On the random G equation with nonzero divergence

William Cooperman

Received: 23 May 2022 / Accepted: 25 July 2023 / Published online: 11 August 2023
© The Author(s), under exclusive licence to Springer-Verlag GmbH Germany, part of Springer Nature 2023

Abstract
We prove a quantitative rate of homogenization for the G equation in a random setting with finite range of dependence and nonzero divergence, with explicit dependence of the constants on the Lipschitz norm of the environment. Inspired by work of Burago–Ivanov–Novikov, the proof uses explicit bounds on the waiting time for the associated metric problem.

Mathematics Subject Classification 49L12 (Primary); 60K35 (Secondary)

1 Introduction
We consider the behavior, as $\varepsilon \to 0^+$, of the family $\{u_\varepsilon\}_{\varepsilon > 0}$ of solutions to the G equation,
\begin{align}
D_t u_\varepsilon(t, x) - |D_x u_\varepsilon(t, x)| + V(\varepsilon^{-1} x) \cdot D_x u_\varepsilon(t, x) &= 0 \quad \text{in } \mathbb{R}_{>0} \times \mathbb{R}^d \\
u_\varepsilon(0, x) &= u_0(x) \quad \text{in } \mathbb{R}^d ,
\end{align}
where $d \geq 2$, $V : \mathbb{R}^d \to \mathbb{R}^d$ is a random vector field and the initial data $u_0 : \mathbb{R}^d \to \mathbb{R}^d$ is Lipschitz. The level sets of $u_\varepsilon$ model a flame front which expands at unit speed in the normal direction while being advected by $V$, which models the wind velocity. When compared with homogenization of other Hamilton–Jacobi equations, the main difficulty with the G equation is that, since we do not assume that $\|V\|_{L^\infty} < 1$, the equation may not be coercive. On the other hand, if $E[V] = 0$, then the equation is still “coercive on average”, so we can hope to recover some large-scale controllability.

When $\text{div} V = 0$, the wind cannot form “traps” where the flame can be contained, and so a controllability bound holds [9]. The main novelty of this paper is a more quantitative controllability bound, which allows for the possibility that $\text{div} V$ is nonzero but small, and rules out the existence of such traps.

Cardaliaguet–Souganidis [9] proved, under the assumption that the environment $V \in C^{1,1}(\mathbb{R}^d; \mathbb{R}^d)$ is stationary ergodic and divergence-free, that the equation homogenizes; i.e. we have the locally uniform convergence of solutions $u_\varepsilon \to \overline{u}$ as $\varepsilon \to 0$ almost surely, where

Communicated by F.-H. Lin.
\( \bar{u} \) is the solution to the effective equation

\[
\begin{align*}
D_t \bar{u}(t, x) &= \overline{H}(D_x \bar{u}(t, x)) \quad \text{in } \mathbb{R}_{>0} \times \mathbb{R}^d \\
\bar{u}(0, x) &= u_0(x) \quad \text{in } \mathbb{R}^d,
\end{align*}
\]

(1.2)

and \( \overline{H} : \mathbb{R}^d \to \mathbb{R} \), called the effective Hamiltonian, is positively homogeneous of degree one and coercive.

Our main result is the following.

**Theorem 1** Let \( V : \mathbb{R}^d \to \mathbb{R}^d \) be a random Lipschitz vector field which has unit range of dependence and is \( \mathbb{Z}^d \)-translation invariant. Then there is a function \( \overline{H} : \mathbb{R}^d \to \mathbb{R} \), which is positively homogeneous of degree one and coercive, and a constant \( C = C(d) > 0 \) such that, if

\[
|\text{div}\, V| \leq C^{-1}(\|V\|_{C^{0,1}} + 1)^{-C}
\]

almost surely, then there is a random variable \( T_0 \), with

\[
\mathbb{E}\left[\exp(C^{-1}(\|V\|_{C^{0,1}} + 1)^{-C} \log^{3/2} T_0)\right] \leq C,
\]

such that

\[
|u^\varepsilon(t, x) - \bar{u}(t, x)| \leq C(\|V\|_{C^{0,1}} + 1)^C (T \varepsilon)^{1/2} \log^2(\varepsilon^{-1} T)
\]

(1.3)

for all \( T \geq \varepsilon T_0 \) and \( t, |x| \leq T \), where \( u^\varepsilon \) is the solution to the G equation 1.1 and \( \bar{u} \) is the solution of the effective equation 1.2.

### 1.1 How quantitative is Theorem 1?

There are two main quantitative features of Theorem 1: the bound on \( |u^\varepsilon - \bar{u}| \), and the random variable \( T_0 \), which represents how long we must wait before the bound takes effect. As for the former, the exponent \( \frac{1}{2} \) of \((t \varepsilon)\) matches with the best known bound for convergence of the limiting shape in first-passage percolation [3]. Indeed, first-passage percolation is an easier problem, since controllability is free and the Hamiltonian is i.i.d., so we cannot hope for a better bound without improving the result for first-passage percolation as well.

As for the bound on \( T_0 \), we note that the distribution of \( T_0 \) has subpolynomial tails and therefore all moments of \( T_0 \) are finite. However, our only bound on the typical value of \( T_0 \) is

\[
\mathbb{E}[T_0] \leq \exp\left(C(\|V\|_{C^{0,1}} + 1)^C\right).
\]

We note that while \( \|V\|_{C^{0,1}} \) appears to be a random variable, the finite range of dependence assumption implies that it is constant almost surely. The exponential dependence on \( \|V\|_{C^{0,1}} \) is an artifact of the fact that the exponent \( \frac{1}{2} \), discussed above, is the tightest possible with our current argument. Indeed, by the same proof it would follow that, if we replace the exponent \( \frac{1}{2} \) in 1.3 with an exponent of \( \frac{1}{2} - \delta \), the corresponding \( T_0 \) would instead depend polynomially on \( \|V\|_{C^{0,1}} \), with the bound

\[
\mathbb{E}[T_0] \leq C(\|V\|_{C^{0,1}} + 1)^{C/\delta}.
\]
1.2 Prior work

There is a rich body of literature studying homogenization of the G equation and enhancement of the front speed (see [5, 6, 8, 9, 13] for example), so we limit our focus to work most closely related to the current situation. Inspired by Cardaliaguet–Souganidis [9], the author [10] showed that, if $V \in C^{1,1}(\mathbb{R}^d; \mathbb{R}^d)$ has unit range of dependence and is divergence-free, then there is a constant $C = C(d, \|V\|_{C^{1,1}}) > 0$ and a random variable $T_0$ with subpolynomial tail bound $\mathbb{E}[\exp(C^{-1} \log^{3/2} T_0)] \leq C$, such that

$$\|u^\varepsilon(t, x) - \bar{u}(t, x)| \leq C\|u_0\|_{C^{0,1}(t\varepsilon)^{1/2}} \log^3(\varepsilon^{-1} t)$$

(1.4)

for all $t \geq \varepsilon T_0$ and $|x| \leq t$.

Because the bound (1.4) is bootstrapped from local controllability estimates by Cardaliaguet–Souganidis [9], the dependence of the constant $C$ on $\|V\|_{C^{1,1}}$ was unspecified. Besides, the work of Cardaliaguet–Souganidis [9] used the divergence-free condition in a critical way, which was necessary under their weaker assumption of stationary ergodicity.

On the other hand, when the environment is periodic instead of random, Cardaliaguet–Nolen–Souganidis [8] proved quantitative homogenization of the G equation without the divergence-free condition. Indeed, they made only the weaker assumption that $|\text{div} V| \leq \varepsilon$ for some $\varepsilon = \varepsilon(d) > 0$, which is related to the constant in the isoperimetric inequality for periodic sets.

