Optimization of Fire Sales and Borrowing in Systemic Risk

Maxim Bichuch ∗  Zachary Feinstein †

Monday 12th February, 2018

Abstract

This paper provides a framework for modeling financial contagion in a network subject to fire sales and price impacts, but allowing for firms to borrow to cover their shortfall as well. We consider both uncollateralized and collateralized loans. The main results of this work are providing sufficient conditions for existence and uniqueness of the clearing solutions (i.e., payments, liquidations, and borrowing); in such a setting any clearing solution is the Nash equilibrium of an aggregation game.

AMS subject classification 91G99, 90B10, 91A06.
JEL subject classification G32.
Keywords Systemic Risk, Networks, Fire Sales, Borrowing, Financial Contagion.

1 Introduction

Traditional financial risk considers each financial firm as separate and individual entities that do not interact or exacerbate each other’s downside events. Systemic risk, in contrast, considers the risk of the distress of a single bank or multiple banks spreading throughout the financial system, up to and including threatening the health of the entire system, due to characteristics of the interactions between firms. This spread of defaults is also called financial contagion. These contagious events can occur through local connections (e.g., contractual obligations) or global connections (e.g., impacts to asset prices). Such a systemic event occurred during the 2007-2009 financial crisis in which the entire financial system was threatened with failure. Due to the threat, this event led to government intervention requiring a significant bailout and directly precipitating a global recession. It is for these reasons that the modeling of systemic events is of paramount importance. This study will advance the modeling of such events by incorporating notions of borrowing and fire sales.

This paper will extend the Eisenberg-Noe network model approach of [11]. That paper considers the network of interbank obligations and finds the equilibrium payments. Central banks and regulators have applied the Eisenberg-Noe model to study cascading failures in the banking systems within their jurisdictions, see, e.g., [3, 19, 6, 12, 26, 16, 4].

The Eisenberg-Noe model has been extended previously to include more realistic structures for contagion; this includes bankruptcy costs, cross-holdings, and fire sales. We refer to [27, 25, 20] for surveys of these extensions. In our work we will consider specifically an extension related to fire sales. Fire sales for a single (representative) illiquid asset have been studied in, e.g., [10, 21, 17, 1, 8, 27, 2], and for multiple illiquid assets in, e.g., [13, 15, 14].

The goal of this paper is to investigate the effects of confidence and liquidity on systemic risk and financial contagion. To do so, a modified Eisenberg-Noe network model is considered under which there is a network of banks with connections between them representing interbank liabilities; additionally, the banks hold illiquid assets that may need to be liquidated in order to raise funds. During the crisis events that are being studied, the asset selling is subject to price impact from fire sales. Moreover, firms are allowed to raise

∗Department of Applied Mathematics and Statistics, Johns Hopkins University 3400 North Charles Street, Baltimore, MD 21218. mbichuch@jhu.edu. Research is partially supported by the Acheson J. Duncan Fund for the Advancement of Research in Statistics.
†Department of Electrical and Systems Engineering, Washington University, St. Louis, MO 63130, USA, zfeinstein@wustl.edu.
funds through short-term borrowing, rather than through the liquidation of assets. The short-term interest rate is postulated to be a function of the confidence in the financial system and of the specific bank; as such the firms may have heterogeneous interest rates. The focus of this paper is on the Nash equilibria of bank decisions over the two methods of raising cash.

The novelty of this project is to consider other avenues of increasing reserve levels and raising cash. Specifically, one area that will be considered is borrowing. Most banks rely on short-term borrowing in order to fund their daily operations. The current project advances the field as it adds another dimension to systemic risk—confidence, and specifically confidence as expressed through borrowing rates. In the financial crisis of 2007-9, an important reason for the failure of the financial system and one of the solutions employed by the regulators was to increase the liquidity of the overnight lending market. To the best of our knowledge, [24] was the first to include confidence when studying systemic risk. [5] proposed a model to quantify and capture this effect in calculations of perceived riskiness of an individual bank.

This paper combines the above features in assessing the systemic risk in the financial system. In particular, the proposed framework is general enough to incorporate all of the aforementioned concepts. This model is an extension of the framework of [11] incorporating borrowing, confidence, liquidity, and fire sales. The organization of this paper is as follows. Section 2 contains the details of the initial modification to the Eisenberg-Noe model that includes fire sales which we will be extending. Section 3 contains the analysis of optimal liquidation through fire sales and borrowing, together with some simple examples. Section 4 incorporates the additional constraint that all borrowing must be collateralized.

2 Original Eisenberg-Noe Fire Sales model

Given $n$ interlinked banks. Denote $L_{ij} \geq 0$ the liability of bank $i$ towards $j$, for $i, j = 1, ..., n$, and denote $\bar{p}_i = \sum_{j=0}^{n} L_{ij}$ to be the total liability of bank $i$. The liability $L_{i0}$, $i = 1, ..., n$ is assumed to be external liability of bank $i$ to an entity outside of the banking network. It will be assumed that $L_{i0} \geq 0$. Under a pro-rata payment scheme, $(\Pi)_{ij} = \pi_{ij}$ will denote the relative liability, which is given by $\pi_{ij} = \frac{L_{ij}}{\bar{p}_i}$ if $\bar{p}_i > 0$ and $\pi_{ij} = 0$ otherwise, for $i = 1, ..., n, j = 0, ..., n$. These define the $n \times (n+1)$ matrix $\Pi$. Moreover, it will be assumed that bank $i$ has liquid endowment $c_i \geq 0$ and illiquid endowment $a_i \geq 0$. While the liquid endowment can be assumed to be cash, the illiquid endowment is in physical units of assets as the liquidation price of these assets remains to be determined and will be assumed to depend on the size of the sale. For now, this price will be denoted by $q$.

