We relate the ground state degeneracy (GSD) of a non-Abelian topological phase on a surface with boundaries to the anyon condensates that break the topological phase to a trivial phase. Specifically, we propose that gapped boundary conditions of the surface are in one-to-one correspondence to the sets of condensates, each being able to completely break the phase, and we substantiate this by examples. The GSD resulting from a particular boundary condition coincides with the number of confined topological sectors due to the corresponding condensation. These lead to a generalization of the Laughlin-Wu-Tao (LWT) charge-pumping argument for Abelian fractional quantum Hall states (FQHS) to encompass non-Abelian topological phases, in the sense that an anyon loop of a confined anyon winding a non-trivial cycle can pump a condensate from one boundary to another. Such generalized pumping may find applications in quantum control of anyons, eventually realizing topological quantum computation.

PACS numbers: 11.15.-q, 71.10.-w, 05.30.Pr, 71.10.Hf, 02.10.Kn, 02.20.Uw

INTRODUCTION

A key feature of intrinsic topological orders is the existence of protected GSD. For phases placed on closed spatial 2-surfaces, this has been thoroughly understood\cite{1-4}. It is well known that the genus number of the closed surface and the fusion rules between anyon excitations determine the GSD\cite{5}. Protected ground states are vital to topological quantum computation, when they are manipulated appropriately, e.g. by the braiding of anyons. Realizing GSD on high genus closed surfaces is, however, not generally achievable in experiments. Quite the contrary, it is much more natural to build finite open systems. To have a well defined notion of GSD in an open system, it is necessary that any boundary massless modes often occurring in topological orders can be gapped. The gapping conditions of Abelian phases have recently been understood in terms of the concept of Lagrangian subsets\cite{6-9}, and subsequently the GSDs of these Abelian phases on open surfaces with multiple boundaries were computed\cite{8, 10}, based on the idea of allowed anyon transport across different boundaries. Nevertheless, non-Abelian phases bear much richer sets of degenerate ground states, and the braiding of non-Abelian anyons serves as the best known candidate that may realize universal topological quantum computing; therefore, to extend these previous studies to the case of non-Abelian topological order is urgent. This paper generalizes the two key ingredients – Lagrangian subsets and admissible charge transport across boundaries – to generic non-Abelian phases, leading to a count of their GSD.

Summarizing our main results, we

• clarify and generalize the concept of the Lagrangian subset to cover the case of non-Abelian phases, explaining the relation between gapping conditions at the boundaries of topological orders and anyon condensation;

• for given boundary conditions, find the GSD, dictated by confined anyons. Typically, on a cylinder,

\[
\text{GSD} = \#\text{confined anyons};
\]

• show that the above counting is equivalent to but more natural than a generalization of prior works that uses the idea of charge transport. This involves generalizing the LWT charge-pump argument in Abelian FQHS\cite{11, 12} to generic anyon-pump in non-Abelian phases;

• demonstrate the first example of a GSD counting of a non-Abelian topological phase on a cylinder whose boundaries satisfy different boundary conditions. This can be readily generalized to more general surfaces with more boundaries and a richer variety of boundary conditions.

We note that we will make heavy use of the technologies studying anyon condensation, which have been developed in a series of works in \cite{13-15}. We shall not review them here but refer the readers to the original references.

GSD OF THE $\mathbb{Z}_2$ TORIC CODE ON OPEN SURFACES REVISITED

To explain the generalization of Lagrangian subsets and the subsequent GSD counting in Abelian phases, we revisit these concepts in the perspective of anyon condensation by taking the $\mathbb{Z}_2$ toric code as an example. Recall that the $\mathbb{Z}_2$ toric code has four topological sectors: \{1, e, m, f\}, where e, m are self-bosons, and f a
fermion. The mutual statistics between any two distinct nontrivial anyons is fermionic. The fusion rules are $x \times x = 1, 1 \times x = x,$ and $x_1 \times x_2 = x_3,$ where $x \in \{1, e, m, f\},$ and $x_1, x_2, x_3 \in \{e, m, f\}$ are all distinct. On each boundary of the system, there exists two different gapped boundary conditions, respectively characterized by the two Lagrangian subsets:

$$L_e = \{1, e\}, \quad L_m = \{1, m\}.$$  \hspace{1cm} (2)

