Fractional advection-diffusion-asymmetry equation

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Fractional kinetic equations employ non-integer calculus to model anomalous relaxation and diffusion in many systems. While this approach is well explored, it so far failed to describe an important class of transport in disordered systems. Motivated by work on contaminant spreading in geological formations we propose and investigate a fractional advection-diffusion equation describing the biased spreading. While usual transport is described by diffusion and drift, we find a third term describing symmetry breaking which is omnipresent for transport in disordered systems. Our work is based on continuous time random walks with a finite mean waiting time and a diverging variance, a case that on the one hand is very common and on the other was missing in the kaleidoscope literature of fractional equations. The fractional space derivative stems from long trapping times while previously it was interpreted as a consequence of spatial Lévy flights.

Fractional calculus is an old branch of mathematics that studies non-integer differential operators. This method is used extensively to model anomalous diffusion and relaxation in a wide variety of systems [1–4]. To recap consider the fractional diffusion equation [5, 6] for the density of spreading particles \( \mathcal{P}(x, t) \)

\[
\frac{\partial^{\beta} \mathcal{P}}{\partial t^{\beta}} = D_{\beta, \mu} V^{\mu} \mathcal{P},
\]

(1)

where \( D_{\beta, \mu} \), with units \( \text{m}^{\mu}/\text{s}^{\beta} \), is a generalised diffusion constant. The fractional time and space derivatives are convolution operators that more intuitively are defined with their respective Laplace and Fourier transforms (see below). This equation, sometimes called the fractional diffusion-wave equation, reduces to the diffusion equation when \( \beta = 1, \mu = 2 \) and the wave equation for \( \beta = 2, \mu = 2 \). \( \mu < 2 \) corresponds to long spatial jumps referred to as Lévy flights, while \( \beta < 1 \) to long dwelling times between jump events [3]. Originally this equation was derived using the continuous time random walk (CTRW) model [3, 7–9]. More recently the fractional diffusion equation with \( \beta = 1 \) was derived for heat transport using models of interacting particles [10, 11]. Such fractional kinetic equations are used to describe time of flight experiments of charge carriers in disordered systems where due to trapping \( \beta < 1, \mu = 2 \) [12, 13] and anomalous diffusion of cold atoms in optical lattices where the atom-laser interaction induces \( \mu < 2 \) and \( \beta = 1 \) [14, 15]. Extensions that include external forces are well studied, within a framework referred to as the fractional Fokker-Planck equation [16–19], and distributed order fractional equations [20, 21]. For an extensive review see [3].

Eq. (1) exhibits reflection symmetry and hence the packet of spreading particles is symmetric around its mean, if the initial condition density is localised. In disordered systems and in the presence of fixed advection, symmetry breaking is found, and the above equation is invalid. Such behavior is found throughout hydrology, for example, for tracer and contaminant spreading in heterogeneous media. For more than two decades, two opposing and competing frameworks developed in this field. One approach advanced by Benson, Schumer, Meerschaert, and Wheatcraft (BSMW) [22, 23] proposed that the mechanism for transport is controlled by non-local spatial jumps of the Lévy type [24, 25]. It was suggested that solute particles may experience long movements in high velocity flow paths, leading to such super diffusive behavior, possibly in the spirit of Lévy flights in rotational flow [26]. Importantly, since field observations exhibit non-symmetric shapes of the spreading packet of particles, the microscopic picture introduces skewed probability density function (PDF) of spatial jump lengths. This approach extensively promoted the use of non-symmetrical fractional space advection-diffusion equations for Lévy flights, see [25] for an overview.

The second approach uses what might be considered the opposite strategy. Instead of long non-local Lévy jumps in space, Berkowitz, Scher and co-workers [27–33] showed that the CTRW framework with power law trapping time PDF is the key feature needed to explain the observed data. Physically this is the result of long trapping events in geometrically induced dead-ends found in strongly disordered porous media. Specifically, based on field experiments and extensive modelling, the trapping time PDF is \( \psi(\tau) \sim \tau^{-1+\beta} \) and importantly in many cases \( 1 < \beta < 2 \) [29, 31]. Here the mean trapping time is finite, while the variance diverges. In this case Eq. (1) is certainly not valid. To see this consider a CTRW with a finite variance of jump lengths, so \( \mu = 2 \) and then as mentioned if we take \( \beta \to 2 \) we get the wave equation, which is completely irrelevant for the transport under study. Thus, so far, the Scher-Berkowitz theoretical framework is based on a random walk picture [30] and not on a governing fractional advection-diffusion equation. Both the CTRW and the BSMW frameworks and the experiments in the field agree on one thing; advection-diffusion is anomalous and non-symmetric [31, 34], however otherwise these schools promote widely different philosophies.

The theoretical challenge is to derive a fractional advection-diffusion equation for the widely applicable CTRW but with \( 1 < \beta < 2 \). The case \( 0 < \beta < 1 \) was treated already extensively [3, 30]. Such a tool will be very powerful...
as it could deal with contaminant sources, time dependent fields. and thus provide a basic tool for transport not only in the field of Hydrology, but rather describe many other disordered systems modelled by CTRW in the relevant regime of parameters. After deriving the fractional equation for the Berkowitz-Scher transport, we will be in the position to compare it to the BSMW Lévy flights. One of the main goals of this letter is to promote a better understanding of the meaning of the fractional space derivative in transport equations. As mentioned in the literature these are associated with Lévy flights, however we show here that they are actually related to the long tailed PDF of trapping times, provided that $1 < \beta < 2$.

