General Couplings of Four Dimensional Maxwell-Klein-Gordon System: Global Existence

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Abstract

In this paper, we consider the multi component fields interactions of the complex scalar fields and the electromagnetic fields (Maxwell-Klein-Gordon system) on four dimensional Minkowski spacetime with general gauge couplings and the scalar potential turned on. Moreover, the complex scalar fields span an internal manifold assumed to be Kähler. Then, by taking the Kähler potential to be bounded by $U(1)^N$ symmetric Kähler potential, the gauge couplings to be bounded functions, and the scalar potential to be the form of either polynomial, sine-Gordon, or Toda potential, we prove the global existence of the system.

1 Introduction

The Maxwell-Klein-Gordon (MKG) system describes the multi-field interaction between the complex scalar fields $\phi^a$ and the electromagnetic fields $A_{\mu}^a$ on 4-dimensional Minkowski spacetime with the standard coordinate $x^\mu = (t, x^i)$ where $\mu = 0, 1, 2, 3$ and $i = 1, 2, 3,$ and metric $\eta_{\mu\nu} = \text{diag}(−1, 1, 1, 1)$. Let $\Lambda, \Sigma, \Gamma = 1, 2, 3,..., N_V$, denotes the number of gauge fields and the Roman index $a, b, c = 1, 2, 3,..., N_C$, describes the number of complex scalar fields. The complex scalar fields span an internal manifold assumed to be Hermitian manifold endowed with metric $g_{ab}$.\textsuperscript{*}

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The Lagrangian of this MKG system with general couplings can be written down as

\[
\mathcal{L} = -\frac{1}{4} h_{\Lambda \Sigma} (\phi, \bar{\phi}) \mathcal{F}^\Lambda_{\mu \nu} \mathcal{F}^{\Sigma \mu \nu} + \frac{1}{4} k_{\Lambda \Sigma} (\phi, \bar{\phi}) \mathcal{F}^\Lambda_{\mu \nu} \mathcal{F}^{\Sigma \mu \nu} - g_{ab} (\phi, \bar{\phi}) D_\mu \phi^a D^\mu \phi^b - V (\phi, \bar{\phi}) ,
\]

(1.1)

where \( \mathcal{F}^\Lambda_{\mu \nu} \equiv \partial_\mu A^\Lambda_{\nu} - \partial_\nu A^\Lambda_{\mu} \) is the gauge field strength, \( D_\mu \phi^a \equiv \partial_\mu \phi^a - i q_1 A^\mu_{\nu} \phi^a \) is the covariant derivative, and \( \mathcal{F}^{\Sigma \mu \nu} \) is the Hodge dual of the \( \mathcal{F}^\Lambda_{\mu \nu} \). The real functions \( h_{\Lambda \Sigma} (\phi, \bar{\phi}) \) and \( k_{\Lambda \Sigma} (\phi, \bar{\phi}) \) denote the gauge couplings, and the real function \( V (\phi, \bar{\phi}) \) is the scalar potential of MKG system. Since the Lagrangian (1.1) is invariant under local \( U(1)^N \) symmetry where \( U(1)^N = \bigotimes_{n=1}^N U(1) \), without loss of generality we could take \( h_{\Lambda \Sigma} (\phi, \bar{\phi}) = h_{\Lambda \Sigma} (|\phi|^2) \), \( k_{\Lambda \Sigma} (\phi, \bar{\phi}) = k_{\Lambda \Sigma} (|\phi|^2) \), and \( V (\phi, \bar{\phi}) = V(|\phi|^2) \) where \( |\phi|^2 \equiv \delta_{ab} \phi^a \bar{\phi}^b \). Moreover, to simplify the case, the scalar manifold has to be Kähler manifold equipped with metric \( g_{ab} = \partial_a \partial_b K \) where \( K \equiv K(\phi, \bar{\phi}) \) is the Kähler potential admitting \( U(1)^N \) isometry.

In recent years, the analysis of the global solution of the Maxwell Klein-Gordon system has been considered in detail. These systems obey the local gauge transformation. Then, we have the freedom to choose the gauge condition according to the method and problem considered. The well-known gauge choices are the Coulomb gauge, \( \partial^A A^a = 0 \), and the Lorenz gauge, \( \partial_\mu A^\mu = 0 \). Considering the Coulomb gauge condition, Klainerman and Machedon proved that the solution exists globally for the finite initial data [1]. As for the global well-posedness of the MKG systems in the Lorentz gauge condition, we suggest an interested reader to consult [2].

Another gauge condition we can take into account is the temporal gauge condition, \( A_0 = 0 \) as in [3][4]. In this choice, the local and global existence of the Yang-Mills-Higgs equation is also obtained [5][6]. In the previous studies, we consider the temporal gauge conditions to prove the local existence of the bosonic part of \( N = 1 \) supersymmetric Yang-Mills-Higgs with general couplings and the scalar potential turned on [7]. In particular, we take the Kähler potential to be bounded above by the \( U(n) \) symmetric Kähler potential, the first derivative of the scalar potential to be locally Lipshitz, and the first derivative of gauge couplings to be at most linear growth functions.

It is of interest to complete the proof of our previous study by showing its global existence. However, we are facing a problem to regularize such as the three vertex terms of the gauge field \( A^\Lambda_{\mu} \) when the gauge couplings \( h_{\Lambda \Sigma} (\phi, \bar{\phi}) \) and \( k_{\Lambda \Sigma} (\phi, \bar{\phi}) \) are no longer constants and moreover, the internal manifold of the \( \sigma \)-model is not flat. Therefore, to evade such a problem, in this paper we only establish the proof of the global existence of the MKG system for multi-field interactions with the addition of general gauge coupling in temporal gauge and scalar potential turned on. In particular, we assume that the gauge couplings \( h_{\Lambda \Sigma} (\phi, \bar{\phi}) \) and \( k_{\Lambda \Sigma} (\phi, \bar{\phi}) \) are bounded smooth functions, the Kähler potential \( K \) has to be bounded by \( U(1)^N \) symmetric Kähler potential which generalizes the case of [7], and the scalar potential has the form of either polynomial, sine-Gordon, or Toda potential.

We organize the paper as follows. In Section 2, we derive the field equations of motions for both gauge and complex scalar fields. We also discuss some assumptions regarding the gauge coupling functions. In Section 3, we provide the discussion of properties of the scalar internal manifold which is Kähler. In Section 4, we discuss some estimates for the gauge and complex scalar fields which are the important ingredients for proving the global existence. Finally, the final proof and the main theorem are presented in Section 5.
2 The Field Equations of Motions

Let us first consider the equations of motions of Lagrangian (1.1)

\[
\partial^\alpha F^\Sigma_{\alpha\gamma} = h^{\Lambda\Sigma}\left\{i\gamma_{\Lambda}g_{ab}\left[D_\gamma\phi^a\tilde{\phi}^b - \phi^a\tilde{D}_\gamma\phi^b\right] - \partial^\alpha h_{\Lambda\Gamma}F^\Gamma_{\alpha\gamma} + \partial^\alpha k_{\Lambda\Gamma}\tilde{F}^\Gamma_{\alpha\gamma}\right\}, \tag{2.1}
\]

\[
D^\alpha D_\alpha \phi^b = g^{ab}\left(\frac{1}{4}F^\Lambda_{\alpha\beta}\partial_dG^{\alpha\beta}_d - \partial_dg_{ab}D_\alpha\phi^a\tilde{D}_\gamma\phi^b - \partial^\alpha g_{ab}\tilde{D}_\gamma\phi^b - \partial_dV\right), \tag{2.2}
\]

with \(h^{\Lambda\Sigma}\) is the inverse of \(h_{\Lambda\Sigma}\), and

\[
G^{\alpha\beta}_\Lambda = -h_{\Lambda\Sigma}F^{\Sigma[\alpha\beta} + k_{\Lambda\Sigma}\tilde{F}^{\Sigma]\alpha\beta}. \tag{2.3}
\]

The gauge field strength also yields the Bianchi identity

\[
\partial_\gamma F^\Sigma_{\alpha\mu} + \partial_\alpha F^\Sigma_{\gamma\mu} + \partial_\mu F^\Sigma_{\gamma\alpha} = 0. \tag{2.4}
\]

Taking the covariant divergence of (2.1) and use the Bianchi identity of the gauge field strength, we have

\[
\Box F^\Sigma_{\gamma\mu} = -i\gamma_{\Lambda}h^{\Lambda\Sigma}\left\{\partial_\mu g_{ab}D_\gamma\phi^a\tilde{\phi}^b + g_{ab}\partial_\mu\left(D_\gamma\phi^a\tilde{\phi}^b\right)\right\} + h^{\Lambda\Sigma}(\partial^\alpha h_{\Lambda\Gamma})\partial_\alpha F^\Gamma_{\gamma\mu} - h^{\Lambda\Sigma}(\partial^\alpha \partial_\gamma h_{\Lambda\Gamma})F^\Gamma_{\alpha\mu} + h^{\Lambda\Sigma}(\partial^\alpha \partial_\gamma k_{\Lambda\Gamma})\tilde{F}^\Gamma_{\alpha\mu}. \tag{2.5}
\]

In a similar way, we can obtain

\[
\Box D_\mu \phi^\alpha = i\gamma_{\Gamma}\partial^\alpha F^\Gamma_{\mu\alpha} + i\gamma_{\Gamma}F^\Gamma_{\mu\alpha}\partial^\alpha \phi^\alpha + i\gamma_{\Gamma}\partial^\alpha\left(A_\Gamma\partial_\alpha \phi^\alpha\right) - i\gamma_{\Gamma}\partial^\alpha\left(A_\Gamma\partial_\alpha \phi^\alpha\right) + \partial_\mu \partial^\alpha D_\alpha \phi^\alpha, \tag{2.6}
\]

where \(\Box \equiv -\partial^2_t + \partial^2_r + \partial^2_\theta + \partial^2_\phi\) is the d’Alembert operator.

