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Linearized Impulsive Fixed-Time Fuel-Optimal Space rendezvous: A New Numerical Approach

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Abstract: This paper focuses on the fixed-time minimum-fuel rendezvous between close elliptic orbits of an active spacecraft with a passive target spacecraft, assuming a linear impulsive setting and a Keplerian relative motion. Following earlier works developed in the 1960s, the original optimal control problem is transformed into a semi-infinite convex optimization problem using a relaxation scheme and duality theory in normed linear spaces. A new numerical convergent algorithm based on discretization methods is designed to solve this problem. Its solution is then used in a general simple procedure dedicated to the computation of the optimal velocity increments and optimal impulses locations. It is also shown that the semi-infinite convex programming has an analytical solution for the out-of-plane rendezvous problem. Different realistic numerical examples illustrate these results.

Keywords: Impulsive optimal control, elliptic rendezvous, primer vector, semi-infinite convex programming, discretization methods

1. INTRODUCTION

Since the first space missions (Gemini, Apollo, Vostok) involving more than one vehicle, space rendezvous between two spacecraft has become a key technology raising relevant open control issues. Formation flight (PRISMA), on-orbit satellite servicing or supply missions to the International Space Station (ISS) are all examples of projects that require adequate rendezvous planning tools. A main challenge is to achieve autonomous far range rendezvous on elliptical orbits while preserving optimality in terms of fuel consumption. In short, the far range rendezvous is an orbital transfer between an active chaser spacecraft and a passive target spacecraft, with specified initial and final conditions, over a fixed or a free time period. Searching for the guidance law that achieves the maneuver with the lowest possible fuel consumption leads to define a minimum-fuel optimal control problem.

In this article, the fixed-time linearized fuel-optimal impulsive space rendezvous problem as defined in Carter and Brient (1995), is studied assuming a linearized Keplerian relative motion. The impulsive approximation for the thrust means that instantaneous velocity increments are applied to the chaser whereas its position is continuous. Indirect approaches, based on the optimality conditions derived from the Pontryagin’s maximum principle and leading to the so-called primer vector theory (Lawden (1963)), have been extensively studied. For a fixed number of impulses, necessary and sufficient conditions can be derived (Carter and Brient (1995)). However due to the nonconvex and polynomial nature of these conditions, a numerical solution is still difficult to compute and would only be suboptimal for the original rendezvous problem for which the number of possible maneuvers is free. An iterative algorithm based on the calculus of variations, originally developed in Lion and Handelsman (1968), has been designed to address the problem of determining the optimal number of impulses. In this algorithm, Davidon-Fletcher-Powell penalty minimization step is proposed in order to move the impulses and achieve a smooth optimal trajectory as detailed in the modern account given in Prussing (2010). In Arzelier et al. (2013), a mixed iterative algorithm combines variational tests with sophisticated numerical tools from algebraic geometry to solve these polynomial necessary and sufficient conditions of optimality and avoid the local optimization step. However, these two algorithms remain heuristic with no proof of convergence in all cases and may exhibit only suboptimal solutions on some instances.

Neustadt (1964) proposed an important theoretical contribution for the optimal control problem: it is recast to a semi-infinite optimization problem, using a relaxation scheme and the duality theory in minimum-norm problems. Claeys et al. (2013) revisit his approach from the angle of generalized moment problems, by formulating it as a linear programming problem on measures. In this approach, the numerical solving is rather cumbersome since one needs high degree polynomial approximations for building hierarchies of linear–matrix inequalities (LMIs). Also, they consider only the case of ungimbled identical thrusters, which gives a linear problem.

Following Neustadt (1964), we propose a new numerical algorithm to solve the fixed-time impulsive linear rendezvous without fixing a priori the number of impulses, and whose convergence is rigorously shown. Firstly, we
focus on the moment problem formulation (Sec. 2) and recall topological duality theory results from Lunberger (1969) and Neustadt (1964), which allow for the moment problem to be transformed into a Semi-Infinite Convex Programming (SICP) (Sec. 4). The novelty of our approach is to use differential methods Resende and Rückman (1998) to solve the SICP problem. A convergent numerical algorithm is designed in Sec. 4, whose solution is the optimal primer vector of the original rendezvous problem. An estimation of the numerical error made on the optimal cost of the original problem, is also provided.

Typically, in a rendezvous situation, a spacecraft is in the target vehicle orbit. The equations of relative motion are written in a moving Local-Vertical-Local-Horizontal (LVLH) frame located at the center of gravity of a passive target and which rotates with its angular velocity. In this frame, the state vector \( X' = [x, y, z, v_x, v_y, v_z] \) is composed of the positions and velocities of a chaser satellite in the in-track, cross-track and radial axes, respectively. Under the previous assumptions and using the true anomaly of the target-vehicle orbit as the independent variable, a system of linear differential equations with periodic coefficients is easily obtained and the considered minimum-fuel linearized rendezvous problem may be reformulated as the following optimal control problem:

