COINVARIANT ALGEBRAS AND FAKE DEGREES FOR SPIN WEYL GROUPS OF EXCEPTIONAL TYPE

CONSTANCE BALTERA AND WEIQIANG WANG

Abstract. We compute all the spin fake degrees for the exceptional Weyl groups, which are by definition the graded multiplicities of the simple modules of a spin Weyl group in its spin coinvariant algebra. The spin fake degrees are all shown to be palindromic polynomials.

1. Introduction

We formulated in [BW] (also see [WW]) the notion of spin coinvariant algebras and spin fake degrees for every Weyl group $W$, which is a spin analogue of coinvariant algebras and fake degrees for Weyl groups [Lu]. The spin fake degrees were computed in [WW] for type $A$ and in [BW] for the remaining classical types. The goal of this sequel to [BW] is to compute all the spin fake degrees for the exceptional Weyl groups of types $G_2, F_4, E_6, E_7, E_8$.

The spin Weyl group algebra $\mathbb{C}W^-$ (associated to a distinguished double cover of $W$) is naturally a superalgebra, and a $\mathbb{C}W^-$-module is always understood to be $\mathbb{Z}_2$-graded (see [BW, §2] for more detail). Informally speaking, the spin fake degrees are the graded multiplicities of the simple modules of $\mathbb{C}W^-$ in its spin coinvariant algebra.

We shall denote by $|\mathbb{C}W^-|$ the underlying ungraded algebra. For $W$ of exceptional type, the split classes of $W$ were classified (built on the work of Carter [Ca]), the simple ungraded $|\mathbb{C}W^-|$-characters constructed, and the spin character tables computed by Morris [Mo].

After a review in Section 2 of the formulation of spin fake degrees in the framework of the ($\mathbb{Z}_2$-graded) module theory of the superalgebra $\mathbb{C}W^-$ (see [BW, §1, §3] for detail), we establish in Section 3 a spin variant of Molien’s formula, a basic formula used in the computation of spin fake degrees in this paper.

In Section 4, we classify the simple $\mathbb{C}W^-$-modules and determine their types, refining the classification of ungraded simple modules due to Morris [Mo]. Then we write new code using CHEVIE [CHE] and [GAP] to compute the spin fake degrees in all exceptional cases, where as an input we use the spin character tables computed in [Mo] (in which we note a typo in the $E_8$ spin character table). The GAP code for computing all spin fake degrees for both exceptional and classical Weyl groups is available in [Ba]. The spin fake degrees of all exceptional types are tabulated (see Tables 1-5) in Section 5. We also take this opportunity to add Tables 6-9 for spin fake degrees of some classical Weyl groups of low rank. Combining with the results for the classical types in [BW],

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we observe that all the spin fake degrees for all Weyl groups are palindromic. A similar palindromicity was observed for the usual fake degrees by Beynon-Lusztig [BL].

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2. Spin coinvariant algebras and spin fake degrees

In this section we recall from [BW] the notion of spin Weyl groups, spin coinvariant algebras, and spin fake degrees.

2.1. Spin Weyl groups. Let $W$ be a Weyl group with the following presentation:

$$\langle s_1, \ldots, s_n \mid (s_is_j)^{m_{ij}} = 1, m_{ii} = 1, m_{ij} = m_{ji} \in \mathbb{Z}_{\geq 2}, \text{for } i \neq j \rangle. \quad (2.1)$$

We shall label the vertices of the Coxeter diagrams of the exceptional Weyl groups as in [BW, §2.1]. In this paper (as in [KW, BW]), we shall be concerned exclusively with a distinguished double covering $\tilde{W}$ of $W$:

$$1 \longrightarrow \mathbb{Z}_2 \longrightarrow \tilde{W} \longrightarrow W \longrightarrow 1. \quad (2.2)$$

We denote by $\mathbb{Z}_2 = \{1, z\}$. The quotient algebra of $\mathbb{C}W$ by the ideal generated by $z + 1$ is denoted by $\mathbb{C}W^-$ and called the spin Weyl group algebra associated to $W$. The double cover $\tilde{W}$ is chosen so that the spin Weyl group algebra $\mathbb{C}W^-$ has the following uniform presentation: $\mathbb{C}W^-$ is the algebra generated by $t_i, 1 \leq i \leq n$, subject to the relations

$$\left( t_it_j \right)^{m_{ij}} = (-1)^{m_{ij}+1}, \quad \forall i, j. \quad (2.3)$$

The algebra $\mathbb{C}W^-$ has a natural superalgebra structure by letting each $t_i$ be odd.

2.2. Spin coinvariant algebras and spin fake degrees. Denote by $V$ the irreducible reflection representation of the Weyl group $W$. Note that $V$ carries a $W$-invariant nondegenerate bilinear form $(\cdot, \cdot)$, and let $\mathcal{Cl}_V$ be the Clifford algebra associated to $(V, (\cdot, \cdot))$. The action of $W$ on $V$ preserves the bilinear form $(\cdot, \cdot)$ and thus $W$ acts as automorphisms of the algebra $\mathcal{Cl}_V$. This gives rise to a semi-direct product $\mathcal{H}_c^W := \mathcal{Cl}_V \rtimes W$, which is called the Hecke-Clifford algebra for $W$. Note that $\mathcal{Cl}_V$ is naturally an $\mathcal{H}_c^W$-module. The algebra $\mathcal{H}_c^W$ is endowed with a superalgebra structure by letting each element in $W$ be even and each $\beta_i$ be odd.