We also note that Feldman [11] extended work of Burago–Ivanov–Novikov [5] to prove quantitative estimates on the waiting time in an environment which satisfies a mixing condition in both space and time variables, under the assumption $\text{div} V = 0$.

In this paper, we extend the author’s work [10] to the case where $\text{div} V$ may be nonzero but small and $V$ is only Lipschitz. Along the way, we quantify the dependence of the constant in (1.4) on the Lipschitz norm of $V$. The proof adapts an argument of Burago–Ivanov–Novikov [4], as well as a new argument to show that, even in the presence of nonzero divergence, the reachable set continues to grow quickly.

1.3 Definitions, assumptions, and conventions

We use $C > 0$ to denote a (large) constant which may vary from line to line, but (unless otherwise specified) depends only on the dimension, $d$. We write $Q_R$ to denote the cube of side length 2 centered at the origin. For the sake of brevity, we write $\text{Lip}(V)$ to denote the maximum of 1 and the smallest Lipschitz constant for $V$.

For convenience, we will assume that $V \in C^{1,1}(\mathbb{R}^d; \mathbb{R}^d)$ qualitatively; since every bound we prove depends only on the Lipschitz norm of $V$, this condition can be dropped by approximating $V$ by its mollification. We also assume that there there is zero average drift, i.e. $\mathbb{E}[V] = 0$.

If $E \subseteq \mathbb{R}^d$, we write $\mathcal{G}(E)$ to denote the $\sigma$-algebra generated by $V$ restricted to $E$. That is, $\mathcal{G}(E)$ is the smallest $\sigma$-algebra such that the random variables $V(x)$ are $\mathcal{G}(E)$-measurable for each $x \in E$. We assume that $V$ has unit range of dependence, which means that if $A, B \subseteq \mathbb{R}^d$ are sets with $\text{dist}(A, B) \geq 1$, then $\mathcal{G}(A)$ and $\mathcal{G}(B)$ are independent.

Given $t > 0$ and a measurable function $\alpha: [0, t] \to B_1$, define the controlled path $X_\alpha^0: [0, t] \to \mathbb{R}^d$ to be the solution to the initial-value problem

$$\begin{cases}
\dot{X}_\alpha^0 = \alpha + V(X_\alpha^0) \\
X_\alpha^0(0) = x.
\end{cases}$$

(1.5)
For each \( x \in \mathbb{R}^d \), define the \textit{reachable set} at time \( t \) by
\[
\mathcal{R}_t(x) := \left\{ y \in \mathbb{R}^d \mid \exists \alpha : [0, t] \to B_1 \text{ such that } X^x_\alpha(t) = y \right\}.
\] (1.6)

Note that this definition still makes sense for \( t < 0 \), if we interpret \([0, t]\) as \([t, 0]\). For convenience, we also define the sets
\[
\mathcal{R}_t^-(x) := \bigcup_{0 \leq s \leq t} \mathcal{R}_s(x)
\]
for \( t \geq 0 \) and
\[
\mathcal{R}_t^+(x) := \bigcup_{t \leq s \leq 0} \mathcal{R}_s(x)
\]
for \( t \leq 0 \). Define the \textit{first passage time}
\[
\theta(x, y) := \inf \{ t \mid y \in \mathcal{R}_t(x) \}.
\]
Finally, if \( E \subseteq \mathbb{R}^d \) is a set, we define
\[
\mathcal{R}_t(E) = \bigcup_{e \in E} \mathcal{R}_t(e),
\]
and we do the same for \( \mathcal{R}_t^- \) and \( \mathcal{R}_t^+ \).

\section{Local waiting time estimates}

In this section, we adapt the proof of Burago–Ivanov–Novikov [4] to estimate the waiting time for the metric problem associated to the \( G \) equation.

\subsection{The incompressible case}

First, we prove that, with high probability, sufficiently large \((d - 1)\)-dimensional cubes have very little flux.

Let \( E(R_1, R_0, \varepsilon) \) be the event that every axis-aligned \((d - 1)\)-dimensional cube \( B \), of radius between \( R_0 \) and \( R_1 \), which intersects \( Q_{R_1} \), satisfies
\[
\left| \int_B V(x) \cdot \nu(x) \, dx \right| \leq \varepsilon |B|,
\] (2.1)
where \( \nu : B \to \mathbb{R}^d \) denotes a unit normal to \( B \).

\begin{lemma}
\label{lem:incompressible}
The event \( E(R_1, R_0, \varepsilon) \) has probability at least
\[
\mathbb{P}[E(R_1, R_0, \varepsilon)] \geq 1 - C \left( \frac{R_1 \text{ Lip}(V)}{\varepsilon} \right)^d \left( \frac{R_1(1 + \|V\|_{L^\infty})}{\varepsilon} \right) \exp \left( \frac{-\varepsilon^2 R_0^{d-1}}{C \|V\|_{L^\infty}} \right).
\]
\end{lemma}

\begin{proof}
Step 1. Let \( B \) be an axis-aligned \((d - 1)\)-dimensional cube of radius \( r \geq R_0 \). Partition \( B \) into at least \( C^{-1} r^{d-1} \) equally sized \((d - 1)\)-dimensional cubes, called \( B_1, \ldots, B_n \), of radius between 1 and 2. For each \( i \), the random variable \( \int_{B_i} V(x) \cdot \nu(x) \, dx \) has expectation zero and absolute value at most \( C \|V\|_{L^\infty} \). The random variables for non-neighboring cubes are independent, so we can group the sum
\[
\int_B V(x) \cdot \nu(x) \, dx = \sum_{i=1}^n \int_{B_i} V(x) \cdot \nu(x) \, dx
\]
into \(2^d - 1\) separate sums, each of which contains mutually independent random variable summands, which correspond to non-neighboring cubes. By Azuma’s inequality, we conclude that (2.1) holds with probability at least
\[
1 - \exp\left(-\varepsilon^2 R_0^{d-1} \frac{1}{C \|V\|_{L^\infty}}\right).
\]

Step 2. Use the union bound to apply Step 1 to every cube \(B\) which has a vertex in \((\varepsilon C_{\text{Lip}}(V)) Z_0^d \cap [R_0 - 1, R_1 + 1]\), and conclude using the Lipschitz bound on \(V\), translating and rescaling any cube \(B\) so that it has a vertex in this set.

Next, we show that a subset of \(\partial Q_R\) which has small boundary must also have small flux.

**Lemma 2.2** In the event \(E(R_1, R_0, \varepsilon)\), if \(D \subseteq \partial Q_R\) has a \((d - 2)\)-rectifiable boundary for some \(0 \leq R \leq R_1\), then
\[
\left| \int_D V(x) \cdot \nu(x) \, dx \right| \leq C \|V\|_{L^\infty} R_0 |\partial D| + \varepsilon |\partial Q_R|,
\]
where \(\nu: \partial Q_R \to \mathbb{R}^d\) denotes the outward unit normal to \(\partial Q_R\).

