BEAUVILLE SURFACES AND PROBABILISTIC GROUP THEORY

SHELLY GARION

ABSTRACT. A Beauville surface is a complex algebraic surface that can be presented as a quotient of a product of two curves by a suitable action of a finite group. Bauer, Catanese and Grunewald have been able to intrinsically characterize the groups appearing in minimal presentations of Beauville surfaces in terms of the existence of a so-called "Beauville structure". They conjectured that all finite simple groups, except $A_5$, admit such a structure. This conjecture has recently been proved by Guralnick-Malle and Fairbairn-Magaard-Parker.

In this survey we demonstrate another approach towards the proof of this conjecture, based on probabilistic group-theoretical methods, by describing the following three works. The first is the work of Garion, Larsen and Lubotzky, showing that the above conjecture holds for almost all finite simple groups of Lie type. The second is the work of Garion and Penegini on Beauville structures of alternating groups, based on results of Liebeck and Shalev, and the third is the case of the group $\text{PSL}_2(p^r)$, in which we give bounds on the probability of generating a Beauville structure. We also discuss other related problems regarding finite simple quotients of hyperbolic triangle groups and present some open questions and conjectures.

1. Beauville surfaces and Beauville structures

A Beauville surface $S$ (over $\mathbb{C}$) is a particular kind of surface isogenous to a higher product of curves, i.e., $S = (C_1 \times C_2)/G$ is a quotient of a product of two smooth curves $C_1$ and $C_2$ of genus at least two, modulo a free action of a finite group $G$ which acts faithfully on each curve. For Beauville surfaces the quotients $C_i/G$ are isomorphic to $\mathbb{P}^1$ and both projections $C_i \to C_i/G \cong \mathbb{P}^1$ are coverings branched over three points. A Beauville surface is in particular a minimal surface of general type.

Beauville [4] constructed a minimal surface of general type $S$ with $K_S^2 = 8$ and $p_g = q = 0$ in the following way: take two curves $C_1 = C_2$ given by the Fermat equation $x^5 + y^5 + z^5 = 0$ and $G$ the group $(\mathbb{Z}/5\mathbb{Z})^2$ acting on $C_1 \times C_2$ by

$$(a, b) \cdot ([x : y : z], [u : v : w]) = ([\xi^a x : \xi^b y : z], [\xi^{a+3b} u : \xi^{2a+4b} v : w]),$$

where $\xi = e^{2\pi i/5}$ and $a, b \in \mathbb{Z}/5\mathbb{Z}$. Then define $S$ by the quotient $(C_1 \times C_2)/G$. Moreover $C_i \to C_i/G \cong \mathbb{P}^1$ and both covers are branched in exactly three points. Curves with such properties are said to be triangle curves.

Inspired by this construction Catanese [5] observed that in general if $C_1$ and $C_2$ are two triangle curves with group $G$, if the action of $G$ on the product $C_1 \times C_2$ is free, then $S = (C_1 \times C_2)/G$ is a strongly rigid surface, i.e., if $S'$ is another surface homotopically

\begin{flushright}
2000 Mathematics Subject Classification. 20D06, 20H10, 14J10, 14J29, 30F99.

The author was supported by the SFB 878 “Groups, Geometry and Actions”.
\end{flushright}
equivalent to $S$ then $S'$ is either biholomorphic or antibiholomorphic to $S$. He proposed to name these surfaces *Beauville surfaces*.

A Beauville surface $S$ is either of *mixed* or *unmixed* type according respectively as the action of $G$ exchanges the two factors (and then $C_1$ and $C_2$ are isomorphic) or $G$ acts diagonally on the product $C_1 \times C_2$. The subgroup $G_0$ (of index $\leq 2$) of $G$ which preserves the ordered pair $(C_1, C_2)$ is then respectively of index 2 or 1 in $G$. Any Beauville surface $S$ can be presented in such a way that the subgroup $G_0$ of $G$ acts effectively on each of the factors $C_1$ and $C_2$. Catanese called such a presentation *minimal* and proved its uniqueness in [3]. In this survey we shall consider only Beauville surfaces of unmixed type so that $G_0 = G$.

An extensive research on Beauville surfaces was initiated by the collaboration of Bauer, Catanese and Grunewald [1,2]. They have been able to intrinsically characterize the groups appearing in minimal presentations of unmixed Beauville surfaces in terms of the existence of the so-called unmixed *Beauville structure*.

**Definition 1.** An *unmixed Beauville structure* for a finite group $G$ is a quadruple $(x_1, y_1; x_2, y_2)$ of elements of $G$, which determines two triples $(x_1, y_1, z_1)$ and $(x_2, y_2, z_2)$ satisfying:

(i) $x_1y_1z_1 = 1$ and $x_2y_2z_2 = 1$,

(ii) $(x_1, y_1) = G$ and $(x_2, y_2) = G$,

(iii) $\Sigma(x_1, y_1, z_1) \cap \Sigma(x_2, y_2, z_2) = \{1\}$,

where $\Sigma(x, y, z)$ is the union of the conjugacy classes of all powers of $x$, all powers of $y$, and all powers of $z$.

Moreover denoting the order of an element $g$ in $G$ by $|g|$, we define the *type* $\tau$ of $(x, y, z)$ to be the triple $(|x|, |y|, |z|)$. In this situation, we say that $G$ admits an *unmixed Beauville structure of type* $(\tau_1, \tau_2)$.

The question whether a finite group admits an unmixed Beauville structure of a given type is closely related to the question whether it is a quotient of certain triangle groups. More precisely, a necessary condition for a finite group $G$ to admit an unmixed Beauville structure of type $(\tau_1, \tau_2) = ((r_1, s_1, t_1), (r_2, s_2, t_2))$ is that $G$ is a quotient with torsion-free kernel of the triangle groups $T_{r_1,s_1,t_1}$ and $T_{r_2,s_2,t_2}$, where $$T_{r,s,t} = \langle x, y, z : x^r = y^s = z^t = xyz = 1 \rangle.$$

Indeed, conditions (i) and (ii) of Definition [1] are equivalent to the condition that $G$ is a quotient of each of the triangle groups $T_{|x|,|y|,|z|}$, for $i \in \{1, 2\}$, with torsion-free kernel.

