D-particle polarizations with multipole moments of higher-dimensional branes

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Abstract

We study the polarization states of the D0-brane in type IIA string theory. In addition to states with angular momentum and magnetic dipole moments, there are polarization states of a single D0-brane with nonzero D2-brane dipole and magnetic H-dipole moments, as well as quadrupole and higher moments of various charges. These fundamental moments of the D0-brane polarization states can be determined directly from the linearized couplings of background fields to the D0-brane world-volume fermions. These couplings determine the long range supergravity fields produced by a general polarization state, which typically have non-zero values for all the bosonic fields of type IIA supergravity. We demonstrate the precise cancellation between spin-spin and magnetic dipole-dipole interactions and an analogous cancellation between 3-form and H-field dipole-dipole interactions for a pair of D0-branes. The first of these cancellations follows from the fact that spinning D0-brane states have gyromagnetic ratio $g = 1$, and the second follows from the fact that the ratio between the 3-form and H-field dipole moments is also 1 in natural units. Both of these relationships can be seen immediately from the couplings in the D0-brane world-volume action.
1 Introduction

Understanding the relationship between black brane solutions of supergravity carrying R-R charges and Dirichlet branes in string theory has been one of the principal themes in the second superstring revolution, following Polchinski’s seminal paper relating these objects \[1\]. A substantial literature is now devoted to the construction of BPS-saturated supergravity solutions describing various brane configurations including multiple branes, branes with angular momentum, and other interesting features (see for example \[2, 3\] for reviews of some of the earlier literature on this subject).

The most basic supersymmetric black brane solutions of supergravity are the bosonic R-R charged black \(p\)-brane solutions first described in \[4\]. These supergravity solutions correspond to the field configurations around some number of coincident Dirichlet \(p\)-branes in string theory. A particularly simple example of these solutions is the 10-dimensional supersymmetric extremal black hole in type IIA supergravity which is charged under the R-R vector field. This corresponds to the supergravity field configuration around a D-particle (D0-brane) of type IIA string theory.

Type IIA supergravity is related to 11-dimensional supergravity by dimensional reduction on a single circle of radius \(R = g l_s\), where \(g\) is the string coupling and \(l_s\) is the string length. Via this reduction, the D0-brane of type IIA string theory is naturally related to the KK particle associated with a supergraviton of momentum \(1/R\) around the compact direction of the 11D theory. The 11-dimensional supergraviton has 256 polarizations, and these lead to 256 polarization states of a D0-brane in the IIA theory.

From the point of view of the D0-brane worldvolume theory, the polarization states of the D0-brane arise from quantizing the fermions on the world-volume of the brane. These worldvolume fermions have couplings to the background fields of type IIA supergravity, and these couplings give rise to different physical properties for the different polarization states. In this paper we classify the various polarization states of the D0-brane and study their physical properties. In particular, we will see that different polarization states generate different long range fields, and determine the long range supergravity fields corresponding to an arbitrary D0-brane polarization state.

In a previous paper \[5\] we found an explicit description of the world-volume action of a system of multiple D0-branes in terms of the component fields on the world-volume, up to terms describing the linear coupling to a general supergravity background. Restricting this action to a single D0-brane located at the origin with zero velocity, we find that the world-volume fermions couple to small fluctuations in the background metric \(h_{\mu\nu}\), NS-NS 2-form field \(B_{\mu\nu}\) and R-R fields \(C_\mu, C_{(3)}^\mu\) through the terms

\[
S_{linear} = m_0(h_{0i,j}(0) + C_{i,j}(0)) \frac{i}{8} \theta \gamma^{ij} \theta + m_0(B_{ij,k}(0) + C_{0ij,k}^{(3)}(0)) \frac{i}{16} \theta \gamma^{ijk} \theta + O(\theta^4) \tag{1}
\]

where \(m_0 = \frac{1}{g_s \sqrt{\alpha'}}\) is the D0-brane mass, \(\theta\) is the 16-component spinor and \(\gamma^i\) generate the 16-dimensional representation of the \(SO(9)\) Clifford algebra. These fermionic couplings for a
single D0-brane were also previously found by Morales, Scrucca and Serone. From (1) it is clear that certain choices of polarization for the D0-brane fermions will have nonvanishing spin angular momentum (coupling to $h_{0,i,j}$), magnetic D0-brane dipole moment (coupling to $C_{i,j}$), D2-brane dipole moment (coupling to $C_{0(i,j)}^{(3)}$) and magnetic H-dipole moment (coupling to $B_{i,j,k}$). Furthermore, it is manifestly clear that the gyromagnetic ratio between the magnetic D0-brane dipole moment and spin angular momentum is always $g = 1$ and that similarly the D2-brane dipole moment and magnetic H-dipole moment have a ratio $g_{2H} = 1$.

It was shown in [7] using supergravity methods that some D0-brane polarization states carry angular momentum and magnetic D0-dipole moment, and that the gyromagnetic ratio of 1 implies that the spin-spin and dipole-dipole interactions precisely cancel so that the BPS zero force condition is satisfied. As we show here, the agreement between the D2-dipole moment and magnetic H-dipole moment for any polarization state similarly guarantees that these two types of dipole-dipole interactions will also precisely cancel for pairs of D0-branes.

Since the different polarization states of a D0-brane couple differently to the supergravity background fields, they give rise to different supergravity solutions. These solutions are the “superpartners” of the bosonic D0-brane solution, obtained by acting on the bosonic solution with broken supersymmetry generators. Whereas the bosonic D0-brane solution has only dilaton, metric and RR one-form fields turned on, the solutions corresponding to arbitrary D0-brane polarizations generically have non-zero values for all the bosonic fields of type IIA supergravity including the RR three-form and NS-NS two form. The couplings we derive allow us to directly determine the long range supergravity fields corresponding to an arbitrary polarization state of the D0-brane.

In this paper we will focus on the couplings between a single D0-brane and the background supergravity fields. In [5], we determined the analogous couplings for a system of multiple D0-branes. In the case of many branes, the appearance of noncommuting matrices describing the space-time brane configuration can lead to multipole moments of higher-dimensional branes encoded in the bosonic matrix-valued brane coordinates. For example, a system of many D0-branes couples to the RR 3-form through a term of the form

$$C_{0(i,j)}^{(3)} \text{Tr } X^i [X^j, X^k].$$

It was pointed out by Myers [9] that as a consequence of this coupling, the presence of a background 4-form field strength $C_{0(i,j)}^{(3)}$ produces a polarized D0-brane configuration corresponding to a fuzzy D2-brane sphere. In this paper we show that such polarization can occur even for a single D0-brane, since in the presence of a background 4-form, the couplings (4) will break the degeneracy of the 256 states in the multiplet associated with even a single D0-brane. We emphasize that for a single D0-brane, the multipole moments we find are fundamental moments, like the magnetic dipole moment of an electron, whereas in Myers’ dielectric effect, the configurations exhibiting D2-brane dipole moments are spatially extended.

The paper is organized as follows. In section 2, we discuss the action for a single D0-brane in type IIA supergravity, and in particular, derive the linear couplings of the bulk fields to
the worldvolume fermions of a static D0-brane. In addition to the quadratic terms found in previous work, we derive higher order terms with up to 8 fermions which correspond to higher moments of various conserved quantities. We also provide an alternate derivation of the quadratic fermion terms from the $\kappa$-symmetric abelian D0-brane action, including an extension of the linearized result to the full non-linear D0-brane action to second order in the worldvolume fermion fields. In section 3 we discuss the supergravity solutions corresponding to the various states and in particular calculate the leading long range fields for an arbitrary polarization state. The expressions that we obtain depend on operators built out of the worldvolume fermions. In order to obtain the physical values of the fields for a given state, it is necessary to take the expectation values of these operator valued expressions in the state of interest. In section 4, we describe an explicit representation of the D0-brane states and develop the machinery necessary to evaluate expectation values of any worldvolume fermion operators for arbitrary states. We also classify the states by providing an orthonormal basis of eigenstates for a maximally commuting set of fermion operators. Section 5 contains a discussion of the interaction between a pair of D0-branes, and in particular describes the complete cancellation of dipole-dipole forces. We conclude in Section 6 with a brief discussion.

2 D0-background couplings

At low energies, a single D0-brane is described by simple non-relativistic quantum mechanics, with nine coordinates $X^i$ and dual momenta $P_i$ obeying the canonical commutation relations

$$[X^i, P_j] = i\delta^i_j$$

In addition, there are 16 fermionic operators $\theta_\alpha$ which obey

$$\{\theta_\alpha, \theta_\beta\} = \delta_{\alpha\beta}$$

This is the 16-dimensional Clifford algebra, so we may represent the $\theta$'s by $2^{16/2} = 256$ dimensional gamma matrices, showing that the D0-brane has 256 independent states.

The flat space Hamiltonian governing the low-energy D0-brane is the nonrelativistic action for a free particle

$$H = \frac{1}{2} P_i P_i$$

which makes no reference to the fermions, so the 256 states are degenerate and apparently indistinguishable.

However, as we will discuss in the next subsection, the fermionic operators $\theta$ do couple to background fields, even for a static D0-brane. Thus, turning on various background supergravity fields will break the degeneracy between the D0-brane states. Conversely, a D0-brane in flat space-time will generate a different set of supergravity fields depending on the state it is in. Thus, it is important to determine the couplings of worldvolume fermions...
to background fields in order to understand the different physical properties of the various polarization states.

2.1 Couplings to background fields

Ignoring the worldvolume fermions, the classical action for a single D0-brane in the presence of background type IIA supergravity fields is given in string frame by

\[ S_{\text{bos}} = -\mu_0 \int e^{-\phi} ds + \mu_0 \int C \]

where \( \mu_0 = (\alpha')^{-\frac{3}{2}} \) is the D0-brane charge, \( \phi \) is the dilaton field, \( C \) is the RR one-form field, and \( ds = \sqrt{g_{\mu\nu}x^\mu x^\nu} d\tau \). Since the D0-brane mass is given by \( \frac{\mu_0}{g_s} = \mu_0 e^{-\phi} \), this is essentially the action for a relativistic charged particle in a gravitational field. Together with the bulk type IIA supergravity action, this gives rise to the usual D0-brane supergravity solution, given in Einstein frame by

\[ ds^2 = -H^{-\frac{7}{8}} dt^2 + H^{\frac{1}{8}} d\vec{x}^2, \quad e^\phi = H^{-\frac{3}{4}}, \quad C_0 = H^{-1} \]

where \( H \) is a harmonic function,

\[ H = 1 + \frac{60\pi^3 g_s (\alpha')^\frac{7}{2}}{r^7} \]

In particular, we note that only the metric, dilaton, and RR one-form field are involved in the bosonic worldvolume action and supergravity solution.

The story becomes more interesting with the inclusion of fermion fields. In a previous paper \[5\], we have derived leading terms in the D0-brane action coupling linearly to all type IIA supergravity background fields up to quadratic order in the fermion fields. At this order, the Lagrangian for a static D0-brane contains four terms involving fermions, namely (omitting an overall factor of \( \mu_0 \))

\[ (\partial_i h_{0j} + \partial_i C_j) \frac{i}{8} \theta \gamma^{ij} \theta + (\partial_i B_{jk} + \partial_i C_{0jk}) \frac{i}{16} \theta \gamma^{ijk} \theta \]

where \( \gamma^{ij} \) and \( \gamma^{ijk} \) are antisymmetrized products of 16 × 16 symmetric Dirac matrices. These couplings agree with those found earlier in \[3\]. The operator \( J^{ij} \equiv -\frac{i}{4} \theta \gamma^{ij} \theta \) represents the fermionic part of the angular momentum (intrinsic spin) of the D0-brane \[10\] and has the expected coupling to the first spatial derivative of the metric component \( h_{0i} \). The coupling of \( J^{ij} \) to \( \partial_i C_j \) indicates that D0-branes carry a magnetic dipole moment (with respect to the RR one-form field) proportional to the angular momentum, so a D0-brane has gyromagnetic ratio 1. This fact, derived earlier as a property of D0-brane supergravity solutions \[7\], is transparent from the form of the worldvolume couplings.

