Article

Tsallis Distribution Decorated With Log-Periodic Oscillation

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Abstract: In many situations, in all branches of physics, one encounters power-like behavior of some variables which are best described by a Tsallis distribution characterized by a nonextensivity parameter $q$ and scale parameter $T$. However, there exist experimental results which can be described only by a Tsallis distributions which are additionally decorated by some log-periodic oscillating factor. We argue that such a factor can originate from allowing for a complex nonextensivity parameter $q$. The possible information conveyed by such an approach (like the occurrence of complex heat capacity, the notion of complex probability or complex multiplicative noise) will also be discussed.

Keywords: scale invariance, log-periodic oscillation, complex nonextensivity parameter, complex multiplicative noise

1. Introduction

In many situations, in all branches of physics, one encounters behavior of some variables $X$ which become pure power distributions for large values of $X$ and exponential for $X \to 0$. Because of this they are known as power-like distributions and in many cases they are identified with a Tsallis distribution \cite{1},

$$F(X) = A \left[ 1 - (1 - q) \frac{X}{T} \right]^{1/(1-q)},$$

(1)
characterized by a scale parameter $T$ and parameter $q$ known as nonextensivity parameter ($A$ is normalization)$^1$. Obviously, for $X \to 0$ distribution (1) becomes the usual Boltzmann-Gibbs exponential formula with temperature $T$, but it becomes pure exponential (i.e., BG) also for $q \to 1$. For $q \neq 1$ and large values of $X$ it becomes pure power distribution not sensitive to scale parameter $T$.

To fully recognize the nontrivial character of distribution (1), one must realize that, usually, in different parts of phase space of the variable $X$, one encounters (or, rather, one expects) a dominance of different (if not completely disparate) dynamical factors. This is best seen in the processes of multiparticle production at high energies (the best known to us). They will serve here to exemplify our further consideration concerning some specific log-periodic oscillations, apparently visible in such processes, which must be therefore somehow hidden in the original distribution (1).

Before proceeding further, we shall briefly summarize the present status of application of Tsallis distributions in this context, concentrating only on multiparticle production processes. They comprise of many different mechanisms in different parts of phase space. Limiting ourselves only to particle production in the central rapidity region and to distribution of their transverse momenta $p_T$, it is customary to divide this production into independent soft and hard processes populating different parts of the transverse momentum space$^2$ separated by a momentum scale $p_0$. As a rule of thumb, the spectra of the soft processes in the low-$p_T$ region are (almost) exponential, $F(p_T) \sim \exp(-p_T/T)$, and are usually associated with the thermodynamical description of the hadronizing system. The $p_T$ spectra of the hard process in the high-$p_T$ region are regarded as essentially power-like, $F(p_T) \sim p_T^{-n}$, and are usually associated with the hard scattering process (for relevant literature concerning both parts see [2]). However, it was very soon recognized that both descriptions could be replaced by a simple interpolating formula [3],

$$F(p_T) = A \left( 1 + \frac{p_T}{p_0} \right)^{-n},$$

(2)

that becomes power-like for high $p_T$ and exponential-like for low $p_T$. The reasoning was that for high $p_T$, where we are usually neglecting the constant term, the scale parameter $p_0$ becomes irrelevant, whereas for low $p_T$ it becomes, together with the power index $n$, an effective temperature $T = p_0/n$. The same formula re-emerged later to become known as the QCD-based Hagedorn formula [4]. It was used for the first time in [5] and became one of the standard phenomenological formulas for $p_T$ data analysis [6–10]. In the mean time it was realized that both formulas are, in fact, identical once

$$n = \frac{1}{q - 1} \quad \text{and} \quad p_0 = nT, \quad (3)$$

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1 The reason being the fact that Eq. (1) is also emerging from nonextensive statistical mechanics [1].

