On sampling of scattering phase functions

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Abstract

Sampling the scattering phase function is a basic but important part in Monte Carlo radiative transfer (MCRT). For complicated scattering phase functions, tabulated method is widely used for sampling in Monte Carlo radiative transfer. In this paper, it is noted that the previous tabulated method in MCRT produces random deviates from a piecewise constant approximation of the exact scattering phase function. Considering this, improvements can be made by using piecewise linear or log-linear approximation instead. Particularly, combined with the accept-reject method, a new tabulated is also presented for sampling scattering phase functions. Complicated phase function like Fournier-Forand phase function, can be sampled exactly with this new tabulated method. Furthermore, a two-stage Gibbs sampling method is introduced for sampling complicated approximate analytic phase functions, for which it was previously supposed that direct sampling method can not be used. In addition, a new phase function proposed for dust particles. The comparison with measured data indicates that the new phase function can provide a nearly uniform better fit than other approximate phase functions over all detection angles.

Keywords: Sampling methods; Gibbs sampling; Monte Carlo radiative transfer; Approximate phase function.

1. Introduction

Monte Carlo radiative transfer [1] is a widely used tool for studying the spectroscopic properties of scattering medium such as interstellar dust [2, 3], atmosphere [4] and ocean [5] et. al.. Monte Carlo simulations make use of pseudo random numbers to approximate some integral as a expectation under a probability distribution. To implement a Monte Carlo simulation, however, it is necessary to produce sequences of random numbers. In Monte Carlo radiative transfer, one of the most important part is to sample scattering phase functions.

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And the scattering phase function plays a fundamental role in determining radiative properties of scattering medium. Hence, it is necessary and useful to study efficient sampling methods.

For simple analytic phase function like the Henyey-Greenstein (HG) phase function, it is efficient to use the direct sampling method by taking the inverse transform of the corresponding cumulative distribution function (CDF). For other common phase functions, the inverse CDF may not exist or too expensive to be calculated, sampling has to be done in different ways. One of such examples is Mie phase function, which does not even have a analytic form. Another example is the Fournier-Forand (FF) phase function, which provides a good fit for Petzold’s measurement in water. Although having an analytic CDF, finding the inverse transform for FF phase function leads to a rather complicated formula. The tabulated method is often applied to sample such complicated phase functions by constructing a look-up table and then obtain the corresponding random variate using linear interpolation. More recently, a nonlinear lookup table method has been proposed by et. al.. However, as an approximate method, accuracy of tabulated method relies on the number of interpolation points. It is extremely difficult to sample phase functions near forward and backward directions accurately, due to the peaks at $0^\circ$ and $180^\circ$ as well as their relative difference (can reach more than 6 orders in magnitude). There are a huge amount of studies on the impact of the forward and backward peaks of scattering phase function. Poor quality of samples may have a significant impact on the accuracy of simulated radiance, irradiance.

Accept-reject method may be another choice for solving this problem. For Rayleigh phase function, Frisvad reviewed several sampling methods and concluded that the simple accept-reject method is the most efficient way. However, since common phase functions are often with strongly forward-scattering peaks, the simple accept-reject method is of limited use because of the low acceptance rate. Therefore, accept-reject method was seldom used for other scattering phase functions.

In this paper, analysis of the previous tabulated methods is conducted and a new tabulated method is proposed for sampling complicated phase functions by combining with accept-reject method. Remarkable improvement can be achieved with this new method. Particularly, FF phase function which were often considered as impossible to sample exactly, could be sampled exactly with this new method. This new method might be useful in various settings that involve simulating multiple light scattering with Monte Carlo method. In addition, the Gibbs sampling method is also proposed for generating random deviates from some complicated analytic phase functions. Particularly, Gibbs sampling procedures are given for the Draine phase function and a new phase function with exponential decay terms near forward and backward directions. The new phase function provides a good fit to the measured aerosol phase function.
2. Phase function

