Simplified convergence proof of Bézier finite elements on D-dimensional simplex

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Abstract
By using a general formalism, we expose a simplified proof of the convergence of the Bézier polynomials attached to a continuous function defined in arbitrary dimensional simplex. We obtain an error estimate that contains the error in approximating by exponential functions. Our new proof is based on the topological Stone-Weierstrass theorem.

1 Introduction
For the numerical simulation of the wall touching kink modes (WTKM) in the thin wall approximation [1] one of the problem is to extend the existing code, designed for the smooth tokamak wall using triangular finite elements, to the case when to the tokamak wall is attached a limiter. In order to reduce the modification in the previously elaborated programme and input data, in the generic cases we are forced to solve the problem of nonconforming finite elements. Despite the construction of the finite elements on the limiter alone does not pose complex problems, on the contact line on the tokamak wall special problems appears due to nonconforming position of the finite element simplexes. In this work we study the convergence rate of Bézier simplicial finite elements, in order to elaborate the mathematical foundations for the forthcoming work intended to elaborate codes with nonconforming finite elements. We give a new, simplest proof of the convergence of Bernstein polynomials toward a $D$ variable continuous function defined on a $D$ dimensional simplex and estimate the speed of convergence.
2 The Bézier polynomials on D dimensional simplex

Our goal is to give a new and simple proof that the Bézier polynomials attached to a continuous function defined on a simplicial finite element, that is an $D$ dimensional simplex, is really an approximant of the function and to estimate the error. We stress that we are not intend to give a new proof of some extension of the Weierstrass theorem on simplexes, rather we are interested in better understanding the approximation mechanism in order to handle the more complex geometries that appears in the case of nonconforming Bézier finite element analysis.

Let a compact set $K \subset \mathbb{R}^D$. Here $\mathbb{R}^D$ is considered with standard real vector space structure and scalar product $\cdot$. We denote by $\mathcal{C}(K)$ the Banach algebra of real continuous functions on $K$ with the topology given by the norm

$$\|f\| := \sup_{x \in K} |f(x)|; \quad f \in \mathcal{C}(K)$$  \hspace{1cm} (1)

Remark: Recall the Stone-Weierstrass theorem [7] that will be used: Let $K$ a compact topological space and $A \subset \mathcal{C}(K)$ a subalgebra of $\mathcal{C}(K)$ with containing constant function. If the subalgebra $A$ contains functions that distinguish every pair of points, that is for each pair $x_1, x_2 \in K$ there exits $f \in A$ such that $f(x_1) \neq f(x_2)$ then $A$ is dense in $\mathcal{C}(K)$ in the norm topology. In the our setting $K$ is a simplex from.

2.1 Definitions and notations

Consider a general $D$ dimensional closed simplex $T$

$$x_i \in T \subset \mathbb{R}^D, 1 \leq i \leq D$$  \hspace{1cm} (2)

Denote by $\mathcal{C}(T)$ the normed space of continuous functions on $T$ with uniform convergence norm

$$\|f\| := \sup_{x \in T} |f(x)|; \quad f \in \mathcal{C}(T)$$  \hspace{1cm} (3)
For a given point \( x \in \mathbf{T} \) we denote by \( s_i(x), \ i = 0, D \) its barycentric coordinates:

\[
x = \sum_{i=0}^{D} s_i(x)x_i
\]

\[
\sum_{i=0}^{D} s_i(x) = 1; \ s_i(x) \geq 0, \ i = 0, D
\]

Consider only the case when the simplex is non degenerated. The correspondence defined in Eqs. (4, 5),

\[
\mathbb{R}^D \ni x \rightarrow (s_0(x), s_2(x), \cdots, s_D(x)) := S(x) \in \mathbb{R}^{D+1}
\]

transform the simplex \( T \) in \( T_0 \), the standard \( D \) dimensional simplex from \( \mathbb{R}^{D+1} \), that is the convex hull of the unit vectors from \( \mathbb{R}^{D+1} \). The map \( S \) is one to one; denote its inverse by \( R \):

\[
\mathbb{R}^{D+1} \ni T_0 \ni (t_0, t_1, \cdots, t_D) \rightarrow \mathbb{R} \ni x = \sum_{i=0}^{D} t_i x_i := R(t_0, t_1, \cdots, t_D) \in \mathbf{T}
\]

\[
\sum_{i=0}^{D} t_i = 1; \ t_i \geq 0, \ i = 0, D
\]

The one dimensional analogue of the Eq. (5) was the key starting point in the probabilistic proof of the classical Weierstrass theorem by S. Bernstein in the case of functions of one variable [2], [3].

