Limit theorems for branching processes with immigration in a random environment

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Abstract
We investigate branching processes with immigration in a random environment. Using Goldie’s implicit renewal theory we prove that under a generalized Cramér condition the stationary distribution of such processes has a power law tail. We further show how several methods familiar in the extreme value theory provide a natural and elegant path to their mathematical analysis. In particular, we rely on the point processes theory and the concept of tail process to determine the limiting distribution for the corresponding extremes and partial sums. Since Kesten, Kozlov and Spitzer seminal 1975 paper, it is known that one class of these processes has a close relation with random walks in a random environment. Even in that well studied context, the method we follow yields new results. For instance, we are able to i) move away from the conditions used by Kesten et al., ii) provide precise form of the limiting distribution in their main theorem, and iii) characterize the long term behavior of the worst traps a random walk in random environment encounters when drifting away from the origin.

Keywords Branching process in a random environment · Regularly varying stationary sequences · Tail process · Implicit renewal theory

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1 Introduction and notation

We investigate branching processes with immigration in a random environment aiming to give a precise description of their long term behavior. It is known that in an i.i.d. random environment, a branching process with immigration, \((X_n)\) say, has a Markovian structure i.e. it satisfies recursion \(X_n = \psi(X_{n-1}, Z_n)\) for an i.i.d. sequence of random elements \((Z_n)\). Although, its evolution seems somewhat more involved than the one described by the standard stochastic recurrence equations (as studied in Buraczewski et al. (2016) for instance), we show in Sect. 2 that the implicit renewal theory of Goldie (1991) can be readjusted to characterize the tails of the corresponding stationary distribution. Next, in Sect. 3 we give detailed description of the asymptotic limits of properly normalized values \((X_i)\) for \(i = 1, \ldots, n\), and of their partial sums \(S_n = X_1 + \cdots + X_n\), showing that the latter can exhibit both Gaussian and non–Gaussian limits under appropriate conditions. Observe that Roitershtein (2007) studied partial sums of certain multitype branching processes with immigration in a more general random environment concentrating on Gaussian domain of attraction.

Besides the straightforward epidemiological interpretation of the processes considered here, it is known that they have a close relation with random walks in a random environment, see for instance Kesten et al. (1975). As a corollary of our main result, in Sect. 4 we obtain an independent proof of the limiting result by Kesten et al. (1975) for a random walk in a random environment model with positive drift. We explain how their conditions yield various limiting distributions of such walks with the emphasis on the arguably more interesting non–Gaussian case. Moreover, we show that one can relax original conditions in Kesten et al. (1975) to a certain extent, and also write out the characteristic function of the limiting distributions. The precise form of the characteristic function which follows from our analysis seems to be new. However, some constants in the formula are implicit. Finally, our method additionally yields the long term behavior of the worst traps such a random walk encounters when moving to the right. It is intuitively understood that it is exactly those traps, that is the edges visited over and over again as the walk moves to the right, which give rise to the non–Gaussian limits for random walks in a random environment.

Throughout the article, the random environment is modeled by a sequence of i.i.d. random variables \(\xi, (\xi_t)_{t \in \mathbb{Z}}\) with values in a measurable space \(\mathcal{X}\). It may help in the sequel to specify \(\mathcal{X} = \Delta^2\), where \(\Delta\) denotes the space of probability measures on \(\mathbb{N} = \{0, 1, \ldots\}\). In that case we may write \(\xi = (\nu_\xi, \nu_\xi)\). Alternatively, we assume that there exists a measurable function which maps each \(\xi\) to a pair \((\nu_\xi, \nu_\xi) \in \Delta^2\). The two components of the pair \((\nu_\xi, \nu_\xi)\) are called the offspring and the immigration distribution, respectively. The Galton–Watson process with immigration in the random environment \(\mathcal{E} = \sigma\{\xi_t : t \in \mathbb{Z}\}\) evolves as follows. Let \(X_0 = x \in \mathbb{N}\), and then set
where we assume that, conditioned on the environment $\mathcal{E}$, the variables $\{A_i^{(n)}, B_n : n \in \mathbb{Z}, i \geq 1\}$ are independent, moreover, for $n$ fixed, $(A_i^{(n)})_{i \geq 1}$ are i.i.d. with distribution $\nu_{\xi_n}$, and $B_n$ has distribution $\nu_{\xi_n}^\circ$. Note that given the environment the random variables are independent however, $\nu_{\xi_n}$ and $\nu_{\xi_n}^\circ$ may be dependent. We interpret the variable $A_i^{(n)}$ as the number of offspring of the $i$th element in the $(n-1)$st generation, and $B_n$ as the number of immigrants in the $n$th generation. Hence, $\theta_n$ as in (1) denotes a random operator on nonnegative integers which act as follows

$$\theta_n \circ x = \sum_{i=1}^x A_i^{(n)}.$$ 

In the sequel, we abuse the notation, by writing $\theta_n \circ (x+y) = \theta_n \circ x + \theta_n \circ y$, keeping in mind that the two random operators on the right-hand side (r.h.s.) are not really the same, but independent conditioned on the environment.

For an $\mathbb{X}$–valued random element $\xi$ denote by

$$m(\xi) = \sum_{i=1}^\infty i \nu_{\xi}(\{i\}), \quad m^\circ(\xi) = \sum_{i=1}^\infty i \nu_{\xi}^\circ(\{i\}),$$

the expectation of its offspring and immigration distribution. This is clearly a nonnegative random variable, potentially equal to $+\infty$. We assume in the sequel that the process is subcritical, i.e.

$$\mathbb{E}[\log m(\xi)] < 0,$$

and that the following Cramér’s condition holds

$$\mathbb{E}[m(\xi)^\kappa] = 1 \quad \text{for some } \kappa > 0.$$ 

Finally, to avoid trivial situations, we assume that $\nu_{\xi}^\circ$ is not concentrated at 0.

Note that by our assumptions the sequence of random elements $Z_n = (\xi_n, B_n, A_1^{(n)}, A_2^{(n)}, \ldots), \ n \in \mathbb{Z}$ with values in $\mathbb{X} \times \mathbb{N}^\mathbb{N}$ is an i.i.d. sequence. Therefore, one can represent the evolution of the process $(X_n)$ using a measurable mapping $\psi$ and i.i.d. sequence $(Z_n)$ as

$$X_{n+1} = \psi(X_n, Z_{n+1}) = \Psi_{n+1}(X_n) = \theta_{n+1} \circ X_n + B_{n+1},$$ 

emphasizing the Markovian character of the process $(X_n)$. Here $\Psi_n$ denotes a random map $x \mapsto \psi(x, Z_n)$. By iterating (1) backward one may expect that the stationary distribution of the process can be found as

$$X_\infty = B_0 + \theta_0 \circ B_{-1} + \theta_0 \circ \theta_{-1} \circ B_{-2} + \ldots + \sum_{i=0}^\infty \theta_0 \circ \theta_1 \circ \ldots \circ \theta_{i-1} \circ B_i,$$ 

where

$$X_{n+1} = \theta_{n+1} \circ X_n + B_{n+1} = \theta_{n+1} \circ \left( X_n + B_n \right), \quad n \geq 0,$$
provided that the right-hand side converges a.s. to a finite limit (cf. Lemma 1). Clearly, if well-defined, such a distribution satisfies the distributional fixed point equation

$$X \overset{d}{=} \psi(X, Z) = \sum_{i=1}^{X} A_i + B = \theta \circ X + B,$$

(6)

where \( Z = (\xi, B, A_1, A_2, \ldots) \) and \( X \) on the r.h.s. are independent.

2 Implicit renewal theory

2.1 Moments of the stationary distribution

One of the main steps in the analysis of the process \((X_n)\) is to determine whether the random variable \(X_\infty\) in (5) has finite moments of order \(\alpha > 0\) say. If this holds, \(X_\infty\) would be clearly finite with probability one. On the other hand, by the conditional Jensen inequality

$$E[m(\xi^t)] \leq E[A^t]$$

for \( t \geq 1 \), while

$$E[m(\xi^t)] \geq E[A^t]$$

for \( t \leq 1 \).

Observe, for any fixed \( t > 0 \), \( E[A^t] = \infty \) is possible with the assumptions (2) and (3) still being satisfied for some \( \kappa > 0 \).

In deterministic environment for multitype processes, the existence and explicit expression for the moments of order \(\alpha\) were subject of Quine (1970) for \(\alpha = 1, 2\), and of Barczy et al. (2018) for \(\alpha = 3\). Necessary and sufficient condition for the existence of moments for multitype processes with general \(\alpha > 0\) was obtained by Szűcs (2014), see also Kevei and Wiandt (2021). In our case, under the condition that \(\alpha > 0\) satisfies

$$E[m(\xi^\alpha)] < 1, \quad E[A^\alpha] < \infty, \quad \text{and} \quad E[B^\alpha] < \infty,$$

(7)

using Lemma 3.1 in Buraczewski and Dyszewski (2019) one can easily deduce that there exist constants \( c > 0 \) and \( 0 < \rho < 1 \) such that

$$E[(\theta_0 \circ \theta_1 \circ \cdots \circ \theta_{i-1} \circ B_i)^\alpha] \leq c \rho^i.$$

Therefore, for \(\alpha \geq 1\) by Minkowski’s inequality

$$E[X_\infty^\alpha] \leq \sum_{i=0}^{\infty} \left( E[(\theta_0 \circ \theta_1 \circ \cdots \circ \theta_{i-1} \circ B_i)^\alpha] \right)^{1/\alpha} \leq c^{1/\alpha} \sum_{i=0}^{\infty} \rho^{i/\alpha} < \infty,$$

while for \(\alpha < 1\), simply by subadditivity

$$E[X_\infty^\alpha] \leq \sum_{i=0}^{\infty} E[(\theta_0 \circ \theta_1 \circ \cdots \circ \theta_{i-1} \circ B_i)^\varphi] \leq c \sum_{i=0}^{\infty} \rho^i < \infty,$$

which immediately yields the following useful result.
Lemma 1 If (7) holds for some $\alpha > 0$, then the random variable $X_\infty$ in (5) satisfies $E[X_\infty^\alpha] < \infty$.

Lemma 1 implies in particular that (5) represents a solution of the distributional equation in (6).

For easier reference we state a simple moment bound, which we use later with the choice $Z_i = A_i - m(\xi)$ and $n = X_\infty$, conditionally on the environment $\xi$. Put $a \lor b = \max\{a, b\}$.

Lemma 2 Let $Z, Z_1, Z_2, \ldots$ be i.i.d. mean zero random variables, and $n \in \mathbb{N}$.

(i) Let $\alpha \in (0, 2], \eta \in [0, 1]$ be such that $2\eta \leq \alpha \leq 1 + \eta$. Then there is a constant $c > 0$ depending only on $\alpha$ and $\eta$ such that

$$E\left[ \left| \sum_{i=1}^{n} Z_i \right|^{\alpha} \right] \leq c n^{\alpha-\eta} E\left[ \left| Z \right|^{\alpha-\eta} \right].$$

(ii) Let $\alpha \geq 1$. Then there is a $c > 0$ depending only on $\alpha$ such that

$$E\left[ \left| \sum_{i=1}^{n} Z_i \right|^{\alpha} \right] \leq c n^{1+\alpha/2} E[|Z|^{\alpha}].$$

Proof (i) First using Jensen’s inequality ($\alpha - \eta \leq 1$), then the Marcinkiewicz–Zygmund inequality [Petrov (1995), 2.6.18], and finally the subadditivity ($\alpha/[2(\alpha - \eta)] \leq 1$) we obtain

$$E\left[ \left| \sum_{i=1}^{n} Z_i \right|^{\alpha} \right] \leq \left( E\left[ \left( \sum_{i=1}^{n} Z_i^{\alpha-\eta} \right)^{\frac{\alpha}{\alpha-\eta}} \right] \right)^{\alpha-\eta} \leq c \left( E\left[ \left( \sum_{i=1}^{n} Z_i^{2(\alpha-\eta)} \right)^{\frac{\alpha}{2(\alpha-\eta)}} \right] \right)^{\alpha-\eta} \leq c n^{\alpha-\eta} \left( E\left[ \left| Z \right|^{\frac{\alpha}{\alpha-\eta}} \right] \right)^{\alpha-\eta}.$$

(ii) Follows from (i) with $\eta = \alpha - 1$, whenever $\alpha \in [1, 2]$. For $\alpha \geq 2$ this is simply Rosenthal’s inequality [Petrov (1995), Theorem 2.10].
2.2 Goldie's condition

In (4) we described the evolution of the Markov chain \((X_n)\) using an i.i.d. sequence of random functions \((\Psi_n), \Psi_n : \Omega \times \mathbb{N} \to \mathbb{N}, n \in \mathbb{Z}\), having the following general form

\[
\Psi_n(k) = \sum_{i=1}^{k} A_i^{(n)} + B_n, \quad k \in \mathbb{N}.
\]

Clearly, distributional fixed point equation in (6) can be written as

\[
X \overset{d}{=} \Psi(X),
\]

with \(\Psi\) and \(X\) independent on the right-hand side.

