Harnack Inequalities for SDEs with Multiplicative Noise and Non-regular Drift

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Abstract

The log-Harnack inequality and Harnack inequality with powers for semigroups associated to SDEs with non-degenerate diffusion coefficient and non-regular time-dependent drift coefficient are established, based on the recent papers \[7, 20\]. We consider two cases in this work: (1) the drift fulfills the LPS-type integrability, and (2) the drift is uniformly Hölder continuous with respect to the spatial variable. Finally, by using explicit heat kernel estimates for the stable process with drift, the Harnack inequality for the stochastic differential equation driven by symmetric stable process is also proved.

Keywords: Harnack inequality, LPS condition, Hölder continuity, Zvonkin-type transformation

Mathematics Subject Classification (2010): 60H10

1 Introduction and Main Results

The dimension-free Harnack inequality with powers introduced in \[13\] and the log-Harnack inequality introduced in \[10\] have been intensively investigated for various stochastic (partial) differential equations. They are efficiently applied to study heat kernel estimates, functional inequalities, transportation-cost inequalities and properties of invariant measures, see e.g. \[15\] and references therein. Consider the following stochastic differential equation (SDE) on \(\mathbb{R}^d\):

\[
dX_t = \sigma(t, X_t) \, dW_t + b(t, X_t) \, dt, \quad X_0 = x,\tag{1.1}
\]

where \(\sigma : [0, \infty) \times \mathbb{R}^d \to \mathbb{R}^d \otimes \mathbb{R}^d\) and \(b : [0, \infty) \times \mathbb{R}^d \to \mathbb{R}^d\) are two Borel measurable functions, and \((W_t)_{t \geq 0}\) is a standard \(d\)-dimensional Brownian motion defined on a complete filtered probability space \((\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_t)_{t \geq 0})\). When the equation (1.1) has a unique solution for any starting point \(x\), we denote it by \(X_t(x)\) and define the associated Markov semigroup \((P_t)_{t \geq 0}\) as follows:

\[
P_t f(x) = \mathbb{E} f(X_t(x)), \quad t \geq 0, x \in \mathbb{R}^d, f \in B_b(\mathbb{R}^d).
\]

If the coefficients \(\sigma\) and \(b\) are semi-Lipschitz continuous with respect to the spatial variable locally uniformly in the time variable, Harnack inequalities for \(P_t\) have been established in \[14\];
see Theorem 3.1 below for the explicit statement. Recently, Harnack inequalities for (1.1) with log-Lipschitz continuous coefficients have been studied in [12]. The aim of this paper is to consider Harnack inequalities for SDE (1.1) with non-regular time-dependent drift coefficient.

1.1 Drift satisfying the LPS-type condition

We first consider the case where the drift $b$ satisfies an integrability condition, which is known in fluid dynamics as the Ladyzhenskaya–Prodi–Serrin condition (LPS condition for short). More precisely, assume that the drift coefficient $b \in L^q_{\text{loc}}([0, \infty), L^p(\mathbb{R}^d))$ for some $p > d$ and $q > 2$ such that

$$\frac{d}{p} + \frac{2}{q} < 1,$$

that is, for all $T > 0$, it holds

$$\int_0^T \left( \int_{\mathbb{R}^d} |b(t,x)|^p \, dx \right)^{q/p} \, dt < +\infty. \tag{1.3}$$

The SDE (1.1) with diffusion coefficient $\sigma = \text{Id}$ and drift coefficient $b$ satisfying (1.3) was first studied by Krylov and Röckner [8], where the existence of the unique strong solution was proved. X. Zhang [19] extended their result to the more general SDE (1.1) with variable diffusion coefficient $\sigma$. Furthermore, it was shown in [20] (see also [5]) that SDE (1.1) generates a unique stochastic flow $X_t$ of homeomorphisms on $\mathbb{R}^d$, provided that the diffusion coefficient $\sigma$ satisfies the following conditions:

\begin{enumerate}
  \item \((H^{\sigma}_1)\) $\sigma(t,x)$ is uniformly continuous in $x \in \mathbb{R}^d$ locally uniformly with respect to $t$, and there exist positive constants $K$ and $\delta$ such that for all $(t,x) \in [0, \infty) \times \mathbb{R}^d$,
    $$\delta|y|^2 \leq |\sigma(t,x)^* y|^2 \leq K|y|^2, \tag{1.4}$$
  \item \((H^{\sigma}_2)\) $|\nabla \sigma(t, \cdot)| \in L^q_{\text{loc}}([0, \infty), L^p(\mathbb{R}^d))$ with the same $p, q$ as in (1.2), where $\nabla$ denotes the generalized gradient with respect to $x$.
\end{enumerate}

In the recent paper [3], Beck et al. considered SDE (1.1) with $\sigma = \text{Id}$ and a drift $b$ satisfying the generalized LPS condition (i.e., ‘$<$’ in (1.2) is replaced by ‘$\leq$’). They first established the well-posedness of the corresponding stochastic continuity (and also transport) equation, from which they deduced the existence of a unique Lagrangian flow associated to (1.1). Notice that, however, in the limit case $(p, q) = (d, \infty)$, they assumed in addition that the $L^\infty_{\text{loc}}([0, \infty), L^d(\mathbb{R}^d))$-norm of the drift $b$ is small enough (see [3, Condition 8]), thus leaving the general case as an open question.

Inspired by X. Zhang’s work [20], J. Shao established in [11, Theorem 2.1] the Harnack inequalities for SDE (1.1) by using the coupling method, under some additional assumptions

\[ \delta|y|^2 \leq \sum_{i,k} |\sigma^{ik}(t,x)y_i|^2 \leq K|y|^2. \tag{1.5} \]

It is clear that (1.4) implies (1.5), but the inverse implication does not hold. To see the latter, simply look at the $(2 \times 2)$ constant matrix

$$\sigma = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}.$$

Nevertheless, the arguments of [20] still work and the results all hold true.
(see (H\textsuperscript{3})), (H\textsuperscript{4}) and (H\textsuperscript{b}) in [11]). However, there are some extra constants on the right hand sides of the inequalities [11, (2.3) and (2.4)]. In the next theorem, we remove the additional constant in the log-Harnack inequality [11, (2.4)], since we do not have the extra conditions (H\textsuperscript{3}) and (H\textsuperscript{b}) in [11].

**Theorem 1.1.** Assume that \( \sigma \) fulfills (H\textsuperscript{7}) and (H\textsuperscript{2}), and \( b \in L^4\text{loc}([0, \infty), L^{p}(\mathbb{R}^d)) \) with \( p, q \) verifying (1.2). Let \( P_t \) be the semigroup associated to (1.1). Then, for any \( T > 0 \), there is a positive constant \( C > 0 \) such that the following log-Harnack inequality

\[
P_t \log f(y) \leq \log P_t f(x) + \frac{C|y-x|^2}{\delta(t-s)}, \quad x,y \in \mathbb{R}^d
\]

(1.6)

holds for \( 0 < t \leq T \) and \( f \in B_b(\mathbb{R}^d) \) with \( f \geq 1 \), where \( \delta \) is the constant in (H\textsuperscript{7}).

Our method is based on the \( L^2 \)-gradient estimate of the semigroup given in the proof of [20, Theorem 3.5] (see also (2.4) of the current paper). For the moment, we are unable to remove the extra constant in the Harnack inequality [11, (2.4)], since we do not have the \( L^1 \)-gradient estimate.

