Horizons in Robinson–Trautman spacetimes

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Abstract
The past quasi-local horizons in vacuum Robinson–Trautman spacetimes are described. The case of a null (non-expanding) horizon is discussed. It is shown that the only Robinson–Trautman spacetime admitting such a horizon with sections diffeomorphic to $S^2$ is the Schwarzschild spacetime. Weakening this condition leads to the horizons of the C-metric. Properties of the hypersurface $r = 2m$ are examined.

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1. Introduction

A vacuum Robinson–Trautman (RT) metric can be determined by a function of three variables, satisfying a fourth-order partial differential equation [1]. Very few of its solutions are known in an explicit form [2]. The only ones representing asymptotically flat spacetimes correspond to the Minkowski and the Schwarzschild metrics. Nontrivial asymptotically flat RT spacetimes were shown to exist and tend to the Schwarzschild solution in the limit of infinite retarded time [3].

In this paper, we describe quasi-local horizons [4] in RT spacetimes. We derive equations obtained previously by Tod [5] and Chow and Lun [6]. Using the equation of Chow and Lun we prove that the past horizon is null if and only if its sections admit a shear-free inward pointing null normal field.

It is a well-known property of a nonexpanding horizon that its null generator is the repeated principal null direction of the Weyl tensor [4]. Since the repeated principal null direction responsible for algebraic properties of the Weyl tensor of RT spacetime is transverse to the horizon, the Weyl tensor is of Petrov-type D on the horizon. We prove that this property also holds in the neighbourhood of the horizon and no nonexpanding horizons of the form $R \times S^2$ exist unless the RT spacetime is the Schwarzschild spacetime. If no restriction on topology of sections is made, one obtains null horizons of the C-metric. Their sections contain conical singularities [7].
In section 5 we consider the metric induced on the hypersurface \( r = 2m \) for finite \( u \), and examine its signature (proposition 5.2). We also find the relation between this hypersurface and trapped surfaces (proposition 5.1).

The standard form of the Robinson–Trautman metric reads as \([1, 2]\)

\[
g = 2du(H \, du + dr) - \frac{r^2}{P^2} \, d\xi \, d\bar{\xi}. \tag{1}
\]

Vacuum Einstein’s equations give

\[
H = P^2(\ln P),\bar{\xi} - r(\ln P),u - \frac{m}{r} \tag{2}
\]

and the Robinson–Trautman equation

\[
P^2(\ln P),\bar{\xi} + 3m(\ln P),u - m,\bar{u} = 0. \tag{3}
\]

Here \( m \) is a function of \( u \) which, when non-vanishing, can be transformed to \( m = \pm 1 \). In this case, equation (3) takes the form

\[
K,\bar{\xi} - 3m(P^{-2}),u = 0, \tag{4}
\]

where

\[
K = 2P^2(\ln P),\bar{\xi} \tag{5}
\]

is the Gaussian curvature of the surface given by \( u = \text{const}, r = 1 \). Point transformations preserving metric (1) and \( m \) have the following form [2]:

\[
u \rightarrow u + \text{const}, \tag{6}
\]

\[
\xi \rightarrow h(\xi), \tag{7}
\]

\[
P \rightarrow |h'|^{-1}P, \tag{8}
\]

\[
r \rightarrow r. \tag{9}
\]

The induced metric of a surface of constant \( u \) and \( r \) is given by

\[
g' = \frac{r^2}{P^2} \, d\xi \, d\bar{\xi}. \tag{10}
\]

If the function

\[
\hat{P} = P/(1 + \xi \bar{\xi}/2) \tag{11}
\]

is regular (smooth and positive) for all \( \xi \in \mathbb{C} \) and \( \xi \rightarrow \infty \), then the surface \( u = \text{const}, r = \text{const} \) is diffeomorphic to \( S_2 \) and \( \xi \) can be interpreted as the complex stereographic coordinate on \( S_2 \). The function \( P = (1 + \xi \bar{\xi}/2) \) giving the standard metric of \( S_2 \) will be denoted by \( P_S \).

Two-dimensional surfaces admitting regular \( \hat{P} \) will be referred to as regular surfaces.

