Resolution of the Cauchy problem for the Toda lattice with non-stabilized initial data

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Abstract

This paper is the continuation of the work "On an inverse problem for finite-difference operators of second order" ([1]). We consider the Cauchy problem for the Toda lattice in the case when the corresponding $L$-operator is a Jacobi matrix with bounded elements, whose spectrum of multiplicity 2 is separated from its simple spectrum and contains an interval of absolutely continuous spectrum. Using the integral equation of the inverse problem for this matrix, obtained in the previous work, we solve the Cauchy problem for the Toda lattice with non-stabilized initial data.

Introduction

1. In paper [1] we explained the importance of the extension of classes of initial data for which the Cauchy problem for nonlinear evolutionary equations can be solved. Due to this reason the goal of many investigations is the search of new inverse problems for linear $L$-operators, which can be applied to solve the corresponding Cauchy problems with new and possibly wider classes of initial data.

In this connection in paper [1] we considered the Cauchy problem for the equation of oscillation of the doubly-infinite Toda lattice

$$\frac{d^2 x_k}{dt^2} = e^{x_{k+1} - x_k} - e^{x_{k-1} - x_k}, \quad k \in \mathbb{Z},$$

(0.1)

$$x_k(0) = v_k, \quad \dot{x}_k(0) = w_k.$$  (0.2)
The Jacobi matrix

\[
J = \begin{pmatrix}
\ddots & \ddots & \ddots & \ddots \\
\ddots & b_{-2} & a_{-1} & b_{-1} \\
& b_{-1} & a_{0} & b_{0} \\
& & b_{0} & a_{1} & b_{1} \\
& & & b_{1} & a_{2} & b_{2} \\
& & & & \ddots & \ddots & \ddots \\
\end{pmatrix},
\]  

(0.3)

with the coefficients \(a_k, b_k\), defined by the formula

\[
a_k = w_k, \quad b_{k-1} = \exp \frac{v_k - v_{k-1}}{2},
\]

(0.4)
is the \(L\)-operator that corresponds to this problem.

The main result of paper [1] is the finding and solving of a new inverse spectral problem for Jacobi matrix \(J\) when it satisfies the following main conditions: its elements are bounded, its spectrum of multiplicity 2 is separated from its simple spectrum and contains an interval \([a,b]\) of absolutely continuous spectrum of multiplicity 2. (Besides, we imposed some additional technical conditions on the behavior of the arguments of the Weyl functions of the matrix \(J\) in the neighborhood of its spectrum.)

In the present work we use the obtained inverse problem (more precisely the integral equation of the inverse problem) to solve the Cauchy problem (0.1), (0.2) for the Toda lattice in the case when the Jacobi matrix (0.3), defined from the initial conditions by (0.4), satisfies the described conditions on the type of spectrum.

2. Before starting the solving of the Cauchy problem let us outline the main results of paper [1]. We denote by \(P_k(\lambda), Q_k(\lambda)\) the solutions of the finite-difference equation

\[
b_{k-1}\omega_{k-1} + (a_k - \lambda)\omega_k + b_k\omega_{k+1} = 0, \quad k \in \mathbb{Z},
\]

(0.5)

with initial data \(P_0(\lambda) = 1, P_{-1}(\lambda) = 0, Q_0(\lambda) = 0, Q_{-1}(\lambda) = 1\).

As it is known, for nonreal \(\lambda\) equation (0.5) has the Weyl solutions

\[
\varphi^R(k, \lambda) = m^R(\lambda)P_k(\lambda) - \frac{Q_k(\lambda)}{b_{-1}}, \quad k \in \mathbb{Z},
\]

\[
\varphi^L(k, \lambda) = -\frac{P_k(\lambda)}{b_{-1}} + m^L(\lambda)Q_k(\lambda), \quad k \in \mathbb{Z},
\]

such that \(\sum_{k=N}^{\infty} |\varphi^R(k, \lambda)|^2 < \infty\), \(\sum_{k=-\infty}^{k=N} |\varphi^L(k, \lambda)|^2 < \infty\) for any finite \(N\). Here \(m^R(\lambda)\) and \(m^L(\lambda)\) are the Weyl functions of the matrix \(J\).

The functions \(m^R(\lambda), m^L(\lambda)\) and the number \(b_{-1}\) play the role of spectral data from which the Jacobi matrix \(J\) is reconstructed. Without restriction of generality we consider the case when \([a,b] = [-2,2]\) and \(b_{-1} > 0\).\footnote{Namely, if \([a,b] \neq [-2,2]\), we can consider, instead of \(J\) and its Weyl functions \(m^R(\lambda), m^L(\lambda)\), the matrix}

\[
\tilde{J} = \frac{4}{b-a}(J - \frac{a+b}{2}I),
\]

Instead of the two Weyl functions and two Weyl solutions, which are defined in the plane of spectral parameter \(\lambda\), we introduce one function and one parameter defined in
the z-plane, where the variable \( z \) and the parameter \( \lambda \) are connected by the relation \( z + z^{-1} = \lambda \):

\[
    n(z) = \begin{cases} 
        -b_{-1}m^R(z + z^{-1}), & |z| < 1, \\ 
        -b_{-1}m^L(z + z^{-1}), & |z| > 1, 
    \end{cases}
\]

\[
    \psi(k, z) = n(z)P_k(z + z^{-1}) + Q_k(z + z^{-1}) = \begin{cases} 
        -b_{-1}\varphi^R(k, z + z^{-1}), & |z| < 1, \\ 
        \frac{\varphi^L(k, z + z^{-1})}{m^L(z + z^{-1})}, & |z| > 1.
    \end{cases}
\]

(0.6)

The key step of the inverse problem is the suitable factorization of the function

\[
    (z - z^{-1})\left(n(z) - n(z^{-1})\right)^{-1}
\]

and the choice of the factorizing function \( R(z) \), which are obtained by the following theorem:

The function \((z - z^{-1})\left(n(z) - n(z^{-1})\right)^{-1}\) in its domain of holomorphy can be represented in the form of the product of two functions \( R(z) \), \( R(z^{-1}) \):

\[
    \frac{z - z^{-1}}{n(z) - n(z^{-1})} = R(z)R(z^{-1}).
\]

The function \( R(z) \) may only have singularities at such points \( z \), that \( z + z^{-1} \) belongs to the spectrum of the matrix \( J \). If \( z + z^{-1} \) belongs to the absolutely continuous spectrum of multiplicity 2, then \( R(z) \) may have singularities at both points \( z, z^{-1} \), and if \( z + z^{-1} \) belongs to the simple spectrum, then \( R(z) \) may only have singularities at one of the points \( z, z^{-1} \). (Paper [1] represents the explicit expression for the function \( R(z) \). This theorem is strictly proved in paper [2] in a more general form.)

Further for all \( k \in \mathbb{Z} \) we introduce the functions

\[
    g(k, z) = \frac{R(z)}{R(\infty)}z^{-(k+1)}h_k\psi(k, z),
\]

where

\[
    h_k = \begin{cases} 
        b_{-1} \ldots b_{k-1}, & k \geq 0, \\ 
        1, & k = -1, \\ 
        \frac{1}{b_{k-1} \ldots b_{-2}}, & k \leq -2.
    \end{cases}
\]

(0.8)

The following theorem is the main result of paper [1]:

where \( I \) is the identity matrix, and its Weyl functions

\[
    \tilde{m}^R(\lambda) = m^R\left(\frac{a + b}{2} + \frac{b - a}{4}\lambda\right), \quad \tilde{m}^L(\lambda) = m^L\left(\frac{a + b}{2} + \frac{b - a}{4}\lambda\right).
\]

