Lipschitz and biLipschitz Maps on Carnot Groups

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Abstract

Suppose $A$ is an open subset of a Carnot group $G$ and $H$ is another Carnot group. We show that a Lipschitz function from $A$ to $H$ whose image has positive Hausdorff measure in the appropriate dimension is biLipschitz on a subset of $A$ of positive Hausdorff measure. We also construct Lipschitz maps from open sets in Carnot groups to Euclidean space that do not decrease dimension. Finally, we discuss two counterexamples to explain why Carnot group structure is necessary for these results.

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1 Introduction

In 1988, Guy David proved in [5] that if $f$ is a Lipschitz function from the unit cube in $\mathbb{R}^n$ to a subset of some Euclidean space with positive $n$-dimensional Hausdorff measure, there exists a subset $K$ of the domain of $f$ with positive $n$-dimensional Hausdorff measure such that $f$ is biLipschitz on $K$.

Shortly thereafter, Peter Jones proved the following stronger result in [9]: if $f$ is a Lipschitz function from the unit cube in $\mathbb{R}^n$ to a subset of some Euclidean space, then the unit cube can be broken up into the union of a 'garbage' set (whose image under $f$ has arbitrarily small $n$-dimensional Hausdorff content) and a finite number of sets $K_1, \ldots, K_N$ such that $f$ is biLipschitz on each $K_i$.

Three years later, Guy David in [6] translated this proof into the language of wavelets, which are more readily generalizable to the Heisenberg (and especially other Carnot) groups. The proof as written in [6] only depends on a few general properties, all but one of which hold for Heisenberg (and other Carnot) groups.

The next section of this paper is primarily devoted to proving the first result mentioned in the abstract after adopting some of the ideas in [6] and [9] to Carnot groups. As a consequence, we show that Problems 22 and 24 in
are logically equivalent when restricted to maps whose domains are open sets.

The third section will investigate the question of how big, in terms of dimension, Lipschitz images of Carnot groups in Euclidean space can be. Finally, the fourth section will explore two counterexamples explaining why Carnot group structure is necessary for these results.

The author would also like to thank Raanan Schul for showing him Guy David’s proof of the decomposition of Lipschitz functions from subsets of $\mathbb{R}^m$ to $\mathbb{R}^n$ into biLipschitz pieces. Further, the author would like to thank Jeremy Tyson for showing him the usefulness of Cantor set constructions for creating maps on Carnot groups.

2 Jones-Type Decomposition for Carnot Groups

2.1 Brief Outline

This section is organized as follows. In Section 2.2 we shall give some definitions concerning Carnot groups and their various properties while also justifying some notational conventions. In Section 2.3 we shall state the five properties of Euclidean space on which David’s argument rests and show how the first four of them work for Heisenberg groups. In Section 2.4 we will explain why these properties work just as well for any Carnot group. In Section 2.5, we shall prove our main result: if $A$ is an appropriate subset of the $k$th Heisenberg group $H_k$ corresponding roughly to the unit cube in $\mathbb{R}^n$, and $F$ is a Lipschitz function from $A$ to another Heisenberg group whose image has positive Hausdorff $(2k + 2)$-dimensional measure, then there exists $B \subset A$ with positive Hausdorff $(2k + 2)$-dimensional measure such that $F$ is biLipschitz on $B$. Finally, Section 2.6 will derive some further results of interest as corollaries of the main theorem from Section 2.5.

Although our main focus is on the Heisenberg groups (especially $H_1$), all of the results in this paper apply equally well to Carnot groups in general. To exploit this fact, the results in Section 2.5 will be stated and proved in the more general context of Carnot groups.

2.2 Definitions

We begin by defining the Heisenberg groups as follows.

Definition 2.1. The $n$th Heisenberg group $H_n$ is defined as the set

$$\{(z_1, \ldots, z_n, t) : z_j \in C, t \in \mathbb{R}\}$$
equipped with the following group law:

\[(z_1, \ldots, z_n, t)(w_1, \ldots, w_n, s) = (z_1 + w_1, \ldots, z_n + w_n, t + s + \Im \sum_{j=1}^{n} z_j \bar{w}_j)\]

where \(\Im\) denotes imaginary part.

The Heisenberg group is a special example of a Carnot group, which is defined as follows:

**Definition 2.2.** A Carnot group \(G\) is a connected, simply connected, nilpotent Lie group whose Lie algebra \(\mathfrak{g}\) is graded, i.e.

\[\mathfrak{g} = \bigoplus_{j=1}^{d} \mathfrak{g}_j\]

where

\[[\mathfrak{g}_1, \mathfrak{g}_j] = \mathfrak{g}_{j+1}\]

and

\[\mathfrak{g}_{d+1} = \{0\}\]

We call \(\mathfrak{g}_1\) the horizontal component of \(\mathfrak{g}\).

By standard properties of Lie group theory, (see, for example, [15]), the exponential map gives a diffeomorphism between a Carnot group and its Lie algebra. Further, the standard definition of a Lie algebra in terms of vector fields provides a canonical identification between the tangent space of a Lie group at a given point and the Lie group itself. (When \(g \in G\) is fixed, for every tangent vector \(v\) there is a unique \(X \in \mathfrak{g}\) such that \(X(g) = v\) and we can identify \(\exp(X)\) with \(v\).)

Throughout this paper we shall freely use these canonical identifications between a Carnot group, its Lie algebra, and its tangent space. For example, this shall give us a coordinate structure for our Lie groups induced by their Lie algebras. We have already done this for the Heisenberg groups above. In this case, \(\mathfrak{g}_1\) consists of the points of the form \((z_1, \ldots, z_n, 0)\) with final coordinate equal to zero.

**Definition 2.3.** Let \(G\) be a Carnot group. We say that \(G\) is rationalizable if, writing its Lie algebra \(\mathfrak{g}\) as

\[\mathfrak{g} = \bigoplus_{j=1}^{d} \mathfrak{g}_j\]

where

\[[\mathfrak{g}_1, \mathfrak{g}_j] = \mathfrak{g}_{j+1}\]

and

\[\mathfrak{g}_{d+1} = \{0\}\]
and letting \( n_i \) be the dimension of \( g_i \) as a vector space, each \( g_i \) has a basis \( \{ X_{(i,m)} \}_{1 \leq m \leq n_i} \), such that letting \( \mathfrak{X} \) be the collection
\[
\{ X_{(i,m)} : 1 \leq i \leq d, 1 \leq m \leq n_i \},
\]
the Lie bracket of any two elements of \( \mathfrak{X} \) is a linear combination of elements of \( \mathfrak{X} \) with rational coefficients.

For example, Euclidean space and the Heisenberg groups are rationalizable.

We next define the following family of homomorphisms on a Carnot group.

**Definition 2.4.** Let \( \lambda > 0 \), let \( G \) be a Carnot group and let \( g \in G \), where
\[
g = \sum g_i
\]
with \( g_i \in g_i \). Define the **dilation**
\[
\delta_\lambda(g) = \sum \lambda^i g_i.
\]

We shall next equip each Carnot group with a metric structure, defined as follows (see [3]).

**Definition 2.5.** The **Carnot-Carathéodory distance** \( d_{\text{CC}} \) on a Carnot group \( G \) is defined as follows: given two points \( g, h \) in \( G \) and setting \( \Gamma_{g,h} \) to be the set of all curves \( \gamma : [0, 1] \to G \) with \( \gamma(0) = g \), \( \gamma(1) = h \), and \( \gamma'(t) \in g_1 \) for each \( t \in [0, 1] \),
\[
d_{\text{CC}}(g,h) = \inf_{\gamma \in \Gamma_{g,h}} \int_0^1 |\gamma'(t)| dt
\]
(where the absolute value refers to a Euclidean metric placed on the real vector space \( g_1 \)).

Because \( \Gamma_{g,h} \) in the above definition is nonempty (cf [13]), \( d_{\text{CC}}(g,h) < \infty \) whenever \( g, h \in G \).

**Note 2.6.** It is often easier to work with a comparable \( L^\infty \) quasidistance function \( d \) based on the Carnot metric. For the first Heisenberg group \( H_1 \), this is done by defining distance to the origin as
\[
d((x,y,z),(0,0,0)) = \max(|x|, |y|, |z|^{5})
\]
and for an arbitrary \( g, h \) in this group, defining
\[
d(g, h) = d(h^{-1}g, (0,0,0)).
\]
There is of course a completely analogous construction in an arbitrary Carnot group: if \( G \) is a Carnot group, we use the grading of its Lie algebra \( g \) as in the definition of Carnot groups:

\[
g = \bigoplus_{j=1}^{d} g_j
\]

and put a norm \( \| \cdot \|_j \) on \( g_j \) for \( j = 1, \ldots, d \).

