On the position uncertainty measure on the circle

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Abstract

New position uncertainty (delocalization) measures for a particle on the circle are proposed
and illustrated on several examples, where the previous measures (based on 2π-periodic position
operators) appear to be unsatisfactory. The new measures are suitably constructed using the
standard multiplication angle operator variances. They are shown to depend solely on the state
of the particle and to obey uncertainty relations of the Schrödinger–Robertson type.

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1. Introduction

Recently there is a renewed interest to the old problem of the uncertainties and uncertainty relations
for a particle on the circle [1, 2, 3, 4, 5, 6]. Due to the controversial commutation relation between
the angle and angular momentum operators most attention have been paid to position operators that
are invariant under translation \( \varphi \rightarrow \varphi + a, a \in \mathbb{R} \). The relation \([\hat{l}_z, \hat{\varphi}] = -i\) stems from the Dirac
correspondence rule

\[
\{f, g\} \longrightarrow i[\hat{f}, \hat{g}]
\]

between Poisson bracket \(\{f, g\}\) of two classical quantities \(f\) and \(g\) and the commutator of the corre-
sponding operators \(\hat{f}\) and \(\hat{g}\) for quantum observables. This rule is formally satisfied with

\[
\hat{\varphi} = \varphi \quad \text{and} \quad \hat{l}_z = -id/d\varphi.
\]

However on the eigenstates

\[
\psi_m(\varphi) = \exp(im\varphi)/\sqrt{2\pi}, \quad m = 0, \pm1, \ldots,
\]

the above commutation relation breaks down, together with the associated standard Heisenberg-
Robertson uncertainty relation \((\Delta l_z)^2(\Delta \varphi)^2 \geq 1/4\). Therefore authors try to adopt another position
operator \([4, 5, 7]\), or even another definition of the uncertainty on the circle \([3]\).

In this letter we provide an approach to the issue with minimal (in our opinion) deviation from the
standard commutation relation and standard measure of uncertainty. The main idea has been sketched
in [2]. Here we develop it in greater detail, providing some proofs and further examples. After a brief
review of the properties of main previous position uncertainty measures in section 2, two different
new measures are constructed and discussed in section 3. The new measures are constructed using
suitably the standard expressions of the first and second moments of the angle variable, calculated by
integration over 2π intervals. They are of the form of positive state functionals, the values of which
depend solely on the state considered.

The terms position uncertainty measure and state delocalization measure are used here as syn-
onyms. The uncertainty measure of states is also called measure of spread of corresponding wave
functions (more precisely of the corresponding probability distributions \( p(\varphi) = |\psi(\varphi)|^2 \)). It is worth noting that all uncertainty measures are maps of the infinite dimensional state space into the positive part of the real line. It is impossible in such a way to distinguish between all states. Therefore different measures should be considered not only as competitive, but as complementary as well.

2. A brief review of previous delocalization measures

For a particle on the real line the standard measure of the position uncertainty is given by the second moment \((\Delta x)^2 := \langle (x - \langle x \rangle)^2 \rangle\) of the position operator \(\hat{x} = x\), or equivalently by the standard deviation \(\Delta x\). Mathematically both \(\Delta x\) and \(\langle x \rangle\) are one-to-one functionals on state space. The quantity \((\Delta x)^2\) is also called variance, or dispersion, of \(x\) and is also denoted as \(Dx\) or \(M^{(2)}x\). The variance of \(x\) is regarded as a measure of spread, or delocalization, of the wave function \(\psi(x)\). From (5) and (4) one encounters some unsatisfactory results. For example, it produces the same maximal delocalization \((\Delta x)^2 = 1/4\) for the eigenstates \(\psi_m(\varphi)\) of \(\hat{\varphi}\) and for all states \(\psi(\varphi)\) with the property \(|\psi(\varphi + \pi)| = |\psi(\varphi)|\). The centroid for those \(\pi\)-periodic distributions \(\psi(\varphi)^2\) is in the center of the ring. On figure 1 graphics of three \(\pi\)-periodic distributions are shown: uniform one \(p_{\text{u}}(\varphi) = 1/2\pi = |\psi_m(\varphi)|^2\), \(p_x(\varphi) = |\psi_x(\varphi)|^2 = \sin^2 \varphi / \pi\) and \(p_\alpha(\varphi) = |\sin(2\varphi)|^2 / \pi\). It is clear that the localization of that distributions is quite different, and it is desirable to have an uncertainty measure that distinguishes between them.

