E(5) and X(5) critical point symmetries obtained from Davidson potentials through a variational procedure

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Abstract

Davidson potentials of the form $\beta^2 + \beta_0^4/\beta^2$, when used in the E(5) framework (i.e., in the original Bohr Hamiltonian after separating variables as in the E(5) model, describing the critical point of the U(5) to O(6) shape phase transition) bridge the U(5) and O(6) symmetries, while they bridge the U(5) and SU(3) symmetries when used in the X(5) framework (i.e., in the original Bohr Hamiltonian after separating variables as in the X(5) model, corresponding to the critical point of the U(5) to SU(3) transition). Using a variational procedure, we determine for each value of angular momentum $L$ the value of $\beta_0$ at which the rate of change of various physical quantities (energy ratios, intraband B(E2) ratios, quadrupole moment ratios) has a maximum, the collection of the values of the physical quantity formed in this way being a candidate for describing its behavior at the relevant critical point. Energy ratios lead to the E(5) and X(5) results (whose correspond to an infinite well potential in $\beta$), while intraband B(E2) ratios and quadrupole moments lead to the E(5)-$\beta^4$ and X(5)-$\beta^4$ results, which correspond to the use of a $\beta^4$ potential in the relevant framework. A new derivation of the Holmberg–Lipas formula for nuclear energy spectra is obtained as a by-product.

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1. Introduction

The recently introduced E(5) [1] and X(5) [2] models are supposed to describe shape phase transitions in atomic nuclei, the former being related to the transition from U(5) (vibrational) to O(6) (γ-unstable) nuclei, and the latter corresponding to the transition from U(5) to SU(3) (rotational) nuclei. In both cases the original Bohr collective Hamiltonian [3] is used, with an infinite well potential in the collective β-variable. Separation of variables is achieved in the E(5) case by assuming that the potential is independent of the collective γ-variable, while in the X(5) case the potential is assumed to be of the form \( u(\beta) + u(\gamma) \).

We are going to refer to these two cases as “the E(5) framework” and “the X(5) framework” respectively. The selection of an infinite well potential in the β-variable in both cases is justified by the fact that the potential is expected to be flat around the point at which a shape phase transition occurs. Experimental evidence for the occurrence of the E(5) and X(5) symmetries in some appropriate nuclei is growing ([4, 5] and [6, 7] respectively).

In the present work we examine if the choice of the infinite well potential is the optimum one for the description of shape phase transitions. For this purpose, we need one-parameter potentials which can span the U(5)-O(6) region in the E(5) framework, as well as the U(5)-SU(3) region in the X(5) framework. It turns out that the exactly soluble [8, 9] Davidson potentials [10]

\[
    u(\beta) = \beta^2 + \frac{\beta_0^4}{\beta^2},
\]

where \( \beta_0 \) is the position of the minimum of the potential, do possess this property.

Taking into account the fact that various physical quantities should change most rapidly at the point of the shape phase transition [11], we locate for each value of the angular momentum \( L \) the value of \( \beta_0 \) for which the rate of change of the physical quantity is maximized. The collection of the values of the physical quantity formed in this way should then correspond to the behavior of this physical quantity at the critical point. As appropriate physical quantities we have used energy ratios within the ground state band, as well as intraband B(E2) ratios and quadrupole moment ratios within the ground state band, and within excited bands.

The energy ratios within the ground state band lead to results very similar to these provided by the infinite well potential in both the E(5) and the X(5) frameworks, thus indicating that the choice of the infinite well potential in both cases is the optimum one. Intraband B(E2) ratios and quadrupole moment ratios lead to results close to the ones provided by the E(5)-\( \beta^4 \) [12, 13] and X(5)-\( \beta^4 \) [14] models, which use a \( u(\beta) = \beta^4/2 \) potential in the E(5) and X(5) framework respectively. Further discussion of these results is deferred to Section 4.

The variational procedure used here is analogous to the one used in the framework of the Variable Moment of Inertia (VMI) model [15], where the energy is minimized with respect to the (angular momentum dependent) moment of inertia for each value of the angular momentum \( L \) separately.
In the framework of the Interacting Boson Model (IBM) [16], there have been attempts to consider the U(5) to O(6) transition [12], as well as the U(5) to SU(3) transition [17] in terms of one-parameter schematic Hamiltonians. In the present approach, in contrast, Davidson potentials are used directly in the original Bohr Hamiltonian, without the intervention of any IBM approximations.

On the other hand, exactly soluble models have been constructed by using the Coulomb and Kratzer [18] potentials in the original Bohr Hamiltonian in the E(5) [19] and X(5) [20] frameworks, while a two-parameter quasi-exactly soluble model [21, 22, 23] has been constructed by using [24] the sextic oscillator with a centrifugal barrier [25] in the E(5) framework. Some of these potentials will be further commented below.

In Section 2 the E(5) case is considered, while the X(5) case is examined in Section 3, in which a new derivation of the Holmberg–Lipas formula [26] for nuclear energy spectra, as well as quadrupole moments for the X(5) and related models are obtained as by-products. Finally, Section 4 contains a discussion of the present results and plans for further work. A preliminary version of this work, limited to the variational study of the spectra of ground state bands, has been reported in Ref. [27].

2. Davidson potentials in the E(5) framework

2.1 Spectra and B(E2) transition rates

The original Bohr Hamiltonian [3] is

\[ H = -\frac{\hbar^2}{2B} \left[ \frac{1}{\beta^4} \frac{\partial}{\partial \beta} \beta^4 \frac{\partial}{\partial \beta} + \frac{1}{\beta^2 \sin 3\gamma} \frac{\partial}{\partial \gamma} \sin 3\gamma \frac{\partial}{\partial \gamma} - \frac{1}{4\beta^2} \sum_{k=1,2,3} \frac{Q_k^2}{\sin^2 \left( \gamma - \frac{2}{3} \pi k \right)} \right] + V(\beta, \gamma), \]

(2)

where \( \beta \) and \( \gamma \) are the usual collective coordinates describing the shape of the nuclear surface, \( Q_k \ (k = 1, 2, 3) \) are the components of angular momentum, and \( B \) is the mass parameter.

Assuming that the potential depends only on the variable \( \beta \), i.e. \( V(\beta, \gamma) = U(\beta) \), one can proceed to separation of variables in the standard way [3, 28], using the wave function \( \Psi(\beta, \gamma, \theta_i) = f(\beta)\Phi(\gamma, \theta_i) \), where \( \theta_i \ (i = 1, 2, 3) \) are the Euler angles describing the orientation of the deformed nucleus in space.

In the equation involving the angles, the eigenvalues of the second order Casimir operator of SO(5) occur, having the form \( \Lambda = \tau(\tau + 3) \), where \( \tau = 0, 1, 2, \ldots \) is the quantum number characterizing the irreducible representations (irreps) of SO(5), called the “seniority” [29]. This equation has been solved by Bès [30].

The “radial” equation can be simplified by introducing [1] reduced energies \( \epsilon = \frac{2B}{\hbar^2} E \) and reduced potentials \( u = \frac{2B}{\hbar^2} U \), leading to

\[ \left[ -\frac{1}{\beta^4} \frac{\partial}{\partial \beta} \beta^4 \frac{\partial}{\partial \beta} + \frac{\tau(\tau + 3)}{\beta^2} + u(\beta) \right] f(\beta) = \epsilon f(\beta). \]

(3)
When plugging the Davidson potentials of Eq. (1) in the above equation, the \( \beta_0^4/\beta^2 \) term is combined with the \( \tau(\tau + 3)/\beta^2 \) term appearing there and the equation is solved exactly [8, 9], the eigenfunctions being Laguerre polynomials of the form

\[
F_\tau^n(\beta) = \left[ \frac{2n!}{\Gamma(n + p + \frac{5}{2})} \right]^{1/2} \beta^p L_n^{p+\frac{3}{2}}(\beta^2)e^{-\beta^2/2},
\]

where \( \Gamma(n) \) stands for the \( \Gamma \)-function, while \( p \) is determined by [8]

\[
p(p + 3) = \tau(\tau + 3) + \beta_0^4,
\]

leading to

\[
p = -\frac{3}{2} + \left[ (\tau + \frac{3}{2})^2 + \beta_0^4 \right]^{1/2}.
\]

The energy eigenvalues are then [8, 9] (in \( \hbar \omega = 1 \) units)

\[
E_{n,\tau} = 2n + p + \frac{5}{2} = 2n + 1 + \left[ (\tau + \frac{3}{2})^2 + \beta_0^4 \right]^{1/2}.
\]

For \( \beta_0 = 0 \) the original solution of Bohr [3, 31], which corresponds to a 5-dimensional (5-D) harmonic oscillator characterized by the symmetry \( U(5) \supset SO(5) \supset SO(3) \supset SO(2) \) [32], is obtained. The values of angular momentum \( L \) contained in each irrep of \( SO(5) \) (i.e. for each value of \( \tau \)) are given by the algorithm [16] \( \tau = 3\nu_\Delta + \lambda \), where \( \nu_\Delta = 0, 1, \ldots \) is the missing quantum number in the reduction \( SO(5) \supset SO(3) \), and \( L = \lambda, \lambda + 1, \ldots, 2\lambda - 2, 2\lambda \) (with \( 2\lambda - 1 \) missing).

The levels of the ground state band are characterized by \( L = 2\tau \) and \( n = 0 \). Then the energy levels of the ground state band are given by

\[
E_{0,L} = 1 + \frac{1}{2} \left[ (L + 3)^2 + 4\beta_0^4 \right]^{1/2},
\]

while the excitation energies of the levels of the ground state band relative to the ground state are

\[
E_{0,L,exc} = E_{0,L} - E_{0,0} = \frac{1}{2} \left[ (L + 3)^2 + 4\beta_0^4 \right]^{1/2} - \left[ 9 + 4\beta_0^4 \right]^{1/2}.
\]

For \( u(\beta) \) being a 5-D infinite well

\[
u(\beta) = \begin{cases} 
0 & \text{if } \beta \leq \beta_W \\
\infty & \text{for } \beta > \beta_W
\end{cases}
\]

one obtains the \( E(5) \) model of Iachello [1] in which the eigenfunctions are Bessel functions \( J_{\tau+3/2}(z) \) (with \( z = \beta k \), \( k = \sqrt{\epsilon} \)), while the spectrum is determined by the zeros of the Bessel functions

\[
E_{\xi,\tau} = \frac{\hbar^2}{2B} k_{\xi,\tau}^2, \quad k_{\xi,\tau} = \frac{x_{\xi,\tau}}{\beta_W}
\]

(11)
where \( x_{\xi,\tau} \) is the \( \xi \)-th zero of the Bessel function \( J_{\tau+3/2}(z) \). The spectra of the E(5) and Davidson cases become directly comparable by establishing the formal correspondence \( n = \xi - 1 \).

