ICTP Lectures on Theoretical Aspects of Neutrino Masses and Mixings

R. N. Mohapatra

Department of Physics, University of Maryland, College Park, MD, 20742, USA
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Abstract

Recent neutrino oscillation experiments are yielding valuable information on the nature of neutrino masses and mixings even though we are far from a complete understanding of the new physics implied by them. In these lectures, I summarize the current theoretical status of neutrino mass physics.
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I.1: Introduction

For a long time, it was believed that neutrinos are massless, spin half particles, making them drastically different from their other standard model spin half cousins such as the charged leptons \((e, \mu, \tau)\) and the quarks \((u, d, s, c, t, b)\), which are known to have mass. In fact the masslessness of the neutrino was considered so sacred in the 1950s and 1960s that the fundamental law of weak interaction physics, the successful V-A theory for charged current weak processes proposed by Sudarshan, Marshak, Feynman and Gell-Mann was considered to be intimately linked to this fact. The argument went as such: a massless fermion field equation is invariant under the \(\gamma_5\) transformation; since neutrino is one such particle and it participates exclusively in weak interactions, the weak interactions must somehow reflect invariance under the \(\gamma_5\)-transformation of all fermionic matter (i.e. quarks, charged leptons and the neutrinos) participating in weak interactions. The argument is obviously very heuristic; but it is not hard to see its profound implication: it leads to the resulting four-Fermi weak interaction to involve only V-A currents. This argument remained persuasive for a long time since there was no evidence for neutrino mass for almost next 50 years and became a celebrated myth in particle theory.

This myth has however been shattered by the accumulating evidence for neutrino mass from the solar and atmospheric neutrino data compiled in the nineties and still ongoing. One must therefore now be free to look beyond the \(\gamma_5\) invariance idea for exploring new physics as we proceed to understand the neutrino mass.

The possibility of a nonzero neutrino mass at phenomenological level goes back almost 50 years [1]. In the context of gauge theories, they were discussed extensively in the 70’s and 80’s long before there was any firm evidence for it. For instance the left-right symmetric theories of weak interactions introduced in 1974 and discussed in those days in connection with the structure of neutral current weak interactions, predicted nonzero neutrino mass as a necessary consequence of parity invariance and quark lepton symmetry.

The existence of a nonzero neutrino mass makes neutrinos more like the quarks, and allows for mixing between the different neutrino species leading to the phenomenon of neutrino oscillation, an idea first discussed by Pontecorvo [2] and Maki et al. [3] in the 1960’s, unleashing a whole new realm of particle physics phenomena to explore. More importantly, the simple fact that neutrino masses vanish in the standard model implies that, evidence for neutrino mass is a solid evidence for the existence of new physics beyond the standard model.

We are of course far from a complete picture of the masses and mixings of the various neutrinos and cannot therefore have a full outline of the theory of neutrino masses at present. However there exist enough information and indirect indications that constrain the masses and mixings among the neutrinos that we can see a narrowing of the possibilities for the theories beyond the standard model. Combined with other ideas outside the neutrino arena such as supersymmetry and unification, the possibility narrows even further. Many clever experiments now under way will soon clarify or rule out many of the allowed models. It will be one of the goals of this article to give a panoramic view of the most likely scenarios for new physics that explain what is now known about neutrino masses [4]. Such discussions are
of course by nature very subjective and therefore a sincere apology is due at the beginning
of this discussion to all those whose ideas are not cited in this lecture.

We hope to emphasize two kinds of ideas: one which provides a general framework for
understanding of the small neutrino masses, the seesaw mechanism and predicts the existence
of superheavy right handed neutrinos. In the opinion of this author, these ideas are likely to
be part of the final theory of neutrino masses. We then touch briefly on some specific models
that are based on the above general framework but attempt to provide an understanding of
the detailed mass and mixing patterns. These works are instructive for several reasons: first
they provide proof of the detailed workability of the general ideas described above (sort of
existence proofs that things will work); second they often illustrate the kind of assumptions
needed and thru that provide a unique insight into which directions the next step should be;
finally of course nature may be generous in picking one of those models as the final message
bearer.

In discussing the neutrino mass, it is instructive to compare it with the other well known
fermion, the electron. The electron and the neutrino are in many ways very similar parti-
cles: they are both spin half objects; they both participate in weak interactions with same
strength; in fact they are so similar that in the limit of exact gauge symmetries they are
two states of the same object and therefore in principle indistinguishable. Yet there are
profound differences between them in the standard model: after gauge symmetry break-
ing, only electron has electric charge but the neutrinos come out electrically neutral as they
should to match observations. Another difference is that only the lefthanded neutrinos of
each generation are included in the standard model and not its righthanded counterpart
whereas for the e, µ, τ and all the quark flavors, both helicity states are included. The fact
that the righthanded neutrino is excluded from the standard model coupled with the fact
that $B - L$ is an exact symmetry of the model implies that neutrino remains massless to
all orders in perturbation theory as well as nonperturbatively, as we will discuss later on in
this review.

The fact that the neutrino has no electric charge endows it with certain properties not
shared by other fermions of the standard model. One can write two kinds of Lorentz invar-
iant mass terms for the neutrino, the Dirac and Majorana masses, whereas for the charged
fermions, conservation of electric charge allows only Dirac type mass terms. In the four
component notation for the fermions, the Dirac mass has the form $\bar{\psi}\psi$, whereas the Majo-
rana mass is of the form $\psi^TC^{-1}\psi$, where $\psi$ is the four component spinor and $C$ is the charge
conjugation matrix. One can also discuss the two different kinds of mass terms using the
two component notation for the spinors, which provides a very useful way to discuss neu-
trino masses. We therefore present some of the salient concepts behind the two component
description of the neutrino.

### I.2 Two component notation for neutrinos

Before we start the discussion of the 2-component neutrino, let us write down the Dirac
equation for an electron $\psi$:

$$i\gamma^\lambda \partial_\lambda \psi - m\psi = 0$$

This equation follows from a free Lagrangian.
\[ \mathcal{L} = i \bar{\psi} \gamma^\lambda \partial_\lambda \psi - m \bar{\psi} \psi \]  \hspace{2cm} (2)

and leads to the relativistic energy momentum relation \( p^\lambda p_\lambda = m^2 \) for the spin-half particle only if the four \( \gamma_\lambda \)'s anticommute. If we take \( \gamma_\lambda \)'s to be \( n \times n \) matrices, the smallest value of \( n \) for which four anticommuting matrices exist is four. Therefore \( \psi \) must be a four component spinor. The physical meaning of the four components is as follows: two components for particle spin up and down and same for the antiparticle.

A spin-half particle is said to be a Majorana particle if the spinor field \( \psi \) satisfies the condition of being self charge conjugate, i.e.

\[ \psi = \psi^c \equiv C\bar{\psi}^T, \]  \hspace{2cm} (3)

where \( C \) is the charge conjugation matrix and has the property \( C\gamma_\lambda C^{-1} = -\gamma^\lambda \). This constraint reduces the number of independent components of the spinor by a factor of two, since the particle and the antiparticle are now the same particle. Using this condition, the mass term in the Lagrangian in Eq. (2) can be written as \( \psi^T C^{-1} \psi \), where we have used the fact that \( C \) is a unitary matrix. Writing the mass term in this way makes it clear that if a field carries a \( U(1) \) charge and the theory is invariant under those \( U(1) \) transformations, then the mass term is forbidden. This means that one cannot impose the Majorana condition on a particle that has a gauge charge. Since the neutrinos do not have electric charge, they can be Majorana particles unlike the quarks, electron or the muon. It is of course well known that the gauge boson interactions in a gauge theory Lagrangian conserve a global \( U(1) \) symmetry known as lepton number with the neutrino and electron carrying the same lepton number. If lepton number were to be established as an exact symmetry of nature, the Majorana mass for the neutrino would be forbidden and the neutrino, like the electron, would be a Dirac particle.

The properties of a Majorana fermion can be seen in its free field expansion in terms of creation and annihilation operators:

\[ \psi(x) = \int \frac{d^3p}{\sqrt{(2\pi)^3 2E_p}} \sum_s \left( a_s(p) u_s(p) e^{-ip \cdot x} + a_s^\dagger v_s(p) e^{ip \cdot x} \right). \]  \hspace{2cm} (4)

In the gamma matrix convention where \( \gamma_i = \begin{pmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{pmatrix} \) and \( \gamma_0 = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} \), the \( u_s \) and \( v_s \) are given by

\[ u_s(p) = \frac{m}{\sqrt{E}} \left( \frac{\alpha_s}{E - \sigma \cdot p} \right) \]  \hspace{2cm} (5)

and

\[ v_s(p) = \frac{m}{\sqrt{E}} \left( -\frac{E + \sigma \cdot p}{m} \alpha_s' \right). \]  \hspace{2cm} (6)

\( \alpha_s \) and \( \alpha_s' \) are two component spinors.

If we choose \( \alpha_s' = \sigma_2 \alpha_s \), we get the relation among the spinors \( u_s(p) \) and \( v_s(p) \)

\[ C\gamma_0 u^*_s(p) = v_s(p) \] \hspace{2cm} and the Majorana condition follows. Note that if \( \psi \) were to describe
a Dirac spinor, then we would have had a different creation operator $b^\dagger$ in the second term in the free field expansion above.

The origin of the two component neutrino is rooted in the isomorphism between the Lorentz group and the SL(2, C) group. The latter is defined as the set of $2 \times 2$ complex matrices with unit determinant, whose generators satisfy the same Lie algebra as that of the Lorentz gtroup. Its basic representations are 2 and $2^*$ dimensional. These are the spinor representations and can be used to describe spin half particles.

We can therefore write the familiar 4-component Dirac spinor used in the text books to describe an electron can be written as

$$\psi = \left( \begin{array}{c} \phi \\ i\sigma_2 \chi^* \end{array} \right),$$

where $\chi$ and $\phi$ two two component spinors. A Dirac mass is the given by $\chi^T \sigma_2 \phi$ whereas a Majorana mass is given by $\chi^T \sigma_2 \chi$, where $\sigma_a$ are the Pauli matrices. To make correspondence with the four component notation, we point out that $\phi$ and $i\sigma_2 \chi^*$ are nothing but the $\psi_L$ and $\psi_R$ respectively. It is then clear that $\chi$ and $\phi$ have opposite electric charges; therefore the Dirac mass $\chi^T \sigma_2 \phi$ maintains electric charge conservation (as well as any other kind of charge like lepton number etc.).

2-component neutrino is described by the following Lagrangian:

$$L = \bar{\nu} i\sigma^\lambda \partial_\lambda \nu - \frac{im}{2} e^{i\delta} \nu^T \sigma_2 \nu + \frac{im}{2} e^{-i\delta} \nu^\dagger \sigma_2 \nu^*. \quad (7)$$

This leads to the following equation of motion for the field $\chi$

$$i\sigma^\lambda \partial_\lambda \chi - im\sigma_2 \chi^* = 0 \quad (8)$$

As is conventionally done in field theories, we can now give a free field expansion of the two component Majorana field in terms of the creation and annihilation operators:

$$\chi(x,t) = \sum_{p,s} \left[ a_{p,s} \alpha_{p,s} e^{-ip.x} + a^\dagger_{p,s} \beta_{p,s} e^{ip.x} \right], \quad (9)$$

where the sum on $s$ goes over the spin up and down states.

**Exercise 1:** Using the field equations for a free massive two component Majorana spinor, show that its expansion in terms of the creation and annihilation operators and two component spinors $\alpha = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\beta = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ is given by the following expression:

$$\chi(x,t) = \sum_p \left[ a_{p,+} e^{-ip.x} - a_{p,-} e^{ip.x} \right] \alpha \sqrt{E + p}$$

$$+ \sum_p \left[ a_{p,-} e^{-ip.x} + a^\dagger_{p,+} e^{ip.x} \right] \beta \sqrt{E - p}. \quad (10)$$

Note that in a beta decay process, where a neutron is annihilated and proton is created, the leptonic weak current that is involved is $\bar{e}\nu$ (dropping gamma matrices); therefore, along with the electron, what is created predominantly is a right handed particle (with a wave function $\alpha$), the amplitude being of order $\sqrt{E + p} \approx \sqrt{2E}$. This is the right handed anti neutrino. The left handed neutrino is produced with a much smaller amplitude $\sqrt{E - p} \approx m\nu/E$. Similarly, in the fusion reaction in the core of the Sun, what is produced is a left handed state of the neutrino with a very tiny i.e. $O(m\nu/E)$ admixture of the right handed helicity.
A. Neutrinoless double beta decay and neutrino mass

As already noted a Majorana neutrino breaks lepton number by two units. This has the experimentally testable prediction that it leads to the process of neutrinoless double beta decay, where a nucleus (generally even-even nuclei) \((Z, N) \rightarrow (Z + 2, N - 2) + 2e^-\). We will now show by using the above property of the Majorana neutrino that if light neutrino exchange is responsible for this process, then the amplitude is proportional to the neutrino mass.

Double beta decay involves the change of two neutrons to two protons and therefore has to be a second order weak interaction process. Since each weak interaction process emits an antineutrino, in second order weak interaction, the final state will involve two anti-neutrinos. But in neutrinoless double beta decay, there are no neutrinos in the final state; therefore the two neutrinos must go into the vacuum state. Vacuum state by definition has no spin whereas the antineutrino emitted in a beta decay has spin. Consider the antineutrino from one of the decays: it must be predominantly right handed. But to disappear into vacuum, it must combine with a lefthanded antineutrino so that the left and right handed spin projections add up to zero. In the previous paragraph, we showed that the fraction of left handed spin projection in a neutrino emitted in beta decay is \(\frac{m_\nu}{E}\). Therefore, \(\bar{\nu}_e \bar{\nu}_e \rightarrow |0\rangle\) must be proportional to the neutrino mass. Thus neutrinoless double beta decay is a very sensitive measure of neutrino mass.

B. Neutrino mass in two component notation

Let us now discuss the general neutrino mass for Majorana neutrinos. We saw earlier that for a Majorana neutrinos, there are two different ways to write a mass term consistent with relativistic invariance. This richness in the possibility for neutrino masses also has a down side in the sense that in general, there are more parameters describing the masses of the neutrinos than those for the quarks and leptons. For instance for the electron and quarks, dynamics (electric charge conservation) reduces the number of parameters in their mass matrix. As an example, using the two component notation for all fermions, for the case of two two component spinors, a charged fermion mass will be described only by one parameters whereas for a neutrino, there will be three parameters. This difference increases rapidly e.g. for 2N spinors, to describe charged fermion masses, we need \(N^2\) parameters (ignoring CP violation) whereas for neutrinos, we need \(\frac{2N(N+1)}{2}\) parameters. What is more interesting is that for a neutrino like particle, one can have both even and odd number of two component objects and have a consistent theory.

In this article, we will use two component notation for neutrinos. Thus when we say that there are N neutrinos, we will mean N two-component neutrinos.

In the two component language, all massive neutrinos are Majorana particles and what is conventionally called a Dirac neutrino is really a very specific choice of mass parameters for the Majorana neutrino. Let us give some examples: If there is only one two component neutrino (we will drop the prefix two component henceforth), it can have a mass \(m\nu^T \sigma_2 \nu\) (to be called \(m\nu\nu\) in shorthanded notation). The neutrino is now a self conjugate object which can be seen if we write an equivalent 4-component spinor \(\psi\):
\[ \psi = \left( \begin{array}{c} \nu \\ i\sigma_2\nu^* \end{array} \right) \]  

(11)

Note that this 4-component spinor satisfies the condition

\[ \psi = \psi^c \equiv C\bar{\psi}^T \]  

(12)

This condition implies that the neutrino is its own anti-particle, a fact more transparent in the 4- rather than the two-component notation. The above exercise illustrates an important point i.e. given any two component spinor, one can always write a self conjugate (or Majorana) 4-component spinor. Whether a particle is really its own antiparticle or not is therefore determined by its interactions. To see this for the electrons, one may solve the following exercise i.e. if we wrote two Majorana spinors using the two two component spinors that describe the charged fermion (electron), then until we turn on the electromagnetic interactions and the mass term, we will not know whether the electron is its own antiparticle or not. Once we turn on the electromagnetism, this ambiguity is resolved.

Let us now go one step further and consider two 2-component neutrinos (\(\nu_1, \nu_2\)). The general mass matrix for this case is given by:

\[ M_{2 \times 2} = \begin{pmatrix} m_1 & m_3 \\ m_3 & m_2 \end{pmatrix} \]  

(13)

Note first that this is a symmetric matrix and can be diagonalized by orthogonal transformations. The eigenstates which will be certain admixtures of the original neutrinos now describe self conjugate particles. One can look at some special cases:

**Case i:**

If we have \(m_{1,2} = 0\) and \(m_3 \neq 0\), then one can assign a charge +1 to \(\nu_1\) and -1 to \(\nu_2\) under some \(U(1)\) symmetry other than electromagnetism and the theory is invariant under this extra \(U(1)\) symmetry which can be identified as the lepton number and the particle is then called a Dirac neutrino. The point to be noted is that the Dirac neutrino is a special case of the Majorana neutrinos. In fact if we insisted on calling this case one with two Majorana neutrinos, then the two will have equal and opposite (in sign) mass as can be seen diagonalizing the above mass matrix. Thus a Dirac neutrino can be thought of as two Majorana neutrinos with equal and opposite (in sign) masses. Since the argument of a complex mass term in general refers to its C transformation property (i.e. \(\psi^c = e^{i\delta_m} \psi\), where \(\delta_m\) is the phase of the complex mass term), the two two component fields of a Dirac neutrino have opposite charge conjugation properties.

**Case ii:**

If we have \(m_{1,2} \ll m_3\), this case is called pseudo-Dirac neutrino since this is a slight departure from case (i). In reality, in this case also the neutrinos are Majorana neutrinos with their masses \(\pm m_0 + \delta\) with \(\delta \ll m_0\). The two component neutrinos will be maximally mixed. Thus this case is of great current physical interest in view of the atmospheric (and perhaps solar) neutrino data.

**Case iii:**

There is third case where one may have \(m_1 = 0\) and \(m_3 \ll m_2\). In this case the eigenvalues of the neutrino mass matrix are given respectively by: \(m_{\nu} \simeq -\frac{m_3^2}{m_2}\) and \(M \simeq m_2\). One may wonder under what conditions such a situation may arise in a realistic gauge model.
out that if $\nu_1$ transforms as an $SU(2)_L$ doublet and $\nu_2$ is an $SU(2)_L$ singlet, then the value of $m_3$ is limited by the weak scale whereas $m_2$ has no such limit and $m_1 = 0$ if the theory has no $SU(2)_L$ triplet field (as for instance is the case in the standard model). Choosing $m_2 \gg m_3$ then provides a natural way to understand the smallness of the neutrino masses. This is known as the seesaw mechanism. Since this case is very different from the case (i) and (ii), it is generally said that in grand unified theories, one expects the neutrinos to be Majorana particles. The reason is that in most grand unified theories there is a higher scale which under appropriate situations provides a natural home for the large mass $m_2$.

While we have so far used only two neutrinos to exemplify the various cases including the seesaw mechanism, these discussions generalize when $m_{1,2,3}$ are each $N \times N$ matrices (which we denote by $M_{1,2,3}$). For example, the seesaw formula for this general situation can be written as

$$M_{\nu} \simeq -M_D^T M_R^{-1} M_D \quad (14)$$

where the subscripts $D$ and $R$ are used in anticipation of their origin in gauge theories where $M_D$ turns out to be the Dirac matrix and $M_R$ is the mass matrix of the right handed neutrinos and all eigenvalues of $M_R$ are much larger than the elements of $M_D$. It is also worth pointing out that Eq. (13) can be written in a more general form where the Dirac matrices are not necessarily square matrices but $N \times M$ matrices with $N \neq M$. We give such examples below.

