On Euclidean random matrices in high dimension

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Abstract

In this note, we study the $n \times n$ random Euclidean matrix whose entry $(i,j)$ is equal to $f(\|X_i - X_j\|)$ for some function $f$ and the $X_i$'s are i.i.d. isotropic vectors in $\mathbb{R}^p$. In the regime where $n$ and $p$ both grow to infinity and are proportional, we give some sufficient conditions for the empirical distribution of the eigenvalues to converge weakly. We illustrate our result on log-concave random vectors.

Keywords: Euclidean random matrices ; Marcenko-Pastur distribution ; Log-concave distribution.

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1 Introduction

Let $Y$ be an isotropic random vector in $\mathbb{R}^p$, i.e. $EY = 0$, $E[YY^T] = I/p$, where $I$ is the identity matrix. Let $(X_1, \cdots, X_n)$ be independent copies of $Y$. We define the $n \times n$ matrix $A$ by, for all $1 \leq i,j \leq n$,

$$A_{ij} = f(\|X_i - X_j\|),$$

where $f : [0, \infty) \rightarrow \mathbb{R}$ is a measurable function and $\| \cdot \|$ denotes the Euclidean norm. The matrix $A$ is a random Euclidean matrix. It has already attracted some attention see e.g. Mézard, Parisi and Zhee [16], Vershik [18] or Bordenave [7] and references therein.

If $B$ is a symmetric matrix of size $n$, then its eigenvalues, say $\lambda_1(B), \cdots, \lambda_n(B)$ are real. The empirical spectral distribution (ESD) of $B$ is classically defined as

$$\mu_B = \frac{1}{n} \sum_{i=1}^{n} \delta_{\lambda_i(B)},$$

where $\delta_x$ is the Dirac delta function at $x$. In this note, we are interested in the asymptotic convergence of $\mu_A$ as $p$ and $n$ converge to $+\infty$. This regime has notably been previously considered in El Karoui [10] and Do and Vu [9]. More precisely, we fix a sequence $p(n)$ such that

$$\lim_{n \rightarrow \infty} \frac{p(n)}{n} = y \in (0, \infty). \quad (1.1)$$

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Throughout this note, we consider, on a common probability space, an array of random variables \((X_k(n))_{1 \leq k \leq n}\) such that \((X_1(n), \ldots, X_n(n))\) are independent copies of \(Y(n)\), an isotropic vector in \(\mathbb{R}^p(n)\). For each \(n\), we define the Euclidean matrix \(A(n)\) associated. For ease of notation, we will often remove the explicit dependence in \(n\): we write \(p, Y, X_k\) or \(A\) in place of \(p(n), Y(n), X_k(n)\) or \(A(n)\).

The Marcenko-Pastur probability distribution with parameter \(1/y\) is given by

\[
\nu_{MP}(dx) = (1 - y)^+ \delta_0(dx) + \frac{y}{2\pi x} \sqrt{(y_+ - x)(x - y_-)} 1_{(y_- y_+)}(x) dx,
\]

where \(x^+ = (x \vee 0), y_\pm = (1 \pm \frac{1}{\sqrt{y}})^2\) and \(dx\) denotes the Lebesgue measure. Since the celebrated paper of Marcenko and Pastur [15], this distribution is known to be closely related to empirical covariance matrices in high-dimension.

We say that \(Y\) has a log-concave distribution, if \(Y\) has a density on \(\mathbb{R}^p\) which is log-concave. Log-concave random vectors have an increasing importance in convex geometry, probability and statistics (see e.g. Barthe [5]). For example, uniform measures on convex sets are log-concave. We will prove the following result.

**Theorem 1.1.** If \(Y\) has a log-concave distribution and \(f\) is three times differentiable at 2, then, almost surely, as \(n \to \infty\), \(\mu_A\) converges weakly to \(\mu\), the law of \(f(0) - f(2) + 2f''(2)\) \(S\), where \(S\) has distribution \(\nu_{MP}\).