In what follows the ratios
\[
R_{n,L} = \frac{E_{n,L} - E_{0,0}}{E_{0,2} - E_{0,0}},
\]
(12)
and
\[
\tilde{R}_{n,L} = \frac{E_{n,L} - E_{n,0}}{E_{n,2} - E_{n,0}}
\]
(13)
with the notation \( E_{n,L} \), will be used. In the former case energies in all bands are measured relative to the ground state and normalized to the excitation energy of the \( L = 2 \) state of the ground state band (as in Ref. [1]), while in the latter case energies in each band are measured relative to the bandhead (\( L = 0 \)) of this band and normalized to the excitation energy of the \( L = 2 \) state of this band. From Eq. (7) it is clear that Eq. (13) yields identical results for all bands with \( L = 2\tau \), irrespectively of \( n \). For the ground state band \( (n = 0) \) the simplified notation
\[
R_{L} \equiv R_{0,L}
\]
(14)
will also be used.

The quadrupole operator has the form [28]
\[
T_{\mu}^{(E2)} = t\alpha_{\mu} = t\beta \left[ D^{(2)}_{\mu,0}(\theta_i) \cos \gamma + \frac{1}{\sqrt{2}} (D^{(2)}_{\mu,2}(\theta_i) + D^{(2)}_{\mu,-2}(\theta_i)) \sin \gamma \right],
\]
(15)
where \( t \) is a scale factor and \( D(\theta_i) \) denote Wigner functions of the Euler angles, while the \( B(E2) \) transition rates are given by
\[
B(E2; \varrho_i L_i \rightarrow \varrho_f L_f) = \frac{1}{2L_i + 1} \left| \langle \varrho_f L_f | T^{(E2)} | \varrho_i L_i \rangle \right|^2
\]
\[
= \frac{2L_f + 1}{2L_i + 1} B(E2; \varrho_f L_f \rightarrow \varrho_i L_i),
\]
(16)
where by \( \varrho \) quantum numbers other than the angular momentum \( L \) are denoted. The calculation of \( B(E2) \) rates proceeds as in [1, 13].

In what follows, the intraband ratios
\[
R^{B(E2)}_{n,L} = \frac{B(E2; (L+2)_n \rightarrow L_n)}{B(E2; 2_0 \rightarrow 0_0)}
\]
(17)
and
\[
\tilde{R}^{B(E2)}_{n,L} = \frac{B(E2; (L+2)_n \rightarrow L_n)}{B(E2; 2_n \rightarrow 0_n)}
\]
(18)
will be used. In the former case the \( B(E2) \) intraband transition rates of all bands are normalized to the \( B(E2) \) transition rate between the two lowest states of the ground state.
band (as in Ref. [1]), while in the latter case the B(E2) intraband transition rates within each band are normalized to the B(E2) transition rate between the two lowest states of this band.

It should be noted at this point that quadrupole moments in this framework vanish, if one is limited to the quadrupole operator of Eq. (15), because of a $\Delta \tau = \pm 1$ selection rule. This case is reminiscent of the vanishing (to lowest order) of the quadrupole moments in the O(6) limit of IBM [16]. Non-vanishing quadrupole moments can be obtained by including the next order terms in the quadrupole operator of Eq. (15).

2.2 The U(5) and O(6) limits

For $\beta_0 = 0$ it is clear that the original vibrational model of Bohr [3, 31] (with $R_4 = 2$) is obtained, while for large $\beta_0$ the O(6) limit of the Interacting Boson Model (IBM) [16] for large boson numbers, which coincides with the $\gamma$-unstable rotator (with $R_4 = 2.5$) is approached [8]. The latter fact can be seen in Table 1, where the $R_L$ energy ratios within the ground state band for two different values of the parameter $\beta_0$ are shown, together with the O(6) predictions for large boson numbers, which correspond to [33]

$$E(L) = AL(L + 6), \quad R_L = \frac{L(L + 6)}{16},$$

with $A$ constant. It is clear that the O(6) limit is approached as $\beta_0$ is increased, the agreement being already very good at $\beta_0 = 8$.

In Table 1 the intraband B(E2) ratios $R_{0,L}^{B(E2)}$ (within the ground state band, which has $n = 0$) and $R_{1,L}^{B(E2)}$ (within the next band, which is characterized by $n = 1$) are also shown. In the O(6) limit (for infinite number of bosons) one has [16]

$$B(E2; L + 2 \rightarrow L) = a \frac{L + 2}{L + 5},$$

where $a$ constant, therefore in this case

$$R_{n,L}^{B(E2)} = \frac{5 (L + 2)}{2 (L + 5)},$$

for all values of $n$. In Table 1 we remark that the $n = 0$ and $n = 1$ results still differ a little at $\beta_0 = 4$, becoming almost identical to the O(6) behavior at $\beta_0 = 8$.

The gradual evolution from the U(5) to the O(6) limit, as $\beta_0$ is increased, is depicted in Fig. 1, where the energy ratios $R_L$ within the ground state band are depicted, and in Fig. 2, where the intraband B(E2) ratios $R_{0,L}^{B(E2)}$ (within the ground state band) and $R_{1,L}^{B(E2)}$ (within the $n = 1$ band) are shown. The limiting values at the right hand side are in agreement with the values given in Table 1 (taking into account the difference between the $R_{n,L}^{B(E2)}$ and $R_{n,L}^{B(E2)}$ ratios, defined in Eqs. (17) and (18)) , while at the left hand side the U(5) values, corresponding to [13, 16]

$$E(L) = AL, \quad R_L = \frac{L}{2},$$

(22)
\[ B(E2; (L+2)_0 \rightarrow L_0) = a(L+2), \quad R_{0,L}^{B(E2)} = \frac{L+2}{2}, \quad (23) \]

\[ B(E2; (L+2)_1 \rightarrow L_1) = a'(L+2)(L+7), \quad E_{1,L}^{B(E2)} = \frac{5}{14} \frac{(L+2)(L+7)}{L+5}, \quad (24) \]

where \( A, a, a' \) constants, are obtained.

It should be noticed that the O(6) limit of IBM (with large boson numbers) is also obtained [19] by using in the E(5) framework the exactly soluble Kratzer potential [18], which has the form

\[ u(\beta) = -2D \left( \frac{\beta_0}{\beta} - \frac{1}{2} \frac{\beta_0^2}{\beta^2} \right) = -\frac{A}{\beta} + \frac{B}{\beta^2}, \quad (25) \]

where \( D \) is the depth of the minimum of the potential, which is located at \( \beta_0 \), while \( A = 2\beta_0 D \) and \( B = \beta_0^2 D \). The O(6) limit is obtained for large values of \( D \) or large values of \( \beta_0 \). In the case of the Kratzer potential, however, it is clear that small values of \( D \) or small values of \( \beta_0 \) lead [19] to the Coulomb potential.

2.3 Variational procedure applied to energy ratios

It is useful to consider the ratios \( R_L \) within the ground state band, defined above, as a function of \( \beta_0 \). As seen in Fig. 1, where the ratios \( R_4, R_{12} \) and \( R_{20} \) are shown, these ratios increase with \( \beta_0 \), the increase becoming very steep at some value \( \beta_{0,m}(L) \) of \( \beta_0 \), where the first derivative \( \frac{dR_L}{d\beta_0} \) reaches a maximum value, while the second derivative \( \frac{d^2R_L}{d\beta_0^2} \) vanishes. Numerical results for \( \beta_{0,m} \) are shown in Table 2, together with the values of \( R_L \) occuring at these points, which are compared to the \( R_L \) ratios occuring in the ground state band of the E(5) model [1]. Very close agreement of the values determined by the procedure described above with the E(5) values is observed in Table 2, as well as in Fig. 3(a), where these ratios are also shown, together with the corresponding ratios of the U(5) (Eq. (22)) and O(6) (Eq. (19)) limits. Finally, the potentials obtained for different angular momenta \( L \) are depicted in Fig. 4.

The work performed here is reminiscent of a variational procedure. Wishing to determine the critical point in the shape phase transition from U(5) to O(6), one chooses a potential (the Davidson potential) with a free parameter \( (\beta_0) \), which helps in covering the whole range of interest. Indeed, as we have seen in the previous subsection, for \( \beta_0 = 0 \) the U(5) picture is obtained, while large values of \( \beta_0 \) lead to the O(6) limit. One then needs a physical quantity which can serve as a “measure” of collectivity. For this purpose one considers the ratios \( R_L \), encouraged by the fact that these ratios are well-known indicators of collectivity in nuclear structure [34]. Since at the critical point (if any) one expects the collectivity to change very rapidly, one looks, for each \( R_L \) ratio separately, for the value of the parameter at which the change of \( R_L \) is maximum. Indeed, the first derivative of the ratio \( R_L \) with respect to the parameter \( \beta_0 \) exhibits, as seen in the upper panel of Fig. 1, a sharp maximum, which is then a good candidate for being the critical point for this particular value of the angular momentum \( L \). The \( R_L \) values at the critical points corresponding to each value of \( L \) form a collection, which should correspond to the behaviour of the ground state band of a nucleus.
at the critical point. The infinite well potential used in E(5) succeeds in reproducing all these “critical” $R_L$ ratios in the ground state band for all values of the angular momentum $L$, without using any free parameter. It is therefore proved that the infinite well potential is indeed the optimum choice for describing the ground state bands of nuclei at the critical point of the U(5) to O(6) shape phase transition.

