An asymptotic vanishing theorem for the cohomology of thickenings

Bhargav Bhatt\textsuperscript{1} · Manuel Blickle\textsuperscript{2} · Gennady Lyubeznik\textsuperscript{3} · Anurag K. Singh\textsuperscript{4} · Wenliang Zhang\textsuperscript{5}

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Abstract

Let \( X \) be a closed equidimensional local complete intersection subscheme of a smooth projective scheme \( Y \) over a field, and let \( X_t \) denote the \( t \)-th thickening of \( X \) in \( Y \). Fix an ample line bundle \( \mathcal{O}_Y(1) \) on \( Y \). We prove the following asymptotic formulation of the Kodaira vanishing theorem: there exists an integer \( c \), such that for all integers \( t \geq 1 \), the cohomology group \( H^k(X_t, \mathcal{O}_{X_t}(j)) \) vanishes for \( k < \dim X \) and \( j < -ct \). Note that there are no restrictions on the characteristic of the field, or on the singular locus of \( X \). We also construct examples illustrating that a linear bound is indeed the best possible, and that the constant \( c \) is unbounded, even in a fixed dimension.

1 Introduction

Let \( Y \) be a projective scheme over a field, and let \( X \) be a closed subscheme defined by an ideal sheaf \( \mathcal{I} \subset \mathcal{O}_Y \). For integers \( t \geq 1 \), let \( X_t \) denote the \( t \)-th thickening of \( X \) in \( Y \), i.e., the closed subscheme of \( Y \) defined by \( \mathcal{I}^t \). In [2], we proved the following version of the Kodaira vanishing theorem for thickenings of local complete intersection (lci) subvarieties of projective space \( \mathbb{P}^n \):

**Theorem 1.1** [2, Theorem 1.4] Let \( X \) be a closed lci subvariety of \( \mathbb{P}^n \) over a field of characteristic zero. Then, for each \( t \geq 1 \) and \( k < \text{codim}(\text{Sing } X) \), one has

\[
H^k(X_t, \mathcal{O}_{X_t}(j)) = 0 \quad \text{for } j < 0.
\]

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Extended author information available on the last page of the article
When $X$ is smooth and $t = 1$, this is precisely what is obtained from the Kodaira vanishing theorem. There are well-known counterexamples in the case of positive characteristic $[9,12]$; the condition on the singular locus is needed as well in view of the examples from $[1]$. Nonetheless, as we prove here, there is an asymptotic version of the above vanishing theorem that holds in good generality:

**Theorem 1.2** Let $Y$ be a smooth projective scheme over a field, equipped with an ample line bundle $\mathcal{O}_Y(1)$. Let $X$ be a closed equidimensional lci subscheme of $Y$. Then there exists an integer $c \geq 0$, such that for each $t \geq 1$ and $k < \dim X$, one has

$$H^k(X_t, \mathcal{O}_{X_t}(j)) = 0 \quad \text{for all} \quad j < -ct,$$

where, for a closed subscheme $Z \subset Y$ and integer $j$, we write $\mathcal{O}_Z(j) := \mathcal{O}_Y(1)^{\otimes j}|_Z$.

Unlike Theorem 1.1 that relies on Hodge-theoretic input (via Kodaira vanishing), the proof of Theorem 1.2 only uses Serre vanishing; this is why we do not need any assumption on the characteristic of the field in Theorem 1.2.

In the case where $Y = \mathbb{P}^n$, with $\mathcal{O}_Y(1)$ the standard ample line bundle, Theorem 1.2 answers [6, Questions 7.1 and 7.2] in the lci case; see Corollaries 3.3 and 3.4. The linear bound in Theorem 1.2 is best possible in view of Example 4.1 where, for each integer $c \geq 2$, we construct an lci scheme $X$ of dimension 1 such that, for each $t \geq 1$, the cohomology group $H^0(X_t, \mathcal{O}_{X_t}(j))$ vanishes for $j \leq -ct$, and is nonzero for $j = -ct + 1$. Theorem 1.2 may fail—even in characteristic zero—when $X$ is not lci, see Example 4.2, or when $X$ is lci but not equidimensional, see Example 4.3.