**Problem 1. (Optimal control problem)**

Find \( u \in \mathcal{L}_1,p([v_0, v_f], \mathbb{R}^r) \) solution of the optimal control problem:

\[
\inf_u \| u \|_{L_1,p} = \inf_u \int_{v_0}^{v_f} \| u(v) \|_0 \, dv \\
\text{s.t. } X'(v) = A(v)X(v) + Bu(v), \ \forall v \in [v_0, v_f] \\
X(v_0) = X_0, \ X(v_f) = X_f \in \mathbb{R}^r, \ v_0, v_f \text{ fixed}
\]

where matrices \( A(v) \) and \( B \) define the state-space model of relative dynamics given by Tschauner (1967):

\[
A(v) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} \Omega_{13 	imes 3} \end{pmatrix}
\]

The form of these matrices shows that the equations describing motion in the plane of the target-vehicle orbit and those describing motion normal to the orbit plane can be decoupled and handled separately. Therefore, the out-of-plane and in-plane rendezvous will be dealt with independently hereafter in the article. Indeed, the state vector dimension and the number of inputs in (1) are denoted \( n \) and \( r \), respectively with \( n = 2, \ r = 1 \) for the out-of-plane case and \( n = 4, \ r = 2 \) for the in-plane case.

**Remark 1.** In Problem 1, the 1-norm cost captures indirectly the consumption of fuel used. In fact, the performance index used in Problem 1 is an upper-bound expressed as an angular velocity, on the usual characteristic velocity expressed in m/s.

### 2.2 A minimum norm moment problem

Following the approach from Neustadt (1964), Problem 1 is now transformed into an equivalent problem of moment by integrating equation (1). As \( A \in \mathcal{C}(\mathbb{R}, \mathbb{R}^{n \times n}) \), the equation (1) has a unique solution that exists for every \( X_0 \in \mathbb{R}^r \) and for all \( \nu \in \mathbb{R} \) and \( u(\nu) \in \mathcal{L}_1,p([v_0, v_f], \mathbb{R}^r) \), Antsaklis and Michel (2003):

\[
X(\nu) = \Phi(\nu, v_0)X_0 + \int_{v_0}^{v_f} \Phi(\nu, \sigma)Bu(\sigma)\,d\sigma,
\]

where \( \Phi(\nu, \nu_0) = \varphi(\nu)\varphi^{-1}(\nu_0) \) and \( \varphi(\nu) \) are respectively the transition and Yamanaka-Ankersen fundamental matrices of Keplerian relative motion. Let us define the matrix \( Y(\nu) = \varphi^{-1}(\nu) \mathbb{I} = [y_1(\nu) \cdots y_n(\nu)]^T \in \mathbb{R}^{n \times r} \),

\[
c := \varphi^{-1}(\nu_f)X(\nu_f) - \varphi^{-1}(v_0)X_0 = \int_{v_0}^{v_f} \varphi^{-1}(\sigma)Bu(\sigma)\,d\sigma = \int_{v_0}^{v_f} Y(\nu)u(\sigma)\,d\sigma.
\]

It is important to notice for the remainder of the analysis that for the specific matrices \( Y(\nu) \) encountered in the rendezvous problem, \( y_1(\nu) \cdots y_n(\nu) \) are linearly independent elements of \( \mathcal{C}([v_0, v_f], \mathbb{R}^r) \). This will be assumed in the
rest of the paper. It follows from (4) that Problem 1 can be equivalently written as:

**Problem 2.** (Minimum norm moment problem) Find \( \bar{u}(t) \in L_{1,p}(\nu_0, \nu_f, \mathbb{R}^r) \) solution of the minimum norm moment problem:

\[
\begin{align*}
\inf_{u} & \quad \|u\|_{1,p} = \inf_{u} \int_{\nu_0}^{\nu_f} \|u(\nu)\|_p d\nu \\
\text{s.t.} & \quad \int_{\nu_0}^{\nu_f} Y(\sigma)u(\sigma)d\sigma = c, \quad \nu_0, \nu_f \text{ fixed.}
\end{align*}
\] (5)

It is well-known that Problem 2 may not reach its optimal solution due to concentration effects (see the reference Roubíček (2006)). This is mainly due to the fact that the functional space \( \mathcal{L}_{1,p}(\nu_0, \nu_f, \mathbb{R}^r) \) in which the optimal solution is sought, is not the topological dual of any other functional space Luenberger (1969). It is then necessary to resort to a relaxation scheme by embedding the space \( \mathcal{L}_{1,p}(\nu_0, \nu_f, \mathbb{R}^r) \) in the dual space \( \mathcal{C}^*(\nu_0, \nu_f, \mathbb{R}^r) \) of the Banach space \( \mathcal{C}(\nu_0, \nu_f, \mathbb{R}^r) \).

### 3. A CLASSICAL APPROACH REVISITED

In this section, the theoretical framework used to transform the original optimal control problem into a semi-infinite optimization program is recalled. We consider the formalism based on functions of bounded variation, developed in Neustadt (1964) and Luenberger (1969), rather than the ones in Roubíček (2006) or Claeyts et al. (2013), which are more rooted in the measure theory setup.

