Coherence and Confluence

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Abstract
Proofs of coherence in category theory, starting from Mac Lane’s original proof of coherence for monoidal categories, are sometimes based on confluence techniques analogous to what one finds in the lambda calculus, or in term-rewriting systems in general. This applies to coherence results that assert that a category is a preorder, i.e. that “all diagrams commute”. This note is about this analogy, paying particular attention to cases where the category for which coherence is proved is not a groupoid.

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1 Introduction

This note is about a connection between the categorial notion of coherence and the notion of confluence found in term-rewriting systems. By coherence we understand the following:

Coherence is a completeness result for an axiomatization of a brand of category, usually with respect to a particular category as a model.

In cases when one expects from coherence to decide whether two terms stand for the same arrow, the model category should be manageable in the sense that there is a decision procedure, preferably elementary, for equality of arrows in it. By varying the model category, we can cover with the notion above the results of Mac Lane and Kelly concerning coherence of monoidal, symmetric monoidal
and symmetric monoidal closed categories (see [14], [15] and [10]), as well as many other coherence results (see [5] and [6]).

This notion of coherence is made more precise by taking in the particular brand of category that interests us a category $\mathcal{K}$ freely generated by a set of objects (this set may be understood as a discrete category). This free category will always exist if our axiomatization is purely equational. Then coherence amounts to showing the following:

There is a faithful functor $G$ from the free category $\mathcal{K}$ to a particular model category $\mathcal{M}$.

In logical terms, the existence of the functor $G$ from $\mathcal{K}$ to $\mathcal{M}$ is soundness, and the faithfulness of $G$ is completeness proper.

Proofs of coherence in category theory, starting from Mac Lane’s original proof of coherence for monoidal categories of [14], are sometimes based on confluence techniques analogous to what one finds in the lambda calculus, or in term-rewriting systems in general. This applies to coherence results that assert that a category is a preorder, i.e. that “all diagrams commute”. (A preordering relation is a reflexive and transitive relation; a category that is a preorder is a preordering relation on the set of its objects.) To make such coherence results accord with the notion of coherence above, in many cases one can take that the image of $\mathcal{K}$ in $\mathcal{M}$ is a discrete category. In this note we will make some comments on the analogy between proofs of coherence and proofs of confluence, paying particular attention to cases where the category for which coherence is proved is not a groupoid.

## 2 Coherence and proof theory

If one envisages a deductive system as a graph whose nodes are formulae:
and whose arrows are derivations from the sources understood as premises to the targets understood as conclusions, then equality of derivations usually transforms this deductive system into a category of a particular brand. This category has a structure induced by the connectives of the deductive system. Although equality of derivation is dictated by logical concerns, usually the categories we end up with are of a kind that categorists have already introduced for their own reason. The prime example here is given by the deductive system for the conjunction-implication fragment of intuitionistic propositional logic. After derivations in this deductive system are equated according to ideas about normalization of derivations that stem from Gentzen, one obtains the cartesian closed category $\mathcal{K}$ freely generated by a set of propositional variables.

Equality of proofs in intuitionistic logic has not led up to now to a coherence result—a coherence theorem is not forthcoming for cartesian closed categories. If we take that the model category $\mathcal{M}$ is a category whose arrows are graphs like the graphs of [10], then we do not have a faithful functor $G$ from the free cartesian closed category $\mathcal{K}$ to $\mathcal{M}$.

If $\eta_{p,q}$ is the canonical arrow from $q$ to $p \to (p \times q)$, where $A \to B$ stands for $B^A$, and $w_A$ is the diagonal arrow from $A$ to $A \times A$, then $G(w_{p \to (p \times q)} \circ \eta_{p,q})$:

$$
\begin{array}{c}
q \\
\downarrow \\
(p \to (p \times q)) \times (p \to (p \times q))
\end{array}
$$

which is obtained from

$$
\begin{array}{c}
q \\
\downarrow \\
(p \to (p \times q))
\end{array}
$$

$G(\eta_{p,q})$

$G(w_{p \to (p \times q)})$

is different from $G((\eta_{p,q} \times \eta_{p,q}) \circ w_q)$:

3
which is obtained from

\[
\begin{array}{c}
\quad q \\
\quad \downarrow \\
\quad (p \rightarrow (p \times q)) \times (p \rightarrow (p \times q))
\end{array}
\]

\[
\begin{array}{c}
q \\
\downarrow \\
q \times q
\end{array}
\]

\[
\begin{array}{c}
\quad G(w_q) \\
\quad \downarrow \\
\quad G(\eta_{p,q} \times \eta_{p,q})
\end{array}
\]

So, if \( w \) is a natural transformation, then \( G \) is not a functor.