Light scattering properties of small particles [24] are characterized by the scattering cross section and the scattering phase function. And the scattering phase function is defined as the normalized differential scattering cross section [24]:

\[ p(\Omega, \Omega') = \frac{1}{\sigma_{sc}(\Omega)} \frac{d\sigma_{sc}(\Omega)}{d\Omega'} \]

where \( \Omega \) and \( \Omega' \) are the incident and scattered direction of light. Under this definition, the scattering phase function \( p(\Omega, \Omega') \) describes the angular distribution of scattered radiance at a wavelength. If we further assume the medium to be isotropic, the scattering phase function is just a function of the angle \( \theta \) between \( \Omega \) and \( \Omega' \), or more specifically \( \cos \theta \).

A widely used analytic phase function was proposed by Henyey and Greenstein:

\[ p_{HG}(\theta | g) = \frac{1}{2} \frac{1 - g^2}{(1 + g^2 - 2g \cos \theta)^{3/2}} \]

where \( g \) is the parameter with \(-1 \leq g \leq 1\). This one parameter phase function has been used to simulating light scattering properties of various medium. And the parameter \( g \) can be estimated easily with the method of moment ( \( g = < \cos \theta > \) ). Although the HG phase function is simple and successful, it fails in reproducing observations for high and low frequency case. To remedy this drawback, Draine proposed a phase function:

\[ p_{D}(\cos \theta | g, \alpha) = \frac{1 - g^2}{2(1 + g^2 - 2g \cos \theta)^{3/2}} \frac{1 + \alpha \cos^2 \theta}{1 + \alpha(1 + 2g^2)^{3/2}} \]

with \( C = \frac{1}{1 + \alpha(1 + 2g^2)^{3/2}} \). The Draine phase function reduces to Rayleigh phase function at low frequency and reduces to Cornette-Shanks phase function when \( \alpha = 1 \).

Another approach for modeling the true phase function is by taking the ensemble average with respect to a particle size distribution. For a specific particle size, the scattering cross section and phase function are calculated by solving the rigorous Maxwell equations with numerical methods like Lorenz-Mie theory [24] et. al.. In many cases, this approach can reproduce the observed quantities quite well. One widely used particle size distribution for atmosphere and water [25] [5] is the power-law (Junge) distribution:

\[ n(r) = Ar^{-\alpha}, \quad r \geq r_{\text{min}} \]

where \( \alpha \) is the parameter, \( r \) is the value of the radius for the particle volume-equivalent sphere, \( r_{\text{min}} (> 0) \) is the minimum of \( r \), \( A = (\alpha - 1)r_{\text{min}}^{\alpha - 1} \) is the normalization constant. It should be noted that \( r_{\text{min}} \) is necessary to make the distribution reasonable, but somehow often ignored in the literature. The average scattering phase function in the medium can be represented by the ratio
of ensemble average differential scattering cross section and ensemble average scattering cross section:

$$p(\theta) = \frac{\int_{r_{\text{min}}}^{\infty} n(r)\sigma_{sc}(r)p(\theta, r)dr}{\int_{r_{\text{min}}}^{\infty} n(r)\sigma_{sc}(r)dr} \quad (5)$$

For underwater environment, Fournier and Forand presented a two-parameter approximate analytic phase function with the particle size obeying a power-law distribution and each particle scatters light with the anomalous diffraction approximation. The FF phase function was considered to be appropriate for underwater particles with a relative refractive index close to 1. The latest form of FF phase function can be written as

$$p_{\text{FF}}(\theta) = \frac{1}{2(1 - \delta)^{2\nu}} \left[ \nu(1 - \delta) - (1 - \delta^{\nu}) \right] + \frac{1 - \delta^{\nu}}{8(\delta_{\pi} - 1)\delta_{\nu}} (3\cos^2 \theta - 1) \quad (6)$$

