On a bound of Heath-Brown for Dirichlet $L$-functions on the critical line

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Abstract

Let $\chi$ a primitive character (mod $q$) and consider the Dirichlet $L$-function

$$L(s, \chi) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s}.$$ 

We give a new proof of an upper bound of Heath-Brown for $|L(s, \chi)|$ on the critical line $s = 1/2 + it$.  

1 Introduction

For integer $q$, let $\chi$ be a primitive character (mod $q$). We consider the order of the Dirichlet $L$-function

$$L(s, \chi) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s},$$

on the critical line $s = 1/2 + it$. This problem has been considered in a number of previous works and is essentially equivalent to bounding sums of the form

$$\sum_{M<n\leq M+N} \chi(n)n^it. \quad (1)$$
Burgess \cite{3} showed that for bounded $t$

$$L(1/2 + it, \chi) \ll q^{3/16 + o(1)},$$

which was improved by Heath-Brown in \cite{7} by developing a $q$-analogue of the van der Corput method, to obtain the bound

$$L(1/2 + it, \chi) \ll (q^{1/4} + (qt)^{1/6}) (qt)^{o(1)}. \quad (2)$$

Later in \cite{8}, Heath-Brown adapted the Burgess method \cite{3} to deal with sums of the form (1), leading to the bound

$$L(1/2 + it, \chi) \ll (qt)^{3/16 + o(1)}, \quad (3)$$

which gives an improvement on (2) for values of $t \leq q^{1/3}$. When $q$ is prime, Huxley and Watt \cite{10} extended the Bombieri and Iwaniec method \cite{1} to deal with sums of the form (1). This gives an improvement on (2) and (3) when $t$ is not too small, although they remark it does not seem possible to use the method of \cite{10} to improve on the results of Heath-Brown when $t \leq q^{15}$.

We give a new proof of the bound (3). As in \cite{8}, we reduce the problem to bounding the double mean value

$$\int_A^B \sum_{\lambda=1}^q \max_{V/2 < Q \leq V} \left| \sum_{V/2 < v \leq Q} \chi(\lambda + v)(x + v)^i e^{2\pi i v} \right|^4 \, dx. \quad (4)$$

This is done using some ideas of Chang \cite{4}, Friedlander and Iwaniec \cite{6} and Heath-Brown \cite{8}. We note that although this part of the argument is different to that of Heath-Brown, we still rely on some key ideas from \cite{8}, mainly the use of the Sobolev-Gallagher inequality (see Lemma 4). To bound the double mean value (4), we follow the argument of Heath-Brown \cite{8} to reduce the problem to bounding

$$\int_A^B \sum_{\lambda=1}^q \left| \sum_{V/2 < v \leq V/2 + C} \chi(\lambda + v)(x + v)^i e^{2\pi i v} \right|^4 \, dx,$$

which we dealt with differently to Heath-Brown \cite{8}, using results of Burgess \cite{2}, Huxley \cite{9} and Phong, Stein and Sturm \cite{13}.
2 Main Results

Theorem 1. For integer $q$, let $\chi$ be a primitive character (mod $q$). Let $t \geq 1$ and let the integers $M$ and $N$ satisfy

$$q^{1/4} t^{1/4} \leq N \leq q^{5/8} t^{1/8},$$

and

$$N \leq M \leq 2N.$$

Then we have

$$\left| \sum_{M < n \leq M+N} \chi(n)n^{it} \right| \leq N^{1/2} (qt)^{3/16 + o(1)}.$$

Using Theorem 1 as in [8] gives,

Theorem 2. For integer $q$, let $\chi$ be a primitive character (mod $q$) and let $t \geq 1$. Then we have

$$L(1/2 + it, \chi) \leq (qt)^{3/16 + o(1)}.$$ 

3 Preliminary Results

The proof of the following can be found in the proof of [5, Lemma 7].

Lemma 3. For integers $q, M, N, U$ let $I$ denote the number of solutions to the congruence

$$n_1 u_1 \equiv n_2 u_2 \pmod{q},$$

with

$$1 \leq u_1, u_2 \leq U, \quad M < n_1, n_2 \leq M + N,$$

and

$$(u_1, q) = 1, \quad (u_2, q) = 1.$$

Then

$$I \leq NU \left( \frac{NU}{q} + 1 \right) q^{o(1)}.$$

As in [8] we use the Sobolev-Gallagher inequality, for the proof see [12, Lemma 1.1].
Lemma 4. Let $b > a$ and suppose $f(x)$ has continuous derivative for $a \leq x \leq b$. For any $a \leq u \leq b$ we have

$$|f(u)| \leq \frac{1}{b - a} \int_a^b |f(x)| dx + \int_a^b |f'(x)| dx.$$  

The following is a special case of [13, Theorem 1]. We reproduce the proof for the special case relevant to us.

Lemma 5. Let $F(x) = Lx^2 + Mx + N$ be a polynomial of degree 2 with real coefficients and two distinct roots $\alpha_1, \alpha_2$. Then

$$\mu(\{x \in \mathbb{R} : |F(x)| \leq \varepsilon\}) \leq \frac{8\varepsilon}{|L(\alpha_1 - \alpha_2)|},$$

where $\mu(.)$ denotes the Lebesgue measure.

Proof. Writing

$$F(x) = L(x - \alpha_1)(x - \alpha_2),$$

we split $\mathbb{R}$ into two sets

$$I_1 = \{x \in \mathbb{R} : |x - \alpha_1| \leq |x - \alpha_2|\},$$
$$I_2 = \{x \in \mathbb{R} : |x - \alpha_2| \leq |x - \alpha_1|\}.$$

Then if $x \in I_1$ we have

$$|\alpha_1 - \alpha_2| \leq |\alpha_1 - x| + |\alpha_2 - x| \leq 2|\alpha_2 - x|,$$

and if $x \in I_2$

$$|\alpha_1 - \alpha_2| \leq 2|\alpha_1 - x|.$$

Hence

$$|F(x)| = |L(x - \alpha_1)(x - \alpha_2)| \geq \frac{|L(x - \alpha_1)(\alpha_1 - \alpha_2)|}{2}, \quad \text{if } x \in I_1,$$

and

$$|F(x)| \geq \frac{|L(x - \alpha_2)(\alpha_1 - \alpha_2)|}{2}, \quad \text{if } x \in I_2,$$

which gives the set inclusions

$$I_1 \cap \{x \in \mathbb{R} : |F(x)| \leq \varepsilon\} \subseteq \left\{x \in \mathbb{R} : \frac{|L(x - \alpha_1)(\alpha_1 - \alpha_2)|}{2} \leq \varepsilon\right\},$$

$$I_2 \cap \{x \in \mathbb{R} : |F(x)| \leq \varepsilon\} \subseteq \left\{x \in \mathbb{R} : \frac{|L(x - \alpha_2)(\alpha_1 - \alpha_2)|}{2} \leq \varepsilon\right\},$$

