MEASURES OF POLYNOMIAL GROWTH AND CLASSICAL
CONVOLUTION INEQUALITIES

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Abstract. We study $L^p(\mu) \to L^q(\nu)$ mapping properties of the convolution operator $T_\lambda f(x) = \lambda \ast (f \mu)(x)$ and of the corresponding maximal operator $T_\lambda f(x) = \sup_{t > 0} |\lambda_t \ast (f \mu)(x)|$, where $\lambda$ is a tempered distribution, and $\mu$ and $\nu$ are compactly supported measures satisfying the polynomial growth bounds $\mu(B(x, r)) \leq Cr^s\mu$ and $\nu(B(x, r)) \leq Cr^s\nu$. As a result, we prove variants of the classical $L^p$-improving (Littman; Strichartz) and maximal (Stein) inequalities in a setting where the Plancherel formula is not available. Connections with the David-Semmes conjecture are also discussed.

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Date: October 2, 2014.
This work was partially supported by the NSF Grant DMS10-45404.
Convolution inequalities play an important role in modern harmonic analysis, partial differential equations, and related areas. Many key problems come down to the estimation, on various function spaces, of operators of the form

\[ T f(x) = \int K(x - y) f(y) dy, \]

where \( K \) is a suitable kernel. Even a sketchy survey can easily fill up a book, so let us mention a couple of key examples that are particularly relevant to this paper. Let

\[ A_t f(x) = \int_{S^{d-1}} f(x - ty) d\sigma(y), \]

\( t \in \mathbb{R} \), where \( \sigma \) is the Lebesgue measure on the unit sphere \( S^{d-1} \subset \mathbb{R}^d \), \( d \geq 2 \). A result due to Littman ([15]) and Strichartz ([27]) says that for a fixed \( t \),

\[ A_t : L^p(\mathbb{R}^d) \to L^q(\mathbb{R}^d) \]

if \( \left( \frac{1}{p}, \frac{1}{q} \right) \) is contained in the triangle with the endpoints \((0, 0), (1, 1), \left( \frac{d}{d+1}, \frac{1}{d+1} \right) \).

Let \( A f(x) = \sup_{t>0} |A_t f(x)| \). A result due to Stein ([25]) in dimensions three and higher, and to Bourgain ([2]) in two dimensions (see also results by Schlag and Sogge in [20] and [21] for the \( L^p - L^q \) variants), says that

\[ A : L^p(\mathbb{R}^d) \to L^p(\mathbb{R}^d) \quad \text{for} \quad p > \frac{d}{d-1}. \]

The key estimate used to establish (1.1) and (1.2), at least in dimensions three and higher, is the fact that for a fixed \( t \),

\[ A_t : L^2(\mathbb{R}^d) \to L^{2 \frac{d}{d-1}}(\mathbb{R}^d), \]

where \( L^{2 \frac{d}{d-1}}(\mathbb{R}^d) \) denotes the inhomogeneous Sobolev space of \( L^2(\mathbb{R}^d) \) functions with \( \frac{d-1}{2} \) derivatives in \( L^2(\mathbb{R}^d) \). This estimate follows from Plancherel and the classical fact (see e.g. [26], [23]) that

\[ |\hat{\sigma}(\xi)| \leq C (1 + |\xi|)^{- \frac{d-1}{2}}. \]

In this paper we show that bounds for these and other convolution operators extend to the setting where \( L^p(\mathbb{R}^d) \) is replaced by \( L^p(\mu) \), where \( \mu \) is a compactly supported Borel measure satisfying

\[ \mu(B(x, r)) \leq C r^{s_\mu} \]

for some \( d \geq s_\mu > 0 \) and every \( x \in \text{supp}(\mu) \) and \( r \in [0, \text{diam}(\text{supp}(\mu))] \).
Let $\lambda$ be a tempered distribution, and denote by $\lambda^\epsilon$ the convolution of $\lambda$ with 
\[ \rho_\epsilon(x) \equiv \epsilon^{-d} \rho(\frac{x}{\epsilon}) , \quad \epsilon > 0, \] with $\rho \in C_0^\infty(\mathbb{R}^d)$, and $\int \rho(x) dx = 1$. Then $\lambda^\epsilon$ is a $C^\infty(\mathbb{R}^d)$ function. Define

\begin{equation}
T_{\lambda^\epsilon}f(x) = \int \lambda^\epsilon(x - y)f(y)d\mu(y),
\end{equation}

where $\mu$ is a compactly supported Borel measure satisfying (1.5) above, and $\lambda$ is a tempered distribution whose Fourier transform is a locally integrable function satisfying

\begin{equation}
|\hat{\lambda}(\xi)| \leq C|\xi|^{-\alpha}
\end{equation}

for some $\alpha \in [0, \frac{d}{2})$.

If $\mu$ is the Lebesgue measure on $\mathbb{R}^d$, the Plancherel theorem says that the $L^2(\mathbb{R}^d)$ bound of $T_{\lambda^\epsilon}$ holds if and only if $\hat{\lambda}$ is bounded. If $\mu$ is not the Lebesgue measure, Plancherel is not available. As a substitute, we have the following result.

**Theorem 1.1.** Let $T_{\lambda^\epsilon}$ be as in (1.6) above with $\lambda$ satisfying (1.7). Let $\mu$ be a compactly supported Borel measure satisfying (1.5) and suppose that $\nu$ is another compactly supported Borel measure satisfying (1.5) with $s_\mu$ replaced by $s_\nu$.

