Approximate conservation laws in perturbed integrable lattice models

Marcin Mierzejewski, Tomaz Prosen, and Peter Prelovšek

1Institute of Physics, University of Silesia, 40-007 Katowice, Poland
2Faculty of Mathematics and Physics, University of Ljubljana, SI-1000 Ljubljana, Slovenia
3J. Stefan Institute, SI-1000 Ljubljana, Slovenia

We develop a numerical algorithm for identifying approximately conserved quantities in models perturbed away from integrability. In the long-time regime, these quantities fully determine correlation functions of local observables. Applying the algorithm to the perturbed XXZ model we find that the main effect of perturbation consists in expanding the support of conserved quantities. This expansion follows quadratic dependence on the strength of perturbation. The latter result together with correlation functions of conserved quantities obtained from the memory function analysis confirm feasibility of the perturbation theory.

PACS numbers: 75.10.Pq, 72.10.-d, 05.60.Gg, 75.10.Jm

I. INTRODUCTION

A considerable interest has recently been attracted by integrable quantum models which, in contrast to generic systems, have macroscopic number of local conserved quantities (CQ). Due to their presence, the isolated integrable systems don’t thermalize but instead relax towards non-thermal steady states. It has been suggested that such unusual steady states are fully specified by local as well as quasilocal CQ. Inclusion of the latter ones is necessary at least in several systems which cannot be mapped to noninteracting particles. The research on integrable systems has been motivated not only by such fundamental problems like that concerning mechanisms of thermalization/relaxation. The integrable systems are interesting also because they show dissipationless (ballistic) energy, spin and charge transport which might be important for the future applications.

However, real systems are never perfect and their description in terms of integrable models should be viewed, at best, as reasonable approximations. Therefore, it is important to understand the properties of systems which are weakly perturbed away from integrability. In the case of classical mechanics, relevant formalism has been developed for more than fifty years, whereas for quantum systems such understanding is still missing or, at least, remains largely incomplete. It is not evident to what extent the breaking of integrability may be described within a single universal picture and which properties are specific for particular model and/or perturbation. It is also not quite clear which hallmarks of integrability disappear abruptly and which decay smoothly when the perturbation gradually increases.

Recent studies allow to formulate several general expectations. At least at the infinite time-scale and for sufficiently strong perturbation the ballistic transport should be replaced by large but finite conductivity. However, after turning on the perturbation, the system should evolve towards a quasi-steady state (prethermalization) which is analogous to the generalized Gibbs ensembles. Hence, at least for a finite time window and for sufficiently weak perturbation, the system maintains the main property of integrable systems, i.e., the existence of local or quasilocal operators which for perturbed systems are conserved only approximately. These quantities are not identical with CQ of the integrable parent model but rather they are modified by the perturbation. Finally, since quasilocal CQ play important role for strictly integrable models they should also be included in the studies on perturbed systems.

In this paper we develop an algorithm which captures/verifies all these properties of the perturbed integrable systems. The approach yields approximately conserved quantities (ACQ) which completely determine the long-time correlation functions of all local operators supported on assumed subsystem. The algorithm captures cases ranging from strict integrability with local and quasilocal CQ to generic systems where CQ are generally nonlocal linear combinations of projections on eigenstates of the Hamiltonian . In the first part of this paper we show the general approach, while in the second part we apply it to the perturbed anisotropic Heisenberg (XXZ) model. We also find that the ACQ at weak perturbation can be described as quantities decaying exponentially in time with the characteristic rate depending quadratically on the perturbation strength, consistent with specific findings of Ref. This result as well as the memory function analysis confirm nonsingular behavior at weak perturbation strength and the feasibility of the perturbation theory for a generic class of integrability-breaking perturbations.

II. GENERAL METHOD

We study a one-dimensional tight-binding Hamiltonian on a lattice of sites with periodic boundary conditions. We consider the space of local, extensive, translationally-invariant observables with the scalar
product (see Appendix A)

\[ (A|B) = \frac{1}{\mathcal{L}} \langle A|B \rangle = \frac{1}{\mathcal{L}} \sum_{mn} A^*_m B_{mn} p(E_n). \]  

where \( A_{mn} = \langle m|A|n \rangle \), \( H|n \rangle = E_n|n \rangle \) and the weights are assumed to satisfy \( \sum_n p(E_n) = 1 \) and \( p(E_n) > 0 \) for all \( n \). The latter assumption excludes the zero-temperature case but, at least in finite systems, accounts for the thermal states \( p(E_n) \propto \exp(-\beta E_n) \), where \( \beta \) is the inverse temperature.