**Lemma 3** Assume that there exist \(\kappa > 0\) such that \(\mathbb{E}[m(\xi)^{\kappa}] = 1, \mathbb{E}[A^{\kappa}] < \infty, \mathbb{E}[B^{\kappa}] < \infty\). Then the law in (5) represents the unique stationary distribution for the Markov chain \((X_n)\). Suppose further that at least one of the following three conditions holds

(i) \(\kappa > 1\) and \(\mathbb{E}[m(\xi)^{\kappa-1}(\mathbb{E}[A^2|\xi])^{1/2}] < \infty\);

(ii) \(\kappa > 1\) and there exists \(\delta > 0\) such that \(\mathbb{E}[A^{\kappa+\delta}] < \infty\);

(iii) \(\kappa \leq 1\) and there exists \(\eta \in (0, \kappa/2]\) such that

\[
\mathbb{E}\left[\left(\mathbb{E}[A^{\kappa-\eta}|\xi]\right)^{\kappa-\eta}\right] < \infty.
\]

Then the random variable \(X = X_\infty\) further satisfies

\[
\mathbb{E}[|\Psi(X)^{\kappa} - (m(\xi)X)^{\kappa}|] < \infty.
\]

**Proof** It is straightforward to see that the assumptions of the lemma imply (7) for any \(\alpha \in (0, \kappa)\), which, by Lemma 1 further proves that the Markov chain in (1) has a stationary distribution. Denote by

\[
d_0 = \min\{k \geq 0 : \mathbb{P}(A_1 = 0, B_1 = k) > 0\}.
\]

The nonnegative integer \(d_0\) is well defined since \(\mathbb{P}(A_1 = 0, B_1 = k) > 0\) for at least one \(k \geq 0\) by the subcriticality assumption 2. Thus \(\mathbb{P}(A_1 = 0, B_1 = d_0) = \mathbb{E}[\mathbb{P}(A_1 = 0, B_1 = d_0|\xi)] > 0\), implying, for some \(\delta > 0\) that

\[
\mathbb{P}(\mathbb{P}(A_1 = 0, B_1 = d_0|\xi) > \delta) > \delta.
\]

Therefore, by conditional independence

\[
\mathbb{P}(X_{n+1} = d_0|X_n = x) \geq \delta^{x+1}, \quad x \in \mathbb{N},
\]
showing that $d_0$ is an accessible atom for the Markov chain $(X_n)$. This makes the chain irreducible, and the stationary distribution unique, see [Douc et al. (2018), Theorem 7.2.1] for instance.

First consider the $\kappa > 1$ case. We use that for any $\alpha \geq 1$ for some $c = c_\alpha > 0$

$$|x^\alpha - y^\alpha| \leq c|x - y| (y^{\alpha - 1} + |x - y|^{\alpha - 1}).$$

Therefore

$$E\left[|\Psi(X)^\kappa - (m(\xi)X)^\kappa|\right] \leq c\left(E\left[|\Psi(X) - m(\xi)X|(m(\xi)X)^{\kappa - 1}\right] + E\left[|\Psi(X) - m(\xi)X|^\kappa\right]\right). \tag{9}$$

For the second term in (9) by Minkowski’s inequality

$$(E\left[|\Psi(X) - m(\xi)X|^\kappa\right])^{\frac{1}{\kappa}} \leq \left(E\left[\sum_{i=1}^{X} (A_i - m(\xi))^\kappa\right]\right)^{\frac{1}{\kappa}} + (E[B^\kappa])^{\frac{1}{\kappa}}. \tag{10}$$

The second term in (10) is finite according to our assumptions. The finiteness of the first term in (10) follows from Lemmas 1 and 2 (ii) (the latter applied conditionally on $\xi$ and $X$).

For the first term in (9) we have

$$E\left[|\Psi(X) - m(\xi)X|(m(\xi)X)^{\kappa - 1}\right] \leq E\left[\sum_{i=1}^{X} (A_i - m(\xi))(m(\xi)X)^{\kappa - 1}\right] \tag{11}
+ E[B(m(\xi)X)^{\kappa - 1}].$$

By independence and Hölder’s inequality

$$E[B(m(\xi)X)^{\kappa - 1}] = E[X^{\kappa - 1}]E[Bm(\xi)^{\kappa - 1}]$$

$$\leq E[X^{\kappa - 1}](E[B^\kappa])^{1/\kappa}(E[m(\xi)^\kappa])^{(\kappa - 1)/\kappa} < \infty,$$

so the second term on the r.h.s. in (11) is finite. Therefore, it only remains to show the finiteness of the first term on the r.h.s. in (11).

Assume (i). Applying first the Marcinkiewicz–Zygmund inequality and then Jensen’s inequality

$$E\left[\left|\sum_{i=1}^{n} (A_i - m(\xi))\right|\xi\right] \leq c E\left[\left(\sum_{i=1}^{n} (A_i - m(\xi))^2\right)^{1/2}\xi\right]$$

$$\leq c \left(E\left[\sum_{i=1}^{n} (A_i - m(\xi))^2\xi\right]\right)^{1/2}$$

$$\leq c n^{1/2} (E[A^2|\xi])^{1/2},$$

where $c$ does not depend on $n$. Here, and later on, $c > 0$ is a generic constant, does not depend on relevant quantities, and its actual value may change from line to line. Substituting back into the first term in (11)
\[ E \left[ \sum_{i=1}^{X} (A_i - m(\xi)) \right] (m(\xi)X)^{k-1} \leq cE[X^{k-1/2}]E[m(\xi)^{k-1}(E[A^2|\xi])^{1/2}], \]

which is finite whenever (i) holds.

Assume (ii). For the first term in (11) by Hölder’s inequality we have
\[ E \left[ \left| \sum_{i=1}^{X} (A_i - m(\xi)) \right| (m(\xi)X)^{k-1} \right] \leq \left( E \left[ \left| \sum_{i=1}^{X} (A_i - m(\xi)) \right|^p \right] \right)^{1/p} \left( E[(m(\xi)X)^{q(k-1)}] \right)^{1/q}, \]

with \( 1/p + 1/q = 1 \). Choose \( p = \kappa + \epsilon \), for some \( 0 < \epsilon < \delta \), with \( \delta > 0 \) given in the condition (ii). Then an easy computation shows that \( q = \kappa/(\kappa - 1) - \epsilon'/\epsilon \), where \( \epsilon' \downarrow 0 \) as \( \epsilon \downarrow 0 \). Since \( q(\kappa - 1) < \kappa \), the second factor is finite by the independence of \( X \) and \( \xi \), and by Lemma 1. The finiteness of the first factor in (12) follows from Lemmas 1 and 2. Indeed, for \( \kappa \geq 2 \) this is immediate. For \( \kappa \in (1, 2) \) choose \( p = \kappa + \epsilon \leq 2 \) and apply Lemma 2 (ii).

The case \( \kappa \leq 1 \) is simpler. By the inequality
\[ |x^\kappa - y^\kappa| \leq |x - y|^\kappa, \]
we have
\[ |\Psi(X)^\kappa - (m(\xi)X)^\kappa| \leq |\Psi(X) - m(\xi)X|^\kappa. \]

Thus by subadditivity
\[ E[|\Psi(X)^\kappa - (m(\xi)X)^\kappa|] \leq E \left[ \left| \sum_{i=1}^{X} (A_i - m(\xi)) \right|^\kappa \right] + E[B^\kappa]. \]

Since the second term is finite by assumption, it is enough to show that
\[ E \left[ \left| \sum_{i=1}^{X} (A_i - m(\xi)) \right|^\kappa \right] < \infty. \]

This follows from Lemma 2 (i), with \( \eta \) given in condition (iii).

**Remark 1** For special classes of offspring distributions the conditions of the lemma can be simplified. If,
\[ E[A^2|\xi] \leq c(m(\xi)^2 + 1) \quad \text{a.s. for some } c > 1, \]
then both the condition for \( \kappa \leq 1 \) and condition (i) reduce to \( E[m(\xi)^{\kappa}] < \infty \), which holds since \( E[m(\xi)^{\kappa}] = 1 \).

In particular if, conditionally on \( \xi \), \( A \) has Poisson distribution with parameter \( \lambda(\xi) > 0 \), then \( m(\xi) = E[A|\xi] = \lambda(\xi) \) and \( E[A^2|\xi] = \lambda(\xi)^2 + \lambda(\xi) \leq 2(m(\xi)^2 + 1) \). While if \( A \), conditionally on \( \xi \), has geometric distribution with parameter \( p(\xi) \in (0, 1) \), i.e. \( P(A = k | \xi) = (1 - p(\xi))^k p(\xi), k \geq 0 \), then \( m(\xi) = E[A|\xi] = (1 - p(\xi))/p(\xi) \) and
\[ E[A^2[\xi]] = (2 - 3p(\xi) + p(\xi)^2)/p(\xi)^2 \leq 3(1 - p(\xi))^2/p(\xi)^2 + 3 = 3m(\xi)^2 + 3. \] Thus, in both cases (13) holds.

From Corollary 2.4 in Goldie (1991) we obtain the following. Put \( \log^+ x = 0 \vee \log x \).

**Theorem 1** Assume the conditions of Lemma 3, \( E[m(\xi)^{x} \log^+ m(\xi)] < \infty \), and that the law of \( \log m(\xi) \) given \( m(\xi) > 0 \) is nonarithmetic. For \( \kappa < 1 \) additionally assume that \( E[m(\xi)^{x} A \log^+ A] < \infty \). Then the law of \( X_\infty \) in (5) represents the unique stationary distribution for the Markov chain \( (X_n) \) and

\[ P(X_\infty > x) \sim Cx^{-\kappa} \quad \text{as } x \to \infty, \] 

(14)

where

\[ C = \frac{1}{\kappa E[m(\xi)^{x} \log m(\xi)]} E[\Psi(X_\infty)^{\kappa} - m(\xi)^{\kappa}X_\infty] > 0. \]

**Proof** According to Lemma 3, the Markov chain \( (X_n) \) has unique stationary distribution given in (5). By the same token, we can conclude that the conditions of Corollary 2.4 in Goldie (1991) hold which yields the tail asymptotics in (14). It remains to show the strict positivity of the constant \( C \) above.

For \( \kappa \geq 1 \) this can be deduced directly from its form. Indeed, since \( X_\infty \) is independent of \( \Psi \) and \( \xi \)

\[ E[\Psi(X_\infty)^{\kappa} - m(\xi)^{\kappa}X_\infty] = \sum_{n=1}^{\infty} P(X_\infty = n)E\left[\left( \sum_{i=1}^{n} A_i + B \right)^{\kappa} - m(\xi)^{\kappa}n^{\kappa}\right] \]

\[ > \sum_{n=1}^{\infty} P(X_\infty = n)E\left[\left( \sum_{i=1}^{n} A_i \right)^{\kappa} - m(\xi)^{\kappa}n^{\kappa}\right], \]

(15)

where in the last step we used that \( B \) is not identically 0. By Jensen’s inequality

\[ E\left[\left( \sum_{i=1}^{n} A_i \right)^{\kappa} - m(\xi)^{\kappa}n^{\kappa}\right] = E\left[\left( \sum_{i=1}^{n} A_i \right)^{\kappa} \right] - m(\xi)^{\kappa}n^{\kappa} \geq E[n^{\kappa}m(\xi)^{\kappa} - m(\xi)^{\kappa}n^{\kappa}] = 0, \]

from which the strict positivity follows.

For \( \kappa < 1 \) we deduce that \( C > 0 \) from Theorem 1 in Afanasyev (2001) as follows. Note that the extra assumption in the theorem is the same as therein.

