Resummation of semiclassical short folded string

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ABSTRACT: We reconsider semiclassical quantization of folded string spinning in $AdS_3$ part of $AdS_5 \times S^5$ using integrability-based (algebraic curve) method. We focus on the “short string” (small spin $S$) limit with the angular momentum $J$ in $S^5$ scaled down according to $J = \rho \sqrt{S}$ in terms of the variables $J = J/\sqrt{x}$, $S = S/\sqrt{x}$. The semiclassical string energy in this particular scaling limit admits the double expansion $E = \sum_{n=0}^{\infty} \sum_{p=0}^{\infty} (\sqrt{x})^{1-n} a_{n,p}(\rho) S^{p+1/2}$. It behaves smoothly as $J \to 0$ and partially resums recent results by Gromov and Valatka. We explicitly compute various one-loop coefficients $a_{1,p}(\rho)$ by summing over the fluctuation frequencies for integrable perturbations around the classical solution. For the simple folded string, the result agrees with what could be derived exploiting a recent conjecture of Basso. However, the method can be extended to more general situations. As an example, we consider the $m$-folded string where Basso’s conjecture fails. For this classical solution, we present the exact values of $a_{1,0}(\rho)$ and $a_{1,1}(\rho)$ for $m = 2, 3, 4, 5$ and explain how to work out the general case.

KEYWORDS: AdS/CFT spectrum, folded string, algebraic curve approach

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1 Introduction and results

AdS/CFT duality [1] predicts the equivalence between the spectrum of the planar \( \mathcal{N} = 4 \) supersymmetric gauge theory and the spectrum of free closed quantum superstring propagating in \( AdS_5 \times S^5 \). States are labeled both in the gauge and string theory by the five conserved spins \( C = (S_{1,2}, J_{1,2,3}) \) corresponding to the bosonic subgroup \( SO(2,4) \times SO(6) \) of the symmetry group \( PSU(2,2|4) \) as well as by higher hidden charges. AdS/CFT correspondence can be expressed as the general relation

\[
E_{\text{gauge}}(\lambda, C) = E_{\text{string}}(\sqrt{\lambda}, C),
\]

where \( \frac{\sqrt{\lambda}}{2\pi} = \frac{E}{2\pi\alpha'} \) is the \( AdS_5 \times S^5 \) string tension in terms of the ’t Hooft coupling \( \lambda \).

In the strong-coupling regime \( (\lambda \gg 1) \), massive quantum string states probe a near-flat region of \( AdS_5 \times S^5 \) and thus should have \( E \sim \sqrt{\lambda} \) [2]. More generally, considerations based on solving the two dimensional marginality condition [3] perturbatively in \( \sqrt{\lambda} \ll 1 \) for fixed charges suggest [4] that (up to a possible shift of \( E \) by a constant)\(^1\)

\[
E^2 = 2N\sqrt{\lambda} + b_0 + \frac{b_1}{\sqrt{\lambda}} + \frac{b_2}{(\sqrt{\lambda})^2} + \ldots,
\]

\(^1\)Here we assume that the 1-loop string correction does not contain “non-analytic” terms, cf. [4]. This will be indeed so in the cases discussed below treated in the algebraic curve approach.
where $N$ is the flat-space level. As was argued in [4–6], one can attempt to find quantum string energies by starting with the semiclassical strings with fixed parameters $C = \frac{C}{\sqrt{\lambda}}$ and then take the short string limit $C \to 0$. Indeed, for quantum strings with fixed charges $C$ the limit $\sqrt{\lambda} \gg 1$ implies $C = \frac{C}{\sqrt{\lambda}} \to 0$. Assuming commutativity of the limits, that suggests the possibility to compute the subleading terms in the above expansion by using the semiclassical string theory methods. The semiclassical string expansion gives

$$
E = \sqrt{\lambda} E_0(C) + E_1(C) + \frac{1}{\sqrt{\lambda}} E_2(C) + \ldots
$$

Replacing $C$ by $\frac{C}{\sqrt{\lambda}}$ and re-expanding in large $\lambda$ for fixed $C$ one should find that $E$ takes the form consistent with (1.2):

$$
E = \sqrt{\lambda} \left( k_1 + \frac{k_2}{\sqrt{\lambda}} + \frac{k_3}{(\sqrt{\lambda})^2} + \ldots \right).
$$

This semiclassical approach was successfully applied to the case of the short string states representing members of the Konishi multiplet [4, 6–9], matching the results of the weak-coupling TBA approach extrapolated to strong coupling [10, 11]. Also, the structure of the one-loop semiclassical energy $E_1$ is very rich and can be exploited to make non trivial higher order predictions [9, 12].

In this paper, we reconsider the specific case of the folded string spinning in $AdS_5$ and rotating in $S^5$ [13] with the aim of giving some additional information about the one-loop correction $E_1$. For the quantization of this classical solution, the only charges appearing in (1.3) are the spin $S = \frac{S}{\sqrt{\lambda}}$ and the angular momentum $J = \frac{J}{\sqrt{\lambda}}$. The short string expansion of the classical term $E_0$ can be derived easily starting from the results in [13] and reads (at fixed $J$)

$$
E_0(S, J) = J + \frac{\sqrt{J^2 + 1}}{J} S - \frac{J^2 + 2}{4 \sqrt{J^2 + 1}} S^2 + \frac{2 J^6 + 4 J^2 + 4}{64 J^5} S^3 + \ldots
$$

Following the analysis of [9], it is interesting to re-expand all the coefficients of the powers of $S$ at $J \to 0$. We obtain

$$
E_0(S, J) = J + \left( \frac{1}{J} + \frac{J + 3}{2} + \frac{J^3}{8} + \frac{J^5}{16} + \cdots \right) S + \left( -\frac{1}{2 J^3} + \frac{1}{4 J} - \frac{3}{4} J + \frac{3}{4} J^3 + \cdots \right) S^2 + \left( \frac{1}{2 J^5} - \frac{1}{8 J} + \frac{11}{32} J - \frac{155}{256} J^3 + \cdots \right) S^3 + \ldots
$$

The main remark of this paper is that this expansion breaks down at $J \to 0$, but can be partially resummed by considering the scaling limit $S \to 0$ with fixed ratio

$$
\frac{J}{\sqrt{S}} = \rho.
$$

Indeed, in the regime (1.7) the most singular terms in (1.6) are all proportional to $\sqrt{S}$ with inverse powers of $\rho$ labeling the contributions coming from different powers of $1/J$. 

