Ternary Virasoro - Witt Algebra

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Abstract

A 3-bracket variant of the Virasoro-Witt algebra is constructed through the use of su(1,1) enveloping algebra techniques. The Leibniz rules for 3-brackets acting on other 3-brackets in the algebra are discussed and verified in various situations.

1 Introduction

Recently there has been some progress in constructing a 2 + 1-dimensional local quantum field theory with SO(8) superconformal symmetry [1, 2, 7]. This is a useful stepping-stone to obtain a world-volume Lagrangian description for coincident M2-branes. Crucial to the construction is the use of 3-algebras (a term originally coined by Filippov [6], following up on earlier work of Nambu [11]) which are built around antisymmetrized products of three operators. It now seems possible to incorporate an arbitrary Lie algebra into a 3-algebra through the use of a small number of auxiliary charges.

While it may be sufficient to restrict attention to finite Lie and 3-algebras for model-building purposes, e.g. in M-theory, nevertheless it is intrinsically interesting to also consider infinite dimensional 3-algebras. Previous studies [9] have already considered the infinite 3-algebra that follows from the classical Nambu 3-bracket, on a 3-torus, in the form

\[ [E_a, E_b, E_c] = a \cdot (b \times c) E_{a+b+c} . \]  

This is easily remembered as just a Leibniz rule for 3-brackets acting on other 3-brackets. (However, this does not necessarily mean \([E_a, E_b, AB]\) is the same as \([E_a, E_b, A]B + A[E_a, E_b, B]\) for arbitrary \(A\) and \(B\). While these are the same for the classical Nambu bracket, in general these two expressions differ for an operator algebra.)

A 3-algebraic variant of the su(2) Kac-Moody algebra has also been studied [10]. Moreover, it is straightforward to embed an infinite Onsager 3-algebra in this framework, just as it is to embed the usual infinite Onsager Lie algebra [12] (also see [5]) into that for su(2). Nonetheless, it is not completely clear from these previous examples how infinite conformal 3-algebras should be constructed. We consider this problem here. We construct an infinite 3-algebra variant of the centerless Virasoro (i.e. Witt) algebra. We leave the inclusion of central charges as well as supersymmetric extensions of our results for future work.
2 Three-brackets from Enveloping Algebras

The “BL algebra” presented by Bagger and Lambert \[1\] is actually just a minor modification of Nambu’s original results on 3-brackets for the \(su(2)\) angular momentum algebra, \([L_x, L_y] = iL_z\), etc. Of course, Nambu’s work motivated and inspired both Filippov (12 years later) and Takhtajan (20 years later) \[13\], as well as more recent studies (see \[3\] and references therein).

After introducing the 3-bracket, Nambu noted that the \(su(2)\) Casimir followed immediately from it (see Eqn(49) in \[11\]).

\[
[L_x, L_y, L_z] \equiv L_x [L_y, L_z] + L_y [L_z, L_x] + L_z [L_x, L_y] = i \left( L_x^2 + L_y^2 + L_z^2 \right) = iL^2 .
\] (3)

From this it follows that the BL algebra can be easily found in the enveloping algebra for \(su(2)\). If we rescale the angular momenta by a fourth root of the Casimir,

\[
Q_x = \frac{L_x}{\sqrt[4]{L^2}} , \quad Q_y = \frac{L_y}{\sqrt[4]{L^2}} , \quad Q_z = \frac{L_z}{\sqrt[4]{L^2}} ,
\] (4)

as well as define a fourth charge as that fourth root,

\[
Q_t = \sqrt[4]{L^2} ,
\] (5)

then we obtain \([Q_t, Q_x] = 0\) etc. Hence

\[
[Q_x, Q_y, Q_z] = iQ_t , \quad [Q_t, Q_x, Q_y] = iQ_z , \quad [Q_t, Q_y, Q_z] = iQ_x , \quad [Q_t, Q_z, Q_x] = iQ_y .
\] (6)

