Hawking–Moss tunneling in non-commutative eternal inflation

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Abstract. The quantum behavior of non-commutative eternal inflation is quite different from the usual scenario. Unlike the usual eternal inflation, non-commutative eternal inflation has quantum fluctuation suppressed by the Hubble parameter. Because of this, we need to reconsider many conceptions of eternal inflation. In this paper we study the Hawking–Moss tunneling in non-commutative eternal inflation using the stochastic approach. We obtain a brand new form of tunneling probability for this process and find that the Hawking–Moss tunneling is more unlikely to take place in the non-commutative case than in the usual one. We also conclude that the lifetime of a metastable de Sitter vacuum in the non-commutative spacetime is longer than that in the commutative case.

Keywords: string theory and cosmology, inflation, physics of the early universe
Hawking–Moss tunneling in non-commutative eternal inflation

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1. Introduction

Inflation has been widely considered as a remarkably successful theory in explaining many problems in the very early universe, such as the flatness, horizon and monopole problems [1]–[5]. During inflation, quantum effects play a crucial role and may bring the universe into a self-reproducing process which is dubbed ‘eternal inflation’ [6]–[8]. The string theory landscape indicates that there are a huge number of metastable vacua surrounded by various kinds of effective potentials [9]–[12]. The realization of a string landscape provides an important arena for eternal inflation.

In eternal inflation driven by the false vacuum, the false vacuum is a metastable state and would decay through a mix of semiclassical tunneling and stochastic evolution. The probability of finding the inflaton at the top of the plateau in its potential decreases exponentially with time [13]. However, the false vacuum is also expanding exponentially while decaying. When the rate of exponential expansion is larger than the decay rate during this process, the total volume of the false vacuum will grow eternally although the false vacuum is decaying. In this case, bubbles form by random nucleation and then start to expand. Every growing bubble can be viewed as an open FRW universe [14], and we are living in one such ‘pocket universe’ [15]. Another approach to eternal inflation is achieved in chaotic inflation when the quantum fluctuation of the inflaton dominates over its classical motion. As the inflaton is rolling down the potential classically, its change during one Hubble time ($\delta t = 1/H$) can be divided into $\delta \varphi = \Delta \varphi + \delta_q \varphi$, where $\Delta \varphi$ denotes the classical value and $\delta_q \varphi$ represents the quantum one. For a Gaussian probability distribution, when $\delta_q \varphi > 0.61 \Delta \varphi$, the quantum behavior overwhelms the classical evolution and inflation becomes eternal.

When dealing with the decaying process of false vacua, one has to make the inflaton tunnel from one false vacuum to another. One method was provided by Coleman and De Luccia (CDL) [14]; another method was investigated by Hawking and Moss [16]. During the Hawking–Moss tunneling, the potential between two vacua is so flat that a CDL instanton cannot exist. It was shown that the probability of tunneling from one false vacuum $\varphi_0$ to another is given by

$$P_C \sim \exp \left( -\frac{24\pi^2}{V(\varphi_0)} + \frac{24\pi^2}{V(\varphi_{\text{top}})} \right)$$

$$\sim \exp \left( -8\pi^2 \frac{H(\varphi_{\text{top}})^2 - H(\varphi_0)^2}{H(\varphi_0)^2H(\varphi_{\text{top}})^2} \right).$$

(1)
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which is related to the value of the top barrier. A proper scenario of this tunneling can be
realized in the stochastic approach to inflation [17]–[22]. Here the potential of the scalar
field is flat enough for slow rolling which requires $V'' < V$. In the stochastic description,
the quantum fluctuations can be simulated by stochastic noise and random scalar field
walks.

As is known, eternal inflation happens when the energy scale of the universe is
extremely high. Thus we would like to take into consideration a more fundamental theory
in logic, namely, the string theory. Recently, we have considered the effects of spacetime
non-commutativity on slow roll eternal inflation in [23]. In non-commutative inflation, it
is generally assumed that the background evolution of inflaton is not modified but the
fluctuations are affected by non-commutativity (see [24]–[27], and for a review see [28] and
references therein). In order to introduce the non-commutativity [29,30] into the four-
dimensional flat Friedmann–Robertson–Walker universe, we would like to define another
time coordinate $\tau$,

$$ds^2 = dt^2 - a^2(t) dx^2 = a^{-2}(\tau) d\tau^2 - a^2(\tau) dx^2,$$

(2)

where $a$ is the scale factor. Then the spacetime uncertainty relation can be realized via
the commutation relation

$$[\tau, x]_* = iM_N^{-2},$$

(3)

where $M_N$ is the energy scale of non-commutativity and the $\ast$-product is defined as

$$(f \ast g)(x, \tau) = \exp \left(-\frac{i}{2}M_N^{-2}(\partial_x \partial_{\tau'} - \partial_{\tau} \partial_y)\right)f(x, \tau)g(y, \tau') |_{y=x, \tau'=\tau}. $$

