Quantum corrections for BTZ black hole via 2D reduced model

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ABSTRACT

The one-loop quantum corrections for BTZ black hole are considered using the dimensionally reduced 2D model. Cases of 3D minimal and conformal coupling are analyzed. Two cases are considered: minimally coupled and conformally coupled 3D scalar matter. In the minimal case, the Hartle-Hawking and Unruh vacuum states are defined and the corresponding semiclassical corrections of the geometry are found. The calculations are done for the conformal case too, in order to make the comparison with the exact results obtained previously in the special case of spinless black hole. Beside that we find exact corrections for AdS_2 black hole for 2D minimally coupled scalar field in the Hartle-Hawking and Boulware state.
1 Introduction

For a long time it was believed that black hole solutions do not exist in three dimensions, and therefore the discovery of Bañados, Teitelboim, Zanelli [1] came as a surprise. This solution has many properties which the familiar black hole solutions in four dimensions (4D) do not possess. BTZ black hole can be obtained by identifications of points in 3D anti-de Sitter (AdS) space [4], the space of constant negative curvature. BTZ black hole is locally anti-de Sitter space, and therefore its singularity is not a curvature singularity. Obviously, this solution is not asymptotically flat, although the asymptotic region can be identified. On the other hand, the fact that BTZ black hole is three dimensional enables one to work out exactly many computations which in 4D can be done only approximately. Among these, the thermal Green function of the conformally coupled scalar field is found in the framework of the procedure developed by Avis, Isham and Storey [3] which resolves the problem of time-like infinity of AdS. Along with that, various dimensional reductions from BTZ black hole to two dimensions are formulated [4].

One of the most interesting questions in the analysis of black holes is the Hawking radiation. A considerable work has been done in the last couple of years in an effort to find 2D effective models which can describe the properties of 4D black holes and radiated field. The main idea of this approach is to consider the effective action obtained by functional integration of scalar field as semiclassical correction to the gravitational action. There are a couple of different variants of 2D effective action but usually it describes the effects of s-modes of scalar field to the one-loop order. A similar analysis has been recently extended [3] to the reduction of BTZ black hole from three to two dimensions in the case of minimal 3D coupling with scalar matter.

The purpose of this paper is twofold. Our first goal is to define the Unruh vacuum by means of dimensionally reduced model. The definition of the Unruh vacuum seems still to be an open question for BTZ black hole. On the other hand, the analogy with Schwarzchild and Reisner-Nordström case offers a possibility of straightforward generalization of the nonsingularity of energy-momentum tensor (EMT) on the future horizon of black hole.

The other important point is to use the advantage of low-dimensionality of BTZ solution in order to analyze the dimensional-reduction procedure. This problem is of great heuristic importance, as dimensional reduction is repeatedly done in different scenarios of string and brane theories, although the mechanism is fully understood only at the classical level. The study of dimensional reduction from four to two dimension in the case of Schwarzchild black hole was done previously [3, 4, 8, 9]. There are also some new ideas in the literature, as dimensional-reduction anomaly [10, 11]. However, the analysis is far from complete. In order to compare with the results obtained for 3D BTZ [12, 13, 14], we formulated a dimensionally reduced theory for the conformal matter. We defined the Hartle-Hawking vacuum and calculated the backreaction effects.

And finally, for the sake of completeness, we discuss the most frequently used
effective action, Polyakov-Liouville for the case 2D minimally coupled scalar field. As dimensionally reduced spinless BTZ black hole is, in fact, two-dimensional black hole with constant negative curvature, we obtain the full discussion of quantum corrections of 2D AdS black hole as a subcase.

The plan of the paper is the following. In Sect. 2 we introduce the general setting of the problem. Sect. 3 gives the analysis of the Unruh vacuum for the minimally coupled case, while the conformal coupling is discussed in Sect. 4. Sect. 5 is devoted to Polyakov-Liouville action and 2D Anti-de Sitter black hole.

2 General setting

We start with the three dimensional gravitational action with negative cosmological constant \((-2\Lambda = -2l^{-2} < 0)\) coupled to the scalar field \(f\):

\[
\Gamma_0^{(3)} = \frac{1}{16\pi G} \int d^3x \sqrt{-g^{(3)}} \left( R^{(3)} + \frac{2}{l^2} \right) - \frac{1}{16\pi G} \int d^3x \sqrt{-g^{(3)}} \left( (\nabla f)^2 + \xi R^{(3)} f^2 \right). \tag{1}
\]

The case \(\xi = 0\) describes the minimal coupling in 3D, while \(\xi = \frac{1}{8}\) is the conformal coupling. This action admits the vacuum solution \(f = 0\). We consider the BTZ black hole solution which is locally AdS\(_3\) space:

\[
d s^2 = \left( r^2 - \frac{l^2}{2} \right) dt^2 + Jldtd\theta + r^2 d\theta^2 + \left( r^2 - \frac{l^2}{2} - lM + \frac{J^2l^2}{4r^2} \right)^{-1} dr^2. \tag{2}
\]

If we construct the metric reduced from (2) to two-dimensional \(t, r\) hypersurface by the standard procedure \([15]\), we obtain

\[
d s^2 = -g_{ct} dt^2 + \frac{1}{g_{ct}} dr^2, \tag{3}
\]

where the metric function \(g_{ct}(r)\) is given by

\[
g_{ct}(r) = \frac{r^2}{l^2} - lM + \frac{J^2l^2}{4r^2} = \frac{(r^2 - r_{+}^2)(r^2 - r_{-}^2)}{r^2l^2}. \tag{4}
\]

As showed in \([2]\), quantities \(M\) and \(J\) have the meaning of mass and angular momentum. The last equality holds when \(Ml \geq J\); the case \(Ml = J\) is the extremal BTZ black hole. One can see from the Penrose diagram that this space shows great resemblance with Reisner-Nordström black hole. The outer and inner horizons \(r_\pm\) are given by

\[
r_\pm^2 = \frac{l^2}{2} \left( Ml \pm \sqrt{M^2l^2 - J^2} \right). \tag{5}
\]

Inversely

\[
M = \frac{r_+^2 + r_-^2}{l^3}, \quad J = \frac{2r_+r_-}{l^2}. \tag{6}
\]
Acchucarro and Ortiz \[4\] showed that the metric (3) can be obtained from dimensionally reduced action in the following way. Let us assume the axially symmetric metric ansatz in three dimensions:

\[ds^2_{(3)} = g_{\mu\nu} dx^\mu dx^\nu + l^2 \Phi^2 (\alpha d\theta + A_\mu dx^\mu)^2 ,\]  

where \(g_{\mu\nu}, \Phi, A_\mu\) are two-dimensional metric, dilaton and U(1) gauge field. All quantities do not depend on \(\theta\). The constant \(\alpha\) will be fixed later. 3D scalar curvature for ansatz (7) is

\[R^{(3)} = R - \frac{l^2 \Phi^2}{4} F_{\mu\nu} F^{\mu\nu} - \frac{2 \Box \Phi}{\Phi} ,\]  

where \(F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu\) while \(R\) is 2D curvature. Also, \(\sqrt{-g^{(3)}} = \sqrt{-g} l \alpha \Phi\). Introducing the reduction formula (8) into the action (1) and integrating over the angular variable \(\theta\), we obtain 2D action

\[\Gamma_0 = \Gamma_g + \Gamma_m .\]  

Its gravitational part is, up to a total divergence, given by

\[\Gamma_g = \frac{l \alpha}{8G} \int d^2 x \sqrt{-g} \Phi \left( R - \frac{l^2 \Phi^2}{4} F^2 + \frac{2}{l^2} \right) ,\]  

while the part describing the matter is

\[\Gamma_m = -\frac{l \alpha}{8G} \int d^2 x \sqrt{-g} \Phi \left( (\nabla f)^2 + \xi f^2 (R - \frac{l^2 \Phi^2}{4} F^2 - \frac{2 \Box \Phi}{\Phi}) \right) .\]  

In the following, we will choose \(\alpha\) such that \(\frac{l \alpha}{8G} = 1\). Also, instead of the dilaton field \(\Phi\), we will use its logarithm \(\varphi = \log \Phi\).