A module over a superalgebra $A$ is always understood in this paper as a $\mathbb{Z}_2$-graded $A$-module $M = M_0 \oplus M_1$ whose grading is compatible with the action of $A$; that is, $A_iM_j \subseteq M_{i+j}$. We shall denote by $|A|$ the underlying algebra of $A$ with $\mathbb{Z}_2$-grading forgotten, and by $|M|$ the $|A|$-module with the $\mathbb{Z}_2$-grading of $M$ forgotten. We denote by $A\text{-mod}$ the category of finite-dimensional $A$-modules.

Accordingly to [KW, Theorem 2.4], there exists an explicit isomorphism of superalgebras:

$$\Phi : \mathcal{H}_c^W = \mathcal{Cl}_V \rtimes W \xrightarrow{\sim} \mathcal{Cl}_V \otimes \mathbb{C}W^-, \quad \text{which extends the identity map on } \mathcal{Cl}_V. \quad \text{We say the superalgebras } \mathcal{H}_c^W \text{ and } \mathbb{C}W^- \text{ are Morita super-equivalent, as there exists a functor } \mathcal{G} : B\text{-mod} \longrightarrow A\text{-mod},$$

which is
almost an equivalence of categories (see [BW, Proposition 3.3]). In particular, it follows by [BW, Theorem 3.5] that we have \( \mathcal{G}(\text{Cl}_V) \cong \mathcal{B}_W \) as \( CW^- \)-modules, where \( \mathcal{B}_W \) is the basic spin module of \( CW^- \) (see [BW, §3.3]).

The Weyl group \( W \) acts on \( V \) as its reflection representation, and then on the symmetric algebra \( S^*V \). The coinvariant algebra \( (S^*V)_W = S^*V/(S^*V)_W^+ \), where \( (S^*V)_W^+ \) denotes the ideal generated by the homogeneous \( W \)-invariants of positive degrees, is a graded regular representation of \( W \). Following [BW], we call \( \text{Cl}_V \otimes (S^*V)_W \) the spin coinvariant algebra for \( W \). Note that

\[
\text{Cl}_V \otimes (S^*V)_W = \bigoplus_k \text{Cl}_V \otimes (S^kV)_W
\]

is a graded regular representation of the Hecke–Clifford superalgebra \( \mathcal{H}_W \), where \( \text{Cl}_V \) acts by left multiplication on the first tensor factor and \( W \) acts diagonally. The functor \( \mathcal{G} \) sends the \( \mathcal{H}_W \)-module \( \text{Cl}_V \otimes (S^*V)_W \) to the \( CW^- \)-module \( \mathcal{B}_W \otimes (S^*V)_W \).

Let \( \chi \) be a simple \( \mathcal{H}_W \)-module or its character, and let \( \chi^- \) be a simple \( CW^- \)-module or its character corresponding to \( \chi \) under the Morita super-equivalence \( \mathcal{G} \). Define

\[
P_W(\chi, t) = \sum_k \dim \text{Hom}_{\mathcal{H}_W}(\chi, \text{Cl}_V \otimes (S^kV)_W)t^k,
\]

\[
P_W^-(\chi^-, t) = \sum_k \dim \text{Hom}_{CW^-}(\chi^-, \mathcal{B}_W \otimes (S^kV)_W)t^k,
\]

\[
H_W^- (\chi^-, t) = \sum_k \dim \text{Hom}_{CW^-}(\chi^-, \mathcal{B}_W \otimes S^kV)t^k.
\]

The precise relations among \( P_W(\chi, t) \), \( P_W^-(\chi^-, t) \), and \( H_W^- (\chi^-, t) \) were determined in [BW, §3]. By [BW, Lemma 3.11], we have

\[
P_W^-(\chi^-, t) = H_W^- (\chi^-, t) \prod_{i=1}^n (1 - t^{d_i}),
\]

where the \( d_i \)'s are the degrees of the Weyl group \( W \). Following [BW], we call the polynomial \( P_W^-(\chi^-, t) \) the spin fake degree of the simple \( CW^- \)-character \( \chi^- \).

The main goal of this paper is to compute the spin fake degrees \( P_W^- (\chi^-, t) \) for every exceptional Weyl group \( W \) and every simple \( CW^- \)-character \( \chi^- \).

3. Spin Molien’s Formula

In this section we formulate a spin version of Molien’s formula for later use in computing the spin fake degrees.

Recall the double cover \( \theta : \tilde{W} \rightarrow W \) from (2.2). We now consider the conjugacy classes of \( W \) and \( \tilde{W} \). All the elements of a given conjugacy class have the same parity, so a given conjugacy class in \( W \) is either even or odd. Let \( K \) be a conjugacy class of \( W \). Then \( \theta^{-1}(K) \) is either a single conjugacy class of \( \tilde{W} \), or splits into two as \( \theta^{-1}(K) = \tilde{K} \cup z\tilde{K} \); in the latter case, we say that \( \tilde{K} \), \( K \), and \( z\tilde{K} \) are split classes. We say \( x \in W \) is split if it belongs to a split conjugacy class. If we denote \( \theta^{-1}(x) = \{\tilde{x}, z\tilde{x}\} \), \( x \) is split if and only if \( \tilde{x} \) and \( z\tilde{x} \) are not conjugate in \( \tilde{W} \).
Lemma 3.1. Let \( \psi \) be a simple character of \( \mathbb{C}W^- \) (or, equivalently, of \( \widetilde{W} \) where \( z \) acts as \(-1\)), and let \( \bar{x} \in \widetilde{W} \). Then \( \psi(\bar{x}) = 0 \), unless \( \bar{x} \in W \) is even split.

