Dynamics of a 1-D model for the emergence of the plasma edge shear flow layer with momentum conserving Reynolds stress

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Abstract

A one-dimensional version of the second-order transition model based on the sheared flow amplification by Reynolds stress and turbulence suppression by shearing is presented. The model discussed in this paper includes a form of the Reynolds stress which explicitly conserves momentum. A linear stability analysis of the critical point is performed. Then, it is shown that the dynamics of weakly unstable states is determined by a reduced equation for the shear flow. In the case in which the flow damping term is diffusive, the stationary solutions are those of the real Ginzburg-Landau equation.
I. INTRODUCTION

The existence of a shear flow layer at the tokamak edge and in ohmic discharges has been known for a long time [1, 2]. It was later found that similar edge shear flow layers existed in other confinement devices. This seems to be a generic feature of confined plasmas. In the last years, a great deal of attention has been directed to the formation of shear flow layers and the corresponding region of radial electric field gradient to understand the improved confinement regimes. Many of the recent theoretical developments in this direction have been focused on barrier formation [3] or zonal flows [4]. On the experimental side, much progress has been done in the visualization of edge turbulence and flows ([5], [6]), including specific applications to the emergence of the shear flow layer in the TJ-II stellarator ([7]), in which we are especially interested.

Here, we want to turn back to the basic plasma edge shear flow layer. In stellarators, unlike in tokamaks, one can operate at densities for which no shear layer is present in the plasma edge, being thus possible to study its formation.

The emergence of the plasma edge shear flow layer as the density increases in the TJ-II stellarator [8] is shown [9, 10] to have the characteristic properties of a second order phase transition. It is consistent [11] with a simple transition model that couples shear flow amplification by turbulence [12, 13] with turbulence suppression by sheared flows [14]. The model used in interpreting the TJ-II results is based on a transition model [15] initially introduced to explain the transition from the low confinement mode (L mode) to the high confinement mode (H mode) [16] in magnetically confined plasmas. This model consists of two envelope equations for the fluctuation level and mean poloidal flow. A later extension of the model [17] included a third equation to account for the pressure gradient contribution to the radial electric field. This second model shows the existence of two critical points, the second one causing the first order transition that has been associated with the L to H transition. For this transition there is a hysteresis cycle and the transition is characterized by an S-curve (see for instance [18]). The first critical point leads to a second order transition (consequently, it does not have a hysteresis cycle), which has been identified with the emergence of the plasma edge shear flow layer.

In this paper, we focus on this second order transition. By excluding the diamagnetic term in the momentum balance equation, only this transition is included in the model. This
simplification is reasonable because there is a large range of densities separating the two critical points. For the same reason, the range of plasma parameters considered is not yet in an L-mode regime. Therefore, we do not expect avalanche-like transport \cite{19, 20} and the transport terms can be represented by purely diffusive terms. Concretely, we discuss an extension to 1-D of the original model used in comparison with the experimental data \cite{11}. In contrast with previous 1-D extensions of the transition model \cite{21, 22}, here we formulate the Reynolds stress term as a momentum conserving term. The model is defined by three 1-D partial differential equations describing the evolution of the turbulent fluctuation level $E$, the averaged poloidal velocity shear $U$ and (minus) the pressure gradient $N$. It predicts a second-order phase transition with order parameter $U$ and control parameter $\Gamma$, the particle flux, which enters the model through the boundary conditions. For $\Gamma$ below a critical value $\Gamma_c$ the stable stationary solutions have $U = 0$, whereas for $\Gamma > \Gamma_c$ such solutions are unstable and undergo a transition to states with $U \neq 0$ and reduced turbulence fluctuations.

After performing a detailed stability analysis of the model we study two interesting special cases depending on the form of the flow-damping term: collisional drag and collisional diffusion. For both of them we find reduced equations describing the dynamics of weakly unstable states. In the latter case, the reduced equation is closely related to the Ginzburg-Landau equation for second-order phase transitions. In particular, the stationary solutions of our equation are exactly those of the Ginzburg-Landau one.

