PERIOD AND INDEX OF GENUS ONE CURVES OVER
NUMBER FIELDS

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Abstract. The period of a curve is the smallest positive degree of Galois-invariant divisor classes. The index is the smallest positive degree of rational divisors. We construct examples of genus one curves with prescribed period and index over certain number fields.

1. Introduction

Let $X$ be a nonsingular, projective, geometrically integral curve over a field $K$. Define the index $I(X)$ to be the greatest common divisor of $[L : K]$, where $L$ varies over finite algebraic field extensions for which $X(L) \neq \emptyset$. In particular, if $X$ has a rational point over $K$ already, the index is 1. Define the period $P(X)$ of $X$ to be the smallest positive degree amongst Galois-invariant divisor classes on $X$. That is, if $K$ is an algebraic closure for $K$, then we consider divisor classes on $X := X \times K$ which are fixed points for the Galois action $\text{Gal}(\overline{K}/K)$. We have $P(X) \mid I(X)$, for the Galois orbit of any $x \in X(L)$ yields an invariant divisor class. However, the two need not be equal: take the example of a conic without rational points, say $(the\ projective\ curve\ given\ by)\ x^2 + y^2 = -1$ over $\mathbb{R}$. Clearly, the index is 2, but the class of a single point is Galois invariant.

In [10], Lichtenbaum showed that

**Theorem 1.1.** For $X/K$ as above, if the genus, period, and index of $X$ are $g$, $P$, and $I$ respectively, then $P \mid I \mid 2P^2$, $I \mid (2g - 2)$, and if either $(2g - 2)/I$ or $P$ is even, then $I \mid P^2$.

However, over local fields and $C_1$ fields, the above divisibility conditions are not sharp. For example, for genus 1 curves over local fields, the period and index are always equal [10]. One may then ask what triples $(g, P, I)$ actually occur as the genus, period, and index of a curve over a fixed $K$. Over local fields of characteristic not 2, the problem is solved in [13]. We consider the case $g = 1$ and prove the following:

**Theorem 1.2.** Let $K$ be a number field and $E$ an elliptic curve over $K$. Let $\ell$, $n$ be positive integers such that $\ell \mid n$. Suppose either that $K$ contains the $n$th roots of unity, or that $E$ has an order $n$ Galois-stable cyclic subgroup. If $n$ is even and $4 \nmid \ell$, suppose further that $K$ contains the $(2n)$th roots of unity, or that $E$ has a Galois-stable cyclic subgroup of order $2n$. Then there exists a genus 1 curve $X$ with Jacobian $E$ for which $P(X) = n$ and $I(X) = n\ell$.

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Note that for $g = 1$, $(2g - 2)/I$ is always even; therefore by Theorem 1.1 we have $I \mid P^2$. For $K$ satisfying the hypotheses of Theorem 1.2, then, we have covered every possible period and index for genus 1 curves.

The significance of period and index lies in computations of Tate-Shafarevich and Brauer groups. Grothendieck [7, §4 et seq.] is the canonical source for these computations. Gonzalez-Avilés [6] showed that given a (suitably nice) curve $X$ over a global field $K$, under certain hypotheses the finiteness of the Brauer group of a model for $X$ is equivalent to the finiteness of the Tate-Shafarevich group of the Jacobian of $X$. The relationship between the sizes of these groups is then computable, and depends on the indices and periods of $X$ both over $K$ and over the completions of $K$. Liu, Lorenzini, and Raynaud extended this result ([11, Theorem 4.3] and [12]). They showed that for $K$ the function field of a curve over a finite field, and assuming that for some prime $\ell$ the $\ell$-part of $Br_X$ is finite, or that the Tate-Shafarevich group of the Jacobian of $X$ is finite, we have $Br_X$ is finite and has square order.

Lastly, in a forthcoming paper [4] the author with Pete Clark uses a similar result as the one in this paper to show that if $E$ is an elliptic curve over a number field $K$, $p$ is a prime and $N$ any integer, then there exists a $p$-extension $L/K$ such that $\#(E/L)[p] \geq N$; that is, the order of the $p$-torsion of the Tate-Shafarevich group is unbounded over $p$-extensions.

For other results exhibiting curves with various periods and indices, see Cassels [1], Lang-Tate [8], Stein [20], O’Neil [15], and Clark [2], [3] and [5]. The strongest previous results in this direction are Clark’s, who proved that when $E[p] \subset E(K)$ for $p$ prime, there are principal homogeneous spaces for $E$ with period $p$ and index $p^2$, and that there are curves of every index over every number field.

Recently, Jakob Stix [21] has shown that if $X$ is a curve over $\mathbb{Q}$ and there is a section for the canonical map $\pi_1(X) \to \pi_1(\mathbb{Q})$, then the period and index of $X$ are equal.

2. Preliminary Results

2.1. Basic properties of period and index. One can alternatively define the index to be the smallest positive degree of a divisor on $X$ over $K$. If $x \in X(\overline{K})$, the Galois orbit of $x$ written as a divisor $\sum(\sigma x - x)$ furnishes a rational divisor of positive degree. Any prime divisor (of $\text{Div}_X$, not $\text{Div}(X)$) is necessarily of this form. This shows the equivalence of the two definitions.

There is also an alternate definition of the period. Choose any $x \in X(\overline{K})$. The cocycle $\sigma \mapsto \sigma x - x$ furnishes a cohomology class in $\xi \in H^1(K, J)$, where $J$ is the Jacobian variety of $X$. Let $\text{Pic}^1_{X/K}$ be the connected component of the Picard scheme of $X$ representing degree 1 invertible sheaves; it is a principal homogeneous space for $J = \text{Pic}^0_{X/K}$. As such, there is a class in $H^1(K, J)$ representing $\text{Pic}^1_{X/K}$, and in fact one sees that this class is $\xi$. Similarly, $n\xi$ represents $\text{Pic}^n_{X/K}$. We know that $\text{Pic}^0_{X/K}$ is a trivial principal homogeneous space for $J$ precisely when it possesses a $K$-point. Such a point occurs when there is a Galois-invariant element of $\text{Pic}^0_{X/K}$. Thus we may define the period as the order of $\xi$.

Note that when $X$ is a genus one curve, $X$ itself equals $\text{Pic}^1_{X/K}$, and $\xi$ represents $X$ as an element of $H^1(K, E)$, where $E$ is the Jacobian of $X$. We say a field $L$ splits
if and only if
Lemma 2.1. Let \( P_1, P_2, I_1, \) and \( I_2 \) be positive integers such that the \( P_i \) are relatively prime and \( P_1 \mid I_1 \mid P_2^2 \). Let \( E \) be an elliptic curve over a field \( K \). If there are genus one curves \( X_1 \) and \( X_2 \) over \( K \) with Jacobian \( E \) such that \( X_1 \) has period \( P_i \) and index \( I_i \), then there is also a curve \( X \) with Jacobian \( E \) having period \( P_1 P_2 \) and index \( I_1 I_2 \).

