Matzoh ball soup in spaces of constant curvature

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Abstract

In this paper, we generalize Magnanini-Sakaguchi’s result [23] from Euclidean space to spaces of constant curvature. More precisely, we show that if a conductor satisfying the exterior geodesic sphere condition in the space of constant curvature has initial temperature 0 and its boundary is kept at temperature 1 (at all times), if the thermal conductivity of the conductor is inverse of its metric, and if the conductor contains a proper sub-domain, satisfying the interior geodesic cone condition and having constant boundary temperature at each given time, then the conductor must be a geodesic ball. Moreover, we show similar result for the wave equations and the Schrödinger equations in spaces of constant curvature.

1 Introduction

Klamkin’s conjecture [17] (also referred to by L. Zalcman in [29] as the Matzoh ball soup problem) states that, in a bounded domain Ω (i.e., the Matzoh ball in $\mathbb{R}^n$), if the normalized temperature $u = u(t, x)$ satisfies the heat equation:

$$\begin{cases}
\frac{\partial u}{\partial t} = \Delta u & \text{in } (0, +\infty) \times \Omega, \\
u = 1 & \text{on } (0, +\infty) \times \partial \Omega, \\
u = 0 & \text{on } \{0\} \times \Omega,
\end{cases}$$

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and if all spatial isothermic surfaces of \(u\) are invariant with time (the values of \(u\) vary with time on its spatial isothermic surfaces), then \(\Omega\) must be a ball.

In [3]-[4], this conjecture had been settled affirmatively by G. Alessandrini (also see [25] for a different method, by which Klamkin’s conjecture can be proved). A stronger result has also been obtained by Magnanini and Sakaguchi in [23], which says that Klamkin’s conjecture holds only if one spatial isothermic surface of \(u\) is invariant with time.

It is a natural question to ask whether Magnanini-Sakaguchi’s stronger result still hold in the space \(M_k\) of constant curvature \(k\) \((k \in \mathbb{R}^1)\)?

The main purpose of this paper is to prove the following:

**Theorem 1.1.** Let \(\Omega\) be a bounded domain in the \(n\)-dimensional space \(M_k\) of constant curvature \(k\) with metric \(g_{ij} = \frac{4\delta_{ij}}{(1 + k|x|^2)^2}\) (\(\delta_{ij}\) is Kronecker’s symbol; in case of \(k > 0\), \(\Omega\) is required to lie in a hemisphere), \(n \geq 2\). Let \(\Omega\) satisfy the exterior geodesic sphere condition and assume that \(D\) is a domain, with boundary \(\partial D\), satisfying the interior geodesic cone condition, and such that \(\bar{D} \subset \Omega\).

Let \(u\) be the solution to the problem

\[
\frac{\partial u}{\partial t} = \sum_{i=1}^{n} \left( \frac{1 + k|x|^2}{2} \right)^2 \frac{\partial^2 u}{\partial x_i^2} \quad \text{in} \quad (0, +\infty) \times \Omega, \tag{1.1}
\]

and the two conditions:

\[
u = 1 \quad \text{on} \quad (0, +\infty) \times \partial \Omega, \tag{1.2}
\]

\[
u = 0 \quad \text{on} \quad \{0\} \times \Omega. \tag{1.3}
\]

If \(u\) satisfies the extra condition:

\[
u(t, x) = a(t), \quad (t, x) \in (0, +\infty) \times \partial D, \tag{1.4}
\]

for some function \(a : (0, +\infty) \to (0, +\infty)\), then \(\Omega\) must be a geodesic ball in \(M_k\).

It is well-known (see, for example, [24, p.79]) that in a solid medium, the heat flow is governed by two characteristics, conductivity and capacity, which may vary over the medium. A general mathematical model is provided by a manifold \(M\), in which the conductivity, or rather its inverse, the resistance, corresponds to a Riemannian metric, and the capacity corresponds to a Borel measure. The above theorem means that in the space of constant curvature with the metric \(\frac{4\delta_{ij}}{(1 + k|x|^2)^2}\), if the thermal conductivity of the conductor is inverse of its metric, and if one spatial isothermic surface is invariant with time (of course, its boundary is kept at temperature 1), then the conductor takes the shape of a geodesic ball. Clearly, Theorem 1.1 reduces to Magnanini-Sakaguchi’s result [23] when \(k = 0\).
The proof of our main theorem is essentially based on three ingredients: The first ingredient is Varadhan’s theorem, which not only implies that (1.1) is the correct form of the heat equation on \( M_k \), but also tells us that \( \partial D \) and \( \partial \Omega \) are equidistant surfaces. The second ingredient is a new method which is due to Magnanini and Sakaguchi (see [23]). This method contains an integral transform with respect to time variable, two kinds of balance laws and an asymptotic formula. In order to apply Magnanini-Sakaguchi’s method to fit our manifold setting, we use two techniques: One is the invariance property of operator \( \sum_{i,j=1}^{n}(\frac{1+k|x|^2}{2})\frac{\partial^2}{\partial x_i \partial x_j} \) under isometries. The other is an orthogonal projection from the sphere \( S_{1/\sqrt{k}}^n \) or hyperboloid model \( \mathbb{H}_{1/\sqrt{-k}}^n \) to the Euclidean space \( \{(x,0) \in \mathbb{R}^{n+1} | x \in \mathbb{R}^n \} \), which allows us to derive a formula for the principal curvatures (see Lemma 4.1). This is also a key step toward the proof of our main theorem. The last ingredient is Alexandrov’s theorem [2] that provides a characteristic property of geodesic spheres in the spaces of constant curvature.

Finally, we show similar result for the wave equations and the Schrödinger equations in spaces of constant curvature.

2 Preliminaries

Let \( M_k \) be a complete, simply connected \( n \)-dimensional Riemannian manifold of constant curvature \( k \). Then \( M_k \) is uniquely determined, up to isometric equivalence (see [14], [18] or [28]). Of course, when \( k = 0 \) we may take \( M_k = \mathbb{R}^n \) with the usual Euclidean metric\( ds^2 = dx_1^2 + dx_2^2 + \cdots + dx_n^2 \). For \( k \neq 0 \), various realizations are possible. Thus, if \( k > 0 \) we may take for \( M_k \) the \( n \)-dimensional sphere \( S_\rho^n = \{ y \in \mathbb{R}^{n+1} | \sqrt{y_1^2 + \cdots + y_{n+1}^2} = \rho \} \) of radius \( \rho = 1/\sqrt{k} \), centered at the origin in \( \mathbb{R}^{n+1} \), with the induced Euclidean metric; equivalently, \( S_\rho^n \) may be realized by stereographic projection from the north pole. This is a map \( \sigma : S_\rho^n \setminus \{(0, \cdots, 0, \rho)\} \ni y \mapsto x = \sigma y \in \mathbb{R}^n \), which maps a point \( y \in S_\rho^n \) into the intersection \( x = y \in \mathbb{R}^n \) of the line jointing \( y \) and the north pole \((0, \cdots, 0, \rho)\) with the equatorial hyperplane \( \mathbb{R}^n \). Clearly, the south pole \((0, \cdots, 0, -\rho)\) is mapped into the origin, and one has (see [8, p.59])

\[
y_{n+1} = \rho \frac{|x|^2 - \rho^2}{|x|^2 + \rho^2}, \quad (y_1, \cdots, y_n) = \frac{2\rho^2 x}{\rho^2 + |x|^2} \quad \left( x = \frac{\rho}{\rho - y_{n+1}} (y_1, \cdots, y_n) \right).
\]

The map \( \sigma \) induces a metric on \( \mathbb{R}^n \):

\[
ds^2 = \frac{4|dx|^2}{(1 + |x|^2/\rho^2)^2}, \tag{2.1}
\]
i.e.,
\[ g_{ij} = \left\langle \frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_k} \right\rangle = \frac{4\delta_{ij}}{(1 + |x|^2/\rho^2)^2}. \] (2.2)

Let \( M \) and \( N \) be two manifolds with metrics \( g \) and \( h \), respectively. We say that a diffeomorphism \( \Phi : (M, g) \rightarrow (N, h) \) is an isometry if \( \Phi^* h = g \). It is well-known that every isometry of \( S^n_\rho \) is an element of \( O(n + 1) \). For \( y \in S^n_\rho \setminus \{ \text{the north pole} \} \), take \( R_y \in O(n + 1) \) satisfying \( R_y(y) = \text{the south pole} \) \( (R_y(-y) = \text{the north pole}) \).

Setting \( z = \sigma y \in \mathbb{R}^n \) and \( z^* := \rho^2 z/|z|^2 \) yields \( \sigma(-y) = -z^* \). Let us consider the map \( f = \sigma \circ R_y \circ \sigma^{-1} \). One has \( f(z) = 0, f(-z^*) = \infty \). Note that a M"obius transformation with such property is of the form
\[ f(x) = \lambda A((x + z^*)^{*} - (z + z^{*})^{*}) \]
with \( \lambda > 0 \) and a constant orthogonal matrix \( A \) (see [Ah, p.21]). Similar to the method of [5, p.1106], we get that
\[ f(x) = \frac{\rho^2(\rho^2 + |z|^2)}{|z|^2} A((x + z^*)^{*} - (z + z^{*})^{*}) \] with \( A \in O(n) \). (2.3)