To the original process \( X_n \), one can couple a process \( Y_n \), starting at \( Y_0 = 1 \), and satisfying

\[ Y_{n+1} = \sum_{i=1}^{Y_n} A_i^{(n+1)}, \quad n \geq 0, \]
where the sequence of environments \((\xi_1, \xi_2, \ldots)\) are the same as in (1). Note, \((Y_n)\) is a subcritical branching process in random environment, which by Theorem 1 in Afanasyev (2001) satisfies

\[
\lim_{x \to \infty} x^k P(\sup_{n \geq 1} Y_n > x) = c > 0. \tag{16}
\]

Consider the following stationary two dimensional process

\[
(X_n - B_n, B_n) = \left( \sum_{i=1}^{X_{n-1}} A_i^{(n)}, B_n \right), \quad n \geq 0.
\]

It is also a Markov chain with a unique stationary distribution clearly related to that of the chain \((X_n)\). Recall \(d_0\) from the proof of Lemma 3. Then, for the two dimensional process \((0, d_0)\) is an accessible atom. Denote by \(\tau_{d_0} = \min\{n \geq 1 : (X_n - B_n, B_n) = (0, d_0)\}\). If we start the chain at \((0, d_0)\) observe that by stationarity we have \(E_{(0,d_0)}[\tau_{d_0}] < \infty\). Moreover, if we start \((X_0 - B_0, B_0)\) at \((0, d_0)\), by the construction \(X_n - B_n \geq Y_n\) a.s. for every \(n \geq 1\). Let \(\tau_Y = \min\{n \geq 1 : Y_n = 0\}\), clearly \(\tau_Y \leq \tau_{d_0}\). In particular, the process \(Y_n\) dies out almost surely. By the standard theory of Markov chains, see Theorem 7.2.1 in Douc et al. (2018) or Theorem 10.4.9 in Meyn and Tweedie (2009), we have

\[
P(X_\infty > x) = \frac{1}{E_{(0,d_0)}[\tau_{d_0}]} E_{(0,d_0)} \left[ \sum_{i=0}^{\tau_{d_0} - 1} 1 \{ X_i > x \} \right] \geq \frac{1}{E_{(0,d_0)}[\tau_{d_0}]} E \left[ \sum_{i=0}^{\tau_Y - 1} 1 \{ Y_i > x \} \right] \geq \frac{1}{E_{(0,d_0)}[\tau_{d_0}]} P(\sup_{n \geq 1} Y_n > x),
\]

which by (16) implies

\[
\liminf_{x \to \infty} x^k P(X_\infty > x) > 0,
\]

as claimed. \qed

**Remark 2** Observe that the assumption \(E[m(\xi)^{k-1}A \log^+ A] < \infty\) for \(k < 1\) is rather weak. It directly follows from the moment assumption on \(m(\xi)\) and (13) for instance. To see this, let \(Y\) be a nonnegative integer valued random variable with finite mean, and let \(\hat{Y}\) denote its size-biased version, that is \(P(\hat{Y} = k) = kP(Y = k)/E[Y]\), \(k = 1, 2, \ldots\). Then, by Jensen’s inequality

\[
E[Y \log^+ Y] = E[Y]E[\log \hat{Y}] \leq E[Y] \log E[\hat{Y}] = E[Y] \log \frac{E[Y^2]}{E[Y]}.
\]

Therefore, applying the inequality above conditionally on the environment \(\xi\) together with (13), we obtain that \(E[m(\xi)^{k-1}A \log^+ A] < \infty\) follows from \(E[m(\xi)^{k} \log^+ m(\xi)] < \infty\).
Remark 3 As a consequence of the applied machinery, apart from some special cases, the constant C in the theorem is merely implicit; the formula contains the limit law itself. However, for \( \kappa = 1 \) we simply have
\[
E[\Psi(X_\infty) - m(\xi)X_\infty] = E[E(\sum_{i=1}^{X_\infty} A_i + B - m(\xi)X_\infty | X_\infty)] = E[B],
\]
so C can be computed.

Remark 4 Analogous results hold in the arithmetic case. Using Theorem 2 in Kevei (2017) (see also Theorem 3.7 by Jelenković and Olvera-Cravioto (2012)) one can show the following. If the conditions of Lemma 3 hold, and the law of \( \log m(\xi) \) given \( m(\xi) > 0 \) is arithmetic with span \( h > 0 \), then there exists a function \( q \) such that whenever \( x \) is a continuity point of \( q \). The function \( x^{-\kappa}q(x) \) is nonincreasing, and
\[
q(xe^h) = q(x) \text{ for all } x > 0.
\]

2.3 Relaxes Cramér’s condition

In what follows, we weaken condition (3) in two ways, such that the tail of the stationary distribution is still regularly varying. We use a slight extension of Goldie’s renewal theory by Kevei (2016).

First we consider weakening the assumption
\[
E[m(\xi)^{\kappa} \log m(\xi)] < \infty.
\]
The condition \( E[m(\xi)^{\kappa}] = 1 \) ensures that
\[
F_\kappa(x) = E[1\{\log m(\xi) \leq x\}m(\xi)^{\kappa}]
\]
is a distribution function. The additional logarithmic moment condition in (17) is equivalent to the finiteness of the expectation of the distribution \( F_\kappa \). This condition is needed to use the standard key renewal theorem. However, strong renewal theorems in the infinite mean case have been known by Garsia and Lamperti (1962/63). They showed that the strong renewal theorem holds if the underlying distribution belongs to the domain of attraction of a \( \gamma \)-stable law with \( \gamma \in (1/2, 1] \), while for \( \gamma \leq 1/2 \) extra conditions are needed. Recently, Caravenna and Doney (2019) obtained necessary and sufficient conditions for the strong renewal theorem to hold, solving a 50-year old open problem.

Assume that \( F_\kappa \) belongs to the domain of attraction of a stable law of index \( \gamma \in (0, 1] \), i.e., for some \( \kappa > 0 \) and \( \gamma \in (0, 1] \), for a slowly varying function \( \ell' \)
\[
E[m(\xi)^{\kappa}] = 1, \quad 1 - F_\kappa(x) = \frac{\ell'(x)}{x^{\gamma'}}.
\]

Define the truncated expectation as
\[ M(x) = \int_0^x (1 - F_k(y))dy \sim \frac{\ell'(x)x^{1-\gamma}}{1 - \gamma}, \]  

(19)

where the asymptotic equality holds for \( \gamma < 1 \). If \( \gamma \in (0, 1/2] \) further assume the Caravenna–Doney condition

\[ \lim \lim_{\delta \to 0} \lim_{x \to \infty} x[1 - F_k(x)] \int_1^{\delta x} \frac{1}{y[1 - F_k(y)]^2} F_k(x - dy) = 0. \]

(20)

This is a kind of uniformity condition on the tail, and it is satisfied by the common distributions with regularly varying tail. A sufficient condition for the strong renewal theorem (and thus to (20)) is that

\[ \lim_{x \to \infty} x(1 - F_k(x))[F_k(x + h) - F_k(x)] < \infty, \]

for any \( h > 0 \), see Doney (1997) for the arithmetic and Vatutin and Topchii (2014) for the nonarithmetic case. Counterexamples, i.e. distributions with regularly varying tails for which (20) does not hold, are constructed in [Caravenna and Doney (2019), Section 10].

For our second extension, assume now that \( \mathbb{E}[m(\xi)^\kappa] = \varphi < 1 \) for some \( \kappa > 0 \). If \( \mathbb{E}[m(\xi)^t] < \infty \) for some \( t > \kappa \), then by Lemma 1 the tail of \( X_\infty \) cannot be regularly varying with index \( \kappa \). Therefore we assume \( F_k \) is heavy-tailed, i.e. \( \mathbb{E}[m(\xi)^t] = \infty \) for any \( t > \kappa \). Define now the distribution function

\[ F_k(x) = \varphi^{-1}\mathbb{E}[1\{\log m(\xi) \leq x\}]m(\xi)^\kappa. \]

(21)

The analysis of the stochastic fixed point equation (6) leads to a defective renewal equation. To understand the asymptotic behavior of the solution of these equations we need to introduce \textit{locally subexponential distributions}.

For \( T \in (0, \infty] \) let \( V_T = (0, T] \) and for a distribution function (df) \( H \) we put \( H(x + V_T) = H(x + T) - H(x) \). Let \( * \) denote the usual convolution operator. A df \( H \) is \textit{locally subexponential}, \( H \in S_{loc} \), if for each \( T \in (0, \infty] \) we have (i) \( H(x + t + V_T) \sim H(x + V_T) \) as \( x \to \infty \) uniformly in \( t \in [0, 1] \), (ii) \( H(x + V_T) > 0 \) for \( x \) large enough, and (iii) \( H * H(x + V_T) \sim 2H(x + V_T) \) as \( x \to \infty \). For more details see [Foss et al. (2013), Section 4.7]. Informally, a locally subexponential distribution is a subexponential distribution with well-behaved density function.

Our assumptions on \( m(\xi) \) are the following:

\[ \mathbb{E}[m(\xi)^\kappa] \leq \varphi < 1, \quad \kappa > 0, \quad F_k \in S_{loc}, \]

for each \( T \in (0, \infty] \)

\[ \sup_{y > x} F_k(y + V_T) = O(F_k(x + V_T)) \text{ as } x \to \infty. \]

(22)

\[ \mathbf{Theorem 2} \] Assume that condition (i) or (iii) in Lemma 3 holds, the law of \( \log m(\xi) \) given \( m(\xi) > 0 \) is nonarithmetic, \( \mathbb{E}[B^v] < \infty \) for some \( v > \kappa \), and one of the following two conditions is satisfied:
(i) condition (18) holds, and if $\gamma \in (0, 1/2]$ also (20) holds;
(ii) condition (22) holds.

Then the law of $X_\infty$ in (5) represents the unique stationary distribution for the Markov chain $(X_n)$ and

$$P(X_\infty > x) \sim Cx^{-\kappa}L(x) \quad \text{as } x \to \infty,$$

where

$$L(x) = \begin{cases} \frac{(\Gamma(a)\Gamma(2-a)M(\log x))^{-1}}{F_\kappa(1 + \log x) - F_\kappa(\log x))\varphi/(1 - \varphi)^2}, & \text{in case (ii)}, \\ \frac{1}{\kappa} E[\Psi(X_\infty)^\kappa - m(\xi)^\kappa X_\infty^\kappa] \geq 0, & \text{in case (i)}, \end{cases}$$

is a slowly varying function, and

$$C = \frac{1}{\kappa} E[\Psi(X_\infty)^\kappa - m(\xi)^\kappa X_\infty^\kappa] \geq 0,$$

with $C > 0$ for $\kappa \geq 1$.

**Proof** The result follows from Theorems 2.1 and 2.3 in Kevei (2016). We only have to check that

$$\mathbb{E} \left[ \left( \sum_{i=1}^{X_\infty} A_i + B \right)^{\kappa + \delta} - (m(\xi)X_\infty)^{\kappa + \delta} \right] < \infty$$

for some $\delta > 0$. This can be done exactly in the same way as in the proof of Lemma 3 case (i) and (iii).

For $\kappa \geq 1$ the strict positivity of the constant $C$ in the theorem follows exactly as in the proof of Theorem 1.

**Remark 5** Note that in general we only proved the strict positivity of $C$ above for $\kappa \geq 1$. However, in some special cases it is possible to show that $C > 0$ for a general $\kappa > 0$. In particular, if the expectation in the infinite sum in the first line of (15) is strictly positive for each $n$, then clearly $C > 0$.

**Remark 6** Note that both Remarks 3 and 4 after Theorem 1 apply in this setup as well.

**Remark 7** Local subexponentiality imposes a strong condition on the local behavior of the distributions, and it is not implied by tail asymptotics, such as regular variation. Typical examples from this class are the Pareto, lognormal and Weibull distributions. In the Pareto case, i.e. if for large enough $x$ we have $1 - F_\kappa(x) = cx^{-\beta}$, for some $c > 0$, $\beta > 0$, then $P(X_\infty > x) \sim c'x^{-\kappa}(\log x)^{-\beta-1}$. In the lognormal case, when $F_\kappa(x) = \Phi(\log x)$ for $x$ large enough, with $\Phi$ being the standard normal df, we have $P(X_\infty > x) \sim cx^{-\kappa}e^{-(\log \log x)^2/2}\log x$, $c > 0$. While, for Weibull tails $1 - F_\kappa(x) = e^{-x^\beta}$, $\beta \in (0, 1)$, we obtain $P(X_\infty > x) \sim cx^{-\kappa}(\log x)^{\beta-1}e^{-(\log x)^{\beta}}$, $c > 0$. 

\[ Springer \]
3 Asymptotic behavior of the process \((X_t)\)

3.1 Dependence structure of the process

In the sequel we only need that our process has a stationary distribution with regularly varying tail. In the previous section we derived several conditions which ensures regularly varying tail. Since in Theorem 2 we only have the strict positivity of the underlying constant \(C\) for \(\kappa \geq 1\), we assume that one of the following holds:

\[
\text{conditions of Theorem 1, or conditions of Theorem 2 and } C > 0. \tag{23}
\]

Then in particular, there exist a stationary Galton–Watson process with immigration in random environment \((X_n)_{n \in \mathbb{Z}}\) which satisfies

\[
X_{n+1} = \sum_{i=1}^{X_n} A_i^{(n+1)} + B_{n+1}, \quad n \in \mathbb{Z}, \tag{24}
\]

with the same interpretation of the random variables \(\{A_i^{(n)}, B_n : n \in \mathbb{Z}, i \geq 1\}\). Here again by \(\xi, (\xi_i)_{i \in \mathbb{Z}}\) we denote i.i.d. random variables representing the environment. The offspring and immigration distributions are governed by the environment \(\mathcal{E}\) in the same way as before.