With the weaker assumption that \(f\) is differentiable at 2, Theorem 1.1 is conjectured in Do and Vu [9]. (For more background, we postpone to the end of the introduction). Their conjecture has motivated this note. It would follow from the thin-shell hypothesis which asserts that there exists \(c > 0\), such that for any isotropic log-concave vector \(Y\) in \(\mathbb{R}^p\), \(\mathbb{E}(\|Y\| - 1)^2 \leq c/p\) (see Anttila, Ball and Perissinaki [3] and Bobkov and Koldobsky [6]). Klartag [14] has proved the thin-shell hypothesis for isotropic unconditional log-concave vectors.

The proof of Theorem 1.1 will rely on two recent results on log-concave vectors. Let \(X = X(n)\) be the \(n \times n\) matrix with columns given by \((X_1(n), \ldots, X_n(n))\). Pajor and Pastur have proved the following:

**Theorem 1.2 ([17]).** If \(Y\) has a log-concave distribution, then, in probability, as \(n \to \infty\), \(\mu_{X^T X}\) converges weakly to \(\nu_{MP}\).

We will also rely on a theorem due to Guédon and Milman.

**Theorem 1.3 ([12]).** There exist positive constants \(c_0, c_1\) such that if \(Y\) is an isotropic log-concave vector in \(\mathbb{R}^p\), for any \(t \geq 0\),

\[
\mathbb{P}(\|Y\| - 1 \geq t) \leq c_1 \exp \left( -c_0 \sqrt{t} (t \wedge t^3) \right).
\]

With Theorems 1.2 and 1.3 in hand, the heuristic behind Theorem 1.1 is simple. Theorem 1.3 implies that \(\|X_i\|^2 \sim 1\) with high probability. Hence, since \(\|X_i - X_j\|^2 = \|X_i\|^2 + \|X_j\|^2 - 2X_i^T X_j\), a Taylor expansion of \(f\) around 2 gives

\[
A_{ij} \simeq \begin{cases} 
  f(2) - 2f''(2)X_i^T X_j & \text{if } i \neq j \\
  f'(0) & \text{if } i = j.
\end{cases}
\]

In other words, the matrix \(A\) is close to the matrix

\[
M = (f(0) - f(2) + 2f''(2))I + f(2)J - 2f''(2)X^T X,
\]

where \(I\) is the identity matrix and \(J\) is the matrix with all entries equal to 1. From Theorem 1.2, \(\mu_{X^T X}\) converges weakly to \(\nu_{MP}\). Moreover, since \(J\) has rank one, it is
negligible for the weak convergence of ESD. It follows that $\mu_M$ is close to $\mu$. The actual proof of Theorem 1.1 will be elementary and it will follow this heuristic. We shall use some standard perturbation inequalities for the eigenvalues. The idea to perform a Taylor expansion was already central in [10, 9].

Beyond Theorems 1.2-1.3, the proof of Theorem 1.1 is not related to log-concave vectors. In fact, it is nearly always possible to linearize $f$ as soon as the norms of the vectors concentrate around their mean. More precisely, let us say that two sequences of probability measures $(\mu_n), (\nu_n)$, are asymptotically weakly equal, if for any bounded continuous function $f$, $\int f \, d\mu_n = \int f \, d\nu_n$ converges to 0.

**Theorem 1.4.** Assume that there exists an integer $\ell \geq 1$ such that $E||Y|| - 1|^{2\ell} = O(p^{-1})$, and that for any $\varepsilon > 0$,

$$\lim_{n \to \infty} \mathbb{P}\left( \max_{1 \leq i, j \leq n} \left\{ \|X_i - X_j\|^2 - 2\|X_i\|^2 - 1 \right\} \leq \varepsilon \right) = 1. \quad (1.3)$$

Then, if $f$ is $\ell$ times differentiable at 2, almost surely, $\mu_A$ is asymptotically weakly equal to the law of $f(0) - f(2) + 2f'(2) - 2f'(2) S$, where $S$ has distribution $E\mu_X^T X$.

The case $\ell = 1$ of Theorem 1.4 is contained in Do and Vu [9, Theorem 5]. Besides Theorem 1.2, some general conditions on the matrix $X$ guarantee the convergence of $\mu_X^T X$, see Yin and Krishnaiah [19], Götze and Tikhomirov [11] or Adamczak [1].