Let \( \mathbb{R}^{n+1} \) be equipped with the Lorentzian metric
\[ \langle y, y \rangle = -y_{n+1}^2 + y_1^2 + \cdots y_n^2. \]
For \( k < 0 \), let \( \rho = 1/\sqrt{-k} \) and
\[ \mathbb{H}^n_\rho = \{ y \in \mathbb{R}^{n+1} | \langle y, y \rangle = -\rho, \ y_{n+1} > 0 \} \]
with the Riemannian metric induced from the Lorentzian metric. \( \mathbb{H}^n_\rho \) is called the hyperboloid model or Lobachevskian pseudo-sphere (see [18, p.38-42]). By regarding \( \mathbb{R}^n \) as \( \{ (x, 0) \in \mathbb{R}^{n+1} \} \), we consider the hyperbolic stereographic projection \( \zeta : \mathbb{H}^n_\rho \ni y \rightarrow x = \zeta y \in \mathbb{R}^n \), which map a point \( y \in \mathbb{H}^n_\rho \) into the intersection \( x \in \mathbb{B}_\rho := \{ x \in \mathbb{R}^n | |x| < \rho \} \) of line joining \( y \) and the point \( (0, \cdots, -\rho) \) with \( \mathbb{R}^n \). Then, the point \( (0, \cdots, 0, \rho) \) is mapped into the origin, and we have
\[ y_{n+1} = \rho \frac{\rho^2 + |x|^2}{\rho^2 - |x|^2}, \quad (y_1, \cdots, y_n) = \frac{2\rho^2 x}{\rho^2 - |x|^2}, \quad (x = \frac{\rho}{\rho + y_{n+1}}(y_1, \cdots, y_n)). \]
This map induces the metric on \( \mathbb{B}_\rho \):
\[ ds^2 = \frac{4|dx|^2}{(1 - |x|^2/\rho^2)^2}, \] (2.4)
i.e.,
\[
g_{ij} = \left\langle \frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_k} \right\rangle = \frac{4\delta_{ij}}{(1 - |x|^2/\rho^2)^2}. \tag{2.5}
\]
Conformal transformations of the plane are holomorphic mappings, whereas in higher dimensions \((n \geq 3)\) the only possibilities are rotations, dilations, inversions \(x \rightarrow x^* = \frac{x}{|x|^2}\) and their compositions (Liouville’s theorem, see [10, §15]). Every isometry of \((\mathbb{B}^n_\rho, g)\) is a conformal map \(f : \mathbb{B}^n_\rho \rightarrow \mathbb{B}^n_\rho\). Similar to [1], we can verify that the general form of such a map is:
\[
f(x) = T_z(x) := A \frac{\rho^2 (\rho^2 - |z|^2)(x - z) - |x - z|^2 z}{\rho^4 + |x|^2 |z|^2 - 2\rho^2 x \cdot z} \quad \text{with} \ A \in O(n). \tag{2.6}
\]
Obviously, \(T_z(z) = 0\), and the isometries of \((\mathbb{B}^n_\rho, g)\) transform spheres into spheres.

Throughout this paper (except for the proofs of Lemma 4.1 and Theorem 4.2), \(M_k\) can be regarded as \(\mathbb{R}^n\) with metric (2.2) when \(k > 0\); \(M_k = \mathbb{R}^n\) with the Euclidean metric when \(k = 0\); \(M_k\) as \(\mathbb{B}^n_\rho\) with metric (2.5) when \(k < 0\). As \(k \rightarrow 0\), the metric (2.1) and (2.4) approach the flat (Euclidean) metric. For (2.1) this is geometrically obvious, and in any case can be seen from the equivalent form
\[
ds^2_k = \frac{4|dx|^2}{(1 + k|x|^2)^2}, \tag{2.7}
\]
valid for all \(k\). The pair \((D, ds^2_k)\) with \(D \subset \mathbb{R}^n\) is call the canonical representation of the domain \(D\) in \(M_k\).

Let \(M\) be an \(n\)-dimensional Riemannian manifold with the metric \(g_{ij}(x)\), and let \(L\) be the following differential operator acting on smooth functions on \(M\):
\[
Lu = \frac{1}{2} \sum_{i,j=1}^n g^{ij}(x) \frac{\partial^2 u}{\partial x_i \partial x_j},
\]
where \((g^{ij}(x))\) is the matrix inverse to \((g_{ij}(x))\).

Let \(p(\tau), 0 \leq \tau \leq 1\), be a smooth path in \(M\). Then the length of such a path is defined as
\[
l(p) = \int_0^1 [\dot{p}(\tau)g(p(\tau))\dot{p}(\tau)]^{1/2} d\tau,
\]
where \(\dot{p}(\tau)\) stands for \(dp(\tau)/d\tau\) and \((\theta g \theta)\) for the quadratic form \(\sum_{i,j=1}^n g_{ij}(x)\theta_i \theta_j\); \(l(p)\) is the natural length in a metric defined locally as
\[
ds^2 = \sum_{i,j=1}^n g_{ij} dx_i dx_j.
\]
The global distance \( d(x, y) \) induced by this metric is defined as
\[
d(x, y) = \inf_{\{p \mid p(0) = x, p(1) = y\}} l(p).
\]

**Lemma 2.1 (Varadhan’s theorem, see [27]).** Let \( \Omega \) be a bounded domain in a Riemannian manifold \( M \) with uniform Hölder continuous metric \( g_{ij}(x) \). Let \( \phi(s, x) \) be the solution of the equation
\[
\frac{1}{2} \sum_{i,j=1}^{n} g^{ij}(x) \frac{\partial^2 \phi}{\partial x_i \partial x_j} = s \phi \quad \text{for } x \in \Omega \tag{2.8}
\]
with the boundary value \( \phi = 1 \) on the boundary \( \partial \Omega \) of \( \Omega \). Then
\[
\lim_{s \to \infty} \left[ -\frac{1}{\sqrt{2s}} \log \phi(s, x) \right] = \tilde{F}(x),
\]
for \( x \) is any point of \( \tilde{\Omega} \) and
\[
\tilde{F}(x) = \text{dist}(x, \partial \Omega) \quad \text{(2.10)}
\]
is the shortest distance to the boundary \( \partial \Omega \) from \( x \).

**Lemma 2.2 (Alexandrov’s theorem, see [2]).** Let \( \Gamma \) be a closed \((n-1)\)-dimensional surface in an \( n \)-space \( \mathbb{M}_k \) of constant curvature \( k \) (in case of sphere, \( \Gamma \) is required to lie in a hemisphere). Suppose that \( \Gamma \) has no multiple points and is of class \( C^2 \). Let \( \lambda_1 \geq \cdots \geq \lambda_{n-1} \) denote its principal curvatures, at an arbitrary point \( p \in \Gamma \). Assume that \( F = F(\beta_1, \cdots, \beta_{n-1}) \) is a continuous differentiable function, defined for \( \beta_1, \cdots, \beta_{n-1} \), and subject to the condition
\[
\text{const} > \frac{\partial F(\beta_1, \cdots, \beta_{n-1})}{\partial \beta_j} > \text{const} > 0, \quad (j = 1, \cdots, n-1),
\]
at least on \( \Gamma \), i.e., \( \beta_j = \lambda_j \) \((j = 1, \cdots, n-1)\). Then, if \( F(\lambda_1, \cdots, \lambda_{n-1}) \equiv \text{constant on } \Gamma \), \( \Gamma \) is a geodesic sphere.

Proof. When \( n = 2 \), we have that
\[
\text{const} > \frac{dF(\beta_1)}{d\beta_1} > \text{const} > 0 \quad \text{on } \Gamma,
\]
i.e., \( F(\lambda_1) \) is increasing in \( \lambda_1 \in \Gamma \). Thus, from \( F(\lambda_1) \equiv \text{constant on } \Gamma \), we get that \( \lambda_1 \) (i.e., the curvature of \( \Gamma \)) must be a constant on \( \Gamma \), which implies that \( \Gamma \) is the boundary of a geodesic disk in \( \mathbb{M}_k \). When \( n \geq 3 \), the theorem had been proved by A. D. Alexandrov (see [2, Theorem and (I2) of Remark (6)]). \( \square \)
3 Isometric invariance and balance law

In this section, we shall prove some lemmas, which are needed for proving our main theorem. First, we prove a simple invariance property of the operator (3.1) below. If \((U, \phi)\) is a local chart on \(M\) and \(f \in C^2(M)\), we often write \(f^*\) for the composite function \(f \circ \phi^{-1}\).

Lemma 3.1. Let \(\Phi\) be a diffeomorphism of the Riemannian manifold \(M\) with metric \(g_{ij}\). Then \(\Phi\) leaves the operator \(L\) invariant if and only if it is an isometry, where

\[
Lf = \sum_{i,j=1}^{n} g^{ij}(x) \frac{\partial^2 f}{\partial x_i \partial x_j}. \tag{3.1}
\]

Proof. Let \(p \in M\) and let \((V, \psi)\) be a local chart around \(p\). Then \((\Phi(V), \psi \circ \Phi^{-1})\) is a local chart around \(\Phi(p)\). For \(x \in V\), let \(y = \Phi(x)\) and

\[
\psi(x) = (x_1, \cdots, x_n), \quad x \in V,
\]

\[
(\psi \circ \Phi^{-1})(y) = (y_1, \cdots, y_n), \quad y \in \Phi(V).
\]

Then

\[
x_i(x) = y_i(\Phi(x)), \quad d\Phi_x \left( \frac{\partial}{\partial x_i} \right) = \left( \frac{\partial}{\partial y_i} \right)_{\Phi(x)} \quad (1 \leq i \leq n),
\]

where \(d\Phi_x\) is the tangent map. For each function \(f \in C^2(M)\),

\[
((Lf)^{\Phi^{-1}})(x) = (Lf)(\Phi(x)) = \sum_{i,j=1}^{n} g^{ij}(y) \frac{\partial^2 f^*}{\partial y_i \partial y_j}, \tag{3.2}
\]

\[
(Lf^{\Phi^{-1}})(x) = \sum_{i,j=1}^{n} g^{ij}(x) \frac{\partial^2 (f \circ \Phi)^*}{\partial x_i \partial x_j}. \tag{3.3}
\]

Now if \(\Phi\) is an isometry, then \(g_{ij}(x) = g_{ij}(y)\) for all \(i, j\). Because of the choice of coordinates, we have

\[
\frac{\partial^2 f^*}{\partial y_i \partial y_j} = \frac{\partial^2 (f \circ \Phi)^*}{\partial x_i \partial x_j}, \quad 1 \leq i, j \leq n.
\]

Thus the right-hand sides of (3.2) and (3.3) coincide and \(L^{\Phi} = L\), which implies that the operator \(L\) is invariant. On the other hand, if (3.2) and (3.3) agree, then we find by equating coefficients that \(g_{ij}(x) = g_{ij}(y)\), which shows that \(\Phi\) is an isometry. \(\square\)
The following Lemma is the so-called *balance law*, which has been proved by Magnanini and Sakaguchi in Euclidean case (see [21, 22, 23]) and by Sakaguchi in $\mathbb{M}_k$ with the Laplace-Beltrami operator instead of $L$ (see [26]).

