Nonlinear Schrödinger equation on the half-line with nonlinear boundary condition

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Abstract. In this paper, we study the initial boundary value problem for nonlinear Schrödinger equations on the half-line with nonlinear boundary conditions of type $u_x(0,t) + \lambda |u(0,t)|^r u(0,t) = 0$, $\lambda \in \mathbb{R} - \{0\}$, $r > 0$. We discuss the local well-posedness when the initial data $u_0 = u(x,0)$ belongs to an $L^2$-based inhomogeneous Sobolev space $H^s(\mathbb{R}_+)$ with $s \in (\frac{1}{2}, \frac{7}{2}) - \{\frac{3}{2}\}$. We deal with the nonlinear boundary condition by first studying the linear Schrödinger equation with a time-dependent inhomogeneous Neumann boundary condition $u_x(0,t) = h(t)$ where $h \in H^{\frac{2s-1}{4}}(0,T)$. This latter problem is studied by adapting the method of Bona-Sun-Zhang [3] to the case of inhomogeneous Neumann boundary conditions.

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1. Introduction and Main Result

The nonlinear Schrödinger equation (NLS) is a fundamental dispersive partial differential equation. NLS can be used in many physical nonlinear systems such as quantum many body systems, optics, hydrodynamics, acoustics, quantum condensates, and heat pulses in solids.

In this article, we consider the following nonlinear Schrödinger equation with nonlinear boundary condition on the (right) half-line:

$$\begin{cases}
i \partial_t u + \partial_x^2 u + k |u|^p u = 0, & x \in \mathbb{R}_+, t \in (0,T), \\
u(x,0) = u_0(x), \\
\partial_x u(0,t) + \lambda |u(0,t)|^r u(0,t) = 0,
\end{cases}$$

where $u(x,t)$ is a complex valued function, the real variables $x$ and $t$ are space and time coordinates, and $\partial_t, \partial_x$ denote partial derivatives with respect to time and space. The constant parameters satisfy $k, \lambda \in \mathbb{R} - \{0\}$, and $p, r > 0$. 

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When $\lambda = 0$, the boundary condition reduces to the classical homogeneous Neumann boundary condition. When $r = 0$, the boundary condition is the classical homogeneous Robin boundary condition. When $\lambda$ and $r$ are both non-zero as in the present case, the boundary condition can be considered as a nonlinear variation of the Robin boundary condition.

Our main goal is to solve the classical local well-posedness problem for (1.1). More precisely, we will prove the local existence and uniqueness for (1.1) together with the continuous dependence of solutions on the initial data $u_0$, which is taken from an $L^2$-based inhomogeneous Sobolev space $H^s(\mathbb{R}_+) \setminus \{\frac{3}{2}\}$. We will also deduce a blow-up alternative for the solutions of (1.1) in the $H^s$-sense.

The well-posedness problem will be considered in the function space $X^s_T$, which is the set of those elements in

$$C([0,T];H^s(\mathbb{R}_+)) \cap C(\mathbb{R}_+^s;\mathcal{H}^{2s+1}(0,T))$$

that are bounded with respect to the norm $\| \cdot \|_{X^s_T}$. This norm is defined by

$$\| f \|_{X^s_T} := \sup_{t \in [0,T]} \| f(\cdot,t) \|_{H^s(\mathbb{R}_+)} + \sup_{x \in \mathbb{R}_+} \| f(x,\cdot) \|_{H^{2s+1}(0,T)}.$$ (1.2)

It is well-known that the trace operators $\gamma_0 : u_0 \rightarrow u_0(0)$ and $\gamma_1 : u_0 \rightarrow u_0'(0)$ are well-defined on $H^s(\mathbb{R}_+)$ when $s > \frac{3}{2}$ and $s > \frac{3}{2}$, respectively. Therefore, both $u_0(0)$ and $u_0'(0)$ make sense if $s > \frac{3}{2}$. Hence, we will assume the compatibility condition $u_0'(0) = -\lambda|u_0(0)|^ru_0(0)$ when $s > \frac{3}{2}$ on the initial data to comply with the desire that the solution be continuous at $(x,t) = (0,0)$.

Now we can state our main result.

**Theorem 1.1 (Local well-posedness).** Let $T > 0$ be arbitrary, $s \in \left(\frac{1}{2},\frac{3}{2}\right) - \{\frac{3}{2}\}$, $p, r > 0$, $k, \lambda \in \mathbb{R} - \{0\}$, $u_0 \in H^s(\mathbb{R}_+)$ together with $u_0'(0) = -\lambda|u_0(0)|^ru_0(0)$ whenever $s > \frac{3}{2}$. We in addition assume the following restrictions on $p$ and $r$:

(A1) If $s$ is integer, then $p \geq s$ if $p$ is an odd integer and $[p] \geq s - 1$ if $p$ is non-integer.

(A2) If $s$ is non-integer, then $p > s$ if $p$ is an odd integer and $[p] \geq [s]$ if $p$ is non-integer.

(A3) $r > \frac{2s-1}{4}$ if $r$ is an odd integer and $[r] \geq \left[\frac{2s-1}{4}\right]$ if $r$ is non-integer.

Then, the following hold true.

(i) Local Existence and Uniqueness: There exists a unique local solution $u \in X^s_{T_0}$ of (1.1) for some $T_0 = T_0 \left(\| u_0 \|_{H^s(\mathbb{R}_+)}\right) \in (0,T]$.

(ii) Continuous Dependence: If $B$ is a bounded subset of $H^s(\mathbb{R}_+)$, then there is $T_0 > 0$ (depends on the diameter of $B$) such that the flow $u_0 \rightarrow u$ is Lipschitz continuous from $B$ into $X^s_{T_0}$.

(iii) Blow-up Alternative: If $S$ is the set of all $T_0 \in (0,T]$ such that there exists a unique local solution in $X^s_{T_0}$, then whenever $T_{\text{max}} := \sup_{T_0 \in S} T_0 < T$, it must be true that $\lim_{t \uparrow T_{\text{max}}} \| u(t) \|_{H^s(\mathbb{R}_+)} = \infty$. 

Remark 1.2. If $s = 1$ or $p$ is even, then the assumptions on $p$ given in (A1)-(A2) in Theorem 1.1 are redundant. The same remark applies to $r$ when $s = 5/2 - \epsilon$ or $r$ is even.

Remark 1.3. In the above theorem, when $s \geq 2$, the equation is understood in the $L^2$-sense. However, if $s < 2$, the equation should be understood in the distributional sense, namely in the sense of $H^{s-2}(\mathbb{R}_+)$. For low values of $s$, the boundary and the initial condition can be understood in the sense of Definition 2.2 in [3].

Literature Overview

To the best of our knowledge, the model given in (1.1) has only been studied in the case that $k = 0, \lambda = 1$, and $r > 0$ by Ackleh-Deng [1]. In [1], the main equation was only linear. More precisely, the authors studied the following. 

\begin{equation}
\begin{cases}
i\partial_t u + \partial_x^2 u = 0, & x \in \mathbb{R}_+, t \in (0, T), \\
u(x, 0) = u_0(x), \\
\partial_x u(0, t) + |u(0, t)|^r u(0, t) = 0.
\end{cases}
\end{equation}

(1.3)

Ackleh-Deng [1] proved that if $u_0 \in H^3(\mathbb{R}_+)$, then there is $T_0 > 0$ such that (1.3) possesses a unique local solution $u \in C([0, T_0]; H^1(\mathbb{R}_+))$. Moreover, it was shown in [1] that (large) solutions with negative initial energy blow-up if $r \geq 2$ and are global otherwise. Therefore, $r = 2$ was considered the critical exponent for (1.3). Obtaining local existence and uniqueness consisted of two steps. First, the authors studied the linear Schrödinger equation with an inhomogeneous Neumann boundary condition on the half-line. Secondly, they used a contraction argument once the representation formula for solutions was restricted to the boundary point $x = 0$. In other words, the contraction argument was used on a function space which included only time dependent elements. Unfortunately, the same technique cannot be applied in the presence of the nonlinear source term $f(u) = k|u|^p u$ in the main equation. The reason is that even if the representation formula can still be restricted to the point $x = 0$, the sought after fixed point in the representation formula would also depend on the space variable. Therefore, one can no longer use a simple contraction argument on a function space which includes only time dependent elements. We are thus motivated to use a contraction argument on a function space which includes elements that depend on both time and space variables. Of course, this requires nice linear and nonlinear space-time estimates.

The drawback of the technique used in [1] is that the initial data has been assumed to be too smooth compared to the regularity of the solutions obtained. It is well-known from the theory of the linear Schrödinger equation that solutions are of the same class as the initial state. From this point of view, the generation of $H^1$ solutions with $H^3$ data seems far from optimal. We are thus inclined to obtain a regularity theory which shows that $H^s$ initial data generates $H^s$ solutions.

Regarding nonlinear boundary conditions, we are aware of very few other results for Schrödinger equations, see for example [19] and [12]. In [19],
the authors study the Schrödinger equation with nonlinear, attractive, and
dissipative boundary conditions of type $\frac{\partial u}{\partial \nu} = ig(u)$ where $g$ is a monotone
function with the property that the corresponding evolution operator gener-
ates a strongly continuous contraction semigroup on the $L^2$-level. The more
recent paper [12] studies Schrödinger equation with Wentzell boundary con-
ditions. This work also uses the fact that the Wentzell boundary condition
provide a semigroup in an appropriate topology. In the present case, due to
the fact that $\lambda$ is not a purely imaginary number, the problem does not have
a monotone structure, and the method of [19], [12] cannot be applied here.

A common strategy for proving well-posedness of solutions to PDEs
with nonlinear terms relies on two classical steps: (1) obtain a good linear
type with non-homogeneous terms; (2) establish local well-posedness for
the nonlinear model by a fixed point argument.

