Special $L$-values of cusp forms on $GL(2)$ over imaginary quadratic fields

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Abstract. We prove the non-vanishing of special $L$-values of cusp forms on $GL(2)$ over imaginary quadratic fields of class number one twisted by Hecke characters with split prime power conductors. Also we show that newforms can be determined by their twisted $L$-values. One of the main ingredients of the proof is estimating the Galois averages of fast convergent series expressions of special $L$-values.

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1. Introduction

Studying the complex or mod $p$ non-vanishing of special $L$-values is interesting as fascinating mathematical techniques and ingredients are involved. There are plenty of the non-vanishing results from which important arithmetic consequences arise.

Rohrlich [11] shows the non-vanishing of the special values of cyclotomic modular $L$-values by estimating the Galois averages of fast convergent series expressions of the special $L$-values. As a consequence of the non-vanishing result, Rohrlich [11] deduces that the Mordell-Weil groups of CM elliptic curves over the cyclotomic $\mathbb{Z}_p$-extension of $\mathbb{Q}$ are finitely generated. Luo-Ramakrishnan [5] prove that cusp forms can be determined by their twisted $L$-values by introducing the twisted Galois averages on both the set of primitive wild characters and the set of quadratic characters. Kim and Sun [4] obtain partial results toward the mod $p$ non-vanishing of cyclotomic modular $L$-values by studying the homological nature of modular symbols. Also Sun [13] proves that the Hecke field of a newform can be generated
by a single special value of the modular $L$-function, by studying an additive variant of the Galois average, which supersedes the non-vanishing results.

One can consider more general settings. Rohrlich [10] proves the non-vanishing of the twisted $L$-values of automorphic forms on $GL(2)$ over a general number field by taking a certain average on an appropriate set of ray class characters. For imaginary quadratic fields, Namikawa [7] shows that the special values of $L$-functions of automorphic forms on $GL(2)$ over an imaginary quadratic field is non-vanishing for infinitely many Hecke character twists when the class number is one. Also Namikawa [8] presents the non-vanishing result for cyclotomic $L$-values of automorphic forms over general number fields when the weight is greater than or equal to four.

The main result of the present paper is the non-vanishing result for the special $L$-values of automorphic forms on $GL(2)$ over imaginary quadratic fields twisted by $p$-power order Hecke characters of split prime power conductors, which is an analogue of the result of [11] or [5]. Also we prove the determination of a newform by its twisted special $L$-values, which is a generalization of one of the results of [5].

1.1. Main Theorems. Let us give some notations and settings. Let $k$ be a positive integer. Let $F$ be an imaginary quadratic field of class number one, $\mathfrak{n}$ an integral ideal of $F$, $S_k(\mathfrak{n}, \chi)$ the space of cuspidal automorphic forms over $F$ of parallel weight $k$ and level $\mathfrak{n}$ with a central character $\chi$. Let $p$ be a split prime ideal of $F$ lying above $\mathfrak{p}$ and $\Xi_p$ the set of Hecke characters of $p$-power order with $p$-power conductors. Let $f \in S_k(\mathfrak{n}, \chi)$, $\varphi \in \Xi_p$, $L(s, f \otimes \varphi)$ the $L$-function attached to $f$ and $\varphi$, $f(\varphi)$ the conductor of $\varphi$, $K_f$ the Hecke field of $f$, which is the field generated by all the Fourier-Whittaker coefficients of $f$, and $n_0 := \max\{m \in \mathbb{Z} | \mu_{\mathfrak{p}^m} \subset K_f\}$.

To obtain fast convergent series expressions of twisted modular $L$-values, namely the approximate functional equation of the $L$-function, we have to find the Fourier-Whittaker expansions of automorphic forms and its Mellin transforms. Also we need Atkin-Lehner theory for automorphic forms over imaginary quadratic fields. These ingredients will be developed in Section 2 and 3 together with a brief introduction to automorphic forms over imaginary quadratic field. In order to focus on generalizing the results of Rohrlich and Luo-Ramakrishnan and to avoid technical details, we assume that the class number of $F$ is one.

For a suitable $\gamma \in \mathcal{O}_F$, define the $\gamma$-twisted Galois averages of special $L$-values by

$$L_{av}(f \otimes \varphi, \gamma) := \frac{1}{[K_f(\varphi) : K_f]} \sum_{\sigma \in \text{Gal}(K_f(\varphi)/K_f)} \overline{\varphi(\gamma)}^{\sigma} L\left(\frac{k}{2}, f \otimes \varphi^{\sigma}\right).$$

From Section 4 to 6, we will discuss the Galois averages of twisted $L$-values and obtain their estimations, which play the key role in the proofs of main theorems. Also we will estimate the bounds of ‘Kloosterman-like’ exponential sums which appears in the Galois average of the dual term of our fast convergent series expressions of twisted special $L$-values. We would like to remark that for an inert prime $p$ of $F$, the corresponding estimations of the Galois averages over the characters are worse than ones in the present paper due to the existence of the corank one group of $p$-adic units.

We set $\theta \in [0, 1]$ as a bound of an exponent of the eigenvalues of a Hecke eigenform, i.e., $\theta$ is a number in $[0, 1]$ such that for any $\varepsilon > 0$ and integral ideals $\mathfrak{a}$ of $F$, ...
one has
\begin{equation}
|a_f(a)| \ll N(a)^{\frac{1}{12}+\theta+\varepsilon}
\end{equation}
where $a_f(a)$ is the $a$-th Hecke eigenvalue of $f$, and $N$ is the norm map of $F/\mathbb{Q}$.

Then we obtain an estimation of the twisted Galois averages of special $L$-values which depends on $\theta$:

**Theorem 1.1.** Let $\gamma \in \mathcal{O}_F$ which is coprime to $p$ and $N(\gamma) \leq (N(f(\varphi)))p^{-n(\varphi)}1/|\mathcal{W}|$.

If $\theta < \frac{3+\sqrt{17}}{8}$, then an estimation of $L_{av}(f \otimes \varphi, \gamma)$ is given by

\begin{equation}
L_{av}(f \otimes \varphi, \gamma) = \frac{a_f(\gamma \mathcal{O}_F)}{w_F N(\gamma)^{k/2}} + o(1)
\end{equation}
as $N(f(\varphi))$ tends to the infinity.

Note that the more improved $\theta$ we have, the more enhanced estimations on the Galois averages we have.

By applying Theorem 1.1 which is the generalization of Rohrlich’s estimation on the Galois average of special $L$-values, together with the bound $\theta = 7/64$ obtained by Nakasuji \cite{nakasuji} less than $\frac{3+\sqrt{17}}{8}$, we obtain the following non-vanishing result:

**Theorem 1.2.** For a normalized Hecke eigenform $f \in S_k(n, \chi)$, we have

\[ L\left(\frac{k}{2}, f \otimes \varphi\right) \neq 0 \]

for almost all $\varphi \in \Xi_p$.

As an another consequence of the estimation 1.2, we have the following main theorem about the determination of automorphic forms:

**Theorem 1.3.** Let $f \in S_k(n, \chi)$ and $g \in S_{k'}(n', \chi')$ be newforms. Assume that there exist constants $B$ and $C$ such that for infinitely many $\varphi \in \Xi_p$ and $n(\varphi) \in \mathbb{Z}_{\geq 0}$,

\[ L\left(\frac{k}{2}, f \otimes \varphi\right) = B^{n(\varphi)}C \cdot L\left(\frac{k'}{2}, g \otimes \varphi\right) \]

where $n(\varphi) \to \infty$ as $N(f(\varphi)) \to \infty$. Then we have $f = g$.

The first named author plans to generalize the results in \cite{fried} and \cite{silver} to the current setting.

1.2. Notations. Let us provide some notations which will be used globally in this paper. For $N \in \mathbb{Z}_{>0}$, denote $\zeta_N$ by a primitive $N$-th root of unity and $\mu_N$ by the set of $N$-th roots of unity. Let $F$ be an imaginary quadratic field, $\mathbb{A}_F$ the adele ring of $F$, $\delta_F$ the different of $F/\mathbb{Q}$, $d_F$ a generator of $\delta_F$, $D_F$ the discriminant of $F$, $h_F$ the class number of $F$, $\mathcal{O}_F$ the integer ring of $F$, and $w_F$ the number of the unit elements of $\mathcal{O}_F$. Let $p$ be an odd prime splits over $F$, $N$ the norm map of $F/\mathbb{Q}$ (or $\prod_{p \nmid p} N_{F_p/\mathbb{Q}_p}$) and $\operatorname{Tr}$ the trace map of $F/\mathbb{Q}$ (or $\sum_{p \mid p} \operatorname{Tr}_{F_p/\mathbb{Q}_p}$). For an integral ideal $n$ of $F$, define

\[ U_0(n) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{GL}_2(\mathcal{O}_F \otimes \hat{\mathbb{Z}}) : c \in n \otimes \hat{\mathbb{Z}} \right\}, \]

\[ U_1(n) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{GL}_2(\mathcal{O}_F \otimes \hat{\mathbb{Z}}) : c, d-1 \in n \otimes \hat{\mathbb{Z}} \right\}. \]
For \( x \in \mathbb{C} \) and \( y \in \mathbb{R}_{>0} \), let \((x; y) := \left( \begin{array}{cc} y^{1/2} & xy^{-1/2} \\ 0 & y^{-1/2} \end{array} \right)\) and \( H_3 := \{(x; y) \in \text{SL}_2(\mathbb{C}) : x \in \mathbb{C}, y \in \mathbb{R}_{>0}\}\). For \( z \in \mathbb{C} \), denote \( e(z) := \exp(2\pi iz) \) and \( e_\infty(z) := e(\text{Tr}_\mathbb{C}/\mathbb{R}(z)) \).