Because the identity element in a Carnot group is the image of the origin under the exponential map, we shall refer to it as 0. Now, letting \( g \) be an arbitrary point in \( G \) we first define its quasidistance to the identity element, \( d(g, 0) \), by recalling the direct sum decomposition

\[
\exp^{-1}(g) = \Sigma g_j
\]

with \( g_j \in g_j \) and setting

\[
d(g, 0) = \max_{1 \leq j \leq d}(\|g_j\|_j)^{1/2}.
\]

Finally, for an arbitrary \( g, h \in G \), we finish by setting

\[
d(g, h) = d(h^{-1}g, 0).
\]

For the duration of this paper, \( d_{CC} \) shall refer to Carnot-Carathéodory distance and \( d \) shall refer to quasidistance.

One fundamental operation for Carnot groups is the Pansu differential, defined as follows (following the treatment in [3]):

**Definition 2.7.** Let \( F : G \to H \) be a function from one Carnot group \( G \) to another Carnot group \( H \). Then the **Pansu differential** \( DF(g) \) of \( F \) at \( g \in G \) is the map

\[
DF(g) : G \to H
\]

defined at \( g' \in G \) as the limit

\[
DF(g)(g') = \lim_{s \to 0} \delta_{s^{-1}}[F(g)^{-1}F(g\delta_sg')]
\]

whenever it exists.

Using the canonical identifications stated above, we can view the Pansu differential as a map between Lie algebras or as a map from the tangent space at \( g \) to the tangent space at \( F(g) \) for \( g \in G \). We shall take advantage of this fact throughout the paper.

Further, using the tangent vector interpretation, the Pansu differential \( DF(g) \) induces a linear map between the horizontal component of the tangent space of \( G \) at \( g \) and the horizontal component of the tangent space of \( H \) at
$F(g)$ (see [14]). Calling this linear map $MF(g)$, we can view $MF$ as a matrix-valued map sending $g$ to $MF(g)$.

Finally, we define a sub-Riemannian manifold as follows.

**Definition 2.8.** A sub-Riemannian manifold is a triple $(M, \Delta, g)$ where $M$ is a smooth manifold, $\Delta$ is a distribution (i.e. sub-bundle of the tangent bundle $TM$) on $M$ which is smooth and satisfies the property that for each $p \in M$, $(TM)_p$ is generated as a Lie algebra by $\Delta_p$, and $g$ is a smooth section of positive-definite quadratic forms on $\Delta$ (i.e. $g_p$ defines an inner product on $\Delta_p$ which varies smoothly in $p$).

Recall (see [18]) that the set $S$ is said to generate a Lie algebra if the set of finite Lie brackets of elements of $S$ spans $g$ as a vector space.

We shall consider $M$ to be naturally equipped with a metric $d_{CC}$ defined as follows: for $x, y \in M$,

$$d_{CC}(x, y) = \inf_{\gamma \in \Gamma_{x,y}} \int_0^1 \sqrt{g(\gamma'(t), \gamma'(t))} dt$$

where $\Gamma_{x,y}$ is the family of all curves

$$\gamma : [0, 1] \to M$$

with $\gamma(0) = x$, $\gamma(1) = y$, and $\gamma'(t) \in \Delta_{\gamma(t)}$ for all $t$.

The notion of a sub-Riemannian manifold will not occur in the proof of our main theorem. It will only be used when we mention partial generalizations at the end of Section 2 and give a counterexample in Section 4.2.

2.3 Five key properties

2.3.1 Dyadic decomposition

There exists a dyadic decomposition for Euclidean space defined as follows:

For each nonnegative integer $k$ we let $A_k$ be the set of all “cubes” of the form

$$(a_1 \cdot 2^{-k}, (a_1 + 1) \cdot 2^{-k}) \times \cdots \times (a_n \cdot 2^{-k}, (a_1 + 1) \cdot 2^{-k})$$

contained in the unit cube, where the $a_i$ are all integers. Then the elements of $A_k$ are disjoint open sets. Further, each element of $A_k$ is (up to a set of measure zero) a disjoint union of elements of $A_{k+1}$, the $A_k$’s are all translates of each other, and one can transform an arbitrary element of $A_k$ into an arbitrary element of $A_{k+1}$ by a dilation (by a factor of $2^{-k}$) followed by translation. Finally, fixing a cube $Q \in A_k$ and letting $d$ be its diameter (i.e. $d = \sqrt{72}^{-k}$), the number of cubes in $A_k$ whose distance from $Q$ is at most $d$ is bounded above by a constant depending only on $n$. 

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Our immediate goal is to generalize this decomposition to the Heisenberg group $H_1$. To do this we loosely follow Christ’s construction of Theorem 11 in [4]. Temporarily adopting Christ’s notation to create this dyadic structure, we shall create a tree for the entire Heisenberg group. We let $B_0$ denote the discrete Heisenberg group, which is generated by $(1, 0, 0)$ and $(0, 1, 0)$. Analogously, for the $k$th Heisenberg group, we begin by rewriting the elements of $H_k$ to mirror the above construction for $H_1$: in other words, writing $z_j = x_{2j-1} + ix_{2j}$ where $x_{2j-1}, x_{2j} \in \mathbb{R}$, we let $B_0$ be the subgroup of $H_k$ generated by

$$\{(x_1, \ldots, x_{2n}, 0) : x_j = \pm \delta_{j,l}, 1 \leq l \leq 2k\}$$

where $\delta_{j,l}$ is the Kronecker delta.

For each positive integer $n$, we let $B_n$ be the image of $B_0$ under the dilation $\delta_{10^{-n}}$ (in particular, the first $2k$ coordinates are multiplied by $10^{-n}$; the final coordinate is multiplied by $10^{-2n}$). Equivalently, $B_n$ is the subgroup of the $k$th Heisenberg group generated by $$\{10^{-n}e_j : 1 \leq j \leq 2k\}.$$ If $x$ is a point in $B_n$, we give it the label $(x, n)$ and note that $x$ has a different label for each $B_n$ containing $x$. We form a tree by defining an order relation $\leq$ on the set of all such pairs $(x, n)$. We start this procedure with the following definition.

**Definition 2.9.** $(x, \alpha)$ is a parent of $(y, \beta)$ if $\beta = \alpha + 1$ and $y = xg$ where the first $2k$ components of $g$ all lie in $(-\frac{1}{2}10^{-\alpha}, \frac{1}{2}10^{-\alpha}]$ and the real component lies in $(-\frac{1}{2} \cdot 10^{-2\alpha}, \frac{1}{2} \cdot 10^{-2\alpha}]$.

Using the obvious analogies from family trees (‘ancestor’, ‘descendant’, ‘grandparent’, ‘sibling’, etc.) for both the tree points and corresponding dyadic cubes, we say $(x, \alpha) \leq (y, \beta)$ if $(y, \beta)$ is an ancestor of $(x, \alpha)$. Following along exactly as in Definition 14 of [4], we create from this tree a family of dyadic ‘cubes’; each cube $Q(x, \alpha)$ is the cube at scale $\alpha$ based at $x$ while $A_\alpha$ consists of all the cubes with $\alpha$ as second component. All the cubes in $A_\alpha$ are translates of each other by elements of the discrete Heisenberg group of the appropriate scale; further, each member of each $A_\alpha$ is an open set while each element of $A_\alpha$ is (up to a set of measure zero) the disjoint union of elements of $A_{\alpha+1}$. Also, one can transform an arbitrary element of $A_\alpha$ into an arbitrary element of $A_{\alpha+1}$ by a dilation (by a factor of $10^{-1}$) followed by translation. Finally, the number of cubes in $A_\alpha$ within diam$(Q(x, \alpha))$ of $Q(x, \alpha)$ is bounded by a constant independent of $\alpha$.

In this construction, the analogue to the unit cube in Euclidean space is the unique cube of scale 0 containing the identity element; according to the notation defined in the preceding paragraph, the name for this cube is $Q(0, 0)$. 7
Remark: In making this decomposition we are saying nothing about the boundaries of the elements of the $A_\alpha$ other than that they are closed sets of Hausdorff measure zero in the appropriate dimension. Also, this decomposition is not the same as the decomposition of the Heisenberg group found in [17].

