Analysis of fuzzy priority queues using heptagonal fuzzy numbers

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Abstract. In this study, a procedure to find the various performance measures in terms of crisp values for non-preemptive priority queuing system, in which the arrival rates and service rate are heptagonal fuzzy numbers has been proposed. DSW algorithm is used to define membership functions of the performance measures of priority queuing system. The α – cut representation of fuzzy sets in a standard interval analysis is used. The validity of the model is checked by a numerical example.

1. Introduction

Queuing models have a great extent of applications in service organizations, manufacturing firms, telecommunication networks, computer systems, inventory systems, etc., The services are done in any one of the following queue discipline namely first in first out, last in first out, service in random order and selection in priority order. In this priority discipline customers are given service according to some priorities over others. The two types in this priority discipline are preemptive and non-preemptive. In the preemptive case the customer with the highest priority is allowed to enter service immediately by suspending the service in progress to a lower priority customer. In the non-preemptive priority case, the highest priority customer goes to the head of the queue but cannot get into service until the current service is completed.

Priority queue models play a vital role in a gamut of real, daily situations, particularly in cases when preferential treatment is guaranteed for certain individuals, as in emergency hospital medical treatment. Rough queuing models with priority discipline was analyzed by M. Thangaraj and P. Rajendran[4]. Fuzzy queue with Erlang service model using DSW algorithm was done by Mohammed Shapique[1]. Fuzzy queuing model using DSW algorithm for multi server queuing model with hexagonal fuzzy number was studied by S. Narayanamoorthy and L. Ramya[2]. Two class priority queuing system with restricted number of priority customers has been analyzed by A.M.K. Tarabia[3].

In this paper, we develop an approach that can provide the system characteristics of fuzzy priority queues in which the three arrival rates and the service rate are heptagonal fuzzy numbers. The
α - cut approach and fuzzy arithmetic operations are used to derive the membership functions. As α varies, the family of crisp queues are then described and solved by using DSW algorithm.

2. Basic Concepts

2.1. Heptagonal Fuzzy Number

A heptagonal fuzzy number \( \tilde{A}(x) \) can be represented by \( \tilde{A}(a_1, a_2, a_3, a_4, a_5, a_6, a_7) \) with membership function \( \eta_{\tilde{A}}(x) \) given by

\[
\eta_{\tilde{A}}(x) = \begin{cases} 
0 & \text{if } x < a_1 \\
\frac{x - a_1}{2(a_2 - a_1)} & \text{if } a_1 \leq x < a_2 \\
\frac{1}{2}, & \text{if } a_2 \leq x < a_3 \\
\frac{1}{2} + \frac{x - a_3}{2(a_4 - a_3)} & \text{if } a_3 \leq x < a_4 \\
\frac{1}{2} + \frac{a_4 - x}{2(a_5 - a_4)} & \text{if } a_4 \leq x < a_5 \\
\frac{1}{2}, & \text{if } a_5 \leq x < a_6 \\
\frac{a_7 - x}{2(a_7 - a_6)} & \text{if } a_6 \leq x < a_7 \\
0, & \text{otherwise.}
\end{cases}
\]

2.2. α - cut

The α - cut of a fuzzy number \( \tilde{A} \) is defined as

\[
\tilde{A}_{\alpha} = \left[ a_1 + \frac{\alpha}{2}(a_2 - a_1), a_7 - \frac{\alpha}{2}(a_7 - a_6) \right], \quad \alpha \in \left[ 0, \frac{1}{2} \right],
\]

\[
\tilde{A}_{\frac{\alpha}{2}} = \left[ a_1 + (2\alpha - 1)(a_2 - a_1), a_5 - (2\alpha - 1)(a_5 - a_4) \right], \quad \alpha \in \left[ \frac{1}{2}, 1 \right].
\]

3. Non-Preemptive Fuzzy Priority Queue Model

Consider a queuing system in which the priority customers arrive according to Poisson process and the service rate follow exponential distribution regardless of priority. The fuzzified arrival rate of higher priority \( \tilde{\lambda}_1 \), of middle priority \( \tilde{\lambda}_2 \), of lower priority \( \tilde{\lambda}_3 \) and service rate \( \tilde{\mu} \) are all heptagonal fuzzy numbers. Also the arrival rate \( \tilde{\lambda} = \tilde{\lambda}_1 + \tilde{\lambda}_2 + \tilde{\lambda}_3 \) and \( \rho = \frac{\tilde{\lambda}_1 + \tilde{\lambda}_2 + \tilde{\lambda}_3}{\tilde{\mu}} = \frac{\tilde{\lambda}}{\tilde{\mu}} \) Suppose arrival rate and service rate are approximately known and can be represented by convex fuzzy sets.

