A MULTIPLICATIVE PROPERTY OF QUANTUM FLAG MINORS II

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Abstract. Let $U^+$ be the plus part of the quantized enveloping algebra of a simple Lie algebra and let $B^*$ be the dual canonical basis of $U^+$. Let $b, b'$ be in $B^*$ and suppose that one of the two elements is a $q$-commuting product of quantum flag minors. We show that $b$ and $b'$ are multiplicative if and only if they $q$-commute.

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1. Introduction

Let $B^*$ denote the dual of the canonical basis $[8, 13 \S 14.4.6]$ of a quantized enveloping algebra of a simple Lie algebra $g$. Two elements of $B^*$ are said to be multiplicative if their product also lies in $B^*$ up to a power of $q$. They are said to $q$-commute if they commute up to a power of $q$. The Berenstein-Zelevinsky conjecture states that two elements of $B^*$ are multiplicative if and only if they $q$-commute. While Reineke [17, 4.5] has shown that if two elements are multiplicative then they $q$-commute, the conjecture is now known to be false: a counter-example was provided by Leclerc [10], who showed that for all but finitely many types of simple Lie algebra, there exist elements $b \in B^*$ whose square does not lie in $B^*$ even up to a power of $q$ (such elements obviously $q$-commute with themselves). Such elements are called imaginary; elements of $B^*$ whose square does lie in $B^*$ up to a power of $q$ are known as real. We consider the following question:

**Question 1.1.** Let $b, b'$ be in $B^*$ and suppose that $b$ is real. Is it the case that $b$ and $b'$ are multiplicative if and only if they $q$-commute?

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Leclerc has made a conjecture [10, Conjecture 1] concerning the expansion of the product of two dual canonical basis elements, and has remarked that if true, it would imply that the answer to this question is yes in general. In this paper, we prove that the answer to this question is yes if one of the two elements involved is a $q$-commuting product of quantum flag minors (in type $A$). We note that such products are known to be real [11]. A key part of our proof involves results concerning the piecewise-linear reparametrization function $R_i^{k}$ associated by Lusztig to a pair $i, i'$ of reduced decompositions for the longest word $w_0$ in the Weyl group of $\mathfrak{g}$. Lusztig has defined the canonical basis via the bases of Poincaré-Birkhoff-Witt type associated to such reduced decompositions, and such reparametrization functions arise from taking two of the Lusztig parametrizations of the canonical basis. We show that these functions share some of the properties known to be possessed by the reparametrization functions arising from string parametrizations for the canonical basis [1]. In particular we show that if two dual canonical basis elements are parametrized by $\mathfrak{g}$-commuting product of quantum flag minors (in type $A$) basis [1]. Let $\mathfrak{g}$ be a compatible triangular $\mathfrak{h}$-commuting product of quantum flag minors. This generalizes a theorem of [5].

This enables us to prove that the answer to Question 1.1 is yes in type $A_1$. Note that results in section 1-4 can be easily generalized to all simply-laced types. Note also that Lemma 5.3 is only true in type $A_n$. Hence, Theorem 5.5 needs this assumption.

2. Background and notation

We use the set-up of [1]. Let $\mathfrak{g} = sl_{n+1}(\mathbb{C})$ denote the simple Lie algebra of type $A_n$. Let $\mathfrak{h}$ be a Cartan subalgebra and let $\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+$ be a compatible triangular decomposition. Let $\alpha_1, \alpha_2, \ldots, \alpha_n$ be the corresponding positive roots, and $\Delta^+$ the corresponding set of positive roots. Let $W$ be the Weyl group associated to the root system, $P$ be the weight lattice generated by the fundamental weights $\varpi_i$, $1 \leq i \leq n$, and $\langle , \rangle$ be the $W$-invariant form on $P$. Let $U = U_q(\mathfrak{g})$ be the simply connected Drinfel’d-Jimbo quantized enveloping algebra over $\mathbb{Q}(q)$ associated to $\mathfrak{g}$ as defined in [7]. Let $U^-$, $U^0$ and $U^+$ be the subalgebras associated to the sub-Lie algebras $\mathfrak{n}^-$, $\mathfrak{h}$ and $\mathfrak{n}^+$ respectively; we have the triangular decomposition $U \cong U^- \otimes U^0 \otimes U^+$. The subalgebra $U^+$ is generated over $\mathbb{Q}(q)$ by canonical generators $E_1, E_2, \ldots, E_n$, subject to the quantized Serre relations; the subalgebra $U^-$ is isomorphic to $U^+$, with corresponding generators $F_1, F_2, \ldots, F_n$, and the subalgebra $U^0$ is isomorphic to $\mathbb{Q}(q)[P]$, the element corresponding to $\lambda \in P$ being denoted by $K_\lambda$. Let $\mathfrak{b}^+ = \mathfrak{h} \oplus \mathfrak{n}^+$ and $\mathfrak{b}^- = \mathfrak{h} \oplus \mathfrak{n}^-$ and we define $U(b^+) = U^0 U^+$ and $U(b^-) = U^- U^0$. The algebra $U$ is a Hopf algebra with comultiplication $\Delta$, antipode $S$ and augmentation $\varepsilon$ given by:

$$\Delta(E_i) = E_i \otimes 1 + K_{\alpha_i} \otimes E_i, \quad \Delta(F_i) = F_i \otimes K_{-\alpha_i} + 1 \otimes F_i, \quad \Delta(K_\lambda) = K_\lambda \otimes K_\lambda,$$
we can naturally identify $U$ corresponding element $u \leq \text{associated to}$

where Birkhoff-Witt basis of $\mathbb{L}$ and let $\psi$ canonical basis) of $X$, all decomposition $U$ (Lusztig) Fix a reduced decomposition $\delta_\iota(1 - q^2)^{-1}$, 

$$ (u^+, u^-_1 u^-_2) = (\Delta(u^+_1), u^-_1 \otimes u^-_2), \quad u^+ \in U(b^+), u^-_1, u^-_2 \in U(b^-), $$

$$ (u^+_1 u^-_2, u^-) = (u^+_2 \otimes u^+_1, \Delta(u^-)), \quad u^- \in U(b^-), u^+_1, u^+_2 \in U(b^+), $$