- i) Let $s = \frac{s_\mu + s_\nu}{2}$. Suppose that $\alpha > d - s$. Then

\begin{equation}
\|T_{\lambda^\epsilon}f\|_{L^2(\nu)} \leq C\|f\|_{L^2(\mu)}
\end{equation}

with constant $C$ independent of $\epsilon$.

- ii) If $\alpha \leq d - s$, then $T_{\lambda^\epsilon}$ does not in general map $L^2(\mu)$ to $L^2(\nu)$ with constants independent of $\epsilon > 0$.

**Remark 1.2.** The proof of Theorem 1.1 reduces the estimation of $\|T_{\lambda^\epsilon}f\|_{L^2(\nu)}$ to the estimations of the form

\begin{equation}
\int |\hat{f}\mu(\xi)|^2 |\xi|^{\alpha_\mu} d\xi \leq C\|f\|_{L^2(\mu)}^2
\end{equation}

for $\alpha_\mu > d - s_\mu$, with the notation as above. Inequalities of this type can be restated as the classical trace inequalities completely characterized by Mazya ([16]) using potential theoretic methods, and by Kerman and Sawyer ([12]) using testing conditions.

**Remark 1.3.** Note that if $\lambda$ is, say, a compactly supported Borel measure, then the $\epsilon$-mollification above is not necessary. Moreover, we can interpret (1.8) without $\epsilon$ in the standard way. The tempered distribution $\lambda * (f\mu)$ is a weak limit of the $C^\infty$
functions $\lambda^\epsilon (f_\mu)$ that are uniformly bounded in $L^2(\nu)$ by (1.8). Thus the tempered distribution $T_\lambda f \equiv \lambda^\epsilon (f_\mu)$ can be interpreted as an $L^2(\nu)$ function and the $\epsilon$ can be removed.

**Remark 1.4.** With a bit of work, one can prove a "micro-local" version of part i) above. More precisely, we can show that if on the left hand side of (1.8) $T_\lambda f$ is replaced by $P_k T_\lambda^k f$, where $P_k$ is the standard Littlewood-Paley projection operator on $L^2(\mathbb{R}^d)$, then the constant $C$ on the right hand side can be replaced by $C2^{k(d-s-\alpha)}$. When $\mu$ and $\nu$ are both Lebesgue measures on $\mathbb{R}^d$, we recover the $C2^{-k\alpha}$ Plancherel bound. However, it is in general false, that $\|f\|_{L^2(\nu)} \leq C \sum_k \|P_k f\|_{L^2(\nu)}$.

**Remark 1.5.** In the case when $\lambda = \sigma$, the Lebesgue measure on the unit sphere in $\mathbb{R}^d$, $d \geq 2$, $\mu = \nu$ and $f \equiv 1$, the conclusion of Theorem 1.1 was established by the third listed author in her Ph.D. dissertation.

**Remark 1.6.** It is interesting to note that while part ii) of Theorem 1.1 establishes the sharpness of the $\frac{s\mu + s\nu}{2} > d - \alpha$ condition in general, it may not be sharp in particular cases. For example, it turns out that if $Tf(x) = \sigma * f\sigma(x)$, where $\sigma$ is the Lebesgue measure on the unit circle in $\mathbb{R}^2$, $S^1$, then $T : L^p(S^1) \to L^p(S^1)$, $1 \leq p \leq \infty$, even though in this case $s = 1$ and $\alpha = \frac{1}{2}$. Moreover, consider the same operator where $\sigma$ is the Lebesgue measure on the sphere in $\mathbb{R}^d$. This operator satisfies better Sobolev estimates than those afforded by Theorem 1.1. To see this, observe that since all the action takes place on the unit sphere, $T$ is just the averaging operator over geodesic spheres on $S^{d-1}$. It is known ([23]; chapter 7) that such an operator maps $L^2(S^{d-1})$ to $H^{\frac{d-2}{2}}(S^{d-1})$, where $H^{s}(S^{d-1})$ is the standard inhomogeneous Sobolev space of order $s$ on $S^{d-1}$. On the other hand, Theorem 1.1 affords the operator $T$ only $\alpha + s - d = \frac{d-1}{2} + d - 1 - d = \frac{d-3}{2}$ derivatives on $L^2(S^{d-1})$.

**Remark 1.7.** In establishing the sharpness of Theorem 1.1 part ii), we will use the notion of Ahlfors David regularity of a set; the definition of such sets is included here.

**Definition 1.8.** We say that a Borel set $E \subset \mathbb{R}^d$ is Ahlfors-David regular if there exists $C > 0$ such that for all $x \in E$

\begin{equation}
C^{-1} \delta^{s_E} \leq \mu(B(x, \delta)) \leq C \delta^{s_E},
\end{equation}

where $s_E$ is the Hausdorff dimension of $E$, $\mu$ is the Hausdorff measure restricted to $E$ and $B(x, \delta)$ is the ball of radius $0 < \delta < \text{diam}(E)$ centered at $x$. 
Theorem 1.1 above provides a substitute for the Plancherel identity in the context of $L^2(\mu) \rightarrow L^2(\nu)$ bounds. In the next two sections we give a sampler of applications of this result. The list is not meant to be comprehensive in any sense, and the systematic study of the inequalities studied below will certainly be needed. Our aim is to illustrate the flexibility and broad range of applicability of the method.

