Harder-Narasimhan stacks for principal bundles in higher dimensions

Sudarshan Gurjar and Nitin Nitsure

Abstract

Let $G$ be a connected split reductive group over a field $k$ of arbitrary characteristic, chosen suitably. Let $X \to S$ be a smooth projective morphism of locally noetherian $k$-schemes, with geometrically connected fibers. We show that for each Harder-Narasimhan type $\tau$ for principal $G$-bundles, all pairs consisting of a principal $G$-bundle on a fiber of $X \to S$ together with a given canonical reduction of HN-type $\tau$ form an Artin algebraic stack $Bun_{X/S}^\tau(G)$ over $S$. Moreover, the forgetful 1-morphism $Bun_{X/S}^\tau(G) \to Bun_{X/S}(G)$ to the stack of all principal $G$-bundles on fibers of $X \to S$ is a schematic morphism, which is of finite type, separated and injective on points.

The notion of a relative canonical reduction that we use was defined earlier in arXiv:1505.02236, where we showed that a stronger result holds in characteristic zero, namely, the 1-morphisms $Bun_{X/S}^\tau(G) \to Bun_{X/S}(G)$ are locally closed imbeddings which stratify $Bun_{X/S}(G)$ as $\tau$ varies.

1 Introduction

Let $G$ be a split reductive group over a field $k$, such that the following hypothesis ($\ast$) is satisfied (we will say more about this hypothesis at the end of the Introduction).

($\ast$) Preservation of canonical reductions under field extensions: If $L/K/k$ are extension fields of $k$, if $H = P/R_u(P)$ where $P$ is a standard parabolic in $G$ and if $E$ a semistable principal $H$-bundle on a geometrically irreducible smooth projective curve $X$ over $K$, then the base change $E_L$ is a semistable principal $H$-bundle on $X_L$.

Let $S$ be a locally noetherian scheme over $k$ and let $X \to S$ be a smooth projective morphism with geometrically connected fibers. Let there be chosen a split maximal torus in $G$ and a Borel containing it, and let $\tau$ be an element of the resulting closed positive Weyl chamber. For any $S$-scheme $T$ and a principal $G$-bundle $E$ on $X_T$, we defined in [G-N-2] the notion of a relative canonical reduction $[L, \phi]$ of $E$ of Harder-Narasimhan type $\tau$ (which is recalled later). We define an $S$-groupoid $Bun_{X/S}^\tau(G)$ which attaches to $T$ the groupoid whose objects are pairs consisting of a principal $G$-bundle $E$ on $X_T$ and a relative canonical reduction of $E$ of HN-type $\tau$. We denote by $Bun_{X/S}(G)$ the algebraic stack of all $G$-bundles on $X/S$. The main result of this note is the following.

Theorem 1.1 The $S$-groupoid $Bun_{X/S}^\tau(G)$ is an algebraic stack over $S$. The natural forgetful 1-morphism $Bun_{X/S}^\tau(G) \to Bun_{X/S}(G)$ is a schematic morphism, which is of finite type, separated and injective on points.
The above theorem can be equivalently re-formulated as follows.

**Theorem 1.2** Let $E$ be a principal $G$-bundle on $X$. There exists a scheme $S^\tau(E)$ over $S$ which has the universal property that for any $S$-scheme $T$, the set of all relative canonical reductions of type $\tau$ of the pullback $E_T/X_T/T$ is in a natural bijection with the set of all $S$-morphisms from $T$ to $S^\tau(E)$. Moreover, the morphism $S^\tau(E) \to S$ is of finite type, separated and injective.

If $k$ is of characteristic zero, it can be shown (see [G-N-2] Theorem 1.1 and Proposition 7.4) that each $S^\tau(E)$ is a locally closed subscheme of $S$, and as $\tau$ varies over the closed positive Weyl chamber $\overline{C}$, these subschemes stratify $S$. Correspondingly in the Theorem above, $Bun^\tau_{X/S}(G) \to Bun_{X/S}(G)$ is a locally closed substack, and these stratify $Bun_{X/S}(G)$ as $\tau$ varies (see [G-N-2] Theorem 7.7).