Proof. Let \( \xi_i \in H^1(K, E) \) represent \( X_i \). I claim that \( \xi := \xi_1 + \xi_2 \) has period \( P_1 P_2 \) and index \( I_1 I_2 \).

Since the period is the order of \( \xi \) in \( H^1(K, E) \), the first part of the claim is obvious. Now suppose that \( \xi \) has index \( I \). If \( L \) is a finite extension of \( K \) which splits \( \xi \), then \( \xi \) lies in the kernel of the restriction map \( H^1(K, E) \to H^1(L, E) \). Since the orders of the \( \xi_i \) are relatively prime and res is a homomorphism, \( L \) splits both of the \( \xi_i \) as well. Therefore \( I_1 I_2 \) divides \( I \). On the other hand, any field which splits both of the \( \xi_i \) splits \( \xi \) as well. In particular, we can choose fields of the form \( L_1 \cdot L_2 \) where each \( L_i \) splits \( \xi_i \). Varying over all such choices, one sees that \( I \mid I_1 I_2 \). □

2.2. O’Neil’s obstruction map. Our main tool for computing the index is O’Neil’s obstruction map. In order to define it, we first construct a theta group. Let \( E[n] \) be the \( n \)-torsion points of the elliptic curve \( E \), viewed as a group scheme over \( K \). Then our theta group is a central extension \( \mathcal{G}(n) \) in the category of group schemes over \( K \) given by the exact sequence

\[
0 \to \mathbb{G}_m \to \mathcal{G}(n) \to E[n] \to 0,
\]

where the commutator is given by the Weil pairing. That is, \( \mathcal{G}(n) \) as a scheme is just \( \mathbb{G}_m \times E[n] \), but with multiplication satisfying

\[
(x, S)(y, T)(x, S)^{-1}(y, T)^{-1} = (e_n(S, T), O),
\]

where \( e_n \) is the Weil pairing on \( E[n] \) and \( O \) is the identity of \( E \); see [14, §23] or [12] for more details. Taking (nonabelian) Galois cohomology, one obtains a map

\[
\text{Ob} : H^1(K, E[n]) \to H^2(K, \mathbb{G}_m) = \text{Br} K.
\]

This is the obstruction map. Note that it is a quadratic map, not a homomorphism. (Quadratic essentially means that \( \text{Ob}(\xi) = b(\xi, \xi) \) for some bilinear map \( b \).) This results from the fact that \( \mathcal{G}(n) \) is a noncommutative group scheme, and hence our cohomology sequence takes place in nonabelian Galois cohomology; see Zarhin [22].

Recall the Kummer sequence for \( E \) gives rise to the exact sequence in Galois cohomology

\[
0 \to E(K)/nE(K) \to H^1(K, E[n]) \to H^1(K, E)[n] \to 0.
\]

Thus, if \( X \) has period dividing \( n \), there exist (usually many) \( \xi \in H^1(K, E[n]) \) representing it; that is, the image of \( \xi \) in \( H^1(K, E)[n] \) represents \( X \) in the usual sense.

In [15], O’Neil showed
Proposition 2.2. The image of the obstruction map lies in $(\text{Br}K)[n]$. If $\text{Ob}(\xi)$ has order $\ell$, then the index of the curve $X$ represented by $\xi$ divides $n\ell$. If $X$ has index $n$, then there is a class $\xi$ representing $X$ such that $\text{Ob}(\xi) = 0$.

The idea is that $H^1(K, E[n])$ classifies principal homogeneous spaces $X$ for $E$ over $K$ along with a choice of Galois-invariant degree $n$ invertible sheaf $\mathcal{L} \in (\text{Pic}X)^G$, up to isomorphism over $K$. Specifically, any cocycle in the class of $\xi$ gives rise to a $\overline{\mathcal{K}}$-isomorphism $\varphi : E \to X$. Then $\mathcal{L} = \mathcal{L}(n\varphi(O))$. One sees that $\Gamma(X, \mathcal{L})/\overline{\mathcal{K}}^\times$ forms a twist of $\mathbb{P}^{n-1}$ over $K$—that is, a Brauer-Severi variety. The obstruction map takes the pair $(X, \mathcal{L})$ to the class of this Brauer-Severi variety in $\text{Br}K$. From this characterization, the proposition is easy: If $X$ has index $n$, then there is a rational divisor $D$ of degree $n$. The class $\xi$ representing $(X, \mathcal{L}(D))$ satisfies $\text{Ob}(\xi) = 0$. If $\text{Ob}(\xi)$ has order $\ell$ and $\xi$ represents the pair $(X, \mathcal{L})$, then the Brauer-Severi variety arising from $\mathcal{L}$ is split by an extension of degree $\ell$. Then the tensor product $\mathcal{L}^\ell$ is a class of degree $n\ell$ containing a rational divisor. See [15] for more details.

One would hope that the converse held: if $X$ has index $n\ell$, then there ought to be $\xi$ representing $X$ such that $\text{Ob}(\xi)$ has order $\ell$. However, this is not known to be true. We will use a trick in proving the main theorem to construct $X$ for which this does hold.

Let $\delta$ be the composition $E(K) \to E(K)/nE(K) \to H^1(K, E[n])$, where the latter map is given by the map in (2.2). For $x \in E(K)$ and $X$ a principal homogeneous space for $E$ over $K$, let $T(x, X)$ denote the Tate pairing of $X$ with the image of $x$ in $E(K)/nE(K)$. Then

$\text{Proposition 2.3.} \quad \text{Ob} (\xi + \delta x) = \text{Ob} (\xi) + T(x, X).$

Proof. See [15, §5].

O’Neil’s introduction of the obstruction map is useful because in many cases it may be computed using a Hilbert symbol, which we now define. Let $K$ be a field of characteristic not dividing $n$ which contains $\mu_n$, the $n$th roots of unity. Let $a, b \in K^\times/K^{\times n}$. By Kummer theory, we know that $K^\times/K^\times_n = H^1(K, \mu_n)$. The cup product gives a map

$K^\times/K^\times_n \times K^\times/K^\times_n \to H^2(K, \mu_n \otimes \mu_n)$,

Fix a primitive $n$th root of unity $\zeta$. Define an isomorphism $\mu_n \otimes \mu_n \to \mu_n$ by $\zeta^i \otimes \zeta^j \mapsto \zeta^{ij}$, which induces an isomorphism $H^2(K, \mu_n \otimes \mu_n) \to H^2(K, \mu_n)$. Using the fact that $H^2(K, \mu_n) = (\text{Br}K)[n]$, we see that the composition gives a pairing

$K^\times/K^\times_n \times K^\times/K^\times_n \to (\text{Br}K)[n]$

$$(a, b) \mapsto \langle a, b \rangle$$

which we call the Hilbert symbol. If we let $w = (a, b)$, then we will alternatively write $\langle w \rangle$ for $\langle a, b \rangle$. Since the cup product is bilinear and skew-symmetric, so is the Hilbert symbol.