where $$\nu = \frac{3 - \alpha}{2}$$ and $$\delta = \frac{4}{3(m-1)^2}\sin^2(\theta/2)$$, $$\alpha$$ is the exponent of the power-law distribution, $$m$$ is the real part of the relative refractive index for the particle. The FF phase function successfully models marine environment. However, the value of $$p_{\text{FF}}(\theta)$$ at $$\theta = 0^\circ$$ can not be defined, which is rather peculiar. Hence, a regularized FF phase function is proposed:

$$p'_{\text{FF}}(\theta) = P(\theta \leq \theta_0)I(\theta \leq \theta_0) \frac{1}{1 - \cos \theta_0} + P(\theta > \theta_0)I(\theta > \theta_0)p_{\text{FF}}(\theta) \quad (7)$$

where $$P(\Theta \leq \theta_0)$$ represents the probability of scattering angle is smaller than or equal to $$\theta_0$$.

Although analytic phase functions like DPF can provide good approximation to true at low frequency. For higher frequency, the true phase function of particles becomes strongly forward-peaked and modeling with analytic phase functions can result in noticeable error. Considering two peaks of true phase functions at forward and backward direction, a phase function is proposed with exponential decay terms:

$$f(\cos \theta | g, a, b, k, k') = \frac{1 - g^2}{2(1 + g^2 - 2g\cos \theta)^{3/2}} \left( 1 + a \exp(-k\theta) + b \exp(-k'(|\pi - \theta|)) \right) \quad (8)$$

Fig. 1 compares the ensemble averaged phase function (modeled with roughened spheroid) and the new phase function.

### 3. Sampling methods

Monte Carlo simulations are based on the generalization of realizations of random variables distributed according to a specific distribution. The most
Figure 1: Scattering phase function and various approximations for feldspar particles. The measured phase function is from the Amsterdam Light Scattering Database\cite{26} (orange triangle) at wavelength 632.8 nm. The refractive index is $1.5 + 0.001i$. Model averaged phase function is calculated with roughened spheroids as described in \cite{10}.

A straightforward way for producing random variates is perhaps the inverse CDF method \cite{14}, because of the fact that the CDF value of a continuous random variable is a random variable uniformly distributed on $(0, 1)$. Hence, if the inverse CDF exists, a random variable $\mu$ can be generated by taking the inverse transform of a uniform random variable $\xi$ on $(0, 1)$, that is

$$\mu = F^{-1}(\xi), \quad F(\mu) = \int_{-1}^{\mu} p(\mu')d\mu' \quad (9)$$

where $\mu = \cos \theta$ and $p(\mu)$ denotes scattering phase function and $F(\mu)$ denotes the CDF. For example, the HG phase function can be sampled easily with the inverse CDF method. Because of its simplicity, the inverse CDF method will not be discussed further in this paper.

3.1. Accept-reject method

There are some cases that it is extremely difficult or even impossible to directly obtain the inverse CDF. For such phase functions, we cannot derive a direct sampling method. Thus, we may use a simpler distribution and simulate the phase function with the accept-reject method. To generate realizations of a random variable $X$ from density $f_X$, the accept-reject method makes use of another random variable $Y$ whose probability density function $f_Y$ is similar to $f_X$ and is readily to draw samples. And the maximum $M$ of the ratio of these

$$M = \max_{\mu} \frac{f_Y(\mu)}{f_X(\mu)}$$

where $f_Y(\mu)$ represents the density of $Y$ and $f_X(\mu)$ represents the density of $X$. The accept-reject method generates a random vector $(X, Y)$ from the region $Y < M X$ with density $f_Y$. If $Y < M X$, we accept $(X, Y)$ and otherwise, we reject it.
two density function is also needed

\[ M = \max f_X(x) / f_Y(x) \]  \hspace{1cm} (10)