Proof. Let \( M \) be the \((\mathbb{Z}_2\text{-graded})\) module of \( \widetilde{W} \) underlying \( \psi \). For \( \bar{x} \) odd, \( \bar{x} \) switches the even and odd parts of \( M \), and hence the trace of \( \bar{x} \) on \( \bar{M} \) is zero, i.e., \( \psi(\bar{x}) = 0 \).

If \( \bar{x} \) is even and non-split, then \( \bar{x} \) is conjugate to \( z\bar{x} \) by definition. So we have the trace identity on \( M \): \( \text{tr}(\bar{x}) = \text{tr}(z\bar{x}) = -\text{tr}(\bar{x}) \), since \( z \) acts by \(-1\) on \( M \). Hence again \( \psi(\bar{x}) = \text{tr}(\bar{x}) = 0 \). \( \square \)

We shall denote by \( \langle x \rangle \) the conjugacy class of \( x \in W \), and \( C_x \) the centralizer of \( x \) in \( W \). The following proposition is a variation on Molien’s formula. For a finite group \( G \), we denote by \( G^e \) the set of even split conjugacy classes of \( G \); cf. [Joz], [CW, §3.1.4], and [BW, §2].

Proposition 3.2 (Spin Molien’s formula). Let \( W \) be a Weyl group, and let \( \chi^- \) be a simple \( \mathbb{C}W^- \)-character. Then the graded multiplicity of \( \chi^- \) in the \( \mathbb{C}W^- \)-module \( B_W \otimes S^*V \) is

\[
H^-_{W}(\chi^-, t) = \sum_{\langle x \rangle \in W^e} \frac{\chi^-(\bar{x})\text{tr}(\bar{x})|_{B_W}}{|C_x|\det(1-tx)},
\]

where \( \bar{x} \in \widetilde{W} \) is chosen such that \( x = \theta(\bar{x}) \).

Proof. The character values of the \( \widetilde{W} \)-modules \( B_W \) and \( S^*V \) are all real. Using the standard \( \widetilde{W} \)-character inner product \((\cdot, \cdot)\) (see [Joz, Theorem 4.12]), we compute

\[
H^-_{W}(\chi^-, t) = (\chi^-, B_W \otimes S^jV)
= (\chi^-, B_W \otimes \sum_{j \geq 0} t^j(S^jV))
= \frac{1}{|W|} \sum_{\bar{x} \in \widetilde{W}} \chi^-(\bar{x})\text{tr}(\bar{x}^{-1})|_{B_W} \left( \sum_{j \geq 0} \text{tr}(x^{-1})|_{S^jV} t^j \right)
= \frac{1}{2|W|} \sum_{\bar{x} \in \widetilde{W}} \chi^-(\bar{x})\text{tr}(\bar{x})|_{B_W} \frac{1}{\det(1-tx)}.
\]

We first reorganize this sum over conjugacy classes \( \langle \bar{x} \rangle \) of \( \widetilde{W} \). By Lemma 3.1, we may restrict \( \langle \bar{x} \rangle \) to even split classes from now on. The split conjugacy classes in \( \widetilde{W} \) are obtained as half of the inverse image of split conjugacy classes in \( W \), that is, \( \langle \bar{x} \rangle \cup z\langle \bar{x} \rangle = \theta^{-1}(\langle x \rangle) \). The classes \( \langle \bar{x} \rangle \) and \( z\langle \bar{x} \rangle \) are of the same size (as the class \( \langle x \rangle \)), but have opposite character values. We may reorganize the sum now over even split conjugacy classes of \( W \). The details are as follows:

\[
\frac{1}{2|W|} \sum_{\bar{x} \in \widetilde{W}} \frac{\chi^-(\bar{x})\text{tr}(\bar{x})|_{B_W}}{\det(1-tx)} = \frac{1}{2} \sum_{\langle x \rangle \in W^e} \frac{\chi^-(\bar{x})\text{tr}(\bar{x})|_{B_W}}{|C_x|\det(1-tx)}
= \sum_{\langle x \rangle \in W^e} \frac{\chi^-(\bar{x})\text{tr}(\bar{x})|_{B_W}}{|C_x|\det(1-tx)}.
\]
The proposition is proved.

We have the following corollary of Proposition 3.2 thanks to (2.5).

**Corollary 3.3.** Let \( W \) be a Weyl group with degrees \( d_1, \ldots, d_n \), and let \( \chi^- \) be a simple \( CW^- \)-character. Then the spin fake degree of \( \chi^- \) is

\[
P_W(\chi^-, t) = \sum_{\langle x \rangle \in W_{\chi^-}} \frac{\chi^-(\tilde{x})}{|C_x|} \det(1 - tx) \prod_{i=1}^{n} (1 - t^{d_i}),
\]

where \( \tilde{x} \in \tilde{W} \) is chosen such that \( x = \theta(\tilde{x}) \).

4. **The spin fake degrees of exceptional Weyl groups**

Let \( W \) be an exceptional Weyl group throughout this section.

