COMMENTS ON THREE-BRANES

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\textbf{ABSTRACT}

The Born–Infeld-like effective world-volume theory of a single 3-brane is deduced from a manifestly space-time supersymmetric description of the corresponding $D$-brane. This is shown to be invariant under $SL(2, \mathbb{R})$ transformations that act on the abelian gauge field as well as the bulk fields. The effective theory of two nearby parallel three-branes involves massive world-volume supermultiplets which transform under $SL(2, \mathbb{Z})$ into the dyonic solitons of four-dimensional $N = 4$ spontaneously broken $SU(2)$ Yang–Mills theory.

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The $p$-brane solitonic solutions of string theory play an important rôle in determining the systematics of the duality relationships between string theories. These $p$-branes carry charges associated with $p+1$-form potentials in the Neveu–Schwarz–Neveu–Schwarz (NS⊗NS) sector and the Ramond–Ramond R⊗R sector. According to the $D$-brane description the R⊗R sector $p$-branes are described by configurations of open superstrings with endpoints tethered on a $(p + 1)$-dimensional hypersurface embedded in ten dimensions. The surface may, in principle, be curved but it is simplest in the first instance to consider flat $p$-branes. The coordinates of the open-string end-points are restricted to the plane $X^i = y^i$, where the directions labelled $i$ are transverse to the brane world-volume ($i = p+1, \ldots, 9$). The dynamics of the brane is therefore prescribed by the open superstring theory with Neumann boundary conditions in the directions labelled by $\alpha = 0, \ldots, p$ and Dirichlet boundary conditions in the transverse directions [2, 3]. The usual kind of ‘effective’ world-volume field theory emerges only as an approximation to this ‘underlying’ open superstring theory. A characteristic feature of the world-volume theory of a R⊗R $p$-brane is the occurrence of a world-volume vector field [4, 5]. In the $D$-brane description this arises as a ground state of the open-string theory. The fact that such $D$-branes are described by open superstring theory means that they preserve half the supersymmetry of the fundamental type II string theory, which means that they are BPS solitons of the type II theories. The supersymmetries of the type II theory that are broken by the open-string truncation are associated with the Goldstone fermions living in the world-volume [6].

Source equations and effective actions

The effective action for superstring theory in the presence of a solitonic R⊗R $p$-brane can, in principle, be written in the form

$$S = S_{\text{bulk}} + S^{(p)}_{\text{source}},$$  

(1)

where the bulk action is a ten-dimensional integral while the source is restricted to $p + 1$ dimensions. The equations of motion that follow from this action arise in the underlying string theory from the consistency conditions that ensure conformal invariance of the world-sheet theory. In the case of the type IIB theory there is no obvious covariant bulk action (due to the presence of a five-form field strength that satisfies a self-duality condition) but the source-free equations of motion are known [7, 8]. The bulk terms arise from the usual closed-string sector of the theory but the source terms come from world-sheets with boundaries on which there can be a condensate of the open-string fields. These actions, which are of the Born–Infeld type, reproduce the equations of motion that follow from the consistency conditions for string theory in the background that includes the open-string boundary condensate, $F = dA$, of the electromagnetic field where $A$ is the massless open-string abelian vector potential. Such conditions on open string theory with Neumann conditions in all directions were originally derived [9, 10, 11] in the ten-dimensional type I theory assuming that $F_{\alpha\beta}$ is constant. Imposing Dirichlet conditions in $9 - p$ directions defines a $D$-brane boundary state.
An effective action of the form \( S_{\text{source}}^{(1)} \) is obtained in the case of the \( R \otimes R \) one-brane (the \( D \)-string) by considering the consistency conditions for type IIB string theory in the presence of world-sheet boundaries with Neumann conditions in two directions and fixed Dirichlet boundary conditions in the remaining eight directions. The form of \( S_{\text{source}}^{(1)} \) is determined in the case that \( F \) is constant to be a version of the Born–Infeld theory coupled to the metric (\( G \)), the background antisymmetric potentials (\( B^N \) and \( B^R \)) and scalars (\( \phi \) and \( \chi \)) in the NS \( \otimes \) NS and R \( \otimes \) R sectors. It was shown in [12, 13] that this is the appropriate world-sheet action to describe a type IIB string carrying charges in the NS \( \otimes \) NS and R \( \otimes \) R sectors. Furthermore, the action of \( SL(2, \mathbb{Z}) \) on the background closed-string fields induces the transformation among the infinite set of dyonic type IIB strings that arise as solitons in the usual field-theoretic construction [7].