#### 3.1 Relaxation of the original problem

A so-called relaxed problem is considered, whose solutions are thought of as generalized solutions of the original Problem 2.

**Problem 3.** (Relaxed problem) Determine \( \check{g} \in BV([\nu_0, \nu_f], \mathbb{R}^r) \) solution of the following problem:

\[
\begin{align*}
\inf_{\check{g}} & \quad \|\check{g}\|_{1,v_p} = \inf_{\check{g}} \sup_{\nu_0 < \nu_1 < \cdots < \nu_n = \nu_f} \|g(\nu_i) - g(\nu_{i-1})\|_{v_p} \\
\text{s.t.} & \quad \int_{\nu_0}^{\nu_f} Y(\nu)dg(\nu) = c.
\end{align*}
\] (6)

The two problems defined in eq. (10) may be considered as dual through the equality of the optimal values of their respective objectives and the relation between their solutions thanks to the alignment condition in eq. (11). This results in a significant simplification: The infinite-dimensional optimization Problem 4 has been converted to a search of an optimal vector \( \lambda \) in a finite-dimensional vector space submitted to a continuum of constraints, yielding a semi-infinite convex problem (SICP):

**Problem 5.** (SICP problem) Find \( \lambda \in \mathbb{R}^n \) solution of

\[
\check{\lambda} = \min_{\lambda \in \mathbb{R}^n} -c^T \lambda \\
\|Y^T(\nu)\lambda\|_q \leq 1.
\] (12)

### 3.2 A semi-infinite programming problem

The following seminal and important result has been originally given in Neustadt (1964) in its complete form and partially in Krasovskii (1957) for particular optimization problems. Here, we follow the lines developed in the textbook of (Luenberger, 1969, Chapter 5).

**Theorem 1.** (Luenberger (1969))

Let \( y_i(\cdot) \in C([\nu_0, \nu_f], \mathbb{R}) \), \( i = 1, \ldots, n \) and suppose that

\[
D = \{ l \in C^* : \langle y_i(\cdot), l \rangle = c_i, \quad i = 1, \ldots, n \} \neq \emptyset
\] then

\[
\check{\eta} = \min_{l \in D} \|l\|_q = \max_{\|Y^T(\nu)\lambda\|_{v_p} \leq 1} c^T \lambda.
\] (10)

### Problem 4. (Linear minimum norm problem)

Find a linear functional \( l \in \mathcal{C}^*(\nu_0, \nu_f, \mathbb{R}^r) \) solution of the linear minimum norm problem:

\[
\begin{align*}
\check{\eta} = \inf_{l} \|l\|_q = \inf_{l} \sup_{\|g(\cdot)\|_q \leq 1} \|l(\cdot)\|_q \\
\text{s.t.} & \quad l(g_i(\cdot)) = c_i, \forall i = 1, \cdots, n.
\end{align*}
\] (8)

Despite the fact that Problem 4 is an infinite-dimensional optimization problem, it is particularly appealing due to its simplicity and the possibility to use a duality principle based on the extension form of the Hahn-Banach theorem. This establishes the equivalence between two optimization problems respectively defined in a Banach space and its dual. The result is summarized in the next subsection.
\[ \ddot{g}_s(\dot{\nu}_j) - \ddot{g}_s(\dot{\nu}_j^*) = \alpha_{\dot{\nu}_j} \text{sgn}(\ddot{g}_s(\dot{\nu}_j)) \chi_j, \quad \alpha_{\dot{\nu}_j} > 0, \]
when \( p = 1, \)
or
\[ \ddot{g}_s(\dot{\nu}_j) - \ddot{g}_s(\dot{\nu}_j^*) = \alpha_{\dot{\nu}_j} |\ddot{g}_s(\dot{\nu}_j)|^{q-1} \text{sgn}(\ddot{g}_s(\dot{\nu}_j)), \]
when \( 1 < p < \infty, \)

for \( s = 1, \ldots, r \) and \( \alpha_{\dot{\nu}_j} \) solutions of the linear system:
\[ \sum_{j=1}^{N} \beta_i(\dot{\nu}_j) \alpha_{\dot{\nu}_j} = c_i, \quad i = 1, \ldots, n \]
where \( \beta_i(\dot{\nu}_j) \) are given by:
\[ \beta_i(\dot{\nu}_j) = \sum_{s=1}^{r} y_{s,i}(\dot{\nu}_j) \text{sgn}(\ddot{g}_s(\dot{\nu}_j)), \quad \text{when } p = 1, \]
or
\[ \beta_i(\dot{\nu}_j) = \sum_{s=1}^{r} y_{s,i}(\dot{\nu}_j) |\ddot{g}_s(\dot{\nu}_j)|^{q-1} \text{sgn}(\ddot{g}_s(\dot{\nu}_j)), \]
when \( 1 < p < \infty, \)

for all \( j = 1, \ldots, N. \)

This theorem states important results that have been known for a while in the aerospace community but whose value has not been completely exploited to derive efficient numerical algorithms for impulsive maneuvers design. First, it says that the optimal controlled trajectory for the minimum-fuel Keplerian linearized elliptic rendezvous problem is purely impulsive and that the number of impulses is upper-limited by \( n \) which is the dimension of the fixed final conditions of the optimal control problem.