**Proof** This is exactly Lemma 3.3 from Burago–Ivanov–Novikov [4]; for completeness, we include the proof here.

Partition \(\partial Q_R\) into at least \(C^{-1} \left(\frac{R}{R_0}\right)^{d-1}\) equally sized \((d - 1)\)-dimensional cubes, called \(B_1, \ldots, B_n\), of radius between \(R_0\) and \(2R_0\). For each \(i\), define \(P_i := |\partial D \cap B_i|\) and \(S_i := \min(|B_i \cap D|, |B_i \setminus D|)\). The isoperimetric inequality says that
\[
S_i \leq CP_i R_0^{d-1}\]
and the fact that \(S_i \subseteq B_i\) implies that
\[
S_i \leq CR_0^{d-1}.
\]
Interpolating between these bounds, we conclude that
\[
S_i \leq CP_i R_0.
\]

On the other hand, in the event \(E(R_1, R_0, \varepsilon)\) we have
\[
\left| \int_{B_i \cap D} V(x) \cdot \nu(x) \, dx + \int_{B_i \setminus D} V(x) \cdot \nu(x) \, dx \right| \leq \varepsilon |B_i|.
\]
Since one of the sets \(B_i \cap D\) or \(B_i \setminus D\) has measure \(S_i\), we conclude that one of the integrals above has absolute value at most \(\|V\|_{L^\infty} S_i\), so they both have absolute value at most \(\varepsilon |B_i| + \|V\|_{L^\infty} S_i\). We conclude that
\[
\left| \int_{B_i \cap D} V(x) \cdot \nu(x) \, dx \right| \leq \varepsilon |B_i| + C \|V\|_{L^\infty} P_i R_0.
\]
We conclude by summing over \(i\). \(\square\)

The next lemma shows a weak form of controllability. It can be found in Cardaliaguet–Souganidis [9] and we include it here as well for completeness.
Lemma 2.3 The reachable set $\mathcal{R}^-_1(x_0)$ contains the cone

$$\left\{ x_0 + tv \mid v \in B_{1/2}(V(x_0)), \ 0 \leq t \leq (2 \text{Lip}(V)(1 + \|V\|_{L\infty}))^{-1} \right\}.$$  

Furthermore, suppose that $x_0 \notin \left(\mathcal{R}^-_{T+1}(x_0)\right)^0$. Then $\mathcal{R}^-_T(x)$ is disjoint from the cone

$$\left\{ x_0 + tv \mid v \in B_{1/2}(V(x_0)), \ -(2 \text{Lip}(V)(1 + \|V\|_{L\infty}))^{-1} \leq t < 0 \right\}.$$  

Proof Let $v \in B_{1/2}(V(x_0))$ and $t \in (0, (2 \text{Lip}(V)(1 + \|V\|_{L\infty}))^{-1})$. For any $t' \in (0, t)$, we have

$$|V(x_0) - V(x_0 + t'v)| \leq t'|v| \text{Lip}(V) \leq \frac{1}{2}.$$  

Therefore, $v \in B_1(V(x_0 + t'v))$ for all $t \in (0, t)$, so $x_0 + tv \in \mathcal{R}^-_T(x_0) \subseteq \mathcal{R}^-_1(x_0)$, which was the first claim. The contrapositive of the second claim follows by the same argument. \(\square\)

Finally, we show that, on most of the boundary of the reachable set, the vector field $V$ points toward the interior of the reachable set.

Lemma 2.4 Let $R > 0$ and $T_0 \geq 0$. Then for every $\varepsilon > 0$, there is some $T_0 < T \leq T_0 + C R^d/\varepsilon$ such that

$$\left| \left\{ x \in (\partial \mathcal{R}^-_T(0)) \cap Q_R \mid V(x) \cdot \nu(x) \geq -\frac{1}{2} \right\} \right| \leq \varepsilon,$$

where $\cdot$ above denotes the Hausdorff $(d - 1)$-measure.

Proof We use the fact that $t \mapsto |\mathcal{R}^-_T(0) \cap Q_R|$ is Lipschitz with derivative

$$\partial_t |\mathcal{R}^-_T(0) \cap Q_R| = \int_{(\partial \mathcal{R}^-_T(0)) \cap Q_R} (1 + V(x) \cdot \nu(x))_+ \, dx$$

almost everywhere, where $\nu$ denotes the outward unit normal to $\mathcal{R}^-_T(0)$. Also, for any $t \geq 0$ we have $|\mathcal{R}^-_T(0) \cap Q_R| \leq |Q_R| \leq (2R)^d$. The claim follows with $C = 2^{d+1}$ by the mean value theorem. \(\square\)

All the ingredients for the proof of our local waiting time estimate are now in place.

Theorem 2.5 Suppose that $\text{div}V = 0$ almost surely. Let $W := \inf\{t > 0 \mid \mathcal{R}^-_T(0) \supseteq B_{1/2}\}$. Then for any $\lambda \geq 1$,

$$\mathbb{P}[W \geq \lambda] \leq C \exp\left(-C^{-1} \lambda^{(d-1)/d} \text{Lip}(V)^{3-3d}(1 + \|V\|_{L\infty})^{3-5d}\right).$$

Proof Assume $d \geq 3$ (if $d = 2$, just add another dimension in which everything is constant). We follow the proof of Burago–Ivanov–Novikov [4], keeping track of an extra error term to get a quantitative estimate.

Let $T > 0$ and $R > 0$. The boundary $\partial(\mathcal{R}^-_T(0) \cap Q_R)$ has two main parts: we define

$$S_R := (\partial \mathcal{R}^-_T(0)) \cap Q_R$$

and

$$D_R := \mathcal{R}^-_T(0) \cap (\partial Q_R).$$
Further, we let

\[ L_R := (\partial \mathcal{R}_T^{-}(0)) \cap (\partial Q_R). \]

Generically, \( S_R \) and \( D_R \) are \((d-1)\)-dimensional and \( L_R \) is \((d-2)\)-dimensional. Cannarsa–Frankowska [7] proved that the boundary of the reachable set \( \mathcal{R}_T^{-}(0) \) is \( C^{1,1} \) everywhere except at the origin, so, as in Burago–Ivanov–Novikov [4] (see the remark after Lemma 2.4), it can be equipped with a continuous unit normal and the divergence theorem holds (Fig. 1).

If \( x \in S_R \), let \( \nu(x) \) denote the outward unit normal to \( \mathcal{R}_T^{-}(0) \) and if \( x \in \partial Q_R \), let \( \nu(x) \) denote the outward unit normal to \( Q_R \). We also define the subset \( P_R \) of \( S_R \) to be the part of the boundary of the reachable set which is growing at a speed of more than \( \frac{1}{2} \):

\[ P_R := \left\{ x \in S_R \mid V(x) \cdot \nu(x) \geq -\frac{1}{2} \right\}. \]

Integrate \( \text{div} V = 0 \) in \( \mathcal{R}_T^{-}(0) \cap Q_R \) to find

\[ |S_R \setminus P_R| \leq 2 \int_{P_R \cup D_R} V(x) \cdot \nu(x) \, dx \leq 2 \left( \|V\|_{L^\infty} |P_R| + \int_{D_R} V(x) \cdot \nu(x) \, dx \right), \tag{2.2} \]

where \(|\cdot|\) denotes the Hausdorff measure of appropriate dimension (here it’s \( d-1 \)).

On the other hand, the co-area inequality yields

\[ |S_R| \geq \int_0^R |L_r| \, dr. \tag{2.3} \]

We also apply the isoperimetric inequality in \( \partial Q_R \):

\[ \min (|D_R|, |\partial Q_R \setminus D_R|) \leq C |L_R|^\frac{d+1}{2d}, \tag{2.4} \]
and so the divergence theorem applied to \( Q_R \) yields

\[
\left| \int_{D_R} V(x) \cdot \nu(x) \, dx \right| \leq C|L_R|^\frac{d-1}{2}. \tag{2.5}
\]

Combining (2.2), (2.3), and 2.5 yields

\[
|L_R| \geq C^{-1} \left( \int_0^R |L_r| \, dr - (1 + 2\|V\|_{L^\infty})|P_R| \right)^\frac{d-2}{d-1}. \tag{2.6}
\]

Everything so far has only used incompressibility and boundedness of \( V \) in \( L^\infty \), and applies for all \( T, R > 0 \). From now on, we start selecting parameters to show that \( B_{1/2} \subseteq \mathcal{R}_T(0) \).

