Superextension of Jordanian Deformation  
for $U(osp(1|2))$  
and its Generalizations  

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Abstract  

We describe Jordanian “nonstandard” deformation of $U(osp(1|2))$ by  
employing the twist quantization technique. An extension of these results  
to $U(osp(1|4))$ describing deformed graded $D=4$ AdS symmetries and to  
their super-Poincaré limit is outlined.  

1 Introduction  

It is well-known that the $U(sl(2))$ algebra with basic commutators  

$$[h,e_\pm] = \pm e_\pm, \quad [e_+, e_-] = 2h,$$  

(1)  
can be endowed with the following two inequivalent quantum deformations:  
i) Drinfeld-Jimbo “standard” $q$-deformation with the following classical $r$-  
matrix ($q = 1 - \gamma$)  

$$r_{DJ} = \gamma e_+ \wedge e_-,$$  

(2)
ii) Jordanian “nonstandard” quantum deformation, generated by the classical $r$-matrix
\[ r_j = \xi h \wedge e_+. \]

The Hopf-algebraic structure of the Drinfeld-Jimbo deformation $U_q(sl(2))$ was given firstly in [1–3], and the Hopf algebra describing Jordanian deformation of $U(sl(2))$ was presented in [4].

The classical $r$-matrix (3) satisfies classical YB equation and its quantization can be described by so-called twist quantization method [5]. We recall that twist quantization of a Hopf algebra $H= (A, m, \Delta, S, \varepsilon)$ is given by the twisting two-tensor $F = \sum f_i^{(1)} \otimes f_i^{(2)}$ modifying the coproduct $\Delta$ and antipode $S$ as follows:
\[ \Delta \rightarrow \Delta_F = F \Delta F^{-1}, \]
\[ S \rightarrow S_F = uS u^{-1}, \quad u = \sum_i f_i^{(1)} S(f_i^{(2)}). \]

It should be stressed that the algebraic sector $A$ of $H$ remains unchanged.

The twisting two-tensor $F$ for the Jordanian deformation of $U(sl(2))$ (see (3)) has been given firstly by Ogievetsky [6] in the following closed form
\[ F_j = \exp(\xi h \otimes E_+), \]
where
\[ E_+ = \frac{1}{\xi} \ln(1 + \xi e_+) = e_+ + O(\xi). \]

The deformations (2) and (3) of $U(sl(2))$ provide important building blocks in the theory of quantum deformations of arbitrary Lie algebra. Similar role in the deformation theory of Lie superalgebras is played by the deformations of rank 1 superalgebra $osp(1|2)$, which is the supersymmetric extension of $sl(2) \simeq sp(2)$. Our first aim here is to generalize the Jordanian deformation of $U(sl(2))$ to the $U(osp(1|2))$ case. Further we present briefly the Jordanian deformation of $U(osp(1|4))$ as a special example of general framework presented by one of the authors in [7]. By interpreting $osp(1|4)$ as $D = 4$ AdS superalgebra we were able to obtain via contraction a new $\kappa$-deformation of $D = 4$ Poincaré superalgebra [8].

2 Jordanian Deformation of $U(osp(1|2))$

The classical $r$-matrices (2) and (3) are supersymmetrically extended as follows:
\[ r_{\text{DI}}^{\text{Susy}} = \gamma (e_+ \wedge e_- + 2v_+ \wedge v_-), \]  
\[ r_f^{\text{Susy}} = \xi (h \wedge e_+ - v_+ \wedge v_+), \]  
(7a, 7b)

where the odd generators \( v_{\pm} \) of \( osp(1,2) \) extend the \( SL(2) \) algebra as follows:

\[ [h, v_{\pm}] = \pm \frac{1}{2} v_{\pm}, \quad \{ v_+, v_- \} = -\frac{1}{2} h, \quad e_{\pm} = \pm 4 (v_{\pm})^2, \]  
(8)

and in (7a−7b) for odd generators we define \( a \wedge b = a \otimes b + b \otimes a \).