When investigating the existence of an unmixed Beauville structure for a finite group, one can consider only types $(\tau_1, \tau_2)$, where for $i \in \{1, 2\}$, $\tau_i = (r_i, s_i, t_i)$ satisfies $1/r_i + 1/s_i + 1/t_i < 1$. Then $T_{r_i,s_i,t_i}$ is a (infinite non-soluble) hyperbolic triangle group and we may say that $\tau_i$ is *hyperbolic*. Indeed, if $1/r_i + 1/s_i + 1/t_i > 1$ then $T_{r_i,s_i,t_i}$ is a finite group, and moreover, it is either dihedral or isomorphic to one of $A_4$, $A_5$ or $S_4$. By [1] Proposition 3.6 and Lemma 3.7, in these cases $G$ cannot admit an unmixed Beauville structure. If $1/r_i + 1/s_i + 1/t_i = 1$ then $T_{r_i,s_i,t_i}$ is one of the (soluble infinite) “wall-paper” groups, and by [1] §6, none of its finite quotients can admit an unmixed Beauville structure.

Observe that condition (iii) of Definition [1] is clearly satisfied under the assumption that $r_1s_1t_1$ is coprime to $r_2s_2t_2$. However this assumption is not always necessary, as demonstrated by many examples, such as Beauville’s original construction, abelian groups [1] Theorem 3.4, alternating groups [19] Theorem 1.2 and the group $\text{PSL}_2(p^e)$ [17].
2. Beauville surfaces and finite simple groups

A considerable effort has been made to classify the finite simple groups which admit an unmixed Beauville structure. We recall that by the classification theorem of finite simple groups, any finite simple group belongs to one of the following families: the cyclic groups $\mathbb{Z}_p$ of prime order; the alternating permutation groups $A_n (n \geq 5)$; the finite simple groups of Lie type, defined over finite fields (e.g. $\text{PSL}_n(q)$); and finally the 26 so-called sporadic groups.

A finite abelian simple group clearly does not admit an unmixed Beauville structure as given a prime $p$, any pair $(a, b)$ of elements of the cyclic group $\mathbb{Z}_p$ of prime order $p$ generating it satisfies $\Sigma(a, b, c) = \mathbb{Z}_p$. In fact Bauer, Catanese and Grunewald showed in [1, Theorem 3.4] that the only finite abelian groups admitting an unmixed Beauville structure are the abelian groups of the form $\mathbb{Z}_n \times \mathbb{Z}_n$ where $n$ is a positive integer coprime to 6. (Here $\mathbb{Z}_n$ denotes a cyclic group of order $n$.)

In [1] the authors also provide the first results on finite non-abelian simple groups admitting an unmixed Beauville structure. More precisely they show that the alternating groups of sufficiently large order admit an unmixed Beauville structure, as well as the projective special linear groups $\text{PSL}_2(p)$ where $p > 5$ is a prime. Moreover using computational methods, they checked that every finite non-abelian simple group of order less than 50000 admits an unmixed Beauville structure with the exception of the alternating group $A_5$. Based on these results and the latter observation, they conjectured that all finite non-abelian simple groups admit an unmixed Beauville structure with the exception of $A_5$.

This conjecture has received much attention and has recently been proved to hold. Concerning the simple alternating groups, it was established in [15] that $A_5$ is indeed the only one not admitting an unmixed Beauville structure. In [16] the conjecture is shown to hold for the projective special linear groups $\text{PSL}_2(q)$ (where $q > 5$), the Suzuki groups $2B_2(q)$ and the Ree groups $2G_2(q)$ as well as other families of finite simple groups of Lie type of small rank. More precisely, the projective special and unitary groups $\text{PSL}_3(q)$, $\text{PSU}_3(q)$, the simple groups $G_2(q)$ and the Steinberg triality groups $3D_4(q)$ are shown to admit an unmixed Beauville structure if $q$ is large (and the characteristic $p$ is greater than 3 for the simple exceptional groups of type $G_2$ or $3D_4$). The next major result concerning the investigation of the conjecture with respect to the finite simple groups of Lie type was pursued by Garion, Larsen and Lubotzky who showed in [18] that the conjecture holds for finite non-abelian simple groups of sufficiently large order. The final step regarding the investigation of the conjecture was carried out by Guralnick and Malle [22] and Fairbairn, Magaard and Parker [13] who established its veracity in general, namely,

**Theorem 2.** [22] Any finite non-abelian simple group, except $A_5$, admits an unmixed Beauville structure.

There has also been an effort to classify the finite quasisimple groups and almost simple groups which admit an unmixed Beauville structure. Recall that a finite group $G$ is quasisimple provided $G/Z(G)$ is a non-abelian simple group and $G = [G, G]$. In [16] it was shown that $\text{SL}_2(q)$ (for $q > 5$) admits an unmixed Beauville structure. Fairbairn, Magaard and Parker [13] proved the following general result.

**Theorem 3.** [13] With the exceptions of $\text{SL}_2(5)$ and $\text{PSL}_2(5) \cong \text{SL}_2(4) \cong A_5$, every finite quasisimple group admits an unmixed Beauville structure.
Recall that a group $G$ is called almost simple if there is a non-abelian simple group $G_0$ such that $G_0 \leq G \leq \text{Aut}(G_0)$. By [2, 15] the symmetric groups $S_n$ (where $n \geq 5$) admit an unmixed Beauville structure, and by [17] the group $\text{PGL}_2(p^e)$ admits such a structure.

Moreover for the alternating and symmetric groups Garion and Penegini [19] proved another conjecture that Bauer, Catanese and Grunewald proposed in [1], that almost all of these groups admit a Beauville structure with fixed type, namely,

**Theorem 4.** [19, Theorem 1.2]. If $\tau_1 = (r_1, s_1, t_1)$ and $\tau_2 = (r_2, s_2, t_2)$ are two hyperbolic types, then almost all alternating groups $A_n$ admit an unmixed Beauville structure of type $(\tau_1, \tau_2)$.