First, choose \( \varepsilon > 0 \) such that

\[
\int_0^1 |L_r| \, dr \geq \varepsilon.
\]

By Lemma 2.3 and the isoperimetric inequality, we can choose

\[
\varepsilon := C_0^{-1}(\text{Lip}(V)(1 + \|V\|_{L^\infty}))^{1-d}, \tag{2.7}
\]

where \( C_0 = C_0(d) > 2^{d-1} \) will be chosen by the end of the proof.

Next, choose

\[
R_1 := \left( \frac{\lambda \varepsilon}{1 + \|V\|_{L^\infty}} \right)^{1/d},
\]

and

\[
R_0 := \frac{R_1}{C_1(1 + \|V\|_{L^\infty})^2},
\]

where \( C_1 = C_1(d) > 0 \) will also be chosen by the end of the proof. By Lemma 2.1, the event \( E(R_1, R_0, \varepsilon) \) has probability at least

\[
\mathbb{P}[E(R_1, R_0, \varepsilon)] \geq 1 - C R_0^{d+1}\|V\|_{C_0,1} \exp \left( -C^{-1} \lambda^{(d-1)/d} \varepsilon^{2+(d-1)/d}(1 + \|V\|_{L^\infty})^{-2d} \right) \geq 1 - C \exp \left( -C^{-1} \lambda^{(d-1)/d} \text{Lip}(V)^{3-3d}(1 + \|V\|_{L^\infty})^{3-5d} \right).
\]

Work in the event \( E(R_1, R_0, \varepsilon) \). By Lemma 2.4, we can choose \( 1 \leq T \leq CR_1^d(1 + \|V\|_{L^\infty}) \varepsilon^{-1} \leq C \lambda \) such that \( (1 + 2\|V\|_{L^\infty})|P_R| \leq \frac{\varepsilon}{2} \). Since we assume that \( B_{1/2} \not\subseteq \mathcal{R}_{T+1}(0) \), our choice of \( T \) does not affect \( \varepsilon \). Plug our choice of \( \varepsilon \) and \( T \) into (2.6) to see that

\[
\frac{d}{dR} \int_0^R |L_r| \, dr \geq C^{-1} \left( \int_0^R |L_r| \, dr \right)^\frac{d-2}{d-1}
\]

for all \( 1 \leq R \leq R_1 \) and

\[
\int_0^1 |L_r| \, dr \geq \frac{\varepsilon}{2},
\]

which implies

\[
\int_0^R |L_r| \, dr \geq C^{-1}(R - 1)^{d-1} \tag{2.8}
\]

for all \( 1 \leq R \leq R_1 \). We make a note that the constant \( C \) in 2.8 does not depend on \( C_0 \) or \( C_1 \).
Combining (2.8) with (2.2) and (2.3), we conclude that

\[ C^{-1}(R - 1)^{d-1} \leq (1 + 2\|V\|_{L^\infty})|P_R| + \int_{D_R} V(x) \cdot \nu(x) \, dx. \tag{2.9} \]

Apply Lemma 2.2 to \( D_R \) and combine with (2.9) to obtain

\[ C^{-1}(R - 1)^{d-1} \leq (1 + 2\|V\|_{L^\infty})|P_R| + C(1 + \|V\|_{L^\infty})R_0|L_R| + 2d\varepsilon R^{d-1}. \tag{2.10} \]

As long as \( \varepsilon \leq \frac{1}{2\pi C dC} \) and \( |P_R| \leq \frac{1}{\pi C (1 + 1\|V\|_{L^\infty})} \), which we ensure by choosing \( C_0 > 0 \) sufficiently large in (2.7), then for \( R \geq 2 \) we have

\[ R^{d-1} \leq CR_0(1 + \|V\|_{L^\infty})|L_R|. \tag{2.11} \]

We integrate and apply (2.3) to conclude that

\[ |S_R| \geq \frac{R^d - 1}{CR_0(1 + \|V\|_{L^\infty})} \tag{2.12} \]

for every \( 2 \leq R \leq R_1 \).

At \( R = R_1 \), this yields

\[ |S_{R_1}| \geq C_1 C^{-1}(1 + \|V\|_{L^\infty})R_1^{d-1}. \]

To conclude, we choose \( C_1 \) large enough so that \( |S_{R_1}| \geq 2(1 + \|V\|_{L^\infty})(1 + d2^d R_1^{d-1}) \), which contradicts (2.2), as \( |P_{R_1}| \leq 1 \) and \( D_{R_1} \subseteq \partial Q_{R_1} \) and hence \( |D_{R_1}| \leq |\partial Q_{R_1}| = d2^d R_1^{d-1} \). \( \square \)

### 2.2 The compressible case

Next, we adapt the proof in the incompressible case to allow \( |\text{div} V| \) to be nonzero but small.

**Proposition 2.6** Let \( W := \inf\{ t > 0 \mid R_t^-(0) \supseteq B_{1/2} \} \). For each \( p > 0 \), there is \( \varepsilon(\text{Lip}(V), \|V\|_{L^\infty}, p, d) > 0 \) such that, if \( |\text{div} V| \leq \varepsilon \) almost surely, then

\[ \mathbb{P}\left[ W \geq C\text{Lip}(V)^3(1 + \|V\|_{L^\infty})^{5d+4}(\log p^{-1})^{d/(d-1)} \right] \leq p. \]

Furthermore, we can choose

\[ \varepsilon \geq C^{-1}\text{Lip}(V)^{-3}(1 + \|V\|_{L^\infty})^{-6}(\log p^{-1})^{1/(d-1)}. \]

**Proof** The proof is nearly identical to the proof of Theorem 2.5. The only difference is the addition of \( \varepsilon CR^d \) error terms in (2.2)

\[ |S_R \setminus P_R| \leq 2 \int_{P_R \cup D_R} V(x) \cdot \nu(x) \, dx + \varepsilon CR^d \leq 2 \left( \|V\|_{L^\infty}|P_R| + \int_{D_R} V(x) \cdot \nu(x) \, dx \right) + \varepsilon CR^d, \tag{2.13} \]

and (2.5)

\[ \left| \int_{D_R} V(x) \cdot \nu(x) \, dx \right| \leq C|L_R|^\frac{d-1}{2} + \varepsilon CR^d. \tag{2.14} \]

As long as \( \varepsilon \leq C^{-1}R_1^{-1} \), where \( R_1 \) is defined in the proof of Theorem 2.5, the extra error term is at most \( C^{-1}R_1^{d-1} \), and therefore does not affect any of the calculations. \( \square \)
We conclude the section by showing that, even without assuming incompressibility, the reachable set at time $t$ grows proportionally to $t^d$. This lemma plays a key role in ensuring that homogenization occurs in the sense of uniform convergence, by showing that no traps can arise where the reachable set stays bounded for a long time.

**Proposition 2.7** There is some $\varepsilon = \varepsilon(d) > 0$ such that, if $|\text{div} V| \leq \varepsilon$ almost surely, then

$$|\mathcal{R}_t^-(x_0)| \geq \frac{t^d}{2d}$$

for every $x_0 \in \mathbb{R}^d$ and $t \geq 0$ almost surely.

**Proof** Assume for simplicity that $x_0 = 0$ and let $K := t(1 + \|V\|_{L^\infty})$. Then $\mathcal{R}_t^-(0) \subseteq Q_K$.

