TEMPERLEY-LIEB PFAFFINANTS AND SCHUR $Q$-POSITIVITY CONJECTURES

THOMAS LAM AND PAVLO PYLYAVSKYY

ABSTRACT. We study pfaffian analogues of immanants, which we call pfaffinants. Our main object is the TL-pfaffinants which are analogues of Rhoades and Skandera’s TL-immanants. We show that TL-pfaffinants are positive when applied to planar networks and explain how to decompose products of complementary pfaffians in terms of TL-pfaffinants. We conjecture in addition that TL-pfaffinants have positivity properties related to Schur $Q$-functions.

1. Introduction

An immanant of an $n \times n$ matrix $X = (x_{ij})$ is an expression of the form

$$\sum_{w \in S_n} f(w)x_{1,w(1)} \cdots x_{n,w(n)}$$

where $f : S_n \to \mathbb{R}$ is a function. The well-known examples of immanants are determinants and permanents. Desarmenien, Kung and Rota [DKR] gave a standard basis of the space $I(X)$ of immanants, labeled by standard bitableaux while recently Pylyavskyy [Pyl] introduced a basis labeled by non-crossing bitableaux.

Immanants with certain positivity properties, most notably the irreducible immanants, had been studied earlier in [GJ, Gre, Hai, Ste92, SS]. In a series of papers [Ska, RS05a, RS05b], Rhoades and Skandera studied the dual canonical basis of $I(A)$, also called Kazhdan-Lusztig immanants, labeled by permutations. These immanants possess remarkable positivity properties: (a) they are non-negative when applied to totally non-negative matrices [RS05a, RS05b], and (b) they are Schur-positive when applied to Jacobi-Trudi matrices [RS05b]. This second property was used in [LPP] to resolve several Schur-positivity conjectures. The subset of the dual canonical basis corresponding to 321-avoiding permutations can be given a purely combinatorial interpretation and were called Temperley-Lieb immanants, or TL-immanants, in [RS05a]. Rhoades and Skandera also gave a simple positive combinatorial rule for writing a product of two complementary minors of $A$ in terms TL-immanants.

The pfaffian $\text{pf}(A)$ of a skew symmetric $2n \times 2n$ matrix $A$ (see Section 2.1) replaces the symmetric group $S_{2n}$ in the determinant with the set of matchings of $2n$ points. Replacing the symmetric group in (1) with matchings one also obtains a pfaffian analogue of immanants, which we call pfaffinants. The main object of this paper are the TL-pfaffinants denoted $\text{Pf}_{\text{TL}}(A)$, which are analogues of the TL-immanants.

Stembridge [Ste90] interpreted the pfaffian $\text{pf}(A(N))$ in terms of non-intersecting path families in a planar network $N$, where $A(N)$ is a skew-symmetric matrix.
obtained from $N$. Separately, it is also known (JP [Ma]) that the Schur $Q$-function $Q_\lambda$ is equal to the pfaffian $\text{pf}(A_\lambda)$ for a particular skew symmetric matrix $A_\lambda$, which we call a $Q$-Jacobi-Trudi matrix. Our search for the TL-pfaffinants revolves around the following three properties:

1. a product of complementary pfaffians should decompose positively and simply in terms of the TL-pfaffinants;
2. a TL-pfaffinant should be positive when evaluated on the skew symmetric matrix $A(N)$ associated to a planar network;
3. a TL-pfaffinant should be Schur $Q$-positive when evaluated on a $Q$-Jacobi-Trudi matrix.

The pfaffinants $\text{Pf}_D(A)$ that we define satisfy properties (1) and (2), and we conjecture that they satisfy property (3). The positivity properties (2) and (3) are subtly different from the situation with TL-immanants. Our definition of the pfaffinants $\text{Pf}_D(A)$ requires the intermediate definition of a diagram pfaffinant $\text{Pf}_D'(A)$. It appears rather mysteriously that it is the diagram pfaffinants that describe network and (conjecturally) Schur $Q$-positivity. We should point out that the correct pfaffian analogue of the entire dual canonical basis is still missing. A basis of this entire space of pfaffinants (without the positivity properties we desire) is given by DeConcini and Procesi [DP] from the point of view of invariant theory.

One of the Schur $Q$-positivity conjectures (Conjecture 50) that we state is a Schur $Q$-function version of a sequence of positivity results we call cell transfer: the monomial positivity version is established in [LP05], the fundamental quasi-symmetric function version in [LP06] and the Schur positivity version in [LPP].

We now briefly describe the organization of the paper. In Section 2 we define diagram pfaffinants and Temperley-Lieb pfaffinants, and show that the latter form a basis for the space of products of pairs of complementary pfaffians. In Section 3 we explain Stembridge’s work on pfaffians and planar networks and show that TL-pfaffinants are non-negative when applied to planar networks. We characterize the linear combinations of products of pairs of complementary pfaffinants that are network-nonnegative. In Section 4 we explore the relationship between TL-immanants and TL-pfaffinants when applied to certain matrices. In Section 5 we state a number of conjectures concerning Schur $Q$-positivity properties of pfaffinants, and in addition we prove a number of intermediate results.

2. Pfaffians and Pfaffinants

2.1. Preliminaries. A skew-symmetric matrix $A = (a_{ij})_{i,j=1}^n$ is a matrix satisfying $A^t = -A$ or alternatively $a_{ij} = -a_{ji}$. These matrices are in bijection with arrays $(a_{ij})_{1\leq i<j\leq n}$ obtained by taking the part of $A$ above the diagonal. We denote the corresponding array also by $A$ and will not usually distinguish the skew-symmetric matrix from the upper-triangular array.

Now suppose $A$ is a skew-symmetric $2n \times 2n$ matrix. Define the pfaffian $\text{pf}(A)$ of $A$ by

$$\text{pf}(A) = \sum_{\pi \in F_{2n}} \epsilon(\pi) \prod_{(i,j) \in \pi} a_{ij},$$

where the sum is taken over the set $F_{2n}$ of matchings $\pi$ on $2n$ vertices, and $\epsilon(\pi)$ is the sign or crossing number of a matching. It can be determined by the following rule: place the $2n$ vertices on a straight line and draw all the edges in $\pi$ as arcs
above this line. Let \( cn(\pi) \) denote the number of crossings between the arcs. Then 
\( \epsilon(\pi) = (-1)^{cn(\pi)} \). For convenience we write \( a_\pi := \prod_{(i,j) \in \pi} a_{ij} \) for any \( \pi \in F_{2n} \). We will generally think of the matching \( \pi \) as a set of unordered pairs of elements of \([2n]\). For example, if \( n = 2 \) we have \( \text{pf}(A) = a_{12}a_{34} - a_{13}a_{24} + a_{14}a_{23} \).

Let \( I \subset [2n] \) be a \( 2m \)-element subset and let \( A_I \) be the corresponding submatrix, obtained by taking only the rows and columns with indices in \( I \). We denote by \( \text{pf}_I(A) \) the pfaffian of this submatrix. More generally, for disjoint subsets \( I_1, I_2, \ldots \) of \([2n]\) we denote \( \text{pf}_{I_1,I_2,\ldots}(A) = \text{pf}_{I_1}(A)\text{pf}_{I_2}(A)\cdots \) the product of the corresponding pfaffians. If \( I \subset [2n] \) we let \( \bar{I} = [2n] \setminus I \) denote the complement of \( I \) in \([2n]\).

Two special cases of \( \text{pf}_{I_1,I_2,\ldots}(A) \) are particular important to us. One is the complementary pfaffians \( \text{pf}_{I,\bar{I}}(A) \), which are the products of pfaffians of two complementary subarrays. The second one is the monomials \( \text{pf}_\pi(A) = a_\pi \prod_{(i,j) \in \pi} a_{ij} \). Thus one may also write the definition of the pfaffian as \( \text{pf}(A) = \sum_{\pi \in F_{2n}} \epsilon(\pi)\text{pf}_\pi(A) \).

Next, for an arbitrarily function \( f : F_{2n} \to \mathbb{R} \) we define the pfaffinant \( Pf_f(A) = \sum_{\pi \in F_{2n}} f(\pi)\text{pf}_\pi(A) \). It is easy to see that if \( I, J, \ldots \) is a partitioning of \([2n]\) into disjoint sets (of even size), then \( \text{pf}_{I,J,\ldots} \) is a pfaffinant. In particular, each \( \text{pf}_\pi \) is a pfaffinant with \( f(\rho) = \delta_{\rho \pi} \).

### 2.2. The complementary Pfaffian subspace

Let \( \mathbb{R}[A] = \mathbb{R}[a_{12}, a_{13}, \ldots] \) denote the \( \mathbb{R} \)-vector space of polynomials in the variables \( \{a_{ij}\}_{1 \leq i < j \leq 2n} \). Now let \( P_n := \mathbb{R}[\text{pf}_{I,\bar{I}}(A)] \subset \mathbb{R}[A] \) denote the subspace spanned by the complementary pfaffians \( \text{pf}_{I,\bar{I}} \), for all possible pairs \((I, \bar{I})\), including the case \( I = \emptyset \).

We call a partitioning \((I, \bar{I})\) of \([2n]\) standard if \( I = \{i_1 < i_2 < \cdots < i_a\} \) and \( \bar{I} = \{j_1 < j_2 < \cdots < j_b\} \) where \( a \geq b \) and \( i_k < j_k \) for each \( k \in [1, a] \). Alternatively, \((I, \bar{I})\) is standard if \( I \) and \( \bar{I} \) form the first and second rows of a standard Young tableau. We say \( \text{pf}_{I,\bar{I}} \) is standard if \((I, \bar{I})\) is.

**Theorem 1** ([DP]). A basis of \( P_n \) is given by the set \( \{\text{pf}_{I,\bar{I}}(A) \mid (I, \bar{I}) \text{ is standard}\} \) of standard complementary pfaffians. The dimension of \( P_n \) over \( \mathbb{R} \) is equal to the number of standard Young tableaux of size \( 2n \) with at most \( 2n \) rows, each row of even size.

**Proof.** In [DP], a product of several complementary pfaffians is associated to any (possibly non-standard) tableau \( T \) with even parts. It is shown ([DP Theorem 6.5]) that the set of such products indexed by standard tableaux forms a basis for the space of all pfaffiants. The straightening algorithm showing that any tableau can be expressed in terms of standard ones ([DP, Lemma 6.1-6.3]) involves quadratic relations among products of pfaffians. Since the number of parts in the tableaux involved do not increase in such straightenings, the statement of the theorem follows.

We will give another proof of Theorem [11] later.

**Remark 2.** The following is the natural generalisation. Let \( P_{k,n} \subset \mathbb{R}[A] \) denote the subspace spanned by \( k \) complementary pfaffians of a \( 2n \times 2n \) skew-symmetric matrix. Then the dimension of \( P_{k,n} \) is equal to the number of standard tableaux of size \( 2n \) with at most \( k \) rows such that each row has even length.

### 2.3. Symmetric Temperley-Lieb diagrams

Consider a rectangle with the \( 2n \) points \( 1, 2, \ldots, 2n \) on the left side and \( 2n \) points \( 1', 2', \ldots, 2n' \) on the right side.
Proposition 3. For any integer \( n \geq 1 \) we have

\[
|T_n| = \binom{2n}{n} \quad \text{and} \quad |T_n^e| = \frac{1}{2} \binom{2n}{n} = \binom{2n-1}{n}.
\]

Proof. We show that \( T_n \) is in bijection with \( n \)-subsets of a 2\( n \) element set. One possible such correspondence is obtained as follows: for \( D \in T_n \) color all \( i \in [2n] \) such that \( i < j \) in \( D \) black. Among the remaining points color black the largest ones so that we get \( n \) black points in total. The inverse map from a coloring of 2\( n \) points black and white, \( n \) of each color, can be described as follows. Start reading the points in reverse order, from 2\( n \) to 1. For each black point \( i \) one encounters we find the smallest \( j > i \) colored white which has not yet been used and include the edge \((i, j)\) in \( D \). If no such \( j \) exists, we include the edge \((i, i')\) in \( D \). After doing this for all the black points, we include an edge \((j, j')\) for each unmatched white point \( j \).

Now let \( T_n^o = T_n \setminus T_n^e \) denote the set of odd symmetric TL-diagrams. We define an involution \( \omega \) on \( T_n \) which sends \( T_n^e \) to \( T_n^o \). Let \( D \in T \). If \((1, 1') \in D\), there exists some smallest \( i \in [2n] \) where \( i \neq 1 \) so that \((i, i') \in D\). We define \( \omega(D) \) by removing the edges \((1, 1')\) and \((i, i')\) from \( D \) and including the edges \((1, i)\) and \((1', i')\). Otherwise \((1, k) \in D \) for some \((\text{even}) \ k \in [2n] \). We define \( \omega(D) \) by removing the edges \((1, k)\) and \((1', k')\) and including the edges \((1, 1')\) and \((k, k')\). The involution \( \omega \) shows that \(|T_n^o| = |T_n^e| = \frac{1}{2} |T_n|\). \(\Box\)

2.4. Diagram pfaffinants. For each \( D \in T_n \) we now define a function \( f_D : F_{2n} \rightarrow \mathbb{Z} \) which in turn gives us the diagram pfaffinant \( \text{Pfaf}_D(A) := \text{Pfaf}_{f_D}(A) \).

