Geometric Pluripotential Theory on Sasaki Manifolds

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Abstract
We extend profound results in pluripotential theory on Kähler manifolds (Darvas in arXiv:1902.01982, 2019) to Sasaki setting via its transverse Kähler structure. As in Kähler case, these results form a very important piece to solve the existence of Sasaki metrics with constant scalar curvature in terms of properness of $\mathcal{K}$-energy, considered by the first named author in He (arXiv:1802.03841, 2019). One main result is to generalize Darvas’ theory on the geometric structure of the space of Kähler potentials in Sasaki setting. Along the way we extend most of corresponding results in pluripotential theory to Sasaki setting via its transverse Kähler structure.

Keywords Sasaki structure · Transverse Kähler potential · Orlicz–Finsler geometry · Pythagorean formula

Mathematics Subject Classification 53C25 · 32U15

1 Introduction
Sasaki manifolds have gained their prominence in physics, algebraic geometry, and Riemannian geometry [13]. There are tremendous work in the last two decades in Sasaki geometry, in particular on Sasaki–Einstein manifolds, see [13,14,27,37,39,50,54] and reference therein. On the other hand, Sasaki geometry is an odd-dimensional analog of Kähler geometry and almost all results in Kähler geometry have their counterparts in Sasaki geometry. Calabi’s extremal metric [17,18] (and csck) has played a very important role in Kähler geometry and it has a direct adaption in Sasaki set-

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In 1997, Donaldson [34] proposed an extremely fruitful program to approach existence of csck (extremal metrics) on a compact Kähler manifold with a fixed Kähler class. Donaldson’s program has also been extended to Sasaki setting, see [42,46] for example.

A major problem in Kähler geometry is to characterize exactly when a Kähler class contains a csck (extremal). The analytic part for existence of csck is to solve a fourth-order highly non-linear elliptic equation, the scalar curvature-type equation. This problem is regarded as a very hard problem in the field. Recently Chen and Cheng [22–24] have solved a major conjecture that existence of csck is equivalent to well-studied conditions such as properness of Mabuchi’s $K$-energy, or geodesic stability. The first named author [49] proved the following counterpart in Sasaki setting,

**Theorem 1** [49] *There exists a Sasaki metric with constant scalar curvature if and only if the $K$-energy is reduced proper with respect to $\text{Aut}_0(\xi, J)$, the identity component of automorphism group which preserves the Reeb vector field and transverse complex structure.*

The proof of Theorem 1 is a adaption of recent breakthrough of Chen–Cheng [24] on the existence of csck in Kähler setting to Sasaki setting. Technically, the arguments consist of two major parts: a priori estimates of non-linear PDE and pluripotential theory. Building up on previous development of pluripotential theory, Darvas [28,29] has developed profound theory to study the geometric structure of space of Kähler potentials. Among others, he introduced a Finsler metric $d_1$, and proved very effective estimates of distance function $d_1$ in terms of well-studied energy functionals such as Aubin’s $I$-functional. Darvas’s results turn out to be very useful to understand the geometric structure of space of Kähler potentials, in particular in the study of csck [6,24,32]. In this paper, we extend many results in pluripotential theory on Kähler manifolds, notably in [28,29,44] to Sasaki setting. These results play an important role in the proof of Theorem 1. To prove these results, we need to explore the geometric structures of Sasaki manifolds, in particular the Kähler cone structure and transverse Kähler structure.

Let $(M, g)$ be a compact Riemannian manifold of dimension $2n + 1$, with a Riemannian metric $g$. Sasaki manifolds have very rich geometric structures and have many equivalent descriptions. A probably most straightforward formulation is as follows: its metric cone

$$X = M \times \mathbb{R}_+, \quad g_X = dr^2 + r^2 g$$

is a Kähler cone. Hence there exists a complex structure $J$ on $X$ such that $(g_X, J)$ defines a Kähler structure. We identify $M$ with its natural embedding $M \to \{ r = 1 \} \subset X$. The 1-form $\eta$ is given by $\eta = J(r^{-1} dr)$ and it defines a contact structure on $M$. The vector field $\xi := J(r \partial_r)$ is a nowhere vanishing, holomorphic Killing vector field and it is called the Reeb vector field when it is restricted on $M$. The integral curves of $\xi$ are geodesics, and give rise to a foliation on $M$, called the Reeb foliation. Then there is a Kähler structure on the local leaf space of the Reeb foliation, called the transverse Kähler structure. A standard example of a Sasaki manifold is the odd-dimensional
round sphere $S^{2n+1}$. The corresponding Kähler cone is $\mathbb{C}^{n+1}\setminus\{0\}$ with the flat metric and its transverse Kähler structure descends to $\mathbb{C}P^n$ with the Fubini-Study metric.

We can also formulate Sasaki geometry, in particular the transverse Kähler structure via its contact bundle $D = \text{Ker}(\eta) \subset TM$. The complex structure $J$ on the cone descends to the contact bundle via $\Phi_1 := J|_D$. The Sasaki metric can be written as follows,

$$g = \eta \otimes \eta + g^T,$$

where $g^T$ is the transverse Kähler metric, given by $g^T := 2^{-1} d\eta(\Phi_1 \otimes I)$. The transverse Kähler form is denoted by $\omega^T = 2^{-1} d\eta$. We shall study the transverse Kähler geometry of Sasaki metrics, with the Reeb vector field $\xi$ and transverse complex structure (equivalently the complex structure $J$ on the cone) both fixed. This means that we fix the basic Kähler class $[\omega^T]$ with $\omega^T = 2^{-1} d\eta$ and study the Sasaki structures induced by the space of transverse Kähler potentials,

$$\mathcal{H} = \{ \phi \in C^\infty_B(M) : \omega_\phi = \omega^T + d_B d_B^c \phi > 0 \},$$

where $C^\infty_B(M)$ is the space of smooth basic functions. The main result in the paper is:

**Theorem 2** ($\mathcal{E}_p(M, \xi, \omega^T), d_p$) is a complete geodesic metric space for $p \in [1, \infty)$, which is the metric completion of $(\mathcal{H}, d_p)$. For any $u, v \in \mathcal{E}_p(M, \xi, \omega^T)$, $d_p(u, v)$ is realized by a unique finite-energy geodesic in $\mathcal{E}_p(M, \xi, \omega^T)$ connecting $u$ and $v$. There exists a uniform constant $C = C(n, p) > 1$ such that

$$C^{-1} I_p(u, v) \leq d_p(u, v) \leq C I_p(u, v),$$

where the energy functional $I_p$ is given by

$$I_p(u, v) = \|u - v\|_{p,u} + \|u - v\|_{p,v}.$$

Moreover, we have

$$d_p \left( u, \frac{u + v}{2} \right) \leq C d_p(u, v).$$

We refer to Sect. 3 for notions such as $\mathcal{E}_p(M, \xi, \omega^T), d_p$. Theorem 2 is the counterpart of main results in [28] in Sasaki setting. An important notion in the study of csck is the convexity of $\mathcal{K}$-energy along $C^{1,1}$ geodesics [3] (see also [25]), which was generalized to Sasaki setting by [51,58]. Given the results above, one can then extend $\mathcal{K}$-energy to $\mathcal{E}_1$-class and keep its convexity along finite energy geodesics as in [7]. Moreover, this allows to define precisely the properness of $\mathcal{K}$-energy in terms of the distance $d_1$. One can then prove Theorem 1 using a priori estimates of scalar curvature-type equation together with properness assumption, where the effective estimates of $d_1$ in Theorem 2 play an important role; for details, see [49].

