Evolution Kernels for Light-Ray Operators:
Twist 2 and Twist 3 Contributions

Johannes Blümlein*, Bodo Geyer† and Dieter Robaschik*

*DESY – Zeuthen, Platanenallee 6, D – 15735 Zeuthen, Germany
† Naturwissenschaftlich-Theoretisches Zentrum der Universität Leipzig,
Augustusplatz 10, D–04109 Leipzig, Germany

Abstract. The general evolution kernels of the twist 2 light-ray operators for unpolarized and polarized deep inelastic scattering are calculated in $O(\alpha_s)$. From these evolution kernels a series of special evolution equations can be derived, among them the Altarelli-Parisi equations and the evolution equation for the meson wave function. In the case of twist 3 the results of Balitzki and Braun are confirmed.

INTRODUCTION

The study of the Compton amplitude for scattering of a virtual photon off a hadron is one of the basic tools in QCD to understand the short-distance behavior of the theory. The Compton amplitude for the general case of non-forward scattering is given by

$$T_{\mu\nu}(p_+, p_-, Q) = i \int d^4 x e^{i Q x} \langle p_2 | T(J_{\mu}(x/2) J_{\nu}(-x/2)) | p_1 \rangle,$$  \hspace{1cm} (1)

where $p_+ = p_2 + p_1$, $p_- = p_2 - p_1 = q_1 - q_2$ and $q = (q_1 + q_2)/2$. The time-ordered product in eq. (1) can be represented in terms of the operator product expansion. Here we use the representation derived in ref. [1,2]

$$T(J_{\mu}(x/2) J_{\nu}(-x/2)) \approx \int_{-\infty}^{+\infty} d\kappa_+ \int_{-\infty}^{+\infty} d\kappa_- [C_a(x^2, \kappa_i, \mu^2) S_{\mu\nu}^{\rho\sigma} \ddot{x}_\rho O_\sigma^{a}(\kappa_i, \ddot{x}, \mu^2) + C_{a,5}(x^2, \kappa_i, \mu^2) \varepsilon_{\mu\nu}^{\rho\sigma} \ddot{x}_\rho O_{5,\sigma}^{a}(\kappa_i, \ddot{x}, \mu^2)],$$ \hspace{1cm} (2)

with $S_{\mu\nu}^{\rho\sigma} = g_{\mu\rho}g_{\nu\sigma} - g_{\mu\sigma}g_{\nu\rho} - g_{\mu\nu}g_{\rho\sigma}$ and $\varepsilon_{\mu\nu}^{\rho\sigma}$ denoting the Levi-Civita symbol. The light-like vector $\ddot{x} = x + r(x.r/r.r) \left[ \sqrt{1 - x.x.r/(x.r)^2} - 1 \right]$ is
related to $x$ and a subsidiary four–vector $r$, $C_a$ and $C_{a5}$ denote the respective coefficient functions. Different kinematic situations in the Bjorken region are described by different matrix–elements of the involved light–ray operators $O^a_\sigma(\kappa_+ \tilde{x}, \kappa_- \tilde{x}, \mu^2)$ and $O^a_{5, \sigma}(\kappa_+ \tilde{x}, \kappa_- \tilde{x}, \mu^2)$. Let us illustrate this with the help of the non–singlet quark operator, given in the axial gauge $\bar{x}_\mu A^\mu(x) = 0$ by

$$O^\text{NS}(\kappa_1, \kappa_2) = i \left[ \psi_a(\kappa_1 \bar{x}) \lambda_f \gamma_\mu \bar{x}^\mu \psi_a(\kappa_2 \bar{x}) \right].$$

(3)

For deep inelastic forward scattering, $p_1 = p_2 = p$, the parton distribution reads

$$q(z, \mu^2) = \int_{-\infty}^{+\infty} \frac{d(\bar{x} p)}{2\pi \bar{x} p} e^{2i\kappa \bar{x} p t} \langle p | \bar{\psi}_a(-\kappa \bar{x}) \lambda_f \gamma \bar{x} \psi_a(+\kappa \bar{x}) | p \rangle$$