- 2 -
Similarly, the next-to-leading singular terms would be resummed by the $S^{3/2}$ contribution and so on. From the exact results of [13] it is a straightforward exercise to derive the following expansion

$$E_0(S, \rho) = \sqrt{\rho^2 + 2 \sqrt{S}} + \frac{2 \rho^2 + 3}{4 \sqrt{\rho^2 + 2}} S^{3/2} - \frac{4 \rho^6 + 20 \rho^4 + 34 \rho^2 + 21}{32 (\rho^2 + 2)^{3/2}} S^{5/2} + \ldots. \quad (1.8)$$

Consistency of (1.8) with (1.6) can be checked by expanding at large \( \rho \) the various coefficients

$$\sqrt{\rho^2 + 2} = \rho + \frac{1}{\rho} - \frac{1}{2 \rho^2} + \frac{1}{2 \rho^5} + \ldots, \quad (1.9)$$

$$\frac{2 \rho^2 + 3}{4 \sqrt{\rho^2 + 2}} = \frac{\rho}{2} + \frac{1}{4 \rho} - \frac{1}{8 \rho^5} + \ldots, \quad (1.10)$$

$$- \frac{4 \rho^6 + 20 \rho^4 + 34 \rho^2 + 21}{32 (\rho^2 + 2)^{3/2}} = -\frac{\rho^3}{8} - \frac{\rho}{4} - \frac{1}{8 \rho^5} + \frac{1}{32 \rho^3} + \ldots, \quad (1.11)$$

and clearly (1.8) includes an infinite number of terms in (1.6) with arbitrary high powers of \( S \).

A very natural question is whether the limit (1.7) is also able to resum the one-loop contribution \( E_1(S, J) \) in a similar way. In the recent paper [9], Gromov and Valatka evaluated \( E_1 \) in the double limit \( S \to 0 \) followed by \( J \to 0 \). The computation is performed working in the algebraic curve framework developed in [14–16]. An important result of [9] is the following structure of the result

$$E_1(S, J) = \sum_{n=1}^{\infty} S^n \sum_{p=0}^{\infty} \frac{c_{n,p}}{J^{2n-1-2p}}, \quad (1.12)$$

where the coefficients \( c_{n,p} \) are rational combinations of zeta numbers \( \zeta(n) \) and are computed at a certain fixed order in the \( S \) expansion. In the scaling regime (1.7), we then obtain an expansion in half-integer powers of \( S \)

$$E_1(S, \rho) = \sum_{p=0}^{\infty} a_{1,p}(\rho) S^{p+1/2}, \quad (1.13)$$

where the large \( \rho \) expansion of \( a_{1,p}(\rho) \) generates all the coefficients \( c_{n,p} \) with varying \( n \) according to the relation

$$a_{1,p}(\rho) = \sum_{n=1}^{\infty} \frac{c_{n,p}}{\rho^{2n-1-2p}}. \quad (1.14)$$

In more general terms, we expect the same structure to hold for all contributions \( E_n \), i.e.

$$E_n(S, \rho) = \sum_{p=0}^{\infty} a_{n,p}(\rho) S^{p+1/2}. \quad (1.15)$$

\^2 Notice that (1.8) is not a trivial consequence of (1.5), but is instead a genuine (partial) resummation thereof.
The explicit resummation can be performed by exploiting the recent Basso’s conjecture [12]. According to this conjecture, the squared energy admits the expansion (compatible with semiclassical calculations)

$$E^2 = J^2 + \left( A_1 \sqrt{\lambda} + A_2 + \frac{A_3}{\sqrt{\lambda}} + \cdots \right) S + \left( B_1 + \frac{B_2}{\sqrt{\lambda}} + \frac{B_3}{\lambda} + \cdots \right) S^2 + \left( \frac{C_1}{\sqrt{\lambda}} + \frac{C_2}{\lambda} + \frac{C_3}{\lambda^{3/2}} + \cdots \right) S^3 + \cdots \quad (1.16)$$

where the following exact formula for the constants $A_i$ is claimed

$$A_1 \sqrt{\lambda} + A_2 + \frac{A_3}{\sqrt{\lambda}} + \cdots = 2 \sqrt{\lambda} Y_j(\sqrt{\lambda}), \quad Y_j(x) = \frac{d}{dx} \log I_j(x). \quad (1.17)$$

Expanding at large $\lambda$, we find the first values

$$A_1 = 2, \quad A_2 = -1, \quad A_3 = \frac{1}{4}, \quad A_4 = \frac{1}{4}, \quad A_5 = -\frac{1}{4} J^4 + \frac{13}{8} J^2 - \frac{25}{64}, \quad A_6 = -J^4 + \frac{7}{4} J^2 - \frac{16}{15}, \quad A_7 = \frac{4}{8} - \frac{115}{32} J^4 + \frac{1187}{128} - \frac{1073}{512}. \quad (1.18)$$

Also, we know that $B_1 = \frac{3}{2}$ and $B_2 = \frac{3}{8} - 3 \zeta(3)$ [9]. Setting in (1.16) the scaling relation (1.7) and comparing with (1.3) and (1.15) and find immediately the following results

$$a_{1,0}(\rho) = -\frac{1}{2 \sqrt{\rho^2 + 2}}, \quad (1.19)$$

$$a_{1,1}(\rho) = \frac{8 \rho^4 + 23 \rho^2 + 12}{16 (\rho^2 + 2)^{3/2}} - \frac{3 \zeta(3)}{2 \sqrt{\rho^2 + 2}}, \quad (1.20)$$

$$a_{2,0}(\rho) = -\frac{\rho^2 + 3}{8 (\rho^2 + 2)^{3/2}}, \quad (1.21)$$

$$a_{3,0}(\rho) = \frac{2 \rho^4 + 9 \rho^2 + 11}{16 (\rho^2 + 2)^{5/2}}, \quad (1.22)$$

and so on. Notice also that the scaling (1.7) can be continued to the related regime where $\mathcal{J}/S = J/S$ is kept fixed as in [7]. This continuation cannot clearly be done at the level of (1.12), at least without resorting to Basso’s conjecture.

The proposed resummation is based on Basso’s results. In order to be able to treat more general cases, like for instance the $m$-folded string [17] where the conjecture is not valid, we want to show how to compute the functions $a_{1,n}(\rho)$ from the one-loop algebraic curve calculation in the regime (1.7). This computation is interesting in itself since it shows how to compute the short string expansion of a non trivial (elliptic) semiclassical solution by explicitly summation over the frequencies. This raises various technical points (missing poles, summation shifts) that could be important for the extension of such analysis to

\footnote{We write them explicitly in order to emphasize the fact that the constants $A_i$ are polynomials in $J^2$. In general, also the constants $B_i, C_i$ etc will be dependent on $J^2$.}

\footnote{See also [18] for a similar approach in a simpler case.
other cases like ABJM or rigid circular strings. In particular, we shall derive from a direct computation the functions $a_{1,0}(\rho)$ and $a_{1,1}(\rho)$ finding perfect agreement with the above results. We shall also provide the following conjecture for $a_{1,2}(\rho)$

$$a_{1,2}(\rho) = \frac{64c - \rho^2 (42 \rho^4 + 212 \rho^2 + 335)}{64 (\rho^2 + 2)^{5/2}} + \frac{3 (4 \rho^4 + 15 \rho^2 + 13)}{8 (\rho^2 + 2)^{3/2}} \zeta(3) + \frac{15 (\rho^2 + 1)}{8 \sqrt{\rho^2 + 2}} \zeta(5),$$

(1.23)

depending on a single unfixed coefficient $c$.