In order to analyze the vacuum fluctuations of the scalar field \(f\), one has to integrate it functionally to the first order in \(\bar{h}\). Our approximation consists of the fact that we do the functional integration of \(f\) in 2D action (11) and not in the full 3D action. We use the methods developed in \[16, 17\]. The result which we obtained for the one-loop effective action is

\[\Gamma_1 = \frac{1}{96\pi} \int d^2 x \sqrt{-g} (12\xi - 1) R \frac{1}{\Box} R + \frac{1}{8\pi} \int d^2 x \sqrt{-g} \left( \frac{1}{4} - 2\xi \right) R \frac{1}{\Box} (\nabla \varphi)^2 + \left( \frac{1}{2} - 2\xi \right) R \varphi - \frac{\xi l^2}{4} R \frac{1}{\Box} e^{2\varphi} F^2 \right) .\]  

Note, that the effective actions for 2D dilaton models are analyzed in various papers \[18, 16, 17, 19, 20, 21, 22, 23, 24, 25, 26\].

It is easier to use the local form of the action (12); it can be obtained by a suitable introduction of auxiliary fields \[8, 5\]. The local form, however, differs in the cases we are going to discuss. Therefore, we proceed with the minimal case.
3 3D minimal coupling

For $\xi = 0$, the effective action $\Gamma_1$ can be rewritten in the local form as

$$\Gamma_{1,min} = -\frac{1}{96\pi} \int d^2x \sqrt{-g} \left( 2R(\psi - \frac{3}{2}\chi) + (\nabla \psi)^2 - 3(\nabla \psi)(\nabla \chi) - 3(\nabla \varphi)^2 \psi - 6R\varphi \right),$$

(13)

where the auxiliary fields $\psi$ and $\chi$ satisfy equations

$$\Box \psi = R,$$

(14)

$$\Box \chi = (\nabla \varphi)^2.$$

(15)

The full semiclassical action for the minimally coupled field is

$$\Gamma_{min} = \Gamma_g + \Gamma_{1,min}$$

$$= \int d^2x \sqrt{-g} e^{\varphi} \left( R + \frac{2}{l^2} - \frac{l^2}{4} e^{2\varphi} F_{\mu\nu} F^{\mu\nu} \right) - \kappa \int d^2x \sqrt{-g} \left( 2R(\psi - \frac{3}{2}\chi) + (\nabla \psi)^2 - 3(\nabla \psi)(\nabla \chi) - 3(\nabla \varphi)^2 - 6R\varphi \right).$$

(16)

We introduced the constant $\kappa = \frac{1}{96\pi}$ which will be the small perturbation parameter in the following. The equations of motion obtained from the action (16), are (14-15) and

$$\nabla_\mu \left( e^{3\varphi} F^{\mu\nu} \right) = 0,$$

(17)

$$R + \frac{2}{l^2} - \frac{3l^2}{4} e^{2\varphi} F^2 = 6\kappa e^{-\varphi} \left( -R + \nabla_\mu (\psi \nabla_\nu \varphi) \right),$$

(18)

$$g_{\alpha\beta} \Box \Phi - \nabla_\alpha \nabla_\beta \Phi - \Phi g_{\alpha\beta} \left( \frac{1}{l^2} - \frac{l^2}{8} \Phi^2 F_{\mu\nu} F^{\mu\nu} \right) - \frac{l^2}{2} \Phi^2 F_{\beta\mu} F_{\alpha}^\mu$$

$$= \frac{T_{\alpha\beta}}{2}$$

$$= \kappa \left( \nabla_\alpha \psi \nabla_\beta \psi - \frac{3}{2} \nabla_\alpha \psi \nabla_\beta \chi - \frac{3}{2} \nabla_\alpha \chi \nabla_\beta \psi \right)$$

$$- \frac{3}{2} 3\varphi \nabla_\alpha \varphi \nabla_\beta \varphi - 2 \nabla_\beta \nabla_\alpha (\psi - 3\varphi - \frac{3}{2}\chi)$$

$$- \frac{1}{2} g_{\alpha\beta} \left( (\nabla \psi)^2 - 3\nabla \psi \nabla \chi - 3\psi (\nabla \varphi)^2 \right) + 2g_{\alpha\beta} \Box (\psi - 3\varphi - \frac{3}{2}\chi).$$

(19)

$T_{\alpha\beta}$ is the energy-momentum tensor of the radiated matter

$$T_{\mu\nu} = -\frac{2}{\sqrt{-g}} \frac{\delta \Gamma_1}{\delta g^{\mu\nu}}.$$

(20)

For $\kappa = 0$ we obtain the classical vacuum equations of motion. The classical solution is

$$\Phi = e^\varphi = \frac{r}{l}, \quad F^{\mu\nu} = \frac{\epsilon^{\mu\nu}}{\sqrt{-g \frac{J_1}{r^3}}} = \frac{E^{\mu\nu}}{\sqrt{-g \frac{J_1}{r^3}}},$$

(21)

Note that our auxiliary fields differ slightly from those introduced in [5].
where $E^{\mu\nu}$ is the covariant antisymmetric tensor. The zero-th order solution of (19) is the BTZ metric (3). Note that $T_{\mu\nu}$ defined in (19) being of the first order in $\kappa$, is determined by the zero-th order solution for the fields $\psi, \chi$ and $\varphi$.

The Hartle-Hawking vacuum state for the minimally coupled scalar field in the BTZ background in the framework of dimensionally reduced model was analyzed in details in the work of Medved and Kunstatter, [5]. Here we will outline some of their results briefly, in order to compare them to the other results. In the Hartle-Hawking vacuum all functions are independent of the time. The solution of (14-15) is

$$\psi(r) = - \log g_{cl}(r) + Cr_*, \quad (22)$$

$$\chi(r) = \int \frac{dr}{g_{cl}(r)} \left( \int \frac{g_{cl}(r)}{r^2} dr \right) + Dr_*, \quad (23)$$

where the tortoise coordinate $r_*$ for nonextremal BTZ metric is given by

$$r_* = \int \frac{dr}{g_{cl}(r)} = \frac{l^2}{2(r_+^2 - r_-^2)} \left( r_- \log \frac{r + r_-}{r - r_-} - r_+ \log \frac{r + r_+}{r - r_+} \right). \quad (24)$$

The assumption that the energy-momentum tensor is regular on the outer horizon, $r = r_+$, in the freely falling frame means that [27]

$$T_{uu} < \infty, \quad \frac{T_{uu}}{g_{cl}} < \infty, \quad \frac{T_{uu}}{g_{cl}^2} < \infty \quad \text{for } r = r_+, \quad (25)$$

where the components of EMT are given in the null $u, v$ coordinates [5]. Using (25), for the constants $C$ and $D$ we obtain

$$C = 2 \frac{r_+^2 - r_-^2}{l^2 r_+^2}, \quad D = - \frac{6r_+^2 + 2r_-^2}{3l^2 r_+^2}. \quad (26)$$

Introducing these values, we get

$$\psi(r) = - \log \frac{(r + r_+)^2 (r^2 - r_-^2)}{r^2 l^2} + \frac{r_-}{r_+} \log \frac{r + r_-}{r - r_-}, \quad (27)$$

$$\chi(r) = \frac{3r_+^2 + r_-^2}{3(r_+^2 - r_-^2)} \log \frac{(r + r_+)^2}{r^2 - r_-^2} - \frac{(3r_+^2 + r_-^2)r_-}{3(r_+^2 - r_-^2)r_+} \log \frac{r + r_-}{r - r_-} + \frac{1}{3} \log \frac{(r^2 - r_-^2)^2}{r}. \quad (28)$$