Lemma 3.2. Let $\Omega$ be a domain in the $n$-dimensional space $\mathbb{M}_k$ of constant curvature $k$ (in case of sphere, $\Omega$ is required to lie in a hemisphere), $n \geq 2$. Let $x_0$ be a point in $\Omega$ and set $d_* = \text{dist}(x_0, \partial \Omega)$. Assume that $v = v(t, x)$ is a solution of

$$\frac{\partial v}{\partial t} = \sum_{i,j=1}^{n} \left( \frac{1+k|x|^2}{2} \right)^2 \frac{\partial^2 v}{\partial x_i \partial x_j} \quad \text{in} \ (0, +\infty) \times \Omega. \tag{3.4}$$

Then, the following two assertions hold:

(i) $v(t, x_0) = 0$ for every $t \in (0, +\infty)$ if and only if

$$\int_{\partial B_r(x_0)} v(t, x) dA_r = 0 \quad \text{for every} \ (t, r) \in (0, +\infty) \times [0, d_*], \tag{3.5}$$

where $\partial B_r(x_0)$ denotes the geodesic sphere centered at $x_0$ with radius $r > 0$ and $dA_r$ denotes its area element;

(ii) $\nabla v(t, x_0) = 0$ for every $t \in (0, +\infty)$ if and only if

$$\int_{\partial B_r(x_0)} \exp_{x_0}^{-1} x v(t, x) dA_r = 0 \quad \text{for every} \ (t, r) \in (0, +\infty) \times [0, d_*], \tag{3.6}$$

where $\exp_{x_0}$ is the exponential map at $x_0$.

Proof. (i) If (3.5) holds, then we immediately get that $v(t, x_0) = 0$ for every $t \in (0, +\infty)$. Conversely, for any two points $x'$ and $x''$ in $\mathbb{M}_k$, we can find an isometry $\Phi$ that maps $\mathbb{M}_k$ onto itself such that $\Phi x' = x''$ (cf. section 2). It follows from Lemma 3.1 that the operator $L$ is invariant under the isometry $\Phi$ (here $g^{ij}(x) = \left( \frac{1+k|x|^2}{2} \right)^2 \delta_{ij}$), that is, $(Lu) \circ \Phi = L(u \circ \Phi)$. Thus, by an isometry we may put $x_0 = 0$ in the canonical representation. Note that spherical coordinates are valid about any point in $\Omega \subset \mathbb{M}_k$ for each fixed $k$ (see [7, p.37-39]). Therefore, about the origin in the canonical representation, there exists a coordinate system $(r, \theta) \in [0, d_*) \times \mathbb{S}^{n-1}$, relative to which the Riemannian metric reads as

$$ds^2 = (dr)^2 + (h_k(r))^2 |d\theta|^2, \tag{3.7}$$

where

$$h_k(r) = \begin{cases} 
(1/\sqrt{k}) \sin \sqrt{k} r, & k > 0, \\
r, & k = 0, \\
(1/\sqrt{-k}) \sinh \sqrt{-k} r, & k < 0, 
\end{cases} \tag{3.8}$$
\(|d\theta|^2\) denotes the metric on the Euclidean sphere \(S^{n-1}\) of radius 1, and \(r\) the geodesic distance from \(x_0 = 0\).

Let \(C(r) := \{ \theta \in T_0M_k | |\theta| = 1 \text{ and } \gamma_\theta(s) = \exp_0(s\theta) , s \in [0,r] \text{, is minimizing} \} \). In view of \(T_0M_k = \mathbb{R}^n\), we see that \(C(r) = S^{n-1}\) for all \(r \in [0,d_*)\). Denote \(x = x(r,\theta) \in M_k\), where \(d(.,0) = r\), \(\theta \in S^{n-1}\). It follows that

\[
\int_{B_r(0)} v(t,x)d\mu(x) = \int_0^r \left( \int_{S^{n-1}} v(t,\exp_0(\bar{r},\theta))J(\bar{r},\theta)d\Theta(\theta) \right) d\bar{r},
\]

which implies

\[
\int_{\partial B_r(0)} v(t,x)dA_r = \int_{S^{n-1}} v(t,\exp_0(r,\theta))J(r,\theta)d\Theta(\theta) \tag{3.9}
\]

\[
= \int_{S^{n-1}} v(t,\exp_0(r\theta))(h_k(r))^{n-1}d\Theta(\theta),
\]

where \(d\Theta\) is the volume form of the unit \((n-1)\)-sphere, \(J(r,\theta) = \sqrt{\det(g_{ij})} = (h_k(r))^{n-1}\) (see [9, p.74-76]). Then (3.5) is equivalent to

\[
\int_{S^{n-1}} v(t,\exp_0(r\theta))d\Theta(\theta) = 0 \text{ for any } (t,r) \in (0, +\infty) \times [0,d_*).
\]

We define

\[
U(t,r) := \int_{S^{n-1}} v(t,\exp_0(r\theta))d\Theta(\theta), \tag{3.10}
\]

for all \((t,r) \in (0, +\infty) \times [0,d_*)\). Since

\[
L = \sum_{i=1}^{n} \left( \frac{1 + k|x|^2}{2} \right)^2 \frac{\partial^2}{\partial x_i^2} \tag {3.11}
\]

\[
= \left( \frac{1 + kr^2}{2} \right)^2 \left( \frac{\partial^2}{\partial r^2} + \frac{n-1}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \triangle_{S^{n-1}} \right),
\]

where \(\triangle_{S^{n-1}}\) denotes the Laplace-Beltrami operator on \(S^{n-1}\), by substituting this into

\[
0 = \int_{S^{n-1}} \left[ \left( \frac{\partial}{\partial t} - L \right) v(t,\exp_0(r\theta)) \right] d\Theta(\theta),
\]

and using \(\int_{S^{n-1}} (\triangle_{S^{n-1}} v(t,\exp_0(r\theta)))d\Theta(\theta) = 0\), we obtain

\[
\frac{\partial U}{\partial t} = \left( \frac{1 + kr^2}{2} \right)^2 \left( \frac{\partial^2 U}{\partial r^2} + \frac{n-1}{r} \frac{\partial U}{\partial r} \right) \text{ in } [0,d_*) \times (0, +\infty). \tag{3.12}
\]
It follows from the local regularity result of parabolic equations (see [12, 13], [19, 20] or [26, p.404-405]) that $U$ and $\frac{\partial U}{\partial r}$ are real analytic in $(0, +\infty) \times [0, d_*)$. Therefore

$$4r \frac{\partial U}{\partial t} = \left(k^2 r^5 + 2kr^3 + r\right) \frac{\partial^2 U}{\partial r^2} + (n - 1)(k^2 r^4 + 2kr^2 + 1) \frac{\partial U}{\partial r}$$

(3.13)

for all $(t, r) \in (0, +\infty) \times [0, d_*)$. Obviously, $U(t, 0) = 0$ for any $t > 0$. From (3.12), we have

$$\frac{\partial U}{\partial r}(t, 0) = \lim_{r \to 0} r^{n - 1} \left[\left(\frac{2}{1 + kr^2}\right)^2 \frac{\partial U}{\partial t} - \frac{\partial^2 U}{\partial r^2}\right] = 0$$

for all $t \in (0, +\infty)$.

We shall show by induction that

$$\frac{\partial^m U}{\partial r^m}(t, 0) = 0 \quad \text{for all } t > 0 \text{ and any integer } m \geq 0.$$  (3.14)

Suppose that

$$U(t, 0) = \frac{\partial U}{\partial r}(t, 0) = \cdots = \frac{\partial^m U}{\partial r^m}(t, 0) = 0 \quad \text{for all } t > 0.$$  

By differentiating both sides of (3.13) for $m$ times with respect to $r$, we get

$$\sum_{j=0}^{m} C_m^j \frac{\partial^j (4r)}{\partial r^j} \left(\frac{\partial}{\partial t} \frac{\partial^{m-j} U}{\partial r^j}\right)$$

$$= \sum_{j=0}^{m} C_m^j \frac{\partial^j (k^2 r^5 + 2kr^3 + r)}{\partial r^j} \frac{\partial^{m+2-j} U}{\partial r^{m+2-j}}$$

$$+ (n - 1) \sum_{j=0}^{m} C_m^j \frac{\partial^j (k^2 r^4 + 2kr^2 + 1)}{\partial r^j} \frac{\partial^{m+1-j} U}{\partial r^{m+1-j}},$$

where $C_m^j = \frac{m!}{j!(m-j)!}$. Thus, letting $r = 0$ and using the above assumption, we have

$$0 = m \frac{\partial^{m+1} U}{\partial r^{m+1}} + (n - 1) \frac{\partial^{m+1} U}{\partial r^{m+1}},$$

i.e., $\frac{\partial^{m+1} U}{\partial r^{m+1}}(t, 0) = 0$. It follows from induction that (3.14) holds. From the analyticity of $U$, we obtain that

$$U \equiv 0 \quad \text{in } (0, +\infty) \times [0, d_*).$$

Therefore, we conclude that (3.5) is true.
As in the argument of (i), by putting \( x_0 = 0 \) we get that (3.6) is equivalent to
\[
\int_{\mathbb{S}^{n-1}} (r\theta) v(t, \exp_0(r\theta))(h_k(r))^{n-1} d\Theta(\theta) = 0 \quad \text{for all} \quad (t, r) \in (0, +\infty) \times [0, d_*) ,
\]
i.e.,
\[
\int_{\mathbb{S}^{n-1}} \theta v(t, \exp_0(r\theta)) d\Theta(\theta) = 0 \quad \text{for all} \quad (t, r) \in (0, +\infty) \times [0, d_*) .
\]
If (3.6) holds, then, by the divergence theorem, we get that \( \nabla v(t, 0) = 0 \) for every \( t > 0 \).

