HAUSDORFF DIMENSION OF LIMSUP SETS OF RECTANGLES IN THE HEISENBERG GROUP

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Abstract. The almost sure value of the Hausdorff dimension of limsup sets generated by randomly distributed rectangles in the Heisenberg group is computed in terms of directed singular value functions.

1. Introduction

Dimensional properties of subsets of Heisenberg groups have attained a lot of interest recently. Due to the non-trivial relation between the Hausdorff dimensions with respect to the Euclidean and the Heisenberg metrics [4, 5], one cannot directly transfer dimensional results in Euclidean spaces into Heisenberg groups. Indeed, it turns out that some theorems concerning dimensions have a special flavour or even an essentially different form in the Heisenberg setting. These include, for example, dimensional properties of self-affine sets, projections and slices.

In the Heisenberg group Hausdorff dimensions of self-similar and self-affine sets have been studied in [1, 5, 6]. Even though the class of affine iterated function systems is quite restrictive — every such system is a horizontal lift of an affine iterated function system on the plane — the dimension calculations involve some subtleties. The behaviour of the Hausdorff dimension under projections and slicing transpires to be interesting, see [2, 3, 13, 20]. There are two kinds of natural projections (and slices) in Heisenberg groups — the horizontal and vertical ones. The vertical projections possess an exceptional feature: they are not Lipschitz continuous. This indicates that the methods developed in the Euclidean setting cannot be utilised. For related questions concerning Sobolev maps and the foliations generated by the horizontal subspaces, see [4].

In this paper, we initiate a new direction of research in Heisenberg groups by investigating dimensions of limsup sets generated by rectangles. Let $X$ be a space and let $(A_n)$ be a sequence of subsets of $X$. The limsup set generated by the sequence $(A_n)$ consists of those points of $X$ which are covered by infinitely many of the sets $A_n$, that is,

$$\limsup_n A_n = \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} A_k.$$ 

Limsup sets are encountered in many fields of mathematics — one of the earliest appearances being the Borel-Cantelli lemma [9, 10]. They play a central role in the study of Besicovitch-Eggleston sets concerning the $k$-adic expansions of real numbers [8, 12] as well as in Diophantine approximation [21, 24]. For more information on different aspects of limsup sets, we refer to [19] and the references therein.

Dimensional properties of random limsup sets have been actively studied, see for example [11, 14, 16, 17, 19, 22, 23, 25, 26, 28]. Combining the results of these
papers, the almost sure value of dimension of random limsup sets is known in the following cases:

- the underlying space $X$ is a Riemann manifold and the driving measure determining the randomness is not singular with respect to the Lebesgue measure,
- $X$ is the Euclidean or the symbolic space, the generating sets $(A_n)$ are balls and the driving measure has special properties like being a Gibbs measure,
- $X$ is an Ahlfors regular metric space, randomness is given by the natural measure and $(A_n)$ is a sequence of balls.

In [12] a dimension formula for limsup sets generated by rectangles in products of Ahlfors regular metric spaces is derived. In this paper, we address the problem of determining the Hausdorff dimension of random limsup sets generated by rectangles in the first Heisenberg group (see Theorem 1 below). In [13] the Lipschitz continuity of projections is utilised to a great extent, and because of that, the same methods cannot be used in our setting. Instead, we will extend some results known in the Euclidean setting to unimodular groups or to compact metric spaces, and make calculations specific to the Heisenberg group to complete the argument.

We proceed by introducing our notation. The Heisenberg group $H$ is the set $\mathbb{R}^3$ with the non-commutative group operation

$$pp' = (x + x', y + y', z + z' + 2(xy' - yx')),$$

where $p = (x, y, z)$ and $p' = (x', y', z')$. The unit in $H$ is $(0, 0, 0)$ and the inverse of $p$ is $p^{-1} = (-x, -y, -z)$. There is a norm on $H$ given by

$$\|p\| = \left( (x^2 + y^2)^2 + z^2 \right)^{1/4},$$

which gives rise to a left-invariant metric

$$d_H(p, p') = \|p^{-1}p'\| = \left( (x' - x)^2 + (y' - y)^2 + (z' - z - 2(xy' - yx'))^2 \right)^{1/4}.$$

Both left and right translation in $H$ move vertical lines to vertical lines in such a way that the Euclidean distance between lines is preserved, and the image of the Lebesgue measure on a vertical line under translation is the Lebesgue measure on the image line. Thus Fubini’s theorem implies that the Lebesgue measure on $B(0, r)$ in $H$ is proportional to $r^4$, and by translation invariance the same is true for every ball of radius $r$. In particular, the metric space $(H, d_H)$ has Hausdorff dimension 4.

Let $L(p)$ be the vertical line through $p$ and

$$H(p) = \{ p' ; z' = z + 2(xy' - yx') \};$$

this is the plane through $p$ that has slope 0 in the direction $(x, y)$ and slope $2(x^2 + y^2)^{1/2}$ in the orthogonal direction $(-y, x)$. Then

$$d_H(p, p') \geq \left( (x' - x)^2 + (y' - y)^2 \right)^{1/2},$$

with equality if and only if $p' \in H(p)$. It follows that the distance to $L(p')$ from any point on $L(p)$ equals the Euclidean distance from $(x, y)$ to $(x', y')$, and vice versa by symmetry. Thus vertical lines are parallel in the Heisenberg metric, and the distance between them is the same as the Euclidean distance. The symmetry of the metric implies that $p' \in H(p)$ if and only if $p \in H(p')$, and the translation invariance of the metric implies that $pH(p') = H(pp')$. If $p' \in L(p)$ then $H(p')$ is parallel in the Euclidean sense to $H(p)$.

A closed rectangle in $H$ is a set of the form

$$\overline{R}(p, r) = \{ p' ; (x' - x)^2 + (y' - y)^2 \leq r^2, \; |z' - z - 2(xy' - yx')| \leq r^2 \},$$
where \( r = (r_1, r_2) \). This is the set of points that can be reached from \( p \) by moving “horizontally” in \( H(p) \) a distance at most \( r_1 \) and then vertically a distance at most \( r_2 \), or by moving first vertically a distance at most \( r_2 \) and then horizontally a distance at most \( r_1 \). A rectangle centred at \( p \) is the left translation by \( p \) of a rectangle centred at 0, that is, \( \overline{R}(p,r) = pR(0,r) \).

Let \( p = (p_n) \) be a sequence of points in \( H \) and let \( x = (r_n) \) be a bounded sequence of pairs of positive numbers, and define

\[
E_x(p) = \limsup_n \overline{R}(p_n, r_n).
\]

The purpose of this article is to give a formula for the Hausdorff dimension of such a set when the centres of the rectangles are chosen randomly. Let \( \lambda \) be the Lebesgue measure on \( H \) and let \( W \) be a bounded open subset of \( H \). Let \( \lambda_W = \lambda(W)^{-1}\lambda|_W \) and define the probability space \((\Omega, \mathcal{P}) \) by \( \Omega = H^N \) and \( \mathcal{P} = \lambda_N^N \). Then \( \omega \mapsto E_x(\omega) \) can be considered as a random set defined on \((\Omega, \mathcal{P}) \). The directed singular value function is defined as follows: for \( r = (r_1, r_2) \), if \( r_1 \leq r_2 \) let

\[
\Phi^t(r) = \begin{cases} r_2 & \text{if } t \in [0, 2], \\ r_1^{t-2}r_2 & \text{if } t \in [2, 4], \\ \end{cases}
\]

and if \( r_1 \geq r_2 \) let

\[
\Phi^t(r) = \begin{cases} r_1 & \text{if } t \in [0, 3], \\ r_1^{t-3}r_2^{2(t-3)} & \text{if } t \in [3, 4]. \\ \end{cases}
\]

**Theorem 1.** Almost surely,

\[
\dim_H E_x = \inf \left\{ t : \sum_n \Phi^t(r_n) < \infty \right\} \wedge 4.
\]

Let \( X \) be a metric space. The t-dimensional Hausdorff content of a subset \( A \) of \( X \) is defined by

\[
H^t_\infty(A) = \left\{ \sum_n |C_n|^t ; \{C_n\} \text{ is a cover of } A \right\}.
\]

If \( \mu \) is a Borel measure on \( X \) and \( t > 0 \), the t-energy of \( \mu \) is defined by

\[
I_t(\mu) = \iint d(x,y)^{-t} \, d\mu(y) \, d\mu(x).
\]

The t-capacity of a Borel subset \( A \) of \( X \) is defined by

\[
\operatorname{Cap}_t(A) = \sup \left\{ \frac{1}{I_t(\mu)} ; \mu \in \mathcal{P}(A) \right\},
\]

where \( \mathcal{P}(A) \) denotes the set of Borel probability measures on \( X \) that give full measure to \( A \). It can be shown that

(1) \[
\operatorname{Cap}_t(A) \leq H^t_\infty(A)
\]

always holds (see [10, Remark 3.3]).

