Controllability of the Strongly Damped Wave Equation with Impulses and Delay

Abstract: Evading fixed point theorems we prove the interior approximate controllability of the following semilinear strongly damped wave equation with impulses and delay

\[
\begin{cases}
    w_{tt} + \eta(-\Delta)^{1/2} w_t + \gamma(-\Delta) w = 1_\omega u(t,x) + f(t,w,w_t,w_{tt}), & \text{in } (0,\tau) \times \Omega, \\
    w = 0, & \text{in } (0,\tau) \times \partial\Omega, \\
    w(0,x) = w_0(x), & \text{in } (0,\tau), \\
    w_t(t_0^+,x) = w_t(t_0^-,x) + I_k(t_k,w(t_k,x),w_t(t_k,x),u(t_k,x)), & x \in \Omega,
\end{cases}
\]

in the space \(Z_{1/2} = D((-\Delta)^{1/2}) \times L^2(\Omega),\) where \(\tau > 0\) is the delay, \(\Gamma = (0,\tau) \times \Omega, \partial\Gamma = (0,\tau) \times \partial\Omega, \Gamma_r = [-\tau,0] \times \Omega, \) \((\phi,\psi) \in C([-\tau,0];Z_{1/2}), \) \(k = 1,2,\ldots,p,\) \(\omega\) is a bounded domain in \(\mathbb{R}^N(N \geq 1),\) \(\omega\) is an open nonempty subset of \(\Omega,\) \(1_\omega\) denotes the characteristic function of the set \(\omega,\) the distributed control \(u \in L^2(0,\tau;U),\) with \(U = L^2(\Omega),\) \(\eta,\gamma\) are positive numbers and \(f,I_k \in C([0,\tau] \times \mathbb{R} \times \mathbb{R};\mathbb{R}), k = 1,2,3,\ldots,p.\) Under some conditions we prove the following statement: For all open nonempty subsets \(\omega\) of \(\Omega\) the system is approximately controllable on \([0,\tau].\) Moreover, we exhibit a sequence of controls steering the nonlinear system from an initial state \((\phi(0),\psi(0))\) to an \(e\)-neighborhood of the final state \(z_1\) at time \(\tau > 0.\)

Keywords: semilinear strongly damped wave equation, impulses and delay, approximate controllability, strongly continuous semigroups

MSC: Primary: 93B05; secondary: 93C10

1 Introduction

This work has been motivated by the work done in [23] where the Rothe’s fixed point theorem was applied to prove the interior approximate controllability of the following semilinear impulsive strongly damped wave equation with Dirichlet boundary conditions

\[
\begin{cases}
    w_{tt} + \eta(-\Delta)^{1/2} w_t + \gamma(-\Delta) w = 1_\omega u(t,x) + f(t,w,w_t,w_{tt}), & \text{in } (0,\tau) \times \Omega, \\
    w = 0, & \text{in } (0,\tau) \times \partial\Omega, \\
    w(0,x) = w_0(x), & \text{in } (0,\tau), \\
    w_t(t_0^+,x) = w_t(t_0^-,x) + I_k(t_k,w(t_k,x),w_t(t_k,x),u(t_k,x)), & x \in \Omega,
\end{cases}
\]

in the space \(Z_{1/2} = D((-\Delta)^{1/2}) \times L^2(\Omega),\) \(k = 1,2,\ldots,p,\) \(\Omega\) is a bounded domain in \(\mathbb{R}^N(N \geq 1),\) \(\omega\) is an open nonempty subset of \(\Omega,\) \(1_\omega\) denotes the characteristic function of the set \(\omega,\) the distributed control \(u \in L^2(0,\tau;U),\) with \(U = L^2(\Omega),\) \(\eta,\gamma\) are positive numbers and \(f,I_k \in C([0,\tau] \times \mathbb{R} \times \mathbb{R};\mathbb{R}), k = 1,2,3,\ldots,p,\) such that