Since $|\mathcal{R}_t^-(0)| \geq \frac{1}{2}|B_1|$ for sufficiently small $t \geq 0$, it suffices to show that

$$\partial_t|\mathcal{R}_t^-(0)| = \int_{\partial \mathcal{R}_t^-(0)} (1 + V(x) \cdot \nu(x))_+ \, dx \geq \frac{1}{2}|\partial \mathcal{R}_t^-(0)|,$$

where $\nu$ denotes the outward unit normal to $\mathcal{R}_t^-(0)$.

By the divergence theorem,

$$\int_{\partial \mathcal{R}_t^-(0)} (1 + V(x) \cdot \nu(x))_+ \, dx = |\partial \mathcal{R}_t^-(0)| + \int_{\mathcal{R}_t^-(0)} \text{div} V(x) \, dx.$$

To get rid of small parts of the boundary of the reachable set, we define its discretized version by

$$E := \bigcup \{x + \frac{Q_1}{2} : x \in \mathbb{Z}^d \text{ and } |\mathcal{R}_t^-(0) \cap (x + Q_1/2)| \geq \frac{1}{2}\}.$$

We will estimate the divergence term by integrating over $E$ instead and using the unit range of dependence. First, we bound the symmetric difference by

$$|(E \setminus \mathcal{R}_t^-(0)) \cup (\mathcal{R}_t^-(0) \setminus E)| \leq C|\partial \mathcal{R}_t^-(0)|,$$

by the isoperimetric inequality applied in each integer-centered unit cube. The bound on $\text{div} V$ then implies

$$\left| \int_E \text{div} V(x) \, dx - \int_{\mathcal{R}_t^-(0)} \text{div} V(x) \, dx \right| \leq \varepsilon C|\partial \mathcal{R}_t^-(0)|. \tag{2.15}$$

On the other hand, the isoperimetric inequality also yields

$$|\partial E| \leq C|\partial \mathcal{R}_t^-(0)|. \tag{2.16}$$

We claim that $\int_E \text{div} V(x) \, dx \leq \varepsilon C|\partial E|$ almost surely. Indeed, if this is true, then since there are only countably many possible values for $E$, the inequality holds for all possible $E$ almost surely. We finish by combining the claim with (2.15) and (2.16) to conclude that

$$\int_E \text{div} V(x) \, dx \leq \varepsilon C|\partial \mathcal{R}_t^-(0)|,$$

so choosing $\varepsilon := \frac{1}{2} C^{-1}$ allows us to conclude.

It remains to prove the claim. Let

$$D := \{x \in E \mid \text{dist}(x, \partial E) \leq 1\}.$$
Then $|D| \leq C|\partial E|$ (as before, we abuse notation by using $| \cdot |$ to denote the $d$-dimensional measure on the left and $(d - 1)$-dimensional measure on the right-hand side), since every integer-centered unit cube in $E$ which intersects $\partial E$ is adjacent to an integer-centered unit cube in $E$ which has at least one of its faces contained in $\partial E$.

By the divergence theorem,

$$\int_E \text{div} V(x) \, dx = \int_{\partial E} V(x) \cdot \nu(x) \, dx,$$

where $\nu$ denotes the outward unit normal to $E$. The integral on the right-hand side depends only on $V$ restricted to $\partial E$, and is therefore independent from the random variable

$$\int_E \text{div} V(x) \, dx.$$

However, we have

$$\int_E \text{div} V(x) \, dx = \int_D \text{div} V(x) \, dx + \int_{E \setminus D} \text{div} V(x) \, dx \leq \varepsilon C|\partial E| + \int_{E \setminus D} \text{div} V(x) \, dx.$$

Taking the conditional expectation with respect to $G(E \setminus D)$ and using independence yields the claim. \hfill \Box

### 3 Global waiting time estimates

Next, we improve our local waiting time bounds to global bounds, by showing that the region where the local waiting time is small contains a supercritical percolation cluster. The argument is identical to that in [10], so we only sketch the proofs.

**Lemma 2.8** There is a constant $C = C(d) > 0$ such that for each $0 < p < 1$, if

$$|\text{div} V| \leq C^{-1}(\|V\|_{C^{0.1}} + 1)^{-C}\left(\log(1 - p)^{-1}\right)^{-1/(d - 1)}$$

almost surely, then the function $G : \mathbb{Z}^d \to \{0, 1\}$, defined by

$$G(v) = \begin{cases} 1 & \text{if } \theta(x, y) \leq C(\|V\|_{C^{0.1}} + 1)^C \log(1 - p)^{-1} \text{ for all } x, y \in B_{\sqrt{d}}(v), \\ 0 & \text{otherwise,} \end{cases}$$

is $\mathbb{Z}^d$-translation invariant with finite range of dependence

$$C_{\text{dep}} \leq C(\|V\|_{C^{0.1}} + 1)^C \log(1 - p)^{-1}$$

and $\mathbb{P}[G(0) = 1] \geq p$.

**Proof sketch** By Proposition 2.6, applied to $V$ and $-V$ separately, there is $C > 0$ such that $\mathbb{P}[G(0) = 1] \geq p$. The $\mathbb{Z}^d$-translation invariance of $G$ follows from that of $\mathbb{P}$. Finite range of dependence follows from the fact that the value of $G(v)$ depends only on controlled paths starting in $B_{\sqrt{d}}(v)$ which run for time at most $C\|V\|_{C^{0.1}} \log(1 - p)^{-1}$. The bound on $C_{\text{dep}}$ follows, noting that the top speed of a path is $1 + \|V\|_{L^\infty}$.

**Definition 2.9** To translate between $\mathbb{Z}^d$ and $\mathbb{R}^d$, for each set $E \subseteq \mathbb{Z}^d$ we introduce the “solidification”

$$\sigma(E) := E + \left[\frac{1}{2}, \frac{1}{2}\right]^d.$$
Theorem 2.10 There is a constant $C = C(d) > 0$ such that, if

$$|\text{div}V| \leq C^{-1}(\|V\|_{C^{0.1}} + 1)^{-C},$$

then for each $R \geq 1$, the “extra waiting time”

$$\mathcal{E}(R) := \sup_{x,y \in Q_R} \frac{\theta(x,y) - C(1+|x-y|)}{C(\|V\|_{C^{0.1}} + 1)^C}$$

satisfies

$$\mathbb{P}[\mathcal{E}(R) > n] \leq CR^d \exp(-C^{-1}(\|V\|_{C^{0.1}} + 1)^{-C} n).$$

Proof sketch We partition $\mathbb{R}^d$ into cubes of side length 1, centered at points in $\mathbb{Z}^d$. Fix $p := 1 - \exp(-CC_{\text{dep}}^d)$, where $C_{\text{dep}}$ is given by Lemma 2.8. By Lemma 2.8, if

$$|\text{div}V| \leq C^{-1}(\|V\|_{C^{0.1}} + 1)^{-C},$$

then $\mathbb{P}[G(v) = 1] \geq p$. For $v \in \mathbb{Z}^d$, we say that the site $v$ is open if $G(v) = 1$ and closed otherwise. We say that a point $x \in \mathbb{R}^d$ lies near an open site if there is some $v \in \mathbb{Z}^d$ such that $x \in \sigma([v])$. Let $S = [R]$ and $x, y \in Q_s$. For convenience, we will prove that

$$\mathbb{P}[\mathcal{E}(R) > Cn] \leq CR^d \exp(-C^{-1}(\|V\|_{C^{0.1}} + 1)^{-C} n);$$

this easily implies the original claim by changing $C$ to $C^2$.