The proof of Theorem 4 is based on estimating the Hausdorff content and capacity of rectangles in the Heisenberg group. It is shown in Section 4 that there is a constant \( c_t \) such that

\[
c_t^{-1}H^t_\infty(R(x,r)) \leq \Phi^t(r) \leq c_t \operatorname{Cap}_t(R(x,r))
\]

for every \( x \) and \( r \). This immediately implies that if \( \sum_n \Phi^t(r_n) < \infty \) then \( H^t(E_x(\omega)) = 0 \) for every \( \omega \), since every tail of the sequence of rectangles is a cover of \( E_x^t(\omega) \). The almost sure lower bound for \( \dim_H E_x \) then follows from the estimate of the capacity of a rectangle together with Theorem 2 below.
A Borel measure \( \mu \) on a metric space \( X \) is \((c, d)\)-regular if
\[
c^{-1}r^d \leq \mu(B(x, r)) \leq cr^d
\]
for every \( x \in X \) and \( r \in [0, |X|] \). A locally compact group \( G \) is unimodular if the left invariant Haar measure is also right invariant, or equivalently, if it is invariant under inversion. For example, the Heisenberg group is \((c, 4)\)-regular for some \( c \) and unimodular.

Theorem \( \text{[2]} \) is a variant of \([19 \text{ Theorem } 1(b)] \) and \([25 \text{ Theorem } 1] \). We give a somewhat different proof even though the main philosophy follows the same lines as the proofs in \([19, 25] \).

**Theorem 2.** Let \( G \) be a unimodular group with left invariant metric and \((c, d)\)-regular Haar measure \( \lambda \), and let \( W \) be a bounded open subset of \( G \) such that \( \lambda(W) = 1 \). Define the probability space \( (\Omega, \mathcal{P}) \) by \( \Omega = G^\mathbb{N} \) and \( \mathcal{P} = (\lambda|_W)^\mathbb{N} \). Let \( (V_n) \) be a bounded sequence of open subsets of \( G \) (bounded meaning that there is a ball in \( G \) that contains \( V_n \) for every \( n \)). For \( \omega = (\omega_n) \in \Omega \) let
\[
E(\omega) = \limsup_n (\omega_n V_n).
\]
If \( t \in (0, d) \) and
\[
\sum_n \text{Cap}_t(V_n) = \infty
\]
then almost every \( \omega \) is such that \( \mathcal{H}^t (U \cap E(\omega)) = \infty \) for every open subset \( U \) of \( W \).

The assumption that \( \lambda(W) = 1 \) makes the proof slightly easier to read, but is not essential since a constant multiple of a \((c, d)\)-regular Haar measure is a \((c', d)\)-regular Haar measure for some \( c' \).

The proof of Theorem \( \text{[2]} \) is based on the following deterministic lemma (compare \([15 \text{ Lemma } 7], [19 \text{ Proposition } 4.6] \) and \([27 \text{ Theorem } 1] \)). If \( X \) is a compact metric space, the weak*-topology on the space of finite Borel measures on \( X \) is the topology generated by the maps \( \{ \mu \mapsto \mu(\varphi) \} \), where \( \varphi \) ranges over the continuous functions \( X \to \mathbb{R} \). It is not difficult to see that \( \lim_{n \to \infty} \mu_n = \mu \) in the weak*-topology if and only if \( \lim_{n \to \infty} \mu_n(\varphi) = \mu(\varphi) \) for every continuous function \( \varphi \).

**Lemma 3.** Let \( \nu \) be a finite Borel measure on a compact metric space \( X \) and let \( (\varphi_n)_{n=1}^\infty \) be a sequence of non-negative continuous functions on \( X \) such that \( \lim_{n \to \infty} \varphi_n d\nu = \nu \) in the weak*-topology and \( \liminf_{n \to \infty} I_t(\varphi_n \rho d\nu) \leq I_t(\rho d\nu) \) whenever \( \rho \) is a product of finitely many of the functions \( \{ \varphi_n \} \). Then for every \( t > 0 \),
\[
\text{Cap}_t \left( \sup_n \nu(\text{sup} \sup \varphi_n) \right) \geq \frac{\nu(X)^2}{I_t(\nu)}.
\]

**Notation and conventions.** All measures appearing below are Borel measures, but this will not be explicitly stated. Thus “measure” below means “Borel measure”. If \( X \) is a metric space with a measure \( \mu \) and \( \varphi \) is a non-negative continuous function on \( X \), then \( I_t(\varphi) \) means \( I_t(\varphi d\mu) \). Similarly, if \( A \) is a Borel subset of \( X \) then \( I_t(A) \) means \( I_t(\mu|_A) \). If \( X \) is a compact metric space, then the space of finite measures on \( X \) is considered as a topological space under the weak*-topology.

2. **Proof of Lemma 3**

Let \( Y \) be a topological space. A function \( f : Y \to (-\infty, \infty] \) is lower semi-continuous if \( f^{-1}((a, \infty]) \) is open for every \( a \in \mathbb{R} \), or equivalently if \( f(y_0) \leq \liminf_{y \to y_0} f(y) \) for every \( y_0 \in Y \). The following lemma is well known, but a proof is included for the convenience of the reader.
Lemma 4. Let $X$ be a compact metric space and let $t > 0$. Then $\mu \mapsto I_t(\mu)$ is lower semicontinuous.

Proof. Let $\mathcal{M}(X)$ be the space of finite measures on $X$. It will first be shown that the map $\mathcal{M}(X) \to \mathcal{M}(X) \times \mathcal{M}(X)$, $\mu \mapsto \mu \times \mu$ is continuous. Let $\eta$ be a continuous function on $X \times X$ and let $\mu_0 \in \mathcal{M}(X)$. It suffices to show that for every $\varepsilon > 0$, the set

$$A = \{ \mu \in \mathcal{M}(X); |(\mu \times \mu)(\eta) - (\mu_0 \times \mu_0)(\eta)| < 3\varepsilon \}$$

contains an open neighbourhood of $\mu_0$.

Let $K > \mu_0(X)$. By Stone–Weierstrass’ theorem, there are functions $\{\eta_i\}_{i=1}^n$ of the form $\eta_i = \eta_{i,1} \times \eta_{i,2}$, where $\eta_{i,j}$ are continuous, such that $\|\eta - \sum_{i=1}^n \eta_i\|_{\infty} < \varepsilon/K^2$. The set

$$V = \{ \mu \in \mathcal{M}(X); |(\mu \times \mu)(\eta_i) - (\mu_0 \times \mu_0)(\eta_i)| < \frac{\varepsilon}{n} \text{ for every } i \text{ and } \mu(X) < K \}$$

is open since $(\mu \times \mu)(\eta_i) = \mu(\eta_{i,1})\mu(\eta_{i,2})$, and $\mu_0 \in V$. If $\mu \in V$ then

$$|(\mu \times \mu)(\eta) - (\mu_0 \times \mu_0)(\eta)| \leq |(\mu \times \mu)(\eta) - (\mu \times \mu)\left(\sum_{i=1}^n \eta_i\right)|$$

$$+ \sum_{i=1}^n |(\mu \times \mu)(\eta_i) - (\mu_0 \times \mu_0)(\eta_i)|$$

$$+ |(\mu_0 \times \mu_0)\left(\sum_{i=1}^n \eta_i\right) - (\mu_0 \times \mu_0)(\eta)| < 3\varepsilon,$$

so that $V$ is a subset of $A$.

Define

$$I_t^M(\mu) = \int \int d(x, y)^{-t} \wedge M \, d\mu(y) \, d\mu(x).$$

Let $a \in \mathbb{R}$ and let $\mu_0$ be a measure in $\mathcal{M}(X)$ such that $I_t(\mu_0) > a$. Then there exists $M$ such that $I_t^M(\mu_0) > a$, and the set $\{ \mu; I_t^M(\mu) > a \}$ is open, contains $\mu_0$ and is contained in $\{ \mu; I_t(\mu) > a \}$.

