Abstract

This paper is concerned with the problem of top-$K$ ranking from pairwise comparisons. Given a collection of $n$ items and a few pairwise binary comparisons across them, one wishes to identify the set of $K$ items that receive the highest ranks. To tackle this problem, we adopt the logistic parametric model—the Bradley-Terry-Luce model, where each item is assigned a latent preference score, and where the outcome of each pairwise comparison depends solely on the relative scores of the two items involved. Recent works have made significant progress towards characterizing the performance (e.g. the mean square error for estimating the scores) of several classical methods, including the spectral method and the maximum likelihood estimator (MLE). However, where they stand regarding top-$K$ ranking remains unsettled.

We demonstrate that under a random sampling model, the spectral method alone, or the regularized MLE alone, is minimax optimal in terms of the sample complexity—the number of paired comparisons needed to ensure exact top-$K$ identification. This is accomplished via optimal control of the entrywise error of the score estimates. We complement our theoretical studies by numerical experiments, confirming that both methods yield low entrywise errors for estimating the underlying scores. Our theory is established based on a novel leave-one-out trick, which proves effective for analyzing both iterative and non-iterative optimization procedures. Along the way, we derive an elementary eigenvector perturbation bound for probability transition matrices, which parallels the Davis-Kahan $\sin \Theta$ theorem for symmetric matrices. This further allows us to close the gap between the $\ell_2$ error upper bound for the spectral method and the minimax lower limit.

Keywords. top-$K$ ranking, pairwise comparisons, spectral method, regularized MLE, perturbation analysis, Erdős–Rényi random graph, leave-one-out analysis, reversible Markov chain.

1 Introduction

Imagine we have a large collection of $n$ items, and we are given partially revealed comparisons between pairs of items. These paired comparisons are collected in a non-adaptive fashion, and could be highly noisy and incomplete. The aim is to aggregate these partial preferences so as to identify the $K$ items that receive the highest ranks. This problem, which is called top-$K$ rank aggregation, finds applications in numerous contexts, including web search (Dwork et al., 2001), recommendation systems (Baltrunas et al., 2010; Park et al., 2015), sports competition (Masse, 1997), and so on. The challenge is both statistical and computational: how can one achieve reliable top-$K$ ranking from a minimally possible number of pairwise comparisons, while retaining computational efficiency?

1.1 Popular approaches

To address the aforementioned challenge, many prior approaches have been put forward based on certain statistical models. Arguably one of the most widely used parametric models is the Bradley-Terry-Luce

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(BTL) model (Bradley and Terry, 1952; Luce, 1959; Agresti, 2013), which assigns a latent preference score $w_i^*$ ($1 \leq i \leq n$) to each of the $n$ items. The BTL model posits that: the chance of each item winning a paired comparison is determined by the relative scores of the two items involved, or more precisely,

$$\Pr \{\text{item } j \text{ is preferred over item } i \} = \frac{w_j^*}{w_i^* + w_j^*}$$

in each comparison of item $i$ against item $j$. The items are repeatedly compared in pairs according to this parametric model. The task then boils down to identifying the $K$ items with the highest preference scores, given these pairwise comparisons.

Among the ranking algorithms tailored to the BTL model, there are two procedures that have received particular attention, both of which rank the items in accordance with appropriate estimates of the latent preference scores:

1. **The spectral method.** By connecting the winning probability in (1) with the transition probability of a reversible Markov chain, the spectral method attempts recovery of $\{w_i^*\}_{1 \leq i \leq n}$ via the leading left eigenvector of a sample transition matrix. This spectral ranking procedure, which is also known as Rank Centrality (Negahban et al., 2017a), bears some similarity to the PageRank algorithm (Page et al., 1999).

2. **The maximum likelihood estimator (MLE).** This approach proceeds by finding the score assignment that maximizes the likelihood function (Ford, 1957). When parameterized appropriately, solving the MLE becomes a convex program, and hence is computationally feasible. There are also important variants of the MLE that enforce additional regularization; see Negahban et al. (2017a).

Details are postponed to Section 2.2. In addition to their remarkable practical applicability, these two ranking paradigms are appealing in theory as well. For instance, both of them provably achieve near-optimal mean square errors when estimating the latent preference scores (Negahban et al., 2017a; Hajek et al., 2014).

Nevertheless, the $\ell_2$ error for estimating the latent scores merely serves as a “meta-metric” for the ranking task, which does not necessarily reveal the accuracy of top-$K$ identification. Given that the $\ell_2$ loss only reflects the estimation error in some average sense, it is certainly possible that an algorithm obtains minimal $\ell_2$ estimation loss but incurs (relatively) large errors when estimating the scores of the highest ranked items. Interestingly, a recent work Chen and Suh (2015) demonstrates that: a careful combination of the spectral method and the coordinate-wise MLE is optimal for top-$K$ ranking. This leaves open the following natural questions: **where does the spectral alone, or the MLE alone, stand in top-$K$ ranking? Are they capable of attaining exact top-$K$ recovery from minimal samples?** These questions are poorly understood and form the primary objectives of our study.

As we will elaborate later, the spectral method part of the preceding question has recently been explored in an interesting paper Jang et al. (2016), for a regime where a relatively large fraction of item pairs have been compared. However, it remains unclear how well these two algorithms can perform in a much broader—and often more challenging—regime.

### 1.2 Main contributions

The central focal point of the current paper is to assess the accuracy of both the spectral method and the regularized MLE in top-$K$ identification. Assuming that the pairs of items being compared are randomly selected, our paper delivers a somewhat surprising message:

**Both the spectral method and the regularized MLE achieve perfect identification of top-$K$ ranked items under optimal sample complexity (up to some constant factor)!**

It is worth emphasizing that these two algorithms work even under the sparsest possible regime, a scenario where only an exceedingly small fraction of pairs of items have been compared. This calls for precise control of the entrywise error—as opposed to the $\ell_2$ loss—for estimating the scores. To this end, our theory is established upon a novel leave-one-out argument, which might shed light on how to analyze the entrywise error for more general optimization problems. As a byproduct of the analysis, we derive an
elementary eigenvector perturbation bound for probability transition matrices, which parallels Davis-Kahan’s sin Θ theorem for symmetric matrices. This simple perturbation bound leads to an improved ℓ₂ error bound for the spectral method, which matches the minimax lower limit derived in the prior literature.

Finally, we remark that both methods can be performed in time nearly proportional to reading all samples. In fact, the near-optimal computational cost of the spectral method has already been demonstrated in Negahban et al. (2017a). When it comes to the regularized MLE, we will show that this method—which only exhibits a fairly weak level of strong convexity—can be computed by the gradient descent in nearly linear time. In summary, both methods provably achieve optimal efficiency from both statistical and computational perspectives.

1.3 Notation

Before continuing, we introduce a few notations that will be useful throughout. To begin with, vectors and matrices are in boldface whereas scalars are not.

For any strictly positive probability vector \( \pi \in \mathbb{R}^n \), we define the inner product space indexed by \( \pi \) as a real vector space in \( \mathbb{R}^n \) endowed with the inner product (Brémaud, 1999)

\[
\langle x, y \rangle_{\pi} = \sum_{i=1}^{n} \pi_i x_i y_i.
\]

The corresponding vector norm and the induced matrix norm are defined respectively as

\[
\| x \|_{\pi} = \sqrt{\langle x, x \rangle_{\pi}} \quad \text{and} \quad \| A \|_{\pi} = \sup_{\| x \|_{\pi} = 1} \| x^T A \|_{\pi}.
\]

If \( 0 < \pi_{\min} \leq \pi_i \leq \pi_{\max} \) for all \( 1 \leq i \leq n \), then one has the following elementary inequalities

\[
\sqrt{\pi_{\min}} \| x \|_2 \leq \| x \|_{\pi} \leq \sqrt{\pi_{\max}} \| x \|_2 \quad \text{and} \quad \sqrt{\frac{\pi_{\min}}{\pi_{\max}}} \| A \|_2 \leq \| A \|_{\pi} \leq \sqrt{\frac{\pi_{\max}}{\pi_{\min}}} \| A \|_2.
\]

Additionally, the standard notation \( f(n) = O(g(n)) \) or \( f(n) \lesssim g(n) \) means that there exists a constant \( c > 0 \) such that \( |f(n)| \leq c |g(n)| \), \( f(n) = \Omega(g(n)) \) or \( f(n) \gtrsim g(n) \) means that there exists a constant \( c > 0 \) such that \( |f(n)| \geq c |g(n)| \), \( f(n) = \Theta(g(n)) \) or \( f(n) \sim g(n) \) means that there exists constants \( c_1, c_2 > 0 \) such that \( c_1 |g(n)| \leq |f(n)| \leq c_2 |g(n)| \), and \( f(n) = o(g(n)) \) means that \( \lim_{n \to \infty} \frac{f(n)}{g(n)} = 0 \).

Given a graph \( G \) with vertex set \( \{1, 2, \ldots, n\} \) and edge set \( E \), we denote by \( L_G = \sum_{i,j \in E, i > j} (e_i - e_j) (e_i - e_j)^T \) the (unnormalized) Laplacian matrix (Chung, 1997) associated with it, where \( \{e_i\}_{1 \leq i \leq n} \) are the standard basis vectors in \( \mathbb{R}^n \). For a matrix \( A \in \mathbb{R}^{n \times n} \) with \( n \) real eigenvalues, we let \( \lambda_1(A) \geq \lambda_2(A) \geq \cdots \geq \lambda_n(A) \) be the eigenvalues sorted in descending order.

2 Statistical models and main results

2.1 Problem setup

We begin with a formal introduction of the Bradley-Terry-Luce parametric model for binary comparisons.

Preference scores. As introduced earlier, we assume the existence of a positive latent score vector

\[
w^* = [w_1^*, \ldots, w_n^*]^T
\]

that comprises the underlying preference score \( \{w_i^*\}_{1 \leq i \leq n} \) assigned to each of the \( n \) items. Alternatively, it is sometimes more convenient to reparameterize the score vector by

\[
\theta^* = [\theta_1^*, \ldots, \theta_n^*] = \log w_i^*.
\]

These scores are assumed to fall within a fixed dynamic range, namely,

\[
w_i^* \in [w_{\min}, w_{\max}], \quad \text{or} \quad \theta_i^* \in [\theta_{\min}, \theta_{\max}]
\]
for all $1 \leq i \leq n$, where $w_{\text{min}} > 0$, $w_{\text{max}} > 0$, $\theta_{\text{min}} = \log w_{\text{min}}$, and $\theta_{\text{max}} = \log w_{\text{max}}$ are some absolute constants independent of $n$. Without loss of generality, it is assumed that

$$w_{\text{max}} \geq w_1^* \geq w_2^* \geq \ldots \geq w_n^* \geq w_{\text{min}},$$

meaning that items 1 through $K$ are the desired top-$K$ ranked items. For notational simplicity, we define

$$\kappa := \frac{w_{\text{max}}}{w_{\text{min}}},$$

which is assumed to be a constant independent of $n$.

**Comparison graph.** Let $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ stand for a comparison graph, where the vertex set $\mathcal{V} = \{1, 2, \ldots, n\}$ represents the $n$ items of interest. The items $i$ and $j$ are compared if and only if $(i, j) \in \mathcal{E}$. Unless otherwise noted, we assume that $\mathcal{G}$ is drawn from the Erdős–Rényi random graph $\mathcal{G}_{n, p}$ (Durrett, 2007), such that an edge between any pair of vertices is present independently with some probability $p$. In words, $p$ captures the fraction of item pairs being compared. In order to ensure that the comparison graph is connected (otherwise there is no basis to rank items belonging to two disconnected components of $\mathcal{G}$), we assume throughout that

$$p \geq c_0 \frac{\log n}{n}$$

for some constant $c_0 > 1$.

**Pairwise comparisons.** For each $(i, j) \in \mathcal{E}$, we obtain $L$ independent paired comparisons between items $i$ and $j$. Let $y_{i,j}^{(l)}$ be the outcome of the $\ell$-th comparison, which is independently drawn as

$$y_{i,j}^{(l)} \text{ ind.} = \begin{cases} 1, & \text{with probability } \frac{w_j^*}{w_i^* + w_j^*} = \frac{e^{\theta_j^*}}{e^{\theta_i^*} + e^{\theta_j^*}}, \\ 0, & \text{else} \end{cases}$$

By convention, we set $y_{j,i}^{(l)} = 1 - y_{i,j}^{(l)}$ for all $(i, j) \in \mathcal{E}$ throughout the paper. This is also known as the *logistic* pairwise comparison model, due to its strong resemblance to logistic regression (Agresti, 2013). It is self-evident that the sufficient statistics under this model are given by

$$y := \{y_{i,j} \mid (i, j) \in \mathcal{E}\}, \quad y_{i,j} := \frac{1}{L} \sum_{l=1}^{L} y_{i,j}^{(l)}.$$ 

To simplify the notation, we shall also take

$$y_{i,j}^* := \frac{w_j^*}{w_i^* + w_j^*} = \frac{e^{\theta_j^*}}{e^{\theta_i^*} + e^{\theta_j^*}}.$$ 

**Goal.** The goal is to identify the set of top-$K$ ranked items—that is, the set of $K$ items that exhibit the largest preference scores—from the pairwise comparisons $y$.

### 2.2 Algorithms

#### 2.2.1 The spectral method: Rank Centrality

The spectral ranking algorithm, or Rank Centrality (Negahban et al., 2017a), is motivated by the connection between the pairwise comparisons and a random walk over a directed graph. The algorithm starts by converting the pairwise comparison data $y$ into a transition matrix $P = [P_{i,j}]_{1 \leq i, j \leq n}$ in such a way that

$$P_{i,j} = \begin{cases} \frac{1}{d} y_{i,j}, & \text{if } (i, j) \in \mathcal{E}, \\ 1 - \frac{1}{d} \sum_{k:(i,k) \in \mathcal{E}} y_{i,k}, & \text{if } i = j, \\ 0, & \text{otherwise}, \end{cases}$$

for some constant $c_0 > 1$. 

### 4
Algorithm 1 Spectral method (Rank Centrality).

\begin{itemize}
  \item **Input** the comparison graph $\mathcal{G}$, the sufficient statistics $\mathbf{y}$, and the normalization factor $d$.
  \item **Define** the probability transition matrix $P = [P_{i,j}]_{1 \leq i,j \leq n}$ as in (11).
  \item **Compute** the leading left eigenvector $\pi$ of $P$.
  \item **Output** the $K$ items that correspond to the $K$ largest entries of $\pi$.
\end{itemize}

for some given normalization factor $d$, and then proceeds by computing the stationary distribution $\pi \in \mathbb{R}^n$ of the Markov chain induced by $P$. As we shall see later, the parameter $d$ is taken to be on the same order of the maximum vertex degree of $\mathcal{G}$ while ensuring the non-negativity of $P$. As asserted by Negahban et al. (2017a), $\pi$ is a faithful estimate of $w^*$ up to some global scaling. The algorithm is summarized in Algorithm 1.

To develop some intuition regarding why this spectral algorithm gives a reasonable estimate of $w^*$, it is perhaps more convenient to look at the population transition matrix $P^* = [P^*_{i,j}]_{1 \leq i,j \leq n}$:

$$
P^*_{i,j} = \begin{cases} 
\frac{1}{d} \frac{w^*_j}{w^*_i + w^*_j}, & \text{if } (i,j) \in \mathcal{E}, \\
1 - \frac{1}{d} \sum_{k: (i,k) \in \mathcal{E}} \frac{w^*_k}{w^*_i + w^*_k}, & \text{if } i = j, \\
0, & \text{otherwise,}
\end{cases}
$$

which coincides with $P$ by taking $L \to \infty$. It can be seen that the normalized score vector $\pi^* := \frac{1}{\sum_{i=1}^n w^*_i} \begin{bmatrix} w^*_1, w^*_2, \ldots, w^*_n \end{bmatrix}^\top$ (12)
is the stationary distribution of the Markov chain induced by $P^*$, since $P^*$ and $\pi^*$ are in detailed balance (Brémaud, 1999), namely,

$$
\pi^*_i P^*_{i,j} = \pi^*_j P^*_{j,i}, \quad \forall (i,j). \tag{13}
$$

As a result, one expects the stationary distribution of the sample version $P$ to form a good estimate of $w^*$, provided that the sample size is sufficiently large.