A rather nonstandard expressions for position and angular momentum uncertainties for a particle on the circle were introduced and discussed in [3]:

\[
\Delta^2(\hat{\varphi}) = \frac{1}{4} \ln \langle e^{-2i\varphi} \rangle, \quad \Delta^2(\hat{\varphi}) = -\frac{1}{4} \ln |\langle U(\varphi)^2 \rangle|^2.
\]
For a large sets of states these quantities obey the inequality $\Delta^2(\hat{l}) + \Delta^2(\hat{\phi}) \geq 1$, the equality being reached in the eigenstates $|\xi\rangle$ of the operator $Z = \exp(-\hat{l} + \frac{1}{2})U(\phi)$. The family of $|\xi\rangle$ is overcomplete and the states $|\xi\rangle$ are called CS on the circle [3, 4, 5, 6].

The functional $\Delta^2(\hat{\phi})$ was proposed as a position uncertainty on the circle. However this uncertainty measure was found [2] to be not quite consistent with state localization: on CS $|\xi\rangle$ it equals 1/2, while on the visually more localized states $|\xi\rangle - |\xi\rangle$ (Schrödinger cat states on the circle) it can take rather less value of 0.33 (see [2] and figure 2 therein). On the above noted states $\psi_s(\phi)$, $\psi_{s2}(\phi)$ and $\psi_m(\phi)$ it takes values 0.346, $\infty$, $\infty$. Thus it makes distinction between $\psi_{s1}(\phi)$ and $\psi_{s2}(\phi)$ and $\psi_m(\phi)$, but identifies $\psi_{s2}(\phi)$ with the uniform state $\psi_m(\phi)$ (see figure 1). Another unsatisfactory property of $\Delta^2(\hat{\phi})$ is, that it takes the smaller value of 0.143 on the two-peak state $\psi_{s4}(\phi) = (0.2 + \sin^2(\phi))/N$, while on the CS $|z=1\rangle$ (one-peak state) it assumes the much larger value of 0.5 (see figure 2).

The authors of [1, 3] do not consider the above noted properties as a deficiency of $\Delta^2(\hat{\phi})$ and in support of such opinion provide [11] the example of two step functions $\psi(x)$ and $\phi(x)$ on the real line: $\psi(x)$ is different from 0 in the interval $(-L/2, L/2)$ only (where it takes the value $1/\sqrt{L}$): $\phi(x)$ is different from 0 in the two smaller intervals $(-L/2, -L/4)$, $(L/4, L/2)$ (where it takes the value $\sqrt{2/L}$) (see figure 1 in [11]). The standard second moment $\langle \Delta(\phi)^2 \rangle$ for $\psi(x)$ is lesser than that for $\phi(x)$. However the authors of [11] write "the state $|\psi\rangle$ is much worse localized on the interval $|x| < L/2$, than the state $|\phi\rangle$. In fact, we know that in the state $|\phi\rangle$ the particle is not in the region $|x| < L/4$". My remark is that the step function $\phi(x)$ can not be regarded as a one particle state on the interval $|x| < L/2$ exactly due to the fact that $\phi(x) = 0$ in the region $|x| < L/4$. Due to this fact particle never can jump from the left region $(-L/2, -L/4)$ to the right one $(L/4, L/2)$, and vice versa. Therefore this example can not be interpreted against the reliability of $\Delta x$ as an uncertainty measure on the real line.