2.3.2 Orthogonal decomposition of $L^2$

Looking back at Euclidean space $\mathbb{R}^n$ for inspiration, we note that the Hilbert space $L^2([0,1]^n)$ of square-integrable functions on the unit cube, i.e. the set of all Lebesgue measurable functions $f : [0,1]^n \to \mathbb{R}$ such that $\|f\|_2 < \infty$ where

$$\|f\|_2^2 = \int_{[0,1]^n} |f|^2,$$

can be decomposed into orthogonal subspaces as follows: we let $C_0$ consist of the set of all constant functions on the unit cube and then for each integer $\beta \geq 1$, we let $C_\beta \subset L^2([0,1]^n)$ be the set of functions

$$\{f : f|_Q \text{ is constant for } Q \text{ of scale } \beta \text{ and } \int_Q f = 0 \text{ for } Q \text{ of scale } \beta - 1\}.$$

The space of $L^2$ functions on the unit cube in Euclidean space can be written as the orthogonal direct sum of the spaces $C_\beta$ for $\beta \geq 0$; in other words, if $f \in C_\beta$, $g \in C_\gamma$ with $\beta \neq \gamma$, $\int_{[0,1]^n} fg = 0$ while for each $h \in L^2([0,1]^n)$ there exists $h_\beta \in C_\beta$ for $\beta$ a nonnegative integer with

$$h = \sum_{\beta=0}^{\infty} h_\beta$$

with the sum in question converging in $L^2([0,1]^n)$ to $h$.

For the Heisenberg groups we can mimic this procedure as follows: here, our ‘base’ cube shall be denoted as $Q(0,0)$ where the first zero denotes the origin and the second zero denotes scale. Similarly, we define the $C_\beta$ (as subspaces of the Hilbert space $L^2(Q(0,0))$ of real-valued, square-integrable functions) identically to the way we did with Euclidean space. In other words, $C_0$ still consists of constant functions; for $\beta$ a strictly positive integer, $C_\beta$ is the set

$$\{f : f|_Q \text{ is constant for } Q \text{ of scale } \beta \text{ and } \int_Q f = 0 \text{ for } Q \text{ of scale } \beta - 1\}.$$

nj For $\beta > 0$, $C_\beta$ has a spanning set consisting of the functions $f_{Q,Q'}$ for $Q, Q'$ sibling cubes in $B_\beta$ defined as follows: $f_{Q,Q'}$ is equal to 1 on $Q$, $-1$ on $Q'$, and 0 everywhere else; we shall call this spanning set $S_\beta$. $S_\beta$ is approximately
orthogonal in the following sense: there exists some universal constant $K$ (independent of $\beta$) such that for each $f \in S_\beta$ there are at most $K$ elements $g$ in $S_\beta$ such that the integral of $fg$ over $Q(0,0)$ is nonzero.

When we proceed to the proof, we will wish to ensure that if $g$ and $g'$ are elements of $Q(0,0)$ there exists some dyadic cube $Q$ whose diameter is less than $\epsilon d_{CC}(g,g')$ (for some fixed $\epsilon > 0$) with $g, g' \in Q$.

This is arranged by considering not just the cube families $A_\alpha$ discussed in the previous section but also using related cube families $A'_\alpha$ defined as follows: for $\alpha > 0$, $A'_\alpha$ consists of the left-translates of the $A_\alpha$ mesh by elements in the discrete Heisenberg group of scale $\alpha + 2$; in the first Heisenberg group there are a hundred million such translates and in general, the number of such translates for the case of the $j$th Heisenberg group is $10000^j + 1$. We then let $C'_\beta$ consist of all of the functions of the form $f \circ (g \cdot g^{-1})$ where $f \in C_\beta$ and $g$ is an element in the discrete Heisenberg group of scale $\beta + 2$ such that the horizontal components of $g$ lie in $[-10^{-\beta}, 10^{-\beta}]$ and the last component of $g$ lies in $[-10^{-2\beta}, 10^{-2\beta}]$ and construct an approximately orthogonal basis for each of the $C'_\beta$ analogously to the way we did for each $C_\beta$ (by looking at each translate separately).

Finally, for both Euclidean space and the Heisenberg group, it is occasionally necessary to work with sets on a slightly larger scale than the unit cube. To do this, one fixes some integer $k \leq 0$, denotes our base cube to be the cube of scale $k$ which contains $Q(0,0)$, and then defines $C_\beta$ and $C'_\beta$ appropriately for $\beta \leq 0$ (for example, $C_k$ will consist of the constant functions on our new base cube here).

### 2.3.3 Differentiability

On the Euclidean unit cube, there exists a Jacobian map that sends each Lipschitz function $f$ (which may be either scalar-valued or Euclidean vector-valued) on the unit cube to the almost-everywhere-defined function $Jf$, the Jacobian of $f$. At almost every point, the Jacobian is a linear map from the tangent space of the domain to the tangent space of the image. Further, the partial derivative of each component is bounded above by the Lipschitz coefficient of $f$. Finally, a Lipschitz function $f$ with almost everywhere constant Jacobian defined on a connected open set is uniquely determined by this Jacobian and its value at a single point: if $Jf$ is equal to the linear map $T$ almost everywhere and $f(x_0) = y_0$ then

$$f(x) = T(x - x_0) + y_0$$

for all $x$ where $f(x)$ is defined.

Similarly, if $G$ and $H$ are two Heisenberg groups and $F : G \to H$ is Lipschitz, then by [14] the Pansu differential $DF$ satisfies these three properties:
(i) At almost every point in the domain, $DF$ induces a Lie algebra homomorphism from the tangent space of the domain to the tangent space of the image.

(ii) The magnitude of each component of $DF$ is bounded above (up to a constant depending on normalization) by the Lipschitz coefficient of $F$.

(iii) For almost every $g$ with respect to Haar measure on $G$, $DF(g)$ (which was defined as an $H$-valued function defined on $G$) is equal to the Lie group homomorphism $\phi : G \rightarrow H$ and $g_0 \in G$, $h_0 \in H$ with $F(g_0) = h_0$ then

$$F(g) = h_0 \phi(g_0^{-1}g)$$

for all $g$ where $F(g)$ is defined.

The only property which is not a completely trivial consequence of [14, (iii), is a direct consequence of the following fact concerning uniqueness of Lipschitz maps:

**Fact 2.10.** Suppose $G$ and $H$ are Carnot groups, $U \subset G$ is connected and open, $g_0 \in U$ and $F_1 : U \rightarrow G$ and $F_2 : U \rightarrow G$ are two Lipschitz maps such that $DF_1(g) = DF_2(g)$ for almost all $g \in U$ with respect to Haar measure and $F_1(g_0) = F_2(g_0)$. Then $F_1 = F_2$.

**Proof.** Defining $Z = F_2F_1^{-1}$ we note that $Z : U \rightarrow G$ is Lipschitz with $Z(g_0) = 0$ and $DZ = 0$ almost everywhere. Suppose there exists $u \in U$ with $Z(u) \neq 0$. Letting $\gamma$ be a piecewise horizontal curve in $U$ joining $g_0$ to $u$ we note that there exists $g' \in G$ sufficiently close to the identity such that the left translation of $\gamma$ by $g'$ lies in $U$ (which of course implies that $g'g_0, g'u \in U$) with $Z(g'g_0) \neq Z(g'u)$ and almost everywhere on this translation, $DZ = 0$. However, integration then implies $Z(g'g_0) = Z(g'u)$ producing a contradiction, so we conclude that $Z$ is the zero map and therefore $F_1 = F_2$ as desired.

In fact, because each linear map $\psi$ from the horizontal component of $G$ to the horizontal component of $H$ has at most one extension (which we call $\tilde{\psi}$) to a Lie group homomorphism from $G$ to $H$, we can go one step further and say that if $MF$ is equal to the linear map $\psi$ almost everywhere and $g_0 \in G$, $h_0 \in H$ with $F(g_0) = h_0$ then

$$F(g) = h_0 \tilde{\psi}(g_0^{-1}g)$$

for all $g$ where $F(g)$ is defined.

### 2.3.4 Weak convergence

If $f_n$ is a sequence of uniformly Lipschitz functions on a bounded Euclidean region converging uniformly to some function $f$ then $f$ is Lipschitz, and moreover the Jacobians $Jf_n$ converge weakly in $L^2$ to the Jacobian of $f$.

In other words, we have the following fact:
Fact 2.11. Let $U \subset \mathbb{R}^k$ be a bounded open set and let $\{f_n\} : U \to \mathbb{R}^m$ be a sequence of uniformly Lipschitz functions which converges uniformly to the function $f : U \to \mathbb{R}^m$. If $g : U \to \mathbb{R}$ is an $L^2$ function (i.e. Lebesgue measurable with $\int_U |g|^2 < \infty$ where the integral is taken with respect to Lebesgue measure) and $D$ represents partial differentiation with respect to a fixed vector in $\mathbb{R}^k$ then
\[ \int_U (Df_n)g \to \int_U (Df)g, \]
where the integrals are with respect to Lebesgue measure and the derivatives in question are defined almost everywhere.