### 2.2.2 The regularized MLE

Under the BTL model, the negative log-likelihood function conditioned on the graph is given by (up to some global scaling)

$$
L(\theta; \mathbf{y}) := -\sum_{(i,j) \in \mathcal{E}, i > j} \left\{ y_{j,i} \log \frac{e^{\theta_i}}{e^{\theta_i} + e^{\theta_j}} + (1 - y_{j,i}) \log \frac{e^{\theta_j}}{e^{\theta_i} + e^{\theta_j}} \right\} \\
= -\sum_{(i,j) \in \mathcal{E}, i > j} \left\{ -y_{j,i} (\theta_i - \theta_j) + \log (1 + e^{\theta_i - \theta_j}) \right\}. \tag{14}
$$

The regularized MLE (Negahban et al., 2017a, Section 3.3) then amounts to solving the following convex program

$$
\text{minimize}_{\theta \in \mathbb{R}^n} \quad L_\lambda(\theta; \mathbf{y}) := L(\theta; \mathbf{y}) + \frac{1}{2} \lambda \|\theta\|^2,
$$

for a regularization parameter $\lambda > 0$. As will be discussed later, we shall adopt the choice $\lambda \approx \sqrt{\frac{np \log n}{L}}$ throughout this paper. For the sake of brevity, we let $\theta$ represent the resulting penalized maximum likelihood estimate whenever it is clear from the context.

### 2.3 Main results

The most challenging part of top-$K$ ranking is to distinguish the $K$-th and the $(K+1)$-th items. In fact, the score difference of these two items captures the distance between the item sets $\{1, \ldots, K\}$ and
{K + 1, \ldots, n}. Unless their latent scores are sufficiently separated, the finite-sample nature of the model would make it infeasible to distinguish these two critical items. With this consideration in mind, we define the following separation measure

\[ \Delta_K := \frac{w_K^* - w_{K+1}^*}{w_{\max}}. \]  

(16)

This metric turns out to play a crucial role in determining the minimal sample complexity for perfect top-

\[ K \]  

identification.

The main finding of this paper concerns the optimality of both the spectral method and the regularized MLE. Recall that under the BTL model introduced before, the total number \( N \) of samples we collect concentrates sharply around its mean, namely,

\[ N = (1 + o(1))E[N] = (1 + o(1))\frac{1}{2}n^2pL \]  

(17)

occurs with high probability. Our main result is stated in terms of the sample complexity required for exact top-

\[ K \]  

identification.

**Theorem 1.** Consider the pairwise comparison model specified in Section 2.1. Suppose that \( p > \frac{c_0 \log n}{n} \) and that

\[ \frac{n^2pL}{2} \geq c_1 n \log n \Delta_K^2 \]  

(18)

for some sufficiently large constants \( c_0, c_1 > 0 \). With probability exceeding \( 1 - O(n^{-5}) \), the set of top-

\[ K \]  

ranked items can be recovered exactly by the spectral method given in Algorithm 1, and by the regularized MLE given in (15). Here, we take \( d = c_0 np \) in the spectral method and \( \lambda = c_1 \sqrt{\frac{n \log n}{npL}} \) in the regularized MLE, where \( c_d \geq 2 \) and \( c_\lambda > 0 \) are some absolute constants.

Theorem 1 asserts that both the spectral method and the regularized MLE achieve a sample complexity on the order of \( n \log n \). Encouragingly, this sample complexity coincides with the minimax limit identified in (Chen and Suh, 2015, Theorem 2) up to a constant factor:

**Theorem 2 (Chen and Suh (2015)).** Fix \( \epsilon \in (0, 1/2) \), and suppose that

\[ n^2pL \leq 2c_2 \frac{(1 - \epsilon)n \log n - 2}{\Delta_K^2}, \]  

(19)

where \( c_2 = w_{\min}^4 / (4w_{\max}^4) \). Then for any ranking procedure \( \psi \), one can find a score vector \( w^* \) with separation \( \Delta_K \) such that \( \psi \) fails to retrieve the top-

\[ K \]  

items with probability at least \( \epsilon \).

Moving beyond the sample complexity issue, we remark that the spectral method can be carried out by means of the power method, while the regularized MLE can be computed via the standard gradient descent (as we will elaborate in Section 4.2.1). Both procedures are computationally feasible and can be accomplished within nearly linear time—in time proportional to reading all data. Consequently, these two algorithms are optimal in terms of both computational complexity and sample complexity. We emphasize that it is nontrivial to establish the nearly linear-time computational complexity of the regularized MLE, given that the log-likelihood function (or the logistic link function) is not strongly convex. We will elaborate on this point in Section 4.2.1.

We are now positioned to compare our results with Jang et al. (2016), which also investigates the accuracy of the spectral method for top-

\[ K \]  

ranking. Specifically, (Jang et al., 2016, Theorem 3) establishes the optimality of the spectral method for the relatively dense regime where

\[ p \gtrsim \sqrt{\frac{\log n}{n}}. \]

In this regime, however, the total sample size necessarily exceeds

\[ \frac{n^2pL}{2} \geq \frac{n^2p}{2} \gtrsim \sqrt{n^3 \log n}, \]  

(20)
which rules out the possibility of achieving minimal sample complexity if $\Delta_K$ is sufficiently large. For instance, consider the case where $\Delta_K \approx 1$, then the optimal sample size—as revealed by Theorem 1 or (Chen and Suh, 2015, Theorem 1)—is on the order of

$$\frac{n \log n}{\Delta_K^2} \asymp n \log n,$$

which is a factor of $\sqrt{n/\log n}$ lower than the bound in (20). In contrast, our results hold all the way down to the sparsest possible regime where $p \approx \log n$, confirming the optimality of the spectral method even for the most challenging scenario. Furthermore, we establish that the regularized MLE shares the same optimality guarantee as the spectral method, which was previously out of reach.

### 2.4 Optimal control of entrywise estimation errors

In order to establish the ranking accuracy as asserted by Theorem 1, the key is to obtain precise control of the $\ell_\infty$ loss of the score estimates. Our results are as follows.

**Theorem 3 (Entrywise error of the spectral method).** Suppose that $p > c_0 \log n/\sqrt{n}$ for some sufficiently large constant $c_0 > 0$. Choose $d = c_d n p$ for some constant $c_d \geq 2$ in Algorithm 1. Then the spectral estimate $\pi$ satisfies

$$\|\pi - \pi^*\|_\infty \lesssim \sqrt{\log n / npL}$$

with probability exceeding $1 - O(n^{-5})$, where $\pi^*$ is the normalized score vector as defined in (12).

**Theorem 4 (Entrywise error of the regularized MLE).** Suppose that $p > c_0 \log n/\sqrt{n}$ for some sufficiently large constant $c_0 > 0$, and that the regularization parameter is $\lambda = c_\lambda \sqrt{np \log n / L}$ for some absolute constant $c_\lambda > 0$. Then the regularized MLE $\theta$ satisfies

$$\|e^{\theta} - e^{\theta^*} - \theta^*\|_\infty \lesssim \sqrt{\log n / npL}$$

with probability exceeding $1 - O(n^{-5})$, where $\theta^* := \frac{1}{n} 1^\top \theta^*$ and $e^{\theta} := [e^{\theta_1}, \ldots, e^{\theta_n}]^\top$.

Theorems 3-4 indicate that if the number of comparisons associated with each item—which concentrates around $npL$—exceeds $\log n$ to some sufficiently large extent, then both methods are able to achieve a small $\ell_\infty$ error when estimating the scores.

Recall that the $\ell_2$ estimation error of the spectral method has been characterized by Negahban et al. (2017a) (or Theorem 6 of this paper with an improved upper bound), which obeys

$$\|\pi - \pi^*\|_2 \lesssim \sqrt{\log n / npL}$$

with high probability. Similar theoretical guarantees have been derived for another variant of the MLE (the constrained version) under a uniform sampling model as well (Negahban et al., 2017a). In comparison, our results indicate that the estimation errors for both algorithms are almost evenly spread out across all coordinates rather than being localized or clustered. Notably, the pointwise errors revealed by Theorems 3-4 immediately lead to exact top-$K$ identification as claimed by Theorem 1.

**Proof of Theorem 1.** In what follows, we prove the theorem for the spectral method part. The regularized MLE part follows from an almost identical argument and hence is omitted.

Since the spectral algorithm ranks the items in accordance with the score estimate $\pi$, it suffices to demonstrate that

$$\pi_i - \pi_j > 0, \quad \forall 1 \leq i \leq K, \ K + 1 \leq j \leq n.$$
To this end, we first apply the triangle inequality to get

\[
\pi_i - \pi_j \geq \frac{\pi_i^* - \pi_j^*}{\| \pi^* \|_\infty} \geq \frac{\pi_i - \pi_j^*}{\| \pi^* \|_\infty} - \frac{| \pi_j - \pi_j^* |}{\| \pi^* \|_\infty} \geq \Delta_K - \frac{2\| \pi - \pi^* \|_\infty}{\| \pi^* \|_\infty}.
\]  \hspace{1cm} (24)

In addition, it follows from Theorem 3 as well as our sample complexity assumption that

\[
\frac{\| \pi - \pi^* \|_\infty}{\| \pi^* \|_\infty} \lesssim \sqrt{\frac{\log n}{npL}} \quad \text{and} \quad n^2 pL \gtrsim \frac{n \log n}{\Delta_K^2}.
\]

These conditions taken collectively imply that

\[
\frac{\| \pi - \pi^* \|_\infty}{\| \pi^* \|_\infty} < \frac{1}{2} \Delta_K
\]

as long as \( \frac{npL \Delta_K^2}{\log n} \) is sufficiently large. Substitution into (24) reveals that \( \pi_i - \pi_j > 0 \), as claimed. \( \square \)

2.5 Heuristic arguments

We pause here to develop some heuristic explanation as to why the score estimation errors are expected to be spread out across all entries. For simplicity, we focus on the case where \( p = 1 \) and \( L \) is sufficiently large, so that \( y \) and \( P \) sharply concentrate around \( y^\ast \) and \( P^\ast \), respectively. For simplicity of presentation, we introduce the notation \( \approx \) to represent asymptotic equivalence up to logarithmic factors.

We begin with the spectral algorithm. Since \( \pi \) and \( \pi^\ast \) are respectively the invariant distributions of the Markov chains induced by \( P \) and \( P^\ast \), we can decompose

\[
(\pi - \pi^\ast)^\top = \pi^\top P - \pi^\ast^\top P^\ast = (\pi - \pi^\ast)^\top P + \pi^\ast^\top (P - P^\ast).
\]  \hspace{1cm} (25)

When \( p = 1 \) and \( \frac{\| \pi^\ast \|_\infty}{\| \pi^\ast \|_2} \asymp 1 \), the entries of \( \pi^\ast \) (resp. the off-diagonal entries of \( P^\ast \) and \( P - P^\ast \)) are all of the same order and, as a result, the energy of the uncertainty term \( \xi \) is spread out. In fact, we will demonstrate in Section 3.2 that

\[
\frac{\| \xi \|_\infty}{\| \pi^\ast \|_\infty} \lesssim \sqrt{\frac{\log n}{npL}} \cdot \frac{\| \pi - \pi^\ast \|_2}{\| \pi^\ast \|_2},
\]  \hspace{1cm} (26)

which coincides with the optimal rate. Further, if we look at each entry of (25), then for all \( 1 \leq m \leq n \),

\[
\pi_m - \pi^\ast_m = [(\pi - \pi^\ast)^\top P]_m + \xi_m,
\]

\[
= (\pi_m - \pi^\ast_m) P_{m,m} + [P_{1,m}, \cdots, P_{m-1,m}, 0, P_{m+1,m}, \cdots, P_{n,m}] \begin{bmatrix} \pi_1 - \pi^\ast_1 \\ \vdots \\ \pi_n - \pi^\ast_n \end{bmatrix} + \xi_m.
\]  \hspace{1cm} (27)

By construction of the transition matrix, one can easily verify that \( P_{m,m} \) is bounded away from 1 and \( P_{j,m} \approx 1/n \) for all \( j \neq m \). As a consequence, the identity \( \pi^\top P = \pi^\top \) allows one to treat each \( \pi_m - \pi^\ast_m \) as a mixture of three effects: (i) the first term of (27) behaves as a entrywise contraction of the error; (ii) the second term of (27) is a (nearly uniformly weighted) average of the errors over all coordinates, which can essentially be treated as a smoothing operator applied to the error components; and (iii) the uncertainty term \( \xi_m \). Rearranging terms in (27), we are left with

\[
(1 - P_{m,m}) |\pi_m - \pi^\ast_m| \lesssim \frac{1}{n} \sum_{i=1}^n |\pi_i - \pi^\ast_i| + \xi_m,
\]  \hspace{1cm} (28)
or equivalently,
\[ \| \pi - \pi^* \|_\infty \lesssim \frac{1}{n} \sum_{i=1}^n |\pi_i - \pi_i^*| + \|\xi\|_\infty. \] (29)

There are two possibilities compatible with this bound: (1) \( \| \pi - \pi^* \|_\infty \lesssim \frac{1}{n} \sum_{i=1}^n |\pi_i - \pi_i^*| \), and (2) \( \| \pi - \pi^* \|_\infty \lesssim \|\xi\|_\infty \) by (26). In either case, the errors are fairly delocalized, revealing that
\[ \frac{\| \pi - \pi^* \|_\infty}{\|\pi^*\|_2} \lesssim \frac{\| \pi - \pi^* \|_2}{\|\pi^*\|_2}. \]

We now move on to the regularized MLE, which follows a very similar argument. By the optimality condition that \( \nabla \mathcal{L}_\lambda (\theta) = 0 \), one can derive (for some \( \eta \) to be specified later)
\begin{align*}
\theta - \theta^* &= \theta - \eta \nabla \mathcal{L}_\lambda (\theta) - \theta^* \\
&= \theta - \eta \nabla \mathcal{L}_\lambda (\theta) - (\theta^* - \eta \nabla \mathcal{L}_\lambda (\theta^*)) - \eta \nabla \mathcal{L}_\lambda (\theta^*) \\
&\approx (I - \eta \nabla^2 \mathcal{L}_\lambda (\theta^*)) (\theta - \theta^*) - \zeta.
\end{align*}

Write \( \nabla^2 \mathcal{L}_\lambda (\theta^*) = D - A \), where \( D \) and \( A \) denote respectively the diagonal and off-diagonal parts. Under our assumptions, one can check that \( D_{m,m} \approx n \) for all \( 1 \leq m \leq n \) and \( A_{j,m} \approx 1 \) for any \( j \neq m \). With these notations in place, one can write the entrywise error as follows
\[ \theta_m - \theta^*_m = (1 - \eta D_{m,m}) (\theta_m - \theta^*_m) + \sum_{j:j \neq m} \eta A_{j,m} (\theta_j - \theta^*_j) - \zeta_m. \]

By choosing \( \eta = c_2/n \) for some sufficiently small constant \( c_2 > 0 \), we get \( 1 - \eta D_{m,m} < 1 \) and \( \eta A_{j,m} \approx 1/n \). Therefore, the right-hand side of the above relation also comprises a contraction term as well as an error smoothing term, similar to (27). Carrying out the same argument as for the spectral method, we see that the estimation errors of the regularized MLE are expected to be spread out.

### 2.6 Numerical experiments

It is worth noting that extensive numerical experiments have already been conducted in Negahban et al. (2017a) to justify the practicability of both the spectral method and the regularized MLE. This section provides some additional simulations to complement their experimental results and our theory. Throughout the experiments, we set the number of items \( n \) to be 200, while the number of repeated comparisons \( L \) and the edge probability \( p \) can vary with the experiments. For the tuning parameters, we choose \( d = 2d_{\max} \) in the spectral method where \( d_{\max} \) is the maximum degree of the graph and \( \lambda = 2\sqrt{np \log n \over L} \) in the regularized MLE, which are consistent with the configurations considered in the main theorems. All of the results are averaged over 100 Monte Carlo simulations.

We first investigate the relative \( \ell_\infty \) error of the spectral method and the regularized MLE when estimating the preference scores. To this end, we generate the latent scores \( w^*_i \) \((1 \leq i \leq n)\) independently and uniformly at random over the interval \([0, 1]\). Figure 1(a) (resp. Figure 1(b)) displays the entrywise error in the score estimation as the number of repeated comparisons \( L \) (resp. the edge probability \( p \)) varies. As can be seen from the plots, the \( \ell_\infty \) error of both methods gets smaller as \( p \) and \( L \) increase, confirming our results in Theorems 3-4. Next, we show in Figure 2 the relative \( \ell_\infty \) error and top-\( K \) ranking accuracy while fixing the total number of samples \((i.e., n^2pL)\). It can be seen that the performance almost does not change if the total sample complexity \( n^2pL \) remains the same. In addition, Figure 3 illustrates the relative \( \ell_\infty \) error and the relative \( \ell_2 \) error in score estimation for both methods. As we can see, the relative \( \ell_\infty \) errors are not much larger than the relative \( \ell_2 \) errors (recall that \( n = 200 \)), thus providing empirical evidence that the errors in the score estimates are spread out across all entries.