4.1. **List of split classes.** We first briefly recall Carter’s parametrization of conjugacy classes of Weyl groups by admissible diagrams [Ca] as follows. Given a conjugacy class, choose a representative element \( w \), and decompose it into a product of two involutions subject to certain conditions (see [Ca, Section 3]). Each involution is a product of reflections corresponding to mutually orthogonal roots, and the admissible diagram is a graph whose nodes correspond to (the roots associated to) the reflections, with the edge between nodes corresponding to roots \( r \) and \( s \) having weight \( 4 \frac{[r,s]}{[r,r][s,s]} \). Many of the possible graphs resemble Dynkin diagrams and are named accordingly. Carter shows that if \( w \) has an admissible diagram \( \Gamma \), then so must all its conjugates [Ca, p.6], and that we may describe the conjugacy classes of any Weyl group \( W \) by such admissible diagrams [Ca, p.45].

Morris [Mo] has determined the split conjugacy classes for the exceptional Weyl groups \( W \) (with the double covers \( \tilde{W} \)), and their descriptions are given in terms of Carter’s parametrization by admissible diagrams.

We also need to determine the parity of the split classes, and this can be done in a simple and precise manner. It turns out that the nodes of the admissible diagram that labels a conjugacy class of \( W \) correspond to the reflections in a certain decomposition of an element in that class. Since all reflections in \( W \) are odd, the parity of each split conjugacy class of \( W \) may be read off from the number of nodes in the corresponding admissible diagram. Below we summarize Morris’ classification of split classes, enhanced by the parity separation.

**Proposition 4.1.** A complete list of split classes of every exceptional Weyl group \( W \) is as follows.

- **(E₆)** There are 9 even split classes: \( \emptyset, A_2, A_4, 2A_2, D_4(a_1), 3A_2, E_6(a_1), E_6(a_2) \).
  There are 4 odd split classes: \( D_5, D_5(a_1), D_3 + D_2, \) and \( A_4 + A_1 \).

- **(E₇)** There are 13 even split classes: \( \emptyset, D_4(a_1), A_2, 2A_2, 3A_2, E_6(a_2), E_6, E_6(a_1), A_4, A_4 + A_2, D_6(a_1), A_6, \) and \( D_6 \);
  There are 13 odd split classes: \( 7A_1, 2A_3 + A_1, D_4 + 3A_1, D_6(a_2) + A_1, E_7(a_4), A_5 + A_2, E_7(a_2), E_7, D_6 + A_1, E_7(a_3), A_7, E_7(a_1), A_4 + A_1 \).

- **(GFE)** All 3, 9, 30 split classes of \( G_2, F_4, \) and \( E_8 \) listed in [Mo, §8,§9] are even.
4.2. Classification of simple modules. Recall a simple module of a superalgebra $A$ always has endomorphism algebra of dimension 1 or 2, cf. [Joz] and [BW, §2]. A simple $A$-module is called type $\mathcal{M}$ (and respectively, type $\mathcal{Q}$) if its endomorphism algebra is of dimension 1 (and respectively, 2). Moreover, there is a simple recipe which determines the numbers of simple $\mathbb{C}W^-$-modules of type $\mathcal{M}$ and of type $\mathcal{Q}$ by knowing the numbers of even split and odd split classes of $W$, cf. [Joz, Proposition 4.14], [CW, Proposition 3.8], or [BW, Proposition 2.5].

Now Proposition 4.1 allows us to determine the numbers of simple $\mathbb{C}W^-$-modules of type $\mathcal{M}$ and type $\mathcal{Q}$. Morris [Mo] (also cf. Read [Re] for $E_8$) has determined the spin character table of all the ungraded simple characters of $|\mathbb{C}W^-|$. It turns out that we may determine which pairs of ungraded simple characters add to become type $\mathcal{Q}$ characters of $\mathbb{C}W^-$ since the character values for any $\mathbb{C}W^-$-module on the odd split classes must be 0.

We will follow Morris’s notation [Mo], referring to the irreducible spin characters for the exceptional Weyl groups by their degrees with a subscript $s$ (or $ss$, $sss$, $ssss$, for multiple characters of the same degree). In cases where two ungraded simple characters of the same degree add to become a type $\mathcal{Q}$ character (which only happens in $E_6, E_7$), we will denote the type $\mathcal{Q}$ character by its degree with the shortest possible subscript and a superscript $Q$.

The Weyl group $E_7$ is the direct product of the Chevalley group $B_3(2)$ and a cyclic group $\langle \xi \rangle$ of order 2, where all elements in $B_3(2)$ are even while the generator $\xi$ is odd. Hence the simple $E_7$-modules are exactly the tensor products of simple $B_3(2)$-modules with the unique two-dimensional simple module of $\langle \xi \rangle$ (of type $\mathcal{Q}$), and they are manifestly all of type $\mathcal{Q}$. Table E summarizes the conversion between the new type $\mathcal{Q}$ notation and Morris’s ungraded simple characters.