The corresponding values of EMT in the Hartle-Hawking vacuum are

$$T_{uu} = \frac{\kappa}{2l^4 r_+^6} \left( (r - r_+)^2 \left( -3r_+^6 r_-^2 - 6r_+^5 r_+ (2r_+^2 - r_-^2) - r_+^4 r_- (3r_+^2 + 2r_-^2) \right) \right)$$

$$\quad - 2r^3 r_+ r_-^2 (5r_+^2 - 3r_-^2) - 3r_+^2 r_-^2 (2r_+^2 - 3r_-^2) + 10r_+^3 r_+^2 + 5r_+^4 r_+^2 \right)$$

$$\quad + 3r_+^2 (r^2 - r_+^2)^2 (r^2 - r_-^2)^2 \left( \log \frac{r + r_+^2 (r^2 - r_-^2)}{r^2 l^2} - \frac{r_-}{r_+} \log \frac{r + r_-}{r - r_-} \right), \quad (29)$$

In the further text we will often switch among the three common choices of coordinates. These are: Schwarzschild coordinates $t, r$, null coordinates $u, v$ ($u = t-r, v = t+r$) and Eddington-Finkelstein coordinates $v, r$. 

6
\[ T_{uv} = \frac{\kappa}{2l^4r^6}(r^2 - r_+^2)(r^2 - r_-^2)(13r^4 + 3r^2(r_+^2 + r_-^2) - 3r_+^2r_-^2), \quad (30) \]

\[ T_{vv} = T_{uu}. \quad (31) \]

For the energy density of the radiation, \( T_{00} = T_{tt} \), we get

\[ T_{tt} = \frac{\kappa}{l^4r^6r_+}(10r^8 r_+ - 6r^7(r_+^2 - r_-^2) + 8r^6(r_+^3 - 3r_+r_-^2) - 6r^5(r_+^4 - r_-^4)
- r^4(6r_+^5 - 16r_+^3r_-^2 + 6r_+r_-^4) + 2r^3r_+^2r_-^2(r_+^2 - r_-^2) + 2r^5r_-^4
+ 3(r^2 - r_+^2)(r^2 - r_-^2)^2r_+(\log \left( \frac{r + r_+}{r_+ - r_-} \right) - \frac{r_-}{r_+} \log \left( \frac{r + r_+}{r_+ - r_-} \right) - \frac{3}{r_+} \log \left( \frac{r + r_+}{r_+ - r_-} \right)) + 3(r^2 - r_+^2)^2(r^2 - r_-^2)^2r_+ \left( \frac{r + r_+}{r^2} \right) - \frac{r_-}{r_+} \log \left( \frac{r + r_+}{r_+ - r_-} \right) \right). \]

(32)

There is an important comment on the values of energy density in the asymptotic region. One can notice that \( T_{tt} \) diverges asymptotically \((r \to \infty)\) as \( r^2 \log r \), a feature which is not present in the Schwarzschild case. However, the Schwarzschild metric is asymptotically flat, while the BTZ metric has nonzero curvature and \( g_{cd}(r) \) behaves like \( r^2 \) as \( r \to \infty \). In order to understand the properties of the Hawking radiation better, we can transform to the locally flat coordinates \( t', r' \) at some distant fixed point \((t, L)\). We get the asymptotics assuming that \( r \sim L \to \infty \). The transformation of coordinates which we need is

\[ t' = \sqrt{g_{cd}(L)}t, \quad r' = \frac{1}{\sqrt{g_{cd}(L)}}r. \quad (33) \]

We see that asymptotically, \( t' \sim \frac{t}{L} \sim \frac{r}{L} \), and therefore

\[ T_{vv} \sim \frac{L^2}{r^2} T_{tt} \sim \frac{\kappa}{L^2}(10 + 6 \log \frac{r}{L}), \quad (34) \]

so the energy density diverges logarithmically in the asymptotic region. This is a rather unexpected behavior of the minimally coupled radiation in BTZ background and we will see that the conformal coupling will improve it.

Having fixed the components of EMT, one can find the first correction of the metric, i.e. solve the equations \((19)\) in the first order in \( \kappa \). The one-loop corrected static ansatz for the metric is

\[ ds^2 = -g(r)e^{2\kappa \omega(r)}dt^2 + \frac{1}{g(r)}dr^2. \quad (35) \]

The function \( g(r) \) we take in the form

\[ g(r) = g_{cd}(r) - \kappa m(r), \quad (36) \]

and the equations \((19)\) to the first order read:

\[ 2\frac{\kappa}{l} \omega' = T_{11} + \frac{T_{00}}{g_{cd}}, \quad (37) \]

\[ \kappa m' = \frac{T_{00}}{g_{cd}}. \quad (38) \]
Their solution is
\[ m(r) = \frac{4r^2 - 6(r_+^2 + r_-^2)}{l^2 r} + \frac{16r_+ - 16r_-}{l^2 r} \log \frac{r + r_-}{r - r_-} \]
\[ + \frac{3r^4 - r_+^2 r_-^2 + 3r_+^2 r_-^2}{l^2 r^3} \left( \log \frac{(r + r_+)^2 (r_+^2 - r_-^2)}{r^2 l^2} - \frac{r_-}{r_+} \log \frac{r + r_-}{r - r_-} \right), \tag{39} \]
\[ \omega(r) = F(r) - F(L), \tag{40} \]

where the function \( F(r) \) is given by
\[ F(r) = l \left( \frac{1}{r} + \frac{(r_+ - 3r_-)(r_+ + r_-)}{r_+(r_+ - r_-)(r + r_-)} + \frac{(r_+ + 3r_-)(r_+ - r_-)}{r_+(r_+ + r_-)(r - r_-)} \right) \]
\[ - \frac{2(3r_+^2 + r_-^2)}{(r + r_+)(r_+^2 - r_-^2)} + \frac{32r_+ r_-^2}{(r_+^2 - r_-^2)^2} \log(r + r_-) \]
\[ - \left( \frac{3r_-}{r_+ + r_-} - \frac{8r_-}{(r_+ + r_-)^2} \right) \log(r - r_-) + \frac{(3r_- - 8r_-)}{r_+ + r_-} \frac{8r_-}{(r_+ - r_-)^2} \log(r + r_-) \]
\[ - \frac{3}{r} \log \frac{(r + r_+)^2 (r_+ - r_-)^2}{r^2 l^2} \right). \tag{41} \]

Here \( L \) is the integration constant. We have assumed that our system is in a 1D box of size \( L \).