2. CUSP FORMS

2.1. Cusp forms over imaginary quadratic field. In this section, we will briefly give the definitions of cuspidal automorphic forms on \( \text{GL}_2(\mathbb{A}_F) \) and \( \text{GL}_2(\mathbb{C}) \), respectively. From now on, these are called cusp forms for simplicity. All the settings in this section follow Hida [2] or Namikawa [7].

We have the Fourier-Whittaker expansion of a cusp form on \( \text{GL}_2(\mathbb{A}_F) \). Associating cusp forms on \( \text{GL}_2(\mathbb{A}_F) \) and \( \text{GL}_2(\mathbb{C}) \) by using the strong approximation theorem, one can obtain the Fourier-Whittaker expansion of a cusp form on \( \text{GL}_2(\mathbb{C}) \).

Let us give some notations. Denote \( s = {}^t(S \ T) \) where \( S, T \) are indeterminates. For a commutative ring \( R \) with unity, let \( L(m, R) \) be the space of homogeneous polynomials of variable \( s \) and degree \( m \) with coefficients \( R \). Note that there is the usual action \( \text{GL}_2(R) \) on \( L(m, R) \). Set \( I_F = \text{Gal}(F/\mathbb{Q}) = \{ i, c \} \) where \( i \) and \( c \) are the identity map and the complex conjugation, respectively. Let \( \mathbb{Z}[I_F] \) be the free \( \mathbb{Z} \)-module generated by \( I_F \) and \( k = k_{id} id + k_c c \in \mathbb{Z}[I_F] \). Let \( \mathfrak{n} \) be an integral ideal of \( F \) and \( \chi : F^x \backslash \mathbb{A}_F^x \to \mathbb{C}^x \) a Hecke character over \( F \) of finite order with conductor dividing \( \mathfrak{n} \).

**Definition 2.1.** A cusp form on \( \text{GL}_2(\mathbb{A}_F) \) of weight \( k \) with level \( U_1(\mathfrak{n}) \) is a \( C^\infty \)-function \( f : \text{GL}_2(\mathbb{A}_F) \to L(k_{id} + k_c - 2, \mathbb{C}) \) such that

1. \( D_\sigma f = \left( \frac{(k_{id} - 2)^2}{2} + k_\sigma - 2 \right) f \) for \( \sigma \in \mathbb{Z}[I_F] \), where \( D_\sigma \) is the Casimir operator for \( \text{SL}_2 \) corresponding to \( \sigma \in \mathbb{Z}[I_F] \).
2. \( f(\gamma z_\infty g u) (s) = z_\infty^{-k + 2(i id + c)} f(g)(u_\infty s) \) for \( \gamma \in \text{GL}_2(F) \), \( z_\infty \in \mathbb{C}^x \) and \( u = u_\infty u(\infty) \) where \( u_\infty \in \text{SU}_2(\mathbb{C}) \) and \( u(\infty) \in U_1(\mathfrak{n}) \).
3. \( \int_{U(F) \backslash U(\mathbb{A}_F)} f(v)(g)(s) du = 0 \) for \( g \in \text{GL}_2(\mathbb{A}_F) \), where

\[
U(R) = \left\{ \left( \begin{array}{cc} 1 & u \\ 0 & 1 \end{array} \right) : u \in R \right\}.
\]

The space of cusp forms on \( \text{GL}_2(\mathbb{A}_F) \) of weight \( k \) with level \( U_1(\mathfrak{n}) \) is denoted by \( S_k(U_1(\mathfrak{n})) \).

Let \( W_k : \mathbb{C}^x \to L(k_{id} + k_c - 2, \mathbb{C}) \) be the Whittaker function defined by

\[
W_k(z)(s) := \sum_{j=0}^{k_{id} + k_c - 2} \binom{k_{id} + k_c - 2}{j} \left( \frac{z}{i|z|} \right)^{k_c - 1 - j} K_{j+1-k_c}(4\pi|z|) S^{k_{id} + k_c - 2-j} T^j
\]

where \( K_j \) is the \( j \)-th modified Bessel function. Following proposition describes the Fourier-Whittaker expansion of an adelic cusp form:

**Proposition 2.2** (Hida [2] Theorem 6.1). Let \( \mathcal{F} \) be the group of fractional ideals of \( F \). For \( f \in S_k(U_1(\mathfrak{n})) \), there exists a function \( a_f : \mathcal{F} \to \mathbb{C} \) satisfying following properties:

1. \( a_f(\mathfrak{a}) = 0 \) if \( \mathfrak{a} \in \mathcal{F} \) is not integral.
2. We have the Fourier-Whittaker expansion of \( f \) by

\[
f \left( \begin{array}{cc} y & x \\ 0 & 1 \end{array} \right) (s) = |y|_F^{k_x} \sum_{\xi \in F^x} a_f(\xi y(\infty)) \delta_F W_k(\xi y(\infty))(s) e_F(\xi x)
\]
where $y = y_{\infty}y^{(\infty)}$, $(y_{\infty}, y^{(\infty)}) \in \mathbb{C}^\times \times \mathbb{A}_F^{(\infty), \times}$, and $e_F : \mathbb{A}_F/F \rightarrow \mathbb{C}$ is the additive character such that

$$e_F(z^{(\infty)}) = e_\infty(z_{\infty}) \cdot \prod_{p \mid p} e_p \circ \text{Tr}_{F_p/Q_p}(z^{(\infty)}).$$

**Definition 2.3.** A cusp form on $GL_2(\mathbb{C})$ of weight $k$ with level $\Gamma$ is a $C^\infty$-function $f : GL_2(\mathbb{C}) \rightarrow L(k_{id} + k_c - 2, \mathbb{C})$ such that

1. $D_\sigma f = ((k_{\sigma} - 2)^2 + k_{\sigma} - 2)f$ for $\sigma \in \mathbb{Z}[I_F]$, where $D_\sigma$ is the Casimir operator for $SL_2$ corresponding to $\sigma \in \mathbb{Z}[I_F].$
2. $f(\gamma zgu)(s) = z^{-k_{\gamma} + 2(id + c)}f(g)(us)$ for $\gamma \in \Gamma, z \in \mathbb{C}^\times$, and $u \in SU_2(\mathbb{C}).$
3. $\int_{\xi^{-1} \in \Gamma \cap U(\mathbb{C}) \setminus U(\mathbb{C})} f(\xi^g)(s)du = 0$ for $\xi \in SL_2(F), g \in GL_2(\mathbb{C}),$ where

$$U(\mathbb{C}) = \left\{ \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} : u \in \mathbb{C} \right\}.$$

The space of cusp forms on $GL_2(\mathbb{C})$ of weight $k$ with level $\Gamma$ is denoted by $S_k(\Gamma)$.

Fix a representative $(a_i)_{i=1}^{h_F} \subset \mathbb{A}_F^{(\infty), \times}$ of $F^\times \setminus \mathbb{A}_F^{(\infty)} / K(\mathcal{O}_F) \cong \text{Cl}(F)$ where $K(\mathcal{O}_F) := \text{the image of } \left( \mathbb{C}^\times \times \prod_{v \text{ finite place of } F} \mathcal{O}_F^\times \right)$ in $F^\times \setminus \mathbb{A}_F^{(\infty)}$.

Without loss of generality, we can assume that $a_i$’s are coprime to $n$ for $i = 1, \cdots, h_F$. By the strong approximation theorem (cf. [1] Section 5.2) which is given by

$$GL_2(\mathbb{A}_F) = \prod_{i=1}^{h_F} GL_2(F) \cdot t_iGL_2(\mathbb{C})U_1(n)$$

we have an isomorphism of cusp form spaces [2] Section 3] by

$$S_k(U_1(n)) \cong \bigoplus_{i=1}^{h_F} S_k(\Gamma_i^0(n)), f \mapsto (f^{(i)})_{i=1}^{h_F}$$

where

$$\Gamma_i^0(n) := GL_2(F) \cap t_i GL_2(\mathbb{C})U_1(n)t_i^{-1}, f^{(i)}(g) := f(t_ig).$$

By combining [2] Corollary 2.2 and Eichler-Shimura-Harder isomorphism [2] Proposition 3.1, we know that if $k_{id} \neq k_c$, then $S_k(\Gamma_i^0(n)) = \{0\}$ for any $i$. Therefore, from now on, we consider parallel weights $k = k \cdot id + k \cdot c$ and let us denote

$$S_k(\Gamma_1^0(n)) := S_k(\Gamma_1^0(n)).$$

**Definition 2.4.** Define a right action of $\gamma \in GL_2(F)$ on $f \in S_k(\Gamma_i^0(n))$ by $(f \gamma)(g) := f(\gamma g)$. A cusp form on $GL_2(\mathbb{C})$ of parallel weight $k$ and level $\Gamma_1^0(n)$ with a central character $\chi$ is an element of $S_k(\Gamma_1^0(n))$ such that

$$(f \gamma)(g) = \chi(\gamma)f(\gamma g)$$

for any $\gamma \in \Gamma_0^0(n) := GL_2(F) \cap t_i GL_2(\mathbb{C})U_0(n)t_i^{-1}$ where $\chi(\gamma)$ is defined via the following isomorphism

$$\Gamma_0(n)/\Gamma_1(n) \cong (\mathcal{O}_F/n)^\times, \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto d \mod n.$$

The space of cusp forms of parallel weight $k$ of level $n$ with a central character $\chi$ is denoted by $S_k(n, \chi)$. 
2.2. Complex Fourier-Whittaker expansion. From now on, we assume that the class number $h_F$ of $F$ is 1 and we denote $\Gamma_1^1(n)$ by $\Gamma_1(n)$.

Let $f \in S_k(\Gamma_1(n))$. Note that the set $H_3 = \{(x; y) \in \text{SL}_2(\mathbb{C}) : x \in \mathbb{C}, y \in \mathbb{R}_{>0}\}$ can be identified with the hyperbolic 3-fold
\[
\mathcal{H}_3 := \left\{ \begin{pmatrix} x & -y \\ y & \tau \end{pmatrix} : x \in \mathbb{C}, y \in \mathbb{R}_{>0} \right\}
\]
by the map $(x; y) \mapsto \left( \begin{smallmatrix} x \\ y \end{smallmatrix} \right)$ induced by the usual action of $\text{SL}_2(\mathbb{C})$ on $\mathcal{H}_3$.