Such \( \tilde{J} \) has the necessary form. So, we can solve the inverse problem, described in [1], for this new matrix and apply the inverse problem to solve the Cauchy problem for the Toda lattice with the initial data that correspond (by (0.4)) to the matrix \( J \). If \( \tilde{x}_n(t) \) is the solution of the equation of the Toda lattice with these initial data, then the functions

\[
    x_n(t) = \tilde{x}_n\left(\frac{b - a}{4}t\right) + 2n\ln\left(\frac{b - a}{4}\right) + \frac{a + b}{2}t,
\]

as it can be verified, are the solution of the Toda equation with the original initial data (i.e. corresponding to the matrix \( J \)).
Theorem. The function $g(k, z)$ is representable in the form

$$g(k, z) = 1 + \int_{-\infty}^{\infty} \frac{u(k, \alpha)s(\alpha)}{1 - \alpha z^{-1}} d\sigma(\alpha) + \frac{1}{\pi} \int_{-\pi}^{\pi} \frac{\hat{r}(e^{i\theta})u(k, e^{i\theta})}{1 - ze^{-i\theta}} d\theta,$$

(0.9)

where the function $u(k, \beta)$ is the solution of the integral equation

$$\beta^{2(k+1)}u(k, \beta) - \frac{\beta}{|\beta|} \frac{m(\beta)}{2q(\beta^{-1})} u(k, \beta^{-1}) +$$

$$+ \nu.p. \int_{-\infty}^{\infty} \frac{u(k, \alpha)}{1 - \beta^{-1}\alpha^{-1}} s(\alpha) \alpha d\sigma(\alpha) + \frac{1}{\pi} \nu.p. \int_{-\pi}^{\pi} \frac{\hat{r}(e^{i\theta})u(k, e^{i\theta})}{1 - \beta^{-1}e^{-i\theta}} d\theta = -1,$$

(0.10)

and $s(\alpha) = \begin{cases} 1, & |\alpha| > 1, \\ -1, & |\alpha| < 1. \end{cases}$ The functions $m(\alpha)$, $q(\alpha)$, the measure $d\sigma(\alpha)$, defined on the real line, and the function $\hat{r}(e^{i\theta})$, defined on the unit circle, are explicitly expressed in terms of the function $n(z)$ (see [1]). If the matrix $J$ satisfies the imposed conditions, the equation (0.10) is uniquely solvable for any $k$ in the class $L^2(\mathbb{R}_\sigma \cup \mathbb{T}_m)$. (Here $\frac{m}{\pi}$ is the Lebesgue measure on the unit circle $\mathbb{T}$, divided by $\pi$.)

In equation (0.10) the parameter $\beta$ belongs to the unit circle and the real line. We observe that the support of the measure $d\sigma(\alpha)$ and the unit circle $\mathbb{T}$ coincide with the set of such points $z$, that $z + z^{-1}$ belongs to the spectrum of the operator $J$.

The collection $\{\hat{r}(e^{i\theta}), \frac{m(\alpha)}{q(\alpha^{-1})}, d\sigma(\alpha)\}$, consisting of the functions $\hat{r}(e^{i\theta})$, $\frac{m(\alpha)}{q(\alpha^{-1})}$ and the measure $d\sigma(\alpha)$, determines integral equation (3.39) and representation (3.40). This collection is called the reduced spectral data of the Jacobi matrix $J$. Evidently, they play the same role in our inverse problem as the scattering data in the inverse scattering problem. According to the last theorem for each $k \in \mathbb{Z}$, the equations (0.10), reconstructed according to these data, have a unique solution in the space $L^2(\mathbb{R}_\sigma \cup \mathbb{T}_m)$.

In order to reconstruct the Jacobi matrix $J$ according to the given spectral data it is necessary to solve equation (0.10), then find the functions $g(k, z)$ using formula (0.9) and then find the elements $a_k$, $b_k$ of the Jacobi matrix using formulas

$$b_{k-1}^2 = \frac{g(k, 0)}{g(k - 1, 0)}, \quad a_k = \lim_{z \to \infty} \frac{z(g(k, z) - g(k + 1, z))}{g(k, z)}.$$

(0.11)

(The elements $b_k$ can be found up to their sign.) We will prove the last two formulas later on (see lemma 3).

The map

$$J \mapsto n(z) \mapsto \{\hat{r}(e^{i\theta}), \frac{m(\alpha)}{q(\alpha^{-1})}, d\sigma(\alpha)\},$$

described in the previous sections, is the solution of the direct spectral problem, and the map

$$\{\hat{r}(e^{i\theta}), \frac{m(\alpha)}{q(\alpha^{-1})}, d\sigma(\alpha)\} \mapsto J$$

solves the inverse spectral problem. Such are the main results of paper [1].

3. In order to solve the Cauchy problem for the equation of the Toda lattice we introduce the operator

$$\hat{\Gamma}(k, t) = \beta^{2(k+1)}e^{\beta t}I + e^{\beta^{-1}t}C_2,$$

(0.12)
where $C_2$ is the operator in the space $L^2(R_\sigma \cup T_{\frac{m}{\pi}})$, defined by the formula
\[
(C_2u)(\beta) = \frac{\beta}{2q(\beta-1)} u(\beta^{-1}) + v.p. \int_{-\infty}^{\infty} \frac{u(\alpha)}{1 - \beta^{-1} \alpha^{-1}} s(\alpha) d\sigma(\alpha) + \frac{1}{\pi} v.p. \int_{-\pi}^{\pi} \frac{r(e^{i\theta}) u(e^{i\theta})}{1 - \beta^{-1} e^{-i\theta}} d\theta.
\]
According to theorem 4 of paper [1], the operator $\tilde{\Gamma}(k,t)$ is always invertible and the inverse is bounded. In these notations the equation (0.10) of the inverse problem can be rewritten in the form
\[
\tilde{\Gamma}(k,o)u(k,\beta) + 1 = 0.
\]

Let us define in the space $L^2(R_\sigma \cup T_{\frac{m}{\pi}})$ the operator $P$, which project the whole space into its subspace consisting of the constant functions:
\[
Pw = k \int_{-\infty}^{\infty} \alpha w(\alpha) s(\alpha) d\sigma(\alpha) + \frac{1}{\pi} \int_{-\pi}^{\pi} e^{i\theta} \tilde{r}(e^{i\theta}) w(e^{i\theta}) d\theta, \quad w \in L^2(R_\sigma \cup T_{\frac{m}{\pi}}).
\]
(The coefficient $k \neq 0$ is chosen so that the condition $P(1) = 1$ is satisfied.)

Let us also denote by $N$ the operator of multiplication by the variable $\beta$ in the space $L^2(R_\sigma \cup T_{\frac{m}{\pi}})$:
\[
(Nw)(\beta) = \beta w(\beta).
\]

The furthest part of the work is organized in the following way. In section 1 we prove that for arbitrary parameters $\{\tilde{r}(e^{i\theta}), \frac{\tilde{m}(\alpha)}{\tilde{q}(\alpha-1)}, d\sigma(\alpha)\}$, which guarantee the invertibleness of the operator $\tilde{\Gamma}(k,t)$, and for the operators $\tilde{\Gamma}(k,t)$, $P$ and $N$, defined as above, the functions
\[
\tilde{x}_k(t) = \ln P^{-1} \tilde{\Gamma}(k,t)^{-1} N^{-1} \tilde{\Gamma}(k + 1, t)(1) - c_1,
\]
where $c_1$ is a constant number, are the solutions of the Toda equation (0.1) without the initial condition (0.2). In section 2 we prove that if we define a Jacobi matrix $J$ according to (0.3), (0.4) from the initial conditions (0.2), find its reduced spectral data $\{\tilde{r}(e^{i\theta}), \frac{\tilde{m}(\alpha)}{\tilde{q}(\alpha-1)}, d\sigma(\alpha)\}$ and substitute them into the expression of the operators $\tilde{\Gamma}(k,t)$ and $P$, then the last formula gives us the solution of the Cauchy problem (0.1), (0.2).

We note that there is a short presentation of this work in paper [3].