Table E: Type $\mathcal{Q}$ characters as sums of ungraded simple characters

| $E_6$ | Type $\mathcal{Q}$ character | Ungraded characters |
|-------|-----------------------------|--------------------|
|       | $120^s_6$, $160^s_8$, $40^s_{ss}$, $128^s_6$ | $60_s + 60_{ss}$, $80_s + 80_{ss}$, $20_s + 20_{ss}$, $64_s + 64_{ss}$ |
| $E_7$ | Type $\mathcal{Q}$ character | Ungraded characters |
|       | $16^s_8$, $96^s_8$, $336^s_8$, $560^s_8$, $224^s_8$ | $8_s + 8_{ss}$, $48_s + 48_{ss}$, $168_s + 168_{ss}$, $280_s + 280_{ss}$, $112_s + 112_{ss}$ |
|       | $224^s_{ss}$, $1024^s_8$, $1440^s_8$, $1120^s_8$, $890^s_8$ | $112_s + 112_{ss}$, $512_s + 512_{ss}$, $720_s + 720_{ss}$, $560_s + 560_{ss}$, $448_s + 448_{ss}$ |
|       | $240^s_8$, $128^s_8$, $128^s_8$ | $120_s + 120_{ss}$, $64_s + 64_{ss}$, $64_s + 64_{ss}$ |

In this way, we have upgraded the results of Morris [Mo] into the following.

**Proposition 4.2.** The types of the simple $\mathbb{C}W^-$ modules, for $W$ an exceptional Weyl group, are as follows.

1. $\mathbb{C}E_6^-$ has 5 type $\mathcal{M}$ simple modules $8_s, 40_s, 72_s, 40_{ss}, 120_s$, and 4 type $\mathcal{Q}$ simple modules $120^s_{ss}, 160^s_8, 40^s_{ss}$, $128^s_6$.
2. All 13 simple $\mathbb{C}E_7^-$-modules are of type $\mathcal{Q}$.
3. All 30 simple $\mathbb{C}E_8^-$-modules are of type $\mathcal{M}$, as are all 9 simple $\mathbb{C}F_4^-$-modules and all 3 simple $\mathbb{C}G_2^-$-modules.
4.3. **Spin fake degrees.** The main computational tool for the spin fake degrees of the exceptional Weyl groups is the spin Molien’s formula, see Proposition 3.2 or Corollary 3.3. We implement this using CHEVIE; code is available upon request. To that end, as inputs we shall need the spin character tables of $\mathbb{C}W^-$, which were computed by Morris [Mo] in the ungraded setting.

More precisely, we use the spin character tables of Morris for $E_6$ [Mo, Table III], $E_7$ [Mo, Table IV], $E_8$ [Mo, Table V], and $G_2$ [Mo, Table VI]. In cases of $E_6$ and $E_7$, we need to add suitable pairs of columns in the spin character tables to form the characters of type $Q$ simple modules.

**Remark 4.3.** There is a typo in the $E_8$ spin character table in [Mo]: the thirteenth entry of the last character should be 2 rather than $-2$. This is detected and corrected using the orthogonality relations of simple characters.

For $W$ of type $F_4$, Read has labeled its 9 simple characters as $\phi_1, \ldots, \phi_9$. See Table F for a comparison between Read’s notation [Re, Table 1] and Morris’s [Mo, Table VII].

**Table F: Two labelings of the spin character table for $F_4$**

| Morris’s labels | 4s | 4ss | 8sss | 8ssss | 12ss | 12s | 8s | 24s | 8ss |
|-----------------|----|-----|------|-------|------|-----|----|-----|-----|
| Read’s labels   | $\phi_1$ | $\phi_2$ | $\phi_3$ | $\phi_4$ | $\phi_5$ | $\phi_6$ | $\phi_7$ | $\phi_8$ | $\phi_9$ |

The character values which we use in the computation of the spin fake degrees for $F_4$ are those given by Read [Re, Table 1]. They differ by a sign in the conjugacy classes $A_2, \tilde{A}_2$, and $B_2$ from those given by Morris [Mo, Table VII] because of a different choice of these $\tilde{F}_4$-conjugacy classes (differing by a factor $z = -1$). However this does not affect the computation of the spin fake degrees.

We summarize our CHEVIE computations in Proposition 4.4 and Theorem 4.5 below. Denote by $N$ the number of reflections in $W$. The following can be observed by inspection case-by-case from our CHEVIE computation.

**Proposition 4.4.** Let $W$ be an arbitrary exceptional Weyl group. Then for every simple $\mathbb{C}W^-$-character $\chi$, we have

$$P_W^{-}(\chi, t) = t^N P_W^{-}(\chi, t^{-1}).$$

(The statement already holds for classical Weyl groups [BW].)

We exploit this property when presenting the spin fake degrees for the exceptional groups in Table 1 through Table 5 in Section 5. In each column of the tables, we list only the coefficients of the spin fake degrees $P_W^{-}(\chi, t)$ (which are polynomials in $t$) for degrees 0 through $\frac{N}{2}$. The remaining halves of coefficients can be determined via palindromicity in Proposition 4.4.

**Theorem 4.5.** For $W$ exceptional, the coefficients of the spin fake degrees $P_W^{-}(\chi, t)$ are given in Table 1 through Table 5 in Section 5.

**Proof.** These values are computed using Corollary 3.3. The computations are performed using the GAP 3 package CHEVIE [GAP, CHE].
5. Tables for spin fake degrees

We use the notation and convention as specified in Section 4. In particular, the characters with superscripts $Q$ in Table 1 through Table 5 are of type $Q$, and those without are of type $M$. Note that in all tables, the character of the basic spin module $B_W$ is listed first.

In Table 6 through Table 9, we also list the spin fake degrees of classical Weyl groups of type $B_2, A_4, B_4,$ and $D_4$ (see [BW] for notation).

