Remarks on Interior Regularity Criterion for an Axially Symmetric Suitable Weak Solution to the Navier Stokes Equations

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Abstract

We show that if $v$ is an axially symmetric suitable weak solution to the Navier Stokes equations (in the sense of L. Caffarelli, R. Kohn & L. Nirenberg) such that the radial component of $v$ has a higher regularity (i.e. satisfies weighted Serrin-Prodi type condition), then all components of $v$ are regular.

Introduction. In paper [2] is proved a result concerning conditional regularity of an axially symmetric suitable weak solutions of Navier-Stokes equations. The authors show that if $v_r$ is the radial component of the velocity and satisfies Serrin-Prodi type condition, i.e.

$$\int_0^T \left( \int_\Omega |v_r|^a \, dx \right)^\frac{b}{a} \, dt < \infty,$$

with $\frac{3}{2} + \frac{2}{b} \leq 1$, $a \in (3, \infty]$, $b \in [2, \infty]$, then $v$ is regular. In this paper we modify the proof and obtain the same result under a more general assumption: weighted Serrin-Prodi type condition.

We suppose that $\Omega$ is either $\mathbb{R}^3$ or an axially symmetric bounded domain with smooth boundary and denote $Q_T = \Omega \times (0, T)$ for $T > 0$. We consider the following problem

$$\frac{\partial v}{\partial t} + (v \cdot \nabla) v = f - \nabla p + \nu \Delta v \quad \text{in} \quad Q_T \quad (1)$$

$$\text{div} \, u = 0 \quad \text{in} \quad Q_T \quad (2)$$
\( \mathbf{v} = 0 \) on \( \partial \Omega \times (0, T) \)

(3)

\( \mathbf{v}_{t=0} = \mathbf{v}_0 \).

(4)

We will further suppose for simplicity that \( \mathbf{f} = 0 \). Proceeding similarly as in [2] we can reduce the above problem to the problem on \( B_2 \times (t_0 - \tau, t_0) \). Then we have \( \mathbf{u} \) which satisfies in a classical sense the equations

\[
\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla) \mathbf{u} = \mathbf{h} - \nabla (\eta \rho) + \nu \Delta \mathbf{u} \tag{5}
\]

\[
\text{div} \, \mathbf{u} = 0 \tag{6}
\]

in \( B_2 \times (t_0 - \tau, t_0) \). Our goal is to proved that \( \mathbf{u} \) do not blow up at \( t = t_0 \). Therefore we have to prove appropriate estimates for solution \( \mathbf{u} \) under the assumption that \( \mathbf{u} \) is axially symmetric and \( u_\rho \) (the radial component of the velocity) has a higher regularity. It is convenient to write the equation (5) in cylindrical coordinates

\[
\frac{\partial u_\rho}{\partial t} + u_\rho \frac{\partial u_\rho}{\partial \rho} + u_z \frac{\partial u_\rho}{\partial z} - \frac{1}{\rho} u_\rho^2 + \frac{\partial (\eta \rho)}{\partial \rho} = h_\rho + \nu \left[ \frac{1}{\rho} \frac{\partial}{\partial \rho} \left( \rho \frac{\partial u_\rho}{\partial \rho} \right) + \frac{\partial^2 u_\rho}{\partial z^2} - \frac{u_\rho}{\rho^2} \right] \tag{7}
\]

\[
\frac{\partial u_\theta}{\partial t} + u_\rho \frac{\partial u_\theta}{\partial \rho} + u_z \frac{\partial u_\theta}{\partial z} + \frac{1}{\rho} u_\theta u_\rho = h_\theta + \nu \left[ \frac{1}{\rho} \frac{\partial}{\partial \rho} \left( \rho \frac{\partial u_\theta}{\partial \rho} \right) + \frac{\partial^2 u_\theta}{\partial z^2} - \frac{u_\theta}{\rho^2} \right] \tag{8}
\]

\[
\frac{\partial u_z}{\partial t} + u_\rho \frac{\partial u_z}{\partial \rho} + u_z \frac{\partial u_z}{\partial z} + \frac{\partial (\eta \rho)}{\partial z} = h_z + \nu \left[ \frac{1}{\rho} \frac{\partial}{\partial \rho} \left( \rho \frac{\partial u_z}{\partial \rho} \right) + \frac{\partial^2 u_z}{\partial z^2} \right]. \tag{9}
\]

The equation of continuity has the following form in cylindrical coordinates

\[
\frac{\partial u_\rho}{\partial \rho} + \frac{u_\rho}{\rho} + \frac{\partial u_z}{\partial z} = 0. \tag{10}
\]

We put

\[
\omega = \text{curl} \, \mathbf{u}, \quad \mathbf{g} = \text{curl} \, \mathbf{h}. \tag{11}
\]

Then we have

\[
\omega_\rho = -\frac{\partial u_\rho}{\partial z}, \quad \omega_\theta = \frac{\partial u_\rho}{\partial z} - \frac{\partial u_z}{\partial \rho}, \quad \omega_z = \frac{1}{\rho} \frac{\partial (\rho u_\theta)}{\partial \rho} \tag{12}
\]

