Interactions of Massive Integer-Spin Fields

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We review the interactions of massive fields of arbitrary integer spins with the constant electromagnetic field and symmetrical Einstein space in the gauge invariant framework. The problem of obtaining the gauge-invariant Lagrangians of integer spin fields in an external field is reduced to purely algebraic problem of finding a set of operators with certain features using the representation of the higher-spin fields in the form of vectors in a pseudo-Hilbert space. Such a construction is considered up to the second order for the electromagnetic field and at linear approximation for symmetrical Einstein space. The results obtained are valid for space-time of arbitrary dimensionality.

Introduction. At present only the superstring theory claims to have the consistent description of interaction of the higher-spin fields. But interacting strings describe the infinite set of fields and the question about interaction of finite number of fields is still open.

In this paper we consider the interaction of an arbitrary massive spin-$s$ field with a homogeneous electromagnetic field up to the second order in the strength and a symmetrical Einstein space at linear approximation in curvature.

We consider the massive higher-spin fields in gauge invariant way representing them as states on an auxiliary Fock space. We regard any interaction as a deformation of initial free gauge algebra. The preservation of gauge symmetry and absence of terms with higher derivatives ensure the right number of physical degrees of freedom.

Free massive field with integer spins. Let us consider the Fock space generated by creation-annihilation operators $\bar{a}_\mu$, $a_\mu$ and $\bar{b}$, $b$ which are vectors and scalars, correspondingly, in the $D$-dimensional Minkowski space $\mathcal{M}_D$ and form the following algebra

$$[a_\mu, \bar{a}_\nu] = g_{\mu\nu}, \quad a_\mu^\dagger = \bar{a}_\mu, \quad [b, \bar{b}] = 1, \quad b^\dagger = \bar{b}. \quad (1)$$

where $g_{\mu\nu}$ is the metric tensor with signature $\|g_{\mu\nu}\| = \text{diag}(-1, 1, 1, ..., 1)$. Since the metric is indefinite, the Fock space, which realizes the representation of the Heisenberg algebra $[\mathbb{H}]$, is a Pseudo-Hilbert space.

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We shall describe the massive field of spin \( s \) as the following vector in the Fock space:

\[
|\Phi^s\rangle = \sum_{n=0}^{s} \Phi_{\mu_1...\mu_n}(x) \bar{b}^{s-n} \prod_{i=1}^{n} \bar{a}_{\mu_i} |0\rangle.
\]  

(2)

Coefficient function \( \Phi_{\mu_1...\mu_n}(x) \) is a symmetrical tensor of rank \( n \) in space \( \mathcal{M}_D \).

Let us introduce the following operators in our pseudo-Hilbert space

\[
L_1 = p \cdot a + mb, \quad L_2 = \frac{1}{2} \left(a \cdot a + b^2\right), \quad L_0 = p^2 + m^2, \quad L_{-n} = L_n^\dagger.
\]  

(3)

Here \( p_\mu = i\partial_\mu \) is the momentum operator that acts in the space of the coefficient functions, while \( m \) is an arbitrary parameter that has dimensionality and sense of mass. In non-interacting case one can consider such a transition as the dimensional reduction \( \mathcal{M}_{D+1} \to \mathcal{M}_D \otimes S^1 \) with the radius of sphere \( R \sim 1/m \) (refer also to [1, 2]).

Operators (3) satisfy the commutation relations:

\[
\begin{align*}
[L_1, L_{-2}] &= L_{-1}, \\
[L_2, L_{-2}] &= N + \frac{D+1}{2}, \\
[L_1, L_{-1}] &= L_0, \\
[L_1, L_2] &= 0, \\
[L_0, L_n] &= 0, \\
[N, L_n] &= -nL_n, \quad n = 0, \pm 1, \pm 2.
\end{align*}
\]  

(4)

Here \( N = \bar{a} \cdot a \) is a level operator that defines the spin of states. So, for instance, for the state (2) we have

\[
N|\Phi^s\rangle = s|\Phi^s\rangle.
\]

For state (2) to describe the state with spin \( s \) one has to imposes the condition:

\[
(L_2)^2|\Phi^s\rangle = 0.
\]  

(5)

Lagrangian for massive states can be written as an expectation value of a Hermitian operator in the state (3):

\[
\mathcal{L}_s = \langle \Phi^s | \mathcal{L}(\mathcal{L}) | \Phi^s \rangle, \quad \langle \Phi^s | = | \Phi^s \rangle^\dagger,
\]  