So, any matrix rep of the angular momentum algebra yields a matrix rep of this 3-bracket algebra. This algebra can be summarized as

\[
[Q_a, Q_b, Q_c] = i\varepsilon_{abc}^d Q_d ,
\] (7)

where \(\varepsilon_{x y z t} = +1\) with a \([-1, -1, -1, +1]\) Lorentz signature used to raise indices. In this form it is easily verified that the FI conditions are satisfied: \([Q_d, Q_e, [Q_a, Q_b, Q_c]] = [[Q_d, Q_e, Q_a], Q_b, Q_c] + [Q_a, [Q_d, Q_e, Q_b], Q_c] + [Q_a, Q_b [Q_d, Q_e, Q_c]]\) \(= [Q_d, Q_e, AB] = [Q_d, Q_e, B] + A [Q_d, Q_e, B]\) for arbitrary \(A\) and \(B\).

Let us now pursue such enveloping algebra techniques to consider infinite conformal algebras as realized nonlinearly with \(su(1, 1)\) generators. This gives the centerless version of the usual Virasoro algebra,

\[
[L_n, L_m] = (n - m) L_{n+m} .
\] (8)

An explicit realization is \[4\]

\[
L_n = (L_0 + n\beta) \frac{\Gamma (L_0 + \beta)}{\Gamma (L_0 + \beta + n)} (L_1)^n ,
\]

\[
L_{-n} = (L_0 - n\beta) \frac{\Gamma (L_0 + 1 - \beta - n)}{\Gamma (L_0 + 1 - \beta)} (L_{-1})^n ,
\] (9)

where the \(su(1, 1)\) algebra and Casimir are given by

\[
[L_{+1}, L_{-1}] = 2L_0 , \quad [L_{\pm 1}, L_0] = \pm L_{\pm 1} , \quad \beta (1 - \beta) \equiv C = L_{+1}L_{-1} - (L_0 + 1) L_0 .
\] (10)

From this construction, we compute 3-brackets and abstract from them a 3-algebra. An interesting question is: Does the algebra so obtained satisfy the 3-bracket Leibniz rule when acting on other 3-brackets?

First we compute the 3-bracket for charges with positive indices.

\[
[L_k, L_m, L_n] = (m - n) L_k L_{m+n} + (n - k) L_m L_{n+k} + (k - m) L_n L_{k+m} = C \frac{(k - m)(m - n)(k - n)}{L_0 + (k + n + m) \beta} L_{k+m+n} .
\] (11)
Here we have used \[\text{(8)}\] and an identity specific to the realization at hand,

\[
L_k L_{m+n} = (L_0 + k \beta) \left(1 + \frac{k (1 - \beta)}{L_0 + (k + n + m) \beta}\right) L_{k+m+n},
\]

plus cyclic permutations of \(k, m, n\). Note the simplification when \(\beta = 0\) or when \(\beta = 1\). Similarly, we compute

\[
[L_p, L_q, [L_k, L_m, L_n]] = C (k - m) (m - n) (k - n) (p - q) \left(1 + \frac{(k + m + n) (1 - 2 \beta)}{L_0 + (k + m + n + p + q) \beta}\right) L_{k+m+n+p+q}.
\]

The case \(\beta = 1/2\) stands out as particularly tasteful.

\[
[L_p, L_q, [L_k, L_m, L_n]]|_{\beta=1/2} = \frac{1}{4} (k - m) (m - n) (k - n) (p - q) L_{k+m+n+p+q}.
\]

This case allows us to quickly check the FI condition for the \(L_s\). We find that it holds true.

\[
[L_p, L_q, [L_k, L_m, L_n]] - [[L_p, L_q, L_k], L_m, L_n] - [L_k, [L_p, L_q, L_m], L_n] - [L_k, L_m, [L_p, L_q, L_n]]|_{\beta=1/2} = 0.
\]

Before checking the FI for other values of \(\beta\), it is useful to first make some general observations.