Following the network models of [11, 10, 2, 13], the notional payments are given by $p = \bar{p} \wedge (c + sq + \Pi^T p)$, where $s_i \in [0, a_i]$ is the quantity of illiquid assets being sold by firm $i$ evaluated by mark-to-market valuation with price $q$, and $c_i$ is its cash reserve. Throughout this work we will denote $x \wedge y = (\min(x_1, y_1), ..., \min(x_n, y_n))^\top$ for $x, y \in \mathbb{R}^n$. There is an implicit no short selling constraint in this model. It is assumed that the inverse demand curve for the illiquid asset provides the equilibrium price via $q = f(\sum_{i=1}^{n} s_i)$. The following assumptions are assumed about the inverse demand function $f$:

**Assumption 2.1.** Let $M \geq \sum_{i=1}^{n} a_i$ be the total initial market capitalization of the illiquid asset. The inverse demand function $f : \mathbb{R}_+ \rightarrow [0, 1]$ is strictly decreasing and twice continuously differentiable, with $f(0) = 1$ and $f(s) > 0$ for any $s \in [0, M]$. Additionally it will be assumed that the first derivative $f' : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ is nondecreasing. Further assume that the mapping $s \in [0, M] \mapsto sf(s)$ is strictly increasing and $s \in [0, M] \mapsto \frac{d^2}{ds^2}(sf(s)) = 2f'(s) + sf''(s) < 0$ is strictly negative.

The intuition being that during normal times, the price of the asset is one. However, during a fire sale the price of the illiquid asset is artificially depressed due to the lack of liquidity. Hence the book price of the asset is one, assuming the bank does not need to liquidate it, otherwise, the liquidation price will be set to be $q \in (0, 1)$. Further, we incorporate the notion that the price drops slower at lower prices. Finally, if a bank were to sell an extra unit, it would obtain positive, but decreasing marginal, cash. This is consistent with the construction of an order book.

In this project the liquidation rule of [2] will be used. That is, a firm must liquidate illiquid assets in order to have enough reserves to satisfy its liabilities, i.e., $s = a \wedge \frac{\bar{p} - c - \Pi^T p}{\bar{p}}$. We note that, under a regular network [11, Definition 5] and Assumption 2.1, the joint equilibrium between payments $p$ and prices $q$ is unique with this liquidation rule.

2
3 Optimal Tradeoff between Fire Sales and Borrowing

Assume that bank $i$’s interest rate is $r_i$, which may be a function of parameters such as the LIBOR rate. For example, these parameters include quantifiable proxies of confidence as suggested in [5]. However, since the model is static, these parameters will be assumed to be fixed, and a static interest rate will be assumed. Additionally, it is convenient to define $s_{-i} = \sum_{j=1, j \neq i}^{n} s_j$. Thus there are three cases to consider for each bank $i$:

Case I: Bank $i$ is solvent with no borrowing nor asset liquidation is required. This is the case if $\bar{p}_i \leq c_i + \sum_{j=1}^{n} \pi_{ji} p_j$.

Case II: Bank $i$ is fundamentally insolvent. In this case, the book value of the bank is below its obligation, that is, even if the bank were to sell its illiquid asset $a_i$ at the hypothetical price 1 it will still default on its debt. That is $p_i > c_i + a_i + \sum_{j=1}^{n} L_{ji}$. There is no borrowing in this case, as it will be assumed that it is imprudent to lend money to a fundamentally insolvent bank. Including a notion of bankruptcy costs as in, e.g., [22, 9], it is assumed that for the duration of the crisis being studied, no obligations will be paid by any fundamentally insolvent firms. Such a firm will not participate in fire sales as their assets will be distributed or liquidated in bankruptcy court at a later time.

Case III: Bank $i$ is fundamentally solvent with borrowing and asset liquidations required. This is the case if $c_i + \sum_{j=1}^{n} \pi_{ji} p_j < \bar{p}_i \leq c_i + a_i + \sum_{j=1}^{n} L_{ji}$. In this case, the bank can decide how much to borrow and liquidate in order to optimize its cash flow, i.e. minimize expenses due to the interest payment and the loss due to fire sale. In this case, it will be assumed that the bank can borrow funds as needed. The set of all such banks will be denoted $C_3$.

Borrowing can, and will, only happen in the last case, when the bank is fundamentally solvent and at the asset’s nominal price of 1. As such, the remainder of this section, and much of this paper will focus on banks in Case III. Namely, firm $i$ in Case III seeks to optimally liquidate

$$s_i^*(s_{-i}) = \arg\min_{s \in [0, a_i]} s(1 - f(s_{-i} + s) + r_i (h_i - sf(s_{-i} + s)))^+$$

units of the illiquid asset where $h_i = \bar{p}_i - (c_i + \sum_{j} \pi_{ji} p_j)$ is the liquid shortfall for bank $i$. Note that the total required borrowing given sales of $s_i$ is $(h_i - s_i f(s_{-i} + s))^+$.

**Theorem 3.1** (Existence of Nash Equilibrium). Under Assumption 2.1 there exists Nash equilibrium liquidating strategy $s^{**} = (s^*_1(s^{**}_1), ..., s^*_n(s^{**}_n))^\top \in \prod_{i=1}^{n} [0, a_i]$ with resulting equilibrium price $q^{**} = f(\sum_{i=1}^{n} s_i^{**})$.

