The asymptotics of an eigenfunction-correlation
determinant for Dirac-δ perturbations

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Abstract. We prove the exact asymptotics of the scalar product of the ground
states of two non-interacting Fermi gases confined to a 3-dimensional ball \( B_L \)
of radius \( L \) in the thermodynamic limit, where the underlying one-particle
operators differ by a Dirac-δ perturbation. More precisely, we show the algebraic
decay of the correlation determinant

\[
\left| \det \left( \langle \phi^L_j, \psi^L_k \rangle \right)_{j,k=1,...,N} \right|^2 = L^{-\zeta(E)+O(1)},
\]

as \( N, L \to \infty \) and \( N/|B_L| \to \rho > 0 \), where \( \phi^L_j \) and \( \psi^L_k \) denote the lowest-
energy eigenfunctions of the finite-volume one-particle Schrödinger operators.
The decay exponent is given in terms of the s-wave scattering phase shift
\( \zeta(E) := \delta^2(\sqrt{E})/\pi^2 \). For an attractive Dirac-δ perturbation we conclude that the
decay exponent \( \frac{1}{\pi^2} \| \arcsin \left| T(E) / 2 \right| \|_{\text{HS}}^2 \) found in [GKMO14] does not provide a
sharp upper bound on the decay of the correlation determinant.

1. Introduction

We consider the asymptotics of the scalar product of the ground states of two
non-interacting finite-volume \( N \)-particle Schrödinger operators in the thermody-
namic limit approaching the particle density \( \rho(E) > 0 \) corresponding to the Fermi
energy \( E > 0 \). Here, the underlying one-particle Schrödinger operators are the neg-
ative Laplacian in 3-dimensional Euclidean space and the negative Laplacian with a
Dirac-δ or zero-range perturbation sitting at the origin. We restrict this pair to the
ball \( B_L(0) \) of radius \( L \) and are interested in the \( L \)-asymptotics of the scalar product
of the ground states of the corresponding two non-interacting \( N \)-particle operators,
which we call ground-state overlap in the sequel. Using the representation of the
ground states as Slater determinants, we see that the ground-state overlap is the
following correlation determinant

\[
S^N_L := \det \left( \langle \phi^L_j, \psi^L_k \rangle \right)_{1 \leq j,k \leq N}.
\]  

In this note, we are interested in its thermodynamic limit, i.e. increasing \( L \) and
\( N \in \mathbb{N} \) simultaneously such that \( N/|B_L(0)| \to \rho(E) > 0 \), where \( \rho(E) \) denotes the
integrated density of states of the negative Laplacian at the energy \( E > 0 \). Here, \( \phi^L_j \) and \( \psi^L_k \) are the eigenfunctions belonging to the \( N \) lowest eigenvalues of the
restricted operators, which we call \( H_L \) and \( H_{\alpha,L} \), and \( \langle \cdot, \cdot \rangle \) denotes the scalar

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product in $L^2(B_L(0))$. Anderson claimed in [And67] that in the case of a Dirac-$\delta$-perturbation the determinant admits the asymptotics

$$|S^N_L|^2 \sim L^{-\zeta(E)}$$

as $N, L \to \infty$, $N/|B_L(0)| \to \rho(E) > 0$, where $\zeta(E) := \frac{1}{2\pi^2}\delta^2(\sqrt{E})$ and $\delta$ refers to the s-wave scattering phase shift. This algebraic decay of the ground-state overlap is called Anderson’s orthogonality catastrophe in the physics literature and we refer to [GKM14] for further references.

The starting point of the proofs of previous rigorous results is the following expansion of the determinant

$$\ln |S^N_L|^2 = -\sum_{n=1}^{\infty} \frac{1}{n} \text{tr} \left\{ \left(1_{(-\infty,\lambda^L_k)}(H_L)1_{[\mu^L_{k+1},\infty)}(H_{\alpha,L}) \right)^n \right\},$$

valid for appropriate choices of $N$, where $\lambda^L_k$ and $\mu^L_{N+1}$ denote the $n$th and $(N+1)$th eigenvalue of the finite-volume operators $H_L$ and $H_{\alpha,L}$, see [GKMO14]. Thus, estimates on the correlation determinant $S^N_L$ are closely related to asymptotics of products of spectral projections given in (1.3). Considering only the $n=1$ term in (1.3), the first rigorous bounds on $S^N_L$ were proved in [KOS13] valid for 1-dimensional systems and short-range perturbations. They found the upper bound on $|S^N_L|^2 \lesssim L^{-\gamma}$ with the decay exponent $\gamma(E) := \frac{1}{2\pi^2}||T(E)/2||^2_{\text{HS}}$, where $T$ refers to the scattering $T$-matrix of the corresponding infinite-volume operators, and a non-optimal lower bound. Later in [GKM14] the same upper bound $\gamma(E) := \frac{1}{2\pi^2}||T(E)/2||^2_{\text{HS}}$ was deduced for quite general pairs of Schrödinger operators in arbitrary dimension, which differ by a sign-definite potential. Taking all summands in (1.3) into account, [GKMO14] proved an upper bound with the decay exponent $\gamma(E) := \frac{1}{2\pi^2}||\arcsin(T(E)/2)||^2_{\text{HS}}$ in the general setting discussed in [GKM14]. Let us point out that these previous results concern upper bounds and are also valid for special choices of thermodynamic limits only. Here, in the toy model of a Dirac-$\delta$ perturbation we provide the exact asymptotics of the correlation determinant and we consider arbitrary thermodynamic limits approaching a particle density $\rho > 0$, see Theorem 2.1 below. We show this using a representation of the ground-state overlap other than (1.3), which is valid for rank-1-perturbations, i.e.

$$|S^N_L|^2 = \prod_{j=1}^N \prod_{k=N+1}^\infty \frac{|\mu^L_k - \lambda^L_j|}{|\lambda^L_k - \lambda^L_j|} \frac{|\lambda^L_k - \mu^L_j|}{|\mu^L_k - \mu^L_j|},$$

where $\lambda^L_k$ and $\mu^L_j$ are the eigenvalues of the pair of the finite-volume Schrödinger operators, see Section 3. This formula is known in physics literature and goes back at least to [TOS5]. Using the latter formula, we give a straightforward proof of the algebraic decay (1.2) with the exponent $\zeta(E) = \frac{1}{2\pi^2}\delta^2(\sqrt{E})$, as Anderson predicted. It turns out that the decay exponent is equal to the one found in [GKMO14] in the case of a repulsive Dirac-$\delta$ perturbation only, i.e. $\zeta(E) = \frac{1}{2\pi^2}||\arcsin(T(E)/2)||^2_{\text{HS}}$. On the other hand, we obtain $\zeta(E) > \frac{1}{2\pi^2}||\arcsin(T(E)/2)||^2_{\text{HS}}$, for an attractive Dirac-$\delta$ see Remark 2.3 below. Hence, the decay exponent $\frac{1}{2\pi^2}||\arcsin(T(E)/2)||^2_{\text{HS}}$ does not provide the exact asymptotics of (1.4).

Recently, [KOS15] proved the asymptotics of a shifted correlation determinant for one-dimensional models with a perturbation by a magnetic field. A related
problem, which we mention for completeness, is considering the asymptotics of products of spectral projections of infinite-volume operators, similar to \([1,3]\). This was done in the proof of \([GKMO14]\) and extended in \([FP14]\).

2. Model and results

We start with the operator \(- \Delta_0 : C_c^\infty(\mathbb{R}^3 \setminus \{0\}) \rightarrow L^2(\mathbb{R}^3)\), which has deficiency indices \((1,1)\). Therefore, \(- \Delta_0\) gives rise to a one-parameter family of self-adjoint extensions which we index by \(\alpha \in \mathbb{R}\) and denote by \(- \Delta_\alpha\), see \([AGHH05]\), Chapter 1. We refer to \(- \Delta_\alpha\) as the negative Dirichlet Laplacian with a Dirac-\(\delta\) perturbation sitting at the origin \(0\) of strength \(\alpha\). Throughout, we consider for \(\alpha \in \mathbb{R}\) the pair of Schrödinger operators

\[
H := -\Delta \quad \text{and} \quad H_\alpha := -\Delta_\alpha
\]

on the Hilbert space \(\mathcal{H} = L^2(\mathbb{R}^3)\), where \(-\Delta\) is the negative Laplacian. More precisely, following \([AGHH05]\), Chapter 1, the operators \(H\) and \(H_\alpha\) admit a decomposition with respect to angular momentum. Thus, there exists a unitary \(U\) such that both operators transform into the direct sum

\[
U H U^* = \bigoplus_{\ell \in \mathbb{N}_0} \bigoplus_{-\ell \leq m \leq \ell} h_\ell^\alpha \quad \text{and} \quad U H_\alpha U^* = \bigoplus_{\ell \in \mathbb{N}_0} \bigoplus_{-\ell \leq m \leq \ell} h_\ell^\alpha,
\]

where \(h_\ell^\alpha : L^2((0, \infty)) \supset \text{dom}(h_\ell^\alpha) \rightarrow L^2((0, \infty))\) and \(h_\ell^\alpha\) for all \(\ell \geq 1\). In the \(\ell = 0\) case the operators are given by

\[
h^0 = -\frac{d^2}{dx^2}, \quad \text{dom}(h^0) = \{ f \in L^2((0, \infty)) : f, f' \in AC_{\text{loc}}((0, \infty)); f(0+) = 0; f'' \in L^2((0, \infty)) \}\]

\[
h_\alpha^0 = -\frac{d^2}{dx^2}, \quad \text{dom}(h_\alpha^0) = \{ f \in L^2((0, \infty)) : f, f' \in AC_{\text{loc}}((0, \infty)); -4\pi \alpha f(0+) + f'(0+) = 0; f'' \in L^2((0, \infty)) \},
\]

where we denote by \(AC_{\text{loc}}((0, \infty))\) the set of all locally absolutely continuous functions. Thus, the difference of \(H\) and \(H_\alpha\) takes place in the lowest angular momentum channel via a different boundary condition at \(0\) which we parametrise by \(\alpha \in \mathbb{R}\). In the following we are interested in the restrictions of these operators to the ball \(B_L(0)\) of radius \(L\) around the origin

\[
H_L := -\Delta_L \quad \text{and} \quad H_{\alpha,L} := -\Delta_{\alpha,L}.
\]