Furthermore, we examine the top-\( K \) ranking accuracy of both methods. Here, we fix \( p = 0.25 \) and \( L = 20 \), set \( K = 10 \), and let \( w^*_i = 1 \) for all \( 1 \leq i \leq K \) and \( w^*_j = 1 - \Delta \) for all \( K+1 \leq j \leq n \). By construction, the score separation satisfies \( \Delta_K = \Delta \). Figure 4 illustrates the accuracy of both methods in identifying the top-\( K \) ranked items. The performance of both methods improves when the score separation becomes larger, which matches our theory in Theorem 1.
Figure 1: Empirical performance of both the spectral method and the regularized MLE: (a) $\ell_\infty$ error vs. $L$, and (b) $\ell_\infty$ error vs. $p$.

Figure 2: Empirical performance of both the spectral method and the regularized MLE when fixing the number of samples $n^2pL$: (a) relative $\ell_\infty$ error, and (b) top-$K$ ranking accuracy.

Figure 3: Comparisons between the relative $\ell_\infty$ error and the relative $\ell_2$ error for (a) the spectral method, and (b) the regularized MLE.
2.7 Other related works

The problem of ranking based on partial preferences has received much attention during the past decade. Two types of observation models have been considered: (1) the cardinal-based model, where users provide explicit numerical ratings of the items being compared, (2) the ordinal-based model, where users are asked to make comparative measurements. See Ammar and Shah (2011); Shah et al. (2014) for detailed experimental and theoretical comparisons between these two distinct models.

In terms of the ordinal-based model—and in particular, ranking from pairwise comparisons—both parametric and nonparametric models have been extensively studied. For example, Hunter (2004) examined variants of the parametric BTL model, and established the convergence properties of the minorization-maximization algorithm for solving the corresponding MLE problem. The BTL model was also investigated in Borkar et al. (2016), where the authors proposed a randomized Kaczmarz algorithm to infer the latent scores with provable $\ell_2$ error guarantees. Moreover, the BTL model falls under the category of low-rank parametric models, since the preference matrix is generated by passing a rank-2 matrix through the logistic link function (Rajkumar and Agarwal, 2016). Additionally, the work Jiang et al. (2011); Xu et al. (2012) proposed a least-squares type method, called HodgeRank, to estimate the full ranking, which generalizes the simple Borda count algorithm (Borda, 1784; Ammar and Shah, 2011; Rajkumar and Agarwal, 2014). For many of these algorithms, the sample complexities needed for perfect total ranking have been determined by Rajkumar and Agarwal (2014), although the top-$K$ ranking accuracy was not considered there.

Going beyond the parametric models, a recent line of works Shah et al. (2017); Shah and Wainwright (2015); Chen et al. (2017); Pananjady et al. (2017) considered the nonparametric stochastically transitive model, where the only model assumption is that the comparison probability matrix follows certain transitivity rules. This type of models subsumes the BTL model as a special case. For instance, Shah and Wainwright (2015) suggested an exceedingly simple counting-based algorithm which can reliably recover the top-$K$ ranked items for a wide range of models. However, the sampling paradigm considered therein is quite different from ours in the sparse regime; for instance, their model does not come close to the setting where $p$ is small but $L$ is large, which is the most challenging regime of the model adopted in our paper and Negahban et al. (2017a); Chen and Suh (2015); Jang et al. (2016).

All of the aforementioned papers concentrate on the case where there is a single ground-truth ordering. It would also be interesting to investigate the scenarios where different users might have different preference scores (Negahban et al., 2017b; Lu and Negahban, 2014; Wu et al., 2015; Oh and Shah, 2014). To this end, Negahban et al. (2017b); Lu and Negahban (2014) imposed the low-rank structure on the underlying preference matrix and adopted the nuclear-norm relaxation approach to recover the users' preferences. The paper Wu et al. (2015) proposed a two-step procedure, where a clustering step was first applied to label different users, followed by an estimation step built upon the previous clustering outcome. Additionally, several papers explored the ranking problem for the more general Plackett-Luce model (Hajek et al., 2014; Soufiani et al., 2013, 2014), in the presence of adaptive sampling (Braverman and Mossel, 2008; Jamieson and Nowak, 2011; Ailon, 2012; Busa-Fekete et al., 2013; Mohajer and Suh, 2016; Heckel et al., 2016; Agarwal et al., 2017), for the crowdsourcing scenario (Chen et al., 2013; Osting et al., 2016), and in the adversarial
setting (Suh et al., 2017). These are beyond the scope of the present paper.

When it comes to the technical tools, it is worth noting that the leave-one-out idea has been invoked to analyze random designs for other high-dimensional problems as well, e.g. robust M-estimators (El Karoui et al., 2013; El Karoui, 2017), confidence intervals for Lasso (Javanmard and Montanari, 2015), likelihood ratio test (Sur et al., 2017). In particular, Zhong and Boumal (2017) and Abbe et al. (2017) use this technique to precisely characterize entrywise behavior of eigenvectors of a large class of symmetric random matrices, which improves upon the prior analysis on \( \ell_\infty \) eigenvector perturbation (Fan et al., 2016). As a consequence, they are able to show the sharpness of spectral methods—a type of methods based on the eigen-structure of a certain data matrix—in many popular models. Our introduction of leave-one-out auxiliary quantities are motivated by these papers.

Finally, we remark that the family of spectral methods has been successfully applied in numerous applications, e.g. matrix completion (Keshavan et al., 2010; Jain et al., 2013), phase retrieval (Candès et al., 2015; Chen and Candès, 2017), graph clustering (Rohe et al., 2011; Chaudhuri et al., 2012), synchronization and joint alignment (Singer, 2011; Chen and Candès, 2016). All of them are designed based on the eigenvectors of some symmetric matrix, or the singular vectors if the matrix of interest is asymmetric. Our paper contributes to this growing literature by establishing a sharp eigenvector perturbation analysis framework for an important class of asymmetric matrices—the probability transition matrices.

3 Analysis for the spectral method

This section is devoted to proving Theorem 3, which characterizes the pointwise error of the spectral estimate.

3.1 Preliminaries

Here, we gather some preliminary facts about reversible Markov chains as well as the Erdős–Rényi random graph.

The first important result concerns the eigenvector perturbation for probability transition matrices, which can be treated as the analogue of the celebrated Davis-Kahan sin \( \Theta \) theorem (Davis and Kahan, 1970) for asymmetric matrices. Due to its potential importance for other problems, we promote it to a theorem as follows.

**Theorem 5 (Eigenvector perturbation).** Suppose that \( P, \hat{P}, \) and \( P^\star \) are probability transition matrices with stationary distributions \( \pi, \hat{\pi}, \pi^\star \), respectively. Also, assume that \( P^\star \) represents a reversible Markov chain. Then it holds that

\[
\|\pi - \hat{\pi}\|_{\pi^\star} \leq \frac{\|\pi^\top (P - \hat{P})\|_{\pi^\star}}{1 - \max \left\{ \lambda_2(P^\star), -\lambda_n(P^\star) \right\} - \|P - \hat{P}\|_{\pi^\star}}.
\]

**Proof.** See Appendix A.1.

Several remarks regarding Theorem 5 are in order. First, in contrast to standard perturbation results like Davis-Kahan’s sin \( \Theta \) theorem, our theorem involves three matrices in total, where \( P, \hat{P}, \) and \( P^\star \) can all be arbitrary. For example, one may choose \( P^\star \) to be the population transition matrix, and \( P \) and \( \hat{P} \) as two finite-sample versions associated with \( P^\star \). Second, we only impose reversibility on \( P^\star \), whereas \( P \) and \( \hat{P} \) need not induce reversible Markov Chains. Third, Theorem 5 allows one to derive the \( \ell_2 \) estimation error in Negahban et al. (2017a) directly without resorting to the power method; in fact, our \( \ell_2 \) estimation error bound improves upon Negahban et al. (2017a) by some logarithmic factor.

**Theorem 6.** Suppose \( p \geq c_0 \frac{\log n}{n} \) for some sufficiently large constant \( c_0 > 0 \) and \( d \geq c_d np \) for \( c_d \geq 2 \) in Algorithm 1. With probability exceeds \( 1 - O(n^{-5}) \), one has

\[
\frac{\|\pi - \pi^\star\|_2}{\|\pi^\star\|_2} \lesssim \frac{1}{\sqrt{npL}}.
\]

**Proof.** See Appendix D.
Notably, Theorem 6 matches the minimax lower bound derived in (Negahban et al., 2017a, Theorem 3). As far as we know, this is the first result to demonstrate the orderwise optimality of the spectral method when measured by the \( \ell_2 \) loss.

The next result is concerned with the concentration of the vertex degrees in an Erdős–Rényi random graph.

**Lemma 1** (Degree concentration). Suppose that \( G \sim G_{n,p} \). Let \( d_i \) be the degree of node \( i \), \( d_{\text{min}} = \min_{1 \leq i \leq n} d_i \) and \( d_{\text{max}} = \max_{1 \leq i \leq n} d_i \). If \( p \geq \frac{c_0 \log n}{n} \) for some sufficiently large constant \( c_0 > 0 \), then the following event

\[
A_0 = \left\{ \frac{np}{2} \leq d_{\text{min}} \leq d_{\text{max}} \leq \frac{3np}{2} \right\}
\]

obeys

\[
P(A_0) \geq 1 - O(n^{-10}).
\]

**Proof.** This lemma follows from the standard Chernoff bound, and hence is omitted.

Since \( d \) is chosen to be \( c_d np \) for some constant \( c_d \geq 2 \), we have, by Lemma 1, that the maximum vertex degree obeys \( d_{\text{max}} < d \) with high probability.

### 3.2 Proof outline of Theorem 3

In this subsection, we outline the proof of Theorem 3. Recall that \( \pi = [\pi_1, \cdots, \pi_n]^\top \) and \( \pi^* = [\pi^*_1, \cdots, \pi^*_n]^\top \) are the stationary distributions associated with \( P \) and \( P^* \), respectively. This gives

\[
\pi^\top P = \pi^\top \quad \text{and} \quad \pi^{*\top} P^* = \pi^{*\top}.
\]

For each \( 1 \leq m \leq n \), one can decompose

\[
\pi_m - \pi^*_m = \pi^\top P_m - \pi^{*\top} P^*_m = \pi^{*\top} (P_m - P^*_m) + (\pi - \pi^*)^\top P_m
\]

\[
= \sum_j \pi^*_j (P_j, m - P^*_j, m) + (\pi_m - \pi^*_m) P_{m, m} + \sum_{j \neq m} (\pi_j - \pi^*_j) P_{j, m},
\]

where \( P_m \) denotes the \( m \)-th column of \( P \). Then it boils down to controlling \( I^m_1, I^m_2 \) and \( \sum_{j \neq m} (\pi_j - \pi^*_j) P_{j, m} \).

1. Since \( \pi^* \) is deterministic while \( P \) is random, we can easily control \( I^m_1 \) using Hoeffding’s inequality. The bound is the following.

**Lemma 2.** With probability exceeding \( 1 - O(n^{-5}) \), one has

\[
\max_m |I^m_1| \lesssim \sqrt{\frac{\log n}{L_d}} \|\pi^*\|_\infty.
\]

**Proof.** See Appendix A.2.

2. Next, the term \( I^m_2 \) behaves as a contraction of \( |\pi_m - \pi^*_m| \), as demonstrated below.

**Lemma 3.** With probability exceeding \( 1 - O(n^{-5}) \), there exists some constant \( c > 0 \) such that for all \( 1 \leq m \leq n \),

\[
|I^m_2| \leq \left( 1 - \frac{np}{2(1 + \kappa)d} + c \sqrt{\frac{\log n}{L_d}} \right) |\pi_m - \pi^*_m|.
\]

Here, we recall that \( \kappa = w_{\text{max}}/w_{\text{min}} \).

**Proof.** See Appendix A.3.
3. The statistical dependency between $\pi$ and $P$ introduces difficulty in obtaining a sharp estimate of the third term $\sum_{j\neq m} (\pi_j - \pi_j^*) P_{j,m}$. Nevertheless, the leave-one-out technique helps us decouple the dependency and obtain effective control of this term. The key component of the analysis is the introduction of a new probability transition matrix $P^{(m)}$, which is a leave-one-out version of the original matrix $P$. More precisely, $P^{(m)}$ replaces all of the transition probabilities involving the $m$-th item with their expected values (unconditional on $G$); that is, for any $i \neq j$,

$$P^{(m)}_{i,j} := \begin{cases} P_{i,j}, & i \neq m, j \neq m, \\ \frac{1}{m} y_{i,j}^*, & i = m \text{ or } j = m, \end{cases}$$

and for any $1 \leq i \leq n$,

$$P^{(m)}_{i,i} := 1 - \sum_{j \neq i} P^{(m)}_{i,j}$$

in order to ensure that $P^{(m)}$ is a probability transition matrix. In addition, we let $\pi^{(m)}$ be the stationary distribution of the Markov chain induced by $P^{(m)}$. As to be demonstrated later, the main advantages of introducing $\pi^{(m)}$ are two-fold: (1) the original spectral estimate $\pi$ can be very well approximated by $\pi^{(m)}$, and (2) $\pi^{(m)}$ is statistically independent of the connectivity of the $m$-th node and the comparisons with regards to the $m$-th item. Now we further decompose $\sum_{j \neq m} (\pi_j - \pi_j^*) P_{j,m}$:

$$\sum_{j \neq m} (\pi_j - \pi_j^*) P_{j,m} = \sum_{j \neq m} (\pi_j - \pi_j^{(m)}) P_{j,m} + \sum_{j \neq m} (\pi_j^{(m)} - \pi_j^*) P_{j,m}.$$  

4. For $I^m_3$, we apply the Cauchy-Schwarz inequality to obtain that with probability at least $1 - O(n^{-10})$,

$$|I^m_3| \leq \|\pi^{(m)} - \pi\|_2 \left( \sum_{j \neq m} \frac{P^2_{j,m}}{m} \right)^{\frac{1}{2}} \leq \frac{1}{\sqrt{d}} \|\pi^{(m)} - \pi\|_2,$$

where (i) follows from the fact that $P_{j,m} \leq \frac{1}{d}$ for all $j \neq m$ and $d_{\text{max}} \leq d$ on the event $A_0$. Consequently, it suffices to control the $\ell_2$ difference between the original spectral estimate $\pi$ and its leave-one-out version $\pi^{(m)}$. This is accomplished in the following lemma.

**Lemma 4.** With probability at least $1 - O\left(n^{-10}\right)$,

$$\|\pi^{(m)} - \pi\|_2 \leq \frac{16\sqrt{\kappa}}{\gamma} \sqrt{\frac{\log n}{Ld}} \|\pi^*\|_\infty + \|\pi - \pi^*\|_\infty,$$

(31)

where $\kappa = w_{\text{max}}/w_{\text{min}}$ and $\gamma = 1 - \max\{\lambda_2(P^*), -\lambda_n(P^*)\} - \|P - P^*\|_{\pi^*}$.  

**Proof.** See Appendix A.4.

As shown in (Negahban et al., 2017a, Section 4.2), $\gamma$ is lower bounded by some positive constant.