Although there is no experimental proof that the neutrino is a Majorana particle, the general opinion is that since the seesaw mechanism provides such a simple way to understand the glaring differences between the masses of the neutrinos and the charged fermions and since it implies that the neutrinos are Majorana fermions, that they indeed are most likely to be Majorana particles.

Even though for most situations, the neutrino can be treated as a two component object regardless of whether its mass is of Dirac or Majorana type, there are certain practical situations where differences between the Majorana and Dirac neutrino becomes explicit: one case is when the two neutrinos annihilate. For Dirac neutrinos, the particle and the antiparticle are distinct and therefore there annihilation is not restricted by Pauli principle in any manner. However, for the case of Majorana neutrinos, the identity of neutrinos and antineutrinos plays an important role and one finds that the annihilation to the $Z$-bosons occurs only via the P-waves. Similarly in the decay of the neutrino to any final state, the decay rate for the Majorana neutrino is a factor of two higher than for the Dirac neutrino.

**Exercise 2**: Derive explicit expressions for the four component momentum dependent spinors $u_s(p)$ and $v_s(p)$ when $\psi$ is a Majorana spinor.

### I.3 Experimental indications for neutrino masses

There have been other lectures at this school on the experimental evidences for neutrino masses and their analyses to determine the current favorite values for the various madd differences as well as mixing angles. I will therefore only summarize the main points that are relevant for our present understanding of neutrino masses and for the sake of completeness. (For detailed discussion, see [1] and lectures by Akhmedov, Fogli, Lipari).
At present there is no conclusive evidence for neutrino masses from direct search experiments for neutrino masses using tritium beta decay and neutrinoless double beta decay (see later).

There are however many experiments that have measured the flux of neutrinos from the Sun and the cosmic rays and which have provided clear evidences for neutrino oscillations i.e. neutrinos of one flavor transmuting to neutrinos of another flavor. Since such transmutation can occur only if the neutrinos have masses and mixings, these experiments provide evidence for neutrino mass. The expression for vacuum oscillation probability for neutrinos of a given energy $E$ that have travelled a distance $L$ is given by:

$$P_{\alpha\beta} = \sum_{i,j} |U_{\alpha i} U_{\beta i}^* U_{\alpha j}^* U_{\beta j}| \cos \left( \frac{\Delta_{ij} L}{2E} - \phi_{\alpha\beta,ij} \right)$$ (15)

From this it is clear that neutrino oscillation data yields information about the mass difference squares of the neutrinos ($\Delta_{ij} = m_i^2 - m_j^2$) and mixing angles $U_{\alpha i}$. If the neutrino propagates in dense matter, Mikheyev-Smirnov-Wolfenstein effect will change this formula but it also depends on the same parameters mass difference square and the $U_{\alpha i}$. The analysis of the data for the atmospheric neutrinos where the neutrino propagates in vacuum and that for solar neutrinos where the effect of dense matter in the Sun is important therefore leads to the following picture for masses and mixings:

Atmospheric neutrinos:

in the Super-Kamiokande experiment [8] which confirms the indications of oscillations in earlier data from the Kamiokande, IMB experiments. More recent data from Soudan II and MACRO experiments provide further confirmation of this evidence. The observation here is the following: in the standard model with massless neutrinos, all the muon and electron neutrinos produced at the top of the atmosphere would be expected to reach detectors on the earth and would be isotropic; what has been observed is that while that is true for the electron neutrinos, the muon neutrino flux observed on earth exhibit a strong zenith angle dependence. A simple way to understand this would be to assume that the muon neutrinos oscillate into another undetected species of neutrino on their way to the earth, with a characteristic oscillation length of order of ten thousand kilometers. Since the oscillation length is roughly given by $E(GeV)/\Delta m^2(eV^2)$ kilometers, for a GeV neutrino, one would expect the particle physics parameter $\Delta m^2$ corresponding to the mass difference between the two neutrinos to be around $10^{-3}$ eV$^2$ corresponding to maximal mixing.

From the existing data several important conclusions can be drawn: (i) the data cannot be fit assuming oscillation between $\nu_\mu$ and $\nu_\tau$ nor $\nu_\mu - \nu_s$, where $\nu_s$ is a sterile neutrino which does not any direct weak interaction; (ii) the oscillation scenario that fit the data best is $\nu_\mu - \nu_\tau$ for the mass and mixing parameters

$$\Delta m_{\mu-\tau}^2 \simeq (2 - 8) \times 10^{-3} \text{ eV}^2;$$

$$\sin^2 2\theta_{\mu-\tau} \simeq 0.8 - 1$$ (16)

Solar neutrinos

The second evidence for neutrino oscillation comes from the seven experiments that have observed a deficit in the flux of neutrinos from the Sun as compared to the predictions of the standard solar model championed by Bahcall and his collaborators [6] and more recently
studied by many groups. The experiments responsible for this discovery are the Chlorine, Kamiokande, Gallex, SAGE, Super-Kamiokande, SNO, GNO \[7\] experiments conducted at the Homestake mine, Kamioka in Japan, Gran Sasso in Italy and Baksan in Russia and Sudbery in Canada. The different experiments see different parts of the solar neutrino spectrum. The details of these considerations are discussed in other lectures. The oscillation interpretation of the solar neutrino deficit has more facets to it than the atmospheric case: first the final state particle that the $\nu_e$ oscillates into and second what kind of $\Delta m^2$ and mixings fit the data. At the moment there is a multitude of possibilities. Let us summarize them now.

As far as the final state goes, it can either be one of the two remaining active neutrinos, $\nu_\mu$ and $\nu_\tau$ or it can be the sterile neutrino $\nu_s$. SNO neutral current data announced recently has very strongly constrained the second possibility (i.e. the sterile neutrino in the final state). The global analyses of all solar neutrino data seem to favor the so called large mixing angle MSW solution with parameters: $\Delta m^2 \simeq 1.2 \times 10^{-5} - 3.1 \times 10^{-4} eV^2$; $\sin^2 2\theta \simeq 0.58 - 0.95$.

Finally, we come to the last indication of neutrino oscillation from the Los Alamos Liquid Scintillation Detector (LSND) experiment \[13\], where neutrino oscillations both from a stopped muon (DAR) as well as the one accompanying the muon in pion decay (known as the DIF) have been observed. The evidence from the DAR is statistically more significant and is an oscillation from $\bar{\nu}_\mu$ to $\bar{\nu}_e$. The mass and mixing parameter range that fits data is:

$$LSND : \Delta m^2 \simeq 0.2 - 2 eV^2; \sin^2 2\theta \simeq 0.003 - 0.03$$

(17)

There are also points at higher masses specifically at 6 eV$^2$ which are also allowed by the present LSND data for small mixings. KARMEN experiment at the Rutherford laboratory has very strongly constrained the allowed parameter range of the LSND data \[14\]. Currently the Miniboone experiment at Fermilab is under way to probe the LSND parameter region \[15\].

**Neutrinoless double beta decay and Tritium decay experiment**

Oscillation experiments only depend on the difference of mass squares of the different neutrinos and the mixing angles. Therefore, in order to have a complete picture of neutrino masses, we need other experiments. Two such experiments are the neutrinoless double beta decay searches and the search for neutrino mass from the analysis of the end point of the electron energy spectrum in tritium beta decay.

Neutrinoless double beta decay measures the following combination of masses and mixing angles:

$$< m >_{\beta\beta} = \sum_i U^2_{ei} m_i$$

(18)

Therefore naively speaking it is sensitive to the overall neutrino mass scale. But in practice, as we will see below, for the case of both normal and inverted hierarchies, it is unlikely to settle the question of the overall mass scale at the presently contemplated level of sensitivity in double beta decay searches. Only if the neutrino mass patterns are hierarchical does one expect a visible signal in $\beta\beta_0$ decay. We do not get into great details into this issue except to mention that in drawing any conclusions about neutrino mass from this process, one has to first have a good calculation of nuclear matrix elements of the various nuclei involves
such as $^{76}$Ge, $^{136}$Xe, $^{100}$Mo etc.; secondly, another confusing issue has to do with alternative physics contributions to $\beta\beta_{0\nu}$ which are unrelated to neutrino mass. Nevertheless, neutrinoless double beta decay is a fundamental experiment and a nonzero signal will establish a fundamental result that neutrino is a Majorana particle and that lepton number symmetry is violated. Regardless of whether it tells us anything about the neutrino masses, it would provide a fundamental new revelation about physics beyond the standard model. Presently two experiments Heidelberg-Moscow and IGEX that use enriched $^{76}$Ge have published limits of $\leq 0.3$ eV [10]. More recently, evidence for a double beta signal in the Heidelberg-Moscow data has been claimed [11].

Another important result in further understanding of neutrino mass physics could come from the tritium end point searches for neutrino masses. This experiment will measure the parameter $m_\nu = \sqrt{\sum_i |U_{ei}|^2 m_i^2}$. This involves a different combination of masses and mixing angles than $< m >_{\beta\beta}$. Presently, the KATRIN proposal for a high sensitive search for for $m_\nu$ has been made and it is expected that it can reach a sensitivity of 0.3 eV.

A third source of information on neutrino mass will come from cosmology, where more detailed study of structure in the universe is expected to provide an upper limit on $\sum_i m_i$ of less than an eV.

Our goal now is to study the theoretical implications of these discoveries. We will proceed towards this goal in the following manner: we will isolate the mass patterns that fit the above data and then look for plausible models that can first lead to the general feature that neutrinos have tiny masses; then we would try to understand in simple manner some of the features indicated by data in the hope that these general ideas will be part of our final understanding of the neutrino masses. As mentioned earlier on, to understand the neutrino masses one has to go beyond the standard model. First we will sharpen what we mean by this statement. Then we will present some ideas which may form the basic framework for constructing the detailed models.

\section*{I.4 Patterns and textures for neutrinos}

As already mentioned, we will assume two component neutrinos and therefore their masses will in general be Majorana type. Let us also give our notation to facilitate further discussion: the neutrinos emitted in weak processes such as the beta decay or muon decay are weak eigenstates and are not mass eigenstates. The mass eigenstates determine how a neutrino state evolves in time. Similarly, in the detection process, it is the weak eigenstate that is picked out. This is of course the key idea behind neutrino oscillation and the formula presented in the last section. To set the notation, let us express the weak eigenstates in terms of the mass eigenstates. We will denote the weak eigenstate by the symbol $\alpha, \beta$ or simply $e, \mu, \tau$ etc whereas the mass eigenstate will be denoted by the symbols $i, j, k$ etc. To relate the weak eigenstates to the mass eigenstates, let us start with the mass terms in the Lagrangian for the neutrino and the charged leptons:

$$\mathcal{L}_m = \nu_L^\dagger M \nu_L + E_L M E_R + h.c.$$  \hspace{1cm} (19)

Here the $\nu$ and $E$ which denote the column vectors for neutrinos and charged leptons are in the weak basis. To go to the mass basis, we diagonalize these matrices as follows:
The physical neutrino mixing matrix is then given by:

$$U = V_L U_L$$  \quad (21)$$

$U_{\alpha i}$ and relate the two sets of eigenstates (weak and mass) as follows:

$$\begin{pmatrix}
\nu_e \\
\nu_\mu \\
\nu_\tau
\end{pmatrix} = U
\begin{pmatrix}
\nu_1 \\
\nu_2 \\
\nu_3
\end{pmatrix} \quad (22)$$

Using this equation, one can derive the well-known oscillation formulae for the survival probability of a particular weak eigenstate $\alpha$ discussed in the previous section.

To see the general structure of the mixing matrix $U$, let us recall that the matrix $M_\nu$ is complex and symmetric and therefore has six complex parameters describing it for the case of three generations. But since the neutrino is described by a complex field, we can redefine the phases of three fields to remove three parameters. That leaves nine parameters. In terms of observables, there are three mass eigenvalues ($m_1, m_2, m_3$) and three mixing angles and phases in the mixing matrix $U$. The three phases can be split into one Dirac phase, which is analogous to the phase in the quark mixing matrix and two Majorana phases. We can then write the matrix $U$ as

$$U = U^{(0)} \begin{pmatrix}
1 \\
e^{i\phi_1} \\
e^{i\phi_2}
\end{pmatrix} \quad (23)$$

The matrix $U^{(0)}$ has three real angles $\theta_{12}, \theta_{23}, \theta_{13}$ and a phase. The goal of experiments is to determine all nine of these parameters. The knowledge of the nine observables allows one to construct the mass matrix for the neutrinos and from there one can go in search of the new physics beyond the standard model that leads to such a mass matrix.

The neutrino mass observables given above can be separated into two classes: (i) oscillation observables and (ii) non-oscillation observables. The first class of observables are those accessible to neutrino oscillation experiments and are the two mass differences $\Delta m^2_\odot$ and $\Delta m^2_A$; three mixing angles $\theta_{12}$ (or $\theta_{1\odot}$); $\theta_{23}$ (or $\theta_A$) and $\theta_{13}$ (the reactor angle, also called $U_{e3}$) and the CP phase $\delta$ in $U^{(0)}$. The remaining three observables which can only be probed by nonoscillation experiments are the lightest mass of the three neutrinos and the two Majorana phases $\phi_{1,2}$.

I.5 Neutrino mixing matrix and mass patterns

Let us first discuss to what extent the oscillation observables are known. Our discussion will focus on the three neutrino fits to all neutrino data which give as central value for $\Delta m^2_A \simeq 0.0025$ eV$^2$; for solar neutrinos, it gives $\Delta m^2_\odot \simeq (2-20) \times 10^{-5}$ eV$^2$. It also provides information on the angles in $U$ which can be summarized by the following mixing matrix (neglecting all CP phases):
\[
U = \begin{pmatrix}
\frac{c}{\sqrt{2}} & \frac{s}{2} & \frac{\epsilon}{\sqrt{2}} \\
\frac{s + cs}{\sqrt{2}} & \frac{c - sc}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\
\frac{s - cs}{\sqrt{2}} & \frac{-c - sc}{\sqrt{2}} & \frac{1}{\sqrt{2}}
\end{pmatrix}
\]

(24)

where \( \epsilon \leq 0.16 \) from the CHOOZ and PALO-VERDE reactor experiments \[12\]. \( s \) is the solar neutrino mixing angle. Present experiments allow the range \( 0.6 \leq \sin^2 2\theta_\odot \leq 0.96 \) with the central being near 0.8. The atmospheric mixing angle \( \theta_A \) is close to maximal i.e. \( \sin^2 2\theta_A \simeq 0.8 - 1 \).

As far as the mass pattern goes however, there are three possibilities:

- (i) normal hierarchy: \( m_1 \ll m_2 \ll m_3 \);
- (ii) inverted hierarchy: \( m_1 \simeq -m_2 \gg m_3 \) and
- (iii) approximately degenerate pattern \[10\] \( m_1 \simeq m_2 \simeq m_3 \),

where \( m_i \) are the eigenvalues of the neutrino mass matrix. In the first case, the atmospheric and the solar neutrino data give direct information on \( m_3 \) and \( m_2 \) respectively.

On the other hand, in the last case, the mass differences between the first and the second eigenvalues will be chosen to fit the solar neutrino data and the second and the third to fit the atmospheric neutrino data.

Since Majorana masses violate lepton number, a very important constraint on any discussion of neutrino mass patterns arises from the negative searches for neutrinoless double beta decay \[9\]. The most stringent present limits are obtained from the Heidelberg-Moscow enriched Germanium-76 experiment at Gran Sasso and implies an upper limit on the following combination of masses and mixings:

\[
< m_\nu > \equiv \sum_i U_{e i}^2 m_{\nu_i} \leq 0.35 \text{ eV} \quad 95\% \text{ c.l.}
\]

(25)

This upper limit depends on the nuclear matrix element calculated by the Heidelberg group \[9\]. There could be an uncertainty of a factor of two in this estimate. This would then relax the above upper bound to at most 0.7 eV in the worst case scenario. This is still a very useful limit and becomes especially relevant when one considers whether the neutrinos constitute a significant fraction of the hot dark matter of the universe. A useful working formula is \( \sum_i m_{\nu_i} \simeq 24\Omega_\nu \text{ eV} \) where \( \Omega_\nu \) is the neutrino fraction that contributes to the dark matter of the universe. For instance, if the neutrino contribution to dark matter fraction is 20%, then the sum total of neutrino masses must be 4.8 eV. Such large values are apparently in disagreement with present upper limits from structure surveys such as 2dF survey and others.

Neutrinoless double beta decay limits also imply very stringent constraints on the mixing pattern in the degenerate case.

The inverted hierarchy case (ii) is quite an interesting one and will be discussed from a theoretical perspective in more detail later on; but at the moment we simply note that in this case the value of \( m_1 \) is nothing but the \( \sqrt{\Delta m^2_\beta} \). The solar mass difference is an additional parameter in the mass matrix.

At this point, it is appropriate to stress the theoretical challenges raised by the existing neutrino oscillation data.
- **Ultralight neutrinos**: Why are the neutrino masses so much lighter than the quark and charged lepton masses?

- **Near bimaximal mixing**: How to understand simultaneously two large mixing angles one for the $\mu - \tau$ and another for $e - \mu$?

- **Smallness of $\Delta m^2_\odot / \Delta m^2_A$**: Experimentally, $\Delta m^2_\odot \simeq 10^{-2} \Delta m^2_A$. How does one understand this in a natural manner?

- **Smallness of $U_{e3}$**: The reactor results also seem to indicate that the angle $\theta_{13} \equiv U_{e3}$ is a very small number. One must also understand this in a framework that simultaneously explains all other puzzles.

Possible other puzzles include a proper understanding of neutrino mass degeneracy if there is a large positive signal for the neutrinoless double beta decay and of course, when we have evidence for CP violating phases in the mass matrix, we must understand their magnitude.

### I.6 Neutrino mass textures

From the mixing matrix in Eq. (24), we can write down the allowed neutrino mass matrix for any arbitrary mass pattern assuming the neutrino is a Majorana fermion. Denoting the matrix elements of $M_\nu$ as $\mu_{\alpha\beta}$ for $\alpha, \beta = 1, 2, 3$, we have (Recall that $\mu_{\alpha\beta} = \mu_{\beta\alpha}$):

$$
\mu_{11} = [c^2 m_1 + s^2 m_2 + \epsilon^2 m_3] \\
\mu_{12} = \frac{1}{\sqrt{2}}[-c(s + \epsilon)m_1 + s(c - \epsilon)m_2 + \epsilon m_3] \\
\mu_{13} = \frac{1}{\sqrt{2}}[-c(s - \epsilon)m_1 - s(c + \epsilon)m_2 + \epsilon m_3] \\
\mu_{22} = \frac{1}{2}[(s + \epsilon)^2 m_1 + (c - \epsilon)^2 m_2 + m_3] \\
\mu_{23} = \frac{1}{2}[-(s^2 - c^2 \epsilon^2)m_1 - (c^2 - s^2 \epsilon^2)m_2 + m_3] \\
\mu_{33} = \frac{1}{2}[(s - \epsilon)^2 m_1 + (c + \epsilon)^2 m_2 + m_3].
$$

One can use Eq. (26) to get information on the nature of the neutrino mass matrix and use it to get clues to the nature of physics beyond the standard model. One technique is to rewrite this mass matrix in a way that reflects some new underlying symmetry of physics beyond the standard model. Below we give some examples of mass matrices that are closely related to this mass matrix but reflect some symmetries of the lepton world. A second utility of Eq. (26) is to search for mass matrices that lead to testable predictions and thereby test models.

As an example consider the case when $\mu_{11} = \mu_{22} = \mu_{33} = 0$. This is the prediction from the Zee model [17]. Using Eq. (26), one can easily see that in the leading order the vanishing of all diagonal entries implies that $\Delta m^2_\odot = 0$ i.e. one must keep higher order terms in $\epsilon$ to get a nonzero $\Delta m^2_\odot$. 