The stronger results in characteristic zero are made possible by the uniqueness and the infinitesimal uniqueness of a canonical reduction in characteristic zero. It is known (see [He]) that the property of infinitesimal uniqueness does not necessarily hold in the finite characteristic case (failure of the Behrend conjecture [Be], which in the context of principal bundles says that the canonical reduction of a principal $G$-bundle over a curve has no infinitesimal deformations). Instead, one has the (weaker) results of this note.

The importance of the hypothesis ($\ast$) is that it allows the definition of a moduli functor for $G$-bundles of a given HN-type on curves (and also on higher dimensional projective varieties). In fact (see [He]), ($\ast$) is a consequence of the Behrend conjecture. It is known that if $G = GL_{n,k}$ or $SL_{n,k}$, then the Behrend conjecture is satisfied for all $k$, and if $\text{char}(k) = 0$, then it is satisfied for all $G$. Moreover, the conjecture always holds for classical groups, and holds for exceptional simple groups whenever $\text{char}(k)$ is large enough (see Theorem 1 in [He]).

**Question:** More generally, if $S$ is a quasi-finite, flat scheme over Spec $\mathbb{Z}$ and if $G$ is obtained by base change from a reductive group scheme $\mathcal{G}$ defined over $S$, then one may ask whether there exists a nonempty open subscheme $S' \subset S$ such that the Behrend conjecture (or at least the hypothesis ($\ast$)) holds whenever Spec $k$ factors via $S'$.

## 2 Preliminaries

Let $G$ be a reductive group over a field $k$ of arbitrary characteristic, such that $G$ is split over $k$, together with a chosen split maximal torus and a Borel containing it.

Let $K$ be an extension field over $k$, and $X$ a smooth irreducible projective variety over $K$ with a very ample line bundle $\mathcal{O}_X(1)$. Let $E$ be a principal $G$-bundle defined on $X$ (or defined on a big open subscheme $U$ of $X$, where bigness of $U$ signifies that $X - U$ is of codimension $\geq 2$). Recall that $E$ is said to be semistable w.r.t. the choice of $\mathcal{O}_X(1)$ if for any standard parabolic $P \subset G$, any section $\sigma : W \to E/P$ defined
on a big open subscheme $W$ of $U$, and any dominant character $\chi : P \rightarrow G_{m,K}$, we have
\[
\deg(\chi_*\sigma^*E) \leq 0
\]
where $\sigma^*E$ is the principal $P$-bundle on $W$ defined by the reduction $\sigma$, and $\chi_*\sigma^*E$ is the $G_{m}$-bundle obtained by extending its structure group via $\chi : P \rightarrow G_{m}$, which is equivalent to a line bundle on $W$. This line bundle extends uniquely (up to a unique isomorphism) to a line bundle on $X$, denoted again by $\chi_*\sigma^*E$, and $\deg(\chi_*\sigma^*F)$ is its degree w.r.t. $\mathcal{O}_X(1)$. A rational reduction of the structure group to a standard parabolic $P$ is a section $\sigma : U \rightarrow E/P$ of $E/P \rightarrow X$ over a big open subscheme $U \subset X$. Recall that a canonical reduction of $E$ is a rational reduction of structure group of $E$ to a standard parabolic $P \subset G$ for which the following two conditions hold:

1. If $\rho : P \rightarrow L = P/R_u(P)$ is the Levi quotient of $P$ (where $R_u(P)$ is the unipotent radical of $P$) then the principal $L$-bundle $\rho_*\sigma^*E$ is a semistable principal $L$-bundle defined on the big open subscheme $U$ on which $\sigma$ is defined.

2. For any non-trivial character $\chi : P \rightarrow G_{m}$ whose restriction to the chosen maximal torus $T \subset B \subset P$ is a linear combination $\sum n_i\alpha_i$ of simple roots $\alpha_i \in \Delta$ where $n_i \geq 0$, and at least one $n_i \neq 0$, we have $\deg(\chi_*\sigma^*E) > 0$.