$$I_1 \cap \{x \in \mathbb{R} : |F(x)| \leq \varepsilon\} \subseteq \left\{x \in \mathbb{R} : \frac{|L(x - \alpha_1)(\alpha_1 - \alpha_2)|}{2} \leq \varepsilon\right\},$$

$$I_2 \cap \{x \in \mathbb{R} : |F(x)| \leq \varepsilon\} \subseteq \left\{x \in \mathbb{R} : \frac{|L(x - \alpha_2)(\alpha_1 - \alpha_2)|}{2} \leq \varepsilon\right\}.$$
and
\[
I_2 \cap \{ x \in \mathbb{R} : |F(x)| \leq \varepsilon \} \subseteq \left\{ x \in \mathbb{R} : \frac{|L(x - \alpha_2)(\alpha_1 - \alpha_2)|}{2} \leq \varepsilon \right\}.
\]
Since
\[
I_1 \cup I_2 = \mathbb{R},
\]
we get
\[
\mu \left( \{ x \in \mathbb{R} : |F(x)| \leq \varepsilon \} \right) \leq \mu \left( \left\{ x \in \mathbb{R} : \frac{|L(x - \alpha_1)(\alpha_1 - \alpha_2)|}{2} \leq \varepsilon \right\} \right)
+ \mu \left( \left\{ x \in \mathbb{R} : \frac{|L(x - \alpha_2)(\alpha_1 - \alpha_2)|}{2} \leq \varepsilon \right\} \right),
\]
and the result follows since
\[
\mu \left( \left\{ x \in \mathbb{R} : \frac{|L(x - \alpha_i)(\alpha_1 - \alpha_2)|}{2} \leq \varepsilon \right\} \right) \leq \frac{4\varepsilon}{|L(\alpha_1 - \alpha_2)|}, \quad i = 1, 2.
\]

Lemma 6. Let \( A, B, V \) be real numbers satisfying
\[
0 \leq A < B, \quad B \ll V, \quad V \geq 1,
\]
and let \( t \geq 1 \). Let the integers \( v_1, v_2, v_3, v_4 \) satisfy
\[
V/2 < v_i \leq V, \quad (v_1 - v_3)(v_1 - v_4)(v_2 - v_3)(v_2 - v_4) \neq 0,
\]
and let
\[
F(x) = \frac{(x + v_1)(x + v_2)}{(x + v_3)(x + v_4)}.
\]
and
\[
\Delta = (v_1 - v_3)(v_1 - v_4)(v_2 - v_3)(v_2 - v_4).
\]
Then if
\[
v_1 + v_2 \neq v_3 + v_4,
\]
we have
\[
\int_A^B F(x)^{it} \, dx \ll \frac{V^2}{t^{1/2}|\Delta|^{1/4}}.
\]
and if
\[ v_1 + v_2 = v_3 + v_4, \]
we have
\[ \int_A^B F(x)^t \, dx \ll \frac{V^4}{t|(v_1 - v_4)(v_2 - v_4)|}. \]

Proof. Consider first when
\[ v_1 + v_2 \neq v_3 + v_4. \]

Let
\[ F(x) = \frac{(x + v_1)(x + v_2)}{(x + v_3)(x + v_4)}, \]
and
\[
L = (v_1 + v_2) - (v_3 + v_4), \\
M = v_3 v_4 - v_1 v_2, \\
N = (v_1 + v_2)v_3 v_4 - (v_3 + v_4)v_1 v_2,
\]
so that
\[ M^2 - LN = \Delta = (v_1 - v_3)(v_1 - v_4)(v_2 - v_3)(v_2 - v_4), \]
and
\[ F'(x) = \frac{Lx^2 + 2Mx + N}{(x + v_3)^2(x + v_4)^2}. \]

Since the discriminant of the polynomial occurring in the numerator of (7) is
\[ 4\Delta = 4(v_1 - v_3)(v_1 - v_4)(v_2 - v_3)(v_2 - v_4), \]
by the assumption that \( \Delta \neq 0 \) we see that the polynomial
\[ f(x) = Lx^2 + 2Mx + N, \]
has two distinct roots \( \alpha_1, \alpha_2 \). Hence by Lemma [3] for any fixed \( \varepsilon > 0 \)
\[
\left| \int_A^B F(x)^t \, dx \right| \leq \left| \int_{A \leq x \leq B} F(x)^t \, dx \right| + \left| \int_{A \leq x \leq B} F(x)^t \, dx \right| \\
\ll \frac{\varepsilon}{|L(\alpha_1 - \alpha_2)|} + \left| \int_{A \leq x \leq B} F(x)^t \, dx \right|. 
\]
Integrating the second integral by parts gives

\[
\int_{A \leq x \leq B} F(x)^{it} dx = \int_{A \leq x \leq B} \frac{F'(x)}{|f(x)|} F(x)^{it} dx
\]

\[
= \left[ \frac{F(x)^{1+it}}{(1 + it)F'(x)} \right]_{A \leq x \leq B}^{B \leq x \leq A} - \frac{1}{1 + it} \int_{A \leq x \leq B} \frac{F''(x)}{|f(x)|} F(x)^{1+it} dx.
\]

Since

\[
\frac{V}{2} < v_i \leq V, \quad 0 \leq A < B \ll V,
\]

we have for \( A \leq x \leq B \) and \( |f(x)| > \varepsilon \)

\[
|F(x)| \ll \frac{(x + V)^2}{(2x + V)} \ll 1,
\]

\[
\frac{1}{|F'(x)|} = \frac{(x + v_3)^2(x + v_4)^2}{Lx^2 + 2Mx + N} \gg \frac{V^4}{\varepsilon},
\]

so that

\[
\left| \int_{A \leq x \leq B} F(x)^{it} dx \right| \leq \frac{V^4}{t\varepsilon} + \frac{1}{t} \int_{A \leq x \leq B} \left| \frac{F''(x)}{|f(x)|^2} \right| dx.
\]

For the last integral, since

\[
F''(x) = \frac{d((Lx^2 + 2Mx + N))(x + v_3)^2(x + v_4)^2 - (Lx^2 + 2Mx + N)d((x + v_3)(x + v_4)^2)}{(x + v_3)^4(x + v_4)^4},
\]

and the polynomial occurring in the numerator of (8) has degree at most 5, we see that for real \( x \) the function

\[
\frac{F''(x)}{F'(x)^2},
\]

has at most 5 sign changes. Hence we may break the integral

\[
\int_{A \leq x \leq B} \left| \frac{F''(x)}{F'(x)^2} \right| dx,
\]

into \( O(1) \) integrals of the form

\[
\int_{A_i}^{B_i} \frac{F''(x)}{F'(x)^2} dx,
\]

\[
\int_{A_i}^{B_i} \frac{F''(x)}{F'(x)^2} dx,
\]

