Exploring the Dynamics of the Circumcenter Map

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Using simulation and visualization techniques, we explore interesting plane curves implied by a certain map applied to a generic polygon.¹ Let us first define this map, called here the “circumcenter map.” Referring to Figure 1, we make the following definition.

Definition 1 (Circumcenter map). Given a point \( M \) and a polygon \( \mathcal{P} \) with vertices \( P_1, \ldots, P_n \), the circumcenter map \( C_M(\mathcal{P}) \) yields a new polygon \( \mathcal{P}' \) whose vertices are the respective circumcenters of \( MP_1P_2, MP_2P_3, \ldots, MP_nP_1 \).

We review classical results that underpin the phenomena studied herein, namely:

(i) \( C_M(\mathcal{P}) \) yields the (half-sized) \( M \)-antipedal polygon; see Figure 2.
(ii) As a consequence of a result proved in [9], \( n \) consecutive applications of this map, i.e., \( C_M^n(\mathcal{P}) \), yields a new polygon that is the image of \( \mathcal{P} \) under a rigid rotation about \( M \) by \( \alpha \), and a homothety (uniform scaling) with ratio \( s \); see Figure 3.
(iii) \( n \) subsequent applications of the map results in a transformation with identical parameters \( \alpha \) and \( s \).

Given a starting polygon \( \mathcal{P} \), we will study the locus of \( M \) such that after \( n \) applications of the map, the resulting polygon has the same area (\( s = 1 \)) and/or angle (\( \alpha = 0 \)) as the original one. Experiments reveal an interesting structure of such loci and beautiful symmetries when \( \mathcal{P} \) is regular; see Figure 4.

Overview
We begin by stating the main results, a bit of background, and the organization of the rest of the paper.

Main Results
The claims below were initially evidenced by simulation. Most are proved with a computer algebra system, and their proofs will be omitted.

(1) We explicitly derive the locus of \( M \) such that \( s = 1 \) when \( \mathcal{P} \) is an equilateral triangle (respectively a square): it is a degree-6 (respectively degree-8) polynomial on the coordinates of \( M \).
(2) If \( \mathcal{P} \) is a regular \( n \)-gon, the locus of \( M \) such that \( \alpha = 0 \) is the union of \( n \) lines through the centroid of \( \mathcal{P} \), rotated about each other by \( \pi/n \).
(3) In all cases, there is a discrete set of locations such that both \( s = 1 \) and \( \alpha = 0 \). We study the map with \( \mathcal{P} \) an equilateral triangle, showing that if \( M \) is on any one of these locations, the map is 3-periodic. If \( M \) is at the centroid of the equilateral triangle, the map is 2-periodic.
(4) Based on compactified visualizations of the \( s = 1 \) boundary, we conjecture that for all \( n \), (i) there is only one connected region such that \( s > 1 \), and (ii) if \( n \) is odd (respectively even), the number of connected regions such that \( s < 1 \) is given by \( 1 + n(n+1)/2 \) (respectively \( 1 + n^2/2 \)), i.e., in both cases it is of order \( O(n^2) \).
(5) We informally investigate topological changes in the \( s = 1 \) locus with respect to affine stretching of the initial polygon.

Background
In [10], properties of the “central” subtriangle defined by a fourfold subdivision of a reference triangle (using cevians) are studied. A map based on the second isodynamic point of a polygon’s subtriangles is described in [7].

Article Organization
In the next section we review classical results that show that \( n \) applications of the circumcenter map result in a similarity transform with parameters \( s \) and \( \alpha \) that depend only on \( M \). In the following section, we describe properties of the map for the initial polygon a triangle or a square. We then extend the analysis to regular polygons of any number of sides. Conclusions and suggestions for further exploration appear in the final section. To facilitate reproducibility, explicit expressions for the circumcenter map appear in an appendix.

¹For a preview of the graphical results, see Figures 13 and 14 below.
The Mathematical Intelligencer

Review: Stewart’s Result

Referring to Figure 5, recall that (i) the M-pedal polygon of a polygon \( \mathcal{P} \) has vertices at the intersections of perpendiculars dropped from \( M \) onto the sides of \( \mathcal{P} \); (ii) the M-antipedal polygon of \( \mathcal{P} \) is such that \( \mathcal{P} \) is its M-pedal. Finally, (iii) the M-reflection polygon of \( \mathcal{P} \) has vertices at the reflections of \( M \) about the sidelines of \( \mathcal{P} \).