We want to compute the complex Fourier-Whittaker expansion of a cusp form on $H_3$ to obtain an integral representation of the $L$-function of $f$ to estimate the special $L$-values. For $(x; y) \in H_3$, we have the following:

**Proposition 2.5.** The Fourier-Whittaker expansion of a cusp form $f$ on $(x; y) \in H_3$ is given by
\[
f(x; y)(s) = \sum_{j=0}^{2k-2} \binom{2k-2}{j} y^k \sum_{\alpha \in \mathcal{O}_F} \left( \frac{\alpha}{|\alpha|} \right)^{k-1-j} a_f(\alpha)
\]
\[
\times K_{j+1-k} \left( \frac{4\pi N(\alpha)^{1/2}}{|D_F|^{1/2}} \right) e_\infty \left( \frac{\alpha x}{d_F} \right) S^{2k-2-jT^j}
\]
where $a := \alpha \mathcal{O}_F$.

**Proof.** By Proposition 2.2 and Definition 2.3 we have
\[
y^{-k+2} f(x; y)(s) = y^2 \sum_{\xi \in F^*} a_f(\xi \delta_F) W_k(\xi y)(s) e_F(\xi x)
\]
where
\[
W_k(z)(s) = \sum_{j=0}^{2k-2} \binom{2k-2}{j} \left( \frac{z}{|z|} \right)^{k-1-j} K_{j-k+1}(4\pi |z|) S^{2k-2-jT^j}.
\]

Thus, $f$ on $H_3$ is given by
\[
f(x; y)(s) = \sum_{j=0}^{2k-2} \binom{2k-2}{j} y^k \sum_{\xi \in F^*} \left( \frac{\xi}{|\xi|} \right)^{k-1-j} a_f(\xi \delta_F) K_{j+1-k} \left( \frac{4\pi |\xi| y}{|D_F|^{1/2}} \right) e_F(\xi x) S^{2k-2-jT^j}.
\]

Note that $(\xi x)_\infty = \xi x$ and $(\xi^\infty) = 0$, thus we have $e_F(\xi x) = e_\infty(\xi x)$. By the change of variable $\alpha = \xi d_F$, we obtain
\[
f(x; y)(s) = \sum_{j=0}^{2k-2} \binom{2k-2}{j} y^k \sum_{\alpha \in \mathcal{O}_F \setminus \{0\}} \left( \frac{\alpha}{|\alpha|} \right)^{k-1-j} a_f(\alpha \mathcal{O}_F)
\]
\[
\times K_{j+1-k} \left( \frac{4\pi |\alpha| y}{|D_F|} \right) e_\infty \left( \frac{\alpha x}{d_F} \right) S^{2k-2-jT^j}.
\]

Hence we are done.

\[
\square
\]

3. Special $L$-values

Let $f \in S_k(\Gamma_1(n))$. Define the $L$-function $L(s, f)$ of $f$ by the analytic continuation of the following Dirichlet series
\[
\sum_{\alpha \neq 0 \in \mathcal{O}_F} \frac{a_f(\alpha)}{N(\alpha)^s}
\]

where $\alpha \in \mathcal{O}_F$ and $N(\alpha)$ is the norm of $\alpha$. This function is an analytic function on the complex plane with a simple pole at $s = 1$.
where \( a \) runs over the nonzero integral ideals of \( \mathcal{O}_F \). Note that (3.1) converges absolutely in \( s \in \{ \Re(s) > \frac{k}{2} + 1 \} \).

3.1. Integral representation of the special value. To express the special \( L \)-value \( L \left( \frac{k}{2}, f \right) \) of \( f \) as a fast convergent infinite series, we need to compute the Mellin transform of the \( f \)-th component \( f_{k-1} \) of \( f(s) = \sum_{j=0}^{2k-2} f_j S^{2k-2-j} T_j \) is given by

\[
\int_0^\infty y^{2s-k} f_{k-1}(0; y) \frac{dy}{y} = \left( \frac{2k-2}{k-1} \right) \int_0^\infty y^{2s} \sum_{0 \neq a \in \mathcal{O}_F} a_f(a) K_0 \left( \frac{4\pi N(a)^{1/2} y}{|D_F|^{1/2}} \right) \frac{dy}{y}.
\]

By the integral formula for the \( j \)-th modified Bessel function which is given by

\[
\int_0^\infty y^j K_j(ay) \frac{dy}{y} = 2^{s-j} a^{-j} \Gamma \left( \frac{s+j}{2} \right) \Gamma \left( \frac{s-j}{2} \right) \text{ if } \Re(s+j) > 0,
\]

we can obtain an integral representation of the \( L \)-function of \( f \) from (3.2) by

\[
\int_0^\infty y^{2s-k} f_{k-1}(0; y) \frac{dy}{y} = \left( \frac{2k-2}{k-1} \right) \frac{w_F |D_F|^{-\nu}}{16} \Gamma_C(s)^2 L(s, f)
\]

for \( \Re(s) > \frac{k+1}{2} \) where \( \Gamma_C(s) = 2(2\pi)^{-s} \Gamma(s) \).

3.2. Fricke involution. To find an integral representation of the \( L \)-function which converges uniformly on the entire complex plane, we need to cut the integral representation (3.4) of completed \( L \)-function of \( f \) into two parts by using the Fricke involution similar with the classical case.

As we have set \( h_F = 1 \), we can choose a generator \( \nu \in \mathcal{O}_F \) of \( \mathfrak{n} \). Let us define the Fricke involution \( W_\nu \) on \( S_k(\Gamma_1(\mathfrak{n})) \) by

\[
W_\nu f(g)(s) := (-1)^{k-1} |\nu|^{k-2} f \left( \begin{pmatrix} 0 & -1 \\ \nu & 0 \end{pmatrix} g \right)(s).
\]

We can easily see that \( (W_\nu f)_{k-1} \) does not depend on the choice of a generator \( \nu \).

**Proposition 3.1.** If \( f \in S_k(\Gamma_0(\mathfrak{n}), \chi) \) is a Hecke eigenform, then

\[
W_\nu f \in S_k(\Gamma_0(\mathfrak{n}), \chi)
\]

and it is also a Hecke eigenform.

**Proof.** Let \( W_\nu := \left( \begin{pmatrix} 0 & -1 \\ \nu & 0 \end{pmatrix} \right) \). For \( \sigma \in \mathbb{Z}[I_F] \) and \( g \in \text{GL}_2(\mathbb{C}) \), we have

\[
D_\sigma(W_\nu f)(g)(s) = \frac{d}{dt} f(W_\nu g \cdot \exp(D_\sigma t))(s) \bigg|_{t=0} = \left( \frac{(k_\sigma - 2)^2}{2} + k_\sigma - 2 \right) f(W_\nu g)(s).
\]

Therefore \( W_\nu f \) is an eigenform of the Casimir operators. For \( \gamma = (a \ b \ c \ d) \in \Gamma_0(\mathfrak{n}) \), we have

\[
\begin{pmatrix} 0 & -1 \\ \nu & 0 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} d & -c/\nu \\ -\nu b & a \end{pmatrix} \begin{pmatrix} 0 & -1 \\ \nu & 0 \end{pmatrix},
\]

thus for \( g \in \text{GL}_2(\mathbb{C}) \), \( u \in \text{SU}_2(\mathbb{C}) \) and \( z \in \mathbb{C}^\times \), we have

\[
(W_\nu f)(\gamma zgu)(s) = |z|^{-k+2} \chi(d)(W_\nu f)(g)(us).
\]

Also as \( W_\nu \) normalizes \( \Gamma_0(\mathfrak{n}) \) by (3.5), we can show that \( W_\nu f \) is an eigenform of the Hecke operators due to the standard argument (the definition of the adelic version of Hecke operators is given in [2, Chapter 4]).
Clearly the constant term of \( W_n f \) at each cusp \( \xi \in \text{SL}_2(F) \) is 0 by the equation (2.1). \( \square \)

It is easy to see that
\[
(W_n f)_{k-1}(0; y) = f_{k-1}(0; |\nu|^{-1} y^{-1}).
\]
From this expression, we can obtain a fast convergent series expression of the special \( L \)-value.

**Theorem 3.2.** For any \( t > 0 \), we have
\[
\frac{(4\pi)^k w_F}{16} \Gamma_C \left( \frac{k}{2} \right)^2 L \left( \frac{k}{2}, f \right) = \sum_{0 \neq \alpha \in \mathcal{O}_F} \frac{a_F(a)}{N(\alpha)^{k/2}} \int_{4\pi N(\alpha)^{1/2}|D_F\nu|^{-1/4} t}^{\infty} y^k K_0(y) \frac{dy}{y}
+ \sum_{0 \neq \alpha \in \mathcal{O}_F} \frac{a_{W_n f}(\alpha)}{N(\alpha)^{k/2}} \int_{4\pi N(\alpha)^{1/2}|D_F\nu|^{-1/4} t}^{\infty} y^k K_0(y) \frac{dy}{y}
\]
where \( a = \alpha \mathcal{O}_F \).

