Invariant bilinear forms under the rigid motions of a regular polygon

Dilchand Mahto and Jagmohan Tanti*

Department of Mathematics, Central University of Jharkhand, Ranchi-835205, Jharkhand, India
Department of Mathematics, Babasaheb Bhimrao Ambedkar University, Lucknow-226025, Uttar Pradesh, India
*Corresponding author.
E-mail: dilchandiitk@gmail.com; jagmohan.t@gmail.com

Abstract

In this paper, for \( n \) a positive integer, we compute the number of \( n \) degree representations for a dihedral group \( G \) of order \( 2m, \ m \geq 3 \) and the dimensions of the corresponding spaces of \( G \) invariant bilinear forms over a complex field \( \mathbb{C} \). We explicitly discuss about the existence of a non-degenerate invariant bilinear form. The results are important due to their applications in the studies of physical sciences.

Keywords: Bilinear forms; Representation theory; Vector spaces; Direct sums; Semi direct product.

2010 MSC: 15A63, 11E04, 06B15, 15A03.

1. Introduction

Representation theory enables the study of a group as operators on certain vector spaces and an orthogonal group with respect to a corresponding bilinear space. Also since last several decades the search of non-degenerate invariant bilinear forms has remained of great interest. Such type of studies acquires an important place in quantum mechanics and other branches of physical sciences.

Let \( G \) be a finite group and \( \mathbb{V} \) a vector space over a field \( \mathbb{F} \), then we have the following.

**Definition 1.1.** A homomorphism \( \rho : G \to GL(\mathbb{V}) \) is called a representation of the group \( G \). \( \mathbb{V} \) is also called a representing space of \( G \). The dimension of \( \mathbb{V} \) over \( \mathbb{F} \) is called degree of the representation \( \rho \).

Let \( \rho : G \to GL(\mathbb{V}) \) be a representation.

**Definition 1.2.** A bilinear form on \( \mathbb{V} \) is said to be invariant under the representation \( \rho \) if

\[
\mathbb{B}(\rho(g)x, \rho(g)y) = \mathbb{B}(x, y), \ \forall \ g \in G \text{ and } x, y \in \mathbb{V}.
\]

For the basic properties of a bilinear form one can refer to [8].

Let \( \mathbb{Xi} \) denotes the space of bilinear forms on the vector space \( \mathbb{V} \) over \( \mathbb{F} \).

**Definition 1.3.** The space of invariant bilinear forms under the representation \( \rho \) is given by

\[
\mathbb{Xi}_G = \{ \mathbb{B} \in \mathbb{Xi} \mid \mathbb{B}(\rho(g)x, \rho(g)y) = \mathbb{B}(x, y), \ \forall \ g \in G \text{ and } x, y \in \mathbb{V} \}.
\]

It is obvious that \( \mathbb{Xi}_G \) is a subspace of \( \mathbb{Xi} \).
The representation \((\rho, \mathbb{V})\) is irreducible of degree \(n\) if and only if \(\{0\}\) and \(\mathbb{V}\) are the only invariant subspaces of \(\mathbb{V}\) under \(\rho\). Let \(r\) be the number of conjugacy classes of \(G\). If \(\mathbb{F}\) is algebraically closed and \(\text{char}(\mathbb{F}) = 0\) or not dividing \(|G|\), by Frobenius (see pp 319, Theorem (5.9)) there are \(r\) irreducible representations \(\rho_i\) (say), \(1 \leq i \leq r\) of \(G\) and \(\chi_i\) (say) is the corresponding character of \(\rho_i\). Also by Maschke’s theorem (see pp 316, corollary (4.9)) every \(n\) degree representation of \(G\) can be written as a direct sum of copies of irreducible representations. For \(\rho = \bigoplus_{i=1}^{r} k_i \rho_i\) an \(n\) degree representation of \(G\), the coefficient of \(\rho_i\) is \(k_i\), \(1 \leq i \leq r\), so that \(\sum_{i=1}^{r} d_i k_i = n\), and \(\sum_{i=1}^{r} d_i^2 = |G|\), where \(d_i\) is the degree of \(\rho_i\) and \(d_i||G|\) with \(d_i' \geq d_i\) when \(i' > i\). It is already well understood in the literature that the invariant space \(\Xi_G\) under \(\rho\) can be expressed by the set \(\Xi_G = \{X \in \mathbb{M}_n(\mathbb{F})| C^g_{\rho(g)} X C_{\rho(g)} = X, \forall g \in G\}\) with respect to an ordered basis \(\mathcal{E}\) of \(\mathbb{V}\), where \(\mathbb{M}_n(\mathbb{F})\) is the set of square matrices of order \(n\) with entries from \(\mathbb{F}\) and \(C_{\rho(g)} = [\rho(g)]_{\mathcal{E}}\) is the matrix representation of the linear transformation \(\rho(g)\) with respect to \(\mathcal{E}\).

In this paper our investigation pertains to the following questions.

**Question.** How many \(n\) degree representations (upto isomorphism) of \(G\) can be there? Distinguish which of them are faithful representations? What is the dimension of \(\Xi_G\) for every \(n\) degree representation? What are the necessary and sufficient conditions for the existence of a non-degenerate invariant bilinear form.