First, we sample \( Y \) from \( f_Y \) and then draw \( \xi \sim U(0, 1) \), accept \( X = Y \) if \( \xi < f_X(Y) / (Mf_Y(Y)) \) otherwise go back to the first step. Iterations of above procedure will produce random deviates from density \( f_X \). And the average number of iterations for generating a random variate is proportional to \( M \) (\( M \) is larger than 1.). More generally, if \( f_X(x) \) can be written in the form:

\[ f_X(x) = Cg(x)p(x) \] \hspace{1cm} (11)

where \( p(x) \) is another density function and \( C \) is a positive constant that makes \( g(x) \in [0, 1] \). Then, we could obtain random deviates from \( f(x) \) by the following procedure:

1: procedure Accept-Reject Sampling
2: do
3: sample \( x \sim f(x) \)
4: sample \( \xi \sim U(0, 1) \)
5: while (\( \xi > g(x) \))
6: return \( x \)
7: end procedure

3.2. Tabulated method

In many settings, the CDFs associated with common phase functions are difficult to simulate with simple accept-reject algorithm due to the strongly forward-scattering peak (hence, low acceptance rate). Then, the tabulated method is often applied for arbitrary scattering phase function by constructing a look-up table and then obtain the corresponding random variate using linear interpolation. It is a fast but approximate method. In the following, new tabulated methods will be presented.

In the tabulated CDF method, first we draw a random number \( \xi \sim U(0, 1) \). Then, finding the interval where \( \xi \) locates,

\[ F_k \leq \xi \leq F_{k+1} \]  \hspace{1cm} (12)

The index of the interval can be found with the bisection search algorithm[27].

\[ \mu = \frac{\mu_{k+1} - \mu_k}{F_{k+1} - F_k} (\xi - F_k) + \mu_k \] \hspace{1cm} (13)

where \(-1 = \mu_1 < \mu_2 < ... < \mu_{K+1} = 1 \). It should be pointed out that using this tabulated method, the phase function is actually approximated by a piecewise constant function. That is the phase function is approximated by a finite sum:

\[ p(\mu) \approx \sum_{k=1}^{K} c_k \mathbb{I}(\mu \in A_k) \] \hspace{1cm} (14)
Figure 2: HG phase function (red dashed) compared with the normalized histogram of 1000000 samples generated using the tabulated method with PCA.

Figure 3: HG phase function (red dashed) compared with the normalized histogram of 1000000 samples generated using the new tabulated method with PLA.
Figure 4: Exact CDF of HG scattering (red dashed line) compared with the empirical CDF of HG scattering with PLA (blue step solid line) and PCA (black step solid line).
Figure 5: CDF of Mie scattering (red dashed line) compared with the empirical CDF of Mie scattering with PLA (blue step solid line) and PCA (black step solid line). Mie phase function was calculated for a spherical particle with radius 2\(\mu m\) refractive index 1.42 buried in the medium with refractive index 1.352 illuminated by the incident light at wavelength 600nm.
where \( I \) is the indicator function, \( A_k \) represents the interval \((\mu_k, \mu_{k+1})\) and 
\[ c_k = \frac{(F_{k+1} - F_k)(\mu_{k+1} - \mu_k)}{2s} \] with \( s = \sum_k c_k (\mu_{k+1} - \mu_k) \). Fig. 2 illustrates a normalized histogram with samples generated with tabulated method and PCA. Values of CDF were uniformly distributed over \([0, 1]\) and corresponding values \( \mu \) were determined by interpolation. Observing this equivalence, it is clear that the quality of the random numbers generated from tabulated method highly depends on the choice of partition. And piecewise constant approximation (PCA) sometimes is not accurate enough to expand a regular phase function, especially near forward and backward regions. An improvement can be made by substituting the piecewise constant approximation with piecewise linear approximation (PLA) or log-linear approximation (PLLA). The corresponding approximate phase function can be represented as
\[
p(\mu) \approx \sum_{k=1}^{K} (a_k \mu + b_k)I(\mu \in A_k)
\] (15)
where \( a_k = \frac{(p(\mu_{k+1}) - p(\mu_k))(\mu_{k+1} - \mu_k)}{\mu_{k+1} - \mu_k} \) and \( b_k \). Instead of a piecewise constant approximation, the phase function has been better approximated by a piecewise linear function that agrees well with it at each of the points \(-1 = \mu_1 < \mu_2 < \ldots < \mu_{K+1} = 1\). The sampling procedure is similar with above, except by solving the quadratic equation:
\[
\frac{1}{2}a_k \mu^2 + b_k \mu - \frac{1}{2}a_k \mu_k^2 - b_k \mu_k = \xi
\] (16)
The procedure for sampling is shown in algorithm 1. Fig. 3 illustrates a normalized histogram of \( \mu \) sampled from PLA of a HG phase function. And PLA outperforms PCA, as shown in Fig. 4 and Fig. 5. Further acceleration, as shown in algorithm 3, can be made if we choose the values of CDF to be uniform on \([0, 1]\) [17, 18]. In such setting, the index of subinterval can be found easily. The tradeoff would be intervals must be fine enough to approximate the scattering phase function.