**Proof.** In a standard manner, one can obtain the following from (3.4):
\[
\left( 2k-2 \right) \frac{w_F |D_F|^s}{k-1} \int_0^{\infty} dy \left( \frac{y}{|\nu|^{\frac{1}{2}}} \right)^s L(s, f) = \int_0^{\infty} dy \left( \frac{y}{|\nu|^{\frac{1}{2}}} \right)^s f_{k-1}(0; |\nu|^{-\frac{1}{2}} y) + \int_0^{\infty} dy \left( \frac{y}{|\nu|^{\frac{1}{2}}} \right)^s (W_n f)_{k-1}(0; |\nu|^{-\frac{1}{2}} y).
\]
Note that by Proposition 2.5 we have the following expansion of \( f_{k-1} \):
\[
f_{k-1}(0; |\nu|^{-\frac{1}{2}} y) = \left( 2k-2 \right) \frac{w_F |D_F|^s}{k-1} y^{k-1} |\nu|^{-\frac{k}{2}} \sum_{0 \neq \alpha \in \mathcal{O}_F} a_F(\alpha) K_0 \left( \frac{4\pi N(\alpha)^{1/2} y}{|D_F\nu|^{1/2}} \right).
\]
By Proposition 3.1 and the equation (3.6), we can also obtain a similar expansion for \( W_n f \). Setting \( s = \frac{k}{2} \), we obtain the desired one. \( \square \)

### 3.3. Twisted cusp forms

We will define and discuss a cusp form twisted by a Hecke character. In this subsection, we follow and generalize the process introduced in the book of Shimura [12].

Let \( \mathfrak{c} \) be an integral ideal of \( F \) coprime to \( D_F \), \( c_0 \) a generator of \( \mathfrak{c} \), and \( \varphi : F^\times \backslash \mathbb{A}_F^\times \to \mathbb{C}^\times \) a Hecke character over \( F \) of finite order with conductor \( \mathfrak{c} \). As \( h_F = 1 \), we can consider \( \varphi \) as a character of \( (\mathcal{O}_F/\mathfrak{c})^\times \) via a surjective homomorphism \( (\mathcal{O}_F/\mathfrak{c})^\times \to \text{Cl}_F(\mathfrak{c}) = F^\times \backslash \mathbb{A}_F^\times / \ker(\varphi) \) where \( \text{Cl}_F(\mathfrak{c}) \) is the ray class group of \( \mathfrak{c} \) over \( F \). We extend \( \varphi \) to \( \mathcal{O}_F \) by defining \( \varphi(\alpha) = 0 \) if \( \alpha \in \mathcal{O}_F \) is not coprime to \( \mathfrak{c} \).

Define the Gauss sum \( G(\varphi) \) for \( \varphi \) by
\[
G(\varphi) := \varphi(d_F)^{-1} \sum_{u \in (\mathcal{O}_F/\mathfrak{c})^\times \times \mathfrak{c}} \varphi(u) e_{\infty} \left( \frac{u}{c_0 d_F} \right).
\]
One can easily check that it does not depend on the choice of \( c_0, d_F \), and a representative set \( \{ u \} \) of \( (\mathcal{O}_F/\mathfrak{c})^\times \). Then we have the following lemma for the Gauss sums:

**Lemma 3.3.** (1) For any \( \alpha \in \mathcal{O}_F \), we have
\[
G(\varphi(\alpha)) = \varphi(d_F) \sum_{u \in (\mathcal{O}_F/\mathfrak{c})^\times} \varphi(u) e_{\infty} \left( \frac{\alpha u}{c_0 d_F} \right).
\]
From the equality (3.11), one can see that

\[ f \otimes \varphi := G(\overline{\varphi})^{-1} \varphi(d_F) \sum_{u \in (O_F/c)^\times} \overline{\varphi}(u)f \begin{pmatrix} 1 & u/c_0 \\ 0 & 1 \end{pmatrix}. \]

Then we have the following proposition:

**Proposition 3.4.** We have \( f \otimes \varphi \in S_k(\Gamma_0(q), \chi\varphi^2) \) where \( q = n \cap m \cap c^2 \) and \( m \mid n \) is the conductor of \( \chi \). Also we have \( a_{f \otimes \varphi}(a) = a_f(a)\varphi(a) \).

**Proof.** For \( \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(q) \), an easy computation tells us that

\[
\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a + \frac{cn}{c_0} & b - a + \frac{cd|d-1|u}{c_0} - \frac{cd^2u}{c_0} \\ c & d - \frac{cd^2u}{c_0} \end{pmatrix} \begin{pmatrix} 1 & d^2u/c_0 \\ 0 & 1 \end{pmatrix}.
\]

Note that \( c_0^2 | c \), hence all the entries of the third matrix in (3.10) are integral. By (3.11), (3.10), and the change of variable by \( d^2u \mapsto u \) (note that \( d \in (O_F/c)^\times \)), we have

\[
f \otimes \varphi \left( \begin{pmatrix} a & b \\ c & d \end{pmatrix} \right) = G(\overline{\varphi})^{-1} \varphi(d_F) \sum_{u \in (O_F/c)^\times} \overline{\varphi}(u)\varphi^2(d_f) \begin{pmatrix} 1 & u/c_0 \\ 0 & 1 \end{pmatrix} = \chi\varphi^2(d)(f \otimes \varphi).
\]

The remaining parts are clear (cf. Proposition 3.1), thus we have \( f \otimes \varphi \in S_k(\Gamma_0(q), \chi\varphi^2) \).

By Proposition 2.5 and Lemma 3.3 (1), we obtain

\[
G(\overline{\varphi})(f \otimes \varphi)_j(x; y)(s) = \varphi(d_F) \begin{pmatrix} 2k - 2 \\ j \end{pmatrix} y^k \sum_{0 \neq a \in O_F} \frac{\alpha}{|\alpha|} \begin{pmatrix} k-j \\ j \end{pmatrix} a_f(a) 
\]

\[
\times K_{j+1-k} \left( \frac{4\pi N(\alpha)^{\frac{1}{2}}y}{|D_F|^{\frac{1}{2}}} \right) \sum_{u \in (O_F/c)^\times} \overline{\varphi}(u)e_{\infty} \left( \frac{\alpha(x + \frac{d}{d_F})}{d_F} \right).
\]

From the equality (3.11), one can see that \( a_{f \otimes \varphi}(a) = a_f(a)\varphi(a) \).

Also we have a relation between the Fricke involution and the twisting by Hecke characters:

**Proposition 3.5.** If \( c \) and \( n \) are coprime, then we have

\[ W_{n^2}(f \otimes \varphi) = \iota(\varphi) \cdot (W_n f) \otimes \overline{\varphi} \]

where \( \iota(\varphi) := \chi(c_0)\varphi(D_F\nu)G(\varphi^2N(c_0))^{-1}. \)
Then we can write (3.14) as

$$L_{nc2}^k(f \otimes \varphi) = \frac{(-1)^{k-1}c_0^2 \nu^{k-2} \varphi(d_F)}{G(\varphi)} \sum_{u \in (\mathcal{O}_F / c)^\times} \varphi(u)f \begin{pmatrix} 1 & u/c_0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ c_0^2 \nu & 0 \end{pmatrix}.$$  

Put the following identity of matrices

$$\begin{pmatrix} 1 & u/c_0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ c_0^2 \nu & 0 \end{pmatrix} = \begin{pmatrix} c_0 & 0 \\ 0 & c_0 \end{pmatrix} \begin{pmatrix} \nu & -v \\ 0 & d \end{pmatrix} \begin{pmatrix} 1 & u/c_0 \\ 0 & 1 \end{pmatrix}$$

into (3.12), then by the properties of $f$ and Proposition 3.1 we obtain

$$W_{nc2}^k(f \otimes \varphi) = G(\varphi)^{-1} \varphi(d_F) \sum_{u \in (\mathcal{O}_F / c)^\times} \chi(d) \varphi(W_n f) \begin{pmatrix} 1 & v/c_0 \\ 0 & 1 \end{pmatrix},$$

Note that $v$ runs over $(\mathcal{O}_F / c)^\times$ when $u$ runs over $(\mathcal{O}_F / c)^\times$ since $\nu v \equiv c_0 - 1 \pmod{c}$ and $\nu, c_0 - 1$ are coprime to $c$. By (3.13) and Lemma 3.3, we can write $W_{nc2}^k(f \otimes \varphi)$ as

$$W_{nc2}^k(f \otimes \varphi) = \frac{\varphi(c_0 - 1)}{G(\varphi)} \chi(c_0) \varphi(D_F \nu) G(\varphi)^2 \times G(\varphi)^{-1} \varphi(d_F) \sum_{v \in (\mathcal{O}_F / c)^\times} \varphi(v)(W_n f) \begin{pmatrix} 1 & v/c_0 \\ 0 & 1 \end{pmatrix} = \chi(c_0) \varphi(D_F \nu) G(\varphi)^2 N(c_0)^{-1} (W_n f) \otimes \varphi,$$

which concludes the proof of proposition.

Finally, we can find a fast convergent series expression of the twisted special $L$-value: Let $c$ and $n$ be coprime. By Theorem 3.2 and Proposition 3.5, we have

$$L\left(\frac{k}{2}, f \otimes \varphi \right) = \sum_{0 \neq \alpha \in \mathcal{O}_F} \frac{a_f(\alpha) \varphi(\alpha)}{N(\alpha)^{k/2}} C_{k,F} \int_{4 \pi N(\alpha)^{-\frac{k}{4}} D_F \nu c_0^{-2}}^{\infty} \frac{y^k K_0(y)}{y} dy + \sum_{0 \neq \alpha \in \mathcal{O}_F} \frac{a_{W_n f}(\alpha) \varphi(\alpha)}{N(\alpha)^{k/2}} C_{k,F} \int_{4 \pi N(\alpha)^{-\frac{k}{4}} D_F \nu c_0^{-2}}^{\infty} \frac{y^k K_0(y)}{y} dy$$

where $\iota(\varphi) = \chi(c_0) \varphi(D_F \nu) G(\varphi)^2 N(c_0)^{-1}$ and $C_{k,F} := \frac{(4 \pi)^{k/2} \nu^{k/2}}{16} \Gamma_c \left(\frac{k}{2}\right)^2$. Define

$$V_{k,F}(t) := \frac{1}{C_{k,F}} \int_{4 \pi |D_F \nu|^{-\frac{k}{4}} t}^{\infty} y^{k-1} K_0(y) dy$$

Then we can write (3.12) as

$$L\left(\frac{k}{2}, f \otimes \varphi \right) = \sum_{0 \neq \alpha \in \mathcal{O}_F} \frac{a_f(\alpha) \varphi(\alpha)}{N(\alpha)^{k/2}} V_{k,F} \left(\frac{N(\alpha)^{\frac{k}{2}}}{N(c_0)^{\frac{k}{2}}}\right) + \sum_{0 \neq \alpha \in \mathcal{O}_F} \frac{a_{W_n f}(\alpha) \iota(\varphi) \varphi(\alpha)}{N(\alpha)^{k/2}} V_{k,F} \left(\frac{N(\alpha)^{\frac{k}{2}}}{N(c_0)^{\frac{k}{2}}}\right).$$

The behavior of the function $V_{k,F}$ can be described as following:
Proposition 3.6. For any $k > 0$, we have $V_{k,F}(y) \ll_{D_F,k,\nu,w_F} y^{k-\frac{2}{e}} e^{-y}$ as $y \to \infty$, and $V_{k,F}(y) - 1 \ll_{D_F,k,\nu,w_F} y^{k} \log(y)$ as $y \to 0$. Also $-V'_{k,F}$ is a positive and strictly decreasing function on $\mathbb{R}_{\geq 0}$ with $-V'_{k,F} \ll_{D_F,k,\nu,w_F} y^{k-\frac{2}{e}} e^{-y}$ as $y \to \infty$.