In this paper we will use the constraints of space-time supersymmetry (formulated in a light-cone gauge) to determine the effective world-volume theory of the three-brane which is a four-dimensional supersymmetric theory with \( N = 4 \) global supersymmetry. The bosonic part of the three-brane theory will be described and the \( SL(2, \mathbb{R}) \) invariance of its equations of motion demonstrated. Any vacuum configuration of \( n \) parallel \( p \)-branes describes the moduli space of supersymmetric \( U(n) \) Yang–Mills theory [3] dimensionally reduced from ten dimensions to \( p + 1 \) dimensions where the \( U(1) \) symmetry of a particular brane can be viewed as a subgroup of \( U(n) \). The system of two closely separated parallel three-branes has excited states in which the branes are joined by a string which may be either a fundamental type IIB string or any one of the dyonic solitonic type IIB strings [14]. We will discuss how \( SL(2, \mathbb{Z}) \) acts to reveal the usual spectrum of solitonic dyonic states in the four-dimensional world volume.

**Light-cone boundary states and D-branes.**

A closed-string state in the \( R \otimes R \) or NS \( \otimes \) NS sector coupling to a world-sheet with a boundary can be represented as a semi-infinite cylinder that describes the evolution of a closed string state \( |\Phi\rangle \) from \( \tau = -\infty \) to the boundary at \( \tau = 0 \),

\[
\langle \Phi \rangle_{\text{Disk}} = \langle \Phi | B, \eta \rangle,
\]

where \( | B, \eta \rangle \) denotes the boundary state and \( \eta = \pm 1 \) labels whether the state is BPS or anti-BPS. In the formalism with manifest world-sheet supersymmetry there are separate boundary states in each sector, \( | B, \eta \rangle_N \) and \( | B, \eta \rangle_R \), which both satisfy the boundary conditions,

\[
(\partial X^i - \bar{\partial} X^i) | B, \eta \rangle = 0, \quad (\partial X^\alpha + \bar{\partial} X^\alpha) | B, \eta \rangle = 0
\]

which impose Neumann conditions on \( p + 1 \) directions and Dirichlet conditions on the rest (a state \( | B, \eta \rangle \) with no subscript is in either of the two possible sectors). Implicit in [8] is the fact that \( X^i(\sigma) | B, \eta \rangle = y^i | B, \eta \rangle \), where \( y^i \) is the constant transverse position of the brane. In order to ensure that half the world-sheet supersymmetry is preserved the
world-sheet fermions must satisfy the boundary conditions
\[
(\psi^i + i\eta \tilde{\psi}^i)|B, \eta\rangle = 0, \quad (\psi^\alpha - i\eta \tilde{\psi}^\alpha)|B, \eta\rangle = 0. \tag{4}
\]

It is easy to see that these conditions ensure that an infinite sequence of local supersymmetry gauge conditions are preserved \[13, 1\],
\[
(F + i\eta \tilde{F})|B, \eta\rangle = 0, \tag{5}
\]
where \( F = \psi \cdot \partial X, \tilde{F} = \bar{\psi} \cdot \bar{\partial} X \).

Space-time supersymmetry, which is not manifest in the above approach, relates the two kinds of boundary state, \(|B\rangle_N\) and \(|B\rangle_R\). However, it is also possible to formulate the problem in a light-cone gauge in which space-time supersymmetry is manifest \[13, 16\]. In this gauge \( \tau = X^+ = (X^0 + X^9)/\sqrt{2} \) so that the boundary state is at a fixed time, i.e., \( X^+ \) is required to satisfy a Dirichlet condition. Furthermore, \( X^- = (X^0 - X^9)/\sqrt{2} \) is determined in terms of the transverse \( X^I \) coordinates \( (I = 1, 2, \cdots, 8) \) and their fermionic partners and also satisfies a Dirichlet condition (whether \( X^I \) are Neumann or Dirichlet coordinates). This means that there are at least two Dirichlet directions and that one of these is time-like. This kinematics describes a \'(p+1)-instanton'\ rather than a \(D\)-brane, where time is one of the Neumann directions. It is related to the \(D\)-brane by a double Wick rotation. The coordinates transverse to the \( \pm \) directions satisfy the boundary conditions
\[
(\partial X^I + M_{IJ} \tilde{\partial} X^J)|B, \eta\rangle = 0, \tag{6}
\]
where \( M_{IJ} \) is an element of \(SO(8)\). The Neumann directions will be chosen to be \( \alpha = I = 1, \cdots, p + 1 \) while the Dirichlet directions will be chosen to be \( i = I = p + 2, \cdots, 8 \). In the absence of a boundary condensate this matrix can be written in block diagonal form,
\[
M_{IJ} = \begin{pmatrix}
I_{p+1} & 0 \\
0 & -I_{7-p}
\end{pmatrix}, \tag{7}
\]
(\text{where } I_q \text{ indicates the } (q \times q)\text{-dimensional unit matrix}). In the presence of a boundary condensate of the open-string vector potential \( M_{IJ} \) is a more general \(SO(8)\) matrix and can be written as,
\[
M_{IJ} = -\exp \left\{ \Omega_{KL} \Sigma_{IJ}^{KL} \right\}, \tag{8}
\]
where \( \Sigma_{IJ}^{KL} = (\delta^K_L \delta^I_J - \delta^K_J \delta^I_L) \) are generators of \(SO(8)\) transformations in the vector representation. The parameters \( \Omega_{IJ} \) depend on the open-string boundary condensate and can be written in block off-diagonal form in a particular basis as,
\[
\Omega_{\alpha\beta} = \sum_{m=0}^{(p-1)/2} c_m \Sigma_{\alpha\beta}^{2m+1} 2m+2, \tag{9}
\]
and $\Omega_{ij} = \Omega_{ai} = 0$ if the brane is static. In the absence of a condensate $c_m = \pi$.

