Constrained low-rank quaternion approximation for color image denoising by bilateral random projections

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Abstract—In this letter, we propose a novel low-rank quaternion approximation (LRQA) model by directly constraining the quaternion rank prior for effectively removing the noise in color images. The LRQA model treats the color image holistically rather than independently for the color space components, thus it can fully utilize the high correlation among RGB channels. We design an iterative algorithm by using quaternion bilateral random projections (Q-BRP) to efficiently optimize the proposed model. The main advantage of Q-BRP is that the approximation of the low-rank quaternion matrix can be obtained quite accurately in an inexpensive way. Furthermore, color image denoising is further based on nonlocal self-similarity (NSS) prior. The experimental results on color image denoising illustrate the effectiveness and superiority of the proposed method.

Index Terms—Color image denoising, quaternion, bilateral random projections, low-rank, non-local similarity priors.

I. INTRODUCTION

Image denoising is an important task in the field of image processing. Among numerous image denoising methods, low-rank matrix approximation (LRMA) methods have made a great success [1]–[4]. These methods generally adopt certain rank approximation regularizers, e.g., the nuclear norm which has been proven the tightest convex relaxation of the NP-hard rank minimization function [5]. To better approximate the rank function, the authors in [2] considered assigning different weights to different singular values and proposed the weighted nuclear norm minimization (WNNM) algorithm. Although most of the existing LRMA methods have achieved excellent performance for grayscale image denoising, when handling color images, they may suffer from performance degradation. These methods are inherently designed for grayscale image denoising, when extending them to color images, they usually processes each color channel independently using the monochromatic model or processes the concatenation of three color channels using the concatenation model [6], [7]. However, these two schemes ignore the inter-relationship among RGB channels, thus they may produce hue distortions in the reconstruction results [8].

Recently, quaternion, as an elegant color image representation tool, has achieved great success for color image processing [9]–[11]. Adopting quaternion algebra, a color image encoded as a pure quaternion matrix is processed holistically and the coupling between the color channels is handled naturally [12]. More recently, the authors in [8] and [7] extended the traditional LRMA methods to quaternion field, and respectively proposed LRQA (including nuclear norm, Laplace function, Ge-
the proposed approach compared to the state-of-the-art methods in color image denoising task.

The remainder of this letter is organized as follows. Section II briefly introduces some notations and preliminaries for quaternion algebra. Section III gives the proposed model and algorithm. Section IV provides some experiments to illustrate the performance of our algorithm, and compare it with several state-of-the-art methods. Finally, some conclusions are drawn in Section V.

II. NOTATIONS AND PRELIMINARIES

A. Notations

In this letter, \( \mathbb{R} \) and \( \mathbb{H} \) respectively denote the real space and quaternion space. A scalar, a vector, and a matrix are written as \( a \), \( \mathbf{a} \), and \( \mathbf{A} \), respectively. \( \mathbf{a} \), \( \mathbf{a}^* \), and \( \mathbf{A}^* \) respectively represent a quaternion scalar, a quaternion vector, and a quaternion matrix. \((\cdot)^*\), \((\cdot)^{-1}\), and \((\cdot)^H\) denote the conjugation, inverse, and conjugate transpose, respectively. \(|\cdot|\) and \(|\cdot|_F\) are respectively the modulus and the Frobenius norm. \(\text{tr}\{\cdot\}\) and \(\text{rank}(\cdot)\) denote the trace and rank operators, respectively.

B. Basic knowledge of quaternion algebras

Quaternion space \( \mathbb{H} \) was first introduced by W. Hamilton [14] in 1843. A quaternion \( \mathbf{q} \in \mathbb{H} \) is defined as

\[
\mathbf{q} = q_0 + q_1 i + q_2 j + q_3 k,
\]

where \( q_l \in \mathbb{R} \) \((l = 0, 1, 2, 3)\), and \( i, j, k \) are imaginary number units and obey the quaternion rules that

\[
\begin{aligned}
i^2 &= j^2 = k^2 = ijk = -1, \\
i j &= -ji = k, j k &= -kj = i, k i &= -ki = j.
\end{aligned}
\]

