Random field induced order in low dimension

Nicholas Crawford

The Technion - Haifa, Israel

received 17 October 2012; accepted in final form 17 April 2013
published online 17 May 2013

PACS 64.60.Cn - Order-disorder transformations
PACS 64.60.De – Statistical mechanics of model systems (Ising model, Potts model, field-theory models, Monte Carlo techniques, etc.)
PACS 05.70.Fh – Phase transitions: general studies

Abstract – We address an unresolved issue in the physics of low-dimensional many-body systems: the question of whether or not a random field can produce order at low temperatures for statistical mechanical systems possessing continuous internal symmetries. Concretely, we verify that the XY model in a uniaxial random field orders in two and three dimensions. The direction the system orders is perpendicular to the randomness for any choice of symmetry breaking field with nonzero projection perpendicular to the randomness. The result is particularly relevant in two dimensions, where there are a number of competing effects — quasi-long-range order of the pure system and strong fluctuations of the random field. While we consider only classical systems explicitly, the effect is robust and our work has implications for quantum systems as well, producing ordered phases in any dimension.

Copyright © EPLA, 2013

Introduction. – It is generally understood that disorder plays an important role in the behavior of one- and many-body physical systems. The most well-known example stems from the work of Anderson [1] on transport properties of electrons in crystals with impurities; disorder can localize electron wave functions making conductivity negligible. Another important example is the “rounding” of first-order transitions in low-dimensional classical equilibrium systems [2,3], recently extended to quantum systems in [4].

The rounding effect aside, the behavior of interacting many-body systems in the presence of disorder is less well understood. For example, it was only recently demonstrated (in the physics literature) that Anderson localization persists in the presence of weak interactions between electrons [5]. The issue is still largely under investigation in the mathematics literature and we refer to the recent papers [6–8] for more on this and related topics. Other interesting recent work concerns the localization of Cooper pairs in the vicinity of the BCS superconducting transition [9].

In these examples, as well as in subjects such as quantum computing, the existence of quenched randomness in a system is viewed as an unwanted effect which disrupts ordering and coherence. It is less well known that randomness can itself create ordering for systems in which the order parameter has a continuous symmetry. We refer to this effect below as random field induced order (RFIO).

The basic model for RFIO is an XY model in a uniaxial random field. It is a classical equilibrium statistical mechanics model in which spins take values on the unit circle and interact in a ferromagnetic way. There is additionally a random field acting in the vertical direction only. In more precise terms and generalizing the setting, we will below consider the following family of models. Let \( \Lambda_N = \{-N/2, \ldots, N/2-1\}^d \). For \( x \in \Lambda_N \), spin variables \( \sigma_x \) lie in the unit sphere \( S^{n-1} \subseteq \mathbb{R}^n \), \( n \geq 2 \). We define the (random) Hamiltonian via

\[
-\mathcal{H}_N(\sigma) = - \sum_{\langle xy \rangle} [\sigma_x - \sigma_y]^2 + \epsilon \sum_x \alpha_x \cdot \sigma_x + \sum_{x \in \partial \Lambda_N} u \cdot \sigma_x. \tag{1}
\]

The first sum is over adjacent pairs \( x, y \) in \( \Lambda_N \), the second is over \( x \) in \( \Lambda_N \) and the third is over vertices \( x \) in the interior boundary of \( \Lambda_N \) — the set of vertices in \( \Lambda_N \) which have vertices in \( \mathbb{Z}^d \setminus \Lambda_N \) as nearest neighbors. The \( \alpha_x \)'s form a family of independent, identically distributed \( k \)-dimensional standard Gaussian vectors, where we have taken \( k < n \). The last term is introduced for symmetry breaking purposes with \( u \in S^{n-1} \) a fixed direction. When defining Gibbs measures and states, the \( \text{a priori} \) measure on spin space is the uniform surface measure on \( S^{n-1} \). We refer below to these models collectively as RFO(\( n; k \)) models. In this notation, the XY model with a uniaxial field is the RFO(\( n; n - 1 \)) model. The latter will be shortened to the RFO(\( n \)) model.