$$(K_\lambda, K_\mu) = q^{-\lambda(\mu)}, \quad (K_\lambda, F_i) = 0, \quad (E_i, K_\lambda) = 0, \quad \lambda, \mu \in P.$$ The form $(,)$ is nondegenerate on $U^\alpha_+ \otimes U^\alpha_-\lambda$ for all $\alpha \in Q^+$. Since $U^+$ and $U^-$ are isomorphic algebras (with isomorphism preserving their weight spaces), we can naturally identify $U^-\alpha$ with $U^\alpha_+$ so for each element $u \in U^\alpha_+$ there is a corresponding element $u^* \in U^\alpha_+$ (corresponding to $u$ using the form). Let $w_0$ be the longest element of $W$: denote by $R(w_0)$ the set of all reduced decompositions for $w_0$. Fix a reduced decomposition $i = (i_1, i_2, \ldots, i_N)$ of $w_0$, and for $1 \leq t \leq N$, let $\beta_t = s_{i_t}s_{i_{t+1}}\cdots s_{i_1}(\alpha_{i_t})$; we get an ordering $\beta_1 < \beta_2 < \cdots < \beta_N$ of $\Delta^+$. For $1 \leq i \leq n$, let $T_i$ denote the Lusztig braid automorphism of $U^+$ [13, 37.1.3], [19] associated to $i$. For $1 \leq t \leq N$ let $E_{\beta_t} = T_{i_t}T_{i_{t+1}}\cdots T_{i_{t-1}}(E_{i_t})$. The Poincaré-Birkhoff-Witt basis of $U^+$ is the basis $B_i = \{ E(m) : m \in \mathbb{Z}_{\geq 0}^N \}$, where $E(m) = E_i(m) = \prod_{\lambda=1}^{N} \frac{1}{[m_\lambda]} E_{\beta_\lambda}^{m_\lambda}$, with the product taken in the ordering given above. Here $|m|! = |m|[m-1] \cdots [1]$, where $[m] = \frac{q^{m} - q^{-m}}{q-q^{-1}}$. By [12] we have that $E(m)^\ast = \prod_{\lambda=1}^{N} \psi_{m_\lambda}(q^2) E(m)$, where $\psi_{m}(z) = \prod_{k=1}^{m} (1 - z^k)$. Let $\mathcal{L}$ be the sub-$\mathbb{Z}[q]$-lattice of $U^+$ generated by $B_i$, and let $\mathcal{L}^\ast$ be the sub-$\mathbb{Z}[q]$-lattice of $U^+$ generated by $B_i^\ast = \{ E_i(m)^\ast : m \in \mathbb{Z}_{\geq 0}^N \}$. Lusztig has shown that both $\mathcal{L}$ and $\mathcal{L}^\ast$ are independent of the choice of reduced decomposition $i$. Let $\eta$ be the $\mathbb{Q}$-algebra automorphism of $U$ fixing the generators $E_i$ and $F_i$, with $\eta(K_\lambda) = K_{-\lambda}$ and $\eta(q) = q^{-1}$. Lusztig [13] also shows:

**Theorem 2.1.** (Lusztig) Fix a reduced decomposition $i$ of $w_0$. Then, for each $m \in \mathbb{Z}_{\geq 0}$, there is a unique element $B(m) = B_i(m) \in U^+$ such that $\eta(B(m)) = B(m)$ and $B(m) \in E(m) + qL$. The set $B = \{ B(m) : m \in \mathbb{Z}_{\geq 0} \}$ is a basis (called the canonical basis) of $U^+$ which does not depend on $i$.  

\[ S(E_i) = -K_{-2\alpha_i}E_i, \quad S(F_i) = -F_i K_{2\alpha_i}, \quad S(K_\lambda) = K_{-\lambda}, \quad \varepsilon(E_i) = \varepsilon(F_i) = 0, \quad \varepsilon(K_\lambda) = 1. \]
We'll call the parametrization $\mathbb{Z}_N^N \to \mathcal{B}$, $\mathbf{m} \mapsto B(\mathbf{m})$, Lusztig's parametrization of $\mathcal{B}$ (arising from $i$). If $b \in \mathcal{B}$, we denote by $L_i(b)$ its Lusztig parameter (so $L_i$ is the inverse of $B_i$). For any $i, i' \in R(w_0)$, we also have Lusztig's piecewise-linear reparametrization function $\tilde{R}_i' = L_{i'} L_i^{-1}$ (which can be regarded as a function from $\mathbb{R}^N \to \mathbb{R}^N$ via the same formula in terms of coordinates). The canonical basis was discovered independently by Kashiwara [8], who called it the global crystal basis. Let $\mathcal{B}^* = \{b^* : b \in \mathcal{B}\}$ denote the dual canonical basis of $U^+$. For $b \in \mathcal{B}$ we define $L_i(b^*)$ to be $L_i(b)$. Let $\sigma$ be the antihomomorphism of $U$ (as $\mathbb{Q}(q)$-algebra) taking $E_i$ to $E_i$, $F_i$ to $F_i$ and $K_\lambda$ to $K_{-\lambda}$. As in [11] Proposition 16] we have:

**Proposition 2.2.** Fix a reduced decomposition $i$ of $w_0$. Then, for each $\mathbf{m} \in \mathbb{Z}_{\geq 0}^N$, the element $B(\mathbf{m})^*$ is the unique element $X$ of $U^+$ with weight $\sum_i m_i \alpha_i$ such that

\[
\eta(X) = (-1)^{\text{tr}(X)} q^{-\langle \text{wt}(X), \text{wt}(X) \rangle / 2} q_X^{-1} \sigma(X), \quad X \in E(\mathbf{m})^* + q\mathcal{L}^*,
\]

where $q_X = \prod q^{m_i}$, $\text{wt}(X) = \sum_i m_i \alpha_i$.

Fix a reduced decomposition $i$ in $R(w_0)$. Let $\mathbf{e}_k, 1 \leq k \leq N$, be the canonical generators of $\mathbb{Z}_N^N$. We can define an ordering on the semigroup $\mathbb{Z}_{\geq 0}^N$ associated to $i$ in the following way. The ordering $\prec i$ is generated by $\mathbf{m} \leq \mathbf{e}_k + \mathbf{e}_{k'}$, $1 \leq k < k' \leq N \Rightarrow E_i(\mathbf{m})$ is a term of the PBW decomposition of $E_{\beta_k} E_{\beta_{k'}} - q^{\langle \beta_{k'}, \beta_k \rangle} E_{\beta_k} E_{\beta_{k'}}$. It will be denoted by $\prec$ if no confusion occurs.