Let \( \eta_1(a), \eta_2(b), \eta_3(c), \eta_4(d) \) denote the membership functions of the arrival rates with higher, middle and lower priority and service rate respectively.

\[
P = \{(a, \eta_1(a)) | a \in A\}
\]
\[
Q = \{(b, \eta_2(b)) | b \in B\}
\]
\[
R = \{(c, \eta_3(c)) | c \in C\}
\]
\[
S = \{(d, \eta_4(d)) | d \in D\}
\]

The α - cut of arrival rates with higher, middle and lower priority and service rate are represented as
\[ P(\alpha) = \{ p \in A / \eta_p(p) \geq \alpha \} \]
\[ Q(\alpha) = \{ q \in B / \eta_q(q) \geq \alpha \} \]
\[ R(\alpha) = \{ r \in C / \eta_r(r) \geq \alpha \} \]
\[ S(\alpha) = \{ s \in D / \eta_s(s) \geq \alpha \} \]

Let \( f(p, q, r, s) \) denote the system performance of interest. Hence \( p, q, r, s \) are all fuzzy numbers and likewise \( f(p, q, r, s) \) are fuzzy numbers.

4. DSW Algorithm

The DSW algorithm consists of the following steps:
1) Select a \( \alpha \) – cut value where \( 0 \leq \alpha \leq 1 \).
2) Find the intervals in the input membership functions that correspond to this \( \alpha \).
3) Using standard binary interval operations compute the interval for the output membership function for the selected \( \alpha \) – cut level.
4) Repeat steps 1-3 for different values of \( \alpha \) to compute \( \alpha \) – cut representation of the solution.

5. Numerical Example

The arrival rate of higher priority \( \hat{\lambda}_1 \), middle priority \( \hat{\lambda}_2 \), lower priority \( \hat{\lambda}_3 \) and service rate \( \mu \) are all heptagonal fuzzy numbers represented by \( \hat{\lambda}_1 = [2, 3, 4, 5, 6, 7, 8], \hat{\lambda}_2 = [4, 5, 6, 7, 8, 9, 10], \hat{\lambda}_3 = [6, 7, 8, 9, 10, 11, 12], \hat{\mu} = [31, 32, 33, 34, 35, 36, 37] \) respectively. The system manager wants to evaluate the performance measures of the system such as the expected number of customers in the higher, middle and lower priority queues, expected waiting time of customers in the higher, middle and lower priority queues.

\[
L_{q_1} = \frac{\rho \frac{\hat{\lambda}_1}{\mu}}{1 - \frac{\hat{\lambda}_1}{\mu}}
\]
\[
L_{q_2} = \left( \frac{\rho \frac{\hat{\lambda}_2}{\mu}}{1 - \frac{\hat{\lambda}_1 + \hat{\lambda}_2}{\mu}} \right) \left( 1 - \frac{\hat{\lambda}_1}{\mu} \right)
\]
\[
L_{q_3} = \left( \frac{\rho \frac{\hat{\lambda}_3}{\mu}}{1 - \frac{\hat{\lambda}_1 + \hat{\lambda}_2}{\mu}} \right) (1 - \rho)
\]
\[
W_{q_1} = \frac{\hat{\lambda}}{\mu (\mu - \hat{\lambda}_1)}
\]
\[
W_{q_2} = \frac{\hat{\lambda}}{(\mu - (\hat{\lambda}_1 + \hat{\lambda}_2))(\mu - \hat{\lambda}_1)}
\]
\[
W_{q_3} = \frac{\hat{\lambda}}{(\mu - (\hat{\lambda}_1 + \hat{\lambda}_2))(\mu - \hat{\lambda})}
\]