A similar theorem also applies for symmetric groups, see [19], and a similar conjecture was raised in [19], replacing $A_n$ by a finite simple classical group of Lie type of sufficiently large Lie rank, namely,

**Conjecture 5.** [19, Conjecture 1.7]. Let $\tau_1 = (r_1, s_1, t_1)$ and $\tau_2 = (r_2, s_2, t_2)$ be two hyperbolic types. If $G$ is a finite simple classical group of Lie type of Lie rank large enough, then it admits an unmixed Beauville structure of type $(\tau_1, \tau_2)$.

In contrast, when the Lie rank is very small, as in the case of $\text{PSL}_2(q)$, such a conjecture does not hold, as demonstrated in [17], where there is a characterization of the types of Beauville structures for these groups.

It is well known that almost all pairs of elements in a finite simple (non-abelian) group are generating pairs [11, 25, 31], hence the following question was raised in the workshop “Beauville surfaces and groups 2012”.

**Question 6.** Let $G$ be a finite (non-abelian) simple group. What is the probability $P(G)$ that for four random elements $x_1, y_1, x_2, y_2 \in G$ the quadruple $(x_1, y_1; x_2, z_2)$ is an unmixed Beauville structure for $G$?

In particular, is it true that if $G = A_n$ or $G = G_n(q)$, a finite simple group of Lie type of Lie rank $n$, then $P(G) \to 1$ as $n \to \infty$?

Two interesting comments were made during the workshop regarding this question. The first comment, due to Malle, is that for finite simple groups of Lie type of bounded Lie rank, $P(G)$ does not go to 1, and it is bounded above by a function of the rank. The second comment, due to Magaard, is that the techniques in [13] demonstrate that one can generate many unmixed Beauville structures for the finite simple groups of Lie type, allowing to obtain a constant lower bound on $P(G)$, when $G$ is a finite simple classical group. In the specific case where $G = \text{PSL}_2(q)$ we give the following bounds on the probability of $P(G)$ (see [4,2] for the proof).

**Theorem 7.** Let $G = \text{PSL}_2(q)$.

- If $q$ is odd then $\frac{1}{32} - \epsilon_q \leq P(G) \leq \frac{15}{32} + \epsilon_q$,
- If $q$ is even then $\frac{1}{32} - \epsilon_q \leq P(G) \leq \frac{35}{36} + \epsilon_q$,

where $\epsilon_q \to 0$ as $q \to \infty$. 

3. Hyperbolic triangle groups and their finite quotients

Since for a finite group $G$ which admits an unmixed Beauville structure there exists an epimorphism from a hyperbolic triangle group to $G$, we recall in this section some results on finite quotients of hyperbolic triangle groups.

A hyperbolic triangle group $T$ is a group with presentation

$$T = T_{r,s,t} = \langle x, y, z : x^r = y^s = z^t = xyz = 1 \rangle,$$

where $(r, s, t)$ is a triple of positive integers satisfying the condition $1/r + 1/s + 1/t < 1$. Geometrically, let $\Delta$ be a hyperbolic triangle group having angles of sizes $\pi/r$, $\pi/s$, $\pi/t$, then $T$ can be viewed as the group generated by rotations of angles $\pi/r$, $\pi/s$, $\pi/t$ around the corresponding vertices of $\Delta$ in the hyperbolic plane $\mathbb{H}^2$. Moreover, a hyperbolic triangle group $T_{r,s,t}$ has positive measure $\mu(T_{r,s,t})$ where $\mu(T_{r,s,t}) = 1 - (1/r + 1/s + 1/t)$. As hyperbolic triangle groups are infinite and non-soluble it is interesting to study their finite quotients, particularly the simple ones.

A hyperbolic triangle group $T_{r,s,t}$ has minimal measure when $(r, s, t) = (2, 3, 7)$. The group $T_{2,3,7}$ is also called the $(2, 3, 7)$-triangle group and its finite quotients are also known as Hurwitz groups. These are named after Hurwitz who showed in the late nineteenth century that if $S$ is a compact Riemann surface of genus $h \geq 2$ then $|\text{Aut } S| \leq 84(h - 1)$ and this bound is attained if and only if $\text{Aut } S$ is a quotient of the triangle group $T_{2,3,7}$. Following this result, much effort has been given to classify Hurwitz groups, especially the simple ones, see for example [8] for a historical survey, and [9, 46] for the current state of the art.

Most alternating groups are Hurwitz as shown by Conder (following Higman) who proved in [7] that if $n > 167$ then the alternating group $A_n$ is a quotient of $T_{2,3,7}$. Concerning the finite simple groups of Lie type, there is a dichotomy with respect to their occurrence as quotients of $T_{2,3,7}$ depending on whether the Lie rank is large or not. Indeed as shown in [35] many classical groups of large rank are Hurwitz (and there is no known example of classical groups of large rank which are not Hurwitz). As an illustration by [36] if $n \geq 267$ then the projective special linear group $\text{PSL}_n(q)$ is Hurwitz for any prime power $q$. The behavior of finite simple groups of Lie type of relatively low rank with respect to the Hurwitz generation problem is rather sporadic. As an illustration by respective results of [6, 46, 38, 37], $\text{PSL}_3(q)$ is Hurwitz if and only if $q = 2$, $\text{PSL}_4(q)$ is never Hurwitz, $G_2(q)$ is Hurwitz for $q \geq 5$, and $\text{PSL}_2(p^e)$ is Hurwitz if and only if $e = 1$ and $p \equiv 0, \pm 1 \mod 7$, or $e = 3$ and $p \equiv \pm 2, \pm 3 \mod 7$. Therefore, unlike the alternating groups, there are finite simple groups of Lie type of large order which are not quotients of $T_{2,3,7}$. As for the 26 sporadic finite simple groups, 12 of them are Hurwitz (including the Monster [49]) while the other 14 groups are not.

Turning to general hyperbolic triples $(r, s, t)$ of integers, Higman had already conjectured in the late 1960s that every hyperbolic triangle group has all but finitely many alternating groups as quotients. This was eventually proved by Everitt [12], namely,

**Theorem 8.** [12] For any hyperbolic triangle group $T = T_{r,s,t}$, if $n \geq n_0(r, s, t)$ then the alternating group $A_n$ is a quotient of $T$.