The first sum is over adjacent pairs \( x, y \) in \( \Lambda_N \), the second is over \( x \) in \( \Lambda_N \) and the third is over vertices \( x \) in the interior boundary of \( \Lambda_N \) — the set of vertices in \( \Lambda_N \) which have vertices in \( \mathbb{Z}^d \setminus \Lambda_N \) as nearest neighbors. The \( \alpha_x \)'s form a family of independent, identically distributed \( k \)-dimensional standard Gaussian vectors, where we have taken \( k < n \). The last term is introduced for symmetry breaking purposes with \( u \in S^{n-1} \) a fixed direction. When defining Gibbs measures and states, the \( \text{a priori} \) measure on spin space is the uniform surface measure on \( S^{n-1} \). We refer below to these models collectively as RFO(\( n; k \)) models. In this notation, the XY model with a uniaxial field is the RFO(\( n; n - 1 \)) model. The latter will be shortened to the RFO(\( n \)) model.
The RFO(2) model was first studied in the 1970s via mean field theory and a renormalization group calculation [10]. The system appears to order, but only perpendicular to the axis on which the randomness acts. In low dimensions, particularly on the two-dimensional square lattice $\mathbb{Z}^2$, the mean field and renormalization group calculations are less reliable, as the conclusions run counter to other effects, such as the Mermin-Wagner theorem and the Imry-Ma argument that strong fluctuations disrupt ordering [3] (more about the relation between RFIO and these results appears below). Thus in the 1980s multiple groups addressed two-dimensional behavior by more specialized renormalization group analyses [11–13]. One group [11] concluded there is a low-temperature paramagnetic phase while the other [13] concluded there is an intermediate-temperature ordered phase from which they extrapolate the low-temperature behavior. Interesting related rigorous work was done in the 1990s on ground states in the strong field regime in [14,15].

Both of the aforementioned analyses of the RFIO(2) model in a weak field are problematic. The work [11,12] is based on an approximation via a random field sine-Gordon model, which is extremely unstable with respect to the noise (see our related discussion of the random field Gaussian model below). The paper [13] suggests that the ordered phase persists at a fixed temperature even as the field strength $\epsilon$ is taken to 0. Such a conclusion is dubious as simpler versions of the model, such as a model which replaces the random field by a periodic field with period two, have a critical inverse temperature which diverges as the field strength tends to 0 (in two dimensions).

With this background it seems useful to provide a definitive analysis of the RFO(2) model in two dimensions. We do this below. In Theorem 1, we state, for dimension $d = 2, 3$, that ordering in the direction perpendicular to randomness does occur at low temperature. Moreover, we show that this is the only direction in which ordering occurs in the following sense: it is independent of the direction of infinitesimal field used to break symmetry. This is part of a more general result for RFO($n$) models.

Next we state a second result, Theorem 2. This provides a small step toward the mathematical understanding of RFO($n$; $k$) models for $k \leq n - 2$. The general statement is that the projection of a (spatially averaged) spin variable at any fixed vertex onto the subspace which supports the randomness is necessarily small. After formulating our main results, we give the main ideas of their rigorous proofs. Full details are rather lengthy and are presented elsewhere [16,17].

Let us remark that the issue of RFIO resurfaced over the last five years in the work of three separate groups. With the advent of controllable interacting Bose-Einstein condensates in optical traps, the effect was suggested as a possible response to the presence of certain kinds of experimentally realizable disorder [18–22]. Here the (pseudo-)spin variables arise from internal structure of the atoms, the tuning of interactions and the structure of the optical lattice. Building on [23], the same sort of mechanism was put forward as the reason for the splitting of Landau level degeneracy in experiments on graphene [24], though the particulars of this model, namely the fact that the derived random field is the curl of random coupling constants, means that the fluctuations of the disorder are subcritical from the perspective of the Imry-Ma argument. Finally, in [25], investigation of the RFO(2) model was emphasized as an intermediate step toward the understanding of whether (and when) real-space coarse-graining procedures preserved a version of the spatial Markov property which characterizes Gibbs measures.

To explain the subtlety of our results in two dimensions and to put them in context we recall, for the convenience of the reader, the behavior of related models.

The pure O(2) model. In the setup of (1), this is the case $n = 2$ and $\epsilon = 0$, so there is no randomness. When $d = 2$ all Gibbs states are rotationally invariant in the thermodynamic limit (this is the content of the Mermin-Wagner theorem, see [26,27] among many other works) and, in particular, there is no residual magnetic order. There is however a Kosterlitz-Thouless phase transition [28,29] expressed by a change in the behavior of the spin-spin correlation function $\langle \sigma_x \sigma_y \rangle$ from exponential to algebraic decay in terms of the lattice distance $|x - y|$. If $d \geq 3$ residual magnetic ordering does occur [30].