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for some $A_i, B_i$ with

$$|f(x)| \geq \varepsilon \quad \text{for} \quad A_i \leq x \leq B_i.$$ 

Since

$$\left| \int_{A_i}^{B_i} \frac{F''(x)}{F'(x)^2} \, dx \right| \leq \frac{1}{|F'(A_i)|} + \frac{1}{|F'(B_i)|} \ll \frac{V^4}{\varepsilon},$$

we get

$$\int_{A \leq x \leq B} \frac{F''(x)}{|f(x)|} \, dx \ll \frac{V^4}{\varepsilon},$$

so that

$$\left| \int_A^B F(x)^2 \, dx \right| \ll \frac{\varepsilon}{|L(\alpha_1 - \alpha_2)|} + \frac{V^4}{t\varepsilon}.$$ 

Since $\alpha_1$ and $\alpha_2$ are the roots of the polynomial

$$f(x) = Lx^2 + 2Mx + N,$$

we see that

$$|L(\alpha_1 - \alpha_2)| = 2|M^2 - LN|^{1/2}$$

$$= 2|(v_1 - v_3)(v_1 - 4)(v_2 - v_3)(v_2 - v_4)|^{1/2}$$

$$= 2|\Delta|^{1/2},$$

so that taking

$$\varepsilon = \frac{|\Delta|^{1/4}V^2}{t^{1/2}},$$

gives

$$\left| \int_A^B F(x)^2 \, dx \right| \ll \frac{V^2}{|\Delta|^{1/4}t^{1/2}}.$$ 

Next consider when

$$v_1 + v_2 = v_3 + v_4,$$

so that (7) becomes

$$F'(x) = \frac{(2x + v_1 + v_2)(v_1 - v_4)(v_2 - v_4)}{(x + v_3)^2(x + v_4)^2}, \quad (9)$$

and we see that $F'(x)$ has one zero at $x = -(v_1 + v_2)/2$. Since

$$0 \leq A < B, \quad (v_1 + v_2)/2 \geq V/2,$$

(10)
integrating by parts as above gives
\[ \left| \int_A^B F(x)^it \right| \ll \frac{1}{t} \max_{A \leq x \leq B} \frac{1}{|F'(x)|} \]
so that by (9) and (11) we have
\[ \left| \int_A^B F(x)^it \right| \ll \frac{1}{t} \max_{A \leq x \leq B} \frac{1}{|F'(x)|} \ll \frac{V^4}{t|v_1 - v_4|}. \]

\[ \text{Lemma 7. Let } A, B, V \text{ be real numbers satisfying } \]
\[ 0 \leq A < B, \quad B \ll V, \]
and let \( t \geq 1 \). For integers \( v_1, v_4 \) satisfying
\[ \frac{V}{2} < v_1, v_4 \leq V, \quad v_1 \neq v_4, \]
we have
\[ \int_A^B \left( \frac{x + v_1}{x + v_4} \right)^it \, dx \ll \frac{V^2}{t|v_1 - v_4|}. \]

Proof. Let
\[ F(x) = \frac{x + v_1}{x + v_4}, \]
so that
\[ F'(x) = \frac{v_4 - v_1}{(x + v_4)^2}. \]
Integrating by parts as in the proof of Lemma 6 gives
\[ \left| \int_A^B \left( \frac{x + v_1}{x + v_4} \right)^it \right| \ll \max_{A \leq x \leq B} \frac{1}{t|F'(x)|} \ll \frac{V^2}{t|v_1 - v_4|}. \]

The following is due to Burgess [2, Lemma 2,3,4].
Lemma 8. Let $p$ be prime and $\alpha$ be an integer. For integers $v_1, v_2, v_3, v_4$ let $N(p^\alpha)$ denote the number of solutions to the congruence

$$((v_1+v_2)-(v_3+v_4))x^2+(v_3v_4-v_1v_2)x+(v_1+v_2)v_3v_4-(v_3+v_4)v_1v_2 \equiv 0 \pmod{p^\alpha},$$

with $1 \leq x \leq p^\alpha$ and let $\chi$ be a primitive character $(\mod{p^\alpha})$. Then if $p$ is odd

$$\left| \sum_{n=1}^{p^\alpha} \chi \left( \frac{(x+v_1)(x+v_2)}{(x+v_3)(x+v_4)} \right) \right| \leq \begin{cases} N(p^{\alpha/2})p^{\alpha/2}, & \text{if } \alpha \text{ is even}, \\ N(p^{(\alpha-1)/2})p^{\alpha/2} + N(p^{(\alpha+1)/2})p^{(\alpha-1)/2}, & \text{if } \alpha \text{ is odd}, \end{cases}$$

and if $p = 2$

$$\left| \sum_{n=1}^{2^\alpha} \chi \left( \frac{(x+v_1)(x+v_2)}{(x+v_3)(x+v_4)} \right) \right| \leq \begin{cases} N(2^{\alpha/2})2^{\alpha/2}, & \text{if } \alpha \text{ is even}, \\ N(2^{(\alpha+1)/2})2^{(\alpha+1)/2}, & \text{if } \alpha \text{ is odd}. \end{cases}$$

The following Lemma is due to Huxley [9], see [14, Section 3] for related results.

Lemma 9. Let $F(x) \in \mathbb{Z}[x]$ be a polynomial of degree $r \geq 2$ and let $\Delta$ denote the discriminant of $F$. For prime $p$ and integer $\alpha$, let $N(F, p^\alpha)$ denote the number of solutions to the congruence

$$F(x) \equiv 0 \pmod{p^\alpha}, \quad 1 \leq x \leq p^\alpha.$$

Then if $\Delta \neq 0$ we have

$$N(F, p^\alpha) \leq r(p^\alpha, \Delta)^{1/2}.$$

Using the proof of [2, Lemma 7] with Lemma 8 and Lemma 9 gives,

Lemma 10. For integer $q$, let $\chi$ be a primitive character $(\mod{q})$ and suppose the integers $v_1, v_2, v_3, v_4$ satisfy

$$\Delta = (v_1-v_3)(v_1-v_4)(v_2-v_3)(v_2-v_4) \neq 0.$$

Then if

$$v_1 + v_2 \neq v_3 + v_4,$$
we have
\[ \left| \sum_{\lambda=1}^{q} \chi \left( \frac{(\lambda + v_1)(\lambda + v_2)}{(\lambda + v_3)(\lambda + v_4)} \right) \right| \leq (q, \Delta)^{1/2} q^{1/2 + o(1)}, \]
and if
\[ v_1 + v_2 = v_3 + v_4, \]
we have
\[ \left| \sum_{\lambda=1}^{q} \chi \left( \frac{(\lambda + v_1)(\lambda + v_2)}{(\lambda + v_3)(\lambda + v_4)} \right) \right| \leq (q, (v_1 - v_4)(v_2 - v_4)) q^{1/2 + o(1)}. \]

