Massive Relativistic Particle Models with Bosonic Counterpart of Supersymmetry

Sergey Fedoruk* and Jerzy Lukierski†
Institute for Theoretical Physics,
University of Wrocław, pl. Maxa Borna 9, 50-204 Wrocław, Poland

Abstract
We consider the massive relativistic particle models on four-dimensional Minkowski space extended by $N$ commuting Weyl spinors for $N = 1$ and $N = 2$. The $N = 1$ model is invariant under the most general form of bosonic counterpart of simple $D = 4$ supersymmetry, and provides after quantization the bosonic counterpart of chiral superfields, satisfying Klein–Gordon equation. In massless case these fields do satisfy the Fierz-Pauli equations. For $N = 2$ we obtain after quantization the free massive higher spin fields for arbitrary spin satisfying linear Bargman–Wigner equations. Finally the problem of statistics in presented framework for half–integer classical spin fields is discussed.

Keywords: higher spins; supersymmetry.

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1 Introduction

Important extensions of the relativistic symmetries were considered in the following two directions:

1. Supersymmetric extension, relating by supersymmetry (SUSY) transformations integer and half–integer spin fields (see e. g. [1, 2]). The geometric way of describing supersymmetric multiplets is realized in terms of superfields – the functions on superspace $Y_A = (x_\mu, \theta^i_\alpha, \bar{\theta}^i_{\dot{\alpha}})$ where $\theta^i_\alpha$ are anticommuting Grassmann spinors.

2. Introduction of higher spin (HS) algebras, which act on infinite spin multiplets or if $m = 0$ on infinite helicity multiplets (see e. g. [3]-[5]). The representation spaces of HS algebras are described by the functions on ‘bosonic’ superspace $Z_A = (x_\mu, \lambda^i_\alpha, \bar{\lambda}^i_{\dot{\alpha}})$ with additional commuting spinor variables $\lambda^i_\alpha$. The bosonic counterparts of superfields one can call the spinorial Kaluza–Klein (KK) fields, with spinorial additional dimensions. We obtain

$$\Phi_A(Z_A) = \sum_{n,k=0}^{\infty} \sum_{(\alpha_1...\alpha_n)} \sum_{(\beta_1...\beta_k)} \varphi^{\alpha_1...\alpha_n\beta_1...\beta_k}(x) \lambda^i_\alpha_1...\lambda^i_\alpha_n \bar{\lambda}^j_{\dot{\alpha}_1}...\bar{\lambda}^j_{\dot{\alpha}_k}.$$ (1)

The auxiliary commuting spinorial variables $(\lambda^i_\alpha, \bar{\lambda}^i_{\dot{\alpha}})$ occurs in several geometric frameworks, for example in twistor approach to the space–time geometry [6]-[8] or in the models with double (target and world volume) supersymmetry [9]-[11].

In this note we would like to study the group-theoretic and dynamical consequences of introducing bosonic counterpart of supersymmetry, obtained by supplementing the Poincaré algebra by bosonic spinorial charges. We recall the general $N=1$ SUSY relation with tensorial charges [12, 13]

$$\{Q_a, Q_b\} = 2(\gamma^\mu C)_{ab} P_\mu + (\sigma^{\mu\nu} C)_{ab} Z_{\mu\nu}$$ (2)

where in Majorana representation $C = \gamma_0$ and

- $Q_a$ is a four–component Majorana spinor of supercharges
- $Z_{\mu\nu} = -Z_{\nu\mu}$ describe six Abelian tensorial charges.

The bosonic counterpart of general $N=1$ SUSY takes the form

$$[R_a, R_b] = 2(\gamma^\mu \gamma_5 C)_{ab} P_\mu + 2C_{ab}Z^{(1)} + 2(\gamma_5 C)_{ab}Z^{(2)}$$ (3)

where

- $R_a$ is a four–component spinor of bosonic charges
- $Z^{(1)}$ ($Z^{(2)}$) are scalar (pseudoscalar) central charges.

In order to obtain in [3] the standard inversion properties of the fourmomentum generator one should assume suitable transformation properties of the spinor $R_a$.\(^{1}\)

\(^{1}\)Spinorial supercharges transform under space–time inversions in standard way $(Q'_a = (\gamma_0 Q)_a$ for the space inversion $P$, $Q'_a = (\gamma_0 \gamma_5 Q)_a$ for the time inversion $T$). The bosonic spinorial charges $R_a$ are so–called pseudospinor [14, 15] transforming under inversion in alternative way $(R'_a = (\gamma_0 \gamma_5 R)_a$ under $P$, $R'_a = (\gamma_0 R)_a$ under $T$).
Our aim is to study the massive relativistic particle models invariant under bosonic counterpart of SUSY and perform their quantization. Contrary to the case of simple SUSY the $N = 1$ relation contains scalar and pseudoscalar central charges, which can be related with the mass parameter. In Sect. 2 we describe (using two–component Weyl notation) the particle model describing the trajectory in the spinorial KK space $\mathcal{M}^{4,4}$ with the coordinates $Z_A = (x_\mu, \lambda_\alpha, \bar{\lambda}_{\dot{\alpha}})$. After calculating the complete set of constraints we perform the quantization using either Heisenberg picture or the Gupta–Bleuler method (Schrödinger picture). We shall obtain the wave function $\Psi(Z_A)$ satisfying the KG equation and the bosonic counterpart of the chirality condition. In Sect. 3 we analyze the massless limit of our model, with massless fields with arbitrary helicity satisfying Fierz-Pauli equations. In Sect. 4 we consider the relativistic particle in $N = 2$ spinorial KK space $\mathcal{M}^{4,8}$ with the coordinates $(x_\mu, \lambda_{\alpha i}, \bar{\lambda}_{\dot{\alpha} i})$ $(i = 1, 2)$. It appears that for the particular choice of bosonic counterpart of $N = 2$ SUSY, with internal symmetry $O(1,1)$, one can obtain the linear Bargman–Wigner equations for $D = 4$ massive higher spin fields. In Sect. 5 we shall discuss the problem of nonstandard relation between spin and statistics for the field components of spinorial KK fields.

2 Massive particle model with $N = 1$ bosonic counterpart of SUSY.

2.1 Classical model

We consider the following action

$$S = \int dt \mathcal{L},$$

$$\mathcal{L} = -m(\dot{\omega}_\mu \dot{\omega}^\mu)^{1/2} - i(z\dot{\lambda}_\alpha \lambda_\alpha - \bar{z}\bar{\lambda}_{\dot{\alpha}} \dot{\lambda}_{\dot{\alpha}})$$

where

$$d\omega^\mu = \dot{\omega}^\mu d\tau = dx^\mu - i\lambda_\alpha \sigma^\mu_{\alpha\beta}\bar{\lambda}_{\dot{\beta}} + i\bar{\lambda}_{\dot{\alpha}} \sigma^\mu_{\dot{\alpha}\dot{\beta}} \dot{\lambda}_{\dot{\beta}}.$$

The action describes the particle trajectory in Minkowski space extended by two commuting complex Weyl spinor coordinates $\lambda^\alpha(\tau)$, $\bar{\lambda}_{\dot{\alpha}} = (\bar{\lambda}^\alpha)$ and invariant under the following spinorial bosonic transformation

$$\delta x^\mu = i\lambda^\alpha \sigma^\mu_{\alpha\beta}\bar{\epsilon}^\beta - i\bar{\epsilon}^\alpha \sigma^\mu_{\dot{\alpha}\dot{\beta}} \dot{\lambda}_{\dot{\beta}}, \quad \delta \lambda_\alpha = \epsilon_\alpha, \quad \delta \bar{\lambda}_{\dot{\alpha}} = \bar{\epsilon}^\dot{\alpha}$$

where $\epsilon_\alpha$ is a constant commuting Weyl spinor. The constant $m$ is the mass of particle whereas $z$ is an arbitrary complex parameter with the dimension of mass. It is easy to see that performing the suitable phase transformation $\lambda_\alpha = e^{ia} \lambda_\alpha$, $\bar{\lambda}_{\dot{\alpha}} = e^{-ia} \bar{\lambda}_{\dot{\alpha}}$, where $a = \frac{1}{2} \text{arg} z$ one gets the real parameter $z$.