The random field Ising model. In this case we constrain the spins to point only in the vertical direction, the same as the random field. When $d \geq 3$ and the disorder is weak, it turns out that residual magnetic ordering persists [31,32]. In dimension $d = 2$ there is no magnetic order for any inverse temperature $\beta$ and any strength $\epsilon$ [2]. This is the rounding effect alluded to in the introduction. Physically, see [3], the reason for this is that at all scales there are random fluctuations strong enough to overcome the loss in energy due to mismatched spins on the interior and exterior of domains.

The random field Gaussian model. The last model we wish to mention is obtained by replacing the vector valued spins $\sigma_x \in S^1$ by a field $\phi_x \in \mathbb{R}$ and otherwise retaining the setup described above. This model appears, among other places, in [33,34]. In a finite volume $\Lambda$ with 0 boundary conditions, we let $-\Delta_\Lambda$ denote the discrete Laplace operator on $\Lambda$ with Dirichlet boundary conditions. We have

$$E[(\phi_x \phi_y)_\Lambda] = -\Delta_\Lambda^{-1}(x,y) + \epsilon^2 \Delta_\Lambda^2(x,y). \quad (2)$$

Here and below $E$ denotes the average over the randomness in the model of interest.

If $\epsilon > 0$ and if $d \leq 4$, the second term on the RHS grows with $\Lambda$ while when $d = 2$, it grows even after taking gradients in both arguments $x, y$.

If one believes in magnetic ordering for the RFO(2) model in two dimensions, the random field Gaussian model
provides serious difficulties: indeed the spin wave Hamiltonian obtained expanding the RFO(2) Hamiltonian (1) around a fixed direction in angular coordinates leads to this model. Thus the instability of the random field Gaussian model when $d = 2$ suggests that either the ansatz of having order is flawed or that one must introduce additional arguments to explain why the fact that spins are constrained to lie on $S^1$ keep the influence of field fluctuations under control. This instability of the random field Gaussian model presumably is responsible for the paramagnetic low-temperature phase prediction in [11]. On the other hand, this effect seems to have been neglected in the arguments presented in [13,22].

Order-by-disorder. The phenomenon of “order-by-disorder” provides a class of systems which exhibit ordering due to various types of fluctuations. The most relevant example, first considered by Henley [35], concerns a model Hamiltonian on $\mathbb{Z}^2$ of the form

$$-\mathcal{H}(\sigma) = \sum_{|x-y|^2=1} J_1[\sigma_x - \sigma_y]^2 + \sum_{|x-y|^2=2} J_2[\sigma_x - \sigma_y]^2$$

with $|J_1| < 2J_2$. The ground states for this (frustrated) system are obtained by choosing a purely anti-aligned configuration of spins on each of the even and odd sublattices of $\mathbb{Z}^2$ and are thus parameterized by two angles: an angle between the spin at $(0,0)$ and the $e_1$-axis and relative angle between the spin at $(0,1)$ and $(0,0)$. The degeneracy of ground states is partially lifted under the introduction of two types of “disorder”. The first type consists in passing from 0 to positive temperature. More relevant to the RFIO is a second mechanism: site dilution. Vertices of $\mathbb{Z}^2$ are deleted from the system independently with probability $p \ll 1$. According to the calculations in [35], at 0 and low temperature the system prefers the ground states with relative angle between spins at $(0,0)$ and $(0,1)$ to be fixed at $\pm \frac{\pi}{2}$.

The crucial difference between this site-diluted model and the RFO($n$) models we consider concerns fluctuations due to randomness. These are substantially weaker in the site-diluted order-by-disorder setting. In particular, there is an analog to the field $g_\alpha$ introduced below but the fluctuations of this field in two dimensions are about as singular as the four-dimensional version of the RFO(2) model. While the site-diluted order-by-disorder problem has not been rigorously addressed, this feature suggests the conclusions in [35] are reliable. It would be interesting to see if our methods can be adapted to this case.

Main results. – In the following statements, $P$ denotes the projection operator onto the $k$-dimensional subspace supporting the distribution of $\alpha_z$.

Recall (1) and let $(\hat{\sigma}^{3,u}_N)$ denote the corresponding random Gibbs state on $\Lambda_N$ at inverse temperature $\beta$. For the statement of our main result, let us define the block average observables

$$M_z = \epsilon^d \sum_{y:|y-z|\leq(2\epsilon)^{-1}} \sigma_y,$$

where we recall that $\epsilon$ is the strength of the random field in (1) and the dependence of $M_z$ on this parameter is suppressed in our notation.