3. Generalized uncertainty measures based on the variance

The state space of a particle on the circle consists of $2\pi$-periodic square-integrable functions $\psi(\phi)$. (In fact periodicity is up to a phase factor). In view of this periodicity the scalar product of two states $\psi_1(\phi)$ and $\psi_2(\phi)$ can be calculated by integration with respect to $\phi$ within any interval of length $2\pi$.

Since $\hat{p}\psi(\phi)$ is no more periodic in $\phi$ the standard second moment $D\phi \equiv \langle \Delta(\phi)^2 \rangle$ of $\phi$ would naturally depend on the interval of integration (here specified by the reference point $\phi_0$),

$$D\phi = \int_{\phi_0-\pi}^{\phi_0+\pi} \langle \phi - \langle \phi | \phi_0 \rangle \rangle |\psi(\phi)|^2 d\phi = D\phi(\phi_0),$$

$$\langle \phi | \phi_0 \rangle = \int_{\phi_0-\pi}^{\phi_0+\pi} \phi |\psi(\phi)|^2 = M\phi(\phi_0).$$

This $\phi_0$-dependence of the standard moments of $\phi$ is the main reason authors to abandon $D\phi$ and to look for other expressions to simulate quantum position uncertainties on the circle or, equivalently, the spread of the related periodic probability distributions $p(\phi)$. It turns out however, that the variance [5] could still be useful in construction of relevant uncertainty measures.

First of all we note, that if one defines the $\phi_0$-dependent covariance $\Delta l_\phi(\phi_0)$ of $\phi$ and $\hat{l}$ as the real part of the matrix element $G_{l_\phi} := \langle \hat{l} - \langle \hat{l} \rangle \rangle (|\phi - \langle \phi | \phi_0 \rangle \rangle \psi)$, $\Delta l_\phi(\phi_0) = \text{Re} G_{l_\phi}(\phi_0)$, (where the means are taken by integration as in [5, 6]), one obtains the inequality (see also [2] and [9])

$$D\phi(\phi_0)D\hat{l}_\phi - \langle \Delta l_\phi(\phi_0) \rangle^2 \geq (\text{Im} G_{l_\phi}(\phi_0))^2,$$

which is a generalization of the Schrödinger (or Schrödinger–Robertson) uncertainty relation [10]. For a particle on the real line the latter relation read $D\hat{x} D\hat{p} - \langle \text{Cov}(x, p) \rangle^2 \geq 1/4$, where $\text{Cov}(x, p) \equiv \Delta xp$ is the covariance of $\hat{x}$ and $\hat{p}$. The problem remains however to define on the circle uncertainty (or delocalization, or spread) measure $\Delta_{\phi_0}^2(\chi)$ of the state $|\psi\rangle$ (or of the distribution $p(\phi)$) that depends solely on the state $|\psi\rangle$, and not on the limit of integration in [5]. It turned out that this problem can be resolved by a suitable use of $D\phi(\phi_0)$ due to the $2\pi$-periodic property of the functional [5],

$$D\phi(\phi_0 + 2\pi) = D\phi(\phi_0).$$
The property \( \text{(10)} \) can be easily proved, using the state periodicity \(|\psi(\varphi + 2\pi)| = |\psi(\varphi)|\) and the definition of \( D\phi(\varphi_0) \). In fact one can show that all moments \( M^{(n)}\phi(\varphi_0) = \langle (\varphi - \langle \varphi \rangle)^n \rangle, \quad n = 1, \ldots \), of \( \varphi \) are \( 2\pi \)-periodic in \( \varphi_0 \). In view of this periodic property the \( \varphi_0 \)-dependent uncertainty measure can be defined in two different ways:

(a) as an arithmetic mean of \( D\varphi(\varphi_0) \) with respect to \( \varphi_0 \in I_{2\pi} \), and