The paper is organized as follows:

In Section II we introduce the one-dimensional transition model with momentum conservation. Section III is devoted to the study of the fixed points of the model, their linear stability properties and a general discussion of the critical conditions. In Section IV we consider the dynamics near the critical point. Section V contains the conclusions and an outline of future research lines.

II. THE ONE-DIMENSIONAL TRANSITION MODEL

The relevant plasma edge region to which this model is applied corresponds to $r \in [r_0, a]$, $(a - r_0)/a \approx 0.1$, where $a$ is the minor radius of the plasma. We take the slab geometry approximation in representing this region and use as coordinate $x := (r - r_0)/(a - r_0)$, so that $x \in [0, 1]$. 
The fields of our model will be the fluctuation level envelope $E := \langle (\tilde{n}_k / n_0)^2 \rangle^{1/2}$, the averaged poloidal shear flow $U := \partial \langle V_\theta \rangle / \partial r$ and (minus) the averaged pressure gradient $N := -\partial \langle p \rangle / \partial r$, where $\langle \cdot \rangle$ denotes ensemble average. A suitable one-dimensional generalization of the model discussed in Ref. [11] requires a form of the Reynolds stress which conserves momentum. Using a pressure-gradient-driven turbulence model and assuming densely packed turbulence, a quasi-linear calculation yields the following form for the Reynolds stress:

$$
\langle \tilde{V}_x \tilde{V}_\theta \rangle = \bar{D}_0 E^2 \partial_x \langle V_\theta \rangle + \bar{D}_2 E^2 \partial_x^3 \langle V_\theta \rangle,
$$

which is very similar to the expression previously derived in Ref. [23] in the context of zonal flow dynamics. Notice that $\bar{D}_3$ measures the strength of the Reynolds stress and non-zero $\bar{D}_2$ is needed for the spectrum of the instability to be bounded.

The model discussed in the present work is:

$$
\partial_t E = \gamma_0 N E - \alpha_1 E^2 - \alpha_2 U^2 E + \partial_x \left[ (D_0 + D_1 E) \partial_x E \right],
$$

$$
\partial_t U = -\bar{\mu}_1 U + \bar{\mu}_2 \partial_x^2 U - \bar{\alpha}_3 \partial_x^2 \langle E^2 U \rangle - D_2 \partial_x^3 \langle E^2 \partial_x^2 U \rangle,
$$

$$
\partial_t N = \partial_x^2 \left[ (D_3 E + D_4) N \right],
$$

Here, $\gamma_0 N$ is the linear growth rate of the characteristic instability and $\alpha_1$ is computed from the nonlinear saturation condition of the instability, both in the absence of sheared flow. The $\alpha_2$ coefficient is derived from the condition of turbulence suppression by sheared flow. $D_0$ and $D_4$ are neoclassical diffusivity coefficients, whereas $D_1$ and $D_3$ are the coefficients of anomalous diffusivity multiplying the fluctuation level. Finally, $\bar{\mu}_1$ and $\bar{\mu}_2$ are the coefficients of the collisional flow-damping terms.

We can eliminate the explicit dependence on the parameters $\gamma_0$, $\alpha_1$ and $\alpha_2$ by means of the following change of variables:

$$
E := \frac{\alpha_1}{\gamma_0} \xi, \quad U := \sqrt{\frac{\alpha_2}{\gamma_0}} \eta, \quad t := \gamma_0 \tilde{t}, \quad N := N,
$$

and Eqs. (2) read in terms of $E(x, t)$, $U(x, t)$ and $N(x, t)$:

$$
\partial_t E = N E - E^2 - U^2 E + \partial_x \left[ (D_0 + D_1 E) \partial_x E \right],
$$

$$
\partial_t U = -\bar{\mu}_1 U + \bar{\mu}_2 \partial_x^2 U - \bar{\alpha}_3 \partial_x^2 \langle E^2 U \rangle - D_2 \partial_x^3 \langle E^2 \partial_x^2 U \rangle,
$$

