Spontaneous symmetry breaking (SSB) has long played a vital role in our understanding of Nature \cite{1}. Examples include ferromagnetism, superconductivity, Bose-Einstein condensation in QCD \cite{2}, and unification of the fundamental forces \cite{3}. Both static and dynamic properties of symmetry broken phases at low temperature are governed by a mean field, which is usually identified in a heuristic manner. Prime examples include the molecular field theory of ferromagnetism by Weiss \cite{4}, the Ginzburg-Landau theory of superconductors \cite{5}, the Gross-Pitaevskii theory of Bose-Einstein condensates \cite{6,7}, culminating in the discovery of the pairing mean field by Bardeen, Cooper and Schrieffer \cite{8}. The identification of a mean field is equivalent to that of an order parameter and this amounts to the identification of an operator which supports a long-range order (LRO) in quantum field theory \cite{9}.

In this Letter, we propose a systematic method of finding mean fields of quantum many-body systems based on the Lie algebra. We construct a building block of an ordered ground state from a weight vector of the representation of the Lie algebra, and present a systematic method to identify the unbroken symmetry, calculate the number and type of Nambu-Goldstone modes (NG modes) and find the homotopy group of topological excitations. We apply this method to a $U(N)$-symmetric system which has recently been realized in ultracold atomic gases \cite{10,11}. The theoretical analysis of this systems becomes increasingly complicated as $N$ grows because the number of generators of the underlying Lie group and possible types of symmetry breaking become dauntingly large.

We consider a quantum field theory whose symmetry is represented by unitary transformations $\phi_i \mapsto e^{iT_\alpha} \phi_i$, where $\{\phi_i\}_i$ is a set of fields of particles, and $g = \{T_\alpha\}_\alpha$ is the Lie algebra of the generators of the symmetry group. We do not specify quantum statistics of the particles and the system can be defined either on a lattice or in a continuous space. We assume that the generator of the symmetry group belongs to a simple compact Lie algebra or its product with $u(1)$, such as $su(N)$, $so(N)$ and $u(N)$. We do not consider a symmetry breaking of the space-time symmetry.

We begin by identifying the building blocks of ordered ground states. Consider the irreducible decomposition by the Lie algebra $g$ of the space of polynomials of the fields $\{\phi_i\}_i$. We express each irreducible representation in terms of its highest weight $\mu_H$ and denote the set of weight vectors $\mu$ by $W[\mu_H]$. Let $a_\mu$ be the field corresponding to the weight vector $|\mu\rangle$. We define simple $\mu$-symmetry breaking (simple $\mu$-SB) as the symmetry breaking into a broken phase that has either an off-diagonal long-range order (ODLRO) with respect to $a_\mu$ or a diagonal long-range order (DLRO) with respect to the adjoint representation of $a_\mu$. S-wave superconductors and ferromagnets are examples of simple $\mu$-SB. An $s$-wave superconductor has an ODLRO with respect to a pairing field $\psi_\uparrow \psi_\downarrow$, which corresponds to the weight vector of the one-dimensional irreducible representation of $su(2)$. A ferromagnet has a DLRO with respect to a spin operator $S$, which corresponds to the adjoint representation of the fundamental representation of $su(2)$. There are two types of ordered phases which cannot be described by simple $\mu$-SB. One is the case in which the system has a LRO with respect to more than one irreducible representation. In this case, different types of order coexist. Another is the case in which the system is condensed into a superposition of more than one weight vector in one irreducible representation. We call them composite $\mu$-symmetry breaking (composite $\mu$-SB). Examples include a biaxial-nematic phase and a cyclic phase in spin-2 BEC \cite{12,13} and a $d_{x^2-y^2}$ superconductor. Here, we focus on simple $\mu$-SB and briefly touch upon composite $\mu$-SB at the end of this Letter.