**Remark 2.** It is also shown in Neustadt (1964) that a sequence of functions \( u_i() \in L_{1,p}([\nu_0, \nu_f], \mathbb{R}^r) \) converges to a linear combination of \( \delta() \) functions corresponding to the function \( \bar{g}() \) with equal norms. Let \( \Delta V(\hat{\nu}_j) = \bar{g}(\hat{\nu}_j) - \bar{g}(\hat{\nu}_j) \), then roughly speaking, this may be described by:
\[ \bar{u}_i(\nu) \to \sum_{i=1}^{N} \Delta V(\hat{\nu}_i) \delta(\hat{\nu}_i - \nu), \quad \epsilon \to 0. \]

Indeed, the initial optimal control problem amounts to find the sequences of optimal impulse locations \( \{\hat{\nu}_i\}_{i=1, \ldots, N} \) and optimal impulsive vectors \( \{\Delta V(\hat{\nu}_i)\}_{i=1, \ldots, N} \) verifying the boundary equation:
\[ c = \sum_{i=1}^{N} Y(\hat{\nu}_i) \Delta V(\hat{\nu}_i). \]

### 3.3 Primer-vector interpretation and relation with the mixed algorithm in Arzelier et al. (2013)

The vector \( y(\nu) = Y^T(\nu) \lambda \) involved in (12) is nothing but the primer vector initially defined in the seminal work of Lawden (Lawden, 1963). In this reference, the primer vector \( y(\nu) \) is defined as the velocity adjoint vector arising from applying the Pontryagin Maximum Principle to optimal trajectory problems or Lagrangian duality as in Carter and Brient (1995) where the vector \( \lambda \) is the optimal Lagrange multiplier. For an optimal impulsive trajectory, the primer vector \( y(\nu) \) must satisfy the well-known Lawden’s necessary and sufficient optimality conditions recalled in Carter and Brient (1995) or in Arzelier et al. (2013). In this last reference, a mixed iterative algorithm aiming at converging to the minimum-fuel solution over the number of impulses via an iterative process is designed by taking advantage of the polynomial nature of the underlying optimality conditions. Although efficient in practice on some instances, this last algorithm suffers from the lack of proof of convergence of the iterative procedure based on simple heuristic rules. As will be shown in the Section 5 dedicated to numerical examples, this algorithm may fail and may only exhibit a suboptimal solution. The next section proposes a new procedure based on a discretization algorithm for the solution of the semi-infinite programming Problem 5 whose convergence may be rigorously established.

### 4. A CONVERGENT DISCRETIZATION APPROACH

#### 4.1 General solving procedure

Based on Problem 5 and Theorem 2, a convergent numerical method is presented. Firstly, the SICP Problem 5 is solved using Algorithm 1 given in Section 4.2 together with its convergence proof. Algorithm 1 provides a numerical value for the optimal cost. Secondly, one identifies the impulse locations and velocity increments based on Theorem 2 in Algorithm 2 in Section 4.3.

#### 4.2 Convergent discretization algorithms for SICP

Consider the general formulation of Problem 5 as a semi-infinite programming problem \( \mathcal{P}(\Theta) \):
\[ \text{Minimize } f(\lambda) \]
subject to \( g(\lambda, \nu) \leq 0, \nu \in \Theta \)

Note that in our case \( f \) is a linear function of \( \lambda, g(\lambda, \nu), \nu \in \Theta \) is convex and \( \Theta \) is a compact set (a closed interval). Efficient discretization methods have been developed for such problems (Reemtsen and Rückman, 1998, Chap.7). They consider a sequence of finite subsets \( \Theta_i \subseteq \Theta \) and solve \( \mathcal{P}(\Theta_i) \) respectively. Let \( M(\Theta_i) \) be the set of feasible points for problem \( \mathcal{P}(\Theta_i) \): \( M(\Theta_i) = \{ \lambda : g(\lambda, \nu) \leq 0, \nu \in \Theta_i \} \).

The advantage is that for finite programs \( \mathcal{P}(\Theta_i) \), feasibility can usually be checked easily and accurately.

Under certain conditions, one chooses an initial set \( \Theta_0 \), and obtains an initial solution \( \lambda_0 \) of \( \mathcal{P}(\Theta_0) \). Then \( \Theta_i \) is chosen as: \( \Theta_i = \Theta_{i-1} \cup \{ \text{arg max}_{\nu \in \Theta} g(\lambda_{i-1}, \nu) \} \). One has to ensure that the sequence of solutions of \( \mathcal{P}(\Theta_i) \) converges to the solution of \( \mathcal{P}(\Theta) \).

In the following, we summarize results from (Reemtsen and Rückman, 1998, Lemma 2.4, Chap.7), (Reemtsen and Rückman, 1998, Theorem 2.8, Chap.7), (Reemtsen and Rückman, 1998, Corollary 2.9, Chap.7) which prove that this procedure is convergent. Algorithm 1 details the implementation for our particular case.