### 2. Marginally trapped surfaces

Let \( S \) be a two-dimensional spacelike surface given by

\[
u = \text{const}, \quad r = R(\xi, \bar{\xi}). \tag{12}
\]

We introduce four null forms,

\[
\theta^0 = du, \quad \theta^1 = \left( H + \frac{P^2}{r^2} |R,\xi|^2 \right) \, du + dr - \left( R, \xi \, d\xi + \text{c.c.} \right), \tag{13}
\]

\[
\theta^2 = \frac{r}{P} \, d\xi - \frac{P}{r} R, \xi \, du, \quad \theta^3 = \bar{\theta}^2,
\]

\[\]
and the dual null tetrad,

\[ e_0 = \partial_u - \left( H - \frac{p^2}{r^2} |R,\xi|^2 \right) \partial_r + \frac{p^2}{r^2} (R,\xi) \partial_\xi + \text{c.c.}, \]

\[ e_1 = \partial_r, \]

\[ e_2 = \frac{p}{r} (\partial_\xi + R,\xi \partial_r), \quad e_3 = \bar{e}_2. \]

In appendix A we list the non-vanishing Newman–Penrose spin coefficients and Weyl scalars related to tetrad (14), needed in further calculations. The ingoing and outgoing null vectors normal to \( S \) are, respectively, \( l = e_0, k = e_1 \), while \( m = e_2 \) and \( \bar{m} = e_3 \) are tangent to \( S \).

Using the formulae

\[ -m^a \bar{m}^b l_{(a;b)} = \frac{H}{r} + (\ln P),_u - \frac{p^2}{r^2} R,\ddot{\xi} + \frac{p^2}{r^2} |R,\xi|^2, \]

\[ -m^a \bar{m}^b k_{(a;b)} = r^{-1} \]

from appendix A and substituting for \( H \) from (2) we find the expansion scalars of the vectors \( k \) and \( l \) on \( S \) [5]

\[ \theta(l) = \frac{1}{R} \left( \frac{K}{2} - \frac{p^2 (\ln R),\ddot{\xi}}{2} - \frac{m}{R} \right), \]

\[ \theta(k) = R^{-1}. \]

Therefore, the condition that the expansion of \( l \) vanishes takes the form of the following equation for \( R \):

\[ -P^2 (\ln R),\ddot{\xi} + \frac{K}{2} - \frac{m}{R} = 0. \]

The existence and uniqueness of positive, smooth solutions of (19) for regular spherical surfaces \( u = \text{const}, r = \text{const} \) has been proved by Tod [5]. If \( m = 1 \), the area of these surfaces is independent of \( u \) due to (3) and Stokes’ theorem, and can be chosen to be equal to \( 4\pi \). Note that \( u \) enters (19) as a parameter via the function \( P \).

3. Horizons

In this section, we consider hypersurfaces foliated by marginally trapped surfaces. Let \( \mathcal{H} \) be defined by

\[ r = R(u, \xi, \bar{\xi}), \]

where \( R \) satisfies (19). It follows that

\[ \nabla_t (m^a \bar{m}^b l_{(a;b)}) = 0 \]

on \( \mathcal{H} \), where \( t \) is tangent to \( \mathcal{H} \) and orthogonal to its sections. Using the maximum principle and (17), Tod proved that a marginally trapped surface described by (12) and (19) is outermost [5]. Results of Andersson et al [8] assure that there exists a horizon such that \( S \) is its section. Using (14) we obtain

\[ t = l + \chi k, \quad \chi = R,_{uu} + H + \frac{p^2}{R^2} R,\xi R,\ddot{\xi}. \]
Following [6] we rewrite equation (21) in terms of the spin coefficients:

\[
\frac{P^2}{R} \left( \frac{\chi}{R} \right)_{,\bar{\xi} \bar{\xi}} + \chi \Psi_2 - \lambda \bar{\lambda} = 0,
\]

where \( \lambda \) is the shear of \( \ell \). Equation (23) corresponds to equation (15) in [9], where the general case of constraint equations on a marginally trapped tube was considered.