Note that the Hilbert symbol depends on $n$ and $\zeta$. Frequently we will abuse notation and define the Hilbert symbol as a map $(K^\times)^2 \to \text{Br}K$, by composing with the obvious quotient map.

We return to the obstruction map. Assume that the $n$-torsion of $E$ is rational over $K$. By the theory of the Weil pairing, $\mu_n \subset K$. Let $\zeta$ be the previously chosen primitive $n$th root of unity. Fix a basis $(S, T)$ for $E[n]$ such that $e(S, T) = \zeta$, where $e$ is the Weil pairing. The choice of basis, and our fixed generator $\zeta$ of $\mu_n$,
Proposition 2.6. For \( u \) which by Hensel’s Lemma also equals the order of \( \text{im} \) immediately for the first claim, and from bilinearity for the second claim.

Proof. The case for that \( n \) is odd see [15, Prop. 3.4] and [10]. A proof in the even case, including an explicit computation of \( \text{Ob}(\xi) = \langle \kappa(\xi) \rangle \), can be found in [4, §2].

Assume from now on that \( K \) is a number field. If \( v \) is a place of \( K \), write \( K_v \) for the completion of \( K \) at \( v \). In order to use the Hilbert symbol, we will reduce to the local case using the fact that, if \( K \) is a global field, \( \text{Br} K = \bigoplus_v \text{Br} K_v \). That is, in order to compute \( \langle a, b \rangle \), it suffices to compute \( \langle a, b \rangle_v \), where the latter symbol is computed in \( K_v \) and \( a, b \) are considered as elements of \( K_v \).

(More generally, for any \( K \)-group scheme \( M \) and integer \( q \), we have a natural localization map
\[
\text{H}^q(K,M) \to \text{H}^q(K_v,M)
\]
in étale cohomology. We will denote this map by adding the subscript \( v \); e.g. \( \xi \mapsto \xi_v \).

If \( v \) is a place of \( K \), we also use \( v \) to denote a fixed corresponding valuation.

Lemma 2.5. Let \( K_v \) be a nonarchimedean local field such that \( v(n) = 0 \). Let \( \pi \) be a uniformizing parameter and \( u, u' \) units; that is, \( v(u) = v(u') = 0 \). Let \( F \) be the residue field of \( K_v \).

1. \( \langle u, u' \rangle_v = 0 \).
2. The order of \( \langle u, \pi \rangle_v \) equals the order of the image of \( u \) in \( \mathbb{F}^\times \mathbb{F}^\times_n \).

Proof. According to [17, Ch. XIV], \( \langle a, b \rangle_v = 0 \) if and only if \( b \) is a norm from the extension \( K_v(a^{1/n})/K_v \). The extension \( K_v(a^{1/n}) \) is unramified with degree, say, \( d \), which by Hensel’s Lemma also equals the order of \( u \) in \( \mathbb{F}^\times \mathbb{F}^\times_n \). According to local class field theory, the norm from an unramified extension of degree \( d \) is precisely the set of elements \( x \) such that \( v(x) \) is a multiple of \( dv(\pi) \). The result follows immediately for the first claim, and from bilinearity for the second claim.

We also have the following:

Proposition 2.6. For \( a, b \in K^\times \), \( \sum_v \langle a, b \rangle_v = 0 \).

Here, we use the fact that \( \text{Br} K_v \) is canonically isomorphic to \( \mathbb{Q}/\mathbb{Z}, \mathbb{Z}/\mathbb{Z}, \) or \( 0 \) via the invariant map so that the sum makes sense. For the remainder of this paper, we identify \( \text{Br} K_v \) with the appropriate subgroup of \( \mathbb{Q}/\mathbb{Z} \). Note that the sum is finite: there are finitely many archimedean primes, and for all but finitely many nonarchimedean \( v \), \( v(a) = v(b) = v(n) = 0 \), so that by Lemma 2.5 \( \langle a, b \rangle_v = 0 \).

Proof. This follows from the fact that \( \text{Br} K \) is isomorphic to the set of \( (t_v) \), \( t_v \in \text{Br} K_v \), such that almost all \( t_v = 0 \) and \( \sum t_v = 0 \); and the fact that cup products, and hence the Hilbert symbol, commute with localization.

In order to deal with the fact that the obstruction map does not necessarily give a sharp bound on index, we will use obstruction maps of different levels. We define \( \text{Ob}_n \) to be the corresponding obstruction map \( \text{H}^1(K, E[n]) \to \text{Br} K \).
Proposition 2.7. Let $n$ and $m$ be positive integers. The following diagrams commute:

(1) \[ \begin{array}{ccc}
H^1(K, E[n]) & \xrightarrow{\text{Ob}_n} & \text{Br} K \\
\downarrow j_* & & \downarrow m \\
H^1(K, E[mn]) & \xrightarrow{\text{Ob}_{mn}} & \text{Br} K
\end{array} \]

where $j_*$ is induced by the canonical inclusion $j : E[n] \to E[mn]$, and $m$ is multiplication by $m$.

(2) \[ \begin{array}{ccc}
H^1(K, E[mn]) & \xrightarrow{\text{Ob}_{mn}} & \text{Br} K \\
\downarrow [m] & & \downarrow m \\
H^1(K, E[n]) & \xrightarrow{\text{Ob}_n} & \text{Br} K
\end{array} \]

where $[m]$ is the map induced by the multiplication by $m$ map $E[mn] \to E[n]$.

(3) \[ \begin{array}{ccc}
H^1(K, E[n]) & \xrightarrow{\text{Ob}_n} & \text{Br} K \\
\downarrow \text{res} & & \downarrow \text{res} \\
H^1(L, E[n]) & \xrightarrow{\text{Ob}_n} & \text{Br} L
\end{array} \]

where $L/K$ is a field extension and $\text{res}$ is the restriction map.

Proof. Let $\xi \in H^1(K, E[n])$ represent the pair $(X, \mathcal{L})$, where $\mathcal{L}$ is the divisor class of $n(x)$ for some $x \in X(K)$. Then $j_*(\xi)$ represents $(X, \mathcal{L}^m)$. The map which takes elements of $(\text{Pic} \, \overline{X})^G$ to the class of the associated Brauer-Severi variety is a homomorphism; indeed, it arises from the Leray spectral sequence $H^p(K, H^q(\overline{X}, \mathbb{G}_m)) \Rightarrow H^{p+q}(X, \mathbb{G}_m)$, which yields the exact sequence

$$0 \to \text{Pic} \, X \to H^0(K, \text{Pic} \, \overline{X}) \to H^2(K, \mathbb{G}_m).$$

The last map sends an invertible sheaf to the class of its associated Brauer-Severi variety; see for example [9]. The first part of the proposition follows.

For the second part, Mumford showed [13, p. 309–310] that multiplication by $m$ extends to a homomorphism on theta groups $\mathcal{G}(mn) \to \mathcal{G}(n)$. The restriction of this homomorphism to $\mathbb{G}_m$ is also multiplication by $m$. Taking the long cohomology sequence associated to (2.1), we obtain the commutative diagram.