A Lagrangian subset is a maximal collection of anyons that are self-bosons sharing trivial mutual statistics and excludes any anyon with non-trivial mutual statistics with at least one of its elements, which is clearly satisfied by both sets in $L_e$. One crucial observation is that for the boundary condition characterized by a set $L_i$, an anyon in the set ceases to be conserved and can disappear at the boundary, or be created there. Therefore, a gapped boundary condition can be understood as the condensation of a maximal set of mutually local anyons right at the boundary. Any non trivial anyon not in a condensed $L_i$ would be confined at the boundary. For example, $m$ and $f$ are confined in the $L_e$ condensate and thus are mobile in the bulk but fails to cross the boundary into the vacuum. Equally importantly, in the vicinity of the $L_e$ condensate, $m$ and $f$ are indistinguishable, much like 1 and $e$ become identified, by fusion with arbitrary number of $e$’s freely supplied by the boundary condensate. This leads to an easy counting of the GSD. Consider for simplicity a cylinder with both boundaries characterized by $L_e$. A convenient basis for ground states can be constructed by uncontractible anyon loops wrapping the cylinder. There are only two distinct possibilities for such anyon loops for the $L_e$ boundary: 1 and $m$. One then infers that

$$GSD_{L_e \text{ boundaries on cylinder}}^{Z_{2}\text{toric code}} = 2,$$  \hspace{1cm} (3)

in perfect agreement with the result of 9. This is illustrated in Fig. 1. The LWT charge-pump argument for FQHS applies here. That is, if one threads a magnetic flux loop around the cylinder and adiabatically increase it from zero to a unit flux, a charge $e$ can be pumped from one boundary to the other of the cylinder, as depicted in Fig. 1.

To deal with more boundaries potentially characterized by different $L_i$’s, we need only to work out the remaining conserved (i.e. confined) anyon lines that can wind non-trivial cycles, and that if multiple cycles merge to form a larger cycle, the anyon lines winding the merging cycles should fuse to the anyon line winding the larger cycle. The next example illustrates how to generalize the procedure described above to non-Abelian phases.

**EXAMPLE I: DOUBLED FIBONACCI**

We consider the doubled Fibonacci system, $\text{Fibo} \times \text{Fibo}$, which has four anyons $\{1, \tau \bar{\tau}, 1\bar{\tau}, \tau \bar{1}\}$ of quantum dimensions $d_1 = 1, d_{\tau} = 1 + d_\tau,$ and $d_{\bar{\tau}} = d_{\tau \bar{1}} = d_\tau$, with $d_\tau = \frac{1+\sqrt{5}}{2}$. Their self statistics are $\theta_1 = 1 - \theta_{\tau \bar{\tau}} = 1, \theta_{\tau \bar{1}} = 1 + \theta_{\bar{\tau} \tau} = 2\pi/10$. Since $\theta_{\tau \bar{1}} = 1, \tau \bar{\tau}$ is the only nontrivial anyon that may condense. Thus, the only admissible non-Abelian Lagrangian subset would be $L = \{1, \tau \bar{\tau}\}$. Moreover, because $d_{\tau \bar{1}} > 1, \tau \bar{\tau}$ must split in order to condense $\{1, \tau \bar{\tau}\}$: $\tau \bar{\tau} = 1 + \chi$, where the trivial part 1 is the actual condensate, and $\chi$ with $d_\chi = d_{\tau \bar{1}}$ is some topological sector in the phase $T_L$ after the condensation of $L$. It is easy to check that in $T_L$ all nontrivial anyons are confined and that $\chi, 1\bar{\tau},$ and $\tau \bar{1}$ are indistinguishable. Namely, $T_L = \{1, \chi\}$ with $\chi \times \chi = 1 + \chi$, behaving just like the Fibonacci anyon $\tau$.

