We study the 3D topology of Rayleigh-Taylor (RT) and Richtmyer-Meshkov (RM) single-modes, which includes bubbles, jets and saddle points. We present an analytic description of the interface as a whole, for arbitrary time-dependant acceleration $g(t)$. The dependance of morphology on the lattice - Hexagonal, square or triangular -of bubbles are investigated. RM accelerations in the case of a large density ratio produce jets well separated from each other while, in RT case, jets are connected by liquid sheets. We compare our analytic results to numerical simulations.
The RT and RM instabilities (RTI and RMI) play an important role in astrophysics, inertial confinement fusion (ICF), in shock-tube mixing, and in chemical, nuclear or thermonuclear combustion. The asymmetry caused by RTI and RMI in spherical implosions strongly alters the neutron yield and energy gain in ICF targets. Recently, it was proposed to use experiments on very powerful existing and future laser systems such as Omega and the National Ignition Facility in the USA or the Laser Mégajoule in France for modeling the unstable explosion of supernovae (SN) or unstable expansion of SN-remnants. Flow with two shocks and mixing between them (similar to SN-flow) takes place during an explosion of detonation products after an explosion. RTI/RMI are also significant for other astrophysical applications such as planetary nebulae, Wolf-Rayet stars and magnetospheres of neutron stars. The physical origins of exchange instabilities (RTI and RMI) are connected with baroclinic generation of vorticity. RTI is driven by buoyancy (see reviews). RMI occurs after the passage of a shock wave through surface corrugations.

Configurations of 3D single-mode perturbations can be represented by bubble lattices having various geometrical symmetries: hexagonal (B6), square (B4) or triangular (B3). The B stands for “bubble” and the digits 6, 4 and 3 correspond to the number of bubbles adjacent to the chosen one. Saddles and jets also form lattices. In the lattice B6 a jet J has three neighbouring jets. Therefore the lattice B6 is the same time as the lattice J3.

Our goal is to describe 3D phenomena and their dependence on both lattice symmetry and acceleration profile, $g(t)$. This approach is needed to understand phenomena occurring in ICF, astrophysics, etc., because real unstable flows are three-dimensional and the acceleration, $g(t)$, satisfies neither the RTI nor the RMI conditions. To begin our study, let us compare the initial conditions with density $\rho$ and velocity perturbation $\phi$. The lattice B6 is the same time as the lattice J3. Periodic cell structure in the horizontal plane or two-dimensional crystal formed from points of tips of bubbles B, jets J and saddles S marked by circles.

The six direct numerical simulations (DNS) in Fig. 3 (3D) are presented below. In the 2D case (Fig. 4) [resp. 3D case (Fig. 5)], the points B and J (resp. B, S and J) are very important since they correspond to stagnation points. At these points, the velocity of the fluid becomes zero relative to the surface $\eta$.

**FIG. 1.** Rippled structure of 2D solutions. Periodic sequence of parallel valleys. There is a chain ...-B-J-B-J-... of tips of bubbles B and jets J.

**FIG. 2.** Geometry of a single 3D mode in a square lattice (B4). Periodic cell structure in the horizontal plane or two-dimensional crystal formed from points of tips of bubbles B, jets J and saddles S marked by circles.

The 3D/B4 flow is invariant relative to 90°-rotations around B and J vertical axis and to 180°-rotations around S axis (Fig. 3).

For small density ratios, $\mu \ll 1$, the whole flow is clearly divided into two qualitatively very different parts. The first is the bubble envelope imprinting into the dense fluid. Bubbles brake the initially continuous dense fluid and produce jets. Above the envelope, the dense fluid is still in a contiguous state. The second part is the ejecta (jets) pinched and driven down by the imprinting bubbles. The points B belong to the contiguous fluid and the points S and J to the ejecta. The 3D pattern of the ejecta is rather complicated (Fig. 3). The ejecta consists of 1) wall type jets going down from B to S and 2) leg type jets going from S to J. The bubble has the shape of a well in the dense fluid. This well transforms into walls or skirts around the points B.

The relevant topological measure of the form of the ejecta is the geometrical ratio

$$\Gamma(t) = \Delta z_{BS}/\Delta z_{BJ},$$

with $\Delta z_{MN} = z_M(t) - z_N(t)$, (1)

where $z_B$, $z_S$ and $z_J$ are the vertical positions of the points (Fig. 3). $\Gamma$ is the ratio of the length of “skirt” to the length of “legs”.