The fractional advection-diffusion-asymmetry equation investigated in this letter reads

$$\frac{\partial P}{\partial t} = D \frac{\partial^2 P}{\partial x^2} - V \frac{\partial P}{\partial x} + S \frac{\partial^\beta}{\partial (-x)^\beta} P. \tag{2}$$

The first two terms on the right hand side of this equation are the standard diffusion and drift terms, the last term is the modification we propose. The operator $\partial^\beta/\partial (-x)^\beta$ is a Weyl fractional derivative of order $1 < \beta < 2$ [3]. The Fourier transform of this operator acting on some test function is $F \left[\partial^\beta g(x)/\partial (-x)^\beta\right] = (-ik)^\beta \tilde{g}(k)$ where $\tilde{g}(k)$ is the Fourier transform of $g(x)$. In contrast to the spatial Weyl derivative in Eq. (1), the generalised Laplacian in Eq. (2) is the symmetric Riesz fractional derivative [5], where $F[\nabla^\beta g(x)] = -|k|^\beta \tilde{g}(k)$. Further in Eq. (2) we have no fractional time derivative and hence obviously it is very different from the standard fractional diffusion Eq. (1).

The parameters $D, V$ and $S$ are transport coefficients. $D$ with units $m^2/s$ describes normal diffusion, $V$ controls the magnitude of the drift which is found for example in response to an external force field. Finally $S$, whose units are $m^\beta/s$, is what we may call a symmetry breaking transport coefficient that gives the anomalous behaviour of the process. In what follows, after presenting the fundamental solution of the fractional advection-diffusion equation, we discuss its meaning and its applications. Deriving the equation we will give the relation between the transport process.

When initially $P(x,t)|_{t=0} = \delta(x)$, namely the packet of particles is localised on the origin, and when the transport coefficients are time independent, the solution is obtained using Fourier transform. Let $\tilde{P}(k,t)$ be the Fourier pair of $P(x,t)$ then using Eq. (2) the solution is

$$\tilde{P}(k,t) = \exp \left[-Dk^2t - ikVt + S(-ik)^\beta t\right]. \tag{3}$$

Thus the solution is a convolution of a Gaussian and a non-symmetric Lévy density. These correspond to limit distributions of sums of independent identically distributed random variables described by thin and fat tailed densities respectively. More specifically we denote $L_\beta(y)$ as the asymmetrical Lévy density whose Fourier transform is $\exp[(-ik)^\beta]\), and hence $P(x,t) = L_\beta[x/(St)^\beta](St)^{-\beta} \otimes \exp[-(x-Vt)^2/4Dt]/\sqrt{4\pi Dt}$ where $\otimes$ is the convolution symbol, see further details below and in the Supplementary Material (SM).

Model. We treat the problem using the assumption that the particle will wait for some random time $\tau$ between two successive jumps. This is exactly the framework of the CTRW that describes a particle performing random independent steps $x$, determined by the PDF $f(x)$, and between two successive steps the particle waits for time $\tau$ that is distributed according to $\psi(\tau)$ [3, 34, 35]. All the waiting times and the jump lengths are independent. The probability of observing $N$ jumps is fully determined by $\psi(\tau)$ and is well studied [36, 37]. We consider, $\psi(\tau) \sim \beta(\tau_0)^\beta \tau^{1-\beta}$ and as mentioned $1 < \beta < 2$. Here $\tau_0$ has units of time, and this time scale together with the finite mean waiting time $\langle \tau \rangle = \int_0^\infty \tau \psi(\tau) d\tau$ are important. The probability of observing $N$ steps at time $t$ is

$$Q_\beta(N) \sim \frac{1}{(t/\bar{t})^{1/\beta}} L_\beta \left[\frac{N - t/\langle \tau \rangle}{(t/\bar{t})^{1/\beta}}\right] \tag{4}$$

with $\bar{t} = \langle \tau \rangle^{1+\beta}/\langle (\tau_0)^\beta |(1-\beta)\rangle$. This equation is valid for large times and large $N$, for example the mean number of jumps $\langle N \rangle \sim t/\langle \tau \rangle$ is large. Eq. (4) means that Lévy statistics are applicable for the shifted observable $N - \langle N \rangle$, somewhat similarly to Gaussian statistics found when the second moment of $\psi(\tau)$ is finite. For the jump length distribution $f(x)$, we assume that the mean size of the jumps is $a$ and the variance is $\sigma$. For example, in simulations below, the PDF of jump size is Gaussian

$$f(x) = \frac{1}{\sqrt{2\pi \sigma}} \exp \left[-\frac{(x-a)^2}{2\sigma^2}\right]. \tag{5}$$

The parameter $a$ is the bias, and the mean position of the particle after $N$ steps, is $Na$ hence on average the packet of particles starting on the origin will be on $at/\langle \tau \rangle$. Clearly this modelling implies that we do not assume fat tailed jump length distribution, unlike the Lévy flight picture in BSMW.
In CTRW the position of the particle after \( N \) steps is \( X = \sum_{i=1}^{N} x_i \), and thus it depends both on the microscopic displacements \( x_i \) and the random number of steps \( N \). By conditioning on a specific outcome of \( N \) displacements, the PDF of finding the particle at \( X \) at time \( t \) is

\[
\mathcal{P}_{\text{CTRW}}(X,t) = \sum_{N=0}^{\infty} Q_t(N)P(X|N).
\]  

We are interested in the long time limit, hence we replace \( P(X|N) \) with the Gaussian since in this limit \( N \) is large, and similarly replace \( Q_t(N) \) with the Lévy distribution Eq. (4). Switching from summation to integration, in the long time limit we find

\[
\mathcal{P}_{\text{CTRW}}(X,t) \sim \frac{1}{(t/t)_{1/\beta}} \int_{0}^{\infty} L_{1/\beta} \left( \frac{N-t/(\tau)}{(t/t)_{1/\beta}} \right) \times \exp \left( \frac{-\{X-aN\}^2}{2\sigma^2 N} \right) dN.
\]  

This idea is also known as the subordination of the spatial process \( X \) by the temporal process for \( N \) and is routinely considered in the literature for \( \beta < 1 \), see [13, 38]. We already mentioned our intention to derive the spatial derivative usually associated with Lévy spatial jumps using the perfectly Gaussian jump statistics in space, and that is what we do next.