Along the way to prove Theorem 2, it is necessary to extend results as in [30,44] to Sasaki setting. Certainly the essential ideas lie in results in Kähler setting and many results are rather straightforward extensions from Kähler setting; we refer to Darvas’ lecture notes [30] for an excellent reference. However, we should also emphasize that in
Sasaki setting, there are several substantial new difficulties when the Reeb foliation is irregular. To overcome these difficulties, the Sasaki structure (the Kähler cone structure and transverse Kähler structure) plays an essential role. Lemma 3.1 is an extension of Blocki–Kolodziej’s approximation of plurisubharmonic functions by smooth decreasing sequence. For this proof we construct explicit holomorphic charts on the Kähler cone out of its transverse Kähler structure, see Lemma 2.1. This very explicit relation between the holomorphic charts and foliation charts of transverse Kähler structure seems to appear in literature for the first time, to the authors’ knowledge. This explicit construction of holomorphic charts builds a very straightforward relation between plurisubharmonic functions on cone and (transverse) plurisubharmonic functions via transverse Kähler structure. Lemmas 3.2 and 3.3 give Darvas’ volume partition formula for rooftop construction. This decomposition is a very important technical result for Darvas’ theory and the proof in Kähler setting does not carry over for irregular Sasaki structures. We overcome this difficulty using Type-I deformation (see Theorem 6.1, Lemmas 6.1 and 6.2). (Of course there are many other places that there are substantial new difficulties; for example, the geodesic equation solved by Guan–Zhang is harder.) For completeness we include the details of almost all arguments, even in the case when the proof follows rather straightforwardly from the Kähler setting. The pluripotential theory in Sasaki setting has few references (see [51,58] for example) and we hope that our presentation is helpful.

We organize the paper as follows. In Sect. 2 we introduce basic notations and concepts of Sasaki geometry. We study the geometric structure of the space of transverse Kähler potentials using geodesic equation and pluripotential theory in Sect. 3. In Sect. 4 we prove the main theorem. We include a brief discussion of Sasaki-extremal metric in Sect. 5. Appendix contains various topics in pluripotential theory, including complex Monge–Ampere operator and various energy functionals on $E_1$; we prove various results which are stated in [49, Section 2.2].

2 Preliminary on Sasaki Geometry

A good reference on Sasaki geometry can be found in the monograph [13] by Boyer–Galicki. Let $M$ be a compact differentiable manifold of dimension $2n + 1 (n \geq 1)$. A Sasaki structure on $M$ is defined to be a Kähler cone structure on $X = M \times \mathbb{R}_+$, i.e., a Kähler metric $(g_X, J)$ on $X$ of the form

$$g_X = dr^2 + r^2 g,$$

where $r > 0$ is a coordinate on $\mathbb{R}_+$, and $g$ is a Riemannian metric on $M$. We call $(X, g_X, J)$ the Kähler cone of $M$. We also identify $M$ with the link $\{ r = 1 \}$ in $X$ if there is no ambiguity. Because of the cone structure, the Kähler form on $X$ can be expressed as

$$\omega_X = \frac{1}{2} \sqrt{-1} \partial \bar{\partial} r^2 = \frac{1}{2} \ddc r^2.$$
We denote by \( r \partial_r \) the homothetic vector field on the cone, which is easily seen to be a real holomorphic vector field. A tensor \( \alpha \) on \( X \) is said to be of homothetic degree \( k \) if

\[
\mathcal{L}_{r \partial_r} \alpha = k \alpha.
\]

In particular, \( \omega \) and \( g \) have homothetic degree two, while \( J \) and \( r \partial_r \) has homothetic degree zero. We define the Reeb vector field

\[
\xi = J (r \partial_r).
\]

Then \( \xi \) is a holomorphic Killing field on \( X \) with homothetic degree zero. Let \( \eta \) be the dual one-form to \( \xi \):

\[
\eta(\cdot) = r^{-2} g_X(\xi, \cdot) = 2d^c \log r = \sqrt{-1} (\partial - \bar{\partial}) \log r.
\]

We also use \( (\xi, \eta) \) to denote the restriction of them on \( (M, g) \). Then we have

\begin{itemize}
  \item \( \eta \) is a contact form on \( M \), and \( \xi \) is a Killing vector field on \( M \) which we also call the Reeb vector field;
  \item \( \eta(\xi) = 1 \), \( \iota_{\xi} d\eta(\cdot) = d\eta(\xi, \cdot) = 0 \);
  \item the integral curves of \( \xi \) are geodesics.
\end{itemize}

The Reeb vector field \( \xi \) defines a foliation \( \mathcal{F}_\xi \) of \( M \) by geodesics. There is a classification of Sasaki structures according to the global property of the leaves. If all the leaves are compact, then \( \xi \) generates a circle action on \( M \), and the Sasaki structure is called \textit{quasiregular}. In general, this action is only locally free, and we get a polarized orbifold structure on the leaf space. If the circle action is globally free, then the Sasaki structure is called \textit{regular}, and the leaf space is a polarized Kähler manifold. If \( \xi \) has a non-compact leaf, the Sasaki structure is called \textit{irregular}.

One can also understand Sasaki structure through contact metric structure. There is an orthogonal decomposition of the tangent bundle

\[
TM = L\xi \oplus D,
\]

where \( L\xi \) is the trivial bundle generated by \( \xi \), and \( D = \text{Ker}(\eta) \). The metric \( g \) and the contact form \( \eta \) determine a \((1, 1)\) tensor field \( \Phi \) on \( M \) by

\[
g(Y, Z) = \frac{1}{2} d\eta(Y, \Phi Z), \ Y, Z \in \Gamma(D)
\]

\( \Phi \) restricts to an almost complex structure on \( D \):

\[
\Phi^2 = -\mathbb{I} + \eta \otimes \xi.
\]