**Lemma 5 (Spectral gap Negahban et al. (2017a)).** Under the model specified in Section 2.1, if $p \geq c_0 \frac{\log n}{n}$ for some sufficiently large constant $c_0 > 0$, then with probability at least $1 - O(n^{-5})$, one has

$$\gamma := 1 - \max\{\lambda_2(P^*), -\lambda_n(P^*)\} - \|P - P^*\|_{\pi^*} \geq \frac{1}{2\kappa^2} \frac{d_{\text{min}}}{d_{\text{max}}} \geq \frac{1}{6\kappa^2}.$$  

With this spectral gap in mind, Lemma 4 reads

$$\|\pi^{(m)} - \pi\|_2 \lesssim \sqrt{\frac{\log n}{Ld}} \|\pi^*\|_\infty + \|\pi - \pi^*\|_\infty.$$  

(32)

5. In order to control $I^m_3$, we exploit the statistical independence between $\pi^{(m)}$ and $P_m$. Specifically, we demonstrate that:
Lemma 6. Suppose that \( p > \frac{c_0 \log n}{n} \) for some sufficiently large constant \( c_0 > 0 \). With probability at least \( 1 - O(n^{-10}) \),

\[
|I_q^n| \lesssim \frac{1}{\sqrt{n}} \|\pi^{(m)} - \pi\|_2 + \sqrt{\frac{\log n}{Ld}} \|\pi^*\|_\infty + \frac{\sqrt{np \log n + \log n}}{\sqrt{d}} \|\pi^{(m)} - \pi^*\|_\infty.
\]

Proof. See Appendix A.5. \( \square \)

The above bound depends on both \( \|\pi^{(m)} - \pi\|_2 \) and \( \|\pi^{(m)} - \pi^*\|_\infty \). We can invoke Lemma 4 and the inequality \( \|\pi^{(m)} - \pi^*\|_\infty \leq \|\pi^{(m)} - \pi\|_2 + \|\pi - \pi^*\|_\infty \) to reach

\[
|I_q^n| \lesssim \left( \frac{1}{\sqrt{n}} + \frac{\sqrt{np \log n + \log n}}{d} \right) \|\pi^{(m)} - \pi\|_2 + \sqrt{\frac{\log n}{Ld}} \|\pi^*\|_\infty + \frac{\sqrt{np \log n + \log n}}{\sqrt{d}} \|\pi - \pi^*\|_\infty
\]

\[
\lesssim \left\{ \left( \frac{1}{\sqrt{n}} + \frac{\sqrt{np \log n + \log n}}{d} \right)^\gamma \right\} \|\pi^{(m)} - \pi\|_2 + \sqrt{\frac{\log n}{Ld}} \|\pi^*\|_\infty + \frac{1}{\sqrt{d}} \|\pi^*\|_\infty
\]

6. Finally, putting the preceding bounds together, we conclude that with high probability, for some absolute constants \( c_1, c_2, c_3 > 0 \) one has

\[
\left( \frac{np}{2(1+\kappa)d} - c_1 \sqrt{\frac{\log n}{Ld}} - 1 \right) \|\pi_m - \pi^*_m\|_2 \leq \left\{ c_2 + \left( \frac{1}{\sqrt{d}} + \frac{c_3 \sqrt{np \log n + \log n}}{d} \right) \frac{16\sqrt{\kappa}}{\gamma} \right\} \sqrt{\frac{\log n}{Ld}} \|\pi^*\|_\infty
\]

simultaneously for all \( 1 \leq m \leq n \). By taking the maximum over \( m \) on the left-hand side and combining terms, we get

\[
\left( \frac{np}{2(1+\kappa)d} - c_1 \sqrt{\frac{\log n}{Ld}} - \frac{1}{\sqrt{d}} - \frac{c_3 \sqrt{np \log n + \log n}}{d} \right) \|\pi - \pi^*\|_\infty
\]

\[
\leq \left\{ c_2 + \left( \frac{1}{\sqrt{d}} + \frac{c_3 \sqrt{np \log n + \log n}}{d} \right) \frac{16\sqrt{\kappa}}{\gamma} \right\} \sqrt{\frac{\log n}{Ld}} \|\pi^*\|_\infty.
\]

Hence, as long as \( np/\log n \) is sufficiently large, one has \( d \gg np \gg \sqrt{np \log n} > \log n \). This implies that \( \alpha_1 > 0 \) and \( \alpha_1 \approx \alpha_2 \approx 1 \), which further leads to

\[
\|\pi - \pi^*\|_\infty \lesssim \sqrt{\frac{\log n}{Ld}} \|\pi^*\|_\infty \gtrsim \sqrt{\frac{\log n}{npL}} \|\pi^*\|_\infty.
\]

This finishes the proof of Theorem 3.

4 Analysis for the regularized MLE

The goal of this section is to establish the \( \ell_\infty \) error of the regularized MLE as claimed in Theorem 4. Recall that in Theorem 4, we compare the regularized MLE \( \theta \) with \( \theta^* = \theta - \bar{\theta}^* 1 \), and hence without loss of generality we can assume that

\[
1^\top \theta^* = 0.
\] (33)
4.1 Preliminaries and notation

Before proceeding to the proof, we gather some simple facts. To begin with, the gradient and the Hessian of \( L(\theta; y) \) in (14) can be computed as

\[
\nabla L(\theta; y) = \sum_{(i,j) \in E, i > j} \left\{ -y_{j,i} + \frac{e^{\theta_i} + e^{\theta_j}}{e^{\theta_i} + e^{\theta_j}} (e_i - e_j) \right\} (e_i - e_j)
\]

(34)

and

\[
\nabla^2 L(\theta; y) = \sum_{(i,j) \in E, i > j} \frac{e^{\theta_i} e^{\theta_j}}{(e^{\theta_i} + e^{\theta_j})^2} (e_i - e_j) (e_i - e_j)^\top,
\]

(35)

where \( e_1, \ldots, e_n \) stand for canonical basis vectors in \( \mathbb{R}^n \).

When evaluated at the truth \( \theta^* \), the size of the gradient can be controlled as follows.

**Lemma 7.** The following event

\[
A_2 := \left\{ \|\nabla L_\lambda(\theta^*; y)\|_2 \lesssim \sqrt{\frac{n^2 p \log n}{L}} \right\}.
\]

occurs with probability exceeding \( 1 - O(n^{-10}) \).

**Proof.** See Appendix B.1.

The following lemmas characterize the smoothness and the strong convexity of the function \( L_\lambda(\theta; y) \).

In the sequel, we denote by \( L_G = \sum_{(i,j) \in E, i > j} (e_i - e_j) (e_i - e_j)^\top \) the (unnormalized) Laplacian matrix (Chung, 1997) associated with \( G \), and for any matrix \( A \) we let

\[
\lambda_{\min,\perp}(A) := \left\{ \mu \mid z^\top A z \geq \mu \|z\|^2 \text{ for all } z \text{ with } 1^\top z = 0 \right\}.
\]

(37)

**Lemma 8.** Suppose that \( p > \frac{c_0 \log n}{n} \) for some sufficiently large constant \( c_0 > 0 \). Then on the event \( A_0 \) as defined in (30), one has

\[
\lambda_{\max}(\nabla^2 L_\lambda(\theta; y)) \leq \lambda + np, \quad \forall \theta \in \mathbb{R}^n.
\]

**Proof.** Note that \( \frac{e^{\theta_i} e^{\theta_j}}{(e^{\theta_i} + e^{\theta_j})^2} \leq \frac{1}{4} \). It follows immediately from the Hessian in (35) that

\[
\lambda_{\max}(\nabla^2 L_\lambda(\theta; y)) \leq \lambda + \frac{1}{4} \|L_G\| \leq \lambda + \frac{1}{2} \lambda_{\max},
\]

where \( \lambda_{\max} \) is the maximum vertex degree in the graph \( G \). In addition, on the event \( A_0 \) we have \( \lambda_{\max} \leq 2np \), which completes the proof.

**Lemma 9.** For all \( \|\theta\|_\infty \leq C \) where \( C > 0 \) is some absolute constant, we have

\[
\lambda_{\min,\perp}(\nabla^2 L_\lambda(\theta; y)) \geq \lambda + \frac{e^{-2C}}{1 + e^{-2C}} \lambda_{\min,\perp}(L_G).
\]

**Proof.** The proof is straightforward and hence is omitted.

**Lemma 10.** Let \( G \sim G_{n,p} \), and suppose that \( p > \frac{c_0 \log n}{n} \) for some sufficiently large constant \( c_0 > 0 \). Then one has

\[
P \left( \lambda_{\min,\perp}(L_G) \geq \frac{1}{2} np \right) \geq 1 - O(n^{-10}).
\]

**Proof.** Note that \( \lambda_{\min,\perp}(L_G) \) is exactly the spectral gap of the Laplacian matrix. See (Tropp, 2015, Sec 5.3.3) for the derivation of this lemma.
Algorithm 2 Gradient descent for regularized MLE

Initialize $\theta^0 = 0$.

for $t = 0, 1, 2, \ldots, T - 1$ do
  $\theta^{t+1} = \theta^t - \eta_t \left( \nabla L (\theta^t; y) + \lambda \theta^t \right)$;
end for

Output $\theta^T$.

By combining Lemma 9 with Lemma 10, we reach the following result.

Corollary 1. Under the assumptions of Lemma 10, with probability exceeding $1 - O \left( n^{-10} \right)$ one has

$$\lambda_{\min} \left( \nabla^2 \mathcal{L}_\lambda (\theta; y) \right) \geq \lambda + \frac{e^{-2C}}{2(1 + e^{-2C})^2} np$$

simultaneously for all $\theta$ obeying $\|\theta\|_\infty \leq C$.

4.2 Proof outline of Theorem 4

This subsection outlines the main steps for establishing Theorem 4. Before continuing, we single out an important fact that $\mathbf{1}^\top \theta = 0$, which is an immediate consequence of the shift-invariance property of $\mathcal{L} (\theta; y)$.

Fact 1. The regularized MLE $\theta$ obeys $\mathbf{1}^\top \theta = 0$.

Proof. Suppose instead that $\mathbf{1}^\top \theta \neq 0$. Then setting $\theta = \frac{1}{n} \mathbf{1}^\top \theta$ one has

$$\|\theta\|_2^2 > \|\theta - \mathbf{1}\|_2^2 \quad \text{and} \quad \mathcal{L} (\theta; y) = \mathcal{L} (\theta - \mathbf{1}; y),$$

where the last identity arises since $\mathcal{L} (\theta; y)$ is invariant under global shift. This implies that $\mathcal{L}_\lambda (\theta; y) > \mathcal{L}_\lambda (\theta - \mathbf{1}; y)$, which contradicts the optimality of $\theta$.

The proof relies heavily on the strong convexity of the regularized negative log-likelihood function. Notably, while $\mathcal{L}_\lambda$ is a strongly convex function, its curvature might be arbitrarily close to $\lambda$ for those points far away from the optimizer. In contrast, if we restrict our attention to a set of parameters that are reasonably close to the optimizer, a much stronger level of convexity can be exploited. In light of this observation, we divide our analysis into two parts: (1) a coarse-level analysis demonstrating that $\theta$ is not far from $\theta^*$ in an entrywise manner; (2) a fine-scale analysis that reveals the optimal performance of the regularized MLE using refined strong convexity.

4.2.1 A coarse-level analysis

In this section, our goal is to obtain the crude bound

$$\|\theta - \theta^*\|_\infty \lesssim 1. \tag{38}$$

Rather than directly resorting to the optimality condition, we adopt an algorithmic perspective to analyze the regularized MLE $\theta$. Specifically, we consider the standard gradient descent algorithm that is expected to converge to the minimizer $\theta$, and analyze the trajectory of this iterative algorithm instead. The algorithm is stated in Algorithm 2.

Our proof can be separated into three parts. In what follows, we shall take

$$\eta_t \equiv \eta = \frac{1}{\lambda + np}, \quad t = 0, 1, 2, \ldots. \tag{39}$$

1. It is seen that the sequence $\{\theta^t\}_{t=1}^\infty$ converges to the regularized MLE $\theta$, a property that is standard in convex optimization literature. This claim is summarized in the following lemma.
Lemma 11. On the event $A_0$ as defined in (30), one has
\[
\|\theta^t - \theta\|_2^2 \leq c_1^t \|\theta^0 - \theta\|_2^2,
\]
where $c_1 = 1 - \frac{\lambda}{\lambda + np}$.

Proof. The result directly follows from Lemma 8, Corollary 1, and the property of the gradient descent algorithm (e.g. (Bubeck, 2015, Theorem 3.10)). 

With this convergence property in place, if we can further show that $\|\theta^t - \theta^*\|_\infty \leq C$ for all $t \in \mathbb{N}$, then we obtain the desired result $\|\theta - \theta^*\|_\infty \leq C$.

2. Before moving to the analysis for $\ell_\infty$ errors, we develop an upper bound on $\|\theta^t - \theta^*\|_2$.

Lemma 12. On the event $A_2$ as defined in (36), there exists some constant $C_2 > 0$ such that
\[
\|\theta^t - \theta^*\|_2 \leq C_2 \sqrt{n}, \quad t = 0, 1, 2, \cdots.
\]

Proof. See Appendix B.2.

3. We then turn to the $\ell_\infty$ error of $\theta^t$. To this end, we proceed by building a contraction relationship between $|\theta^t_{m+1} - \theta_m^*|$ and $|\theta^t_m - \theta_m^*|$. More precisely, the aim is to prove that
\[
|\theta^t_{m+1} - \theta_m^*| \leq \rho |\theta^t_m - \theta_m^*| + c_2
\]
for some $\rho < 1$ and $c_2 > 0$, as detailed in the following lemma.

Lemma 13. On the event $A_0 \cap A_2$ that occurs with probability at least $1 - O(n^{-10})$, we have
\[
|\theta^t_{m+1} - \theta_m^*| \leq \left(1 - 0.3 \eta np \frac{e^{-2\sqrt{3}C_2}}{(1 + e^{-2\sqrt{3}C_2})^2}\right) |\theta^t_m - \theta_m^*| + 8 \eta np + \eta \lambda \|\theta^*\|_\infty + \frac{C_2}{4} \eta np
\]
simultaneously for all $1 \leq m \leq n$ and $t \in \mathbb{N}$. Here, $C_2 > 0$ is the same constant as in Lemma 12.

Proof. See Appendix B.3.

Notice that $\theta^0 = 0$ and hence $\|\theta^0 - \theta^*\|_\infty = \|\theta^*\|_\infty \lesssim 1$. In view of Lemma 13, by picking $C > 0$ to be any constant satisfying
\[
\left|1 - 0.3 \eta np \frac{e^{-2\sqrt{3}C_2}}{(1 + e^{-2\sqrt{3}C_2})^2}\right| C + 8 \eta np + \eta \lambda \|\theta^*\|_\infty + \frac{C_2}{4} \eta np \leq C,
\]
(40)
one can conclude that
\[
\|\theta^t - \theta^*\|_\infty \leq C \quad \implies \quad \|\theta^{t+1} - \theta^*\|_\infty \leq C.
\]

4. By induction, the above bounds taken together guarantee that on the event $A_0 \cap A_2$, the iterates satisfy $\|\theta^t - \theta^*\|_\infty \leq C$ throughout all iterations. In fact, this holds for all $C > 0$ compatible with the condition (40), or equivalently,
\[
C \geq \frac{8 + \lambda/np + \frac{C_2}{4} (1 + e^{-2\sqrt{3}C_2})^2}{0.3 \ e^{-2\sqrt{3}C_2}} \approx 1.
\]
This establishes our claim (38).
Remark 1 (Near-optimal computational cost). Before we move on to refine the analysis, we point out a byproduct of the preceding analysis: the near-optimal computational complexity of the gradient descent algorithm. Specifically, the above analysis reveals that
\[ \|\theta^t - \theta^*\|_\infty \lesssim 1 \quad t = 0, 1, \ldots, \]
provided that the starting point is \( \theta^0 = 0 \). This, combined with Corollary 1, suggests a much stronger level of convexity within a region of interest, namely,
\[ \lambda_{\min, \perp} \left( \nabla^2 \mathcal{L}_\lambda (\tilde{\theta}; y) \right) \gtrsim \lambda + np, \quad \forall \tilde{\theta} \in \mathcal{B} := \{ \tilde{\theta} \mid \|\tilde{\theta} - \theta^*\|_\infty \lesssim 1 \}, \]
where the entire trajectory of \( \{\theta^t \mid t \geq 0\} \) stays within this region \( \mathcal{B} \). Therefore, one can immediately improve Lemma 11 to
\[ \|\theta^t - \theta\|_2^2 \leq c_2^t \|\theta^0 - \theta\|_2^2 \]
for some absolute constant \( 0 < c_2 < 1 \). This reveals an interesting phenomenon: even though the negative log-likelihood function is not strongly convex (due to the possibly flat curvature of the logistic function) and the regularization parameter \( \lambda \) is small, the gradient descent algorithm converges geometrically fast to the optimizer, provided that we start from \( \theta^0 = 0 \). Given that the gradient update in each iteration takes linear time—in time proportional to reading the data, the whole algorithm has nearly linear runtime.

4.2.2 A fine-scale analysis

This subsection further refines our analysis using our results in Section 4.2.1. A key ingredient of the analysis is to introduce a leave-one-out version of the MLE that replaces all log-likelihood components involving the \( m \)-th item with their expected values (unconditional on \( G \)); that is,
\[ \theta^{(m)} := \arg \min_{\theta \in \mathbb{R}^n} \mathcal{L}^{(m)}_\lambda (\theta; y), \]
where
\[ \mathcal{L}^{(m)}_\lambda (\theta; y) := \sum_{(i,j) \in E, i \neq j, i \neq m, j \neq m} \{-y_{j,i} (\theta_{i} - \theta_{j}) + \log (1 + e^{\theta_{i} - \theta_{j}}) \}
+ \sum_{i \neq m} p \left\{ -\frac{e^{\theta_{m}^*}}{e^{\theta_{i}} + e^{\theta_{m}}} (\theta_{i} - \theta_{m}) + \log (1 + e^{\theta_{i} - \theta_{m}}) \right\} + \frac{1}{2} \lambda \|\theta\|_2^2. \tag{41} \]

As before, this auxiliary vector \( \theta^{(m)} \) serves as a reasonably good proxy for \( \theta \), while being statistically independent of \( \{y_{i,m} \mid (i, m) \in E\} \).