### 1.2 Hölder continuous drift

Next we consider the case where the drift coefficient \( b \) is Hölder continuous with respect to the spatial variable. The motivation for considering this type of drift comes from the papers [6, 7] of Flandoli, Gubinelli and Priola. In the influential work [6], the authors considered the SDE (1.1) with diffusion coefficient \( \sigma = \text{Id} \) and drift \( b \in L^\infty\text{loc}([0, \infty), C^\theta_0(\mathbb{R}^d, \mathbb{R}^d)) \) for some \( \theta \in (0,1) \). They proved that, in this case, equation (1.1) generates a stochastic flow of diffeomorphisms, from which they constructed an explicit solution to the corresponding stochastic transport equation. Flandoli et al. extended in [7] the property of flow of diffeomorphisms to more general SDEs with non-constant diffusion coefficient \( \sigma \), satisfying a uniform non-degeneracy. Their proof is based on a modified Zvonkin transformation (called the Itô–Tanaka trick in [7]). Notice that the main part of [7] is focused on the SDE (1.1) with coefficients independent on time, but the authors mentioned in [7, Remark 9] that their method works as well in the time-dependent case; moreover, they outlined the essential steps needed for transforming the proofs to the time-dependent case.

Here are our assumptions in this case (see Subsection 3.1 for the definition of the functional spaces):

(H3) \( b \in L^\infty\text{loc}([0, \infty), C^\theta_b(\mathbb{R}^d, \mathbb{R}^d)) \) and \( \sigma \in L^\infty\text{loc}([0, \infty), C^{1+\theta}_b(\mathbb{R}^d, \mathbb{R}^d \otimes \mathbb{R}^d)) \) for some constant \( \theta \in (0,1) \);

(H4) for any \( (t,x) \in [0, \infty) \times \mathbb{R}^d \), the inverse of \( a(t,x) := \sigma(t,x)\sigma(t,x)^* \) exists, and

\[
\sup_{t \in [0,T], x \in \mathbb{R}^d} \|a^{-1}(t,x)\|_{HS} < \infty \quad \text{for all } T > 0,
\]

where \( \| \cdot \|_{HS} \) is the Hilbert–Schmidt norm of matrices.

Under these conditions we shall prove

**Theorem 1.2.** Assume the hypotheses (H3) and (H4). Let \( P_t \) be the semigroup associated to the Itô SDE (1.1). Then, for any \( T > 0 \), there are three positive constants \( K, \kappa \) and \( \delta \) such that
(1) for any $0 < t \leq T$ and $f \in B_b(\mathbb{R}^d)$ with $f \geq 1$, it holds
\[
P_t \log f(y) \leq \log P_t f(x) + \frac{2K|x - y|^2}{\kappa^2(1 - e^{-Kt})} \quad \text{for all } x, y \in \mathbb{R}^d;
\]
(1.7)

(2) for $p > (1 + \delta / \kappa)^2$ and $\delta_p := \max\{\delta, \kappa(\sqrt{p} - 1)/2\}$, it holds
\[
(P_t f(y))^p \leq (P_t f^p(x)) \exp \left[ \frac{K\sqrt{p}(\sqrt{p} - 1)|x - y|^2}{\delta_p((\sqrt{p} - 1)\kappa - \delta_p)(1 - e^{-Kt})} \right] \quad \text{for all } f \in B_b^+(\mathbb{R}^d), x, y \in \mathbb{R}^d, 0 < t \leq T.
\]
(1.8)

Unfortunately, the explicit expressions of the constants $K, \kappa$ and $\delta$ are a little complicated, as can be seen from the proof in Section 3. To point out the main difference between the Hölder continuous situation and the Lipschitz continuous setting, we first explain the idea of coupling for (1.1) with semi-Lipschitz continuous drift. For simplicity, we consider the time-independent case where $\sigma = \text{Id}$ and for some $K \in \mathbb{R}$,
\[
\langle b(x) - b(y), x - y \rangle \leq K|x - y|^2, \quad x, y \in \mathbb{R}^d.
\]
For $x \neq y \in \mathbb{R}^d$ and $T > 0$, let $X_t$ solve (1.1) with $X_0 = x$, and $Y_t$ solve
\[
dY_t = dB_t + b(Y_t) dt + \frac{(X_t - Y_t)|x - y|e^{-Kt}}{|X_t - Y_t|_0^T e^{-2Ks} ds} dt, \quad Y_0 = y.
\]
Then, $Y_t$ is well defined up to the coupling time
\[
\tau = \inf\{t \geq 0 : X_t = Y_t\}.
\]
Let $Y_t = X_t$ for $t \geq \tau$. We have
\[
d|X_t - Y_t| \leq K|X_t - Y_t| dt - \frac{|x - y|e^{-Kt}}{\int_0^T e^{-2Ks} ds} dt, \quad t \leq \tau.
\]
That is,
\[
d(|X_t - Y_t|e^{-Kt}) \leq -\frac{|x - y|e^{-2Kt}}{\int_0^T e^{-2Ks} ds} dt, \quad t \leq \tau.
\]
This implies $\tau \leq T$ and hence, $X_T = Y_T$. Combining it with the Girsanov theorem yields the desired Harnack inequalities; see for instance the proof of [1, Theorem 2] or that of [14, Theorem 1.1]. However, due to the poor Hölder regularity of the drift vector field $b$, it seems that in the present setting one cannot directly use the coupling method above to establish the Harnack inequalities.

Now we briefly describe our strategy to help the readers understand better the proof of Harnack inequalities with Hölder continuous drift. Following the ideas in the proof of [7, Theorem 7], we can transform the equation (1.1) into a new SDE (3.5) which has smooth coefficients with bounded derivatives; moreover, there is a simple relationship between their corresponding semigroups (see (3.6) below). For this new equation (3.5), we can check that the assumptions (A1)–(A3) in [14] are satisfied under our hypotheses (H3)–(H4). In this way we first get Harnack inequalities for the semigroup associated with the new equation (3.5), then the relationship (3.6) between the semigroups allows us to prove Theorem 1.2.
This paper is organized as follows. By making use of the $L^2$-gradient estimate in [20, p.1109], we establish in Section 2 the log-Harnack inequality (1.6) by applying the semigroup interpolation scheme (see e.g. [2] for intensive studies on Markov Triples) and the Zvonkin transformation. In Section 3, we first recall some necessary results from the references [7, 14], then the main part is devoted to check that the coefficients of the transformed SDE (3.5) verify the hypotheses (A1)–(A3) in [14]. With the key relation (3.6) in hand, it is easy to give the proof of Theorem 1.2. Finally, by using explicit heat kernel estimates, we establish in Section 4 the Harnack inequality for the SDE driven by $\alpha$-stable process.

2 Log-Harnack inequality for SDE with LPS-type drift

This section is devoted to the proof of Theorem 1.1. We shall first prove the log-Harnack inequality (1.6) for the semigroup associated to the following Itô SDE without drift:

$$dY_t = \sigma(t, Y_t) dW_t, \quad Y_0 = x,$$

where $\sigma$ verifies ($H^1_\sigma$) and ($H^2_\sigma$). In the sequel, we denote by $T_{s,t}$ the two-parameter semigroup associated to (2.1) defined by

$$T_{s,t}f(x) = \mathbb{E}\{f(Y_t) | Y_s = x\}, \quad 0 \leq s \leq t.$$

For simplicity, we set $T_t f(x) = T_{0,t} f(x)$. For $t \geq 0$, define the time-dependent second order differential operator associated with $Y_t$ as follows

$$L_t f(x) = \frac{1}{2} \text{Tr} \left[ a(t,x) \nabla^2 f(x) \right], \quad f \in C^2_b(\mathbb{R}^d),$$

where $a(t,x) = \sigma(t,x) \sigma(t,x)^*$ and $\nabla^2 f$ is the Hessian matrix of $f$. Then we have the well-known Kolmogorov equations:

$$\partial_s T_{s,t} f = -L_s T_{s,t} f, \quad \partial_t T_{s,t} f = T_{s,t} L_t f,$$

where $\partial_s = \frac{\partial}{\partial s}$. For $f, g \in C^2(\mathbb{R}^d)$, define

$$\Gamma(t)(f,g) = \frac{1}{2} \{L_t (fg) - gL_t f - fL_t g\},$$

and set $\Gamma(t)(f) = \Gamma(t)(f,f)$ for short. Then

$$\Gamma(t)(f,g) = \frac{1}{2} \langle \sigma(t,x)^* \nabla f, \sigma(t,x)^* \nabla g \rangle.$$