## 2 Preliminaries

Let $X$ be a projective scheme over a field $\mathbb{P}$. Set $d := \dim X$. We use $D^\text{coh}(X)$ to denote the derived category of complexes

$$\cdots \longrightarrow P^{i-1} \longrightarrow P^i \longrightarrow P^{i+1} \longrightarrow \cdots$$

of $\mathcal{O}_X$-modules with coherent cohomology, and $D^{b}_{\text{coh}}(X)$ for the full triangulated subcategory of bounded complexes, i.e., those with only finitely many nonzero cohomology groups. We use $D^{\leq a}_{\text{coh}}(X)$ (resp. $D^{\geq a}_{\text{coh}}(X)$) for complexes whose cohomology vanishes for $i > a$ (resp. $i < a$). It is straightforward that each complex in $D^{\leq a}_{\text{coh}}(X)$ (resp. $D^{\geq a}_{\text{coh}}(X)$) is quasi-isomorphic to a complex $P^\bullet$ such that $P^i = 0$ for $i > a$ (resp. $i < a$). In particular, each complex in $D^{b}_{\text{coh}}(X)$ is quasi-isomorphic to a complex $P^\bullet$ such that $P^i \neq 0$ only for finitely many integers $i$.

We use $D^{\leq a}(\mathbb{P})$ to denote the derived category of complexes of $\mathbb{P}$-vector spaces whose cohomology vanishes for $i > a$, with $D^{\leq a}(\mathbb{P})$ defined analogously.

Since the global section functor $R \Gamma(X, -)$ sends a coherent sheaf $E$ on $X$ to a complex in $D^{\leq d}(\mathbb{P})$, and since each element $P$ in $D^{b}_{\text{coh}}(X) \cap D^{\leq a}_{\text{coh}}(X)$ is represented by a complex $P^\bullet$ such that $P^i \neq 0$ only for finitely many $i$ and $P^i = 0$ for $i > a$, it follows by applying the hypercohomology spectral sequence to $P^\bullet$ that the complex...
$R\Gamma(X, P^\bullet)$ lies in $D^{\geq a+d}(\mathcal{F})$; while we do not need it here, this is true even without the boundedness assumption.

A key technical ingredient is the derived $m$-th divided power functor

$$\Gamma^m : D^\leq_{coh}(X) \longrightarrow D^\leq_{coh}(X)$$

constructed in [8], see also [10, Chapter 25] or [11]. We summarize the properties of $\Gamma^m$ that we use in this paper. For a locally free sheaf $E$ of finite rank, $\Gamma^m$ is the usual $m$-th divided power of $E$. In particular, one has in this case,

$$\Gamma^m(E) = \text{Sym}^m(E^\vee)^\vee,$$

where $(-)^\vee = \mathcal{H}om(-, \mathcal{O}_X)$. By [10, 25.2.4.1], the functor $\Gamma^m$ preserves $D^\leq_{coh}(X)$ for all integers $a \leq 0$. Just as divided powers are not an additive functor, neither is $\Gamma^m$; the functor $\Gamma^m$ does not preserve shifts or exact triangles in general. However, $\Gamma$ is compatible with direct sums in the following sense: if $P = \bigoplus P^i$ is a (finite) direct sum, then

$$\Gamma^m(P) \cong \bigoplus_{a_i \geq 0} \bigotimes a_i = m \Gamma^m(P^i).$$

More generally, by [8, 5.4] or [10, 25.2], $\Gamma^m := \bigoplus_m \Gamma^m$ extends to a monoidal functor on the filtered derived category, which is compatible with the formation of the associated graded object in the above sense. In particular, if $P^\bullet$ is a complex with a finite filtration whose associated graded object is $\bigoplus P^i$, then $\Gamma^m(P^\bullet)$ has a finite filtration with the associated graded object given by

$$\bigoplus_{a_i \geq 0, \sum a_i = m} \bigotimes \Gamma^m(P^i).$$

In our applications, an ample line bundle $\mathcal{O}_X(1)$ on $X$ is usually fixed at the outset. Thus, for $E \in D_{coh}(X)$ and any integer $n$, we write $E(n) := E \otimes_{\mathcal{O}_X} (\mathcal{O}_X(1))^\otimes_n$ as expected.

