A sharp threshold for minimum bounded-depth and bounded-diameter spanning trees and Steiner trees in random networks

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Abstract

In the complete graph on \(n\) vertices, when each edge has a weight which is an exponential random variable, Frieze proved that the minimum spanning tree has weight tending to \(\zeta(3) = \frac{1}{1^3} + \frac{1}{2^3} + \frac{1}{3^3} + \cdots\) as \(n \to \infty\). We consider spanning trees constrained to have depth bounded by \(k\) from a specified root. We prove that if \(k \geq \log_2 \log n + \omega(1)\), where \(\omega(1)\) is any function going to \(\infty\) with \(n\), then the minimum bounded-depth spanning tree still has weight tending to \(\zeta(3)\) as \(n \to \infty\), and that if \(k < \log_2 \log n\), then the weight is doubly-exponentially large in \(\log_2 \log n - k\). It is NP-hard to find the minimum bounded-depth spanning tree, but when \(k \leq \log_2 \log n - \omega(1)\), a simple greedy algorithm is asymptotically optimal, and when \(k \geq \log_2 \log n + \omega(1)\), an algorithm which makes small changes to the minimum (unbounded depth) spanning tree is asymptotically optimal. We prove similar results for minimum bounded-depth Steiner trees, where the tree must connect a specified set of \(m\) vertices, and may or may not include other vertices. In particular, when \(m = \text{const} \times n\), if \(k \geq \log_2 \log n + \omega(1)\), the minimum bounded-depth Steiner tree on the complete graph has asymptotically the same weight as the minimum Steiner tree, and if \(1 \leq k \leq \log_2 \log n - \omega(1)\), the weight tends to \((1 - 2^{-k}) \sqrt{\frac{8m}{n}} \left[ \frac{\sqrt{2mn}}{2^k} \right]^{1/(2^k - 1)}\) in both expectation and probability. The same results hold for minimum bounded-diameter Steiner trees when the diameter bound is \(2k\); when the diameter bound is increased from \(2k\) to \(2k + 1\), the minimum Steiner tree weight is reduced by a factor of \(2^{1/(2^k - 1)}\).

1 Introduction

For graphs with random edge weights, we study minimum spanning trees and Steiner trees in which there is a bound on the diameter or else a bound on the depth from a specified root vertex. We obtain precise estimates of the weight of the optimal tree, and some of the results are surprising. There is a sharp cutoff in the depth/diameter constraint, above which the constrained minimum spanning tree has almost the same weight as the unconstrained minimum spanning tree, and below which the weight blows up.

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1.1 Definitions

The minimum spanning tree (MST) of an edge-weighted undirected simple graph $G$ on $n$ vertices is the spanning tree which minimizes the sum of the edge weights. The Steiner tree problem also specifies a set $T$ of $m = |T| \leq n$ terminal vertices that are to be connected by the tree; the tree may or may not contain the other vertices in the graph $G$. We denote the minimum spanning tree of $G$ by MST$(G)$, and the minimum Steiner tree by MST$(G, T)$. Prim’s algorithm and Kruskal’s algorithm are two classic efficient algorithms for finding the MST$(G)$, but computing the minimum Steiner tree MST$(G, T)$ is well-known to be NP-hard [GJ79].

The bounded-depth Steiner tree problem, also known as the Steiner tree problem with “hop constraints,” is an abstraction of several important real-world combinatorial optimization problems, including designing telecommunications networks with a maximum transmission delay bound [GM03] and solving lot sizing problems with a limits on the number of periods goods can in stock [Voß99]. We denote by $\text{MST}_d(r) \leq k(G, T)$ the minimum weight Steiner tree of the graph $G$ connecting the terminal vertices $T$ such that each vertex is within distance $k$ from the root vertex $r$. Similarly, $\text{MST}_{\text{diam}} \leq k(G, T)$ denotes the minimum weight Steiner tree of $G$ connecting the vertices in $T$ with diameter bounded by $k$. The minimum bounded-depth and bounded-diameter spanning trees are of course the special case where $T$ is the set of all vertices.

For a tree $T$, we let wt$(T)$ denote its weight.

There has been extensive research (which we describe below) in computer science, mathematics, operations research, and physics on the bounded-diameter and bounded-depth versions of these problems. In the bounded-diameter version, the minimization is only over trees which satisfy a bound on their diameter (maximum number of edges within the tree connecting a pair of vertices), and in the bounded-depth version, the minimization is over trees satisfying a bound on the maximum distance from a pre-specified root vertex. (The bounded-diameter and bounded-depth versions are closely related.)

1.2 The MST on random graphs

There has been a lot of research on the properties of MST’s on random graphs. Two of the most well-studied ensembles of random graphs are the following:

1. The vertices of $G$ are the points of a Poisson point process in Euclidean space, with the edge weights being the Euclidean distance between the points. The MST for these geometric graphs $G$ has been studied in [AB92] [Pen03 Chapter 13] [CIL+07] and other articles.

2. The graph $G$ is the complete graph $K_n$, with the edge weights being i.i.d. copies of a random variable, such as an exponential with mean 1, or a uniform number between 0 and 1. (It turns out to matter very little which random variate occurs on the edges.)
In 1985, Frieze showed that the expected cost of the minimum spanning tree on the complete graph with edge weights distributed independently and uniformly between 0 and 1 tends to a constant as \( n \) tends to \( \infty \), and the constant is \( \zeta(3) = 1/1^3 + 1/2^3 + 1/3^3 + \cdots = 1.202\ldots \) \cite{Fri85}. In our notation, this says \( \mathbb{E}[\text{wt}(\text{MST}(K_n))] \to \zeta(3) \) as \( n \to \infty \) (we abuse notation by making the edge weight distribution implicit in the graph \( K_n \)). A concentration result was also proven, so the actual weight is with high probability close to \( \zeta(3) \) \cite{Fri85}. Even more precise results are known: the distribution of \( \text{wt}(\text{MST}(K_n)) \) converges to a Gaussian with mean \( \zeta(3) \) and variance \( (6\zeta(4) - 4\zeta(3))/n \) \cite{Jan95, JW06}. Since most edges in the optimal tree have weight close to 0, so long as the weight random variables have a density function that is 1 at weight 0, \( \text{wt}(\text{MST}(K_n)) \to \zeta(3) \) with high probability \cite{Ste87, FM89}.

Regarding the structure of \( \text{MST}(K_n) \), it is known that, with high probability, the diameter of \( \text{MST}(K_n) \) is \( \Theta(n^{1/3}) \), and the expected diameter is also \( \Theta(n^{1/3}) \) \cite{ABBR08}. (This is in contrast to the uniformly random spanning tree on the complete graph, which has diameter \( \Theta(n^{1/2}) \) in probability and in expectation \cite{RS67, Sze83}.) From this diameter bound, it follows that \( \text{wt}(\text{MST}_{\text{depth} \leq \omega(n^{1/3})}(K_n)) \to \zeta(3) \) and \( \text{wt}(\text{MST}_{\text{diam} \leq \omega(n^{1/3})}(K_n)) \to \zeta(3) \) in probability.

We prove that this convergence still holds for a much more restrictive diameter bound or depth bound:

**Theorem 1.1.** For the complete graph \( K_n \) with \( \text{Exp}(1) \) edge weights, if \( k = \log_2 \log n + \omega(1) \), where \( \omega(1) \) is any quantity that tends to \( \infty \), however slowly, then

\[
\text{wt}(\text{MST}_{\text{depth} \leq k}(K_n)) \to \zeta(3) \quad \text{and} \quad \text{wt}(\text{MST}_{\text{diam} \leq 2k}(K_n)) \to \zeta(3)
\]

(1) in both probability and expectation. This is tight in the sense that when \( k = \log_2 \log n - \Delta \),

\[
\text{wt}(\text{MST}_{\text{depth} \leq k}(K_n)) = \exp(2^{\Delta+\Theta(1)}) \quad \text{and} \quad \text{wt}(\text{MST}_{\text{diam} \leq 2k}(K_n)) = \exp(2^{\Delta+\Theta(1)}).
\]

(2)

Thus there is a sharp cutoff at depth \( \log_2 \log n \pm \Theta(1) \) (diameter \( 2 \log_2 \log n \pm \Theta(1) \)).

**Remark 1.2.** Any tree of depth \( k \) has diameter \( \leq 2k \). Any tree of diameter \( 2k \) is also a tree of depth \( k \), rooted at a uniquely defined central vertex. Similarly, any tree of diameter \( 2k+1 \) is a tree of depth \( k \) rooted at a central edge. Thus, the principal difference between \( \text{MST}_{\text{depth} \leq k} \) and \( \text{MST}_{\text{diam} \leq 2k} \) is that in the first case, the root vertex is pre-specified, and in the second case, any vertex may serve as the root. Thus,

\[
\text{wt}(\text{MST}_{\text{diam} \leq 2k}(G, m)) \leq \text{wt}(\text{MST}_{\text{depth} \leq k}(G, m)),
\]

(3) and there are some parameter values for which the bounded-diameter tree is quite a bit lighter than the bounded-depth tree. For example, if \( m = 2 \) and \( k = 1 \), we have \( \text{wt}(\text{MST}_{\text{diam} \leq 2}(K_n, 2)) = \Theta(1/\sqrt{n}) \) while \( \text{wt}(\text{MST}_{\text{depth} \leq 1}(K_n, 2)) = \Theta(1) \). But for the parameter values covered by our theorems, it turns out *a posteriori* that \( \text{wt}(\text{MST}_{\text{diam} \leq 2k}(K_n, m)) = (1 + o(1)) \text{wt}(\text{MST}_{\text{depth} \leq k}(K_n, m)). \)
Remark 1.3. This theorem also holds when the edge weights come from other distributions, as discussed below.

