ON THE RANK FUNCTION OF GENERIC LINEARLY CONSTRAINED FRAMEWORKS IN $\mathbb{R}^2$

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Abstract. We study the rank function of generic rigidity of bar-and-joint frameworks in $\mathbb{R}^2$ such that some vertices are constrained to lie in some lines. The rigidity of such frameworks is characterised by Streinu and Theran [10]. In this paper we characterise the rank function of such frameworks in terms of vertex covers by extending Lovász and Yemini’s characterisation of the rank function of the 2-dimensional generic bar-and-joint rigidity matroid [8].

1. Introduction

A (2-dimensional) bar-and-joint framework is a pair $(G, p)$ where $G = (V, E)$ is a simple graph and $p : V \rightarrow \mathbb{R}^2$ is a map which is called the realisation map of the framework $(G, p)$. The framework $(G, p)$ is called rigid if every continuous motion which fixes the edge lengths is a congruence of $\mathbb{R}^2$. Determining whether a framework $(G, p)$ is rigid is NP-hard [1]. The problem is easier to deal with when $p$ is generic, that is algebraically independent over $\mathbb{Q}$. The rigidity of such frameworks only depends on the underlying graph $G$ [2].

The rigidity matrix $R(G, p)$ of $(G, p)$ is a $|E| \times 2|V|$ matrix such that the row indexed by an edge $e = uv$ has $p(u) - p(v)$ and $p(v) - p(u)$ in the two columns indexed by $u$ and $v$ and zeros elsewhere. We can construct a matroid $\mathcal{R}(G, p)$ on $E$ from $R(G, p)$ by defining a set $F \subseteq E$ to be independent in $\mathcal{R}(G, p)$ if the set of rows of $R(G, p)$ corresponding to the edges in $F$ is linearly independent. A framework $(G, p)$ is infinitesimally rigid if rank $R(G, p) = 2|V| - 3$ or equivalently if $E$ has rank $2|V| - 3$ in $\mathcal{R}(G, p)$. Since the rigidity of $(G, p)$ depends only on the underlying graph $G$ when $p$ is generic, we can talk about the rigidity of a graph $G$. A graph $G$ is rigid as a bar-and-joint framework if there exists a generic $p$ for which $(G, p)$ is infinitesimally rigid. 2-dimensional rigid graphs are characterised by Pollaczek-Geiringer [9] and rediscovered by Laman [7]. Using the fact that generic frameworks give rise to the same matroid, we can define the generic rigidity matroid $\mathcal{R}_2(G)$ of $G$ by setting $\mathcal{R}_2(G) = \mathcal{R}(G, p)$ for some generic $p$. A characterisation of the rank function $r_2$ of $\mathcal{R}_2$ is given by Lovász and Yemini [8] which is given below. The technical definitions in the theorem below will be given in Section 2.

Theorem 1.1. [8] Let $G = (V, E)$ be a simple graph. Then

$$r_2(G) = \min \sum_{i=1}^{k} (2|X_i| - 3)$$

where the minimum is taken over all 1-thin covers of $G$.

With the help of Theorem 1.1, they also show the following.

Theorem 1.2. [8] Every 6-connected graph is rigid as a bar-and-joint framework in $\mathbb{R}^2$. 

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In this paper we will consider the frameworks that have some extra constraints. These constraints will force some vertices to lie in some lines. A looped simple graph $G = (V, E, L)$ is a graph such that $(V, E)$ is a simple graph and $L$ is a set of loops. Note that there are no multiple edges in $E$, but $L$ is allowed to have multiple loops. A 2-dimensional linearly constrained framework is a triple $(G, p, q)$ where $G = (V, E, L)$ is a looped simple graph, $p : V \to \mathbb{R}^2$ and $q : L \to \mathbb{R}^2$ are maps. We interpret this definition as follows: $(G, p)$ is a usual 2-dimensional (bar-and-joint) framework and $q(l)$, where $l$ is a loop at $v$, constrains $v$ to lie in the line which contains the point $p(v)$ and has $q(l)$ as its normal vector. The rigidity matrix $R(G, p, q)$ of $(G, p, q)$ is a $(|E| + |L|) \times 2|V|$ matrix which is obtained from $R(G, p)$ by adding $|L|$ new rows such that the row corresponding to a loop $l$ at $v$ has $q(l)$ in the two columns corresponding to $v$ and zeros elsewhere. A (2-dimensional) linearly constrained framework is infinitesimally rigid if $\text{rank } R(G, p, q) = 2|V|$. A looped simple graph is rigid as a linearly constrained framework if rank $R(G, p, q) = 2|V|$ for some $(p, q)$. Similar to the bar-and-joint case we can construct a generic linearly constrained rigidity matrix $R_{lc}^2(G)$ of a looped simple graph $G = (V, E, L)$ from a linearly constrained rigidity matrix $R(G, p, q)$ for which $(p, q)$ is generic. For more information on the linearly constrained frameworks as well as some higher dimensional results we refer the interested readers to [3, 6].