\( \mathbf{q} \) can be decomposed into a real part \( \Re(\mathbf{q}) := q_0 \) and an imaginary part \( \Im(\mathbf{q}) := q_1 i + q_2 j + q_3 k \) such that \( \mathbf{q} = \Re(\mathbf{q}) + \Im(\mathbf{q}) \). If the real part \( \Re(\mathbf{q}) = 0 \), \( \mathbf{q} \) is named a pure quaternion. Given two quaternions \( \mathbf{p} \) and \( \mathbf{q} \in \mathbb{H} \), the sum and multiplication of them are respectively

\[
\begin{aligned}
\mathbf{p} + \mathbf{q} &= (p_0 + q_0) + (p_1 + q_1)i + (p_2 + q_2)j + (p_3 + q_3)k, \\
p \mathbf{q} &= (p_0 q_0 - p_1 q_1 - p_2 q_2 - p_3 q_3) + (p_0 q_1 + p_1 q_0 + p_2 q_3 - p_3 q_2)i \\
&\quad + (p_0 q_2 - p_1 q_3 + p_2 q_0 + p_3 q_1)j \\
&\quad + (p_0 q_3 + p_1 q_2 - p_2 q_1 + p_3 q_0)k.
\end{aligned}
\]

It is noticeable that the multiplication of two quaternions is not commutative so that in general \( p \mathbf{q} \neq \mathbf{q} p \). The conjugate and the modulus of a quaternion \( \mathbf{q} \) are, respectively, defined as follows:

\[
\mathbf{q}^* = q_0 - q_1 i - q_2 j - q_3 k, \\
|\mathbf{q}| = \sqrt{q_0^2 + q_1^2 + q_2^2 + q_3^2}.
\]

Analogously, a quaternion matrix \( \mathbf{Q} = (q_{mn}) \in \mathbb{H}^{M \times N} \) is written as \( \mathbf{Q} = \mathbf{Q}_0 + \mathbf{Q}_1 i + \mathbf{Q}_2 j + \mathbf{Q}_3 k \), where \( \mathbf{Q}_l \in \mathbb{R}^{M \times N} \) \((l = 0, 1, 2, 3)\). \( \mathbf{Q} \) is named a pure quaternion matrix when \( \Re(\mathbf{Q}) := \mathbf{Q}_0 = 0 \). The Frobenius norm of quaternion matrix is defined as

\[
|\mathbf{Q}|_F = \sqrt{\sum_{m=1}^{M} \sum_{n=1}^{N} |q_{mn}|^2} = \sqrt{\text{tr}((\mathbf{Q})^H \mathbf{Q})}.
\]

More details about quaternion algebra can be found in [15], [16] and their references.

III. MAIN RESULTS

A. The proposed model

Color images are represented as pure quaternion matrix. We aim to recover the clear image \( \mathbf{X} \in \mathbb{H}^{M \times N} \) from its noisy observation

\[
\mathbf{Y} = \mathbf{X} + \tilde{\mathbf{G}},
\]

where \( \tilde{\mathbf{G}} \) is assumed to be Gaussian noise [7], [8]. Supposing that \( \mathbf{X} \) is low-rank with \( \text{rank}(\mathbf{X}) \leq r \), we formulate the following model

\[
\begin{aligned}
\min_{\mathbf{X}} & \quad \|\mathbf{Y} - \mathbf{X}\|_F^2, \\
\text{s.t.} & \quad \text{rank}(\mathbf{X}) \leq r.
\end{aligned}
\]

B. The proposed algorithm

The optimal \( \mathbf{X} \) in (4) can be obtained by truncated QSVD of \( \mathbf{Y} \). The QSVD in [7] and [8] was calculated by their equivalent complex matrices with twice sizes, which generally requires \( \min(O(MN^2), O(M^2 N)) \) flops, thus it is impractical when \( \mathbf{Y} \) is of large size. To efficiently solve the problem (4) we develop a Q-BRP algorithm, which replaces the truncated QSVD, and significantly reduces the time cost.

Definition 1. (Q-BRP) For \( \mathbf{Q} \in \mathbb{H}^{M \times N} \) (w.l.o.g., \( M > N \)), the Q-BRP of \( \mathbf{Q} \) can be constructed, i.e., \( \mathbf{P}_1 = \mathbf{Q} \mathbf{A}_1 \), and \( \mathbf{P}_2 = \mathbf{Q}^H \mathbf{A}_2 \), wherein \( \mathbf{A}_1 \in \mathbb{H}^{N \times r} \) and \( \mathbf{A}_2 \in \mathbb{H}^{M \times r} \) are random quaternion matrices.