Let $\{\rho_n\}_{n \geq 1}$ be a family of mollifiers on $\mathbb{R}^d$ and set $\sigma^n(t,x) = (\sigma(t,\cdot) * \rho_n)(x)$, $n \geq 1$. We consider the following Itô SDE with smooth coefficient:

$$dY^n_t = \sigma^n(t, Y^n_t) dW_t, \quad Y_0^n = x.$$

Following the arguments in the proof of [20, Theorem 3.5], we can show that

$$C_1 := \sup_{n \geq 1} \sup_{0 \leq t \leq T} \sup_{x \in \mathbb{R}^d} \mathbb{E}(|\nabla Y^n_t(x)|^2) < +\infty, \quad T > 0.$$

(2.2)

Then for any $f \in C^1_b(\mathbb{R}^d)$, $x, y \in \mathbb{R}^d$ and $t > 0$, by the mean value formula,

$$f(Y^n_t(x)) - f(Y^n_t(y)) = \int_0^1 \langle \nabla f [Y^n_t(y + r(x-y))], [\nabla Y^n_t(y + r(x-y))] (x-y) \rangle dr.$$
Therefore,

$$|f(Y_t^n(x)) - f(Y_t^n(y))| \leq |x - y| \int_0^1 |\nabla f [Y_t^n(y + r(x - y))]| \cdot |\nabla Y_t^n(y + r(x - y))| \, dr.$$  

Cauchy’s inequality and (2.2) imply that for any $f \in C_b^1(\mathbb{R}^d)$, $x, y \in \mathbb{R}^d$ and $0 < t \leq T$,

$$|\mathbb{E}f(Y_t^n(x)) - \mathbb{E}f(Y_t^n(y))| \leq \sqrt{C_1} |x - y| \int_0^1 \left( \mathbb{E}|\nabla f [Y_t^n(y + r(x - y))]|^2 \right)^{1/2} \, dr. \quad (2.3)$$

Moreover, by [20, (3.7)], we have

$$\lim_{n \to \infty} \mathbb{E}|Y_t^n(x) - Y_t(x)| = 0.$$

Thus, by the dominated convergence theorem, letting $n$ tend to $\infty$ in (2.3) yields that for any $f \in C_b^1(\mathbb{R}^d)$, $x, y \in \mathbb{R}^d$ and $0 < t \leq T$,

$$|\mathbb{E}f(Y_t(x)) - \mathbb{E}f(Y_t(y))| \leq \sqrt{C_1} |x - y| \int_0^1 \left( \mathbb{E}|\nabla f [Y_t(y + r(x - y))]|^2 \right)^{1/2} \, dr.$$

Now we let $y \to x$ and obtain

$$|\nabla T_t f(x)|^2 \leq C_1 T_t |\nabla f|^2(x), \quad x \in \mathbb{R}^d, 0 < t \leq T.$$

Similarly, we have for all $0 \leq s \leq t \leq T$,

$$|\nabla T_{s,t} f(x)|^2 \leq C_1 T_{s,t} |\nabla f|^2(x), \quad x \in \mathbb{R}^d. \quad (2.4)$$

Now standard arguments lead to the log-Harnack inequality for the semigroup $T_{s,t}$.

**Proposition 2.1.** Assume that $\sigma$ verifies (H$_1^\sigma$) and (H$_2^\sigma$). Then for any $T > 0$, there is a constant $C_1 > 0$ such that for all $f \in B_b(\mathbb{R}^d)$ with $f \geq 1$,

$$T_{s,t} \log f(y) \leq \log T_{s,t} f(x) + \frac{C_1 |y - x|^2}{2\delta(t - s)}, \quad x, y \in \mathbb{R}^d, s \leq t, \quad (2.5)$$

where $\delta$ is the constant in (H$_1^\sigma$).

**Proof.** Take $f \geq 1$. Applying Itô’s formula, we have

$$d \log T_{u,t} f(Y_u) = \langle \nabla \log T_{u,t} f(Y_u), \sigma(u, Y_u) \, dW_u \rangle + L_u \log T_{u,t} f(Y_u) \, du - \frac{L_u T_{u,t} f(Y_u)}{T_{u,t} f(Y_u)} \, du$$

$$= \langle \nabla \log T_{u,t} f(Y_u), \sigma(u, Y_u) \, dW_u \rangle - \frac{\Gamma(u)(T_{u,t} f)(Y_u)}{(T_{u,t} f)^2(Y_u)} \, du,$$

where the last equality follows by

$$L_u \log T_{u,t} f = \frac{L_u T_{u,t} f}{T_{u,t} f} - \frac{\Gamma(u)(T_{u,t} f)}{(T_{u,t} f)^2}.$$

Then, by integrating from $s$ to $u$, we get

$$\log T_{u,t} f(Y_u) - \log T_{s,t} f(Y_s) = \int_s^u \langle \nabla \log T_{r,t} f(Y_r), \sigma(r, Y_r) \, dW_r \rangle - \int_s^u \frac{\Gamma(r)(T_{r,t} f)(Y_r)}{(T_{r,t} f)^2(Y_r)} \, dr.$$  

6
Taking expectation with respect to \( \{Y_s = x\} \), we have
\[
T_{s,u} \log T_{u,t} f(x) - \log T_{s,t} f(x) = - \int_s^u T_{s,r} \left( \frac{\Gamma(r)(T_{r,t} f)}{(T_{r,t} f)^2} \right)(x) \, dr, \quad u \in [s,t].
\] (2.6)

Now for \( x, y \in \mathbb{R}^d \), let \( \gamma_u = (y - x)\frac{u - s}{t - s} + x \). The identity (2.6) implies that \([s,t] \ni u \mapsto T_{s,u} \log T_{u,t} f(\gamma_u)\) is absolutely continuous; thus by \((H^p)\) and the definition of \( \Gamma(u) \), for any \( 0 \leq s \leq t \leq T \), we have
\[
\frac{d}{du} T_{s,u} \log T_{u,t} f(\gamma_u) = - T_{s,u} \left( \frac{\Gamma(u)(T_{u,t} f)}{(T_{u,t} f)^2} \right)(\gamma_u) + \langle \nabla (T_{s,u} \log T_{u,t} f)(\gamma_u), \gamma_u \rangle \\
\leq - \frac{\delta}{2} T_{s,u} \| \nabla \log T_{u,t} f \|^2(\gamma_u) + \| \gamma_u \| \left( C_1 T_{s,u} \| \nabla \log T_{u,t} f \|^2(\gamma_u) \right)^{\frac{1}{2}} \\
\leq \frac{C_1}{2\delta} \| \gamma_u \|^2, \quad \mathcal{L}^1\text{-a.e. } u \in [s,t],
\]
where in the first inequality we have used (2.4). Integrating from \( s \) to \( t \) gives us the log-Harnack inequality (2.5).

It remains to transfer the above result to the general Itô SDE (1.1) with drift. Before moving on, we introduce two function spaces: for \( p, q \geq 1 \) and \( s < t \), let
\[
L^q_p(s,t) = L^q([s,t], L^p(\mathbb{R}^d)) \quad \text{and} \quad \mathbb{H}^q_{2,p}(s,t) = L^q([s,t], W^{2,p}(\mathbb{R}^d)),
\]
where \( W^{2,p}(\mathbb{R}^d) \) is the standard Sobolev space.