(6)

where

\[
\mathcal{L}(\mathcal{L}) = L_0 - L_{-1}L_1 - 2L_{-2}L_0L_2 - L_{-2}L_{-1}L_1L_2 + (L_{-2}L_1L_1 + h.c.).
\]  

(7)

Lagrangian (6) is invariant under gauge transformations

\[
\delta|\Phi^s\rangle = L_{-1}|\Lambda^{s-1}\rangle.
\]  

(8)
as a consequence of the relation $\mathcal{L}(L) L_{-1} = (...) L_2$. Here, the gauge state

$$|\Lambda^{s-1}\rangle = \sum_{n=0}^{s-1} \Lambda_{\mu_1...\mu_n} b^{s-n-1} \prod_{i=1}^{n} \bar{\partial}_{\mu_i} |0\rangle,$$

satisfies the condition

$$L_2 |\Lambda\rangle = 0. \quad (9)$$

For convenience, hereinafter we assume $m = 1$.

**Interactions of massive integer-spin fields.** It is worth noting that the massive gauge higher-spin fields are described by the first-class constraints only from point of view of the Hamiltonian formulation. As is well-known the “minimal” coupling prescription breaks the right number of physical degrees of freedom. In the considered here gauge manner of description this is represented as breaking the gauge invariance and, as a consequence, breaking the algebra of the first-class constraints. But if we can restore the gauge invariance by some deformation of the Lagrangian and the transformation, hence we can restore the algebra of the constraints. Of course, any local interaction of physical fields (both consistent and inconsistent) can be rewritten in a gauge invariant way with the help of Stueckelberg fields and therefore the Stueckelberg gauge symmetry principle alone cannot serve as a fundamental principle for fixing interactions. But the important point is that, in general, such a procedure leads to gauge invariant interactions with higher derivatives what is the reason why the number of degrees of freedom may be changed despite that the gauge symmetry is preserved.

We introduce interactions by means of the “minimal” coupling prescription, i.e. we replace usual momentum operators with covariant ones $p_\mu \rightarrow P_\mu$. The commutator of the covariant momenta is proportional to the strength of an external field: $[P_\mu, P_\nu] \sim F_{\mu\nu}$, where $F$ is strength of the electromagnetic field or Riemann tensor.

The $L$-operators cease to obey algebra (4) after substitution of the usual momenta with the covariant ones in definitions (3). Therefore, Lagrangian (7) loses the invariance under transformations (8).

To restore algebra (4) we represent operators (3) as normal ordered functions of creation and annihilation operators as well as of $F$. The particular form of the operators $L_i$ can be defined from the condition of recovering of commutation relations (4) by these operators. We should note that it is enough

\footnote{The Hamiltonian formulation for the gauge-invariant description of the massive fields with spins 2 and 3 was considered in Ref. [3]}

\[ \]
to define the form of the operators $L_1$ and $L_2$, since one can take following expressions:

$$L_0 \overset{\text{def}}{=} [L_1, L_{-1}], \quad N \overset{\text{def}}{=} [L_2, L_{-2}] - \frac{D+1}{2}. \quad (10)$$

as definitions of the operators $L_0$ and $N$.

Since we have turned to the extended universal enveloping algebra, the arbitrariness in the definition of operators\(^3\) $a$ and $b$ appears. Besides, in the right-hand side of (1), we can admit the presence of the arbitrary operator functions depending on $a$, $b$, and $F$. Here, such a modification of the operators must not lead to breaking the Jacob’s identity. Besides, these operators must restore the initial algebra in limit $F \to 0$. However, one can make sure that using the arbitrariness in the definition of creation and annihilation operators, we can restore algebra (1) for both the e.m. and gravitational fields at corresponding orders.

We shall search for the operators $L_1$ and $L_2$ in the form of an expansion in $F$.

**Electromagnetic interaction of massive spin-$s$ field.** Here we consider the interaction between massive gauge fields with arbitrary integer spin $s$ and a constant e.m. field.

The commutator of covariant momenta defines the e.m. field strength

$$[\mathcal{P}_\mu, \mathcal{P}_\nu] = F_{\mu\nu}. \quad (11)$$

For convenience we included the imaginary unit and coupling constant into the definition of the strength tensor.