The conditions on the fermionic coordinates will be derived by requiring the boundary state to be annihilated by a linear combination of the space-time supercharges,

$$Q^{+a}_\eta \equiv (Q^a + i\eta M^a_\eta \tilde{Q}^\eta)|B, \eta\rangle = 0, \quad Q^{+\hat{a}}_\eta \equiv (Q^{\hat{a}} + i\eta M^{\hat{a}}_\eta \tilde{Q}^{\hat{\eta}})|B, \eta\rangle = 0,$$

where $Q^a$ and $Q^{\hat{a}}$ are two inequivalent $SO(8)$ left-moving supercharges and $\tilde{Q}^a$, $\tilde{Q}^{\hat{a}}$ are the right-moving ones. These conditions enforce the requirement that the boundary state preserves one half of the space-time supersymmetries, generalizing one of the arguments in [15] to the context of the $D$-brane. This means that the $D$-brane is a BPS saturated state.

It is easy to deduce from (10) that the modes of the $SO(8)$ fermionic world-sheet fields, $S^a$ and $S^{\hat{a}}$, must satisfy the boundary conditions,

$$(S^a_n + i\eta M_{ab}S^b_{-n})|B, \eta\rangle = 0, \quad (S^{\hat{a}}_n + i\eta M_{\hat{a}\hat{b}}S^{\hat{b}}_{-n})|B, \eta\rangle = 0.$$

The bispinor $SO(8)$ matrix $M_{ab}$ is determined by consistency with the superalgebra to satisfy the condition

$$M_{ab}M_{cb} = \delta_{ac},$$

and $M_{\hat{a}\hat{b}}$ satisfies,

$$\gamma^I_{\hat{a}\hat{b}} - M_{\hat{a}\hat{b}}M_{I\hat{J}}\gamma^J_{\hat{a}\hat{b}} = 0.$$

These conditions imply that $M_{ab}$ and $M_{\hat{a}\hat{b}}$ describe the same $SO(8)$ rotations as $M_{I\hat{J}}$ but acting on the spinors rather than on vectors. In the absence of a condensate these equations are solved by

$$M_{ab} = (\gamma^1\gamma^2\ldots\gamma^{p+1})_{ab}, \quad M_{\hat{a}\hat{b}} = (\gamma^1\gamma^2\ldots\gamma^{p+1})_{\hat{a}\hat{b}}.$$

More generally, in the presence of a condensate,

$$M_{ab} = e^{i\frac{1}{2}\Omega_{I\hat{J}}\gamma^I_{ab}},$$

with the same $SO(8)$ rotation $\Omega_{I\hat{J}}$ as before.

The boundary state that satisfies these conditions is given by

$$|B\rangle = \exp \sum_{n>0} \left(\frac{1}{n}M_{I\hat{J}}\alpha^I_{-n}\tilde{\alpha}^J_{-n} + iM_{ab}S^a_n\tilde{S}^a_{-n}\right) |B_0\rangle,$$

where the zero-mode factor is

$$|B_0\rangle = C \left(M_{I\hat{J}}|I\rangle\tilde{J}\rangle + iM_{\hat{a}\hat{b}}|\hat{a}\rangle\tilde{\hat{b}}\rangle\right).$$
The overall normalization constant $C$ of the boundary state can easily be determined by relating the cylinder diagram to an annulus (as in [11, 17]) which is here interpreted as the free energy of a gas of open strings and is given as a trace over physical open-string states. This boundary state defines source terms for all the massless and massive closed-string fields since it describes the relative couplings of the boundary to the closed-string states. The source terms for the massless fields in the effective action can be obtained from the zero-mode factor $C$. The results agree with calculations in the covariant boundary state formalism in the purely Neumann case considered in [18] and the $D$-string considered in [19, 13].

The non-linearly realized supersymmetries are those that are generated by the supercharges that are not annihilated by the boundary state, $Q^-_a \equiv (Q^a - i\eta M^a_b \tilde{Q}^b)$ and $Q^-_{\bar{a}} \equiv (Q^{\bar{a}} - i\eta M^{\bar{a}}_{\bar{b}} \tilde{Q}^{\bar{b}})$. The $Q^-$ supercharges are zero-momentum fermion emission vertices so that $Q^{-a}_a |B\rangle$ and $Q^{-\bar{a}}_{\bar{a}} |B\rangle$ are equivalent to zero-momentum fermion insertions and a soft Goldstino theorem can be demonstrated [16] as required for a non-linearly realized supersymmetry.

