-hard · treewidth

1 Introduction

In real life, an alliance is a collection of people, groups, or states such that the union is stronger than individual. The alliance can be either to achieve some common purpose, to protect against attack, or to assert collective will against others. This motivates the definitions of defensive and offensive alliances in graphs. The properties of alliances in graphs were first studied by Kristiansen, Hedetniemi, and Hedetniemi [16]. They introduced defensive, offensive and powerful alliances. The alliance problems have been studied extensively during last twenty years [9,11,19,22], and generalizations called $r$-alliances are also studied [20]. Throughout this article, $G = (V, E)$ denotes a finite, simple and undirected graph of order $|V| = n$. The subgraph induced by $S \subseteq V(G)$ is denoted by $G[S]$. For a vertex $v \in V$, we use $N_G(v) = \{u : (u, v) \in E(G)\}$ to denote the (open) neighbourhood of vertex $v$ in $G$, and $N_G[v] = N_G(v) \cup \{v\}$ to denote the closed neighbourhood of $v$. The degree $d_G(v)$ of a vertex $v \in V(G)$ is $|N_G(v)|$. For a subset $S \subseteq V(G)$, we define its closed neighbourhood as $N_G[S] = \bigcup_{v \in S} N_G[v]$ and its open neighbourhood as $N_G(S) = N_G[S] \setminus S$. For a non-empty subset $S \subseteq V$ and a vertex $v \in V(G)$, $N_S(v)$ denotes the set of neighbours of $v$ in $S$, that is, $N_S(v) = \{u \in S : (u, v) \in E(G)\}$. We use $d_S(v) = |N_S(v)|$ to denote the degree of vertex $v$ in $G[S]$. The complement of the vertex set $S$ in $V$ is denoted
by $S^c$.

A non-empty set $S \subseteq V$ is a **defensive alliance** in $G$ if $d_S(v) + 1 \geq d_{S^c}(v)$ for all $v \in S$. Since each vertex in a defensive alliance $S$ has at least as many vertices from its closed neighbor in $S$ as it has in $S^c$, by strength of numbers, we say that every vertex in $S$ can be **defended** from possible attack by vertices in $S^c$.

**Definition 1.** A non-empty set $S \subseteq V$ is an **offensive alliance** in $G$ if $d_S(v) \geq d_{S^c}(v) + 1$ for all $v \in N(S)$.

Since each vertex in $N(S)$ has more neighbors in $S$ than in $S^c$, we say that every vertex in $N(S)$ is **vulnerable** to possible attack by vertices in $S$. Equivalently, since an attack by the vertices in $S$ on the vertices in $V \setminus S$ can result in no worse than a “tie” for $S$, we say that $S$ can effectively attack $N(S)$.

**Definition 2.** A non-empty set $S \subseteq V$ is a **strong offensive alliance** in $G$ if $d_S(v) \geq d_{S^c}(v) + 2$ for all $v \in N(S)$.

In this paper, we consider **Offensive Alliance**, **Exact Offensive Alliance** and **Strong Offensive Alliance** problems under structural parameters. We define these problems as follows:

**Offensive Alliance**

**Input:** An undirected graph $G = (V, E)$ and an integer $k \geq 1$.

**Question:** Is there an offensive alliance $S \subseteq V(G)$ such that $1 \leq |S| \leq k$?

**Exact Offensive Alliance**

**Input:** An undirected graph $G = (V, E)$ and an integer $k \geq 1$.

**Question:** Is there an offensive alliance $S \subseteq V(G)$ such that $|S| = k$?

**Strong Offensive Alliance**

**Input:** An undirected graph $G = (V, E)$ and an integer $k \geq 1$.

**Question:** Is there a strong offensive alliance $S \subseteq V(G)$ such that $1 \leq |S| \leq k$?

For standard notations and definitions in graph theory, we refer to West [23]. For the standard concepts in parameterized complexity, see the recent textbook by Cygan et al. [3]. The graph parameters we explicitly use in this paper are vertex cover number, feedback vertex set number, pathwidth, treewidth and treedepth.

**Definition 3.** The **vertex cover number** is the size of a minimum vertex cover in a graph $G$ and it is denoted by $vc(G)$.