Dually, if \( \varepsilon_{p,q} \) is the canonical arrow from \( p \times (p \rightarrow q) \) to \( q \), and \( k_{A,B}^1 \) is the first projection from \( A \times B \) to \( A \), then \( G(k_{r,q}^1 \circ (1_r \times \varepsilon_{p,q})) \):

\[
\begin{array}{c}
\quad r \times \left( p \rightarrow (p \times q) \right) \\
\quad \downarrow \\
\quad r
\end{array}
\]

which is obtained from

\[
\begin{array}{c}
r \times \left( p \times (p \rightarrow q) \right) \\
r \times q \\
r
\end{array}
\]

\[
\begin{array}{c}
\quad G(1_r \times \varepsilon_{p,q}) \\
\quad \downarrow \\
\quad G(k_{r,q}^1)
\end{array}
\]

is different from \( G(k_{r,p \times (p \rightarrow q)}^1) \):

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So, if $k^1$ is a natural transformation, then $G$ is not a functor. The faithfulness of $G$ fails because of a counterexample in [19]. This does not exclude that with a more sophisticated model category $\mathcal{M}$ we might still be able to obtain coherence for cartesian closed categories (for an attempt along these lines see [17]).

Equality of proofs in classical logic may, however, lead to coherence with respect to model categories that catch up to a point the idea of generality of proofs. Such is in particular the category $\text{Rel}$, whose arrows are relations between finite ordinals, i.e. relations between occurrences of the same propositional letters in the premises and conclusions. The idea that generality of proofs may serve as a criterion for identity of proofs stems from Lambek’s pioneering papers in categorial proof theory of the late 1960s (see [11] for references). This criterion says, roughly, that two derivations represent the same proof when their generalizations with respect to diversification of variables (without changing the rules of inference) produce derivations with the same source and target, up to a renaming of variables.

It is shown in [5] that coherence with respect to the model category $\text{Rel}$ could justify plausibly equality of derivations in various systems of propositional logic, including classical propositional logic. The goal of that book was to explore the limits of coherence with respect to the model category $\text{Rel}$. This does not exclude that other coherence results may involve other model categories, and, in particular, with a model category different from $\text{Rel}$, classical propositional logic may induce a different notion of Boolean category than the one introduced in Chapter 14 of [5]. That notion of Boolean category was not motivated a priori, but was dictated by coherence with respect to $\text{Rel}$. The definition of that notion was however not given via coherence, but via an equational axiomatization. We take such definitions as being proper axiomatic definitions.

We could easily define nonaxiomatically a notion of Boolean category with respect to graphs of the Kelly-Mac Lane kind (see [10]). Equality of graphs would dictate what arrows are equal. In this notion, conjunction would not be a product, because the diagonal arrows and the projections would not make natural transformations (see above), and, analogously, disjunction would not be a coproduct (cf. [5], Section 14.3.) The resulting notion of Boolean category would not be trivial—the freely generated categories of that kind would not be preorders—but its nonaxiomatic definition would be trivial. There might
exist a nontrivial equational axiomatic definition of this notion. Finding such a
definition is an open problem.

We are looking for nontrivial axiomatic definitions because such definitions
give information about the combinatorial building blocks of our notions, as
Reidemeister moves give information about the combinatorial building blocks
of knot equivalence (see [3], Chapter 1). Our axiomatic equational definition of
Boolean category in [5] is of the nontrivial, combinatorially informative, kind.
Coherence of these Boolean categories with respect to $Rel$ is a theorem, whose
proof in [5] requires considerable effort.

Another analogous example is provided by the notion of monoidal category,
which was introduced in a not entirely axiomatic way, via coherence, by Bénabou
in [1], and in the axiomatic way, such as we favour, by Mac Lane in [14]. For
Bénabou, coherence is built into the definition, and for Mac Lane it is a theorem.
One could analogously define the theorems of classical propositional logic as
being the tautologies (this is done, for example, in [4], Sections 1.2-3), in which
case completeness would not be a theorem, but would be built into the definition.