In analogy to the single variable case of Bernstein polynomial attached to the 1-simplex \([0, 1]\),

\[
B^n_k(s) = \frac{n!}{k!(n-k)!} s^k (1-s)^{n-k}; \ 0 \leq k \leq n; \ k, n \ \text{integer}
\]

the Bernstein-Bézier polynomials [4], [5], [6] attached to the \( D \) dimensional simplex \( \mathbf{T} \) from (2) are defined as follows. First, we use the notation \( \mathbf{k} \) for the sequence of \( D + 1 \) integers \((k_0, k_1, \cdots, k_D) \in \mathbb{N}^{(D+1)}\) and denote

\[
|\mathbf{k}| := \sum_{j=0}^{D} k_j
\]
We use the standard notation for the Newton multinomial coefficient $s$ with $k \in \mathbb{N} \times (D+1)$
\[
\begin{pmatrix}
 n \\
 (k_0, k_1, \ldots, k_D)
\end{pmatrix} = \begin{pmatrix}
 n \\
 k
\end{pmatrix} := \frac{n!}{\prod_{j=0}^{D} k_j!}
\]

$|k| = n$

With these notations the Bézier polynomials of order $|k| = n$, defined on the simplex $T \subset \mathbb{R}^D$, indexed by $k$, are defined as follows

\[
B^n_k(x) := \begin{pmatrix}
 n \\
 (k)
\end{pmatrix} \prod_{j=0}^{D} [s_j(x)]^{k_j}
\]

$|k| = n; \quad 0 \leq k_j \leq n; \quad j = 0, D; \quad k_j \in \mathbb{N}$ (8)

In the following, we will denote by $M_n$, the set whose elements are sequence of integers with property:

\[
\mathbb{N} \times (D+1) \ni M_n := \{k|k \in \mathbb{N} \times (D+1); \quad |k| = n\}
\]

This subset $M_n$ of sequences appears in the Newton multinomial formula:

\[
\left(\sum_{j=0}^{D} a_j\right)^n = \sum_{k \in M_n} \begin{pmatrix}
 n \\
 k
\end{pmatrix} \prod_{j=0}^{D} [a_j]^{k_j}
\]

(11)

The set of Bernstein-Bézier polynomials of given order $n$, in the case of a real continuous function of $D$ real variables defined on the simplex $T$

\[
\mathbb{R}^D \ni T \ni y \mapsto f(y) \in \mathbb{R}
\]

generate a polynomial $B_n(f; y)$ in $D$ real variables $y = (y_1, y_2, \ldots, y_D)$, that is, by anticipating, an uniform polynomial approximant of the function $f(y)$

\[
B_n(f; y) := \sum_{k \in M_n} f \left( R \left( \frac{k_1}{n}, \frac{k_2}{n}, \ldots, \frac{k_D}{n} \right) \right) B^n_k(y) = \sum_{k \in M_n} f \left( \sum_{j=1}^{D} \frac{k_j}{n} x_j \right) B^n_k(y)
\]

(12)
The previous equations (12), (13) define a sequence linear operators $\hat{B}_n f := B_n(f;\cdot)$, indexed by $n$, on the space $C(T)$

$$\left(\hat{B}_n f\right)(x) = B_n(f;x) \quad (14)$$

The set of points,

$$C_n := \left\{ R \left( \frac{k_1}{n}, \frac{k_2}{n}, \cdots, \frac{k_D}{n} \right) \mid k \in M_n \right\} \subset T \quad (15)$$

are called the set of control points on the simplex $T$. We have

$$C_n = R(M_n/n) \quad (16)$$

### 2.2 Convergence proof and error estimate for exponential functions

We denote by $C(K)$ space of continuous functions on a compact set $K$ in a finite dimensional real vector space, with the uniform convergence topology and denote by $E(K)$ the subspace of $C(K)$ generated by all finite linear combinations of the functions

$$\exp(a \cdot x); \ a \in \mathbb{R}^D \quad (17)$$

We start with a proposition that simplifies the proof.

