Multiple classical limits in relativistic and nonrelativistic quantum mechanics

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The existence of a classical limit describing interacting particles in a second-quantized theory of identical particles with bosonic symmetry is proved. This limit exists in addition to a previously established classical limit with a classical field behavior, showing that the limit \( \hbar \to 0 \) of the theory is not unique. An analogous result is valid for a free massive scalar field: two distinct classical limits are proved to exist, describing a system of particles or a classical field. The introduction of local operators in order to represent kinematical properties of interest is shown to break the permutation symmetry under some localizability conditions, allowing the study of individual particle properties.

I. INTRODUCTION

Quantum Field Theory was born in the attempt to conciliate Quantum Mechanics and Relativity and in the attempt to deal with the so-called “particle-wave duality” underlying quantum phenomena by making the corpuscular character of matter compatible with the classical notion of field. Its main purpose is to describe particle physics, but the way in which its fundamental principles admit or recognize the notion of particle is rather indirect, if not obscure. A fundamental attempt to understand when quantum field theories describe particles was the work of Haag and Schwinger \cite{1}, further developed by Buchholz and Wichmann in \cite{2} and in subsequent works. In a nutshell, these works point to the fact that any relativistic quantum field theory describing particles must have some specific limitations on the number of degrees of freedom at finite volume and limited energy.

Quantum Field Theory is believed to be, in some sense, a fundamental theory, but the notions of particles and of fields are derived from our sensorial experiences in a classical macroscopic world. It is, therefore, of great importance to have a precise understanding on how the classical scenario can be reached from that more fundamental quantum starting point. In particular, one should naturally expect that the particle-field duality manifests itself in any general attempt to reach the classical limit of quantum fields. The existence of these two different limits (particles or fields) lies deeply in the structure of quantum field theory and its physical interpretation. It was first remarked in the fundamental work of Hepp \cite{3} on the classical limit of quantum systems and the main purpose of the present work is to clarify certain aspects of this remark, specially in the relativistic regime. We believe that the analysis of these multiple classical limits could have some conceptual importance in the context of quantum field theories formulated in curved spacetimes, where no natural concept of particle states is available.

There are several ways to approach the formulation of a classical limit of Quantum Mechanics. In this work we follow the ideas introduced by Hepp in \cite{3}, which can be applied to a wide range of systems and can be understood in a simple and precise way. The central result of his work combines an old observation of Schrödinger \cite{4}, which led to the discovery of coherent states \cite{5,6}, with the intuitive explicit content of Ehrenfest theorem. Schrödinger observed that in a harmonic oscillator a Gaussian wavefunction moves without distortion along a classical orbit, what led him to discover the concept of coherent states \cite{5,6}, with the intuitive explicit content of Ehrenfest theorem. Schrödinger observed that in a harmonic oscillator a Gaussian wavefunction moves without distortion along a classical orbit, what led him to discover the concept of coherent states \cite{5,6}, with the intuitive explicit content of Ehrenfest theorem.

In order to briefly describe Hepp’s analysis, consider a simple example. The classical nonrelativistic motion of a single particle of mass \( m \) moving in one-dimension under an external potential \( V \) is described in Classical Mechanics by the Hamiltonian \( H = \frac{\hat{p}^2}{2m} + V(x) \), leading to the canonical equations \( \dot{p} = -\frac{\partial V}{\partial x} \) and \( \dot{x} = \frac{p}{m} \). Let \( (\xi(t), \pi(t)) \) be the solution for initial conditions \( x(0) = \xi, p(0) = \pi \). The dynamics of the corresponding quantum system is defined by the Hamilton operator

\[
H = -\frac{\hbar^2}{2m} \frac{\partial^2}{\partial q^2} + V(q),
\]

where \( q \) is the position operator. Let \( U(t) \) be the propagator associated to this Hamiltonian. The question we face is how to recover the classical trajectory defined by \( \xi(t) \) and \( \pi(t) \) from the quantum Hamiltonian. Following Hepp, there are three steps involved in the solution of this problem. In the first step we go to the Weyl representation, replacing position and momentum operators by their exponentiated versions \( \exp(i a q), \exp(i b p) \), \( a, b \in \mathbb{R} \), so that only bounded
operators are involved. In the second step we consider averages of general Weyl operators $W(a, b) := e^{iaq + b p}$ on
time-evolved coherent states $|\alpha\rangle$, with $\alpha = (\xi + i \pi)/(\sqrt{2} \hbar)$. Finally, the limit $h \to 0$ is taken. As established in [3], it
turns out that under natural regularity requirements, and for $t$ restricted to a finite interval $|t| < T$, one has
\[
\lim_{h \to 0} \langle \alpha | U(t)^* W(a, b) U(t) | \alpha \rangle = e^{i[a \xi(t) + \pi \sigma(t)]},
\]
from which the classical trajectory $(\xi(t), \pi(t))$ can be recovered (for the above mentioned time-interval). The result
is valid for a large class of potentials, as stated in [3].

The steps and results leading to (2) can be easily extended in order to indicate how the classical limit of a quantum
nonrelativistic system with finitely many degrees of freedom describing distinguishable particles has to be performed
in order that this limit describes a classical mechanical system of finitely many particles (at least for a short time).
In this case one can use coherent states as those of Eq. (2) in connection with individual position and momentum
operators $p_i, q_i$ for each particle to find the classical limit of interacting point particles. The procedure presented
above have also been extended in [3] to some quantum systems with infinitely many degrees of freedom, leading to
classical limits describing classical fields.

An important quantum system considered in [3] for which the classical limit is a classical field is a system of second-
quantized interacting bosonic particles moving in one dimension, described in a bosonic Fock space $\mathcal{F}(\mathcal{H}) : = \bigoplus_{n=1}^{\infty} \mathcal{H}_{s}^{(n)}$, where $\mathcal{H}_{s}^{(n)}$ is the usual Hilbert space of symmetric square-integrable functions over $\mathbb{R}^n$, with the Hamiltonian
\[
H = -\frac{\hbar^2}{2} \int dx a^*(x) \nabla^2 a(x) + \frac{1}{2} \int dxdy a^*(x)a^*(y)V(x-y)a(x)a(y),
\]
where the creation and annihilation operators $a^*$ and $a$ satisfy the usual commutation rules,
\[
[a(x), a(y)] = [a^*(x), a^*(y)] = 0, \quad [a(x), a^*(y)] = \delta(x-y),
\]
and $V(x) = V(-x) = V(x)^*$ is some real Kato potential. The classical limit of this system was studied in [7] as a
model for superfluid Helium, and that work was one of the motivations for Hepp’s. In this case the classical field
behavior is taken as an approximate description for the dynamics of the matter density in the fluid. That limit can be
achieved with the same strategy adopted for the case of the one-particle dynamics of (1). Coherent states of bosonic
systems are usually defined in order to make field aspects become more evident, as can be seen from many examples
in Quantum Optics (see e.g. [8] for an introduction). Accordingly, if one considers the time-evolution of the average
of exponentiated smeared local fields evaluated on an initially coherent state, one gets a classical field theory in the
limit $\hbar \to 0$. As shown in [3], the classical field $\alpha$ satisfies the non-linear partial integro-differential equation
\[
\frac{\partial \alpha}{\partial \beta} (\beta, t, x) = \frac{i}{2\mu} \nabla^2 \alpha(\beta, t, x) + i \int dy V(x-y)|\alpha(\beta, t, y)|^2 \alpha(\beta, t, x),
\]
where $\beta(x)$ is the initial condition (which must be taken in $D(\nabla^2)$), and the constant $\mu$ is related to the mass $m$ of
the bosonic particles. The particular case when $V(x-y) = g\delta(x-y)$ leads in [4] to the well-known Gross–Pitaevskii
equation (or non-linear Schrödinger equation), widely employed in the study of Bose-Einstein condensates. The field
behavior exposed in [4] is not expected to hold along an arbitrary time interval.