For each feasible point \( \lambda_\Theta \in M(\Theta) \) (if such point exists) and \( \Theta_i \subseteq \Theta \), define the level set
\[ L(\lambda_\Theta, \Theta_i) = M(\Theta_i) \cap \{ \lambda : f(\lambda) \leq f(\lambda_\Theta) \} \]

### Theorem 3. (Reemtsen and Rückman, 1998, Chap.7) Let \( f \) and \( g(., \nu) \), \( \nu \in \Theta \), be convex. Let a sequence of compact sets \( (\Theta_i)_{i \in \mathbb{N}} \) s.t. \( \Theta_0 \) is finite, \( \Theta_i \subseteq \Theta_{i+1} \subseteq \Theta \) and dist \( (\Theta_i, \Theta) = 0 \) where dist is the classical Hausdorff distance.

(3.1) Suppose there exists \( \lambda_\Theta \in M(\Theta) \) s.t. \( L(\lambda_\Theta, \Theta_i) \) is bounded.

Then the set of solutions of \( \mathcal{P}(\Theta_i) \) is nonempty and compact. Algorithm 1 generates an infinite sequence \( \lambda_i \) such that \( \lambda_i \) has an accumulation point and each such point solves \( \mathcal{P}(\Theta) \). Moreover the sequence \( \inf_{\lambda \in M(\Theta)} f(\lambda) \) converges monotonically increasingly to \( \inf_{\lambda \in M(\Theta)} f(\lambda) \) when \( \lambda \to \infty \).
In what follows, we consider two cases which arise in practice and which specify the norms for Problem 5:

1. for a gimbaled single thruster one has \( p = q = 2 \), which gives a semi-infinite positive semi-definite (SDP) problem:

    \[
    \inf_{\lambda \in \mathbb{R}^m} -c^T \lambda \\
    \text{s.t.} \quad \left[ -1 \quad Y^T(\nu) \right]^{\top} \lambda^T Y(\nu) - 1 \leq 0, \quad \forall \nu \in [\nu_0, \nu_f];
    \]  

(20)

2. for \( N \) unimodal identical thrusters, one has \( p = 1, \ Q = \infty \) which gives a semi-infinite linear programming (LP) problem:

    \[
    \inf_{\lambda \in \mathbb{R}^m} -c^T \lambda \\
    \text{s.t.} \quad \sum_{i=1}^n \lambda_i y_{i,s}(\nu) \leq 1, \quad \forall \nu \in [\nu_0, \nu_f], \ s = 1, \ldots, r.
    \]  

(21)

Both problems defined by (21) and (20) are particular instances of \( \mathcal{P}(\Theta) \) for which discretized versions can be efficiently numerically solved. For the convergence proof, Assumption A1 in Theorem 3 is verified in what follows.

**Lemma 1.** Let \( \Theta_0 \equiv \{\theta_0, \theta_1\} \subseteq [\nu_0, \nu_f], \ \theta_1 - \theta_0 \neq k\pi, \ k \in \mathbb{N} \). Assumption A1 holds for both Problems in eqs. (20) and (21) for \( L(0, \Theta_0) \).

**Proof.** First, it is easily checked that \( \lambda_{\Theta} = 0 \) is an interior feasible (Slater) point for Problems in eqs. (20) and (21) (and any of their discretizations). Second, the set \( M(\Theta_0) \) is closed by the definition of the discretized SDP/LP problems. Finally, for (21), one can prove that if \( \det[Y(\theta_0)Y(\theta_1)] = 0 \) then \( M(\Theta_0) \) and hence \( L(0, \Theta_0) \) is bounded. Similarly, for (20), the condition translates to \( \ker(Y^T(\theta_0) \cap \ker(Y^T(\theta_1)) = \{0\} \). The sufficient condition on \( \theta_1 - \theta_0 \) follows by computation.

Thus, Algorithm 1 is initialized based on Lemma 1 and an initial \( \lambda(0) \) (and primer vector \( Y(\nu)^T(\nu) \lambda(0) \)) is computed by solving eq. (17) for \( \nu \in [\theta_0, \theta_1] \).

**Input:** interval \( \Theta = [\nu_0, \nu_f] \), matrix \( Y(\nu) \), initial condition \( c, \) accuracy \( \varepsilon \)

**Output:** \( \mu(i) \) and \( \lambda(i) \) numerical solution of Pb. 5

**Init:**

\( i \leftarrow 0; \theta_0 \leftarrow \{\theta_0, \theta_1\} \subset \Theta \) s.t. \( \theta_0 - \theta_1 \neq k\pi \); Solve eq. (17) for \( \Delta V_0 \) and \( \Delta Y_1 \); Solve for \( \lambda(0) \) the system \( Y^T(\theta_0) \lambda(0) = \Delta V_0/\| \Delta V_0 \|_p \), \( k = 0, 1 \).