Proof of Lemma 4 Let $\varepsilon \in (0, \nu(X))$ and let $(\varepsilon_k)$ be a sequence of positive numbers such that $\sum_k \varepsilon_k \leq \varepsilon$. Define recursively a sequence $(\rho_k)_{k=1}^\infty$ of natural numbers as follows, using the notation $\rho_k = \varphi_{n_k} \cdots \varphi_{n_1}$ (in particular, $\rho_0 = 1$). For $k \geq 1$, assume that $n_1, \ldots, n_{k-1}$ are defined. Then since $\lim_{n \to \infty} \nu(\varphi_{n_k} \rho_{k-1}) = \nu(\rho_{k-1})$ and

$$\liminf_{n \to \infty} I_t(\varphi_{n_k} \rho_{k-1}) \leq I_t(\rho_{k-1}),$$

it is possible to find $n_k > n_{k-1}$ such that

$$\nu(\rho_k) \geq \nu(\rho_{k-1}) - \varepsilon_k$$

and

$$I_t(\rho_k) \leq I_t(\rho_{k-1}) + \varepsilon_k.$$

Let $\mu$ be an accumulation point of the sequence of measures $(\rho_k \, d\nu)$ — then there is a strictly increasing sequence $(k_i)$ such that $\mu = \lim_{i \to \infty} \rho_{k_i} \, d\nu$. Thus

$$\mu(X) = \lim_{i \to \infty} \nu(\rho_{k_i}) \geq \liminf_{k \to \infty} \nu(\rho_k) \geq \nu(X) - \sum_k \varepsilon_k \geq \nu(X) - \varepsilon,$$

and by Lemma 4

$$I_t(\mu) \leq \liminf_{i \to \infty} I_t(\rho_{k_i}) \leq \limsup_{k \to \infty} I_t(\rho_k) \leq I_t(\nu) + \sum_k \varepsilon_k \leq I_t(\nu) + \varepsilon.$$
Moreover,
\[
\text{supp} \, \mu \subset \text{supp} \, \nu \cap \bigcap_k \text{supp} \, \varphi_{n_k} \subset \text{supp} \, \nu \cap \lim \sup_n \left( \text{supp} \, \varphi_n \right)
\]
and thus
\[
\text{Cap}_t \left( \text{supp} \, \nu \cap \lim \sup_n \left( \text{supp} \, \varphi_n \right) \right) \leq \frac{\mu(X)^2}{I_t(\mu)} \geq \frac{(\nu(X) - \varepsilon)^2}{I_t(\nu) + \varepsilon}.
\]
Letting \( \varepsilon \to 0 \) concludes the proof. \( \square \)

3. Proof of Theorem 2

The proof of Theorem 2 is based on constructing a sequence \((\varphi_n^\omega)\) of random continuous functions supported on a compact neighbourhood of \( W \), satisfying the hypothesis of Lemma 3, such that
\[
\lim_n \left( \text{supp} \, \varphi_n^\omega \right) \subset \text{supp} \, (\omega_n V_n).
\]
A few lemmas are needed. The first one is used in the proofs of Lemma 6 and Theorem 2.

Lemma 5. Let \((X, \lambda)\) be a \((c, d)\)-regular space and let \( t \in (0, d) \). Then there is a constant \( C \) such that for every \( x \in X \) and \( r > 0 \),
\[
I_t(B(x, r)) \leq C r^{2d-t}.
\]

Proof. If \( \varphi \) is a non-negative function on \( X \) then
\[
\int \varphi \, d\lambda = \int_0^\infty \lambda \{ z \in X; \varphi(z) > \gamma \} \, d\gamma,
\]
since both sides equal the \( \lambda \times \text{Leb} \)-measure of the set \( \{ (z, u) \in X \times [0, \infty); u \in (0, \varphi(z)) \} \) (note that the boundary of this set has measure 0). Thus for \( y \in X \),
\[
\int_{B(x, r)} d(y, z)^{-t} \, d\lambda(z) = \int_0^\infty \lambda \{ z \in B(x, r); d(y, z)^{-t} > \gamma \} \, d\gamma
\]
\[
= \int_0^\infty \lambda \left( B(x, r) \cap B \left( y, \gamma^{-1/t} \right) \right) \, d\gamma \leq c \int_0^\infty \min \left( r^d, \gamma^{-d/t} \right) \, d\gamma
\]
\[
= c \int_0^{r^{-t}} r^d \, d\gamma + c \int_{r^{-t}}^\infty \gamma^{-d/t} \, d\gamma = \frac{cd}{d-t} r^{d-t},
\]
and thus
\[
I_t(B(x, r)) \leq \lambda(B(x, r)) \cdot \frac{cd}{d-t} r^{d-t} \leq \frac{c^2 d}{d-t} r^{2d-t}.
\]
The next lemma is a variant of [19, Proposition 3.8].

Lemma 6. Let \((X, \lambda)\) be a compact \((c, d)\)-regular space and let \( \theta \) be a finite measure on \( X \). Then there is a sequence \((\varphi_n)\) of non-negative continuous functions on \( X \) such that \( \lim_{n \to \infty} \varphi_n \, d\lambda = \theta \) and \( \limsup_{n \to \infty} I_t(\varphi_n) \leq I_t(\theta) \) for every \( t \in (0, d) \).

Proof. For each \( n \), let \( \{ x_{n, i} \}_{i=1}^{N_n} \) be a maximal \( 1/n \)-separated subset of \( X \). Then \( \{ B(x_{n, i}, 1/2n) \} \) are disjoint and \( \{ B(x_{n, i}, 1/n) \} \) is a cover of \( X \). Let \( Q_{n, i} \) be the set of points \( x \) in \( X \) for which \( i \) is the first index such that \( d(x, x_{n, i}) = \min_j d(x, x_{n, j}) \), that is,
\[
Q_{n, i} = \{ x \in X; d(x, x_{n, i}) < d(x, x_{n, j}) \text{ for } j = 1, \ldots, i - 1 \text{ and } d(x, x_{n, i}) \leq d(x, x_{n, j}) \text{ for } j = i, \ldots, N_n \}.
\]
Then \( \{Q_{n,i}\}_{i=1}^{N_n} \) is a partition of \( X \) into Borel sets and
\[
B\left( x_{n,i}, \frac{1}{2n} \right) \subset Q_{n,i} \subset B\left( x_{n,i}, \frac{1}{n} \right)
\]
for every \( i \).

For each \( i \), let
\[
\varphi_{n,i}(x) = a_{n,i} \max(0, 1 - 4nd(x, x_{n,i}))
\]
where \( a_{n,i} \) is such that \( \lambda(\varphi_{n,i}) = 1 \). Then \( \varphi_{n,i} \) is supported on \( \overline{B}(x_{n,i}, 1/4n) \), and
\[
a_{n,i} \leq 2^{1+3d}cn^d
\]
since
\[
\int_{B(x_{n,i}, \frac{1}{2n})} (1 - 4nd(x, x_{n,i})) \, d\lambda(x) \geq \frac{1}{2} \lambda\left( B\left( x_{n,i}, \frac{1}{8n} \right) \right) \geq 2^{-(1+3d)}c^{-1}n^{-d}.
\]
Let
\[
\varphi = \sum_{i=1}^{N_n} \theta(Q_{n,i})\varphi_{n,i}.
\]

Let \( \eta \) be a continuous function on \( X \) and let \( \varepsilon > 0 \). Since \( X \) is compact, \( \eta \) is uniformly continuous, and thus there is some \( n_0 \) such that
\[
|\eta(x) - \eta(x_{n,i})| \leq \frac{\varepsilon}{2\theta(X)}
\]
whenever \( n \geq n_0 \) and \( x \in Q_{n,i} \). Then for \( n \geq n_0 \),
\[
|\theta(\eta) - (\varphi_n \, d\lambda)(\eta)| \leq \sum_{i=1}^{N_n} \left( \int_{Q_{n,i}} |\eta - \eta(x_{n,i})| \, d\theta + \theta(Q_{n,i}) \int_{Q_{n,i}} |\eta(x_{n,i}) - \eta(\varphi_{n,i})| \, d\lambda \right) \leq \varepsilon,
\]
using that \( (\varphi_{n,i} \, d\lambda)(Q_{n,i}) = 1 \) for every \( i \). Thus \( \lim_{n \to \infty} \varphi_n \, d\lambda = \theta \). It remains to show that \( \limsup_{n \to \infty} I_t(\varphi_n) \leq I_t(\theta) \), and for this it may be assumed that \( I_t(\theta) < \infty \).