Steiner trees in networks with uniformly random edges weights on the complete graph $K_n$ were investigated in [BGRS04]. Since $K_n$ is symmetric, it is only necessary to specify the number $m$ of terminals rather than the precise set. There it was shown that when $2 \leq m \leq o(n)$,

$$\mathbb{E}[\text{wt}(\text{MST}(K_n, m))] = (1 + o(1)) \frac{m - 1}{n} \log \frac{n}{m}.$$  

When $m$ is of the same order as $n$, say $m = \alpha n$, the value of $\mathbb{E}[\text{wt}(\text{MST}(K_n, \alpha n))]$ is $\Theta(1)$, but its precise limiting value is not known except when $\alpha = 1$ (where it is $\zeta(3)$). We prove that this same weight can be achieved when suitably restricting the diameter:

**Theorem 1.4.** If $k \geq \log_2 \log n + \omega(n/(m \log(en/m)))$, and the edge weight probability distribution has density 1 at 0, then

$$\frac{\text{wt}\left(\text{MST}(K_n, m)\right)}{\text{wt}(\text{MST}(K_n, m))} \to 1 \text{ and } \frac{\text{wt}\left(\text{MST}(K_n, m)\right)}{\text{wt}(\text{MST}(K_n, m))} \to 1$$

in probability, and if the expected edge weight is finite, convergence holds in expectation too.

In the case $m = n$, Theorem 1.4 yields the first part of Theorem 1.1, but for Steiner trees with general $m$ we do not know if there is a sharp cutoff in the same sense that there is for the minimum spanning tree, though whenever $m = \Theta(n)$ there is still a sharp cutoff at $\log_2 \log n \pm \Theta(1)$.

The next theorem gives the weight when the depth is smaller than $\log_2 \log n$.

**Theorem 1.5.** If $2 \leq k < \log_2 \log n - \log_2 \log(en/m) - \omega(1)$, and the edge weight probability distribution has density 1 at 0, then

$$\begin{align*}
\frac{\text{wt}\left(\text{MST}(K_n, m)\right)}{\text{wt}(\text{MST}(K_n, m))} &= (1 - 2^{-k} \pm o(1)) \sqrt{\frac{8m}{n}} \left(\frac{\sqrt{2mn}}{2^k}\right)^{\frac{1}{2^{k-1}}} \\
\frac{\text{wt}\left(\text{MST}(K_n, m)\right)}{\text{wt}(\text{MST}(K_n, m))} &= (1 - 2^{-k} \pm o(1)) \sqrt{\frac{8m}{n}} \left(\frac{\sqrt{mn/2}}{2^k}\right)^{\frac{1}{2^{k-1}}} \\
\frac{\text{wt}\left(\text{MST}(K_n, m)\right)}{\text{wt}(\text{MST}(K_n, m))} &= (1 - 2^{-k} \pm o(1)) \sqrt{\frac{8m}{n}} \left(\frac{\sqrt{mn/2}}{2^k}\right)^{\frac{1}{2^{k-1}}}
\end{align*}$$

in probability, and if the expected edge weight is finite, convergence holds in expectation too. (These formulas are valid for $k = 1$ too, when the weights are Exp(1) random variables.)

For example, when $k = 2$ and $m = \alpha n$, Theorem 1.5 implies that with high probability

$$\text{wt}\left(\text{MST}(K_n, \alpha n)\right) = (1 + o(1)) \frac{3}{2} \alpha^{2/3} n^{1/3}.$$  

The second part of Theorem 1.1 follows from Theorem 1.5 upon specializing to the case $m = n$ and $k = \log_2 \log n - \Delta$. 
1.3 Computational intractability and approximation algorithms

Obtaining optimal trees is computationally intractable in general. The minimum bounded-diameter spanning tree problem is NP-hard for any diameter between 4 and \(n - 2\) \cite[pg. 206]{GJ79}, and the minimum bounded-depth spanning tree problem is NP-hard even for depth 2 (this can be shown by a reduction from the facility location problem, see \cite{DGR06}). Of course the bounded-diameter and bounded-depth Steiner tree problems are only harder, so they too are NP-hard. In fact, for any fixed diameter \(\geq 4\), it is NP-hard to even approximate the minimum bounded-diameter spanning tree to within an approximation ratio of better than \(O(\log n)\) \cite{BIKP01}.

Because of this complexity, numerous algorithms have been investigated, including exact (but time-consuming) integer programming formulations \cite{AC92,AC93,GR05a}, fast rigorous approximation algorithms \cite{BIKP01,KP99,AFHP+05}, and heuristic approximation algorithms \cite{DGR06,Vof99,Gou95,Mon01,Gou96,CCL08b,CCL08a,RJ03,GR05b,GvHR06,Kop06,Pui07,Zau08,BBB+08}.

These intractability and inapproximability results are of course for worst-case graphs, and for random graphs one can do better. One of the heuristic algorithms, based on “survey propagation” and “the cavity method,” was recently tested on random graphs \cite{BBB+08}, which led us to investigate the weight of the true optimal minimum bounded-depth and bounded-diameter spanning tree and Steiner tree on random graphs. There were also some earlier investigations of the minimum bounded-diameter spanning tree on random graphs \cite{ADF99,AD02} which consisted of testing the performance of several other heuristic algorithms. In the present paper we rigorously analyze the asymptotic weight of these trees. Indeed, we also describe two algorithms that approximate well the constrained spanning (or Steiner) tree problem.

1.4 Proof strategy

In the remaining sections of this paper, we present the proofs of these theorems. We saw already that Theorem 1.1 is implied by Theorems 1.4 and 1.5.

The upper bound in Theorem 1.5 follows from analyzing (in §2) the Steiner tree produced by a simple greedy heuristic algorithm, which estimates how many vertices should occur in each level of the tree, and picks the cheapest set of this many vertices connected to the previous level of the tree. The proof of the upper bound in Theorem 1.4 is also algorithmic, and appears in §3. There, the strategy is to start with the minimum unconstrained spanning or Steiner tree, delete a small number of edges to break the tree apart into pieces, and then splice these pieces back together using the greedy algorithm from Theorem 1.5. The resulting tree has almost the same weight as the original tree, and it has very small depth. Both of these algorithms produce trees rooted at a pre-specified vertex, so they yield both the bounded-depth and bounded-diameter upper bounds.

The lower bounds in Theorem 1.4 are self-evident, since the weight of the unconstrained
minimum Steiner tree is an obvious lower bound on the weight of a constrained Steiner tree. The lower bounds in [Theorem 1.5] are proved in [§5] and make use of a tight concentration inequality that is derived in [§4]. The lower bound applies to any Steiner tree (where any vertex can be the root in a tree of diameter $2k$, or any edge can be the root in a tree of diameter $2k + 1$), so recalling [3], the proof yields both the bounded-diameter and bounded-depth lower bounds.

When carrying out the above calculations in [§2], [§3], [§4], and [§5], we assume that the edge weights are distributed according to exponential random variables with mean 1, since this distribution is quite natural and simplifies many of the calculations. In [§6] we show how our results for exponential random variables imply the corresponding results for other distributions (when the depth bound is bigger than 1).

## 2 Greedy tree

### 2.1 Construction

Let us consider the following greedy method for algorithmically growing a low-weight spanning or Steiner tree with bounded depth or diameter. We will build a tree in which every vertex is within distance $k$ from a particular root vertex or root edge. (For bounded-diameter trees, the root may be chosen arbitrarily.) This greedy tree $T_\ell$ depends on a sequence of non-negative numbers $\ell = (\ell_0, \ell_1, \ldots, \ell_k)$ such that $\ell_0 = 1$ (if the root is a vertex) or $\ell_0 = 2$ (if the root is an edge), and $\sum \ell_i = n$ if the desired tree is a spanning tree, and $\ell_k = m$ if the desired tree is a Steiner tree. The idea is that the greedy tree $T_\ell$ will have exactly $\ell_i$ vertices at distance $i$ from the root, except that for Steiner trees there may be fewer than $\ell_k = m$ vertices at the $k^{th}$ level if the terminal vertices were used closer to the root. The construction is inductive. Level 0 is the root vertex, or pair of vertices if the root is an edge. For convenience, we let $s_i = \ell_0 + \cdots + \ell_i$. For $i < k$, suppose we have chosen the $s_{i-1}$ vertices at level $i - 1$ and below. For each of the unchosen $n - s_{i-1}$ vertices, we look at the lightest edge connecting it to level $i - 1$, and, to form level $i$ of the tree $T_\ell$, we choose the $\ell_i$ vertices of these vertices that have the lightest edges to level $i - 1$. For level $i = k$, we connect the terminal vertices (that have not already been included in the tree) using their lightest edge to level $k - 1$.

### 2.2 Approximate weight

The choice of the sequence $\ell$ has great influence on the total weight of the resulting tree $T_\ell$. Let $\text{wt}_i(T_\ell)$ be the (random) total weight of the edges within the greedy tree connecting level $i - 1$ to level $i$. For each of the $n - s_{i-1}$ vertices not in the tree up to level $i - 1$, the weight of the lowest weight edge leading to it from the $\ell_{i-1}$ vertices at level $i - 1$ is an exponential random variable with mean $1/\ell_{i-1}$. When picking the $\ell_i$ vertices at level $i < k$,
we pick the $\ell_i$ smallest of these random variables. The $j^{th}$ smallest has expectation

$$\frac{1}{\ell_{i-1}} \left[ \frac{1}{n-s_{i-1}} + \frac{1}{n-s_{i-1}-1} + \cdots + \frac{1}{n-s_{i-1}-(j-1)} \right],$$

and so

$$\mathbb{E}[\text{wt}_i(T_\ell)] = \frac{1}{\ell_{i-1}} \left[ \frac{\ell_i}{n-s_{i-1}} + \frac{\ell_i-1}{n-s_{i-1}-1} + \cdots + \frac{1}{n-s_{i-1}-(\ell_i-1)} \right].$$

In §4 we derive a concentration result for these random variables.