Applying operator curl to equation (5) we obtain the system

\[
\frac{\partial \omega_\rho}{\partial t} + u_\rho \frac{\partial \omega_\rho}{\partial \rho} + u_z \frac{\partial \omega_\rho}{\partial z} - \frac{\partial u_\rho}{\partial \rho} \omega_\rho - \frac{\partial u_z}{\partial z} \omega_z = g_\rho + \nu \left[ \frac{1}{\rho} \frac{\partial}{\partial \rho} \left( \rho \frac{\partial \omega_\rho}{\partial \rho} \right) + \frac{\partial^2 \omega_\rho}{\partial z^2} - \frac{\omega_\rho}{\rho^2} \right] \tag{13}
\]

\[
\frac{\partial \omega_\theta}{\partial t} + u_\rho \frac{\partial \omega_\theta}{\partial \rho} + u_z \frac{\partial \omega_\theta}{\partial z} - \frac{u_\rho}{\rho} \omega_\rho + 2 \frac{u_\theta}{\rho} u_\rho \omega_\rho - \frac{\partial \omega_\theta}{\partial \rho} = g_\theta + \nu \left[ \frac{1}{\rho} \frac{\partial}{\partial \rho} \left( \rho \frac{\partial \omega_\theta}{\partial \rho} \right) + \frac{\partial^2 \omega_\theta}{\partial z^2} - \frac{\omega_\theta}{\rho^2} \right] \tag{14}
\]

\[
\frac{\partial \omega_z}{\partial t} + u_\rho \frac{\partial \omega_z}{\partial \rho} + u_z \frac{\partial \omega_z}{\partial z} - \frac{\partial u_\rho}{\partial \rho} \omega_\rho - \frac{\partial u_z}{\partial z} \omega_z = g_z + \nu \left[ \frac{1}{\rho} \frac{\partial}{\partial \rho} \left( \rho \frac{\partial \omega_z}{\partial \rho} \right) + \frac{\partial^2 \omega_z}{\partial z^2} \right]. \tag{15}
\]

Our result is following
Theorem 1. Let \( \mathbf{v} \) be an axially symmetric suitable weak solution to the problem (1)-(4) with \( f = 0 \). Suppose that there exists a sub-domain \( D \) of \( Q_T \) such that the radial component \( v_\rho \) of \( \mathbf{v} \) has its negative part \( v_\rho^- \) in \( L^{b,a}_\gamma(D) \) for some \( a \in \left( \frac{3}{2}, \infty \right), b \in (1, \infty) \) such that \( \frac{3}{a} + \frac{2}{b} + \gamma \leq 1 \) and \( \frac{3}{2} + \frac{2}{b} < 2 \). Then \( \mathbf{v} \) is regular in \( D \).

The condition \( v_\rho^- \in L^{b,a}_\gamma(U \times (0,T)) \) means that
\[
\int_0^T \left( \int_U |u_\rho^\gamma \cdot \rho^\gamma| \, dx \right)^\frac{a}{b} \, dt < \infty.
\]

(16)

The prove will be given in several steps.

**Step 1.** Assume that \( q \) is even, \( t \in (t_0 - \tau, t_0) \) and multiply equation (8) by \( u_\theta^{q-1} \) and integrate over \( B_2 \). Then we get
\[
\int_{B_2} \frac{\partial u_\theta}{\partial t} u_\theta^{q-1} + \int_{B_2} u_\rho \frac{\partial u_\theta}{\partial \rho} u_\theta^{q-1} + \int_{B_2} u_z \frac{\partial u_\theta}{\partial z} u_\theta^{q-1} + \int_{B_2} \frac{1}{\rho} u_\rho u_\theta^q
\]
\[
= \int_{B_2} h_\theta u_\theta^{q-1} + \nu \int_{B_2} \left[ \frac{1}{\rho} u_\theta^{q-1} \frac{\partial}{\partial \rho} (\rho \frac{\partial u_\theta}{\partial \rho}) + u_\theta^{q-1} \frac{\partial^2 u_\theta}{\partial z^2} - \frac{u_\theta^q}{\rho^2} \right].
\]

We have
\[
\int_{B_2} \frac{\partial u_\theta}{\partial t} u_\theta^{q-1} = \frac{1}{q} \int_{B_2} u_\theta^q,
\]
\[
\int_{B_2} u_\rho \frac{\partial u_\theta}{\partial \rho} u_\theta^{q-1} = \frac{1}{q} \int_{B_2} u_\rho u_\theta^q,
\]
\[
\int_{B_2} u_z \frac{\partial u_\theta}{\partial z} u_\theta^{q-1} = \frac{1}{q} \int_{B_2} u_z u_\theta^q,
\]
\[
\int_{B_2} \frac{1}{\rho} u_\theta^{q-1} \frac{\partial}{\partial \rho} (\rho \frac{\partial u_\theta}{\partial \rho}) = \int_{B_2} u_\theta^{q-1} \frac{\partial}{\partial \rho} (\rho \frac{\partial u_\theta}{\partial \rho}) + \int_{B_2} u_\theta^{q-1} \frac{\partial^2 u_\theta}{\partial z^2} - \frac{u_\theta^q}{\rho^2} = (1 - q) \int_{B_2} \frac{\partial u_\theta}{\partial \rho} u_\theta^{q-2},
\]
\[
\int_{B_2} \nu (1 - q) \int_{B_2} \frac{\partial u_\theta}{\partial \rho} u_\theta^{q-2}.
\]