**Proof.** Clearly, for $i \notin C_3$, $s_i^{**} = 0$. For $i \in C_3$, two distinct scenarios will be considered: The first scenario we consider is when the bank does not borrow and only liquidates. In this case $h_i = sf(s_{-i} + s)$. The solution $s_i^L(s_{-i})$, if it exists, is unique and satisfies

$$s_i^L(s_{-i}) f(s_{-i} + s_i^L(s_{-i})) = h_i.$$  

If no solution to (3.2) exists then $s_i^L(s_{-i}) = +\infty$. Under the condition that the firm holds enough assets to possibly cover their entire shortfall, the existence and the uniqueness $s_i^L(s_{-i})$ follows from Assumption 2.1. The second scenario considered here is when the bank does a mix of borrowing and liquidations. From Assumption 2.1, this value can be found by equating the derivative of (3.1) to zero and solving for $s_i^0(s_{-i})$ which satisfies

$$1 - (1 + r_i)(f(s_{-i} + s_i^0(s_{-i})) + s_i^0 f'(s_{-i} + s_i^0(s_{-i}))) = 0,$$

where $s_i^0(s_{-i}) = +\infty$ if no such solution exists. Note that a priori this might not be a solution to the optimization problem (3.1), as it additionally needs to be compared with $s_i^L(s_{-i})$ from (3.2) as well as the bounds 0 and $a_i$. 

3
• If \( f(s_{-i}) \geq (1+r_i)^{-1} \) then the solution (3.3), if it exists is unique, since by Assumption 2.1 \( \text{card}\{s \geq 0 \mid f(s_{-i} + s) + sf'(s_{-i} + s) = (1+r_i)^{-1}\} \leq 1 \) in this case. Moreover, the solution to the original optimization problem (3.1) is given by \( s^*_i(s_{-i}) = \min\{s^b_i(s_{-i}), s^b_i(s_{-i}), a_i\} \). If \( s^0_i(s_{-i}) \in [0,s^b_i(s_{-i}) \land a_i] \) then it is optimal and the bank should liquidate \( s^0_i(s_{-i}) \) shares of the illiquid asset and borrow the remainder of the liquid shortfall. If \( s^0_i(s_{-i}) > s^b_i(s_{-i}) \land a_i \) then it follows that

\[
f(s_{-i} + s) + sf'(s_{-i} + s) > (1+r_i)^{-1} \quad \text{for every } s \in [0,s^b_i(s_{-i}) \land a_i].
\]

Thus if \( s^b_i(s_{-i}) \leq a_i \), the bank only needs to liquidate \( s^b_i(s_{-i}) \) number of illiquid assets, and no borrowing is needed. Whereas if \( s^b_i(s_{-i}) > a_i \), the bank will liquidate all of its illiquid asset holdings \( a_i \) and borrow the rest. This is the optimal behavior by (3.4). From a financial perspective, it is optimal to liquidate as little as possible, so it is sufficient to liquidate \( s^b_i(s_{-i}) \land a_i \) if \( s^0_i(s_{-i}) > s^b_i(s_{-i}) \land a_i \).

• If \( f(s_{-i}) < (1+r_i)^{-1} \) then it follows that \( s^*_i(s_{-i}) = 0 \), i.e. the optimal solution \( s^*_i \) of (3.1) is pure borrowing, and liquidating nothing.

To summarize, each bank \( i \) will choose to liquidate \( s^*_i(s_{-i}) \) shares of the illiquid asset provided that in aggregate all other firms are liquidating \( s_{-i} \), where \( s^*_i(s_{-i}) \) is given by:

\[
s^*_i(s_{-i}) = \begin{cases} 
\min\{s^b_i(s_{-i}), s^b_i(s_{-i}), a_i\} & \text{if } f(s_{-i}) \geq (1+r_i)^{-1} \text{ and } i \in C_3, \\
0 & \text{otherwise}.
\end{cases}
\]

In fact, we can see that \( s^*_i : [0,M - a_i] \to [0,a_i] \) is continuous due to continuity of each of its components. Indeed, \( s^*_i \) is continuous as a function of \( s_{-i} \) in the (possibly empty) regions where \( f(s_{-i}) < (1+r_i)^{-1} \) and \( f(s_{-i}) > (1+r_i)^{-1} \). It also can be seen that \( s^*_i(s_{-i}) \to 0 \) as \( f(s_{-i}) \to (1+r_i)^{-1} \), which establishes the continuity at the point where \( f(s_{-i}) = (1+r_i)^{-1} \). Therefore, by Brouwer’s fixed point theorem there exists an equilibrium liquidation strategy \( s^{**} = (s^{**}_1, \ldots, s^{**}_n)^\top \in \prod_{i=1}^n [0,a_i] \).