**while** \( \max_{\theta,\Theta} \| Y(\theta)^T I(i) \|_q - 1 > \varepsilon \) **do**

\( i \leftarrow i + 1; \Theta_i \leftarrow \Theta_{i-1} \cup \left\{ \arg \max_{\theta,\Theta} \| Y(\theta) \lambda(i) \|_q \right\} \)

**Find** \( \lambda(i) \) solution of discretized problem:

\[
\mu(i) = \inf_{\lambda \in \mathbb{R}^m} -c^T \lambda \\
\text{s.t.} \quad \| Y^T(\theta_k) \lambda \|_q \leq 1 \quad \text{for all} \ \theta_k \in \Theta_i
\]

**end**

**Return** \( \mu(i), \lambda(i) \).

**Algorithm 1:** Numerical procedure for solving Problem 5

We give in what follows an estimation of the accuracy of the obtained numerical value \( \mu(i) \) with respect to the optimal cost \( \eta \) in Problem 4. The discretization method produces outer approximations of a solution of the SIP problem, i.e. the approximate solutions of \( \mathcal{P}(\Theta_i) \) are not feasible for \( \mathcal{P}(\Theta) \), but provide increasing lower bounds for its solution. A global solution \( \tilde{\lambda}(i) \) of \( \mathcal{P}(\Theta_i) \) which is feasible for \( \mathcal{P}(\Theta) \), solves \( \mathcal{P}(\Theta) \), since:

\[
\inf_{\lambda \in \mathbb{R}^m} f(\lambda) \leq \inf_{\lambda \in \mathbb{R}^m} f(\lambda) \leq \lambda(i) \]

Thus, if the discretized problem \( \mathcal{P}(\Theta_i) \) is accurately solved, one has: \( \mu(i) \leq \inf_{\lambda \in \mathbb{R}^m} f(\lambda) \). This gives an upper bound for \( \eta \), using equation (10):

\[
\eta = \max_{\| Y(\nu)^T \lambda \|_q \leq 1} \| c^T \lambda \| - \min_{\| Y(\nu)^T \lambda \|_q \leq 1} -c^T \lambda \leq -\mu(i).
\]

A lower bound can also be obtained.

**Lemma 2.** Suppose one can rigorously check that when Algorithm 1 stops,

\[
\max_{\theta,\Theta} \| Y(\theta)^T \lambda \|_q \leq 1 + \varepsilon,
\]

where \( \varepsilon \) is a user defined input parameter. Then

\[
\frac{-\mu(i)}{1 + \varepsilon} \leq \eta.
\]

**Proof.** One can prove (see e.g. Neustadt (1964)) that in Theorem 1, equation (10) can be replaced by

\[
\eta = \max_{\theta,\Theta} \| Y(\theta)^T \lambda \|_q \leq 1 + \varepsilon,
\]

and \( -c^T \lambda(i) = \mu(i) \).

Thus, given \( \varepsilon \), the output \( \mu(i), \lambda(i) \) of Algorithm 1 provides a good numerical approximation for the optimal cost of the original problem, \( \eta \). The impulse locations and impulse vectors are recovered as follows.

**4.3 Reconstruction of the solution**

The impulse locations can be identified based on Theorem 2 i.e., by finding \( \Gamma = \{\theta_k \in [\nu_0, \nu_f] : \| Y(\theta_k)^T \lambda(i) \|_q = 1\} \). This is done numerically on a grid of \([\nu_0, \nu_f]\). Then one solves the system given in eq. (17). This is always possible, since, according to Neustadt, the following holds: if at most \( N \) locations are found in \( \Gamma \), the system is under-determined/determined and it has at least one solution; if more than \( N \) locations are found in \( \Gamma \), one can select \( N \) among them such that the system has a solution. The detailed numerical procedure is given in Algorithm 2.

**4.4 Analytical results for out-of-plane maneuvers**

We present a simple geometrical interpretation which leads to the analytical solution for the out-of-plane rendezvous problem. For \( n = 2 \) and \( r = 1 \), the vector \( Y^T(\nu) \lambda \) reduces to the scalar function \( \lambda_1 \sin \nu + \lambda_2 \cos \nu \) and problem (12) simplifies to a semi-infinite LP:

\[
\min_{\lambda \in \mathbb{R}^m} -c^T \lambda \\
\text{s.t.} \quad \lambda_1 \sin \nu + \lambda_2 \cos \nu \leq 1
\]

\[
\lambda_1 \sin \nu + \lambda_2 \cos \nu \geq -1, \ \forall \nu \in [\nu_0, \nu_f].
\]

In the plane \((\lambda_1, \lambda_2)\), the feasible set of (25) is defined by two families of lines delimiting half-spaces when \( \nu \) varies in \([\nu_0, \nu_f]\) :

\[
d_1(\nu) \colon \cos \nu \lambda_2 = \lambda_1 \sin \nu + 1 + \epsilon \cos \nu
\]

\[
d_2(\nu) \colon \cos \nu \lambda_2 = \lambda_1 \sin \nu - 1 - \epsilon \cos \nu
\]

(26)
Input: interval \( \Theta = [\nu_0, \nu_f] \), matrix \( Y(\nu) \), initial condition \( c \), accuracy \( \varepsilon \), numerical solution \( \lambda(0) \in \mathbb{R}^n \) of Pb. 5