Thus we get
\[
\frac{1}{q} \frac{d}{dt} \int_{B_2} u_\theta^q + \frac{1}{q} \int_{B_2} u_\rho u_\theta^q + \frac{1}{q} \int_{B_2} u_z u_\theta^q + \frac{1}{\rho} \int_{B_2} u_\rho u_\theta^q + \nu (q - 1) \int_{B_2} \left[ \left( \frac{\partial u_\theta}{\partial \rho} \right)^2 + \left( \frac{\partial u_\theta}{\partial z} \right)^2 \right] u_\theta^{q-2} + \nu \int_{B_2} \frac{u_\theta^q}{\rho^2}
\]

\( \text{This is the ball } B_2 \text{ given in cylindrical coordinates.} \)
Using (10) we get

\[
\int_{B_2} u_\rho \frac{\partial u_\theta}{\partial \rho} + \int_{B_2} u_z \frac{\partial u_\theta}{\partial z} = \int_{B_2} (u_\rho \rho) \frac{\partial u_\theta}{\partial \rho} + \int_{B_2} u_z \frac{\partial u_\theta}{\partial z} = -\int_{B_2} (\rho \frac{\partial u_\rho}{\partial \rho} + u_\rho) u_\theta - \int_{B_2} \frac{\partial u_z}{\partial z} u_\theta^q
\]

\[
= -\int_{B_2} \left( \frac{\partial u_\rho}{\partial \rho} + \frac{u_\rho}{\rho} + \frac{\partial u_z}{\partial z} \right) u_\theta^q = 0. \tag{18}
\]

Clearly we have

\[
[(\frac{\partial u_\theta}{\partial \rho})^2 + (\frac{\partial u_\theta}{\partial z})^2] u_\theta^{-q} = (u_\theta^{q/2})^2 (\frac{\partial u_\theta}{\partial \rho})^2 + (u_\theta^{q/2})^2 (\frac{\partial u_\theta}{\partial z})^2 = (2/q) [(\frac{\partial u_\theta^{q/2}}{\partial \rho})^2 + (\frac{\partial u_\theta^{q/2}}{\partial z})^2],
\]

thus equality (17) has the following form

\[
\frac{1}{q} \frac{d}{dt} \int_{B_2} u_\theta^q + \frac{1}{q} \int_{B_2} \rho u_\rho u_\theta + \nu (q-1) (q/2) \int_{B_2} [(\frac{\partial u_\theta^{q/2}}{\partial \rho})^2 + (\frac{\partial u_\theta^{q/2}}{\partial z})^2] + \nu \int_{B_2} \frac{u_\theta^q}{\rho^2} = \int_{B_2} h_\theta u_\theta^{q-1}. \tag{19}
\]

Applying Young inequality\(^4\) we get \(\int_{B_2} h_\theta u_\theta^{q-1} \leq \frac{1}{q} (\frac{q-1}{q}) q^{-1} \int_{B_2} h_\theta^q + \int_{B_2} u_\theta^q.\) Hence from (19) we get the estimate\(^5\)

\[
\frac{d}{dt} \|u_\theta\|^q + \frac{4(q-1)}{q} \int_{B_2} [(\frac{\partial u_\theta^{q/2}}{\partial \rho})^2 + (\frac{\partial u_\theta^{q/2}}{\partial z})^2] + \nu q \int_{B_2} \frac{u_\theta^q}{\rho^2} \leq q \int_{B_2} \frac{1}{\rho} u_\rho u_\theta^q + q \|u_\theta\|^q + \|h_\theta\|^q. \tag{20}
\]

Now we shall estimate the first term on the right hand side. We set

\[
p = 1 + \frac{2a + 3b}{2ab - 2a - 3b}, \quad s = \frac{2a}{b} + 3. \tag{21}
\]

Then \(s > 3\) and \(p > 1\), because from the assumption (\(?\)) we get \(3b + 2a < 2ab\). Therefore we may write

\[
\int_{B_2} \frac{1}{\rho} u_\rho u_\theta^q = \int_{B_2} u_\rho^{-q(p-1)/p} \rho^{(2-p)/p} u_\theta^{q/p} \rho^{-2/p} \leq \left( \int_{B_2} \left| u_\rho^{-\frac{p}{p-1}} u_\theta^{\frac{p-q}{p-1}} \right|^{(p-1)/p} \right)^{1/(p-1)} \left( \int_{B_2} \frac{u_\theta^q}{\rho^2} \right)^{1/p}. \tag{22}
\]

\(^4\)\(ab \leq \frac{1}{q} (\frac{q-1}{q}) q^{-1} a^q + b^{q-1}.\)

\(^5\)\(\| \cdot \|_q\) denotes \(\| \cdot \|_{L^q(B_2)}.\)

\(^6\)\(\text{Hölder: } (\frac{p}{p-1}, p).\)
Thus we get

\[
\int_{B_2} |u_\rho|^{\frac{p}{p-1}} u_\theta |u_\rho|^{\frac{2-p}{p-1}} \leq \left( \int_{B_2} |u_\rho|^{\frac{sp}{2(p-1)}} \rho^{\frac{(2-p)s}{2(p-1)}} \right)^{2/s} \left( \int_{B_2} u_\theta^{\frac{q}{q-2}} \right)^{1-s/2} \]

