Quantum algorithms for subset finding

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Recently, Ambainis gave an \(O(N^{2/3})\)-query quantum walk algorithm for element distinctness, and more generally, an \(O(N^{L/(L+1)})\)-query algorithm for finding \(L\) equal numbers. We point out that this algorithm actually solves a much more general problem, the problem of finding a subset of size \(L\) that satisfies any given property. We review the algorithm and give a considerably simplified analysis of its query complexity. We present several applications, including two algorithms for the problem of finding an \(L\)-clique in an \(N\)-vertex graph. One of these algorithms uses \(O(N^{2L/(L+1)})\) edge queries, and the other uses \(O(N^{(5L−2)/(2L+4)})\), which is an improvement for \(L \leq 5\). The latter algorithm generalizes a recent result of Magniez, Santha, and Szegedy, who considered the case \(L = 3\) (finding a triangle). We also pose two open problems regarding continuous time quantum walk and lower bounds.

I. INTRODUCTION

In this paper, we address the following rather general query complexity problem:

**Problem (L-subset finding).**

*Input:* (i) Black box function \(f : D \to R\), where the domain \(D\) and the range \(R\) are finite sets, and \(|D| = N\) is the problem size. (ii) Property \(\mathcal{P} \subset (D \times R)^L\).

*Output:* Some \(L\)-subset \(\{x_1, \ldots, x_L\} \subset D\) such that \((x_1, f(x_1)), \ldots, (x_L, f(x_L)) \in \mathcal{P}\), or reject if none exists.

In the general case, for the purpose of query complexity, we can assume that the property \(\mathcal{P}\) is given explicitly as a list. However, for specific versions of this problem, there will generally be a much more compact way to specify the property.

The quantum query complexity of the \(L\)-subset finding problem is well-understood in the cases \(L = 1, 2\). The \(1\)-subset finding problem is nothing but the well-known unstructured search problem. This problem can be solved in \(O(\sqrt{N})\) queries \([1]\), which is optimal \([2]\). The element distinctness problem (see \([3]\) and references therein) is a particular case of 2-subset finding. This problem can be solved using \(O(N^{2/3})\) queries using a recent algorithm of Ambainis \([4]\), which is also optimal \([3, 7, 8, 8]\).

In \([4]\), Ambainis also gave an \(O(N^{L/(L+1)})\)-query algorithm for the \(L\)-element distinctness problem, where the goal is to find \(L\) inputs that give the same output. This is a particular case of \(L\)-subset finding with \(\mathcal{P} = \{((x_1, y), \ldots, (x_L, y))\}\). However, the algorithm is actually even more general: it solves the \(L\)-subset finding problem using \(O(N^{L/(L+1)})\) queries, regardless of \(\mathcal{P}\).

We review the algorithm in this context in Section II. We also present a simplified proof that the algorithm works in Section III.

Note that for \(L\)-element distinctness, any \(L\)-subset of inputs can potentially satisfy \(\mathcal{P}\), so we could have described \(\mathcal{P}\) as a subset of \(R^L\). Many variants of element distinctness could also be described this way. However, for other problems such as finding a clique in a graph, we require the generality provided by letting \(\mathcal{P} \subset (D \times R)^L\).

Because the property \(\mathcal{P}\) can be any subset of \((D \times R)^L\), the subset finding algorithm can be applied to a wide variety of related problems. We discuss some of these applications in Section IV. In particular, we consider using the algorithm to find \(L\)-vertex subgraphs in \(N\)-vertex graphs, assuming the graph is given as a black box allowing edge queries. The most straightforward application of the \(L\)-subset finding algorithm gives an \(O(N^{2L/(L+1)})\)-query algorithm for finding an \(L\)-clique. This upper bound is the best we know for \(L \geq 6\). However, for \(L \leq 5\) we improve it to \(O(N^{(5L−2)/(2L+4)})\) using the recursive approach of \([8]\), which considers the particular case \(L = 3\) (finding a triangle).

We conclude in Section V by suggesting some open problems. We discuss why the subset finding algorithm is well-suited to discrete rather than continuous time quantum walks, and we pose an open problem related to the simulation of continuous time quantum walks. We also discuss the known lower bounds for \(L\)-subset finding, and we suggest a specific case where the subset finding algorithm might be optimal for general \(L\).