Since (3) is assumed to describe a quantum system of interacting particles, it is natural to expect that a second kind
of classical limit exists which describes classical particles instead of classical fields. In fact, Hepp observes that these
limits should depend on the way in which certain physical parameters are scaled when $\hbar \to 0$. There are, however,
some implicit difficulties in applying Hepp’s program to systems of indistinguishable particles, as in the case of the
nonrelativistic quantum many-body system described in [3] or a relativistic quantum field model. The main problems
are that: (i) the coherent states are not invariant under permutation symmetry, and (ii) observables describing individual kinematical properties may not be defined. The problem (i) is circumvented with the use of symmetrized products of single-particle coherent states, but the second problem requires some variation of the general technique. We will show how the introduction of local operators acting on a class of essentially localized states defined herein can be used to address this problem. A classical limit of $N$ interacting particles is found when $N$ apparatuses situated at distinct regions are used to observe an essentially localized state with $N$ localization centers coinciding with the apparatuses’ positions.

In the relativistic regime, one has to face the additional problem of defining the notion of quantum particle-like
states in a proper way, since this is required for a particle classical limit. We adopt the notion of essentially localized
states discussed by Haag and Swieca in [1] as a suitable representation of the intuitive idea of a particle, and adjust it
to our purposes. Position averages will be evaluated with the Newton-Wigner operator [6]. An explicit construction
based on the single-particle relativistic coherent states of [10] will be shown to lead to the desired particle classical
II. COHERENT STATES AND ESSENTIALLY LOCALIZED STATES

Coherent states $|\alpha\rangle$ for a one-dimensional quantum mechanical system are usually defined as eigenstates of the annihilation operator, $a |\alpha\rangle = \alpha |\alpha\rangle$, with $\alpha = (\xi + i\pi)/\sqrt{2\hbar} \in \mathbb{C}$. It follows from this definition that $|\alpha\rangle$ is a minimum uncertainty state centered at $x = \xi$ and $p = \pi$ with equal uncertainties $\Delta p = \Delta q$, and this property is one of the main motivations for the study of such states, since it allows one to think of them as the “most classical states” in some natural sense. An alternative and equivalent definition is given in terms of the action of a translation operator $U(\alpha)$ on the harmonic oscillator ground state $|0\rangle$,

$$|\alpha\rangle := U(\alpha) |0\rangle, \quad U(\alpha) := \exp(\alpha a^* - \bar{\alpha} a),$$

where one has $U(\alpha) a U(\alpha)^* = a - \alpha$. The scalar products are given by

$$|\langle \alpha|\beta\rangle|^2 = \exp(-|\alpha - \beta|^2),$$

thus coherent states are not orthogonal; nevertheless, the overlap decreases rapidly with the distance $|\alpha - \beta|$.

Coherent states can be defined in much more general systems (see [11, 12]). The case of $n$-dimensional systems is straightforward, as well as the case of many-particles systems when no symmetrization is required. Let $d$ be the number of spatial dimensions, and $n$ the number of particles. In this case the coordinate space is $N$-dimensional, with $N = nd$. All that is needed is to define a series of $N$ annihilation operators $a_r = (q_r + ip_r)/\sqrt{2\hbar}$, one for each spatial dimension and particle of the system, and consider the simultaneous eigenstates of all annihilation operators, $a_r |\alpha_1, \ldots, \alpha_N\rangle = \alpha_r |\alpha_1, \ldots, \alpha_N\rangle$, $r = 1, \ldots, N$. The eigenstates $|\alpha_1, \ldots, \alpha_N\rangle$ are the coherent states for this system. Writing the labels $\alpha$ in terms of real and imaginary parts as $\alpha_r = (\xi_r + i\pi_r)/\sqrt{2\hbar}$, one finds that $|\alpha_1, \ldots, \alpha_N\rangle$ describes a Gaussian wavefunction centered at $x = (\xi_1, \ldots, \xi_N)$ in coordinate space and $p = (\pi_1, \ldots, \pi_N)$ in momentum space with minimum uncertainty for each canonical pair, $\Delta q_r = \Delta p_r = \sqrt{\hbar}/2$. So everything goes as in the previous case.

The case of many-body systems of identical particles, as the one described by the Hamiltonian [3] or in Quantum Field Theory, demands special care. Consider an $N$-particle state of a many-body system of identical particles in one space dimension and with bosonic symmetry (to which we restrict ourselves in this work). Because of the permutation symmetry, one cannot introduce individual position and momentum operators for each of the particles as in the case of distinguishable particles and a different procedure is required. The most natural choice, and the one adopted in this work, is to consider symmetrized products of single-particle coherent states $|\alpha_r\rangle$,

$$|\alpha_1, \ldots, \alpha_N\rangle_S = \frac{1}{N!} \sum_{\pi} |\alpha_{\pi(1)}, \ldots, \alpha_{\pi(N)}\rangle = \frac{1}{N!} \sum_{\pi} |\alpha_{\pi(1)}\rangle \otimes \cdots \otimes |\alpha_{\pi(N)}\rangle,$$

as the analog of $N$-particle coherent states, where the sum is over all elements of the permutation group of $N$ elements and where the normalization constant is

$$N' = N! \sum_{\pi} \langle \alpha_1 |\alpha_{\pi(1)}\rangle \cdots \langle \alpha_N |\alpha_{\pi(N)}\rangle.$$

These states will be of special interest in what follows, as examples of what will be called “essentially localized states”. It will be argued that such states reproduce within nonrelativistic quantum mechanics some basic intuitive properties of states so denoted in Algebraic Quantum Field Theory [1].

In order to discuss localization properties consider the example of one-dimensional single-particle states first, with the Hilbert space $\mathcal{H} = L^2(\mathbb{R}, dx)$. Denote by $\langle A \rangle_{\psi} := \langle \psi |A|\psi\rangle$ the expectation value of an operator $A$ acting on $\mathcal{H}$ in some vector state for $\psi \equiv |\psi\rangle \in \mathcal{H}$ with $||\psi||^2 \equiv \langle \psi |\psi\rangle = 1$.

Let $A$ be a bounded operator acting on $\mathcal{H}$ with norm $||A||$. For any finite open region $O \subset \mathbb{R}$, define the local version of $A$ corresponding to the region $O$ by

$$A_O := \frac{1}{2} \left( \chi_O A + A \chi_O \right),$$

where $\chi_O$ is the characteristic function of $O$, i.e., $\chi_O(x) = 1$ if $x \in O$ and $\chi_O(x) = 0$ if $x \notin O$. The operator $A_O$ is also a bounded operator acting on $\mathcal{H}$ and one has $A - A_O = \frac{1}{2} \left[ (1 - \chi_O) A + A (1 - \chi_O) \right]$. Hence, for expectation values
of $A - A_O$ on some normalized vector state $\psi$ one has

$$\langle A \rangle_\psi - \langle A_O \rangle_\psi = \frac{1}{2} \langle (1 - \chi_O) \psi | A \rangle + \frac{1}{2} \langle \psi | A (1 - \chi_O) \psi \rangle,$$

from which we get, by the Cauchy-Schwarz inequality,

$$\left| \langle A \rangle_\psi - \langle A_O \rangle_\psi \right| \leq \|A\| \| (1 - \chi_O) \psi \|.$$

Let us consider the concrete case where $O = (\xi - R, \xi + R)$, the interval of radius $R > 0$ around the center $\xi \in \mathbb{R}$. We say that $\psi \in \mathcal{H}$ is essentially localized around $\xi$ if $\|\psi\|_{O^c} = \| (1 - \chi_O) \psi \|$, the fraction of the norm of the state lying outside $O$, satisfies

$$\|\psi\|_{O^c} := \| (1 - \chi_O) \psi \| \leq K e^{-R^2/2\hbar}, \quad (10)$$

for all $R > 0$, $K$ being some positive constant. For vector states satisfying (10), one has

$$\langle A \rangle_\psi - \langle A_O \rangle_\psi \leq K \|A\| e^{-R^2/2\hbar}. \quad (11)$$

Thus the operator $A_O$ can in fact be understood as a local version of the operator $A$ in the region $O$, since the expectation $\langle A_O \rangle_\psi$ is a good approximation for $\langle A \rangle_\psi$, up to an error which decreases rapidly with $R$.