(b) as an extremal value \(^1\) of \( D\varphi(\varphi_0) \) in \( I_{2\pi} \), where \( I_{2\pi} \) is any interval of length \( 2\pi \),

\[
\begin{align*}
(a) \quad & a\Delta^2 \varphi = \frac{1}{2\pi} \int_{I_{2\pi}} D\varphi(\varphi_0) d\varphi_0, \\
(b) \quad & b\Delta^2 \varphi = \min_{\varphi_0 \in I_{2\pi}} D\varphi(\varphi_0).
\end{align*}
\]

We introduce also the arithmetic mean squared covariance (by integration in any \( 2\pi \) interval \( I_{2\pi} \))

\[
a(\Delta l_{2\varphi})^2 = \frac{1}{2\pi} \int_{I_{2\pi}} (\Delta l_{2\varphi}(\varphi_0))^2 d\varphi_0.
\]

Then taking into account eqs. \( \text{(9)}, \text{(11)} - \text{(13)} \) and the fact that minimum of \( D\varphi(\varphi_0) \) is achieved at some \( \varphi_0 = \varphi_{\text{min}} \), we arrive at two Schrödinger type uncertainty relations \( (\Delta l_z = D l_z = (\Delta l_z)^2) \)

\[
i\Delta^2 \varphi \Delta^2 l_z - i(\Delta l_z \varphi)^2 \geq i(\text{Im} G_{l_z \varphi})^2,
\]

where \( i = a, b \) and \( a(\text{Im} G_{l_z \varphi})^2 \) is the arithmetic mean of \( (\text{Im} G_{l_z \varphi}(\varphi_0))^2 \). Thus both measures \( a\Delta \varphi \) and \( b\Delta \varphi \) are supported by inequalities of the type of Schrödinger uncertainty relation. It follows from this analogy that the quantities \( i\Delta^2 \varphi \), \( \Delta^2 l_z \), and \( i(\Delta l_z \varphi)^2 \) could be regarded as (generalized) second moments of \( \varphi \) and \( l_z \).

The examinations show that in a variety of examples the quantities \( a\Delta^2 \varphi \) and \( b\Delta^2 \varphi \) behave as relevant position uncertainty measures on the circle. Both measures distinguish between all states presented on figure 1 and figure 2, their value for the uniform distribution being greater than that for the other distributions. On the states in figure 1 and figure 2 we have a satisfactory arrangement of the spread measures, consistent with the visualized localization. The values of \( b\Delta^2 \varphi \), for example, read

\[
\begin{align*}
&b\Delta^2 \varphi|_{p_0(\varphi)} = \frac{\pi^2}{3} > b\Delta^2 \varphi|_{p_{2\pi}(\varphi)} = 3.16 > b\Delta^2 \varphi|_{p_{\varphi}(\varphi)} = 2.79, \quad (\text{figure 1}), \\
&\frac{\pi^2}{3} > b\Delta^2 \varphi|_{p_{\varphi}(\varphi)} = 2.61 > b\Delta^2 \varphi|_{p_{\varphi}(\varphi)} = 0.5, \quad (\text{figure 2}).
\end{align*}
\]

Compare the results \( \text{(15)} \) and \( \text{(16)} \) with the corresponding values of measures \( \text{(5)} \) and \( \text{(7)} \). For example compare \( \text{(13)} \) with \( \Delta^2(\varphi)|_{p_{\varphi}} = 0.346 < \Delta^2(\varphi)|_{p_{\varphi}} = 0.5 \).

