A note on the reconstruction of a 3D-3C velocity field for box turbulence

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Abstract. We examine the reconstruction of a three-dimensional three-component (3D-3C) snapshot velocity field in a torus using a denoising method of compressed sensing. Using 30\% of the total grid points, we could reconstruct the velocity with a relative error of approximately 10\%.

1. Motivation

Velocimetry, observations for weather or engineering experiments, are spatially and temporally limited, and, moreover, noise contamination cannot be avoided. Our prime interest is how to detect turbulent coherent structure from measurements. This document examines a reconstruction problem for three-dimensional three-component (3D-3C) snapshot velocity from limited data using a denoising algorithm of compressed sensing.

The reconstruction problem has been attracting a number of researcher’s interests. Proper Orthogonal Decomposition (POD) proposed by Lumley \cite{1} is one of the representative examples on low-dimensional model description for turbulence \cite{2, 3, 4, 5}. This method is also called as Principal Component Analysis (PCA), Karhunen-Loève procedure \cite{6}, or Empirical Orthogonal Function (EOF) \cite{7} depending on research topics. Considering the singular value decomposition (SVD) for matrix consisting of spatiotemporal data, we could extract structure in the sense of root-mean-square (RMS) fields from average field. The left and right singular vectors represent fluctuations from time- and space-averaging fields, respectively. Especially, the fluctuations from time-mean field are called POD modes. Gappy POD is a reconstruction method using POD modes \cite{8}, where reconstructed field consists of superposition of POD modes. Because the POD modes are linearly independent, key topics are how to obtain coefficients of each POD mode using input data; that is to say, finding optimal observing valuables, appropriate locations of probes, the required number of probes and so on.

However, POD modes represent the second moments of time series, RMS, and it is difficult to characterise unsteady structure of turbulence. Schmid \cite{9, 10} proposed an approximation method for time development of turbulence, named Dynamical Mode Decomposition (DMD). From the viewpoint of data-driven science, DMD is a tractable method and has been applied to lots of practical interests \cite{11, 12, 13, 14, 15, 16}. DMD approximates the time-shift matrix from time series data. The eigenvalues and eigenmodes of this matrix are called DMD eigenvalues and modes, respectively. The imaginary part of DMD eigenvalues represents the frequency of time. Therefore, the DMD modes express a certain structure arising from the frequency. We could
find several developments such as non-uniform data with respect to time [17], approximation of inertial manifold in phase space [18], and so on. From viewpoints of data science, detailed explanations are shown in [19].

Rowley et al. [20] pointed out relationships between DMD and Koopman analysis, which was proposed for Hamiltonian dynamical systems [21]. Even if the dynamical systems are nonlinear, a linear time-shift map for observable is formally defined. Here, the observable is a function of the state variables. This map and its eigenvalues are called Koopman operator and Koopman eigenvalues, respectively. Koopman analysis considers an expansion of variables using the Koopman eigenvalues, where the weights of the Koopman eigenvalues are named Koopman modes. We should note that the Koopman eigenvalues and modes correspond to DMD eigenvalues and modes. Theoretical descriptions and extended studies are found in [22, 23, 24].

As for turbulence, a dissipative dynamical system, it is open to finding appropriate observables for Koopman analysis. Page & Kerswell [25] studied DMD and Koopman expansion for the 1D Burgers equation. They reported that Koopman modes are linearly dependent and the profile of each Koopman mode depends on the initial conditions. Kutz et al. [26] showed that the observables including the nonlinear term arising from the dynamical systems improve accuracy of the time-map model.

In POD and DMD, we reduce the number of their modes by selecting dominant singular values, and, therefore, we ignore leftover information. Behind these methods, there are expectations: turbulence, a fully nonlinear dynamical system, possesses a sparsity. In other words, if turbulence could be expressed by a few characteristic structures, the flow field could be estimated using limited measurements. Here, we consider the reconstruction problem. In the sense of compressed sensing, image pixels correspond to the components of a matrix. They carry out the extraction of characteristic set of pixels and the noise reduction; detecting structure and reconstructing data [27]. This document examines a typical matrix completion algorithm [28] to reconstruct snapshot velocities of turbulence by reshaping the 3D-3C vector to 2D matrix in a facile manner.