II. ALGORITHM

In this section, we review Ambainis’s algorithm for element distinctness \([4]\) in the context of subset finding. The algorithm is based on the idea of a discrete time quantum walk on a graph \([1, 7, 8, 8, 9]\). A quantum walk is simply a way of formulating local quantum dynamics on a graph. If the walk takes discrete steps that only move...
amplitude between neighboring vertices, then in general the walk must include an ancillary state space \( \Theta \), sometimes referred to as a “coin,” that can be used to indicate the direction of the next step of the walk. Thus the quantum walk can be built as a sequence of transformations, some of which act on the coin register and some of which use the state of the coin register to update the location in the graph.

The graph used in Ambainis’s construction is a bipartite graph whose vertices are all subsets of the domain \( D \) of size either \( M \) or \( M + 1 \). We will choose \( M \) to be \( \Theta(N^q) \) for some \( 0 < q < 1 \) (recall \( |D| = N \)). Let \( A \subset D \) with \( |A| = M \), and let \( B \subset D \) with \( |B| = M + 1 \). Vertices \( A \) and \( B \) are connected in \( G \) iff \( |A \cap B| = M \), i.e., \( B = A \cup \{k\} \) for some \( k \in D \setminus A \).

To define the quantum walk on \( G \), we define an orthonormal basis of quantum states \( |A\rangle \), one for each subset \( A \). For any subset \( A \) with \( |A| = M \), there are \( M \) associated function values \( f(A) \in \mathbb{R}^M \), and similarly, for any subset \( B \) with \( |B| = M + 1 \), there are \( M + 1 \) associated function values \( f(B) \in \mathbb{R}^{M+1} \). A key idea of the algorithm is to store these function values along with the subset. The only parts of the algorithm that require queries will be those that manipulate the function values.

The full state of the quantum computer has the form \( |A, f(A), k\rangle \) where \( |A, f(A)\rangle \) is the state described previously, including the function values, and \( |k\rangle \) is the coin register, where \( k \in D \). If \( |A| = M \) then \( k \) indicates an element to be added to \( A \), so we must have \( k \notin A \). Similarly, if \( |B| = M + 1 \) then \( k \) indicates an element to be removed from \( B \), so we must have \( k \in B \).

One step \( W \) of the quantum walk (which actually involves two steps on \( G \)) is a product of four unitary transformations, \( W = SC_2SC_1 \). The shift operation \( S \) acts as

\[
S|A, f(A), k\rangle = |A \cup \{k\}, f(A \cup \{k\}), k\rangle \tag{1}
\]

\[
S|B, f(B), k\rangle = |B \setminus \{k\}, f(B \setminus \{k\}), k\rangle \tag{2}
\]

and can be implemented using one query of the black box. The coin operations \( C_1 \) and \( C_2 \) are Grover diffusion operators on \( k \notin A \) and \( k \in B \), respectively. In other words, we have

\[
C_1|A, f(A), k\rangle = |A, f(A), k\rangle - \frac{2}{N - M} \sum_{k' \notin A} |A, f(A), k'\rangle \tag{3}
\]

\[
C_1|B, f(B), k\rangle = |B, f(B), k\rangle \tag{4}
\]

and

\[
C_2|A, f(A), k\rangle = |A, f(A), k\rangle \tag{5}
\]

\[
C_2|B, f(B), k\rangle = |B, f(B), k\rangle - \frac{2}{M + 1} \sum_{k' \in B} |B, f(B), k'\rangle \tag{6}
\]

These unitary transformations do not change the subset, so they also do not affect the function values, and consequently do not require any queries.

The algorithm also involves a phase flip operation that distinguishes subsets \( A \) that include an \( L \)-subset satisfying \( \mathcal{P} \). For simplicity, we suppose there is exactly one special subset \( \mathcal{S} = \{x_1, \ldots, x_L\} \subset D \) of inputs such that \( (x_1, f(x_1)), \ldots, (x_L, f(x_L)) \in \mathcal{P} \). The general case can be handled by modifying the algorithm using standard sampling techniques \cite{4}. The phase flip operation is

\[
P|A, f(A)\rangle = \begin{cases} -|A, f(A)\rangle & \mathcal{S} \subset A \\ |A, f(A)\rangle & \mathcal{S} \not\subset A \end{cases}.	ag{7}
\]

Given \( A \) and \( f(A) \), the property \( \mathcal{P} \) can be checked without any further queries of the black box, so no queries are required to implement \( P \).

The initial state for the algorithm is the symmetric state on subsets of size \( M \),

\[
|s\rangle = \frac{1}{\sqrt{c}} \sum_{|A| = M} |A, f(A)\rangle \sum_{k \notin A} |k\rangle , \tag{8}
\]

where \( c = \binom{N}{M}(N-M) \) is a normalization constant. This state can be made using \( M \) queries to the black box.