Proof. Note that $K_0(z) = O(\log(z))$ as $|z| \to 0$, hence by the formula (3.3) and the change of variable, we have

$$V_{k,F}(y) - 1 = -\frac{1}{C_{k,F}} \int_0^{4\pi |D_F\nu|^{-\frac{2}{e}} y} x^{k-1} K_0(x) dx \ll_{D_F,\nu,w_F} \int_{-\infty}^{\log(y)} t e^{kt} dt \ll_k y^{k} \log(y).$$

On the other hand, $K_0$ is a positive and strictly decreasing function on $\mathbb{R}_{\geq 0}$ and $K_0(z) = O(z^{-\frac{2}{e}} e^{-z})$ as $|z| \to \infty$. By using the definition of $V_{k,F}$ together with these facts, we can conclude the proof. \hfill \Box

4. Galois Averages of Hecke Characters

In this section, we are going to discuss the Galois averages of Hecke characters and an exponential sum which have a crucial role to estimate the Galois average of the $L$-values. Recall that $p$ is a split prime ideal of $F$ lying above $p$, thus $p$ is coprime to $D_F$. We assume that $n$ and $p$ are coprime which allows us to use the discussion in Subsection 3.3. Choose generators $\pi_p$ and $\nu$ of $p$ and $n$, respectively. Let $K/Q$ be a finite extension and set

$$n_0 := \max \{ m \in \mathbb{Z} | \mu_p^m \subset K \}.$$ 

Set $\mathcal{O}_{F,p} := \varprojlim_m (\mathcal{O}_F/p^m)$ and $\Gamma_n := 1 + p^n \mathcal{O}_{F,p}$, then we have a noncanonical decomposition

$$\mathcal{O}_{F,p}^{\times} = W \cdot \Gamma_1$$

via the following split exact sequence

$$1 \longrightarrow 1 + p\mathcal{O}_{F,p} \longrightarrow \mathcal{O}_{F,p}^{\times} \mod p \longrightarrow (\mathcal{O}_F/p)^{\times} \longrightarrow 1$$

where $W$ is the image of a section to mod $p$.

Proposition 4.1. For $\alpha \in \mathcal{O}_{F,p}^{\times}$, there exists a unique pair $(\kappa, \gamma) \in W \times \Gamma$ such that $\alpha = \kappa \gamma$, i.e.,

$$\mathcal{O}_{F,p}^{\times} \cong W \times \Gamma_1$$

Proof. Note that $W \cap (1 + p\mathcal{O}_{F,p}) = \{ 1 \}$. Hence by (4.1), we are done. \hfill \Box

For a Hecke character $\varphi$ over $F$ of finite order, we define the Galois averages of $\varphi$ over $K$ by

$$\varphi_{av} := \frac{1}{[K(\varphi) : K]} \sum_{\sigma \in \text{Gal}(K(\varphi)/K)} \varphi^\sigma$$

$$\varphi_{av}' := \frac{1}{[K(\varphi) : K]} \sum_{\sigma \in \text{Gal}(K(\varphi)/K)} \iota(\varphi^\sigma)^{\overline{\sigma}}$$

where $K(\varphi)$ is the field determined by adjoining all the values of $\varphi$ to $K$. We suppose that the image of $\varphi$ is sufficiently large so that $\text{Gal}(K(\varphi)/K)$ is non-trivial. The following proposition tells us that nonzero integral elements which makes the Galois averages zero are distributed sparsely.
Let us fix an embedding $F \hookrightarrow F_p$ and a Hecke character $\varphi$ over $F$ of $p$-power order with conductor $p^n$. Then $\varphi$ can be considered as a continuous map on $\mathcal{O}^\times_{F_p}$ so that $\ker \varphi = W \times \Gamma_n$ by Proposition 4.1.

**Proposition 4.2.** Let $\varphi$ be a Hecke character over $F$ of $p$-power order with conductor $p^n$. Then for $\alpha \in \mathcal{O}^\times_{F_p}$, $\varphi_{av}(\alpha) \neq 0$ if and only if $\alpha \in W \cdot \Gamma_{n-n_0}$.

**Proof.** Assume that $\varphi_{av}(\alpha) \neq 0$, then $\alpha$ is coprime to $p$, thus we can consider $\alpha \in (\mathcal{O}_F/p^n)^\times$. Also we have $\varphi(\alpha) \neq 0$ and $\varphi$ is of finite order, hence $\varphi(\alpha) = \zeta_{p^{n-1}}^r$ for some $r \in \mathbb{Z}_{\geq 0}$. Our assumption $\varphi_{av}(\alpha) \neq 0$ says that

$$\text{Tr}_{K_0} \zeta_{p^{n-1}}^{r}/K(\zeta_{p^{n-1}}^r) \neq 0,$$

thus we have $n-1-v_p(r) \leq n_0$. By Proposition 4.1 we have

$$\varphi(\gamma) = \varphi(\kappa \gamma) = \varphi(\kappa) \in W \times \Gamma_n,$$

which implies that $\gamma \in \Gamma_{n-n_0}/\Gamma_n$ since $\Gamma_{n-n_0}/\Gamma_n \cong \mathbb{Z}/p^{n_0} \mathbb{Z}$.

Conversely, if $\alpha \in W \cdot \Gamma_{n-n_0}$, then $\varphi(\alpha) = \varphi(\kappa \gamma) = \varphi(\kappa) \in p^{n_0}$ for some $(\kappa, \gamma) \in W \times \Gamma$. Thus, $\varphi(\alpha) = \zeta_{p^{n_0}}^r$ for some $r \in \mathbb{Z}_{\geq 0}$. Hence the Galois average of $\varphi(\alpha)$ is given by

$$\varphi_{av}(\alpha) = \frac{1}{[K(\varphi):K]} \sum_{\sigma \in \text{Gal}(K(\varphi)/K)} (\zeta_{p^{n_0}}^r)^{\sigma} = \zeta_{p^{n_0}}^r \neq 0.$$

This concludes the proof of proposition. \qed

Following lemma is a generalization of [4, Lemma 4.2] to our setting, which is about an estimation of the 'Kloosterman-like' exponential sum. It allows us to estimate the size of $\varphi_{av}$.

**Lemma 4.3.** Let $v, n \in \mathbb{Z}_{\geq 0}$ with $v \leq \left[\frac{n}{2}\right]$. For $a, b \in \mathcal{O}_F$ with $w \leq n - \left[\frac{n}{2}\right] - v$, we have

$$(4.2) \quad \sum_{r \in \Gamma_v/\Gamma_n} e_{\infty} \left( \frac{ar + b r' + \pi_p^d_F}{\pi_p^d F} \right) \leq 2p^{\left[\frac{n}{2}\right]+w}$$

where $r'$ is the mod $p^n$ inverse of $r$, and $w = \min(v_p(a), v_p(b))$.

**Proof.** Let us set $Q(j) := aj + b j'$ and $h = \left[\frac{n}{2}\right]$. Then for $\beta \in \mathcal{O}_F$, $r \in \Gamma_v$, we have

$$Q(r + \pi_p^h r; \beta) \equiv ar + a\pi_p^h r; \beta + br' (1 - \pi_p^h \beta + O(\pi_p^2 h)) \equiv Q(r) + \pi_p^h (ar - br' \beta) \pmod{p^n}.$$  

We have the following group isomorphism:

$$r(1 + \pi_p^h \beta) \in \Gamma_v/\Gamma_n \cong \Gamma_v/\Gamma_h \times \Gamma_h/\Gamma_n \cong \Gamma_v/\Gamma_h \times \mathcal{O}_F/p^{n-h} \ni (r, \beta).$$

From the above two relations, we can rewrite (4.2) as

$$(4.3) \quad \sum_{r \in \Gamma_v/\Gamma_h} e_{\infty} \left( \frac{Q(r)}{\pi_p^d F} \right) \sum_{\beta \in \mathcal{O}_F/p^{n-h}} e_{\infty} \left( \frac{(ar - br') \beta}{\pi_p^d F} \right).$$

By the orthogonality of characters of finite group, (4.3) becomes

$$(4.4) \quad p^{n-h} \sum_{r \in \Gamma_v/\Gamma_h} e_{\infty} \left( \frac{Q(r)}{\pi_p^d F} \right).$$

...
Proposition 4.4. Let \( \varphi \) be a Hecke character over \( F \) of \( p \)-power order with conductor \( p^n \). Then we have

\[
\varphi'^{\dagger}(\alpha) \leq 2|W|^2 \cdot p^\left\lceil \frac{n}{2} \right\rceil - n + n_0
\]