In the case of spanning trees ($m = n$), the above formula also holds for level $i = k$, and simplifies to $\mathbb{E}[\text{wt}_i(T_\ell)] = \ell_i/\ell_{i-1}$. For general Steiner trees, at level $k$ we have $\mathbb{E}[\text{wt}_i(T_\ell)] \leq \ell_i/\ell_{i-1}$ because some of the terminals may have been selected already.

Thus the expected weight $\mathbb{E}[\text{wt}_i(T_\ell)]$ of the $i^{th}$ level may be approximated by

$$\mathbb{E}[\text{wt}_i(T_\ell)] \approx \begin{cases} \frac{\ell_i^2}{2nl_{i-1}}, & \text{if } s_i \ll n; \\ \frac{\ell_i^2}{ml_{i-1}}, & \text{if } s_i = n, \text{ i.e., } i = k. \end{cases}$$

If $s_{k-1} \ll n$ then the above approximation holds for all $i$. Next we choose a good sequence $\ell$ that makes $\mathbb{E}[\text{wt}(T_\ell)] = \sum_i \mathbb{E}[\text{wt}_i(T_\ell)]$ small.

### 2.3 An optimization problem

It is convenient to define $f_n(a,b) = b^2/(2na)$ and

$$f_{n,c}(\ell) = f_{n,c}(\ell_0, \ell_1, \ldots, \ell_k) = f_n(\ell_0, \ell_1) + \cdots + f_n(\ell_{k-2}, \ell_{k-1}) + cf_n(\ell_{k-1}, \ell_k)$$

$$= \frac{\ell_0^2}{2nl_0} + \cdots + \frac{\ell_{k-1}^2}{2nl_{k-2}} + c \frac{\ell_k^2}{2nl_{k-1}},$$

(a factor of $c$ appears only in the last level). We have argued that the greedy tree with level sizes $\ell$ has expected weight approximately $f_{n,2n/m}(\ell)$, provided that most of the nodes occur in the last level. Next we optimize $f_{n,c}(\ell)$, which is a deterministic function of $\ell$, allowing the level sizes to be real numbers rather than constraining them to be integers. We further relax the constraints on the sum of the level sizes ($\ell_0 + \cdots + \ell_k = n$ for MST and the corresponding constraint for the Steiner tree), and instead fix $\ell_0(=1)$ and $\ell_k$, and optimize the intermediate level sizes. In §2.4 and §2.5 we return to the question of how well this approximates the weight of the greedy tree with constrained integer level sizes and random edge weights.

For $\ell_{i-1}$ and $\ell_{i+1}$ held fixed, let us find the choice of $\ell_i$ which minimizes $f_n(\ell)$. If $i+1 < k$ then the two terms of $f_n(\ell)$ involving $\ell_i$ are

$$f_n(\ell_{i-1}, \ell_i) + f_n(\ell_i, \ell_{i+1}) = \frac{\ell_i^2}{2nl_{i-1}} + \frac{\ell_{i+1}^2}{2nl_i},$$
which is minimized when
\[
\frac{2\ell_i}{2n\ell_{i-1}} - \frac{\ell_{i+1}^2}{2n\ell_i^2} = 0,
\]
i.e., when \(f_n(\ell_i, \ell_{i+1}) = 2f_n(\ell_{i-1}, \ell_i)\). In the last level, with \(i = k - 1\) we still have that the optimal choice of \(\ell_i\) yields \(f_n(\ell_i, \ell_{i+1}) = 2f_n(\ell_{i-1}, \ell_i)\).

An optimal sequence of \(\ell_i\)'s should satisfy this for all \(i\), so we wish to solve the recursion subject to \(\ell_0 = 1\) with given \(\ell_k\). Let \(r_i = \ell_i/\ell_{i-1}\). If \(i + 1 < k\) we have \(2\ell_i^3 = \ell_{i-1}\ell_{i+1}^2\), i.e.,
\[
2r_i = r_{i+1}^2.
\]
For \(i = k - 1\) we have \(2\ell_i^3 = c\ell_{i-1}\ell_{i+1}^2\) so
\[
2r_i = cr_{i+1}^2.
\]
When the ratios \(r_i\) satisfy these equations, we have, for all \(i < k\),
\[
r_i = 2(r_k\sqrt{c}/2)^{2^{k-i}}.
\]
Multiplying, we find for \(i < k\)
\[
\ell_k/\ell_i = \prod_{j=i+1}^{k} r_j = 2^{k-i}(r_k\sqrt{c}/2)^{2^{k-i}-1}/\sqrt{c}.
\]
In particular,
\[
\ell_k = \ell_k/\ell_0 = 2^k(r_k\sqrt{c}/2)^{2^{k-1}-1}/\sqrt{c},
\]
so
\[
r_k = \frac{2}{\sqrt{c}} \left( \frac{\ell_k\sqrt{c}}{2^k} \right)^{\frac{1}{2^{k-1}}}
\]
and for \(i < k\),
\[
\ell_i = 2^i \left( \frac{\ell_k\sqrt{c}}{2^k} \right)^{1-\frac{2^{k-i}-1}{2^{k-1}}}.
\]
On this optimal sequence \(\ell\), we have
\[
f_{n,c}(\ell) = (2 - 2^{1-k})\frac{c\ell_k^2}{2n\ell_{k-1}} = \frac{2\ell_k\sqrt{c}}{n}(1 - 2^{-k}) \left( \frac{\ell_k\sqrt{c}}{2^k} \right)^{1-\frac{2^{k-1}}{2^{k-1}}}.
\]
2.4 Bounded-depth trees

How do we relate this sequence to the greedy minimum bounded-depth spanning tree or Steiner tree? Let $\hat{\ell}$ denote this optimal sequence when $\ell_k = m$ and $c = 2n/m$. Let us greedily place $\lceil \hat{\ell}_i \rceil$ nodes in level $i$ for $i < k$, and then connect the remaining nodes to the last level of the tree. For fixed $m/n$, so long as $k \leq \log_2 \log n - \omega(1)$, we have $r_k \gg 1$, so for each $i < k$ we have $s_i \ll n$, so the expected weight of level $i$ of the tree is $(1 + o(1)) f_n(\lceil \hat{\ell}_i - 1 \rceil, \lceil \hat{\ell}_i \rceil)$ for $i < k$, and is at most $c f_n(\lceil \hat{\ell}_{i-1} \rceil, \lceil \hat{\ell}_i \rceil)$ for $i = k$. Furthermore, for each $i > 0$ we have $\hat{\ell}_i \gg 1$, so the rounding to integers only causes a $(1 + o(1))$ multiplicative correction. Thus the expected weight of this greedy minimum spanning tree or Steiner tree is at most

$$E[\text{wt}(\hat{T}_k)] \leq (1 + o(1)) f_{2n/m}(\hat{\ell}) = (1 + o(1))\sqrt{8m/n(1 - 2^{-k})} \left(\frac{\sqrt{2mn}}{2^k}\right)^{1/2^{k-1}}.$$ 

This is the upper bound of the in-expectation part of Theorem 1.5 for bounded-depth Steiner trees when the edge weights are Exp(1) random variables. Concentration will follow when we prove the lower bound in §5.

For example, when $k = 2$ and $m = \alpha n$, the best choice is $\ell_1 \approx \alpha^{1/3}n^{2/3}$, yielding a total expected weight of about $\frac{3}{2} \alpha^{2/3}n^{1/3}$.

We will also be interested in taking larger $k$'s, namely $k = \log_2 \log n + \Theta(1)$ and larger. If we simply substitute this $k$ (or any larger $k$) into the estimate for the weight of the greedy tree, we would get $E[\text{wt}(\hat{T}_k)] \approx \sqrt{8} = \Theta(1)$. This estimate for $E[\text{wt}(\hat{T}_k)]$ is not valid, because for $i = k - \Theta(1)$ we have $s_i = \Theta(n)$, so the $E[W_i]$'s are larger than the above formula gives. However, these $E[W_i]$'s are only a constant factor larger than predicted, so we still have $E[\text{wt}(\hat{T}_k)] = \Theta(1)$.

Intuitively, all the choices that we made when constructing the greedy tree are close to optimal. We will see in the next section that it is in fact possible to build a better tree by making non-greedy choices when $k$ is larger than $\log_2 \log n + \omega(1)$ (the construction there makes use of this greedy tree, combining it with the optimal minimum spanning tree with unbounded depth). However, we will prove in the lower bound section that the weight of this greedy tree is within a factor of $(1 + o(1))$ of the weight of the optimal tree so long as $k \leq \log_2 \log n - \omega(1)$.

2.5 Bounded-diameter trees

Having proved the upper bound for $\text{wt}(\text{MST}(K_n, m))$, we now consider the case of trees with a bounded diameter. As noted, the primary difference is that now there is no fixed root from which to measure distances. The following is an easy observation (which will be more useful for the lower bounds).

Lemma 2.1. A tree with diameter $2k$ contains a unique root vertex, from which the tree
has depth $k$. A tree of diameter $2k + 1$ contains a unique edge so that all vertices are within distance $k$ of an endpoint of the edge.

Proof. Take a path of maximal length in the tree, and take as the root vertex or edge the central vertex or edge of the path. The bounds on the depth follow from the maximality of the path.