We now turn our attention to uniqueness of the Nash equilibrium strategy. From the above computations we can conclude that the optimal liquidations can be provided by the equivalent minimization problem

\[
s^*_i(s_{-i}) = \arg\min_{s \in [0,a_i]} \{ s(1 - f(s_{-i} + s)) + r_i (h_i - sf(s_{-i} + s)) \mid sf(s_{-i} + s) \leq h_i \}, \quad i \in C_3.
\]

(3.6)

Now let’s consider a modified equilibrium problem between liquidations and the resultant prices, where the price in the constraint is replaced with the variable \( q \in (0,1] \).

\[
s^*_i(q^i) = \arg\min_{s \in [0,\min\{s_i, a_i\}]} \left( 1 - f \left( \sum_{j \neq i} s_j + s \right) \right) + r_i \left( h_i - sf \left( \sum_{j \neq i} s_j + s \right) \right), \quad i \in C_3
\]

(3.7)

\textbf{Theorem 3.2} (Uniqueness of Nash Equilibrium). Under Assumption 2.1 and for a fixed price \( q \in [f(M),1] \) there exists a unique equilibrium liquidation strategy \( \hat{s}(q) = s_1(\hat{s}(q), q) \). Additionally, if

\[
f'(s) + sf''(s) \leq 0 \quad \text{and} \quad -Mf'(0) < f(M)
\]

(3.8)

then there exists a unique joint liquidation-price equilibrium \( s^\dagger = s_1(q^\dagger) \) and \( q^\dagger = f(\sum_{i=1}^n s_i) \).

Before we can prove this theorem we require the following auxiliary lemma.

\textbf{Lemma 3.3.} The function \( H(s; \rho) = \sum_{i=1}^n \rho_i \left( s_i \left( 1 - f \left( \sum_{j=1}^n s_j \right) \right) + r_i \left( h_i - s_i f \left( \sum_{j=1}^n s_j \right) \right) \right) \), \( \rho \in \mathbb{R}^n_+ \) is diagonally strictly convex.

\textbf{Proof.} Recall from [23] that the function \( H(s; \rho) \) (as a function of \( s \)) is diagonally strictly convex, if for some (fixed) \( \rho \in \mathbb{R}^n_+ \) and for every \( s^0, s^1 \in \mathbb{R}^n \), \( s^0 \neq s^1 \), we have \( (s^1 - s^0)^\top g(s^0; \rho) - (s^1 - s^0)^\top g(s^1; \rho) < 0 \), where the \( i \)-th component \( g_i(s; \rho) = \frac{\partial}{\partial s_i} H_i(s; \rho) \). Additionally, [23, Theorem 6] shows that a sufficient condition for \( H \) to be strictly convex, is if \( G(s; \rho) + G(s; \rho)^\top \) is a symmetric positive definite matrix for every \( s \in \mathbb{R}^n \) and some \( \rho \in \mathbb{R}^n_+ \), where \( G \) is the Jacobian matrix of \( g \) with respect to \( s \).
Set \( \rho_i = \frac{1}{1+\tau_i} \), then \((G(s; \rho) + G(s; \rho)^\top)\) 
\( = -(2+21_{i=j})f'(\sum_{k=1}^n s_k) - (s_i + s_j)f''(\sum_{k=1}^n s_k) \). Thus, write \( G(s; \rho) + G(s; \rho)^\top = G_1(s; \rho) + G_2(s; \rho) + f''(\sum_{k=1}^n s_k) G_3(s; \rho) \), where the matrix valued functions 
\( G_1(s; \rho) = -(2f'(\sum_{k=1}^n s_k) + (\sum_{k=1}^n s_k)f''(\sum_{k=1}^n s_k)) I_{n \times n}, G_2(s; \rho) = -\text{diag}(2f'(\sum_{k=1}^n s_k) 1_n + s_i f''(\sum_{k=1}^n s_k)), \) 
and \( (G_3(s; \rho))_{i,j} = \sum_{k \neq i,j} s_k. \) Here, and elsewhere throughout this manuscript we will use the notation that \([\cdot]_{i=1, \ldots, n} = [\cdot]_{j=1, \ldots, m} \) and \([\cdot]_{i=1, \ldots, n} \) to specify \( n \times m \) dimensional matrix and \( n \)-dimensional vector respectively, where \( \text{diag}((i, i)) \) is a \( n \times n \) matrix, with diagonal element \((i, i)\) the \( i \)-th coordinate of the vector. Additionally, \( 1_{n \times n} \) is a \( n \times n \) matrix and \( 1_n \) is a \( n \times 1 \) vector with all elements one.

By construction, the matrix \( G_1 \) is positive semidefinite and \( G_2 \) is positive definite. \( G_3 \) is positive semidefinite, which would then give the desired result, as \( f'' \geq 0 \) by Assumption 2.1. Note that \( G_3(s; \rho) = \sum_{i=1}^n s_i 1_{n \times n}^{(i)} \), where the matrices \( 1_{n \times n}^{(i)}, i = 1, \ldots, n \) are given by \( (1_{n \times n}^{(i)})_{j,k} = 1_{j \neq i, k \neq i}. \) It is readily seen that \( 1_{n \times n}^{(i)} \) are positive semidefinite. It then follows that so is \( G_3. \)

**Proof of Theorem 3.2.** We first fix \( q \in [f(M), 1] \), and look for an equilibrium \( \bar{s}_i(q) = \bar{s}_i^{(i)}(\sum_{j \neq i} \bar{s}_j(q)) \). That is, for the modified Nash equilibrium given by (3.7). For a fixed \( q \), the existence of such an equilibrium follows from the logic of Theorem 3.1 and uniqueness of \( \bar{s}(q) \) is a result of Lemma 3.3 as shown in [23, Theorem 2].