We have recently developed a procedure for identifying a complete set of local and quasi-local CQ in integrable lattice systems.\( ^{24} \) The main step is to construct scalar products of all time-averaged operators \( (A|B) \), where

\[ A = \lim_{\tau \to \infty} \frac{1}{\tau} \int_0^\tau dt A(t) = \sum_{E_n = E_m} A_{mn} |m\rangle\langle n| . \]  

The identity \( (A|B) = (A|B) \) is essential since it allows to distinguish between local, quasi-local and generic nonlocal CQ. In order to determine ACQ in the perturbed system one should consider a finite time–window \([0, \tau] \), however, a simple omission of the limit \( \tau \to \infty \) violates the latter essential relation. Therefore, we define an effective operator time-average with a time-scale \( \tau \) as

\[ \tilde{A}^\tau = \int_{-\infty}^{\infty} dt A(t) \frac{\sin(t/\tau)}{\pi t} \]  

which in spectral representation amounts to cutting off quickly oscillating (in time) matrix elements

\[ \tilde{A}^\tau = \sum_{mn} \theta \left( \frac{1}{\tau} - |E_n - E_m| \right) A_{mn} |m\rangle\langle n| . \]  

It is quite obvious that the truncated operators are approximately conserved at the time-scale \( t \ll \tau \). Since \( \theta^2(x) = \theta(x) \), this simplified time-averaging maintains the property \( \tilde{A}^\tau \tilde{B}^\tau = (A^\tau|B^\tau) \). Moreover, it becomes identical with the actual time-averaging over an infinite time–window, \( \lim_{\tau \to \infty} \tilde{A}^\tau = \tilde{A} \), whereas for finite \( \tau \) it is related with the low-frequency spectrum of standard correlation functions:

\[ (A^\tau|B^\tau) = \lim_{\varepsilon \to 0} \int_{-\varepsilon}^\varepsilon d\omega \frac{1}{2\pi} \int_{-\infty}^{\infty} dt e^{i\omega t - \varepsilon|t|} \frac{(A^\dagger B(t))}{\mathcal{L}} . \]  

Since physically interesting observables are usually supported on few sites only, we define a subspace \( B^M_L \) of \( A_L \) which contains operators supported on up to \( M \) consecutive lattice sites. The choice of interesting operators determines \( M \). We introduce also the basis of \( B^M_L \) composed of operators \( O_s \) which are orthonormal \( (\bar{O}_s|O_{s'}) = \delta_{s,s'} \). After constructing \( O_s \) we solve the eigenproblem for the matrix \( K^\tau_{ss'} = (\bar{O}_s^\dagger \bar{O}_{s'}) \)

\[ \sum_{s,s'} U_{s's} K^\tau_{ss'} U_{s't} = \delta_{ss'} \lambda_l, \quad U U^\dagger = U^\dagger U = 1 , \]  

which generates orthogonal set of ACQ consisting of \( Q_l = \sum_{s} U_{sd} \bar{O}_d^\tau \). Generally, the truncation \( \mathcal{L} \) modifies the support of operators and transforms local operators \( O_s \) into quasi-local ones \( \bar{O}_s^\tau \). Therefore, we split \( Q_l \) into two orthogonal components \( Q_l = Q^M_l + Q^L_l \) such that \( Q^M_l = \sum_{s} (O_s|Q_l) O_s \in B^M_L \) while \( (Q^L_l|O_s) = 0 \) for all \( s \). The eigenvalues \( \lambda_l \) obtained from Eq. (8) bear important information on the support of \( Q_l \) (see Appendix B),

\[ ||Q_l||^2 = (Q_l|Q_l) = \lambda_l, \quad \frac{||Q^M_l||^2}{||Q^L_l||^2} = \lambda_l. \]