**Proof.** Consider first when
\[ v_1 + v_2 \neq v_3 + v_4, \tag{11} \]
then if \( \chi \) is a primitive character (mod \( p^\alpha \)), as in [2, Lemma 5], since the discriminant of the polynomial
\[ ((v_1 + v_2) - (v_3 + v_4))x^2 + (v_3v_4 - v_1v_2)x + (v_1 + v_2)v_3v_4 - (v_3 + v_4)v_1v_2, \tag{12} \]
is
\[ 4\Delta = 4(v_1 - v_3)(v_1 - v_4)(v_2 - v_3)(v_2 - v_4), \]
we see from Lemma 8 and Lemma 9 that
\[ \left| \sum_{n=1}^{p^\alpha} \chi \left( \frac{(x + v_1)(x + v_2)}{(x + v_3)(x + v_4)} \right) \right| \ll (p^\alpha, \Delta)^{1/2} p^{\alpha/2}, \tag{13} \]
which under the assumption (11) gives the desired result when \( q \) is a prime power. For the general case, suppose \( \chi \) is a primitive character (mod \( q \)) and let \( q = p_1^{\alpha_1} \ldots p_k^{\alpha_k} \) be the prime factorization of \( q \). By the Chinese remainder theorem, there exists \( \chi_1, \ldots, \chi_k \), where each \( \chi_i \) is a primitive character (mod \( p_i^{\alpha_i} \)) such that
\[ \chi = \chi_1 \ldots \chi_k. \]
Let
\[ F(x) = \frac{(x + v_1)(x + v_2)}{(x + v_3)(x + v_4)}, \]
and writing \( q_i = q/p_i^{\alpha_i} \) we have

\[
\sum_{n=1}^{p^\alpha} \chi(F(n)) = \sum_{n_1=1}^{p_1^{\alpha_1}} \ldots \sum_{n_k=1}^{p_k^{\alpha_k}} \chi_1(F(n_1q_1 + \ldots n_kq_k)) \ldots \chi_k(F(n_1q_1 + \ldots n_kq_k)) = \prod_{i=1}^{k} \left( \sum_{n_i=1}^{p_i^{\alpha_i}} \chi_i(F(n_i)) \right) = \prod_{i=1}^{k} \left( \sum_{n_i=1}^{p_i^{\alpha_i}} \chi_i(F(n_i)) \right). \tag{14}
\]

Letting \( \omega(q) \) denote the number of distinct prime factors of \( q \), by (13) and (14) we have for some absolute constant \( C \),

\[
\left| \sum_{n=1}^{p^\alpha} \chi(F(n)) \right| \leq C \omega(q) (q, \Delta) \frac{1}{2} q^{1/2} \leq (q, \Delta)^{1/2} q^{1/2+o(1)}.
\]

Next suppose

\[
v_1 + v_2 = v_3 + v_4,
\]

so that (12) becomes

\[
(2x + v_1 + v_2)(v_1 - v_4)(v_2 - v_4).
\]

For \( \chi \) be a primitive character \((\bmod p^\alpha)\), since the number of solutions to the congruence

\[
(2x + v_1 + v_2)(v_1 - v_4)(v_2 - v_4) \equiv 0 \pmod{p^\alpha}, \quad 1 \leq x \leq p^\alpha,
\]

is bounded by

\[
(p^\alpha, 2(v_4 - v_1)(v_4 - v_2)) \leq 2(p^\alpha, (v_1 - v_4)(v_2 - v_4)),
\]

we have from Lemma \[8\]

\[
\left| \sum_{n=1}^{p^\alpha} \chi \left( \frac{(x + v_1)(x + v_2)}{(x + v_3)(x + v_4)} \right) \right| \ll (p^\alpha, (v_1 - v_4)(v_2 - v_4)) p^{\alpha/2},
\]

so that using the Chinese remainder theorem as above gives

\[
\left| \sum_{n=1}^{p^\alpha} \chi(F(n)) \right| \leq (q, (v_1 - v_4)(v_2 - v_4)) q^{1/2+o(1)}.
\]

\[\square\]
Lemma 11. For integer $q$, let $\chi$ be a primitive character (mod $q$) and let $A, B, V$ be real numbers satisfying

$$0 \leq A < B, \quad B \ll V,$$

and let $t \geq 1$. For any real number $\alpha$ we have

$$\int_A^B \sum_{\lambda=1}^q \sum_{V/2<v\leq V} \chi(\lambda + v)(x + v)^i e^{2\pi i \alpha v} \left| \frac{1}{x} \right| dx \leq \left( qV^3 + \frac{q^{1/2}V^5}{t^{1/2}} \right)(qV)^{o(1)}.$$

Proof. We first show that for $C \leq V/2$

$$\int_A^B \sum_{\lambda=1}^q \sum_{V/2<v\leq V/2+C} \chi(\lambda + v)(x + v)^i e^{2\pi i \alpha v} \left| \frac{1}{x} \right| dx \leq \left( qV^3 + \frac{q^{1/2}V^5}{t^{1/2}} \right)(qV)^{o(1)},$$

then we complete the proof using the argument of Heath-Brown [8, Section 5].

Expanding the inner sum in (15)

$$\int_A^B \sum_{\lambda=1}^q \sum_{V/2<v\leq V/2+C} \chi(\lambda + v)(x + v)^i e^{2\pi i \alpha v} \left| \frac{1}{x} \right| dx \leq \sum_{V/2<v_1,v_2,v_3,v_4\leq V/2+C} \int_A^B \left( \frac{(x + v_1)(x + v_2)}{(x + v_3)(x + v_4)} \right)^it dx \left| \sum_{\lambda=1}^q \chi \left( \frac{(\lambda + v_1)(\lambda + v_2)}{(\lambda + v_3)(\lambda + v_4)} \right) \right|.$$

We break the outer summation over $(v_1, v_2, v_3, v_4)$ into two sets. In the first set $V_1$, we put all $(v_1, v_2, v_3, v_4)$ that contain at most two distinct integers and in the second set $V_2$, we put the remaining $(v_1, v_2, v_3, v_4)$. Estimating the inner sum and integral trivially for the first set, since

$$C \leq V/2, \quad 0 \leq A < B \ll V,$$
By Lemma 6 and Lemma 10 we have

\[ \int_{A}^{B} \sum_{\lambda=1}^{q} \left| \sum_{V/2<v<2V+C} \chi(\lambda + v)(x + v)^{it} \right|^{4} dx \ll \]

\[ CqV^{2} + \sum_{(v_{1},v_{2},v_{3},v_{4}) \in V_{2}} \int_{0}^{B} \left( \frac{(x + v_{1})(x + v_{2})}{(x + v_{3})(x + v_{4})} \right)^{it} dx \ll \sum_{\lambda=1}^{q} \chi \left( \frac{(\lambda + v_{1})(\lambda + v_{2})}{(\lambda + v_{3})(\lambda + v_{4})} \right). \]