Table 2. Spin fake degrees for type $F_4$

| $4_s$ | $4_{ss}$ | $8_{ss}$ | $8_{sss}$ | $12_{ss}$ | $12_s$ | $8_s$ | $24_s$ | $8_{ss}$ |
|------|----------|----------|----------|----------|--------|-------|--------|--------|
| 0    | 1        |          |          |          |        |       |        |        |
| 1    | 1        |          |          |          |        |       |        |        |
| 2    | 0        | 1        |          |          |        |       |        |        |
| 3    | 0        | 1        |          |          |        |       |        |        |

Table 1. Spin fake degrees for type $G_2$

| $2_s$ | $2_{ss}$ | $2_{sss}$ |
|-------|----------|-----------|
| 0     | 1        |           |
| 1     | 1        | 1         |
| 2     | 0        | 1         |
| 3     | 0        | 2         |

Table 3. Spin fake degrees for type $E_6$

| $8_s$ | $40_s$ | $72_s$ | $40_{ss}$ | $120_s$ | $120^Q_{ss}$ | $160^Q_s$ | $40^Q_{ss}$ | $128^Q_s$ |
|-------|--------|--------|-----------|---------|-------------|----------|------------|---------|
| 0     | 1      |        |           |         |             |          |            |         |
| 1     | 1      | 1      |           |         |             |          |            |         |
| 2     | 0      | 1      |           |         |             |          |            |         |
| 3     | 0      | 1      |           |         |             |          |            |         |
| 4     | 1      | 3      | 1         |         |             |          |            |         |
| 5     | 2      | 4      | 3         |         |             |          |            |         |
| 6     | 1      | 4      | 4         |         |             |          |            |         |
| 7     | 1      | 5      | 5         | 1       | 11          | 14       | 12         | 6       |
| 8     | 2      | 8      | 9         | 3       | 13          | 18       | 18         | 4       |
| 9     | 2      | 9      | 13        | 6       | 18          | 24       | 24         | 8       |
| 10    | 1      | 9      | 14        | 8       | 27          | 30       | 34         | 6       |
| 11    | 2      | 11     | 19        | 8       | 34          | 34       | 44         | 8       |
| 12    | 4      | 14     | 27        | 13      | 39          | 38       | 52         | 14      |
| 13    | 3      | 15     | 29        | 19      | 47          | 44       | 64         | 16      |
| 14    | 1      | 14     | 30        | 18      | 55          | 52       | 74         | 18      |
| 15    | 2      | 16     | 35        | 20      | 59          | 56       | 80         | 24      |
| 16    | 4      | 19     | 39        | 28      | 62          | 56       | 88         | 26      |
| 17    | 3      | 18     | 40        | 26      | 67          | 58       | 92         | 24      |
| 18    | 2      | 16     | 40        | 20      | 70          | 60       | 92         | 24      |
Table 4. Spin fake degrees for type $E_7$

| $16^g$ | $96^g$ | $336^g$ | $560^g$ | $224^g$ | $224^g_{1,2}$ | $1024^g$ | $1440^g$ | $1120^g$ | $896^g$ | $240^g$ | $128^g$ | $128^g_{1,2}$ |
|--------|--------|---------|---------|---------|----------------|--------|---------|--------|-------|--------|--------|----------------|
| 0      | 2      |         |         |         |                |        |         |        |       |        |        |                |
| 1      | 2      | 2       |         |         |                |        |         |        |       |        |        |                |
| 2      | 0      | 2       | 2       |         |                |        |         |        |       |        |        |                |
| 3      | 0      | 0       | 4       | 2       |                |        |         |        |       |        |        |                |
| 4      | 0      | 2       | 4       | 4       | 2              |        |         |        |       |        |        |                |
| 5      | 2      | 4       | 6       | 4       | 2              | 4      | 2       |        |       |        |        |                |
| 6      | 2      | 6       | 8       | 6       | 4              | 6      | 6       | 2      |       |        |        |                |
| 7      | 2      | 6       | 10      | 10      | 4              | 12     | 10      | 2      | 4     |        |        |                |
| 8      | 2      | 6       | 14      | 16      | 6              | 18     | 16      | 4      | 4     | 2      |        |                |
| 9      | 2      | 8       | 20      | 22      | 8              | 2      | 26      | 24     | 12    | 6      |        |                |
| 10     | 2      | 10      | 26      | 28      | 10             | 4      | 38      | 36     | 18    | 14     | 6      |                |
| 11     | 2      | 12      | 30      | 36      | 14             | 4      | 52      | 56     | 26    | 22     | 10     | 2      2     |
| 12     | 4      | 14      | 36      | 46      | 20             | 8      | 68      | 80     | 40    | 32      | 14     | 4      4     |
| 13     | 4      | 18      | 44      | 58      | 26             | 14     | 90      | 104    | 60    | 50      | 18     | 4      4     |
| 14     | 4      | 18      | 54      | 72      | 30             | 18     | 116     | 136    | 86    | 68      | 22     | 8      8     |
| 15     | 2      | 18      | 64      | 90      | 34             | 22     | 142     | 178    | 116   | 90      | 28     | 12     12    |
| 16     | 4      | 22      | 72      | 108     | 40             | 30     | 176     | 222    | 150   | 122     | 36     | 14     14    |
| 17     | 6      | 28      | 82      | 124     | 52             | 40     | 212     | 274    | 192   | 154     | 44     | 20     20    |
| 18     | 6      | 30      | 94      | 142     | 62             | 48     | 250     | 334    | 240   | 188     | 56     | 26     26    |
| 19     | 4      | 30      | 104     | 166     | 66             | 58     | 294     | 396    | 288   | 234     | 68     | 30     30    |
| 20     | 4      | 32      | 116     | 190     | 74             | 68     | 338     | 462    | 346   | 278     | 76     | 38     38    |
| 21     | 6      | 36      | 130     | 212     | 86             | 82     | 382     | 530    | 410   | 322     | 86     | 48     48    |
| 22     | 6      | 40      | 140     | 234     | 94             | 96     | 430     | 600    | 468   | 376     | 100    | 52     52    |
| 23     | 6      | 40      | 148     | 256     | 102            | 104    | 476     | 676    | 526   | 424     | 114    | 60     60    |
| 24     | 6      | 42      | 158     | 278     | 112            | 116    | 518     | 748    | 588   | 468     | 124    | 70     70    |
| 25     | 8      | 46      | 170     | 298     | 120            | 132    | 562     | 806    | 648   | 520     | 134    | 74     74    |
| 26     | 6      | 48      | 180     | 316     | 126            | 142    | 600     | 866    | 702   | 562     | 144    | 82     82    |
| 27     | 6      | 46      | 186     | 332     | 130            | 146    | 632     | 926    | 748   | 594     | 154    | 90     90    |
| 28     | 6      | 48      | 190     | 346     | 136            | 156    | 662     | 968    | 784   | 632     | 164    | 92     92    |
| 29     | 8      | 52      | 196     | 354     | 144            | 166    | 684     | 1000   | 818   | 656     | 168    | 96     96    |
| 30     | 8      | 52      | 200     | 362     | 148            | 168    | 696     | 1026   | 842   | 668     | 170    | 102    102   |
| 31     | 6      | 50      | 200     | 368     | 142            | 168    | 706     | 1038   | 848   | 682     | 174    | 100    100   |
Table 5. Spin fake degrees for type $E_8$ (Part 1)