**Proposition 2.3.** Fix $i$ in $R(w_0)$. Then, for all $\mathbf{m}, \mathbf{n} \in \mathbb{Z}_{\geq 0}^N$:

(i) if $\mathbf{m} \prec_i \mathbf{n}$ then $\mathbf{m}$ is lower than $\mathbf{n}$ for the lexicographical ordering,

(ii) $B_i(\mathbf{m}) = E_i(\mathbf{m}) + \sum_{\mathbf{n} \prec_i \mathbf{m}} d_{\mathbf{m}}(\mathbf{n}) E_i(\mathbf{n})$, $d_{\mathbf{m}}(\mathbf{n}) \in \mathbb{q}\mathbb{Z}[q]$,

(iii) $B_i(\mathbf{m})^* = E_i(\mathbf{m})^* + q \sum_{\mathbf{n} \prec_i \mathbf{m}} c_{\mathbf{m}}(\mathbf{n}) E_i(\mathbf{n})^*$, $c_{\mathbf{m}}(\mathbf{n}) \in \mathbb{Z}$.

**Proof:** See [5, 2.1], $\Box$

Let $q^\mathcal{B}^*$ denote the set $\{q^b : b \in \mathcal{B}, n \in \mathbb{Z}\}$. We therefore have:

**Corollary 2.4.** Let $\alpha \in Q^+$, and suppose $u \in U^+$. Fix $i \in R(w_0)$. Then $u \in q^\mathcal{B}^*$ if and only if

(i) There is $f \in \mathbb{Z}[q, q^{-1}]$ such that $\sigma f(u) = f u$, and

(ii) $u = q^k (E(\mathbf{m})^* + q \sum_{\mathbf{n} \prec_i \mathbf{m}} c_{\mathbf{m}}(\mathbf{n}) E_i(\mathbf{n})^*)$, where $k \in \mathbb{Z}$, $c_{\mathbf{m}}(\mathbf{n}) \in \mathbb{Z}[q]$ and $\prec_i$ denotes the lexicographical ordering.

**Proof:** It is immediate from Proposition 2.2 that if $u \in q^\mathcal{B}^*$ then it satisfies (i) and (ii). If $k = 0$ in part (ii), this ensures that, in the expansion of $u$ in terms of the dual canonical basis, $B_i(\mathbf{m})^*$ occurs with coefficient 1 (see Proposition 2.3).

It follows that the eigenvalue $f$ in part (i) must be the same as that appearing in (2.1), and it follows from Proposition 2.2 that $u \in \mathcal{B}^*$. The result for arbitrary $k$ follows. $\Box$

**Remark 2.5.** We note that, from the proof, we can see that $u \in \mathcal{B}^*$ if and only if $u$ satisfies (i) and (ii) with $k = 0$.

Two elements of the dual canonical basis $\mathcal{B}^*$ are said to be multiplicative if their product also lies in the dual canonical basis up to a power of $q$. They are said to $q$-commute if they commute up to a power of $q$. The following Corollary of Proposition 2.2 is due to Reineke [17, 4.5].

**Corollary 2.6.** If two elements of the dual canonical basis are multiplicative then they $q$-commute.
Definition 2.7. There is a natural action of $U^+$ on itself $\mathbb{1}$, in which each generator $E_i$ of $U^+$ acts on $U^+$ as a $q$-differential operator $\delta_i$. This is the unique action of $U^+$ on itself satisfying the following properties:

(a) (Homogeneity) If $E \in U^+_\alpha$, $x \in U^+_\gamma$, then $E(x) \in U^+_{\gamma - \alpha}$.

(b) (Leibnitz formula)

$$\delta_i(xy) = \delta_i(x)y + q^{-(\gamma,\alpha)}x\delta_i(y), \quad \text{for } x \in U^+_\gamma, \ y \in U^+.$$ 

(c) (Normalization) $\delta_i(E_j) = (1 - q^2)^{-1}\delta_{ij}$ for $i, j = 1, 2, \ldots, n$.

We remark that formula (b) implies that for all $i$ and for $r \in \mathbb{Z}_{\geq 0}$, we have $\delta_i(E_i^{(r)}) = q^{-r+1}(1 - q^2)^{-1}E_i^{(r-1)}$, which is easily checked by induction on $r$ ($E_i^{(0)}$ is interpreted as 1 and $E_i^{(-1)}$ as zero). For $u \in U^+$, set $\varphi_i(u) = \max\{r \mid \delta_i^r(u) \neq 0\}$, and for $r \in \mathbb{Z}_{\geq 0}$, let $\delta_i^{(r)}$ denote the divided power $\delta_i^{(r)}(u) := \delta_i^{(\varphi_i(b^*))}(u)$. It is known that for $b \in \mathcal{B}$, $\delta_i^{(max)}(b^*) \in \mathcal{B}^*$ (see [1, §1]). If $i \in R(w_0)$, set $a_1 = \varphi_i(b^*)$, $a_2 = \varphi_i(b^*(\delta_i^{(1)}(b^*)), \ldots, a_N = \varphi_i(\delta_i^{(N-1)}(b^*) \ldots \delta_i^{(1)}(b^*)$; then $(a_1, a_2, \ldots, a_N)$ is known as the string of $b$ (or $b^*$) in direction $i$; this coincides with the string of $b$ arising from Kashiwara’s approach to $\mathcal{B}$ (see [1] and the end of Section 2 in [18]).

3. PBW-strings

In this section we will show how the Lusztig parametrization of a dual canonical basis element can also be regarded as a string (in a similar sense to the above).

We define new operators $\Delta_i$, $i = 1, 2, \ldots, n$, depending on a choice of reduced decomposition for $w_0$, which play the role of the operators $\delta_i^{(max)}$ for the PBW parametrization. We first of all note that $\delta_i^{(max)}$ is a well-defined operator on all of $U^+$.

Definition 3.1. Let $w = s_1 s_2 \cdots s_i$ be a reduced decomposition for $w \in W$. Let

$$\Delta_1 = \delta_1^{(max)}, \ \Delta_2 = T_1 \delta_2^{(max)} T_1^{-1}, \ldots, \Delta_i = (T_1 \cdots T_{i-1}) \delta_{i-1}^{(max)} (T_1 \cdots T_{i-1})^{-1},$$

operators on $U^+$ (with codomain $U$).

We will use the following:

Lemma 3.2. (Saito) Let $i \in \{1, 2, \ldots, n\}$. Then $T_i(U^+) \cap U^+ = \ker \delta_i$.