Later, Liebeck and Shalev [32] gave an alternative proof to Higman’s Conjecture based on probabilistic group theory, and moreover they have conjectured the following.
Conjecture 9. [33]. For any hyperbolic triangle group $T = T_{r,s,t}$, if $G = G_n(q)$ is a finite simple classical group of Lie rank $n \geq n_0(r,s,t)$, then the probability that a randomly chosen homomorphism from $T$ to $G$ is an epimorphism tends to 1 as $|G| \to \infty$.

This conjecture has been proved by Marion [41, 42] for certain families of groups of small Lie rank and certain triples $(r,s,t)$. For example, take $(r,s,t)$ to be a hyperbolic triple of odd primes and $G = \text{PSL}_3(q)$ or $\text{PSU}_3(q)$ containing elements of orders $r, s$ and $t$, then the conjecture holds. As another example, if $(r,s,t)$ is a hyperbolic triple of primes and $G = 2^2B_2(q)$ or $2^2G_2(q)$ contains elements of orders $r, s$ and $t$, then the conjecture also holds.

However, for finite simple groups of small Lie rank such a conjecture does not hold in general, and it fails to hold in the case of $\text{PSL}_2(q)$. Indeed, Langer and Rosenberger [27] and Levin and Rosenberger [29] had generalized the aforementioned result of Macbeath, and determined, for a given prime power $q = p^e$, all the triples $(r,s,t)$ such that $\text{PSL}_2(q)$ is a quotient of $T_{r,s,t}$, with torsion-free kernel. It follows that if $(r,s,t)$ is hyperbolic, then for almost all primes $p$, there is precisely one group of the form $\text{PSL}_2(p^e)$ or $\text{PGL}_2(p^e)$ which is a homomorphic image of $T_{r,s,t}$ with torsion-free kernel. We note that this result can also be obtained by using other techniques. Firstly, Marion [39] has recently provided a proof for the case where $r, s, t$ are primes relying on probabilistic group theoretical methods. Secondly, it also follows from the representation theoretic arguments of Vincent and Zalesski [48, Theorems 2.9 and 2.11]. Such methods can be used for dealing with other families of finite simple groups of Lie type, see for example [40, 43, 45, 47, 48].

Recently, a new approach was presented by Larsen, Lubotzky and Marion [28], based on the theory of representation varieties (via deformation theory). They prove a conjecture of Marion [40] showing that various finite simple groups are not quotients of $T_{r,s,t}$, as well as positive results showing that many finite simple groups are quotients of $T_{r,s,t}$.

4. Beauville structures for the group $\text{PSL}_2(q)$

In this section we discuss the specific case of $\text{PSL}_2(q)$, and briefly sketch the proof of Garion and Penegini [19] for the following theorem, which is based on results of Macbeath [37].

Theorem 10. [16, 19]. Let $p$ be a prime number, and assume that $q = p^e$ is at least 7. Then the group $\text{PSL}_2(q)$ admits an unmixed Beauville structure.

In addition, we bound the probability that four random elements in $\text{PSL}_2(q)$ generate an unmixed Beauville structure and prove Theorem 7.

4.1. Sketch of the proof of Theorem 10. In order to construct an unmixed Beauville structure for $\text{PSL}_2(q)$ one needs to find a quadruple $(A_1, B_1; A_2, B_2)$ of elements of $\text{PSL}_2(q)$ satisfying the three conditions given in Definition 1. This can be done directly by finding specific elements in the group satisfying these conditions (see [16]) or indirectly by using the following results of Macbeath [37] (see [17, 19]).

Theorem 11. [37, Theorem 1]. For every $\alpha, \beta, \gamma \in \mathbb{F}_q$ there exist three matrices $A, B, C \in \text{SL}_2(q)$ satisfying $\text{tr}(A) = \alpha$, $\text{tr}(B) = \beta$, $\text{tr}(C) = \gamma$ and $ABC = I$.

This theorem immediately implies Condition (i).

Moreover, Macbeath [37] classified the pairs of elements in $\text{PSL}_2(q)$ in a way which makes it easy to decide what kind of subgroup they generate. By [37, Theorem 2], a triple
(α, β, γ) ∈ ℱ3 is singular, namely α² + β² + γ² − αβγ − 4 = 0, if and only if for the corresponding triple of matrices (A, B, C), the group generated by the images of A and B is a structural subgroup of PSL₂(q), that is a subgroup of the Borel or a cyclic subgroup.

Hence, in order to verify Condition (ii), one needs to show that the subgroup generated by A, B ∈ PSL₂(q) is neither a structural subgroup (using the aforementioned result of Macbeath), not a dihedral subgroup, not one of the small subgroups A₄, S₄ or A₅, and not a subfield subgroup (namely, isomorphic to PSL₂(q₁) or to PGL₂(q₁), where q = qᵢⁿ) hence it must be PSL₂(q) itself, as the subgroup structure of PSL₂(q) is well-known (see e.g. [10, 44]). For example, Condition (ii) is always satisfied when q ≥ 13 and the orders |A| = |B| = |C| = (q − 1)/d or (q + 1)/d, where d = gcd(2, q − 1).

Condition (iii) is clearly satisfied under the assumption that the product of the orders |A₁| · |B₁| · |C₁| is coprime to |A₂| · |B₂| · |C₂|. For example, for any q > 7 the group PSL₂(q) admits unmixed Beauville structures of types

\[ \left( \left( \frac{q-1}{d}, \frac{q-1}{d}, \frac{q+1}{d}, \frac{q+1}{d} \right), \left( \frac{q-1}{d}, \frac{q-1}{d}, \frac{q+1}{d}, \frac{q+1}{d} \right) \right), \]

and

\[ \left( \left( \frac{q-1}{d}, \frac{q-1}{d}, \frac{q+1}{d}, \frac{q+1}{d} \right), \left( \frac{q-1}{d}, \frac{q-1}{d}, \frac{q+1}{d}, \frac{q+1}{d} \right) \right), \]

appearing in [19] and [16] respectively.

However this assumption is not always necessary. Indeed, by [17], PSL₂(q) (where q = p²e, p an odd prime) always admits unmixed Beauville structures of types ((p, p, t₁), (p, p, t₂)) for certain t₁, t₂ dividing (q − 1)/2, (q + 1)/2 respectively.