$$
\partial_t N = \partial_x^2 \left[ (D_3 E + D_4) N \right],
$$

and Eqs. (3) read in terms of $E(x, t)$, $U(x, t)$ and $N(x, t)$:

$$
\partial_t E = N E - E^2 - U^2 E + \partial_x \left[ (D_0 + D_1 E) \partial_x E \right],
$$

$$
\partial_t U = -\bar{\mu}_1 U + \bar{\mu}_2 \partial_x^2 U - \bar{\alpha}_3 \partial_x^2 \langle E^2 U \rangle - D_2 \partial_x^3 \langle E^2 \partial_x^2 U \rangle,
$$

$$
\partial_t N = \partial_x^2 \left[ (D_3 E + D_4) N \right],
$$

and Eqs. (4) read in terms of $E(x, t)$, $U(x, t)$ and $N(x, t)$:
with $D_0 = D_0/\gamma_0$, $D_1 = D_1/\alpha_1$, $D_3 = D_3/\alpha_1$, $D_4 = D_4/\gamma_0$, $\mu_1 = \bar{\mu}_1/\gamma_0$, $\mu_2 = \bar{\mu}_2/\gamma_0$, $\alpha_3 = \bar{\alpha}_3\gamma_0/\alpha_1^2$, $D_2 = D_2\gamma_0/\alpha_1^2$. Finally, we choose the boundary conditions:

\begin{align}
\partial_x U(u, t) &= \partial_x^2 U(u, t) = 0, \quad u = 0, 1, \forall t, \quad (5a) \\
\partial_x E(u, t) &= 0, \quad u = 0, 1, \forall t, \quad (5b) \\
(D_3 E + D_4) N|_{(0, t)} &= \Gamma, \quad \partial_x N|_{(1, t)} = 0, \quad (5c)
\end{align}

where $\Gamma$ is the particle flux and is the natural control parameter of the model.

### III. FIXED POINTS AND LINEAR STABILITY ANALYSIS

It is obvious from Eq. (4b) that any fixed point of the model must satisfy $U = 0$. Then, from Eq. (4a) we find that $NE = E^2$. It only remains to use the boundary condition of $N$ at $x = 0$ and we finally have that

(i) There always exists a fixed point

$$U_f = 0, \quad E_f(\Gamma) = N_f(\Gamma) = \frac{1}{2D_3} \left( \sqrt{D_1^2 + 4D_3\Gamma - D_4} \right).$$  \hspace{1cm} (6)

(ii) If $D_4 \neq 0$ there exists a second fixed point (always unstable, see below)

$$U'_f = 0, \quad E'_f = 0, \quad N'_f(\Gamma) = \Gamma/D_4.$$  \hspace{1cm} (7)

Let $(E_0, U_0, N_0)$ be a fixed point and linearize the equations (4) around it:

$$E(x, t) = E_0 + \xi_E e^{\gamma t + ikx},$$
$$U(x, t) = \xi_U e^{\gamma t + ikx},$$
$$N(x, t) = N_0 + \xi_N e^{\gamma t + ikx}.$$

For the fixed point $(E'_f, U'_f, N'_f)$ the eigenvalue condition leads to the dispersion relations:

$$\gamma - N'_f + D_0 k^2 = 0,$$
$$\gamma + \mu_1 + \mu_2 k^2 = 0,$$
$$\gamma + D_4 k^2 = 0.$$  \hspace{1cm} (8)

It is clear that this fixed point is unstable for any $\Gamma > 0$, since the first dispersion relation gives, for $k = 0$, $\gamma = \Gamma/D_4$. 

