Past and future gauge in numerical relativity

Maurice H.P.M. van Putten

MIT 2-378, Cambridge, MA 02139-4307

mvp@schauder.mit.edu

ABSTRACT

Numerical relativity describes a discrete initial value problem for general relativity. A choice of gauge involves slicing space-time into space-like hypersurfaces. This introduces past and future gauge relative to the hypersurface of present time. Here, we propose solving the discretized Einstein equations with a choice of gauge in the future and a dynamical gauge in the past. The method is illustrated on a polarized Gowdy wave.

1. Introduction

The initial value problem for general relativity is receiving much attention in the prediction of wave-forms from candidate sources for the upcoming gravitational wave detectors LIGO/VIRGO (1; 2). The structure of gravitational waves has recently been elucidated in new hyperbolic formulations of general relativity (see, e.g., (3) for references), which holds promise for accurate integration schemes. Long-time integrations also require accurate conservation of the associated elliptic constraints, representing energy and momentum conservation (4). While analytically these constraints are conserved under dynamical evolution, the nonlinear nature of the equation typically tends to introduce some numerical departure from exact conservation.

Gauge choice in numerical relativity involves slicing space-time into space-like hypersurfaces. Data from one hypersurface are evolved numerically onto the next, for example using a hyperbolic formulation. Algebraic slicing is particularly illustrative, which considers prescribed lapse and shift functions $N_a = g_{ta}$ for the metric tensor $g_{ab}$.

Here, we propose an new approach for numerical relativity with preservation of the constraints by through choice of gauge in the future and a dynamical gauge in the past. With prescribed gauge
and dynamical space-like components of the metric in the future, this obtains a complete system of ten equations in ten unknowns.

The presented approach recognizes two types of constraints in Hamiltonian approaches within the context of the underlying four-covariant theory of general relativity. Recall that Hamiltonian formulations assign the three-metric \( h_{ij}(t) \) and its canonical momentum \( \pi_{ij}(t) \) as dynamical variables to a hypersurface of constant coordinate time \( t \). This implicitly carries a geometric constraint, in having \( h_{ij}(t) \) as exact projections of the four-covariant metric \( g_{ab} \) onto the hypersurfaces of constant \( t \). In addition, general relativity obeys energy-momentum conservations by the Bianchi identity, which translates into the Gauss-Codacci relations on the same hypersurfaces. These observations may be contrasted with the nonlinear wave equations of (6), which describe the four-covariant hyperbolic evolution of the Riemann-Cartan connections \( \omega_{\mu ab} \) first, with subsequent slicing of space-time for the purpose of numerical implementation. In the continuum limit, both the projections and energy-momentum constraints are exact and consistent. This need not carry over to the discrete case, given the nonlinear nature of the Einstein equations. This may be illustrated by the observed violations of energy-momentum constraints in present numerical simulations (4).

A discretization of the Einstein equations introduces finite deviations from the continuum limit. For the purpose of numerical relativity, these deviations are acceptable, provided they permit stable evolution and convergence upon refinement of the discretization. The question therefore is: where are discretizations allowed to introduce deviations from the continuum theory? The above suggests to consider seeking a trade off between the geometric constraint of exact projections and energy-momentum conservation. This points towards a numerical scheme in which the past slicing is adjusted within the chosen discretization level, so as to satisfy energy-momentum conservation identically for a given definition of the discretized Einstein tensor. This obtains a complete system of evolution equations, as follows from simple counting: the contracted Bianchi identity shows that the Einstein equations provide six constraints on the second time-derivatives of the 3-metric \( h_{ij} \) and four constraints on the first time-derivatives of the lapse and shift functions \( N_a = g_{at} \). The first tells us that the discretized Einstein equations live on three time-slices, whereas the latter indicates that the lapse and shift functions are constraint on a pair of them, e.g., the first (past) and the third (future) time-slice. Thus, choosing a gauge for one of these leaves the gauge on the other determined self-consistently by the Einstein equations. It seems natural to maintain control over the future gauge, i.e., the future gauge is by choice of the the user. This leaves the past gauge as a dynamical variable, to be determined implicitly during numerical evolution. This can be achieved using Newton’s method.
Non-exact projections naturally permit an uncertainty between the three-metric and its canonical momentum within the underlying context of a four-covariant theory, i.e.: also in regards to the association with the hypersurface at hand. In the covariant approach of (6), this would thus reflect an uncertainty in the tetrad elements, which define the projection, and their connections. This points towards a potential connection to quantum gravity. Indeed, soon after this work was proposed (5), the author learned of a very interesting independent discussion on the problem of consistent discretizations in this context (7).