Let \( \alpha > 1 \), and if \( \psi_1, \psi_2 \) are continuous functions on \( X \) let
\[
J_t(\psi_1, \psi_2) = \int \int d(x, y)^{-\alpha} \psi_1(x)\psi_2(y) \, d\lambda(x) \, d\lambda(y).
\]
Then
\[
I_t(\varphi_n) = S_1 + S_2 + S_3,
\]
where
\[
S_1 = \sum_{d(x_{n,i}, x_{n,j}) \geq \frac{1}{n}} \theta(Q_{n,i})\theta(Q_{n,j})I_t(\varphi_{n,i}, \varphi_{n,j}),
\]
\[
S_2 = \sum_{d(x_{n,i}, x_{n,j}) < \frac{1}{n}, i \neq j} \theta(Q_{n,i})\theta(Q_{n,j})I_t(\varphi_{n,i}, \varphi_{n,j}),
\]
\[
S_3 = \sum_{i} \theta(Q_{n,i})^2I_t(\varphi_{n,i}).
\]

If \( x \in \text{supp} \varphi_{n,i} \) and \( y \in \text{supp} \varphi_{n,j} \) and \( d(x_{n,i}, x_{n,j}) \geq \alpha/n \), then
\[
|d(x, y) - d(x_{n,i}, x_{n,j})| \leq \frac{1}{n} \leq \frac{d(x_{n,i}, x_{n,j})}{\alpha},
\]
so that
\[
\frac{\alpha - 1}{\alpha} \leq \frac{d(x, y)}{d(x_{n,i}, x_{n,j})} \leq \frac{\alpha + 1}{\alpha}.
\]
Thus for $i, j$ appearing in $S_1$,
\[
\theta(Q_{n,i})\theta(Q_{n,j})J_t(\varphi_{n,i}, \varphi_{n,j}) \leq \left( \frac{\alpha}{\alpha - 1} \right)^t \theta(Q_{n,i})\theta(Q_{n,j})d(x_{n,i}, x_{n,j})^{-t} \\
\leq \left( \frac{\alpha + 1}{\alpha - 1} \right)^t \int_{Q_{n,i} \times Q_{n,j}} d(x, y)^{-t} d\theta(x) d\theta(y),
\]
and it follows that
\[
S_1 \leq \left( \frac{\alpha + 1}{\alpha - 1} \right)^t I_t(\theta).
\]

If $x \in \text{supp} \varphi_{n,i}$ and $y \in \text{supp} \varphi_{n,j}$ then $i \neq j$ implies that $d(x, y) \geq 1/2n$ and if $x \in Q_{n,i}$ and $y \in Q_{n,j}$ then $d(x_{n,i}, x_{n,j}) \leq \alpha/n$ implies that $d(x, y) \leq (\alpha + 2)/n$.

Thus
\[
S_2 \leq 2^t \sum_{i \neq j} \int_{d(x_{n,i}, x_{n,j}) \leq \frac{1}{2n}} \theta(Q_{n,i})\theta(Q_{n,j})n^t \\
\leq 2^t(\alpha + 2)^t \sum_{d(x_{n,i}, x_{n,j}) \leq \frac{1}{2n}} \int_{Q_{n,i} \times Q_{n,j}} d(x, y)^{-t} d\theta(x) d\theta(y) \\
\leq 2^t(\alpha + 2)^t \int_{d(x, y) \leq \frac{1}{2n}} d(x, y)^{-t} d\theta(x) d\theta(y).
\]

By Lemma 5 there is a constant $C$ such that
\[
I_t(B(x, r)) \leq Cr^{2d-t}
\]
for every $x \in X$ and $r > 0$. It follows that
\[
I_t(\varphi_{n,i}) \leq a_{n,i}^2 I_t \left( B \left( x_{n,i}, \frac{1}{4n} \right) \right) \leq C'n^t,
\]
where $C' = 4^{1+3d}d^2C$, so that
\[
S_3 \leq C' \sum_i \theta(Q_{n,i})^2 n^t \leq 2^t C' \sum_i \int_{Q_{n,i} \times Q_{n,i}} d(x, y)^{-t} d\theta(x) d\theta(y) \\
\leq 2^t C' \int_{d(x, y) \leq \frac{1}{2n}} d(x, y)^{-t} d\theta(x) d\theta(y).
\]

Given $\varepsilon > 0$ it is possible to choose $\alpha$ large enough so that $S_1 \leq I_t(\theta) + \varepsilon$, and then $n_0$ large enough so that $S_2 + S_3 \leq \varepsilon$ for every $n \geq n_0$. It follows that $\limsup_{n \to \infty} I_t(\varphi_n) \leq I_t(\theta) + 2\varepsilon$, and letting $\varepsilon \to 0$ concludes the proof. \hfill \Box

The following lemma is a modification of the argument from [19, p. 39].
Lemma 7. Let \((X, \lambda)\) be a \((c, d)\)-regular space and let \((V_n)\) be a sequence of open subsets of an open ball \(B\) in \(X\) such that \(\sum_n \text{Cap}_t(V_n) = \infty\). Then there is a sequence \((V'_n)\) of open subsets of \(X\) such that

1. \(V'_n \subset V_n\) for every \(n\),
2. \(\lim_{n \to \infty} |V'_n| = 0\), and
3. \(\sum_n \text{Cap}_t(V'_n) = \infty\).

Proof. Let \(B_1 = \{x \in X; d(x, B) < 1\}\) and for each \(n\), let \(\mu_n\) be a probability measure on \(V_n\) such that \(I_t(\mu_n) \leq 2 \text{Cap}_t(V_n)^{-1}\). Consider any \(r \in (0, 1)\) and let \(A_n = \{x \in X; \mu_n(B(x, r)) > 0\}\), and for \(x \in A_n\) let \(\mu_n^x = \mu_n(B(x, r))^{-1}\mu_n\big|_{B(x, r)}\).

Using Cauchy–Schwarz’ inequality, Fubini’s theorem and then that \(B(y, r) \subset B_1\) for \(\mu_n\)-almost every \(y\) for every \(n\),

\[
\int_{B_1} \mu_n(B(x, r))^2 \, d\lambda(x) \geq \frac{1}{\lambda(B_1)} \left( \int_{B_1} \mu_n(B(x, r)) \, d\lambda(x) \right)^2 = \frac{1}{\lambda(B_1)} \left( \int \lambda(B_1 \cap B(y, r)) \, d\mu_n(y) \right)^2 \geq \frac{e^{-2r^{2d}}}{\lambda(B_1)}. \]

Thus for any natural number \(a\),

\[
\int_{B_1} \sum_{n=a}^\infty \text{Cap}_t(V_n \cap B(x, r)) \, d\lambda(x) \geq \int_{B_1} \sum_{n \geq a} \frac{1}{I_t(\mu_n^x)} \, d\lambda(x) \geq \sum_{n=a}^\infty \frac{1}{I_t(\mu_n)} \int_{B_1 \cap A_n} \mu_n(B(x, r))^2 \, d\lambda(x) = \sum_{n=a}^\infty \frac{1}{I_t(\mu_n)} \int_{B_1} \mu_n(B(x, r))^2 \, d\lambda(x) = \infty.
\]

It follows that \(\sum_{n=a}^\infty \text{Cap}_t(V_n \cap B(x, r))\) is unbounded as a function of \(x\), and hence there exist \(x \in B_1\) and a natural number \(b\) such that

\[
\sum_{n=a}^b \text{Cap}_t(V_n \cap B(x, r)) \geq 1.
\]

It is now possible to recursively define a strictly increasing sequence \((n_k)\) of natural numbers and a sequence \((x_k)\) of points in \(B_1\), such that \(n_1 = 1\) and for every \(k\),

\[
\sum_{n=n_k}^{n_{k+1}-1} \text{Cap}_t(V_n \cap B(x_k, 2^{-k})) \geq 1.
\]

The sequence \((V'_n)\) defined by \(V'_n = V_n \cap B(x_k, 2^{-k})\) for \(n = n_k, \ldots, n_{k+1} - 1\) has the properties i)--iii).