Carrying out the finite size scaling \( L \to \infty \) of \( \lambda_l \) we distinguish between local ACQ when \( \lambda_l = 1 \) for sufficiently large \( M \), quasi-local ACQ when \( 0 < \lambda_l < 1 \) for any \( M \), and generic nonlocal ACQ when \( \lambda_l \to 0 \). One can also show (see Appendix C) that the correlation function of arbitrary local observables \( A, B \in B^M_L \) is completely determined by their projections on \( Q_l \),

\[ (\tilde{A}^\tau|\tilde{B}^\tau) = \sum_l \frac{(A|Q_l) (Q_l|B)}{(Q_l|Q_l)} \]

\[ = \sum_l \lambda_l \frac{(A|Q^M_l) (Q^M_l|B)}{(Q^M_l|Q^M_l)} \]  

where the latter equation comes from identity \( (A|Q_l^M) = 0 \). Choosing \( A = B \) as a current operator and taking the limit \( \tau \to \infty \) we recognize that Eq. (8) becomes the saturated Mazur bound \( ^{23,24} \) for the charge/spin stiffness. Note that Eq. (8) involves normalized operators \( Q_l||Q_l|| \). Therefore, the key point is to follow how the supports of ACQ (and not their norms) depend on the time-scale \( \tau \) and the strength of perturbation.\( ^{2} \) Our approach gives complete set of ACQ, which are sorted from the most relevant local operators with \( \lambda_l = 1 \) to the least relevant ACQ with the smallest \( \lambda_l \).

### III. PERTURBED ANISOTROPIC HEISENBERG MODEL

Next, we apply this approach to the extended XXZ model

\[ H = J \sum_{j=1}^{L} \left[ \frac{1}{2} (S_j^+ S^-_{j+1} + S^{-}_j S^+_j) + \Delta S_j^x S_{j+1}^x \right] + \alpha H' \]

\[ H' = J \sum_{j=1}^{L} S_j^z S_{j+2}^z, \]

where \( S_j^{\pm,z} \) are spin-1/2 operators and the integrability is broken by the last term when \( \alpha \neq 0 \). We take \( J = 1 \) as the energy unit. For concreteness, we study the infinite–temperature limit \( p(E_n) = \text{const} \) when Eq. (1) becomes the Hilbert–Schmidt scalar product. Then, the orthonormal basis of \( B^M_L \) is composed of operators\( ^{24} \)

\[ O_s = \sum_j \sigma_j^{s_1} \sigma_{j+1}^{s_2} \cdots \sigma_{j+m-1}^{s_m}, \quad m = 2, ..., M \]  

(11)
where $\sigma_j^+ \equiv 2S_j^x$, $\sigma_j^- \equiv \sqrt{2}S_j^z$, $\sigma_j^z \equiv 1$, $S_\lambda = (s_1, \ldots, s_m)$, $s_j \in \{+, -, z, 0\}$ while $s_{1,m} \in \{+, -, z\}$. We introduce symbols “R” and “I” to distinguish between real $(O_l+iO_l)$ and imaginary $(iO_l-iO_l)$ combinations of basis operators, respectively. We use also letters “E” and “O” to distinguish between operators which respectively are odd and even under the spin–flip transformation. Since the Hamiltonian is invariant under spin–flip and time–reversal transformation, we separately study four orthogonal sectors of operators denoted as RE (includes, e.g., the Hamiltonian), IE (includes, e.g., the energy current), IO (includes, e.g., the spin current), and RO. In the integrable parent model, the local and quasilocal CQ exist in all four sectors provided $\Delta < 1$. In order not to exclude any symmetry sector from our considerations, we take $\Delta =\frac{1}{2}$. We restrict also the Hilbert space to the states with $\mathcal{S}_\text{tot} = 0$.

As follows from Eq. (9), breaking of integrability affects the correlation functions by either changing the support of $Q_l$ (parameterized by $\lambda_l$) or by changing the projected operators, $Q^M_l$. In order to quantify the latter changes we have calculated the projections

$$ P_l = \frac{(Q^M_l|Q^{M\dagger}_l)}{||Q^M_l|| \ ||Q^{M\dagger}_l||}. \quad (12) $$

where $Q_{0,l}$ are CQ obtained for the integrable parent model. Figure 1 shows size–dependence of $\lambda_l$ and $P_l$ for various $\tau$. We observe that the perturbation strongly reduces $\lambda_l$ (except for $\lambda_1$ in RE sector discussed below), whereas the projected operators $Q^M_l$ do not change significantly. Therefore, we conclude that the main effect of perturbation consists in expanding the support of ACQ.