The full algorithm is \((W^t_1 P)^{t_2}\), which uses \( 2t_1 t_2 \) additional queries, where \( t_1, t_2 \) will be determined in the analysis. Thus, using \( M + 2t_1 t_2 \) queries, we produce the state \((W^t_1 P)^{t_2}|s\rangle \). Our goal is to choose \( t_1, t_2 \) so that a measurement of this state is likely to provide a solution to the problem.

Note that in the description of this algorithm, we have focused only on the query complexity of the various steps. For the algorithm to be efficient in terms of the number of computational steps as well as the number of queries, it must be possible to use \( A \) and \( f(A) \) to efficiently determine whether there is an \( \{x_1, \ldots, x_L\} \subset A \) such that \( (x_1, f(x_1)), \ldots, (x_L, f(x_L)) \in \mathcal{P} \). In doing so it may be helpful to maintain the function values in some particular data structure, such as a hash table, for the element distinctness problem (see Section 6 of \cite{3}). However, for the general problem we will not be concerned with how \( \mathcal{P} \) is given or how efficiently it can be checked.

### III. ANALYSIS OF THE ALGORITHM

In this section, we give an analysis of the subset finding algorithm that is simpler, as well as tighter, than that of \cite{3}. The main result is the following:

**Theorem 1.** The quantum query complexity of \( L \)-subset finding is \( O(N^{L/(L+1)}) \).

To prove this theorem, we need to understand the spectrum of \( W^{t_1} P \), which we will see is close to a small rotation in a two-dimensional subspace spanned by \( |s\rangle \) and a state that gives a solution to the problem. We will begin by proving Lemma 2 below, which characterizes the spectrum of the walk step \( W \) alone and shows how to choose \( t_1 \). Then we will use techniques from \cite{14} to
prove Lemma 3 below, which describes the spectrum of \( W^t \), and shows how to choose \( t_2 \). Finally, we will give the proof of Theorem 2.

To analyze the algorithm, we will use the fact that the evolution takes place in a \((2L + 1)\)-dimensional subspace of the full Hilbert space. Let \( S_1 = S \) and \( S_0 = D \setminus S \). Define the states

\[
|A_{j,p}\rangle = \frac{1}{\sqrt{c_{j,p}}} \sum_{|A|=M, A \cap S = j} |A, f(A)\rangle \sum_{k \in A} |k\rangle
\]

for \( j = 0, 1, \ldots, L - 1 \), \( p = 0, 1 \) and for \( j = L, p = 0, 1 \), and the states

\[
|B_{j,p}\rangle = \frac{1}{\sqrt{d_{j,p}}} \sum_{|B|=M+1, B \cap S = j} |B, f(B)\rangle \sum_{k \in B} |k\rangle
\]

for \( j = 0, p = 0 \) and for \( j = 1, \ldots, L, p = 0, 1 \). Here the normalization constants are given by

\[
c_{j,0} = \left( \frac{N - L}{M - j} \right)^{L-j} \left[ (N - L) - (M - j) \right]
\]

\[
c_{j,1} = \left( \frac{N - L}{M - j} \right)^{L-j} (L - j)
\]

\[
d_{j,0} = \left( \frac{N - L}{M + 1 - j} \right)^{L-j} (M + 1 - j) = c_{j,0}
\]

\[
d_{j,1} = \left( \frac{N - L}{M + 1 - j} \right)^{L-j} j = c_{j-1,1}
\]

In terms of these states, the shift operation acts as

\[
S|A_{j,0}\rangle = |B_{j,0}\rangle
\]

\[
S|A_{j,1}\rangle = |B_{j+1,1}\rangle
\]

\[
S|B_{j,0}\rangle = |A_{j,0}\rangle
\]

\[
S|B_{j,1}\rangle = |A_{j-1,1}\rangle
\]

By explicit calculation, the coin transformations have the following matrix elements (with others determined by the fact that \( C_1, C_2 \) are Hermitian as well as unitary):

\[
\langle A_{j,0}|C_1|A_{j,0}\rangle = 1 - 2\alpha(L - j)
\]

\[
\langle A_{j,0}|C_1|A_{j,1}\rangle = 2\sqrt{\alpha(L - j)}[1 - \alpha(L - j)]
\]

\[
\langle A_{j,1}|C_1|A_{j,1}\rangle = 2\alpha(L - j) - 1
\]

where \( \alpha = 1/(N - M) \), and

\[
\langle B_{j,0}|C_2|B_{j,0}\rangle = 1 - 2\beta j
\]

\[
\langle B_{j,0}|C_2|B_{j,1}\rangle = 2\sqrt{\beta j(1 - \beta j)}
\]

\[
\langle B_{j,1}|C_2|B_{j,1}\rangle = 2\beta j - 1
\]

where \( \beta = 1/(M + 1) \).