**The three-brane source equations**

In the example of the three-brane (or four-instanton) there are four transverse Dirichlet directions and four Neumann (recall that in this discussion the $\pm$ light-cone directions are identified with two directions transverse to the $(p + 1)$-dimensional world-volume).

In the absence of a boundary condensate it is easy to see that the boundary state (17) does not couple to the dilaton. This follows simply from the fact that $\text{Tr} M_{IJ} = 0$ (with $M_{IJ}$ defined in (7)). This agrees with the ansatz that defines the three-brane soliton of the supergravity in [3] where the only non-trivial fields are the graviton and the fourth-rank antisymmetric potential $A^{(4)}$. Furthermore, since $M_{ab} = \frac{1}{2}(\gamma^{1234} + \gamma^{5678})$ the boundary state couples equally to $A^{(4)}$ and $\hat{*}A^{(4)}$ (where $\hat{*}$ denotes the Poincaré dual in the eight-dimensional space transverse to the $\pm$ directions) and therefore it respects the self-duality of $F^{(5)}$.

More generally, there will be a boundary condensate of the vector potential with field strength $F_{\alpha\beta}$, which will here be assumed to be constant. In this case the $SO(8)$ matrix, $M_{IJ}$, may be written in $4 \times 4$ block diagonal form in a suitable basis [19, 16],

$$M_{IJ} = \begin{pmatrix} (1-F) & 0 & 0 \\ (1+F) & 0 & -14 \end{pmatrix}$$

where $F \equiv F_{\alpha\beta}$ is the constant $4 \times 4$ field strength in the world-volume of the brane. An orthogonal transformation brings this to the form,

$$F_{\alpha\beta} = \begin{pmatrix} 0 & f_1 & 0 & 0 \\ -f_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & f_2 \\ 0 & 0 & -f_2 & 0 \end{pmatrix}$$

5
so that,
\[
\frac{1 - F}{1 + F} = \begin{pmatrix} M(f_1) & 0 \\ 0 & M(f_2) \end{pmatrix}
\]  
(20)

where
\[
M(f) = \frac{1}{1 + f^2} \begin{pmatrix} 1 - f^2 & -2f \\ 2f & 1 - f^2 \end{pmatrix}
\]  
(21)

The matrix \( M_{IJ} \) can be expressed in the form (8) with \( c_n = \pi + \alpha_n \), where \( \cos \alpha_n = (1 - f_n^2)/(1 + f_n^2) \) (with \( n = 1, 2 \)). Since \( M_{IJ} \) has both symmetric and antisymmetric parts it is a source for \( B^N \) as well as the metric and dilaton.

Likewise the matrix \( M_{ab} \) is modified by the boundary condensate and can be written in the same basis as
\[
M_{ab} = \frac{1}{\sqrt{(1 + f_1^2)(1 + f_2^2)}} (f_1 f_2 \delta_{ab} + f_2 \gamma_{ab}^{12} + f_1 \gamma_{ab}^{34} + \gamma_{ab}^{1234})
\]  
(22)

The normalization constant is determined by considering the equivalence of the annulus and cylinder diagrams and is \( C = \sqrt{(1 + f_1^2)(1 + f_2^2)} = \sqrt{\det(1 + F)} \).

The effective three-brane action and \( SL(2, R) \) symmetry.

It is straightforward to verify that the covariant (euclidean) effective action that reproduces these source terms has the form,
\[
S_{\text{source}}^{(3)} = \int d^4x \left( e^{-\phi} \sqrt{\det(G + F)} + \frac{i}{4} \chi \mathcal{F} \ast \mathcal{F} + \frac{i}{2} B^R \ast \mathcal{F} + \frac{i}{24} \epsilon^{\alpha \beta \gamma \delta} A^{(4)}_{\alpha \beta \gamma \delta} \right),
\]  
(23)

where \( \mathcal{F} = F - B^N \) and \( \ast F_{\gamma \delta} = \frac{1}{2} \epsilon_{\gamma \delta} F_{\alpha \beta} \). The sources generated by the boundary state (17) match up with the expansion of (23) in small fluctuations of the bulk closed-string fields to linearized order, keeping all orders in \( F \). The \( R \otimes R \) sector terms, proportional to \( \chi \), \( B^R \) and \( A^{(4)} \), match in an obvious manner. The terms in the NS \( \otimes \) NS sector are proportional to \( \phi \) and \( h_{\alpha \beta} = \eta_{\alpha \beta} - G_{\alpha \beta} + B^N_{\alpha \beta} \). It is important to remember that in the light-cone gauge \( \phi = h^I_I/4 = (h^i_i + h^\alpha_\alpha)/4 \). The expansion of the determinant factor in (23) gives
\[
-\phi \sqrt{\det(1 + F)} + \frac{1}{2} \sqrt{\det(1 + F)} h^{\alpha \beta}(1 + F)^{-1}_{\alpha \beta} = \frac{1}{4} \sqrt{\det(1 + F)} \left( -h_{ii} + \left( \frac{1 - F}{1 + F} \right)_{\alpha \beta} h^{\alpha \beta} \right),
\]  
(24)

which is precisely the same as the source obtained from the boundary state (17).