2. Method

We consider the Navier–Stokes flow in a three-dimensional torus:

\[
\frac{\partial u}{\partial t} + (u \cdot \nabla) u = -\frac{1}{\rho} \nabla p + \nu \nabla^2 u + F, \tag{1}
\]

\[
\nabla \cdot u = 0, \tag{2}
\]

where \( p(x, t) \) and \( u(x, t) \) are the pressure and velocity, respectively, at time \( t \) and position \( x = (x_1, x_2, x_3)^T \). The superscript \( T \) refers to the transpose. The pressure and velocity are \( 2\pi \) periodic with respect to the \( x_i \) directions. Moreover, \( \rho \) is the constant density, \( \nu \) is the kinematic viscosity, and \( F \) is a body force. This paper considers two types of body forcing: a steady forcing and an isotropic one [29, 30],

\[
F_I(x, t) = c(t) u_{k_c}(x, t), \tag{3}
\]

\[
F_S(x) = \begin{pmatrix}
-\sin x_1 \cos x_2 \\
\cos x_1 \sin x_2 \\
0
\end{pmatrix}, \tag{4}
\]

Here \( u_{k_c} \) is the low-pass filtered velocity, \( k_c \) is a cutoff wavenumber, and \( c(t) \) is a coefficient to maintain a constant energy,

\[
c(t) = \frac{\nu \| \nabla \times u \|_2^2}{\| u_{k_c} \|_2^2}, \tag{5}
\]
Table 1. Setup for direct numerical simulation. Here, \( \nu \) is the kinematic viscosity, \( N^3 = N_1 N_2 N_3 (N_1 = N_2 = N_3) \) is the total number of grid points, \( R_\ell \) is Taylor-micro Reynolds number, and \# of snapshots is the number of snapshots in the one eddy turnover time. Moreover \((\lambda_0, \Delta \lambda)\) is the initial soft-threshold value and its multiplier while \( J \) is the number of SVD iterations in the case of \( p = 0.7 \).

| Forcing type | \( \nu \) | \( N^3 \) | \( R_\ell \) | \# of snapshots | \((\lambda_0, \Delta \lambda)\) | \( J \) |
|--------------|----------|----------|-----------|----------------|------------------|-----|
| Steady       | \( 3 \times 10^{-2} \) | \( 96^3 \) | 46.2      | 10             | \((50,0.7)\)     | 100 |
| Steady       | \( 10^{-2} \)          | \( 128^3 \) | 65.2      | 10             | \((50,0.7)\)     | 100 |
| Steady       | \( 3 \times 10^{-3} \) | \( 256^3 \) | 130       | 1              | \((100,0.98)\)   | 200 |
| Isotropic    | \( 10^{-2} \)          | \( 128^3 \) | 55.0      | 10             | \((50,0.7)\)     | 100 |
| Isotropic    | \( 5 \times 10^{-3} \) | \( 128^3 \) | 74.6      | 10             | \((50,0.7)\)     | 100 |
| Isotropic    | \( 3 \times 10^{-3} \) | \( 256^3 \) | 194       | 1              | \((100,0.98)\)   | 200 |

where \( \| \cdot \|_2 \) is the \( L_2 \) norm. We fix \( k_c = 1.5 \). A flow driven by \( F_S(x) \) is called a Taylor-Green flow.

As for direct numerical simulation, we employ the Fourier spectral method:

\[
\omega_j(x,t) = \sum_{|k| \leq T_K} \hat{\omega}_{k,j}(t) \exp(i k \cdot x), \quad j = 1, 2, 3, (6)
\]

where, \( \hat{\omega}_{k,j} \) is the Fourier coefficient of the \( j \)-th component of vorticity, \( k = (k_1, k_2, k_3)^T \) is the Fourier wave vector, and \( T_K \) is the truncation wavenumber, respectively. The 4th-order Runge–Kutta method is used to carry out time integration for the vorticity. We use the phase-shift method for dealising of the nonlinear term and the integrating factor method for the viscous term. We use the number of grid points as \( N_1 = N_2 = N_3 \) for direct numerical simulation.

To calculate the time average in approximately one eddy turnover time. Table 1 lists the parameters for direct numerical simulation.