Obtaining a good linear theory with non-homogeneous terms is a sub-
tle point for boundary value problems, especially those with low-regularity
boundary data. One might attempt to extend the boundary data into the
domain and homogenize the boundary condition. However, this approach in
general requires a high regularity boundary data ([5], [7]), as opposed to the
rough boundary situation as in the present paper for low values of $s$. There
are different approaches one can follow to study a linear PDE with an inho-
mogeneous boundary data on the half-line without employing an extension-
homogenization approach, though. For example, Colliander-Kenig [13] used a
technique on the KdV equation by replacing the given initial-boundary value
problem with a forced initial value problem where the forcing is chosen in
such a way that the boundary condition is satisfied by inverting a Riemann-
Liouville fractional integral. Holmer [14] applied this technique on nonlinear
Schrödinger equations with inhomogeneous Dirichlet boundary conditions on
the half line. A second approach is to obtain norm estimates on solutions
by using a representation formula, which can be easily obtained through
a Laplace/Fourier transform. This technique has been used for example by
Kaikina in [15] for nonlinear Schrödinger equations with inhomogeneous Neu-
mann boundary conditions and by Bona-Sun-Zhang in [3] for inhomogeneous
Dirichlet boundary conditions. In [15], the well-posedness result assumes the
smallness of the given initial-boundary data while the results of [3] have global
character in this sense.

Although nonlinear Schrödinger equations with inhomogeneous boundary-
conditions have been studied to some extent, most of these papers were
devoted to inhomogeneous Dirichlet boundary conditions; see [10], [4], [5], [8],
[6], [26], [9], [14], [21], [16], [3], [22], [24]. There are relatively less results on in-
homogeneous Neumann boundary conditions; see [5], [7], [15], [23], [24]. In [5]
and [7], well-posedness is obtained under smooth boundary data. Relatively
less smooth boundary data was treated in [24] using Strichartz estimates,
but the regularity results were not optimal. In [15], the smoothness of initial
and boundary data was crucial. In [23], the focus was on the existence of
weak solutions, and questions concerning continuity in time, uniqueness, and
continuous dependence on data were not studied. In the present paper, we
draw a more complete and optimal well-posedness picture where the spatial
domain is half-line.

Orientation

In this paper, we will follow a step-by-step approach to prove Theorem 1.1:

**Step 1:** We will first study the linear Schrödinger equation with inhomogeneous terms both in the main equation and in the boundary condition. This problem is written in (2.1). Our aim in this step is to derive optimal norm estimates with respect to regularities of the initial state \( u_0 \), boundary data \( h \), and nonhomogeneous source term \( f \). This linear theory is constructed in Section 2 by adapting the method of [3] to nonhomogeneous Neumann boundary conditions.

**Step 2:** In the second step, we will replace the nonhomogeneous source term \( f = f(x, t) \) in (2.1) with \( f = f(u) = k|u|^p u \) as in (3.1). We will use a contraction mapping argument to prove the existence and uniqueness of local solutions together with continuous dependence on data. The blow-up alternative will be obtained via a classical extension-contradiction argument. This step is treated in Sections 3.1 - 3.4.

**Step 3:** In this step, we will replace the boundary data \( h = h(x, t) \) in (2.1) with \( h = h(u) = -\lambda|u_0(t)|^r u_0(t) \), and \( f \) with \( k|u|^p u \). Arguments similar to those in Step 2 will eventually give the well-posedness in the presence of nonlinear boundary conditions. The only difference is that the contraction argument must be adapted to deal with the nonlinear effects due to the nonlinear boundary source. This is given in Section 3.5.

**Remark 1.4.** Step 2 is indeed optional. One can directly run the contraction and blow-up arguments with nonlinear boundary conditions. However, it is useful to include the general theory of nonlinear Schrödinger equations with inhomogeneous Neumann boundary conditions to study other related problems in the future.

2. Linear nonhomogeneous model

In this section, we study the nonhomogeneous linear Schrödinger equation with nonhomogeneous Neumann boundary condition. We will later apply this linear theory to obtain the local well-posedness for nonlinear Schrödinger equations first with inhomogeneous Neumann boundary conditions and then with nonlinear boundary conditions. In order to obtain a sufficiently nice linear theory, we adapt the method presented for nonhomogeneous Dirichlet boundary conditions in [3] to the case with nonhomogeneous Neumann boundary conditions.

We consider the following linear model

\[
\begin{align*}
&i\partial_t u + \partial_x^2 u + f = 0, x \in \mathbb{R}_+, t \in (0, T), \\
&u(x, 0) = u_0, \partial_x u(0, t) = h(t),
\end{align*}
\]

(2.1)

where \( f \) and \( h \) lie in appropriate function spaces.
2.1. Compatibility conditions

Suppose \( u_0 \in H^s(\mathbb{R}_+) \), \( h \in H_{2s-1/4}^{2s-1/4}(0, T) \) in (2.1). It is well-known from the trace theory that both \( u_0'(0) \) and \( h(0) \) make sense when \( s > \frac{3}{2} \). Therefore, one needs to assume the zeroth order compatibility condition

\[
u_0'(0) = h(0)
\]

when \( s \in \left( \frac{3}{2}, \frac{7}{2} \right) \) in order to get continuous solutions at \((x, t) = (0, 0)\). As the value of \( s \) gets higher, one needs to consider more compatibility conditions. For example, if \( s \in \left( 2k + \frac{3}{2}, 2(k + 1) + \frac{3}{2} \right) \) \((k \geq 1)\), then the \( k\)-th order compatibility condition is defined inductively:

\[
\varphi_0 = u_0, \varphi_{n+1} = i(\partial_t^m f|_{t=0} + \partial_x^2 \varphi_n),
\]

\[
\partial_t^n h|_{t=0} = \partial_x \varphi_k|_{x=0}
\]

provided that \( f \) is also smooth enough for traces to make sense. If one wants to add the end point cases \( s = 2k + \frac{3}{2} \) to the analysis, then global compatibility conditions must be assumed (see for example [2] for a discussion of local and global compatibility conditions in the case of Dirichlet boundary conditions).

2.2. Boundary operator

We will first deduce a representation formula for solutions of the following linear model with an inhomogeneity on the boundary.

\[
\begin{cases}
i\partial_t u + \partial_x^2 u = 0, x \in \mathbb{R}_+, t \in (0, T), \\
u(x, 0) = 0, \partial_x u(0, t) = h(t).
\end{cases}
\]

We will study the above model by constructing an evolution operator which acts on the boundary data. We will start by taking a Laplace (in time) - Fourier (in space) transform of the given model. In order to do that, we will first extend the boundary data to the whole line utilizing the following lemma.

**Lemma 2.1 (Extension).** Let \( s \in \left( \frac{1}{2}, \frac{7}{2} \right) - \{ \frac{3}{2} \} \), \( h \in H_{2s-1/4}^{2s-1/4}(0, T) \) with \( h(0) = 0 \) if \( s > \frac{3}{2} \). Then, there exists \( h_e \in H_{2s-1/4}^{2s-1/4} \) with compact support in \([0, 2T + 1]\) which extends \( h \) so that \( H(t) := \int_{-\infty}^t h_e(s)ds \) also has compact support in \([0, 2T + 1]\) and \( \|H\|_{H_{2s-1/4}^{2s+1/4}} \leq C(1 + T)\|h\|_{H_{2s-1/4}^{2s-1/4}(0, T)} \) for some \( C > 0 \) which is independent of \( T \).

**Proof.** If \( \frac{1}{2} < s < \frac{3}{2} \), we have \( 0 < \frac{2s-1}{4} < \frac{3}{2} \). Now we take the zero extension of \( h \) onto \( \mathbb{R} \), say we get \( h_0 \). Then we set \( h_e(t) := h_0(t) - h_0(t - T) \).

If \( s \in \left( \frac{3}{2}, \frac{7}{2} \right) \), then \( \frac{1}{2} < \frac{2s-1}{4} < \frac{3}{2} \). In this case, we first take an extension \( h_A \) of \( h \) onto \( \mathbb{R} \) so that \( \|h_A\|_{H_{2s-1/4}^{2s+1/4}} \leq 2\|h\|_{H_{2s-1/4}^{2s-1/4}(0, T)} \) by using the fact that

\[
\|h\|_{H_{2s-1/4}^{2s+1/4}(0, T)} := \inf \left\{ \|\phi\|_{H_{2s-1/4}^{2s-1/4}} : \phi \in H_{2s-1/4}^{2s+1/4}, \phi|_{(0, T)} = h_0 \right\}.
\]

Secondly, the restriction \( h_B := h_A|_{(0, \infty)} \in H_{2s-1/4}^{2s+1/4}(0, \infty) \) will satisfy \( \|h_B\|_{H_{2s-1/4}^{2s-1/4}(0, \infty)} \leq \|h_A\|_{H_{2s-1/4}^{2s-1/4}} \). Now we can take the zero extension, say \( h_C \), of \( h_B \) onto \( \mathbb{R} \) so
that $\|h_C\|_{H^{2s+3/4}} \leq C\|h_B\|_{H^{2s+3/4}(\mathbb{R}^+)}$ with $C$ independent of $T$. By the previous inequalities, we get $\|h_C\|_{H^{2s+3/4}(\mathbb{R}^+)} \leq C\|h\|_{H^{2s+1/4}(0,T)}$ with $C$ independent of $T$. Then we pick a function $\eta \in C_c^\infty(\mathbb{R})$ so that $\eta = 1$ on $(0,T)$ and $\eta = 0$ on $[T + 1/2, \infty)$. Now we consider $h_1 = \eta h$, which is of course in $H^{2s+1/4}$, since $H^{2s+1/4}$ is a Banach algebra when $s > \frac{3}{2}$. Finally, we set $h_e(t) = h_1(t) - h_1(t - T - 1/2)$.