### 3 Proof of the main theorem, and some consequences

To prove Theorem 1.2, we shall need a result which, very roughly speaking, is a variant of Serre vanishing where tensor powers of a sufficiently ample line bundle are replaced by divided powers of a sufficiently ample vector bundle. To make the proof flow better, it is convenient to formulate a more general statement involving complexes as follows:

**Proposition 3.1** Let $X$ be a projective scheme over a field $\mathbb{F}$, equipped with an ample line bundle $\mathcal{O}_X(1)$. Fix a coherent sheaf $F$ and $E \in D^b_{coh}(X) \cap D^\leq_{coh}(X)$. Then, for $c \gg 0$, one has

$$R\Gamma(X, \Gamma^m(E(c)) \otimes F(l)) \in D^{\leq 0}(\mathbb{F}).$$
for all integers $l \geq 0$ and $m > 0$.

The idea of the proof is to choose a representative of $E$ where each term is a direct sum of twists of the structure sheaf $\mathcal{O}_X$, and then use Serre vanishing. However, to avoid working with unbounded complexes, we only choose an “approximate representative” for $E$, i.e., one that does not change cohomology in a certain range of degrees. The key point is Lemma 3.2, which ensures that applying derived divided powers to a shift of a “positive” complex can only increase “positivity.”

**Proof** Fix a coherent sheaf $F$ on $X$ as in the statement of the proposition. By Serre vanishing, there exists an integer $j_0 > 0$ such that $H^i(X, F(j)) = 0$ for all $i > 0$ and $j \geq j_0$. Stated differently, $R\Gamma(X, F(j)) \in D^{\leq 0}(\mathbb{F})$ for $j \geq j_0$.

For the purpose of the proof, we may replace $E$ by any complex quasi-isomorphic to $E$. By constructing a resolution of $E$ whose terms consist of finite direct sums of twists of $\mathcal{O}_X$, we may hence assume that $E$ is bounded above by zero, and that each $E^i$ is a finite direct sum of twists of $\mathcal{O}_X$. Set $d := \dim X$. For an integer $r$ with $r > d$, set $P^\bullet$ to be

$$0 \longrightarrow E^{-r} \longrightarrow E^{-(r-1)} \longrightarrow \cdots \longrightarrow E^{-1} \longrightarrow E^0 \longrightarrow 0.$$

Then each $P^i$ is a finite direct sum of twists of $\mathcal{O}_X$, and the cokernel $Q^\bullet$ of the injective map $P^\bullet \hookrightarrow E^\bullet$ lies in $D^{b}_{coh}(X) \cap D^{\leq-r}_{coh}(X)$.

For each integer $c$, we view

$$\varphi : P^\bullet(c) \hookrightarrow E^\bullet(c)$$

as a one-step decreasing filtration of $E^\bullet(c)$, normalized so that $\text{gr}^1(E^\bullet(c)) = P^\bullet(c)$ and $\text{gr}^0(E^\bullet(c)) = Q^\bullet(c)$. By the compatibility of $\Gamma^m$ with filtrations, as discussed in §2, we obtain an induced filtration on $\Gamma^m(E^\bullet(c))$ with the associated graded pieces given by

$$\text{gr}^a(\Gamma^m(E^\bullet(c))) = \Gamma^a(P^\bullet(c)) \otimes \text{gr}^b(Q^\bullet(c)), \quad \text{with } a + b = m,$$

where negative divided powers are understood to be 0. Thus, the graded pieces vanish unless $0 \leq a \leq m$, and $a = 0$ gives the “top” graded piece (i.e., the quotient) while $a = m$ gives the “bottom” graded piece (i.e., a subobject). In particular, the map

$$\Gamma^m(\varphi) : \Gamma^m(P^\bullet(c)) \longrightarrow \Gamma^m(E^\bullet(c))$$

identifies with the inclusion

$$\text{gr}^m(\Gamma^m(E^\bullet(c))) \xleftarrow{\sim} \text{Fil}^m(\Gamma^m(E^\bullet(c))) \longrightarrow \Gamma^m(E^\bullet(c)),$$

and hence its cokernel (which we regard as a representative for its cone in the derived category) carries a filtration whose graded pieces have the form

$$\Gamma^a(P^\bullet(c)) \otimes \text{gr}^b(Q^\bullet(c)), \quad \text{with } a + b = m \text{ and } b > 0.$$
Since \( \Gamma^a \) preserves \( D^{\leq i}_{\text{coh}}(X) \) for \( i \leq 0 \), we have \( \Gamma^a(P^\bullet) \in D^{\leq 0}_{\text{coh}}(X) \) and \( \Gamma^b(Q^\bullet) \in D^{\leq -d}_{\text{coh}}(X) \) provided \( b > 0 \), and hence their tensor product lies in \( D^{\leq -d}_{\text{coh}}(X) \). Since tensoring with \( F(j) \) preserves \( D^{\leq -d}_{\text{coh}}(X) \), we see that the cone of

\[
\Gamma^m(P^\bullet(c)) \otimes F(j) \longrightarrow \Gamma^m(E(c)) \otimes F(j)
\]

also lies in \( D^{\leq -d}_{\text{coh}}(X) \) for all \( m \geq 0 \) and \( c, j \in \mathbb{Z} \).