The next goal is to show \( q \mapsto \Phi(q) = f(\sum_{j=1}^n \bar{s}_j(q)) \) is a contraction mapping. Indeed, for \( q_1 \neq q_2 \):

\[
\left| \frac{\Phi(q_1) - \Phi(q_2)}{|q_1 - q_2|} \right| = \frac{1}{|q_1 - q_2|} \left| f(\sum_{j=1}^n \bar{s}_j(q_1)) - f(\sum_{j=1}^n \bar{s}_j(q_2)) \right| \leq -f''(0) \max_{q \in [f(M), 1]} \sum_{j=1}^n \bar{s}_j''(q) < 1.
\]

Thus to be a contraction mapping, it is sufficient to show that \(-f''(0) \max_{q \in [f(M), 1]} \sum_{j=1}^n \bar{s}_j''(q) < 1.\)

In order to show this, consider the sensitivity of \( \bar{s}(q) \) with respect to \( q \). Recall the construction of \( s^* \) given by (3.5); here we will replace \( s_i^*(s_{-i}) \) with \( h_i/q \).

Assume that \( a_i, 1/q, h_i^{(0)}(\sum_{j \neq i} \bar{s}_j(q)) \) are all different for all \( i = 1, \ldots, n \), so that together with the continuity of \( s^0 \) it follows that \( s^* \) is differentiable with respect to \( q \) and its derivative for a given bank \( i \) is given by

\[
\bar{s}_i''(q) = \begin{cases} 
\frac{h_i}{a_i} & 
\text{if } i = 1, \ldots, n \\
(\sum_{j \neq i} \bar{s}_j(q)) \frac{h_i}{q} + (s_i^{(0)}(\sum_{j \neq i} \bar{s}_j(q))) (\sum_{j \neq i} \bar{s}_j''(q)) \| \bar{s}_i(q) \|_0 \leq s_i^{(0)}(\sum_{j \neq i} \bar{s}_j(q)) < h_i^{(0)} \wedge a_i 
\end{cases}.
\]

Here, the derivative of the optimal liquidations \( s_i^{(0)}(s_{-i}) \) can be found via implicit differentiation: \( (s_i^{(0)}(s_{-i}) = -f'_{s_i} f_{s_{-i}}(s_{-i} + s_i^{(0)}(s_{-i})) + s_i^{(0)}(s_{-i}) f_{s_{-i}}(s_{-i} + s_i^{(0)}(s_{-i})). \) Therefore \( (s_i^{(0)}(s_{-i}) \in (-1, 0) \) for all \( i \in C_3 \) if \( f'(s_i) + s_i f''(s_i) \leq 0 \) for every \( s \in [0, M]. \)

Noting that \( \bar{s} \) is zero for Case I and II institutions, solving the system (3.10), it follows that

\[
\bar{s}(q) = -\left( I - \text{diag} \left( \left( s_i^{(0)}(\sum_{j \neq i} \bar{s}_j(q)) \| \bar{s}_i(q) \|_0 \leq s_i^{(0)}(\sum_{j \neq i} \bar{s}_j(q)) < h_i^{(0)} \wedge a_i \right) \mathbb{I}_{i \in C_3} \right) \right)^{-1} \left( \begin{array}{c} \bar{s}_i(q) \\
\end{array} \right)
\]

\[
\times \text{diag} \left( \left( \sum_{j \neq i} \bar{s}_j(q) \| \bar{s}_i(q) \|_0 \leq s_i^{(0)}(\sum_{j \neq i} \bar{s}_j(q)) < h_i^{(0)} \wedge a_i \right) \mathbb{I}_{i \in C_3} \right) \left( \begin{array}{c} \bar{s}_i(q) \\
\end{array} \right).
\]

Using the fact that \( (s_i^{(0)}(s_{-i}) \in (-1, 0) \) for \( i = 1, \ldots, n \) as follows from the sufficient assumption of the theorem, it thus follows that

\[
\left| \frac{\bar{s}''(q)}{\bar{s}_i^{(0)}(s_{-i})} \right| \leq \max_{d \in [0, 1]} \left| \left( \begin{array}{c} \bar{s}_i(q) \\
\end{array} \right) \right| \mu \left( \begin{array}{c} \bar{s}_i(q) \\
\end{array} \right) \frac{h_i}{q} \right| < 1.
\]

To compute this maximum, let \( B(d) := I + \text{diag}(d)(1_n - I) \). By the Sherman-Morrison formula \( B(d)^{-1} = (1_n - d)^{-1} - \frac{1}{1+\text{diag}(1_n - d)^{-1} d} \left( 1_n - d \right)^{-1} d \text{diag} (1_n - d)^{-1} \). It now follows that for any \( j = 1, \ldots, n \)

\[
\sum_{i=1}^n (B(d)^{-1})_{ij} 1_{d_i = 0} = \frac{1}{1+\sum_{k=1}^n \frac{d_k}{1-d_k}} \left( 1+\sum_{k=1}^n \frac{d_k}{1-d_k} - \sum_{k \neq j} \frac{d_k}{1-d_k} \right) 1_{d_j = 0} = \frac{1}{1+\sum_{k=1}^n \frac{d_k}{1-d_k}}.
\]
We conclude that
\[
\max_{q \in \mathcal{f}(M),1} \left| \mathbf{1}_n^\top \bar{s}'(q) \right| \leq \max_{q \in \mathcal{f}(M),1, a \in [0,1]} \left| \mathbf{1}_n^\top B(d)^{-1} \text{diag} \left( \left[ \mathbb{1}_{d_i=0, \frac{a_i}{q} < a_i} \right]_{i=1, \ldots, n} \right) \frac{h}{q^2} \right| \quad (3.14)
\]
\[
\leq \max_{q \in \mathcal{f}(M),1} \left| \mathbf{1}_n^\top \frac{h}{q} \wedge q \right| \leq \max_{q \in \mathcal{f}(M),1} \frac{\sum_{i=1}^n a_i}{q} \leq \frac{M}{f(M)}.
\]
Recalling (3.9), we conclude that $\Phi$ is a contraction mapping if $-M f'(0) < f(M)$.