|   | 16,112 | 320,448 | 224,448,1680,2592,1344,5600,4800,2016,5600,9072,800,000 |
|---|-------------------|-------------------|-------------------|-------------------|-------------------|-------------------|-------------------|-------------------|
| 1 | 1                  | 1                  | 1                  | 1                  | 1                  | 1                  | 1                  | 1                  |
| 2 | 0                  | 0                  | 0                  | 0                  | 0                  | 0                  | 0                  | 0                  |
| 3 | 1                  | 1                  | 1                  | 1                  | 1                  | 1                  | 1                  | 1                  |
| 4 | 2                  | 2                  | 2                  | 2                  | 2                  | 2                  | 2                  | 2                  |
| 5 | 0                  | 0                  | 0                  | 0                  | 0                  | 0                  | 0                  | 0                  |
| 6 | 1                  | 1                  | 1                  | 1                  | 1                  | 1                  | 1                  | 1                  |
| 7 | 2                  | 2                  | 2                  | 2                  | 2                  | 2                  | 2                  | 2                  |
| 8 | 0                  | 0                  | 0                  | 0                  | 0                  | 0                  | 0                  | 0                  |
| 9 | 1                  | 1                  | 1                  | 1                  | 1                  | 1                  | 1                  | 1                  |
| 10| 2                  | 2                  | 2                  | 2                  | 2                  | 2                  | 2                  | 2                  |
| 11| 0                  | 0                  | 0                  | 0                  | 0                  | 0                  | 0                  | 0                  |
| 12| 1                  | 1                  | 1                  | 1                  | 1                  | 1                  | 1                  | 1                  |
| 13| 2                  | 2                  | 2                  | 2                  | 2                  | 2                  | 2                  | 2                  |
| 14| 0                  | 0                  | 0                  | 0                  | 0                  | 0                  | 0                  | 0                  |
| 15| 1                  | 1                  | 1                  | 1                  | 1                  | 1                  | 1                  | 1                  |
| 16| 2                  | 2                  | 2                  | 2                  | 2                  | 2                  | 2                  | 2                  |
| 17| 0                  | 0                  | 0                  | 0                  | 0                  | 0                  | 0                  | 0                  |
| 18| 1                  | 1                  | 1                  | 1                  | 1                  | 1                  | 1                  | 1                  |
| 19| 2                  | 2                  | 2                  | 2                  | 2                  | 2                  | 2                  | 2                  |
| 20| 0                  | 0                  | 0                  | 0                  | 0                  | 0                  | 0                  | 0                  |
| 21| 1                  | 1                  | 1                  | 1                  | 1                  | 1                  | 1                  | 1                  |
| 22| 2                  | 2                  | 2                  | 2                  | 2                  | 2                  | 2                  | 2                  |
| 23| 0                  | 0                  | 0                  | 0                  | 0                  | 0                  | 0                  | 0                  |
| 24| 1                  | 1                  | 1                  | 1                  | 1                  | 1                  | 1                  | 1                  |
| 25| 2                  | 2                  | 2                  | 2                  | 2                  | 2                  | 2                  | 2                  |
| 26| 0                  | 0                  | 0                  | 0                  | 0                  | 0                  | 0                  | 0                  |
| 27| 1                  | 1                  | 1                  | 1                  | 1                  | 1                  | 1                  | 1                  |
| 28| 2                  | 2                  | 2                  | 2                  | 2                  | 2                  | 2                  | 2                  |
| 29| 0                  | 0                  | 0                  | 0                  | 0                  | 0                  | 0                  | 0                  |
| 30| 1                  | 1                  | 1                  | 1                  | 1                  | 1                  | 1                  | 1                  |
| 31| 2                  | 2                  | 2                  | 2                  | 2                  | 2                  | 2                  | 2                  |
| 32| 0                  | 0                  | 0                  | 0                  | 0                  | 0                  | 0                  | 0                  |
| 33| 1                  | 1                  | 1                  | 1                  | 1                  | 1                  | 1                  | 1                  |
| 34| 2                  | 2                  | 2                  | 2                  | 2                  | 2                  | 2                  | 2                  |
| 35| 0                  | 0                  | 0                  | 0                  | 0                  | 0                  | 0                  | 0                  |
| 36| 1                  | 1                  | 1                  | 1                  | 1                  | 1                  | 1                  | 1                  |
| 37| 2                  | 2                  | 2                  | 2                  | 2                  | 2                  | 2                  | 2                  |
| 38| 0                  | 0                  | 0                  | 0                  | 0                  | 0                  | 0                  | 0                  |
| 39| 1                  | 1                  | 1                  | 1                  | 1                  | 1                  | 1                  | 1                  |
| 40| 2                  | 2                  | 2                  | 2                  | 2                  | 2                  | 2                  | 2                  |
| 41| 0                  | 0                  | 0                  | 0                  | 0                  | 0                  | 0                  | 0                  |
| 42| 1                  | 1                  | 1                  | 1                  | 1                  | 1                  | 1                  | 1                  |
| 43| 2                  | 2                  | 2                  | 2                  | 2                  | 2                  | 2                  | 2                  |
| 44| 0                  | 0                  | 0                  | 0                  | 0                  | 0                  | 0                  | 0                  |
| 45| 1                  | 1                  | 1                  | 1                  | 1                  | 1                  | 1                  | 1                  |
| 46| 2                  | 2                  | 2                  | 2                  | 2                  | 2                  | 2                  | 2                  |
| 47| 0                  | 0                  | 0                  | 0                  | 0                  | 0                  | 0                  | 0                  |
| 48| 1                  | 1                  | 1                  | 1                  | 1                  | 1                  | 1                  | 1                  |
| 49| 2                  | 2                  | 2                  | 2                  | 2                  | 2                  | 2                  | 2                  |
| 50| 0                  | 0                  | 0                  | 0                  | 0                  | 0                  | 0                  | 0                  |
| 51| 1                  | 1                  | 1                  | 1                  | 1                  | 1                  | 1                  | 1                  |
| 52| 2                  | 2                  | 2                  | 2                  | 2                  | 2                  | 2                  | 2                  |
| 53| 0                  | 0                  | 0                  | 0                  | 0                  | 0                  | 0                  | 0                  |
| 54| 1                  | 1                  | 1                  | 1                  | 1                  | 1                  | 1                  | 1                  |
| 55| 2                  | 2                  | 2                  | 2                  | 2                  | 2                  | 2                  | 2                  |
| 56| 0                  | 0                  | 0                  | 0                  | 0                  | 0                  | 0                  | 0                  |
| 57| 1                  | 1                  | 1                  | 1                  | 1                  | 1                  | 1                  | 1                  |
| 58| 2                  | 2                  | 2                  | 2                  | 2                  | 2                  | 2                  | 2                  |
| 59| 0                  | 0                  | 0                  | 0                  | 0                  | 0                  | 0                  | 0                  |
| 60| 1                  | 1                  | 1                  | 1                  | 1                  | 1                  | 1                  | 1                  |
Table 5. Spin fake degrees for type $E_8$ (Part 2)