Recall that we have 4\( n \) vertices on the sides of the rectangle: 1, \ldots, 2\( n \) on the left side and 1', \ldots, 2\( n' \) on the right. Given a matching \( \pi \in F_{2n} \), let \( \nu(\pi) \) be the matching on \([2n] \cup [2n']\) such that \((i, j')\) and \((i', j)\) are in \( \nu(\pi) \) if and only if \((i, j) \in \pi\). Pick a planar embedding of \( \nu(\pi) \) such that all edges lie inside the rectangle, and every pair of edges intersect at most once. We assume the embedding is chosen (a) to have mirror symmetry, (b) no pair of edges have a point of tangency, and (c) that no 3 edges cross at a single point. Call an embedding satisfying these conditions nice. Such an embedding is far from unique, however we will show that the construction does not depend on the choice of embedding. We assume for now one such presentation has been chosen for each \( \pi \), which we will (abusing notation) denote by \( \nu(\pi) \) as well.

The set of intersections among the edges of \( \nu(\pi) \) can be divided into two kinds: the unpaired crossings, which are the crossings between pairs of edges of the form \((i, j')\) and \((i', j)\); and the paired crossings, which are the pairs of crossings between
(p', q) and (r', s) and between (p, q') and (r, s'), where inequalities q < s and r < p either both fail or both hold.

Given π ∈ F_{2n} we define a set X(π) of uncrossings of ν(π). Each embedded graph x ∈ X(π) is obtained from ν(π) by uncrossing every intersection, where each intersection can be uncrossed in two ways: as a vertical uncrossing “\( \downarrow \) \( \uparrow \)” or as a horizontal uncrossing “\( \cdots \)”. In addition, we require that paired crossings are uncrossed in the same way. With this additional restriction, the uncrossed diagram \( x \) is mirror symmetric. Thus \( x \) is topologically equivalent to an element \( D(x) \in T_n \) union a number of closed loops.

We define the weight \( \text{wt}(x) \) of an uncrossed embedded graph \( x \in X(\pi) \) as

\[
\text{wt}(x) = 2^{l(x)}(-1)^{\text{uv}(x)+\text{ph}(x)}.
\]

Here \( l(x) \) is the number of closed loops in \( x \), where pairs of mirror symmetric loops are counted only once; \( \text{uv}(x) \) is the number of unpaired vertical uncrossings in \( x \); and \( \text{ph}(x) \) is the number of paired horizontal uncrossings in \( x \).

Now we define \( f_D : F_{2n} \rightarrow \mathbb{Z} \) by

\[
f_D(\pi) = \sum_{x \in X(\pi) \atop D(x) = D} \text{wt}(x).
\]

**Theorem 4.** The function \( f_D \) obtained in this way does not depend on the particular embedding we have picked for each \( \nu(\pi) \).

Theorem 4 is in fact not logically required for the rest of the paper. Its proof is delayed to Section 6.

**Example 5.** For \( n = 2 \) and \( \pi = \{(1, 4), (2, 3)\} \), there are essentially two different embeddings \( A \) and \( B \) of \( \nu(\pi) \), shown in Figure 1. The embeddings are reflections of each other about a horizontal axis. These embeddings have two pairs of mirror-symmetric crossings and two unpaired crossings, so the set \( X(\pi) \) has cardinality 16 in each case. The following table shows the calculation of \( f_D(\pi) \).

| Diagram D | \( f_D(\pi) \) for embedding A | \( f_D(\pi) \) for embedding B |
|----------|-------------------------------|-------------------------------|
| \( \emptyset \) | 1 | 1 |
| \( \{(1, 2)\} \) | \(-1\) | \(1\) |
| \( \{(3, 4)\} \) | \(2 + 2 - 1 - 1 - 2 - 1 = -1\) | \(-1\) |
| \( \{(1, 2), (3, 4)\} \) | \(1 - 2 + 1 + 2 = 2\) | \(1 - 2 + 1 + 2 = 2\) |
| \( \{(2, 3)\} \) | \(1 + 1 - 2 = 0\) | \(1 + 1 - 2 = 0\) |
| \( \{(2, 3), (1, 4)\} \) | \(-1\) | \(-1\) |

Thus for example for embedding \( A \) there are 6 uncrossings \( x \in X(\pi) \) with \( D(x) = \{(3, 4)\} \).

**Example 6.** For \( n = 2 \) the diagram pfaffinants are given in the following table. The diagrams are described by the sets of their vertical edges. The reader can verify that the coefficients of \( a_{14}a_{23} \) agree with the calculations in Example 5.
Figure 1. Two different choices of the embedding $\nu(\pi)$ for $\pi = \{(1, 4), (2, 3)\}$.

| Diagram $D$ | Diagram pfaffinant $\text{Pfaff}_D(A)$ |
|-------------|---------------------------------------|
| $\emptyset$ | $a_{12}a_{34} + a_{14}a_{23} - a_{13}a_{24}$ |
| $\{(1, 2)\}$ | $-a_{14}a_{23} + a_{13}a_{24} - a_{12}a_{34}$ |
| $\{(3, 4)\}$ | $-a_{14}a_{23} + a_{13}a_{24} - a_{12}a_{34}$ |
| $\{(1, 2), (3, 4)\}$ | $a_{12}a_{34} + 2a_{14}a_{23} - a_{13}a_{24}$ |
| $\{(2, 3)\}$ | $0$ |
| $\{(2, 3), (1, 4)\}$ | $a_{13}a_{24} - a_{14}a_{23}$ |

We now state the main property of diagram pfaffinants. Let $I \subseteq [2n]$ and recall that $\bar{I} = [2n] \setminus I$ denotes the complement of $I$ in $[2n]$. The $I$-coloring of $[2n] \cup [2n]'$ is obtained by coloring the elements of $I \cup I'$ black, and the elements $I' \cup \bar{I}$ white. We call a diagram $D \in T_n$ compatible with $I$ (or simply $I$-compatible) if each edge of $D$ has ends of different color in the $I$-coloring. Denote by $D(I) \subset T_n$ the set of $I$-compatible diagrams.

**Theorem 7.** Let $I \subset [2n]$ be a subset of even cardinality. Then

$$ pf_{I, \bar{I}}(A) = \sum_D \text{Pfaff}_D(A) $$

where the sum is over all $I$-compatible diagrams of $T_n$.

The following proof imitates a proof in [LPP].

**Proof.** Let $\pi \in F_{2n}$. Then the monomial $a_\pi$ occurs in $pf_{I, \bar{I}}$ if no edge of $a_\pi$ connects an element of $I$ with an element of $\bar{I}$. In other words, $\pi$ must be the union of the two matchings $\pi_I$ and $\pi_{\bar{I}}$ obtained by restricting the vertex set. The coefficient of $a_\pi$ in $pf_{I, \bar{I}}$ is then equal to $(-1)^{c_n(\pi_I) + c_n(\pi_{\bar{I}})}$.

Now suppose $x \in X(\pi)$ is an uncrossing of $\nu(\pi)$ such that $D(x) \in D(I)$. We direct all the strands and loops in $x$ so that the initial vertex of each strand belongs to $I \cup I'$ (and, thus the end vertex belongs to $\bar{I} \cup I'$). We allow the closed loops to be directed in either direction. Thus the coefficient of $a_\pi$ in $\sum_D \text{Pfaff}_D(A)$ is equal to the sum of $(-1)^{uv(y) + ph(y)}$ over all orientations $y$ of the uncrossings $\{x \in X(\pi) \mid D(x) \in D(I)\}$.

Now we define a sign-reversing partial involution $\iota$ on this set of oriented graphs. A **misaligned uncrossing** is an uncrossing of the form “$\,\,\,\,\,$”, “$\,\,\,\,\,$”, “$\,\,\,\,\,$”, or “$\,\,\,\,\,$”. We say that we **switch** a misaligned uncrossing if we apply one of the following transformations: $\,\,\,\,\,$ or $\,\,\,\,\,$. If $y$ contains any misaligned uncrossings then we let $\iota$ switch the leftmost such uncrossing. If this uncrossing is a paired uncrossing, we also switch its mirror image. If all the uncrossings are aligned, then $\iota$ is not defined. Since $\iota$ is a sign-reversing involution on the set of oriented graphs where it is defined, we need only consider the contribution of
\[ (-1)^{uv(y)} + ph(y) \] for oriented graphs \( y \) where \( \iota \) is undefined. An example of the application of \( \iota \) for \( n = 3 \) and \( I = \{1, 3\} \) is given in Figure 2. We switch the leftmost misaligned uncrossing, which in this case happens to be paired.

Now suppose that \( y_\pi \) is an oriented diagram with only aligned uncrossings (see for example Figure 3). Then converting the uncrossings back into crossings, keeping the orientation the same, we obtain an orientation \( \mu(\pi) \) of \( \nu(\pi) \) such that all edges start in \( I \) end in \( I' \) or start in \( \bar{I} \) and end in \( \bar{I}' \). Thus \( \pi \) is the union of two matchings \( \pi_I \) and \( \pi_{\bar{I}} \). It is also clear that one can recover \( y_\pi \) from \( \mu(\pi) \) and that \( \mu(\pi) \) is completely determined by \( \nu(\pi) \). Thus \( y_\pi \), if it exists, is unique.

Finally, we calculate the sign of \( y_\pi \). Each unpaired crossing of \( \nu(\pi) \) corresponds to the intersection of \( (i, j') \) with \( (i', j) \) for an edge \( (i, j) \) in \( \pi_I \) or \( \pi_{\bar{I}} \). These crossings are always uncrossed horizontally to obtain \( y_\pi \), and so contributes no sign to \( y_\pi \). Each paired crossing \( (c, c') \) in \( \nu(\pi) \) arises from a crossing \( \xi \) of \( \pi \). To obtain \( y_\pi \), the pair \( (c, c') \) is uncrossed horizontally if \( \xi \) is a crossing in \( \pi_I \) or \( \pi_{\bar{I}} \), and \( (c, c') \) is uncrossed vertically otherwise. Thus \( (-1)^{uv(y_\pi)} + ph(y_\pi) = (-1)^{cn(\pi_I) + cn(\pi_{\bar{I}})} \), and we have checked that the monomial \( a_\pi \) appears in both sides with the same coefficient.

\[ \square \]
2.5. Temperley-Lieb pfaffinants. Let $D \in \mathcal{T}_n$. For $i, j \in \{1, \ldots, 2n\}$ satisfying $i < j$ we call the edge $(i, j)$ of $D$ odd if $i$ is odd and even otherwise. For $D \in \mathcal{T}_n$ let $S(D)$ be the set of all diagrams in $\mathcal{T}_n$ that can be obtained from $D$ by erasing several odd edges (and their mirror images) and matching the resulting unmatched vertices by horizontal edges of the form $(i, i')$. In particular, $D \in S(D)$.

Lemma 8. If $D_1, D_2 \in \mathcal{T}_n$ and $D_1 \in S(D_2)$ then $S(D_1) \subset S(D_2)$. The size of $S(D)$ is a power of 2.

Proof. The first statement is clear since after obtaining $D_1$ out of $D_2$ by removing several odd edges, we can keep removing the remaining odd edges, and the result belongs to $S(D_2)$ by definition. For the second part, note that if $(i, j)$ is an odd edge, that is if $i$ is odd, then all the edges inside $[i, j]$ cannot be removed either because they are even or because they are contained within the segment bounded by ends of an even edge. Thus all odd edges that can be removed can be removed independently one from another, which implies the statement of the lemma. □

Lemma 9. Suppose $D \in \mathcal{T}_n$ and $I \subset [2n]$ is a subset of even cardinality. If $D \in D(I)$ then $D' \in D(I)$ for every $D' \in S(D)$. Conversely, if $D' \in D(I)$ then there exists a unique $D_{\text{max}} \in \mathcal{T}_n \cap D(I)$ such that $D' \in S(D_{\text{max}})$ and $D_{\text{max}}$ is maximal in the following sense: if $D' \in S(D)$ for some other $D \in D(I)$ then $S(D) \subset S(D_{\text{max}})$.