Assume now that the surfaces \( u = \text{const}, r = \text{const} \) have spherical topology. As pointed out in [6], it follows from the maximum principle for (23) that \( H \) is a non-timelike surface. If \( H \) is null (\( \chi = 0 \)), then \( \lambda = 0 \). Conversely, if \( \lambda \) vanishes, multiplying both sides of (23) by \( \chi/R^2 \) and using Stokes’ theorem for \( S \) we get

\[
- \int_S \left| \nabla \frac{\chi}{R} \right|^2 d\sigma = \int_S \frac{\chi^2}{R^2} |\Psi_2| d\sigma,
\]

where \( d\sigma = (P/R)^2 d\xi \wedge d\bar{\xi} \) is the natural measure on \( S \) and \( \Psi_2 = -|\Psi_2| \) follows from (A.9) and the assumption \( m > 0 \). This shows that \( \chi \) vanishes. Thus,

\[
H \text{ is null } \iff \lambda = 0.
\]

4. Null case

We would like to discuss conditions that \( R \) and \( P \) should satisfy for \( H \) to be a null (nonexpanding) horizon. In this case \( \lambda = \chi = \nu = 0 \) (see (25) and (A.11) and the vector tangent to \( H \) and normal to sections is just \( \ell \) (see (22)). Consider the following Newman–Penrose equations:

\[
\delta \lambda - \bar{\delta} \mu = (\rho - \bar{\rho}) v + (\mu - \bar{\mu}) \pi + \mu(\alpha + \beta) + \lambda(\bar{\alpha} - 3\beta) - \Psi_3 + \Phi_{21},
\]

\[
\Delta \lambda - \bar{\delta} v = (\bar{\rho} - 3\gamma - \mu - \bar{\mu}) \lambda + (3\alpha + \beta + \pi - \bar{\tau}) v - \Psi_4.
\]

Note that derivatives present in (26) and (27) are \( \Delta = \nabla_i, \delta = \nabla_m \) and \( \bar{\delta} = \nabla_{\bar{m}} \). These directions are tangent to \( H \); this allows us to conclude that conditions \( \Psi_4 = \Psi_3 = 0 \) must be satisfied on \( H \). Hence \( g \) is of Petrov-type D on \( H \) (\( \Psi_2 \neq 0 \) for non-vanishing \( m \), see (A.9) in appendix A). From equation (A.10) in appendix A it follows that on \( H \)

\[
\frac{3Pm}{R^2} R_{,\bar{\xi} \bar{\xi}} + \frac{P}{2R^2} K_{,\bar{\xi} \bar{\xi}} = 0.
\]

This equation can be easily integrated to give \( R \) in terms of \( K \):

\[
R = 6m(K + a)^{-1},
\]

where \( a = a(u) \) is real. Substituting (29) into the condition \( \lambda = 0 \) (see (A.4) in appendix A) we get

\[
(P^2 K_{,\bar{\xi} \bar{\xi}})_{,\bar{\xi}} = 0.
\]

Note that due to (29), (30) is a condition on \( P \) only. Due to the fact that \( r \)-dependence of the metric is known, further results hold for arbitrary \( r > 0 \). It turns out that (30) restricts substantially a set of solutions of the RT equation (3), regardless of whether the trapped surfaces are spherical or not.

Proposition 4.1. The only vacuum Robinson–Trautman metrics admitting a nonexpanding horizon are the Schwarzschild metric and the C-metric.
Proof. It follows from (30) that
\[ P^2 \xi f(u, \xi) = f(u, \xi). \]  
(31)

Substituting (5) and (31) into (3) yields
\[ (f P^{-2})_\xi = 3m(P^{-2})_u. \]  
(32)

Applying twice the operator \( P^2 \partial_\xi + 2PP_\xi \) to (32) we obtain
\[ P^2(K, \bar{\xi})^2 = 6m(P^2(\ln P), \bar{\xi}). \]  
(33)

Conditions (30) and (33) assure that the corresponding metric is of Petrov-type D [2]. For \( f = 0 \) equations (31) and (32) lead to \( P, u = 0 \) and \( K = \text{const} \), hence one obtains the Schwarzschild metric. According to [2], an inspection of Kinnersley’s list of type D vacuum solutions of the Einstein equations [10] shows that for \( f \neq 0 \) solutions of (31) and (32) necessarily yield the C-metrics. We present an independent proof of this property in appendix B.