2Gupta–Bleuler method has been applied to massive relativistic superparticle e. g. in [15, 17].

3We use following notations. The metric has mostly minus $\eta_{\mu\nu} = \text{diag}(+ - - -)$. The Weyl two–spinor indices are risen and lowered by $\varphi^\alpha = \epsilon^{\alpha\beta} \varphi_{\beta}$, $\varphi_\alpha = \varphi^\beta \epsilon_{\beta\alpha}$, $\bar{\varphi}^\dot{\alpha} = \epsilon^{\dot{\alpha}\dot{\beta}} \bar{\varphi}_{\dot{\beta}}$, $\bar{\varphi}_{\dot{\alpha}} = \bar{\varphi}^{\dot{\beta}} \epsilon_{\dot{\beta}\dot{\alpha}}$ where $\epsilon^{\alpha\beta} \epsilon_{\beta\gamma} = -\delta^\alpha_\gamma$, $\epsilon^{\dot{\alpha}\dot{\beta}} \epsilon_{\dot{\beta}\dot{\gamma}} = -\delta^\dot{\alpha}_{\dot{\gamma}}$. Algebra $\sigma$–matrices $\sigma^\mu_{\alpha\beta} = (\overline{\sigma}^\mu_{\alpha\dot{\beta}})$ and $\sigma^\mu_{\alpha\dot{\beta}} = \epsilon^{\alpha\beta} \epsilon^{\dot{\alpha}\dot{\beta}} \sigma^\mu_{\beta\dot{\beta}}$ is $\sigma_{\mu\dot{\gamma}} \sigma^\mu_{\nu\gamma} + \sigma_{\nu\gamma} \sigma^\mu_{\mu\dot{\gamma}} = 2\eta_{\mu\nu} \delta^\beta_\gamma$. Also we define $p_{\alpha\dot{\beta}} = p_\mu \sigma^\mu_{\alpha\beta}$, $p^{\alpha\dot{\beta}} = p_\mu \sigma^\mu_{\alpha\beta}$ for any vector $p_\mu$. 
Conserved Noether spinorial charges corresponding to the transformations (7) are

\[ R_\alpha \equiv \pi_\alpha - ip_{\alpha \dot{\beta}} \bar{\lambda}^{\dot{\beta}} - iz\lambda_\alpha, \]  
\[ \bar{R}_{\dot{\alpha}} \equiv \bar{\pi}_{\dot{\alpha}} + i\lambda^{\dot{\beta}} p_{\beta \dot{\alpha}} + i\bar{z}\bar{\lambda}_{\dot{\alpha}}, \]  
where the canonical momenta are defined by

\[ p_\mu = \frac{\partial L}{\partial \dot{x}_\mu} = -m(\dot{\omega}_\nu \dot{\omega}^\nu)^{-1/2} \dot{\omega}_\mu, \]  
\[ \pi_\alpha = \frac{\partial L}{\partial \dot{\lambda}^\alpha} = -ip_{\alpha \dot{\beta}} \bar{\lambda}^{\dot{\beta}} - iz\lambda_\alpha, \]  
\[ \bar{\pi}_{\dot{\alpha}} = \frac{\partial L}{\partial \dot{\bar{\lambda}}_{\dot{\alpha}}} = i\lambda^{\dot{\beta}} p_{\beta \dot{\alpha}} + i\bar{z}\bar{\lambda}_{\dot{\alpha}}. \]  

Using the canonical Poisson brackets

\[ \{x^\mu, p_\nu\} = \delta^\mu_\nu, \quad \{\lambda^\alpha, \pi_\beta\} = \delta^\alpha_\beta, \quad \{\bar{\lambda}^{\dot{\alpha}}, \bar{\pi}_{\dot{\beta}}\} = \delta^{\dot{\alpha}}_{\dot{\beta}}, \]  
we obtain the algebra

\[ \{R_\alpha, \bar{R}_{\dot{\beta}}\} = -2ip_{\alpha \dot{\beta}}, \]  
\[ \{R_\alpha, R_\beta\} = 2iz\epsilon_{\alpha \beta}, \quad \{\bar{R}_{\dot{\alpha}}, \bar{R}_{\dot{\beta}}\} = -2i\bar{z}\epsilon^{\dot{\alpha} \dot{\beta}}, \]  
which is equivalent to the algebra (3) with \( Z = Z^{(1)} + iZ^{(2)} = z. \)

From (10)–(12) follow the mass shell constraint and the set of four spinorial constraints

\[ T \equiv p^2 - m^2 \approx 0, \]  
\[ D_\alpha \equiv \pi_\alpha + ip_{\alpha \dot{\beta}} \bar{\lambda}^{\dot{\beta}} + iz\lambda_\alpha \approx 0, \]  
\[ \bar{D}_{\dot{\alpha}} \equiv \bar{\pi}_{\dot{\alpha}} - i\lambda^{\dot{\beta}} p_{\beta \dot{\alpha}} - i\bar{z}\bar{\lambda}_{\dot{\alpha}} \approx 0. \]  

Using the formulae (10)–(12) we confirm that the canonical Hamiltonian vanishes\(^4\)

\[ \mathcal{H} = \dot{x}^\mu p_\mu + \dot{\lambda}^\alpha \pi_\alpha + \bar{\lambda}^{\dot{\alpha}} \bar{\pi}_{\dot{\alpha}} - \mathcal{L} = 0, \]

and the total Hamiltonian is the linear combination of first class constraints multiplied by Lagrange multipliers.

The constraints (16)–(18) satisfy the following Poisson brackets

\[ \{D_\alpha, \bar{D}_{\dot{\beta}}\} = 2ip_{\alpha \dot{\beta}}, \]
\[ \{D_\alpha, D_\beta\} = -2i\epsilon_{\alpha \beta}, \quad \{\bar{D}_{\dot{\alpha}}, \bar{D}_{\dot{\beta}}\} = 2i\bar{z}\epsilon^{\dot{\alpha} \dot{\beta}}. \]

The scalar constraint \( T \approx 0 \) is first class and all the spinorial constraints (17), (18) are second class. Indeed we find that the determinant of the matrix

\[ \mathcal{C} = \begin{pmatrix}
\{D_\alpha, D_\beta\} & \{D_\alpha, \bar{D}_{\dot{\beta}}\} \\
\{\bar{D}_{\dot{\alpha}}, D_\beta\} & \{\bar{D}_{\dot{\alpha}}, \bar{D}_{\dot{\beta}}\}
\end{pmatrix} = \begin{pmatrix}
-2i\epsilon_{\alpha \beta} & 2ip_{\alpha \dot{\beta}} \\
-2i\epsilon^{\dot{\alpha} \dot{\beta}} & 2i\bar{z}\epsilon^{\dot{\alpha} \dot{\beta}}
\end{pmatrix}. \]

\(^4\)The vanishing of Hamiltonian follows from the invariance of the action (4–6) under the arbitrary local rescaling \( \tau \to \tau' = \tau'(\tau). \)
is equal to\(^5\)
\[ \det C = 16(p^2 + |z|^2)^2. \]  
(22)

We see from (22) that the matrix \(C\) (see (21)) is invertible for any \(z\), and the constraints \((17, 18)\) are second class.