2 A few words of definition concerning this phase space is necessary. A produced particle has some momentum $\vec{p} = [p_L, \vec{p}_T]$. Its longitudinal part, $p_L$, is defined as parallel to the axis of collision, its transverse part, $\vec{p}_T$, as perpendicular to that axis. They are defined by means of rapidity $y$ variable, $y = \frac{1}{2} \ln \frac{E + p_T}{E - p_T}$, as, respectively, $p = |\vec{p}| = m \sinh y$, whereas energy of particle, $E = \sqrt{m^2 + p^2} = m \cosh y$. Central rapidity means $y = 0$. In what follows, our $X$ from Eq. (1) will be identified with transverse momentum, $X = p_T$. 
and therefore they can be used interchangeably.\(^3\)

This distribution is usually used in a thermodynamical content in which the scale parameter \(T\) is identified with the usual temperature (although such identification cannot be solid [25]) and with a real power index \(n = 1/(q - 1)\) (or real nonextensivity parameter \(q\)). Actually, a Tsallis distribution can be regarded as a generalization to real power \(n\) (or \(q\)) of such well known distributions as the Snedecor distribution (with \(n = (\nu + 2)/2\) and integer \(\nu\), which for \(\nu \to \infty\) it becomes exponential distribution).

In [26] we investigate the case when \(q\) is a complex number. We shall review our results in this field in the next section adding examples where log-periodic oscillations occur at different energies and for different collision systems. In Section 3 we discuss the possible consequences of complex nonextensivity parameter including some new recent developments in this field (as complex probability and complex multiplicative noise). The final section contains our conclusions and summary.

2. Log-periodic oscillations in Tsallis distribution - complex power index

Recently, the experiments [8–10] at the Large Hadron Collider (LHC) at CERN provided new data in a very large domain of transverse momenta, \(p_T\), phase space. They turned out to be extremely interesting because of the following:

- They allow us to test the standard Tsallis formula, Eq. (1), over \(\sim 14\) orders of magnitude. As can be seen in Fig. 1a, the observed \(p_T\) distributions of secondaries produced in proton-proton collisions in these experiments can be very well reproduced (cf. also [23])\(^4\).

- And, what is of special importance to us, they disclose some features which suggest a departures from the single form of Eq. (1), cf.Figs. 1 b-c.. Apparently they could not be seen in previous experiments because they seem to be connected with rather large values of transverse momenta, not available earlier..

However, whereas fits to Eq. (1) look pretty good, closer inspection shows that the ratio of data/fit is not flat. It shows some kind of visible oscillations, cf. Fig. 1b. These are the oscillations we have mentioned before.

It turns out that these oscillations cannot be compensated, or erased, by any reasonable change of fitting parameters. Moreover, they are visible by all three experiments CMS, ATLAS, ALICE. The only condition for such an effect to be visible is that the experiment covers a sufficiently large domain of transverse momenta \(p_T\), cf. Fig. 1b. It is also seen at all energies covered by these experiments, cf. Fig. 1c. And, finally, as Fig. 1d shows, this effect is also visible (and is even more pronounced) in nuclear collisions. When taken seriously, it turns out that to account for these oscillations one has to "decorate"

\[^3\] Both Eqs. (1)) and (2) have been widely used in the phenomenological analysis of multiparticle productions, including situations where the nowadays observed spectra extend over many orders of magnitude, [6–23]. Up to now such possibility of testing Tsallis distribution offered only cosmic ray fluxes, cf. [24].

\[^4\] These secondaries were produced at midrapidity, i.e., for \(y \simeq 0\) for which, for large transverse momentum, \(p_T > m\) (where \(m\) is the mass of the particle), one has that, approximately, the energy of particle, \(E \simeq p_T\), i.e., it practically coincides with \(p_T\).
distribution \( f(p_T) \) from Eq. (1) (i.e., one has to multiply it) with some log-periodic oscillating factor. It is usually taken in the form [29]:

\[
R(E) = a + b \cos [c \ln(E + d) + f].
\] (4)

**Figure 1.** (Color online) Examples of log-periodic oscillations. (a) \( dN/dp_T \) for highest energy 7 TeV, the Tsallis behavior is evident. Only CMS data are shown [8], others behave essentially in an identical manner. (b) Log-periodic oscillations showing up in different experimental data like CMS [8] or ATLAS [9] taken at 7 TeV. (c) Results from CMS [8] for different energies. (d) Results for different systems \((p+p)\) collisions compared with \(Pb+Pb\) taken for 5% centrality [27]. Results from ALICE [28] are very similar. Fits for \(p+p\) collision at 7, 2.76 and 0.9 TeV are performed with \(q = 1.139 + i \cdot 0.0385\), \(1.134 + i \cdot 0.0269\) and \(1.117 + i \cdot 0.0307\), respectively. Fit for central \(Pb+Pb\) collisions at 2.76 TeV is done with \(q = 1.135 + i \cdot 0.0321\). See text for more details.