The action (23) encodes the information about the equations of motion that follow from the string consistency conditions. However, it is only these equations that we shall make use of in what follows since it is well known that the presence of the self-dual five-form field strength means that the bulk part of the action \([11]\) cannot be written in
a covariant manner. This action is presented in the string frame in (23). In order to
discuss the duality properties of the theory we shall transform this to the Einstein frame
by transforming $G \rightarrow G_E = G e^{-\phi/2}$. This replaces (23) with

$$S_{source}^{(3)} = \int d^4x \left( \sqrt{\text{det}(G_E + e^{-\phi/2} F)} + \frac{i}{4} \chi F \ast F + \frac{i}{2} B_R \ast F + \frac{i}{24} \epsilon^{\alpha\beta\gamma\delta} A^{(4)}_{\alpha\beta\gamma\delta} \right),$$

with the corresponding change of metric in the bulk part of the total action.

When continued to lorentzian signature the action (23) is remarkably similar to an
action studied in [20] which was not related to consideration of $D$-branes. There, the bulk
term was supposed to be a four-dimensional $SL(2, R)$-invariant theory while the Born–
Infeld part, $S_{source}^{(3)}$, was precisely of the form (23) with the fields $B^N$ and $B^R$ set equal to zero. The pseudoscalar was referred to as the axion in [20] while in (25) it is identified
with the $R \otimes R$ scalar, $\chi$.

In the absence of the antisymmetric tensor fields the $SL(2, R)$ transformations act
on the fields in the action of [20] as follows,

$$\lambda \rightarrow \frac{p \lambda + q}{r \lambda + s}, \quad F_{\mu\nu} \rightarrow s F_{\mu\nu} + r * G_{\mu\nu}, \quad G_{\mu\nu} \rightarrow p G_{\mu\nu} - q * F_{\mu\nu}, \quad \Lambda \equiv \begin{pmatrix} p & q \\ r & s \end{pmatrix} \in SL(2, R),$$

where $\lambda = \chi + ie^{-\phi}$, $G$ is defined by

$$G_{\mu\nu} = -2 \frac{\delta S_{source}^{(3)}}{\delta F_{\mu\nu}},$$

(and $** = -1$ with Lorentzian signature).

The invariance of the equations of motion can be seen by considering infinitesimal
$SL(2, R)$ variations with $p = 1 + \alpha$, $q = \beta$, $r = \gamma$ and $s = 1 - \alpha$,

$$\delta \chi = 2 \alpha \chi + \beta - \gamma(\chi^2 - e^{-2\phi}), \quad \delta \phi = 2(\chi \gamma - \alpha)$$

$$\delta F_{\mu\nu} = \gamma * G_{\mu\nu} - \alpha F_{\mu\nu}, \quad \delta G_{\mu\nu} = \alpha G_{\mu\nu} - \beta * F_{\mu\nu}$$

The steps outlined in [20] that demonstrate the $SL(2, R)$ invariance of the equations
of motion of the theory may now be generalized to include the extra background fields
$B^N$ and $B^R$ in (25) which transform as,

$$\begin{pmatrix} B^N \\ B^R \end{pmatrix} \rightarrow (\Lambda^{-1})^T \begin{pmatrix} B^N \\ B^R \end{pmatrix} = \begin{pmatrix} s & -r \\ -q & p \end{pmatrix} \begin{pmatrix} B^N \\ B^R \end{pmatrix}$$

7
and the fourth-rank potential, $A^{(4)}$, which is invariant. The infinitesimal form of these transformations is,

$$
\delta B^N = -\gamma B^R_{\mu\nu} - \alpha B^N_{\mu\nu}, \quad \delta B^R_{\mu\nu} = \alpha B^R_{\mu\nu} - \beta B^N_{\mu\nu}
$$

(34)

The $SL(2, R)$ symmetry of the equations of motion is dependent on the invariance of $G$ which was defined by (30). Its variation is given by

$$
\delta G_{\mu\nu} = \left( \frac{\partial G_{\mu\nu}}{\partial F_{\rho\sigma}} \delta F_{\rho\sigma} + \frac{\partial G_{\mu\nu}}{\partial B^N_{\rho\sigma}} \delta B^N_{\rho\sigma} + \frac{\partial G_{\mu\nu}}{\partial \chi} \delta \chi + \frac{\partial G_{\mu\nu}}{\partial \phi} \delta \phi \right) + \delta * B^R_{\mu\nu}
$$

(35)

There are three independent terms in the transformations with coefficients $\beta, \gamma, \alpha$. The $\beta$ term in (35) gives

$$
- * F_{\mu\nu} = \frac{\partial G_{\mu\nu}}{\partial \chi} - * B^N_{\mu\nu},
$$

(36)

and since $\partial G_{\mu\nu}/\partial \chi = - * F_{\mu\nu} = - * (F - B^N)_{\mu\nu}$, this equation is identically satisfied.