| 2800, 5600, 7168, 1120, 8400, 11200, 6720, 2800, 1344, 6480, 8192, 2016, 20160, 7168, 856 | }
| Table 6. Spin fake degrees for type $B_2$ |
|---------------------------------------|
| (2) (1, 1)                           |
| 0 1                                  |
| 1 1 1                                |
| 2 0 2                                |

| Table 7. Spin fake degrees for type $A_4$ |
|---------------------------------------|
| (5) $^2$ (4, 1) (3, 2)                |
| 0 1                                  |
| 1 1 2                                |
| 2 1 4 2                              |
| 3 2 6 4                              |
| 4 2 8 6                              |
| 5 2 8 8                              |

| Table 8. Spin fake degrees for type $B_4$ |
|---------------------------------------|
| (4) (3, 1) (2, 2) (2, 1, 1) (1, 1, 1) |
| 0 1                                  |
| 1 1 1                                |
| 2 0 2 1                              |
| 3 1 2 1 2 1 2 1                     |
| 4 1 3 3 2 2                         |
| 5 1 4 3 4 1                         |
| 6 1 5 2 6 1                         |
| 7 1 5 4 6 2                         |
| 8 2 4 6 6 2                         |

| Table 9. Spin fake degrees for type $D_4$ |
|---------------------------------------|
| \{(1, 1, 1, 1), (4)\} \{(2, 1, 1, (3, 1)\} \{(2, 2)\} $+$ \{(2, 2)\} $-$ |
| 0 1                                  |
| 1 1 1                                |
| 2 0 3                                |
| 3 2 4 1 1                            |
| 4 2 5 3 3                            |
| 5 1 7 3 3                            |
| 6 2 8 2 2                            |
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