Then, the Q-BRP based \( r \) rank approximation of \( \mathbf{Q} \) is

\[
\mathbf{Q} = \mathbf{P}_1 (\mathbf{A}_2^H \mathbf{P}_1)^{-1} \mathbf{P}_2^H.
\]

For random quaternion matrices \( \mathbf{A}_1 \) and \( \mathbf{A}_2 \), we pick matrices with \( i.i.d. \) Gaussian entries for real and imaginary parts. These two random quaternion matrices are
used to iteratively project $\hat{Q}$ to $r$-dimensional subspaces. And note that $A_2^H P_1$ is invertible with probability one. Step (5) is an approximation of the truncated QSVD, which is similar to the case in the real setting as explained in [17]. Then, following the framework in [18], the constrained low-rank quaternion approximation algorithm by bilateral random projections (CLQA-BRP) is summarized in TABLE I.

Table I

**CLQA-BRP Algorithm.**

| Input: $Y$, $r$, maximum iterations $T$. |
|-----------------------------------------|
| 1: Initialize $X^0 = 0$, iteration index $t = 0$. |
| 2: for $t = 1 : T$ do |
| 3: $P_1 = Y A_1$, $A_2 = P_1$. |
| 4: $P_2 = Y H P_1$, $P_1 = Y P_2$. |
| 5: if $\text{rank}(A_2^H P_1) < r$ then |
| 6: $r := \text{rank}(A_2^H P_1)$, regenerate the random quaternion matrix $A_1$, and go to the first step. |
| 7: end if |
| 8: $X^t = P_1 (A_2^H P_1)^{-1} P_2^H$ |
| 9: end for |
| Output: $X^t$. |

**Remark 1:** (Computation complexity) The computation of $X^t$ consists of an inverse of an $r \times r$ ($r \ll \text{min}(M, N)$) quaternion matrix and three quaternion matrix multiplications. Hence, for $Y \in \mathbb{H}^{M \times N}$ with $\text{rank}(X) \leq r$, $O(MN^2)$ flops are required to perform Q-BRP, $O(r^2(M + r) + MNr)$ flops are required to compute $X$. We can find that the computational complexity is much less than QSVD-based approximation method.

**Remark 2:** In the denoising task, we let the maximum iterations $T = 1$, which is enough since increasing the number of iterations would not noticeably improve the result.

**C. CLQA-BRP for color image denoising**

Except for low-rank prior, we further consider the NSS property for image denoising tasks. We follow the similar procedure of NSS used in [7]. Consequently, given a noisy color image $Y \in \mathbb{H}^{M \times N}$, the whole procedure of our color image denoising method is listed as follows:

**Step 1:** Divide the noisy image $Y$ into overlapped patches with size $w \times w$, and vectorize each patch as a quaternion column vector $\hat{y}_i \in \mathbb{H}^w$, then find its $n$ nearest neighbor patches (including $\hat{y}_i$ itself) within its local searching window. At last, the $n$ similar patches are stacked as quaternion column vectors of quaternion matrix $\hat{Y}_i \in \mathbb{H}^{w^2 \times n}$. 

**Step 2:** For each $\hat{Y}_i$, adopt the proposed CLQA-BRP algorithm to estimate the clear color image patch $\hat{X}_i$. 

**Step 3:** Aggregate $\{\hat{X}_i\}$ together to form the final clear image $X$.

Generally, to obtain a better result, several rounds (denoted by $K$) for Step 1 and Step 2 are needed before going to Step 3.

**IV. EXPERIMENTAL RESULTS**

Some experiments on 8 widely used color images (see Fig.1) are conducted to evaluate the effectiveness of the proposed method. The additive white Gaussian noise with zero mean and variance $\sigma_n^2$ ($\sigma_n = 50, 70$ are considered in our experiments) is added to these clean color images. We compare the proposed method with several latest state-of-the-art methods including WNNM [2] (a weighted nuclear norm minimization algorithm), QNNM [7] (a quaternion nuclear norm minimization algorithm labeled as LRQA-1 in [7]), QWNNM [8] (a quaternion weighted nuclear norm minimization algorithm) and LRQA-WSNN [7] (a quaternion weighted Schatten norm minimization algorithm labeled as LRQA-4 in [7]). We set the same parameters in the NSS procedure for all the methods. For instance, when $\sigma_n = 50, 70$, we set patch size to $8 \times 8$ and $9 \times 9$ respectively. The number of similar patches group is set to 120 and 140, respectively. The parameters of each compared algorithm are optimally set or selected as suggested in the source papers.