Proof. The first statement follows immediately from the definitions of the set $S(D)$ and of $I$-compatibility. Now let $D' \in D(I)$. We say that a vertex $i \in [2n]$ is free if $(i, i')$ is an edge in $D$. It is clear that there are the same number of black and white vertices in the $I$-coloring amongst the non-free vertices. Also, one checks that the free vertices alternate in parity beginning with an odd vertex and ending with an even vertex. If there are two vertices $i < j$ such that between $i$ and $j$ there are no free vertices, $i$ is odd, $j$ is even and they have different colors then we call the pair $(i, j) \in [2n] \times [2n]$ addable. Removing $(i, i')$ and $(j, j')$ from $D'$ and adding $(i, j)$ and $(i', j')$ gives some $D \in \mathcal{T}_n \cap D(I)$ such that $D' \in S(D)$. The unique maximal such $D = D_{\text{max}}$ is obtained by performing the above operation for every pair of addable vertices. Since $I$ is required to have even cardinality and all the free vertices of $D_{\text{max}}$ has the same color, $D_{\text{max}}$ must be even. □

We say that $D \in \mathcal{T}_n^c$ is $I$-maximal if it has the form $D_{\text{max}}$ as in Lemma 9. We denote the set of $I$-maximal diagrams by $D_{\text{max}}(I)$. By Lemma 8 if $D_1, D_2 \in D_{\text{max}}(I)$ then $D_1 \notin S(D_2)$ and $D_2 \notin S(D_1)$.

Definition 10. Let $D \in \mathcal{T}_n^c$. Define the TL-pfaffinant $\text{Pfaf}_D(A)$ by

$$\text{Pfaf}_D(A) = \sum_{D' \in S(D)} \text{Pfaf}_{D'}(A).$$

Example 11. For $n = 2$ the TL-pfaffinants are given in the following table, calculated using Example 6. The even diagrams are described by the sets of their vertical edges.

| Even diagram $D$ | TL-pfaffinant $\text{Pfaf}_D(A)$ |
|------------------|-----------------------------------|
| $\emptyset$     | $a_{12}a_{34} + a_{14}a_{23} - a_{13}a_{24}$ |
| $(1, 2), (3, 4)$| $a_{13}a_{24} - a_{12}a_{34}$          |
| $(2, 3), (1, 4)$| $a_{13}a_{24} - a_{14}a_{23}$          |
Example 12. Let $I = [2n]$ and let $D \in \mathcal{T}_n^e$ be the even symmetric TL-diagram with all edges horizontal. Then $\text{pf}_{I, i}(A) = \text{Pfaf}_D(A)$.

Theorem 13. Suppose $I \subset [2n]$ is a subset with even cardinality. Then
$$\text{pf}_{I, i}(A) = \sum_{D \in \mathcal{D}_{\text{max}}(I)} \text{Pfaf}_D(A).$$

Proof. By Theorem 7, it suffices to show that the set of $I$-compatible diagrams $D(I) \subset \mathcal{T}_n$ is the disjoint union of the sets $S(D)$ for $D \in \mathcal{D}_{\text{max}}(I)$. This follows from Lemmas 8 and 9. \qed

Suppose $D \in \mathcal{T}_n$ is a (possibly odd) symmetric TL-diagram on $4n$ vertices. We define a subset $I(D) \subset [2n]$ by
$$I(D) = \{ i \in [2n] \mid (i < j) \in D \text{ or } (i, i') \in D \}.$$ Note that $|I(D)| = 2n - |D|$, so that $I(D)$ has even cardinality whenever $D \in \mathcal{T}_n^e$. Recall from before Theorem 1 the definition of a standard partition of $[2n]$.

Lemma 14. The map $D \mapsto (I(D), \overline{I(D)})$ is a bijection with image equal to the set of standard partitions of $[2n]$ with at most 2 parts.

Proof. We describe how to recover $D$ from $I(D)$. Let $\overline{I(D)} = \{ j_1 < j_2 < \cdots < j_k \}$. Then it must be the case that $(j_1 - 1, j_1) \in D$. More generally suppose we know all the edges of $D$ connected to $\{ j_1, j_2, \ldots, j_{l-1} \}$ for some $l \leq k$. Then $(i, j_i)$ is an edge of $D$, where $i \in I(D)$ is the maximum number in $I(D)$ which is less than $j_i$ and which is not connected to $\{ j_1, j_2, \ldots, j_{l-1} \}$. Furthermore, it is clear that this algorithmic definition of the inverse map $(I, \bar{I}) \mapsto D$ terminates successfully if and only if $(I, \bar{I})$ is a standard partitioning. \qed

Corollary 15. The dimension of $P_n$ is $\binom{2n-1}{n}$.

Proof. This is an immediate corollary of Theorem 1, Proposition 3 and Lemma 14. \qed

Let $I, J \subset [2n]$ be two subsets of the same cardinality. We say $I = \{ i_1 < \cdots < i_k \}$ is lexicographically smaller than $J = \{ j_1 < \cdots < j_k \}$ and write $I \prec_{\text{lex}} J$ if for some $1 \leq l \leq k$ we have $i_1 = j_1, i_2 = j_2, \ldots, i_{l-1} = j_{l-1}, i_l < j_l$. We now define a total order $\prec$ on subsets of $[2n]$. Suppose $I, J \subset [2n]$. We define $I \prec J$ if $|I| > |J|$ or $|I| = |J|$ and $I \prec_{\text{lex}} J$. We use the map $D \mapsto I(D)$ to give an induced total order on $\mathcal{T}_n$: we have $D \prec D'$ if $I(D) \prec I(D')$.

Figure 4. The order $\prec_{\text{lex}}$ on $\mathcal{T}_2$.

Lemma 16. Let $D, D' \in \mathcal{T}_n$. If $D$ is $I(D')$ compatible then $D \prec D'$.
Proof. Suppose $D$ is $I(D')$ compatible. Then $I(D')$ must have at least $|D| = |I(D)|$ elements, so we have $|I(D')| \leq |I(D)|$. Thus we may suppose $|D| = |D'| = k$. Let $\{(i_1 < j_1), \ldots, (i_k < j_k)\}$ be the vertical edges of $D$ (on the left side) and suppose that $j_1 < j_2 < \cdots < j_k$. If $D$ is $I(D')$-compatible then $I(D') \cap (i_l, j_l) = 1$ for each $l \in [1, k]$, so we must have $I(D') \prec_{\text{lex}} I(D)$. This in turn implies that $I(D) \prec_{\text{lex}} I(D')$, so $D \prec D'$.

Example 17. For $n = 2$ we get $\{1, 2, 3, 4\} \prec_{\text{lex}} \{1, 2, 3\} \prec_{\text{lex}} \{1, 2, 4\} \prec_{\text{lex}} \{1, 3, 4\} \prec_{\text{lex}} \{1, 2\} \prec_{\text{lex}} \{1, 3\}$ which gives us the order on $T_2$ as shown in Figure 4.

Proposition 18. The transition matrix (given by Theorem 13) from the set $\{\text{pf}_{I, \bar{I}} \mid (I, \bar{I}) \text{ is standard}\}$ of standard complementary pfaffians to the set $\{\text{Pfaf}_D(A) \mid D \in T_n^+\}$ is upper triangular with 1’s along the diagonal, under the order $\prec$.

Proof. Clearly $D \in D_{\text{max}}(I(D))$ so the matrix of the Proposition has 1’s along the diagonal. Since $D_{\text{max}}(I) \subset D(I)$, by Theorem 7 the coefficient of $\text{Pfaf}_D$ in $\text{pf}_{I, \bar{I}}$ is non-zero if and only if $D$ is $I$-compatible. By Lemma 16 $D$ is $I(D')$ compatible only if $D \prec D'$, giving the upper triangularity.

Example 19. For $n = 2$ one obtains the transition matrix

\[
\begin{pmatrix}
1 & 1 & 1 \\
0 & 1 & 1 \\
0 & 0 & 1
\end{pmatrix}
\]

We have labeled the rows by the standard complementary pfaffians $(I, \bar{I}) = ((1, 2), (3, 4))$, $(\emptyset, (1, 2), (3, 4))$ and $(\emptyset, (1, 2), (3, 4))$ from top to bottom and we label the columns by the symmetric even TL-diagrams with vertical edges $\{(1, 2), (3, 4), (1, 4), (2, 3), \emptyset\}$ from left to right.

Theorem 20. The TL-pfaffinants $\{\text{Pfaf}_D(A) \mid D \in T_n^+\}$ form a basis for $P_n$.

Proof. This follows from Theorem 1 and Proposition 18.

We will obtain another proof of Theorem 20 in Section 3.3.

Problem 21. Do the diagram pfaffinants $\{\text{Pfaf}_D(A) \mid D \in T_n\}$ always lie in $P_n$? If so, how are they expressed in the basis of TL-pfaffinants and in the basis of standard complementary pfaffians?

By Examples 6 and 11 the answer to the first question is affirmative for $n = 2$. Note also that by Proposition 8 the number of diagram pfaffinants is twice larger than the dimension of $P_n$, so if the diagram pfaffinants $\{\text{Pfaf}_D(A)\}$ do lie in $P_n$ there must be non-trivial relations among them.

3. Pfaffians and non-intersecting paths in networks

3.1. Stembridge’s network interpretation of Pfaffians. John Stembridge in [Ste90] introduced an interpretation of pfaffians in terms of networks. Let $G = (V, E)$ be a finite acyclic directed graph. We say that two directed paths in $G$ intersect if they have a common vertex. If $W$ and $U$ are ordered sets of vertices of $G$, we say that $W$ is $G$-compatible with $U$ if whenever $u < u'$ in $W$ and $v > v'$ in $U$, every path from $u$ to $v$ intersects every path from $u'$ to $v'$.
Let us suppose that a weight function $w : E \to R$, where $R$ is some ring, has been fixed. For a $G$-path $p$, let $w(p) = \prod_{e \in p} w(e)$ where the product is taken over all edges in $p$. For $u \in V$, $W \subset V$ let $P(u, W)$ denote the set of $G$-paths from $u$ to any $v \in I$, and let $Q(u, W)$ be the associated weight function $Q(u, W) = \sum_{p \in P(u, W)} w(p)$. Similarly, for an $r$-tuple $u = (u_1, \ldots, u_r)$ let $P(u, W)$ denote the set of $r$-tuples of paths $(p_1, \ldots, p_r)$ such that $p_i \in P(u_i, W)$. The weight $w(p_1, \ldots, p_r)$ of a $r$-tuple of paths is the product of the weights of each of the paths. Let $P_0(u, W) \subset P(u, W)$ denote the subset of non-intersecting tuples of paths. We define $Q(u, W) = Q_0(u, W)$ to be the sum of the weights of the elements of $P_0(u, W)$.

**Theorem 22** ([Ste90, Theorem 3.1]). Let $u = (u_1, \ldots, u_r)$ be an $r$-tuple of vertices in an acyclic digraph $G$, and assume that $r$ is even. If $W \subset V$ is an ordered subset of vertices such that $u$ is $G$-compatible with $W$, then

$$Q(u, W) = pf ((Q((u_i, u_j), W)]_{1 \leq i < j \leq r}).$$

For convenience, if $G$ is an acyclic directed graph and ordered vertex sets $u = (u_1, \ldots, u_{2n}) \subset V$ and $W \subset V$ have been chosen we call the triple $N = (G, u, I)$ a network. For a network $N$, we define $P(N) = P(u, W)$ and $P_0(N) = P_0(u, W)$. We also let $Q(N)$ denote the weight sum $Q(u, W)$, and let $A(N) = A(G, u, I)$ denote the array $(a_{ij} = Q((u_i, u_j), W)]_{1 \leq i < j \leq r}$.

If $I \subset [2n]$, we let $u_I = \{u_{ij}\}_{1 \leq i \leq n}$ denote the corresponding set of vertices. Then we set $P_I(N) \subset P(N)$ to be the subset of paths $p = (p_1, \ldots, p_{2n})$ such that $p_i$ and $p_j$ do not intersect if both $i, j \in I$ or both $i, j \in I^c$. We call the paths $p \in P_I(N)$ compatible with $I$. Then $P_0(N) = P_0(N) = P_{[2n]}(N)$. We finally define $Q_I(N)$ to be the sum of the weights of the paths in $P_I(N)$.

The following statement is immediate from Theorem 22 and the definitions we have made.

**Corollary 23.** Let $N = (G, u, W)$ be a $G$-compatible network and $I \subset [2n]$ be of even cardinality. Then

$$Q_I(N) = pf_{I, I}(A(N)).$$

### 3.2. Planar network definition of Pfaffians

Let $N = (G, u, W)$ be a fixed network. We assume that $G$ is planar and that a Jordan curve $C$ passes through the sets $u$ and $W$ of vertices so that $G$ is contained completely in the interior of $C$. We also assume that $u$ and $W$ are contained in disjoint segments of $C$ so that the ordering of $u$ and $W$ is consistent with the arrangement of these vertices on $C$. With this assumption, the $G$-compatibility of $u$ and $W$ is immediate. For short we will call a network $N$ satisfying these assumptions a planar network.