**Theorem 1.** Let $d \in \{2,3\}$ be fixed and consider the RFO($n$; $n-1$) model on $\mathbb{Z}^d$. Let $\epsilon_n$ be a fixed unit vector so that $P \cdot \epsilon_n = 0$. There exists $\epsilon_0 \in (0,1)$ and functions $\beta_0: (0,\epsilon_0) \to (0,\infty)$ and $\xi: (0,\epsilon_0) \to (0,1)$ with $\lim_{\epsilon \downarrow 0} \xi(\epsilon) = 0$ so that the following holds for all $\epsilon \in (0,\epsilon_0)$:

If $u \in S^{n-1}$, $u \cdot \epsilon_n > \xi(\epsilon)$

and $\beta > \beta_0(\epsilon)$, we have

$$\liminf_{N \to \infty} \mathbb{E} \left[ \langle M_z \cdot \epsilon_n \rangle^2_N \right] \geq 1 - \xi(\epsilon)$$

for any $z \in \mathbb{Z}^d$.

Let us make some remarks regarding Theorem 1. First, a similar effect was previously observed in XY models with a weak uniaxial field alternating direction in a chessboard fashion in classical [10,25] and also in quantum models [36]. Also, at 0 temperature, Theorem 1 strengthens the picture presented in [13,22]. Our methods show that, as $\epsilon$ tends to 0, in the ground state the length of the projection of most spins onto the direction perpendicular to the randomness tends to 1 as $\epsilon$ tends to 0.

Second, the condition $u \cdot \epsilon_n > \xi(\epsilon)$ is technical. Ideally, one would like to prove that there exists $\epsilon_0 > 0$ so that for all $\epsilon \in (0,\epsilon_0)$ the conclusion holds whenever the boundary condition $u \cdot \epsilon_n > 0$. It is possible that further analysis will remove this restriction.

Third, the proof requires $\xi(\epsilon) \geq C|\log \epsilon|^{-1/2}$ and $\beta_0(\epsilon) \geq C\epsilon^{-2}$ for some universal constant $C > 0$. The latter bound is of particular interest because [13] suggests the critical temperature is $\epsilon$-independent. It is an open issue to investigate the behavior of the RFO($n$; $n-1$) models at intermediate temperatures when $d = 2$.

Finally, intuition and [10] suggest similar statements should hold at a rigorous level for $d \geq 4$. Moreover we expect that the effects of large fluctuations should be easier to control as the dimension of the lattice increases. Our technical estimates bear this out, but currently we cannot provide a complete proof. While we discuss this point further below, let us mention here that this is connected with the fact that control of the Dirichlet energy of functions provides better pointwise control in low dimension than in high dimension.

Next we present a weaker result for general RFO($n$; $k$) models in any dimension $d \geq 2$. Let $\tau$ be a fixed spin configuration on $\mathbb{Z}^d$ and let $(\hat{\sigma}^{3,\tau}_N)$ denote the Gibbs state on $\Lambda_N$ with boundary condition $\tau$. 

36003-p3
Theorem 2. Let $d \geq 2$ and $k < n$ be fixed and consider the RFO($n; k$) model on $\mathbb{Z}^d$. There exists $\epsilon_0 \in (0, 1)$ and functions $\beta_0: (0, \epsilon_0) \to (0, \infty)$ and $\xi: (0, \epsilon_0) \to (0, 1)$ with $\lim_{\epsilon \to 0} \xi(\epsilon) = 0$ so that the following holds for all $\epsilon \in (0, \epsilon_0)$:

If $\beta > \beta_0(\epsilon)$,

$$\limsup_N E \left[ \|P \cdot M_N \|_{L^2}^{2/\beta} \right] \leq \xi(\epsilon)$$

for any $z \in \mathbb{Z}^d$.

For $d \in \{2, 3, 4\}$, Theorem 2 bears some resemblance to Theorem 4.4 of [2]. In fact, from that result, one can deduce the qualitative statement that ordering does not occur for the RFO($n; k$) model in directions contained in the support of the random field. However quantitative, local estimates are not easily derivable from the proof technique presented there. We further emphasize that the main point of Theorem 4.4 of [2], and also its antecedent [3], was to determine the lower critical dimension for ordering in various spin models including, in our notation, the RFO($n; n$) model (in which case the dimension is $d = 4$).

We make no rigorous statements regarding magnetic ordering in RFO($n; k$) models for $k \leq n - 2$, although ordering should occur if $d \geq 3$. If $d = 2$ we expect the RFO($n; n - 2$) models to have a Kosterlitz-Thouless transition and that if $k < n - 2$, the model exhibits exponential decay of correlations for all $\beta$ in correspondence with the expected behavior of pure $O(n - k)$ models when $d = 2$.