Here, \(-\Delta_L\) is the negative Dirichlet Laplacian on \(B_L(0)\). The operator \(-\Delta_{\alpha,L}\) corresponds to the restriction of the operator \(-\Delta_\alpha\) imposing Dirichlet boundary condition at \(L\) in each angular momentum channel, i.e. also the restriction of \(-\Delta_\alpha\) to \(B_L(0)\) with Dirichlet boundary conditions. Thus, \(H_L\) and \(H_{\alpha,L}\) differ as well as before in the lowest angular momentum channel only by a different boundary condition at \(0\). We call the corresponding operators in the \(\ell = 0\) channel, i.e. the restrictions of \(h^0\) and \(h_\alpha^0\) to the interval \((0, L)\) with Dirichlet boundary condition at \(L\),

\[
h^0_L \quad \text{and} \quad h_\alpha^0_L.
\]
Using standard results for regular Sturm-Liouville operators, we obtain for all \( z \in \rho(h^0_{\alpha,L}) \cap \rho(h^0_{\alpha,L}) \) a vector \( \eta^0_{L,z} \in L^2(B_L(0)) \) such that the resolvents satisfy
\[
\frac{1}{h^0_L - z} - \frac{1}{h^0_{\alpha,L} - z} = |\eta^0_{L,z}\rangle \langle \eta^0_{L,z}|.
\]
(2.7)

Thus, \( h^0_{\alpha,L} \) is a rank-1-perturbation of \( h^0_L \) in the resolvent, and the same is true for the pair \( H_{\alpha,L} \) and \( H_L \). We point out that the perturbation is not compactly supported since \( \eta^0_{L,z} \) is \( L \) dependent. Moreover, the compactness of the resolvents of \( H_L \) and \( H_{\alpha,L} \) imply that both \( H_L \) and \( H_{\alpha,L} \) have discrete spectra. We write
\[
\lambda^L_1 \leq \lambda^L_2 \leq \cdots \quad \text{and} \quad \mu^L_1 \leq \mu^L_2 \leq \cdots
\]
for their non-decreasing sequences of eigenvalues, counting multiplicities, and \( \langle \varphi_j^L \rangle_{j \in \mathbb{N}} \) and \( \langle \psi_k^L \rangle_{k \in \mathbb{N}} \) for the corresponding sequences of normalised eigenfunctions, where we choose the same eigenvectors for \( H_L \) and \( H_{\alpha,L} \) in any angular momentum channel \( \ell \geq 1 \). This choice ensures that the eigenfunctions of \( H_L \) and \( H_{\alpha,L} \) differ in the lowest angular momentum channel only. Let us point out that in the case of \( \alpha < 0 \) there exists precisely one negative eigenvalue \( \mu_1 = -(4\pi\alpha)^2 \) for the infinite-volume operator \( H_{\alpha} \), respectively \( h^0_{\alpha} \), see \[AGHH05, \text{Chapter 1}\]. Dirichlet-Neumann bracketing implies \( h^0_{\alpha,L} \leq h^0_{\alpha,L} \oplus h^0_{\ell} \), where \( h^0_{\ell} \) denotes the negative Laplacian on \((L, \infty)\) with Dirichlet boundary condition at \( L \). Thus, in the case of \( \alpha < 0 \) we obtain the uniform lower bound on the finite-volume operators
\[
H_{\alpha,L} \geq -(4\pi\alpha)^2 \quad \text{and equivalently} \quad h^0_{\alpha,L} \geq -(4\pi\alpha)^2.
\]
(2.9)

Let \( N \in \mathbb{N} \). In the following we are interested in the correlation determinant
\[
S^N_L := \det(\langle \varphi_j^L, \psi_k^L \rangle)_{1 \leq j, k \leq N}.
\]
(2.10)

The main result concerning \( S^N_L \) is the following.

**Theorem 2.1.** Let \( \alpha \in \mathbb{R} \), \( E > 0 \) and \( N(\cdot)(E) : \mathbb{R}^+ \rightarrow \mathbb{N} \) an arbitrary function subject to
\[
\frac{N_L(E)}{|B_L(0)|} \rightarrow \rho(E) := \frac{E^{3/2}}{8\pi^3},
\]
(2.11)
i.e. \( \rho \) denotes the integrated density of states of the operator \(-\Delta\). Then, the correlation determinant corresponding to the pair \( H_L \) and \( H_{\alpha,L} \) admits the asymptotics
\[
|S^N_L(E)|^2 = L^{-\frac{1}{2}} \delta^2_{\alpha}(\sqrt{E}) + o(1),
\]
(2.12)
as \( L \rightarrow \infty \), equivalently,
\[
\lim_{L \rightarrow \infty} \frac{\ln |S^N_L(E)|}{\ln L} = \frac{1}{2\pi^2} \delta^2_{\alpha}(\sqrt{E}),
\]
(2.13)
and \( \delta_{\alpha} \) is given by Definition 2.2 below.

**Definition 2.2** (Scattering phase shift). Let \( k > 0 \). Then, the scattering phase shift is defined by
\[
\delta_{\alpha}(k) := \begin{cases} \arctan \left( \frac{k}{4\pi\alpha} \right) & \text{for } \alpha \geq 0, \\ \pi - \arctan \left( \frac{k}{4\pi|\alpha|} \right) & \text{for } \alpha \leq 0, \end{cases}
\]
(2.14)
where we use the convention \( \arctan \left( \frac{k}{0} \right) := \frac{\pi}{2} \) for \( k > 0 \).

**Remarks 2.3.**

(i) The separate definitions of the phase shift are reminiscent to the existence of a negative eigenvalue whenever \( \alpha < 0 \) and Levinson’s theorem.

(ii) Due to the nature of a Dirac-\( \delta \) perturbation in 3 dimensions the same result is apparently valid for the corresponding problem on the half-axis.

(iii) We emphasise that we allow arbitrary thermodynamic limits approaching the particle density \( \rho > 0 \).

(iv) The \( o(1) \)-error in (2.12) depends on the particular choice of the thermodynamic limit. To see this, we refer to equations (4.38) and (4.39) in the proof of Theorem 2.1. In particular, we think that the error cannot be improved allowing arbitrary thermodynamic limits.

(v) In [GKMO14, Theorem 2.2] an upper bound on the ground-state overlap is proved for quite general pairs of Schrödinger operators which is valid for subsequences only. More precisely, they prove for a subsequence

\[
\limsup_{L \to \infty} \frac{\ln |S_{N_L}(E)|}{\ln L} \leq -\gamma(E),
\]

(2.15)

where

\[
\gamma(E) := \frac{1}{\pi^2} \| \text{arcsin} |T(E)/2| \|_{HS}^2
\]

and \( T \) denotes the scattering \( T \)-matrix. Since we consider here s-wave scattering, we restrict ourselves to the lowest angular momentum channel. In this case, \( T(E) \) is a complex number and \( |T(E)/2| = \sin(\delta_\alpha(\sqrt{E})) \). Now, computing \( \gamma(E) \) yields

\[
\gamma(E) = \begin{cases} 
\frac{1}{\pi^2} \delta^2_\alpha(\sqrt{E}) & \text{for } |\delta_\alpha(\sqrt{E})| \leq \frac{\pi}{2} \\
\frac{1}{\pi^2} \left( \text{arcsin} \left( \sin(\delta_\alpha(\sqrt{E})) \right) \right)^2 & \text{for } |\delta_\alpha(\sqrt{E})| \geq \frac{\pi}{2}.
\end{cases}
\]

(2.17)

Thus, in general the decay exponent \( \gamma(E) \) does not provide a sharp upper bound on the correlation determinant whenever the phase shift is bigger than \( \pi/2 \). In our model this is equivalent to \( \alpha < 0 \) which we refer to as the attractive case.