We shall need the following preparations which are taken from [20, pp.1110–1111]. Assume that \( \sigma \) satisfies \((H^p)\) and \( b \in L^q_0(0,T) \) with \( p, q \) verifying (1.2) for any \( T > 0 \). Fix \( 0 < T_0 \leq T \). For any \( 0 \leq s < t \leq T \) with \( t - s \leq T_0 \), let \( (u(r,x))_{s \leq r \leq t} \) with \( u(r,x) := (u^1(r,x), \ldots, u^d(r,x)) \) be the solution to the backward parabolic equation
\[
\partial_r u^i(r,x) + L_r u^i(r,x) + b^i(r,x) = 0, \quad u^i(t,x) = 0, \quad 1 \leq i \leq d,
\]
where
\[
L_r u^i(r,x) = \frac{1}{2} \text{Tr} [a(r,x) \nabla^2 u^i(r,x)] + b(r,x) \nabla u^i(r,x),
\]
\( a(r,x) = \sigma(r,x) \sigma(r,x)^* \) and \( b(r,x) := (b^1(r,x), \ldots, b^d(r,x)) \). Then by [20, Theorem 5.1] (see also [8, Theorem 10.3 and Remark 10.4]), one has
\[
C_2 := \sup_{s \in [0, t-T_0], t} \left( \| \partial_r u \|_{L^q_p(s,t)} + \| u \|_{\mathbb{H}^q_{2,p}(s,t)} \right) < +\infty. \tag{2.7}
\]

It follows from [8, Lemma 10.2] that the function \((r,x) \mapsto \nabla u(r,x)\) is Hölder continuous and for fixed \( \delta \in (0, \frac{1}{2} - \frac{d}{2p} - \frac{1}{q}) \), there exists a constant \( C_3 > 0 \) depending on \( p, q, \delta \) and \( T \) such that
\[
\sup_{(r,x) \in [s,t] \times \mathbb{R}^d} |\nabla u(r,x)| \leq C_3 T_0^\delta. \tag{2.8}
\]

Define \( \Phi_r(x) = x + u(r,x) \), \((r,x) \in [s,t] \times \mathbb{R}^d \). It is easy to see that
\[
\partial_r \Phi_r(x) + L_r \Phi_r(x) = 0, \quad \Phi_t(x) = x. \tag{2.9}
\]
Moreover, if \( T_0 \) is small enough, we deduce from (2.8) that for all \( r \in [s,t] \),
\[
\frac{1}{2} |x - y| \leq |\Phi_r(x) - \Phi_r(y)| \leq \frac{3}{2} |x - y| \quad \text{for all } x, y \in \mathbb{R}^d. \tag{2.10}
\]
Therefore \( \Phi_r \) is a diffeomorphism on \( \mathbb{R}^d \). The following result is proved in [20, Lemma 4.3].
Lemma 2.2 (Zvonkin transformation). Let $X_r$ be an $\mathbb{R}^d$-valued $(\mathcal{F}_r)_{r \geq 0}$-adapted continuous process satisfying

$$\mathbb{P}\left\{ \omega \in \Omega : \int_s^t \left( |b(r,X_r(\omega))| + |\sigma(r,X_r(\omega))| \right) dr < +\infty \right\} = 1.$$ 

Then $X_r$ solves the equation (1.1) on the time interval $[s,t]$ if and only if $Y_r = \Phi_r(X_r)$ solves the following SDE on $[s,t]$

$$dY_r = \Sigma(r,Y_r) dW_r,$$

where $\Sigma(r,y) = (\nabla \Phi_r \cdot \sigma(r,\cdot)) \circ \Phi_r^{-1}(y)$.

With Proposition 2.1 and Lemma 2.2 in mind, we can now present

Proof of Theorem 1.1. We first check that the matrix valued function $\Sigma$ given in Lemma 2.2 satisfies (H1) and (H2) with $\sigma$ replaced by $\Sigma$. To this end, we fix some $0 < s < t \leq T$ with $t-s \leq T_0$. By (2.10),

$$\frac{1}{2} \leq |\nabla \Phi_r(x)| \leq \frac{3}{2} \quad \text{for all } (r,x) \in [s,t] \times \mathbb{R}^d.$$ 

From the definition of $\Sigma$ and (H1), we deduce that

$$\frac{1}{4} |y|^2 \leq |\Sigma(r,x)^* y|^2 \leq \frac{9}{4} |y|^2 \quad \text{for all } (r,x) \in [s,t] \times \mathbb{R}^d \text{ and } y \in \mathbb{R}^d.$$ 

Thus (H1) holds with new constants $\frac{1}{4}$ and $\frac{9}{4}$. For the second condition (H2), we note that

$$\partial_t \Sigma_{ik}(r,y) = \left[ (\partial_t \partial_j \Phi^i_r \cdot \sigma^j_k(r,\cdot) + \partial_j \Phi^i_r \cdot \partial_t \sigma^j_k(r,\cdot)) \circ \Phi_r^{-1}(y) \right] \cdot \partial_t \Phi_r^{-1}(y).$$

By (2.7), (2.10) and (H2), we conclude that $\| \partial_t \Sigma_{ik} \|_{L^2([s,t])} < +\infty$. That is, (H2) also holds on the small interval $[s,t]$.

Denote by $\tilde{T}_{s,t}$ the semigroup associated to the new SDE (2.11) without drift. We can apply Proposition 2.1 to obtain that, for any $f \in \mathcal{B}_0(\mathbb{R}^d)$ with $f \geq 1$ and any $0 \leq s < t \leq T$ with $t-s \leq T_0$, it holds

$$\tilde{T}_{s,t} \log f(y) \leq \log \tilde{T}_{s,t} f(x) + \frac{\tilde{C}_1 |y-x|^2}{\delta(t-s)}, \quad x,y \in \mathbb{R}^d,$$ 

where $\tilde{C}_1 > 0$ is some constant. We have to transfer the above log-Harnack inequality (2.12) to the semigroup $P_{s,t}$ associated to (1.1). This process is summarized in the next result.

Lemma 2.3. For any $f \in \mathcal{B}_0(\mathbb{R}^d)$ with $f \geq 1$ and any $0 \leq s < t \leq T$ with $t-s \leq T_0$, it holds

$$P_{s,t} \log f(y) \leq \log P_{s,t} f(x) + \frac{\tilde{C}_1 |y-x|^2}{\delta(t-s)} \quad \text{for all } x,y \in \mathbb{R}^d.$$ 

Proof. Fix $0 \leq s < t \leq T$ with $t-s \leq T_0$. For $g \in \mathcal{B}_0(\mathbb{R}^d)$, by the definition of the semigroup $P_{s,t}$ and Lemma 2.2, we have

$$P_{s,t} g(x) = \mathbb{E}(g(X_t) | X_s = x) = \mathbb{E}(g(\Phi_t^{-1}(Y_t)) | \Phi_s^{-1}(Y_s) = x) = \mathbb{E}(g(Y_t) | Y_s = \Phi_s(x)) = \tilde{T}_{s,t} g(\Phi_s(x)).$$


where the third equality follows from (2.9). Therefore, for \( f \in \mathcal{B}_b(\mathbb{R}^d) \) with \( f \geq 1 \), by (2.12),

\[
P_{s,t} \log f(y) = \tilde{T}_{s,t} \log f(\Phi_s(y)) \leq \log \tilde{T}_{s,t} f(\Phi_s(x)) + \frac{\tilde{C}_1|y - x|^2}{\delta(t - s)}
\]

\[
= \log P_{s,t} f(x) + \frac{\tilde{C}_1|y - x|^2}{\delta(t - s)},
\]

which is the desired inequality.