We shall prove the converse assertion. Let us introduce an \( \mathbb{R}^n \)-valued function \( Q(t, r) \) by
\[
Q(t, r) = \int_{\mathbb{S}^{n-1}} \theta v(t, \exp_0(r\theta)) d\Theta(\theta) \quad (t, r) \in (0, +\infty) \times [0, d_*) .
\]
(3.15)

By putting (3.11) into
\[
0 = \int_{\mathbb{S}^{n-1}} \theta \left[ \left( \frac{\partial}{\partial t} - L \right) v(t, \exp_0(r\theta)) \right] d\Theta(\theta) ,
\]
and using \( -\Delta_{\mathbb{S}^{n-1}} \theta = (n-1)\theta \) together with integration by parts, we obtain that in \( (0, +\infty) \times [0, d_*) \),
\[
\frac{\partial Q}{\partial t} = \left( \frac{1 + kr^2}{2} \right)^2 \left( \frac{\partial^2 Q}{\partial r^2} + \frac{n-1}{r} \frac{\partial Q}{\partial r} - \frac{n-1}{r^2} Q \right) .
\]
(3.16)
Thus
\[
4r^2 \frac{\partial Q}{\partial t} = \left( k^2 r^6 + 2kr^4 + r^2 \right) \frac{\partial^2 Q}{\partial r^2} + (n-1)(k^2 r^5 + 2kr^3 + r) \frac{\partial Q}{\partial r} - (n-1)Q
\]
for all \( (t, r) \in (0, +\infty) \times [0, d_*) \). In view of \( \nabla v(t, 0) = 0 \), we find by the divergence theorem that \( Q(t, 0) = \frac{\partial Q}{\partial r}(t, 0) = 0 \). It follows from the method of induction that \( \frac{\partial^m Q(t, 0)}{\partial r^m} = 0 \) for all \( t \in (0, +\infty) \) and \( m = 1, 2, \ldots \). Therefore, the analyticity of \( Q(t, r) \) implies that \( Q(t, r) \equiv 0 \) in \( (0, +\infty) \times [0, d_*) \), and the desired result holds. □

Lemma 3.3. Let \( \Omega \) be a domain with \( C^2 \) boundary in the \( n \)-dimensional space \( \mathbb{M}_k \) of constant curvature \( k \), \( n \geq 2 \), and let \( W(s, x) \) be the solution of the following elliptic boundary value problem
\[
\sum_{i=1}^{n} \left( \frac{1 + k|x|^2}{2} \right)^2 \frac{\partial^2 W}{\partial x_i^2} = sW \quad \text{in} \quad \Omega ,
\]
(3.18)
\[
W = 1 \quad \text{on} \quad \partial \Omega .
\]
(3.19)
Then, for every $\epsilon > 0$, there exists a positive number $s_\epsilon$ such that

$$W_\epsilon^-(s, x) \leq W(s, x) \leq W_\epsilon^+(s, x)$$  \hspace{1cm} (3.20)

for every $x \in \tilde{\Omega}$ and every $s \geq s_\epsilon$, where

$$W_\epsilon^\pm(s, x) = \exp\{-\sqrt{s(1 \mp \epsilon)} \tilde{g}(x)\},$$  \hspace{1cm} (3.21)

and $\tilde{g}(x)$ is defined by (2.10).

**Proof.** We can take $\delta > 0$ small enough such that the function $\tilde{g} = \tilde{g}(x)$ defined in (2.10) is of class $C^2$ in the set $\Omega_\delta$ where

$$\Omega_\delta = \{x \in \Omega : \tilde{g}(x) < \delta\}.$$  \hspace{1cm} (3.22)

It is easy to calculate

$$\left(\sum_{i=1}^{n} \left(\frac{1+k|x|^2}{2}\right)^2 \frac{\partial^2 W_\epsilon^\pm}{\partial x_i^2}\right) - s W_\epsilon^\pm$$

$$= \left(\exp\{-\sqrt{s(1 \mp \epsilon)} \tilde{g}(x)\}\right) \left\{\left(\frac{1+k|x|^2}{2}\right)^2\right\}$$

$$\times \sum_{i=1}^{n} \left[-\sqrt{s(1 \mp \epsilon)} \frac{\partial^2 \tilde{g}}{\partial x_i^2} + s(1 \mp \epsilon) \left(\frac{\partial \tilde{g}}{\partial x_i}\right)^2\right] - s \right\}$$

$$= \mp \epsilon \sqrt{s} \left\{\sqrt{s} \pm \frac{\sqrt{(1 \mp \epsilon)}}{\epsilon} \sum_{i=1}^{n} \left(\frac{1+k|x|^2}{2}\right)^2 \frac{\partial^2 \tilde{g}}{\partial x_i^2}\right\} W_\epsilon^\pm \text{ in } \Omega_\delta.$$

Here we have used the fact that

$$\sum_{i=1}^{n} \left(\frac{1+k|x|^2}{2}\right)^2 \left(\frac{\partial \tilde{g}}{\partial x_i}\right)^2 = \sum_{i=1}^{n} \frac{\partial \tilde{g}}{\partial x_i} \left[\left(\frac{1+k|x|^2}{2}\right)^2 \delta_{ij}\right] = \langle \nabla \tilde{g}, \nabla \tilde{g} \rangle = 1.$$

Setting $M_\delta = \max_{\tilde{\Omega}} \left|\sum_{i=1}^{n} \left(\frac{1+k|x|^2}{2}\right)^2 \frac{\partial^2 \tilde{g}}{\partial x_i^2}\right|$, we get that if $s \geq \frac{1+\epsilon}{\epsilon} M_\delta^2$, then in $\Omega_\delta$

$$\sum_{i=1}^{n} \left(\frac{1+k|x|^2}{2}\right)^2 \frac{\partial W_\epsilon^+}{\partial x_i^2} - s W_\epsilon^+ \leq 0$$  \hspace{1cm} (3.23)

$$\sum_{i=1}^{n} \left(\frac{1+k|x|^2}{2}\right)^2 \frac{\partial W_\epsilon^-}{\partial x_i^2} - s W_\epsilon^- \geq 0.$$  \hspace{1cm} (3.24)
Since the function $-\frac{1}{\sqrt{s}} \log W(s, x)$ converges uniformly on $\bar{\Omega}$ to $\mathfrak{F}(x)$ as $s \to +\infty$, by Lemma 2.1 (Varadhan’s theorem) there exists a real number $s^* > 0$ such that for every $s \geq s^*$,

$$-\delta(1 - \sqrt{1-\epsilon}) \leq -\frac{1}{\sqrt{\epsilon}} \log W(s, x) - \mathfrak{F}(x) \leq \delta(\sqrt{1+\epsilon}-1), \quad x \in \bar{\Omega}.$$ 

Put $s_\epsilon = \max\{s^*, \frac{1+\epsilon}{\epsilon^2} M^2\}$. Completely similar to [23, p.938], we get (3.23). □

### 4 Principal curvatures and asymptotic formulas

We introduce some notations and definitions for the principal curvatures of $\partial \Omega$. Let $M$ be an $m$-dimensional submanifold of the $n$-dimensional Riemannian manifold $N$. The metric $\langle \cdot, \cdot \rangle$ on $N$ induces a metric on $M$. Then one has

$$\nabla_x^M Y = (\nabla_x^N Y)^\top \quad \text{for } X, Y \in \Gamma(TM),$$

where $\nabla^N$ is the Levi-Civita connection of $N$, $\nabla^M$ is the induced connection, and $\top: T_x N \to T_x M$ for $x \in M$ denotes the orthogonal projection.

Let $\nu(x)$ be a vector field in a neighborhood of $x_0 \in M \subset N$, that is orthogonal to $M$, i.e.,

$$\langle \nu(x), X \rangle = 0 \quad \text{for all } X \in T_x M. \quad (4.1)$$

We denote by $T_x M^\perp$ the orthogonal complement of $T_x M$ in $T_x N$. The bundle $TM^\perp$ with fiber $T_x M^\perp$ at $x \in M$ is called normal bundle of $M$ in $N$. (4.1) means $\nu(x) \in T_x M^\perp$. For a fixed normal field $\nu(x) \in T_x M^\perp$, we write $A_{\nu}(X) = (\nabla_X^N \nu)^\top$. Clearly, $A_{\nu}: T_x M \to T_x M$ is selfadjoint with respect to the metric $\langle \cdot, \cdot \rangle$. Suppose $\langle \nu(x), \nu(x) \rangle \equiv 1$; i.e., $\nu$ is a unit normal field. The $m$ eigenvalues of $A_{\nu}$ which are all real by self adjointness are called the principal curvatures of $M$ in the direction $\nu$, and the corresponding eigenvectors are called principal curvature vectors.

For any point $x \in \bar{\Omega} \subset M_k$, let $\mathfrak{F}(x)$ be defined by (2.10). Then $\mathfrak{F}(x) = 0$ is the hypersurface $\partial \Omega$. Since $\nu(x) = \nabla \mathfrak{F}(x)$ for any $x \in \partial \Omega$, we know that $\nabla_X^\Omega \nabla \mathfrak{F}(x)$ is always tangential to $\partial \Omega$ for any $X \in T_x(\partial \Omega)$, where $\nabla \mathfrak{F}(x) = \sum_{j,l} \frac{\partial \mathfrak{F}}{\partial x_j} g^{jl} \frac{\partial}{\partial x_l}$. In the local coordinates, the Hessian of $\mathfrak{F}(x)$ is

$$\nabla^\Omega \nabla \mathfrak{F} = \left( \frac{\partial^2 \mathfrak{F}}{\partial x_i \partial x_j} - \frac{\partial \mathfrak{F}}{\partial x_k} \Gamma^k_{ij} \right), \quad (4.2)$$

and we have

$$\nabla^\Omega \nabla \mathfrak{F}(X, Y) = \langle \nabla_X^\Omega \nabla \mathfrak{F}, Y \rangle, \quad X, Y \in T_x, \quad (4.3)$$

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where $\Gamma^k_{ij}$ is the Christoffel symbols. Therefore, $-\nabla^\Omega \nabla F$ has $n$ eigenvalues at $x \in \partial \Omega$, one of which is 0 (corresponding to the eigenvector $\nabla F(x)$), and the others are the principal curvatures of $\partial \Omega$ at $x$.