Streinu and Theran characterised the rigid looped simple graphs in $\mathbb{R}^2$ [10]. Before giving their result we first need to give some definitions. Let $G = (V, E, L)$ be a looped simple graph. For a set $T \subseteq E \cup L$, we use $V(T)$ to denote the set of vertices incident with the edges and loops in $T$. We say that $G$ is $(2,0,3)$-graded-sparse if

1. $|T| \leq 2|V(T)| - 3$ for all $\emptyset \subset T \subseteq E$ and
2. $|T| \leq 2|V(T)|$ for all $T \subseteq E \cup L$.

A $(2,0,3)$-graded-sparse graph is called $(2,0,3)$-tight if $|E| + |L| = 2|V|$.

**Theorem 1.3.** [10] Let $G = (V, E, L)$ be a looped simple graph. Then $G$ is rigid (as a linearly constrained framework) in $\mathbb{R}^2$ if and only if it contains a spanning $(2,0,3)$-tight subgraph.

Let $r_{lc}^2$ denote the rank function of $R_{lc}^2$. In this paper we will characterise $r_{lc}^2$ in a similar way to Lovász and Yemini used for Theorem 1.1. Since every simple graph $(V, E)$ can be considered as a looped simple graph $(V, E, L)$ with $L = \emptyset$, our characterisation of $r_{lc}^2$ will be an extension of the characterisation given by Lovász and Yemini. The following theorems are our main results. The technical definitions will be given in the corresponding sections.

**Theorem 1.4.** Let $G = (V, E, L)$ be a looped simple graph. Then

$$r_{lc}^2(G) = \min\{\text{val}(\mathcal{X}) : L' \subseteq L, \mathcal{X} \text{ is an admissible 1-thin cover of } G - L'\}.$$

Using Theorem 1.4 we shall show the following.

**Theorem 1.5.** Every 6-balanced looped simple graph is rigid as a linearly constrained framework in $\mathbb{R}^2$.

2. **The Rank Function**

Given a looped simple graph $G = (V, E, L)$ let us define a function $f : 2^{E \cup L} \to \mathbb{Z}$ by

$$f(T) := \begin{cases} 2|V(T)| - 3, & T \subseteq E \\ 2|V(T)|, & T \cap L \neq \emptyset \end{cases}$$

where $V(T)$ is the set of all vertices incident with the edges or loops in $T$. 

Lemma 2.1. The function $f$ is submodular (i.e., $f(T_1) + f(T_2) \geq f(T_1 \cup T_2) + f(T_1 \cap T_2)$ for all $T_1, T_2 \subseteq E \cup L$).

Proof. Let $T_1, T_2 \subseteq E \cup L$, and $a, b \in \{0, 3\}$ such that we have $f(T_1) = 2|V(T_1)| - a$ and $f(T_2) = 2|V(T_2)| - b$. By symmetry we may assume $a \leq b$. Then we have $f(T_1 \cup T_2) = 2|V(T_1 \cup T_2)| - a$ (since if $T_1$ or $T_2$ contain some loops then so does $T_1 \cup T_2$); and $f(T_1 \cap T_2) \leq 2|V(T_1 \cap T_2)| - b$ (since $T_1 \cap T_2$ does not necessarily contain loops when $T_1$ and $T_2$ do so). This gives

$$f(T_1) + f(T_2) = 2|V(T_1)| - a + 2|V(T_2)| - b$$
$$= 2|V(T_1) \cup V(T_2)| - a + 2|V(T_1) \cap V(T_2)| - b$$
$$\geq 2|V(T_1 \cup T_2)| - a + 2|V(T_1 \cap T_2)| - b$$
$$\geq f(T_1 \cup T_2) + f(T_1 \cap T_2)$$

where the first inequality follows from $V(T_1 \cap T_2) \subseteq (V(T_1) \cup V(T_2))$ since there may be vertices $x, y \in V(T_1) \cap V(T_2)$ such that $xy \notin E$. □

By Lemma 2.1 and Edmonds [4] we have the following.