**Definition 4.** For a graph $G = (V, E)$, the parameter **feedback vertex set** is the cardinality of a smallest set $S \subseteq V(G)$ such that the graph $G - S$ is a forest and it is denoted by $fvs(G)$.
We now review the concept of a tree decomposition, introduced by Robertson and Seymour in [18]. Treewidth is a measure of how “tree-like” the graph is.

**Definition 5.** [18] A tree decomposition of a graph \( G = (V, E) \) is a tree \( T \) together with a collection of subsets \( X_t \) (called bags) of \( V \) labeled by the vertices \( t \) of \( T \) such that \( \bigcup_{t \in T} X_t = V \) and (1) and (2) below hold:

1. For every edge \( (u, v) \in E(G) \), there is some \( t \) such that \( \{u, v\} \subseteq X_t \).
2. (Interpolation Property) If \( t \) is a vertex on the unique path in \( T \) from \( t_1 \) to \( t_2 \), then \( X_{t_1} \cap X_{t_2} \subseteq X_t \).

**Definition 6.** [5] The width of a tree decomposition is the maximum value of \( |X_t| - 1 \) taken over all the vertices \( t \) of the tree \( T \) of the decomposition. The treewidth \( tw(G) \) of a graph \( G \) is the minimum width among all possible tree decomposition of \( G \).

**Definition 7.** If the tree \( T \) of a tree decomposition is a path, then we say that the tree decomposition is a path decomposition, and similarly the pathwidth of a graph \( G \) is the minimum width among all possible path decomposition of \( G \).

A rooted forest is a disjoint union of rooted trees. Given a rooted forest \( F \), its transitive closure is a graph \( H \) in which \( V(H) \) contains all the nodes of the rooted forest, and \( E(H) \) contain an edge between two vertices only if those two vertices form an ancestor-descendant pair in the forest \( F \).

**Definition 8.** The treedepth of a graph \( G \) is the minimum height of a rooted forest \( F \) whose transitive closure contains the graph \( G \). It is denoted by \( td(G) \).

### 1.1 Our Main Results

We show that the Offensive Alliance and Exact Offensive Alliance problems are W[1]-hard parameterized by any of the following parameters: the feedback vertex set number, treewidth, pathwidth, and treedepth of the input graph.

### 1.2 Known Results

The decision version for several types of alliances have been shown to be NP-complete. For an integer \( r \), a nonempty set \( S \subseteq V(G) \) is a defensive \( r \)-alliance if for each \( v \in S \), \( |N(v) \cap S| \geq |N(v) \setminus S| + r \). A set is a defensive alliance if it is a defensive \((-1)\)-alliance. A defensive \( r \)-alliance \( S \) is global if \( S \) is a dominating set. The defensive \( r \)-alliance problem is NP-complete for any \( r \) [20]. The defensive alliance problem is NP-complete even when restricted to split, chordal and bipartite graph [13]. For an integer \( r \), a nonempty set \( S \subseteq V(G) \) is an offensive \( r \)-alliance if for each \( v \in N(S) \), \( |N(v) \cap S| \geq |N(v) \setminus S| + r \). An offensive 1-alliance is called an offensive alliance. An offensive \( r \)-alliance \( S \) is global if \( S \) is a dominating set. Fernau et al. showed that the offensive \( r \)-alliance and global offensive \( r \)-alliance problems are NP-complete for any fixed \( r \) [8]. They also proved that
for \( r > 1 \), \( r \)-offensive alliance is NP-hard, even when restricted to \( r \)-regular planar graphs. There are polynomial time algorithms for finding minimum alliances in trees \([11,13]\). A polynomial time algorithm for finding minimum defensive alliance in series parallel graph is presented in \([12]\). Fernau and Raible showed in \([7]\) that the defensive and offensive alliance problems and their global variants are fixed parameter tractable when parameterized by the solution size \( k \). Kiyomi and Otachi showed in \([14]\), the problems of finding smallest alliances of all kinds are fixed-parameter tractable when parameterized by the vertex cover number. The problems of finding smallest defensive and offensive alliances are also fixed-parameter tractable when parameterized by the neighbourhood diversity \([10]\). Enciso \([6]\) proved that finding defensive and global defensive alliances is fixed parameter tractable when parameterized by domino treewidth.