4. As it was already said, the main theorems of paper [1] are proved with some additional conditions for the matrix $J$. Since in what follows we will use the invertibleness of the operator $\tilde{\Gamma}(k,t)$, defined from the reduced spectral data, we will also need these conditions for the main theorem of the present work. So, we rewrite them here. Put the arguments of the Weyl functions:
\[
\eta^R(\tau) = \lim_{\varepsilon \downarrow 0} \arg m^R(\tau + i\varepsilon), \quad \eta^L(\tau) = \lim_{\varepsilon \downarrow 0} \frac{-1}{m^L(\tau + i\varepsilon)}.
\]
Let $\hat{\Omega}^R$ and $\hat{\Omega}^L$ be the sets of singularities of the functions $m^R(\lambda)$ and $\frac{1}{m^L(\lambda)}$, resp.\(^2\)

\(^2\)That means that $\hat{\Omega}^R$ and $\hat{\Omega}^L$ are the sets of the points where the functions $m^R(\lambda)$ and $\frac{1}{m^L(\lambda)}$ are not holomorphic. It is easy to see that $\hat{\Omega}^R$ and $\hat{\Omega}^L$ can be defined as the supports of the measures $d\rho_R(\lambda)$ and $d\rho_L(\lambda)$, where $\rho_R(\lambda)$, $\rho_L(\lambda)$ are such nondecreasing functions that
\[
b_{-1} m^R(\lambda) = \int_{-\infty}^{\infty} \frac{d\rho_R(\tau)}{\tau - \lambda}, \quad \frac{1}{b_{-1} m^L(\lambda)} = \frac{\lambda}{b_{-1}} + \beta + \int_{-\infty}^{\infty} \frac{d\rho_L(\tau)}{\tau - \lambda},
\]
with $\beta \in R$. 
Let, further,
\[ \tilde{\Omega}_2 \equiv \tilde{\Omega}^R \cap \tilde{\Omega}^L, \quad \tilde{\Omega}_1 \equiv (\tilde{\Omega}^R \setminus \tilde{\Omega}^L) \cup (\tilde{\Omega}^L \setminus \tilde{\Omega}^R), \]
and \( \tilde{\Omega}_2^s \subset \tilde{\Omega}_2 \) be the set of the common poles of the functions \( m^R(\lambda) \) and \( m^{-1}L(\lambda) \), and let \( \tilde{\Omega}_2^a \equiv \tilde{\Omega}^R \setminus \tilde{\Omega}_2^s \).

We assume that:
A) All the three sets \( \tilde{\Omega}_1, \tilde{\Omega}_2^s, \tilde{\Omega}_2^a \) have positive mutual distances, \([−2, 2] \subset \tilde{\Omega}_2^a \), and the set \( \tilde{\Omega}_2^s \) is finite or empty.
B) For some \( \varepsilon > 0 \) almost everywhere (with respect to Lebesque measure) on the set \( \tilde{\Omega}_2^a \)
\[ 0 < \varepsilon < \eta^R(\alpha) < \pi - \varepsilon, \quad 0 < \varepsilon < \eta^L(\alpha) < \pi - \varepsilon. \]
C) In some neighborhood of the set \( \tilde{\Omega}_2^a \) the function \( \eta^R(\alpha) - \eta^L(\alpha) \) satisfies the Hölder condition.
D) The set \( \tilde{\Omega}_2^a \setminus [−2, 2] \) can be covered with mutually disjoint intervals \( \delta_i \) on each of which the following inequalities are true:
\[ \text{ess sup}_{\alpha \in \delta_i} \eta^R(\alpha) - \text{ess inf}_{\alpha \in \delta_i} \eta^L(\alpha) < \pi, \]
E) For some small \( \varepsilon > 0 \) and \( 0 < \alpha < \varepsilon \)
\[ \eta^R(2 + \alpha) = \eta^L(2 + \alpha) = 0, \quad \eta^R(-2 - \alpha) = \eta^L(-2 - \alpha) = \pi, \]
and the functions \( \eta^R(\tau), \eta^L(\tau) \) satisfy the Hölder condition on the interval \([−2, 2] \).

1. The construction of solutions of the Toda equation from the integral operators of the form \( \hat{\Gamma}(k, t) \)

We denote the associative ring of the bounded operators in the space \( L^2(\mathbb{R}_\sigma \cup T_{\pi}) \) by \( K(L^2(\mathbb{R}_\sigma \cup T_{\pi})) \equiv K. \) We suppose that these operators depend on the real parameter \( t \) and on the integer parameter \( k \). Let us introduce in this ring operators \( \partial \) and \( \partial_\alpha \) of the differentiation by \( t \) and by \( k \)
\[ \partial x(k, t) = x_t(k, t), \quad \partial_\alpha x(k, t) = x(k + 1, t) - x(k, t), \quad x(k, t) \in K, \]
and an automorphism \( \alpha \):
\[ \alpha(x(k, t)) = x(k + 1, t). \]

It is evident that
\[ \partial_\alpha = \alpha - I, \]
where \( I \in K \) is a unit operator in \( L^2(\mathbb{R}_\sigma \cup T_{\pi}) \) (\( Ix = x \)), and that
\[ \alpha(xy) = \alpha(x)\alpha(y). \]

\[ \text{The considerations of this section are the application of more common scheme of constructing of solutions of nonlinear equations presented in [4] to the case of the Toda lattice and to our form of the operator} \ \hat{\Gamma}(k, t) \]
It is also clear that the operators of differentiation \( \partial \) and \( \partial_\alpha \) are commutative. So are the operators \( \partial \) and \( \alpha \). We will denote by \( e \) the identical operator in \( L^2(\mathbb{R}_\sigma \cup T_{\hat{m}}) \), which is the unit element of the ring \( K \).

Instead of \( \hat{\Gamma}(k,t) \) we will for now consider the operator

\[
\Gamma(k,t) = \beta^{k+2}e^{\beta t}I + \beta^{-k}e^{\beta^{-1}t}C_2 = \beta^{-k}\hat{\Gamma}(k,t),
\]

with the operator \( C_2 \), defined in (0.13). It is simply to verify immediately that this operator satisfies the differential equations

\[
\partial \Gamma = (\partial_\alpha + I)\Gamma, \quad \partial^2 \Gamma + \Gamma = A(\partial_\alpha + I)\Gamma,
\]

where \( A \in K \) is the operator of multiplication by \( (\beta + \beta^{-1}) \) in the space \( L^2(\mathbb{R}_\sigma \cup T_{\hat{m}}) \).

The operator \( A \) is a constant operator (i.e. it does not depend on \( t \) and \( k \)). As it is seen from (1.2), the logarithmic derivative

\[
\gamma = \Gamma^{-1}\partial \Gamma = \Gamma^{-1}(\partial_\alpha + I)\Gamma = \Gamma^{-1}\alpha(\Gamma)
\]

of the operator \( \Gamma \) is invertible:

\[
\gamma^{-1} = \alpha(\Gamma^{-1})\Gamma.
\]

**Lemma 1.** (see [3]) The logarithmic derivative \( \gamma = \Gamma^{-1}\partial \Gamma \) of the operator \( \Gamma \) satisfies the equation

\[
\partial(\gamma^{-1}\partial \gamma) = \gamma^{-1}\alpha(\gamma) - \alpha^{-1}(\gamma^{-1})\gamma.
\]

**Proof.** 1) Let us first suppose that \( \gamma \) satisfies the equations

\[
\partial \gamma - \gamma \partial_\alpha(\gamma) = 0, \quad \partial^2 \gamma + 2\gamma \partial \gamma - (\gamma^2 + \partial \gamma + e)\partial_\alpha(\gamma) = 0.
\]

Then

\[
\gamma^{-1}\partial \gamma = \partial_\alpha(\gamma), \quad 
\gamma^{-1}\partial^2 \gamma + 2\partial \gamma - (\gamma - \gamma^{-1}\partial \gamma + \gamma^{-1})\partial_\alpha(\gamma) = 0.
\]

Substituting \( \gamma^{-1}\partial \gamma \) instead of \( \partial_\alpha(\gamma) \) into the second equality, we have

\[
\gamma^{-1}\partial^2 \gamma + 2\partial \gamma - \gamma \gamma^{-1}\partial \gamma + \gamma^{-1}\partial \gamma \gamma \gamma^{-1}\partial \gamma - \gamma^{-1}\partial_\alpha(\gamma) = 0,
\]

or

\[
\gamma^{-1}\partial^2 \gamma - \gamma^{-1}\partial \gamma \gamma^{-1}\partial \gamma = \gamma^{-1}\partial_\alpha(\gamma) - \partial \gamma.
\]

