Striped patterns for generalized antiferromagnetic functionals with power law kernels of exponent smaller than $d + 2$

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Pattern formation

Spontaneous formation of regular structures is observed in nature at different scales

- Biology, Material Science
- Experiments and Simulations
- Examples: polymers, colloidal suspensions, Langmuir monolayers, thin magnetic films, amphiphile solutions
- Applications: memory storage device; artificial photonic crystals; biological devices for separation of biomolecules on microchips, nanolithography
Pattern formation

Figure: Black-and-white striped zebra and polka-dot zebra [Caters News Agency Ltd.]
Generalized antiferromagnetic model in sharp interface version

Striped patterns are expected to emerge as minimizers/ground states of the following free-energy functional

\[
\tilde{\mathcal{F}}_{\alpha,J,L}(E) = \frac{1}{L^d} (J \text{Per}_1(E, L) - \int_{[0,L]^d} \int_{\mathbb{R}^d} K_{\alpha,1}(\zeta) \left| \chi_E(x) - \chi_E(y) \right|) 
\]

1. \( J > 0, J \in [J_C - \tau, J_C) \) where \( \tau > 0 \) and \( J_c := \int_{\mathbb{R}^d} |\zeta_1| K_{\alpha,1}(\zeta) \, d\zeta \) is critical constant above which the minimizers are trivial
2. \( \text{Per}_1(E, [0, L]^d) := \int_{\partial E \cap [0, L]^d} \| \nu_E^E(x) \|_1 \, d\mathcal{H}^{d-1}(x) \), where \( \| y \|_1 = \sum_{i=1}^d |y_i| \) is the 1-norm
3. \( K_{\alpha,1} \) is the power kernel, 
   \[ K_{\alpha,1}(\zeta) = \frac{1}{(\| \zeta \|_1 + 1)^{p(\alpha)}}, \quad p(\alpha) = d + 2 - \alpha \]
Variety of cases

• $p = d - 2 \ (\alpha = 4)$  
diblock copolymer
• $p = d \ (\alpha = 2)$  
micromagnetics
• $p = d + 1 \ (\alpha = 1)$  
thin magnetic films
• $p > d + 1 \ (\alpha < 1)$  
generalized antiferromagnetic model

Aim: to improve current results towards the physical cases
Rescaled model

Rescale the functional in such a way that optimal stripes width and energy are of order $O(1)$

$$\mathcal{F}_{\alpha, \tau, L}(E) := \left\{ \begin{array}{l}
\frac{1}{L^d} \left[ \text{Per}_1(E; [0, L)^d) \left(-1 + \int_{\mathbb{R}^d} K_{\alpha, \tau}(\zeta) |\zeta_1| \, d\zeta \right) 
- \int_{\mathbb{R}^d} \int_{[0, L)^d} |\chi_E(x) - \chi_E(x + \zeta)| K_{\alpha, \tau}(\zeta) \, dx \, d\zeta \right] \\
\end{array} \right. \tag{1}$$

where

$$K_{\alpha, \tau}(\zeta) = \frac{1}{\left( \|\zeta\|_1 + \tau^{1/(1-\alpha)} \right)^{p(\alpha)}}. \tag{2}$$
Daneri and Runa '18 showed that, for $p \geq d + 2$ (i.e. $\alpha \leq 0$) global minimizers of $\mathcal{F}_{\tau,L}$ are, when $\tau$ is sufficiently small, periodic unions of stripes. Moreover, their results hold in the large volume limit, namely for arbitrarily large $L$ whenever $L$ is an even multiple of $h^*_\tau$, which is the optimal stripes’ period among all possible periods.

For the discrete analogue of the above functional and exponents $p > 2d$ a characterization of minimizers was first proved by Giuliani, Seiringer '16.

One-dimensionality and periodicity for a diffuse interface version of the model for exponents $\alpha \leq 0$ has been proved by Daneri, K., Runa '19 and Daneri, Runa '21 in the large volume limit.
Theorem (K. '21)

Let \( d \geq 1, \, L > 0 \). Then there exists \( 0 < \bar{\alpha} \ll 1 \) and \( \bar{\tau}_L > 0 \) such that \( \forall 0 < \alpha \leq \bar{\alpha} \) and \( \forall 0 < \tau \leq \bar{\tau}_L \) the minimizers of the functional \( \mathcal{F}_{\alpha,\tau,L} \) are periodic unions of stripes of width \( h_{\tau,L,\alpha} \).

For \( \tau, \alpha > 0 \), let \( h_{\tau,\alpha}^* \) be optimal among all widths of stripes for \( \mathcal{F}_{\alpha,\tau,L} \) as \( L \) varies.