Before proceeding any further let us remember that such log-periodic oscillations are widely known in all situations in which one encounters power distributions. In fact, such behavior has been found in earthquakes [30], escape probabilities in chaotic maps close to crisis [31], biased diffusion of tracers on random systems [32], kinetic and dynamic processes on random quenched and fractal media [33], when considering the specific heat associated with self-similar [34] or fractal spectra [35],
diffusion-limited-aggregate clusters [36], growth models [37], or stock markets near financial crashes [38], to name only a few examples. However, in all these cases the basic distributions were scale free power law, without any scale parameter (here $T$) and without a constant term governing their $X < nT$ behavior.

In the context of nonextensive statistical mechanics log-periodic oscillations have first been observed and discussed while analyzing the convergence dynamics of $z$-logistic maps [39]. In this paper we shall propose another way of introducing such oscillations to Tsallis distributions. It will be based on allowing the power index $n$ (or nonextensivity parameter $q$) in a Tsallis distribution to become complex. For completeness of the presentation we start from the simple pure power law distribution,

$$O(x) = C \cdot x^{-m}. \quad (5)$$

This function is scale invariant, i.e.,

$$O(\lambda x) = \mu O(x), \quad (6)$$

with $m = -\ln \mu / \ln \lambda$. However, because $1 = \exp(\imath 2\pi k)$, one can as well write that

$$\mu \lambda^m = 1 = \exp(\imath 2\pi k), \quad k = 0, 1, \ldots \quad (7)$$

It means therefore that, in general, the index $m$ can become complex,

$$m = -\frac{\ln \mu}{\ln \lambda} + \frac{2\pi k}{\ln \lambda}. \quad (8)$$

As will be obvious from further, general considerations, such a form of the power index results in $R$ as given by Eq. (4) when one only keeps $k = 0, 1$ terms (which is the usual assumption customary applied in all applications [29–33]).

However, Tsallis distribution is only a power-like, not a power distribution. Therefore, to explain the origin of such a dressing factor in this case one has to find a right variable in which the real scaling holds. We start from the observation that, whereas the Boltzmann-Gibbs (BG) distribution,

$$f(E) = \frac{1}{T} \exp \left( -\frac{E}{T} \right), \quad (9)$$

comes from the simple equation,

$$\frac{df(E)}{dE} = -\frac{1}{T} f(E), \quad (10)$$

with the scale parameter $T$ being constant, the same equation, but with variable scale parameter in the form

$$T = T(E) = T_0 + \frac{E}{n}, \quad (11)$$

(known as preferential attachment in networks [14,40]5),

$$\frac{df(E)}{dE} = -\frac{1}{T(E)} f(E) = -\frac{1}{T_0 + E/n} f(E), \quad (12)$$

5 It is worth recalling here that this very same form, $T(E) = T_0 + (1 - q)E$, also appears in [22] within a Fokker-Planck dynamics applied to the thermalization of quarks in a quark-gluon plasma by collision processes.
results in the Tsallis distribution

\[ f(E) = \frac{n-1}{nT_0} \left(1 + \frac{E}{nT_0}\right)^{-n}. \]  

(13)

We shall write now Eq. (12) in finite difference form,

\[ f(E + \delta E) = \frac{-n\delta E + nT + E}{nT + E} f(E). \]  

(14)

In practical sense this means a first-order Taylor expansion for small \( \delta E \ll E \) (from Eq. (14) on, we use \( T \) instead of \( T_0 \)). We shall now consider a situation in which \( \delta E \) always remains finite (albeit, depending on the value of the new scale parameter \( \alpha \), it can be very small) and equal to

\[ \delta E = \alpha nT(E) = \alpha (nT + E). \]  

(15)

Because one expects that changes \( \delta E \) are of the order of the temperature \( T \), the scale parameter must be limited by \( 1/n \), i.e., \( \alpha < 1/n \). In this case, substituting (15) into (14), we have,

\[ f[E + \alpha(nT + E)] = (1 - \alpha n) f(E). \]  

(16)