Since both \( g \) and \( \eta \) are invariant under \( \xi \), there is a well-defined Kähler structure \((g^T, \omega^T, J^T)\) on the local leaf space of the Reeb foliation. We call this a \textit{transverse Kähler structure}. In the quasiregular case, this is the same as the Kähler structure on...
the quotient. Clearly, $\omega^T = 2^{-1}d\eta$. The upper script $T$ is used to denote both the transverse geometric quantity, and the corresponding quantity on the bundle $D$. For example, we have on $M$

$$g = \eta \otimes \eta + g^T.$$ 

From the above discussion it is not hard to see that there is an intrinsic formulation of a Sasaki structure as a compatible integrable pair $(\eta, \Phi)$, where $\eta$ is a contact one-form and $\Phi$ is an almost CR structure on $D = \text{Ker} \eta$. Here “compatible” means first that $d\eta(\Phi U, \Phi V) = d\eta(U, V)$ for any $U, V \in D$, and $d\eta(U, \Phi U) > 0$ for any non-zero $U \in D$. Further, we require $\mathcal{L}_\xi \Phi = 0$, where $\xi$ is the unique vector field with $\eta(\xi) = 1$, and $d\eta(\xi, \cdot) = 0$. $\Phi$ induces a splitting $D \otimes \mathbb{C} = D_{-1,0} \oplus D_{0,1}$, with $\mathcal{D}^{1,0} = D^{0,1}$. “Integrable” means that $[D^{0,1}, D^{0,1}] \subset D^{0,1}$. This is equivalent to that the induced almost complex structure on the local leaf space of the foliation by $\xi$ is integrable. For more discussions on this, see [13, Chapter 6].

**Definition 2.1** A $p$-form $\theta$ on $M$ is called basic if

$$\iota_\xi \theta = 0, \quad L_\xi \theta = 0.$$ 

Let $\Lambda^p_B$ be the bundle of basic $p$-forms and $\Omega^p_B = \Gamma(S, \Lambda^p_B)$ the set of sections of $\Lambda^p_B$.

The exterior differential preserves basic forms. We set $d_B = d|\Omega^p_B$. Thus the subalgebra $\Omega^p_B(\mathcal{F}_\xi)$ forms a subcomplex of the de Rham complex, and its cohomology ring $H^p_B(\mathcal{F}_\xi)$ is called the basic cohomology ring. When $(M, \xi, \eta, g)$ is a Sasaki structure, there is a natural splitting of $\Lambda^p_B \otimes \mathbb{C}$ such that

$$\Lambda^p_B \otimes \mathbb{C} = \bigoplus \Lambda^i_j B,$$

where $\Lambda^i_j B$ is the bundle of type $(i, j)$ basic forms. We thus have the well-defined operators

$$\partial_B : \Omega^i_{j} B \to \Omega^{i+1,j} B,$$

$$\bar{\partial}_B : \Omega^i_{j} B \to \Omega^{i,j+1} B.$$ 

Then we have $d_B = \partial_B + \bar{\partial}_B$. Set $d^c_B = \frac{1}{2} \sqrt{-1} (\bar{\partial}_B - \partial_B)$. It is clear that

$$d_B d^c_B = \sqrt{-1} \partial_B \bar{\partial}_B, \quad d_B^2 = (d^c_B)^2 = 0.$$ 

We shall recall the transverse complex (Kähler) structure on local coordinates. Let $U_\alpha$ be an open covering of $M$ and $\pi_\alpha : U_\alpha \to V_\alpha \subset \mathbb{C}^n$ submersions such that

$$\pi_\alpha \circ \pi^{-1}_\beta : \pi_\beta(U_\alpha \cap U_\beta) \to \pi_\alpha(U_\alpha \cap U_\beta)$$

We have

$$\partial_B d_B = \sqrt{-1} \partial_B \bar{\partial}_B.$$ 

$$d_B^2 = (d^c_B)^2 = 0.$$ 

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We have

$$\partial_B d_B = \sqrt{-1} \partial_B \bar{\partial}_B.$$ 

$$d_B^2 = (d^c_B)^2 = 0.$$
is biholomorphic when \( U_\alpha \cap U_\beta \) is not empty. One can choose local coordinate charts \((z_1, \ldots, z_n)\) on \( V_\alpha \) and local coordinate charts \((x, z_1, \ldots, z_n)\) on \( U_\alpha \subset M\) such that \( \xi = \partial_x \), where we use the notations
\[
\partial_x = \frac{\partial}{\partial x}, \quad \partial_i = \frac{\partial}{\partial z_i}, \quad \partial j = \frac{\partial}{\partial \bar{z}_j} = \frac{\partial}{\partial z_j}.
\]

The map \( \pi_\alpha : (x, z_1, \ldots, z_n) \to (z_1, \ldots, z_n) \) is then the natural projection. There is an isomorphism, for any \( p \in U_\alpha \),
\[
d\pi_\alpha : D_p \to T_{\pi_\alpha(p)}V_\alpha.
\]

Hence the restriction of \( g \) on \( D \) gives an Hermitian metric \( g_\alpha^T \) on \( V_\alpha \) since \( \xi \) generates isometries of \( g \). One can verify that there is a well-defined Kähler metric \( g_\alpha^T \) on each \( V_\alpha \) and
\[
\pi_\alpha \circ \pi_\beta^{-1} : \pi_\beta(U_\alpha \cap U_\beta) \to \pi_\alpha(U_\alpha \cap U_\beta)
\]
gives an isometry of Kähler manifolds \((V_\alpha, g_\alpha^T)\). The collection of Kähler metrics \( \{g_\alpha^T\} \) on \( \{V_\alpha\} \) can be used as an alternative definition of the transverse Kähler metric. The (local) transverse holomorphic (Kähler) structure is essential for us and we shall use these charts enormously. We summarize as follows:

**Definition 2.2 (Local foliation charts)** We can choose the open covering \( \{U_\alpha\} \) of \( M \) such that there exists a local product structure for each \( \alpha \), determined by its foliation structure and transverse complex structure. That is, there are charts
\[
\Psi_\alpha : U_\alpha \to W_\alpha \subset \mathbb{R} \times \mathbb{C}^n,
\]
where \( W_\alpha = (-\delta, \delta) \times V_\alpha \). For a point \( p \in W_\alpha \), we write \( p = (x, z_1, \ldots, z_n) \) with \( \xi = \partial_x \) and \( V_\alpha = B_r(0) \subset \mathbb{C}^n \) for \( 0 < r \). We assume that \( \delta, r \) are sufficiently small depending only on \( (M, \xi, \eta, g) \), and \( \omega_\alpha^T \) is uniformly equivalent to an Euclidean metric on each \( V_\alpha = B_r \subset \mathbb{C}^n \),
\[
\frac{1}{2} \delta_{ij} \leq \omega_\alpha^T \leq 2 \delta_{ij}.
\]