5
The fixed point \((E_f, U_f, N_f)\) is more interesting. In this case we have:

\[
\gamma = -\mu_1 - \mu_2 k^2 + \alpha_3 E_f^2 k^2 - D_2 E_f^2 k^4,
\]

\[
(\gamma + E_f + (D_0 + 2D_1 E_f)k^2)(\gamma + (D_1 + D_3 E_f)k^2) + D_3 E_f^2 k^2 = 0.
\] (9)

The second equation does not give solutions with \(\gamma > 0\). However, the first one can yield an instability. Then, we ask under which conditions

\[
\gamma(k) = -\mu_1 - \mu_2 k^2 + \alpha_3 E_f^2 k^2 - D_2 E_f^2 k^4
\] (10)

is positive. The neutral modes, i.e. the values of \(k\) for which \(\gamma(k) = 0\) are given by:

\[
k^2 = \frac{\alpha_3 E_f^2 - \mu_2 \pm \sqrt{(\alpha_3 E_f^2 - \mu_2)^2 - 4\mu_1 D_2 E_f^2}}{2D_2 E_f^2},
\] (11)

which has real solutions \(k_-\), \(k_+\) if and only if

\[
\alpha_3 E_f^2 - \mu_2 - 2E_f \sqrt{\mu_1 D_2} \geq 0.
\] (12)

Now observe that the boundary conditions imply the quantization of \(k\). Namely,

\[
k = n\pi, \ 0 \leq n \in \mathbb{Z}.
\] (13)

Therefore (12) is only a necessary condition for the existence of instabilities. In addition, there must exist some \(k = n\pi\) such that \(k_- < n\pi < k_+\). The critical point is defined by the minimum value of the flux, \(\Gamma_c\), for which there exists an unstable mode \(k_c = n_c\pi\).

We define \(E_c := E_f(\Gamma_c)\). From (10) we find that

\[
E_c^2 = \frac{\mu_1 + \mu_2 k_c^2}{k_c^2(\alpha_3 - D_2 k_c^2)}.
\] (14)

Thus, in particular, if \(\alpha_3/D_2 \leq \pi^2\) there are no unstable modes, no matter how much we increase \(\Gamma\). If \(\alpha_3/D_2 > \pi^2\) at least \(k = \pi\) can become unstable. Actually, if \(\sqrt{\alpha_3/D_2} \in (\pi, n\pi)\), there exist \(n-1\) potentially unstable modes \(k = \pi, 2\pi, \ldots, (n-1)\pi\).

IV. DYNAMICS NEAR MARGINAL STABILITY

Our aim is to find approximate equations for the dynamics of Eqs. (4) near (and above) the critical point. To that end we perform an expansion with parameter

\[
\delta = \sqrt{\frac{\Gamma}{\Gamma_c} - 1}
\] (15)
for small $\delta$. Explicitly, we take:

\[ E = E_c + \delta^2 E_2 + \ldots \]  \hspace{1cm} (16a)
\[ U = \delta U_1 + \ldots \]  \hspace{1cm} (16b)
\[ N = N_c + \delta^2 N_2 + \ldots \]  \hspace{1cm} (16c)
\[ \Gamma = \Gamma_c(1 + \delta^2) \]  \hspace{1cm} (16d)

(recall that $N_c = E_c$) and we perform a rescaling of the coordinates:

\[ \bar{\eta} = \delta x, \quad \bar{\tau} = \delta^4 t. \]  \hspace{1cm} (17)

Expanding the equation of $N$ gives:

\[ N_2 = E_c \left(1 - \frac{D_3 E_2}{D_4 + D_3 E_c}\right) \]  \hspace{1cm} (18)

whereas the equation for $E$ yields:

\[ N_2 - E_2 - U_1^2 = 0 \]  \hspace{1cm} (19)

and using (18) we obtain for $E_2$:

\[ E_2 = \frac{D_4 + D_3 E_c}{D_4 + 2D_3 E_c} (E_c - U_1^2). \]  \hspace{1cm} (20)

Consequently, for weakly unstable states, the problem of studying the dynamics of our model consists in finding an approximate equation for the dynamics of $U$, $E$ and $N$ being determined at the end of the day from the slaving conditions (20), (18). The form of the reduced equation for $U$ is quite different for $\mu_1 \neq 0$ and $\mu_1 = 0$. That is why in the next sections we study the cases $\mu_2 = 0$ (collisional drag) and $\mu_1 = 0$ (collisional diffusion) separately. If the collisionality at the plasma edge is high enough, the damping is essentially diffusive and the limit $\mu_1 = 0$ is a good approximation. At low collisionality the magnetic pumping dominates and the relevant limit is $\mu_2 = 0$.