2. A discretized initial value problem

We illustrate our this approach on the vacuum Einstein equations, described by the vanishing Ricci tensor

\[ R_{ab} = 0. \] (1)

The Ricci tensor \( R_{ab} \) is a second-order expression in the metric \( g_{ab} \). Hence, (1) defines a relationship between metric data \( (g_{ab}^{n-1}, g_{ab}^n, g_{ab}^{n+1}) \) on a triple of time-slices \( t_{n-1} < t_n < t_{n+1} \):

\[ R_{ab} (g_{ab}^{n+1}, g_{ab}^n, g_{ab}^{n-1}) = 0. \] (2)

Here, \( R_{bd} = R^a_{\ bcd} \) derived from the Riemann tensor

\[ R^a_{\ bcd} = \partial_d \Gamma^a_{\ bc} - \partial_c \Gamma^a_{\ bd} + \Gamma^e_{\ bc} \Gamma^a_{\ ed} - \Gamma^e_{\ bd} \Gamma^a_{\ ec}. \] (3)

This expression (3) can be discretized by finite differencing on a triple of time-slices with preservation of the quasi-linear second-order structure of \( R_{ab} \).

Algebraic gauge-fixing takes the form of specifying the components \( N_a = g_{ta} \) in coordinates \( \{x^a\}_{a=1}^4 \) with \( t = x^1 \) time-like. A gauge-choice on a triple of time-slices, therefore, amounts to a choice of \( (N_a^{n-1}, N_a^n, N_a^{n+1}) \). Recall that this gauge-choice in the metric arises explicitly in the Gauss-Codacci relations for energy-momentum conservation. The components \( h_{ij} = g_{ij} \), where \( i, j = 2, 3, 4 \) refer to projections of the metric into the time-slice \( t = \text{const.} \), which describe the dynamical part of the metric. The combination \( (h_{ij}, N_a) \) reflects the mixed hyperbolic-elliptic structure in numerical relativity and (1) represents ten evolution equations in these variables on a triple of time-slices.

In algebraic gauge-fixing, we prescribe \( N_a^{n+1} \) as a future gauge in computing \( h_{ij}^{n+1} \) on a future hypersurface \( t = t_{n+1} \) from data at present and past hypersurfaces \( t = t_{n-1} \) and \( t = t_n \). We propose
closure by re-introducing \( N_{n-1}^n \) as dynamical gauge in the past, leaving \( h_{ij}^{n-1} \) fixed. Combined, this defines an advanced hyperbolic-retarded elliptic evolution of the metric. The partitioning of the metric in past and future variables as

\[
g_{ab} = (h_{ij}^{n+1}, N_a^{n-1}) = \begin{pmatrix}
N_1^{n-1} & N_2^{n-1} & N_3^{n-1} & N_4^{n-1} \\
N_2^{n-1} & h_{xx}^{n+1} & h_{xy}^{n+1} & h_{xz}^{n+1} \\
N_3^{n-1} & h_{yx}^{n+1} & h_{yy}^{n+1} & h_{yz}^{n+1} \\
N_4^{n-1} & h_{zx}^{n+1} & h_{zy}^{n+1} & h_{zz}^{n+1}
\end{pmatrix}
\]  

(4)

thus obtains ten dynamical variables in the ten equations

\[
R_{ab}(h_{ij}^{n+1}, N_a^{n-1}, \cdots) = 0 \quad \text{at} \quad t = t_n.
\]  