The 3-bracket of \(L_s\) introduces a new operator given in the present realization as a product. To check the FIs for the 3-algebra then requires knowing how this new operator is affected under the action of a 3-bracket. Since the 3-bracket is not a derivation, in general, this requires some new input, even when the new operator is realized as a product. Therefore, we require several independent statements for the various 3-brackets. From the \(su(1,1)\) enveloping algebra expressions, we find

\[
[L_k, L_m, L_n] = C (k - m) (m - n) (k - n) M_{k+m+n},
\]

\[
[L_p, L_q, M_k] = (p - q) \left(L_{k+p+q} + (1 - 2 \beta) k M_{k+p+q}\right),
\]

where, in that realization,

\[
M_k = \frac{1}{L_0 + k \beta} L_k = \frac{\Gamma(L_0 + \beta)}{\Gamma(L_0 + \beta + k)} (L_1)^k.
\]

In particular, \(M_0 = 1\). The Casimir-dependent factors are tantamount to operators, of course, and will eventually be absorbed into normalizations, but we leave them explicit for now. To close the algebra, we also need to consider two additional 3-brackets: \([L_p, M_q, M_k]\) and \([M_p, M_q, M_k]\).

So we compute some more in the enveloping algebra, to find

\[
[M_q, M_k] = 0, \quad [L_p, M_q] = -q M_{p+q},
\]

and then,

\[
[L_p, M_q, M_k] = L_p [M_q, M_k] + M_k [L_p, M_q] - M_q [L_p, M_k]
\]

\[
= -q M_k M_{p+q} + k M_q M_{p+k}
\]

\[
= (k - q) M_{k+p+q},
\]

upon using another identity valid in the \(su(1,1)\) realization. Namely\[\text{(13)}\]

\[
M_k M_{p+q} = M_{k+p+q}.
\]

Thus we arrive at the remaining 3-brackets.

\[
[M_p, M_q, M_k] = 0, \quad [L_p, M_q, M_k] = (k - q) M_{k+p+q}.
\]

\[\text{Footnote:}\] An alternative presentation of the Lie algebra for the \(L_s\) and \(M_s\) is given by taking generating elements \(L_0\) and \(M_1\), with the condition \([L_0, M_1] = -M_1\). Then \(M_0 \equiv (M_1)^{1/2}\) and \(L_n \equiv M_n \left(L_0 + n (\beta - 1)\right)\) leads to the algebra \[\text{(9)}, \text{(13)}\]. From this it is clear the \(L_s\) depend on \(\beta\), but the \(M_s\) do not.
The complete 3-algebra found through use of the $su(1,1)$ enveloping algebra is then

$$[L_k, L_m, L_n] = C (k-m)(m-n)(n-k) \ M_{k+m+n} ,$$

$$[L_p, L_q, M_k] = (p-q) (L_{k+p+q} + (1-2\beta) k M_{k+p+q} ) ,$$

$$[L_p, M_q, M_k] = (k-q) M_{k+p+q} ,$$

$$[M_p, M_q, M_k] = 0 .$$

(22)

In addition, the $su(1,1)$ realization gives the interesting result that all higher N-brackets (i.e. totally antisymmetrized products of N operators) are null. This follows from explicit calculation of the five possible forms for 4-brackets, three of which exhibit nontrivial cancellations of terms, to obtain

$$0 = [L_j, L_k, L_m, L_n] = [M_j, L_k, L_m, L_n] = [M_j, M_k, M_m, L_n] = [M_j, M_k, M_m, M_n] .$$

(23)

Consequently, all 5-brackets vanish upon being resolved into 4-brackets [3], etc. Thus, all ($N \geq 4$)-brackets vanish for the $su(1,1)$ realization of this infinite algebra.

From the first two relations in (22), it is now straightforward to check

$$[L_p, L_q, [L_k, L_m, L_n]] = [[L_p, L_q, L_k], L_m, L_n] + [L_k, [L_p, L_q, L_m], L_n] + [L_k, L_m, [L_p, L_q, L_n]] ,$$

(24)

for any value of $C$ (and $\beta$). But of course, there are more possible FIs involving both $L$s and $M$s, and these should also be checked. We proceed to do this, after a bit of stream-lining.

The overall Casimir factor in the first relation of (22) is easily eliminated by rescaling the charges by a fourth root of the Casimir, as in the original $su(2)$ example of Nambu: $Q_k \equiv \frac{1}{\sqrt{C}} L_k, R_k \equiv \frac{1}{\sqrt{C}} M_k$. However, the remaining Casimir-dependent factor in the second relation of (22) (i.e. a central charge $z \equiv (1 - 2\beta) / \sqrt{C}$ after the rescalings) is not so easily removed. In any case we now abstract from the $su(1,1)$ enveloping algebra results the following.