The proof of Theorem 2.1 follows from a different approach than the one made in [GKMO14] and [GKMO14], i.e. we do not use the representation (1.3) in this article. Here, the key is the following remarkable product representation of the determinant in terms of the eigenvalues of the finite-volume Schrödinger operators. To our knowledge, this was first stated in [TO85].

**Lemma 2.4.** Let \( N \in \mathbb{N} \). Then,

\[
\left| \det \left( \langle \varphi_j^L, \psi_k^L \rangle \right)_{1 \leq j, k \leq N} \right|^2 = \prod_{j=1}^{N} \prod_{k=N+1}^{\infty} \frac{|\mu_k^L - \lambda_j^L| |\lambda_k^L - \mu_j^L|}{|\lambda_k^L - \lambda_j^L| |\mu_k^L - \mu_j^L|}.
\]

(2.18)

We start with proving this product representation for general pairs of compact operators which differ by a rank-1-perturbation in Section 3. We apply this to our setting in Section 4 and prove Theorem 2.1.
3. Representation of the ground-state overlap

In this section we prove a quite general representation for determinants of eigenvectors of pairs of operators which differ by a rank-1-perturbation. The main result in this section, Theorem 3.1, will be the key to the proof of Theorem 2.1.

Let $\mathcal{H}$ be a separable infinite-dimensional Hilbert space and $A : \mathcal{H} \to \mathcal{H}$ be a compact, linear and self-adjoint operator. Moreover, we assume $A \geq 0$ with $\ker(A) = \{0\}$. We define

$$B := A + |\phi\rangle\langle \phi|$$

for some $0 \neq \phi \in \mathcal{H}$. We write $\alpha_1 \geq \alpha_2 \geq \cdots$ and $\beta_1 \geq \beta_2 \geq \cdots$ for the non-increasing sequences of eigenvalues of $A$, respectively $B$ and denote by $(\varphi_j)_{j \in \mathbb{N}}$ and $(\psi_k)_{k \in \mathbb{N}}$ the corresponding normalised eigenvectors. Since $A$ and $B$ differ by a rank-1-perturbation, the min-max theorem implies that the eigenvalues interlace. We assume the following condition on the eigenvalues

$$\sum_{n=1}^{\infty} |\beta_n - \alpha_n| < \infty. \quad (3.2)$$

Moreover, for simplicity we also assume the following strict interlacing condition

$$\beta_1 > \alpha_1 > \beta_2 > \alpha_2 > \cdots \quad (3.3)$$

In particular, $\beta_k \neq \alpha_j$ for all $j, k \in \mathbb{N}$. Furthermore, the above implies cyclicity of $\phi$. Assumption (3.3) is not necessary but simplifies notation and computations. In the general case one has to consider the restriction to the cyclic subspace generated by the perturbation $\phi$. But the application in mind will satisfy the interlacing condition (3.3), therefore, we assume it.

**Theorem 3.1.** Let $N \in \mathbb{N}$. We assume conditions (3.2) and (3.3) to hold. Then,

$$|\det(\langle \varphi_j, \psi_k \rangle)_{1 \leq j, k \leq N}|^2 = \prod_{j=1}^{N} \prod_{k=1}^{\infty} \frac{|\beta_k - \alpha_j| |\alpha_k - \beta_j|}{|\alpha_k - \alpha_j| |\beta_k - \beta_j|}. \quad (3.4)$$

**Proof of Theorem 3.1.** We use the eigenvalue equations and assumption (3.3) to obtain for all $j, k \in \mathbb{N}$

$$\langle \varphi_j, \psi_k \rangle = \frac{\langle \varphi_j, \phi \rangle \langle \phi, \psi_k \rangle}{\beta_k - \alpha_j}. \quad (3.5)$$

Hence, the multi-linearity of the determinant implies

$$|\det(\langle \varphi_j, \psi_k \rangle)_{1 \leq j, k \leq N}|^2 = \left| \det \left( \frac{\langle \varphi_j, \phi \rangle \langle \phi, \psi_k \rangle}{\beta_k - \alpha_j} \right)_{1 \leq j, k \leq N} \right|^2 = \left( \prod_{j=1}^{N} \prod_{k=1}^{N} |\langle \varphi_j, \phi \rangle \langle \phi, \psi_k \rangle|^2 \right) \left| \det \left( \frac{1}{\beta_k - \alpha_j} \right)_{1 \leq j, k \leq N} \right|^2. \quad (3.6)$$
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Now, the remaining determinant can be computed explicitly. We use the Cauchy determinant formula to evaluate this, see e.g. [Wey13 Lem. 7.6.A], and end up with

\[\det_{j,k=1}^{N} \left|\langle \varphi_{j}, \phi \rangle \langle \phi, \psi_{k} \rangle \right|^{2} \prod_{j,k=1,j\neq k}^{N} \frac{\beta_{k} - \beta_{j}}{\alpha_{k} - \alpha_{j}} \left|\beta_{j} - \beta_{k}\right| \left|\alpha_{j} - \alpha_{k}\right|, \tag{3.7}\]

Corollary 3.3 below yields

\[\det_{j=1}^{N} \prod_{l=1}^{\infty} \left|\beta_{l} - \beta_{j}\right| \prod_{l=1}^{N} \prod_{l\neq j}^{\infty} \left|\beta_{l} - \alpha_{j}\right| \prod_{j,k=1}^{N} \frac{\beta_{k} - \beta_{j}}{\beta_{k} - \alpha_{j}} \left|\alpha_{j} - \alpha_{k}\right|. \tag{3.8}\]

This gives the claim, where we remark that by assumption (3.2) all products in the latter converge absolutely. \(\square\)

To complete the proof, we continue with computing the resolvents of the operators \(A\) and \(B\) in terms of their eigenvalues.

**Lemma 3.2.** We assume (3.2) and (3.3). Then, there exist \(a, b \in \mathbb{R}\) with \(ab = -1\) such that

(i) for all \(z \in \varrho(A)\)

\[\langle \phi, \frac{1}{A - z} \phi \rangle + 1 = a \prod_{k=1}^{\infty} \frac{\beta_{k} - z}{\alpha_{k} - z}, \tag{3.9}\]

(ii) for all \(z \in \varrho(B)\)

\[\langle \phi, \frac{1}{B - z} \phi \rangle - 1 = b \prod_{n=1}^{\infty} \frac{\alpha_{n} - z}{\beta_{n} - z}. \tag{3.10}\]

**Corollary 3.3.** Let \(j, k \in \mathbb{N}\). Under the assumption (3.2) and (3.3)

\[\left|\langle \varphi_{j}, \phi \rangle \langle \phi, \psi_{k} \rangle \right|^{2} = |\beta_{j} - \alpha_{j}| |\alpha_{k} - \beta_{k}| \left(\prod_{l=1}^{\infty} \frac{|\beta_{l} - \alpha_{j}|}{|\alpha_{l} - \alpha_{j}|}\right) \left(\prod_{l \neq k}^{\infty} \frac{|\alpha_{l} - \beta_{k}|}{|\beta_{l} - \beta_{k}|}\right). \tag{3.11}\]

**Proof of Corollary 3.3** Using Lemma 3.2 we compute the residue of the resolvents

\[\left|\langle \varphi_{j}, \phi \rangle \right|^{2} = \lim_{z \to \alpha_{j}} \left(\langle \phi, \frac{1}{A - z} \phi \rangle \right) \]

\[= \lim_{z \to \alpha_{j}} \left(\alpha_{j} z - \alpha_{j}\right) a \prod_{l=1}^{\infty} \frac{\beta_{l} - z}{\alpha_{l} - z} = a (\beta_{j} - \alpha_{j}) \prod_{l=1}^{\infty} \frac{\beta_{l} - \alpha_{j}}{\alpha_{l} - \alpha_{j}} \tag{3.12}\]

and along the same line

\[\left|\langle \psi_{k}, \phi \rangle \right|^{2} = b (\alpha_{k} - \beta_{k}) \prod_{l \neq k}^{\infty} \frac{\alpha_{l} - \beta_{k}}{\beta_{l} - \beta_{k}} \tag{3.13}\]
Taking the absolute value and using $|ab| = 1$, we get the result. □

**Proof of Lemma 3.2** First note that by assumption 3.2, the sequences

$$
\left( \prod_{k=1}^{N} \frac{\beta_k - z}{\alpha_k - z} \right)_{N \in \mathbb{N}} \quad \text{and} \quad \left( \prod_{n=1}^{N} \frac{\alpha_n - z}{\beta_n - z} \right)_{N \in \mathbb{N}}
$$

(3.14)

converge locally uniformly for all $z \in \varrho(A) \cap \varrho(B)$, see [Kno96, Thm. 252]. Therefore, the limits

$$F(z) := \prod_{n=1}^{\infty} \frac{\alpha_n - z}{\beta_n - z} \quad \text{and} \quad G(z) := \prod_{k=1}^{\infty} \frac{\beta_k - z}{\alpha_k - z}
$$