Since \( R\Gamma(X, -) \) takes \( D^{\leq -d}_{\text{coh}}(X) \) to \( D^{\leq 0}(\mathbb{F}) \), the cone of

\[
R\Gamma(X, \Gamma^m(P^\bullet(c)) \otimes F(j)) \longrightarrow R\Gamma(X, \Gamma^m(E(c)) \otimes F(j))
\]

lies in \( D^{\leq 0}(\mathbb{F}) \) for all \( m \geq 0 \) and \( c, j \in \mathbb{Z} \). It is thus sufficient to prove the proposition when \( E \) is replaced by \( P^\bullet \); indeed, for the remainder of the proof, we take \( E \) to be \( P^\bullet \).

By construction, \( P^i = 0 \) for \( i > 0 \) and \( i < -r \). Consider the filtration on \( P^\bullet(c) \) with the \( i \)-th filtered piece given by

\[
0 \longrightarrow P^{-i}(c) \longrightarrow \cdots \longrightarrow P^0(c) \longrightarrow 0.
\]

By the compatibility of \( \Gamma^m \) with filtrations, we get that \( \Gamma^m(P^\bullet(c)) \) has a filtration with associated graded object

\[
\bigoplus_{a_i \geq 0, \sum a_i = m} \Gamma^{a_0}(P^0(c)) \otimes \Gamma^{a_1}(P^{-1}(c)[1]) \otimes \cdots \otimes \Gamma^{a_r}(P^{-r}(c)[r])
\]

for each \( m \geq 0 \) and \( c \in \mathbb{Z} \). Tensoring with \( F(j) \), we see that for each \( c, j \in \mathbb{Z} \) and \( m \geq 0 \), the complex \( \Gamma^m(P^\bullet(c)) \otimes F(j) \) has a finite filtration with associated graded object

\[
\bigoplus_{a_i \geq 0, \sum a_i = m} \Gamma^{a_0}(P^0(c)) \otimes \Gamma^{a_1}(P^{-1}(c)[1]) \otimes \cdots \otimes \Gamma^{a_r}(P^{-r}(c)[r]) \otimes F(j).
\]

It is thus enough to show: for \( m > 0, j \geq 0, \) and \( c \gg 0 \), applying \( R\Gamma(X, -) \) to each of the terms in the direct sum above produces an object in \( D^{\leq 0}(\mathbb{F}) \). Fix such a term corresponding to an index of the form \( m = \sum_i a_i \) with \( a_i \geq 0 \).
As each $P^{-i}$ is a finite direct sum of twists of the structure sheaf, and only finitely many terms $P^{-i}$ are nonzero, we know that for $c \gg 0$, each $P^{-i}(c)$ is a direct sum of line bundles of the form $O_X(j)$ for $j \geq j_0$, where $j_0$ was the integer chosen at the start of the proof. By Lemma 3.2 below, there are now two possibilities for the term $\Gamma^{a_i}(P^{-i}(c)[i])$ appearing above: if $a_i = 0$, we simply get $O_X$, while for $a_i > 0$, we get a complex which is a direct sum of complexes of the form $O_X(j) \otimes F V$ with $V \in D^{\leq 0}(F)$. Since $m = \sum a_i$ is positive, we must have $a_i > 0$ for at least one $i$. Thus, the complex displayed above is a direct sum of complexes of the form $F(j) \otimes F V$ for some $j \geq j_0$ and $V \in D^{\leq 0}(F)$. By our choice of $j_0$, we know that

$$R\Gamma(X, F(j) \otimes F V) \in D^{\leq 0}(F)$$

if $j \geq j_0$ and $V \in D^{\leq 0}(F)$, which completes the proof. \qed

**Lemma 3.2** Let $X$ be a projective scheme over a field $F$, equipped with an ample line bundle $O_X(1)$. Let $b, j_1, \ldots, j_s$ be integers, where $b \geq 0$, and set