The third part is obvious, since $\text{res} \, \xi$ represents the same pair $(X, \mathcal{L})$ as $\xi$ does, but over $L$. \square

Similar to $\text{Ob}_n$, we define $\delta_n$ to be the composition $E(K) \to E(K)/nE(K) \to H^1(K, E[n])$, where the second map in the composition is the coboundary coming from the Kummer sequence for $E$. 


3. The case of rational torsion

By Lemma \[2.1\] we may assume that \( n \) is a prime power \( p^r \); this assumption holds for the remainder of the paper. In this section, we will prove the theorem under the additional assumption that \( E[n] \subset E(K) \) in the case that \( p \) is odd, and \( E[2n] \subset E(K) \) if \( p = 2 \).

3.1. Choosing a pair of primes. Our first step is to find a pair of distinct nonarchimedean primes \( v, v' \) satisfying certain splitting conditions. (We use primes and places interchangeably.) We state the conditions below, after which we show that infinitely many pairs \( v, v' \) satisfying the conditions exist. The proof is essentially repeated use of the Cebotarev density theorem. Let \( S \) be the union of the primes \( w \) of \( K \) such that \( E \) has bad reduction at \( w \), archimedean primes, and primes dividing \( n \). The conditions are

A1. The primes \( v, v' \) are principal with totally positive generators \( \pi \) and \( \pi' \) respectively.

A2. Let \( E(K) \) embed in \( E(K_v) \) in the usual manner. Then \( E(K) \) lies in \( nE(K_v) \).

A3. For each \( w \in S \), \( \pi \) and \( \pi' \) lie in \( K_w^{\times n} \).

A4. The order of the image of \( \pi' \) in \( K_v^{\times n} / K_v^{\times n} \) is exactly \( n \).

Lemma 3.1. There exist infinitely many pairs of distinct primes \( v, v' \) satisfying conditions \( A1–A4 \).

Proof. Condition A1 is equivalent to \( v \) and \( v' \) splitting completely in the Hilbert class field of \( K \).

Condition A2 is the same as requiring \( v \) to split completely in \( K([n]^{-1}E(K)) \); that is, the field obtained by adjoining to \( K \) all \( x \in E(K) \) such that \( [n]x \in E(K) \). By [12, p.194], \( K([n]^{-1}E(K)) \) is a finite abelian extension which is unramified outside \( S \).

Let \( \mathfrak{m} \) be the modulus given by the product of \( n^2 \) and all primes where \( E \) has bad reduction. Let \( K_{\mathfrak{m}} \) be the ray class field for \( K \) with modulus \( \mathfrak{m} \). Then condition A3 holds if \( v \) and \( v' \) split completely in \( K_{\mathfrak{m}} \); for in that case, the Frobenius for \( v \) (say) is trivial and, by class field theory, \( v \) has a generator \( \pi \) which is congruent to 1 (mod \( \mathfrak{m} \)). Then Hensel’s Lemma implies A3.

We now choose \( v \) to be any prime which splits completely in the compositum of the three fields named above. Next we tackle A4.

By abuse of notation, let \( v \) be a valuation corresponding to the prime \( v \). Let \( \alpha \in K \) be any element such that \( v(\alpha) = 0 \) and whose image in \( K_v^{\times n} / K_v^{\times n} \) has order \( n \); since \( \pi \equiv 1 \pmod{\mathfrak{m}} \), and \( n \mid \mathfrak{m} \), there exists such \( \alpha \) in \( K_v \). But \( K \) is dense in \( K_v \), so we may find such \( \alpha \) in \( K \). Let \( F' \) be the ray class field with modulus \( v \). By class field theory, the Galois group \( \text{Gal}(F'/K) \) is isomorphic to the class group with modulus \( v \). In particular, if \( v' \) and \( (\alpha) \) lie in the same class in this class group, then \( v' \) has a generator \( \pi' \) which is congruent to \( \alpha \pmod{v} \), and hence satisfies A4.

Let \( F \) be the compositum of \( K_{\mathfrak{m}} \) and \( K([n]^{-1}E(K)) \). We see that \( F' \) is unramified outside \( v \), while \( F \) is unramified at \( v \). Hence \( F \cap F' \) lies in the Hilbert class field of \( K \). To satisfy A1–A3, we wish \( v' \) to split completely in \( F \), while to satisfy A4, we wish \( v' \) to lie in the class of \( (\alpha) \) in the appropriate ray class group. These conditions are compatible since they both imply that \( v' \) splits completely in the Hilbert class field of \( K \). Thus we may apply the Cebotarev density theorem to find infinitely many such \( v' \) and \( \pi' \).

3.2. Construction of curve. As mentioned earlier, a choice of ordered basis $(S, T)$ for $E[n]$ with $e(S, T) = \zeta$ yields an isomorphism

$$\kappa : H^1(K, E[n]) \to K^\times/K^\times n \times K^\times/K^\times n.$$ 

Choose $\xi \in H^1(K, E[n])$ such that $\kappa(\xi) = (\pi, \pi^m/\ell)$. Let $X$ be the corresponding principal homogeneous space for $E$.

**Proposition 3.2.** The curve $X$ has period $n$ and index $n\ell$.

**Proof.** For any positive integer $m$, we have $\kappa(m\xi) = (\pi^m, \pi^{mn/\ell})$. Let $v$ be the place of $K$ lying over $(\pi)$, and suppose $m < n$. If the curve associated to $m\xi$ is a trivial principal homogeneous space, then there exists $x \in E(K)$ such that $\kappa(\delta x + m\xi) = (1, 1)$, where we recall that $\delta$ is the composition

$$E(K) \to E(K)/nE(K) \to H^1(K, E[n]).$$

But by condition A2, $x$ lies in $nE(K)$. Hence $(\delta x)_v = 0$. Therefore for any choice of $x$, $\kappa(\delta x + m\xi)_v = (\pi^m, \cdot)$, and so $m\xi$ yields a trivial principal homogeneous space if and only if $n \mid m$. Therefore the period of $X$ is $n$.

By Proposition 2.4, $\text{Ob}(\xi) = (\pi, \pi^m/\ell)$. We compute the Hilbert symbol locally. For places $w$ satisfying $w(n) > 0$, condition A3 shows that

$$\langle \pi, \pi^m/\ell \rangle_w = 0.$$ 

Let $\nu'$ be the place corresponding to $\pi'$. For $w \neq v, \nu'$ and $w(n) = 0$, $\pi$ and $\pi'$ are both units in $K_w$, and so by Lemma 2.6 the local Hilbert symbol is zero. By Proposition 2.2 the order of the Hilbert symbol at $v$ equals that at $\nu'$, so we need only consider $v$. Combining condition A4 with Lemma 2.6, we see that $\langle \pi, \pi^m/\ell \rangle$ has order $\ell$. Hence $\text{Ob}(\xi)$ has order $\ell$, and the index of $X$ divides $n\ell$.