In Sasaki geometry, it is often mostly convenient to work with these charts when we need to consider the Sasaki structure locally. For each \( U_\alpha \), we assume it is contained in the geodesic normal neighborhood of its “center,” \( \Psi_\alpha^{-1}(0, 0, \ldots, 0) \), by choosing \( \delta, r \) small enough. We call these charts foliation charts. The existence of foliation charts is well known in the subject, see [40]; in particular, any Sasaki metric \( g \) can be locally expressed in terms of a real function of \( 2n \) variables. Given a foliation chart \( W_\alpha = (-\delta, \delta) \times V_\alpha \), for \( (x, z_1, \ldots, z_n) \in U_\alpha \), locally there exists a strictly plurisubharmonic function \( h : V_\alpha \to \mathbb{R} \), and the Sasaki structure reads
\[ \xi = \partial_x; \quad \eta = dx - \sqrt{-1} \sum_i (h_i dz^i - h^i \bar{z}^j) \]

\[ \omega^T = \sqrt{-1} h_{ij} dz^i \wedge d\bar{z}^j; \quad g = \eta \otimes \eta + 2 h_{ij} dz^i \otimes d\bar{z}^j. \] (2.1)

If we consider a Sasaki structure induced by a transverse Kähler potential \( \phi \), then locally we have \( h \rightarrow h + \phi \). In particular, we have

\[ \eta_{\phi} = \eta + \sqrt{-1} (\bar{\partial} - \partial) \phi, \quad \omega_{\phi} = \omega^T + \sqrt{-1} \partial \bar{\partial} \phi. \]

We shall also use holomorphic charts on its Kähler cone \( X \). There exist indeed holomorphic charts on the Kähler cone \( X \) which are closely related to foliation charts on \( M \). This seems to be much less well known and we shall describe them now.

**Lemma 2.1** (Holomorphic coordinates on the Kähler cone) *For a Sasaki structure locally generated by a plurisubharmonic function \( h \): \( V_\alpha \rightarrow \mathbb{R} \) in foliation charts on \( M \), then the following gives a local holomorphic structure on its Kähler cone \( X \), for \( w = (w_0, \ldots, w_n) \in \tilde{U}_\alpha \subset \mathbb{C} \times V_\alpha \),

\[ w_0 = \log r - h(z, \bar{z}) + \sqrt{-1} x, \quad w_i = z_i, \quad i = 1, \ldots, n, \quad z = (z_1, \ldots, z_n). \] (2.2)

The holomorphic structure \( J \) is given by the holomorphic coordinates \( w = (w_0, \ldots, w_n) \),

\[ J \frac{\partial}{\partial w_i} = \sqrt{-1} \frac{\partial}{\partial w_i}, \quad i = 0, \ldots, n. \] (2.3)

**Proof** Given (2.1), it is straightforward to check that (2.2) gives a holomorphic chart satisfying (2.3). \( \square \)

**Remark 2.1** These holomorphic charts would be very useful for us later; in particular, when we consider plurisubharmonic functions on \( X \) and transverse plurisubharmonic functions on \( M \). The explicit holomorphic charts given above seem to appear in literature first time to our knowledge, while the foliation charts are well known.

When the Reeb vector field \( \xi \) is irregular, the local foliation charts satisfy cocycle condition but they do not give a manifold (or orbifold) structure of the quotient \( M/F_\xi \).

We shall recall *Type-I deformation* defined in [15]. Let \((M, \xi_0, \eta_0, g_0)\) be a compact Sasaki manifold, denote its automorphism group by \( \text{Aut}(M, \xi_0, \eta_0, g_0) \). We fix a torus

\[ T \subset \text{Aut}(M, \xi_0, \eta_0, g_0) \text{ such that } \xi_0 \in t = \text{Lie algebra}(T). \]

**Definition 2.3** (Type-I deformation) Let \((M, \xi_0, \eta_0, g_0)\) be a \( T \)-invariant Sasaki structure. For any \( \xi \in t \) such that \( \eta_0(\xi) > 0 \). We define a new Sasaki structure on \( M \) explicitly as

\[ \eta = \frac{\eta_0}{\eta_0(\xi)}, \quad \Phi = \Phi_0 - \Phi_0 \xi \otimes \eta, \quad g = \eta \otimes \eta + \frac{1}{2} d\eta(\mathbb{I} \otimes \Phi). \]

Note that under Type-I deformation, the essential change is the Reeb vector field \( \xi_0 \leftrightarrow \xi \) and this construction can be done vice versa.
3 The Space of Transverse Kähler Potentials

In this section, we consider the space of transverse Kähler potentials on a compact Sasaki manifold through its transverse Kähler structure. It turns out to be necessary to consider these objects not only from point of view of PDE, but also from the point of view of pluripotential theory. Geometric pluripotential theory on Kähler manifolds turns out to be one crucial piece in the proof of properness conjecture [6,24]. We refer [30,44] and references therein for details of pluripotential theory. We extend these results to Sasaki manifolds. These results would form a crucial piece for existence of constant scalar curvature (cscs) on Sasaki manifolds as well, see [49] for details.

Using the transverse Kähler structure of a Sasaki structure, many of the extensions of pluripotential theory on Kähler manifolds to Sasaki manifolds are rather straightforward, and the proofs are a direct adaption of Kähler setting with some necessary modifications. On the other hand, there are several exceptions that would need essential inputs from the Sasaki structure. And the proofs are new and substantially different, compared with the Kähler setting. We summarize the main differences as follows. The first is Lemma 3.1, where we will prove a counterpart of an approximation result of plurisubharmonic functions as in Kähler setting by Blocki–Kolodziej [10]. One can apply Blocki–Kolodziej approximation locally to transverse Kähler structure and obtain a local approximation, but such construction has trouble to patch together when the Sasaki structure is irregular. Instead, we need to do the construction on the Kähler cone, and the holomorphic chart on the cone (Lemma 2.1) plays a substantial role in our construction. The second main difference is Lemma 3.2, where we will prove an important property of the rooftop envelop construction $P(u_0, u_1)$ on the non-contact set; this result plays a very important role in Darvas’s theory. The proof as in Kähler setting again does not work directly to Sasaki setting when the Sasaki structure is irregular. Instead, we need to apply a Type-I deformation carefully (Theorem 6.1, Lemma 6.1) to bypass the difficulty.

3.1 The Quasiplurisubharmonic Functions on Sasaki Manifolds

Denote $\mathcal{H} = \{ \phi \in C^\infty_B(M) : \omega_\phi = \omega^T + \sqrt{-1} \partial B \bar{\partial}_B \phi > 0 \}$, the space of transverse Kähler potentials on a Sasaki manifold $(M, \xi, \eta, g)$. Given $\phi \in \mathcal{H}$, it defines a new Sasaki structure, $(M, \xi, \eta_\phi, g_{\eta_\phi})$ as follows,

$$\eta_\phi = \eta + 2d_B^{\phi}, \omega_\phi = \omega^T + \sqrt{-1} \partial B \bar{\partial}_B \phi, g_{\eta_\phi} = \eta_\phi \otimes \eta_\phi + \omega_\phi.$$

The most relevant results in pluripotential theory for us lie in [44], [5, Section 2], [45] and [30]. Part of them has been done by van Covering [58, Section 2], including the Monge–Ampere operator and weak convergence, with main focus on $L^\infty$ and $C^0$ potentials. We shall need most of the results on the energy classes $E$ and $E_p$ (defined below).

