Multivariate copula and co-copula on BL-algebra

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Abstract

This paper studies the functions of copula on BL-algebra on n-dimension. Our main idea for this study focuses on linking some statistical concepts with BL-algebra. We have proposed several concepts such as some definitions, theories, characteristics of these functions and test their terms with respect to algebra. In fact, our work focuses on determining the type of generalization of each classical copula type in such a forced system. The conditions of copula, co-copula and its related properties have been demonstrated. In the end, there are many different examples that have been shown on the definitions of each type of copula we have built on BL-algebra.

Keywords: BL-algebra, copula, co-copula, orthogonal, state.
1 Introduction

(Basic logic -algebra) BL-algebras have been presented by Hájek in 1990 in order to construct an algebra system depending on the theory of basic logic’s complements ,see [14].

Afterwards, Turunen in the 1999 has studied some essential concepts of fuzzy logic with respect to BL-algebras. He has employed them to study some deductive systems, see [26]. While, in 2000, Cignoli has shown that the logic of continuous t-norms and Simultaneously, he has presented a very useful and interest concept of a systematic study on BL-algebra , see [5]. Once again, in 2001, Turunen has demonstrated a new study that combine implicative deductive systems to the Boolean algebra systems of deductive,see [13].

Furthermore, in 2006, Haveshki has proved and explained a topic related to the analysis of BL-algebras systems. when, he has presented some type of algebraic filters. As a result of his study he has shown that the filters and deductive systems are not the same , see [13].

On other hand and in the statistical world, copula function represents one of the most interesting modern topics in the last five decades. It is one of the well-known statistical tools that has been implemented in statistical studies and data analysing. It has a main role in the description of dependence structures between two or more of random variables. The constructions of copula have firstly been appeared implicitly in many statistical studies in different fields and specially in finance and biology. It has a very wide usage in finance studies specially the study of portfolio, risk management, loans, and insurance [14, 16]. It has also been implemented to the biological studies of genes and biomedical studies, [21]. Indeed, there are many other applications that have been used copulas and their concepts in their structures such as in mechanics, physics and quantum logics.

Consequently, many authors have participated their ideas and concepts to copula function. But one of the main results that have a wide impact to copula function is Nelsen (2006). His book entitled "An Introduction to Copulas" represents the major reference to many students, readers and researchers who are interested in copulas and its application [22]. Moreover, there are several applications that have been involved copulas in their structures and analysing of their properties.
2 Basic Concepts

Main concepts that we need to present in this section involved definitions, proposition, properties, and etc. that represent the basic of our constructions on BL-algebra. We firstly begin with basic definition of BL-algebra.

Definition 2.1. [13] An algebra $\mathcal{A} = (\mathcal{A}, \wedge, \vee, \odot, \to, O, I)$ of type $(2, 2, 2, 2, 0, 1)$ is a BL-algebra if the following conditions are satisfied:

1. $(\mathcal{A}, \vee, \wedge, O, I)$ is a bounded lattice;
2. $(\mathcal{A}, \odot, I)$ is a commutative monoid;
3. $u \leq s \to t \iff s \odot u \leq t$;
4. $s \land t = s \odot (s \to t)$;
5. $(s \to t) \lor (t \to s) = I$. for all $s, t, u \in \mathcal{A}$ and consider $s^* = s \to O$.

Proposition 2.1. [5, 8, 9, 24] Let $\mathcal{A}$ be a BL-algebra. For all $s, t, u \in \mathcal{A}$ the following properties are satisfy:

1. $s \odot (s \to t) \leq t$;
2. $s \leq t \to (s \odot t)$;
3. $s \leq t \iff s \to t = I$;
4. $s \to (t \to u) = s \odot t \to u = t \to (s \to u)$;
5. $s \leq t \implies u \to s \leq u \to t$;
6. $t \leq (t \to s) \to s$;
7. $(s \to t) \odot (s \to u) \leq s \to u$;
8. $t \to s \leq (u \to t) \to (u \to s)$;
9. $s \to t \leq (t \to u) \to (s \to u)$;
10. $s \lor t = [(s \to t) \to t] \land [(t \to s) \to s]$;
11. \( s \leq t \implies t^* \leq s^* \);

12. \( I \rightarrow s = s, s \rightarrow s = I, s \rightarrow I = I \);

13. \( s \leq t \rightarrow s \) or equivalently \( s \rightarrow (t \rightarrow s) = I \);

14. \( ((s \rightarrow t) \rightarrow t = s \rightarrow t) \);

15. \( I^* = O, O^* = I \);

16. \( I^{**} = I, O^{**} = O \) that is \( O \) and \( I \) are involution;

17. \( (s \vee t)^* = s^* \land t^*, (s \land t)^* = s^* \lor t^* \);