Remark 1. Clearly, the M-reflection polygon of \( \mathcal{P} \) is the twice-sized M-pedal polygon of \( \mathcal{P} \), with \( M \) as the homothety center.

Let \( \mathcal{P}' = \mathcal{C}_M(\mathcal{P}) \) be the polygon obtained under the circumcenter map of Definition 1. Let \( P'_i, i = 1, \ldots, n \), denote its vertices.

Lemma 1. The polygon \( \mathcal{C}_M^{-1}(\mathcal{P}) \) is the M-reflection of polygon \( \mathcal{P} \).

Proof. Referring to Figure 1, \( P'_i \) (respectively \( P'_{i+1} \)) is the center of a circle \( \mathcal{K}_i \) passing through \( M, P_i, P_{i+1} \) (respectively \( \mathcal{K}_{i+1} \) passing through \( M, P_{i+1}, P_{i+2} \)). We have that \( \mathcal{K}_i \cap \mathcal{K}_{i+1} = \{ M, P_i \} \).

The inverse circumcenter map is \( \mathcal{C}_M^{-1}(\mathcal{P}) = \mathcal{P} = \{ P_i \} \).

For each such pair of consecutive circles, the intersections \( \{ M, P_i \} \) are symmetric about \( P'_iP'_{i+1} \). See Figure 1 for the case \( n = 3 \). \( \square \)

Using Remark 1 and Lemma 1, the following property is illustrated in Figure 2.

Corollary 1. \( \mathcal{C}_M(\mathcal{P}) \) is homothetic to the M-antipedal of \( \mathcal{P} \), with ratio 1/2 and homothety center \( M \).

Recall the following well-known result by Roger Arthur Johnson [2, Theorem 2c].

Result (Johnson, 1918). If two polygons \( \mathcal{F} \) and \( \mathcal{F} \) with no parallel sides are similar, then there exists a point \( M \), called the self-homologous point, such that \( \mathcal{F} \) is a rigid rotation of \( \mathcal{F} \) about \( M \) followed by uniform scaling about the same point.

We shall use the definition of the Miquel map from [9, Construction 1].

Definition 2 (Miquel map). Given a point \( M \) and an angle \( \theta \), let \( \mathcal{M} \) denote a map that sends a polygon \( \mathcal{P} = \{ P_i \} \) to a new polygon \( \mathcal{P}' \) (known as the Miquel polygon) with each vertex \( P'_i \) on the line \( P_iP_{i+1} \) and such that \( \angle P'_i P'_i M = \theta \), for \( i = 1, \ldots, n \).

Let \( \mathcal{M}^k \) denote \( k \) successive applications of the Miquel map. The following key result was proved in [9, Theorem 2].

Theorem (Stewart, 1940). Let \( \mathcal{P} \) be a polygon with \( n \) sides; \( \mathcal{M}(\mathcal{P}) \) is similar to \( \mathcal{P} \) with \( M \) as the self-homologous point. It follows that if \( \mathcal{M}(\mathcal{P}) \) is similar to \( \mathcal{M}(\mathcal{P}) \), then \( i \equiv j \) (mod \( n \)).

Let \( \mathcal{M}_\perp \) denote the Miquel map with \( \theta = \pi/2 \). Definition 2 implies that \( \mathcal{M}_\perp(\mathcal{P}) \) is the pedal polygon of \( \mathcal{P} \) with respect to \( M \). Referring to Figure 6, we have the following corollary.

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**Figure 1.** Given a point \( M \), the circumcenter map sends a triangle \( T = ABC \) to \( T' = A'B'C' \) with vertices at the circumcenters of \( MBC, MCA, MAB \), respectively. Also shown are the vertices \( a'', b'', c'' \) of the M-pedal of \( T' \), which is a half-sized version of \( T \).

**Figure 2.** If \( \mathcal{P} = \{ P_i \} \) is a 5-gon (red), then \( \mathcal{C}_M(\mathcal{P}) \) is another 5-gon (green) that is a half-sized version of the M-antipedal of \( \mathcal{P} \) (magenta). Also shown is the construction for \( P'_i \), which is the center of the circle \( MP_iP_{i+1} \) (dashed gray).
Corollary 2. \( {\mathcal{M}}_1^n (P) \) is self-homologous to \( P \), i.e., it is an image under a rigid rotation and a uniform scaling of \( P \) about \( M \).