The $\gamma$ term in (35) gives

$$
0 = \left( \frac{\partial G_{\mu\nu}}{\partial F_{\rho\sigma}} (\ast G_{\rho\sigma} + B^R_{\rho\sigma}) - (\chi^2 - e^{-2\phi}) \frac{\partial G_{\mu\nu}}{\partial \chi} + 2\chi \frac{\partial G_{\mu\nu}}{\partial \phi} \right),
$$

(37)

and the $\alpha$ term gives

$$
G_{\mu\nu} = \left( \frac{\partial G_{\mu\nu}}{\partial F_{\rho\sigma}} (- F_{\rho\sigma} + B^N_{\rho\sigma}) + 2\chi \frac{\partial G_{\mu\nu}}{\partial \chi} + 2\chi \frac{\partial G_{\mu\nu}}{\partial \phi} \right) + * B^R_{\mu\nu}.
$$

(38)

These variations reduce to the ones given in [20] but with $F$ replaced by $\mathcal{F}$. Therefore, the arguments in [21] generalize to demonstrate the $SL(2, R)$ invariance of the equations in the presence of $B^N$ and $B^R$ which is broken to $SL(2, Z)$ in the quantum theory. Although the action is not uniquely determined by these arguments the requirements of supersymmetry constrain it considerably – not only must the theory possess manifest $N = 4$ supersymmetry but there must also be four non-linearly realized supersymmetries.

The consistency of $p$-branes ending in $(p + 2)$-branes was discussed in general terms in [14, 21]. In this case we consider a dyonic string soliton carrying charges $(Q^N, Q^R)$ associated with the NS $\otimes$ NS sector and the R $\otimes$ R sector terminating on a three-brane. An eight-sphere $S^8$ intersects the string at a point and the three-brane on a two-sphere $S^2$ surrounding the end-point of the string. The equations of motion for the antisymmetric tensor field strengths in the presence of the string and the three-brane are given by (in form notation)

$$
d* H^R = Q^R \delta^8(x) + \frac{\delta S^{(3)}}{\delta B^R} \wedge \delta^6(x)
$$

(39)

$$
d* H^N = Q^N \delta^8(x) + \frac{\delta S^{(3)}}{\delta B^N} \wedge \delta^6(x)
$$

(40)
where the δ⁸ and δ⁶ are the 8 and 6 form delta functions in the space transverse to the string and the three-brane respectively (and ♣ denotes duality with respect to the ten-dimensional space-time). Using the form of the three-brane action and integrating over the S⁸ gives
\[ 0 = Q^R + \int_{S^2} F, \quad \quad 0 = Q^N + \int_{S^2} *G \] (41)
where G is defined by (30). This shows that the string end-point in the four-dimensional world-volume is a dyon carrying electric and magnetic charges that are equal to \( Q^N \) and \( Q^R \), respectively. A \( SL(2, Z) \) duality transformation that acts on the antisymmetric tensor charges by
\[ \begin{pmatrix} Q^N \\ Q^R \end{pmatrix} \rightarrow \Lambda \begin{pmatrix} Q^N \\ Q^R \end{pmatrix} = \begin{pmatrix} p & q \\ r & s \end{pmatrix} \begin{pmatrix} Q^N \\ Q^R \end{pmatrix} \] (42)
also acts on the worldvolume vector potential as a duality transformation of the type considered in [22]. This transforms the fundamental electric charges into the infinite set of dyonic charges.

It is easy to see that the combined system of a string terminating in a three-brane breaks half of the space-time supersymmetry of the single three-brane. This means that it breaks three quarters of the original 32-component type IIB supersymmetry leaving eight unbroken supercharges, which is consistent with the presence of dyonic solitons in four-dimensional \( N = 4 \) super Yang–Mills theory.

**Parallel three-branes and dyons.**

It was emphasized in [3] that the configuration space of \( n \) parallel \( p \)-branes describes the moduli space of ten-dimensional supersymmetric \( U(n) \) Yang–Mills theory dimensionally reduced to \( p + 1 \) dimensions. If the branes are not coincident the \( U(n) \) symmetry breaks to a subgroup and if none of them are coincident it breaks to \( U(1)^n \) – one \( U(1) \) factor for each brane. The broken generators of the group correspond to massive gauge potentials that are the ground states of strings that stretch from one brane to another, \( A_{rs}^\mu \) (where \( r, s \) label the branes on which the endpoints are fixed and \( \mu = 0, \cdots, 9 \)) while the unbroken \( U(1) \)'s are associated with open strings with both end-points terminating on the same brane that have massless ground states that are the potentials \( A_{rs}^\mu \).