The second pair of couplings involves the operator \( D^{ijk} \equiv \frac{i}{4} \theta \gamma^{ijk} \theta \). The coupling to \( \partial_i C_{0jk}^{(3)} \) indicates that certain polarization states of the D0-brane carry dipole moments of D2-brane
charge, while the coupling to $\partial_i B_{jk}$ indicates that these states also have magnetic H-dipole moments, where H is the field strength of the NS-NS two form field. Again, it is clear that the ratio of these dipole moments is 1 in natural units.

The couplings (4) were derived in [3] using the results of a Matrix theory calculation and exploiting the relationship between Matrix theory and D0-branes in type IIA string theory. In [4], these couplings were determined by using the Green-Schwarz boundary state formalism to compute the interaction between a pair of D0-branes. Both of these methods are somewhat indirect. In principle, the extension of the action (2) to include fermions could also be determined from the known $\kappa$-symmetric D0-brane action which is written compactly using a $D = 10$ superspace formalism. However, to derive the couplings of worldvolume fermions to the background fields from this action, it is necessary to expand the superfields in terms of the component fields of type IIA supergravity and the worldvolume fields, a non-trivial procedure which has not yet been carried out. In subsection (2.4), we will perform this expansion explicitly to quadratic order in the worldvolume fermions as a check. These methods also yield the extension of (4) to the complete non-linear D0-brane action up to quadratic order in $\theta$.

Before discussing higher order terms, we note that there is a very natural understanding of the ratios $g = 1$ and $g_{2H} = 1$ based on the 11-dimensional origin of the D0-brane action. In the Matrix theory action describing a single graviton with zero transverse momentum in DLCQ supergravity [11], there are two couplings between background $D = 11$ supergravity fields and bilinears of the Matrix theory fermions (which describe the supergraviton polarizations), proportional to $\partial_j h_{+i} \theta \gamma^{ij} \theta$ and $\partial_k A_{+ij} \theta \gamma^{ijk} \theta$. Recalling the relations (to linear order)

$$C^I I A_{i} = h^1_{10} A_{i}, \quad h^I_{0i} = h^1_{0i}, \quad B^I A_{ij} = A^1_{10 ij}, \quad C^I A_{ij} = A^1_{0ij}$$

we see that the first two couplings in (4) arise from the single term $\partial_j h_{+i} \theta \gamma^{ij} \theta$ term in eleven dimensions, leading to $g = 1$, while the second pair of couplings in (4) arises from the single term $\partial_k A_{+ij} \theta \gamma^{ijk} \theta$ term in eleven dimensions, leading to $g_{2H} = 1$. For a more precise discussion of the relationship between the Matrix theory action and D0-brane action, see [5].

### 2.2 Couplings with more than two fermions

Additional couplings between fermions and background fields exist with four, six, eight and possibly up to sixteen fermions. One way to derive these would be to extend the Matrix theory calculation of [11] to higher orders in $1/r$ and to higher order in the fermion fields. The required calculation would be rather tedious, but we will be able to deduce some of these couplings here based on other considerations.

In general, we may write the linear couplings of background fields to the worldvolume fields in the D0-brane action as

$$\int dt \sum_{n=0}^{\infty} \frac{1}{n!} \left[ \frac{1}{2} (\partial_k \cdots \partial_k h_{\mu\nu}) I^\mu_{\nu(k_1 \cdots k_n)} + (\partial_k \cdots \partial_k \phi) I^{(k_1 \cdots k_n)}_{\phi} \right]$$

(5)
\[
(\partial_{k_1} \cdots \partial_{k_n} C^\mu_{\lambda}) I^{(k_1, \cdots, k_n)}_o + (\partial_{k_1} \cdots \partial_{k_n} \tilde{C}^\mu_{\lambda\rho\sigma\tau\zeta}) I^{\mu\nu\lambda\rho\sigma\tau\zeta(k_1, \cdots, k_n)}_6
\]
\[
+ (\partial_{k_1} \cdots \partial_{k_n} B_{\mu\nu}) I^{\mu\nu(k_1, \cdots, k_n)}_s + (\partial_{k_1} \cdots \partial_{k_n} \tilde{B}_{\mu\nu\lambda}) I^{\mu\nu\lambda\rho\sigma}(k_1, \cdots, k_n)_5
\]
\[
+ (\partial_{k_1} \cdots \partial_{k_n} C^{(3)}_{\mu\nu\lambda}) I^{\mu\nu\lambda(k_1, \cdots, k_n)}_2 + (\partial_{k_1} \cdots \partial_{k_n} \tilde{C}^{(3)}_{\mu\nu\lambda\rho}) I^{\mu\nu\lambda\rho\sigma}(k_1, \cdots, k_n)_4
\]
\]

Here, \( I^{(k_1, \cdots, k_n)}_o \), which couples to \( n \) derivatives of the metric, represents the \( n \)-th spatial moment of the stress energy tensor for a D0-brane, while the remaining operators \( I \) represent moments of various other currents.

We would like to derive expressions for the moments \( I \) for a static D0-brane at the origin. The couplings for a moving D0-brane (velocity dependent terms) could then be deduced by performing a boost on this action. In the static case, the expressions \( I \) are built only out of the fermion fields \( \theta \) as well as \( \gamma \)-matrices, since we have \( X^i = \dot{X}^i = 0 \). There are various constraints that allow us to determine leading terms in the various couplings:

**Symmetries**

Of course, the D0-brane action should have \( SO(9) \) rotational invariance. This requires that all spatial indices in the action be contracted with other spatial indices.

**Consistency with T-duality**

The D0-brane action is related by T-duality to all of the higher-dimensional Dp-brane actions. For a given \( p \), the Dp-brane action must have \( p + 1 \) dimensional Lorentz invariance as well as \( SO(9 - p) \) rotational invariance. The requirement of these symmetries in the dual actions places further constraints on the form of the D0-brane action. In particular, it requires that all operators \( I \) may be written as the dimensional reduction of \( D = 10 \) Lorentz invariant objects, just as the low-energy flat-space part of the action is the dimensional reduction of \( D = 10 \) SYM theory. For example, under a T-duality of all nine spatial directions, a weak background metric transforms as \( h^{I[A} \rightarrow -h^{I[B} \). Thus, the operator coupling to \( h_{ij} \) in the D0-brane action is the T-dual version of the operator coupling to \(-h^{I[IBJ} \) in the D9-brane action. In this case, T-duality acts on the worldvolume operator simply by dimensional reduction, thus, we conclude that the operator coupling to \( h_{ij} \) in the D0-brane action must be the dimensional reduction of a \( D = 10 \) Lorentz covariant object.

The requirement of \( D = 10 \) Lorentz covariance suggests that the D0-brane action should be written most compactly using \( D = 10 \) notation for the fermions, i.e. 32-component Majorana-Weyl spinors and \( 32 \times 32 \) Dirac matrices. Taking into account the Weyl property of the spinors as well as their anticommutation relation, it is evident that all non-vanishing expressions may be built from fermion bilinears with three-index antisymmetric products of \( \Gamma \)-matrices

\[
\bar{\Theta} \Gamma^{abc} \Theta
\]

where \( a, b, c \in \{0, \ldots, 9\} \).

To summarize, the purely fermionic parts of the operators \( I \) should be constructed out of \( D = 10 \) fermion bilinears (6). The operators should display \( D = 10 \) Lorentz covariance,
thus all indices apart from those contracted with background fields or derivatives must be contracted with each other and summed from 0 to 9. In constructing operators with more than two fermions, it is necessary to take into account various Fierz identities listed in the appendix.

In many cases, these requirements completely determine the operators \( I \) up to a coefficient.

**Relation to \( N = 4 \) D = 4 chiral operators**

In some cases, the fermionic terms in the moments may be determined from the known expressions for the purely bosonic terms using supersymmetry and the AdS/CFT correspondence. For a given field \( \phi \), the lowest dimension operator coupling to \( \phi \) in the D3-brane action is essentially the chiral operator of \( \mathcal{N} = 4 \) SYM theory related to the particle \( \phi \) via the AdS/CFT correspondence. These chiral operators may be obtained by acting with supersymmetry generators on chiral primary operators of the form

\[
\text{STr} \left( X^{i_1} \cdots X^{i_n} \right) - \{\text{traces}\}
\]

By determining the correct combination of supersymmetry generators that reproduce the known bosonic terms in an operator, the purely fermionic terms may be easily deduced. This method was used in [3] to determine the four fermion terms in the operator coupling to the Einstein frame dilaton in the D3-brane action. Such purely fermionic terms in the D3-brane action may be T-dualized to give the desired fermionic couplings in the D0-brane action including coefficients.

### 2.3 Results for the D0-brane action to linear order in background fields

Assuming the validity of the considerations above, we find the following terms in the action for a static D0-brane in the presence of arbitrary type IIA supergravity fields. We omit overall factors of D0-brane mass/charge.

**zero fermion terms: charges**

\[
S_0 = \int d\sigma \left( \frac{3}{4} \phi + \frac{1}{2} h_{00} + C_0 \right)
\]

These terms, written in Einstein frame, indicate that the D0-brane is massive and charged under the RR one-form field. The coupling to the dilaton field arises because the D0-brane mass/charge.

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1. Note that the RR couplings in Dp-brane actions include a \( p + 1 \) index antisymmetric epsilon tensor when expressed in terms of components. Thus, for each coupling \( CI_C \) in the D0-brane action, either the RR field \( C \) or the tensor \( I \) must contain a single free 0 index which “contracts” with the trivial \( \epsilon \) tensor \( \epsilon^0 \).