In particular, this approximation works fine for coherent states. It was already mentioned that the position wavefunction $\psi_\alpha(x) = |x\rangle\langle\alpha|$, corresponding to a coherent state $|\alpha\rangle$ is a normalized Gaussian curve with width $\sqrt{\hbar}$ centered around some $\xi \in \mathbb{R}$. Thus $\psi_\alpha(x)$ is essentially contained in a radius of a few $\sqrt{\hbar}$ around $\xi$ and one has

$$\|\psi_\alpha\|_{O^c}^2 := \| (1 - \chi_O) \psi_\alpha \|^2 = 1 - \int_{\xi-R}^{\xi+R} |\psi_\alpha(x)|^2 \, dx \leq Ke^{-R^2/\hbar}, \quad (12)$$

for some $K > 0$, and decreases faster than exponentially with increasing $R$.

There is another sense in which (10) leads to a notion of essentially localized states. Consider two distinct states $|\psi\rangle$ and $|\phi\rangle$ which satisfy (10) with respect to disjoint regions $O$ and $O'$, and let the distance between such regions be $d(O, O') = d > 0$. Hence, $|\langle \psi | \phi \rangle| \leq 2e^{-d^2/8\hbar}$. Therefore, if $|\psi\rangle$ and $|\phi\rangle$ are states essentially localized in disjoint regions, the overlap between the corresponding wavefunctions is small for large distances, and decreases as a Gaussian with the distance between the localization regions. Accordingly, the inequality (10) is used herefrom as the defining property of a “state essentially localized in $O$”.

The concepts of local operator and essentially localized states can be easily translated to $N$-particle states. The symmetrized product of single-particle states $|\psi_i\rangle$ essentially localized in pairwise disjoint regions $O_i$ (with $O_i \cap O_j = \emptyset$ if $i \neq j$) is the natural extension adopted here. These states will be denoted “multiply localized states” or “essentially localized states”. Simple examples are given by the symmetrized product of coherent states as displayed in Eq. (7), as long as the localization centers $\xi_i$ of the coherent states $|\alpha_i\rangle$ are chosen sufficiently far apart. Given a local operator $A_O$ acting on the single-particle subspace we define its $N$-particle counterpart acting on $\mathcal{H}_s^{(N)}$ as

$$A_O^{(N)} = \sum_{i=1}^{N} 1 \otimes \cdots \otimes 1 \otimes A_O \otimes 1 \otimes \cdots \otimes 1, \quad (13)$$

where the operator $A_O$ occupies the $i$-th slot of the tensor product in the $i$-th term of the sum.

The average value of the local operator $A_O^{(N)}$ on a multiply localized state $|\psi\rangle$ is given by

$$\langle A_O^{(N)} \rangle_\psi = \sum_i \frac{N!}{N^2} \langle \psi_1 | \psi_{\pi(1)} \rangle \cdots \langle \psi_i | A_O \psi_{\pi(i)} \rangle \cdots \langle \psi_N | \psi_{\pi(N)} \rangle. \quad (14)$$

The factor $N! / N^2$ goes to 1 when $\hbar \rightarrow 0$. In the case where $|\psi\rangle$ is a multiply localized state with localization centers $O_i$, one has

$$\langle \psi_i | \psi_{\pi(i)} \rangle \leq 2e^{-d^2/8\hbar}, \quad \text{if } i \neq \pi(i), \quad (15)$$

where $d$ is the smallest distance between any two $O_i$, $O_j$, $i \neq j$. Thus, only a few terms in (14) contribute appreciably to the sum, i.e., those for which $i = \pi(i), \forall i$. Each such terms contains a factor $\langle \psi_i | A_O | \psi_i \rangle$. Now, suppose some of
the localization centers coincides with the region where the local operator in question is defined: \( O_k = O \) for some \( k \). Then one has the bounds

\[
|\langle A \rangle_{\psi_k} - \langle A_O \rangle_{\psi_k}| \leq \|A\| e^{-R^2/\hbar},
\]

\[
|\langle A_O \rangle_{\psi_i}| \leq \|A\| e^{-d^2/\hbar}, \quad \text{if } i \neq k.
\]

(16)

(17)

We assume in addition that

\[
|\langle \psi_i | A_O | \psi_j \rangle| \leq \|A\| e^{-d^2/2\hbar}, \quad \text{if } i \neq j.
\]

(18)

This inequality corresponds to the requirement of a local property of the operators. It is automatically satisfied when \( A_O \) is the local version of any operator \( A \) of the form \( A = A(q) \). It is also satisfied when \( A = A(q) \exp(ibp/\hbar) \), with \( b \leq d \), that is, when the operator \( A \) is a function of the coordinates, up to some small translation. In this case, we get from (13) the approximation

\[
\langle A^{(N)}_O \rangle_{\psi} \approx \langle A \rangle_{\psi_k},
\]

(19)

with an error which decreases as \( \exp(-d^2/4\hbar) \exp(-R^2/\hbar) \) for large arguments \( d, R \). Therefore, the average value of a local many-body operator associated with a region \( O \), evaluated on a multiply localized state which has \( O \) as one of its localization centers, reduces to the average value of the corresponding single particle operator evaluated on a state essentially localized above \( O \).

The intuitive picture underlying the definitions and approximations given in this section is the following. In a many-body system of identical particles, one cannot in general disentangle kinematical properties of individual components, since permutation symmetry mixes them. But for some special states, an approximation is feasible where the particles behave as independent subsystems. Such "essentially localized states" look like isolated lumps of matter distributed over distinct regions of space. If a measuring apparatus works in a region where no lump of matter is present it will detect nothing. If there is one lump of matter present, a single particle will be detected, but the observer will be unaware of the existence of other identical particles comprising a larger Hilbert space together with the detected one — his "local operators" break the permutation symmetry since they are not sensible to particles far apart. If the measuring apparatus operates in a region where several lumps of matter are present, the individual properties of the particles become intertwined. The configurations of interest for the existence of a classical limit describing particles are those in which \( N \) observers situated at distinct regions can measure properties of one particle each.

### III. CLASSICAL LIMIT OF INTERACTING BOSONIC PARTICLES

A vector state in \( \mathcal{H}^N_s \) can be represented by a normalized wavefunction \( \psi(x_1, \ldots, x_N) \) symmetric under the exchange of any two coordinates \( x_i, x_j \). Let \( |\alpha_i\rangle \) be single-particle coherent states and \( \psi_i(x) \) be the corresponding wavefunctions. The symmetrized product of such states is represented by

\[
\psi(x_1, \ldots, x_N) = \frac{1}{\mathcal{N}} \sum_{\pi} \psi_{\pi(1)}(x_1) \cdots \psi_{\pi(N)}(x_N),
\]

(20)

where \( \mathcal{N} \) is a normalization constant. The action of the Hamiltonian (3) on such states is given by

\[
H\psi(x_1, \ldots, x_N) = -\frac{\hbar^2}{2N!} \sum_j \sum_{\pi} \nabla^2_j \psi_{\pi(1)}(x_1) \cdots \psi_{\pi(N)}(x_N)
\]

\[
+ \frac{1}{2N!} \sum_{j \neq k} \sum_{\pi} V(x_j - x_k) \psi_{\pi(1)}(x_1) \cdots \psi_{\pi(N)}(x_N).
\]

(21)

Since the Hamiltonian is time-independent, the propagator is simply \( U(t) = \exp(iHt/\hbar) \). The potential \( V(x) \) is required to satisfy the regularity condition \( \int dx |V(x)|^2 \exp(-\rho x^2) < \infty \) for some \( \rho < \infty \). This quantum dynamics is to be compared with the classical Hamiltonian equations

\[
H_c = \sum_j \frac{\pi_j^2}{2} + \frac{1}{2} \sum_{j \neq k} V(\xi_j - \xi_k),
\]

(22)

\[
\dot{\xi}_j = \pi_j, \quad \pi_j = -\sum_{k \neq j} V'(\xi_j - \xi_k).
\]

(23)
Following [3], we assumed that \( \nabla^2 V \) is Lipschitz, so that solutions of the canonical equations exist and are unique in a finite time-interval \(|t| \leq T\). Denote by \( \xi(\alpha,t) \), \( \pi(\alpha,t) \) the solution for initial conditions \( \alpha_j = (\xi_j + i\pi_j)/\sqrt{2\hbar} \), \( j = 1, \ldots, N \), and let \( V \) be Hölder continuous \( C^{2+\epsilon} \) in an open neighborhood of \( \xi(\alpha,t) \), for all \(|t| \leq T\).