Theorem 2.2. Let $G = (V, E, L)$ be a looped simple graph and $f$ be the function defined above. Put

$$I_f := \{T \subseteq E \cup L : |I| \leq f(I) \text{ for all } I \subseteq T \text{ with } I \neq \emptyset\}$$

Then $\mathcal{M}(E \cup L, I_f)$ is a matroid with rank function $\hat{f} : 2^{E \cup L} \to \mathbb{Z}$ given by

$$\hat{f}(T) := \min \{|T'| + \sum_{i=0}^{k} f(T_i) : T' \subseteq T \text{ and } \{T_0, T_1, \ldots, T_k\} \text{ is a partition of } T \setminus T'\}.$$ 

By Theorems 1.3 and 2.2 we have the following.

Corollary 2.3. Let $G = (V, E, L)$ be a looped simple graph. Then $\mathcal{M}(E \cup L, I_f)$ is isomorphic to $\mathcal{R}_2^G(G)$ and so

$$r_2^G(T) = \hat{f}(T) = \min \{|T'| + \sum_{i=0}^{k} f(T_i) : T' \subseteq T \text{ and } \{T_0, T_1, \ldots, T_k\} \text{ is a partition of } T \setminus T'\}.$$ 

Proof. The statement follows by comparing the independent sets in each matroid, that is $(2,0,3)$-graded-sparsity is equivalent to the conditions in the definition of $I_f$. One may think that the empty set seem problematic in this comparison as $f(\emptyset) = -3$. But the condition $I \neq \emptyset$ in the definition of $I_f$ clears this issue. □

Let $G = (V, E, L)$ be a looped simple graph $T \subseteq E \cup L$, $T' \subseteq T$ and $\{T_0, T_1, \ldots, T_k\}$ be a partition of $T \setminus T'$. Consider a simple edge $e = xy \in T$. If $e \in T'$, it contributes one to the minimum in the definition of $r_2^G$. Since $f(\{e\}) = 2|\{x, y\}| - 3 = 1$, we can obtain the same value in the minimum by replacing $T'$ by $T' - e$ and $\{T_0, T_1, \ldots, T_k\}$ by $\{T_0, T_1, \ldots, T_k, \{e\}\}$. Hence the rank can be obtained by considering $T'$ with $T' \subseteq L$.

The rank function $r_2^G$ has no condition on the partition $\{T_0, T_1, \ldots, T_k\}$ of $T \setminus T'$, in particular $|V(T_i) \cap V(T_j)|$ can be arbitrarily large and every $T_i$ is allowed to contain some loops. We will show that the minimum value can be obtained from the partitions which

\begin{footnote}{Another way of dealing with this is to set $f(\emptyset) = 0$. Then $\hat{f}$ will be intersecting submodular instead of submodular. We can still use Edmonds’ result [4] on this new $\hat{f}$ to obtain Theorem 2.2. See, for example [5, Theorem 13.4.2] for the statement of Edmonds’ theorem specifically for the intersecting submodular functions.}


satisfy $|V(T_1) \cap V(T_j)| \leq 1$ and have only one member, say $T_0$, that may contain some loops. We need the following lemma for this.

**Lemma 2.4.** Let $G = (V, E, L)$ be a looped simple graph and $T_1, T_2 \subseteq E \cup L$.

(i) If $|V(T_1) \cap V(T_2)| \geq 2$, then $f(T_1 \cup T_2) \leq f(T_1) + f(T_2)$.

(ii) If $T_1 \cap L$ and $T_2 \cap L$ are both nonempty, then $f(T_1 \cup T_2) \leq f(T_1) + f(T_2)$.

**Proof.** We shall prove (i) and (ii) simultaneously. As in the proof of Lemma 2.1 let $a, b \in \{0, 3\}$ such that we have $f(T_1) = 2|V(T_1)| - a$ and $f(T_2) = 2|V(T_2)| - b$. By symmetry we may assume $a \leq b$. Then we have $f(T_1 \cup T_2) = 2|V(T_1 \cup T_2)| - a$. Note that (i) corresponds to $b = 0$ or $b = 3$ and (ii) corresponds to $b = 0$. We can write

$$f(T_1) + f(T_2) = 2|V(T_1)| - a + 2|V(T_2)| - b$$

$$= 2|V(T_1) \cup V(T_2)| - a + 2|V(T_1) \cap V(T_2)| - b$$

$$= 2|V(T_1 \cup T_2)| - a + 2|V(T_1) \cap V(T_2)| - b$$

$$\geq f(T_1 \cup T_2)$$

where the inequality follows from the fact that $|V(T_1) \cap V(T_2)| \geq 2$ if $b = 3$, and it trivially holds when $b = 0$. \hfill \Box

We can now give the rank function $r^lc_2$ as follows.