From now on we focus on the support of ACQ as parameterized by $\lambda_l$ [see Eq. (9)]. The leading eigenvalues in the parent integrable model (lines with points in Fig. 1) are independent of the time-scale $\tau$ indicating that the corresponding $Q_{0,l}$ are strictly conserved. It holds true both for local CQ with $\lambda_l = 1$ (Figs. 1a and 1b) as well as quasilocal CQ with $0 < \lambda_l < 1$ (Fig. 1c). For the perturbed system, the only strictly conserved quantity is $Q_1$ in the RE sector, which actually represents the Hamiltonian. All other $Q_l$ are quasilocal for finite $\tau$. Their supports visibly depend on the time-scale even for quite large $\tau$, hence they are conserved only approximately. We have verified these conclusions also for other eigenvalues, symmetry sectors, supports $M = 3, 4, 5$ and perturbations $\alpha/\Delta = \frac{1}{5}, \frac{1}{3}, \frac{1}{2}$.

The most relevant and challenging problem is to establish $\lambda_l$ for large $\tau$ and small $\alpha$. It is also important that the finite size scaling $L \to \infty$ precedes the limit $\tau \to \infty$. In Fig. 2 we plot $\lambda_l$ linearly extrapolated to $1/L \to 0$ and normalized to results for integrable parent model. We clearly see that the dependence of extrapolated $\lambda_l$ on $\tau$ and $\alpha$ is universal

$$ R_l(\tau, \alpha) = \frac{\lambda_l(L \to \infty, \tau, \alpha)}{\lambda_l(L \to \infty, \tau \to \infty, \alpha = 0)} \simeq \tilde{R}_l(\tau \alpha^2), \quad (13) $$

and can be well approximated by

$$ R_l(\tau, \alpha) \approx \frac{\pi}{2} \arctan \left( \frac{1}{\tau \alpha^2 \gamma_l} \right). \quad (14) $$

We have found the same type of behavior for other eigenvalues (excluding the Hamiltonian), symmetry sectors and accessible supports $M = 3, 4, 5$ (not shown).

Our main result for the Heisenberg model at infinite
temperature [Eqs. 13 and 14] can be best explained in the projection procedure according to Mori’s [see also Refs. 44, 46], to analyze the relaxation function for CQ of the integrable parent model, $Q_{0,t}$

$$\Phi_{1}(\omega) = \frac{1}{L - \omega} |Q_{0,t}|^2 = -\frac{||Q_{0,t}||^2}{\omega + M(\omega)}.$$  

$$M(\omega) = \frac{PLQ_{0,t}}{PL-P - \omega} |PLQ_{0,t}|,$$  \hspace{1cm} (15)

where $LA = [H, A]$ and $P$ is the projection onto the operator space orthogonal to $Q_{0,t}$.

When the formalism is applied to the perturbed integrable system (10) with $\alpha \ll 1$, it follows directly from Eq. (15) that

$$\mathcal{L}Q_{0,t} = \alpha[H', Q_{0,t}]/\alpha,$$  \hspace{1cm} (16)

so that $M(\omega) = \alpha^2 M(\omega)$. It is plausible, but by no means obvious that the imaginary part of the memory function, $M''(\omega)$, is almost constant for small $|\omega|$. However, if the latter is true then the dynamical relaxation reduces to

$$\Phi_{1}(\omega) \simeq -\frac{||Q_{0,t}||^2}{\omega + i\alpha^2 M''(\omega)} \simeq -\frac{||Q_{0,t}||^2}{\omega + i\alpha^2 \gamma_l}.$$  \hspace{1cm} (17)