In other words, starting from the Davidson potentials and using a variational procedure, according to which the rate of change of the $R_L$ ratios as a function of the parameter $\beta_0$ is maximized for each value of the angular momentum $L$ separately, one forms the collection of critical values of $R_L$ which corresponds to the ground state band of the E(5) model, which is supposed to describe nuclei at the critical point.

Variational procedures in which each value of the angular momentum $L$ is treated separately are not unheard of in nuclear physics. An example is given by the Variable Moment of Inertia (VMI) model [15], in which the energy of the nucleus is minimized with respect to the (angular momentum dependent) moment of inertia for each value of the angular momentum separately. From the cubic equation obtained from this condition, the moment of inertia is uniquely determined (as a function of angular momentum) in each case. The collection of energy levels occurring by using in the energy formula the appropriate value of the moment of inertia for each value of the angular momentum $L$ forms the ground state band of the nucleus.

$L$-dependent potentials are also not unheard of in nuclear physics. They are known to occur in the framework of the optical model potential [35, 36, 37], as well as in the case of quasimolecular resonances, like $^{12}\text{C}+^{12}\text{C}$ [38].

Some comparison of the variational procedure used here with the standard Ritz variational method used in quantum mechanics ([39], for example) is in place. In the (simplest version of the) Ritz variational method a trial wave function containing a parameter is chosen and subsequently the energy is minimized with respect to this parameter, thus determining the parameter value and, after the relevant substitution, the energy value at the minimum. In the present case a trial potential containing a parameter is chosen and subsequently the rate of change of the physical quantity (here the rate of change of the energy ratios) is maximized with respect to this parameter, thus determining the parameter value and, after the relevant calculation, the value of the physical quantity (here the energy ratios) at the maximum (i.e. at the critical point). The main similarity between the two methods is the use of a parameter-dependent trial wave function/trial potential respectively. The main difference between the two methods is that in the former the relevant physical quantity (the energy) is minimized with respect to the parameter, while in the latter the rate of change of the physical quantity (the energy ratios) is maximized with respect to the parameter.

It should be emphasized that the trial potentials to be used for the study of the critical region between two different symmetries should possess the correct limiting behavior, i.e. they should be able to reproduce the two symmetries for special values of the free parameter.
(as in the present case the Davidson potentials reproduce the U(5) symmetry for $\beta_0 = 0$ and the O(6) symmetry for large $\beta_0$).

It is worth mentioning at this point that the consequences of replacing in the E(5) framework the infinite well potential by a well of finite depth have been studied in detail [40], the main conclusion being that many key features of E(5) remain essentially unchanged, even if the depth of the potential is radically changed. This observation implies that the E(5) predictions, reassured above through the variational procedure, are stable and do not depend sensitively on any parameter like the depth of the potential.

The success of the variational procedure when applied to energy ratios within the ground state band encourages its use for excited bands as well. Since it is reasonable to treat each band as a separate entity, the ratios $R_{n,L}$ (defined in Eq. (13)) should be used for this purpose, which involve levels of one band only, in contrast to the ratios $R_{n,L}$ (defined in Eq. (12)) which, except in the case of the ground state band, use levels from two different bands. From Eq. (7) it is clear, however, that in the case of the Davidson potential and for bands with $L = 2\tau$ the $R_{n,L}$ ratios will be identical to $R_L$ for all values of $n$. This is a special feature of the Davidson potentials, due to the their oscillator-like spectrum. This feature is lifted when one considers generalized Davidson potentials of the form

$$u(\beta) = \beta^{2n} + \frac{\beta_0^{4n}}{\beta^{2n}}, \quad n = 2, 3, 4, \ldots \tag{26}$$

which will be shortly discussed in Section 4.

2.4 Variational procedure applied to excitation energies

It is instructive to apply the variational procedure developed in the previous subsection to isolated energy levels instead of energy ratios. For this purpose the excitation energies $E_{0,L,exc}$ of the levels of the ground state band, given by Eq. (9), will be used. The values $\beta_{0,m}$ where the absolute value of the first derivative (since the first derivative is negative in this case) becomes maximum are reported in Table 2, together with the corresponding $E_{0,L,exc}$ values. Using the $E_{0,L,exc}$ values obtained in this way, one can calculate the relevant $R_L$ ratios of Eq. (14). As seen in Table 2, the results obtained in this way are very close to the U(5) results, provided by Eq. (22).

This result is easy to explain: From the experimental data (see Ref. [41], for example) it is known that excitation energies within a series of isotopes drop very rapidly in the region of the vibrational limit, as one moves away from the pure vibrational behaviour (see, for example, the chains of the Sm, Gd, Dy isotopes), while the changes near the rotational limit as one moves from one isotope to the next are minimal (see, for example, the Th, U, Pu isotopes). Trying then to identify a series of energy levels corresponding to the most rapid changes in the excitation energies, one naturally ends up with the vibrational limit. Therefore the application of the variational procedure to isolated energy levels just demonstrates the effectiveness of the method, leading to results physically expected.

2.5 Variational procedure applied to B(E2) ratios
The success of the variational procedure when applied to the energy ratios $R_L$ also encourages its use for intraband B(E2) ratios. Considering each band as a separate entity, it is reasonable to use the ratios $R_{n,L}^{B(E2)}$, defined in Eq. (18), which involve transitions within one band (while the ratios $R_{n,L}^{B(E2)}$, defined in Eq. (17), involve intraband transitions from two different bands, except in the case of the ground state band). It should be emphasized that the different choice of the denominator in the ratios $R_{n,L}^{B(E2)}$ and $R_{n,L}^{B(E2)}$ is not a trivial matter of normalization, since one divides in each case by a different function of $\beta_0$, being led in this way to different results when the variational procedure is applied.

As we have seen in Fig. 2, the B(E2) ratios go down from their U(5) values at $\beta_0 = 0$ to the O(6) limiting values at large $\beta_0$. Therefore in this case we are going to determine the values $\beta_{0,m}$ at which the absolute value of the first derivative, $|dR_{n,L}^{B(E2)}/d\beta_0|$ has a maximum, while the second derivative vanishes. Numerical results for the ground state band ($n = 0$) and the $n = 1$ band are given in Table 2, and depicted in Figs. 3(b) and 3(c) respectively, where the U(5) and O(6) results, calculated from Eqs. (23), (24), and (21), are shown for comparison. In addition, the results given by the original E(5) model [1], as well as by the E(5)-$\beta^4$ model [12, 13], which uses a $u(\beta) = \beta^4/2$ potential in the E(5) framework instead of an infinite well potential, are exhibited. In both bands the variational procedure leads to results which are quite close to the E(5)-$\beta^4$ case. We defer further discussions of these results to Section 4.

It should be mentioned at this point that the first derivative of the $B(E2 : L + 2 \rightarrow L)$ values with respect to $\beta_0$ does not exhibit a maximum, while the second derivative does not vanish at any value other than $\beta_0 = 0$. Therefore the variational procedure cannot be applied to isolated B(E2) values, being applicable to B(E2) ratios only.

3. Davidson potentials in the X(5) framework

3.1 Spectra, B(E2) transition rates, and quadrupole moments

Starting again from the original Bohr Hamiltonian of Eq. (2), one seeks solutions of the relevant Schrödinger equation having the form $\Psi(\beta, \gamma, \theta_i) = \phi_K^L(\beta, \gamma)D_{M,K}^L(\theta_i)$, where $\theta_i$ ($i = 1, 2, 3$) are the Euler angles, $D(\theta_i)$ denote Wigner functions of them, $L$ are the eigenvalues of angular momentum, while $M$ and $K$ are the eigenvalues of the projections of angular momentum on the laboratory-fixed $z$-axis and the body-fixed $z'$-axis respectively.

As pointed out in Ref. [2], in the case in which the potential has a minimum around $\gamma = 0$ one can write the last term of Eq. (2) in the form

$$\sum_{k=1,2,3} \frac{Q_k^2}{\sin^2(\gamma - \frac{2\pi}{3} k)} \approx \frac{4}{3} (Q_1^2 + Q_2^2 + Q_3^2) + Q_3^2 \left( \frac{1}{\sin^2\gamma} - \frac{4}{3} \right).$$

(27)

Using this result in the Schrödinger equation corresponding to the Hamiltonian of Eq. (2), introducing reduced energies $\epsilon = 2BE/h^2$ and reduced potentials $u = 2BV/h^2$, and assuming that the reduced potential can be separated into two terms, one depending on $\beta$
and the other depending on $\gamma$, i.e. $u(\beta, \gamma) = u(\beta) + u(\gamma)$, the Schrödinger equation can be separated into two equations [2, 42], the “radial” one being

$$\left[ -\frac{1}{\beta^4} \frac{\partial}{\partial \beta} \beta^4 \frac{\partial}{\partial \beta} + \frac{1}{4 \beta^2} \frac{4}{3} (L(L+1) - K^2) + u(\beta) \right] \xi_L(\beta) = \epsilon_\beta \xi_L(\beta). \quad (28)$$

When plugging the Davidson potentials of Eq. (1) in this equation, the $\beta^4_0/\beta^2$ term of the potential is combined with the $(L(L+1) - K^2)/3\beta^2$ term appearing there and the equation is solved exactly, the eigenfunctions being Laguerre polynomials of the form

$$F^L_n(\beta) = \left[ \frac{2n!}{\Gamma \left( n + \frac{a}{2} \right)} \right]^{1/2} \beta^a L_n^{a+\frac{3}{2}}(\beta^2) e^{-\beta^2/2} \quad (29)$$

where $a$ is given by

$$a = -\frac{3}{2} + \left[ \frac{1}{3} (L(L+1) - K^2) + \frac{9}{4} + \beta^4_0 \right]^{1/2}. \quad (30)$$

The energy eigenvalues are then (in $\hbar \omega = 1$ units)

$$E^{(K)}_{n,L} = 2n + a + \frac{5}{2} = 2n + 1 + \left[ \frac{1}{3} (L(L+1) - K^2) + \frac{9}{4} + \beta^4_0 \right]^{1/2}. \quad (31)$$