**Theorem 2.5.** Let $G = (V, E, L)$ be a looped simple graph and $f$ be the function defined above. Then

$$r^lc_2(T) = \min \{ |L'| + \sum_{i=0}^{k} f(T_i) : L' \subseteq T \cap L \text{ and } \{T_0, T_1, \ldots, T_k\} \text{ is a partition of } T \setminus L' \}$$

where $T_1, T_2, \ldots, T_k$ have no loops and $|V(T_i) \cap V(T_j)| \leq 1$ for distinct $i, j$.

**Proof.** The condition $L' \subseteq T \cap L$ follows from the paragraph just after the proof of Corollary 2.3 ($T'$ is relabelled as $L'$) and the conditions $T_1, T_2, \ldots, T_k$ have no loops and $|V(T_i) \cap V(T_j)| \leq 1$ for distinct $i, j$. Follow from Lemma 2.4 and by relabelling if necessary. (By relabelling only the set $T_0$ is allowed to contain loops.) \hfill \Box

The rank function in Theorem 2.5 deals with both edges or loops (partitions) and vertices (condition on $|V(T_i) \cap V(T_j)|$). We can replace the partition of edges or loops by some family of subsets of $V$ so that the rank function deals only with some subsets of vertices. In order to do this we need give some definitions first. Let $G = (V, E, L)$ be a looped simple graph. A family $\mathcal{X} = \{X_0, X_1, X_2, \ldots, X_k\}$ of subsets of $V$ satisfying $|X_i| \geq 2$, for all $1 \leq i \leq k$, is said to be a cover of $G$ if every edge $e \in E$ and every loop $l \in L$ is induced by at least one member in $\mathcal{X}$. A cover $\mathcal{X} = \{X_0, X_1, X_2, \ldots, X_k\}$ of $G - L'$ where $L' \subseteq L$ is said to be admissible if every loop in $L \setminus L'$ is induced by the vertices in $X_0$ and every loop in $L'$ is induced by the vertices in $V \setminus X_0$. The set $X_0$, which is allowed to be empty, is called the looped member of $\mathcal{X}$. We say $\mathcal{X}$ is $t$-thin (for some $t$) if $|X_i \cap X_j| \leq t$ for all $0 \leq i < j \leq k$. The value of the admissible cover $\mathcal{X}$ of $G - L'$ which is denoted by $\text{val}(\mathcal{X})$ is defined as

$$\text{val}(\mathcal{X}) := |L'| + 2|X_0| + \sum_{i=1}^{k} (2|X_i| - 3).$$

For a set $X \subseteq V$, we use $E_G(X)$ and $L_G(X)$ to denote the set of simple edges and respectively loops in $G$ which are induced by the vertices in $X$. We can now give a proof of Theorem 1.3.
Theorem 1.4. Let \( G = (V, E, L) \) be a looped simple graph. Then

\[
r_{2}^{lc}(G) = \min \{ \text{val}(\mathcal{X}) : L' \subseteq L, \mathcal{X} \text{ is an admissible 1-thin cover of } G - L' \}.
\]

Proof. Let \( \mathcal{X} = \{X_0, X_1, \ldots, X_k\} \) be an admissible 1-thin cover of \( G - L' \) for some \( L' \subseteq L \). Consider the sets \( T_0 = E_G(X_0) \cup L_G(X_0) \), and \( T_i = E_G(X_i) \), \( 1 \leq i \leq k \). Note that \( V(T_j) \subseteq X_j \) for all \( 0 \leq j \leq k \). Possibly some \( T_i \) are empty. Since the empty set is not allowed in a partition, we discard all such \( T_i \), and by relabelling if necessary, we may assume \( \{T_0, T_1, \ldots, T_m\} \) is a partition of \( (E \cup L) \setminus L' \) for some \( m \leq k \), and the sets \( T_i \), \( 0 \leq i \leq m \), satisfy the conditions in the statement of Theorem 2.5. Now the facts that \( f(T_0) \leq 2|X_0| - 3 \) for all \( 1 \leq i \leq m \), satisfy the conditions in the statement of Theorem 2.5 imply that

\[
r_{2}^{lc}(G) \leq |L'| + \sum_{i=0}^{m} f(T_i) \leq |L'| + 2|X_0| + \sum_{i=1}^{k} (2|X_i| - 3) = \text{val}(\mathcal{X}).
\]

We need to show that there exists an admissible 1-thin cover whose value hits the rank to finish the proof. To do this let \( L' \subseteq L \) and \( T_0, T_1, \ldots, T_k \) be a partition of \( (E \cup L) \setminus L' \) satisfying the conditions in the statement of Theorem 2.5 from which \( r_{2}^{lc}(G) \) can be obtained. We split the proof into two cases.