The inverse map \( {\mathcal{M}}^{-1}_1 (P) \) yields the \( M \)-antipedal polygon of \( P \). Referring to Corollary 1, we make the following remark.

Remark 2. \( C_M (P) \) is a half-sized homothety of \( {\mathcal{M}}^{-1}_1 (P) \) with respect to \( M \).

Since the pedal transformation has an inverse, \( {\mathcal{M}}^{-n}_1 (P) \) is similar to \( P \). Referring to Figure 6, and noting that the scaling in Corollary 1 does not affect similarity, we have the following corollary.

Corollary 3. \( C_M^n (P) \) is similar to \( P \).

Let \( P' = C_M^n (P) \) and \( P'' = C_M^n (P') \). Express these as \( P' = {\mathcal{A}} (P) \) and \( P'' = {\mathcal{I}} (P') \), where \( {\mathcal{I}} \) and \( {\mathcal{A}} \) are similarity transforms (that is, composition of a rotation and scaling about \( M \)).

Proposition 1. \( {\mathcal{I}} = {\mathcal{A}} \).

Proof. Since \( P' = {\mathcal{A}} (P) \), then \( C_M^n (P') = C_M^n (P) \).

By definition (see above), the circumcenter map is based
on the circumcenters of subtriangles of a given triangle. Since circumcenters are triangle centers (see [3]), they are equivariant over similarity transforms, and therefore,

\[ C^n_M(\mathcal{R}(P)) = \mathcal{R}(C^n_M(P)) = \mathcal{R}(\mathcal{P}), \]

i.e., \( C^n_M(\mathcal{P}) = \mathcal{R}(\mathcal{P}) \).

In Figures 3 and 7, we illustrate a statement by Stewart in [9], namely, that intermediate applications of the map produce polygons “as diverse in shape as is imaginable.”

The Cases \( n = 3 \) and \( n = 4 \)

In this section we study the locus of \( M \) such that \( C^n_M(\mathcal{P}) \) is area-preserving (\( s = 1 \)) and/or rotation-neutral (\( u = 0 \)) for the cases in which \( \mathcal{P} \) is an equilateral triangle or a square.

Let \( \mathcal{P} = ABC \) be a triangle, and \( \mathcal{P} = C^1_M(\mathcal{P}) = A'B'C' \).

Let \( \mathcal{A}(\mathcal{P}) \) denote the area of a polygon \( \mathcal{P} \). Via a computer algebra system, we obtain the following proposition.

**Proposition 2.** The ratio of sides \( A'B' \), \( B'C' \), and \( A'C' \) of \( \mathcal{P} \) and \( \mathcal{P} \) is given by

\[
\frac{|A'B'|}{|AB|} = \frac{|A'C'|}{|AC|} = \frac{|B'C'|}{|BC|} = \frac{l_a l_b l_c m_a m_b m_c}{8.\mathcal{A}(ABM)\mathcal{A}(BCM)\mathcal{A}(ACM)},
\]

where \( l_a = |BC|, l_b = |AC|, l_c = |AB|, m_a = |AM|, m_b = |BM|, m_c = |CM|.

Figure 6. Left: Applying the circumcenter map with respect to \( M \) to a starting triangle \( P_1P_2P_3 \) (solid red) yields the (dashed green) triangle \( P'_1P'_2P'_3 \). In turn, applying the map to the latter yields the (dashed blue) triangle \( P''_1P''_2P''_3 \). Finally, a third application of the map yields \( P'''_1 \) (solid red), similar to the original. Right: Likewise, starting from a pentagon \( P_i, i = 1, \ldots, 5 \), we have that \( n = 5 \) applications of the map yields \( P'_1 \) (dashed red), self-homologous to the original. Intermediate generations are colored dashed green, blue, orange, and magenta.
Proposition 3. Let $\alpha$ denote the angle of rotation of $\mathcal{P}$ with respect to $\mathcal{P}$. Then
\[
\cos \alpha = \frac{1}{M_B l_{m} m_{b} m_{b} m_{b}} \left[ m_{a}^{2} (m_{a}^{2} + m_{b}^{2}) s(ABM) + m_{b}^{2} (m_{a}^{2} + m_{c}^{2}) s(ACM) + m_{c}^{2} (m_{b}^{2} + m_{c}^{2}) s(BCM) \right].
\]