We continue the proof of Theorem 1.1. It remains to extend the above result to the case where \( t - s > T_0 \), which follows from the semigroup property. If \( t - s \in (T_0, 2T_0) \), then by the semigroup property and Jensen’s inequality, we have

\[
P_{s,t} \log f(y) = P_{s+T_0,t} \left( P_{s+T_0,0} \log f \right)(y) \leq P_{s+T_0,t} \left[ \log \left( P_{s+T_0,0} f \right) \right](y)
\]

\[
\leq \log \left[ P_{s+T_0,t} \left( P_{s+T_0,0} f \right) \right](x) + \frac{\tilde{C}_1|y - x|^2}{\delta(t - s)}
\]

\[
= \log \left( P_{s,t} f \right)(x) + \frac{\tilde{C}_1|y - x|^2}{\delta(t - s)},
\]

where in the second inequality we have used (2.13). We complete the proof by repeating this procedure.

\[\square\]

3 Harnack inequalities for SDEs with Hölder continuous drift

In this section, we present the proof of Theorem 1.2. In the first subsection, we give some notations of spaces of spatially Hölder continuous functions and preliminary results, then we shall prove Theorem 1.2 in the second subsection.

3.1 Notations and preliminary results

We adopt the notations in [6, p.7]. Let \( T > 0 \) and \( \theta \in (0, 1) \) be fixed. Define the space \( L^\infty([0, T], C^\theta_b(\mathbb{R}^d)) \) as the set of all bounded Borel functions \( f : [0, T] \times \mathbb{R}^d \to \mathbb{R} \) for which

\[
[f]_{\theta,T} = \sup_{t \in [0, T]} \sup_{x \neq y \in \mathbb{R}^d} \frac{|f(t, x) - f(t, y)|}{|x - y|^\theta} < +\infty. \tag{3.1}
\]

This is a Banach space with respect to the usual norm \( \|f\|_{\theta,T} = \|f\|_{0,T} + [f]_{\theta,T} \) where \( \|f\|_{0,T} = \sup_{(t,x) \in [0,T] \times \mathbb{R}^d} |f(t, x)| \) is the supremum norm. If \( f \) is vector-valued or matrix-valued, then we simply replace the absolute value in the numerator of (3.1) and in the definition of \( \|f\|_{0,T} \) by the Euclidean norm or Hilbert–Schmidt norm, which gives us the spaces \( L^\infty([0, T], C^\theta_b(\mathbb{R}^d, \mathbb{R}^d)) \) and \( L^\infty([0, T], C^\theta_b(\mathbb{R}^d, \mathbb{R}^d \otimes \mathbb{R}^d)) \) respectively.

Moreover, for \( n \geq 1 \), \( f \in L^\infty([0, T], C^n_b+\theta(\mathbb{R}^d)) \) if all spatial partial derivatives \( \nabla_i \ldots \nabla_k f \in L^\infty([0, T], C^\theta_b(\mathbb{R}^d)) \) for all \( k = 0, 1, \ldots, n \), where \( \nabla_j = \frac{\partial}{\partial x_j} \). The corresponding norm is defined as

\[
\|f\|_{n+\theta,T} = \|f\|_{0,T} + \sum_{k=1}^{n} \|\nabla^k f\|_{0,T} + [\nabla^n f]_{\theta,T},
\]

in which we have extended the previous notations \( \|\cdot\|_{0,T} \) and \( [\cdot]_{\theta,T} \) to tensors. In the same way, we can define the spaces \( L^\infty([0, T], C^n_b+\theta(\mathbb{R}^d, \mathbb{R}^d)) \) and \( L^\infty([0, T], C^n_b+\theta(\mathbb{R}^d, \mathbb{R}^d \otimes \mathbb{R}^d)) \), and the associated norms. We can also extend these function spaces to \( T = \infty \) (we are considering
functions defined on $[0, \infty) \times \mathbb{R}^d$), and the corresponding norms are simply denoted by $\| \cdot \|_0$, $\| \cdot \|_{n+\theta}$ and so on. These norms will also be used for functions in the spaces $C_b^{n+\theta}(\mathbb{R}^d)$, $C_b^{1+\theta}(\mathbb{R}^d, \mathbb{R}^d)$ and $C_b^{n+\theta}(\mathbb{R}^d, \mathbb{R}^d \otimes \mathbb{R}^d)$ for all $n \geq 0$, which are independent of time. There will be no confusion according to the context.

We now recall the main result in [14] which will play an important role in the proof of Theorem 1.2. To this end, let $\sigma : [0, \infty) \times \mathbb{R}^d \to \mathbb{R}^d \otimes \mathbb{R}^d$ and $b : [0, \infty) \times \mathbb{R}^d \to \mathbb{R}^d$ be two Borel measurable functions. We first list some assumptions which are taken from [14, Introduction]:

(A1) for any $T > 0$, there exists a constant $K_0 > 0$ such that

$$\|\sigma(t,x) - \sigma(t,y)\|_{HS}^2 + 2(b(t,x) - b(t,y), x - y) \leq K_0|x - y|^2, \quad t \in [0,T], x, y \in \mathbb{R}^d;$$

(A2) for any $T > 0$, there is a constant $\kappa_0 > 0$ such that

$$a(t,x) = \sigma(t,x)\sigma(t,x)^T \geq \kappa_0^2 \text{Id}, \quad t \in [0,T], x \in \mathbb{R}^d;$$

(A3) for any $T > 0$, there is a constant $\delta_0 \geq 0$ such that

$$|\langle \sigma(t,x) - \sigma(t,y), x - y \rangle| \leq \delta_0|x - y|, \quad t \in [0,T], x, y \in \mathbb{R}^d.$$

It is well known that assumption (A1) ensures the pathwise uniqueness of solutions to (1.1). For the moment, we assume that SDE (1.1) has a unique strong solution $X_t$. For the associated semigroup. F.-Y. Wang has shown in [14, Theorem 1.1] the following results on the Harnack inequalities for the semigroup $P_t$.

**Theorem 3.1.**

(1) If (A1) and (A2) hold, then

$$P_t \log f(y) \leq \log P_t f(x) + \frac{K_0|x - y|^2}{2\kappa_0^2(1 - e^{-K_0t})}, \quad f \in B_b(\mathbb{R}^d) \text{ with } f \geq 1, x, y \in \mathbb{R}^d, 0 < t \leq T.$$

(2) If (A1), (A2) and (A3) hold, then for $p > (1 + \delta_0/\kappa_0)^2$ and $\delta_p := \max\{\delta_0, \kappa_0(\sqrt{p} - 1)/2\}$,

$$(P_t f(y))^p \leq (P_t f^p(x)) \exp \left[\frac{K_0\sqrt{p}(\sqrt{p} - 1)|x - y|^2}{4\delta_p(\sqrt{p} - 1)\kappa_0 - \delta_p(1 - e^{-K_0t})}\right]$$

holds for all $f \in B_b^+(\mathbb{R}^d), x, y \in \mathbb{R}^d$ and $0 < t \leq T$.

**3.2 Proof of Theorem 1.2.**

We need some more preparations. Fix any $T > 0$. We redefine $b$ and $\sigma$ on the space $[T, \infty) \times \mathbb{R}^d$ by setting

$$b(t,x) = b(T, x) \text{ and } \sigma(t,x) = \sigma(T, x), \quad t \geq T, \quad x \in \mathbb{R}^d.$$

Obviously, we have $b \in L^\infty([0, \infty), C_b^0(\mathbb{R}^d, \mathbb{R}^d))$ and $\sigma \in L^\infty([0, \infty), C_b^{1+\theta}(\mathbb{R}^d, \mathbb{R}^d \otimes \mathbb{R}^d))$. Given $\lambda > 0$ and a function $f \in L^\infty([0, \infty), C_b^\theta(\mathbb{R}^d, \mathbb{R}^d))$, consider the equation

$$\partial_t u_\lambda + L_t u_\lambda - \lambda u_\lambda = f \quad \text{in } [0, \infty) \times \mathbb{R}^d,$$

(3.2)

where the operator $L_t$ associated to SDE (1.1) is defined by

$$L_t f(t, x) = \frac{1}{2} \text{Tr}[(a(t,x)\nabla^2 f(t,x)) + (b(t,x), \nabla f(t,x))],$$

for some regular enough function $f : [0, \infty) \times \mathbb{R}^d \to \mathbb{R}^d$. Note that the solution to (3.2) is understood in the same sense as in [6, p.10].