These questions have been studied by many people in the distinct perspectives. Gongopadhyay and Kulkarni investigated the existence of \(T\)-invariant non-degenerate symmetric (resp. skew-symmetric) bilinear forms. Kulkarni and Tanti investigated the dimension of space of \(T\)-invariant bilinear forms. Gongopadhyay, Mazumder and Sardar investigated for an invertible linear map \(T: V \rightarrow V\), when does the vector space \(V\) over \(\mathbb{F}\) admit a \(T\)-invariant non-degenerate \(c\)-hermitian form. Chen discussed the all matrix representations of the real numbers. Authors investigated the dimensions of invariant spaces and explicitly discussed about the existence of the non-degenerate invariant bilinear forms under \(n\) degree representations of a group of order \(p^3\), with prime \(p\). Sergeichuk studied systems of forms and linear mappings by associating with them self-adjoint representations of a category with involution. Frobenius proved that every endomorphism of a finite dimensional vector space \(V\) is self-adjoint for at least one non-degenerate symmetric bilinear form on \(V\). Later, Stenzel determined when an endomorphism could be skew-adjoint for a non-degenerate quadratic form, or self-adjoint or skew-self adjoint for a symplectic form on complex vector spaces. However his results were later generalized to an arbitrary field. Pazzis tackled the case of the automorphisms of a finite dimensional vector space that are orthogonal (resp. symplectic) for at least one non-degenerate quadratic form (resp. symplectic form) over an arbitrary field of characteristics 2.

Let \(D_m\) be a dihedral group of order \(2m, m \in \mathbb{Z}^+\) and \((\rho, \mathbb{V})\) an \(n\) degree representation of \(D_m\) over \(\mathbb{C}\).

In this paper we investigate about the counting of \(n\) degree representations of \(D_m\) with \(m \geq 3\) over \(\mathbb{C}\), dimensions of their corresponding spaces of invariant bilinear forms and establish a characterization criteria for the existence of a non-degenerate invariant bilinear form. Our investigations are stated in the following four main
Theorem 1.1. The number of \( n \) degree representations (upto isomorphism) of \( D_m, m \geq 3 \) is

\[
\sum_{s=0}^{\left\lfloor \frac{m}{2} \right\rfloor} \left( s + \left\lfloor \frac{m-3}{2} \right\rfloor \right) \left( n - 2s + 2|Z(G)| - 1 \right).
\]

Theorem 1.2. The space \( \mathcal{E}_{D_m} \), with \( m \geq 3 \) under \( n \) degree representation \((\rho, \mathcal{V})\) over \( \mathbb{C} \) is isomorphic to the direct sum of the subspaces \( \mathcal{W}_{(G,k,\rho_i)} \) of \( \mathbb{M}_n(\mathbb{C}) \), i.e., \( \mathcal{E}_{G} = \bigoplus_{i=1}^{r} \mathcal{W}_{(G,k,\rho_i)} \) or \( \mathcal{E}_{G} \cong \bigoplus_{i=1}^{r} \mathbb{M}_n(\mathbb{C}) \), where \( \mathcal{W}_{(G,k,\rho_i)} = \{ X \in \mathbb{M}_n(\mathbb{C}) \mid X = \text{Diag}(O_{d_1k_1}^{1}, \ldots, X_{d_ik_i}^{n}, O_{d_kk_i}^{n}) \} \) with \( X_{d_ik_i}^{n} \) a square matrix of order \( d_ik_i \) satisfying \( X_{d_ik_i}^{n} = C_{k,\rho_i(g)}^{t}X_{d_ik_i}^{n}C_{k,\rho_i(g)} \), \( \forall g \in D_m \). Also dimension of \( \mathcal{W}_{(G,k,\rho_i)} = k_i^2 \).

Theorem 1.3. If \( G \) is a dihedral group, then an \( n \) degree representation \( \sigma \) of \( G \) admits a non-degenerate invariant bilinear form if and only if every irreducible representation admits a non-degenerate invariant bilinear form.

Theorem 1.4. 1. The number of faithful irreducible representations of a dihedral group \( D_m \) is

\[
\left\lfloor \frac{m-1}{2} \right\rfloor - (l_2 + 1)(l_3 + 1) \ldots \ldots \ldots (l_p + 1) + |Z(G)| + 1.
\]

2. If \( \rho \) is a finite degree representation of \( D_m \), then \( \rho(D_m) \) is isomorphic to either \( \mathbb{Z}_1, \mathbb{Z}_2 \) or \( D_{\text{lcm}(\frac{2\pi}{m}, t^\text{th\ irrep})} \), where \( t^\text{th\ irrep} \) stands for the \( t^\text{th} \) irreducible representation of degree 2 appearing in \( \rho \).

Remark 1.1. Thus we get the necessary and sufficient condition for the existence of a non-degenerate invariant bilinear form under an \( n \) degree representation of a dihedral group over \( \mathbb{C} \).