**FIG. 3.** Numerical results for RM (the triad in the upper panel, $t = 20$) and RT (lower panel, $t = 9$) cases for the lattices B6, B4 and B3 from left to right. The ratio is $\mu = 0.1$.

The six direct numerical simulations (DNS) in Fig. 3 were done by a grid-characteristics method. Several interesting works are devoted to DNS of 3D RTI/RMI flows.

Before presenting our results, let us give the boundary conditions and spectral decomposition. The motion is described by a velocity harmonic potential $\varphi$ (the vorticity is concentrated at the interface). Classical kinematic and dynamic conditions are

$$\eta_t = w - \eta_x u - \eta_y v, \quad \bar{v} = \{u, v, w\}, \quad \eta_t \equiv \partial \eta/\partial t,$$  (2)
\[ \varphi = \frac{\partial^2}{\partial t^2} \varphi + g(t) \eta, \quad f \equiv f|_{\eta} \equiv f|_{z=\eta(x,y,t)} \]  

We represent the potential by a Fourier series and we obtain geometrical \( \eta \) near stagnation points by a Taylor series

\[ \varphi(x, z, t) = \sum_{n=1}^{N} \varphi_n(t) c_{n\infty} e^{-n\Delta z}, \]

where \( c_{n\infty} = \cos n\pi \) and \( N \) is the truncation number. It defines the order of approximation of conditions (3).

2D solution. The expansion (3) satisfies \( \Delta \varphi = 0 \). The expressions \( \varphi_0, K_1, ... \) are unknowns for the ordinary differential equation system in our method of asymptotic collocations (MAC). We say asymptotic collocations because of the close connection to the method of ordinary collocations in which boundary conditions are approximated in a set of points \( \{ x_i \} \). The equations of the MAC appear asymptotically when all points tend to point B, or S, or J. In 2D, these equations, for \( N \leq 6 \), were first derived and integrated in (3) in case of bubbles.

The Fig. 1 presents the \( N = 5 \) solution. Here, for the first time, we use high-order MAC to study jets. We carefully describe acceleration of the jet and for the RM case obtain very accurate values for its terminal (asymptotic) velocity, \( w_J(t = \infty, N = 5) = -1.923 \) \( \varphi(t) \equiv \eta \equiv \eta(x, y, t) \), standard initial conditions. The accuracy of this value is \( \varepsilon(1) = 10^{-1.0} \), \( \varepsilon(2) = 10^{-2.3} \), \( \varepsilon(3) = 10^{-3.16} \) and \( \varepsilon(4) = 10^{-3.43} \), where \( \varepsilon(N) = |w_J(\infty, N + 1) - w_J(\infty, N)|/|w_J(\infty, N)| \). Since the error is very small, the MAC may be used to control the accuracy of other methods.

3D solution. For the B4 lattice we have

\[ \varphi^{(B4)}(x, y, z, t) = \sum_{n=1}^{N} \sum_{m=1}^{M} \varphi_{nm}(t) c_{n\infty} c_{m\infty} \eta^{(4)} \]

\[ \eta(x, y, t) = \sum_{n=1}^{N} \sum_{m=1}^{M} K_{nm}(t) x^{2n} y^{2m}/(2n)!(2m)!, \]

where \( 0 \leq n + m \leq N, \) \( e^{(4)}_{nm} = \exp(-\eta_{0}(\Delta z)) \). In the RM case we have \( \eta_{0}(t) = \eta_{0}(x, y, t) \) and \( \varphi^{(4)}_{nm} = \sqrt{n^2 + m^2} \). The unknowns in the system are \( \eta_0, K_{nm}, \varphi^{(4)}_{nm} \). The terms \( \eta_0, K_{nm}, \varphi^{(4)}_{nm} \) are known. For \( N = 1 \), the indices \( (n, m) \) are 10 and 01. For \( N = 2 \), they are 10, 01, 20, 11 and 02. Points B and J (but not S) are symmetric, \( \varphi_{mn} = \varphi_{nm} \) and \( K_{nm} = K_{mn} \). At the lowest order \( N = 1 \) (Lazyer approximation) the unknowns, at points B and J, are \( \eta_0, K, w \), where \( K = K_{10} = K_{01} \) is the curvature and \( w = -\varphi_{10} = -\varphi_{01} \) is the velocity of a bubble or a jet. The system \( N = 1 \) for the B4 lattice, valid in points B and J, has been derived and solved for the RM case in (3). For \( N = 2 \) this system was considered in (3), (4), (5) and 3D \( N = 1 \) systems for B6 and B4 cases were examined in (3), (4), (5).