Output: impulse locations and impulse vectors \( \Gamma_{\text{imp}}(\Delta V_i) \)

\( \Gamma_d \leftarrow \) discretized grid of \( [\nu_0, \nu_f] \)

\( \Gamma \leftarrow \{ \nu_{\hat{k}} \in \Gamma : \| Y(\nu_{\hat{k}})^T \lambda(0) \|_q - 1 \in [-\varepsilon, \varepsilon] \} \)

\( \text{if } (N \leq n) \text{ then} \)

\( \Gamma_{\text{imp}} \leftarrow \Gamma \)

Solve for \( \Delta V_i, i = 1, \ldots, N \), the linear system

\[ c = \sum_{\nu_{\hat{k}} \in \Gamma_{\text{imp}}} Y(\nu_{\hat{k}}) \Delta V_i. \]

else

\( \Gamma_{\text{imp}} \leftarrow \text{Choose } n \) points in \( \Gamma \) s.t. the linear system

\[ c = \sum_{\nu_{\hat{k}} \in \Gamma_{\text{imp}}} Y(\nu_{\hat{k}}) \Delta V_i \text{ has a solution.} \]

end

return \( \Gamma_{\text{imp}}(\Delta V_i) \).

Algorithm 2: Numerical Reconstruction of impulse locations and vectors

For each family, the curve tangent to each member is its envelope, Stoker (1969) and defines a part of the boundary of the feasible set. When \( \nu \) covers the interval \( [\nu_0, \nu_f] \), the two envelopes describe two circle arcs whose corresponding circles equations are given by:

\[ C_1 : \lambda_1^2 + (\lambda_2 + e)^2 = 1 \]
\[ C_2 : \lambda_1^2 + (\lambda_2 + e)^2 = 1 \] (27)

These two circles depend only upon the eccentricity of the reference orbit but the actual feasible region will also depend upon the duration of the rendezvous. To characterize the feasible region, let the following points in the \((\lambda_1, \lambda_2)\)-plane (see also Fig. 1, 3 and 2).

\[ \{P_1\} = d_1(\nu_0) \cap C_1, \quad \{P_2\} = d_1(\nu_f) \cap C_1 \]
\[ \{P_3\} = d_2(\nu_0) \cap C_2, \quad \{P_4\} = d_2(\nu_f) \cap C_2 \]
\[ \{I_1, I_2\} = \{\nu_0\} \cap C_1. \]

The points \( I_1 \) and \( I_2 \) have respectively the coordinates \((-\sqrt{1-e^2}, 0)\) and \((\sqrt{1-e^2}, 0)\) in the \((\lambda_1, \lambda_2)\)-plane. We define also the two anomaly \( \nu_1 \) such that \( d_2(\nu_1) \cap C_1 = \{I_1\} \). Note that all anomalies are defined as the angles between the associated line with the positive real axis as reference. Three different configurations are then possible.

The duration is such that \( \nu_f - \nu_0 < 2\pi \)

- Case I: \( \nu_2 \leq \nu_f \). The feasible set is a convex set bounded by two circle arcs as shown on Fig. 1. In this case, the tangent to the feasible set is not defined uniquely at the points \( I_1 \) and \( I_2 \) due to the lack of differentiability of the boundary at these points. The optimal solution \( \lambda \) is always unique and is either the point of tangency of the line defined by the criterion and \( C_1 \) or \( C_2 \) or it is \( I_1 \) if \( -c_1/c_2 < -\sqrt{1-e^2}/e \) and \( I_2 \) if \( -c_1/c_2 > \sqrt{1-e^2}/e \). Note that \( -\sqrt{1-e^2}/e \) and \( \sqrt{1-e^2}/e \) define respectively the slope of the tangent to \( C_1 \) and \( C_2 \) at the points \( I_1 \) and \( I_2 \).

- Case II: \( \nu_f < \nu_2 \). The arc described by the lines \( \{d_2(\nu) : \nu \in [\nu_0, \nu_f]\} \) does not reach the point \( I_1 \), hence, the feasible set is a convex set bounded by two circle arcs plus two or four line segments depending whether \( d_2(\nu_f) \cap C_1 \) belongs to the circle arc described by the lines \( \{d_1(\nu) : \nu \in [\nu_0, \nu_f]\} \). See for example Fig. 2 and 3. There is either a unique or an infinite number of optimal solutions \( \lambda \) of (25) depending if the line defined by the criterion is tangent to a point of the circle arcs of parallel to one of the lines.

The duration is such that \( \nu_f - \nu_0 \geq 2\pi \)

The feasible set is the same as Case I, Fig. 1.