Since $\text{Fibo} \times \text{Fibo}$ has a unique Lagrangian subset $L = \{1, \tau \bar{\tau}\}$, on a cylinder, both boundaries must be characterized by $L$. As a result, the only conserved nontrivial topological sector must be the confined $\chi \in T_L$, as it cannot leak through the boundaries into the vacuum. We conclude that the 2 distinct sectors in $T_L$ implies that

$$GSD_{\text{Fibo} \times \text{Fibo}}^{\text{cylinder}} = |T_L| = 2,$$  \hspace{1cm} (4)

in accord with the result obtained by counting the fusion channels between condensed anyons on the two boundaries to the trivial sector . Specifically, since there is only one trivial fusion channel in $\tau \bar{\tau} \times \tau \bar{\tau} = 1 + 1\bar{\tau} + \tau \bar{1} + \tau \bar{1}$, together with the obvious trivial fusion channel $1 \times 1 = 1,$ there are 2 trivial fusion channels all told.

The above agreement between the two different ways of counting the GSD shouts for a generalization of the aforementioned LWT argument for FQHS to the case of non-Abelian topological phases, in the sense that an adiabatically changing Wilson, or anyon loop of $\chi$ around the cylinder should be able to pump a unit of the condensed $\tau \bar{\tau}$ from one boundary to the other (Fig. 2(a)). To check our claims, note that it is expected that

$$GSD_{\text{Fibo} \times \text{Fibo}}^{\text{cylinder}} = GSD_{\text{torus}}^{\text{Fibo}}.$$  \hspace{1cm} (5)
This can be understood as follows. First, creating an anyon on, say, the right boundary, and annihilating it on the left boundary is equivalent to creating the anti (time reversed) of the anyon on the left boundary and annihilating it on the right. Second, Fibo $\times \overline{\text{Fibo}}$ ending at a boundary can be thought of as folding up a Fibonacci phase characterized by single copy of the anyons $\{1, \tau\}$. Thus the Fibo $\times \overline{\text{Fibo}}$ on a cylinder is in fact equivalent to the Fibonacci phase itself residing on two different cylinders, yet joined at both boundaries because of the gapped boundary condition we imposed on Fibo $\times \overline{\text{Fibo}}$. That is, we have in fact Fibo on a torus. Identity \[ \text{(5)} \] can be extended to encompass multiple boundaries and higher genus surfaces. We note also that such a correspondence between a “doubled phase” on a surface with gapped boundaries characterized by condensates of all the diagonal pair – the analogues of $\tau\overline{\tau}$ – and the undoubled phase on a closed surface is in fact generic. This correspondence offers a non-trivial check of our methods and our generalization of the LWT charge-pump in large classes of non-Abelian phases expressible as a double phase using well-known results of GSD of phases on closed surfaces, and by which we find perfect agreement.

![FIG. 2. (a) Fibo $\times \overline{\text{Fibo}}$ on a cylinder with both boundaries characterized by the Lagrangian subset $\{1, \tau\}$. (b) Single Fibonacci phase on a torus. The two systems are equivalent.](image)

Now consider placing the Fibo $\times \overline{\text{Fibo}}$ on a surface with three holes, or the “pants diagram” (a special case of Fig. \[ \text{3(a)} \].) The three boundaries must all be characterized by the unique Lagrangian subset $L$. On this surface, there are three nontrivial cycles (holes) around which the anyon loops of 1 or $\chi$ can wind. When any anyon loops respectively around two of the three cycles merge to an anyon loop around the third cycle, it is necessary that the three loops of anyons admit a fusion channel. The GSD counting in this scenario then boils down to the formula:

\[
GSD^\text{pants}_{\text{Fibo} \times \overline{\text{Fibo}}} = \sum_{a,b,c \in \{1, \chi\}} N^a_{bc} = 5, \quad \text{(6)}
\]

where $a$, $b$, and $c$ respectively wraps the three holes, and $N^a_{bc}$ are known as the fusion matrices; for any given $a$, the matrix element $N^a_{bc}$ is the multiplicity of $a$ in the fusion product of $b$ and $c$, which is in complete agreement with the expected result following from

\[
GSD^\text{pants}_{\text{Fibo} \times \overline{\text{Fibo}}} = GSD^\text{genus-2 torus}_{\text{Fibo}}. \quad \text{(7)}
\]

**GENERAL PRINCIPLES**

We now lay down the general two-step procedure, as illustrated by the above example, to obtain the GSD of a generic non-Abelian topological order with boundaries.

