Protecting clean critical points by local disorder correlations

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Abstract – We show that a broad class of quantum critical points can be stable against locally correlated disorder even if they are unstable against uncorrelated disorder. Although this result seemingly contradicts the Harris criterion, it follows naturally from the absence of a random-mass term in the associated order parameter field theory. We illustrate the general concept with explicit calculations for quantum spin-chain models. Instead of the infinite-randomness physics induced by uncorrelated disorder, we find that weak locally correlated disorder is irrelevant. For larger disorder, we find a line of critical points with unusual properties such as an increase of the entanglement entropy with the disorder strength. We also propose experimental realizations in the context of quantum magnetism and cold-atom physics.

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Introduction. – The effects of quenched disorder in condensed matter are various and constitute an interesting and important field of research. For instance, randomness can change the universality class of a critical point and even originate novel phases. A paradigmatic effect of disorder near a phase transition is known as “random mass.” As there is no translational invariance, different regions of the same system can be at different distances from criticality\(^1\). Thus, disorder fluctuations induce large and rare regions which can be “locally” in one phase while the bulk is in the other one. This yields the so-called Griffiths singularities [1–3].

Using the random-mass concept, Harris [4] formulated a simple criterion for the relevance of disorder at continuous phase transitions: If \(dν < 2\) (with \(d\) being the spatial dimension and \(ν\) the clean correlation length exponent), then the clean critical behavior is destabilized by weak disorder. This is the famous Harris criterion. It applies to the case of uncorrelated disorder. Spatial correlations among the random masses modify the criterion. When such correlations decay as \(x^{-a}\) with distance \(x\), the Harris criterion reads \(\min\{d, a\}ν < 2\) [5].

This implies that correlations are relevant only when they decay slower than \(x^{-d}\). Interestingly, when the disorder correlations are relevant, the Griffiths singularities are also enhanced [6]. From the above arguments, one may expect that local (i.e., short-range) disorder correlations do not change the relevance or irrelevance of the disorder.

In this letter, we show that this is not always true. Specifically, we demonstrate that certain types of local correlations can render the disorder perturbatively irrelevant even though \(dν < 2\), thereby stabilizing the clean critical point against weak disorder. Although this result appears to contradict the Harris criterion, it arises naturally when disorder correlations make the mass term spatially uniform. If the strength of the correlated disorder is increased beyond a critical value, where perturbative methods cannot be used anymore, a line of finite-disorder critical points (tuned by disorder strength) appears.

Field theory. – Before turning to a specific microscopic model, let us consider the relevance of disorder within a general field theory framework. Consider a (clean) Euclidean quantum-field theory given by the action \(S_0[φ] = \int dτ dx L_0[φ]\) (\(x\) represents position in \(d\) dimensions and \(τ\) is imaginary time). The action \(S_0\) is perturbed by a disorder term \(S_{\text{dis}} = \sum_φ \int dτ dx λ_0(φ)O(x, τ)\). Here,
the sum is over all possible operators $\mathcal{O}$ to which disorder can couple, and $\lambda_\mathcal{O}(x)$ is a random variable of zero mean. (As we are considering quenched disorder, $\lambda_\mathcal{O}$ depends only on the spatial coordinates.) Using the replica trick and performing a tree level renormalization group (RG) calculation (see, e.g., ref. [7]), the relevance of $\lambda_\mathcal{O}(x)$ can be gauged by computing the scale dimension $|w_\mathcal{O}|$ of the second moment $w_\mathcal{O}$ of $\lambda_\mathcal{O}(x)$ at the clean fixed point. If $|w_\mathcal{O}| = d + 2z_0 - 2|\mathcal{O}| > 0$, where $|\mathcal{O}|$ is the scale dimension of $\mathcal{O}$ and $z_0$ is the dynamical exponent of the clean theory $S_0$, then disorder is perturbatively relevant. This can be viewed as a generalization of the Harris criterion.

Applying these ideas to a $\phi^4$ order parameter theory, $\mathcal{L}_0 = r|\phi|^2 + u|\phi|^4 + |\partial_x \phi|^2 + |\partial_y \phi|^2$, we find that quenched disorder can be relevant at tree level only if it couples to the mass term $|\phi|^2$ (see footnote 2). (We exclude random fields that locally break the symmetry.) This follows from the fact that $|w_\phi| = -d + 2/\nu$, which recovers the original Harris criterion, while disorder coupling to higher order and gradient terms leads to negative scale dimensions and is thus irrelevant. Consequently, if the macroscopic random variables do not produce a random mass term, disorder effects are strongly suppressed.