Applying Hölder inequality we get

\[
\int_{B_2} |u_\rho|^{\frac{p}{p-1}} u_\theta |u_\rho|^{\frac{2-p}{p-1}} \leq \left( \int_{B_2} |u_\rho|^{\frac{sp}{2(p-1)}} \rho^{\frac{(2-p)s}{2(p-1)}} \right)^{2/s} \left( \int_{B_2} u_\theta^{\frac{q}{q-2}} \right)^{1-s/2}
\]

\[
\leq \left( \int_{B_2} |u_\rho|^{\frac{sp}{2(p-1)}} \rho^{\frac{(2-p)s}{2(p-1)}} \right)^{2/s} \|u_\theta\|_q^{\frac{q}{q-2}} \|u_\theta\|_3^2
\]

\[
\leq 3 \epsilon_2 \|u_\theta\|_q^2 + \frac{s - 3}{s} \epsilon_2^{\frac{s}{q}} \left( \int_{B_2} u_\rho^{\frac{sp}{2(p-1)}} \rho^{\frac{(2-p)s}{2(p-1)}} \right)^{2/(s-3)} \|u_\theta\|_q^2.
\]

Thus we get

\[
\int_{B_2} \frac{1}{\rho} u_\rho^q u_\theta \leq \frac{3(p-1)}{sp} \epsilon_1^{1/(1-p)} \epsilon_2 \|u_\theta\|_q^2 + \frac{(p-1)(s-3)}{sp} \epsilon_1^{1/(1-p)} \epsilon_2^{\frac{3}{q}} \left( \int_{B_2} u_\rho^{\frac{sp}{2(p-1)}} \rho^{\frac{(2-p)s}{2(p-1)}} \right)^{2/(s-3)} \|u_\theta\|_q^2 + \|h_\rho\|_q^2.
\]

From Sobolev embedding theorem we have

\[
\|u_\theta\|_3^2 \leq c(q) \int_{B_2} \left[ \left( \frac{\partial u_\theta^{q/2}}{\partial \rho} \right)^2 + \left( \frac{\partial u_\theta^{q/2}}{\partial z} \right)^2 \right],
\]

thus using (20) and (22) we get

\[
\frac{d}{dt} \|u_\theta\|_q^q + \nu \frac{4(q-1)}{q} \int_{B_2} \left[ \left( \frac{\partial u_\theta^{q/2}}{\partial \rho} \right)^2 + \left( \frac{\partial u_\theta^{q/2}}{\partial z} \right)^2 \right] + \nu q \int_{B_2} \frac{u_\theta^q}{\rho^2}
\]

\[
\leq \frac{3(p-1)qc(q)}{sp} \epsilon_1^{1/(1-p)} \epsilon_2 \int_{B_2} \left[ \left( \frac{\partial u_\theta^{q/2}}{\partial \rho} \right)^2 + \left( \frac{\partial u_\theta^{q/2}}{\partial z} \right)^2 \right] + q \|u_\theta\|_q^q + \|h_\theta\|_q^q.
\]

\[
^7 \text{Young: } ab \leq \frac{1}{p} a^{p/(p-1)} + \frac{1}{p} b^p.
\]

\[
^8 \text{Hölder: } (\frac{q}{2}, \frac{s}{2} - 1).
\]

\[
^9 \text{Hölder: } (\frac{q}{2}, \frac{s}{2} - 2).
\]

\[
^{10} \text{Young: } (\frac{q}{2}, \frac{s}{2}).
\]
Now we choose \( \varepsilon_1 \) and \( \varepsilon_2 \) small enough and we obtain
\[
\frac{d}{dt} \| u_\theta \|_q^q + \frac{2(q-1)}{q} \int_{B_2} \left[ \left( \frac{\partial u_\theta^q/2}{\partial \rho} \right)^2 + \left( \frac{\partial u_\theta^q/2}{\partial z} \right)^2 \right] + \nu q/2 \int_{B_2} \frac{u_\theta^q}{\rho^2} \leq \| h_\theta \|_q^q + d(t) \| u_\theta \|_q^q. \tag{23}
\]
From \[21\] and the assumption \[16\] we know, that function \( d(t) \) in integrable on \((t_0 - \tau, t_0)\). In particular we have
\[
\frac{d}{dt} \| u_\theta \|_q^q \leq \| h_\theta \|_q^q + d(t) \| u_\theta \|_q^q.
\]
If we multiply the sides by \( \exp(-\int_{t_0-\tau}^t d(s) ds) \) and integrate over \((t_0 - \tau, t)\), then we obtain the following estimate
\[
\| u_\theta(t) \|_q^q \leq e^{\int_{t_0-\tau}^t d(s) ds} \| u_\theta(t_0 - \tau) \|_q^q + \int_{t_0-\tau}^t d(s) ds \| h_\theta \|_q^q e^{\int_{t_0-\tau}^t d(s) ds},
\]
i.e.
\[
\| u_\theta(t) \|_q \leq \text{const \ for \ } t \in (t_0 - \tau, t_0). \tag{24}
\]
**Step 2.** We take \( \varepsilon \in (0, 1) \) and we multiply the sides of \[14\] by \( \frac{\omega_\theta}{\rho^{2-\varepsilon}} \) and integrate over \( B_2 \)
\[
\int_{B_2} \rho^{2-\varepsilon} \frac{\omega_\theta}{\rho^{2-\varepsilon}} + \int_{B_2} \rho^{2-\varepsilon} \frac{\partial \omega_\theta}{\partial \rho} \frac{\omega_\theta}{\rho^{2-\varepsilon}} + \int_{B_2} \rho^{2-\varepsilon} \frac{\partial \omega_\theta}{\partial z} \frac{\omega_\theta}{\rho^{2-\varepsilon}} - \int_{B_2} \rho \frac{\partial \omega_\theta}{\partial \rho} \frac{\omega_\theta}{\rho^{2-\varepsilon}} - 2 \int_{B_2} \rho^{2-\varepsilon} \frac{\omega_\theta}{\rho^{2-\varepsilon}}
\]
\[
= \int_{B_2} g_\theta \frac{\omega_\theta}{\rho^{2-\varepsilon}} + \nu \int_{B_2} \rho \frac{\partial}{\partial \rho} \left( \frac{\partial \omega_\theta}{\partial \rho} \right) \frac{\omega_\theta}{\rho^{2-\varepsilon}} + \nu \int_{B_2} \rho \frac{\partial^2 \omega_\theta}{\partial z^2} \frac{\omega_\theta}{\rho^{2-\varepsilon}} - \nu \int_{B_2} \rho^{2-\varepsilon} \frac{\omega_\theta}{\rho^{2-\varepsilon}}.
\]
Now we can calculate
Now we shall estimate the right hand side. We recall that