2. Preliminaries

Symmetries are the rigid motions of an \( m \)-sided regular polygon with \( m \in \mathbb{N} \). Considering a circle \( C(0, R) \) with radius \( R \) and centered at origin on a real plane, if the point \((R, 0)\) is rotated counter-clockwise by an angle \( \frac{2\pi}{m} \) radian (or resp. clockwise) around the circle, which returns back to itself after \( m \) successive rotations, while these rotations, the set of points \{(RCos0, RSin0), (RCos\frac{2\pi}{m}, RSin\frac{2\pi}{m}), \ldots, (RCos\frac{2(m-1)\pi}{m}, RSin\frac{2(m-1)\pi}{m})\} \) lies on the circle forming the vertices of an \( m \)-sided regular polygon \( P_m \), say, which is a symmetric regular polygon. Let \( a \) be the rotation by angle \( \frac{2\pi}{m} \) counter-clockwise then for \( 1 \leq s \leq m \), rotating it counter-clockwise by an angle of \( \frac{2\pi}{m} \) radian, we get "a". Thus we get \( m \) symmetric regular \( m \)-gons namely \( a, a^2, a^3, \ldots, a^{m-1}, a^m = P_m \), the last one \( a^m \) is the original regular \( m \)-gon. When \( m \) is odd, for \( 1 \leq s \leq m \), the reflection about the angle bisection of the vertex \((RCos\frac{2\pi}{m}, RSin\frac{2\pi}{m})\) of the regular \( m \)-gon \( a^s \) is also a symmetric regular \( m \)-gon \( (\text{say}) \), whereas if \( m \) is even, the reflections about the side bisections as well as the angle bisections of the vertices \((RCos\frac{2\pi}{m}, RSin\frac{2\pi}{m})\), \( 1 \leq s \leq m \) of the regular \( m \)-gon \( a^s \) is symmetric with \( a^m \). Thus there is a set of \( 2m \) symmetric regular \( m \)-gons \( \{a, a^2, \ldots, a^m, b, ab, a^2b, \ldots, a^{m-1}b\} \), which forms a group with the operation being compositions of \( m \)-gons and known as a dihedral group of order \( 2m \) denoted by \( D_m \). Centre of \( D_m \), \( Z(D_m) = \{a^m\} \) or \( \{a^m, a^\frac{m}{2}\} \) according to \( m \) is odd or even respectively and the number of the conjugacy
classes is \( r = 2|Z(D_m)| + \left\lfloor \frac{m-1}{2} \right\rfloor \), which is equal to the number of irreducible representations (over an algebraically closed field \( \mathbb{F} \), char(\( \mathbb{F} \)) = 0 or not dividing \( 2m \)) with degree \( d_i \). Here \( d_i||G| \) and \( \sum_{i=1}^{r} d_i^2 = |G| \). There are \( 2|Z(D_m)| \) representations of degree 1 and \( \left\lfloor \frac{m-1}{2} \right\rfloor \) representations of degree 2 for \( D_m \). In the next section, we formulate all \( r \) irreducible representations of \( D_m \) such that \( C_{D_m(g)} \) is either a rotation operator or a reflection operator.

**Definition 2.1.** The character of \( \rho \) is a function \( \chi : G \rightarrow \mathbb{F} \), \( \chi(g) = \text{tr}(\rho(g)) \) and is also called character of the group \( G \).

**Theorem 2.1.** (Maschke’s Theorem): If char(\( \mathbb{F} \)) does not divide \( |G| \), then every representation of \( G \) is a direct sum of irreducible representations.

**Proof.** See pp 316, corollary (4.9) \[1\].

**Theorem 2.2.** Two representations \( (\rho, V) \) and \( (\rho', V) \) of \( G \) are isomorphic iff their character tables are same i.e., \( \chi(g) = \chi'(g) \) for all \( g \in G \).

**Proof.** See pp 319, corollary (5.13) \[1\].

In the rest part of this section we take \( \mathbb{F} = \mathbb{C} \).

**2.1. Irreducible representations of \( D_m \) with \( m \geq 3 \).**

In this subsection \( (\rho_t, \mathbb{W}_t) \) stands for an irreducible representation of \( D_m \) with degree 1 or 2 over \( \mathbb{C} \). Let \( \rho_{2|Z(G)|+t} \), \( t \geq 1 \), denotes an irreducible representations of degree 2 of \( G \), where \( 1 \leq t \leq \left\lfloor \frac{m-1}{2} \right\rfloor \). Since \( \rho_{2|Z(G)|+t} \) is a homomorphism from \( D_m \) to \( GL(\mathbb{W}_2) \cong GL(2, \mathbb{C}) \). So by the fundamental theorem of homomorphism \( \frac{G}{\text{Ker}(\rho_{2|Z(G)|+t})} \cong \rho_{2|Z(G)|+t}(G) \). Now we calculate those \( t \) which are co-prime with \( m \). All \( 2m \), regular \( m \)-gons are in

\[
D_m = \{1, a, a^2, \ldots, a^{m-1}, b, ab, a^2b, \ldots, a^{m-1}b | a^m = b^2 = 1, ba = a^{m-1}b\}.
\]

The representation \( \rho_{2|Z(G)|+t} \) rotates a regular \( m \)-gon \( a^s \) in the counter-clockwise direction by an angle \( st \theta_t \), where \( \theta_t = t \frac{2\pi}{m} \) (or resp. clockwise \( (m-t) \frac{2\pi}{m} \) and \( b \) the reflection about \( x = axis \). For \( 1 \leq s \leq m \), the polygon \( a^sb \) is mapped to the composition of rotation by angle \( s \theta_t \) and the reflection under an irreducible representation of degree 2, whose trace is zero. Thus the character of the group is to be calculated on the regular \( m \)-gon \( a^s \), \( 1 \leq s \leq m \).