Suppose that $p = (p_1, p_2, \ldots, p_{2n}) \in P(N)$ is a family of paths such that no three paths in $p$ intersect at the same vertex. Removing all the edges of $N$ that do not lie on any of the paths $p_i \in p$, and in addition marking all the edges of $N$ used twice by $p$ we obtain a marked network $\tilde{N} = \tilde{N}(p)$. Note that by our assumption an edge of $N$ can be used at most twice by the path family $p$. We say that $p$ covers $\tilde{N}$ and denote the set of coverings of $\tilde{N}$ by $P(\tilde{N})$. If $\tilde{N}$ is the marked network obtained from some $p \in P(N)$ we call $\tilde{N}$ a marked subnetwork of $N$ and write $\tilde{N} \ll N$. The weight $w(\tilde{N})$ of a marked subnetwork is the weight $w(p)$ for any path family covering $\tilde{N}$.
Suppose \( p_i \) and \( p_j \) intersect at some vertex \( v \). Then there are two (possibly not distinct) edges \( e_i \in p_i, e_j \in p_j \) entering \( v \) and two edges \( f_i \in p_i \) and \( f_j \in p_j \) leaving \( v \). The \textit{vertical uncrossing} of \( v \) is obtained by detaching \( v \) into two new vertices \( v_e \) and \( v_f \) so that \( v_e \) is incident with \( e_i \) and \( e_j \) while \( v_f \) is incident with \( f_i \) and \( f_j \), as it is illustrated on Figure 5. Alternatively, if the vertices \( u \) are arranged on the left, the vertices \( W \) arranged on the right, and all edges are directed strictly from left to right, then the vertical uncrossings always look like “\( \bigcup \) ”.

Define an undirected graph \( \Theta(\widetilde{N}) \) by vertically uncrossing every intersection point of \( \widetilde{N} \), removing all the marked edges and ignoring all the orientations. Note that \( \Theta(\widetilde{N}) \) does not depend on \( p \), only on \( \widetilde{N} \). The graph \( \Theta(\widetilde{N}) \) is a disjoint union of a number of cycles, together with a number of paths. We define the multiplicity of the marked network \( \widetilde{N} \) by \( \text{mult}(\widetilde{N}) = 2^r \) where \( r \) is equal to the number of connected components of \( \Theta(\widetilde{N}) \) which do not contain any of the vertices in \( u \).

The components of \( \Theta(\widetilde{N}) \) containing one or more of the vertices of \( u \) are a collection of paths which give rise to a matching \( \text{type}(\widetilde{N}) \) of \( [2n] \cup [2n]' \): if \( u_i, u_j \) belong to the same component of \( \Theta(\widetilde{N}) \) then \((i, j), (i', j') \in \text{type}(\widetilde{N})\). If \( u_i \) does not belong in any component with some other \( u_j \), then \((i, i') \in \text{type}(\widetilde{N})\).

\begin{lemma}
Let \( p \in P(N) \) be a family of paths such that no three paths in \( p \) intersect at the same vertex and let \( \widetilde{N} = \widetilde{N}(p) \). Then type(\widetilde{N}) \in T_n.
\end{lemma}

\begin{proof}
We need to check that if \((i, j) \in \text{type}(\widetilde{N})\) and \( i < k < j \) then \((k, l) \in \text{type}(\widetilde{N})\) for some \( i < l < j \). The components of \( \Theta(\widetilde{N}) \) are simple curves in the interior of the Jordan curve \( C \) connecting two points on the boundary of \( C \). The assumption that \( u \) is arranged in order along the boundary of \( C \) immediately implies the required criterion.
\end{proof}

The definition of \( \Theta(\widetilde{N}) \) does not rely on the assumption that the graph is drawn inside a Jordan curve, but Lemma 24 does.

\begin{lemma}
Let \( N \) be a planar network and \( \widetilde{N} \ll N \) a marked subnetwork. Suppose \( I \subset [2n] \). Then the number of path families which cover \( \widetilde{N} \) and are \( I \)-compatible is given by
\[
|P(\widetilde{N}) \cap P_I(N)| = \begin{cases} 
\text{mult}(\widetilde{N}) & \text{if type}(\widetilde{N}) \in \mathcal{D}(I), \\
0 & \text{otherwise}.
\end{cases}
\]
In particular, \( |P(\widetilde{N}) \cap P_I(N)| \) only depends on whether there is some \( I \)-compatible path family covering \( \widetilde{N} \).
\end{lemma}

\begin{proof}
For each \( p \in P(\widetilde{N}) \) we orient \( \Theta(\widetilde{N}) \) in the following manner. If an edge \( e \in \Theta(\widetilde{N}) \) belongs to \( p_i \) where \( i \in I \) we orient \( e \) with the same orientation as in \( N \), that is, from \( u \) to \( W \). If an edge \( e \in \Theta(\widetilde{N}) \) belongs to \( p_j \) where \( j \in \bar{I} \) we orient \( e \)
with the opposite orientation to the one in \( N \). Since we removed all the marked edges when we produced \( \Theta(\tilde{N}) \) no edge \( e \in \Theta(\tilde{N}) \) receives both orientations.

The resulting directed graph \( \Theta(\tilde{N})_p \) is a disjoint union of directed paths and directed cycles. This follows from the fact that every intersection of \( \tilde{N} \) involves a pair of paths \((p_i, p_j)\) where \( i \in I \) and \( j \in \bar{I} \). One now checks that \( p \mapsto \Theta(\tilde{N})_p \) is a bijection between path families in \( p \in P(\tilde{N}) \) and such directed graphs.

In addition, \( p \in P_I(\tilde{N}) \) if and only if the directed path in \( \Theta(\tilde{N})_p \) that \( u_i \) lies on is directed away from \( u_i \) if \( i \in I \) and directed towards \( u_i \) if \( i \in \bar{I} \). This requirement can be satisfied only if \( \text{type}(\tilde{N}) \in D(I) \). The number of orientations of \( \Theta(\tilde{N}) \) satisfying this additional condition is by definition equal to \( \text{mult}(\tilde{N}) \).

For \( D \in T_n \) define the following function \( \hat{\text{Pfaf}}_D : \{\text{planar networks}\} \to \mathbb{R} \) on planar networks:

\[
\hat{\text{Pfaf}}_D(N) = \sum_{\tilde{N} \ll N, \text{type}(\tilde{N}) = D} \text{mult}(\tilde{N})w(\tilde{N}).
\]

**Proposition 26.** Let \( I \subset [2n] \) be of even cardinality and \( N \) be a planar network. Then

\[
\text{pf}_{I,\bar{I}}(A(N)) = \sum_{D \in D(I)} \hat{\text{Pfaf}}'_D(N).
\]

**Proof.** By Corollary \[23\] \( \text{pf}_{I,\bar{I}}(A(N)) \) is the sum of the weights of the \( I \)-compatible families of paths \( P_I(N) \). Thus

\[
\text{pf}_{I,\bar{I}}(A(N)) = \sum_{p \in P_I(N)} w(p)
= \sum_{\tilde{N} \ll N} \left( \sum_{p \in P_I(N) \cap P(\tilde{N})} w(p) \right)
= \sum_{\tilde{N} \ll N, \text{type}(\tilde{N}) \in D(I)} \text{mult}(\tilde{N})w(\tilde{N}) \quad \text{by Lemma } [25]
= \sum_{D \in D(I)} \hat{\text{Pfaf}}'_D(N).
\]

\( \square \)

Now for \( D \in T^e_n \) define \( \text{Pfaf}_D : \{\text{planar networks}\} \to \mathbb{R} \) by

\[
\text{Pfaf}_D(N) = \sum_{D' \in S(D)} \hat{\text{Pfaf}}'_{D'}(N).
\]

**Theorem 27.** Let \( D \in T^e_n \) and \( N \) be a planar network. Then

\[
\text{Pfaf}_D(A(N)) = \text{Pfaf}_D(N).
\]

**Proof.** The proof of Theorem \[13\] and Proposition \[26\] shows that

\[
\text{pf}_{I,\bar{I}}(A(N)) = \sum_{D \in D(I)} \hat{\text{Pfaf}}_D(N).
\]

\( \square \)
Using the statement and the proof of Proposition 18 we see that $P_{\text{pf} D}(A(N))$ and $P_{\text{pf} D}(A(N))$ can be expressed in terms of $\text{pf}_{I, I}(A(N))$ in an identical way so we conclude that $P_{\text{pf} D}(A(N)) = P_{\text{pf} D}(N)$. □

Remark 28. Observe that functions $P_{\text{pf} D}(N)$ do not coincide with the evaluations $P_{\text{pf} D}(A(N))$ of diagram pfaffinants. In particular the diagram pfaffinants $P_{\text{pf} D}(A)$ might take negative values when evaluated at $A(N)$ for a planar network $N$.

3.3. Independence of Temperley-Lieb pfaffinants. We will show directly using Theorem 27 that the elements $\{P_{\text{pf} D}(A) | D \in T_n\}$ are linearly independent. This will give us alternative proofs of Theorems 1 and 20.

Let $D \in T_n$. We will now define a planar network $N(D)$ with the property that $P_{\text{pf} D}(N(D))$ is non-zero if and only if $D = D'$. The network $N(D)$ is embedded into the plane $\mathbb{R}^2$ in a particular way. First, place the vertices $u_1, \ldots, u_{2n}$ on the line $x = 0$ so that $u_i$ has coordinates $(0, 2n - i)$. For an edge $(i < j) \in D$ we call the vertex $i$ outgoing and the vertex $j$ ingoing. The vertices $i$ such that $(i, i') \in D$ are neither outgoing nor ingoing. Now place the “sink” vertices $W$ as follows: for each $i \in [2n]$ such that $(i, i') \in D$ or $(i < j) \in D$ we place $w_i \in W$ at coordinates $(1, 2n - i)$. To obtain the rest of $N(D)$, we first join $u_i$ with $w_i$ with a straight line whenever $w_i$ exists, that is when $i$ is not ingoing. Finally we join $u_j$ with $w_{i_k}$ where $j_1 < j_2 < \cdots$ are the ingoing vertices and $i_1 < i_2 < \cdots$ are the outgoing vertices. The intersection of any of these lines is also defined to be a vertex of $N(D)$ which does not belong to either $u$ or to $W$. All edges are directed so that the $x$-coordinate increases along the edges.

Note that no three of the drawn lines intersect at one point, since by construction the set of these lines is a union of two pairwise non-intersecting families of lines. An example of this construction of $N(D)$ is shown in Figure 6.

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Note that no three of the drawn lines intersect at one point, since by construction the set of these lines is a union of two pairwise non-intersecting families of lines. An example of this construction of $N(D)$ is shown in Figure 6.

![Figure 6. A symmetric TL-diagram D and the corresponding network N(D)](image)

Say that an edge $(i, j) \in D$ is on the outside if one cannot find $(k, l) \in D$ so that $1 \leq k < i < j < l \leq 2n$. The network $N(D)$ is the union of the networks $N(D_{[i,j]})$ for outside edges $(i, j)$ together with the networks $N(D_i)$ for horizontal edges $(i, i')$. Here $D_{[i,j]}$ denotes the obvious restriction of a diagram $D$ to the set
of vertices \([i, j] \cup [i', j'] \subset [2n] \cup [2n]'\). Let \(\tilde{N}(D) \ll N(D)\) denote the marked subnetwork consisting of all edges of \(N(D)\), with no edges marked.

**Lemma 29.** We have \(\text{type}(\tilde{N}(D)) = D\). Let \(p \in P(N)\) be a family of paths such that no three paths intersect at the same vertex. Then \(\tilde{N}(p) = N(D)\).

**Proof.** By the previous comments, it is enough to prove the lemma for each of the networks \(N(D_{[i,j]})\) corresponding to outside edges \((i, j) \in D\). We proceed by induction on \(|j - i|\), the base case being trivial. All vertices in \([i, j]\) are outgoing or ingoing, and there are twice as many source vertices \(u\) as sink vertices \(W\) in \(N(D_{[i,j]})\). Call the edges of \(N(D_{[i,j]})\) incident to the sink vertices the outer skeleton \(\text{Sk}(N(D_{[i,j]}))\).

Now remove the outer skeleton from \(N(D_{[i,j]})\). We obtain a network \(N(D_{[i,j]})'\) isomorphic to \(N(D_{[i+1,j-1]})\), which is the union of the networks \(N(D_{[p,j]})\), where \(\{(i, j)\}\) is the set of outside edges formed when we remove edge \((i, j)\) from \(D_{[i,j]}\). Under this identification, the sink vertices of \(N(D_{[i,j]})'\) are the intersection points of the pairs of segments \([u_{jk}, w_{ik}], [u_{ik+1}, w_{ik+1}]\). By the inductive assumption, we have \(\text{type}(\tilde{N}(D_{[i+1,j-1]})) = D_{[i+1,j-1]}\) and since \(\text{Sk}(N(D_{[i,j]}))\) (after redirecting the edges) is a path from \(u_i\) to \(u_j\), it follows immediately that \(\text{type}(\tilde{N}(D_{[i,j]})) = D_{[i,j]}\).