For a given symmetry group $G$, all the mean fields of simple $\mu$-SB can be found systematically. Since the irreducible representation and the weight vectors have been completely classified for all simple compact Lie algebra \cite{15}, we can derive all mean fields of simple $\mu$-SB in the following way. First, list the irreducible representation and its weight vectors from a low-dimensional representation. When the two weight vectors transform into each other by the action of $G$, the ordered states described by them are physically the same; they merely have different quantization axes. In such a case, we take only one representative weight vector. Table 1 shows classification of...
\(\mu\)-SB for \(U(N)\)-symmetric systems up to two-body pairing of an \(N\)-dimensional representation of \(U(N)\).

The analysis of simple \(\mu\)-SB is done conveniently by taking a special basis of the Lie algebra \(\mathfrak{g}\) called Cartan decomposition \(\mathfrak{g} = \{\{H_\alpha\}^r_{\alpha=1}, \{E^{R}_{\alpha}, E^L_{\alpha}\} \in \mathbb{R}_+\}\) [12]. The Cartan decomposition reveals the symmetry of the ground state and block-diagonalizes an effective Lagrangian of the NG modes. The Cartan decomposition is a generalization of the basis of \(\mathfrak{su}(2)\) Lie algebra \(\{\mathfrak{s}^+, \mathfrak{s}^-\}\), decomposing the basis into two groups, diagonal matrices \(\{H_\alpha\}_{\alpha=1}^r\) and off-diagonal matrices \(\{E^R_{\alpha}, E^L_{\alpha}\} \in \mathbb{R}_+\). Here, \(\{H_\alpha\}_{\alpha=1}^r\) is called the Cartan subalgebra whose elements commute with each other, where \(r = \text{rank}\mathfrak{g}\) is the rank of the Lie algebra \(\mathfrak{g}\); \(E^R_{\alpha}\) and \(E^L_{\alpha}\) are the real and imaginary parts of the raising and lowering operators, where \(\alpha\) is a positive root and \(R_+\) is the set of positive roots. In general, the \(\mathfrak{su}(N)\) algebra is decomposed into the Cartan subalgebra with \(r = N - 1\) and \(N(N - 1)/2\) pairs of raising and lowering operators.

To determine the effective Lagrangian of the NG modes and the homotopy group of the topological excitations, we first prove the following theorem.

**Theorem:** For the \(\mu\)-symmetry breaking with the weight vector \(\mu \in W[\mu_H]\), the Lie algebra of the unbroken symmetry \(\mathfrak{h}\) is determined as follows:

\[
basis(\mathfrak{h}) = \{E^{R}_{\alpha}, E^{L}_{\alpha}|\alpha \in R_+, \mu - \alpha \notin W[\mu_H]\}
\cup \{H_\alpha|t \perp \mu\} \cup \mathfrak{h}_C,
\]

(1)

where \(\basis(\mathfrak{h})\) is the basis of \(\mathfrak{h}\), \(H_\alpha = \sum_{a=1}^r t_a H_a\) is the element of the Cartan subalgebra, and \(\mathfrak{h}_C = \{H_\mu\} \cup \{0\}\) depending on whether the ground state has ODLRO or DLRO.

**Proof:** It follows from the definition of \(\mu\)-SB for ODLRO that the order parameter can be written as \(O = \{(\mu_0)|\mu \in W[\mu_H]\}\). The symmetry of the ground state is determined as the set of elements of \(G\) whose action does not change \(O\). The unitary transformations generated by \(E^{R,L}_{\alpha}\) leave the ground state invariant if and only if \(E^{R,L}_{\alpha}(\mu) = 0\). Thus \(\mu - \alpha \notin W[\mu_H]\). The unitary transformations generated by \(H_\alpha\) act on the field \(a_\mu\) just as a phase shift since it is the eigenstate of the Cartan generators: \(\exp(iH_\alpha \theta)(\mu) = \exp(i(\mu_\alpha \theta))(\mu)\). This phase shift can be eliminated by taking linear combinations of \(H_\alpha\) except for one direction \(\mathfrak{h}_C = \{H_\mu\}\). The broken generators of DLRO in simple \(\mu\)-SB coincide with those of ODLRO in simple \(\mu\)-SB except for the Cartan generator \(H_\mu\). The Cartan generator \(H_\mu\) is broken for ODLRO but it is not broken for DLRO, which completes the proof of Eq. (1).