18. \( s \vee s^* = I, s \land s^* = O \);
Example 2.1. [15] Let $A = \{o,s,t,I\}$. Define $\odot,\rightarrow,\lor,\land$ as follows:

$$
\begin{array}{cccc}
\odot & O & s & t & I \\
O & O & O & O & O \\
s & O & s & O & s \\
t & O & O & t & t \\
I & O & s & t & I \\
\end{array}
$$

$$
\begin{array}{cccc}
\rightarrow & O & s & t & I \\
O & I & I & I & I \\
s & t & I & t & I \\
t & s & s & I & I \\
I & O & s & t & I \\
\end{array}
$$

$$
\begin{array}{cccc}
\lor & O & s & t & I \\
O & O & s & t & I \\
s & s & s & I & I \\
t & t & t & I & I \\
I & I & I & I & I \\
\end{array}
$$

$$
\begin{array}{cccc}
\land & O & s & t & I \\
O & O & O & O & O \\
s & O & s & O & s \\
t & O & O & t & t \\
I & O & s & t & I \\
\end{array}
$$

$$
\begin{array}{cccc}
\ast & O & s & t & I \\
I & t & s & O \\
\end{array}
$$

it is clear that $A$ is BL-algebra.

Definition 2.2. [23] Let $A$ be a BL-algebra. Two elements $s,t \in A$ are said to be orthogonal and denoted by $s \perp t$, if $s \leq t^*$.

Remark 2.1. [23] Simply, it is easy to show that $s \perp t$ if and only if $s \leq t^*$ and $s \odot t = O$. Also, it is clear that $s \perp t$ if and only if $t \perp s$ and $s \perp O$ for each $s \in A$. 
Definition 2.3. Let $A$ be a BL-algebra. A function $\delta : A \to [0, 1]$ is called state if the following conditions are hold:

1. $\delta(I) = 1$;

2. If $s \perp t$ then $\delta(s \vee t) = \delta(s) + \delta(t)$.

It is well-known that $\delta(O) = 0$, $\forall s, t \in A$.

Proposition 2.2. Let $\delta$ be a state on BL-algebra. If $s \leq t$ and $s \perp t$ then $\delta(s) \leq \delta(t)$

proof:

Since $s \leq t$ so $t = s \vee t \Rightarrow \delta(t) = \delta(s \vee t)$

since $\delta$ is a state $\Rightarrow \delta(t) = \delta(s) + \delta(t)$

which implies that $\delta(t) - \delta(s) = \delta(t) \geq 0$

$\Rightarrow \delta(t) - \delta(s) \geq 0 \Rightarrow \delta(t) \geq \delta(s)$

Example 2.2. From (Example 2.1), A will be state by the following table.

| $\delta(.)$ | $O$ | $s$ | $t$ | $I$ |
|-------------|-----|-----|-----|-----|
| .           | 0.4 | 0.6 | 1   |

it is clear that $\delta$ is a state

Definition 2.4. [22] A bivariate copula is a function $C : [0, 1][0, 1] \to [0, 1]$ that satisfies the following conditions:

1. For each $s \in [0, 1]$, $C(s, 0) = C(0, s) = 0$;

2. For each $s \in [0, 1]$, $C(1, s) = s, C(s, 1) = s$;

3. If $s_1 \leq s_2$ and $t_1 \leq t_2$ then $C(s_1, t_1) + C(s_2, t_2) \geq C(s_1, t_2) + C(s_2, t_1)$ which is called $2 -$ increasing.

Definition 2.5. [22] A bivariate co-copula is a function $C^* : [0, 1][0, 1] \to [0, 1]$ with the following conditions:

1. For each $s \in [0, 1]$, $C^*(s, 0) = s, C^*(0, s) = s$;

2. For each $s \in [0, 1]$, $C^*(1, s) = 1, C^*(s, 1) = 1$;
3. If \( s_1 \leq s_2 \) and \( t_1 \leq t_2 \) then \( C(s_1, t_1) + C(s_2, t_2) \leq C(s_1, t_2) + C(s_2, t_1) \).

Example 2.3. \([22]\) \( M(s, t) = \min(s, t) \) is copula functions and \( W(s, t) = \max(s + t - 1, 0) \) is co-copula.

Definition 2.6. \([22]\) Let \( n \geq 2 \). A multivariate copula is a function \( C : [0, 1]^n \to [0, 1] \) that satisfies the following conditions:

1. \( C(s_1, \ldots, s_i, \ldots, s_n) = 0 \) if there exists \( s_i = 0, i = 1, \ldots, n \);
2. \( C(1, \ldots, 1, s_1, 1, \ldots, 1) = s_i, s_i \in [0, 1] \);
3. If \( (s_1, \ldots, s_n), (t_1, \ldots, t_n) \in [0, 1]^n \) such that \( s_i \leq t_i, i = 1, \ldots, n \) then
   \[
   \sum_{k_1=1}^{n} \cdots \sum_{k_n=1}^{n} (-1)^{k_1 + \cdots + k_n} C(w_{1k_1}, \ldots, w_{nk_n}) \geq 0
   \]
   where \( w_{j1} = s_j, w_{j2} = t_j, j = 1, \ldots, n \).