By the inductive assumption applied to each \(N(D_{[p,j]})\), there is only one marked network of \(N(D_{[i,j]})'\) arising from a family of paths \(p \in P(N)\) without triple intersections. Each of the sink vertices of \(N(D_{[i,j]})'\) has incoming degree 2, and thus \(p\) must cover (counted with multiplicity) two of the outgoing edges from each such vertex. However, \(p\) must contain the two paths consisting of the single edge \((u_i, w_i)\) and the single edge \((u_{jk}, w_{ik})\), where \(j = j\). A simple counting argument shows that each sink vertex \(w_{ik}\) is incident with exactly two paths. Combining these facts, one concludes that each edge of \(\text{Sk}(N(D_{[i,j]}))\) is covered by \(p\) exactly once. \(\square\)

An illustration of the proof is shown in Figure 7.

![Figure 7](image-url)

**Figure 7.** A symmetric TL-diagram \(D\), with corresponding network \(N(D)\) and outer skeleton \(\text{Sk}(N(D))\) shown in bold.

**Theorem 30.** The elements \(\{\text{Pfaf}_D(A) \mid D \in T_n^*\} \subset P_n\) are linearly independent.

**Proof.** Suppose there is a non-trivial linear combination \(c = \sum_{D \in T_n^*} c_D \text{Pfaf}_D(A)\) of the \(\text{Pfaf}_D(A)\)-s which evaluates to 0 for any upper triangular array \(A\). Then in
particular it should evaluate to 0 on $A(N)$ for a planar network $N$. Let $D \in \mathcal{T}_n^\times$ be such that $\text{Pfaf}_D(A)$ enters the expression with a non-zero coefficient $c_D$, and $|D|$ is the largest possible satisfying this condition. Then by Lemma 29, $\text{Pfaf}_D'(N(D))$ contributes a non-zero value to $\text{Pfaf}_D(A(N(D)))$ but we have $\text{Pfaf}_D'(N(D)) = 0$ for all other $D' \neq D$ such that $D' \in S(D)$. However, by the choice of $D$ we have $\text{Pfaf}_D'(A(N(D))) = 0$ for all other $D'$ such that $c_{D'} \neq 0$. We conclude $c_D = 0$, obtaining a contradiction. 

Theorem 30 gives alternative proofs of Theorems 1 and 20 without relying on results of [DP].

3.4. Network positivity. Call a skew symmetric matrix $A$ network-positive if it is equal to $A(N)$ for some planar network $N$ with positive weights on edges (we assume the coefficient ring $R = \mathbb{R}$).

The notion of network positivity is a substitute for the notion of total non-negativity of matrices. Recall that an arbitrary matrix $M$ is totally non-negative if all its minors are non-negative. It is known (see for example [Br, Theorem 3.1]) that every totally non-negative matrix arises from a planar network.

It is not clear how to make a similar definition for skew-symmetric matrices. The following example is taken from [Kim]. Take the following skew-symmetric matrix:

$$A = \begin{pmatrix}
0 & 1 & 0 & 0 \\
-1 & 0 & 1 & 0 \\
0 & -1 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}.$$ 

Every skew-symmetric submatrix of $A$ of even size has a non-negative pfaffian. However, as we will now show, $A$ is not equal to $A(N)$ for any planar network $N$. Thus the naive generalization does not seem to be appropriate.

**Lemma 31.** $A$ is not equal to $A(N)$ for any positive planar network $N$.

**Proof.** Indeed, assume $u$ and $W$ are placed on the boundary of a Jordan curve. Since $a_{23} \neq 0$ there should be a pair of non-intersecting paths $p_2$ and $p_3$ from $u_2$ and $u_3$ to $W$ (see Figure 8).

![Figure 8](image)

**Figure 8.** It is impossible to have $a_{12}, a_{23} > 0$ while $a_{13} = 0$ in $A(N)$ for a non-negative planar network $N$.

Since $a_{12} \neq 0$ there should be at least one path $p_1$ from $u_1$ to $W$. Since $a_{13} = 0$, the path $p_1$ must intersect $p_3$, and therefore $p_2$. However, in that case if we traverse
Suppose Lemma 34. cone. Then so does $S$ set $f$ to elements $D$ Lemma says that to find the edge generators of $C$ is maximal if no odd edges can be added to it. By Lemma 9, $I$ the subset $D$ of $C$ possessing some interesting polyhedral geometry and the edge generators of $C$ are rather tricky to describe. Finding generators of the semigroup $C_n \cap \mathbb{Z}[p_{f_{I,J}} \mid I \subset [2n]]$ of integral points is even trickier. Note that by Theorem 13 and Proposition 18, the $Z$-span of $\{Pf_D(A) \mid D \in T^c_n\}$ is equal to the $Z$-span of $\{p_{f_{I,J}} \mid I \subset [2n]\}$.

The description of the edge generators of $C_n$ can be simplified to a combinatorial problem concerning boolean lattices.

Let us call an even symmetric diagram $D \in T^c_n$ maximal if it is $I$-maximal for the subset $I = I_{alt} = \{1, 3, 5, \ldots, 2n - 1\}$. Since $D(I_{alt}) = T^c_n$, a diagram $D \in T^c_n$ is maximal if no odd edges can be added to it. By Lemma 9, $D \in T^c_n$ is maximal if and only if for every $D'$ so that $D \in S(D')$ we have $D = D'$. The following Lemma says that to find the edge generators of $C_n$ we may restrict our attention to elements $f \in P_n$ which are linear combinations of TL-pfaffinants labeled by a set $S(D_m) \cap T^c_n$ for maximal $D_m$.

**Lemma 34.** Suppose $f = \sum_{D \in T^c_n} c_D Pf_D(A) \in C_n$ lies in the network positive cone. Then so does $f_{D_m} = \sum_{D \in S(D_m)} c_D Pf_D(A)$ for each maximal diagram $D_m \in T^c_n$. 

**3.5. Network positivity and pfaffinants.**

**Proposition 32.** For a network-positive $A$ and any $D \in T^c_n$ we have $Pf_D(A) \geq 0$.

**Proof.** We know from Theorem 27 that $Pf_D(A)$ has an interpretation as the weight-multiplicity generating function of certain marked subnetworks of $N$. The statement follows immediately.

For any $K \in P_n$ one can formally write $K$ as a linear combination of the symbols $Pf'_D$. Namely, by Theorem 20 one can express $K = \sum c_D Pf'_D$ in terms of TL-pfaffinants. Now we use the expansions $Pf_D(A) = \sum_{D' \in S(D)} Pf'_D(A)$ to obtain the needed formal presentation $K = \sum c'_D Pf'_D$.

**Theorem 33** (cf. Corollary 3.6, [RS05a]). Let $K \in (P_n)_\mathbb{R}$. The following are equivalent:

1. for any network-positive $A$ one has $K(A) \geq 0$;
2. The coefficients $c'_D$ in $K = \sum_{D \in T_n} c'_D Pf'_D$ are non-negative.

We call an element $f \in P_n$ network positive if it satisfies one of the conditions (and thus both) of Theorem 33.

**Proof.** By Theorem 27, $K(A(N)) = \sum c'_D Pf'_D(N)$ and each Pfaf'_D(N) by definition enumerates sums of weights of certain marked subnetworks of $N$; one direction is obvious. It was shown in Lemma 29 that the networks $N(D)$ possess the property that Pfaf'_{D'}(N(D)) \neq 0 if and only if $D' = D$. This implies the other direction.
Proof. Suppose when expressed in terms of diagram pfaffinants as in Theorem 33 we have $f = \sum_{D' \in \mathcal{T}_n} c_{D'} \operatorname{Pf}_{D'}(A)$ where $c_{D'} \geq 0$ is given by

$$\sum_{\substack{D \in \mathcal{T}_n \setminus D' \in S(D)}} c_{D}.$$ 

Suppose $D_m$ is maximal and $D \in S(D_m)$. By Lemma 9 the summation in (2) can be taken over $D \in (\mathcal{T}_n^c \cap S(D_m))$ satisfying $D' \in S(D)$ instead. Also using Lemma 8, this shows that $f_{D_m}$ lies in $C_n$. \(\square\)

Now let $D_m$ be maximal. By the proof of Lemma 8 the diagrams $D' \in S(D_m)$ form a boolean lattice $B_s = 2^{[s]}$ under the order $D_1 < D_2 \iff D_1 \in S(D_2)$. When $s$ is even, the even diagrams $S(D_m) \cap \mathcal{T}_n^e$ correspond to the even levels $B^e_s$ in $B_s$. When $s$ is odd, the even diagrams $S(D_m) \cap \mathcal{T}_n^e$ correspond to the odd levels $B^o_s$ in $B_s$. The edge generators of $C_n$ can then be calculated by solving the following problem.

**Problem 35.** Let $B_s$ be a boolean lattice and $\alpha \in \{o, e\}$. What is the cone of sequences $\{a_S\}_{S \in \mathcal{B}_s^\alpha}$ of real numbers, indexed by either the odd or the even subsets of $\{1, 2, \ldots, s\}$, which satisfies the condition

$$\sum_{S' \subset S} a_{S'} > 0$$

for every $S' \in B_s$?

**Example 36.** Assign to each element of the lattice $B_3 = 2^{\{a,b,c\}}$ one of the formal variables $\emptyset, a, b, c, ab, ac, bc, abc$. Also, define $a = a + \emptyset$, $b = b + \emptyset$, $c = c + \emptyset$, $\overline{abc} = \overline{abc} = \overline{abc} = \overline{abc}$. We want to characterize the cone of $(t_1, t_2, t_3, t_4) \in \mathbb{R}^4$ such that $t_1 abc + t_2 a + t_3 b + t_4 c$ is non-negative in terms of the original eight formal variables. It turns out that the edges of the this cone are generated by the set $V_3$ of vectors $(0, 1, 0, 0), (0, 0, 1, 0), (0, 0, 0, 1), (1, -1, -1, 1), (1, -1, 1, -1)$. However, if we were to consider the problem restricted to $(1, 1, 1, 1)$, we need to also add the vectors $(1, -1, 1, 1)$, $(1, 0, 0, -1)$ to the above set.

Let $D \in \mathcal{T}_n^e$ be the diagram with vertical edges $(1, 2), (3, 4), (5, 8), (6, 7)$. Label the (removable) odd edges $(1, 2), (3, 4), (5, 8)$ with $a, b$ and $c$ correspondingly. Then elements of $S(D)$ are in bijection with nodes of boolean algebra $B_3$. Thus, a linear combination $\sum_{D' \in S(D)} t_{D'} \operatorname{Pf}_{D'}(A)$ can be network-nonnegative if and only if the coefficients of the four TL-pfaffinants corresponding to the nodes $abc$, $a$, $b$ and $c$ of $B_3$ lies in the cone generated by the set $V_3$.

3.6. **Positive differences of complementary pfaffians.** One obtains a criterion for a linear combination of complementary pfaffians $\operatorname{pf}_{I, \overline{I}}$ to be network positive by combining Theorems 13 and 33. The next result gives one way to produce network positive differences of two complementary pfaffians.

First suppose $I \subset [2n]$ is an even subset and suppose that $|I| \geq n$. Suppose $I = \{i_1 < i_2 < \cdots < i_k\}$ and $\overline{I} = \{j_1 < j_2 < \cdots < j_r\}$ where $r \leq k$. We set $\min(I, \overline{I}) = \{\min(i_1, j_1), \min(i_2, j_2), \ldots, \min(i_r, j_r), i_{r+1}, \ldots, i_k\}$. For convenience, we may let $j_{r+1} = \cdots = j_k = \infty$.

**Proposition 37.** The difference $\operatorname{pf}_{\min(I, \overline{I}), \min(I, \overline{I})} - \operatorname{pf}_{I, \overline{I}}$ is network positive.
Figure 9. The boolean algebra $B_3 = 2^{\{a,b,c\}}$ in Example 36.