Proof: See [19]. □

Lemma 3.3. Let $i \in R(w_0)$, and $m = (m_1, m_2, \ldots, m_N) \in \mathbb{Z}_{\geq 0}^N$. Then

$$\delta_i^{(max)}(E_i(m)^*) = E_i(0, m_2, m_3, \ldots, m_N)^*.$$ 

Proof: Let $\gamma_m = \prod_{i=1}^N \psi_{m_i}(q^2)$, so that $E_i(m)^* = \gamma_mE_i(m)$. Then $E_i(m)^* = E_i(m_1)T_i(y)$, where $y = E_{i_2}(m_2)T_2 \cdots E_{i_{N-1}}(m_{N-1})$. Using Lemma 3.2, the formula for $\gamma_m$ and the fact that $\delta_i(E_i(m)^*) = q^{-m+1}(1 - q^2)^{-1}E_i(m_1)^{m_1-1}$ the result follows. □

We can now prove a Lemma giving basic properties of the Lusztig parametrization with respect to the operators $\Delta_i$. 

Lemma 3.4. Let $b \in B$, and suppose $i \in R(w_0)$ and $\Delta_1, \Delta_2, \ldots, \Delta_N$ are as above. Let $L_i(b) = (m_1, m_2, \ldots, m_N) \in \mathbb{Z}^N_\geq$. Then, for $k = 1, 2, \ldots, N$, we have:

(i) $\Delta_i \Delta_{k-1} \cdots \Delta_1(b^*) \in B^*$,
(ii) $\varphi_b(T_i^{-1} \cdots T_{k-1}^{-1} \Delta_{k-1} \cdots \Delta_1(b^*)) = m_k$,
(iii) $L_i(\Delta_k \Delta_{k-1} \cdots \Delta_1(b^*)) = (0, 0, \ldots, 0, m_{k+1}, m_{k+2}, \ldots, m_N)$.

Proof: Since $\delta_1^{(\text{max})}$ preserves $B^*$, we see that $\Delta_1(b^*)$ lies in $B^*$, so (i) holds for $k = 1$, but we need more precise information. Since $b^* \in B^*$, we know by [5] (see also Proposition 2.3) that

\[ b^* = E_i(m)^* + qx_0, \]

where $x_0$ is a $\mathbb{Z}[q]$-linear combination of dual PBW-basis elements $E_i(n)^*$ with $n < m$ (and in particular, satisfying $n_1 \leq m_1$). Here $<$ denotes the lexicographic ordering. By Lemma 3.3, we see that

\[ \delta_1^{(\text{max})}(b^*) = E_i(0, m_2, m_3, \ldots, m_N)^* + qx_1, \]

where $x_1$ is a $\mathbb{Z}[q]$-linear combination of elements of the form $E_i(n)^*$ with $n = (0, n_2, n_3, \ldots, n_N) < m$ for all $n$ occurring in the sum. By Corollary 2.4(i) we have that $L_i(\Delta_1(b^*)) = (0, m_2, m_3, \ldots, m_N)$, so (ii) holds for $k = 1$. It is also clear that $\varphi_1(b^*) = m_1$, so (iii) holds for $k = 1$. We thus see that (i), (ii) and (iii) all hold for $k = 1$.

From the above form for $\Delta_1(b^*)$, we have that

\[ T_i^{-1}(\Delta_1(b^*)) = E_i(m_2, m_3, \ldots, m_N, 0)^* + qy_1, \]

where $y_1 = s_{i_1} s_{i_2} \cdots s_{i_N} \tilde{s}_{i_1}$ is the index of the simple root $-w(i_1)$ (the Chevalley automorphism applied to $i_1$), and $y_1$ is a $\mathbb{Z}[q]$-linear combination of dual PBW-basis elements $E_i(n)^*$ with $n = (n_2, n_3, \ldots, n_N, 0) < (m_2, m_3, \ldots, m_N, 0)$ and therefore satisfies Corollary 2.4(ii) (with $k = 0$). Since for all $i$, $T_i^{-1} \sigma_\eta$ and $\sigma_\eta T_i^{-1}$ differ on weight spaces by plus or minus a power of $q$, it follows that $T_i^{-1}(\Delta_1(b^*))$ satisfies Corollary 2.4(i), and therefore (see the remark after Corollary 2.4(i)) lies in $B^*$. Since $\delta_1^{(\text{max})}$ preserves $B^*$, we thus obtain that $\delta_1^{(\text{max})}T_i^{-1}\Delta_1(b^*) \in B^*$. Arguing as above, with $i_2$ playing the role of $i_1$, we obtain that

\[ \delta_2^{(\text{max})}T_i^{-1}\Delta_1(b^*) = E_i(0, m_3, m_4, \ldots, m_N, 0)^* + qy_2, \]

where $y_2$ is a $\mathbb{Z}[q]$-linear combination of elements of the form $E_i(n)^*$ with $n = (0, n_3, n_4, \ldots, n_N, 0) < (0, m_3, m_4, \ldots, m_N, 0)$. We also obtain that $\varphi_2(T_i^{-1}\Delta_1(b^*)) = m_2$ (part (ii) for $k = 2$). Applying $T_i$ we obtain that

\[ \Delta_2 \Delta_1(b^*) = T_i \delta_1^{(\text{max})}T_i^{-1}\Delta_1(b^*) = E_i(0, 0, m_3, m_4, \ldots, m_N)^* + qx_2, \]

where $x_2$ is a $\mathbb{Z}[q]$-linear combination of dual PBW-basis elements of the form $E_i(n)^*$ with $n = (0, 0, n_3, n_4, \ldots, n_N) < (0, 0, m_3, m_4, \ldots, m_N)$. Arguing as above for $T_i^{-1}$ and applying Corollary 2.4, we obtain that $\Delta_2 \Delta_1(b^*) \in B^*$ and that $L_i(\Delta_2 \Delta_1(b^*)) = (0, 0, m_3, m_4, \ldots, m_N)$. We thus see that (i) and (iii) hold for $k = 2$. It is now clear that an inductive argument gives (i),(ii) and (iii) for $k = 1, 2, \ldots, N$.