Given a Sasaki structure $(M, \xi, \eta, g)$, we recall the following definition,

**Definition 3.1** An $L^1$, upper semicontinuous (usc) function $u : M \to \mathbb{R} \cup \{-\infty\}$ is called a transverse $\omega^T$-plurisubharmonic (TPSH for short) if $u$ is invariant under
the Reeb flow, and \( u \) is \( \omega^T \)-plurisubharmonic on each local foliation chart \( V_\alpha \), that is
\[
\omega^T_\alpha + \sqrt{-1} \partial \bar{\partial} u \geq 0 \text{ as a (1, 1)-current on } V_\alpha.
\]

It is apparent that the definition above does not depend on the choice of foliation charts. Indeed, \( u \) is invariant along the flow of \( \xi \) and we extend \( u \) trivially in the cone direction to a function on cone. Using the holomorphic structure on the cone (see Lemma 2.1), \( u \) is a TPSH if and only if \( \omega^T + \sqrt{-1} \partial \bar{\partial} u \geq 0 \) is a closed, positive (1,1) current on the cone \( X \). We use the notation,

\[
\text{PSH}(M,\xi,\omega^T) = \{ u \in L^1(M) : u \text{ is usc and invariant under the Reeb flow; } \omega_u \geq 0 \}
\]

One of the cornerstones of Bedford–Taylor theory [2] is to associate a complex Monge–Ampere measure to a bounded psh function. Their construction generalizes to bounded Kähler potentials in a straightforward manner [44] and it has direct adaption to Sasaki setting. We refer to [58, Section 2] and Sect. 1 for definition of complex Monge–Ampere measures

\[
\omega^n_u \wedge \eta = \omega^n_{\max(u,v)} \wedge \eta.
\]

Proposition 3.1 Suppose that the sequence \( u_j \in \text{PSH}(M,\xi,\omega^T) \cap L^\infty \) decreases to \( u \in \text{PSH}(M,\xi,\omega^T) \cap L^\infty \). Then for \( k = 1, \ldots, n \), we have the following weak convergences of complex Monge–Ampere measures,

\[
\omega^k_{u_j} \wedge (\omega^T)^{n-k} \wedge \eta \to \omega^k_u \wedge (\omega^T)^{n-k} \wedge \eta.
\]

Proof By applying a partition of unity subordinated to covering by foliation charts, we need to show that for \( f \in C^\infty \), supported on a foliation chart \( W_\alpha = (-\delta,\delta) \times V_\alpha \)

\[
\int_M f \omega^k_{u_j} \wedge (\omega^T)^{n-k} \wedge \eta \to \int_M f \omega^k_u \wedge (\omega^T)^{n-k} \wedge \eta.
\]

We should emphasize that \( f \) is not a basic function in general. The weak convergence in Kähler setting implies that for each \( x \in (-\delta,\delta) \)

\[
\int_{V_\alpha} f(x,z,\bar{z}) \omega^k_{u_j} \wedge (\omega^T)^{n-k} \to \int_{V_\alpha} f(x,z,\bar{z}) \omega^k_u \wedge (\omega^T)^{n-k}.
\]

Note that for each \( x \), \( f \) is supported on \( V_\alpha \). Taking integration with respect to \( dx \), this leads to (3.2), since on \( W_\alpha \), \( \omega^k_u \wedge (\omega^T)^{n-k} \wedge \eta = \omega^k_u \wedge (\omega^T)^{n-k} \wedge dx \) as a product measure.

The following Bedford–Taylor identity in Sasaki setting would be used numerously:

Proposition 3.2 For \( u, v \in \text{PSH}(M,\xi,\omega^T) \cap L^\infty \),

\[
\chi_{\{u>v\}} \omega^n_{\max(u,v)} \wedge \eta = \chi_{\{u>v\}} \omega^n_u \wedge \eta.
\]
Proposition 3.3

There exists $C$ prove the following well-known result in pluripotential theory. The extension to Sasaki setting, where Proposition 3.2 plays an important role. First we prove this using sub-mean value property of plurisubharmonic functions, similar as Proposition 3.2. We use foliation charts in this argument.

Proof

When $u \in \text{PSH}(M, \xi, \omega^T)$, the Bedford–Taylor constructions to a much larger class on a compact Kähler manifold, see Guedj–Zeriahi [44]. The reference [30, Section 2] is sufficient for our purpose. These constructions in Kähler setting have a direct extension to Sasaki setting, where Proposition 3.2 plays an important role. First we prove the following well-known result in pluripotential theory.

Proposition 3.3

There exists $C = C(M, g)$ such that for any $u \in \text{PSH}(M, \xi, \omega^T)$,

$$
\sup_M u \leq \frac{1}{\text{Vol}(M)} \int_M u \, d\mu_g + C.
$$

Proof

When $u$ is $C^2$ this is obvious by the fact that $\Delta_g u + n \geq 0$. In general, we can prove this using sub-mean value property of plurisubharmonic functions, similar as in [30, Lemma 3.45]. In this proof, we can either use foliation charts on $M$ or Kähler cone structure on $X = C(M)$. We use foliation charts in this argument.

We assume $\sup_M u = 0$ and want to show that the integration of $u$ is uniformly bounded below. We can cover $M$ by finitely many nested foliation charts $U_k \subset W_k \subset M$ such that there exist diffeomorphisms $\phi_k : B(0, 4) \times (-2\delta, 2\delta) \to W_k$ with $\phi_k(B(0, 1) \times (-\delta, \delta)) = U_k$, where $\delta$ is a fixed positive constant and $B(0, 4) \subset B(0, 4) \subset \mathbb{C}^n$ are Euclidean balls centered at the origin in $\mathbb{C}^n$. We assume that $(z, x) \in B(0, 4) \times (-2\delta, 2\delta)$ such that $z \in B(0, 4)$ represents transverse holomorphic charts and $x \in (-2\delta, 2\delta)$ represents the Reeb direction (i.e., $\xi \equiv \partial_z \bar{z} \psi_k$). On each $W_k$, there exists a smooth basic function $\psi_k = \psi_k(z)$ such that $\omega_T = \sqrt{-1} \partial_z \bar{z} \psi_k$. Note that we only need to show that, there exists a uniformly bounded constant $C > 0$, such that

$$
\int_{U_k} u \, d\mu_g \geq -C, \quad k \in \{1, \ldots, N\}.
$$

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Note that $u$ is basic, we have
\[ \int_{B(0,1) \times (-\delta, \delta)} u \circ \varphi_k \, d\mu_{x,z} = 2\delta \int_{B(0,1)} u \circ \varphi_k(z, x_0) \, d\mu_z, \quad x_0 \in (-\delta, \delta) \]
where $d\mu_{x,z}$ and $d\mu_z$ are Euclidean measure on $\mathbb{C}^n \times \mathbb{R}$ and $\mathbb{C}^n$, respectively. Hence we only need to show that
\[ \int_{B(0,1)} u \circ \varphi_k(z, x_0) \, d\mu_z \geq -C, \quad k \in \{1, \cdots, N\}. \quad (3.4) \]