Now we are ready to analyze the spectrum of the walk step \( W = SC_2SC_1 \). Similarly to Lemma 2 of [4], we find

**Lemma 2.** The eigenvalues of \( W \) are 1 and \( \exp[\pm i(2\sqrt{\alpha + \beta - j\alpha \beta})] \) for \( j = 1, \ldots, L \). The eigenvector with eigenvalue 1 is \(|s\rangle\), and the two eigenvectors with eigenvalues \( \exp[\pm i(2\sqrt{\alpha + \beta - j\alpha \beta})] \) are \( \frac{1}{\sqrt{2}}(|A_{L-1,1}\rangle \pm i|A_{L,0}\rangle) + O(\sqrt{M/N}) \).

In fact, the eigenvalues of \(|W - 1\rangle\) are exactly zero and \( \sqrt{(\alpha + \beta - j\alpha \beta)} \) for \( j = 1, 2, \ldots, L \), but we will not require this level of detail.

Note that the notation \(|\psi\rangle = |\phi\rangle + O(\epsilon)\) means \(||\psi\rangle - |\phi\rangle|| = O(\epsilon)\). Similarly, for operators \(X, Y\) we write \(X = Y + O(\epsilon)\) as shorthand for \(||X - Y|| = O(\epsilon)\).

**Proof.** Direct calculation shows that \(|s\rangle\) is an eigenvector of \( W \) with eigenvalue 1.

To understand the rest of the spectrum, we can think of the walk as being composed of two parts, \( C_1 \) and \( SC_2S \), each of which is a unitary transformation in the \( A \) subspace. Let

\[
C_1 = C + \Delta_1
\]

\[
SC_2S = C + \Delta_2
\]

where \( C \) is the diagonal matrix with diagonal elements

\[
\langle A_{j,p}|C|A_{j,p}\rangle = (-1)^p.
\]

The matrices \( \Delta_1, \Delta_2 \) each consist of \( L \times 2 \) blocks and one \( 1 \times 1 \) block (a zero), so their eigenvalues can be computed explicitly, giving \(||\Delta_1|| = O(1/\sqrt{N})\) and \(||\Delta_2|| = O(1/\sqrt{M})\). Now we have

\[
W = 1 + \Delta_2C + C\Delta_1 + \Delta_2\Delta_1
\]

\[
= 1 + \Delta_2C + O(1/\sqrt{N})
\]

Therefore, it is sufficient to calculate the eigenvectors and eigenvalues of the matrix \( \Delta_2C \), which has matrix elements

\[
\langle A_{j,0}|\Delta_2C|A_{j,0}\rangle = -2\beta j
\]

\[
\langle A_{j,0}|\Delta_2C|A_{j,1}\rangle = -2\beta j(1 - \beta j)
\]

\[
\langle A_{j+1,1}|\Delta_2C|A_{j,0}\rangle = 2\beta j(1 - \beta j)
\]

\[
\langle A_{j-1,1}|\Delta_2C|A_{j,1}\rangle = -2\beta j
\]

This matrix also has \( L \times 2 \) blocks and one \( 1 \times 1 \) block, so its eigenvalues and eigenvectors can also be computed explicitly. Using perturbation theory to calculate the size of the small correction from the \( O(1/\sqrt{N}) \) term, we find that the eigenvalues of \( W - 1 \) are 0 and \( \pm 2\sqrt{\beta} + O(1/\sqrt{N}) \) for \( j = 1, \ldots, L \), which implies the claim about the eigenvalues of \( W \). The claim about the two relevant eigenvectors of \( W \) also follows by perturbation theory.

**Lemma 2** shows that in \( W^t \), we want to take \( t_1 = O(\sqrt{M}) \), since this choice implements a rotation by an angle of order unity. In fact, we choose \( t_1 = \lfloor \frac{\sqrt{M}}{L} \rfloor \), where \( \lfloor x \rfloor \) denotes the nearest integer to \( x \), so that the
two extreme eigenvalues of \( W^t \) are \(-1 + O(1/\sqrt{M})\) (here the dominant source of error is actually the fact that \( \frac{2}{\sqrt{M/L}} \) may not be close to an integer).