Let us consider the linear approximation and modify the $L$-operators by appropriate way. So, the operator $L_1$ should be no higher than linear in operator $\mathcal{P}_\mu$, since the presence of its higher number changes the type of gauge transformations and leads to appearance of terms with higher derivatives in the Lagrangian that breaks the right number of physical degrees of freedom. Therefore, at this order we shall search for operators having the form

$$L_1^{(1)} = (\bar{a} F a) h_0(\bar{b}, b)b + (\mathcal{P} F a) h_1(\bar{b}, b) + (\bar{a} F \mathcal{P}) h_2(\bar{b}, b)b^2. \quad (12)$$

At the same time the operator $L_2$ cannot depend on the momentum operators at all, since condition (3) defines purely algebraic constraints\(^4\) on the coefficient

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\(^3\)Such an arbitrariness has been also presented earlier as an internal automorphism of the Heisenberg-Weil algebra defining the Fock space \(^4\). But exactly the transformations depending on $F$ are important for us.

\(^4\)Besides, presence of the derivatives in $L_2$ also leads to appearance of higher derivatives in the Lagrangian.
functions. Therefore, at this order we choose the operator $L_2$ in the following form:

$$L_2^{(1)} = (\bar{a} F a) h_3(\bar{b}, b) b^2,$$

Here $h_i(\bar{b}, b)$ are normal ordered operator functions of the type

$$h_i(\bar{b}, b) = \sum_{n=0}^{\infty} H^n_i \bar{b}^n b^n,$$

where $H^n_i$ are arbitrary real coefficients. We consider only the real coefficients since the operators with purely imaginary coefficients do not give any contribution to the "minimal" interaction.

Let us define the particular form of functions $h_i$ from the condition recovering commutation relations (12) by operators (13). This algebra is entirely defined by (10) and by the following commutators

$$[L_2, L_1] = 0, \quad [L_2, L_{-1}] = L_1, \quad [L_0, L_1] = 0.$$

Having calculated (14) and passing to the normal symbols of creation and annihilation operators, we obtain a system of differential equations for the normal symbols of operator functions $h_i$. For the normal symbols of the operator functions we shall use the same notations. This does not lead to the mess since we consider the operator functions as the functions of two variables while their normal symbols as the functions of one variable. Having solved the systems of the equations we obtain the particular form of functions $h_i$:

$$h_0(x) = 1 - d_2, \quad h_1(x) = d_1 \left( \frac{1}{2} - x \right) e^{-2x} + d_2 \left( \frac{1}{2} + x \right),$$

$$h_2(x) = d_1 e^{-2x} + d_2, \quad h_3(x) = 0.$$

Here $d_1$ and $d_2$ are arbitrary real parameters.

So, normal symbols of operators $L_n$ has the following form in this approximation:

$$L_1^{(1)} = (1 - d_2) (\bar{a} F a) \beta + \left( e^{-2\bar{\beta}} d_1 \left( \frac{1}{2} - \bar{\beta} \right) + d_2 \left( \frac{1}{2} + \bar{\beta} \right) \right) (\mathcal{F} \mathcal{P} a)$$

$$+ \left( e^{-2\bar{\beta}} d_1 + d_2 \right) (\bar{a} F \mathcal{P} a) \beta^2,$$

$$L_0^{(1)} = (1 - 2d_2) (\bar{a} F a) + \left\{ (1 + 2d_2) (\mathcal{F} \mathcal{P} a) \bar{\beta} + \text{h.c.} \right\},$$

$$L_2^{(1)} = 0,$$

where $\bar{a}_\mu$ and $a_\mu$ are the normal symbols of the operators $\bar{a}_\mu$ and $a_\mu$. 
The transition to the operator functions is realized in the conventional manner:

\[ O(\bar{a}, \bar{b}, a, b) := e^{\bar{a} \frac{\partial}{\partial \bar{\alpha}}} e^{\bar{b} \frac{\partial}{\partial \bar{\beta}}} e^{a \frac{\partial}{\partial \alpha}} e^{b \frac{\partial}{\partial \beta}} O(\bar{\alpha}, \bar{\beta}, \alpha, \beta) \bigg|_{\bar{\alpha} \to a, \bar{\beta} \to b} \bigg|_{\bar{\alpha} \to 0, \bar{\beta} \to 0}. \]

Thus, we have obtained the general form of operators \( L_n \) which obey algebra \( (4) \) in linear approximation. This means that Lagrangian \( (7) \) is an invariant under transformations \( (8) \) at this order.

From (17) it is clear that there exists the two-parametric arbitrariness in linear approximation. But one of the arbitrary parameters \( d_1 \) and \( d_2 \) is determined in the second approximation. In this, there are two solutions: when \( d_1 \) vanishes and \( d_2 \) is arbitrary, and vice versa, when \( d_1 \) is a free parameter and \( d_2 \) is equal to \( \frac{1}{2} \). One can verify that the gyromagnetic ratio vanishes in the second case.