We end up with a Lorentzian form which explains the specific dependence of eigenvalues $\lambda_l$ on the time-scale $\tau$ and perturbation $\alpha$ in Eq. (12). Namely, using Eqs. 15 and 17 we find

$$\int_0^\tau d\omega \text{Im} \Phi_{1}(\omega) \propto \arctan \left( \frac{1}{\tau \alpha^2 \gamma_l} \right).$$  \hspace{1cm} (18)

The memory function analysis can be easily generalized also to a quantity $A$ which is not CQ but has substantial overlap with the conserved quantity,

$$\frac{|\langle A |Q_{0,t} \rangle|}{|A||Q_{0,t}|} \sim o(1).$$  \hspace{1cm} (19)

In such case the numerator in Eq. (17) should be renormalized becoming the Drude weight (dissipationless part) of the considered operator $A$

$$\Phi_{1}(\omega) \sim -\frac{||Q_{0,t}||^2 |A|Q_{0,t}|^2}{\omega + i\alpha^2 M''(\omega)} \simeq -\frac{||Q_{0,t}||^2 |A|Q_{0,t}|^2}{\omega + i\alpha^2 \gamma_l}.$$  \hspace{1cm} (20)

The latter equation is valid only for weak enough perturbation $\alpha \ll 1$ and in the low $\omega$ regime.

For the numerical calculations of the memory function we employ the microcanonical Lanczos method well adapted for the studies of dynamics at $\beta \to 0$ where we can evaluate spin systems with up to $L = 32$ sites. The important parameter is the number of Lanczos steps $N_L \leq 20000$ which determines the $\omega$ resolution of the method $\delta \omega \sim \delta E/N_L$ where $\delta E$ is the energy span of the $L$-site spectrum, so that we reach $\delta \omega \sim 10^{-3}$.

**IV. DISCUSSION AND CONCLUSIONS**

We have carried out numerical calculations at $\beta \to 0$ for $Q_{0,1}$ in the IE sector being the energy current $j_E$ in the unperturbed parent model and for the spin current $j_s$ which has large projection on quasilocal $Q_{0,1}$ in the IO sector. From numerically obtained $\Phi_{1}(\omega)$ we have extracted the relevant $M(\omega)$ via Eq. (15). Results presented in Fig. 3 confirm that $M''(\omega)/\alpha^2$ for $j_E$ is indeed very broad, featureless in a wide range $\omega \in [0, \omega_0]$ where $\omega_0 > 1$, and (almost) independent of $\alpha$. On the other hand, $\Phi_{1}(\omega)$ for $j_s$ has a nonzero $M''(\omega)$ even for integrable $\alpha = 0$ case, since $j_s$ is not conserved. Still, in the regime $\omega \leq 1$, $M''(\omega)$ as well $\Phi_{1}(\omega)$ follow the scaling as given by Eq. (17). Note that for the Lorentzian assumption in Eq. (17) it is enough that it holds for $\omega \lesssim \alpha^2 \gamma_l$.

Most importantly, Figs. 3 and 6 show convincing quantitative agreement between the result obtained from our general approach and the formalism of the memory functions. Since the latter results have been obtained for much larger systems (but for two observables only) they can also serve a test of the finite-size scaling of $\lambda_l$.
proximately conserved operators, $Q_l$. For an assumed time-scale $\tau$, these quantities completely determine the correlation functions of all local observables supported on several ($M$) lattice sites. $Q_l$ smoothly depend on $\tau$, $\alpha$, and for the limiting case $\tau \to \infty$, $\alpha \to 0$ coincide with strictly conserved (local or quasilocal) quantities of integrable parent model. We have shown that the perturbation influences the correlation functions mostly by expanding the supports of $Q_l$. In our approach this effect is parameterized by eigenvalues $\lambda_l$ decreasing from $\lambda_l = 1$ (for local operators) down to $\lambda_l = 0$ (for generic nonlocal operators). We have found a scaling $\tau \propto l$ for all local observables supported on finite systems. The latter analysis allowed us also to extend the results also for systems perturbed by 4th nearest-neighbor interaction (not shown). This scaling seems to be typical, however, one cannot exclude that it breaks down for other specially tuned perturbations (see, e.g., Refs. 44 and 52, 54).