1.1. Acknowledgements. The authors are grateful to Fedja Nazarov for several helpful remarks that led to a considerable simplification of the proof of Theorem 1.1.

2. A fractal variant of Stein’s spherical maximal function

One of the key tools in the study of maximal averaging operators in Euclidean space, including the ones described in the introduction above, is the following result, proved independently by several authors in the 80s. See [3], [4] and [24].

**Theorem 2.1.** Let $T^t f(x)$ be defined via its Fourier transform by $\hat{T}^t f(\xi) = m(t\xi)\hat{f}(\xi)$. Let $T f(x) = \sup_{t \in [1,2]} |T^t f(x)|$. Suppose that there exists $\epsilon > 0$ such that

$$|m(\xi)| \leq C|\xi|^{\frac{1}{2} - \epsilon}$$

and

$$|\nabla m(\xi)| \leq C|\xi|^{-\frac{1}{2} - \epsilon}.$$

Then

$$||T f||_{L^2(\mathbb{R}^d)} \leq C||f||_{L^2(\mathbb{R}^d)}.$$

We shall prove that in the polynomial growth setting, the following result holds.

**Theorem 2.2.** Let

$$T^t_\lambda f(x) = \int \lambda_t(x - y)f(y)d\mu(y)$$

with $\mu$ is compactly supported Borel measures satisfying (1.5), $\lambda$ a tempered distribution satisfying (1.7), $\hat{\lambda}_t(\xi) \equiv \hat{\lambda}(t\xi)$ and let

$$T_\lambda f(x) = \sup_{t \in [1,2]} |T^t_\lambda f(x)|.$$

Let $\nu$ be another compactly supported Borel measure satisfying (1.5) with $s_\mu$ replaced by $s_\nu$. Let $s = \frac{s_\mu + s_\nu}{2}$. Suppose, in addition, that

$$|\nabla \hat{\lambda}(\xi)| \leq C|\xi|^{-\alpha}$$

(2.1)
for $\alpha > d - s + 1 \geq 1$. Then
\[
T_\lambda \colon L^2(\mu) \to L^2(\nu)
\]
with constants independent of $\epsilon$.

Remark 2.3. In view of part ii) of Theorem 1.1, the conclusion of Theorem 2.2 does not in general hold if $s \leq d - \alpha$. Further restrictions are discussed in conjunction with Corollary 2.5 below.

Remark 2.4. In Theorem 2.2 we lose a full derivative with respect to the fixed $t$ operator bound, in contrast with the classical result in Theorem 2.1 where only $\frac{1}{2}$ derivative is lost. The reason for this is a lack of a suitable Littlewood-Paley theory in the context of $L^2(\mu)$ for a general Borel measure $\mu$ satisfying the assumptions of our theorem.

We have the following simple consequence which can be viewed as a generalization of Stein’s spherical maximal theorem. See also [6] and [5] where weighted norm inequalities of various type are proved for the spherical maximal operator.

**Corollary 2.5.** Suppose that $\nu = \mu$, $\lambda = \sigma$, the Lebesgue measure on the sphere for $d \geq 4$ and $d \geq 4$.

i) Suppose that $s > \frac{d+3}{2}$, with $s \equiv s_\mu$. Then
\[
T_\sigma : L^p(\mu) \to L^p(\mu) \text{ for } 2 \leq p \geq \frac{2s - (d + 1)}{2s - (d + 2)}. 
\]

ii) If $0 \leq s \leq 1$, for all $p < \infty$, or if $s > 1$, for $p \leq \frac{s}{s-1}$, then $T_\sigma$ is not, in general, bounded on $L^p(\mu)$.

iii) The estimate (2.2) does not in general hold if $s < \frac{d}{2}$. When $d = 2, 3$, the estimate (2.2) does not in general hold if $s < \frac{d + 1}{2}$.

iv) When $d = 2$, (2.2) does not in general hold if $s < 2$. When $d = 3$, (2.2) does not in general hold if $s < \frac{5}{3}$. When $d = 4$, (2.2) does not in general hold if $s < \frac{10}{3}$. When $d = 5$, (2.2) does not in general hold if $s < 4$.

Remark 2.6. The measures giving the counterexamples in part iv) of Corollary 2.5 are all supported on products of a cube and a one-dimensional Cantor set. Note that part iv) of Theorem 2.5 rules out, in general, results in the plane when $s < 2$. When $d = 5$, the exponent 4 equals $\frac{d+3}{2}$, so part i) yields a sharp result in $\mathbb{R}^5$ when $p = 2$. 
When \( d = 3, 4 \), there is still a discrepancy between the threshold \( \frac{d+3}{2} \) afforded by part i) and the counter-examples in iv), but it is interesting that all the exponents are \( > \frac{d+2}{2} \).

3. \( L^p \)-improving measures

We establish the following fractal analog of the results of Littman and Strichartz described in the introduction above.

**Theorem 3.1.** Suppose that \( \lambda = \sigma \), the Lebesgue measure on the sphere in \( \mathbb{R}^d \), \( \lambda \geq 2 \).

i) Suppose that \( s > \frac{d+1}{2} \). Then

\[
\|T_\sigma f\|_{L^{p'}(\mu)} \leq C_p \|f\|_{L^p(\mu)}
\]

for

\[ 2 \geq p > \frac{2s - d + 1}{2s - d} \]

ii) The estimate (3.1) does not in general hold if \( p < \frac{s+1}{s} \).