Proof. If \( \alpha \) is not coprime to \( p \), then clearly the above inequality holds. So we can assume that \( \alpha \) is coprime to \( p \). By the definition of \( \iota(\varphi) \) and \( G(\varphi) \), we have

\[
\iota(\varphi)\overline{\varphi}(\alpha) = \frac{\chi(\pi_p^n)}{N(\pi_p^n)} \sum_{u,v \in (\mathcal{O}_F/p^n)^{\times}} \varphi(\alpha' \nu uv) e_\infty \left( \frac{u + v}{\pi_p^n d_F} \right)
\]

(4.5)

where \( \alpha' \) is the mod \( p^n \)-inverse of \( \alpha \). By taking the Galois average of (4.5), we have

\[
\varphi'^{\dagger}(\alpha) = \frac{\chi(\pi_p^n)}{p^n} \sum_{\beta \in (\mathcal{O}_F/p^n)^{\times}} \varphi(\alpha' \beta \nu) \sum_{uv \equiv \beta (p^n)} e_\infty \left( \frac{u + v}{\pi_p^n d_F} \right).
\]

(4.6)

As \( \nu \) is coprime to \( p \), we can abbreviate \( \alpha' \beta \nu \) to \( \alpha' \beta \). By Proposition 4.1, \( \varphi'^{\dagger}(\alpha' \beta) \neq 0 \) if and only if \( \alpha' \beta = \kappa (1 + \pi_p^{n-n_0} \gamma) \) for some \( \kappa \in W, \gamma \in \mathcal{O}_F/p^{n_0} \). So we can rewrite (4.6) as

\[
\varphi'^{\dagger}(\alpha) = \frac{\chi(\pi_p)}{|G| p^n} \sum_{\kappa \in W} \sum_{\gamma \in \mathcal{O}_F/p^{n_0}} \sum_{\sigma \in G} \varphi(1 + \pi_p^{n-n_0} \gamma) \sum_{uv \equiv \kappa (1 + \pi_p^{n-n_0} \gamma) \alpha (p^n)} e_\infty \left( \frac{u + v}{\pi_p^n d_F} \right)
\]

where \( G = \text{Gal}(K(\varphi)/K) \) and \( u, v \) runs over \( (\mathcal{O}_F/p^n)^{\times} \). Note that we have

\[
u = \omega r \in (\mathcal{O}_F/p^n)^{\times} \cong W \times \Gamma_1/\Gamma_n \equiv (\omega, r).
\]

Hence, (4.7) becomes

\[
\frac{\chi(\pi_p)}{|G| p^n} \sum_{\sigma \in G} \sum_{\omega \in W} \sum_{\gamma \in \mathcal{O}_F/p^{n_0}} \sum_{r \in \Gamma_1/\Gamma_n} \varphi(1 + \pi_p^{n-n_0} \gamma) e_\infty \left( \frac{\omega r + \kappa \omega' (1 + \pi_p^{n-n_0} \gamma) \alpha'}{\pi_p^n d_F} \right).
\]

By Lemma 4.3, the absolute value of (4.8) can be bounded by \( 2|W|^2 p^{n_0} p^\left\lceil \frac{n}{2} \right\rceil - n \).
5. Estimations of Galois averages of the special $L$-values

In this section, we obtain an estimation on the twisted special $L$-values which allows us to prove the non-vanishing statement.

Let us recall that $p$ is a split prime ideal of $F$ lying above $p$, $\pi_p$ is a generator of $p$, $\Xi_p$ is the set of Hecke characters over $F$ of $p$-power order and $p$-power conductors, $\varphi$ is an element of $\Xi_p$, $\varphi(f)$ is the conductor of $\varphi$, $k$ is a positive integer, $f \in S_k(n, \chi)$ is a normalized Hecke eigenform, and $K_f$ is the Hecke field of $f$ over $\mathbb{Q}$ (cf. subsection §3.1). Let us denote $\gamma$ by the integer with $f(\varphi)$, By Proposition 3.3, the twisted form $f \otimes \varphi \in S_k(nf(\varphi)^2, \chi \varphi^2)$ is also a Hecke eigenform. For $\gamma \in \mathcal{O}_F$ with $N(\gamma) \leq (N(f(\varphi))p^{-n_0})^{1/W}$ and being coprime to $p$, define the $\gamma$-twisted Galois average of the special $L$-value by

$$L_{av}(f \otimes \varphi, \gamma) := \frac{1}{|K_f(\varphi) : K_f|} \sum_{\sigma \in \text{Gal}(K_f(\varphi)/K_f)} \varphi(\gamma)^\sigma L\left(\frac{k}{2}, f \otimes \varphi^\sigma\right).$$

Then by (5.1), the $\gamma$-twisted Galois average $L_{av}(f \otimes \varphi, \gamma)$ is given by

$$\sum_{0 \neq \alpha \in \mathcal{O}_F} \frac{a_f(\alpha) \varphi_{av}(\alpha \gamma')}{N(\alpha)^k/2} V_{k,F} \left( \frac{N(\alpha)^{k/2}}{N(f(\varphi))^{1/2}} \right)$$

$$+ \sum_{0 \neq \alpha \in \mathcal{O}_F} \frac{a_{W,WF}(\alpha) \varphi_{av}(\alpha \gamma)}{N(\alpha)^{k/2}} V_{k,F} \left( \frac{N(\alpha)^{k/2}}{N(f(\varphi))^{1/2}} \right)$$

where $\gamma' \equiv 1/0 \bmod f(\varphi)^{-1}$ is the mod $f(\varphi)$ inverse of $\gamma$ and $a = \alpha \mathcal{O}_F$. To estimate the quantity $\gamma_{av}(f \otimes \varphi, \gamma)$, we need a bound for the Hecke eigenvalues of $f$. Let us assume that there exists $\theta \in [0, 1]$ such that

$$|a_f(\alpha)| \leq 2N(p)^{\frac{k-1}{4} + \theta}$$

for any prime ideals $p$ of $\mathcal{O}_F$. Hence, for $\varepsilon > 0$ and each integral ideal $\alpha$ of $\mathcal{O}_F$, we have

$$|a_f(\alpha)| \leq 2d(\alpha)N(\alpha)^{\frac{k-1}{4} + \theta} \leq_{\varepsilon} N(\alpha)^{\frac{k-1}{4} + \theta + \varepsilon}$$

where $d(\alpha)$ is the number of the integral ideals of $F$ dividing $\alpha$.

Now we split (5.2) into two summands;

$$\sum_{\alpha \in \mathcal{O}_F \atop N(\alpha) = N(\gamma)} + \sum_{\alpha \in \mathcal{O}_F \atop N(\alpha) \neq N(\gamma)} \cdot$$

Clearly $N(\alpha) = N(\gamma)$ if and only if $\alpha \mathcal{O}_F = \gamma \mathcal{O}_F$. By Proposition 3.2, $\varphi_{av}(\alpha \gamma') \neq 0$ if and only if $\alpha \in \gamma(\mathcal{W}n_{n_0} \cap \mathcal{O}_F)$. If $\alpha \in \gamma(\mathcal{W}n_{n_0} \cap \mathcal{O}_F)$, then $N(\alpha) = N(\gamma)$ or $N(\alpha)^{1/W} > N(f(\varphi))p^{-n_0}$, as we have

$$N(\alpha)^{1/W} \equiv N(\gamma)n_{n_0}^{1/W} \equiv N(\gamma)^{1/W} \pmod{N(f(\varphi))p^{-n_0}}.$$ 

Hence for sufficiently large $n$, we can rewrite (5.3) as

$$a_f(\gamma \mathcal{O}_F) V_{k,F} \left( \frac{N(\gamma)^{k/2}}{N(f(\varphi))^{1/2}} \right)$$

$$+ \sum_{\alpha \in \mathcal{W} \atop N(\alpha) \equiv (N(f(\varphi))p^{-n_0})^{1/W}} \frac{a_f(\alpha) \varphi_{av}(\alpha \gamma')}{N(\alpha)^{k/2}} V_{k,F} \left( \frac{N(\alpha)^{k/2}}{N(f(\varphi))^{1/2}} \right).$$
where (5.5) and (5.6) turn out to be the main term and the error term of (5.1), respectively.

We have the following estimations about the number of lattices in some region:

**Proposition 5.1.** Let $\Lambda$ be a lattice of $\mathbb{C}$, $P(\Lambda)$ the fundamental parallelogram of $\Lambda$, and $d(\Lambda) := \sup_{x,y \in P(\Lambda)} |x - y|$, a diameter of $P(\Lambda)$. For $\omega_0 \in \mathbb{C}$, we have

$$
\# \{ \alpha \in \omega_0 + \Lambda : |\alpha| \leq R \} \leq \frac{(R + d(\Lambda))^2}{\text{Area}(P(\Lambda))}$$

$$
\# \{ \alpha \in \omega_0 + \Lambda : |\alpha| = R \} \leq \frac{2\pi R}{d(\Lambda)}
$$

**Proof.** By the definition of $d(\Lambda)$, there exists a covering of the disc $|\alpha| \leq R$ by a collection of parallelograms $\{\omega_0 + \lambda_j + P(\Lambda)\}_{\lambda_j \in \Lambda}$ contained in the circle $|\alpha| \leq R + d(\Lambda)$. Therefore, the number of lattice points $\alpha \in \omega + \Lambda$ with $|\alpha| \leq R$ is bounded by $(R + d(\Lambda))^2/\text{Area}(P(\Lambda))$.