Thus,

\[
\partial(\gamma^{-1}\partial \gamma) = \partial(\partial_\alpha(\gamma)) = \partial_\alpha \partial \gamma = \alpha(\partial \gamma) - \partial \gamma.
\]

On the other hand,

\[
\partial(\gamma^{-1}\partial \gamma) = \gamma^{-1}\partial^2 \gamma - \gamma^{-1}\partial \gamma \gamma^{-1}\partial \gamma
\]

\[
= \gamma^{-1}\partial \gamma \gamma^{-1}\partial \gamma + \gamma^{-1}\partial_\alpha(\gamma) - \partial \gamma - \gamma^{-1}\partial \gamma \gamma^{-1}\partial \gamma
\]

\[
= \gamma^{-1}\partial_\alpha(\gamma) - \partial \gamma,
\]

Hence,

\[
\alpha(\partial \gamma) = \gamma^{-1}\partial_\alpha(\gamma) = \gamma^{-1}\alpha(\gamma) - e,
\]

\[
\partial \gamma = \alpha^{-1}(\gamma^{-1})\gamma - e.
\]
Substituting these expressions into the right-hand side of (1.5), we obtain that $\gamma$ satisfies equation (1.3):

$$ \partial(\gamma^{-1}\partial\gamma) = \alpha(\alpha^{-1}(\gamma^{-1})\gamma - e) - (\alpha^{-1}(\gamma^{-1})\gamma - e) = \gamma^{-1}\alpha(\gamma) - \alpha^{-1}(\gamma^{-1})\gamma, $$

which is the abstract form of the Toda lattice equation.

2) Thus, to prove the lemma, we have to show that equations (1.2) imply equations (1.4). Let us prove the first of them. The equation

$$ \partial \Gamma = (\partial_a + I)\Gamma = \alpha(\Gamma) $$

is equivalent to

$$ \Gamma(\Gamma^{-1}\partial\Gamma) = \Gamma(\partial_a + I)\Gamma, $$

or

$$ \Gamma\gamma = \Gamma(\Gamma^{-1}\alpha(\Gamma)). $$

Applying the operator $\partial$ to both sides of the equality, we have

$$ \partial\Gamma\gamma + \Gamma\partial\gamma = \partial\Gamma\Gamma^{-1}\alpha(\Gamma) + \Gamma\partial(\Gamma^{-1}\alpha(\Gamma)), $$

or, multiplying the equality by $\Gamma^{-1}$ from the left-hand side,

$$ \gamma^2 + \partial\gamma = \gamma\Gamma^{-1}\alpha(\Gamma) + \partial(\Gamma^{-1}\alpha(\Gamma)) = \gamma\Gamma^{-1}\alpha(\Gamma) - \Gamma^{-1}\partial\Gamma\Gamma^{-1}\alpha(\Gamma) + \Gamma^{-1}\partial(\alpha(\Gamma)) $$

$$ = \Gamma^{-1}\partial(\alpha(\Gamma)) = \Gamma^{-1}\alpha(\partial\Gamma) = \Gamma^{-1}\alpha(\Gamma) = \Gamma^{-1}(\alpha(\Gamma))\alpha(\gamma) = \gamma\alpha(\gamma). $$

Since $\alpha(\gamma) - \gamma = \partial_a(\gamma)$, this has as a consequence that

$$ \partial\gamma - \gamma\partial_a(\gamma) = 0, $$

which we, actually, wanted to prove.

Let us now prove the second of the equalities (1.4). The equation

$$ \partial^2 \Gamma + \Gamma = A(\partial_a + I)\Gamma = A\alpha(\Gamma) $$

is equivalent to

$$ \Gamma(\Gamma^{-1}\partial^2\Gamma + e) = A\Gamma(\Gamma^{-1}\alpha(\Gamma)). $$

Applying the operator $\partial$ to this equality, we have

$$ \partial\Gamma(\Gamma^{-1}\partial^2\Gamma + e) + \Gamma\partial(\Gamma^{-1}\partial^2\Gamma + e) = A\partial\Gamma(\Gamma^{-1}\alpha(\Gamma)) + A\Gamma\partial(\Gamma^{-1}\alpha(\Gamma)), $$

or, multiplying it by $\Gamma^{-1}$ from the left-hand side,

$$ \gamma(\Gamma^{-1}\partial^2\Gamma + e) + \partial(\Gamma^{-1}\partial^2\Gamma + e) = \Gamma^{-1}A\Gamma\{\Gamma^{-1}\partial\Gamma\Gamma^{-1}\alpha(\Gamma) + \partial(\Gamma^{-1}\alpha(\Gamma))\}. $$

We can eliminate $\Gamma^{-1}A\Gamma$ from the last equality, because

$$ \Gamma^{-1}A\Gamma \cdot \Gamma\alpha(\Gamma) = \Gamma^{-1}A\alpha(\Gamma) = \Gamma^{-1}(\partial^2\Gamma + \Gamma) = \Gamma^{-1}\partial^2\Gamma + e, $$

from which

$$ \Gamma^{-1}A\Gamma = (\Gamma^{-1}\partial^2\Gamma + e)(\Gamma^{-1}\alpha(\Gamma))^{-1} = (\Gamma^{-1}\partial^2\Gamma + e)(\alpha(\Gamma))^{-1}\Gamma. $$
Thus, the analyzed equation is equivalent to the equality

\[
\gamma (\Gamma^{\!-1}\partial^2\Gamma + e) + \partial(\Gamma^{\!-1}\partial^2\Gamma) = (\Gamma^{\!-1}\partial^2\Gamma + e)(\alpha(\Gamma))^{-1}\alpha(\Gamma) \gamma \]

Further, since

\[
\partial^2\gamma = \partial^2(\Gamma^{\!-1}\partial\Gamma) = \partial(-\Gamma^{\!-1}\partial\Gamma \Gamma^{\!-1}\partial\gamma + \Gamma^{\!-1}\partial^2\Gamma) = -\gamma \partial\gamma - \partial\gamma \gamma + \partial(\Gamma^{\!-1}\partial^2\Gamma),
\]

we have

\[
\partial(\Gamma^{\!-1}\partial^2\Gamma) = -\gamma^2 + \Gamma^{\!-1}\partial^2\Gamma,
\]

Substituting this in our equality we obtain

\[
\gamma (\partial\gamma + \gamma^2 + e) + \partial^2\gamma + \gamma \partial\gamma + \partial\gamma \gamma = (\partial\gamma + \gamma^2 + e)(\partial\alpha\gamma + \gamma),
\]

or

\[
\partial^2\gamma + 2\gamma \partial\gamma - (\gamma^2 + \partial\gamma + e)\partial\alpha(\gamma) = 0,
\]

which is the second of equations (1.4). This proves the lemma.

The logarithmic derivative \(\gamma(k,t)\) is an operator in the space \(L^2(\mathbb{R}_\sigma \cup \mathbb{T}_\mathbb{m})\). As a function of \(k\) and \(t\) it is a solution of nonlinear equation (1.3) is the ring \(K\). Now we are going to construct from this solution a solution of the same nonlinear equation for scalar functions of \(k\) and \(t\). Let \(P \in K\) is a projector in the space \(L^2(\mathbb{R}_\sigma \cup \mathbb{T}_\mathbb{m})\), i.e. \(P = P^2\), and let it not depend on \(k\) and \(t\).

**Lemma 2.** If the equation

\[
\partial(y^{-1}\partial y) = y^{-1}\alpha^{-1}(y) - \alpha^{-1}(y^{-1})y
\]

has in the ring \(K\) a solution \(y = x\), which satisfies the condition

\[
Px = Pxp,
\]

then the element \(PxP\) is a solution of the same equation in the subring \(PKP\).