\textbf{A. Collisional drag}

Let us set $\mu_2 = 0$, $\mu_1 \neq 0$. The dispersion relation reads:

\[ \gamma(k) = -\mu_1 + E_j^2 k^2 (\alpha_3 - D_2 k^2) \]  \hspace{1cm} (21)
so that
\[ E_c := \sqrt{\frac{\mu_1}{k_\gamma^2 (\alpha_3 - D_2 k_\gamma^2)}}. \]  

(22)

As pointed out above, if \( \alpha_3 / D_2 < \pi^2 \) there are no unstable modes. If \( \pi^2 < \alpha_3 / D_2 < 4\pi^2 \) only \( k = \pi \) can be unstable and is, of course, the critical mode. If \( \alpha_3 / D_2 > 4\pi^2 \) there are at least two modes which may become unstable and which of them is the critical mode depends on the quotient \( \alpha_3 / D_2 \). Hence, in general, \( k = \pi \) is not the most unstable mode (see Figs. 1 and 2).

In order to find a reduced equation for the weakly non-linear dynamics of \( U \) we expand (4b) keeping terms up to order \( \delta^7 \):

\[
\delta^5 \partial_t U_1 = -\mu_1 \delta U_1 - \delta^3 \alpha_3 E_f(\Gamma)^2 \partial_x^2 U_1 + 2E_c \alpha_3 \frac{D_4 + D_3 E_c}{D_4 + 2D_3 E_c} \delta^5 \partial_{\eta}^2 U_1^3 - \delta^5 D_2 E_f(\Gamma)^2 \partial_{\eta}^4 U_1 \\
+ 2D_2 E_c \frac{D_4 + D_3 E_c}{D_4 + 2D_3 E_c} \delta^7 \partial_{\eta}^2 \left(U_1^2 \partial_{\eta}^2 U_1\right),
\]

(23)

where we have made use of (20). Finding a reduced equation without an explicit dependence in \( \delta \) seems difficult in this case. However, we can still obtain a useful reduced equation by simply taking (23) and going back to the original variable \( U \) and coordinates \( x, t \):

\[
\partial_t U = -\mu_1 U - E_f(\Gamma)^2 \left(\alpha_3 \partial_x^2 U - D_2 \partial_x^4 U\right) + 2E_c \frac{D_4 + D_3 E_c}{D_4 + 2D_3 E_c} \left[\alpha_3 \partial_x^2 U^3 + D_2 \partial_x^2 \left(U^2 \partial_x^2 U\right)\right].
\]

(24)

The time-evolution predicted by this equation is compared to the original model in Fig. 3 for \( \delta = 0.274 \). The dynamics is very sensitive to \( \delta \) and Eq. (24) ceases to describe it accurately for larger values of the expansion parameter. However, it is remarkable that even for values of \( \delta \) of order 1, the stationary solutions of the reduced equation give good approximations of the exact ones (see Fig. 4, where \( \delta = 0.82 \)).

B. Collisional diffusion

In this section we take \( \mu_1 = 0 \) and \( \mu_2 \neq 0 \). In this case

\[
\gamma(k) = (-\mu_2 + \alpha_3 E_f^2 - D_2 E_f^2 k^2)k^2
\]

(25)

and the critical point is given by

\[
E_c = \sqrt{\frac{\mu_2}{\alpha_3 - k_\gamma^2 D_2}}.
\]