Therefore, we have,
\( \hat{\lambda}_1 = [2 + 2\alpha, 8 - 2\alpha], \hat{\lambda}_2 = [9 + 2\alpha, 15 - 2\alpha], \hat{\lambda}_3 = [16 + 2\alpha, 22 - 2\alpha], \hat{\mu} = [31 + 2\alpha, 37 - 2\alpha], \) when \( \alpha \in [0, 0.5] \)
\[
\hat{\alpha}_1 = [3 + 2\alpha, 7 - 2\alpha], \quad \hat{\alpha}_2 = [10 + 2\alpha, 14 - 2\alpha], \quad \hat{\alpha}_3 = [17 + 2\alpha, 21 - 2\alpha], \quad \beta = [32 + 2\alpha, 36 - 2\alpha], \text{when } \alpha \in [0, 1]
\]

Table 1. The \( \alpha \) - cuts of \( Lq_1, Lq_2 \) and \( Lq_3 \) at different \( \alpha \) values.

| \( \alpha \) | \( \hat{L}^{\alpha}_1 \) | \( \hat{U}^{\alpha}_1 \) | \( \hat{L}^{\alpha}_2 \) | \( \hat{U}^{\alpha}_2 \) | \( \hat{L}^{\alpha}_3 \) | \( \hat{U}^{\alpha}_3 \) | \( Lq_1^L \) | \( Lq_1^U \) | \( Lq_2^L \) | \( Lq_2^U \) | \( Lq_3^L \) | \( Lq_3^U \) |
|-------------|----------------|----------------|----------------|----------------|----------------|----------------|----------|----------|----------|----------|----------|----------|
| 0.0         | 2              | 8              | 4              | 10             | 6              | 12             | 31        | 37       | 0.0185   | 0.3367   | 0.0443   | 1.0032   | 0.0929   | 27.6923  |
| 0.1         | 2.2            | 7.8            | 4.2            | 9.8            | 6.2            | 11.8           | 31.2      | 36.8     | 0.0218   | 0.3141   | 0.0503   | 0.9054   | 0.1062   | 14.1716  |
| 0.2         | 2.4            | 7.6            | 4.4            | 9.6            | 6.4            | 11.6           | 31.4      | 36.6     | 0.0253   | 0.2929   | 0.0570   | 0.8181   | 0.1212   | 9.0488   |
| 0.3         | 2.6            | 7.4            | 4.6            | 9.4            | 6.6            | 11.4           | 31.6      | 36.4     | 0.0292   | 0.2729   | 0.0643   | 0.7401   | 0.1380   | 6.3887   |
| 0.4         | 2.8            | 7.2            | 4.8            | 9.2            | 6.8            | 11.2           | 31.8      | 36.2     | 0.0333   | 0.2540   | 0.0724   | 0.6703   | 0.1571   | 4.7792   |
| 0.5         | 3.0            | 7.0            | 5.0            | 9.0            | 7.0            | 11             | 32        | 36       | 0.0379   | 0.2363   | 0.0812   | 0.6075   | 0.1786   | 3.7125   |
| 0.6         | 4.2            | 5.8            | 6.2            | 7.8            | 8.2            | 9.8            | 33.2      | 34.8     | 0.0734   | 0.1492   | 0.1545   | 0.3399   | 0.3859   | 1.1939   |
| 0.7         | 4.4            | 5.6            | 6.4            | 7.6            | 8.4            | 9.6            | 33.4      | 34.6     | 0.0809   | 0.1375   | 0.1710   | 0.3086   | 0.4400   | 1.0222   |
| 0.8         | 4.6            | 5.4            | 6.6            | 7.4            | 8.6            | 9.4            | 33.6      | 34.4     | 0.0889   | 0.1265   | 0.1890   | 0.2801   | 0.5027   | 0.8800   |
| 0.9         | 4.8            | 5.2            | 6.8            | 7.2            | 8.8            | 9.2            | 33.8      | 34.2     | 0.0974   | 0.1162   | 0.2008   | 0.2541   | 0.5756   | 0.7612   |
| 1.0         | 5              | 5              | 7              | 7              | 9              | 9              | 34        | 34       | 0.1065   | 0.1065   | 0.2304   | 0.2304   | 0.6608   | 0.6608   |