**Remark 3.2.** Observe that if \( s = d \), the critical exponent above becomes \( \frac{d+1}{d} \), matching the Littman and Strichartz results up to the endpoint. However, when \( \frac{d+1}{2} < s < d \), there is a gap between the sufficient condition in i) and the necessary condition in ii).

4. A counter-example to the David-Semmes conjecture in the general setting

Suppose that in Theorem 1.1 we are assigned a signed measure \( \lambda \) with a Fourier decay at infinity of order \( \alpha > 0 \), and a probability Borel measure \( \mu \) compactly supported on a set of Hausdorff dimension \( s > 0 \). We are asking under what assumptions the \( L^2(\mu) \) norm of \( \lambda \ast f \mu \) is bounded by a constant multiple of \( \|f\|_{L^2(\mu)} \).

We proved in particular that this is the case if \( \mu \) is a Frostman measure on a set of Hausdorff dimension \( s > d - \alpha \). We now discuss the endpoint case. Define the Riesz transform in the usual way by the relation

\[
Rf(x) = p.v. \int \frac{x - y}{|x - y|^{1+s}} f(y) d\mu(y).
\]
The David-Semmes conjecture says that if
\[ R : L^2(\mu) \to L^2(\mu), \]
then \( s \) is an integer and \( E \) is uniformly rectifiable. See [7], [18], [19], [29] and the references contained therein for the description of the current state of knowledge on this problem.

Examples of operators for which the David-Semmes conjecture does not hold have recently been found by Jaye and Nazarov ([14]). For example they show that the singular operator with the kernel \( z \) has this property, in contrast with the operator with the kernel \( \frac{1}{z} \) for which the David-Semmes conjecture is known to hold.

We shall see in a moment that the analog of the David-Semmes conjecture fails badly in the realm of positive operators. We begin with the following simple example.

Let \( d = 2 \) and let \( \lambda = \mu = \sigma \), the Lebesgue measure on the unit circle in \( \mathbb{R}^2 \). Instead of considering \( \sigma \ast \sigma \) directly, we work with \( \sigma^\epsilon \ast \sigma \), where
\[
\sigma^\epsilon = \sigma \ast \rho^\epsilon,
\]
where \( \rho^\epsilon(x) = \epsilon^{-2} \rho(x/\epsilon) \) with \( \rho \in C^\infty_0(\mathbb{R}^2) \) supported near the origin, \( \rho \geq 0 \) and \( \int \rho(x)dx = 1 \).

We are interested in \( \sigma^\epsilon \ast \sigma \) restricted to the unit circle. The intersection of a circle of radius 1 and a circle of radius 1 shifted by a unit vector is a pair of points. Using the fact that \( \sigma^\epsilon \approx \epsilon^{-1} \) on an \( \epsilon \) annulus of the circle of radius 1, we see that if \( x \) is unit vector, then
\[
\sigma^\epsilon \ast \sigma(x) \leq C\epsilon^{-1} \cdot \epsilon,
\]
which is uniformly bounded independently of \( \epsilon \). It follows that \( \sigma \ast \sigma \in L^\infty(\sigma) \), so in particular, it is in \( L^2(\sigma) \).

In order to contradict the (variant of) the David-Semmes conjecture in two dimensions, with \( \lambda = \sigma \), the measure on the unit circle, we must construct a compactly supported measure \( \mu \) on a set of Hausdorff dimension \( \frac{3}{2} \), such that \( \mu(B(x, r)) \leq Cr^{\frac{3}{2}} \).

To see where the numerology comes from, recall that
\[
|\hat{\sigma}(\xi)| \leq C|\xi|^{-\frac{d-1}{2}}
\]
if \( \sigma \) is the Lebesgue measure on the unit sphere in \( \mathbb{R}^d \). In two dimensions, the decay is \( -\frac{1}{2} \), so \( \alpha \) in Theorem 1.1 equals \( \frac{1}{2} \) and the endpoint case is where \( \frac{1}{2} + s = 2 \), which yields \( s = \frac{3}{2} \).

Let
\[
E = \{ r\omega : r \in U; \omega \in S^1 \},
\]
where \( U \) is a subset of \( \left[ \frac{1}{2}, 1 \right] \) consisting of real numbers with 0s and 2s in their base 4 expansions. We have thus constructed an Ahlfors-David regular set of Hausdorff
Let us now estimate \( \sigma^\epsilon \ast \mu(x) \), with \( x \in E \). This expression is approximately equal to \( \epsilon^{-1} \) times the \( \mu \) measure of the intersection of \( E \) and a piece of the annulus of radius 1 and thickness \( \epsilon \). Divide this piece of the annulus into \( \approx \epsilon^{-1} \) pieces that contain and are contained in a square of side-length \( \approx \epsilon \) by cutting with respect to the vertical coordinate. By construction the \( \mu \)-measure of this intersection \( \approx \epsilon^{-\frac{3}{2}} \) of these pieces is \( \leq C \epsilon^{\frac{3}{2}} \) and the \( \mu \)-measure of the rest is 0. It follows that if \( x \in E \),

\[
\sigma^\epsilon \ast \mu(x) \leq C \epsilon^{-1} \cdot \epsilon^{-\frac{1}{2}} \cdot \epsilon^{\frac{3}{2}} \leq C.
\]

This proves that

(4.5) \quad \sigma^\epsilon \ast \mu \in L^\infty(\mu).