Note that by our construction, $(\psi_k + u) \circ \varphi_k$ is independent of $x$ and is plurisubharmonic on $B(0, 4)$ for each $k$. As $u$ is usc, its supremum is realized at some point $p_1 \in M$ such that $u \leq u(p_1) = 0$. Since $U_k$ covers $M$, we can assume $p_1 \in U_1$ with the coordinate $\varphi_1(z_1, x_1) = p_1$ for some $(z_1, x_1) \in B(0, 1) \times (-\delta, \delta)$. Note that since $u$ is basic, hence it is independent of $x$-coordinate we can also take $x_1 = 0$. Since $B(z_1, 2) \subset B(0, 4)$, we have the following sub-mean value property for $(\psi_1 + u) \circ \varphi_1$,
\[ \psi_1 \circ \varphi_1(z_1, 0) = (\psi_1 + u) \circ \varphi_1(z_1, 0) \leq \frac{1}{\mu(B(z_1, 2))} \int_{B(z_1, 2)} (\psi_1 + u) \circ \varphi_1(z, 0) \, d\mu_z. \]

Since $u \leq 0$ and $B(0, 1) \subset B(z_1, 2)$, there exists $C_1 > 0$, independent of $u$, such that
\[ \int_{B(0,1)} u \circ \varphi_1 \, d\mu_z \geq -C_1. \quad (3.5) \]

Since $\{U_k\}_k$ covers $M$, we can assume $U_1$ intersects $U_2$. We can choose $r_2 > 0$, such that $\varphi_2(B(z_2, r_2) \times (\delta_1, \delta_2)) \subset U_1 \cap U_2$ for some $B(z_2, r_2) \subset B(0, 1)$ and $-\delta < \delta_1 < \delta_2 < \delta$. Since $u \leq 0$, it follows that there exists $\tilde{C}_1 > 0$, independent of $u$ ($\tilde{C}_1$ depends only on $C_1, r_2$, and $\psi_2$), such that
\[ \frac{1}{\mu(B(z_2, r_2))} \int_{B(z_2, r_2)} (u + \psi_2) \circ \varphi_2 \, d\mu_z \geq -\tilde{C}_1. \]

Since $(u + \psi_2) \circ \varphi_2$ is plurisubharmonic in $B(0, 4)$, we can obtain that
\[ \frac{1}{\mu(B(z_2, 2))} \int_{B(z_2, 2)} (u + \psi_2) \circ \varphi_2 \, d\mu_z \geq \frac{1}{\mu(B(z_2, r_2))} \int_{B(z_2, r_2)} (u + \psi_2) \circ \varphi_2 \, d\mu_z \geq -\tilde{C}_1. \]

Since $u \leq 0$ and $B(0, 1) \subset B(z_2, 2)$, we obtain for some $C_2 > 0$
\[ \int_{B(0,1)} u \circ \varphi_2 \, d\mu_z \geq -C_2. \]
We continue this process to consider that $U_1 \cup U_2$ intersects a member, say $U_3$. After at most $N-2$ step, we prove (3.4).

As a direct consequence, we know the following (see [33, Proposition I.5.9]):

**Proposition 3.4** The set $\mathcal{C} = \{ u \in \text{PSH}(M, \xi, \omega^T) : \sup_M u \leq C \}$ is bounded in $L^1$ and it is precompact in $L^1(\text{d}\mu_g)$ topology.

**Proof** By the above-mentioned proposition, we know that $\sup_M u$ bounded above implies that $\int_M |u| \text{d}\mu_g$ is uniformly bounded. By the Motel property of subharmonic functions and plurisubharmonic functionals [33, Propositions I.4.21, I.5.9], $\mathcal{C}$ is precompact with respect to $L^1(\text{d}\mu_g)$ topology. Note that in Sasaki setting we apply the compactness of plurisubharmonic functions to nested foliation charts $U_k \subset W_k$ as above for $\omega^T_k$-plurisubharmonic functions locally, that $\mathcal{C}$ is precompact in $L^1$ topology in each $U_k$. After passing by subsequence if necessary, we can then get weak compactness of $\mathcal{C}$ with respect to $L^1(\text{d}\mu_g)$ topology. \(\Box\)

Let $v \in \text{PSH}(M, \xi, \omega^T)$. For $h \in \mathbb{R}$, we denote $v_h = \max\{v, -h\}$ to be the canonical cutoffs of $v$. It is evident that $v_h$ is invariant under the Reeb flow and hence $v_h \in \text{PSH}(M, \xi, \omega^T) \cap L^\infty$. If $h_1 < h_2$, then Proposition 3.2 implies that

$$\chi_{\{v > -h_1\}} \omega^n_{v_{h_1}} \wedge \eta = \chi_{\{v > -h_2\}} \omega^n_{v_{h_2}} \wedge \eta \leq \chi_{\{v > -h_2\}} \omega^n_{v_{h_2}} \wedge \eta.$$  

Hence $\chi_{\{v > -h\}} \omega^n_{v_h} \wedge \eta$ is an increasing sequence of Borel measure on $M$ with respect to $h$. This leads to the following definition:

**Definition 3.2** We define

$$\omega^n_v \wedge \eta := \lim_{h \to \infty} \chi_{\{v > -h\}} \omega^n_{v_h} \wedge \eta. \quad (3.6)$$

We shall emphasize that by the definition above, we have for any Borel set $B \subset M$,

$$\int_B \omega^n_v \wedge \eta = \lim_{h \to \infty} \int_B \chi_{\{v > -h\}} \omega^n_{v_h} \wedge \eta. \quad (3.7)$$

Hence the convergence in (3.6) is a stronger than the weak convergence of measures.

To proceed, we need the following approximation of TPSH functions. Our proof uses the Kähler cone structure and builds up on Blocki–Kolodziej [10].

**Lemma 3.1** Given $u \in \text{PSH}(M, \xi, \omega^T)$, there exists a decreasing sequence $\{u_k\}_{k \in \mathbb{N}} \subset \mathcal{H}$ such that $u_k$ converges to $u$.