\[
|f(t,w,w_t)| \leq a_0 (w^{a_0} + v^{b_0}) + b_0 |w|^{b_0} + c_0, \quad u, w, v \in \mathbb{R}.
\]
\begin{align}
|I_k(t, w, v, u)| & \leq a_k \left( |w|^{a_k} + |v|^{a_k} \right) + b_k |u|^{\beta_k} + c_k, \quad k = 1, 2, 3, \ldots, p. \\
\frac{1}{2} \leq a_k < 1, \frac{1}{2} \leq \beta_k < 1, \quad k = 0, 1, 2, 3, \ldots, p, \text{ and} \\
w(t_k, x) &= w(t_k^-, x) = \lim_{t \to t_k^-} w(t, x), \quad w(t_k^+, x) = \lim_{t \to t_k^+} w(t, x), \\
w_r(t_k, x) &= w_r(t_k^-, x) = \lim_{t \to t_k^-} w_r(t, x), \quad w_r(t_k^+, x) = \lim_{t \to t_k^+} w_r(t, x).
\end{align}

Now, we will add delay to this equation and evade Fixed Point Theorems by applying the technique employed by A.E. Bashirov and Noushin Gahramanlou ([2]), A.E. Bashirov and Noushin Gahramanlou ([3]) and A.E. Bashirov, N. Mahmudov, N. Semi and H. Etkin ([4]) to study the interior approximate controllability of semilinear evolution equations, in order to prove the approximate controllability of the following semilinear strongly damped wave equation with impulses and delay

\begin{equation}
\begin{cases}
w_{tt} + \eta(-\Delta)^{1/2} w_t + \gamma(-\Delta) w = 1_\omega u(t, x) + f(t, w(t - r), w_r(t - r), u(t)), & t \in \Gamma, \\
w = 0, & \partial \Gamma, \\
w(s, x) = \phi(s, x), \quad w_r(s, x) = \psi(s, x), & \Gamma_r, \\
w_r(t_k^-, x) = w_r(t_k^-, x) + I_k(t_k, w(t_k, x), w_r(t_k, x), u(t_k, x)), & x \in \Omega,
\end{cases}
\end{equation}

in the space $Z_{1/2} = D((-\Delta)^{1/2}) \times L^2(\Omega)$, where $r > 0$ is the delay, $\Gamma = (0, \tau) \times \partial \Omega, \partial \Gamma = (0, \tau) \times \partial \Omega, \Gamma_r = [-r, 0] \times \Omega, (\phi, \psi) \in C([-r, 0]; Z_{1/2}), k = 1, 2, \ldots, p, \Omega$ is a bounded domain in $\mathbb{R}^N (N \geq 1), \omega$ is an open nonempty subset of $\Omega, 1_\omega$ denotes the characteristic function of the set $\omega$, the distributed control $u \in L^2(0, \tau; U)$, with $U = L^2(\Omega), \eta, \gamma$ are positive numbers and $f, I_k \in C([0, \tau] \times \mathbb{R} \times \mathbb{R}; \mathbb{R}), k = 1, 2, 3, \ldots, p$ and

\begin{equation}
|f(t, w, v, u)| \leq a_0 \sqrt{|w|^2 + |v|^2} + b_0, \quad u, w, v \in \mathbb{R}.
\end{equation}

Several evolutionary processes in nature are characterized by the fact that at certain moments in time they experience an abrupt change. This behavior is observed in real-life problems including: mechanics, chemotherapy, population dynamics, optimal control, ecology, industrial robotics, biotechnology, diffusive processes, etc. The theory of impulsive differential equations [6, 24] provides a natural framework to mathematically describe these processes. The controllability of impulsive differential equations has been widely studied, but the main focus has been exact controllability: impulsive partial neutral functional differential equations with infinite delay are studied in [7], the exact controllability of semilinear impulsive integrodifferential evolution systems with nonlocal conditions are discussed in [25], and the exact controllability for impulsive differential systems with finite delay are studied in [26]. To the best of our knowledge, there is only a few works on approximate controllability of impulsive semilinear evolution equations; worth mentioning is [9], where the authors study the approximate controllability of impulsive differential equations with nonlocal conditions, using measure of noncompactness and Monch fixed point theorem, and assuming that the nonlinear terms $f, I_k$ do not depend on the control variable.