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An elementary way to proceed in this direction is to start with ways to understand the maximal mixing using only two flavors as in the case of the atmospheric neutrinos (i.e. the 2-3 sector of the neutrino flavor). It is well known that if we have a matrix of the form

$$M = \begin{pmatrix} A & B \\ B & A \end{pmatrix}$$

then, its eigenstates are maximal admixtures of the original “flavor” states. The eigenvalues are \((A + B), (A - B)\). It is now clear that if we want one of the eigenvalues to be much less than the other, we must have \(A \approx B\). We will now have to generalize this discussion to the case of three generations. For this we note that, the “mixing matrix” for this case can be written as:

$$U = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

We will call this matrix “maximal mixing matrix”.

*Theorem:* The maximal mixing matrix for \(N\)-generations, if \(N\) is a prime number is given by:

$$U_N = \frac{1}{\sqrt{N}} \begin{pmatrix} 1 & 1 & 1 & \cdots \\ 1 & z & z^2 & z^3 & \cdots \\ 1 & z^2 & z^4 & \cdots \\ 1 & z^3 & z^6 & \cdots \\ 1 & z^4 & z^8 & \cdots \end{pmatrix}$$

(29)

where \(z\) is the \(N\)-th root of unity and the rows and extend in an obvious manner to make the matrix \(N \times N\). Note that the general form of an arbitrary element is \(z^{pq}\). When \(pq \geq N\), the power is simply given by \(pq - N\). When the number of generations is not a prime number, then \(N = N_1 \cdot N_2 \cdot N_3 \cdots\). In this case,

$$U_N = U_{N_1} \times U_{N_2} \times U_{N_3} \times \cdots$$

(30)

where \(\times\) stands for a direct product. For example for the case of \(N = 4\), we have for the maximal mixing matrix \(U_4 = U_2 \times U_2\) which is given by

$$U_4 = \frac{1}{2} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix}$$

(31)

The four by four case turns out to be interesting since by decoupling a linear combination of the states from the \(4 \times 4\) system, we can get the exact bimaximal form \[19\]. The corresponding \(3 \times 3\) mass matrix can also be gotten by starting from this discussion. To see how one gets that, note that the matrix in Eq.(24) for the \(2 \times 2\) case has a \(S_{12}\) symmetry. The corresponding symmetry in the \(4 \times 4\) case is then \(S_{12} \times S_{34}\). Using this property and decoupling one linear combination of the fields, we can get the mass matrix for the \(3 \times 3\) case that leads to bimaximal mixing. It is left as an exercise to the reader to show that the corresponding mass matrix is given by: \[19\]:

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\[
M_\nu = \begin{pmatrix}
A + D & F & F \\
F & A & D \\
F & D & A
\end{pmatrix}
\] (32)

Since the present data implies that there are deviations from the exact bimaximal form, this mass matrix must only be treated as a leading order contribution.

The three different mass patterns can emerge from this mass matrix in various limits: e.g. (i) for \( F \ll A \simeq -D \), one gets the normal hierarchy; (ii) for \( F \gg A, D \), one has the inverted pattern for masses and (iii) the parameter region \( F, D \ll A \) leads to the degenerate case. Clearly this mass matrix is the leading order matrix and there will have to be small corrections to this to fit solar neutrino data as well as any possible evidence (or lack of it) for neutrinoless double beta decay. An interesting symmetry of this mass matrix is the \( \nu_\mu \leftrightarrow \nu_\tau \) interchange symmetry, which is obvious from the matrix; but in the limit where \( A = D = 0 \), there appears a much more interesting symmetry i.e. the continuous symmetry \( L_e - L_\mu - L_\tau \) [20]. If the inverted mass matrix is confirmed by future experiments, this symmetry will provide an important clue to new neutrino related physics beyond the standard model (for an example see [21]).

There are other ways to proceed towards the same goal. One way inspired by the studies of quark mass matrices is to consider mass matrices with zeros in it and hope that there are sensible symmetries that will guarantee the zeros. One may then be able to obtain relations between different observables such as masses and mixing angles that can be tested in experiments.

To proceed towards this goal, let us recall that in the absence of CP violation, the neutrino mass matrix has six independent entries. In terms of observables there are five oscillation observables two mass differences \( (\Delta m^2_\odot, \Delta m^2_A) \), and three mixing angles solar, atmospheric and the reactor angle \( U^2_{e3} \). So if we have more than one vanishing element, we will have nontrivial relations between oscillation observables which can be used as tests of the various ansätze.

First point to note is that if there are three zeros or more, the mass matrix cannot describe data. The proof of this is left as an exercise to the reader.

**Exercise** Show that the following “three zero” mass matrix \( M_{3\text{-zero}} = \begin{pmatrix} 0 & a & b \\ a & 0 & c \\ b & c & 0 \end{pmatrix} \) predicts \( \sin^2 2\theta_\odot = 1 - \frac{1}{16} \frac{\Delta m^2_\odot}{\Delta m^2_A} \) and is therefore ruled out by present neutrino data.

Let us therefore consider some typical two zero mass matrices [22]:

\[
M_\nu = \sqrt{\Delta m^2_A} \begin{pmatrix}
0 & 0 & d\epsilon \\
0 & 1 + a\epsilon & 1 \\
d\epsilon & 1 & 1 + b\epsilon
\end{pmatrix}
\] (33)

or alternatively

\[
M_\nu = \sqrt{\Delta m^2_A} \begin{pmatrix}
0 & d\epsilon & 0 \\
d\epsilon & 1 + a\epsilon & 1 \\
0 & 1 & 1 + b\epsilon
\end{pmatrix}
\] (34)
They give a four parameter fit to all data and yield a relation between $\Delta m_\odot^2 / \Delta m_A^2$, $U_{e3}$, $\theta_\odot$ and $\theta_A$ [22].

$$U_{e3}^2 \cos 2\theta_\odot = \frac{\sin^2 2\theta_\odot \Delta m_\odot^2}{4 \Delta m_A^2}$$

This relation predicts that $U_{e3} \geq 0.12$ and can be tested in planned long baseline experiments to be conducted in near future.

There are also some other two zero textures that predict a degenerate spectrum for neutrinos.

So far we have focussed on studying the form of the neutrino mass matrix. The implicit assumption in these discussions has been that the charged lepton mass matrix is diagonal. It may however be that all neutrino mixings are a reflection of structure in the charged lepton mass matrix rather than in the neutrinos. An example of this kind is the democratic mass matrix [24] which also leads to a near bimaximal mixing. As was noted in Ref. [25], in order to get the democratic mixing matrix, one must choose the charged lepton mass matrix in the following form while keeping the neutrino mass matrix diagonal:

$$M_\ell = \begin{pmatrix} a & 1 & 1 \\ 1 & a & 1 \\ 1 & 1 & a \end{pmatrix}$$

This form can be derived from a permutation symmetry $S_3$ operating on the lepton doublets, which may provide another clue to possible gauge model building.

Other models that use charged lepton sector to understand neutrino has been the focus of several papers [26], specially in the context of grand unified theories [26]. We do not discuss this class of models here.

**I.7: Inverted hierarchy and $L_e - L_\mu - L_\tau$ symmetry**

In this section, we consider another interesting clue to model building present in neutrino data if the mass arrangement is inverted. As already noted, if in eq.33, we set $A = D = 0$, this leads to a neutrino mass matrix with two degenerate neutrinos with mass $\pm \sqrt{2} F$ and one massless neutrino. The atmospheric mass difference is given by $\Delta m_A^2 = 2 F^2$ and mixing angle $\theta_A = \pi/4$. As far as the solar $\nu_e$ oscillation is concerned, the $\sin^2 2\theta_\odot = 1$ but $\Delta m_\odot^2 = 0$. While this is unphysical, this raises the hope that as corrections to this mass matrix are taken into account, it may be possible understand the smallness of $\Delta m_\odot^2 / \Delta m_A^2$ naturally.

In fact this hope is fortified by the observation that the $A = D = 0$ limit of the mass matrix in ref.33 has the leptonic symmetry $L_e - L_\mu - L_\tau$; therefore one might hope that as this symmetry is broken by small terms, one will end up with a situation that fits data well.

This question was studied in two papers [27,68]. To proceed with the discussion, let us consider the following mass matrix for neutrinos where small $L_e - L_\mu - L_\tau$ violating terms have been added.

$$\mathcal{M}_\nu = m \begin{pmatrix} z & c & s \\ c & y & d \\ s & d & x \end{pmatrix}.$$
The charged lepton mass matrix is chosen to have a diagonal form in this basis and $L_e - L_\mu - L_\tau$ symmetric.

In the perturbative approximation, we find the following sumrules involving the neutrino observables and the elements of the neutrino mass matrix. The two obvious relations are

$$\sin^2 2\theta_A = \sin^2 2\theta + O(\delta^2)$$

$$D_3 \equiv \Delta m^2_A = -m^2 + 2\Delta m^2_\odot + O(\delta^2)$$

(38)

The nontrivial relations that also hold for this model are:

$$\sin^2 2\theta_\odot = 1 - \left(\frac{\Delta m^2_\odot}{4\Delta m^2_A} - z\right)^2 + O(\delta^3)$$

$$\frac{\Delta m^2_\odot}{\Delta m^2_A} = 2(z + \vec{v} \cdot \vec{x}) + O(\delta^2)$$

$$U_{e3} = \vec{A} \cdot (\vec{v} \times \vec{x}) + O(\delta^3)$$

(39)

where $\vec{v} = (\cos^2 \theta, \sin^2 \theta, \sqrt{2} \sin \theta \cos \theta)$, $\vec{x} = (x, y, \sqrt{2}d)$ and $\vec{A} = \frac{1}{\sqrt{2}}(1, 1, 0)$. $\delta$ in the preceding equations represents the small parameters in the mass matrix. These equations represent one of the main results of this paper. Below we study their implications. Finally, there is the relation $\langle m \rangle_{\beta\beta} = mz$. This is an an exact relation true to all orders in the small parameters.

One of the major consequences of these relations is that (i) there is a close connection between the measured value of the solar mixing angle and the neutrino mass measured in neutrinoless double beta decay; (ii) the present values for the solar mixing angle can be used to predict the $m_{\beta\beta}$ for a value of the $\Delta m^2_\odot$. For instance, for $\sin^2 2\theta_\odot = 0.9$, we would predict $(\frac{\Delta m^2_\odot}{\Delta m^2_A} - z) = 0.3$. For small $\Delta m^2_\odot$, this implies $m_{\beta\beta} \simeq 0.01$ eV. The second relation involving the $\Delta m^2_\odot/\Delta m^2_A$ in terms of $x, y, z, d$ tells us that for this to be the case, we must have strong cancellation between the various small parameters. Given this, the above $m_{\beta\beta}$ value is expected to be within the reach of new double beta decay experiments contemplated [29]. Note however that the $\sin^2 2\theta_\odot$ cannot be larger than 0.9 in the case of approximate $L_e - L_\mu - L_\tau$ symmetry.

If the value of $\sin^2 2\theta_\odot$ is ultimately determined to be less than 0.9, the question one may ask is whether the idea of $L_e - L_\mu - L_\tau$ symmetry is dead. The answer is in the negative since so far we have explored the breaking of $L_e - L_\mu - L_\tau$ symmetry only in the neutrino mass matrix. It was shown in [27] that if the symmetry is broken in the charged lepton mass, one can lower the $\sin^2 2\theta_\odot$ as long as the value of $U_{e3}$ is sizable. However given the present upper limit on $U_{e3}$, the smallest value is somewhere around $\sin^2 2\theta_\odot \simeq 0.8$.

### 1.8 CP violation

A not very well explored aspect of neutrino physics at the moment is CP violation in lepton physics. Unlike the quark sector, CP violation for Majorana neutrinos allows for more phases for neutrinos. Since the Majorana neutrino mass matrix is symmetric, for $N$ generations of neutrinos, there are in general $\frac{N(N+1)}{2}$ phases in it. When the mass matrix is
diagonalized, these phases will appear in the unitary matrix $U_L$ that does the diagonalization (i.e. $U^T - M_\nu U_L = d_\nu$). If we are working in a basis where the charged lepton mass matrix is diagonal, then $U_L$ leptonic weak mixing matrix. As we saw this has $N(N+1)/2$ phases. Out of them, redefinition of the charged lepton fields in the weak current allows the removal of $N$ phases; so there are $N(N-1)/2$ phases in the neutrino masses. In the quark sector, both up and down fields could be redefined allowing for the number of physical phases that appear in the end to be smaller. However for Majorana neutrinos, redefinition of the fields does not remove the phases entirely from the theory but rather shifts them to other places where they can manifest themselves physically \[30\].

Thus for two generations there is one and for three generations there are 3 phases. A convenient parameterization of the mixing matrix with these phases in the mixing matrix for the case of three generations is $U_K$:

$$U = \begin{pmatrix} c & s & \epsilon e^{-i\delta} \\ \frac{s + c\epsilon e^{i\delta}}{\sqrt{2}} & \frac{c - s\epsilon e^{i\delta}}{\sqrt{2}} & 1 \\ \frac{s - c\epsilon e^{i\delta}}{\sqrt{2}} & \frac{-c - s\epsilon e^{i\delta}}{\sqrt{2}} & 1 \end{pmatrix} \quad (40)$$

and

$$K = Diag(1, e^{i\phi_1}, e^{i\phi_2}) \quad (41)$$

The phase $\delta$ is called the Dirac phase and the $\phi_i$ are called Majorana phases. Note that the Dirac phase is always multiplied by the small mixing angle $\epsilon$. Its measurability is therefore very closely tied to the absolute magnitude of $\epsilon$. Coming to the Majorana phases, one of the two phases $\phi_i$ can in principle be probed in neutrinoless double beta decay. To see this, let us note that

$$<m>_{\beta\beta} = |m_1e^2 + m_2s^2e^{i\phi_1} + m_3\epsilon^2e^{-i(\delta - \phi_2)}| \quad (42)$$

Since $\epsilon \ll 1$, the last term in the above equation can be dropped. It is then easy to see that one has some chance of seeing the CP phase $\phi_1$, once one has a precise knowledge of the $\Delta m^2_{31}$ and the solar mixing angle, provided the nuclear matrix elements are known better \[31\]. More optimistically, when the phase is zero, the two terms in the expression for $\beta\beta_0\nu$ add up and the chances for seeing it is enhanced.

On the whole though, there is some chance of measuring the CP phase both for the inverted and the degenerate mass case provided the nuclear matrix elements have much smaller uncertainties than presently known, whereas for the case of normal hierarchy, it depends on how small the smallest neutrino mass is. If it is very close to $m_2$, the so called quasi-degenerate case, then one has a good chance to measure the phase if the $<m>_{\beta\beta}$ is measured to a precision of 0.001 eV.

A very interesting question relates to the observability of truly CP violating leptonic processes. This question has been addressed in \[32\]. For instance although one can probe the CP phase, it is not really a CP violating process; in other words, even in the presence of CP violation, the rate for neutrinoless double beta decay of a nucleus is same as that for the corresponding anti-nucleus. On the other hand, there are genuine CP violating processes where the CP phase can be probed. One is the celebrated example of early universe leptogenesis where one looks at the decay of the Majorana right handed neutrino.
into $\ell + H$ and $\bar{\ell} + H$ and it is their difference that manifests as the lepton asymmetry. Similarly, one can look at rare decays of $K^\pm$ to $\pi^\pm + \mu^\pm + \mu^\pm$ and similar decay modes for the $B$ meson, where the presence of a physical intermediate state leads to the observability of a truly CP violating difference between decay rates.
II. Why neutrino mass requires physics beyond the standard model?

We will now show that in the standard model, the neutrino mass vanishes to all orders in perturbation theory as well as nonperturbatively. The standard model is based on the gauge group $SU(3)_c \times SU(2)_L \times U(1)_Y$ group under which the quarks and leptons transform as described in the Table I.

| Field                        | gauge transformation |
|------------------------------|----------------------|
| Quarks $Q_L$                 | $(3, 2, \frac{1}{3})$|
| Righthanded up quarks $u_R$  | $(3, 1, \frac{2}{3})$|
| Righthanded down quarks $d_R$| $(3, 1, -\frac{2}{3})$|
| Lefthanded Leptons $L$       | $(1, 2, -1)$         |
| Righthanded leptons $e_R$    | $(1, 1, -2)$         |
| Higgs Boson $H$              | $(1, 2, +1)$         |
| Color Gauge Fields $G_a$     | $(8, 1, 0)$          |
| Weak Gauge Fields $W^\pm, Z, \gamma$ | $(1, 3 + 1, 0)$ |

**Table caption:** The assignment of particles to the standard model gauge group $SU(3)_c \times SU(2)_L \times U(1)_Y$.

The electroweak symmetry $SU(2)_L \times U(1)_Y$ is broken by the vacuum expectation of the Higgs doublet $< H^0 > = v_{weak} \approx 246$ GeV, which gives mass to the gauge bosons and the fermions, all fermions except the neutrino. Thus the neutrino is massless in the standard model, at the tree level. There are several questions that arise at this stage. What happens when one goes beyond the above simple tree level approximation? Secondly, do nonperturbative effects change this tree level result? Finally, how to judge how this result will be modified when the quantum gravity effects are included?

The first and second questions are easily answered by using the B-L symmetry of the standard model. The point is that since the standard model has no $SU(2)_L$ singlet neutrino-like field, the only possible mass terms that are allowed by Lorentz invariance are of the form $\nu^T_i C^{-1} \nu_j L$, where $i, j$ stand for the generation index and $C$ is the Lorentz charge conjugation matrix. Since the $\nu_{iL}$ is part of the $SU(2)_L$ doublet field and has lepton number +1, the above neutrino mass term transforms as an $SU(2)_L$ triplet and furthermore, it violates total lepton number (defined as $L = L_e + L_\mu + L_\tau$) by two units. However, a quick look at the standard model Lagrangian convinces one that the model has exact lepton number symmetry after symmetry breaking; therefore such terms can never arise in perturbation theory. Thus to all orders in perturbation theory, the neutrinos are massless. As far as the nonperturbative effects go, the only known source is the weak instanton effects. Such effects could effect the result if they broke the lepton number symmetry. One way to see if such breaking weak instanton effects. Such effects could effect the result if they broke the lepton number symmetry. One way to see if such breaking occurs is to look for anomalies in lepton number current conservation from triangle diagrams. Indeed $\partial_{\mu} J^\mu_\ell = cW\bar{W} + c'B\bar{B}$ due to the contribution of the leptons to the triangle involving the lepton number current and $W$’s
or $B$’s. Luckily, it turns out that the anomaly contribution to the baryon number current nonconservation has also an identical form, so that the $B - L$ current $j^\mu_{B-L}$ is conserved to all orders in the gauge couplings. As a consequence, nonperturbative effects from the gauge sector cannot induce $B - L$ violation. Since the neutrino mass operator described above violates also $B - L$, this proves that neutrino masses remain zero even in the presence of nonperturbative effects.

Let us now turn to the effect of gravity. Clearly as long as we treat gravity in perturbation theory, the above symmetry arguments hold since all gravity coupling respect $B - L$ symmetry. However, once nonperturbative gravitational effects e.g black holes and worm holes are included, there is no guarantee that global symmetries will be respected in the low energy theory. The intuitive way to appreciate the argument is to note that throwing baryons into a black hole does not lead to any detectable consequence except thru a net change in the baryon number of the universe. Since one can throw in an arbitrary number of baryons into the black hole, an arbitrary information loss about the net number of missing baryons would prevent us from defining a baryon number of the visible universe; thus baryon number in the presence of a black hole can not be an exact symmetry. Similar arguments can be made for any global charge such as lepton number in the standard model. A field theoretic parameterization of this statement is that the effective low energy Lagrangian for the standard model in the presence of black holes and worm holes etc must contain baryon and lepton number violating terms. In the context of the standard model, the only such terms that one can construct are nonrenormalizable terms of the form $LHLH/M_{Pl}$. After gauge symmetry breaking, they lead to neutrino masses; however these masses are at most of order $v^2_{wk}/M_{Pl} \simeq 10^{-5}$ eV [33]. But as we discussed in the previous section, in order to solve the atmospheric neutrino problem, one needs masses at least three orders of magnitude higher.