The present study considers a reconstruction problem for a three-dimensional three-component (3D-3C) snapshot velocity field. We reshape the 3D-3C velocity into a matrix of size \( N^2 \times 3N \). A schematic diagram of this reshaping process is illustrated in figure 1. At first, let us consider the first component of velocity \( u_1(x_1, x_2, x_3) \) (see Fig. 1a). If we fix the \( x_3 = \pi \), the grid-point data of this component, \( u_1(x_1, x_2, \pi) \), could be considered as a matrix with the size \( N_2 \times N_1 \) (see Fig. 1b). We call this the sliced matrix. Changing \( x_3 \) from 0 to 2\( \pi \), we obtain \( N_3 \) matrices with this size from each grid point of \( x_3 \). From \( x_3 = 0 \) to 2\( \pi \), we arrange these \( N_3 \) matrices one by one to the row direction with keeping the structure of each sliced matrix (see Fig. 1c). As a result, we obtain a ‘sliced \( u_1 \)’ matrix, long in the row direction, of size \( N_2 N_3 \times N_1 \). Considering this slicing process for each component of velocity, we array three long-in-row matrices one by one to the column direction, and we obtain “reshaped \( u(x) \)” which we call the data matrix \( A \in \mathbb{R}^{N_2 N_3 \times 3 N_1} \) (see Fig. 1d). In our case, \( N_1 = N_2 = N_3 = N \).

We consider the reconstruction problem using the singular value decomposition. The singular
value decomposition of $A$ is given by

$$A = U\Sigma V^T = \sum_{i=1}^{3N_1} \sigma_i u_i v_i^T. \quad (7)$$

Here, $\Sigma \in \mathbb{R}^{N_2 N_3 \times 3N_1}$ has the singular values $\sigma_1 \geq \sigma_2 \cdots \geq \sigma_{3N_1}$ in the diagonal. In addition, $U$ and $V$ are orthogonal matrices consisting of the left and right singular vectors $u_i \in \mathbb{R}^{N_2 N_3}$ and $v_i \in \mathbb{R}^{3N_1}$, respectively, in each column. Note that the matrix $A$ is represented by the superposition of the matrix $u_i v_i^T$, and that large singular values make a large contribution to the matrix $A$. In addition, the singular values are invariant under transformations expressed by a unitary matrix.

From the total number of grid points $N^3$, we randomly choose a fraction $p$ of grid points, and define the three velocity components at these points as the missing data. Using the 3D-3C velocity at the remaining grid points, we attempt to reconstruct the original velocity field as follows:

1. Substitute random values following a normal distribution for the missing components. Here, the mean and variance of the normal distribution are given by the remaining components. We call this matrix $\tilde{A}_0$.
2. Carry out the singular value decomposition for $\tilde{A}_j$.
3. Apply the soft-thresholding function $S_{\lambda_j}$ for the singular values $\sigma_i (i = 1, \cdots, 3N_1)$:

   $$\tilde{\sigma}_i = S_{\lambda_j}(\sigma_i) = \begin{cases} 
   \sigma_i - \lambda_j & \sigma_i > \lambda_j \\
   0 & \sigma_i \leq \lambda_j
   \end{cases} \quad (8)$$

   We obtain the matrix $\tilde{A}_j = U\tilde{\Sigma}V^T$, where $\tilde{\Sigma}$ has $\tilde{\sigma}_i$ in the diagonal.
4. Substitute the remaining original components into matrix $\tilde{A}_j$ and obtain matrix $\tilde{A}_{j+1}$.
5. Update the threshold: $\lambda_{j+1} = \lambda_j + \Delta \lambda$, where $0 < \Delta \lambda < 1$.
6. Repeat steps 2 to 5, $J$ times.

We change $\lambda_0, \Delta \lambda$ and $J$ for different $p$ and $R_\ell$. For instance, in the case of $R_\ell = 130$ under steady forcing, and $p = 0.7$, we use $\lambda_0 = 100, \Delta \lambda = 0.98$ and $J = 200$. Empirically, we find that this moderate change of the threshold $\lambda_j$ increases the accuracy of the reconstruction. In the case of high Reynolds number, the matrix size is $256^2 \times 3 \cdot 256$. It takes approximately two days to carry out the reconstruction, each SVD requires about fifty minutes. Meanwhile, in the case of $R_\ell < 100$, we take the ensemble average using 10 snapshots of the velocity field in approximately one eddy turnover time.
3. Results