This approach can be effectively used to construct many unmixed Beauville structures for PSL₂(q), and in [17] there is a characterization of the types of unmixed Beauville structures for this group.

4.2. Proof of Theorem 7. The proof relies on considering the various types of elements in G = PSL₂(q). Recall that an element in G is called split if its order divides (q − 1)/d (where d = gcd(2, q − 1)), non-split if its order divides (q + 1)/d, and unipotent if its order is 2 (and then the trace of its pre-image in SL₂(q) equals ±2).

It is well-known that there are roughly q³ matrices in SL₂(q), and moreover, for any \( \alpha \in ℱ_q \) the number of matrices A ∈ SL₂(q) with tr(A) = α is roughly \( q^3 \) (see for example [3, Table 1]). In addition, the probability that a random element in \( ℱ_q \) is a trace of a split (respectively, non-split) matrix in SL₂(q) goes to 1/2 as \( q \to \infty \) (see [37, Lemma 2]). Therefore, probabilistically, the number of unipotents in G is negligible, and moreover, if we denote by \( P_q^s \) (respectively \( P_q^n \)) the probability that a random element in G is split (respectively, non-split) then \( P_q^s \to \frac{1}{2} \) and \( P_q^n \to \frac{1}{2} \) as \( q \to \infty \).

By [3] Proposition 7.2 it follows that for any non-singular triple (α, β, γ) ∈ ℱ₃, the number of triples (A, B, C) ∈ SL₂(q³) satisfying tr(A) = α, tr(B) = β, tr(C) = γ and ABC = I is roughly \( q^3 \). By [37] Lemma 3] almost all triples in \( ℱ_q^3 \) are non-singular. Since the probability that a random triple of element in \( ℱ_q^3 \) contains only traces of split (respectively, non-split) matrices goes to \( \frac{1}{4} \) as \( q \to \infty \), it follows that the probability that two random elements A, B ∈ G satisfy that A, B and AB are all split (respectively, non-split), goes to \( \frac{1}{8} \) as \( q \to \infty \).
In order to obtain a lower bound, observe that the probability that two random elements \( A, B \in G \) do not generate \( G \) goes to 0 as \( q \to \infty \) (this can be deduced from the aforementioned results of Macbeath \([37]\), or alternatively, as a specific case of \([25]\)). Namely, 
\[
\frac{\#\{(A, B) \in G^2 : \langle A, B \rangle \neq G\}}{|G|^2} \leq \epsilon'_q,
\]
where \( \epsilon'_q \to 0 \) as \( q \to \infty \). Hence,
\[
\frac{\#\{(A, B) \in G^2 : A, B, AB \text{ are split and } \langle A, B \rangle = G\}}{|G|^2} \geq \frac{1}{8} - \epsilon'_q,
\]
and
\[
\frac{\#\{(A, B) \in G^2 : A, B, AB \text{ are non-split and } \langle A, B \rangle = G\}}{|G|^2} \geq \frac{1}{8} - \epsilon'_q.
\]
Since \( (q - 1)/d \) and \( (q + 1)/d \) are relatively prime then Condition \((iii)\) is immediately satisfied when \( A_1, B_1, A_2B_2 \) are all split and \( A_2, B_2, A_2B_2 \) are all non-split (and vice-versa). Therefore,
\[
\frac{\#\{(A_1, B_1, A_2, B_2) \in G^4 : (A_1, B_1; A_2, B_2) \text{ is a Beauville structure}\}}{|G|^4} \geq \left( \frac{1}{8} - \epsilon'_q \right)^2 + \left( \frac{1}{8} - \epsilon'_q \right)^2 \geq \frac{1}{32} - \epsilon_q,
\]
where \( \epsilon_q \to 0 \) as \( q \to \infty \).

Assume that \( q \) is odd. In order to obtain an upper bound, observe that if the orders of \( A_1 \) and \( A_2 \) are both even then Condition \((iii)\) is not satisfied, see \([17, \text{Lemma 4.2}]\). Denote by \( P_q^e \) the probability that a random element in \( G \) has even order, then \( P_q^e \geq \frac{1}{4} - \epsilon'_q \), where \( \epsilon'_q \to 0 \) as \( q \to \infty \). Indeed, if \( (q - 1)/2 \) (respectively, \( (q + 1)/2 \)) is even, then at least half of the split (respectively, non-split) elements are of even order, since any split (respectively, non-split) element belongs to a cyclic subgroup of order \( (q - 1)/2 \) (respectively, \( (q + 1)/2 \)), and at least half the elements in a cyclic group of even order are of even order. Therefore,
\[
\frac{\#\{(A_1, B_1, A_2, B_2) \in G^4 : (A_1, B_1; A_2, B_2) \text{ is not a Beauville structure}\}}{|G|^4} \geq \left( \frac{1}{4} - \epsilon'_q \right)^2 \geq \frac{1}{16} - \epsilon_q,
\]
where \( \epsilon_q \to 0 \) as \( q \to \infty \).

When \( q \) is even the proof is similar, replacing \( P_q^e \) by the probability that a random element in \( G \) has order divisible by 3, which is at least \( \frac{1}{6} - \epsilon'_q \), where \( \epsilon'_q \to 0 \) as \( q \to \infty \).

### 5. Beauville structures for finite simple groups

In this section we briefly describe the probabilistic group-theoretical methods in proving Theorem \([2]\) and Theorem \([3]\). We shall mainly sketch the proof of Garion, Larsen and Lubotzky \([18]\) that the conjecture of Bauer, Catanese and Grunewald (Theorem \([2]\)) holds for almost all finite simple groups of Lie type, as well as present the proof of Garion and Penegini \([19]\) regarding Beauville structures of alternating groups (Theorem \([4]\)), which is based on the probabilistic results of Liebeck and Shalev \([32]\).
As in the probabilistic approach we can ignore finitely many simple groups, we will not deal here with the sporadic groups, whose unmixed Beauville structures can be found in [13, 22]. Thus we shall consider only the alternating groups and the finite simple groups of Lie type.

Recall that in order to construct an unmixed Beauville structure for a finite simple (non-abelian) group $G$ one needs to find a quadruple $(x_1, y_1; x_2, y_2)$ of elements of $G$ satisfying the three conditions given in [1].