By the sketchy arguments of [7, Remark 9], we have
Lemma 3.2. Assume that conditions (H3) and (H4) hold with some constant \( \theta \in (0,1) \). For any \( f \in L^\infty([0,\infty), C^2_b(\mathbb{R}^d, \mathbb{R}^d)) \), there exists a unique solution \( u_\lambda \) to equation (3.2) in the space \( L^\infty([0,\infty), C^{2+\theta}_b(\mathbb{R}^d, \mathbb{R}^d)) \) such that for any \( \lambda \geq 1 \),

\[
\|\nabla u_\lambda\|_0 \leq C \sqrt{\lambda} \|f\|_\theta,
\]

where the constant \( C > 0 \) is independent of \( \lambda \).

Now for \( \lambda > 0 \), consider the parabolic system

\[
\partial_t \psi_\lambda + L_1 \psi_\lambda - \lambda \psi_\lambda = b, \quad (t,x) \in [0,\infty) \times \mathbb{R}^d.
\]

(3.3)

By Lemma 3.2, there exists a unique solution \( \psi_\lambda \in L^\infty([0,\infty), C^{2+\theta}_b(\mathbb{R}^d, \mathbb{R}^d)) \) to the equation (3.3). Define \( \Psi_\lambda(t,x) = x + \psi_\lambda(t,x) \) for \( t \geq 0 \) and \( x \in \mathbb{R}^d \). Then we have

Lemma 3.3. For \( \lambda \) large enough such that \( \|\nabla \psi_\lambda\|_0 < 1 \), the following statements hold:

(i) \( \Psi_\lambda \) has bounded first and second order spatial derivatives uniformly in \( t \in [0,\infty) \) and, moreover, the second order derivative \( \nabla^2 \Psi_\lambda \) is globally \( \theta \)-Hölder continuous uniformly in \( t \in [0,\infty) \);

(ii) for any \( t \geq 0 \), \( \Psi_\lambda(t, \cdot) \) is a \( C^2 \)-diffeomorphism of \( \mathbb{R}^d \);

(iii) \( \Psi_\lambda^{-1}(t, \cdot) \) has bounded first and second order spatial derivatives uniformly in \( t \in [0,\infty) \) and, moreover,

\[
\nabla \Psi_\lambda^{-1}(t, \cdot)(x) = \sum_{k \geq 0} \left( -\nabla \psi_\lambda(t, \Psi_\lambda^{-1}(t, \cdot)(x)) \right)^k, \quad x \in \mathbb{R}^d, t \geq 0.
\]

We choose \( \lambda \) large enough such that

\[
\|\nabla \psi_\lambda\|_0 \leq \frac{1}{2}.
\]

To simplify the notations, we shall omit the subscript \( \lambda \) and write \( \psi_t(x) \) (resp. \( \Psi_t(x) \)) instead of \( \psi(t,x) \) (resp. \( \Psi_t(x) \)). For \( t \geq 0 \) and \( x \in \mathbb{R}^d \), set

\[
\hat{\sigma}(t,x) = \nabla \Psi_t(\Psi_t^{-1}(x)) \sigma(t, \Psi_t^{-1}(x)), \quad \hat{b}(t,x) = \lambda \psi_t(\Psi_t^{-1}(x)).
\]

(3.4)

Consider the SDE on \( \mathbb{R}^d \):

\[
d\hat{X}_t = \hat{\sigma}(t, \hat{X}_t) \, dW_t + \hat{b}(t, \hat{X}_t) \, dt, \quad \hat{X}_0 = y.
\]

(3.5)

This equation is equivalent to (1.1) in the sense that if \( X_t \) is a solution to (1.1), then \( \hat{X}_t = \Psi_t(X_t) \) satisfies (3.5) with \( y = \Psi_0(x) \); conversely, if \( \hat{X}_t \) is a solution to (3.5), then \( X_t = \Psi_t^{-1}(\hat{X}_t) \) solves (1.1) with \( x = \Psi_0^{-1}(y) \). From this we also deduce the relationship between their semigroups. Indeed, let \( \hat{P}_t \) be the semigroup associated to (3.5), then for any \( f \in B_b(\mathbb{R}^d) \), we have

\[
\hat{P}_tf(x) = \mathbb{E}f(X_t(x)) = \mathbb{E}\left[ f(\Psi_t^{-1}(\Psi_t(X_t(x)))) \right]
\]

\[
= \mathbb{E}\left[ (f \circ \Psi_t^{-1})(\hat{X}_t(\Psi_0(x))) \right] = \hat{P}_t(f \circ \Psi_t^{-1})(\Psi_0(x)).
\]

(3.6)

Now by Lemma 3.3, we can verify that the assumptions (A1)–(A3) in Subsection 3.1 are satisfied by \( \hat{\sigma} \) and \( \hat{b} \). We collect the computations in the next lemma.
Lemma 3.4. Under the hypotheses (H3) and (H4), the coefficients $\hat{\sigma}$ and $\hat{b}$ given by (3.4) satisfy the assumptions (A1)-(A3) in Theorem 3.1. More precisely, for any $T > 0$, there exist positive constants $K_1, \kappa_1$ and $\delta_1$ such that for all $0 \leq t \leq T$ and $x, y \in \mathbb{R}^d$, it holds

1. $\|\hat{\sigma}(t, x) - \hat{\sigma}(t, y)\|_{HS}^2 + 2\langle \hat{b}(t, x) - \hat{b}(t, y), x - y \rangle \leq K_1|x - y|^2$;
2. $\hat{a}(t, x) = \hat{\sigma}(t, x)\hat{\sigma}(t, x)^* \geq \kappa_1^2 \text{Id}$;
3. $|\langle \hat{\sigma}(t, x) - \hat{\sigma}(t, y), (x - y) \rangle| \leq \delta_1|x - y|$.

Proof. (1) For $x, y \in \mathbb{R}^d$,

$$∥\hat{\sigma}(t, x) - \hat{\sigma}(t, y)∥_{HS} = ∥\nabla\Psi_t(\Psi_t^{-1}(x))\sigma(t, \Psi_t^{-1}(x)) - \nabla\Psi_t(\Psi_t^{-1}(y))\sigma(t, \Psi_t^{-1}(y))∥_{HS} \leq ∥\nabla\Psi_t(\Psi_t^{-1}(x)) - \nabla\Psi_t(\Psi_t^{-1}(y))∥_{HS}∥\sigma(t, \Psi_t^{-1}(x)) - \sigma(t, \Psi_t^{-1}(y))∥_{HS}.$$ 

Since $\Psi_t(x) = x + \psi_t(x)$, we have $\nabla\Psi_t(x) = \text{Id} + \nabla\psi_t(x)$, and hence

$$∥\nabla\Psi_t(\Psi_t^{-1}(x)) - \nabla\Psi_t(\Psi_t^{-1}(y))∥_{HS} = ∥\nabla\psi_t(\Psi_t^{-1}(x)) - \nabla\psi_t(\Psi_t^{-1}(y))∥_{HS} \leq ∥\nabla^2\psi_t∥_0|\Psi_t^{-1}(x) - \Psi_t^{-1}(y)| \leq ∥\nabla^2\psi_t∥_0∥\nabla\Psi_t^{-1}∥_0|x - y|.$$ 

As $\sup_{t \geq 0}∥\nabla\psi_t∥_0 \leq 1/2$, the assertion (iii) in Lemma 3.3 implies that

$$∥\nabla\Psi_t^{-1}∥_0 \leq \sum_{k \geq 0}∥\nabla\psi_t∥_0^k \leq 2.$$ 

Thus,

$$∥\nabla\Psi_t(\Psi_t^{-1}(x)) - \nabla\Psi_t(\Psi_t^{-1}(y))∥_{HS} \leq 2∥\nabla^2\psi_t∥_0|x - y|.$$ 