By the fundamental theorem of arithmetic, we have

\[
m = 2^{l_2}3^{l_3}5^{l_5}7^{l_7}11^{l_{11}} \ldots p^{l_p},
\]

where \( p \) is the largest prime divisor of \( m \), so \( l_p \geq 1 \). The number of divisors of \( m \) excluding 1 is \((l_2 + 1)(l_3 + 1)(l_5 + 1)(l_7 + 1)(l_{11} + 1)(l_p + 1) - 1\). We need divisors of \( m \) greater than 1 and less than \( \frac{m}{2} \). These divisors fall inside the range of \( t \) which are not coprime to \( m \), also for each \( t \) we have an irreducible representation of degree 2. If \( t_1 \) is a divisor of \( m \) then \( \frac{m}{(m,t_1)} \) is the order of \( \sigma_2|Z(G)|+t_1(a) \), so kernel of the respective irreducible representation is non-trivial. There may be many such distinct divisors of \( m \), less than \( \frac{m}{2} \). Thus the number of
non-isomorphic images of $G$ under the representations $\sigma_2|Z(G)| + t$ is same as the number of distinct divisors of $m$, less than $\frac{m}{2}$ and whenever $(m, t) = 1$ then we have $|\text{Ker}(\rho_2|Z(G)| + t)| = 1$.

2.2. **Counter-clockwise rotations and their compositions with reflection can be seen as below.**

$$\rho_2|Z(G)| + t(a^s) = \begin{bmatrix} \cos \left( \frac{2\pi m ts}{m} \right) & -\sin \left( \frac{2\pi m ts}{m} \right) \\ \sin \left( \frac{2\pi m ts}{m} \right) & \cos \left( \frac{2\pi m ts}{m} \right) \end{bmatrix} \quad \text{and} \quad \rho_2|Z(G)| + t(a^t b) = \begin{bmatrix} \cos \left( \frac{2\pi m ts}{m} \right) & \sin \left( \frac{2\pi m ts}{m} \right) \\ \sin \left( \frac{2\pi m ts}{m} \right) & -\cos \left( \frac{2\pi m ts}{m} \right) \end{bmatrix}.$$ 

2.3. For $m$ is odd and $1 \leq t \leq \frac{m-1}{2}$, all irreducible representations of $G$ are recorded in the following table.

**Table 1.** Irreducible representations of $D_m$ when $m$ is odd and $1 \leq t \leq \frac{m-1}{2}$.

| $\rho_1$ | $\rho_2$ | $\rho_2+t$ |
|----------|----------|-------------|
| $a$      | 1        | $\begin{bmatrix} \cos(\theta t) & -\sin(\theta t) \\ \sin(\theta t) & \cos(\theta t) \end{bmatrix}$ |
| $b$      | 1        | $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ |

2.4. For $m$ is even and $1 \leq t \leq \frac{m}{2} - 1$, all irreducible representations of $G$ are presented by the following table.

**Table 2.** Irreducible representations of $D_m$ when $m$ is even and $1 \leq t \leq \frac{m}{2} - 1$.

| $\rho_1$ | $\rho_2$ | $\rho_3$ | $\rho_4$ | $\rho_4+t$ |
|----------|----------|----------|----------|-------------|
| $a$      | 1        | $\begin{bmatrix} \cos(\theta t) & -\sin(\theta t) \\ \sin(\theta t) & \cos(\theta t) \end{bmatrix}$ |
| $b$      | 1        | $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ |

Now as

$$\rho = k_1 \rho_1 \oplus k_2 \rho_2 \oplus \ldots \oplus k_r \rho_r,$$

where for every $1 \leq i \leq r$, $k_i \rho_i$ stands for the direct sum of $k_i$ copies of the irreducible representation $\rho_i$.

Let $\chi$ be the corresponding character of the representation $\rho$, then

$$\chi = k_1 \chi_1 + k_2 \chi_2 + \ldots + k_r \chi_r,$$

where $\chi_i$ is the irreducible character of $\rho_i$, for every $i$, $1 \leq i \leq r$. Dimension of the character $\chi$ is being calculated at the identity element of a group. i.e,

$$\text{dim}(\rho) = \chi(1) = \text{tr}(\rho(1)).$$

$$\Rightarrow d_1 k_1 + d_2 k_2 + \ldots + d_r k_r = n. \quad (2)$$

**Note 2.1.** Equation (2) holds in more general case which helps us in finding all possible distinct $r$-tuples $(k_1, k_2, \ldots, k_r)$, which correspond to the distinct $n$ degree representations (up to isomorphism) of a finite group.
The orthonormality condition of characters of irreducible representations of degree 2 is being calculated in the following manner:

\[
\left( \chi_{t_1+2|Z(D_m)|}; \chi_{t_2+2|Z(D_m)|} \right) = \begin{cases} \frac{4}{|D_m|} \sum_{a^* \in <a>} \cos^2(s \theta_{t_1}), & \text{when } t_1 = t_2, \\ \frac{4}{|D_m|} \sum_{a^* \in <a>} \cos(s \theta_{t_1}) \cos(s \theta_{t_2}), & \text{when } t_1 \neq t_2. \end{cases}
\]

3. Faithful representations of a dihedral group of order 2m with \( m \geq 3 \).

In this section we distinguish all \( n \) degree faithful representations of \( G = D_m \) over \( \mathbb{C} \). Here \( m = 2^l 3^l 5^l 7^l \ldots p^l \).

**Theorem 3.1.** The number of irreducible representations of a dihedral group \( G \) of order 2m with non-trivial kernels is \((l_2+1)(l_3+1)\ldots(l_p+1) + |Z(G)| - 1\).

**Proof.** Number of irreducible representations of \( D_m \) is \( r = 2|Z(G)| + \left\lfloor \frac{m-1}{2} \right\rfloor \). For a 2 degree irreducible representation \( \rho_{2|Z(G)|+t} \), if \( (m,t) \neq 1 \) then kernel is non-trivial. The number of divisors of \( m \) excluding 1, less than \( \frac{m}{2} \) is \((l_2+1)(l_3+1)\ldots(l_p+1) - |Z(G)| - 1\) and \( 2|Z(G)| \) representations of degree 1 have non-trivial kernel. Thus the result follows.