Note that $\|h_e\|_{H^{2s+1/4}} \leq C\|h\|_{H^{2s+1/4}(0,T)}$ where the positive constant $C$ does not depend on $T$, since all the extensions in the above paragraph and the multiplication by $\eta$ are continuous operators between corresponding Sobolev spaces whose norms do not depend on the initial domain $(0,T)$. Moreover, we set up $h_e$ in such a way that its average is zero. Hence, its antiderivative $H(t) := \int_{-\infty}^t h_e(s)ds$ is compactly supported and therefore belongs to the space $H^{2s+1/4}$.

Since $H$ is compactly supported with support in $[0,2T+1]$ by the Poincaré inequality we have $\|H\|_{L^2} \leq (2T + 1)\|h_e\|_{L^2}$. Hence

$$\|H\|_{H^{2s+1/4}} \sim \|D^{s+1/3}H\|_{L^2} + \|H\|_{L^2} \leq C\|D^{s+1/3}h_e\|_{L^2} + (2T + 1)\|h_e\|_{L^2} \leq C(1 + T)\|h_e\|_{H^{2s+3/4}} \leq C(1 + T)\|h\|_{H^{2s+1/4}(0,T)} \quad (2.4)$$

for some $C > 0$. □

Now we consider the following model, which is an extended-in-time version of (2.2).

$$\left\{ \begin{array}{ll}
i\partial_t u_e + \partial_x^2 u_e = 0, & x \in \mathbb{R}^+, t > 0, \\
u_e(x,0) = 0, \partial_x u_e(0,t) = h_e(t) & \end{array} \right. \quad (2.5)$$

where $h_e$ is the extension of $h$, as in Lemma 2.4.

We first take the Laplace transform of (2.5) in $t$ to get

$$\left\{ \begin{array}{ll}
i\lambda \tilde{u}_e(x,\lambda) + \partial_x^2 \tilde{u}_e(x,\lambda) = 0, & \\
\tilde{u}_e(+\infty,\lambda) = 0, \partial_x \tilde{u}_e(0,\lambda) = \tilde{h}_e(\lambda) & \end{array} \right. \quad (2.6)$$

with $\lambda > 0$, where $\tilde{u}_e$ denotes the Laplace transform of $u_e$. The solution of (2.6) is

$$\tilde{u}_e(x,\lambda) = \frac{1}{r(\lambda)} \exp(r(\lambda)x)\tilde{h}_e(\lambda)$$

where $\text{Re } r(\lambda)$ solves $i\lambda + r^2 = 0$ together with $\text{Re } r < 0$. Then,

$$u_e(x,t) = \frac{1}{2\pi i} \int_{-\infty i + \gamma}^{+\infty i + \gamma} \exp(\lambda t) \frac{1}{r(\lambda)} \exp(r(\lambda)x)\tilde{h}_e(\lambda) d\lambda,$$

where $\gamma > 0$ (fixed), solves (2.6). By passing to the limit in $\gamma$ as $\gamma \to 0$ and applying change of variables, we can rewrite $u_e(x,t)$ as follows:
Therefore, if $s > \frac{3}{2}$, we can write

$$u_e(x, t) = \frac{1}{i\pi} \int_0^\infty \exp(-i\beta^2 t + i\beta x)\tilde{h}_e(-i\beta^2)d\beta$$

$$- \frac{1}{\pi} \int_0^\infty \exp(i\beta^2 t - \beta x)\tilde{h}_e(i\beta^2)d\beta. \quad (2.7)$$

Note, that $u := u_e|_{[0,T]}$ is a solution of (2.2). We define $\nu_1(\beta) := \frac{1}{i\pi}\tilde{h}_e(-i\beta^2)$ for $\beta \geq 0$ and zero otherwise. Let $\phi_{h_e}$ be the inverse Fourier transform of $\nu_1$, that is $\hat{\phi}_{h_e}(\beta) = \nu_1(\beta)$ for $\beta \in \mathbb{R}$. Similarly, we define $\nu_2(\beta) := -\frac{1}{\pi}\tilde{h}_e(i\beta^2)$ for $\beta \geq 0$ and zero otherwise. Let $\psi_{h_e}$ be the inverse Fourier transform of $\nu_2$, that is $\hat{\psi}_{h_e}(\beta) = \nu_2(\beta)$ for $\beta \in \mathbb{R}$. Now, for $x \in \mathbb{R}_+$, we can write

$$u_e(x, t) = [W_b(t)h_e](x) := [W_{b,1}(t)h_e](x) + [W_{b,2}(t)h_e](x)$$

where

$$[W_{b,1}(t)h_e](x) := \int_{-\infty}^{\infty} \exp(-i\beta^2 t + i\beta x)\hat{\phi}_{h_e}(\beta)d\beta$$

and

$$[W_{b,2}(t)h_e](x) := \int_{-\infty}^{\infty} \exp(i\beta^2 t - \beta x)\hat{\psi}_{h_e}(\beta)d\beta.$$ 

Note that we can extend $W_{b,1}(t)h_e$ to $\mathbb{R}$ without changing its definition. For such an extension we have the following lemma:

**Lemma 2.2.** $u(x, t) = [W_{b,1}(t)h_e](x)$ solves the initial value problem

$$i\partial_t u + \partial_x^2 u = 0, u(x, 0) = \phi_{h_e}(x), x \in \mathbb{R}, t \in \mathbb{R}_+.$$ 

**Proof.** By direct calculation, we have

$$i\partial_t u + \partial_x^2 u = [i(-i\beta^2) + (i\beta)^2][W_{b,1}(t)h_e](x) = 0,$$

and

$$u(x, 0) = \mathcal{F}^{-1}(\hat{\phi}_{h_e})(x) = \phi_{h_e}(x).$$

We deduce from the above lemma that we can get space time estimates on $W_{b,1}(t)h_e$ by using the well-known linear theory of Schrödinger equations on $\mathbb{R}$. These estimates are given in Section 2.4 We extend $[W_{b,2}(t)h_e](x)$ to $\mathbb{R}$ by setting

$$[W_{b,2}(t)h_e](x) := \int_{-\infty}^{\infty} \exp(i\beta^2 t - \beta|x|)\hat{\psi}_{h_e}(\beta)d\beta.$$ 

However, if $s > \frac{3}{2}$, then this extension would not be differentiable at $x = 0$. Therefore, if $s > \frac{3}{2}$, we cannot directly use the linear theory of Schrödinger equations on $\mathbb{R}$ to estimate various norms of the term $W_{b,2}(t)h_e$. This makes it necessary to obtain space-time estimates for $W_{b,2}(t)h_e$ directly by using its definition.

The relation between regularities of $\phi_{h_e}, \psi_{h_e}$ and the regularity of the boundary data $h$ is given by the following lemma.
Lemma 2.3. Let \( s \geq \frac{1}{2}, h \in H^{\frac{2s-1}{4}}(0, T) \) such that \( h(0) = 0 \) if \( s > \frac{3}{2} \). Then, \( \phi_{h_e}, \psi_{h_e} \in H^s \).

Proof.

\[
\|\phi_{h_e}\|^2_{H^s} = \int_{-\infty}^{\infty} (1 + \beta^2)^s |\hat{\phi}_{h_e}(\beta)|^2 \, d\beta \\
= \frac{1}{\pi^2} \int_{0}^{\infty} (1 + \beta^2)^s |\hat{h}_e(-i\beta)|^2 \, d\beta.
\] (2.8)

Upon change of variables, the last term in (2.8) can be rewritten and estimated as follows.

\[
\frac{1}{2\pi^2} \int_{0}^{\infty} \frac{(1 + \beta^2)^s}{\beta^2} |\hat{h}_e(-i\beta)|^2 \, d\beta \lesssim \frac{1}{\pi^2} \int_{0}^{\infty} (1 + \beta^2)^{\frac{s+3}{2}} |\hat{H}(\beta)|^2 \, d\beta \\
\lesssim \|H\|^2_{H^{\frac{s+3}{4}}}.
\] (2.9)

where we use the relationships, \( \hat{h}_e(-i\beta) = \hat{h}_e(\beta) \) and \( \hat{h}_e(\beta) = i\beta \hat{H}(\beta) \) in the first inequality. The last estimate combined with Lemma 2.1 implies that \( \phi_{h_e} \in H^s \). We can repeat the same argument for \( \psi_{h_e} \), too.

\( \square \)

Notation. A given pair \((q, r)\) is said to be admissible if \( \frac{1}{q} + \frac{1}{2r} = \frac{1}{4} \) for \( q, r \geq 2 \).

Now, we will present several space-time estimates for the second part of the evolution operator \( W_b(t) \).

Lemma 2.4 (Space Traces). Let \( s \geq \frac{1}{2} \) and \( T > 0 \). Then, there exists \( C > 0 \) (independent of \( T \)) such that

\[
\sup_{t \in [0, T]} \|W_{b,2}(\cdot)h_e\|_{H^s} \leq C(1 + T)\|h\|_{H^{\frac{2s-1}{4}}(0, T)}
\] (2.10)

for any \( h \in H^{\frac{2s-1}{4}}(0, T) \) with \( h(0) = 0 \) if \( s > \frac{3}{2} \).