For \(\mathfrak{g} = \mathfrak{u}(N)\), \(\mathfrak{h}\) in Eq. (1) can be rewritten into a common form because the subalgebra \(\mathfrak{h}\) in Eq. (1) is a regular subalgebra of \(\mathfrak{g}\)

\[
\mathfrak{h} = \mathfrak{u}(1)_a \oplus (\mathfrak{u}(1)_a)^a \oplus \bigoplus_{j=1}^b \mathfrak{su}(n_j),
\]

(2)

where \(a, b \in \mathbb{N}\) and \(\mathfrak{su}(1)\) is considered to be \(\{0\}\). The basis of \(\mathfrak{u}(1)_a\) is \(I_N\) (the \(N\)-dimensional identity matrix) for DLRO and \(|\mu|^2 I_N - H_\mu\) for ODLRO with \(\mu \neq 0\). Note that \(\mathfrak{u}(1)_a\) is considered to be \(\{0\}\) for ODLRO with \(\mu = 0\). The basis of \(\mathfrak{u}(1)_a\) in Eq. (2) is an element of \(\mathfrak{su}(N)\) and a traceless matrix. The regular subalgebras of a simple Lie algebra are completely classified by Dynkin [10] and it is known that they have such a common form such as shown in Eq. (2) for \(\mathfrak{g} = \mathfrak{u}(N)\).

Next, we determine the numbers of NG modes. We note that in non-relativistic systems the type-2 NG mode with a quadratic dispersion is allowed, in contrast to relativistic systems where only the type-1 NG mode with a linear dispersion is allowed [17]. It has been shown that the numbers of type-1 and type-2 NG modes, \(n_1\) and \(n_2\), are determined only by the ground state \(|\text{vac}\rangle\) and the broken generators of the symmetry group \(\{T_\alpha^e\}_\alpha\) as follows [18] [19]:

\[
n_1 + 2n_2 = \text{dim}(G/H),
\]

(3)

\[
n_2 = \frac{1}{2} \text{rank}(\rho_{WB}),
\]

(4)

\[
(\rho_{WB})_{\alpha'\beta'} = -i \langle \text{vac}| [\hat{T}_{\alpha'}, \hat{T}_{\beta'}]| \text{vac}\rangle, \tag{5}
\]

where \(\rho_{WB}\) is the Watanabe-Brauner matrix [20]. However, since the effective Lagrangian is not diagonalized in these works, the dynamics and the corresponding broken generator of the NG mode are not available from this formalism alone. Here, we show that the Watanabe-Brauner matrix and the effective Lagrangian can be simultaneously block-diagonalized by taking the Cartan decomposition for simple \(\mu\)-SB. Also NG modes are classified into three categories with respect to its dynamics.

Let \(\mu_H^I\) be the irreducible representation of the order parameter \(O\) and \(\mathcal{V}(O)\) be an effective potential of the order parameter \(\mathcal{V}(O)\). The effective Lagrangian of the NG modes is obtained by the differential expansion of \(\mathcal{V}(O)\). We keep \(\nabla O\) and \(\partial O\) to the second order.

We can then prove the following theorem.

**Theorem:** Consider a simple \(\mu\)-SB in a non-relativistic system, let \(R_B\) be the set of the root vectors of the broken raising and lowering operators \(E^{R,L}_{\alpha}\), let \(R_{B,2}\) be the set of the root vectors \(\alpha \in \mathfrak{R}\) with \(\langle \alpha, \mu \rangle = 0\) and let \(R_{B,2}\) be the set of the root vectors \(\alpha \in \mathfrak{R}\) with \(\langle \alpha, \mu \rangle \neq 0\). Then, the effective Lagrangian of NG modes \(\delta \mathcal{L}\) is block-diagonalized as follows:

\[
\delta \mathcal{L} = \sum_{\alpha \in R_{B,1}} \delta \mathcal{L}^\alpha_{\text{osc}} + \sum_{\alpha \in R_{B,2}} \delta \mathcal{L}^\alpha_{\text{pre}} + \delta \mathcal{L}^\mu_{\text{ph}}, \tag{6}
\]

\[
\delta \mathcal{L}^\alpha_{\text{osc}} = \frac{1}{2} \left( \begin{array}{cc} \nabla^R \pi^L_{\alpha} & b_{1,\alpha} - i b_{2,\alpha} \\ \nabla^L \pi^L_{\alpha} & b_{1,\alpha} + i b_{2,\alpha} \end{array} \right) \left( \begin{array}{c} \nabla^R \pi^R_{\alpha} \\ \nabla^L \pi^R_{\alpha} \end{array} \right), \tag{7}
\]

\[
\delta \mathcal{L}^\mu_{\text{pre}} = \left( \begin{array}{c} \nabla^R \pi^L_{\mu} \\ \nabla^L \pi^L_{\mu} \end{array} \right) \left( \begin{array}{c} b_{1,\mu} \\ i b_{2,\mu} \end{array} \right) \left( \begin{array}{c} \nabla^R \pi^R_{\mu} \\ \nabla^L \pi^R_{\mu} \end{array} \right), \tag{8}
\]
\( \delta L_{\text{pha}} = \frac{1}{2} B (\nabla \pi_\mu)^2 + \frac{1}{2} C (\pi_\mu)^2, \) (9)

where the last term in the first equation \( \delta L_{\text{pha}} \) is absent for DLRO, and the matrix elements \( a_\alpha, b_{\alpha,1}, b_{\alpha,2}, c_{\alpha,1}, c_{\alpha,2} \) and the coefficients \( B, C \) are real constants.

**Proof:** We first show that the Watanabe-Brauner matrix is block-diagonalized by taking the basis of the generators as Cartesian decomposition. We decompose the Noether charge \( \left\langle \hat{\mathcal{T}}_a \right\rangle \) into the charge from the coherent order \( \left\langle \hat{\mathcal{T}}_a \right\rangle^\text{coh} \) and the charge from incoherent background \( \left\langle \hat{\mathcal{T}}_a \right\rangle^\text{inc} \): \( \left\langle \hat{\mathcal{T}}_a \right\rangle = \left\langle \hat{\mathcal{T}}_a \right\rangle^\text{coh} + \left\langle \hat{\mathcal{T}}_a \right\rangle^\text{inc} \). Since the simple \( \mu \)-SB with the weight vector \( \mu \) describes the ordered state in the spin direction \( \mu \), we have \( \left\langle \hat{H}_1, \hat{H}_2, \ldots, \hat{H}_3 \right\rangle^\text{coh} = N_c \mu, \left\langle \hat{E}_{\alpha,1}^R \right\rangle^\text{coh} = \left\langle \hat{E}_{\alpha,1}^L \right\rangle^\text{inc} = 0, \) and \( \left\langle \hat{H}_\alpha \right\rangle^\text{inc} = 0 \) when \((\alpha, \mu) = 0\). Here \( N_c \) is the number of particles that contribute to the LRO. It follows from the definition of \( \mu \)-SB and the commutation relation of the Cartesian decomposition that

\[ \rho_{\text{WB}} = \bigoplus_{\alpha \in R_B, \beta} \left( \begin{array}{cc} 0 & \left\langle \hat{H}_\alpha \right\rangle^\text{inc} \\ \left\langle \hat{H}_\alpha \right\rangle^\text{coh} & 0 \end{array} \right) \bigoplus_{\alpha} (0), \] (10)