**Theorem 3.2.** If \( \rho_{2|Z(G)|+t} \) is an irreducible representation of degree 2 of a dihedral group \( G \) of order 2m and \( (m,t) = 1 \), then \( \rho_{2|Z(G)|+t} \) is a faithful representation.

**Proof.** For \( 1 \leq t \leq \left\lfloor \frac{m-1}{2} \right\rfloor \), \( \rho_{2|Z(G)|+t} \) is a homomorphism from \( G \) to \( GL(\mathbb{W}_2) \). Let \( t_1 \in \{1, 2, \ldots, \left\lfloor \frac{m-1}{2} \right\rfloor \} \) such that \( (m,t_1) = 1 \) then \( Ker(\rho_{2|Z(G)|+t_1}) = \{1\} \). Thus the result follows.

**Corollary 3.1.** The number of faithful irreducible representations of a dihedral group \( G \) of order 2m is \( \left\lfloor \frac{m-1}{2} \right\rfloor - (l_2+1)(l_3+1)\ldots(l_p+1) + |Z(G)| + 1 \).

**Proof.** Follows from the theorems 3.1 and 3.2.

**Theorem 3.3.** If \( \sigma \) is a non-trivial one degree representation of \( D_m \), then the image of \( D_m \) under \( \sigma \) is isomorphic to \( \mathbb{Z}_2 \).

**Proof.** Follows from the tables in the subsections 2.3 and 2.4.

**Theorem 3.4.** If \( \sigma \) is an irreducible representation for \( D_m \) of degree 2, then the image of \( D_m \) under \( \sigma \) is isomorphic to \( D_{\frac{m}{(m,t)}} \).

**Proof.** If greatest common divisor of \( m \) and \( t \) is \( (m,t) \) then \( \frac{m}{(m,t)} \) is the order of \( \sigma(a) \) and \( \sigma(b) \) is the reflection operator. Thus the group \( \sigma(D_m) \) is isomorphic to \( D_{\frac{m}{(m,t)}} \).
Corollary 3.2. If $\sigma$ is a non-trivial irreducible representation of $D_m$, then the image of $D_m$ under $\sigma$ is isomorphic to either $\mathbb{Z}_2$ or $D_{\frac{m}{\gcd(m,i)}}$.

Proof. Follows from the theorems 3.3 and 3.4.

Corollary 3.3. The number of non-isomorphic images under irreducible representations of $D_m$ is $2 + (l_2 + 1)(l_3 + 1)\ldots(l_p + 1) - |Z(D_m)|$.

Proof. An irreducible representation sends $D_m$ to either one of $\mathbb{Z}_1$, $\mathbb{Z}_2$, $\mathbb{Z}_3$, and as the number of distinct divisors of $m$, less than $\frac{m}{\gcd(m,i)}$ is $(l_2 + 1)(l_3 + 1)\ldots(l_p + 1) - |Z(D_m)|$, the result follows.

Corollary 3.4. If $\rho$ is a finite degree representation of $D_m$, then $\rho(D_m)$ is isomorphic to either $\mathbb{Z}_1$, $\mathbb{Z}_2$ or $\mathbb{D}_{\text{lcm}\{\frac{m}{(m,t)}, t^i\text{irrep}\}}$, where $t^i\text{irrep}$ stands for the $t^i$ irreducible representation of degree 2 appearing in $\rho$.

Proof. If $\rho$ consists of only degree 1 representations then $\rho(D_m) \cong \mathbb{Z}_1$ or $\mathbb{Z}_2$. So suppose $\rho$ consists of $t^i_{r_1}, t^i_{r_2}, \ldots, t^i_{r_k}$ irreducible representations of degree 2 then from Theorem 3.4 for every $r$, $\rho_r(D_m)$ is isomorphic to $D_{\frac{m}{\gcd(m,i)}}$ i.e., $\frac{m}{\gcd(m,i)}$ is the order of $\rho_r(a)$. Therefore $\text{lcm}\{\frac{m}{(m,t_1)}, \frac{m}{(m,t_2)}, \ldots, \frac{m}{(m,t_k)}\}$ is the order of $\rho(a)$ together with $\rho(b)^2$ is an identity operator. Hence the result follows.

4. Existence of non-degenerate invariant bilinear forms.

An element in the space of invariant bilinear forms under representation of a finite group is either non-degenerate or degenerate. An element of the space is degenerate when at least one irreducible representation consists of degenerate bilinear forms. If all elements of the space is degenerate then the space is called a degenerate invariant space, which is also discussed in [10] for the groups of order 8 and [11] for the groups of order $p^t$, with an odd prime $p$. How many such representations exists out of total representations, is a matter of investigation. Some of the spaces contains both non-degenerate and degenerate invariant bilinear forms under a certain representation. In this section we compute the number of such representations of $D_m$ over $\mathbb{C}$.