The Case of an Equilateral Triangle

Let $\mathcal{R} = ABC$ be the equilateral triangle with vertices $A = (1, 0), B = (-1, \sqrt{3})/2, C = (-1, -\sqrt{3})/2$. Let
\[\mathcal{R}' = \mathcal{C}_{M}^{2} (\mathcal{R}) \text{ with } M = (x_m, y_m). \text{ Let } s = \mathcal{RA}/\mathcal{RB}.\]
Via a computer algebra system, we obtain the following proposition.

Proposition 4. If $\mathcal{P}$ is an equilateral triangle, then $s = 1$ if $M = (x_m, y_m)$ satisfies
\[
3 x_m^6 - y_m^6 - 12 x_m^5 + 9 y_m^4 + (-27 y_m^2 + 9) x_m^4 \\
+ (24 y_m^2 + 6) x_m^3 + (33 y_m^2 + 18 y_m^2) x_m^2 \\
+ (36 y_m^4 - 18 y_m^2) x_m - 6 y_m^2 = 0.
\]

Figures 8 and 9 illustrate sequential applications of the circumcenter map starting from an equilateral triangle for the cases of $M$ in an area-contracting region, an area-expanding region, and the boundary between them, as defined in Proposition 4.

Proposition 5. If $\mathcal{P}$ is an equilateral triangle, then $\alpha = 0$ if $M = (x_m, y_m)$ satisfies
\[
y_m (y_m^2 - 3x_m^2) = 0.
\]

As shown in Figure 10, as the starting triangle is affinely distorted, the number of connected components in the $s = 1$ locus changes, revealing a nontrivial relationship (not studied here).

Referring to Figure 11, we note the following corollary.

Corollary 4. There are six points as in Proposition 4 such that $\alpha = 0$. These are $K_1 = \left(1 + \sqrt{3}, 0\right), K_2 = \left(1 - \sqrt{3}, 0\right)$, and their rotations by $\pm 2\pi/3$.

Let $\mathcal{R} = A_B C$, denote the image of $\mathcal{R}$ under $i$ iterations of the circumcenter map. As shown above, $\mathcal{R}_i$ is similar to $\mathcal{R}$. Referring to Figure 11 (left), we have the following.

Proposition 6. If $M = K_1$, then the vertices of $\mathcal{R}_1$ and $\mathcal{R}_2$ have coordinates
\[
A_1 = (1, 0), \quad B_1 = \left(1 + \frac{\sqrt{3}}{2}, \frac{3}{2} + \sqrt{3}\right), \\
C_1 = \left(1 + \frac{\sqrt{3}}{2}, \frac{3}{2} - \sqrt{3}\right),
\]
\[
A_2 = \left(-2 - \sqrt{3}, 0\right), \quad B_2 = \left(1 + \frac{\sqrt{3}}{2}, \frac{3}{2}\right), \\
C_2 = \left(1 + \frac{\sqrt{3}}{2}, -\frac{3}{2}\right).
\]

The triangles $\mathcal{R}_1$ and $\mathcal{R}_2$ have respective internal angles $30^\circ, 75^\circ, 75^\circ$ and $150^\circ, 15^\circ, 15^\circ$.

Referring to Figure 11 (right), we have the following proposition.

Proposition 7. If $M = K_2$, then the vertices of $\mathcal{R}_1$ and $\mathcal{R}_2$ have coordinates
\[
A_1 = (1, 0), \quad B_1 = \left(1 - \frac{\sqrt{3}}{2}, \frac{3}{2} + \sqrt{3}\right), \\
C_1 = \left(1 - \frac{\sqrt{3}}{2}, \frac{3}{2} - \sqrt{3}\right),
\]
\[
A_2 = \left(\sqrt{3} - 2, 0\right), \quad B_2 = \left(1 - \frac{\sqrt{3}}{2}, \frac{3}{2}\right), \quad C_2 = \left(1 - \frac{\sqrt{3}}{2}, \frac{3}{2}\right).
\]
Figure 8. Left: Nine iterations of the circumcenter map starting from an equilateral triangle (solid red) centered at \(O\); \(M\) is placed in an area-shrinkage region (green). A new, smaller, equilateral triangle (red) is produced every three applications of the map. Right: With \(M\) in the area-expansion region (green), the area expands on every three applications of the map.