We change variables according to \( y = (N-t/(\tau))/(t/\bar{t})_{1/\beta} \) and find

\[
\mathcal{P}_{\text{CTRW}} \sim \int_{-\infty}^{\infty} L_{1/\beta}(y) \exp \left\{ \frac{-\{X-ay/(\tau)\}^2}{2\sigma^2 (t/(\tau) + y(t/t)_{1/\beta})} \right\} \sqrt{2\sigma^2 t/(\tau)} dy,
\]  

where the lower limit of the integration is found when \( t \to \infty \). The typical fluctuations of the process, are found when \( X-ay/(\tau) \sim a(t/\bar{t})_{1/\beta} \), and in this scaling regime we finally find in the long time limit

\[
\mathcal{P}_{\text{CTRW}} \sim \int_{-\infty}^{\infty} L_{1/\beta}(y) \frac{\exp \left\{ -\{X-ay/(\tau)\}^2/2\sigma^2 (t/(\tau) + y(t/t)_{1/\beta}) \right\}}{\sqrt{2\pi\sigma^2 t/(\tau)}} dy.
\]  

Technically this limit is obtained with a change of variables to \( \xi = (X-ay/(\tau))/a(t/\bar{t})_{1/\beta} \) and \( \xi \) is kept fixed while \( t \to \infty \). We now take the time derivative of the Fourier transform of Eq. (9) and find

\[
\frac{\partial \tilde{\mathcal{P}}(k,t)}{\partial t} = -\frac{\sigma^2}{2\langle \tau \rangle} k^2 \tilde{\mathcal{P}}(k,t) - i\frac{a}{\langle \tau \rangle} \tilde{\mathcal{P}}(k,t) + (-ik)^2 \frac{\sigma^2}{\bar{t}} \tilde{\mathcal{P}}(k,t).
\]  

This is the Fourier representation of Eq. (2) when we identify the transport constants:

\[
D = \frac{\sigma^2}{2\langle \tau \rangle}, \quad V = \frac{a}{\langle \tau \rangle}, \quad S = \frac{a^2}{\bar{t}}.
\]  

The two formulas for \( D \) and \( V \) are standard relations in the theory of advection-diffusion. To summarize, the fractional advection-diffusion-asymmetry equation (2) describes the biased CTRW process and this has several consequences which are now discussed.

The importance of bias. An interesting effect is that in the absence of bias, i.e. \( a = 0 \) we get \( S = 0 \), hence the anomaly is present only when we have advection. Since \( S = 0 \) implies normal diffusion, in the case of weak advection the solution exhibits nearly normal behaviour for very long times, an effect crucial for experiments. Further, Eq. (11) shows how the two transport coefficients \( S \) and \( V \) are generally not independent. To see this consider linear response theory. Then we have \( a \sim F \) where \( F \) is the external force field, and we have \( V \sim F \) and \( S \sim F^\beta \), a prediction that could be tested in experiments.

Packets in two dimensions. The fact that the asymmetry constant \( S \) is bias dependent leads to the following interesting prediction in two dimensions. Imagine the bias is directed in the \( x \) direction, then the distortion of the
FIG. 1: Packets of particles released from an origin in two dimensions with \( \beta = 3/2 \), time \( t = 200 \) where the mean waiting time is \( \langle \tau \rangle = 0.3 \) and \( \tau_0 = 0.1 \). The bias is pointing to the \( x \) direction, while it is absent along the \( y \) axis, and this creates packets distorted in the direction of the field. The symmetry breaking effect is visibly stronger as the bias level is increased. Here we show how simulations of the CTRW process and the analytical solutions of the fractional advection-diffusion-asymmetry equation nicely match. For theory we use Eq. (11) which gives \( D = 41.7 \), \( V = a/0.3 \), and \( S = a^{3/2}/0.44 \) the bias \( a \) is provided in the figure while in the \( y \) direction \( D = 41.7 \), and \( S = V = 0 \). For further details on simulations see SM.

packet of particles is found only along the \( x \) axis. In other words the diffusion in the perpendicular \( y \) direction will be perfectly normal. In the SM we extend our mathematical treatment of the problem to two dimensions. Here we present this effect graphically in Fig. 1, the asymmetrical oval like shape of the spreading packet is clearly visible, with the left tail broader than the right one. Similar experimental observations were reported in [29, 39]. The left tail, seen clearly in the figure, is due to trapping of particles far lagging behind the mean position of the packet and this as we showed is modeled with the asymmetry operator \( \partial^{\beta}/\partial(-x)^{\beta} \) in Eq. (2). Thus the physical interpretation of the fractional space derivative in advection-diffusion equations should be made with care, as it does not necessarily mean that the process exhibits Lévy flights.