### 2.2 Quantization

The first quantization of the model can be performed using one of two methods:

**i)** Following the technique of quantization of systems with second class constraints one can introduce Dirac brackets (DB) for the independent phase space degrees of freedom \(Z_M = (x^\mu, p_\mu, \lambda_\alpha, \bar{\lambda}_{\dot{\alpha}})\)

\[ \{Z_M, Z_N\}^* = \{Z_M, Z_N\} - \{Z_M, D_r\}(C^{-1})_{rs}\{D_s, Z_N\} \]  
(23)

where \(D_r = (D_\alpha, \bar{D}_{\dot{\alpha}})\). In particular for suitably normalized spinor coordinates\(^6\)

\[ \eta_\alpha = [2(p^2 + |z|^2)]^{1/2}\lambda_\alpha, \quad \bar{\eta}_{\dot{\alpha}} = [2(p^2 + |z|^2)]^{1/2}\bar{\lambda}_{\dot{\alpha}} \]  
(24)

one obtains the relations

\[ \{\eta_\alpha, \eta_\beta\}^* = -iz\epsilon_{\alpha\beta}, \quad \{\eta_{\dot{\alpha}}, \eta_{\dot{\beta}}\}^* = iz\epsilon_{\dot{\alpha}\dot{\beta}}, \quad \{\eta_\alpha, \eta_{\dot{\beta}}\}^* = ip_{\alpha\dot{\beta}} \]  
(25)

leading after quantization to noncommutative Weyl spinor coordinates. Similarly one can calculate

\[ S^{\mu\nu} = \lambda^\alpha \left[ (\sigma^{\mu\nu})_\alpha^\beta p_{\beta\dot{\gamma}} + p_{\alpha\dot{\beta}}(\bar{\sigma}^{\mu\nu})_\dot{\beta}^\dot{\gamma} \right] \bar{\lambda}^{\dot{\gamma}} + z\lambda^\alpha (\sigma^{\mu\nu})_\alpha^\beta \lambda_\beta + z\bar{\lambda}_{\dot{\alpha}}(\bar{\sigma}^{\mu\nu})_{\dot{\alpha}}^\dot{\beta} \bar{\lambda}_{\dot{\beta}}, \]

\[ (\sigma^{\mu\nu})_\alpha^\beta \equiv \frac{1}{2}(\sigma^{\mu\gamma}\sigma^{\nu\dot{\gamma}} - \sigma^{\nu\gamma}\sigma^{\mu\dot{\gamma}} - \sigma^{\mu\dot{\gamma}}\sigma^{\nu\gamma}), \quad (\bar{\sigma}^{\mu\nu})_{\dot{\alpha}}^\dot{\beta} \equiv \frac{1}{2}(\bar{\sigma}^{\mu\gamma}\bar{\sigma}^{\nu\dot{\gamma}} - \bar{\sigma}^{\nu\dot{\gamma}}\bar{\sigma}^{\mu\gamma}) \]

i.e. we see that the coordinates are becoming also noncommutative.

One can note that after the linear transformation of the form

\[ \eta'_\alpha = \eta_\alpha + cp_{\alpha\dot{\beta}} \bar{\eta}_{\dot{\beta}}, \quad \bar{\eta}'_{\dot{\alpha}} = \bar{\eta}_{\dot{\alpha}} + c\bar{\eta}_{\dot{\gamma}} p_{\gamma\dot{\beta}}. \]  
(26)

we can obtain from the algebra (25) for certain choice of \(c\) the DB relations \(\{\eta'_\alpha, \eta'_{\dot{\beta}}\}^* \sim \epsilon_{\alpha\beta}, \{\eta'_\alpha, \bar{\eta}'_{\dot{\beta}}\}^* = 0\). The algebra of such type is used for description of massless fields with arbitrary helicities in [3,4]. For other choice of \(c\) we obtain alternatively \(\{\eta'_\alpha, \eta'_{\dot{\beta}}\}^* \sim p_{\alpha\beta}\). In such a case \(\eta'_\alpha\) and \(\bar{\eta}'_{\dot{\alpha}}\) can be treated as of suitably rescaled creation and annihilation operators.

**ii)** Other way is the Gupta–Bleuler quantization method. Such a technique implies the split of the second class constraints into complex–conjugated pairs, with holomorphic and antiholomorphic parts forming separately the subalgebras of first class constraints. The algebra (19, 20)

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\(^5\)In calculation it is convenient to use that \(\det C = \det D \det (A - BD^{-1}C)\) for matrix \(C = \begin{pmatrix} A & B \\ C & D \end{pmatrix}\)

\[ \begin{pmatrix} 1 & B \\ 0 & D \end{pmatrix} \begin{pmatrix} A - BD^{-1}C & 0 \\ D^{-1}C & 1 \end{pmatrix}. \]

\(^6\)On the mass shell \(T \approx 0\) and at \(z = m\) we get \(\eta_\alpha = 2m\lambda_\alpha\) and \(\bar{\eta}_{\dot{\alpha}} = 2m\bar{\lambda}_{\dot{\alpha}}.\)
of the constraints \((17), (18)\) does not satisfy these requirements. Let us introduce, however, new constraints as follows
\[
D_\alpha = D_\alpha + \frac{b}{2} p_{\alpha \beta} \bar{D}^\beta, \quad \bar{D}_\dot{\alpha} = \bar{D}_\dot{\alpha} + \frac{b}{2} D^\beta p_{\beta \dot{\alpha}},
\]
\[\bar{D}_\dot{\alpha} = (\bar{D}_\dot{\alpha}).\] If \(b\) satisfies the equation \((b^2 - 2b) \frac{m^2}{|z|^2} - 1 = 0\) (i.e. \(b = (1 \pm \sqrt{1 + \frac{|z|^2}{m^2}})\)) the algebra of the constraints \((27)\) takes the form
\[
\{D_\alpha, D_\beta\} = \frac{2i}{z} \epsilon_{\alpha \beta} T, \quad \{\bar{D}_\dot{\alpha}, \bar{D}_\dot{\beta}\} = -\frac{2i}{\bar{z}} \epsilon_{\dot{\alpha} \dot{\beta}} T, \quad \{D_\alpha, \bar{D}_\dot{\beta}\} = -4b(1 + \frac{m^2}{|z|^2}) i p_{\alpha \beta} - \frac{2b^2}{z^2} p_{\alpha \beta} T.
\]
We see that the constraints \((27)\) are suitable for application of Gupta–Bleuler quantization method. It should be mentioned that the transformation from constraints \((D_\alpha, \bar{D}_\dot{\alpha})\) to constraints \((\bar{D}_{\dot{\alpha}}, D_\alpha)\) is invertible.