Thus one must seek physics beyond the standard model to explain observed evidences for neutrino masses. While there are many possibilities that lead to small neutrino masses of both Majorana as well as Dirac kind, here we focus on the possibility that there is a heavy right handed neutrino (or neutrinos) that lead to a small neutrino mass. The resulting mechanism is known as the seesaw mechanism and leads to neutrino being a Majorana particle.

The nature and origin of the seesaw mechanism can also be tested in other experiments and we will discuss them below. This will be dependent on the kind of operators that play a role in generating neutrino masses. If the leading order operator is of dimension 5, then the scale necessarily is very high (of order $10^{12}$ GeV or greater). On the other hand, in theories with extra space dimensions, this operator may be forbidden and one may be forced to go to higher dimensional operators, in which case the scale could be lower. In the lecture IV, an example of five and six dimensional theory is given, where this indeed happens.

The seesaw mechanism raises a very important question: since we require the mass of the right handed neutrino to be much less than the Planck scale, a key question is “what symmetry keeps the right handed neutrino mass lighter?” We will give two examples of symmetries that can do this.

II.2 Seesaw and the right handed neutrino
The simplest possibility extension of the standard model that leads to nonzero mass for the neutrino is one where only a right handed neutrino is added to the standard model. In this case $\nu_L$ and $\nu_R$ can form a mass term; but apriori, this mass term is like the mass terms for charged leptons or quark masses and will therefore involve the weak scale. If we call the corresponding Yukawa coupling to be $Y_\nu$, then the neutrino mass is $m_D = Y_\nu v / \sqrt{2}$. For a neutrino mass in the eV range requires that $Y_\nu \simeq 10^{-11}$ or less. Introduction of such small coupling constants into a theory is generally considered unnatural and a sound theory must find a symmetry reason for such smallness. As already already alluded to before, seesaw mechanism \[5\], where we introduce a singlet Majorana mass term for the right handed neutrino is one way to achieve this goal. What we have in this case is a $(\nu_L, \nu_R)$ mass matrix which has the form:

$$M = \begin{pmatrix} 0 & m_D \\ m_D & M_R \end{pmatrix}$$ \hspace{1cm} (43)

Since $M_R$ is not constrained by the standard model symmetries, it is natural to choose it to be at a scale much higher than the weak scale. Now diagonalizing this mass matrix for a single neutrino species, we get a heavy eigenstate $N_R$ with mass $M_R$ and a light eigenstate $\nu$ with mass $m_\nu \simeq \frac{-m_D^2}{M_R}$. This provides a natural way to understand a small neutrino mass without any unnatural adjustment of parameters of a theory. In a subsequent section, we will discuss a theory which connects the scale $M_R$ to a new symmetry of nature beyond the standard model.

II.2A Why is $M_{\nu_R} \ll M_{Pl}$?

The question “why $M_{\nu_R} \ll M_{Pl}$?” is in many ways similar to the question in the standard model i.e. “why is $M_{Higgs} \ll M_{Pl}$?” It is well known that searches for answer to this question has led us to consider many interesting possibilities for physics beyond the standard model and supersymmetry appears to be the most promising answer to this question. It is hoped that answering this question for $\nu_R$ can also lead us to new insight into new symmetries beyond the standard model. There are two interesting answers to our question that I will elaborate later on.

$B - L$:

If one adds three right handed neutrinos to implement the seesaw mechanism, the model admits an anomaly free new symmetry i.e. $B - L$. One can therefore extend the standard model symmetry to either $SU(2)_L \times U(1)_{I_{3R}} \times U(1)_{B-L}$ or its left-right symmetric extension $SU(2)_L \times SU(2)_R \times U(1)_{B-L}$. In either case the right handed neutrino carries the B-L quantum number and its Majorana mass breaks this symmetry. Therefore, the mass of the $\nu_R$ can at most be the scale of $B - L$ symmetry breaking, hence answering the question “why $M_{\nu_R} \ll M_{Pl}$?”.

$SU(2)_H$:

While local $B - L$ is perhaps the most straight forward and natural symmetry that keeps $\nu_R$ lighter than the Planck scale, another possibility has recently been suggested in ref. \[21\].
The main observation here is that is the standard model is extended by including a local $SU(2)_H$ symmetry acting on the first two lepton generations including the right handed charged leptons, then global Witten anomaly freedom dictates that there must be at least two right handed neutrinos which transform as a doublet under the $SU(2)_H$ local symmetry. In this class of models, in the limit of exact $SU(2)_H$ symmetry, the $\nu_R$’s are massless and as soon as the $SU(2)_H$ symmetry is broken, they pick up mass. Therefore “lightness” of the $\nu_R$’s compared to the Planck scale in these models is related to an $SU(2)_H$ symmetry. These comments are elaborated with explicit examples later on in this review.

II.2B Small neutrino mass using a double seesaw mechanism with $\nu_R$

As we saw from the previous discussion, the conventional seesaw mechanism requires rather high mass for the right handed neutrino and therefore a correspondingly high scale for B-L symmetry breaking. There is however no way at present to know what the scale of B-L symmetry breaking is. There are for example models bases on string compactification [34] where the $B - L$ is quite possibly is in the TeV range. In this case small neutrino mass can be implemented by a double seesaw mechanism suggested in Ref. [35]. The idea is to take a right handed neutrino $N$ and a singlet neutrino $S$ which has extra quantum numbers which prevent it from coupling to the left handed neutrino. One can then write a three by three neutrino mass matrix in the basis $(\nu, N, S)$ of the form:

$$M = \begin{pmatrix} 0 & m_D & 0 \\ m_D & 0 & M \\ 0 & M & \mu \end{pmatrix}$$  \hspace{1cm} (44)

For the case $\mu \ll M \approx M_{B-L}$, (where $M_{B-L}$ is the $B - L$ breaking scale) this matrix has one light and two heavy neutrinos per generation and the latter two form a pseudo-Dirac pair with mass of order $M_{B-L}$. The important thing for us is that the light mass eigenvalue is given by $m^2_D \mu / M^2$; for $m_D \approx \mu \simeq \text{GeV}$, a 10 TeV $B - L$ scale is enough to give neutrino masses in the eV range. For the case of three generations, the formula for the light neutrino mass matrix is given by:

$$\mathcal{M}_\nu = M_D M^{-1} \mu M^{-1} M_D^T$$  \hspace{1cm} (45)

II.3 High mass Higgs triplet induced neutrino masses

As already discussed, one way to generate nonzero neutrino masses without using the righthanded neutrino is to include in the standard model an $SU(2)_L$ triplet Higgs field with $Y = 2$ so that the electric charge profile of the members of the multiplet is given as follows: $(\Delta^{++}, \Delta^+, \Delta^0)$. This allows an additional Yukawa coupling of the form $f_L L^T \tau_2 \tau L \Delta$, where the $\Delta^0$ couples to the neutrinos. Clearly $\Delta$ field has $L = 2$. When $\Delta^0$ field has a nonzero vev, it breaks lepton number by two units and leads to Majorana mass for the neutrinos. There are two questions that arise now: one, how does the vev arise in a model and how does one understand the smallness of the neutrino masses in this scheme. There are two answers to the first question: One can maintain exact lepton number symmetry in the model and generate the vev of the triplet field via the usual “mexican hat” potential. There are
two problems with this case. This leads to the triplet Majoron which has been ruled out by LEP data on Z-width. Though it is now redundant it may be worth pointing out that in this model smallness of the neutrino mass is not naturally understood.

There is however another way to generate the induced vev keeping a large but positive mass \( M_\Delta \) for the triplet Higgs boson and allowing for a lepton number violating coupling \( M_\Delta^* H H \). In this case, minimization of the potential induces a vev for the \( \Delta^0 \) field when the doublet field acquires a vev:

\[
v_T \equiv \langle \Delta^0 \rangle = \frac{M_\nu_{\text{wk}}}{M_\Delta^2} \tag{46}
\]

Since the mass of the \( \Delta \) field is invariant under \( SU(2)_L \times U(1)_Y \), it can be very large connected perhaps with some new scale of physics. If we assume that \( M_\Delta \sim M \sim 10^{13} \text{ GeV} \) or so, we get \( v_T \sim \text{eV} \). Now in the Yukawa coupling \( f_1 L^T \tau_2 \tau L \Delta \), since the \( \Delta^0 \) couples to the neutrinos, its vev leads to a neutrino mass in the eV range or less depending on the value of the Yukawa couplings [36]. We will see later when we discuss the seesaw models that unlike those models, the neutrino mass in this case is not hierarchically dependent on the charged fermion masses. This model is more in the spirit of models with minimal grand unification and can in fact be implemented in models [3] such as those based on the SU(5) group where there is no natural place for the right handed neutrino.

There are however two potential problems with this kind of models: the first one is again a naturalness question and the second a more detailed cosmological one.

**II.3A Naturalness of the Higgs triplet models**

To fit present neutrino data, one would like the neutrino mass scale to be roughly in the eV range. The neutrino mass formula in the triplet Higgs models is \( m_\nu \simeq \frac{v_{\text{wk}}^2}{M_\Delta} \). To get an eV scale mass therefore requires that \( M_\Delta \simeq 10^{13} \text{ GeV} \). The question then arises as to why scalar mass is of order of \( 10^{13} \text{ GeV} \) rather than the Planck scale. It is the same kind of question that leads to the so called gauge hierarchy problem of the standard model. Since the Higgs triplet is a scalar boson, one cannot use chiral symmetry arguments for this purpose. One must necessarily make it part of a supersymmetric theory. This then leads to a problem with baryogenesis that we discuss now.

**II.3B Baryogenesis problem in models without right handed neutrinos**

The simplest scenario for baryogenesis in these models is via leptogenesis. The only possibility [37] here is that the decay of triplet Higgs to leptons provides a way to generate enough baryons in the model. However for that to happen, one must satisfy one of Sakharov’s three conditions for matter-antimatter asymmetry i.e. the decay particle which leads to baryon or lepton asymmetry must be out of equilibrium. This requires that

\[
\frac{f_1^2 M_\Delta}{12 \pi} < \sqrt{g^*} \frac{M_\Delta^2}{M_\pi} \tag{47}
\]

implying a lower limit on the mass \( M_\Delta \geq \frac{f_1^2 M_\pi}{12 \pi \sqrt{g^*}} \). For \( f_1 \sim 10^{-1} \) as would be required by the atmospheric neutrino data, one gets conservatively, \( M_\Delta \geq 10^{13} \text{ GeV} \). The problem with
such a large mass arises from the fact that in an inflationary model of the universe, the
typical reheating temperature dictated by the gravitino problem of supergravity is at most
$10^9$ GeV. Thus there is an inherent conflict between the standard inflationary picture of
the universe and the baryogenesis in the simple triplet model for neutrino masses. For this
model to work therefore, one must invoke a new scenario to resolve the gravitino reheating
problem.

II.4 A gauge model for Seesaw mechanism: left right symmetric unification

Let us now explore the implications of including the righthanded neutrinos into the exten-
sions of the standard model to understand the small neutrino mass by the seesaw mechanism.
As already emphasized, if we assume that there are no new symmetries beyond the stan-
dard model, the right handed neutrino will have a natural mass of order of the Planck scale
making the light neutrino masses too small to be of interest in understanding the observed
oscillations. We must therefore search for new symmetries that can keep the RH neutrinos
at a lower scale than the Planck scale. A new symmetry always helps in making this natural.

To study this question, let us note that the inclusion of the right handed neutrinos
transforms the dynamics of the gauge models in a profound way. To clarify what we mean,
note that in the standard model (that does not contain a $\nu_R$) the $B - L$ symmetry is
only linearly anomaly free i.e. $Tr[(B - L)Q_a^2] = 0$ where $Q_a$ are the gauge generators
of the standard model but $Tr(B - L)^3 \neq 0$. This means that $B - L$ is only a global
symmetry and cannot be gauged. However as soon as the $\nu_R$ is added to the standard
model, one gets $Tr[(B - L)^3] = 0$ implying that the B-L symmetry is now gaugeable and
one could choose the gauge group of nature to be either $SU(2)_L \times U(1)_{I_{3R}} \times U(1)_{B-L}$ or
$SU(2)_L \times SU(2)_R \times U(1)_{B-L}$, the latter being the gauge group of the left-right symmetric
models [38]. Furthermore the presence of the $\nu_R$ makes the model quark lepton symmetric
and leads to a Gell-Mann-Nishijima like formula for the electric charges [39] i.e.

$$Q = I_{3L} + I_{3R} + \frac{B - L}{2}$$

The advantage of this formula over the charge formula in the standard model charge formula
is that in this case all entries have a physical meaning. Furthermore, it leads naturally to
Majorana nature of neutrinos as can be seen by looking at the distance scale where the
$SU(2)_L \times U(1)_Y$ symmetry is valid but the left-right gauge group is broken. In that case,
one gets

$$\Delta Q = 0 = \Delta I_{3L} :$$

$$\Delta I_{3R} = -\Delta \frac{B - L}{2}$$

We see that if the Higgs fields that break the left-right gauge group carry righthanded isospin
of one, one must have $|\Delta L| = 2$ which means that the neutrino mass must be Majorana
type and the theory will break lepton number by two units.

Let us now proceed to give a few details of the left-right symmetric model and demon-
strate how the seesaw mechanism emerges in this model.

The gauge group of the theory is $SU(2)_L \times SU(2)_R \times U(1)_{B-L}$ with quarks and leptons
transforming as doublets under $SU(2)_{L,R}$. In Table 3, we denote the quark, lepton and Higgs
fields in the theory along with their transformation properties under the gauge group.
Table II

| Fields | SU(2) \times SU(2) \times U(1)_{B-L} |
|--------|-------------------------------------|
| Q_L    | (2,1, + \frac{1}{3})              |
| Q_R    | (1,2, \frac{1}{3})               |
| L_L    | (2,1, -1)                        |
| L_R    | (1,2, -1)                        |
| \phi   | (2,2,0)                           |
| \Delta_L | (3,1, +2)                      |
| \Delta_R | (1,3, +2)                      |

Table caption: Assignment of the fermion and Higgs fields to the representation of the left-right symmetry group.

The first task is to specify how the left-right symmetry group breaks to the standard model i.e. how one breaks the SU(2)_R \times U(1)_{B-L} symmetry so that the successes of the standard model including the observed predominant V-A structure of weak interactions at low energies is reproduced. Another question of naturalness that also arises simultaneously is that since the charged fermions and the neutrinos are treated completely symmetrically (quark-lepton symmetry) in this model, how does one understand the smallness of the neutrino masses compared to the other fermion masses.

It turns out that both the above problems of the LR model have a common solution. The process of spontaneous breaking of the SU(2)_R symmetry that suppresses the V+A currents at low energies also solves the problem of ultralight neutrino masses. To see this let us write the Higgs fields explicitly:

\[ \Delta = \begin{pmatrix} \Delta^+/\sqrt{2} & \Delta^{++} \\ \Delta^0 & -\Delta^+/\sqrt{2} \end{pmatrix}; \quad \phi = \begin{pmatrix} \phi_1^0 & \phi_1^+ \\ \phi_1^- & \phi_2^0 \end{pmatrix} \]  

(50)

All these Higgs fields have Yukawa couplings to the fermions given symbolically as below.

\[ \mathcal{L}_Y = h_1 \bar{L}_L \phi L_R + h_2 \bar{L}_L \tilde{\phi} L_R + h'_1 \bar{Q}_L \phi Q_R + h'_2 \bar{Q}_L \tilde{\phi} Q_R + f(L_L L_L \Delta_L + L_R L_R \Delta_R) + \text{h.c.} \]  

(51)

The SU(2)_R \times U(1)_{B-L} is broken down to the standard model hypercharge U(1)_Y by choosing \( < \Delta_0^R > = v_R \neq 0 \) since this carries both SU(2)_R and U(1)_{B-L} quantum numbers. It gives mass to the charged and neutral righthanded gauge bosons i.e. \( M_{W_R} = g v_R \) and \( M_{Z'} = \sqrt{2} g v_R \cos \theta_W / \sqrt{\cos 2 \theta_W} \). Thus by adjusting the value of \( v_R \) one can suppress the right handed current effects in both neutral and charged current interactions arbitrarily leading to an effective near maximal left-handed form for the charged current weak interactions.

The fact that at the same time the neutrino masses also become small can be seen by looking at the form of the Yukawa couplings. Note that the f-term leads to a mass for the right handed neutrinos only at the scale \( v_R \). Next as we break the standard model symmetry by turning on the vev’s for the \( \phi \) fields as Diag \( < \phi > = (\kappa, \kappa') \), we not only give masses
to the $W_L$ and the $Z$ bosons but also to the quarks and the leptons. In the neutrino sector the above Yukawa couplings after $SU(2)_L$ breaking by $<\phi>\neq0$ lead to the so called Dirac masses for the neutrino connecting the left and right handed neutrinos. In the two component neutrino language, this leads to the following mass matrix for the $\nu,N$ (where we have denoted the left handed neutrino by $\nu$ and the right handed component by $N$).

$$M = \begin{pmatrix} 0 & h\kappa \\ h\kappa & f v_R \end{pmatrix} \quad (52)$$

Note that $m_D$ in previous discussions of the seesaw formula (see Eq. (1)) is given by $m_D = h\kappa$, which links it to the weak scale and the mass of the RH neutrinos is given by $M_R = f v_R$, which is linked to the local B-L symmetry. This justifies keeping RH neutrino mass at a scale lower than the Planck mass. It is therefore fair to assume that seesaw mechanism coupled with observations of neutrino oscillations are a strong indication of the existence of a local B-L symmetry far below the Planck scale.

By diagonalizing this $2 \times 2$ matrix, we get the light neutrino eigenvalue to be $m_\nu \simeq \frac{(h\kappa)^2}{f v_R}$ and the heavy one to be $f v_R$. Note that typical charged fermion masses are given by $h \kappa$ etc. So since $v_R \gg \kappa, \kappa'$, the light neutrino mass is automatically suppressed. This way of suppressing the neutrino masses is called the seesaw mechanism [5]. Thus in one stroke, one explains the smallness of the neutrino mass as well as the suppression of the V+A currents.

In deriving the above seesaw formula for neutrino masses, it has been assumed that the vev of the lefthanded triplet is zero so that the $\nu_L \nu_L$ entry of the neutrino mass matrix is zero. However, in most explicit models such as the left-right model which provide an explicit derivation of this formula, there is an induced vev for the $\Delta^0_L$ of order $<\Delta^0_L> = v_T \simeq \frac{v^2}{v_R}$. In the left-right models, this this arises from the presence of a coupling in the Higgs potential of the form $\Delta_L \phi \Delta^0_R \phi^\dagger$. In the presence of the $\Delta_L$ vev, the seesaw formula undergoes a fundamental change. One can have two types of seesaw formulae depending on whether the $\Delta_L$ has vev or not:

**Type I seesaw formula**

$$M_\nu \simeq -M^T_D M^{-1}_{NR} M_D \quad (53)$$

where $M_D$ is the Dirac neutrino mass matrix and $M_{NR} \equiv f v_R$ is the right handed neutrino mass matrix in terms of the $\Delta$ Yukawa coupling matrix $f$.