In the case of the $m$-folded string, Gromov and Valatka recently computed the expansion (1.12) for various values of the folding number $m$. Their results are not compatible with any trivial modification of Basso’s conjecture whose extension still has to be found. Our method can treat this classical solution with minor effort. In particular, we are able to perform the resummation leading to (1.13) and obtain the following leading one loop results

$$a_{1,0}^{m=2}(\rho) = \frac{16 \rho^2 + 5}{6 \sqrt{\rho^2 + 2}} + \frac{1}{3} \sqrt{\rho^2 + 8} - 3 \rho,$$

$$a_{1,0}^{m=3}(\rho) = \frac{243 \rho^2}{40 \sqrt{\rho^2 + 2}} + \frac{\sqrt{\rho^2 + 18}}{8} + \frac{2}{5} \sqrt{4 \rho^2 + 18} + \frac{53}{20 \sqrt{\rho^2 + 2}} - 7 \rho,$$

$$a_{1,0}^{m=4}(\rho) = \frac{1048 \rho^2}{105 \sqrt{\rho^2 + 2}} + \frac{\sqrt{\rho^2 + 8}}{3} + \frac{\sqrt{\rho^2 + 32}}{15} + \frac{3}{7} \sqrt{9 \rho^2 + 32} + \frac{1007}{210 \sqrt{\rho^2 + 2}} - \frac{35 \rho}{3},$$

$$a_{1,0}^{m=5}(\rho) = \frac{14375 \rho^2}{1008 \sqrt{\rho^2 + 2}} + \frac{\sqrt{\rho^2 + 50}}{24} + \frac{2}{21} \sqrt{4 \rho^2 + 50} + \frac{3}{16} \sqrt{9 \rho^2 + 50} + \frac{4}{9} \sqrt{16 \rho^2 + 50} + \frac{3623}{504 \sqrt{\rho^2 + 2}} - \frac{101 \rho}{6}.$$

(1.24)

In Section (7) we shall explain how to work out a general $m$ without difficulty and provide various additional higher order results. The expressions (1.24) have increasing complexity as $m$ becomes larger and suggest that the above modification is indeed non trivial. Hopefully, they will be a useful constraint in fixing any proposed generalization of Basso’s conjecture.

2 The folded string in the Algebraic Curve framework

The general construction of the algebraic curve for the $AdS_5 \times S^5$ superstring is discussed in [14, 16]. Here, we summarize the main results in the specific case of the folded string under consideration.

2.1 Classical data

The monodromy matrix of the Lax connection for the integrable dynamics of the $AdS_5 \times S^5$ superstring has eigenvalues

$$\{e^{i\tilde{p}_1}, e^{i\tilde{p}_2}, e^{i\tilde{p}_3}, e^{i\tilde{p}_4}, e^{i\tilde{p}_5}, e^{i\tilde{p}_6}, e^{i\tilde{p}_7}, e^{i\tilde{p}_8}\}.$$  

(2.1)
The eigenvalues are roots of the characteristic polynomial and define an 8-sheeted Riemann surface. The classical algebraic curve has macroscopic cuts connecting various pairs of sheets. They impose suitable discontinuities on the quasi-momenta $\tilde{p}_{1,2,3,4}, \tilde{p}_{1,2,3,4}$. In addition, one has to take into account Virasoro constraints and asymptotic properties that are fully discussed in [14, 16] and that we shall not repeat here.

For the folded string the classical solution is associated with an algebraic curve with two symmetric cuts along the real axis

$$(-b, -a) \cup (a, b), \quad 1 < a < b. \quad (2.2)$$

The branch points are function of the charges

$$(S, J, E) = \frac{1}{\sqrt{\lambda}}(S, J, E), \quad (2.3)$$

according to

$$S = \frac{1}{2\pi} \frac{ab + 1}{ab} \left[ b E \left( 1 - \frac{a^2}{b^2} \right) - a K \left( 1 - \frac{a^2}{b^2} \right) \right],$$

$$J = \frac{1}{\pi} \frac{1}{b} \sqrt{(a^2 - 1)(b^2 - 1)} K \left( 1 - \frac{a^2}{b^2} \right), \quad (2.4)$$

$$E = \frac{1}{2\pi} \frac{ab - 1}{ab} \left[ b E \left( 1 - \frac{a^2}{b^2} \right) + a K \left( 1 - \frac{a^2}{b^2} \right) \right].$$

The complete set of quasi-momenta associated with this curve can be found in [7]. We do not repeat their expressions and simply follow the notation of that paper.

### 2.2 One-loop correction to the energy

The one-loop correction is computed in terms of two ingredients to be determined for each physical polarization $I = (\alpha, \beta)$ taking the possible $8 + 8$ bosonic and fermionic values

$$S^5 : (\alpha, \beta) = (\bar{1}, \bar{3}), (\bar{1}, \bar{4}), (\bar{2}, \bar{3}), (\bar{2}, \bar{4}), \quad (2.5)$$

AdS$_5$ : $(\alpha, \beta) = (\bar{1}, \bar{3}), (\bar{1}, \bar{4}), (\bar{2}, \bar{3}), (\bar{2}, \bar{4}), \quad (2.6)$

Fermions : $(\alpha, \beta) = (\bar{1}, \bar{3}), (\bar{1}, \bar{4}), (\bar{2}, \bar{3}), (\bar{2}, \bar{4}), \quad (2.7)$

$$(\bar{1}, \bar{3}), (\bar{1}, \bar{4}), (\bar{2}, \bar{3}), (\bar{2}, \bar{4}).$$

The first ingredient is the on-shell pole associated with mode number $n$. It is computed by solving the equation

$$p_\alpha(x_n^{(I)}) - p_\beta(x_n^{(I)}) = 2n\pi. \quad (2.8)$$

The meaning of $x_n^{(I)}$ is that of an extra quantum pole to be added to the classical cut representing the continuum limit of a dense distribution of classical Bethe roots for the integrable classical solution of the superstring equations of motion.