Our proof proceeds with the following steps, all of which rely on the fact that \( \theta \) and \( \theta^{(m)} \) live in a “good” region near \( \theta^* \) where strong convexity holds, guaranteed by our coarse-level analysis in Section 4.2.1.

1. Show that both \( \theta \) and \( \theta^{(m)} \) are close to the truth \( \theta^* \) when measured by the \( \ell_2 \) loss. Specifically, we prove that:

Lemma 14. With probability exceeding \( 1 - O(n^{-5}) \), one has
\[ \|\theta - \theta^*\|_2 \lesssim \sqrt{\frac{\log n}{pL}}, \tag{42} \]
and
\[ \|\theta^{(m)} - \theta^*\|_2 \lesssim \sqrt{\frac{\log n}{pL}}, \quad 1 \leq i \leq m. \tag{43} \]

When \( \|\theta^*\|_2 \asymp \sqrt{n} \), Lemma 14 yields bound on relative \( \ell_2 \) error \( \|\theta - \theta^*\|_2 / \|\theta^*\|_2 \lesssim \sqrt{\frac{\log n}{npL}} \) of MLE, the same rate (up to log factors) as the one of spectral method in Theorem 6.
2. Control the $\ell_2$ difference between the MLE and its leave-one-out counterpart $\theta^{(m)}$ as follows:

**Lemma 15.** Suppose that $p > \frac{c_0 \log n}{n}$ for some sufficiently large constant $c_0 > 0$. With probability exceeding $1 - O(n^{-10})$, there exists some constant $c_1 > 0$ such that

$$\|\theta^{(m)} - \theta\|_2 \leq c_1 \left( \frac{\log n}{npL} + \frac{1}{3} \|\theta - \theta^*\|_\infty \right), \quad 1 \leq m \leq n.$$  

**Proof.** See Appendix B.5. 

Notably, while $\|\theta^{(m)} - \theta\|_2$ depends also on the pointwise error $\|\theta - \theta^*\|_\infty$, we emphasize the presence of a contraction factor (i.e. $1/3$) in front of $\|\theta - \theta^*\|_\infty$. This factor plays a crucial role in showing error contraction.

3. Show that the leave-one-out estimate $\theta^{(m)}$ is close to the truth when it comes to the $m$-th coordinate:

**Lemma 16.** With probability exceeding $1 - O(n^{-10})$, we have

$$|\theta^{(m)}_m - \theta^*_m| \lesssim \sqrt{\frac{\log n}{npL}}, \quad 1 \leq i \leq m.$$  

**Proof.** See Appendix B.6. 

4. Taken collectively, the above bounds imply the existence of some absolute constant $c_2 > 0$ such that

$$|\theta_m - \theta^*_m| \leq |\theta^{(m)}_m - \theta^*_m| + \|\theta - \theta^{(m)}\|_2 \leq c_2 \left( \frac{\log n}{npL} + \frac{1}{3} \|\theta - \theta^*\|_\infty \right)$$  

with high probability. Since this holds for each $1 \leq m \leq n$, we can take the maximum over $m$ in the left-hand side and apply the union bound to derive

$$\|\theta - \theta^*\|_\infty \leq c_2 \left( \frac{\log n}{npL} + \frac{1}{3} \|\theta - \theta^*\|_\infty \right),$$  

or equivalently,

$$\|\theta - \theta^*\|_\infty \leq \frac{3c_2}{2} \sqrt{\frac{\log n}{npL}}.$$  

(45)

5. It remains to show that

$$\frac{\|e^\theta - e^{\theta^*}\|_\infty}{\|e^{\theta^*}\|_\infty} \lesssim \sqrt{\frac{\log n}{npL}}.$$  

Toward this end, we observe that for each $1 \leq m \leq n$,

$$\frac{|e^{\theta}_m - e^{\theta^*}_m|}{e^{\theta^*_m}} \leq \frac{e^{\theta}_m (\theta_m - \theta^*_m)}{e^{\theta^*_m}} \leq \frac{e^{\theta^*_m + \|\theta - \theta^*\|_\infty} |\theta_m - \theta^*_m|}{e^{\theta^*_m}},$$  

where $\tilde{\theta}_m$ is between $\theta_m$ and $\theta^*_m$, and $\theta^*_m$ is the largest entry of $\theta^*$. Continuing the derivation and using (46), we arrive at

$$\max_{1 \leq m \leq n} |\theta_m - \theta^*_m| \leq \frac{e^{\theta^*_m + \|\theta - \theta^*\|_\infty} |\theta_m - \theta^*_m|}{e^{\theta^*_m}} \lesssim \sqrt{\frac{\log n}{npL}},$$  

as long as $npL/\log n$ is sufficiently large. This completes the proof of Theorem 4.
Connection with the spectral method. Before concluding this section, we remark that the proof strategy for the regularized MLE shares a lot of similarity with that for the spectral method. For instance, the proofs in both cases proceed by introducing appropriate leave-one-out estimates, which are exceedingly close to the original estimates in the sense that

\[ \| \pi^{(m)} - \pi \|_2 \lesssim \sqrt{\frac{\log n}{npL}} \| \pi^* \|_\infty \quad \text{and} \quad \| \theta^{(m)} - \theta \|_2 \lesssim \sqrt{\frac{\log n}{npL}}. \]

Furthermore, a crucial ingredient of the proofs is some sort of the perturbation bound. This corresponds to the counterpart of the Davis-Kahan theorem (Theorem 5) for the spectral method, and the optimality condition for the regularized MLE (see, e.g. (60)) under strong convexity. Both cases take the following form: for an optimal solution \( z \) and its perturbed version \( \tilde{z} \), one has

\[ \| z - \tilde{z} \|_2 \lesssim \frac{\| f(\tilde{z}) \|_2}{\gamma}, \]  

(47)

where \( \gamma \) represents some stability constant (spectral gap in spectral method or strong convexity parameter in MLE). Moreover, \( f(\cdot) \) is some vector-valued function obeying \( f(z) = 0 \) that captures the optimality of \( z \); more precisely, one can take

\[ f(z) : = \begin{cases} z^\top P - z, & \text{for the spectral method;} \\ \nabla L_\lambda(z), & \text{for the regularized MLE.} \end{cases} \]

We expect this form (47) to extend to a much broader class of optimization problems.

5 Discussion

In this paper, we justify the optimality of both the spectral method and the regularized MLE for top-\( K \) rank aggregation. Our theoretical studies are by no means exhaustive, and there are numerous directions that would be of interest for future investigations. We point out a few possibilities as follows.

• **Unregularized MLE.** We have studied the optimality of the regularized MLE with the regularization parameter \( \lambda \approx \sqrt{\frac{np \log n}{L}} \). Our analysis relies on the regularization term to obtain the coarse bound on \( \| \theta - \theta^* \|_\infty \) in Section 4.2.1. It is natural to ask whether such a regularization term is necessary or not. This question remains open.

• **More general comparison graphs.** So far we have focused on a tractable but somewhat restrictive comparison graph, namely, the Erdős–Rényi random graph. It would certainly be interesting and important to understand the performance of both methods under a broader family of comparison graphs, and to see which algorithms would enable optimal sample complexities under general sampling patterns.

• **Beyond pairwise comparisons.** In real world applications, we often encounter the case when we are given more than two items to compare. For instance, in online advertisement scenarios, the users might be asked to compare three to five different advertisements. How to extend the methods and analyses to handle more general listwise comparisons is of significant practical interest.

• **Entrywise perturbation analysis for general convex optimization.** The current paper provides the \( \ell_\infty \) perturbation analysis for the regularized MLE using the leave-one-out trick as well as an inductive argument along the algorithmic updates. We expect this analysis framework to carry over to a much broader family of convex optimization problems, which may in turn offer a powerful tool for showing the stability of optimization procedures in an entrywise fashion.
A Proofs in Section 3

This section collects proofs of the theorems and lemmas that appear in Section 3. By Lemma 1, the event

\[ A_0 = \left\{ \frac{1}{2} np \leq d_{\min} \leq d_{\max} \leq \frac{3}{2} np \right\} \]

happens with probability at least \( 1 - O(n^{-10}) \). Throughout this section, we shall assume that we are on this event without explicitly referring to it each time. An immediate consequence is that \( d_{\max} \leq d \) on this event.

A.1 Proof of Theorem 5

To begin with, we write

\[ \pi^T - \hat{\pi}^T = \pi^T P - \pi^T \hat{P} = \pi^T (P - \hat{P}) + (\pi - \hat{\pi})^T \hat{P}. \]  

(48)

The last term of the above identity can be further decomposed as

\[ (\pi - \hat{\pi})^T \hat{P} = (\pi - \hat{\pi})^T P^* + (\pi - \hat{\pi})^T (\hat{P} - P^*) \]

\[ = (\pi - \hat{\pi})^T (P^* - 1\pi^*^T) + (\pi - \hat{\pi})^T (\hat{P} - P^*), \]

(49)

where we have used the fact that \((\pi - \hat{\pi})^T 1\pi^* = 0\). Combining (48) and (49) we get

\[ \pi^T - \hat{\pi}^T = \pi^T (P - \hat{P}) + (\pi - \hat{\pi})^T (P^* - 1\pi^*^T) + (\pi - \hat{\pi})^T (\hat{P} - P^*), \]

which together with a little algebra gives

\[ \|\pi - \hat{\pi}\|_{\pi^*} \leq \|\pi^T (P - \hat{P})\|_{\pi^*} + \|\pi - \hat{\pi}\|_{\pi^*} \|P^* - 1\pi^*^T\|_{\pi^*} + \|\pi - \hat{\pi}\|_{\pi^*} \|\hat{P} - P^*\|_{\pi^*}. \]

\[ \implies \|\pi - \hat{\pi}\|_{\pi^*} \leq \frac{\|\pi^T (P - \hat{P})\|_{\pi^*}}{1 - \|P^* - 1\pi^*^T\|_{\pi^*} - \|\hat{P} - P^*\|_{\pi^*}}. \]

The theorem follows by recognizing that

\[ \|P^* - 1\pi^*^T\|_{\pi^*} = \max \{\lambda_2(P^*), -\lambda_n(P^*)\}. \]

A.2 Proof of Lemma 2

Observe that

\[ I_1^m = \sum_{j:j \neq m} \pi_j^* (P_{j,m} - P_{j,m}^*) + \pi_m^* (P_{m,m} - P_{m,m}^*) \]

\[ = \sum_{j:j \neq m} \pi_j^* (P_{j,m} - P_{j,m}^*) + \pi_m^* \left\{ \left( 1 - \sum_{j:j \neq m} P_{m,j} \right) - \left( 1 - \sum_{j:j \neq m} P_{m,j}^* \right) \right\} \]

\[ = \sum_{j:j \neq m} \left( \pi_j^* + \pi_m^* \right) (P_{j,m} - P_{j,m}^*) \]

\[ = \frac{1}{Ld} \sum_{j:j \neq m} \sum_{l=1}^L \left( \pi_j^* + \pi_m^* \right) \mathbf{1}_{(j,m) \in E} \left( y_{j,m}^{(l)} - y_{j,m}^* \right), \]

(50)

where (i) follows from the fact that \( P \) and \( P^* \) are both probability transition matrices. By Lemma 17 and the fact that \( d_{\max} \leq d \), one can derive

\[ \mathbb{P}\left\{ |I_1^m| \geq \ell | G \right\} = \mathbb{P}\left\{ \left| \sum_{j:j \neq m} \sum_{l=1}^L \left( \pi_j^* + \pi_m^* \right) \mathbf{1}_{(j,m) \in E} \left( y_{j,m}^{(l)} - y_{j,m}^* \right) \right| \geq Ld \ell \right\} \]

\[ \leq 2 \exp\left( -\frac{2(Ld \ell)^2}{Ld_{\max}(2\|P^*\|_{\infty})^2} \right). \]
This together with the high probability event $A_0$ indicates

$$P\{ |I_1^m| \geq t \} \leq 2\exp\left(\frac{-Ld t^2}{2\|\pi^\star\|_\infty^2}\right),$$

which further implies

$$P\{ |I_1^m| \geq 4\sqrt{\frac{\log n}{Ld}}\|\pi^\star\|_\infty \} \leq 2n^{-8}.$$

The lemma is established by taking the union bound.

### A.3 Proof of Lemma 3

Applying Lemma 17 to the quantity

$$P_{m,m} - P_{m,m}^\star = -\sum_{j:j\neq m} (P_{m,j} - P_{m,j}^\star) = -\frac{1}{Ld} \sum_{j:j\neq m} \sum_{l=1}^L \mathbb{I}_{(j,m)\in \mathcal{E}} \left( y_{j,m}^{(l)} - y_{j,m}^\star \right),$$

we get

$$P\{ \max_m |P_{m,m} - P_{m,m}^\star| \geq 2\sqrt{\frac{\log n}{Ld}}\} \leq 2n^{-7}.$$

On the other hand, we have for all $1 \leq m \leq n$

$$P_{m,m}^\star = 1 - \sum_{j:j\neq m} P_{m,j}^\star \leq 1 - \frac{d_{\min}}{d} \cdot \frac{1}{1 + \kappa} \leq 1 - \frac{np}{2} \cdot \frac{1}{1 + \kappa}.$$

Combining these two pieces completes the proof.

### A.4 Proof of Lemma 4

First, by the relationship between $\| \cdot \|_2$ and $\| \cdot \|_{\pi^\star}$, we have

$$\|\pi^{(m)} - \pi\|_2 \leq \frac{1}{\sqrt{\pi^{\star}_{\min}}} \|\pi^{(m)} - \pi\|_{\pi^\star},$$

where $\pi^{\star}_{\min} := \min_i \pi^\star_i$. Invoking Theorem 5, we obtain

$$\|\pi^{(m)} - \pi\|_{\pi^\star} \leq \frac{1}{\sqrt{\pi^{\star}_{\max}}} \|\pi^{(m)} - P\|_{\pi^\star},$$

where we define $\gamma := 1 - \max \{\lambda_2(P^\star), -\lambda_n(P^\star)\}$ and $\pi^{\star}_{\max} := \max_i \pi^\star_i$.

To facilitate the analysis of $\|\pi^{(m)}\|_{\pi^\star}$, we introduce another Markov chain with transition probability matrices $P^{(m),\mathcal{G}}$, which is also a leave-one-out version of the transition matrix $P$. Similar to $P^{(m)}$, $P^{(m),\mathcal{G}}$ replaces all the transition probabilities involving the $m$-th item with their expected values (conditional on $\mathcal{G}$). Concretely, for $i \neq j$,

$$P^{(m),\mathcal{G}}_{i,j} = \begin{cases} P_{i,j}, & i \neq m, j \neq m, \\ \frac{1}{d} y^\star_{i,j} \mathbb{I}_{(i,j)\in \mathcal{E}}, & i = m \text{ or } j = m. \end{cases}$$

And for each $1 \leq i \leq n$, we define

$$P^{(m),\mathcal{G}}_{i,i} = 1 - \sum_{j:j\neq i} P^{(m),\mathcal{G}}_{i,j}.$$
to make $\mathbf{P}^{(m)}: \mathcal{G}$ a valid probability transition matrix. Hence by the triangle inequality, we see that

$$||\pi^{(m)^\top} (\mathbf{P}^{(m)} - \mathbf{P})||_2 \leq ||\pi^{(m)^\top} (\mathbf{P}^{(m)} - \mathbf{P}^{(m)}: \mathcal{G})||_2 + ||\pi^{(m)^\top} (\mathbf{P}^{(m)} - \mathbf{P}^{(m)}: \mathcal{G})||_2.$$

The next step is then to bound $J^m_1$ and $J^m_2$ separately.