However, for semilinear systems without impulses and delay things are different, there are many results whose proofs are standard and the fixed point methods for semilinear controllability problems are frequently used and known in the literature of control theory. Moreover, different kinds of controllability of many types of nonlinear and semilinear dynamical control systems have been recently considered in several articles, worth mentioning are [12–19] and the references therein. But, as we mention, here we would not use fixed point theorems to get our result.

A Banach space is the natural choice when working with impulsive differential equations.

\begin{align}
PC([0, \tau]; Z_{1/2}) &= \{ z : J = [0, \tau] \to Z_{1/2} : z \in C(J; Z_{1/2}), \exists z(t_k^-, \cdot), z(t_k^+, \cdot) \}
\end{align}

and $z(t_k^-, \cdot) = z(t_k^+, \cdot)$, $J' = [0, \tau] \setminus \{ t_1, t_2, \ldots, t_p \}$ endowed with the norm

\begin{align}
\| z \|_0 &= \sup_{t \in [0, \tau]} \| z(t, \cdot) \|_{Z_{1/2}},
\end{align}
where \( z = (w, v)^T = (w, w_t)^T \)

\[
\|z\|_{Z_{1/2}} = \sqrt{\int_{\Omega} \left( \|(-\Delta)^{1/2} w\|^2 + \|v\|^2 \right) dx}, \quad \forall z \in Z_{1/2} = D((-\Delta)^{1/2}) \times L^2(\Omega).
\]

The interior approximate controllability of the following strongly damped wave equation without impulses and delay has been proved in [20] for all \( t > 0 \):

\[
\begin{cases}
    w_{tt} + \eta(-\Delta)^{1/2} w_t + \gamma(-\Delta) w = 1_w u(t, x), & \text{in } (0, \tau) \times \Omega, \\
    w = 0, & \text{in } (0, \tau) \times \partial\Omega, \\
    w(0, x) = w_0(x), \quad w_t(0, x) = w_1(x), & \text{in } \Omega.
\end{cases}
\]

\[\text{(6)}\]

### 2 Formulation of the Problem

Let \( X = L_2(\Omega) = L_2(\Omega, R) \) and consider the linear unbounded operator \( A : D(A) \subset X \rightarrow X \) defined by \( A\phi = -\Delta \phi \), where

\[
D(A) = H^2(\Omega, R) \cap H^1_0(\Omega, R).
\]

The fractional powered spaces \( X' \) (see details in [20]) are given by:

\[
X' = D(A') = \{ x \in X : \sum_{n=1}^{\infty} \lambda_n^2 \| E_n x \|^2 < \infty \}, \quad r \geq 0,
\]

with \( \| x \|_r = \| A' x \| = \left\{ \sum_{n=1}^{\infty} \lambda_n^{2r} \| E_n x \|^2 \right\}^{1/2} \), \( x \in X' \), and if we let \( Z_r = X' \times X \), the corresponding norm in this Hilbert space is

\[
\left\| \begin{bmatrix} w \\ v \end{bmatrix} \right\|_{Z_r} = \sqrt{\|w\|^2 + \|v\|^2}.
\]

**Proposition 2.1.** The operator \( P_j : Z_r \rightarrow Z_r, \quad j \geq 0 \), defined by

\[
P_j = \begin{bmatrix} E_j & 0 \\ 0 & E_j \end{bmatrix}, \quad j \geq 1,
\]

is a continuous(bounded) orthogonal projections in the Hilbert space \( Z_r \).