It is useful in the sequel to introduce a deterministic sequence \((a_n)_{n \in \mathbb{N}}\) such that

\[
n \mathbb{P}(X_\infty > a_n) \longrightarrow 1 \quad \text{as } n \to \infty. \tag{25}
\]

Observe that by Theorem 1 and 2, \((a_n)\) is a regularly varying sequence with index \(1/\kappa\), i.e. \(a_n = \tilde{c}(n)n^{1/\kappa}\), with an appropriate slowly varying function \(\tilde{c}\). In particular, if the conditions of Theorem 1 hold we may set \(a_n = (Cn)^{1/\kappa}\) and that any other sequence \((a_n)\) in (25) necessarily satisfies \(a_n \sim (Cn)^{1/\kappa}\).

Once we have shown that the marginal stationary distribution of the \(X_n\)’s is regularly varying, it is relatively easy to prove that all the finite dimensional distributions of \((X_n)\) in (24) have multivariate regular variation property, cf. Resnick (2007), i.e. \((X_n)_{n \in \mathbb{Z}}\) is regularly varying sequence in the sense of Basrak and Segers (2009). According to Theorem 2.1 in Basrak and Segers (2009), this is equivalent to the existence of the so-called tail sequence, which is the content of the first theorem below.

Observe first that under Cramér’s condition (3), one can construct a tilted distribution of the following form

\[
\mathbb{P}(\xi^* \in \cdot \mid \xi) = \mathbb{E}[\mathbb{I}\{\xi \in \cdot \}\cdot m(\xi)^\alpha].
\]

Similar change of measure appeared already in [Afanasyev (2001), Lemma 1], see also [Buraczewski and Dyszewski (2019), Lemma 3.1]. By the convexity of the function \(\lambda(\alpha) = \mathbb{E}[m(\xi)^\alpha]\)

\[
\mathbb{E}[\log m(\xi^*)] = \mathbb{E}[m(\xi)^\kappa \log m(\xi)] = \lambda'(\kappa) > 0,
\]

possibly infinite. Therefore, the corresponding branching process in this new random environment \(\xi^*\) is supercritical. This type of measure change, converting a subcritical
process to a supercritical one is typical in branching processes, see e.g. [Afanasyev (2001), Section 2], or the conceptual proofs in the classical setup by Lyons et al. (1995).

To simplify notation, denote \( m = m(\xi), m_i = m(\xi_i), i \geq 1 \), under their original distribution. Introduce an auxiliary i.i.d. sequence \( m^* = m(\xi^*), i \geq 1 \), with common distribution \( \mathbb{P}(m^* \in A) = \mathbb{E}[1\{m(\xi) \in A\}m(\xi)^k] \) and independent of \( (m_i) \).

On the other hand, if (22) holds, i.e. \( \mathbb{E}[m(\xi)^k] = \varphi < 1 \), then \( m_i^*, i \geq 1 \), is a sequence of extended random variables with common distribution \( \mathbb{P}(m^* \in A) = \mathbb{E}[1\{m(\xi) \in A\}m(\xi)^k] \), for \( A \subset \mathbb{R} \), and \( \mathbb{P}(m^* = \infty) = 1 - \varphi \), and independent of \( (m_i) \). In the following result we use the usual convention \( 1/\infty = 0 \).

**Theorem 3** Let \( (X_n)_{n \in \mathbb{Z}} \) be a stationary sequence satisfying (24). Assume (23), and let \( (m_n)_{n \geq 1}, (m_n^*)_{n \geq 1} \) be independent i.i.d. sequences introduced above and independent of \( Y_0 \) with Pareto distribution \( \mathbb{P}(Y_0 > u) = u^{-\kappa}, u \geq 1 \). Then, for any integers \( k, \ell \geq 0 \) as \( x \to \infty \)

\[
\mathcal{L}\left(\frac{X_{-k}}{x}, \ldots, \frac{X_0}{x}, \ldots, \frac{X_{\ell}}{x} \left| X_0 > x \right) \xrightarrow{d} Y_0((m_n^* \cdots m_1^*)^{-1}, \ldots, (m_n^*)^{-1}, 1, m_1, \ldots, m_1 \cdots m_{\ell}).
\]

Writing the random vector on the r.h.s. above as \((Y_t, t = -k, \ldots, \ell)\), note that, in the language of Basrak and Segers (2009), the sequence \((Y_t)_{t \in \mathbb{Z}}\) represents the tail process of the sequence \((X_t)\). Hence, in this case, both the forward and backward tail processes are multiplicative random walks.

**Proof** Observe first that \( \mathcal{L}(X_0/x \mid X_0 > x) \xrightarrow{d} Y_0, \) by (23) and the regular variation property. We will show next that for arbitrary \( \ell \geq 0 \) and \( \varepsilon > 0 \)

\[
\mathbb{P}\left(\left| \frac{X_{\ell}}{X_0} - m_1 \cdots m_{\ell} \right| > \varepsilon \mid X_0 > x \right) \longrightarrow 0, \text{ as } x \to \infty.
\]

Indeed, consider first \( \ell = 1 \), by the independence of \( A_j^{(1)} \)'s and \( B_1 \) from \( X_0 \),

\[
\mathbb{P}\left(\left| \frac{X_1}{X_0} - m_1 \right| > \varepsilon \mid X_0 > x \right) = \sum_{k > x} \mathbb{P}\left(\left| \frac{\sum_{j=1}^{X_0} A_j^{(1)} + B_1}{X_0} - m_1 \right| > \varepsilon ; X_0 = k \right) \frac{1}{\mathbb{P}(X_0 > x)}.
\]

If we condition on the environment, the strong law of large numbers implies \( n^{-1} \sum_{j=1}^{n} (A_j^{(1)} + B_1) \to m_1 \) a.s. Since the convergence in probability holds as well, the r.h.s. above remains below any \( \varepsilon' > 0 \) as \( x \to \infty \), proving (26). Instead of general \( \ell \geq 1 \), consider for simplicity \( \ell = 2 \). One can write \( X_2 = \sum_{j=1}^{X_0} A_j^{(2)} + \sum_{j=1}^{B_1} A_j^{(2)} + B_2 \). Given the environment \( E \), the three terms on the r.h.s. are independent and \( A_j^{(2)} \)'s are i.i.d. with the following distribution

\[ \text{Springer} \]
In particular, $\mathbb{E}[\bar{A}_1^{(2)} \mid \mathcal{E}] = m_1 m_2$, the ergodic theorem applied on the sequence $(\bar{A}_j^{(2)})$ again yields

$$
\mathbb{P}\left( \left| \frac{X_2}{X_0} - m_1 m_2 \right| > \varepsilon \mid X_0 > x \right) = \sum_{k>x} \mathbb{P}\left( \left| \frac{\sum_{j=1}^{k} \bar{A}_j^{(2)} + \sum_{j=1}^{k} A_j^{(2)} + B_2}{k} - m_1 m_2 \right| > \varepsilon \right) \frac{\mathbb{P}(X_0 = k)}{\mathbb{P}(X_0 > x)} \rightarrow 0
$$

as $x \rightarrow \infty$. The same argument works for any $\ell \geq 1$, which implies (26). Observing that $\mathcal{L}(X_0/x \mid X_0 > x) \overset{d}{\longrightarrow} Y_0$, we can conclude that for an arbitrary $\ell$

$$
\mathcal{L}\left( \frac{X_0}{x}, \frac{X_1}{X_0}, \ldots, \frac{X_2}{X_0} \mid X_0 > x \right) \overset{d}{\longrightarrow} (Y_0, m_1, \ldots, m_1 \cdots m_\ell).
$$

This proves the statement of the theorem for $k = 0$. In particular, the sequence $(X_i)$ is regularly varying, and the multiplicative random walk

$$
Y_i = Y_0 \Theta_i = Y_0 m_1 \cdots m_i, \quad i \geq 0
$$

represents the forward part of its tail process. By Theorem 3.1 in Basrak and Segers (2009), this uniquely determines the distribution of the whole tail process, including the negative indices. The past distribution of the tail process has been determined already (see Theorem 5.2 and Example 3.3 in Segers (2007) for instance). It turns out that the backward part of the tail process has the representation

$$
Y_{-k} = Y_0 \Theta_{-k} = Y_0 / (m_k^* \cdots m_1^*), \quad k > 0
$$

for an i.i.d. sequence $(m_n^*)$ as in the statement of the theorem. \qed

Alternatively, the theorem above can be obtained using the results in [Janssen and Segers (2014), Theorem 2.1].

Recall that by our assumptions on progeny and immigrant distributions, there exists $0 < \alpha < \kappa, \alpha \leq 1$, such that

$$
\mathbb{E}[m(\xi)^{\alpha}] < 1 \quad \text{and} \quad \mathbb{E}[m^{\alpha}(\xi)^{\alpha}] < \infty.
$$

Denote the Markov transition kernel of the sequence $(X_n)$ by $P(x, \cdot)$ and consequently, by $P^n(x, \cdot), n \geq 1$ denote the $n$-step Markov transition kernel corresponding to $P$. Next we apply the standard drift method to show that the Markov chain $(X_n)$ is uniformly $V$-geometrically ergodic. First we introduce some notation. For a function $V : \mathbb{N} \rightarrow [1, \infty)$, and any two probability measures $\nu_1, \nu_2 \in \Delta$ put
\[ \| v_1 - v_2 \|_V = \sup_{g \colon |g| \leq V} \left| \sum_{n=0}^{\infty} g(n)(v_1(\{n\}) - v_2(\{n\})) \right|. \]

We use the notation \( E_\xi \) when the initial distribution of \( X_0 \) is \( \xi \), in particular, \( E_x \) means that \( X_0 = x \). The stationary distribution is denoted by \( \pi \).

**Lemma 4** Let \( V(x) = x^\alpha + 1 \), with \( \alpha < \kappa, \alpha \leq 1 \). The Markov chain \( (X_n)_n \) is uniformly \( V \)-geometrically ergodic, that is, there exists \( \rho \in (0, 1) \) and \( c > 0 \) such that for each \( x \in \mathbb{N} \)

\[ \| P^x(x, \cdot) - \pi \|_V \leq cV(x)\rho^x, \]

where \( \pi \) denotes the unique stationary distribution of the Markov chain.