We shall assume that the wave function satisfies the Klein–Gordon equation, what follows from the constraint \((16)\). On the mass shell \((16)\) the constraints \((27)\) have the form
\[
D_\alpha = \pi'_\alpha - 2b(1 + \frac{m^2}{|z|^2}) i p_{\alpha \beta} \bar{\chi}^\beta \approx 0, \quad \bar{D}_{\dot{\alpha}} = \bar{\pi}'_{\dot{\alpha}} + 2b(1 + \frac{m^2}{|z|^2}) i \lambda^\beta p_{\beta \dot{\alpha}} \approx 0
\]
where we introduced new spinor variables via the following canonical transformation
\[
\pi'_\alpha \equiv \pi_\alpha + \frac{b}{2} p_{\alpha \beta} \bar{\pi}^\beta, \quad \bar{\pi}'_{\dot{\alpha}} \equiv \bar{\pi}_{\dot{\alpha}} + \frac{b}{2} \pi^\beta p_{\beta \dot{\alpha}},
\]
\[
\lambda^\alpha \equiv \frac{|z|^2}{|z|^2 + b^2 p^\alpha} (\lambda^\alpha - \frac{b}{2} \bar{\lambda}^\beta D^\beta \lambda^\alpha), \quad \bar{\lambda}_{\dot{\alpha}} \equiv \frac{|z|^2}{|z|^2 + b^2 p^\alpha} (\bar{\lambda}_{\dot{\alpha}} - \frac{b}{2} \bar{\lambda}^\beta \lambda_{\dot{\beta}}).
\]
\[\text{i.e. we obtain the standard canonical commutation relations (compare with } (13))\]
\[
\{\lambda^\alpha, \pi'_\beta\} = \delta^\alpha_\beta, \quad \{\bar{\lambda}_{\dot{\alpha}}, \bar{\pi}'_{\dot{\beta}}\} = \delta^\beta_{\dot{\alpha}}, \quad \{\lambda^\alpha, \bar{\pi}'_{\dot{\beta}}\} = \{\bar{\lambda}_{\dot{\alpha}}, \pi'_\beta\} = 0.
\]

For the quantization of our model we consider the Schrödinger representation of the CCR \((31)\)
\[
\pi'_\alpha = -i \partial / \partial \lambda^\alpha, \quad \bar{\pi}'_{\dot{\alpha}} = -i \partial / \partial \bar{\lambda}_{\dot{\alpha}}
\]
and use the wave function \(\Psi\) in the momentum representation, i.e. \(\Psi = \Psi(p_\mu, \lambda^\alpha, \bar{\lambda}_{\dot{\alpha}})\). The spinorial wave equation \(\bar{D}_\dot{\alpha} \Psi = 0\) takes the following form\(^7\)
\[
(-\partial / \partial \bar{\lambda}_{\dot{\alpha}} + 2b(1 + \frac{m^2}{|z|^2}) \lambda^\beta p_{\beta \dot{\alpha}}) \Psi = 0.
\]
The solution of \((33)\) is given by
\[
\Psi(p_\mu, \lambda^\alpha, \bar{\lambda}_{\dot{\alpha}}) = e^{2b(1 + \frac{m^2}{|z|^2}) \lambda^\beta p_{\beta \dot{\alpha}} \bar{\lambda}_{\dot{\alpha}}} \tilde{\Psi}(p_\mu, \lambda^\alpha)
\]
where the field \(\tilde{\Psi}(p_\mu, \lambda^\alpha)\) depends only on one Weyl spinor \(\lambda^\alpha\) and provides the bosonic counterpart of \(D = 4\ N = 1\) chiral superfield.

Due to the bosonic nature of \(\lambda^\alpha\) in expansion of \(\tilde{\Psi}(p_\mu, \lambda^\alpha)\) there is an infinite number of space–time fields \(\psi_{\alpha_1 \ldots \alpha_n}(p) = \psi(\alpha_1 \ldots \alpha_n)(p), n = 0, 1, \ldots, \infty\). The mass–shell condition \((16)\) after
\[\text{\textsuperscript{7}The choice of } D_\alpha \text{ in place } \bar{D}_{\dot{\alpha}} \text{ is equally well possible.}\]

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the transition by Fourier transformation to the space–time picture, leads to the Klein–Gordon (KG) equation ($\Box \equiv \partial_\mu \partial^\mu$)

$$(\Box + m^2)\Psi(x; \lambda, \bar{\lambda}) = 0 \quad \Leftrightarrow \quad (\Box + m^2)\psi_{\alpha_1 \ldots \alpha_n}(x) = 0 \quad (n = 0, 1, 2, \ldots).$$

(35)

Here we should observe that

i) The half-integer spin fields (n odd) satisfy KG equation, however in massless case the half-integer helicity fields do satisfy linear equations (see Sect. 3),

ii) The spin–statistic theorem is not valid – both integer and half–integer spin fields are bosonic. We shall come back to the question of statistics in Sect. 4.

3 Massless particle model with $N = 1$ bosonic counterpart of SUSY

The model (4), (5) can be described equivalently by the Lagrangian

$$\mathcal{L} = -\frac{1}{2\omega}(\dot{\omega}_\mu \dot{\omega}^{\mu} + e^2m^2) - i(z\dot{\lambda}^{\alpha}\lambda_\alpha - \bar{z}\dot{\bar{\lambda}}^{\dot{\alpha}}\bar{\lambda}_{\dot{\alpha}}).$$

(36)

After eliminating the einbein $e$ by means of its equation of motion, from (36) one obtains the Lagrangian (39). The massless limit of (36) looks as follows

$$\mathcal{L} = -\frac{1}{2\omega}(\dot{\omega}_\mu \dot{\omega}^{\mu}) - i(z\dot{\lambda}^{\alpha}\lambda_\alpha - \bar{z}\dot{\bar{\lambda}}^{\dot{\alpha}}\bar{\lambda}_{\dot{\alpha}}).$$

(37)

Besides the constraint $p_e \approx 0$ which implies pure gauge character of the einbein $e$, from (37) one gets the following constraints

$$T = p^2 \approx 0,$$

$$D_\alpha = \pi_\alpha + ip_{\alpha\dot{\beta}}\dot{\bar{\lambda}}^{\dot{\beta}} + iz\lambda_\alpha \approx 0,$$

$$\bar{D}_{\dot{\alpha}} = \bar{\pi}_{\dot{\alpha}} - i\lambda^{\beta}p_{\beta\dot{\alpha}} - i\bar{z}\bar{\lambda}_{\dot{\alpha}} \approx 0.$$  

(38)

The nonvanishing Poisson brackets are

$$\{D_\alpha, \bar{D}_{\dot{\beta}}\} = 2ip_{\alpha\dot{\beta}}, \quad \{D_\alpha, D_{\beta}\} = -2iz\epsilon_{\alpha\beta}, \quad \{\bar{D}_{\dot{\alpha}}, \bar{D}_{\dot{\beta}}\} = 2i\bar{z}\epsilon_{\dot{\alpha}\dot{\beta}}.$$  

(40)

The mass constraint (38) is of the first class. The determinant of the Poisson brackets matrix (39) characterizing the spinorial constraints (39) is the following

$$\det \mathcal{C} = 16(p^2 + |z|^2)^2 \approx 16|z|^4.$$  

(41)