Expressing Eq. (16) in a new variable \( x \),

\[ x = 1 + \frac{E}{nT}, \]  

(17)

we recognize that the argument of the function on the left-hand side of equality (16) is

\[ E + \alpha(nT + E) = (1 + \alpha)xnT - nT, \]

while the argument of the function on its right-hand side is

\[ E = xnT - nT. \]

Notice that, in comparison with the right-hand side, the variable \( x \) on the left-hand side is multiplied by the additional factor \( (1 + \alpha) \). This means that, formally, Eq.(16), when expressed in \( x \), corresponds to the following scale invariant relation:

\[ g[(1 + \alpha)x] = (1 - \alpha n)g(x). \]  

(18)

This means than that, following the discussion after Eq. (6), its general solution is a power law,

\[ g(x) = x^{-m_k}, \]  

(19)

with exponent \( m_k \) depending on \( \alpha \) and acquiring an imaginary part,

\[ m_k = -\frac{\ln(1 - \alpha n)}{\ln(1 + \alpha)} + ik\frac{2\pi}{\ln(1 + \alpha)}. \]  

(20)
The special case of $k = 0$, i.e., the usual real power law solution with $m_0$ corresponding to fully continuous scale invariance\footnote{In this case power law exponent $m_0$ still depends on $\alpha$ and increases with it roughly as $m_0 \simeq n + \frac{2}{n} (n+1) \alpha + \frac{8}{n^2} (4n^2 + 3n - 1) \alpha^2 + \frac{2}{n^3} (6n^3 + 4n^2 - n + 1) \alpha^3 + \ldots$. Notice also that $\alpha < 1/n$.}, recovers in the limit $\alpha \to 0$ the power $n$ in the usual Tsallis distribution. In general one has

$$g(x) = \sum_{k=0}^\infty w_k \cdot \text{Re} \left( x^{-m_k} \right) = x^{-\text{Re}(m_k)} \sum_{k=0}^\infty w_k \cdot \cos \left[ \text{Im} (m_k) \ln(x) \right]. \quad (21)$$

One therefore obtains a Tsallis distribution decorated by a weighted sum of log-oscillating factors (where $x$ is given by Eq. (17)). Because usually in practice we do not \textit{a priori} know the details of the dynamics of processes under consideration (i.e., we do not known the weights $w_k$), for fitting purposes one usually uses only $k = 0$ and $k = 1$. In this case one has, approximately,

$$g(E) \simeq \left( 1 + \frac{E}{nT} \right)^{-m_0} \left\{ w_0 + w_1 \cos \left[ \frac{2\pi}{\ln(1+\alpha)} \ln \left( 1 + \frac{E}{nT} \right) \right] \right\} \quad (22)$$

and reproduces the general form of a dressing factor given by Eq. (4) and often used in the literature \cite{29}. In this approximation the parameters $a, b, c, d$ and $f$ from Eq. (4) get the following meaning:

$$\frac{a}{b} = \frac{w_0}{w_1}, \quad c = \frac{2\pi}{\ln(1+\alpha)}, \quad d = nT, \quad f = -\frac{2\pi}{\ln(1+\alpha)} \ln(nT). \quad (23)$$

In fact this is not the most general result for in our derivation, Eqs.(15)-(18)), we have so far only accounted for a single step evolution. In real situation one should expect to have a whole hierarchy of evolutions. In such a case consecutive steps of evolution are connected by:

$$E_i = E_{i-1} + \alpha_{i-1} (nT + E_{i-1}) \quad (24)$$

each with its own scale parameter $\alpha_i$. In the simplest situation, neglecting any fluctuations of consecutive scaling parameters, i.e., assuming that all $\alpha_i = \alpha$, one has that after $\kappa$ steps

$$nT + E_\kappa = (1 + \alpha)^\kappa (nT + E_0). \quad (25)$$

This means that, in general, Eq. (18) should be replaced by a new scale invariant equation:

$$g \left[ (1 + \alpha)^\kappa x \right] = (1 - \alpha n)^\kappa g(x). \quad (26)$$