Temporal variations of the mean velocity \( a/\langle \tau \rangle \) is often present in the real world and tested experimentally in [33]. We explore this issue now using a time dependent but piece wise constant bias \( a(t) \) [33]. Indeed in controlled experiments, the velocity \( V \) can be modified, and then theoretical predictions can be tested in a non trivial setting. This example will demonstrate the power of the fractional framework, as it allows for a semi-analytical solution of the rather complex behaviour, and present physical effects related to the magnitude of the bias. We consider four stages of the transport [33]: i) we use bias \( a = 1 \) ii) we then sharply increase \( a \) to a value \( a = 3.6 \), then (iii) decrease the value then bias to a small number \( a = 0.09 \), and finally (iv) return to the bias in state i). All along the second length scale \( \sigma = 5 \) is fixed. The time lapses of each stage clearly indicated in Fig. 2 while the derivation of analytical results is left to the SM. Note that as we modify the bias \( a \) we are effectively modifying \( V \) and \( S \) while \( D \) remains fixed, see
Eq. (11). The essential idea behind the analytical approach is that the final state of each stage serves as an initial condition to the spatial distribution of the next stage. In Fig. 2 (curve A) this analytical method is compared to numerical solution of the CTRW with $\beta = 3/2$, finding excellent agreement. We also present the case of a constant time independent $a = 1$ (curve B). Our initial condition is localised at the origin and in the figure we present the so called breakthrough curve. The latter is the common observation in the field of contaminant/tracer spreading in Hydrology. It is the concentration $P(x_b, t)$ at some fixed $x_b$ as a function of time. The weak bias in stage (i) is not able to bring the particles to the target at $x_b$ so one needs to wait for the strong bias in stage (ii) to see the sharp rise in the density. Then at stage three we have a weak field and hence stagnation of the concentration. Finally after a long time the density decays to zero as the bias sweep the packet to infinity. Fig. 2 clearly demonstrates the excellent quantitative agreement between theory and simulation, in a regime of dynamics which is close to real real life experiments and far from trivial. Hence we are confident that our tool, the fractional advection-diffusion-asymmetry equation is a useful one.

Lévy flights and the interpretation of MADE-2 experiment. The CTRW process with long tailed PDFs is an excellent model for transport in a wide variety of system, including for example porous media, hence the governing diffusion equation (2) is deeply related to transport in many physical systems [3, 9, 12, 40–43], see [35] for an updated list of Refs. Still it is interesting to compare our approach to the fractional model of Lévy flights (LF) that reads [22, 44]

$$\frac{\partial P_{LF}}{\partial t} = -V \frac{\partial P_{LF}}{\partial x} + K \left( q \frac{\partial}{\partial (-x)^{\mu}} + p \frac{\partial}{\partial x^{\mu}} \right) P_{LF}$$


Clearly this equation is very different from ours, still a fascinating twist is found when the LF hypothesis is used to fit experiments. Specifically, Zhang et al. [24, 25] fit a contaminant/tracer experiment called MADE-2 using the Lévy flight picture. What this fit shows is that the parameter $p$ is actually zero, however this means that an interpretation based on the Scher-Berkowitz CTRW picture with long tailed trapping times is a valid description. Specifically using our notation the fit of Zhang et al. means $D = 0$, $\beta = 1.1$, $V = 0.12m/d$ and $S = 0.12m^2/d$ (d=day) which implies that the MADE-2 data display a strong bias in the large time limit of CTRW. This highlights that the data are consistent with Eq. (2).

Extensions with subordination. A key formula is the transformation Eq. (7). It shows how to transform a normal process to an anomalous one, for the case when $1 < \beta < 2$ and as mentioned, this idea is called subordination. In Eq. (7) $t$ is the laboratory time and $N$ is sometimes referred to as operational time. The idea is simple, $N$ which is actually the random number of steps in the process, is distributed according to Lévy statistics, as expected from the generalised central limit theorem. We then transform the Gaussian process in the operational time $N$ to the laboratory framework with what we call a Lévy transformation, see Eq. (7). This method, can be extended to include cases with different boundary conditions, different spatial dependent force fields, stochastic trajectories etc. and hence the mathematical approach we presented is versatile and far more general than what we considered here. Finally, in this manuscript we considered the typical fluctuations of the process, rare events which control the mean square displacement were recently considered in [34, 35].

Summary. The fractional advection-diffusion-asymmetry equation (2) is controlled by three transport coefficients, $D$, $V$ and $S$, given in Eq. (11). This framework is valuable in many CTRW systems, ranging from the field of contaminant spreading and geophysics to transport in quenched random environments, for example, the quenched trap model [40, 45]. What is remarkable is that the long tailed PDF of trapping times, which for $0 < \beta < 1$ implies a fractional time derivative, is transplanted into a spatial space derivative when $1 < \beta < 2$. And long tailed PDFs of jump sizes, like in Lévy flights, are not a basic requirement for fractional space operators in transport equations, rather these are related to Lévy statistics applied to the number of jumps in the process. In this sense we have provided a new physical and widely applicable interpretation of the fractional space derivative, within the context of fractional diffusion.

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Appendix A: Mathematical background

Here we briefly discuss fractional derivatives, the non-symmetric stable probability density function, and convolution used in the main text.
1. Weyl fractional derivatives

There are many excellent texts describing the long history and analytical properties of fractional derivatives, for example see Refs. \cite{46–48}. Here we focus on Weyl fractional derivatives \cite{46, 49}. The Weyl fractional derivative operator is defined through

\[
\frac{d^\beta}{d(-x)^\beta}g(x) = \frac{(-1)^n}{\Gamma(n - \beta)} \frac{d^n}{dx^n} \int_{-\infty}^{\infty} (y - x)^{n - \beta - 1} g(y) dy,
\]  

where \( n \) is the smallest integer larger than \( \beta \). While if the bottom limit of the integral is set to minus infinity, we have another related expression

\[
\frac{d^\beta}{dx^\beta}g(x) = \frac{1}{\Gamma(n - \beta)} \frac{d^n}{dx^n} \int_{-\infty}^{x} (x - y)^{n - \beta - 1} g(y) dy.
\]  

Note that for fractional time derivatives, for example Riemann-Liouville derivatives, the integration is commonly from zero to time \( t \). In Fourier space, the Weyl fractional derivatives obey the following theorem \cite{46, 49}

\[
\begin{cases}
\mathcal{F}\left[\frac{d^\beta}{d(-x)^\beta}g(x)\right] = (-ik)^\beta \hat{g}(k), \\
\mathcal{F}\left[\frac{d^\beta}{dx^\beta}g(x)\right] = (ik)^\beta \hat{g}(k),
\end{cases}
\]

where we denote \( \hat{g}(k) \), \( \tilde{g}(k) = \int_{-\infty}^{\infty} \exp(-ikx)g(x)dx \), as the Fourier transform of \( g(x) \).