Proof. We can rewrite \([W_{b,2}(t)h](x)\) as

\[
[W_{b,2}(t)h_e](x) := \int_{-\infty}^{\infty} K_t(x, y)\psi_{h_e}(y) \, dy =: \mathcal{K}(t)\psi_{h_e}
\]

where \( K_t(x, y) = \int_{0}^{\infty} \exp(i\beta^2 t - \beta|x| - iy\beta) \, d\beta \). It is proven in [3] Proposition 3.8 that

\[
\|\mathcal{K}(t)\psi_{h_e}\|_{L^q(0,T;L^r)} \lesssim \|\psi_{h_e}\|_{L^2}
\]

for an admissible \((q, r)\). Similarly, taking one derivative in \( x \) variable, one gets

\[
\|\partial_x[\mathcal{K}(t)\psi_{h_e}]\|_{L^q(0,T;L^r)} \lesssim \|\partial_x[\psi_{h_e}]\|_{L^2}.
\]

Now, one can interpolate and use the proof of Lemma 2.3 to obtain

\[
\|W_{b,2}(\cdot)h_e\|_{L^q(0,T;W^{s,r})} \lesssim \|\mathcal{H}\|_{H^{\frac{s+3}{4}}}
\] (2.11)

for \( s \in \left[\frac{1}{2}, 1\right] \). For larger \( s \), one can differentiate and interpolate again. Finally, (2.10) follows by taking \( r = 2, q = \infty \) in (2.11). Now, (2.10) follows from (2.11) and Lemma 2.1. \( \square \)
Lemma 2.5 (Time traces). Let $s \geq \frac{1}{2}$ and $T > 0$. Then, there exists $C > 0$ (independent of $T$) such that
\[
\sup_{x \in \mathbb{R}_+} \|W_{b,\cdot}(\cdot)h_c\|_{H^{\frac{2s+1}{4}}(0,T)} \leq C(1+T)\|h\|_{H^{\frac{2s-1}{4}}(0,T)}
\]
for any $h \in H^{\frac{2s-1}{4}}(0,T)$ with $h(0) = 0$ if $s > \frac{3}{2}$.

Proof. This result is an application of Theorem 4.1 in [25]. For $k \geq 0$ (integer)
\[
\|\partial^k_t W_{b,\cdot}(\cdot)h_c\|_{L^2_t}^2 = \int_{\mathbb{R}_+} \beta^{4k} |\hat{w}_{k,\cdot}(\beta)|^2 \frac{d\beta}{2\beta} \lesssim \int_{\mathbb{R}_+} (1 + \beta^2)^{k+\frac{1}{2}} |\hat{H}(\beta)|^2 d\beta \lesssim \|H\|_{H^{k+\frac{1}{2}}}.
\]
Upon interpolation, the result follows in the case that $h$, $h_c$, and $H$ are smooth, then a density argument finishes the proof. Now, (2.12) follows from (2.13) and Lemma 2.1. \hfill \Box

2.3. Representation Formula

We take an extension of $u_0$ to $\mathbb{R}$, say $u_0^* \in H^s$ such that $\|u_0^*\|_{H^s} \lesssim \|u_0\|_{H^s(\mathbb{R}^+)}$. Therefore, $u = W_R(t)u_0^*$ solves the problem
\[
i\partial_t u + \partial^2_x u = 0, \ u(0, t) = u_0^*(x), \ x, t \in \mathbb{R}
\]
where $W_R(t)$ is the evolution operator for the linear Schrödinger equation. Similarly, if $f^*$ is an extension of $f$, then the solution of the non-homogeneous Cauchy problem
\[
iu_t + u_{xx} = f^*(x,t), \ u(x, 0) = 0, \ x, t \in \mathbb{R}
\]
can be written as
\[
u(x, t) = -i \int_0^t W_R(t - \tau)f^*(\tau) d\tau.
\]
Therefore, if we define
\[
u_c(x, t) = W_R(t)u_0^* - i \int_0^t W_R(t - \tau)f^*(\tau) d\tau + W_b([h - g - p]_c(t))
\]
with
\[
g(t) = \partial_x W_R(t)u_0^*|_{x=0}
\]
and
\[
p(t) = -i\partial_x \int_0^t W_R(t - \tau)f^*(\tau) d\tau|_{x=0},
\]
then $u = u_c|_{[0,T)}$ will solve
\[
\begin{cases}
i\partial_t u + \partial^2_x u = f, t \in (0,T), \ x \in \mathbb{R}_+, \\ u(x, 0) = u_0, \partial_x u(0,t) = h(t).
\end{cases}
\]
In the formula we have given, $g(t)$ and $p(t)$ make sense only if $s > 3/2$. In other cases, we take those boundary traces equal to zero in the representation formula (2.14).
2.4. Space-time estimates on $\mathbb{R}$

We will utilize the following space and time estimates on $\mathbb{R}$ for the evolution operator of the linear Schrödinger equation \cite{11}. Note that these estimates can be directly applied to the first part $W_{b,1}$ of the boundary evolution operator.

**Lemma 2.6.** Let $s \in \mathbb{R}$, $T > 0$, $\phi \in H^s$, and $u := W_{\mathbb{R}} \phi$. Then, there exists $C = C(s)$ such that

$$\sup_{t \in [0,T]} \|u(\cdot, t)\|_{H^s} + \sup_{x \in \mathbb{R}} \|u(x, \cdot)\|_{H^{2s+1}((0,T)} \leq C\|\phi\|_{H^s}. \quad (2.16)$$

**Lemma 2.7.** Let $T > 0$, $f \in L^1(0,T;H^s)$, and $u := \int_0^T W_{\mathbb{R}}(t-\tau)f(\tau)d\tau$. Then, for any $s \in \mathbb{R}$, there exists a constant $C = C(s) > 0$ such that

$$\sup_{t \in [0,T]} \|u(\cdot, t)\|_{H^s} + \sup_{x \in \mathbb{R}} \|u(x, \cdot)\|_{H^{2s+1}((0,T)} \leq C\|f\|_{L^1(0,T;H^s)}. \quad (2.17)$$

2.5. Regularity

Combining Lemmas 2.2-2.7, we have the following regularity theorems for the linear model.

**Theorem 2.8.** Let $T > 0$, and $s \geq 1/2$. Then, there exists $C > 0$ (independent of $T$) such that for any $h \in H^{2s-1}((0,T)$ with $h(0) = 0$ if $s > \frac{3}{2}$, $u = W_b(t)h$ satisfies

$$\sup_{t \in [0,T]} \|u(\cdot, t)\|_{H^s(\mathbb{R}_+)} + \sup_{x \in \mathbb{R}_+} \|u(x, \cdot)\|_{H^{2s+1}((0,T)} \leq C(1 + T)\|h\|_{H^{2s-1}((0,T)}, \quad (2.18)$$

**Theorem 2.9.** Let $T > 0$, $s \in (\frac{1}{2}, \frac{3}{2}) - \{3/2\}$, $h \in H^{2s-1}(0,T)$, $f \in L^1(0,T;H^s(\mathbb{R}_+))$, $u_0 \in H^s(\mathbb{R}_+)$, and if $s \not\in (\frac{3}{2}, \frac{5}{2})$, we assume the zeroth order compatibility condition $u_0(0) = h(0)$. Then there exists $C > 0$ (independent of $T$) such that the solution $u$ of (2.15) satisfies

$$\sup_{t \in [0,T]} \|u(x, \cdot)\|_{H^s(\mathbb{R}_+)} + \sup_{x \in \mathbb{R}_+} \|u(x, \cdot)\|_{H^{2s+1}((0,T)} \leq C\left(\|u_0\|_{H^s(\mathbb{R}_+)} + (1 + T)\|h\|_{H^{2s-1}((0,T)} + \|f\|_{L^1(0,T;H^s(\mathbb{R}_+)}\right). \quad (2.19)$$

**Remark 2.10.** The optimal local smoothing estimate for the Schrödinger evolution operator is $\|W_{\mathbb{R}}u_0\|_{L^\infty_{x,t} H^{2s+1}} \lesssim \|u_0\|_{H^s}$; see for instance \cite{25}. This is why we consider the space $X^s_T$ defined in Section 1 as our solution space. It is shown in \cite{14} and \cite{3} that the natural space for the boundary data $h$ is $H^{2s+1}_t((0,T)$, when one considers Dirichlet boundary conditions. Since one can formally think that one derivative in the space variable is equivalent to 1/2 derivatives in the time variable, we are inclined to consider $H^{2s+1}_t((0,T)$ as the natural space for the boundary data $h$ when we consider Neumann boundary conditions.
3. Nonlinear Schrödinger equation

In this section, we study nonlinear Schrödinger equations with nonhomogeneous Neumann type boundary data. More precisely, we consider the following model:

\[
\begin{cases}
i \partial_t u - \partial_x^2 u + f(u) = 0, & x \in \mathbb{R}_+, \ t \in (0, T), \\
u(x, 0) = u_0, \\
\partial_x u(0, t) = h,
\end{cases}
\]

where \(f(u) = k |u|^pu, \ p > 0, \ k \in \mathbb{R} - \{0\}, \ u_0 \in H^s(\mathbb{R}_+), \) and \(s \in \left(\frac{1}{2}, \frac{5}{2}\right) - \left\{\frac{3}{2}\right\}\).

Here, we consider two problems. The first one is the open-loop well-posedness problem when \(h\) is taken as a time dependent function in the Sobolev space \(H^{2s-1}_{2s} (0, T)\). The second one is the closed-loop well-posedness problem when \(h\) is taken as a function of \(u(0, t)\) in the form \(h(u(0, t)) = -\lambda |u(0, t)|^r u(0, t)\) with \(\lambda \in \mathbb{R} - \{0\}\).

3.1. Local existence

In order to prove the local existence of solutions we will use the contraction mapping argument. For the contraction mapping argument, we will use the following operator on a closed ball \(\bar{B}_R(0)\) in the function space \(X^s_{t_0}\) for appropriately chosen \(R > 0\) and \(T_0 \in (0, T]\).

\[
\Psi(u)(t) := W(t)u_0^* - i \int_0^t W(t - \tau)f(u^*(\tau))d\tau \\
+ W_b(t)([h - g - p(u^*)])
\]

with \(g(t) = \partial_x W(t)u_0^*|_{x=0}\) and \([p(u^*)](t) = -i\partial_x \int_0^t W(t - \tau)f(u^*(\tau))d\tau|_{x=0}\).

Here, \(g(t)\) and \(p(t)\) make sense only if \(s > 3/2\). For \(s \in \left(\frac{1}{2}, \frac{5}{2}\right)\), we take these boundary traces equal to zero in (3.2).