**Step 1. Defining boundary conditions**

First we have to decide upon the boundary condition on each boundary. Each boundary whose edge modes could be completely gapped is characterized by a generalized Lagrangian subset $L$, which is a collection of anyons that could condense simultaneously at the boundary, and that the resultant phase after the condensation $T_L$ contains only confined anyons as well as the trivial sector.

**Step 2a. Counting GSD via confined charges**

If all the boundaries are characterized by the same $L$, the GSD is obtained by finding out the fusion rules of all the confined anyons in $T_L$, and count all possible basis states constructed from closed confined anyon lines winding nontrivial cycles, subjected to the consistency condition that anyons wrapping cycles that merge have to fuse to the anyon wrapping the resultant merged cycle. This is to ensure that there is no net charge in the bulk. The boundaries could also be characterized by different $L_i$’s, which would have anyons not mutually local. One would have to work out the condensed phase $T_{L_i}$ at each boundary, and then further reduce the number of conserved anyons, which correspond roughly to finding an intersection of the $T_{L_i}$’s. The fusion between the remaining conserved anyons again determine the GSD. There is not to date a fully systematic procedure dealing with multiple sets of non-mutually local condensates, but we will exemplify how this is to work by a non-trivial example in the next section. In summary (Fig. \[ \text{3(b)} \]), assuming that there are $M > 3$ boundaries, respectively characterized by (potentially identical) Lagrangian subsets $L_i$’s,

\[
GSD\{L_i\} = \sum_{\{a_i, b_i\}} N^a_{a_1 a_2} \prod_{i=3}^{M-1} N^a_{a_i b_i-2} N^a_{a_i b_i M-2}, \quad \text{(8)}
\]

where $\{L_i\} := \cap_{i=1}^{M} L_i$, $b_i$’s are the intermediate fusion channels, and $\{a_i, b_i\}$ refers to summing over $\{L_i\}$.

**Step 2b. Counting GSD via charge transport**

As emphasized in the examples above, we can alternatively count the GSD by considering charge transport across boundaries. This corresponds to counting
the number of fusion channels to the trivial sector, be-
included data about the multiplicity of a condensed anyon—the number of condensed sectors contained in each anyon in $L_1$ that splits under condensation—to allow for a correct GSD counting and restore the correspondence between a charge and its flux pump in a generalized version of the Laughlin-Wu-Tao thought experiment. For this extra twist in the story, a counting of GSD via confined anyons is more natural.

EXAMPLE II: $\mathbb{Z}_2^3$ TWISTED QUANTUM DOUBLE

We would like to present here a fascinating example—the $\mathbb{Z}_2^3$ twisted quantum double (TQD)$^{[19–21]}$—that bears more than one sets of nontrivial gapped boundary conditions. Being a twisted version of the $G = \mathbb{Z}_2^3$ Kitaev model, this model contains 22 distinct anyons. As in other gauge theories, the electric charges are representations of the gauge group $G$, and as such they come in 8 distinct types. We can denote them by $E_{e_1 e_2 e_3}$, where $e_i \in \{0, 1\}$, corresponding to the trivial and nontrivial one dimensional representations of each of the three $\mathbb{Z}_2$ groups in $G$. The rest of the anyons are dyons, denoted by $D_M^\pm$, which are non-Abelian anyons all with quantum dimension 2. Their magnetic charges are labeled by a set of 3 numbers: $M = \{m_1, m_2, m_3 | m_i \in \{0, 1\}, m_1 m_2 m_3 \neq 000\}$. Each given $M$ gives two versions of the dyons, labeled $\pm$. This is summarized in table 1. Their properties, such as fusion rules, will be detailed in the appendix. We find two distinct admissible Lagrangian subsets, $L_E$ and $L_D$. Set $L_E$ contains all the electric charges $L_E = \{E_{e_1 e_2 e_3}\}$. As explained in the appendix, the condensed phase $T_D$ contains 8 anyons, which are all confined except for the new trivial sector. The seven non-trivial confined anyons descend from the dyons, where $D_M^+ \rightarrow 2d_M$. This gives, on a cylinder with both boundaries characterized by $L_E$,