We translate the coordinate system to the lightcone coordinate system centered at \(p\). Then, we can write the equations (2.5) and (2.6) as the vanishing of a surface integral over the interior of the past light cone \(K_p\) from a certain point \(p\) to the initial data surface. By using spherical means method, we can write the equations of MKG theory with general coupling in the integral form of its field strength as

\[
F^\Sigma_{\gamma\mu} = F^\Sigma_{\gamma\mu}^{\text{lin}} + \frac{1}{4\pi}\int_{K_p} \int r dr d\Omega \left( -ih^{\Lambda\Sigma}g_{\Lambda} \left\{ \partial_\mu g_{ab}D_\gamma\phi^a\tilde{\phi}^b + g_{ab}\partial_\mu\left(D_\gamma\phi^a\tilde{\phi}^b\right) \right\} \right) + h^{\Lambda\Sigma}(\partial^\alpha h_{\Lambda\Gamma})\partial_\alpha F^\Gamma_{\gamma\mu}, \tag{2.7}
\]

\[
D_\mu \phi^\alpha = D_\mu \phi^\alpha^{\text{lin}} + \frac{1}{4\pi}\int_{K_p} \int r dr d\Omega \left( i\gamma_{\Gamma}\partial^\alpha F^\Gamma_{\mu\alpha} + i\gamma_{\Gamma}F^\Gamma_{\mu\alpha}\partial^\alpha \phi^\alpha \right) + \partial_\mu \partial^\alpha \left(A_\Gamma\partial_\alpha \phi^\alpha\right) + \partial_\mu \partial^\alpha D_\alpha \phi^\alpha \right|_{t=-r}, \tag{2.8}
\]

with

\[
F^\Sigma_{\gamma\mu}^{\text{lin}} = \frac{1}{4\pi}\int_{S^2} d\Omega \left[ r_0 \frac{\partial \{ F^\Sigma_{\gamma\mu} \}}{\partial t} + r_0 \frac{\partial \{ F^\Sigma_{\gamma\mu} \}}{\partial r} + F^\Sigma_{\gamma\mu} \right]_{t=t_0, r=r_0}, \tag{2.9}
\]

\[
D_\mu \phi^\alpha^{\text{lin}} = \frac{1}{4\pi}\int_{S^2} d\Omega \left[ r_0 \frac{\partial \{ D_\mu \phi^\alpha \}}{\partial t} + r_0 \frac{\partial \{ D_\mu \phi^\alpha \}}{\partial r} + D_\mu \phi^\alpha \right]_{t=t_0, r=r_0}. \tag{2.10}
\]
The energy-momentum tensor of this system is given by
\[ T^{\mu\nu} = \frac{h_{\Lambda\Sigma}}{2} \left( \mathcal{F}^{\Lambda|\mu} \mathcal{F}^{\Sigma|\nu} + \mathcal{F}^{\Lambda|\mu} \tilde{\mathcal{F}}^{\Sigma|\nu} \right) + 2g_{ab}D^\mu \phi^a D^\nu \phi^b - \eta^{\mu\nu} g_{ab}D_\gamma \phi^a D^\gamma \phi^b - \eta^{\mu\nu} V. \] (2.11)

It is common to split the field strength \( F^{\mu\nu} \) into electric and magnetic component as
\[ E^{\Sigma|\alpha} = \mathcal{F}^{\Sigma|\alpha}, \quad H^{\Sigma|\alpha} = \frac{1}{2} \varepsilon^{ijk} \mathcal{F}_{ij}^{\Sigma}. \] (2.12)

Thus, we can write the energy function as
\[ \mathcal{E}_0 = \int_{B_p} r^2 dr d\Omega \left( \frac{h_{\Lambda\Sigma}}{2} \left( E^{\Lambda} E^{\Sigma} + H^{\Lambda} H^{\Sigma} \right) + g_{ab}D^0 \phi^a D^0 \phi^b + g_{ab}D_\gamma \phi^a D^\gamma \phi^b + V \right) \bigg|_{t=t_0}, \] (2.13)

where \( B_p \) represents a solid sphere in the lightcone coordinate that intersect with \( K_p \).

Throughout this paper, we set the Lagrangian (1.1) to be invariant with respect to the local transformation \( U(1)^N \) such that we have
\[ h_{\Lambda\Sigma} \left( \bar{U} \phi, \bar{U} \bar{\phi} \right) = h_{\Lambda\Sigma} \left( \phi, \bar{\phi} \right), \]
\[ k_{\Lambda\Sigma} \left( \bar{U} \phi, \bar{U} \bar{\phi} \right) = k_{\Lambda\Sigma} \left( \phi, \bar{\phi} \right), \]
\[ V \left( \bar{U} \phi, \bar{U} \bar{\phi} \right) = V \left( \phi, \bar{\phi} \right). \] (2.14)

As the result of the gauge ambiguity, we have the freedom to choose the appropriate gauge conditions. In particular, we choose the temporal gauge condition
\[ A_0^\Sigma (x) = 0, \] (2.15)

for all \( \Sigma \) which has been shown in [7] that the solutions of (2.1) and (2.2) satisfy (2.15) for all time. We take

**Assumption 1.**
\[ h_{\Lambda\Sigma} \left( \phi, \bar{\phi} \right) = h_{\Lambda\Sigma} \left( |\phi|^2 \right), \]
\[ k_{\Lambda\Sigma} \left( \phi, \bar{\phi} \right) = k_{\Lambda\Sigma} \left( |\phi|^2 \right), \] (2.16)

with \( |\phi|^2 \equiv \delta_{ab} \phi^a \phi^b \), and both \( h_{\Lambda\Sigma} \left( |\phi|^2 \right) \) and \( k_{\Lambda\Sigma} \left( |\phi|^2 \right) \) are bounded functions for all \( \Lambda, \Sigma \).

### 3 The Internal Manifold

This section is devoted to discuss some properties of the internal scalar manifold which has to be Kähler mentioned in section [11]. In particular, we consider the case of the Kähler potential to be bounded above a \( U(1)^N \) symmetric function. Our estimates derived in this section play an important role in proving the global existence of this system.

First of all, let \( K \) be a Kähler potential satisfying the following condition
\[ K \left( \phi, \bar{\phi} \right) \leq \Phi \left( |\phi| \right), \] (3.1)

with
\[ |\phi| = \left( \delta_{ab} \phi^a \phi^b \right)^{1/4}. \] (3.2)

Then, we have the following lemma
Lemma 1. Suppose $\mathcal{M}$ is a Kähler manifold satisfying (3.1). Let $\Phi$ satisfy the inequality
\[
\left| \frac{Q'}{2|\Phi|} \right| \leq \sum_{n=0}^{N} b_n |\phi|^n , \tag{3.3}
\]
where $b_n > 0$ for all $n$, $Q(|\phi|) = \frac{1}{4|\phi|} \left( \Phi'' - \frac{\Phi'}{|\phi|} \right)$, and $\Phi' = \frac{\partial \Phi}{\partial |\phi|}$. Then, we have
\[
|K| \leq \sum_{n=0}^{N} \frac{8b_n}{(n+4)(n+5)(n+6)} |\phi|^{n+6} + \sum_{n=0}^{N} \frac{12b_n}{(n+2)(n+3)(n+4)} |\phi|^{n+4} + 2C_1|\phi|^3 + C_2 \frac{|\phi|^2}{2} + C_3 , \tag{3.4}
\]
with $C_i \geq 0$ for all $i = 1, 2, 3$.