Case 1. \( T_0 \cap L \neq \emptyset \).

Let \( \mathcal{X} = \{X_0, X_1, \ldots, X_k\} \) such that \( X_i = V(T_i) \) for all \( 0 \leq i \leq k \). Then \( \mathcal{X} \) is an admissible 1-thin cover of \( G - L' \). We have \( f(T_0) = 2|X_0| \) and \( f(T_i) = 2|X_i| - 3 \) for all \( 1 \leq i \leq k \) and so

\[
r_{2}^{lc}(G) = |L'| + \sum_{i=0}^{m} f(T_i) = |L'| + 2|X_0| + \sum_{i=1}^{k} (2|X_i| - 3) = \text{val}(\mathcal{X}).
\]

Case 2. \( T_0 \subseteq E \).

Let \( \mathcal{X} = \{X_0, X_1, X_2, \ldots, X_k, X_{k+1}\} \) where \( X_0 = \emptyset, X_i = V(T_i) \) for all \( 1 \leq i \leq k \) and \( X_{k+1} = V(T_0) \). Then \( \mathcal{X} \) is an admissible 1-thin cover of \( G - L' \) whose looped member is the empty set. We have \( f(T_0) = 2|X_{k+1}| - 3, f(T_i) = 2|X_i| - 3 \) for all \( 1 \leq i \leq k \) and similar to the previous case we can deduce

\[
r_{2}^{lc}(G) = |L'| + \sum_{i=0}^{k} f(T_i) = |L'| + 2|X_0| + \sum_{i=1}^{k+1} (2|X_i| - 3) = \text{val}(\mathcal{X}).
\]

The proof of Case 2 above gives the relation between the covers we use and the covers used by Lovász and Yemini. Basically, if we would like to get the rank of a looped simple graph \( G = (V, E, L) \), for which \( L = \emptyset \) (a simple graph), we should set the looped member as the empty set. Such covers are the 1-thin covers Lovász and Yemini used to characterise the rank function \( r_{2} \) of \( \mathcal{R}_2 \).

3. An Application

Let \( G = (V, E, L) \) be a looped simple graph. We say \( G \) is \( k \)-balanced if every connected component of \( G - T \) where \( T \subseteq V \) with \( |T| \leq k \), has at least \( k - |T| \) vertices with loops. Note that by definition every \( k \)-balanced graph has at least \( k \) loops (by taking \( T = \emptyset \)). Note also that a \( k \)-balanced graph does not need to be connected. See Figure 1 for some examples of \( 3 \)-balanced graphs. The underlying simple graph of the graph drawn on the left in Figure 1 is \( 3 \)-connected, and adding three loops to different vertices makes it \( 3 \)-balanced. (Adding \( k \) loops to distinct vertices in a \( k \)-connected graph always gives a \( k \)-balanced graph.) The graph in the middle is not \( 3 \)-connected, but it is \( 3 \)-balanced. If we take the disjoint union of two copies of the graph in the middle, we obtain the graph on the right. Clearly, the graph
we obtain after this operation is not even connected whereas being 3-balanced is preserved.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{3-balanced_graphs.png}
\caption{Some examples of 3-balanced graphs.}
\end{figure}

**Theorem 1.5.** Every 6-balanced looped simple graph is rigid as a linearly constrained framework in $\mathbb{R}^2$.

**Proof.** Let $G = (V, E, L)$ be a counterexample with $|V|$ being minimum and with respect to this $|E|$ being maximum and subject to this $|L|$ being minimum. Since $G$ is not rigid there exists a set $L' \subseteq L$ and an admissible 1-thin cover $X' = \{X_0, X_1, \ldots, X_k\}$ of $G - L'$ such that

$$r_2^L(G) = \text{val}(X) = |L'| + 2|X_0| + \sum_{i=1}^{k} (2|X_i| - 3) < 2|V|$$

by Theorem [14]. Using the maximality of $E$, $(X_i, E(G[X_i]))$ is a complete graph for all $0 \leq i \leq k$. That is, $xy \in G$ for all distinct $x, y \in X_i$. We see that every vertex has at most one loop by the minimality of $L$.