The distance of each two lattice point of $\omega_0 + \Lambda$ on the circle $|\alpha| = R$ is always less than or equal to $d(\Lambda)$, hence the number of lattice points $\alpha \in \omega_0 + \Lambda$ with $|\alpha| = R$ is bounded by $2\pi R/d(\Lambda)$. $\square$

**Lemma 5.2.** For any $x > 0$, we have the following estimations:

$$
U_n(x) := \sum_{\alpha \in \gamma(n, \mathcal{O}_F) : N(\alpha) \leq x} 1 \ll_D N(f(\varphi))^{1/2} + 1
$$

$$
u(x) := \sum_{0 \neq \alpha \in \mathcal{O}_F : N(\alpha) = x} 1 \ll_D x^{1/2}
$$

**Proof.** We can rewrite $U_n(x)$ as

$$
U_n(x) = \# \{ \alpha \in \gamma(n, \mathcal{O}_F) : N(\alpha) \leq x \}.
$$

So we can apply Proposition 5.1 for $R = x^{1/2}$, $\omega_0 + \Lambda = \gamma(n, \mathcal{O}_F)$. Note that we have

$$
d(\Lambda) \leq \frac{\sqrt{|D_F|} + 9}{2} N(f(\varphi))^{1/2}, \quad \text{Area}(P(\Lambda)) = \frac{\sqrt{|D_F|} N(f(\varphi))}{2}.
$$

Similarly as above, we can rewrite $\nu(x)$ as

$$
\nu(x) = \# \{ \alpha \in \mathcal{O}_F : N(\alpha) = x \}.
$$

Applying Proposition 5.1 for $R = x^{1/2}$, $\omega_0 + \Lambda = \mathcal{O}_F$, and $d(\Lambda) \geq \sqrt{4 + |D_F|}/2$, we are done. $\square$

From now on, we do not consider the variables $D_F$, $w_F$, and $\gamma$ in the estimations. Also we will write $V_{k,F}$ as $V_k$. From Lemma 5.2 we can estimate the last term (5.2) of the averaged L-value by following:

**Lemma 5.3.** Let $\gamma \in \mathcal{O}_F$ such that $N(\gamma) \leq (N(f(\varphi))p^{-n_0})^{1/|W|}$ and coprime to $p$. For $\varepsilon > 0$ and $t > 0$, we have that

$$
\sum_{n \in W \setminus 0 \neq \alpha \in \mathcal{O}_F} \frac{a_{w,F}(\alpha)\varphi_w^{\varepsilon}(\alpha\gamma)}{N(\alpha)^{k/2}} V_k \left( \frac{N(\alpha)^{1/2}}{N(f(\varphi))^{1/2} t} \right) \ll_{\varepsilon,k,n_0,p} N(f(\varphi))^{(1/2 + \theta + \varepsilon)} t^{2(\theta + \varepsilon + 1)} + N(f(\varphi))^{1/2} t^2.
$$
Proof. Note that $W_n f$ is also an Hecke eigenform by Proposition 3.1. Thus we can use the bound (5.3). By Proposition 4.4 and (5.3), we obtain
\[
\sum_{0 \neq \alpha \in \mathcal{O}_F} \frac{aw_n f(a) \varphi_{\alpha \gamma}(\alpha \gamma)}{N(\alpha)^{k/2}} V_k \left( \frac{N(\alpha)^{\frac{1}{2}}}{N(f(\varphi))^{1/2} t} \right)
\]
\[(5.7) \ll \varepsilon, N_0, p \frac{1}{N(f(\varphi))^{1/2}} \sum_{0 \neq \alpha \in \mathcal{O}_F} N(\alpha)^{\theta + \varepsilon - \frac{1}{12}} V_k \left( \frac{N(\alpha)^{\frac{1}{2}}}{N(f(\varphi))^{1/2} t} \right)\]
\[= \frac{1}{N(f(\varphi))^{1/2}} \sum_{m > N(\alpha) = m} m^{\theta + \varepsilon - \frac{1}{12}} V_k \left( \frac{m^{1/2}}{N(f(\varphi))^{1/2} t} \right).\]

By Lemma 5.2, we can provide a bound of (5.7) by
\[(5.8) \ll \varepsilon, N_0, p \frac{1}{N(f(\varphi))^{1/2}} \sum_{m > 0} m^{\theta + \varepsilon} V_k \left( \frac{m^{1/2}}{N(f(\varphi))^{1/2} t} \right)\]

We can split (5.8) into two parts $I + II$:
\[
I = \frac{1}{N(f(\varphi))^{1/2}} \sum_{m > N(f(\varphi)) t^2} m^{\theta + \varepsilon} V_k \left( \frac{m^{1/2}}{N(f(\varphi))^{1/2} t} \right),
\]
\[
II = \frac{1}{N(f(\varphi))^{1/2}} \sum_{0 < m \leq N(f(\varphi)) t^2} m^{\theta + \varepsilon} V_k \left( \frac{m^{1/2}}{N(f(\varphi))^{1/2} t} \right).
\]

Using Proposition 3.6, we have
\[
I \leq \frac{1}{N(f(\varphi))^{1/2}} \int_{N(f(\varphi)) t^2}^{\infty} x^{\theta + \varepsilon} V_k \left( \frac{x^{1/2}}{N(f(\varphi))^{1/2} t} \right) dx\]
\[= 2N(f(\varphi))^{\frac{1}{2} + \theta + \varepsilon} t^2(\theta + \varepsilon + 1) \int_{1}^{\infty} u^{2(\theta + \varepsilon) + 1} V_k(u) du \ll_k N(f(\varphi))^{\frac{1}{2} + \theta + \varepsilon} t^2(\theta + \varepsilon + 1)\]

\[
II \ll \varepsilon, k \frac{1}{N(f(\varphi))^{1/2}} \sum_{0 < m \leq N(f(\varphi)) t^2} 1 = N(f(\varphi))^{1/2} t^2\]
which concludes our proof. \qed

An estimation of (5.6) can be obtained by following:

**Lemma 5.4.** Let $\gamma \in \mathcal{O}_F$ such that $N(\gamma) \leq (N(f(\varphi)) p^{-n_0})^{1/|W|}$ and coprime to $p$. For $\varepsilon, t > 0$ and $y > N(f(\varphi)) p^{-n_0}$, we have that
\[
\sum_{\kappa \in W} \sum_{\alpha \in \mathcal{O}_F} a_f(\alpha) \varphi_{\alpha \gamma}(\alpha \gamma) \frac{N(\alpha)^{k/2}}{N(f(\varphi))^{1/2} t} V_k \left( \frac{N(\alpha)^{\frac{1}{2}}}{N(f(\varphi))^{1/2} t} \right)\]
\[(5.9) \ll \varepsilon, k, N_0, p \frac{1}{N(f(\varphi))^{1/2} t} \frac{1}{N(f(\varphi))} + \frac{y^{\frac{1}{2} + \theta + \varepsilon}}{N(f(\varphi))} + \frac{1}{N(f(\varphi))^{1/2} t^2} + \left( \frac{y}{N(f(\varphi))} + t^{k - \frac{1}{2}} \left( \frac{y}{N(f(\varphi))} \right)^{\frac{\theta + 1}{2}} \right) \exp \left( -\frac{y^{1/2} t}{N(f(\varphi))^{1/2}} \right)\]
as $y^{1/2} t / N(f(\varphi))^{1/2}$ tends to the infinity.
Proof. We decompose (5.4) into three parts \((\ast)+(\ast\ast)+(\ast\ast\ast)\):

\[(\ast) = \sum_{\kappa \in W} \sum_{\alpha \in \gamma(\kappa \Gamma_{n-\eta_0} \cap \mathcal{O}_F)} (N(f(\varphi)))^{p^{-\eta_0}1/|W|} < N(\alpha) \leq N(f(\varphi))^{p^{-\eta_0}} \]

\[(\ast\ast) = \sum_{\kappa \in W} \sum_{\alpha \in \gamma(\kappa \Gamma_{n-\eta_0} \cap \mathcal{O}_F)} \text{ and } (\ast\ast\ast) = \sum_{\kappa \in W} \sum_{\alpha \in \gamma(\kappa \Gamma_{n-\eta_0} \cap \mathcal{O}_F)} .

Using Proposition 3.6, the bound (5.3) and Lemma 5.2, \((\ast)\) is bounded by (5.10)

\[(\ast) \ll_{\varepsilon,p} \sum_{\alpha \in \gamma(\kappa \Gamma_{n-\eta_0} \cap \mathcal{O}_F)} \frac{1}{N(\alpha)^{2/\theta - \varepsilon}} \ll \frac{1}{N(f(\varphi))^{(2/\theta - \varepsilon)/1/|W|}}.

By using Proposition 3.6, the bound (5.3) and the Abel summation formula, \((\ast\ast)\) is bounded by

\[(\ast\ast) \ll_{\varepsilon,k,n_0,p} \sum_{\alpha \in \gamma(\kappa \Gamma_{n-\eta_0} \cap \mathcal{O}_F)} \frac{1}{N(\alpha)^{2/\theta - \varepsilon}} - \sum_{\alpha \in \gamma(\kappa \Gamma_{n-\eta_0} \cap \mathcal{O}_F)} \frac{1}{N(f(\varphi))^{2/\theta - \varepsilon}}

+ \left(\frac{1}{2} - \theta - \varepsilon\right) \int_{N(f(\varphi))}^y \sum_{\alpha \in \gamma(\kappa \Gamma_{n-\eta_0} \cap \mathcal{O}_F)} \frac{1}{x^{2/\theta - \varepsilon}} dx.

By Lemma 5.2, the integral (5.11) is bounded by

\[\ll_{n_0} \frac{y^{\theta + \varepsilon}}{N(f(\varphi))} + \frac{1}{N(f(\varphi))^{2/\theta - \varepsilon}}.