Given a reduced decomposition $\bar{w} = s_{i_1} s_{i_2} \cdots s_{i_N}$ for an element $w \in W$, we define the PBW-string $L_{\bar{w}}(b^*)$ of a dual canonical basis element $b^*$ in direction $\bar{w}$ as follows. Let $i = i_{s_{i_1}} s_{i_2} \cdots s_{i_N}$ be any completion of $\bar{w}$ to a reduced decomposition for $w_0$. Then let $L_{\bar{w}}(b^*) = (m_1, m_2, \ldots, m_1)$ where $L_i(b^*) = (m_1, m_2, \ldots, m_N)$. It is clear from Lemma 3.3 that this is well-defined.
We make definitions for PBW-strings in the same way as Berenstein and Zelevinsky [1] for usual strings.

**Definition 3.5.** Let \(i \in R(w_0)\). A PBW \(i\)-wall is defined to be a hyperplane in \(\mathbb{R}^N\) given by the equation \(a_k = a_{k+2}\) for some index \(k\) such that \(a_k = a_{k+2} = i_{k+1} = 1\). Let \(m \in \mathbb{R}^N\). We say that \(m\) is PBW \(i\)-regular if for every \(i' \in R(w_0)\) the point \(R_i^i(m)\) does not lie on any PBW \(i\)-wall. We define the PBW \(i\)-linearity domains to be the closures of the connected components of the set of PBW \(i\)-regular points.

As in the string case [1, 2.8], we have the following justification for the terminology; the proof is basically the same, as \(R_i^i\) is very similar to the string reparametrization function.

**Proposition 3.6.** Every PBW \(i\)-linearity domain is a polyhedral convex cone in \(\mathbb{R}^N\). Two points \(m, m'\) in \(\mathbb{R}^N\) lie in a single PBW \(i\)-linearity domain if and only if

\[
R_i^i(m + m') = R_i^i(m) + R_i^i(m')
\]

for every \(i' \in R(w_0)\).

We can now prove the analogue of [1, 2.9]:

**Theorem 3.7.** Let \(b^*, b'^*\) be elements of the dual canonical basis that \(q\)-commute. Then, for every \(i \in R(w_0)\), the PBW strings \(m = L_i(b^*)\) and \(m' = L_i(b'^*)\) belong to a single \(i\)-linearity domain.

**Proof:** We follow the proof of Berenstein and Zelevinsky [1]; however there will be some differences, so we include the details. We know by Proposition 3.6 that it is enough to show that, for every \(i' \in R(w_0)\), \(R_i^i(m + m') = R_i^i(m) + R_i^i(m')\). Thus, it is enough to show that \(m\) and \(m'\) are not separated by any PBW \(i\)-wall. Suppose that a \(i\)-wall corresponds to the move \(i = s_i \cdots s_i s_j s_i s_{i+1} \cdots s_N \mapsto s_i \cdots s_i s_j s_i s_{i+1} \cdots s_N = i'\).

By [1, 3.6] and the fact that the \(T_i\) are algebra automorphisms, the elements \((T_i T_{i} - T_i - 1)_{i=1} \cdots (T_i T_{i} - T_i - 1)\Delta_i \Delta_1 \cdots \Delta_1 (b^*)\) and \((T_i T_{i} - T_i - 1)_{i=1} \cdots (T_i T_{i} - T_i - 1)\Delta_i \Delta_1 \cdots \Delta_1 (b'^*)\) q-commute. Replacing \(b^*\) and \(b'^*\) by \((T_i T_{i} - T_i - 1)_{i=1} \cdots (T_i T_{i} - T_i - 1)\Delta_i \Delta_1 \cdots \Delta_1 (b^*)\) we can assume that \(t = 0\) (using Lemma 3.4). Let \(L_{i j} = L_{s_i s_j s_i} (b^*) = (m_1, m_2, m_3)\) and \(L_{i j} = L_{s_i s_j s_i} (b'^*) = (m_1, m_2, m_3')\) be the PBW-strings of \(b^*\) and \(b'^*\) in direction \(s_i s_j s_i\). It is sufficient to show that the integers \(m_1 - m_3\) and \(m_1 - m_3'\) are of the same sign. Now let \(\Delta_1, \Delta_2, \Delta_3\) be the operators on \(B^*\) associated to the reduced decomposition \(s_i s_j s_i\) as in Definition 3.3. Let \(b_0 = \Delta_3 \Delta_2 \Delta_1 (b^*)\), and let \(b_0' = \Delta_3 \Delta_2 \Delta_1 (b'^*)\). Suppose that \(b^* b'^* = q^n b^* b'^*\) and \(b_0 b_0' = q^{n_0} b_0 b_0'\) for integers \(n, n_0\). Then we have

\[
\Delta_3 \Delta_2 \Delta_1 (b^* b'^*) = T_i T_j \delta_{i j}^{(max)} (T_j - T_i)^{-1} \delta_{i j}^{(max)} (T_i - T_j)^{-1} \delta_{i j}^{(max)} (b')
\]

\[
= q^n \Delta_3 \Delta_2 \Delta_1 (b^*) \Delta_3 \Delta_2 \Delta_1 (b'^*)
\]

\[
= q^{n+1} \delta_0 \delta_0',
\]

where

\[
r_1 = \Phi_{i, \gamma}(m_1, m_1') + \Phi_{i, s_j s_i, \gamma - m_1, \alpha_j}(m_2, m_2') + \Phi_{i, s_j s_i, \gamma - m_1, \alpha^j}(m_3, m_3'),
\]

and \(\gamma\) is the degree of \(b^*\). Here \(\Phi_{\mu, \nu}(n, m) = nm - (\mu, \alpha_k) = m(n - (\mu, \alpha_k))\) is defined as in [1 Proposition 3.1] and we are using [1 (3.6)]. We also use the
fact that for all \( \alpha \in Q^+ \), \( T_i^{\pm 1}(U_\alpha^+) \cap U^+ \subset U_{s_i(\alpha)}^+ \). Expanding and simplifying, we obtain
\[
 r_1 = m_1 m'_1 + m_2 m'_2 + m_3 m'_3 + m_1 m'_3 - m_1 m'_3 - m'_1(\gamma, \alpha_i) - m'_2(\gamma, \alpha_i + \alpha_j) - m'_3(\gamma, \alpha_j).
\]
Similarly, we obtain that \( \Delta_3 \Delta_2 \Delta_1 (b'^*) = q^{2s} \Delta_3 \Delta_2 \Delta_1 (b'^*) \Delta_3 \Delta_2 \Delta_1 (b'^*) \), where
\[
 r_2 = m'_1 m_1 + m'_2 m_2 + m'_3 m_3 + m'_1 m_2 + m'_2 m_3 - m'_1 m_3 - m_1(\gamma', \alpha_i) - m_2(\gamma', \alpha_i + \alpha_j) - m_3(\gamma', \alpha_j),
\]
and \( \gamma' \) is the weight of \( b'^* \). We thus have:

\[
(3.1) \quad n_0 - n = r_1 - r_2.
\]