Let us consider the curvature of the boundary of a geodesic ball $B_r(x_0)$ in $\mathbb{M}_k$. Since any two geodesic balls with the same radius in $\mathbb{M}_k$ are isometric, their boundaries have the same curvature. It is easy to check that the geodesic sphere of radius $R$ at the origin has constant curvature $\tau_k(r)$ (see [9, p.66]),

$$
\tau_k(r) = \begin{cases} 
\sqrt{k} \cot \sqrt{k} r & \text{if } k > 0, \\
\frac{1}{r} & \text{if } k = 0, \\
\sqrt{-k} \coth \sqrt{-k} r & \text{if } k < 0.
\end{cases}
$$ (4.4)

Let $\Omega$ be a domain with $C^2$ boundary in either Euclidean space $\mathbb{R}^n$, or the hyperboloid model $\mathbb{H}^n$, or the sphere $S^n$. In the last case, $\Omega$ is required to lie in a hemisphere. Let $\Omega$ contain $x_0$, where $x_0$ is either the origin in Euclidean space $\mathbb{R}^n$, or the south pole $(0, \cdots, 0, -\rho)$ on the sphere $S^n$, or the point $(0, \cdots, 0, \rho)$ on the hyperboloid model $\mathbb{H}^n$. We define the orthogonal projection $P_0$ from $\Omega$ to the Euclidean space $\{(x,0) \in \mathbb{R}^{n+1} | x \in \mathbb{R}^n\}$ by

$$
P_0(y) = \begin{cases} 
(y_1, \cdots, y_n), & \forall \ y = (y_1, \cdots, y_n, 0) \in \Omega \subset S^n, \\
(y_1, \cdots, y_n), & \forall \ y = (y_1, \cdots, y_n) \in \Omega \subset \mathbb{R}^n, \\
(y_1, \cdots, y_n), & \forall \ y = (y_1, \cdots, y_n, 0) \in \Omega \subset \mathbb{H}^n.
\end{cases}
$$ (4.5)

Lemma 4.1. Let $\Omega$, $x_0$ and $P_0$ be as in the above description. Assume that $q \in \partial \Omega \cap \partial B_R(x_0)$, where $B_R(x_0) \subset \Omega$ is an open geodesic ball of $\mathbb{M}_k$ with geodesic radius $R > 0$ center at $x_0$. Denote by $\lambda_i(q)$ (respectively, $\bar{\lambda}_i(P_0(q))$) the principal curvatures of $\partial \Omega$ at $q$ (respectively, $P_0(\partial \Omega)$ at $P_0(q)$). Then

$$
\lambda_i(q) = (\bar{\lambda}_i(P_0(q))) h_k'(R),
$$ (4.6)

where

$$
h_k'(r) = \begin{cases} 
\cos \sqrt{k} r, & k > 0, \\
1 & k = 0, \\
\cosh \sqrt{-k} r & k < 0.
\end{cases}
$$ (4.7)

Proof. It suffice to prove this lemma for spherical and hyperboloid model cases.

(i) For spherical case, recall that $S^n = \{y \in \mathbb{R}^{n+1} | \sqrt{y_1^2 + \cdots + y_{n+1}^2} = \rho\}$ of radius $\rho = 1/\sqrt{k}$, centered at the origin in $\mathbb{R}^{n+1}$, with the induced Euclidean metric. Let $\{e_1, \cdots, e_{n-1}, \nu\}$ be a local orthonormal frame field in a neighborhood of $q$ such that
$e_1, \cdots, e_{n-1}$ are the principal curvature vectors of $\partial \Omega$ and $\nu$ is the exterior unit normal vector to the boundary $\partial \Omega$ of $\Omega$. Since $\langle \nu, e_j \rangle = 0$, we get that

$$0 \equiv e_i \langle \nu, e_j \rangle = \langle \nabla_{e_i} \nu, e_j \rangle + \langle \nu, \nabla_{e_i} e_j \rangle$$

for all $i, j = 1, \cdots, n - 1$, i.e.,

$$II(e_i, e_j) := \langle \nabla_{e_i} \nu, e_j \rangle = -\langle \nu, \nabla_{e_i} e_j \rangle,$$  \hspace{1cm} (4.8)

where $II$ is the second fundamental form of $\partial \Omega$, and the inner product $\langle \cdot, \cdot \rangle$ is taken in the induced Euclidean metric. Similarly, we have

$$II(\tilde{e}_i, \tilde{e}_j) = \langle \nabla_{\tilde{e}_i} \tilde{\nu}, \tilde{e}_j \rangle = -\langle \tilde{\nu}, \nabla_{\tilde{e}_i} \tilde{e}_j \rangle,$$  \hspace{1cm} (4.9)

where $\{\tilde{e}_1, \cdots, \tilde{e}_{n-1}, \tilde{\nu}\}$ is a local orthonormal frame filed in a neighborhood of $P_0(q)$ in Euclidean space $\{(x, 0) \in \mathbb{R}^{n+1} | x \in \mathbb{R}^n\}$ such that $\tilde{e}_1, \cdots, \tilde{e}_{n-1}$ are the principal curvature vectors of $P_0(\partial \Omega)$ and $\tilde{\nu}$ is the exterior unit normal vector to the boundary $P_0(\partial \Omega)$ of $P_0(\Omega)$. For any $y \in \Omega \subset S^n_\rho$, it is obvious (see, for example, [8, p.62]) that

$$y = \left( \left( \frac{1}{\sqrt{k}} \sin \sqrt{k} r \right) \theta, \frac{1}{\sqrt{k}} \cos \sqrt{k} r \right),$$

and hence

$$P_0(y) = \left( \frac{1}{\sqrt{k}} \sin \sqrt{k} r \right) \theta,$$

where $\theta \in S^{n-1}$ and $r$ is the geodesic distance from the south pole $x_0$ to $y$. By our assumption, it follows that $\nu(q) = ((\cos \sqrt{k} R) \theta, -\sin \sqrt{k} R)$. Thus, in the Euclidean space $\mathbb{R}^{n+1}$ we have that

$$e_i(q) = \tilde{e}_i(P_0(q)) \quad \text{for all} \quad i = 1, \cdots, n - 1,$$

and

$$\langle \nu(p), \tilde{\nu}(P_0(q)) \rangle = \langle \nu(q), (\theta, 0) \rangle = \cos \sqrt{k} R.$$ From this and (4.8)–(4.9), we get the corresponding part of (4.6) for $k > 0$.

(ii) Recall that

$$\mathbb{H}^n_\rho = \{ y \in \mathbb{R}^{n+1} | \langle y, y \rangle = -\rho, \ y_{n+1} > 0 \}$$

with the Riemannian metric induced from the Lorentzian metric

$$\langle y, y \rangle = -y_{n+1}^2 + y_1^2 + \cdots + y_n^2,$$

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where $\rho = 1/\sqrt{-k}$. Let $\{e_1, \ldots, e_{n-1}, \nu\}$ be a local orthonormal frame field in a neighborhood of $q$ such that $e_1, \ldots, e_{n-1}$ are the principal curvature vectors of $\partial \Omega$ and $\nu$ is the exterior unit normal vector to the boundary $\partial \Omega$ of $\Omega$. Since $\langle \nu, e_j \rangle = 0$, we get that

$$II(e_i, e_j) = \langle \nabla_{e_i} \nu, e_j \rangle = - \langle \nu, \nabla_{e_i} e_j \rangle,$$

where $\langle \cdot, \cdot \rangle$ is taken in the Lorentzian metric. Similarly, we have

$$II(\tilde{e}_i, \tilde{e}_j) = \langle \nabla_{\tilde{e}_i} \tilde{\nu}, \tilde{e}_j \rangle = - \langle \tilde{\nu}, \nabla_{\tilde{e}_i} \tilde{e}_j \rangle,$$

where $\{\tilde{e}_1, \ldots, \tilde{e}_{n-1}, \tilde{\nu}\}$ is a local orthonormal frame filed in a neighborhood of $P_0(q)$ in Euclidean space $\{(x, 0) \in \mathbb{R}^{n+1} | x \in \mathbb{R}^n \}$ such that $\tilde{e}_1, \ldots, \tilde{e}_{n-1}$ are the principal curvature vectors of $P_0(\partial \Omega)$ and $\tilde{\nu}$ is the exterior unit normal vector to the boundary $P_0(\partial \Omega)$ of $P_0(\Omega)$. Note that for any $y \in \Omega \subset \mathbb{H}_p^n$, one has (see, for example, [10, p.22]) that

$$y = \left(\frac{1}{\sqrt{-k}} \sinh \sqrt{-k} r \theta, \frac{1}{\sqrt{-k}} \cosh \sqrt{-k} r \right),$$

and hence

$$P_0(y) = \left(\frac{1}{\sqrt{-k}} \sinh \sqrt{-k} r \theta \right) \text{ for any } y \in \Omega,$$

where $\theta \in S^{n-1}$ and $r$ is the geodesic distance from the point $x_0 = (0, \ldots, 0, \rho)$ to $y$.

By the assumption, in the Euclidean space $\mathbb{R}^{n+1}$ with Lorentzian metric, we then have that

$$e_i(q) = \tilde{e}_i(P_0(q)) \text{ for all } i = 1, \ldots, n-1,$$

and

$$\nu(q) = ((\cosh \sqrt{-k} R) \theta, \sinh \sqrt{-k} R),$$

which implies

$$\langle \nu(q), \tilde{\nu}(P_0(q)) \rangle = \langle ((\cosh \sqrt{-k} R) \theta, \sinh \sqrt{-k} R), (\theta, 0) \rangle = \cosh \sqrt{-k} R.$$

Therefore, we obtain the corresponding part of (4.6).

**Theorem 4.2.** Let $\Omega$ be a domain with $C^2$ boundary in the n-dimensional space $\mathbb{M}_k$ of constant curvature $k$, $n \geq 2$, and let $\lambda_1, \ldots, \lambda_{n-1}$ denote the principal curvatures of $\partial \Omega$. Assume that $B_R(x_0) \subset \Omega$ is an open geodesic ball with radius $R > 0$ center at $x_0$ and suppose that the set $\partial \Omega \cap \partial B_R(x_0)$ is made of a finite number of points $p_1, \ldots, p_l$ such that $\lambda_j(p_m) < \tau_k(R)$ for every $j = 1, \ldots, n-1$ and every $m = 1, \ldots, l$, where $\tau_k(R)$ is as in (4.4). Let $W = W(s, x)$ be the solution to problem

$$\sum_{i=1}^n \left(\frac{1 + k|x|^2}{2}\right)^{\frac{1}{2}} \frac{\partial^2 W}{\partial x_i^2} - sW = 0 \quad \text{in } \Omega, \quad (4.10)$$

$$W = 1 \quad \text{on } \partial \Omega. \quad (4.11)$$
Then, the following formula holds for every function \( \phi \) continuous on \( \mathbb{M}_k \):

\[
\lim_{s \to +\infty} s^{\frac{n-1}{2}} \int_{\partial B_s(x_0)} \phi(x) W(s, x) \, dA_x
\]

\[
= (2\pi)^{\frac{n-1}{2}} \sum_{m=1}^{l} \phi(p_m) \left[ \frac{1}{(h'_k(R))^{n-1}} \prod_{j=1}^{n-1} \left( \tau_k(R) - \lambda_j(p_m) \right) \right]^{-1/2},
\]

where \( h'_k(R) \) is as in (4.7).