**Transverse field Ising chain.**—A tantalizing example in which the above scenario actually completely removes the random mass and thus stabilizes the clean critical point even though the inequality $d\nu < 2$ is fulfilled is the ferromagnetic quantum phase transition of the 1D transverse-field Ising model. Its Hamiltonian reads

$$H_{\text{Ising}} = -\sum_i J_i \sigma_i^z \sigma_{i+1}^z - h_i \sigma_i^x, \quad (1)$$

where $J_i$’s are the interactions, $h_i$’s are the transverse fields, and $\sigma_i^z$ and $\sigma_i^x$ are Pauli matrices. The correlation-length exponent of the clean ($J_i \equiv J, h_i \equiv h$) critical point is $\nu = 1$, and the Harris criterion predicts that weak disorder in the $h_i$ and $J_i$ is relevant. In agreement with this, uncorrelated disorder has dramatic effects [2,8,9]. The clean critical point is unstable against weak disorder and the RG flows towards an exotic infinite-randomness critical point (IRCP) whose dynamics is so slow that the dynamical exponent $z$ is formally infinite. Surrounding the transition, there are gapless quantum Griffiths phases in which the dynamical exponent $z$ can be arbitrarily large and the correlation length $\xi$ is finite. Moreover, the average entanglement entropy $S$ [10,11] of a subsystem with length $\ell$ embedded in the bulk diverges at the IRCP as $S(\ell) \sim (c_{\text{eff}}/3) \ln \ell$, where $c_{\text{eff}} = (1/2) \ln 2$ is often called the effective central charge [12,13]. The properties of this model are summarized in fig. 1(b).

The order parameter field theory of this transition can be obtained from the Hamiltonian (1) via a Trotter-Suzuki decomposition of the partition function and a Hubbard-Stratanovich transformation of the resulting path integral. After taking the continuum limit, the resulting $(1+1)$-dimensional action takes the form of a disordered $\phi^4$ theory (see above). The (local) coefficient of the mass term $|\phi|^2$ must be a function of the ratios $h_i/J_i$ because these are the only two energy scales in the problem. In fact, it has been shown [14] that the critical point of (1) occurs exactly when $\prod J_i = \prod h_i$, suggesting that $\delta_i = \ln(h_i/J_i)$ is an appropriate measure of the local distance from criticality. Consequently, it is clear that a simple local correlation between $h_i$ and $J_i$, say, $h_i = e^{\delta_i}J_i$, is sufficient to make the mass term uniform rather than random.

The disorder of the higher-order and gradient terms of the renormalized action cannot be inferred from this argument, generically they remain random because they do not depend on the combinations $h_i/J_i$ only. Although translational symmetry is broken, the system is locally at the same distance from criticality everywhere.

**Numerical results.**—To check the prediction that such correlated disorder is irrelevant at the clean critical point, we mapped the Hamiltonian (1) onto free fermions using the Jordan-Wigner transformation [15,16]. We then performed an exact-diagonalization study contrasting the cases of uncorrelated and correlated disorder. The absence of Griffiths phases can be verified by analyzing the gap $\Delta$ in the excitation spectrum as a function of the distance from criticality $\delta = |\delta_i|_{\text{av}} = \ln(h_i/J_i)_{\text{av}}$

3At first glance, this definition seems to break the symmetry between left and right neighbors in (1). However, the concept of a distance from criticality is well defined only for regions large compared to the lattice constant. After averaging $\delta_i$ over such a region, the discrepancy vanishes.

4It is important to distinguish the bare and renormalized actions. Plante’s criticality condition [14] guarantees that the renormalized theory does not have a random mass term. This does not imply that this term must vanish in any bare theory.

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Fig. 1: (Color online) (a) Disorder averaged spectral gap $\Delta$ vs. distance from criticality $\delta$ for different disorder parameters $h_0$ with and without disorder correlations (sizes up to $2^{15}$ sites, averaged over 1000 disorder realizations). (b) RG flow diagram for uncorrelated disorder, the disorder strength renormalizes to infinity. (c) Flow diagram for correlated disorder ($h_i = e^{\delta_i}J_i$), the clean fixed point is stable. The line of fixed points for strong disorder can be only accessed for $h_0 = 0$ (see text).
(see fig. 1(a)). Here, the fields are drawn from a uniform (box) distribution in which \(0 < h < 1\). Hence, \(h_0\) parameterizes the disorder strength. For correlated disorder, we set \(h_i = e^\delta J_i\), while for uncorrelated disorder, the couplings are independent random variables drawn from a uniform distribution such that, for the sake of comparison, \(h_0 e^{-\delta} < J_i < e^{-\delta}\). Figure 1(a) clearly shows that the gap vanishes only at criticality (\(\delta = 0\)) for correlated disorder, signalling the absence of quantum Griffiths phases. In contrast, the gap closes before criticality is reached when the disorder is uncorrelated, a hallmark of quantum Griffiths singularities.