If $z \neq 0$ all spinorial constraints (39) are second class. In the case of vanishing central charges $z = 0$ the four spinorial constraints (39) contain two second class constraints and two first class. Below we analyze massless particle at $z = 0$ with spinorial first class constraints, defined as follows:

$$F^{\dot{\alpha}} = p^{\dot{\alpha}\dot{\beta}}D_\beta \approx 0,$$

$$\bar{F}^\alpha = \bar{D}_{\dot{\beta}}p^{\dot{\beta}\alpha} \approx 0$$

(42)

with the following Poisson brackets

$$\{F^{\dot{\alpha}}, \bar{D}_{\dot{\beta}}\} = 2i\delta^{\dot{\alpha}}_{\dot{\beta}}T \approx 0,$$

$$\{\bar{F}^\alpha, D_\beta\} = -2i\delta^\alpha_\beta T \approx 0.$$
But the first class constraints (42) are reducible: \( p_{\alpha\dot{\alpha}} F^{\beta} \approx 0, \ F^{\beta} p_{\beta\dot{\alpha}} \approx 0 \). The irreducible separation of first and second class constraints is obtained by projecting of the spinorial constraints (39) along spinors \( \lambda^\alpha \) and \( \lambda_\dot{\alpha} \). The constraints
\[
G \equiv \lambda^\alpha D_\alpha \approx 0, \quad \bar{G} \equiv \bar{D}_\dot{\alpha} \bar{\lambda}^{\dot{\alpha}} \approx 0
\]
are second class whereas the constraints
\[
F \equiv \bar{\lambda}_\dot{\alpha} p^{\dot{\alpha}} D_\alpha \approx 0, \quad F \equiv D_\alpha p^{\dot{\alpha}} \lambda^\alpha \approx 0
\]
are of first class. Their Poisson brackets look as follows:
\[
\{G, \bar{G}\} = 2 i \lambda^\alpha p_{\alpha\dot{\alpha}} \bar{\lambda}^{\dot{\alpha}} \neq 0, \quad \{F, \bar{F}\} = - (\lambda^\alpha \pi_\alpha - \bar{\pi}_\dot{\alpha} \bar{\lambda}^{\dot{\alpha}}) T,
\]
\[
\{G, F\} = - \{\bar{G}, F\} = F, \quad \{G, \bar{F}\} = - \{\bar{G}, \bar{F}\} = - \bar{F}.
\]

We carry out quantization of massless particle with \( N = 1 \) bosonic counterpart of SUSY by Gupta–Bleuler method. The wave equations are imposed by the first class constraints (38), (44) \( T \approx 0, \ F \approx 0, \ \bar{F} \approx 0 \) and either \( \bar{G} \approx 0 \) or \( G \approx 0 \). But the pair of constraints \( G \approx 0 \) and \( F \approx 0 \) are equivalent to the constraints \( D_\alpha \approx 0 \); similarly the constraints \( \bar{G} \approx 0 \) and \( \bar{F} \approx 0 \) are equivalent to the constraints \( \bar{D}_\dot{\alpha} \approx 0 \). Thus we have two possible quantizations:

- ‘bosonic chiral’ quantization with the wave equations
\[
T|\Psi\rangle = 0, \quad F|\Psi\rangle = 0, \quad \bar{D}_{\dot{\alpha}}|\Psi\rangle = 0
\]

- ‘bosonic antichiral’ quantization with wave function subjected to the conditions
\[
T|\Psi\rangle = 0, \quad \bar{F}|\Psi\rangle = 0, \quad D_\alpha|\Psi\rangle = 0.
\]

Let us consider the chiral case. In the representation
\[
p_\mu = -i \partial / \partial x^\mu \equiv -i \partial_\mu, \quad \pi_\alpha = -i \partial / \partial \lambda^\alpha \equiv -i \partial_\alpha, \quad \bar{\pi}_{\dot{\alpha}} = -i \partial / \partial \bar{\lambda}^{\dot{\alpha}} \equiv -i \bar{\partial}_{\dot{\alpha}}
\]
the wave function \( \Psi(x, \lambda, \bar{\lambda}) \) satisfies the equations
\[
\square \Psi = 0, \quad \bar{D}_{\dot{\alpha}} \Psi = (-\bar{\partial}_{\dot{\alpha}} - \lambda^\beta \partial_\beta) \Psi = 0 \quad \text{(48)}
\]
\[
- i \bar{\lambda}_\dot{\alpha} \partial^{\dot{\alpha}} D_\alpha \Psi = - \bar{\lambda}_\dot{\alpha} \partial^{\dot{\alpha}} \partial_\alpha \Psi = 0 \quad \text{(49)}
\]
In the variables \( x^\mu_L = x^\mu + i \lambda \sigma^\mu \bar{\lambda}, \lambda^\alpha, \bar{\lambda}^{\dot{\alpha}} \) bosonic SUSY-covariant derivatives take the form
\[
D_\alpha = -i \partial_\alpha + 2 \partial_{L\alpha\dot{\alpha}} \bar{\lambda}^{\dot{\alpha}}, \quad \bar{D}_{\dot{\alpha}} = -i \bar{\partial}_{\dot{\alpha}}. \quad \text{(50)}
\]

\(^8\)This procedure is corrected since spinors \( \lambda^\alpha \) and \( \lambda_\dot{\alpha} p^{\dot{\alpha}} \) are not proportional in considered task. Otherwise, when \( \lambda^\alpha p_{\alpha\dot{\alpha}} \bar{\lambda}^{\dot{\alpha}} = 0 \), we have \( p_{\alpha\dot{\alpha}} \sim \lambda_\alpha \lambda_{\dot{\alpha}} \). Then the spinorial constraints (39), taking the form \( \pi_\alpha \approx 0, \bar{\pi}_{\dot{\alpha}} \approx 0 \), exclude completely the dependence on \( \lambda, \bar{\lambda} \). As result we obtain the system describing only by the variables \( x^\mu, p_\mu \) and the constraint (38) i. e. the massless particle of zero helicity.
Thus due to the chirality condition (48) the wave function does not depend on $\bar{\lambda}^\dot{\alpha}$. It depends only on the left chiral variables $z_L = (x^\mu_L, \lambda^\alpha)$, and commuting spinor $\lambda$. One can write the following expansion

$$\Psi(x_L, \lambda) = \sum_{n=0}^{\infty} \lambda^{\alpha_1} \ldots \lambda^{\alpha_n} \phi_{\alpha_1 \ldots \alpha_n}(x_L)$$

(51)

where the multispinor fields are totally symmetric in spinor indices, i.e. $\phi_{\alpha_1 \ldots \alpha_n} = \phi_{(\alpha_1 \ldots \alpha_n)}$. The usual fields depending on real space–time coordinates $x^\mu$ are obtained by

$$\phi_{\alpha_1 \ldots \alpha_n}(x) = e^{-i\lambda^\sigma \bar{\lambda}^\dot{\alpha}_\mu} \phi_{\alpha_1 \ldots \alpha_n}(x_L).$$

The equation (49) gives Fierz–Pauli equations for the component fields

$$\partial^{\dot{\beta}} \phi_{\beta \alpha_2 \ldots \alpha_n} = 0.$$  

(52)

The Klein–Gordon equation $\Box \phi_{\alpha_1 \ldots \alpha_n} = 0$, resulting from (47), follows also from (52). We see therefore that the expansion of the wave function (51) describes an infinite set of massless particles with helicities $n/2$.

The Gupta-Bleuler quantization procedure presented here is analogous to the one used for the quantization of massless Brink-Schwarz superparticle, but due to the bosonic character of spinorial variable $\lambda$, we get infinite helicity spectrum. We recall that the infinite set of integer and half–integer helicities describes also the spectrum of supersymmetric massless particles propagating in tensorial superspace [13].

4 Massive relativistic particles with $N = 2$ bosonic counterpart of SUSY.

4.1 $N = 2$ action and the constraints

Let us introduce two commuting Weyl spinors $\lambda_i^\alpha$, $\bar{\lambda}_i^{\dot{\alpha}} = (\bar{\lambda}_i^{\dot{\alpha}})$ $(i = 1, 2)$. The natural generalization of the Lagrangian (5) is

$$\mathcal{L} = -m(\dot{\omega}_\mu \dot{\omega}^\mu)^{1/2} - i(z_{ij} \dot{\lambda}_i^\alpha \lambda_{\alpha j} - \bar{z}_{ij} \bar{\lambda}_{\alpha j} \dot{\bar{\lambda}}_i^{\dot{\alpha}}).$$  

(53)

Here the constant matrix $z_{ij}$ is symmetric, $z_{ij} = z_{ji}$; the last terms in (53) are total derivatives, e.g. $z_{ij} \dot{\lambda}_i^\alpha \lambda_{\alpha j} = \frac{1}{2}(z_{ij} \lambda_i^\alpha \lambda_{\alpha j})$ if $z_{ij} = -z_{ji}$.