This is the first time we use MAC to describe the dynamics of all three kinds of points (B, S and J) at order \( N = 2 \). It appears much more complicated than \( N = 1 \) (compare also with 2D, \( N \leq 5 \)). The terminal velocities of RM jet and saddle are \( v_J(\infty, N = 1) = -\sqrt{2}, w_J(\infty, N = 2) = -1.698 \) and \( w_S(\infty, N = 1) = -0.512 \). This gives an accurate asymptotic value of the geometrical ratio \( \eta_0 \), \( \Gamma \rightarrow w_S(\infty)/w_J(\infty) \) as \( t \rightarrow \infty \). Fig. 2 gives an example of second order solution.

For the B6 and B3 lattices, we have in first order

\[ \varphi^{(B6, 3)} = [\varphi^+(t) c^+ + \varphi^-(t) c^- + \varphi^0(t) c^0] e^{-\Delta z}, \]

where \( c^\pm = \cos k^\pm \vec{r}, k^\pm = \{1/2, \pm \sqrt{3}/2, 0\}, \)

\( k^0 = \{0, 0, 0\} \) and \( \varphi^0 = \varphi^0 = \varphi^0 \) for B and J points.

The first order dynamical systems for points B and J for all three lattices B6, 4 and 3 are the same. \( \eta_0 = w, \)

\( \dot{\omega} = -[w^2 + 4g(t) K]/(2(1 + 2K), K = -(1 + 4K) w/2, \)

where \( f(t) = df/dt \). In this system, we emphasize that \( g(t) \) is an arbitrary function. Eliminating \( t \) between first and last equations, we obtain \( dK/d\eta_0 = -1/2 - 2K \). The solution is

\[ K(\eta_0) = -1/4 + \exp(-2\eta_0)/4. \]

The 2D analog is \( dK/d\eta_0 = -1 - 3K \) and \( K = -1/3 + \exp(-3\eta_0)/3 \). The solution (3) has linear asymptotes for \( |\eta_0| \ll 1 \) and tends to \(-1/4 \) for \( |\eta_0| \gg 1 \). In the linear stage we have \( \eta = \eta_0(e^x + e^y)/2 \) (B4), \( \eta = \eta_0(e^x + e^y)/2 \) (B6, 3) and \( K = -\eta_0/2 \). In RM case, the differential system has integrals (10)

\[ \sqrt{1 + 2K} - \frac{1}{\sqrt{2}} \ln \sqrt{2 + 4K} + \sqrt{1 + 4K} \exp(-\sqrt{2}) = \frac{w_0}{t}, \]

\[ \frac{u_0}{w} - 1 + \frac{1}{2\sqrt{2}} \ln \left( \frac{\sqrt{2} + 1}{\sqrt{2} - 1} \frac{u_0 - w}{u_0 + w} \right) = \frac{w_0}{t}. \]

Substituting solution \( K(\eta_0) \) into (7) we obtain

\[ \sqrt{1 + \epsilon^{2\eta_0}} - \frac{1}{\sqrt{2}} \ln \sqrt{1 + e^{-2\eta_0}} = \frac{w_0}{t}. \]

In (3), \( w_0 \) is the initial velocity of B or J points. We will assume that this initial velocity for the tip of bubble B is equal to 1 for all lattices. Then initial velocities of points J are \(-1/2, -1 \) and \(-2 \) for lattices B6, 4 and 3 respectively. Similarly to (7), the 2D solution has been obtained previously (3). (10). The integrals (11) give \( \eta_0(t), K(t), w(t) \) in analytic form for all times from initial to asymptotic state. They are valid in the RM case for B and J points of B6, 4 and 3 lattices.

It is very surprising that the relation \( K(\eta_0) \) between the main geometrical characteristics \( \eta_0 \) and \( K \) for a bubble penetrating into the dense fluid does not depend upon
$g(t)$ for $N = 1$. This means that the relation is only weakly dependent on $g(t)$ in the general case with arbitrary $N$.