**Type II seesaw formula**

$$M_\nu \simeq f \frac{v^2_{wk}}{v_R} - M^T_D M^{-1}_{NR} M_D \quad (54)$$

Note that in the type I seesaw formula, what appears is the square of the Dirac neutrino mass matrix which in general expected to have the same hierarchical structure as the corresponding charged fermion mass matrix. In fact in some specific GUT models such as SO(10), $M_D = M_u$. This is the origin of the common statement that neutrino masses given by the
seesaw formula are hierarchical i.e. $m_{\nu_e} \ll m_{\nu_\mu} \ll m_{\nu_\tau}$ and even a more model dependent statement that $m_{\nu_e} : m_{\nu_\mu} : m_{\nu_\tau} = m_u^2 : m_c^2 : m_t^2$.

On the other hand if one uses the type II seesaw formula, there is no reason to expect a hierarchy and in fact if the neutrino masses turn out to be degenerate as discussed before as one possibility, one possible way to understand this may be to use the type II seesaw formula.

Secondly, the type II seesaw formula is a reflection of the parity invariance of the theory at high energies. Evidence for it would point more strongly towards left-right symmetry at high energies.

**II.5 Understanding detailed pattern for neutrinos using the seesaw formula**

Let us now address the question: to what extent one can understand the details of the neutrino masses and mixings using the seesaw formulae. The answer to this question is quite model dependent. While there exist many models which fit the observations, none (except a few) are completely predictive and almost always they need to invoke new symmetries or new assumptions. The problem in general is that the seesaw formula of type I, has 12 parameters in the absence of CP violation (six parameters for a symmetric Dirac mass matrix and six for the $M_R$) which is why its predictive power is so limited. In the presence of CP violation, the number of parameters double making the situation worse. Specific predictions can be made only under additional assumptions.

For instance, in a class of seesaw models based on the SO(10) group that embodies the left-right symmetric unification model or the SU(4)-color, the mass the tau neutrino mass can be estimated provided one assumes the normal mass hierarchy for neutrinos and a certain parameter accompanying a higher dimensional operator to be of order one. To see this, let us assume that in the SO(10) theory, the B-L symmetry is broken by a $16$-dim. Higgs boson. The RH neutrino mass in such a model arises from the nonrenormalizable operator $\lambda(16_F \bar{16}_H)^2/M_{Pl}$. In a supersymmetric theory, if $16$-Higgs is also responsible for GUt symmetry breaking, then after symmetry breaking, one obtains the RH neutrino mass $M_R \simeq \lambda(2 \times 10^{16})^2/M_{Pl} \simeq 4\lambda 10^{14}$ GeV. In models with $SU(4)_c$ symmetry, $m_{\nu_\tau,D} \simeq m_t(M_U) \sim 100$ GeV. Using the seesaw formula then, one obtains for $\lambda = 1$, tau neutrino mass $m_{\nu_\tau} \simeq 0.025$ eV, which is close to the presently preferred value of 0.05 eV. The situation with respect to other neutrino masses is however less certain and here one has to make assumptions.

The situation with respect to mixing angles is much more complicated. For instance, the striking difference between the quark and neutrino mixing angles makes one doubt whether complete quark lepton unification is truly obeyed in nature. In II.8 and III.3, two examples are given where very few assumptions are made in getting maximal atmospheric mixing angle.

**II.6 General consequences of the seesaw formula for neutrino masses**

In this section, we will consider some implications of the seesaw mechanism for understanding neutrino masses. We will discuss two main points. One is the nature of the right handed neutrino spectrum as dictated by the seesaw mechanism and secondly, ways to get an approximate $L_e - L_\mu - L_\tau$ symmetric neutrino mass matrix using the seesaw mechanism and its possible implications for physics beyond the standard model [51].
For this purpose, we use the type I seesaw formula along with the assumption of a diagonal Dirac neutrino mass matrix to obtain the right handed neutrino mass matrix $M_R$:

$$M_{R,ij} = m_{D,i} \mu_{ij}^{-1} m_{D,j}$$

with

$$
\begin{align*}
\mu_{11}^{-1} &= c^2 + s^2 + \epsilon^2/m_1 \\
\mu_{12}^{-1} &= -c(s + \epsilon) + s(c - s\epsilon)/\sqrt{2m_1} + \epsilon/\sqrt{2m_1} \\
\mu_{13}^{-1} &= c(s - \epsilon) - s(c + s\epsilon)/\sqrt{2m_1} + \epsilon/\sqrt{2m_1} \\
\mu_{22}^{-1} &= (s + \epsilon)^2/2m_1 + (c\epsilon)^2/2m_2 + 1/2m_3 \\
\mu_{23}^{-1} &= -(s^2 - c^2\epsilon^2)/2m_1 - (c^2 - s^2\epsilon^2)/2m_2 + 1/2m_3 \\
\mu_{33}^{-1} &= (s - \epsilon)^2/2m_1 + (c + s\epsilon)^2/2m_2 + 1/2m_3.
\end{align*}
$$

Since for the cases of normal and inverted hierarchy, we have no information on the mass of the lightest neutrino $m_1$, we could assume it in principle to be quite small. In that case, the above equation enables us to conclude that quite likely one of the three right handed neutrinos is much heavier than the other two. The situation is of course completely different for the degenerate case. This kind of separation of the RH neutrino spectrum is very suggestive of a symmetry. In fact we have recently argued that [21], this indicates the possible existence of an $SU(2)_H$ horizontal symmetry, that leads in the simplest case to an inverted mass pattern for light neutrinos. This idea is discussed in a subsequent section.

II.7 $L_e - L_\mu - L_\tau$ symmetry and 3 x 2 seesaw

In this section, we discuss how an approximate $L_e - L_\mu - L_\tau$ symmetric neutrino mass matrix may arise within a seesaw framework. Consider a simple extension of the standard model by adding two additional singlet right handed neutrinos [52], $N_1, N_2$ assigning them $L_e - L_\mu - L_\tau$ quantum numbers of $+1$ and $-1$ respectively. Denoting the standard model lepton doublets by $\psi_{e,\mu,\tau}$, the $L_e - L_\mu - L_\tau$ symmetry allows the following new couplings to the Lagrangian of the standard model:

$$\mathcal{L}' = (h_3 \bar{\psi}_\tau + h_2 \bar{\psi}_\mu)HN_2 + h_1 \bar{\psi}_eHN_1 + M N_1^T C^{-1} N_2 + h.c.$$

where $H$ is the Higgs doublet of the standard model; $C^{-1}$ is the Dirac charge conjugation matrix. We add to it the symmetry breaking mass terms for the right handed neutrinos, which are soft terms, i.e.

$$\mathcal{L}_B = \epsilon(M_1 N_1^T C^{-1} N_1 + M_2 N_2^T C^{-1} N_2) + h.c.$$

with $\epsilon \ll 1$. These terms break $L_e - L_\mu - L_\tau$ by two units but since they are dimension 3 terms, they are soft and do not induce any
with \( \epsilon \ll 1 \). These terms break \( L_e - L_\mu - L_\tau \) by two units but since they are dimension 3 terms, they are soft and do not induce any new terms into the theory.

It is clear from the resulting mass matrix for the \( \nu_L, N \) system that the linear combination \( h_2 \nu_\tau - h_3 \nu_\mu \) is massless and the atmospheric oscillation angle is given by \( \tan \theta_A = h_2/h_3 \); for \( h_3 \sim h_2 \), the \( \theta_A \) is maximal. The seesaw mass matrix then takes the following form (in the basis \((\nu_e, \tilde{\nu}_\mu, N_1, N_2)\) with \( \tilde{\nu}_\mu \equiv h_2 \nu_\mu + h_3 \nu_\tau \)):

\[
M = \begin{pmatrix}
0 & 0 & m_1 & 0 \\
0 & 0 & 0 & m_2 \\
m_1 & 0 & \epsilon M_1 & M \\
0 & m_2 & M & \epsilon M_2
\end{pmatrix}
\]

(59)

The diagonalization of this mass matrix leads to the mass matrix of the form discussed before.

II.8 \( \text{SO}(10) \) realization of the seesaw mechanism

The most natural grand unified theory for the seesaw mechanism is the \( \text{SO}(10) \) model, although as has been mentioned, the gross dissimilarity between quark and lepton mixings makes additional assumptions necessary to reconcile with the inherent quark-lepton unification in such models. Nevertheless, the \( \text{SO}(10) \) models are so natural framework for neutrino masses that some salient features may be instructive for any neutrino model building.

The first interesting point to note about the \( \text{SO}(10) \) models is that the \( 16 \)-dimensional spinor representation contains all the fermions of each generation in the standard model plus the right handed neutrino. Thus the right handed neutrino is necessary for the seesaw mechanism is automatic in these models. Secondly, in order to break the B-L symmetry present in the \( \text{SO}(10) \) group, one may use either the Higgs multiplets in \( 16 \) or \( 126 \) dimensional rep. We will see that either of these representations can be used to implement the seesaw mechanism. To see this note that under the left-right symmetric group \( SU(2)_L \times SU(2)_R \times SU(4)_c \), these fields decompose as follows:

\[
16 = (2, 1, 4) \oplus (1, 2, 4^*) \\
126 = (1, 1, 6) \oplus (3, 1, 10) \oplus (1, 3, 10^*) \oplus (2, 2, 15)
\]

(60)

Note that in order to break the B-L symmetry it is the \((1, 2, 4^*)\) and \((1, 3, 10^*)\) in the respective multiplets whose neutral elements need to pick up a large vev. Note however that the \( 16_H \) does not have any renormalizable coupling with the \( 16 \) spinors which contain the \( \nu_R \) whereas there is a renormalizable \( \text{SO}(10) \) invariant coupling of the form \( 1616 \bar{16}/M_\ell \). This has important implications for the B-L scale. In the former case, the B-L breaking scale is at an intermediate level such as \(~10^{13}\) GeV or so whereas in the latter case, we can have B-L scale coincide with the GUT scale of \( 2 \times 10^{16} \) GeV as in the typical SUSYGUT models \[41\].

In addition to having the right handed neutrino as part of the basic fermion representation and the Higgs representations the \( \text{SO}(10) \) model several other potential advantages for
understanding neutrino masses. For example, if one uses only the $\mathbf{10}$ dimensional representation for giving masses to the quarks and leptons, one has the up quark mass matrix $M_u$ being equal to the Dirac mass matrix of the neutrinos which goes into the seesaw formula. As a result, if we work in a basis where the up quark masses are diagonal so that all CKM mixings come from the down mass matrix, then the number of arbitrary parameters in the seesaw formula goes down from 12 to 6. Thus even though one cannot predict neutrino masses and mixings, the parameters of the theory get fixed by their values as inputs. This may then be testable thru its other predictions. The $\mathbf{10}$ only Higgs models have problems of their own i.e. there are tree level mass relations in the down sector such as $m_d = m_e$ which are renormalization group invariant and are in disagreement with observations. It may be possible in supersymmetric models to generate enough one loop corrections out of the supersymmetry breaking terms (nonuniversal) to save the situation but they will introduce unknown parameters into the theory.

There is one very special class of models where all 12 parameters of the neutrino sector are predicted by the quark and lepton masses [40]. This is the minimal renormalizable SO(10) model with the Higgs content of only one $\mathbf{10}$ and one $\mathbf{126}$ plus two Higgs multiplets $\mathbf{45}+\mathbf{54}$. The last two multiplets break the SO(10) symmetry down to $SU(2)_L \times SU(2)_R \times U(1)_{B-L} \times SU(3)_c$. The $\mathbf{126}$ breaks the $SU(2)_L \times SU(2)_R \times U(1)_{B-L} \times SU(3)_c$ down to the standard model which is then broken by the $\mathbf{10}$ Higgs.

SO(10) has the property that the Yukawa couplings involving the $\mathbf{10}$ and $\mathbf{126}$ Higgs representations are symmetric [41]. Therefore if we ignore CP violation and work in a basis where one of these two sets of Yukawa coupling matrices is diagonal, then it will have only nine parameters. Noting the fact that the $(2,2,15)$ submultiplet of $\mathbf{126}$ has a standard model doublet that contributes to charged fermion masses, one can write the quark and lepton mass matrices as follows:

\begin{align}
M_u &= h\kappa_u + fv_u \\
M_d &= h\kappa_d + fv_d \\
M_\ell &= h\kappa_d - 3fv_d \\
M_\nu_D &= h\kappa_u - 3fv_u
\end{align}

where $h$ is the $\mathbf{10}$ Higgs Yukawa coupling matrix and $f$ is the $\mathbf{126}$ Yukawa coupling matrix; $\kappa_{u,d}$ are the vev’s of the up and down type Higgs doublets in the $\mathbf{10}$ Higgs and $v_{u,d}$ are the corresponding vevs for the standard model doublets in the $\mathbf{126}$ Higgs multiplet. Note that there are 13 parameters in the above equations and there are 12 inputs (six quark masses, three lepton masses and three quark mixing angles). Thus all parameters except one is determined.

Next important observation on this model is that the same $\mathbf{126}$ responsible for the fermion masses also has a vev along the $\nu_R\nu_R$ directions so that it generates the right handed neutrino mass matrix which are proportional to the $f$ matrix. Thus using the seesaw formula (type I), there are no free parameters in the light neutrino sector. This model was extensively analysed in [40] prior to the emergence of all the neutrino oscillation data in late 90’s. It was shown that this minimal SO(10) model without any CP phase cannot fit both the solar and the atmospheric neutrino data simultaneously and is therefore ruled out. It has however
been recently noted that once the CP phases are properly included in the discussion, this model can yield a near bimaximal mixing pattern for neutrinos [42]. A generic prediction of these class of models is a large (close to experimental upper limit) value for $U_{e3}$. Therefore more refined measurements for this parameter under planning can test this model.

A very interesting point regarding these models has recently been noted in Ref. [43], where they point out that if one assume that the direct triplet term in type II seesaw, dominates, then it provides a very natural understanding of the large atmospheric mixing angle. The simple way to see it is to note that when the triplet term dominates the seesaw formula, then we have the neutrino mass matrix $M_{\nu} \propto f$, where $f$ matrix is the 126 coupling to fermions discussed earlier. Using the above equations, one can derive the following sumrule [44]:

$$M_{\nu} = c(M_d - M_\ell)$$

(63)

Now quark lepton symmetry implies that for the second and third generation, the $M_{d,\ell}$ have the following general form:

$$M_d = \begin{pmatrix} \epsilon_1 & \epsilon_2 \\ \epsilon_2 & m_b \end{pmatrix}$$

(64)

and

$$M_\ell = \begin{pmatrix} \epsilon'_1 & \epsilon'_2 \\ \epsilon'_2 & m_\tau \end{pmatrix}$$

(65)

where $\epsilon_i \ll m_{b,\tau}$ as is required by low energy observations. It is well known that in supersymmetric theories, when low energy quark and lepton masses are extrapolated to the GUT scale, one gets approximately that $m_b \simeq m_\tau$. One then sees from the above sumrule for neutrino masses that all entries for the neutrino mass matrix are of the same order leading very naturally to the atmospheric mixing angle to be large.

Few cautionary words are in order:

(i) In this approach, one must make sure that indeed in the seesaw formula for neutrinos, the triplet term indeed dominates.

(ii) This approach does not lead to the atmospheric mixing angle to be maximal as is observed but that is it large.

(iii) Finally, since in this model there are no free parameters, the solar mixing angle should be predicted and one has to see whether it is large. In fact one such attempt before did give a small mixing angle, although there might be other domains of parameters where one may get a large solar mixing angle.

Recently other SO(10) models have been considered where under different assumptions, the atmospheric and solar neutrino data can be explained together [45].

In the minimal supersymmetric left-right model, an analogous situation happens where the neutrino Dirac masses are found to be equal to the charged lepton masses [47]. Thus in this model too, one has only six parameters to describe the neutrino sector and once the neutrino data is fitted all parameters in the model are fixed so that one has predictions that can be tested. For instance, it has been emphasized in Ref. [47] that there is a prediction for the $B(\tau \rightarrow \mu + \gamma)$ in this model that is about two orders of magnitude below the present limits [48] and could therefore ne used to test the model.
Finally let us comment that in models where the light neutrino mass is understood via the seesaw mechanism that uses heavy righthanded neutrinos, there is a very simple mechanism for the generation of baryon asymmetry of the universe. Since the righthanded neutrino has a high mass, it decays at a high temperature to generate a lepton asymmetry \[ [49] \] and this lepton asymmetry is converted to baryon asymmetry via the sphaleron effects \[ [50] \] at lower temperature. It also turns out that one of the necessary conditions for sufficient leptogenesis is that the right handed neutrinos must be heavy as is required by the seesaw mechanism. To see this note that one of Sakharov conditions for leptogenesis is that the right handed neutrino decay must be slower than the expansion rate of the universe at the temperature \( T \sim M_{N_R} \). The corresponding condition is:

\[
\frac{h_i^2 M_{N_R}}{16\pi} \leq \sqrt{g^*} \frac{M_{N_R}^2}{M_{Pl}}
\]

(66)

This implies that \( M_{N_R} \geq \frac{h_i^2 M_{Pl}}{16\pi \sqrt{g^*}} \). For the second generation, it implies that \( M_{N_{2R}} \geq 10^{13} \) GeV and for the third generation a value even higher. In arriving at this conclusion, we have assumed that leptonic Yukawa couplings are of the same order as the up quark sector. Note that the deduced lower limits on the RH neutrino masses are above the inflation reheating upper bound from considerations of the gravitino production alluded to before. However for the first generation, it can be about \( 10^8 \) GeV so that there is no conflict with the gravitino bound on the reheating temperature and therefore leptogenesis can occur. Incidentally, the leptogenesis condition also imposes limits on the matrix elements of the right handed neutrino mass, thereby reducing the arbitrariness of the seesaw predictions slightly.

### II.9 Type II seesaw and Quasi-degenerate neutrinos

In this subsection we like to discuss some issues related to the degenerate neutrino hypothesis, which will be necessary if there is evidence for neutrinoless double beta decay at a significant level (see for example the recent results from the Heidelberg-Moscow group \[ [11] \]) and assuming that no other physics such as R-parity breaking or doubly charged Higgs etc are not the source of this effect). Thus it is appropriate to discuss how such models can arise in theoretical schemes and how stable they are under radiative corrections.

There are two aspects to this question: one is whether the degeneracy arises within a gauge theory framework without arbitrary adjustment of parameters and the second aspect being that given such a degeneracy arises at some scale naturally in a field theory, is this mass degeneracy stable under renormalization group extrapolation to the weak scale where we need the degeneracy to be present. In this section we comment on the first aspect.

It has already been alluded to before and first made in \[ [10] \] is that degenerate neutrinos arise naturally in models that employ the type II seesaw since the first term in the mass formula is not connected to the charged fermion masses. One way that has been discussed is to consider schemes where one uses symmetries such as SO(3) or SU(2) or permutation symmetry \( S_4 \) \[ [33] \] so that the Majorana Yukawa couplings \( f_i \) are all equal. This then leads to the dominant contribution to all neutrinos being equal. This symmetry however must be broken in the charged fermion sector in order to explain the observed quark and lepton masses. Such models consistent with known data have been constructed based on SO(10) as well as other groups. The interesting point about the SO(10) realization is that the
dominant contributions to the $\Delta m^2$’s in this model comes from the second term in the type II seesaw formula which in simple models is hierarchical. It is of course known that if the MSW solution to the solar neutrino puzzle is the right solution (or an energy independent solution), then we have $\Delta m_{\text{solar}}^2 \ll \Delta m_{\text{ATMOS}}^2$. In fact if we use the fact true in SO(10) models that $M_u = M_D$, then we have $\Delta m_{\text{ATMOS}}^2 \simeq m_0^2 \frac{m_{\nu R}^2}{f_{\nu R}}$ and $\Delta m_{\text{SOLAR}}^2 \simeq m_0^2 \frac{m_{\nu R}^2}{f_{\nu R}}$ where $m_0$ is the common mass for the three neutrinos. It is interesting that for $m_0 \sim \text{few eV}$ and $f_{\nu R} \approx 10^{15}$ GeV, both the $\Delta m^2$’s are close to the required values.