The $\omega$–form can be written in general case as follow

$$\dot{\omega}^\mu = \dot{x}^\mu - i\kappa_{ij}(\dot{\lambda}_i^\alpha \sigma^{\mu}_{\alpha \dot{\beta}} \bar{\lambda}_j^{\dot{\beta}} - \lambda_j^\alpha \sigma^{\mu}_{\alpha \dot{\beta}} \bar{\lambda}_i^{\dot{\beta}}).$$

(54)

where $\kappa_{ij} = \kappa_{ji}$ is the $2 \times 2$ Hermitean metric in $N = 2$ unitary space. If we consider possible linear definitions of spinors $\lambda_i^\alpha$ in $N = 2$ internal space one can choose

$$\kappa_{ij} = \begin{pmatrix} 1 & 0 \\ 0 & \kappa \end{pmatrix}$$

(55)
where \( \kappa \) is real.

From expressions for the canonical momenta

\[
p_\mu = \frac{\partial L}{\partial \dot{x}_\mu} = -m(\dot{\omega}_\mu \omega^\nu)^{-1/2}\dot{\omega}_\mu, \tag{56}
\]

\[
\pi_{\alpha i} = \frac{\partial L}{\partial \dot{\lambda}_i^\alpha} = -i\kappa_{ij}p_{\alpha \bar{\beta}}\bar{\lambda}_j^\bar{\beta} - iz_{ij}\lambda_{\alpha j}, \tag{57}
\]

\[
\bar{\pi}_{\bar{\alpha} i} = \frac{\partial L}{\partial \dot{\bar{\lambda}}_i^{\bar{\alpha}}} = i\kappa_{ij}\lambda_j^\beta p_{\beta \bar{\alpha}} + iz_{ij}\bar{\lambda}_{\bar{\alpha} j}
\tag{58}
\]

we obtain the following constraints

\[
T \equiv p^2 - m^2 \approx 0, \tag{59}
\]

\[
D_{\alpha i} \equiv \pi_{\alpha i} + i\kappa_{ij}p_{\alpha \bar{\beta}}\bar{\lambda}_j^\bar{\beta} + iz_{ij}\lambda_{\alpha j} \approx 0, \tag{60}
\]

\[
\bar{D}_{\bar{\alpha} i} \equiv \bar{\pi}_{\bar{\alpha} i} - i\kappa_{ij}\lambda_j^\beta p_{\beta \bar{\alpha}} - iz_{ij}\bar{\lambda}_{\bar{\alpha} j} \approx 0. \tag{61}
\]

Using the canonical Poisson brackets

\[
\{x^\mu, p_\nu\} = \delta^\mu_\nu, \quad \{\lambda_i^\alpha, \pi_{\beta j}\} = \delta_i^\alpha \delta_{ij}, \quad \{\bar{\lambda}_i^{\bar{\alpha}}, \bar{\pi}_{\bar{\beta} j}\} = \delta_\beta^{\bar{\alpha}} \delta_{ij}
\]

\[
\{\lambda_{\alpha i}, \pi_{\beta j}\} = \{\pi_{\alpha i}, \lambda_{\beta j}\} = -\epsilon_{\alpha \beta} \delta_{ij}, \quad \{\bar{\lambda}_{\bar{\alpha} i}, \bar{\pi}_{\bar{\beta} j}\} = \{\bar{\pi}_{\bar{\alpha} i}, \bar{\lambda}_{\bar{\beta} j}\} = -\epsilon_{\alpha \bar{\beta}} \delta_{ij}
\]

we obtain nonzero Poisson brackets of the constraints \([59]-[61]\)

\[
\{D_{\alpha i}, \bar{D}_{\bar{\beta} j}\} = 2i\kappa_{ij}p_{\alpha \bar{\beta}}, \tag{62}
\]

\[
\{D_{\alpha i}, D_{\beta j}\} = -2iz_{ij}\epsilon_{\alpha \beta}, \quad \{\bar{D}_{\bar{\alpha} i}, \bar{D}_{\bar{\beta} j}\} = 2i\bar{z}_{ij}\epsilon_{\alpha \bar{\beta}}. \tag{63}
\]

It should be pointed out that the relations \([62], [63]\) with changed sign on the rhs describe the bosonic counterpart of the generalized \( N = 2 \) superalgebra with the Hermitean metric \( \kappa_{ij} \) in internal \( N = 2 \) space.

The constraint \([59]\) \( T \approx 0 \) is the first class constraint. From the spinor constraints \([60], [61] \) one gets the following \( 4 \times 4 \) matrix of PB

\[
C = \begin{pmatrix}
\{D_{\alpha i}, \bar{D}_{\bar{\beta} j}\} & \{D_{\alpha i}, D_{\beta j}\} \\
\{\bar{D}_{\bar{\alpha} i}, \bar{D}_{\bar{\beta} j}\} & \{\bar{D}_{\bar{\alpha} i}, D_{\beta j}\}
\end{pmatrix} = \begin{pmatrix}
-2iz_{ij}\epsilon_{\alpha \beta} & 2i\kappa_{ij}p_{\alpha \bar{\beta}} \\
-2i\kappa_{ij}p_{\beta \bar{\alpha}} & 2i\bar{z}_{ij}\epsilon_{\alpha \bar{\beta}}
\end{pmatrix}. \tag{64}
\]

We obtain that

\[
\det C = 2^8[\det(\hat{\zeta} \hat{\zeta} + p^2 \hat{\kappa} \hat{\kappa}^{-1} \hat{\zeta} \hat{\zeta})]^2
\]

where ‘hats’ denote the corresponding matrices, i.e. \( \hat{\zeta} = (z_{ij}), \hat{\bar{\zeta}} = (\bar{z}_{ij}) \) and \( \hat{\kappa} = (\kappa_{ij}) \) is given by \([55]\). One can consider two cases:

\( i) \) If matrix \( \hat{\zeta} = (z_{ij}) \) is diagonal, \( (z_{ij}) = \begin{pmatrix} z_1 & 0 \\ 0 & z_2 \end{pmatrix} \), we obtain that \( \det(\hat{\zeta} \hat{\zeta} + p^2 \hat{\kappa} \hat{\kappa}^{-1} \hat{\zeta} \hat{\zeta}) = (|z_1|^2 + p^2)(|z_2|^2 + p^2\kappa^2) \), i.e. it is always nonvanishing. We see therefore that for arbitrary values of \( \kappa \) and \( z_1, z_2 \) all the constraints \([60], [61] \) are second class.
In case of antidiagonal matrix \((z_{ij}) = \begin{pmatrix} 0 & z \\ -z & 0 \end{pmatrix}\) (we remind that matrix \(z_{ij}\) is symmetric), we obtain that \(\det (\hat{\xi}^2 + p^2 \hat{\kappa} \hat{\xi}^{-1} \hat{\kappa}^2) = (|z|^2 + p^2 \kappa)^2\). One gets that the matrix of Poisson brackets of the constraints \([53]\) has vanishing determinant if \(\kappa = -\frac{|z|^2}{m^2} < 0\) and we conclude that in such a case the first class constraints are present in the model. Putting \(z = m\), i.e. \(\kappa = -1\), it is easy to check that the unitary metric tensor \(\hat{\kappa}\) implies the invariance of the form \(\omega_\mu\) (see \([54]\)) under \(U(1,1)\) symmetry. The presence of the central charge reduces however this symmetry to the invariance group \(O(1,1) = U(1,1) \cap O(2;c)\), and only in this case the first class constraints are present in the model \([53]\).