In settings where $E||Y|| - 1|^2 = O(p^{-1})$, statements analogous to Theorem 1.4 were already known, notably in the case where the entries of $Y$ are i.i.d., see El Karoui [10, Theorem 2.2] or Do and Vu [9, Corollary 3]. When the vector $Y$ satisfies a concentration inequality for all Lipschitz functions, see El Karoui [10, Theorem 2.3]. (it applies notably to log-concave vectors which density in $\mathbb{R}^p$ of the form $e^{-V(x)}$ with $\text{Hess}(V) \geq cI$ and $c > 0$).

2. **Proofs**

2.1 **Perturbation inequalities**

We first recall some basic perturbation inequalities of eigenvalues and introduce a good notion of distances for ESD. For $\mu, \nu$ two real probability measures, the Kolmogorov-Smirnov distance can be defined as

$$d_{KS}(\mu, \nu) = \sup \left\{ \int f \, d\mu - \int f \, d\nu : \|f\|_{BV} \leq 1 \right\},$$

where, for $f : \mathbb{R} \to \mathbb{R}$, the bounded variation norm is $\|f\|_{BV} = \sup \sum_{k \in \mathbb{Z}} |f(x_{k+1}) - f(x_k)|$, and the supremum is over all real increasing sequence $(x_k)_{k \in \mathbb{Z}}$. The following inequality is a classical consequence of the interlacing of eigenvalues (see e.g. Bai and Silverstein [4, Theorem A.43]).

**Lemma 2.1** (Rank inequality). *If $B, C$ are $n \times n$ Hermitian matrices, then,

$$d_{KS}(\mu_B, \mu_C) \leq \frac{\text{rank}(B - C)}{n}.$$*

For $p \geq 1$, let $\mu, \nu$ be two real probability measures such that $\int |x|^p \, d\mu$ and $\int |x|^p \, d\nu$ are finite. We define the $L^p$-Wasserstein distance as

$$W_p(\mu, \nu) = \left( \inf_{\pi} \int_{\mathbb{R} \times \mathbb{R}} |x - y|^p \, d\pi \right)^{\frac{1}{p}}.$$
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where the infimum is over all coupling $\pi$ of $\mu$ and $\nu$ (i.e. $\pi$ is probability measure on $\mathbb{R} \times \mathbb{R}$ whose first marginal is equal to $\mu$ and second marginal is equal to $\nu$). Hölder inequality implies that for $1 \leq p \leq q$, $W_p \leq W_q$. Moreover, the Kantorovich-Rubinstein duality gives a variational expression for $W_1$:

$$W_1(\mu, \nu) = \sup \left\{ \int fd\mu - \int fd\nu : \|f\|_L \leq 1 \right\},$$

where $\|f\|_L = \sup_{x \neq y} |f(x) - f(y)|/|x - y|$ is the Lipschitz constant of $f$. The next classical inequality is particularly useful (see e.g. Anderson, Guionnet and Zeitouni [2, Lemma 2.1.19]).

**Lemma 2.2** (Hoffman-Wielandt inequality). If $B, C$ are $n \times n$ Hermitian matrices, then

$$W_2(\mu_B, \mu_C) \leq \sqrt{\frac{1}{n}\text{tr}(B - C)^2}.$$  

We finally introduce the distance

$$d(\mu, \nu) = \sup \left\{ \int fd\mu - \int fd\nu : \|f\|_L \leq 1 \text{ and } \|f\|_{BV} \leq 1 \right\}.$$  

By Lemmas 2.1 and 2.2, we obtain that for any $n \times n$ Hermitian matrices $B, C$,

$$d(\mu_B, \mu_C) \leq \sqrt{\frac{1}{n}\text{tr}(B - C)^2} \land \frac{\text{rank}(B - C)}{n}. \quad (2.1)$$

Notice that $d(\mu_n, \mu) \to 0$ implies that $\mu_n$ converges weakly to $\mu$.