2. This is most clear for minimally coupled scalars which are described by the same fields everywhere. For a discussion of the more general case, see [2].
mass is inversely proportional to the string coupling.

two fermion terms: dipole moments

\[
S_2 = -\frac{i}{8} \partial_j h_{0i} \bar{\Theta} \Gamma^{0ij} \Theta - \frac{i}{8} \partial_j C_i \bar{\Theta} \Gamma^{0ij} \Theta + \frac{i}{16} \partial_k B_{ij} \bar{\Theta} \Gamma^{ijk} \Theta + \frac{i}{16} \partial_k C_{0ij}^{(3)} \bar{\Theta} \Gamma^{ijk} \Theta
\]

(7)

These are exactly the terms (14) discussed above, written now in \( D = 10 \) notation. They indicate that certain polarization states may carry angular momentum, RR one-form magnetic dipole moment, D2-brane dipole moment and magnetic H-dipole moment.

four fermion terms: quadrupole moments

\[
S_4 = \frac{1}{384} \partial_k \partial_l h_{00} \bar{\Theta} \Gamma^{0ak} \Theta \bar{\Theta} \Gamma^{0al} \Theta - \frac{1}{384} \partial_k \partial_l h_{ij} \bar{\Theta} \Gamma^{aik} \Theta \bar{\Theta} \Gamma^{ajl} \Theta - \frac{1}{192} \partial_k \partial_l B_{0i} \bar{\Theta} \Gamma^{a0k} \Theta \bar{\Theta} \Gamma^{a0l} \Theta + \frac{1}{32} \partial_k \partial_l C_{ij}^{(3)} \bar{\Theta} \Gamma^{0il} \Theta \bar{\Theta} \Gamma^{jkm} \Theta + \frac{1}{128} \partial_m \partial_n C_{ijkl}^{(5)} \bar{\Theta} \Gamma^{mij} \Theta \bar{\Theta} \Gamma^{nkl} \Theta
\]

The first two terms here indicate quadrupole moments for the 00 and \( ij \) components of the stress-energy tensor. The third term indicates a quadrupole moment of string electric charge \( (B_{0i}) \). The RR couplings both correspond to a magnetic quadrupole moment of D2-brane charge or equivalently an electric quadrupole moment of D4-brane charge. These quadrupole moments were previously found in [8].

six fermion terms: octupole moments

\[
S_6 = \frac{i}{5760} \partial_j \partial_k \partial_l h_{0i} \bar{\Theta} \Gamma^{a0k} \Theta \bar{\Theta} \Gamma^{b0l} \Theta \bar{\Theta} \Gamma^{abj} \Theta + \frac{i}{11520} \partial_k \partial_l \partial_m B_{ij} \bar{\Theta} \Gamma^{aik} \Theta \bar{\Theta} \Gamma^{bjl} \Theta \bar{\Theta} \Gamma^{abm} \Theta + c_1 \partial_p \partial_q \partial_r C_{jklmn}^{(5)} \bar{\Theta} \Gamma^{0jp} \Theta \bar{\Theta} \Gamma^{kqa} \Theta \bar{\Theta} \Gamma^{mnr} \Theta + c_1 \partial_p \partial_q \partial_r C_{0ijklmn}^{(7)} \bar{\Theta} \Gamma^{ijp} \Theta \bar{\Theta} \Gamma^{kql} \Theta \bar{\Theta} \Gamma^{mnr} \Theta
\]

\(^3\)In expressions containing terms with four or more fermions, there is an ordering ambiguity that arises when promoting a classical action to an operator expression, since the \( \Theta \) operators obey a non-trivial anticommutation relation. The fermion operators described in this section are related to chiral operators of \( N = 4 \) SYM theory which are obtained by acting with supersymmetry transformations on symmetrized traces of bosonic fields. This suggests that the correct resolution of the ordering ambiguity is an antisymmetrization of the \( \Theta \)s in each of the expressions here.
Here, \( c_1 \) is a numerical constant that we have not determined. These terms indicate an octupole moment of the 0\( i \) component of the stress energy tensor, a magnetic H-octupole moment, an octupole moment of D6-brane charge and an octupole moment of D2-brane charge (or magnetic D4-brane charge).

**eight fermion terms: 16-pole moments**

At this order, we only mention one additional term,

\[
S_8 = c_2 \partial_0 \partial_q \partial_s C^{(7)}_{ijklmp} \Theta \Gamma^{0iq} \Theta \Theta \Gamma^{jkr} \Theta \Theta \Gamma^{lms} \Theta \Theta \Gamma^{np} \Theta
\]

which represents a 16-pole moment of magnetic D6-brane charge (or electric D0-brane charge).

The couplings we have listed are those related to nonvanishing terms in the action for Matrix theory in a \( D = 11 \) supergravity background. In particular, we have derived the leading terms coupling linearly to all the background supergravity fields. We expect additional terms that vanish in the Seiberg-Sen limit, possibly up to sixteen fermion terms. The anticommutation relations for the fermions ensure that no terms with more than sixteen fermions exist, so there are certainly only a finite number of moments carried by a single D0-brane, as we would expect for a point particle.

### 2.4 Fermion couplings from the superspace action.

We have derived the fermionic couplings in the D0-brane action by somewhat indirect methods, exploiting various symmetries and dualities as well as connections with Matrix theory and the AdS/CFT correspondence. In principle, we could also have derived all of these terms directly using the known \( \kappa \)-symmetric superspace action for a D0-brane. As an demonstration of this approach and a check of the results in the previous section, we now calculate the two-fermion terms directly from the superspace action.

The \( \kappa \)-symmetric action for a D0-brane is given by [14, 15, 16, 17]

\[
S = -\mu_0 \int dt \left( e^{-\frac{2}{3} \Phi} \sqrt{-\Pi_0^a \Pi_0^a} - \partial_0 Z^M \Gamma^M \right),
\]

where \( \Pi_0^a = \partial_0 Z^M E^a_M \) is the pullback of the supervielbein to the world-line of the D0-brane[4].

In order to find the coupling of the D0-brane to the component background fields, we need to expand the superfields out in terms of the component fields of type IIA supergravity. To do this, we perform an order by order expansion in the fermionic superspace coordinates.

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[4] Type IIA superspace is defined by coordinates \( Z^M \), \( M = \{ \mu, a \} \), where \( Z^\mu = X^\mu \) are the 10 bosonic coordinates and \( Z^a = \Theta^a \) is a pair of Majorana-Weyl spinors of opposite chirality combined into a single Majorana spinor. In this section, we use indices \( M, \mu, \alpha \) (and similar) to represent spacetime superfield, bosonic, and fermionic indices respectively and \( A, r, a \) (and similar) to represent tangent space superfield, bosonic and fermionic indices respectively.
Θ, in a method known as gauge completion. A similar expansion was carried out to second order for eleven-dimensional supergravity in [18], and our analysis will mirror that in their paper closely. The expansion of type IIA superspace fields in terms of component fields has also been discussed in [19]. An alternative approach to expanding the vielbein $E^r_M$ in terms of fermionic component fields was taken in [20], where the authors performed a superspace analog of the Riemann normal coordinate expansion.

The method of gauge completion involves reconciling the component forms of the supersymmetry transformations of the fields with superspace diffeomorphisms at each order in $\Theta$. In this way, starting from order $\Theta^0$ we can build up the superfields in terms of their components in an expansion in $\Theta$.

The supersymmetry transformations of the relevant fields in type IIA supergravity are listed in appendix B. A given component field supersymmetry transformation corresponds to some combination of a superspace diffeomorphism, a Lorentz transformation and a gauge transformation, whose action on the fields appearing in the D0-brane action is

$$
\delta \Phi = \Xi^M \partial_M \Phi \\
\delta E^A_M = \partial_M \Xi^N E^A_N + \Xi^N \partial_N E^A_M + \Lambda^A_B E^B_M \\
\delta B_M = \partial_M \Xi^N B_N + \Xi^N \partial_N B_M + \partial_M \Omega.
$$

Here, $\Xi^M$ is the superdiffeomorphism parameter, $\Omega$ is the gauge transformation parameter for $B$, and $\Lambda^A_B$ is the tangent space Lorentz transformation (with nonvanishing components $\Lambda^{rs}$ and $\Lambda^{a_b} = \frac{1}{4} \Lambda^{rs} (\Gamma^{rs})^{a_b}$). Each of these superspace parameters is some function of the component fields, the component field transformation parameters and the superspace coordinate $\Theta$.

At zeroth order in $\Theta$, we identify the components of the superfields and the transformation parameters as

$$
E^r_\mu = e^r_\mu \\
E^a_\mu = \psi^a_\mu \\
E^r_\alpha = 0 \\
\Phi = \phi \\
\Lambda^{rs} = \lambda^{rs}
$$

and

$$
\Xi^\mu = \xi^\mu \\
\Xi^\alpha = \epsilon^\alpha \\
\Omega = \omega
$$

where $\epsilon_a$, $\xi^\mu$, $\lambda^{rs}$, and $\omega$ parameterize the usual component field supersymmetry transformations, diffeomorphisms, local Lorentz transformations, and gauge transformations on the RR one-form respectively.

We begin by finding the transformation parameters $\Xi, \Lambda, \Omega$ to first order in $\Theta$ by demanding that the $\Theta^0$ terms in the commutator of superspace transformations applied to a superfield match with the commutator of the component field transformations on the appropriate component field. Using this, we find to order $\Theta$

$$
\Xi^\mu = \xi^\mu - \frac{i}{2} \bar{\Theta} \gamma^\mu \epsilon
$$
\[ \Xi^\alpha = \epsilon^\alpha \]
\[ \Omega = \omega + \frac{i}{2} \Theta \gamma^\rho C_\rho + \frac{i}{2} e^{-\frac{1}{2} \phi} \Theta \Gamma^{11} \epsilon \]
\[ \Lambda^{pq} = \chi^{pq} - \frac{i}{2} \Theta \gamma^\mu \epsilon \omega^{pq} - \frac{i}{64} e^{\frac{i}{2} \phi} \Theta \gamma^{pq} \lambda \Gamma^{11} \epsilon F_{\nu \lambda} - \frac{7i}{32} e^{i \phi} \Theta \Gamma^{11} \epsilon F^{pq} \]
\[ + \frac{i}{96} e^{-\frac{1}{2} \phi} \Theta \gamma^{pq} \lambda \sigma \Gamma^{11} \epsilon H_{\nu \lambda \sigma} \]
\[ + \frac{i}{256} e^{\frac{1}{2} \phi} \Theta \gamma^{pq} \lambda \sigma \tau \epsilon F_{\nu \lambda \sigma \tau} + \frac{5i}{64} e^{\frac{1}{2} \phi} \Theta \gamma_{\sigma \tau} \epsilon F^{q \alpha \beta \sigma \tau} \]

Here, and in the following, we set to zero all background fermion fields, except where they are important in determining the desired terms in the final action.

To determine the superspace fields at first order in \( \Theta \), we write down the order \( \Theta^0 \) terms in the superspace transformations (8) and demand that the resulting equations are consistent with the component field supersymmetry transformations. We find, up to ambiguities that can be removed by gauge transformations,

\[ \Phi = \phi + i \sqrt{2} \Theta \Gamma^{11} \lambda \]
\[ E^r_\mu = e^r_\mu + i \Theta \Gamma^r \psi_\mu \]
\[ E^r_\alpha = \frac{i}{2} (\Theta \Gamma^r)_\alpha \]
\[ B_\mu = C_\mu + \frac{3i \sqrt{2}}{4} e^{-\frac{1}{2} \phi} \Theta \gamma_\mu \lambda + i e^{-\frac{3}{4} \phi} \Theta \Gamma^{11} \psi_\mu \]
\[ B_\alpha = -\frac{i}{2} e^{-\frac{3}{4} \phi} (\Theta \Gamma^{11})_\alpha \]

Note that the order \( \Theta \) terms in the bosonic components of the superspace fields are precisely the supersymmetry variations of the order \( \Theta^0 \) terms in these fields with supersymmetry variation parameter \( \Theta \), as we would expect.