5.1. Choosing disjoint conjugacy classes. One usually starts by looking for proper conjugacy classes $X_1, Y_1, Z_1, X_2, Y_2, Z_2$ such that

$$\Sigma(x_1, y_1, z_1) \cap \Sigma(x_2, y_2, z_2) = \{1\}$$

for any $x_i \in X_i$, $y_i \in Y_i$, $z_i \in Z_i$ ($i = 1, 2$), so that Condition (iii) is satisfied.

For finite simple groups of Lie type, one can choose two maximal tori $T_1$ and $T_2$, such that if $C_i$ denotes the set of all conjugates of elements of $T_i$, then $C_1 \cap C_2 = \{1\}$.

For example, let $G = \text{SL}_{r+1}(q)$ ($r > 1$), and let $t_1$ and $t_2$ denote elements of $G$ whose characteristic polynomials are respectively irreducible (of degree $r + 1$) and the product of irreducible polynomials of degree $1$ and $r$. If $T_1$ and $T_2$ denote the centralizers of $t_1$ and $t_2$ respectively, then, by [18] Proposition 7, for all $g \in G$, $T_1 \cap g^{-1}T_2g = Z(G)$, thus Condition (iii) is satisfied for $\text{PSL}_{r+1}(q)$.

In fact, for the finite simple groups of Lie type, one can choose several maximal tori containing $f$ cycles of length $m$ each and any maximal torus is isomorphic to a product of at most $r$ cyclic groups, so a similar argument to the one presented in §4.2 implies that the probability that four random elements generate an unmixed Beauville structure is bounded above by a function of the Lie rank $r$, and any maximal torus is isomorphic to a product of at most $r$ cyclic groups, so a similar argument to the one presented in [1, 2] implies that the probability that four random elements generate an unmixed Beauville structure is bounded above by a function of $r$.

One can also choose proper conjugacy classes for the alternating groups (see [13, 16, 19, 22]). In [19], Garion and Penegini used the so-called almost homogeneous conjugacy classes introduced by Liebeck and Shalev [32].

**Definition 12.** [32]. Conjugacy classes in $S_n$ of cycle-shape $(m^k)$, where $n = mk$, namely, containing $k$ cycles of length $m$ each, are called homogeneous. A conjugacy class having cycle-shape $(m^k, 1^f)$, namely, containing $k$ cycles of length $m$ each and $f$ fixed points, with $f$ bounded, is called almost homogeneous.

By [19] Algorithm 3.5] one can construct for any six integers $k_1, l_1, m_1, k_2, l_2, m_2 \geq 2$, six distinct almost homogeneous conjugacy classes $X_1, Y_1, Z_1, X_2, Y_2, Z_2$ in $A_n$ whose elements are of orders $k_1, l_1, m_1, k_2, l_2, m_2$ respectively, such that all of them have different numbers of fixed points, thus satisfying Condition (iii).

5.2. Frobenius formula and Witten’s zeta function. Condition (i) follows from a classical formula of Frobenius: If $X$, $Y$ and $Z$ are conjugacy classes in a finite group $G$, then the number $N_{X,Y,Z}$ of solutions of $xyz = 1$ with $x \in X$, $y \in Y$ and $z \in Z$ is given by

$$N_{X,Y,Z} = \frac{|X| \cdot |Y| \cdot |Z|}{|G|} \sum_{\chi \in \text{Irr}(G)} \frac{\chi(x)\chi(y)\chi(z)}{\chi(1)},$$

where $\text{Irr}(G)$ denotes the set of complex irreducible characters of $G$. 

Usually, the main contribution to this character sum comes from the trivial character, and the absolute sum on all other characters is negligible. In order to show this, one needs to bound the absolute value $|\chi(g)|$ for an irreducible character $\chi$ and an element $g$, from any of the above conjugacy classes (see §5.3 for details). If this value can be effectively bounded then one deduces that the value of the sum of the contribution of all non-trivial characters to (1) is bounded above by some global constant (depending only on the sizes of $G$ and the conjugacy classes) multiplied by the sum $\sum_{\chi \neq 1} \chi(1)^{-1}$. So it remains to prove that the letter sum converges to 0 as $|G| \to \infty$.

Therefore, a key role in the probabilistic approach is played by the so called Witten zeta function, which is defined by

$$\zeta^G(s) = \sum_{\chi \in \text{Irr}(G)} \chi(1)^{-s}.$$ 

It was originally defined and studied by Witten [50] for Lie groups. For finite simple groups it was studied and applied in detail by Liebeck and Shalev [32, 33, 34], who proved the following desired results.

**Theorem 13.** [34, Theorem 1.1], [33, Corollary 1.3] and [32, Corollary 2.7].

Let $G$ be a finite simple group.

- If $s > 1$ then $\zeta^G(s) \to 1$ as $|G| \to \infty$.
- If $s > 2/3$ and $G \neq \text{PSL}_2(q)$ then $\zeta^G(s) \to 1$ as $|G| \to \infty$.
- If $s > 1/2$ and $G \neq \text{PSL}_2(q), \text{PSL}_3(q), \text{PSU}_3(q)$ then $\zeta^G(s) \to 1$ as $|G| \to \infty$.
- If $s > 0$ and $G = A_n$ then $\zeta^G(s) \to 1$ as $|G| \to \infty$. Moreover, $\zeta^G(s) = 1 + O(n^{-s})$.

An alternative approach to prove Condition (i) for the finite simple groups of Lie type, which was successfully applied in [13, 22] and [18, §4], is based on the following result of Gow [21]: if $X$ and $Y$ are conjugacy classes of regular semisimple elements in a finite Lie type group $G$, then the set $XY$ contains every non-central semisimple element of $G$.

### 5.3. Character estimates in finite simple groups.

A crucial part in the proof is to estimate character values in finite simple groups. More precisely, one needs to bound the absolute value $|\chi(g)|$ for an irreducible character $\chi$ and an element $g$, from any of the conjugacy classes chosen in §5.1. We therefore recall some useful results for the finite simple groups of Lie type and the alternating groups.

**Lemma 14.** [18 Corollary 4]. Let $\chi$ be an irreducible character of $G = \text{SL}_{r+1}(q)$.

- If $r \geq 2$ and the characteristic polynomial of $t_2 \in G$ has an irreducible factor of degree $r$, then $|\chi(t_2)| \leq \frac{2(r+1)^2}{\sqrt{3}}$.