Figure 2 shows the reconstructed field in the case of $R_{\ell} = 130, p = 0.7$, while the relative error is plotted in figure 3. In this case, we reconstruct the 3D-3C velocity with a relative error of approximately 11%. Figure 4 represents the singular values $\sigma_i$. The singular values of the initial matrix $A_0$ have a plateau at high indices, $i$, and they decrease by the soft-threshold function. Note that the dominant singular values at a low indices are also subtracted by $\lambda_j$. However, when we substitute the velocity at the remaining grid points, the dominant singular values could recover the original values. The accuracy of the reconstruction depends on how much essential information the remaining data possesses. Figure 5 shows the relative error as a function of the remaining fraction of grid points, $1 - p$. We find that by using 30% of the total grid points the 3D-3C velocity is reconstructed with a relative error of approximately 10%, independent of the Reynolds number and type of forcing. Note that 30% of the total grid points is $(0.3 \times N^3)^{1/3}/N \approx 0.669 \cdots$, and, therefore, the remaining velocity data randomly exists at approximately 67% of grid points per side of the box. As $1 - p$ decreases, the relative error increases monotonically, faster than $(1 - p)^{-1/3}$.

In general, changing the slice direction is not expressed by an unitary matrix. We consider the oblique slicing shown in Figure 6. Figure 7 plots the singular values. Singular values are changed, but the number of dominant singular values is the same as before. Moreover, the number of zero singular values depends on the slice direction. Figure 8 shows the flow field of the first, second and third singular vectors visualized by the energy isosurface. The structure of the energy isosurface also depend on the slice directions. In both of them we find columnar structure stemming from the body force, but we do not have explanations for their different shapes. In addition, the accuracy of the reconstruction does not change with the slice direction.
4. Discussion

We examined the reconstruction of three-dimensional three-component (3D-3C) velocity using singular value decomposition. We find that using velocity at 30% of the total grid points we could reconstruct the velocity field with a relative error of approximately 10%, independently of the Reynolds number and the types of forcing. However, it seems difficult to find physical meaning for the singular vectors. Moreover, in the case of non-uniform grid points, for example in channel flow, it is an open problem whether this algorithm works as well as in box turbulence. Recently, the reconstruction of flow profiles using DMD modes has been reported [31, 32]. Note
that DMD mode is an eigenfunction of a linear model driven by time-sequence data, and it shows time-periodic motion. However, it is unclear whether the DMD mode describes the nonlinear phenomena. As for SVD, we perform it using serial calculation without MPI library, which should be improved for high Reynolds number turbulence.

Acknowledgments

In this study we used Fujitsu PRIMERGY CX2550/CX2560 M4 computer system, of Research Institute for Information Technology, Kyushu University, and we employed LAPACK in Intel Math Kernel Library (MKL) for singular value decomposition. This study was carried out in the context of the 4th Madrid Turbulence Workshop, funded by the Coturb program of the European Research Council. We thank Dr. J. Garicano Mena (UPM) for giving us fruitful comments and suggestions on this manuscript. The first author thanks Dr. Shingo Motoki (Osaka Univ.) for the visualization technique. This work was partially supported by JSPS Grant-in-Aid for Young Scientists Grant Number 19K14883.

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Figure 6. Two types of slicing: (a) the sections normal to $x_3$ axis shown by the green squares, (b) the oblique sections shown by the blue rectangles. The case of the normal slicing (a) is the same as in Figure 1.

Figure 7. Singular values of raw velocity in the case of $R_{\ell} = 130$. The blue dots and red circles represent the types of the slicing normal to the $x_3$ axis and oblique, respectively.

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Figure 8. Singular vector in the case of $R_e = 130$: (a) slicing normal to $x_3$-axis where $\sigma_1 = 4.43 \times 10^3$, $\sigma_2 = 4.04 \times 10^3$, $\sigma_3 = 1.81 \times 10^3$, and (b) oblique slicing where $\sigma_1 = 4.38 \times 10^3$, $\sigma_2 = 4.25 \times 10^3$, $\sigma_3 = 1.77 \times 10^3$. The red, green, and blue objects represent the singular vector field with respect to the first, second and third singular values. Visualized by the energy isosurface $\frac{1}{2}u^2 = 1.0$

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