**Proof.** We shall show that $D(I) \subset D(\min(I, \bar{I}))$. The result will then follow from Theorems 7 and 33. So let $D \in D(I)$ and suppose that $(i < j) \in D$. Then either $i \in I$ and $j \in \bar{I}$ or $i \in \bar{I}$ and $j \in I$. We need to show that exactly one of $(i, j)$ lies in $\min(I, \bar{I})$. The key fact is that

$$|[i + 1, j - 1] \cap I| = |[i + 1, j - 1] \cap \bar{I}|.$$

Suppose that $i = i_a \in I$ and $j = j_b \in \bar{I}$. If $i_a < j_a$ then $i \in \min(I, \bar{I})$ and furthermore $i_b < j_b$ by (3) so that $j \notin \min(I, \bar{I})$. Otherwise if $i_a > j_a$ we deduce by (3) that $i_b > j_b$; so we conclude again that exactly one of $(i, j)$ lies in $\min(I, \bar{I})$. The case that $i \in \bar{I}$ and $j \in I$ is similar. \hfill $\square$

## 4. Relation between Pfaffinants and Immanants

### 4.1. Rhoades and Skandera’s Temperley-Lieb Immanants

The Temperley-Lieb immanants were discovered by Rhoades and Skandera [RS05a], who gave a number of remarkable positivity properties of these immanants. The exposition we now give is similar to the presentation in [LPP] to which we refer for unexplained notations.

Let $\text{TL}_n$ be the set of Temperley-Lieb diagrams on $2n$ points $\{1, 2, \ldots, 2n\}$, with $\{1, 2, \ldots, n\}$ arranged top to bottom on the left side of a rectangle and $\{n + 1, \ldots, 2n\}$ arranged bottom to top on the right side. Let $w$ be a permutation in $S_n$. By abuse of notation we also denote by $w$ a chosen wiring diagram, thought of as a planar network connecting the $n$ source points on the left to $n$ sink points on the right. Now uncross the crossings of $w$ in all possible ways, each crossing becoming either a vertical uncrossing “$\uparrow$” or a horizontal uncrossing “$\sim$”. Let $X(w)$ be the set of such uncrossings, and for $x \in X(w)$ let $D(x)$ be the element of $\text{TL}_n$ topologically equivalent to $x$ (with any loops removed). Let $h(x)$ be the number of horizontal uncrossings in $x$ and let $l(x)$ be the number of loops formed. Define the weight $\text{wt}(x)$ of $x$ by $\text{wt}(x) = 2^{l(x)}(-1)^{h(x)}$. For $d \in \text{TL}_n$ define $f_d : S_n \to \mathbb{Z}$ by

$$f_d(w) = \sum_{x \in X(w), D(x) = d} \text{wt}(x).$$
Let $B = (b_{ij})$ be a $n \times n$ matrix. Then for $d \in \mathbb{T}L_n$ the $\mathbb{T}L$-immanant $\text{Imm}_d(B)$ is defined as
\[
\text{Imm}_d(B) = \sum_{w \in S_n} f_d(w)b_{1,w(1)} \cdots b_{n,w(n)}.
\]

Let $S \subseteq [2n]$ and recall that $\bar{S} = [2n] \setminus S$ denotes the complement of $S$ in $[2n]$. The $S$-\textit{coloring} of $[2n]$ is obtained by coloring the elements of $S$ black, and the elements $\bar{S}$ white. We call a diagram $d \in \mathbb{T}L_n$ compatible with $S$ (or simply $S$-compatible) if each edge of $d$ has ends of different color in the $S$-coloring. We denote by $\mathcal{D}(S) \subseteq \mathbb{T}L_n$ the set of $S$-compatible diagrams.

For two subsets $I, J \subseteq [n]$ of the same cardinality let $\Delta_{I,J}(B)$ denote the \textit{minor} of an $n \times n$ matrix $B$ in the row set $I$ and the column set $J$. Let $\hat{I} := [n] \setminus I$ and let $I^\wedge := \{2n + 1 - i \mid i \in I\}$.

**Theorem 38.** Rhoades-Skandera [RS05a, Proposition 4.3] For two subsets $I, J \subseteq [n]$ of the same cardinality and $S = J \cup (\hat{I})^\wedge$, we have
\[
\Delta_{I,J}(B) \cdot \Delta_{I,J}(B) = \sum_{d \in \mathcal{D}(S)} \text{Imm}_d(B).
\]

4.2. \textbf{Expressing $\mathbb{T}L$-immanants as $\mathbb{T}L$-pfaffinants.} Let $A = (a_{ij})_{1 \leq i, j \leq 2n}$ be an uppertriangular array such that $a_{ij} = 0$ if $1 \leq i < j \leq n$ or $n + 1 \leq i < j \leq 2n$. Let $B = (b_{ij})$ be the $n \times n$ matrix given by $b_{ij} = a_{i,j+n}$. Our aim is to relate the $\mathbb{T}L$-pfaffinants $\text{Pf}_d(A)$ with the $\mathbb{T}L$-immanants $\text{Imm}_d(B)$.

Call a set $I \subseteq [2n]$ balanced if $|I \cap [n]| = \frac{|I|}{2}$. Let $I_1 = I \cap [n]$ and $I_2 = \{i - n \mid i \in I \cap [n + 1, 2n]\}$.

**Lemma 39.** Let $I \subseteq [2n]$ be an even subset. Then $\text{pf}_{\hat{I},\hat{I}}(A) = 0$ if $I$ (equivalently, $\hat{I}$) is not balanced. If $I$ is balanced then $\text{pf}_{\hat{I},\hat{I}}(A) = (-1)^{|I_1| + |I_2|} \Delta_{I_1,I_2}(B)$.

\textit{Proof.} The first statement is clear since if $I$ is not balanced any matching contains an edge corresponding to a zero entry of $A$. The second statement follows from the observation that $\text{pf}(A) = (-1)^{|I_1|} \text{det}(B)$.

\[ \square \]

Thus non-zero products of complementary pfaffians of $A$ are up to sign equal to products of complementary minors of $B$. Hence one should be able to express the $\mathbb{T}L$-pfaffinants of $A$ in terms of the $\mathbb{T}L$-immanants of $B$.

Let $d \in \mathbb{T}L_n$. Define a matching $\nu(d)$ of $[2n] \cup [2n]'$ as follows: interpret the left side of $d$ (originally labeled $\{1, 2, \ldots, n\}$) as the vertices from 1 to $n$ and the right side of $d$ (originally labeled $\{2n, 2n - 1, \ldots, n + 1\}$) as the vertices from $(n + 1)'$ to $2n'$. Now force $\nu(d)$ to be mirror-symmetric by adding the edge $(i,j)$ (resp. $(i',j')$, $(i,j')$, $(i',j)$) whenever the edge $(i',j')$ (resp. $(i,j)$, $(i',j)$) is present in $d$. Let $X(d)$ be the set of all ways to uncross all crossings in $\nu(d)$, where as in Section 2.3, we always uncross mirror symmetric crossings in the same manner. As usual, we pick the embedding of $\nu(\pi)$ so that no pair of edges intersect more than once or have a point of tangency, and no three edges intersect at a single point.

We define the \textit{weight} $\text{wt}(x)$ of an element $x \in X(d)$ as
\[
\text{wt}(x) = 2^{\nu(x)}(-1)^{uv(x) + ph(x)}
\]
Figure 10. The matching $\nu(d)$ produced from a TL-diagram $d$. 

where $l(x)$, $uv(x)$, $ph(x)$ are as defined in Section 2.4. Similarly we define $D(x) \in \mathcal{T}_n$ to be the symmetric TL-diagram obtained from the uncrossing $x$.

We define $g_D : \mathcal{TL}_n \to \mathbb{Z}$ by

$$g_D(d) = \sum_{x \in X(d)} \sum_{D(x) = D} \text{wt}(x).$$

Denote by $z(d)$ the number of edges in $d$ with both ends in $[n]$. Finally, let $\tilde{g}_D(d) = (-1)^{z(d)} n g_D(d)$.

**Theorem 40.** Let $D \in \mathcal{T}_n^e$. Then

$$\text{Pfaf}_{D}(A) = \sum_{D' \in \mathcal{S}(D)} \sum_{d \in \mathcal{TL}_n} \tilde{g}_{D'}(d) \text{Imm}_d(B).$$

**Proof.** Let $\text{Pfaf}_{D}(A)$ denote the right hand side of the equation in the theorem:

$$\text{Pfaf}_{D}(A) = \sum_{D' \in \mathcal{S}(D)} \sum_{d \in \mathcal{TL}_n} \tilde{g}_{D'}(d) \text{Imm}_d(B).$$

By Proposition [13] it is enough to show that the elements $\{\text{Pfaf}_{D}(A)\}$ satisfy the following decomposition formula (see Theorem [13]):

$$\text{pf}_{I,\bar{I}}(A) = \sum_{D \in \mathcal{D}_{\text{max}}(I)} \text{Pfaf}_{D}(A).$$

We have by definition

$$\sum_{D \in \mathcal{D}_{\text{max}}(I)} \text{Pfaf}_{D}(A) = \sum_{D' \in \mathcal{D}(I)} \sum_{d \in \mathcal{TL}_n} \tilde{g}_{D'}(d) \text{Imm}_d(B)$$

$$= \sum_{d \in \mathcal{TL}_n} (-1)^{z(d)} n \left( \sum_{x \in X(d)} \sum_{D(x) \in \mathcal{D}(I)} \text{wt}(x) \right) \text{Imm}_d(B).$$

Now we proceed as in the proof of Theorem [7]. Suppose $x \in X(d)$ is an uncrossing of $\nu(d)$ such that $D(x) \in \mathcal{D}(I)$. We direct all the strands and loops in $x$ so that the initial vertex of each strand belongs to $I \cup (\bar{I})'$ (and, thus the end vertex belongs
to \( \bar{I} \cup I' \). We allow the closed loops to be directed in either direction. Now define an almost sign-reversing involution on this set of oriented diagrams exactly as in Theorem 7.

Thus the contribution of \( \text{Imm}_d(B) \) to \( \sum_{D \in D_{\max}(I)} Pfaf_D(A) \) is equal to the sum over the aligned uncrossings \( x \in X(d) \) of \((-1)^{z(d)} n(-1)^{uv(x)+ph(x)} \). As in the proof of Theorem 7, such an aligned uncrossing \( x_d \in X(d) \) is unique if it exists – it corresponds to an orientation \( \mu(d) \) of \( \nu(d) \) which connects elements of \( I \cup (\bar{I})' \) to elements of \( \bar{I} \cup I' \). Restricting \( \mu(d) \) to \( \{1, 2, \ldots, n\} \cup \{(n+1)', \ldots, (2n)\} \), we see that \( d \) must be \( S = I_1 \cup (\bar{I}_2)' \)-compatible, and in particular \( I \) must be balanced. Conversely, if \( d \) is \( I_1 \cup (\bar{I}_2)' \)-compatible one obtains a unique such orientation \( \mu(d) \).

Finally we must calculate \((-1)^{uv(x)+ph(x)} \) for \( x(d) \). The unpaired crossings between \( (i, j') \) and \( (j, i') \) are always uncrossed horizontally, so contribute nothing to the sign. The paired crossings which are uncrossed horizontally correspond to pairs of edges \( (i_1 < j_1) \in d \) and \( (i_2 < j_2) \in d \), both of which are horizontal and such that both \( i_1, i_2 \in I_1 \) or both \( i_1, i_2 \in \bar{I}_2 \). Thus for \( d \in D(S) \) the coefficient of \( \text{Imm}_d(B) \) in \( \sum_{D \in D_{\max}(I)} Pfaf_D(A) \) is equal to

\[
(-1)^{z(d)} n(-1)^{(i_1)\bar{I}_2 + (n-i_1)\bar{I}_1} = (-1)^{\frac{n}{2}} \cdot (-1)^{(i_1)\bar{I}_2 + (n-i_1)\bar{I}_1}.
\]

This identity can be proven by induction on \( z(d) \), noting that \((-1)^{(\frac{n}{2})} = 1 \) if \( k \equiv 0, 1 \mod 4 \) and \((-1)^{(\frac{n}{2})} = 1 \) if \( k \equiv 2, 3 \mod 4 \). Now summing over over all \( d \in \text{TL}_n \) and using Theorem 48 and Lemma 39 we see that \( pf_{I,\bar{I}}(A) = \sum_{D \in D_{\max}(I)} Pfaf_D(A) \).

\[\text{Example 41.}\] Let us take a \( 4 \times 4 \) skew-symmetric matrix

\[
A = \begin{pmatrix}
0 & 0 & x & y \\
0 & 0 & z & t \\
-x & -z & 0 & 0 \\
-y & -t & 0 & 0
\end{pmatrix}
\]

and let \( B \) be its \( 2 \times 2 \) minor

\[
B = \begin{pmatrix} x & y \\ z & t \end{pmatrix}.
\]

\[\text{Figure 11.}\] The three even symmetric diagrams in of \( T_2^e \) and the two Temperley-Lieb diagrams in TL_2.