$$E := \bigoplus_{i=1}^s O_X(j_i)[b],$$

which is a shift of a direct sum of twists of $O_X$. Then, for each integer $a \geq 0$, one has

$$\Gamma^a(E) = \bigoplus_{a_i \geq 0, \sum a_i = a} O_X(a_1j_1 + \cdots + a_s j_s) \otimes F \Gamma^{a_1}(F[b]) \otimes F \cdots \otimes F \Gamma^{a_s}(F[b]),$$

where each $\Gamma^{a_i}(F[b])$ is a complex of $F$-vector spaces lying in $D^{\leq 0}(F)$.

**Proof** As $\Gamma^*(-)$ preserves $D^{\leq 0}(F)$, the containment in $D^{\leq 0}(F)$ asserted at the end is automatic. The rest follows from the behavior of $\Gamma^a$ under direct sums, and the fact that

$$\Gamma^a(O_X(j)[b]) \simeq O_X(a j) \otimes F \Gamma^a(F[b])$$

for integers $a, b, j$ with $a, b \geq 0$. \qed

**Proof of Theorem 1.2** Set $d := \dim X$, and let $I \subset O_Y$ be the ideal sheaf of the lci subscheme $X \hookrightarrow Y$, so $I/I^2$ is the conormal bundle of this closed immersion. Since $X$ is lci and equidimensional, its dualizing complex has the form $\omega_X[d]$ for a line bundle $\omega_X$, so Serre duality says

$$H^i(X, O_X(j)) \cong H^{d-i}(X, \omega_X(-j))^\vee.$$ 

By Serre vanishing, there exists an integer $c_0 \geq 1$ such that

$$H^{d-i}(X, \omega_X(-j)) = 0 \quad \text{for all} \quad - j \geq c_0 \text{ and } i < d.$$
Equivalently, we have
\[ R\Gamma(X, \mathcal{O}_X(j)) \in D^{\geq d}(\mathbb{F}) \quad \text{for } j \leq -c_0. \]

We shall reduce the rest of the proof to the following assertion:

There exists an integer \( c_1 \geq 0 \) such that, for each integer \( s \geq 1 \), one has
\[ R\Gamma(X, \text{Sym}^s(\mathcal{I}/\mathcal{I}^2)(j)) \in D^{\geq d}(\mathbb{F}) \quad \text{for } j < -c_1s. \] (3.1)

We claim that (3.1) implies the theorem. Indeed, given an integer \( t \geq 1 \) as in the theorem, summing the conclusion of (3.1) for \( s = 1, \ldots, t - 1 \) implies that
\[ R\Gamma(X_t, \mathcal{I}/\mathcal{I}^t) \in D^{\geq d}(\mathbb{F}) \]
for \( j < -c_1(t-1) = -c_1t + c_1 \), and hence also for \( j < -c_1t \). Taking \( c = \max(c_0, c_1) \) gives the theorem.

It remains to prove (3.1). Let \( \mathcal{N} := (\mathcal{I}/\mathcal{I}^2)^\vee \) denote the normal bundle. Using Serre duality, it suffices to show that there exists \( c_1 \geq 0 \), such that for each \( s \geq 1 \), one has
\[ R\Gamma(X, \Gamma^s(\mathcal{N})(j) \otimes \omega_X) \in D^{\leq 0}(\mathbb{F}) \quad \text{for } j > c_1s. \]

But this follows from Proposition 3.1, since
\[ \Gamma^s(\mathcal{N})(as + b) = \Gamma^s(\mathcal{N}(a))(b) \]
for all integers \( a, b \). \( \square \)

We record implications of Theorem 1.2 for local cohomology modules. By a standard graded ring over a field \( \mathbb{F} \), we mean an \( \mathbb{N} \)-graded ring \( R \) with \( R_0 = \mathbb{F} \) that is generated, as an \( \mathbb{F} \)-algebra, by finitely many elements of \( R_1 \). Let \( R \) be a standard graded polynomial ring over a field, and let \( I \) be a homogeneous ideal. For \( t \geq 1 \), set \( X_t := \text{Proj } R/\mathcal{I}^t \). Let \( j \) be an arbitrary integer. Using \( m \) to denote the homogeneous maximal ideal of \( R \), one has an exact sequence relating local cohomology and sheaf cohomology:
\[
0 \longrightarrow H^0_m(R/\mathcal{I}^t)_j \longrightarrow (R/\mathcal{I}^t)_j \longrightarrow H^0(X_t, \mathcal{O}_{X_t}(j)) \longrightarrow H^1_m(R/\mathcal{I}^t)_j \longrightarrow 0. \] (3.2)