 Whereas this equation does not change the slope parameter $m_0$, it significantly influences the frequency of oscillations which are now $\kappa$ times smaller,

$$c = \frac{2\pi}{\kappa \ln(1+\alpha)} \quad (27)$$

(in Eq.(26) $\lambda = (1 + \alpha)^\kappa$ and $\mu = (1 - \alpha n)^\kappa$; the slope parameter $m_0 = -\ln \mu / \ln \lambda$ is independent of $\kappa$, whereas the frequency of oscillations, $2\pi / \ln \lambda$, decreases with $\kappa$ as $1/\kappa$). For more complex behavior of intermediate scale parameters $\alpha_i$ one gets more complicated expressions (we shall not discuss this here).
3. Other consequences of complex nonextensivity parameter

There are other consequences of allowing the parameter \( m \) to be complex. In what follows we shall discuss shortly three examples: complex heat capacity, complex probability and complex multiplicative noise.

3.1. Complex heat capacity

The complex power exponent in the Tsallis distribution, \( m = m' + i \cdot m'' \), means that

\[
q - 1 = \frac{1}{m} = \frac{m'}{|m|^2} + i \frac{m''}{|m|^2}. \tag{28}
\]

As shown in [18] (cf. also [14,15,41]), the nonextensivity parameter \( q \) can be treated as a measure of the thermal bath heat capacity \( C \) with

\[
C = \frac{1}{q - 1} = m' + i m''. \tag{29}
\]

The complex nonextensive parameter \( q \) must therefore have some profound consequences because now the corresponding heat capacity becomes complex as well. As a matter of fact, such complex (frequency dependent) heat capacities (or generalized calorimetric susceptibilities) are known in the literature [45] and are usually written in the form

\[
C = C_\infty + \frac{C_0 - C_\infty}{1 + (\omega \tau)^2} (1 - i \omega \tau). \tag{30}
\]

Here \( C_\infty \) is the heat capacity related to the infinitely fast degrees of freedom of the system as compared to the frequency \( \omega \), and \( C_0 \) is the total contribution at equilibrium (the frequency is set to zero) of the degrees of freedom, fast and slow, of the sample. The time constant \( \tau \) is the kinetic relaxation time constant of a certain internal degree of freedom.

These complex heat capacities are known as dynamic heat capacities and are intensively explored from both experimental and theoretical perspectives. It is expected that dynamic calorimetry can provide an insight into the energy landscape dynamics, cf., for example, [46–49]. Usually one associates the imaginary part of linear susceptibility with the absorption of energy by the sample from the applied field.

In the case of temperature fluctuations \( \delta T(t) \) the deviation of the energy from its equilibrium value \( \delta U(t) \) is, for a certain linear operator \( \hat{C}(t) \), some linear function of the corresponding variation of the temperature,

\[
\delta U(t) = \hat{C} \delta T(t). \tag{31}
\]

If the temperature of the reservoir changes infinitely slowly in time, then the system can keep up with any changes in the reservoir and its susceptibility is just the specific heat of the system \( C_V \). However, in general, the behavior of the system is described by a generalized susceptibility \( C_V(\omega) \), which can be called the complex and \( \omega \)-dependent heat capacity of the system. The change in the energy of a system in the field of the thermal force can be represented by

\[
\delta U(t) = \int L(t') \delta T(t - t') dt', \tag{32}
\]
where $L(t')$ is the response function of the system describing its relaxation properties given by $\Phi(t) = \int_t^\infty L(t') dt'$. Taking the Fourier transform one gets

$$\delta U(\omega) = C_V(\omega) \delta T(\omega),$$

where

$$C_V(\omega) = \int L(t') e^{i\omega t'} dt'$$

is the generalized susceptibility of the system and is called the complex heat capacity. In practice, the frequency dependent heat capacity is a linear susceptibility describing the response of the system to the small thermal perturbation (occurring on the time scale $1/\omega$) that takes the system slightly away from the equilibrium.

A complex $C_V(\omega)$ means that $\delta U$ and $\delta T$ are shifted in phase and that the entropy production in the system differs from zero [49]. The corresponding fluctuation-dissipation theorem for the frequency dependent heat capacity was established in [48]. According to this result, the frequency-dependent heat capacity may be expressed within the linear response approximation as a linear susceptibility describing the response of the system to arbitrarily small temperature perturbations away from equilibrium,

$$C_V(\omega) = \frac{(U^2)_0}{\langle T \rangle^2} - i \frac{\omega}{\langle T \rangle^2} \int_0^\infty dt e^{-i\omega t} \langle U(0)U(t) \rangle$$

(35)

(the $\omega$ denotes frequency with which temperature field is varying with time).