For $J^m_1$, similar to (50), one has

$$\left[\pi^{(m)^\top} \mathbf{P}^{(m)}: \mathcal{G}\right]_{m} \overset{(i)}{=} \left[\pi^{(m)^\top} (\mathbf{P} - \mathbf{P}^* )\right]_{m} = \frac{1}{Ld} \sum_{j,m \neq m} \sum_{j,m = 1}^L \left( \pi^{(m)}_j + \pi^{(m)}_m \right) I_{(j,m) \in \mathcal{E}} \left( y^{(t)}_{j,m} - y^{(t)}_{j,m} \right),$$

where (i) comes from the fact that $\mathbf{P}^{(m)}: \mathcal{G} = \mathbf{P}^*$. Recognizing that $\pi^{(m)}$ is statistically independent of $\{y_{j,m}\}_{j \neq m}$, by Hoeffding’s inequality (cf. Lemma 17), we get

$$\mathbb{P}\left( \left| \left[\pi^{(m)^\top} (\mathbf{P} - \mathbf{P}^{(m)}: \mathcal{G})\right]_{m} \right| \geq 4 \sqrt{\frac{\log n}{Ld}} \|\pi^{(m)}\|_{\infty} \right) \leq 2n^{-8}. \quad (51)$$

And for $j \neq m$, we have

$$\left[\pi^{(m)^\top} (\mathbf{P} - \mathbf{P}^{(m)}: \mathcal{G})\right]_{j} = \sum_{i} \pi^{(m)}_i \left( P_{i,j} - P_{i,j} \right) = \pi^{(m)}_j \left( P_{j,j} - P_{j,j} \right) + \pi^{(m)}_m \left( P_{m,j} - P_{m,j} \right) = \pi^{(m)}_j \frac{1}{d} \left( y_{j,m} - y_{j,m} \right) I_{(j,m) \in \mathcal{E}} + \pi^{(m)}_m \frac{1}{d} \left( y_{m,j} - y_{m,j} \right) I_{(j,m) \in \mathcal{E}} = \pi^{(m)}_j \left( \mathbf{P}^* - \mathbf{P} \right)_{j,m} + \pi^{(m)}_m \left( \mathbf{P} - \mathbf{P}^* \right)_{m,j}.$$

In addition by Hoeffding’s inequality in Lemma 17, we have

$$\max_{j \neq m} \left| P_{j,m} - \mathbf{P}^*_{j,m} \right| \leq \frac{2}{d} \sqrt{\frac{\log n}{Ld}},$$

with probability at least $1 - O(n^{-5})$. As a consequence,

$$\left[\pi^{(m)^\top} (\mathbf{P} - \mathbf{P}^{(m)}: \mathcal{G})\right]_{j} \leq \begin{cases} \frac{4}{d} \sqrt{\frac{\log n}{Ld}} \|\pi^{(m)}\|_{\infty}, & \text{if } (j,m) \in \mathcal{E}, \\ 0, & \text{else.} \end{cases} \quad (52)$$

Combining (51) and (52) yields

$$J^m_1 \leq 4 \sqrt{\frac{\log n}{Ld}} \|\pi^{(m)}\|_{\infty} + \frac{4d_{\max} - 1}{d} \sqrt{\frac{\log n}{Ld}} \|\pi^{(m)}\|_{\infty} \leq 8 \sqrt{\frac{\log n}{Ld}} \|\pi^{(m)}\|_{\infty},$$

where (i) comes from the fact that $d_{\max} \leq d$.

Regarding $J^m_2$, we invoke the identity $\pi^{(m)^\top} (\mathbf{P}^{(m)} - \mathbf{P}^{(m)}: \mathcal{G}) = 0$ to get

$$\pi^{(m)^\top} (\mathbf{P}^{(m)} - \mathbf{P}^{(m)}: \mathcal{G}) = \left( \pi^{(m)} - \pi^* \right)^\top (\mathbf{P}^{(m)} - \mathbf{P}^{(m)}: \mathcal{G}).$$

Therefore, for $j \neq m$ we have

$$\left[\left( \pi^{(m)} - \pi^* \right)^\top (\mathbf{P}^{(m)} - \mathbf{P}^{(m)}: \mathcal{G})\right]_{j} = \sum_{i} (\pi^{(m)}_i - \pi^*_i) (P_{i,j} - P_{i,j} : \mathcal{G}) = (\pi^{(m)}_j - \pi^*_j) (P^{(m)}_{j,j} - P^{(m)}_{j,j} : \mathcal{G}) + (\pi^{(m)}_m - \pi^*_m) (P^{(m)}_{m,j} - P^{(m)}_{m,j} : \mathcal{G}) = - (\pi^{(m)}_j - \pi^*_j) (P^{(m)}_{j,m} - P^{(m)}_{j,m} : \mathcal{G}) + (\pi^{(m)}_m - \pi^*_m) (P^{(m)}_{m,j} - P^{(m)}_{m,j} : \mathcal{G}).$$
Recognizing that $|P_{j,m}^{(m)} - P_{j,m}^{(m),\mathcal{G}}| \leq \frac{2}{d}$ for $(j, m) \in \mathcal{E}$ and $|P_{j,m}^{(m)} - P_{j,m}^{(m),\mathcal{G}}| \leq \frac{2}{d}$ for $(j, m) \notin \mathcal{E}$, we have
\[
\left|\left[\left(\pi^{(m)} - \pi^*\right) (P^{(m)} - P^{(m),\mathcal{G}})\right]_{j,m}\right| \leq \left\{\begin{array}{ll}
\frac{2}{d} \|\pi^{(m)} - \pi^*\|_\infty, & \text{if } (j, m) \in \mathcal{E}, \\
\frac{2}{d} \|\pi^{(m)} - \pi^*\|_\infty, & \text{else}.
\end{array}\right.
\]
(53)

And for $j = m$, it holds that
\[
\left|\left[\left(\pi^{(m)} - \pi^*\right)^\top (P^{(m)} - P^{(m),\mathcal{G}})\right]_{m,j}\right| = \left|\left(\pi^{(m)}_m - \pi^*_m\right) (P^{(m)}_{m,m} - P^{(m),\mathcal{G}}_{m,m}) + \sum_{j \neq m} \left(\pi^{(m)}_j - \pi^*_j\right) (P^{(m)}_{j,m} - P^{(m),\mathcal{G}}_{j,m})\right|
\leq \left|\left(\pi^{(m)}_m - \pi^*_m\right) (P^{(m)}_{m,m} - P^{(m),\mathcal{G}}_{m,m})\right| + \sum_{j \neq m} \left|\left(\pi^{(m)}_j - \pi^*_j\right) (P^{(m)}_{j,m} - P^{(m),\mathcal{G}}_{j,m})\right|
= \sum_{j \neq m} \left|\left(\pi^{(m)}_m - \pi^*_m\right) (P^{(m)}_{j,m} - P^{(m),\mathcal{G}}_{j,m})\right| + \sum_{j \neq m} \left|\left(\pi^{(m)}_j - \pi^*_j\right) (P^{(m)}_{j,m} - P^{(m),\mathcal{G}}_{j,m})\right|.
\]
(55)

Given that $P^{(m)}_{m,j} - P^{(m),\mathcal{G}}_{m,j} = \frac{\eta_{m-j}}{d} (p - 1)_{(m,j) \in \mathcal{E}}$, we have
\[
J_3^m = \sum_{j \neq m} \left|\left(\pi^{(m)}_m - \pi^*_m\right) \frac{\eta_{m-j}}{d} (p - 1)_{(m,j) \in \mathcal{E}}\right|.
\]
(56)

Since $\|\xi_\infty^{(m)}\|_\infty \leq \frac{1}{d} \|\pi^{(m)} - \pi^*\|_\infty$ and $\|\xi_2^{(m)}\|_2 \leq \frac{1}{d} \|\pi^{(m)} - \pi^*\|_2$, Lemma 18 implies that
\[
|J_3^m| \lesssim \sqrt{np \log n + \log n} \|\pi^{(m)} - \pi^*\|_\infty
\]
with high probability. The same bound holds for $J_4^m$. Combine (53), (55) and (56) to arrive at
\[
J_2^m \lesssim \left(\frac{\sqrt{np \log n + \log n}}{d} + \frac{p \sqrt{n}}{d} + \frac{\sqrt{d}}{d}\right) \|\pi^{(m)} - \pi^*\|_\infty.
\]

Combining all, we deduce that
\[
\|\pi^{(m)} - \pi\|_2 \lesssim \frac{\sqrt{\kappa}}{\gamma} (J_1^m + J_2^m)
\lesssim \frac{\sqrt{\kappa}}{\gamma} \left(8 \sqrt{\frac{\log n}{Ld}} \|\pi^{(m)}\|_\infty + C \left(\frac{\sqrt{np \log n + \log n}}{d} + \frac{p \sqrt{n}}{d} + \frac{\sqrt{d}}{d}\right) \|\pi^{(m)} - \pi^*\|_\infty\right)
\lesssim \frac{\sqrt{\kappa}}{\gamma} \left(8 \sqrt{\frac{\log n}{Ld}} \|\pi^*\|_\infty + C \left(8 \sqrt{\frac{\log n}{Ld}} + \frac{\sqrt{np \log n + \log n}}{d} + \frac{p \sqrt{n}}{d} + \frac{\sqrt{d}}{d}\right) \|\pi^{(m)} - \pi^*\|_\infty\right)
\]
(57)

where (i) holds as long as $p \geq c_0 n \log n$ for $c_0$ sufficiently large. The triangle inequality
\[
\|\pi^{(m)} - \pi^*\|_2 \leq \|\pi^{(m)} - \pi\|_2 + \|\pi - \pi^*\|_2
\]
yields
\[
\|\pi^{(m)} - \pi\|_2 \leq \frac{16\sqrt{\kappa}}{\gamma} \sqrt{\frac{\log n}{Ld}} \|\pi^*\|_\infty + \|\pi - \pi^*\|_\infty,
\]
which concludes the proof.
A.5 Proof of Lemma 6

For ease of presentation, we define

\[ \tilde{y}_{i,j} := \frac{1}{L} \sum_{l=1}^{L} y_{i,j}^{(l)} \]

for all \( i \neq j \), where

\[ y_{i,j}^{(l)} \text{ ind.} \begin{cases} 1, & \text{with probability } \frac{w_i^*}{w_i^* + w_j^*}, \\ 0, & \text{else}. \end{cases} \]

This allows us to write \( y \) as \( y_{i,j} = \tilde{y}_{i,j} \mathbb{1}_{(i,j) \in \mathcal{E}} \) for all \( i \neq j \). With this notation in place, we can obtain

\[ I_4^m = \sum_{j:j \neq m} (\pi_j^{(m)} - \pi_j^*) \left( \frac{1}{Ld} \sum_{l=1}^{L} y_{j,m}^{(l)} \right) \mathbb{1}_{(j,m) \in \mathcal{E}}. \]

We can further decompose \( I_4^m \) into

\[ I_4^m = \mathbb{E} [I_4^m | \mathcal{G}_{-m}, \tilde{y}] + I_4^m - \mathbb{E} [I_4^m | \mathcal{G}_{-m}, \tilde{y}], \]

where \( \mathcal{G}_{-m} \) represent the graph without the \( m \)-th node, and \( \tilde{y} = \{ \tilde{y}_{i,j} \, | \, i \neq j \} \) represents all the binary outcomes.

We start with the expectation term

\[
\mathbb{E} [I_4^m | \mathcal{G}_{-m}, \tilde{y}] = \sum_{j:j \neq m} (\pi_j^{(m)} - \pi_j^*) \left( \frac{1}{Ld} \sum_{l=1}^{L} y_{j,m}^{(l)} \right) \mathbb{P} \{ (j, m) \in \mathcal{E} \}
\]

\[
\leq \left( \begin{array}{l}
(i) \quad \mathbb{P} \| \pi^{(m)} - \pi^* \|_2 \frac{\sqrt{n}}{d} \\
(ii) \quad \frac{1}{2\sqrt{n}} \| \pi^{(m)} - \pi^* \|_2 \\
(iii) \quad \frac{1}{2\sqrt{n}} (\| \pi^{(m)} - \pi \|_2 + \| \pi - \pi^* \|_2),
\end{array} \right)
\]

where (i) comes from the Cauchy-Schwarz inequality, (ii) follows from the choice \( d = c_d np \geq 2np \) and (iii) results from the triangle inequality. By Theorem 6, with high probability we have

\[ \| \pi - \pi^* \|_2 \leq C_N \sqrt{\frac{\log n}{Ld}} \| \pi^* \|_2 \leq C_N \sqrt{\frac{\log n}{Ld}} \| \pi^* \|_\infty, \]

thus indicating that

\[ \mathbb{E} [I_4^m | \mathcal{G}_{-m}, \tilde{y}] \leq \frac{1}{2\sqrt{n}} \| \pi^{(m)} - \pi \|_2 + \frac{C_N}{2} \sqrt{\frac{\log n}{Ld}} \| \pi^* \|_\infty. \quad (58) \]

When it comes to the fluctuation term, one can write

\[ I_4^m - \mathbb{E} [I_4^m | \mathcal{G}_{-m}, \tilde{y}] = \sum_{j:j \neq m} (\pi_j^{(m)} - \pi_j^*) \left( \frac{1}{Ld} \sum_{l=1}^{L} y_{j,m}^{(l)} \right) \mathbb{1}_{(j,m) \in \mathcal{E}} - \mathbb{P} \{ (j, m) \in \mathcal{E} \}. \]

Since \( \| \beta^{(m)} \|_2 \leq \frac{1}{d} \| \pi^{(m)} - \pi^* \|_2 \) and \( \| \beta^{(m)} \|_\infty \leq \frac{1}{d} \| \pi^{(m)} - \pi^* \|_\infty \), one can apply Lemma 18 to derive

\[ |I_4^m - \mathbb{E} [I_4^m | \mathcal{G}_{-m}, \tilde{y}]| \leq \frac{\sqrt{mp \log n} + \log n}{d} \| \pi^{(m)} - \pi^* \|_\infty \]

with high probability.

The bounds (58) and (59) taken together complete the proof.
B Proofs in Section 4

This section gathers the proofs of the lemmas in Section 4.

B.1 Proof of Lemma 7

Observe that

\[
\nabla L_\lambda (\theta^*; y) = \lambda \theta^* + \sum_{(i,j) \in E, i > j} \left( -y_{j,i} + \frac{e^{\theta_{i,j}^*}}{e^{\theta_{i,j}^*} + e^{\theta_{j,i}^*}} \right) (e_i - e_j)
\]

\[
= \lambda \theta^* + \frac{1}{L} \sum_{(i,j) \in E, i > j} \sum_{l=1}^{L} \left( -y_{j,i} + \frac{e^{\theta_{i,j}^*}}{e^{\theta_{i,j}^*} + e^{\theta_{j,i}^*}} \right) (e_i - e_j).
\]

It is seen that \( E[z_{i,j}^{(l)}] = 0, ||z_{i,j}^{(l)}|| \leq \sqrt{2} , \)

\[
E\left[ z_{i,j}^{(l)} z_{i,j}^{(l)\top} \right] = \text{Var}\left[ y_{i,j}^{(l)} \right] (e_i - e_j) (e_i - e_j)^\top \preceq (e_i - e_j) (e_i - e_j)^\top
\]

and \( E\left[ z_{i,j}^{(l)\top} z_{i,j}^{(l)} \right] = \text{Tr} \left( E\left[ z_{i,j}^{(l)\top} z_{i,j}^{(l)} \right] \right) \leq 2 . \)

This implies that with high probability (note that the randomness comes from \( \mathcal{G} \)),

\[
\left| \sum_{(i,j) \in E, i > j} \sum_{l=1}^{L} E\left[ z_{i,j}^{(l)} z_{i,j}^{(l)\top} \right] \right| \leq L \left| \sum_{(i,j) \in E, i > j} (e_i - e_j) (e_i - e_j)^\top \right| = L \| L_\mathcal{G} \| \lesssim Lnp
\]

and

\[
\left| \sum_{(i,j) \in E, i > j} \sum_{l=1}^{L} E\left[ z_{i,j}^{(l)\top} z_{i,j}^{(l)} \right] \right| \leq 2L \left| \sum_{(i,j) \in E, i > j} 1 \right| \lesssim Ln^2p.
\]

Letting \( V := \frac{1}{L^2} \max\left\{ \left| \sum_{(i,j) \in E} \sum_{l=1}^{L} E\left[ z_{i,j}^{(l)} z_{i,j}^{(l)\top} \right] \right|, \left| \sum_{(i,j) \in E} \sum_{l=1}^{L} E\left[ z_{i,j}^{(l)\top} z_{i,j}^{(l)} \right] \right| \right\} \) and \( B := \max_{i,j,l} ||z_{i,j}^{(l)}|| \),

we can invoke the matrix Bernstein inequality (Tropp, 2012, Theorem 1.6) to reach

\[
\| \nabla L_\lambda (\theta^*; y) - E [ \nabla L_\lambda (\theta^*; y) | \mathcal{G} ] \|_2 \lesssim \sqrt{V \log n + B \log n} \lesssim \sqrt{\frac{n^2p \log n}{L}} + \frac{\log n}{L} \lesssim \sqrt{\frac{n^2p \log n}{L}}
\]

with probability at least \( 1 - O(n^{-10}) \). Combining this with the identity \( E [ \nabla L_\lambda (\theta^*; y) | \mathcal{G} ] = \lambda \theta^* \) yields

\[
\| \nabla L_\lambda (\theta^*; y) \|_2 \leq \| E [ \nabla L_\lambda (\theta^*; y) | \mathcal{G} ] \|_2 + \| \nabla L_\lambda (\theta^*; y) - E [ \nabla L_\lambda (\theta^*; y) | \mathcal{G} ] \|_2
\]

\[
\lesssim \lambda \| \theta^* \| + \sqrt{\frac{n^2p \log n}{L}} \approx \sqrt{\frac{n^2p \log n}{L}},
\]

thus concluding the proof.