As will shortly be stated formally in Fact 2.12, Fact 2.11 generalizes to Heisenberg groups when the map $MF$ induced by the Pansu differential (see the definitions section) is used in place of the Jacobian.

In particular, one notes that because $MF$ consists of derivatives of horizontal components of $F$ with respect to horizontal tangent vectors, $MF$ can be viewed as an array of horizontal derivatives of real-valued Lipschitz functions (after postcomposing with the appropriate coordinate functions). Then, the weak convergence in question is the following fact:

Fact 2.12. Let $U \subset H_k$ be a bounded open set and let $\{f_n\} : U \to H_m$ be a sequence of uniformly Lipschitz functions which converges uniformly to the function $f : U \to H_m$. If $g : U \to \mathbb{R}$ is an $L^2$ function (which means $g$ is measurable with respect to Haar measure and $\int_U |g|^2 < \infty$ where the integral is taken with respect to Haar measure) and $D$ represents partial differentiation with respect to a fixed left-invariant horizontal vector field in $H_k$ then
\[ \int_U (Df_n)g \to \int_U (Df)g, \]
where the integrals are with respect to Haar measure and the derivatives in question are defined almost everywhere.

This fact is proved analogously to the above fact for Euclidean space (as the Euclidean result is classical, I merely give a brief sketch below which works for both Fact 2.11 and Fact 2.12).

Proof. (Sketch) Approximate $g$ by a sufficiently smooth test function with compact support and integrate by parts. \qed

2.3.5 Lipschitz extension

If $A$ is a subset of the unit cube and $f$ is a Lipschitz function from $A$ to some Euclidean space, then $f$ can be extended to a Lipschitz function on the entire unit cube (or, in fact, to all of $\mathbb{R}^n$ for that matter). It is not known whether
the same extension property holds for maps from a subset of a Heisenberg group \( G \) to \( G \) (see \[1\] and \[2\]). This is why the result we derive in Section 2.6 requires the domain of our map to be open; if this extension property holds, we can drop this assumption.

It has been shown recently in \[1\] that this extension property does not hold for maps from \( \mathbb{R}^k \) to \( H_n \) with \( n < k \). However, this property trivially holds for maps from any Carnot group to any \( \mathbb{R}^k \). It also holds for maps from \( \mathbb{R}^2 \) to \( H_n \) for \( n \geq 2 \), as was shown in \[7\] and \[12\].

### 2.4 General Carnot groups

With the appropriate scalings of the relevant components, the method in \[4\] can be followed (as in Section 2.3.1) to create a dyadic decomposition for a general rationalizable Carnot group based on the corresponding discrete Carnot groups. In other words, although the scaling constant used in our mesh depends on the specific Carnot group itself (in particular, it depends on the specific relationship between the coordinates of an arbitrary point \( g \) and \( d_{CC}(g,0) \); cf Theorem 2.10 in \[13\]), the procedure for Heisenberg groups can otherwise be copied exactly to create a dyadic decomposition of rationalizable Carnot groups. With this construction, the other properties (approximately orthogonal decomposition of square-integrable functions, differentiability of Lipschitz functions, and weak convergence) follow for rationalizable Carnot groups precisely as for Heisenberg groups (although the constants in the definition of approximate orthogonality will depend on the scaling factor of our mesh and the homogeneous dimension of the Carnot group in question). Because the base cube for our construction will still be a cube based at the origin of scale zero, we can still refer to it as \( Q(0,0) \). Also, differentiability of Lipschitz functions and weak convergence follow for the case of maps from rationalizable Carnot groups to general Carnot groups.

### 2.5 Proof of main theorem

In what follows, \( H^k \) and \( h^k \) shall refer to Hausdorff \( k \)-dimensional measure and Hausdorff \( k \)-dimensional content, respectively (both of which we define with respect to Carnot-Carathéodory distance).

Our goal is to prove the following theorem.

**Theorem 2.13.** Let \( G \) be a rationalizable Carnot group of homogeneous dimension \( k \) and \( H \) be another Carnot group. Suppose \( F: Q(0,0) \subset G \to H \) is Lipschitz. If \( \delta > 0 \), there exists a positive integer \( N \) and subsets \( Z, F_1, \ldots, F_N \) of \( Q(0,0) \) such that

\[
h^k(F(Z)) < \delta, \quad Z \cup F_1 \cup \cdots \cup F_N = Q(0,0),
\]
and \( f \) is bi-Lipschitz on each \( F_i \). Further, \( N \) and the bi-Lipschitz coefficients of \( F_i \) only depend on the data (in particular, the groups \( G \) and \( H \), the Lipschitz coefficient of \( F \), and \( \delta \); not the specific choice of \( F \)).

Before beginning our proof, we shall introduce two notions of nearness.

**Definition 2.14.** Suppose \( Q(x, \alpha) \) and \( Q(y, \alpha) \) are elements of the decomposition from [2.3.1] of a rationalizable Carnot group into cubes of the same scale. Then we say that \( Q(x, \alpha) \) and \( Q(y, \alpha) \) are **adjacent** if the distance from \( Q(x, \alpha) \) to \( Q(y, \alpha) \) is bounded above by the diameter of \( Q(x, \alpha) \).

Note that two coincident cubes of the same scale are considered adjacent.

**Definition 2.15.** Suppose \( Q(x, \alpha) \) and \( Q(y, \alpha) \) are elements of the decomposition of a rationalizable Carnot group into cubes as in the previous definition. Then we say that \( Q(x, \alpha) \) and \( Q(y, \alpha) \) are **semi-adjacent** if \( Q(x, \alpha) \) and \( Q(y, \alpha) \) are not adjacent and the parents of \( Q(x, \alpha) \) and \( Q(y, \alpha) \) are not adjacent, but the grandparents of \( Q(x, \alpha) \) and \( Q(y, \alpha) \) are adjacent.

We turn to the proof of Theorem 2.13. We may assume that \( F \) is 1-Lipschitz and that there exists \( \eta > 0 \) such that \( F \) is defined on the dilation \( \delta_{1+\eta}Q(0,0) \). We let \( W \) be a positive integer such that every cube \( Q' \) of scale \( W - 10 \) such that \( Q' \cap Q(0,0) \neq 0 \) satisfies \( Q' \subset \delta_{1+\eta}Q(0,0) \). For convenience we replace Hausdorff measure by a constant multiple of itself chosen so that \( |Q(0,0)| = 1 \) where \( || \) denotes the replacement measure. Finally, we suppose

\[
|F(Q(0,0))| = K \quad \text{with} \quad 1 \geq K \geq 0.
\]

Our next aim is to prove the following proposition, which provides a partial wavelet decomposition of the linear map \( MF \) induced by the Pansu differential \( DF \) of \( F \).

**Proposition 2.16.** Suppose \( 1 \geq \epsilon > 0 \) and let \( E \) be the ratio of the diameter of an arbitrary “cube” to the diameter of one of its “children” using Carnot-Carathéodory distance. There exists \( n, C > 0 \) such that if \( \alpha \geq W \) and \( Q(a, \alpha) \) and \( Q(b, \alpha) \) are semi-adjacent cubes with

\[
h^k(F(Q(a, \alpha))) \leq \epsilon KE^{-k\alpha} \quad (1)
\]

and

\[
h^k(F(Q(b, \alpha))) \leq \epsilon KE^{-k\alpha} \quad (2)
\]

but

\[
d_{CC}(F(Q(a, \alpha)), F(Q(b, \alpha))) \leq \epsilon KE^{-\alpha} \quad (3)
\]

then there exists \( \beta \in [\alpha - 4, \alpha + n] \) and \( f_{x,y} \in C^\beta_\delta \) and integers \( i,j \) such that
where $C'_\beta$ is the space defined in Section 2.3.2 and $MF$ is the matrix of horizontal components of the Pansu differential $DF$.

Further, $C$ only depends on $G, H, \epsilon, \text{ and } K$ (and, in particular, not on the specific choice of $F$).

Note, for example, that if $G$ is a Heisenberg group (using exactly the “cube” decomposition from Section 2.3.1), then $E = 10$.

Also, the inner product in question is taken in the appropriate $L^2$ space of real-valued functions defined on $G$ and therefore is the integral of the real-valued function $(MF)_{i,j}f_{x,y}$ with respect to the natural Haar measure on $G$.

We also note that the number of possible candidates for elements of $f_{Q,Q'}$ for a given $Q(a, \alpha)$ is uniformly bounded, with a bound that depends only on the specific groups $G$ and $H$.