The second ingredient is the off-shell fluctuation energy $\Omega^{(I)}(x)$. This is a quantity that provides the one-loop correction to the energy from polarization $(I)$ once $x$ is replaced by its on-shell value

$$\omega_n^{(I)} = \Omega^{(I)}(x_n^{(I)}). \quad (2.9)$$
As discussed in full details in [16], the folded string has enough symmetries to allow all off-shell fluctuation energies to be written in terms of only the two basic frequencies

\[ \Omega^{(2)}(x) = \Omega^{\bar{2}}(x) = \Omega^{\bar{3}}(x) = \Omega^{\bar{4}}(x) = \Omega^{-\frac{1}{2}} \left( 1 + \frac{f(x)}{x^2 - 1} \right), \quad (2.10) \]

where

\[ f(x) = \sqrt{x - a} \sqrt{x + a} + \sqrt{x - b} \sqrt{x + b} \quad (2.12) \]

All other frequencies are given by the following expressions

\[ \Omega^{(1)}(x) = \Omega^{-\frac{1}{2}}(x) = -\Omega_A \left( \frac{1}{x} \right) - 2, \quad (2.13) \]

\[ \Omega^{(2)}(x) = \Omega^{\bar{3}}(x) = \Omega^{-\frac{1}{2}}(x) = -\frac{1}{2} \Omega_A(x) - \frac{1}{2} \Omega_A \left( \frac{1}{x} \right) - 1, \quad (2.14) \]

\[ \Omega^{(3)}(x) = \Omega^{\bar{2}}(x) = \Omega^{\bar{4}}(x) = \Omega^{\bar{3}}(x) = \frac{1}{2} \Omega_A(x) + \frac{1}{2} \Omega_\mathbf{F}(x), \quad (2.15) \]

\[ \Omega^{(4)}(x) = \Omega^{\bar{3}}(x) = \Omega^{\bar{4}}(x) = \Omega^{\bar{3}}(x) = \frac{1}{2} \Omega_\mathbf{F}(x) - \frac{1}{2} \Omega_A \left( \frac{1}{x} \right) - 1. \quad (2.16) \]

The one-loop shift of the energy is given in terms of the \( \omega_n^{(l)} \) by the expression

\[ \mathcal{E}_1 = \frac{1}{2} \sum_{n,l} (-1)^{F_l} \omega_n^{(l)}, \quad (2.17) \]

where \( F_l = 1 \) for bosonic and \(-1\) for fermionic polarizations. In principle, this sum is ill-defined and could require separate non trivial shifts in the various terms as discussed in [14]. These ambiguities can be bypassed when there is a definite BMN limit [19] of the classical solution. Amazingly, the sum can also be written as a contour integral as in [7] apparently solving automatically these problems. Here, we shall stick to the above representation as a sum over frequencies in order to provide all the details for a would-be interesting comparison with world-sheet calculations. This comparison is however beyond the scope of this work and shall not be addressed.

3 On-shell frequencies for physical polarizations: Leading order

The explicit calculation of frequencies requires the solution of (2.8). We shall find it perturbatively in the short string limit \( S \to 0 \) in the regime (1.7). This limit can be implemented by using the following parametrization of the branch points

\[ a = 1 + st + \frac{st^2}{2} + \frac{s^3 \left( -4t^3 - 4t^2 + 1 \right)}{16t + 16} - \frac{s^4 \left( t^3 (2t^3 + 6t^2 + 4t - 1) \right)}{16(t + 1)}, \quad (3.1) \]

\[ b = 1 + s(t + 2) + s^2 \left( \frac{t^2}{2} + 2t + 2 \right) + \frac{s^3 \left( 4t^3 + 20t^2 + 31t + 14 \right)}{16t + 16} + \frac{s^4 \left( (t + 2)^2 (2t^3 + 6t^2 + 4t + 1) \right)}{16(t + 1)} + \cdots, \]
where $s$ is the expansion parameter and $t$ is a real constant. The expansion is built in order to have

$$S = \frac{1}{2} s^2, \quad \frac{J}{\sqrt{S}} = \rho = \sqrt{2t(t+2)}.$$  \hspace{1cm} (3.2)

Also, we can compute the expansion of the classical energy

$$\frac{E}{\sqrt{2S}} = t + 1 + \frac{4t^2 + 8t + 3}{16(t+1)} s^2 + \ldots$$  \hspace{1cm} (3.3)

In particular, for $t \to 0$ we recover the well known expansion of the classical energy of the $J = 0$ folded string

$$E = \sqrt{2S} \left( 1 + \frac{3}{8} \, S + \ldots \right). \hspace{1cm} (3.4)$$

The detailed analysis of the on-shell frequencies leads to the following results. We report the leading order expression of the pole position for $\mathcal{O}(s)$ (there is $\mathcal{O}(s^3)$ symmetry) and the full $\mathcal{O}(s)$ contribution to the on-shell energy correction obtained by computing for each polarization $I$ the quantity (2.9) \footnote{Notice that it requires the $\mathcal{O}(s^3)$ expansion of the pole that we do not write for brevity.}. 

- \quad (2\,3) = (2\,4) = (1\,3) = (1\,4)

There is a pole for each $n \geq 1$. It reads

$$x_n^{(S)} = 1 + \frac{\sqrt{t(t+2)}}{n} s + \ldots, \hspace{1cm} (3.5)$$

$$\omega_n^{(S)} = \frac{n}{s(t+1)} - \frac{\sqrt{t(t+2)}}{t+1} + \frac{s \left( n^3(4t^2 + 8t + 5) + 8t(t+1)^2(t+2) \right)}{16n(t+1)^3} + \ldots.$$  \hspace{1cm} (3.6)

For $n = 1$ there are two solutions discussed in the next section. For $n \geq 2$ there is a pole that reads

$$x_n^{(A)} = 1 + \frac{\sqrt{(n-1)^2 t(t+2) + 1 - t - 1}}{n(n-2)} s + \ldots, \hspace{1cm} (3.6)$$

$$\omega_n^{(A)} = \frac{n}{s(t+1)} - 1 + \frac{s \left[ n^3(4t(t+2) + 5) + n^2(4t(t+2)(2t(t+2) + 1) - 6) - 8nt(t+2)(t+1)^2 - 8(t+1)^4 \right]}{16(n-2)n(t+1)^3} - \frac{(n-1) \sqrt{(n-1)^2 t(t+2) + 1}}{2(n-2)n} + \ldots.$$  \hspace{1cm} (3.6)
There is a pole for each \( n \geq 1 \). It reads
\[
x_n^{(1)} = 1 + \frac{\sqrt{(n+1)^2 t (t+2) + 1} + t + 1}{n(n+2)} s + \cdots ,
\]
\[
\omega_n^{(1)} = \frac{n}{s(t+1)} - 1 + s \left[ \frac{(n+1)\sqrt{(n+1)^2 t (t+2) + 1}}{2n^2 + 4n} + \frac{1}{16} \left( 8(n^2 + n - 1) \right) + \frac{n}{(n+2)n} + \frac{4(n+1)}{t+1} + 4 \left( \frac{1}{n+2} + \frac{1}{n-2} \right) \right] + \cdots .
\]

- \((\bar{1} \, \bar{3}) = (\bar{2} \, \bar{4})\)

For large enough \( t \) there is a pole for \( n \geq 2 \). It reads
\[
x_n^{(2)} = 1 + \frac{\sqrt{(n^2 - 1) t (t+2) - 1}}{n \sqrt{n^2 - 1}} s + \cdots ,
\]
\[
\omega_n^{(2)} = \frac{n}{s(t+1)} - 1 + s \left( \frac{n^2 (4t^2 + 8t + 5) + 8(t+1)^4}{16n(t+1)^3} \right) + \cdots .
\]

In the next section, we shall discuss what happens as \( \rho \) decreases and eventually goes to zero.