**Proof** Applying Theorem 16.0.1 in Meyn and Tweedie (2009) (equivalence of (ii) and (iv)), it is enough to prove that for some \( \beta \in (0, 1), b > 0 \), and a petite set \( C \)

\[ E_\xi[V(X_1)] = E[V(X_1)|X_0 = x] \leq \beta V(x) + b1\{x \in C\}. \]

Using Jensen’s inequality and that \( (u + v)^\alpha \leq u^\alpha + v^\alpha, u, v > 0 \), we have

\[ E \left[ \left( \sum_{i=1}^{x} A_i + B \right)^\alpha \right] \leq E \left[ \left( E \left[ \sum_{i=1}^{x} A_i + B \mid \mathcal{F} \right] \right)^\alpha \right] \leq x^\alpha E[m(\xi)^\alpha] + E[m(\xi)^\alpha]. \]

Therefore, there exist \( \beta \in (0, 1), b > 0 \), and \( x_0 \in \mathbb{N} \) such that for all \( x \in \mathbb{N} \)

\[ E_\xi[V(X_1)] = E[V(X_1)|X_0 = x] \leq \beta V(x) + b1\{x \leq x_0\}. \]

Moreover, the level set \( C = \{x : x \leq x_0\} \) is small in terminology of Meyn and Tweedie (2009). Indeed, for any \( x \leq x_0 \) and a measurable set \( D \subseteq \mathbb{N} \)

\[ P(x, D) = P \left( \sum_{i=1}^{x} A_i + B \in D \right) \geq P \left( \sum_{i=1}^{x_0} A_i = 0 ; B \in D \right) =: \mu(D). \]

Since we assumed that the process is subcritical, measure \( \mu \) is not trivial, the set \( C \) is small, and therefore petite as well. Thus the result follows. \( \square \)

Geometric ergodicity implies that \( (X_n)_n \) is \( \beta \)-mixing and strongly mixing therefore (see [Douc et al. (2018), Corollary F.3.4], or Meyn and Tweedie (2009), and Jones (2004)), which further implies by Proposition 1.34 in Krizmanić (2010) the mixing condition \( \mathcal{A}_1(a_n) \) (see Condition 2.2 in Basrak et al. (2012)). That is, there is a sequence \( r_n \to \infty, r_n = o(n) \), such that for any continuous \( f : [0, 1] \times [0, \infty) \to [0, \infty) \) for which there exists a \( \delta > 0 \) such that \( f(x, y) = 0 \) whenever \( y \leq \delta \), we have
\[ E \left[ \exp \left\{ -\sum_{i=1}^{n} f \left( \frac{i}{n}, \frac{X_i}{a_n} \right) \right\} \right] - \prod_{k=1}^{k_n} E \left[ \exp \left\{ -\sum_{i=1}^{r_n} f \left( \frac{kr_n}{n}, \frac{X_i}{a_n} \right) \right\} \right] \to 0, \]

where \( k_n = [n/r_n] \).

Roughly speaking, the last condition ensures that the sequence \((X_t)\) can be split into blocks of consecutive observations

\[ C_i = C_i(n) = (X_{(i-1)r_n+1}, \ldots, X_{ir_n}), \quad i = 1, 2, \ldots, k_n, \]

which are asymptotically independent. Individual blocks could be considered as random elements of the space

\[ l_0 = \{(x_i)_{i \in \mathbb{Z}} : \lim_{|i| \to \infty} |x_i| = 0\}, \]

see also Basrak et al. (2018). This embedding boils down to concatenating infinitely many zeros before and after a given block. We equip the space \( l_0 \) with the sup-norm

\[ \| (x_i)_{i \in \mathbb{Z}} \| := \sup_{i \in \mathbb{Z}} |x_i| \]

and with the corresponding Borel \( \sigma \)-field.

Next we claim that the large values in the sequence \((X_n)\) cannot linger for ‘too long’, i.e. the so called anticlustering condition from Davis and Hsing (1995) holds. Recall the norming sequence \( a_n \) in (25).

**Lemma 5** Let \( r_n \) be a sequence such that \( r_n = o(n) \). Then for any \( u > 0 \)

\[ \lim_{k \to \infty} \limsup_{n \to \infty} \mathbb{P} \left( \max_{k \leq |t| \leq r_n} X_t > a_n u \left| X_0 > a_n u \right. \right) = 0. \]

**Proof** It is not too difficult to give a direct proof following the idea in the proof of Lemma 3.2 in Basrak et al. (2013). However, observe that in the proof of Lemma 4, we showed the geometric drift Assumption 14.3.1 of Kulik and Soulier (2020). Moreover their Assumption 14.3.2 holds for \( g(x) = x, \ c = 1 \) and \( q = \alpha \) of our proof. By Theorem 14.3.5 therein and discussion above that theorem, it follows that the statement of our lemma holds. \( \square \)

Consider now stationary sequence \((X_t)\), and recall that

\[ \mathcal{L} \left( \left( \frac{X_t}{X_0} \right), \left| X_0 > x \right. \right) \overset{d}{\rightarrow} (Y_t), \quad (29) \]

where the convergence in distribution is to be understood here with respect to the product topology (which simply corresponds to the convergence of finite-dimensional distributions). Recall further that the tail process has the form \( Y_k = Y_0 \Theta_k \), \( k \in \mathbb{Z} \), where \( (\Theta_k) \) is the two-sided multiplicative random walk introduced in (27) and (28). Since the walk has negative drift, \( Y_t \to 0 \) a.s. for \( |t| \to \infty \), that is \( (Y_t) \in l_0 \) with probability 1. Moreover, since the anticlustering condition holds by Lemma 5, Proposition 4.2 in Basrak and Segers (2009) (see also the proof of Theorem 4.3 therein) implies that

\[ \mathcal{L} \left( \left( \frac{X_t}{X_0} \right), \left| X_0 > x \right. \right) \overset{d}{\rightarrow} (Y_t), \quad (29) \]

where the convergence in distribution is to be understood here with respect to the product topology (which simply corresponds to the convergence of finite-dimensional distributions). Recall further that the tail process has the form \( Y_k = Y_0 \Theta_k \), \( k \in \mathbb{Z} \), where \( (\Theta_k) \) is the two-sided multiplicative random walk introduced in (27) and (28). Since the walk has negative drift, \( Y_t \to 0 \) a.s. for \( |t| \to \infty \), that is \( (Y_t) \in l_0 \) with probability 1. Moreover, since the anticlustering condition holds by Lemma 5, Proposition 4.2 in Basrak and Segers (2009) (see also the proof of Theorem 4.3 therein) implies that

\[ \mathcal{L} \left( \left( \frac{X_t}{X_0} \right), \left| X_0 > x \right. \right) \overset{d}{\rightarrow} (Y_t), \quad (29) \]

where the convergence in distribution is to be understood here with respect to the product topology (which simply corresponds to the convergence of finite-dimensional distributions). Recall further that the tail process has the form \( Y_k = Y_0 \Theta_k \), \( k \in \mathbb{Z} \), where \( (\Theta_k) \) is the two-sided multiplicative random walk introduced in (27) and (28). Since the walk has negative drift, \( Y_t \to 0 \) a.s. for \( |t| \to \infty \), that is \( (Y_t) \in l_0 \) with probability 1. Moreover, since the anticlustering condition holds by Lemma 5, Proposition 4.2 in Basrak and Segers (2009) (see also the proof of Theorem 4.3 therein) implies that
\[ \theta = \mathbb{P}(\sup_{t<0} Y_t < 1) = \mathbb{P}(\sup_{t>0} Y_t < 1) = \mathbb{P}(\max_{t>0} Y_0 m_1 \cdots m_t < 1) \tag{30} \]

is strictly positive. The second equality above follows from the fact that the so-called anchoring of the tail process at the first or the last exceedence of level 1 leads to the same value of \( \theta \), e.g. see (3.9) in Basrak and Planinić (2021) and the discussion following it. By Theorem 4.3 and Remark 4.6 in Basrak and Segers (2009) there exist two distributions of random elements, \((Z_t)\) and \((Q_t)\) say, in \( l_0 \) such that

\[ (Z_t)_t \overset{d}{=} (Y_t)_t \mid \sup_{t<0} Y_t \leq 1, \quad \text{and} \quad (Q_t)_t \overset{d}{=} (Z_t)_t / \max\{Z_t : t \in \mathbb{Z}\}. \tag{31} \]

### 3.2 Point process convergence and partial maximum

Consider now a branching process with immigration in a random environment started from an arbitrary initial distribution \( \zeta \). Recall that \( P(\cdot, \cdot) \) denotes the Markov transition kernel of the sequence \( (X_n) \). By \( \zeta \mathbb{P}^n \) we represent the distribution of the random variable \( X_n \). In the following theorem we use the notion of convergence in distribution for point process. Following Kallenberg (2017), we endow the space of point measures on the state space \([0, 1] \times (0, \infty)\) with the vague topology. Recall, that (deterministic) measures \( \nu_n \) converge vaguely to \( \nu \) in such a topology if \( \int f \, d\nu_n \to \int f \, d\nu \) for any continuous bounded function \( f \) with a support in some set of the form \([0, 1] \times (x, \infty), x > 0\).

**Theorem 4** Let \((X_t)_{t\geq 0}\) be a branching process with immigration in a random environment – b.p.i.r.e. satisfying (1) with an arbitrary initial distribution. Assume (23), then

\[ N_n = \sum_{i=0}^{n} \delta_{(i/n X_i/a_i)} \overset{d}{\longrightarrow} N = \sum_{i} \sum_{j} \delta_{(T_i, P_i Q_{ij})}, \tag{32} \]

where

i) \( \sum_{i} \delta_{(T_i, P_i)} \) is a Poisson process on \([0, 1] \times (0, \infty)\) with intensity measure \( \text{Leb} \times \nu \) where \( \nu(dy) = \theta \kappa^{y-\kappa-1} dy \) for \( y > 0 \).

ii) \((Q_{ij})_j, i=1, 2, \ldots,\) is an i.i.d. sequence of elements in \( l_0 \) independent of \( \sum_{i} \delta_{(T_i, P_i)} \)

with common distribution equal to the distribution of \((Q_j)\) in (31).

**Remark 8** As we have seen in (30), one can characterize the key constant \( \theta \) in the theorem using the tail process of (29) as

\[ \theta = \mathbb{P}(\sup_{t>0} Y_0 \Theta_t \leq 1) = \mathbb{P}(E_0 + \sup_{t>0} S_t \leq 0), \]

where random variable \( E_0 \) is independent of two sided random walk \( S_t = \log \Theta_t, t \in \mathbb{Z}, \)

and has exponential distribution with parameter \( \kappa \).
To describe the distribution of the \( (Q_j) \) in the theorem, consider the quotient space \( \tilde{l}_0 \) of elements in \( l_0 \) which are shift-equivalent (elements \( (x_i)_i \), \( (y_i)_i \) \( \in l_0 \) are shift-equivalent if for some \( j \in \mathbb{Z} \), \( (x_{i+j})_i = (y_i)_i \)), cf. [Basrak et al. (2018), Section 2]. It is shown in Basrak and Planinić (2021) that in \( \tilde{l}_0 \), \( (Q_j) \) has the same distribution as \( (\Theta_t) \) under the condition that \( \Theta_t < 1 \) for \( t < 0 \) and \( \Theta_t \leq 1 \) for \( t > 0 \). In other words, \( (Q_j) \) has the same distribution as \( (\exp S_t)_t \), with random walk \( (S_t)_t \) conditioned on staying strictly negative for \( t < 0 \) and non positive for \( t > 0 \). We refer to Biggins (2003) for more about random walks conditioned in this way.

**Proof** Assume first that \( (X_t)_{t \geq 0} \) is a stationary b.p.i.r.e. satisfying (24). The statement of the theorem follows immediately from Theorem 3.1 in Basrak and Tafro (2016) together with Lemmas 4 and 5, and discussion following the first lemma.

It remains to prove the convergence of point processes in (32) in the case when \( X_0 \) has an arbitrary initial distribution \( \zeta \). Observe that by the proof of Lemma 4, the function \( V \) is superharmonic for the Markov transition kernel \( P(x, \cdot) \) outside of the level set \( M \). Moreover, by the last argument in the proof of Lemma 4 each level set \( \{ x : x \leq r \} \) is petite. By Theorem 10.2.13 in Douc et al. (2018) the Markov transition kernel \( P \) is Harris recurrent. This, together with Theorem 11.3.1 in Douc et al. (2018) shows that for any initial distribution \( \zeta \) of \( X_0 \),

\[
\| \zeta P^m - \pi \|_{TV} \longrightarrow 0,
\]
as \( m \rightarrow \infty \), where \( \zeta P^m \) denotes the distribution of \( X_m \).