Proof. Assume the contrary. Then, for each $n$ there exists a 1-Lipschitz map $F_n$ and semi-adjacent cubes $Q(a_n, \alpha_n)$ and $Q(b_n, \alpha_n)$ such that

$$h^k(F_n(Q(a_n, \alpha_n))) > \epsilon KE^{-\alpha_n},$$

$$h^k(F_n(Q(b_n, \alpha_n))) > \epsilon KE^{-\alpha_n},$$

$$d_{CC}(F_n(Q(a_n, \alpha_n)), F_n(Q(b_n, \alpha_n))) < \epsilon KE^{-\alpha_n},$$

and

$$\int_{Q(0,0)} \psi f_{Q,Q'} \leq 2^{-n}|Q(x_n, \alpha_n)|^{-\frac{n}{2}}\|f_{Q,Q'}\|_{L^2(Q(0,0))}$$

whenever $\psi$ is a matrix entry of $MF_n$ and

$$f_{Q,Q'} \in C'_\beta$$

where

$$\beta \in [\alpha_n - 4, \alpha_n + n].$$

By rescaling and translating we may suppose $Q(a_n, \alpha_n)$ is constant at $Q(a, \alpha)$ and by passing to a subsequence we suppose $Q(b_n, \alpha_n)$ is constant at $Q(b, \alpha)$. Further, the Arzelà-Ascoli theorem lets us pass to another subsequence such that $F_n$ converges uniformly on $Q(0,0)$ to some Lipschitz map $F$. Moreover, by translation (we can do this because of the expanded $C'$ families)
we can suppose $Q(a, \alpha)$ and $Q(b, \alpha)$ have the same great-great-grandparent $Q(z, \alpha - 4)$. By weak-star convergence, the restriction of each component of $MF$ to $Q(z, \alpha - 4)$ is orthogonal to $C_\beta$ for $\beta > \alpha - 4$ which implies that $MF$ is constant almost everywhere on $Q(z, \alpha - 4)$. From this, the discussion in Section 2.3.3 lets us conclude that there exists a Lie group homomorphism $\phi$ such that 

$$DF = \phi$$

for all $g \in Q(z, \alpha - 4)$. Further, there exist elements $g_0 \in G, h_0 \in H$ such that

$$F(g) = h_0 \phi(g_0^{-1} g)$$

(5)

for all $g \in Q(z, \alpha - 4)$. 

By the change-of-variables formula for Carnot groups (cf the proof of Theorem 7 of [19], which can be directly adapted to this case).

As (5) contradicts our hypotheses, the proposition follows.  

Armed with this proposition, our next goal is to show that a sufficiently large portion of our domain is lies in finitely many such semi-adjacent pairs.
Proposition 2.17. Let $\Omega$ be the set of all pairs of cubes which satisfy the hypotheses of Proposition 2.16 and let

$$\phi(x) = \# \{ \omega = (Q, Q') : x \in Q \cup Q' \}.$$ 

Suppose $N > 0$; then there exists a multiplicative constant $K'$ depending only $G, H, \epsilon,$ and $K$ such that

$$|\{x : \phi(x) \geq N\}| \leq K'N^{-1}.$$ 

Proof. If $(Q, Q') \in \Omega$, there is a wavelet function $f_{Q, Q'}$ corresponding to $(Q, Q')$ such that the projection of $MF$ onto $f_{Q, Q'}$ had $L^2$ magnitude at least $C\epsilon |Q|^{-1/2}$. However, only a bounded number of cubes can correspond to a given wavelet function (the scale must be between $\alpha - 4$ and $\alpha + n$; further, the ancestors at scale $\alpha - 4$ must overlap). By looking at approximately orthogonal components, we conclude that

$$1 \geq ||MF||^2 \geq \Sigma_{(Q, Q') \in \Omega} |Q|$$

up to a multiplicative constant that depends on $G, H, \epsilon,$ and $K$.

The first of these two inequalities (which, of course, depends on the identity of the specific Carnot groups in question but nothing else) follows directly from the fact that $F$ is 1-Lipschitz. The second identity is computed by observing that the projection of $MF$ onto $f_{Q, Q'}$ has $L^2$ norm at most $C\epsilon |Q|^{-1/2}$; therefore, the integral of the square of this projection is at most $C^2 \epsilon^2 |Q|$. Taking the sum of these expressions over each $\Omega$, observing that if

$$s(Q) = \# \{ Q' : Q, Q' \text{ semi-adjacent} \}$$

then $s(Q)$ only depends on $G$, and recalling what approximate orthogonality means indeed allows us to conclude that with the same summation as above,

$$1 \geq L \Sigma |Q|,$$

where $L$ depends quantitatively on $G, H, \epsilon,$ and $K$ (and nothing else).

However, letting

$$\phi(x) = \# \{ Q : x \in Q \},$$

$$\int \phi = \Sigma |Q|;$$

Chebyshev’s inequality therefore tells us that writing $S_N$ as the set

$$\{ x : \phi(x) \geq N \},$$

$$|S_N| \leq LN^{-1}$$

proving the proposition. \qed
Proof of theorem. We complete the theorem through an infinite series of iterations as in [9]. This process is divided into stages (indexed by \( \alpha \geq 0 \)); at stage \( \alpha \) we assign each point \( x \) of each subcube of \( Q(0,0) \) of scale \( \alpha \) a label \( x_{\alpha} \), i.e. a finite string of zeroes and ones, such that every point in a fixed cube of scale \( \alpha \) has the same label.

At stage 0 we apply a leading digit of 0 to every point in the base cube. In other words, for each \( x \in Q(0,0) \), we set \( x_0 = 0 \). Also, we define \( Z_0 = \emptyset \) for future reference.

For \( 0 < \alpha \), we begin by defining the garbage set \( Z_{\alpha} \) by letting \( S_{\alpha} \) be the collection of all cubes \( Q \) of scale \( \alpha + W \) such that

\[
|f(Q)| \leq \delta E^{-k(\alpha+W)}
\]

and set \( Z_{\alpha} = S_{\alpha} \cup Z_{\alpha-1} \).

Next, we run through each pair of cubes at scale \( \alpha + W \) which lie in \( Q(0,0) \) \( \setminus \) \( Z_{\alpha} \) and which satisfy the hypotheses of Proposition 2.16 with \( \epsilon = 0.01 \delta \). Supposing that there are \( n_{\alpha} \) such pairs \((Q_1, Q'_1), \ldots, (Q_{n_{\alpha}}, Q'_{n_{\alpha}})\), we will inductively define the labels \( x_{(\alpha,m)} \) for \( m = 0, 1, \ldots, n_{\alpha} \) as follows:

First, \( x_{(\alpha,0)} = x_{\alpha-1} \) for each \( x \in Q(0,0) \) \( \setminus \) \( Z_{\alpha} \). Then, for \( m > 0 \) we define \( x_{(\alpha,m)} = x_{(\alpha,m-1)} \) for \( x \not\in Q_m \cup Q'_m \). We note that \( x_{(\alpha,m-1)} \), when viewed as a function on \( Q(0,0) \) \( \setminus \) \( Z_{\alpha} \), is constant at a value (call it \( z_1 \), and let \( y_1 \) be its length) on \( Q_m \) and at a possibly different value (call it \( z_2 \), and let \( y_2 \) be its length) on \( Q'_m \); without loss of generality we may assume that \( y_1 \geq y_2 \). There are several cases to consider:

I) If \( y_1 = y_2 \) and \( z_1 \neq z_2 \) we simply define \( x_{(\alpha,m)} = x_{(\alpha,m-1)} \) on both \( Q_m \) and \( Q'_m \).

II) If \( y_1 = y_2 \) and \( z_1 = z_2 \) we then let \( x_{(\alpha,m)} \) be equal to the string created by adding a 0 to the end of \( x_{(\alpha,m-1)} \) on \( Q_m \) and the string created by adding a 1 to the end of \( x_{(\alpha,m-1)} \) on \( Q'_m \).

III) If \( y_1 > y_2 \) and \( z_2 \) is NOT the first \( y_2 \) digits of \( z_1 \) we simply define \( x_{(\alpha,m)} = x_{(\alpha,m-1)} \) on both \( Q_m \) and \( Q'_m \).

IV) If \( y_1 > y_2 \) and \( z_2 \) is the first \( y_2 \) digits of \( z_1 \), we let define \( x_{(\alpha,m)} = x_{(\alpha,m-1)} \) on \( Q_m \); on \( Q'_m \) we let \( y' \) be the element of \( \{0, 1\} \) that is NOT the \((y_2 + 1)\)th digit of \( z_1 \) and define \( x_{(\alpha,m)} \) on \( Q'_m \) to be the string created by adding \( y' \) to the end of \( x_{(\alpha,m-1)} \).