Moreover, for each \( k \geq 1 \), one has
\[ H^k(X_t, \mathcal{O}_{X_t}(j)) = H^{k+1}_m(R/\mathcal{I}^t)_j. \]
The asymptotic behavior of lengths of local cohomology modules has been studied extensively, see [4] and the references therein. For \( R \) an analytically unramified local ring and \( I \) an arbitrary ideal, the limit

\[
\lim_{t \to \infty} \frac{\ell(H^0_m(R/I))}{t^{\dim R}}
\]
exists by [4, Corollary 6.3]. In [5, Theorem 1.2] the authors give an example where this limit is irrational, for \( I \) defining a smooth complex projective curve. In the context of local cohomology, Theorem 1.2 yields the following:

**Corollary 3.3** Let \( R \) be a standard graded polynomial ring over a field, and \( m \) the homogeneous maximal ideal of \( R \). Suppose \( I \) is a homogeneous ideal such that \( R/I \) is equidimensional and \( \text{Proj } R/I \) is lci. Then

\[
\lim \sup_{t \to \infty} \frac{\ell(H^k_m(R/I^t))}{t^{\dim R}} < \infty
\]
for each \( k < \dim R/I \).

**Proof** The case \( k = 0 \) is covered by [4, Corollary 6.3], so assume \( k \geq 1 \). By Theorem 1.2 applied to \( Y = \mathbb{P}^n \), with \( \mathcal{O}_Y(1) \) being the standard ample line bundle, there exists an integer \( c \geq 0 \), such that for each \( t \geq 1 \) and \( k < \dim R/I \), one has

\[
H^k_m(R/I^t)_j = 0 \quad \text{for } j < -ct.
\]

The result now follows from [6, Theorem 5.3].

**Corollary 3.4** Let \( R \) be a standard graded polynomial ring over a field, with homogeneous maximal ideal \( m \). Suppose \( I \) is a homogeneous radical ideal such that \( R/I \) is equidimensional and \( \ell(H^k_m(R/I^t)) < \infty \) for each \( k < \dim R/I \) and \( t \geq 1 \). Then, for each \( k < \dim R/I \),

\[
\lim \sup_{t \to \infty} \frac{\ell(H^k_m(R/I^t))}{t^{\dim R}} < \infty.
\]

**Proof** For a radical ideal \( a \) in a regular local ring \( A \), a theorem of Cowsik and Nori implies that \( A/a^t \) is Cohen–Macaulay for each \( t \geq 1 \) if and only if \( A/a \) is a complete intersection ring, [3, page 219]. The finiteness of the length of each local cohomology module \( H^k_m(R/I^t) \), for \( k < \dim R/I \), implies that \( (R/I^t)_p \) is Cohen–Macaulay for each \( t \geq 1 \) and \( p \in \text{Spec } R \setminus \{m\} \). It follows that \( (R/I)_p \) is a complete intersection ring for each \( p \neq m \), and hence that \( \text{Proj } R/I \) is lci. The desired result is now immediate from Corollary 3.3.

**Remark 3.5** In the recent paper [7], the authors prove the following result: let \( R \) be a standard graded ring over a field of characteristic zero; let \( m \) denote the homogeneous maximal ideal of \( R \). Suppose \( I \) is a homogeneous ideal such that \( R/I \) is Cohen–Macaulay and of dimension at least 2, and \( I \) is locally a complete intersection on...
Spec \( R \setminus \{ m \} \). Fix an integer \( k \) with \( k < \dim R/I \). Then, for \( t \geq 1 \), the lowest degree in which the local cohomology module \( H^k_m(R/I^t) \) is nonzero is bounded below by a linear function of \( t \).

The hypotheses in [7] are somewhat different from those in Theorem 1.2 of the present paper, where there is no assumption on the characteristic, nor do we require the ring \( R/I \) to be Cohen–Macaulay.

### 4 Examples

The following example, which is a variation of [2, Example 5.7], shows that the bound in Theorem 1.2 cannot be better than linear; the example also shows that the constant \( c \) in the theorem may be unbounded, even when \( \dim X \) is fixed.