Proof. If $\bar{\mathcal{M}}$ is a Kähler manifold generated by $\Phi$, then we can write the metric $\tilde{g}_{ab} = \partial_a \partial_b \Phi$ as
\[
\tilde{g}_{ab} = \frac{\Phi'}{2|\phi|} \delta_{ab} + \frac{1}{4|\phi|} \left( \Phi'' - \frac{\Phi'}{|\phi|} \right) \delta_{a\bar{c}} \delta_{\bar{d}c} \phi^c \phi^\bar{d} , \tag{3.5}
\]
with $\Phi' = \frac{\partial \Phi}{\partial |\phi|}$. The inverse and the first derivative of $\tilde{g}_{ab}$ can be written down
\[
\partial_a \tilde{g}_{ab} = Q (\delta_{a\bar{c}} \delta_{\bar{d}c} + \delta_{a\bar{d}} \delta_{\bar{c}c}) \phi^\bar{d} + \frac{Q'}{2|\phi|} \left( \delta_{a\bar{d}} \delta_{\bar{c}c} \phi^c \phi^\bar{d} + \frac{Q'}{2|\phi|^2} \left( \Phi'' - \frac{\Phi'}{|\phi|} \right) \phi^\bar{d} \phi^c \phi^\bar{f} ,
\]
respectively, where
\[
Q(|\phi|) = \frac{1}{4|\phi|^2} \left( \Phi'' - \frac{\Phi'}{|\phi|} \right) , \quad \frac{Q'}{2|\phi|} = \frac{1}{8|\phi|^4} \left( \Phi'' - \frac{3\Phi''}{|\phi|} + \frac{3\Phi'}{|\phi|^2} \right) .
\]
Hence, using the assumption in (3.3) and the integral inequality properties
\[
\left| \int f(x) \, dx \right| \leq \int |f(x)| \, dx ,
\]
we obtain
\[
|Q| \leq \sum_{n=1}^{N} \frac{b_n}{n+1} |\phi|^{n+1} + C_1 ,
\]
which implies
\[
|\Phi| \leq \sum_{n=0}^{N} \frac{8b_n}{(n+4)(n+5)(n+6)} |\phi|^{n+6} + \sum_{n=0}^{N} \frac{12b_n}{(n+2)(n+3)(n+4)} |\phi|^{n+4} + 2C_1|\phi|^3 + C_2 \frac{|\phi|^2}{2} + C_3 ,
\]
with $C_i \geq 0$ for all $i = 1, 2, 3$. Thus, applying (3.1), we get (3.4). \qed
By substituting equation (3.5) into (2.13), we can express the energy of the system in the form of

$$E_0 = \int \int B_p r^2 dr d\Omega \left( \frac{\hbar \Lambda \Sigma}{2} (E^\Lambda E^{\Sigma|i} + H^\Lambda H^{\Sigma|i}) + \frac{\Phi'}{|\phi|} \delta_{ab} D_\mu \phi^a D^\mu \phi^b \right. \left. + \frac{1}{4} |\phi| \left( \Phi'' - \frac{\Phi'}{|\phi|} \delta_{ac} \delta_{bc} \phi^c \phi^d D_\mu \phi^a D^\mu \phi^b + V \right) \right)_{t=t_0}. \tag{3.6}$$

Since the energy of the system must be positive, then we should take

**Assumption 2.** The function $\Phi$ has to be bounded below by

$$|\Phi| \geq c_1 |\phi|^2 + c_2,$$ \tag{3.7}

where $c_1 \geq 0$ and $c_2 \geq 0$.

## 4 Estimates

In this section, we derive some estimates for the complex scalar fields and the gauge fields which are the significant part of the global existence proof.

### 4.1 The Flat Energy Estimate

We define a flat energy functional as

$$J(t) \equiv \|E\|_{L^2} + \|H\|_{L^2} + C_1/2 \|D\phi\|_{L^2} + \|\phi\|_{L^2} + \|V\|_{L^2}, \tag{4.1}$$

where $c_1 \geq 0$ and

$$\|f\|_{L^p} \equiv \left( \int_S |f|^p d\mu \right)^{1/p}, \tag{4.2}$$

is a standard $L^p$ norm. The flat energy functional (4.1) plays an important role for bounding some estimates in the MKG system.

**Preposition 1.** Let $\phi^a, E^\Lambda_i, H^\Lambda_i$ be solutions of MKG system as in (2.1) and (2.2) in temporal gauge (2.13) with

$$J_0 \equiv J(0) < \infty. \tag{4.3}$$

Then, for all $t \geq 0$ there exists a positive constant $C_N > 1$ such that

$$J(t) \leq C_N J_0 (1 + t). \tag{4.4}$$

**Proof.** Clearly that $E_0 \leq J_0^2$. So, the first, second, and last terms of (4.1) are bounded by $E_0^{1/2}$, that is,

$$\|E\|_{L^2} + \|H\|_{L^2} + \|V\|_{L^2} \leq E_0^{1/2} \leq C J_0,$$ \tag{4.5}
where $C \geq 1$, and we have defined $|E| \equiv (E^A E_A)^{1/2}$ and $|H| \equiv (H^A H_A)^{1/2}$. Next, from (3.7), we obtain

$$\left| \frac{\Phi'}{2|\phi|} \right| \geq \frac{c_1}{2},$$

(4.6)

such that the estimate of the third term in (4.1) has the form

$$\frac{c_1}{2} \|D\phi\|_{L^2} \leq \mathcal{J}_0 ,$$

(4.7)

with $|D\phi| \equiv \left( \delta_{ab} D_\mu \phi^a D^\mu \phi^b \right)^{1/2}$. To get the estimate of the fourth term in (4.1), let us consider

$$\frac{\partial}{\partial t} \int_{K_p} r^2 d\Omega |\phi|^2 = 2 \text{Re} \int_{K_p} r^2 d\Omega |\phi D_0 \phi|$$

(4.8)

$$\leq 2 \|\phi\|_{L^2} \|D_0 \phi\|_{L^2} \leq 2\mathcal{J}_0 \|\phi\|_{L^2} .$$

Integrating the inequality, we obtain

$$\|\phi\|_{L^2} \leq \tilde{C} \mathcal{J}_0 (1 + t) ,$$

(4.9)

with $\tilde{C} > 0$ showing that (4.4) is fulfilled. \hfill \Box

### 4.2 Estimate for the gauge fields

Let us rewrite the integral equation (2.7) as

$$\mathcal{F}_{\mu \gamma}^\Sigma = \mathcal{F}_{\mu \gamma}^{\Sigma \text{lin}} + I_{\mu \gamma}^\Sigma + J_{\mu \gamma}^\Sigma + K_{\mu \gamma}^\Sigma + L_{\mu \gamma}^\Sigma + M_{\mu \gamma}^\Sigma ,$$

(4.10)

where

$$I_{\mu \gamma}^\Sigma \equiv - \frac{1}{4\pi} \int_{K_p} r^2 d\Omega h^{\Lambda \Sigma} \partial^\alpha \partial_\gamma h_{\Lambda \Gamma} \mathcal{F}_{\alpha \mu}^\Gamma \bigg|_{t=-r} ,$$

(4.11)

$$J_{\mu \gamma}^\Sigma \equiv \frac{1}{4\pi} \int_{K_p} r^2 d\Omega h^{\Lambda \Sigma} (D_\mu h_{\Lambda \Gamma}) \partial_\gamma \mathcal{F}_{\mu \gamma}^\Gamma \bigg|_{t=-r} ,$$

(4.12)

$$K_{\mu \gamma}^\Sigma \equiv - \frac{i}{4\pi} \int_{K_p} r^2 d\Omega h^{\Lambda \Sigma} q_\Lambda \partial_\mu g_{ab} D_\gamma \phi^a \phi^b \bigg|_{t=-r} ,$$

(4.13)

$$L_{\mu \gamma}^\Sigma \equiv - \frac{i}{4\pi} \int_{K_p} r^2 d\Omega h^{\Lambda \Sigma} q_\Lambda g_{ab} \partial_\mu \left( D_\gamma \phi^a \phi^b \right) \bigg|_{t=-r} ,$$

(4.14)

$$M_{\mu \gamma}^\Sigma \equiv \frac{1}{4\pi} \int_{K_p} r^2 d\Omega h^{\Lambda \Sigma} (\partial^\alpha \partial_\gamma, k_{\Lambda \Gamma}) \mathcal{F}_{\alpha \mu}^\Gamma \bigg|_{t=-r} .$$

(4.15)

First, we consider the first term of the nonlinear part of the equation in (4.10) which satisfies

$$|I_{\mu \gamma}^\Sigma| \leq \int_{K_p} r^2 d\Omega h^{\Lambda \Sigma} \partial^\alpha \partial_\gamma h_{\Lambda \Gamma} \mathcal{F}_{\alpha \mu}^\Gamma \bigg|_{t=-r} .$$

(4.16)
Using the Holder inequality, we have the following estimate

\[ |I_{\mu\gamma}^{\Sigma}| \leq \left( \int_{K_p} r^2 drd\Omega \frac{|x_{\alpha\mu}|^2}{r^2} \right)^{1/2} \left( \int_{K_p} r^2 drd\Omega |h^{\Lambda\Sigma}\partial^\alpha \partial_{\gamma} h_{\Lambda\Gamma}|^2 \right)^{1/2} \leq \left( \int_0^{r_0} dr \|F(-r)\|_{L^\infty}^2 \right)^{1/2} \|h^{\Lambda\Sigma}\partial^\alpha \partial_{\gamma} h_{\Lambda\Gamma}\|_{L^2} . \]

We have then

\[ \|h^{\Lambda\Sigma}\partial^\alpha \partial_{\gamma} h_{\Lambda\Gamma}\|_{L^2} \leq c \left( \|\partial^\alpha \Psi \partial_{\gamma} \Psi\|_{L^2} + \|\partial^\alpha \partial_{\gamma} \Psi\|_{L^2} \right) , \tag{4.17} \]

where \( c \geq 0 \) and we have defined \( \Psi \equiv |\phi|^2 = \delta_{\alpha\beta} \phi^\alpha \phi^\beta \).