Definition 2.7. \([22]\) Let \( n \geq 2 \). A multivariate co-copula is a function \( C^* : [0, 1]^n \to [0, 1] \) that satisfies the following conditions:

1. \( C^*(s_1, \ldots, s_i, \ldots, s_n) = s_i \) if there exists \( s_i \in [0, 1], s_i \neq 0, i = 1, \ldots, n \);
2. \( C^*(s_1, s_2, \ldots, s_i, \ldots, s_n) = 1 \), \( s_i = 1 \);
3. If \( (s_1, \ldots, s_n), (t_1, \ldots, t_n) \in [0, 1]^n \) such that for any \( s_i \leq t_i, i = 1, \ldots, n \) then
   \[
   \sum_{k_1=1}^{n} \cdots \sum_{k_n=1}^{n} (-1)^{k_1 + \cdots + k_n+1} C^*(w_{1k_1}, \ldots, w_{nk_n}) \geq 0 \]
   where, \( w_{j1} = s_j, w_{j2} = t_j, j = 1, \ldots, n \).

Definition 2.8. Let \( A \) be a BL-algebra. A bivariate BL-copula is a function \( BC : A^2 \to [0, 1] \) that satisfies the following conditions:

1. For each \( s \in A \), \( BC(s, O) = BC(O, s) = 0 \);
2. \( BC(I, \cdot), BC(\cdot, I) \) are states on \( A \);
3. For each \( s_1, s_2, t_1, t_2 \in A \), if \( s_1 \leq s_2, t_1 \leq t_2 \), then \( BC(s_1, t_1) + BC(s_2, t_2) \geq BC(s_1, t_2) + BC(s_2, t_1) \).
Definition 2.9. Let $A$ be a BL-algebra. A bivariate BL-co-copula is a function $BH : AA \to [0, 1]$ that satisfies the following conditions:

1. $BH(O, .), BH(., O)$ are states on $A$;
2. For each $s \in A$ $BH(s, I) = BH(I, s) = 1$;
3. If $s_1 \leq s_2$ and $t_1 \leq t_2$, where $s_1, s_2, t_1, t_2 \in A$, then $BH(s_1, t_1) + BH(s_2, t_2) \leq BH(s_1, t_2) + BH(s_2, t_1)$.

Remark 2.2. In [22], Nelsen, has proved that each copula is positive in $[0, 1]^2$, and according to this notion BL-copulas are also positive on the cross product of $A$.

3 Multivariate copula on BL-algebra

In this section, the extension of bivariate function on BL-algebra is generalized. Start with multivariate copula and co-copula functions on BL-algebra.

Definition 3.1. Let $A$ be a BL-algebra and $n \geq 2$. A $n$-dimensional BL-copula is a function $BC : A^n \to [0, 1]$ that satisfies the following properties:

1. $BC(s_1, ..., s_i, ..., s_n) = 0$ if there exists $s_i \in A$ such that $s_i = 0$, for $i = 1, ..., n, (s_i \in A)$.
2. $BC(., I, ..., I), BC(I, ., I, ..., I), BC(I, ..., I, .)$ are states on $A$.
3. if $(s_1, ..., s_n), (t_1, ..., t_n) \in A^n$ such that for any $s_i \leq t_i$, $i = 1, ..., n$ then
   \[
   \sum_{k_1=1}^{n} \cdots \sum_{k_n=1}^{n} (-1)^{(k_1+\cdots+k_n)} BC(u_{1k_1}, ..., u_{nk_n}) \geq 0
   \]
   where $u_{j1} = s_j, u_{j2} = t_j$, $j = 1, ..., n$. for instance if $n = 2$ then the inequality of a two-dimensional copula is satisfied:
   \[
   BC(u_{11}, u_{21}) - BC(u_{12}, u_{21}) - BC(u_{11}, u_{22}) + BC(u_{12}, u_{22}) \geq 0
   \]

Proposition 3.1. Let $BC_1, BC_2$ be BL-copulas on a $A$-algebra $A$. If, for all $k \in [0, 1]$ $BC(s_1, ..., s_n) = kBC_1(s_1, ..., s_n) + (1 - k)BC_2(s_1, ..., s_n)$, then $BC$ is a BL-copula.

Proof:
It is given that $BC_1, BC_2$ are BL-copulas. Then, we show that $BC$ with respect to equation satisfies the conditions of BL-copula. So
1. \( BC(O, s_2, \ldots, s_n) = kB_C(O, s_2, \ldots, s_n) + (k-1)BC_2(O, s_2, \ldots, s_n) = 0 = BC(s_1, O, s_3, \ldots, s_n) = \sum_{k=1}^{n} BC(s_1, s_3, s_4, \ldots, s_n) = \ldots = BC(s_1, \ldots, s_n - 1, O) \)

2. To prove \( BC(I, \ldots, I, I) \) is a state:
   \[ a. BC(I, \ldots, I, I)(I) = BC(I, \ldots, I, I) = kB_C(I, I, \ldots, I) + (1-k)BC_2(I, I, \ldots, I) \]
   \[ = k + 1 - k = 1; \]
   \[ b. Let s \perp t. Then \]
   \[ BC(I, \ldots, I, s)(s \perp t) = BC(I, \ldots, I, s \perp t) = kB_C(I, \ldots, I, s \perp t) + (1-k)BC_2(I, \ldots, I, s \perp t) \]
   \[ = kBC_1(I, \ldots, I, s) + kBC_1(I, t) + (1-k)BC_2(I, \ldots, I, s) + (1-k)BC_2(I, \ldots, I, t) \]
   \[ = BC(I, \ldots, I, s) + BC(I, \ldots, I, t) \]
   Thus \( BC(I, I, \ldots, I) \) is state.
   Similarly \( BC(., I, \ldots, I), BC(., I, \ldots, I), \ldots, BC(., I, \ldots, I) \) are also states.