Writing

\[ \Delta = (v_{1} - v_{3})(v_{1} - v_{4})(v_{2} - v_{3})(v_{2} - v_{4}), \]

we split \( V_{2} \) into three sets,

\[ V_{3} = \{ (v_{1},v_{2},v_{3},v_{4}) \in V_{2} : \Delta = 0 \}, \]
\[ V_{4} = \{ (v_{1},v_{2},v_{3},v_{4}) \in V_{2} : \Delta \neq 0, v_{1} + v_{2} \neq v_{3} + v_{4} \}, \]
\[ V_{5} = \{ (v_{1},v_{2},v_{3},v_{4}) \in V_{2} : \Delta \neq 0, v_{1} + v_{2} = v_{3} + v_{4} \}. \]

By Lemma 6 and Lemma 10 we have

\[ \sum_{(v_{1},v_{2},v_{3},v_{4}) \in V_{4}} \int_{A}^{B} \left( \frac{(x + v_{1})(x + v_{2})}{(x + v_{3})(x + v_{4})} \right)^{it} dx \ll \sum_{\lambda=1}^{q} \chi \left( \frac{(\lambda + v_{1})(\lambda + v_{2})}{(\lambda + v_{3})(\lambda + v_{4})} \right) \ll \]

\[ \frac{q^{1/2+o(1)}V^{2}}{t^{1/2}} \sum_{(v_{1},v_{2},v_{3},v_{4}) \in V_{4}} \frac{(q,\Delta)^{1/2}}{|\Delta|^{1/4}}. \]

Since

\[ \sum_{(v_{1},v_{2},v_{3},v_{4}) \in V_{4}} \frac{(q,\Delta)^{1/2}}{|\Delta|^{1/4}} = \sum_{(v_{1},v_{2},v_{3},v_{4}) \in V_{4}} \frac{(q,(v_{1} - v_{3})(v_{1} - v_{4})(v_{2} - v_{3})(v_{2} - v_{4}))^{1/2}}{|(v_{1} - v_{3})(v_{1} - v_{4})(v_{2} - v_{3})(v_{2} - v_{4})|^{1/4}} \]
\[ \leq \sum_{(v_{1},v_{2},v_{3},v_{4}) \in V_{4}} \frac{(q,(v_{1} - v_{3}))^{1/2}(q,(v_{1} - v_{4}))^{1/2}(q,(v_{2} - v_{3}))^{1/2}(q,(v_{2} - v_{4}))^{1/2}}{|(v_{1} - v_{3})(v_{1} - v_{4})(v_{2} - v_{3})(v_{2} - v_{4})|^{1/4}}, \]

we break the above sum into \( q^{(1)} \) sums of the form

\[ \sum_{(v_{1},v_{2},v_{3},v_{4}) \in V_{4}} \frac{(q,(v_{1} - v_{3}))^{1/2}(q,(v_{1} - v_{4}))^{1/2}(q,(v_{2} - v_{3}))^{1/2}(q,(v_{2} - v_{4}))^{1/2}}{|(v_{1} - v_{3})(v_{1} - v_{4})(v_{2} - v_{3})(v_{2} - v_{4})|^{1/4}}. \]
where each $d_{i,j}$ is a divisor of $q$. Since
\[
\frac{(q, v_1 - v_3)^{1/2}(q, (v_1 - v_4))^{1/2}}{|(v_1 - v_3)(v_1 - v_4)|^{1/4}} \leq \frac{(q, v_1 - v_3)}{|(v_1 - v_3)|^{1/2}} + \frac{(q, v_1 - v_4)}{|(v_1 - v_4)|^{1/2}};
\]
we have
\[
\sum_{\substack{(v_1, v_2, v_3) \in V_4 \\ (q, v_1 - v_3) = d_{i,j} \\ i=1,2,j=3,4}} \frac{(q, (v_1 - v_3))^{1/2}(q, (v_1 - v_4))^{1/2}(q, (v_2 - v_3))^{1/2}(q, (v_2 - v_4))^{1/2}}{|(v_1 - v_3)(v_1 - v_4)(v_2 - v_3)(v_2 - v_4)|^{1/4}}
\leq \sum_{\substack{(v_1, v_2, v_3, v_4) \in V_4 \\ (q, v_1 - v_3) = d_{i,j} \\ i=1,2,j=3,4}} \frac{d_{1,3}(d_{2,4}d_{2,3})^{1/2}}{|(v_1 - v_3)|^{1/2}|(v_2 - v_3)(v_2 - v_4)|^{1/4}}
\]
\[
+ \sum_{\substack{(v_1, v_2, v_3, v_4) \in V_4 \\ (q, v_1 - v_3) = d_{i,j} \\ i=1,2,j=3,4}} \frac{d_{1,4}(d_{2,4}d_{2,3})^{1/2}}{|(v_1 - v_4)|^{1/2}|(v_2 - v_3)(v_2 - v_4)|^{1/4}}.
\]
Consider the first sum, for integers $M_1, M_2, M_3$ since the number of solutions to the equations
\[
v_1 - v_3 = d_{1,3}M_1,
\]
\[
v_2 - v_3 = d_{2,3}M_2,
\]
\[
v_2 - v_4 = d_{2,4}M_3,
\]
with \((v_1, v_2, v_3, v_4) \in \mathcal{V}_4\) is bounded by \(C\), we have

\[
\sum_{(v_1, v_2, v_3, v_4) \in \mathcal{V}_4} \frac{d_{1,3}(d_{2,4}d_{2,3})^{1/2}}{|(v_1 - v_3)|^{1/2}|(v_2 - v_3)(v_2 - v_4)|^{1/4}} \leq C^3.
\]

A similar argument shows

\[
\sum_{(v_1, v_2, v_3, v_4) \in \mathcal{V}_4} \frac{d_{1,4}(d_{2,4}d_{2,3})^{1/2}}{|(v_1 - v_4)|^{1/2}|(v_2 - v_3)(v_2 - v_4)|^{1/4}} \leq C^3,
\]

and since \(C \leq V\), we get

\[
\sum_{(v_1, v_2, v_3, v_4) \in \mathcal{V}_4} \left| \int_A^B \left( \frac{(x + v_1)(x + v_2)}{(x + v_3)(x + v_4)} \right)^{it} dx \right| \left| \sum_{\lambda=1}^q \chi \left( \frac{(\lambda + v_1)(\lambda + v_2)}{(\lambda + v_3)(\lambda + v_4)} \right) \right| \leq \frac{Cq^{1/2+o(1)}V^4}{t^{1/2}}.
\]