Next, it holds that

$$∥\sigma(t, \Psi_t^{-1}(x)) - \sigma(t, \Psi_t^{-1}(y))∥_{HS} \leq ∥\nabla\sigma(t, \cdot)∥_0|\Psi_t^{-1}(x) - \Psi_t^{-1}(y)| \leq ∥\nabla\sigma(t, \cdot)∥_0∥\nabla\Psi_t^{-1}∥_0|x - y| \leq 2∥\nabla\sigma(t, \cdot)∥_0|x - y|.$$ 

Therefore, we have

$$∥\hat{\sigma}(t, x) - \hat{\sigma}(t, y)∥_{HS} \leq 2∥\nabla^2\psi_t∥_0∥\sigma(t, \cdot)∥_0|x - y| + 2∥\nabla\Psi_t∥_0∥\nabla\sigma(t, \cdot)∥_0|x - y| = 2(∥\nabla^2\psi_t∥_0∥\sigma(t, \cdot)∥_0 + ∥\nabla\Psi_t∥_0∥\nabla\sigma(t, \cdot)∥_0)|x - y|.$$ 

On the other hand, by (3.4) and the fact that $∥\nabla\psi_t∥_0∥\nabla\Psi_t^{-1}∥_0 \leq 1$ for all $t \geq 0$,

$$|\langle \hat{b}(t, x) - \hat{b}(t, y), (x - y) \rangle| \leq |\hat{b}(t, x) - \hat{b}(t, y)| |x - y| \leq \lambda∥\nabla\psi_t∥_0|\Psi_t^{-1}(x) - \Psi_t^{-1}(y)| |x - y| \leq \lambda∥\nabla^2\psi_t∥_0∥\nabla\Psi_t^{-1}∥_0|x - y| \leq \lambda|x - y|^2.$$ 

Combining all the estimates above, we get the desired estimate (A1) with

$$K_1 = 4\sup_{t \geq 0}(∥\nabla^2\psi_t∥_0∥\sigma(t, \cdot)∥_0 + ∥\nabla\Psi_t∥_0∥\nabla\sigma(t, \cdot)∥_0)^2 + 2\lambda < +\infty.$$
(2) For any \( x, z \in \mathbb{R}^d \), it follows from the definition of \( \hat{\sigma} \) that
\[
\langle \hat{\sigma}(t, x) \hat{\sigma}(t, x)^* z, z \rangle = \langle \nabla \Psi_t(\Psi_t^{-1}(x)) \sigma(t, \Psi_t^{-1}(x)) \sigma(t, \Psi_t^{-1}(x))^* [\nabla \Psi_t(\Psi_t^{-1}(x))]^* z, z \rangle \\
= \langle a(t, \Psi_t^{-1}(x)) [\nabla \Psi_t(\Psi_t^{-1}(x))]^* z, [\nabla \Psi_t(\Psi_t^{-1}(x))]^* z \rangle.
\]
Denote by \( \lambda_i(t, x), 1 \leq i \leq d \), the eigenvalues of \( a(t, x) \). Under the hypothesis (H4), we have \( \lambda_i(t, x) > 0 \) for all \( i \in \{1, \ldots, d\} \) and \( (t, x) \in [0, \infty) \times \mathbb{R}^d \). By Cauchy’s inequality,
\[
\sum_{i=1}^d \frac{1}{\lambda_i(t, x)} = \text{Tr}(a^{-1}(t, x)) \leq \sqrt{d} \left( \sum_{i=1}^d (a^{-1}_{ii}(t, x))^2 \right)^{1/2} \\
\leq \sqrt{d} \|a^{-1}(t, x)\|_{HS} \leq \sqrt{d} \|a^{-1}\|_0.
\]
Hence
\[
\inf_{(t, x) \in [0, \infty) \times \mathbb{R}^d} \lambda_i(t, x) \geq \frac{1}{\sqrt{d} \|a^{-1}\|_0} > 0, \quad 1 \leq i \leq d.
\]
As a consequence,
\[
\langle \hat{\sigma}(t, x) \hat{\sigma}(t, x)^* z, z \rangle \geq \frac{1}{\sqrt{d} \|a^{-1}\|_0} \|\nabla \Psi_t(\Psi_t^{-1}(x))^* z\|^2.
\]
Noting that \( \nabla \Psi_t = \text{Id} + \nabla \psi_t \), thus for any \( y \in \mathbb{R}^d \),
\[
\|\nabla \Psi_t(y)^* z\| \leq \|z + [\nabla \psi_t(y)]^* z\| \geq |z| - \|\nabla \psi_t(y)^* z\| \geq |z| - \|\nabla \psi_t(y)\|_{HS} |z| \geq |z|/2,
\]
thanks to \( \|\nabla \psi_t\|_0 = \sup_{y \in \mathbb{R}^d} \|\nabla \psi_t(y)\|_{HS} \leq 1/2 \) for all \( t \geq 0 \). Therefore
\[
\langle \hat{\sigma}(t, x) \hat{\sigma}(t, x)^* z, z \rangle \geq \frac{|z|^2}{4\sqrt{d} \|a^{-1}\|_0},
\]
which means that the condition (A2) holds with \( \kappa_1 = (4\sqrt{d} \|a^{-1}\|_0)^{-1/2} \).

(3) Since \( \sigma \) is bounded, it is obvious that for any \( (t, x) \in [0, \infty) \times \mathbb{R}^d \),
\[
\|\hat{\sigma}(t, x)\|_{HS} \leq \|\nabla \Psi_t(\Psi_t^{-1}(x))\|_{HS} \|\sigma(t, \Psi_t^{-1}(x))\|_{HS} \leq (d + \|\nabla \psi_t\|_0) \|\sigma\|_0 \leq (d + 1/2) \|\sigma\|_0.
\]
Then the condition (A3) holds with \( \delta_1 = (2d + 1) \|\sigma\|_0 \).

Now we are ready to prove Theorem 1.2.

**Proof of Theorem 1.2.** By Lemma 3.4 and Theorem 3.1, for any fixed \( T > 0 \) we know that the semigroup \( \hat{P}_t \) corresponding to the SDE (3.5) satisfies the log-Harnack inequality:
\[
\hat{P}_t \log g(y) \leq \log \hat{P}_t g(x) + \frac{K_1 |x - y|^2}{2\kappa_1^2 (1 - e^{-\kappa_1 t})}, \quad 0 < t \leq T, g \in B_b(\mathbb{R}^d) \text{ with } g \geq 1, x, y \in \mathbb{R}^d, \quad (3.7)
\]
and the Harnack inequality with power: for \( p > (1 + \delta_1/\kappa_1)^2 \) and \( \delta_p \) := \max\{\delta_1, \kappa_1(\sqrt{p} - 1)/2\},
\[
(\hat{P}_t g(y))^p \leq (\hat{P}_t g^p(x)) \exp \left[ \frac{K_1 \sqrt{p(\sqrt{p} - 1)} |x - y|^2}{4\delta_p ((\sqrt{\delta_p(\sqrt{p} - 1)} - \kappa_1 - \delta_p(1 - e^{-\kappa_1 t}))} \right]
\]
for all \( 0 < t \leq T, g \in B_b^+(\mathbb{R}^d) \) and \( x, y \in \mathbb{R}^d \). Now we shall apply the relation (3.6) to show that the semigroup \( P_t \) associated with the SDE (1.1) also satisfies the same Harnack inequalities (possibly with different constants).
we did not obtain useful estimates on the time-change factor. After checking the proofs of [6, Theorem 2 and Lemma 4], unfortunately, in this section, we consider the following SDE

\[ 4 \text{ Harnack Inequalities for SDEs Driven by Symmetric Stable Processes} \]