The levels of the ground state band are characterized by $n = 0$ and $K = 0$. Then the excitation energies relative to the ground state are given by

$$E^{(0)}_{0,L,exc} = E^{(0)}_{0,L} - E^{(0)}_{0,0} = \left[ \frac{1}{3} L(L+1) + \frac{9}{4} + \beta^4_0 \right]^{1/2} - \left[ \frac{9}{4} + \beta^4_0 \right]^{1/2}, \quad (32)$$

which can easily be put into the form

$$E'_{0,L,exc} = \frac{E^{(0)}_{0,L,exc}}{\left[ \frac{9}{4} + \beta^4_0 \right]^{1/2}} = \left[ 1 + \frac{L(L+1)}{3 \left( \frac{9}{4} + \beta^4_0 \right)} \right]^{1/2} - 1, \quad (33)$$

which is the same as the Holmberg–Lipas formula [26]

$$E_H(L) = a_H \left( \sqrt{1 + b_H L(L+1)} - 1 \right), \quad (34)$$

with

$$a_H = 1, \quad b_H = \frac{1}{3 \left( \frac{9}{4} + \beta^4_0 \right)}. \quad (35)$$

It is worth remarking at this point that the Holmberg–Lipas formula can be derived [33] by assuming that the moment of inertia $I$ in the energy expression of the rigid rotator ($E(L) = L(L+1)/2I$) is a function of the excitation energy, i.e. $I = \alpha + \beta E(L)$, where $\alpha$
and $\beta$ are constants, the latter being proportional to $b_H$ and acquiring positive values. It is therefore clear that the Holmberg–Lipas formula, as well as the spectrum of the Davidson potentials derived in this section, have built-in the concept of the Variable Moment of Inertia (VMI) model [15], according to which the moment of inertia is an increasing function of the angular momentum.

For $u(\beta)$ being a 5-D infinite well potential (see Eq. (10)) one obtains the X(5) model of Iachello [2], in which the eigenfunctions are Bessel functions $J_\nu(k_{s,L}\beta)$ with $\nu = (L(L + 1) - K^2/3 + 9/4)^{1/2}$, while the spectrum is determined by the zeros of the Bessel functions, the relevant eigenvalues being

$$\epsilon_{\beta; s, L} = (k_{s,L})^2, \quad k_{s,L} = x_{s,L}/\beta_W,$$

where $x_{s,L}$ is the $s$-th zero of the Bessel function $J_\nu(k_{s,L}\beta)$. The eigenfunctions are

$$\xi_{s,L}(\beta) = c_{s,L}\beta^{-3/2}J_\nu(k_{s,L}\beta),$$

where $c_{s,L}$ are normalization constants.

The spectra of the X(5) and Davidson cases become directly comparable by establishing the formal correspondence $n = s - 1$. In addition to the energy ratios $R_{n,L}$ and $R_{n,L'}$, defined in Eqs. (12) and (13), which will be used for $K = 0$ bands, the ratios

$$R'_{n,L} = \frac{E_{n,L}^{(2)} - E_{n,2}^{(2)}}{E_{n,3}^{(2)} - E_{n,2}^{(2)}},$$

defined within the $K = 2$ band, will be used below.

The quadrupole operator is again given by Eq. (15), while the B(E2) transition rates are given by

$$B(E2; L_s \rightarrow L'_s) = \frac{|\langle L_s || T^{(E2)} || L'_s \rangle|^2}{2L_s + 1}. \quad (40)$$

The matrix elements of the quadrupole operator involve an integral over the Euler angles, which is the same as in Ref. [2] and is performed by using the properties of the Wigner $D$ functions, of which only $D^{(2)}_{\mu,0}$ participates, since $\gamma \simeq 0$ in Eq. (15) (as mentioned before Eq. (27)), as well as an integral over $\beta$. After performing the integrations over the angles one is left with

$$B(E2; L_s \rightarrow L'_s) = (L_s2L'_s|000\rangle^2 I_{s,L; s', L'}^2,$$

where the Clebsch–Gordan coefficient $(L_s2L'_s|000\rangle$ appears, which determines the relevant selection rules. In the Davidson case the integral has the form

$$I_{s,L; s', L'} = \int \beta F^L_n(\beta)F_{n'}^{L'}(\beta)\beta d\beta,$$
with \( n = s - 1 \) and \( n' = s' - 1 \), which involves Laguerre polynomials, as seen from Eq. (29).

In addition to the intraband \( \text{B}(\text{E}2) \) ratios defined in Eqs. (17) and (18), interband \( \text{B}(\text{E}2) \) transition rate ratios

\[
R_{n,L,n',L'}^{B(\text{E}2)} = \frac{B(\text{E}2; (L_n \rightarrow L_n'))}{B(\text{E}2; 2_0 \rightarrow 0_0)}
\]  

(43)

will be used.

Quadrupole moments are defined by [16]

\[
Q_{s,L} = \frac{4\sqrt{\pi}}{5} (L_s L_s 2|L_s - L_s 0)\langle L_s ||T(\text{E}2)||L_s \rangle.
\]  

(44)

In what follows, the ratios

\[
R_{n,L}^Q = \frac{Q_{n,L}}{Q_{0,2}},
\]  

(45)

and

\[
R_{n,L}^{Q'} = \frac{Q_{n,L}}{Q_{n,2}}
\]  

(46)

will be used.

### 3.2 The X(5)-\( \beta^2 \) and SU(3) limits

For \( \beta_0 = 0 \) the exactly soluble X(5)-\( \beta^2 \) model (with \( R_4 = 2.646 \)) is obtained, the details of which can be found in Ref. [14], while for large \( \beta_0 \) the SU(3) limit of IBM with large boson numbers, which coincides with the rigid rotator (with \( R_4 = 3.333 \)) is obtained. In what follows the occurrence of the SU(3) limit will be discussed in more detail.

It is clear that the Holmberg–Lipas formula gives rotational spectra for small values of \( b_H \), at which one can keep only the first \( L \)-dependent term in the Taylor expansion of the square root appearing in Eq. (34), leading to energies proportional to \( L(L+1) \). From Eq. (35) it is then clear that rotational spectra are expected for large values of \( \beta_0 \), for which small values of \( b_H \) occur. This can be seen in Table 3, where the \( R_L \) ratios occurring for two different values of \( \beta_0 \) are shown, together with the predictions of the SU(3) limit of IBM at large boson numbers, which correspond to the rigid rotator with

\[
E(L) = AL(L+1), \quad R_L = \frac{L(L+1)}{6},
\]  

(47)

where \( A \) constant [16]. The agreement to the SU(3) results is quite good already at \( \beta_0 = 8 \). The same is seen for the ratios \( R'_{0,L} \), regarding the \( n = 0, K = 2 \) band, also reported in Table 3, which in the rigid rotator case correspond to the limiting values

\[
R'_{0,L} = \frac{L(L+1)}{6} - 1.
\]  

(48)
In Table 3 the intraband B(E2) ratios $R_{0,L}^{B(E2)}$ (within the ground state band, which has $n = 0$) and $R_{1,L}^{B(E2)}$ (within the next band, which is characterized by $n = 1$) are also shown. In the SU(3) limit (for infinite number of bosons) one has for all $K = 0$ bands [16]

$$B(E2; L + 2 \rightarrow L) = a \frac{(L + 2)(L + 1)}{(2L + 3)(2L + 5)},$$  \hfill (49)

where $a$ constant, therefore in this case

$$R_{n,L}^{B(E2)} = \frac{15}{2} \frac{(L + 2)(L + 1)}{(2L + 3)(2L + 5)},$$  \hfill (50)

for all values of $n$. In Table 3 we remark that the $n = 0$ and $n = 1$ results still differ a little at $\beta_0 = 4$, becoming almost identical to the SU(3) behavior at $\beta_0 = 8$.

The gradual evolution from the X(5)-$\beta^2$ to the SU(3) limit, as $\beta_0$ is increased, is depicted in Fig. 5, where the energy ratios $R_L$ within the ground state band are depicted, and in Fig. 6(a), where the intraband B(E2) ratios $R_{0,L}^{B(E2)}$ (within the ground state band) and $R_{1,L}^{B(E2)}$ (within the $n = 1$ band) are shown. The limiting values at the right hand side are in agreement with the results given in Table 3 (taking into account the difference between the ratios $R_{n,L}^{B(E2)}$ and $R_{n,L}^{B(E2)}$), while at the left hand side the X(5)-$\beta^2$ values, given in Ref. [14], are obtained.

In Table 3 interband B(E2) transition rates from the $n = 1$ band to the $n = 0$ band (defined in Eq. (43)) are also shown. These transitions are forbidden in the SU(3) framework [16]. The rapid fall of these transitions towards zero can be seen both in Table 3 and in Fig. 6(b), where the X(5)-$\beta^2$ limiting values on the left are in agreement with the ones given in Ref. [14].

Furthermore in Table 3 ratios of quadrupole moments within the $n = 0$ and $n = 1$ bands are given. In the SU(3) limit (for infinite number of bosons) one has [16] for all values of $n$

$$Q_{n,L} = a \frac{L}{2L + 3},$$  \hfill (51)

where $a$ constant, corresponding to

$$R_{n,L}^{Q} = \frac{7}{2} \frac{L}{2L + 3}.$$  \hfill (52)

It is clear from Table 3 that at $\beta_0 = 4$ some differences between the $n = 0$ and $n = 1$ cases are still visible, while at $\beta_0 = 8$ both cases become almost identical to the SU(3) values.

The evolution of quadrupole moments from the X(5)-$\beta^2$ case to the SU(3) limiting values is depicted in Fig. 6(c). The SU(3) limiting values on the right are in good agreement with the contents of Table 3 (taking into account the difference between the ratios $R_{n,L}^{Q}$ and $R_{n,L}^{Q}$, defined in Eqs. (45) and (46)). Since no results for quadrupole moments for the X(5)-$\beta^2$ case are given in Ref. [14], they are reported here in Table 4. Quadrupole moments for the
X(5) model, as well as for the X(5)-β^4, X(5)-β^6, and X(5)-β^8 models (defined in Ref. [14]) are also listed in Table 4 as a by-product.