(26)
As we already know, if $\alpha_3/D_2 < \pi^2$ there are no unstable modes. Unlike the case of collisional drag, if $\alpha_3/D_2 > \pi^2$ the most unstable (i.e. critical) mode is always $k = \pi$ (see Fig. 5). In addition, for typical values of $\alpha_3$ and $D_2$, $\alpha_3 \gg \pi^2 D_2$, and we can use the approximation

$$E_c = \sqrt{\frac{\mu_2}{\alpha_3}}. \quad (27)$$

We are now ready to derive a reduced equation for the dynamics of weakly unstable states in this case. The expansion of (4b) in powers of $\delta y$ yields:

$$\delta^5 \partial^\tau U_1 = \delta^3 \partial^2 \eta \left[ (\mu_2 - \alpha_3 E_c^2) U_1 - 2\delta^2 \alpha_3 E_c E_2 U_1 - \delta^2 D_2 E_c^2 \partial^2 \eta \right]. \quad (28)$$

Noting that the first term on the right-hand side vanishes if we use the approximation (27), we are left with

$$\partial_\tau U_1 = -\partial^2 \eta \left[ 2\alpha_3 E_c E_2 U_1 + D_2 E_c^2 \partial^2 \eta \right]. \quad (29)$$

Finally, we use (20) and perform one more change of variables:

$$\sigma = \frac{1}{\sqrt{E_c}} U_1, \quad \tau = \frac{1}{D_2} \left( \frac{2\alpha_3 E_c (D_4 + D_3 E_c)}{D_4 + 2D_3 E_c} \right)^2 \bar{\tau}, \quad \eta = \sqrt{\frac{2\alpha_3 (D_4 + D_3 E_c)}{D_2 (D_4 + 2D_3 E_c)}} \bar{\eta} \quad (30)$$

obtaining the definitive form of the equation describing the weakly non-linear dynamics of $U$:

$$\partial_\tau \sigma = -\partial^2 \eta \left[ \sigma - \sigma^3 + \partial^2 \sigma \right]. \quad (31)$$

In Fig. 6 we show a comparison between the dynamics of Eq. (31) and that of the original model, whereas Figs. 7 and 8 are comparisons of the stationary solutions.

A remark is in order at this point. A consequence of using the approximation (27) in Eq. (28) is that the linear growth rate is modified. An easy calculation shows that Eq. (31) gives the actual linear growth rate (at order $\delta^2$) only if we make the replacement

$$\delta^2 \mapsto \delta^2 + \frac{D_2 \pi^2}{\alpha_3} \quad (32)$$

which is natural, since the accuracy of our approximation depends on the quotient $D_2 \pi^2/\alpha_3$. Thus, very close to the critical point one has to take into account the above correction of $\delta$ when comparing the results of (51) with those coming from the integration of the exact equations of the model.

Using that at the boundary, $\partial_x \sigma = \partial^2 \sigma = 0$, and taking an initial condition such that $\int_0^1 U(x)dx = 0$, the stationary solutions of Eq. (31) are exactly the solutions of:

$$\sigma - \sigma^3 + \partial^2 \sigma = 0, \quad (33)$$

\[9\]
which is the time-independent Ginzburg-Landau equation for second-order phase transitions. It is the same equation as in the case of non-momentum-conserving Reynolds stress with a collisional drag (see Ref. [24]). We follow the lines of Ref. [24] to obtain analytical expressions for the stationary solutions. The key observation is that there exists a ‘conserved quantity’. Namely,

\[
\frac{1}{2} (\partial_{\eta} \sigma)^2 + \frac{\sigma^2}{2} \left(1 - \frac{\sigma^2}{2}\right) = C, \quad C \in \mathbb{R}.
\]  

(34)

This allows to reduce the solutions to quadratures:

\[
\int \frac{d\sigma}{\sqrt{(\sigma^2 - b_+)(\sigma^2 - b_-)}} = \int \frac{d\eta}{\sqrt{2}}
\]

(35)

with \(b_{\pm} = 1 \pm \sqrt{1 - 4C}\). Defining \(m = b_-/b_+\) and performing the change of variable \(\sigma = \sqrt{b_-} \sin \phi\) we can recast the solutions in the following form:

\[
\sqrt{\frac{b_+}{2}} \eta = \int_0^\phi \frac{d\theta}{\sqrt{1 - m \sin^2 \theta}} - A,
\]

(36)

where \(A\) is an integration constant.