We have found a qualitative and quantitative agreement between our results and the memory functions obtained numerically for spin and energy currents for much larger systems. The latter analysis allowed us also to explain the origin of the specific scaling of $\lambda_l$. Since this explanation is of perturbative character, we believe that the validity of the obtained scaling extends down to $\alpha \to 0$ well beyond the regime which can be inferred directly from bare numerical results.

Within each symmetry sector we have found that the smaller $\lambda_l$ is (roughly understood as a more extended support of $Q_l$) the larger is the scattering rate $\gamma_l(l)$. A relevant open question emerges: how many independent scattering rates are introduced by a single perturbation? Since all the scattering rates found in our studies are of the same order of magnitude, this problem may pose a challenge.

**ACKNOWLEDGMENTS**

M.M. acknowledges support from the DEC-2013/09/B/ST3/01659 project of the Polish National Science Center. P.P. and T.P. acknowledge support by the program P1-0044 and projects J1-4244 (P. P.) and J1-5349, N1-0025 (T. P.) of the Slovenian Research Agency.

**Appendix A: The choice of the scalar product**

As an alternative to the scalar product defined in Eq. (1) one may consider also other scalar products discussed, e.g. in Ref. 58:

$$
(A|B)_{1} = \frac{1}{2L}(A^\dagger B + BA^\dagger) = \frac{1}{L} \sum_{mn} A^*_{mn} B_{mn} \frac{p(E_n) + p(E_m)}{2},
$$

(A1)

or

$$
(A|B)_{2} = \frac{1}{\beta L} \int_{0}^{\beta} dx(e^{xH}A^\dagger e^{-xH}B) = \frac{1}{L} \sum_{mn} A^*_{mn} B_{mn} p(E_n) e^{\frac{\beta(E_n - E_m)}{\beta}} - \frac{1}{\beta} \frac{1}{E_n - E_m},
$$

(A2)

where $A_{mn} = \langle m|A|n \rangle$, $H|n \rangle = E_n|n \rangle$. Both these scalar products maintain the essential property, i.e., $(A|B)_{1} = (A^\dagger|B)_{1}$ and $(A|B)_{2} = (A^\dagger|B)_{2}$. Calculating the scalar products of operators averaged over infinite time–window we find that the only contribution comes from states with equal energies ($E_n = E_m$), hence

$$
(A|B) = (A|B)_{1} = (A|B)_{2} = \frac{1}{L} \sum_{E_n = E_m} A^*_{mn} B_{mn} p(E_n).
$$

(A3)

Consequently, the stiffness $\frac{1}{\beta}(\Delta A)$ can be expressed in the same way by all considered scalar products as $(A|A)_{1,2}$. However, when discussing the memory function C, at finite temperature $\beta < \infty$ one should use the scalar product defined in Eq. (A2).

**Appendix B: Support of the approximately conserved quantities**

The orthogonal set of ACQ consists of operators

$$
Q_l = \sum_{s} U_s \tilde{O}^*_s,
$$

(B1)

where the unitary matrix $U$ is defined in Eq. (8) and the norm of $Q_l$ can be found as

$$
||Q_l||^2 = \sum_{s,s'} (U_s \tilde{O}^*_s |U_{s'} \tilde{O}^*_{s'}) = \sum_{s,s'} U^\dagger_{s'} (\tilde{O}^*_{s'} |\tilde{O}^*_s) U_{s'} = \lambda_l.
$$

(B2)

We split ACQ into two components $Q_l = Q^M_l + Q^\perp_l$, where the former operator is supported on $M$ sites $Q^M_l = \sum_{s'} (O_{s'}|Q_l)_{s'} \in B^M_l$ while the latter one $(Q^\perp_l|O_s) = 0$. Using Eq. (B1), Eq. (10) and the identity $(A|B) = (B|A)$ we find

$$
Q^M_l = \sum_{s,s'} (O_{s'}|\tilde{O}^*_s) U_{s'} O_{s'} = \sum_{s,s'} (\tilde{O}^*_{s'} |\tilde{O}^*_s) U_{s} O_{s'} = \sum_{s'} \lambda_l U_{s'} O_{s'},
$$

(B3)
The latter result together with the assumption concerning the orthonormal basis, \((O_s'|O_s) = \delta_{s,s'}\) yields

\[ ||Q^M||^2 = (\sum_s \lambda_s U_s'O_s') \sum_s \lambda_s (O_s'O_s) = \lambda_l^2 \sum_{s,s'} U_{ls}^\dagger (O_{s'}|O_s) U_{sl} = \lambda_l^2 \] (B4)

Eqs. (B2) and (B4) lead to Eq. (4) which relates eigenvalue \(\lambda_l\) with the support of \(Q_l\).