It should be noted that the SU(3) limit of IBM (at large boson numbers) is also obtained [20] by using in the X(5) framework the exactly soluble Kratzer potential [18] of Eq. (25). The SU(3) limit is obtained for large values of B. In the case of the Kratzer potential, however, it is clear that small values of B lead [20] to the Coulomb potential.

3.3 Variational procedure applied to energy ratios

The variational procedure used in subsection 2.3 can also be applied here. Wishing to determine the critical point in the shape phase transition from U(5) to SU(3), one chooses a potential (the Davidson potential of Eq. (1)) with a free parameter (β_0), which serves in spanning the range of interest. For large values of β_0 the SU(3) limit of IBM (with large boson numbers) is obtained, while for β_0 = 0 the X(5)-β^2 picture occurs [14], which is not the U(5) limit, but it is located between U(5) and X(5), on the way from U(5) to SU(3). Thus the region of interest around X(5) is covered from X(5)-β^2 to SU(3).

Then the values of β_0 at which the first derivative dR_L/dβ_0 exhibits a sharp maximum, as seen in the upper panel of Fig. 5, are determined for each value of the angular momentum L separately. Numerical results for β_0,m are shown in Table 5, together with the values of R_L occuring at these points, which are compared to the R_L ratios occuring in the ground state band of the X(5) model [2]. Very close agreement of the values determined by the variational procedure with the X(5) results is observed, thus indicating that the choice of the infinite well potential used in the X(5) model is the optimum one for the description of the shape phase transition from U(5) to SU(3).

The results are depicted in Fig. 7(a), where in addition to the bands provided by the variational procedure and the X(5) model, the bands corresponding to the U(5) and SU(3) cases, calculated from Eqs. (22) and (47), as well as the band corresponding to X(5)-β^2 (taken from Ref. [14]) are shown. Finally, the potentials obtained for different angular momenta L are shown in Fig. 8, which looks very similar to Fig. 4, obtained in the E(5) framework.

For excited K = 0 bands the ratios R_{n,L} should be considered. However, because of Eq. (31), it is clear that all these ratios for all K = 0 bands are identical to R_L for all values of n. This feature is due to the oscillator-like form of the spectrum of Davidson potentials and is lifted when using the generalized Davidson potentials of Eq. (26), which will be further discussed in Section 4.

For K = 2 bands the ratios R′_{n,L} (defined in Eq. (39)) should be used. The relevant results for the n = 0, K = 2 band are reported in Table 5 and Fig. 7(b). Not only the energy ratios obtained through the variational procedure are very close to the X(5) results [14, 42], but in addition the β_0,m values obtained for the even values of L are very close to the corresponding ones obtained from the application of the variational procedure to the ground state band, discussed above.

3.4 Variational procedure applied to excitation energies
As in subsection 2.4, it is instructive to apply the variational procedure to the excitation energies of the levels of the ground state band, given by Eq. (32). Numerical results are shown in Table 5, and are seen to be very close to the results provided by the X(5)-$\beta^2$ model [14] (which corresponds to a Davidson potential with $\beta_0 = 0$).

This result is again an indication that the method works efficiently. Indeed, as mentioned in subsection 2.4, it is well known [41] that excitation energies within a series of isotopes drop very rapidly as one moves away from the vibrational behaviour towards the rotational region, where the change is slow. Trying then to identify a series of energy levels corresponding to the most rapid changes in the excitation energies, one naturally ends up with the most vibrational behaviour possible within the realm of the model used, i.e. with the X(5)-$\beta^2$ model in the present case.

3.5 Variational procedure applied to B(E2) ratios

As explained in subsection 2.5, the ratios $R_{n,L}^{B(E2)}$ should be considered in this case. Numerical results for the ground state band ($n = 0, K = 0$) and the $n = 1, K = 0$ band are given in Table 5, and depicted in Figs. 7(c) and 7(d), where the U(5) and SU(3) results, calculated from Eqs. (23), (24) and (50) respectively, are shown for comparison. In addition, the results given by the original X(5) model [2], as well as by the X(5)-$\beta^2$ and X(5)-$\beta^4$ models [14], which use the $u(\beta) = \beta^2/2$ and $u(\beta) = \beta^4/2$ potentials in the X(5) framework instead of an infinite well potential, are exhibited. In both bands the variational procedure leads to results which are quite close to the X(5)-$\beta^4$ case. Further discussion of these results is deferred to Section 4.

It should be mentioned at this point that the first derivative of the $B(E2; L + 2 \rightarrow L)$ values with respect to $\beta_0$ again does not exhibit a maximum, while the second derivative does not vanish at any value other than $\beta_0 = 0$. Therefore the variational procedure cannot be applied to isolated B(E2) values, being applicable to B(E2) ratios only, as in the E(5) framework.

3.6 Variational procedure applied to quadrupole moments

The variational procedure can also be applied to quadrupole moments. Treating each band as a separate entity, one should use the ratios $R_{n,L}^Q$, defined in Eq. (46), which involve quadrupole moments of only one band, in contrast to the ratios $R_{n,L}^Q$, defined in Eq. (45), which involve quadrupole moments from two different bands, except in the case of the ground state band. Numerical results for the ground state band ($n = 0$) and the $n = 1$ band are given in Table 5, and plotted in Figs. 7(e) and 7(f), where the SU(3) results (calculated from Eq. (52)) are shown for comparison. In addition, the results provided by the X(5), X(5)-$\beta^2$ and X(5)-$\beta^4$ models, given in Table 4, are shown. In both cases it is clear that the results of the variational procedure are close to the X(5)-$\beta^4$ values. These results will be further discussed in the next section.

4. Discussion

The main results obtained in the present work are summarized here:
1) A variational procedure for determining the values of physical quantities at the point of shape phase transitions in nuclei has been suggested. Using one-parameter potentials spanning the region between the two limiting symmetries of interest, the parameter values at which the rate of change of the physical quantity becomes maximum are determined for each value of the angular momentum separately and the corresponding values of the physical quantity at these parameter values are calculated. The values of the physical quantity collected in this way represent its behavior at the critical point.

2) The method has been applied in the shape phase transition from U(5) to O(6), using one-parameter Davidson potentials [10] and considering the energy ratios $R_L = E(L)/E(2)$ within the ground state band as the relevant physical quantity, leading to a band which practically coincides with the ground state band of the E(5) model [1]. It has also been applied in the same way in the shape phase transition from U(5) to SU(3), leading to a band which practically coincides with the ground state band of the X(5) model [2]. Energy ratios within the lowest $K = 2$ band in the latter case also lead to the relevant X(5) results [14, 42].

3) The method has also been applied to intraband B(E2) ratios of the ground state band and the first excited band in the U(5)-O(6) transition region, leading to results very close to the ones provided by the E(5)-$\beta^4$ model [12, 13], which uses a $u(\beta) = \beta^4/2$ potential instead of an infinite well potential in the E(5) framework. It has also been applied to intraband B(E2) ratios and ratios of quadrupole moments of the ground state band and the first excited band in the U(5)-SU(3) transition region, leading to results very similar to the ones provided by the X(5)-$\beta^4$ model [14], which uses a $u(\beta) = \beta^4/2$ potential instead of an infinite well potential in the X(5) framework.

4) The method has also been applied to isolated excitation energies of the ground state band, leading to a U(5) band in the E(5) framework and to a X(5)-$\beta^2$ [14] band in the X(5) framework. (The U(5) and X(5)-$\beta^2$ models correspond to the use of a harmonic oscillator potential $u(\beta) = \beta^2/2$ in the E(5) and X(5) frameworks, respectively.) These results are expected, since it is known [41] that the most rapid change (drop) of the excitation energies in a series of isotopes occurs as one starts moving away from the vibrational limit towards the rotational limit.

5) It should be emphasized that the application of the method was possible because the Davidson potentials correctly reproduce the U(5) and O(6) symmetries of IBM (with large boson numbers) in the E(5) framework (for small and large parameter values respectively), as well as the relevant X(5)-$\beta^2$ [14] and SU(3) symmetries in the X(5) framework (for small and large parameter values respectively). The occurrence of SU(3) (with large boson numbers) in the X(5) framework is a new result, which has been proved by considering energy, intraband and interband B(E2), and quadrupole moment ratios, while the occurrence of O(6) in the E(5) framework has essentially been observed earlier [8] and has been corroborated here by considering energy and intraband B(E2) ratios.
6) As a by-product, a derivation of the Holmberg–Lipas formula [26] has been achieved using Davidson potentials in the X(5) framework.

7) As another by-product, quadrupole moments for the X(5) model and the X(5)-β^{2n} models [14] for \( n = 1, 2, 3, 4 \) have been calculated.

The following comments are now in place:

i) The fact that the application of the variational procedure to energy ratios within the ground state band in the E(5) and X(5) frameworks leads to results very close to the ground state bands of the E(5) and X(5) models suggests that the selection of the infinite well potential is the optimum one in both cases.

ii) The fact that the application of the variational procedure to intraband B(E2) transition ratios and to quadrupole moment ratios leads to results close to the E(5)-β^4 [12, 13] and X(5)-β^4 [14] models in the E(5) and X(5) frameworks respectively, suggests that further studies are needed. In particular, it is of interest to apply the variational procedure using the generalized Davidson potentials of Eq. (26) as “trial potentials” in both the E(5) and X(5) frameworks. These potentials with \( \beta_0 = 0 \) are known to approach smoothly the E(5) and X(5) models from the U(5) direction [13, 14], as the power of \( \beta \) in the \( \beta^{2n} \) term increases. It is expected that these potentials with \( \beta_0 \neq 0 \) will be smoothly approaching the E(5) and X(5) models from the O(6) and SU(3) directions, respectively. Then the results of the variational procedure could converge towards the E(5) and X(5) results with increasing \( n \).

iii) The application of the variational procedure to energy ratios involving levels of excited bands with \( n > 0 \) is trivial in the case of the Davidson potential, because of its harmonic oscillator features, but it becomes nontrivial in the case of the generalized Davidson potentials, and therefore this task should be undertaken.

iv) The generalized Davidson potentials are known to possess the appropriate limiting behaviour for \( \beta_0 = 0 \) [13, 14] and are expected to approach the appropriate limits (near to O(6) and SU(3) in the E(5) and X(5) frameworks respectively) for large values of \( \beta_0 \). Any other potential/Hamiltonian bridging the relevant pairs of symmetries (U(5)-O(6) and U(5)-SU(3)) could be equally appropriate.

v) It is interesting that the most general (up to two-body terms) IBM Hamiltonian appropriate for the U(5) to O(6) transition leads [12] to the E(5)-β^4 model, in agreement to the results mentioned in ii). It will also be interesting to examine if appropriate symmetry-conserving higher order terms [43, 44, 45, 46], when added to this Hamiltonian, modify this conclusion.