The choice of standard Gaussian vectors is not crucial. The proof of Theorem 1 applies for variables with sub-Gaussian tails in $d = 2$ and for any distribution having arbitrary exponential moments if $d = 3$. The proof of Theorem 2 requires even weaker assumptions on the tail behavior.

Extensions. The effect described in this note should be very robust, applying to classical and quantum systems and both symmetric and biased disorder. Examples include:

Biased disorder: Suppose the $\alpha_x = \sum_{i=1}^m \alpha_x(i)$ independent Gaussian vectors with nonzero means $E[\alpha_x(i)] = v_i$. For example, if the $v_i$ form an orthonormal basis for $\mathbb{R}^n$, ordering should occur along the directions $\pm v_i$. It is possible this sort of model is treatable by our methods combined with Pirogov-Sinai theory.

Quantum models: Whenever the quantum system without disorder may be expressed in terms of pseudo-spin order parameters with continuous symmetry, effects analogous to those described above are possible. Examples include the models discussed in [18–21,24].

Proofs. For the interested reader, we now give a brief sketch of the proof. For simplicity, we restrict the discussion to the RFO(2) model. Demonstration of the remainder of Theorem 1 follows along the same lines. The proof of Theorem 2 is actually simpler because we do not need to distinguish different phases. For notational purposes set

$$\epsilon_d = \begin{cases} \epsilon \sqrt{\log \epsilon}, & \text{if } d = 2, \\ \epsilon, & \text{if } d \geq 3. \end{cases}$$

We use the notation $|R|$ for the cardinality of a finite set $R \subset \mathbb{Z}^d$.

Calculations for small boxes. – To understand the issues we encountered in the analysis, let us begin by studying the model restricted to boxes $Q_\ell \subset \Lambda$. We take the side-length $\ell$ of order $\epsilon^{-1}$ without being completely precise yet. This scale is the fundamental length scale in the problem as revealed below in formula (6).

In a fixed box $Q_\ell$, the first calculation we do is an optimization of the Hamiltonian $-\mathcal{H}_{Q_\ell}$ assuming the spin variables have small deviations from a fixed direction. Here $-\mathcal{H}_{Q_\ell}$ is the analog of (1) in $Q_\ell$ with free boundary conditions.

Fix an angle $\psi \in [0, 2\pi)$ and let us introduce deviation variables $\hat{\theta}_x \psi = (\cos(\psi + \hat{\theta}_x), \sin(\psi + \hat{\theta}_x))$. For convenience, denote $\alpha_x := \alpha_x - |Q|^{-1} \sum_{z \in Q_\ell} \alpha_z$ and

$$\mathcal{E}_{Q_\ell}(\alpha) := -\sum_{x \in Q_\ell} \alpha_x \cdot \Delta^{-1} \alpha_x.$$  

Expanding $-\mathcal{H}_{Q_\ell}$ to second order in the $\hat{\theta}$ variables,

$$\sup_{(\hat{\theta}_x)_x \in Q_\ell} -\mathcal{H}_{Q_\ell}(\hat{\theta}) = \frac{\epsilon^2}{2} \cos^2(\psi) \mathcal{E}_{Q_\ell} + O\left(\epsilon \sum_{z \in Q_\ell} \alpha_z\right).$$

The first term has a typical order of magnitude $\epsilon^2 \ell^d$ for $d \geq 3$ and $\epsilon^2 \ell^2 \log \ell$ for $d = 2$ while the second term is typically of order $\epsilon \ell^{d/2}$.

Neglecting the second term, directions of presumed ordering are obtained by optimizing the RHS in $\psi$. Further, for $\psi$ fixed, the optimal choice for the deviation variables $\hat{\theta}_x = \hat{\theta}_x = \epsilon \cos(\psi) g_{Q_\ell,x}$ where $g_{Q_\ell,x} = -\Delta^{-1} \cdot \alpha_x$.