In the case of the Schwarzschild metric the trapped surfaces form the past event horizon \( r = 2m \). In the case of the C-metric, by virtue of (29), equation (19) takes the form
\[ P^2(\ln (K + a)), \bar{\xi} + K = 0. \]  
(34)

Substituting (B.18) into (34) and integrating yields
\[ 6mK_x = -\frac{1}{3}K^3 - \left( \frac{a^2}{3} + a_1 \right) K + aa_1 \]  
(35)

where \( a_1 \) is another function of \( u \). Equation (35) coincides with (B.19) if \( a \) and \( a_1 \) are constants, \( a_1 = b - \frac{a}{3} \) and
\[ \frac{a^3}{3} - ab + c = 0. \]  
(36)

Thus, the C-metric admits locally a nonexpanding horizon. It is known that this horizon does not admit regular spherical sections (see e.g. [7]) because of conical singularity on each section \( S \).

5. Properties of the surface \( r = 2m \)

It is interesting to examine the relation between \( S \) and the surface \( r = 2m \) which plays the role of past event horizon in the Schwarzschild case. In this section, we assume that the surfaces of constant \( r \) and \( u \) are diffeomorphic to \( S_2 \) and the function \( \hat{P} = P/P_S \) is regular on these surfaces.

**Proposition 5.1.** Let \( S \) be a regular spheroidal marginally trapped surface given by (12) and (19). Then \( S \) crosses the surface \( r = 2m \).

**Proof.** Let
\[ R = \frac{2m}{1 - h} \]  
(37)

for some smooth function \( h < 1 \). Rewriting (19) in terms of \( h \) we get
\[ -2P^2[\ln(1 - h)]_\xi = K - 1 + h. \]  
(38)
From the Gauss–Bonnet theorem and the regularity of $h$ it follows that
\begin{equation}
\int_{\{r=1, u=\text{const}\}} h = 0, \tag{39}
\end{equation}
hence $h^{-1}(0) \neq \emptyset$ which finishes the proof. $\square$

Another property of $r=2m$ is the signature of its induced metric for finite $u$. Obviously, it is null in the Schwarzschild spacetime. The converse statement is also true.

**Proposition 5.2.** Let $g$ be a vacuum RT metric (1). If the hypersurface $r=2m$ is null, then $g$ is the Schwarzschild metric.

**Proof.** Assume that the surface $r=2m$ is null, then in (1) one has to set $H|_{r=2m} = 0$ which is equivalent to
\begin{equation}
K - 1 = 4m(\ln P). \tag{40}
\end{equation}
Combining (40) with the RT equation (3) we get
\begin{equation}
\partial_u(4m(\ln P), \bar{\xi} \xi - 3m P^{-2}) = 0. \tag{41}
\end{equation}
Hence
\begin{equation}
(\ln P), \bar{\xi} \xi - \frac{3}{4} P^{-2} = f(\xi, \bar{\xi}) \tag{42}
\end{equation}
and
\begin{equation}
K = \frac{3}{2} + 2 P^2 f \tag{43}
\end{equation}
follow. Differentiating $K$ with respect to $u$ and using (40) and (43) we obtain
\begin{equation}
K_u = 4P P_u f = 2(\ln P)_u2 P^2 f = \frac{1}{2m}(K - 1)(K - 3/2). \tag{44}
\end{equation}
This equation can be integrated to give
\begin{equation}
\frac{K - 1}{K - 3/2} = h(\xi, \bar{\xi}) e^{-u/4m} \tag{45}
\end{equation}
for some $u$-independent function $h$. Now from (43) and (45) we have
\begin{equation}
K - 1 = h e^{-u/4m} 2 P^2 f. \tag{46}
\end{equation}
Solving (45) for $K$ and using (43) we have
\begin{equation}
P^2 = \frac{1}{4} h e^{-u/4m} - 1. \tag{47}
\end{equation}
Inserting (47) into the RT equation yields a polynomial of degree 4 in $U = \exp(-u/4m)$ equal to zero:
\begin{equation}
(h, \bar{\xi} \xi - 6 f h) U - (hh, \bar{\xi} \xi - 2|h, \bar{\xi} |^2 - 18 f h^2) U^2 - 18 f (hU)^3 + 6 f (hU)^4 = 0. \tag{48}
\end{equation}
Since $h$ is $u$-independent, it has to vanish identically, and from (46) we get $K = 1$ and $P = P_S$. Hence $g$ is the Schwarzschild metric. $\square$