- \((\bar{2} \, \bar{3}) = (\bar{2} \, \bar{4}) = (\bar{3} \, \bar{1}) = (\bar{3} \, \bar{2})\)

For large enough \( t \) there is a pole for \( n \geq 2 \). It reads
\[
x_n^{(3)} = 1 + \frac{(2n-1)\sqrt{t (t+2) - 1}}{2n(n-1)} s + \cdots ,
\]
\[
\omega_n^{(3)} = \frac{n}{s(t+1)} + \left( \frac{-\sqrt{t (t+2)}}{2(t+1)} - \frac{1}{2} \right) + s \left( \frac{-n^2 + n - 1}{4(n-1)n} \right) + \frac{1}{16(n-1)n(t+1)^2} \left( n^3 (4t(t+2) + 5) + n^2 (2t(t+2)(2t(t+2) + 1) - 3) + 2n(t+1)^2 (2t(t+2) + 1) - 2(t+1)^2 (2t(t+2) + 1) \right) + \cdots .
\]

In the next section, we shall discuss what happens as \( \rho \) decreases and eventually goes to zero.

- \((\bar{1} \, \bar{3}) = (\bar{1} \, \bar{4}) = (\bar{4} \, \bar{1}) = (\bar{4} \, \bar{2})\)

For large enough \( t \) there is a pole for \( n \geq 2 \). It reads
\[
x_n^{(4)} = 1 + \frac{(2n+1)\sqrt{t (t+2) + 1}}{2n(n+1)} s + \cdots ,
\]
\[
\omega_n^{(4)} = \frac{n}{s(t+1)} + \left( \frac{-\sqrt{t (t+2)}}{2(t+1)} - \frac{1}{2} \right) + s \left( \frac{\sqrt{t (t+2)}}{4n(n+1)} + \frac{1}{4} \sqrt{t (t+2)} \right) + \frac{1}{16n(n+1)(t+1)^3} \left( n^3 (4t(t+2) + 5) - n^2 (2t(t+2)(2t(t+2) + 1) - 3) + 2n(t+1)^2 (2t(t+2) + 1) + 2(t+1)^2 (2t(t+2) + 1) \right) + \cdots .
\]
In the next section, we shall discuss what happens as \( \rho \) decreases and eventually goes to zero.

### 3.1 Missing modes

As we have seen, the basic equation (2.8) admits a solution for all \( n \) except special values. We shall refer to these special values as missing modes. In general, one can locate the missing modes by taking a near BMN limit that in our case means large \( \rho \) (or, what is the same, large \( t \)). In this regime, there is only a fixed and small number of missing poles while \( S \rightarrow 0 \). Thus, it is easy to identify them and compute their contribution. Once this is done, it is possible to take \( \rho \rightarrow 0 \). In this process, additional missing solutions to (2.8) do appear. They move through the cuts and end on unphysical polarization planes as explained in [16]. Their positions become generally complex. Nevertheless, their contribution to the energy is continuous through the crossing. So, the trick is to compute the full one-loop correction at sufficiently large \( \rho \) and then continue the result to small \( \rho \). A detailed analysis of the \( n > 0 \) missing modes at large \( \rho \) shows that

a) The bosonic polarization (\( A \)) has missing mode \( n = 1 \). It corresponds to the branch point. The contribution can be smoothly computed as \( \omega_{A,1} \) although the expansion of the pole is singular for \( n = 1 \). The same is obtained by computing \( \Omega^{(A)}(a) = \Omega^{(A)}(b) \). Notice that the multiplicity of this pole is one since a small deformation removes the pole from one of the two cut endpoints.

b) The bosonic polarization (\( 2 \)) has missing mode \( n = 1 \). It can be computed by analytic continuation on the unphysical \((\bar{1}2)\) sheet. If this is done, one has to take into account that 2+2 poles are missing (the physical and unphysical ones at \( n = \pm 1 \)). They appear on the unphysical \((\bar{1}2)\) sheet at fixed positions \( x = 0, \pm i, \infty \). Nevertheless, again it is possible to smoothly compute \( \omega^{(2)}_1 \).

c) The fermionic polarization (\( 3 \)) (associated with \((\bar{2}3)\)) has missing mode \( n = 1 \). These missing modes (\( n = 1 \) in pair with \( n = -1 \)) can be found on the unphysical sheet \((\bar{3}3)\) at fixed positions \( x = 0, \infty \). Their contribution requires the analytic continuation \( \Omega^{(3)} \) of

\[
\Omega^{(3)}(0) = \frac{1}{s(t+1)} - \frac{\sqrt{t(t+2)}}{t+1} + s \frac{8t^4 + 32t^3 + 48t^2 + 32t + 9}{16(t+1)^3} + \cdots , \quad (3.12)
\]

\[
\Omega^{(3)}(\infty) = \frac{1}{s(t+1)} - 1 + s \frac{8t^4 + 32t^3 + 48t^2 + 32t + 9}{16(t+1)^3} + \cdots . \quad (3.13)
\]
4 Sum over frequencies and $\mathcal{O}(\sqrt{S})$ correction

The one loop correction is

$$
\mathcal{E}_1 = \sum_{n=2}^{\infty} (4 \omega_n^{(S)} + \omega_n^{(1)} + 2 \omega_n^{(2)} + \omega_n^{(4)}) - 4 \omega_n^{(3)} - 4 \omega_n^{(4)} + 4 \omega_1^{(S)} + 2 \omega_1^{(2)} + \omega_1^{(4)} - 4 \omega_1^{(3)} + \frac{1}{2} \mathcal{O}^{(3)}(0) + \mathcal{O}^{(5)}(\infty).
$$

(4.1)

Using the previous results, we see that the $\sim 1/s$ and constant term $\sim s^0$ cancel. Instead, the $\mathcal{O}(s)$ contribution is non trivial and reads (here, LO means that we have not computed $\mathcal{O}(s^2)$ terms)

$$
\mathcal{E}^{1\text{LO}}_s = \sum_{n=2}^{\infty} \left[ \frac{n^2 + 6t^2 + 12t + 2}{(n-2)n(n+1)(n+2)(n-1)(t+1)} - \frac{(n-1)\sqrt{(n-1)^2 t^2 + 2(n-1)^2 t + 1}}{2(n-2)n} \right.
\left. + \frac{(n+1)\sqrt{(n+1)^2 t^2 + 2(n+1)^2 t + 1}}{2n(n+2)} + \frac{2\sqrt{t+2}}{n(n+1)(n-1)} \right] +
\frac{10t^2 + 20t + 1}{12(t+1)} - \frac{3}{2}\sqrt{t+2} + \frac{1}{3}\sqrt{4t+2} + 1 + \mathcal{O}(s).
$$

(4.2)

As a first check of this result we can take the $t \to 0$ limit. Then, we find

$$
\mathcal{E}^{1\text{LO}}_s = -\sum_{n=2}^{\infty} \frac{n^2 + n + 1}{(n-1)n(n+1)(n+2)} + \frac{5}{12} = -\frac{2}{3} + \frac{5}{12} = -\frac{1}{4}.
$$

(4.3)

in agreement with [7].