For summation over the set \(\mathcal{V}_5\), we have by Lemma 6 and Lemma 10

\[
\sum_{(v_1, v_2, v_3, v_4) \in \mathcal{V}_5} \left| \int_A^B \left( \frac{(x + v_1)(x + v_2)}{(x + v_3)(x + v_4)} \right)^{it} dx \right| \left| \sum_{\lambda=1}^q \chi \left( \frac{(\lambda + v_1)(\lambda + v_2)}{(\lambda + v_3)(\lambda + v_4)} \right) \right| \leq \frac{V^4q^{1/2+o(1)}}{t} \sum_{(v_1, v_2, v_3, v_4) \in \mathcal{V}_5} \frac{(q, v_1 - v_4)(q, v_2 - v_4)}{|(v_1 - v_4)(v_2 - v_4)|},
\]

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and for integers $M_1, M_2$ since the number of solutions to the equations
\[
\begin{align*}
v_1 - v_4 &= M_1, \\
v_2 - v_4 &= M_2, \\
v_1 + v_2 &= v_3 + v_4,
\end{align*}
\]
with $(v_1, v_2, v_3, v_4) \in \mathcal{V}_5$ is bounded by $C$, we get
\[
\sum_{(v_1,v_2,v_3,v_4)\in\mathcal{V}_5} \frac{(q, v_1 - v_4)(q, v_2 - v_4)}{|(v_1 - v_4)(v_2 - v_4)|} \ll C \sum_{M_1,M_2 \leq C} \frac{(q, M_1)(q, M_2)}{M_1 M_2} \ll C^{1+o(1)} q^{o(1)},
\]
so that
\[
\sum_{(v_1,v_2,v_3,v_4)\in\mathcal{V}_5} \left| \int_{A} \left( \frac{x + v_1}{x + v_3} \right)^{it} dx \right| \left| \sum_{\lambda=1}^{q} \chi \left( \frac{(\lambda + v_1)(\lambda + v_2)}{\lambda + v_3} \right) \right| \ll \frac{C q^{1/2+o(1)} V^{4+o(1)}}{t}.
\]
Considering $\mathcal{V}_3$, since
\[
(v_1 - v_3)(v_1 - v_4)(v_2 - v_3)(v_2 - v_4) = 0,
\]
we have by symmetry,
\[
\sum_{(v_1,v_2,v_3,v_4)\in\mathcal{V}_3} \left| \int_{A} \left( \frac{x + v_1}{x + v_3} \right)^{it} dx \right| \left| \sum_{\lambda=1}^{q} \chi \left( \frac{\lambda + v_1}{\lambda + v_3} \right) \right| \ll V^2 \sum_{d \leq q} \left| \int_{A} \left( \frac{x + v_1}{x + v_4} \right)^{it} dx \right| \left| \sum_{\lambda=1}^{q} \chi \left( \frac{\lambda + v_1}{\lambda + v_4} \right) \right|.
\]
Using [11, Equation 3.5] and [11, Equation 12.51] we have
\[
\sum_{\lambda=1}^{q} \chi \left( \frac{\lambda + v_1}{\lambda + v_4} \right) = \sum_{\lambda=1}^{q} e^{2\pi i (v_1 - v_2) \lambda/q} \ll (q, v_1 - v_2),
\]
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so that by Lemma 7

\[
\sum_{V/2<v_1,v_4\leq V/2+C} \left| \int_A^B \left( \frac{x+v_1}{x+v_4} \right)^it \, dx \right| \sum_{\lambda=1}^q \chi \left( \frac{\lambda+v_1}{\lambda+v_4} \right) \approx \frac{V^2}{t} \sum_{v_4<v_1\leq C} \frac{q, v_1 - v_4}{v_1 - v_4} \leq Cq^{o(1)} V^{2+o(1)}.
\]

hence we get

\[
\sum_{(v_1,v_2,v_3,v_4)\in V_3} \left| \int_A^B \left( \frac{(x+v_1)(x+v_2)}{(x+v_3)(x+v_4)} \right)^it \, dx \right| \sum_{\lambda=1}^q \chi \left( \frac{\lambda+v_1}{\lambda+v_2}, \frac{\lambda+v_3}{\lambda+v_4} \right) \approx \frac{Cq^{o(1)} V^{4+o(1)}}{t}.
\]

Combining the estimates for \( V_1, V_3, V_4, V_5 \) gives

\[
\int_A^B \sum_{\lambda=1}^q \left| \sum_{V/2<v\leq V/2+C} \chi(\lambda+v)(x+v)^it \, e^{2\pi i \alpha v} \right|^4 \, dx \leq C \left( qV^2 + \frac{q^{1/2+o(1)} V^{4+o(1)}}{t^{1/2}} \right) (qV)^{o(1)}.
\]

(16)

Next we use (16) as in the argument of [8, Section 5] to bound

\[
\int_A^B \sum_{\lambda=1}^q \max_{V/2<Q\leq V} \left| \sum_{V/2<v\leq Q} \chi(\lambda+v)(x+v)^it \, e^{2\pi i \alpha v} \right|^4 \, dx.
\]

For each \( 1 \leq \lambda \leq q \) and \( A \leq x \leq B \), let \( Q_{\lambda,x} \) be the integer defined by

\[
\max_{V/2<Q\leq V} \left| \sum_{V/2<v\leq Q} \chi(\lambda+v)(x+v)^it \, e^{2\pi i \alpha v} \right|^4 = \left| \sum_{V/2<v\leq V+2+Q_{\lambda,x}} \chi(\lambda+v)(x+v)^it \, e^{2\pi i \alpha v} \right|^4,
\]

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and let
\[ Q_{\lambda,x} = \sum_{r \leq R} \delta_{\lambda,x}(r) 2^r, \]
be the binary expansion of \( Q_{\lambda,x} \), where \( R \) is the largest integer such that \( 2^R \leq V \) and we define
\[ s_{\lambda,x}(r) = \sum_{r < t \leq R} \delta_{\lambda,x}(t) 2^{t-r}. \]

Then writing
\[ H_{\lambda,x}(C, D) = \sum_{D < v \leq D+C} \chi(\lambda + v)(x + v)^i e^{2\pi i \alpha v}, \]
we have as in [8]
\[ H_{\lambda,x}(Q_{\lambda,x}, V/2) = \sum_{r \leq R} \delta_{\lambda,x}(r) H_{\lambda,x}(2^r, V/2 + s_{\lambda,x}(r)2^r). \]