**Algorithm 1** Tabulated method with PLA

1: **procedure** `SAMPLE(\{\mu_i\}, \{F_i\})`
2: \( \xi \sim U(0, 1) \)
3: Find \( k \) with \( F_k \leq \xi \leq F_{k+1} \) \( \triangleright \) Bisection search
4: Solve \( \frac{1}{2}a_k \mu^2 + b_k \mu - \frac{1}{2}a_k \mu_k^2 - b_k \mu_k = \xi \)
5: \( \text{return } \mu \in [\mu_k, \mu_{k+1}] \)
6: **end procedure**

Another improvement made in this paper is by dividing the entire interval of \( \mu \in [-1, 1] \) into equal probable subintervals as the tabulated method, but for each subinterval, the accept reject method is applied. Specially, if the analytic form of the phase function is known, the exact phase function can be sampled using this approach. The procedure for the tabulated method combined with
accept-reject method is shown in algorithm 2. Fig. 6 compares the sample quality of the new and previous tabulated method with normalized histograms of random cos $\theta$ generated from a FF phase function. As shown in Fig. 6, FF phase function can be sampled exactly with the new tabulated method.

Algorithm 2 New Tabulated Method

1: procedure SAMPLE($\{\mu_i\}, \{F_i\}$)
2: $\xi_1 \sim U(0,1)$
3: Find $k$ with $F_k \leq \xi_1 \leq F_{k+1}$ ▶ Bisection search
4: $M \leftarrow \max(p(\mu_k), p(\mu_{k+1}))$
5: do ▶ Accept-reject sampling
6: $\xi_2, \xi_3 \sim U(0,1)$
7: $\mu \leftarrow \mu_k + \xi_2(\mu_{k+1} - \mu_k)$
8: while $(p(\mu) < \xi_3 * M)$
9: return $\mu$
10: end procedure

Algorithm 3 Tabulated method with PLA and acceleration

1: procedure SAMPLE($\{\mu_i\}, \{F_i\}, \{p_i\}$)
2: $\xi_1 \sim U(0,1)$
3: $k \leftarrow \lfloor \xi_1/(F_2 - F_1) \rfloor$
4: $M \leftarrow \max(p(\mu_k), p(\mu_{k+1}))$
5: do ▶ Accept-reject sampling
6: $\xi_2, \xi_3 \sim U(0,1)$
7: $p \leftarrow (p_{k+1} - p_k)\xi_2 + p_k$ ▶ Piecewise linear approximation
8: while $(p < \xi_3 * M)$
9: return $\mu_k + \xi_2(\mu_{k+1} - \mu_k)$
10: end procedure