In this section, we consider the following SDE

\[ \text{Theorem 1.1, } \text{Priola proved that the solution} X_t \text{ for the constant} \]

\[ \text{motion. In this case, since the vector field} \]

\[ \text{A possible approach to bypass the difficulty is to use the regularization approximations of the underlying subordinator; see [16] for the recent study of dimension-free Harnack inequalities for a class of stochastic equations driven by a Lévy noise containing a subordiante Brownian motion. In this case, since the vector field} b \text{ is time-independent, the drift coefficient in the new SDE is variable-separated, and the diffusion coefficient is the identity matrix (cf. [21, (2.3)] or [16, (3.2)]). At first glance, it seems that one can establish the Harnack inequalities in the same way as above by using the transform in [6, page 14]. However, this requires an explicit expression for the constant} C \text{ in the Schauder estimate (see [6, Theorem 2]), especially its dependence on the time-change factor. After checking the proofs of [6, Theorem 2 and Lemma 4], unfortunately, we did not obtain useful estimates on } C. \]

On the other hand, by using the explicit heat kernel estimates, we can prove

\[ \text{Proposition 4.1. There exists a constant} C > 1 \text{ such that for any} f \in B_0^+(\mathbb{R}^d), T > 0 \text{ and } \]

\[ P_T f(x) \leq C \left( 1 + \frac{|x - y|}{(T \wedge 1)^{1/\alpha}} \right)^{d+\alpha} P_T f(y). \]
Proof. Note that the drift term $b$ is bounded, and so it belongs to the Kato class (see e.g. [4, Definition 1.1]). Then, for $\alpha > 1$, according to [4, Corollary 1.3], the process $X_t$ has a jointly continuous transition density function $p(t, x, y)$ with respect to the Lebesgue measure on $\mathbb{R}^d$. Moreover, there exists a constant $c \geq 1$ such that for all $(t, x, y) \in (0, 1] \times \mathbb{R}^d \times \mathbb{R}^d$,

$$c^{-1} \left( t^{-d/\alpha} \wedge \frac{t}{|x - y|^{d+\alpha}} \right) \leq p(t, x, y) \leq c \left( t^{-d/\alpha} \wedge \frac{t}{|x - y|^{d+\alpha}} \right). \quad (4.3)$$

On the other hand, noticing that $b \in C^\beta_b(\mathbb{R}^d; \mathbb{R}^d)$ with $\beta > 1 - \frac{d}{2}$, we know from [18, Theorem 1.1] (see also [18, Theorem 3.6]) that (4.3) also holds for $\alpha = 1$.

Furthermore, having (4.3) at hand and following the proof of [17, Lemma 2.1], we can get that for any $t \in (0, 1]$ and $x, y, z \in \mathbb{R}^d$,

$$\frac{p(t, x, z)}{p(t, y, z)} \leq 2^{\alpha + 2} c^2 \left( 1 + \frac{|x - y|}{t^{1/\alpha}} \right)^{d+\alpha}. \quad (4.4)$$

Therefore, for any $f \in B^+_b(\mathbb{R}^d)$, $t \in (0, 1]$ and $x, y \in \mathbb{R}^d$,

$$P_t f(x) = \int f(z) p(t, x, z) \, dz = \int f(z) \frac{p(t, x, z)}{p(t, y, z)} p(t, y, z) \, dz \leq \left( \max_{z \in \mathbb{R}^d} \frac{p(t, x, z)}{p(t, y, z)} \right) \int f(z) p(t, y, z) \, dz = C \left( 1 + \frac{|x - y|}{t^{1/\alpha}} \right)^{d+\alpha} P_t f(y),$$

where $C$ is a positive constant independent of $x, y, z$ and $t$. For any $t > 0$, we write

$$P_t f = P_{t \land 1} P_{(t-1)\land 1} f.$$

This, together with the inequality above, yields the required assertion. \hfill \Box

Surely, the Harnack inequality (4.2) is not satisfactory in the sense that $C > 1$, which means that such inequality is not sharp for the case $x = y$. Nonetheless, since the process has the transition density function, it has the strong Feller property, and so even for $C > 1$, we still have some applications, e.g., long time behaviors and properties of invariant measure.

References

[1] M. Arnaudon, A. Thalmaier, F.-Y. Wang, Harnack inequality and heat kernel estimates on manifolds with curvature unbounded below. *Bull. Sci. Math.* 130 (2006), no. 3, 223–233.

[2] D. Bakry, I. Gentil, M. Ledoux, *Analysis and geometry of Markov diffusion operators*, Grundlehren der Mathematischen Wissenschaften 348, Springer, 2014.

[3] L. Beck, F. Flandoli, M. Gubinelli, M. Maurelli, Stochastic ODEs and stochastic linear PDEs with critical drift: regularity, duality and uniqueness. arXiv:1401.1530v1

[4] Z.-Q. Chen, L. Wang, Uniqueness of stable processes with drift. arXiv:1309.6414v1

[5] E. Fedrizzi, F. Flandoli, Hölder flow and differentiability for SDEs with nonregular drift. *Stoch. Anal. Appl.* 31 (2013), no. 4, 708–736.
[6] F. Flandoli, M. Gubinelli, E. Priola, Well-posedness of the transport equation by stochastic perturbation. *Invent. Math.* 180 (2010), no. 1, 1–53.

[7] F. Flandoli, M. Gubinelli, E. Priola, Flow of diffeomorphisms for SDEs with unbounded Hölder continuous drift. *Bull. Sci. Math.* 134 (2010), no. 4, 405–422.

[8] N.V. Krylov, M. Röckner, Strong solutions of stochastic equations with singular time dependent drift. *Probab. Theory Related Fields* 131 (2005), no. 2, 154–196.

[9] E. Priola, Pathwise uniqueness for singular SDEs driven by stable processes. *Osaka J. Math.* 49 (2012), no. 2, 421–447.

[10] M. Röckner, F.-Y. Wang, Log-Harnack inequality for stochastic differential equations in Hilbert spaces and its consequences. *Infin. Dimens. Anal. Quantum Probab. Relat. Top.* 13 (2010), no. 1, 27–37.

[11] J. Shao, Harnack inequalities and heat kernel estimates for SDEs with singular drifts. *Bull. Sci. Math.* 137 (2013), no. 5, 589–601.

[12] J. Shao, F.-Y. Wang, C. Yuan, Harnack inequalities for stochastic (functional) differential equations with non-Lipschitzian coefficients. *Electron. J. Probab.* 17 (2011), Paper no. 100, 1–18.

[13] F.-Y. Wang, Logarithmic Sobolev inequalities on noncompact Riemannian manifolds. *Probab. Theory Related Fields* 109 (1997), no. 3, 417–424.

[14] F.-Y. Wang, Harnack inequality for SDE with multiplicative noise and extension to Neumann semigroup on nonconvex manifolds. *Ann. Probab.* 39 (2011), no. 4, 1449–1467.

[15] F.-Y. Wang, *Harnack Inequalities for Stochastic Partial Differential Equations*, Springer, New York, 2013.

[16] F.-Y. Wang, J. Wang, Harnack inequalities for stochastic equations driven by Lévy noise. *J. Math. Anal. Appl.* 410 (2014), no. 1, 513–523.

[17] J. Wang, Harnack inequalities for Ornstein–Uhlenbeck processes driven by Lévy processes. *Stat. Probab. Letters* 81 (2011), no. 9, 1436–1444.

[18] L. Xie, X. Zhang, Heat kernel estimates for critical fractional operators. arXiv: 1210.7063v2

[19] X. Zhang, Strong solutions of SDEs with singular drift and Sobolev diffusion coefficients. *Stoch. Proc. Appl.* 115 (2005), no. 11, 1805–1818.

[20] X. Zhang, Stochastic homeomorphism flows of SDEs with singular drifts and Sobolev diffusion coefficients. *Electron. J. Probab.* 16 (2011), no. 38, 1096–1116.

[21] X. Zhang, Derivative formulas and gradient estimates for SDEs driven by α-stable processes. *Stoch. Proc. Appl.* 123 (2013), no. 4, 1213–1228.