**Claim 3.1.** Every vertex $v \in V$ without a loop is contained in at least two $X_i$, $0 \leq i \leq k$.

**Proof of Claim.** Suppose the contrary and let $v$ be a loopless vertex that belongs to only one $X_j$ for some $0 \leq j \leq k$. The fact that $v$ has no loops and $G$ being 6-balanced imply that $d(v) \geq 6$. Since $X_j$ is the only set in the cover $X$ that contains $v$, all neighbours of $v$ belong to $X_j$ and this implies $|X_j| \geq 7$. Now consider the graph $G' = G - v$ and let $X' = \{X_0', X_1', \ldots, X_k'\}$ such that $X_j' = X_j - v$ and $X_i' = X_i$ for all $i \neq j$. Then $X'$ is an admissible 1-thin cover of $G' - L'$ and we have

$$\text{val}(X') = |L'| + 2|X_0'| + \sum_{i=1}^{k} (2|X_i'| - 3) = |L'| + 2|X_0| + \sum_{i=1}^{k} (2|X_i| - 3) - 2$$

$$\leq |L'| + 2|X_0| + \sum_{i=1}^{k} (2|X_i| - 3) - 2$$

$$< 2|V| - 2 = 2|V - v|.$$ 

By the minimality of $|V|$, the graph $G'$ cannot be 6-balanced. Then there exists a non-empty set $T \subset V$ with $|T| \leq 5$ such that some connected component(s) of $G' - T$ has less than $6 - |T|$ loops. Let $C_1, C_2, \ldots, C_m, J_1, J_2, \ldots, J_n$ be the connected components of $G' - T$ such that the components $C_i$, $1 \leq i \leq m$ fail to have at least $6 - |T|$ loops, and the components $J_s$, $1 \leq s \leq n$ have at least $6 - |T|$ loops.

If $v$ does not have a neighbour in some $C_i$ for $1 \leq i \leq m$, then $C_i$ will be a connected component of $G - T$ with less than $6 - |T|$ loops, a contradiction to $G$ being 6-balanced. Thus $v$ has a neighbour $v_i \in C_i$ for all $1 \leq i \leq m$. The facts that $(X_j, E(G[X_j]))$ is a complete graph and that all neighbours of $v$ are contained in $X_j$ now imply that these $v_i$,..
1 \leq i \leq m$ are pairwise adjacent in $G' - T$. Thus we may assume $m = 1$ that is, there is only one component $C_1$ of $G' - T$ with less than $6 - |T|$ loops.

If $v$ does not have a neighbour $x \in J_s$, for some $1 \leq s \leq n$, then $G[V(C_1) \cup \{v\}]$ is a connected component of $G - T$ with less than $6 - |T|$ loops, a contradiction to $G'$ being 6-balanced. Therefore $vx \in E$ for some $x \in J_s$. Then since $(X_j, E'(G|X_j))$ is a complete graph and $N(v) \subseteq X_j$, we see that $v_1x \in E(G' - T)$, a contradiction to the fact that $v_1$ is contained in a component of $G' - T$ with less than $6 - |T|$ loops and $x$ is contained in a component of $G' - T$ with at least 6 - $|T|$ loops. □

Let $Y$ denote the set of vertices in $X_0$ whose neighbourhood is contained in $X_0$, and put $Z = X_0 \setminus Y$ (i.e., $Z$ is the set of vertices in $X_0$ with at least one neighbour in $V \setminus X_0$). For a vertex $v \in V$ let $l(v)$ denote the number of loops in $G$ incident with $v$. Since every vertex in $V$ has at most one loop by the minimality of $L$, we have $0 \leq l(v) \leq 1$ and $\sum_{v \in V \setminus X_0} l(v) = |L'|$.  

Claim 3.2. For $v \in V$, we have

\begin{enumerate}[(i)]
\item $\sum_{X_i,v \in X_i} \left(2 - \frac{3}{|X_i|}\right) \geq 2 - \frac{l(v)}{2}$ when $v \in (V \setminus X_0)$,
\item $\sum_{X_i,v \in X_i} \left(2 - \frac{3}{|X_i|}\right) = 2 - \frac{3}{|X_0|}$ when $v \in Y$, and
\item $\sum_{X_i,v \in X_i} \left(2 - \frac{3}{|X_i|}\right) \geq 2 - \frac{3}{|X_0|} + \frac{1}{2}$ when $v \in Z$.
\end{enumerate}

Proof of Claim. (i): By relabelling if necessary, we may assume $v$ is contained in the sets $X_1, X_2, \ldots, X_m$ such that $|X_1| \geq |X_2| \geq \cdots \geq |X_m|$. 

We first consider the case $l(v) = 0$. Then by Claim 3.1, $m \geq 2$. The facts that $G$ is 6-balanced and that $l(v) = 0$ imply $d(v) \geq 6$. Combining $|N(v)| = d(v) \geq 6$ with the fact that every edge incident with $v$ is contained in some $X_i$, we obtain $\sum_{X_i,v \in X_i} (|X_i| - 1) \geq 6$. Then depending on $m$, there are three subcases.