In evaluating the validity of this calculation one has to worry about three issues. First, we must understand in what sense the assumption of small deviation from a fixed angle is valid. If we denote the Dirichlet energy of spin configurations in boxes $Q_\ell$ by

$$\mathcal{E}_{Q_\ell}(\sigma) := \frac{1}{|Q_\ell|} \sum_{x,y \in Q_\ell} |\sigma_x - \sigma_y|^2,$$

the only a priori control of small amplitude deviations we are able to obtain regarding $\mathcal{E}_{Q_\ell}$ is that it is costly energetically for $\mathcal{E}_{Q_\ell}(\sigma) \geq 4\epsilon_d^2 |Q_\ell|$. One cannot ask for better estimates than this because the approximate ground states found as the result of the optimization take the form $\sigma_x = (\cos(\epsilon g_{Q_\ell,x}), \sin(\epsilon g_{Q_\ell,x}))$ and typically have Dirichlet energy on the order $\epsilon_d^2 |Q_\ell|$.  

36003-p4
Second, in order to be self-consistent, the calculation imposes an upper bound on \( \ell \): It is a fact that \( g_{Q_{l},x} \) has typical order of magnitude \( \ell, \sqrt{\ell}, \sqrt{\log \ell} \) in dimensions \( d = 2, 3, 4 \) respectively. This indicates that naive expansions breakdown beyond the length scales \( \epsilon^{-1}, \epsilon^{-2}, \exp(\epsilon^{-2}) \), respectively, since at these scales, \( \epsilon g \) is of constant order. If we take fluctuations into account, in two dimensions this imposes \( \ell \ll \epsilon^{-1}_d \). If one believes in ordering, nonlinear effects must play a role at larger scales.

Conversely, and of particular significance in two dimensions, moderate deviations of the field may make the term \( O\left( \epsilon \sum_{z \in Q_{l}} \alpha_z \right) \) relevant. In two dimensions, this imposes the lower bound \( \ell \gg \epsilon^{-1} |\log \epsilon|^{-1} \). There is just enough room between the two constraints to make sense of the computation. If we take \( \ell = \epsilon^{-1} |\log \epsilon|^{-\frac{1}{2}} \gamma \) for \( \gamma \in (0, 1/2) \) fixed, we are able to keep \( |g_x| \leq |\log \epsilon|^{-\gamma} \) (even taking into account fluctuations) while \( O\left( \epsilon \sum_{z \in Q_{l}} \alpha_z \right) \) is lower order with probability exponential in \( |\log \epsilon|^{-2\gamma} \).

A key computation. – Next we present a computation of both physical and technical interest which allows us to turn the naive analysis presented above into a rigorous Peierls argument. The idea is that low-energy spin configurations are composed of a fast oscillating approximate ground state (like low-energy spin configurations are composed of a fast oscillating approximate ground state (like 

\[
\sigma \in \mathbb{R} \quad \text{(gamma)}
\]

into a more tractable form. A key computation. – Next we explain how to use (6) to implement a Peierls argument. Taking inspiration from [37], we define a second scale

\[
L \sim \begin{cases} 
\epsilon^{-1} |\log \epsilon|, & \text{if } d = 3, \\
\epsilon^{-1} |\log \epsilon|^{-\frac{1}{2}} \gamma, & \text{if } d = 2,
\end{cases}
\]

so that \( \ell \ll \epsilon^{-1}_d \ll L \).

Given a spin configuration \( \sigma \) on \( \Lambda \), a box \( Q_{l} \) is bad for \( \sigma \) if one of two things occurs. The first possibility is that the Dirichlet energy of \( \sigma \) inside \( Q_{l} \) is substantially larger than the energy scale \( E_{\Lambda}(\alpha) \) set by the randomness in \( Q_{l} \) (this is typically of order \( \epsilon_3^{d} \)). If the Dirichlet energy is smaller than this scale we identify a second possible source of bad behavior, namely that the average angle \( \psi_{Q_{l}} \) associated with \( \sigma \) in \( Q_{l} \) is bounded away from \( \{0, \pi\} \) by a chosen cutoff \( \xi(\epsilon) \).

A region of space \( \Gamma \subset \Lambda \) is called a contour for \( \sigma \) if it is maximally connected union of boxes \( Q_{L} \in \{Q_{l} | z \in \mathbb{R}^{d} \} \) so that for each \( Q_{L} \), there is a bad cube \( Q_{l} \) within Hausdorff distance \( 3L/2 \) of \( Q_{L} \). The goal, as with any Peierls argument, is to show that large contours are unlikely to occur at low temperature.