Hence, the equations (4) can be written as an abstract second order ordinary differential equations with impulses and delay as follows

\[
\begin{cases}
    w' + \eta A^{1/2} w' + \gamma A w = 1_w u(t, w(t-r), w'(t-r), u), & t \neq t_k, \\
    w(s) = \phi(s), \quad w'(s) = \psi(s), & s \in [-r, 0], \\
    w'(t_k) = w(t_k) + \int_{t_k}^t f_k(t, w(t), w'(t), u(t)), & k = 1, 2, 3, \ldots, p,
\end{cases}
\]

where for all \( x \in \Omega, k = 1, 2, \ldots, p, \quad I_k : [0, r] \times Z_{1/2} \times U \rightarrow X \) and \( f_k : [0, r] \times C(-r, 0; Z_{1/2}) \times U \rightarrow X \) are defined by

\[
I_k(t, w, v, u)(x) = I_k(t, w(x), v(x), u(x)), \quad f_k(t, \phi, \psi, u)(x) = f(t, \phi(-r, x), \psi(-r, x), u(x)).
\]

With the change of variables \( w' = v \), we can write the second order equation (9) as a first order system of ordinary differential equations with impulses and delay in the space \( Z_{1/2} = X^{1/2} \times X \) as follows:

\[
\begin{cases}
    z' = Az + B w u + F(t, z(t-r), u(t)), & z \in Z_{1/2}, \quad t \neq t_k, \\
    z(s) = \Phi(s), & s \in [-r, 0], \\
    z(t_k) = z(t_k) + \int_{t_k}^t I_k(t, z(t), u(t)), & k = 1, 2, 3, \ldots, p,
\end{cases}
\]

\[\text{(10)}\]
where $u \in L^2(0, \tau; U)$, $U = L^2(\Omega)$, $\Phi \in C(-r, 0; Z_{1/2})$

$$z = \begin{bmatrix} w \\ v \end{bmatrix}, \quad B_\omega = \begin{bmatrix} 0 \\ 1_\omega I \end{bmatrix}, \quad \Phi = \begin{bmatrix} \phi \\ \psi \end{bmatrix} \quad \text{and} \quad A = \begin{bmatrix} 0 & I_X \\ -\gamma A & -\eta A^{1/2} \end{bmatrix} \quad (11)$$

is an unbounded linear operator with domain $D(A) = D(A) \times D(A^{1/2})$ and $J_k : [0, \tau] \times Z_{1/2} \times U \to Z_{1/2}$ and $F : [0, \tau] \times C(-r, 0; Z_{1/2}) \times U \to Z_{1/2}$ are defined by:

$$F(t, \Phi, u) = \begin{bmatrix} f^\tau(t, \phi(-r), \psi(-r), u) \end{bmatrix} \quad \text{and} \quad J_k(t, z, u) = \begin{bmatrix} 0 \\ I^k(t, w, v, u) \end{bmatrix}. \quad (12)$$

From condition (5) and the continuous inclusion $X^{1/2} \subset X$ one can prove the following proposition

**Proposition 2.2.** The function $F$ defined above satisfies the following estimate

$$\|F(t, \Phi, u)\| \leq \hat{a}\|\Phi(-r)\|_{Z_{1/2}} + \hat{b}, \quad \forall (t, \Phi, u) \in [0, \tau] \times C(-r, 0; Z_{1/2}) \times U. \quad (13)$$

It is well known [8] that the operator $A$ generates a strongly continuous semigroup $\{T(t)\}_{t \geq 0}$ in the space $Z = Z_{1/2} = X^{1/2} \times X$, which is also analytic. Now, using Lemma 2.1 in [21], one can get the following representation for this semigroup.