Consider the following estimate

\[ \|\phi\|_{L^3} \leq \|\phi\|_{L^2}^{1/2} \|\nabla \phi\|_{L^2}^{1/2} \leq c_N J_0^{1/2} (1 + t)^{1/2} \|\nabla \phi\|_{L^2} \]

\[ \leq c_N J_0^{1/2} (1 + t)^{1/2} (\|D_i \phi\|_{L^2} + \|\nabla A_i \phi\|_{L^2} \|\phi\|_{L^3})^{1/2} \leq c_N J_0 (1 + t) \left( 1 + \|\phi\|_{L^3} \right)^{1/2} \leq c_N J_0 (1 + t) . \]

with \( c_N > 1 \). Using the results above, we can get the following estimate

\[ \|\nabla \phi\|_{L^2} \leq \|D_i \phi\|_{L^2} + \|\nabla A_i \phi\|_{L^2} \|\phi\|_{L^3} \leq J_0 + J_0 \|\phi\|_{L^3} \leq C_N J_0^2 (1 + t) . \]

Thus, the first term of (4.17) can be bound to the energy and its \( L^\infty \) norm

\[ \|\partial^\alpha \Psi \partial_{\gamma} \Psi\|_{L^2} \leq \|\partial^\alpha \Psi\|_{L^\infty} \|\partial_{\gamma} \Psi\|_{L^2} \leq \tilde{C}_N \|\partial^\alpha \Psi\|_{L^\infty} \|\phi\|_{L^\infty} \|\partial_{\gamma} \phi\|_{L^2} \leq \tilde{C}_N J_0^2 (1 + t) \|\partial^\alpha \Psi\|_{L^\infty} \|\phi\|_{L^\infty} , \tag{4.18} \]

with \( \tilde{C}_N > 1 \).

As for the second term \( \|\partial^\alpha \partial_{\gamma} \phi\|_{L^2} \), we have

\[ \|\partial^\alpha \partial_{\gamma} \phi\|_{L^2} \leq C_N J_0^2 (1 + t) \|\partial \phi\|_{L^\infty} + \|\phi\|_{L^\infty} \|\partial^\alpha \partial_{\gamma} \phi\|_{L^2} , \tag{4.19} \]

where

\[ \|\partial^\alpha \partial_{\gamma} \phi\|_{L^2} \leq \|D^\alpha D_\alpha \phi\|_{L^2} . \tag{4.20} \]

We can use the equation of motion in (2.2) so that

\[ \left| D^\alpha D_\alpha \phi \right|_{L^2} \leq \left| g^{\alpha \beta} x_{\alpha \beta} \phi \right|_{L^2} + \left| g^{\alpha \beta} \partial \phi \partial A_\alpha \phi \partial \phi \right|_{L^2} \leq \left| g^{\alpha \beta} \partial \phi \partial A_\alpha \phi \partial \phi \right|_{L^2} + \left| g^{\alpha \beta} \partial \phi \partial \phi \right|_{L^2} \tag{4.21} \]

In order to have an estimate of (4.21), we have to specify the form of the scalar potential \( V(\phi, \bar{\phi}) \):

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Assumption 3. The scalar potential $V(\phi, \tilde{\phi})$ has to be either of the following form

$$V(\Psi) = \sum_{n=0}^{N} a_n \Psi^n, \quad (4.22)$$

$$V(\Psi) = V_0 (1 - \cos \lambda \Psi), \quad (4.23)$$

$$V(\Psi) = \sum_{n=0}^{N} \tilde{a}_n e^{-\tilde{\lambda}_n \Psi}, \quad (4.24)$$

where $a_n, \tilde{a}_n, V_0, \lambda$ are real constants, while $\tilde{\lambda}_n > 0$ for every $n$.

It is worth mentioning that there three known examples in the case of (4.22), namely, for $\tilde{N} = 1$ it corresponds to the mass term in the Klein-Gordon equation, while for $\tilde{N} = 2$ it describes the $\phi^4$-theory. Equations (4.23) and (4.24) correspond to the sine-Gordon and the Toda field theories, respectively.

Making use of the Holder inequality, equation (3.5), and its derivative, we have then

$$\left\| D^\alpha D_\alpha \bar{\phi}\right\|_{L^2} \leq \tilde{c}_1 \mathcal{J}_0 (1 + \|\bar{\phi}\|_{L^\infty}) \left(\|\phi\|_{L^\infty} + \|\phi\|_{L^\infty}^3 + \sum_{n=1}^{N} \|\phi\|_{L^\infty}^{n+2} + \sum_{n=1}^{N} \|\phi\|_{L^\infty}^{n+4}\right)$$

$$+ \tilde{c}_2 \mathcal{J}_0 \|\phi\|_{L^\infty} \|\bar{\phi}\|_{L^\infty} + \tilde{c}_3 \mathcal{J}_0 \mathcal{I}(\|\phi\|), \quad (4.25)$$

with $\tilde{c}_i > 1$ for all $i = 1, 2, 3,$ and

$$\mathcal{I}(\|\phi\|) \leq \begin{cases} \mathcal{O}(\|\phi\|), & \text{if } V \text{ is of the form (4.22),} \\ \|\phi\|_{L^\infty} \|\bar{\phi}\|_{L^\infty} \mathcal{J}_0, & \text{if } V \text{ is either of the form (4.23) or (4.24),} \end{cases} \quad (4.26)$$

where

$$\mathcal{O}(\|\phi\|) \leq \|\phi\|_{L^\infty} \|\bar{\phi}\|_{L^\infty} \left(1 + \mathcal{J}_0 (1 + t) \sum_{n=1}^{N-2} \|\phi\|_{L^\infty}^{2n+1}\right). \quad (4.27)$$

Thus, we obtain the estimate for the first term of (4.10)

$$|I_{\mu'\gamma}^{\Sigma}| \leq \mathcal{C}_1 \mathcal{J}_0^2 (1 + t) \left(\int_0^{r_0} dr \|\mathcal{F}(-r)\|_{L^\infty}^2\right)^{1/2} \left(\|\partial \Psi\|_{L^\infty} \|\phi\|_{L^\infty} + \|\bar{\phi}\|_{L^\infty}\right)$$

$$+ \left(\|\phi\|_{L^\infty} + \|\phi\|_{L^\infty}^3 + \sum_{n=1}^{N} \|\phi\|_{L^\infty}^{n+2} + \sum_{n=1}^{N} \|\phi\|_{L^\infty}^{n+4}\right) (\|\bar{\phi}\|_{L^\infty} + 1)$$

$$+ \|\phi\|_{L^\infty} \|\partial \phi\|_{L^\infty} + \|\phi\|_{L^\infty} \mathcal{I}(\|\phi\|)\right), \quad (4.28)$$

where $\mathcal{C}_1 > 1$ and

$$\|(4) \mathcal{F}(t)\|_{L^\infty} \equiv \left|\mathcal{F}_{\alpha\beta}^{\Sigma} \mathcal{F}_{\gamma\alpha\beta}(t)\right|_{L^\infty}^{1/2}. \quad (4.29)$$

For the second term of nonlinear part in (4.10), we have

$$|J_{\mu'\gamma}^{\Sigma}| \leq \int_{K_p} r dr d\Omega |h^{\lambda\Sigma}(\partial^\mu h_{\lambda\Gamma}) \partial_\mu \mathcal{F}_{\gamma\mu}^{\Sigma}|_{t=-r}. \quad (4.30)$$
Making use of the Holder inequality again, we get

$$|J^\Sigma_{\mu\gamma}| \leq \left( \int_0^{r_0} dr \|D\phi\|_{L^\infty}^2 \right)^{1/2} \|\phi\|_{L^\infty} \|\partial_\alpha F^\Gamma_{\gamma\mu}\|_{L^2}, \quad (4.31)$$

where

$$\|D\phi\|_{L^\infty} \equiv \left| \delta_{\alpha\beta} D_\alpha \phi^a D_\beta \phi^b \right|^{1/2}_{L^\infty}. \quad (4.32)$$

We can use the equation of motion in (2.1) to get estimate for \(\|\partial_\alpha F^\Gamma_{\gamma\mu}\|_{L^2}\), namely,

$$\|\partial_\alpha F^\Gamma_{\gamma\mu}\|_{L^2} \leq C J_0 (\|\partial_\psi\|_{L^\infty} + \|\phi\|_{L^\infty}) \quad (4.33)$$

with \(C > 1\). Thus, the estimate for the second term is given by

$$|J^\Sigma_{\mu\gamma}| \leq C_2 J_0 \left( \int_0^{r_0} dr \|D\phi\|_{L^\infty}^2 \right)^{1/2} (\|\phi\|_{L^\infty} \|\partial_\psi\|_{L^\infty} + \|\phi\|_{L^\infty}^2) \quad (4.34)$$

with \(C_2 > 1\).