The calculation of order \( \Theta^2 \) terms in the superspace fields and transformation parameters proceeds in the same way as for order \( \Theta \). In order to evaluate the order \( \Theta^2 \) terms in the D0-brane action, it is only necessary to determine \( E^r_\mu, B_\mu, \) and \( \Phi \) to order \( \Theta^2 \), and these may be deduced immediately by demanding that the order \( \Theta \) terms in (8) agree with the component field supersymmetry transformations. The resulting expressions are somewhat complicated, but may be simplified considerably by fixing \( \kappa \)-symmetry. We set half of the 32 components of \( \Theta \) to zero making the gauge choice \( \frac{1}{2} (1 - \Gamma^{11}) \Theta = \Theta \). We have made this choice so that the remaining spinor \( \Theta \) may be identified with the worldvolume \( \Theta \) appearing in earlier expressions for the D0-brane action. The resulting \( \Theta^2 \) components are then given by\(^5\)

\[ \Phi|_{\Theta^2} = \frac{i}{48} e^{-\frac{1}{2} \phi} \Theta \gamma^{\mu \lambda} \Theta H_{\mu \nu \lambda} \]

\(^5\)It is interesting to note that the \( \Theta^2 \) terms in these bosonic components may be obtained more simply by the observation that they are exactly the anticommutator of two supersymmetry variations on the order \( \Theta^0 \) terms, that is

\[ A|_{\Theta^2} = \frac{1}{2} \{ \delta_{\epsilon_1}, \delta_{\epsilon_2} \} a|_{\epsilon_1 = \epsilon_2 = \theta} \]
In terms of these expressions, the complete non-linear action for a D0-brane to order $\Theta^2$ is then given by\footnote{Note that we may set to zero any terms involving derivatives on $\Theta$ since the worldvolume fermions are non-dynamical.}

$$S = -\mu_0 \int d\tau e^{-\frac{3}{4}\Phi}(1 - \frac{3}{4} \Phi|_{\Theta^2} + \ldots) \sqrt{-g_{\mu\nu}} (g_{\mu\nu} + 2\epsilon_{\mu\nu} E_{\mu}|_{\Theta^2} + \ldots) \dot{x}^\mu \dot{x}^\nu + \mu_0 \int d\tau (C_\mu + B_{\mu}|_{\Theta^2} + \ldots) \dot{x}^\mu$$

where dots indicate terms at fourth or higher order in $\Theta$. Choosing the static gauge $X^0 = \tau$ and taking the weak field approximation (keeping only terms linear in the background fields), the velocity independent terms reduce to

$$S_{\text{weak}} = -\frac{i}{8} (\partial_j h_{0i} + \partial_j C_i) \bar{\Theta} \Gamma^{0ij} \Theta + \frac{i}{16} (\partial_k C_{0ij} + \partial_k B_{ij}) \bar{\Theta} \Gamma^{ijk} \Theta$$

These are in agreement with the results from the previous section, providing an alternate derivation and a check of the results previously discussed, as well as an extension to the complete non-linear action to order $\Theta^2$. In principle, the expansion of the superspace fields in terms of components could be extended to higher orders in $\Theta$ to reproduce and check the higher order terms that we have derived, but we will not attempt this here. For an expansion to high order in $\Theta$, it might be that the superspace normal coordinate approach of \cite{20} would lead more efficiently to results which could be compared with those of the previous subsection.

### 3 D0-brane supergravity solutions

An important physical effect of the fermion couplings derived in the previous section is that D0-branes in different polarization states will produce different long range supergravity fields. This phenomenon has been discussed previously in \cite{7, 6} (and also \cite{21} in the context of the M2-brane). In those papers, it was explained that the supergravity solution corresponding to an arbitrary polarization state could be obtained by acting iteratively with broken supersymmetry generators on the fields of the usual bosonic supergravity solution. The result is a supergravity solution with fields depending on the fermionic supersymmetry transformation parameter. In \cite{21}, it was pointed out that this fermionic parameter should be identified with $a$ is the component field $a = A|_{\Theta^0}$. Thus, we could have obtained the desired expressions in one step directly from the supersymmetry transformations.
quantized fermion zero-mode operators, which from the worldvolume point of view are the 16 non-dynamical worldvolume spinors. The classical supergravity solution corresponding to a given state can then be evaluated by taking the expectation value of these operator valued supergravity fields in the state of interest.

The fermionic couplings derived in the previous section give a direct understanding of how different polarization states lead to different supergravity solutions. With our results, it is possible to directly read off the long range supergravity fields corresponding to a given state. In the next subsection, we give the leading long range behavior for each field in type IIA supergravity as a function of the fermionic operators $\theta$.

One important result, already clear from the results of the previous section is that the RR three-form field $C^{(3)}_{\mu\nu\lambda}$ (as well as its dual $C^{(5)}$) and the NS-NS two form field $B_{\mu\nu}$ are generically non-zero in D0-brane supergravity solutions. In [7], it was assumed that these fields were not relevant to the D0-brane solutions, so the D2-brane dipole moments and magnetic H-dipole moments of D0-brane polarization states were not found. In subsection 3.2, we rederive these moments using the methods of [7] by dropping the assumption that $C^{(3)}$ and $B$ vanish.

### 3.1 Long range fields

In this section we write down the leading long range supergravity fields corresponding to an arbitrary D0-brane polarization state. The long range fields are determined by the couplings derived in section 2 as well as type IIA bulk supergravity action, given to quadratic order in the Einstein frame by

$$S_{\text{IIA}} = \frac{1}{2\kappa^2} \int d^{10}x \left\{ R - \frac{1}{2} \partial_\mu \phi \partial^\mu \phi - \frac{1}{12} |dB|^2 - \frac{1}{4} |dC^{(1)}|^2 - \frac{1}{48} |dC^{(3)}|^2 \right\}$$  \hspace{1cm} (9)

Choosing the standard gauges $\partial^\mu (h_{\mu\nu} - (1/2)\eta_{\mu\nu} h_{\lambda\lambda}) = 0$, $\partial^\mu C^{(0)}_{\mu} = 0$, $\partial^\mu B_{\mu\nu} = 0$ and $\partial^\mu C^{(3)}_{\mu\nu\lambda} = 0$, we may use the equations of motion derived from (9) and (5) to determine the following long range fields, expressed in terms of the multipole moment operators $I$ appearing in (5).

$$h^{\alpha\beta} = \sum_n (-1)^n \frac{15\kappa^2}{16\pi^4 n!} \left( I_h^{\alpha\beta (i_1 \ldots i_n)} - \frac{1}{8} \eta^{\alpha\beta} (I_h)_{\mu}^{\mu (i_1 \ldots i_n)} \right) \partial_{i_1} \ldots \partial_{i_n} \left\{ \frac{1}{r^7} \right\}$$

$$\phi = \sum_n (-1)^n \frac{15\kappa^2}{16\pi^4 n!} I_\phi^{(i_1 \ldots i_n)} \partial_{i_1} \ldots \partial_{i_n} \left\{ \frac{1}{r^7} \right\}$$

$$D^{\mu_1 \ldots \mu_k} = \sum_n (-1)^n \frac{15\kappa^2 k!}{16\pi^4 n!} I_{C}^{\mu_1 \ldots \mu_k (i_1 \ldots i_n)} \partial_{i_1} \ldots \partial_{i_n} \left\{ \frac{1}{r^7} \right\}$$

Here, $D$ stands for any of the form fields $B$, $C^{(1)}$, $C^{(3)}$, $C^{(5)}$ or $C^{(7)}$. In terms of the string theory parameters, the gravitational coupling is given by $\kappa^2 = 2^6 \pi^7 g_s^2 (\alpha')^4$.

The expressions for the $I$s may be determined by comparing the couplings derived in section 2 with the general expression (5). Since we are dealing only with the linearized
theory, the long range fields generated by each of the couplings in section 2 may be calculated separately and combined where necessary by superposition. As an example, we find all long range fields at order $r^{-8}$. These fields are associated with dipole moments and arise from the couplings (7) quadratic in fermions. They are given by

\begin{align*}
    h_{0i} &= -\frac{15}{2} \pi^3 i(\alpha')^2 \Theta \Gamma^{0ij} \Theta \partial_j \left\{ \frac{1}{r^7} \right\} \\
    C_i &= \frac{15}{2} \pi^3 i(\alpha')^2 \Theta \Gamma^{0ij} \Theta \partial_j \left\{ \frac{1}{r^7} \right\} \\
    C_{0ij} &= \frac{15}{2} \pi^3 i(\alpha')^2 \Theta \Gamma^{ijk} \Theta \partial_k \left\{ \frac{1}{r^7} \right\} \\
    B_{ij} &= -\frac{15}{2} \pi^3 i(\alpha')^2 \Theta \Gamma^{ijk} \Theta \partial_k \left\{ \frac{1}{r^7} \right\} \\
\end{align*}

These order $r^{-8}$ terms were also found in [6]. The corrections at orders $r^{-9}$ and higher may be similarly determined from the couplings of section 2 involving four or more fermions. Note that these fields depend on the worldvolume fermion operators $\Theta$. To determine the numerical values of a field for a given polarization state, we simply replace these expressions by their expectation value in the particular state. In section 4, we will give an explicit representation of the polarization states and develop the methods to compute these expectation values.

### 3.2 Fermionic couplings from superpartner solutions

In this section, we use the methods of [7] to provide an alternate derivation that the supergravity solution corresponding to an arbitrary D0-brane polarization state has long range fields consistent with the new couplings that we have derived. In particular, by applying broken supersymmetry generators to the bosonic D0-brane solution, we will generate the “superpartner” solutions whose fields depend on the supersymmetry variation parameter. Relating this spinor parameter with the non-dynamical worldvolume fermions, we will show that the new subleading (order $1/r^8$) long range fields in the superpartner solutions match with the results of the previous subsection and therefore correspond precisely to the quadratic fermion couplings we have derived. In particular, we will see the presence of a non-zero RR three-form field corresponding to a D2-brane dipole directly from the supergravity solutions. This will provide an independent check of these results.

We begin with the bosonic D0-brane solution (3) which has a non-vanishing metric, dilaton, and RR one-form field.

If we let $\Phi$ denote the fields of the bosonic D0-brane solution, the superpartner solution is given by $\tilde{\Phi} = e^{\epsilon Q_a} \Phi$, where $Q_a$ are the 16 broken supersymmetry generators. This is analogous to obtaining a spatially translated solution by exponentiating the broken translation generator, $\Phi(\vec{x} + \delta \vec{x}) = e^{\delta x^i P_i} \Phi$, however in our case, the exponential series must truncate at order 16 since $\epsilon$ is a Grassman quantity with 16 independent components. Each application of a pair of $Q$’s results in an additional derivative on the original fields, so the leading effects of the $\epsilon^{2n}$ terms in the long range fields will be at order $r^{-8n}$. Thus, in order to compare
the dipole fields of the supergravity solution with those predicted by the quadratic fermion couplings, we need only compute to order $\epsilon^2$.

The supersymmetry transformations of the fields of type IIA supergravity are listed in appendix B. Acting on the bosonic D0-brane solution, we find that the first supersymmetry variations of the spinor fields are given by

$$
\delta \lambda = \frac{3\sqrt{2}}{8} H^{-\frac{12}{16}} \partial_i H \Gamma^{00} P_+ \epsilon
$$

$$
\delta \psi_0 = -\frac{7}{16} H^{-\frac{3}{2}} \partial_i H \Gamma^{00} P_+ \epsilon + \{\partial_0 \epsilon\}
$$

$$
\delta \psi_i = \frac{1}{16} H^{-1} \partial_j H (\Gamma^{ij} - 7\delta^{ij}) P_+ \epsilon + \left\{ \partial_i \epsilon + \frac{7}{32} H^{-1} \partial_i H \epsilon \right\}
$$

where $P_+ = \frac{1}{2} (1 + \Gamma^0 \Gamma^{11})$ is a projection operator. By choosing $\epsilon = H^{-\frac{1}{32}} \eta$ where $\eta$ is a constant spinor satisfying $P_+ \eta = 0$, the right hand side of these equations vanish, showing that the solution preserves 16 supersymmetries. We are interested in acting with the broken supersymmetry generators, which we take to be $\epsilon = H^{-\frac{1}{32}} \eta$ with $P_+ \eta = \eta$ so that the bracketed terms above vanish.