In order to use the Banach fixed point theorem, we have to show that \(\Psi\) maps \(\bar{B}_R(0)\) onto itself, and moreover that it is a contraction on the same set. Therefore, we will estimate each term in (3.2) with respect to the norm defined in (1.2). By Lemma 2.6

\[
\|W(t)u_0^*\|_{X^s_t} \lesssim \|u_0^*\|_{H^s} \lesssim \|u_0\|_{H^s(\mathbb{R}_+)}.
\]

In order to estimate the second term at the right hand side of (3.2), we will first prove the following lemma:

**Lemma 3.1 (Nonlinearity).** Let \(f(u) = |u|^pu\) and \(s > \frac{1}{2}\). Moreover, let \((p, s)\) satisfy one of the following assumptions:

(a1) If \(s\) is integer, then assume that \(p \geq s\) if \(p\) is an odd integer and \([p] \geq s - 1\) if \(p\) is non-integer.

(a2) If \(s\) is non-integer, then assume that \(p > s\) if \(p\) is an odd integer and \([p] \geq [s]\) if \(p\) is non-integer.

If \(u, v \in H^s\), then

\[
\|f(u)\|_{H^s} \lesssim \|u\|_{H^{s+1}}^{p+1},
\]

\[
\|f(u) - f(v)\|_{H^s} \lesssim (\|u\|_{H^s}^p + \|v\|_{H^s}^p)\|u - v\|_{H^s}.
\]
Proof. See Lemma 4.10.2 [11] for s being an integer and Lemma 3.10(2) [18] for p being a non-integer. Therefore, we will only consider the cases with s being a non-integer, and p being an odd integer or non-integer.

Let us first consider the case $1/2 < s < 1$. By the chain rule (Theorem A.7 [17]), $\|D^s f(u)\|_{L^2} \lesssim \|f'(u)\|_{L^\infty} \|D^s u\|_{L^2}$. Since $|f'(u)| \lesssim |u|^p$, we have $\|f'(u)\|_{L^\infty} \lesssim \|u\|^p \lesssim \|u\|_{H^s}^p$, where the last inequality follows by the Sobolev embedding $H^s \hookrightarrow L^{\infty}$ for $s > 1/2$. Also, $\|D^s u\|_{L^2} \lesssim \|u\|_{H^s}$. It follows that $\|D^s f(u)\|_{L^2} \lesssim \|u\|_{H^s}^{p\beta}$. On the other hand, $\|f(u)\|_{L^2} = \|u\|_{L_{2p+2}}^{p+1} \lesssim \|u\|_{H^s}^{p+1}$, where the inequality follows by the Sobolev’s embedding $H^s \hookrightarrow L^2p+2$ for $s > 1/2$. Hence, we have just shown that $\|f(u)\|_{H^s} \lesssim \|u\|_{H^s}^{p\beta}$.

Now, consider the case $s = \sigma + m > 1$ for some positive integer $m$ and $\sigma \in (0, 1)$. Then, $\|D^s f(u)\|_{L^2} \lesssim \|D^\sigma(D^m f(u))\|_{L^2}$ where $D^m f(u)$ is a sum of the terms of type $f^{(k)}(u) \prod_{j=1}^k D^{\beta_j} u$ where $k$ ranges from $k = 1$ up to $k = m$ and $\sum_{j=1}^k \beta_j = m$.

By the fractional version of the Leibniz rule [17], we can write

$$\|D^\sigma(f^{(k)}(u)) \prod_{j=1}^k D^{\beta_j} u\|_{L^2} \lesssim \|D^\sigma(f^{(k)}(u))\|_{L^{p_1}} \prod_{j=1}^k \|D^{\beta_j} u\|_{L^{p_2}}$$

$$\quad+ \|f^{(k)}(u)\|_{L^{\infty}} \|D^\sigma(\prod_{j=1}^k D^{\beta_j} u)\|_{L^2} = I \cdot II + III \cdot IV. \quad (3.6)$$

together with $\frac{1}{2} = \frac{1}{p_1} + \frac{1}{p_2}$, $p_1, p_2 > 2$. By using the chain rule, the first term is estimated as $I \lesssim \|f^{(k+1)}(u)\|_{L^{q_1}} \|D^\sigma u\|_{L^{q_2}}$ together with $\frac{1}{p_1} = \frac{1}{q_1} + \frac{1}{q_2}$, $q_1, q_2 > p_1 > 2$. Here, we choose $q_1$ sufficiently large so that $q_1(p - k) > 2$. Therefore, $\|f^{(k+1)}(u)\|_{L^{q_1}} \lesssim \|u\|_{L^{q_1}(p-k)}^{p-k} \lesssim \|u\|_{H^s}^{p-k}$ and $\|D^\sigma u\|_{L^{q_2}} \lesssim \|D^\sigma u\|_{H^{m}} \lesssim \|u\|_{H^{s}}$. If $k = 1$ (therefore $\beta_1 = m$), then the second term can be estimated as $II = \|D^m u\|_{L^{p_2}} \lesssim \|D^m u\|_{H^s} \lesssim \|u\|_{H^s}$. In the last estimate, if $\sigma < 1/2$, then we choose $p_2$ as $\frac{1}{p_2} = \frac{1}{2} - \sigma$, otherwise we can use any $p_2 > 2$. If $k > 1$, then using Hölder’s inequality

$$\|\prod_{j=1}^k D^{\beta_j} u\|_{L^{p_2}} \leq \prod_{j=1}^k \|D^{\beta_j} u\|_{L^{q_j}} \lesssim \prod_{j=1}^k \|D^{\beta_j} u\|_{H^{1+\sigma}} \lesssim \|u\|_{H^s}^{\frac{p_1}{p_2}},$$

where $\frac{1}{p_2} = \sum_{j=1}^k \frac{1}{q_j}$ and $q_j > 2$. Hence, it follows that we always have $I \cdot II \lesssim \|u\|_{H^s}^{p\beta}$. The third term can be easily estimated as $III \lesssim \|u\|_{L^\infty}^{p-k+1}$. Regarding the fourth term, the case $k = 1$ is trivial. So let us consider the case $k > 1$.

In this case, applying the Leibniz formula, we have $\|D^\sigma(\prod_{j=1}^k D^{\beta_j} u)\|_{L^2} \lesssim \sum_{l=1}^k \|D^{\sigma+\beta_l} u\|_{L^{q_l}} \prod_{j=1, j\neq l}^k \|D^{\beta_j} u\|_{L^{q_j}}$ for some $\{q_j > 2, j = 1, ..., k\}$ such that $\sum_{j=1}^k \frac{1}{q_j} = \frac{1}{2}$. But the right hand side of the last inequality is dominated by

$$\sum_{l=1}^k \|D^{\sigma+\beta_l} u\|_{H^{m-\beta_l}} \prod_{j=1, j\neq l}^k \|D^{\beta_j} u\|_{H^{s-\beta_j}} \lesssim \|u\|_{H^s}^{k}.$$
Hence, it follows that \( III \cdot IV \lesssim \|u\|^p_{H^s} \). By the above estimates, we deduce \(\text{III} \).

Regarding the differences, let us first consider the case \( 1/2 < s < 1 \) again. Then, by using the fractional chain rule and the fact that \( H^s \hookrightarrow L^\infty \), we get

\[
\|D^s f(u) - D^s f(v)\|_{L^2} \lesssim \|f'(u) - f'(v)\|_{L^\infty} \|D^s u - D^s v\|_{L^2} \\
\lesssim (\|u\|^p_{L^\infty} + \|v\|^p_{L^\infty}) \|u - v\|_{H^s} \lesssim (\|u\|^p_{H^s} + \|v\|^p_{H^s}) \|u - v\|_{H^s}. \tag{3.7}
\]

Now, we consider the case \( s = \sigma + m > 1 \) for some positive integer \( m \) and \( \sigma \in (0,1) \). Then,

\[
\|D^s f(u) - D^s f(v)\|_{L^2} \lesssim \|D^\sigma (D^m (f(u) - f(v)))\|_{L^2}
\]

where \( D^m (f(u) - f(v)) \) is a sum of the terms of type

\[
f^{(k)}(u) \prod_{j=1}^k D^{\beta_j} u - f^{(k)}(v) \prod_{j=1}^k D^{\beta_j} v
\]

\[
= \left(f^{(k)}(u) - f^{(k)}(v)\right) \prod_{j=1}^k D^{\beta_j} u - f^{(k)}(v) \prod_{j=1}^k D^{\beta_j} w_j \tag{3.8}
\]

where \( k \) ranges from \( k = 1 \) up to \( k = m \), \( \sum_{j=1}^k \beta_j = m \), and \( w_j \)'s are equal to \( u \) or \( v \), except one, which is equal to \( u - v \). Now the \( L^2 \)-norm of the term \( D^\sigma (f^{(k)}(v) \prod_{j=1}^k D^{\beta_j} w_j) \) can be estimated in a manner similar to \(\text{III} \) using the fractional Leibniz rule, except we also use several applications of Young's inequality to separate the products involving \( u \) and \( v \). What remains is to estimate the term \( D^\sigma \left[ (f^{(k)}(u) - f^{(k)}(v)) \prod_{j=1}^k D^{\beta_j} u \right] \), which can also be done as in \(\text{III} \) using the fractional Leibniz rule. In order to do this, we also use the observation

\[
\|f^{(k)}(u) - f^{(k)}(v)\|_{L^\infty} \lesssim \left(\|u\|^{p-k}_{H^s} + \|v\|^{p-k}_{H^s}\right) \|u - v\|_{H^s},
\]

which easily follows from the fact that

\[
|f^{(k)}(u) - f^{(k)}(v)| \lesssim (\|u\|^{p-k}) + |u - v|
\]

and the Sobolev embedding \( H^s \hookrightarrow L^\infty \) for \( s > 1/2 \).