We now describe techniques for understanding the spectrum of \( W^t P \). Essentially the same analysis appears in Section IV.A of [14], and we will follow that treatment closely. Our goal is to calculate spectral properties of an operator \( UP \) where \( P = 1 - 2 |w\rangle \langle w| \). In our case \( U \) is \( W^t \), and \( P \) is given by [15], so that \( |w\rangle = |A_{L,0}\rangle \).

An eigenvector \( |\theta_a\rangle \) of \( UP \), with eigenvalue \( e^{i\theta_a} \), satisfies

\[
UP|\theta_a\rangle = (U - 2U|w\rangle \langle w|)|\theta_a\rangle = e^{i\theta_a}|\theta_a\rangle ,
\]

i.e.,

\[
(U - e^{i\theta_a})|\theta_a\rangle = 2U|w\rangle \langle w|\theta_a\rangle .
\]

Define

\[
R_a = |\langle w|\theta_a\rangle|^2
\]

and choose the phase of \( |\theta_a\rangle \) so that

\[
\langle w|\theta_a\rangle = \sqrt{R_a} .
\]

As long as \( R_a > 0 \), we may write

\[
|\theta_a\rangle = \frac{2\sqrt{R_a}}{1 - U e^{i\theta_a}}|w\rangle .
\]

Consistency with (4) then gives the eigenvalue condition

\[
\langle w|\frac{2}{1 - U e^{i\theta_a}}|w\rangle = 1 .
\]

Now let the eigenstates of \( U \) be \( |u_j\rangle \) with eigenvalues \( e^{iu_j} \). Then the eigenvalue condition (39) may be written

\[
\sum_j \frac{2|\langle w|u_j\rangle|^2}{1 - e^{i(\theta_a - u_j)}} = 1 .
\]

The real part of this equation is automatically satisfied, and the imaginary part gives

\[
\sum_j |\langle w|u_j\rangle|^2 \cot \left( \frac{\theta_a - u_j}{2} \right) = 0 .
\]

Roots of this equation determine the values of \( \theta_a \), which specify the eigenvalues of \( UP \).

Similar considerations let us determine properties of the eigenvectors. The normalization condition on \( |\theta_a\rangle \) gives

\[
R_a \langle w|\frac{4}{1 - U e^{i\theta_a}}|w\rangle = 1 ,
\]

i.e.,

\[
R_a = \left[ \sum_j \frac{4|\langle w|u_j\rangle|^2}{|1 - e^{i(\theta_a - u_j)}|^2} \right]^{-1}
\]

\[
= \left[ 1 + \sum_j |\langle w|u_j\rangle|^2 \cot^2 \left( \frac{\theta_a - u_j}{2} \right) \right]^{-1} .
\]

To compute the overlap of \( |\theta_a\rangle \) on the eigenstates \( |u_j\rangle \) of \( U \), we multiply on the left by \( \langle u_j| \), which gives

\[
\langle u_j|\theta_a\rangle = \langle w|\theta_a\rangle \langle u_j|w\rangle \left[ 1 + i \cot \left( \frac{\theta_a - u_j}{2} \right) \right] .
\]

Applying these techniques to our specific problem, we find

**Lemma 3.** \( W^t P \) has two eigenvectors \( |\theta_{\pm}\rangle = \frac{1}{\sqrt{2}}(|w\rangle \pm i|s\rangle) + O(1/M + M/N) \) with eigenvalues \( \exp(\pm 2i(M/N)^{L/2}(1 + O(1/M + M/N)) \).

**Proof.** From (4), we have \( |w\rangle = |A_{L,0}\rangle \). The eigenvalue condition (39) for \( W^t P \) is

\[
0 = |\langle w|s\rangle|^2 \cot \left( \frac{\theta_a}{2} \right) + \sum_{j \neq 0} |\langle w|u_j\rangle|^2 \cot \left( \frac{\theta_a - u_j}{2} \right) .
\]