Once we have finished this process for each cube, we define \( x_{\alpha} = x_{(\alpha,n_{\alpha})} \) on \( Q(0,0) \) \( \setminus \) \( Z_{\alpha} \).

Now, defining \( Y_n \) to be the set of all points \( x \) such that \( x_{\alpha} \) has length at least \( n \) for some \( \alpha \), we conclude from Proposition 2.17 that there exists \( N \) such that

\[
|\{x \in Q(0,0) \setminus \bigcup_{\alpha} Z_{\alpha} : x \in Y_N\}| < 0.01 \delta;
\]

we now define the set \( Z = \bigcup_{\alpha} Z_{\alpha} \cup Y_N \).
If \( x \in Q(0, 0) \setminus \mathbb{Z} \), then the sequence \( \{ x_n \} \) is eventually constant; denote its limiting value by \( x_\infty \). Note that there are at most \( 2^N \) possible values of \( x_\infty \) as there are at most \( 2^n \) strings of length \( n \).

We finish by setting

\[
X_w = \{ x \in Q(0, 0) \setminus \mathbb{Z} : x_\infty = w \}
\]

whenever \( w \) is a string of zeroes and ones of length less than \( N \). For each such \( w \), \( F_w \) must be biLipschitz (if not, there exist \( x_1, x_2 \in X_w \) and a pair of cubes \((Q, Q')\) satisfying the hypotheses of Proposition 2.16 such that \( x_1 \in Q, x_2 \in Q' \), contradicting the definition of \( X_w \)), proving the proposition.

### 2.6 Consequences

**Corollary 2.18.** Suppose \( A \) is an open subset of a rationalizable Carnot group \( G \) (with homogeneous dimension \( k \)), \( H \) is another Carnot group, and \( F : A \to H \) is Lipschitz, and \( H^k(F(A)) > 0 \). Then there exists a subset \( B \subset A \) of positive \( k \)-dimensional Hausdorff measure such that \( F \) restricted to \( B \) is biLipschitz. Further, the measure of \( B \) and the biLipschitz constant of \( F \) are quantitative depending only on \( G, H \), the Lipschitz coefficient of \( F \), and the \( k \) dimensional Hausdorff content of \( F(A) \).

**Proof.** We can express \( A \) as a countable union of translates and dilates of the base cube \( Q(0, 0) \); by countable additivity of Hausdorff measure one of these cubes, which we call \( C \), is sent by \( F \) to a set \( F(C) \) with \( H^k(F(C)) > 0 \). By rescaling we can suppose \( C \) is the base cube \( Q(0, 0) \). The previous theorem divides this cube into the union of a ‘garbage’ set \( Z \) (consisting of those cubes whose image has measure too small, as well as those cubes which are in too many bad pairs), where \( F(Z) \) can be taken to be arbitrarily small (say, with \( h^k(F(Z)) = \frac{1}{2} h^k(F(A)) \)) and a finite union of sets \( F_j \) such that \( F|_{F_j} \) is biLipschitz for each \( j \). As \( H^k(F(\cup_j F_j)) > 0 \), there exists some \( j \) where \( |F_j| > 0 \) and we let \( B = F_j \).

Restricting attention to the first Heisenberg group \( H_1 \), we use this corollary to show that if we only consider maps whose domains are open, two questions from [8] are equivalent. To begin we need two more definitions.

**Definition 2.19.** Suppose \( Q_1 \) and \( Q_2 \) are metric spaces with Hausdorff dimension \( k \). \( Q_1 \) **looks down on** \( Q_2 \) if there exists a Lipschitz function \( f \) from some subset of \( Q_1 \) to \( Q_2 \) such that the image of \( f \) has nonzero Hausdorff \( k \)-measure.

**Definition 2.20.** Suppose \( Q \) is a metric space with Hausdorff dimension \( k \). We say that \( Q \) is **minimal** in looking down if whenever \( Q' \) is a metric space with Hausdorff dimension \( k \) such that \( Q \) looks down on \( Q' \), \( Q' \) also looks down on \( Q \).
Question 22 in [8] asks whether the first Heisenberg group is minimal in looking down and Question 24 asks if every Lipschitz map from $H_1$ to a metric space with nontrivial Hausdorff 4-measure is biLipschitz on some subset with positive Hausdorff 4-measure.

Clearly 24 implies 22. However, we now know from the corollary that 22 implies 24 (when only looking at maps from open sets). This is true because (assuming $H_1$ is minimal in looking down) if $F : E \subset H_1 \to X$ is Lipschitz and $H^4(F(E)) > 0$ then, letting $G : X \to H_1$ be another Lipschitz map with $H^4(G(X)) > 0$ (and supposing, by restricting images, that $X = F(E)$), $G \circ F$ satisfies the conditions of the corollary and therefore is biLipschitz on some subset $E' \subset E$ with $|E'| > 0$. On this set, we therefore have that $F$ is invertible with inverse $(G \circ F)^{-1} \circ G$, which is clearly Lipschitz, which therefore implies that $F|E'$ is biLipschitz. Because $F$ was arbitrary, we can conclude that Question 24, when restricted to maps defined on open sets, is equivalent to Question 22.

Raanan Schul recently proved a statement corresponding to Question 24 for maps where the domain is Euclidean in [15]. In particular, he showed that if $F$ is a Lipschitz function from the $k$-dimensional unit cube $[0,1]^k$ into a general metric space, one can decompose

$$[0,1]^k = G \cup \bigcup_{j=1}^{n} F_j$$

where $F(G)$ has arbitrarily small Hausdorff content and $F$ is bi-Lipschitz on each of the $F_j$. The main reason why Schul’s argument does not generalize to this setting is the dearth of rectifiable curves passing through a given point in a general Carnot group (for example, although the first Heisenberg group has Hausdorff dimension 4, the space of horizontal tangents to rectifiable curves through a given point in that group has dimension two).

We finish this section by discussing the question of Jones-style decompositions for Lipschitz maps on Carnot groups. Just as in the work of Peter Jones in [9], my argument for the main theorem actually implies the following stronger statement:

**Corollary 2.21.** Suppose $U$ is a bounded open subset of a rationalizable Carnot group $G$ with Hausdorff dimension $Q$, $H$ is another Carnot group, $F : U \to H$ is Lipschitz, and $\epsilon > 0$. Then there exists a finite collection $\{A_i\}$ of subsets of $U$ such that $F$ restricted to each $A_i$ is bi-Lipschitz and

$$H^Q(U \setminus \bigcup_i A_i) < \epsilon.$$ 

For unbounded open subsets of rationalizable Carnot groups a diagonalization argument yields the following.
Corollary 2.22. Suppose $U$ is an open subset of a rationalizable Carnot group $G$ with Hausdorff dimension $Q$, $H$ is another Carnot group, $F : U \to H$ is Lipschitz, and $\epsilon > 0$. Then there exists a countable collection \( \{A_i\} \) of subsets of $U$ such that $F$ restricted to each $A_i$ is bi-Lipschitz and

\[ h^Q(U \cup \cup_i A_i) = 0. \]

Because all of these statements are invariant under Lipschitz mappings, the above corollary actually holds for certain sub-Riemannian manifolds as well. In particular, if $G$ and $H$ are replaced by sub-Riemannian manifolds which are locally bi-Lipschitz equivalent to Carnot groups as well (with $G$ locally bi-Lipschitz equivalent to a rationalizable Carnot group), the corollary still holds (because manifolds are second countable).

Of course, the best-known examples of such sub-Riemannian manifolds are Riemannian manifolds themselves. As Riemannian manifolds are locally bi-Lipschitz equivalent to Euclidean spaces, where we have all five properties from Section 2, we do not need to restrict to the case where the domain is open. For example, we can say the following: if $A$ is a subset of a Riemannian manifold $M$ (with dimension $k$), $N$ is another Riemannian manifold, and $F : A \to N$ is Lipschitz, with $H^k(F(A)) > 0$, then there exists a subset $B \subset A$ with $H^k(B) > 0$ such that $f|B$ is biLipschitz.

Note that not all sub-Riemannian manifolds are locally bi-Lipschitz equivalent to Euclidean spaces, so the results from this section need not apply to such cases. In particular, as will be shown in Section 4.2, Corollary 2.21 becomes false if $G$ and $H$ are replaced by the Grushin plane and the Euclidean plane, respectively.

3 Hausdorff Dimension of Lipschitz Images

We begin by observing the following corollary of the results in Section 2.

Corollary 3.1. Suppose that $A$ is an open subset of some rationalizable Carnot group $G$ (with homogeneous dimension $k$), $H$ is another Carnot group, and $f : A \to H$ is Lipschitz, with the $k$ dimensional Hausdorff measure of the image nontrivial. Then there exists an injective Lie group homomorphism from $G$ to $H$.