Step 1. We claim that, without loss of generality, we may assume that $x$ and $y$ lie near sites in the same open cluster. Indeed, if not, then (say) $x$ lies in a connected component, $\mathcal{D}$, of $Q_{S+\delta} \setminus \mathcal{C}$, where $\delta > 0$ is chosen appropriately (to be precise, we can take $\delta = n^d$) and $\mathcal{C}$ is the largest open cluster contained in $Q_{S+\delta}$. Since $p$ can be made arbitrarily close to 1 by choosing $C$ large, Lemma 2.4 in [10] gives a tail bound on the size of connected components of $Q_{S+\delta}$, so with high probability, $\mathcal{D} \cap \partial Q_{S+\delta} = \emptyset$. Then, Proposition 2.7 implies that $\mathcal{R}_{\tau}(x) \cap \sigma(\mathcal{C}) \neq \emptyset$ for some time $t > 0$ which is not too large, and thus we replace $x$ by any member of this intersection. Repeating the same procedure for $y$ if necessary, the claim follows.

Step 2. To show that

$$\theta(x,y) - C|x-y| \leq C(\|V\|_{C^{0.1}} + 1)^C n,$$

we will build a “skeleton” of points $x = x_0, x_1, \ldots, x_k = y$ which all lie near open sites and satisfy $|x_{i+1} - x_i| \leq \sqrt{d}$ for each $0 \leq i < k$. Then, by connecting the points with paths given by Lemma 2.8, we can build a controlled path of length at most $C(\|V\|_{C^{0.1}} + 1)^C k$ which follows the skeleton.

Our strategy is to go from $x$ to $y$ in a straight line, taking necessary detours around closed clusters, as shown in Fig. 2. We omit the remaining details, as they are identical to those in the proof of Theorem 3.2 in [10].

4 Random fluctuations in first-passage time

Next, we consider how much $\theta(0, y)$ deviates from its expectation. Our proof will follow roughly the same path as the proof of Proposition 4.1 from Armstrong–Cardaliaguet–Souganidis [2], with some modifications which are made possible by the controllability estimate. As in the previous section, the proofs are nearly identical to those in [10], with a bit of care taken to keep track of the dependence of the constants on $V$, so we omit them.
To get started, we introduce a “guaranteed” version of first passage time. For any $\rho > 0$, define the $\rho$-guaranteed reachable set recursively by

$$
R^\rho_t(x) := \begin{cases} 
R^\rho_t(x) & \text{if } t < \rho \\
R^\rho_t(x) \cup (R^\rho_t(x) + B_1) & \text{otherwise.}
\end{cases}
$$

The $\rho$-guaranteed reachable set is similar to the reachable set, except that we enforce expansion at a rate of at least $1/\rho$ in a certain discrete sense. We similarly define the $\rho$-guaranteed first passage time

$$
\theta^\rho(x, y) = \min\{t \geq 0 \mid y \in R^\rho_t(x)\}.
$$

Note that the $\rho$-guaranteed first passage time coincides with the usual first passage time if we have sufficient control on the extra waiting time $\mathcal{E}$ (from Theorem 2.10) in a suitable domain.

Fix some $y \in \mathbb{R}^d$ and define the random variable $\{Z^\rho_t\}_{t \geq 0}$ by

$$
Z^\rho_t := \mathbb{E}[\theta^\rho(0, y) \mid \mathcal{F}_t],
$$

where $\mathcal{F}_t$ is the $\sigma$-algebra generated by the environment $V(x)$ restricted to the $\rho$-guaranteed reachable set $R^\rho_t(0)$. In other words, $\mathcal{F}_t$ is the smallest $\sigma$-algebra so that the functions $V(x)1_{x \in R^\rho_t(0)}$ are $\mathcal{F}_t$-measurable for every $x \in \mathbb{R}^d$. Since $R^\rho_t(0)$ are increasing sets, $\{\mathcal{F}_t\}_{t \geq 0}$ is a filtration, so $\{Z^\rho_t\}_{t \geq 0}$ is a martingale.

We first show that $Z^\rho_t$ depends mostly on the shape of $R^\rho_t(0)$, without regard for the values of $V$ inside $R^\rho_t(0)$. In order to condition on the approximate shape of the reachable set, for any $E \subseteq \mathbb{R}^d$ we introduce the discretization

$$
\text{disc}(E) := \{z \in d^{-1/2}\mathbb{Z}^d \mid B(z, 1) \cap E \neq \emptyset\}.
$$
Lemma 2.11 For any \( t \geq 0 \), we have
\[
\left| \max(Z^\rho_t, t) - f(t, \text{disc}(R^\rho_t(0))) \right| \leq 3\rho,
\]
where we define \( f(t, S) \), for any \( t \geq 0 \) and any finite set \( S \subseteq d^{-1/2}Z^d \), by
\[
f(t, S) := t + \mathbb{E}[\theta^\rho(S, y)].
\]
Proof See proof of Lemma 3.5 in [10].

Next, we claim that our approximation for \( Z^\rho_t \), given by \( f(t, \text{disc}(R^\rho_t(0))) \), has bounded increments.

Lemma 2.12 Let \( t, s \geq 0 \). Then
\[
\left| f(t, \text{disc}(R^\rho_t(0))) - f(s, \text{disc}(R^\rho_s(0))) \right| \leq 2\rho + |t - s|(\|V\|_{L^\infty} + \rho + 2).
\]
Proof See proof of Lemma 3.6 in [10].

Together, the previous lemmas show that the martingale \( \{Z^\rho_t\}_{t \geq 0} \) has bounded increments. Applying Azuma’s inequality to \( \{Z^\rho_t\}_{t \geq 0} \), choosing \( \rho \) carefully to balance competing error terms, we deduce a tail bound on the distribution of
\[
Z^\rho_\infty = \mathbb{E}[\theta(0, y) \mid \mathcal{F}_t].
\]

Proposition 2.13 There is a constant \( C = C(d) > 0 \) such that, if \( y_1, y_2 \in \mathbb{R}^d \),
\[
\lambda \geq C(\|V\|_{C^{0,1}} + 1)^C |y_1 - y_2|^{1/2} \log^2 |y_1 - y_2|,
\]
and
\[
|\text{div} V| \leq C^{-1}(\|V\|_{C^{0,1}} + 1)^{-C},
\]
then
\[
\mathbb{P}[|\theta(y_1, y_2) - \mathbb{E}[\theta(y_1, y_2)]| > \lambda] \leq C \exp \left( -C^{-1}(\|V\|_{C^{0,1}} + 1)^{-C} \lambda^{1/2} \right).
\]
Proof See proof of Proposition 3.7 in [10].

5 Nonrandom scaling bias

In this section, we use the bounds on random fluctuations of \( \theta \) to bound the difference between \( \mathbb{E}[\theta(0, y)] \) and \( \lim_{\epsilon \to 0^+} \epsilon \mathbb{E}[\theta(0, \epsilon^{-1}y)] \), which we refer to as the nonrandom scaling bias. We follow a similar argument as in Alexander [1], who proved an analogous result for Bernoulli percolation in two dimensions. Happily, the argument goes through in any dimension with the help of the Hobby–Rice theorem [12], a version of which we quote below. We include their proof, because it is short and beautiful.