\[ \int \frac{\partial \omega_0}{\partial t} \omega_0 \rho^{2-\varepsilon} = \frac{1}{2} d \int \frac{\omega_0^2}{\rho^{2-\varepsilon}}, \]

\[ \int \frac{u}{\rho} \frac{\partial \omega_0}{\partial \rho} \frac{\omega_0}{\rho^{2-\varepsilon}} = \frac{1}{2} \int \frac{u}{\rho} \frac{\partial \omega_0^2}{\partial \rho} = \frac{1}{2} \int \frac{u}{\rho^{1-\varepsilon}} \frac{\partial \omega_0^2}{\partial \rho} = -\frac{1}{2} \int \frac{1}{\rho^{1-\varepsilon}} \frac{\partial u}{\rho} \left( \frac{\varepsilon-1}{2} \right) \frac{\omega_0^2}{\rho^{2-\varepsilon}}, \]

\[ \int u_z \frac{\partial \omega_0}{\partial z} \frac{\omega_0}{\rho^{2-\varepsilon}} = \frac{1}{2} \int u_z \frac{\partial \omega_0^2}{\partial z} = -\frac{1}{2} \int \frac{\partial u_z}{\partial z} \frac{\omega_0^2}{\rho^{2-\varepsilon}}, \]

\[ \int \frac{1}{\rho} \frac{\partial}{\partial \rho} \left( \frac{\partial \omega_0}{\partial \rho} \frac{\omega_0}{\rho^{2-\varepsilon}} \right) = \int \frac{\vartheta}{\rho^{2-\varepsilon}} \frac{\partial \omega_0}{\partial \rho} \frac{\omega_0}{\rho^{2-\varepsilon}} = -\int \frac{1}{\rho^{1-\varepsilon}} \left( \frac{\partial \omega_0}{\partial \rho} \right)^2 - \frac{1}{2} (\varepsilon-2) \int \frac{\partial \omega_0^2}{\partial \rho} \frac{1}{\rho^{1-\varepsilon}} = -\int \frac{1}{\rho^{2-\varepsilon}} \left( \frac{\partial \omega_0}{\partial z} \right)^2. \]

Then we get

\[ \frac{1}{2} \frac{d}{dt} \int \frac{\omega_0^2}{\rho^{2-\varepsilon}} - \frac{1}{2} \int \left[ \frac{\partial u}{\partial \rho} + \frac{u}{\rho} + \frac{\partial u_z}{\partial z} \right] \frac{\omega_0^2}{\rho^{2-\varepsilon}} - \frac{\varepsilon}{2} \int \frac{u}{\rho^{1-\varepsilon}} \frac{\omega_0^2}{\rho^{2-\varepsilon}} - 2 \int \frac{u}{\rho} \frac{\omega_0^2}{\rho^{2-\varepsilon}} \]

\[ = \int g \frac{\omega_0}{\rho^{2-\varepsilon}} + \nu \left[ \frac{1}{2} (2 - \varepsilon) - 1 \right] \int \frac{\omega_0^2}{\rho^{4-\varepsilon}} - \nu \int \left[ \left( \frac{\partial \omega_0}{\partial \rho} \right)^2 + \left( \frac{\partial \omega_0}{\partial z} \right)^2 \right] \frac{1}{\rho^{2-\varepsilon}}. \]

If we use (10), then we have

\[ \frac{1}{2} \frac{d}{dt} \int \frac{\omega_0^2}{\rho^{2-\varepsilon}} + \nu \int \left[ \left( \frac{\partial \omega_0}{\partial \rho} \right)^2 + \left( \frac{\partial \omega_0}{\partial z} \right)^2 \right] \frac{1}{\rho^{2-\varepsilon}} = 2 \int \frac{u}{\rho} \frac{\omega_0}{\rho^{2-\varepsilon}} \frac{\omega_0}{\rho^{2-\varepsilon}} + \frac{\varepsilon}{2} \int \frac{u}{\rho} \frac{\omega_0^2}{\rho^{2-\varepsilon}} = \nu \left[ \frac{1}{2} (2 - \varepsilon) - 1 \right] \int \frac{\omega_0^2}{\rho^{4-\varepsilon}} + \int g \frac{\omega_0}{\rho^{2-\varepsilon}}. \]

(25)