Appendix C: Correlation functions and saturated Mazur bound

We consider an operator supported on \(M\) sites, \(A \in B_M^M\) which can be expressed in terms of basis operators

\[ A = \sum_s a_s O_s \].

Using Eq. (B3) we find

\[ A^\dagger = \sum_s a_s \tilde{O}_s^\dagger = \sum_{ls} a_s U_{ls}^\dagger Q_l \equiv \sum_l v_l Q_l, \] (C1)

and

\[ (Q_l|A) = (Q_l|A^\dagger) = v_l \lambda_l. \] (C2)

Repeating the same calculations for some other operator \(B \in B_M^M\) we arrive at Eq. (8):

\[ (A^\dagger|B^\dagger) = \sum_{l:\lambda_l \neq 0} \frac{(A|Q_l)}{\lambda_l} (Q_l|Q_l) \frac{(Q_l|B)}{\lambda_l} \]
\[ = \sum_{l:\lambda_l \neq 0} \frac{(A|Q_l)(Q_l|B)}{(Q_l|Q_l)} \] (C3)
25. E. Ilievski, J. De Nardis, B. Wouters, J.-S. Caux, F. H. L. Essler, and T. Prosen, “Complete generalized gibbs ensembles in interacting theories,” arXiv:1507.02993 (2015).

26. M.G. Tetelman, “Lorentz group for two-dimensional integrable lattice systems,” Sov. Phys. JETP 55, 306 (1982).

27. M. P. Grabowski and P. Mathieu, “Structure of the conservation laws in quantum integrable spin chains with short range interactions,” Ann. Phys. (N.Y.) 243, 299 (1995).

28. X. Zotos and P. Prelovšek, “Evidence for ideal insulating or conducting state in a one-dimensional integrable system,” Phys. Rev. B 53, 983–986 (1996).

29. X. Zotos, F. Naef, and P. Prelovsek, “Transport and conservation laws,” Phys. Rev. B 55, 11029–11032 (1997).

30. X. Zotos, “Finite temperature Drude weight of the one-dimensional spin-1/2 Heisenberg model,” Phys. Rev. Lett. 82, 1764–1767 (1999).

31. M.S. Hawkins, M.W. Long, and X. Zotos, “Long-time asymptotics and conservation laws in integrable systems,” arXiv:0812.3096v1 (2008).

32. J. Benz, T. Fukui, A. Klümper, and C. Scheeren, “On the finite temperature Drude weight of the anisotropic Heisenberg chain,” Journal of the Physical Society of Japan 74, 181–190 (2005). http://journals.jps.jp/doi/pdf/10.1143/JPSJS.74.S181.

33. F. Heidrich-Meisner, A. Honecker, and W. Brenig, “Transport in quasi one-dimensional spin-1/2 systems,” The European Physical Journal Special Topics 151, 135–145 (2008).

34. J. Sirker, R. G. Pereira, and I. Affleck, “Diffusion and ballistic transport in one-dimensional quantum systems,” Phys. Rev. Lett. 103, 216602 (2009).

35. Tomaz Prosen, “Exact nonequilibrium steady state of a strongly driven open XXZ chain,” Phys. Rev. Lett. 107, 137201 (2011).

36. Tomaz Prosen, “Ergodic properties of a generic nonintegrable quantum many-body system in the thermodynamic limit,” Phys. Rev. E 60, 3949–3968 (1999).

37. Tomaz Prosen, “General relation between quantum ergodicity and fidelity of quantum dynamics,” Phys. Rev. E 65, 036208 (2002).

38. Tomaz Prosen, “Spin-current autocorrelations from single pure-state propagation,” Phys. Rev. Lett. 112, 120601 (2014).

39. Tomaz Prosen, “Exact nonequilibrium steady state of a strongly driven open XXZ chain,” Phys. Rev. Lett. 107, 137201 (2011).