Also \((\ast\ast\ast)\) can be estimated similarly as (5.10) by using the Abel summation formula:

\[(\ast\ast\ast) \ll_p \sum_{\alpha \in \gamma(\kappa \Gamma_{n-\eta_0} \cap \mathcal{O}_F)} V_k \left(\frac{N(\alpha)^{1/2} t}{N(f(\varphi))^{1/2}}\right) - \sum_{\alpha \in \gamma(\kappa \Gamma_{n-\eta_0} \cap \mathcal{O}_F)} V_k \left(\frac{y^{1/2} t}{N(f(\varphi))^{1/2}}\right)

+ \int_y^\infty \sum_{\alpha \in \gamma(\kappa \Gamma_{n-\eta_0} \cap \mathcal{O}_F)} \left(- \frac{d}{dx} V_k \left(\frac{x^{1/2} t}{N(f(\varphi))^{1/2}}\right)\right) dx.

By Lemma 5.2, the integral (5.12) is bounded by

\[\ll_{n_0} \frac{y}{N(f(\varphi))} V_k \left(\frac{y^{1/2} t}{N(f(\varphi))^{1/2}}\right) + \frac{1}{t^2} \int_y^\infty x^{1/2} \left(- V_k \left(\frac{x^{1/2} t}{N(f(\varphi))^{1/2}}\right)\right) dx

= \frac{y}{N(f(\varphi))} V_k \left(\frac{y^{1/2} t}{N(f(\varphi))^{1/2}}\right) + \frac{1}{t^2} \int_{u^{1/2}}^{\infty} u^2 V_k(u) du.\]
By Proposition 6.6 an estimation for \( \text{6.13} \) can be provided by
\[
\ll_k \left( \frac{y}{N(f(\varphi))} + t^{k - \frac{1}{2}} \left( \frac{y}{N(f(\varphi))} \right)^{2k+1} \right) \exp \left( \frac{-y^{1/2}t}{N(f(\varphi))^{1/2}} \right)
\]
as \( \frac{y^{1/2}t}{N(f(\varphi))^{1/2}} \) tends to the infinity.

We have the following estimation on the averaged special \( L \)-values:

**Theorem 5.5.** Let \( \gamma \in \mathcal{O}_F \) which is coprime to \( p \) and \( N(\gamma) \leq (N(f(\varphi)))^{p^{-n_0}1/|W|} \).

For \( a > 1, b > 0 \) with \( a - 2b > 1 \), and \( \varepsilon, j > 0 \), an estimation of \( L_{\text{av}}(f \otimes \varphi, \gamma) \) is given by

\[
L_{\text{av}}(f \otimes \varphi, \gamma) = \frac{a_f(\gamma\mathcal{O}_F)}{w_F N(\gamma)^{k/2}} \ll_{\varepsilon, j, k, n_0, p} \frac{1}{N(f(\varphi))^{(\frac{1}{4}+\varepsilon)k}} \log \left( \frac{1}{N(f(\varphi))^{(\frac{1}{4}+\varepsilon)k}} \right) + \frac{1}{N(f(\varphi))^{(\frac{1}{4}-\theta-\varepsilon)k\varepsilon}} + \frac{1}{N(f(\varphi))^{(\frac{1}{4}+\theta+\varepsilon)}}
\]
as \( N(f(\varphi)) \) tends to the infinity.

**Proof.** Set \( y = N(f(\varphi))^a \) and \( t = N(f(\varphi))^b \) for \( a > 1, b > 0 \) with \( a - 2b > 1 \), then by \( \text{5.10}, \text{5.11} \), Proposition \( \text{5.6} \), Lemma \( \text{5.3} \) and Lemma \( \text{5.4} \) we are done.

\( \square \)

### 6. Non-vanishing of \( L \)-values and Determination of newforms

In this section, let \( \theta = 7/64 \) be the bound of the exponent of Hecke eigenvalues which is obtained by Nakasuji [B]. We have the non-vanishing of the special \( L \)-values as a corollary of Theorem \( \text{5.3} \).

**Corollary 6.1.** For a normalized Hecke eigenform \( f \in S_k(n, \chi) \), we have

\[
L \left( \frac{k}{2}, f \otimes \varphi \right) \neq 0
\]
for almost all \( \varphi \in \Xi_p \).

**Proof.** To make the error terms of \( \text{6.14} \) converges to 0 when \( N(f(\varphi)) \) goes to \( \infty \), \( a \) and \( b \) must satisfy the following inequalities:

\[
a > 1, \quad a + \left( \frac{1}{2} + \theta \right) < 1, \quad a - 2b > 1, \quad 2b(\theta + 1) > \theta + \frac{1}{2}, \quad 4b > 1.
\]

Or equivalently,

\[
\frac{2\theta + 1}{4(\theta + 1)} < b < \frac{a - 1}{2} < \frac{1 - 2\theta}{4\theta + 2}.
\]

We can find \( a \) and \( b \) satisfying the above inequalities if \( \theta \) is less than \( \frac{1 + \sqrt{17}}{8} = 0.140388 \). By [B, Corollary 1.2], one has \( \theta = \frac{7}{64} = 0.109375 \), hence we can choose such \( a \) and \( b \). Note that for any \( \gamma \in \mathcal{O}_F \), there exists \( \varphi \in \Xi_p \) such that \( N(\gamma) < N(f(\varphi))p^{-n_0} \). Therefore, by Theorem \( \text{5.3} \) we have

\[
\lim_{N(f(\varphi)) \to \infty} L_{\text{av}}(f \otimes \varphi, \gamma) = \frac{a_f(\gamma\mathcal{O}_F)}{w_F N(\gamma)^{k/2}}
\]
for any $\gamma \in O_F$ which is coprime to $p$. By Putting $\gamma = 1$ in (6.1), we obtain

$$L_{av}(f \otimes \varphi, 1) = \frac{1}{[K_f(\varphi) : K_f]} \sum_{\sigma \in \text{Gal}(K_f(\varphi)/K_f)} L\left(\frac{k}{2}, f \otimes \varphi^\sigma\right) \neq 0$$

for all but finitely many $\varphi \in \mathbb{F}_p$. On the other hand, by [2, Theorem 11.1], we have

(6.2) $$L\left(\frac{k}{2}, f \otimes \varphi\right)^\sigma = 0 \text{ if and only if } L\left(\frac{k}{2}, f^\sigma \otimes \varphi^\sigma\right) = 0$$

for any $\sigma \in \text{Aut}_Q(C)$. Assume that $L\left(\frac{k}{2}, f \otimes \varphi\right) \neq 0$ for infinitely many $\varphi \in \mathbb{F}_p$. Then by (6.2), we have $L_{av}(f \otimes \varphi, 1) = 0$ for infinitely many $\varphi \in \mathbb{F}_p$, which is a contradiction. \hfill \Box

As a corollary of Theorem 5.5, we can determine a newform by its twisted special $L$-values:

**Corollary 6.2.** Let $f \in S_k(n, \chi)$ and $g \in S_{k'}(n', \chi')$ be newforms. Assume that there exist constants $B$ and $C$ such that for infinitely many $\varphi \in \mathbb{F}_p$ and $n(\varphi) \in \mathbb{Z}_{\geq 0}$,

$$L\left(\frac{k}{2}, f \otimes \varphi\right) = B^n(\varphi) C \cdot L\left(\frac{k'}{2}, g \otimes \varphi\right)$$

where $n(\varphi) \to \infty$ as $N(f(\varphi)) \to \infty$. Then we have $f = g$.

**Proof.** From our assumption and (6.1) we have

(6.3) $$\frac{a_f(\gamma O_F)}{w F N(\gamma)^{k/2}} = \lim_{N(f(\varphi)) \to \infty} B^n(\varphi) C \cdot \frac{a_g(\gamma O_F)}{w F N(\gamma)^{k'/2}}$$

for any $\gamma \in O_F$ coprime to $p$. Letting $\gamma = 1$ in (6.3), we have

$$\lim_{N(f(\varphi)) \to \infty} B^n(\varphi) C = 1$$

which implies that $B = C = 1$. Therefore, by (6.3), we have an equality

$$\frac{a_f(\gamma O_F)}{N(\gamma)^{k/2}} = \frac{a_g(\gamma O_F)}{N(\gamma)^{k'/2}}$$

for any $\gamma \in O_F$ coprime to $p$, which implies that $f = g$ by the strong multiplicity one theorem [3]. \hfill \Box

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**References**

[1] J. Bygott, Modular forms and modular symbols over imaginary quadratic fields, PhD Thesis, University of Exeter, 1998.
[2] H. Hida, On the critical values of $L$-functions of $GL(2)$ and $GL(2) \times GL(2)$, Duke Math. J. 74 (1994), no. 2, 431-529.
[3] H. Jacquet and J. A. Shalika, On Euler products and the classification of automorphic representations. I. Amer. J. Math. 103 (1981), no. 3, 499-558.
[4] M. Kim and H. Sun, Modular symbols and modular $L$-values with cyclotomic twists, Submitted.
[5] W. Luo and D. Ramakrishnan, Determination of modular forms by twists of critical $L$-values, Invent. Math. 130 (1997), no. 2, 371-398.
[6] M. Nakasuji, Generalized Ramanujan conjecture over general imaginary quadratic fields, Forum Math. 24 (2012), no. 1, 85-98
[7] K. Namikawa, On mod $p$ nonvanishing of special values of $L$-functions associated with cusp forms on $GL_2$ over imaginary quadratic fields, Kyoto J. Math. 52 (2012), no. 1, 117-140
[8] K. Namikawa, On $p$-adic $L$-function associated with cusp forms on $GL_2$, Manuscripta Math. 153 (2017), no. 3-4, 563-622
[9] J. Neukirch, Algebraic number theory, Springer-Verlag (1992).
[10] D. E. Rohrlich, Nonvanishing of $L$-functions for $GL(2)$, Invent. Math. 97 (1989), 391-403.
[11] D. E. Rohrlich, On $L$-functions of elliptic curves and cyclotomic towers, Invent. Math. 75 (1984), 409-423.
[12] G. Shimura, Introduction to the arithmetic theory of automorphic functions, Princeton University Press, 1971.
[13] H. Sun, Generation of cyclotomic Hecke fields by modular $L$-values with cyclotomic twists, to appear in Amer. J. Math.

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