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Appendix A. Spin coefficients and Weyl scalars for tetrad (13)

We follow the notation from [2] to derive the spin coefficients and Weyl scalars of (13).

\[
\alpha = \frac{P}{2r} - \frac{PR,\xi}{r^2}, \\
\beta = -\frac{P}{2r}, \\
\tau = \bar{\pi} = -\frac{P}{r^2}R,\xi, \\
\lambda = (\partial_\xi + R,\xi \partial_r) \left( \frac{p^2}{r^2} R,\xi \right), \\
\rho = -\frac{1}{r}, \\
\gamma = \frac{1}{2r} \left[ -2P^2|R,\xi|^2 + r^3 H_r + rP(P,\xi R,\xi - c.c.) \right], \\
\nu = \frac{p}{r} \left( \partial_\xi + R,\xi \partial_r \right) \left( R,u + H + \frac{p^2}{r^2} |R,\xi|^2 \right), \\
\mu = -\frac{H}{r} - (\ln P)_u + \frac{p^2}{r^2} R,\xi \xi = -\frac{p^2}{r^3} |R,\xi|^2, \\
\Psi_2 = -\frac{m}{r^3}, \\
\Psi_3 = 3Pm \left( \frac{r^3}{P} R,\xi \right) + P \frac{2}{r^2} K,\xi. 
\]

Note that from the definition of \( \chi \) in (22) and (A.7) it follows that

\[ \nu = \nabla_{\bar{h}} \chi. \]

Appendix B. Robinson–Trautman metrics of type D

In section 4 we have shown that a necessary condition for \( \mathcal{H} \) to be null is that the Robinson–Trautman metric is of type D. In this appendix we will explicitly solve equations (31) and (32) equivalent to the condition \( \lambda = 0 \) and the Robinson–Trautman equation. Since \( f = 0 \) corresponds to the Schwarzschild metric or a flat one we will assume here that \( f \neq 0 \).

Let

\[ \xi' = h(u, \xi) \]

where

\[ h,\xi = f^{-1} \]

(note that transformation (B.1) does not preserve the Robinson–Trautman equation (3)). In terms of coordinates \( \xi', \xi'' \) and \( u' = u \) equation (31) reads as

\[ p^2 K,\xi' = |f|^2. \]
Hence, $K = K(u', x')$, where $x' = \text{Re} \xi'$ and

$$p^2 = \frac{2}{K'} |h, \xi|^2.$$  \hfill (B.4)

Substituting (B.4) into definition (5) of $K$ yields

$$(\ln K_{x'})_{x'} = -2KK_{x'}.$$  \hfill (B.5)

Integrating twice equation (B.5) we get

$$K_{x'} = -\frac{1}{3}K^3 + bK + c,$$  \hfill (B.6)

where $b$ and $c$ are functions of $u$.

It follows from (B.4) that

$$\left(\ln P\right)_u = -\frac{1}{2}(\ln K_{x'})_{x'} - \frac{1}{2}(\ln K_{x'})_{x'} \text{Re} h_u - (\ln |h, \xi|)_u$$  \hfill (B.7)

and the Robinson–Trautman equation takes the form

$$(\ln K_{x'})_{x'} + (\ln K_{x'})_{x'} \left(\text{Re} h_u - \frac{1}{6m}\right) + 2(\ln |h, \xi|)_u = 0.$$  \hfill (B.8)

By virtue of (B.6) acting on (B.8) with the operator $\partial_{\xi'}$$\partial_{\overline{\xi}}$ yields

$$\left(\ln(KK_{x'})_{x'} + (\ln(KK_{x'})_{x'} \left(\text{Re} h_u - \frac{1}{6m}\right) + 2(\ln |h, \xi|)_u = 0.$$  \hfill (B.9)