In order to compare the classical and quantum equations, consider for each \( j \) the localized operators \( \mathcal{W}(a, b)_{O_j(t)} \) associated to the Weyl operators \( \mathcal{W}(a, b) := \exp(iaq + bp) \), acting on the single particle space, given as in [13] by

\[
\mathcal{W}(a, b)_{O_j(t)} := \frac{1}{2} \left( \chi_{O_j(t)} \exp i(aq + bp) + \exp i(aq + bp)\chi_{O_j(t)} \right),
\]

where \( O_j(t) := (\xi_j(\alpha, t) - R, \xi_j(\alpha, t) + R) \), where \( \chi_{O_j(t)} \) is the characteristic function of the set \( O_j(t) \), and \( q, p \) are the usual position and momentum operators. For each \( j \), let \( \mathcal{W}(a, b)_{O_j(t)}^{(N)} \) be the corresponding \( N \)-body operator defined as in [13]. The desired classical limit is encoded in the expression

\[
\lim_{\hbar \to 0} \left[ \langle \psi | U(t) \ast \mathcal{W}(a, b)_{O_j(t)}^{(N)} U(t) | \psi \rangle - e^{i[a\xi_j(t) + h\pi_j(t)]} \right] = 0, \tag{24}
\]

valid for each \( j = 1, \ldots, N \), where \( |\psi\rangle \) is the state defined in [20]. This limit is proved as follows.

Consider the left side of the equation (24). The state \( |\psi\rangle \) is a superposition of \( N! \) unsymmetrized coherent-states in \( \mathcal{H}^{\otimes N} \). Let \( |\phi\rangle \) be one of these states. The Hamiltonian [21] can be understood as acting on \( \mathcal{H}^{\otimes N} \), and in this case it is known from Hepp’s work [3] that, under the assumed hypotheses,

\[
\lim_{\hbar \to 0} \left\| U(t) |\phi\rangle - U(\alpha(t))W(t) |0\rangle \right\| = 0,
\]

where \( W(t) = T \exp \left[ -i/\hbar \int_{t_0}^{t} dt' H'(t') \right] \) (here, \( T \) denotes the usual time-ordering prescription) is the propagator associated with the second-order Hamiltonian

\[
H' = -\frac{\hbar^2}{2} \sum_j \nabla_j^2 + \frac{1}{2} \sum_{j \neq k} V''(\xi_j(t) - \xi_k(t))(x_j - x_k)^2,
\]

obtained from the linearization of the operator in (21) around the classical orbit \( \xi(\alpha, t), \pi(\alpha, t) \). The state \( |0\rangle \) is the coherent state centered at zero position and momentum. To the state \( W(t) |0\rangle \) there corresponds a wavefunction which is Gaussian and centered at zero in each coordinate \( \alpha_i \) (an explicit expression is derived in Appendix A). Thus one can write

\[
\phi'(x, t) = \langle x | U(\alpha(t))W(t) |0\rangle = \prod_j \phi_j'(x_j, t) \tag{25}
\]

\[
\phi_j'(x_j, t) = \left( \frac{\omega_j}{\pi\hbar} \right)^{1/4} \exp \left[ -\frac{1}{2\hbar} \omega_j(t)(x_j - \xi_j(\alpha, t))^2 + i\frac{\pi_j(\alpha, t)}{\hbar} x_j \right]. \tag{26}
\]

The positive coefficients \( \omega_j(t) \) are continuous functions on \([0, T]\) determined by the Hamiltonian \( H' \). The state \( U(t) |\psi\rangle \) is the symmetric part of \( U(t) |\phi\rangle \), which for small \( \hbar \) is well approximated by

\[
\psi(x, t) \approx \frac{1}{N!} \sum_{\pi} \phi_{\pi(1)}(x_1, t) \ldots \phi_{\pi(N)}(x_N, t), \tag{27}
\]

with an error of order \( O(\hbar^{5/2}) \). Define \( \Omega := \min \{ \omega_j(t) \} j = 1, \ldots, N \) and \( t \in [0, T] \}. \) Then (24) ensures that the states (27) are essentially localized around the classical orbit \( \xi(\alpha, t), \pi(\alpha, t) \), i.e., \( \| \phi_j'(x, t) \|_{O_j(t)} \leq e^{-R^2/2\hbar'} \), with \( \hbar' = \hbar/\Omega \).

Now observe that the Weyl operators can be rewritten as \( \mathcal{W}(a, b) := \exp(iab\hbar/2) \exp(iaq) \exp(i(bh)p/\hbar) \), so there is some \( \hbar \) such that \( \hbar' \) is less than the smallest distance between the particles. Thus the inequality [13] is valid, and then the approximation [19] yields, for each \( k = 1, \ldots, N \),

\[
\langle \psi | U(t) \ast \mathcal{W}(a, b)_{O_k(t)}^{(N)} U(t) | \psi \rangle \approx \langle \phi_k'(t) | \mathcal{W}(a, b) | \phi_k'(t) \rangle.
\]

This approximation is valid as long as the classical orbits \( \xi_j(\alpha, t), j = 1, \ldots, N \), do not cross, i.e., \( |\xi_j(t) - \xi_l(t)| > 2R + \kappa \), for all \( j \neq l \) and \( t \in [0, T] \), \( \kappa > 0 \), in order that a finite minimal distance \( \kappa \) exists between the localization regions centers \( \xi_j(\alpha, t) \). The error involved in the approximation vanishes for \( \hbar \to 0 \), and the average values on the right side can be easily evaluated, leading to the limit (24), and completing the proof.
Some comments are in order. First, it is clear that there is a lot of freedom in the choice of the local operators, since there are many ways to define localization regions $O_j(t)$. In the proof displayed above, these regions follow the classical trajectory with some fixed radius $R$. This is not necessary — it is sufficient that the classical trajectory of the $j$-th particle is in the interior of $O_j(t)$. If there are disjoint regions $O_j$ such that $\xi(t) \in O_j$ for all $t$, then the time-dependence can be removed. In this case one would have a physical situation where the measuring apparatuses are placed at fixed regions where the particles are confined, a situation which is likely to happen for small time-intervals. Another remark concerns the possibility of collisions between particles. In this case the classical limit does not exist in the sense of the proved theorem, so the classical limit of scattering processes are not considered here.

Regarding the existence of two distinct classical limits in the same quantum system, it is seen that the existence of two kinds of coherent states for bosonic systems is responsible for the fact. The usual field coherent states lead to a classical field equation obeying $\mathfrak{H}$ for a time-interval of order $t \sim \hbar^2$, while the essentially localized states defined herein lead to a particle dynamics for a time-interval of order $\hbar^{3/2}$. The introduction of local operators is necessary in the case of the particle limit in order to break the permutation symmetry and allow for the study of individual trajectories.

A natural question connected with Hepp’s analysis is to what extent it helps understanding the existence of a ‘classical world’ as a consequence of more fundamental quantum laws. In short, what it states is that under a certain condition – the existence of a coherent state – and for a small time-interval, operator averages obey the expected classical laws of motion. Now, this result raises two natural questions. The first is to understand why a classical behavior is usually met with in macroscopic scales, that is, why states other than coherent states usually do not show up in macroscopic scales. Another question concerns the possibility of removing the restriction to small time-intervals, in order that this classical limit exists in time-scales compatible with all classical dynamics.