The first correction of the scalar curvature, \( R = R_0 + \kappa R_1 \), where
\[ R_0 = -\frac{2}{l^2} - \frac{6r_+^2 r_-^2}{l^2 r^4}, \tag{42} \]
and \( R_1 \) can be expressed in terms of \( m, \omega \) as
\[ R_1 = -3g_{ij} \omega' + lm'' - 2g_{ij} \omega'', \tag{43} \]
is regular on the horizon \( r = r_+ \). If one would not have fixed \( C \) and \( D \) previously, the same values (26) would have been obtained assuming the regularity of \( R_1 \) on \( r_+ \). We find
\[ R_1 = \frac{6}{l^2 r_+ r_+^5} (2r_+^3 (r_+^2 - r_-^2) + 8r_+^3 r_-^2) \]
\[ - r_+ (r^4 - r_+^2 (r_+^2 + r_-^2) - 3r_+^2 r_-^2) \left( \log \frac{(r + r_+)^2 (r_+^2 - r_-^2)}{r^2 l^2} - \frac{r_-}{r_+} \log \frac{r + r_-}{r - r_-} \right). \tag{44} \]

The corrected value of the metric gives us the possibility to find how the horizon of the black hole changes due to the backreaction of the Hawking radiation. The apparent horizon of the black hole (which in the static case coincides with the event horizon) in 2D is defined by
\[ g^{\mu \nu} \partial_\mu r \partial_\nu r = 0. \tag{45} \]
In the corrected null coordinates $\bar{u}, \bar{v}$, in the general case the metric is

$$ds^2 = -g(v, r)e^{2\kappa \omega(v, r)} dv^2 + 2e^{\kappa \omega(v, r)} dv dr$$

with $d\bar{v} = dv$, $d\bar{u} = \mu dv - \frac{2\mu}{g(v, r)} e^{-\kappa \omega} dr$ and $\mu$ is the integration factor [9]. Analyzing the condition (45) in $\bar{u}, \bar{v}$ coordinates, we come to

$$\partial_{\bar{u}} r|_{r_{AH}} = 0, \quad \partial_{\bar{v}} r|_{r_{AH}} = 0,$$

which is equivalent to

$$e^{\kappa \omega} g(v, r)|_{r_{AH}} = 0.$$

Taking the position of the apparent horizon in the form

$$r_{AH} = r_+ + \kappa r_1,$$

we get that the corrected value is

$$r_{AH} = r_+ + \frac{k^3 m(v, r_+) r_+}{2(r_+^2 - r_-^2)}.$$

In the Hartle-Hawking case (50) gives the one-loop corrected value of the event horizon:

$$r_{AH} = r_+ + \kappa \frac{l r_+ r_-}{(r_+^2 - r_-^2)} \left( \frac{5r_+^2 - r_-^2}{r_+^2} \log \frac{r_+ + r_-}{r_+ - r_-} + \frac{3r_+^2 + r_-^2}{r_+ r_-} \log \frac{4(r_+^2 - r_-^2)}{l^2} - \frac{r_+^2 + 3r_-^2}{r_+ r_-} \right).$$

Having found $\psi, \chi$ and the one-loop corrections of the metric, one can easily calculate the corrected thermodynamical quantities, temperature and entropy. Entropy is defined as [28]

$$S = -2\pi \epsilon_{\alpha\beta\gamma\delta} \frac{\partial L}{\partial R_{\alpha\beta\gamma\delta}} \bigg|_{r_{AH}}.$$

For the action (16) for entropy we get

$$S = 4\pi \left( \frac{r_+}{l} - \kappa (2\psi - 3\chi - 6 \log \frac{r_+}{l}) \right) \bigg|_{r_{AH}}.$$

In Hartle-Hawking state we obtain

$$S = 4\pi \left( \frac{r_+}{l} + \kappa \left( -\frac{r_+^2 + 3r_-^2}{r_+^2 - r_-^2} + \frac{5r_+^2 - r_-^2}{r_+^2 - r_-^2} \log \frac{4(r_+^2 - r_-^2)}{l^2} - \frac{r_+^2 + 3r_-^2}{r_+ r_-} \right) - \frac{2r_+^2 + r_-^2}{r_+^2 - r_-^2} \log \frac{r_+^2 - r_-^2}{4r_+^2} + \log (4r_+(r_+^2 - r_-^2)) + 6 \log \frac{r_+}{l} \right),$$

for entropy while the temperature is given by

$$T_H = \frac{r_+^2 - r_-^2}{2\pi l^2 r_+^2} \left( 1 - \kappa F(L) \right) - \kappa \left( \frac{r_+^4 + 9r_+^4 + 6r_+^2 r_-^2}{2\pi l^2 r_+^2 (r_+^2 - r_-^2)} - \frac{8r_+^2}{\pi l (r_+^2 - r_-^2)} \log \frac{16r_+^2}{l^2} \right).$$
The terms proportional with small parameter $\kappa$ are one-loop corrections for the entropy and Hawking temperature.

We will now analyze the Unruh vacuum. The Unruh vacuum can be defined as the state of matter whose energy-momentum tensor is regular on the future event horizon. As it is easily seen, the region $-\infty < t < \infty, r_+ \leq r < \infty$ of the $t, r$ plane transforms into the interior of the triangle $v = -\infty, u = \infty, u = v$ in the $u, v$ plane. The line $u = v$ is the time-like boundary (asymptotic region) of BTZ, $u = \infty$ is the future event horizon, while $v = -\infty$ is the past event horizon of BTZ black hole. In order to find the energy-momentum tensor, we need to solve equations (14-15) for the general case. Those equations can be transformed into the system of partial linear equations which is similar to the one obtained in [9] for the SSG model. For details we refer the reader to [9]. The general solution in the minimal BTZ case reads:

$$\psi(v, r) = -\log g_{cl}(r) + \mathcal{C}(r_+ - \frac{v}{2}) + \mathcal{G}(v),$$  

$$\chi(v, r) = \int \frac{dr}{g_{cl}(r)} \left( \int \frac{g_{cl}(r)}{r^2} dr \right) + \mathcal{D}(r_+ - \frac{v}{2}) + \mathcal{H}(v),$$

where $\mathcal{C}, \mathcal{G}, \mathcal{D}, \mathcal{H}$ are arbitrary functions of their arguments. Note that the arguments in (55-56) are written in such a way that the regularity on the future horizon $u \to \infty, v = \text{const}$ is equivalent to the regularity on $r \to r_+ (r_+ \to -\infty)$, as the values of $v$ and its functions are constant on the future horizon.

While the expression for $T_{uv}$ in the general case is the same as (30), for $T_{uu}$ and $T_{vv}$ we obtain

$$T_{uu} = \frac{\kappa}{2l^4 r^6} \left( -3r^8 + 2r^6(r_+^2 + r_-^2) - 3r^4(r_+^4 - 4r_+^2 r_-^2 + r_-^4) - 6r^2 r_+^2 r_-^2(r_+^2 + r_-^2) + 5r_+^4 r_-^4 - 3(r^2 - r_+^2)^2(r^2 - r_-^2)^2(\mathcal{G} + \mathcal{C} - \log \frac{(r^2 - r_+^2)(r^2 - r_-^2)}{r^2 l^2}) \right),$$

$$T_{vv} = \frac{\kappa}{2l^4 r^6} \left( -3r^8 + 2r^6(r_+^2 + r_-^2) - 3r^4(r_+^4 - 4r_+^2 r_-^2 + r_-^4) - 6r^2 r_+^2 r_-^2(r_+^2 + r_-^2) + 5r_+^4 r_-^4 - 3(r^2 - r_+^2)^2(r^2 - r_-^2)^2(\mathcal{G} + \mathcal{C} - \log \frac{(r^2 - r_+^2)(r^2 - r_-^2)}{r^2 l^2}) \right),$$

From these expressions one can see that in order to enforce the regularity of $T_{uv}/g_{cl}$ on the future horizon, one needs to put the functions $\mathcal{C}$ and $\mathcal{D}$ linear in their arguments, $\mathcal{C}(x) = Cx, \mathcal{D}(x) = Dx$ with the Hartle-Hawking values of constants $C, D$ given by (26). The functions $\mathcal{G}, \mathcal{H}$ cannot be fixed in this manner. In order to analyze this in more details let us assume that $\mathcal{G}$ and $\mathcal{H}$ are also linear, which is in accordance with the request of constant luminosity of the black hole. Under this
assumption we see that the difference of outgoing and ingoing fluxes in the asymptotic region \( r \to \infty \) \( (r_* \to 0) \) has the leading behavior

\[
T_{uu} - T_{vv} \sim -\frac{\kappa r_+ r^2}{2l^4 r_+^2} \left( 2 \log r + C - G - 1 \right),
\]  

and it is much smaller than the asymptotic value of the flux

\[
T_{uu} \sim \frac{3 \kappa r_+ r^2}{2l^4 r_+^2} \left( 2 \log r + \frac{C}{2} v - G - 1 \right).
\]

In fact, the asymptotic value of the flux is not dominated by the function \( G(v) = G(t + r_*) \) for \( r_* \to 0 \), although it fixes the luminosity of the black hole. The dominant term is the \( r^2 \log r \)-term, and it is the same for \( T_{uu} \) and \( T_{vv} \). This is a rather peculiar characteristic of BTZ if we keep in mind that in the Unruh vacuum for the Schwarzschild black hole the outgoing flux is asymptotically constant, \( T_{uu} \to \text{const} \), while the ingoing flux vanishes, \( T_{vv} \to 0 \) as \( r \to \infty \).