Now we shall estimate the right hand side. We recall that \( \omega_\rho = -\frac{\partial \omega_0}{\partial z} \) and we get

\[ 2 \int \frac{u}{\rho} \frac{\omega_0}{\rho^{2-\varepsilon}} = -2 \int \frac{u}{\rho} \frac{\partial u}{\partial z} \frac{\omega_0}{\rho^{2-\varepsilon}} = -\int \frac{\partial u^2}{\partial z} \frac{\omega_0}{\rho^{3-\varepsilon}} = \int \frac{u^2}{\rho^{3-\varepsilon}} \frac{\partial \omega_0}{\partial z}. \]
We notice that

\[
\int_{B_2} \left( \frac{\partial}{\partial \rho} \frac{\omega_\theta}{\rho^{1-\epsilon}} \right)^2 \frac{1}{\rho^\epsilon} = \int_{B_2} \left[ \frac{1}{\rho^{1-\epsilon}} \frac{\partial \omega_\theta}{\partial \rho} + (\epsilon - 1) \frac{\omega_\theta}{\rho^{2-\epsilon}} \right]^2 \frac{1}{\rho^\epsilon}
\]

\[
= \int_{B_2} \frac{1}{\rho^{2-\epsilon}} \left( \frac{\partial \omega_\theta}{\partial \rho} \right)^2 + (\epsilon - 1) \int_{B_2} \frac{1}{\rho^{2-\epsilon}} \frac{\partial \omega_\theta^2}{\partial \rho} + (\epsilon - 1)^2 \int_{B_2} \frac{\omega_\theta^2}{\rho^{4-\epsilon}}
\]

\[
= \int_{B_2} \frac{1}{\rho^{2-\epsilon}} \left( \frac{\partial \omega_\theta}{\partial \rho} \right)^2 + (\epsilon - 1) \int_{B_2} \frac{\omega_\theta^2}{\rho^{4-\epsilon}}.
\]

If we notice that

\[
\int_{B_2} \frac{1}{\rho^{2-\epsilon}} \left( \frac{\partial \omega_\theta}{\partial \rho} \right)^2 = \int_{B_2} \left[ \frac{\partial}{\partial z} \left( \frac{\omega_\theta}{\rho^{1-\epsilon}} \right) \right]^2 \frac{1}{\rho^\epsilon},
\]

then from (25) we get

\[
\frac{1}{2} \frac{d}{dt} \int_{B_2} \frac{\omega_\theta^2}{\rho^{2-\epsilon}} + \nu \int_{B_2} \left\{ \left[ \frac{\partial}{\partial \rho} \left( \frac{\omega_\theta}{\rho^{1-\epsilon}} \right) \right]^2 + \left[ \frac{\partial}{\partial z} \left( \frac{\omega_\theta}{\rho^{1-\epsilon}} \right) \right]^2 \right\} \frac{1}{\rho^\epsilon}
\]

\[
\leq \frac{\nu}{2} \int_{B_2} \frac{1}{\rho^{2-\epsilon}} \left( \frac{\partial \omega_\theta}{\partial z} \right)^2 + \frac{1}{2 \nu} \int_{B_2} \frac{u_\theta^4}{\rho^{4-\epsilon}} + \frac{\epsilon}{2} \int_{B_2} \frac{|u_\rho| \omega_\theta^2}{\rho^{2-\epsilon}}
\]

\[
+ \nu \frac{\epsilon}{2} (\epsilon - 2) \int_{B_2} \frac{\omega_\theta^2}{\rho^{4-\epsilon}} + \int_{B_2} |g_\theta| \frac{|\omega_\theta|}{\rho^{2-\epsilon}}.
\]  

(26)

Clearly we have

\[
\int_{B_2} |g_\theta| \frac{|\omega_\theta|}{\rho^{2-\epsilon}} \leq \| g_\theta \|_{L^6/5} \| \omega_\theta \|_{L^6} \leq c_1 \| \nabla (\omega_\theta) \|_{L^6} \leq c_1 \left( \int_{B_2} \left[ \frac{\partial}{\partial \rho} \left( \frac{\omega_\theta}{\rho^{1-\epsilon}} \right) \right]^2 + \left[ \frac{\partial}{\partial z} \left( \frac{\omega_\theta}{\rho^{1-\epsilon}} \right) \right]^2 \right) \frac{1}{\rho^\epsilon} \right)^{1/2}
\]

\[
\leq c_3 + \frac{\nu}{4} \int_{B_2} \left\{ \left[ \frac{\partial}{\partial \rho} \left( \frac{\omega_\theta}{\rho^{1-\epsilon}} \right) \right]^2 + \left[ \frac{\partial}{\partial z} \left( \frac{\omega_\theta}{\rho^{1-\epsilon}} \right) \right]^2 \right\} \frac{1}{\rho^\epsilon}.
\]

If we use this estimate in (26), then we obtain

\[
\frac{1}{2} \frac{d}{dt} \int_{B_2} \frac{\omega_\theta^2}{\rho^{2-\epsilon}} + \frac{\nu}{4} \int_{B_2} \left\{ \left[ \frac{\partial}{\partial \rho} \left( \frac{\omega_\theta}{\rho^{1-\epsilon}} \right) \right]^2 + \left[ \frac{\partial}{\partial z} \left( \frac{\omega_\theta}{\rho^{1-\epsilon}} \right) \right]^2 \right\} \frac{1}{\rho^\epsilon}
\]

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\[ \leq \frac{1}{2\nu} \int_{B_2} \frac{u_\theta^4}{\rho^{4-e}} + \frac{\varrho}{2} \left( \int_{B_2} \frac{|u_\rho|^2}{\rho} \frac{\omega_\theta^2}{\rho^{2-e}} + \frac{\nu}{2} (\varepsilon - 2) \int_{B_2} \frac{\omega_\theta^2}{\rho^{4-e}} + c_3. \right) \quad (27) \]