40. R. Steinigeweg, J. Herbrchy, P. Prelovšek, and M. Mierzejewski, “Coexistence of anomalous and normal diffusion in integrable mott insulators,” Phys. Rev. B 85, 214409 (2012).

41. L. Vidmar, S. Langer, I. P. McCulloch, U. Schneider, U. Schollwöck, and F. Heidrich-Meisner, “Sudden expansion of Mott insulators in one dimension,” Phys. Rev. B 88, 235117 (2013).

42. D. Crivelli, P. Prelovšek, and M. Mierzejewski, “Energy and particle currents in a driven integrable system,” Phys. Rev. B 90, 195119 (2014).

43. J. J. Mendoza-Arenas, S. R. Clark, and D. Jaksch, “Coexistence of energy diffusion and local thermalization in nonequilibrium XXZ spin chains with integrability breaking,” Phys. Rev. E 91, 042129 (2015).

44. P. Jung, R. W. Helmes, and A. Rosch, “Transport in almost integrable models: Perturbed Heisenberg chains,” Phys. Rev. Lett. 96, 067202 (2006).

45. Peter Jung and Achim Rosch, “Spin conductivity in almost integrable spin chains,” Phys. Rev. B 76, 245108 (2007).

46. Robert Bamler and Achim Rosch, “Equilibration and approximate conservation laws: Dipole oscillations and perfect drag of ultracold atoms in a harmonic trap,” Phys. Rev. A 91, 063604 (2015).

47. X. Zotos, “High temperature thermal conductivity of two-leg spin-1/2 ladders,” Phys. Rev. Lett. 92, 067202 (2004).

48. Maxim Olshanii, “Geometry of quantum observables and thermodynamics of small systems,” Phys. Rev. Lett. 114, 060401 (2015).

49. F. H. L. Essler, S. Kehrein, S. R. Manmana, and N. J. Robinson, “Quench dynamics in a model with tuneable integrability breaking,” Phys. Rev. B 89, 165104 (2014).

50. Yichen Huang, C. Karrasch, and J. E. Moore, “Scaling of electrical and thermal conductivities in an almost integrable chain,” Phys. Rev. B 88, 115126 (2013).

51. Maxim Olshanii, Kurt Jacobs, Marcos Rigol, Vanja Dunjko, Harry Kennard, and Vladimir A. Yurovsky, “An exactly solvable model for the integrability-chaos transition in rough quantum billiards,” Nat. Commun. 3, 641 (2012).

52. Marcos Rigol and B. Stram Schraa, “Drude weight in systems with open boundary conditions,” Phys. Rev. B 77, 161101 (2008).

53. J. Sirker, R. G. Pereira, and I. Affleck, “Diffusion and ballistic transport in one-dimensional quantum systems,” Phys. Rev. Lett. 103, 216602 (2009).

54. Robin Steinigeweg, Jochen Gemmer, and Wolfram Brenig, “Spin-current autocorrelations from single pure-state propagation,” Phys. Rev. Lett. 112, 120601 (2014).

55. G. P. Berman and F. M. Izrailev, “The Fermi – Pasta – Ulam problem: Fifty years of progress,” Chaos 15, 015104 (2005).

56. V.I. Arnold, “Proof of a theorem by a.n.kolmogorov on the invariance of quasi-periodic motions under small perturbations of the hamiltonian,” Usp. Math. Nauk. 18, 13–40 (1963).

57. V. F. V. Unruh, “Universe as a black hole,” Phys. Rev. D 14, 870 (1976).

58. V. F. V. Unruh, “Black hole, temperature, and thermodynamics,” Phys. Rev. D 14, 870 (1976).

59. P. Mazur, “Non-ergodicity of phase functions in certain systems,” Physica 43A, 533 – 545 (1969).

60. Hazime Mori, “Transport, collective motion, and brownian motion,” Progress of Theoretical Physics 33, 423–455 (1965).

61. M. W. Long, P. Prelovšek, S. El Shawish, J. Karadamoglou, and X. Zotos, “Finite-temperature dynamical correlations using the microcanonical ensemble and the lanczos algorithm,” Phys. Rev. B 68, 235106 (2003).