More generally, by [13, Proposition 7], there exist an absolute constant $c$ such that for every sufficiently large group of Lie type $G$ (of Lie rank $r$), there exist maximal tori $T_1$ and $T_2$ as in §5.1 and for every regular $t \in T_1 \cup T_2$ and every irreducible character $\chi$ of $G$,

$$|\chi(t)| \leq cr^3.$$ 

**Lemma 15.** [32, Proposition 2.12]. Let $\pi \in S_n$ have cycle-shape $(m^k, 1^f)$. Then for any $\chi \in \text{Irr}(S_n)$ we have

$$|\chi(\pi)| \leq c \cdot (2n)^{\frac{k-1}{2}} \chi(1)^{\frac{1}{2}}.$$
where $c$ depends only on $m$.

It is also interesting to provide an upper bound on the character ratio $|\chi(g)/\chi(1)|$, where $G$ is a finite simple group, $g \in G$ and $\chi$ is an irreducible character. Gluck and Magaard [20] computed these bounds for the finite classical groups. Such bounds play a crucial role in the proof of a conjecture of Guralnick and Thompson [24], which is related to the inverse Galois problem, namely, which finite groups occur as Galois groups of algebraic number fields (for details see [14]).

5.4. Finding generating pairs. In order to prove Condition (ii) one should show that the set of solutions of $xyz = 1$ with $x \in X$, $y \in Y$ and $z \in Z$ contains a generating pair of $G$, namely, one should avoid pairs $(x, y)$ contained in maximal subgroups of $G$. Probabilistically, one expects such a result to hold since almost all pairs of elements in a finite simple (non-abelian) group are generating pairs (see [11, 25, 31]).

Namely, one should estimate the sum

$$\sum_{M \in \text{max} G} |\{(x, y, z) \mid x \in X \cap M, y \in Y \cap M, z \in Z \cap M, xyz = 1\}|,$$

where $\text{max} G$ denotes the set of maximal proper subgroups of $G$. This quantity is bounded above by

$$\sum_{M \in \text{max} G} |M|^2 = |G|^2 \sum_{M \in \text{max} G} \frac{1}{|G : M|^2} \leq \frac{|G|^2}{m(G)^{1/2}} \sum_{M \in \text{max} G} |G : M|^{-3/2},$$

where $m(G)$ is the minimal index of a proper subgroup of $G$ or, equivalently, the minimal degree of a non-trivial permutation representation of $G$. By estimates of Landazuri and Seitz [26], there exists an absolute constant $c$ such that if $G$ is a finite simple group of Lie type $G$ of Lie rank $r$ then $m(G) \geq cq^r$. Hence, again one should estimate a ‘zeta function’ encoding the indices of maximal subgroups of finite simple groups of Lie type, which was investigated by Liebeck, Martin and Shalev [30], who proved the following desired result.

**Theorem 16.** [30]. If $G$ is a finite simple group, and $s > 1$, then

$$\lim_{|G| \to \infty} \sum_{M \in \text{max} G} |G : M|^{-s} \to 0.$$

Alternatively, in [13, 22] they relied on more delicate results about maximal subgroups in finite simple groups of Lie type containing special elements, called primitive prime divisors, of Guralnick, Pentilla, Praeger and Saxl [23].

For sufficiently large alternating groups, Conditions (i) and (ii) follow from the following result of Liebeck and Shalev [32]. If $(k, l, m)$ is hyperbolic then the probability that three random elements $x, y, z \in A_n$, with product $1$, from almost homogeneous classes $X, Y, Z$, of orders $k, l, m$ will generate $A_n$, tends to $1$ as $n \to \infty$. Moreover, this probability is $1 - O(n^{-\mu})$, where $\mu = 1 - (1/k + 1/l + 1/m)$.

**Acknowledgement.** The author would like to thank her co-organizers, Ingrid Bauer and Alina Vdovina, and all the participants in the workshop “Beauville surfaces and groups 2012” for their assistance and useful conversations, as well as the University of Newcastle for hosting the workshop. The author is grateful to the referee for pointing out further relevant references.
References

[1] I. Bauer, F. Catanese, F. Grunewald, Beauville surfaces without real structures, Geometric methods in algebra and number theory, Progr. Math., vol. 235, Birkhäuser Boston, (2005), 1–42.

[2] I. Bauer, F. Catanese, F. Grunewald, Chebycheff and Belyi polynomials, dessins d’enfants, Beauville surfaces and group theory, Mediterr. J. Math. 3 No.2, (2006), 121–146.

[3] T. Bandman and S. Garion, Surjectivity and Equidistribution of the word $x^ay^b$ on $\mathrm{PSL}(2,q)$ and $\mathrm{SL}(2,q)$, Internat. J. Algebra Comput. 22 No. 2, (2012).

[4] A. Beauville, Surfaces algébriques complexes, Astérisque 54, Paris (1978).

[5] F. Catanese, Fibred surfaces, varieties isogenous to a product and related moduli spaces, Amer. J. Math. 122, (2000), 1–44.

[6] J. Cohen, On non-Hurwitz groups and non-congruence subgroups of the modular group, Glasg. Math. J. 22 (1981) 1–7.

[7] M.D.E. Conder, Generators for alternating and symmetric groups, J. London Math. Soc. 22 (1980), 75–86.

[8] M.D.E. Conder, Hurwitz groups: a brief survey, Bull. Amer. Math. Soc. 23 (1990), 359-370.

[9] M.D.E. Conder, An update on Hurwitz groups, Groups, Complexity and Cryptology 02 (2010), 35–40.

[10] L.E. Dickson, Linear Groups with an Exposition of the Galois Field Theory, Teubner (1901).

[11] J.D. Dixon, The probability of generating the symmetric group, Math. Z. 110 (1969), 199–205.

[12] B. Everitt, Alternating quotients of Fuchsian groups, J. Algebra 223 (2000), 457–476.

[13] B. Fairbairn, K. Magaard, C. Parker, Generation of finite simple groups with an application to groups acting on Beauville surfaces, to appear in Proc. London Math. Soc.