### 2.2 Concentration inequality

For $x = (x_1, \ldots, x_n) \in \mathcal{M}_{p,n}(\mathbb{R})$, define $a(x)$ as the Euclidean matrix obtained from the columns of $x$: $a(x)_{ij} = f(\|x_i - x_j\|^2)$. In particular, we have $A = a(X)$. Let $i \in \{1, \ldots, n\}$, $x'(x'_1, \ldots, x'_n) \in \mathcal{M}_{p,n}(\mathbb{R})$ and assume that $x'_j = x_j$ for all $j \neq i$. Then $a(x)$ and $a(x')$ have all entries equal but the entries on the $i$-th row or column. We get

$$\text{rank}(a(x) - a(x')) \leq 2.$$

It thus follows from Lemma 2.1 that for any function $f$ with $\|f\|_{BV} < \infty$,

$$\left| \int fd\mu_a(x) - \int fd\mu_a(x') \right| \leq \frac{2\|f\|_{BV}}{n}.$$

Using Azuma-Hoeffding’s inequality, it is then straightforward to check that for any $t \geq 0$,

$$\mathbb{P} \left( \int fd\mu_A - \mathbb{E} \int fd\mu_A \geq t \right) \leq \exp \left( -\frac{nt^2}{8\|f\|_{BV}^2} \right). \quad (2.2)$$

(For a proof, see [8, proof of Lemma C.2] or Guntuboyina and Leeb [13]). Using the Borel-Cantelli Lemma, this shows that for any such function $f$, a.s.

$$\int fd\mu_A - \int fd\mathbb{E} \mu_A \to 0. \quad (2.3)$$

Now, recall that $M$ was defined by (1.2). Note that the matrix $J$ has rank one. We get from Theorem 1.2 and Lemma 2.1 that $d(\mathbb{E} \mu_M, \mu)$ converges weakly to $\mu$.

**Proposition 2.3.** Under the assumptions of Theorem 1.1, we have

$$\lim_{n \to \infty} d(\mathbb{E} \mu_A, \mathbb{E} \mu_M) = 0.$$
Theorem 1.1 is a corollary of Proposition 2.3. Indeed, it implies that $E_{\mu_A}$ is a tight sequence of probability measures. Hence, a.s. $\mu_A$ is also tight. Then, since the set of continuous functions on an interval endowed with the uniform norm is separable, from (2.3) we get that a.s. $\mu_A$ and $E_{\mu_A}$ are asymptotically weakly equal. Now, Theorem 1.1 follows from a new application of Proposition 2.3.

### 2.3 Proof of Proposition 2.3

The idea is to perform a multiple Taylor expansion which takes the best out of (2.1).

**Step 1 : concentration of norms**

By assumption, there exists an open interval $K = (2 - \delta, 2 + \delta)$ such that $f$ is $C^1$ in $K$ and, for any $x \in K$,

$$f(x) = f(2) + f'(2)(x - 2) + \frac{f''(2)}{2}(x - 2)^2 + \frac{f'''(2)}{6}(x - 2)^3 + o(1).$$

For any $i \neq j$, $(X_i - X_j)/\sqrt{2}$ is an isotropic log-concave vector. Define the sequence $\varepsilon(n) = n^{-\kappa} \wedge (\delta/2)$ with $0 < \kappa < 1/6$. It follows from Theorem 1.3 and the union bound that the event

$$E = \left\{ \max_{i,j} \left\{ \|X_i - X_j\|^2 - 2 \|X_i\|^2 \right\} \leq \varepsilon(n) \right\}$$

has probability tending to 1 as $n$ goes to infinity.

**Step 2 : Taylor expansion around $\|X_i\|^2 + \|X_j\|^2$**

We consider the matrix

$$B_{ij} = \begin{cases} f(\|X_i\|^2 + \|X_j\|^2) - 2f'(\|X_i\|^2 + \|X_j\|^2)X_i^T X_j & \text{if } i \neq j \\ f(0) & \text{if } i = j. \end{cases}$$