(5)

Here the dots refer to the remaining data \((h_{ij}^{n-1}, h_{ij}^n, N_a^n, N_a^{n+1})\), which are kept fixed while solving for \((h_{ij}^{n+1}, N_a^{n-1})\). Thus, (5) which takes into account all ten Einstein equations with no reduction of variables. Time-stepping by (5) evolves the metric into the future with dynamical gauge in the past, in an effort to satisfy energy-momentum conservation within the definition of the discretized Ricci tensor. Because (5) comprises derivatives of \( N_a \) only to first-order in time, numerically though the data \( N_a^{n+1} \) and \( N_a^{n-1} \), we anticipate that the evolution of \( N_a \) is of first-order in the \( t - \)discretization \( \Delta t \). This introduces non-exactness in \( h_{ij}^{n-1} \) as projections of \( g_{ab} \) on \( t - \Delta t \) to within the same order of accuracy. It may result in a first-order drift in the \( t - \)labeling of the hypersurfaces – permitted by coordinate invariance.

The presented approach can be illustrated on a polarized Gowdy wave. Gowdy cosmologies are an extensively studied class of universes with compact space-like hypersurfaces with two Killing vectors \( \partial_\sigma \) and \( \partial_\delta \). With cyclic boundary conditions, the space-like hypersurfaces are homeomorphic to the three-torus as a model universe collapsing into a singularity. The associated line-element is (see, e.g., (8))

\[
ds^2 = e^{(\tau - \lambda)/2} (-e^{-2\tau} d\tau^2 + d\theta^2) + d\Sigma^2,
\]  

(6)

where \( \lambda = \lambda(\tau, \theta) \) and \( d\Sigma \) denotes the surface element in the space supported by the two Killing vectors. Polarized Gowdy waves form a special case, which permit a reduction to

\[
d\Sigma^2 = e^{-\tau} (e^P d\sigma^2 + e^{-P} d\delta^2).
\]  

(7)

Here \( P \) satisfies a linear wave-equation \( P_{\tau\tau} = e^{-2\tau} P_{\theta\theta} \); a long wave-length solution is

\[
P_0(\tau, \theta) = Y_0(e^{-\tau}) \cos \theta,
\]  

(8)
where $Y_0$ is the Bessel function of the second kind of order zero. This leaves

$$
\lambda(\tau, \theta) = \frac{1}{2} Y_0(e^{-\tau}) Y_1(e^{-\tau}) e^{-\tau} \cos 2\theta + \frac{1}{2} \int_{e^{-\tau}}^{1} (Y_0^2(s) + Y_0^2(s)) sds.
$$

(9)

A spectrally accurate numerical integration is described in (9).

The implicit equation (5) for the dynamical variables $(h_{ij}^{n+1}, N_n^{a-1})$ has been implemented numerically. We have done so by solving for the all ten components $(h_{ij}^{n+1}, N_n^{a-1})$ using Newton iterations on these variables. This procedure uses a numerical evaluation of the Jacobian

$$
J_{AB} = \frac{\partial R_A}{\partial U_B}
$$

(10)

where the capital indices $A, B = 1, 2, \ldots, 10$ refer to the labeling

$$
R_A = (R_{11}, R_{22}, R_{33}, R_{44}, R_{12}, R_{13}, R_{14}, R_{23}, R_{24}, R_{34}),
$$

$$
U_B = (N_n^{a-1}, h_{11}^{n+1}, h_{22}^{n+1}, h_{33}^{n+1}, N_2^{a-1}, N_3^{a-1}, N_4^{a-1}, h_{23}^{n+1}, h_{24}^{n+1}, h_{34}^{n+1}).
$$

(11)

The Ricci tensor (3) has been implemented by second-order finite differencing, such that it remains quasi-linear in the second derivatives. In particular, the Christoffel symbols

$$
\Gamma^c_{ab} = \frac{1}{2} g^{ce} (g_{cb,a} + g_{ac,b} - g_{ab,c})
$$

(12)

is obtained by symmetric finite-differencing on the metric components, and itself differentiated by the product rule following individual numerical differentiations of $g^{ab}$ and $(g_{cb,a} + g_{ac,b} - g_{ab,c})$. The choice of future gauge $N_a^{n+1}$ is provided by the the components

$$
g_{at} = (e^{(\tau-\lambda)/2}, 0, 0, 0)
$$

(13)

of the analytical line-element (6-9), which facilitates error analysis by direct comparison of the numerical results with the analytic expression for the line-element (6). It will be appreciated that in principle other choices of $N_a^{n+1}$ can be made.