Let $\ell'$ strictly divide $\ell$, and let $j_\ast$ be the canonical map

$$H^1(K, E[n]) \to H^1(K, E[n\ell']).$$

If $X$ had index $n\ell'$, then by Lemma 2.5 there would exist $x$ in $E(K)$ such that

$$\text{Ob}_{n\ell'}(\delta_{n\ell'}x + j_\ast(\xi)) = 0.$$ 

According to Proposition 2.5 the above equals

$$\text{Ob}_{n\ell'}(j_\ast(\xi)) + T(x, X).$$

By condition A2, $x \in nE(K_v)$ and hence $T(x, X)_v = 0$. But by Proposition 2.7 $\text{Ob}_{n\ell'} j_\ast(\xi) = \ell' \text{Ob}_n \xi \neq 0$ if $\ell' < \ell$. Therefore the index of $X$ is precisely $n\ell$. □

4. Proof of Main Theorem, odd $n$

For now, we assume $n$ is odd. We suppose either: a) $\mu_n \subset K$, or b) $E$ contains a Galois-stable subgroup of order $n$. Over $L := K(E[n])$, we may use the arguments of the previous section to construct a cohomology class $\xi \in H^1(L, E[n])$ with the desired properties. Let cores be the corestriction map

$$\text{cores} : H^1(L, E[n]) \to H^1(K, E[n]).$$

We will show that cores $\xi$ represents a curve over $K$ with period $n$ and index $n\ell$. In order to compute the index, we will need to base extend back to $L$; in other words, we’ll compute $\text{res} \circ \text{cores} \xi$, where $\text{res}$ is the restriction map

$$\text{res} : H^1(K, E[n]) \to H^1(L, E[n]).$$
4.1. Pairs of primes. As in the previous section, we wish to choose principal primes of \( L \), \((\pi)\) and \((\pi')\), such that analogues to conditions A1–A4 hold, along with a new condition. For any place \( w \) of \( L \), let \( w_K \) denote the place of \( K \) lying below \( w \). Let \( S \) be the set of primes \( w \) of \( L \) such that \( E \) has bad reduction at \( w_K \), \( w_K \) is archimedean, or \( w_K(n) > 0 \).

We now state the relevant conditions.

B1. The primes \( v = (\pi) \) and \( v' = (\pi') \) are principal, with totally positive generators \( \pi \) and \( \pi' \).

B2. Under the usual embedding, \( E(K) \) lies in \( nE(K_{vK}) \).

B3. The generators \( \pi \) and \( \pi' \) lie in \( L_v^n \) for all \( w \in S \).

B4. The order of the image of \( \pi' \) in \( L_v^x / L_v^n \) is \( n \). Additionally, \( \sigma \pi' \) lies in \( L_v^x \) for all nontrivial \( \sigma \in \text{Gal}(L/K) \).

B5. The primes \( v_K, v_K' \) split completely in \( L \).

**Lemma 4.1.** There are infinitely many pairs of primes \((\pi), (\pi')\) satisfying conditions B1–B5.

**Proof.** Let \( \mathfrak{m} \) be the modulus over \( L \) given as the product of \( n^2 \) and all primes \( w \) in \( S \). Let \( L_{\mathfrak{m}} \) be the ray class field of \( L \) with modulus \( \mathfrak{m} \). The modulus \( \mathfrak{m} \) is rational over \( K \), so that \( L_{\mathfrak{m}} \) is Galois over \( K \). Let \( F \) be the compositum of \( L_{\mathfrak{m}} \), \( K([n^{-1}]E(K)) \), and the Hilbert class field of \( K \). By the Cebotarev density theorem, there exists infinitely many primes \( v_K \) of \( K \) which split completely in \( F \). Setting \( v \) equal to any prime of \( L \) lying over \( v_K \), we use the same reasoning as in Lemma 3.1 to see that \( v \) satisfies conditions B1–B5.

Choose any unit \( \beta \) in \( L \), which has order \( n \) in \( L_v^x / L_v^n \). By the Chinese Remainder Theorem, there exists \( \alpha \in L \) such that

\[ \alpha \equiv \beta \pmod{v} \]

(4.1)

\[ \alpha \equiv 1 \pmod{\sigma v} \quad \forall \sigma \neq 1 \in \text{Gal}(L/K) \]

Let \( F' \) be the ray class field for \( L \) with modulus \( \prod \sigma v \). The modulus is rational over \( K \), so that \( F' \) is Galois over \( K \). The Galois group \( \text{Gal}(F'/L) \) is isomorphic to the class group with modulus \( \prod \sigma v \) via the Artin reciprocity map. Let \( \gamma \) in \( \text{Gal}(F'/L) \) map to the class of \((\alpha)\) under this isomorphism. Then under the inclusion \( \text{Gal}(F'/L) \hookrightarrow \text{Gal}(F'/K) \), \( \gamma \) maps to, say, \( \gamma \). Let \([\gamma]\) be the conjugacy class of \( \gamma \) in \( \text{Gal}(F'/K) \).

One can find \( \tau \in \text{Gal}(F'F/K) \) such that \( \tau|_{F'} \) lies in \([\gamma]\), and \( \tau|_{F} \) is trivial. This follows because \( F'/L \) is ramified only at \( \sigma v \), while \( F/L \) is unramified at those places; therefore \( F \cap F' \) is contained in the Hilbert class field of \( L \). But \( \gamma \) acts trivially on the Hilbert class field since \((\alpha)\) is principal. Therefore, \( \tau \) as prescribed exists.

Now we apply the Cebotarev density theorem again to find \( v'_K \) corresponding to the conjugacy class of \( \tau \). Choosing \( v' \) to be any place of \( L \) lying over \( v'_K \), we use the same reasoning as before to conclude that \( v' \) satisfies the conditions. \( \square \)

4.2. The corestriction map. As before, choose a basis \((S,T)\) for \( E[n] \) such that \( e(S,T) = \zeta \). If we are under the hypothesis that \( E \) contains a Galois-stable order \( n \) subgroup, choose our basis so that \( S \) generates this subgroup. As before, our choice of basis yields an isomorphism

\[ \kappa : H^1(L, E[n]) \to L^x / L^{x_n} \times L^x / L^{x_n} \]
Let $\pi, \pi'$ be as in the previous section, and choose $\xi$ so that $\kappa(\xi) = (\pi, \pi')$. Let $\text{cores} : H^1(L, E[n]) \to H^1(K, E[n])$ be the corestriction map, and $\eta = \text{cores}(\xi)$. We wish to show that the curve corresponding to $\eta$ satisfies the conditions of our theorem. But it is too difficult to compute the obstruction map $\text{Ob}(\eta)$, so we will instead use $\text{Ob}(\text{res} \eta)$, where $\text{res}$ is the restriction map $H^1(K, E[n]) \to H^1(L, E[n])$.