It is worth noting that the construction obtained is free from pathologies. Indeed, the gauge invariance and absence of higher derivatives ensure the appropriate number of physical degree of freedom. In this, the model is causal in linear approximation since by virtue of the antisymmetry and homogeneity of \( F_{\mu \nu} \) the characteristic determinant\(^5\) for equations of motion of any massive state has the form \( D(n) = (n^2)^p + O(F^2) \), where \( n_\mu \) is a normal vector to the characteristic surface and the integer constant \( p \) depends on the spin of massive state. The equations of motion will be causal (hyperbolic) if the solutions \( n^0 \) to \( D(n) = 0 \) are real for any \( \vec{n} \). In our case condition \( D(n) = 0 \) corresponds to the ordinary light cone at this order.

One can consider the next approximation in the similar manner \(^6\). The only essential difference of this order is that the operator \( L_2 \) depends on \( F_{\mu \nu} \).

Thereby, we have restored algebra \( (4) \) up to the second order in the electromagnetic field strength. It means that we have restored the gauge invariance\(^4\) of Lagrangian \( (7) \) at the same order as well. Here we have not used the dimensionality of space-time anywhere explicitly, i.e. the obtained expressions do not depend on it.

**Propagation of massive higher-spin field in symmetrical Einstein space.** Now we consider an arbitrary \( D \)-dimensional symmetrical Einstein space. The determinant is entirely determined by the coefficients of the highest derivatives in equations of motion after gauge fixing and resolving of all the constraints \(^6\).

\(^5\) Which ensures the right number of physical degree of freedom since in this order there are no terms with higher derivatives either.
space, i.e. the Riemann space defined by the following equations:

\[ R^{\mu\nu} - \frac{1}{2} g^{\mu\nu} R + g^{\mu\nu} \lambda = 0, \quad \mathcal{D}^{(R)}_\mu R^{\nu\alpha\beta\gamma} = 0, \]

where \( \mathcal{D}^{(R)}_\mu \) is the covariant derivative with the Cristofel connection \( \Gamma^\alpha_{\nu\mu} \). We assume that the Greek indexes are global while the Latin ones are local. As usual, the derivative \( \mathcal{D}^{(R)}_\mu \) acts on tensor fields with global indexes only.

To describe the massive higher-spin fields in the Riemann background, we must replace the ordinary derivatives with the covariant one, i.e. we make the substitution:

\[ p_\mu \rightarrow P_\mu = i \left( \mathcal{D}^{(T)}_\mu + \omega_\nu^{ab} \bar{a}_a a_b \right), \quad (17) \]

where \( \omega_\nu^{ab} \) is the Lorentz connection. We imply that the creation and annihilation operators primordially carry the local indexes. We also have to introduce the non-degenerate vielbein \( e_\mu^a \) for the transition from the local indexes to the global ones and vice versa. As usual, we impose the conventional requirement on the vielbein

\[ \mathcal{D}^{(T+\omega)}_\mu e_\nu^a = \partial_\mu e_\nu^a - \Gamma_\nu^\lambda e_\lambda^a + \omega_\mu^{ab} e_\nu^b = 0. \]

By means of this relation one can transfer from expressions with one connection to those with other. Besides, we should note that due to this relation the operator \( P_\mu \) commutes with the vector creation-annihilation operators carrying global indexes \( \bar{a}_\nu = e_\nu^b \bar{a}_b \) and \( a_\nu = e_\nu^b a_b \). This allows us not to care about the ordering of operators (3).

One can verify that the covariant momentum operator defined in this way properly acts on the states of type (2), indeed

\[ P_\mu |\Phi\rangle = i \mathcal{D}^{(\omega)}_\mu \Phi_b^{b_1...b_n} \prod_{i=1}^n \bar{a}_{b_i} |0\rangle = i \mathcal{D}^{(T)}_\mu \Phi^{\nu_1...\nu_n} \prod_{i=1}^n \bar{a}_{\nu_i} |0\rangle. \]

The commutator of the covariant momenta defines the Riemann tensor:

\[ [P_\mu, P_\nu] = R^{ab}_\mu \omega_\nu^{ab} |\bar{a}_a a_b\rangle. \quad (18) \]

where \( R^{ab}_\mu (\omega) = \partial_\mu \omega_\nu^{ab} + \omega_\mu^{ac} \omega_\nu^{cb} - (\mu \leftrightarrow \nu). \)

In the definition of operators (3), we replace the ordinary momenta with the covariant ones as well. As a result, the operators cease to obey algebra (4). Therefore, Lagrangian (7) loses the invariance under gauge transformations (8).