Lemma 8. Let \((b_n)\) be a sequence of positive numbers bounded away from 0, such that \(\sum_n b_n^{-1} = \infty\). Then there are non-negative numbers \((a_{n,k})_{n,k \in \mathbb{N}}\) such that

1. \((a_{n,k})\) has finite support for every \(n\) and \(\lim_{n \to \infty} \min\{k; a_{n,k} \neq 0\} = \infty\),
2. \(\sum_k a_{n,k} = 1\) for every \(n\) and \(\sum_{n,k} a_{n,k}^2 < \infty\), and
3. \(\lim_{n \to \infty} \sum_k a_{n,k}^2 b_k = 0\).

Proof. Let

\[
a_{n,k} = \begin{cases} a_n b_k^{-1} & \text{if } M_n \leq k \leq N_n, \\ 0 & \text{otherwise}, \end{cases}
\]

where \(M_n, N_n\) are some sequences to be determined.
where
\[ a_n = \frac{1}{\sum_{k=M_n}^{N_n} b_k^{-1}}. \]

Since \( \sum_n b_n^{-1} = \infty \) it is possible to choose \((M_n)\) and \((N_n)\) such that (i) holds and \( \sum_n a_n < \infty \). Then clearly \( \sum_n a_{n,k} = 1 \) for every \( n \), and
\[ \sum_{n,k} a_{n,k}^2 \leq B \sum_n \left( a_n \sum_k a_{n,k} \right) = B \sum_n a_n < \infty, \]
where \( B = \sup_k b_k^{-1} \). Finally,
\[ \sum_{k} a_{n,k}^2 b_k = \frac{\sum_{k=M_n}^{N_n} b_k^{-1}}{(\sum_{k=M_n}^{N_n} b_k^{-1})^2} = a_n, \]
which converges to 0 when \( n \to \infty \). □

**Lemma 9.** Let \( \mu \) be a probability measure on a compact metric space \( X \), and define the probability space \((\Omega, P)\) by \( \Omega = X^\mathbb{N} \) and \( P = \mu^\mathbb{N} \). Let \((a_{n,k})\) be non-negative numbers such that \( \sum_k a_{n,k} = 1 \) for every \( n \) and \( \sum_{n,k} a_{n,k}^2 < \infty \). For \( \omega \in \Omega \), let
\[ \mu_\omega n = \sum_{k} a_{n,k} \delta_{\omega k}. \]
Then almost surely \( \lim_{n \to \infty} \mu_\omega n = \mu \).

**Proof.** Let \( \eta \) be a continuous function on \( X \). Then for every \( k \),
\[ \mathbb{E} \eta(\omega_k) = \mu(\eta) \]
and
\[ \text{Var} \eta(\omega_k) = \mu(\eta^2) - \mu(\eta)^2 \leq \| \eta \|_{\infty}^2, \]
and hence
\[ \mathbb{E}(\mu_\omega n(\eta)) = \mu(\eta) \]
and
\[ \text{Var}(\mu_\omega n(\eta)) = \sum_{k} a_{n,k}^2 \text{Var} \eta(\omega_k) \leq \| \eta \|_{\infty}^2 \sum_{k} a_{n,k}^2. \]

Let \( \varepsilon > 0 \). Then by Chebyshev’s inequality,
\[ \sum_n \mathbb{P}(|\mu_\omega n(\eta) - \mu(\eta)| \geq \varepsilon) \leq \sum_n \frac{\| \eta \|_{\infty}^2 \sum_{k} a_{n,k}^2}{\varepsilon^2} < \infty, \]
so that Borel–Cantelli’s lemma implies that
\[ \limsup_{n \to \infty} |\mu_\omega n(\eta) - \mu(\eta)| \leq \varepsilon \]
a almost surely. Since the space of continuous functions on \( X \) is separable, it follows that \( \lim_{n \to \infty} \mu_\omega n = \mu \) almost surely. □

**Lemma 10.** Let \( (\xi_n) \) be a sequence of independent random variables. Then almost surely
\[ \liminf_{n \to \infty} \xi_n \leq \liminf_{n \to \infty} \mathbb{E}\xi_n. \]

**Proof.** By taking a subsequence it may be assumed that \( \lim_{n \to \infty} \mathbb{E}\xi_n \) exists. Let \( (\varepsilon_n) \) be a sequence of positive numbers converging to 0, such that
\[ \sum_n \frac{\varepsilon_n}{1 + \varepsilon_n} = \infty. \]
By Markov’s inequality,

\[ \mathbb{P} (\xi_n \leq (1 + \varepsilon_n) \mathbb{E} \xi_n) = 1 - \mathbb{P} (\xi_n > (1 + \varepsilon_n) \mathbb{E} \xi_n) \geq 1 - \frac{\mathbb{E} \xi_n}{(1 + \varepsilon_n) \mathbb{E} \xi_n} = \frac{\varepsilon_n}{1 + \varepsilon_n}. \]

Then by Borel–Cantelli’s lemma there is almost surely a strictly increasing sequence \((n_k)\) of natural numbers such that \(\xi_{n_k} \leq (1 + \varepsilon_{n_k}) \mathbb{E} \xi_{n_k}\) for every \(k\), and thus

\[ \liminf_{n \to \infty} \xi_n \leq \liminf_{k \to \infty} \xi_{n_k} \leq \limsup_{k \to \infty} \xi_{n_k} \leq \limsup_{n \to \infty} \mathbb{E} \xi_n = \liminf_{n \to \infty} \mathbb{E} \xi_n. \quad \square \]

**Proof of Theorem 2.** By Lemma 7 it may be assumed that \(\lim_{n \to \infty} |V_n| = 0\).

Define a sequence \((\varphi_n)\) of random continuous functions on \(G\) in the following way. For each \(n\), there is a probability measure \(\theta_n\) on \(V_n\) such that \(I_n(\theta_n) \leq 2 \text{Cap}(V_n)^{-1}\), and an open subset \(A_n\) of \(V_n\) such that \(d(A_n, V_n^c) > 0\) and \(\theta_n(A_n) \geq 1/2\). By Lemma 8 there is then a non-negative continuous function \(\psi_n\) on \(G\) such that \((\psi_n, d\lambda)(A_n) \geq \theta_n(A_n)/2 \geq 1/4\) and \(I_n(\psi_n) \leq 2I_n(\theta_n) \leq 4 \text{Cap}(V_n)^{-1}\). Let \(\psi_n\) be a continuous function on \(G\) such that \(\chi_{A_n} \leq \psi_n \leq \chi_{V_n}\) and let \(\varphi_n = c_n \psi_n \varphi_n^2\), where \(c_n\) is such that \(\psi_n d\lambda\) is a probability measure. Then \(c_n \leq 4\) and hence

\[ I_1(\psi_n) \leq c_n^2 I_1(\psi_n^2) \leq 64 \text{Cap}(V_n)^{-1}, \]

so that the hypothesis of the theorem implies that \(\sum_n I_1(\psi_n) = \infty\). Let \((a_{n,k})\) be as in Lemma 8 with respect to \(b_n = I_1(\psi_n) (\geq |V_n|^{-2})\), and let

\[ \varphi_n^\infty = \sum_k a_{n,k} \varphi_n^k, \]

where \(\varphi_n^k(x) = \psi_k(\omega_k^{-1} x)\).