3. Let \( (s_1, \ldots, s_n), (t_1, \ldots, t_n) \in A^n \) such that for any \( i = 1, \ldots, n, s_i \leq t_i. \) Then we should prove the increasing property that means:
   \[ \sum_{k=1}^{n} \sum_{k_n=1}^{n} (-1)^{k_1+\ldots+k_n} BC(u_{1k_1}, \ldots, u_{nk_n}) \geq 0. \]
   \[ = k[BC_1(t_1, \ldots, t_n) - BC_1(s_1, \ldots, s_n)] + (1-k)[BC_2(t_1, \ldots, t_n) - BC_2(s_1, \ldots, s_n)] \]
   \[ = k \{ \sum_{k=1}^{n} \sum_{k_n=1}^{n} (-1)^{k_1+\ldots+k_n} BC_1(u_{1k_1}, \ldots, u_{nk_n}) \} \]
   \[ + k-1 \{ \sum_{k=1}^{n} \sum_{k_n=1}^{n} (-1)^{k_1+\ldots+k_n} BC_2(u_{1k_1}, \ldots, u_{nk_n}) \} \]
   since \( BC_1, BC_2 \) are BL-copula, then
   \[ k \{ \sum_{k=1}^{n} \sum_{k_n=1}^{n} (-1)^{k_1+\ldots+k_n} BC_1(u_{1k_1}, \ldots, u_{nk_n}) \} \]
   \[ + k-1 \{ \sum_{k=1}^{n} \sum_{k_n=1}^{n} (-1)^{k_1+\ldots+k_n} BC_2(u_{1k_1}, \ldots, u_{nk_n}) \} \]
   \[ = k \{ \sum_{k=1}^{n} \sum_{k_n=1}^{n} (-1)^{k_1+\ldots+k_n} BC_1(u_{1k_1}, \ldots, u_{nk_n}) \} \]
   \[ + k-1 \{ \sum_{k=1}^{n} \sum_{k_n=1}^{n} (-1)^{k_1+\ldots+k_n} BC_2(u_{1k_1}, \ldots, u_{nk_n}) \} \]
   \[ \geq 0 \]
   Hence \( \sum_{k=1}^{n} \sum_{k_n=1}^{n} (-1)^{k_1+\ldots+k_n} BC(u_{1k_1}, \ldots, u_{nk_n}) \geq 0. \)
4 Multivariate co-copula on BL-algebra

Definition 4.1. Let $\mathcal{A}$ be a BL-algebra and $n \geq 2$. A $n$-dimensional BL-co-copula is a function $BH : \mathcal{A}^n \to [0,1]$ that satisfies the following properties:

1. $BH(s_1, ..., s_i, ..., s_n) = 1$ if there is $s_i = I, i \in 1, ..., n$.

2. $BH(., O, ., O), BH(., ., O, O), BH(., O, ., .)$ are states on $\mathcal{A}$.

3. If $(s_1, ..., s_n), (t_1, ..., t_n) \in \mathcal{A}^n$ such that for any $s_i \leq t_i$, $i = 1, ..., n$ then
   \[
   \sum_{k_1=1}^{n} \sum_{k_{n}=1}^{n} (-1)^{k_1 + \ldots + k_{n}+1} BH(u_{k_1}, \ldots, u_{k_n}) \geq 0
   \]
   where $u_{j_1} = s_j, u_{j_2} = t_j, j = 1, ..., n$.

For instance if $n = 2$ then we can see that the inequality of a two-dimensional BL-co-copula property is satisfied
\[
BH(u_{11}, u_{21}) - BH(u_{12}, u_{21}) - BH(u_{11}, u_{22}) + BH(u_{12}, u_{22}) \geq 0.
\]
\[
BH(s_1, t_2) + BH(s_2, t_1) - BH(s_1, s_2) - BH(t_1, t_2) \geq 0.
\]

Example 4.1. Consider a three dimensional function
\[
BH(s, t, u) = \max(\delta(s), \delta(t), \delta(u))
\]
such that $\delta$ is a state. then $BH$ is a BL-copula.