The quantization of the deformation (7a) is well known as a particular case of the extension of Drinfeld-Jimbo quantization method to Lie superalgebras. The Jordanian quantization of \( U(osp(1|2)) \) generated by (7b) has been obtained quite recently, by the superextension of twist quantization procedure. It should be mentioned that incomplete discussion of twist quantization of \( U(osp(1|2)) \) was presented earlier, but explicit formulae for the twist tensor and all coproduct formulae have been given firstly in [12].

Let us recall that twisting element \( F \) should satisfy the cocycle equation

\[ F^{12} (\Delta \otimes \text{id})(F) = F^{23} (\text{id} \otimes \Delta)(F), \]  
(9)

and the “unital” normalization condition

\[ (\varepsilon \otimes \text{id})(F) = (\text{id} \otimes \varepsilon)(F) = 1. \]  
(10)

We assume that the twisting two-tensor \( F_{sj} \) describing the quantization of the classical \( r \)-matrix (7b) can be factorized as follows

\[ F_{sj} = F_s F_j, \]  
(11)

where the “supersymmetric part” \( F_s \) depend on the odd generators \( v_{\pm} \). Substituting (11) into (9) provides the following twisted cocycle condition for \( F_s \)

\[ F_s^{12} (\Delta_j \otimes 1)(F_s) = F_s^{23} (1 \otimes \Delta_j)(F_s), \]  
(12)

where

\[ \Delta_j(a) = F_j \Delta^{(0)}(a) F_j^{-1}, \]  
(13)
and \( \Delta^{(0)}(a) = a \otimes 1 + 1 \otimes a \) for \( a \in osp(1|2) \). Taking into consideration that the twist \( F_{SJ} \) for small values of \( \xi \) should have a form describing classical \( r \)-matrix \( \text{(1b)} \)

\[
F_{SJ} = 1 + \xi(h \otimes e_+ - v_+ \otimes v_+) + \mathcal{O}(\xi^2),
\]

one can write the solution of \( \text{(12)} \) in the following explicit form:

\[
\tilde{F}_S = 1 - 4\xi \frac{v_+}{e^{2\sigma} + 1} \otimes \frac{v_+}{e^{2\sigma} + 1} = 1 - \xi \frac{v_+ e^{-\frac{1}{2}\sigma}}{\cosh \frac{1}{2}\sigma} \otimes \frac{v_+ e^{-\frac{1}{2}\sigma}}{\cosh \frac{1}{2}\sigma},
\]

where \( \sigma = \frac{\xi}{2} E_+ = \frac{1}{2} \ln(1 + \xi e_+) \) and \( \Delta_j(\sigma) = \sigma \otimes 1 + 1 \otimes \sigma \). One can show that

\[
F_{S}^{-1} = \frac{\cosh \frac{1}{2}\sigma \otimes \cosh \frac{1}{2}\sigma + \xi v_+ e^{-\frac{1}{2}\sigma} \otimes v_+ e^{-\frac{1}{2}\sigma}}{\cosh \frac{1}{2}\Delta_j(\sigma)}. \tag{16}
\]

Modifying \( \text{(15)} \) by a factor \( \Phi = \Phi(\sigma) \) as follows

\[
\tilde{F}_S = \Phi F_S,
\]

where

\[
\Phi = \sqrt{\frac{(e^{2\sigma} + 1) \otimes (e^{2\sigma} + 1)}{2(e^{\sigma} \otimes e^{\sigma} + 1)}}, \tag{18}
\]

one obtains the unitary twist factor, i.e. \( \tilde{F}_S \tilde{F}_S^* = \tilde{F}_S^* \tilde{F}_S = 1 \), provided that the parameter \( \xi \) is purely imaginary. The choice \( \text{(17)} \) provides the following deformed coproducts of \( U_\xi(osp(2|1)) \)

\[
\tilde{\Delta}_{SJ}(h) = h \otimes e^{-2\sigma} + 1 \otimes h + \xi v_+ e^{-\sigma} \otimes v_+ e^{-2\sigma}, \tag{19a}
\]