Given a spin configuration \( \sigma \) and an associated contour \( \Gamma \), to extract energy cost from the existence of \( \Gamma \) we compare \( \sigma \) with a new spin configuration \( \tilde{\sigma} \) which agrees with either \( \sigma \) or the reflection of \( \sigma \) across the \( e_2 \) axis on each component of \( \Lambda \setminus N_{L}(\Gamma) \), where \( N_{L}(\Gamma) = \{ x \in \mathbb{Z}^{d} : \text{dist}(x, \Gamma) \leq L \} \) and \( \text{dist} \) is the Hausdorff distance on finite subsets of \( \mathbb{Z}^{d} \) associated with the Euclidean norm. We require that \( \tilde{\sigma} \) is within some \( \delta \ll \xi(\epsilon) \) of either 0 or \( \pi \) on the whole of \( \Gamma \).

We construct \( \tilde{\sigma} \) in a few steps. First, we find a layer \( \mathcal{L} \) surrounding \( \Gamma \) which itself has “thickness” of order \( L \). \( \mathcal{L} \) is chosen so that the restriction of \( \sigma \) to \( \mathcal{L} \) has two properties: at the (exterior) boundary of \( \mathcal{L} \), \( |\sigma_x \cdot e_1| > 1/2 \) and also \( \sigma_x \cdot e_1 \) is of constant sign on each connected component of \( \mathcal{L} \). In what follows, a contour will be called a \( \pm \) contour depending on the sign of \( \sigma_x \cdot e_1 \) on the unique component of \( \mathcal{L} \) which separates \( \Gamma \) from \( \infty \).

It is worth remarking that our restriction on the dimension enters here. By definition, at the boundary of a contour, the Dirichlet energy \( E_{\mathcal{L}} \) and the spin average \( \ell^{-d} \sum_{z \in Q_{l}} \sigma_z \) are under control for all boxes \( Q_{l} \subset \mathcal{L} \). The lower the dimension, the more strongly this control restricts the size of “defects”, i.e. connected subsets \( A \subset N_{L}(\Gamma) \) in which \( |\sigma_x \cdot e_1| < \frac{1}{2} \) (say).

Let \( \sigma_{L,\pm} \) denote the spin configuration in \( \mathcal{L} \) obtained by applying the inverse of \( f \) in (5) to the optimizer of (6), where we take as a boundary condition the angular coordinates of \( \sigma \) on the exterior boundary of \( \mathcal{L} \). Consider the spin configuration \( \sigma' \) obtained by replacing, for \( x \in \mathcal{L} \), \( \sigma_x \) with \( \sigma_{L,\pm} \) with \( \sigma_{L,\pm} \).

This will cost us energetically (because \( K_{\mathcal{L}} \) is only approximately the correct transformed Hamiltonian), but
less than we ultimately gain by extracting energy from the bad behavior of $\sigma$ in $\Gamma$. The key point in this step of our construction is that one can show, for typical realizations of disorder, $\sigma^s_2$ is uniformly close to one of $\pm e_1$ as long as $x \in \mathcal{L}$ and $\text{dist}(x, \mathcal{L}') > L/4$. Moreover, the sign is constant over connected components of $\mathcal{L}$. This follows from the form of (6) as long as $L \gg e_1^{-1}$.

The final step is to modify $\sigma'$ as follows. We first optimize $K_A(\phi)$ in the domain

$$\mathcal{A} := \{ x \in \mathcal{L} \cup \Gamma : \text{dist}(x, \Lambda_N \setminus (\mathcal{L} \cup \Gamma)) > L/2 \}$$

with free boundary conditions. Typically the optimizer $\phi_A$ is close to 0 throughout $\mathcal{A}$. We want to “glue” $\sigma'$ in $\mathcal{A}$ to the spin configuration $\eta$ determined by $\phi_A$ using minimal cost in energy. To do this, on each internal component of $\Lambda_N \setminus \mathcal{A}$ we replace, as necessary, $\sigma'$ by its reflection $\sigma''$ across the $e_2$ axis so that $\sigma''_2, e_1$ has the same sign on all components of $\Lambda \setminus \mathcal{A}$. $\sigma$ is defined as $\sigma''$ on $\Lambda_N \setminus \mathcal{A}$ and is $\eta$ (or its reflection) on $\mathcal{A}$.

The big hurdle in all of these considerations is to make sure regions where fluctuations of the disorder invalidate the above reasoning are sparse. This is rather delicate in two dimensions. Details may be found in [16,17].

**Conclusion.** — In this letter we addressed a 25 year old controversy regarding randomness induced ordering, showing rigorously that the random field can actual select ordered phases in two and three dimensions. The results demonstrate that the low-dimensional qualitative behavior of such models agrees with the first work on the subject [10] and elucidates contradictory results in the papers [11,13].