(3.15)

are well-defined analytic functions on $\varrho(A) \cap \varrho(B)$, which fulfil $FG = 1$. Due to the locally uniform convergence, the derivative of $F$ satisfies

$$F'(z) = \lim_{N \to \infty} \sum_{l=1}^{N} \prod_{n \neq l}^{N} \frac{\alpha_n - z}{\beta_n - z} \frac{\alpha_l - z}{\beta_l - z} = \lim_{N \to \infty} \sum_{l=1}^{N} \prod_{n \neq l}^{N} \frac{\alpha_n - z}{\beta_n - z} \left( \frac{\alpha_l - z}{\beta_l - z} \right)^2 F(z) = \lim_{N \to \infty} \sum_{l=1}^{N} \left( \frac{1}{\beta_l - z} - \frac{1}{\alpha_l - z} \right)
$$

(3.16)

for all $z \in \varrho(A) \cap \varrho(B)$. We apply Lemma 3.4 below and obtain

$$F'(z) = - F(z) \left\langle \frac{1}{A - z}, \frac{1}{B - z} \phi \right\rangle.
$$

(3.17)

Now, the resolvent identity implies for all $z \in \varrho(A) \cap \varrho(B)$

$$\frac{1}{B - z} - \frac{1}{A - z} = - \frac{1}{A - z} \phi \left( \frac{1}{B - z} \phi, \cdot \right)
$$

(3.18)

which provides the equality

$$\frac{1}{A - z} \phi = \frac{1}{1 - \left( \frac{1}{B - z} \phi, \phi \right)} \frac{1}{B - z} \phi.
$$

(3.19)

Inserting this into (3.17), we see that $F$ solves the differential equation

$$F'(E) = F(E) \frac{1}{\left\langle \phi, \frac{1}{B - E} \phi \right\rangle} - 1 \left( \frac{1}{B - E} \right)^2 \phi
$$

(3.20)

at least for all $E \in \varrho(A) \cap \varrho(B) \cap \mathbb{R}$. On the other hand the resolvent of $B$ is analytic in $\varrho(B)$ and the function $t \mapsto \left\langle \phi, \frac{1}{B - t} \phi \right\rangle - 1$, $t < 0$, solves the above ODE (3.20) as well. Now, the general solution to this ODE is $f(t) = x_0 \exp \left( \int_{t_0}^{t} ds \left\langle \phi, \frac{1}{(B - s) \phi} - 1 \right\rangle \left( \frac{1}{B - s} \right)^2 \phi \right)$, for some initial condition $(t_0, x_0)$. Note that the functions $t \mapsto F(t)$ and $t \mapsto \left\langle \phi, \frac{1}{B - t} \phi \right\rangle - 1$ are non-zero, thus $\left\langle \phi, \frac{1}{B - t} \phi \right\rangle - 1 = c F(t)$ for some $c \neq 0$. This and the identity theorem for analytic functions give the claim. Equation (3.20) follows from $F(z)G(z) = 1$ and the identity

$$\left( \left\langle \phi, \frac{1}{B - z} \phi \right\rangle - 1 \right) \left( \left\langle \phi, \frac{1}{A - z} \phi \right\rangle + 1 \right) = -1,
$$

(3.21)

for all $z \in \varrho(A) \cap \varrho(B)$ which is a consequence of (3.18). □
Lemma 3.4. Let \( z \in \varrho(A) \cap \varrho(B) \). Assume (3.3). Then, we obtain the following identity
\[
\lim_{N \to \infty} \sum_{l=1}^{N} \left( \frac{1}{\beta_l - z} - \frac{1}{\alpha_l - z} \right) = -\langle \frac{1}{A - z} \phi, \frac{1}{B - z} \phi \rangle.
\] (3.22)

Let us point out that in the finite-dimensional case the above equality follows directly from the resolvent equation, (3.18). Nevertheless, the infinite-dimensional case is a bit more involved due to convergence issues.

Proof. For \( \lambda \in \mathbb{R} \) we define the operator
\[
A(\lambda) := A + \lambda \langle \phi \rangle \langle \phi \rangle
\] (3.23)
and write \( \alpha_l(\lambda) \) for the \( l \)th eigenvalue counted from above and \( \psi_l(\lambda) \) for the corresponding eigenvector. Moreover, we remark that \( \alpha_l(1) \) and \( \varphi_l(1) \) correspond to \( \beta_l \) and \( \psi_l \). Assumption (3.3) and the definite sign of the perturbation imply that the eigenvalues of \( A(\lambda) \) are non-degenerate for all \( \lambda \in [0,1] \). Thus, standard results, see [RS78, Chap. XII], give differentiability of the eigenvalues for all \( \lambda \in (0,1) \) and we apply the Feynman-Hellmann theorem, see e.g. [IZ88], to deduce for all \( l \in \mathbb{N} \) and \( \lambda \in (0,1) \)
\[
\alpha_l'(\lambda) = |\langle \varphi_l(\lambda), \phi \rangle|^2.
\] (3.24)

Hence, we compute using the latter
\[
\lim_{N \to \infty} \sum_{l=1}^{N} \left( \frac{1}{\beta_l - z} - \frac{1}{\alpha_l - z} \right) = -\lim_{N \to \infty} \sum_{l=1}^{N} \int_{0}^{1} \, d\lambda \left( \frac{1}{\alpha_l(\lambda) - z} \right)^2 \alpha_l'(\lambda)
\]
\[
= -\lim_{N \to \infty} \sum_{l=1}^{N} \int_{0}^{1} \, d\lambda \left( \frac{1}{\alpha_l(\lambda) - z} \right)^2 |\langle \varphi_l(\lambda), \phi \rangle|^2.
\] (3.25)

The eigenvalue equation implies
\[
\langle \varphi_l(\lambda), \phi \rangle = \langle \varphi(1), \varphi_l(\lambda) \rangle \langle \varphi_l(\lambda), \phi \rangle,
\]
\[
= -\int_{0}^{1} \, d\lambda \langle \phi, \left( \frac{1}{A(\lambda) - z} \right)^2 \phi \rangle,
\] (3.26)
where we used Fubini’s theorem to interchange the integral with the sum and the fact that the vectors \( \{ \varphi_l(\lambda) \}_{l \in \mathbb{N}} \) form an ONB. The resolvent identity (3.18) implies
\[
\frac{1}{A(\lambda) - z} \phi = \frac{1}{1 + \lambda \langle \phi, \frac{1}{A - z} \phi \rangle} \frac{1}{A - z} \phi.
\] (3.27)
Therefore, we continue

\[
\text{(3.26)} = - \int_0^1 d\lambda \langle \phi, \left(\frac{1}{A-z}\right)^2 \phi \rangle \left(1 + \frac{\lambda}{A-z} \right)^2
\]

\[
= - \frac{\langle \phi, \left(\frac{1}{A-z}\right)^2 \phi \rangle}{\langle \phi, \frac{1}{A-z} \phi \rangle} \int_0^1 d\lambda \left(1 + \frac{\lambda}{A-z} \phi \right)
\]

\[
= \frac{\langle \phi, \left(\frac{1}{A-z}\right)^2 \phi \rangle}{\langle \phi, \frac{1}{A-z} \phi \rangle} \left(1 - \left(1 + \frac{1}{A-z} \phi \right)\right) = - \frac{\langle \phi, \left(\frac{1}{A-z} \right)^2 \phi \rangle}{1 + \langle \phi, \frac{1}{A-z} \phi \rangle}. \tag{3.28}
\]

Equation \text{(3.27)} with \(\lambda = 1\) provides the assertion

\[
\text{(3.28)} = - \left(\frac{1}{A-z}, \frac{1}{B-z} \phi \right). \tag{3.29}
\]

\[\square\]

4. Proof of Theorem 2.1

We decompose the determinant according to the angular momentum decomposition \text{(4.2)}. This implies

\[
\left| \det \left( \langle \varphi^L_j, \psi^L_k \rangle \right) \right|_{1 \leq j, k \leq N_L(E)}^2 = \prod_{l \in \mathbb{N}_0} \left| \det \left( \langle \varphi^L_j (\ell), \psi^L_k (\ell) \rangle \right) \right|_{1 \leq j, k \leq N^L_{\ell}(E)}^{2(2\ell+1)},
\]

where \(\varphi^L_j(\ell)\) and \(\psi^L_k(\ell)\) correspond to the radial part of the eigenfunctions lying in the \(\ell\)-th angular momentum channel and \(N^L_{\ell}(E)\) to the relative particle number in the \(\ell\)-th angular momentum channel. More precisely,

\[
N^L_{\ell}(E) := \# \left\{ k \in \mathbb{N} : \exists j \in \{1, \cdots, N_L\} \text{ with } \lambda^L_k(\ell) = \lambda^L \right\} \quad \tag{4.2}
\]

where \(\lambda^L_k(\ell)\) denote the eigenvalues of \(h^L_k\). Since we chose the eigenfunctions of \(H_L\) and \(H_{\alpha,L}\) to be the same in every angular momentum channel \(\ell \geq 1\) we obtain that only the \(\ell = 0\) term in the product \text{(4.1)} is different from 1. Hence,

\[
\left| \det \left( \langle \varphi^L_j, \psi^L_k \rangle \right) \right|_{1 \leq j, k \leq N_L(E)}^2 = \left| \det \left( \langle \varphi^L_j (0), \psi^L_k (0) \rangle \right) \right|_{1 \leq j, k \leq N^L_0(E)}^2. \tag{4.3}
\]

Thus, we reduced our problem to a problem on the half-axis, where the relative particle number satisfies

\textbf{Lemma 4.1.} Given \(E > 0\). Let \(L\) and \(N_L(E) \in \mathbb{N}\) such that \(\frac{N_L(E)}{|B_L(0)|} \to \rho(E)\) as \(L \to \infty\). Then,

\[
\frac{N^0_{\ell}(E)}{L} \to \frac{\sqrt{E}}{\pi} =: \rho(E), \quad \tag{4.4}
\]

as \(L \to \infty\).