(4)

and obeys the Altarelli–Parisi equation. The meson wave function, on the other hand, has the representation

$$\Phi(t, \mu^2) = \int_{-\infty}^{+\infty} \frac{d(\bar{x} p)}{2\pi \bar{x} p} e^{i\kappa \bar{x} p t} \langle 0 | \bar{\psi}_a(-\kappa \bar{x}) \lambda_f \gamma \bar{x} \psi_a(+\kappa \bar{x}) | p \rangle$$

(5)

and satisfies the Brodsky–Lepage equation. For general non–forward processes the distribution function is defined [2] by

$$F(z_+, z_-, \mu^2) = \int_{-\infty}^{+\infty} \frac{d(\bar{x} p_+)}{2\pi \bar{x} p_+} \frac{d(\bar{x} p_-)}{2\pi \bar{x} p_-} e^{i\kappa \bar{x} p_+ z_+ + i\kappa \bar{x} p_- z_-} \times \langle p_2 | \bar{\psi}_a(-\kappa \bar{x}) \lambda_f \gamma \bar{x} \psi_a(+\kappa \bar{x}) | p_1 \rangle.$$  

(6)

The corresponding evolution equations are given by eq. (25), see also [6]. If we restrict the general non–forward process by the condition $\tau = \bar{x} p_- / \bar{x} p_+$, $|\tau| \leq 1$, then a one-variable distribution function [2]

$$q(t, \tau, \mu^2) = \int_{-\infty}^{+\infty} \frac{d(\bar{x} p_+ \kappa)}{2\pi \bar{x} p_+} e^{i\kappa \bar{x} p_+ t} \langle p_2 | \bar{\psi}_a(-\kappa \bar{x}) \lambda_f \gamma \bar{x} \psi_a(+\kappa \bar{x}) | p_1 \rangle |_{\bar{x} p_- = \tau \bar{x} p_+}$$

$$= \int_{-\infty}^{+\infty} dz_+ F(t - \tau z_-, z_-).$$

(7)

is obtained. Note that the partition functions $q(z, \mu^2)$ and $\Phi(t, \mu^2)$ can be obtained as limits $\tau \to 0$, $\tau = \pm 1$, respectively, see ref. [3].

**THE EVOLUTION KERNELS**

A consequence of the relation of different distribution functions to a single operator is, that the evolution kernel, i.e. the anomalous dimension, of this

1) For brevity we put $\kappa_1 = -\kappa_-, \kappa_2 = \kappa_- \equiv \kappa$ in eqs. (4–7).
operator allows the derivation of the corresponding kernels which emerge in a variety of different processes. On the other hand, it underlines the importance to determine the general kernels for all relevant operators themselves. In the following we consider the flavor singlet operators only.

\[ O^q(k_1, k_2) = \frac{i}{2} \left[ \overline{\psi}_a(k_1 \bar{x}) \gamma_\mu \bar{x}^\mu \psi_a(k_2 \bar{x}) - \overline{\psi}_a(k_2 \bar{x}) \gamma_\mu \bar{x}^\mu \psi_a(k_1 \bar{x}) \right] \]  

(8)

\[ O^G(k_1, k_2) = \frac{i}{2} \left[ \overline{\psi}_a(k_1 \bar{x}) \gamma_5 \gamma_\mu \bar{x}^\mu \psi_a(k_2 \bar{x}) + \overline{\psi}_a(k_2 \bar{x}) \gamma_5 \gamma_\mu \bar{x}^\mu \psi_a(k_1 \bar{x}) \right] \]  

(9)

\[ O^G(k_1, k_2) = \bar{x}^\mu F_{\mu \nu}^s(k_1 \bar{x}) \bar{x}^\nu F_{\mu \nu}^s(k_2 \bar{x}) \]  

(10)

\[ O^G(k_1, k_2) = \frac{1}{2} \left[ \bar{x}^\mu F_{\mu \nu}^s(k_1 \bar{x}) \bar{x}^\nu F_{\mu \nu}^s(k_2 \bar{x}) - \bar{x}^\nu F_{\mu \nu}^s(k_2 \bar{x}) \bar{x}^\mu F_{\mu \nu}^s(k_1 \bar{x}) \right] \]  

(11)

where \( \psi_a \) denotes the quark and \( F_{\mu \nu}^s \) the gluon field strength operators, respectively. The operator dual to \( F_{\mu \nu}^s \) is \( \bar{F}_{\mu \nu}^s \), and \( k_1 = k_+ - k_-, k_2 = k_+ + k_- \).