Hence, the solution for \(\sigma(\eta)\) may be expressed as

\[
\sigma(\eta) = \sqrt{b_-} \text{sn} \left( \sqrt{\frac{b_+}{2}} \eta + A|m\right)
\]

(37)

where \(\text{sn}(y|m)\) stands for the Jacobi elliptic function, which is periodic in \(y\). Its period is \(P = 4K(m)\), with

\[
K(m) = 4 \int_0^{\pi/4} \frac{d\theta}{\sqrt{1 - m \sin^2 \theta}}
\]

(38)

the complete elliptic integral of the first kind. In order to write explicitly the solution in terms of the original variable \(U\) and coordinate \(x\), recall that

\[
U = \delta \sqrt{E_c} \sigma, \quad \eta = \delta \sqrt{\frac{2\alpha_3(D_4 + D_3E_c)}{D_2(D_4 + 2D_3E_c)}} x.
\]

(39)

Then,

\[
U(x) = \delta \sqrt{E_c} b_- \text{sn} \left( \sqrt{\frac{b_+ \alpha_3(D_4 + D_3E_c)}{D_2(D_4 + 2D_3E_c)}} \delta x + A|m\right).
\]

(40)

Thus, the wavelength of the solutions is

\[
\lambda = \frac{4K(m)}{\delta} \sqrt{\frac{D_2(D_4 + 2D_3E_c)}{b_+ \alpha_3(D_4 + D_3E_c)}}.
\]

(41)
The boundary conditions determine the integration constants $C$ and $A$. In particular, they imply the quantization condition
\[ n\lambda^2 = 1, \quad 0 \leq n \in \mathbb{Z}. \] (42)

V. CONCLUSIONS AND FURTHER WORK

We have introduced a one-dimensional version of the phase transition model considered in Ref. [11] including a Reynolds stress term with manifest momentum conservation. The model consists of three envelope equations for the fluctuation level $E$, the poloidal shear flow $U$ and the density gradient $N$. It possesses a critical point corresponding to a second order transition whose natural control parameter is $\Gamma$, the particle flux. Below the critical value, $\Gamma_c$, the model has a non-trivial fixed point with $U = 0$. Through a linear stability analysis we have shown that if $\Gamma > \Gamma_c$ the stationary solutions have non-zero shear flow and reduced turbulent fluctuations. We have also studied the dynamics of the model near (and above) the critical point. Defining a suitable expansion around the critical point we have derived slaving conditions for $E$ and $N$, so that they are determined from the value of $U$. Then, we have found reduced equations for the weakly non-linear dynamics of $U$. In the case of diffusive shear flow damping the reduced equation is related to the Ginzburg-Landau equation, which allows us to work out analytical expressions for the stationary solutions of weakly unstable states.

The analysis performed in this work shows some interesting differences with respect to previous one-dimensional versions of the model in which momentum conservation is not implemented (see Ref. [24]), the most relevant of them concerning the nature of the fixed points and the instabilities.

In Ref. [24] the most unstable mode is always $k = 0$, which is related to the fact that in the model studied therein there exist two (non-trivial) fixed points, one of them with $U = 0$ and the other one with $U \neq 0$. The introduction of a momentum-conserving Reynolds stress makes the fixed point with non-zero $U$ disappear, so that all stationary solutions with non-zero shear flow have non-trivial spatial structure. This is connected to the results derived in previous sections regarding the linear stability analysis of the critical point. In the model with momentum conservation $k = 0$ is always stable and the discussion of the structure of the most unstable (critical) mode is more complicated. In general, the critical mode depends
on the values of the parameters. The presence of non-zero critical mode is in agreement with
the experimental findings reported in Ref. [10].