There are two ways to understand these results in two dimensions. On the one hand, if one thinks along the lines of Anderson localization, the randomness localizes the spin wave and vortex dipole excitations which enable the Kosterlitz-Thouless phase. On the other hand, if one thinks about the homogenization theory of classical fields, the behavior we discuss is a consequence of imposing the hard constraint that spins lie on the unit sphere.

Our work gives a very precise picture for the behavior of these systems at low temperature, bounds on transition temperatures in terms of the strength of the randomness and tools to extend the stated results to a broad class of classical and quantum systems.

**REFERENCES**

[1] Anderson P. W., Phys. Rev., 109 (1958) 1992.
[2] Aizenman M. and Wehr J., Commun. Math. Phys., 130 (1990) 489.
[3] Imry Y. and Ma S. K., Phys. Rev. Lett., 35 (1975) 1399.
[4] Aizenman M., Greenblatt R. L. and Lebowitz J. L., Phys. Rev. Lett., 103 (2009) 197201.
[5] Basko D. M., Aleiner I. L. and Altshuler B. L., Ann. Phys. (N.Y.), 321 (2006) 1126.
[6] Aizenman M. and Warzel S., Commun. Math. Phys., 290 (2009) 903.
[7] Hamza E., Sims R. and Stolz G., Commun. Math. Phys., 315 (2012) 215.
[8] Hove neers F., Energy fluctuations in simple conduction models, preprint http://arxiv.org/abs/1112.0512.
[9] Sacépe et al., Nat. Phys., 7 (2011) 239.
[10] Aharony A., Phys. Rev. B, 18 (1978) 3328.
[11] Dotsenko V. S. and Feigelman M. V., J. Phys. C, 14 (1981) L823.
[12] Dotsenko V. S. and Feigelman M. V., J. Phys. C, 15 (1982) L565.
[13] Minchau B. J. and Pelcovits R. A., Phys. Rev. B, 32 (1985) 3081.
[14] Feldman D. E., J. Phys. A, 31 (1998) L177.
[15] Feldman D. E., J. Exp. Theor. Phys., 88 (1999) 1170.
[16] Crawford N., Random Field Induced Order in Low Dimension I, preprint http://arxiv.org/abs/1208.3149.
[17] Crawford N., in preparation.
[18] Niederberger A., Schulte T., Wehr J., Lewenstein M., Sanchez-Palencia L. and Sacha K., Phys. Rev. Lett., 100 (2008) 1.
[19] Niederberger A., Wehr J., Lewenstein M. and Sacha K., EPL, 86 (2009) 26004.
[20] Niederberger A., Rams M. M., Dziarmaga J., Cucchietti F. M., Wehr J. and Lewenstein M., Phys. Rev. A, 82 (2010) 013630.
[21] Sanchez-Palencia L. and Lewenstein M., Nat. Phys., 6 (2010) 87.
[22] Wehr J. et al., Phys. Rev. B, 74 (2006) 224448.
[23] Jordaniski S. V. and Koshelev A. E., Pisma Zh. Eksp. Teor. Fiz., 41 (1985) 471 (JETP Lett., 41 (1985) 574).
[24] Arabin D. A., Lee P. A. and Levitov L. S., Phys. Rev. Lett., 98 (2007) 156801.
[25] van Enter A. C. D. et al., Braz. J. Probab. Stat., 24 (2010) 226.
[26] Dobrushin R. L. and Shlosman S. B., Commun. Math. Phys., 42 (1975) 31.
[27] Mermin D. and Wagner H., Phys. Rev., 17 (1966) 1133.
[28] Fröhlich J. and Spencer T., Commun. Math. Phys., 81 (1981) 527.
[29] Kosterlitz J. M. and Thouless D. J., J. Phys. C, 6 (1973) 1181.
[30] Fröhlich J., Simon B. and Spencer T., Commun. Math. Phys., 50 (1976) 79.
[31] Bricmont J. and Kupiainen A., Commun. Math. Phys., 116 (1988) 539.
[32] Imirie J. Z., Commun. Math. Phys., 98 (1985) 145.
[33] Bézivin E. et al., J. Stat. Phys., 51 (1988) 1.
[34] van Enter A. C. D. and Kühleske C., Ann. Appl. Probab., 18 (2008) 109.
[35] Henley C. L., Phys. Rev. Lett., 62 (1989) 2056.
[36] Aizenman M. et al., Phys. Rev. A, 70 (2004) 023612.
[37] Presutti E., Scaling Limits in Statistical Mechanics and Microstructures in Continuum Mechanics (Springer Verlag) 2008.