Similarly, by employing some computation, we obtain the estimates for the third and fourth term of (4.11)

$$|K^\Sigma_{\mu\gamma}| \leq C_3 J_0^2 (1 + t) \left( \int_0^{r_0} dr \|D\phi\|_{L^\infty}^2 \right)^{1/2} \|\partial_k g_{\alpha\beta}\|_{L^\infty} \|\phi\|_{L^\infty}, \quad (4.35)$$

$$|L^\Sigma_{\mu\gamma}| \leq C_4 J_0 (1 + t) \left( \int_0^{r_0} dr \|\phi\|_{L^\infty}^2 \right)^{1/2} \left\{ \sum_{n=1}^N \|\phi\|_{L^\infty}^{n+5} + \sum_{n=1}^N \|\phi\|_{L^\infty}^{n+4} + \sum_{n=1}^N \|\phi\|_{L^\infty}^{n+3} + \sum_{n=1}^N |\phi|_{n+2} \right\}$$

$$+ \|\phi\|_{L^\infty} + \|\phi\|_{L^\infty} + 1 \times (1 + \|\partial_\phi\|_{L^\infty}) \left( \|\phi\|_{L^\infty} + \|\phi\|_{L^\infty}^3 + \sum_{n=1}^N \|\phi\|_{L^\infty}^{n+2} + \sum_{n=1}^N \|\phi\|_{L^\infty}^{n+4} \right)$$

$$+ \|\phi\|_{L^\infty} \|\partial_\phi\|_{L^\infty} + c_4 I (|\phi|) + \|\phi\|_{L^\infty} + \|A\|_{L^\infty} \right\}$$

$$+ C_4 J_0 (1 + t) \left( \int_0^{r_0} dr \|D\phi\|_{L^\infty}^2 \right)^{1/2} \left\{ \sum_{n=1}^N \|\phi\|_{L^\infty}^{n+5} + \sum_{n=1}^N \|\phi\|_{L^\infty}^{n+4} + \sum_{n=1}^N \|\phi\|_{L^\infty}^{n+3} \right\}$$

$$+ \sum_{n=1}^N |\phi|_{n+2}^2 + \|\phi\|_{L^\infty}^2 + \|\phi\|_{L^\infty} + 1 \right), \quad (4.36)$$

respectively, with \(C_3, C_4 > 1\). It is important to notice the estimate of the fifth term \(M^\Sigma_{\mu\gamma}\) gives the same result as the estimate of the first term \(J^\Sigma_{\mu\gamma}\).

Next, we need to estimate the linear term of (2.7). The linear term can be estimated using the initial data

$$|F^\Sigma_{\mu\gamma}|_{\text{lin}} \leq \frac{1}{4\pi} \int_{S^2} d\Omega \left| r_0 \frac{\partial \{ F^\Sigma_{\mu\gamma} \}}{\partial t} + r_0 \frac{\partial \{ F^\Sigma_{\mu\gamma} \}}{\partial r} + F^\Sigma_{\mu\gamma} \right|_{t=r_0} \leq \tilde{C}_0 + \tilde{C}_1 r_0, \quad (4.37)$$
with $\tilde{C}_0, \tilde{C}_1 \geq 0$. Therefore, the estimate for the gauge field is given by

\[
\|F_{\mu\gamma}\| \leq \tilde{C}_0 + \tilde{C}_1 r_0 + J_0^2 (1 + t) \left\{ \left( \int_0^{r_0} |F(-r)|^2 \right)^{1/2} \right. \\
\left. + \left( \int_0^{r_0} |D\phi|^2 \right)^{1/2} \right\} \tilde{C}_2 M(t) + \left( \int_0^{r_0} |\phi|^2 \right)^{1/2} \tilde{C}_3 N(t),
\]

(4.38)

where $\tilde{C}_i > 1$ for all $i = 1, 2, 3$, and

\[
L(t) = \|\phi\|_{L^\infty} + \|\phi\|^3_{L^\infty} + \sum_{n=1}^{N} \|\phi\|_{L^\infty}^{n+2} + \sum_{n=1}^{N} \|\phi\|_{L^\infty}^{n+4} (\|\partial\phi\|_{L^\infty} + 1) + \|\partial\phi\|_{L^\infty} + \|\partial\phi\|_{L^\infty}^2 + \|\phi\|_{L^\infty}^2 + \|\phi\|_{L^\infty}^4 \\
+ \sum_{n=1}^{N} \|\phi\|_{L^\infty}^{n+5} + \sum_{n=1}^{N} \|\phi\|_{L^\infty}^{n+4} + \sum_{n=1}^{N} \|\phi\|_{L^\infty}^{n+3} + \sum_{n=1}^{N} \|\phi\|_{L^\infty}^{n+2} + 1,
\]

(4.39)

\[
M(t) = \|\phi\|_{L^\infty}^2 + \sum_{n=1}^{N} \|\phi\|_{L^\infty}^{n+3} + \sum_{n=1}^{N} \|\phi\|_{L^\infty}^{n+4} + \sum_{n=1}^{N} |\phi|^n + \|\phi\|_{L^\infty}^2 + \|\phi\|_{L^\infty}^4 \\
+ \sum_{n=1}^{N} \|\phi\|_{L^\infty}^{n+5} + \sum_{n=1}^{N} \|\phi\|_{L^\infty}^{n+4} + \sum_{n=1}^{N} \|\phi\|_{L^\infty}^{n+3} + \sum_{n=1}^{N} \|\phi\|_{L^\infty}^{n+2} + \|\phi\|_{L^\infty} + 1
\]

(4.40)

\[
N(t) = \sum_{n=1}^{N} \|\phi\|_{L^\infty}^{n+5} + \sum_{n=1}^{N} \|\phi\|_{L^\infty}^{n+4} + \sum_{n=1}^{N} \|\phi\|_{L^\infty}^{n+3} + \sum_{n=1}^{N} \|\phi\|_{L^\infty}^{n+2} + \|\phi\|_{L^\infty} + 1
\]

(4.41)

4.3 Estimate for the complex scalar fields

Let us rewrite the integral equation (2.8) as

\[
D_\mu \phi^a = D_\mu \phi^a_{lin} + N_\mu^a + R_\mu^a,
\]

(4.42)

where $N_\mu^a$ and $R_\mu^a$ are the nonlinear part of (2.8) whose forms are defined as

\[
N_\mu^a \equiv \frac{i}{4\pi} \int_{K_p} r dr d\Omega \quad q_F \left( \partial^a f_{\mu a} F_{\nu a} + \partial^a \phi_{\mu a} \partial^\nu \phi_{\mu a} + \partial^a \left( A^\Gamma_{\mu a} \partial_\mu \phi^a - A^\Gamma_{\mu a} \partial_\mu \phi^a \right) \right)_{t=-r},
\]

(4.43)

\[
R_\mu^a \equiv \frac{1}{4\pi} \int_{K_p} r dr d\Omega \partial_\mu \partial^a D_\alpha \phi^a_{lim} |_{t=-r},
\]

(4.44)