To any such reduction, one associates a Harder-Narasimhan type $\tau \in \mathcal{N}$ (see [G-N-2], section 4 for an exposition).

We recall the following well known fact (originally proved by Behrend [Be] for curves).

**Proposition 2.1** Let $G$ be a reductive group over a field $k$ of arbitrary characteristic, such that $G$ is split over $k$, together with a chosen a split maximal torus and a Borel containing it. Let $K$ be an extension field over $k$, and $X$ a smooth irreducible projective variety over $K$ with a very ample line bundle. Let $E$ be a principal $G$-bundle on $X$. Then $E$ admits a unique canonical reduction.

Let $T \subset B \subset G$ be the chosen split torus and Borel. Given a standard parabolic $P \supset B$, let $\lambda_P \in X^*(T)$ be a chosen dominant weight such that $\lambda_P$ is a character on $P$ which lies in the negative ample cone for $G/P$. Let $V_{\lambda_P}$ be a chosen irreducible representation of $G$ with highest weight $\lambda_P$, and let $0 \neq v \in V_{\lambda_P}$ be a chosen highest weight vector. Then for the action of $G$ on the projective space $\mathbf{P}(V_{\lambda_P})$ of lines in $V_{\lambda_P}$, the isotropy subgroup scheme at the point $[v] \in \mathbf{P}(V_{\lambda_P})$ is $P$, and we get a closed $G$-equivariant embedding $G/P \hookrightarrow \mathbf{P}(V_{\lambda_P})$ under which $eP \mapsto [v]$.

With the above notation, we recall the definition of a relative canonical reduction made in [G-N-2]. Let $X \rightarrow S$ be a smooth projective morphism with geometrically connected fibers, where $S$ is a noetherian scheme over $k$, with a given relatively very ample line bundle $\mathcal{O}_{X/S}(1)$ on $X$. For any principal $G$-bundle $E$ on $X$, let $E(V_{\lambda_P})$ denote the associated vector bundle on $X$ corresponding to the representation $V_{\lambda_P}$. As defined in [G-N-2], a relative rational reduction of structure group of $E$ from $G$
to $P$ is an equivalence class $[L, f]$ of pairs $(L, f)$, where $L$ is a line bundle on $X$ and $f : L \to E(V_{\lambda_p})$ is an injective $\mathcal{O}_X$-linear homomorphism of sheaves, such that

(i) the open subscheme $U = \{x \in X \mid \text{rank}(f_x) = 1\} \subset X$ is relatively big over $S$, that is, for each $s \in S$ the fiber $U_s$ has complementary codimension $\geq 2$ in the fiber $X_s$, and

(ii) the section $U \to \mathbf{P}(E(V_{\lambda_p}))$ defined by $f$ factors via the natural closed embedding $E/P \to \mathbf{P}(E(V_{\lambda_p}))$.

Two such pairs $(L, f)$ and $(L', f')$ are equivalent if there exists an isomorphism $\phi : L \to L'$ such that $f = f' \circ \phi$.

In the special case where $S = \text{Spec} \, K$ for a field $K$, the above definition is equivalent to the usual definition of a rational reduction to $P$ that we recalled earlier ([G-N-2] Proposition 3.2). Finally, we say that a pair $(L, f)$ as above defines a canonical reduction $[L, f]$ of type $\tau$ if its restriction to each fiber $X_s$ of $X \to S$ is a canonical reduction of $E_s$ of constant type $\tau$ (this is well-defined).

The following remark shows that if $S$ has a Zariski open cover $(W_i)$ and we have relative rational $P$-reductions $[L_i, f_i]$ of $E_{W_i}/X_{W_i}/W_i$ which are represented by pairs $(L_i, f_i)$ which are equivalent over each $W_i \cap W_j$, then there exists a unique relative rational reduction $[L, f]$ of $E/X/S$ which restricts to these.