One can verify that the given energy-momentum tensor really describes the Unruh vacuum, because it is regular on the future horizon but divergent on the past event horizon \( (v \to -\infty, u = \text{const}) \). If we express EMT \([\text{78}, \text{58}]\) in terms of \( r \) and \( u \), a logarithmically divergent term for \( u = \text{const}, r_* \to -\infty \) appears independently on the choice of the functions \( G \) and \( H \). I. e., excepting for the case \( G(v) = \frac{r^2 - v^2}{4 r_+} v \) which gives the time independence of EMT and therefore the Hartle-Hawking vacuum state.

Taking the above discussion into account, we conclude that the functions \( G \) and \( H \) cannot be fixed by the properties of EMT only. The simplest choice for the Unruh vacuum would be \( G = H = 0 \). In that case

\[
\psi(v, r) = -\frac{r^2}{l^2 r_+} v - \left( \log \left( \frac{r + r_+}{r_+} \right) - \frac{r_+}{r} \log \left( \frac{r + r_+}{r - r_-} \right) \right),
\]

\[
\chi(v, r) = \frac{3 r^2 + r_+}{3 l^2 r_+} v + \frac{3 r^2}{3 (r_+^2 - r_-^2)} \log \left( \frac{r + r_+}{r_+^2 - r_-^2} \right) \log \left( \frac{r + r_+}{r - r_-} \right) + \frac{1}{3} \log \left( \frac{r^2 - r_-^2}{r} \right)
\]

The final expressions for \( T_{\mu \nu} \) are

\[
T_{uu} = \frac{\kappa}{2 l^6 r_+^2} \left( l^2 (r - r_+)^2 (-3 r r_+^2 - 6 (2 r_+^2 - r_-^2) r^5 - r_+ (3 r_+^2 + 2 r_-^2) r^4 - 2 r_+^2 (5 r_+^2 - 3 r_-^2) r^3 - 3 r_+ r_-^2 (2 r_+^2 - 3 r_-^2) r^2 + 10 r_+^2 r_-^4 r + 5 r_+^3 r_-^4) + 3 (r^2 - r_+^2)^2 (r^2 - r_-^2) (r_+ - 2) r v + 3 l^2 r_+ (r^2 - r_+^2) (r^2 - r_-^2) (\log \left( \frac{r + r_+}{r_+^2 - r_-^2} \right) - \frac{r_+}{r} \log \left( \frac{r + r_+}{r - r_-} \right)) \right),
\]

\[
T_{vv} = \frac{\kappa}{2 l^6 r_+^2} \left( l^2 r_+ (-3 r^6 + 2 (r_+^2 + r_-^2) r^6) \right)
\]

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\[-3\left(r_+^4 + r_-^4 - 4r_+^2r_-^2\right)r^4 - 6r_+^2r_-^2\left(r_+^2 + r_-^2\right)r^2 + 5r_+^4r_-^4\]
\[+ 3\left(r_+^2 - r_-^2\right)^2 (r_+^2 - r_-^2)^2 (r_+^2 - r_-^2) v\]
\[+ 3l^2 r_+ (r_+^2 - r_-^2)^2 (r_+^2 - r_-^2)^2 \left(\log \frac{(r_+ + r_-)^2 (r_+^2 - r_-^2)}{r^2 l^2} - \frac{r_-}{r_+} \log \frac{r_+ + r_-}{r - r_-}\right)\]  

\[(64)\]

The same values of the energy-momentum tensor are obtained applying the procedure which is developed by Balbinot and Fabbri, [4].

Now we will find the corrected geometry. The one-loop ansatz for the metric is
\[ds^2 = -g(v, r)e^{2\kappa \omega(v, r)} dv^2 + 2e^{\kappa \omega(v, r)} dvdr ,\]  

where \(g(v, r) = g_d(r) - \kappa lm(v, r)\). Putting this ansatz in the equation \((13)\) we get
\[
\frac{\kappa}{l} \frac{\partial \omega}{\partial r} = \frac{T_{rr}}{2},
\]
\[
-\frac{\kappa}{l} \frac{\partial m}{\partial r} = T_{rv},
\]
\[
\frac{\kappa}{l} \frac{\partial m}{\partial v} = T_{vv} + g_d(r)T_{vr}.
\]

Introducing the values \((33), (34), (35)\) in the system of equations for \(m(v, r)\) and \(\omega(v, r)\) we obtain the one-loop correction for the metric:
\[
m(v, r) = -v \frac{r_+^2 - r_-^2}{l^2 r_+ r^3} \left(-3r_+^4 + 8r_+ r^3 - 3(r_+^2 + r_-^2)r^2 + r_+^2 r_-^2\right)
\[+ \frac{4r_+^2 - 6(r_+^2 + r_-^2)}{l^2 r_+} + 16 \frac{r_-}{l^2} \log \frac{r_+ + r_-}{r - r_-}\]
\[+ \frac{3r_+^4 + 3(r_+^2 + r_-^2)r_+^2 - r_+^2 r_-^2}{l^2 r_+^3} \left(\log \frac{(r_+ + r_-)^2 (r_+^2 - r_-^2)}{r^2 l^2} - \frac{r_-}{r_+} \log \frac{r_+ + r_-}{r - r_-}\right),\]  

\[(69)\]

\[
\omega(v, r) = \frac{l(3r_+ - r_-)(r_+ + r_-)}{r_+ (r + r_-)(r_+ - r_-)} - \frac{l(r_- - r_+)(r_+ + 3r_-)}{r_+ (r_+ - r_-)(r_+ + r_-)} - \frac{2l(3r_+^2 + r_-^2)}{(r + r_-)(r_+^2 - r_-^2)}
\[+ \frac{8l r_-}{(r_+^2 - r_-^2)^2} \left((r_+^2 + r_-^2) \log \frac{r_- - r_+}{r_+ + r_-} + 2r_+ r_- \log \frac{(r + r_-)^2}{r_-^2 - r_+^2}\right)
\[+ \frac{3l}{r} \left(\log \frac{(r + r_-)^2 (r_+^2 - r_-^2)}{r^2 l^2} - \frac{r_-}{r_+} \log \frac{r_+ + r_-}{r - r_-}\right)
\[+ \frac{l}{r} - 3v \frac{r_+^2 - r_-^2}{l r_+ r_-}.\]  

The value for the apparent horizon in this case is
\[
r_{AH} = r_+ + \kappa \frac{l}{r_+ (r_+^2 - r_-^2)} \left[r_+ (3r_+^2 + r_-^2) \log \frac{4(r_+^2 - r_-^2)}{l^2}\right]
\[+ r_-(r_+^2 - 5r_-^2) \log \frac{r_+ + r_-}{r_+ - r_-} - r_+ (r_+^2 + 3r_-^2) - \frac{v}{l^2} (r_+^2 - r_-^2)^2\right].\]  

\[(71)\]
Entropy for Unruh state is given by
\[
S = 4\pi \left(\frac{r_+}{l} + k \left(\frac{r_+^2 + 3r_-^2}{r_+^2 - r_-^2} + \frac{4r_+}{r_+^2 - r_-^2} \log \frac{4(r_+^2 - r_-^2)}{l^2} + \frac{2r_+^2 + r_-^2}{r_+^2 - r_-^2} \log \frac{4r_+}{r_+^2 - r_-^2} + \log(4r_+(r_+^2 - r_-^2)) + 6 \log \frac{r_+}{l}\right)\right). \tag{72}
\]