On the event $E$, $\|X_i\|^2 + \|X_j\|^2 \in K$. Since $f$ is $C^1$ in $K$, we may perform a Taylor expansion of $f(\|X_i - X_j\|^2)$ around $\|X_i\|^2 + \|X_j\|^2$. It follows that for $i \neq j$,

$$|A_{ij} - B_{ij}| = o(\|X_i - X_j\|^2 - \|X_i\|^2 - \|X_j\|^2) \leq \delta(n)|X_i^T X_j|,$$

where $\delta(n)$ is a sequence going to 0. From (2.1) and Jensen’s inequality, we get

$$d(E_{\mu_A}, E_{\mu_B}) \leq d(\mu_A, \mu_B) \leq P(E^c) + \left( \frac{1}{n} \sum_{i \neq j} E |A_{ij} - B_{ij}|^2 1_E \right)^{1/2} \leq P(E^c) + \delta(n) \left( nE \|X_1\|^2 \right)^{1/2}.$$

Now, from the assumption that $X_1$ and $X_2$ are independent and isotropic, we find

$$E |X_1^T X_2|^2 = E \left( \sum_{k=1}^p X_{k1} X_{k2} \right)^2 = \sum_{k=1}^p (E X_{k1}^2) = 1/p.$$

By assumption (1.1), we deduce that

$$\lim_{n \to \infty} d(E_{\mu_A}, E_{\mu_B}) = 0.$$

It thus remains to compare $E_{\mu_B}$ and $E_{\mu_M}$.
Step 3: Taylor expansion around 2

We define the matrix

\[ C_{ij} = \begin{cases} 
    f(\|X_i\|^2 + \|X_j\|^2) - 2f'(2)X_i^T X_j & \text{if } i \neq j \\
    f(0) & \text{if } i = j.
\end{cases} \]

We now use the fact that \( f' \) is locally Lipschitz at 2. It follows that if \( E \) holds, for \( i \neq j, \)

\[ |B_{ij} - C_{ij}| = O(X_i^T X_j(\|X_i\|^2 + \|X_j\|^2 - 2)) \leq c\varepsilon(n)|X_i^T X_j|. \]

The argument of step 2 implies that

\[ \lim_{n \to \infty} d(E_{\mu_B}, E_{\mu_C}) = 0. \]

It thus remains to compare \( E_{\mu_C} \) and \( E_{\mu_M} \).

Step 4: Taylor expansion around 2 again

We now consider the matrix

\[ D_{ij} = \begin{cases} 
    f(2) + f'(2)(\|X_i\|^2 + \|X_j\|^2 - 2) + \frac{f''(2)}{2}(\|X_i\|^2 + \|X_j\|^2 - 2)^2 & \text{if } i \neq j \\
    f(0) + \frac{f''(2)}{6}(\|X_i\|^2 + \|X_j\|^2 - 2)^3 - 2f'(2)X_i^T X_j & \text{if } i = j.
\end{cases} \]

We are going to prove that

\[ \lim_{n \to \infty} d(E_{\mu_C}, E_{\mu_D}) = 0. \]  \hspace{1cm} (2.4)

We perform a Taylor expansion of order 3 of \( f(\|X_i\|^2 + \|X_j\|^2) \) around 2. It follows that if \( E \) holds, for \( i \neq j, \)

\[ |C_{ij} - D_{ij}| = o(\|X_i\|^2 + \|X_j\|^2 - 2)^3 \leq \delta(n)|X_i^2 + \|X_j\|^2 - 2|^3, \]

where \( \delta(n) \) is a sequence going to 0. Using (2.1) and arguing as in step 2, in order to prove (2.4), it thus suffices to show that

\[ \frac{1}{n} \sum_{i \neq j} E(||X_i||^2 + ||X_j||^2 - 2^6 1_\varepsilon = O(1). \]

Since, for \( \ell \geq 1, |x + y|^\ell \leq 2^{\ell-1}(|x|^\ell + |y|^\ell), \) it is sufficient to show that

\[ nE(||X_1||^2 - 1)^6 1_\varepsilon = O(1). \]

To this end, for integer \( \ell \geq 1, \) we write

\[ E(||X_1||^2 - 1)^\ell 1_\varepsilon = E(||X_1|| - 1)^\ell ||X_1||^\ell 1_\varepsilon \leq 3\ell E(||X_1|| - 1)^\ell. \]