**Proposition 2.3.** The semigroup $\{T(t)\}_{t \geq 0}$ generated by the operator $A$ is compact and has the following representation

$$T(t)z = \sum_{j=1}^\infty e^{At_j}P_jz, \quad z \in Z_{1/2}, \quad t \geq 0, \quad (14)$$

where $\{P_j\}_{j \geq 0}$ is a complete family of orthogonal projections in the Hilbert space $Z_{1/2}$ given by (8) and

$$A_j = R_jP_j, \quad R_j = \begin{bmatrix} 0 & 1 \\ -\gamma A_j & -\eta A_j^{1/2} \end{bmatrix}, \quad j \geq 1. \quad (15)$$

Moreover, $e^{A_jt} = e^{R_jt}P_j$, the eigenvalues of $R_j$ are:

$$\lambda_j = -\lambda_j^{1/2} \left( \eta z^2 - \frac{\lambda_j^2 - \lambda_j^{1/2}}{2} \right), \quad j = 1, 2, \ldots, \quad (16)$$

and

$$\lambda_j^+ = R_j^*P_j, \quad R_j^* = \begin{bmatrix} 0 & -1 \\ \gamma A_j & -\eta A_j^{1/2} \end{bmatrix}. \quad (17)$$

**Definition 2.1.** (Approximate Controllability) The system (4) is said to be approximately controllable on $[0, \tau]$ if for every $\Phi \in C$, $z_1 \in Z_{1/2}$, $\varepsilon > 0$ there exists $u \in L^2(0, \tau; U)$ such that the solution $z(t)$ of (4) corresponding to $u$ verifies:

$$z(0) = \Phi(0) \quad \text{and} \quad \|z(\tau) - z_1\|_{Z_{1/2}} < \varepsilon. \quad (18)$$

### 3 Controllability of the Linear Equation

In this section we present some characterization of the interior approximate controllability of the linear strongly damped wave equations without impulses and delay. To this end, we note that, for all $z_0 \in Z_{1/2}$ and $u \in L^2(0, \tau; U)$ the initial value problem

$$\begin{cases}
  z' = Az + B_\omega u, & z \in Z_{1/2}, \\
  z(t_0) = z_0,
\end{cases} \quad (19)$$

admits only one mild solution given by

\[ z(t) = T(t-t_0)z_0 + \int_{t_0}^{t} T(t-s)B_\omega u(s)ds, \quad t \in [t_0, t]. \]  (17)

**Definition 3.1.** For system (16) we define the following concept: The controllability map \( G_{t_\delta} : L^2(\tau - \delta, \tau; U) \to Z_{1/2} \) defined by

\[ G_{t_\delta}u = \int_{\tau-\delta}^{\tau} T(t-s)B_\omega u(s)ds, \quad u \in L^2(\tau - \delta, \tau; U). \]  (18)

The adjoint of this operator \( G_{t_\delta}^*: Z_{1/2} \to L^2(\tau - \delta, \tau; U) \) is given by

\[ (G_{t_\delta}^*z)(t) = B_\omega^* T^*(\tau - t)z, \quad t \in [\tau - \delta, \tau]. \]

The Gramian controllability operator is given by:

\[ Q_{t\delta} = G_{t\delta}G_{t\delta}^* = \int_{\tau-\delta}^{\tau} T(t-s)B_\omega B_\omega^* T^*(\tau - t)dt. \]  (19)

The following lemma holds in general for a linear bounded operator \( G : W \to Z_{1/2} \) between Hilbert spaces \( W \) and \( Z \) (see [1], [5], [10],[11] and [22]).

**Lemma 3.1.** The following statements are equivalent to the approximate controllability of the linear system (16) on \([\tau - \delta, \tau]\).

- a) \( \text{Rang}(G_{t\delta}) = Z_{1/2} \).
- b) \( \text{Ker}(G_{t\delta}^*) = \{0\} \).
- c) \( \langle Q_{t\delta}z, z \rangle > 0, z \neq 0 \) in \( Z_{1/2} \).
- d) \( \lim_{a \to 0} a(aI + Q_{t\delta})^{-1}z = 0 \).
- e) For all \( z \in Z \) we have \( G_{t\delta}u_a = z - a(aI + Q_{t\delta})^{-1}z \), where

\[ u_a = G_{t\delta}^*(aI + Q_{t\delta})^{-1}z, \quad a \in (0, 1]. \]