By applying Holder inequality, Sobolev estimates, and the result in (4.33), the estimate
of \( N_\mu^a \) has the form

\[
|N_\mu^a| \leq K_1 J_0 \left( \int_0^{r_0} dr \|\phi\|_{L^\infty}^2 \right)^{1/2} \|\phi\|_{L^\infty} (1 + \|\partial\phi\|_{L^\infty})
\]

\[+ K_2 J_0^2 (1 + t) \left( \int_0^{r_0} dr \|A\|_{L^\infty}^2 \right)^{1/2} \]

\[+ K_3 J_0 \left( \int_0^{r_0} dr \|A\|_{L^\infty}^2 \right)^{1/2} \left\{ \left( \|\phi\|_{L^\infty} + \|\phi\|_{L^\infty}^3 + \sum_{n=1}^N \|\phi\|_{L^\infty}^{n+2} + \sum_{n=1}^N \|\phi\|_{L^\infty}^{n+4} \right) (1 + \|\phi\|_{L^\infty}) \right\}
\]

\[+ \|\phi\|_{L^\infty} \|\partial\phi\|_{L^\infty} + c_4 I (\|\phi\|) + K_3 J_0 \left( \int_0^{r_0} dr \|\partial\phi\|_{L^\infty}^2 \right)^{1/2},
\]

(4.45)

where \( K_i > 1 \) for all \( i = 1, 2, 3 \). As for \( R_\mu^a \), we first consider

\[
|R_\mu^a| \leq \int_{K_r} r dr d\Omega |\partial_\mu (\overline{D^aD_\alpha\phi^a}) - i q_\Gamma \partial_\mu (A_\Gamma^a D_\alpha\phi^a)|_{t=-r}.
\]

(4.46)

To estimate this term, we must derive an estimate for the first derivative of the equation (2.2), that is,

\[
\partial_\mu (\overline{D^aD_\alpha\phi^a}) = g^{ab} \left( \frac{1}{4} \partial_\mu F_{\alpha\beta}^\Lambda D_\alpha^\beta G_\Lambda^\beta + \frac{1}{4} F_{\alpha\beta}^\Lambda \partial_\mu g_{ac} D_\alpha^b D_\alpha^c \right) - \partial_d g_{ac} \partial_\mu D_\alpha^b \phi^c - \partial_d g_{ac} D_\alpha^b \partial_\mu D_\alpha^c \phi^c - \partial_d g_{ac} \partial_\mu D_\alpha^b \phi^c - \partial_d g_{ac} \partial_\mu D_\alpha^b \phi^c - \partial_d g_{ac} \partial_\mu D_\alpha^b \phi^c - \partial_d g_{ac} \partial_\mu D_\alpha^b \phi^c - \partial_d g_{ac} \partial_\mu D_\alpha^b \phi^c - \partial_d g_{ac} \partial_\mu D_\alpha^b \phi^c
\]

(4.47)

which can be bound by an expression involving the energy and \( L^\infty \) norm. Hence, the
estimate of $R^a_\mu$ is given by

$$
| R^a_\mu | \leq B_1 \mathcal{J}_0 \left( \int_0^{r_0} dr \left| \partial \phi \right|^2_{L^\infty} \right)^{1/2} \left\{ \left( \| \phi \|_{L^\infty} + \| \phi \|_{L^\infty}^3 \phi + \sum_{n=1}^N \| \phi \|_{L^\infty}^{n+2} + \sum_{n=1}^N \| \phi \|_{L^\infty}^{n+4} \right)^2 \right. \\
\times \left( 1 + \| | \partial \phi |_{L^\infty} + \| | \phi \|_{L^\infty} | \partial \phi |_{L^\infty} + c_4 \mathcal{I}(\| \phi \|) \right) \\
+ B_2 \mathcal{J}_0 \left( \int_0^{r_0} dr \left| (a F(-r))_{L^\infty} \right|^2 \right)^{1/2} \left( \| \partial \phi \|_{L^\infty}^2 \phi \|_{L^\infty}^2 + \| \partial \phi \|_{L^\infty} \| \phi \|_{L^\infty}^2 \right) \\
+ B_3 \mathcal{J}_0 \left( \int_0^{r_0} dr \left| D \phi (-r) \right|^2_{L^\infty} \right)^{1/2} \left\{ \| \partial \phi \|_{L^\infty} \left( \sum_{n=1}^N \| \phi \|_{L^\infty}^{n+2} + \sum_{n=1}^N \| \phi \|_{L^\infty}^{n+1} + 1 \right) \\
+ \| \phi \|_{L^\infty} (1 + \| \phi \|_{L^\infty} + (1 + t) \| A \|_{L^\infty}) \right\} \\
+ B_4 \left( \int_0^{r_0} dr \left| \partial \phi \right|^2_{L^\infty} \right)^{1/2} \mathcal{H}(\| \phi \|)
$$

(4.48)

where $B_i > 1$ for all $i = 1, \ldots, 4$,

$$
\mathcal{H}(\| \phi \|) \leq \frac{D(\| \phi \|)}{\mathcal{J}_0(\| \partial \phi \|_{L^\infty}^2 + 1)} , \quad \text{if } V \text{ is of the form } (4.22), \quad \mathcal{J}_0(\| \partial \phi \|_{L^\infty}^2 + 1) \quad \text{if } V \text{ is either of the form } (4.23) \text{ or } (4.24),
$$

(4.49)

and

$$
D(\| \phi \|_{L^\infty}) \leq \sum_{n=0}^{N-1} \| \phi \|_{L^\infty}^{2n} \mathcal{J}_0(1 + t) \| \partial \phi \|_{L^\infty} \| \phi \|_{L^\infty}^{2n} \sum_{n=0}^{N-2} \| \phi \|_{L^\infty}^{2n} .
$$

(4.50)

In the same way as the gauge field case, the linear term of (2.8) is bounded by the initial data. So, the total estimate for the complex scalar field is given by

$$
|D_\phi \phi^a| \leq \mathcal{K}_1 \mathcal{J}_0 \left( \int_0^{r_0} dr \left| D \phi (-r) \right|^2_{L^\infty} \right)^{1/2} \mathcal{S}(t) + \mathcal{K}_2 \mathcal{J}_0^2 (1 + t) \left( \int_0^{r_0} dr \left| (a F(-r))_{L^\infty} \right|^2 \right)^{1/2} \mathcal{X}(t) \\
+ \mathcal{K}_3 \mathcal{J}_0 \left( \int_0^{r_0} dr \left| \phi \right|^2_{L^\infty} \right)^{1/2} \| \phi \|_{L^\infty} (1 + \| \phi \|_{L^\infty}) + \mathcal{K}_4 \mathcal{J}_0 \left( \int_0^{r_0} dr \left| A \right|^2_{L^\infty} \right)^{1/2} \mathcal{U}(t) \\
+ \mathcal{K}_5 \mathcal{J}_0 \left( \int_0^{r_0} dr \left| \partial \phi \right|^2_{L^\infty} \right)^{1/2} \mathcal{W}(t) + k_0 + k_1 r_0
$$

(4.51)
where $K_i > 1$ for all $i = 1, \ldots, 5$, and

\[
S(t) = \|\partial \phi\|_{L^\infty} \left( \sum_{n=1}^{N} \|\phi\|_{L^\infty}^{n+2} + \sum_{n=1}^{N} \|\phi\|_{L^\infty}^{n+1} + 1 \right) + I(\|\phi\|) + \|\phi\|_{L^\infty} + (1 + t) \|A\|_{L^\infty} + \|\partial \phi\|_{L^\infty} + \|\phi\|_{L^\infty} + \|\phi\|_{L^\infty} + \|\phi\|_{L^\infty} + \|\phi\|_{L^\infty} + \|\phi\|_{L^\infty} \right),
\]

\[
X(t) = 1 + \|\partial \phi\|_{L^\infty}^2 + \|\partial \phi\|_{L^\infty} + \|\phi\|_{L^\infty} + \|\phi\|_{L^\infty} + \|\phi\|_{L^\infty} + \|\phi\|_{L^\infty} + \|\phi\|_{L^\infty} \right),
\]

\[
U(t) = \left( \|\phi\|_{L^\infty} + \|\phi\|_{L^\infty} + \sum_{n=1}^{N} \|\phi\|_{L^\infty}^{n+2} + \sum_{n=1}^{N} \|\phi\|_{L^\infty}^{n+4} \right) \left( 1 + \|\partial \phi\|_{L^\infty} + \|\phi\|_{L^\infty} \right)
\]

\[
W(t) = \left\{ \left( \|\phi\|_{L^\infty} + \|\phi\|_{L^\infty} + \sum_{n=1}^{N} \|\phi\|_{L^\infty}^{n+2} + \sum_{n=1}^{N} \|\phi\|_{L^\infty}^{n+4} \right) \right\} \left( \|\phi\|_{L^\infty} + \|\phi\|_{L^\infty} + \sum_{n=1}^{N} \|\phi\|_{L^\infty}^{n+2} + \sum_{n=1}^{N} \|\phi\|_{L^\infty}^{n+4} \right) + H(\|\phi\|_{L^\infty}) \right.
\]

\[
(4.52)
\]

\[
(4.53)
\]

\[
(4.54)
\]

\[
(4.55)
\]

5 The Global Existence

In this section we put the final argument and prove the global existence of the MKG system with general coupling.