Recall that it was assumed that $a_i, \frac{a_i}{q}, s^0_i(\sum_{j \neq i} \bar{s}_j(q))$ are all different. If this assumption is violated, say $s^0_i(\sum_{j \neq i} \bar{s}_j(q)) > a_i = \frac{h_i}{q}$, then we need to consider one-sided derivatives. In that case, the derivative from the left $\partial_- \bar{s}_i(q) = 0$, while the derivative from the right $\partial_+ \bar{s}_i(q) = -\frac{h_i}{q}$. In this case, both one-sided derivatives would satisfy (3.14). The other cases, can be treated similarly.

In the following examples, we show that the conditions (3.8) of Theorem 3.2 are true for a broad range of inverse demand functions $f$.

**Example 3.4.** Linear price impact: $f(s) = 1 - \alpha s$ for $0 < \alpha < \frac{1}{2M}$ satisfies all conditions for an inverse demand function in Assumption 2.1. The first condition of (3.8) is true for $\alpha > 0$. While the second condition of (3.8) is true when $\alpha < \frac{1}{2M}$.

That is, existence and uniqueness are guaranteed if $0 < \alpha < \frac{1}{2M}$.

**Example 3.5.** Exponential price impact: $f(s) = \exp(-\alpha s)$ for $0 < \alpha < \frac{1}{2}$ satisfies all conditions for an inverse demand function in Assumption 2.1. The first condition of (3.8) is true when $0 < \alpha < \frac{1}{M}$. While the second condition of (3.8) is true when $\alpha < \frac{W(1)}{M} = 0.507$ where $W$ is the Lambert W function. That is, existence and uniqueness are guaranteed if $0 < \alpha < \frac{W(1)}{M}$.

**Example 3.6.** Hyperbolic price impact: $f(s) = \frac{\epsilon}{\sqrt{s^2 + \epsilon}}$ for $\epsilon > 0$ satisfies all conditions for an inverse demand function in Assumption 2.1. The first condition of (3.8) is true when $\epsilon \geq M$. While the second condition of (3.8) is true when $\epsilon > \frac{1 + \sqrt{5}}{2} M$. That is, existence and uniqueness are guaranteed if $\epsilon > \frac{1 + \sqrt{5}}{2} M$.

## 4 Optimal Fire Sales with Collateralized Borrowing

One of the key assumptions of the previous section was that the bank can borrow funds without the need for collateral. In other words, once the determination is made that the bank is fundamentally solvent, there is no restriction on the size of the loan. The original interpretation can be that the loan is made by the lender of the last resort, who initially determines if the bank is solvent or not. Once it is determined that the bank is solvent, the bank is allowed to borrow as needed.

A more realistic and better alternative to consider is that the short-term loan needs to be collateralized, and that this collateral is in the form of the illiquid asset. The collateralized value of the illiquid asset will be assumed to be one. This is done in order to guarantee that a firm doing collateralized borrowing will have positive equity using the book value price for the illiquid asset. More precisely, in Section 3, it is possible that a firm that is required to both liquidate a portion of its holdings and borrow will fundamentally have negative equity after it liquidates assets and then borrows additional cash, which would be in contrast to the notion of “solvency” as it is generally considered. Thus the current setup demonstrates more completely that firms required to liquidate and borrow remain fundamentally solvent, and bounds their actions so as to keep such a firm in this solvency region. More specifically, this splits the original third case to be as follows

**Case III:** Bank $i$ is fundamentally solvent with borrowing and asset liquidations, but unable to pass a stress test. This is the case if $c_i + \sum_{j=1}^n \pi_{ij} h_j < \bar{h}_i \leq c_i + a_i \sum_{j=1}^n L_{ji}, \ a_i(1 - \nu) < h_i$. The first condition is the same as in the original Case III, but an additional condition is now added, that under a stress test scenario, when the illiquid asset loses a proportion $\nu \in (0, 1)$ of its value because of a shock, the bank becomes insolvent. It will be assumed that in this scenario this bank will be taken over by the regulator, who will also honor the bank’s obligations and will ultimately be sold to a solvent bank. In this scenario the illiquid asset will not be liquidated.
Case IV: Bank $i$ is fundamentally solvency with borrowing and asset liquidations, and able to pass a stress test. This is the case if $c_i + \sum_{j=1}^{n} x_j p_j < \tilde{p}_i \leq c_i + \sum_{j=1}^{n} L_{ij}$, $a_i (1 - \nu) \geq h_i$. In this case, the bank can decide how much to borrow in order to optimize its cash flow, and minimize expenses due to the interest payment and the loss due to fire sale. The maximum borrowing amount is constrained by the collateral requirement. The set of all such banks will be denoted $C_4$.

As in the prior section, the last case of banks (Case IV firms) are those of primary interest for us. In this case, the value of the collateral will be assumed to be the asset’s book price, i.e., one. Thus, for bank $i \in C_4$, the maximum amount that can be borrowed is the solution to

$$s_i^b(s_{-i}) (1 - f(s_{-i} + s_i^b(s_{-i}))) = a_i - h_i. \quad (4.1)$$

The solution $s_i^b(s_{-i})$ to (4.1) is unique if it exists, otherwise set $s_i^b(s_{-i}) = \infty$.