There is a third invariantly defined state characteristic point on the circle (the first two are the points, where \( D\varphi(\varphi_0) \) attains its extrema). This third point is the center of the packet \( p(\varphi) \), denoted here as \( \varphi_c \). For a large set of distributions the center of the packet \( \varphi_c \) can be defined and determined as the angle of the centroid of \( p(\varphi) \). The cartesian coordinates of the centroid are \( x = \langle \cos \varphi \rangle \) and \( y = \langle \sin \varphi \rangle \). We define the third measure of spread of \( p(\varphi) \) as \( c\Delta^2 \varphi \)

\[
c\Delta^2 \varphi = D\varphi(\varphi_0 = \varphi_c),
\]

where \( D\varphi(\varphi_0) \) is the second moment \( \text{(8)} \).

The choice of \( \varphi_0 = \varphi_c \) in the limits of integration in \( \text{(8)} \) was wrongly interpreted in \( \text{(11)} \) as introduction of a definition of average values depending on the particular state. To reveal this misinterpretation suffice it to recall that \( \varphi_c \) is a characteristic point of the distribution \( p(\varphi) \), therefore of the state \( |\psi(\varphi)\rangle \): the value of \( \varphi_c \), and thereby the value of \( c\Delta^2 \varphi \) and \( \langle \varphi \rangle_{\varphi_c} \) are determined solely by the state \( |\psi(\varphi)\rangle \). Thus \( c\Delta^2 \varphi \), first proposed in \( \text{(2)} \), is a correct positive functional of the state and may be examined as an uncertainty measure.

\(^1\)We consider the minimal value only, since the maximal one may be greater than that of the uniform distribution.
A problem with the definition (17) appears in the case of $\pi$-periodic distributions $p(\varphi)$, since in such cases centroid’ angle is not determined (the centroid is in the origin). The subtle however is easily overcome if one note [4] that the centroid is a natural measure of the mean of the distribution. This gives a hint to define more generally the center of the packet $\varphi_c$ as solution of the equation

$$M\varphi(\varphi_0) = \varphi_0,$$

where $M\varphi(\varphi_0)$ is the limit-dependent mean of $\varphi$ given by (8a). The examination shows that the centroid’ angle $\varphi_c$, when exits, is a solution of eq. (18). For $\pi$-periodic distributions the centroid is in the origin, and $\varphi_c$ remains undefined. It turned out that for such distributions eq. (18) has more than one solution, i.e. there are several equivalent points $\varphi_{c,i}$. We will say that in such cases several points $\varphi_{c,i}$ on the circle may serve as "centers of the packet", or the packet is "multi-centered". If $p(\varphi + \pi/k) = p(\varphi)$, $k = 1, \ldots, n$ then equation (18) should have $2n$ different solutions $\varphi_{c,i}$, $i = 1, \ldots, 2n$. For $p_1(\varphi), p_2(\varphi)$ on figure 1 (and $p_{c1}(\varphi), p_{c2}(\varphi)$ on figure 2) we have solutions $\varphi_c = \pm \pi/2, \varphi_c = \pm \pi/4, \pm 3\pi/4$ (and $\varphi_c = 0, \varphi_c = 0, \pi$). For the uniform distribution eq. (18) degenerates to the identity $\varphi_0 = \varphi_0$, i.e. for $p_0(\varphi)$ all points on the circle are equivalent.

The equation (18) may be difficult for analytical handling, but solutions can be easily found numerically, or by the following rule/anzatz: $\varphi_{c,i}$ are points $\varphi_{\text{min}}$ of the global minimum of the second moment $D\varphi(\varphi_0)$, eq. (8), as a function of $\varphi_0$. This means that $\Delta^2 \varphi(\varphi_{c,i})$, $i = 1, \ldots, n$, coincide, and

$$D\varphi(\varphi_{c,i}) = \Delta^2 \varphi, \quad i = 1, \ldots, n.$$  

The rule works (is confirmed) on the example of a variety of distributions $p(\varphi)$, in particular on all examples in figures 1 and 2. Since the global minimum of $D\varphi(\varphi_0)$ can be calculated invariantly in any interval $I_{2\pi} \ni \varphi_0$ the above coincidence confirms again that the measure $\Delta^2 \varphi$ depends solely on the state.