To prove the upper bound for $\mathbb{E} \left[ \text{wt} \left( \text{MST} \left( K_n, m \right) \right) \right]$, just note that any tree of depth $k$ is also a tree of diameter $2k$. Thus

$$\text{wt} \left( \text{MST} \left( K_n, m \right) \right) \leq \text{wt} \left( \text{MST} \left( K_n, m \right) \right).$$

To prove the upper bound for $\mathbb{E} \left[ \text{wt} \left( \text{MST} \left( K_n, m \right) \right) \right]$, we fix a root edge with small weight, then repeat the argument for the weight of the greedy tree, except that level 0 now has size $\ell_0 = 2$ rather than 1. There is an easy way to relate the optimal costs without repeating the optimization problem.

The key is that the cost $f_{n,c}(\ell)$ is homogeneous in the sequence $\ell$. Consider the optimal sequence $\hat{\ell}$ for spanning $m/2$ terminals among $n/2$ vertices, then $2\hat{\ell}$ is the optimal sequence for spanning $m$ terminals among $n$ vertices, except that it starts with two vertices at level 0. Thus we can repeat the greedy construction to find that

$$\mathbb{E} \left[ \text{wt} \left( \text{MST} \left( K_n, m \right) \right) \right] \leq (1 + o(1)) f_{n,2n/m}(2\hat{\ell}) = (1 + o(1)) f_{n/2,2n/m}(\hat{\ell}) = \text{our upper bound on } \mathbb{E} \left[ \text{wt} \left( \text{MST} \left( K_{n/2}, m/2 \right) \right) \right].$$

This is the upper bound of the in-expectation part of Theorem 1.5 for bounded-diameter Steiner trees when the edge weights are Exp(1) random variables.

3 Sliced-and-spliced tree

To construct a small-diameter spanning tree with weight close to $\zeta(3)$, the idea is to take the true (unconstrained) minimum spanning tree (or Steiner tree), break it apart into small subtrees which each still contain many vertices, and then splice the subtrees together using the greedy tree approach from §2. The resulting tree is locally the same as the MST, so the weight is about the same, but globally it has been rewired to have much smaller diameter.

Recall that the weight of the unconstrained Steiner tree is $(1 - o(1))(m - 1)/n \log(n/m)$ (w.h.p. and in expectation) when $m \ll n$ [BGRS04], and that (using also Frieze’s result on spanning trees [Fri85]) consequently $\text{wt} (\text{MST}(K_n, m)) = \Theta(m/n \log(en/m))$ for $2 \leq m \leq n$. 

Theorem 3.1. Suppose $2 \leq m \leq n$ and $k = \log_2 \log n + \Delta$ where $\Delta \geq n/(m \log(en/m))$, and the edge weights are exponential random variables with mean 1. Then

$$
\mathbb{E} \left[ \text{wt} \left( \text{MST} \left( K_n, m \right) \right) - \text{wt}(\text{MST}(K_n, m)) \right] \leq O \left( \sqrt{\frac{m \log(en/m)}{n\Delta}} \right).
$$

Proof. Each edge weight $w_e$ is given by an exponential Exp(1) with mean 1. For $0 < \epsilon < 1$, we can write $w_e = \min(w'_e, w''_e)$ where $w'_e$ and $w''_e$ are independent exponentials with mean $1/(1-\epsilon)$ and $1/\epsilon$. Let $X = \text{wt}(\text{MST}(K_n, m))$, i.e., the weight of the unconstrained minimum Steiner tree with the $w_e$ edge weights. Let us take the unconstrained minimum spanning or Steiner tree $T$ using the $w'_e$ weights. The weight of this tree satisfies $X \leq \text{wt}(T) \geq X/(1-\epsilon)$. The edge weights $w''_e$ are independent of $T$, which will soon be useful. We will eventually choose $0 < \epsilon \ll 1$. Assuming $\epsilon \leq 1/2$, we have $\mathbb{E}[\text{wt}(T) - X] \leq O(\epsilon \mathbb{E}[X])$.

Any tree with diameter greater than $\Delta$ may be broken up into two trees with diameter $\geq \lfloor \Delta/2 \rfloor$ by removing the middle edge of any path realizing the diameter. Let us repeatedly break up the Steiner tree $T$ in this way until we are left with a collection of subtrees with diameters between $\lfloor \Delta/2 \rfloor$ and $\Delta$, of total weight $\leq \text{wt}(T)$.

We view these subtrees as meta-nodes. We will construct a spanning meta-tree of depth $\lfloor \log_2 \log n \rfloor$ on these meta-nodes, with the property that its edges are attached to a single vertex within each meta-node. This implies that the depth of the resulting tree (rooted at the marked vertex in the meta-root) is at most $\log_2 \log n + \Delta$. This is done using the greedy tree construction. We start with any subtree as the root meta-node, and mark an arbitrary vertex within it. When building a layer of the greedy tree, we only consider edges from the marked vertices of the previous layer of the tree, but these edges may connect to any vertex of a subtree not yet in the greedy tree. For each sub-tree there are at least $\lfloor \Delta/2 \rfloor + 1 \geq \Delta/2$ such edges, and each one has weight at most $w''_e$ (here we use that the $w''_e$ are independent of the spanning or Steiner tree). Thus the weight of the lightest connection between a marked vertex and a sub-tree is dominated by an Exponential with mean $2/(\epsilon \Delta)$. When a sub-tree gets connected, the vertex by which it gets connected becomes the marked vertex. Let $S$ denote the total weight of the splice edges within the greedy meta-tree. By our results from §2, $\mathbb{E}[S] \leq O(1/(\epsilon \Delta))$.

The weight of the sliced-and-spliced tree is at most $\text{wt}(T) + S$. To optimize this bound, we pick $\epsilon$ to minimize $O(\epsilon \mathbb{E}[X]) + O(1/(\epsilon \Delta))$, i.e., $\epsilon = \Theta(1/\sqrt{\mathbb{E}[X] \Delta})$, and since $\Delta \geq n/(m \log(en/m)) = \Theta(1/\mathbb{E}[X])$ was one of our assumptions, we can pick such an $\epsilon \leq 1/2$. Thus, $\text{wt}(\text{MST} \left( K_n, m \right))$ is at most the sliced-and-spliced tree weight, which is at most $X = \text{wt}(\text{MST}(K_n, m))$ plus a quantity which in expectation is at most $\Theta \left( \sqrt{\frac{m \log(en/m)}{n\Delta}} \right)$. □

The proof is wasteful in the separation of edges (or weights) into two independent components. If this is not done, then the edges between sub-trees are likely not to have come
from the MST, and so tend to be heavier. It seems that this should only give a constant factor to the cost of the meta-tree.

**Proof of Theorem 1.4 for exponential weights.** Using Theorem 3.1, if $k = \log_2 \log n + \Delta$ where $\Delta \geq \omega(n/(m \log(en/m)))$, then

$$
E \left[ \text{wt} \left( \text{MST} \left( K_n, m \right) \right) - \text{wt}(\text{MST}(K_n, m)) \right] \leq o(E[\text{wt}(\text{MST}(K_n, m))]),
$$

so the convergence in probability for bounded-depth Steiner trees is an immediate consequence of Markov’s inequality, and convergence in expectation is also immediate. The bounded-diameter statements are a consequence of the bounded-depth statements.

The case of other distributions is handled in §6.

## 4 Concentration of level weights

Let $U_1, \ldots, U_p$ be a pool (set) of $p$ i.i.d. exponential random variables, and for $b \leq p$, let $W_{b,p}$ be the sum of the $b$ best (smallest) $U_i$’s. The distribution of $W_{b,p}$ plays a key role in the behavior of the weights of bounded-depth minimum spanning trees and bounded-depth Steiner trees. The total weight $W_i$ of the edges connecting levels $i-1$ and $i$ in the greedy tree from §2.1 is given by

$$
\text{wt}_i(T_\ell) = \frac{1}{\ell_{i-1}} W_{\ell_{i-1} - \ell_0 - \ell_1 - \ldots - \ell_{i-1}}.
$$

We derive here some basic properties of $W_{b,p}$, including its expected value, and the probability that it deviates far from its expected value.

Let $Y_i$ be the $i^{th}$ smallest of the $U_i$’s. Since the minimum of independent exponentials is again an exponential, and since an exponential conditioned to be larger than some value is a translated exponential, we have $Y_{i+1} - Y_i = \frac{1}{p-i} X_i$, where the $X_i$’s are i.i.d. Exp(1) random variables (where by convention $Y_0 = 0$). It follows that

$$
W_{b,p} = \sum_{i=1}^{b} Y_i = \sum_{i=0}^{b-1} \frac{b - i}{p - i} X_i.
$$

Thus

$$
E[W_{b,p}] = \sum_{i=0}^{b-1} \frac{b - i}{p - i}.
$$

Let us approximate the expected value by

$$
\bar{W}_{b,p} = \int_0^b \frac{b - i}{p - i} \, di = b + (p - b) \log \left( 1 - \frac{b}{p} \right) = \frac{b^2}{2p} + \frac{b^3}{6p^2} + \frac{b^4}{12p^3} + \cdots \begin{cases} \leq \frac{b^2}{p}, \\ \geq \frac{b^2}{2p}; \end{cases}
$$
we have
\[W_{b,p} \leq \mathbb{E}[W_{b,p}] \leq \frac{b}{p} + W_{b,p}.
\]

**Lemma 4.1.** For any \(\delta > 0\) and \(b \leq p\),
\[\Pr \left[ W_{b,p} < (1 - \delta) \frac{b^2}{2p} \right] \leq \Pr \left[ W_{b,p} < (1 - \delta) \mathbb{E}[W_{b,p}] \right] \leq \exp \left[ -\frac{1}{8} \delta^2 b \right].\]

