Saturating Sperner Families

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Received: 31 May 2011 / Published online: 26 May 2012 © Springer 2012

Abstract A family $\mathcal{F} \subseteq 2^{[n]}$ saturates the monotone decreasing property $\mathcal{P}$ if $\mathcal{F}$ satisfies $\mathcal{P}$ and one cannot add any set to $\mathcal{F}$ such that property $\mathcal{P}$ is still satisfied by the resulting family. We address the problem of finding the minimum size of a family saturating the $k$-Sperner property and the minimum size of a family that saturates the Sperner property and that consists only of $l$-sets and $(l + 1)$-sets.

Keywords Extremal set theory · Sperner property · Saturation

The research of D. Gerbner, B. Keszegh and C. Palmer was supported by Hungarian National Scientific Fund, Grant number: OTKA NK-78439.
The European Union and the European Social Fund have provided financial support to the project under the Grant Agreement No. TÁMOP 4.2.1./B-09/1/KMR-2010-0003 to D. Pálvölgyi.
The research of B. Patkós’s was supported by Hungarian National Scientific Fund, Grant Numbers: OTKA K-69062 and PD-83586.

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1 Introduction

One of the most basic and most studied problems of extremal combinatorics is that how many edges a (hyper)graph can have if it possesses some prescribed property $P$. If this property $P$ is monotone decreasing (i.e. if $G$ possesses $P$, then $F \subseteq G$ implies $F$ possesses $P$), then there exists a “dual” problem to the above one: we say that a (hyper)graph $G$ saturates property $P$ if it possesses property $P$ but adding any (hyper)edge $E$ to $G$ would result in a graph not having property $P$. The problem is to determine the minimum size that such a saturating (hyper)graph can have. Many researchers have dealt with this kind of problems both for graphs [2–7,14,17,20,24,25,31–33] and hypergraphs [1,13,15,16,22,23]. To our knowledge all papers so far have considered Turán-type properties or the intersecting property (with the exception of [10]): properties defined through some forbidden (hyper)graphs.

In the present paper we investigate the saturation of Sperner-type properties. To introduce our main definitions let $k$ and $n$ be positive integers and let $F \subseteq 2^{[n]}$ be a family of sets such that,

1. there do not exist $k + 1$ distinct sets $F_1, \ldots, F_{k+1} \in F$ that form a $(k + 1)$-chain, i.e. $F_1 \subset F_2 \subset \cdots \subset F_{k+1}$ holds,
2. for every set $S \in 2^{[n]} \setminus F$ there exist $k$ distinct sets $F_1, \ldots, F_k \in F$ such that $S$ and the $F_i$’s form a $(k + 1)$-chain.

A family $F$ is called $k$-Sperner if it satisfies Property 1, weakly saturating $k$-Sperner if it satisfies 2 and (strongly) saturating $k$-Sperner if it satisfies both. The maximum size of a $k$-Sperner family $F \subset 2^{[n]}$ was determined by Sperner [29] in the special case $k = 1$ and by Erdős [11] for arbitrary $k$.

In Sect. 2 we will derive bounds on $\text{sat}(n,k)$ (wsat$(n,k)$) the minimum number of sets that strongly saturating $k$-Sperner (weakly saturating $k$-Sperner) family $F \subset 2^{[n]}$ can contain. By definition, we have $\text{wsat}(n,k) \leq \text{sat}(n,k)$. The following product construction shows that there is an upper bound on both of these numbers that is independent of $n$, namely $\text{wsat}(n,k) \leq \text{sat}(n,k) \leq 2^{k-1}$. Let $F \subset 2^{[n]}$ be defined by

$F := 2^{[k-2]} \times \{\emptyset, [n]\setminus[k-2]\} = 2^{[k-2]} \cup \{F \in 2^{[n]} : [n]\setminus[k-2] \subseteq F\}$.

It is easy to see that $F$ is indeed strongly saturating $k$-Sperner. It is natural to formulate the following conjecture.

Conjecture 1 For every positive integer $k$ there exists an $n_0 = n_0(k)$ such that for any $n \geq n_0$ we have $\text{sat}(n,k) = 2^{k-1}$.

It is trivial to verify that $n_0(k) = k$ for $k = 1, 2, 3$. By giving constructions we will prove the following two upper bounds.

Theorem 2 For integers $6 \leq k \leq n$ we have the following inequalities:

(i) $\text{sat}(k,k) \leq \frac{15}{16} 2^{k-1}$,
(ii) $\text{wsat}(n,k) = O\left(\frac{\log k}{k} 2^k\right)$.