2. Non-symmetric Lévy distribution

As mentioned in the main text, in the long time limit, the number of jumps \( N \) in the renewal process, shifted by its mean \( \langle N \rangle \), follows the Lévy stable distribution, which is defined by

\[
L_\beta(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \exp(ikx) \exp(-(ik)^\beta) dk.
\]  

If \( 0 < \beta < 1 \), the above equation reduces to the one-sided Lévy distribution, namely \( L_\beta(x) = 0 \) for \( x < 0 \). In the literature, some authors prefer to use characteristic function to define the Lévy distribution, for example, see Refs. \cite{50–53}, while we use the Fourier transform (the difference is simply a sign in front of \( k \)). Eq. (A4) is plotted in Fig. A1 using Mathematica command, i.e., PDF[\texttt{StableDistribution}[1, \beta, -1, 0, \texttt{Abs}[\texttt{Cos}[\texttt{Pi}+\beta/2]^1/\beta], x]; see also the inset in linear-linear scale. Rewriting Eq. (A4), we have

\[
L_\beta(x) = \frac{1}{\pi} \int_{0}^{\infty} \exp\left(\frac{\pi^\beta}{2} \cos\left(\frac{\pi^\beta}{2} k\right) \cos\left(kx - k^\beta \sin\left(\frac{\pi^\beta}{2}\right)\right)\right) dk,
\]  

which can be used to plot the PDF of the Lévy distribution. We are interested in the decay of the far left and right tails, which are briefly presented below to emphasize the difference between them.

In the large deviation regime, namely \( x \) is large, from Eq. (A4) the saddle-point method yields

\[
L_\beta(x) \sim \frac{\exp\left(-\frac{(\beta - 1)(x/\pi^\beta)^{\pi^\beta}}{2 \pi^\beta - (\beta - 1)x^{\pi^\beta}}\right)}{\sqrt{2 \pi^\beta - (\beta - 1)x^{\pi^\beta}}},
\]  

which was found originally in Ref. \cite{52} using other methods. This indicates that the right tail of the distribution decays as \( \exp(- (\beta - 1)(x/\pi^\beta)^{\pi^\beta}) \), approaching zero rapidly.

Splitting the integral Eq. (A4) into two parts at \( k = 0 \) and changing variables \( k' = k \) for the negative \( k \), we have

\[
L_\beta(x) = \Re e \left[ \frac{1}{\pi} \int_{0}^{\infty} \exp(-ikx) \exp((ik)^\beta) dk \right],
\]  

(A7)
where \( \Re [g(x)] \) means the real part of \( g(x) \). Expanding the integrand \( \exp[(ik)^{\beta}] \) in the right hand side of Eq. (A7) as a Taylor series

\[
\exp[(ik)^{\beta}] = \sum_{n=0}^{\infty} \frac{(-ik)^{\beta n}}{n!} = \sum_{n=0}^{\infty} \frac{(\frac{n+\frac{1}{2} - \frac{1}{\beta}}{n!})}{n!},
\]

and substituting Eq. (A8) into Eq. (A7), we get the asymptotic behavior of the left tail

\[
L_\beta(x) \sim \sum_{n=0}^{\infty} \Re \left[ \exp \left( \frac{(n\beta + 1)n\pi}{2} \right) \Gamma(n\beta + 1) \frac{1}{n!(-x)^{n\beta+1}} \right] = \sum_{n=1}^{\infty} \frac{-\sin(n\beta\pi)\Gamma(n\beta + 1)}{\pi n!(-x)^{n\beta+1}} \sim \frac{1}{\Gamma(-\beta)(-x)^{\beta+1}}
\]

with \( x \to -\infty \). The well known leading term, i.e., the last line of Eq. (A9), is plotted in Fig. A1 using the symbols ‘+’. Note that here we used method of stationary phase when calculating Eq. (A9) from the integral Eq. (A7). As shown in Fig. A1, we can see that two tails of the asymmetry Lévy distribution under study show different behaviors. The left one decays as a power law, i.e., \( L_\beta(x) \sim (-x)^{-\beta-1} \) with \( x \to -\infty \), tending to zero slowly if compared with the right one. This indicates that the variance of \( x \) is infinity but the mean is finite. The exact expression of Lévy distribution, exists in terms of Fox function, which can also be expressed in the form of Mellin-Barnes type of integral. See Ref. [51] for a review. There are a number of cases of analytically expressible stable distribution, for example, when \( \beta = 3/2 \), the Lévy distribution is related to Whittaker function [54].