Outside the seesaw framework, there could also be electroweak symmetries that guarantee the mass degeneracy. For a recent model of this type see, ref [18].

The second question of stability under RGE of such a pattern is discussed in a subsequent section.
III.

III.1 Lepton flavor violation and neutrino masses

In the standard model, the masslessness of the neutrino implies that there is no lepton flavor changing effects unlike in the quark sector. Once one includes the right handed neutrinos $N_R$ one for each family, there is lepton mixing and therefore lepton flavor changing effects such as $\mu \rightarrow e + \gamma$, $\tau \rightarrow e, \mu + \gamma$ etc. However, a simple estimate of the one loop contribution to such effects shows that the amplitude is of order

$$A(\ell_j \rightarrow \ell_i + \gamma) \approx \frac{eG_F m_{\ell} m_\nu^2}{\pi^2 m_W^2} \mu_B$$ (67)

This leads to an unobservable branching ratio (of order $\sim 10^{-40}$) for the rare radiative decay modes for the leptons.

The situation however changes drastically as soon as the seesaw mechanism for neutrino masses is embedded into the supersymmetric models. It has been noted in many papers already that in supersymmetric theories, the lepton flavor changing effects get significantly enhanced. They arise from the the mixings among sleptons (superpartners of leptons) of different flavor caused by the renormalization group extrapolations which via loop diagrams lead to lepton flavor violating (LFV) effects at low energies \[55\].

The way this happens is as follows. In the simplest N=1 supergravity models \[54\], the supersymmetry breaking terms at the Planck scale are taken to have only few parameters: a universal scalar mass $m_0$, universal $A$ terms, one gaugino mass $m_{1/2}$ for all three types of gauginos. Clearly, a universal scalar mass implies that at Planck scale, there is no flavor violation anywhere except in the Yukawa couplings (or when the Yukawa terms are diagonalized, in the CKM angles). However as we extrapolate this theory to the weak scale, the flavor mixings in the Yukawa interactions induce non universal flavor violating scalar mass terms (i.e. flavor violating slepton and squark mass terms). In the absence of neutrino masses, the Yukawa matrices for leptons can be diagonalized so that there is no flavor violation in the lepton sector even after extrapolation down to the weak scale. On the other hand, when neutrino mixings are present, there is no basis where all leptonic flavor mixings can be made to disappear. In fact, in the most general case, of the three matrices $Y_\ell$, the charged lepton coupling matrix, $Y_\nu$, RH neutrino Yukawa coupling and $M_{N_R}$, the matrix characterizing the heavy RH neutrino mixing, only one can be diagonalized by an appropriate choice of basis and the flavor mixing in the other two remain. In a somewhat restricted case where the right handed neutrinos do not have any interaction other than the Yukawa interaction and an interaction that generates the Majorana mass for the right handed neutrino, one can only diagonalize two out of the three matrices (i.e. $Y_\nu, Y_\ell$ and $M_R$). Thus there will always be lepton flavor violating terms in the basic Lagrangian, no matter what basis one chooses. These LFV terms can then induce mixings between the sleptons of different flavor and lead to LFV processes. If we keep the $M_\ell$ diagonal by choice of basis, searches for LFV processes such as $\tau \rightarrow \mu + \gamma$ and/or $\mu \rightarrow e + \gamma$ can throw light on the RH neutrino mixings/or family mixings in $M_D$, as has already been observed.

Since in the absence of CP violation, there are at least six mixing angles (nine if $M_D$ is not symmetric) in the seesaw formula and only three are observable in neutrino oscillation,
to get useful information on the fundamental high scale theory from LFV processes, it is assumed that $M_{N_R}$ is diagonal so that one has a direct correlation between the observed neutrino mixings and the fundamental high scale paramters of the theory. The important point is that the flavor mixings in $Y_\nu$ then reflect themselves in the sneutrino mixings that lead to the LFV processes via the RGEs.

From the point of view of the LFV analysis, there are essentially two classes of neutrino mass models that need to be considered: (i) the first class is where it is assumed that the RH neutrino mass $M$ is either a mass term in the basic Lagrangian or arises from nonrenormalizable terms such as $\nu^c \chi^2 / M_{PL}$, as in a class of SO(10) models; and (ii) a second class where the Majorana mass of the right handed neutrino itself arises from a renormalizable Yukawa coupling e.g. $f \nu^c \nu^c \Delta$. In the first class of models, in principle, one could decide to have all the flavor mixing effects in the right handed neutrino mass matrix and keep the $Y_\nu$ diagonal. In that case, RGEs would not induce any LFV effects. However we will bar this possibility and consider the case where all flavor mixings are in the $Y_\nu$ so that RGEs can induce LFV effects and estimate them in what follows. In class two models on the other hand, there will always be an LFV effect, although its magnitude will depend on the choice of the seesaw scale $(v_{BL})$.

Examples of class two models are models for neutrino mixings such as SO(10) with a Higgs field \cite{11} or left-right model with a triplet Higgs, whose vev is the seesaw scale.

In both these examples, the key equations that determine the extent of lepton flavor violation are:

**Case (i):**

$$\frac{dm_{\ell}^2}{dt} = \frac{1}{4\pi^2}[(m_L^2 + 2m_{H_d}^2)Y_\ell Y_\ell^\dagger + (m_L^2 + 2H_u^2)Y_\nu Y_\nu^\dagger + 2Y_\ell^\dagger m_{\ell c}^2 Y_\ell + Y_\ell^\dagger Y_\ell m_{\ell}^2] \quad (68)$$

$$\frac{dA_\ell}{dt} = \frac{1}{16\pi^2}A_\ell[Tr(3Y_d^\dagger Y_d + Y_\ell^\dagger Y_\ell) + 5Y_\ell^\dagger Y_\ell + Y_\nu^\dagger Y_\nu - 3g_2^2 - ag_R^2 - bg_{B-L}^2]$$

$$+ Y_\ell[Tr(6A_d Y_d^\dagger + A_\ell Y_\ell^\dagger) + 4Y_\ell^\dagger A_\ell + 2Y_\nu^\dagger A_\nu + 6g_2^2 M_2 + \ldots]$$

**Case (ii)** In addition to the above two equations, two more equations are necessary:

$$\frac{dY_\nu}{dt} = \frac{Y_\nu}{16\pi^2}[Tr(3Y_u^\dagger Y_u + Y_\nu^\dagger Y_\nu^\dagger) + 3Y_\nu^\dagger Y_\nu + Y_\ell^\dagger Y_\ell + 4f^\dagger f - 3g_2^2 - crg_R^2 - c_{B-L}g_{B-L}^2] \quad (69)$$

$$\frac{dA_\nu}{dt} = \frac{1}{16\pi^2}A_\nu[Tr(3Y_u^\dagger Y_u + Y_\nu^\dagger Y_\nu^\dagger) + 5Y_\nu^\dagger Y_\nu + Y_\ell^\dagger Y_\ell + 4f^\dagger f - 3g_2^2 - ag_R^2 - bg_{B-L}^2]$$

$$+ Y_\ell[Tr(6A_d Y_d^\dagger + A_\ell Y_\ell^\dagger) + 4Y_\ell^\dagger A_\ell + 6g_2^2 M_2 + \ldots]$$

In order to apply these equations, we note that in the basis where the charged lepton masses are diagonal, the seesaw formula involves the right handed neutrino mass matrix $M_R$ and the neutrino Dirac mass matrix $M_D$. Assuming the the Dirac mass matrix is symmetric, there are 12 parameters (for the case with CP conservation). Since neutrino masses and mixings only provide six observables, there are several different ways that can lead to the observed neutrino mixings. Two distinct extreme ways are as follows: (i) the first case is where the neutrino mixings arise primarily from the off diagonal elements of the $M_D$ assuming the $M_R$ is diagonal or even an extremely simplified case where it is a unit matrix and (ii) a second
case where we can keep the $M_D$ diagonal and all mixings arise from $M_R$ having mixings. In the first case, the RGEs always lead to lepton flavor violation whereas in the second case, flavor violations arise only if the $M_R$ arises from a Majorana Yukawa coupling of the form $\nu^c \nu \Delta$ after $\Delta \neq 0$ as already explained. We will call this the Majorana case and case (i) as the Dirac case.

In the Dirac case, starting with the simplest supersymmetry breaking assumption of universal scalar masses and proportional $A$ terms, the scalar sleptons develop off diagonal terms due to the flavor violation in $Y_\nu$ and these mass terms have the form

$$m^2_{L,ij} \propto \frac{3 + a^2}{16\pi^2} \ell n \frac{M_{PL}}{M_{B-L}} Y^\dagger_\nu Y_\nu + \cdots$$

(70)

where \cdots denote the diagonal terms that cannot cause flavor mixing. If the $M_R$ is diagonal, the $Y^\dagger_\nu Y_\nu$ is nothing but the neutrino mass matrix up to a constant $= M_R$ Using this we can get the Dirac mass dependence of the $B(\ell_j \rightarrow e + \gamma)$ (for $\ell_j = \tau, \mu$) to be:

$$B(\ell_j \rightarrow e + \gamma) \sim \frac{(m_{2c}s + m_{3}\epsilon)^2}{G^2_F v^4_B - L} \cdot c_j$$

(71)

where $c_\tau = 1/6$ and $c_\mu = 1$ whereas for $B(\tau \rightarrow \mu + \gamma) \propto 6 \frac{m^2_{10} m^2_{D2}}{G^2_F v^4_B - L} m^4_0$.

Now coming to the second case with Majorana-Yukawa couplings, starting with universal scalar masses at the Planck scale and in the Majorana-Yukawa case all couplings flavor diagonal except the $f$ coupling, it is easy to see that the above equations will induce flavor changing effects in $m^2_L$ and $A_\ell$. The strength of the slepton flavor mixings depends sensitively on the neutrino mass texture and the resulting the texture in the coupling matrix $f$. Below we give the branching ratio expressions for one extreme case where we assume that the lightest neutrino mass dominates the $f$ matrix elements. We caution the reader that this is by no means the most typical case and results differ significantly as different mass textures are considered.

The slepton flavor mixings, via a one loop diagram involving the winos ($\tilde{W}^+$ and $\tilde{W}^3$ lead to the lepton flavor changing radiative amplitudes. Keeping only the contribution of the $m^2_L$ term which dominates for larger $\tan \beta$, we find that roughly speaking the three branching ratios are given by:

$$B(\mu \rightarrow e + \gamma) \propto \frac{1}{G^2_F m^4_0 v^4_{B-L}} \frac{m^4_{D3} m^6_{D2} m^2_{D1} \tan^2 \beta}{m^4_1 v^4_{w_k}};$$

(72)

$$B(\tau \rightarrow e + \gamma) \propto \frac{1}{G^2_F m^4_0 v^4_{B-L}} \frac{m^{10}_{D3} m^2_{D1} \tan^2 \beta}{m^4_{1} v^4_{w_k}};$$

(73)

$$B(\tau \rightarrow \mu + \gamma) \propto \frac{1}{G^2_F m^4_0 v^4_{B-L}} \frac{m^{10}_{D3} m^2_{D2} \tan^2 \beta}{m^4_{1} v^4_{w_k}}.$$

(74)

Note an important difference between the predictions for the Dirac case and the Majorana one. Both $\tau \rightarrow e + \gamma$ and $\mu \rightarrow e + \gamma$ are of the same order of magnitude for the Dirac
case whereas they are very different for the Majorana case where $B(\mu \to e + \gamma)/B(\tau \to e + \gamma) \simeq (m_D/\mu)^6$. Therefore, even a mild hierarchy in the Dirac mass sector can make the $\tau \to e + \gamma$ branching ratio much larger than the $\mu \to e + \gamma$ branching ratio. This kind of discrepancy could in principle be used to test the origin of the seesaw mechanism.

It must be pointed out that the predictions for the Majorana case are extremely texture sensitive. In a recent paper, [23], calculations have been carried out for a different but consistent texture where $B(\mu \to e + \gamma)$ though much lower than the $\tau$ rare decay branching ratios, is still found to be within the range of currently planned experiments at PSI.

For completeness, we give the formula for calculating the radiative decay of the leptons. If we express the amplitude for the decay as:

$$\mathcal{L} = i e m_j \left( \bar{\ell}_j L \sigma_{\mu \nu} \ell_i R \right) F^{\mu \nu} + h.c. \quad (75)$$

then the Branching ratio for the decay $\ell_j \to \ell_i + \gamma$ is given by the formula

$$B(\ell_j \to \ell_i + \gamma) = \frac{48 \pi^3 \alpha_{em}}{G_F^2} (|C_L|^2 + |C_R|^2) B(\ell_j \to \ell_i + 2\nu) \quad (76)$$

### III.2 Renormalization group evolution of the neutrino mass matrix

In the seesaw models for neutrino masses, the neutrino mass arises from the effective operator

$$\mathcal{O}_\nu = -\frac{1}{4} \kappa_{\alpha \beta} \frac{L_\alpha H L_\beta H}{M} \quad (77)$$

after symmetry breaking $< H^0 > \neq 0$; here $L$ and $H$ are the leptonic and weak doublets respectively. $\alpha$ and $\beta$ denote the weak flavor index. The matrix $\kappa$ becomes the neutrino mass matrix after symmetry breaking i.e. $< H^0 > \neq 0$. This operator is defined at the scale $M$ since it arises after the heavy field $N_R$ is integrated out. On the other hand, in conventional oscillation experiments, the neutrino masses and mixings being probed are at the weak scale. One must therefore extrapolate the operator down from the seesaw scale $M$ to the weak scale $M_Z$ [10]. The form of the renormalization group extrapolation of course depends on the details of the theory. For simplicity we will consider only the supersymmetric theories, where the only contributions come from the wave function renormalization and is therefore easy to calculate. The equation governing the extrapolation of the $\kappa_{\alpha \beta}$ matrix is given in the case of MSSM by:

$$\frac{d\kappa}{dt} = [-3g_2^2 + 6T r(Y_u^\dagger Y_u)]\kappa + \frac{1}{2}[\kappa(Y_e^\dagger Y_e) + (Y_e^\dagger Y_e)\kappa] \quad (78)$$

We note two kinds of effects on the neutrino mass matrix from the above formula: (i) one that is flavor independent and (ii) a part that is flavor specific. If we work in a basis where the charged leptons are diagonal, then the resulting correction to the neutrino mass matrix is given by:

$$\mathcal{M}_\nu(M_Z) = (1 + \delta)\mathcal{M}(M_{B-L})(1 + \delta) \quad (79)$$

where $\delta$ is a diagonal matrix with matrix elements $\delta_{\alpha \alpha} \simeq -\frac{m_\alpha \tan^2 \beta}{16 \pi^2 v^2}$. In more complicated theories, the corrections will be different. Let us now study some implications of this corrections. For this first note that in the MSSM, this effect can be sizable if $\tan \beta$ is large (of order 10 or bigger).
III.3 Radiative magnification of neutrino mixing angles

A major puzzle of quark lepton physics is the diverse nature of the mixing angles. Whereas in the quark sector the mixing angles are small, for the neutrinos they are large. One possible suggestion in this connection is that perhaps the mixing angles in both quark and lepton sectors at similar at some high scale; but due to renormalization effects, they may become magnified at low scales. It was shown in ref. [57] that this indeed happens if the neutrino spectrum os degenerate. This can be seen in a simple way for the $\nu_\mu - \nu_\tau$ sector [57].

Let us start with the mass matrix in the flavor basis:

$$ M_F = U^* M_D U^\dagger $$

(80)

Let us examine the situation when $\phi = 0$ (i.e. CP is conserved), which corresponds to the case when the neutrinos $\nu_1$ and $\nu_2$ are in the same $CP$ eigenstate. Due to the presence of radiative corrections to $m_1$ and $m_2$, the matrix $M_F$ gets modified to

$$ M_F \to \begin{pmatrix} 1 + \delta_\alpha & 0 \\ 0 & 1 + \delta_\beta \end{pmatrix} M_F \begin{pmatrix} 1 + \delta_\alpha & 0 \\ 0 & 1 + \delta_\beta \end{pmatrix} $$

(81)

The mixing angle $\bar{\theta}$ that now diagonalizes the matrix $M_F$ at the low scale $\mu$ (after radiative corrections) can be related to the old mixing angle $\theta$ through the following expression:

$$ \tan 2\bar{\theta} = \tan 2\theta \left(1 + \delta_\alpha + \delta_\beta\right) \frac{1}{\lambda} $$

(82)

where

$$ \lambda \equiv \frac{(m_2 - m_1)C_{2\theta} + 2\delta_\beta(m_1S_{\theta}^2 + m_2C_{\theta}^2) - 2\delta_\alpha(m_1C_{\theta}^2 + m_2S_{\theta}^2)}{(m_2 - m_1)C_{2\theta}}. $$

(83)

If

$$ (m_1 - m_2) C_{2\theta} = 2\delta_\beta(m_1S_{\theta}^2 + m_2C_{\theta}^2) - 2\delta_\alpha(m_1C_{\theta}^2 + m_2S_{\theta}^2), $$

(84)

then $\lambda = 0$ or equivalently $\bar{\theta} = \pi/4$; i.e. maximal mixing. Given the mass hierarchy of the charged leptons: $m_{l_\alpha} \ll m_{l_\beta}$, we expect $|\delta_\alpha| \ll |\delta_\beta|$, which reduces (84) to a simpler form:

$$ \epsilon = \frac{\delta m C_{2\theta}}{(m_1S_{\theta}^2 + m_2C_{\theta}^2)} $$

(85)

In the case of MSSM, the radiative magnification condition can be satisfied provided provided

$$ h_\tau(MSSM) \approx \sqrt{\frac{8\pi^2|\Delta m^2(\Lambda)|C_{2\theta}}{ln(\Lambda_m^2)\Delta m^2}}. $$

(86)

For $\Delta m^2|\approx \Delta m^2_{\chi}$, this condition can be satisfied for a very wide range of $\tan\beta$.

It is important to emphasize that this magnification occurs only if at the seesaw scale the neutrino masses are nearly degenerate. A similar mechanism using the righthanded neutrino Yukawa couplings instead of the charged lepton ones has been carried out recently [58]. Here two conditions must be satisfied: (i) the neutrino spectrum must be nearly degenerate (i.e. $m_1 \simeq m_2$ as in ref. [57]) and (ii) there must be a hierarchy between the righthanded neutrinos.
III.4 An explicit example of a neutrino mass matrix unstable under RGE

In this section, we give an explicit example of a neutrino mass matrix unstable under RGE effects [59]. Consider the following mass matrix with degenerate neutrino masses and a bimaximal mixing [60].

\[ M_\nu = \begin{pmatrix} 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{\sqrt{2}} & \frac{1}{2} & \frac{1}{2} \end{pmatrix} \]  \hspace{1cm} (87)

The eigenvalues of this mass matrix are \((1, -1, 1)\) and the eigenvectors:

\[ V_1 = \begin{pmatrix} 0 \\ \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}; V_2 = \begin{pmatrix} \frac{1}{2} \\ -\frac{1}{2} \\ -\frac{1}{2} \end{pmatrix}; V_3 = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix} \]  \hspace{1cm} (88)

After RGE to the weak scale, the mass matrix becomes

\[ M_\nu = \begin{pmatrix} 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}(1 + \delta) \\ \frac{1}{\sqrt{2}} & \frac{1}{2} & -\frac{1}{2}(1 + \delta) \\ \frac{1}{\sqrt{2}}(1 + \delta) & \frac{1}{2}(1 + \delta) & \frac{1}{2}(1 + 2\delta) \end{pmatrix} \]  \hspace{1cm} (89)

It turns out that the eigenvectors of this matrix become totally different and are given by:

\[ V_1 = \begin{pmatrix} \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \\ 0 \end{pmatrix}; V_2 = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{2} \\ -\frac{1}{2} \end{pmatrix}; V_3 = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix} \]  \hspace{1cm} (90)

We thus see that the neutrino mixing pattern has become totally altered. although the eigenvalues are only slightly perturbed from their unperturbed value.