Theorem 2.14 (Hobby–Rice) Let \( \gamma : [0, 1] \to \mathbb{R}^d \) be continuous. Then there is a partition
\[
0 = t_0 < t_1 < \cdots < t_{d+1} = 1,
\]
along with signs
\[ \delta_1, \ldots, \delta_{d+1} \in \{-1, +1\}, \]
such that
\[ \sum_{k=1}^{d+1} \delta_k (\gamma(t_k) - \gamma(t_{k-1})) = 0. \]

**Proof** We parameterize signed partitions by points on the \( d\)-sphere as follows. Given a point \( x \in S^d \), we define associated signs by \( \delta^x_k = \text{sgn}(x_k) \) and define \( \{t_k\}_k \) to be the unique partition of \([0, 1]\) such that \( t_k^x - t_{k-1}^x = x_k^2 \). As such, we define the map \( f : S^d \to \mathbb{R}^d \) by
\[
 f(x) := \sum_{k=1}^{d+1} \delta^x_k (\gamma(t^x_k) - \gamma(t^x_{k-1})).
\]

By the Borsuk–Ulam theorem, there is some \( x \in S^d \) such that \( f(x) = f(-x) \). However, \( f \) is odd, so \( f(x) = 0 \), which proves the claim. \( \Box \)

We now bound the nonrandom scaling bias. Given a function \( f : \mathbb{R}^d \to \mathbb{R} \), we define the large-scale limit \( \overline{f} : \mathbb{R}^d \to \mathbb{R} \) by
\[
 \overline{f}(x) := \lim_{\varepsilon \to 0^+} \varepsilon f(\varepsilon^{-1} x).
\]

**Proposition 2.15** Assume that the law of \( V \) is \( \mathbb{Z}^d \)-translation invariant and that
\[ |\text{div}V| \leq C^{-1}(\|V\|_{C^{0,1}} + 1)^{-C}. \]
Let \( f(x) := \mathbb{E}[\theta(0, x)] \). Then
\[ |f(x) - \overline{f}(x)| \leq C(\|V\|_{C^{0,1}} + 1)^C |x|^{1/2} \log^2 |x| \]
for all \( |x| \geq 1 \).

**Proof** First, note that translation invariance and the controllability bound in Theorem 2.10 implies that \( f \) is subadditive up to a constant, that is,
\[ f(x + y) \leq f(x) + f(y) + C(\|V\|_{C^{0,1}} + 1)^C \]
for all \( x, y \in \mathbb{R}^d \), so it follows immediately that
\[ f(y) \geq \overline{f}(y) - C(\|V\|_{C^{0,1}} + 1)^C. \]

Our goal is to show that \( f \) is superadditive up to some small error, after which we apply an argument similar to that in Fekete’s lemma to bound the difference between \( f \) and its large-scale limit.

Fix any \( y \in \mathbb{R}^d \). By Proposition 2.13, Theorem 2.10, and the union bound, the event that
\[ |\theta(v, w) - \mathbb{E}[\theta(v, w)]| \leq C(\|V\|_{C^{0,1}} + 1)^C |y|^{1/2} \log^2 |y| \] (5.1)
for all
\[ |v|, |w| \leq C(\|V\|_{C^{0,1}} + 1)^C |y| \]
has positive probability. By translation invariance, this implies that

\[ |\theta(w, x) - \theta(y, z)| \leq C(\| V \|_{C^{0,1}} + 1)^C |y|^{1/2} \log^2 |y| \]  

whenever \(|(x - w) - (z - y)| \leq C\) and

\[ |w|, |x|, |y|, |z| \leq C(\| V \|_{C^{0,1}} + 1)^C |y|. \]

In an instance of this event, let \( \gamma : [0, \theta(0, y)] \to \mathbb{R}^d \) be a controlled path from 0 to \( y \). Applying Theorem 2.14 to \( \gamma \), we conclude that there are points

\[ 0 \leq s_1 < t_1 < s_2 < t_2 < \cdots < s_\ell < t_\ell \leq 1, \]

where \( \ell \leq \frac{d+1}{2} \), such that

\[ \sum_{k=1}^\ell \gamma(t_k) - \gamma(s_k) = \frac{1}{2} y. \]

Applying (5.1) and (5.2), we conclude that

\[ 2 f \left( \frac{1}{2^n} y \right) \leq C(\| V \|_{C^{0,1}} + 1)^C |y|^{1/2} \log^2 |y| + \theta \left( 0, \frac{1}{2^n} y \right) + \theta \left( \frac{1}{2^n} y, y \right) \]

\[ \leq C(\| V \|_{C^{0,1}} + 1)^C |y|^{1/2} \log^2 |y| + \left( \sum_{k=1}^\ell \theta(\gamma(s_k), \gamma(t_k)) \right) \]

\[ + \left( \theta(0, \gamma(s_1)) + \theta(\gamma(t_1), y) + \sum_{k=2}^\ell \theta(\gamma(t_{k-1}), \gamma(s_k)) \right) \]

\[ \leq C(\| V \|_{C^{0,1}} + 1)^C |y|^{1/2} \log^2 |y| + \theta(0, y) \]

\[ \leq C(\| V \|_{C^{0,1}} + 1)^C |y|^{1/2} \log^2 |y| + f(y). \]

It follows by induction that

\[ 2^n f(y) \leq f \left( 2^n y \right) + \sum_{k=0}^{n-1} 2^{n-1-k} C(\| V \|_{C^{0,1}} + 1)^C |2^k y|^{1/2} \log^2 |2^k y|. \]

Dividing by \( 2^n \) on both sides and taking the limit as \( n \to \infty \) yields

\[ f(y) \leq \overline{f}(y) + C(\| V \|_{C^{0,1}} + 1)^C |y|^{1/2} \log^2 |y|. \]

\[ \square \]

6 Homogenization

In this section, we prove our main homogenization results for the shape of the reachable set and for solutions of the G equation, using our bounds on convergence of first-passage time to the large-scale average.
6.1 The reachable set

We combine the random fluctuation bound and nonrandom bias bound to deduce a rate of convergence of the rescaled reachable sets.

**Proposition 2.16** Assume that the law of $V$ is $\mathbb{Z}^d$-translation invariant and that

$$|\text{div} V| \leq C^{-1}(\|V\|_{C^{0.1}} + 1)^{-C}.$$  

Then there is a closed set $S \subseteq \mathbb{R}^d$ (in fact $S = \{x \in \mathbb{R}^d \mid \vartheta(x) \leq 1\}$) such that, for all $t \geq 0$,

$$\mathbb{P}\left[ \text{dist}_H(\mathcal{R}_t(0), tS) > C(\|V\|_{C^{0.1}} + 1)^C t^{1/2} \log^2 t + \lambda \right] \leq C \exp\left( -\frac{C^{-1}(\|V\|_{C^{0.1}} + 1)^{-C} \lambda^{1/2}}{t^{1/4}} \right),$$  

(6.1)

where dist$_H$ denotes the Hausdorff distance. Furthermore, there is a random variable $T_0$, with

$$\mathbb{E}[\exp(C^{-1}(\|V\|_{C^{0.1}} + 1)^C \log^{3/2} T_0)] < \infty,$$

such that

$$\sup_{(t,x) \in [0,T] \times B_r} \frac{\text{dist}_H(\mathcal{R}_t(x), x + tS)}{T^{1/2} \log^2 T} \leq C(\|V\|_{C^{0.1}} + 1)^C$$

for all $T \geq T_0$.