We are interested in solutions where \( \theta_a \) is small (which we will indeed find for large \( N \)), so Taylor expansion in \( \theta_a \) gives

\[
0 = |\langle w|s\rangle|^2 \left( \frac{2}{\theta_a} + O(1) \right)
\]

\[
- \sum_{j \neq 0} |\langle w|u_j\rangle|^2 \left( \cot \left( \frac{u_j}{2} + \frac{\theta_a}{2} \right) \csc \left( \frac{u_j}{2} \right) + O(\theta_a^2) \right)
\]

\[
= |\langle w|s\rangle|^2 \left( \frac{2}{\theta_a} + O(1) \right)
\]

\[
- \sum_{p \text{ pairs of } j} |\langle w|u_j\rangle|^2 \left( \theta_a \csc \left( \frac{u_j}{2} \right) + O(\theta_a^2) \right)
\]

where in the second equality we have used the fact that the values \( u_j \) come in \pm \) pairs, with identical values of \( |\langle w|u_j\rangle|^2 \), since \( W \) is a real unitary matrix. From Lemma 2 we see that the only nonnegligible contributions to this condition come from \( |s\rangle \) and the two eigenstates with \( |\langle w|u_j\rangle|^2 = \frac{1}{2} + O(M/N) \). We have

\[
0 = |\langle w|s\rangle|^2 \left( \frac{2}{\theta_a} + O(1) \right)
\]

\[
- \left( \frac{1}{2} + O(M/N) \right) \theta_a \left[ 1 + O(1/M) \right] + O(\theta_a^2)
\]

\[
+ O(M/N) O(\theta_a) ,
\]

which gives two solutions,

\[
\theta_{\pm} = \pm 2|\langle w|s\rangle| + O((M/N)^{1+L/2}) ,
\]

where the leading order contribution as well as the error term follow from

\[
|\langle w|s\rangle| = \sqrt{e_{L,0}/c}
\]

\[
= \frac{(N - L)!M!}{N!(M - L)!}
\]

\[
= (M/N)^{L/2} \left[ 1 + O(1/M) \right] .
\]
Having found the eigenvalues of $W^t \psi$, we can now calculate its eigenstates. Applying Lemma 2 to (4) gives

$$\langle w|\theta_\pm \rangle = \frac{1}{\sqrt{2}} + O(1/M + M/N),$$

(54)

and a similar calculation using (45) gives

$$\langle s|\theta_\pm \rangle = \pm \frac{i}{\sqrt{2}} + O(1/M + M/N),$$

(55)

which completes the proof.

From Lemma 3 we see that we should choose $t_2 = \lceil \frac{\pi}{2} (N/M)^{L/2} \rceil$ so that $(W^t \psi)^{t_2}$ implements a rotation from $|s\rangle$ to a state near $|w\rangle$.

We now complete the proof of the main result:

**Proof of Theorem 4.** Recall that $t_1 = \lceil \frac{\pi}{2} \sqrt{M/L} \rceil$ and $t_2 = \lceil \frac{\pi}{2} (N/M)^{L/2} \rceil$. From Lemma 3 it is easy to see that

$$|\langle w|(W^t \psi)^{t_2}|s\rangle|^2 = 1 - O(1/M + M/N).$$

(56)

Therefore, a measurement of the state $(W^t \psi)^{t_2}|s\rangle$ yields a subset $A$ for which $S \subset A$, together with the associated function values $f(A)$, with probability close to 1. As discussed in Section III the state $|s\rangle$ can be prepared using $M$ queries, and the rest of the algorithm uses $2t_1 t_2$ queries. Thus the total query complexity is $M + 2t_1 t_2 = O(M + N^{L/2} M^{(1-L)/2})$. Choosing $M = \lceil N^{L/(L+1)} \rceil$ shows that the algorithm uses $O(N^{L/(L+1)}$ queries.

Equation (55) can be contrasted with Lemma 3 of 3, which only shows that the success probability is $O(1)$.

**IV. APPLICATIONS**

As discussed in 3, the quantum walk subset finding algorithm solves a natural generalization of element distinctness, the problem of finding $L$ inputs to $f$ that give the same output, using $O(N^{L/(L+1)}$ queries. However, we have seen that the same algorithm finds $L$-subsets satisfying an arbitrary property $P$. Therefore, we can also solve many variants of $L$-element distinctness in $O(N^{L/(L+1)}$ queries—for example, finding a set of $L$ consecutive function values, relatively prime function values, or sum-free function values (no two of the $L$ values sum to one of the other $L$ values).