As such, the new problem that replaces (3.6) for bank $i$ in Case IV is to optimize the tradeoff in the number of sold shares and borrowing:

$$s_i^b(s_{-i}) = \arg\min_{s \in [0, a_i]} \{ s(1 - f(s_{-i} + s)) + r_i (h_i - sf(s_{-i} + s)) \mid sf(s_{-i} + s) \leq h_i, s \leq s_i^b(s_{-i}) \}. \quad (4.2)$$

The additional constraint in (4.2) as compared with (3.6) necessitates only trivial modifications to the proof of existence as done in Theorem 3.1. The equivalent of Theorem 3.2 is:

**Theorem 4.1** (Uniqueness of Nash Equilibrium). Under Assumption 2.1 and for a fixed price $q \in [f(M), 1]$ there exists a unique equilibrium liquidation strategy $\bar{s}(q) = s^1(\bar{s}(q), q)$ where $s^1_i(s_I^1, q) = 0$ for $i \not\in C_4$ and $s^1_i(s_I^1, q) = \arg\min_{s \in [0, \min\{a_i, b_i, \frac{a_i - b_i}{q}\}]} s \left(1 - f \left(\sum_{j \neq i} s_j^1 + s\right)\right) + r_i \left(h_i - sf \left(\sum_{j \neq i} s_j^1 + s\right)\right)$ for $i \in C_4$. Additionally, if $f'(s) + sf''(s) \leq 0$ and $-Mf'(0) < \nu \land f(M)$ then there exists a unique joint liquidation-price equilibrium $s^1 = s^1(s_I^1, q_I^1)$ and $q_I^1 = f(\sum_{i=1}^n s_i^1)$.

**Proof.** This proof remains largely unchanged from those of Theorems 3.1 and 3.2. The mapping $q \mapsto \Phi(q) = f(\sum_{j=1}^n \bar{s}_j(q))$ is a contraction mapping if (3.9) is dominated by 1. We need to change $\bar{s}^1(q)$ in (3.10). Similar to the original proof of Theorem 3.2 we assume that the quantities $a_i, \frac{a_i - b_i}{q}, \frac{a_i - b_i}{q^2}, s_i^0(s_{-i})$ $i = 1, ..., n$ are all different. The modification to the proof if that is not the case, is the same as in the original proof. Under this assumption $\bar{s}_i^1(q) = \Pi_{i \in C_4} \left( - \frac{a_i}{q^2} \frac{a_i - b_i}{q} < a_i, \frac{a_i - b_i}{q} \right) \left( \frac{a_i - b_i}{q} \frac{a_i - b_i}{q} \frac{a_i - b_i}{q} \right) + (\Pi_{j \neq i} \bar{s}_j(q)/(\Pi_{j \neq i} \bar{s}_j(q)))_{\Pi_{0 \leq s \in [0, \Pi_{j \neq i} \bar{s}_j(q)] < \frac{a_i - b_i}{q}}_{i=1, ..., n}$ for $i = 1, ..., n$. Then, similar to (3.11)

$$\bar{s}'(q) = - \left( I - \text{diag} \left( \left( \Pi_{j \neq i} \bar{s}_j(q)/(\Pi_{j \neq i} \bar{s}_j(q)))_{\Pi_{0 \leq s \in [0, \Pi_{j \neq i} \bar{s}_j(q)] < \frac{a_i - b_i}{q}}_{i=1, ..., n} \right) \left(1_{n \times n} - I\right) \right)^{-1} \times \text{diag} \left( \left[ \frac{\Pi_{i \in C_4} \frac{b_i}{q^2} \frac{a_i - b_i}{q} \frac{a_i - b_i}{q} \frac{a_i - b_i}{q}}{a_i} \right]_{i=1, ..., n} \right) \left[ \frac{h_i}{q^2} \frac{a_i - b_i}{q} < \frac{a_i - b_i}{q} \frac{a_i - b_i}{q} \frac{a_i - b_i}{q} \right]_{i=1, ..., n}. \right)$$

Then similar to (3.12) we recover

$$|1_n^T \bar{s}'(q)| \leq \max_{d \in [0,1]^n} \left| 1_n^T B(d)^{-1} \text{diag} \left( \left[ \Pi_{d_i=0} \left( \frac{\Pi_{a_i} a_i - b_i}{q} \frac{a_i - b_i}{q} \frac{a_i - b_i}{q} \frac{a_i - b_i}{q} \right)_{i=1, ..., n} \right) \left( \frac{h_i}{q^2} \frac{a_i - b_i}{q} < \frac{a_i - b_i}{q} \frac{a_i - b_i}{q} \frac{a_i - b_i}{q} \right)_{i=1, ..., n} \right)^{-1} \times \text{diag} \left( \left[ \frac{\Pi_{i \in C_4} \frac{b_i}{q^2} \frac{a_i - b_i}{q} \frac{a_i - b_i}{q} \frac{a_i - b_i}{q}}{a_i} \right]_{i=1, ..., n} \right) \left[ \frac{h_i}{q^2} \frac{a_i - b_i}{q} < \frac{a_i - b_i}{q} \frac{a_i - b_i}{q} \frac{a_i - b_i}{q} \right]_{i=1, ..., n}. \right.$$
finally conclude
\[
\max_{q \in [f(M), 1]} \left| \mathbf{1}_n^T \tilde{z}'(q) \right| \leq \max_{d \in [0, 1]^n} \left| \mathbf{1}_n^T B(d)^{-1} \right| \left| \text{diag} \left( \frac{h_i}{d_i - q \wedge \alpha} \right) \right| \frac{a - h_i}{1 - q} \wedge \frac{a}{1 - q} \leq \max_{q \in [f(M), 1]} \left| \mathbf{1}_n^T \tilde{z}'(q) \right| \leq \sum_{i=1}^n \frac{a_i}{\nu} \wedge \frac{\sum_{i=1}^n a_i}{f(M)} \leq \frac{M}{\nu \wedge f(M)}.
\]