4 Conformal coupling

We will now discuss the case of the conformally coupled matter. The coupling constant for the conformal coupling in three dimensions is \(\xi = \frac{1}{8}\). The local form of the effective action (12) for this value is
\[
\Gamma_{\text{conf}} = \kappa \int d^2x \sqrt{-g} R(2\psi + \chi) + (\nabla \psi)^2 + (\nabla \psi)(\nabla \chi) + \frac{3l^2}{4} \psi e^{2\varphi} F^2 + 6R\varphi, \tag{73}
\]
and the full action reads
\[
\Gamma_{\text{conf}} = \Gamma_g + \Gamma_{\text{conf}} = \frac{\kappa}{2} \int d^2x \sqrt{-g} e^\varphi \left( R + \frac{2}{l^2} - \frac{l^2}{4} e^{2\varphi} F_{\mu\nu} F^{\mu\nu} \right) + \frac{\kappa}{2} \int d^2x \sqrt{-g} R(2\psi + \chi) + (\nabla \psi)^2 + (\nabla \psi)(\nabla \chi) + \frac{3l^2}{4} \psi e^{2\varphi} F^2 + 6R\varphi. \tag{74}
\]

The equations which follow from the variational principle for (74) are
\[
\Box \psi = R, \tag{75}
\]
\[
\Box \chi = -\frac{3l^2}{4} e^{2\varphi} F^2, \tag{76}
\]
\[
\nabla_\mu \left((1 + \frac{3}{2} \kappa \psi e^{-\varphi}) e^{3\varphi} F^{\mu\nu} \right) = 0, \tag{77}
\]
\[
R + \frac{2}{l^2} - \frac{3l^2}{4} e^{2\varphi} F^2 = -\kappa e^{-\varphi} \left(3R - \frac{3l^2}{4} \psi e^{2\varphi} F^2\right) \tag{78}
\]
and
\[
g_{\alpha\beta} \Box \Phi - \nabla_\alpha \nabla_\beta \Phi - \Phi g_{\alpha\beta} \left(\frac{1}{l^2} - \frac{l^2}{8} \Phi^2 F_{\mu\nu} F^{\mu\nu}\right) - \frac{l^2}{2} \Phi^3 F_{\mu\beta} F^\mu_\alpha = T_{\alpha\beta}/2 \\
= -\frac{\kappa}{2} \left(\nabla_\alpha \psi \nabla_\beta \psi + \frac{1}{2} \nabla_\alpha \psi \nabla_\beta \chi + \frac{1}{2} \nabla_\alpha \chi \nabla_\beta \psi - \frac{3l^2}{2} \psi e^{2\varphi} F_{\beta\nu} F^\nu_\alpha - \nabla_\beta \nabla_\alpha (2\psi + \chi + 6\varphi) - \frac{1}{2} g_{\alpha\beta} ((\nabla \psi)^2 + \nabla \psi \nabla \chi - \frac{3l^2}{4} \psi e^{2\varphi} F^2) + g_{\alpha\beta} \Box (2\psi + \chi + 6\varphi) \right). \tag{79}
\]
We can again take that the solution of (78) for dilaton is $e^\varphi = \frac{r}{l}$, and this in fact represents our choice of the radial coordinate. Then (77) can also be solved exactly
\[ F_{\mu \nu} = E_{\mu \nu} e^{-3\varphi} \frac{J}{l^2} (1 + 3\kappa \frac{l^2}{2r})^{-1} . \] (80)

We proceed with the static case in order to find the values of fields in thermal equilibrium. The zero-th order solution for $\psi$ is, as before
\[ \psi(r) = -\log g_{cd}(r) + C r_+ , \] (81)
while for $\chi$ we have
\[ \chi(r) = \int \frac{dr}{g_{cd}(r)} \left( \int \frac{3J^2 l^2}{2r^4} dr \right) + Dr_+ . \] (82)

Our goal is to solve the equation (79) determining the backreaction to the metric, i.e. to extract the equations for the functions $m(r)$ and $\omega(r)$ from it. Let us note that, as it can be seen from (80), in the conformal case the "electromagnetic field" $F_{\mu \nu}$ has the corrections of the first order in $\kappa$. This means that in three dimensions the angular part of the metric has also to be corrected. Technically, there are the first-order terms on the both sides of equation (79). We will collect all first-order terms on the right hand side. Then the equations for the metric read
\[ 2\kappa \frac{l}{l^2} \omega' = T_{11} + \frac{T_{00}}{g_{cd}} , \] (83)
\[ \kappa m' = T_{00} - \frac{3\kappa J^2 l^2}{2r^4} , \] (84)
under the same ansatz (84) for $g_{\mu \nu}$ as before.

The procedure to determine the integration constants is as for the minimal coupling. The values of constants for the Hartle-Hawking vacuum are
\[ C = \frac{2r_+^2 - r_-^2}{l^2 r_+} , \quad D = \frac{2r_-^2}{l^2 r_+} . \] (85)

For the auxiliary field $\chi$ we get
\[ \chi(r) = \frac{r_+^2}{r_+^2 - r_-^2} \log \frac{(r + r_-)(r - r_-)}{r^2} \]
\[ + \frac{r_-^3}{r_+(r_+^2 - r_-^2)} \log \frac{(r + r_-)}{(r - r_-)} - \frac{2r_-}{r_+^2 - r_-^2} \log \frac{r + r_+}{r} , \] (86)
while $\psi$ is the same as in 3D minimal case (and as it will be for the Polyakov-Liouville action). The energy-momentum tensor in the Hartle-Hawking vacuum reads:
\[ T_{uu} = T_{vv} = -\kappa \frac{(r - r_-)^2}{2l^4 r^6} \left(3r^6 + 6r^5 r_+ + r^4(3r_+^2 - 10r_-^2) \right. \]
\[ - 20r_+^3 r_-^2 + 3r^2 r_+^2 (-3r_+^2 + r_-^2) + 8rr_+ r_-^4 + 4r_+^2 r_-^4 \) , \] (87)
\[ T_{\mu\nu} = \frac{\kappa}{24\pi^2} \left( r^2 - r^2_+ r^2_- \right) \left( r^4 + 3r^2(r^2_+ + r^2_-) - 12r^2_+ r^2_- \right) \]
\[ - 3r^2_+ r^2_- \left( \log \left( \frac{r + r^2_+}{r_+} \frac{r^2_+}{r^2} \right) - \frac{r_-}{r_+} \log \frac{r + r_-}{r - r_-} \right), \]

while the energy density is

\[ T_{tt} = -\frac{\kappa}{24\pi^2} \left( 2r^8 - 4r^6(2r^2_+ + 3r^2_-) + r^4(6r^4_+ + 38r^2_+ r^2_- + 6r^4_-) \right) \]
\[ - 2r^2_+ r^2_-(r^2_+ - r^2_-) - 24r^2_+ r^2_- (r^2_+ + r^2_-) + 16r^4_+ r^4_- \]
\[ - 3\kappa \frac{(r^2 - r^2_+)(r^2 - r^2_-)}{24\pi^2} \left( \log \left( \frac{r + r^2_+}{r_+} \frac{r^2_+}{r^2} \right) - \frac{r_-}{r_+} \log \frac{r + r_-}{r - r_-} \right). \]