Using these fermion transformation rules we may now compute the bosonic fields of the superpartner solution to quadratic order in $\epsilon$. We find that the long range fields are given by

$$
(e_0^i)_c \epsilon = \frac{1}{2} (\epsilon Q)^2 e_0^i \epsilon = -\frac{7i}{32} H^{-\frac{31}{16}} \partial_j H \eta \Gamma^{ij0} \eta
$$

$$
(e_i^0)_c \epsilon = \frac{1}{2} (\epsilon Q)^2 e_i^0 \epsilon = \frac{1}{32} H^{-\frac{31}{16}} \partial_j H \eta \Gamma^{ij0} \eta
$$

$$
(C_i)_c \epsilon = \frac{1}{4} (\epsilon Q)^2 C_i \epsilon = \frac{i}{4} H^{-2} \partial_j H \eta \Gamma^{ij0} \eta
$$

$$
(B_{ij})_c \epsilon = \frac{1}{2} (\epsilon Q)^2 B_{ij} \epsilon = -\frac{i}{4} H^{-1} \partial_k H \eta \Gamma^{ijk0} \eta
$$

$$
(C_{0ij})_c \epsilon = \frac{1}{2} (\epsilon Q)^2 C_{0ij} \epsilon = \frac{i}{4} H^{-2} \partial_k H \eta \Gamma^{ijk0} \eta
$$

Note that these are precisely the fields that we found coupling to worldvolume fermion bilinears.

We would like to understand the relationship between the supersymmetry variation parameter $\eta$ in these expressions and the worldvolume fermion operator $\Theta$ appearing in (10). From equation (11), we see that $\eta$ is related to the “values” of the bulk fermion fields in the superpartner solutions. Quantum mechanically, these fermion zero-modes are operator valued, and from the supergravity point of view, it is these fermion zero-mode operators that are responsible for creating the 256-dimensional multiplet of D0-brane states. Thus, the parameters $\eta$ appearing in the superpartner solutions should be viewed as operators acting on the Fock space of D0-brane states, and (as may be deduced from the bulk fermion anticommutation relations) these operators satisfy a Clifford algebra equivalent to that of the worldvolume fermions $\Theta$.

In order to directly relate the parameters $\eta$ with the worldvolume fermions considered in the previous sections, we need to make a change of variables, since we had defined $\frac{1}{2} (1 - \frac{1}{2} H^{-1} 1 - 1)$.

---

7In calculating these expressions, we take the RR one-form field of the bosonic solution to be $C_0 = H^{-1} - 1$. The extra factor of -1 relative to (3) is physically unimportant since $C$ is a potential, but is chosen so that $C$ vanishes at infinity.
\( \Gamma^{11}) \Theta = \Theta \) while the broken supersymmetry generators satisfy \( \frac{1}{2}(1 + \Gamma^0 \Gamma^{11}) \eta = \eta \). The transformation relating these two is

\[
\eta = \frac{1}{4}(\Gamma^{11} + \Gamma^{11} \Gamma^0) \Theta.
\]

Since the normalization of \( \eta \) was arbitrary, we were free to choose the overall normalization on the right hand side of this equation. Using this transformation, we find

\[
\bar{\eta} \Gamma^{ij0} \eta = \frac{1}{2} \bar{\Theta} \Gamma^{0ij} \Theta, \quad \bar{\eta} \Gamma^{ijk0} \eta = \frac{1}{2} \bar{\Theta} \Gamma^{ijk} \Theta
\]

With this substitution, the long range fields from (12) precisely match those calculated earlier (10) from the linear couplings.

### 4 Classification of D0-brane polarizations

In the previous sections, we have shown that the action for a single D0-brane contains many couplings between the worldvolume fermions and the background type IIA supergravity fields. These couplings will cause the various polarization states to behave differently in the presence of non-zero background fields, and as discussed in the previous section lead to different long range supergravity fields. All of the operators coupling linearly to bosonic supergravity fields are built out of the basic operators

\[
J^{ij} \equiv -\frac{i}{4} \theta \gamma^{ij} \theta \quad D^{ijk} \equiv \frac{i}{4} \theta \gamma^{ijk} \theta
\]

These are also the objects that appear in the operator valued expressions for the long range fields of the supergravity solution. Thus, all information necessary to understand the physical properties of the various polarization states is determined by the action of the \( J \) and \( D \) operators on the 256 polarization states.

In this section, we provide an explicit representation of the polarization states. We show that the operators \( J \) and \( D \) together generate SO(16) and that the bosonic and fermionic states lie in opposite chirality 128 spinor representations of this group. Finally, we write down explicitly the actions of \( J \) and \( D \) on an arbitrary polarization state and use these to classify the states according to their eigenvalues for a physically interesting maximally commuting set of generators.

#### 4.1 Fock space of D0-brane polarizations

In order to label the D0-brane states, it is useful to rearrange the \( \theta \)'s into creation and annihilation operators

\[
\lambda_\alpha = \frac{1}{\sqrt{2}} (\theta_\alpha - i \theta_{8+\alpha}) \quad (13)
\]

\[
\lambda_\alpha^\dagger = \frac{1}{\sqrt{2}} (\theta_\alpha + i \theta_{8+\alpha})
\]
with \( \alpha = 1, \ldots, 8 \). These obey
\[
\{ \lambda_\alpha, \lambda_\beta^\dagger \} = \delta_{\alpha\beta}, \quad \{ \lambda_\alpha, \lambda_\beta \} = \{ \lambda_\alpha^\dagger, \lambda_\beta^\dagger \} = 0.
\]
The polarization states of the D0-brane may then be constructed by acting with creation operators \( \lambda^\dagger \) on a state \( | - \rangle \) which is annihilated by all \( \lambda \)'s. A given state may be labeled by
\[
| c_1 \cdots c_8 \rangle = (\lambda_1^\dagger)^{c_1} \cdots (\lambda_8^\dagger)^{c_8} | - \rangle
\]
where \( c_\alpha \in \{0, 1\} \).

These 256 states of a D0-brane may be understood naturally in terms of their M-theory origins. They are simply the polarization states of the 11-dimensional supergraviton. These form a representation of \( SO(9) \), the little group for massless particles in 11 dimensions, and may be divided into three irreducible representations corresponding to the graviton, three-form, and gravitino of \( D = 11 \) supergravity.

Let us focus first on the bosonic states. There are a total of 128 independent bosonic states of a D0-brane, 44 arising from the polarization states of the 11-dimensional graviton and 84 arising from the polarization states of the 11-dimensional three-form field. A general bosonic state will be some linear combination of these 128 states, and we may represent it by polarization tensors
\[
\{ h_{ij}, A_{ijk} \}, \quad \{ i, j, k = 1, \ldots, 9 \}
\]
where \( h_{ij} \) is a complex symmetric traceless tensor and \( A_{ijk} \) is a complex antisymmetric tensor.

The remaining 128 independent D0-brane states are fermionic and arise from the 128 polarization states of the \( D = 11 \) gravitino field. The most general fermionic D0-brane state is a linear combination of these states, and we may represent it by a gravitino polarization tensor
\[
\psi_{i\alpha}, \quad \{ i = 1, \ldots, 9; \alpha = 1 \ldots 16 \}
\]
where \( \psi \) is a complex vector-spinor satisfying the 16 constraints \( \gamma_{\alpha\beta}^{i} \psi_{i\beta} = 0 \).

An explicit construction of the general states \( | h_{ij}, A_{ijk} \rangle \) and \( | \psi_{i\alpha} \rangle \) in terms of the creation and annihilation operators \( \lambda \) may be found in [22]. It is possible to choose conventions such that the inner product between arbitrary states is given by
\[
\langle \tilde{h}, \tilde{A} | h, A \rangle = \tilde{h}_{ij}^* h_{ij} + \tilde{A}_{ijk}^* A_{ijk} \quad \langle \tilde{\psi} | \psi \rangle = \tilde{\psi}^* \psi_{i\alpha} \psi_{i\alpha}
\]
The two sets of bosonic states constitute the irreducible 44 and 84 representations of \( SO(9) \), the little group for massive particles in 10 dimensions, while the fermionic states form a single irreducible 128 representation. In this sense, we have three different types of D-particles.

\(^8\)Of course, this is also the little group for massive particles in 10 dimensions.
However, in the presence of non-zero background supergravity fields, the two types of bosonic states mix with each other.

From the action (4), we see that the effects of background fields on the D0-brane states, as well as the background fields generated by a given state will be governed by the two operators

\[ J^{ij} \equiv -\frac{i}{4} \theta \gamma^{ij} \theta \]
\[ D^{ijk} \equiv \frac{i}{4} \theta \gamma^{ijk} \theta. \]  

(15)

The commutation relations of these operators may be determined in a straightforward way from the anticommutation relations for the \( \theta \)'s. The operators \( J^{ij} \) have the algebra of \( SO(9) \):

\[ [J^{ij}, J_{kl}] = 4i \delta^{[i}_{[k} J^{j]}_{l]} \]

This is to be expected since \( J^{ij} \) are precisely the fermionic parts of the operators which generate spatial rotations in the theory. The commutation relations between \( D \) and \( J \) operators reflect the property that the \( D \)'s are in the 3-index antisymmetric tensor representation of the \( SO(9) \) generated by \( J \)'s. They are

\[ [J^{ij}, D_{klm}] = 6i \delta^{ij}_{[k} D^{j]}_{lm]} \]

Finally, the commutation relations for the \( D \)'s are given by

\[ [D^{ijk}, D_{lmn}] = 18i \delta^{ij}_{[l} D^{k]}_{m]n} - i \epsilon^{ijklmnpqr} D_{pqr} \]

It turns out that the \( J \)'s and \( D \) together generate \( SO(16) \), however, they form a somewhat unusual set of generators. The standard \( SO(16) \) generators are simply the fermion bilinears

\[ A_{\alpha\beta} = \theta_\alpha \theta_\beta \]

as may be easily checked. The \( J \)'s and \( D \) form a different basis of the 120 independent bilinears, and we see from (13) that the matrices \( \gamma_{ij}^{\alpha\beta} \) and \( \gamma_{ijk}^{\alpha\beta} \) are the coefficients which relate the ordinary basis to the \( J, D \) basis.

The bosonic D0-brane states which formed a 44 and 84 of \( SO(9) \) combine into a single 128 chiral spinor representation of this \( SO(16) \). It is interesting to note that this representation of \( SO(16) \) is precisely the one which appears when we consider the action of \( SO(16) \) on the coset space \( E_8/SO(16) \) [23]. This suggests that perhaps the D0-brane polarization states may correspond to broken symmetry generators in some more symmetric phase of string theory, but we will not pursue this connection further here. The fermionic states also lie in a 128 chiral spinor representation of \( SO(16) \), but of the opposite chirality.

