Scattering of a scalar relativistic particle by the hyperbolic tangent potential

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Abstract

We solve the Klein-Gordon equation in the presence of the hyperbolic tangent potential. The scattering solutions are derived in terms of hypergeometric functions. The reflection $R$ and transmission $T$ coefficients are calculated in terms of Gamma function and, superradiance is discussed, when the reflection coefficient $R$ is greater than one.

1 Introduction

The study of the solution of the Klein-Gordon equation with different potentials has been extensively studied in recent years, for bound states and scattering solutions [1, 2, 3, 4, 5, 6, 7]. The scattering solutions to the Dirac equation also have been studied for several potentials, a particular case is the Hulthén potential [8]. The superradiance phenomenon, when the reflection coefficient $R$ is greater than one, has been widely discussed. Manogue [9] discussed the superradiance on a potential barrier for Dirac and Klein-Gordon equations. Sauter [10] and Cheng [11] have studied the same phenomenon for the hyperbolic tangent potential with the Dirac equation. Superradiance for the Klein-Gordon equation with this particular potential has been studied for Cheng too [11]. The hyperbolic tangent potential even wake interest, recently has discussed the bound states of scalar particle in presence of the truncated hyperbolic tangent potential [12] and the hyperbolic tangent potential [13].
In this paper we have calculated the scattering solutions of the Klein-Gordon equation in terms of hypergeometric functions in presence of the hyperbolic tangent potential. The reflection $R$ and transmission $T$ coefficients are calculated in terms of Gamma function. The behaviour of the reflection $R$ and transmission $T$ coefficients is studied for five different regions of energy. We have observed for some region that $R > 1$ and $T < 0$, so the phenomenon of superradiance is observed in this potential [11, 14].

This paper is organized of the following way. Section 2 shows the one-dimensional Klein-Gordon equation. In section 3 the hyperbolic tangent potential is shown. Section 4 shows the scattering solutions and the behaviour of the reflection $R$ and transmission $T$ coefficients. Finally, in section 5 conclusions are discussed.

2 The Klein-Gordon equation

The one-dimensional Klein-Gordon equation to solve is, in natural units $\hbar = c = 1$ [15]

$$\frac{d^2 \phi(x)}{dx^2} + \{[E - V(x)]^2 - m^2\} \phi(x) = 0, \quad (1)$$

where $E$ is the energy, $V(x)$ is the potential and, $m$ is the mass of the particle.

3 The hyperbolic tangent potential

The hyperbolic tangent potential is defined as

$$V(x) = a \tanh(b x), \quad (2)$$

where $a$ represents the height of the potential and $b$ gives the smoothness of the curve. The form of the hyperbolic tangent potential is showed in the Fig. 1. From Fig. 1 we can note that the hyperbolic tangent potential reduces to a step potential for $b \to \infty$. 

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Figure 1: Hyperbolic tangent potential for $a = 5$ with $b = 2$ (solid line) and $b = 50$ (dotted line).

4 Scattering States

In order to consider the scattering solutions, we solve the differential equation

$$\frac{d^2 \phi(x)}{dx^2} + \{[E - a \tanh(bx)]^2 - m^2\} \phi(x) = 0. \quad (3)$$

On making the substitution $y = -e^{2bx}$, Eq. (3) becomes

$$4b^2 y \frac{d}{dy} \left[ y \frac{d\phi(y)}{dy} \right] + \left[ (E + a \frac{1+y}{1-y})^2 - m^2 \right] \phi(y) = 0. \quad (4)$$

Putting $\phi(y) = y^\alpha(1-y)^\beta f(y)$, Eq. (4) reduces to the hypergeometric differential equation

$$y(1-y)f'' + [(1+2\alpha) - (2\alpha + 2\beta + 1)y]f' - (\alpha + \beta - \gamma)(\alpha + \beta + \gamma)f = 0, \quad (5)$$

where the primes denote derivate with respect to $y$ and the parameters $\alpha$, $\beta$, and $\gamma$ are
\[ \alpha = i\nu \text{ with } \nu = \sqrt{(E+a)^2 - m^2}, \]  
\[ \beta = \lambda \text{ with } \lambda = \frac{b + \sqrt{b^2 - 4a^2}}{2b}, \]  
\[ \gamma = i\mu \text{ with } \mu = \sqrt{(E-a)^2 - m^2}. \]  