3.3. Gibbs sampling

It should be emphasized that although approximate methods by tabulation are easy to implement, the accuracy of simulated results is significantly dependent on the number of tabulated points. It is also noted that such approximate methods may take too much memory to run on GPUs, which degrade its usefulness further. In the following, the Gibbs sampling, a two-state sampling will be introduced for generating random deviates from some complicated scattering phase functions exactly. The Gibbs sampling for random number generation relies upon building a joint density with conditional distributions which are easy for sampling. This means an auxiliary random variable is needed in this method and random deviates are iteratively generated from a sequence of conditional distributions. Consequently, the limit distribution of this sequence converges to the desired density that we would like to sample from. The algorithm for the Gibbs sampler can be summarized as follows:
Algorithm 4 \( f_{X|Y} \) and \( f_{Y|X} \) denote the conditional distributions of \( X \) given \( Y \) and of \( Y \) given \( X \), respectively.

1: \textbf{procedure} \textsc{Gibbs Sampler}(\( x_i \))
2: \hspace{1em} sample \( y_{i+1} \sim f_{Y|X}(y|X = x_i) \);
3: \hspace{1em} sample \( x_{i+1} \sim f_{X|Y}(x|Y = y_{i+1}) \)
4: \hspace{1em} return \( x_{i+1} \)
5: \textbf{end procedure}

As an example, let me introduce how to sample the Draine phase function with Gibbs sampling method. Its sampling scheme can be summarized as follows:

1: \textbf{procedure} \textsc{Sample Draine}(\( \mu_i \))
2: \hspace{1em} sample \( \xi_1 \) from \( U(0, 1 + \alpha \mu_i^2) \) \hspace{1em} \( \triangleright \) sample \( Y \sim f_{Y|X} \)
3: \hspace{1em} do \hspace{1em} \( \triangleright \) sample \( X \sim f_{X|Y} \)
4: \hspace{1em} \( \xi_2, \xi_3 \sim U(0, 1) \)
5: \hspace{1em} \( \mu \leftarrow \frac{1}{2g}(1 + g^2) - ((1 - g^2)/(1 + g(2\xi_2 - 1))^2) \)
6: \hspace{1em} while \( (1 + \alpha \mu^2 < \xi_3) \)
7: \hspace{1em} return \( \mu_{i+1} \leftarrow \mu \)
8: \textbf{end procedure}

The Fig. 7 shows the comparison for the Draine phase function with different parameters and the Fig. 8 shows normed histograms for the Draine phase function samples generated with Gibbs sampling method. As we have seen, Gibbs sampling is an extremely powerful method and allows us to sample some complicated phase functions exactly. The sampling procedure for the new phase function can be summarized as follows:

1: \textbf{procedure} \textsc{Sample NPF}(\( \mu_i \))
2: \hspace{1em} \( \theta_i \leftarrow \arccos \mu_i \)
3: \hspace{1em} sample \( \xi_1 \) from \( U(0, 1 + a \exp(-k\theta_i) + b \exp(-k'(\pi - \theta_i))) \) \hspace{1em} \( \triangleright \) sample \( Y \sim f_{Y|X} \)
4: \hspace{1em} do \hspace{1em} \( \triangleright \) sample \( X \sim f_{X|Y} \)
5: \hspace{1em} \( \xi_2, \xi_3 \sim U(0, 1) \)
6: \hspace{1em} \( \mu \leftarrow \frac{1}{2g}(1 + g^2) - ((1 - g^2)/(1 + g(2\xi_2 - 1))^2) \)
7: \hspace{1em} \( \theta \leftarrow \arccos \mu \)
8: \hspace{1em} while \( (1 + a \exp(-k\theta) + b \exp(-k'(\pi - \theta)) < \xi_3) \)
9: \hspace{1em} return \( \mu_{i+1} \leftarrow \mu \)
10: \textbf{end procedure}
4. Simulations and results

As a simple test the performance of Gibbs sampling and the new phase function, a backward Monte Carlo simulations is conducted to compute reflected and transmitted radiance through plane parallel medium. Each photon launched from the detector has been traced before it was forced back to the direction of light source. The radiance was calculated by adding up the contribution of each light path. The albedo of surface was set to zero, so only the volume scattering event was considered. All simulations were conducted on MacBook Pro with 2.7 GHz Intel Core i7 processor. In the simulation, single scattering properties are calculated with invariant imbedding T-Matrix method [28]. Fig. 9 illustrates the normed samples histogram from the new phase function with Gibbs sampling method.