[14] D. Frohardt, K. Magaard, About a conjecture of Guralnick and Thompson, In: Arasu et al. (eds), Groups, Difference Sets, and the Monster, Proc. Ohio State Conference on Groups and Geometries, Walter de Gruyter, Berlin, New York, (1996) 43-54.

[15] Y. Fuertes, G. González-Diez, On Beauville structures on the groups $S_n$ and $A_n$, Math. Z. 264 (2010), 959–968.

[16] Y. Fuertes, G. Jones, Beauville surfaces and finite groups, J. Algebra 340 (2011), 13–27.

[17] S. Garion, On Beauville Structures for PSL(2,q), arXiv:1003.2792.

[18] S. Garion, M. Larsen, A. Lubotzky, Beauville surfaces and finite simple groups, J. Reine Angew. Math. 666 (2012), 225–243.

[19] S. Garion, M. Penegini, New Beauville surfaces and finite simple groups, Manuscripta Math. 142 (2013), 391–408.

[20] D. Gluck, K. Magaard, Character and fixed point ratios in finite classical groups, Proc. London Math. Soc., 71 (1995), 547–584.

[21] R. Gow, Commutators in finite simple groups of Lie type, Bull. London Math. Soc. 32 (2000), no. 3, 311–315.

[22] R. Guralnick, G. Malle, Simple groups admit Beauville structures, J. London Math. Soc. 85 (2012), no. (3), 694–721.

[23] R. Guralnick, C. Praeger, T. Penttila, J. Saxl, Linear groups with orders having certain large prime divisors, Proc. London Math. Soc. 78 (1999), 167–214.

[24] R. Guralnick, J. Thompson, Finite groups of genus zero, J. Algebra, 131 (1990), 303–341.

[25] W. M. Kantor, A. Lubotzky, The probability of generating a finite classical group, Geom. Dedicata 36 (1990), 67–87.

[26] V. Landazuri, G.M. Seitz, On the minimal degrees of projective representations of the finite Chevalley groups, J. Algebra 32 (1974), 418–443.

[27] U. Langer, G. Rosenberger, Erzeugende endlicher projektiver linearer Gruppen, Results Math. 15 (1989), no. 1-2, 119–148.

[28] M. Larsen, A. Lubotzky, C. Marion, Deformation theory and finite simple quotients of triangle groups I, arXiv:1301.2049.

[29] F. Levin, G. Rosenberger, Generators of finite projective linear groups. II., Results Math. 17 (1990), no. 1-2, 120–127.

[30] M.W. Liebeck, B.M.S. Martin, A. Shalev, On conjugacy classes of maximal subgroups of finite simple groups, and a related zeta function, Duke Math. J. 128 (2005), no. 3, 541–557.
[31] M.W. Liebeck, A. Shalev, *The probability of generating a finite simple group*, Geom. Ded. **56** (1995), 103–113.
[32] M.W. Liebeck, A. Shalev, *Fuchsian groups, coverings of Riemann surfaces, subgroup growth, random quotients and random walks*, J. Algebra **276** (2004), 552–601.
[33] M. W. Liebeck, A. Shalev, *Character degrees and random walks in finite groups of Lie type*, Proc. London Math. Soc. (3) **90** (2005), no. 1, 61–86.
[34] M.W. Liebeck, A. Shalev, *Fuchsian groups, finite simple groups and representation varieties*, Invent. Math. **159** (2005), no. 2, 317–367.
[35] A. Lucchini, M.C. Tamburini, *Classical groups of large rank as Hurwitz groups*, J. Algebra **219** (1999), 531–546.
[36] A. Lucchini, M.C. Tamburini, J.S. Wilson, *Hurwitz groups of large rank*, J. London Math. Soc. **61** (2000), 81–92.
[37] A.M. Macbeath, *Generators of the linear fractional groups*, Number Theory (Proc. Sympos. Pure Math., Vol. XII, Houston, Tex., 1967), Amer. Math. Soc., Providence, R.I. (1969), 14–32.
[38] G. Malle, *Hurwitz groups and $G_2(q)$*, Canad. Math. Bull. **33** (1990), 349–357.
[39] C. Marion, *Triangle groups and $\text{PSL}_2(q)$*, J. Group Theory **12** (2009), 689–708.
[40] C. Marion, *On triangle generation of finite groups of Lie type*, J. Group Theory **13** (2010), 619–648.
[41] C. Marion, *Triangle generation of finite exceptional groups of low rank*, J. Algebra **332** (2011), 35–61.
[42] C. Marion, *Random and deterministic triangle generation of three-dimensional classical groups I*, Comm. Algebra. **41** (2013), No. 3, 797–852.
[43] L. Di Martino, M.C. Tamburini, A.E. Zalesski, *On Hurwitz groups of low rank*, Comm. Algebra **28** (2000), no. 11, 5383–5404.
[44] M. Suzuki, *Group Theory I*, Springer, Berlin (1982).
[45] M.C. Tamburini, M. Vsemirnov, *Irreducible $(2,3,7)$-subgroups of $\text{PGL}_n(F)$ for $n \leq 7$*, J. Algebra **300** No. 1 (2006), 339–362.
[46] M.C. Tamburini, M. Vsemirnov, *Hurwitz groups and Hurwitz generation*, Handbook of Algebra, vol. 4, edited by M. Hazewinkel, Elsevier (2006), 385–426.
[47] M.C. Tamburini, A.E. Zalesski, *Classical groups in dimension 5 which are Hurwitz*, Finite groups 2003, 363–371, Walter de Gruyter, Berlin, 2004.
[48] R. Vincent and A.E. Zalesski, *Non-Hurwitz classical groups*, LMS J. Comput. Math. **10** (2007), 21–82.
[49] R.A. Wilson, *The Monster is a Hurwitz group*, Journal of Group Theory **4** (4), (2001), 367–374.
[50] E. Witten, *On quantum gauge theories in two dimensions*, Comm. Math. Phys. **141** (1991), 153–209.

FACHBEREICH MATHEMATIK UND INFORMATIK, UNIVERSITÄT MÜNSTER, EINSTEINSTRASSE 62, D-48149 MÜNSTER, GERMANY

E-mail address: shelly.garion@uni-muenster.de