The above uniform distributions of fields and couplings represent weak or moderate disorder (as long as \(h_0 \neq 0\)). To study the fate of the critical point for stronger correlated disorder (and, therefore, study nonperturbative effects of disorder not accomplished by the conventional field-theoretic analysis), we now consider a family of gapless disorder distributions.

\[
\mathcal{P}(h) = (1/D) h^{-1+1/D}, \text{ with } 0 < h < 1, \tag{2}
\]

with the bonds given by \(J_i = e^\delta h_i\) as before. \(D\) parameterizes the disorder strength. In this case, the system is gapless for any \(\delta\); the absence of a gap simply follows from the low-energy tail of the distribution \(\mathcal{P}\).

We first focus on the correlation length \(\xi\) as a function of \(\delta\) which is computed by fitting the spin-spin correlation function \(\langle \sigma_i^z(1) \sigma_i^z(n) \rangle \sim e^{-x/\xi} x^{-\eta}\). Figure 2(a) shows that \(\xi\) takes its clean value regardless of disorder strength \(D\). The correlation length exponent is thus \(\nu = \nu_{\text{clean}} = 1\) for all \(D\), i.e., it violates the inequality \(d \nu > 2\) [17]. This implies that the spin correlations are governed by the clean physics analogous to gapped quantum magnets doped with non-magnetic impurities [18]. We now explore the critical point \(J_c = h_i\), upon increasing the disorder strength \(D\). The dynamical exponent \(z\), computed using the finite-size scaling of the gap, \(\Delta \propto L^{-z}\), is shown in fig. 2(b). We see that \(z = z_{\text{clean}} = 1\) for \(D < D_c \approx 0.3\), whereas for stronger disorder, \(z\) becomes a monotonically increasing function of \(D\).

The data in figs. 1 and 2 confirm our prediction that the ferromagnetic quantum phase transition is governed by the clean fixed point for weak correlated disorder. For stronger disorder, it is governed by a line of finite-disorder fixed points as shown in fig. 1(c). This line of fixed points also emerges from an extension of the strong-disorder RG (SDRG) [9] to correlated disorder [19]. Moreover, for very strong disorder, the result \(z \approx D\) can be understood as arising from the singularity of the bare disorder distribution via local fluctuations of weakly connected spin clusters. This line of fixed points belongs to a novel universality class dominated by disorder effects that are perturbatively irrelevant. In this class, some critical exponents (such as \(\nu\)) are universal and take their clean values, while others are nonuniversal, such as \(z\).

**Entanglement entropy.** – We now turn to the ground-state (GS) entanglement properties at criticality in the case of correlated disorder, \(J_i = h_i\). We first study the von Neumann entanglement entropy \(S(\ell) = -\text{Tr}(\rho_A \ln \rho_A)\), where \(\rho_A\) is the reduced density matrix of a subsystem A of size \(\ell\). Because we wish to relate \(S(\ell)\) to local fluctuations of a globally conserved operator, we consider the spin-1/2 random XX chain,

\[
H_{XX} = 4 \sum_i t_i \left( S_i^z S_{i+1}^z + S_i^x S_{i+1}^x \right), \tag{3}
\]

from now on. It can be mapped onto the Hamiltonian (1) by setting \(t_{2i-1} = h_i\) and \(t_{2i} = J_i\). The two systems share the *same* entanglement properties at criticality with \(2S_{\text{sing}}(\ell) = S_{XX}(\ell)\) [20].

Figure 3 shows our numerical results for the average entanglement entropy \(\bar{S}(\ell)\), computed via standard methods [13]. As expected, for weak correlated disorder, \(D < D_c\), \(\bar{S}(\ell) \sim (c_{\text{eff}}/\ell) \ln \ell\), is universal and \(c_{\text{eff}}\) takes the clean value \(c_{\text{eff}} = c = 1\). Surprisingly, for \(D > D_c\), the entanglement entropy *increases* monotonically with \(D\), i.e., \(\bar{S} \sim \frac{1}{c_{\text{eff}}} \ln \ell\), with \(c_{\text{eff}} > c\), as shown in inset (a). In contrast, for uncorrelated disorder, \(c_{\text{eff}} = c \ln 2 < c\) [12]. The increase of \(\bar{S}(\ell)\) with correlated disorder was first noticed in ref. [21], but the different nature of the GS, its dependence on the disorder strength and its universality...
class were not considered. We also point out that a small deviation from perfect disorder correlations will drive the system back to the IRCP physics, as revealed by the SDRG and checked numerically [19].