For the proposed CLQA-BRP algorithm, we set $T = 1$, and select $r$ from $\{7, 9, 15\}$. We employ two widely used quantitative quality indexes (including the peak signal-to-noise ratio (PSNR) and the structure similarity (SSIM)) for performance evaluation. For WNNM, we perform it on each channel of the test color images individually. For quaternion-based methods, we use the same quaternion toolbox.

![Fig. 1. The 8 color images (from left to right, Image(1) ~ Image (8)), are selected form Kodak PhotoCD Dataset (Kodak) all with size 512 × 768 × 3.](https://sourceforge.net/projects/qtfm/)

**TABLE II** lists the quantitative PSNR and SSIM values (and the average values of PSNR) of all denoising methods. Fig.2 displays the visual comparison between the proposed method and all compared methods on the Image(3) with $\sigma_n = 70$. Fig.3 shows the runtime comparison of all the quaternion-based methods. According to the obtained results, the following conclusions can be found:

1. https://sourceforge.net/projects/qtfm/
2. http://r0k.us/graphics/kodak/
TABLE II
QUANTITATIVE ASSESSMENT INDEXES (PSNR/SSIM) OF DIFFERENT METHODS ON THE EIGHT COLOR IMAGES (BOLD FONTS DENOTE THE BEST PERFORMANCE; UNDERLINE ONES REPRESENT THE SECOND-BEST RESULTS).

| Methods | WNNM [2]  | QNNM [7]  | QWNNM [8]  | LRQA-WSNN [7] | CLQA-BRP |
|---------|-----------|-----------|------------|---------------|----------|
| Images  | σ_n = 50  | σ_n = 70  | σ_n = 50   | σ_n = 70      | σ_n = 70 |
| Image(1)| 23.936/0.753 | 23.956/0.753 | 24.215/0.751 | 24.540/0.786 | 24.653/0.774 |
| Image(2)| 30.025/0.912  | 29.500/0.923  | 30.270/0.943 | 30.240/0.874 | 30.294/0.950 |
| Image(3)| 25.880/0.945  | 28.310/0.944  | 29.035/0.953 | 29.031/0.952 | 29.002/0.959 |
| Image(4)| 24.150/0.740  | 23.372/0.754  | 24.437/0.793 | 24.410/0.783 | 24.420/0.789 |
| Image(5)| 25.210/0.800  | 23.500/0.765  | 24.040/0.786 | 24.390/0.804 | 24.450/0.810 |
| Image(6)| 25.360/0.758  | 24.840/0.756  | 25.557/0.769 | 25.570/0.766 | 25.579/0.768 |
| Image(7)| 26.001/0.822  | 25.392/0.818  | 26.240/0.843 | 26.390/0.833 | 26.010/0.831 |
| Image(8)| 30.130/0.946  | 29.521/0.939  | 30.356/0.956 | 30.211/0.953 | 30.225/0.950 |
| Av.     | 26.589       | 26.052       | 26.771      | 26.818        | 26.858    |

Images: the observed image (b) is the observed image (σ_n = 50), (b) is the recovery results of WNNM, QNNM, QWNNM, LRQA-WSNN and CLQA-BRP, respectively.

Fig. 2. Color image denoising results on Image(3). (a) is the original image. (b) is the observed image (σ_n = 70). (c)-(g) are the recovery results of WNNM, QNNM, QWNNM, LRQA-WSNN and CLQA-BRP, respectively.

Fig. 3. The runtime comparison of all the quaternion-based methods. (a) σ_n = 50, (b) σ_n = 70.

o The quaternion-based methods (QWNNM, LRQA-WSNN, and CLQA-BRP) outperform WNNM in all color images. The QNNM has a relatively poor performance, since it, relative to WNNM, QWNNM, and LRQA-WSNN, does not assign different singular values with different weights.

o The performance of QWNNM, LRQA-WSNN, and our proposed CLQA-BRP are very close, even so, CLQA-BRP has the best average PSNR values in both σ_n = 50 and σ_n = 70.

o From Figure 3, we can see that the runtime of CLQA-BRP is much shorter than that of other quaternion-based methods (about half of them). This means that CLQA-BRP is more practical and efficient.

V. CONCLUSION

We introduced a novel constrained low-rank quaternion approximation model for removing the noise in color images. Then we design an iterative algorithm by using Q-BRP for efficiently solving the proposed model, which can significantly accelerate the approximation of the low-rank quaternion matrix. Experimental results on color image denoising demonstrate the effectiveness of the proposed CLQA-BRP. In the future, we tend to apply the CLQA-BRP algorithm to color image inpainting and other color image processing tasks.

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