**Proof.** We use the method of bounded differences (see e.g., [McD89]). For \(\beta > 0\), we have
\[
\Pr[\mathbb{E}[W_{b,p}] < x] = \Pr[e^{-\beta W_{b,p}} > e^{-\beta x}] \leq e^{\beta x} \mathbb{E}[e^{-\beta W_{b,p}}]
\]
\[
= e^{\beta x} \prod_{i=0}^{b-1} \mathbb{E}[e^{-\beta (b-i)/(p-i)} X_i]
\]
\[
= e^{\beta x} \prod_{i=0}^{b-1} \frac{1}{1 + \beta (b-i)/(p-i)}
\]
\[
= \exp \left[ \beta x - \sum_{i=0}^{b-1} \log \left( 1 + \frac{\beta (b-i)}{p-i} \right) \right].
\]

Because \(-\log(1 + u) \leq -u + u^2/2\) for \(u > 0\), we have
\[
\Pr[W_{b,p} < x] \leq \exp \left[ \beta x + \sum_{i=0}^{b-1} \left( -\beta \frac{b-i}{p-i} + \frac{\beta^2 (b-i)^2}{2(p-i)^2} \right) \right]
\]
\[
\leq \exp \left[ \beta(x - \mathbb{E}[W_{b,p}]) + \frac{\beta^2 b^3}{2p^2} \right].
\]

Letting \(x = (1 - \delta) \mathbb{E}[W_{b,p}]\), we obtain
\[
\Pr[W_{b,p} < (1 - \delta) \mathbb{E}[W_{b,p}]] \leq \exp \left[ -\beta \delta \mathbb{E}[W_{b,p}] + \frac{\beta^2 b^3}{2p^2} \right],
\]
and setting \(\beta = \delta \mathbb{E}[W_{b,p}] p^2 / b^3\), we obtain
\[
\Pr[W_{b,p} < (1 - \delta) \mathbb{E}[W_{b,p}]] \leq \exp \left[ -\delta^2 \mathbb{E}[W_{b,p}]^2 \frac{p^2}{2b^3} \right]
\]
and since \(\mathbb{E}[W_{b,p}] \geq b^2/(2p)\), we conclude
\[\Pr[W_{b,p} < (1 - \delta) \mathbb{E}[W_{b,p}]] \leq \exp \left[ -\delta^2 \frac{b}{8} \right].\]

When bounding \(\Pr[W_{b,p} < (1 - \delta) b^2/(2p)]\), it is possible to get a constant of 3/8 rather than 1/8, and 3/8 is tight. But we also use \(\Pr[W_{b,p} < (1 - \delta) \mathbb{E}[W_{b,p}]]\) in \(\S\ 5\), and in the end this constant does not affect the asymptotic lower bound that we prove there.
5 MST lower bounds

5.1 Strategy

For sets $A$ and $B$ of vertices, let $F(A, B)$ be the minimal total weight of a set of edges connecting each vertex in $B$ to some vertex in $A$. Note that $F(A, B)$ is increasing in $B$ and non-increasing in $A$. Define

$$F(a, b) = \min_{|A| \leq a, |B| \geq b, A \cap B = \emptyset} F(A, B),$$

i.e., the minimal cost for connecting at least $b$ vertices to at most $a$ vertices. Next let

$$F(\ell) = F(\ell_0, \ell_1, \ldots, \ell_k) = \sum_{i=1}^{k} F(\ell_{i-1}, \ell_i);$$

this is a lower bound on the cost of any spanning tree of depth $k$ whose level sizes are given by $(\ell_0, \ell_1, \ldots, \ell_k)$. (Mnemonically, the $F$’s are random variables determined by the edge weights of the graph, and the $f$’s from §2.3 are deterministic quantities which we argue are likely to closely approximate the $F$’s.) We can obtain a sharper lower bound by treating the last level differently. In particular, for the last level of Steiner trees, we need only consider sets $B$ which contain only terminal nodes of the Steiner tree. The sharper bound is then

$$F_m(\ell) = \sum_{i=1}^{k-1} F(\ell_{i-1}, \ell_i) + F_m(\ell_{k-1}, \ell_k),$$

where $F_m(a, b)$ is defined as $F(a, b)$ was, but with the set $B$ restricted to be a subset of the $m$ terminals of the Steiner tree.

Let $(\ell_0, \ell_1, \ldots, \ell_k)$ be the “greedy sequence” of level sizes that optimizes $f_{n, 2n/m}(\ell)$ and which we used for the greedy tree in §2. Our strategy is to show that $F_m(\ell)$ is approximately minimized at $F_m(\ell)$ for $n$ large enough, and that $F_m(\ell)$ is within a factor $(1 - \delta)$ of $f_{n, 2n/m}(\ell)$. It will follow that for $n$ large enough, the weight of the greedy tree is close to the weight of the optimal tree.

Lemma 5.1. For any $\delta > 0$, $a \in \{1, \ldots, n\}$ and $b \in \{1, \ldots, n - a\}$ we have

$$\Pr \left[ F(a, b) < (1 - \delta) \frac{b^2}{2na} \right] \leq \exp \left[ a \log \frac{ne}{a} - \frac{1}{8} \delta^2 b \right].$$

Proof. Fix a set $A$ of size at most $a$. For each $x \notin A$ the minimal weight of an edge connecting $x$ to $A$ is an independent $\frac{1}{|A|}$ $\text{Exp}(1)$. Let $W_A$ be the total weight of edges connecting $b$ vertices to $A$. By Lemma 4.1

$$\Pr \left[ W_A < (1 - \delta) \frac{b^2}{2na} \right] \leq \Pr \left[ W_A < (1 - \delta) \frac{b^2}{2(n - |A|)|A|} \right] \leq \exp \left[ -\frac{1}{8} \delta^2 b \right].$$
Finally, the number of sets $A$ of size at most $a$ is

$$\binom{n}{0} + \binom{n}{1} + \cdots + \binom{n}{a} \leq \left(\frac{ne}{a}\right)^a,$$

and a union bound yields the claim. \hfill \square

**Lemma 5.2.** For any $\delta > 0$, $a \in \{1, \ldots, n\}$ and $b \in \{1, \ldots, m-a\}$ we have

$$\Pr\left[F_m(a, b) < (1-\delta) \left(1 - \frac{m-b}{b} \log \frac{m}{m-b}\right) \frac{b}{a}\right] \leq \exp\left[-a \log \frac{ne}{a} - \frac{1}{8} \delta^2 b\right].$$

Note that the bound on $F_m(a, b)$ is $(1-o(1))b^2/(ma)$ in the limit $\delta \to 0$ and $b/m \to 1$.

**Proof.** The proof is essentially the same proof used for Lemma 5.1, except that we use the more precise estimate from Lemma 4.1. \hfill \square

In other words, if $b$ is large enough compared to $a$, then it is unlikely that $F(a, b)$ is much smaller than $f_n(a, b) = b^2/(2na)$, and $F_m(a, b)$ is unlikely to be much smaller than $f_{n,2n/m}(a, b) = b^2/(ma)$. Let us define

$$R_{\delta,n}(a) = 16\delta^2 a \log \frac{ne}{a}.$$

Let

$$f_n^{(R_{\delta,n})}(a, b) = \mathbf{1}_{b>R_{\delta,n}(a)} \frac{b^2}{2na}.$$

Then with high probability, for each $a$ and $b$ we have $F(a, b) \geq f_n^{(R_{\delta,n})}(a, b)$, indeed, the probability that this fails is at most

$$\frac{1}{\binom{n}{0} + \binom{n}{1}} + \frac{1}{\binom{n}{0} + \binom{n}{1} + \binom{n}{2}} + \cdots \leq \frac{1}{n} + (n-1) \frac{2}{n(n-1)} = \frac{3}{n}.$$

Let

$$f_n^{(R_{\delta,n})}(\ell) = f_n^{(R_{\delta,n})}(\ell_0, \ell_1, \ldots, \ell_k) = \sum_{i=1}^k f_n^{(R_{\delta,n})}(\ell_{i-1}, \ell_i) = \sum_{i=1}^k \frac{\ell_i^2}{2n\ell_i} \mathbf{1}_{\ell_i>R_{\delta,n}(\ell_{i-1})},$$

and $f_n^{(R_{\delta,n})}(\ell)$ be defined similarly, but with an extra factor of $c$ in the $k$th term of the sum.

**Corollary 5.3.** With high probability $(\geq 1-3/n)$, any spanning tree with level sizes given by $(\ell_0, \ldots, \ell_k)$ has weight at least $(1-\delta)f_n^{(R_{\delta,n})}(\ell)$. If $\ell_k = (1-o(1))m$, then any Steiner tree connecting a given set of $m$ terminals with level sizes $(\ell_0, \ldots, \ell_k)$ has weight at least $(1-\delta-o(1))f_{n,2n/m}^{(R_{\delta,n})}(\ell)$. 
Thus we are done if we show that $f_{n,c}^{(R_\delta,n)}(\ell)$ constrained to $\sum_i \ell_i \geq m$ is almost minimized at the sequence $\hat{\ell} = (\hat{\ell}_0, \ldots, \hat{\ell}_k)$ that minimizes $f_{n,c}(\hat{\ell})$ constrained to $\ell_k = m$ (and which we used in the greedy tree construction).