Remark 1. Due to the constraints on liquidations and the possibility of borrowing we can immediately conclude that the (unique) equilibrium price provided by collateralized borrowing is greater than that in the uncollateralized case, which in turn is above the pure fire sale setting of [2]. Economically, this recovers the observations of [18, 7] that the freezing of the repo market can generate excess systemic risk since we would, in some sense, move from the equilibrium under collateralized borrowing to the pure fire sale setting.

References

[1] H. Amini, D. Filipović, and A. Minca. Systemic risk with central counterparty clearing. Swiss Finance Institute Research Paper No. 13-34, Swiss Finance Institute, 2013.
[2] H. Amini, D. Filipović, and A. Minca. Uniqueness of equilibrium in a payment system with liquidation costs. Operations Research Letters, 44(1):1–5, 2016.
[3] K. Anand, B. Craig, and G. Von Peter. Filling in the blanks: Network structure and interbank contagion. Quantitative Finance, 15(4):625–636, 2015.
[4] M. Bardoscia, P. Barucca, A. Brinley Codd, and J. Hill. The decline of solvency contagion risk. Bank of England Staff Working Paper, 662, 2017.
[5] M. Bichuch and K. Chen. Systemic Risk: the Effect of Market Confidence. 2017. Working paper.
[6] M. Boss, H. Elsinger, M. Summer, and S. Thurner. Network topology of the interbank market. Quantitative Finance, 4(6):677–684, 2004.
[7] M. K. Brunnermeier. Deciphering the liquidity and credit crunch 2007-2008. The Journal of Economic Perspectives, 23(1):77–100, 2009.
[8] N. Chen, X. Liu, and D. D. Yao. An optimization view of financial systemic risk modeling: The network effect and the market liquidity effect. Operations Research, 64(5), 2016.
[9] C. Chong and C. Klüppelberg. Contagion in financial systems: A Bayesian network approach. SIAM Journal on Financial Mathematics, 9(1):28–53, 2018.
[10] R. Cifuentes, H. S. Shin, and G. Ferrucci. Liquidity risk and contagion. Journal of the European Economic Association, 3(2-3):556–566, 2005.
[11] L. Eisenberg and T. H. Noe. Systemic risk in financial systems. Management Science, 47(2):236–249, 2001.
[12] H. Elsinger, A. Lehar, and M. Summer. Network models and systemic risk assessment. In Handbook on Systemic Risk, pages 287–305. Cambridge University Press, 2013.
[13] Z. Feinstein. Financial contagion and asset liquidation strategies. Operations Research Letters, 45(2):109–114, 2017.
[14] Z. Feinstein. Obligations with physical delivery in a multi-layered financial network. 2017. Working paper.
[15] Z. Feinstein and F. El-Masri. The effects of leverage requirements and fire sales on financial contagion via asset liquidation strategies in financial networks. Statistics & Risk Modeling, 34(3-4):109–114, 2017.
[16] P. Gai, A. Haldane, and S. Kapadia. Complexity, concentration and contagion. Journal of Monetary Economics, 58(5):453–470, 2011.
[17] P. Gai and S. Kapadia. Contagion in financial networks. Bank of England Working Papers 383, Bank of England, 2010.
[18] G. B. Gorton and A. Metrick. Journal of Financial Economics, 104(3):425–451, 2012.
[19] G. Halaj and C. Kok. Modelling the emergence of the interbank networks. *Quantitative Finance*, 15(4):653–671, 2015.

[20] A.-C. Hüser. Too interconnected to fail: A survey of the interbank networks literature. *Journal of Network Theory in Finance*, 1(3):1–50, 2015.

[21] E. Nier, J. Yang, T. Yorulmazer, and A. Alentorn. Network models and financial stability. *Journal of Economic Dynamics and Control*, 31(6):2033–2060, 2007.

[22] L. C. Rogers and L. A. Veraart. Failure and rescue in an interbank network. *Management Science*, 59(4):882–898, 2013.

[23] J. B. Rosen. Existence and uniqueness of equilibrium points for concave n-person games. *Econometrica*, 33(3):520–534, 1965.

[24] N. Sarin and L. H. Summers. *Have big banks gotten safer?* Brookings Institution, 2016.

[25] J. Staum. Counterparty contagion in context: Contributions to systemic risk. In *Handbook on Systemic Risk*, pages 512–548. Cambridge University Press, 2013.

[26] C. Upper. Simulation methods to assess the danger of contagion in interbank markets. *Journal of Financial Stability*, 7(3):111–125, 2011.

[27] S. Weber and K. Weske. The joint impact of bankruptcy costs, fire sales and cross-holdings on systemic risk in financial networks. *Probability, Uncertainty and Quantitative Risk*, 2(1):9, June 2017.