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Figure captions

**Fig. 1** (Color online) $R_L$ energy ratios (Eq. (14)) for the ground state band (for $L = 4, 12, 20$) and their derivatives $dR_L/d\beta_0$ vs. the parameter $\beta_0$, calculated using Davidson potentials (Eq. (1)) in the E(5) framework. The $R_L$ curves demonstrate the evolution from the U(5) symmetry (on the left) to the O(6) limit of IBM with large boson numbers (on the right). See subsections 2.2 and 2.3 for further details.

**Fig. 2** (Color online) Intraband B(E2) ratios (Eq. (17)) $R_{0,L}^{B(E2)}$ (for the ground state band) and $R_{1,L}^{B(E2)}$ (for the $n = 1$ band) for $L = 2, 10, 18$, vs. the parameter $\beta_0$. The curves show the evolution from the U(5) symmetry (on the left) to the O(6) limit of IBM with large boson numbers (on the right). See subsections 2.2 and 2.5 for further discussion.

**Fig. 3** (Color online) Energy ratios $R_L$ [Eq. (14)] (a) and intraband B(E2) ratios $R_{0,L}^{B(E2)}$ [Eq. (18)] (b) for the ground state band, as well as intraband B(E2) ratios $R_{1,L}^{B(E2)}$ [Eq. (18)] (c) for the $n = 1$ band, obtained through the variational procedure (labeled by “var”) using Davidson potentials in the E(5) framework, compared to the values provided by the U(5), O(6), E(5), and E(5)-$\beta^4$ models. See subsections 2.3 and 2.5 for further details.

**Fig. 4** (Color online) Davidson potentials (Eq. (1)) obtained for different angular momenta $L$ through the application of the variational procedure to energy ratios within the ground state band in the E(5) framework. The $\beta_0$ values corresponding to these potentials are listed in Table 2. See subsection 2.3 for further discussion.

**Fig. 5** (Color online) $R_L$ energy ratios (Eq. (14)) for the ground state band (for $L = 4, 12, 20$) and their derivatives $dR_L/d\beta_0$ vs. the parameter $\beta_0$, calculated using Davidson potentials (Eq. (1)) in the X(5) framework. The $R_L$ curves demonstrate the evolution from the X(5)-$\beta^2$ symmetry (on the left) to the SU(3) limit of IBM with large boson numbers (on the right). See subsections 3.2 and 3.3 for further details.

**Fig. 6** (Color online) (a) Intraband B(E2) ratios (Eq. (17)) $R_{0,L}^{B(E2)}$ (for the ground state band) and $R_{1,L}^{B(E2)}$ (for the $n = 1, K = 0$ band) for $L = 2, 10, 18$, vs. the parameter $\beta_0$. (b) Interband B(E2) ratios $R_{1,L,0,L}^{B(E2)}$ (Eq. (43)) vs. $\beta_0$. (c) Quadrupole moment ratios (Eq. (45)) $R_{0,L}^{Q}$ (for the ground state band) and $R_{1,L}^{Q}$ (for the $n = 1, K = 0$ band) for $L = 4, 12, 20$, vs. $\beta_0$. In all cases the curves show the evolution from the X(5)-$\beta^2$ model (on the left) to the SU(3) limit of IBM with large boson numbers (on the right). See subsections 3.2, 3.5, and 3.6 for further discussion.

**Fig. 7** (Color online) Energy ratios $R_L$ [Eq. (14)] (a), intraband B(E2) ratios $R_{0,L}^{B(E2)}$ [Eq. (18)] (c), and quadrupole moment ratios $R_{0,L}^{Q}$ [Eq. (46)] (e) for the ground state band, as well as intraband B(E2) ratios $R_{1,L}^{B(E2)}$ [Eq. (18)] (d) and quadrupole moment ratios $R_{1,L}^{Q}$ [Eq. (46)] (f) for the $n = 1, K = 0$ band, and energy ratios $R_{0,L}^{'}$ [Eq. (39)] (b) for the $n = 0, K = 2$ band obtained through the variational procedure (labeled by “var”) using Davidson potentials in the X(5) framework, compared to the values provided by the U(5), SU(3), X(5), X(5)-$\beta^2$ and X(5)-$\beta^4$ models. See subsections 3.3, 3.5 and 3.6 for further details.
Fig. 8 (Color online) Davidson potentials (Eq. (1)) obtained for different angular momenta $L$ through the application of the variational procedure to energy ratios within the ground state band in the $X(5)$ framework. The $\beta_0$ values corresponding to these potentials are listed in Table 5. See subsection 3.3 for further discussion.
Table 1: $R_L$ energy ratios (Eq. (14)) and $R_{0,L-2}^{E2}$ intraband B(E2) ratios (Eq. (18)) for the ground state band, as well as $R_{1,L-2}^{E2}$ intraband B(E2) ratios (Eq. (18)) for the $n = 1$ band of the Davidson potentials in the E(5) framework for different values of the parameter $\beta_0$, compared to the O(6) results for large boson numbers. See subsection 2.2 for further details.

| $L$ | $R_L$ | $R_L$ | $R_L$ | $R_{0,L-2}^{E2}$ | $R_{0,L-2}^{E2}$ | $R_{0,L-2}^{E2}$ | $R_{1,L-2}^{E2}$ | $R_{1,L-2}^{E2}$ | $R_{1,L-2}^{E2}$ |
|-----|-------|-------|-------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|
|     | $\beta_0 = 4.$ | $\beta_0 = 8.$ | O(6) | $\beta_0 = 4.$ | $\beta_0 = 8.$ | O(6) | $\beta_0 = 4.$ | $\beta_0 = 8.$ | O(6) |
| 2   | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 |
| 4   | 2.486 | 2.499 | 2.500 | 1.441 | 1.429 | 1.429 | 1.439 | 1.429 | 1.429 |
| 6   | 4.441 | 4.496 | 4.500 | 1.702 | 1.669 | 1.667 | 1.697 | 1.669 | 1.667 |
| 8   | 6.846 | 6.990 | 7.000 | 1.885 | 1.823 | 1.818 | 1.876 | 1.823 | 1.818 |
| 10  | 9.677 | 9.978 | 10.000 | 2.030 | 1.930 | 1.923 | 2.015 | 1.930 | 1.923 |
| 12  | 12.909 | 13.459 | 13.500 | 2.154 | 2.011 | 2.000 | 2.133 | 2.010 | 2.000 |
| 14  | 16.515 | 17.430 | 17.500 | 2.268 | 2.074 | 2.059 | 2.240 | 2.073 | 2.059 |
| 16  | 20.469 | 21.888 | 22.000 | 2.376 | 2.125 | 2.105 | 2.341 | 2.124 | 2.105 |
| 18  | 24.743 | 26.831 | 27.000 | 2.482 | 2.167 | 2.143 | 2.438 | 2.166 | 2.143 |
| 20  | 29.311 | 32.254 | 32.500 | 2.587 | 2.204 | 2.174 | 2.535 | 2.203 | 2.174 |

Table 2: Parameter values $\beta_{0,m}$ where the absolute value of the first derivative of a physical quantity has a maximum, while the second derivative vanishes, for the $R_L$ energy ratios (Eq. (14)), the $R_{0,L-2}^{E2}$ intraband B(E2) ratios (Eq. (18)), and the $E_{0,L,exc}$ excitation energies (Eq. (9)) of the ground state band, as well as for the $R_{1,L-2}^{E2}$ intraband B(E2) ratios (Eq. (18)) for the $n = 1$ band of the Davidson potentials in the E(5) framework, together with the corresponding values of each physical quantity (labeled by “var”), which are compared to appropriate E(5), E(5)-$\beta^4$, or U(5) results. In the case of $E_{0,L,exc}$, the corresponding $R_L$ ratios (Eq. (14)) are also shown. See subsections 2.3, 2.4, and 2.5 for further details.

| $L$ | $\beta_{0,m}$ | $R_L$ | $R_L$ | $\beta_{0,m}$ | $R_{0,L-2}^{E2}$ | $R_{0,L-2}^{E2}$ | $R_{0,L-2}^{E2}$ | $\beta_{0,m}$ | $R_{1,L-2}^{E2}$ | $R_{1,L-2}^{E2}$ | $R_{1,L-2}^{E2}$ |
|-----|----------------|-------|-------|----------------|-----------------|-----------------|-----------------|----------------|-----------------|-----------------|-----------------|
|     | var E(5)       |       |       | var E(5)-$\beta^4$ | var E(5)-$\beta^4$ | var E(5)-$\beta^4$ | var E(5)-$\beta^4$ |       |       |       |       |
| 4   | 1.421 | 2.185 | 2.199 | 1.213 | 1.801 | 1.832 | 1.124 | 1.697 | 1.757 |
| 6   | 1.522 | 3.549 | 3.590 | 1.266 | 2.538 | 2.564 | 1.184 | 2.300 | 2.406 |
| 8   | 1.609 | 5.086 | 5.169 | 1.299 | 3.252 | 3.227 | 1.224 | 2.868 | 2.990 |
| 10  | 1.687 | 6.793 | 6.934 | 1.321 | 3.960 | 3.841 | 1.250 | 3.421 | 3.529 |
| 12  | 1.759 | 8.667 | 8.881 | 1.336 | 4.667 | 4.417 | 1.269 | 3.967 | 4.035 |
| 14  | 1.825 | 10.705 | 11.009 | 1.347 | 5.375 | 4.961 | 1.283 | 4.510 | 4.513 |
| 16  | 1.888 | 12.906 | 13.316 | 1.355 | 6.085 | 5.480 | 1.293 | 5.051 | 4.969 |
| 18  | 1.947 | 15.269 | 15.799 | 1.361 | 6.798 | 5.978 | 1.300 | 5.593 | 5.407 |
| 20  | 2.004 | 17.793 | 18.459 | 1.365 | 7.511 | 6.457 | 1.306 | 6.135 | 5.828 |
Table 2: (continued)

| $L$ | $\beta_{0,m}$ | $E_{0,L,exc}$ | $R_L$ (var) | $R_L$ (U(5)) |
|-----|---------------|---------------|-------------|--------------|
| 2   | 1.399         | 0.709         | 1.000       | 1.000        |
| 4   | 1.538         | 1.423         | 2.008       | 2.000        |
| 6   | 1.658         | 2.142         | 3.021       | 3.000        |
| 8   | 1.766         | 2.862         | 4.037       | 4.000        |
| 10  | 1.866         | 3.583         | 5.054       | 5.000        |
| 12  | 1.960         | 4.303         | 6.070       | 6.000        |
| 14  | 2.049         | 5.022         | 7.084       | 7.000        |
| 16  | 2.135         | 5.739         | 8.096       | 8.000        |
| 18  | 2.217         | 6.455         | 9.105       | 9.000        |
| 20  | 2.297         | 7.168         | 10.111      | 10.000       |