So, \( \lim_{a \to 0} G_{t\delta}u_a = z \) and the error \( E_{t\delta}z \) of this approximation is given by the formula

\[ E_{t\delta}z = a(aI + Q_{t\delta})^{-1}z, \quad a \in (0, 1]. \]

f) Moreover, if we consider for each \( v \in L^2(\tau - \delta, \tau; U) \) the sequence of controls given by

\[ u_a = G_{t\delta}^*(aI + Q_{t\delta})^{-1}z + (v - G_{t\delta}^*(aI + Q_{t\delta})^{-1}G_{t\delta}v), \quad a \in (0, 1], \]

we get that:

\[ G_{t\delta}u_a = z - a(aI + Q_{t\delta})^{-1}(z + G_{t\delta}v) \]

and

\[ \lim_{a \to 0} G_{t\delta}u_a = z, \]

with the error \( E_{t\delta}z \) of this approximation is given by the formula

\[ E_{t\delta}z = a(aI + Q_{t\delta})^{-1}(z + G_{t\delta}v), \quad a \in (0, 1]. \]

**Remark 3.1.** The foregoing Lemma implies that the family of linear operators \( \Gamma_{t\delta} : Z_{1/2} \to W \), defined for \( 0 < \alpha \leq 1 \) by

\[ \Gamma_{t\delta}z = G_{t\delta}^*(aI + Q_{t\delta})^{-1}z, \]  (20)
is an approximate inverse for the right of the operator $W$, in the sense that

$$\lim_{\alpha \to 0} G_{r,s}I_{r,s} = I. \quad (21)$$

in the strong topology.

**Lemma 3.2.** $Q_{r,s} > 0$ if, and only if, the linear system (16) is controllable on $[\tau - \delta, \tau]$. Moreover, given an initial state $y_0$ and a final state $z^1$, we can find a sequence of controls $\{u^\delta_a\}_{0 < \delta \leq 1} \subset L^2(\tau - \delta, \tau; U)$

$$u_a = u_{a,\delta} = G_{r,s}(\alpha I + G_{r,s}^{-1}(z^1 - T(\tau)y_0), \quad \alpha \in (0, 1],$$

such that the solutions $y(t) = y(t, \tau - \delta, y_0, u_a)$ of the initial value problem

$$\begin{cases}
y' = Ay + Bu(t), \quad y \in Z_1/2, \quad t > 0, \\
y(\tau - \delta) = y_0,
\end{cases} \quad (22)$$

satisfies

$$\lim_{\alpha \to 0^+} y(\tau, \tau - \delta, y_0, u_a) = z^1.$$ e.i.,

$$\lim_{\alpha \to 0^+} y(\tau) = \lim_{\alpha \to 0^+} \left\{ T(\delta)y_0 + \int_{\tau - \delta}^{\tau} T(\tau - s)Bu_a(s)ds \right\} = z^1.$$

## 4 Controllability of the Semilinear System with Impulses and Delay

In this section we prove the main result of this paper: the interior approximate controllability of the semilinear strongly damped wave equation with impulses and delay (4), which is equivalent to prove the approximate controllability of the system (9). To this end, for all $\Phi \in C(-r, 0; Z_1/2)$ and $u \in L^2(0, \tau; U)$ the initial value problem

$$\begin{cases}
z' = Az + Bzw + F(t, z(t - \tau), u(t)) \quad z \in Z_1/2, \quad t \in [0, \tau], \quad t \neq t_k, \\
z(s) = \Phi(s), \quad s \in [-r, 0], \\
z(t_k) = z(t_{k-1}^+ + J_k(t_k, z(t_k), u(t_k))), \quad k = 1, 2, 3, \ldots, p,
\end{cases} \quad (23)$$

admits only one mild solution given by

$$\begin{align*}
z_u(t) &= T(t)\Phi(0) + \int_0^t T(t - s)Bzwu(s)ds + \int_0^t T(t - s)F(s, z_u(s), u(s))ds \\
&\quad + \sum_{0 < t_k < t} T(t - t_k)J_k(t_k, z(t_k), u(t_k)), \quad t \in [0, \tau], \\
z(s) &= \Phi(s), \quad s \in [-r, 0].
\end{align*} \quad (24, 25)$$

**Theorem 4.1.** If the functions $f, J_k$ are smooth enough and satisfy the hypothesis (5) the system (4) is approximately controllable on $[0, \tau]$, for all $\tau > 0$.