**Proof.** According to the assumption, the element \(P = P^2\) is constant. Hence, the following identities are true:

\[
P\partial(y) \equiv \partial(Py), \quad P\alpha(y) \equiv \alpha(Py),
\]

\[
\partial(y)P \equiv \partial(yP), \quad \alpha(y)P \equiv \alpha(yP),
\]

\[
0 = (P^{-1}\partial^2P + e)(\alpha(P))^{-1}\alpha(P) \gamma.
\]
from which we have that for the element \( x \), satisfying (1.6),

\[
P\partial(x) = \partial(Px) = \partial(PxP) = \partial(PxP)P,
\]

\[
P\alpha^{\pm}(x) = \alpha^{\pm}(Px) = \alpha^{\pm}(PxP) = \alpha^{\pm}(PxP)P.
\]

Besides, from \( PxP = Px \) it follows that

\[
PxP^{-1} = P,
\]

and, subsequently, in the subring \( PKP \) the equality holds:

\[
Px^{-1} = (PxP)^{-1}P.
\]

(Here \( PxP \) is scalar. Hence, \( (PxP)^{-1} \) makes sense.) Thus, if the element \( x \) satisfies the second equation of (1.3), then

\[
0 = P(\partial(x^{-1}\partial x) - x^{-1}\alpha(x)x^{-1}) = \partial(Px^{-1}\partial x) - (PxP)^{-1}P\alpha(x) + \alpha^{-1}(Px^{-1})x
\]

\[
= \partial((PxP)^{-1}\partial(PxP)) - (PxP)^{-1}\alpha(PxP) + \alpha^{-1}((PxP)^{-1})PxP.
\]

According to the lemma, to construct the solutions of the Toda equation (0.1) from \( \gamma \), we only need to satisfy the condition (1.6). Now we remark that the equation (1.5) has a multiplicative group of transformations, that is, if \( \gamma \) satisfies this equations, then so does the element \( N^{-1}\gamma \), where \( N \) is an arbitrary constant element. Hence, if the element \( \gamma \) satisfies the equation

\[
\gamma = \gamma P + N(e - P),
\]

with some constant elements \( N \) and \( P = P^2 \), then

\[
\gamma(e - P) = N(e - P),
\]

\[
N^{-1}\gamma(e - P) = (e - P),
\]

\[
PN^{-1}\gamma(e - P) = 0.
\]

Thus,

\[
PN^{-1}\gamma = PN^{-1}\gamma P,
\]

that is, the element \( PN^{-1}\gamma = PN^{-1}\gamma P \) satisfies the conditions of lemma 2 and it is a solution of equation (1.5) in the subring \( PKP \).

Relation (1.7), which guarantees the possibility of projecting, is equivalent to the following equation for \( \Gamma \):

\[
\partial\Gamma(e - P) = \Gamma N(e - P).
\]

So, if we choose the operator \( N \) and the projector \( P \) in the space \( L^2(\mathbb{R}_\sigma \cup \mathbb{T}_{\pi}) \) so that the operator \( \Gamma \) satisfies relation (1.8), then the scalar function \( z(k, t) = PN^{-1}\gamma(k, t)P \) will satisfy the equation

\[
(z(k, t)^{-1}z_t(k, t))_t = z(k, t)^{-1}z(k + 1, t) - z(k - 1, t)^{-1}z(k, t).
\]

In this case the functions \( x_k(t) = \ln z(k, t) \) will satisfy equation (0.1) of the oscillation of the Toda lattice, which is the goal of this section.
Substituting into (1.8) the expression (1.1) for \( \Gamma(k, t) \), we reduce the equation to the form

\[(\beta \beta^{2(k+1)} e^{\beta t} I + \beta^{-1} e^{\beta^{-1} t} C_2)(I - P) = (\beta^{2(k+1)} e^{\beta t} I + e^{\beta^{-1} t} C_2)N(I - P).\]

This equation will hold if the relations

\[\beta I(I - P) = IN(I - P),\]
\[\beta^{-1} C_2(I - P) = C_2(I - P).\]  

are satisfied. Let us define \( N \in K \) as the operator of multiplication by the variable \( \beta \) in the space \( L^2(\mathbb{R}_\sigma \cup T_{\pi}) \):

\[(Nw)(\beta) = \beta w(\beta).\]  

Then the first of these relations is always fulfilled. Let us define the projector \( P \in K \) as

\[Pw = \frac{1}{k} \int_{-\infty}^{\infty} \alpha w(\alpha) s(\alpha) d\sigma(\alpha) + \frac{1}{\pi} \int_{-\pi}^{\pi} e^{i\theta} \hat{\tau}(e^{i\theta}) w(e^{i\theta}) d\theta, \quad w \in L^2(\mathbb{R}_\sigma \cup T_{\pi}).\]  

(The coefficient \( k \neq 0 \) is chosen so that the requirement \( P^2 = P \) is satisfied, i.e. so that \( P(1) = 1 \). Do not mix the real coefficient \( \hat{k} \) with the integer index \( k \).) We have to verify the relation

\[\beta^{-1} C_2 - C_2 N = (\beta^{-1} C_2 - C_2 N)P.\]

For this let us calculate, according to the definition of \( C_2 \) in (0.13),

\[((\beta^{-1} C_2 - C_2 N)w)(\beta)\]

\[= \beta^{-1} \text{v.p.} \int_{\Omega_0} \frac{w(\alpha)}{1-\beta^{-1} \alpha^{-1}} s(\alpha) d\sigma(\alpha) + \frac{1}{\pi} \int_{-\pi}^{\pi} \frac{\hat{\tau}(e^{i\theta}) w(e^{i\theta})}{1-\beta^{-1} e^{-i\theta}} d\theta\]

\[= \text{v.p.} \int_{\Omega_0} \frac{\beta^{-1} - \alpha}{1-\beta^{-1} \alpha^{-1}} w(\alpha) s(\alpha) d\sigma(\alpha) - \frac{1}{\pi} \int_{-\pi}^{\pi} \frac{\hat{\tau}(e^{i\theta}) w(e^{i\theta}) e^{i\theta}}{1-\beta^{-1} e^{i\theta}} d\theta\]

\[= -\int_{\Omega_0} \alpha w(\alpha) s(\alpha) d\sigma(\alpha) - \frac{1}{\pi} \int_{-\pi}^{\pi} e^{i\theta} \hat{\tau}(e^{i\theta}) w(e^{i\theta}) d\theta = -\frac{1}{k} Pw,\]

because

\[\frac{\beta^{-1} - \alpha}{1-\beta^{-1} \alpha^{-1}} = \alpha^{-1} \beta^{-1} - 1 = -\alpha.\]

Thus,

\[\beta^{-1} C_2 - C_2 N = -\frac{1}{k} P.\]

Therefore, the relation to verify is equivalent to the equality

\[-\frac{1}{k} P = -\frac{1}{k} PP\]

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and is always trivially satisfied.

**Conclusion:** The functions

\[ x_k(t) = \ln \left( PN^{-1} \Gamma(k, t)^{-1} \Gamma_t(k, t) P \right) \]

\[ = \ln \left( PN^{-1} \tilde{\Gamma}(k, t)^{-1} \tilde{\Gamma}_t(k, t)(1) \right) = \ln \left( PN^{-1} \tilde{\Gamma}(k, t)^{-1} N^{-1} \Gamma(k + 1, t)(1) \right) \]

are solutions of the Toda lattice equation (0.1). Hence, if \( x_k(0) \) and \( \dot{x}_k(0) \) satisfy the initial data, then \( x_k(t) \) will be solutions of the corresponding Cauchy problem for the Toda lattice equation. So, to solve this Cauchy problem it is sufficient to choose the Jacobi matrix \( J \) in such way that the functions \( x_k(t) = \ln \left( PN^{-1} \tilde{\Gamma}(k, t)^{-1} \tilde{\Gamma}_t(k, t)(1) \right) \), where \( \tilde{\Gamma}(k, t) \) are the operators defined from the inverse problem for the matrix \( J \), satisfy the given initial data when \( t = 0 \).