Fig. 1 shows numerical results for evolution of initial data on $0 \leq \tau \leq 4$. The results show that all Einstein equations in the form of $R_{ab} = 0$ are satisfied with arbitrary precision, while the metric components are solved accurately to within one percent. The asymptotic behavior of the implicit corrections to the lapse functions are shown in Fig. 2. Note that these corrections are finite to first-order in $\Delta t$, a testimonial to the appearance of the lapse function in the Einstein equations to first-order in time.

In summary, a dynamical gauge in the past gives a complete number of ten degrees of freedom in evolving to a future hypersurface, permitting all ten Einstein equations to be satisfied with arbitrary
precision. The Einstein equations are hereby satisfied numerically on a triple of hypersurfaces of past, present and future within the definition of the discretized Riemann tensor. The simulation of a nonlinear one-dimensional Gowdy wave by implicit time-stepping according to the ten discretized vacuum Einstein equations (5) serves to illustrate a numerical implementation. The presented approach ensures conservation of energy and momentum, within the definition of the discretization used. Satisfying these constraints is generally a necessary condition for stability in long-time integrations. It is an important open question if satisfying these constraints is a sufficient condition for stability. It would be of interest to study this method in evolving Schwarzschild black holes with singularity-avoiding slicings of spacetime, and its behavior in the presence of outgoing radiative boundary conditions. More generally, it would be of interest to consider this approach in higher dimensions, including a self-consistent integration of any of the modern hyperbolic formulations and efficient elliptic solvers.

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Figure Captions

FIGURE 1. Shown is the simulation for $0 \leq \tau \leq 4$ of the polarized Gowdy wave. The solutions $P(\tau, \theta)$ and $\lambda(\tau, \theta)$ are displayed as a function of $(\tau, \theta)$ (upper windows). The middle windows display the solutions for $\tau = 4$, wherein the circles denote the numerical solution and the solid lines the analytical solution. The $\tau$-evolution of the errors (lower windows) are computed relative to the analytical solution to Gowdy’s reduced wave equation. The simulations discretize $\theta$ by $m_1 = 64$ points and the $\tau$-interval by $m_2 = 1024$ time-steps. Particular to the proposed numerical algorithm is a dynamical gauge in the past and a prescribed gauge in the future. This permits satisfying all of the discretized Einstein equations $R_{ab} = 0$ to within arbitrary precision by Newton iterations. The slight increase in the error of about $10^{-10}$ reflects the exponential growth of the analytic solution, because the Gowdy cosmology evolves towards a singularity.

FIGURE 2. Shown are the self-consistent corrections on the slicing $t = t_{n+1}$, introduced by the
difference between the past gauge $N^{n-1}(t_{n+2})$ to the hypersurface $t = t_{n+2}$ and the earlier future gauge $N^{n+1}(t_n)$ to the hypersurface $t = t_n$. The three curves refer to different discretizations $m_1 = 16, 32$ and $64$ points with, respectively, $m_2 = 256, 512$ and $1024$ time-steps. These similar results for various discretizations indicate asymptotic behavior consistent with the first-order appearance of the lapse function in the Einstein equations. A first-order accuracy in lapse introduces a commensurate offset in slicing or, equivalently, an offset in the coordinate $t$.

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FIGURE 1
\[ \frac{|N_{n+1}(\tau_n) - N_{n-1}(\tau_{n+2})|}{N_n(\tau_{n+1})} \Delta \tau \]

FIGURE 2