To recover the gauge invariance, we do not need to restore the whole algebra (4), it is enough to ensures the existence of the two first commutators in
To restore these relations, we shall search for the operators $L_1$ and $L_2$ as series in the Riemann tensor and scalar curvature.

Let us consider linear approximation.

Using the very arguments as for the case with constant e.m. field we choose the following ansatz

$$L_1^{(1)} = R \left( h_0(\bar{b}, b) b + h_1(\bar{b}, b) b (\bar{a} \cdot a) + \bar{b} h_2(\bar{b}, b) a^2 + h_3(\bar{b}, b) b^3 \bar{a}^2 \\
+ h_4(\bar{b}, b) (\mathcal{P} \cdot a) + h_5(\bar{b}, b) b^2 (\bar{a} \cdot \mathcal{P}) \right) + R^{\mu \nu ab} \left( h_6(\bar{b}, b) b \bar{a}_\mu \bar{a}_a a_\nu a_b \\
+ h_7(\bar{b}, b) \bar{a}_\mu a_\nu \mathcal{P}_a a_b + h_8(\bar{b}, b) b^2 \bar{a}_\mu \mathcal{P}_\nu \bar{a}_a a_b \right).$$

(19)

for operator $L_1$ at this approximation and

$$L_2^{(1)} = R \left( h_9(\bar{b}, b) b^2 + h_{10}(\bar{b}, b) a^2 + h_{11}(\bar{b}, b) b^2 (\bar{a} \cdot a) + h_{12}(\bar{b}, b) b^4 \bar{a}^2 \\
+ h_{13}(\bar{b}, b) b^2 \bar{a}^\mu \bar{a}^\nu a^\mu a^\nu a^b R^{\mu \nu ab} \right).$$

(20)

Here $h_i(\bar{b}, b)$ are normal ordered operator functions of type (14).

Let us define a particular form of the functions $h_i$ from the condition of recovering two first commutation relations in (15) by the operators $L_1$ and $L_2$.

We have to note that these operators can obey the two first relations in (15) up to the terms proportional to $L_2^{(0)} = \frac{1}{2} (a^2 + b^2)$ at linear order, since this does not break the gauge invariance due to constraint (9).

Having calculated the commutators under consideration and passing to normal symbols of the creation and annihilation operators, we obtain a system of differential equations in the normal symbols of operator functions $h_i$. Having solved that, we obtain the particular form of the operators $L_1$ and $L_2$:

$$L_1^{(1)} = \frac{1}{6} R^{\mu \nu \alpha \beta} \bar{a}_\alpha a_\mu \left\{ \mathcal{P}_\nu \alpha_\beta \left( 1 + 2 \bar{\beta} \beta \right) - 5 \bar{\alpha}_\nu \alpha_\beta \beta + 2 \bar{\alpha}_\nu \mathcal{P}_\beta \beta^2 \right\} \\
+ R \left\{ c_2 (\mathcal{P} \cdot \alpha) + c_4 \beta \right\},$$

$$L_2^{(1)} = \frac{R}{12D} \left\{ \alpha^2 (1 + 2 \bar{\beta} \beta) + 4 \bar{\beta} \beta^3 \right\},$$

(21)

where $c_1$ and $c_2$ are arbitrary real parameters.

Thus, we have obtained the general form of the operators $L_n$, which provide the gauge invariance of Lagrangian (7) at this order. The form of the operator $L_2$ has changed in this approximation, hence, the conditions

$$(L_2)^2 |\Phi^s\rangle = 0, \quad L_2 |\Lambda^{s-1}\rangle = 0$$

undergo the nontrivial modifications in terms of the coefficient functions.
Conclusion. In this paper we have constructed the Lagrangian describing the interaction of massive fields of arbitrary integer spins with the homogeneous electromagnetic field up to the second order in the strength. It is noteworthy that unlike the string approach \[8\] our consideration does not depend on the space-time dimensionality, and, moreover, we have described the interaction of the single field with spin $s$ while in the string approach the presence of constant electromagnetic field leads to the mixing of states with different spins \[9\] and one cannot consider any states separately.

Consistency of the obtained results is based on the two following principles: i) the absence of terms with higher-derivatives and ii) gauge symmetry. Item i) provides preservation of dimensionality of the unreduced phase space when an interaction is switched on, while item ii) in addition to i) guarantees preservation of dimensionality of the physical phase space. So, these requirements provide the right number of physical degrees of freedom in the interacting case.

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