Remark 4.1. The space $\Xi_G$ of invariant bilinear forms under an $n$ degree representation $\rho$ contains only those $X \in M_n(\mathbb{C})$ whose $(i,j)^{th}$ block is a $O$ sub-matrix of order $d_ik_i \times d_jk_j$ when $i \neq j$ whereas the $(i,i)^{th}$ block of $X$, for $1 \leq i \leq 2|Z(D_m)|$, is given by

$$X_{ix}^{ji} = \begin{bmatrix} x_{11}^{ji} & x_{12}^{ji} & \ldots & \ldots & x_{1k_i}^{ji} \\ x_{21}^{ji} & x_{22}^{ji} & \ldots & \ldots & x_{2k_i}^{ji} \\ \vdots & \vdots & \ddots & \ldots & \vdots \\ \vdots & \vdots & \ldots & \ddots & \vdots \\ x_{k_i1}^{ji} & x_{k_i2}^{ji} & \ldots & \ldots & x_{k_ik_i}^{ji} \end{bmatrix}$$

and for $i \geq 2|Z(D_m)| + 1$. 

Note 4.1. \( X \in \Xi_G \) is an invariant bilinear form under \( \rho \) if and only if for every \( i, 1 \leq i \leq r, X_{d_i, k_i}^{ii} \) is an invariant bilinear form under \( k_i \rho_i \).

4.1. Characterization of invariant bilinear forms under an \( n \) degree representation of a dihedral group of order \( 2m \), with \( m \geq 3 \).

Lemma 4.1. If \( X \in \Xi_G \), and for \( 1 \leq i \leq r, X_{d_i, k_i}^{ii} \) is a non-singular sub-matrix, then \( X \) must be non-singular.

Proof. With reference to the above remark, for every \( X \in \Xi_G \), we have \( X = \text{Diag}[X_{d_1, k_1}^{11}, X_{d_2, k_2}^{22}, \ldots, X_{d_r, k_r}^{rr}] \) with \( X_{d_i, k_i}^{ii} = C_{k_i \rho_i (g)}^{l_i} X_{d_i, k_i}^{ii} C_{k_i \rho_i (g)} \) and for \( 1 \leq i \leq r, X_{d_i, k_i}^{ii} \) is a non-singular sub-matrix, so there exists \( Y = \text{Diag}[(X_{d_1, k_1}^{11})^{-1}, (X_{d_2, k_2}^{22})^{-1}, \ldots, (X_{d_i, k_i}^{ii})^{-1}, \ldots, (X_{d_r, k_r}^{rr})^{-1}] \) in \( M_n(\mathbb{C}) \) such that \( XY = I_n = YX \). Thus the result follows.

To prove the next lemma we will choose only those \( X \in M_n(\mathbb{C}) \) whose \((i, j)^{th}\) block matrix is zero for \( i \neq j \) and for the \((i, i)^{th}\) block-diagonal matrix \( X_{d_i, k_i}^{ii} \) is non-singular.

Lemma 4.2. For \( n \in \mathbb{Z}^+ \), every \( n \)-degree representation of a dihedral group \( G \), has a non-degenerate invariant bilinear form.

Proof. From equation (2) we have \( d_1 k_1 + d_2 k_2 + \ldots + d_r k_r = n \) and \( X \in M_n(\mathbb{C}) \) such that \( X = \text{Diag}[X_{d_1, k_1}^{11}, X_{d_2, k_2}^{22}, \ldots, X_{d_r, k_r}^{rr}] \). If for every \( i, 1 \leq i \leq r \), the block diagonal sub-matrix \( X_{d_i, k_i}^{ii} \) of \( X \) is chosen (from the above remark [4.1]) to be non-singular, then \( X_{d_i, k_i}^{ii} = C_{k_i \rho_i (g)}^{l_i} X_{d_i, k_i}^{ii} C_{k_i \rho_i (g)} \), \( \forall g \in G \). Therefore \( X \in \Xi_G \) and is non-singular.

Lemma 4.3. Every irreducible representation of a dihedral group consists of a non-degenerate invariant bilinear form.

Proof. Follows from the proof of the lemma 4.2.

Remark 4.2. Since \( \mathbb{C} \) contains infinitely many non-zero elements, hence if there is one non-degenerate invariant bilinear form in the space \( \Xi_G \), it has infinitely many.

Thus from Lemma [4.2] we find that every \( n \) degree representation of a dihedral group \( G \) of order \( 2m \), with \( m \geq 3 \), consists of a non-degenerate invariant bilinear form.

Lemma 4.4. Let \( G \) be a dihedral group of order \( 2m \) and \( \rho = \bigoplus_{i=1}^{r} k_i \rho_i \) an \( n \) degree representation of \( G \), then \( \rho \) has a degenerate invariant bilinear form iff at least one block-diagonal matrix is singular.

Proof. Easy to see.
Definition 4.1. The space \( \Xi_G \) of invariant bilinear forms is called degenerate if it’s all elements are degenerate.

We will discuss about the degenerate invariant space in the later section.

5. Dimensions of spaces of invariant bilinear forms under representations of a dihedral group of order \( 2m \).

The space of invariant bilinear forms under an \( n \) degree representation is generated by finitely many vectors so its dimension is finite along with its symmetric subspace and the skew-symmetric subspace. In this section we calculate the dimension of the space of invariant bilinear forms under a representation over \( \mathbb{C} \) of a dihedral group \( D_m \) of order \( 2m \), with \( m \geq 3 \).

Theorem 5.1. If \( \Xi_G \) is the space of invariant bilinear forms under an \( n \) degree representation \( \rho = \oplus_{i=1}^{r} k_i \rho_i \) of \( D_m \), then \( \dim(\Xi_G) = \sum_{i=1}^{2|Z(D_m)|+\lceil \frac{m-1}{2} \rceil} k_i^2 \).