where \( \left\langle \hat{H}_\alpha \right\rangle^\text{coh} = N_c (\mu, \alpha) + \left\langle \hat{H}_\alpha \right\rangle^\text{inc} \). Here each \( 2 \times 2 \) block corresponds to a pair of the raising and lowering operators \( E_{\alpha,1}^R \) and \( E_{\alpha,1}^L \). Since \( \left\langle \hat{H}_\alpha \right\rangle^\text{inc} \) depends on microscopic parameters and thermodynamic parameters, \( \left\langle \hat{H}_\alpha \right\rangle \neq 0 \) without a fine-tuning of the parameters. Next, we block-diagonalize the effective Lagrangian. Let \( \pi_\alpha (x) \) be the NG mode associated with the generator \( T_\alpha \). We first discuss the case of ODLRO. Consider the field corresponding to the order parameter \( a'_\mu \) with small fluctuations

\[ a'_\mu (x) = \bigg[ \exp (i \sum_a T_a \pi_a (x)) \bigg]_{\mu} \langle a_\mu \rangle \] (11)

(12)

and substitute them into the effective potential. The effective Lagrangian \( \delta \mathcal{L} \) is obtained up to second order of \( \nabla \) and \( \partial_x \) as

\[ \delta \mathcal{L} = \mathcal{V} (a'_\mu (x)) = \mathcal{V} (\pi_\mu (x)) \] (13)

where \( \mathcal{F}^{(i)}(\pi_\mu (x)) = \sum_{\mu} a_\mu \partial_x \mu \)]

(14)

where \( \mathcal{F}^{(i)}(\pi_\mu (x)) = \sum_{\mu} a_\mu \partial_x \mu \)]

(15)

where we choose the basis of generators \( T_a, T_b \) as \( E_{\alpha,1}^R, E_{\alpha,1}^L, H_\alpha \) and used \((\mu, T_\alpha T_\beta, \mu) = 0\) except for \((T_a, T_b) = (E_{\alpha,1}^R, H_\alpha, \alpha), (E_{\alpha,1}^L, E_{\alpha,1}^R), (E_{\alpha,1}^L, H_\alpha, \alpha), (H_\alpha, H_\alpha, \alpha), (H_\alpha, H_\beta, \alpha), \) and \( \hat{b}_{1,\alpha} \) and \( \hat{b}_{2,\alpha} \) are real constants. Therefore the second term on the right-hand side of Eq. (14) is block-diagonalized into a similar form as Eqs. (6), (7), (8) and (9). The third term is block-diagonalized

\( (\mu_1, \mu_2) \) and \( (\mu_1, \mu_2) \) be the NG mode associated with the generator \( T_\mu \). We first discuss the case of ODLRO. Consider the field corresponding to the order parameter \( a'_\mu \) with small fluctuations

\[ a'_\mu (x) = \bigg[ \exp (i \sum_a T_a \pi_a (x)) \bigg]_{\mu} \langle a_\mu \rangle \] (11)
calculation can be done for \( \mu \) case of \( G \). The systematic way to calculate them when the Lie algebra is explained by considering the example of an \( NG \) modes is shown in Table 1. The number of type-2 NG modes arising from a pair of the raising and lowering operators which satisfy \( \langle \mu, \alpha \rangle = 0 \) and these modes are the generalization of magnons in an antiferromagnet. These modes represent harmonic oscillations of the magnetization axis or population imbalance: \( \left[ \langle \hat{H}_{\alpha}(x), \langle \hat{E}_{\alpha}^{R}(x), \langle \hat{E}_{\alpha}^{I}(x) \rangle \right] = (\cos(k \cdot x), \sin(k \cdot x) \cos \phi, \sin(k \cdot x) \sin \phi) \). The precession is the type-2 NG mode arising from a pairing of the raising and lowering operators \( E_{\alpha}^{R} \) and \( E_{\alpha}^{I} \) that satisfy \( \langle \mu, \alpha \rangle \neq 0 \). This mode is the generalization of a magnon in a ferromagnet. It arises from the asymmetry in the populations between different flavors of the ground state and represents the precession of the magnetization axis or population imbalance: \( \left[ \langle \hat{H}_{\alpha}(x), \langle \hat{E}_{\alpha}^{R}(x), \langle \hat{E}_{\alpha}^{I}(x) \rangle \right] = (\cos \Delta \sin \Delta \cos(k \cdot x), \sin \Delta \sin(k \cdot x)) \). As is shown in Table 1, the number of type-2 NG modes in \( U(N) \)-symmetric systems grows linearly with \( N \), in sharp contrast to relativistic systems where only type-1 NG modes appear. The growth of the number of type-2 NG modes is explained by considering the example of an \( SU(N) \) ferromagnet. A type-2 NG modes arises from the difference in the flavor density of the condensate ground state. Only the flavor-1 is condensed in the \( SU(N) \) ferromagnetic ground state and there are the differences in the flavor density between flavor-1 and flavor-2, 3, \cdots, \( N \). Therefore \( N - 1 \) type-2 NG modes exist.