Let \( L_{ijk} = L_{s_is_js_j}(b^*) = (n_1, n_2, n_3) \) and \( L'_{ijk} = L_{s_is_js_j}(b'^*) = (n'_1, n'_2, n'_3) \) be the PBW-strings of \( b^* \) and \( b'^* \) in direction \( s_is_js_j \). Then we know that \( R(m_1, m_2, m_3) = (n_1, n_2, n_3) \) and that \( R(m'_1, m'_2, m'_3) = (n'_1, n'_2, n'_3) \), where \( R = R_{s_is_js_j}^{s_is_js_j} \) is Lusztig’s piecewise reparametrization function associated to the canonical basis in type \( A_n \) (see [13]). Let \( \Delta'_1, \Delta'_2, \Delta'_3 \) be the operators on \( B^* \) associated to the reduced decomposition \( s_is_js_j \) in Definition 3.1. It follows from Lemma 3.3 that \( b'_3 = \Delta'_1 \Delta'_2 \Delta'_1(b'^*) \), and that \( b''_3 = \Delta'_2 \Delta'_3 \Delta'_2(b'^*) \). Hence, a similar argument to that given above shows that

\[
(3.2) \quad n_0 - n = r'_1 - r'_2,
\]
where
\[
r'_1 = n'_1 n_1 + n'_2 n_2 + n'_3 n_3 + n'_1 n'_3 + n'_2 n'_3 - n_1 n'_3 - n'_1(\gamma, \alpha_i) - n'_2(\gamma, \alpha_i + \alpha_j) - n'_3(\gamma, \alpha_j)
\]
and
\[
r'_2 = n'_1 n_1 + n'_2 n_2 + n'_3 n_3 + n'_1 n'_2 + n'_2 n'_3 - n'_1 n'_3 - n_1(\gamma', \alpha_i) - n_2(\gamma', \alpha_i + \alpha_j) - n_3(\gamma', \alpha_j).
\]
Suppose now that \( m_1 > m_3 \) and \( m'_1 < m'_3 \). Then \((n_1, n_2, n_3) = R(m_1, m_2, m_3) = (m_2, m_3, m_1 + m_2 - m_3) \) and \((n'_1, n'_2, n'_3) = R(m'_1, m'_2, m'_3) = (m'_2 + m'_3 - m'_1, m'_1, m'_2) \). Equating (3.1) and (3.2) we obtain: \((m_2 - m_3)(m'_3 - m'_1) = 0\), which is a contradiction, as each of \( m_1 - m_3 \) and \( m'_3 - m'_1 \) has been assumed to be positive. A similar argument shows that we cannot have \( m_1 < m_3 \) and \( m'_1 > m'_3 \). It follows that \( \mathbf{m} \) and \( \mathbf{m}' \) cannot be separated by a PBW \( i \)-wall, and the Theorem is proved. \( \square \)

4. Properties of PBW \( i \)-Linearity Domains

In this section we show that the set of \( i \)-linearity domains forms a fan. We start with a key proposition (which we shall also need in section 5 when we discuss \( q \)-commuting properties of the dual canonical basis). We call the connected components of the set of PBW \( i \)-regular points PBW \( i \)-chambers, so that PBW \( i \)-linearity domains are the closures of PBW \( i \)-chambers (see \( \square \) [88]).

**Lemma 4.1.** Let \( \mathbf{m}, \mathbf{m}' \in \mathbb{R}_{\geq 0}^N \). Then \( \mathbf{m} \) and \( \mathbf{m}' \) lie in a single PBW \( i \)-linearity domain if and only if for all \( i' \in R(u_0) \), \( R_{i'}^i(\mathbf{m}) \) and \( R_{i'}^i(\mathbf{m}') \) are weakly on the same side of all PBW \( i' \)-walls.
Proof: By definition, \( m \) and \( m' \) lie in a single PBW \( i \)-chamber if and only if for all \( i' \in R(w_0) \), \( R_i^j(m) \) and \( R_i^j(m') \) are strictly on the same side of all PBW \( i' \)-walls. Since PBW \( i \)-linearity domains are the closures of PBW \( i \)-chambers, it follows from the continuity of the functions \( R_i^j \) that if \( m \) and \( m' \) lie in a single PBW \( i \)-linearity domain, then for all \( i' \in R(w_0) \), \( R_i^j(m) \) and \( R_i^j(m') \) are weakly on the same side of all PBW \( i' \)-walls. Conversely, if for all \( i' \in R(w_0) \), \( R_i^j(m) \) and \( R_i^j(m') \) are weakly on the same side of all PBW \( i' \)-walls, then for all \( i' \in R(w_0) \),

\[
R_i^j(m + m') = R_i^j(m) + R_i^j(m').
\]

This can be proved by induction on the number of braid relations needed to take \( i \) to \( i' \) (see the proof of [1, 8.1]). By Proposition 3.6, this implies that \( m \) and \( m' \) lie in a single PBW \( i \)-linearity domain. \( \square \)

**Proposition 4.2.** Let \( i \in R(w_0) \). Suppose that \( m_i, 1 \leq i \leq k \) and \( \sum_{i=1}^k m_i \) all lie in a single PBW \( i \)-linearity domain \( X \). Suppose also that \( \sum_{i=1}^k m_i \) and \( q \) both lie in a single PBW \( i \)-linearity domain (not necessarily the same as \( X \)). Then \( m_i \), \( i = 1, 2, \ldots, k \), \( \sum_{i=1}^k m_i \), and \( q \) all lie in a single PBW \( i \)-linearity domain.