\(\square\)

**Remark 3.2.** The assumption \((a1)\) and \((a2)\) are needed to guarantee that \( f \) is sufficiently smooth. The assumption \((a1)\) guarantees that \( f \) is at least \( C^m(C,\mathbb{C}) \), which is what one needs in the case \( s \) is an integer (see Remark 4.10.3 [11]). Since \( f \) is \( C^\infty(C,\mathbb{C}) \) when \( p \) is even, no assumption was necessary in this case. If \( s \) is fractional, the proof uses the \( m + 1 \)-th derivative, which forces us to make the second assumption \((a2)\).
It follows from Lemma 2.7 and Lemma 3.1 that
\[
\left\| -i \int_0^t W_\mathbb{R}(t - \tau) f(u^*(\tau)) d\tau \right\|_{X_T^p} \leq \int_0^T \| f(u^*(\tau)) \|_{H^s} d\tau \\
\lesssim \int_0^T \| u^*(\tau) \|_{H^{s+1}} d\tau \lesssim \int_0^T \| u(\tau) \|_{H^{s+1}(\mathbb{R}_+)} d\tau \leq T \| u \|_{X_T^{p+1}}. \tag{3.9}
\]

Similarly,
\[
\left\| -i \int_0^t W_\mathbb{R}(t - \tau) [f(u^*(\tau)) - f(v^*(\tau))] d\tau \right\|_{X_T^p} \leq \int_0^T \| f(u^*(\tau)) - f(v^*(\tau)) \|_{H^s} d\tau \\
\lesssim \int_0^T \| u^*(\tau) \|_{H^{s+1}} + \| v^*(\tau) \|_{H^{s+1}} \| u^*(\tau) - v^*(\tau) \|_{H^s} d\tau \\
\lesssim \int_0^T (\| u(\tau) \|_{H^{s}(\mathbb{R}_+)} + \| v(\tau) \|_{H^{s}(\mathbb{R}_+)}) \| u(\tau) - v(\tau) \|_{H^{s}(\mathbb{R}_+)} d\tau \\
\lesssim T (\| u \|_{X_T^p} + \| v \|_{X_T^p}) \| u - v \|_{X_T^p}. \tag{3.10}
\]

For \( s \in \left( \frac{1}{2}, \frac{3}{2} \right) \), since \( g = p = 0 \), the last term in (3.2) is estimated as follows by using Theorem 2.8
\[
\| W_b(\cdot) h \|_{X_T^p} \leq C(1 + T) \| h \|_{H^{2p-1}(0,T)}. \tag{3.11}
\]

For \( s \in \left( \frac{3}{2}, \frac{7}{2} \right) \), we have the assumption \( h(0) = u_0^*(0) \), and therefore \( h - g - p \) vanishes at \( x = 0 \). Moreover, the following estimate holds true.
\[
\| W_b(\cdot)([h - g - p]_e) \|_{X_T^p} \leq C(1 + T) \| h - g - p(u^*) \|_{H^{2p-1}(0,T)} \\
\leq C(1 + T) \left( \| h \|_{H^{2p-1}(0,T)} + \| g \|_{H^{2p-1}(0,T)} + \| p(u^*) \|_{H^{2p-1}(0,T)} \right). \tag{3.12}
\]

Note that,
\[
\| g \|_{H^{2p-1}(0,T)} = \| \partial_x W_\mathbb{R}(t) u_0^* \|_{x = 0} \| H^{2p-1}(0,T) \\
\leq \sup_{x \in \mathbb{R}_+} \| \partial_x W_\mathbb{R}(t) u_0^* \|_{H^{2p-1}(0,T)} \\
\leq \| \frac{d}{dx} u_0^* \|_{H^{s-1}} \leq \| u_0^* \|_{H^s} \lesssim \| u_0 \|_{H^s(\mathbb{R}_+)}. \tag{3.13}
\]

In (3.13), the second inequality follows from Lemma 2.6 and the fact that \( \partial_x W_\mathbb{R}(t) u_0^* \) is a solution of the linear Schrödinger equation on \( \mathbb{R} \) with initial condition \( \frac{d}{dx} u_0^* \).
Similarly,
\[
\|p(u^*)\|_{H^{\frac{2s-1}{4}}(0,T)} = -i\partial_x\int_0^t W_R(t-\tau)f(u^*(\tau))d\tau|_{x=0}\|_{H^{\frac{2s-1}{4}}(0,T)}
\leq \sup_{x \in \mathbb{R}_+} -i\partial_x\int_0^t W_R(t-\tau)f(u^*(\tau))d\tau\|_{H^{\frac{2s-1}{4}}(0,T)}
\leq \|\partial_x f(u)\|_{L^1(0,T;H^{s-1})} \leq \|f(u^*)\|_{L^1(0,T;H^s)} \lesssim T\|u\|_{X^s_T}. \tag{3.14}
\]

The last term in (3.12),
\[
\|p(u^*) - p(v^*)\|_{H^{\frac{2s-1}{4}}(0,T)} \lesssim T(\|u(\tau)\|_{X^s_T}^p + \|v(\tau)\|_{X^s_T}^p)\|u - v(\tau)\|_{X^s_T}. \tag{3.15}
\]

Combining above estimates, we obtain
\[
\|\Psi(u)\|_{X^s_T} \leq C \left(\|u_0\|_{H^s(\mathbb{R}_+)} + (1 + T)\|h\|_{H^{\frac{2s-1}{4}}(0,T)} + T\|u\|_{X^s_T}^{p+1}\right).
\]

Similarly, regarding the differences, again by above estimates, we have
\[
\|\Psi(u) - \Psi(v)\|_{X^s_T} \leq C \left(T(\|u(\tau)\|_{X^s_T}^p + \|v(\tau)\|_{X^s_T}^p)\|u(\tau) - v(\tau)\|_{X^s_T}\right).
\]

Now, let \(A := C \left(\|u_0\|_{H^s(\mathbb{R}_+)} + (1 + T)\|h\|_{H^{\frac{2s-1}{4}}(0,T)}\right), R = 2A\) and \(T\) be small enough that \(\frac{A}{R} < 2A\). Now, if necessary we can choose \(T\) even smaller so that \(\Psi\) becomes a contraction on \(\widetilde{B}_R(0) \subset X^s_T\), which is a complete space. Hence, \(\Psi\) must have a unique fixed point in \(\widetilde{B}_R(0)\) when we look for a solution whose lifespan is sufficiently small.

We conclude this section with the proposition below.

**Proposition 3.3.** Let \(T > 0\), \(s \in \left(\frac{1}{2}, \frac{7}{2}\right) - \left\{\frac{3}{2}\right\}\), \(p, r > 0\), \(u_0 \in H^s(\mathbb{R}_+), h \in H^{\frac{2s-1}{4}}(0,T)\), and \(u_0(0) = h(0)\) whenever \(s > \frac{3}{2}\). We in addition assume (a1)-(a2) given in Lemma 3.1. Then, (3.1) has a local solution \(u \in X^s_{T_0}\) for some \(T_0 \in (0, T]\).

### 3.2. Uniqueness

In the previous section, we have proved uniqueness in a fixed ball in the space \(X^s_T\). This does not immediately tell us that the solution must also be unique in the entire space. Fortunately, this latter statement is also true. In order to show this, let \(u_1, u_2 \in X^s_{T_0}\) be two solutions of (3.1). Then,
\[
u_1(t) - u_2(t) = -i \int_0^t W_R(t-s)[f(u_1^*(s)) - f(u_2^*(s))]ds + W_b(t)[p(u_2^*) - p(u_1^*)]_e \tag{3.16}\]
for a.a. \(t \in [0, T_0]\).
Since $s > 1/2$,

$$\|u_1(t) - u_2(t)\|_{H^s} \leq \int_0^{T_0} \|f(u_1^*(s)) - f(u_2^*(s))\|_{H^s} + C(1 + T_0)\|p(u_1^*) - p(u_1^*)\|_{H^{\frac{2s-1}{2}}} \leq C(1 + T_0) \int_0^{T_0} \|u_1(s) - u_2(s)\|_{H^s} \|u_1(s)\|_{H^s} + \|u_2(s)\|_{H^s} ds \leq C(1 + T_0)(\|u_1(s)\|_{X^s_{T_0}} + \|u_2(s)\|_{X^s_{T_0}}) \int_0^{T_0} \|u_1(s) - u_2(s)\|_{H^s} ds. \quad (3.17)$$

By Gronwall’s inequality, $\|u_1(t) - u_2(t)\|_{H^s} = 0$, which implies $u_1 \equiv u_2$.

Now, we can state the uniqueness statement as follows.

**Proposition 3.4.** If $u_1, u_2$ are two local solutions of (3.11) in $X^s_{T_0}$ as in Proposition 3.3, then $u_1 \equiv u_2$.

### 3.3. Continuous dependence

Regarding continuous dependence on data, let $B$ be a bounded subset of $H^s(\mathbb{R}_+) \times H^{\frac{2s-1}{2}}(0, T)$. Let $(u_0, h_1) \in B$ and $(v_0, h_2) \in B$. Let $u, v$ be two solutions on a common time interval $(0, T_0)$ corresponding to $(u_0, h_1)$ and $(v_0, h_2)$, respectively. Then $w = u - v$ satisfies

$$\begin{cases}
i \partial_t w + \partial^2_x w = F(x, t) \equiv f(v) - f(u), \quad x \in \mathbb{R}_+, \ t \in (0, T), \\
w(x, 0) = w_0(x) \equiv (u_0 - v_0)(x), \\
\partial_x w(0, t) = h(t) \equiv (h_1 - h_2)(t). \quad (3.18)
\end{cases}$$

Now, using the linear theory together with the nonlinear $H^s$ estimates on the differences, we have

$$\|w\|_{X^s_{T_0}} \leq C \left(\|w_0\|_{H^s(\mathbb{R}_+)} + (1 + T_0)\|h\|_{H^{\frac{2s-1}{2}}(0, T)} + \|F\|_{L^1(0, T_0; H^s(\mathbb{R}_+))} \right),$$

where

$$\|F\|_{L^1(0, T_0; H^s(\mathbb{R}_+))} \leq CT_0 \left(\|u\|_{X^s_{T_0}} + \|v\|_{X^s_{T_0}} \right)\|u - v\|_{X^s_{T_0}}.$$  

Choosing $R$, which depends on $u_0$ and $h$ (i.e., on the bounded set $B$), as in the proof of the local existence, and $T_0$ accordingly small enough, we obtain

$$\|u - v\|_{X^s_{T_0}} \leq C \left(\|u_0 - v_0\|_{H^s(\mathbb{R}_+)} + \|h_1 - h_2\|_{H^{\frac{2s-1}{2}}(0, T)} \right). \quad (3.19)$$

Hence, we have the following result.