$$GSD_{L_E}^{Z_2^3 TQD} \text{ boundaries on cylinder} = 8. \quad (9)$$

This immediately agrees with the result obtained by considering allowed charge transport across the boundaries—i.e. number of fusion channels between the condensed anyons in the top and bottom boundary that fuse to the trivial sector. More interesting is the boundary characterized by $L_D$, where

$$L_D = \{(D_M^{100})^2, E_{0 e_2 e_3}\}. \quad (10)$$

The resultant condensed phase $T_D$ contains again 8 distinct sectors, 7 of which confined and descended from the other dyons and electric charges of the form $E_{0 e_2 e_3}$. Something very special arises here, namely, the confined anyon $D_M^{100} \rightarrow 1 + 1$, splitting into two copies of the vacuum in the condensed phase $T_D$. This is unlike any other examples encountered above, where each sector appearing in the condensate only splits into the trivial sector once! To make that information explicit, we have included the dyon $D_M^{100}$ twice in defining the set $L_D$ above. Now consider the GSD on a cylinder. The number of confined sectors in $T_D$ would indicate that

$$GSD_{L_D}^{Z_2^3 TQD} \text{ boundaries on cylinder} = 8. \quad (11)$$

again. However, a naive count of allowed charge transport across the boundaries gives only 5 channels, if we count $D_M^{100}$ only once. The only way to recover a match between these two ways of counting is to take $D_M^{100}$ literally as appearing twice, so that they alone contribute $2^2 = 4$ fusion channels between the top and bottom boundaries to the trivial sector, instead of only one as in the naive count. Then we would recover a GSD $= 4 + 4 = 8$. We therefore postulate that the generalization of the Lagrangian subset in non-Abelian phases must include specifying the multiplicity of a condensed anyon—the number of condensates that is actually contained in the splitting of the anyon after anyon condensation. We have tested this postulate in this model in surfaces with more boundaries and found that a match between the counting via charge-transport across boundaries and the analysis via confined sectors in the condensed phase continue to hold. We have also checked our postulate in a different quantum double model, with group $G = D_3$. The GSD following from a nontrivial condensate involving multiplicities greater than 1 again supports our postulate.

To end the section, we return to the $Z_3^2$ TQD model on a cylinder, with, however, the left boundary characterized by $L_E$ and the right by $L_D$. The two sets of anyons have exactly 4 fusion channels that can fuse to one, corresponding to the fusion of the four shared electric charges. So the charge transport reasoning leads to the following interesting result:

$$GSD_{Z_2^3 TQD}^{L_E \text{ left}, \ L_D \text{ right}} = 4. \quad (12)$$

We could arrive at the same conclusion via considerations of confined sectors, as depicted in Fig. 4. As detailed in the appendix and briefly described above, $T_D$ has 8 sectors, conveniently denoted

$$\{1, d_{01}^{12}, d_{10}^{12}, d_{11}^{12}, d\}. \quad (13)$$
satisfying the following important fusion rule
\[ d_{M}^{(2)} \otimes d = d_{M}^{(1)}. \]  
(14)

Since \( d \) descends from \( E_{1e_{1}e_{2}} \), this charge is no longer conserved in the other boundary where all electric charges are contained in \( L_{E} \) and condenses. Therefore, \( d_{M}^{\pm} \) becomes indistinguishable, and the GSD is characterized by the following four states each with an anyon line winding the non-trivial cycle
\[ \{ |1\rangle, |d_{0,1}\rangle, |d_{1,0}\rangle, |d_{1,1}\rangle \}, \]
(15)
leading again to a precise match. This is the first example of a non-Abelian phase in which we can compute its GSD on an open surface with multiple boundary conditions.