The system for $N = 2$ is rather long and can not be given here. However, for saddles and $N = 1$, we have:

$$\dot{\gamma}(0) = 2 \frac{c}{\beta}$$

$$\ddot{K} = -(1 + 3K)\alpha + K\beta,$$  \hspace{1cm} (10)

$$\dot{Q} = -Q\alpha + (1 + 3Q)\beta,$$ \hspace{1cm} (11)

$$-(1 + K)\dot{\alpha} + K\dot{\beta} = \alpha^2 + g(t)K,$$ \hspace{1cm} (12)

$$-Q\dot{\alpha} + (1 + Q)\dot{\beta} = \beta^2 + g(t)Q,$$ \hspace{1cm} (13)

in the B4 case with $\varphi = (\alpha c_1 + \beta c_1) e^{-\Delta z}$. For B6,3 we have

$$\dot{\eta}_0 = 2\alpha - \gamma,$$ \hspace{1cm} (14)

$$2\dot{K} = -(1 + 6K)\alpha + 2(1 + 3K)\gamma,$$ \hspace{1cm} (15)

$$2\dot{Q} = -(3 + 10Q)\alpha + 2Q\gamma,$$ \hspace{1cm} (16)

$$-2(1 + 4K)\dot{\alpha} + 4(1 + K)\dot{\gamma} = (\alpha - 2\gamma)^2 + 4g(t)K,$$ \hspace{1cm} (17)

$$-2(3 + 4Q)\dot{\alpha} + 4Q\dot{\gamma} = 9\alpha^2 + 4g(t)Q,$$ \hspace{1cm} (18)

where $\alpha$ and $\gamma$ are the amplitudes of potential $\varphi = (\alpha c^+ + \alpha c^-) + \gamma c e^{-\Delta z}$ written with respect to the point S. In the standard case, initial conditions are $\alpha(0) = \gamma(0) = -1/3$ (B6), $\alpha(0) = \beta(0) = -1/2$ (B4), $\alpha(0) = \gamma(0) = 2/3$ (B3) and $\eta_0(0) = K(0) = Q(0) = 0$ (B6,4,3). From (10) and (15), initial velocities of the saddles are $-1/3$ (B6), 0 (B4) and 2/3 (B3) - to be compared with initial velocities of jets. Although, the systems are the same for B6 and B3, the initial conditions differ. From systems (14) or (15) we found numerically the trajectories of saddles $\eta_0(t)$ and the evolution of the curvatures $K(t), Q(t)$. The terminal velocities of saddles in RM case are $w_S(\infty, N = 1) = -0.748$ (B6), $w_S(\infty, 1) = -0.572$ (B4) and $w_S(\infty, 1) = -0.196$ (B3).

In Fig. 4 we present the time variation of the geometrical ratio $\Gamma$. Initial values of the ratio are $\Gamma_{B6}(t = 0) = 8/9$, $\Gamma_{B4}(0) = 1/2$ and $\Gamma_{B3}(0) = 1/9$. Simulations $(\mu = 0.1)$ fit rather well with the theory $(\mu \to 0)$ although $\mu$ differs. We observe that the agreement between theory and simulation is better for the RMI than for the RTI. The increase of $N$ significantly improves the accuracy (for the B4 system, the $N = 2$ curve and the one coming from simulation are very close for the RMI). The function $g(t)$ influences therefore the evolution of this ratio. The morphology of the ejecta mainly depends on the type of lattice. The shortest skirts are obtained for the B3 lattice (lower set of curves) which produces powerful and fast jets (Fig. 3). Previously, it has been shown [16] that for random (turbulent) cases, the patterns are similar to the B6 and B4 lattices (B3 lattices occur only for special conditions). Moreover, B3 type structures may appear after reshock and the corresponding rephasing because B6 bubbles transform into B3 jets.

In summary, we have considered the effects of lattice and time-dependant acceleration on the evolution of the interface. We found that the shape of imprinting bubbles is very weakly dependent on these factors, but the position of the bubbles depends on the history of the acceleration. At the same time, the shape and dynamics of the ejecta appear very sensitive to both factors.

The authors would like to thank J.F. Haas for many interesting discussions and Douglas Wilson for careful reading. A.M.O. and N.A.I. are grateful for the support from RBRF (00-01-00250, 99-02-16666) and from Landau Institute - CNRS jumelage (N.A.I.).

FIG. 4. Effect of lattice symmetry and acceleration history on the evolution of the shape of $\eta$. The upper, middle and lower sets of curves correspond respectively to the B6, B4 and B3 lattices. The grey and black curves correspond respectively to numerical simulations - see Fig. 3 - and to the analytical approach with $N = 1$. The dashed curves have been obtained from theory with $N = 2$. The curves labelled by 1 (resp. 2) describe the RTI (resp. RMI).

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