The renormalization group equation implies the following evolution equations for these operators:

\[ \mu^2 \frac{d}{d \mu^2} \left( \frac{O^q(k_i)}{O^G(k_i)} \right) = \frac{\alpha_s(\mu^2)}{2\pi} \int_0^1 d\alpha_1 \int_0^{1-\alpha_1} d\alpha_2 K(\alpha_1, \alpha_2) \left( \frac{O^q(k'_i)}{O^G(k'_i)} \right), \]  

(12)

with \( \alpha_s = g_s^2/(4\pi) \) the strong coupling constant, \( \mu \) the renormalization scale, and \( k'_1 = k_1(1 - \alpha_1) + k_2 \alpha_1, k'_2 = k_2(1 - \alpha_2) + k_1 \alpha_2 \). \( K \) denotes the matrix of the singlet evolution kernels in the unpolarized case. The singlet evolution equations for the polarized case are obtained replacing \( O^q,G \) by \( O^G \) and \( K \) by \( \Delta K \). As far as relations are concerned which are valid both for the unpolarized and polarized case under this replacement, we will only give that for the unpolarized case in the following. The matrices of the singlet kernels are

\[ K = \begin{pmatrix} K^{qq} & K^{qG} \\ K^{Gq} & K^{GG} \end{pmatrix} \quad \text{and} \quad \Delta K = \begin{pmatrix} \Delta K^{qq} & \Delta K^{qG} \\ \Delta K^{Gq} & \Delta K^{GG} \end{pmatrix}, \]  

(13)

respectively. In the unpolarized case the kernels are given by

\[ K^{qq}(\alpha_1, \alpha_2) = C_F \left\{ 1 - \delta(\alpha_1) - \delta(\alpha_2) + \delta(\alpha_1) \left[ \frac{1}{\alpha_2} \right]_+ + \delta(\alpha_2) \left[ \frac{1}{\alpha_1} \right]_+ \right\} + \frac{3}{2} \delta(\alpha_1) \delta(\alpha_2) \]  

(14)

\[ K^{qG}(\alpha_1, \alpha_2) = -2N_f T_R K_- \{ 1 - \alpha_1 - \alpha_2 + 4\alpha_1 \alpha_2 \} \]  

(15)

2) The non–singlet operators are discussed in [3].
\[ K^G(\alpha_1, \alpha_2) = -C_F \frac{1}{\kappa_- - i\varepsilon} \{ \delta(\alpha_1)\delta(\alpha_2) + 2 \} \]  

\[ K^G(\alpha_1, \alpha_2) = C_A \left\{ 4(1 - \alpha_1 - \alpha_2) + 12\alpha_1\alpha_2 + \delta(\alpha_1) \left[ \frac{1}{\alpha_2^+} - 2 + \alpha_2 \right] + \delta(\alpha_2) \left[ \frac{1}{\alpha_1^+} - 2 + \alpha_1 \right] \right\} + \frac{\beta_0}{2} \delta(\alpha_1)\delta(\alpha_2), \]  

in \( \mathcal{O}(\alpha_s) \), where \( C_F = (N_c^2 - 1)/2N_c \equiv 4/3, T_R = 1/2, C_A = N_c \equiv 3, \beta_0 = (11C_A - 4T_R N_f)/3, N_f \) being the number of active quark flavors. The +\- prescription is defined as

\[ \int_0^1 \! dx \left[ f(x, y) \right]_+ \varphi(x) = \int_0^1 \! dx \varphi(x) \left[ f(x, y) - \varphi(y) \right], \]  

where the singularity of \( f \) is of the type \( \sim 1/(x - y) \).