A possible improvement of the theorem in order to address these problems could be the inclusion of an external system interacting with the system of interest. That would bring some contact with the widely studied decoherence approach to the emergence of classical behavior in quantum systems \cite{14}. It is known that several new effects can show up in this case, such as Zeno effect \cite{15} and non-unitary time-evolution \cite{16}, for instance, and it is not unreasonable that they may play an important role in the existence of a classical limit for large time-scales. Some experimental evidence points in this direction. A clean example is found in a series of papers on interference of matter waves (fullerenes) in a Talbot-Lau interferometer \cite{17, 18}. It was observed that both interaction with gas particles as well as emission of radiation, i.e., interaction with a quantized electromagnetic field, helps preserving classical behavior. In both cases a process of localization of the particles is supposed to happen with a certain frequency in consequence of the interaction with the external system, thus naturally enforcing the regular occurrence of the conditions required for the validity of Hepp’s analysis.

\section{CLASSICAL PARTICLES AND THE KLEIN-GORDON FIELD}

In this section we extend our results to the relativistic regime. We will discuss the existence of two distinct kinds of classical limits in a system of bosonic relativistic particles, one of them describing a classical field, the other one describing classical systems with finitely many particles. For simplicity, we consider a scalar field theory with mass $m > 0$ in $1 + 1$-dimensional Minkowski spacetime. The existence of a classical field limit for this system in the presence of a polynomial interaction was proved in \cite{3}, roughly in the same way as for nonrelativistic mechanics: one studies the time-evolution of average values evaluated in an initially coherent state, and verifies that these averages obey the expected classical equations of motion. Here we show that a classical particle limit can also be reached, using methods analog to those applied in the nonrelativistic case, i.e., by studying the time-evolution of the average of local operators evaluated on essentially localized states. This will be done explicitly for the case of the relativistic position operator, in order to show how classical trajectories arise from the quantum dynamics.

It is well-known that the problem of localizability of relativistic particles is rather more intricate than in the nonrelativistic case. The basic difference is that here the construction of a wavepacket cannot involve arbitrary superpositions of states, being restricted to the space of positive energy solutions. It turns out that a strictly localized state cannot be constructed, in contrast with the nonrelativistic case where arbitrarily localized states can be easily written down. A relativistic particle is at best “nearly” localized, i.e., concentrated in a small region of space. The questions of how well localized the particle can be, and how to characterize these localized states, were dealt with in the classical work of Newton and Wigner \cite{4}, where a set of natural conditions were stated which any localized state should satisfy. The solutions for these conditions are the so-called Newton-Wigner states. These states can be characterized as infinite norm eigenstates of a relativistic version of the position operator. In the following, we adopt this notion of localizability and use the Newton-Wigner operator to evaluate position averages where necessary.

The existence of a particular classical limit relies on the existence of an adequate kind of coherent state. So, in order to formulate a classical particle limit, one must first look for the analogous of particle-like coherent states in
relativistic quantum theory. This problem was addressed in several previous works and we refrain from giving a complete list of references here. The states introduced by G. Kaiser in [10] (which are also particular cases of the formalism developed by S. Ali et alii in [20]) proved to be a good starting point for our purposes. In order to simplify the calculations, we actually work with a simple modification of these states. Before we introduce the coherent states we will deal with, let us recall some aspects of the theory of Newton and Wigner.

Let $\varphi(x, t)$ be a classical scalar field satisfying the Klein-Gordon equation with mass $m$ in $1 + 1$-dimensional Minkowski spacetime, and $\phi(p, \omega)$ its momentum space representation, and assume that $\phi(p, \omega) \equiv \phi(p)$ is restricted to the positive mass shell, with $\omega = \sqrt{p^2 + m^2}$ (we adopt $c = 1$ throughout the paper). One can view $\varphi(x, t)$ as a first quantized particle wavefunction or as a state in the one-particle sector of the Fock space of the corresponding quantum field theory.

In this context, there are two relevant Hilbert spaces to be considered: $\mathcal{H}^1 = L^2(\mathbb{R}, dp/\omega)$, with the relativistically invariant scalar product $\langle \phi | \psi \rangle_{\mathcal{H}^1} := \int_{\mathbb{R}} dp \phi(p) \overline{\psi(p)}$, and $\mathcal{H}^2 = L^2(\mathbb{R}, dp)$, with the usual scalar product $\langle \phi | \psi \rangle_{\mathcal{H}^2} := \int_{\mathbb{R}} dp \phi(p) \overline{\psi(p)}$ (both scalar products interpreted in momentum space representation). The map $M_{\sqrt{\omega}} : \mathcal{H}^1 \rightarrow \mathcal{H}^1$ defined by $(M_{\sqrt{\omega}} \phi)(p) := \sqrt{\omega} \phi(p)$ (multiplication operator by $\sqrt{\omega}$) is unitary. The usual position operator on $\mathcal{H}^2$ is $i\hbar \frac{\partial}{\partial p}$, and is self-adjoint in some adequate domain. Its counterpart in $\mathcal{H}^1$ is $q := M_{\sqrt{\omega}} \left(i\hbar \frac{\partial}{\partial p}\right) M_{1/\sqrt{\omega}}$, the so-called Newton-Wigner position operator (at time zero) [19, 10]. $q$ is also self-adjoint, since $M_{\sqrt{\omega}}$ is unitary and $(q \phi)(p) := i\hbar \left(\frac{\partial}{\partial p} - \frac{p}{2m^2} \right) \phi(p)$. The Newton-Wigner position operator at time $t$, denoted here by $q_t$, is given by $q_t := e^{itq/\hbar} = e^{i\omega t/\hbar} \frac{\partial}{\partial p} - \frac{p}{2\omega} \frac{itp}{\hbar}$, for $t \in \mathbb{R}$, i.e.,

$$
(q \phi)(p) = i\hbar e^{itq/\hbar} \frac{\partial}{\omega \partial p} \left(\frac{e^{-i\omega t/\hbar} \phi(p)}{\sqrt{\omega}}\right) = i\hbar \left(\frac{\partial}{\partial p} - \frac{p}{2\omega^2} \frac{itp}{\hbar} \right) \phi(p),
$$

and is also self-adjoint in $\mathcal{H}^1$. The momentum operator both in $\mathcal{H}^1$ and $\mathcal{H}^2$ is just multiplication by $p$.

Recall that the Newton-Wigner states [9] localized at $x$ at time $t$, denoted here by $\psi(x, t)$, are given in momentum representation in $\mathcal{H}^1$ by $\langle p | \psi(x, t) \rangle = \psi(x, t)(p) = \sqrt{\omega/2\pi} e^{i\omega x/\hbar} e^{i\omega t/\hbar} e^{i\omega x/\hbar}$, where $\pi \equiv \sqrt{\omega}$ are the usual momentum eigenstates of the Newton-Wigner operator at time $t$ given in [28], i.e., $q_t \psi(x, t) = x \psi(x, t)$, that allows to express $q_t$ in its spectral representation form in $\mathcal{H}^1$ as $q_t = \int_{\mathbb{R}} x | \psi(x, t) \rangle \langle \psi(x, t) | \ dx$. Notice that $\int_{\mathbb{R}} | \psi(x, t) \rangle \langle \psi(x, t) | \ dx$ is formally the identity operator on $\mathcal{H}^1$ and that $\langle \psi(x, t) | \psi(x', t) \rangle = \delta(x - x')$, as one easily checks. The Newton-Wigner wavefunction [19] associated to a state $\phi$ is given by

$$
\phi_{NW}(x, t) := \langle \psi(x, t) | \phi \rangle = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} dp \frac{e^{-i\omega t/\hbar + ipx/\hbar}}{\sqrt{\omega}} \phi(p).
$$

For each $t$, one has $(q_t \phi)(x, t) = x \phi_{NW}(x, t)$. Therefore, we can interpret the space of the Newton-Wigner wavefunctions $\phi_{NW}(x, t)$ as the spectral representation space of $q_t$, i.e., the space where it acts as a multiplication operator [28]. In other words, the wavefunction $\phi_{NW}(x, t)$ is the description of the state $\phi$ in the spectral representation of the Newton-Wigner operator $q_t$. Notice also that $\langle \phi | \phi \rangle = \int_{\mathbb{R}} \phi_{NW}(x, t) \phi_{NW}^*(x, t) \ dx$, and we are allowed to regard $|\phi_{NW}(x, t)|^2$ as the probability density to find a particle at the position $x$ at the instant $t$.