(4)

From the result of the paper [23], we can see that the quantum fluctuation still satisfies
the Gaussian distribution, but the form of its amplitude in the IR region is changed to
$\delta_q \varphi \simeq (1/2\pi)(M_N^2/H)$. Therefore, when the Hubble parameter is lifted high enough,
eternal inflation will cease. This is strongly different from the normal scenario of eternal
inflation. In this paper, we shall use the analysis of non-commutativity mentioned above
(especially the IR region) and the stochastic approach to study Hawking–Moss tunneling
of eternal inflation.

This paper is organized as follows. In section 2, we study the Hawking–Moss tunneling
in non-commutative eternal inflation using the stochastic approach. In section 3, we draw
the conclusions that Hawking–Moss tunneling is more unlikely to happen in the non-
commutative case than in the usual one and that the lifetime of a metastable de Sitter
vacuum in the non-commutative spacetime is longer than that in the commutative case.

2. Stochastic approach to non-commutative inflation

One can describe inflation by analyzing the stochastic probability distribution $P(\varphi, t)$,
which represents the probability of finding the inflaton field $\varphi$ at the time $t$. In our
paper we consider the probability distribution averaged in a Hubble volume observed by
a comoving observer. The inflaton evolves as a Brownian particle. Consequently, the
probability distribution $P(\varphi, t)$ satisfies the Fokker–Planck (FP) equation (see a detailed
introduction in [20,31]):
\[
\frac{\partial P}{\partial t} = \frac{\partial}{\partial \varphi} \left( \frac{\partial (DP)}{\partial \varphi} + \gamma \frac{dV}{d\varphi} P \right),
\]
where \( D \) is the diffusion coefficient and \( \gamma \) is the mobility coefficient. A general form of the FP equation is given by
\[
\frac{\partial P}{\partial t} = \frac{\partial}{\partial \varphi} \left( D^{1-d} \frac{\partial (D^d P)}{\partial \varphi} + \gamma \frac{dV}{d\varphi} P \right).
\]
However, the choice of the parameter \( d \) does not affect the calculation of probability distribution a lot. Therefore we do not plan to discuss it in our paper but only focus on the case \( d = 1 \).

Using the slow roll approximation \( \dot{\varphi} \simeq -V'/(3H) \), we can establish that \( \gamma = 1/3H \). In the following we need to derive the form of the coefficient \( D \).

Following the usual view of inflation, the background evolution of non-commutative inflation can be described by
\[
3H^2 \simeq V(\varphi),
\]
where we take the normalization \( M_p^2 = 1/8\pi G = 1 \). According to the calculation in [23], the IR quantum fluctuation in the momentum space \( \delta_q \varphi_k \) is linked to the canonical perturbation \( u_k \) by \( u_k \simeq a\delta_q \varphi_k \), and when the perturbation begins to be generated the initial conditions require \( u_k \) to be canonically normalized as \( u_k \simeq 1/\sqrt{2k} \) with \( a \simeq Hk/M_N^2 \).

Therefore, the IR quantum fluctuation in momentum space can be generally given by
\[
\delta_q \varphi_k \simeq \frac{1}{\sqrt{2k}} \frac{M_N^2}{Hk}. \tag{7}
\]

After that, the fluctuations outside the horizon are nearly frozen. It can be shown that the initial wavelength for the \( k \) mode is \( \lambda_k = H/M_N^2 \), so it is appropriate to do the spatial averaging at a length scale \( H/M_N^2 \). During one Hubble time, we can calculate the IR quantum fluctuation in coordinate space \( \delta_q \varphi \) as follows:
\[
\delta_q \varphi|_H \equiv \sqrt{\langle \delta_q \varphi^2 \rangle}_H \left( \frac{dk}{k} \frac{k^3}{2\pi^2} \delta_q \varphi_k \delta_q \varphi_{-k} \right)^{1/2} \simeq \frac{1}{2\pi} \frac{M_N^2}{H}. \tag{8}
\]

Note that due to the nearly scale invariance of the spectrum, the result of (8) is not sensitive to the length scale \( H/M_N^2 \) where we do the spatial averaging. We might have chosen the Hubble length \( H^{-1} \) or the non-commutativity scale \( M_N^{-1} \), and the result of the integration in leading order does not change.