Subtracting (B.8) from (B.9) implies

$$\frac{K_{x'}}{K_{x'}} + \text{Re} h_u - \frac{1}{6m} = 0.$$  \hfill (B.10)

Applying $\partial_{\xi'}$$\partial_{\overline{\xi}}$ to (B.10) shows that $K_{x'}/K_{x'}$ is linear in $x'$ and

$$h_{u'} = c_1(u)h + c_2(u),$$  \hfill (B.11)

where $c_1$ and $c_2$ are, respectively, a real and a complex function of $u$. Hence

$$h = \frac{2}{c_3(u)}h'(\xi) + c_4(u),$$  \hfill (B.12)

where $c_3$ is a real function of $u$ and $h'$ is a function of $\xi$. Note that transformation $\xi \to h'(\xi)$, accompanied by $P \to P|h, \xi|$ preserves the Robinson–Trautman metric. Thus, locally, without loss of generality we can assume $h'(\xi) = \xi$. In this case substituting (B.12) into (B.4) yields

$$p^2 = \frac{c_3}{K},$$  \hfill (B.13)

where now $K = K(u, x)$ is considered as a function of $u$ and $x = \text{Re} \xi$. Given (B.13) we see that $P = P(u, x)$ and equation (32) takes the form

$$c_3 P_x - 12mP_u = 0.$$  \hfill (B.14)

It follows from (B.14) that $P$ is a function of the variable $s = x + d(u)$, where

$$d_{u'} = \frac{c_3}{12m}.$$  \hfill (B.15)

Equation (B.13) shows that

$$K = c_3(K'(s) + c_3(u))$$  \hfill (B.16)

$$p^2 = \frac{1}{K'},$$  \hfill (B.17)
where $K'$ and $c_5$ are functions of $s$ and $u$, respectively. Substituting (B.16) and (B.17) into relation (5) implies that $c_3$ and $c_5$ are constants. Due to (B.15) and the remaining freedom of linear transformations of $\xi$ we can assume that

$$K = K(x + u), \quad P^2 = \frac{12m}{K_x}.$$  \hspace{1cm} (B.18)

Now equation (B.6) takes the form

$$6m K_x = -\frac{4}{3} K^3 + bK + c,$$  \hspace{1cm} (B.19)

where $b$ and $c$ are constants. Robinson–Trautman metrics satisfying conditions (B.18) and (B.19) are equivalent to the C-metrics [11]. Thus, the Schwarzschild metric and the C-metric are the only vacuum Robinson–Trautman metrics which satisfy condition (30).

References
[1] Robinson I and Trautman A 1962 Some spherical gravitational waves in general relativity Proc. R. Soc. London A 265 463
[2] Stephani H, Kramer D, MacCallum M A H, Hoenselaers C and Herlt E 2003 Exact Solutions to Einstein’s Field Equations 2nd edn (Cambridge: Cambridge University Press)
[3] Chruściel P T 1992 On the global structure of Robinson–Trautman spacetimes Proc. R. Soc. London A 436 299
[4] Ashtekar A and Krishnan B 2004 Isolated and dynamical horizons and their applications Living Rev. Rel. 7 10 cited 14 July 2008 http://www.livingreviews.org/lrr-2004-10
[5] Tod P 1989 Analogues of the past horizon in Robinson–Trautman spacetimes Class. Quantum Grav. 8 1159
[6] Chow E W M and Lun A W-C 1999 Apparent horizons in vacuum Robinson–Trautman spacetimes J. Aust. Math. Soc. B 41 217 (Preprint arXiv:gr-qc/9503065)
[7] Griffiths J B, Krtouš P and Podolsky J 2006 Interpreting the C-metric Class. Quantum Grav. 23 6745
[8] Andersson L, Mars M and Simon W 2005 Local existence of dynamical and trapping horizons Phys. Rev. Lett. 95 111102
[9] Korzyński M 2006 Isolated and dynamical horizons from a common perspective Phys. Rev. D 74 104029
[10] Kinnersley W 1969 Type D vacuum metrics J. Math. Phys. 10 1195
[11] Kinnersley W 1969 Field of an arbitrarily accelerating point mass Phys. Rev. 186 1335