**Proposition 3.5.** If $B$ is a bounded subset of $H^s(\mathbb{R}_+) \times H^{\frac{2s-1}{2}}(0, T)$, then there is $T_0 > 0$ such that the flow $(u_0, h) \to u$ is Lipschitz continuous from $B$ into $X^s_{T_0}$.
3.4. Blow-up alternative

In this section, we want to obtain a condition which guarantees that a given local solution on $[0, T_0]$ can be extended globally. Let’s consider the set $S$ of all $T_0 \in (0, T]$ such that there exists a unique local solution in $X^s_{T_0}$. We claim that if $T_{\text{max}} := \sup_{T_0 \in S} T_0 < T$, then $\lim_{t \uparrow T_{\text{max}}} \|u(t)\|_{H^s(\mathbb{R}^+)} = \infty$. In order to prove the claim, assume to the contrary that $\lim_{t \uparrow T_{\text{max}}} \|u(t)\|_{H^s(\mathbb{R}^+)} = \infty$. Then $\exists M$ and $t_n \in S$ such that $t_n \to T_{\text{max}}$ and $\|u(t_n)\|_{H^s(\mathbb{R}^+)} \leq M$. For a fixed $n$, we know that there is a unique local solution $u_1$ on $[0, t_n]$. Now, we consider the following model.

\[
\begin{aligned}
&i \partial_t u - \partial_{xx} u + f(u) = 0, \quad x \in \mathbb{R}^+, \; t \in (t_n, T), \\
&u(x, t_n) = u_1(x, t_n), \\
&\partial_x u(0, t) = h(t).
\end{aligned}
\]  

We know from the local existence theory that the above model has a unique local solution $u_2$ on some interval $[t_n, t_n + \delta]$ for some $\delta = \delta(M, \|h\|_{H^s_{2s-1}(0, T)}) \in (0, T - t_n]$. Now, choose $n$ sufficiently large that $t_n + \delta > T_{\text{max}}$. If we set

\[
\begin{aligned}
u := \begin{cases}
u_1, & t \in [0, t_n], \\
\nu_2, & t \in [t_n, t_n + \delta],
\end{cases}
\end{aligned}
\]

then $u$ is a solution of (3.1) on $[0, t_n + \delta]$ where $t_n + \delta > T_{\text{max}}$, which is a contradiction.

We have the theorem.

**Proposition 3.6.** Let $S$ be the set of all $T_0 \in (0, T]$ such that there exists a unique local solution in $X^s_{T_0}$. If $T_{\text{max}} := \sup_{T_0 \in S} T_0 < T$, then $\lim_{t \uparrow T_{\text{max}}} \|u(t)\|_{H^s(\mathbb{R}^+)} = \infty$.

3.5. Nonlinear boundary data

In this section, we study the most general nonlinear model given in (1.1). We define the operator $\Psi$ as in (3.2), except that we take $h(t) = h(u(0, t)) = -\lambda|u(0, t)|^r u(0, t)$. Therefore, the solution operator we have to use for the contraction argument takes the following form.

\[
[\Psi(u)](t) := W_\mathbb{R}(t)u_0^* - i \int_0^t W_\mathbb{R}(t - \tau)f(u^*(\tau))d\tau \\
+ W_b(t)[h(u(0, \cdot)) - g - p(u^*)]_c.
\]  

The proofs of local well-posedness and blow-up alternative now follows similar to the proofs in Sections 3.1 - 3.4. The only additional work in this part would be to get nonlinear $H^s$ estimates on the boundary trace $-\lambda|u(0, t)|u(0, t)$, which is of course possible with assumptions on $r$, which are almost equivalent to the assumptions we made on $p$. Indeed, we will assume that $r > \frac{2s-1}{4}$ if $r$ is an odd integer and $[r] \geq \left[\frac{2s-1}{4}\right]$ if $r$ is non-integer.
We will need the following lemma to get useful estimates on the boundary operator for the contraction argument.

**Lemma 3.7.** Let \( h \in H^{\sigma+\epsilon}(0,T) \), \( \sigma, \epsilon > 0 \). Then \( \|h\|_{H^{\sigma}(0,T)} \leq T^{1+\frac{\sigma}{\sigma+\epsilon}} \|h\|_{H^{\sigma+\epsilon}(0,T)} \).

**Proof.** Let \( H(t) := \int_0^t h(s)ds \). Applying the Cauchy-Schwartz inequality we get \( \|H\|_{L^2(0,T)} \leq \int_0^T (\int_0^T |h(s)|ds)^2 dt \leq T^2 \|h\|_{L^2(0,T)}^2 \). On the other hand \( H' = h \), which implies \( \|h\|_{H^{-1}(0,T)} \leq \|H\|_{L^2(0,T)} \), hence \( \|h\|_{H^{-1}(0,T)} \leq T \|h\|_{L^2(0,T)} \). By interpolation theorem [20, Theorem 12.4, Proposition 2.3], \( \|h\|_{H^{\sigma}} \leq \|h\|^\theta_{H^{-1}} \cdot \|h\|^{1-\theta}_{H^{\sigma+\epsilon}} \), in which \( \theta = \frac{\epsilon}{1+\sigma+\epsilon} \). Hence we obtain \( \|h\|_{H^{\sigma}} \leq T^\theta \|h\|^\theta_{H^{-1}} \cdot \|h\|^{1-\theta}_{H^{\sigma+\epsilon}} \leq T^\theta \|h\|_{H^{\sigma+\epsilon}} \). \( \square \)

Let us first consider the case \( r \) being an odd integer. In this case, we assume \( r > \frac{2s-1}{4} \). Now, if \( \frac{2s-1}{4} < \frac{1}{2} \), then we can choose \( \epsilon = \frac{1}{2} \) so that \( \frac{1}{2} < \frac{2s-1}{4} + \epsilon < 1 \leq r \). If \( \frac{2s-1}{4} > \frac{1}{2} \), then we can choose \( \epsilon \) sufficiently small so that we again have \( \frac{1}{2} < \frac{2s-1}{4} + \epsilon \leq r \).

Secondly, let us consider the situation for \( r > 0 \) being a non-integer. In this case, we assume \( [r] \geq \frac{2s-1}{4} \). If \( \frac{2s-1}{4} < \frac{1}{2} \) then we choose \( \epsilon = \frac{1}{2} \) so that \( \frac{1}{2} < \frac{2s-1}{4} + \epsilon \). If \( \frac{2s-1}{4} > \frac{1}{2} \), then we choose \( \epsilon \) sufficiently small so that \( \frac{1}{2} < \frac{2s-1}{4} + \epsilon \).

If \( r \) is even and \( \frac{2s-1}{4} < \frac{1}{2} \), then again we choose \( \epsilon = \frac{1}{2} \).

Now, given \( u \in X_T^r \), we know that \( u(0, \cdot) \) in particular belongs to the space \( H^{\frac{2s-1}{4}+\epsilon}(0,T) \) for any \( \epsilon \in (0, \frac{1}{2}) \). So, let us take an extension of \( u(0, \cdot) \in H^{\frac{2s-1}{4}+\epsilon}(0,T) \), say \( U \in H^{\frac{2s-1}{4}+\epsilon} \), so that

\[
\|U\|_{H^{\frac{2s-1}{4}+\epsilon}} \leq 2\|u(0, \cdot)\|_{H^{\frac{2s-1}{4}+\epsilon}(0,T)},
\]

see (2.3). Now, \( |U|^rU \) is an extension of \( |u(0, \cdot)|^ru(0, \cdot) \), and therefore

\[
\|u(0, \cdot)|^ru(0, \cdot)\|_{H^{\frac{2s-1}{4}+\epsilon}(0,T)} \leq \||U|^rU\|_{H^{\frac{2s-1}{4}+\epsilon}} \lesssim \|U\|_{H^{\frac{2s-1}{4}+\epsilon}^r}^r \|U\|_{H^{\frac{2s-1}{4}+\epsilon}}^{1-r}
\]

by Lemma 3.7. By using the inequality (3.23), we have

\[
\|U\|_{H^{\frac{2s-1}{4}+\epsilon}^r} \lesssim \|u(0, \cdot)\|_{H^{\frac{2s-1}{4}+\epsilon}}^{1+\epsilon}(0,T).
\]

Combining the above estimates, we arrive at

\[
\|u(0, \cdot)|^ru(0, \cdot)\|_{H^{\frac{2s-1}{4}+\epsilon}(0,T)} \lesssim \|u(0, \cdot)|^{r+\frac{1}{4}}_{H^{\frac{2s-1}{4}+\epsilon}} \lesssim \|u(0, \cdot)\|^{r+\frac{1}{4}}_{H^{\frac{2s-1}{4}+\epsilon}}(0,T).
\]
Finally, we deduce that
\[
\|W_b(\cdot)[h(u(0, \cdot))_e]\|_{X^s_T} \leq C(1 + T)\|u(0, \cdot)^\tau u(0, \cdot)^{\tau^s}(0, T) \\
\leq C(1 + T)T^{\frac{4}{2s+3+4\epsilon}}\|u(0, \cdot)^\tau u(0, \cdot)^{\tau^s}(0, T) \\
\leq C(1 + T)T^{\frac{4}{2s+3+4\epsilon}}\|u(0, \cdot)^\tau u(0, \cdot)^{\tau^s}(0, T) \\
\leq C(1 + T)T^{\frac{4}{2s+3+4\epsilon}} \sup_{x \in \mathbb{R}_+} u(x, \cdot)^\tau u(0, \cdot)^{\tau^s}(0, T) \\
\leq C(1 + T)T^{\frac{4}{2s+3+4\epsilon}} \|u\|_{X^s_T}^{\tau^s}. \tag{3.24}
\]