**Proof.** Let \( p_m \in \{ p_1, \ldots, p_l \} \); by applying a partition of unity, we can suppose that \( \text{supp} \phi \) does not contain any \( p_i \) different from \( p_m \).

Since there exists an isometry \( \Phi \) that maps \( \mathbb{M}_k \) onto itself such that \( \Phi x_0 = 0 \) and the equation (4.10) is invariant under the isometry map \( \Phi \), we may assume that \( x_0 = 0 \) and use the spherical coordinates about the point \( x_0 = 0 \). As in (3.9), we have

\[
\int_{\partial B_R(0)} \phi(x) e^{-\sqrt{s} \mathfrak{g}(x)} \, dA_x
\]

\[
= \int_{\mathbb{S}^{n-1}} (h_k(R))^{n-1} \phi(\exp_0(R\theta)) e^{-\sqrt{s} \mathfrak{g}(\exp_0(R\theta))} \, d\Theta(\theta)
\]

\[
= \int_{\mathbb{S}^{n-1}} \phi(P_0^{-1}(\tilde{x})) e^{-\sqrt{s} \mathfrak{g}(P_0^{-1}(\tilde{x}))} \, d\Theta(\tilde{x}),
\]

where \( \mathbb{S}^{n-1} \) is the sphere of radius \( h_k(R) \) with center at the origin in Euclidean space \( \mathbb{R}^n \), \( P_0^{-1} \) is the inverse of \( P_0 \). Here \( P_0 \) is the orthogonal projection from \( \Omega \) to the Euclidean space \( \mathbb{R}^n \) as before (Note that in order to say \( P_0 \), we must regard \( \mathbb{M}_k \) as either \( \mathbb{R}^n \), or the sphere \( \mathbb{S}^n \), or the hyperboloid model \( \mathbb{H}^n_0 \)). For convenience, we denote by \( \tilde{x} \) the point \( P_0(x) \) for any \( x \in \Omega \). Also, we can suppose that \( P_0^{-1}(\text{supp} \phi) \) does not contain the point \( -P_0^{-1}(p_m) \). As in [23, p.938], let \( \mathbb{R}^{n-1} \ni \eta = (\eta_1, \ldots, \eta_{n-1}) \mapsto \tilde{x}(\eta) \in \mathbb{S}^{n-1}_{h_k(R)} \) be the stereographic projection from the point \( -p_m \) onto the tangent space to \( \mathbb{S}^{n-1}_{h_k(R)} \) at \( \tilde{p}_m \). More precisely, take an orthogonal basis \( \xi^1, \ldots, \xi^n \) of \( \mathbb{R}^n \) with \( \xi^n = -\tilde{p}_m / h_k(R) \), and put

\[
\tilde{x}(\eta) = \frac{2h_k(R)|\eta|^2}{(2h_k(R))^2 + |\eta|^2} \xi^n + \frac{(2h_k(R))^2}{(2h_k(R))^2 + |\eta|^2} \sum_{j=1}^{n-1} \eta_j \xi^j + \tilde{p}_m.
\]

Thus, we have

\[
\int_{\partial B_R(0)} \phi(x) e^{-\sqrt{s} \mathfrak{g}(x)} \, dA_x = \int_{\mathbb{R}^{n-1}} \phi(P_0^{-1}(\tilde{x}(\eta))) e^{-\sqrt{s} \mathfrak{g}(P_0^{-1}(\tilde{x}(\eta)))} J(\eta) \, d\eta,
\]

where

\[
J(\eta) := \sqrt{\det \left( \frac{\partial \tilde{x}(\eta)}{\partial \eta_i} \cdot \frac{\partial \tilde{x}(\eta)}{\partial \eta_j} \right)} = \left( \frac{(2h_k(R))^2}{(2h_k(R))^2 + |\eta|^2} \right)^{n-1}.
\]
Set \( \tilde{\mathcal{S}}^*(\eta) = (\tilde{\mathcal{S}} \circ P_0^{-1})(\tilde{x}(\eta)) \). Then \( \tilde{\mathcal{S}}^*(0) = 0 \), and \( \tilde{\nabla} \tilde{\mathcal{S}}^*(0) = 0 \) and \( \tilde{\nabla}^2 \tilde{\mathcal{S}}^*(0) \) is positive definite, where \( \tilde{\nabla} \) and \( \tilde{\nabla}^2 \) is in the sense of Euclidean metric. In fact, differentiating \( \tilde{\mathcal{S}}^*(\eta) \) twice yields (cf. [23, p.938]):

\[
\frac{\partial^2 \tilde{\mathcal{S}}^*}{\partial \eta_i \partial \eta_j}(\eta) = \frac{\partial \tilde{x}}{\partial \eta_i}(\eta) \cdot \left( (\tilde{\nabla}^2 (\tilde{\mathcal{S}} \circ P_0^{-1})(\tilde{x}(\eta))) \frac{\partial \tilde{x}}{\partial \eta_j}(\eta) \right) + (\tilde{\nabla} (\tilde{\mathcal{S}} \circ P_0^{-1})(\tilde{x}(\eta))) \cdot \frac{\partial^2 \tilde{x}}{\partial \eta_i \partial \eta_j}(\eta), \quad i, j = 1, \ldots, n - 1,
\]

for every \( \eta \in \mathbb{R}^{n-1} \), where the dot denotes scaler product of vectors in \( \mathbb{R}^n \). It follows from \( \tilde{x}(\eta) \in S^{n-1}_{h_k(R)} \) for every \( \eta \in \mathbb{R}^{n-1} \) that

\[
\frac{\partial \tilde{x}}{\partial \eta_i}(\eta) \cdot ((\tilde{x}(\eta)) - 0) = 0, \quad i = 1, \ldots, n - 1,
\]

\[
\frac{\partial^2 \tilde{x}}{\partial \eta_i \partial \eta_j}(\eta) \cdot ((\tilde{x}(\eta)) - 0) + \frac{\partial \tilde{x}}{\partial \eta_i}(\eta) \cdot \frac{\partial \tilde{x}}{\partial \eta_j}(\eta) = 0, \quad i, j = 1, \ldots, n - 1,
\]

for all \( \eta \in \mathbb{R}^{n-1} \). Clearly, as in [23, p.939] we have

\[
-(\tilde{\nabla} (\tilde{\mathcal{S}} \circ P_0^{-1})) (\tilde{p}_m) = \frac{(\tilde{x}(0) - 0)}{h_k(R)}.
\]

Thus

\[
((\tilde{\nabla} (\tilde{\mathcal{S}} \circ P_0^{-1})) (\tilde{p}_m)) \cdot \frac{\partial \tilde{x}}{\partial \eta_i}(0) = 0, \quad i = 1, \ldots, n - 1, \quad (4.16)
\]

\[
[((\tilde{\nabla} (\tilde{\mathcal{S}} \circ P_0^{-1})) (\tilde{p}_m)) \cdot \frac{\partial^2 \tilde{x}}{\partial \eta_i \partial \eta_j}(0)] = \left( \frac{1}{h_k(R)} \right) \frac{\partial \tilde{x}}{\partial \eta_i}(0) \cdot \frac{\partial \tilde{x}}{\partial \eta_j}(0), \quad (4.17)
\]

\[
i, j = 1, \ldots, n - 1.
\]

We find from this and (4.15) that

\[
\tilde{\nabla} \tilde{\mathcal{S}}^*(0) = 0,
\]

\[
\frac{\partial^2 \tilde{\mathcal{S}}^*}{\partial \eta_i \partial \eta_j}(0) = \frac{1}{h_{k}^2(R)} \left\{ \frac{\partial \tilde{x}}{\partial \eta_i}(0) \cdot \left( (h_k(R))((\tilde{\nabla}^2 (\tilde{\mathcal{S}} \circ P_0^{-1})) (\tilde{p}_m)) + \frac{h_{k}'(R)}{h_k(R)} I \right) \frac{\partial \tilde{x}}{\partial \eta_j}(0) \right\},
\]

where \( I \) is the \((n - 1) \times (n - 1)\) identity matrix. Since

\[
\frac{\partial \tilde{x}(\eta)}{\partial \eta_i} \frac{\partial \tilde{x}(\eta)}{\partial \eta_j} = \left( \frac{(2h_k(R))^2}{(2h_k(R))^2 + |\eta|^2} \right) \delta_{ij}, \quad i, j = 1, \ldots, n - 1,
\]

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we see that the vectors $\frac{\partial x}{\partial n} (0), \ i = 1, \cdots, n-1$, make an orthogonal basis of the tangent space $T_{\tilde{p}_m} (P_0(\partial \Omega)) = T_{\tilde{p}_m} (P_0(\partial B_R(0)))$. Thus

$$\nabla^2 \tilde{\mathcal{F}} (0) = 0, \quad \det \nabla^2 \tilde{\mathcal{F}} (0) = \frac{1}{(h'_{\lambda_k}(R))^n} \det \left( h'_{\lambda_k}(R) \left( (\nabla^2 \tilde{\mathcal{F}} (P_0^{-1}) (\tilde{p}_m)) + \tau_k(R) I \right) \right).$$