Take an arbitrary continuous nonnegative function \( f \) with support in \([0, 1] \times (\varepsilon, \infty) > 0 \) for some \( \varepsilon > 0 \). Consider Laplace functional (see [Kallenberg (2017), Chapter 4]) of the point process \( N_n \) in (32) under initial distribution \( X_0 \sim \zeta \)

\[
E_\zeta \left[ \exp \left\{ - \sum_{i=1}^{n} f \left( \frac{i}{n}, \frac{X_i}{a_n} \right) \right\} \right] = E_\zeta \left[ \exp \left\{ - \sum_{i=m+1}^{n} f \left( \frac{i}{n}, \frac{X_i}{a_n} \right) \right\} \right] + r_{n,m},
\]

where for \( m \) fixed, \( r_{n,m} \rightarrow 0 \) as \( n \rightarrow \infty \), because \( X_i/a_n \rightarrow 0 \) a.s. for \( i = 1, 2, \ldots, m \). Thus

\[
E_\zeta \left[ \exp \left\{ - \sum_{i=m+1}^{n} f \left( \frac{i}{n}, \frac{X_i}{a_n} \right) \right\} \right] = E_\zeta P^m \left[ \exp \left\{ - \sum_{i=1}^{n-m} f \left( \frac{i+m}{n}, \frac{X_i}{a_n} \right) \right\} \right] = : \int_{H_n} d\zeta P^m,
\]

for a suitably chosen function \( H_n \), which is nonnegative and bounded by 1. Similarly, for the stationary Markov chain \( (X_n) \).
\[
\mathbb{E}_\pi \left[ \exp \left\{ - \sum_{i=1}^{n} f \left( \frac{i}{n}, \frac{X_i}{a_n} \right) \right\} \right] = \int_{\mathbb{N}} H_n \, d\pi + r'_{n,m}
\]

\[
\to \mathbb{E} \left[ \exp \left\{ - \sum_{i} \sum_{j} f(T_i, P_{ij}) \right\} \right].
\]

Observing that \( \left| \int_{\mathbb{N}} H_n \, d\xi \right| - \int_{\mathbb{N}} H_n \, d\pi \leq \| \xi \|_{TV} \), uniformly over \( n \in \mathbb{N} \), and \( r'_{n,m} \to 0 \) for fixed \( m \) as \( n \to \infty \), yields the statement. \( \square \)

We consider next partial maxima of the sequence \((X_i)_{i \in \mathbb{N}}\), namely we define \( M_n = \max\{X_1, \ldots, X_n\} \), for any \( n \geq 1 \). Observe that event \( \{M_n/a_n \leq x\} \) corresponds to the event \( \{N_n([0, 1] \times (x, \infty)) = 0\} \). Moreover, for any \( x > 0 \), the limit point process \( N = \sum_i \sum_j \delta_{T_i, P_{ij}} \) has probability 0 of hitting the boundary of the set \([0, 1] \times (x, \infty)\), thus \( \mathbb{P}(N_n([0, 1] \times (x, \infty))) = 0 \). However, by (31), \( (Q_{ij}) \) in (32) satisfy \( Q_{ij} \leq 1 \) with at least one point exactly equal to 1. Thus \( \mathbb{P}(N([0, 1] \times (x, \infty))) = 0 \) implies \( \mathbb{P}(\sum_i \delta_{T_i, P_i}(10, 1) \times (x, \infty)) = 0 \). Therefore, for any initial distribution of \( X_0 \), partial maxima converge to a rescaled Fréchet distribution.

**Corollary 1** Let \((X_i)_{i \geq 0}\) be a b.p.i.r.e. satisfying (1) with an arbitrary initial distribution. Suppose that (23) holds. Then for any \( x \geq 0 \)

\[
\mathbb{P} \left( \frac{M_n}{a_n} \leq x \right) \to e^{-\theta x^{-\kappa}} \quad \text{as} \quad n \to \infty.
\]

### 3.3 Partial sums

Denote by \((X_t)\) a stationary branching process with immigration in a random environment and assume that one of the conditions in (23) holds. To derive limit theorem for the partial sums from the point process convergence is immediate for \( \kappa \in (0, 1) \), but needs an extra condition for \( \kappa \in [1, 2) \).

**Lemma 6 (Vanishing small values)** Assume that \( \kappa \in [1, 2) \). Then for any \( \varepsilon > 0 \)

\[
\lim_{\gamma \downarrow 0} \lim_{n \to \infty} \sup \mathbb{P} \left( \left| \sum_{i=1}^{n} \left[ X_i \mathbb{1}_{\{X_i \leq a_n \gamma\}} - \mathbb{E}[X_0 \mathbb{1}_{\{X_0 \leq a_n \gamma\}}] \right] \right| > a_n \varepsilon \right) = 0.
\]

**Proof** Choose \( \alpha < 1 \) such that \( 1 + \alpha > \kappa \). (For \( \kappa > 1 \) we may choose \( \alpha = 1 \).) Lemma 4 holds with \( V(x) = 1 + x^\alpha \). Let \( \varepsilon' = 1 - \alpha \). Then,

\[
h(x) := \frac{x}{(a_n \gamma)^{\varepsilon'}} \mathbb{1}_{\{x \leq a_n \gamma\}} \leq x^\alpha, \quad \text{for} \quad x \in \mathbb{N}.
\]

Thus \(|h| \leq V\), therefore by Lemma 4, for some \( \rho \in (0, 1) \) and \( c > 0 \) we have
Thus
\[
\mathbb{E}\left[ X_i \mathbb{1}\{ X_i \leq a_n \gamma \} \right] X_0 \mathbb{1}\{ X_0 \leq a_n \gamma \}
\]
\[
= \sum_{m \leq a_n \gamma} m \mathbb{P}(X_0 = m)(a_n \gamma)^\epsilon \mathbb{E}\left[ \frac{X_i}{(a_n \gamma)^\epsilon} \mathbb{1}\{ X_i \leq a_n \gamma \} \mid X_0 = m \right]
\]
\[
\leq \sum_{m \leq a_n \gamma} m \mathbb{P}(X_0 = m)(a_n \gamma)^\epsilon \left( \mathbb{E}\left[ \frac{X_0}{(a_n \gamma)^\epsilon} \mathbb{1}\{ X_0 \leq a_n \gamma \} \right] + c(1 + m^\alpha)\rho^i \right)
\]
\[
\leq \left( \mathbb{E}[X_0 \mathbb{1}\{ X_0 \leq a_n \gamma \}] \right)^2 + 2c\rho(a_n \gamma)^\epsilon \mathbb{E}[X_0^2 \mathbb{1}\{ X_0 \leq a_n \gamma \}]\mathbb{E}\left[ X_i \right]
\]
where, recall that \( c > 0 \) is a constant, whose value may differ from line to line. For \( \beta > \kappa \) by the regular variation and Karamata's theorem as \( u \to \infty \)
\[
\mathbb{E}[X_0^\beta \mathbb{1}\{ X_0 \leq u \}] \sim \frac{\beta}{\beta - \kappa} u^\beta \mathbb{P}(X_0 > u).
\]
Substituting back into (33), and using that \( \mathbb{P}(X_0 > a_n \gamma) \sim \gamma^{-\kappa}n^{-1} \) we have
\[
\text{Cov} \left( X_0 \mathbb{1}\{ X_0 \leq a_n \gamma \}, X_i \mathbb{1}\{ X_i \leq a_n \gamma \} \right) \leq c\rho^i \frac{a_n^2}{n} \gamma^{2-\kappa}.
\]
Therefore, using the stationarity, (33), and (34), we obtain
\[
\mathbb{E}\left[ \left( \sum_{i=1}^n [X_i \mathbb{1}\{ X_i \leq a_n \gamma \} - \mathbb{E}[X_0 \mathbb{1}\{ X_0 \leq a_n \gamma \}]] \right)^2 \right]
\]
\[
= n \text{Var} \left( X_0 \mathbb{1}\{ X_0 \leq a_n \gamma \} \right)
\]
\[
+ \sum_{i=1}^{n-1} 2(n - i) \text{Cov} \left( X_0 \mathbb{1}\{ X_0 \leq a_n \gamma \}, X_i \mathbb{1}\{ X_i \leq a_n \gamma \} \right)
\]
\[
\leq n \text{Var} \left( X_0 \mathbb{1}\{ X_0 \leq a_n \gamma \} \right) + c \sum_{i=1}^{n-1} (n - i)\rho^i \frac{a_n^2}{n} \gamma^{2-\kappa}
\]
\[
\leq ca_n^2 \gamma^{2-\kappa}.
\]
Therefore, the claim follows from Chebyshev’s inequality. \( \square \)

**Remark 9** In fact, we proved condition (9) in Davis (1983).

**Theorem 5** Let \( (X_i)_{i \geq 0} \) be a branching process with immigration in random environment – b.p.i.r.e. satisfying (1) with an arbitrary initial distribution. Assume (23) and denote by \( (b_n) \) a sequence of real numbers given by
\[ b_n = 0, \text{ for } \kappa < 1, \text{ and } b_n = n\mathbb{E}\left[\frac{X_\infty}{a_n} \mathbb{I}\left\{\frac{X_\infty}{a_n} \leq 1\right\}\right], \text{ for } \kappa \in [1, 2). \]

Then

\[ V_n = \sum_{k=1}^{n} \frac{X_k}{a_n} - b_n \xrightarrow{d} V, \quad n \to \infty, \tag{35} \]

where \( V \) is a \( \kappa \)-stable random variable. For \( \kappa > 2 \), as \( n \to \infty \)

\[ \frac{1}{\sqrt{n\sigma}} \sum_{j=1}^{n} (X_i - \mathbb{E}[X_\infty]) \xrightarrow{d} Z, \]

where \( Z \) is a standard normal random variable, and \( \sigma^2 = \frac{1 + \mathbb{E}[\lambda]}{1 - \mathbb{E}[\lambda]} \Var(X_\infty) > 0 \).

**Remark 10** Using Theorem 3.4 in Basrak et al. (2012), for \( \kappa < 1 \) the result above immediately extends to the functional convergence in the space of càdlàg functions \( D[0, 1] \) endowed with the \( M_1 \) topology. However, to show the same for \( 1 \leq \kappa < 2 \), one would need to establish Condition 3.3 in Basrak et al. (2012), which amounts to a uniform version of the result in Lemma 6, see also condition ANSJU\( (a_n) \) in Kulik and Soulier (2020). We leave this as an open question. However, it might be even more interesting to see if one can apply any of this to show the functional version of the results in Sect. 4 concerning random walks in random environment.

Note finally, that for \( \kappa > 2 \) one can easily check that the conditions of Theorem 19.1 in Billingsley (1999) hold, therefore the functional version of the CLT also holds.

**Remark 11** Recall that under the conditions of Theorem 1 for the normalizing sequence we may choose \( a_n = (Cn)^{1/\kappa} \), in general we have \( a_n = \tilde{c}(n)n^{1/\kappa} \), for some slowly varying \( \tilde{c} \), as observed after (25). Concerning the centering constants \( (b_n) \), recall that the the law of \( X_\infty \) represents the unique stationary distribution for b.p.i.r.e. satisfying (1). For \( \kappa \in (1, 2) \), the mean of \( X_\infty \) is finite, so one could substitute centering constants \( (b_n) \) by \( n\mathbb{E}[X_\infty/a_n] \) to show

\[ \frac{1}{a_n} \sum_{j=1}^{n} (X_i - \mathbb{E}[X_\infty]) \xrightarrow{d} V - \frac{\kappa}{\kappa - 1}. \tag{36} \]

Thus, we again get a \( \kappa \)-stable limit with different location parameter, cf. Remark 3.1 in Davis and Hsing (1995). Moreover, under conditions of Theorem 1, in the case \( \kappa = 1, b_n \sim C^{-1} \log n \).

**Remark 12** One can also describe the limiting stable distribution in (35) in terms of parameters \( \kappa, \theta \) and the distribution of \( (Q_j) \), see Remark 3.2 in Davis and Hsing (1995) for details. In the case \( \kappa < 1 \) for instance, the limiting random variable \( V \) has a characteristic function of relatively simple form

\[ \mathbb{E}[e^{itV}] = \exp\left\{-\frac{\kappa}{\kappa - 1} \left(1 - \frac{\theta}{\kappa t^\kappa}\right)\right\} \]

for \( \kappa \neq 0 \), and \( \mathbb{E}[e^{itV}] = \exp\left\{-\theta \left(1 - \frac{\theta}{\kappa t^\kappa}\right)\right\} \) for \( \kappa = 0 \).
\[ E[e^{itV}] = \exp \left( -d|t|^{\kappa} \left( 1 - i \text{sgn}(t) \tan \frac{\pi \kappa}{2} \right) \right), \]  
\tag{37}
where \( d = \theta \Gamma(1 - \kappa) E[(\sum_j Q_j)^{\kappa}] \cos(\pi \kappa / 2) \), which is known to be finite since for \( \kappa \leq 1 \), \( E[(\sum_j Q_j)^{\kappa}] \leq E[\sum_j Q_j] < \infty \), see [Davis and Hsing (1995), Remark 3.2, Theorem 3.2].