1. $BH(s, t, I) = \max[\delta(s), \delta(t), \delta(I)] = 1$

2. To prove $BH(., O, .), BH(., ., O), BH(., ., .)$ are states
   a. $BH(., O, .)(I) = BH(O, O, I) = \max[\delta(O), \delta(O), \delta(I)] = 1$
   \[
   = BH(., O, O) = BH(., ., O)
   \]
   b. Let $s, t \in \mathcal{A} \ni s \perp t$ then
   \[
   BH(O, O, .)(s + t) = BH(O, O, s + t) = \max[\delta(O), \delta(O), \delta(s + t)]
   \]
   \[
   = \max[\delta(O), \delta(O), \delta(s) + \delta(t)]
   \]
   \[
   = \max[\delta(O), \delta(O), \delta(s)] + \max[\delta(O), \delta(O), \delta(t)]
   \]
   \[
   = BH(O, O, s) + BH(O, O, t)
   \]

3. $s_1 \leq t_1, s_2 \leq t_2$ and $s_3 \leq t_3$
   \[
   -BH(s_1, s_2, t_3) - BH(s_1, t_2, s_3) - BH(t_1, s_2, s_3) - BH(t_1, t_2, t_3) + BH(s_1, s_2, s_3) + BH(s_1, t_2, t_3) + BH(t_1, s_2, t_3) + BH(t_1, t_2, s_3) \geq 0.
   \]
   Substitute each term with the general form of the considered function:
   
   $BH(s_1, s_2, t_3) = \max[\delta(s_1), \delta(s_2), \delta(t_3)]$
\[ BH(s_1, t_2, s_3) = \max[\delta(s_1), \delta(t_2), \delta(s_3)] \]
\[ BH(t_1, s_2, s_3) = \max[\delta(t_1), \delta(s_2), \delta(s_3)] \]
\[ BH(t_1, t_2, s_3) = \max[\delta(t_1), \delta(t_2), \delta(s_3)] \]
\[ BH(s_1, s_2, s_3) = \max[\delta(s_1), \delta(s_2), \delta(s_3)] \]
\[ BH(s_1, t_2, t_3) = \max[\delta(s_1), \delta(t_2), \delta(t_3)] \]
\[ BH(t_1, s_2, t_3) = \max[\delta(t_1), \delta(s_2), \delta(t_3)] \]
\[-BH(s_1, s_2, t_3) - BH(t_1, t_2, s_3) - BH(t_1, t_2, t_3) + BH(s_1, s_2, s_3) + BH(s_1, t_2, t_3) + BH(t_1, s_2, t_3) + BH(t_1, t_2, s_3) \]

According to (Remark 2.2), we also obtain that
\[ -\max[\delta(s_1), \delta(s_2), \delta(t_3)] - \max[\delta(s_1), \delta(t_2), \delta(s_3)] - \max[\delta(t_1), \delta(s_2), \delta(s_3)] \]
\[ -\max[\delta(t_1), \delta(t_2), \delta(t_3)] + \max[\delta(s_1), \delta(s_2), \delta(s_3)] + \max[\delta(s_1), \delta(t_2), \delta(t_3)] \]
\[ +\max[\delta(t_1), \delta(s_2), \delta(t_3)] + \max[\delta(t_1), \delta(t_2), \delta(t_3)] \geq 0. \]

**Proposition 4.1.** Let \( BH_1, BH_2 \) be BL-co-copulas on a BL-algebra \( A \). Then for all \( k \in [0, 1] \)
\[ BH(s_1, ..., s_n) = kBH_1(s_1, ..., s_n) + (1 - k)BH_2(s_1, ..., s_n) \] is also a BL-co-copula.

**Proof:**

Since \( BH_1, BH_2 \) are BL-co-copulas. So we prove that BH is also BL-co-copula:

1. \( BH(I, s_2, ..., s_n) = kBH_1(I, s_2, ..., s_n) + (k-1)BH_2(I, s_2, ..., s_n) = 1 = BH(s_1, I, s_3, ..., s_n) \)
   \[ BH(s_1, s_2, I, s_4, ..., s_n) = ... = BC(s_1, s_2, s_3, s_4, ...I) \]