**CONCLUSION**

In this paper we have achieved the long sought goal to count GSD of a generic non-Abelian topological order with boundaries, making use of insights and technologies developed in the past \[6, 8, 13, 17, 22]. Given the importance of robust GSD as a resource in topological quantum computation, the new understanding here will be crucial toward finding applications of topological orders in experimentally more realistic settings, in which boundaries are essentially unavoidable. Our results also suggest an interesting generalization of the Laughlin-Wu-Tao charge-pump thought experiment even in non-Abelian phases. This in fact has the potential of supplying novel ways of braiding and controlling non-Abelian anyons. Further implications in Mathematics, such as the building and classification of modular invariants in CFTs, will be explored in future work.

We thank Juven Wang for explaining his work on Abelian phases to us in great detail, and for many subsequent discussions. We also thank William Witczak-Krempa for his helpful comments on the manuscript. YW owes gratitude to his mentor, Prof. Xiao-gang Wen, for his constant support and inspirational conversations. YW is grateful to Prof. Yong-Shi Wu for his hospitality at Fudan University, where the work was completed. YW also appreciates his cousin, Prof. Wanxing Wang, who helped drawing the figures in the paper. LYH is supported by the Croucher Fellowship. This research was supported in part by Perimeter Institute for Theoretical Physics. Research at Perimeter Institute is supported by the Government of Canada through Industry Canada and by the Province of Ontario through the Ministry of Economic Development & Innovation.

**The \( \mathbb{Z}_{3}^{2} \) Twisted Quantum Double**

We review the details of the \( \mathbb{Z}_{3}^{2} \) TQD model here. The TQD model we are considering is characterized by the 3-cocycle \( \omega \in H^{3}(\mathbb{Z}_{2} \times \mathbb{Z}_{2} \times \mathbb{Z}_{2}, U(1)) \), given by
\[ \omega(x, y, z) = e^{i\pi x_{1}y_{2}z_{3}}, \]
(16)
where each of \( x, y, z \) is a three component vector, and each of its component takes values in \( \{0, 1\} \).

The anyons in this theory fall into two categories: pure charges and dyons. Each pure charge is labelled by a three component vector \( E = (e_{1}, e_{2}, e_{3}) \), where \( e_{i} \in \{1, 0\} \). The dyons are each two dimensional irreducible projective representation of \( \mathbb{Z}_{2} \times \mathbb{Z}_{2} \times \mathbb{Z}_{2} \). A dyon is labeled by a 3-vector \( M = (m_{1}, m_{2}, m_{3}) \in \mathbb{Z}_{2} \times \mathbb{Z}_{2} \times \mathbb{Z}_{2} \), and for each \( M \) there are two corresponding distinct representations, labeled \( \pm \), which can be taken as the charge label. The table summarises the distinct topological sectors in the theory and their self-statistics\[19].

| Top. Sector | Self-statistics | q.d. = d_{M} |
|-------------|-----------------|--------------|
| \( E_{E=(e_{1},e_{2},e_{3})} \) | \pm 1 \( m = (m_{1}, m_{2}, m_{3}) \neq (0,0,0) \) \( M \) | 2 |
| \( D_{M}^{\pm} | M = (1,1,1) \) | \pm \( M \) | 2 |

**TABLE I. Different topological sectors of \( D^{\pm} (\mathbb{Z}_{2} \times \mathbb{Z}_{2} \times \mathbb{Z}_{2}) \)**

We will explore two boundary conditions characterized by two different generalized Lagrangian subsets and work out the corresponding condensed phase for each of them.

**Electric condensate**

This is the simplest scenario, in which
\[ L_{E} = \{ E_{e_{1},e_{2}} \}, \]
(17)
meaning that the boundary is characterized by the simultaneous condensation of all the electric charges, all having trivial self statistics and mutual statistics among themselves.

All members in \( L_{E} \) have quantum dimension unity, so they do not split any further in the condensed phase \( T_{E} \). The dyonic anyons have quantum dimension 2, and might potentially split into two pieces.