By Hölder’s inequality
\[ |H_{\lambda,x}(Q(\lambda, x), V/2)|^4 \leq R^3 \left( \sum_{r \leq R} \delta_{\lambda,x}(r) |H_{\lambda,x}(2^r, V/2 + s_{\lambda,x}(r)2^r)|^4 \right) \]
\[ \leq V^{o(1)} \sum_{r \leq R} |H_{\lambda,x}(2^r, V/2 + s_{\lambda,x}(r)2^r)|^4, \]
and since \( s(r, \lambda, x) \leq 2^{R-r} \), we have
\[ |H_{\lambda,x}(Q(\lambda, x), V/2)|^4 \leq V^{o(1)} \sum_{r \leq R} \sum_{s \leq 2^{R-r}} |H_{\lambda,x}(2^r, V/2 + 2s)|^4. \]

Hence by (16)
\[ \int_{A}^{B} \sum_{\lambda=1}^{q} \max_{V/2 < Q \leq V} \left| \sum_{V/2 < v \leq Q} \chi(\lambda + v)(x + v)^i e^{2\pi i \alpha v} \right|^4 dx \leq \]
\[ q^{o(1)} \sum_{r \leq R} \sum_{s \leq 2^{R-r}} \int_{A}^{B} \sum_{\lambda=1}^{q} |H_{\lambda,x}(2^r, V/2 + s2^r)|^4 dx \leq \]
\[ \left( qV^2 + \frac{q^{1/2+o(1)} V^{4+o(1)}}{t^{1/2}} \right) (qV)^{o(1)} \sum_{r \leq R} \sum_{s \leq 2^{R-r}} 2^r, \]
so that
\[ \sum_{r \leq R} \sum_{s \leq 2^{R-r}} 2^r \leq \sum_{r \leq R} 2^R \leq 2RV^{o(1)}, \]
which gives
\[ \int_A^B \sum_{\lambda=1}^q \max_{V/2 < Q \leq V} \left| \sum_{V/2 < v \leq Q} \chi(\lambda + v)(x + v)^i e^{2\pi i \alpha v} \right|^4 dx \leq \left( qV^3 + \frac{q^{1/2+o(1)}V^{5+o(1)}}{t^{1/2}} \right) (qV)^{o(1)}. \]

4 Proof of Theorem 1

We begin with some ideas from the proof of [6, Theorem 1]. Let
\[ f(x) = \begin{cases} \min(x - M, 1, M + N - x), & \text{if } M \leq x \leq M + N, \\ 0, & \text{otherwise}, \end{cases} \]
so that \( f(x) \) is a continuous function equal to 1 for integers \( M < n \leq M + N \) and 0 otherwise, hence
\[ \sum_{M < n \leq M + N} \chi(n)n^it = \sum_{M - N < n \leq M + N} f(n)\chi(n)n^it. \]

We define the integers
\[ U = \lfloor N \frac{t^{-1/4}q^{-1/4}} \rfloor, \quad V = \lfloor t^{1/4}q^{1/4} \rfloor, \quad \text{(17)} \]
and the sets
\[ \mathcal{U} = \{ U/2 < u \leq U, \ (u, q) = 1 \}, \quad \mathcal{V} = \{ V/2 < v \leq V \}, \]
so by assumption on \( N \) and \( t \) we have \( U, V \geq 1 \) and \( UV \leq N \). Since
\[ N \geq UV, \]

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we have
\[
\sum_{M < n \leq M+N} \chi(n)n^it = \frac{1}{\#U\#V} \sum_{M-N < n \leq M+N} \sum_{u \in U} \sum_{v \in V} f(n+uv)\chi(n+uv)(n+uv)^it.
\]

As in [6], let \( g(y) \) denote the Fourier transform of \( f(x) \), so that
\[
f(x) = \int_{-\infty}^{\infty} g(y)e^{-2\pi ixy}dy,
\]
and
\[
\sum_{M < n \leq M+N} \chi(n)n^it = \frac{1}{\#U\#V} \sum_{M-N < n \leq M+N} \sum_{u \in U} \int_{-\infty}^{\infty} g(y) \left( \sum_{v \in V} \chi(n+uv)(n+uv)^it e^{2\pi iuv} \right) dy.
\]
The change of variable \( x = uy \) in the above integral gives
\[
\left| \sum_{M < n \leq M+N} \chi(n)n^it \right| \leq \frac{1}{\#U\#V} \sum_{M-N < n \leq M+N} \sum_{u \in U} \int_{-\infty}^{\infty} \frac{1}{u} \left| g\left( \frac{y}{u} \right) \right| \left| \sum_{v \in V} \chi(nu^* + v)(nu^{-1} + v)^it e^{2\pi iuv} \right| dy,
\]
where \( u^* \) denotes the multiplicative inverse of \( u \) (mod \( q \)). As in [6, Theorem 2], we have
\[
\frac{1}{u} \left| g\left( \frac{y}{u} \right) \right| \ll \min \left( N, \frac{1}{|y|}, \frac{U}{|y|^2} \right),
\]
so that
\[
\left| \sum_{M < n \leq M+N} \chi(n)n^it \right| \leq \frac{1}{\#U\#V} \int_{-\infty}^{\infty} \min \left( N, \frac{1}{|y|}, \frac{U}{|y|^2} \right) \times 
\sum_{M-N < n \leq M+N} \sum_{u \in U} \sum_{v \in V} \chi(nu^* + v)(nu^{-1} + v)^it e^{2\pi iuv} \bigg| dy.
\]
Let $\alpha$ be defined by

$$\max_{y \in \mathbb{R}} \sum_{M - N < n \leq M + N} \sum_{u \in U} \sum_{v \in V} \chi(n + uv)(n + uv)^i e^{2\pi i ny}$$

$$= \sum_{M - N < n \leq M + N} \sum_{u \in U} \sum_{v \in V} \chi(n + uv)(n + uv)^i e^{2\pi i ny},$$

so that

$$\left| \sum_{M < n \leq M + N} \chi(n)n^i \right| \leq \left( \int_{-\infty}^{\infty} \min \left( N, \frac{1}{|y|}, \frac{U}{|y|^2} \right) dy \right) \times \frac{1}{\#U \#V} \sum_{M - N < n \leq M + N} \sum_{u \in U} \sum_{v \in V} \chi(nu^* + v)(nu^{-1} + v)^i e^{2\pi i \alpha \nu}.$$

Since

$$\int_{-\infty}^{\infty} \min \left( N, \frac{1}{|y|}, \frac{U}{|y|^2} \right) dy \leq (qt)^{o(1)},$$

we get

$$\left| \sum_{M < n \leq M + N} \chi(n)n^i \right| \leq \frac{(qt)^{o(1)}}{\#U \#V} \sum_{M - N < n \leq M + N} \sum_{u \in U} \sum_{v \in V} \chi(nu^* + v)(nu^{-1} + v)^i e^{2\pi i \alpha \nu}.$$