Therefore, we need to compute $\text{res} \circ \text{cores}(\xi)$. Note that $\text{cores} \circ \text{res}$ is well-known and equal to $[L:K]$, but the same is not true of $\text{res} \circ \text{cores}$.

For any $G_K$-module $M$ and nonnegative integer $r$, one has an action of $\text{Gal}(L/K)$ on $H^r(L, M)$. This action is induced by the action on homogeneous $r$-cochains

$$c^\sigma(\gamma_1, \ldots, \gamma_r) = \sigma c(\sigma^{-1} \gamma_1 \sigma, \ldots, \sigma^{-1} \gamma_r \sigma)$$

where $\sigma \in \text{Gal}(L/K)$ and $\gamma_i \in G_L$. (Actually, we must lift $\sigma$ to any fixed element of $G_K$ when acting by conjugation on the $\gamma_i$.) Define $\text{Nm} : H^r(L, M) \to H^r(L, M)$ by

$$\text{Nm}(\theta) = \sum_{\sigma \in \text{Gal}(L/K)} \theta^\sigma.$$  

**Lemma 4.2.** For $\theta \in H^r(L, M)$, $\text{res} \circ \text{cores} \theta = \text{Nm} \theta$, where $\text{res}$ and $\text{cores}$ are the obvious restriction and corestriction maps.

**Proof.** This follows from the definition of corestriction and dimension shifting; see [17, p. 119]. \qed

Therefore cores $\eta = \text{Nm} \xi$. Unfortunately, $\kappa$ is not $\text{Gal}(L/K)$-equivariant, so that $\kappa(\text{Nm} \xi) \neq (\text{Nm} \pi, \text{Nm} \pi')$. Instead, we have a twisted norm on $L^\times / L^\times n \times L^\times / L^\times n$, which we now describe. Let

$$\text{Gal}(L/K) \to \text{GL}_2(\mathbb{Z}/n\mathbb{Z})$$

$$\sigma \mapsto M_\sigma$$

be the representation of $\text{Gal}(L/K)$ on $E[n]$ with respect to the basis $(S, T)$. The determinant of $M_\sigma$ is given by the $n$th cyclotomic character. If $\mu_n \subset K$, then the determinant is identically $1$, and hence the representation lies in $\text{SL}_2(\mathbb{Z}/n\mathbb{Z})$. If $S$ generates a Galois-stable subgroup, then the representation lies in the set of upper-triangular matrices.

**Proposition 4.3.** Suppose $\sigma \in \text{Gal}(L/K)$, $\xi \in H^1(L, E[n])$ and $\kappa(\xi) = (a, b)$. Then

$$\kappa(\xi^\sigma) = \frac{M_\sigma}{\det M_\sigma}(\sigma a, \sigma b)$$

where for $M \in \text{GL}_2(\mathbb{Z}/n\mathbb{Z})$, if

$$M = \begin{bmatrix} i & j \\ k & l \end{bmatrix},$$

then $M(a, b) = (a^i b^j, a^k b^l)$.

**Proof.** Let $\rho : E[n] \to \mu_n \times \mu_n$ be the isomorphism given by the basis $(S, T)$. Write $(\mu_n \times \mu_n)_\rho$ for the $\text{Gal}(L/K)$-module with underlying group $\mu_n \times \mu_n$, but with
module-structure making \( \rho \) an isomorphism of Galois modules; that is
\[
\sigma(\zeta_1, \zeta_2)_{\rho} = \rho \sigma \rho^{-1}(\zeta_1, \zeta_2) = M_{\sigma}(\zeta_1, \zeta_2).
\]

Consider the diagram
\[
\begin{array}{c}
\text{(4.2)} \quad L^x / L^{x_n} \times L^x / L^{x_n} \xrightarrow{\psi} H^1(L, \mu_n \times \mu_n) \\
\downarrow \quad \downarrow \quad \downarrow \quad \downarrow \quad \downarrow \quad \downarrow \\
H^1(L, (\mu_n \times \mu_n)_{\rho}) \xrightarrow{\varphi} H^1(L, E[n])
\end{array}
\]

The isomorphism \( \psi \) comes from the usual Kummer isomorphism. The map \( i_* \) is induced by the canonical isomorphism of the underlying groups \( i : \mu_n \times \mu_n \to (\mu_n \times \mu_n)_{\rho}. \) The map \( \varphi \) is induced by the map sending \((\zeta, 1)\) to \(S\) and \((1, \zeta)\) to \(T\) (compare to \(\kappa^{-1}\)). Then the horizontal arrows are both \(\text{Gal}(L/K)\)-equivariant, and the composition \(\varphi i_* \psi\) is equal to \(\kappa^{-1}\).

Let \(\gamma \in \text{Gal}(\overline{L}/L)\) and lift \(\sigma\) arbitrarily to an element, also written \(\sigma\), of \(\text{Gal}(\overline{K}/K)\). We have
\[
[i_* \psi(a, b)]^{\sigma}(\gamma) = \sigma i(\psi(a, b)(\sigma^{-1} \gamma) \sigma) \quad (4.3)
\]
\[
= \sigma i(\sigma^{-1} \psi(a, b)(\sigma^{-1} \gamma)) = \sigma i(\sigma^{-1} \psi(a, b) \sigma(\gamma)) = \sigma i(\sigma^{-1} \psi(\sigma a, \sigma b)(\gamma)) \quad (4.4)
\]
\[
= \frac{M_{\sigma}}{\det M_{\sigma}} i[\psi(\sigma a, \sigma b)(\gamma)] = i_* \psi \left( \frac{M_{\sigma}}{\det M_{\sigma}} (\sigma a, \sigma b) \right)(\gamma). \quad (4.5)
\]

The equality \(4.4\) follows by the definition of the \(\text{Gal}(L/K)\)-action on \(H^1(L, \mu_n \times \mu_n)\). Since \(\psi\) is a \(\text{Gal}(L/K)\)-morphism, we obtain \(4.5\). Finally, \(4.7\) follows from the action of \(\sigma\) on \(E[n]\) by the matrix \(M_{\sigma}\), and the action of \(\sigma^{-1}\) on \(\mu_n \times \mu_n\) by \((\det M_{\sigma})^{-1}\).

Now apply \(\varphi\) to both sides of the equation to obtain
\[
[k^{-1}(a, b)]^{\sigma} = \kappa^{-1} \left( \frac{M_{\sigma}}{\det M_{\sigma}} (\sigma a, \sigma b) \right)
\]
from which the lemma follows.

\[\square\]

**Corollary 4.4.** Let \(\kappa(\xi) = (a, b)\). Then \(\kappa(\text{Nm} \xi)\) equals
\[
\left( \prod (\det M_{\sigma})^{-1} \sigma a^{i(\sigma)} \sigma b^{j(\sigma)}, \prod (\det M_{\sigma})^{-1} \sigma a^{k(\sigma)} \sigma b^{l(\sigma)} \right)
\]
where the product is taken over \(\sigma \in \text{Gal}(L/K)\).