\[
\tilde{\Delta}_{SJ}(v_+) = v_+ \otimes 1 + e^\sigma \otimes v_+, \tag{19b}
\]

\[
\tilde{\Delta}_{SJ}(v_-) = v_- \otimes e^{-\sigma} + 1 \otimes v_- + \frac{\xi}{4} \left\{ \{ h, e^\sigma \} \otimes v_+ e^{-2\sigma} - \{ h, v_+ \} \otimes (e^\sigma - 1)e^{-2\sigma} + 2v_+ \otimes h - \left\{ h, \frac{v_+ e^\sigma}{e^\sigma + 1} \right\} \otimes (e^\sigma - 1)e^{-\sigma} \right\},
\]
\[(e^\sigma - 1) \otimes \left\{ h; \frac{v_+}{e^\sigma + 1} \right\}, \frac{1}{e^\sigma \otimes e^\sigma + 1} \right\}, \tag{19c} \]

where

\[\Delta_{sj} = \tilde{F}_s F_j \Delta^{(0)} F_j^{-1} \tilde{F}_s^{-1}. \tag{20}\]

Besides we get

\[
\begin{align*}
\tilde{S}_{sj}(h) &= -he^{2\sigma} + \frac{1}{4}(e^{2\sigma} - 1), \tag{21a} \\
\tilde{S}_{sj}(v_+) &= -e^{-\sigma}v_+, \tag{21b} \\
\tilde{S}_{sj}(v_-) &= -v_- e^\sigma + \xi h v_+ e^\sigma - \frac{\xi}{4} v_+ e^\sigma, \tag{21c}
\end{align*}
\]

where the formula (4b) with \(F = \tilde{F}_s F_j\) has been used.

Following the general framework of twist quantization the universal \(R\)-matrix is given by the formula \(\left( F^{21} \equiv \sum_i f^{(2)}_i \otimes f^{(1)}_i \right) \)

\[R = \tilde{F}_s^{21} F_j^{21} F_j^{-1} \tilde{F}_s^{-1} = \tilde{F}_s^{21} R_j F_s^{-1}, \tag{22} \]

where \(R_j\) is the Jordanian \(R\)-matrix describing the quantum algebra \(U_\xi(sl(2))\):

\[R_j = F_j^{21} F_j^{-1} = e^{2\sigma \otimes h} e^{-2h \otimes \sigma}. \tag{23} \]

It can be added that

a) We discuss here the complex Lie superalgebra \(osp(1|2)\). It appears that one can consider also the Jordanian deformation \(U_\xi(osp(1|2; R))\) of the real form of \(osp(1|2)\), in a way consistent with the real form of \(sl(2)\) providing \(sl(2; R) \simeq o(2, 1)\) \(\dagger\).

b) The real form of \(U_\xi(osp(1|2))\) extending supersymmetrically the algebra \(U_\xi(o(2, 1))\) describes deformed D=1 conformal superalgebra \(\dagger\dagger\). It appears that possibly for the physical applications it is useful to use the new basis in \(U(osp(1|2; R))\), with deformed \(osp(1|2; R)\) superalgebra relations (see e.g. \(\dagger\dagger\))

### 3 Beyond \(osp(1|2)\)

The next case of Jordanian deformation which is of physical interest in the \(osp(1|2n)\) serie is \(n=2\) \(\dagger\dagger\), providing new quantum deformation of graded anti-de-Sitter al-
gebra [8]. In such a case using the Cartan-Weyl basis of \(osp(1|4)\) (see also [7], where the notation is explained)

(a) the rising generators: \(e_{1-2}, e_{12}, e_{11}, e_{22}, e_{01}, e_{02}\);

(b) the lowering generators: \(e_{2-1}, e_{-2-1}, e_{-1-1}, e_{-2-2}, e_{-10}, e_{-20}\);