Table 3: $R_L$ energy ratios (Eq. (14)), $R_{0,L-2}^{B(E2)}$ intraband B(E2) ratios (Eq. (18)), and $R_{0,L}^Q$ quadrupole moment ratios (Eq. (46)) for the ground state band, as well as $R_{1,L-2}^{B(E2)}$ intraband B(E2) ratios (Eq. (18)) and $R_{1,L}^Q$ quadrupole moment ratios (Eq. (46)) for the $n = 1$, $K = 0$ band, and $R_{0,L}^U$ energy ratios (Eq. (39)) for the $n = 0$, $K = 2$ band of the Davidson potentials in the X(5) framework for different values of the parameter $\beta_0$, compared to the SU(3) results at large boson numbers. In addition, $R_{1,L,0,L}^{B(E2)}$ interband B(E2) ratios (Eq. (43)) between the $n = 1$, $K = 0$ band and the ground state band are given. See subsection 3.2 for further details.

| $L$ | $R_L$ ($\beta_0 = 4.$) | $R_L$ ($\beta_0 = 8.$) | $R_L$ (SU(3)) | $R_{0,L-2}^{B(E2)}$ ($\beta_0 = 4.$) | $R_{0,L-2}^{B(E2)}$ ($\beta_0 = 8.$) | $R_{0,L-2}^{B(E2)}$ (SU(3)) | $R_{1,L-2}^{B(E2)}$ ($\beta_0 = 4.$) | $R_{1,L-2}^{B(E2)}$ ($\beta_0 = 8.$) | $R_{1,L-2}^{B(E2)}$ (SU(3)) |
|-----|-------------------------|-------------------------|--------------|----------------------------------|----------------------------------|--------------------------|---------------------------------|----------------------------------|--------------------------|
| 2   | 1.000                   | 1.000                   | 1.000       | 1.000                            | 1.000                             | 1.000                    | 1.000                           | 1.000                            | 1.000                    |
| 4   | 3.318                   | 3.333                   | 3.333       | 1.437                            | 1.429                             | 1.429                    | 1.436                           | 1.429                             | 1.429                    |
| 6   | 6.921                   | 6.995                   | 7.000       | 1.599                            | 1.575                             | 1.573                    | 1.595                           | 1.575                             | 1.573                    |
| 8   | 11.756                  | 11.984                  | 12.000      | 1.699                            | 1.651                             | 1.647                    | 1.690                           | 1.650                             | 1.647                    |
| 10  | 17.759                  | 18.295                  | 18.333      | 1.777                            | 1.698                             | 1.692                    | 1.764                           | 1.697                             | 1.692                    |
| 12  | 24.856                  | 25.921                  | 26.000      | 1.849                            | 1.731                             | 1.722                    | 1.829                           | 1.730                             | 1.722                    |
| 14  | 32.967                  | 34.856                  | 35.000      | 1.919                            | 1.756                             | 1.743                    | 1.892                           | 1.755                             | 1.743                    |
| 16  | 42.011                  | 45.091                  | 45.333      | 1.991                            | 1.776                             | 1.760                    | 1.957                           | 1.775                             | 1.760                    |
| 18  | 51.904                  | 56.616                  | 57.000      | 2.066                            | 1.793                             | 1.772                    | 2.023                           | 1.793                             | 1.772                    |
| 20  | 62.570                  | 69.421                  | 70.000      | 2.143                            | 1.809                             | 1.782                    | 2.092                           | 1.808                             | 1.782                    |
Table 3: (continued)

| $L$ | $L'$ | $R_{1,L,0,L'}^{B(2)}$ | $100R_{1,L,0,L'}^{B(2)}$ | $\beta_0 = 4.$ | $\beta_0 = 8.$ | $\beta_0 = 4.$ | $\beta_0 = 8.$ | SU(3) | $\beta_0 = 4.$ | $\beta_0 = 8.$ | SU(3) |
|-----|-----|---------------------|------------------------|----------------|----------------|----------------|----------------|--------|----------------|----------------|--------|
| 0   | 2   | 8.261  | 1.983     | 2   | 1.000  | 1.000  | 1.000  | 1.000  | 1.000  | 1.000  |
| 2   | 0   | 1.284  | 0.373     | 4   | 1.278  | 1.273  | 1.273  | 1.278  | 1.278  | 1.278  |
| 2   | 2   | 2.089  | 0.549     | 6   | 1.415  | 1.401  | 1.400  | 1.414  | 1.401  | 1.400  |
| 2   | 4   | 4.940  | 1.062     | 8   | 1.503  | 1.476  | 1.474  | 1.500  | 1.476  | 1.474  |
| 4   | 2   | 1.524  | 0.510     | 10  | 1.568  | 1.525  | 1.522  | 1.564  | 1.525  | 1.522  |
| 4   | 4   | 1.899  | 0.499     | 12  | 1.622  | 1.560  | 1.556  | 1.617  | 1.560  | 1.556  |
| 4   | 6   | 5.006  | 0.977     | 14  | 1.671  | 1.587  | 1.581  | 1.664  | 1.587  | 1.581  |
| 6   | 4   | 1.377  | 0.537     | 16  | 1.716  | 1.608  | 1.600  | 1.707  | 1.608  | 1.600  |
| 6   | 6   | 1.861  | 0.489     | 18  | 1.760  | 1.626  | 1.615  | 1.749  | 1.626  | 1.615  |
| 6   | 8   | 5.365  | 0.963     | 20  | 1.803  | 1.641  | 1.628  | 1.789  | 1.641  | 1.628  |

Table 3: (continued)

| $L$ | $R_{0,L}'$ | $R_{0,L}'$ | $R_{0,L}'$ | SU(3) |
|-----|------------|------------|------------|--------|
| $\beta_0 = 4.$ | $\beta_0 = 8.$ | SU(3) |
| 3   | 1.000      | 1.000      | 1.000      |        |
| 4   | 2.327      | 2.333      | 2.333      |        |
| 5   | 3.977      | 3.999      | 4.000      |        |
| 6   | 5.944      | 5.996      | 6.000      |        |
| 7   | 8.219      | 8.326      | 8.333      |        |
| 8   | 10.797     | 10.987     | 11.000     |        |
| 9   | 13.667     | 13.978     | 14.000     |        |
| 10  | 16.821     | 17.299     | 17.333     |        |

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Table 4: Quadrupole moments (Eq. (44)) of the X(5)-$\beta^4$, X(5)-$\beta^6$, and X(5)-$\beta^8$ models, compared to the predictions of the X(5) and X(5)-$\beta^2$ models for some $K = 0$ bands. See subsection 3.1 for details.

| band | $L$ | X(5)-$\beta^2$ | X(5)-$\beta^4$ | X(5)-$\beta^6$ | X(5)-$\beta^8$ | X(5) |
|------|-----|----------------|----------------|----------------|----------------|------|
|      | $s = 1$ |                |                |                |                |      |
| 0    | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 |
| 2    | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 |
| 4    | 1.466 | 1.413 | 1.391 | 1.380 | 1.358 |      |
| 6    | 1.823 | 1.699 | 1.648 | 1.622 | 1.572 |      |
| 8    | 2.127 | 1.925 | 1.842 | 1.800 | 1.719 |      |
| 10   | 2.395 | 2.114 | 2.000 | 1.941 | 1.828 |      |
| 12   | 2.638 | 2.277 | 2.132 | 2.058 | 1.913 |      |
| 14   | 2.862 | 2.422 | 2.247 | 2.158 | 1.982 |      |
| 16   | 3.070 | 2.552 | 2.348 | 2.245 | 2.038 |      |
| 18   | 3.266 | 2.671 | 2.439 | 2.322 | 2.086 |      |
| 20   | 3.450 | 2.781 | 2.522 | 2.391 | 2.127 |      |
| 22   | 3.626 | 2.884 | 2.598 | 2.454 | 2.162 |      |
| 24   | 3.794 | 2.980 | 2.669 | 2.512 | 2.193 |      |
| 26   | 3.954 | 3.070 | 2.734 | 2.565 | 2.221 |      |
| 28   | 4.109 | 3.155 | 2.795 | 2.615 | 2.245 |      |
| 30   | 4.258 | 3.236 | 2.853 | 2.661 | 2.268 |      |
|      | $s = 2$ |                |                |                |                |      |
| 0    | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 |
| 2    | 1.245 | 1.107 | 1.034 | 0.991 | 0.894 |      |
| 4    | 1.742 | 1.521 | 1.411 | 1.346 | 1.197 |      |
| 6    | 2.095 | 1.796 | 1.655 | 1.573 | 1.379 |      |
| 8    | 2.387 | 2.012 | 1.840 | 1.743 | 1.508 |      |
| 10   | 2.643 | 2.191 | 1.992 | 1.880 | 1.608 |      |
| 12   | 2.875 | 2.347 | 2.120 | 1.995 | 1.689 |      |
| 14   | 3.088 | 2.486 | 2.233 | 2.095 | 1.756 |      |
| 16   | 3.287 | 2.612 | 2.333 | 2.182 | 1.813 |      |
| 18   | 3.473 | 2.727 | 2.423 | 2.260 | 1.863 |      |
| 20   | 3.651 | 2.833 | 2.505 | 2.331 | 1.906 |      |
Table 5: Parameter values $\beta_{0,m}$ where the absolute value of the first derivative of a physical quantity has a maximum, while the second derivative vanishes, for the $R_L$ energy ratios (Eq. (14)), the $R^{B(E2)}_{0,L-2}$ intraband B(E2) ratios (Eq. (18)), the $E^{(0)}_{0,L,exc}$ excitation energies (Eq. (32)), and the $R^{Q}_{0,L}$ quadrupole moment ratios (Eq. (46)) of the ground state band, as well as for the $R^{B(E2)}_{1,L-2}$ intraband B(E2) ratios (Eq. (18)) and the $R^{Q}_{1,L}$ quadrupole moment ratios (Eq. (46)) of the $n = 1, K = 0$ band, and for the $R^{B(E2)}_{1,L}$ energy ratios (Eq. (39)) of the $n = 0, K = 2$ band of the Davidson potentials in the X(5) framework, together with the corresponding values of each physical quantity (labeled by “var”), which are compared to appropriate X(5), X(5)-$\beta_2$, or X(5)-$\beta_4$ results. In the case of $E_{0,L,exc}$, the corresponding $R_L$ ratios (Eq. (14)) are also shown. See subsections 3.3, 3.4, 3.5, and 3.6 for further details.