Finally, if we take the limit \( \varepsilon \to 0^+ \), then we have
\[ \frac{1}{2} \frac{d}{dt} \int_{B_2} \omega_\theta^2 \rho^2 + \nu \left[ \frac{\partial}{\partial \rho} \left( \frac{\varrho}{2} \omega_\theta^2 \rho^2 \right) \right]^2 + \nu \left( \frac{\partial}{\partial z} \left( \frac{\varrho}{2} \omega_\theta^2 \rho^2 \right) \right)^2 \leq \frac{1}{2\nu} \int_{B_2} u_\theta^4 + c_3. \quad (28) \]

**Step 3.** We multiply (28) by \( \frac{u_\theta^4}{\rho^2} \) and integrate over \( B_2 \)
\[ \int_{B_2} \frac{\partial u_\theta u_\theta^3}{\partial t} \rho^2 + \int_{B_2} \frac{u_\rho}{\rho} \frac{\partial u_\theta u_\theta^3}{\partial \rho} \rho^2 + \int_{B_2} \frac{u_z}{\rho} \frac{\partial u_\theta u_\theta^3}{\partial z} \rho^2 + \int_{B_2} \frac{u_\theta^5}{\rho^3} u_\theta = \int_{B_2} \frac{h_\theta u_\theta^3}{\rho^2} + \nu \int_{B_2} \frac{u_\theta^3}{\rho^2} \frac{\partial}{\partial \rho} \left( \rho \frac{\partial u_\theta}{\partial \rho} \right) + \nu \int_{B_2} \frac{\partial^2 u_\theta u_\theta^3}{\partial z^2 \rho^2} - \nu \int_{B_2} \frac{u_\theta^5}{\rho^3}. \]

Now we calculate
\[ \int_{B_2} \frac{\partial u_\theta u_\theta^3}{\partial t} \rho^2 = \frac{1}{4} \int_{B_2} \frac{d}{dt} \int_{B_2} u_\theta^4, \]
\[ \int_{B_2} \frac{u_\rho}{\rho} \frac{\partial u_\theta u_\theta^3}{\partial \rho} \rho^2 = \frac{1}{4} \int_{B_2} \frac{u_\rho u_\theta^4}{\rho \rho^2} = -\frac{1}{4} \int_{B_2} \frac{\partial u_\theta u_\theta^4}{\partial \rho} \rho^2, \]
\[ \int_{B_2} \frac{u_z}{\rho} \frac{\partial u_\theta u_\theta^3}{\partial z} \rho^2 = -\frac{1}{4} \int_{B_2} \frac{\partial u_z u_\theta^4}{\partial z} \rho^2, \]
\[ \int_{B_2} \frac{u_\theta^3}{\rho^2} \frac{\partial}{\partial \rho} \left( \rho \frac{\partial u_\theta}{\partial \rho} \right) = \int_{B_2} \frac{u_\theta^3}{\rho^2} \frac{\partial}{\partial \rho} \left( \rho \frac{\partial u_\theta}{\partial \rho} \right) = -3 \int_{B_2} \frac{u_\theta^2}{\rho^2} (\frac{\partial u_\theta}{\partial \rho})^2 + \int_{B_2} \frac{u_\theta^4}{\rho^2}, \]
\[ \int_{B_2} \frac{\partial^2 u_\theta u_\theta^3}{\partial z^2 \rho^2} = -3 \int_{B_2} \frac{u_\theta^2}{\rho^2} (\frac{\partial u_\theta}{\partial \rho})^2. \]

Thus we have
\[ \frac{1}{4} \frac{d}{dt} \int_{B_2} \frac{u_\theta^4}{\rho^2} = \int_{B_2} \frac{h_\theta u_\theta^3}{\rho^2} - 3\nu \int_{B_2} \frac{u_\theta^2}{\rho^2} (\frac{\partial u_\theta}{\partial \rho})^2 + \nu \int_{B_2} \frac{u_\theta^4}{\rho^2} - 3 \int_{B_2} \frac{u_\theta^2}{\rho^2} (\frac{\partial u_\theta}{\partial \rho})^2, \]

hence
\[ \frac{1}{4} \frac{d}{dt} \int_{B_2} \frac{u_\theta^4}{\rho^2} = \frac{1}{4} \int_{B_2} \left( \frac{\partial u_\theta}{\partial \rho} + \frac{u_\theta}{\rho} + \frac{\partial u_z}{\partial z} \right)^4 \rho^2 + 3 \int_{B_2} \frac{u_\theta^4 u_\rho}{\rho^3} + 3\nu \int_{B_2} \left( \frac{\partial u_\theta}{\partial \rho} \right)^2 + \left( \frac{\partial u_\theta}{\partial z} \right)^2 \frac{u_\theta^2}{\rho^2} = \int_{B_2} h_\theta u_\theta^3. \]
If we use (10) then we have

\[
\frac{1}{4} \frac{d}{dt} \int_{B_2} u_\theta^4 \rho^2 + 3 \int_{B_2} \frac{u_\theta^4 u_\rho}{\rho^3} + 3 \nu \int_{B_2} \left( (\frac{\partial u_\theta}{\partial \rho})^2 + (\frac{\partial u_\theta}{\partial z})^2 \right) \frac{u_\theta^2}{\rho^2} = \int_{B_2} h_\theta \frac{u_\theta^3}{\rho^2}. \tag{29}
\]