**Proof:** We use Lemma 3.1 throughout. By the assumptions in the Proposition, we have:

(i) \( R_i^j(m_i) \), \( 1 \leq i \leq k \) and \( R_i^j(\sum_{i=1}^k m_i) \) are weakly on the same side of all PBW \( i' \)-walls, and:

(ii) \( R_i^j(q) \) and \( R_i^j(\sum_{i=1}^k m_i) \) are weakly on the same side of all PBW \( i' \)-walls. Let \( H \) be a PBW \( i' \)-wall. Firstly, let \( i' \in R(w_0) \) be such that \( R_i^j(\sum_{i=1}^k m_i) \) does not lie on \( H \). Then \( R_i^j(m_i), i = 1, 2, \ldots, k, R_i^j(\sum_{i=1}^k m_i) \) and \( R_i^j(q) \) are all weakly on the same side of \( H \). Next, suppose that \( i' \in R(w_0) \) is such that \( R_i^j(\sum_{i=1}^k m_i) \) does lie on \( H \). By (i) and Proposition 3.6, \( R_i^j(m_i) \) lies on \( H \) for \( 1 \leq i \leq k \). Hence \( R_i^j(m_i), 1 \leq i \leq k, R_i^j(\sum_{i=1}^k m_i) \) and \( R_i^j(q) \) are all weakly on the same side of \( H \). It follows (using Lemma 3.1 again) that the \( m_i, i = 1, 2, \ldots, k, \sum_{i=1}^k m_i \), and \( q \) all lie in a single PBW \( i \)-linearity domain. \( \square \)

**Remark 4.3.** If, in the assumptions and conclusion of the proposition, the property "lie in a single PBW \( i \)-linearity domain" is replaced by the property "\( q \)-commute" (and the tuples involved are assumed to lie in \( \mathbb{Z}_{\geq 0}^N \)), the result is not clear.

Recall that a strongly convex polyhedral cone is a convex polyhedral cone \( C \) for which \( v \in C \) implies \( -v \notin C \). A set of strongly convex polyhedral cones is said to form a fan if the face of every cone in the set lies in the set and if the intersection of any two cones in the set lies again in the set.

**Corollary 4.4.** Let \( i \in R(w_0) \). Then the set of PBW \( i \)-linearity domains (together with all their faces) forms a fan in \( \mathbb{R}^N \).

**Proof:** We first of all note that each PBW \( i \)-linearity domain is a convex polyhedral cone (by Proposition 3.6) and thus is in fact a strongly convex polyhedral cone as it contains no point with negative coordinates. The same is therefore true for all of its faces. We now show that if \( C \) and \( C' \) are distinct PBW \( i \)-linearity domains, then \( C \cap C' \) is a face of \( C \) and of \( C' \). Suppose that \( C \cap C' \) is not a face of \( C \). Then \( C \cap C' \subset F \), where \( F \) is a face of \( C \) (since \( C \) and \( C' \) are convex polyhedral cones). Let \((C')^\circ\) denote the interior of \( C' \); we will use similar notation for the
occurs. The form $d$ (see [4]) associated to canonical basis.

**Proposition 5.2.** We next consider the case where $C, C'$ are faces of distinct PBW-linearity domains, $P$ and $P'$. We show that $C \cap C'$ is a face of $C$ and of $C'$. Suppose that $C \cap C'$ is not a face of $C$. Then, as above, $C \cap C' \subsetneq F$, where $F$ is a face of $C$. As above, we can then choose $q \in (P')^\circ$, $p \in F \setminus C'$ and $m \in C \cap C'$ such that $m + p \in C \cap C'$, since $C \cap C'$ and $F$ are convex polyhedral cones. Note that the first two properties imply that $p \notin P'$. This is because $p$ lies in the linear span of $C' \cap C$, so $p$ lies in the linear span of $C'$. Since $P'$ is convex, if $p$ was in $P'$ and in the linear span of its face $C'$, it would have to lie in $C'$, a contradiction to the assumption that $p \notin P'$. So $q$ and $m + p$ lie in a single PBW $i$-linearity domain $P'$, while $m, p$ and $m + p$ all lie in a single PBW $i$-linearity domain $P$. Proposition 5.2 asserts that $q$ and $p$ both lie in a single PBW $i$-linearity domain, but as $q \in (P')^\circ$, $P'$ is the only PBW $i$-linearity domain containing $q$, while $p \notin P'$, so we have a contradiction. Hence, $C \cap C'$ is a face of $C$, and similarly, we can see that it is a face of $C'$. Finally, suppose that $C, C'$ are distinct faces of a single PBW $i$-linearity domain $P$. Then $C \cap C'$ is a face of $C$, and of $C'$, since $P$ is a polyhedral cone. We have therefore shown that the set of PBW $i$-linearity domains, together with all their faces, forms a fan in $\mathbb{R}^N$ as required. □

5. $q$-COMMUTING PRODUCTS OF QUANTUM FLAG MINORS

For any reduced decomposition $\hat{w} = s_{i_1} \ldots s_{i_k}$ of an element $w$ in $W$, we define as in [5], see also [2], the quantum flag minors $\Delta^*_w$. Roughly speaking, $\Delta^*_w$ is the element of the dual canonical basis corresponding to the extremal vector of weight $w\subset_i$ in the Weyl module with highest weight $\subset_i$. By [5], we have:

**Proposition 5.1.** Fix a reduced decomposition $i = (i_1, i_2, \ldots, i_N)$ in $R(w_0)$. Set $\Delta^*_k = \Delta^*_{\hat{w}_k}$, where $\hat{w}_k = s_{i_1} \ldots s_{i_k}$, $1 \leq k \leq N$. Then, the algebra $A_i$ generated by the $\Delta^*_k$ is a $q$-polynomial algebra, spanned (as a space) by a part of the dual canonical basis.

Let $d_i$ be the form on $\mathbb{Z}^N$ defined by

$$d_i(m, n) = \sum_{j < i} (\beta_i, \beta_j) m_i n_j + \sum_i m_i n_i,$$

where the $\beta_k$’s are the roots associated to $i$. We denote it by $d$ if no confusion occurs. The form $d_i$ encodes the $q$-commutations in the graded algebra $\text{Gr}_i(U^+)$ (see [4]) associated to $i$. To be more precise, set

$$n_k = \sum_{t, t \leq k, i_t = i_k} e_t.$$

Then:

**Proposition 5.2.** Fix $i$ in $R(w_0)$. For $k$, $1 \leq k \leq N$, we have $\Delta^*_k = B_i(n_k)^*$. Moreover, for all $m, n$ in $\mathbb{Z}^N_{\geq 0}$.
(i) \( q^{d(m,n)}B(m)^*B(n)^* + \sum_{1 \leq m+n} Z[q, q^{-1}] B(1)^* \)
(ii) \( q^{d(n,m)} \Delta^*_q E(m)^* \in E(n_k + m)^* + qL^* \).

**Proof:** The first assertion is \[3\] 2.1. Part (i) is a straightforward consequence of \[3\] Corollary 2.1 (iii) and Theorem 1.2. Part (ii) is given by \[3\] Theorem 3.2.  