There are also other examples where the RGE of the neutrino mass matrix can totally destabilize the mass pattern, see [61] for an example with the sterile neutrino.

III.5 Generation of solar mass difference square in the case of \(L_e - L_\mu - L_\tau\) symmetric models

In this section, we give another example where the renormalization group evolution (RGE) of the neutrino mass matrix plays an important role. Consider the following neutrino mass matrix defined at a (unification) scale much above the weak scale as in Eq. (6).

\[ M_\nu^0 = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} m. \]  \hspace{1cm} (91)

Note that as already emphasized earlier, this matrix has several deficits as far as accommodating neutrino data is concerned. To remind the reader, the problems are two fold: (i) it leads to \(\Delta m^2_{\odot} = 0\) and (ii) solar neutrino mixing angle is maximal. As we show below, RGEs can help to overcome both these defects.
The important point to note is that the RGE depends on the charged lepton mass matrix. Let us assume that the charged lepton mass matrix has the form:

$$ M_{\ell^+} = \begin{pmatrix} 0 & 0 & x \\ 0 & y & 0 \\ x' & 0 & 1 \end{pmatrix} m_\tau. \quad (92) $$

For the most part we will take $x' = x$ so that $|x| \simeq \sqrt{m_e/m_\tau}$ and $y \simeq m_\mu/m_\tau$. We will also consider the possibility that $x \gg x'$ (the right handed singlet leptons multiply the matrix in this equation on the right). In the case $x = x'$, note that there is only parameter in the leptonic sector (both the charged leptons and neutrinos). We will show that this model can lead to a realistic description of the neutrino oscillations.

At scales below the unification scale, through the renormalization of the effective $d = 5$ neutrino mass operator the form of $M_\nu^0$ \cite{27} will be modified. (Analogous corrections in $M_{\ell^+}$ is negligible.) The modified neutrino mass matrix at the weak scale is given by

$$ M_\nu \simeq M_\nu^0 + \frac{c}{16\pi^2} \ln\left(\frac{M_U}{M_Z}\right) \left(Y_\ell Y_\ell^\dagger M_\nu^0 + M_\nu^0 (Y_\ell Y_\ell^\dagger)^T\right). \quad (93) $$

Here $c = -3/2$ for SM while $c = 1$ for SUSY, and $Y_\ell$ is the charged lepton Yukawa coupling matrix. We have absorbed the flavor–independent renormalization factor into the definition of $m$ in the above equation. Explicitly,

$$ M_\nu \simeq \begin{pmatrix} 2rx & 1 + r(x^2 + y^2) & 1 + r(1 + 2x^2) \\ 1 + r(x^2 + y^2) & 0 & rx \\ 1 + r(1 + 2x^2) & rx & 2rx \end{pmatrix} m. \quad (94) $$

This mass matrix has both an acceptable $\Delta m^2_{\odot}$ as well as large (but not maximal) solar neutrino mixing angle. This another example where the renormalization group equations do make a difference in our understanding of the neutrino oscillations.

III.6 A horizontal symmetry approach to near bimaximal mixing

In this section, I present a model \cite{21} which motivates the existence of an $SU(2)_H$ horizontal symmetry acting on leptons to understand the near bimaximal mixing pattern and yields the softly broken $L_e - L_\mu - L_\tau$ model discussed earlier.

Suppose, there is an $SU(2)_H$ horizontal symmetry that acts only on leptons. As already discussed, freedom from global Witten anomaly requires that there must be two right handed neutrinos that trasform as a doublet of $SU(2)_H$. The local $SU(2)_H$ symmetry then implies that the masses of those two right handed neutrinos are protected and must be at the scale of $SU(2)_H$ breaking. If there is a the third right handed neutrino for reasons of quark lepton symmetry, then it will acquire a mass of order of the Planck or string scale and decouple from neutrino physics at lower energies. This therefore provides a physically distinct way of implementing the seesaw mechanism. One has a $3 \times 2$ seesaw rather than the usual $3 \times 3$ one.

Furthermore the $SU(2)$ horizontal symmetry restricts both the Dirac mass of the neutrino as well as the righthanded neutrino mass matrix to the forms \cite{21}. 

44
\[
M_{\nu L, \nu R} = \begin{pmatrix}
0 & 0 & 0 & h_0 \kappa_0 & 0 \\
0 & 0 & 0 & 0 & h_0 \kappa_0 \\
0 & 0 & 0 & h_1 \kappa_1 & h_1 \kappa_2 \\
h_0 \kappa_0 & h_1 \kappa_1 & 0 & f v_H' \\
0 & h_0 \kappa_0 & h_1 \kappa_2 & f v_H' & 0
\end{pmatrix}
\] (95)

After seesaw diagonalization, it leads to the light neutrino mass matrix of the form:

\[
M_{\nu} = -M_D M_R^{-1} M_D^T
\] (96)

where \( M_D = \begin{pmatrix}
h_0 \kappa_0 & 0 \\
0 & h_0 \kappa_0 \\
h_1 \kappa_1 & h_1 \kappa_2
\end{pmatrix};
M_R^{-1} = \frac{1}{f v_H'} \begin{pmatrix}
0 & 1 \\
0 & 0
\end{pmatrix}.
\]

The resulting light Majorana neutrino mass matrix \( M_{\nu} \) is given by:

\[
M_{\nu} = -\frac{1}{f v_H'} \begin{pmatrix}
0 & (h_0 \kappa_0)^2 & h_0 h_1 \kappa_0 \kappa_2 \\
(h_0 \kappa_0)^2 & h_0 h_1 \kappa_0 \kappa_1 & 2 h_1^2 \kappa_1 \kappa_2
\end{pmatrix}
\] (97)

To get the physical neutrino mixings, we also need the charged lepton mass matrix defined by \( \bar{\psi}_L M_{\ell} \psi_R \). This is given in our model by:

\[
M_{\ell} = \begin{pmatrix}
h_2' \kappa_0 & 0 & -h_1' \kappa_2 \\
0 & h_2' \kappa_0 & h_1' \kappa_1 \\
h_1' \kappa_1 & h_2' \kappa_2 & h_3' \kappa_0
\end{pmatrix}
\] (98)

Note that in the limit of \( \kappa_1 = 0 \), the neutrino mass matrix has exact \( (L_e - L_\mu - L_\tau) \) symmetry whereas the charged lepton mass matrix breaks this symmetry. This is precisely the class of inverted hierarchy models that was discussed earlier which provides a realistic as well as a testable model for neutrino oscillations. In particular, this model leads to a relation between the neutrino parameters \( U_{e3} \) and the ratio of solar and atmospheric mass difference squared i.e.

\[
U_{e3}^2 \cos 2 \theta_\odot = \frac{\Delta m^2_\odot}{2 \Delta m^2_A} + O(U_{e3}^4, m_e/m_\mu)
\] (99)

which is testable in proposed long baseline experiments such as NUMI off-axis plan at Fermilab or JHF in Japan.
IV.

IV.1 Neutrino masses in models with large extra dimensions

One of the important predictions of string theories is the existence of more than three space dimensions. For a long time, it was believed that these extra dimensions are small and are therefore practically inconsequential as far as low energy physics is concerned. However, recent progress in the understanding of the nonperturbative aspects of string theories have opened up the possibility that some of these extra dimensions could be large without contradicting observations. In particular, models where some of the extra dimensions have sizes as large as a millimeter and where the string scale is in the few TeV range have attracted a great deal of phenomenological attention in the past two years. The basic assumption of these models, inspired by the D-branes in string theories, is that the space-time has a brane-bulk structure, where the brane is the familiar (3+1) dimensional space-time, with the standard model particles and forces residing in it, and the bulk consists of all space dimensions where gravity and other possible gauge singlet particles live. One could of course envision (3+d+1) dimensional D-branes where d-space dimensions have miniscule (≤ TeV⁻¹) size. The main interest in these models has been due to the fact that the low string scale provides an opportunity to test them using existing collider facilities.

A major challenge to these theories comes from the neutrino sector, the first problem being how one understands the small neutrino masses in a natural manner. The conventional seesaw explanation which is believed to provide the most satisfactory way to understand this, requires that the new physics scale (or the scale of $SU(2)_R \times U(1)_{B-L}$ symmetry) be around $10^9$ to $10^{12}$ GeV or higher, depending on the Dirac masses of the neutrinos whose magnitudes are not known. If the highest scale of the theory is a TeV, clearly the seesaw mechanism does not work, so one must look for alternatives. The second problem is that if one considers only the standard model group in the brane, operators such as $LHLH/M^*$ could be induced by string theory in the low energy effective Lagrangian. For TeV scale strings this would obviously lead to unacceptably large neutrino masses.

One mechanism suggested in Ref. is to postulate the existence of one or more gauge singlet neutrinos, $\nu_B$, in the bulk which couple to the lepton doublets in the brane. After electroweak symmetry breaking, this coupling can lead to neutrino Dirac masses, which are suppressed by the ratio $M_\star/M_{Pl}$, where $M_{Pl}$ is the Planck mass and $M_\star$ is the string scale. This is sufficient to explain small neutrino masses and owes its origin to the large bulk volume that suppresses the effective Yukawa couplings of the Kaluza-Klein (KK) modes of the bulk neutrino to the brane fields. In this class of models, naturalness of small neutrino mass requires that one must assume the existence of a global B-L symmetry in the theory, since that will exclude the undesirable higher dimensional operators from the theory.

To discuss the mechanisms in a concrete setting, let us first focus on TeV scale models. Here, one postulates a bulk neutrino, which is a singlet under the electroweak gauge group. Let us denote the bulk neutrino by $\nu_B(x^\mu, y)$. The bulk neutrino is represented by a four-component spinor and can be split into two chiral Weyl 2-component spinors as $\nu_B^T = (\chi^T, -i\phi^T\sigma_2)$. The 2-component spinors $\chi$ and $\phi$ can be decomposed in terms of 4-dimensional Fourier components as follows:

$$\chi(x, y) = \frac{1}{\sqrt{2R}} \chi_{+0} + \frac{1}{\sqrt{R}} \sum_{n=1}^{\infty} \left( \chi_{+n}\cos\frac{n\pi y}{R} + i\chi_{-n}\sin\frac{n\pi y}{R} \right).$$
There is a similar expression for $\phi$. It has a five dimensional kinetic energy term and a coupling to the brane field $L(x^\mu)$. The full Lagrangian involving the $\nu_B$ is

$$\mathcal{L} = i\bar{\nu}_B \gamma_\mu \partial^\mu \nu_B + \kappa \bar{L} H \nu_{BR}(x, y = 0) + i \int dy \bar{\nu}_{BL}(x, y) \partial_5 \nu_{BR}(x, y) + h.c.,$$

(101)

where $H$ denotes the Higgs doublet, and $\kappa = \frac{M_\star}{M_{Pl}}$ is the suppressed Yukawa coupling. This leads to a Dirac mass for the neutrino \[64\] given by:

$$m = \frac{hv_{wk} M_\star}{M_{Pl}},$$

(102)

where $v_{wk}$ is the scale of $SU(2)_L$ breaking. In terms of the 2-component fields, the mass term coming from the fifth component of the kinetic energy connects the fields $\chi_+$ with $\phi_-$ and $\chi_-$ with $\phi_+$, whereas it is only the $\phi_+$ (or $\nu_{B,R,+}$) which couples to the brane neutrino $\nu_{e,L}$. Thus as far as the standard model particles and forces go, the fields $\phi_-$ and $\chi_+$ are totally decoupled, and we will not consider them here. The mass matrix that we will write below therefore connects only $\nu_{e,L}$, $\phi_+$, and $\chi_-$.

From Eq. \[102\], we conclude that for $M_\star \sim 10$ TeV, this leads to $m \simeq 10^{-4} h$ eV. It is encouraging that this number is in the right range to be of interest in the discussion of solar neutrino oscillation if the Yukawa coupling $h$ is appropriately chosen. Furthermore, this neutrino is mixed with all the KK modes of the bulk neutrino, with a mixing mass $\sim \sqrt{2}m$; since the $n$th KK mode has a mass $nR^{-1} \equiv n\mu$, the mixing angle is given by $\sqrt{2}mR/n$. Note that for $R \sim 0.1 mm$, this mixing angle is of the right order to be important in MSW transitions of solar neutrinos.

It is worth pointing out that this suppression of $m$ is independent of the number and radius hierarchy of the extra dimensions, provided that our bulk neutrino propagates in the whole bulk. For simplicity, we will assume that there is only one extra dimension with radius of order of a millimeter.

Secondly, the above discussion can be extended in a very straightforward manner to the case of three generations. The simplest thing to do is to add three bulk neutrinos and consider the Lagrangian to be:

$$\mathcal{L} = i\bar{\nu}_{B,\alpha} \gamma_\mu \partial^\mu \nu_{B,\alpha} + \kappa_{\alpha\beta} \bar{L}_\alpha H \nu_{BR}(x, y = 0) + i \int dy \bar{\nu}_{BaL}(x, y) \partial_5 \nu_{BaR}(x, y) + h.c.,$$

(103)

One can now diagonalize $\kappa_{\alpha\beta}$ by rotating both the bulk and the active neutrinos. The mixing matrix then becomes the neutrino mixing matrix $U$ discussed in the text. In this basis (the mass eigenstate basis), one can diagonalize the mass matrix involving the $\nu_i$’s and the bulk neutrinos to get the mixings between the active and the bulk tower. There are now three mixing parameters, one for each mass eigenstate denoted by $\xi_i \equiv \sqrt{2}m_i R$ and mixing angle for each mass eigenstate to the $n$th KK mode of the corresponding bulk neutrinos is given by $\xi_i/n$. The observed oscillation data can then be used to put limits on $\xi_i$; we discuss this in a subsequent section.

It is also worth noting that due to the presence of the infinite tower mixed with the active neutrino, the oscillation probabilities are distorted in a way which is very different from the case of oscillation to a single neutrino level. This has the implication that if at some point the complete oscillation of a neutrino (as opposed to just the overall suppression
as is the case now) is observed, it will be possible to put stronger limits on the parameter $\xi_i$ from data.

### IV. 1A: Neutrino propagation in matter with a bulk neutrino tower

Let us discuss the propagation of a neutrino in matter in the bulk tower neutrino models. For this purpose we have to consider the neutrino mass matrix in the flavor basis in the presence of matter effect, to be denoted by $\delta_{ee}$. This looks as follows in the basis:

$\begin{pmatrix}
\nu_e, \nu_{BR,1}^{(1)}, \nu_{BL,1}^{(1)}, & \nu_{BR,2}^{(2)}, \nu_{BL,2}^{(2)}, & \cdots
\end{pmatrix}$

is given by:

$$
M = M_{\text{TeV}} \equiv 
\begin{pmatrix}
\delta_{ee} & m & 0 & \sqrt{2}m & 0 & \sqrt{2}m & \cdots \\
0 & 0 & 0 & 0 & 0 & 0 & \cdots \\
\sqrt{2}m & 0 & \mu & 0 & 0 & \cdots \\
0 & 0 & 0 & 0 & 2\mu & \cdots \\
\sqrt{2}m & 0 & 0 & 2\mu & 0 & \cdots \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots
\end{pmatrix}.
$$

(104)

where $\delta_{ee}$ is a possible matter effect. One can evaluate the eigenvalues and the eigenstates of this matrix. The former are the solutions of the transcendental equation:

$$
m_n = \delta_{ee} + \frac{\pi m_n^2}{\mu} \cot \left( \frac{\pi m_n}{\mu} \right). 
$$

(105)

The equation for eigenstates is

$$
\tilde{\nu}_n = \frac{1}{N_n} \left[ \nu_e + \frac{m}{m_n} \nu_{BR,1}^{(0)} + \sum_k \sqrt{2}m \left( \frac{m_n}{m_n^2 - k^2 \mu^2} \nu_{BL,1}^{(k)} + \frac{k \mu}{m_n^2 - k^2 \mu^2} \nu_{BR,2}^{(k)} \right) \right],
$$

(106)

where we have used the notation $\pm$ for the left- and right-handed parts of the KK modes of the bulk neutrino in the two-component notation and dropped the $L, R$ subscripts, the sum over $k$ runs through the KK modes, and $N_n$ is the normalization factor given by

$$
N_n^2 = 1 + m^2 \pi^2 R^2 + \frac{(m_n - \delta_{ee})^2}{m_n^2}.
$$

(107)

One of the very striking effects of the KK tower of bulk neutrinos is the presence of multiple MSW resonances each time the neutrino energy is such that one satisfies the resonance condition in a medium. As in the two neutrino matter resonance, each time there is a level crossing by the $\nu_e$ due to its matter effect, there will be a dip in the survival probability of the electron neutrino.

To understand the origin of the dips, note that the MSW resonance condition is given by

$$
m_{\text{res}}^2 \approx \frac{4G_F \rho}{\sqrt{2}m_p} E_{\text{res}}
$$

(108)

As a result, as the neutrino energy $E$ increases, the resonance condition is satisfied for higher and higher KK modes of the bulk neutrino. The survival probability at and after the resonance is given by
for $E \geq E_{\text{res}}$ and $P_{ee} \sim 1$ for $E < E_{\text{res}}$. The cumulative survival probability is given by $P = P^{(1)} P^{(2)} \ldots P^{(n)}$. Note that due the exponent being larger than one at the resonance (which can be checked by putting numbers), as soon as a new KK level is crossed, the $P_{ee}$ dips to a value much less than one and then starts to rise as $E$ increases giving rise to the dip structure. The net effect of the combined dip structure is to flatten the spectrum.

The typical values of the survival probability within the $^8B$ region ($\sim 6$ to $\sim 14$ MeV) are quite sensitive to the value of $m_R$. As can be seen from Eq. 107, higher $m_R$ increases $1/N_n \approx m/m_n \approx m_R/n$ for various $n$, and thereby increases $\nu_e$ coupling to higher mass eigenstates, strengthens MSW resonances, and lowers $\nu_e$ survival probability. Thus searching for dips in solar neutrino spectrum is one way to probe the size of extra dimensions. The same dip phenomena could also appear in higher energy atmospheric spectra. Since the resonance condition given by the above formula implies that the higher the energy the higher the mass difference squared (or lower the extra dimension radius) probed, search for dips in higher energy neutrinos can in principle reveal the existence of extra dimensions of smaller sizes that cannot be probed by solar neutrinos.

IV.2 Phenomenological and cosmological constraints on bulk neutrino models

A generic feature of understanding neutrino mass via bulk neutrinos in models with large flat extra dimension is the presence of an infinite tower of closely spaced sterile neutrinos mixed with active neutrino (say $\nu_e$). Information about the nature of extra dimensions can therefore be obtained by looking at the phenomenological as well as cosmological effects of this mixing with the infinite tower.