A closely related problem is that of finding an $L$-clique in an $N$-vertex graph given an edge query oracle. For $L = 2$ this is simply the problem of finding any edge in the graph, which is just the unstructured search problem on $\binom{N}{2}$ items, and can be solved using $O(N)$ quantum queries. For $L = 3$ it is the problem of finding a triangle, for which the exact quantum complexity is not known. A straightforward Grover search on the $\binom{N}{3}$ triples of vertices gives an algorithm that uses $O(N^{3/2})$ queries 3, which was recently improved to $O(N^{10/7} \log^2 N)$ in 15 and subsequently to $O(N^{13/10})$ in 8. However, the best known lower bound is only $O(N)$, which follows by straightforward reduction from the unstructured search problem. For $L > 3$, the query complexity of $L$-clique finding seems not to have been widely studied.

We describe two algorithms for finding $L$-cliques. One algorithm applies subset finding in a straightforward way, and another uses it recursively. The simple algorithm is faster than the recursive algorithm for large $L$ ($L \geq 6$), and the recursive algorithm is faster than the simple algorithm for small $L$ ($L \leq 5$). The simple algorithm is a straightforward application of the quantum walk subset finding algorithm where each subset $A$ over which we walk consists of $M$ vertices, and where the algorithm also stores all of the edges between them, i.e., their induced subgraph. The analysis of the algorithm proceeds as before, except now initialization requires $O(M^2)$ queries to determine the subgraph, and each step of the walk requires $O(M)$ queries to compute (or uncompute) the edges incident on the newly added (or removed) vertex. The total number of queries is therefore

$$O(M^2 + (N/M)^{L/2} \times \sqrt{M} \times M),$$

(57)

and choosing $M = \lceil N^{L/(L+1)} \rceil$ gives an overall query complexity of $O(N^{2L/(L+1)})$. This result is no better than straightforward Grover search for $L = 3$, but it gives results that are better than previously known algorithms for all fixed $L \geq 4$.

However, we can improve upon this algorithm for $L \leq 5$ using the recursive approach from 3. Here we again walk over induced subgraphs of $M$ vertices, but now we search for a subgraph that includes $L - 1$ vertices from an $L$-clique in the full graph. Recall that to implement the phase flip $P$ we must determine whether a subgraph satisfies this property. If the subgraph does satisfy the property, then either the $L$-clique lies entirely within the subgraph, or all but one of the $L$-clique vertices form an $(L - 1)$-clique in the subgraph. If the clique falls entirely within the subgraph of size $M$, then no queries are necessary. Otherwise, if one of the vertices of the clique

| $L$ | $\frac{2L}{L+1}$ | $\frac{5L-2}{2L+4}$ |
|-----|-----------------|-----------------|
| 2   | $\frac{4}{3}$  | 1               |
| 3   | $\frac{3}{2}$  | $\frac{13}{10}$ |
| 4   | $\frac{5}{6}$  | $\frac{3}{2}$  |
| 5   | $\frac{2}{5}$  | $\frac{16}{7}$ |
| 6   | $\frac{12}{7}$ | $\frac{7}{4}$  |
| 7   | $\frac{1}{2}$  | $\frac{32}{18}$ |

TABLE I: Query complexities of two algorithms for finding an $L$-clique in an $N$-vertex graph.
falls outside the subgraph, then the clique can be found using a Grover search for the outer vertex, where each Grover iteration uses \((L - 1)\)-subset finding to identify the \(L - 1\) vertices inside the subgraph of size \(M\). The implementation of the phase flip \(P\) using this Grover search requires a total of \(r = \tilde{O}(M^{(L-1)/L} \sqrt{N})\) queries. Thus each of the \(t_2 = O((N/M)^{(L-1)/2})\) iterations of \(W^{t_2} P\) uses \(O(r + t_1 \times M) = \tilde{O}(M^{(L-1)/L} \sqrt{N} + \sqrt{M} \times M)\) queries, where \(M\) queries are needed for each walk step as in the simple algorithm above. Therefore, since \(O(M^2)\) queries are required for initialization, the total number of queries is

\[
\tilde{O}(M^2 + (N/M)^{(L-1)/2}(M^{(L-1)/L} \sqrt{N} + M^3/2)),
\]

and choosing \(M = \lfloor N^{L/(L+2)} \rfloor\) gives an overall query complexity of \(\tilde{O}(N^{(5L-2)/(2L+4)})\).

The numbers of queries used by these algorithms for \(2 \leq L \leq 7\) are summarized in Table \ref{table1}. Of course the algorithms are not really specific to finding cliques, but could be used to find any desired subgraph consisting of \(L\) vertices.