**Definition 5.4.** We say that the $k$th level of a sequence $\ell$ is large if $\ell_k \geq (1 - \delta^2/16)m$. For $t \geq 1$, we say that the $(k-t)$th level is large if

$$\ell_{k-t} \geq R_{\delta,n}^{-1}(\cdots R_{\delta,n}^{-1}(m)\cdots).$$

Since $R_{\delta,n}$ is monotone increasing in $a$ up to $a = n$, the inverse function $R_{\delta,n}^{-1}(b)$ is well-defined for $1 \leq b \leq n$. Since $R_{\delta,n}^{-1}(b) \leq (\delta^2/16)b$, and $\sum_i \ell_i \geq m$, it follows that there must be at least one large level.

We may enlarge the set of $\ell$'s over which we are minimizing; so long as $f_{n,c}^{(R_\delta,n)}(\ell)$ is still almost minimized at $\hat{\ell}$, we will have our desired lower bound. Naturally we relax the constraint $\ell_i \in \mathbb{N}$ to $\ell_i \in \mathbb{R}^+$. We keep the constraint $\ell_0 = 1$. We shall drop the $\sum_i \ell_i \geq m$ constraint, and replace it with a constraint that there is a large level in the above sense, since this only increases the set of sequences that we are optimizing over.

### 5.2 No small jumps

Call a jump from $\ell_i$ to $\ell_{i+1}$ large if $\ell_{i+1} \geq R_{\delta,n}(\ell_i)$ and small otherwise. Suppose that a sequence contains a small jump $(\ell_{i-1}, \ell_i, \ell_{i+1})$. If $\ell_{i+1} < R_{\delta,n}(\ell_i)$, then we may increase $\ell_{i+1}$ or decrease $\ell_i$, and each term of $f_{n,c}^{(R_\delta,n)}(\ell)$ either stays the same or decreases. Thus we may assume that all small jumps are from $a$ to $R_{\delta,n}(a)$. If two consecutive jumps are large then the intermediate value must satisfy $2\ell_i^2 = \ell_{i-1}\ell_{i+1}^2$. We wish to show that a sequence achieving the minimum value of $f_{n,c}^{(R_\delta,n)}$ in fact has no small jumps, which will allow us to find the best sequence. We start by showing that it does not have a small jump followed by a large jump.

**Lemma 5.5.** There is a $\delta_0 > 0$ so that whenever $c \geq 1$ and $0 < \delta \leq \delta_0$, and a sequence $\ell$ has a small jump $(\ell_{i-1}, \ell_i)$ followed by a large jump $(\ell_i, \ell_{i+1})$, it is possible to change $\ell_i$ so as to reduce $f_{n,c}^{(R_\delta,n)}(\ell)$.

**Proof.** Let $\ell_{i-1} = a$ and $\ell_{i+1} = b$. Let $C = 16/\delta^2$. We may assume $\ell_i = R_{\delta,n}(\ell_{i-1})$, since otherwise replacing $\ell_i$ with $R_{\delta,n}(\ell_{i-1})$ reduces $f_{n,c}^{(R_\delta,n)}(\ell)$. There is a slight difference when $i + 1 = k$, as opposed to $i + 1 < k$, since the last $(k^{th})$ summand of $f_{n,c}^{(R_\delta,n)}(\ell)$ contains a factor of $c$. We deal below with the case $i + 1 = k$. The case $i + 1 < k$ differs only in that $c$ does not appear, and is derived by replacing all $c$'s by 1's. We consider two possible replacements $\ell_i'$ and $\ell_i''$ defined by

$$2(\ell_i')^3 = c\ell_{i-1}\ell_{i+1}^2,$$  \hspace{1cm}  $$R_{\delta,n}(\ell_i'') = \ell_{i+1}.$$
The contributions to $f_{n,c}(R_{\delta,n})$ from the two jumps in the three cases are

\[ U = c \frac{\ell_{i+1}^2}{2n\ell_i} = \frac{cb^2}{2nR(a)} = \frac{cb^2}{2nCa \log(ne/a)}. \]

\[ U' = \frac{\ell_i^2}{2n\ell_{i-1}} + c \frac{\ell_{i+1}^2}{2n\ell_i} = \frac{3}{2n} \left( \frac{c^2b^4}{4a} \right)^{1/3}, \]

\[ U'' = \frac{(\ell''_i)^2}{2n\ell_{i-1}}. \]

Since $b = R_{\delta,n}(\ell''_i) = C\ell''_i \log(ne/\ell''_i) \geq \ell''_i$, we have $b \geq C\ell''_i \log(ne/b)$, so

\[ U'' \leq \frac{b^2}{2naC^2 \log(ne/b)^2}. \]

If $U'' > U$ then

\[ Cc \log(ne/b)^2 < \log(ne/a). \]

If $U' > U$, then

\[ b < C'c^{-1/2}a \log(ne/a)^{3/2}, \]

where $C' = (3C)^{3/2}/2 = 96\sqrt{3}/\delta^3$.

If both $U' > U$ and $U'' > U$ then we find

\[ \log(ne/a) > Cc \log^2 \frac{ne}{C'c^{-1/2}a \log(ne/a)}. \]

If we denote $q = \log(ne/a)$, this becomes

\[ q > Cc(q - \log(C'c^{-1/2}q))^2. \]

Since $b \leq m$ and each jump increases the level size by at least $16/\delta^2$, we have $a \leq (\delta^4/256)m$, so $q \geq \log[(256\delta^4)/m]$. Since $C' = 96\sqrt{3}/\delta^3$, we have $q - \log C' \geq \frac{1}{4} \log q - \text{const}$. Thus

\[ q > \frac{16}{\delta^2} c \left( \frac{1}{4} q - \log q + \frac{1}{2} \log c - \text{const} \right)^2. \]

But $q \to \infty$ as $\delta \to 0$ and $c \geq 1$, so this equation cannot be true for small enough $\delta$. Thus, provided $\delta < \delta_0$, we have $\min\{U', U''\} \leq U$, so replacing $\ell_i$ with either $\ell'_i$ or $\ell''_i$ reduces $f_{n,c}(R_{\delta,n})(\ell)$.

Consider the maximal $t$ such that level $k - t$ of the sequence $\ell$ is large. The jump from $\ell_{k-t-1}$ to $\ell_{k-t}$ must be a large jump, since otherwise level $k - t - 1$ would also be a large level. Because (for small enough $\delta$) there are no small jumps followed by large jumps, it follows that the subsequence $\ell_0, \ell_1, \ldots, \ell_{k-t}$ consists only of large jumps. If $t \neq 0$ this would imply that the total cost up to level $k - t$ is too high for $\ell$ to be optimal. Our next step is to obtain a lower bound on $\ell_{k-t}$, for which we need the following lemma.
Lemma 5.6. Assume $\delta \leq 1$. Recall that $C = 16/\delta^2$, and that $\Gamma$ is the gamma function (generalized factorial). If for some $r \geq 2$

\[
b \geq \frac{en}{(C\log C)^{r-1}\Gamma(r)^2},\]

then

\[
R_{\delta,n}^{-1}(b) \geq \frac{en}{(C\log C)^r\Gamma(r)^2}.
\]

Proof. If

\[
a \leq \frac{en}{(C\log C)^r\Gamma(r)^2},
\]

then

\[
R_{\delta,n}(a) \leq C\frac{en}{(C\log C)^r\Gamma(r)^2} \log \frac{ne}{(C\log C)^r\Gamma(r)^2}
\]

\[
\leq \frac{r \log(r^2C\log C)}{r^2\log C} \times \frac{en}{(C\log C)^{r-1}\Gamma(r)^2}.
\]

The first term is

\[
\frac{r \log(r^2C\log C)}{r^2\log C} = \frac{2 \log r}{r \log C} + \frac{\log C}{r \log C} + \frac{\log \log C}{r \log C},
\]

and since $(\log r)/r \leq 1/e$ and $(\log \log C)/\log C \leq 1/e$ and $r \geq 2$, we have

\[
\leq \frac{2}{e \log C} + \frac{1}{2} + \frac{1}{2e}.
\]

As long as $\delta \leq 1$, we have $C \geq 16$, so that this quantity is bounded by 1.

Thus we get a lower bound on the size $\ell_{k-t}$ of the first large level.

Lemma 5.7. If $\delta \leq 1$ and $t \geq 1$ and level $k-t$ is a large level, then

\[
\ell_{k-t} \geq \frac{m}{(t \log \frac{m}{\delta})^{\Theta(t)}}.
\]

Proof. By our definition of “large,”

\[
\ell_{k-t} \geq R_{\delta,n}^{-1}(\cdots R_{\delta,n}^{-1}((1 - \delta^2/16)m) \cdots).
\]

Next we find the smallest $r \geq 2$ satisfying

\[
(1 - \delta^2/16)m \leq \frac{en}{(C\log C)^r\Gamma(r)^2},
\]

(9)
the relevant \( r \) satisfies \( r \leq O(\log \frac{m}{n} / \log \log \frac{m}{n}) \), so we can bound
\[
\ell_{k-t} \geq \frac{en}{(C \log C)^{r+1} (r + t + 1)^2}
\]
which, if \( r > 2 \) (so that (9) is tight), can be bounded by
\[
\ell_{k-t} \geq \frac{(1 - \delta^2/16) m}{(C \log C (r + t)^2)^t}.
\]
Whether or not \( r > 2 \), we can bound
\[
\ell_{k-t} \geq \frac{m}{(t \log \frac{m}{\delta})^{\Theta(t)}}.
\]

**Lemma 5.8.** If \( 0 < \delta < \delta_0 \), there is a constant \( \Delta \) such that whenever \( k \leq \log_2 \log m - \log_2 \log(en/m) - \Delta \), any sequence \( \ell \) optimizing \( f_{n,c}(\ell_n) \) contains no small jumps.