**Proof** Given $\Phi \in C$, a final state $z^1$ and $\epsilon > 0$, we want to find a control $u^\delta_a \in L^2(0, \tau; U)$ steering the system from $\Phi(0)$ to an $\epsilon$-neighborhood of $z^1$ on time $\tau$. Precisely, for $\alpha > 0$ and $0 < \delta < \min\{\tau - t_p, \tau\}$ small enough, there exists control $u^\delta_a \in L^2(0, \tau; U)$ such that corresponding of solutions $z^\delta_a$ of (4) satisfies:

$$\|z^\delta_a(\tau) - z^1\| \leq \epsilon.$$

In fact, consider any $u \in L^2(0, \tau; U)$ and the corresponding solution $z(t) = z(t, 0, \Phi, u)$ of the initial value problem (23). For $\alpha \in (0, 1]$ we define the control $u^\delta_a \in L^2(0, \tau; U)$ as follows

$$u^\delta_a(t) = \begin{cases} u(t), & \text{if } 0 \leq t \leq \tau - \delta, \\
u_a(t), & \text{if } \tau - \delta < t \leq \tau. \end{cases}$$
where
\[ u_a(t) = B_\omega^* T^*(\tau - t)(\alpha I + G_{\tau e} G_{\tau e})^{-1}(z_{1} - T(\delta)z(\tau - \delta)), \quad \tau - \delta < t \leq \tau. \]

Since \( 0 < \delta < \tau - t_0 \), then \( \tau - \delta > t_0 \) and the corresponding solution \( z^{\delta, a}(t) = z(t, 0, \phi, u_0^a) \) of the initial value problem (23) at time \( \tau \) can be written as follows:

\[
z^{\delta, a}(\tau) = T(\tau)\Phi(0) + \int_0^\tau T(\tau - s)B_\omega u_0^a(s)ds + \int_0^\tau T(\tau - s)f(s, z^{\delta, a}(s - r), u_0^a(s))ds
\]

\[
+ \sum_{0 \leq t_k < \tau} T(\tau - t_k)I_k^\delta(t_k, z(t_k), u_0^a(t_k))
\]

\[
= T(\delta) \left\{ T(\tau - \delta)\Phi(0) + \int_0^{\tau - \delta} T(\tau - \delta - s)B_\omega u_0^a(s)ds + \int_0^{\tau - \delta} T(\tau - \delta - s)F(s, z^{\delta, a}(s - r), u_0^a(s))ds
\]

\[
+ \sum_{0 \leq t_k < \tau - \delta} T(\tau - \delta - t_k)I_k^\delta(z^{\delta, a}(t_k), u_0^a(t_k)) \right\}
\]

\[
+ \int_{\tau - \delta}^\tau T(\tau - s)B_\omega u_0^a(s)ds
\]

\[
+ \int_{\tau - \delta}^\tau T(\tau - s)F(s, z^{\delta, a}(s - r), u_0^a(s))ds
\]

\[
= T(\delta)z(\tau - \delta) + \int_{\tau - \delta}^\tau T(\tau - s)B_\omega u_a(s)ds + \int_{\tau - \delta}^\tau T(\tau - s)F(s, z^{\delta, a}(s - r), u_a(s))ds.
\]

So,
\[
z^{\delta, a}(\tau) = T(\delta)z(\tau - \delta) + \int_{\tau - \delta}^\tau T(\tau - s)B_\omega u_a(s)ds + \int_{\tau - \delta}^\tau T(\tau - s)F(s, z^{\delta, a}(s - r), u_a(s))ds.
\]