We are almost done and will be with the help of the following result which is interesting in its own right.

**Theorem 4.1.** With the notation above, suppose that \( \lambda \ast \mu \in L^\infty(\mu) \). Then

\[
T_\lambda : L^p(\mu) \to L^p(\mu) \text{ for } 1 \leq p \leq \infty.
\]

In view of (4.5) and Theorem 4.1, with \( p = 2 \), \( \lambda = \sigma \) and taking \( \mu \) to be the restriction of the Hausdorff measure to the set \( E \) in (4.4) above, we see that we have obtained a counter-example to the variant of the David-Semmes conjecture for the operator

\[
Tf(x) = \int \sigma(x - y)f(y)d\mu(y).
\]

**4.1. Proof of Theorem 4.1.** Taking \( f \) to be non-negative, as we may, yields

\[
||Tf||_{L^1(\mu)} = \int \int \lambda^\epsilon(x - y)f(y)d\mu(y)d\mu(x)
= \int \lambda^\epsilon \ast \mu(y)f(y)d\mu(y) \leq C||f||_{L^1(\mu)}
\]

since \( \lambda^\epsilon \ast \mu \) is bounded by assumption. Thus we conclude that

\[
T : L^1(\mu) \to L^1(\mu).
\]

Using the calculation above once again, we see that

\[
T : L^\infty(\mu) \to L^\infty(\mu)
\]

since
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|\lambda^\epsilon \ast f \mu(x)| \leq ||f||_{L^\infty(\mu)} \cdot ||\lambda^\epsilon \ast \mu||_{L^\infty(\mu)}.

It follows that

T : L^2(\mu) \to L^2(\mu)

by interpolation.

Alternatively, we may argue as follows. We may assume, by scaling, that

||f||_{L^p(\mu)} = ||g||_{L^p'(\mu)} = 1.

Observe that

<T f, g> = \int \int \lambda(x - y)|f(y)| g(x)d\mu(y)d\mu(x)

\leq \frac{1}{p} \int \int \lambda(x - y)|f(y)|^p d\mu(y)d\mu(x) + \frac{1}{p'} \int \int \lambda(x - y)|g(x)|^{p'} d\mu(y)d\mu(x)

= \frac{1}{p} \int \lambda \ast \mu(y)|f(y)|^p d\mu(y) + \frac{1}{p'} \int \lambda \ast \mu(x)|g(x)|^{p'} d\mu(x)

\leq ||\lambda \ast \mu||_{L^\infty(\mu)} \left( \frac{1}{p} + \frac{1}{p'} \right) = ||\lambda \ast \mu||_{L^\infty(\mu)}.

We conclude that

||Tf||_{L^p(\mu)} = \sup_{||g||_{L^p'(\mu)} = 1} <T f, g> \leq ||\lambda \ast \mu||_{L^\infty(\mu)}

which completes the alternate proof.

This completes the proof of Theorem 4.1 and establishes the counter-example above. It is not difficult to see that a similar counter-example can be constructed for a variety of other \lambda's corresponding to, say, surface measures on smooth surfaces. For example, the counter-example above works just as well if \lambda is the Lebesgue measure on any compact strictly convex curve. Higher dimensional analogs can be constructed in a similar fashion.

5. Proof of Theorem 1.1

5.1. Proof of part i). It is enough to show that if g \in L^2(\nu), then

(5.1) | <T_{\lambda^\epsilon} f, g(\nu) > | \leq C ||f||_{L^2(\mu)} \cdot ||g||_{L^2(\nu)},

where the constant C is independent of \epsilon.
The left hand side of (5.1) equals

\[
\int \lambda^\varepsilon * (f \mu)(x) g(x) d\nu(x).
\]

Indeed,

\[
\lambda^\varepsilon * (f \mu)(x) = \int e^{2\pi i x \cdot \xi} \hat{\lambda}(\xi) \hat{\rho}(\varepsilon \xi) \hat{f} \mu(\xi) d\xi
\]

for every \( x \in \mathbb{R}^d \) because the left hand side is a continuous \( L^2(\mathbb{R}^d) \) function and \( \hat{\lambda}(\cdot) \hat{\rho}(\cdot) \hat{f} \mu(\cdot) \in L^1 \cap L^2(\mathbb{R}^d) \).

It follows that (5.2) equals

\[
\int \int e^{2\pi i x \cdot \xi} \hat{\lambda}(\xi) \hat{\rho}(\varepsilon \xi) \hat{f} \mu(\xi) d\xi g(x) d\nu(x).
\]

Applying Fubini, we see that this expression equals

\[
\int \int e^{2\pi i x \cdot \xi} g(x) d\nu(x) \hat{\lambda}(\xi) \hat{\rho}(\varepsilon \xi) \hat{f} \mu(\xi) d\xi
\]

\[
= \int \hat{\lambda}(\xi) \hat{\rho}(\varepsilon \xi) \hat{f} \mu(\xi) \hat{g} \nu(\xi) d\xi.
\]

The modulus of this expression is bounded by an \( \varepsilon \)-independent constant multiple of

\[
\int |\xi|^{-\alpha} |\hat{f} \mu(\xi)| \cdot |\hat{g} \nu(\xi)| d\xi.
\]

By Cauchy-Schwartz, this expression is bounded by

\[
(\int |\hat{f} \mu(\xi)|^2 |\xi|^{-\alpha_\mu} d\xi)^{\frac{1}{2}} \cdot (\int |\hat{g} \nu(\xi)|^2 |\xi|^{-\alpha_\nu} d\xi)^{\frac{1}{2}} = \sqrt{A} \cdot \sqrt{B},
\]

where \( \alpha_\mu, \alpha_\nu > 0 \) and \( \frac{\alpha_\mu + \alpha_\nu}{2} = \alpha \).