We emphasize that in this section we have never used the specific type of the measure \( d\sigma(\alpha) \) and the functions \( m(\alpha), q(\alpha), \hat{r}(e^{i\theta}) \). The only thing we needed was that \( \tilde{\Gamma}(k, t) \) has the common form (5.2'), (5.2''), and is invertible. This means that all the speculations of this section are true for arbitrary measure \( d\sigma(\alpha) \) and functions \( m(\alpha), q(\alpha), \hat{r}(e^{i\theta}) \), provided they guarantee the invertibleness of \( \tilde{\Gamma}(k, t) \). In particular, we can take an arbitrary Jacobi matrix, satisfying the conditions A)–E), then find its reduced spectral data \( d\sigma(\alpha), m(\alpha), q(\alpha), \hat{r}(e^{i\theta}) \), then define the operators \( \tilde{\Gamma}(k, t) \) and \( P \) from them. The solutions \( x_k(t) \), constructed from these operators, will be solutions of the Toda lattice equation (0.1) anyway. But in order to satisfy the given initial data, it is necessary to find appropriately the matrix \( J \).

2. Satisfying if the initial data

So, if we have the initial data \( v_k, w_k \) we define the Jacobi matrix \( J \) of the form (0.3) by (0.4). Let \( \tilde{\Gamma}(k, t) \) is the operator of the inverse problem equation for the matrix \( J \). We define from this operator the solution \( x_k(t) = \ln \left( PN^{-1} \tilde{\Gamma}(k, t)^{-1} N^{-1} \Gamma(k + 1, t)(1) \right) \) of the Toda lattice equation.

In this section we are going to prove that \( x_k \) satisfy the initial data

\[ x_k(0) = v_k + c_1, \quad \dot{x}_k(0) = w_k, \quad (2.1) \]

where the number \( c_1 \in \mathbb{R} \) does not depend on \( k \). Then the solution of the Cauchy problem will be given by

\[ \tilde{x}_k(t) = x_k(t) - c_1. \]

It is evident that to verify (2.1) we have to prove

**Lemma 3.** The functions \( x_k(t) \) satisfy the relations

\[ \exp(x_k(0) - x_{k-1}(0)) = b_{k-1}^2, \quad \dot{x}_k(0) = a_k. \]

\[ ^4 \text{We remark that since } \tilde{\Gamma}(k, t) = N^k \Gamma(k, t), \text{ we have} \]

\[ \tilde{\Gamma}(k, t)^{-1} \tilde{\Gamma}_t(k, t) = \Gamma(k, t)^{-1} N^{-k} N^k \Gamma_t(k, t) = \Gamma(k, t)^{-1} \Gamma_t(k, t). \]

Besides, it is seen that \( \tilde{\Gamma}_t(k, t) = N^{-1} \tilde{\Gamma}(k + 1, t) \).
P r o o f. 1) Let us first see how the elements \( b_k \) and \( a_k \) of the matrix \( J \) are expressed in terms of \( g(k, z) \). We remind that the Weyl solutions \( \psi(k, z) \) satisfy the finite-difference equation
\[
b_{k-1} \psi(k-1, z) + a_k \psi(k, z) + b_k \psi(k, z) = (z + z^{-1}) \psi(k, z) \tag{2.2}
\]
We also considered the functions
\[
g(k, z) = z^{-(k+1)} h_k \frac{R(z)}{R(\infty)} \psi(k, z).
\]
According to theorem 1 of paper [1], they satisfy the asymptotic formula
\[
g(k, z) \to \frac{R(0)}{R(\infty)} h_k^2, \quad z \to 0,
\]
from which it follows that
\[
\frac{g(k, 0)}{g(k - 1, 0)} = \frac{h_k^2}{h_{k-1}^2} = b_{k-1}.
\]
Let us now derive a finite-difference equation for \( g(k, z) \). Multiplying the equation (2.2) by \( z^{-(k+1)} h_k \frac{R(z)}{R(\infty)} \), we obtain
\[
b_{k-1} h_k \frac{z^{-(k+1)}}{z^{-(k-1)+1}} g(k-1, z) + a_k g(k, z) + b_k h_k \frac{z^{-(k+1)}}{z^{-(k-1)+1}} g(k+1, z) = (z + z^{-1}) g(k, z)
\]
or
\[
b_{k-1}^{-1} z^{-1} g(k-1, z) + a_k g(k, z) + z g(k+1, z) = (z + z^{-1}) g(k, z),
\]
which is equivalent to
\[
z^{-1} (b_{k-1}^{-1} g(k-1, z) - g(k, z)) + a_k g(k, z) = z (g(k, z) - g(k+1, z)).
\]
Letting \( z \to \infty \), we have
\[
a_k = \lim_{z \to \infty} \frac{z (g(k, z) - g(k+1, z))}{g(k, z)}.
\]
2) Using formula (0.9)
\[
g(k, z) = 1 + \int_{\Omega_0} \frac{u(k, \alpha) s(\alpha)}{1 - z \alpha^{-1}} d\sigma(\alpha) + \frac{1}{\pi} \text{v.p.} \int_{-\pi}^{\pi} \frac{\hat{r}(e^{i\theta}) u(k, e^{i\theta}) d\theta}{1 - z e^{-i\theta}}
\]
we find that
\[
b_{k-1}^2 = \frac{1 + \int_{\Omega_0} u(k-1, \alpha) s(\alpha) d\sigma(\alpha) + \frac{1}{\pi} \int_{-\pi}^{\pi} \hat{r}(e^{i\theta}) u(k-1, e^{i\theta}) d\theta}{1 + \int_{\Omega_0} u(k-1, \alpha) s(\alpha) d\sigma(\alpha) + \frac{1}{\pi} \int_{-\pi}^{\pi} \hat{r}(e^{i\theta}) u(k-1, e^{i\theta}) d\theta} = \frac{1 + \hat{P} N^{-1} u_k}{1 + PN^{-1} u_{k-1}},
\]
where \( \hat{P} = \frac{k}{k} P \) (i.e. it is the "non-normalized" projector), and \( u_k \equiv u(k, \beta) \) is the solution of the inverse problem equation (for \( t = 0 \)). We also see that
\[
a_k = \lim_{z \to \infty} z (g(k, z) - g(k+1, z))
\]
\[
\lim_{z \to \infty} \left\{ \int_{\Omega_0} \frac{z}{1 - z\alpha} s(\alpha)(u(k, \alpha) - u(k + 1, \alpha))d\sigma(\alpha)
+ \frac{1}{\pi} \int_{-\pi}^{\pi} \frac{z}{1 - ze^{-i\theta}} \hat{r}(e^{i\theta})(u(k, e^{i\theta}) - u(k + 1, e^{i\theta}))d\theta \right\}
\]

\[
= \left\{ - \int_{\Omega_0} \alpha s(\alpha)(u(k, \alpha) - u(k + 1, \alpha))d\sigma(\alpha) + \frac{1}{\pi} \int_{-\pi}^{\pi} e^{i\theta} \hat{r}(e^{i\theta})(u(k, e^{i\theta}) - u(k + 1, e^{i\theta}))d\theta \right\}
\]

\[
= \left\{ -\hat{P}u_k + \hat{P}u_{k+1} \right\}.
\]

3) Thus, we have to verify that

\[
\frac{1 + \hat{P}N^{-1}u_k}{1 + \hat{P}N^{-1}u_{k-1}} = e^{x_k - x_{k-1}}
\]

and

\[
\dot{x}_k = \left\{ \hat{P}u_{k+1} - \hat{P}u_k \right\}.
\]

Let us check the first of these equalities.

\[
b_{k-1} \equiv \frac{1 + \hat{P}N^{-1}u_k}{1 + \hat{P}N^{-1}u_{k-1}} = \frac{k\hat{P}1 + \hat{P}N^{-1}\hat{\Gamma}^{-1}_k(-1)}{kP1 + \hat{P}N^{-1}\hat{\Gamma}^{-1}_{k-1}(-1)}
\]

where we denoted, for the sake of brevity, \(\hat{\Gamma}_k = \hat{\Gamma}(k, 0)\). We remind that \(\hat{\Gamma}_k u_k \equiv \hat{\Gamma}(k, 0)u(k, \beta) = -1\).