Finally, we determine homotopy groups to determine the topological excitations in symmetry broken phases. In general, it is difficult to calculate the homotopy groups \( \pi_{i}(G/H) \) for a given subgroup \( H \). However, there is a systematic way to calculate them when the Lie algebra of \( H \) is a regular subalgebra. Therefore, a systematic calculation can be done for \( U(N) \). Here, we consider the case of \( G = U(N) \).

**Theorem:** For \( \mu \)-SB in \( G = U(N) \), the homotopy group \( \pi_{1}(G/H) \) and \( \pi_{2}(G/H) \) are given as follows:

\[
\pi_{1}(G/H) = \begin{cases} Z & \text{ODLRO with } \mu = 0; \\ \{0\} & \text{DLRO or ODLRO with } \mu \neq 0; \end{cases}
\]

\[
\pi_{2}(G/H) = \begin{cases} \{0\} & a = 0; \\ \mathbb{Z}^a & a \geq 1, \end{cases}
\]

where \( a \) is the integer in Eq. (2).

**Proof:** From the homotopy exact sequence [21], we obtain \( \pi_{1}(G/H) = \text{Coker}\{i_{1}^{*}: \pi_{1}(H) \to \pi_{1}(G)\} \) and \( \pi_{2}(G/H) = \text{Ker}\{i_{1}^{*}: \pi_{1}(H) \to \pi_{1}(G)\} \), where \( i_{1}^{*} \) is the induced homomorphism of the inclusion map \( i: H \to G \) and \( \text{Coker}\{f: X \to Y\} = Y/\text{Im}\{f: X \to Y\} \). The homotopy group of \( G \) is given by \( \pi_{1}(G) = \{[g]^{n} | n \in \mathbb{Z} \} \). The homotopy group of \( H \) is \( \pi_{1}(H) = \{(g_{0})^{n} | n \in \mathbb{Z} \} \). Also, the homotopy group of \( H \) is \( \pi_{1}(H) = \{([g_{0}])^{n} | n \in \mathbb{Z} \} \). The group \( \pi_{0}\) and \( \pi_{1}\) are the nontrivial loops generated by the exponential mappings of \( u(1)_{0} \) and \( u(1)_{n} \), respectively. When \( g_{0} \) and \( g_{a} \) are embedded into \( \pi_{1}(G) \), we have \( i_{1}^{*}(g_{0}) \sim g, i_{1}^{*}(g_{a}) \sim 0 \). The corresponding homotopy equivalence. Thus, we obtain Eqs. (16) and (17). The nontrivial elements in Eq. (16) correspond to quantum vortices. Since quantum vortices do not arise when \( \mu \neq 0 \), in \( U(N) \)-symmetric systems they do not appear in the ground states in s-wave and \( p_{x} + ip_{y} \) superfluids with \( N \geq 3 \) and ferromagnetic BEC with \( N \geq 2 \). This is in sharp contrast to the cases of two-component superfluids and scalar BECs.