Proof. For every \( X \in \Xi_G \), we have \( X = \text{Diag}[X_{d_i,k_1}, X_{d_2,k_2}, \ldots, X_{d_i,k_1}, \ldots, X_{d_r,k_r}] \) with \( X_{d_i,k_i} = C_{k_i,\rho_i(g)}^{d_i,k_i}C_{k_i,\rho_i(g)}^{d_i,k_i} \), for \( 1 \leq i \leq r \) and to generate these sub-matrices it needs \( k_i^2 \) vectors from \( \Xi_G \). Thus the result follows. \( \square \)

Corollary 5.1. The space of invariant skew-symmetric bilinear forms under an \( n \) degree representation \( \rho = \oplus_{i=1}^{r} k_i \rho_i \) of \( D_m \) has dimension \( \sum_{i=1}^{2|Z(D_m)|+\lceil \frac{m-1}{2} \rceil} k_i(k_i+1)/2 \).

Proof. Follows from the proof of theorem 5.1 \( \square \)

Corollary 5.2. The space of invariant skew-symmetric bilinear forms under an \( n \) degree representation \( \rho = \oplus_{i=1}^{r} k_i \rho_i \) of \( D_m \) has dimension \( \sum_{i=1}^{2|Z(D_m)|+\lceil \frac{m-1}{2} \rceil} k_i(k_i-1)/2 \).

Proof. Follows from the proof of theorem 5.1 \( \square \)

6. Main results

Here we present the proofs of the main theorems stated in the Introduction section.

Proof of theorem 5.1 Given \( G \) is the dihedral group of order \( 2m, m \geq 3 \) (so \( r = \lceil \frac{m-1}{2} \rceil + 2|Z(G)| \)) and degree of the representation \( \rho \) is \( n \). Also \( d_i = 1 \) for \( 1 \leq i \leq 2|Z(G)| \) \& \( d_i = 2 \) for \( 2|Z(G)| + 1 \leq i \leq \lceil \frac{m-1}{2} \rceil + 2|Z(G)| \).

Now from equation (2), we have

\[
k_1 + \ldots + k_{2|Z(G)|} + 2k_{2|Z(G)|+1} + \ldots + 2k_{\lceil \frac{m-1}{2} \rceil + 2|Z(G)|} = n.
\]

i.e.,

\[
k_1 + \ldots + k_{2|Z(G)|} = n - 2(k_{2|Z(G)|+1} + \ldots + k_{\lceil \frac{m-1}{2} \rceil + 2|Z(G)|}).
\]

As \( (k_{2|Z(G)|+1} + \ldots + k_{\lceil \frac{m-1}{2} \rceil + 2|Z(G)|}) \leq \lceil \frac{m}{2} \rceil \), we have \( \lceil \frac{m}{2} \rceil \) equations stated below

\[
k_{2|Z(G)|+1} + \ldots + k_{\lceil \frac{m-1}{2} \rceil + 2|Z(G)|} = 0.
\]
Thus the number of all distinct \( 2 |Z(G)| + 1 \) + \( \cdots + k_{[\frac{m-1}{2}]+2|Z(G)|} \) = 1. \( \quad \) (4)

\[ k_{2|Z(G)|+1} + \cdots + k_{[\frac{m-1}{2}]+2|Z(G)|} = 2. \] \( \quad \) (5)

\[ \sum_{s=0}^{[\frac{n}{2}]} \binom{s+\left[\frac{m-3}{2}\right]}{\left[\frac{m-3}{2}\right]} \left( n-2s+2|Z(G)|-1 \right) \]

Thus the number of all distinct \( 2 |Z(G)| + 1 \) + \( \cdots + k_{[\frac{m-1}{2}]+2|Z(G)|} \) is

\[ \sum_{s=0}^{[\frac{n}{2}]} \binom{s+\left[\frac{m-3}{2}\right]}{\left[\frac{m-3}{2}\right]} \left( n-2s+2|Z(G)|-1 \right). \]

Thus from equation (2) and Theorem 2.2 the number of \( n \) degree representations (up to isomorphism) of the dihedral group \( D_m, m \geq 3 \) is \( \sum_{s=0}^{[\frac{n}{2}]} \binom{s+\left[\frac{m-3}{2}\right]}{\left[\frac{m-3}{2}\right]} \left( n-2s+2|Z(G)|-1 \right). \) \( \square \)

**Proof of theorem 1.2** Let \( X \) be an element of \( \Xi'_G \) then we have

\[ C^t_{\rho(g)} X C_{\rho(g)} = X \text{ and } X = \text{Diag}[X_{d_1 k_1}^{11}, X_{d_2 k_2}^{22}, \ldots, X_{d_r k_r}^{rr}, \ldots]. \]

Existence:

Let \( X \in \Xi'_G \) then for \( 1 \leq i \leq r \), there exists at least one \( X_i = \text{Diag}[O_{d_1 k_1}^{11}, O_{d_2 k_2}^{22}, \ldots, O_{d_r k_r}^{rr}] \in \mathbb{W}(G, k_i, \rho_i) \), such that \( \sum_{i=1}^{r} X_i = X \).

Uniqueness:

For \( 1 \leq i \leq r \), suppose there exists \( Y_i \in \mathbb{W}(G, k_i, \rho_i) \), such that \( \sum_{i=1}^{r} Y_i = X \), then \( \sum_{i=1}^{r} X_i = \sum_{i=1}^{r} Y_i \) i.e., \( Y_j - X_j = \sum_{i=1, i \neq j}^{r} (X_i - Y_i) \). Therefore \( Y_j - X_j \in \mathbb{W}(G, k_i, \rho_i) \) hence \( Y_j - X_j = 0 \) or \( Y_j = X_j \) for all \( j \).