Proof. By the preceding results, $f$ is biLipschitz on some $B \subset A$ with positive $k$-dimensional Hausdorff measure. Then the Pansu differential of $f$ at any Lebesgue point of $B$ is such a map.

Because the converse of this result is trivial (the Lie group homomorphism in question is locally Lipschitz), this corollary effectively reduces the question
of whether one Carnot group ‘looks down’ on another to a question of the
algebras of Lie groups.

An easy consequence of Corollary 3.1 is that if $G$ is a rationalizable Carnot
group with homogeneous dimension $k$ and $U \subset G$ then every Lipschitz im-
age of $U$ in any Euclidean space is a nullset with respect to $k$-dimensional
Hausdorff measure; this follows because there are no injective group homo-
morphisms from a nonabelian group to an abelian group. In fact, we do not
even need $U$ to be open here because we are using Euclidean space, where
Lipschitz extension properties hold, as our image space.

Despite this, the Lipschitz image of $U$ in $\mathbb{R}^k$ can still be quite large. For
example, we have the following theorem, which answers a question asked by
Enrico Le Donne (cf [11]):

**Theorem 3.2.** Suppose that $G$ is a rationalizable Carnot group with homo-
genous dimension $k$, and let $\epsilon > 0$. There exists a bounded open $U \subset G$ and
a Lipschitz map $F: U \to \mathbb{R}^k$ such that $H^{k-\epsilon}(F(U)) > 0$.

**Proof.** As in our results in Section 2, we illustrate the case $G = H^1$ in de-
tail and remark that the construction in the general case is analogous. The
construction is based on the procedure from [10].

We begin by setting

$$\gamma = 16^{\frac{1}{2\epsilon}}$$

which tells us that

$$\gamma < \frac{1}{2} \text{ and } \log_{\gamma-1} 16 = 4 - \epsilon.$$  

We next fix $\beta \in [\gamma, \frac{1}{2})$ and define

$$\lambda = \frac{20}{\frac{1}{4} - \beta^2};$$

in particular,

$$\lambda(\frac{1}{4} - \beta^2) = 20 > 10.$$  

With this data, we then set our initial box

$$I^0 = [-1, 1] \times [-1, 1] \times [-\lambda, \lambda] \subset H^1$$

and $I^1$ to be the union of the sixteen boxes

$$(a, b, c) \times \delta \beta I_0$$

where

$$a \in \{-0.5, 0.5\}, b \in \{-0.5, 0.5\}, \text{ and } c \in \{-0.75\lambda, -0.25\lambda, 0.25\lambda, 0.75\lambda\}.$$  

We arbitrarily label these boxes $I^1_j$ for $j = 1, \ldots, 16$.

The point of this construction is to find $\eta > 0$ such that
\[ d_{CC}(I_1^j, I_k^j) > \eta \text{ for } j \neq k \]
and
\[ d_{CC}(I_1^j, \delta(I^0)) > \eta \text{ for all } j. \]

Clearly, if two of the boxes in \( I^1 \) have different horizontal components, then they are at least \( 1 - 2\beta \) apart; similarly, every box in \( I^1 \) is at a distance of exactly \( .5 - \beta \) away from the nearest horizontal edge of \( I_0 \).

The only issue is vertical distance. To find the minimum distance between a vertical edge of \( I^0 \) and a box in \( I^1 \), it suffices to consider a box in \( I^1 \) where \( c = -.75\lambda \) and look at the bottom edge of \( I_0 \). Every point on the bottom edge of such a box has a vertical coordinate which is at least
\[ -.75\lambda - \beta^2\lambda - 2 \times .5\beta > -\lambda + 10 - 2 = -\lambda + 8. \]
Now, we recall that if \( g = (x_1, y_1, 0) \) and \( h = (x_2, y_2, 0) \) are points in \( H^1 \) with \( x_1, y_1, x_2, y_2 \in [-1, 1] \), then writing the product \( g^{-1}h \) as \( (x_3, y_3, z_3) \) we note that \( |z_3| < 2 \).

Consequently, if \( p = (p_1, p_2, p_3) \) is a point in \( I_1 \) and \( q = (q_1, q_2, -\lambda) \) is a point on the bottom edge of \( I_0 \), we note that the vertical coordinate of \( p^{-1}q \) is at most
\[ \lambda + 8 - (\lambda - 8) - 2 = 14, \]
implying a separation of 14 between any two such boxes.

In subsequent stages we replace each box of the form
\[ p \times \delta \lambda I^0 \]
(there are \( 16^k \) such boxes in stage \( k \)) with the sixteen boxes
\[ p \times \delta \lambda (a, b, c) \times \delta \beta \lambda I^0 \]
and call the union of all the boxes produced in stage \( k \) \( I^k \).

In stage \( k \), each box has a label of the form \( I_{(a_1,\ldots,a_k)}^k \) where each \( a_i \) ranges from one to sixteen; we extend this process to stage \( k + 1 \) by labeling the
subboxes from $I^k_{(a_1,\ldots,a_k)}$ as $I^{k+1}_{(a_1,\ldots,a_k,v)}$ where $v = 1, 2, \ldots, 16$. The intersection of the $I^k$'s, to be defined as $I$, is a Cantor set in $H^1$ of dimension

$$\log_{\beta^{-1}} 16 \geq 4 - \epsilon.$$ 

Each point in $x \in I$ has a unique label of the form $(a_1,\ldots,a_n,\ldots)$ where each $a_i$ ranges from one to sixteen such that for each $n \in \mathbb{N}$, $x \in I^n_{(a_1,\ldots,a_n)}$; if $v = (a_1,\ldots,a_n,\ldots)$ and $w = (b_1,\ldots,b_n,\ldots)$ with $m$ being the smallest integer where $a_m \neq b_m$, the distance between the points corresponding to $v$ and $w$ is (up to a multiplicative constant independent of $m$) equal to $\beta^{-m}$. Similarly, we set $J^0$ to be the box $[-1,1]^4$ in Euclidean space $\mathbb{R}^4$ and $J^1$ to be the union of the sixteen boxes $
abla((a,b,c,d) + \gamma I^0)$ where $a,b,c,d$ can each equal $-0.5$ or $0.5$. We arbitrarily label these boxes $J^1_j$ for $j = 1,\ldots,16$.

The point of this construction is to find $\eta' > 0$ such that

$$d_{CC}(J^1_j, J^1_k) > \eta'$$

for $j \neq k$ and

$$d_{CC}(J^1_j, \delta(J^0)) > \eta$$

for all $j$.

Clearly, any two of the boxes in $J^1$ are at least $1 - 2\gamma$ apart; similarly, each such box is at a distance of exactly $0.5 - \gamma$ away from the boundary of $J^0$.

In subsequent stages we replace the box $p + \lambda J^0$ with the sixteen boxes

$$p + \lambda((a,b,c,d) + \gamma J^0)$$

and call the union of all boxes produced in stage $k$ $J^k$.

In stage $k$, each box has a label of the form $J^k_{(a_1,\ldots,a_k)}$ where each $a_i$ ranges from one to sixteen; we extend this process to stage $k+1$ by labeling the sub-boxes from $J^k_{(a_1,\ldots,a_k)}$ as $J^{k+1}_{(a_1,\ldots,a_k,v)}$ where $v = 1, 2, \ldots, 16$. The intersection of the $J^k$'s, to be defined as $J$, is a Cantor set in $H^1$ of dimension

$$\log_{\beta^{-1}} 16 = 4 - \epsilon.$$ 

Each point in $x \in J$ has a unique label of the form $(a_1,\ldots,a_n,\ldots)$ where each $a_i$ ranges from one to sixteen such that for each $n \in \mathbb{N}$, $x \in J^n_{(a_1,\ldots,a_n)}$; if $v = (a_1,\ldots,a_n,\ldots)$ and $w = (b_1,\ldots,b_n,\ldots)$ with $m$ being the smallest
integer where $a_m \neq b_m$, the distance between the points corresponding to $v$ and $w$ is (up to a multiplicative constant independent of $m$) equal to $\gamma^{-m}$.

We can define a Lipschitz map $F$ from $I^0 \subset H^1$ to $\mathbb{R}^4$ whose image contains $J$ (and therefore has Hausdorff dimension $\log_{\gamma-1}(16)$) via the following three-step process.

**Step 1**: Map $I$ to $J$. This is done by mapping a point in $I$ with a label of the form $(a_1, \ldots, a_n, \ldots)$ to the point with the same label in $J$. By construction, one notes that if $\beta = \gamma$ then this map is biLipschitz.