We can estimate the differences similarly. Namely, for any given \(u, v \in X^s_T\), we have
\[
\|W_b(\cdot)[h(u(0, \cdot)) - h(v(0, \cdot))_e]\|_{X^s_T} \\
\leq C(1 + T)T^{\frac{4}{2s+3+4\epsilon}} (\|u\|_{X^s_T} + \|v\|_{X^s_T}) \|u - v\|_{X^s_T}. \tag{3.25}
\]

**Local Existence.** Following the arguments in Section 3.1 and using the estimate (3.24), we have
\[
\|\Psi(u)\|_{X^s_T} \leq C \left(\|u_0\|_{H^s(\mathbb{R}_+)} + (1 + T)T^{\frac{4}{2s+3+4\epsilon}} \|u\|_{X^s_T}^{\tau^s} + T\|u\|_{X^s_T}^{p+1}\right).
\]

On the other hand, using the estimate (3.25) and the arguments in Section 3.1 for differences, we obtain
\[
\|\Psi(u) - \Psi(v)\|_{X^s_T} \leq C \left(T(\|u(\tau)\|_{X^s_T}^{p} + \|v(\tau)\|_{X^s_T}^{p}) \|u(\tau) - v(\tau)\|_{X^s_T} + (1 + T)T^{\frac{4}{2s+3+4\epsilon}} (\|u\|_{X^s_T} + \|v\|_{X^s_T}) \|u - v\|_{X^s_T}\right). \tag{3.26}
\]

Now, let \(A := C \left(\|u_0\|_{H^s(\mathbb{R}_+)}, R = 2A\) and \(T\) small enough that
\[
A + C(1 + T)T^{\frac{4}{2s+3+4\epsilon}} R^{p+1} + CTR^{p+1} < 2A.
\]

Now, if necessary we can choose \(T\) even smaller so that \(\Psi\) becomes a contraction on \(\hat{B}_R(0) \subset X^s_T\), which is a complete space. Hence, \(\Psi\) must have a unique fixed point in \(\hat{B}_R(0)\) when we look for a solution whose lifespan is sufficiently small.

**Uniqueness.** In order to prove uniqueness, we proceed as in Section 3.2 taking into account that the boundary forcing now depends on the solution itself. So, let \(u_1, u_2 \in X^s_{T_0}\) be two solutions of (1.1). Then,
\[
u_1(t) - u_2(t) = -i \int_0^t W_R(t - s)[f(u_1^s(s)) - f(u_2^s(s))]ds \\
+ W_b(t) (\{h(u_1(0, \cdot)) - h(u_2(0, \cdot)) + p(u_2^s) - p(u_1^s)\}_e) \tag{3.27}
\]
for a.a. $t \in [0,T_0]$. Then,
\[
\|u_1(t) - u_2(t)\|_{H^s} \leq \int_0^{T_0} \|f(u_1^*(s)) - f(u_2^*(s))\|_{H^s} + C(1+T_0)\|p(u_2^*) - p(u_1^*)\|_{H^{2s+1}}(0,T)
\]
\[+ C(1+T_0)T_0 \left( \|u_1\|_{X_{T_0}^s} + \|u_2\|_{X_{T_0}^s} \right) \|u_1 - u_2\|_{H^s}. \tag{3.28}\]

Now, choosing $T_0$ sufficiently small, we can subtract the last term above from the left hand side, estimate the rest of terms at the right hand side as in Section 3.2, and then use the Gronwall’s inequality to obtain $\|u_1(t) - u_2(t)\|_{H^s} = 0$.

**Continuous Dependence.** The proof of continuous dependence can be done as in Section 3.3 by taking into account that $h$ is now a function of $u(0,t)$. For this closed loop problem, the estimate (3.19) takes the following form.
\[
\|u - v\|_{X_{T_0}^s} \leq C\|u_0 - v_0\|_{H^s(\mathbb{R}_+)} \tag{3.29}\]
for sufficiently small $T_0$. Of course, for the closed loop problem $B$ is taken as a subset of $H^s(\mathbb{R}_+)$ with finite diameter.

**Blow-up Alternative.** The proof of the blow-up alternative is almost identical to the proof given in Section 3.4 and is therefore omitted here. The only modification is that the parameter $\delta$ in the proof given in Section 3.4 now depends only on $M$.

**References**

[1] A.S. Ackleh, K. Deng, *On the critical exponent for the Schrödinger equation with a nonlinear boundary condition*, Differential and Integral Equations, Vol. 17, No.11–12, 1293-1307, 2004

[2] C. Audiard, *On the boundary value problem for the Schrödinger equation: compatibility conditions and global existence*, preprint.

[3] J.L. Bona, S.M. Sun, B.Y. Zhang, *Nonhomogeneous boundary-value problems for one-dimensional nonlinear Schrödinger equations*, arXiv:1503.00065 [math.AP].

[4] Q. Bu, *On well-posedness of the forced nonlinear Schrödinger equation*, Appl. Anal. 46 (1992), no. 3-4, 219–239.

[5] C. Bu, *An initial-boundary value problem of the nonlinear Schrödinger equation*, Appl. Anal. 53 (1994), no. 3-4, 241–254.

[6] C. Bu, *Forced cubic Schrödinger equation with Robin boundary data: continuous dependency result*, J. Austral. Math. Soc. Ser. B 41 (2000), no. 3, 301–311.

[7] Q. Bu, *The nonlinear Schrödinger equation on the semi-infinite line*. Chinese Ann. Math. Ser. A 21 (2000), no. 4, 437–448.

[8] C. Bu, R. Shull, H. Wang, M. Chu, *Well-posedness, decay estimates and blow-up theorem for the forced NLS*, J. Partial Differential Equations 14 (2001), no. 1, 61–70.

[9] C. Bu, K. Tsutaya, C. Zhang, *Nonlinear Schrödinger equation with inhomogeneous Dirichlet boundary data*, J. Math. Phys. 46 (2005), no. 8, 083504, 6 pp.
[10] R. Carroll, Q. Bu, Solution of the forced nonlinear Schrödinger (NLS) equation using PDE techniques, Appl. Anal. 41 (1991), no. 1-4, 33–51.

[11] T. Cazenave, Semilinear Schrödinger equations, Courant Lecture Notes in Mathematics, 10. New York University, Courant Institute of Mathematical Sciences, New York; American Mathematical Society, Providence, RI, 2003.

[12] M. Cavalcanti, W. Corrêa, I. Lasiecka, C. Lefler, Well-posedness and Uniform Stability for Nonlinear Schrödinger Equations with Dynamic/Wentzell Boundary Conditions, preprint.

[13] J.E. Colliander, C.E. Kenig, The generalized Korteweg-de Vries equation on the half-line, Comm. Partial Diff. Equations 27 (2002) 2187–2266.

[14] J. Holmer, The initial-boundary value problem for the 1-d nonlinear Schrödinger equation on the half-line, Diff. Integral Equations 18 (2005), 647–668.

[15] E.I. Kaikina, Inhomogeneous Neumann initial-boundary value problem for the nonlinear Schrödinger equation, J. Differential Equations 255 (2013), no. 10, 3338–3356.

[16] E.I. Kaikina, Asymptotics for inhomogeneous Dirichlet initial-boundary value problem for the nonlinear Schrödinger equation, J. Math. Phys. 54 (2013), no. 11, 111504, 15 pp.

[17] C.E. Kenig, G. Ponce, and L. Vega, Well-posedness and scattering results for the generalized Korteweg-de Vries equation via the contraction principle, Comm. Pure Appl. Math. 46 (1993), no. 4, 527–620.

[18] D. Kriventsov, Local wellposedness for the nonlinear Schrödinger equation, preprint

[19] I. Lasiecka, R. Triggiani, Well-posedness and sharp uniform decay rates at the $L^2(\Omega)$-level of the Schrödinger equation with nonlinear boundary dissipation, J. Evol. Equ. 6 (2006), no. 3, 485–537.

[20] J.L. Lions, E. Magenes, Non-homogeneous boundary value problems and applications. Vol. I. Translated from the French by P. Kenneth. Die Grundlehren der mathematischen Wissenschaften, Band 181. Springer-Verlag, New York-Heidelberg, 1972.

[21] T. Özsarı, V.K. Kalantarov, I. Lasiecka, Uniform decay rates for the energy of weakly damped defocusing semilinear Schrödinger equations with inhomogeneous Dirichlet boundary control, Journal of Differential Equations, Volume 251, Issue 7, 1 October 2011, Pages 1841-1863

[22] T. Özsarı, Weakly-damped focusing nonlinear Schrödinger equations with Dirichlet control. Journal of Mathematical Analysis and Applications, Volume 389, Issue 1, 1 May 2012, Pages 84-97

[23] T. Özsarı, Global existence and open loop exponential stabilization of weak solutions for nonlinear Schrödinger equations with localized external Neumann manipulation, Nonlinear Analysis: Theory, Methods & Applications, Volume 80, March 2013, Pages 179-193

[24] T. Özsarı, Well-posedness for nonlinear Schrödinger equations with boundary forces in low dimensions by Strichartz estimates, Journal of Mathematical Analysis and Applications, Volume 424, Issue 1, 1 April 2015, Pages 487-508

[25] C.E. Kenig, G. Ponce, L. Vega, Oscillatory Integrals and Regularity of Dispersive Equations, Indiana University Mathematics Journal, Vol. 40, No. 1, 1991.
[26] W. Strauss, C. Bu, *An inhomogeneous boundary value problem for nonlinear Schrödinger equations*, J. Differential Equations 173 (2001), no. 1, 79–91.

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