**Proof** For the first claim (6.1), let $t \geq 0$. Apply Proposition 2.13 to every $x \in \mathbb{Z}^d \cap \overline{B_r(1+\|V\|_{L^\infty})}$ and use Theorem 2.10 to extend the estimate to all of $\overline{B_r(1+\|V\|_{L^\infty})}$ (bounding the passage time from a point in $\mathbb{R}^d$ to its nearest integer-coordinate neighbor). Using the union bound, we see that for $\lambda \geq C(\|V\|_{C^{0.1}} + 1)^C t^{1/2} \log^2 t$ we have

$$\mathbb{P}\left[ \forall x \in \overline{B_r(1+\|V\|_{L^\infty})} : |\theta(0, x) - \mathbb{E}[\theta(0, x)]| > \lambda \right] \leq C \exp\left( -\frac{C^{-1}(\|V\|_{C^{0.1}} + 1)^{-C} \lambda^{1/2}}{t^{1/4}} \right),$$  

(6.2)

where we absorbed polynomials into the exponential by enlarging the constant $C$. Note also that Theorem 2.10 implies that if $0 \leq r \leq s$, then

$$\mathbb{P}\left[ \mathcal{R}_r(0) \not\subseteq \mathcal{R}_s(0) + B_s \right] \leq C r^d \exp\left( -\frac{C^{-1}(\|V\|_{C^{0.1}} + 1)^{-C} \lambda}{} \right).$$  

(6.3)

From (6.2) we see that $\mathcal{R}_r(0)$ and $\{x \mid \mathbb{E}[\theta(0, x)] \leq t\}$ are close in the Hausdorff distance, and from (6.3) we see that $\mathcal{R}_r(0)$ and $\mathcal{R}_r(0)$ are close in the Hausdorff distance. Combining these estimates bounds the random error between $\mathcal{R}_r(0)$ and $\{x \mid \mathbb{E}[\theta(0, x)] \leq t\}$.

It remains to bound the nonrandom error between $\{x \mid \mathbb{E}[\theta(0, x)] \leq t\}$ and $tS$. Indeed, Proposition 2.15 shows that

$$0 \leq \mathbb{E}[\theta(0, x)] - \vartheta(x) \leq C(\|V\|_{C^{0.1}} + 1)^C |x|^{1/2} \log^2 |x|,$$

(6.4)

where

$$\vartheta(x) := \lim_{\epsilon \to 0^+} \epsilon \mathbb{E}[\theta(0, \epsilon^{-1} x)].$$
which, along with the fact that $\theta$ is positively homogeneous of degree one, i.e. $\tilde{\theta}(tx) = t\tilde{\theta}(x)$ for $t \geq 0$, yields (6.1).

For the second claim in the proposition, apply (6.1) to every $(t, x) \in (\mathbb{Z} \cap [0, T]) \times (\mathbb{Z}^d \cap B_T)$ and the union bound to conclude that

$$P \left[ \sup_{t \in \mathbb{Z} \cap [0, T]} \sup_{x \in \mathbb{Z}^d \cap B_T} \text{dist}_H(R_t(x), x + tS) > C(\|V\|_{C^0,1} + 1)^{C T^{1/2} \log^2 T + \lambda} \right] \leq C T^{d+1} \exp \left( \frac{-C^{-1}(\|V\|_{C^0,1} + 1)^{-C \lambda^{1/2}}}{T^{1/4}} \right).$$

Next, apply Theorem 2.10 in $B_T$ to see that the same holds for all $(t, x) \in [0, T] \times B_T$, by enlarging the constant $C$. Plugging in $\lambda = CT^{1/2} \log^2 T$ shows that

$$P \left[ \sup_{(t, x) \in [0, T] \times B_T} \text{dist}_H(R_t(x), x + tS) > C(\|V\|_{C^0,1} + 1)^C \right] \leq C \exp(-C^{-1}(\|V\|_{C^0,1} + 1)^{-C \log^3/2 T}),$$

and the conclusion follows.

$$\square$$

6.2 Solutions of the G equation

We now turn to the proof of Theorem 1.

**Proof** Let $u^\varepsilon$ be a solution to the G equation (1.1) with initial data $u_0$, and let $\overline{u}$ be the solution the the effective equation (1.2) with the same initial data. The effective Hamiltonian is given by

$$H(p) := \sup_{v \in S} p \cdot v \quad (6.5)$$

The optimal control formulations are

$$u^\varepsilon(t, x) = \sup_{\varepsilon R_{-1}(\varepsilon^{-1}x)} u_0 \quad (6.6)$$

and

$$u^\varepsilon(t, x) = \sup_{x + tS} u_0 \quad (6.7)$$

respectively.

Using the representation formulas (6.7) and (6.6), we see that for every $0 \leq t \leq T$ and $x \in B_T$ we have

$$|u^\varepsilon(t, x) - \overline{u}(t, x)| = \left| \sup_{\varepsilon R_{-1}(\varepsilon^{-1}x)} u_0 - \sup_{x + tS} u_0 \right| \leq \text{Lip}(u_0) \text{dist}_H(\varepsilon R_{-1}(\varepsilon^{-1}x), x + tS).$$

Rescaling by $\varepsilon^{-1}$ and applying Proposition 2.16 yields

$$\sup_{(t, x) \in [0, T] \times B_T} \text{dist}_H(\varepsilon R_{-1}(\varepsilon^{-1}x), x + tS) \leq C(\|V\|_{C^0,1} + 1)^C (T\varepsilon)^{1/2} \log^2(\varepsilon^{-1} T)$$

for all $T \geq \varepsilon T_0$, and the result follows.

$$\square$$
Acknowledgements  I would like to thank my advisor, Charles Smart, for suggesting the problem and for many helpful conversations.

References

1. Alexander, K.S.: Lower bounds on the connectivity function in all directions for Bernoulli percolation in two and three dimensions. Ann. Probab. 18(4), 1547–1562 (1990)
2. Armstrong, S., Cardaliaguet, P., Souganidis, P.: Error estimates and convergence rates for the stochastic homogenization of Hamilton–Jacobi equations. J. Am. Math. Soc. 27(2), 479–540 (2014)
3. Auffinger, A., Damron, M., Hanson, J.: 50 Years of First-passage Percolation. University Lecture Series, vol. 68. American Mathematical Society, Providence, RI (2017)
4. Burago, D., Ivanov, S., Novikov, A.: A survival guide for feeble fish. Algebra i Analiz 29(1), 49–59 (2017)
5. Burago, D., Ivanov, S., Novikov, A.: Feeble fish in time-dependent waters and homogenization of the G-equation. Commun. Pure Appl. Math. 73(7), 1453–1489 (2020)
6. Caffarelli, L.: The homogenization of surfaces and boundaries. Bull. Braz. Math. Soc. New Ser. 44(4), 755–775 (2013)
7. Cannarsa, P., Frankowska, H.: Interior sphere property of attainable sets and time optimal control problems. ESAIM: Control Optim. Calc. Var., 12(2), 350–370 (2006)
8. Cardaliaguet, P., Nolen, J., Souganidis, P.E.: Homogenization and Enhancement for the G-equation. Arch. Ration. Mech. Anal. 199(2), 527–561 (2011)
9. Cardaliaguet, P., Souganidis, P.E.: Homogenization and enhancement of the g-equation in random environments. Commun. Pure Appl. Math. 66(10), 1582–1628 (2013)
10. Cooperman, W.: Quantitative stochastic homogenization of the G-equation. arXiv:2111.05221 [math], November 2021. arXiv: 2111.05221
11. Feldman, W.M.: Recovering coercivity for the G-equation in general random media. arXiv:1911.00781 [math] (2019)
12. Hobby, C.R., Rice, J.R.: A moment problem in l1 approximation. Proc. Am. Math. Soc. 16(4), 665–670 (1965)
13. Nolen, J., Novikov, A.: Homogenization of the g-equation with incompressible random drift. Commun. Math. Sci. 9(2), 561–582 (2011)

Publisher’s Note  Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.

Springer Nature or its licensor (e.g. a society or other partner) holds exclusive rights to this article under a publishing agreement with the author(s) or other rightsholder(s); author self-archiving of the accepted manuscript version of this article is solely governed by the terms of such publishing agreement and applicable law.