| $L$ | $\beta_{0,m}$ | $R_L$ | $R_L$ var | $R^{B(E2)}_{0,L-2}$ | $R^{B(E2)}_{0,L-2}$ var | $R^{B(E2)}_{1,L-2}$ | $R^{B(E2)}_{1,L-2}$ var | $\beta_{0,m}$ | $R^{Q}_{0,L}$ | $R^{Q}_{0,L}$ var | $R^{Q}_{1,L}$ | $R^{Q}_{1,L}$ var |
|-----|----------------|--------|-----------|----------------------|------------------------|----------------------|------------------------|----------------|----------------|------------------|----------------|------------------|
| 4   | 1.334          | 2.901  | 2.904     | 1.234                | 1.658                  | 1.690                | 1.059                  | 1.517          | 1.539          |
| 6   | 1.445          | 5.419  | 5.430     | 1.275                | 2.210                  | 2.262                | 1.206                  | 1.892          | 1.966          |
| 8   | 1.543          | 8.454  | 8.483     | 1.312                | 2.771                  | 2.799                | 1.289                  | 2.264          | 2.382          |
| 10  | 1.631          | 11.964 | 12.027    | 1.337                | 3.346                  | 3.305                | 1.345                  | 2.641          | 2.781          |
| 12  | 1.711          | 15.926 | 16.041    | 1.355                | 3.933                  | 3.783                | 1.383                  | 3.024          | 3.162          |
| 14  | 1.785          | 20.330 | 20.514    | 1.367                | 4.530                  | 4.237                | 1.411                  | 3.413          | 3.527          |
| 16  | 1.855          | 25.170 | 25.437    | 1.375                | 5.135                  | 4.671                | 1.431                  | 3.807          | 3.877          |
| 18  | 1.922          | 30.442 | 30.804    | 1.381                | 5.745                  | 5.087                | 1.446                  | 4.205          | 4.214          |
| 20  | 1.985          | 36.146 | 36.611    | 1.386                | 6.361                  | 5.489                | 1.458                  | 4.606          | 4.539          |

Table 5: (continued)

| $L$ | $\beta_{0,m}$ | $E^{(0)}_{0,L,exc}$ | $R_L$ | $R_L$ var | $R^{Q}_{0,L}$ | $R^{Q}_{0,L}$ var | $R^{Q}_{1,L}$ | $R^{Q}_{1,L}$ var |
|-----|----------------|---------------------|--------|-----------|----------------|-------------------|----------------|------------------|
| 2   | 1.329          | 0.397               | 1.000  | 1.000     | 1.401          | 1.395             | 1.413          | 1.354            |
| 4   | 1.470          | 1.055               | 2.655  | 2.646     | 1.472          | 1.670             | 1.699          | 1.582            |
| 6   | 1.604          | 1.804               | 4.540  | 4.507     | 1.524          | 1.894             | 1.925          | 1.818            |
| 8   | 1.726          | 2.591               | 6.517  | 6.453     | 1.562          | 2.090             | 2.114          | 1.979            |
| 10  | 1.840          | 3.394               | 8.539  | 8.438     | 1.591          | 2.267             | 2.277          | 2.120            |
| 12  | 1.947          | 4.206               | 10.582 | 10.445    | 1.613          | 2.431             | 2.422          | 2.246            |
| 14  | 2.049          | 5.022               | 12.634 | 12.465    | 1.630          | 2.584             | 2.552          | 2.360            |
| 16  | 2.146          | 5.839               | 14.690 | 14.494    | 1.644          | 2.730             | 2.671          | 2.463            |
| 18  | 2.240          | 6.656               | 16.745 | 16.529    | 1.654          | 2.868             | 2.781          | 2.559            |
| 20  | 2.330          | 7.472               | 18.798 | 18.568    | 1.654          | 2.868             | 2.781          | 2.559            |
Table 5: (continued)

| $L$ | $\beta_{0,m}$ | $R'_{0,L}$ | $R''_{0,L}$ |
|-----|---------------|------------|-------------|
|     |               | var        | X(5)        |
| 4   | 1.339         | 2.157      | 2.163       |
| 5   | 1.404         | 3.454      | 3.472       |
| 6   | 1.463         | 4.881      | 4.919       |
| 7   | 1.517         | 6.431      | 6.497       |
| 8   | 1.567         | 8.101      | 8.205       |
| 9   | 1.614         | 9.886      | 10.037      |
| 10  | 1.658         | 11.786     | 11.994      |
intraband $B(E2)$ ratios $R_{n,L}^{B(E2)}$

- $4_0 \rightarrow 2_0$
- $12_0 \rightarrow 10_0$
- $20_0 \rightarrow 18_0$
- $4_1 \rightarrow 2_1$
- $12_1 \rightarrow 10_1$
- $20_1 \rightarrow 18_1$
Energy ratios $R_L$ vs. angular momentum $L$:

- U(5)
- O(6)
- E(5)
- var
(b) intraband $B(E2)$ ratios $R_{0,L}^{(E2)}$ vs angular momentum $L$
intraband $B(E^2)$ ratios $R_{1,L}$

$B(E^2)$

$U(5)$

$E(5)$

$E(5)-\beta^4$

$\text{var}$

$O(6)$

angular momentum $L$

(c)
The graph shows the function $u(\beta)$ for different values of $L$. The curves are labeled with $L=4$, $L=8$, $L=12$, $L=16$, and $L=20$. The $\beta$ axis ranges from 1 to 3.5, and the $u(\beta)$ axis ranges from 0 to 16.
The figure shows the energy ratios $R_L$ and their derivatives $dR_L/d\beta_0$ as functions of $\beta_0$ for different values of $L$:

- **Energy Ratios $R_L$:**
  - $L=4$: The curve is nearly flat and horizontal.
  - $L=12$: The curve is slightly higher than $L=4$ and shows a slight increase as $\beta_0$ increases.
  - $L=20$: The curve is the highest among the three and shows a significant increase as $\beta_0$ increases.

- **Derivatives $dR_L/d\beta_0$:**
  - $L=4$: The derivative is very low and nearly zero.
  - $L=12$: The derivative is slightly higher but still relatively low.
  - $L=20$: The derivative is the highest, showing a sharp peak at a certain value of $\beta_0$.
The intraband B(E2) ratios $R_{n,L}$ are plotted as a function of $\beta_0$. The graph shows the following transitions:

- $4_0 \rightarrow 2_0$
- $12_0 \rightarrow 10_0$
- $20_0 \rightarrow 18_0$
- $4_1 \rightarrow 2_1$
- $12_1 \rightarrow 10_1$
- $20_1 \rightarrow 18_1$
interband $B(E2)$ ratios $R_{n,L,n',L'}$

- $0_1 \rightarrow 2_0$
- $2_1 \rightarrow 4_0$
- $2_1 \rightarrow 2_0$

(b)
quadrupole moment ratios $R_{n,L}$
Energy ratios $R_L$ as a function of angular momentum $L$. The graph shows the following trends:

- **U(5)**: Represented by black squares.
- **SU(3)**: Represented by red circles.
- **X(5)**: Represented by green triangles.
- **var**: Represented by blue triangles.
- **X(5)-$\beta^2$**: Represented by cyan diamonds.

The graph is labeled as (a).
energy ratios $R_{0,L}^i$

angular momentum $L$

(b)
Intraband $B(E2)$ ratios $R_{0,L}$

- $X(5) - \beta^2$
- $X(5) - \beta^4$
- $X(5)$
- $U(5)$
- $SU(3)$
- $\text{var}$

Angular momentum $L$

(c)
intraband $B(E2)$ ratios $R_{1,L}$

- $X(5)-\beta^2$
- $X(5)-\beta^4$
- $X(5)$
- SU(3)
- var
- U(5)

(d)

angular momentum L
quadrupole moment ratios $R_{0,L}$ vs. angular momentum $L$. The plot shows different curves for $X(5) - \beta^2$, $X(5) - \beta^4$, $X(5)$, $SU(3)$, and a variable labeled 'var'.
The diagram shows the quadrupole moment ratios $R_{1,L}$ as a function of the angular momentum $L$. The plot includes four different trends labeled as $X(5)-\beta^2$, $X(5)-\beta^4$, $X(5)$, and $SU(3)$, along with a variable trend marked as "var". The x-axis represents the angular momentum $L$, while the y-axis shows the quadrupole moment ratios $R_{1,L}$. The graph indicates a general increasing trend with $L$. The label "(f)" is placed at the bottom right corner of the diagram.