4. Conclusion

In this paper we have introduced and discussed new position uncertainty (delocalization) measures for a particle on the circle. The relevant measure properties are illustrated on several examples, where the previous measures (based on position operators sin $\varphi$, cos $\varphi$, or exp($i2\varphi$)) appear to be unsatisfactory. The new measures resort on multiplication angle operator variance (see eqs. (11), (12), (17)) and obey uncertainty relations of the Schrödinger–Robertson type (with appropriate generalizations of the notions of covariance and mean commutator for the angle and angular momentum observables).

The first two measures are defined as arithmetic mean of the angle variance or as minimal value of the second moment $\Delta^2 \varphi(l_2)$. These possibilities stem from eqs. (9), (11). From (9) and (11) we also derive the uncertainty relations

$$\Delta^2 l_z + \Delta^2 \varphi \geq 2|\text{Im} G_{\varphi l_z}|, \quad i = a, b, c.$$  

The counterpart of this inequality on the real line is $\Delta^2 x + \Delta^2 p_x \geq 1$, which is minimized in the canonical CS $|\alpha\rangle$ only [11]. There are no periodic wave functions on the circle, that precisely minimize (19). Calculations show that they are approximately minimized in the CS on the circle $|\xi\rangle$ [5] [6]: in $|\xi\rangle$ the sum $\Delta^2 l_z + \Delta^2 \varphi$ attains the minimal value, which is very close to 1. In this sense $|\xi\rangle$ are most localized states in the phase space. Let us note, that in $|\xi\rangle$ one also has $\Delta^2(l_z) + \Delta^2(\varphi) = 1$ [3].
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Figure Captions

Figure 1. π-periodic, π/2-periodic and uniform distributions on the circle $p_\pi(\phi) = |\psi_\pi(\phi)|^2$, $p_{\pi/2}(\phi) = |\psi_{\pi/2}(\phi)|^2$ and $p_0(\phi) = |\psi_0(\phi)|^2$. The functional $\Delta \varphi$, eq. (5), on all these distributions takes the same maximal value of 1, while $\Delta^2(\hat{\varphi})$, eq. (7), takes the values $0.346$, $\infty$ and $\infty$ respectively.

Figure 2. One- and two-peak $\varphi$-distributions $p_{\omega}\phi(\varphi), p_{\omega4}(\varphi)$, corresponding to the CS $|\xi_1=1|$ and to state $\psi_{\omega4}(\varphi)$ = const. $(0.2 + \sin^2 \varphi)^2$ on the circle. Here $\Delta_{p_{\omega}}\varphi < \Delta_{p_{\omega4}}\varphi = 1$, while $\Delta^2_{p_{\omega}}(\hat{\varphi}) > \Delta^2_{p_{\omega4}}(\hat{\varphi}) = 0.143$. 
Figure 1. \(\pi\)-periodic, \(\pi/2\)-periodic and uniform distributions on the circle \(p_0(\phi) = |\psi_0(\phi)|^2\), \(p_{3/2}(\phi) = |\psi_{3/2}(\phi)|^2\) and \(p_1(\phi) = |\psi_1(\phi)|^2\). The functional \(\Delta_\phi\), eq. (6), on all these distributions takes the same maximal value of 1, while \(\Delta_s^2(\phi)\), eq. (7), takes the values 0.346, \(\infty\) and \(\infty\) respectively.

Figure 2. One- and two-peak \(\varphi\)-distributions \(p_{cs}(\varphi)\), \(p_{al}(\varphi)\), corresponding to the CS \(k = 1\) and to state \(\psi_{al}(\varphi) = \text{const.} \cdot (0.2 + \sin^2 \varphi)^2\) on the circle. Here \(\Delta_\varphi, \varphi < \Delta_{p_1} \varphi = 1\), while \(\Delta_{al}^2(\varphi) > \Delta_{cs}^2(\varphi) = 0.143\).