**Example 4.1** Consider the polynomial ring \( R := \mathbb{F}[x, y, u, v, w] \), where \( \mathbb{F} \) is a field of arbitrary characteristic. Fix an integer \( c \geq 2 \), and set

\[
I := (uy - vx, vy - wx) + (u, v, w)^c.
\]

The ring \( R/I \) has dimension 2, and the elements \( x, y \) form a system of parameters. Since

\[
(R/I)_x = \mathbb{F}[x, x^{-1}, y, u]/(u^c) \quad \text{and} \quad (R/I)_y = \mathbb{F}[x, y, y^{-1}, w]/(w^c),
\]

one sees that \( X := \text{Proj } R/I \) is lci. We prove that for all integers \( t \geq 1 \), the asymptotic vanishing in this example takes the form \( H^0(X_t, \mathcal{O}_{X_t}(j)) = 0 \) for \( j \leq -ct \), whereas

\[
H^0(X_t, \mathcal{O}_{X_t}(-ct + 1)) \neq 0.
\]

The argument is via local cohomology; the sequence (3.2) shows that for \( j < 0 \), one has

\[
H^0(X_t, \mathcal{O}_{X_t}(j)) = H^1_m(R/I^t)_j.
\]

We analyze \( H^1_m(R/I^t) \) using the Čech complex

\[
0 \longrightarrow R/I^t \longrightarrow (R/I^t)_x \oplus (R/I^t)_y \longrightarrow (R/I^t)_{xy} \longrightarrow 0,
\]

and claim that

\[
\left[ \left( \frac{u}{x^2} \right)^{ct-1}, \left( \frac{w}{y^2} \right)^{ct-1} \right] \in (R/I^t)_x \oplus (R/I^t)_y \quad (4.1)
\]

determines a nonzero element of \( H^1_m(R/I^t)_{-ct+1} \). To verify that the displayed element is indeed a Čech cocycle, it suffices to verify that

\[
(uy^2)^{ct-1} - (wx^2)^{ct-1} \in I^t.
\]
Since the ideal $I$ contains $uy^2 - wx^2$ as well as $(uy^2)^c$, it suffices to check that

$$(uy^2)^{ct-1} - (wx^2)^{ct-1} \in (uy^2 - wx^2, (uy^2)^c)$$

in the polynomial ring $\mathbb{F}[x, y, u, v, w]$, and hence in its subring $\mathbb{F}[uy^2, wx^2]$. Setting $a := uy^2$ and $b := wx^2$ for notational simplicity, it suffices to check that

$$a^{ct-1} - b^{ct-1} \in (a - b, a^c)$$

in the polynomial ring $\mathbb{F}[a, b]$. Replacing $b$ by $a - b$, we need to show

$$a^{ct-1} - (a - b)^{ct-1} \in (b, a^c),$$

which is evident by considering the binomial expansion of $(a - b)^{ct-1}$. This completes the argument that (4.1) is indeed a Čech cocycle.

To verify that $(u/x^2)^{ct-1}$ is nonzero in $(R/I')_x$, note that its image under the surjection

$$(R/I')_x \twoheadrightarrow \left( R/ \left( \frac{R}{uy - vx, vy - wx} + (u, v, w)^{ct} \right) \right)_x = \mathbb{F}[x, x^{-1}, y, u]/(a^{ct})$$

is nonzero. As it has negative degree, the element (4.1) cannot be in the image of

$$R/I' \twoheadrightarrow (R/I')_x \oplus (R/I')_y,$$

which completes the argument that

$$H^0(X_t, \mathcal{O}_X, (-ct + 1)) = H^1_m(R/I')_{-ct+1} \neq 0.$$

Next, we examine the intersection of $(R/I')_x$ and $(R/I')_y$ in $(R/I')_{xy}$. For this, consider the $\mathbb{Z}^3$-grading with

$$\deg u = (2, 0, -1), \quad \deg x = (1, 0, 0), \quad \deg v = (1, 1, -1), \quad \deg y = (0, 1, 0), \quad \deg w = (0, 2, -1).$$