## 2 Hardness Results

In this section we show that **Offensive Alliance** is \( W[1] \)-hard parameterized by a vertex deletion set to trees of height at most seven, that is, a subset \( D \) of the vertices of the graph such that every component in the graph, after removing \( D \), is a tree of height at most seven. On the way towards this result, we provide hardness results for several interesting versions of the **Offensive Alliance** problem which we require in our proofs. The problem **Offensive Alliance** \(^F\) generalizes **Offensive Alliance** that some vertices are forced to be outside the solution; these vertices are called forbidden vertices. This variant can be formalized as follows:

**Offensive Alliance** \(^F\)

**Input:** An undirected graph \( G = (V, E) \), an integer \( r \) and a set \( V_{\square} \subseteq V(G) \) of forbidden vertices such that each degree one forbidden vertex is adjacent to another forbidden vertex and each forbidden vertex of degree greater than one is adjacent to a degree one forbidden vertex.

**Question:** Is there an offensive alliance \( S \subseteq V \) such that \( (i) \ 1 \leq |S| \leq r \), and \( (ii) \ S \cap V_{\square} = \emptyset \)?

**Strong Offensive Alliance** \(^F_N\) is a generalization of **Strong Offensive Alliance** \(^F\) that, in addition, requires some “necessary” vertices to be in \( S \). This variant can be formalized as follows:

**Strong Offensive Alliance** \(^F_N\)

**Input:** An undirected graph \( G = (V, E) \), an integer \( r \), a set \( V_{\triangle} \subseteq V \), a set \( V_{\square} \subseteq V(G) \) of forbidden vertices such that each degree one forbidden vertex is adjacent to another forbidden vertex and each forbidden vertex of degree greater than one is adjacent to a degree one forbidden vertex.

**Question:** Is there a strong offensive alliance \( S \subseteq V \) such that \( (i) \ 1 \leq |S| \leq r \), \( (ii) \ S \cap V_{\square} = \emptyset \), and \( (iii) \ V_{\triangle} \subseteq S \)?
While the **Offensive Alliance** problem asks for offensive alliance of size at most \( r \), we also consider the **Exact Offensive Alliance** problem that concerns offensive alliance of size exactly \( r \). Analogously, we also define exact versions of **Strong Offensive Alliance** presented above. To prove Lemma 2 we consider the following problem:

**Lemma 1.** MRSS is \( \text{W}[1] \)-hard when parameterized by the combined parameter \( k + k' \), even if all integers in the input are given in unary.

We now show that the **Strong Offensive Alliance** problem is \( \text{W}[1] \)-hard when parameterized by the size of a vertex deletion set into trees of height at most 5, via a reduction from MRSS.

**Lemma 2.** The **Strong Offensive Alliance** problem is \( \text{W}[1] \)-hard when parameterized by the size of a vertex deletion set into trees of height at most 5.

**Proof.** To prove this we reduce from MRSS, which is known to be \( \text{W}[1] \)-hard when parameterized by the combined parameter \( k + k' \), even if all integers in the input are given in unary [11]. Let \( I = (k, k', S, t) \) be an instance of MRSS. We construct an instance \( I' = (G, r, V_\Delta, V_\square) \) of **Strong Offensive Alliance** in the following way. See Figure 1 for an illustration. First, we introduce a set of \( k \) forbidden vertices \( U = \{u_1, u_2, \ldots, u_k\} \). For each vector \( s = (s(1), s(2), \ldots, s(k)) \in S \), we introduce a tree \( T_s \) into \( G \). We define \( \max(s) = \max_{1 \leq i \leq k} \{s(i)\} \). The vertex set of tree \( T_s \) is defined as follows:

\[
V(T_s) = A_s \cup B_s \cup A_s^{\square} \cup B_s^{\square} \cup C_s \cup Z_s \cup \{x_s, y_s, z_s\}
\]