A next step in the development of the model would be to incorporate the diamagnetic term
in order to study the L to H transition. In the L-mode regime the model must incorporate the
transport mesoscale, which can be achieved by formulating transport equations in terms of
fractional derivative operators [25]. Such modification of the model will lead to the dynamics
of a reaction-diffusion system [26]. These issues will be addressed in future publications.

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Figure 1: $\gamma$ as a function of $\Gamma$ near the critical point. The values of the parameters are $\alpha_3 = 10^{-2}$, $D_2 = 10^{-4}$, $D_3 = 10^{-2}$, $D_0 = D_1 = D_4 = 0$, $\mu_1 = 1$, $\mu_2 = 0$. The critical mode is $k = 2\pi$ and the critical point is given by $\Gamma_c = 0.041853$. $k = 3\pi$ becomes unstable at $\Gamma = 0.1008$ and $k = \pi$ at $\Gamma = 0.1124$. 
Figure 2: $\gamma(k)$ as a function of $k$ near the critical point. The values of the parameters are $\alpha_3 = 10^{-2}$, $D_2 = 10^{-4}$, $D_3 = 10^{-2}$, $D_0 = D_1 = D_4 = 0$, $\mu_1 = 1$, $\mu_2 = 0$. The critical mode is $k = 2\pi$ and the critical point is given by $\Gamma_c = 0.041853$. 
Figure 3: Time-evolution of $U(0)$ computed from the integration of the original model (solid) and from the reduced equation (24) (dashed). The values of the parameters are $\alpha_3 = 10^{-2}$, $D_2 = 10^{-4}$, $D_3 = 10^{-2}$, $D_0 = D_1 = D_4 = 0$, $\mu_1 = 1$, $\mu_2 = 0$, $\Gamma = 0.045$. The critical point is $\Gamma_c = 0.041853$. 
Figure 4: Stationary solutions computed from the integration of the original model (solid) and from the reduced equation (dashed). The values of the parameters are $\alpha_3 = 10^{-2}$, $D_2 = 10^{-4}$, $D_3 = 10^{-2}$, $D_0 = D_1 = D_4 = 0$, $\mu_1 = 1$, $\mu_2 = 0$, $\Gamma = 0.07$. The critical point is $\Gamma_c = 0.041853$. 


Figure 5: $\gamma(k)$ as a function of $k$ near the critical point. The values of the parameters are $\alpha_3 = 10^{-2}$, $D_2 = 10^{-4}$, $D_3 = 10^{-2}$, $D_0 = D_1 = D_4 = 0$, $\mu_1 = 0$, $\mu_2 = 1$. The critical point is $\Gamma_c = 1.1095$. 
Figure 6: Time-evolution of $U(0)$ computed from Eq. (31) (dashed) and from the integration of the original model (solid). The values of the parameters are $\alpha_3 = 10^{-2}$, $D_2 = 10^{-4}$, $D_3 = 10^{-2}$, $D_0 = D_1 = D_4 = 0$, $\mu_2 = 1$, $\mu_1 = 0$, $\Gamma = 1.2$. The critical point is $\Gamma_c = 1.1095$. 
Figure 7: Structure of the bifurcation at the critical point. The values of the parameters are \( \alpha_3 = 10^{-2}, \ D_2 = 10^{-4}, \ D_3 = 10^{-2}, \ D_0 = D_1 = D_4 = 0, \mu_2 = 1, \mu_1 = 0. \) The critical point is \( \Gamma_c = 1.1095. \) The solid curve corresponds to the analytical solutions of Eq. 33. The points have been computed from the numerical integration of the original model. As we can see, the reduced equation gives very accurate results even at values of \( \delta \) of order one.
Figure 8: Stationary solutions computed from the integration of the original model (solid) and from the GL equation (33) (dashed). The values of the parameters are $\alpha_3 = 10^{-2}$, $D_2 = 10^{-4}$, $D_3 = 10^{-2}$, $D_0 = D_1 = D_4 = 0$, $\mu_2 = 1$, $\mu_1 = 0$, $\Gamma = 1.5$. The critical point is $\Gamma_c = 1.1095$. 