2. To show the second condition of BL-copula, we should prove that BC has the properties of being state. Then
   a. \( BH(O, ..., O,).(I) = BH(O, ..., O, I) = kBH_1(O, ..., O, I) + (1 - k)BH_2(O, ..., O, I) \)
   \[ = k + 1 - k = 1; \]
   b. If \( s \perp t \), then
   \[ BH(O, ..., O,)(s\vee t) = BH(O, ..., O, s\vee t) = kBH_1(O, ..., O, s\vee t) + (1 - k)BH_2(O, ..., O, s\vee t) \]
   \[ = kBH_1(O, ..., O, s) + kBH_1(O, ..., O, t) + (1 - k)BH_2(O, ..., O, s) + (1 - k)BH_2(O, ..., O, t) \]
   \[ = kBH_1(O, ..., O, s) + (1 - k)BH_2(O, ..., O, s) + kBH_1(O, ..., O, t) + (1 - k)BH_2(O, ..., O, t) \]
   \[ = BH(O, ..., O, s) + BC(O, ..., O, t) \]
   Thus \( BC(O, ..., O, .) \) is state.
   Similarly \( BC(., O, ..., O), BC(O, ., O, ..., O), ..., BC(O, ..., , O) \) are also states.
3. If \((s_1, \ldots, s_n), (t_1, \ldots, t_n) \in A^n\) such that for any \(s_i \leq t_i\), \(i = 1, \ldots, n\) then
\[
\sum_{k_1=1}^{n} \cdots \sum_{k_n=1}^{n} (1 - \ell_{k_1+\ldots+k_n}) BH(u_{1k_1}, \ldots, u_{nk_n})
\]
\[
= \sum_{k_1=1}^{n} \cdots \sum_{k_n=1}^{n} (1 - \ell_{k_1+\ldots+k_n}) [kBH_1(u_{1k_1}, \ldots, u_{nk_n}) + (1 - k)BH_2(u_{1k_1}, \ldots, u_{nk_n})]
\]
\[
= \sum_{k_1=1}^{n} \cdots \sum_{k_n=1}^{n} (1 - \ell_{k_1+\ldots+k_n}) kBH_1(u_{1k_1}, \ldots, u_{nk_n})
+ \sum_{k_1=1}^{n} \cdots \sum_{k_n=1}^{n} (1 - \ell_{k_1+\ldots+k_n}) (1 - k)BH_2(u_{1k_1}, \ldots, u_{nk_n})
\]
\[
= k\sum_{k_1=1}^{n} \cdots \sum_{k_n=1}^{n} (1 - \ell_{k_1+\ldots+k_n}) BH_1(u_{1k_1}, \ldots, u_{nk_n})
+ (1 - k)\sum_{k_1=1}^{n} \cdots \sum_{k_n=1}^{n} (1 - \ell_{k_1+\ldots+k_n}) BH_2(u_{1k_1}, \ldots, u_{nk_n})
\]
\[
\text{since } BH_1, BH_2 \text{ are BL-co-copula and } k \in [0, 1], \text{ so}
\sum_{k_1=1}^{n} \cdots \sum_{k_n=1}^{n} (1 - \ell_{k_1+\ldots+k_n}) BH(u_{1k_1}, \ldots, u_{nk_n}) \geq 0.
\]

**Proposition 4.2.** Let \(A\) be a BL-algebra. If \(BC\) is a BL-copula then a function \(BH : A^n \to [0, 1]\) is a BL-co-copula for all \(s_1, s_2, \ldots, s_n \in A\) and \(BH(s_1, \ldots, s_n) = 1 - BC(s_1^*, s_2^*, \ldots, s_n^*)\).

**Proof:**

1. Let \(s_i \in A\) for all \(i = 1, \ldots, n\) then \(BH(I, s_2, \ldots, s_n) = 1 - BC(O, s_2^*, \ldots, s_n^*) = 1\).

2. To prove that \(BH(\ldots, O, \ldots, O), BH(O, \ldots, O)\) are states on \(A\):
   a. \(BH(O, \ldots, O) = 1 - BC(I, \ldots, I) = 1 - 1 = 0\).
   b. \(s_1 \perp s_2 \implies BH(O, \ldots, O)(s_1 \vee s_2) = BH(s_1 \vee s_2, O, \ldots, O) = BH(s_1, O, \ldots, O) + BH(s_2, O, \ldots, O)\).
   c. \(BH(s_1 \vee s_2, O, \ldots, O) = 1 - BC((s_1 \vee s_2)^*, I, \ldots, I)\).
   d. But, \(BC(\ldots, I, \ldots, I)\) is a state, so
   \(BC((s_1 \vee s_2)^*, I, \ldots, I) = 1 - BC((s_1 \vee s_2), I, \ldots, I)\)
since \(BH(s_1 \vee s_2, O, \ldots, O) = 1 - BC((s_1 \vee s_2), I, \ldots, I)\)
\[
= BC(s_1 \vee s_2), I, \ldots, I
\]
the other hand,
\[
BH(s_1, O, \ldots, O) + BH(s_2, O, \ldots, O) = 1 - BC(s_1^*, I, I, \ldots, I) + 1 - BC(s_2^*, I, I, \ldots, I)
\]
\[
= 1 - (1 - BC(s_1, I, I, \ldots, I)) + 1 - (1 - BC(s_2, I, I, \ldots, I))
\]
\[
= BC(s_1, I, I, \ldots, I) + BC(s_2, I, I, \ldots, I)
\]
therefore
\(BH(s_1 \vee s_2, O, O, ..., O) = BH(s_1, O, O, ..., O) + BH(s_2, O, O, ..., O)\)

similarly \(BH(O, O, ..., O, .)\) is a state too.

3. Let \((s_1, ..., s_n), (t_1, ..., t_n) \in \mathcal{A}^n\) such that \(s_i \leq t_i, i = 1, ..., n\).

To show that \(\sum_{k=1}^{2} \sum_{h=1}^{2} (-1)^{(k_1+...+k_2)+1} BH(u_1, ..., u_n) \geq 0\)

Now,
\[
\sum_{k=1}^{2} \sum_{h=1}^{2} (-1)^{(k_1+...+k_2)+1} BH(u_1, ..., u_n)
= \sum_{k=1}^{2} \sum_{h=1}^{2} (-1)^{(k_1+...+k_2)+1} [1 - BC(u_1^*, ..., u_n^*)]
= \sum_{k=1}^{2} \sum_{h=1}^{2} (-1)^{(k_1+...+k_2)+1} - \sum_{k=1}^{2} \sum_{h=1}^{2} (-1)^{(k_1+...+k_2)+1} BC(u_1^*, ..., u_n^*)
\]