**4.3. Computation of the obstruction map.** Let \(\xi \in H^1(L, E[n])\) satisfy \(\kappa(\xi) = (\pi, \pi'^n/\ell)\), where \(\pi\) and \(\pi'\) were chosen as in Lemma 4.1. Let \(\eta = \text{cores}\xi\). In order to compute \(\text{Ob}(\eta)\), we instead compute \(\text{res}\text{Ob}(\eta) = \text{Ob}(\text{res} \eta)\), or \(\text{Ob}(\text{Nm} \xi)\). But
the $n$-torsion $E[n]$ is rational over $L$, so by Proposition 2.4 we may use the Hilbert symbol to compute $\text{Ob}(\text{Nm} \xi)$. Let
\[
c = \prod (\det M_\sigma)^{-1} \sigma \pi \tilde{i}(\sigma) \\
d = \prod (\det M_\sigma)^{-1} \sigma \pi \tilde{k}(\sigma) \\
c' = \prod (\det M_\sigma)^{-1} \sigma \pi \tilde{\pi} \tilde{j}(\sigma) \\
d' = \prod (\det M_\sigma)^{-1} \sigma \pi \tilde{\pi} \tilde{l}(\sigma).
\]
Thus, $\text{Nm} \kappa^{-1}(\pi, 1) = \kappa^{-1}(c, d)$, $\text{Nm} \kappa^{-1}(1, \pi^{n/\ell}) = \kappa^{-1}(c', d')$, and $\kappa(\text{Nm} \xi) = (cc', dd')$. We wish to compute the Hilbert symbol $\langle cc', dd' \rangle$.

**Lemma 4.5.** Let $w$ be any place of $L$ satisfying $w(\pi \pi') = 0$. Then the local Hilbert symbol $\langle cc', dd' \rangle_w$ is trivial.

**Proof.** If $w$ is non-archimedean and $w(n) = 0$, then $cc'$ and $dd'$ are both units in $L_w$. Therefore by Lemma 2.5 the Hilbert symbol is trivial.

If $w(n) \neq 0$, then $w$ lies in $S$. According to condition B3, $\pi$, $\pi'$ and their conjugates lie in $L_{w, n}$. Again, the Hilbert symbol is trivial.

Lastly, since $n$ is an odd prime power and $\mu_n \subset L$, $L$ has no real embeddings. Therefore the Hilbert symbol at all the archimedean places is automatically trivial. \(\square\)

**Lemma 4.6.** Let $v$ be the place of $L$ corresponding to $\pi$. Then $\langle cc', dd' \rangle_v$ has exact order $\ell$.

**Proof.** By bilinearity, we have
\[
\langle cc', dd' \rangle = \langle c, d \rangle + \langle c, d' \rangle + \langle c', d \rangle + \langle d, d' \rangle.
\]
Since $c', d, d'$ are all units in $L_{v, n}$, the last two terms are zero by Lemma 2.5. Now $c = u\pi$, where $u$ is some unit in $L_{v, n}$, so that $\langle c, d' \rangle_v = \langle \pi, d' \rangle_v$. By condition B4, $d' \equiv \pi^{n/\ell} \mod L_{v, n}$, so that $\langle c, d' \rangle_v = \langle \pi, \pi^{n/\ell} \rangle_v$. Again applying condition B4 and Lemma 2.5, we see that $\langle c, d' \rangle_v$ has order $\ell$.

We now show that $\langle c, d \rangle_v = 0$. One sees that it suffices to show that $\langle \pi, d \rangle_v = 0$.

Recall that $(S, T)$ was our chosen basis for $E[n]$. If the subgroup generated by $S$ is Galois-stable over $K$, then the $M_\sigma$ are all upper triangular. In particular, $k(\sigma) = 0$ for all $\sigma \in \text{Gal}(L/K)$, which implies that $d = 1$. Therefore $\langle \pi, d \rangle_v = 0$.

Let us assume instead that $\mu_n \subset K$. This immediately implies that $\det M_\sigma = 1$. Since $n$ is odd, it suffices to show that $2\langle \pi, d \rangle_v = 0$. By expanding out $d$ and using bilinearity, we may instead show that
\[
\langle \pi, \pi \pi^{k(\sigma)} \rangle = -\langle \pi, \pi^{-1} \pi^{k(\sigma)^{-1}} \rangle_v.
\]
Recall that we write $M_\sigma$ by
\[
M_\sigma = \begin{bmatrix} i(\sigma) & j(\sigma) \\ k(\sigma) & l(\sigma) \end{bmatrix}.
\]
But $M_{\sigma^{-1}} = M_\sigma^{-1}$. Using $\det M_\sigma = 1$, we obtain
\[
M_{\sigma^{-1}}^{-1} = \begin{bmatrix} l(\sigma) & -j(\sigma) \\ -k(\sigma) & i(\sigma) \end{bmatrix}.
\]
In other words, \( k(\sigma^{-1}) = -k(\sigma) \). Therefore it suffices to show that \( \langle \pi, \sigma \pi \rangle_v = \langle \pi, \sigma^{-1} \pi \rangle_v \). But we have

\[
\langle \pi, \sigma \pi \rangle_v = \langle \sigma^{-1} \pi, \pi \rangle_{\sigma^{-1} v} = -\langle \pi, \sigma^{-1} \pi \rangle_{\sigma^{-1} v} = \langle \pi, \sigma^{-1} \pi \rangle_v.
\]

The last equality follows from Proposition 2.6 and the fact that the symbol \( \langle \pi, \sigma^{-1} \pi \rangle \) is trivial everywhere but at \( v \) and \( \sigma^{-1} v \). This proves the lemma. \( \square \)

Let \( v' \) be the place of \( L \) corresponding to \( \pi' \).

**Proposition 4.7.** \( \text{Ob} \eta \) has order \( \ell \) at \( v_K \) and \( v'_K \), and is trivial at all other places.

**Proof.** First we prove triviality. For any place \( w_K \) of \( K \), the corestriction map on \( H^1(L, E[n]) \) induces a homomorphism

\[
\oplus \text{cores}_w : \oplus H^1(L_w, E[n]) \to H^1(K_{w_K}, E[n])
\]

where the sum is over all places \( w \) of \( L \) lying over \( w_K \). Recall that \( \eta = \text{cores} \xi \). By construction, \( \xi \) is locally trivial everywhere except at \( v \), \( v' \), and their conjugates. Triviality at \( w_K \neq v_K, v'_K \) follows.