Since the right hand side of (4.38) and (4.51) independent of the spatial coordinates of $p$ and reversing the steps which shifted the origin of coordinates, we can write the result as

\[
\|^{(4)} F(t) \|_{L^\infty} \leq \mathcal{J}_0^2 (1 + t) \left\{ C_1 L(t) \left( \int_0^t ds \|D(s)\|_{L^\infty}^2 \right)^{1/2} + C_2 M(t) \left( \int_0^t ds \|D(s)\|_{L^\infty}^2 \right)^{1/2} \right\} + c_0 + c_1 t
\]

\[
(5.1)
\]

\[
\|D\phi\|_{L^\infty} \leq K_1 J_0 \left( \int_0^r dr \|D(-r)\|_{L^\infty}^2 \right)^{1/2} S(t) + K_2 J_0^2 (1 + t) \left( \int_0^t ds \|^{(4)} F(s) \|_{L^\infty}^2 \right)^{1/2} X(t)
\]

\[
+ K_3 J_0 \left( \int_0^t ds \|\phi(s)\|_{L^\infty}^2 \right)^{1/2} \|\phi\|_{L^\infty} \left( 1 + \|\partial \phi\|_{L^\infty} \right) + K_4 J_0 \left( \int_0^t ds \|A(s)\|_{L^\infty}^2 \right)^{1/2} U(t)
\]

\[
+ K_5 J_0 \left( \int_0^t ds \|\partial \phi(s)\|_{L^\infty}^2 \right)^{1/2} W(t) + k_0 + k_1 t
\]

\[
(5.2)
\]
Furthermore, to prove the global existence of the MKG equation, we must show that the norm $\|D\phi(t)\|_{L^\infty}$ and $\|^{(4)}F(t)\|_{L^\infty}$ is finite. Therefore, we define a function as follows

$$G(t) = \|^{(4)}F(t)\|_{L^\infty} + \|D\phi(t)\|_{L^\infty} .$$

(5.3)

Restating the equation in the form of Gronwall inequality, we obtain

$$G^2(t) \leq \mathcal{N}(t) + \mathcal{Q}(t) \left( \int_0^t ds \left\{ \|F(s)\|^2_{L^\infty} + \|D\phi(s)\|^2_{L^\infty} + \|\phi(s)\|^2_{L^\infty} + \|\partial\phi(s)\|^2_{L^\infty} + \|A(s)\|^2_{L^\infty} \right\} \right) ,$$

(5.4)

with

$$Q(t) = \mathcal{K}_1\mathcal{J}_0S(t) + \mathcal{K}_2\mathcal{J}_0^2 (1 + t) \mathcal{N}(t) + \mathcal{K}_3\mathcal{J}_0\|\phi\|_{L^\infty} (1 + \|\partial\phi\|_{L^\infty})$$

$$+ \mathcal{K}_4\mathcal{J}_0\mathcal{J}_0^2 M(t) + \mathcal{K}_5\mathcal{J}_0W(t) + \mathcal{J}_0^2 (1 + t) \{ \mathcal{C}_1L(t) + \mathcal{C}_2M(t) + \mathcal{C}_3N(t) \} ,$$

$$N(t) = c_0 + c_1t + k_0 + k_1t .$$

To get a bound of $G^2(t)$, we only need to prove that $G^2(t)$ continuous. Continuity of $G^2(t)$ depend on continuity of $\|^{(4)}F(t)\|_{L^\infty}$, $\|\phi(t)\|_{L^\infty}$, $\|D\phi(t)\|_{L^\infty}$, $\|A(t)\|_{L^\infty}$ and $\|D\phi(t)\|_{L^\infty}$. Using the triangle inequality and the Sobolev estimate

$$\|f\|_{L^\infty} \leq \|f\|_{H^2} ,$$

(5.5)

we can get the continuity of $\|^{(4)}F(t)\|_{L^\infty}$. Let $\varepsilon > 0$, then we get

$$\|^{(4)}F(t + \varepsilon)\|_{L^\infty} \leq \|^{(4)}F(t)\|_{L^\infty}$$

(5.6)

hence, we have

$$\|^{(4)}F(t + \varepsilon)\|_{H^2} \to 0 , \quad \varepsilon \to 0 .$$

(5.7)

The last step follows from continuity of $^{(4)}F(t)$ as a curve in $H^2$. The same reason clearly applies to $\|D\phi(t)\|_{L^\infty}$.

So far we have proved that $\|^{(4)}F(t)\|_{L^\infty}$ and $\|D\phi(t)\|_{L^\infty}$ cannot blow up in a finite time. Another estimate we need to prove are $\|A_i(t)\|_{L^\infty}$ and $\|\phi(t)\|_{L^\infty}$. By considering the temporal gauge condition, we have

$$A_i(t, x) = A_i(0, x) + \int_0^t E_i(s) \, ds ,$$

(5.8)

then we get

$$\|A_i(t, x)\|_{L^\infty} = \|A_i(0, x)\|_{L^\infty} + \int_0^t \|E_i(s)\|_{L^\infty} \, ds ,$$

(5.9)

$$\|\phi(t, x)\|_{L^\infty} = \|\phi(0, x)\|_{L^\infty} + \int_0^t \|\partial\phi(s)\|_{L^\infty} \, ds ,$$

(5.10)
which are the key points to complete the proof of the global existence of Maxwell Klein-Gordon system.

To prove the global existence, we must show that the norm \((H_2 \times H_1)^2\) of \((A_i, E_i, \phi, \partial_t \phi)\) does not blow up for a finite time. Therefore, we define a functions that are elements of \((H_2 \times H_1)^2\) as

\[
E_0 = \frac{1}{2} \int dx \left( \delta_{\Lambda \Sigma} \left\{ E_i^\Lambda E_i^{\Sigma} + \partial_j A_i^\Lambda \partial_j A_i^{\Sigma} + m A_i^\Lambda A_i^{\Sigma} \right\} + |\partial_0 \phi|^2 + |\partial_i \phi|^2 + m|\phi|^2 \right) ,
\]

\[
E_1 = \frac{1}{2} \int dx \left( \delta_{\Lambda \Sigma} \left( \partial_j E_i^\Lambda \partial_j E_i^{\Sigma} + \partial_j \partial_k A_i^\Lambda \partial_j \partial_k A_i^{\Sigma} \right) + |\partial_i \partial_0 \phi|^2 + |\partial_i \partial_j \phi|^2 \right) ,
\]

where \(m > 0\) is a positive constant. The function \((E_0 + E_1)^{1/2}\) meets the norm \((H_2 \times H_1)^2\), so that to obtain the global existence of MKG equations with general coupling is sufficient by showing that \(E_0\) and \(E_1\) are not blow-up for a finite time.

The first derivative with respect to time of \(E_0\),

\[
\left| \frac{dE_0}{dt} \right| \leq C_0 \left\{ Y (t) \left( \|D\phi\|_{L^\infty} + 1 + \|\phi\|_{L^\infty} \right) + \|^{(4)} F(t) \|_{L^\infty} \|\partial_\phi\|_{L^\infty} (1 + \|\phi\|_{L^\infty}) \right.
\]
\[
+ \left( \|\partial_\phi\|_{L^\infty} + \|D\phi\|_{L^\infty} \right) Z (t) + I \left( \|\phi\|_{L^\infty} \right) + 1 \} E_0 ,
\]

with

\[
Y (t) = 8 \sum_{n=1}^{N} b_n \|\phi\|_{L^\infty}^{n+6} + \sum_{n=1}^{N} b_n \|\phi\|_{L^\infty}^{n+5} + 12 \sum_{n=1}^{N} b_n \|\phi\|_{L^\infty}^{n+3}
\]
\[+ 6C_1 \left( \|\phi\|_{L^\infty}^2 + \|\phi\|_{L^\infty}^3 \right) + (C_2 + C_3) \|\phi\|_{L^\infty} + C_3 \]
\[
Z (t) = \|\phi\|_{L^\infty} + \|\phi\|_{L^\infty}^3 + \sum_{n=1}^{N} \|\phi\|_{L^\infty}^{n+2} + \sum_{n=1}^{N} \|\phi\|_{L^\infty}^{n+4}
\]

where we used the Holder inequality. Integrating the inequality above, we obtain

\[
E_0 (t) \leq E_0 (0) \exp \left( \int_0^t \mathcal{P} (t) \, dt \right) ,
\]

with

\[
\mathcal{P} (t) = Y (t) \left( \|D\phi\|_{L^\infty} + 1 + \|\phi\|_{L^\infty} \right) + \|^{(4)} F(t) \|_{L^\infty} \|\partial_\phi\|_{L^\infty} (1 + \|\phi\|_{L^\infty})
\]
\[+ \left( \|\partial_\phi\|_{L^\infty} + \|D\phi\|_{L^\infty} \right) Z (t) + I \left( \|\phi\|_{L^\infty} \right) + 1 .
\]

Since all of \(\|^{(4)} F(t) \|_{L^\infty}, \|\phi(t)\|_{L^\infty}, \|\partial_\phi(t)\|_{L^\infty}, \|A(t)\|_{L^\infty}\) and \(\|D\phi(t)\|_{L^\infty}\) does not blow-up for a finite time, therefore, \(E_0 (t)\) is bounded for all time.