In case \(\text{ii})\) we will consider a simple choice \(z = m\), i.e. \(\kappa = -\frac{|z|^2}{mt} = -1\). Introducing the notations \(\lambda_\alpha^a \equiv \lambda^a\) and \(\lambda_2^a \equiv \eta^a\) the Lagrangian \([53]\) and \(\omega\)-form \([54]\) are

\[
\mathcal{L} = -m(\bar{\psi} \dot{\psi})^{1/2} - im(\bar{\lambda}_\alpha \eta_\alpha + i \bar{\eta}^\alpha \lambda_\alpha - \bar{\lambda}_\dot{\alpha} \dot{\bar{\eta}}^\alpha - \bar{\eta}_\dot{\alpha} \dot{\lambda}^\alpha),
\]

(65)

\[
\bar{\psi} \dot{\psi} = \dot{\bar{\psi}} \dot{\psi} = \bar{\lambda}^\alpha \eta_\alpha + i \bar{\eta}^\alpha \lambda_\alpha - \bar{\lambda}_\dot{\alpha} \dot{\bar{\eta}}^\alpha - \bar{\eta}_\dot{\alpha} \dot{\lambda}^\alpha,
\]

(66)

### 4.2 Description of the model in terms of Dirac spinors

The formulation \([65]\) has an attractive interpretation if we pass to the commuting four–component Dirac spinor

\[
\psi_a = \begin{pmatrix} \lambda^a_x \\ \bar{\eta}^a \end{pmatrix}
\]

where \(a = 1, 2, 3, 4\). The Dirac matrices \((\gamma_\mu)_{ab}\) in Weyl representation are as follows

\[
(\gamma_\mu)_{ab} = \begin{pmatrix} 0 & \sigma_{ab}^\mu \\ \sigma_{\dot{a}\dot{b}}^\mu & 0 \end{pmatrix}, \quad \{\gamma_\mu, \gamma_\nu\} = 2 \eta_{\mu\nu}
\]

where \(\sigma_{\alpha\beta}^0 = \sigma_{\dot{a}\dot{b}}^0 = 1_2\) and \(\sigma_{\alpha\beta}^i = -\sigma_{\dot{a}\dot{b}}^i\) \((i = 1, 2, 3)\) are the Pauli matrices. Then

\[
\bar{\psi} \psi = (\bar{\psi}^+ \gamma_0) \psi = (\eta^a, \bar{\lambda}_\alpha)
\]

and we obtain

\[
\bar{\psi} \psi - \bar{\psi} \psi = \lambda^\alpha \eta_\alpha + i \bar{\eta}^\alpha \lambda_\alpha - \bar{\lambda}_\dot{\alpha} \dot{\bar{\eta}}^\alpha - \bar{\eta}_\dot{\alpha} \dot{\lambda}^\alpha,
\]

\[
\bar{\psi} \gamma^\mu \psi - \bar{\psi} \gamma^\mu \psi = i \sigma_{\alpha\beta}^a \eta^a \bar{\lambda}_\dot{\beta} - \bar{\lambda}^\alpha \sigma_{\dot{a}\dot{b}}^a \dot{\bar{\eta}}^\beta - (\eta^a \sigma_{\alpha\beta}^a \dot{\bar{\eta}}^\beta - \lambda^\alpha \sigma_{\dot{a}\dot{b}}^a \eta^\beta).
\]

Thus the Lagrangian \([65]\) takes in the notation using Dirac spinor \(\psi\) the following simple form

\[
\mathcal{L} = -m(\bar{\psi} \dot{\psi})^{1/2} - im(\bar{\psi} \dot{\psi} - \bar{\psi} \dot{\psi}),
\]

(67)

\[\text{We recall that in case of standard } N = 2 \text{ superparticle when spinor variables are Grassmannian and the matrix } z_{ij} \text{ is skew–symmetric, } (z_{ij}) = \begin{pmatrix} 0 & z \\ -z & 0 \end{pmatrix}, \text{ the first class constraints are presented (the matrix of Poisson brackets of the constraints has vanishing determinant) if } \kappa = \frac{|z|^2}{m^2} > 0 \text{ and the internal } N = 2 \text{ symmetry in the presence of central charges } z = m \text{ is } U(2) \cap Sp(2;c) = SU(2).\]
where

\[ \dot{\omega}^\mu = \dot{x}^\mu + i(\bar{\psi} \gamma^\mu \psi - \bar{\psi} \gamma^\mu \dot{\psi}) . \]  

(68)

We would like to point out that the model with spinorial variables described by Dirac spinor corresponds to the choice of noncompact internal sector, with the metric \( \kappa_{ij} = \text{diag}(1, -1) \). It should be added that the model (67) in different context has been firstly proposed in [19].

4.3 Gupta-Bleuler quantization of the model

The constraints (59)–(61) for \( z = m \) or equivalently \( \kappa = -1 \), written in Dirac notation, are the following

\[ T \equiv p^2 - m^2 \approx 0 , \]
\[ D^a \equiv \pi^a + i \bar{\psi}^b (\hat{\rho} - m)_b^a \approx 0 , \]
\[ \bar{D}_a \equiv \bar{\pi}_a - i (\hat{\rho} - m)_a \psi_b \approx 0 . \]

(69)  
(70)  
(71)

Here \( \pi^a \) and \( \bar{\pi}_a \) defined as \( \pi^a = \partial \mathcal{L} / \partial \dot{\psi}_a \) and \( \bar{\pi}_a = \partial \mathcal{L} / \partial \bar{\dot{\psi}}^a \) are conjugate momenta of \( \psi_a \) and \( \bar{\psi}^a \), their Poisson brackets are \( \{ \psi_a, \pi^b \} = \delta_a^b \) and \( \{ \bar{\psi}^a, \bar{\pi}_b \} = \delta_b^a \). Also we shall use notation \( \hat{\rho} \equiv \gamma^\mu \rho_a \).

From Poisson brackets of the constraints

\[ \{ \bar{D}_a, D^b \} = -2i (\hat{\rho} - m)_a^b , \quad \{ D^a, D^b \} = 0 , \quad \{ \bar{D}_a, \bar{D}_b \} = 0 , \]
\[ \{ T, D_a \} = \{ T, \bar{D}_a \} = 0 \]

(72)  
(73)

we obtain directly that the constraint (69) and the half of the spinorial constraints (70), (71) are first class constraints.