(6) \[ \beta = \lambda \]  
(7) \[ \gamma = i\mu \]  
(8) 

Eq. [5] has the general solution in terms of Gauss hypergeometric functions \( _2F_1(\mu, \nu, \lambda; y) \) \[ \phi(y) = C_1 y^\alpha (1 - y)^\beta _2F_1(\alpha + \beta - \gamma, \alpha + \beta + \gamma, 1 + 2\alpha; y) + C_2 y^{-\alpha} (1 - y)^\beta _2F_1(-\alpha + \beta - \gamma, -\alpha + \beta + \gamma, 1 - 2\alpha; y). \]  

(9) 

In terms of variable \( x \) Eq. [9] becomes \[ \phi(x) = c_1 (-e^{2bx})^{i\nu} (1 + e^{2bx})^\lambda _2F_1(i\nu + \lambda - i\mu, i\nu + \lambda + i\mu, 1 + 2i\nu; -e^{2bx}) + c_2 (-e^{2bx})^{-i\nu} (1 + e^{2bx})^\lambda _2F_1(-i\nu + \lambda + i\mu, -i\nu + \lambda - i\mu, 1 - 2i\nu; -e^{2bx}). \]  

(10) 

From Eq. [10] the incident and reflected waves are \[ \phi_{\text{inc}}(y) = d_1 (1 + e^{2bx})^{\lambda} e^{2ib\nu x} _2F_1(i\nu + \lambda - i\mu, i\nu + \lambda + i\mu, 1 + 2i\nu; -e^{2bx}). \]  

(11) \[ \phi_{\text{ref}}(y) = d_2 (1 + e^{2bx})^{\lambda} e^{-2ib\nu x} _2F_1(-i\nu + \lambda + i\mu, -i\nu + \lambda - i\mu, 1 - 2i\nu; -e^{2bx}). \]  

(12) 

Using the relation \[ _2F_1(a, b, c; z) = \frac{\Gamma(c)\Gamma(b-a)}{\Gamma(b)\Gamma(c-a)}(-z)^{(-a)} _2F_1(a, 1-c+a, 1-b+a; z^{-1}) + \frac{\Gamma(c)\Gamma(a-b)}{\Gamma(a)\Gamma(c-b)}(-z)^{(-b)} _2F_1(b, 1-c+b, 1-a+b; z^{-1}). \]  

(13)
The transmitted wave becomes

$$\phi_{\text{trans}}(x) = d_3 e^{2b\lambda x} \left(1 + e^{2b\lambda x}\right)^\lambda e^{2ib\mu x} {}_2F_1 \left(i\nu + \lambda - i\mu, -i\nu + \lambda - i\mu, 1 - 2i\mu; -e^{-2b\lambda x}\right).$$

(14)

As the incident wave is equal to the sum of the transmitted wave and the reflected wave

$$\phi_{\text{inc}}(x) = A \phi_{\text{trans}}(x) + B \phi_{\text{ref}}(x).$$

(15)

We used again the relation (13) and the equation for $\phi_{\text{trans}}(x)$ to find

$$\phi_{\text{inc}}(x) = A \left(1 + e^{2b\lambda x}\right)^\lambda e^{2ib\nu x} {}_2F_1 \left(i\nu + \lambda - i\mu, i\nu + \lambda + i\mu, 1 + 2i\nu; -e^{2b\lambda x}\right).$$