**Lemma 5.1.** With the notation above, we have

\[
A \leq C ||f||_{L^2(\mu)}^2; B \leq C ||g||_{L^2(\nu)}^2
\]

if

\[
\alpha_\nu > d - s_\nu, \alpha_\mu > d - s_\mu, \text{ respectively.}
\]
Lemma 5.1 can be deduced, via a dyadic decomposition, from the following fact due to Strichartz ([28]). With the notation above,

\[
\left| \sup_{r \geq 1} r^{-(d-s_\mu)} \int_{B(x,r)} |\hat{f}_{\mu}(\xi)|^2 \ d\xi \right| \leq C\|f\|^2_{L^2(\mu)}.
\]

Instead, we give a direct argument in the style of the proof of Lemma 7.4 in [30]. It is enough to prove the estimate for \(A\) since the estimate for \(B\) follows from the same argument. By Proposition 8.5 in [30],

\[
A = \int \int f(x)f(y)|x-y|^{-d+\alpha_\mu} d\mu(x)d\mu(y) = \langle f, Uf \rangle,
\]

where

\[
Uf(x) = \int |x-y|^{-d+\alpha_\mu} f(y) d\mu(y).
\]

Observe that

\[
\int |x-y|^{-d+\alpha_\mu} d\mu(y) = \int |x-y|^{-d+\alpha_\mu} d\mu(x)
\]

\[
\leq C \sum_{j>0} 2^{j(d-\alpha_\mu)} \int_{2^{-j} \leq |x-y| \leq 2^{-j+1}} d\mu(y)
\]

\[
\leq C' \sum_{j>0} 2^{j(d-\alpha_\mu-s_\mu)} \leq C'' \text{ if } \alpha_\mu > d - s_\mu.
\]

It follows by Schur’s test (see Lemma 7.5 in [30] and the original argument in [22]) that

\[
\|Uf\|_{L^2(\mu)} \leq C'' \|f\|_{L^2(\mu)}
\]

and we are done in view of (5.6) and the Cauchy-Schwartz inequality.

5.2. Proof of part ii). We shall consider the case \(s_\mu = s_\nu = s\), but an interested reader can easily generalize this example. Let \(\lambda(x) = |x|^{-d+\alpha} \chi_B(x)\), where \(B\) is the unit ball, and suppose that \(\mu\) is the restriction of the \(s\)-dimensional Hausdorff measure to an Ahlfors-David regular set of dimension \(s\). Then

\[
Tf(x) = \int |x-y|^{-d+\alpha} f(y) d\mu(y).
\]

Let \(f \equiv 1\) and observe that

\[
T1(x) \approx \sum_{j} 2^{j(d-\alpha)} \int_{2^{-j} \leq |x-y| \leq 2^{-j+1}} d\mu(y)
\]
\[
\geq \sum_j 2^{j(d-\alpha)} \cdot \mu(B(x, 2^{-j})) \approx \sum_j 2^{j(d-(s+\alpha))}
\]
and this quantity is infinite if \(s \leq d - \alpha\).

6. Proof of Theorem 2.2

The key step is the following calculus lemma.

**Lemma 6.1.** Let \(F \in C^1([1, 2])\). Then

\[
\sup_{t \in [1,2]} |F(t)|^2 \leq |F(1)|^2 + 2 \left( \int_1^2 |F(t)|^2 \, dt \right)^{\frac{1}{2}} \cdot \left( \int_1^2 |F'(t)|^2 \, dt \right)^{\frac{1}{2}}.
\]

The lemma follows by observing that the derivative of \(F^2(t)\) is \(2F(t)F'(t)\), using the Fundamental Theorem of Calculus and the Cauchy-Schwartz inequality.

We wish to apply the lemma to \(F_{\epsilon,x}(t) = T^\epsilon_x f(x)\). Observe that

\[
F_{\epsilon,x}(t) = \int e^{2\pi i t \cdot \xi} \hat{\lambda}(t\xi) \hat{\mu}(\epsilon \xi) \, d\xi
\]
by Fourier inversion. It follows that

\[
F_{\epsilon,x}(t+h) - F_{\epsilon,x}(t) = \int e^{2\pi i t \cdot \xi} \left( \hat{\lambda}((t+h)\xi) - \hat{\lambda}(t\xi) \right) \hat{\mu}(\epsilon \xi) \, d\xi.
\]

Observe that as \(h \to 0\), the integrand tends to 0 and

\[
|F_{\epsilon,x}(t+h) - F_{\epsilon,x}(t)| \leq \int |\hat{\mu}(\epsilon \xi)| \, d\xi \leq C\epsilon^{-d}.
\]