In all the applications mentioned in this section so far (with \(L \geq 3\)), the problem has some structure that may allow us to learn something about whether a given subset of fewer than \(L\) items could be part of an \(L\)-subset satisfying \(P\). For example, in \(L\)-element distinctness, if we find two inputs with different outputs, we know these inputs cannot be part of an \(L\)-subset of inputs that all give the same output. Similarly, if we find two vertices in a graph that are not connected by an edge, then we know that no triangle can include these two vertices. However, such structure is not used by the algorithm; it can just as well solve a problem in which it is impossible to obtain any information about whether fewer than \(L\) inputs might satisfy \(P\). For example, consider the \(L\)-zero sum problem with \(D = \{1, \ldots, N\}\), \(R = \{0, 1\}^m\), and \(\mathcal{P} = \{(y_1, y_2, \ldots, y_L) : y_1 + \cdots + y_L = 0^m\}\); or alternatively, suppose \(R = \{0, 1, \ldots, q - 1\}\) and \(\mathcal{P} = \{(y_1, y_2, \ldots, y_L) : y_1 + \cdots + y_L \equiv 0 \mod q\}\). These particular \(L\)-subset finding problems seem to be examples of the hardest cases, since there is no way to determine anything about whether a subset of \(L - 1\) inputs might be part of an \(L\)-subset satisfying \(P\).

V. OPEN PROBLEMS

The subset finding algorithm we have described is based on a discrete time quantum walk, but there is a more natural formulation of quantum walk as a continuous time process that avoids the need to introduce a coin register [16]. For the present application the coin register seems to be essential since it tells us what to query when we take a step. Nevertheless, we could consider a continuous time quantum walk (without a coin) on the Johnson graph \(J(N, M)\) whose vertices are all subsets \(A \subset D\) of size \(|A| = M\), and in which two vertices are connected if they differ in exactly one element. If we label the vertices \(A\) of \(J(N, M)\) with the corresponding black box function values \(f(A)\), and if we could simulate the continuous time quantum walk on this labeled graph efficiently enough, then we could have an algorithm that works in a similar way to the search algorithm in [14]. However, most of the techniques we currently know for simulating such a walk are based on edge coloring [15, 18, 19, 20] and do not work well when the degree is large. It is an open problem whether the walk on this graph can be simulated efficiently enough to give a competitive algorithm for the \(L\)-subset finding problem.

It would be interesting to know which of the algorithms we have presented are optimal. For the \(L = 1\) problem there is a well-known lower bound of \(\Omega(\sqrt{N})\) that matches the performance of the subset finding algorithm. Of course for this problem the simpler Grover algorithm also achieves this query complexity [1]. For \(L = 2\) there is an \(\Omega(N^{2/3})\) lower bound on element distinctness that follows by reduction from the collision problem [3] that the author has independently established. Again this matches the performance of the subset finding algorithm [4]. Therefore it is natural to conjecture an \(\Omega(N^{L/(L+1)})\) lower bound for the general \(L\)-subset finding problem. To be concrete, we propose

Conjecture. The \(L\)-zero sum problem, i.e., the \(L\)-subset finding problem with domain \(D = \{1, \ldots, N\}\), range \(R = \{0, 1\}^m\), and property \(\mathcal{P} = \{((x_1, y_1), \ldots, (x_L, y_L)) : y_1 \oplus \cdots \oplus y_L = 0^m\}\), where \(m\) is some function of \(N\), requires \(\Omega(N^{L/(L+1)})\) quantum queries.

While we believe this to be the case, the best lower bound we know is \(\Omega(N^{2/3})\), independent of \(L\), for any \(L \geq 2\).

Of course, it would also be interesting to prove lower bounds for problems where something can be learned about subsets of size smaller than \(L\), such as \(L\)-element distinctness or finding an \(L\)-clique in a graph. For the former it is not so clear whether the algorithm could be improved, whereas for the latter there is no particular reason to expect that the known algorithms are optimal for all values of \(L\).

Note added. After this paper was posted to the quant-ph archive, we learned through private communication that Magniez, Santha, and Szegedy independently considered the problem of finding an \(L\)-vertex subgraph in an \(N\)-vertex graph, and found an algorithm using \(\tilde{O}(N^{2L/(L+1)})\) queries. This result is now described in [4]. In the context of the present paper, this can be seen by choosing \(M = \lfloor N^{(L-1)/L} \rfloor\) in [23]. Note that this result subsumes the simple algorithm [17], and also improves the \(\tilde{O}(N^{(5L-2)/(2L+4)})\)-query algorithm for \(L = 5\).

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