Here we will consider two parallel three-branes with world-volume coordinates in directions \( \alpha = 0, \cdots, 3 \) and transverse coordinates in directions \( i = 4, \cdots, 9 \). The \( U(2) = SU(2) \times U(1)_D \) symmetry is broken to \( U(1)_A \times U(1)_D \) when the branes are separated. The diagonal abelian group, \( U(1)_D \), is associated with an overall phase and has gauge connection \( A_{\mu}^+ = \frac{1}{2}(A_{\mu}^{11} + A_{\mu}^{22}) \) while \( U(1)_A \) transforms the relative phase and its connection is \( A_{\mu}^- = \frac{1}{2}(A_{\mu}^{11} - A_{\mu}^{22}) \). The \( A_{\alpha}^a \) (\( \alpha = 0, \cdots, 3 \)) are the massless world-volume potentials in each of the two branes while \( \phi_i^+ = A_i^+ \) (\( i = 4, \cdots, 9 \)) are the massless scalar fields that describe the transverse positions of the two branes. Hence \( \phi_i^- \) is the center of mass position and \( \phi_i^- \) is the relative position of the branes.
The supersymmetric ground state of this system is parameterized by the separation between the branes in a particular transverse direction, which will be taken to be \( i = 9 \). We will be interested in closely spaced branes with separation \( R << (\alpha')^{1/2} \). In other words \( |\langle \phi_9 \rangle| = R \). This non-zero expectation value of a scalar field is a symptom of the fact that the \( U(2) \) symmetry is broken spontaneously. The fact that this scalar field is massless is characteristic of a BPS saturated system which has flat directions in the potential. None of these massless fields is charged under the \( U(1) \)'s.

The system also possesses excited states in which the two branes are joined by a stretched fundamental string with endpoints moving in the branes. The string world-sheets lie in the 0–9 plane. A fundamental string stretching between the three-branes has ground-state vector potentials, \( A^\pm_I(x^\alpha) = \frac{1}{2}(A^{21}_I(x^\alpha) \pm A^{12}_I(x^\alpha)) \), where \( I = 1, \cdots, 8 \) are the eight directions transverse to the open-string world-sheet. Five of these directions are transverse to both the world-volume of the three-brane and the world-sheet of the stretched string and the other three directions lie in the three-brane. The latter three components are interpreted as the physical components of a massive four-dimensional world-volume vector \( W^\pm(x^\alpha) \) – a \( W \)-boson – and the other components, \( \rho^{\pm A}(x^\alpha) \) \( (A = 1, \cdots, 5) \) are the five scalar fields needed to fill out the bosonic components of a massive short world-volume \( N = 4 \) supermultiplet. There are also eight massive ground-state fermionic fields. Therefore the world-volume theory contains two charged massive supermultiplets, in addition to the massless fields considered above. These massive multiplets carry charges \( \pm 1 \) under \( U(1)_A \). The mass of these states depends on the separation of the branes,

\[
M = (\alpha')^{-1} R. \tag{43}
\]

But this is precisely the content of the usual \( N = 4 \) four dimensional supersymmetric \( U(2) \) Yang–Mills theory with spontaneous symmetry breaking, where the scale of the breaking is determined by the vacuum value of a massless scalar field, \( \langle \phi_9 \rangle \).

Instead of the fundamental string joining the branes a solitonic string can join them. This can be any of the infinite number of ‘dyonic’ type IIB strings \([23]\) with string tensions depending on the values of the charges \( m \) and \( n \) that couple to the antisymmetric tensors, \( B^N_{09} \) and \( B^R_{09} \), respectively,

\[
T_{m,n} = \frac{1}{\alpha'} \left( (m - n\chi)^2 + \frac{n^2}{g^2} \right)^{1/2}. \tag{44}
\]

Since the tension of the solitonic string is very large in perturbation theory \( (T_{m,n} \sim 1/g) \) the string length is the minimum possible length, which is \( R \) and so the positions of the end-points are at \( x_{1a} = x_{2a} = x^a \). Using Gauss’ law in the form \([14]\) at either end-point of the stretched string shows that \( e_1 = -e_2 \), where \( e_r \) is the charge that couples to the \( U(1) \) potential \( A^r \). Therefore the diagonal \( U(1)_D \) charge \( (e_1 + e_2) \) vanishes but the relative \( U(1)_A \) charge \( (e_1 - e_2) \) is non-zero. This means that the effective four-volume theory
contains massive solitonic dyons that arise from the composite two-brane system. The action of \( SL(2, \mathbb{Z}) \) transforms the dyons among themselves, at the same time transforming the strings that connect the branes.