Each homogeneous element of $(R/I')_x$ has degree $(i, j, k)$ with $j \geq 0$ and $k > -ct$, whereas, in $(R/I')_y$, each homogeneous element has degree $(i, j, k)$ with $i \geq 0$ and $k > -ct$. Thus, a homogeneous element in the intersection must have degree $(i, j, k)$ satisfying $i \geq 0$, $j \geq 0$, and $k > -ct$. But the $\mathbb{Z}^3$-grading specializes to the standard $\mathbb{N}$-grading on $R$ under the map

$$\mathbb{Z}^3 \rightarrow \mathbb{Z} \quad \text{with} \quad (i, j, k) \mapsto i + j + k,$$
implying that each homogeneous element in the kernel of
\[(R/I^t)_x \oplus (R/I^t)_y \longrightarrow (R/I^t)_{xy}\]
has degree greater than \(-ct\). It follows that
\[H^0(X_t, \mathcal{O}_{X_t}(j)) = H^1_m(R/I^t)_j = 0 \quad \text{for } j \leq -ct.\]

Theorem 1.2 may fail if \(X\) is not lci:

**Example 4.2** Let \(Z\) denote the Segre embedding of \(\mathbb{P}^1 \times \mathbb{P}^2\) in \(\mathbb{P}^5\), over a field \(\mathbb{F}\) of characteristic zero, and set \(X \subset \mathbb{P}^6\) to be the projective cone over \(Z\). Then \(X\) has dimension 4, and is Cohen–Macaulay though not lci. If \(t \geq 2\), we claim that
\[H^3(X_t, \mathcal{O}_{X_t}(j)) \neq 0 \quad \text{for each } j < 0.\]

By [2, Example 5.1], if \(t \geq 2\), then \(H^2(Z_t, \mathcal{O}_{Z_t}) \neq 0\), i.e., \(H^3_{mR}(R/I^t)_0 \neq 0\), where \(R/I\) is the homogeneous coordinate ring for \(Z \subset \mathbb{P}^5\). But then \(X \subset \mathbb{P}^6\) has homogeneous coordinate ring \(S/IS\), where \(S := R[y]\) with \(y\) being a new indeterminate, so
\[H^4_{mS}(S/I^t S) \cong H^3_{mR}(R/I^t) \otimes_{\mathbb{F}} H^1_{(y)}(\mathbb{F}[y])\]
has a nonzero graded component in each negative degree, which proves the claim.

Lastly, Theorem 1.2 may fail if \(X\) is lci but not equidimensional:

**Example 4.3** Consider the polynomial ring \(R := \mathbb{F}[x, y, z]\), where \(\mathbb{F}\) is a field of arbitrary characteristic, and set \(I := (xy, xz)\). Then \(R/I\) has dimension 2, and \(X := \text{Proj } R/I\) is smooth, hence lci. Fix \(t \geq 1\). The exact sequence
\[
0 \longrightarrow R/I^t \longrightarrow R/(x^t) \oplus R/(y, z)^t \longrightarrow R/(x^t + (y, z)^t) \longrightarrow 0
\]
induces an isomorphism
\[H^1_m(R/I^t)_j \cong H^1_m(R/(y, z)^t)_j \quad \text{for } j < 0\]
which shows that \(H^1_m(R/I^t)\) has a nonzero graded component in each negative degree, so
\[H^0(X_t, \mathcal{O}_{X_t}(j)) \neq 0 \quad \text{for each } j < 0.\]
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Affiliations

Bhargav Bhatt$^1$ · Manuel Blickle$^2$ · Gennady Lyubeznik$^3$ · Anurag K. Singh$^4,\dagger$ · Wenliang Zhang$^5$

$\dagger$ Anurag K. Singh
singh@math.utah.edu

Bhargav Bhatt
bhargav.bhatt@gmail.com

Manuel Blickle
blicklem@uni-mainz.de

Gennady Lyubeznik
gennady@math.umn.edu

Wenliang Zhang
wlzhang@uic.edu

1 Department of Mathematics, University of Michigan, 530 Church Street, Ann Arbor, MI 48109, USA
2 Institut für Mathematik, Fachbereich 08, Johannes Gutenberg-Universität Mainz, 55099 Mainz, Germany
3 Department of Mathematics, University of Minnesota, 206 Church St, Minneapolis, MN 55455, USA

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Department of Mathematics, University of Utah, 155 S 1400 E, Salt Lake City, UT 84112, USA

Department of Mathematics, Statistics, and Computer Science, University of Illinois at Chicago, 851 S. Morgan St, Chicago, IL 60607, USA