$m \geq 4$: Since $|X_i| \geq 2$, $\sum_{X_i,v \in X_i} \left(2 - \frac{3}{|X_i|}\right) \geq 4 \cdot \frac{1}{2} = 2$.

$m = 3$: $\sum_{X_i,v \in X_i} (|X_i| - 1) \geq 6$ implies $|X_1| \geq 3$. Then $\sum_{X_i,v \in X_i} \left(2 - \frac{3}{|X_i|}\right) \geq 1 + \frac{1}{2} + \frac{1}{2} = 2$.

$m = 2$: $\sum_{X_i,v \in X_i} (|X_i| - 1) \geq 6$ implies $|X_1| \geq 4$. Then either $|X_1| = |X_2| = 4$ or $|X_1| = 5$, $|X_2| \geq 3$ or $|X_1| \geq 6$, $|X_2| \geq 2$. Then $\sum_{X_i,v \in X_i} \left(2 - \frac{3}{|X_i|}\right)$ is at least $\frac{5}{4} + \frac{5}{4} = 2$ or $\frac{5}{4} + 1 = \frac{9}{4}$ or $\frac{5}{4} + \frac{1}{2}$, respectively. Hence $\sum_{X_i,v \in X_i} \left(2 - \frac{3}{|X_i|}\right) \geq 2 - \frac{l(v)}{2} = 2$ is satisfied.

Now let us consider the case $l(v) = 1$. Then $|N(v)| \geq 5$ since $G$ is 6-balanced. Similarly as above we can obtain $\sum_{X_i,v \in X_i} (|X_i| - 1) \geq 5$. Then there are three subcases depending on $m$.

$m \geq 3$: Since $|X_i| \geq 2$, $\sum_{X_i,v \in X_i} \left(2 - \frac{3}{|X_i|}\right) \geq 3 \cdot \frac{1}{2} = \frac{3}{2}$.

$m = 2$: $\sum_{X_i,v \in X_i} (|X_i| - 1) \geq 5$ implies $|X_1| \geq 4$. Then $\sum_{X_i,v \in X_i} \left(2 - \frac{3}{|X_i|}\right) \geq \frac{5}{4} + \frac{1}{2} \geq \frac{3}{2}$.  

ON THE RANK FUNCTION OF GENERIC LINEARLY CONSTRAINED FRAMEWORKS IN $\mathbb{R}^2$  

7
Then we have $X$ we see that $G$ follows from Claim 3.2(ii) and the last equality follows from the facts in (ii):

\[ X = 1 \times 2 = 2 \]

Therefore $\sum_{X_i \subseteq X_i} (2 - \frac{3}{|X_i|}) \geq 2 - \frac{l(v)}{2} = \frac{3}{2}$ as required.

(ii): Follows from the fact that $X_0$ is the only member of $X$ which contains $v$. (This is implied by the facts that $v \in Y \subseteq X_0$ and that $X$ is $1$-thin.)

(iii): Since $v \in Z \subseteq X_0$, there is an edge $vx$ for some $x \in V \setminus X_0$. Since $X$ covers this edge $vx$, there exists an $X_i$ with $x, v \in X_i$, for some $1 \leq i \leq k$. Hence $v$ is contained in at least two members of $X$ such that one of these members is $|X_0|$. Then the statement follows from the fact that $|X_i| \geq 2$ for all $1 \leq i \leq k$. 

Note that since the loops in $L'$ are incident with vertices in $V \setminus X_0$ and $G$ is $6$-balanced $|Z| + |L'| \geq 6$ holds. We now split the proof into two cases depending on the set $X_0$.

Case 1. $k = 0$ and so $X = \{X_0\}$.

Then since $(X_0, E(G[X_0]))$ is a complete graph and $G$ has at least six vertices with loops we see that $G$ is rigid, a contradiction. Hence $k \geq 1$.

Case 2. $X_0 = \emptyset$.

Then we have

\[
\text{val}(X) = |L'| + \sum_{i=1}^{k} (2|X_i| - 3) = |L'| + \sum_{i=1}^{k} |X_i|(2 - \frac{3}{|X_i|})
\]

\[
= |L'| + \sum_{v \in V} \sum_{X_i \subseteq X_i} (2 - \frac{3}{|X_i|})
\]

\[
\geq |L'| + 2|V| - \frac{|L'|}{2}
\]

\[
= 2|V| + \frac{|L'|}{2} = 2|V| + 3,
\]

where the inequality follows from Claim 3.2(i) and the last equality follows from the facts that $|Z| + |L'| \geq 6$ and $Z \subseteq X_0 = \emptyset$.