\textbf{Proof.} For any \(E > 0\)

\[
\lim_{L \to \infty} \frac{\# \{ k : \lambda^L_k \leq E \}}{|B_L(0)|} = \rho(E) = \lim_{L \to \infty} \frac{N_L(E)}{|B_L(0)|}, \tag{4.5}
\]
where the first equality follows from e.g. [RS78]. Hence, we obtain for an arbitrary \( \epsilon > 0 \) the inequalities
\[
\# \{ k : \lambda^L_k \leq E - \epsilon \} \leq N_L(E) \leq \# \{ k : \lambda^L_k \leq E + \epsilon \}
\]
for \( L \) large enough. Since \( \rho \) is strictly increasing, we obtain \( \lambda^L_{N_L(E)} \to E \). Therefore, \( \lambda^L_{N_L(E)} \to E \) as well because otherwise there would be a gap in the spectrum of \( h^0 \) by the definition of the relative particle number \( N_L^0(E) \). This implies for an arbitrary \( \epsilon > 0 \) and \( L \) large enough
\[
\left| \frac{N_L^0(E)}{L} - \frac{\# \{ k : \lambda^L_k(0) \leq E \}}{L} \right| \leq \frac{c}{\sqrt{E}} \epsilon,
\]
for some constant \( c \). Since \( \# \{ k : \lambda^L_k(0) \leq E \} / L \to \rho_0(E) \), as \( L \to \infty \), this yields the claim.

Given (4.1) and Lemma 4.1, Theorem 2.1 will follow from

**Theorem 4.2.** Let \( E > 0 \). Then,
\[
\left| \det \left( \frac{\langle \varphi^L_j(0), \psi^L_k(0) \rangle}{1 \leq j, k \leq N_L} \right) \right|^2 = L^{-\zeta(E)+o(1)}
\]
as \( L \to \infty \), \( N_L \in \mathbb{N} \) and \( N_L / L \to \frac{\sqrt{E}}{\pi} \), where
\[
\zeta(E) := \frac{1}{\pi^2} \delta^2_\alpha(\sqrt{E})
\]
and \( \delta_\alpha \) is given by Definition 2.2.

From now we shorten the notation and drop the 0 and \( L \)-index of the eigenfunctions and eigenvalues.

Apart from the product representation discussed in Section 3 the main ingredient to the proof of Theorem 4.2 is a elementary formula expressing the non-negative eigenvalues of the perturbed operator \( h^0_{\alpha,L} \) in terms of the eigenvalues of the operator \( h^0_L \) plus corrections depending on the scattering phase shift \( \delta_\alpha \). First, note that the eigenvalues of \( h^0_L \) can be computed explicitly, see [RS78], i.e. for \( n \in \mathbb{N} \)
\[
\lambda_n = \left( \frac{n\pi}{L} \right)^2.
\]

**Lemma 4.3.** Let \( \delta_\alpha \) be given by Definition 2.2. Then,

(i) for \( \alpha \geq 0 \) and \( n \in \mathbb{N} \) the \( n \)th eigenvalues of \( h^0_L \) and \( h^0_{\alpha,L} \) satisfy
\[
0 \leq \sqrt{\mu_n} = \sqrt{\lambda_n} - \frac{\delta_\alpha(\sqrt{\mu_n})}{L},
\]

(ii) for \( \alpha \leq 0 \) and \( n > 1 \) the \( n \)th eigenvalues of \( h^0_L \) and \( h^0_{\alpha,L} \) satisfy
\[
0 \leq \sqrt{\mu_n} = \sqrt{\lambda_n} - \frac{\delta_\alpha(\sqrt{\mu_n})}{L},
\]
(iii) and \( \delta \) exhibits the following expansion
\[
\delta_\alpha(\sqrt{\mu_n}) = \delta_\alpha(\sqrt{\lambda_n}) - \delta_\alpha(\sqrt{\lambda_n}) \frac{\delta_\alpha(\sqrt{\lambda_n})}{L} + o\left(\frac{1}{L}\right),
\]
which is valid for all \( \mu_n \geq 0 \), and the error term depends on \( \alpha \) but is independent of \( n \).

**Proof.** Let \( k > 0 \). Consider the eigenvalue problem
\[
-\mu''_k = k^2 \mu_k, \quad -4\pi \alpha \mu_k(0+) + \mu_k'(0+) = 0.
\]
(4.14)
Introducing Prüfer variables
\[
\mu_k(x) = \rho_k u_k(x) \sin(\theta_k(x)), \quad \mu_k'(x) = k \rho_k u_k(x) \cos(\theta_k(x)),
\]
(4.15)
we see that any non-zero solution of (4.14) is of the form
\[
\mu_k(x) := a \sin\left(kx + \arctan\left(\frac{k}{4\pi\alpha}\right)\right),
\]
(4.16)
for some \( 0 \neq a \in \mathbb{C} \). Since any eigenfunction \( \mu_k \) to an eigenvalue \( k^2 \) of \( h^0_{a,L} \) is a solution of (4.14) in \((0, L)\) and additionally satisfies \( \mu_k(L^-) = 0 \), we obtain that
\[
\mu_k(L) = a \sin\left(kL + \arctan\left(\frac{k}{4\pi\alpha}\right)\right) = 0.
\]
(4.17)
On the other hand, all \( k^2 \) such that (4.17) is satisfied are eigenvalues of \( h^0_{a,L} \). Since the function \( k \mapsto kL + \arctan\left(\frac{k}{4\pi\alpha}\right) \) is strictly increasing we obtain for any \( n \in \mathbb{N} \) an unique eigenvalue \( \mu_n \geq 0 \) of \( h^0_{a,L} \) such that
\[
\sqrt{\mu_n} L + \arctan\left(\frac{\sqrt{\mu_n}}{4\pi\alpha}\right) = n\pi,
\]
(4.18)
where \( \mu_1 < \mu_2 < \cdots \). This proves (i). For the case \( \alpha < 0 \) note that \( h^0_{a,L} \) admits a single negative eigenvalue. Therefore, (4.18) is only valid starting from the second eigenvalue of \( h^0_{a,L} \). This implies for all \( n \in \mathbb{N} \)
\[
\sqrt{\mu_{n+1}} L + \arctan\left(\frac{\sqrt{\mu_{n+1}}}{4\pi\alpha}\right) = \sqrt{\lambda_{n+1}} - \pi - \arctan\left(\frac{\sqrt{\lambda_{n+1}}}{4\pi|\alpha|}\right).
\]
(4.19)
(iii) follows directly from (i), (ii) and Definition (2.2) from the phase shift. \( \square \)

**Corollary 4.4.** The eigenvalues of \( h^0_{L} \) and \( h^0_{a,L} \) satisfy
\[
\mu_1 < \lambda_1 < \mu_2 < \lambda_2 < \cdots.
\]
(4.20)
**Proof.** Note that \( |\delta_\alpha(k)| < \pi \) for all \( k > 0 \). Thus, (4.10) and (4.12) imply the corollary. \( \square \)

Next we apply the results from Section 3 to the determinant:

**Lemma 4.5.** Let \( N \in \mathbb{N} \). Then,
\[
|\det\left(\langle \varphi_j, \psi_k \rangle\right)_{1 \leq j, k \leq N}|^2 = \prod_{j=1}^N \prod_{k=N+1}^\infty \frac{|\mu_k - \lambda_j| |\lambda_k - \mu_j|}{|\lambda_k - \lambda_j| |\mu_k - \mu_j|}.
\]
(4.21)
Proof. First note that $h_{\alpha,L}^0$ is bounded from below by (2.7). This and $h_{L}^0 \geq 0$ imply $-E \in \rho(h_L) \cap \rho(h_{\alpha,L}^0)$ for some $E > 0$. Moreover, (2.7) provides
\[
\frac{1}{h_{L}^0 + E} - \frac{1}{h_{\alpha,L}^0 + E} = |\eta_{L,\alpha}^E \rangle \langle \eta_{L,\alpha}^E |,
\]
for some $\eta_{L}^E \in L^2((0,L))$ and Corollary 1.4 gives
\[
\frac{1}{\mu_1 + E} > \frac{1}{\lambda_1 + E} > \frac{1}{\mu_2 + E} > \frac{1}{\lambda_2 + E} > \cdots,
\]
the eigenvalues satisfy assumption (3.2). Furthermore, the operators $\frac{1}{h_{L}^0 + E}$ and $\frac{1}{h_{\alpha,L}^0 + E}$ are non-negative with trivial kernel and compact. Therefore, we are in position to apply Theorem 3.1 and obtain
\[
\left| \det \left( \langle \varphi_j, \psi_k \rangle \right) \right|_{1 \leq j, k \leq N}^2 = \prod_{j=1}^{N} \prod_{k=N+1}^{\infty} \left| \frac{\mu_k + E}{\lambda_k + E} - \frac{1}{\lambda_k + E} \right| \left| \frac{1}{\mu_k + E} - \frac{1}{\mu_j + E} \right|
\]
\[
= \prod_{j=1}^{N} \prod_{k=N+1}^{\infty} \frac{\left| \mu_k - \lambda_j \right| \left| \lambda_k - \mu_j \right|}{\left| \lambda_k - \lambda_j \right| \left| \mu_k - \mu_j \right|}.
\]

Proof of Theorem 4.2. We start with the product representation given in Lemma 4.4. Note that for $\alpha < 0$ there is an ambiguity since there exists precisely one negative eigenvalue $\mu_1$. Therefore, we treat the $j = 1$ term in the product separately. We define
\[
A_N^\alpha := \prod_{k=N+1}^{\infty} \frac{\left| \mu_k - \lambda_1 \right| \left| \lambda_k - \mu_1 \right|}{\left| \lambda_k - \lambda_1 \right| \left| \mu_k - \mu_1 \right|} = \prod_{k=N+1}^{\infty} \frac{1 + \frac{(\mu_k - \lambda_k)(\lambda_1 - \mu_1)}{(\lambda_k - \lambda_1)(\mu_k - \mu_1)}}{1 + \frac{(\mu_1 - \lambda_1)(\lambda_k - \mu_1)}{(\lambda_k - \lambda_1)(\mu_1 - \mu_1)}}
\]
and estimate using Corollary 4.4
\[
\sum_{k=N+1}^{\infty} \left| \frac{(\mu_k - \lambda_k)(\lambda_1 - \mu_1)}{(\lambda_k - \lambda_1)(\mu_k - \mu_1)} \right| \leq |\lambda_1 - \mu_1| \sum_{k=N+1}^{\infty} \frac{\left( \frac{k\pi}{L} \right)^2 - \left( \frac{(k-1)\pi}{L} \right)^2}{\left( \frac{(k-1)\pi}{L} \right)^2 - \left( \frac{\pi}{L} \right)^2}
\]
\[
\leq \frac{L^2}{\pi^2} |\lambda_1 - \mu_1| \sum_{k=N+1}^{\infty} \frac{(2k-1)}{(2k-1)(2k-2k)}
\]
\[
\leq c \left( \frac{L}{N} \right)^2 |\lambda_1 - \mu_1|.
\]
Since $h_L^0$ is uniformly bounded from below with respect to $L$, see Lemma 2.9
\[
\ln A_N^\alpha = \ln \left( \prod_{k=N+1}^{\infty} \left| \frac{\mu_k - \lambda_1}{\lambda_k - \lambda_1} \right| \left| \frac{\lambda_k - \mu_1}{\mu_k - \mu_1} \right| \right) = O(1)
\]
as $N, L \to \infty$ and $N/L \to \rho(E) > 0$. Therefore, we are left with a product consisting of the non-negative eigenvalues and apply Lemma 4.8 use Lemma 2.6 (i) and $\sqrt{\lambda_n} = \ldots$
\[
\frac{n}{L}, \ n \in \mathbb{N}, \text{ to obtain}
\]
\[
\ln \left| \det \left( \langle \varphi_j, \psi_k \rangle \right)_{1 \leq j, k \leq N} \right|^2 = \ln A_N^\mu
\]
\[
\quad + \sum_{j=2}^{N} \sum_{k=N+1}^{\infty} \ln \left( \frac{(k\pi - \delta(\sqrt{\mu_k}))^2 - (j\pi)^2}{(k\pi)^2 - (j\pi)^2} \right) \ln \left( \frac{(k\pi - \delta(\sqrt{\mu_j}))^2 - (j\pi)^2}{(k\pi - \delta(\sqrt{\mu_j}))^2} \right).
\]
(4.28)

In the following the O(1) and o(1) terms refer to the asymptotics \( L, N \to \infty, N/L \to \rho_0(E) > 0 \). Equation (4.27) above, Lemma A.1 below and the abbreviation \( g_k := -\frac{1}{\pi} \delta(\sqrt{\mu_k}) \) for \( k \in \mathbb{N} \) yield
\[
(4.29)
\]
\[
\quad = - \sum_{j=2}^{N} \sum_{k=N+1}^{\infty} \frac{(2jg_j + g_j^2)}{(k + g_k)^2 - (j + g_j)^2} + O(1).
\]

Using Lemma A.2 and the abbreviation \( \delta_k := -\frac{1}{\pi} \delta(\sqrt{\lambda_k}) \) for \( k \in \mathbb{N} \), we have
\[
(4.30)
\]
\[
\quad = - \sum_{j=2}^{N} \sum_{k=N+1}^{\infty} \frac{(2j\delta_j + \delta_j^2)}{(k + \delta_k)^2 - (j + \delta_j)^2} + O(1).
\]

Lemma A.3 implies
\[
(4.31)
\]
\[
\quad = - \sum_{j=2}^{N} \sum_{k=N+1}^{\infty} \frac{4jk\delta_j \delta_k}{(k^2 - j^2)^2} + O(1).
\]

Lemma A.4 yields
\[
(4.32)
\]
\[
\quad = \frac{1}{\pi^2} \int_{0}^{\frac{\pi}{L}} \int_{0}^{\frac{\pi}{L}} dy \ x y \frac{4xy\delta(x\pi)\delta(y\pi)}{(y^2 - x^2)^2} + O(1).
\]

We define for \( 0 \leq x < y \)
\[
\quad = \frac{4xy\delta(x\pi)\delta(y\pi)}{(y + x)^2}
\]
(4.33)

The explicit representation of \( \delta_\alpha \) implies for all \( \epsilon > 0 \)
\[
\sup_{b > \epsilon} \sup_{(x, y) \in (0, b) \times (b, \infty)} \| \nabla g(x, y) \|_2 := c(\epsilon) < \infty.
\]
(4.34)

Therefore, using the mean value theorem and the Cauchy-Schwarz inequality, we compute for a \( 0 < \epsilon < \sqrt{E} \) and \( N, L \) big enough
\[
\quad \leq c(\epsilon) \int_{0}^{\frac{\pi}{L}} \int_{\frac{N}{L} + \frac{\epsilon}{L}}^{\frac{2N}{L}} dy \ \frac{1}{(y - x)^2} \| \nabla/N - x, y - N/L \|_2 \frac{1}{(y - x)^2}
\]
\[
\quad \leq 2c(\epsilon) \int_{0}^{\frac{\pi}{L}} \int_{\frac{N}{L} + \frac{\epsilon}{L}}^{\frac{2N}{L}} dy \ \frac{1}{(y - x)} = O(1),
\]
(4.35)

where we used the inequality
\[
\frac{|x - N/L| + |y - N/L|}{(y - x)^2} \leq 2 \frac{1}{(y - x)}.
\]
(4.36)
which is valid for all \( x < N/L < y \). Moreover, since \( \frac{N}{L} \to \sqrt{E} \pi > 0 \), we compute
\[
\int_0^{\frac{N}{L}} dx \int_{\frac{N}{L}x}^{\frac{N}{L}x+\frac{1}{N}} dy \frac{1}{(y-x)^2} = \ln L + O(1).
\]  
Hence, combining equation (4.35) and (4.37), we end up with
\[
(4.32) = -\ln L \frac{1}{\pi} \delta_\alpha^2 (\pi N/L) + O(1)
\]
\[
= -\ln L \frac{1}{\pi} \delta_\alpha^2 (\sqrt{E}) + o(\ln L),
\]
where the last line follows from \( \pi \frac{N}{L} \to \sqrt{E} \). This gives the assertion. \( \square \)

Appendix A. Proof of the auxiliary lemmata

In this section we prove the missing lemmata used in the proof of Theorem 4.2. We do not claim to give optimal or very elegant estimates. Throughout this section we drop the index \( \alpha \) in the scattering phase shift and restrict ourselves to the case \( \alpha < 0 \). This implies the following estimate on the phase shift
\[
\delta(x) - \delta(y) \geq 0,
\]  
for \( x < y \), which we use in the sequel. The case \( \alpha \geq 0 \) is even simpler since in that case the Definition (2.2) of the phase shift implies the uniform bound
\[
\|\delta\|_{\infty} \leq \frac{\pi}{2},
\]  
which simplifies some of the following estimates. Moreover, we use the elementary asymptotics
\[
\sum_{j=1}^{N} \sum_{k=N+1}^{\infty} \frac{1}{(k-j)^\beta} = O(\ln N), \quad \beta > 2
\]  
\[
\sum_{j=1}^{N} \sum_{k=N+1}^{\infty} \frac{1}{(k-j)^\beta} = O(1)
\]  
as \( N \to \infty \), where \( \beta > 2 \).