B.2 Proof of Lemma 12

First, Lemma 11 reveals that

\[
\| \theta^k - \theta \|_2 \leq \| \theta^0 - \theta \|_2 = \| \theta \|_2 \leq \| \theta - \theta^* \|_2 + \| \theta^* \|_2
\]

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for all $t \geq 0$. Thus, by the triangle inequality we have

$$
\|\theta^t - \theta^*\|_2 \leq \|\theta^t - \theta\|_2 + \|\theta - \theta^*\|_2 \leq 2\|\theta - \theta^*\|_2 + \|\theta^*\|_2.
$$

Next, it follows from the optimality of $\theta$ as well as the mean value theorem that

$$
\mathcal{L}_\lambda(\theta^*; y) \geq \mathcal{L}_\lambda(\theta; y) = \mathcal{L}_\lambda(\theta^*; y) + \langle \nabla \mathcal{L}_\lambda(\theta^*; y), \theta - \theta^* \rangle + \frac{1}{2} (\theta - \theta^*)^\top \nabla^2 \mathcal{L}_\lambda(\theta; y) (\theta - \theta^*),
$$

where $\tilde{\theta}$ is between $\theta$ and $\theta^*$. This together with the Cauchy-Schwarz inequality gives

$$
\frac{1}{2} (\theta - \theta^*)^\top \nabla^2 \mathcal{L}_\lambda(\tilde{\theta}; y) (\theta - \theta^*) \leq - \langle \nabla \mathcal{L}_\lambda(\theta^*; y), \theta - \theta^* \rangle \leq \|\nabla \mathcal{L}_\lambda(\theta^*; y)\|_2 \|\theta - \theta^*\|_2.
$$

Recall from Fact 1 and the assumption (33) that $1^\top \theta = 1^\top \theta^* = 0$. Then the above inequality gives

$$
\|\theta - \theta^*\|_2 \leq \frac{2 \|\nabla \mathcal{L}_\lambda(\theta^*; y)\|_2}{\lambda_{\min,\perp}(\nabla^2 \mathcal{L}_\lambda(\tilde{\theta}; y))},
$$

(60)

where $\lambda_{\min,\perp}(A)$ is the largest quantity of $\mu$ obeying $x^\top A x \geq \mu \|x\|_2^2$ for all $x$ obeying $1^\top x = 0$.

From the trivial lower bound $\lambda_{\min,\perp}(\nabla^2 \mathcal{L}_\lambda(\tilde{\theta}; y)) \geq \lambda$, the preceding inequality gives

$$
\|\theta - \theta^*\|_2 \leq \frac{2 \|\nabla \mathcal{L}_\lambda(\theta^*; y)\|_2}{\lambda},
$$

and therefore

$$
\|\theta^t - \theta^*\|_2 \leq \frac{4 \|\nabla \mathcal{L}_\lambda(\theta^*; y)\|_2}{\lambda} + \|\theta^*\|_2.
$$

On the event $\mathcal{A}_2 = \left\{ \|\nabla \mathcal{L}_\lambda(\theta^*; y)\|_2 \lesssim \sqrt{n^2 p \log n} \right\}$ and in the presence of the choice $\lambda \approx \sqrt{np \log n}$, we obtain $\|\theta^t - \theta^*\|_2 \leq C_2 \sqrt{n}$ for all $t \geq 0$.

B.3 Proof of Lemma 13

The update rule of Algorithm 2 is given by

$$
\theta_{m+1} = \theta_m - \eta \left[ \nabla \mathcal{L}(\theta^t; y) + \lambda \theta \right]_m.
$$

From the gradient (34), we can rewrite (62) as

$$
\theta_{m+1}^t - \theta^*_m = \theta_m^t - \theta^*_m - \eta \sum_{i:\{m,i\} \in \mathcal{E}} \left\{ -y_{i,m} + \frac{e_{i,m}^t}{e_{i,m}^t + e_{m}^t} \right\} - \eta \lambda \theta_m^t
$$

$$
= I_1^m + I_2^m + I_3^m + I_4^m,
$$

(63)

where the four terms are defined as

$$
I_1^m = -\eta \sum_{i:\{m,i\} \in \mathcal{E}} \left\{ -y_{i,m} + y_{i,m}^t \right\},
$$

$$
I_2^m = -\eta \left( \sum_{i:\{m,i\} \in \mathcal{E}} \left\{ -y_{i,m} + \frac{e_{i,m}^t}{e_{i,m}^t + e_{m}^t} \right\} - p \sum_{i \neq m} \left\{ -y_{i,m} + \frac{e_{i,m}^t}{e_{i,m}^t + e_{m}^t} \right\} \right),
$$

$$
I_3^m = \theta_m^t - \theta^*_m - \eta p \sum_{i \neq m} \left\{ -y_{i,m} + \frac{e_{i,m}^t}{e_{i,m}^t + e_{m}^t} \right\} - \eta \lambda (\theta_m^t - \theta^*_m),
$$

$$
I_4^m = -\eta \lambda \theta_m^t.
$$
Here, we set
\[ y_{i,m}^* := \frac{e^{\theta_{i,m}^*}}{e^{\theta_i^*} + e^{\theta_{m}^*}}. \]
It then comes down to analyzing the preceding four terms.

First of all, on the event \( \mathcal{A}_0 := \left\{ \frac{np}{2} \leq d_{\text{min}} \leq d_{\text{max}} \leq \frac{3np}{2} \right\} \) we have
\[ |I_1^m| \leq 2\eta d_{\text{max}} \leq 3\eta np \]
and
\[ |I_2^m| \leq \eta (2d_{\text{max}} + 2np) \leq 5\eta np. \]
In addition, \( I_4^m \) is simple and bounded by
\[ |I_4^m| \leq \eta \lambda \| \theta^* \|_\infty. \]
Hence, we are only left with \( I_3^m \).

To analyze \( I_3^m \), we define the population loss
\[ \mathcal{L}^* (\theta) := \sum_{i > j} \left\{ -y_{i,j} \left( \theta_i - \theta_j \right) + \log \left( 1 + e^{\theta_i - \theta_j} \right) \right\}, \]
whose gradient and Hessian can be computed as
\begin{align*}
\nabla \mathcal{L}^* (\theta) &= \sum_{i > j} \left\{ -y_{i,j} + \frac{e^{\theta_j}}{e^{\theta_i} + e^{\theta_j}} \right\} (e_i - e_j); \\
\nabla^2 \mathcal{L}^* (\theta) &= \sum_{i > j} \frac{e^{\theta_j}(e_i - e_j)}{(e^{\theta_i} + e^{\theta_j})^2} (e_i - e_j)(e_i - e_j)^\top. \tag{64}
\end{align*}
With these notation in place, one can simplify \( I_3^m \) as
\[ I_3^m = \theta^*_m - \theta^*_m - \eta \lambda \left[ \nabla \mathcal{L}^* (\theta^*_m) \right]_{m,m} - \eta \lambda \left( \theta^*_m - \theta^*_m \right). \tag{66} \]
Furthermore, it is straightforward to see that \( \nabla \mathcal{L}^* (\theta^*_m) = 0 \), which together with Taylor’s theorem yields
\[ \left[ \nabla \mathcal{L}^* (\theta^*_m) \right]_{m,m} = \left[ \nabla \mathcal{L}^* (\theta^*_m) \right]_{m,m} = \left[ \nabla^2 \mathcal{L}^* (\tilde{\theta}^*_m) \right]_{m,m} \left( \theta^*_m - \theta^*_m \right), \]
where \( \tilde{\theta}^* \) lies between \( \theta^*_m \) and \( \theta^*_m \). Combine the preceding two identities to get
\[ I_3^m = \left( 1 - \eta \lambda \right) - \eta \lambda \left[ \nabla \mathcal{L}^* (\tilde{\theta}^*_m) \right]_{m,m} \left( \theta^*_m - \theta^*_m \right) + \sum_{i \neq m} \left( -\eta \lambda \left[ \nabla^2 \mathcal{L}^* (\tilde{\theta}^*_m) \right]_{m,m} \right) \left( \theta^*_m - \theta^*_m \right). \]
Therefore, it remains to control \( J_1^m \) and \( J_2^m \).

For \( J_1^m \), we have
\[ |J_1^m| \leq \left| 1 - \eta \lambda \right| - \eta \lambda \left| \left[ \nabla^2 \mathcal{L}^* (\tilde{\theta}^*_m) \right]_{m,m} \right| \cdot |\theta^*_m - \theta^*_m|. \]
Hence we need to lower bound \[ \left[ \nabla^2 \mathcal{L}^* (\tilde{\theta}^*_m) \right]_{m,m} \]. Recall from Lemma 12 that we have \( \| \theta^* - \theta^* \|_2 \leq C_2 \sqrt{n} \) on the event \( \mathcal{A}_2 \) and, as a result,
\[ \| \tilde{\theta}^* - \theta^* \|_2 \leq C_2 \sqrt{n}. \]
A straightforward consequence is that\(^1\) at least 60% of the entries in \( \tilde{\theta}^* - \theta^* \) must obey \( |\tilde{\theta}^*_i - \theta^*_i| \leq \sqrt{3}C_2 \).
With this observation in mind, one has
\[ \left[ \nabla^2 \mathcal{L}^* (\tilde{\theta}^*_m) \right]_{m,m} = \sum_{i \neq m} \frac{e^{\theta_i^*}e^{\theta_m^*}}{(e^{\theta_i^*} + e^{\theta_m^*})^2} \geq 0.6 (n - 1) \frac{e^{-2\sqrt{3}C_2}}{(1 + e^{-2\sqrt{3}C_2})^2}. \]
\(^1\)Otherwise, we would have \( \| \tilde{\theta}^* - \theta^* \|_2 \geq 0.4n \times \left( \sqrt{3}C_2 \right)^2 = 1.2C_2^2 n > C_2^2 n, \) which is contradictory to the fact that \( \| \tilde{\theta}^* - \theta^* \|_2 \leq C_2 \sqrt{n} \).
This gives
\[
|J^m_1| \leq \left( 1 - 0.3\eta np \frac{e^{-2\sqrt{3}C_2}}{1 + e^{-2\sqrt{3}C_2}} \right) |\theta^*_m - \theta^*_m|.
\]

When it comes to \( J^m_2 \), we see that
\[
|J^m_2| = \eta p \left| \sum_{i: i \neq m} \left( \left[ \nabla^2 \mathcal{L}^* (\theta^i) \right]_{m,i} \right) (\theta^i - \theta^*_i) \right|
\leq \eta p \left( \sum_{i: i \neq m} \left[ \nabla^2 \mathcal{L}^* (\theta^i) \right]_{m,i}^2 \right)^{1/2} \left( \sum_{i: i \neq m} (\theta^i - \theta^*_i)^2 \right)^{1/2}
\leq \eta p \|\theta^i - \theta^*_i\|_2 \left( \sum_{i: i \neq m} \left[ \nabla^2 \mathcal{L}^* (\theta^i) \right]_{m,i}^2 \right)^{1/2}.
\]

Hence we need to upper bound \( \left| \left[ \nabla^2 \mathcal{L}^* (\theta^i) \right]_{m,i} \right| \) for \( i \neq m \). By (65), we have
\[
\left| \left[ \nabla^2 \mathcal{L}^* (\theta^i) \right]_{m,i} \right| = \frac{e^{\tilde{\theta}_i} e^{\tilde{\theta}_m}}{(e^{\tilde{\theta}_i} + e^{\tilde{\theta}_m})^2} \leq \frac{1}{4},
\]
which gives us
\[
|J^m_2| \leq \frac{1}{4} \eta p \sqrt{n - 1} C_2 \sqrt{n} \leq \frac{C_2}{4} \eta np.
\]

Combing all of the above bounds, we deduce that
\[
|\theta^*_m - \theta^*_m| \leq \left( 1 - 0.3\eta np \frac{e^{-2\sqrt{3}C_2}}{1 + e^{-2\sqrt{3}C_2}} \right) |\theta^*_m - \theta^*_m| + 8\eta np + \eta \lambda_\infty \theta^*_m + \frac{C_2}{4} \eta np
\]
on the event \( A_0 \cap A_2 \). This finishes the proof.

**B.4 Proof of Lemma 14**

We first provide here a proof for the bound on \( \|\theta - \theta^*\|_2 \).

Recall from (60) that
\[
\|\theta - \theta^*\|_2 \leq \frac{2 \|\nabla \mathcal{L}_\lambda (\theta^*; y)\|_2}{\lambda_{\min, \perp} \left( \nabla^2 \mathcal{L}_\lambda (\tilde{\theta}; y) \right)}, \tag{67}
\]
where \( \tilde{\theta} \) is between \( \theta \) and \( \theta^* \). We have also learned from Section 4.2.1 that \( \|\theta - \theta^*\|_\infty \lesssim 1 \), and hence
\[
\left\| \tilde{\theta} - \theta^* \right\|_\infty \lesssim \|\theta - \theta^*\|_\infty \lesssim 1. \tag{68}
\]

This together with Corollary 1 reveals that
\[
\lambda_{\min, \perp} \left( \nabla^2 \mathcal{L}_\lambda (\tilde{\theta}; y) \right) \gtrsim np,
\]
and therefore
\[
\|\theta - \theta^*\|_2 \lesssim \frac{\|\nabla \mathcal{L}_\lambda (\theta^*; y)\|_2}{np}. \tag{69}
\]

Therefore, combining Lemma 7 with (67) leads to
\[
\|\theta^* - \theta\|_2 \lesssim \frac{\|\nabla \mathcal{L}_\lambda (\theta^*; y)\|_2}{np} \lesssim \sqrt{\frac{\log n}{pL}}.
\]
as claimed.

The proof concerning \( \|\theta^{(m)} - \theta^*\|_2 \) follows an almost identical argument, except that we need to establish that
\[
\|\theta^{(m)} - \theta^*\|_\infty \lesssim 1.
\]

And this can be easily checked using the same argument in Section 4.2.1.

**B.5 Proof of Lemma 15**

Repeating the argument in Appendix B.4 gives
\[
\|\theta^{(m)} - \theta\|_2 \leq \frac{2}{\lambda_{\min} \left( \nabla^2 \mathcal{L}_\lambda \left( \theta^{(m)}; y \right) \right)} \| \nabla \mathcal{L}_\lambda \left( \theta^{(m)}; y \right) \|_2.
\]

It then suffices to upper bound \( \| \nabla \mathcal{L}_\lambda \left( \theta^{(m)}; y \right) \|_2 \). Recognizing that
\[
\nabla \mathcal{L}_\lambda \left( \theta^{(m)}; y \right) = 0,
\]
we can deduce that
\[
\nabla \mathcal{L}_\lambda \left( \theta^{(m)}; y \right) = \nabla \mathcal{L}_\lambda \left( \theta^{(m)}; y \right) - \nabla \mathcal{L}_\lambda \left( \theta^{(m)}; y \right)
\]
\[
= \sum_{i : \tilde{y} \neq m} \left\{ \left( -y_{m,i} + \frac{\theta^{(m)}_i}{\hat{e}_i^{(m)} + \hat{e}_m^{(m)}} \right) \mathbb{I}_{\{i, m \in \mathcal{E}\}} - p \left( -\frac{\theta^{(m)}_i}{\hat{e}_i^{(m)} + \hat{e}_m^{(m)}} + \frac{\theta^{(m)}_i}{\hat{e}_i^{(m)} + \hat{e}_m^{(m)}} \right) \right\} (e_i - e_m)
\]
\[
= \frac{1}{L} \sum_{i : (i, m) \in \mathcal{E}} \sum_{l = 1}^{L} \left( -y_{m,i} + \frac{\theta^{(m)}_i}{\hat{e}_i^{(m)} + \hat{e}_m^{(m)}} \right) (e_i - e_m)
\]
\[
+ \sum_{i : \tilde{y} \neq m} \left\{ \left( -\frac{\theta^{(m)}_i}{\hat{e}_i^{(m)} + \hat{e}_m^{(m)}} + \frac{\theta^{(m)}_i}{\hat{e}_i^{(m)} + \hat{e}_m^{(m)}} \right) \mathbb{I}_{\{i, m \in \mathcal{E}\}} - p \right\} (e_i - e_m).
\]

In the sequel, we control the two terms of (71) separately.