The evaluation of the above sum for generic $t$ is apparently hopeless. However, due to the telescopic property of the most complicated terms of the summand (those with $n$ inside the square roots), we can obtain the exact sum after some simple manipulations.

The result is remarkably simple and reads

$$
\mathcal{E}^{1\text{LO}}_s = -\frac{1}{4(t+1)}.
$$

(4.5)

In terms of the scaled variable $S$, $\mathcal{J}$ and trading $t$ for $\rho$ using the second equation in (3.2) we finally find

$$
\mathcal{E}^{1\text{LO}}_s = \frac{1}{2\sqrt{\mathcal{J}^2 + 2}}
$$

(4.6)

\[\text{In some details, we are dealing with a sum of the form}

$$
\sum_{n=a}^{\infty} (f_n - f_{n-a}) = \lim_{N \to \infty} \sum_{n=a}^{N} (f_n - f_{n-a}) = -\sum_{n=0}^{a-1} f_n + a f_\infty,
$$

(4.4)

\[\text{that reduces to a finite sum of a terms plus a boundary contribution.}\]
5 \( \mathcal{O}(S) \) correction

In general, we can expand the various on-shell energies \( \omega_n^{(I)} \) in powers of \( s \)

\[
\omega_n^{(I)} = \omega_{n-1}^{(I)} \frac{1}{s} + \omega_{n,0}^{(I)} + \omega_{n,1}^{(I)} s + \omega_{n,2}^{(I)} s^2 + \cdots .
\]  

(5.1)

The \( \mathcal{O}(S) \) correction is associated to the contributions \( \omega_n^{(I)} \). This correction is rather simple and reads

\[
\begin{align*}
\omega_{n,2}^{(S)} &= -\sqrt{\frac{t(t+2)}{16(t+1)^3}} (4t^2 + 8t + 5), \\
\omega_{n,2}^{(A)} &= 0, \\
\omega_{n,2}^{(1)} &= 0, \\
\omega_{n,2}^{(2)} &= 0, \\
\omega_{n,2}^{(3)} &= -\sqrt{\frac{t(t+2)}{32(t+1)^3}} (4t^2 + 8t + 5), \\
\omega_{n,2}^{(4)} &= -\sqrt{\frac{t(t+2)}{32(t+1)^3}} (4t^2 + 8t + 5).
\end{align*}
\]  

(5.2)  

(5.3)  

(5.4)  

(5.5)  

(5.6)  

(5.7)

Adding the missing mode contribution, the full sum vanishes. This means that the one-loop correction to the energy has no \( \mathcal{O}(s^2) \) term, i.e. no \( \mathcal{O}(S) \) term. Of course, this is consistent with the general expansion (1.13).

6 \( \mathcal{O}(S^{3/2}) \) correction and a conjecture for the \( \mathcal{O}(S^{5/2}) \) contribution

The \( \mathcal{O}(S^{3/2}) \) correction is associated with the \( \omega_n^{(I)} \) terms. They are rather involved and we shall not report them explicitly. Repeating the same kind of analysis of the LO correction\(^7\) we are able to resum them and the final result is

\[
E_1 = a_{1,0}(\rho) S^{1/2} + a_{1,1}(\rho) S^{3/2} + a_{1,2}(\rho) S^{5/2} + \cdots .
\]  

(6.1)

with

\[
\begin{align*}
a_{1,0}(\rho) &= -\frac{1}{2 \sqrt{\rho^2 + 2}}, \\
a_{1,1}(\rho) &= \frac{8\rho^4 + 23\rho^2 + 12}{16(\rho^2 + 2)^{3/2}} - \frac{3\zeta(3)}{2\sqrt{\rho^2 + 2}}.
\end{align*}
\]  

(6.2)

Thus, we have recovered with an explicit calculations the results that follow upon using Basso’s conjecture.

\(^7\)There is only one remarkable technical point. The \( n = 1 \) bosonic missing mode with polarization (23) cannot be computed by taking \( n \to 1 \) in the general \( n \geq 2 \) expression. Instead, one has to anlytically continue in the unphysical plane (21) as explained.
6.1 Matching the Gromov-Valatka expansion

As a further check, we can expand at large $\rho$ finding

$$a_{1,0}(\rho) = -\frac{1}{2\rho} + \frac{1}{2\rho^3} - \frac{3}{4\rho^5} + \frac{5}{4\rho^7} - \frac{35}{16\rho^9} + \cdots,$$

$$a_{1,1}(\rho) = \frac{\rho}{2} + \frac{3\zeta(3)}{2} \frac{1}{\rho} + \frac{3\zeta(3)}{2} \frac{1}{\rho^3} + \frac{9\zeta(2)}{4} \frac{1}{\rho^5} + \frac{15\zeta(3)}{4} \frac{1}{\rho^7} + \frac{5}{32} + \frac{21}{128} \frac{105\zeta(3)}{16} + \cdots,$$

in full agreement with (and extending !) equation (B.5) of [9]. The remarkably simple structure of these exact results suggest the following very reasonable conjecture for the function $a_{1,2}(\rho)$

$$a_{1,2}(\rho) = \frac{a_0 + a_1 \rho^2 + a_2 \rho^4 + a_3 \rho^6}{(\rho^2 + 2)^{5/2}} + \frac{\rho_0 + \rho_1 \rho^2 + \rho_2 \rho^4}{(\rho^2 + 2)^{3/2}} \zeta(3) + \frac{\gamma_0 + \gamma_1 \rho^2}{\sqrt{\rho^2 + 2}} \zeta(5).$$

Matching the Gromov-Valatka expansion we find immediately

$$a_{1,2}(\rho) = \frac{64c - \rho^2 (42 \rho^4 + 212 \rho^2 + 335)}{64 (\rho^2 + 2)^{5/2}} + \frac{3(4 \rho^4 + 15 \rho^2 + 13)}{8 (\rho^2 + 2)^{3/2}} \zeta(3) + \frac{15 (\rho^2 + 1)}{8 \sqrt{\rho^2 + 2}} \zeta(5),$$

where $c$ is an undetermined rational constant. Of course, this is nothing but a conjecture and should be proved by an explicit computation at the needed order. This computation can be done along the lines of the previous sections.