On the other hand we can write

\[
\int_{B_2} \left( \frac{\partial}{\partial \rho} \left( \frac{u_\theta^2}{\rho} \right) \right)^2 = \int_{B_2} \frac{1}{\rho^2} \left( \frac{\partial u_\theta^2}{\partial \rho} \right)^2 = 4 \int_{B_2} \frac{u_\theta^2}{\rho^2} \left( \frac{\partial u_\theta}{\partial z} \right)^2 \\
\int_{B_2} \left( \frac{\partial}{\partial \rho} \left( \frac{u_\theta^2}{\rho} \right) \right)^2 = \int_{B_2} \left( \frac{1}{\rho^2} \left( \frac{\partial u_\theta^2}{\partial \rho} \right)^2 \right) - \int_{B_2} \frac{1}{\rho^2} \left( \frac{\partial u_\theta^2}{\partial \rho} \right)^2 + \int_{B_2} \frac{u_\theta^4}{\rho^2} = 4 \int_{B_2} \frac{u_\theta^2}{\rho^2} \left( \frac{\partial u_\theta}{\partial z} \right)^2 - \int_{B_2} \frac{u_\theta^4}{\rho^2}.
\]

Thus using these equalities in (29) we get

\[
\frac{1}{4} \frac{d}{dt} \int_{B_2} u_\theta^4 \rho^2 + 3 \int_{B_2} \left( \frac{\partial}{\partial \rho} \left( \frac{u_\theta^2}{\rho} \right) \right)^2 + \frac{3}{4} \nu \int_{B_2} \left( \frac{\partial}{\partial \rho} \left( \frac{u_\theta^2}{\rho} \right) \right)^2 + \frac{3}{4} \nu \int_{B_2} \frac{u_\theta^4}{\rho^2} = \frac{3}{2} \int_{B_2} \frac{u_\theta^4 u_\rho}{\rho^3} + \int_{B_2} h_\theta \frac{u_\theta^3}{\rho^2}. \tag{30}
\]

From Young inequality we have

\[
\int_{B_2} h_\theta \frac{u_\theta^3}{\rho^2} = \int_{B_2} \frac{u_\theta^3}{\rho^2} \cdot \rho h \leq \frac{\nu}{4} \int_{B_2} \frac{u_\theta^4}{\rho^2} + c \int_{B_2} \rho^4 h_\theta^4,
\]

hence

\[
\frac{1}{4} \frac{d}{dt} \int_{B_2} u_\theta^4 \rho^2 + \frac{3}{4} \nu \int_{B_2} \left( \frac{\partial}{\partial \rho} \left( \frac{u_\theta^2}{\rho} \right) \right)^2 + \frac{1}{2} \nu \int_{B_2} \frac{u_\theta^4}{\rho^2} \leq \frac{3}{2} \int_{B_2} \frac{u_\theta^4 |u_\rho|}{\rho^3} + c. \tag{30}
\]

**Remark 0.1.** In steps 2 and 3 we do not use the assumption on higher regularity of \(u_\rho\).

**Step 4.** We multiply (30) by \(\frac{2}{\rho^2} \)

\[
\frac{1}{2\nu^2} \frac{d}{dt} \int_{B_2} u_\theta^4 \rho^2 + \frac{3}{2\nu} \int_{B_2} \left( \frac{\partial}{\partial \rho} \left( \frac{u_\theta^2}{\rho} \right) \right)^2 + \frac{1}{\nu^2} \int_{B_2} \frac{u_\theta^4}{\rho^2} \leq \frac{3}{\nu^2} \int_{B_2} \frac{u_\theta^4 |u_\rho|}{\rho^3} + c
\]

We add this inequality to (28) and we obtain

\[
\frac{1}{2\nu^2} \frac{d}{dt} \int_{B_2} u_\theta^4 \rho^2 + \frac{1}{2\nu} \frac{d}{dt} \int_{B_2} \omega_\theta^2 \rho^2 + \frac{3}{2\nu} \int_{B_2} \left( \frac{\partial}{\partial \rho} \left( \frac{u_\theta^2}{\rho} \right) \right)^2 + \frac{1}{\nu^2} \int_{B_2} \frac{u_\theta^4}{\rho^2} \leq \frac{3}{\nu^2} \int_{B_2} \frac{u_\theta^4 |u_\rho|}{\rho^3} + c
\]

\[
+ \frac{1}{2\nu} \int_{B_2} \frac{u_\theta^4}{\rho^2} \leq \frac{3}{\nu^2} \int_{B_2} \frac{u_\theta^4 |u_\rho|}{\rho^3} + c
\]

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Proceeding similarly as in [2] we deduce that $\|\omega\|_{L^2}$ is integrable on $(t_0 - \tau, t_0)$, thus it implies the boundedness of $\|Du\|_{L^2}$ on $(t_0 - \tau, t_0)$, therefore $(x_0, t_0)$ cannot be a singular point for $u$.

**Remark 0.2.** The Theorem can be proved also in the case $b = \infty$. Then we have to assume that $\frac{3}{a} + \gamma < 1$. In the proof we put $p = \frac{2a}{2a - \delta a - 3}$, $s = 3 + \delta a$, where $\delta$ is such that $\frac{3}{a} + \gamma = 1 - \delta$ and $\delta \in (0, \frac{2a - 3}{a})$.

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