With these ingredients we can now define localized versions of the position operator $q_t$ associated to (measurable) regions $O \subset \mathbb{R}$ by the spectral representation $q_{t, O} := \int_{\mathbb{R}} x | \psi(x, t) \rangle \langle \psi(x, t) | \ dx$, or, equivalently, by

$$
(q_{t, O} \phi)(x, t) := x \chi_O(x) \phi_{NW}(x, t),
$$

where $\chi_O(x)$ is the characteristic function of $O$. The operator $q_{t, O}$ is self-adjoint and is bounded for bounded $O$.

We now turn to the definition of the coherent states introduced by Kaiser in [10] and discuss their more relevant properties. For each $z := (\xi - i\pi, \tau - i\epsilon) \in \mathbb{C}^2$, with $\xi, \pi, \tau, \epsilon \in \mathbb{R}$ and $\epsilon > |\pi|$, define a coherent state by

$$
\phi_z(p) := N^{-1} \sqrt{\omega} \exp \left[ i\frac{\omega}{\hbar} (\tau + i\epsilon) - i\frac{p}{\hbar} (\xi + i\pi) \right].
$$

The states originally considered by Kaiser differ from those in [31] in that the factor $\sqrt{\omega}$ is absent in his formulation. This change corresponds to taking the original states by Kaiser as elements of $\mathcal{H}^2$, with the states in [31] being the corresponding elements in $\mathcal{H}^1$ obtained by applying the unitary map $M_{\sqrt{\omega}}$. More comments about this are found below. The normalization constant $N$ is fixed by the condition

$$
1 = \int_{\mathbb{R}} dp \omega |\phi_z(p)|^2 = \frac{1}{N^2} \int_{\mathbb{R}} dp e^{-2(\omega_0 - p\pi)/\hbar} = \frac{2m\epsilon}{N^2\lambda} K_1(2m\lambda/\hbar),
$$

where $K_1$ is the modified Bessel function of the first kind.
with \( \lambda \equiv \sqrt{\varepsilon^2 - \pi^2} \), where \( K_\nu \), here and below, are the modified Bessel functions of \( \nu \)-th order (MacDonald’s functions). This leads to \( N = \sqrt{2m\varepsilon K_1(2m\lambda/\hbar)/\lambda} \). Momentum and position averages can be calculated in explicit form.

Their expectation values of \( p \) and \( q_t \) in the states \( \phi_z \) are given by

\[
\langle p \rangle_{\phi_z} = m \frac{\pi K_2(2m\lambda/\hbar)}{\lambda K_1(2m\lambda/\hbar)}, \quad (33)
\]

\[
\langle q_t \rangle_{\phi_z} = \xi + v(t - \tau), \quad \text{for} \quad v = \frac{\pi}{\epsilon}. \quad (34)
\]

From the expressions above we see that the average position of the wavefunction moves with a constant velocity \( v \) determined by the parameters \( \epsilon, \pi \), which also determine the average momentum of the state. The parameter \( \xi \) is the initial position of the coherent state labeled by \( z \), and \( \tau \) the initial instant of time. The usual relativistic relation between momentum and velocity is obtained in the limit \( \hbar \to 0 \), as we will show later.

It is interesting at this point to discuss the localization properties of the solution of the Klein-Gordon equation associated to the coherent states \( \phi_z(p) \). We will denote these solutions by \( \varphi_z(x, t) \). They are given by the Fourier transform

\[
\varphi_z(x, t) = \frac{2}{\sqrt{2\pi}} \int_{\mathbb{R}} dp \, e^{-i(p_0 t - px)/\hbar} \phi_z(p_0, p) \theta(p_0) \delta(p^2 - m^2) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \frac{dp}{\omega} e^{-i\omega t/\hbar + ipx/\hbar} \phi_z(p). \quad (35)
\]

This function depends on \( x \) and only through the combinations \( x - \xi \) and \( t - \tau \). Hence, for simplicity, we set \( \tau = \xi = 0 \) and define \( z_0 \equiv (-i\pi, -i\epsilon) \). We are interested in the asymptotic behavior of this wavefunction when \( x \to \infty \). In order to find it, write

\[
\varphi_{z_0}(x, t) = \int_{\mathbb{R}} dp \, e^{ipx/\hbar} \left\{ \frac{1}{\sqrt{2\pi}} \omega^{-1/2} \right\} \left\{ \frac{1}{N} \exp\left[ -\frac{\omega}{\hbar}(\epsilon + it) + \frac{p\pi}{\hbar} \right] \right\} \quad (36)
\]

and consider each of the two factors in curly brackets in the integrand independently, so that the transform can be computed by the convolution theorem. The Fourier transform of the first factor is just the Newton-Wigner state localized at \( x = 0 \),

\[
\int_{\mathbb{R}} dp \, e^{ipx/\hbar} \frac{1}{\sqrt{2\pi}} \omega^{-1/2} \propto \left( \frac{2m\hbar}{|x|} \right)^{1/4} K_{1/4}(m|x|/\hbar). \quad (37)
\]

The second factor, after changing to hyperbolic coordinates \( p = m \sinh s, \omega = m \cosh s \), and substituting

\[
\epsilon + it = \rho \cosh y, \quad \pi + ix = \rho \sinh y, \quad (38)
\]

with \( \Re(\rho) > 0 \) and \( \Im(y) \in [-\pi/2, \pi/2] \), becomes

\[
\frac{m}{N} \int_{-\infty}^{\infty} ds \exp\left[ -\frac{m\rho}{\hbar} \cosh(s - y) \right] \cosh s = \frac{2m \epsilon + it}{\rho} K_1(-m\rho/\hbar). \quad (39)
\]

Thus,

\[
\varphi_{z_0}(x, t) \propto \int_{\infty}^{\infty} du \frac{\epsilon + it}{\rho} K_1(-m\rho/\hbar) \left( \frac{1}{|x - u|} \right)^{1/4} K_{1/4}(m|x - u|/\hbar), \quad (40)
\]

where \( \rho \) is calculated with \( u \) replacing \( x \) in (38). But \( \rho \approx u + i\pi \) for large \( u \), and the MacDonald functions \( K_\nu \) display an exponential decay for large arguments. This can be used to prove that the overlap integral in (40) has an exponential decay for large values of \( x \), i.e., that

\[
|\varphi_{z_0}(x, t)| \leq k e^{-m|x|/\hbar}, \quad |x| > R, \quad (41)
\]

for some positive \( k, R \). This shows that the coherent state \( \phi_z \) is in fact a wavepacket, localized in some finite region, outside of which it falls to zero exponentially with a mass-dependent rate. This asymptotic behavior is characteristic of localized relativistic particles. The Newton-Wigner states satisfy exactly the same inequality (3). Moreover, the notion of essentially localized state in Algebraic Quantum Field Theory is also based on an analog inequality (1).

As we mentioned, the states originally considered by Kaiser differ from those in (41) by the factor \( \sqrt{\omega} \). The basic properties of the coherent states are not affected by the introduction of this factor, since in both cases the result is
a localized wavepacket moving with constant velocity. The advantage in introducing this factor is that the velocity \( v \) acquires a simple interpretation in terms of the parameters \( \pi, \epsilon \), what not only will be useful when dealing with the classical limit, but also gives a more direct physical interpretation of these coefficients. Moreover, the expression of the associated Newton-Wigner wavefunction is severely simplified, as will be discussed below. The payoff is that the spacetime wavefunction \( \varphi_t(x, t) \) gets more complicated. The reason for our choice is that in order to define the relativistic version of local operators it is natural to use the Newton-Wigner representation, and in consequence of this we will work mainly in this representation.