For \( \kappa \in (1, 2) \), multiplicative random walk \((\Theta_j)\) given by (27) and (28) satisfies \( E[(\sum_j \Theta_j)^{\kappa-1}] \leq E[\sum_j \Theta_j] < \infty \). By Proposition 5.6.5 in Kulik and Soulier (2020), this yields \( E[(\sum_j Q_j)^{\kappa}] < \infty \). Applying Proposition 8.2.3 and Theorem 8.3.5 in Kulik and Soulier (2020), and observing that they use \( n E[X_\infty]/a_n \) as centering constants, one can deduce that
\[ E[e^{itV}] = \exp \left( -d|t|^{\kappa} \left( 1 - i \text{sgn}(t) \tan \frac{\pi \kappa}{2} \right) + i ct \right), \]
with the same expression for the scale parameter \( d \) and with \( c = \kappa/(\kappa - 1) \), cf. (36).

For \( \kappa = 1 \), \((\Theta_j)\) satisfies \( E[\log(\sum_j \Theta_j)] \leq E[\sum_j \Theta_j]^{1/2} < \infty \). By Chapter 5 in Kulik and Soulier (2020) (see Problem 5.33 therein), this implies that \( E[\sum_j Q_j \log Q_j^{-1}] < \infty \). This time, Theorem 8.3.5 in Kulik and Soulier (2020) yields
\[ E[e^{itV}] = \exp \left( -d|t| \left( 1 + i \frac{2}{\pi} \text{sgn}(t) \log |t| \right) + i ct \right), \]
where \( d = \pi/2 \) while an expression for the location parameter \( c \) can be found in Proposition 8.2.4 of Kulik and Soulier (2020).

**Proof of Theorem 5** Assume first that \( \kappa < 2 \). As in the proof of Theorem 4, we first assume that \((X_t)\) is a stationary b.p.i.r.e. process satisfying the conditions of the theorem. The claim then follows directly from Theorem 4 and Theorem 3.1 in Davis and Hsing (1995). Observe that the condition (3.2) therein follows from Lemma 6.

To prove the theorem when \( X_0 \) has an arbitrary initial distribution \( \zeta \) we can use a similar argument as in the proof of Theorem 4. Observe first, that
\[ V_n = \sum_{k=1}^{n} \frac{X_k}{a_n} - b_n = \sum_{k=1}^{n-m} \frac{X_{k+m}}{a_n} - b_n + r_{n,m}, \]
with \( r_{n,m} \to 0 \) in probability as \( n \to \infty \) for any fixed \( m \). Denote by \( \zeta P^m \) the distribution of \( X_m \), and note that for any \( s \in \mathbb{R} \)
\[ E_\zeta \left[ \exp \left( \text{i}s \left( \sum_{k=1}^{n-m} \frac{X_{k+m}}{a_n} - b_n \right) \right) \right] = E_\zeta P^m \left[ \exp \left( \text{i}s \left( \sum_{k=1}^{n-m} \frac{X_k}{a_n} - b_n \right) \right) \right] = E_\zeta \left[ \exp \left( \text{i}s \left( \sum_{k=1}^{n-m} \frac{X_k}{a_n} - b_n \right) \right) \right] + u_{n,m}, \]
where for all \( n \)
\[ u_{n,m} \leq \| \xi P^n - \pi \|_{TV}, \]

with the right hand side tending to 0 as \( m \to \infty \). Observe now that a stationary b.p.i.r.e. \((X_i)\) satisfies \( \gamma'_{n,m} = \sum_{k=n-1}^{n} \frac{X_k}{a_n} \to 0 \) in probability as \( n \to \infty \). Therefore, it also satisfies

\[ \sum_{k=1}^{n-m} \frac{X_k}{a_n} - b_n = \sum_{k=1}^{n} \frac{X_k}{a_n} - b_n - \gamma'_{n,m} \to V. \]

Let now \( \kappa > 2 \). Then \( \text{Var} (X_\infty) < \infty \), and the result follows from standard Markov chain theory. To apply Theorem 1 (i) in Jones (2004) we have to prove the drift condition as in Lemma 4 with \( V(x) = 1 + x^2 \). Simple calculation gives

\[ \mathbb{E} \left[ \left( \sum_{i=1}^x A_i + B \right)^2 \right] = x^2 \mathbb{E}[m(\xi)^2] + x(\mathbb{E}[A^2] - \mathbb{E}[m(\xi)^2] + 2\mathbb{E}[AB]) + \mathbb{E}[B^2]. \]

Since \( \mathbb{E}[m(\xi)^2] < 1 \), there exist \( \beta \in (0, 1), b > 0 \), and \( x_0 \in \mathbb{N} \) such that for all \( x \in \mathbb{N} \)

\[ \mathbb{E}_x[V(X_1)] = \mathbb{E}[V(X_1)|X_0 = x] \leq \beta V(x) + b \mathbb{P} \{ x \leq x_0 \}. \]

Moreover, the level set \( M = \{ x : x \leq x_0 \} \) is small, so the conditions in Theorem 1 (i) in Jones (2004) hold.

We only have to check that \( \sigma^2 = \text{Var}_\pi(X_0) + 2 \sum_{i=1}^\infty \text{Cov}_\pi(X_0, X_i) > 0 \). Consider the decomposition

\[ X_t = \sum_{i=1}^{x_0} \tilde{A}^{(i)} + \sum_{k=0}^{t-1} C_{t,k}, \]  \hspace{1cm} (38)

where the first sum stands for the descendants of generation 0, while the second sum stands for the descendants of the immigrants arrived after generation 0, i.e. \( C_{t,k} = \theta_0 \circ \cdots \circ \theta_{t-k+1} \circ B_{t-k} \) denotes the number of descendants in generation \( t \) from immigrants arrived in generation \( t-k \). Then \( X_0 \) and \( (C_{i,k})_{k=0,1,\ldots,t-1} \) are independent, and \( (\tilde{A}^{(i)}_{i})_{i \in \mathbb{N}} \) are independent given the environments \( \xi_1, \ldots, \xi_t \). Observe now

\[ \text{Cov}_\pi(X_0, X_i) = \text{Cov}_\pi \left( X_0, \sum_{i=1}^{x_0} \tilde{A}^{(i)} \right), \]

where \( \tilde{A}^{(i)} \) stands for the number of descendants in generation \( t \) of a single individual in generation 0. Thus

\[ \mathbb{E} \left[ X_0 \sum_{i=1}^{x_0} \tilde{A}^{(i)} \right] = (\mathbb{E}[A])^t \mathbb{E}[X_0^2]. \]

Therefore

\[ \text{Cov}_\pi(X_0, X_i) = \text{Var}_\pi(X_0) (\mathbb{E}[A])^t, \]
and the formula \( \sigma^2 = \text{Var}_\pi(X_0)(1 + E[A])/(1 - E[A]) \) follows.

To show that \( \sigma > 0 \) we have to prove that the stationary solution cannot be deterministic. Indeed, the fixed point equation (6) (or (8)) has no deterministic solution, since Cramér's condition (3) implies \( P(A > 1) > 0 \), thus \( P(\sum_{i=1}^m A_i + B > m) > 0 \) for any positive integer \( m \).

\[ \square \]

4 Random walks in a random environment

Connection between random walks in a random environment and branching processes with immigration was made already in the seminal paper by Kesten et al. (1975). To describe the setup of their model, we can use again a sequence of i.i.d. random variables \( \xi, (\xi_x)_{x \in \mathbb{Z}} \) with values in the interval \((0, 1)\). Consider the process \( W_0 = 0 \) and

\[ W_t = W_{t-1} + \eta_{W_{t-1}, t} \quad (39) \]

where conditioned on the environment \( \mathcal{E} \), \( \eta_{x,t} \)'s are independent random variables taking value +1 with probability \( \xi_x \) and value −1 with probability \( \xi_x' = 1 - \xi_x \). The process \( (W_t) \) is called random walk in a random environment – r.w.r.e.

Following Kesten et al. (1975) we assume throughout that

\[ E\left[ \log \frac{\xi'}{\xi} \right] < 0 \quad (40) \]

and

\[ E\left[ \left( \frac{\xi'}{\xi} \right)^\kappa \right] = 1 \quad \text{and} \quad E\left[ \left( \frac{\xi'}{\xi} \right)^\kappa \log \frac{\xi'}{\xi} \right] < \infty, \quad (41) \]

for some \( \kappa > 0 \). The first of these conditions ensures that \((W_t)\) drifts to \(+\infty\) and the second one corresponds to (3). As observed in Kesten et al. (1975), for the asymptotic analysis of \((W_t)\), it is crucial to understand asymptotic behavior of the random variables

\[ T_n = \min\{t : W_t = n\}, \quad n \geq 1, \]

which are all finite a.s. Observe that on the way to the state \( n \), process \((W_t)\) visits each state \( k = 0, 1, \ldots, n-1 \) at least once. For \( i = 1, \ldots, n \) denote

\[ L_i^n = 1 + U_i^n = 1 + \#\{\text{visits to the state } n-i \text{ from the right before } T_n\}. \]

Note that until time \( T_n \), each of the visits to \( n-i \) from the right is canceled by one movement back to the right. Note \( U_i^n \) are well defined and possibly different from 0 also for \( i > n \). Observe next that the total number of visits by the process \((W_t)\) to the left of 0, say \( 2U_\infty \), is a.s. finite. Thus, \( R_n = 2 \sum_{i>n} U_i^n \leq 2U_\infty \). Thus in
\[ T_n = n + 2 \sum_{i \geq 1} U^n_i = n + 2 \sum_{i=1}^{n} U^n_i + R_n, \]

the term \( R_n \) remains bounded as \( n \to \infty \).

Observe next that \( U^n_1 = 0 \), so that \( L^n_1 = 1 \) and the sequence \( L^n_i \) evolves as follows

\[ L^n_2 = G_{n,1,1} + 1, \]

where \( G_{n,1,1} \) represents the number of right visits to \( n - 2 \) from \( n - 1 \) before eventual move to the right. Conditioned on \( E \), \( G_{n,1,1} \) has a geometric distribution with mean \( \frac{\xi_{n-1}}{\xi_{n-1}} \). Similarly, for each \( i \geq 2 \)

\[ L^n_i = \sum_{j=1}^{L^{n-1}_i} G_{n,i-1,j} + 1, \quad \text{(42)} \]

where conditionally on \( E \), \( G_{n,i-1,j} \), \( j = 1, 2, \ldots \), are i.i.d. with a geometric distribution with mean \( \frac{\xi_{n-i+1}}{\xi_{n-i+1}} \).

The finite sequences

\[ L^n_1, L^n_2, \ldots, L^n_k, \ldots, L^n_n, \]

represent a special case of branching process in a random environment. Moreover, the initial part \( L^n_1, L^n_2, \ldots, L^n_k \) has the same distribution for each \( n \geq k \), hence if we want to understand the limiting distribution of \( T_n = 2 \sum_{i=1}^{n} L^n_i - n + R_n \), we can simply skip the index \( n \) in (42) and analyze \( \sum_{i=1}^{n} L_i \), which is exactly the content of Theorem 5. Denote in the sequel by \( L_\infty \) the random variable which has the stationary distribution of the Markov chain in (42).

The following corollary of our work corresponds to the seminal theorem in Kesten et al. (1975) in the case \( \kappa < 2 \). Although its three statements are known, note that we can, in contrast to Kesten et al. (1975), determine the exact distribution of the limit \( \tilde{V} \) in all three cases as explained in Remark 12. In particular, location and scale parameters of the stable law \( \tilde{V} \) can be given in terms of the values \( \kappa, \theta \) and the conditional multiplicative random walk \( (Q_j) \). The corresponding statement for \( \kappa > 2 \) follows from our Theorem 5 too, in that case \( T_n \) after centering and normalization with \( \sqrt{n} \) converges to a normal distribution. Observe that in the case \( \kappa = 2 \), \( T_n \) converges to a normal limit again, but additional care has to be taken about normalizing and centering constants, see Kesten et al. (1975).

**Theorem 6** (Kesten, Kozlov, Spitzer) Let \( (W_t)_{t \geq 0} \) be a random walk in random environment satisfying (39). Suppose that the conditions of (40) and (41) hold for some \( \kappa > 0 \). Suppose further that the law of \( \log(\xi'_i/\xi_i) \) is nonarithmetic.

(i) For \( \kappa \in (0, 1) \) as \( n \to \infty \)

\[ \quad \]
where $\widetilde{V}$ has a strictly positive $\kappa$–stable distribution, while, as $t \to \infty$

$$\frac{1}{t^\kappa} W_t \xrightarrow{d} \widetilde{V}^{-\kappa}.$$

(ii) For $\kappa = 1$

$$\frac{1}{n} T_n - 2Cb_n \xrightarrow{d} \widetilde{V},$$

where $b_n \sim \log n$ and $\widetilde{V}$ has a 1-stable distribution. Moreover, as $t \to \infty$

$$\frac{(\log t)^2}{t} (W_t - \delta(t)) \xrightarrow{d} -\frac{\widetilde{V}}{(2C)^2},$$

with $\delta(t) \sim t/(2C \log t)$.