where \( A_s = \{a_1^s, \ldots, a_{\max(s)+1}^s\} \), \( B_s = \{b_1^s, \ldots, b_{\max(s)+1}^s\} \), \( A_s^{\square} = \{a_1^s, \ldots, a_{\max(s)+1}^s\} \), \( B_s^{\square} = \{b_1^s, \ldots, b_{\max(s)+1}^s\} \) and \( C_s = \{c_1^s, \ldots, c_{\max(s)+2}^s\} \) are five sets of vertices, and the set \( Z_s = \{z_1^s, z_2^s, z_3^s, z_4^s, z_5^s, z_6^s\} \) contains five necessary vertices and one forbidden vertex. We now create the edge set of \( T_s \).

\[
E(T_s) = \bigcup_{i=1}^{\max(s)+1} \left\{(a_i^s, b_i^s), (a_i^s, a_i^s), (a_i^s, b_i^s), (x_s, a_i^s)\right\}
\]

\[
\bigcup_{i=1}^5 \left\{(z_s, z_1^s), (z_s, z_2^s)\right\} \bigcup \left\{(x_s, z_s), (z_s, y_s)\right\} \bigcup_{i=1}^{2\max(s)+2} (y_s, c_i^s)
\]
Next we introduce a vertex $a$ and a set of four vertices $A = \{a_1^\triangle, a_2^\triangle, a_3^\triangle, a_4^\square\}$ containing three necessary vertices and one forbidden vertex. Make $a$ adjacent to all the vertices in $A$. For each $i \in \{1, 2, \ldots, k\}$ and for each $s \in S$, we make $u_i$ adjacent to exactly $s(i)$ many vertices of $A_s$ in arbitrary manner. For each $s \in S$, we make $a$ adjacent to all the vertices of $A_s \cup B_s \cup C_s$. For every $u_i \in U$, we create a set $V_{u_i}^\square$ of $\sum_{s \in S} s(i)$ forbidden vertices and a set $V_{u_i}^\triangle$ of $2 \sum_{s \in S} s(i) - 2t(i) + 2$ necessary vertices; and make $u_i$ adjacent to every vertex of $V_{u_i}^\square \cup V_{u_i}^\triangle$. We define

$$V_{\Delta} = \bigcup_{i=1}^{k} V_{u_i}^\triangle \cup A \setminus \{a_4^\square\} \bigcup_{s \in S} Z_s \setminus \{z_{s}^\square\}$$
and

\[ V_{\Box} = U \cup \{a, a^{\Box}\} \bigcup_{i=1}^{k} V_{i, \Box} \bigcup_{s \in S} A_s^{\Box} \cup B_s^{\Box} \cup \{z_s, z^{\Box}\}. \]

We set \( r = \frac{k}{2} \left( \sum_{s \in S} s(i) - t(i) + 1 \right) + \sum_{s \in S} 2(\max(s) + 1) + 5n + 3 + k'. \) Observe that if we remove the set \( U \cup \{a\} \) of \( k + 1 \) vertices from \( G \), each connected component of the resulting graph is a tree with height at most 5. Note that, \( I' \) can be constructed in polynomial time. The reason is this. As all integers in \( I \) are bounded by a polynomial in \( n \), the number of vertices in \( G \) is also polynomially bounded in \( n \).

It remains to show that \( I \) is a yes instance if and only if \( I' \) is a yes instance. Towards showing the forward direction, let \( S' \) be a subset of \( S \) such that \( |S'| \leq k' \) and \( \sum_{s \in S'} s \geq t \). We claim

\[ R = V_{\triangle} \bigcup_{s \in S'} A_s \cup B_s \cup \{x_s\} \bigcup_{s \in S \setminus S'} C_s - \] is a strong offensive alliance of \( G \) such that \( |R| \leq r \), \( V_{\triangle} \subseteq R \), and \( V_{\Box} \cap R = \emptyset \). Observe that \( N_G(R) = U \cup \{a\} \bigcup_{s \in S \setminus S'} \{z_s\} \bigcup_{s \in S'} \{y_s\} \bigcup_{s \in S'} A_s^{\Box} \). Let \( u_i \in U \), then we show that \( d_R(u_i) \geq d_{R'}(u_i) + 2 \). As \( \sum_{s \in S'} s(i) - t(i) \geq 0 \), we get