For the Newton-Wigner wavefunction associated to the coherent states \( \phi_{z_0} \) one gets by \((\ref{equation:39})\) the same integral solved in \((\ref{equation:38})\), so that

\[
\phi_{z_0}^{NW}(x, t) = \frac{1}{\sqrt{2\pi N}} \frac{2m}{\rho} e^{it} K_1(m\rho/\hbar),
\]

with \( \rho \) given by \((\ref{equation:38})\). The expressions for general \( \xi, \tau \) are obtained with translations. It follows that the asymptotic behavior is given in the new representation by the same expression \((\ref{equation:10})\).

We have proved that the wavefunction decays exponentially with large spatial arguments, for a fixed time. On the other hand, from \((\ref{equation:34})\) it is seen that the average position moves with constant velocity. Now we want to discuss how the wavepacket spreads about this average motion, so consider the variance of the position distribution. It turns out that

\[
\langle q^2 \rangle_{\phi_{z_0}} - \langle q \rangle_{\phi_{z_0}}^2 = -v^2 t^2 - \pi^2 + (\pi^2 + \epsilon^2) \frac{1}{N} \int dp \frac{p^2}{\omega^2} e^{-2(\omega \xi - \omega \pi)/2\hbar} =: D(h)^2,
\]

and that \( \lim_{h \to 0} \sigma_{q^2} = 0 \) (uniformly for \( t \) in compacts). Therefore, the wavepacket is well concentrated about the average motion for small \( h \). The momentum is also well-determined in this limit,

\[
\sigma_p^2 = (p^2)_{\phi_{z_0}} - \langle p \rangle_{\phi_{z_0}}^2 = \frac{m^2}{4} \left[ \frac{K_3(2m\lambda/\hbar) - K_1(2m\lambda/\hbar)}{K_1(2m\lambda/\hbar)} \right] + \frac{m^2\pi^2}{\lambda^2} \left[ \frac{K_3(2m\lambda/\hbar) - (2m\lambda/\hbar)}{K_1(2m\lambda/\hbar)} \right]^2,
\]

what leads to \( \lim_{h \to 0} \sigma_p^2 = 0 \). In this limit, \( p = m\pi/\lambda = m\gamma(v)v \), for \( \gamma(v) := (1 - v^2)^{-1/2} \), and the usual relativistic relation between momentum and velocity is obtained.

The fact that \( \lim_{h \to 0} \sigma_{q^2} = 0 \) can be used to determine a nice property of the Newton-Wigner wavefunctions of the coherent states. Let us now take \( \xi \neq 0 \) but \( \tau = 0 \). In the Newton-Wigner representation one has \( \sigma_n^2 = \int dR \left| \phi_{z_0}^{NW}(x, t) \right|^2 \left[ x - (\xi + vt) \right]^2 \). For \( R > 0 \), define the time-dependent region \( O_{t, \xi} := \{ x; \left| x - (\xi + vt) \right| < R \} \) with \( O_{t, \xi}^c \) being its complementary set. Hence,

\[
R^2 \int_{O_{t, \xi}} dR \left| \phi_{z_0}^{NW}(x, t) \right|^2 \leq \int_{O_{t, \xi}} dR \left| \phi_{z_0}^{NW}(x, t) \right|^2 \left[ x - (\xi + vt) \right]^2 \leq \sigma_n^2,
\]

implying that \( \int_{O_{t, \xi}} dR \left| \phi_{z_0}^{NW}(x, t) \right|^2 \leq D(h)^2/R^2 \). Since \( \lim_{h \to 0} D(h) = 0 \), we may claim that, for each fixed \( R \), the fraction of the \( L^2 \) norm of \( \phi_{z_0}^{NW}(x, t) \) outside the time-dependent region \( O_{t, \xi} \) is smaller than any desired bound \( D_0 > 0 \) as \( h \to 0 \). This is a relativistic analog of \((\ref{equation:10})\). Besides that, it follows by analogous computations that the average of the local operator is a good approximation for the average of \( q_t \):

\[
\left| \langle q_t \rangle_{\phi_{z_0}} - \langle q_t, O_{t, \xi} \rangle_{\phi_{z_0}} \right| \leq \frac{D(h)}{R} \left[ ||\xi + vt|| + D(h) \right].
\]

Summing up, the situation is very similar to that found in the nonrelativistic case: coherent states are states essentially localized in some region \( O \), outside of which the wavefunction is as small as desired when \( h \to 0 \), and the average of the position operator can be approximated by a local version. We will now show that the same steps followed there can be repeated here, and an approximation scheme can be devised for the free scalar field which leads to a classical limit describing a system of relativistic particles.

First, define \( N \)-body local operators \( q_{i, O}^{(N)} \) acting on the \( N \)-particle sector of the Fock space generated by \( \mathcal{H} \) by the recipe given in \((\ref{equation:13})\). At each time \( t \), essentially localized \( N \)-particle states \( \Phi^N \) are defined as symmetrized products of one-particle coherent states \( \phi_{z_i} \), situated at disjoint regions \( O_i = [a_i - R, a_i + R] \), where \( a_i = \xi_i + vt_i \). Let \( d \) be the smallest distance between these regions, and put \( A := \max\{|a_i|\} \). Then,

\[
\langle |\phi_i| \phi_j \rangle \leq \frac{2D(h)}{R},
\]

(46)
and for the operator $q_{t,O_k}$,

$$|\langle \phi_i | q_{t,O_k} | \phi_j \rangle| \leq (A + R) \frac{D(h)}{R + d}, \quad \text{if } i \neq j,$$

(47)

$$|\langle q_{t,O_k} \rangle_{\phi_i}| \leq \frac{AD(h)}{R + d}, \quad \text{if } i \neq k,$$

(48)

$$|\langle q_t \rangle_{\phi_k} - \langle q_{t,O_k} \rangle_{\phi_k}| \leq (A + D(h)) \frac{D(h)}{R}.$$  

(49)

These are the analogs of Eqs. (15)–(17). Then, it turns out that

$$\langle q^{(N)}_{t,O_k} \rangle_{\Phi} \simeq \langle q_t \rangle_{\phi_k},$$

up to an error of order $D(h)$. But the average at the right can be evaluated in the classical limit $\hbar \to 0$, leading to

$$\lim_{\hbar \to 0} \langle q^{(N)}_{t,O_k} \rangle_{\Phi} = \xi_k + v_k t.$$  

This completes our argument. The complete result is the following: there are localized position operators defined in each $N$-particle sector for the free scalar field whose mean values, when evaluated at essentially localized states constructed as the symmetrized products of relativistic coherent states, follow the expected classical trajectories in the limit $\hbar \to 0$.

We have restricted the definition of the local operators to some $N$-particle sector, but this is not necessary. One can define local operators $\bigoplus_{N=1}^{\infty} q^{(N)}_{t,O}$ in Fock space which measure positions inside a specified region $O$ of space. The average of this operator for an essentially localized state $|\psi\rangle$ with any number of localization centers $O_i$ is approximately zero if $O_i \cap O = \emptyset$ for all $i$, and is approximated by $\langle \phi_k | q_t | \phi_k \rangle$ if $O_k = O$ for some $k$. The number of particles of the state $|\psi\rangle$ is irrelevant; the local observer at $O$ cannot determine how many particles are there in regions of space which are not accessible to it.

The above results can be viewed in connection with the question of characterizing which quantum field theories can be described in terms of particles. The relativistic coherent states we have discussed provide a quantum representation for the classical concept of a relativistic particle in free motion. Whenever one uses a quantum field theory to describe scattering processes between interacting particles, it is somehow assumed that states which correspond to particles do exist. This requirement is usually formulated as the condition of existence of essentially localized states whose wavefunction decays exponentially with a mass-dependent coefficient at any fixed instant of time. This is the basic idea underlying the approach of [1]. If one is willing to be more restrictive, there is the possibility of imposing the additional condition that localized states remain localized at all times, and this line of argument was pursued in [24]. The discussion above suggests an alternative approach. One may try to associate coherent states to particle tracks observed in experiments. These states must be essentially localized, but besides that, one should impose the condition that the average of suitable local position operators must follow classical trajectories, with a negligible dispersion about this average, at least for a time-interval compatible with the experiment in question. In this case the question of the existence of a particle interpretation could be recast as the question of existence of a classical limit describing particles, and Hepp’s analysis would be a natural tool to deal with this problem.