**Remark 2.2 (Sheaf property.)** Let $Y$ be a scheme, $\mathcal{E}$ be a sheaf of $\mathcal{O}_Y$-modules, and $(U_i)$ be an open cover of $Y$. Let for each $i$ there be given a line bundle $L_i$ on $U_i$ together with an injective $\mathcal{O}_{U_i}$-linear homomorphism of sheaves $f_i : L_i \to \mathcal{E}|_{U_i}$. Suppose that for each $U_{ij} = U_i \cap U_j$, there exists an element $g_{ij} \in \mathbb{G}_m(U_{ij})$ (that is, a nowhere vanishing regular function on $U_{ij}$) such that $f_i = g_{ij} \cdot f_j$ (we do not assume any cocycle condition on the $g_{ij}$’s). Then there exists a line bundle $L$ on $Y$ and an injective $\mathcal{O}_Y$-linear homomorphism $f : L \to \mathcal{E}$, such that for any $i$, there exists an isomorphism $h_i : L|_{U_i} \to L_i$ with $f|_{U_i} = f_i \circ h_i$. Moreover, if $(L', f')$ is another such pair, then there exists a unique $\mathcal{O}_Y$-linear isomorphism $\phi : L' \to L$ such that $f' = f \circ \phi$. For, the image subsheaves $\text{im}(f_i) \subset \mathcal{E}|_{U_i}$ coincide over $U_i \cap U_j$, so they glue together to define a global subsheaf $L \subset \mathcal{E}$. Take $f : L \hookrightarrow \mathcal{E}$ to be the inclusion. Then the pair $(L, f)$ has the desired property. Given any other such $(L', f')$, the image of the homomorphism $f'$ is the subsheaf $L \subset \mathcal{E}$, so $f'$ factors through $L$ to give rise to a homomorphism $\phi : L' \to L$ with the desired property.

## 3 Proofs

If $T$ is a scheme and $\mathcal{E}$ is a sheaf of $\mathcal{O}_T$-modules, we will denote by $\Gamma(T, \mathcal{E})^\times \subset \Gamma(T, \mathcal{E})$ the subset which consists of all nowhere vanishing global sections of $\mathcal{E}$. In particular, $\Gamma(T, \mathcal{O}_T)^\times = \mathbb{G}_m(T)$ is the group of all invertible regular functions on $T$. Note that the group $\mathbb{G}_m(T)$ acts on the set $\Gamma(T, \mathcal{E})^\times$ by scalar multiplication.

The following lemma is a projective version of the result of Grothendieck on the representability by a linear scheme for sections of direct images (see [EGA III 7.7.8,
7.7.9], and [Ni-2, 5.8] for an exposition). We expect this lemma, though elementary, to be of independent interest.

**Lemma 3.1** Let \( X \to S \) be a proper morphism of noetherian schemes, and let \( \mathcal{F} \) be a coherent \( \mathcal{O}_X \)-module that is flat over \( S \). Consider the contravariant functor \( \Psi' : (\text{Schemes}/S)^{\text{op}} \to \text{Sets} \) which associates to any \( S \)-scheme \( T \) the quotient set

\[
\Psi'(T) = \frac{\Gamma(T, \pi_T^* \mathcal{F}_T)^\times}{\mathbb{G}_m(T)}
\]

where \( \Gamma(T, \pi_T^* \mathcal{F}_T)^\times \) is the set of all nowhere vanishing sections of \( \pi_T^* \mathcal{F}_T \), on which \( \mathbb{G}_m(T) \) acts by scalar multiplication. Then the sheafification \( \Psi \) of \( \Psi' \) in the big Zariski site over \( S \) is representable by the \( S \)-scheme

\[
\mathbb{P}(\mathcal{Q}) = \text{Proj}_S \text{Sym}_S^*(\mathcal{Q}),
\]

where \( \mathcal{Q} \) denotes the Grothendieck \( \mathcal{Q} \)-sheaf of \( \mathcal{F}/X/S \) (locally over \( S \), we can take \( \mathcal{Q} \) to be the cokernel of the transpose of the 0th differential of a Grothendieck semi-continuity complex for \( \mathcal{F}/X/S \)).