**Step 2**: For each ordered $n$-tuple $(a_1, \ldots, a_n)$ with each $a_i$ in $\{1, \ldots, 16\}$ (this includes the zero-tuple, where we would be mapping the boundary of $I^0$) we choose a point $p(a_1, \ldots, a_n)$ in $J^n((a_1, \ldots, a_n))$ and then send all of the points in the boundary of $J^n((a_1, \ldots, a_n))$ to $p(a_1, \ldots, a_n)$.

**Step 3**: The remaining region of $I^0$ consists of sets of the form $S^n((a_1, \ldots, a_n))$ defined as the set of all points in $I^n((a_1, \ldots, a_n))$ which do not lie in $I^{n+1}((a_1, \ldots, a_n, v))$ for $v = 1, 2, \ldots, 16$. The closure of this region includes the boundary of $I^n((a_1, \ldots, a_n))$ and of $I^{n+1}((a_1, \ldots, a_n, v))$ for $v = 1, \ldots, 16$. Fixing $(a_1, \ldots, a_n)$ (we may work on each $S^n((a_1, \ldots, a_n))$ separately) we define the map $f$ from the interval $[0, 16]$ to $\mathbb{R}^4$ to be a smooth function sending $0$ to $p(a_1, \ldots, a_n)$ and $v = 1, \ldots, 16$ to $p(a_1, \ldots, a_n, v)$. We can suppose $f$ has Lipschitz norm comparable to $\gamma^n$. We then define $g$ to be a smooth, real-valued, Lipschitz function (with Lipschitz coefficient comparable to $\beta^{-n}$) on the closure of $S^n((a_1, \ldots, a_n))$ which sends the boundary of $I^n((a_1, \ldots, a_n))$ to $0$ and the boundary of $I^{n+1}((a_1, \ldots, a_n, v))$ to $v$. We can create such a $g$ by the Whitney extension theorem (the construction is more straightforward if we do not require smoothness). On the closure of $S^n((a_1, \ldots, a_n))$ (the construction merely repeats the existing one on the boundary) set $F = f \circ g$; $F|S^n((a_1, \ldots, a_n))$ has Lipschitz norm comparable to $(\frac{\gamma}{\beta})^n$.

Note that if $\gamma < \beta$, $(\frac{\gamma}{\beta})^n$ goes to zero as $n$ goes to infinity, which means that $F$ is differentiable (in the Pansu sense) at each point of $I$ with derivative zero. Further, by construction $F$ is $C^1$ outside of $I$ where the Pansu differential always has rank zero or one (and this differential approaches zero as we approach points of $I$); in fact, it is locally constant near the boundaries of the relevant cubes if we use the Whitney extension, so the construction here is indeed an appropriate analogue of [10].

In fact, because the constructed map is constant on the boundary of $I^0$, nesting appropriately-rescaled examples of this form inside each other yield the following corollary.

**Corollary 3.3.** Suppose that $G$ is a rationalizable Carnot group with homogeneous dimension $k$. There exists a bounded open $U \subset G$ and a Lipschitz map $F : U \to \mathbb{R}^k$ such that $F(U)$ has Hausdorff dimension $k$. 

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4 Counterexamples

In this section we develop two counterexamples to show why Carnot group structure, or something close to it, is necessary for the results of the previous two sections.

4.1 A Space-Filling Curve

Theorem 4.1. There exists an Ahlfors 2-regular metric space $X$ and a Lipschitz map $F : X \to \mathbb{R}^2$ whose image has positive 2-dimensional Hausdorff measure but $F$ is not biLipschitz on any set of positive 2-dimensional measure.

Proof. The function in question will be the space-filling curve $F$ from $[0, 1]$ (equipped with the square root distance metric) to the unit square in $\mathbb{R}^2$ mentioned in Section 7.3 of [16]. Although this function is a surjective map of spaces with Hausdorff dimension 2 and Lipschitz, it is not biLipschitz on any subset with Hausdorff 2-measure. To see this, suppose that the space-filling curve $f$ is biLipschitz on a set $A$ with $H^2(A) > 0$. As $f(A)$ has positive Lebesgue measure, it contains a point $x$ of Lebesgue density one. Letting $\epsilon > 0$ there exists $\delta > 0$ such that $|B(x; \delta) \cap f(A)| > (1 - \epsilon)|B(x; \delta)|$.

Writing out the binary expansion of the components of $x$ and of $\delta$, $B(x; \delta)$ contains a dyadic cube $Q$ of side at least $\frac{1}{2}\delta$; as $\epsilon|B(x; \delta)| \leq 1000\epsilon|Q|$, $|Q \cap f(A)| > (1 - 1000\epsilon)|Q|$.

As $F$ is measure-preserving, letting $J$ be the preimage of $Q$ we conclude $|J \cap A| > (1 - 1000\epsilon)|J|$.

By rescaling and translating we can suppose $F$ is therefore biLipschitz on a set $A$ of Hausdorff 2-measure arbitrarily close to 1 (although the rescaled $F$ is not identical to our space-filling curve, it preserves all the relevant properties, such as being Lipschitz in the appropriate metric, measure-preserving, and sending a pair of points whose ‘square root’ distance is at least $\frac{1}{2}$ to the same point).

Let $x, x'$ be two points which are at least $\frac{1}{4}$ apart in Euclidean distance (and therefore $\frac{1}{2}$ away with respect to square root distance) such that $F(x) = F(x')$. We can suppose that $y, y' \in A$ are arbitrarily close to $x, x'$ respectively; therefore, $|y - y'| \geq \frac{1}{4}$; however,

$$|F(y) - F(y')| \leq |F(x) - F(y)| + |F(x') - F(y')|$$

which can be made arbitrarily small by the Lipschitz property (all distances use the square root metric in the domain and the Euclidean metric in the image) showing that $F$ cannot be biLipschitz on $A$ with any coefficient. \qed
In this example, the third and fourth properties (involving differentiability) from Section 2.3 fail, which suggests that some notion of differentiability is necessary for the results in [9] to extend to other spaces.

4.2 The Grushin Plane

**Theorem 4.2.** There exists a 2-dimensional sub-Riemannian manifold $M$ with Hausdorff dimension 2, an open $U \subset M$, and a Lipschitz map $F : U \to \mathbb{R}^2$

which is not decomposable in the following sense:

There does not exist a countable collection $\{A_i\}$ of sets such that

$$H^2(F(U \setminus \bigcup_i A_i)) = 0$$

and $F|A_i$ is biLipschitz for each $i$.

**Proof.** We use the Grushin plane $M$ as our sub-Riemannian manifold.

To construct the Grushin plane we define a Riemannian metric on the following region of $\mathbb{R}^2$: $\{(x, y) : y \neq 0\}$.

This metric is defined as $ds^2 = dx^2 + x^{-2}dy^2$. We then use this metric to induce a geodesic structure on all of $\mathbb{R}^2$, where a rectifiable curve must have horizontal tangent at each point that it crosses the $y$-axis.

One can observe that off of the vertical axis, the Grushin plane is locally bi-Lipschitz to Euclidean space (but with a constant that blows up as we get closer to the axis). However, the distance between two points on the vertical axis is proportional to the square root of their Euclidean distance.

In other words, the Grushin plane is a union of a (disconnected) Riemannian manifold and a line of Hausdorff dimension two, making it a sub-Riemannian manifold of both Euclidean and Hausdorff dimension two.

To construct our counterexample, we consider an open neighborhood of the segment $S$ joining $(0,0)$ to $(0,1)$, say: $U_\epsilon = (-\epsilon, \epsilon) \times (-\epsilon + 1 + \epsilon)$ for $\epsilon > 0$. The space-filling curve previously constructed as in Chapter 7 of [16] has already been shown to be Lipschitz when defined as a function from a set which is bi-Lipschitz to $S$ with image the unit square. We can extend this mapping to a Lipschitz mapping $F$ from $U_\epsilon$ to $\mathbb{R}^2$ by standard constructions (note the importance of having a Euclidean target space here).

However, there does not exist a countable collection of sets $A_1, \ldots, A_n, \ldots$ such that $G := U \setminus \bigcup_n A_n$ is sent to a set of arbitrarily small Hausdorff content by $F$ and $F$ is biLipschitz when restricted to the $A_n$. This is because $A_n \cap S$ must be a nullset (by the previous arguments concerning the space-filling curve for each $G$) which implies that $G$ must contain almost all of $S$, in the sense of Hausdorff measure. Therefore, $F(G)$ must contain almost all of the
unit square in the sense of Hausdorff measure (or Hausdorff content, which is equivalent in this case), producing our desired contradiction.

In this example, the first and second properties (involving homogeneity) from Section 2.3 fail, which suggests that some notion of homogeneity is also necessary for the results in [9] to extend to other spaces.

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