**Proposition 1** The subspace $E(K)$ contains functions that distinguish every pair of points $b_1, b_2 \in K$.

**Proof.** We have to prove that there exists a function $g \in E(K)$ such that $g(b_1) \neq g(b_2)$. Consider

$$a = b_2 - b_1$$

The function $g(x)$ defined by

$$g(x) := \exp \left[ (b_2 - b_1) \cdot (x - b_1) \right]$$

is also contained in $E(K)$ and $g(b_2) > g(b_1)$ that completes the proof. ■
Starting from this result we can justify our main lemma, which states that for every \( f \in C(K) \) and \( \varepsilon > 0 \) there exists an exponential polynomial of the form

\[
P_\varepsilon(x) = \sum_{i=1}^{N(\varepsilon)} c_i^\varepsilon \exp(a_i^\varepsilon \cdot x)
\]

such that

\[
\|f - P\| := \sup_{x \in K} |f(x) - P(x)| \leq \frac{\varepsilon}{2}
\]

where we used the notation Eq. (3). This property can be reformulated as follows:

**Lemma 2** The subspace \( E(K) \) is dense in \( C(K) \) in the uniform convergence topology on the compact set \( K \).

**Proof.** The subspace \( E(K) \) is closed under multiplication, contains identity element so it is a Banach subalgebra of \( C(K) \) with identity, and according to the previous Proposition 1 distinguishes all of the points on the compact set \( K \). So, according to Stone-Weierstrass theorem [7], \( E(K) \) is dense in \( C(K) \).

Now, according with the previous Lemma, in order to prove that \( B_n^n(x) \xrightarrow{n \to \infty} f(x) \), it is sufficient to prove the convergence on the generators \( \exp(a \cdot x) \) of the space \( E(K) \). Denote

\[
e_{a,n}(x) := B^n(f, x); \quad f(x) \equiv \exp(a \cdot x)
\]

For the sake of clarity we decompose the proofs in several steps.

2.2.1 The Bézier polynomial for \( \exp(a \cdot x) \) for \( D \) dimensional simplex, error estimate

**Proposition 3** The Bézier polynomial associated to \( D \) dimensional simplex for exponential function from Eq. (20) can be expressed as follows

\[
e_{a,n}(x) = \left[ \sum_{j=1}^{D} s_j(x) \exp \left( \frac{a \cdot x_j}{n} \right) \right]^n
\]
Proof. We use here a shorter notation: \( s_j(x) \) denote simply as \( s_j \). From Eqs. (13, 8, 17 9) and Newton formula Eq.(11) results

\[
e_{a,n}(x) = \sum_{k \in M_n} \exp \left[ a \cdot \left( \sum_{j=1}^{D} k_j \frac{x_j}{n} \right) \right] B_{i,j,k}^{n}(x) \tag{22}
\]

\[
= \sum_{k \in M_n} \prod_{j=0}^{D} \left[ \exp \left( \frac{a \cdot x_j}{n} \right) \right]^{k_j} \left( \frac{n}{k} \right) \prod_{j=0}^{D} [s_j]^{k_j}
\]

\[
= \left[ \sum_{j=1}^{D} s_j \exp \left( \frac{a \cdot x_j}{n} \right) \right]^n
\]

where was used the definition (10) and the multinomial Newton formula. ■

By using Eq. (21) we have the following

Proposition 4 In the limit of large \( n \) we have the following convergence result of the Bézier polynomial \( e_{a,n}(x) \) on the simplex \( T \)

\[
|e_{a,n}(x) - \exp(a \cdot x)| \leq \left[ \frac{K}{n} + \mathcal{O} \left( \frac{1}{n^2} \right) \right] \exp(a \cdot x) \tag{23}
\]

where \( K \) is a constant, independent of \( n \) and \( x \).