**Proof.** If \( \mathcal{E} \) is any coherent sheaf \( S \), then \( \mathbb{P}(\mathcal{E}) = \text{Proj}_S \text{Sym}_S^*(\mathcal{E}) \) represents the functor \( \varphi : (\text{Schemes}/S)^{\text{op}} \to \text{Sets} \) which is the sheafification in the big Zariski site over \( S \) of the functor that associates to any \( S \)-scheme \( T \) the quotient set

\[
\frac{\text{Hom}(\mathcal{E}_T, \mathcal{O}_T)^{\text{sur}}}{\mathbb{G}_m(T)}
\]

where \( \text{Hom}(\mathcal{E}_T, \mathcal{O}_T)^{\text{sur}} \subset \text{Hom}(\mathcal{E}_T, \mathcal{O}_T) \) consists of all surjective homomorphisms \( \mathcal{E}_T \to \mathcal{O}_T \), and \( \mathbb{G}_m(T) \) acts on it by scalar multiplication (which is the restriction of the action of \( \mathbb{G}_m(T) \) on \( \text{Hom}(\mathcal{E}_T, \mathcal{O}_T) \)). The Grothendieck sheaf \( \mathcal{Q} \) has the universal property (see [EGA III 7.7.8, 7.7.9]) that we have a natural bijection

\[
\Gamma(T, \pi_T^* \mathcal{F}_T) \cong \text{Hom}(\mathcal{Q}_T, \mathcal{O}_T).
\]

This bijection is equivariant under the action of \( \mathbb{G}_m(T) \) on both sides by scalar multiplication. As the bijection is functorial in \( T \), by pull-back to any point of \( T \) it follows that nowhere vanishing sections of \( \pi_T^* \mathcal{F}_T \) exactly correspond to surjective homomorphisms \( \mathcal{Q}_T \to \mathcal{O}_T \), and hence we get a natural \( \mathbb{G}_m(T) \)-equivariant bijection

\[
\Gamma(T, \pi_T^* \mathcal{F}_T)^\times \cong \text{Hom}(\mathcal{Q}_T, \mathcal{O}_T)^{\text{sur}}.
\]

The lemma follows on passage to the quotient sets under \( \mathbb{G}_m(T) \). \( \square \)

**Remark 3.2** Unlike the functor \( T \mapsto \Gamma(T, \pi_T^* \mathcal{F}_T) \), which is representable in the projective case if and only if \( \mathcal{F} \) is flat over \( S \) (see [Ni-1]), the functor \( \Psi \) may or may not be representable when \( \mathcal{F} \) is not flat. For example, let \( X = S = \text{Spec} \ k[t] \) for a field \( k \), and let \( \mathcal{F}_1 = (k[t]/(t))^\sim \) and \( \mathcal{F}_2 = \mathcal{F}_1 \oplus \mathcal{O}_X \), which are coherent sheaves
on $X = S$ which are not flat over $S$. We leave it to the reader to verify that the corresponding functor $\Psi$ is representable for $F_1$, and it is not representable for $F_2$.

**Proof of Theorem 1.2.** An exposition of the basic facts about the relative Picard scheme $Pic_{X/S}$ that we need can be found in [K]. We first treat the case where $\pi : X \to S$ admits a global section $\sigma : S \to X$. Under this assumption, there exists a Poincaré line bundle on $X \times_S Pic_{X/S}$, fixed up to (a non-unique) isomorphism by the requirement that its pullback to $Pic_{X/S}$ under the section $(\sigma, \text{id}_{Pic_{X/S}}) : Pic_{X/S} \to X \times_S Pic_{X/S}$ is a trivial line bundle. The choice of $\tau \in \mathcal{C}$ determines the Hilbert polynomial $h \in \mathbb{Q}[t]$ of any line bundle $L$ which occurs in a canonical reduction $[L, f : L \to E_s(V_{\lambda_P})]$ of type $\tau$ of the principal $G$-bundle $E_s = E|_{X_s}$ on a fiber $X_s$. Let $J = Pic^0_{X/S} \subset Pic_{X/S}$ be the open and closed subscheme where the Hilbert polynomial of the line bundle is $h$. Let $\mathcal{L}$ denote the restriction of the Poincaré line bundle to $X \times_S J \subset X \times_S Pic_{X/S}$. Let $F$ be the coherent $\mathcal{O}$-module on $X \times_S J$ defined by