[9] Here and in the rest of this paper, symmetrization (denoted by \((i_1 \cdots i_n)\)) and antisymmetrization (denoted by \([i_1 \cdots i_n]\)) of indices are taken with weight 1. For example, \( M_{[ij]} \equiv \frac{1}{2}(M_{ij} - M_{ji}) \).
4.2 Action of spin and dipole operators on polarization states

The action of the \( SO(16) \) generators \( J \) and \( D \) on the general bosonic and fermionic states may be determined using the explicit representation of the states found in [22]. We first determine the action of a single operator \( \theta^\alpha \) on the general state. We find

\[
\theta^\alpha: |\psi\rangle \rightarrow |\tilde{\psi}\rangle
\]

\[
\tilde{\psi}_{i\beta} = \frac{\sqrt{2}}{2} \gamma^i_{\beta\alpha} h_{ij} - \frac{\sqrt{6}}{36} (\gamma^i_{\beta\alpha} - 6 \delta^i_{\beta} \gamma^k_{\alpha}) A_{jkl}
\]

(16)

\[
\theta^\alpha: |\psi\rangle \rightarrow |\tilde{h}, \tilde{A}\rangle
\]

\[
\tilde{h}_{ij} = \frac{\sqrt{2}}{2} \gamma^i_{\alpha\beta} \psi^j_{\beta}
\]

\[
\tilde{A}_{ijk} = -\frac{\sqrt{6}}{4} \gamma^i_{\alpha\beta} \psi^j_{\beta}
\]

Using these relations, we find (as expected) that the rotation generators act as

\[
J^{ij}: |\psi\rangle \rightarrow |\tilde{\psi}\rangle
\]

\[
\tilde{\psi}^k = -2i \delta^k_{[i} \gamma^j_{\beta} \psi^l_{\beta] - \frac{i}{2} \gamma^{ij} \psi^k
\]

\[
\tilde{h}_{kl} = -4i \delta^l_{(k} \gamma^j_{l)}
\]

\[
\tilde{A}_{klm} = -6i \delta^l_{[k} A^m_{l]}
\]

(17)

\[
D^{ijk}: |\psi\rangle \rightarrow |\tilde{\psi}\rangle
\]

\[
\tilde{\psi}^l = -4i \delta^l_{[i} \gamma^j_{\beta} \psi^k_{\beta] + i \gamma^{[ij} \psi^k - \frac{i}{2} \gamma^{ijk} \psi^l
\]

The operators \( D \) mix states arising from graviton and three-form polarizations. We have

\[
D^{ijk}: |\psi\rangle \rightarrow |\tilde{h}, \tilde{A}\rangle
\]

\[
\tilde{h}_{im} = 6i \sqrt{3} \left( \delta^i_{[l} A^j_{k]m} - \frac{1}{9} \delta_{lm} A^{ijk} \right)
\]

\[
\tilde{A}_{ilm} = -6i \sqrt{3} \delta^i_{[l} \delta^j_{m} h^k_{n]} - \frac{i}{6} \epsilon^{ijklmnqr} A_{pqr}
\]

(18)

The relations (17) and (18), together with the inner product (14) may be used to compute the action of an arbitrary operator built from \( \theta^s \) on a general bosonic state, as well as the expectation values of arbitrary operators.

For example, we find

\[
\langle J^{ij} \rangle = 4 \text{ Im}(h^i_\alpha h_{jl}) + 6 \text{ Im}(A^i_{lm} A_{jlm})
\]

\[
\langle D^{ijk} \rangle = 12 \sqrt{3} \text{ Im}(h_{m[i} A^*_{j]l} A^*_{lm}) + \frac{1}{3} \text{ Im}(\epsilon^{ijklmnqr} A^*_{l} A^*_{pq})
\]
Recalling the couplings (4), we see that the first of these expressions gives the expectation value of the angular momentum in the \{ij\} plane (equal to the RR one-form magnetic moment), while the second gives the D2-brane dipole moment in the \{ijk\} directions (equal to the NS-NS two form magnetic moment). These expectation values are exactly those needed to evaluate the long range dipole fields (10) for a given polarization state.

4.3 Classification in terms of \(J\) and \(D\) eigenstates.

To understand the physical properties of the various D0-brane states, it is useful to choose a maximal mutually commuting set of generators (Cartan subalgebra) and then write the states in a basis of simultaneous eigenstates for these generators.

A physically interesting choice of commuting generators is the set

\[
\{J_1 \equiv J^{12}, J_3 \equiv J^{34}, J_5 \equiv J^{56}, J_7 \equiv J^{78}, D_1 \equiv D^{129}, D_3 \equiv D^{349}, D_5 \equiv D^{569}, D_7 \equiv D^{789}\}
\]

containing four rotation operators and four dipole operators.

We may now describe an orthogonal basis of the bosonic and fermionic states which are simultaneous eigenstates of these 8 generators. A useful property of these basis elements is that the expectation values of all other generators not in the Cartan subalgebra vanish.

4.3.1 Basis for bosonic states

We begin with the 128 independent bosonic states. We will write bosonic eigenstates in terms of a simpler (non-normalized and non-orthogonal) basis given by

\[
|ij\rangle \equiv |h_{ij} = h_{ji} = 1\rangle, \quad i \neq j
\]

\[
|i\rangle \equiv |h_{ii} = -h_{99} = 1\rangle, \quad i = 1, \ldots, 8
\]

\[
|ijk\rangle \equiv |A_{ijk} = -A_{ikj} = A_{jki} = -A_{jik} = A_{kij} = -A_{kji} = 1\rangle
\]

The action of each of the 8 Cartan generators on this basis may be read off from the relations (17) and (18). Using these, we first diagonalize the \(J\)'s and then diagonalize the \(D\)'s in each subspace of states of fixed \(J\)'s. Below we will use the indices \(a, b, c, d\) to represent distinct elements of \(\{1, 3, 5, 7\}\), labeling the generators in our Cartan subalgebra. Also, for \(a = 2l - 1\), we let \(\hat{a} = 2l\).

**group 1:** \(J = 0\) \(D = \pm 2\)

There are 8 basis states for which all \(J\)'s vanish. These include four graviton states

\[
|a_0\rangle \equiv |aa\rangle + |\hat{a}\hat{a}\rangle
\]

and four three-form states

\[
|a9\rangle \equiv |a\hat{a}9\rangle.
\]

---

\textsuperscript{10}This follows since all other generators may be written as the commutator of some generator with a Cartan subalgebra generator.
Diagonalizing the $D$ generators on the subspace generated by these states, we find that the diagonal basis consists of states for which a single $D$ generator has the eigenvalue $\pm 2$. Explicitly, the normalized eigenstates are:

$$|D_a = \pm 2\rangle \equiv \frac{1}{6} \left(2|a_0\rangle - |b_0\rangle - |c_0\rangle - |d_0\rangle \mp \sqrt{3} |a_9\rangle \right)$$

**group 2:** $J = \pm 2$, $D = 0$

There are 8 states for which a single $J_a$ has a value of $\pm 2$. These all come from graviton polarizations and are given by

$$|J_a = \pm 2\rangle = \frac{1}{2} |a\hat{a}\rangle \mp \frac{i}{2} |a\hat{a}\rangle \pm \frac{i}{2} |a\hat{a}\rangle.$$

For these states, $J_a = \pm 2$ and all other Cartan generators including $D$s vanish

**group 3:** $J_a = \pm 1$, $D_b, D_c, D_d = \pm 1$

There are 32 states for which a single $J_a$ has the value $\pm 1$. These include 8 graviton states

$$|a_{\pm 1}9\rangle \equiv |a9\rangle \pm i|\hat{a}9\rangle$$

as well as 24 three-form states

$$|a_{\pm 1}b\rangle \equiv |ab\rangle \pm i|\hat{a}b\rangle$$

These are not eigenstates of the $D_a$’s, however we may combine them into normalized eigenstates

$$|J_a = \epsilon, D_b = \lambda_b, D_c = \lambda_c, D_d = \lambda_d\rangle \equiv \frac{1}{4} |a_{\epsilon}9\rangle - \frac{i\lambda_b}{4\sqrt{3}} |a_{\epsilon}b\rangle - \frac{i\lambda_c}{4\sqrt{3}} |a_{\epsilon}c\rangle - \frac{i\lambda_d}{4\sqrt{3}} |a_{\epsilon}d\rangle$$

where $\epsilon, \lambda_b\lambda_c, \lambda_d \in \pm 1$ are the eigenvalues of $J_a, D_b, D_c$, and $D_d$ respectively, with the constraint that $\lambda_b\lambda_c\lambda_d = \epsilon$. Recalling that $a, b, c, d$ must be distinct, we may verify that this gives a total of 32 states.

**group 4:** $J_a = \pm 1, J_b = \pm 1, D_a, D_b = \pm 1$

There are 48 states for which two different $J_a$’s have (uncorrelated) eigenvalues of $\pm 1$. These include 24 graviton states

$$|a_{\epsilon a}b_{\epsilon b}\rangle \equiv |ab\rangle + i\epsilon_a|\hat{a}b\rangle + i\epsilon_b|\hat{a}b\rangle - \epsilon_a\epsilon_b|\hat{a}b\rangle$$

and 24 three-form states

$$|a_{\epsilon a}b_{\epsilon b}9\rangle \equiv |ab9\rangle + i\epsilon_a|\hat{a}b9\rangle + i\epsilon_b|\hat{a}b9\rangle - \epsilon_a\epsilon_b|\hat{a}b9\rangle$$
where $\epsilon_a, \epsilon_b \in \pm 1$ are the eigenvalues for $J_a$ and $J_b$. These states are mixed by the $D_a$'s but we may combine them into eigenstates as

$$|J_a = \epsilon_a, J_b = \epsilon_b, D_a = \epsilon_a \delta, D_b = -\epsilon_b \delta\rangle \equiv \frac{1}{4} |a\epsilon_a b\epsilon_b\rangle + \frac{\delta}{4\sqrt{3}} |a\epsilon_a b\epsilon_b\rangle$$

where $\delta = \pm 1$.

**group 5:** $J_a = \pm 1, J_b = \pm 1, J_c = \pm 1, D_d = \pm 1$

The remaining 32 states have three distinct $J_a$'s equal to $\pm 1$. These states all arise from three-form polarizations and are given by

$$|J_a = \epsilon_a, J_b = \epsilon_b, J_c = \epsilon_c, D_d = \epsilon_a \epsilon_b \epsilon_c\rangle = \frac{1}{4\sqrt{3}} \left(|abc\rangle + i\epsilon_a |\hat{a}bc\rangle + \ldots - i\epsilon_a \epsilon_b \epsilon_c |\hat{a}\hat{b}\hat{c}\rangle\right)$$

Note that each of these states is also an eigenstate of the $D$'s with $D_a = D_b = D_c = 0$ and $D_d = \epsilon_a \epsilon_b \epsilon_c$

### 4.3.2 Fermionic states

To describe the fermionic eigenstate basis, we introduce projection operators

$$P^a_{\pm} = \frac{1}{2} (1 \pm i\gamma^a)$$

where, as above, $a \in \{1, 3, 5, 7\}$ and $\hat{a} = a + 1$. Each of these independently reduces the number of independent components of a 16-component spinor by half, so the state $P^1_{\epsilon_1} P^3_{\epsilon_3} P^5_{\epsilon_5} P^7_{\epsilon_7} \chi$ has only a single independent component. Each state in the fermionic basis has an eigenvalue of $\pm \frac{3}{2}$ for exactly one of the 8 Cartan subalgebra generators, with eigenvalues of $\pm \frac{1}{2}$ for the remaining 7 generators. We now describe these states explicitly.