The above results for heat capacity can now be used to a new phenomenological interpretation of the complex $q$ parameter discussed before. Namely, one can argue that

$$q - 1 = \frac{Var(T)}{\langle T \rangle^2} - i \frac{S(T)}{\langle T \rangle^2},$$

were

$$S(T) = \omega \int Cov[T(0), T(t)] e^{-i\omega t} dt$$

is the spectral density of temperature fluctuations (i.e., the Fourier transform of the covariance function averaging over the nonequilibrium density matrix).

We would like to stress at this point that, in a sense, Eq. (36) can be regarded as a generalization of our old proposition for interpreting $q$ as a measure of nonstatistical intrinsic fluctuations in the system [43,44] (which corresponds to the real part of (36)) by adding the effect of spectral density of such fluctuations (via the imaginary part of (36)). Notice that (36) follows from (29) and the relation $U = C_VT$, allowing to write (35) in the form of (36).

### 3.2. Complex probability

From the point of view of superstatistics [42,43], in our particular case complex parameter $q$ corresponds to a complex probability distribution. Namely, one uses the property that gamma-like fluctuation of the scale parameter $T$ in an exponential BG distribution (9) results in the $q$-exponential Tsallis distribution (1) with $q > 1$. The parameter $q$ is given here by the strength of these fluctuations, $q = 1 + Var(X)/\langle X \rangle^2$. From the thermal perspective, it corresponds to situation in which the heat
bath is not homogeneous but has different temperatures in different parts, which are fluctuating around some mean temperature \( T_0 \). It must be therefore described by two parameters: a mean temperature \( T_0 \) and the mean strength of fluctuations given by \( q \).

We now perform the same procedure, but using two gamma distributions, one with a real power index, \( m_0 - 1 \), and one with a complex power index, \( m_0 + im_1 - 1 \),

\[
g(1/T) = w_0 \frac{1}{\Gamma(m_0)nT_0} \left( \frac{T_0}{nT} \right)^{m_0-1} \exp \left( -\frac{nT_0}{T} \right) + \\
+w_1 \frac{1}{\Gamma(m_0 + im_1)nT_0} \left( \frac{T_0}{nT} \right)^{m_0+im_1-1} \exp \left( -\frac{nT_0}{T} \right). \tag{38}
\]

As the result one gets a complex distribution (complex pdf):

\[
h_q(E) = \int_0^\infty f(E)g(1/T)d(1/T) = Cw_0 \left( 1 + \frac{E}{nT_0} \right)^{-m_0} + Cw_1 \left( 1 + \frac{E}{nT_0} \right)^{\neg m_0 - im_1}, \tag{39}
\]

the real part of which is pdf in form of a Tsallis distribution decorated with log-periodic oscillations of the type of Eq. (22),

\[
Re \left[ h_q(E) \right] = C \left( 1 + \frac{E}{nT_0} \right)^{-m_0} \cdot \left\{ w_0 + w_1 \cos \left[ m_1 \ln \left( 1 + \frac{E}{nT_0} \right) \right] \right\}. \tag{40}
\]

The complex pdf has a number of interesting properties [50,51]. It plays an important role in the interference among resonance states during scattering experiments. It is associated with the phase of the resonance channel probability amplitudes (in non-Hermitian quantum mechanics). In wireless communication systems it is generated by a superposition of finite random variables and usually involves the movement, scattering, diffusion or diffraction. The imaginary part is proportional to the degree of the correlation. The imaginary part is then a function of a correlation coefficient or other parameters that state the degree of the relationship of each individual random variable of the superposition of the random variable having a complex pdf. The real and imaginary part have diverse properties, i.e. one for real valued pdf and the other for elementary correlation, respectively.