6.2 The short string limit at fixed $r = J/S$ and the three-loop short state energy

It is interesting to remark that it is possible to set $\rho = r \sqrt{S}$ in (6.1) and discuss the short string limit with fixed ratio $r = J/S$ [7]. The classical energy can be expanded at order $S^3$ and is

$$E_0(S,r) = \sqrt{2S} \left( 1 + \frac{2r^2 + 3}{8} S - \frac{4r^4 - 20r^2 + 21}{128} S^2 + \frac{8r^6 - 28r^4 - 146r^2 + 187}{1024} S^3 + \cdots \right),$$

while the one-loop result reads

$$E_1(S,r) = \sqrt{2S} \left[ -\frac{1}{4} + \left( \frac{r^2 + 3}{16} - \frac{3}{4} \zeta(3) \right) S + \left( -\frac{3r^4 + 28r^2 + 16c}{128} + \left( \frac{3r^2}{16} + \frac{39}{32} \right) \zeta(3) + \frac{15}{16} \zeta(5) \right) S^2 + \cdots \right].$$

This expression gives some information about the three-loop strong coupling energy of short $(S,J)$ states. To this aim, we parametrize the higher order contributions $E_n$

$$E_n = \sqrt{2S} \left( \tilde{a}_{n,0}(r) + \tilde{a}_{n,1}(r) S + \tilde{a}_{n,2}(r) S^2 + \cdots \right).$$
Using the information we gathered on $E_{0,1}$ the result is

$$\frac{E}{\sqrt{2S}} = \lambda^{1/4} + \frac{1}{\lambda^{1/4}} \left( \frac{J^2}{4S} + \frac{3}{8} S - \frac{1}{4} \right) +$$

$$+ \frac{1}{\lambda^{3/4}} \left( -\frac{J^4}{32S^2} + \frac{J^2}{16S} + \frac{5J^2}{32} - \frac{21S^2}{128} - \frac{3\zeta(3)}{4} S + \frac{3}{16} S + \tilde{a}_{3,0} \right) +$$

$$+ \frac{1}{\lambda^{5/4}} \left( \tilde{a}_{3,1} S + \tilde{a}_{4,0} - \frac{3J^4}{128S^2} + J^2 \left( \frac{3\zeta(3)}{16} + \frac{7}{32} \right) + S^2 \left( \frac{c}{8} + \frac{15\zeta(5)}{16} + \frac{39\zeta(3)}{32} \right) \right) + \ldots$$

This expression can be compared with the two-loop prediction [9] and matching is perfect upon setting $\tilde{a}_{3,0} = -\frac{3}{32}$. Thus the three-loop prediction contains the only undetermined constant $c$ and the two functions $\tilde{a}_{3,1}(r)$ and $\tilde{a}_{4,0}(r)$.

### 7 Beyond Basso’s conjecture: The $m$-folded string

The considered folded string solution can be made more interesting by including the possibility of a higher folding, i.e. assuming that the string bounces $m$ times back and forth around the center of $AdS$ with a total of $2m$ spikes [17]. At the classical level, the modifications with respect to the simple $m = 1$ case are trivial. In particular, we shall write again (1.13) where now

$$S = \frac{S}{m\sqrt{\lambda}}, \quad J = \frac{J}{m\sqrt{\lambda}} \quad (7.1)$$

The definitions of $s$, $\rho$, and $t$ are still given by (1.7) and (3.2). A naive attempt to modify in this way Basso’s conjecture is known to be wrong already at one-loop [9].

In the Algebraic Curve approach, the higher folding $m$ is simply introduced by multiplying the charges and the quasi-momenta by $m$. Once this is done, the solutions of (2.8) are obtained by replacing $n \rightarrow n/m$. This replacement does not affect the higher modes with $n \geq m$, but changes the low-lying ones. Let us consider in details the evaluation of $a_{n,0}^m(\rho)$ for the first cases $m = 2, 3, 4, 5$.

- $m = 2$

The bulk contribution is obtained as the first line of (4.1) with $n \rightarrow n/2$ summed over $n \geq 3$. The missing modes appear now at $n = 2$ and their contribution is simply the contribution of the $n = 1$ missing modes we computed for $m = 1$. Finally, we have to add the genuine $n = 1$ modes for the $m = 2$ problem. These are computed separately without problems (actually, all of them are obtained by setting $n = 1/2$ in the $m = 1$ expressions with the single exception of the $A$ polarization that has to be recomputed). Summing up, the final result is

$$\frac{E_{LO,m=2}}{s} = \sum_{n=3}^{\infty} \left[ \frac{(n+2)\sqrt{(n+2)^2+2(n+2)^2t+4}}{2n^2+8n} + \frac{8(n^2+24t^2+48t+8)}{(n-4)n(n+2)(n+4)(n-2)(t+1)} \right] +$$

$$+ \frac{306t^2+612t+101}{60t+60} +$$

$$- \frac{41}{6} \sqrt{t(t+2)} - \frac{1}{6} \sqrt{t(t+2)} + 4 + \frac{1}{3} \sqrt{4t(t+2)} + 1 + \frac{3}{10} \sqrt{9t(t+2) + 4} \quad (7.2)$$
Using again the telescopic property of the most complicated terms in the summand, we are able to compute the exact infinite sum and obtain

\[
\frac{\mathcal{E}_{1,0}^{L,m=2}}{s} = \frac{8}{3}(t + 1) - \frac{9}{4}(t + 1) - 3\sqrt{t(t + 2)} + \frac{1}{3}\sqrt{t(t + 2)} + 4. \tag{7.3}
\]

Replacing \( t = \frac{1}{2}(\sqrt{2\rho^2 + 4} - 2) \) we can write the exact result

\[
a_{1,0}^{m=2}(\rho) = \frac{16\rho^2 + 5}{6\sqrt{\rho^2 + 2}} + \frac{1}{3}\sqrt{\rho^2 + 8} - 3\rho. \tag{7.4}
\]

Expanding at large \( \rho \) we find

\[
a_{1,0}^{m=2}(\rho) = -\frac{1}{2\rho} + \frac{1}{2\rho^3} + \frac{21}{4\rho^5} - \frac{175}{4\rho^7} + \frac{4501}{16\rho^9} - \frac{28161}{16\rho^{11}} + \frac{358545}{32\rho^{13}} + \cdots,
\]

in full agreement with (and extending) the leading singular terms of (B.6) of [9]. The same kind of computation can be repeated for the next higher order term in the \( S \) expansion with the rather simple result

\[
a_{1,1}^{m=2}(\rho) = \frac{15}{4}(\rho^3 + \rho) - \frac{85\rho^4 + 721\rho^2 + 940}{108\sqrt{\rho^2 + 8}} - \frac{1280\rho^6 + 3720\rho^4 + 1635\rho^2 - 884}{432(\rho^2 + 2)^{3/2}} + \frac{12\zeta(3)}{\sqrt{\rho^2 + 2}}. \tag{7.5}
\]

Again, expanding at large \( \rho \) we find

\[
a_{1,1}^{m=2}(\rho) = \frac{\rho}{2} - \frac{12\zeta(3)}{\rho} - \frac{17}{16} + \frac{12\zeta(3)}{\rho^3} + \frac{15}{16} - \frac{18\zeta(3)}{\rho^5} - \frac{727}{32} + \frac{30\zeta(3)}{\rho^7} + \frac{8365}{32} \cdots, \tag{7.6}
\]

in full agreement with (and extending) the next-to-leading singular terms of (B.6) of [9].