Theorem (K. '21)

There exists \( 0 < \bar{\alpha} \ll 1 \) and \( \bar{\tau} > 0 \) such that \( \forall 0 < \alpha \leq \bar{\alpha}, \forall 0 < \tau \leq \bar{\tau} \) and for all \( k \in \mathbb{N} \), \( L = 2kh_{\tau,\alpha}^* \), minimizers of \( \mathcal{F}_{\alpha,\tau,L} \) are periodic stripes of width \( h_{\tau,\alpha}^* \).
Idea for strategy of the proof

We know by the results of Daneri, Runa that minimizers for $\alpha \leq 0$ and $\tau$ small are stripes. Such a result is obtained by:

► for fixed $\alpha \leq 0$ let $\tau \to 0$, a rigidity estimate ensures that minimizers of the limit functional are stripes

► once close to stripes, for $\tau > 0$ show a stability result proving that minimizers are exactly stripes

Difficulties:

► for a fixed $\alpha > 0$, and $\tau \to 0$ no rigidity result is available, this result is sharp for $p = d + 2$, ($\alpha = 0$)

Instead:

► we consider $(\alpha, \tau) \downarrow (0, 0)$

► show $\Gamma$-convergence of $F_{\alpha, \tau, L}$ to $F_{0,0,L}$, where by the above $F_{0,0,L}$ satisfies the rigidity estimate

► show a stability result for $0 < \alpha < 1$, $0 < \tau \ll 1$
Strategy of the proof

1. Decomposition of the functional penalizing derivations from one-dimensional profiles
2. $\Gamma$-convergence as $(\alpha, \tau) \downarrow (0, 0)$ to a functional which is finite only on stripes
3. Stability estimates for $0 < \alpha < 1$, $0 < \tau \ll 1$ showing that once close to stripes minimizers are exactly stripes
4. One-dimensional optimization through reflection positivity showing periodicity
Decomposition of the functional

\[ \mathcal{F}_{\alpha,\tau,L}(E) \geq \frac{1}{L^d} \left( -\sum_{i=1}^{d} \text{Per}_{1i}(E, [0, L)^d) + \sum_{i=1}^{d} G_{\alpha,\tau,L}^i(E) + \sum_{i=1}^{d} I_{\alpha,\tau,L}^i(E) \right). \]

Splitting of functional into three parts, where \( \text{Per}_{1i} \) and \( G_{\alpha,\tau,L}^i \) depend only on oscillations of the characteristic function of \( E \) along the direction \( e_i \) (“one-dimensional” terms) and \( I_{\alpha,\tau,L}^i \) is a cross-interaction term given by

\[ \frac{2}{d} \int_{[0, L)^d \times \mathbb{R}^d} K_{\alpha,\tau}(\zeta) |\chi_E(x) - \chi_E(x + \zeta_i)| |\chi_E(x) - \chi_E(x + \zeta_i^\perp)| \, d\zeta \, dx \]
One-dimensional estimates

Let $E \subset \mathbb{R}$ an $L$-periodic set. Then,

$$-\text{Per}_{1,i}(E, [0, L]) + \mathcal{G}_{\alpha, \tau, L}^{1d,i}(E) \geq$$

$$\geq \sum_{x \in \partial E \cap [0, L]} -1 + C_\alpha^1 C_\alpha^2 \min((x^+ - x)^{(1-\alpha)}, \tau^{-1})$$

$$+ \min((x - x^-)^{(1-\alpha)}, \tau^{-1})),$$

where

$$x^+ = \inf \{x^+ \in \partial E, \text{ with } x^+ > x\},$$

$$x^- = \sup \{x^- \in \partial E, \text{ with } x^- < x\}$$

and $\min_{\alpha < 1} C_\alpha^1 C_\alpha^2 > \bar{C} > 0$. 
Fix any $\bar{\alpha} < 1$ and any $M > 0$. Then there exists $\eta_0 = \eta_0(M, \bar{\alpha})$ such that whenever

$$\exists x \in \partial E : \quad |x - x^-|, |x - x^+| < \eta_0,$$

then for any $\alpha \leq \bar{\alpha}$

$$-\text{Per}_{1,i}(E) + G_{\alpha,\tau,L}^{1d,i}(E) > M > 0.$$
Monotonicity of kernel

It is used to show $\Gamma$-convergence as $(\alpha, \tau) \downarrow (0, 0)$.

Various $K_{\alpha, \tau}$

Let $\bar{\tau}, \tau, \bar{\alpha}, \alpha \in (0, 1)$. Assume $\bar{\alpha} > \alpha$ and $\bar{\tau} > \tau > 0$. Then $\hat{K}_{\bar{\tau}, \bar{\alpha}}(z) < \hat{K}_{\tau, \alpha}(z)$ and $\hat{K}_{0, \bar{\alpha}}(z) < \hat{K}_{0, \alpha}(z)$. 
Stability

Let $E$ be $[0, L)^d$-periodic and $L^1$-close to stripes in direction $e_1$. Assume that $E$ is not exactly a union of stripes, and the following happens

Then, for all $\alpha < 1$ (i.e. $p(\alpha) > d + 1$) and $\tau$ sufficiently small, the interaction term $\mathcal{I}_{\alpha, \tau, L}^1(E)$ is large, thus making the functional positive and the above configuration not energetically convenient.
Thanks for listening.
Any questions?