For the first term \( \mathbf{u}^m \) in (71), we make the observation that
\[
\mathbf{u}^m = \begin{cases} \frac{1}{L} \sum_{i = 1}^{L} \left( -y_{m,i} + \frac{\theta^{(m)}_i}{\hat{e}_i^{(m)} + \hat{e}_m^{(m)}} \right), & \text{if } (i, m) \in \mathcal{E}; \\ \frac{1}{L} \sum_{i : (i, m) \in \mathcal{E}} \sum_{l = 1}^{L} \left( y_{m,i} - \frac{\theta^{(m)}_i}{\hat{e}_i^{(m)} + \hat{e}_m^{(m)}} \right), & \text{if } i = m; \\ 0, & \text{else}. \end{cases}
\]

Since \( \left| \frac{\partial}{\partial m} \frac{\theta^{(m)}_i}{\hat{e}_i^{(m)} + \hat{e}_m^{(m)}} \right| \leq 1 \) and \( \text{card} \left( \{i : (i, m) \in \mathcal{E}\} \right) \asymp np \), we can apply Hoeffding’s inequality and union bounds to get for all \( 1 \leq m \leq n \),
\[
|\mathbf{u}^m| \lesssim \sqrt{\frac{np \log n}{L}} \quad \text{and} \quad |\mathbf{u}^m| \lesssim \sqrt{\frac{\log n}{L}} \quad \forall i \text{ obeying } (i, m) \in \mathcal{E},
\]
which further gives
\[
\|\mathbf{u}^m\|_2 \leq |\mathbf{u}^m| + \sqrt{\sum_{i : (i, m) \in \mathcal{E}} (\mathbf{u}^m)^2} \leq \sqrt{\frac{np \log n}{L}}, \quad \forall 1 \leq m \leq n.
\]

We then turn to the second term \( \mathbf{v}^m \) of (71). This is a zero-mean random vector that satisfies
\[
\mathbf{v}^m = \begin{cases} \xi_i (1 - p), & \text{if } (i, m) \in \mathcal{E}, \\ \sum_{i : \tilde{y} \neq m} \xi_i (\mathbb{I}_{\{(i, m) \in \mathcal{E}\}} - p), & \text{if } i = m, \\ \xi_i \tilde{p}, & \text{else}, \end{cases}
\]
where
\[ \xi_i := -\frac{e^{\theta_i^{(m)}}}{e^{\theta_i} + e^{\theta_i^{(m)}}} + \frac{e^{\theta_i}}{e^{\theta_i} + e^{\theta_i^{(m)}}} = -\frac{1}{1 + e^{\theta_i - \theta_i^{(m)}}} + \frac{1}{1 + e^{\theta_i^{(m)} - \theta_i^{(m)}}}. \]

The first step is to bound the size of the coefficient \( \xi_i \). Define \( g(x) = (1 + e^x)^{-1} \) for \( x \in \mathbb{R} \). We have \( |g'(x)| \leq 1 \) and thus
\[ |\xi_i| = |g(\theta_i^{(m)} - \theta_i^{(m)} - g(\theta_i^* - \theta_i^*)| \leq |(\theta_i^{(m)} - \theta_i^{(m)}) - (\theta_i^* - \theta_i^*)| \leq |\theta_i^* - \theta_i^{(m)}| + |\theta_i^* - \theta_i^{(m)}|. \]

This indicates that
\[ |\xi_i| \leq 2\|\theta^{(m)} - \theta^*\|_\infty \quad \text{and} \quad \sum_{i=1}^n \xi_i^2 \leq 4n\|\theta^{(m)} - \theta^*\|_2^2. \]

Applying the Bernstein inequality in Lemma 18 we obtain
\[ |v_m| \lesssim \left( \max_{1 \leq i \leq n} |\xi_i| \log n \right) \lesssim \left( \sqrt{n p \log n + \log n} \right) \|\theta^{(m)} - \theta^*\|_\infty \]

with high probability. As a consequence,
\[ \|v^{(m)}\| \leq \left( \sum_{i=1}^n |\xi_i|^2 \right)^{\frac{1}{2}} \left( \sum_{i=1}^n (v_i^{(m)})^2 \right)^{\frac{1}{2}} \lesssim \left( \sqrt{n p \log n + \log n} \right) \|\theta^{(m)} - \theta^*\|_\infty + \sqrt{n p} \|\theta^{(m)} - \theta^*\|_\infty + 3 \sqrt{n p} \|\theta^{(m)} - \theta^*\|_\infty \]

Putting the above results together, we see that
\[ \|\theta^{(m)} - \theta\|_2 \lesssim \frac{\|u_m\|_2 + \|v_m\|_2}{np} \]
\[ \lesssim \frac{\sqrt{n p \log n} + (\sqrt{n p \log n + \log n}) \|\theta^{(m)} - \theta^*\|_\infty}{np}. \]

Upper bounding \( \|\theta^{(m)} - \theta^*\|_\infty \) by \( \|\theta - \theta^*\|_\infty + \|\theta - \theta^{(m)}\|_\infty \), one has
\[ \|\theta^{(m)} - \theta\|_2 \leq c_1 \left\{ \sqrt{\frac{n p \log n}{L}} + \sqrt{\frac{n p \log n + \log n}{n p}} \|\theta - \theta^*\|_\infty + \frac{(\sqrt{n p \log n + \log n}) \|\theta^{(m)} - \theta\|_2}{n p} \right\} \]

for some constant \( c_1 > 0 \). When \( p > \frac{c_0 \log n}{n} \) for some sufficiently large constant \( c_0 > 0 \) so that \( np > 4c_1 (\sqrt{n p \log n + \log n}) \), the above inequality guarantees that
\[ \|\theta^{(m)} - \theta\|_2 \leq \frac{c_1 \sqrt{\frac{n p \log n}{L}} + c_1 (\sqrt{n p \log n + \log n}) \|\theta - \theta^*\|_\infty}{1 - c_1 (\sqrt{n p \log n + \log n})} \leq \frac{4c_1 \sqrt{\frac{n p \log n}{L}}}{3np} + \frac{1}{3} \|\theta - \theta^*\|_\infty. \]

### B.6 Proof of Lemma 16

From the definition of \( \theta^{(m)} \), we know that \( \theta^{(m)}_m \) is also the coordinate-wise optimizer when given all other entries of \( \theta^{(m)} \), that is,
\[ \theta^{(m)}_m = \arg \min \ell^{(m)}(\tau; y), \]

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where
\[ \ell^{(m)}(\tau; y) = p \sum_{i: i \neq m} \left\{ -\frac{e^{\theta^*_i}}{e^{\theta^*_i} + e^{\theta^*_m}} \left( \theta^{(m)}_i - \tau \right) + \log \left( 1 + e^{\theta^*_m - \tau} \right) \right\} + \frac{\lambda}{2} \tau^2 \]
is the coordinate-wise MLE when fixing all other entries of \( \theta^{(m)} \). Similar to (60), we obtain from the optimality condition that
\[
\left| \theta^{(m)}_m - \theta^*_m \right| \leq 2 \frac{d\ell^{(m)}(\tau; y)}{d\tau} \bigg|_{\tau = \theta^*_m} - \frac{d^2\ell^{(m)}(\tau; y)}{d\tau^2} \bigg|_{\tau = \tilde{\theta}_m} \tag{74}
\]
where \( \tilde{\theta}_m \) is some quantity between \( \theta^*_m \) and \( \theta^{(m)}_m \).

Observing that
\[
\frac{d\ell^{(m)}(\tau; y)}{d\tau} \bigg|_{\tau = \theta^*_m} = p \sum_{i: i \neq m} \left\{ \frac{e^{\theta^*_i}}{e^{\theta^*_i} + e^{\theta^*_m}} - \frac{e^{\theta^*_m}}{e^{\theta^*_i} + e^{\theta^*_m}} \right\} + \lambda \theta^*_m
\]
and that
\[
\left| \frac{e^{\theta^*_i}}{e^{\theta^*_i} + e^{\theta^*_m}} - \frac{e^{\theta^*_m}}{e^{\theta^*_i} + e^{\theta^*_m}} \right| = \left| \frac{1}{1 + e^{\theta^*_m - \theta^*_i}} - \frac{1}{1 + e^{\theta^*_m - \theta^*_i}} \right| \leq \left| \theta^*_i - \theta^*_m \right|
\]
we arrive at
\[
\left| \frac{d\ell^{(m)}(\tau; y)}{d\tau} \bigg|_{\tau = \theta^*_m} \right| \leq p \left\{ \sum_{i: i \neq m} |\theta^*_i - \theta^{(m)}_i| \right\} + \lambda \theta^*_m \leq np \left\| \theta^* - \theta^{(m)} \right\|_2 + \lambda \theta^*_m \tag{ii}
\]
where (i) follows from Cauchy-Schwarz and (ii) arises from Lemma 14. Additionally, \( \ell^{(m)}(\tau; y) \) is strongly convex in \( \tau \) since
\[
\frac{d^2\ell^{(m)}(\tau; y)}{d\tau^2} \bigg|_{\tau = \tilde{\theta}_m} = p \sum_{i: i \neq m} \frac{e^{\tau \theta^*_i}}{(e^{\tau} + e^{\theta^*_i})^2} + \frac{\lambda}{2} \geq np.
\]
These taken collectively with (74) give
\[
\left| \theta^{(m)}_m - \theta^*_m \right| \leq \sqrt{\frac{\log n}{npL}}.
\]

C Hoeffding’s and Bernstein’s inequalities

This section collects two standard concentration inequalities used throughout the paper, which can be easily found in textbooks such as Boucheron et al. (2013). The proofs are omitted.

Lemma 17 (Hoeffding’s inequality). Let \( \{X_i\}_{1 \leq i \leq n} \) be a sequence of independent random variables where \( X_i \in [a_i, b_i] \) for each \( 1 \leq i \leq n \), and \( S_n = \sum_{i=1}^{n} X_i \). Then
\[
\mathbb{P}(|S_n - \mathbb{E}[S_n]| \geq t) \leq 2e^{-2t^2/\sum_{i=1}^{n} (b_i - a_i)^2}.
\]

The next lemma is about a user-friendly version of the Bernstein inequality.

Lemma 18 (Bernstein’s inequality). Consider \( n \) independent random variables \( z_l \ (1 \leq l \leq n) \), each satisfying \( |z_l| \leq B \). For any \( a \geq 2 \), one has
\[
\left| \sum_{l=1}^{n} z_l - \mathbb{E}\left\{ \sum_{l=1}^{n} z_l \right\} \right| \leq \sqrt{2a \log n \sum_{l=1}^{n} \mathbb{E}[z_l^2]} + \frac{2a}{3} B \log n
\]
with probability at least \( 1 - 2n^{-a} \).
**D Proof of Theorem 6**

By Theorem 5, we obtain

\[ \| \pi^* - \pi \|_{\pi^*} \leq \frac{\| \pi^*^T (P^* - P) \|_{\pi^*}}{1 - \max \{ \lambda_2(P^*), -\lambda_n(P^*) \}} \| P^* - P \|_{\pi^*} \]

\[ \overset{(i)}{\leq} \| \pi^*^T (P^* - P) \|_{2} \overset{(ii)}{\leq} \frac{1}{\sqrt{npl}} \| \pi^* \|_{2}, \]

where (i) is a consequence of Lemma 5, (ii) follows from the relationship between \( \| \cdot \|_{2} \) and \( \| \cdot \|_{\pi^*} \), and (iii) follows as long as one can justify that

\[ \| \pi^*^T (P - P^*) \|_{2} \leq \frac{1}{\sqrt{npl}} \| \pi^* \|_{2}. \]

Therefore, the rest of the proof is devoted to establishing (75). To simplify the notations hereafter, we denote \( \Delta := P - P^* \). In fact, it is easy to check that for any \( i \neq j \),

\[ \Delta_{i,j} = P_{i,j} - P^*_{i,j} = 1_{\{ (i,j) \in \mathcal{E} \}} \frac{1}{d} \left( \frac{1}{L} \sum_{l=1}^{L} y_{i,j}^{(l)} - y_{i,j}^{*} \right) \]

and for \( 1 \leq i \leq n \), one has

\[ \Delta_{i,i} = P_{i,i} - P^*_{i,i} = \left( 1 - \sum_{j:j \neq i} P_{i,j} \right) - \left( 1 - \sum_{j:j \neq i} P^*_{i,j} \right) \]

\[ = - \sum_{j:j \neq i} \Delta_{i,j} = - \sum_{j:j \neq i} \Delta_{i,j} = - \Delta_{\text{lower}} - \Delta_{\text{upper}} \]

where

\[ \Delta_{\text{lower}} := \sum_{j:j < i} \Delta_{i,j} \quad \text{and} \quad \Delta_{\text{upper}} := \sum_{j:j > i} \Delta_{i,j}. \]

Towards proving (75), we decompose \( \Delta \) into four parts \( \Delta = \Delta_{\text{lower}} + \Delta_{\text{upper}} + \Delta_{\text{diag},l} + \Delta_{\text{diag},u} \), where \( \Delta_{\text{lower}} \) is the lower triangular part (excluding the diagonal) of \( \Delta \), \( \Delta_{\text{upper}} \) is the upper triangular part, and

\[ \Delta_{\text{diag},l} := -\text{diag}\left( \{ \Delta_{i,i}^{\text{lower}} \}_{1 \leq i \leq n} \right) \quad \text{and} \quad \Delta_{\text{diag},u} := -\text{diag}\left( \{ \Delta_{i,i}^{\text{upper}} \}_{1 \leq i \leq n} \right). \]

The triangle inequality then gives

\[ \| \pi^*^T \Delta \|_{2} \leq \| \pi^*^T \Delta_{\text{lower}} \|_{2} + \| \pi^*^T \Delta_{\text{upper}} \|_{2} + \| \pi^*^T \Delta_{\text{diag},l} \|_{2} + \| \pi^*^T \Delta_{\text{diag},u} \|_{2}. \]

In what follows, we will focus on controlling the first term \( I_{\text{lower}} \). The other three terms can be bounded using nearly identical arguments.

Note that the \( j \) component of \( \pi^*^T \Delta_{\text{lower}} \) can be expressed as

\[ \left[ \pi^*^T \Delta_{\text{lower}} \right]_{j} = \sum_{i:j > i} \pi_{i,j} \Delta_{i,j}. \]

Recall that for any pair \( (i,j) \in \mathcal{E}, \Delta_{i,j} \) is a sum of \( L \) independent zero-mean random variables, and hence \( \left[ \pi^*^T \Delta_{\text{lower}} \right]_{j} \) is a sum of \( Ld_{\text{lower}} \) independent zero-mean random variables, where

\[ d_{\text{lower}} := \{ (i,j) \mid (i,j) \in \mathcal{E} \text{ and } i > j \}. \]
In view of Hoeffding’s inequality (Lemma 17), one has, when conditional on $G$, that
\[ P \left( \left| \pi^* \Delta_{\text{lower}} \right| \geq t \right) \leq 2 \exp \left( - \frac{2t^2}{(Ld_{\text{lower}} \| \pi^* \|_{\infty})^2} \right). \]

Hence $[\pi^* \Delta_{\text{lower}}]_j$ can be treated as a sub-Gaussian random variable with variance proxy
\[ \sigma^2 \simeq \frac{d_{\text{lower}}}{dL} \| \pi^* \|_{\infty}^2 \lesssim \frac{1}{dL} \| \pi^* \|_{\infty}^2. \]

Given that the entries of $\pi^* \Delta_{\text{lower}}$ are independent, we see that
\[ (I_{\text{lower}})^2 = \sum_{j=1}^{n} [\pi^* \Delta_{\text{lower}}]_j^2, \]

is a quadratic form of a sub-Gaussian vector. On the one hand, $E[I_{\text{lower}}^2] \lesssim n\sigma^2$. On the other hand, we invoke (7, Theorem 1.1) to reach
\[ P \{ I_{\text{lower}}^2 - E[I_{\text{lower}}^2] \geq t \} \leq 2 \exp \left\{ -c \min \left( \frac{t^2}{n\sigma^4}, \frac{t}{\sigma^2} \right) \right\} \]
for some constant $c > 0$. By choosing $t \asymp \sigma^2 \sqrt{n \log n}$, we see that with probability at least $1 - O(n^{-10})$,
\[ I_{\text{lower}}^2 \lesssim E[I_{\text{lower}}^2] + \sigma^2 \sqrt{n \log n} \lesssim n\sigma^2 \lesssim \frac{n}{dL} \| \pi^* \|_{\infty}^2 \lesssim \frac{1}{dL} \| \pi^* \|_{\infty}^2. \]

The same upper bounds can be derived for other terms using the same arguments. We have thus established (75) by recognizing that $d \gtrsim np$.

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