(16)

$$\phi_{\text{ref}}(x) = B \left(1 + e^{2b\lambda x}\right)^\lambda e^{-2ib\nu x} {}_2F_1 \left(-i\nu + \lambda + i\mu, -i\nu + \lambda - i\mu, 1 - 2i\nu; -e^{2b\lambda x}\right).$$

(17)

Where the coefficients $A$ and $B$ in Eqs. (16) and (17) can be expressed in terms of the Gamma function as

$$A = \frac{\Gamma(1 - 2i\mu)\Gamma(-2i\nu)}{\Gamma(-i\nu + \lambda - i\mu)\Gamma(1 - i\nu - \lambda - i\mu)}.$$  

(18)

$$B = \frac{\Gamma(1 - 2i\mu)\Gamma(2i\nu)}{\Gamma(i\nu - \lambda - i\mu)\Gamma(1 + i\nu - \lambda - i\mu)}.$$  

(19)

When $x \rightarrow \pm \infty$ the $V \rightarrow \pm a$ and the asymptotic behaviour of Eqs. (14), (16) and (17) are plane waves with the relation of dispersion $\nu$ and $\mu$,

$$\phi_{\text{inc}}(x) = Ae^{2ib\nu x},$$

(20)

$$\phi_{\text{ref}}(x) = Be^{-2ib\nu x},$$

(21)

$$\phi_{\text{trans}}(x) = e^{2ib\mu x}.$$  

(22)

In order to find $R$ and $T$, we used the definition of the electrical current density for the one-dimensional Klein-Gordon equation (1)
\[ \vec{j} = \frac{i}{2} \left( \phi^* \nabla \phi - \phi \nabla \phi^* \right). \]  

(23)

The current as \( x \to -\infty \) can be decomposed as \( j_L = j_{inc} - j_{refl} \) where \( j_{inc} \) is the incident current and \( j_{refl} \) is the reflected one. Analogously we have that, on the right side, as \( x \to \infty \) the current is \( j_R = j_{trans} \), where \( j_{trans} \) is the transmitted current [1].

The reflection coefficient \( R \), and the transmission coefficient \( T \), in terms of the incident \( j_{inc} \), reflected \( j_{refl} \), and transmission \( j_{trans} \) currents are

\[ R = \frac{j_{refl}}{j_{inc}} = \frac{|B|^2}{|A|^2}. \]  

(24)

\[ T = \frac{j_{trans}}{j_{inc}} = \frac{\mu}{\nu} \frac{1}{|A|^2}. \]  

(25)

The reflection coefficient \( R \), and the transmission coefficient \( T \) satisfy the unitary relation \( T + R = 1 \) and are expresses in terms of the coefficients \( A \) and \( B \), therefore are expressed in terms of the Gamma function and they are determined with the software Maple 18.

The dispersion relation \( \nu \) and \( \mu \) must be positive because it correspond to an incident particle moved from left to right and, their sign depends on the group velocity, define by [17]

\[ \frac{dE}{d\nu'} = \frac{\nu'}{E + a} \geq 0. \]  

(26)

\[ \frac{dE}{d\mu'} = \frac{\mu'}{E - a} \geq 0. \]  

(27)

For the hyperbolic tangent potential we have five different regions, this regions are observed in table [1]

It is important to note that in the regions \( a + m > E > a - m \) and \( -a + m > E > -a - m \) the dispersion relations \( \mu \) and \( \nu \) are imaginary pure and the transmitted wave if attenuates, so \( R = 1 \). In the region \( a - m > E > -a + m \), \( \mu' < 0 \) and, \( \nu' > 0 \) we have that \( R > 1 \), so superradiance occurs.

Figs. 2(a) and 2(b) show the reflection \( R \) and transmission \( T \) coefficients for \( E > m, a = 5 \) and \( b = 2 \). Figs. 3(a) and 3(b) show the reflection \( R \) and
transmission $T$ coefficients $R$ for $E > m$, $a = 5$ and $b = 50$. We observed in the figures that in the region $a - m > E > m$ the reflection coefficient $R$ is bigger than one whereas the coefficient of transmission $T$ becomes negative, so we observed superradiance [11, 14] and that the coefficients $R$ and $T$ satisfy the unitary condition $T + R = 1$. The hyperbolic tangent potential is useful to study the superradiance phenomenon.