Lemma A.1. Set \( g_k := -\frac{1}{\pi} \delta(\sqrt{\mu_k}) \) for \( k \in \mathbb{N} \). Then,
\[
\sum_{j=2}^{N} \sum_{k=N+1}^{\infty} \ln \left( \frac{(k+g_k)^2 - j^2}{(k+g_k)^2 - (j+g_j)^2} \right)
\]  
\[
= -\sum_{j=2}^{N} \sum_{k=N+1}^{\infty} \frac{(2jg_j + g_j^2)(2kg_k + g_k^2)}{(k+g_k)^2 - (j+g_j)^2 (k^2 - j^2)} + O(1)
\]  
as \( N, L \to \infty \), \( \frac{N}{L} \to \sqrt{E} \).
We estimate using $x_N$ as $N \to \infty$ along the same line using (A.7)

Thus, for $j = N$ and $k = N + 1$

\[
\lim_{N,L \to \infty \atop N/L \to \sqrt{E/\pi}} g_N = \lim_{N,L \to \infty \atop N/L \to \sqrt{E/\pi}} g_{N+1} = -\frac{\delta(\sqrt{E})}{\pi} > -1. \tag{A.7}
\]

We prove the assertion in two steps. First we consider the summand. Note that Lemma 4.1 above and $E > 0$ imply

\[
\lim_{N,L \to \infty \atop N/L \to \sqrt{E/\pi}} \ln \left( \frac{(N + 1 + g_{N+1})^2 - N^2}{(N + 1 + g_N)^2 - (N + g_N)^2} \right) = \lim_{N,L \to \infty \atop N/L \to \sqrt{E/\pi}} \ln \left( \frac{1 + g_{N+1}(1 - g_N)}{1 + g_{N+1} - g_N} \right) = \ln \left( 1 - \frac{\delta^2(\sqrt{E})}{\pi^2} \right). \tag{A.8}
\]

Moreover, along the same line using (A.7)

\[
\lim_{N,L \to \infty \atop N/L \to \sqrt{E/\pi}} \frac{(2N g_N + g_N^2)(2(N + 1)g_{N+1} + g_{N+1}^2)}{(N + 1 + g_{N+1})^2 - (N + g_N)^2((N + 1)^2 - N^2)} = -\frac{\delta^2(\sqrt{E})}{\pi^2}. \tag{A.9}
\]

Therefore, the $j = N$ and $k = N + 1$ term is of order 1.

For $j \leq N < N + 1 < k$ we want to apply the bound

\[
|\ln(1 + x) - x| \leq \frac{x^2}{2} \frac{1}{1 - |x|} \quad \text{for } x \in \mathbb{R} \text{ with } |x| < 1, \text{ to } x = x_jk \tag{A.10}
\]

where

\[
x_{jk} := -\frac{(2jg_j + g_j^2)(2kg_k + g_k^2)}{(k + g_k)^2 - (j + g_j)^2 (k^2 - j^2)}. \tag{A.11}
\]

We estimate using $|g_n| \leq 1$ for all $n \in \mathbb{N}$ and $g_k - g_j \geq 0$

\[
|x_{jk}| \leq \frac{(2jg_j)(2kg_k)}{(j + g_j + k + g_k)(k + j)} \frac{1}{(k - j)^2} = 2 \frac{1}{(k - j)^2}. \tag{A.12}
\]

Since $j \leq N < N + 1 < k$, this implies in particular $|x_{jk}| \leq \frac{1}{2}$, and we continue using (A.10) and (A.12)

\[
\sum_{j=1}^{N} \sum_{k=N+2}^{\infty} |\ln(1 + x_{jk}) - x_{jk}| \leq \sum_{j=1}^{N} \sum_{k=N+2}^{\infty} x_{jk}^2 \\
\leq \sum_{j=2}^{N} \sum_{k=N+1}^{\infty} 4 \left( \frac{1}{k-j} \right)^4 = O(1), \tag{A.13}
\]

as $N \to \infty$, where we used (A.4) in the last line. \qed
Lemma A.2. Define $\delta_k := -\frac{1}{\pi}\delta(\sqrt{\lambda_k})$ for $k \in \mathbb{N}$. Then,

$$\sum_{j=2}^{N} \sum_{k=N+1}^{\infty} \left| \frac{(2j\delta_j + \delta_j^2)(2k\delta_k + \delta_k^2)}{(k + \delta_k)^2 - (j + \delta_j)^2} \right| \frac{1}{(k + g_k)(k + g_k^2)} = o(1) \quad (A.14)$$

as $N, L \to \infty$, $N/L \to \sqrt{\pi}$. 

Proof. First, using the expansion of Lemma A.3, we obtain for all $n \in \mathbb{N}$, $n > 1$,

$$|g_n - \delta_n| \leq \frac{1}{\pi}\delta(\sqrt{\mu_n} - \delta(\sqrt{\lambda_n})) \leq \frac{\|\delta\|_\infty}{\pi L} := \frac{c}{L}, \quad (A.15)$$

where the constant $c > 0$ depends only on $\alpha$. We prove the assertion in two steps. In the first step we consider the numerator only in the second step we consider the denominator. Using (A.15) we estimate

$$\sum_{j=2}^{N} \sum_{k=N+1}^{\infty} \left| \frac{(2j\delta_j + \delta_j^2)(2k\delta_k + \delta_k^2) - (2jg_j + g_j^2)(2kg_k + g_k^2)}{(k + g_k)^2 - (j + g_j)^2}(k^2 - j^2) \right| \leq \frac{C}{L} \sum_{j=2}^{N} \sum_{k=N+1}^{\infty} (j + 1)(k + 1) \frac{(j + 1)(k + 1)}{(k + j - 2)(k + j)(k - j)^2} = O\left(\frac{\ln N}{L}\right) \quad (A.16)$$

as $N, L \to \infty$, $N/L \to \sqrt{\pi}$, where we used $|g_j + g_k| \leq 2$, $g_k - g_j > 0$ for $j < k$ and (A.3). In order to estimate the denominator we use (A.15) to obtain some constant $c > 0$ independent of $j, k$ such that

$$\left| \frac{(k + g_k)^2 - (j + g_j)^2}{(k + \delta_k)^2 - (j + \delta_j)^2} \right| \leq c \frac{k + j}{L}. \quad (A.17)$$

Thus,

$$\sum_{j=2}^{N} \sum_{k=N+1}^{\infty} \left| \frac{(2j\delta_j + \delta_j^2)(2k\delta_k + \delta_k^2)}{(k + \delta_k)^2 - (j + \delta_j)^2}(k + j) \right| \frac{1}{(k^2 - j^2)} \leq \frac{4c}{L} \sum_{j=2}^{N} \sum_{k=N+1}^{\infty} \frac{jk(k + j)}{(k - j)^2((k + g_k)^2 - (j + g_j)^2)((k + \delta_k)^2 - (j + \delta_j)^2)}$$

$$\leq \frac{4c}{L} \sum_{j=2}^{N} \sum_{k=N+1}^{\infty} \frac{jk}{(k - j)^2(k + j)^2} = o(1) \quad (A.18)$$

as $N, L \to \infty$, $N/L \to \sqrt{\pi}$, where we used $|g_k + g_j| \leq 2$, $|\delta_k + \delta_j| \leq 2$, $g_k - g_j > 0$ and $\delta_k - \delta_j > 0$ for $j < k$. □
Lemma A.3. The estimate
\[
\left| \sum_{j=2}^{N} \sum_{k=N+1}^{\infty} \frac{(2j\delta_j + \delta_j^2) (2k\delta_k + \delta_k^2)}{((k + \delta_k)^2 - (j + \delta_j)^2)(k^2 - j^2)} - \sum_{j=2}^{N} \sum_{k=N+1}^{2N} 4j\delta_j\delta_k \right| = O(1)
\] (A.19)
holds as \( N, L \to \infty, \frac{N}{L} \to \frac{\sqrt{L}}{\pi}. \)