In Fig. 10, it shows that simulated radiance obtained with the new phase function fit nearly uniformly better than HG phase function. Since HG phase function are monotone function, this fact makes it less flexible and can not fit both forward and backward directions well. The left panel of Fig. 10 illustrates the relative error for simulated transmitted radiance with HG phase function and the new phase function. The error with new phase function are below 2% over nearly all angles and about 1/2 of the error with HG phase function. The right panel of Fig. 10 illustrates the relative error for simulated reflected radiance with these two phase function. And the new phase function also gives much better fit than HG phase phase function except for only a small range of angles, where their errors are about the same. For observation angles from $-30^\circ$ to $30^\circ$, the relative error associated with the new phase function are roughly constant with a value about 1%. Conversely, relative error associated with the HG phase function reaches the maximum at the backward direction and about 5 times larger than the relative error associated with the new phase function. As it has shown, the new phase function fits both forward scattering and backward scattering peak well. Consequently, the simulated radiance with the new phase function fit the exact much better than the radiance simulated with the widely used HG phase function.

5. Discussion and Conclusion

In this paper, it is identified that the widely used tabulated method actually samples a piecewise constant approximation of the exact phase function. Using this tabulated method, there may be significant sampling errors for phase function near the forward and backward region. The main cause if such errors would be from piecewise constant approximate phase functions. Thus, improvements on the quality of random numbers can be made if the PCA is extended to PLA or PLLA. This extension requires minor revisions and does not add much burden to the computer. Moreover, a new tabulated method was proposed by substituting the interpolation procedure with accept-reject sampling. As an example, the regularized Fournier-Forand has been sampled exactly with this new tabulated method. In addition, Gibbs sampling is also proposed for generating random
numbers from some complicated analytic phase functions exactly. The Draine phase function has been sampled as an example. To match the measured data, a new phase function with two exponential day terms is also proposed and sampled exactly with Gibbs sampling method. Since the spectroscopic properties of scattering medium are usually inferred by comparing the observed reflectance or transmittance with simulation results with model particle ensemble. The performance of one new phase function was validated using Monte Carlo simulations. The relative error of simulated transmitted and reflected radiance with the new phase function is much smaller than the one simulated with HG phase function, especially on the backscattering region. This fact may have practical use in lidar remote sensing or biomedical diagnostics. One of drawback of the new phase function needs 4 or 5 parameters, which may difficult to fixed. Empirical relationships for these parameters are needed in further researches.

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Figure 6: FF phase function (red dashed) compared with the normalized histogram of 1000000 samples generated using the new tabulated method with PCA (top panel). The parameters are set as $m = 1.1, \alpha = 3.62$. 

Figure 7: The Draine scattering phase function with different values of parameters. It reduces to the Henyey-Greenstein phase function when $\alpha = 0$ and the Cornette-Shanks phase function when $\alpha = 1$.

Figure 8: Normed sample histograms for the Draine scattering phase function compared with its analytic curve.
Figure 9: Normed sample histograms for the new scattering phase function compared with its analytic curve.

Figure 10: Relative error as a function of observation zenith angle for simulated transmitted and reflected radiance of a turbid medium. The positive angle means azimuth angle is 0°, while the negative angle means azimuth angle is 180°. The red circles represent the relative error for results simulated with HG phase function, and the black squares represent the relative error for results simulated with the new phase function. The exact phase function was sampled with tabulated method. The new phase function was sampled using the Gibbs sampling method. And the HG phase function was sampled with inverse CDF method. 