V. CONCLUSION

We have defined essentially localized states and local versions of Weyl operators for nonrelativistic bosonic particles, and used these concepts to prove the existence of a particle classical limit for a system of $N$ interacting bosonic particles in an external potential. The quantum theory of a massive scalar field was proved to have two distinct classical limits, describing classical particles or a classical field. Local versions of the Newton-Wigner position operator were defined for this theory, and the particle classical limit was obtained as the $\hbar \to 0$ limit of the average of these local operators evaluated on a class of essentially localized states constructed as symmetrized products of relativistic coherent states introduced herein.

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APPENDIX A: TIME-EVOLUTION OF COHERENT STATES FOR LINEARIZED EQUATIONS OF MOTION

A system of \( N \) distinguishable particles of the same mass \((m = 1, \text{for convenience})\) is described by the (unsymmetrized) Hilbert space \( \mathcal{H}^{\otimes N} \) of square-integrable functions on \( \mathbb{R}^N \). Consider the time-evolution generated by the time-dependent quadratic Hamilton operator

\[
H(t) = \sum_{i=1}^{N} \frac{p_i^2}{2} + \frac{1}{2} \sum_{i \neq j} V''(\xi_i(t) - \xi_j(t))(q_i - q_j)^2,
\]

where the \( \xi_i(t), \ j = 1, \ldots, N, \) describe the classical trajectories of \( N \) interacting particles for a solution of the corresponding classical system with initial conditions \( \alpha = [\xi + i\pi]/\sqrt{2\hbar} \). In the Heisenberg picture, the time-dependent position and momentum operator satisfy linear equations

\[
i\hbar \dot{q}_i(t) = p_i(t), \quad i\hbar \dot{p}_i(t) = -\sum_{i \neq j} V''(\xi_i(t) - \xi_j(t))q_j(t),
\]

identical to the classical equations of motion. These are solved by

\[
\begin{bmatrix}
q(t) \\
p(t)
\end{bmatrix} = S(t) \begin{bmatrix} q \\ p \end{bmatrix},
\]

where \( S(t) \) is a \((2N) \times (2N)\) symplectic matrix (in order that the canonical commutation relations be preserved) whose entries depend continuously on \( t \). Writing \( S(t) \) in block form as

\[
S(t) = \begin{bmatrix} A(t) & B(t) \\ C(t) & D(t) \end{bmatrix},
\]

the symplectic condition is equivalent to \( AD^t - BC^t = 1, \ AB^t = BA^t, CD^t = DC^t \) \([21]\). Here, \( A, B, C, D \) and \( 1 \) are \( N \times N \) matrices, \( 1 \) being the identity matrix.

Now, let the initial state of the system be the coherent state \( |0\rangle \) of zero position and momentum in \( \mathcal{H}^{\otimes N} \), and \( \psi_0(x) = \langle x|0 \rangle \) be the corresponding wavefunction. The time-evolved state at instant \( t \) in the Schrödinger representation is \( \psi(x, t) = \langle x|W(t)|0 \rangle \) as usual, where \( W(t) = T \exp \left[ -i/\hbar \int_0^t dt' H'(t') \right] \). The initial state satisfies the differential equations

\[
[q_j + ip_j] \psi_0(x) = 0, \quad j = 1, \ldots, N,
\]

what in turn implies that \( [q_j(t) + ip_j(t)]\psi(x, t) = 0, \forall j \). In terms of the block-components of \( S(t) \), one has

\[
\left[ (A_s^r(t) + iC_s^r(t))x - \hbar(D_s^r(t) - iB_s^r(t))\nabla_s \right] \psi(x, t) = 0.
\]

The indices \( r, s = 1, \ldots, N \) label matrix coefficients and the Einstein sum convention is used. Some results about symplectic matrices allow for an exact solution of this linear system of differential equations. First, it is advantageous to break \( S \) into simpler factors, and solve successively for the action of each factor. It was proved in \([22]\) that a so-called structured singular value decomposition exists for any symplectic matrix \( S \): one can write

\[
S = O'DO,
\]

where \( O, O' \) are orthogonal, \( D = \text{diag}(\omega_1, \ldots, \omega_N, \omega_1^{-1}, \ldots, \omega_N^{-1}) \), with \( \omega_j > 0 \) for all \( j \), and the three factors in the decomposition are symplectic matrices. That any square matrix \( M \) admits a singular value decomposition (i.e., \( M \) can be written as \( M = O_1D_1O_2 \), with \( O_1 \) and \( O_2 \) orthogonal and \( D_1 \) diagonal with non-negative entries) is a well-known result; a structured decomposition is one in each all three factors are symplectic. Second, an orthogonal symplectic matrix \( O \) has a very simple form,

\[
O = \begin{bmatrix} U & V \\ -V & U \end{bmatrix},
\]

where \( U, V \) are real matrices such that \( U - iV \) is unitary.
Hence, for a given time $t$ with $|t| < T$ write $S(t)$ in the form of a singular value decomposition \((A3)\), with $O$ as in \((A1)\). Consider first the action of the factor $O$. One can define new operators

$$
\begin{bmatrix}
q' \\
p'
\end{bmatrix} = O \begin{bmatrix}
q \\
p
\end{bmatrix}
$$

which satisfy the canonical commutation relations $[q'_j, p'_k] = i\hbar\delta_{jk}$. But, by the Stone-von Neumann Theorem, there is, up to unitary equivalence, only one representation of the canonical commutation relations \((A2)\), hence the pair $q'$, $p'$ must be unitarily equivalent to the original pair $q$, $p$. Let $U_O$ be a unitary operator such that $q'_j = U_O q_j U_O$, and $p'_j = U_O p_j U_O$. Use this operator to define the state $\psi'(x) = U_O \psi(x)$ which, from \((A2)\), must satisfy

$$(U'_s - iV'_s) (x + \hbar \nabla_s) \psi'(x) = 0.$$ 

Multiplying by $U^t + iV^t$ on the left, one gets $(x + \hbar \nabla_s) \psi^O(x) = 0$, whose solution is $\psi'(x) = \psi(x)$. The initial state is not changed by the action of $O$. The action of the factors $D$ and $O'$ can be studied in the same manner. Define

$$
\begin{bmatrix}
q'' \\
p''
\end{bmatrix} := D \begin{bmatrix}
q' \\
p'
\end{bmatrix}, \quad \begin{bmatrix}
q''' \\
p'''
\end{bmatrix} := O' \begin{bmatrix}
q'' \\
p''
\end{bmatrix},
$$

and let $U_D$, $U_{O'}$ be unitary operators which accomplish the corresponding unitary transformations. The state $\psi'' = U_D \psi$ satisfies the system of differential equations

$$(\omega_r x_r + \frac{\hbar}{2} \omega_r^{-1} \nabla_r) \psi''(x) = 0,$$

which is solved by the product of Gaussian functions $\psi''(x) = \prod_j (\pi \hbar)^{-1/4} \omega_j^{1/2} \exp[-(\omega_j x_j)^2 / 2 \hbar]$. This state is not changed by the action of $O'$, so that $U_{O'} U_D U_O \psi(x) = \psi''(x)$. But $U_{O'} U_D U_O = U_S$, and the unitary operator which correspond to the symplectic matrix $S$ is just the propagator $W(t)$. Therefore,

$$
\psi(x, t) = \prod_j (\pi \hbar)^{-1/4} \omega_j^{1/2} \exp[-(\omega_j x_j)^2 / 2 \hbar].
$$

The time-evolution of a coherent state under a quadratic Hamiltonian can always be solved via a singular value decomposition. An algorithm giving the decomposition of an arbitrary symplectic matrix was developed in \((A2)\).
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[26] Strictly speaking, the uniqueness statement of the S.-von N. Theorem refers to the representations of the Weyl form of the canonical commutation relations.