Recently, it has been noted that $S(t)$ can be related to the local fluctuations of certain thermodynamic quantities [22]. Here, we use this relation in order to gain further insight into the GS of (3). Hence, we compute the fluctuations $F_m(t)$ of the magnetization $S^m_A = \sum_{i \in A} S^z_i$ of a subsystem A with length $t$. $F_m(t) = (\langle S^m_A \rangle^2 - \langle S^m_A \rangle^2)^{1/2}$. In the clean case ($t = 0$), the fluctuations were shown to be proportional to the von Neumann entropy, $F(t) = \kappa \ln t$, with $\kappa = \pi^{-2}$, yielding $c = \pi^2 \kappa$ [23]. For correlated disorder ($t_{2\nu - 1} = t_{2\nu}$), $F(t) = \kappa_{eff} \ln t$ where $\kappa_{eff}$ takes its clean value for $D < D_c$ and increases similarly to $c_{eff}$ for $D > D_c$, as shown in inset (b) of fig. 3. Interestingly, the ratio $c_{eff}/\kappa_{eff} = \pi^2$ remains identical to its clean value in the entire $D$-range studied. Thus even though the line of critical points at $D > D_c$ is dominated by disorder, the relation between fluctuations and entanglement entropy strongly resembles the delocalized clean system. In contrast, in the random-singlet state arising for uncorrelated disorder, it is easy to show that $F(t) = \kappa_{eff} \ln t$ with $\kappa_{eff} = 1/12$ [24].

Currently, to the best of our knowledge, there is only one general framework in which $c_{eff}$ can be understood in terms of random-singlet phases of 1D systems [25]. We remark that such framework does not apply to our correlated-disorder case as there is no corresponding random-singlet state. Thus, a fundamental understanding of $c_{eff}$ here reported is still lacking and further fundamental understanding of the entanglement entropy in random systems is desirable. Studying the correlated-disorder effects in the many random-singlet states considered in ref. [25], and comparing their entanglement entropy with their clean counterparts seems to be a helpful direction of research.

**Experimental realizations.** – Consider a quantum $S = 1/2$ chain with nearest-neighbor exchange $J_{S_{-i}S_i}$. Doping the chain with a small concentration of $S_{imp} = 1/2$ impurities having $J_{S_{imp}S} \neq J_{S_{-i}S_i}$ would realize a slightly modified $SU(2)$ (Heisenberg) version of a correlated double-bond Hamiltonian (3), closely related to the Kondo problem in spin chains [26,27]. Based on the Matsubara-Matsuda representation of spin-1/2 degrees of freedom by hard-core bosons [28], one could also implement the corresponding hard-core bosonic model (with now correlated random hoppings) using cold-atom systems. Finally, one-dimensional polymers such as the family of polyaniline [29] are modeled as random-dimer tight-binding chains [30] which can defy Anderson localization. This random-dimer chain can be mapped to the Hamiltonian (3). Our results thus provide an alternative view of the absence of localization in certain 1D electronic systems.

**Discussion and conclusions.** – In summary, we have presented a general mechanism by which local correlations between the random variables render a clean critical point stable against weak disorder even though it violates the inequality $d\nu > 2$. Although this appears to contradict the Harris criterion, we emphasize that it merely violates one of its preconditions, namely the spatial variation of the distance from criticality. Indeed, Harris [4] argued that a clean critical point is stable if the mean (local) distance from criticality $|r|_{av}$ is larger than the width $\Delta r$ of its distribution. For uncorrelated disorder, this yields $\Delta r/|r|_{av} \sim \xi^{-\nu} < 1$ for $\xi \to \infty$, recovering the $(d\nu > 2)$-form of the Harris criterion. However, for our correlated disorder, $\Delta r \equiv 0$, and $|r|_{av} \gg \Delta r$ is satisfied regardless of the values of $d$ and $\nu$.

Our mechanism for suppressing the disorder effects by local correlations will also operate in $d > 1$. The question under what general conditions the random mass term can be removed completely will be relegated for future research. Interestingly, Yao et al. [31] recently reported apparent violations of the Harris criterion in several disordered dimerized spin models. In one system (random dimer model), this has been attributed to the fact that the quantum critical point does not depend on the disorder strength, which strongly resembles our mechanism for the absence of random mass. A similar argument was given for the Mott-insulator to superfluid transition at the tip of the Mott lobes [32]. Another system of ref. [31] (random plaquette model) shows a dependence of the critical coupling on the disorder strength, albeit a very weak one. We emphasize that in such a case, our mechanism would restrict the deviations from clean critical behavior to a narrow interval around the critical point that may well be unobservable.
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