Thus we have

\[ \Xi'_G = \oplus_{i=1}^{r} \mathbb{W}(G, k_i, \rho_i) \text{ and } \dim(\Xi'_G) = \sum_{i=1}^{r} \dim(\mathbb{W}(G, k_i, \rho_i)). \] \( \quad \) (7)

Now as \( \mathbb{W}(G, k_i, \rho_i) = \{ X \in M_n(\mathbb{C}) \mid X = \text{Diag}[O_{d_1 k_1}^{11}, \ldots, X_{d_i k_i}^{ii}, \ldots, O_{d_r k_r}^{rr}] \} \) with \( X_{d_i k_i}^{ii} \) a square sub-matrix of order \( d_i k_i \) satisfies \( X_{d_i k_i}^{ii} = C^t_{\rho_i(g)} X_{d_i k_i}^{ii} C_{\rho_i(g)} \), \( \forall g \in G \), from the remark 4.1 we see that for \( 1 \leq i \leq 2|Z(G)| + [\frac{m-3}{2}] \), the sub-matrices \( X_{d_i k_i}^{ii} \) in \( \mathbb{W}(G, k_i, \rho_i) \) have \( k_i^2 \) free variables \& \( \mathbb{W}(G, k_i, \rho_i) \cong M_{k_i}(\mathbb{C}) \). Thus

\[ \Xi'_G \cong \oplus_{i=1}^{r} M_{k_i}(\mathbb{C}) \text{ and } \dim(\mathbb{W}(G, k_i, \rho_i)) = k_i^2. \]

Thus substituting this in equation (7) we get the dimension of \( \Xi'_G \).

**Proof of theorem 1.3** Follows immediately from Lemmas 3.1 to 3.2. \( \square \)

**Proof of theorem 1.4** Follows immediately from the corollaries 3.1 and 3.4.
6.1. Degenerate invariant spaces

From Theorem 1.1 and Lemma 4.2, for every \( n \in \mathbb{Z}^+ \), an \( n \) degree representation of \( D_m \) has a non-degenerate invariant bilinear forms. If the block-diagonal matrix \( X_{d_i,k_i} \) of \( X \in \mathbb{Z}_t^2 \) is singular at least for one \( i \) then the invariant bilinear form \( X \) is singular.

Thus here we have completely characterised the representations of a dihedral group of order \( 2m, m \geq 3 \) to admit a non-degenerate invariant bilinear form over complex field. The authors hope to return in future to the same work over an arbitrary field and for some different finite groups.

**Funding:** Not applicable.

**Conflicts of interest/Competing interests:** Not applicable.

**Availability of data and material:** The manuscript has no associated data.

**Code availability:** Not applicable.

**Acknowledgement** The first author would like to thank UGC, India for providing the research fellowship and the Central University of Jharkhand, India for support to carry out this research work. The second author is thankful to the Babasaheb Bhimrao Ambedkar university for providing excellent environment to finalize this work.

**References**

1. Artin M, Algebra, First Edition, Prentice Hall Inc. (1991) pp. 307 – 344.

2. Chen Y, Matrix representations of the real numbers, Linear Algebra Appl. **536** (2018) 174 – 185.

3. Conrad K, Dihedral groups, https://kconrad.math.uconn.edu/blurbs/grouptheory/dihedral.pdf.

4. Frobenius G, Uber die mit einer Matrix vertauschbaren matrizen, Sitzungsber. Preuss. Akad. Wiss. (1910) 3 – 15.

5. Gongopadhyay K and Kulkarni R S, On the existence of an invariant non-degenerate bilinear form under a linear map, Linear Algebra Appl., **434**(1) (2011) 89 – 103.

6. Gow R and Laffey T J, Pairs of alternating forms and products of two skew-symmetric matrices, Linear Algebra Appl. **63** (1984) 119 – 132.

7. Gongopadhyay K, Mazumder S and Sardar S K, Conjugate Real Classes in General Linear Groups, Journal of Algebra and Its Applications, **18**(3), https://doi.org/10.1142/S0219498819500543 (2019).

8. Hoffman K and Kunze R, Linear Algebra, First Edition, Prentice Hall Inc (1961).

9. Kulkarni R S and Tanti J, Space of invariant bilinear forms, Proc. Math. Sc. Ind. Acad. Sc., **128**(4) (2018) 47.
[10] Mahto D and Tanti J, Space of Invariant bilinear forms under representation of group of order 8, arXive, https://arxiv.org/abs/1910.09850.

[11] Mahto D and Tanti J, Invariant bilinear forms under the operator group of order $p^3$ with prime $p$, arXive, https://arxiv.org/pdf/2006.05307.

[12] Pazzis C H, When does a linear map belong to at least one orthogonal or symplectic group? Linear Algebra Appl. 436(5) (2012) 1385 – 1405.

[13] Stenzel H, Uber die Darstellbarkeit einer Matrix als Produkt von zwei symmetrischer matrizen, als Produkt von zwei alternierenden matrizen und als Produkt von einer symmetrischen und alternierenden matrix, Mat. Zeitschrift 15 (1922) 1 – 25.

[14] Serre J P, Linear representations of finite groups, Springer-Verlag (1977).

[15] Sergeichuk V V, Classification problems for systems of forms and linear map, Izv. Akad. Nauk SSSR Ser. Mat. 51(6) (1987) 1170 – 1190.