Let

$$W = \sum_{M - N < n \leq M + N} \sum_{u \in U} \sum_{v \in V} \chi(nu^* + v)(nu^{-1} + v)^i e^{2\pi i \alpha \nu}, \quad (18)$$

and

$$H = tV^{-1}. \quad (19)$$

For integer $h$, we consider the intervals

$$I_h = \left[ \frac{h}{H}, \frac{h + 1}{H} \right),$$

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and define the sets $\Omega_h$ by

$$\Omega_h = \{(n, u) : M - N < n \leq M + N, \ u \in U, \ \frac{n}{u} \in I_h \},$$

so that by assumption on $M$, $\Omega_h$ is empty for $h < 0$ and $h > 6HN/U$. Using ideas from [8], let

$$F(n, u, x) = \left( \sum_{V/2 < v \leq V} \chi(nu^* + v)(x + v)^it e^{2\piiva} \right)^4,$$

so that we may rewrite (18) as

$$W = \sum_{M-N<n\leq M+N} \sum_{u\in U} |F(n, u, nu^{-1})|^{1/4}. \quad (20)$$

If $nu^{-1} \in \Omega_h$, by Lemma 4 we have

$$|F(n, u, nu^{-1})| \leq H \int_{h/H}^{(h+1)/H} |F(n, u, x)|dx + \int_{h/H}^{(h+1)/H} |F'(n, u, x)|dx. \quad (21)$$

Let $I_h(\lambda)$ denote the number of solutions to the congruence

$$nu^* \equiv \lambda \pmod{q}, \quad (n, u) \in \Omega_h,$$

and writing

$$G(\lambda, x) = \sum_{V/2 < v \leq V} \chi(\lambda + v)(x + v)^it e^{2\piiva}, \quad (22)$$

we have by (20) and (21),

$$|W| \ll \sum_{h \leq 6HN/U} \sum_{\lambda=1}^{q} I_h(\lambda) \left( H \int_{h/H}^{(h+1)/H} |G(\lambda, x)|^4 dx \right)^{1/4}$$

$$+ \sum_{h \leq 6HN/U} \sum_{\lambda=1}^{q} I_h(\lambda) \left( \int_{h/H}^{(h+1)/H} |G'(\lambda, x)||G(\lambda, x)|^3 dx \right)^{1/4}. \quad (23)$$
By Hölder’s inequality
\[
\int_{h/H}^{(h+1)/H} |G'(\lambda, x)| |G(\lambda, x)|^3 \, dx \leq \left( \int_{h/H}^{(h+1)/H} |G(\lambda, x)|^4 \, dx \right)^{3/4} \times \left( \int_{h/H}^{(h+1)/H} |G'(\lambda, x)|^4 \, dx \right)^{1/4},
\]
and since
\[
|G'(\lambda, x)| = t \left| \sum_{V/2<v\leq V} \chi(\lambda+v)(x+v)^{it-1}e^{2\pi iv\alpha} \right|,
\]
we have by partial summation
\[
|G'(\lambda, x)| \leq tV^{-1} \max_{V/2<v\leq V} \left| \sum_{V/2<v\leq Q} \chi(\lambda+v)(x+v)^{it}e^{2\pi iv\alpha} \right| \quad \text{for} \quad x \geq 0.
\]
Hence by (19), (22), (23) and (24)
\[
|W| \ll \left( \frac{t}{V} \right)^{1/4} \sum_{h \leq 6NH/U} \sum_{\lambda=1}^q I_h(\lambda) \left( \int_{h/H}^{(h+1)/H} \max_{V/2<v\leq Q} \left| \sum_{V/2<v\leq Q} \chi(\lambda+v)(x+v)^{it}e^{2\pi iv\alpha} \right|^4 \, dx \right)^{1/4}.
\]
Two applications of Hölder’s inequality gives,
\[
|W|^4 \leq \frac{t}{V} \left( \sum_{h \leq 6NH/U} \sum_{\lambda=1}^q I_h(\lambda) \right)^2 \left( \sum_{h \leq 6NH/U} \sum_{\lambda=1}^q I_h(\lambda)^2 \right) \times \left( \sum_{\lambda=1}^q \int_0^{6N/U} \max_{V/2<v\leq Q} \left| \sum_{V/2<v\leq Q} \chi(\lambda+v)(x+v)^{it}e^{2\pi iv\alpha} \right|^4 \, dx \right).
\]
Since
\[ \sum_{h \leq 6NH/U} \sum_{\lambda=1}^{q} I_h(\lambda), \]
is equal to the number of solutions to the congruence
\[ nu \equiv \lambda \pmod{q}, \]
with
\[ u \in U, \quad M - N < n \leq M + N, \quad 1 \leq \lambda \leq q, \]
we have
\[ \sum_{h \leq 6NH/U} \sum_{\lambda=1}^{q} I_h(\lambda) \leq NU. \] (25)
The term
\[ \sum_{h \leq 6NH/U} \sum_{\lambda=1}^{q} I_h(\lambda)^2, \]
is equal to the number of solutions to the congruence
\[ n_1 u_1 \equiv n_2 u_2 \pmod{q}, \]
with
\[ u_1, u_2 \in U, \quad M - N < n_1, n_2 \leq M + N, \]
so that from Lemma 3 we have,
\[ \sum_{h \leq 6NH/U} \sum_{\lambda=1}^{q} I_h(\lambda)^2 \leq NUq^{o(1)}. \] (26)

From (25) and (26) we get
\[ |W|^4 \leq \frac{t}{V} (NU)^3 q^{o(1)} \left( \sum_{\lambda=1}^{q} \max_{V/2 < Q \leq V} \left| \sum_{V/2 < v \leq Q} \chi(\lambda + v)(x + v)^it e^{2\pi i ov} \right|^4 \right). \]

By Lemma 11
\[ \sum_{\lambda=1}^{q} \max_{V/2 < Q \leq V} \left| \sum_{V/2 < v \leq Q} \chi(\lambda + v)(x + v)^it e^{2\pi i ov} \right|^4 dx \ll \frac{t}{V} (V^2 q + q^{1/2} t^{-1/2} V^4) (gt)^{o(1)}, \]
so that

$$|W|^4 \leq t(NU)^3 \left( V^2 q + q^{1/2} t^{-1/2} V^4 \right) (qt)^{o(1)},$$

which gives

$$\left| \sum_{M<n \leq M+N} \chi(n)n^{it} \right| \leq t^{1/4} N^{3/4} U^{-1/4} \left( q^{1/4} V^{-1/2} + q^{1/8} t^{-1/8} \right) (qt)^{o(1)}.$$ 

Recalling the choices of $U$ and $V$ in (17) gives

$$\left| \sum_{M<n \leq M+N} \chi(n)n^{it} \right| \leq N^{1/2} (qt)^{3/16 + o(1)}.$$

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