Let us assume that
\[ D_{M}^{\pm} = d_{M}^{\pm,1} \oplus d_{M}^{\pm,2}. \]
(18)
Now using the fusion rules
\[
 D_M^\pm \otimes E_{e_1e_2e_3} = D_M^{\pm(-1)^{e_1+e_2+e_3+1}}. \tag{19}
\]
Replacing these anyons by their decomposition in \( T_E \) immediately implies that \( d_M^\pm = d_M^\pm \equiv d_M \). Now recall that \( D_M^\pm \) and \( D_M \) have different topological spins (Table 4), implying that \( d_M \), descended from both sectors, have ill-defined statistics, and thus must be confined in the condensate. This is a proof that the choice of the set of condensate \( L_E \) leads to complete confinement of the phase, and thus a gapped boundary. Now,
\[
 D_{m_1m_2m_3}^\pm \otimes D_{m_1m_2m_3}^\pm = E_{E_1} + E_{E_2} + E_{E_3} + E_{E_4}, \tag{20}
\]
where the precise value of \( E_i \) depends on the vectors \( \{m_1, m_2, m_3\} \). What is important however is that as soon as we replace it by their decompositions, one concludes that
\[
d_M^1 \otimes d_M^3 = 1 = d_M^2 \otimes d_M^1. \tag{21}
\]
For a unitary theory in which the conjugate of an anyon is unique, this implies that
\[
d_M^1 = d_M^2 = d_M. \tag{22}
\]
We therefore end up with exactly 8 distinct sectors in \( T_E \).

**Dyonic condensate**

Now we move on to the more interesting gapped boundary condition characterized by a dyonic condensate \( L_D \).

To begin with we would like to pick \( D_{100}^+ \) as a condensate. This has to split into two parts:
\[
 D_{100}^+ \rightarrow 1 \oplus c. \tag{23}
\]
Taking into account that other dyons also potentially split into two pieces, we write
\[
 D_M^\pm \rightarrow c_1^{M,\pm} \oplus c_2^{M,\pm}. \tag{24}
\]
As we will conclude below, \( D_M^\pm \) should split if \( T_D \) remains unitary, such that each anyon and its conjugate only fuse to the trivial sector once.

Using the fusion relations
\[
 D_{100}^+ \otimes E_{e_1e_2e_3} = D_{100}^+, \tag{25}
 D_{100}^+ \otimes D_{100}^+ = \sum_{e_2,e_3} E_{0e_2e_3}, \tag{26}
\]
one can conclude that
\[
 E_{0e_2e_3} \rightarrow 1, \ c = 1. \tag{27}
\]
Then the following fusion rules
\[
 D_{100}^+ \otimes D_{100}^- = \sum_{e_2,e_3} E_{e_2e_3}, \tag{28}
 D_{100}^+ \otimes E_{e_1e_3} = D_{100}^+, \tag{29}
 D_{100} \otimes D_{100} = \sum_{e_2,e_3} E_{0e_2e_3}, \tag{30}
\]
immediately imply that
\[
 E_{1e_2e_3} \rightarrow d, \tag{31}
 D_{100} \rightarrow c_1^{100,-} \oplus c_2^{100,-} = 2d, \tag{32}
 d \otimes d = 1, \tag{33}
\]
the last equality following also from \( E_{1e_2e_3}^2 = 1 \). This immediately suggests that \( E_{1e_2e_3} \) and \( D_{100} \) are confined because they have different topological spins and yet decompose to the same anyon. Next we have
\[
 D_{0m_1m_2}^\pm \otimes E_{e_1e_2} = D_{0m_1m_2}^{\pm(-1)^{e_1+e_2}}, \tag{34}
 \gamma_i = \frac{2(m_i - 1)(2e_i - 1) + 1}{2}. \tag{35}
\]
This implies that
\[
 c_1^{M,+} \oplus c_2^{M,-} = c_1^{M,-} \oplus c_2^{M,+} \tag{36}
 M \in \{(10, m_1, m_2)\}, \text{excluding } m_1=m_2=0. \tag{37}
\]
Therefore, we are identifying
\[
 c_i^{M,+} = c_i^{M,-}, \tag{38}
\]
and again these anyons become confined. This implies that all non-trivial sectors are confined in the condensate, and
\[
 L_D = \{(D_{100}^+)^2, E_0e_2e_3\} \tag{39}
\]
satisfies the desired condition of a generalized Lagrangian subset.