**Proof.** If \( t \geq 1 \), then we may bound \( f_{n,c}(\ell_n) \) from below by the cost of the first \( k-t \) levels, which by our earlier calculation (8) is
\[
f_n(\ell_0, \ldots, \ell_{k-t}) = \frac{2\ell_{k-t}}{n} (1 - 2^{t-k}) \left( \frac{\ell_{k-t}}{2^{k-t}} \right)^{\frac{1}{2^{k-t-1}}}
\]
Let us assume \( t < k \). Upon substituting our lower bound for \( \ell_{k-t} \) from Lemma 5.7 and \( c = 2n/m \), we may compare this (LHS) to the sequence from the greedy construction (RHS):
\[
\frac{2m}{n(\frac{1}{2} \log \frac{n}{m})^{\Theta(t)}} (1 - 2^{t-k}) \left( \frac{m/(\log \frac{n}{m})^{\Theta(t)}}{2^{k-t}} \right)^{\frac{1}{2^{k-t-1}}} \leq \sqrt{\frac{8m}{n}} (1 - 2^{-k}) \left( \frac{\sqrt{2mn}}{2^k} \right)^{\frac{1}{2^{k-t-1}}}
\]
\[
\left( \frac{m}{2^{k-t}} \right)^{\frac{1}{2^{k-t-1}}} \leq \left( \frac{t \log \frac{n}{m}}{\frac{1}{2} \log \frac{n}{m}} \right)^{\Theta(t)} \sqrt{\frac{m}{n}} \left( \frac{\sqrt{2mn}}{2^k} \right)^{\frac{1}{2^{k-1}}}
\]
\[
\Theta \left( \frac{2^t}{2^k} \right) (\log m - O(k)) \leq \Theta \left( t \log \frac{t}{\delta} \right) + \Theta \left( t \log \log \frac{n}{m} \right) + \Theta \left( \log \frac{n}{m} \right)
\]
Let us assume \( k \leq \log_2 \log m - \Delta \), where \( \Delta \) is a suitably large constant depending on \( \delta \). Then the \( O(k) \) term in the LHS may be neglected, and the LHS is \( \Omega(2^\Delta 2^t) \), which is larger than the \( \Theta(t \log t) \) term on the RHS. If in addition, \( k \leq \log_2 \log m - \log_2 \log(en/m) - \Delta \), then half the LHS is also \( \Omega(2^\Delta 2^t \log(n/m)) \), which dominates the second and third terms in the RHS.

Thus under these conditions on \( k \), any sequence \( \ell \) with \( t \neq 0 \) is not optimal, and so any optimizing sequence for \( f_{n,\log_2(n/m)}(\ell) \) contains no small jumps. \( \square \)
5.3 Lower bounds

Proof of Theorem 1.5, lower bounds, for exponential weights. We start with the bound on \( \text{wt}(\text{MST}(K_n, m)) \). Combining Corollary 5.3 with Lemma 5.8, we find that w.h.p. the cost of a Steiner tree with level sizes given by \( \ell \) is at least \((1 - \delta)f(R_{\delta,n})\), and that this is minimized by the sequence constructed in §2 to give the upper bound. Thus we find the upper bound is tight.

As noted, \( \text{wt}(\text{MST}(K_n, m)) \geq \text{wt}(\text{MST}(K_n, m)) \), so its bound is also tight.

Finally substituting \( m/2 \) and \( n/2 \) throughout, we find that no tree can improve by more than \((1 + o(1))\) on the greedy tree construction for the odd diameter case.

Proof of Theorem 1.5, concentration, for exponential weights. For each of the random variables \( \text{wt}(\text{MST}(K_n, m)) \), \( \text{wt}(\text{MST}(K_n, m)) \), and \( \text{wt}(\text{MST}(K_n, m)) \), the random variable is almost never smaller than a factor of \( 1 + o(1) \) smaller than our upper bound on their expected values. It follows that for each of these random variables, the expected value is within a factor of \( 1 + o(1) \) of our upper bound on it, and that these random variables are with high probability within a factor of \( 1 + o(1) \) of their expected values.

6 Other weight distributions

In the proofs up to this point, we assumed that the edge weights of \( K_n \) are distributed according to an exponential random variable with mean 1. In this section we prove the parts of Theorems 1.1, 1.4, and 1.5 that pertain to more general weight distributions. In our notation up until now we suppressed the weight distribution, but here we make it more explicit: we let \( K_n^\tilde{W} \) denote the complete graph where each edge weight is an i.i.d. copy of a non-negative random variable \( \tilde{W} \).

The key observation (which was made earlier in the context of unconstrained minimum spanning trees [Fri85, Ste87] and Steiner trees [BGRS04]) is that w.h.p. only edges with weights \( o(1) \) are ever used (except when the depth is 1). Thus, it is principally the density of the distribution near 0 that is significant. In this section we assume that the edge weights are i.i.d., and are distributed according to some non-negative random variable \( \tilde{W} \) that has density 1 near 0, i.e., for positive \( t \) near 0,

\[ \Pr[\tilde{W} < t] = t + o(t). \]

(If the density near 0 exists and is not 1, then linearity in the weights gives a multiplicative constant in the theorems.) We let \( W \) denote an exponential random variable with mean 1,

\[ W \sim \text{Exp}(1), \]
and for \( \varepsilon > 0 \) define

\[
W_\varepsilon = \begin{cases} 
W & W \leq \varepsilon \\
\varepsilon & W > \varepsilon,
\end{cases}
\]

and

\[
W^\varepsilon = \begin{cases} 
W & W \leq \varepsilon \\
\infty & W > \varepsilon.
\end{cases}
\]

For a generally distributed non-negative weight \( \tilde{W} \) with density 1 at 0, for any \( \delta > 0 \) there is an \( \varepsilon > 0 \) such that

\[
(1 - \delta)W_\varepsilon \prec \tilde{W} \prec (1 + \delta)W^\varepsilon,
\]

i.e., \( \tilde{W} \) is stochastically sandwiched between \( (1 - \delta)W_\varepsilon \) and \( (1 + \delta)W^\varepsilon \). Since \( \text{wt}(\text{MST}) \) is monotone in the edge weights, it follows that bounded-depth/diameter tree weight distributions are also stochastically sandwiched.

Let us call an edge of the weighted graph \( \varepsilon \)-light if its weight is at most \( \varepsilon \), and otherwise let us call it \( \varepsilon \)-heavy. The idea is to show that w.h.p. the greedy, sliced-and-spliced, and optimal trees use only light edges (when \( k \geq 2 \)), so that it makes little difference whether the edge weights are distributed according to \( W \) or \( \tilde{W} \).

As we shall see, the density-1-at-0 assumption is enough to get convergence in probability, but some additional assumption to rule out the possibility of very fat tails in the distribution is required to get the upper bounds for the convergence in expectation of the tree weights. We shall assume

\[
\mathbb{E}[\tilde{W}] < \infty
\]

when deriving convergence in expectation.

### 6.1 Upper bounds

We start by proving that, in the greedy tree and sliced-and-spliced tree, heavy edges are rare.

**Lemma 6.1.** If \( 2 \leq k \leq \frac{1}{4} \log_2 n \), the expected number of \( \varepsilon \)-heavy edges contained within the greedy Steiner tree from \( \overline{s} \) is at most \( \exp[\Theta(\log n) - \Theta(\min(\varepsilon, 1)n)] + kn \exp[-\varepsilon n^{1/8}] \).

**Proof.** The level sizes \( \lceil \hat{\ell}_i \rceil \) from (7) are monotone increasing in \( i \), and monotone decreasing in \( k \). At level 1, since \( k \geq 2 \) we have

\[
(n/4^k)^{1/4} < 2\sqrt{2nm/4^k} \leq \hat{\ell}_1 \leq 2\sqrt{nm/2} < 2n^{2/3}.
\]

The number of light edges emanating from the root is a binomial with parameters \( n - 1 \) and \( p = 1 - e^{-\varepsilon} \geq \varepsilon - \varepsilon^2/2 \). A standard large-deviation formula (see [McD89 Eqn. 5.6]) tells us that for any binomial random variable \( D \),

\[
\Pr[D < \mathbb{E}[D]/2] \leq e^{-\mathbb{E}[D]/8}.
\]
Assuming \( \varepsilon \geq 5n^{-1/3} \) and \( n \) is large, so that \( \mathbb{E}[D]/2 \geq 2n^{2/3} + 1 \), we deduce that the expected number of heavy edges in the first level of the greedy tree is at most
\[
\left[ 2n^{2/3} \right] e^{-(n-1)p/8} = \exp[\Theta(n) - \Theta(pn)].
\]
(If \( \varepsilon < 5n^{-1/3} \) or \( n \) is not large, the conclusions of the lemma are trivially true.) For any subsequence level of the tree, the number of heavy edges is at most the number of vertices not connected to it via a light edge, and since there are at least \( (n/4^k)^{1/4} \) vertices in the previous level, the expected number of such vertices not reachable by a light edge is at most
\[
O(n(1-p)^{(n/4^k)^{1/4}}) = n \exp[-\varepsilon(n/4^k)^{1/4}] \leq n \exp[-\varepsilon n^{1/8}].
\]
Upon multiplying by \( k-1 \) (since there are \( k-1 \) levels after the first) and adding the heavy edges from the first level, we obtain the desired bound.

**Proof of Theorem 1.5, upper bounds, other weight distributions.** The upper bounds for convergence in probability are an immediate consequence of the fact that for fixed \( \varepsilon \), w.h.p. there are no heavy edges. The upper bounds for convergence in expectation follow from the fact that the expected number of heavy edges is \( o(1) \), and the fact that \( \mathbb{E}[\tilde{W}] \) is finite.