Let \( L, M, N \) be the elements of \( T_2^e \) and \( P, Q \) be the elements of TL_2 as shown in Figure 11. Then \( Pfaf_L(A) = yz - xt \), \( Pfaf_M(A) = xt - yz \), \( Pfaf_N(A) = xt \), \( \text{Imm}_P(B) = xt - yz \), and \( \text{Imm}_Q(B) = yz \). One can check that the coefficients of immanants in pfaffinants agree with Theorem 40.
4.3. Quadratic relations between TL-pfaffinants and TL-immanants. Let $A$ be a skew-symmetric $2n \times 2n$ matrix. The following formula is well known, see for example [Ste90].

**Theorem 42.** We have $\text{pf}(A)^2 = \det(A)$.

More generally, we have the equation $\text{pf}_{I,\bar{I}}(A)^2 = \Delta_{I,\bar{I}}(A)\Delta_{\bar{I},I}(A)$. Let $S = I \cup (\bar{I})^\wedge$. Applying Theorems 13 and 38 we obtain the quadratic relationship

$$(4) \quad \left( \sum_{D \in \mathcal{D}(I)} \text{Pfaf}_D(A) \right)^2 = \sum_{d \in \mathcal{D}(S)} \text{Imm}_d(B).$$

It seems interesting to ask whether it is possible to refine (4) to obtain simple quadratic relationships between TL-pfaffinants and TL-immanants. The simplest case arises from Theorem 42: we have $\text{Pfaf}_D(A)^2 = \text{Imm}_d(A)$ where $D \in \mathcal{T}_n$ and $d \in \mathcal{T}_n$ are the respective diagrams containing only horizontal edges (see Example 12). The following table describes how to express TL-immanants of a skew-symmetric $4 \times 4$ matrix in terms of the TL-pfaffinants. Here we preserve the labeling of TL-pfaffinants as on Figure 11.

| TL-Immanant | Sums of products of TL-pfaffinants |
|-------------|-----------------------------------|
| (4,5), (3,6), (2,7), (1,8) | $\text{Pfaf}_I^2$ |
| (1,2), (4,5), (3,6), (7,8) | $-\text{Pfaf}_I^2$ |
| (1,2), (4,5), (6,7), (3,8) | $-\text{Pfaf}_L\text{Pfaf}_M$ |
| (1,2), (5,6), (4,7), (3,8) | $-\text{Pfaf}_L^2 - \text{Pfaf}_L\text{Pfaf}_N$ |
| (2,3), (4,5), (6,7), (1,8) | $2\text{Pfaf}_L\text{Pfaf}_M$ |
| (2,3), (1,4), (6,7), (5,8) | $\text{Pfaf}_M^2$ |
| (1,2), (3,4), (6,7), (5,8) | $\text{Pfaf}_I^2 + \text{Pfaf}_L\text{Pfaf}_M + \text{Pfaf}_L\text{Pfaf}_N + \text{Pfaf}_M\text{Pfaf}_N$ |
| (1,2), (3,4), (5,6), (7,8) | $2\text{Pfaf}_I^2 + 2\text{Pfaf}_L\text{Pfaf}_N + \text{Pfaf}_N^2$ |

However for $n > 2$ the TL-immanants cannot be expressed in a similar manner through TL-pfaffinants. For example, with $n = 3$ the immanant corresponding to the diagram with edge set $\{(2,3), (4,5), (6,7), (8,9), (10,11), (1,12)\}$ does not lie in the span of the products of the TL-pfaffinants. It remains unclear if any relation between TL-immanants and TL-pfaffinants of a skew symmetric matrix can be established in general.

5. Schur $Q$-positivity

In this section we discuss some conjectural applications of TL-pfaffinants to positivity properties of Schur $Q$-functions. Many of our results and conjectures can be stated alternatively in terms of Schur $P$-functions, but we will not do so explicitly.

5.1. Shifted tableaux. For further details concerning the material of this section we refer the reader to [Mac].

Let $\lambda = \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_t > 0$ be a strict partition of integers. We will not distinguish between $\lambda$ and its shifted diagram $S(\lambda)$ obtained by shifting the $i$-th row of the usual (Young) diagram $(i-1)$ squares to the right, for each $i$. More generally, if $\lambda$ and $\mu$ are two strict partitions so that $S(\mu) \subset S(\lambda)$ then the skew shifted diagram is denoted $\lambda/\mu$. Our notation for diagrams follows the English notation, so that Young diagrams are top-left justified.
A shifted tableaux $T$ with shape $\text{sh}(T) = \lambda / \mu$ is a filling of the shifted diagram $\lambda / \mu$ with the numbers $1', 1, 2', 2, \ldots$ so that

1. the rows and columns are weakly increasing under the order $1' < 1 < 2' < 2 < \ldots$
2. there is at most one occurrence of $i'$ in a row
3. there is at most one occurrence of $i$ in a column.

The weight $\text{wt}(T)$ of a shifted tableau is the composition $\alpha = (\alpha_1, \alpha_2, \ldots)$ where $\alpha_i$ is equal to the combined number of the letters $i$ and $i'$ used in $T$. The Schur $Q$-function $Q_{\lambda/\mu}(x)$ is defined as

$$Q_{\lambda/\mu}(x) = \sum_{T : \text{sh}(T) = \lambda / \mu} x^{\text{wt}(T)}$$

where $x^\alpha = x_1^{\alpha_1} x_2^{\alpha_2} \cdots$. Though it is not immediate from the definition, the function $Q_{\lambda/\mu}(x)$ is a symmetric function in the variables $x_1, x_2, \ldots$.

5.2. Schur $Q$-functions and pfaffians. Schur $Q$-functions can be expressed as pfaffians, as follows. First extend the notation of Schur $Q$-functions by defining $Q_{-r} = 0$ for $r > 0$ and $Q_{(r,s)} = -Q_{(s,r)}$. Define the $l \times l$ skew symmetric matrix $A_\lambda = [Q_{(\lambda_i, \lambda_j)}]_{1 \leq i,j \leq l}$, where $\lambda_i$ is the $i$-th part of $\lambda$, $l = l(\lambda)$ is the number of parts of $\lambda$. By possibly adding an extra zero part to $\lambda$, we may assume that $l$ is even. The following theorem can be found in [Mac].

**Theorem 43.** Let $\lambda$ be a strict partition. Then $Q_{\lambda} = \text{pf}(A_\lambda)$.

A skew version of this formula was proved by Józefiak and Pragacz [JP]. Let $\lambda / \mu$ be a skew shifted shape where $\lambda = \lambda_1 > \cdots > \lambda_l > 0$ and $\mu = \mu_1 > \mu_2 > \cdots > \mu_r \geq 0$. We assume that $l + r$ is even. Let $H = (h_{ij})$ be the $l \times r$ matrix with $h_{ij} = Q_{\lambda_i - \mu_j + 1 - j}$. Define a skew symmetric matrix

$$A_{\lambda/\mu} = \begin{pmatrix} A_\lambda & H \\ -H^t & 0 \end{pmatrix}$$

We call the matrix $A_{\lambda/\mu}$ a $Q$-Jacobi-Trudi matrix. If we allow in the definition $\lambda$ and $\mu$ to possibly be non-strict partitions then we call $A_{\lambda/\mu}$ a generalized $Q$-Jacobi-Trudi matrix.

**Theorem 44 ([JP, Ste90]).** Let $\lambda / \mu$ be a shifted skew shape. Then $Q_{\lambda/\mu} = \text{pf}(A_{\lambda/\mu})$.

**Remark 45.** In [JP] the matrix Let $\tilde{H} = (\tilde{h}_{ij})$ be the $l \times r$ matrix with $\tilde{h}_{ij} = Q_{\lambda_i - \mu_j}$ is used. Using $\tilde{H}$ to define $A_{\lambda/\mu}$, one then has $\text{pf}(A_{\lambda/\mu}) = (-1)^{\binom{r}{2}} \text{pf}(\tilde{A}_{\lambda/\mu})$.

5.3. Schur $Q$-positivity and pfaffinants. As we saw in Section 3.4, network positivity of an element $f \in P_n$ depends on the decomposition of $f$ into diagram pfaffians $\text{Pf}^D$. Somewhat more surprisingly, we conjecture that this decomposition is also related to Schur $Q$-positivity.

**Conjecture 46.** Suppose $f \in P_n$ can be expressed as $f = \sum c_D \text{Pf}^D$ with non-negative coefficients $c_D$. Then for any generalized $Q$-Jacobi-Trudi matrix $A_{\lambda/\mu}$, the evaluation $f(A_{\lambda/\mu})$ is a nonnegative linear combination of Schur $Q$-functions.
Conjecture 46 parallels known Schur-positivity properties of the TL-immanants Imm_{A}(A). It is known [RS05, Ste90] that Imm_{A}(A) is nonnegative whenever A arises from a planar network and that the evaluations Imm_{A}(H_{\lambda/\mu}) on Jacobi-Trudi matrices H_{\lambda/\mu} are nonnegative.

We can prove a weaker version of Conjecture 46.

**Theorem 47.** Suppose f \in P_{n} can be expressed as f = \sum c_{D}Pf_{D} with non-negative coefficients c_{D}. Then for any shifted shape \lambda/\mu, the evaluation f(A_{\lambda/\mu}) is a nonnegative linear combination of monomial symmetric functions.

**Proof.** Stembridge has constructed a network N_{\lambda/\mu} such that \lambda_{A_{\lambda/\mu}} = A(N_{\lambda/\mu}) (see [Ste90, Theorem 6.2]). By Theorem 47 pf(A_{\lambda/\mu}) = Q_{\lambda/\mu} also calculates the Schur Q-function. It remains to note that for each D \in T_{n} the evaluation Pf_{D}(N_{\lambda/\mu}) is a monomial positive formal power series. Finally, since f \in P_{n} we know f(A_{\lambda/\mu}) must be a linear combination of Schur Q-functions and thus symmetric. We conclude that f(A_{\lambda/\mu}) is a nonnegative linear combination of monomial symmetric functions.

5.4. **Schur Q-functions and cell transfer.** In [LP05], we introduced the notion of a \text{T}-\text{labeled poset}. Let Z be a totally ordered set (in [LP05] we chose Z = \mathbb{N}, but the results there generalize easily). Let P be a poset and O be an assignment of a weakly increasing function O(s < t) : Z \to Z \cup \infty to each cover relation s < t of P. A (P, O)-tableau T is a function T : P \to Z so that for each cover relation s < t in P we have T(t) \leq O(s, t)(T(s)).

The boxes in a shifted diagram form a poset also denoted \lambda/\mu. The cover relations s < t correspond to boxes s, t \in \lambda/\mu such that s is immediately above or immediately to the left of t. Let Z = \{1' < 1 < 2' < 2 < \ldots\}. For a letter z \in Z we denote by z' the letter obtained by either removing or adding a prime and for z \in Z we let z \mapsto z + 1 denote the obvious operation which preserves primes. Define the functions f^{r}, f^{c}

\[
    f^{r}(z) = \begin{cases} 
        z & \text{if } z \text{ is not primed,} \\
        z' & \text{if } z \text{ is primed.}
    \end{cases}
\]

\[
    f^{c}(z) = \begin{cases} 
        z & \text{if } z \text{ is primed,} \\
        z' + 1 & \text{if } z \text{ is primed.}
    \end{cases}
\]

Let O_{\lambda/\mu} be the (edge) labeling of \lambda/\mu such that every cover relation along a row is labeled with f^{r} and every cover relation along a column is labeled with f^{c}.

The following result is immediate.

**Proposition 48.** Shifted tableaux of shape \lambda/\mu are \((\lambda/\mu, O_{\lambda/\mu})\)-tableaux.

Now we define the operations \wedge and \vee on pairs of strict partitions (see [LP05, LPP]). Namely, for partitions \lambda = (\lambda_{1}, \lambda_{2}, \ldots, \lambda_{n}) and \mu = (\mu_{1}, \mu_{2}, \ldots, \mu_{n}), define \lambda \vee \mu := (\max(\lambda_{1}, \mu_{1}), \max(\lambda_{2}, \mu_{2}), \ldots) and \lambda \wedge \mu := (\min(\lambda_{1}, \mu_{1}), \min(\lambda_{2}, \mu_{2}), \ldots). We may have to add trailing zeroes before applying these operations. These operations send pairs of strict partitions to strict partitions. The definition can be extended to skew shifted diagrams as follows: \((\lambda/\mu) \vee (\nu/\rho) := (\lambda \vee \nu)/(\mu \vee \rho)\) and \((\lambda/\mu) \wedge (\nu/\rho) := (\lambda \wedge \nu)/(\mu \wedge \rho)\).