(since \(\sum_{k=1}^{2} \sum_{h=1}^{2} (-1)^{(k_1+...+k_2)+1} = 0\), we have:
\[
\sum_{k=1}^{2} \sum_{h=1}^{2} (-1)^{(k_1+...+k_2)+1} - \sum_{k=1}^{2} \sum_{h=1}^{2} (-1)^{(k_1+...+k_2)+1} BC(u_1^*, ..., u_n^*)\]

(Definition 3.3.1)
\[
= \sum_{k=1}^{2} \sum_{h=1}^{2} (-1)^{(k_1+...+k_2)} BC(u_1^*, ..., u_n^*) \geq 0.
\]

Hence
\[
\sum_{k=1}^{2} \sum_{h=1}^{2} (-1)^{(k_1+...+k_2)+1} BH(u_1, ..., u_n) \geq 0,
\]

which means that \(BH\) is a BL-co-copula on \(\mathcal{A}\).

**Proposition 4.3.** Let \(\mathcal{A}\) be a BL-algebra. If \(BH\) is a BL-co-copula, a function \(BC: \mathcal{A}^n \rightarrow [0, 1]\)
is a BL-copula for all \(s_1, ..., s_n \in \mathcal{A}, \text{ and } BC(s_1, ..., s_n) = 1 - BH(s_1^*, ..., s_n^*)\).

**proof:**

1. Let \(s_i \in \mathcal{A}\) then \(BC(O, s_2, ..., s_n) = 1 - BH(I, s_2^*, ..., s_n^*)\)
\[
BC(s_1, s_2, ..., O) = 1 - BH(s_1^*, s_2^*, ..., I).
\]

2. To prove that \(BC(., I, ..., I), BC(.,..., I, .)\) are states on \(\mathcal{A}\):

**a.** \(BC(I, ..., I) = 1 - BH(O, ..., O) = 1 - 0 = 1\).

**b.** \(s_1 \perp s_2\) we show that
\[
BC(., I, ..., I)(s_1 \per s_2) = BC(s_1 \per s_2, I, ..., I)
= BC(s_1, I, ..., I) + BC(s_2, I, ..., I)
\]

\(\Rightarrow\) \(BC(s_1 \per s_2, I, ..., I) = 1 - BH((s_1 \per s_2)^*, O, ..., O)\)

But, \(BH(., O, ..., O)\) is a state. then
\[
BH((s_1 \per s_2)^*, O, ..., O) = 1 - BH((s_1 \per s_2), O, ..., O)
\]
then \( BC(s_1 \lor s_2, I, \ldots, I) = 1 - (1 - BH((s_1 \lor s_2), O, \ldots, O)) \)
\[
= BH((s_1 \lor s_2), O, \ldots, O) = BH(s_1, O, \ldots, O) + BH(s_2, O, \ldots, O)
\]
\(\iff\) on the other hand: \( BC(s_1, I, \ldots, I) + BC(s_2, I, \ldots, I) \)
\[
= 1 - BH(s_1^*, O, \ldots, O) + 1 - BH(s_2^*, O, \ldots, O)
\]
\(= BH(s_1, O, \ldots, O) + BH(s_2, O, \ldots, O) \).

Hence
\[
BC(s_1 \lor s_2, I, \ldots, I) = BC(s_1, I, \ldots, I) + BC(s_2, I, \ldots, I)
\]
which means that
\[
BC(s_1 \lor s_2, I, \ldots, I) = BC(s_1, I, \ldots, I) + BC(s_2, I, \ldots, I)
\]
Similarly we obtain that \( BC(I, \ldots, I,.) \) is a state .

3. Let \((s_1, \ldots, s_n), (t_1, \ldots, t_n) \in A^n \) such that \( s_i \leq t_i, \ i = 1, \ldots, n \), to show that

\[
\sum_{k_1=1}^{s_1} \ldots \sum_{k_n=1}^{s_n} (-1)^{(k_1+\ldots+k_n)}BC(u_{1k_1}, \ldots, u_{nk_n}) \geq 0, \text{ we have:}
\]

\[
\sum_{k_1=1}^{s_1} \ldots \sum_{k_n=1}^{s_n} (-1)^{(k_1+\ldots+k_n)}BC(u_{1k_1}, \ldots, u_{nk_n})
\]
\[
= \sum_{k_1=1}^{s_1} \ldots \sum_{k_n=1}^{s_n} (-1)^{(k_1+\ldots+k_n)}[1 - BH(u_{1k_1}, \ldots, u_{nk_n})]
\]
\[
= \sum_{k_1=1}^{s_1} \ldots \sum_{k_n=1}^{s_n} (-1)^{(k_1+\ldots+k_n)} - \sum_{k_1=1}^{s_1} \ldots \sum_{k_n=1}^{s_n} (-1)^{(k_1+\ldots+k_n)+1} BH(u_{1k_1}, \ldots, u_{nk_n})
\]
(since \( \sum_{k_1=1}^{s_1} \ldots \sum_{k_n=1}^{s_n} (-1)^{(k_1+\ldots+k_n)} = 0 \))
\[
\sum_{k_1=1}^{s_1} \ldots \sum_{k_n=1}^{s_n} (-1)^{(k_1+\ldots+k_n)} - \sum_{k_1=1}^{s_1} \ldots \sum_{k_n=1}^{s_n} (-1)^{(k_1+\ldots+k_n)+1} BH(u_{1k_1}, \ldots, u_{nk_n})
\]
\[
= - \sum_{k_1=1}^{s_1} \ldots \sum_{k_n=1}^{s_n} (-1)^{(k_1+\ldots+k_n)} BH(u_{1k_1}, \ldots, u_{nk_n}) (\text{Definition 3.2.1})
\]
\[
= \sum_{k_1=1}^{s_1} \ldots \sum_{k_n=1}^{s_n} (-1)^{(k_1+\ldots+k_n)+1} BH(u_{1k_1}, \ldots, u_{nk_n}) \geq 0
\]