As in \[13\], we associate a reduced decomposition \( i(Q) \) (up to commutation) of \( w_0 \) to each quiver \( Q \) whose underlying graph is \( A_n \). Let \( R(Q) \subset R(w_0) \) be the set of all reduced decompositions associated to \( Q \).

**Lemma 5.3.** All quantum flag minors can be realized in the form \( B_i(n_k) \), with \( i \in R(Q) \), for some quiver \( Q \) of type \( A_n \).

**Proof:** See \[5\] 4.3. □

Remark that if \( i \) is associated to a quiver orientation, then \( \prec_i \) is the so-called degeneration ordering, \( \prec_i \), and \( d_i \) is Reineke's homological form, see \[17\]. As a consequence, we have:

**Proposition 5.4.** Let \( Q \) be a quiver of type \( A_n \). Fix a reduced decomposition \( i \) in \( R(Q) \). Then, for all \( k, 1 \leq k \leq N \),
(i) \( d_i(n_k,?) \) is increasing for \( \prec_i \), i.e. \( m \prec_i n \Rightarrow d_i(n_k, m) \leq d_i(n_k, n) \),
(ii) \( q^{d(n_k,m)}B(n_k)^* B(m)^* \in B(n_k + m)^* + q \sum_{l \leq n_k + m} Z[q] B(1)^* \).

**Proof:** (i) is \[5\] Proposition 4.2. (ii) follows from (i) together with Proposition 5.2 and Proposition 4.2 (iii). □

We can now prove our main theorem.

**Theorem 5.5.** Let \( c \) be a \( q \)-commuting product of quantum flag minors and let \( b \) be an element of \( B^* \) which \( q \)-commutes with \( c \). Then, \( cb \) is an element of \( B^* \) up to a power of \( q \).

**Proof:** In the sequel, the equalities are given "up to a power of \( q \)." Set \( c = b_m b_{m-1} \cdots b_1 \), where the \( b_k \), \( 1 \leq k \leq m \), are \( q \)-commuting quantum flag minors. Using an induction on \( m \), we know that \( c \) is an element of \( B^* \) up to a power of \( q \).

Now, by Lemma 5.3 for each \( k \) we can fix a reduced decomposition \( i_k \) in \( R(Q) \) (for some quiver \( Q \) of type \( A_n \)) associated to the quantum flag minor \( b_k \). Let \( p_k \), resp. \( p \), be such that \( b_k = B_{i_k}(p_k) \), resp. \( b = B_{i_1}(p) \). We have \( c \in q^Z B_{i_1}(\sum_k p_k) \).

Hence, by Theorem 5.7 \( c \) and \( \sum_k p_k \) are in a single \( i_1 \)-linearity domain, since \( b \) and \( c \)-commute. Moreover, this theorem implies also that \( p_r \), \( 1 \leq r \leq m \), and \( \sum_{k=1}^s p_k \), \( 1 \leq s \leq m \), are in a single \( i_1 \)-linearity domain. By Proposition 4.2 \( p_r \), \( p_s \), \( \sum_{k=1}^s p_k \), \( 1 \leq r, s \leq m \), are in a single \( i_1 \)-linearity domain. Hence :

\[
R_{i_1}^{i_1} (p_k + \ldots + p_1 + p) = R_{i_1}^{i_1} (p_k) + \ldots + R_{i_1}^{i_1} (p_1) + R_{i_1}^{i_1} (p), \quad 1 \leq k, s \leq m.
\]

By Proposition 5.4 (ii),

\[
B_{i_1} (p_1)^* B_{i_1} (p)^* \in B_{i_1} (p_1 + p)^* + q \sum_{l \prec_i p_1 + p} Z[q] B_{i_1} (l)^*.
\]

Suppose by induction on \( k \) that

\[
B_{i_1} (p_1)^* B_{i_1} (p)^* \in B_{i_k} (p_k + \ldots + p_1 + p)^* + q \sum_{l_k \prec_i p_k + \ldots + p_1 + p} Z[q] B_{i_k} (l_k)^*.
\]
Then,
\[
b_{k+1} \ldots b_1 b \in B_{i_{k+1}}(R_{i_{k+1}}^{i_k}(p_{k+1}))^*(B_{i_{k+1}}(R_{i_{k+1}}^{i_k}(p_k + \ldots p_1 + p))^* + q \sum_{l_k < l_k} Z[q] B_{i_{k+1}}(l_k)^*)
\]
\[
= B_{i_{k+1}}(R_{i_{k+1}}^{i_k}(p_{k+1}))^*(B_{i_{k+1}}(R_{i_{k+1}}^{i_k}(p_k + \ldots p_1 + p))^* + q \sum_{l_k < l_k} Z[q] B_{i_{k+1}}(R_{i_{k+1}}^{i_k}(l_k))^*).
\]

In the previous sum, we have, by Proposition 5.2 (i) and (*):
\[
R_{i_{k+1}}^{i_k}(l_k) \prec_{i_{k+1}} R_{i_{k+1}}^{i_k}(p_k) + \ldots + R_{i_{k+1}}^{i_k}(p_1) + R_{i_{k+1}}^{i_k}(p) = R_{i_{k+1}}^{i_k}(p_k + \ldots p_1 + p).
\]

Now, by Proposition 5.4 and the linearity property (*), we obtain the induction.

We can now show that \( cb \) satisfies the properties (i) and (ii) of Corollary 2.4. Taking \( k = m \), and using Proposition 2.3 and the transitivity of \( \prec_{i_k} \), the previous results imply that, up to a power of \( q \), \( cb = E_{i_m}(R_{i_k}^{i_m}(p_m + \ldots p_1 + p))^*+qx \), where \( x \) is a \( Z[q] \)-linear combination of dual PBW-basis elements \( E_{i_m}(n)^* \) with \( n < R_{i_k}^{i_m}(p_m + \ldots p_1 + p) \), and therefore \( cb \) satisfies property Corollary 2.4 (ii). Now, Part (i) of Corollary 2.4 is clear because \( c \) and \( b \) are \( q \)-commuting elements of the dual canonical basis.

This implies that the answer to Question 1.1 is yes when the real element is in the adapted algebra associated to any reduced decomposition.

**Corollary 5.6.** Let \( i \) in \( R(w_0) \) and suppose that \( b \) in \( B^* \) and \( c \) in \( B^* \cap A_1 \) \( q \)-commute. Then, \( b \) and \( c \) are multiplicative.

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