**Proof** First we assume that $u$ has zero Lelong number. Recall $X$ is the Kähler cone and we identify $M$ with the link $\{r = 1\} \subset X$. For $u \in \text{PSH}(M, \xi, \omega^T)$, we extend $u$ to be a function on $X$ such that $u(r, p) = u(p)$, for any $r > 0$. We recall that $\omega^T = \frac{1}{2} \text{d}\eta = \text{d}d^c(\log r) = \sqrt{-1} \partial \bar{\partial}(\log r)$. Hence for $u \in \text{PSH}(M, \xi, \omega^T)$, we have the following,

$$\sqrt{-1} \partial \bar{\partial}(\log r + u) \geq 0$$
In other words, \( v = u + \log r \) is a plurisubharmonic function on \( X \). This is transparent in foliations charts and corresponding holomorphic charts as inLemma 2.1. Let \( h_\alpha \) be a local potential for \( \omega^T \) in a foliation chart \( V_\alpha \), and we write \( h = h(w_1, \bar{w}_1, \ldots, w_n, \bar{w}_n) \) in the holomorphic chart on cone, then \( \log r = h_\alpha + \text{Re}(w_0) \). Denote \( \omega_X \) to be the Kähler form on \( X \). Since \( u \) has zero Lelong number, applying Blocki–Kolodziej [10, Theorem 2], we get a sequence of smooth functions \( v_k \) converges to \( u \), decreasing in \( k \), such that on \( X' = \{2^{-1} \leq r \leq 2\} \subset X \)

\[
\sqrt{-1} \partial \bar{\partial} (v_k) + \omega^T + k^{-1} \omega_X \geq 0. \tag{3.8}
\]

We can assume in addition that \( v_k \) is invariant under the flow of \( \xi \), by taking average with respect to the torus action generated by \( \xi \in \text{Aut}(\xi, \eta, g) \). We define a basic function \( u_k \) on \( M \) such that, by taking \( r = 1, u_k = v_k|_{r=1} \).

Now for any point on \( X' \), we choose holomorphic charts \( \tilde{U}_\alpha \) as in Lemma 2.1 to cover \( X' \). We write the function in a holomorphic chart as

\[
v_k = v_k(\text{Re}(w_0), x, w_1, \bar{w}_1 \ldots, w_n, \bar{w}_n). \]

We recall the relation between the holomorphic charts and the foliation charts,

\[
w_0 = \log(r) + \sqrt{-1}x - h_\alpha(z, \bar{z}), w_i = z_i, i = 1, \ldots, n. \tag{3.9}
\]

Note we assume that \( v_k \) is invariant under the flow of \( \xi \), hence \( v_k \) is independent of \( x = \text{Im}(w_0) \). We write \( v_k \) as follows, using (3.9),

\[
v_k(\text{Re}(w_0), w_1, \bar{w}_1, \ldots, w_n, \bar{w}_n) = v_k(\log r - h(z, \bar{z}), z, \bar{z}).
\]

Locally, this gives

\[
u_k(z, \bar{z}) = v_k(-h_\alpha(z, \bar{z}), z, \bar{z}). \tag{3.10}
\]

The tangent space \( T_pX \) is given by, in terms of coordinate \( (r, x, z_1, \ldots, z_n) \),

\[
T_pX \otimes \mathbb{C} = \text{span} \left\{ \frac{\partial}{\partial r}, r^{-1} \frac{\partial}{\partial x}, X_i = \frac{\partial}{\partial z_i} + \sqrt{-1} h_i \frac{\partial}{\partial \bar{z}_i}, \bar{X}_j = \frac{\partial}{\partial \bar{z}_j} - \sqrt{-1} h_j \frac{\partial}{\partial x} \right\}.
\]

Note that the contact bundle \( D_p = \text{span}(X_i, \bar{X}_j, i = 1, \ldots, n) \). For \( p \in M \subset X \), we can assume that \( h(z, \bar{z}) = \partial h = \bar{\partial} h = 0, h_{ij} = \delta_{ij} \) at \( p \), and hence

\[
T_pX = T_pM \oplus \left\{ \frac{\partial}{\partial r} \right\} = \text{span} \left\{ \frac{\partial}{\partial z_i}, \frac{\partial}{\partial \bar{z}_j}, r^{-1} \frac{\partial}{\partial x}, \frac{\partial}{\partial r} \right\}.
\]

By (3.8), we compute (at \( p \)),

\[
\left( \sqrt{-1} \partial \bar{\partial} v_k + \omega^T + k^{-1} \omega_X \right) \left( \frac{\partial}{\partial z_i}, -\sqrt{-1} \frac{\partial}{\partial \bar{z}_i} \right) = -\partial_r v_k + 1 + k^{-1} + (v_k)_{ii} \geq 0,
\]

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where $t$ stands for the first argument of $v_k$. This is equivalent to the following, on $M$ we have,

$$\sqrt{-1} \partial_B \bar{\partial}_B u_k + (1 + k^{-1}) \omega^T \geq 0.$$ 

It is clear that $u_k$ converges to $u$, deceasing in $k$. Without loss of generality, we can assume that $u \leq -1$ and $u_k \leq 0$. It follows that $k(k + 2)^{-1} u_k \in \mathcal{H}$ such that $k(k + 2)^{-1} u_k$ converges to $u$, decreasing in $k$. This completes the proof when $u$ has zero Lelong number.

Now suppose $u \in \text{PSH}(M, \xi, \omega^T)$. We consider the canonical cutoffs $u_j = \max\{u, -j\} \in \text{PSH}(M, \xi, \omega^T) \cap L^\infty$. By the above statements, we know that for each $j$, there exists a sequence of smooth functions $\{v_k^j\}_{k}$ which decreases to $u_j$.

By adding a small constant $k^{-1}$ to each $v_k^j$, we can assume that $\{v_k^j\}_{k}$ strictly decreases (for each $j$). Then for each $k$, we can find $k_{j+1}$ such that

$$v_{j+1}^{k_{j+1}} < v_k^j.$$  \hspace{1cm} (3.12)

Indeed we consider the open set $U^l := \{x \in M : v_{j+1}^l < v_k^j\}$. Clearly $\{U^l\}_l$ is an increasing sequence of open sets such that $\bigcup_l U^l = M$, since

$$\lim_{l \to \infty} v_{j+1}^l = u_{j+1} \leq u_j < v_k^j.$$ 

Since $M$ is compact, there exists $k_{j+1}$ such that $U^{k_{j+1}} = M$. By (3.12), we can find a sequence $\{v_j^k\}_j \subset \mathcal{H}$ inductively such that $v_j^k \searrow u$. This completes the proof. \hfill \Box

**Remark 3.1** The Kähler cone structure, in particular, the relation between holomorphic charts and foliation charts as in Lemma 2.1, plays a very important role in Sasaki setting. If the Reeb vector field is irregular, the approximation from transverse Kähler structure can produce local approximation. But it seems to be hard to patch such a local construction together when the Reeb vector field is irregular. Instead we do approximation on the Kähler cone. We shall mention that in (3.12), the assumption that each sequence $\{v_j^k\}_k$ strictly decreases is necessary. For example, we can take $u = 1$ over $[0, 1]$, $v = 0$ over $[0, 1)$ and $v(1) = 1$. We can choose $u_k = 1$ for each $k$, and $v_k(x) = x^k + k^{-1}$. Then $v \leq u$ and $\{u_k\}_k$ decreases to $u$ and $v_k$ (strictly) decreases to $v$. But for $\{u_k\}_k$ and $\{v_k\}_k$, (3.12) does not hold: given $u_k$, there does not exist $l$ such that $v_l \leq u_k$ since $v_l(1) > 1$ for all $l$.