We see now that the asymptotic behavior of EMT is improved, as the leading term for \( r \to \infty \) is \( T_{tt} \sim -\frac{\kappa^2}{4\pi^2} \). This means that the energy density of radiation in the locally Minkowskian frame is constant. The solution for the functions \( m \) and \( \omega \) also turns out to be nonsingular on the horizon \( r = r_+ \). It is given by:

\[ m(r) = \frac{1}{2^2\omega^3} \left( -2r^4 - 6r^2(r^2_+ + r^2_-) + 6r^2_+ r^2_- - 2r^3 \log \frac{r + r_-}{r - r_-} \right) \]
\[ - r^2_+ r^2_- \left( \log \left( \frac{r + r^2_+}{r_+} \frac{r^2_+}{r^2} \right) - \frac{r_-}{r_+} \log \frac{r + r_-}{r - r_-} \right), \]

\[ \omega(r) = F(r) - F(L), \]

where \( F(r) \) is given by

\[ F(r) = 4 \frac{l}{r} - \frac{l(r + 2r_-)}{2(r + r_-)(r_+ + r_-)} - \frac{l(r_+ + 2r_-)}{2(r - r_-)(r_+ + r_-)} + \frac{lr_-}{(r + r_+)(r^2_+ - r^2_-)} \]
\[ - \frac{2lr_+ r^2_-}{(r^2_+ - r^2_-)^2} \log \left( \frac{r + r^2_+}{r^2} \frac{r^2_+}{r^2} \right) + \frac{lr_- (r^2_+ + r^2_-)}{(r^2_+ - r^2_-)^2} \log \frac{r + r_-}{r - r_-}. \]

For the first correction of the curvature we obtain

\[ R_1 = \frac{6}{l^5} \left( r^4 + 3r^2_+ r^2_- - 2r^2_+ r^2_- \left( \log \left( \frac{r + r^2_+}{r_+} \frac{r^2_+}{r^2} \right) - \frac{r_-}{r_+} \log \frac{r + r_-}{r - r_-} \right) \right). \]

We can now compare our results with results in the literature. The Green functions for BTZ black hole were calculated in [12, 13, 14]. The starting point of this calculation is the Green function for the scalar field in AdS$_3$ space. However, as AdS space has a time-like infinity, it does not have a Cauchy surface. The prescription to fix the boundary conditions for the wave equation and define the orthonormal basis of eigenfunctions for the quantization is the following [3]. One conformally maps AdS into the half of the Einstein static universe (ESU), which is spatially compact and has a well defined Cauchy problem. The solutions for the conformally coupled scalar field in ESU can be mapped back into the solutions for the conformally coupled scalar field in AdS, and hence from the basis of eigenfunctions in ESU one inherits the basis in AdS. The use of the complete basis in ESU gives the so-called "transparent boundary
conditions”. Transparent boundary conditions have the feature that the energy of scalar field is not conserved. Also, it is possible to define two types of ”reflective boundary conditions” (Dirichlet and Neumann), such that the energy in both cases is conserved. The final step of the construction of Green functions for BTZ black hole is to apply the method of images.

The Green function for spinning BTZ black hole for the transparent boundary conditions is given by Steif [14], the backreaction to the metric was discussed by Martinez and Zanelli [13]. We will not compare our results to those, as the transparent boundary conditions are not appropriate for description of the Hartle-Hawking state because of nonconservation of energy. Lifschytz, Ortiz [12], and Shiraishi, Maki [13] found the Green functions for reflective boundary conditions in the spinless case, \( J = 0 \). In both of these papers some aspects of the behavior of the energy-momentum tensor and of backreaction effects were extracted and we will quote shortly keeping in mind that our results, obtained by dimensional reduction, are approximate.

The expectation value of the energy-momentum tensor for the spinless BTZ black hole is given in [12, 13]. Even in the spinless case the components of EMT have relatively complicated form of infinite sum and nonpolynomial behavior, so it is not easy to compare them with results [5,7,8] which look much simpler. In [12] was shown that the energy density is positive for Dirichlet boundary conditions, while for Neumann boundary conditions it is not. We obtained \( T_{tt} \sim -\frac{\kappa}{r^2} \) for \( r \to \infty \), or in the locally flat frame, \( T'_{tt} \sim -\frac{\kappa}{r^2} \). However, we know from the analysis of the Schwarzchild case that the dimensional reduction can change the sign of the energy, as it takes into account not all but only a part of the modes of scalar field. EMT is regular for \( r = r_+ \) and singular as \( r \to 0 \) both in [12, 13] and in our calculation.

Since the metric ansatz in [12] is not the same as the one we used and it does not seem to be correct [32], we will compare the corrections for the curvature , only. In [12] was obtained that the curvature scalar \( R^2 \) diverges like \( \frac{1}{r^6} \) near \( r = 0 \). For \( J = 0 \) in (93) we see that \( R_1 = \frac{6}{l^2} \). However, this is only the correction of the two-dimensional piece of the curvature scalar. In order to find the full three-dimensional correction, we should employ the reduction formula (8). Using the solutions written to the first order in \( \kappa \)

\[
\Phi = \frac{r}{l}, \quad F^2 = -2\frac{J^2l^2}{r^6}(1 - 3\kappa\frac{l'}{r}),
\]

and

\[
\Box \Phi = \frac{1}{\sqrt{-g}}\partial_\mu(\sqrt{-g}g^{\mu\nu}\partial_\nu \Phi) = \frac{1}{l}(g'_{tt} + \kappa g_{tt}\omega' - \kappa lm'),
\]

we find the first correction of 3D curvature:

\[
R_1^{(3)} = \frac{8}{lr} + 6\frac{Ml^2}{r^3}.
\]

From this expression it can be seen that \( R^2 \) also diverges like \( 1/r^6 \) near \( r = 0 \). Note that in the zero-th order, the reduction formula gives \( R_0^{(3)} = -\frac{6}{l^2} \), while \( R_0 = -\frac{2}{l^2} - \frac{3J^2l^2}{2r^4} \). We will later use the fact that for \( J = 0 \) we get AdS$_2$ black hole.
Finally, we can compare the metric corrections which are in [13] given in the large mass limit. The function $\mu(r)$ used in [13] is proportional to our $m(r)$. It behaves as

$$\mu(r) \sim \frac{r_+}{r} - 1, \quad \text{Neumann b.c.} \quad (97)$$

while our result for $m(r)$ is

$$m(r) = -2 \frac{r^2 + 3r_+^2}{l^2r}. \quad (99)$$

If the limit $M \to \infty$ can be understood as $r_+ \gg r$, than the behavior of $m(r)$ is the same as the one obtained in [13] for the Neumann boundary conditions (up to an integration constant which we, for the sake of simplicity, discarded in the expression (90) for $m$).

Coming back to the 2D conformal matter model, we want to add some remarks, skipping the details of calculations. It is always interesting to give a particular analysis of the extremal black hole, and this was done in [1] for the case of minimally coupled matter. The conclusion was that BTZ black hole behaves similarly to dimensionally reduced Reisner-Nordström black hole [29]. Namely, the Hartle-Hawking EMT for extremal black hole is different from the limit $r_+ \to r_-$ of nonextremal black hole. E.g., it behaves differently on the event horizon: while we have the regularity for the extremal black hole, it is not present in the nonextremal limit. Surprisingly, this is not so in the conformally coupled case: nonextremal and extremal black holes behave similarly. One can check that exactly $r_+ \to r_- (C \to 0)$ gives the best regularity properties to the energy momentum tensor.

The other peculiar thing for the conformal case is that one cannot define the Unruh vacuum obeying all regularity conditions on the future horizon, as it was possible for the minimal case.