The nontrivial elements in Eq. (17) correspond to hedgehog configurations of flavor, \( \left( \left( \hat{E}_{\beta}^{R}, \hat{E}_{\beta}^{I}, \hat{H}_{j} \right) \right) = (\cos \theta \cos \phi, \cos \theta \sin \phi, \sin \theta) \). The number of types of these hedgehog configurations coincides that of \( u(1)_{\alpha} \) in \( h, a \).

In conclusion, we have proposed Lie-algebraic method to systematically find mean fields of quantum many-body systems. We have introduced a notion of simple \( \mu \)-symmetry breaking and used it to calculate the NG modes and topological excitation. In particular, we find the dominance of type-2 NG modes for large \( N \) in nonrelativistic systems which makes a sharp contrast to relativistic systems. Also, the absence of quantized vortices in the ground state in \( U(N)\)-symmetric systems for \( N = 2, 3 \) is a remarkable feature when compared with fewer-component systems. The systematic treatment can be done for NG modes and topological excitations in simple \( \mu \)-SB because the symmetry of the ground state becomes a regular subalgebra.

In this Letter, we have not discussed the general condition for simple \( \mu \)-SB and composite \( \mu \)-SB to emerge. Many of the known symmetry broken phases belong to simple \( \mu \)-SB of low-dimensional irreducible representations and composite \( \mu \)-SB appears in higher-dimensional irreducible representations. The generalization to such cases merits further study.
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Supplemental Materials

Here we show the following formula of the form of the Lie algebra \( \mathfrak{h} \) in \( \mu \)-SB and the homotopy groups of a homogeneous space \( \pi_i(G/H) \) \((i = 1, 2)\) used in the main text.

**Theorem (1):** Consider \( G = U(N) \) and \( \mu \)-SB with the weight vector \( \mu \).

1. The Lie algebra of unbroken symmetry \( \mathfrak{h} \) can be written as follows:
   - For DLRO
     \[
     \mathfrak{h} = \mathfrak{u}(1)_a \oplus (\mathfrak{u}(1)_a)^a \oplus \bigoplus_{j=1}^{b} \mathfrak{su}(n_j), \tag{S1}
     \]
   - For ODLRO with \( \mu = 0 \)
     \[
     \mathfrak{h} = (\mathfrak{u}(1)_a)^a \oplus \bigoplus_{j=1}^{b} \mathfrak{su}(n_j). \tag{S2}
     \]
   - For ODLRO with \( \mu \neq 0 \)
     \[
     \mathfrak{h} = \mathfrak{u}(1)_a \oplus (\mathfrak{u}(1)_a)^a \oplus \bigoplus_{j=1}^{b} \mathfrak{su}(n_j), \tag{S3}
     \]
     where the basis of \( \mathfrak{u}(1)_a \) is \( |\mu|^2I_N - \tilde{H}_{\mu} \). The basis of \( \mathfrak{u}(1)_a \) in Eqs. (S1), (S2), (S3) is an element of \( \mathfrak{su}(N) \) and a traceless matrix, and \( a, b, n_j \) in Eqs. (S1), (S2), (S3) are integers with \( a, b \geq 0, n_j \geq 2 \).

2. Let \( i^*_n : \pi_n(H) \to \pi_n(G) \) be the induced homomorphism of the inclusion map \( i : H \to G \) and \( p^*_n : \pi_n(G) \to \pi_n(G/H) \) be the induced homomorphism of the projection map \( p : G \to G/H \). The homotopy groups of the homogeneous space \( \pi_i(G/H) \) \((i = 1, 2, 3, 4)\) are calculated as follows:

1. For a Lie group \( G \) and its connected subgroup \( H \),
   \[
   \pi_1(G/H) = \text{Coker}\{i^*_1 : \pi_1(H) \to \pi_1(G)\}, \tag{S4}
   \]
   where \( \text{Coker}\{f : X \to Y\} = Y/\text{Im}\{f : X \to Y\} \)
2. For a compact Lie group $G$ and its subgroup $H$,

$$\pi_2(G/H) = \text{Ker}\{i_1^*: \pi_1(H) \to \pi_1(G)\}. \tag{5}$$

**Proof of the theorem (1):**

The proof of the theorem (1) can be done by the decomposition of the Lie algebra $\mathfrak{h}$ into the Lie algebra of the overall phase $u(1)_o$ and the subalgebra of the $\mathfrak{su}(N)$.