3. Convolution of Lévy and Gaussian distribution

Now let us consider the solution of the fractional advection-diffusion equation with an initial condition on the origin. As mentioned in the text the solution \( P(x,t) \) in one dimension is the convolution with respect to Lévy and Gaussian distributions

\[
P(x,t) = \int_{-\infty}^{\infty} \frac{1}{(St)^{1/\beta}} L_\beta \left( \frac{y}{(St)^{1/\beta}} \right) \exp \left( \frac{-((x-y-Vt)^2)}{4Dt} \right) \frac{1}{\sqrt{4\pi dt}} dy.
\]

Such a convolution is sometimes called the Voigt profile, see related discussions in Ref. [55]. The scaling behavior of \( \xi = (x-Vt)/(St)^{1/\beta} \) gives

\[
P(\xi,t) = \int_{-\infty}^{\infty} L_\beta (z) \exp \left( \frac{-((\xi-z)^2)}{4Dt(St)^{-2/\beta}} \right) \frac{1}{\sqrt{4\pi dt(St)^{-1/\beta}}} dz.
\]

Below we investigate two limits:

- When \( Z = 4Dt(St)^{-2/\beta} \to 0 \), i.e., \( 4Dt \ll (St)^{2/\beta} \), the term marked by under-brace in Eq. (A11) reduces to a delta function using \( \lim_{x \to 0^+} \exp(-x^2/(4t))/\sqrt{4\pi x} = \delta(x) \). Thus in the very long time limit the Lévy distribution describes the dynamics. This means that we can use the asymptotic behavior of Lévy distribution to study the properties of far tails of the distribution of the position.

- If \( Z \to \infty \) which is the short time limit, we have that the width of the Lévy distribution is narrow [see Eq. (A10)] if compared with the width of Gaussian distribution. In this case, the Lévy distribution approaches to a “delta function”. As expected, we get the packet of spreading particles, following Gaussian distribution.

Notice that for a finite constant \( Z \) we use the integral Eq. (A10) to show the solution of fractional advection-diffusion equation (2) in the main text.

To demonstrate these properties, in Fig. A2 we plot the solution \( P(x_b,t) \) at site \( x_b = Vt \) versus \( t \), namely we focus on the probability of reaching the mean position. In other words, here \( x_b \) is changing with time. Based on our setting in Fig. A2, we choose \( D = 41.7, V = 3.3 \), and \( S = 2.27 \) which are transport constants of the fractional equation. If
$t = 10$, we have $Z = 26$. Thus we have $P(x = V, t) \sim 1/\sqrt{4\pi Dt}$ for small $t$ since the packet of particles follows nearly Gaussian distribution. While, with the increase of the time $t$, as mentioned the Gaussian distribution fails, deviating from the solution of fractional advection-diffusion-asymmetry equation. Then at large times the solution using the Lévy stable law is valid, i.e., $P(x_b, t) \sim L_\beta(0)/(St)^{1/\beta}$. This transition is presented in Fig. A2.

![Lévy distribution plot](image)

**FIG. A1:** Plot of the Lévy distribution with $\beta = 3/2$ and $\beta = 1.1$, showing asymmetric far tails. The Lévy distribution is tabulated with Mathematica (see the solid lines) and the asymptotic behavior of the left tail, plotted by the symbols ‘+’, is obtained from Eq. (A9) [the last line].

**Appendix B: Fractional advection-diffusion-asymmetry equation in two dimensions**

Now we study advection-diffusion equation in two dimensions and present a generalization of Eq. (2) in the main text. We further use a CTRW formalism in two dimensions to explain the meaning of the equation.

Motivated by previous studies of the CTRW, we consider here the probability density function capturing a fat tail

$$
\psi(\tau) = \begin{cases} 
0, & \tau < \tau_0; \\
\beta (\tau_0) \beta \tau^{\beta-1}, & \tau \geq \tau_0
\end{cases}
$$

with $1 < \beta < 2$. Clearly, the waiting time $\tau$ has a finite mean $\langle \tau \rangle$ but a infinite variance. In our simulations, we choose $\tau_0 = 0.1$ and $\beta = 3/2$ and hence $\langle \tau \rangle = 0.3$. Thus, if the observation time $t = 1000$, the average number of renewals is $\langle N \rangle \sim t/\langle \tau \rangle \approx 3333$.

The joint PDF of jump length follows

$$
f(x, y) = \frac{1}{\sqrt{2\pi}^2} \exp \left( -\frac{(x - a_x)^2}{2(\sigma_x)^2} \right) \times \frac{1}{\sqrt{2\pi}^2} \exp \left( -\frac{y^2}{2(\sigma_y)^2} \right),
$$

where $a_x, \sigma_x, \sigma_y \neq 0$ are constants. This indicates that the drift is only in $x$ direction. In double Fourier spaces, $x \to k_x$ and $y \to k_y$, we get

$$
\tilde{f}(k_x, k_y) = \exp \left( -ik_x a_x - \frac{1}{2}(\sigma_x)^2(k_x)^2 - \frac{1}{2}(\sigma_y)^2(k_y)^2 \right),
$$

(B3)
which gives the PDF of finding the particle on site \((X,Y)\) after exactly \(N\) steps by taking the inverse Fourier transform of \(\tilde{f}^N(k_x,k_y)\). Restarting from Eq. (7) in the main text, in the long time limit, \(\mathcal{P}(X,Y,t)_{\text{CTR}}\) becomes

\[
\mathcal{P}(X,Y,t)_{\text{CTR}} \sim \int_{-\infty}^{\infty} L_\beta(z) \frac{\exp\left(-\frac{(X - \frac{a_x t}{\tau} - a_x z t^{1/\beta})^2}{2(\sigma_x)^2 \frac{t}{\tau}}\right)}{\sqrt{2\pi(\sigma_x)^2 \frac{t}{\tau}}} \frac{\exp\left(-\frac{Y^2}{2(\sigma_y)^2 \frac{t}{\tau}}\right)}{\sqrt{2\pi(\sigma_y)^2 \frac{t}{\tau}}} dz.
\]  

(B4)

Taking double Fourier transforms with respect to \(X\) and \(Y\), respectively, we get a useful expression

\[
\tilde{P}(k_x,k_y,t)_{\text{CTR}} \sim \exp\left(-\frac{(k_y)^2(\sigma_y)^2 t}{2(\tau)} - \frac{(k_x)^2(\sigma_x)^2 t}{2(\tau)} - ia_x k_x \frac{t}{\tau}\right) + \left(-ia_x k_x \left(\frac{t}{\tau}\right)^{1/\beta}\right)^\beta.
\]  