\[
\begin{align*}
d_R(u_i) &= \sum_{s \in S'} s(i) + |V_{u_i, \triangle}| \\
&= \sum_{s \in S'} s(i) + 2 \sum_{s \in S} s(i) - 2t(i) + 2 \\
&= \left( \sum_{s \in S'} s(i) - t(i) \right) + \sum_{s \in S} s(i) - t(i) + \sum_{s \in S} s(i) + 2 \\
&\geq \sum_{s \in S} s(i) - t(i) + \sum_{s \in S} s(i) + 2 \\
&= \sum_{s \in S \setminus S'} s(i) + \left( \sum_{s \in S'} s(i) - t(i) \right) + \sum_{s \in S} s(i) + 2 \\
&\geq \sum_{s \in S \setminus S'} s(i) + \sum_{s \in S} s(i) + 2 = \sum_{s \in S \setminus S'} s(i) + |V_{u_i, \Box}| + 2 \\
&= d_{R'}(u_i) + 2.
\end{align*}
\]

For the remaining vertices \( x \) in \( N(R) \), it is easy to see that \( d_R(x) \geq d_{R'}(x) + 2 \). Therefore, \( R \) is a strong offensive alliance.

Towards showing the reverse direction of the equivalence, suppose \( G \) has a strong offensive alliance \( R \) of size at most \( r \) such that \( V_{\triangle} \subseteq R \) and \( V_{\Box} \cap R = \emptyset \). From the definition of \( V_{\triangle} \) and \( V_{\Box} \), it is easy to note that \( U \subseteq N(R) \). We know
We have the following corollaries from Lemma 2.

\[ V_\Delta \] contains \( \sum_{i=1}^{k} \left( \sum_{s \in S} 2s(i) - 2t(i) \right) + 5n + 3 \) vertices; thus besides the vertices of \( V_\Delta \), there are at most \( \sum_{s \in S} 2(\max(s) + 1) + k' \) vertices in \( R \). Since \( a \in N(V_\Delta) \) and \( d_{G}(a) = \sum_{s \in S} 4(\max(s) + 1) + 4 \) where \( a \) is adjacent to three necessary vertices, it must have at least \( \sum_{s \in S} 2(\max(s) + 1) \) many neighbours in \( R \) from the set \( \bigcup (A_s \cup B_s \cup C_s') \). It is to be noted that if a vertex from the set \( A_s \cup B_s \) is in the solution then the whole set \( A_s \cup B_s \cup \{x_s\} \) lie in the solution. Otherwise \( v \in A_i^{N} \subseteq N(R) \) will have \( d_R(v) < d_{R'}(v) + 2 \) which is a contradiction as \( R \) is a strong offensive alliance. This shows that at most \( k' \) many sets of the form \( A_s \cup B_s \cup \{x_s\} \) contribute to the solution as otherwise the size of solution exceeds \( r \). Therefore, any strong offensive alliance \( R \) of size at most \( r \) can be transformed to another strong offensive alliance \( R' \) of size at most \( r \) as follows:

\[
R' = V_\Delta \bigcup_{x_s \in R} A_s \bigcup B_s \bigcup \{x_s\} \bigcup_{x_s \in V(G) \setminus R} C_s.
\]

We define a subset \( S' = \{ s \in S \mid x_s \in R' \} \). Clearly, \( |S'| \leq k' \). We claim that \( \sum_{s \in S'} s(i) \geq t(i) \) for all \( 1 \leq i \leq k \). Assume for the sake of contradiction that \( \sum_{s \in S'} s(i) < t(i) \) for some \( i \in \{1, 2, \ldots, k\} \). Then, we have

\[
d_{R'}(u_i) = \sum_{s \in S'} s(i) + |V_{u_i \Delta}|
\]

\[
= \sum_{s \in S'} s(i) + 2 \sum_{s \in S} s(i) - 2t(i) + 2
\]

\[
= \sum_{s \in S'} s(i) - t(i) + \sum_{s \in S} s(i) - t(i) + s(i) + 2
\]