(iii) For $\kappa \in (1, 2)$

$$\frac{1}{n^{1/\kappa}} (T_n - n(2E[L_\infty] - 1)) \xrightarrow{d} \widetilde{V},$$

where $\widetilde{V}$ has a $\kappa$–stable distribution. Moreover, as $t \to \infty$

$$\frac{1}{t^{1/\kappa}} \left( W_t - \frac{t}{2E[L_\infty] - 1} \right) \xrightarrow{d} -\frac{\widetilde{V}}{(2E[L_\infty] - 1)^{1+1/\kappa}}.$$

**Proof** Recall that $T_n \xrightarrow{d} \sum_{i=1}^n (2L_i - 1) + R_n$ , with random variables $R_n$ a.s. bounded and where $(L_n)$ represents a special b.p.i.r.e. which is initialized at one, with progeny conditionally geometric given the environmental variables $\xi_j$’s and with the immigration identically equal to 1. Recall further that the conditions of Theorems 1 and 5 are met by assumptions (40), (41) and Remarks 1 and 2.

Define now

$$c_n = \begin{cases} 0, & \text{for } \kappa \in (0, 1), \\ 2E[L_\infty] \{L_\infty \leq (Cn)^{1/\kappa}\} - 1, & \text{for } \kappa = 1, \\ 2E[L_\infty] - 1, & \text{for } \kappa \in (1, 2). \end{cases}$$

Theorem 5 and Remark 11 imply that $2^{-1}(Cn)^{-1/\kappa} \sum_{i=1}^n (2L_i - 1 - c_n)$ converges in distribution to a $\kappa$-stable random variable for all $\kappa \in (0, 2)$. After multiplication by $2C^{1/\kappa}$, this yields

$$\frac{1}{n^{1/\kappa}} (T_n - nc_n) \xrightarrow{d} \widetilde{V},$$
for a certain $\kappa$-stable random variable $\widetilde{V}$. Using the particular form of constants $c_n$ gives the centering sequence in each of the three cases. The convergence in distribution of the r.w.r.e. $W_t$ now follows as in Kesten et al. (1975).

Using our methods one can extend Theorem 6 to obtain a new and analogous result about RWRE under the second condition in (23). Recall first that normalizing sequence $(a_n)$ from (25) satisfies $a_n = n^{1/\kappa} \tilde{c}(n)$ for some slowly varying function $\tilde{c}$, and extend this to a function $a(t) = t^{1/\kappa} \tilde{c}(t)$ on $(0, \infty)$.

**Theorem 7** Let $(W_t)_{t \geq 0}$ be a random walk in random environment satisfying either condition (i) or (ii) in Theorem 2 for $\kappa \in (1, 2)$ with $m(\xi) = \xi'/\xi$. Suppose further that the law of $\log(\xi'/\xi)$ is nonarithmetic. Then

$$
\frac{1}{a_n}(T_n - n(2E[L_\infty] - 1)) \xrightarrow{d} \widetilde{V} \quad \text{as } n \to \infty.
$$

Moreover, as $t \to \infty$

$$
\frac{1}{a(t)} \left( W_t - \frac{t}{2E[L_\infty] - 1} \right) \xrightarrow{d} -\widetilde{V} \quad \text{as } (2E[L_\infty] - 1)^{1+1/\kappa}.
$$

**Proof** The first part follows again from Theorem 5, while the second part follows from the inverse relation between $T_n$ and $W_t$; see (2.38) in Kesten et al. (1975). We omit the details.

**Example 1** Assume that the random variable $\xi$ satisfies $\mathbb{E}(\xi'/\xi)^\kappa = 1$, and has a density function $f$ such that for all $u > 0$ small enough

$$
f(u) = u^{\kappa-1} \alpha \frac{\log \log u^{-1}}{(\log u^{-1})^{\alpha+1}},
$$

where $\kappa \geq 1$, $\alpha \in (1/2, 1)$. Then straightforward calculation shows that

$$
\mathbb{E} \left\{ \log \left( \frac{1 - \xi}{\xi} > x \right) \left( \frac{1 - \xi}{\xi} \right)^\kappa \right\} = \int_0^{(1+e')^{-1}} \left( \frac{1 - u}{u} \right)^\kappa f(u) du \sim \frac{\log x}{x^\alpha}.
$$

Thus, the conditions of the theorem above holds, and for $M$ in (19)

$$
M(x) \sim \frac{x^{1-\alpha} \log x}{1 - \alpha}.
$$

Therefore by Theorem 2, the stationary distribution $L_\infty$ satisfies

$$
\mathbb{P}(L_\infty > x) \sim \frac{C(1 - \alpha)}{x^\kappa (\log x)^{1-\alpha} \log \log x}.
$$

Then
According to Kesten et al. (1975) even before they gave the proof of the theorem, it was conjectured by A.N. Kolmogorov and F. Spitzer that \( T_n \) might exhibit the behavior described above. The intuitive reason behind this observation may be the existence of so-called traps between 0 and \( n \), i.e. sites \( j \in \{0, \ldots, n\} \) where corresponding \( \xi_j \) is atypically small, which makes it very difficult for the random walk \((W_t)\) to cross over to the right. It is interesting that our other main result, Theorem 4, provides a very simple argument characterizing the asymptotic distribution of the worst of such traps. Denote now for \( k < n \) by

\[
\mathcal{V}_k^n = \#\{\text{crossings over the edge } (k, k+1) \text{ before } T_n\}.
\]

Clearly, for \( k = 0, \ldots, n-1 \), we have \( \mathcal{V}_k^n = 1 + 2U_{n-k}^n \) and while for \( k < 0 \) \( \mathcal{V}_k^n = 2U_{n-k}^n \). Observing that \( \max_{k<0} \mathcal{V}_k^n \) remains bounded a.s. again, from Corollary 1 we can deduce the following result concerning the most visited edge until time \( T_n \).

**Corollary 2** Under the assumptions of Theorem 6 or 7

\[
P(\frac{\max_{k<n} \mathcal{V}_k^n}{2a_n} \leq \lambda) \to e^{-\theta e^{-\lambda}},
\]

as \( n \to \infty \).

Recall that by Remark 8, \( \theta \) in either case can be obtained as

\[
\theta = P\left(E_0 + \sup_{t>0} \sum_{i=1}^{t} \log(\xi'_i/\xi_i) \leq 0\right),
\]

where \( E_0 \) stands for an exponential random variable with parameter \( \kappa \) independent of the environment sequence \((\xi)\).

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**References**

Afanasyev, V.I.: On the maximum of a subcritical branching process in a random environment. Stochastic Process. Appl. 93(1), 87–107 (2001)

Barczy, M., Nedényi, F.K., Pap, G.: On aggregation of multitype Galton-Watson branching processes with immigration. Mod. Stoch. Theory Appl. 8(1), 53–79 (2018)

Basrak, B., Krizmanić, D., Segers, J.: A functional limit theorem for dependent sequences with infinite variance stable limits. Ann. Probab. 40(5), 2008–2033 (2012)

Basrak, B., Kulik, R., Palmowski, Z.: Heavy-tailed branching process with immigration. Stoch. Models 29(4), 413–434 (2013)

Basrak, B., Planinić, H.: Compound Poisson approximation for random fields with application to sequence alignment. Bernoulli 27(2), 1371–1408 (2021)
Basrak, B., Planinić, H., Soulier, P.: An invariance principle for sums and record times of regularly varying stationary sequences. Probab. Theory Related Fields 172(3–4), 869–914 (2018)
Basrak, B., Segers, J.: Regularly varying multivariate time series. Stochastic Process. Appl. 119(4), 1055–1080 (2009)
Basrak, B., Tafro, A.: A complete convergence theorem for stationary regularly varying multivariate time series. Extremes 19(3), 549–560 (2016)
Biggins J.D.: Random walk conditioned to stay positive. J. London Math. Soc. (2) 67(1), 259–272 (2003)
Billingsley, P.: Convergence of probability measures, 2nd edn. Wiley Series in Probability and Statistics: Probability and Statistics. John Wiley & Sons, Inc., New York (1999)
Buraczewski, D., Damek, E., Mikosch, T.: Stochastic models with power-law tails. The equation $X = AX + B$. Springer Series in Operations Research and Financial Engineering. Springer (2016)
Buraczewski, D., Dyszewski, P.: Precise large deviation estimates for branching process in random environment. https://arxiv.org/abs/1706.03874v1 (2019)
Caravenna, F., Doney, R.: Local large deviations and the strong renewal theorem. Electron. J. Probab. 24(Paper No. 72), 48 (2019)
Davis, R.A.: Stable limits for partial sums of dependent random variables. Ann. Probab. 11(2), 262–269 (1983)
Davis, R.A., Hsing, T.: Point process and partial sum convergence for weakly dependent random variables with infinite variance. Ann. Probab. 23(2), 879–917 (1995)
Doney, R.A.: One-sided local large deviation and renewal theorems in the case of infinite mean. Probab. Theory Related Fields 107(4), 451–465 (1997)
Douc, R., Moulines, E., Priouret, P., Soulier, P.: Markov chains. Springer Series in Operations Research and Financial Engineering. Springer, Cham (2018)
Foss, S., Korshunov, D., Zachary, S.: An introduction to heavy-tailed and subexponential distributions, 2nd edn. Springer Series in Operations Research and Financial Engineering. Springer, New York (2013)
Garsia, A., Lamperti, J.: A discrete renewal theorem with infinite mean. Comment. Math. Helv. 37, 221–234 (1962/63)
Goldie, C.M.: Implicit renewal theory and tails of solutions of random equations. Ann. Appl. Probab. 1(1), 126–166 (1991)
Janssen, A., Segers, J.: Markov tail chains. J. Appl. Probab. 51(4), 1133–1153 (2014)
Jelenković, P.R., Olvera-Cravioto, M.: Implicit renewal theorem for trees with general weights. Stochastic Process. Appl. 122(9), 3209–3238 (2012)
Jones, G.L.: On the Markov chain central limit theorem. Probab. Surv. 1, 299–320 (2004)
Kallenberg, O.: Random measures, theory and applications, volume 77 of Probability Theory and Stochastic Modelling. Springer, Cham (2017)
Kesten, H., Kozlov, M.V., Spitzer, F.: A limit law for random walk in a random environment. Compositio Math. 30, 145–168 (1975)
Kevei, P.: A note on the Kesten-Grincevičius-Goldie theorem. Electron. Commun. Probab. 21(51), 1–12 (2016)
Kevei, P.: Implicit renewal theory in the arithmetic case. J. Appl. Probab. 54(3), 732–749 (2017)
Kevei, P., Wiandt, P.: Moments of the stationary distribution of subcritical multitype Galton-Watson processes with immigration. Statist. Probab. Lett. 173, 109067, 6 (2021)
Krizmanić, D.: Functional limit theorems for weakly dependent regularly varying time series. PhD thesis, University of Zagreb (2010)
Kulik, R., Soulier, P.: Heavy-tailed time series. Springer-Verlag, New York (2020)
Lyons, R., Pemantle, R., Peres, Y.: Conceptual proofs of $L \log L$ criteria for mean behavior of branching processes. Ann. Probab. 23(3), 1125–1138 (1995)
Meyn, S., Tweedie, R.L.: Markov chains and stochastic stability, 2nd edn. Cambridge University Press, Cambridge (2009)
Petrov, V.V.: Limit theorems of probability theory, volume 4 of Oxford Studies in Probability. The Clarendon Press, Oxford University Press, New York (1995)
Quine, M.P.: The multi-type Galton-Watson process with immigration. J. Appl. Probability 7, 411–422 (1970)
Resnick, S.I.: Heavy-tail phenomena: probabilistic and statistical modeling. Springer Verlag (2007)
Roitershtein, A.: A note on multitype branching processes with immigration in a random environment. Ann. Probab. 35(4), 1573–1592 (2007)
Segers, J.: Multivariate regular variation of heavy-tailed Markov chains. Available on arXiv: https://arxiv.org/abs/math/0701411 (2007)

Szűcs, G.: Ergodic properties of subcritical multitype Galton-Watson processes. Available on arXiv: https://arxiv.org/abs/1402.5539 (2014)

Vatutin, V., Topchii, V.: A key renewal theorem for heavy tail distributions with $\beta \in (0,0.5]$. Theory Probab. Appl. 58(2), 333–342 (2014)

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