Let \(\eta\) be a non-negative continuous function on \(G\) such that \(\text{supp} \eta \subset W\) and let \(\nu = \eta d\lambda\). Property (ii) of Lemma 8 together with Lemma 9 applied in the space \((\mathcal{W}, \lambda|_W)\) implies that for almost every \(\omega\),

\[ \lim_{n \to \infty} \sum_k a_{n,k} \delta_{\omega_k} = \lambda|_W. \]

Since \(\lim_{n \to \infty} |V_n| = 0\), it follows for such \(\omega\) that \(\lim_{n \to \infty} \varphi_n^\infty d\lambda = \lambda|_W\) and thus

\[ \lim_{n \to \infty} \varphi_n^\infty d\nu = \eta d\lambda|_W = \nu. \]

Let \(\rho\) be a continuous function on \(G\) with compact support. Using that \(\psi_n^\infty(x)\) and \(\psi_j^\infty(y)\) are independent for \(i \neq j\),

\[ \mathbb{E} I_1(\varphi_n^\infty \rho d\nu) = \sum_k a_{n,k}^2 \mathbb{E} I_1(\psi_k^\infty \rho d\nu) \]

\[ + \sum_{i \neq j} a_{n,i} a_{n,j} \int d(x, y)^{-1} \mathbb{E} \psi_i^\infty(x) \mathbb{E} \psi_j^\infty(y) \rho(x) \rho(y) d\nu(x) d\nu(y). \]

Since \(I_1(\psi_k^\infty \rho d\nu) \leq \|\rho\|^2 \|\eta\|^2 I_1(\psi_k)\), it follows from property (iii) of Lemma 8 that the first sum converges to 0 when \(n \to \infty\). Next,

\[ \mathbb{E} \psi_i^\infty(x) = \int \psi_i(\omega_i^{-1} x) d\lambda|_W(\omega_i) \leq \int \psi_i(\omega_i^{-1} x) d\lambda(\omega_i) = \lambda(\psi_i) = 1, \]

using that \(\lambda\) is invariant under right translation and inversion. Thus the integral in the second sum is less than or equal to \(I_1(\rho d\nu)\) and it follows by property (ii) in Lemma 8 that the second sum is less than or equal to \(I_1(\rho d\nu)\) as well. Thus by Lemma 10 almost surely

\[ \liminf_{n \to \infty} I_1(\varphi_n^\infty \rho d\nu) \leq I_1(\rho d\nu). \]
Lemma 3 applied in the space \((W, \nu)\) together with (1) now implies that almost surely
\[
\mathcal{H}^t(\text{supp}\, \eta \cap E(\omega)) \geq \frac{\lambda(\eta)^2}{I_t(\eta)}.
\]

For \(m = 1, 2, \ldots\), let \(B_m\) be a maximal collection of disjoint open balls in \(W\) of radius \(2^{-m}\). For each \(B \in \bigcup_m B_m\), let \(\eta_B\) be a non-negative continuous function on \(G\) such that \(\chi_{\frac{1}{2}B} \leq \eta_B \leq \chi_B\), where \(\frac{1}{2}B\) is the ball concentric with \(B\) having half the radius. Since \(\bigcup_m B_m\) is countable, almost every \(\omega\) is such that whenever \(B \in B_m\) then
\[
\mathcal{H}^t(B \cap E(\omega)) \geq \frac{\lambda(\eta_B)^2}{I_t(\eta_B)} \geq C 2^{-tm},
\]
where the last inequality holds by Lemma 5 for some constant \(C\) that is independent of \(m\).

Let \(\omega\) be such and let \(U\) be an open subset of \(W\). Since \((G, \lambda)\) is \(d\)-regular, there is a positive constant \(C'\) and some \(m_0\) such that if \(m \geq m_0\) then
\[
\#\{B \in B_m; B \subset U\} \geq C' 2^{dm}.
\]
Thus for \(m \geq m_0\),
\[
\mathcal{H}^t(U \cap E(\omega)) \geq \sum_{B \in B_m; B \subset U} \mathcal{H}^t(B \cap E(\omega)) \geq C C' 2^{(d-t)m},
\]
and letting \(m \to \infty\) shows that \(\mathcal{H}^t(U \cap E(\omega)) = \infty\). \(\square\)

4. Hausdorff Content and Energy of Rectangles in \(H\)

The purpose of this section is to estimate the Hausdorff content and energy of a rectangle \(R(x, r)\) in the Heisenberg group, up to multiplicative constants. Only upper bounds are provided, but it follows from (1) that they are the best possible ones. The multiplicative constants will mostly be implicit, using the following notation. If \(e_1\) and \(e_2\) are expressions depending on some parameters, then \(e_1 \lesssim e_2\) means that there is a constant \(C\) such that \(e_1 \leq C e_2\) for all parameter values. Often some of the parameters in \(e_1\) and \(e_2\) will be considered as constants—then \(C\) may depend on those parameters. For example, the implicit constants always depend on \(t\).

**Upper bound for the Hausdorff content of a rectangle.**

**Lemma 11.** Let \(R = \overline{R}(0, r)\). If \(r_1 \leq r_2\) then
\[
\mathcal{H}^t_\infty(R) \lesssim \begin{cases} r_2^t & \text{if } t \in [0, 2], \\ r_1^{t-2} r_2^2 & \text{if } t \in [2, 4], \end{cases}
\]
and if \(r_1 \geq r_2\) then
\[
\mathcal{H}^t_\infty(R) \lesssim \begin{cases} r_1^t & \text{if } t \in [0, 3], \\ r_1^{t-2} r_2^2 (t-3) & \text{if } t \in [3, 4], \end{cases}
\]
where the implicit constants depend on \(t\) but not on \(r\).

**Proof.** For \(t \geq 0\),
\[
\mathcal{H}^t_\infty(R) \leq |R|^t \lesssim \max \{r_1^t, r_2^t\} = \begin{cases} r_2^t & \text{if } r_1 \leq r_2, \\ r_1^t & \text{if } r_1 \geq r_2. \end{cases}
\]
The vertical segment \(S = \{0\} \times [-r_2^2, r_2^2]\) is \(r_1\)-dense in \(R\), and
\[
D = \left\{(0, kr_1^2); k = -\left\lfloor \frac{r_2^2}{r_1^2} \right\rfloor, \ldots, \left\lceil \frac{r_2^2}{r_1^2} \right\rceil \right\}.
\]
is $r_1$-dense in $S$ — thus $D$ is $2r_1$-dense in $R$. If $r_1 \leq r_2$ then $\# D \lesssim r_2^2/r_1^2$, and hence

$$\mathcal{H}^{t}_{\infty}(R) \lesssim \frac{r_2^2}{r_1^2}, \quad r_1^t \mathcal{H}^{t}_{\infty}(R) \lesssim r_1^{t-2} r_2^2.$$ 

It remains to show that $\mathcal{H}^{t}_{\infty}(R) \lesssim r_1^{6-t} r_2^{2(t-3)}$ for $r_1 \geq r_2$ and $t \in [3, 4]$. This is done by estimating the Hausdorff content of annuli of the form

$$A_p = \left\{ (x, y, z) : \left| (x^2 + y^2)^{1/2} - \rho \right| \leq \frac{r_2^2}{2\rho}, \quad |z| \leq r_2^2 \right\}.$$ 

Let

$$C_p = \{(x, y, 0) ; \, x^2 + y^2 = \rho^2 \}.$$ 

**Sublemma.** For $\rho \geq r_2$, the set $C_p$ is $\sqrt{7}r_2^2/\rho$-dense in $A_p$.

**Proof.** Let $p = (\rho, 0, 0)$ and let

$$S = \{(x, y, z) ; x \geq 0, \, z = 2\rho y\};$$

this is the part of the plane $H(p)$ defined in the introduction where the $x$-coordinate is non-negative. Let $q = (x, y, z)$ be a point in $A_p \cap S$. Then

$$x \leq \rho + \frac{r_2^2}{2\rho} \quad \text{and} \quad |y| = \frac{|z|}{2\rho} \leq \frac{r_2^2}{2\rho},$$

and

$$x^2 + y^2 \geq \left( \rho - \frac{r_2^2}{2\rho} \right)^2,$$

so that

$$x \geq \left( \rho - \frac{r_2^2}{2\rho} \right)^{1/2} \geq \left( \rho - \frac{r_2^2}{2\rho} \right)^{1/2} \geq \left( \rho - \frac{r_2^2}{2\rho} \right)^{1/2} \geq \rho - \frac{r_2^2}{\rho}$$

(the last inequality is proved by squaring both sides and using that $\rho \geq r_2$). Thus

$$d_H(p, q) = \left( (x - \rho)^2 + y^2 \right)^{1/2} \leq \sqrt{2} \max(|x - \rho|, |y|) \leq \frac{\sqrt{7}r_2^2}{\rho}.$$ 

Let $R_\alpha$ be the rotation by $\alpha$ around the vertical axis $\{0\} \times \mathbb{R}$. The statement follows since $d_H$ is invariant under $R_\alpha$ and

$$C_p = \bigcup_\alpha R_\alpha(p) \quad \text{and} \quad A_p = \bigcup_\alpha R_\alpha(A_p \cap S). \quad \square$$

**Sublemma.** Let $\varepsilon > 0$. The set

$$D_\varepsilon = \left\{ \left( \rho \cos \left( \frac{k\pi}{2\rho^2} \right), \rho \sin \left( \frac{k\pi}{2\rho^2} \right), 0 \right) : k = 0, \ldots, \left\lfloor \frac{4\pi\rho^2}{\varepsilon^2} \right\rfloor \right\}$$

is $\varepsilon$-dense in $C_p$.