The separation of first and second class spinorial constraints in (70), (71) is achieved by the projectors \( \mathcal{P}_\pm \equiv \frac{1}{2m} (m \pm \hat{\rho}) \) where \( 1 = (\mathcal{P}_+ + \mathcal{P}_-) \). One can check that on mass shell \( p^2 = m^2 \) we obtain \( \mathcal{P}_+ \mathcal{P}_- = \mathcal{P}_- \mathcal{P}_+ = 0 \). From eight real spinorial constraints (70), (71) we construct the following sets of reducible constraints

\[ F^a = D^b (\hat{\rho} + m)_a^b , \quad \bar{F}_a = (\hat{\rho} + m)_a^b \bar{D}_b ; \]
\[ G^a = D^b (\hat{\rho} - m)_a^b , \quad \bar{G}_a = (\hat{\rho} - m)_a^b \bar{D}_b . \]

(74)  
(75)

Due to the relations

\[ F^b (\hat{\rho} - m)_a^b = 0 , \quad (\hat{\rho} - m)_a^b \bar{F}_b = 0 ; \]
\[ G^b (\hat{\rho} + m)_a^b = 0 , \quad (\hat{\rho} + m)_a^b \bar{D}_b = 0 \]

on the mass–shell (69) in the set of the constraints \( (F^a, \bar{F}_a) \) there are only four real independent constraints. Analogously, the constraints \( (G^a, \bar{G}_a) \) contain as well four real independent constraints. Expressing the constraints (70), (71) in term of the constraints (74), (75) we get

\[ D^a = \frac{1}{2m} (F^a - G^a) , \quad \bar{D}_a = \frac{1}{2m} (\bar{F}_a - \bar{G}_a) . \]

The constraints (74), (75) satisfy the following Poisson brackets algebra

\[ \{ \bar{F}_a, F^b \} = -2i (\hat{\rho} + m)_a^b T , \quad \{ F^a, F^b \} = \{ \bar{F}_a, \bar{F}_b \} = 0 , \]

11
\{ \bar{F}_a, G^b \} = \{ \bar{G}_a, F^b \} = -2i(\hat{p} - m)_a{}^b T, \quad \{ \bar{F}_a, \bar{G}_b \} = \{ F^a, G^b \} = 0, \\
\{ \bar{G}_a, G^b \} = -8im^2(\hat{p} + m)_a{}^b - 2i[2m\delta_a{}^b + (\hat{p} + m)_a{}^b] T, \quad \{ G^a, G^b \} = \{ \bar{G}_a, \bar{G}_b \} = 0.

From eight real spinorial constraints present in (70), (71) four independent constraints in \( \{ F^a, \bar{F}_a \} \) are first class whereas four independent constraints contained in \( \{ G^a, \bar{G}_a \} \) are second class.

We shall employ the Gupta–Bleuler quantization method by imposing on the wave function all first class constraints \( (T, F^a, \bar{F}_a) \) and half of the second class constraints being in involution \( (G^a \text{ or } \bar{G}_b) \). We have two quantizations:

- bosonic chiral quantization, with the wave function satisfying the following wave equations
  \[
  T|\Psi\rangle = 0, \quad F^a|\Psi\rangle = 0, \quad \bar{F}_a|\Psi\rangle = 0, \quad \bar{G}_a|\Psi\rangle = 0 \quad (76)
  \]

- bosonic antichiral quantization with the wave function submitted to the following equations
  \[
  T|\Psi\rangle = 0, \quad F^a|\Psi\rangle = 0, \quad \bar{F}_a|\Psi\rangle = 0, \quad G^a|\Psi\rangle = 0. \quad (77)
  \]

The reducible constraints \( \bar{F}_a \) and \( \bar{G}_a \) are equivalent to primary constraint \( \bar{D}_a \); similarly the constraints \( F^a \) and are \( G^a \) equivalent to \( D^a \). Therefore one can express the wave equations (76), (77) in other equivalent way

- bosonic chiral quantization:
  \[
  T|\Psi\rangle = 0, \quad \bar{D}_a|\Psi\rangle = 0, \quad F^a|\Psi\rangle = 0 \quad (78)
  \]

- bosonic antichiral case in which wave function is subjected the following constraints
  \[
  T|\Psi\rangle = 0, \quad D^a|\Psi\rangle = 0, \quad \bar{F}_a|\Psi\rangle = 0. \quad (79)
  \]

Let us consider chiral case (78) in more details. Using the realization
\[
\pi^a = -i\partial/\partial\psi_a, \quad \bar{\pi}_a = -i\partial/\partial\bar{\psi}^a
\]
and the momentum-dependent wave function \( \Psi(p, \psi, \bar{\psi}) \) one can write down the relations (78) as follows
\[
\bar{D}_a|\Psi\rangle = -i\left[ \frac{\partial}{\partial\bar{\psi}^a} + (\hat{p} - m)_a{}^b \psi_b \right]|\Psi\rangle = 0, \quad (80)
\]
\[
F^a|\Psi\rangle = -i\frac{\partial}{\partial\psi^b}(\hat{p} + m)_b{}^a|\Psi\rangle = 0, \quad (81)
\]
\[
T|\Psi\rangle = (p^2 + m^2)|\Psi\rangle = 0. \quad (82)
\]
The equation (80) has the general solution
\[
\Psi(p, \psi, \bar{\psi}) = e^{-\psi(\hat{p} - m)\psi}\bar{\Psi}(p, \psi) \quad (83)
\]

where the reduced wave function $\tilde{\Psi}(p, \psi)$ depends only on $\psi$, i.e. we have the expansion

$$
\tilde{\Psi}(p, \psi) = \sum_{n=0}^{\infty} \psi_{a_1} \cdots \psi_{a_n} \phi^{a_1 \cdots a_n}(p).
$$

(84)

Due to commuting nature of spinor $\psi_a$ the component fields $\phi^{a_1 \cdots a_n}(p)$ are totally symmetric

$$
\phi^{a_1 \cdots a_n}(p) = \phi^{(a_1 \cdots a_n)}(p).
$$

(85)

The equations (81) provide the Dirac equations for these fields

$$
(\hat{p} + m)_{a_1} \phi^{a_1 a_2 \cdots a_n}(p) = 0.
$$

(86)

We see that the multispinorial fields (85) are Bargman–Wigner fields describing massive particles of spins $n/2$. Obviously the Klein–Gordon equation (82) is the consequence of (86).

5 Conclusion

The classical $c$-number higher spin fields (85–86) for any spin are mathematically correct, and provide the relativistic quantum–mechanical description of one–particle states with arbitrary mass and spin (see e.g. [20]). The concept of bosons and fermions is related with the symmetry properties of multiparticle states, obtained in quantum field theory by quantum fields acting on the vacuum state. The description of higher spin fields presented here (see (84), (85)) does not take into consideration the spin–statistics theorem, however in the framework of first–quantized one-particle classical mechanics we need not to specify the statistics. The transition to the proper spin–statistic relation can be achieved in two way:

i) By introducing classical theory as a suitable limit $\hbar \to 0$ of quantized higher spin fields. In such a case the half–integer spin fields will have the Grassmann nature (we recall that fermionic quantum fields are described by infinite–dimensional Clifford algebras which become in the limit $\hbar \to 0$ an infinite–dimensional Grassmann algebra).

ii) One can pass from one–particle wave function to the wave function describing multiparticle states by suitable symmetrization procedure (besides bosonic and fermionic multiparticle states one can introduce also parabosonic and parafermionic multiparticle states, with ‘mixed’ symmetry properties).

The wave functions obtained in this paper if used for the description of multiparticle states should be therefore suitably symmetrized: one introduces symmetric products of one–particle wave functions for integer spin fields, and totally antisymmetric products if spin is half–integer. Such a procedure is well-known from the description of multi–particle states in quantum mechanics. If we wish to construct the quantum fields which generate multiparticle states from the vacuum we should multiply the $c$-number wave functions by respective bosonic and fermionic creation and annihilation operators. Such a procedure for obtaining fermionic fields with half–integer spin or helicity can be applied to $N = 1$ massless case (Sect. 3) and $N = 2$ massive case (Sect. 4), due to the presence of linear field equations.
It should be added that \(c\)-number massive higher spin fields have been obtained also in other papers from different relativistic particle models \cite{10, 21, 22}. We should also add that the realizations of ‘bosonic’ superalgebra was used in \cite{23} for description of physical degrees of freedom of the critical open string with \(N = 2\) conformal symmetry in \(2 + 2\) dimensions. Further one can point out that if one introduces fields on twistor spaces (see e. g. \cite{6, 7}) usually they are also commutative for any spin, or any helicity (in massless case).

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