**group 1:**

The first set of 64 states has eigenvalues

$$J_a = \frac{3}{2} \epsilon_a, J_b = \frac{1}{2} \epsilon_b, J_c = \frac{1}{2} \epsilon_c, J_d = \frac{1}{2} \epsilon_d$$

$$D_a = -\frac{1}{2} \epsilon_b \epsilon_c \epsilon_d, D_b = -\frac{1}{2} \epsilon_a \epsilon_c \epsilon_d, D_c = -\frac{1}{2} \epsilon_a \epsilon_b \epsilon_d, D_d = -\frac{1}{2} \epsilon_a \epsilon_b \epsilon_c$$

where as above, $\{a, b, c, d\} = \{1, 3, 5, 7\}$. Since each $\epsilon$ can be $\pm 1$ and we have 4 choices for $a$, this gives 64 states, so the states are specified uniquely by their eigenvalues. The polarization vectors for these basis elements are given by

$$\psi^a = -i\epsilon_a \psi^{\hat{a}} = P^a_{-\epsilon_a} P^b_{-\epsilon_b} P^c_{-\epsilon_c} P^d_{-\epsilon_d} \chi, \quad \psi^i = 0 \quad \{i \neq a, \hat{a}\}$$

Note that the choice of $\chi$ is irrelevant since there is only a single independent component after acting with the four projection operators.
The remaining 64 fermionic states have eigenvalues

\[ J_a = \frac{1}{2} \epsilon_a \quad J_b = \frac{1}{2} \epsilon_b \quad J_c = \frac{1}{2} \epsilon_c \quad J_d = \frac{1}{2} \epsilon_d \]

\[ D_a = \frac{3}{2} \epsilon_b \epsilon_c \epsilon_d \quad D_b = -\frac{1}{2} \epsilon_a \epsilon_c \epsilon_d \quad D_c = -\frac{1}{2} \epsilon_a \epsilon_b \epsilon_d \quad D_d = -\frac{1}{2} \epsilon_a \epsilon_b \epsilon_c \]

Again, these states are specified uniquely by their eigenvalues. The polarization vectors for these basis elements are given by

\[
\psi^a = -i \epsilon_a \bar{\psi} \bar{a} = P^a_{+ \epsilon_a} P^b_{- \epsilon_b} P^c_{+ \epsilon_c} P^d_{- \epsilon_d} \chi \\
\psi^b = -i \epsilon_b \bar{\psi} \bar{b} = -\frac{1}{2} P^a_{- \epsilon_a} P^b_{+ \epsilon_b} P^c_{- \epsilon_c} P^d_{+ \epsilon_d} \gamma^{ba} \chi \\
\psi^c = -i \epsilon_c \bar{\psi} \bar{c} = -\frac{1}{2} P^a_{- \epsilon_a} P^b_{+ \epsilon_b} P^c_{- \epsilon_c} P^d_{- \epsilon_d} \gamma^{ca} \chi \\
\psi^d = -i \epsilon_d \bar{\psi} \bar{d} = -\frac{1}{2} P^a_{+ \epsilon_a} P^b_{- \epsilon_b} P^c_{- \epsilon_c} P^d_{+ \epsilon_d} \gamma^{da} \chi \\
\psi^9 = \epsilon_a \epsilon_b \epsilon_c \epsilon_d \gamma^a \psi \bar{a}
\]

5 Static interactions between polarized D0-branes

In this section, we apply our results and consider the long-range interactions between a pair of D0-branes in fixed polarization states. In particular, we would like to consider the interactions between a pair of D0-branes in identical polarization states. Since each D0-brane is in a state which preserves the same 16 supersymmetries, a pair of D0-branes in identical polarization states should be a BPS configuration in which the force between the branes is identically 0.

The long-range force between a pair of D0-branes with spin and magnetic D0 dipole moment was considered in [7]. These authors showed that the spin-spin and dipole-dipole interactions precisely cancel since the gyromagnetic ratio of the D0-brane is \( g = 1 \). We show here that this result follows immediately from the couplings (4), and furthermore that the dipole-dipole interactions mediated by the R-R 3-form and NS-NS 2-form field also cancel identically due to the equality of the D2-brane dipole and magnetic H-dipole moments.

The cancellation of all dipole-dipole forces between static D0-branes follows immediately from the formulae for long range fields derived in the previous section. We simply let one of the D-branes act as a source which generates the long range fields calculated in the previous section, and treat the other brane as a probe which feels a potential obtained by plugging in the fields generated by the first brane into the action (7). Examining the expressions (10) and (7), it is obvious that the spin-spin potential mediated by the graviton

\[
V_{\text{spin}} = \frac{15}{32} \pi^3 \tilde{\theta}_1 \Gamma^{0ij} \Theta_1 \tilde{\Theta}_2 \Gamma^{0ik} \Theta_2 \partial_j \partial_k \left\{ \frac{1}{r^3} \right\}
\]
is precisely canceled by the RR one-form mediated dipole-dipole potential

\[ V_{dipole} = -\frac{15}{32} \pi^3 i \bar{\Theta}_1 \Gamma^{0ij} \Theta_1 \bar{\Theta}_2 \Gamma^{0ik} \Theta_2 \partial_j \partial_k \left\{ \frac{1}{r^7} \right\} , \]

as was demonstrated previously in [7]. In a similar way, the interaction between D2-brane dipole moments gives rise to a potential

\[ V_{C(3)} = \frac{15}{32} \pi^3 \bar{\Theta}_1 \Gamma^{ijk} \Theta_1 \bar{\Theta}_2 \Gamma^{ijkl} \Theta_2 \partial_i \partial_j \partial_k \left\{ \frac{1}{r^7} \right\} \]

which is precisely canceled by the NS-NS two-form mediated force between magnetic H-dipole moments,

\[ V_B = -\frac{15}{32} \pi^3 \bar{\Theta}_1 \Gamma^{ijk} \Theta_1 \bar{\Theta}_2 \Gamma^{ijkl} \Theta_2 \partial_i \partial_j \partial_k \left\{ \frac{1}{r^7} \right\} \]

Recalling that the leading order gravitational and dilaton mediated forces also cancel precisely with the repulsive force between the two D0-brane charges, we see that the forces between two static D0-branes cancel completely up to order \( \frac{1}{r^9} \).

It is interesting that we did not need to make any assumptions about the polarizations of the two D0-branes. We should point out, however, that we do not expect all forces between two static D0-branes to cancel in general. In fact, the general expression for the long range static potential has been worked out [24, 25], and is given by

\[ V_{D0-D0} = -\frac{5}{43008} \Gamma^{mij} \theta \Gamma^{nk} \theta \partial_i \partial_j \partial_k \left\{ \frac{1}{r^7} \right\} \]

where \( \theta = \theta_1 - \theta_2 \) is the relative polarization of the two D0-branes in the 16-component spinor notation. This is a \( \frac{1}{r^9} \) potential, which should be reproducible using our results through a combination of quadrupole-quadrupole, octupole-dipole and 16pole-monopole forces. It is only when the two D0-branes have the same polarization that we expect to obtain a classical BPS state and thus observe the complete cancellation of all forces.

### 6 Discussion

In this paper, we have investigated the physical properties of the different polarization states of a D0-brane. By analyzing the couplings of the D0-brane worldvolume fermions to the type IIA supergravity bulk fields, we have seen that in addition to mass and RR one-form charge, D0-branes may carry moments of a variety of other conserved quantities. The dipole moments include angular momentum (spin), RR one-form magnetic moment, D2-brane dipole moment, and NS-NS two-form magnetic dipole moment. We have shown a complete cancellation between dipole-dipole forces for a pair of static D0-branes owing to the fact that D0-branes have a gyromagnetic ratio of 1 and that the ratio between D2-brane dipole moment and NS-NS two-form magnetic moment is also 1.

We have determined the leading long range supergravity fields corresponding to an arbitrary polarization state of the D0-brane. For each of these states, there should also be
a corresponding exact supergravity solution, thus we expect there to be a many parameter family of black hole solutions in $D = 10$ preserving 16 supersymmetries and carrying multipole moments for various types of conserved quantities. For example, we expect supergravity solutions corresponding to each element of the eigenstate basis of section 4, including solutions with zero (classical) angular momentum but with dipole moments of D2-brane charge. It would be interesting to find these exact solutions using the knowledge of their long range fields.

One approach to deriving the exact supergravity solutions would be to work in the context of $D = 11$ supergravity. Since type IIA supergravity is related to eleven-dimensional supergravity by dimensional reduction, any exact solution of type IIA supergravity is related to a solution of $D = 11$ supergravity with translational invariance along one of the spatial directions. In this way, the bosonic D0-brane solution of type IIA supergravity is related to the Aichelburg-Sexl solution of $D = 11$ supergravity \[26\] corresponding to the gravitational fields around a massless particle. The superpartners of the D0-brane solution should then be identified with superpartners of the Aichelburg-Sexl solution, and these should take a particularly simple form since the Aichelburg-Sexl solution has only a single non-vanishing field, $h_{--} \propto \frac{1}{r^7}$. Given the Aichelburg-Sexl superpartner solutions, the complete D0-brane superpartner solutions would then be obtained by reducing to $D = 10$ in the standard way.

The couplings of bulk fields to worldvolume fermions that we have derived have interesting implications for the study of systems of two types of branes, for example in the investigation of bound states between D0-branes and other types of branes. If we consider one set of branes to be a source for various bulk fields, the couplings of these fields to the fermions on the worldvolume of the probe D0-branes will result in different energies for the different polarization states of the D0-branes. The minimum energy configuration will likely be one in which the D0-branes lie in a specific polarization state, and the long range supergravity fields corresponding to this system of branes should exhibit any non-zero moments associated with this polarization state. This may be related to an interesting observation made in \[27\]. In that paper, it was found that the wavefunction of a D0-brane in the D0-D4 bound state is indicative of a specific combination of polarization states (representations of Spin(5) in their case).\[27\] It is possible that an understanding of this phenomenon would result from considering the couplings of D0-brane fermions to the supergravity fields generated by the D4-brane.

Finally, we note that almost identical considerations apply to the various other types of branes in theories with 32 supersymmetries (as well as theories with less supersymmetry). Just like the type IIA D0-brane, configurations of parallel BPS branes in these theories generally preserve half the supersymmetry of the relevant theory. The remaining 16 broken supersymmetry generators generate a BPS multiplet of 256 states which splits into two 128 representations of the $SO(16)$ generated by bilinears of the broken generators. In

\[11\] More precisely, the lifted D0-brane solution becomes, after a coordinate transformation, the Aichelburg-Sexl solution smeared along the direction of particle motion.

\[12\] We thank Savdeep Sethi for pointing this out to us.
the case where the spatial dimensions of the brane are compactified, these branes may
be viewed as particle states from the point of view of the uncompactified directions, and
the various states will appear as different polarizations of some number of particle types
(representations of the rotation group of the uncompactified space). For particles obtained
from toroidally compactified Dp-branes, the physical properties of the various polarization
states can be investigated using a similar approach to that used in this paper. The couplings
of worldvolume fermions to background fields for these particles may be obtained from the
fermion terms in the Dp-brane actions derived in [12]. These more general particle states
will correspond to a large class of black hole solutions in various dimensions, and it would
be interesting to investigate their properties. One potentially interesting related direction for
further work would be to use the methods of this paper to study the breaking of degeneracy
between fermionic states in the world-volume theory on a system of 3-branes forming a
dielectric 5-brane sphere (as studied, for example, in [28]).