**Proof.** The points $p = (\rho, 0, 0)$ and $q = (\rho \cos \alpha, \rho \sin \alpha, 0)$ satisfy

$$d_H(p, q) = \rho \left( (1 - \cos \alpha)^2 + \sin^2 \alpha \right)^{1/4} + (2 \sin \alpha)^2 \right)^{1/4} \leq 2^{3/4} \rho (1 - \cos \alpha)^{1/4} \leq \rho |2\alpha|^{1/2},$$

using that $\cos \alpha \geq 1 - \alpha^2/2$. The same bound holds for any pair of points $p, q$ on $C_p$ making an angle $\alpha$, and taking $\alpha = \varepsilon^2/2\rho^2$ gives $d_H(p, q) \leq \varepsilon$. \square
Proof of Lemma 11 (continued). Take \( \epsilon = r_2^2/\rho \) in the definition of \( D_\rho \). Then for \( \rho \geq r_2 \) the set \( D_\rho \) is \( 3r_2^2/\rho \)-dense in \( A_\rho \) and \( \#D_\rho \lesssim \rho^4/r_2^4 \), and thus
\[
\mathcal{H}^t(\rho) \lesssim \frac{\rho^4}{r_2^4} \left( \frac{r_2^2}{\rho} \right)^t = r_2^{2t-4} \rho^{4-t}.
\]
Let \( \rho_k = r_2\sqrt{k} \). Then
\[
\rho_{k+1} - \rho_k = r_2 \int_k^{k+1} \frac{1}{2\sqrt{u}} \, du \leq \frac{r_2^2}{2\sqrt{k}} = \frac{r_2^2}{2\rho_k},
\]
so that \( A_\rho \) and \( A_{\rho_{k+1}} \) overlap. Thus
\[
R \subset \mathbb{R}(0, (r_2, r_2)) \cup \bigcup_{k=1}^{[\frac{r_2}{r_1}]} A_{\rho_k}.
\]
It follows that
\[
\mathcal{H}^t(\rho) \lesssim r_2^4 + \sum_{k=1}^{[\frac{r_2}{r_1}]} r_2^{2t-4} \rho_k^{4-t} = r_2^4 \left( 1 + \sum_{k=1}^{[\frac{r_2}{r_1}]} k^{(4-t)/2} \right)
\]
\[
\lesssim r_2^4 \left( 1 + \left( \frac{r_1}{r_2} \right)^6 \right) \lesssim r_1^{6-t} r_2^{2(t-3)},
\]
using in the last step that \( (r_1/r_2)^{6-t} \geq 1 \).

\[ \square \]

Upper bound for the energy of a rectangle.

Lemma 12. Let \( R = \mathbb{R}(0, r) \). If \( r_1 \leq r_2 \) then
\[
I_t(R) \lesssim \begin{cases} 
    r_1^4 r_2^{4-t} & \text{if } t \in (0, 2), \\
    r_1^{6-t} r_2^2 & \text{if } t \in (2, 4), 
\end{cases}
\]
and if \( r_1 \geq r_2 \) then
\[
I_t(R) \lesssim \begin{cases} 
    r_1^{4-t} r_2^4 & \text{if } t \in (0, 3) \setminus \{1\}, \\
    r_1^{2-t} r_2^{2(6-t)} & \text{if } t \in (3, 4), 
\end{cases}
\]
where the implicit constants depend on \( t \) but not on \( r \).

Proof. Let
\[
R_t(p) = \int_R d_H(p, q)^{-t} \, d\lambda(q),
\]
so that
\[
I_t(R) = \int_R R_t(p) \, d\lambda(p).
\]
Since \( R, d_H \) and \( \lambda \) are invariant under rotation around the vertical axis, the integral defining \( R_t(p) \) does not depend on the angle of \( p \) in the horizontal plane. To estimate \( R_t(p) \) it is therefore enough to consider \( p \) of the form \( p = (\rho, 0, z_0) \). Assume that \( \rho \in [0, r_1] \) and \( z_0 \in [-r_2^2, r_2^2] \), so that \( p \in R \).

Let
\[
f_\rho(x, y, z) = \max \left( |x|, |y|, |z - 2\rho y|^{1/2} \right).
\]
Then for \( q = (x, y, z) \),
\[
d_H(p, q) = \left( ((x - \rho)^2 + y^2)^{1/4} + (z - z_0 - 2\rho y)^2 \right)^{1/4} \approx f_\rho(x - \rho, y, z - z_0).
\]
Define the Euclidean rectangles

\[ A = [-2r_1, 2r_1] \times [-2r_1, 2r_1] \times [-2r_2^2, 2r_2^2], \]
\[ A' = (\rho, 0, z_0) + A, \]

where + means Euclidean translation, and let \( B(a) = \{ q; f_\rho(q) \leq a \} \). Since \( R \subset A' \),

\[
R_\epsilon(p) \leq \int_{A'} d_H(p, q)^{-\epsilon} d\lambda(q) \lesssim \int_{A'} f_\rho(x - \rho, y, z - z_0)^{-\epsilon} d\lambda(x, y, z)
\]
(2)

\[
= \int_A f_\rho(x, y, z)^{-\epsilon} d\lambda(x, y, z) = \int_0^\infty \lambda \{ q \in A; f_\rho(q)^{-\epsilon} \geq \gamma \} \, d\gamma
\]

\[
= \int_0^\infty \lambda \left( A \cap B \left( \gamma^{-1/\epsilon} \right) \right) \, d\gamma \approx \int_0^\infty \lambda (A \cap B(a)) a^{-(\epsilon+1)} \, da.
\]

To estimate \( R_\epsilon(p) \) it is useful to have an upper bound for \( \lambda (A \cap B(a)) \).

The set \( B(a) \) is the intersection of the vertical cylinder \([-a, a] \times [-a, a] \times \mathbb{R} \) with the set of points having vertical Euclidean distance at most \( a^2 \) to the plane \( z = 2py \).

In particular, the projection of \( B(a) \) to the \( yz \)-plane is the intersection of the strips

\[ S_1 = \{(y, z); \; -a \leq y \leq a\}, \quad S_2 = \{(y, z); \; 2py - a^2 \leq z \leq 2py + a^2\}. \]

The projection of \( A \) to the \( yz \)-plane is the intersection of the strips

\[ S_3 = \{(y, z); \; -2r_1 \leq y \leq 2r_1\}, \quad S_4 = \{(y, z); \; -2r_2^2 \leq z \leq 2r_2^2\}. \]

It is used in the computations below that if \( u \neq -1 \) and \( 0 \leq v \leq w \leq \infty \) then

(3)

\[
\int_v^w a^u \, da \lesssim \max (v^{u+1}, w^{u+1}),
\]

with the convention that \( 1/0 = \infty \), and where the implicit constant depends on \( u \) but not on \( v, w \).