$$F = \overline{\text{Hom}}(\mathcal{L}, E_J(V_{\lambda_P}))$$

and let $Q$ denote the coherent $\mathcal{O}_J$-module which is the Grothendieck $Q$-sheaf for $F$. Let $Y = \mathbb{P}(Q)$ be the corresponding projective scheme over $J$, which has the universal property given by Lemma 3.1. Over $Y$, we have a universal element $f \in \Psi(Y)$ in the notation of Lemma 3.1 which can be represented by a Zariski open cover $(U_i)$ of $Y$ together with a family of homomorphisms $f_i : \mathbb{L}_{U_i} \to E_{U_i}(V_{\lambda_P})$. Note that on $U_i \cap U_j$, the homomorphisms $f_i$ and $f_j$ differ by scalar multiplication by an element of $G_m(U_i \cap U_j)$. Let $Y_1 \subset Y$ be the union of the open subschemes of $U_i$ where $f_i : \mathbb{L}_{U_i} \to E_{U_i}(V_{\lambda_P})$ is fiberwise injective in a relatively large open subscheme of $X_{U_i}$. Let $Y_2 \subset Y$ be the closed subscheme which is the union of the closed subschemes of $U_i$ where the homomorphism $f_i : \mathbb{L}_{U_i} \to E_{U_i}(V_{\lambda_P})$ factors via the cone $\mathcal{E}/P \subset \mathcal{E}_{U_i}(V_{\lambda_P})$ over $E/P \subset \mathbb{P}(E_{U_i}(V_{\lambda_P}))$. Let $Y_3 = Y_1 \cap Y_2 \subset Y$, which is the locally closed subscheme, where the $f_i : \mathbb{L}_{U_i} \to E_{U_i}(V_{\lambda_P})$ define a rational reduction of structure group to $P \subset G$. Let $Y_4 \subset Y_3$ be the open and closed subscheme of $Y_3$ where the topological type of the reduction is given by $\tau$. Finally, let $Y_5 \subset Y_4$ be the open subscheme of $Y_4$ where the extension under the Levi quotient $P \to P/R_u(P)$ is semistable. By Lemma 3.1 is immediate from its construction that the $S$-scheme $Y_5$ represents the functor $T \mapsto \Phi^r_{E/X/S}(T)$ which is the set of all relative canonical reductions of type $\tau$ of the pullback $E_T/X_T/T$.

Now we come to a general case, where $X$ may not necessarily admit a global section over $S$. As $X \to S$ is by assumption smooth, there exists a surjective separated étale morphism $p : S' \to S$ such that the base change $X' = X_{S'}$ admits a global section $S' \to X'$. Let $E' = E_{S'}$. Hence by the above special case, there exists a scheme $Y' \to S'$ which represents the functor $\Phi^r_{E'/X'/S'} : (\text{Schemes}/S')^\text{op} \to \text{Sets}$. Let $S'' = S' \times_S S'$, and let $p_1, p_2 : S'' \to S'$ be the two projections. We write $\pi = p \circ p_1 = p \circ p_2$, and $X'' = \pi^*X$ and $E'' = (\text{id} \times \pi)^*E$, so that we have natural identifications

$$p_1^*X' = X'' = p_2^*X'$$
and

\[(\text{id} \times p_1)^* E' = E'' = (\text{id} \times p_2)^* E'.\]