(B5)

Note that if \(k_x = k_y = 0\), we can check that \(\mathcal{P}(X,Y,t)_{\text{CTR}}\) is normalized for any time \(t\). Similar to the calculation in the main text, we take the time derivative of this solution, perform the inverse Fourier transform and then find using Eqs. (A3) and (B5)

\[
\frac{\partial}{\partial t}\mathcal{P}(x,y,t) = \frac{(\sigma_y)^2}{2(\tau)} \frac{\partial^2}{\partial y^2}\mathcal{P}(x,y,t) + \frac{(\sigma_x)^2}{2(\tau)} \frac{\partial^2}{\partial x^2}\mathcal{P}(x,y,t) - \frac{a_x}{(\tau)} \frac{\partial}{\partial x} \mathcal{P}(x,y,t) + \frac{a_x^3}{t^2} \frac{\partial^3}{\partial (-x)^3}\mathcal{P}(x,y,t).
\]  

(B6)

This is the fractional advection-diffusion equation in two dimensions, where the symmetry breaking takes place only in the \(x\) direction, where the bias is pointing to. The solution of the above equation is plotted in Fig. 1 in the main text showing the packet of spreading particles. In this figure, we denote \(a\) as \(a_x\) to simplify our expression. Note that Eq. (B6) is not just valid for Gaussian displacement given in Eq. (B4) but the displacement should have a finite mean and a finite variance. In particular, when \(a_x = 0\), the above equation reduces to the classical diffusion equation. Clearly, the marginal density \(\mathcal{P}(x,t)\) is the corresponding one dimensional solution Eq. (9) in the main text.

Appendix C: Breakthrough curves

Here the aim is to use the fractional advection-diffusion-asymmetry equation found in the main text to predict breakthrough curves. For that, the first step is to obtain \(\mathcal{P}(x,t)\) from Eq. (2) with time-dependent bias and then use
it to compare with simulations of CTRW breakthrough curves.

1. Theory of propagator with time-dependent but piece wise constant bias

Motivated by [33], we consider time-dependent bias determined by four stages. We suppose that the rapid injection of particles is done immediately after starting observing the process. In other words, the initial condition of the particle is \( P(x, t = 0) = \delta(x) \). As mentioned in the main text we simulate the spreading of the particles consisting of four states: (i) after the injection of the particles, they are moving with a constant bias which is determined by \( a_1 = a \), (ii) in the time interval \( t_1 < t < t_2 \), we increase the bias sharply to \( a_2 = 4a\gamma/(\gamma + 1) \) with \( \gamma \geq 1/3 \), (iii) decrease the bias abruptly to \( a_3 = a/(2/3 + \gamma) \), and (iv) then finally starting at \( t_3 \) return to the state (i) with \( a_4 = a \). Here \( \gamma \) is a constant that controls the strength of bias or the average of “velocity”. In particular, when \( \gamma = 1/3 \), and hence all the states mentioned above are the same. In Fourier space, the initial condition satisfies \( \mathcal{P}(k, 0) = 1 \). In view of the special expression of \( \mathcal{P}(k, t) \), \( \mathcal{P}(k, t) \) for different states can be cast as

\[
\mathcal{P}(k, t) = \begin{cases} 
\exp(-c_{11}k^2 - ic_{12}k + c_{13}(-ik)^\beta), & 0 < t \leq t_1; \\
\exp(-c_{21}k^2 - ic_{22}k + c_{23}(-ik)^\beta), & t_1 < t \leq t_2; \\
\exp(-c_{31}k^2 - ic_{32}k + c_{33}(-ik)^\beta), & t_2 < t \leq t_3; \\
\exp(-c_{41}k^2 - ic_{42}k + c_{43}(-ik)^\beta), & t_3 < t 
\end{cases}
\]  

(C1)

with

\[
c_{m1} = \begin{cases} 
\frac{t_{m1}^2}{2\sigma^2}, & m = 1; \\
\frac{t_{m2}^2}{2\sigma^2}, & m = 2; \\
\frac{t_{m3}^2}{2\sigma^2}, & m = 3; \\
\frac{t_{m4}^2}{2\sigma^2}, & m = 4,
\end{cases}
\]

(C2)

\[
c_{m2} = \begin{cases} 
\frac{a_1 t_1}{\langle \tau \rangle}, & m = 1; \\
[a_1 t_1 + a_2(t - t_1)]\frac{1}{\langle \tau \rangle}, & m = 2; \\
[a_1 t_1 + a_2(t - t_1) + a_3(t - t_2)]\frac{1}{\langle \tau \rangle}, & m = 3; \\
[a_1 t_1 + a_2(t - t_1) + a_3(t - t_2) + a_4(t - t_3)]\frac{1}{\langle \tau \rangle}, & m = 4,
\end{cases}
\]

(C3)

\[
c_{m3} = \begin{cases} 
\frac{a_1^2 t_1}{\langle \tau \rangle}, & m = 1; \\
[a_1^2 t_1 + a_2^2(t - t_1)]\frac{1}{\langle \tau \rangle}, & m = 2; \\
[a_1^2 t_1 + a_2^2(t - t_1) + a_3^2(t - t_2)]\frac{1}{\langle \tau \rangle}, & m = 3; \\
[a_1^2 t_1 + a_2^2(t - t_1) + a_3^2(t - t_2) + a_4^2(t - t_3)]\frac{1}{\langle \tau \rangle}, & m = 4.
\end{cases}
\]

(C4)