\[
< \sum_{s \in S'} s(i) - t(i) + \sum_{s \in S} s(i) + 2
\]

\[
= \sum_{s \in S \setminus S'} s(i) + \left( \sum_{s \in S'} s(i) - t(i) \right) + \sum_{s \in S} s(i) + 2
\]

\[
< \sum_{s \in S \setminus S'} s(i) + \sum_{s \in S} s(i) + 2 = \sum_{s \in S \setminus S'} s(i) + |V_{u_i \Delta}| + 2
\]

\[
d_{R'}(u_i) + 2
\]

and we also know \( u_i \in N(R') \), which is a contradiction to the fact that \( R' \) is a strong offensive alliance. This shows that \( I \) is a yes instance.

We have the following corollaries from Lemma 2.

**Corollary 1.** The Strong Offensive AllianceFN problem is \( \text{W}[1] \)-hard when parameterized by the size of a vertex deletion set into trees of height at most 5, even when \( |V_\Delta| = 1 \).
Proof. Given an instance $I = (G, r, V_\Delta, V_{\square})$ of STRONG OFFENSIVE ALLIANCE$^\text{FN}$, we construct an equivalent instance $I' = (G', r', V'_\Delta, V'_{\square})$ with $|V'_\Delta| = 1$. See Figure 2 for an illustration.

Let $v_1, v_2, \ldots, v_\ell$ be vertices of $V_\Delta$ where we assume that $\ell > 1$. We introduce two vertices $x$ and $y$ where $x$ is a forbidden vertex and $y$ is a necessary vertex; and make $x$ and $y$ adjacent. We make $x$ adjacent to all the vertices in $V_\Delta$. We also introduce a set $V_x^{\square}$ of $\ell - 1$ forbidden vertices and make them adjacent to $x$. Set $r' = r + 1$. Define $V'_\Delta = \{y\}$ and $V'_x = \{x\} \cup V_x^{\square} \cup V_{\square}$. We also define $G'$ as follows

$$V(G') = V(G) \cup \{x, y\} \cup V_x^{\square}$$

and

$$E(G') = E(G) \cup \{(x, y), (x, \alpha), (x, \beta) \mid \alpha \in V_x^{\square}, \beta \in V_\Delta\}.$$ 

Let $H$ be a vertex deletion set of $G$ into trees of height at most 5. Clearly, if $H$ has at most $k$ vertices then the set $H \cup \{x\}$ has at most $k + 1$ vertices and is a vertex deletion set of $G'$ into trees of height at most 5. It is easy to see that $I$ and $I'$ are equivalent instances. \qed

We can get an analogous result for the exact variant.

**Corollary 2.** The Exact STRONG OFFENSIVE ALLIANCE$^\text{FN}$ problem is $W[1]$-hard when parameterized by the size of a vertex deletion set into trees of height at most 5 even when $|V_\Delta| = 1$.

Next, we give an FPT reduction that eliminates necessary vertices.

**Lemma 3.** The OFFENSIVE ALLIANCE$^\text{F}$ problem is $W[1]$-hard when parameterized by the size of a vertex deletion set into trees of height at most 5.

**Proof.** To prove this we reduce from the STRONG OFFENSIVE ALLIANCE$^\text{FN}$ problem, which is $W[1]$-hard when parameterized by the size of a vertex deletion set into trees of height at most 5, even when $|V_\Delta| = 1$. See Corollary 1. Given
an instance $I = (G, r, V_{\Delta} = \{x\}, V_{\Box})$ of STRONG OFFENSIVE ALLIANCE$_{FN}$, we construct an instance $I' = (G', r', V_{\Box}')$ of OFFENSIVE ALLIANCE$^F$ the following way. See Figure 3 for an illustration. Let $n$ be the number of vertices in $G$ and