Proof. By separating an \( \mathcal{O}(1/n^2) \) term

\[
\exp \left( \frac{a \cdot x_j}{n} \right) = 1 + \frac{a \cdot x_j}{n} + \left[ \exp \left( \frac{a \cdot x_j}{n} \right) - 1 - \frac{a \cdot x_j}{n} \right]
\]

with the use of Eq.(5), the term in the right hand side from Eq.(21) we rewrite as follows

\[
\sum_{j=1}^{D} s_j(x) \exp \left( \frac{a \cdot x_j}{n} \right) = 1 + \sum_{j=1}^{D} s_j(x) \frac{a \cdot x_j}{n} + r_n(x)
\]

\[
= 1 + \frac{a \cdot x}{n} + r_n(x) \tag{24}
\]

where in the last equality we used Eq.(3) and we denoted the residual \( O(1/n^2) \) term as

\[
r_n(x) := \sum_{j=1}^{D} s_j(x) \exp \left( \frac{a \cdot x_j}{n} \right) - 1 - \frac{a \cdot x}{n} \tag{25}
\]
For large $n$, by using the remainder formula for Taylor series we have the inequality

$$|r_n(x)| \leq \frac{K_{1,n}}{n^2}$$  \hspace{1em} (26)

$$K_{1,n} = \frac{1}{2} \sum_{j=1}^{D} (\mathbf{a} \cdot \mathbf{x}_j)^2 \exp \left( \frac{\mathbf{a} \cdot \mathbf{x}_j}{n} \right) < C$$  \hspace{1em} (27)

where the constant $C$ can be optimized by suitable coordinate change. From Eqs. (21, 24) results

$$e_{a,n}(x) = \left[ 1 + \frac{\mathbf{a} \cdot \mathbf{x}}{n} + r_n(x) \right]^n$$  \hspace{1em} (28)

In order to find the speed of convergence by using Eqs. (26, 27) we compute

$$\left| \log \frac{e_{a,n}(x)}{\exp(\mathbf{a} \cdot \mathbf{x})} \right| < \frac{K}{n} + \mathcal{O} \left( \frac{1}{n^2} \right)$$  \hspace{1em} (29)

$$K = C + \frac{1}{2} \max_{\mathbf{x} \in \mathbf{T}} (\mathbf{a} \cdot \mathbf{x})$$  \hspace{1em} (30)

From Eq. (29) we obtain the relative error bound

$$\left| \frac{e_{a,n}(x) - \exp(\mathbf{a} \cdot \mathbf{x})}{\exp(\mathbf{a} \cdot \mathbf{x})} \right| < \frac{K}{n} + \mathcal{O} \left( \frac{1}{n^2} \right)$$  \hspace{1em} (31)

that completes the proof.

**Corollary 5** In the special case when $f(x) = \exp(\mathbf{a} \cdot \mathbf{x})$ we have the uniform convergence in the simplex $\mathbf{T}$

$$\|B_n(f; \cdot) - f(\cdot)\| := \sup_{x \in \mathbf{T}} |B_n(f; x) - f(x)| \leq \left[ \frac{K}{n} + \mathcal{O} \left( \frac{1}{n^2} \right) \right] \sup_{x \in \mathbf{T}} \exp(\mathbf{a} \cdot \mathbf{x})$$

or by notation (14)

$$\| \hat{B}_n f - f \| \leq \left[ \frac{K}{n} + \mathcal{O} \left( \frac{1}{n^2} \right) \right] \| f \|$$

For all exponential polynomials $P(x) := \sum_{i=1}^{N} c_i \exp(\mathbf{a}_i \cdot \mathbf{x})$ and every $\varepsilon$ there exists an $N(P, \varepsilon)$ such that for all $n \geq N$,

$$\| \hat{B}_n P - P \| \leq \frac{\varepsilon}{4}$$  \hspace{1em} (32)

\[ \blacksquare \]
2.3 Convergence proof, and error estimate, general case.