In [LP05] it was shown that the operations \wedge and \vee (called cell transfer operations) for \text{T}-labeled posets give rise to a range of monomial positivity results. By a slight modification of [LP05, Theorem 3.6] one obtains a bijective proof that

\[
    Q_{(\lambda/\mu) \vee (\nu/\rho)}Q_{(\lambda/\mu) \wedge (\nu/\rho)} - Q_{\lambda/\mu}Q_{\nu/\rho} \text{ is monomial positive.}
\]

The proof of the following stronger result will appear elsewhere.
Theorem 49. Let \( \lambda/\mu \) and \( \nu/\rho \) be skew shifted shapes. Then the difference
\[
Q_{(\lambda/\mu)\lor(\nu/\rho)} - Q_{\lambda/\mu}Q_{\nu/\rho}
\]
is a nonnegative sum of Stembridge’s peak functions \( K_\alpha \).

We will not give the definition of the peak functions \( K_\alpha \) here and refer the reader to [Ste97] for full details. The \( K_\alpha \) form a basis for a subalgebra \( \Pi \) of the algebra quasi-symmetric functions and the \( K_\alpha \) take the place of the fundamental quasi-symmetric functions in \( \Pi \). The Schur \( Q \)-functions \( Q_{\lambda/\mu} \) lie in this subalgebra \( \Pi \) and are known to be positive in the basis \( \{K_\alpha\} \). We now make the following stronger conjecture.

Conjecture 50. Let \( \lambda/\mu \) and \( \nu/\rho \) be skew shifted shapes. Then the difference
\[
Q_{(\lambda/\mu)\lor(\nu/\rho)} - Q_{\lambda/\mu}Q_{\nu/\rho}
\]
is a non-negative combination of Schur \( Q \)-functions.

Conjecture 50 is a Schur \( Q \)-function analogue of what we call the cell transfer theorem. The monomial positivity version was proved in [LP05], the fundamental quasi-symmetric function version in [LPP] and the Schur positivity version in [LPP]. As explained in the introduction of [LP06], these positivity phenomena arise from a collection of data: (a) a class of posets, (b) a ring containing the generating functions of “tableaux”, (c) a basis of this ring, and (d) a set of skew functions. In our case, (a) the posets are shifted Young diagrams, (b) the ring is the subalgebra of the ring of symmetric functions generated by the odd power sums, (c) the basis is the set of Schur \( Q \)-functions for non-skew shifted shapes, and (d) the skew functions are the Schur \( Q \)-functions labeled by skew shifted shapes.

Proposition 51. Conjecture 46 implies Conjecture 50 when \( \mu = \rho = \emptyset \).

Proof. Let \( \pi \) be the (possibly no longer strict) partition obtained from taking the union of the parts of \( \lambda \) and \( \nu \). While \( \pi \) is not necessarily a strict partition, we can still formally define the matrix \( A_\pi \) as above. Clearly, \( \text{pf}_{I,J}(A_\pi) = Q_\lambda Q_\nu \) for the appropriate choice of \( I \). Now recall the definition of \( \min(I,J) \) from Section 3.6. We have
\[
\text{pf}_{\min(I,J),\min(I,J)}(A_\pi) - \text{pf}_{I,J}(A_\pi) = Q_{(\lambda/\mu)\lor(\nu/\rho)}Q_{(\lambda/\mu)\land(\nu/\rho)} - Q_{\lambda/\mu}Q_{\nu/\rho}.
\]
By the proof of Proposition 47, the difference \( \text{pf}_{\min(I,J),\min(I,J)} - \text{pf}_{I,J} \) is a nonnegative linear combination of the TL-pfaffians \( \text{Pfaf}_D \). Conjecture 46 implies that \( \text{Pfaf}_D(A_\pi) \) is Schur \( Q \)-positive, from which the result follows. \( \square \)

5.5. Further Schur \( Q \)-positivity conjectures. The usual Schur function analogue of Conjecture 50 was established in [LPP].

Theorem 52 ([LPP]). Let \( \lambda/\mu \) and \( \nu/\rho \) be shifted shapes. Then the difference
\[
s_{(\lambda/\mu)\lor(\nu/\rho)}s_{(\lambda/\mu)\land(\nu/\rho)} - s_{\lambda/\mu}s_{\nu/\rho}
\]
is a non-negative combination of Schur functions.

Theorem 52 was used to resolve a number of conjectures of Fomin, Fulton, Li, Poon [FFLP], of Lascoux, Leclerc, Thibon [LLT] and of Okounkov [Ok]. We now state the shifted analogue of the Fomin-Fulton-Li-Poon conjecture.

For two partitions \( \lambda \) and \( \mu \), let \( \lambda \cup \mu = (\nu_1, \nu_2, \nu_3, \ldots) \) be the partition obtained by rearranging all parts of \( \lambda \) and \( \mu \) in the weakly decreasing order. Let \( \text{sort}_1(\lambda, \mu) := (\nu_1, \nu_3, \nu_5, \ldots) \) and \( \text{sort}_2(\lambda, \mu) := (\nu_2, \nu_4, \nu_6, \ldots) \). It is not hard to see that if \( \lambda \) and \( \mu \) are strict then so are \( \text{sort}_1(\lambda, \mu) \) and \( \text{sort}_2(\lambda, \mu) \).
Conjecture 53. Let \( \lambda, \mu \) be two shifted shapes. Then \( Q_{\text{sort}_1(\lambda, \mu)}Q_{\text{sort}_2(\lambda, \mu)} - Q_\lambda Q_\mu \) is a nonnegative linear combination of Schur Q-functions.

Proposition 54. Conjecture 53 implies Conjecture 55.

Proof. First note that if \( \lambda/\mu \) and \( \nu/\rho \) are skew shifted diagrams obtained from each other via a translation then \( Q_{\lambda/\mu} = Q_{\nu/\rho} \). For a shifted shape \( \lambda \), let \( \lambda_i \) denote the skew shifted shape obtained by translating \( \lambda \) down one row and hence also one step to the right. We will assume that \( \lambda_i \) is presented as \( \nu/\rho \) where \( \nu_1 = \rho_1 \) is very large (much larger than any other parts involved in the proof). If \( \nu/\rho \) is a shifted shape so that \( \nu_1 = \rho_1 \) we let \( (\nu/\rho)_\tau \) denote the shifted shape obtained by translating one row up (and hence also one step to the left).

We can construct \( \pi = \text{sort}_1(\lambda, \mu) \) and \( \theta = \text{sort}_2(\lambda, \mu) \) from \( \lambda \) and \( \mu \) by a sequence of the operations \( \lor \) and \( \land \). Suppose that we have (strict) partitions \( (\rho, \nu) \) so that \( \rho \lor \nu = \lambda \lor \mu \). Let us suppose that \( \rho \) agrees with \( \pi \) up to the \( i \)-th part and that \( \nu \) agrees with \( \theta \) up to the \( i \)-th part. If \( \rho_{i+1} \neq \pi_{i+1} \), then it must be the case that \( \nu_{i+1} = \pi_{i+1} \). In this case we replace \( (\rho, \nu) \) by \( (\rho^*, \nu^*) = (\rho \lor \nu, \rho \land \nu) \). One checks that \( \rho^* \) agrees with \( \pi \) up to the \( (i+1) \)-th part and \( \nu^* \) agrees with \( \theta \) up to the \( i \)-th part. If \( (\nu^*)_i+1 \neq \theta_{i+1} \) then \( (\rho^*)_{i+2} = \theta_{i+1} \). We now replace \( (\rho^*, \nu^*) \) by \( (\rho^{**}, \nu^{**}) = (\rho^* \land (\nu^*)_i, (\rho^* \lor (\nu^*)_i)_\tau) \). One checks that \( \rho^{**} \) still agrees with \( \pi \) up to the \( (i+1) \)-th part and \( \nu^{**} \) now agrees with \( \theta \) up to the \( (i+1) \)-th part. After a finite number of iterations of the map \( (\rho, \nu) \mapsto (\rho^{**}, \nu^{**}) \) applied to \( (\lambda, \mu) \), one obtains \( (\pi, \theta) \).

If we apply Conjecture 55 to the Schur Q-functions indexed by the pairs of partitions \( (\rho, \nu) \) we see that for each iteration of the above map \( Q_{\rho^{**}} Q_{\nu^{**}} - Q_{\rho} Q_{\nu} \) is Schur Q-positive. This proves the theorem.

Our proof here is very similar to an analogous proof in [LPP], where left and right shifts are used instead of our up and down translations. It would be interesting to generalize other Schur positivity results and conjectures to the shifted case.

We note the following result, which follows from Theorem 49 and the proof of Proposition 53.

Proposition 55. Let \( \lambda, \mu \) be two shifted shapes. Then \( Q_{\text{sort}_1(\lambda, \mu)}Q_{\text{sort}_2(\lambda, \mu)} - Q_\lambda Q_\mu \) is a nonnegative linear combination of peak functions.

6. Proof of Theorem 11

Let \( A \) and \( B \) denote two nice embeddings of \( \nu(\pi) \) and denote by \( f_D(A) \) and \( f_D(B) \) the weight generating function of uncrossings defined by \( A \) and \( B \) respectively (see Section 2.4). By replacing \( A \) or \( B \) with a small deformation which is combinatorially equivalent we may assume even if we draw all the edges of \( A \) and \( B \) that (a) no two edges have a point of tangency and (b) no three strings cross at a single point. However, an edge of \( A \) and an edge of \( B \) may intersect more than once.

We now argue that \( A \) and \( B \) are connected by a sequence of three types of Reidenmester-like moves, denoted \( R_\alpha \), \( R_\beta \) and \( R_\gamma \), as shown in Figure 12. Let \( (i, j) \) be an edge in \( \nu(\pi) \). To change \( A \) to \( B \), we move the embedding of \( (i, j) \) in \( A \) continuously until it agrees with the embedding of \( (i, j) \) in \( B \); and we repeat for each edge of \( \nu(\pi) \). Note that we will always move the mirror symmetric edge simultaneously so that the diagram is always mirror symmetric. There are three types of “singularities” which may occur during this process, changing the combinatorial type of the embedding. These singularities violate the conditions (a) and (b) above.
\(R_\alpha\): If the singularity occurs on the vertical axis of symmetry then one obtains a quadruple intersection between two pairs of mirror symmetric edges, violating both conditions (a) and (b). The Reidemeister move \(R_\alpha\) allows one to pass from one side of the singularity to the other.

\(R_\beta\): If the singularity is a paired singularity, it may involve three edges crossing at the same point, giving the move \(R_\beta\).

\(R_\gamma\): If the singularity is a paired singularity, it may involve a point of tangency, giving the move \(R_\gamma\).

Note that \(R_\alpha\) allows us to permute the crossing points on the vertical axis of symmetry, while \(R_\beta\) and \(R_\gamma\) allow us to do all the other required changes. During this process the rule that no two edges crossing more than once can be violated (by moves \(R_\alpha\) or \(R_\gamma\)).

\[\text{Figure 12. The three types of Reidemeister-like moves } R_\alpha, R_\beta, R_\gamma.\]

To complete the proof we show that \(f_D(A) = f_D(A')\) if \(A\) and \(A'\) are related by a Reidemeister-like move.

\(R_\alpha\): We may use the move \(R_\gamma\) (preserving \(f_D\)) to replace the initial and final pictures with the two intermediate ones shown in Figure 13. Now using the calculation of Example 5 we may obtain the one intermediate picture from the other while again preserving \(f_D\).

\(R_\beta\): There are three pairs of (mirror-symmetric) crossings, giving a total of 8 uncrossings for the initial and final pictures. Denote the three edges coming from the left by \(a, b, c\) from top to bottom and the three edges exiting to the right by \(a', b', c'\). An uncrossing of this local picture will give a matching of \(a, a', b, b', c, c'\) together with a weight. One obtains the following table for the weights of the 8 uncrossings, showing that the weight generating functions agree for each matching. The “initial” embedding here is the top picture in Figure 12.
Figure 13. Verification of the $R_\alpha$ move.

| Matching                  | Initial embedding | Final embedding |
|---------------------------|-------------------|-----------------|
| $\{(a,a'),(b,b'),(c,c')\}$ | 1                 | 1               |
| $\{(a,b),(a',b'),(c,c')\}$ | $1 + 1 + 1 - 2 = 1$ | 1               |
| $\{(a,a'),(b,c),(b',c')\}$ | 1                 | $1 + 1 + 1 - 2 = 1$ |
| $\{(a,c'),(a',b'),(b,c)\}$ | $-1$              | $-1$            |
| $\{(a,b),(c,a'),(b',c')\}$ | $-1$              | $-1$            |

$R_\gamma$: For the initial (top) embedding, the picture has 4 uncrossings. Three of these 4 uncrossings give a matching (the vertical one) which does not occur for the final (bottom) embedding, but their weights (respectively 2, $-1$, $-1$) cancel out. For the other (horizontal) matching we obtain the same contribution of 1 for both the initial and final embeddings.

This completes the proof of Theorem 4.

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E-mail address: tfylam (at) math (dot) harvard (dot) edu

E-mail address: pasha (at) math (dot) mit (dot) edu