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure3.png}
\caption{The reduction from STRONG OFFENSIVE ALLIANCE$_{FN}$ to OFFENSIVE ALLIANCE$^F$ in Lemma 4. Note that the set $\{v_1, \ldots, v_{n'}\}$ may contain forbidden vertices of degree greater than one.}
\end{figure}

let $V(G) = \{x, v_1, v_2, \ldots, v_{n'}, f_1, \ldots, f_{n''}\}$ where $F = \{f_1, \ldots, f_{n''}\}$ is the set of degree one forbidden vertices in $V(G)$. We create set $V_{\Box} = \{t_1, \ldots, t_{4n}\}$ of $4n$ forbidden vertices into $G'$ and make them adjacent to $t_{\Box}$. We introduce a set $V_{x_{\Box}}$ of $n$ forbidden vertices and make them adjacent to $x_{\Box}$. Finally we create a set $T = \{t_1, \ldots, t_{4n}\}$ of $4n$ vertices and make the vertices in $T$ adjacent to $t_{\Box}$ and $x_{\Box}$, and make the vertices in $V(G) \setminus F$ adjacent to $x_{\Box}$. We also add an edge $(x, x_{\Box})$. Set $r' = r + 4n$. We define $G'$ as follows:

\[ V(G') = V(G) \cup T \cup V_{t_{\Box}} \cup V_{x_{\Box}} \cup \{t_{\Box}, x_{\Box}\} \]

and

\[ E(G') = E(G) \cup \{(t_{\Box}, \alpha) : \alpha \in T \cup V_{t_{\Box}} \cup \{x\}\} \]
\[ \cup \{(x_{\Box}, \beta) : \beta \in T \cup V_{x_{\Box}} \cup V(G) \setminus F\} \]

We define $V_{t_{\Box}} = V_{t_{\Box}} \cup V_{x_{\Box}} \cup V_{x_{\Box}} \cup \{t_{\Box}, x_{\Box}\}$. Observe that there exists a set of at most $k + 2$ vertices in $G'$ whose deletion makes the resulting graph a forest containing trees of height at most 5. We can find such a set because there exists a vertex deletion set $H$ of $G$ into trees of height at most 5. We just add $\{x_{\Box}, t_{\Box}\}$ to the set $H$, then the resulting set is of size $k + 2$ whose deletion makes the resulting graph a forest containing trees of height at most 5.

We now claim that $I$ is a yes-instance if and only if $I'$ is a yes-instance. Assume first that $R$ is a strong offensive alliance of size at most $r$ in $G$ such that $\{x\} \subseteq R$ and $V_{\Box} \cap R = \emptyset$. We claim $R' = R \cup T$ is an offensive alliance of size at most $r + 4n$ in $G'$ such that $V_{\Box} \cap R' = \emptyset$. Clearly, $N(R') = \{t_{\Box}, x_{\Box}\} \cup N(R)$. For each $v \in N(R)$, we know that $d_R(v) \geq d_{R'}(v) + 2$ in $G$. Therefore in graph
in $G'$, we get $\Delta_{R'}(v) \geq \Delta_{R^+(v)} + 1$ for each $v \in N(R)$ due to the vertex $x$. For $v \in \{x, t\}$, it is clear that $\Delta_{R'}(v) \geq \Delta_{R^+(v)} + 1$. This shows that $I'$ is a yes instance.

To prove the reverse direction of the equivalence, suppose $R'$ is an offensive alliance of size at most $r' = r + 4n$ in $G'$ such that $R' \cap V_\boxdot = \emptyset$. We claim that $x \in R'$. In fact, we show that $T \cup \{x\} \subseteq R'$. Since $R'$ is non empty, it must contain a vertex from the set $V(G) \cup T$. Then $x \in N(R')$. Due to $n$ forbidden vertices in the set $V_\boxdot$, node $x$ must have at least $n + 1$ neighbours in $R'$. This implies that $R'$ contains at least one vertex from $T$. Then $t \in N(R')$ and it satisfies the condition $\Delta_{R'}(t) \geq \Delta_{R^+(t)} + 1$. Since $|V_\boxdot| = 4n$, the condition $\Delta_{R'}(t) \geq \Delta_{R^+(t)} + 1$ forces the set $\{x\} \cup T$ to be inside the solution. Consider $R = R' \cap V(G)$. Clearly $|R| \leq r$, $x \in R$, $R \cap V_\boxdot = \emptyset$ and we show that $R$ is a strong offensive alliance in $G$. For each $v \in N(R') \cap V(G) = N(R)$, we have $N_{R'}(v) \geq N_{R^+(v)} + 1$ in $G'$. Notice that we do not have $x$ in $G$ which is adjacent to all vertices in $N(R)$. Thus for each $v \in N(R)$, we get $N_R(v) \geq N_{R'}(v) + 2$ in $G$. Therefore $R$ is a strong offensive alliance of size at most $r$ in $G$ such that $x \in R$ and $R \cap V_\boxdot = \emptyset$. This shows that $I$ is a yes instance.