Let 0 and $\{\tilde{\lambda}_j(\tilde{p}_m)\}_{j=1}^{n-1}$ be the eigenvalues of matrix $- (\nabla^2 \tilde{\mathcal{F}} (P_0^{-1}) (\tilde{p}_m))$. Clearly, $\{\tilde{\lambda}_j(\tilde{p}_m)\}_{j=1}^{n-1}$ are the principal curvatures of the boundary $P_0(\partial \Omega)$ of $P_0(\Omega)$ at $\tilde{p}_m$. It follows from Lemma 4.1 that, under map $P_0$, $\{\tilde{\lambda}_j(\tilde{p}_m)\}_{j=1}^{n-1}$ and the principal curvature $\{\lambda_j(p_m)\}_{j=1}^{n-1}$ of the boundary $\partial \Omega$ of $\Omega$ at $p_m$ have the following formula:

$$\lambda_j(p_m) = (\tilde{\lambda}_j(\tilde{p}_m)) h'_{\lambda_j}(R), \quad j = 1, \cdots, n-1,$$

where $R = \text{dist}(0, p_m)$, and $h'_{\lambda_k}(R)$ is as in (4.7). Note that $(\tilde{\mathcal{F}} (P_0^{-1})) \tilde{x} = \tilde{\mathcal{F}} (x)$ for every $x \in \tilde{\Omega}$. Since the eigenvalues of matrix $-(\nabla^2 \tilde{\mathcal{F}} (p_m) + \tau_k(R) I)$ are 0 and $(\tau_k(R) - \lambda_j(p_m))$, $j = 1, \cdots, n-1$, where $\tau_k(R)$ is the constant curvature of the geodesic sphere $\partial B_R(0)$ in $M_k$, it follows that

$$\det \nabla^2 \tilde{\mathcal{F}} (0) = \frac{1}{(h'_{\lambda_k}(R))^n} \prod_{j=1}^{n-1} [(\tau_k(R) - \lambda_j(p_m)).$$

i.e.,

$$\det \nabla^2 \tilde{\mathcal{F}} (0) = \frac{1}{(h'_{\lambda_k}(R))^n} \det(\nabla^2 \tilde{\mathcal{F}} (p_m) + \tau_k(R) I). \quad (4.18)$$

(Note that 0 and the principal curvatures $\{\lambda_j(p_m)\}_{j=1}^{n-1}$ of $\partial \Omega$ at $p_m$ are the eigenvalues of the Hessian matrix [16, p.139]

$$-\nabla^2 \tilde{\mathcal{F}}(p_m) := - \left( \left. \frac{\partial^2 \tilde{\mathcal{F}}}{\partial x_i \partial x_j} - \frac{\partial \tilde{\mathcal{F}}}{\partial x_q} \Gamma^q_{ij} \right) \right|_{p_m}. \)
Since supp $\phi$ does not contain any $p_i$ different from $p_m$, we may assume that $\mathcal{F}^*(\eta) > 0$ if $\eta \neq 0$. Hence by Laplace’s method (see [6, p. 71]), or by the stationary phase method (see [11, p. 208-217] or [23] for example),

$$\lim_{s \to +\infty} s^{n-1} \int_{\mathbb{R}^{n-1}} \phi(P_0^{-1}(\tilde{x}(\eta))) e^{-\sqrt{s}(\mathcal{F} \circ P_0^{-1})(\tilde{x}(\eta))} J(\eta) d\eta$$

$$= (2\pi)^{n-1} \phi(p_m) J(0) \left( \text{det} \nabla^2 \mathcal{F}^*(0) \right)^{-\frac{1}{2}}. \quad (4.19)$$

From $J(0) = 1$, (4.14), (4.18) and (4.19), we get

$$\lim_{s \to +\infty} s^{n-1} \int_{\partial B_R(0)} \phi(x) e^{-\sqrt{s}\mathcal{F}(x)} dA_x \quad (4.20)$$

$$= (2\pi)^{n-1} \phi(p_m) \left( \frac{1}{(h'_k(R))^{n-1}} \prod_{j=1}^{n-1} \left[ \tau_k(R) - \lambda_j(p_m) \right] \right)^{-1/2}. \quad (4.21)$$

Finally, we prove formula (4.11). It is sufficient to prove it for any nonnegative function $\phi$ (see [23, p.940]). From Lemma 3.3, one has that for all $s \geq s_\epsilon$ and any nonnegative $\phi$,

$$\int_{\partial B_R(x_0)} \phi(x) W^-_\epsilon(s, x) dA_x \leq \int_{\partial B_R(x_0)} \phi(x) W(s, x) dA_x \leq \int_{\partial B_R(x_0)} \phi(x) W^+_\epsilon(s, x) dA_x.$$

Therefore, (4.19) and the definition (3.20) implies that

$$\left( \frac{2\pi}{\sqrt{1+\epsilon}} \right)^{n-1} \sum_{m=1}^{l} \phi(p_m) \left( \frac{1}{(h'_k(R))^{n-1}} \prod_{j=1}^{n-1} \left[ \tau_k(R) - \lambda_j(p_m) \right] \right)^{-1/2} \leq \liminf_{s \to +\infty} s^{n-1} \int_{\partial B_R(x_0)} \phi(x) W(s, x) dA_x$$

$$\leq \limsup_{s \to +\infty} s^{n-1} \int_{\partial B_R(x_0)} \phi(x) W(s, x) dA_x$$

$$\leq \left( \frac{2\pi}{\sqrt{1-\epsilon}} \right)^{n-1} \sum_{m=1}^{l} \phi(p_m) \left( \frac{1}{(h'_k(R))^{n-1}} \prod_{j=1}^{n-1} \left[ \tau_k(R) - \lambda_j(p_m) \right] \right)^{-1/2}.$$

for every $\epsilon > 0$. By letting $\epsilon$ tend to 0, we get (4.11) and the proof is completed. \hfill \Box
5 Proof of main results

In this section, we shall prove the analyticity of the boundary $\partial \Omega$ and the main theorem. A domain $\Omega$ is said to satisfy the exterior geodesic sphere condition if for every $y \in \partial \Omega$ there exists a geodesic ball $B_r(y)$ such that $B_r(y) \cap \bar{\Omega} = y$. A domain $D$ satisfies the interior geodesic cone condition if for every $x \in \partial D$ there exists a finite geodesic spherical cone $K_x$ with vertex $x$ such that $K_x \subset D$ and $K_x \cap \partial D = \{x\}$.

Lemma 5.1. Let $\Omega$ be a bounded domain in $n$-dimensional space $\mathbb{M}_k$ of constant curvature $k$. Let $\Omega$ satisfy the exterior geodesic sphere condition and suppose that $D$ is a domain satisfying the interior geodesic cone condition and such that $\bar{D} \subset \Omega$. Assume that the solution $u = u(t,x)$ of problem (1.1)–(1.3) satisfies condition (1.4). Let $R$ be the positive constant given by

$$R = \lim_{s \to +\infty} \left(-\frac{1}{\sqrt{s}} \log A(s)\right), \quad (5.1)$$

where

$$W(s,x) = s \int_0^{+\infty} a(t)e^{-st}dt := A(s), \quad x \in \partial D. \quad (5.2)$$

Then the following assertions hold:

(i) for every $x \in \partial D$, $\mathfrak{z}(x) = R$, where $\mathfrak{z}$ is defined by (2.10);

(ii) $\partial D$ is real analytic;

(iii) $\partial \Omega$ is real analytic and $\partial \Omega = \{x \in \mathbb{M}_k \mid \text{dist}(x,\partial D) = R\}$;

(iv) Let $\lambda_j(y), j = 1, \cdots, n-1$ denote the $j$th principal curvature at $y \in \partial \Omega$ of the real analytic surface $\partial \Omega$; then $\lambda_j(y) < \tau_k(R), j = 1, \cdots, n-1$, for every $y \in \partial \Omega$, where $\tau_k(R)$ is given by (4.4).

Proof. (i) Let

$$W(s,x) = s \int_0^{+\infty} u(t,x)e^{-st}dt, \quad s > 0. \quad (5.3)$$

Then $W(s,x)$ satisfies elliptic boundary value problem (3.18)–(3.19). Applying Lemma 2.1 (Varadhan’s theorem), we have

$$\lim_{s \to +\infty} \left(-\frac{1}{\sqrt{s}} \log W(s,x)\right) = d(x,\partial \Omega) = \mathfrak{z}(x). \quad (5.4)$$

Since $u$ satisfies (1.4), it follows that for fixed $s > 0$, $A(s)$ is constant on $\partial D$. Therefore, $\mathfrak{z}(x) = R$ for every $x \in \partial D$. 

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(ii) It follows from the interior regularity of parabolic equations (see [20]) that $u(t,x)$ is real analytic on any compact subdomain in $\Omega$. By the implicit function theorem for real analytic function (see, for example, [15, p.69]), it suffices to prove that, for every point $x \in \partial D$, there exists a time $t^* > 0$ such that $\nabla u(t^*,x) \neq 0$. Suppose by contradiction that there exists a point $x_0 \in \partial D$ such that $\nabla u(t,x_0) = 0$ for every $t > 0$. It follows from Lemma 3.2 (ii) that

$$\int_{\partial B_R(x_0)} \exp_{x_0}^{-1}(x - x_0)u(t,x) dA_x = 0 \quad \text{for every } t > 0. \quad (5.5)$$

We may put $x_0 = 0$ by an isometry of $M_k$. Thus, (5.5) is equivalent to

$$\int_{\mathbb{S}_{h_k}^{n-1}(\mathbf{r})} \xi u(t,\xi) d\Theta(\xi) = 0 \quad \text{for every } t > 0,$$

and hence

$$\int_{\mathbb{S}_{h_k}^{n-1}(\mathbf{r})} \xi W(s,\xi) d\Theta(\xi) = 0 \quad \text{for every } s > 0. \quad (5.6)$$

On the other hand, as in [23, p.941-942] one can show that

$$\int_{\mathbb{S}_{h_k}^{n-1}(\mathbf{r})} \xi W(s,\xi) d\Theta(\xi) > 0$$

for $s > 0$ sufficiently large. This is a contradiction.