5. NUMERICAL EXAMPLES

The algorithms were implemented in C language and the discretized SDP problems are solved with SDPA developed by Yamashita et al. (2011).
5.1 Out-of-plane maneuvers for a GTO Mission

Let consider the numerical example from Zhou et al. (2011), for which the target spacecraft is in the geostationary orbit transfer (GTO). It is a highly elliptical Earth orbit with apogee of 42,164 km. The rendezvous characteristics are summarized in the Table 1.

| Semi-major axis | a = 22616 km |
|----------------|--------------|
| Eccentricity | ε = 0,73074 |
| Initial state anomaly | ν₀ = 0,1377 rad |
| Initial state vector | X₀ = [10000 -3] m - m/s |
| Initial state vector | X₀ = [16941,75 -5022.57] m/s |
| Final anomaly | ν_f = 9,2 rad |
| Duration | t_f - t₀ = 29888 s |
| Final state vector | X_f = [0 0] m - m/s |

Table 1. Rendezvous parameters: Zhou et al. (2011)

For ε = 10⁻⁴ and after 6 iterations, Algorithm 1 gives an optimal solution 0.73074 0,1377 rad and Algorithm 2 allows to build a minimum-fuel solution with N = 2 impulses ΔV₁ = 3.115020 m/s, ΔV₂ = 3.154741 m/s, located at ν₁ = [ν₁, ν_f] = [2.3883 3.8919] rad., and an optimal fuel consumption given by η = 6.2725 m/s. Fig. 9, 10 and 11 respectively show the graph of the norm of the primer vector at the initialization step, after the first iteration and at the final and optimal configuration.

5.2 Coplanar maneuvers for ATV Mission

The second numerical example is dedicated to the in-plane motion case and based on some example of the Automated Transfer Vehicle (ATV) setup, Labourdette et al. (2008). The parameters of the reference orbit and of the rendezvous are given in Table 2.

| Semi-major axis | a = 6763 km |
|----------------|--------------|
| Inclination | i = 92 deg. |
| Argument of perigee | ω = 0 deg. |
| Longitude of the ascending node | 1 = 0 deg. |
| Eccentricity | ε = 0,0052 |
| Initial time | t₀ = 0 rad. |
| Initial state vector | X₀ = [-30 0.5 8.514] km - m/s |
| Initial state vector | X₀ = [-51.9222 0.0865 0.95734 0] m/s |
| Final anomaly | ν_f = 8.1831 rad. |
| Duration | t_f - t₀ = 7200 s |
| Final state vector | X₀ = [-100 0 0 0] m - m/s |
| Final state vector | X₀ = [-76.3818 0 69.1519] m - m/s |

Table 2. Parameters of the ATV example

The optimal solution falls into the category of optimal solution described by Case III in the previous section. Therefore, the optimal solution is unique and is given by 0.73074 0,1377 rad corresponding to point I₂ since -c₁/c₂ > 1 - ε²/ε. This solution is illustrated on Fig. 8. This solution is exactly the one obtained analytically with an alternate method in Serra et al. (2014) and defined by the optimal locations ν₊, ν₋ and optimal maneuvers ΔV(ν₊), ΔV(ν₋):

\[
ν₊ = \min \left\{ ν \geq ν₀ / \cos(ν) = -e, \sin(ν) = ±\sqrt{1 - ε²} \right\},
\]

\[
ΔV(ν₊) = ±\sqrt{1 - ε²} \left( 2ε \right)(εe₁ ± \sqrt{1 - ε²}c₂).
\]

The color code is the following: newly added discretized constraint is shown in green; initial and final location of the impulses are shown in red; other intermediary discretized constraints are black. Note that the proposed example falls into the category of optimal solution described by Case III in the previous section. Therefore, the optimal solution is unique and is given by 0.73074 0,1377 rad corresponding to point I₂ since -c₁/c₂ > 1 - ε²/ε. This solution is illustrated on Fig. 8. This solution is exactly the one obtained analytically with an alternate method in Serra et al. (2014) and defined by the optimal locations ν₊, ν₋ and optimal maneuvers ΔV(ν₊), ΔV(ν₋):

\[
ν₊ = \min \left\{ ν \geq ν₀ / \cos(ν) = -e, \sin(ν) = ±\sqrt{1 - ε²} \right\},
\]

\[
ΔV(ν₊) = ±\sqrt{1 - ε²} \left( 2ε \right)(εe₁ ± \sqrt{1 - ε²}c₂).
\]

Fig. 10. Primer vector norm: Initial state vector ̂X₀ = [0 0] m - m/s. - m/s.
A new convergent numerical algorithm has been proposed to solve the linearized impulsive fixed-time fuel-optimal space rendezvous problem. Beside its convergence proof, the algorithm features simplicity, speed and reliability: it makes use of state of the art linear/SDP solvers; on classical rendezvous mission examples, for accuracies of $\varepsilon = 10^{-4}$, no more than 10 iterations are necessary, which accounts for few milliseconds on a modern computer; the numerical error bounds provide guarantees that the input accuracy $\varepsilon$ is met at algorithm’s output. Moreover, the presented theoretical overview allows both for a concise explanation of state-of-the-art results and for a new simple geometrical construction of the analytical solution for the elliptic out-of-plane rendezvous problem. As future works, firstly, we intend to investigate the convergence speed of the given algorithm. Secondly, we intend to use a more intricate geometric interpretation for the in-plane case in order to obtain an analytic solution. Finally, we intend to certify the implementation of our algorithm for on-board embedding purposes.

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