Case 3. $X_0 \neq \emptyset$.

Then we have

\[
\text{val}(X) = |L'| + 2|X_0| + \sum_{i=1}^{k} (2|X_i| - 3) = |L'| + \sum_{i=0}^{k} (2|X_i| - 3) + 3 = |L'| + \sum_{i=0}^{k} |X_i|(2 - \frac{3}{|X_i|}) + 3
\]

\[
= |L'| + \sum_{v \in V} \sum_{X_i \subseteq X_i} (2 - \frac{3}{|X_i|}) + 3
\]

\[
= |L'| + \sum_{v \in V} \sum_{X_i \subseteq X_i} (2 - \frac{3}{|X_i|}) + \sum_{v \in Z} \sum_{X_i \subseteq X_i} (2 - \frac{3}{|X_i|}) + \sum_{v \in V \setminus X_0} \sum_{X_i \subseteq X_i} (2 - \frac{3}{|X_i|}) + 3
\]

\[
\geq |L'| + 2|Y| - \frac{3|Y|}{|X_0|} + 2|Z| - \frac{3|Z|}{|X_0|} + \frac{|Z|}{2} + 2|V \setminus X_0| - \frac{|L'|}{2} + 3
\]

\[
= 2|V| - \frac{3(|Y| + |Z|)}{|X_0|} + \frac{|Z|}{2} + \frac{|L'|}{2} + 3 = 2|V| + \frac{|Z| + |L'|}{2} \geq 2|V| + 3
\]

where the first inequality follows from Claim 3.2 and the last inequality follows from the fact that $|Z| + |L'| \geq 6$. 

Since we obtain $\text{val}(X) \geq 2|V| + 3$ in the proof of the previous theorem we can remove three edges or loops and preserve rigidity.

**Corollary 3.3.** Let $G = (V, E, L)$ be a 6-balanced looped simple graph and $e_1, e_2, e_3 \in E \cup L$. Then $G - e_1 - e_2 - e_3$ is rigid as a linearly constrained framework in $\mathbb{R}^2$.

4. Further Remarks and Examples

We cannot replace the number 6 (i.e. being 6-balanced) in Theorem 1.5 by a smaller number. To see this consider the looped simple graph $G = (V, E, L)$ in Figure 2. Its underlying simple graph is 5-connected. Combining this with the fact that it has five distinct vertices with loops we deduce that $G$ is 5-balanced. However, it is not rigid as a linearly constrained framework in $\mathbb{R}^2$. To see this let $X_0$ be the set of vertices with loops, let $X_1, X_2, \ldots, X_7$ be the vertex sets of copies of the other $K_5$’s, and let $X_8, X_9, \ldots, X_{27}$ be the sets of the endpoints of the simple edges that connect distinct copies of $K_5$’s. Then $X = \{X_0, X_1, \ldots, X_{27}\}$ is an admissible 1-thin cover of $G$ ($L' = \emptyset$). Thus Theorem 1.4 gives

$$r_{lc}^e(G) \leq \text{val}(X) = |L'| + 2|X_0| + \sum_{i=1}^{27} (2|X_i| - 3) = 0 + 10 + 7 \cdot 7 + 20 = 79 < 80 = 2|V|$$

implying that $G$ is not rigid as a linearly constrained framework in $\mathbb{R}^2$.

![Figure 2. A non-rigid 5-balanced looped simple graph.](image)

Now consider Corollary 3.3. If we remove more than three edges from a 6-balanced looped simple graph, we may end up with a non-rigid graph. To see this let $G = (V, E, L)$ be a graph obtained from a 6-connected simple graph by adding a single loop to six distinct
vertices. Then clearly, \( G \) is 6-balanced. Let \( l_1, l_2, \ldots, l_6 \) denote loops of \( G \). Consider the graph \( H = G - l_1 - l_2 - l_3 - l_4 \), and let \( L' = \{l_5, l_6\} \), \( X_0 = \emptyset \) and \( X_1 = V \). Then \( \mathcal{X} = \{X_0, X_1\} \) is an admissible 1-thin cover of \( H - L' \) whose looped member is the empty set. Then Theorem 1.4 gives

\[
\text{val}(\mathcal{X}) = |L'| + 2|X_0| + 2|X_1| - 3 = 2 + 0 + 2|V| - 3 = 2|V| - 1 < 2|V|
\]

implying that \( H \) is not rigid as a linearly constrained framework in \( \mathbb{R}^2 \).

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