The kernels for the polarized case are

\[ \Delta K^{qq}(\alpha_1, \alpha_2) = K^{qq}(\alpha_1, \alpha_2) \]  
\[ \Delta K^{qG}(\alpha_1, \alpha_2) = -2N_f T_R \kappa_- \{ 1 - \alpha_1 - \alpha_2 \} \]  
\[ \Delta K^{Gq}(\alpha_1, \alpha_2) = -C_F \frac{1}{\kappa_- - i\varepsilon} \{ \delta(\alpha_1)\delta(\alpha_2) - 2 \} \]  
\[ \Delta K^{GG}(\alpha_1, \alpha_2) = K^{GG}(\alpha_1, \alpha_2) - 12C_A\alpha_1\alpha_2. \]  

Whereas the kernels for the polarized case have been derived for the first time in ref. [3] recently, those for the unpolarized case were found several years ago already in refs. [4,5]. The kernels \( K^{ij} \) and \( \Delta K^{ij} \) determine the respective evolutions of the operators \( \mathcal{O}_{(5)}^i \) and \( \mathcal{O}_{(5)}^G \) in \( \mathcal{O}(\alpha_s) \).

The evolution kernels, eq. (14–17) and eq. (19–22), determine the evolution of all physically interesting partition functions. As an example we consider the general two–variable partition functions, introduced in the introduction, which are defined by

\[ \langle p_1 | O^q | p_2 \rangle_{(i\bar{x}p_+)^2} = e^{-i\kappa_+ \bar{x}p_+} \int_{-\infty}^{+\infty} dz_+ \int_{-\infty}^{+\infty} dz_- e^{-i\kappa_- (\bar{x}p_+ z_+ + \bar{z}p_- z_-)} F_q(z_+, z_-), \]  
\[ \langle p_1 | O^G | p_2 \rangle_{(i\bar{x}p_+)^2} = e^{-i\kappa_+ \bar{x}p_+} \int_{-\infty}^{+\infty} dz_+ \int_{-\infty}^{+\infty} dz_- e^{-i\kappa_- (\bar{x}p_+ z_+ + \bar{z}p_- z_-)} F_G(z_+, z_-). \]  

The singlet evolution equations for these functions read

\[ \mu^2 \frac{d}{d\mu^2} \left( \frac{F^q(z_+, z_-)}{F^G(z_+, z_-)} \right) = \frac{\alpha_s(\mu^2)}{2\pi} \int_{-\infty}^{+\infty} \frac{dz_+'}{|z_+'|} \int_{-\infty}^{+\infty} dz_-' \widetilde{K}(z_+, z_-, z_+'; z_+', z_-') \left( \frac{F^q(z_+', z_-')}{F^G(z_+', z_-')} \right), \]
where
\[
\bar{K}^{ij}(\alpha_1, \alpha_2) = \frac{1}{2} \int_0^1 dz''_+ \bar{O}^{ij}(z_+, z''_+) \theta(1 - \alpha_+) \theta(\alpha'_+ + \alpha'_-) \theta(\alpha'_+ - \alpha'_-) K^{ij}(\alpha'_1, \alpha'_2),
\]
with \(\alpha'_\rho = \alpha_\rho(z_+ \to z''_+)\), and \(\alpha_1 = (\alpha_+ + \alpha_-)/2\), \(\alpha_2 = (\alpha_+ - \alpha_-)/2\),
\(\alpha_+ = 1 - z_+/z'_+, \quad \alpha_- = (z_+ z'_- - z_- z'_+)/z'_+\), and
\[
\bar{O}^{ij}(z_+, z''_+) = \left( \begin{array}{cc}
\delta(z_+ - z''_+) & \partial_{z_+} \delta(z_+ - z''_+)
\end{array} \right).
\]

The determination of the corresponding Brodsky-Lepage and Efremov-Radyushkin kernels, the extended kernels, as well as the Altarelli-Parisi limiting case and the relation to other investigations [6–8] are discussed in ref. [3].

**TWIST 3**

The consideration of the renormalization properties of the twist 3 operators is much more complicated. The reason is that several operators exist which are related by each other due to the equations of motion. If one introduces the Shuryak–Vainshtein operators and applies the equations of motion consequently, then results by Balitzki and Braun, as well as by Bukhvostov et al. and others are reproduced. These results were already published in ref. [9]. However, this may yet not be the complete solution of the problem and further investigations are needed.

**Acknowledgement.** We would like to thank Paul Söding for his constant support of the project and D. Müller for discussions on the present topic.

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