Note that \(p_1^* Y'\) and \(p_2^* Y'\) respectively represent the two functors \(\Phi^*_{(\text{id} \times p_1)^* E'/X''/S''}\) and \(\Phi^*_{(\text{id} \times p_2)^* E'/X''/S''}\), and both these functors have a natural isomorphism with the functor \(\Phi^*_{E'/X''/S''}\) which comes from the above identifications \((\text{id} \times p_1)^* E' = E'' = (\text{id} \times p_2)^* E'\). Hence we get an isomorphism of the representing \(S''\)-schemes

\[g : p_1^* Y' \sim p_2^* Y'.\]

As the bundle \(E\) is defined over the base \(S\), the functor \(\Phi^*_{E/X/S}\) is defined over \(S\)-schemes, and the other functors \((\Phi^*_{(\text{id} \times p_1)^* E'/X''/S''}\) etc.) are obtained from it by base changes. It follows that the above isomorphism \(g\) between the representative schemes for these functors satisfies the cocycle condition when pulled back to \(S' \times_S S' \times_S S'\). Hence \(Y'\) descends to an algebraic space over \(S\). By its construction, \(Y\) represents \(\Phi^*_{E/X/S}\). As the square

\[
\begin{array}{ccc}
Y' & \rightarrow & Y \\
\downarrow & & \downarrow \\
S' & \rightarrow & S
\end{array}
\]

is cartesian, as \(S' \rightarrow S\) is an étale cover and as \(Y' \rightarrow S'\) is of finite type, it follows that \(Y \rightarrow S\) is of finite type. By the same reasoning, as \(Y' \rightarrow S'\) is separated by its construction, it follows that \(Y \rightarrow S\) is separated.

Next, we note that the Proposition 2.1 implies that \(Y' \rightarrow S'\) is injective at the level of underlying sets. By the above reasoning, this implies that \(Y \rightarrow S\) too is injective at the level of underlying sets. In particular, \(Y \rightarrow S\) is quasi-finite.

Given the above properties of \(Y \rightarrow S\), the Proposition 3.3 below implies that \(Y\) is a scheme. This is the desired scheme \(S^r(E)\) by its construction.

\[\square\]

**Proposition 3.3** ([Stacks Project] Tag 03XX, Proposition 55.47.2.) Let \(S\) be a scheme. Let \(f : X \rightarrow T\) be a morphism of algebraic spaces over \(S\). Assume that \(T\) is representable, \(f\) is locally quasi-finite, and \(f\) is separated. Then \(X\) is representable.

**Proof of Theorem 1.1** We begin by recalling that the stack \(Bun_{X/S}(G)\) of \(G\)-bundles on fibers of \(X/S\) is algebraic. To see this, choose a closed embedding \(G \hookrightarrow GL_{n,k}\) as group schemes over \(k\), and consider the induced 1-morphism of stacks \(Bun_{X/S}(G) \rightarrow Bun_{X/S}(GL_{n,k})\). The stack \(Bun_{X/S}(GL_{n,k})\) is just the stack of rank \(n\) vector bundles on fibers of \(X/S\), so it is an algebraic stack (see [L-MB]). Given any \(GL_{n,k}\)-bundle \(E\) on \(X\), the reductions of its structure group to \(G\) are the sections of \(E/G \rightarrow X\), so they are parameterized by a suitable open subscheme of the Hilbert scheme \(Hilb(E/G)/S\) (see for example [Ni-2] section 5.6.2 for an exposition). This shows the 1-morphism \(Bun_{X/S}(G) \rightarrow Bun_{X/S}(GL_{n,k})\) is schematic, which implies that the stack \(Bun_{X/S}(G)\) is algebraic.

Next, given \(\tau \in \mathcal{O}\), consider the 1-morphism from the stack \(\wedge_{X/S}(G)\) of Corollary 1.2 to the stack \(Bun_{X/S}(G)\). The Theorem 1.2 shows that this 1-morphism is schematic and has the desired properties. \[\square\]
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Sudarshan Gurjar
Department of Mathematics
Indian Institute for Technology, Bombay
Powai
Mumbai 400 076
India
srgurjar1984@gmail.com

Nitin Nitsure
School of Mathematics
Tata Institute of Fundamental Research
Homi Bhabha Road
Mumbai 400 005
India
nitsure@math.tifr.res.in