Figure 9. Starting from an equilateral triangle (thick red) with centroid \(O\), a point \(M\) is selected on the boundary between the area-expansion (light red) and area-contraction (light green) regions. Sequential applications of the circumcenter map are shown in green, blue, and back to red. Since \(M\) is on the boundary, every three applications of the map are area-preserving and result in a constant net rotation about \(M\).

Triangles \(\mathcal{R}_1\) and \(\mathcal{R}_2\) have respective internal angles 30°, 75°, 75° and 150°, 15°, 15°.

Referring to Figure 12, we offer the following remark.

Remark 3. When \(M\) is the centroid of \(\mathcal{R}\), repeated applications of the circumcenter are 2-periodic, where the first triangle is \(\mathcal{R}\) and the second one is a reflection of \(\mathcal{R}\) about the centroid \(M\).

The Case of a Square

Let \(\mathcal{Q} = ABCD\) be a square with vertices \(A = (1, 0)\), \(B = (0, 1)\), \(C = (-1, 0)\), \(D = (0, -1)\). Let \(\mathcal{Q}' = \mathcal{Q}^\mathcal{M}(\mathcal{Q})\), with \(M = (x_m, y_m)\). With the help of a computer algebra system, we obtain the following propositions.

Proposition 8. Starting from the square \(\mathcal{Q}\), \(s = 1\) occurs when \(M = (x_m, y_m)\) satisfies

\[
15x_m^8 - 68x_m^6y_m^2 + 90x_m^4y_m^4 - 68x_m^2y_m^6 + 15y_m^8 - 64x_m^6 + 64x_m^4y_m^2 + 64x_m^2y_m^4 - 64y_m^6 + 98y_m^4 + 52y_m^2 + 98y_m^4 - 64x_m^2 - 64y_m^2 + 15 = 0.
\]
Proposition 9. Starting from the square $Q$, we have $\alpha = 0$ when $M = (x_m, y_m)$ satisfies $x_m y_m (x_m^2 - y_m^2) = 0$.

Regular Polygons, $n \geq 3$

In this section we assume that $\mathcal{P}$ is a regular $n$-gon, $n \geq 3$. Without loss of generality, let the centroid be $O = (0, 0)$, and the first vertex, $P_1 = (1, 0)$. Figure 13 illustrates the partitioning of the plane into area-contracting and area-expanding regions by $n$ applications of the map for $n = 3, 4, 5, 6$.

Proposition 10. As $M$ approaches a sideline of $\mathcal{P}$, $s$ approaches infinity.

The proof below was kindly contributed by a referee.

Proof. If $M$ is on a sideline of $\mathcal{P}$, say $P_i P_{i+1}$, then $P'_i$ is at infinity (since $P'_i$ is the intersection of the bisectors $MP'_i$ and $MP_{i+1}$), but the rest of the vertices $P'_j$ are finite. If $a$
polygon $\mathcal{P}$ has a vertex at infinity, it will remain at infinity under the map $C_M$. \hfill $\square$

Let $\mathcal{P} = C_M^n(\mathcal{P})$. Let $\alpha$ be the angle of rotation of the similarity that takes $\mathcal{P}$ to $\mathcal{P}$.

**Conjecture 1.** The locus of $M$ such that $\alpha = 0$ is the union of $n$ lines along directions $k\pi/n$, $k = 0, \ldots, n-1$.

To facilitate region counting, in Figure 14 the plane is compactified into a single hemisphere via stereographic projection. Table 1 shows the counts of the area-contracting regions. This suggests the following conjecture.

**Conjecture 2.** There is a single connected area-expanding region. Let $k$ denote the number of area-contracting connected regions. Then

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**Figure 12.** Left: 36 iterations of the circumcenter map with $M$ close to the centroid $O$ of the starting equilateral triangle, resulting in a sequence that is slightly expanding. Right: If $M = O$, the sequence becomes 2-periodic, containing only the original $ABC$ (blue) and its reflection about $O$ (green).