Here recall that \( \tilde{T} = (\tau)^{1+\beta}/[(\tau_0)^{\beta}(\Gamma(1 - \beta))] \) which is fixed, since in our simulations of the CTRW we only change the bias. The main idea of the analytical calculation is that the final position of each stage will be treated as an “initial condition” for the next stage. In particular, if \( a_1 = a_2 = a_3 = a_4 \), Eq. (C1) reduces to a constant bias case calculated in the main text. For the four-stage process, from Eq. (C1) the solution is of the form

\[
P(x, t) = \int_{-\infty}^{\infty} \frac{1}{\sqrt{4\pi c_{m1}}} \exp\left[-\frac{(x - y - c_{m2})^2}{4c_{m1}}\right] \frac{1}{(c_{m3})^{1/\beta}} L_\beta\left(\frac{y}{(c_{m3})^{1/\beta}}\right) dy
\]

(C5)

with \( m = 1, 2, 3, 4 \) being the number of the state. This is plotted in Fig. A3 for different times \( t \). Note that when \( 0 < t < t_1 \), the solution Eq. (C5) reduces to Eq. (9) in the main text.

2. Breakthrough curves

As mentioned in the main text breakthrough curves are measured considering a source that passes through the absorbent fixed bed sample, which is a method to analyze the adsorption properties of tracers in porous materials.
An example of experiments is the transport through layers of different media, see Refs. \[33, 39\]. With Eq. (C5) we constructed the analytical solution presented in Fig. 2 of the main text. In Fig. 2 we use $\gamma = 10$ for curve A, and further choose $a = 1$, so there we have in the four stages $a = \{1, 3, 6, 0.09, 1\}$. Curve B in Fig. 2 corresponds to the case $\gamma = 1/3$, where we get $a = \{1, 1, 1, 1\}$. For the time interval of each state we use $t_1 = t_2 = t_3 = t_2 = 100$ and $t - t_3 = 700$. Note that the detection wall is set on $x_3 = 1800$. In addition to Fig. 2 in the main text here we present results for the density with $t = 300$, which are the same as Fig. 2 in the main text. Here the time interval is 10 for the first three stage, given that $\langle r \rangle = 0.3$ we have roughly 33 steps in each of the first three time interval. The figure illustrates that even for these relatively short time intervals the approximation works nicely.

FIG. A3: Simulations of propagators for different states with theoretical predictions Eq. (C5). Waiting times of particles are drawn from Eq. (B1) and displacements generated according to $f(x) = \exp((-x - a)^2/2\sigma^2)/\sqrt{2\pi\sigma^2}$, where $a$ is time dependent obtained for $\gamma = 1/3$ (the first row), 10 (the second). The red solid lines are the theory and the blue dashed lines are the simulation results generated from $2 \times 10^6$ trajectories. The time interval of the first three states is 100 and the total time of the last state is 700. Note that here it is not easy to see the asymmetry of the packet of spreading particles in a linear-linear plot for large $t$, at least this figure can be treated as an optical illusion. However, semi-log scales are presented to show the symmetric properties of the packet and the heavy tail; see the left and the right insets. The parameters are $\beta = 3/2$, $\tau_0 = 0.1$, and $\sigma = 5$, which are the same as Fig. 2 in the main text.

Similarly, one can deal with time-dependent bias in two dimensions using Eq. (B5) like Eq. (C1). In Fig. A5, the time-dependent bias in two dimensions is investigated and the asymmetry packet with respect to $x$ is clearly observed. Here we have the drift in the $x$ direction and no bias in the $y$ direction, i.e.,

$$a_x = \begin{cases} 
    a, & 0 < t \leq t_1; \\
    \frac{4a}{\gamma + 1}, & t_1 < t \leq t_2; \\
    \frac{2a}{\gamma + 2}, & t_2 < t \leq t_3; \\
    a, & t_3 < t. 
\end{cases} \quad (C6)$$

It can be seen that Figs. A3 and A5 are complementary and yield a better understanding of breakthrough curves and the asymmetric packets of spreading particles.

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FIG. A4: A plot of breakthrough curves for a weak bias. Here we choose $\beta = 3/2$, $\tau_0 = 0.1$, $t_1 = 10$, $t_2 = 20$, $t_3 = 30$, $t = 1000$, $\gamma = 1/3$ (solid lines), and $\gamma = 10$ (dashed lines). The detection wall for breakthrough curves is $x_b = 10^3$; see the dash-dotted line. For both cases, after reaching a peak, we see a slow decay of the far right tail to zero, if compared to the left one (see the inset). Clearly the theoretical predictions Eq. (C1) (the solid and the dashed lines) are consistent with simulations results plotted by the symbols.

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FIG. A5: Theoretical prediction $P(x, y, t)/P_{\text{max}}$ showing spatial contour maps under the time-dependent bias in two dimensions, where $P_{\text{max}} = \max_{x,y,t} \in (-\infty, \infty) \times (-\infty, \infty) (P(x, y, t))$. The colorbar represents the relative concentration, $P(x, y, t)/P_{\text{max}}$. Here four stages are $0 < t < 100$, $100 < t < 200$, $200 < t < 300$, and $300 < t < 1000$. The joint PDF of displacements follows Eq. (B2) with a changing $\alpha_x$, namely we have $\alpha_x = \{1, 3.6, 0.9, 1\}$ ($\alpha_x = \{1, 1, 1, 1\}$) when $\gamma = 10$ ($\gamma = 1/3$); see Eq. (C6). The theoretical results are obtained from Eqs. (C1) and (B5), namely $P(x, y, t)$ factorizes into a product of Eq. (C5) and $\exp(-Y^2/(2\sigma_y^2(t/(\tau)))$/$\sqrt{2\pi\sigma_y^2(t/(\tau))}$. In our setting, we choose $\sigma_x = \sigma_y = 5$, and the other parameters are the same as Fig. A3.

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