In the right-hand side of the obtained equality let us express \(k\hat{P}1\) in the form

\[
\hat{P}N^{-1}\hat{\Gamma}^{-1}_kN^{-1}\tilde{\omega}_k,
\]

where \(\tilde{\omega}_k \in L^2(R_\sigma \cup T_\hat{\omega})\). For this it is sufficient to solve the following equation for a unknown function \(\tilde{\omega}_k\):

\[
\hat{k}1 = N^{-1}\hat{\Gamma}^{-1}_kN^{-1}\tilde{\omega}_k.
\]

The solution of this equation is the function

\[
\tilde{\omega}_k = N\hat{\Gamma}_kN \cdot \hat{k}1 = \hat{k}\beta^{2(k+2)}1 + \hat{k}\beta C_2 N1.
\]

Besides, we remark now that \(-1 = -N^{-1} \cdot \beta 1\). Hence,

\[
b_{k-1} = \frac{\hat{P}N^{-1}\hat{\Gamma}^{-1}_kN^{-1}\{\hat{k}\beta^{2(k+2)}1 + \hat{k}\beta C_2 N1 - \beta 1\}}{\hat{P}N^{-1}\hat{\Gamma}^{-1}_{k-1}N^{-1}\{\hat{k}\beta^{2(k+1)}1 + \hat{k}\beta C_2 N1 - \beta 1\}}.
\]

To simplify this expression, we show that

\[
\hat{\Gamma}_{k+1}(k1) = \hat{k}\beta^{2(k+2)}1 + \hat{k}\beta C_2 N1 - \beta 1, \quad \hat{\Gamma}_k(k1) = \hat{k}\beta^{2(k+1)}1 + \hat{k}\beta C_2 N1 - \beta 1,
\]

which is equivalent (taking into account the specific of \(\hat{\Gamma}_k\)) to

\[
\hat{k}\beta C_2 N1 - \beta 1 = C_2(\hat{k}1),
\]

or

\[
\hat{k}(C_2 N - \beta^{-1} C_2) \cdot 1 = 1. \quad (2.3)
\]
But, according to equality (1.9), \( C_2N - \beta^{-1}C_2 = \hat{P} \). Further, since \( \hat{k}\hat{P}1 = 1 \), equality (2.3) always holds and

\[
b_{k-1} = \frac{\hat{P}N^{-1}\hat{\Gamma}^{-1}_kN^{-1}\{\hat{\Gamma}_{k+1}(k)\}}{PN^{-1}\hat{\Gamma}^{-1}_{k-1}N^{-1}\{\hat{\Gamma}_k(k)\}} = \frac{P\hat{\Gamma}^{-1}_kN^{-1}\hat{\Gamma}_{k+1} \cdot 1}{PN^{-1}\hat{\Gamma}^{-1}_{k-1}N^{-1}\hat{\Gamma}_k P} = \frac{P\hat{\Gamma}^{-1}_kN^{-1}\hat{\Gamma}_{k+1} P}{PN^{-1}\hat{\Gamma}^{-1}_kN^{-1}\hat{\Gamma}_k P} = \frac{P^{-1}\gamma_k P}{P^{-1}\gamma_k P} = e^{x_k - x_{k-1}}
\]

(with \( \gamma_k \equiv \gamma(k,0) \)), what we needed. Let us now verify that \( \dot{x}_k = a_k \). First, \( (\hat{\Gamma}^{-1}_k)_t = -\hat{\Gamma}^{-1}_kN^{-1}\hat{\Gamma}_{k+1}\hat{\Gamma}^{-1}_k \). So,

\[
(\gamma_k)_t = (\hat{\Gamma}^{-1}_kN^{-1}\hat{\Gamma}_{k+1})_t = -\hat{\Gamma}^{-1}_kN^{-1}\hat{\Gamma}_{k+1}\hat{\Gamma}^{-1}_kN^{-1}\hat{\Gamma}_{k+1} + \hat{\Gamma}^{-1}_kN^{-2}\hat{\Gamma}_{k+1}.
\]

Hence,

\[
\dot{x}_k \equiv \ln(PN^{-1}\gamma_k P)_t = \frac{PN^{-1}\gamma_k P}{PN^{-1}\gamma_k P} = \frac{PN^{-1}\gamma_k \cdot 1}{PN^{-1}\gamma_k \cdot 1} = \frac{PN^{-1}\{\hat{\Gamma}^{-1}_kN^{-1}\hat{\Gamma}_{k+1}\hat{\Gamma}^{-1}_kN^{-1}\hat{\Gamma}_{k+1} + \hat{\Gamma}^{-1}_kN^{-2}\hat{\Gamma}_{k+2}\} \cdot 1}{PN^{-1}\hat{\Gamma}^{-1}_kN^{-1}\hat{\Gamma}_{k+1} \cdot 1}.
\]

Calculate separately the braces \( \{ \} \) in the nominator:

\[
(I) = \{ -\hat{\Gamma}^{-1}_kN^{-1}\hat{\Gamma}_{k+1}\hat{\Gamma}^{-1}_kN^{-1}\hat{\Gamma}_{k+1} + \hat{\Gamma}^{-1}_kN^{-2}\hat{\Gamma}_{k+2} \} \cdot 1 = -\hat{\Gamma}^{-1}_kN^{-1}\hat{\Gamma}_{k+1}\hat{\Gamma}^{-1}_k\{\hat{\Gamma}_k \cdot \beta - \hat{k}^{-1} \} + \hat{\Gamma}^{-1}_kN^{-1}\{\beta^{2(k+2)-1} + \beta^{-1}C_2 \cdot 1 \} + \hat{\Gamma}^{-1}_kN^{-1}\{\beta^{2(k+3)-1} + \beta^{-1}C_2 \cdot 1 \}.
\]

But \( \beta^{-1}C_2 \cdot 1 - C_2N \cdot 1 = -\hat{k}^{-1} \), from where

\[
\beta^{2(k+2)-1} + \beta^{-1}C_2 \cdot 1 = \beta^{2(k+2)-1} + C_2 \cdot \beta - \hat{k}^{-1} = \hat{\Gamma}_k \cdot \beta - \hat{k}^{-1}.
\]

Thus,

\[
(I) = -\hat{\Gamma}^{-1}_kN^{-1}\hat{\Gamma}_{k+1}\hat{\Gamma}^{-1}_k\{\hat{\Gamma}_k \cdot \beta - \hat{k}^{-1} \} + \hat{\Gamma}^{-1}_kN^{-1}\{\hat{\Gamma}_k \cdot \beta - \hat{k}^{-1} \} = \hat{\Gamma}^{-1}_kN^{-1}\hat{\Gamma}_{k+1}\{\beta \hat{\Gamma}^{-1}_k + \beta^{-1} \} + \hat{\Gamma}^{-1}_kN^{-1}\{\hat{\Gamma}_k \cdot \beta - \hat{k}^{-1} \}.
\]

But \( \hat{\Gamma}_ku_k = -1 \), so \( -\hat{k}^{-1}\hat{\Gamma}_ku_k = \hat{k}^{-1} \) and \( \hat{\Gamma}^{-1}_k\hat{k}^{-1} = \hat{k}^{-1}u_k \). Hence,

\[
(I) = \hat{\Gamma}^{-1}_kN^{-1}\hat{\Gamma}_{k+1}\{-\beta - \hat{k}^{-1}u_k \} + \hat{\Gamma}^{-1}_kN^{-1}\{\hat{\Gamma}_k \cdot \beta - \hat{k}^{-1} \} = \hat{\Gamma}^{-1}_kN^{-1}\{-\hat{\Gamma}_{k+1}\beta - \hat{k}^{-1}\hat{\Gamma}_{k+1}u_k + \hat{\Gamma}_{k+1}\beta - \hat{k}^{-1} \} = -\frac{1}{\hat{\Gamma}_k}\hat{\Gamma}^{-1}_kN^{-1}\{1 + \hat{\Gamma}_{k+1}u_k \}.
\]

Let us calculate the denominator in the expression for \( x_k \):

\[
PN^{-1}\hat{\Gamma}^{-1}_kN^{-1}\hat{\Gamma}_{k+1} \cdot 1 = PN^{-1}\hat{\Gamma}^{-1}_kN^{-1}\hat{\Gamma}_{k+1} \frac{Pu_k}{Pu_k} = \frac{1}{Pu_k}PN^{-1}\hat{\Gamma}^{-1}_kN^{-1}\hat{\Gamma}_{k+1}Pu_k = \frac{1}{Pu_k}PN^{-1}\hat{\Gamma}^{-1}_kN^{-1}\hat{\Gamma}_{k+1} \cdot u_k.
\]