Since the local invariants of a global Brauer class sum to zero, it suffices to show that \( (\text{Ob} \eta)_{v_K} \) has order \( \ell \). The diagram

\[
\begin{array}{ccc}
H^1(K, E[n]) & \longrightarrow & \text{Br } K \\
\downarrow & & \downarrow \\
H^1(K_{v_K}, E[n]) & \longrightarrow & \text{Br } K_{v_K} \\
\oplus \text{res} & & \oplus \text{res} \\
\oplus H^1(L_v, E[n]) & \longrightarrow & \oplus \text{Br } L_v
\end{array}
\]

commutes. The horizontal maps are the relevant obstruction maps. Commutativity follows from functoriality of localization and restriction in nonabelian cohomology. But \( v_K \) splits completely in \( L \) by condition B5, so that \( K_{v_K} \cong L_v \). Thus if we consider the restriction map onto a single factor

\[
\begin{array}{ccc}
H^1(K_{v_K}, E[n]) & \longrightarrow & \text{Br } K_{v_K} \\
\downarrow & & \downarrow \\
H^1(L_v, E[n]) & \longrightarrow & \text{Br } L_v
\end{array}
\]

then the vertical maps are isomorphism. Therefore the order of \( (\text{Ob} \eta)_{v_K} \) equals the order of \( (\text{Ob res} \eta)_{v} \), or rather \( (\text{Ob Nm} \xi)_{v} \).

Since \( E[n] \subset E(L) \subset E(L_v) \), we may use the Hilbert symbol to evaluate the obstruction map. By construction, \( \kappa(\text{Nm} \xi) = (cc', dd') \). The result follows from Lemma 4.6. \( \square \)
4.4. End of proof. Let $X$ be the genus 1 curve over $K$ represented by the class $\eta$ in $H^1(K, E[n])$. Clearly the period of $X$ divides $n$. Suppose that the period is smaller. Then there is some positive integer $m$ with $m < n$ and some $x \in E(K)$ such that
\[ \delta x + m\eta = 0. \]
In particular, this holds locally at $v_K$. But by condition B2, $\delta x$ is trivial at $v_K$ for all $x \in E(K)$. Therefore we must have $m\eta = 0$ at $v_K$. In particular, this must hold when we restrict to $L_v$. But this cannot be, for
\[
\kappa(\res_{L/K} m\eta) = m\kappa(\res\eta) = m(u_1 \cdot \pi, u_2) = (u_1^m \cdot \pi^m, u_2^m)
\]
for some $u_1, u_2$ which are units in $L_v$. We conclude that $X$ has exact period $n$.

By Proposition 1.7 Ob $\eta$ has order $\ell$. Applying Proposition 2.2 we see that the index of $X$ divides $n\ell$. Suppose the index is $n\ell'$, with $\ell' < \ell$. By Proposition 2.2 there is a class $\eta'$ in $H^1(K, E[n\ell'])$ representing $X$ such that $\Ob_{n\ell'} \eta' = 0$. Let $j_*$ be the canonical homomorphism
\[ H^1(K, E[n]) \to H^1(K, E[n\ell']). \]
Since $\eta$ also represents $X$, there exists $x \in E(K)$ such that
\[ \eta' = j_*(\eta) + \delta x. \]
By Proposition 2.8 $\Ob_{n\ell'} \eta' = \Ob_{n\ell'} j_*(\eta) + T(x, X)$ where $T$ is the Tate pairing. Consider the latter equation locally at $v_K$. By condition B2, $x$ lies in $nE(K_{v_K})$, so the Tate pairing is zero. The left hand term is zero by hypothesis. But by Proposition 2.7 $\Ob_{n\ell'} j_*(\eta) = \ell' \Ob_n \eta$, and the latter is not zero since $\Ob_n \eta$ has order $\ell$ and $\ell' < \ell$. This yields a contradiction. Therefore the index of $X$ must be exactly $n\ell$.

5. Even period

Assume now that $n$ is a power of 2. Three problems now arise in the previous arguments.

The first problem is that the obstruction map need not equal the Hilbert symbol; instead, according to Proposition 2.4 $\Ob(\xi) - \langle \kappa(\xi) \rangle$ is killed by 2. The second problem is that in the proof of Lemma 1.3 $\langle cc', dd' \rangle_w$ need not be trivial at real $w$; but in any case $2\langle cc', dd' \rangle_w = 0$. Finally, in our proof of Lemma 1.6 we showed that $\langle \pi, d \rangle_v = 0$ when $\mu_n \subset K$ by instead showing that $2\langle \pi, d \rangle_v = 0$; if $n$ is even, of course, this is insufficient.

Suppose we undertake our construction anyway, yielding $\eta \in H^1(K, E[n])$. The three problems above imply that our calculation of $\Ob(\eta)$ may be off by an element of $\langle \Br K \rangle^2$. If $\ell \geq 4$, then the difference is immaterial, and the proof works. We now consider the cases $\ell = 1$ and $\ell = 2$.

Assume that $\ell = 1$. According to our main theorem, we are under the weakened hypothesis that either $\mu_{2n} \subset K$ or $E$ contains a Galois-stable cyclic subgroup of order $2n$. Using the construction from the previous section, we can come up with a curve $X'$ with period $2n$ and index either $2n$ or $4n$, though we are not able to determine which of the two holds. Suppose $\eta \in H^1(K, E[2n])$ represents $X'$ with
Ob₂₂ₙ(η) ∈ (Br K)[2]. Let X be the curve with class 2η. Note that we may view 2η as an element of H¹(K, E[n]); in particular, X has period n. To show that X has index n, it suffices to show that Obₙ(2η) = 0. But by Proposition 2.7, Obₙ(2η) = 2 Ob₂₂ₙ η = 0.

The procedure for ℓ = 2 is similar: construct X′ as before with period 2n and index 8n, so that Ob₂₂ₙ(η) has order 4. Then Obₙ(2η) has order 2, and the curve X represented by 2η has period n and index 2n.

6. Final remarks

The hypothesis on the main theorem amounts to a statement about the Galois representation on E[n]. The hypothesis is needed at precisely one place: the calculation of ⟨π, d⟩ v appearing in the proof of Lemma 4.6. Recall that d is a product of the conjugates of π, each appearing with some multiplicity. If one could, for example, choose π so that ⟨π, σπ⟩ = 0 for all σ, then one could generalize the theorem to arbitrary number fields.

One can get around this to some extent, at least when we wish the index to be maximal (I = P²); see [3], which deals with this case over arbitrary number fields.

The results of the paper should generalize to function fields, that is, finite extensions of F_p(T). The only wrinkle occurs when p | n, for then E[n] is no longer an étale group scheme. Clark, in a personal communication, has results making use of a so-called flat Hilbert symbol. One suspects that, with greater care, the results of this paper can be duplicated using the flat symbol.

Lastly, the description of the Gal(L/K) action on H¹(L, E[n]) yields an explicit description, in most cases, of H¹(K, E[n]) for K an arbitrary number field, in the following manner. Given E/K, for n not divisible by a finite set of primes, we know that Gal(L/K) surjects onto Aut(E[n]). One can show that H¹(L/K, E[n]) = 0 for all q, whence the inflation-restriction sequence yields an isomorphism

\[ H¹(K, E[n]) \cong H¹(L, E[n])^{\text{Gal}(L/K)}. \]

From Proposition 5.6, one obtains the desired description of H¹(K, E[n]) as pairs of elements of L×/L×n.

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