This description of the dyonic spectrum of the \( N = 4 \) supersymmetric Born–Infeld theory is quite different from that of the version of \( N = 4 \) supersymmetric Yang–Mills theory obtained from toroidal compactification of the ten-dimensional heterotic theory\(^{22}\). There the \( W \) boson mass has a scale \( 1/R \), where \( R \) is the radius of a compactified dimension, and the theory does not have an obvious non-linearly realized supersymmetry. The representation of spontaneous symmetry breaking in terms of two copies of the four-dimensional world-volume is somewhat reminiscent of\(^{24}\).

Effective world-volume theories obtained from \( D \)-branes (such as the three-brane considered in this paper) are determined by the underlying open superstring theory. Intriguingly, the quantum corrections to the world-volume theory are given by open-string loop diagrams which unavoidably induce the closed-string gravitational sector with states that propagate in the ten-dimensional embedding space. Thus, the distinction between the effective world-volume and the embedding space should disappear in the full quantum theory.

*Note Added:* During the preparation of this manuscript a paper appeared in the hep-th archive that has some overlap with the material presented here\(^{25}\).

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**References**

[1] J. Polchinski, *Dirichlet-branes and Ramond–Ramond charges*, hep-th/9510017, Phys. Rev. Lett. 75 (1995) 4724.

[2] J. Dai, R.G. Leigh and J. Polchinski, *New connections between string theories*, Mod.Phys.Lett A4 (1989) 2073.

[3] E. Witten, *Bound states of strings and p-branes*, hep-th/9510135, IASSNS-HEP-95-83.

[4] C.G. Callan, J.A. Harvey and A. Strominger, *Worldbrane actions for string solitons*, Nucl. Phys B367 (1991) 60.

[5] M.J. Duff and J.X. Lu, *The selfdual type IIB superthreebrane*, Phys. Lett. 273B (1991) 409.

[6] J. Hughes, J. Liu and J. Polchinski, *Supermembranes*, Phys. Lett. 180B (1986) 370.
[7] J.H. Schwarz, *Covariant field equations of chiral $N = 2$ $D = 10$ supergravity*, Nucl. Phys. **B226** (1983) 269.

[8] P.S. Howe and P.C. West, *The complete $N = 2, d = 10$ supergravity* Nucl. Phys. **B238** (1984) 181.

[9] E.S. Fradkin and A.A. Tseytlin, *Nonlinear electrodynamics from quantized strings*, Phys. Lett. **163B** (1985) 123.

[10] C.G. Callan, C. Lovelace, C.R. Nappi and S.A. Yost, *Adding holes and cross-caps to the superstring*, Nucl. Phys. **B293** (1987) 83.

[11] J. Polchinski and Y. Cai, *Consistency of open superstring theories*, Nucl. Phys. **B296** (1988) 91.

[12] C. Schmidhuber, *D-brane actions*, hep-th/9601003. PUPT-1585.

[13] S.P. de Alwis and K. Sato, *D-strings and F-strings from string loops*, COLO-HEP-368, hep-th/9601167.

[14] A. Strominger, *Open P-branes*, hep-th/9512059.

[15] M.B. Green, *Point-like states for type IIB superstrings*, Phys. Lett. **329B** (1994) 435.

[16] M.B. Green and M. Gutperle, *Space-time supersymmetry for D-branes and D-instantons*, (in preparation).

[17] A. Abouelsaood et al., *Open strings in background gauge fields*, Nucl. Phys. **B280** (1987) 599.

[18] C.G. Callan, C. Lovelace, C.R. Nappi and S.A. Yost, *Loop corrections to the superstring equations of motion*, Nucl.Phys. **B308** (1988) 221.

[19] Miao Li, *Boundary states of D-branes and Dy strings*, hep-th/9510161, BROWN-HET-1020.

[20] G.W. Gibbons and D.A. Rasheed, *SL(2, R) invariance of non-linear electrodynamics coupled to an axion and a dilaton*, hep-th/9509141. Phys. Lett. **365B** (1996) 46.

[21] P.K. Townsend, *D-Branes from M-branes*, hep-th/9512062. DAMTPP-R-95-59.

[22] A. Sen, *Dyon-monopole bound states, self-dual harmonic forms on the multi-monopole moduli space, and SL(2, Z) invariance in string theory*, Phys. Lett. **329B** (1994) 217.

[23] J.H. Schwarz, *an SL(2, Z) multiplet of type IIB superstrings*, hep-th/9508143. Phys. Lett. **360B** (1995) 13.
[24] A. Connes and J. Lott, *Particle models and non-commutative geometry*, Nucl. Phys. Proc. Suppl. 18B (1991) 29.

[25] A.A. Tseytlin, *Self duality of the Born-Infeld action and Dirichlet 3-branes in type IIB superstring theory*, hep-th/9602064, Imperial/TP/95-96/26.