Finally, computing the time derivative of \(E_1\), and after some calculations, we have

\[
\left| \frac{dE_1}{dt} \right| \leq \{ X (t) + W (t) + P (t) + U (t) \} E_1 ,
\]
with
\[
X(t) = Y(t) \left\{ (\| D\phi \|_{L^\infty} \| \partial \phi \|_{L^\infty} + \| D\phi(t) \|_{L^\infty} + \| \phi \|_{L^\infty} + \| \partial \phi \|_{L^\infty} + 1) E_0^{1/2} + 1 \right\} \\
+ E_0^{1/2} \| (4) F(t) \|_{L^\infty} (\| \partial \phi \|_{L^2} + \| \partial \phi \|_{L^\infty} + \| \phi \|_{L^\infty} + \| \partial \phi \|_{L^\infty} + \| \phi \|_{L^\infty}),
\]
\[
W(t) = \| D\phi \|_{L^\infty} Y(t) \left( (\| \partial \phi \|_{L^\infty} E_0^{1/2} + \| \phi \|_{L^\infty} + \| \partial \phi \|_{L^\infty} E_0^{1/2}) + \| (4) F(t) \|_{L^\infty} \right) \\
+ \| D\phi(t) \|_{L^\infty} \| \partial \phi \|_{L^\infty} \left\{ \tilde{Z} \| \partial \phi \|_{L^\infty} E_0^{1/2} + \| \phi \|_{L^\infty} \right\} \\
+ \| \partial \phi \|_{L^\infty} Y(t) \left( \| \phi \|_{L^\infty} + E_0^{1/2} \| \phi \|_{L^\infty}^2 + E_0^{1/2} \| \partial \phi \|_{L^\infty} \| \phi \|_{L^\infty} + \| D\phi(t) \|_{L^\infty} E_0^{1/2} \right) \\
+ \| (4) F(t) \|_{L^\infty} \| \partial \phi \|_{L^\infty}^2 \| \phi \|_{L^\infty}^3 \left( \| \partial \phi \|_{L^\infty}^2 + \| \phi \|_{L^\infty} \right) E_0^{1/2} + \| \partial \phi \|_{L^\infty} \| \phi \|_{L^\infty}^2,
\]
\[
P(t) = \| \phi \|_{L^\infty}^2 \| \partial \phi \|_{L^\infty}^2 + \tilde{Z} \| D\phi(t) \|_{L^\infty} \| \partial \phi \|_{L^\infty} \| \phi \|_{L^\infty} + \| (4) F(t) \|_{L^\infty} \| \partial \phi \|_{L^\infty} \| \phi \|_{L^\infty}^2 \\
+ \| (4) F(t) \|_{L^\infty} \| \partial \phi \|_{L^\infty} + \| \phi \|_{L^\infty} T(t) + Y(t) (T(t) + \| \phi \|_{L^\infty} + \| A \|_{L^\infty} + 1) \\
+ \| \phi \|_{L^\infty} S(t) + \| \phi \|_{L^\infty} Z \left( \| \phi \|_{L^\infty} + Y(t) S(t) + E_0^{1/2} Y(t) \right) \| \partial \phi \|_{L^\infty} + \| \phi \|_{L^\infty} \right) \\
+ Y(t) Z \left( \| \phi \|_{L^\infty} + \| \phi \|_{L^\infty}^2 \| \partial \phi \|_{L^\infty}^3 \| \partial \phi \|_{L^\infty} \| \phi \|_{L^\infty} \| \phi \|_{L^\infty} \| \phi \|_{L^\infty} \right) \\
+ \| (4) F(t) \|_{L^\infty} \left( \| \partial \phi \|_{L^\infty} \| \partial \phi \|_{L^\infty} \| \phi \|_{L^\infty} \| \phi \|_{L^\infty} \| \phi \|_{L^\infty} \| \phi \|_{L^\infty} \| \phi \|_{L^\infty} \right) \\
+ \| (4) F(t) \|_{L^\infty} \left( \| \partial \phi \|_{L^\infty} \right) \right),
\]
\[
U = S(t) + T(t) + Z (\| \phi \|_{L^\infty}),
\]
where
\[
\tilde{Z} = \sum_{n=1}^{N} \frac{(n + 2)}{n + 1} b_n \| \phi \|_{L^\infty}^{n + 2} + C_1 \| \phi \|_{L^\infty},
\]
\[
\tilde{Z} = \sum_{n=0}^{N} \frac{n + 2}{n + 1} b_n \| \phi \|_{L^\infty}^{n + 1} + \sum_{n=0}^{N} (n + 3) b_n \| \phi \|_{L^\infty}^{n + 2} + C_1.
\]
\[
S(t) = \| \partial \phi \|_{L^\infty} \tilde{Z} \left( \| \phi \|_{L^\infty} \right) \left( (4) F(t) \right) \| \phi \|_{L^\infty} E_0^{1/2} + (\| \phi \|_{L^\infty}) \right) \\
+ E_0^{1/2} \| \partial \phi \|_{L^\infty} \tilde{Z}^2 \left( 1 + \| \phi \|_{L^\infty} \right) \| D\phi(t) \|_{L^\infty} + \| \partial \phi \|_{L^\infty} \right) \\
+ E_0^{1/2} \| (4) F(t) \|_{L^\infty} \| \partial \phi \|_{L^\infty} + \| D\phi(t) \|_{L^\infty} \| \phi \|_{L^\infty} \| \phi \|_{L^\infty} \| \phi \|_{L^\infty} \| \phi \|_{L^\infty} \| \phi \|_{L^\infty} E_0^{1/2} \left( 1 + \| \phi \|_{L^\infty} \right) \\
+ \tilde{Z} \| \partial \phi \|_{L^\infty} (1 + \| \phi \|_{L^\infty}) E_0,
\]
\[
T(t) = E_0^{1/2} \left( 1 + \| \phi \|_{L^\infty} \right) \tilde{Z} + \| (4) F(t) \|_{L^\infty} \| \phi \|_{L^\infty} + Z \| D\phi(t) \|_{L^\infty} + 1,
\]
with

\[
Z (\|\phi\|) \leq \begin{cases} 
\|\phi\|_{L^\infty} \sum_{n=1}^{N} (n - 1) \|\Psi\|^{-2}_{L^\infty} E_0^{1/2} + \sum_{n=1}^{N} n \|\Psi\|_{L^\infty}^{-1}, & \text{if } V \text{ is of the form (4.22),} \\
1 + E_0^{1/2} \|\phi\|_{L^\infty}, & \text{if } V \text{ is either of the form (4.23) or (4.24)},
\end{cases}
\]

(5.27)

\[
\chi (\|\phi\|) \leq \begin{cases} 
E_0^{1/2} \sum_{n=1}^{N} n \|\Psi\|_{L^\infty}^{-1}, & \text{if } V \text{ is of the form (4.22),} \\
E_0^{1/2}, & \text{if } V \text{ is either of the form (4.23) or (4.24)},
\end{cases}
\]

(5.28)

Integrating the inequality, we get

\[
E_1 (t) \leq E_1 (0) \exp \left( \int_0^t \{X (t) + W (t) + P (t) + U (t) + \} \, dt \right).
\]

(5.29)

The right hand side of (5.29) is a mixed expression of \( \|F (t)\|_{L^\infty}, \|\phi (t)\|_{L^\infty}, \|\partial \phi (t)\|_{L^\infty}, \|A (t)\|_{L^\infty}, \|D \phi (t)\|_{L^\infty}, \) and \( E_0 (t) \) which is a bounded with respect to time. Therefore, by applying the Gronwall inequality, we find that \( E_1 (t) \) also cannot blow up in a finite time.

Thus we have proven the global existence of MKG system with general coupling in temporal gauge condition

**Main Theorem.** Let \( u_0 = (A_1^\Sigma (0), E_1^\Sigma (0), \phi_0 (0), \partial \phi_0 (0)) \) be the initial data on \((H_2 \times H_1)^2\) such that the initial flat energy function on \((4.1)\) is finite. If the internal scalar manifold satisfies Lemma[7] and Assumption[2] the gauge couplings satisfy Assumption[7] and the scalar potential is of the form given in Assumption[3] then there exist a unique global solution \( u (t) \in (H_2 \times H_1)^2 \) of MKG equation with general gauge couplings in temporal gauge which solves the corresponding equations \((2.1)\) and \((2.2)\) for all \( t \in (0, \infty)\).

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