The dominated convergence implies that for every \(x \in \mathbb{R}^d\) and \(\epsilon > 0\), \(F_{\epsilon,x}(t)\) is a Lipschitz function of \(t \in [1,2]\). Thus Lemma 6.1 applies. Integrating both sides of this inequality with respect to \(d\mu(x)\) and applying the Cauchy-Schwartz followed by Fubini and the triangle inequality shows that

\[
\int \sup_{t \in [1,2]} |T^\epsilon_x f(x)|^2 \, d\nu(x)
\]
is bounded by

\[
\int |T^1_x f(x)|^2 \, d\nu(x)
\]

(6.2)
\begin{align}
(6.3) \quad & \leq \sum_{j=1}^{d} \left( \int_{1}^{2} \int |T_{\lambda} f(x)|^2 \, d\nu(x) \, dt \right)^{\frac{1}{2}} \cdot \left( \int_{1}^{2} \int |T_{\gamma_j} f(x)|^2 \, d\nu(x) \, dt \right)^{\frac{1}{2}}, \\
\end{align}

where
\begin{equation}
\hat{\gamma}_j^\epsilon (\xi) = \frac{\partial \hat{\lambda}^\epsilon}{\partial \xi_j}(\xi) \xi_j.
\end{equation}

The term in (6.2) is bounded by $C ||f||^2_{L^2(\mu)}$ for $\alpha > d - s$ by Theorem 1.1.

Note that $\gamma_j$ is a tempered distribution and
\[ |\hat{\gamma}_j^\epsilon (\xi)| \leq C |\xi|^{-\alpha + 1} \]
with constants independent of $\epsilon$ by the formula above and the condition (2.1).

Since Theorem 1.1 applies to tempered distributions, we see that $T_{\gamma_j}$ maps $L^2(\mu)$ to $L^2(\mu)$ with constants independent of $\epsilon$ if $\alpha > d - s + 1$. Using this and (6.3), we see that $T_{\lambda}$ is bounded on $L^2(\mu)$ with constants independent of $\epsilon$. This completes the proof of Theorem 2.2.

7. Proof of Corollary 2.5

We use the classical analytic families argument. A similar approach was used in the original proof of Stein’s spherical maximal theorem ([25]). Let
\[ T_{\sigma}^z f(x) \equiv \int \sigma^z(x - y) f(y) \, d\mu(y), \]
where
\[ \sigma^z(x) = \frac{1}{\Gamma(z)} \left(1 - |x|^2\right)_+^{z-1}. \]

One can check (see e.g. [26], [27]) that
\[ T_{\sigma}^z f(x) \equiv T_{\sigma} f(x) \]
and
\[ \hat{\sigma}^z(\xi) = \hat{\sigma}(\xi)|\xi|^{-z}. \]

Define
\[ T_{\sigma}^{d,z} f(x) \equiv T_{\sigma}^z f(x), \]
where, as before
Theorem 2.2 implies that
\[ |\hat{\sigma}_t(\xi)| = |\hat{\sigma}(t\xi)|. \]

with constants independent of \( \epsilon \) provided that
\[ \text{Re}(z) < \frac{d+3}{2} - s. \]

On the other hand, if \( \text{Re}(z) = 1 \),
\[ |T^{t,z}_\sigma f(x)| \leq C \int |f(y)|d\mu(y) = C\|f\|_{L^1(\mu)}. \]

Since the maximal operator above is sub-linear, we may apply Stein’s analytic interpolation theorem (see e.g. [26], [23]) and the conclusion of Corollary 2.5 follows.

7.1. Proof of part ii) of Corollary 2.5. First assume \( s > 0 \). Let \( d\mu(x) = |x|^{-d+s}dx \), \( t = |x| \) and
\[ f(x) = |x|^{1-s}\log^{-1}\left(\frac{1}{|x|}\right) \chi_{\frac{1}{2}B}(x). \]

Then
\[ \int |x|^{p(1-s)}\log^{-p}\left(\frac{1}{|x|}\right) d\mu(x) = \int |x|^{p(1-s)-d+s}\log^{-p}\left(\frac{1}{|x|}\right) \chi_{\frac{1}{2}B}(x)dx < \infty \]
if \( p \leq \frac{s}{s-1} \) in case \( s > 1 \), and for all \( p < \infty \) if \( 0 < s \leq 1 \).

On the other hand,
\[ T^{|x|}_\sigma(f\mu)(x) = \int |x - |x||^{1-s}(|x - |x||)^{-d+s}\log^{-1}\left(\frac{1}{|x - |x||}\right) d\sigma(y) \equiv \infty \]
since the sphere is a \( d-1 \)-dimensional surface. We conclude that (2.2) does not in general hold if \( p \leq \frac{s}{s-1} \) when \( s > 1 \), or for any \( p < \infty \) when \( 0 < s \leq 1 \).

Now assume \( s = 0 \). Let \( d\mu(x) = |x|^{-d}\log^{-u}\left(\frac{1}{|x|}\right) dx \), \( t = |x| \) and
\[ f(x) = |x|\log^{-u}\left(\frac{1}{|x|}\right) \chi_{\frac{1}{2}B}(x), \]
for some $u = \beta > 1$. Then the measure $\mu$ has 0-polynomial growth, and the same argument as the case $s > 0$ gives that $f \in L^p(\mu)$ for any $1 \leq p < \infty$, but $T^{|x|}_\sigma(f \mu)(x) = \infty$.