- \( m = 3 \)

The same kind of manipulations leads to the result

\[
a_{1,0}^{m=3}(\rho) = \frac{243\rho^2}{40\sqrt{\rho^2 + 2}} + \frac{\sqrt{\rho^2 + 18}}{8} + \frac{2}{5}\sqrt{4\rho^2 + 18} + \frac{53}{20\sqrt{\rho^2 + 2}} - 7\rho. \tag{7.7}
\]

Expanding at large \( \rho \) we find

\[
a_{1,0}^{m=3}(\rho) = -\frac{1}{2\rho} - \frac{5}{8\rho^3} + \frac{1245}{32\rho^5} - \frac{258785}{512\rho^7} + \frac{13235411}{2048\rho^9} + \cdots, \tag{7.8}
\]

in full agreement with (and extending) the leading singular terms of (B.7) of [9]. At the next order, we find a rather complicated closed expression for \( a_{1,1}^{m=3}(\rho) \). Its expansion at large \( \rho \) is

\[
a_{1,1}^{m=3}(\rho) = \frac{\rho}{2} - \frac{81\zeta(3)}{2\rho} - \frac{7}{4} + \frac{81\zeta(3)}{2\rho^3} + \frac{39}{16} - \frac{243\zeta(3)}{4\rho^5} - \frac{251423}{1024} + O\left(\frac{1}{\rho^{11/2}}\right). \tag{7.9}
\]
in full agreement with the next-to-leading singular terms of (B.7) of [9].

- **m = 4**

The exact result is

\[
a^{m=4}_{1,0}(\rho) = \frac{1048\rho^2}{105\sqrt{\rho^2 + 2}} + \frac{\sqrt{\rho^2 + 8}}{3} + \frac{\sqrt{\rho^2 + 32}}{15} + \frac{3\sqrt{9\rho^2 + 32}}{7} + \frac{1007}{210\sqrt{\rho^2 + 2}} - \frac{35\rho}{3} \tag{7.11}
\]

Expanding at large \(\rho\) we find

\[
a^{m=4}_{1,0}(\rho) = -\frac{1}{2\rho} - \frac{55}{18\rho^3} + \frac{43109}{324\rho^5} - \frac{8049175}{2916\rho^7} + \frac{6448461173}{104976\rho^9} + \cdots \tag{7.12}
\]

At the next order, we find a complicated closed expression for \(a^{m=4}_{1,1}(\rho)\) whose expansion at large \(\rho\) reads

\[
a^{m=4}_{1,1}(\rho) = \frac{\rho}{2} + \frac{96\zeta(3)}{\rho} - \frac{281}{144} + \frac{96\zeta(3)}{\rho^3} + \frac{185}{48} + \frac{-144\zeta(3)}{\rho^5} - \frac{31420351}{23328} + \cdots \tag{7.13}
\]

It implies that the analog of Eqs. (B.5, B.6, B.7) of [9] for \(m = 4\) should read (in the notation of that paper)

\[
\Delta^{m=4}_{1-\text{loop}} = \left( -\frac{1}{2}\mathcal{J} + \frac{\mathcal{J}}{2} + \cdots \right) S + \left[ \frac{55}{18\mathcal{J}^3} - \frac{1}{\mathcal{J}} \left( -96\zeta(3) - \frac{281}{144} \right) + \cdots \right] S^2 + \left[ \frac{43109}{324\mathcal{J}^5} + \frac{1}{\mathcal{J}^3} \left( 96\zeta(3) + \frac{185}{48} \right) + \cdots \right] S^3 + \cdots \tag{7.14}
\]

- **m = 5**

The exact result is

\[
a^{m=5}_{1,0}(\rho) = \frac{14375\rho^2}{1008\sqrt{\rho^2 + 2}} + \frac{\sqrt{\rho^2 + 50}}{24} + \frac{2}{21}\sqrt{4\rho^2 + 50} + \frac{3}{16}\sqrt{9\rho^2 + 50} + \frac{4}{5}\sqrt{16\rho^2 + 50} + \frac{3623}{504\sqrt{\rho^2 + 2}}\frac{1}{6} \tag{7.15}
\]

Expanding at large \(\rho\) we find

\[
a^{m=5}_{1,0}(\rho) = -\frac{1}{2\rho} - \frac{1981}{288\rho^3} + \frac{13823521}{41472\rho^5} - \frac{246936383285}{23887872\rho^7} + \frac{1230239204504695}{3439853568\rho^9} + \cdots \tag{7.16}
\]

At the next order, we compute \(a^{m=5}_{1,1}(\rho)\) whose expansion at large \(\rho\) is

\[
a^{m=5}_{1,1}(\rho) = \frac{\rho}{2} + \frac{375\zeta(3)}{2\rho} - \frac{911}{576} + \frac{375\zeta(3)}{2\rho^3} + \frac{259}{48} + \frac{1125\zeta(3)}{4\rho^5} - \frac{242572445069}{47776244} + \cdots \tag{7.17}
\]

It implies that the analog of Eqs. (B.5, B.6, B.7) of [9] for \(m = 5\) reads (in the notation of that paper)

\[
\Delta^{m=5}_{1-\text{loop}} = \left( -\frac{1}{2}\mathcal{J} + \frac{\mathcal{J}}{2} + \cdots \right) S + \left[ -\frac{1981}{288\mathcal{J}^3} - \frac{1}{\mathcal{J}} \left( -\frac{375\zeta(3)}{2} - \frac{911}{576} \right) + \cdots \right] S^2 + \left[ \frac{13823521}{41472\mathcal{J}^5} + \frac{1}{\mathcal{J}^3} \left( \frac{375\zeta(3)}{2} + \frac{259}{48} \right) + \cdots \right] S^3 + \cdots \tag{7.18}
\]
7.1 Resummed values at $J = 0$

It is interesting to compute the values of the $a(\rho)$ functions at $\rho = 0$, which is $J = 0$. They are a non trivial result of the resummation. Using the closed expressions we have computed, one obtains immediately the following table\(^8\)

| $m$ | $\sqrt{2}a_{1,0}^m(0)$ | $\sqrt{2}a_{1,1}^m(0)$ |
|-----|-------------------------|-------------------------|
| 1   | $-\frac{1}{2}$          | $\frac{3}{8} - \frac{3}{2} \zeta_3$ |
| 2   | $\frac{13}{6}$          | $-\frac{719}{216} - 12 \zeta_3$ |
| 3   | $\frac{29}{5}$          | $-\frac{89547}{8000} - \frac{81}{2} \zeta_3$ |
| 4   | $\frac{2119}{210}$      | $-\frac{230235893}{9261000} - 96 \zeta_3$ |
| 5   | $\frac{3749}{252}$      | $-\frac{5894598851}{128024064} - \frac{375}{2} \zeta_3$ |

As a final check, we remark that the values $a_{1,0}^m(0)$ agree with a general analysis of the short $m$-folded string in the fixed $J/S$ regime [20] as they should.

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\(^8\)Notice that $a_{1,1}^{m-1}(0)$ must not be identified with the similar constant $B_2$. Instead, the precise relation from (1.16) is $B_2 = 2 \sqrt{2} a_{1,1}^{m-1}(0) - 3/8$. Replacing the value of the table, we get back the result of Gromov and Valatka.
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