It is interesting to note that entropy

\[
H = - \left| \int \int (a \ln a + i \cdot b \ln b)dx_1dx_2 \right| = \tag{41}
\]

corresponding to complex joint probability,

\[
f \left( x_1, x_2 \right) = a \left( x_1, x_2 \right) + i \cdot b \left( x_1, x_2 \right), \tag{42}
\]

consists of two components:

\[
H_1 = - \int \int a \ln a \, dx_1dx_2, \quad H_2 = - \int \int b \ln b \, dx_1dx_2; \quad H = |H_1 + iH_2| \sqrt{H_1^2 + H_2^2} \geq H_1. \tag{43}
\]

The imaginary part of entropy is proportional to the degree of incompatibility of the correlated stochastic processes. The incompatibility increases the entropy of correlated stochastic processes.
3.3. Complex multiplicative noise

It is known that multiplicative noise leads to a Tsallis distribution \([44]\). It is then natural to expect that multiplicative complex noise should result in complex \(q\) and in log-periodic oscillations in Tsallis distributions. It can be defined by a Langevin equation

\[
\frac{dp}{dt} + \gamma(t)p = \xi(t), \quad \text{where} \quad \gamma(t) = \gamma_0(t) + i\gamma_1.
\]

(44)

The resulting distribution \([44]\) is now

\[
f(p) = \left( 1 + \frac{q - 1}{T} p^2 \right)^{\frac{q-1}{q}} \quad \text{where} \quad T = \frac{2\text{Var}(\xi)}{\langle \gamma \rangle}, \quad q = 1 + \frac{2\text{Var}(\gamma)}{\langle \gamma \rangle}.
\]

(45)

The parameter \(q\) is now complex because \(\langle \gamma \rangle\) is complex. Even more importantly, \((q - 1)/T = \text{Var}(\gamma)/\text{Var}(\xi)\) is real (it tends to zero for \(q \to 1\)). This is because the complex term \(\gamma_1\) added to the noise is constant. Notice that we could just as well replace in Eq. (45) \((q - 1) (p^2/T)\) by \((p^2/p_0^2)\) where \(p_0^2 = \text{Var}(\xi)/\text{Var}(\gamma)\). The examples and discussion of the systems characterized by the appearance of "imaginary" multiplicative noise terms in an effective Langevin-type description can be found in \([52]\).7

4. Summary and conclusions

In many places in physics, and especially in the realm of high energy multiparticle production processes we are particularly interested in, it became a standard procedure to fit the data on transverse momentum distributions by means of the quasi-power Tsallis formula. The usual interpretation in such cases is that the scale parameter \(T\) is a kind of "temperature" whereas additional nonextensivity parameter \(q\) describes intrinsic, nonstatistical fluctuations existing in the system \([11–22,24,42–44,53]\). However, with increasing range of transverse momenta measured in recent experiments \([8–10]\) two things happened:

- \((i)\) That they still can be fitted by the same formula (which came as surprise because fits now cover \(\sim 14\) orders of magnitude of the measured cross sections \([2,23]\)).
- \((ii)\) That new data revealed weak but persistent oscillation of log-periodic character (discussed already shortly in \([26]\)).

If taken seriously, such log-periodic structures in the data indicate that the system and/or the underlying physical mechanisms have characteristic scale invariant behavior. This is interesting as it provides important constraints on the underlying physics. The presence of log-periodic features signals the existence of important physical structures hidden in the fully scale invariant description. It is important to recognize that Eq. (12) represents an averaging over highly 'non-smooth' processes and, in its present form, suggests rather smooth behavior. In reality, there is a discrete time evolution for the number of steps. To account for this fact, one replaces a differential Eq. (10) by a difference quotient and expresses

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7 In fact, this is not exactly Tsallis formula from Eq. (1). To get it one has to allow for correlation between noises and drift term due to additive noise, i.e., for \(\text{Cov}(\xi, \gamma) \neq 0\) and \(\langle \xi \rangle \neq 0\) (see \([53]\) for details). One obtains then Eq. (1) but with, in general, complex \(T = T(q)\). We shall not discuss it here.
\(dt\) as a discrete step approximation given by Eq. (15) with parameter \(\alpha\) being a characteristic scale ratio. It can also be shown that discrete scale invariance and its associated complex exponents can appear spontaneously, without a pre-existing hierarchical structure. Finally, a complex nonextensivity parameter promises new perspectives in future phenomenological applications being connected to complex heat capacity, to notion of complex probability or to complex multiplicative noise, to mention only a few examples discussed shortly in our paper.

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Conflicts of Interest

The authors declare no conflict of interest.

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