Then, Theorem 1.3 implies that there exists \( c_\ell \) such that

\[ E(||X_1|| - 1)^\ell \leq c_\ell p^{-\ell/6}. \]

It follows that

\[ E(||X_1||^2 - 1)^\ell 1_\varepsilon = O(p^{-\ell/6}). \]  \hspace{1cm} (2.5)

This proves (2.4). It finally remains to compare \( E_{\mu_D} \) and \( E_{\mu_M} \).
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Step 5: End of proof

We set

\[ z_i = (||X_i||^2 - 1). \]

We note that for \( i \neq j \),

\[ D_{ij} = M_{ij} + \sum_{1 \leq k + \ell \leq 3} c_{k\ell} z_i^k z_j^\ell, \]

for some coefficients \( c_{k\ell} \) depending on \( f'(2), f''(2), f'''(2) \). Note that \( c_{10} = c_{01} = f'(2) \). Similarly,

\[ D_{ii} = M_{ii} + 2f'(2)z_i = M_{ii} + c_{10}z_i + c_{01}z_i. \]

Define the matrix \( E \), for all \( 1 \leq i, j \leq n \),

\[ E_{ij} = M_{ij} + \sum_{1 \leq k + \ell \leq 3} c_{k\ell} z_i^k z_j^\ell. \]

If \( E \) holds, then \( \max_i |z_i| \leq \varepsilon(n) \) and we find

\[ |E_{ij} - D_{ij}| = 1(i = j) \left| \sum_{2 \leq k + \ell \leq 3} c_{k\ell} z_i^k z_j^\ell \right| \leq c_1(i = j)\varepsilon(n)^2. \]

It follows from (2.1) that

\[ d(E_{\mu_D}, E_{\mu_E}) \leq Ed(\mu_D, \mu_E) \leq \mathbb{P}(\mathcal{E}^c) + \left( \frac{1}{n} \sum_{i,j} \mathbb{E}|E_{ij} - D_{ij}|^2 1_{\mathcal{E}} \right)^{1/2} \leq \mathbb{P}(\mathcal{E}^c) + c\varepsilon(n)^2. \]

We deduce that

\[ \lim_{n \to \infty} d(E_{\mu_D}, E_{\mu_E}) = 0. \]

We notice finally that the matrix \( E - M \) is equal to

\[ \sum_{1 \leq k + \ell \leq 3} c_{k\ell} Z_k Z_\ell^T, \]

where \( Z_k \) is the vector with coordinates \( (z_i^k)_{1 \leq i \leq n} \). It implies in particular that \( \text{rank}(E - M) \leq 9 \), indeed the rank is subadditive and \( \text{rank}(Z_k Z_\ell^T) \leq 1 \). In particular, it follows from (2.1) that

\[ d(E_{\mu_E}, E_{\mu_M}) \leq Ed(\mu_E, \mu_M) \leq \frac{9}{n}. \]

This concludes the proof of Proposition 2.3 and of Theorem 1.1.

2.4 Proof of Theorem 1.4

The isotropy implies that

\[ \int x^2 E_{\mu_{X^TX}}(dx) = \frac{1}{n} \text{Etr}(X^TX) = 1. \]

It follows that \( E_{\mu_{X^TX}} \) and \( E_{\mu_M} \) are tight sequences of probability measures. Note also that the concentration inequality (2.2) holds. It is thus sufficient to prove the analog of Proposition 2.3. If \( \ell \geq 2 \), the proof is essentially unchanged. In step 1, the assumption (1.3) implies the existence of a sequence \( \varepsilon = \varepsilon(n) \) going to 0 such that \( \mathbb{P}(\mathcal{E}) \to 1 \). Then, in step 4, it suffices to extend the Taylor expansion up to \( \ell \).

For the case \( \ell = 1 \), in step 2, we perform directly the Taylor expansion around 2, for \( i \neq j \) we write \( f(||X_i - X_j||^2) = f(2) - 2f'(2)X_i^T X_j + o(1)) \). We then move directly to step 5. (As already pointed, this case is treated in [9]).
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