Besides, since \( M_N^{-1} \) arises as another important time scale rather than \( H^{-1} \), it is worth calculating the quantum fluctuations during the time \( M_N^{-1} \). We have
\[
\delta_q \varphi|_{M_N} \equiv \sqrt{\langle \delta_q \varphi^2 \rangle}_{M_N} = \left( \int_{k=aM_N^2/H}^{k=\pi M_N^2/H} \frac{dk}{k} \frac{k^3}{2\pi^2} \delta_q \varphi_k \delta_q \varphi_{-k} \right)^{1/2} \simeq \frac{1}{2\pi} \frac{M_N^3}{H^{1/2}}, \tag{9}
\]
which is a bit different from equation (8). We will show later, however, that this is also insensitive to the simulation of the Langevin equation.

We can simulate the quantum fluctuation in the IR region using the Langevin equation which is expressed as

\[
\dot{\phi} = -\frac{V'}{3H} - \frac{M_N^2}{3HV^3/2} \eta.
\]  

Here \( \eta \) is a stochastic noise term added to simulate the quantum fluctuation of the inflaton and we make its form a Gaussian satisfying

\[
\langle \eta(t) \rangle = 0, \quad \langle \eta(t) \eta(t') \rangle = \frac{9}{4\pi^2} \delta(t-t').
\]

This simulation is quite general and very efficient in the IR region of non-commutative inflation no matter what time scale we use. When we consider the quantum fluctuations on one Hubble scale, we can recover

\[
\langle \delta_q \varphi^2 \rangle \approx \frac{M_N^4}{4\pi^2 H^2} \langle \varphi^2 \rangle, \quad \delta_q \varphi = -H \int \frac{M_N^2 \eta}{3H^{3/2}} \, dt; \quad (12)
\]

moreover, when we consider the quantum fluctuations on one non-commutative scale, then we can recover

\[
\langle \delta_q \varphi^2 \rangle \approx \frac{M_N^3}{4\pi^2 H} \langle \varphi^2 \rangle, \quad \delta_q \varphi = -M_N \int \frac{M_N \eta}{3H^{1/2}} \, dt. \quad (13)
\]

Therefore, we have the expression \((d/dt)\langle \varphi^2 \rangle \approx M_N^4/(4\pi^2 H^2)\) both for the Hubble scale and for the non-commutative scale. According to the property of Brownian motion, the diffusion coefficient \(D\) is approximately equal to half of \((d/dt)\langle \varphi^2 \rangle\), and thus we have

\[
D = \frac{M_N^4}{8\pi^2 H^2}.
\]

For simplicity, we study the stationary ansatz of equation (5): \(\partial_t P_N = 0\). Consequently the FP equation (5) in the IR region of the non-commutative case can be solved as

\[
P_N(\varphi, t) \sim \exp \left\{ -\frac{8\pi^2}{3M_N^4} (V - V_0) \right\},
\]

where \(V_0\) (and \(H_0\)) appears from a proper normalization.

Keeping in mind that \(H > M_N\) in the IR region of non-commutative eternal inflation, and comparing the denominator on the exponential of the non-commutative result (14) with the commutative result (1), we conclude that the tunneling probability is more suppressed by the spacetime non-commutativity. This suppression has a clear physical interpretation. Since the spacetime non-commutativity generally suppresses the quantum fluctuation of the inflaton, it should make quantum behavior of the inflaton, such as the Hawking–Moss tunneling, less likely to happen.

We also note that the equations (1) and (14) can be linked smoothly, describing the energy density crossing the non-commutative UV/IR boundary. It is known that when
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Figure 1. A sketch map of Hawking–Moss tunneling from a false vacuum $\varphi_0$ to the true one $\varphi_v$. When the inflaton $\varphi$ lies above the green dot line, its distribution function satisfies equation (14) and the probability of tunneling is suppressed exponentially with respect to $V$; meanwhile, if $\varphi$ is placed below the green line, the form of distribution function returns the usual one (1).

$H < M_N$, the non-commutative inflation is in the UV region, and it is in the IR region when $H > M_N$. The probability distribution of the inflaton in the UV region is described by equation (1) in which the maximal value of the potential is $V \simeq 3H^2 = 3M_N^2$. This is just the minimal value of the potential in the IR region. Consequently the distribution function of non-commutative eternal inflation in the whole parameter space is continuous, and hence, there is no pathology when the field $\varphi$ tunnels through the UV/IR boundary (see figure 1).