7.2. Proof of part iii) of Corollary 2.5. In [8], Falconer proves that the inequality

$$\mu \times \mu\{(x, y) : t \leq |x - y| \leq t + \epsilon\} \leq C(t) \epsilon,$$

which is equivalent to the inequality

$$\int \sigma_t^\epsilon \ast \mu(x) d\mu(x) \leq C(t),$$

does not in general hold if $s < \frac{d}{2}$. When $d = 2$, Mattila ([13]) improved Falconer example to $s < \frac{3}{2} = \frac{d+1}{2}$. Mattila’s example was generalized to the case $d = 3$ by the first listed author and Steven Senger in [11]. Since

$$\int \sigma^\epsilon \ast \mu(x) d\mu(x) \leq \int |\sigma^\epsilon \ast \mu(x)|^2 d\mu(x),$$

part iii) of Corollary 2.5 is established.

7.3. Proof of part iv) of Corollary 2.5. In dimension 2, 3, 4, 5, let $E = [0, 1]^{d-1} \times C_\beta$, where $C_\beta$ is a Cantor set (contained in the unit interval) of dimension $\beta \in [0, 1]$. Let $\mu$ be the restriction of the $s = 1 + \beta$ dimensional Hausdorff measure to $E$. Let $t = |x_d|$, $f(y) = |y|^{-\frac{s}{2} + \delta}$ for some (small) $\delta > 0$ and observe that $f \in L^2(\mu)$. Then the maximal function of $f$ at $x$ is bounded from below by

$$\epsilon^{-1} \int_{|x_d| \leq |x - y| \leq |x_d| + \epsilon} |f(y)| d\mu(y),$$

where we will consider $x$s such that $|x_d| \geq \frac{1}{2}$.

Consider the $\sqrt{\epsilon} \times \sqrt{\epsilon} \times \cdots \times \sqrt{\epsilon} \times \epsilon$ box with sides parallel to the axis contained in the annulus $\{y : |x_d| \leq |x - y| \leq |x_d| + \epsilon\}$. Then the integral above is bounded from below by

$$\epsilon^{-1} (\sqrt{\epsilon})^{d-1} \epsilon^{-\beta} \epsilon^{-\frac{s}{2} + \delta} = \epsilon^{\frac{d}{2} - d - 1 + \frac{s}{2}} + \frac{s}{2},$$

since $s = d - 1 + \beta$.

Since $\delta > 0$ is arbitrarily small, this shows that the maximal function blows up as $\epsilon \to 0$ if $s < \frac{2}{3}(d+1)$, but this only makes sense if $\frac{2}{3}(d+1) \geq d - 1$ since $s > d - 1$ by construction. This yields the threshold $\frac{2}{3}(d+1)$ in dimensions 2, 3, 4, 5, as claimed.
8. Proof of Theorem 3.1

We first prove part i) where we follow the classical analytics families approach. See, for example, Strichartz ([27]). Recall that $s > \frac{d+1}{2}$ and define

$$T_\sigma f(x) \equiv \int \sigma(x-y)f(y)d\mu(y),$$

where

$$\sigma(x) = \frac{1}{\Gamma(z)}(1-|x|^2)^{\frac{z-1}{+}}.$$

One can check (see e.g. [26], [27]) that

$$T_0 f(x) \equiv T f(x)$$

and

$$\hat{\sigma}(\xi) = \hat{\sigma}(\xi)|\xi|^{-z}.$$  

When $Re(z) = 1$, $\sigma(x)$ is clearly bounded, which implies that

$$T_\sigma^z : L^1(\mu) \to L^\infty(\nu).$$

If $Re(z) < \frac{d+1}{2} - s$, then Theorem 1.1 implies that

$$T_\sigma^z : L^2(\mu) \to L^2(\nu),$$

since the decay rate of the multiplier in this range is $> d - s$, consistent with the requirements of Theorem 1.1.

Stein’s analytic interpolation theorem yields the conclusion of Theorem 3.1.

8.1. Proof of part ii) of Theorem 3.1. Let $d\mu(x) = |x|^{-d+s} \chi_B(x)\,dx$. Let $\mu = \nu$. It is easy to see that $\mu$ and $\nu$ satisfy (1.5) with $s_\mu = s_\nu = s$. Let $f(x) = \chi_{B_\delta}(x)$ with $\delta$ small.

On one hand,

$$||\chi_{B_\delta}||_{L^p(\mu)} = \left(\int |x|^{-d+s} \chi_{B_\delta}(x)\,dx\right)^{\frac{1}{p}}$$

$$= \left(|S^{d-1}| \int_0^\delta r^{s-1}dr\right)^{\frac{1}{p}} = C\delta^{\frac{d}{p}}.$$  

On the other hand,

$$T\chi_{B_\delta}(x) = \int |x-y|^{-d+s} \chi_{B_\delta}(x-y)d\sigma(y).$$
Thus $T \chi_{B_\delta}(x) \gtrsim \delta^{s-1}$ on an annulus of radius 1 and thickness $\delta$. It follows that
$$||T \chi_{B_\delta}||_{L^q(\mu)} \gtrsim \delta^{s-1} \cdot \delta^{\frac{1}{q}}.$$ 

We conclude that
$$\delta^{s-1} \cdot \delta^{\frac{1}{q}} \leq C \delta^{\frac{s}{p}},$$

hence
$$\frac{s}{p} \leq s - 1 + \frac{1}{q}.$$ 

The conclusion of part ii) of Theorem 3.1 follows.

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