The corresponding solution \( y^{\delta, a}(t) = y(t, \tau - \delta, z(\tau - \delta), u_a) \) of the initial value problem (22) at time \( \tau \) is given by:
\[
y^{\delta, a}(\tau) = T(\delta)z(\tau - \delta) + \int_{\tau - \delta}^\tau T(\tau - s)B_\omega u_a(s)ds.
\]

On the other hand, from Lemma 3.2 there exists \( a > 0 \) such that
\[ \|y_\delta^a(\tau) - z_1^1\| \leq \frac{\epsilon}{2}. \]

Therefore,
\[ \|z^{\delta, a}(\tau) - y^{\delta, a}(\tau)\| \leq \int_{\tau - \delta}^\tau \|T(\tau - s)\|\|F(s, z^{\delta, a}(s - r), u_0^a(s))\|ds. \]

Now, since \( 0 < \delta < r \) and \( \tau - \delta \leq s \leq \tau \), then \( s - r \leq \tau - r < \tau - \delta \) and
\[ z^{\delta, a}(s - r) = z(s - r). \]

Therefore, there exists \( \delta \) small enough such that \( 0 < \delta < \min\{r, \tau - t_0\} \) and
\[
\|z^{\delta, a}(\tau) - y^{\delta, a}(\tau)\| \leq \int_{\tau - \delta}^\tau \|T(\tau - s)\|\|F(s, z(s - r), u_0^a(s))\|ds \leq \int_{\tau - \delta}^\tau \|T(\tau - s)\|\{\tilde{a}\|z(s - r)\|_{Z_{opt}} + \tilde{b}\}ds < \frac{\epsilon}{2}.
\]
Hence,
\[
\|z^{\delta,a}(\tau) - z^1\| \leq \int_{\tau-\delta}^{\tau} \|T(t-s)\|\|F(s, z(s-r), u^\delta(s))\|ds + \|y^{\delta,a}(\tau) - z^1\|
\]
\[
< \frac{\epsilon}{2} + \frac{\epsilon}{2} < \epsilon.
\]

Geometrically, the proof goes as follows:

This completes the proof of the Theorem.

\[\square\]

5 Final Remarks

Our technique can be applied to those system involving compact semigroups like some control system governed by diffusion processes.

Moreover, Our result can be formulated in a more general setting. Indeed, we can consider the following semilinear evolution equation with impulses and delay in a general Hilbert space \(Z\)

\[
\begin{cases}
  z' = Az + Bu + F(t, z(t-r), u(t)) & z \in Z_{1/2}, \ t \in (0, \tau), \ t \neq t_k, \\
  z(s) = \Phi(s), & s \in [-r, 0] \\
  z(t_k^+) = z(t_k^-) + J_k(t_k, z(t_k), u(t_k)), & k = 1, 2, 3, \ldots, p,
\end{cases}
\]  

(26)

where \(u \in L^2(0, \tau; U)\), \(U\) is another Hilbert space, \(B : U \rightarrow Z_{1/2}\) is a bounded linear operator, \(F : [0, \tau] \times C(-r, 0; Z_{1/2}) \times U \rightarrow Z\), \(A : D(A) \subset Z \rightarrow Z\) is an unbounded linear operator in \(Z\) that generates a strongly continuous semigroup according to Lemma 2.1 from [21]:

\[
T(t)z = \sum_{n=1}^{\infty} e^{At_n}P_nz, \ z \in Z_{1/2}, \ t \geq 0,
\]  

(27)

where \(\{P_n\}_{n=0}\) is a complete family of orthogonal projections in the Hilbert space \(Z\) and

\[
\|F(t, \Phi, u)\|_Z \leq \tilde{a}\|\Phi(-r)\|_{Z_{1/2}} + \tilde{b}, \ \forall (t, \Phi, u) \in [0, \tau] \times C(-r, 0; Z_{1/2}) \times U.
\]  

(28)

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