By Eqs. (12, 13) we defined a family of linear operators $B^n(f; y)$ on the space of continuous functions $C(T)$, indexed by integer $n$, that assign to every $f \in C(T)$ a polynomial of degree $n$. These family of operators has the following properties

\[ f(x) \equiv 1 \Rightarrow B_n(f; x) \equiv 1 \quad (33) \]
\[ f(x) > g(x) \Rightarrow B_n(f; x) > B_n(f; x) \quad (34) \]

The property (33) results from Eqs. (5, 8, 11, 13), while Eq. (34) results from (8) (the Bézier polynomials are positive) and Eq. (13). From Eqs. (33, 34) results that

\[ \|f\| := \sup_{x \in T} |f(x)| \leq A \Rightarrow \|B_n(f; .)\| := \sup_{x \in T} |B_n(f; x)| \leq A \quad (35) \]

The last relation means that the operator $\hat{B}_n$ defined by Eq. (14) on $C(T)$ is a bounded linear operator, whose norm is less or equal to 1, and by Eq. (33) its norm is 1, attained by constant functions:

\[ \|\hat{B}_n\| = 1 \]
\[ \|\hat{B}_n f\| \leq \|f\| \]

Consequently

\[ f \in C(T) \Rightarrow \left\| (\hat{B}_n - \hat{1}) f \right\| \leq 2 \|f\| \quad (36) \]

where $\hat{1}$ is the unit operator on $C(T)$. Now we can state and prove our main result:

**Theorem 6** For all $f \in C(T)$ we have

\[ \lim_{n \to \infty} \sup_{x \in T} |f(x) - B_n(f; x)| = 0 \quad (37) \]

which means that the polynomials $B_n(f; x)$ formed from Bézier polynomials approximate uniformly the function $f \in C(T)$. In term of linear operator notation the equivalent form is

\[ \lim_{n \to \infty} \left\| \hat{B}_n f - f \right\| = 0 \quad (38) \]
Proof. Let \( f \in C(T) \) and \( \varepsilon > 0 \). Then by Lemma (2) there exists an exponential polynomial \( P_\varepsilon(x) \in E(T) \) of the form

\[
\sum_{i=1}^{N} c_i \exp(a_i \cdot x)
\]

such that

\[
\|f - P_\varepsilon\| \leq \frac{\varepsilon}{4} \quad (39)
\]

By Corollary 5 and Eq. (32), there exists an \( N := N(P_\varepsilon, \varepsilon) \) such that for all \( n \geq N(P_\varepsilon, \varepsilon) \)

\[
\left\| (\hat{B}_n - \hat{1}) P_\varepsilon \right\| \leq \frac{\varepsilon}{2} \quad (40)
\]

By using the triangle inequality for norm and Eqs. (36, 39, 40) we obtain

\[
\left\| (\hat{B}_n - \hat{1}) f \right\| = \left\| (\hat{B}_n - \hat{1}) (f - P_\varepsilon) + (\hat{B}_n - \hat{1}) P_\varepsilon \right\| \leq \\
\left\| (\hat{B}_n - \hat{1}) (f - P_\varepsilon) \right\| + \left\| (\hat{B}_n - \hat{1}) P_\varepsilon \right\| \leq \left\| (\hat{B}_n - \hat{1}) \right\| \|f - P_\varepsilon\| + \frac{\varepsilon}{2} < \varepsilon
\]

that completes the proof. ■

3 Conclusions

We presented a new proof of the convergence of the Bézier polynomials attached to a simplex in \( D \) dimensions. We proved that the relative error is dominated by an asymptotic term of the form \( O(dk/n) \) where \( n \) is the order of the Bézier polynomial, \( d \) is the largest side of the simplex and \( k \) is the largest wave number that appear in the Fourier expansion of the function to be approximated.

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