(c) the Cartan generators: \(h_1 := e_{1-1}, h_2 := e_{2-2}\), \(24\)

one can write the following general \(r\)-matrix with its support in Borel sub-sualgebra

\[
\begin{align*}
  r(\xi_1, \xi_2) &= r_1(\xi_1) + r_2(\xi_2), \quad 25 \\
  r_1(\xi_1) &= \xi_1 \left( \frac{1}{2} e_{1-1}^1 \wedge e_{11} + e_{1-2} \wedge e_{12} - 2 e_{01} \otimes e_{01} \right), \quad 26 \\
  r_2(\xi_2) &= \xi_2 \left( \frac{1}{2} e_{2-2}^2 \wedge e_{22} - 2 e_{02} \otimes e_{02} \right). \quad 27
\end{align*}
\]

The twist quantization generated by the classical \(r\)-matrix \(25\) has the form [7, 19]

\[
F(\xi_1, \xi_2) = \tilde{F}_2(\xi_2) F_1(\xi_1), \quad 28
\]

where \(F_1\) is the twisting two-tensor corresponding to the classical \(r\)-matrix \(26\) and \(\tilde{F}_2\) is the two-twisting tensor corresponding to the \(r\)-matrix \(27\) with generators modified by suitable similarity map \(\tilde{e}_{ik} = \omega_{\xi_i} e_{ik} \omega_{\xi_i}^{-1}\), where

\[
\omega_{\xi_i} = \exp \left( \frac{\xi \sigma_{11} e_{1-2} e_{12}}{1 - e^{2\sigma_{11}}} \right) \exp \left( \frac{1}{4} \sigma_{11} \right), \quad 29
\]

and \(\sigma_{11} = \frac{1}{2} \ln(1 + \xi_1 e_{11}).\)

The 10 bosonic generators \(e_{mn}\) (see \(24\)) if \(m, n = \pm 1, \pm 2\) describe the AdS \(O(3, 2)\) generators, and the generators \(e_{0m}\) \((m = \pm 1, \pm 2)\) define four odd supercharges. Introducing the AdS radius \(R\) and performing the limit \(R \to \infty\) one can show [8] that the classical \(r\)-matrix \(25\) has the finite limit if \(\xi = \xi_1 = \xi_2\) and \(\xi\) depends on \(R\) in the following way

\[
\xi(R) = \frac{i}{\kappa R}, \quad 30
\]

In particular one obtains in the limit \(R \to \infty\) from the classical \(r\)-matrix \(r(\xi(R), \xi(R))\) (see \(25\)) the following super-Poincaré classical \(r\)-matrix:

\[
r_{\kappa}^{SUSY} = \frac{1}{\kappa} r_{\kappa}^{LC} + \frac{2}{\kappa} (Q_1 \wedge Q_1 + Q_2 \wedge Q_2), \quad 31
\]
where \( Q_m = \lim_{R \to \infty} (iR)^{-\frac{1}{2}} e_{0m} (m = 1, 2) \) and \( \kappa^{LC} \) describes the light-cone \( \kappa \)-deformation of Poincaré algebra \([20, 21]\). It appears that such a contraction limit \( R \to \infty \) can be applied also to the twisted coproducts and twisted antipode of \( U(osp(1|4)) \) what provides new deformation of \( D = 4 \) Poincaré superalgebra.

In conclusion we would like to state that one can introduce by the contraction procedure two \( \kappa \)-deformations of \( D = 4, N = 1 \) supersymmetries

Drinfeld-Jimbo deformation \( U_q(osp(1,4)) \) \( (q = \frac{1}{kR}; \ R \to \infty) \) Standard \( \kappa \)-deformed \( D = 4 \) Poincaré superalgebra \([22]\)

Jordanian type deformation \( U_{\xi_1, \xi_2}(osp(1|4)) \) \( (\xi_1 = \xi_2 = iR; \ R \to \infty) \) Light-cone \( \kappa \)-deformation of \( D = 4 \) Poincaré superalgebra

More detailed description of the second deformation is provided by the authors of the present report in \([8]\).

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