3 All diagrams commute

The simplest case of coherence is when it asserts that “all diagrams commute”,
which means that the free category $K$ is a preorder, i.e. a preordering relation
on its objects. In this case, some techniques used for proving coherence are
related to those developed in connection with term-rewriting systems (cf. [7]
and [9]). The difference is that with coherence we are not interested in proving
that starting from an object all paths, i.e. all sequences, of arrows (reductions)
obtained by composing terminate in the same normal form. (This may obtain
sometimes, but is not essential.) Instead, we are interested in proving that the
equality of such paths follows from some basic equations assumed for arrows.
So, the level of our interest is not the same. (This is why we need not go so
high as [9] in the $n$-categorial hierarchy.)

Reductions here differ also from reductions in the lambda calculus, where
the lambda terms, which correspond to our arrows, are reduced. We do not
reduce arrows, but their types.

If all the arrows in question are isomorphisms, then proving that all paths
of arrows from the same source to the same target:
are equal amounts to proving that the space between all these paths could be filled in by a complex of commutative diagrams homeomorphic to an \( n \)-dimensional sphere. Such is, for example, the following complex, called the associahedron, or Stasheff polytope:

whose vertices are all five-letter terms made with one binary operation, and whose edges correspond to single applications of the associativity law. Then the equality of two paths follows from the fact that they are homotopic in the complex. This is the global approach to coherence, which stems from [18] (see also Stasheff’s papers in [13], and references therein).

There is also a local approach to coherence, which stems from [14]. In the term-rewriting terminology, we have to prove that for any two paths of arrows that terminate in the same normal form:
one can tile the space in between by commuting diagrams of arrows (reductions). For this tiling we proceed inductively in the following manner (see [8], Lemma 4.3, where the assumption that we deal with isomorphisms is replaced by the weaker assumption that we deal with monomorphisms; cf. also [5], Section 4.3):

At this place, we are faced with all the difficulties that appear in proofs of the Church-Rosser property for a notion of reduction, which consist in listing all the critical pairs of reductions. The difference with what we have in term-rewriting systems is that we must always verify that our tiles are commuting diagrams of arrows. In term-rewriting systems we usually do not deal with that (but cf. [16], and references therein; the procedure sketched above works when all the paths starting from the same vertex are bounded in length).

It is not however true that all the interesting cases of coherence where “all diagrams commute” involve only arrows that are isomorphisms (see [12], [8], Lemma 4.2, and [5], Section 4.2; remark that the four-dimensional associahedron has 42 vertices). Consider, for example, arrows whose type

\[ A \land (B \lor C) \vdash (A \land B) \lor C \]
has something to do both with distributivity and associativity, and which in [5] are called *dissociativity* arrows (in the literature, the same principle is also called weak or linear distribution; see [5], Section 7.1, for references from category theory, logic and universal algebra). These arrows need not be isomorphisms, and they are of particular interest because they underlie the cut principle in multiple-conclusion (plural) sequent systems.

If we have such arrows, which are not isomorphisms, then the global approach to coherence is not open any more, and we have to take the local approach. When we want to show that two paths of arrows are equal by closing the initial forking of arrows by a commutative diagram, as in the following picture:

```
  A
 /   \   \
|     |    |
C     B
 |   |   |
|   |   |
... |   | ...
|   |   |
|   |   |
... |   | ...
|   |   |
|   |   |
  B
```

we need an efficient criterion for showing that the object $C$ is still “above” the object $B$ (here $B$ need not to be in normal form); i.e., we need to show that we have a path of arrows from $C$ to $B$:

```
  A
 /   \   \
|     |    |
C     B
 |   |   |
|   |   |
|   |   |
|   |   |
|   |   |
|   |   |
  B
```

A criterion for the existence of such a path in the case where we have associativity isomorphisms and dissociativity arrows, which are not isomorphisms, is spelled out in [5] (Section 7.3, Theoremhood Proposition).
Coherence in this case could perhaps also be deduced from a very general theorem of [2] (Theorem 5.2.4), whose proof is only sketched in that paper, with substantial parts missing. It is not clear whether the proof of [5] (Section 7.3) was envisaged in [2], and judging by the complexity of particular criteria, as the one mentioned in the preceding paragraph, this seems unlikely.

In cases where such a criterion is not available, the paths of arrows should first be normalized, according to some normalization procedure (this is often a procedure inspired by cut elimination), and then, in order to establish coherence, one has to compare such normalized arrows (see, for example, [10], [5], Chapters 7-14, and [6]). The normal form of paths of arrows need not be unique.

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