## 5 2D minimal coupling

In this section we will consider minimal coupled scalar field to gravity in two dimensions. Performing the functional integration of 2D scalar field in the path integral we will obtain Polyakov-Liouville effective action. It is very often used for the exact or qualitative description of one-loop quantum effects of the scalar field. This action was widely discussed in the context of string theory and 2D dilaton gravity and it is given by

$$\Gamma_{1,PL} = -\frac{1}{96\pi} \int d^2x \sqrt{-g} R \frac{1}{\Box} R, \quad (100)$$

or in the local form

$$\Gamma_{1,PL} = -\kappa \int d^2x \sqrt{-g} \left( (\nabla \psi)^2 + 2R \psi \right). \quad (101)$$
An auxiliary scalar field $\psi$ satisfies the equation $\Box \psi = R$. The energy-momentum tensor determined by (101) is

$$T_{\mu \nu} = 2\kappa \left( \nabla_\mu \psi \nabla_\nu \psi - 2 \nabla_\mu \nabla_\nu \psi - \frac{1}{2} g_{\mu \nu} (\nabla \psi)^2 + 2 g_{\mu \nu} \Box \psi \right). \quad (102)$$

We see that in the Polyakov-Liouville case the effective action looks much simpler, being expressed in terms of only one auxiliary field.

Let us see which results do we get for the action

$$\Gamma_{PL} = \Gamma_g + \Gamma_{1,PL}. \quad (103)$$

For the auxiliary field $\psi$ we have the same result (22), with the same value for the integration constant $C = 2 \frac{r_+^2 - r_-^2}{r_+^2 - l^2}$. The regular values of energy-momentum tensor are

$$T_{uv} = 2\kappa \frac{(r_+^2 - r_-^2)(r^2 - r_-^2)(r^4 + 3r_+^2 r_-^2)}{r^6 l^4}, \quad (104)$$

and

$$T_{uu} = T_{vv} = -2\kappa \frac{(r_+^2 - r_-^2)^2 r^2 (r^2(3r_+^2 - r_-^2) - 2r_+^2 r_-^2)}{r^6 l^4 r_+^2}. \quad (105)$$

The energy density is positive and has regular behavior in the asymptotic region

$$T_{tt} = 4\kappa \frac{(r^2 - r_+^2)(r^6 r_+^2 - r^6 r_-^2 - 4r_+^2 r_-^2 + r^2 r_+^2(6r_+^2 + r_+^2) - 5r_+^2 r_-^2)}{r^6 l^4 r_+^2}. \quad (106)$$

The asymptotic value of the energy density in the locally flat frame is $\frac{4\kappa}{l^2}$.

$$m(r) = \frac{2}{3r^2 r_+^2 r_-^2} \left( 2r_- (3r^4 + 3r^2 (r_+^2 + r_-^2) - 5r_+^2 r_-^2) + 3r^3 (r_+^2 - r_-^2)^2 \log \frac{r - r_-}{r + r_-} \right) \quad (107)$$

$$\omega(r) = \frac{l (-2r_+^2 (3r_+^2 + r_-^2) + 8 r_+^2 r_+^2)}{r_+^2 r_-^2 (r^2 - r_-^2)} + \frac{l (3r_+^2 + r_-^2)}{r_+^2 r_-^2} \log \frac{r + r_-}{r - r_-}. \quad (108)$$

The correction of curvature is $R_1 = 0$.

The results given above are particularly interesting because they can be interpreted as corrections for AdS$_2$ black hole. Namely, in the spinless case, the action (103) describes dilaton gravity with negative cosmological constant with the quantum corrections produced by 2D minimally coupled scalar field. The classical part of this action is Jackiw-Teitelboim model 2D gravity [34]. The classical solution of the equations of motion is AdS$_2$ geometry

$$ds^2 = -\left( \frac{r^2}{l^2} - lM \right) dt^2 + \left( \frac{r^2}{l^2} - lM \right)^{-1} dr^2, \quad (109)$$

with the curvature $R_0 = -\frac{2}{l^2}$. AdS$_2 \times S^2$ geometry appears as the near horizon geometry of the external Reisner-Nordströmm solution and it is analyzed in [30, 31, 32]. For the value of constant $C = 2r_+^2/l^2$ we get the components of EMT:

$$T_{uv} = 2\kappa \frac{r^2 - r_+^2}{l^4}, \quad T_{uu} = T_{vv} = 0, \quad T_{tt} = 4\kappa \frac{r^2 - r_+^2}{l^4}. \quad (110)$$
for black hole in Hartle-Hawking state. AdS$_2$ line element (109) can be rewritten in the null form:

$$ds^2 = -\frac{l M}{\sinh^2\left(\frac{M}{l} \sqrt{u - \frac{v}{2}}\right)} du dv . \tag{111}$$

In this case $r_*$ is

$$r_* = -\sqrt{\frac{l}{M}} \text{Arccoth} \frac{r}{\sqrt{l^3} M} .$$

The Kruskal coordinates,

$$U = -\sqrt{\frac{l}{M}} e^{-\sqrt{M/l} \ u}, V = \sqrt{\frac{l}{M}} e^{\sqrt{M/l} \ v},$$

are regular on the horizon, $r = r_+ = \sqrt{l M^3}$. The line element in these coordinates is

$$ds^2 = -\frac{4 l M}{(1 + M U V/l)^2} dU dV .$$

As we know, the Hartle-Hawking state is the conformal state $|UV\rangle$. It is easy to find the components of EMT in this state using the law of transformation the components of EMT from the Boulware, $|uv\rangle$ to Hartle-Hawking state, $|UV\rangle$. One can check that, performing this transformation the previous result is obtained. Different vacuum states were, in the framework of Reisner-Nordstr"om geometry, discussed by Spradelin and Strominger [30]. Fabbri, Navarro and Navarro-Salas considered the one-loop corrections for evaporating AdS$_2$ black hole [31, 32], but again in the connection with Reisner-Nordström geometry.

Now, we want to find the one-loop solution of this model. The equations of motion take form:

$$R = -\frac{2}{l^2} , \tag{112}$$

$$g_{\alpha\beta} \Box \Phi - \nabla_{\alpha} \nabla_{\beta} \Phi - \Phi g_{\alpha\beta} \frac{1}{l^2} = \frac{1}{2} T_{\alpha\beta} , \tag{113}$$

where EMT is given by (102). This equations can be solved exactly. If we assumed that the one-loop metric is given by (109) than we will obtain that the dilaton is given by

$$\Phi = \frac{r}{l} - 2\kappa , \tag{114}$$

for Hartle-Hawking state. It is interesting to note that in the case of the Boulware vacuum (where $C = 0$) there is again exact solution:

$$\Phi = \frac{r}{l} + \kappa \frac{r}{r_+} \log \frac{r + r_+}{r - r_+} . \tag{115}$$

The integration constants in previous results are chosen in agreement with classical limit $\kappa \to 0$. We see that the one-loop corrected metric is the AdS black hole again - the quantum corrections neither change the character of the space nor they produce the singularity at $r = 0$. 

19
6 Conclusions

In this paper we treated the one-loop corrections of dimensionally reduced BTZ black hole. We analyzed three types of effective actions, corresponding to different couplings of scalar matter (3D minimal, 3D conformal and 2D minimal couplings).

One of the main result is the analysis of the Unruh vacuum for reduced BTZ model. This state is defined demanding that EMT is regular on the future horizon. It has peculiar properties.

The other point was to compare 2D reduced model with exact 3D results in the conformal case. Here we found that the corrections of geometry 2D reduced model is in a relatively good agreement with Neumann boundary conditions for the scalar field. Let us note that due to the ansatz (7) it is not possible to compare the values of EMT directly.

Note that the energy density in the asymptotic region does not obey Stefan-Boltzman law. This is not surprising if we keep in mind that the Hawking radiation is not a free boson gas in this region. The one-loop correction of entropy are logarithmic as it is often the case.

Finally, in the last section we found exact result for JT model in the case Hartle-Hawking and Boulware vacuum. In both cases there is not the correction of AdS$_2$ geometry in the quantum level. The backreaction change dilaton field only. These two solutions will be analyzed in the future publications.

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