A. Neutrino oscillation constraints

To see the kind of constraints one can derive, let us take a simple model where there are three bulk neutrinos (i.e. three infinite towers), each giving mass to one family of neutrinos. In this minimal model, a simple rotation of the active neutrinos takes the neutrinos to the mass basis and separates three towers. It has been shown in recent papers through detailed analysis [66], that while this minimal model can adequately explain both the solar and atmospheric neutrino oscillation phenomena essentially by arranging the active bulk neutrino mixing, it is not possible to accommodate the LSND results. Furthermore, since generic mixing of the various mass eigenstate neutrinos to the nth KK mode of the corresponding bulk neutrino goes like $m_i R/n$, if $R$ is sizable, the observable effect simulating a multitude of sterile neutrinos should be present. Since both present atmospheric as well as solar neutrino current data from SNO severely constrain the oscillation of the active neutrinos to the sterile ones, this puts a constraint on $R$. For instance, considering the muon neutrino oscillation to bulk neutrinos in the atmospheric neutrino data, we can take $m_3 R \leq 10\%$. This implies that $R \leq 2 \text{ eV}^{-1}$ or $R$ less than 40 microns. This constraint is more stringent than the one derived from direct searches for deviations from the inverse square law [37].
B. Big Bang Nucleosynthesis constraints

On the cosmological front, since big bang nucleosynthesis is a very sensitive measure of the number extra neutrino species mixed with the active neutrinos, one should be able to get information about this class of models from this data. The basic idea here is that at the epoch of nucleosynthesis, neutrino oscillation to the bulk modes can bring in the modes into thermodynamic equilibrium with all other relativistic species. If that happens each mode will contribute an amount $\rho_\nu$ to the energy density of the universe and speed up the expansion. The faster the expansion of the universe, the earlier the weak interactions go out of equilibrium (to be called freeze-out). Since the neutron to proton ratio depends very sensitively on the temperature of freeze-out i.e. $n/p \sim e^{-\frac{m_n - m_p}{T^*}}$, the higher the freeze-out temperature $T^*$, the higher the neutron fraction and hence the Helium fraction of the universe.

Higher the mixing strength of the active modes with the bulk modes, the more modes from the bulk that get into thermal equilibrium with the electron. Therefore, to be consistent with observed Helium abundance, one must have a restriction on the mixing parameter $\sqrt{2mR}$.

Very careful analysis of this has been done in two papers [68] by solving the Boltzman equation for the generation of sterile neutrinos from active neutrinos at the BBN epoch. For the case of one space dimension, it was concluded in [68] that the active-bulk mixing and the inverse radius of the extra dimension $\mu$ satisfy the constraint:

$$\left(\frac{\mu}{eV}\right)^{0.92} \sin^2 2\theta \leq 7.06 \times 10^{-4} \quad (110)$$

Applying this to the tau neutrino, we find an even more stringent limit on the size of the extra dimensions $R$ than the one just listed i.e. $R \leq 1.5$ micron.

C. Enhancement of magnetic moments

Another interesting consequence of the presence of the bulk tower is in its effect on the magnetic moment of the active neutrinos. As is wellknown [69], if one adds a singlet right handed neutrino to the standard model to give a Dirac mass to the neutrino, this induces a magnetic moment $\mu_\nu = 10^{-19} \frac{m_\nu}{eV} \mu_B$, where $\mu_B$ is the Bohr magneton ($\mu_B = e/2m_ec$). Since in the bulk neutrino models, there is an infinite tower of neutrinos mixed with the active neutrinos, there is a $\mu_{\nu_e\nu_{b,p}} = 10^{-19} \frac{\sqrt{2m}}{eV} \mu_B$ connecting the active neutrino with each KK mode of the bulk neutrino. In a neutrino scattering process $\nu_e + e$ with neutrino energy $E_\nu$, all bulk neutrino modes up to $E_\nu$ will be excited [70]. Thus the effective neutrino magnetic moment will appear to be $\mu_{eff} \approx 10^{-19} \frac{m}{eV}(ER)^{1/2}$ where $R$ is the radius of the extra dimension. For $R \sim$ millimeters, this enhances the magnetic moment by almost a factor of a million. Thus this could provide an interesting way to probe the existence of extra dimensions. A note of caution is that this apparent enhancement is effective only when the KK modes of the bulk neutrino can be excited. For instance, when neutrino spin precesses in a magnetic field, the most of the KK modes do not get excited due to energy momentum conservation and therefore, the magnetic moment enhancement does not take place.
D. Other implications

Another interesting implication of the bulk neutrino tower appears if the brane model is not the standard model with one Higgs doublet but with two. In this case, the physical charged Higgs can decay into charged lepton and the bulk tower but with the difference that unlike in the normal two Higgs extension, the final state charged lepton emitted in this process will have left-handed helicity whereas in models without any bulk neutrino, the final state charged lepton will have right handed helicity [71].

The presence of the bulk neutrino tower also leads to new contributions to flavor changing leptonic rare decays such as $\mu \rightarrow e + \gamma$ etc. They in turn lead to constraints on the fundamental scale of nature [72].

IV.3 Neutrino mass in low scale gravity models without bulk neutrinos

Since the presence of light tower of bulk neutrinos is so constraining and adhoc forbidding of fully allowed operators is theoretically unappealing for the brane bulk picture with the standard model in the brane, it is important to search for alternative ways to solve the neutrino mass problem. In these kind of scenarios, the strategy is to search for higher dimensional models that will lead to the standard model after compactification of the extra dimensions and yet allow for the possibility of a low fundamental scale. One such alternative has recently been proposed by having the left-right symmetric model or an extension of the standard model with only a local $B-L$ symmetry in the bulk and looking for the standard model in the zero mode part of the spectrum [73]. It is assumed that all fermions are in the bulk. The number of extra space dimensions can either be one or two. The model with two extra space dimensions has additional symmetry which also helps top solve the proton decay problem. So we first give the example of the six dimensional model with the gauge group $SU(2)_L \times U(1)_{B-R} \times U(1)_{B-L}$. Such models are have been called universal extra dimension models [74].

The minimal fermion content of this model is dictated by gravitational anomaly cancellation to be [73]:

$$Q_+(2, 0, 1/3), \psi_+(2, 0, -1), U_-(1, 1/2, 1/3), D_-(1, -1/2, 1/3)$$

$$E_-(1, -1/2, -1), N_-(1, +1/2, -1)$$

where $Q = (u, d)$ and $\psi = (\nu, e)$ and $\pm$ denote the six dimensional chirality; the numbers in the parentheses are the gauge quantum numbers. Note that each fermion field is a four component field with two 4-dimensional 2 component spinors with opposite chirality e.g. $Q$ has a left chiral $Q_L$ and a right chiral field $Q_R$. As such the theory is vectorlike at this stage and we will need orbifold projections to obtain a chiral theory. We choose one Higgs doublet $\phi(2, -1/2, 0)$ and a singlet $B - L$ carrying Higgs boson $\chi(1, 1/2, -1)$. We compactify the theory on a $T_2/Z_2$ orbifold; where $T_2$ is defined by the extra coordinates $y_{1,2}$ satisfying the following conditions: $y_{1,2} = y_{1,2} + 2\pi R$ and $Z_2$ operates on the two extra coordinates as follows: $(y_1, y_2) \rightarrow (-y_1, -y_2)$. We now impose the orbifold conditions on the fields as follows: We choose the following fields to be even under the $Z_2$ symmetry: $Q_L, \psi_L, U_R, D_R, E_R, N_L$; the kinetic energy terms then force the opposite chirality states to
be odd under $Z_2$. Note specifically that, in contrast with the $U,D,E$ fields, it is the $N_L$ which is chosen even under $Z_2$. This is crucial to our understanding of neutrino masses. As is well known, the even fields when Fourier expanded involve only the $\cos \frac{\pi y}{R}$ and the odd fields only $\sin \frac{\pi y}{R}$. As a result only the $Z_2$ even fields have zero modes. Thus, with the above compactification, below the mass scale $R^{-1}$, the only fermionic modes are those of the standard model plus the $N_L^0$. When we give vev to the field $<\chi> = v_{BL}$, it breaks the group down to the standard model. We will choose $v_{B-L} \sim 800$ GeV to a TeV.

Before discussing the implications of the model for neutrino masses and proton decay, let us study the extra symmetries of the 4-dimensional theory implied by the fact that it derives from a 6-dimensional one.

First, the discrete translational symmetry insures the conservation of the fifth and sixth momentum components, $p_a$, which are quantized in integer factors of $1/R$.

Secondly, in the full uncompactified six dimensional theory, there is an extra $U(1)_{45}$ symmetry associated with the rotations in the $y_1,y_2$ plane. After compactification, the $U(1)_{45}$ invariance reduces to a $Z_4$ symmetry. Therefore invariance under the $SO(1,3) \times Z_4$ space-time Lorentz transformations must be imposed on all possible operators allowed in the effective four dimensional theory i.e. the allowed operators will be those that are invariant under the whole $SO(1,5)$ symmetry, plus probably those for which the sum of fermion $U(1)_{45}$ charges is equal to zero modulus 8. The reasoning is as follows: the $Z_4$ spatial symmetry, actually translates into a $Z_8$ symmetry group for the spinorial representation. In fact under a $\pi/2$ rotation of the $x_4-x_5$ plane a fermion transforms as $\Psi(x') = U\Psi(x)$; with $U = \exp[\ii (\pi/2)\Sigma_{45}/2]$; where $\Sigma_{45} = \ii[\Gamma^4,\Gamma^5]/2$ is the generator of the $U(1)_{45}$ group.

To see which operators are allowed, we need to know the $U(1)_{45}$ quantum numbers of the theory which can be easily read off from the six dimensional theory and are given in the table below.

| $Q_L,\psi_L$ | +1/2 |
|------------|------|
| $U_R, D_R, E_R, N_R$ | +1/2 |
| $Q_R, \psi_R, U_L$ | -1/2 |
| $E_L, D_L, N_L$ | -1/2 |

**Table caption:** $U(1)_{45}$ charges of the various fermions in the $SU(2)_L \times U(1)_{I_{3R}} \times U(1)_{B-L}$ model.

We will use these quantum numbers below.

Let us now turn to understanding the small neutrino mass in this model. Note that in this model due to our orbifold assignments and choice of the gauge group coupled with the residual $Z_8$ symmetry discussed above, we only have one term that to leading order can contribute to neutrino masses and the term is: $\lambda^{\psi_L^2 C^{-1} N_L \phi(\chi)^2} M_5^2$.

The following potentially dangerous terms are forbidden for various reasons in this 6-dimensional theory:

- $(\psi_L \phi)^2/M_*$ is forbidden by $B-L$ symmetry.
orbifold compactification. For this purpose, one compactifies the 5th dimension on an orbifold, the operator that leads to neutrino mass has the form \( \frac{\psi_L^T C^{-1} \psi_L (\chi^0)^2}{M_s^2 R^3} \). Using \( M_s \simeq 100 \text{ TeV} \) and \( R^{-1} \simeq \text{TeV} \) and using \( \lambda \sim 0.1 \), we find for that it leads to \( m_\nu \sim eV \), which is in the right range without any fine tuning. Furthermore, the neutrinos in this model are Dirac particles since all Majorana terms are forbidden to leading order by the \( Z_8 \) symmetry.

E. Left-right symmetric model in five dimensions and neutrino mass

In this section, we provide a left-right symmetric embedding of the local B-L symmetry and show in a five dimensional example how it can lead to small neutrino masses despite the fact that the fundamental scale of the model is in the TeV range. These considerations are easily extended to the six space-time dimensions.

Two basic new ingredients of the model are: \( SU(2)_L \times SU(2)_R \times U(1)_{B-L} \) gauge group and a slightly different orbifold compactification based on \( S_1 / (Z_2 \times Z_2') \) for the five dimensional model. In this model, one needs two sets of 4-component spinors for each family of quarks and leptons. A requirement is that the left-right gauge symmetry is broken by orbifold compactification. For this purpose, one compactifies the 5th dimension on an \( S_1 / Z_2 \times Z_2' \). It is well known that for this compactification, the Fourier modes of the five dimensional fields can be labelled as \((+,+), (+,-), (-,+), (-,-) \) where \(+\) and \(-\) denote the \( Z_2 \times Z_2' \) parity of a given mode. They are associated with \( \cos \frac{2\pi y}{R}, \cos \frac{(2n-1)\pi y}{R}, \sin \frac{(2n-1)\pi y}{R}, \sin \frac{2\pi y}{R} \) respectively. It is then easy to see that only the \((+,+\) models have zero mass and all other modes have masses proportional to \( R^{-1} \). The various fermion fields are assigned the following \( Z_2 \times Z_2' \) quantum numbers:

\[
Q_{1L} \equiv \begin{pmatrix} u_{1L}(+,+) \\ d_{1L}(+,+) \end{pmatrix} ; \\
Q_{1R} \equiv \begin{pmatrix} u_{1R}(-,-) \\ d_{1R}(-,-) \end{pmatrix} ; \\
Q_{2L} \equiv \begin{pmatrix} u_{2L}(-,-) \\ d_{2L}(-,-) \end{pmatrix} ; \\
Q_{2R} \equiv \begin{pmatrix} u_{2R}(+,+) \\ d_{2R}(+,+) \end{pmatrix} ;
\]

(111)

and for leptons:

\[
\psi_{1L} \equiv \begin{pmatrix} \nu_{1L}(+,+) \\ e_{1L}(+,+) \end{pmatrix} ; \\
\psi_{1R} \equiv \begin{pmatrix} \nu_{1R}(-,-) \\ e_{1R}(-,-) \end{pmatrix} ; \\
\psi_{1L}' \equiv \begin{pmatrix} \nu_{1L}'(-,+), \\ e_{1L}'(-,+), \end{pmatrix} ; \\
\psi_{1R}' \equiv \begin{pmatrix} \nu_{1R}'(+,-) \\ e_{1R}'(+,-) \end{pmatrix} ;
\]
\[
\psi_{2,L} \equiv \begin{pmatrix} \nu_{2L}(-,+) \\ e_{2L}(-,-) \end{pmatrix}; \quad \psi'_{2,L} \equiv \begin{pmatrix} \nu'_{2L}(+,+) \\ e'_{2L}(+,-) \end{pmatrix}; \\
\psi_{2,R} \equiv \begin{pmatrix} \nu_{2R}(+,-) \\ e_{2R}(+,+) \end{pmatrix}; \quad \psi'_{2,R} \equiv \begin{pmatrix} \nu'_{2R}(-,-) \\ e'_{2R}(-,+) \end{pmatrix}.
\] (112)

We see from the above equation that the process of compactification leaves us for \( E \ll R^{-1} \), with the standard model fermions plus a neutrino like zero mode that has only right handed weak interaction, corresponding to a right handed doublet field dubbed as \( \psi'_2 \). As far as the gauge bosons go, they have the following “parity” assignments:

\[
W^{3,\pm}_{1,\mu}(+,+) ; \quad B_{\mu}(+,+); \quad W^{3}_{2,\mu}(+,+); \quad W^{\pm}_{1,\mu}(+,+); \\
W^{3,\pm}_{1,5}(-,-); \quad B_5(-,-); \quad W^{3}_{2,5}(-,-); \quad W^{\pm}_{2,5}(-,+).
\] (113)

Finally, the Higgs fields i.e. bidoublet field \( \phi(2,2,0) \) and the unidoublet fields \( \chi_{L,R} \) have the following \( Z_2 \times Z'_2 \) parities:

\[
\phi \equiv \begin{pmatrix} \phi^0_u(+,+); \phi^+_d(+,-) \\ \phi^-_d(+,+); \phi^0_u(+,-) \end{pmatrix}; \quad \chi_L \equiv \begin{pmatrix} \chi^0_L(+,+); \chi^-_L(+,-) \\ \chi^+_L(+,+); \chi^0_L(+,-) \end{pmatrix}; \quad \chi_R \equiv \begin{pmatrix} \chi^0_R(+,+); \chi^-_R(+,-) \\ \chi^+_R(+,+); \chi^0_R(+,-) \end{pmatrix}.
\] (114)

To understand the origin of neutrino mass operators given below, note that the gauge symmetry is broken by Higgs pair \( \chi_{L,R} \) which are \( SU(2) \) doublets and the usual bidoublet of the left-right model \( \phi(2,2,0) \). The left handed neutrino is part of the \( SU(2)_L \) doublet called \( \psi_1 \).

To see how neutrino mass arises in this theory, let us note that there are no renormalizable operators (in the 4-D sense) that can generate neutrino mass. Secondly, there are three classes of non-renormalizable operators of higher dimensions that remain invariant under all the symmetries of the theory, and which contribute to neutrino masses.

(i) There are operators connecting the active left handed neutrinos to themselves: i.e. \( O_1 \equiv \psi^T_1 C_5 \psi_1 \phi \chi \chi_{L,R}/M^2_5 \), where \( C_5 \equiv \gamma^0 \gamma^2 \gamma^5 \), where we have omitted the family index. Notice that it has dimension 10 on 5D. It generates, at the four dimensional theory, the effective couplings

\[
\frac{h}{(M_* R)^2} \left( \frac{L \phi \nu}{M^2_5} \right)^2;
\] (115)

with \( h \) the dimensionless coupling. This operator induces a sufficiently small Majorana neutrino mass,

\[
m_\nu = \frac{h v^2_{\nu L} v^2_R}{(M_* R)^2 M^3_5} \approx h \cdot 1 \text{ eV}.
\] (116)

where the right hand side has been estimated using \( v_R \approx 1/R \approx 1 \text{ TeV} \) and \( M_* \approx 100 \text{ TeV} \). A soft hierarchy in the couplings (say \( h \sim 0.01 \)) should provide the right spectrum on neutrino masses.

(ii) The second class of operators connect \( \nu \) to \( \nu_1 \) and have the form in lowest order \( O_2 \equiv \psi^T_1 T_2 \phi \chi \chi_{L,R} T_2 C_5 \psi_2 / M^7_5/2 \). This operator after compactification has a magnitude \( \approx \frac{v_{\nu L} v^2_R}{M^7_5/(M_* R)^3/2} \approx 10 \text{ keV} \)
(iii) The last class connects the left handed neutrinos that transform under the $SU(2)_R$ group to themselves and have the form $O_3 \equiv (\psi_L^2 \chi_R)^2 / M^2$. They contribute to the $\nu_s - \nu_s$ entry and have magnitude after compactification estimated to be $\simeq \frac{v^2}{M^2_R} \simeq 1 - 10$ GeV. The full $6 \times 6$ $\nu - \nu_s$ mass matrix has a seesaw like form and on diagonalization, leads to an effective mass for the light neutrino in the range of 0.1 eV or so.

This model has no twoer of bulk neutrinos with mass gap of milli eV type; secondly, there is no need to invoke global $B - L$ symmetry to prevent undesirable terms. This argument also carries over to the 6-D extension of this model, which one may want for the purpose of suppressing proton decay.

V. CONCLUSIONS AND OUTLOOK

At the moment, neutrino oscillation experiments have provided the first evidence for new physics beyond the standard model. The field of neutrino physics therefore has become central to the study of new physics at the TeV scale and beyond. The other area which most theorists believe will be the next one to emerge from experiments is supersymmetry. We have therefore assumed supersymmetry in most of our discussions, although in the last section, we consider low scale extra dimensional models without supersymmetry.

What have we learned so far? One thing that seems very clear is that there is probably a set of three right handed neutrinos which restore quark lepton symmetry to physics; secondly there must be a local $B - L$ symmetry at some high scale beyond the standard model that is responsible for the RH neutrinos being so far below the Planck scale. While there are very appealing arguments that the scale of $B - L$ symmetry is close to $10^{14}$-10^{16} GeV’s, in models with extra dimensions, you cannot rule out the possibility that it is around a few TeVs. Third thing that one may suspect is that the right handed neutrino spectrum may be split into a heavier one and two others which are nearby. If this suspicion is confirmed, that would point towards an $SU(2)_H$ horizontal symmetry or perhaps even an $SU(3)_H$ symmetry which breaks into an $SU(2)_H$ symmetry (although simple anomaly considerations prefer the first alternative).

The correct theory should explain:

(i) Why both the solar and atmospheric mixing angles are maximal?
(ii) Why the $\Delta m^2_{\odot} \ll \Delta m^2_{\text{A}}$ and what is responsible for the smallness of $U_{e3}$? While in the inverted hierarchy models, the smallness of $U_{e3}$ is natural, in general it is not.
(iii) What is the nature of CP phases in the lepton sector and what is their relation to the CP phases possibly responsible for baryogenesis via leptogenesis?
(iv) What is the complete mass spectrum for neutrinos?

These and other questions is likely to prove to be very exciting challenges to both theory and experiment in neutrino physics for the next two decades.

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some of the important topics which is my only defense for the rather incomplete selection of topics and cited papers.
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