**Sublemma.** It holds that \( \lambda (A \cap B(a)) \lesssim \min (a^2, r_1^2a^2, r_1^2r_2^2) \).

**Proof.** The projection of \( A \cap B(a) \) to the \( yz \)-plane is contained in each of the parallelograms

\[ S_1 \cap S_2, \quad S_3 \cap S_1, \quad S_2 \cap S_3. \]

The first two have area \( 4a^3 \) and \( 16r_1r_2^2 \), respectively, and the third has vertices

\[ (-2r_1, -4\rho r_1, \pm a^2), \quad (2r_1, 4\rho r_1, \pm a^2), \]

and hence area \( 8r_1a^2 \). The extension of \( A \) in the \( x \)-direction is \( 4r_1 \) and the extension of \( B(a) \) in the \( x \)-direction is \( 2a \). Thus

\[
\lambda (A \cap B(a)) \lesssim \min (a^3, r_1^2a^2, r_1^2r_2^2) \cdot \min (r_1, a) \leq \min (a^4, r_1^2a^2, r_1^2r_2^2). \quad \Box
\]

**Proof of Lemma (continued).** The case \( r_1 \leq r_2 \). Let \( t \in (0, 4) \setminus \{2\} \). Using (2), the sublemma and (3),

\[
R_\epsilon(p) \lesssim \int_0^\infty \min (a^4, r_1^2a^2, r_1^2r_2^2) a^{-(\epsilon+1)} \, da
\]

\[
\lesssim \int_0^{r_1} a^{3-\epsilon} \, da + \int_{r_1}^{r_2} r_1^{2a^{1-\epsilon}} \, da + \int_{r_2}^\infty r_1^2r_2^2a^{-(\epsilon+1)} \, da
\]

\[
\lesssim \max (r_1^{4-\epsilon}, r_1^{2r_2^2-\epsilon}).
\]

It follows that

\[
I_\epsilon (R) \lesssim \lambda (R) \cdot \max (r_1^{4-\epsilon}, r_1^{2r_2^2-\epsilon}) = \max (r_1^{4-\epsilon}, r_1^{2r_2^2-\epsilon}) = \begin{cases} r_1^{4-\epsilon}, & \text{if } t \in (0, 2), \\ r_1^{2r_2^2-\epsilon}, & \text{if } t \in (2, 4). \end{cases}
\]

**Sublemma.** It holds that \( \lambda (A \cap B(a)) \lesssim \min (a^4, r_1^2a^3/p, r_1^2a, r_1^2r_2^2). \)
Proof. The projection of $A \cap B(a)$ to the $yz$-plane is contained in each of the parallelograms 

$$S_1 \cap S_2, \quad S_3 \cap S_4, \quad S_2 \cap S_4.$$ 

The first two have area $4a^3$ and $16r_1^2$, respectively, and the third has vertices 

$$\left(\frac{-2r_2^2 \pm a^2}{2\rho}, -2r_2^2\right), \quad \left(\frac{2r_2^2 \pm a^2}{2\rho}, 2r_2^2\right),$$

and hence area $4r_2^2a^2/\rho$. The extension of $A$ in the $x$-direction is $4r_1$ and the extension of $B(a)$ in the $x$-direction is $2a$. Thus 

$$\lambda(A \cap B(a)) \lesssim \min \left(\frac{a^3}{r_2}, r_1 r_2, \frac{r_2^3 a^2}{\rho}\right) \cdot \min (r_1, a)$$



\[\lesssim \min \left(\frac{a^4}{r_2}, \frac{r_2^3 a^3}{\rho}, r_1 r_2^2 a, r_1^2 r_2^2\right).\] 

Proof of Lemma 7.3 (continued). The case $r_1 \geq r_2$. Again, $R_t(p)$ can be estimated using (2), the sublemma and (3), but the estimate now depends on $\rho$. Let $t \in (0, 4) \setminus \{1, 3\}$. Then for every $\rho$, 

$$R_t(p) \lesssim \int_0^{r_1} \min \left(\frac{a^4}{r_2}, r_1 r_2^2 a, r_1^2 r_2^2\right) a^{-t} \, da$$

(4) 

$$\lesssim \int_0^{r_1} a^{-t} \, da + \int_{r_1}^{r_1 r_2^2} r_1 r_2^2 a^{-t} \, da + \int_{r_1 r_2^2}^{\infty} r_1 r_2^2 a^{-t} \, da$$

\[\lesssim \max \left(\frac{r_1}{r_1 r_2^2}, \frac{r_1^2}{r_1 r_2^2}\right) \quad \text{if } t \in (0, 1),
\]

\[\text{if } t \in (1, 4) \setminus \{3\}.
\]

For $t \in (1, 4) \setminus \{3\}$ and $\rho \geq \left(\frac{r_2^2}{r_1}\right)^{1/3}$, the estimate can be made sharper. For such $\rho$, the interval $[r_2^2/\rho, \sqrt{\rho r_1}]$ is non-empty, and when $a$ lies in this interval the minimum in the expression given by the sublemma is achieved by the second option. Thus 

$$R_t(p) \lesssim \int_0^{r_1} \min \left(\frac{a^4}{r_2}, \frac{r_2^3 a^3}{\rho}, r_1 r_2^2 a, r_1^2 r_2^2\right) a^{-t} \, da$$

$$\lesssim \int_0^{r_2^2/\rho} a^{-t} \, da + \int_{r_2^2/\rho}^{r_1 r_2^2} \frac{r_2^3 a^{-t}}{\rho} \, da + \int_{r_1 r_2^2}^{\infty} r_1 r_2^2 a^{-t} \, da \approx \int_{r_1 r_2^2}^{\infty} r_1 r_2^2 \, a^{-t} \, da$$

\[\lesssim \max \left(\frac{r_2^3}{r_1 r_2^2}, \frac{r_1}{r_1 r_2^2}\right) \quad \text{if } t \in (1, 3),
\]

\[\text{if } t \in (3, 4).
\]

Denoting the options in the maximum by $O_1, O_2, O_3$, 

$$\frac{O_1^2}{O_2^2} = \left(\frac{r_1}{r_2^2}\right)^{t-3}, \quad \frac{O_2^2}{O_3^2} = \left(\frac{r_1 r_2^2}{\rho}\right)^{t-4},$$

and the equality * follows using that the expressions in parentheses are greater than or equal to 1.

Let $g_t(p) = \sup_{\rho \leq 2} R_t(p)$ where $p = (\rho, 0, z_0)$. Then 

$$I_t(R) \lesssim r_2^2 \int_0^{r_1} g_t(p) \rho \, dp.$$ 

For $t \in (0, 1)$, the estimate (4) gives 

$$I_t(R) \lesssim r_2^2 \int_0^{r_1} r_1^{2-t} r_2^2 \rho \, dp \approx r_1^{1-t} r_2.$$
For \( t \in (1, 4) \setminus \{3\} \), the first part of the integral \( \int_0^t g_t(\rho) \rho \, d\rho \) is again estimated using (11). Let
\[
I_0 = r_2^2 \int_0^{r_2^2/(r_1)} \left( r_1 r_2 \right)^{(4-t)/3} \rho \, d\rho \approx r_1^{(2-t)/3} r_2^{2(11-t)/3}.
\]
Then for \( t \in (1, 3) \),
\[
I_t(R) \lessapprox I_0 + \frac{r_2^2}{2} \int_0^{r_1} r_1^{(3-t)/3} r_2^{2(3-t)/2} \rho \, d\rho \approx I_0 + r_1^{4-t} r_2^4 \approx r_1^{4-t} r_2^4,
\]
using that \( I_0 \approx r_1^{(2-t)/3} r_2^2 \). For \( t \in (3, 4) \),
\[
I_t(R) \lessapprox I_0 + \frac{r_2^2}{2} \int_0^{r_1} r_2^{2(4-t)} \rho^{13} \, d\rho \approx I_0 + r_1^{4-t} r_2^2 \approx r_1^{4-t} r_2^2,
\]
using that \( I_0 \approx r_1^{(2-t)/3} r_2^{2(1-t)/3} \). \( \square \)

5. Proof of Theorem 1

It is clear that \( \dim H E_\omega(\omega) \leq \dim H = 4 \). Let \( t \in (0, 4) \setminus \{1, 2, 3\} \).

For every \( \omega \) and \( n_0 \),
\[
E_\omega(\omega) \subset \bigcup_{n=n_0}^{\infty} R(\omega_n, r_n),
\]
and by Lemma 11 \( H^t_\infty(R(\omega_n, r_n)) \lessapprox \Phi^t(r_n) \). Thus for every \( n_0 \)
\[
H^t_\infty(E_\omega(\omega)) \lessapprox \sum_{n=n_0}^{\infty} \Phi^t(r_n).
\]

It follows that if \( \sum_n \Phi^t(r_n) < \infty \) then \( H^t_\infty(E_\omega(\omega)) = 0 \) so that \( \dim H E_\omega(\omega) \leq t \).

By Lemma 12
\[
\text{Cap}_t(R(0, r_n)) \geq \frac{\lambda(R(0, r_n))^2}{I_t(R(0, r_n))} \gtrapprox \Phi^t(r_n).
\]
Thus if \( \sum_n \Phi^t(r_n) = \infty \) then Theorem 2 with \( \lambda \) replaced by \( \lambda(W)^{-1} \lambda \) implies that \( \dim H E_\omega(\omega) \geq t \) for almost every \( \omega \).

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