The following lemma essentially appears in [BGRS04].

**Lemma 6.2.** For \( n \geq 3 \) and \( \varepsilon > 0 \), the expected number of \( \varepsilon \)-heavy edges in \( \text{MST}(K_n, m) \) is at most \( O(e^{-\varepsilon n^4 \log^2 n}) \).

**Proof.** Consider any edge of the Steiner tree \( \text{MST}(K_n, m) \) with weight greater than \( \varepsilon \). If its endpoints are connected by a path with total weight \( \leq \varepsilon \), then we could delete the heavy edge and replace it with a portion of or all of the low-weight path connecting that edge’s endpoints, obtaining a lighter Steiner tree. (This argument appeared in [BGRS04].) Janson proved that in the complete graph \( K_n \) with exponential edge weights, w.h.p. every pair of vertices is connected by a path of weight at most \( (3 + o(1))n^{-1} \log n \) [Jan99]. In fact, it follows from [Jan99, Eqn. 2.8] that, when \( n \geq 3 \) and \( \varepsilon \geq 0 \), the expected number of pairs of vertices not connected by a path of weight \( \leq \varepsilon \) is at most \( O(e^{-\varepsilon n^4 \log^2 n}) \). Thus, the expected number of heavy edges in \( \text{MST}(K_n, m) \) is at most \( O(e^{-\varepsilon n^4 \log^2 n}) \).

**Lemma 6.3.** If \( k = \log_2 \log n + \Delta \) where \( \Delta \geq n/(m \log(en/m)) \), the expected number of \( \varepsilon \)-heavy edges in the sliced-and-spliced Steiner tree from §3 is at most \( \exp[\Theta(n) - \varepsilon n] + O(1/\varepsilon) \).

**Proof.** There could be as many as \( \exp[\Theta(n) - \varepsilon n] \) heavy edges in the starting Steiner tree \( \text{MST}(K_n, m) \). Of course, when we do the slicing of \( \text{MST}(K_n, m) \), no heavy edges are introduced, but heavy edges could be introduced when we splice the subtrees using the greedy-tree construction. In the construction, recall that the total weight of the splice edges was in expectation at most
\[
O \left( \sqrt{\frac{m \log(en/m)}{n \Delta}} \right) \leq O(1/\Delta).
\]

The expected number of \( \varepsilon \)-heavy splice edges can be at most \( 1/\varepsilon \) times as large as this.
Proof of Theorem 1.4. other weight distributions. As above, the upper bounds for convergence in probability are an immediate consequence of the fact that for fixed $\varepsilon$, w.h.p. there are not any heavy edges, and the upper bounds for convergence in expectation follow from the fact the expected number of heavy edges is $o(1)$, and the fact that $E[\tilde{W}]$ is finite.

The lower bounds follow from the fact that the unrestricted Steiner tree MST($K_n, m$) w.h.p. has no heavy edges, and is at most as heavy as the bounded-depth/diameter Steiner trees.

6.2 Lower bounds

Proof of Theorem 1.5. lower bounds, other weight distributions. Fix some $\delta > 0$. Let $F_\varepsilon(a, b)$ be defined as $F(a, b)$ but using $(W_\varepsilon)_\varepsilon$. We argue that w.h.p., for every pair $a \leq b$, we have either $F_\varepsilon(a, b) \geq (1 - \delta)F(a, b)$ or $F_\varepsilon(a, b) > \sqrt{n}$. Thus, the cost of a tree with given level sizes is either within $(1 - \delta)$ of the unmodified cost, or else is at least $\sqrt{n}$. Since the optimal choice is smaller than $\sqrt{n}$ (here we use $k > 1$), the proof of the lower bound carries over unchanged.

We consider the graph of light edges, which is $G_{n,p}$ with $p = \varepsilon + o(\varepsilon)$. If every set $A$ of size $|A| = a$ has at least $b$ neighbors in the light-edge graph, then $F_\varepsilon(a, b) = F(a, b)$. (To see this, consider the sets $A$ and $B$ for which $|A| = a$, $|B| = b$, and $F_\varepsilon(A, B) = F_\varepsilon(a, b)$. If there were a heavy edge from $A$ to $B$, then we could delete the endpoint of that edge from $B$, and replace it with a vertex not already in $B$ which is connected to $A$ via a light edge, and $F_\varepsilon(A, \text{modified } B) < F_\varepsilon(A, B)$, a contradiction. Hence there is no heavy edge from $A$ to $B$, so $F(a, b) \leq F(A, B) = F_\varepsilon(A, B) = F_\varepsilon(a, b) \leq F(a, b)$.)

We consider several cases.

Case $b \leq \varepsilon n/4$: For any vertex, its degree $D$ in the light-edge graph is a binomial distribution with parameters $n - 1$ and $p = (1 + o(1))\varepsilon$. Since $E[D] = (1 + o(1))n\varepsilon$, the standard large-deviation formula (see [McD89] Eqn. 5.6)] that we used earlier tells us that $Pr[D < \varepsilon n/2] \leq e^{-(1+o(1))E[D]/8} = e^{-n\varepsilon/(8+o(1))}$. A union bound then tells us that w.h.p. the minimal degree is at least $\varepsilon n/2$. Conditional on this event, any set $A$ has at least $\varepsilon n/2 - |A|$ neighbors, so if $a \leq b \leq \varepsilon n/4$, it follows that $F_\varepsilon(a, b) = F(a, b)$.

Case $a \geq n^{1/3}, b \leq n - n^{3/4}$: We argue that any disjoint sets $A$ and $C$ of sizes at least $n^{1/3}$ and $n^{3/4}$ have an edge between them. This is a union bound over all pairs of sets: the number of pairs of sets is at most $3^n$, but each pair has no edge with probability $(1 - p)^n/n^{13/12}$. This implies that, for $n$ large enough, w.h.p. $F_\varepsilon(a, b) = F(a, b)$ for any $a \geq n^{1/3}$ and $b \leq n - n^{3/4}$, since any such set $A$ has at most $n^{3/4}$ non-neighboring vertices in the light-edge graph.
Case $a \geq n^{1/3}, b > n - n^{3/4}$: By the monotonicity of $F$ and $F_\varepsilon$, and the above case, for large enough $n$ we have w.h.p.

$$F(a, b) \geq F_\varepsilon(a, b) \geq F_\varepsilon(a, n - n^{3/4}) = F(a, n - n^{3/4}).$$

However, based on our bounds on $F$ from Lemmas 5.1 and 5.2 we have

$$F(a, n - n^{3/4}) \geq (1 - \delta) F(a, b)$$

for any $\delta$ given $n$ large enough.

Case $a \leq n^{1/3}, b > \varepsilon n/4$: By the monotonicity of $F$ and $F_\varepsilon$, and the second case above, for large enough $n$ we have w.h.p.

$$F_\varepsilon(a, b) \geq F_\varepsilon(n^{1/3}, \varepsilon n/4) = F(n^{1/3}, \varepsilon n/4).$$

By Lemma 5.1 w.h.p. this is at least

$$(1 - \delta) \frac{\varepsilon^2 n^2/16}{2mn^{1/3}} \gg \sqrt{n} \gg \frac{3}{2} n^{1/3},$$

i.e., it exceeds the weight of the greedy spanning tree.

\[\square\]

7 Open problems

We identified a sharp cutoff of depth $\log_2 \log n \pm \Theta(1)$ above which the minimum bounded-depth spanning tree has weight that is asymptotically equal to the value of the unconstrained minimum spanning tree, and below which it is much larger. This same cutoff at $\log_2 \log n \pm \Theta(1)$ holds for minimum bounded-depth Steiner trees with $m$ terminals when $m = \Theta(n)$, but we do not know the location of the cutoff (or indeed if there is one) when $m$ is much smaller than $n$. If there is a cutoff, we know that it occurs when the depth $k$ is in the interval

$$\log_2 \log m - \log_2 \log(en/m) - o(1) \leq k \leq \log_2 \log n + o\left(\frac{n}{m \log(en/m)}\right),$$

but we do not know where in the interval. It would be interesting to better understand the weights of bounded-depth Steiner trees for these parameter values.

It would be interesting to understand better the large-$n$ behavior of the weight of the bounded-depth MST near depth $\log_2 \log n + \Delta$ as a function of $\Delta$. The precise behavior could be complicated, and is perhaps a periodic function of the fractional part of $\log_2 \log n$, but there are more basic open problems. For example, our construction in \S 3 shows that when the depth bound is $\log_2 \log n + \Delta$, the bounded-depth MST has weight $\leq \zeta(3) + O(1/\sqrt{\Delta})$. 


while our best lower bound is $\zeta(3)$. We do not know how fast the approach to $\zeta(3)$ is when $\Delta$ is increased, or indeed, if $\zeta(3)$ is reached for some finite $\Delta$.

The weight of the minimum weight Steiner tree (with unbounded depth), as a function of $\alpha = m/n$ (the ratio of the number of terminals to the number of vertices) goes from 0 at $\alpha = 0$ to $\zeta(3)$ at $\alpha = 1$. As mentioned in [BGRS04], it would be interesting to understand how the weight varies from 0 to $\zeta(3)$ for intermediate values of $\alpha$.

There was an experimental study aimed at sharpening our estimate of $(1 - o(1)) \frac{3}{2} n^{1/3}$ for the asymptotic weight of minimum bounded-depth spanning trees when with depth bound $k = 2$ [BBB⁺08], suggesting $\frac{3}{2} n^{1/3} - \text{const}$. This constant will depend on the weight distribution; it may be interesting to rigorously determine the constant.

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