**Figure 13.** Area-contraction (green) and area-expansion (red) regions for the circumcenter map applied to a regular triangle (top left), square (top right), pentagon (bottom left), and hexagon (bottom right). Notice that in all cases except $n = 3$, an area-contracting region exists interior to the original polygon. Also shown are the zero-rotation lines (dashed black).
Table 1. Region count according to Figure 14.

| n   | Interior | Noncompact | Compact | Total |
|-----|----------|------------|---------|-------|
| 3   | 0        | 2n         | 0       | 2n    |
| 4   | 1        | n          | n       | 1 + 2n|
| 5   | 1        | 2n         | n       | 1 + 3n|
| 6   | 1        | n          | 2n      | 1 + 3n|
| 7   | 1        | 2n         | 2n      | 1 + 4n|
| 8   | 1        | n          | 3n      | 1 + 4n|
| 9   | 1        | 2n         | 3n      | 1 + 5n|
| 10  | 1        | n          | 4n      | 1 + 5n|
| 11  | 1        | 2n         | 4n      | 1 + 6n|

Conjecture 3. Given a positive integer $n$, the number of area-contracting regions for a simple $n$-gon $Q$ (no self-intersections) is maximal if $Q$ is regular.

Conclusion

A video walk-through of our experiments appears in [4]. The circumcenter map can be generalized to the $X_k$-map, where $X_k$ is some triangle center (see [3]). For example, the $X_2$-map sends a polygon with vertices $P_i$ to a new polygon whose vertices are the barycenters of $MP_iP_{i+1}$. In such a case, an iteration produces a sequence of ever-shrinking polygons that converges to $M$. If the starting polygon is a triangle, a few notable cases include the following: (i) the $X_4$-map (orthocenter) is area-preserving for all $M$, and the sequence of triangles tends to an infinite line; (ii) the $X_{16}$-map (second isodynamic point) induces regions of the plane such that three applications of the map yields the identity (no rotation and no scaling); see [7].

Figure 14. Area-expansion (red) and area-contraction (green) regions for regular $n$-gons, compactified (via stereographic projection) to a single hemisphere; the south pole (center) is at “infinity.” From top to bottom, left to right, $N = 3, \ldots, 11$. For interactive examples, see [4].
A question not addressed here is whether a certain $X_k$-map is integrable in the sense of [1, 5].

Appendix: The Explicit Circumcenter Map

Let $\mathcal{P}$ be a generic $n$-gon with vertices $(x_i, y_i)$, $i = 1, \ldots, n$, and let $M = (x_m, y_m)$. The $M$-circumcenter map $\mathcal{C}_M(\mathcal{P})$ yields a new polygon with vertices $(p_i, q_i)$ given by

$$p_i = \frac{(y_{i+1} - y_m)x_i + y_{i+1}y_m + (x_i - x_m)x_{i+1} + (x_i - x_m)^2y_{i+1} + (x_{i+1} - x_m)^2y_i}{2(y_m - x_{i+1})y_i + 2(y_m - x_m)y_{i+1} + 2(x_m - x_{i+1})y_m},$$

$$q_i = \frac{(x_m - x_{i+1})x_i - (x_i - x_m)x_{i+1} + (x_i - x_m)^2y_{i+1} + (x_{i+1} - x_m)^2y_i}{2(y_m - y_{i+1})x_i + 2(y_m - y_m)x_{i+1} + 2(y_m - y_m)x_m},$$

where $p_i = |M|^2 - |B|^2 = x_i^2 + y_i^2 - x_{i+1}^2 - y_{i+1}^2$. Likewise, the inverse of the $M$-circumcenter map $\mathcal{C}_M^{-1}(\mathcal{P})$ yields a new polygon with vertices $(u_i, v_i)$ given by

$$u_i = \frac{r_i'x_m + 2(y_{i+1} - y_i)x_{i+1} - x_iy_{i+1}}{r_i},$$

$$v_i = \frac{-r_i'y_m + 2(y_{i+1} - y_i)y_{i+1} - x_iy_{i+1}}{r_i},$$

where $r_i = (x_{i+1} - x_i)^2 + (y_{i+1} - y_i)^2$ and $r_i' = (x_{i+1} - x_i)^2 + (y_{i+1} - y_i)^2$.

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