### 3 Virasoro-Witt 3-Algebra

$$[Q_k, Q_m, Q_n] = (k-m)(m-n)(n-k) \ R_{k+m+n} ,$$

$$[Q_p, Q_q, R_k] = (p-q) (Q_{k+p+q} + z k R_{k+p+q} ) ,$$

$$[Q_p, R_q, R_k] = (k-q) R_{k+p+q} ,$$

$$[R_p, R_q, R_k] = 0 ,$$

(25)

where the central charge $z$ is effectively a parameter. For generic values of $z$, by direct application of this ternary algebra, we now find that 3-brackets acting on 3-brackets satisfy the usual Leibniz-like rules (FIs) except when only one $R$ is involved. There are two such exceptions out of twelve 3-on-3 possibilities.

In an obvious notation, the twelve FI possibilities stem from each of the following:

$$[R, R, [R, R, R]] , \ [R, R, [Q, R, R]] , \ [Q, R, [R, R, R]] , \ [R, R, [Q, Q, R]] , \ [Q, R, [Q, R, R]] , \ [Q, Q, [R, R, R]] ,$$

$$[Q, Q, [Q, Q, Q]] , \ [Q, Q, [R, Q, Q]] , \ [R, Q, [Q, Q, Q]] , \ [Q, Q, [R, Q, Q]] , \ [R, R, [Q, Q, Q]] .$$

Ten of the FIs hold and behave as Filippov and Leibniz would dictate. For example, for any $z$, the 3-algebra (25) gives

$$[Q_p, Q_q, [Q_k, Q_m, Q_n]] = [[Q_p, Q_q, Q_k], Q_m, Q_n] + [Q_k, [Q_p, Q_q, Q_m], Q_n] + [Q_k, Q_m, [Q_p, Q_q, Q_n]] ,$$

(26)

as stated earlier in terms of $L$s. For another example,

$$[R_p, R_q, [Q_k, Q_m, Q_n]] = [[R_p, R_q, Q_k], Q_m, Q_n] + [Q_k, [R_p, R_q, Q_m], Q_n] + [Q_k, Q_m, [R_p, R_q, Q_n]] .$$

(27)

From (26) the LHS trivially vanishes in this case, while the three terms on the RHS cancel.

The two exceptions, which for generic $z$ do not obey FI conditions, give instead

$$[Q_p, Q_q, [Q_k, Q_m, R_n]] = [[Q_p, Q_q, Q_k], Q_m, R_n] - [Q_k, [Q_p, Q_q, Q_m], R_n] - [Q_k, Q_m, [Q_p, Q_q, R_n]] = (4 + z^2)(p-q)(k-m)(m-p-q+k)n R_{k+m+n+p+q} ,$$

$$[Q_p, R_q, [Q_k, Q_m, Q_n]] = [[Q_p, R_q, Q_k], Q_m, Q_n] - [Q_k, [Q_p, R_q, Q_m], Q_n] - [Q_k, Q_m, [Q_p, R_q, Q_n]] = (4 + z^2)(n-k)(k-m)(m-n)q R_{k+m+n+p+q} .$$

(28)
Nevertheless, for the special cases \( z = \pm 2i \) the RHSs of (28) also vanish. Hence in these special cases all the FIs hold for the algebra of (24), making it a bona fide 3-algebra.

It is interesting that the special values \( z = \pm 2i \) are obtained in the \( su(1,1) \) realization only in the “classical” limit of large Casimirs, \( C = \beta (1 - \beta) \to -\infty \), for which

\[
z^2 = \frac{(1-2\beta)^2}{\beta(1-\beta)} \to -\infty - 4.
\]

(29)

Perhaps this removes some of the mystery surrounding the FIs, which are all true statements for the proposed Virasoro-Witt 3-algebra for these special values of \( z \), since the FIs are known to hold for classical Nambu 3-brackets (as in (23) above). In this context, we note the coefficient on the RHS of the first relation in (25) is given by \(-a \cdot (b \times c)\) for three vectors \( a = (1, k, k^2) \), \( b = (1, m, m^2) \), and \( c = (1, n, n^2) \).