Hence
\[
\sum_{k_1=1}^{s_1} \ldots \sum_{k_n=1}^{s_n} (-1)^{(k_1+\ldots+k_n)}BC(u_{1k_1}, \ldots, u_{nk_n}) \geq 0
\]
Therefore \( BC \) is a BL–copula on \( A \)
References

[1] AL-Adilee .A,: A Generalization of Probability theory in Quantum Logics, in Slovak University, (2010).

[2] Bělohlávek R.: Some properties of residual lattices, Czechoslovak Math. J. 53(123), 161–171(2003).

[3] Busneang, D., Piciu, D.: On the lattice of deductive systems of a BL-algebra. Central Eur. J.Math., 1 (2):221-238, (2003).

[4] Cignoli, R., I.M.L. D’ottaviano, Mundici, D.: Algebraic Foundations of Many-Valued Reasoning, Kluwer Academic Publ., Dordrecht, (2000).

[5] Cignoli, R., F. Esteva, L. Godo and Torrens, A.: Basic fuzzy logic is the logic of continuous t-norm and their residua. Soft Comput., 4: 106-112, (2002).

[6] Cohen, David. An Introduction to Hilbert Space and Quantum Logic. New York: Springer-Verlag, (1989).

[7] Davey, B.A.; Priestley, H. A.: Introduction to Lattices and Order, Cambridge University Press, ISBN 978-0-521-7845, (2002).

[8] Di Nola, A., Georgescu, G., Leustean, L.: Boolean products of BL-algebras, J. Math. Anal. Appl. 106-131(2000).

[9] Di Nola, A., Leustean, L.: Compact representations of BL-algebras, University Aarhuy, BRICS Report Series, 2002.

[10] Greechie: Orthogonal lattices admitting no states, J. Combin. Theory Ser. A10,119 -132, (1971).

[11] Halmos, P. R.: Measure theory, New York, Van Nostrand Reinhold, (1950).

[12] Hájek, P.: Mathematics of fuzzy logic, Kluwer Academic Publishers, (1998). logic?, Fuzzy Sets and Systems, 157: 597-603.8, (2006).

[13] Haveshki, M., A.: Borumand Saeid and E. Eslami, Some types of filters in BL-algebras, Soft Computing, 10: 657-664, (2006).
[14] Kondo, M. and W.A.: Dudek, Filter theory of BL-algebras. Soft Computing, 12: 419-423, (2007).

[15] Meng, B.L. and Xin X.L.: Ideal theory of BL-algebras,” Submitted to Journal of Applied Mathematics.

[16] Molina, Q. J. J. and Rodreguez Lallena, J. A.: Bivariate copulas with quadratic sections, J. Nonpara. Statistic. 5, 323-337, (1995).

[17] Navara, M.: An orthomodular lattice admitting no group-valued measure, Proc. of the Amer. Math. Soc., 122, 7 - 12, (1994).

[18] Nanasiova, O., Valašková, L.: Marginality and triangle inequality, (2009).

[19] Nánásiová, O., Pulmannova, S.: \textit{S-function and tracial states}, Infom. Sci., 179: 5, 515 – 520, (2009).

[20] Nánásiová, O., Valašková, L.: \textit{functions on a quantum logic}, Soft Comp. 1 – 6, (2009).

[21] Nelsen, R. B., Quesada-Molina: Rodriguez Lallena, J. A. Ubeda Flores, M.: Bounds on bivariate distribution functions with given margins and measures of association, Comm. Statist., 30, 1155 - 1162, (2001).

[22] Nelsen, R. B.: An introduction to copulas, Springer Series in Statistics, 139-270, (2006).

[23] Riččan, B.: On the probability on BL-algebras, Acta Math. Nitra, 4-313, (2000).

[24] Saeid, A.B and Motamed, S.: Normal filters in BL-algebras, world Applied sciences Journal, Vol.7, pp. 70-76, (2009).

[25] Turunen, E.: BL-algebras of basic fuzzy logic. Mathware and Soft Computing, 6: 49-61, 11(1999).

[26] Turunen, E.: Mathematics behind fuzzy logic. Physica-Verlag, (1999). J. Mult. Val. logic, 6: 229-249 (2001).