Now
\[
 D_{100}^+ \otimes D_{0m_1m_2}^+ = D_{1m_1m_2}^+ \otimes D_{1m_1m_2}^-, \tag{40}
\]
leading to identifying
\[
 c_{i_{10m_1m_2,\pm}} = c_{i_{1m_1m_2,\pm}} \equiv d_{m_1m_2}, \text{ (excluding } m_1=m_2=0), \tag{41}
\]
where we further simplify notations by introducing \( d_{m_1m_2} \). Using
\[
 D_{0m_1m_2}^+ \otimes E_{e_1e_2e_3} = D_{0m_1m_2}^{+(-1)^{e_1+e_2}}, \tag{42}
 D_{0m_1m_2}^+ \otimes D_{0m_1m_2}^+ = \sum_{e_2,e_3} E_{e_2e_3}, \tag{43}
\]
one concludes that
\[
 (d_{m_1m_2})^2 = 1, \ d_{m_1m_2} \otimes d_{m_1m_2} = d, \tag{44}
 d_{m_1m_2}^{1m_1m_2} \otimes d = d_{m_1m_2}^2, \tag{45}
 d_{m_1m_2}^{1m_2m_4} \otimes d_{m_3m_4} = d_{m_1m_3}(m_2m_4), \tag{46}
 d_{m_1m_2}^{1m_1m_4} \otimes d_{m_3m_4} = d_{m_1m_4}(m_2m_3), \tag{47}
\]
where the addition appearing in the subscripts are defined only modulo 2.

[1] X. Wen, Phys. Rev. B 40, 7387 (1989).
[2] X. Wen and Q. Niu, Phys. Rev. B 41, 9377 (1990).
[3] X. Wen, Phys. Rev. B 44, 2664 (1991).
[4] X. Wen, Int. J. Mod. Phys. B 239 (1990).
[5] C. Nayak, A. Stern, M. Freedman, and S. Das Sarma, Rev. Mod. Phys. 80, 1083 (2008).
[6] A. Kitaev and L. Kong, Commun. Math. Phys. 313, 351 (2012).
[7] M. Levin, Phys. Rev. X 3, 021009 (2013), 1301.7355.
[8] J. Wang and X.-g. Wen, p. 4 (2012), 1212.4863.
[9] L.-y. Hung and Y. Wan, Phys. Rev. B 87, 195103 (2013).
[10] T. Iadecola, T. Neupert, C. Chamon, and C. Mudry, p. 11 (2014), 1407.4129.
[11] R. B. Laughlin, Phys. Rev. Lett. 50, 1395 (1983).
[12] R. Tao and Y.-S. Wu, Phys. Rev. B 30, 1097 (1984).
[13] F. A. Bais, B. Schroers, and J. Slingerland, Phys. Rev. Lett. 89, 181601 (2002).
[14] F. A. Bais, J. Slingerland, and S. Haaker, Phys. Rev. Lett. 102, 220403 (2009), arXiv:0812.4596v1.
[15] F. A. Bais and J. Slingerland, Phys. Rev. B 79, 045316 (2009).
[16] L.-Y. Hung and Y. Wan, Int. J. Mod. Phys. B p. 1450172 (2014), 1308.4673.
[17] Y. Gu, L.-Y. Hung, and Y. Wan, p. 18 (2014), 1402.3356.
[18] F. A. Bais and S. M. Haaker (2014), 1407.5790.
[19] M. Propitius, Ph.D. thesis, University of Amsterdam (1995), 9511195v1.
[20] Y. Hu, Y. Wan, and Y.-S. Wu, Phys. Rev. B 87, 125114 (2013), 1211.3695.
[21] A. Mesaros and Y. Ran, Phys. Rev. B 87, 155115 (2013), 1212.0835.
[22] X. Wen, Phys. Rev. B 41, 12838 (1990).