Proof. First, we bound the tail, i.e. using \( \delta_k - \delta_j > 0 \) for \( k > j \) and \( |\delta_n| \leq 1 \) for all \( n \in \mathbb{N} \) we estimate
\[
\sum_{j=2}^{N} \sum_{k=2N+1}^{\infty} \frac{(2j\delta_j + \delta_j^2) (2k\delta_k + \delta_k^2)}{((k + \delta_k)^2 - (j + \delta_j)^2)(k^2 - j^2)} \leq \sum_{j=2}^{N} \sum_{k=2N+1}^{\infty} \frac{1}{(k - j)^2}
\]
\[
\leq \frac{N}{(k - N)^2} = O(1), \quad (A.20)
\]
as \( N \to \infty. \) We insert \( \sum_{j=2}^{N} \sum_{k=2N+1}^{2N} \frac{4j\delta_j\delta_k}{((k + \delta_k)^2 - (j + \delta_j)^2)(k^2 - j^2)} \) in (A.19). Thus, in the next step \( \delta_k - \delta_j > 0 \) yields
\[
\leq \sum_{j=2}^{N} \sum_{k=2N+1}^{2N} \left| \frac{(2j\delta_j + \delta_j^2) (2k\delta_k + \delta_k^2) - 4j\delta_j\delta_k}{((k + \delta_k)^2 - (j + \delta_j)^2)(k^2 - j^2)} \right|
\]
\[
\leq \sum_{j=2}^{N} \sum_{k=2N+1}^{2N} \frac{2(j + k) + 1}{(k - j)^2 (k + j)(k + j - 2)}
\]
\[
\leq 3 \sum_{j=2}^{N} \sum_{k=2N+1}^{2N} \left| \frac{1}{(k - j)^2 (k + j - 2)} \right| = O\left( \frac{\ln N}{N} \right), \quad (A.21)
\]
as \( N \to \infty, \) where we used (A.3) in the last line. In the third step, again \( |\delta_n| \leq 1 \) for \( n \in \mathbb{N} \) yields
\[
\leq \sum_{j=2}^{N} \sum_{k=2N+1}^{2N} \frac{4jk}{(k^2 - j^2)} \left| \frac{1}{((k + \delta_k)^2 - (j + \delta_j)^2)} - \frac{1}{(k^2 - j^2)} \right|
\]
\[
\leq \sum_{j=2}^{N} \sum_{k=2N+1}^{2N} \frac{9jk(k + j)}{(k^2 - j^2)(k + j)(k + j - 2)(k - j)}
\]
\[
\leq 9 \sum_{j=2}^{N} \sum_{k=2N+1}^{2N} \frac{1}{(k - j)^3} = O(1), \quad (A.22)
\]
as \( N \to \infty, \) where we used (A.4).

Lemma A.4. The asymptotics
\[
\left| \sum_{j=2}^{N} \sum_{k=N+1}^{2N} \frac{4j\delta_j\delta_k}{(k^2 - j^2)^2} - \frac{1}{\pi^2} \int_{0}^{\pi} dx \int_{\frac{N+1}{\pi}}^{\frac{2N}{\pi}} dy \frac{4x\delta(x\pi)\delta(y\pi)}{(y^2 - x^2)^2} \right| = O(1) \quad (A.23)
\]
holds as \( N, L \to \infty, \frac{N}{L} \to \frac{\sqrt{L}}{\pi}. \)
Proof. We recall that \( \delta_k := -\frac{i}{2} \delta(\sqrt{\lambda_k}) \) and we rewrite
\[
\sum_{j=2}^N \sum_{k=N+1}^{2N} \frac{4jk \delta_0(j) \delta_0(k)}{(k^2 - j^2)^2} = \frac{1}{L^2 \pi^2} \sum_{j=2}^N \sum_{k=N+1}^{2N} \frac{4jk \delta(j \pi) \delta(k \pi)}{(\frac{j}{L})^2 - (\frac{k}{L})^2)^2}.
\] (A.24)
Thus, we estimate
\[
\left| \frac{1}{L^2} \sum_{j=2}^N \sum_{k=N+1}^{2N} \frac{4jk \delta(j \pi) \delta(k \pi)}{(\frac{j}{L})^2 - (\frac{k}{L})^2)^2} \right| \leq \int_{\frac{L}{2}}^{\frac{L}{2}} dx \int_{\frac{L}{2}}^{\frac{L}{2}} dy \left| f\left( \frac{j}{L}, \frac{k}{L} \right) - f(x, y) \right|,
\] (A.25)
where
\[
f(x, y) := \frac{xy \delta(x \pi) \delta(y \pi)}{(y^2 - x^2)^{\frac{3}{2}}}.
\] (A.26)
Using the mean-value theorem and the Cauchy-Schwarz inequality we obtain
\[
(A.25) \leq \sum_{j=2}^N \sum_{k=N+1}^{2N} \sup_{(x, y) \in \left( \frac{L}{2}, \frac{L}{2} \right) \times \left( \frac{L}{2}, \frac{L}{2} \right)} |(\nabla f)(x, y)|_2 \times \int_{\frac{L}{2}}^{\frac{L}{2}} dx \int_{\frac{L}{2}}^{\frac{L}{2}} dy \left| \left( \frac{j}{L} - x, \frac{k}{L} - y \right) \right|_2,
\] (A.27)
where \(| \cdot |_2\) denotes the Euclidean norm. We compute
\[
(\nabla f)(x, y) = \frac{1}{(y^2 - x^2)^{\frac{3}{2}}}
\] (A.28)
\[
\times \left( (y^2 - x^2)(y \delta(x \pi) \delta(y \pi) + xy \delta(x \pi) \delta(y \pi) \pi) + 4x^2 y \delta(x \pi) \delta(y \pi) \right)
\] (A.29)
\[
\times \left( (y^2 - x^2)(x \delta(x \pi) \delta(y \pi) + xy \delta(x \pi) \delta(y \pi) \pi) - 4y^2 \delta(x \pi) \delta(y \pi) \right)
\] (A.29)
We estimate for \((x, y) \in \left( \frac{k}{L}, \frac{k+1}{L} \right) \times \left( \frac{k}{L}, \frac{k+1}{L} \right), j \leq N < k, \)
\[
\left( \frac{1}{y^2 - x^2} \right)^{\frac{3}{2}} \leq \frac{L^6}{(k + j - 1)^{\frac{3}{2}} (k - j)^{\frac{3}{2}}} \leq \frac{L^6}{N^3 (k - j)^{\frac{3}{2}}}
\] (A.30)
and, using \( \delta, \delta' \in L^\infty((0, \infty)), \)
\[
\sup_{(x, y) \in \left( \frac{L}{2}, \frac{L}{2} \right) \times \left( \frac{L}{2}, \frac{L}{2} \right)} |g(x, y)|_2 \leq \sup_{(x, y) \in \left( \frac{L}{2}, \frac{L}{2} \right) \times \left( \frac{L}{2}, \frac{L}{2} \right)} |g(x, y)|_2 = O(1)
\] (A.31)
as \( N, L \to \infty, \frac{N}{L} \to \frac{\sqrt{T}}{\pi}. \) Thus, (A.30) and (A.31) imply
\[
(A.27) \leq O\left( \sum_{j=2}^N \sum_{k=N+1}^{2N} \frac{1}{(y - x)^{\frac{3}{2}}} \right) = O(1)
\] (A.32)
as $N, L \to \infty$, $\frac{N}{L} \to \frac{\sqrt{E}}{\pi}$. □

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References

[AGHH05] S. Albeverio, F. Gesztesy, R. Høegh-Krohn, and H. Holden, Solvable models in quantum mechanics, 2nd ed, American Mathematical Society, Providence, RI, 2005.
[And67] P. W. Anderson, Ground state of a magnetic impurity in a metal, Phys. Rev. 164, 352–359 (1967).
[FP14] R. L. Frank and A. Pushnitski, The spectral density of a product of spectral projections, arXiv:1409.1206 (2014).
[GKM14] M. Gebert, H. Küttler, and P. Müller, Anderson’s Orthogonality Catastrophe, Comm. Math. Phys. 329, 979–998 (2014).
[GKMO14] M. Gebert, H. Küttler, P. Müller, and P. Otte, The decay exponent in the orthogonality catastrophe in Fermi gases, arXiv:1407.2512 (2014).
[IZ88] M. E. H Ismail and Ruiming Zhang, On the Hellmann-Feynman theorem and the variation of zeros of certain special functions, Adv. Appl. Math. 9, 439–446 (1988).
[Knu96] K. Knopp, Theorie und Anwendung der unendlichen Reihen, 6th ed., Springer-Verlag, Berlin, 1996.
[KOS15] H. K. Knörr, P. Otte, and W. Spitzer, Anderson’s orthogonality catastrophe in one dimension induced by a magnetic field, arXiv:1502.07507 (2015).
[KOS13] H. Küttler, P. Otte and W. Spitzer, Anderson’s orthogonality catastrophe for one-dimensional systems, Ann. H. Poincaré 15, 1655–1696 (2014).
[RS78] M. Reed and B. Simon, Methods of modern mathematical physics. IV. Analysis of operators, Academic Press, New York, 1978.
[Sim05] B. Simon, Trace ideals and their applications, 2nd ed., Mathematical Surveys and Monographs, vol. 120, American Mathematical Society, Providence, RI, 2005.
[TO85] Y. Tanabe and K. Ohtaka, Orthogonality catastrophe and the x-ray photoemission spectrum, Phys. Rev. B 32, 2036–2048 (1985).
[Wey13] H. Weyl, The classical groups. Their invariants and representations, Princeton University Press, Princeton, NJ, 1939.