As emphasized in (3) there is nothing fundamental about the FIs conditions so far as associativity is concerned. Generally these “identities” fail to hold due to quantum corrections for associative operator algebras \( h \)-deformed from the Jacobian-like limit of the classical Nambu brackets (CNBs) to become full-fledged quantum Nambu brackets (QNBs). Although the FIs fail for (28) for generic \( z \), there is an alternate identity following only from associativity of the charge multiplication. This alternate identity must hold if the associative algebra is consistent. The identity is

\[
[[A, B, C], D, E] + [[A, D, E], B, C] + [[D, B, E], A, C] + [[D, E, C], A, B]
- [[D, B, C], A, E] - [[E, B, C], D, A] - [[A, D, C], B, E]
- [[A, E, C], D, B] - [[A, B, D], C, E] - [[A, B, E], D, C]
= 3[A, B, C, D, E].
\]

(30)

This is the prototypical generalization of the Jacobi identity for odd QNBs (3), and like the Jacobi identity, it is antisymmetric in all of its elements. The RHS here is an “inhomogeneity” that illustrates a more general result: The totally antisymmetrized action of odd \( n \) QNBs on other odd \( n \) QNBs results in \((2n-1)\)-brackets. For the case at hand, by explicit calculation using (25), we find for any value of \( z \),

\[
[Q_p, Q_q, [Q_k, Q_m, R_n]] + [Q_m, R_n, [Q_k, Q_p, Q_q]] + [Q_k, R_n, [Q_p, Q_m, Q_q]] + [Q_k, Q_m, [Q_p, Q_q, R_n]]
- [Q_k, Q_q, [Q_p, Q_m, R_n]] - [Q_p, Q_k, [Q_q, Q_m, R_n]] - [Q_m, Q_q, [Q_k, Q_p, R_n]]
- [Q_p, Q_m, [Q_k, Q_q, R_n]] - [R_n, Q_q, [Q_k, Q_m, Q_p]] - [Q_p, R_n, [Q_k, Q_m, Q_q]]
= 0.
\]

(31)

Remarkably, there is no inhomogeneity for this or any other 3-on-3 situation computed using (25). Perhaps this is not too surprising given that all \((N \geq 4)\)-brackets vanish for the \( su(1,1) \) realization that led to (25), although in any other realization it would be necessary to specify all the 5-brackets, either by direct calculation where possible, or by definition if necessary. Still, it is reassuring that the proposed 3-algebra (25) satisfies these associativity-required consistency checks.

4 Discussion

Just as there are a countably infinite number of copies of \( su(1,1) \) embedded in the centerless Virasoro-Witt algebra, so there are an infinite number of BL algebras embedded in the (complexified) ternary Virasoro-Witt algebra proposed here. At any given level, \( L_{2n}/\sqrt{n} \) and \( L_0 \) obey the \( su(1,1) \) commutation relations (10), with invariant \( C_n = L_{n+1}L_{-n} - (L_0 + n)L_0 \). The construction of the BL algebra at that level then proceeds as in (4) and (5) above, after complexifying the level’s \( su(1,1) \) to obtain \( su(2) \) in the well-known way.

It is a straightforward extension to include central charges in the Virasoro algebra, as well as the \( L \) and \( M \) commutators,

\[
[L_n, L_m] = (n-m) L_{n+m} + cn^3 \delta_{n,-m}, \quad [L_n, M_k] = -k M_{k+n} + bn^3 \delta_{k,-n},
\]

where \( b \) and \( c \) are central, and to investigate their consequences for the 3-algebra. This will be discussed in detail elsewhere. Similar remarks apply to the supersymmetric extension of (25).
The astute reader will have noticed that (8) and (18) form a sub-algebra of the Schrödinger-Virasoro algebra [13]. The extension of our 3-bracket results to include the remaining charges of that larger algebra could be of interest.

Finally, we re-emphasize that our construction leading to (25) used only 3-brackets defined as totally antisymmetrized operator products: \[ A, B, C \equiv ABC - BAC + BCA - CPA + CAB - ACB. \] Any other definition of the 3-brackets might lead to a ternary algebra different from that proposed here.

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