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A Properties of spinors

In this paper we have used two different types of spinors, 16-component spinors denoted
by θ and 32-component Majorana-Weyl fermions denoted by Θ. With our conventions, the
spinors are related by

$$\Theta = \begin{pmatrix} \theta \\ 0 \end{pmatrix}$$

For the 16-component spinors, we use a real symmetric set of $16 \times 16$ Dirac matrices denoted
by $\gamma^i, \ i = 1 \ldots 9$. The $32 \times 32$ Dirac matrices denoted by $\Gamma$ are given in terms of the $\gamma$’s by

$$\Gamma^i = \begin{pmatrix} \theta \\ 0 \end{pmatrix} \quad \Gamma^0 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \quad \Gamma^{11} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Because of the anticommutation relations obeyed by the fermions, the only nonvanishing
fermion bilinears are

$$\theta \gamma^{ijk} \theta, \quad \theta \gamma^{ij} \theta, \quad \theta \gamma^{ijklmn} \theta = -\frac{1}{3!} \epsilon^{ijklmnop} \theta \gamma^{pqrs} \theta \gamma^{qr} \theta, \quad \theta \gamma^{ijklmnp} \theta = -\frac{1}{2} \epsilon^{ijklmnop} \theta \gamma^{qr} \theta$$
in the 16-component notation and

\[ \bar{\Theta} \Gamma^{abc} \Theta \quad \bar{\Theta} \Gamma^{abcdefg} \Theta = \frac{1}{3!} \epsilon^{abcdefg} hij \bar{\Theta} \Gamma_{hij} \Theta \]

for 32-component spinors. In dealing with expressions containing more than two fermions, various Fierz identities must be taken into account. For the 16-component spinors, a list of these identities may be found in [11, 29]. For the \( D = 10 \) fermions, we have

\[ \bar{\Theta} \Gamma^{abc} \Theta \bar{\Theta} \Gamma_{ab} d \Theta = 0 \]
\[ \bar{\Theta} \Gamma_{a \beta} [c \Theta \bar{\Theta} \Gamma^{def}] \Theta \]
\[ +12 \bar{\Theta} \Gamma_{abc} \bar{\Theta} \Gamma^{def} \Theta + \bar{\Theta} \Gamma_{abcqp} [ef \Theta \bar{\Theta} \Gamma^{defpq}] \Theta + \bar{\Theta} \Gamma_{[abc]pq} \Theta - 24 \bar{\Theta} \Gamma_{[a \beta} \Theta \delta_{\epsilon} \delta_{\epsilon} \Theta \Gamma_{c]f]} \Theta = 0 \]

It is important to note that these identities apply for classical anticommuting fermions. For operators satisfying anticommutation relations \( \{ \Theta_\alpha, \Theta_\beta \} = \delta_{\alpha \beta} \), we should replace the right hand sides of the first and third expressions by \((-320 \eta^{cd} - 896 \eta^{0d} \eta^{0a})\) and \((-768 \delta_{a \beta} \delta_{\epsilon} \delta_{\epsilon} \delta_{c]f] - 4608 \delta^{0a \beta} \delta_{\epsilon} \delta_{\epsilon} \delta_{c]f] 0})\) respectively.

### B Type IIA supergravity conventions

The fields of \( D = 10 \) type IIA supergravity are the vielbein \( e_\mu^m \), the dilaton \( \phi \), the NS-NS two form \( B_{\mu \nu} \), the RR one and three forms \( C_\mu \) and \( C_{\mu \nu \lambda} \), the dilatino \( \lambda \) and the gravitino \( \psi_\mu \). The supersymmetry transformations for these fields are [30, 31]

\[
\delta e_\mu^m = ie\Gamma^m \psi_\mu \\
\delta \phi = i\sqrt{2}e\Gamma^{11} \lambda \\
\delta C_\mu = ie^{-\frac{3}{4}\phi}e\Gamma^{11} \psi_\mu + \frac{3i\sqrt{2}}{4}e^{-\frac{3}{4}\phi}e\gamma_\mu \lambda \\
\delta B_{\mu \nu} = 2ie^{\frac{3}{2}\phi}e\gamma_{\mu \nu} \Gamma^{11} \psi_\lambda - \frac{\sqrt{2}}{2}ie^{\frac{3}{2}\phi}e\gamma_{\mu \nu} \lambda \\
\delta C_{\mu \nu \lambda} = -3ie^{-\frac{3}{4}\phi}e\gamma_{\mu \nu} \psi_\lambda + 6ie^{\frac{3}{2}\phi}eC_{\mu} e\gamma_\mu \Gamma^{11} \psi_\lambda \]

\[+ \frac{3\sqrt{2}}{4}ie^{-\frac{3}{4}\phi}e\gamma_{\mu \nu \lambda} \Gamma^{11} \lambda - \frac{3\sqrt{2}}{2}ie^{\frac{3}{2}\phi}eC_{\mu} e\gamma_{\nu \lambda} \lambda \]

\[\delta \lambda = \frac{\sqrt{2}}{4}e^{\frac{3}{4}\phi}e\gamma_{\mu \nu \lambda} \theta_\mu \phi - \frac{3\sqrt{2}}{32}e^{\frac{3}{4}\phi}e\gamma_{\mu \nu} \epsilon F_{\mu \nu} \]

\[+ \frac{\sqrt{2}}{48}e^{-\frac{3}{4}\phi}e\gamma_{\mu \nu \lambda} \epsilon H_{\mu \nu \lambda} + \frac{\sqrt{2}}{384}e^{\frac{3}{4}\phi}e\gamma_{\mu \nu \lambda} \epsilon \Gamma^{11} F_{\mu \nu \lambda} \epsilon + \ldots \]

\[\delta \psi_\mu = \partial_\mu \epsilon + \frac{1}{4} \omega_{mmn} \Gamma^{mn} \epsilon - \frac{1}{64}e^{\frac{3}{4}\phi} \{ \gamma_\mu \epsilon \lambda - 14 \delta_{\mu \nu} \gamma_\lambda \} \Gamma^{11} e \epsilon F_{\mu \nu} \lambda \]

\[+ \frac{1}{96} e^{-\frac{3}{4}\phi} \{ \gamma_\mu \epsilon \lambda - 9 \delta_{\mu \nu} \gamma_\lambda \} \Gamma^{11} e \epsilon H_{\mu \nu \lambda} + \frac{1}{256} e^{\frac{3}{4}\phi} \{ \gamma_\mu \epsilon \lambda \epsilon \sigma - \frac{20}{3} \delta_{\mu \nu} \gamma_\lambda \epsilon \sigma \} \epsilon F_{\nu \lambda \epsilon \sigma} + \ldots \]
Here, Dirac matrices with tangent space indices are denoted by $\Gamma$ while Dirac matrices with spacetime indices are denoted by $\gamma$ (not to be confused with the $16 \times 16$ Dirac matrices used elsewhere), so that

$$\gamma_\mu = e_\mu^m \Gamma_m$$

Note that the supersymmetry variations of the fermionic fields contain additional terms cubic in the fermion fields which are not relevant for our purposes. The additional terms may be found in [30, 31]. Finally, the field strengths are defined as

$$F_{\mu\nu} = 2 \partial_{[\mu} C_{\nu]}$$
$$H_{\mu\nu\lambda} = 3 \partial_{[\mu} B_{\nu\lambda]}$$
$$F'_{\mu\nu\lambda\sigma} = 4 \partial_{[\mu} C_{\nu\lambda\sigma]} + 8 C_{[\mu} H_{\nu\lambda\sigma]}$$

References

[1] J. Polchinski, “Dirichlet-Branes and Ramond-Ramond Charges,” Phys. Rev. Lett. 75 (1995) 4724, [hep-th/9510017].

[2] K. S. Stelle, “BPS branes in supergravity,” [hep-th/9803116].

[3] D. Youm, “Black holes and solitons in string theory,” Phys. Rept. 316, 1 (1999), [hep-th/9710046].

[4] G. T. Horowitz and A. Strominger, “Black strings and p-branes,” Nucl. Phys. B360 (1991) 197.

[5] W. Taylor and M. Van Raamsdonk, “Multiple D0-branes in weakly curved backgrounds,” [hep-th/9904093].

[6] J. F. Morales, C. A. Scrucca and M. Serone, “Scale independent spin effects in D-branes dynamics,” Nucl. Phys. B534, 223 (1998), [hep-th/9801183].

[7] M. J. Duff, J. T. Liu and J. Rahmfeld, “g = 1 for Dirichlet 0-branes,” Nucl. Phys. B524, 129 (1998), [hep-th/9801072].

[8] P. C. Aichelburg and F. Embacher, “The Exact Superpartners Of N=2 Supergravity Solitons,” Phys. Rev. D34, 3006 (1986).

[9] R. C. Myers, “Dielectric-branes,” JHEP 9912 (1999) 022, [hep-th/9910053].

[10] P. Kraus, “Spin-Orbit interaction from Matrix theory,” Phys. Lett. B419 (1998) 73, [hep-th/9709199].

[11] W. Taylor and M. Van Raamsdonk, “Supergravity currents and linearized interactions for matrix theory configurations with fermion backgrounds,” JHEP 9904 (1999) 013, [hep-th/9812239].

28
[12] W. Taylor and M. Van Raamsdonk, “Multiple Dp-branes in weak background fields,” hep-th/9910052.

[13] I. R. Klebanov, W. Taylor and M. Van Raamsdonk, “Absorption of dilaton partial waves by D3-branes,” hep-th/9905174.

[14] M. Aganagic, C. Popescu, and J. H. Schwarz, “D-brane Actions with Local Kappa Symmetry,” Phys. Lett. B393 (1997) 311-315 hep-th/9610249.

[15] M. Aganagic, C. Popescu, and J. H. Schwarz, “Gauge-Invariant and Gauge-Fixed D-brane Actions,” hep-th/9612080.

[16] M. Cederwall, A. von Gussich, B. E.W. Nilsson, P. Sundell, A. Westerberg, “The Dirichlet Super-p-Branes in Ten-Dimensional Type IIA and IIB Supergravity,” Nucl.Phys. B490 (1997) 179-201 hep-th/9611159.

[17] E. Bergshoeff, P. K. Townsend, “Super D-branes,” Nucl.Phys. B490 (1997) 145-162 hep-th/9611173.

[18] B. de Wit, K. Peeters and J. Plefka “Superspace geometry for supermembrane backgrounds,” hep-th/9803203.

[19] M. Cvetic, H. Lu, C. N. Pope and K. S. Stelle, “T-duality in the Green-Schwarz formalism, and the massless/massive IIA duality map,” Nucl. Phys. B573, 149 (2000), hep-th/9907202.

[20] M. T. Grisaru, M. E. Knutt and W. Siegel, “A superspace normal coordinate derivation of the density formula” Nucl. Phys. B523, 663-679 (1998) hep-th/9711120.

[21] V. Balasubramanian, D. Kastor, J. Traschen and K. Z. Win, “The spin of the M2-brane and spin-spin interactions via probe techniques,” hep-th/9811037.

[22] J. Plefka and A. Waldron, “On the quantum mechanics of M(atrix) theory,” Nucl. Phys. B512, 460 (1998), hep-th/9710104.

[23] F. Adams, Lectures on exceptional Lie groups, University of Chicago Press, 1996.

[24] M. Barrio, R. Helling and G. Polhemus, “Spin-spin interaction in M(atrix) theory ,” JHEP 9805 (1998) 012, hep-th/9801189.

[25] J. Plefka, M. Serone and A. Waldron, “D = 11 SUGRA as the low energy effective action of Matrix Theory: three form scattering,” JHEP 9811 (1998) 010, hep-th/9809070.

[26] P. C. Aichelburg and R. U. Sexl, “On The Gravitational Field Of A Massless Particle,” Gen. Rel. Grav. 2, 303 (1971).
[27] S. Sethi and M. Stern, “The structure of the D0-D4 bound state,” Nucl. Phys. B578, 163 (2000), hep-th/0002131.

[28] J. Polchinski and M. Strassler, “The string dual of a confining four-dimensional gauge theory,” hep-th/0003136.

[29] S. Hyun, Y. Kiem and H. Shin, “Supersymmetric completion of supersymmetric quantum mechanics,” Nucl. Phys. B558, 349 (1999), hep-th/9903022.

[30] M. Huq and M. A. Namazie, “Kaluza-Klein Supergravity In Ten-Dimensions,” Class. Quant. Grav. 2, 293 (1985).

[31] I. C. Campbell and P. C. West, “N=2 D = 10 Nonchiral Supergravity And Its Spontaneous Compactification,” Nucl. Phys. B243, 112 (1984).