We are now ready to show our main hardness result for Offensive Alliance using a reduction from Offensive Alliance$^F$.

**Theorem 1.** The Offensive Alliance problem is W[1]-hard when parameterized by the size of a vertex deletion set into trees of height at most 7.

**Proof.** We give a parameterized reduction from Offensive Alliance$^F$ which is W[1]-hard when parameterized by the size of a vertex deletion set into trees of height at most 5. Let $I = (G, r, V_\boxdot)$ be an instance of Offensive Alliance$^F$. Let $n = |V(G)|$. We construct an instance $I' = (G', r')$ of Offensive Alliance the following way. We set $r' = r$. Recall that each degree one forbidden vertex is adjacent to another forbidden vertex and each forbidden vertex of degree greater than one is adjacent to a degree one forbidden vertex. Let $u$ be a degree one forbidden vertex in $G$ and $u$ is adjacent to another forbidden vertex $v$. For each degree one forbidden vertex $u \in V_\boxdot$, we introduce a tree $T_u$ rooted at $u$ of height 2 as shown in Figure 4. The forbidden vertex $v$ has additional neighbours from the original graph $G$ which are not shown here. We define $G'$ as follows:

$V(G') = V(G) \cup \{V(T_u) \mid u \text{ is a degree one forbidden vertex in } G\}$

and

$E(G') = E(G) \cup \bigcup_{u' \in V_\boxdot} E(T_u)$.

We claim $I$ is a yes instance if and only if $I'$ is a yes instance. It is easy to see that if $R$ is an offensive alliance of size at most $r$ in $G$ then it is also an offensive alliance of size at most $r' = r$ in $G'$. 
To prove the reverse direction of the equivalence, suppose that $G'$ has an offensive alliance $R'$ of size at most $r' = r$. We claim that no vertex from the set $V \bigcup_{u \in V} V(T_u)$ is part of $R'$. It is easy to see that if any vertex from the set $V \bigcup_{u \in V} V(T_u)$ is in $R'$ then the size of $R'$ exceeds $2r$. This implies that $R = R' \cap G$ is an offensive alliance such that $R \cap V = \emptyset$ and $|R| \leq r$. This shows that $I$ is a yes instance.

We have the following consequences.

**Corollary 3.** The Exact Offensive Alliance problem is W[1]-hard when parameterized by the size of a vertex deletion set into trees of height at most 7.

Clearly trees of height at most seven are trivially acyclic. Moreover, it is easy to verify that such trees have pathwidth [15] and treedepth [17] at most seven, which implies:

**Theorem 2.** The Offensive Alliance and Exact Offensive Alliance problems are W[1]-hard when parameterized by any of the following parameters:

- the feedback vertex set number,
- the treewidth and pathwidth of the input graph,
- the treedepth of the input graph.

### 3 Conclusions

In this work we proved that the Offensive Alliance problem is W[1]-hard parameterized by a wide range of fairly restrictive structural parameters such as the feedback vertex set number, treewidth, pathwidth, and treedepth of the input graph thus not FPT (unless FPT = W[1]). This is especially interesting because most “subset problems” that are FPT when parameterized by solution size turned out to be FPT for the parameter treewidth [4], and moreover Offensive Alliance is easy on trees. In the future it may be interesting to study if our ideas can be useful for different kinds of alliances from the literature such as powerful alliances and defensive alliances. It would be interesting to consider the parameterized complexity with respect to twin cover. The parameterized
complexity of offensive and defensive alliance problems remain unsettled when parameterized by other important structural graph parameters like clique-width and modular-width.

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