In order to make this result more explicit and to investigate the details of the physics, we consider the example $V(\varphi) = \lambda \varphi^4$ when $\varphi$ is near one minimum of the potential. To solve the equations (6) and (10), we define for simplicity $\sigma \equiv \varphi^2$; then we have

$$\dot{\sigma} + \alpha \sigma + \beta \eta = 0, \quad \alpha \equiv 8 \sqrt{\frac{\lambda}{3}}, \quad \beta \equiv \frac{2}{3} M_N^2 \sqrt{\frac{3}{\lambda}}. \quad (15)$$

The solution of (15) can be written as

$$\sigma(t) = \sigma_0 e^{-\alpha t} + \beta e^{-\alpha t} \int_0^t e^{\alpha t_1} \eta(t_1) \, dt_1, \quad (16)$$

where $\sigma_0 \equiv \sigma(0)$ sets the initial condition at $t = 0$.

Following [32], the distribution function for $\sigma$ can also be given by

$$P_N(\sigma_m) \int [d\eta] \, dt \exp \left( -\frac{2}{9} \pi^2 \int_0^\infty dt_1 \eta^2(t_1) \right) \delta (\sigma(t) - \sigma_m), \quad (17)$$

which denotes the number of times the universe arrives at the $\sigma(t) = \sigma_m$ surface during infinite time. By using $\delta(y) = \int (dx/2\pi) e^{ixy}$, and doing Gaussian integration twice, we
obtain the integral
\[ P_N(\sigma_m) \sim \int dt \sqrt{\frac{8\pi \lambda}{3M_N^4(1 - e^{-2\alpha t})}} \exp \left( -\frac{8\pi^2 \lambda}{3M_N^4} \sigma_m^2 \left( e^{\alpha t} - \sigma_0 / \sigma_m \right)^2 \right). \quad (18) \]

This integral seems problematic because there is a divergence when \( t \to \infty \). However, note that the energy scale of eternal inflation along a comoving world line will eventually drop. As \( t \) becomes larger, inflation enters the UV region, and the behavior of evolution returns to the commutative case. Consequently, the full integral does not suffer from the divergence, so this measure is well defined.

Further, what we care about is the tunneling probability which corresponds to the case \( \sigma_m > \sigma_0 \). By using the saddle point approximation on the exponential as in [32], we obtain
\[ P_N(\sigma_0) \sim \exp \left( -8\pi^2 M_N^{-4} (H_m^2 - H_0^2) \right) \]
\[ \simeq \exp \left\{ -\frac{8\pi^2}{3M_N^4} (V_m - V_0) \right\}, \quad (19) \]
which is consistent with equation (14).

3. Conclusion and discussion

To take the discussion further, we would like to compare the difference of dS decaying processes, whether or not the spacetime non-commutativity is present. According to the work of Coleman and De Luccia [14], the decay time of a metastable dS vacuum has an approximate expression \( T \sim P^{-1} \). By neglecting all the sub-exponential factors, we have
\[ T_C = \exp \left\{ 24\pi^2 M_p^4 \left( \frac{1}{V_0} - \frac{1}{V_{\text{top}}} \right) \right\}; \quad (20) \]
\[ T_N = \exp \left\{ \frac{8\pi^2}{3M_N^4} (V_{\text{top}} - V_0) \right\}, \quad (21) \]
which represent the dS decay times without the spacetime non-commutativity and the IR region with non-commutativity respectively. To be clear, we have written \( M_p \) explicitly here. It is clear that \( T_N > T_C \), which is a very general result indicating that the lifetime of a metastable dS vacuum with non-commutativity is longer than that without non-commutativity.

To summarize, from the results obtained in this paper we learn that the Hawking–Moss tunneling effect of non-commutative eternal inflation in the IR region is greatly different from the usual one. Its probability distribution is exponentially suppressed by the top barrier value of the potential and makes Hawking–Moss tunneling more difficult than in the usual case. This is because the quantum fluctuation is suppressed by spacetime non-commutativity. Consequently, we may expect the application of non-commutativity to provide closer insight into the high energy physics of eternal inflation. On the basis of the new form of the probability distribution, we find that the lifetime of a metastable dS vacuum in the non-commutative case is longer than in the usual one. This may leave more clues for investigating the new physics of non-commutativity which is worthy of further study.
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