Figure 1. Graph of \( Lq_1 \)

Figure 2. Graph of \( Lq_2 \)

Figure 3. Graph of \( Lq_3 \)
Table 2. The α – cuts of Wq₁, Wq₂ and Wq₃ at different α values.

| α  | α₁         | α₂         | α₃         | α₄         | α₅         | μ⁻ | μ⁺ | Wq₁⁻ | Wq₁⁺ | Wq₂⁻ | Wq₂⁺ | Wq₃⁻ | Wq₃⁺ |
|----|-------------|-------------|-------------|-------------|-------------|----|----|------|------|------|------|------|------|
| 0  | 2           | 8           | 4           | 10          | 6           |    |    | 0.0093 | 0.0421 | 0.0111 | 0.1003 | 0.0155 | 2.3077 |
| 0.1| 2.2         | 7.8         | 4.2         | 9.8         | 6.2         |    |    | 0.0099 | 0.0403 | 0.0120 | 0.0924 | 0.0171 | 1.2010 |
| 0.2| 2.4         | 7.6         | 4.4         | 9.6         | 6.4         |    |    | 0.0106 | 0.0385 | 0.0130 | 0.0852 | 0.0189 | 0.7801 |
| 0.3| 2.6         | 7.4         | 4.6         | 9.4         | 6.6         |    |    | 0.0112 | 0.0369 | 0.0140 | 0.0788 | 0.0209 | 0.5604 |
| 0.4| 2.8         | 7.2         | 4.8         | 9.2         | 6.8         |    |    | 0.0119 | 0.0353 | 0.0151 | 0.0729 | 0.0231 | 0.4267 |
| 0.5| 3.0         | 7.0         | 5.0         | 9.0         | 7.0         |    |    | 0.0126 | 0.0338 | 0.0162 | 0.0675 | 0.0255 | 0.3375 |
| 0.6| 4.2         | 5.8         | 6.2         | 7.8         | 8.2         |    |    | 0.0175 | 0.0257 | 0.0249 | 0.0436 | 0.0471 | 0.1218 |
| 0.7| 4.4         | 5.6         | 6.4         | 7.6         | 8.4         |    |    | 0.0184 | 0.0246 | 0.0267 | 0.0406 | 0.0524 | 0.1065 |
| 0.8| 4.6         | 5.4         | 6.6         | 7.4         | 8.6         |    |    | 0.0193 | 0.0234 | 0.0286 | 0.0378 | 0.0585 | 0.0936 |
| 0.9| 4.8         | 5.2         | 6.8         | 7.2         | 8.8         |    |    | 0.0203 | 0.0223 | 0.0307 | 0.0353 | 0.0654 | 0.0827 |
| 1.0| 5           | 5           | 7           | 7           | 9           |    |    | 0.0213 | 0.0213 | 0.0329 | 0.0329 | 0.0734 | 0.0734 |

Figure 4. Graph of Wq₁

Figure 5. Graph of Wq₂

Figure 6. Graph of Wq₃
With the help of MATLAB software we perform α-cuts of arrival, service, and fuzzy expected number of jobs in queue and fuzzy expected waiting time of jobs in queue. The α-cut represents the possibility that these six performance measure will lie in the associated range. Specially, when α = 0 the range, the performance measures could appear and for α = 1 the range, the performance measures are likely to be. For α varying from 0 to 1, the corresponding membership function graphs of fuzzy characteristics namely \( L_{q1}, L_{q2}, L_{q3}, W_{q1}, W_{q2} \) and \( W_{q3} \) are represented in figures 1, 2, 3, 4, 5 and 6 respectively.

6. Conclusion
The α – cut approach is used to analyze a non-preemptive priority fuzzy queue model. The DSW algorithm greatly simplifies manipulation. A numerical example was given to illustrate the effectiveness of the proposed technique. The proposed method can be employed to analyze the other fuzzy queuing systems.

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