We first show the case of DLRO. The generator $I_N$ is unbroken and $I_N$ commutes with the other elements of $\mathfrak{h}$. Therefore we have $\mathfrak{h} = u(1)_o \oplus \mathfrak{h}'$, where $\mathfrak{h}'$ is a regular subalgebra of the $\mathfrak{su}(N)$. Any regular subalgebra of the $\mathfrak{su}(N)$ Lie algebra can be written into

$$\mathfrak{su}(N) = (u(1)_s)^a \oplus \bigoplus_{j=1}^b \mathfrak{su}(n_j), \tag{6}$$

where $a, b, n_j$ are integers with $a, b \geq 0, n_j \geq 2$ \cite{10}, which complete the proof of Eq. (S1).

Next we show the case of ODLRO with $\mu = 0$. The generator $I_N$ is broken and $\mathfrak{h}$ is a regular subalgebra of the $\mathfrak{su}(N)$. Using a similar argument as above, we obtain Eq. (S2).

Finally we show the case of ODLRO with $\mu \neq 0$. For $\alpha \in R_+ \setminus R_B$, we have $(\alpha, \mu) = 0$ because $(\alpha, \mu) = \langle \mu | H_\alpha | \mu \rangle = -i \langle \mu | [E^{R}_\alpha, E^I_\alpha] | \mu \rangle = 0$. Therefore the unbroken generator $|\mu|^2 I_N - H_\mu$ commutes with the other elements of $\mathfrak{h}$ because $[E^{R}_\alpha, |\mu|^2 I_N - H_\mu] = \pm i (\mu, \alpha) E^I_\alpha R = 0$. Using a similar argument as above, we obtain Eq. (S3).

**Proof of (2):**

The proof is straightforward by using the homotopy exact sequence \cite{21},

$$\pi_2(G) \xrightarrow{i_2^*} \pi_2(G/H) \xrightarrow{p_2^*} \pi_1(H) \xrightarrow{i_1^*} \pi_1(G) \xrightarrow{p_1^*}, \tag{7}$$

$$\pi_1(H) \xrightarrow{i_1^*} \pi_1(G) \xrightarrow{p_1^*} \pi_1(G/H) \xrightarrow{i_1^*} \pi_0(H) \xrightarrow{j_0^*}. \tag{8}$$

To show the theorem, we show the following proposition:

**Proposition:** Consider an exact sequence,

$$\cdots \xrightarrow{f_3} A \xrightarrow{f_2} B \xrightarrow{f_1} C \xrightarrow{f_0} D \xrightarrow{f_0} \cdots \tag{9}$$

(1) If $A = \{0\}$, we have

$$B = \text{Ker}\{f_3 : C \to D\}. \tag{10}$$

(2) If $D = \{0\}$, we have

$$C = \text{Coker}\{f_2 : A \to B\}. \tag{11}$$

The proof of the proposition (1) goes as follows. From $A = \{0\}$, $\text{Ker}f_3 = \text{Im}f_2 = \{0\}$ follows. Therefore, $f_3$ is an injection and

$$B = \text{Im}f_3 = \text{Ker}f_4. \tag{12}$$

The proof of the proposition (2) goes as follows. From $D = \{0\}$, $\text{Im}f_3 = \text{Ker}f_4 = C$ follows. Therefore, from the homomorphism theorem, we have

$$C = \text{Im}f_3 = B/\text{Ker}f_3 = B/\text{Im}f_2 = \text{Coker}f_2. \tag{13}$$

The first theorem is proven by combining the proposition (2) with the assumption $\pi_0(H) = \{0\}$. The second theorem is proven by combining the proposition (1) with the fact that $\pi_2(G) = \{0\}$ for any compact Lie group $G$. 