5 Conclusion

In this paper we have discussed the scattering solutions of the Klein-Gordon equation in presence of the hyperbolic tangent potential. The solutions are determined in terms of hypergeometric function. The reflection $R$ and transmission $T$ coefficients are determined in terms of the Gamma function. We have shown that for the region where $a - m > E > m$, the phenomenon of superradiance occurs.

References

[1] C. Rojas and V. M. Villalba. Scattering of a Klein-Gordon particle by a Woods-Saxon potential. *Phys. Rev. A*, 71:052101, 2005.

[2] C. Rojas and V. M. Villalba. The Klein-Gordon equation with the Woods-Saxon potential well. *Rev. Mex. Fis*, 52:127, 2006.
Figure 2: The reflection $R$ and transmission $T$ coefficients varying energy $E$ for the relativistic hyperbolic tangent potential for $a = 5$, $b = 2$ and, $m = 1$.

[3] V. M. Villalba and C. Rojas. Bound states of the Klein-Gordon equation in the presence of short range potentials. *Int. J. Mod. Phys. A*, 21:313, 2006.

[4] V. M. Villalba and C. Rojas. Scattering of a relativistic scalar particle by a cusp potential. *Phys. Lett. A*, 362:21, 2007.

[5] O. Aydogdu, A. Arda and, R. Sever. Scattering and bound state solutions of the asymmetric Hulthén potential. *Phys. Scr.*, 84:025004, 2011.
Figure 3: The reflection $R$ and transmission $T$ coefficients varying energy $E$ for the relativistic hyperbolic tangent potential for $a = 5$, $b = 50$ and, $m = 1$.

[6] O. Aydogdu, S. Alpdogan and, A. Havare. Relativistic spinless particles in the generalized asymmetric Woods-Saxon potential. *J. Phys. A: Math. Theor.*, 46:015301, 2013.

[7] J.-Y. Guo and X.-Z. Fang. Scattering of a Klein-Gordon particle by a Hulthen potential. *Can. J. Phys.*, 87:1021, 2009.

[8] J.-Y. Guo, Y. Yu and, W. Jin. Transmission resonance for a Dirac particle in a one-dimensional Hulthen potential. *Cent. Eur. J. Phys.*, 9
[9] C. A. Manogue. The Klein paradox and superradiance. *Ann. Phys.*, 181:261, 1988.

[10] F. Sauter. Zum ”Kleinschen paradoxon“. *Z. Phys.*, 73:547, 1931.

[11] T. Cheng, M. R. Ware, Q. Su and, R. Grobe. Pair creation rates for one-dimensional fermionic and bosonic vacua. *Phys. Rev. A*, 80:062105, 2009.

[12] L. A. González-Díaz and V. M. Villalba. Bound states of scalar particles in the presence of a short range potential. *Mod. Phys. Lett. A*, 20:2245, 2005.

[13] W.-J. Tian. Bound states for spin-0 and spin-1/2 particles with vector and scalar hyperbolic tangent and cotangent potentials. [http://www.paper.edu.cn/en_releasepaper/content/36868](http://www.paper.edu.cn/en_releasepaper/content/36868), 2009.

[14] R. E. Wagner, M. R. Ware, Q. Su and, R. Grobe. Bosonic analog of the Klein paradox. *Phys. Rev. A*, 81:024101, 2010.

[15] W. Greiner. *Relativistic Quantum Mechanics. Wave equations*. Springer, 1987.

[16] M. Abramowitz and I. A. Stegun. *Handbook of Mathematical Functions*. Dover, New York, 1965.

[17] A. Calogeracos and N. Dombey. History and physics of the Klein paradox. *Contemp. Phys.*, 40:313, 1999.