(iii) Set $\Gamma = \{ y \in M_k | d(y,D) = R \}$. For each $y \in \Gamma$ there exists a point $x \in \partial D$ such that $d(y,D) = d(y,x)$. Let $\gamma(r)$ be a geodesic starting from $x$ and ending at $y$. We claim that $\dot{\gamma}(0) := \frac{d\gamma}{dr}(0)$ is orthogonal to the tangent space of $\partial D$. In fact, let $\zeta(\mu)$ be a smooth curve in $\partial D$ with $\zeta(0) = x$. For each $\zeta(\mu)$, let $\gamma_{\mu}(r)$ be the geodesic starting from $\zeta(\mu)$ and ending at $y$, and let $L(\mu)$ be the length of the geodesic $\gamma_{\mu}(r)$ between $\zeta(\mu)$ and $y$. Then

$$L(\mu) = \int_0^{L(\mu)} \langle \dot{\gamma}_{\mu}(r), \dot{\gamma}_{\mu}(r) \rangle^{1/2} dr.$$ 

It is easy to check that $L(\mu)$ has the following variational formula (cf. [8, p.67]):

$$\left. \frac{dL(\mu)}{d\mu} \right|_{\mu=0} = \left[ \langle \dot{\gamma}_{\mu}(L(\mu)), \dot{\gamma}_{\mu}(L(\mu)) \rangle^{1/2} \right] \left( \frac{dL}{d\mu}(0) \right)$$

$$+ \left[ \langle \dot{\gamma}_0(r), \frac{\partial \gamma_{\mu}(r)}{\partial \mu} \rangle \bigg|_{\mu=0} \right]_{0}^{L(0)} - \int_0^{L(0)} \langle \dot{\gamma}_0(r), \frac{\partial \gamma_{\mu}(r)}{\partial \mu} \rangle \bigg|_{\mu=0} \right) dr$$

$$= \langle \dot{\gamma}_0(L(0)), \frac{\partial \gamma_{\mu}(L(0))}{\partial \mu} \rangle \bigg|_{\mu=0} - \langle \dot{\gamma}_0(0), \frac{\partial \gamma_{\mu}(0)}{\partial \mu} \rangle \bigg|_{\mu=0}.$$
Proof of Theorem 1.1. From Lemma 5.1, we see that $\partial \Omega$ and $\partial D$ are analytic. Let $p_1$ and $p_2$ be two distinct points in $\partial \Omega$. Then $\nabla \Phi_i$, is the unit interior normal vector of $\partial \Omega$ at $p_i$, $i = 1, 2$. Let $\gamma_i(r)$ be the geodesic satisfying $\gamma_i(0) = p_i$ and $\dot{\gamma}_i(0) = \nabla \Phi_i$, $i = 1, 2$. It follows from Lemma 5.1 that $\gamma_i(R) \in \partial D$ and $\gamma_1(R) \neq \gamma_2(R)$. Let us denote $\gamma_i(R)$ by $P_i$, and by $\Phi_i$ the isometric map of $\mathbb{M}_k$ satisfying $\Phi_i 0 = P_i$, $i = 1, 2$. Then for $x \in B_R(0)$, define the function $v(t, x)$ by

$$v(t, x) = u(t, \Phi_1 x) - u(t, \Phi_2 x). \quad (5.7)$$

Lemma 3.1 implies that $v(t, x)$ satisfies equation (1.1) in $(0, +\infty) \times B_R(0)$. By (1.4), we have

$$v(t, 0) = u(t, P_1) - u(t, P_2) = 0 \quad \text{for all } t > 0.$$ 

It follows from Lemma 3.2 (i) that

$$\int_{\partial B_R(0)} v(t, x)dA_x = 0 \quad \text{for all } t > 0,$$

and hence

$$\int_{\partial B_R(P_1)} u(t, x)dA_x = \int_{\partial B_R(P_2)} u(t, x)dA_x \quad \text{for all } t > 0.$$
Thus, by the definition of (5.3), we obtain
\[
\int_{\partial B_R(p_1)} W(s, x) dA_x = \int_{\partial B_R(p_2)} W(s, x) dA_x \quad \text{for all } s > 0. \tag{5.8}
\]
Multiplying both sides of (5.8) by \(s^{n-1} \frac{\partial}{\partial r} \) and letting \(s \to +\infty\), by (4.12) of Theorem 4.2 (with \(\phi \equiv 1\)) we get
\[
\prod_{j=1}^{n-1} [\tau_k(R) - \lambda_j(p_1)] = \prod_{j=1}^{n-1} [\tau_k(R) - \lambda_j(p_2)],
\]
which implies
\[
\prod_{j=1}^{n-1} [\tau_k(R) - \lambda_j(x)] = \text{constant}, \quad \text{for every } x \in \partial \Omega. \tag{5.9}
\]
Let us put
\[
F = F(\beta_1, \cdots, \beta_{n-1}) = -\prod_{j=1}^{n-1} [\tau_k(R) - \beta_j]. \tag{5.10}
\]
Clearly, \(F\) is of class \(C^1\), and \(F(\lambda_1, \cdots, \lambda_{n-1}) = \text{const}\) on \(\partial \Omega\). Since \(\lambda_j(x) < \tau_k(R)\) on \(\partial \Omega, \ j = 1, \cdots, n - 1\), we have
\[
\text{const} > \frac{\partial F(\beta_1, \cdots, \beta_{n-1})}{\partial \beta_j} > \text{const} > 0 \quad (j = 1, \cdots, n - 1),
\]
at least on \(\partial \Omega\), i.e., for \(\beta_j = \lambda_j (i = 1, \cdots, n - 1)\). It follows from Lemma 2.2 (Alexandrov's theorem) that \(\partial \Omega\) must be a geodesic sphere in \(\mathbb{M}_k\). \(\square\)

For the wave equations and the Schrödinger equations, we have the following:

**Theorem 5.2.** Let \(\Omega\) be a bounded domain in the \(n\)-dimensional space \(\mathbb{M}_k\) of constant curvature \(k\) with the metric \(g_{ij} = 4(1 + k|x|^2)^2 \frac{\delta_{ij}}{(1 + k|x|^2)^2}\) (in case of \(k > 0\), \(\Omega\) is required to lie in a hemisphere), \(n \geq 2\). Let \(\Omega\) satisfy the exterior geodesic sphere condition and assume that \(\bar{D}\) is a domain, with boundary \(\partial \bar{D}\), satisfying the interior geodesic cone condition, and such that \(\bar{D} \subset \Omega\).

Suppose \(v\) satisfies the following wave equation (5.11) or Schrödinger's equation (5.12):
\[
\begin{cases}
\frac{\partial^2 v}{\partial t^2} = \sum_{i=1}^n \frac{(1+k|x|^2)^2}{4} \frac{\partial^2 v}{\partial x_i^2} & \text{in } (0, +\infty) \times \Omega \\
v = 1 & \text{on } (0, +\infty) \times \partial \Omega, \\
u = 0, \quad \frac{\partial v}{\partial r} = 0 & \text{on } \{0\} \times \Omega.
\end{cases} \tag{5.11}
\]
\[
\begin{aligned}
&\left\{
\begin{array}{l}
-i\frac{\partial v}{\partial t} = \sum_{i=1}^{n} \frac{(1+k|x|^2)^2}{4} \frac{\partial^2 v}{\partial x_i^2} \\
v = b(t) \\
u = 0
\end{array}
\right. \\
&\text{in} \ (0, +\infty) \times \Omega \\
&\text{on} \ (0, +\infty) \times \partial\Omega, \\
&\text{on} \ \{0\} \times \Omega,
\end{aligned}
\] (5.12)

If \(v\) satisfies the extra condition:
\[
v(t, x) = a(t), \quad (t, x) \in (0, +\infty) \times \partial D,
\] (5.13)

for some function \(a : (0, +\infty) \rightarrow (0, +\infty)\), then \(\Omega\) must be a geodesic ball in \(\mathbb{M}_k\).

Proof. It is easily verified that the balance law still holds for the wave equation (5.11) and Schrödinger’s equation (5.12).

Now, the proof is similar to that of Theorem 1.1, only noticing the following two techniques: For the wave equation, let us write
\[
V(s, x) = \sqrt{s} \int_{0}^{+\infty} v(t, x)e^{-\sqrt{s}t}dt, \quad s > 0.
\] (5.14)

From (5.11), we get that for any fixed \(s > 0\), \(V(s, x)\) satisfies the elliptic boundary value problem:
\[
\left\{
\begin{array}{l}
\sum_{i,j=1}^{n} \frac{(1+k|x|^2)^2}{4} \frac{\partial^2 V}{\partial x_i^2} - sV(s, x) = 0 \\
V = 1
\end{array}
\right. \quad \text{in} \ \Omega, \\
\text{on} \ \partial\Omega.
\] (5.15)

Moreover, by (5.13) it follows that \(V\) is constant on \(\partial D\). Indeed,
\[
V(s, x) = \sqrt{s} \int_{0}^{+\infty} a(t)e^{-\sqrt{s}t}dt := c_1(s), \quad \forall x \in \partial D, \quad s > 0.
\]

For the Schrödinger equation (5.12), by putting
\[
V(s, x) = \int_{0}^{+\infty} v(t, x)e^{-ist}dt, \quad s > 0,
\] (5.16)
we get elliptic equation
\[
\left\{
\begin{array}{l}
\sum_{i,j=1}^{n} \frac{(1+k|x|^2)^2}{4} \frac{\partial^2 V}{\partial x_i^2} - sV(s, x) = 0 \\
V(s, x) = \int_{0}^{+\infty} b(t)e^{ist}dt
\end{array}
\right. \quad \text{in} \ \Omega, \\
\text{on} \ \partial\Omega.
\] (5.17)

for any fixed \(s > 0\). Let us replace \(V(s, x)\) by \(\int_{0}^{+\infty} \frac{V(s, t)}{b(t)e^{ist}dt}\) (still denote it by \(V(s, x)\)), we also obtain the form of (5.15). Similarly, we have
\[
V(s, x) = \int_{0}^{+\infty} a(t)e^{-ist}dt := c_2(s), \quad \forall x \in \partial D, \quad s > 0.
\]
Since $\Omega$ is a bounded domain in $\mathbb{M}_k$ (in the case $k \neq 0$, $|x| < \rho$ for any $x \in \overline{\Omega}$), there exist two constants $\alpha > 0$ and $\beta > 0$ such that

$$\alpha \leq \frac{1 + k|x|^2}{4} \leq \beta$$

for all $x \in \overline{\Omega}$.

By using the maximum principle to elliptic equation (5.15), we obtain $V \leq 1$ on $\overline{\Omega}$. Thus, Varadhan’s theorem can be applied. □

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