Thus,

\[
\dot{x}_k \equiv \frac{PN^{-1}\gamma_k P}{PN^{-1}\gamma_k P} = \frac{1}{Pu_k}PN^{-1}\hat{\Gamma}^{-1}_kN^{-1}\{1 + \hat{\Gamma}_{k+1}u_k \} = -\frac{Pu_k}{k}\left(1 + \frac{PN^{-1}\hat{\Gamma}^{-1}_kN^{-1}\hat{\Gamma}_{k+1}u_k}{PN^{-1}\hat{\Gamma}^{-1}_kN^{-1}\hat{\Gamma}_{k+1}u_k} \right).
\]
We calculate separately the second term:

\[
\frac{PN^{-1}\hat{\Gamma}_k^{-1}N^{-1} \cdot 1}{PN^{-1}\hat{\Gamma}_k^{-1}N^{-1} \cdot u_k} = \frac{Pu_k}{Pu_k + PN^{-1}\hat{\Gamma}_k^{-1}N^{-1} \cdot u_k}
\]

\[
= \frac{Pu_{k+1}}{Pu_k} \cdot \frac{PN^{-1}\hat{\Gamma}_k^{-1}N^{-1} \cdot 1}{PN^{-1}\hat{\Gamma}_k^{-1}N^{-1}(-1)} = \frac{Pu_{k+1}}{Pu_k}.
\]

Finally, substituting this into the expression for \(x_k\), we obtain

\[
x_k = -\frac{Pu_k}{k} \left\{1 - \frac{Pu_{k+1}}{Pu_k}\right\} = -\frac{1}{k} \{Pu_{k+1} - Pu_k\} = -\frac{1}{k} \{\hat{P}u_{k+1} - \hat{P}u_k\} = a_k,
\]

which ends the proof of lemma 3.

In this section we proved the main theorem.

**Theorem (the main theorem)** (on the existence of the solution of the Cauchy problem for the Toda lattice). Let the Jacobi matrix \(J\), defined by formulas (0.3), (0.4), satisfies conditions A-E.

Then problem (0.1), (0.2) has a solution

\[
\hat{x}_k(t) = \ln PN^{-1}\hat{\Gamma}_k^{-1}(k,t)N^{-1}\hat{\Gamma}(k+1,t) (1) - c_1,
\]

where the operators \(P, N, \hat{\Gamma}(k,t)\) in the space \(L^2(\mathbb{R}_\sigma \cup T_{\mathbb{R}})\), are defined by formulas (1.11), (1.10), (0.12), and \(c_1\) is a constant number.\footnote{In order to make clearer the way of finding \(\hat{x}_k(t)\) from the operators \(\hat{\Gamma}(k,t), P \cdot N\), we comment the last formula. Evidently, when the operator \(\hat{\Gamma}(k+1,t)\) operates the constant function 1, we obtain another function in the space \(L^2(\mathbb{R}_\sigma \cup T_{\mathbb{R}})\). Not only is it not constant on the variable \(\beta\) of the space, but also it depends on \(k, t\). The operator \(N^{-1}\) multiplies it by \(\beta^{-1}\). The operator \(\hat{\Gamma}^{-1}(k,t)\) makes it a new function in the space \(L^2(\mathbb{R}_\sigma \cup T_{\mathbb{R}})\), depending on \(k\) and \(t\); \(N^{-1}\) multiplies the new function by \(\beta^{-1}\). Finally, the projector \(P\) makes of the function a constant in the space \(L^2(\mathbb{R}_\sigma \cup T_{\mathbb{R}})\). However, it depends on \(k\) and \(t\), as before. Taking the logarithm of this constant, we find \(\hat{x}_k(t) = \ln PN^{-1}\hat{\Gamma}_k^{-1}(k,t)N^{-1}\hat{\Gamma}(k+1,t) (1) - c_1\).}.

The constant number \(c_1\), evidently, can be found from the formula

\[
c_1 = \hat{x}_0(0) - \ln PN^{-1}\hat{\Gamma}_0^{-1}(0,0)N^{-1}\hat{\Gamma}(1,0) (1).
\]

**Remark.** The main result can be exposed in more usual terms of the evolution of the spectral data. The inverse problem equation with spectral data depending on time, is rewritten in the form

\[
\beta^{2k+1}u(k, \beta) - \frac{\beta}{|\beta|} \frac{1}{\bar{\rho}(\beta,t)} u(k, \beta^{-1}) + \nu_p \int_{-\infty}^{\infty} \frac{u(k, \alpha)}{1 - \beta^{-1}e^{i\theta}} d\sigma(\alpha, t) + \frac{1}{\pi} \int_{-\pi}^{\pi} \frac{\hat{\rho}(e^{i\theta}, t)u(k, e^{i\theta})}{1 - \beta^{-1}e^{i\theta}} d\theta = e^{-\beta^{-1}t},
\]

where

\[
\hat{\rho}(\beta, 0) = \frac{2q(\beta^{-1})}{m(\beta)}, \quad d\sigma(\alpha, 0) = d\sigma(\alpha), \quad \hat{\rho}(e^{i\theta}, 0) = \hat{\rho}(e^{i\theta})
\]

\[\text{(2.5)}\]
are the reduced spectral data of the matrix $J$, defined by the initial data.

It is evident that for $t = 0$ equation (2.5) coincide with (0.10).

The time evolution of the reduced spectral data is determined by the formula

$$\tilde{r}(\beta, t) = e^{(\beta - \beta^{-1})t} \tilde{r}(\beta, 0) = \frac{2q(\beta^{-1})}{m(\beta)},$$

$$d\sigma(\alpha, t) = e^{(\beta - \beta^{-1})t} d\sigma(\alpha, 0) = e^{(\beta - \beta^{-1})t} d\sigma(\alpha),$$

$$\hat{r}(e^{i\theta}, t) = e^{(\beta - \beta^{-1})t} \hat{r}(e^{i\theta}, 0) = e^{(\beta - \beta^{-1})t} \hat{r}(e^{i\theta}).$$

In order to find the solution of the Cauchy problem for the Toda lattice (0.1), (0.2) at the time $t$, one need:

1) to find the reduced spectral data $\{\frac{2q(\alpha^{-1})}{m(\beta)}, d\sigma(\alpha), \hat{r}(e^{i\theta})\}$ of the matrix $J$, defined by (0.3), (0.4);

2) multiplying the reduced spectral data by $e^{(\beta - \beta^{-1})t}$, to find the data $\{\tilde{r}(\beta, t), d\sigma(\alpha, t), \hat{r}(e^{i\theta}, t)\}$;

3) to solve equation (2.5) for given $t$ and parameters $\tilde{r}(\beta, t), d\sigma(\alpha, t), \hat{r}(e^{i\theta}, t)$, with right-hand side $e^{-(\beta - \beta^{-1})t}$;

4) to reconstruct according to formulas (0.9), (0.11) the matrix $J(t)$ from the solutions and to find

$$\tilde{x}_k(t) = \tilde{x}_0(t) + 2 \sum_{j=1}^{k} \ln b_{j-1}(t).$$

Conclusions

A new type of inverse problem is introduced for the Jacobi matrices with bounded elements, whose spectrum of multiplicity 2 is separated from the simple spectrum and contains an interval of absolutely continuous spectrum. The spectral data in this problem are explicitly expressed in terms of the Weyl functions $m^R(\lambda), m^L(\lambda)$ of the Jacobi matrix on the right and left semiaxis. These spectral data play the same role in the inverse problem as the scattering data in the classical inverse scattering problem. The uniquely solvable integral equation allowing to reconstruct the matrix by these data is obtained.

The Jacobi matrices that satisfy the condition on the spectrum written above, are the $L$-operators of the equation of the oscillation of the Toda lattice with quite a wide class of initial data, which are not stabilized. (This class includes the already investigated cases of fast-stabilized and periodic initial data.) This allowed to apply the inverse problem integral equation obtained to solve the Cauchy problem for the Toda lattice with non-stabilized initial data.

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