As a direct consequence, we have the following (just as in Kähler setting, see [30, Lemma 2.2]),

**Proposition 3.5** For $u \in \text{PSH}(M, \xi, \omega^T) \cap L^\infty$,

$$\text{Vol}(M) := \int_M \omega_n^u \wedge \eta = \int_M \omega_n^\omega^T \wedge \eta.$$  \hspace{1cm} (3.13)
Proof By Lemma 3.1, we can choose a smooth sequence $u_k$ converges to $u$ as a decreasing sequence. It then follows from Bedford–Taylor theory (see Proposition 3.1) that $\omega^n_{u_k} \wedge \eta$ converges to $\omega^n_u \wedge \eta$ weakly, we obtain (3.13).

It is then clear that, given (3.6), we have only $\int_M \omega^n_v \wedge \eta \leq \text{Vol}(M)$ for $v \in \text{PSH}(M, \xi, \omega^T)$.

Definition 3.3 We define the full-mass elements in $\text{PSH}(M, \xi, \omega^T)$ as

$$\mathcal{E}(M, \xi, \omega^T) := \{v \in \text{PSH}(M, \xi, \omega^T) : \int_M \omega^n_v \wedge \eta = \text{Vol}(M)\} \quad (3.14)$$

As in Kähler case, many of the properties that hold for bounded TPSH functions hold for elements of $\mathcal{E}(M, \xi, \omega^T)$ as well. We include the comparison principle, monotonicity property, and generalized Bedford–Taylor identity as follows. These properties are proved in [44] for Kähler setting. Given (3.3) and (3.13), our proof follows almost identical as in Kähler setting (see [44, Theorem 1.5, Proposition 1.6, Corollary 1.7]). Nevertheless, we include the details.

Proposition 3.6 (Comparison principle) Suppose $u, v \in \mathcal{E}(M, \xi, \omega^T)$. Then

$$\int_{\{v<u\}} \omega^n_u \wedge \eta \leq \int_{\{v<u\}} \omega^n_v \wedge \eta. \quad (3.15)$$

Proof Our proof is similar to Kähler case, see [30, Proposition 2.3]. First we show (3.15) for $u, v$ bounded. Using Propositions 3.2 and 3.5, we write

$$\int_{\{v<u\}} \omega^n_u \wedge \eta = \int_{\{v<u\}} \omega^n_{\max\{u,v\}} \wedge \eta = \int_M \omega^n_{\max\{u,v\}} \wedge \eta - \int_{\{u\leq v\}} \omega^n_{\max\{u,v\}} \wedge \eta \leq \text{Vol}(M) - \int_{\{u\leq v\}} \omega^n_{\max\{u,v\}} \wedge \eta$$

$$= \int_M \omega^n_v \wedge \eta - \int_{\{u\leq v\}} \omega^n_v \wedge \eta$$

$$= \int_{\{v\leq u\}} \omega^n_v \wedge \eta.$$ 

Replacing $v$ by $v + \epsilon$, we have

$$\int_{\{v+\epsilon<u\}} \omega^n_u \wedge \eta \leq \int_{\{v+\epsilon\leq u\}} \omega^n_v \wedge \eta.$$ 

Recall that

$$\{v<u\} = \cup_{\epsilon>0} \{v+\epsilon<u\} = \cup_{\epsilon>0} \{v+\epsilon\leq u\}.$$ 

Hence (3.15) for bounded potentials follows immediately by letting $\epsilon \to 0$.  

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In general, let \( u_l = \max\{u, -l\} \), \( v_k = \max\{v, -k\} \), \( l, k \in \mathbb{N} \) be the canonical cutoffs of \( u \), \( v \) respectively. We apply (3.15) for bounded potentials to get

\[
\int_{\{u_l < u_k\}} \omega^n_{u_k} \wedge \eta \leq \int_{\{v < u_k\}} \omega^n_{v} \wedge \eta.
\]

Together with the inclusions \( \{v < u_l\} \subset \{v < u_k\} \subset \{v < u\} \), we have

\[
\int_{\{v < u\}} \omega^n_{u_k} \wedge \eta \leq \int_{\{v < u\}} \omega^n_{v} \wedge \eta.
\]

(3.16)

Letting \( l \to \infty \), using the definition (3.6) on \( \omega^n_{v} \wedge \eta \) and \( \{v < u\} = \cup_{l \in \mathbb{N}} \{v < u_l\} \), (3.16) gives

\[
\int_{\{v < u\}} \omega^n_{u_k} \wedge \eta \leq \int_{\{v < u\}} \omega^n_{v} \wedge \eta.
\]

Letting \( k \to \infty \), using the definition (3.6) on \( \omega^n_{u} \wedge \eta \) and \( \{v \leq u\} = \cap_{k \in \mathbb{N}} \{v < u_k\} \), we get

\[
\int_{\{v < u\}} \omega^n_{u_k} \wedge \eta \leq \int_{\{v \leq u\}} \omega^n_{v} \wedge \eta.
\]

The replacing \( v \) by \( v + \epsilon \) in the above inequality, we can then argue as in the bounded case, taking the limit \( \epsilon \to 0 \) yields (3.15).

**Proposition 3.7** (Monotonicity property) Suppose \( u \in \mathcal{E}(M, \xi, \omega^T) \) and \( v \in \text{PSH}(M, \xi, \omega^T) \). If \( u \leq v \), then \( v \in \mathcal{E}(M, \xi, \omega^T) \).

**Proof** This is proved in [44, Proposition 1.6] in Kähler case and the Sasaki case is almost identical. First we show that \( \psi = v/2 \in \mathcal{E}(M, \xi, \omega^T) \). We can assume that \( u \leq v < -2 \), hence \( \psi < -1 \). This normalization gives the following inclusions for the canonical cutoffs \( u_j, v_j, \psi_j \),

\[
\{\psi \leq -j\} = \{\psi_j \leq -j\} \subset \{u_{2j} < \psi - j + 1\} \subset \{u_{2j} \leq -j\}.
\]

By Proposition 3.15 and the inclusions above, we have

\[
\int_{\{\psi \leq -j\}} \omega^n_{\psi_j} \wedge \eta \leq \int_{\{u_{2j} < \psi - j + 1\}} \omega^n_{\psi_j} \wedge \eta \leq \int_{\{u_{2j} < \psi - j + 1\}} \omega^n_{u_{2j}} \wedge \eta
\]

\[
\leq \int_{\{u_{2j} \leq -j\}} \omega^n_{u_{2j}} \wedge \eta.
\]

Note that we have

\[
\int_{\{u_{2j} \leq -j\}} \omega^n_{u_{2j}} \wedge \eta = \text{Vol}(M) - \int_{\{u_{2j} > -j\}} \omega^n_{u_{2j}} \wedge \eta.
\]
Applying Proposition 3.2 to \( \max\{u_{2j}, -j\} = u_j \) on the set \( \{u_{2j} > -j\} = \{u > -j\} \), we have

\[
\int_{\{u_{2j} > -j\}} \omega^n_{u_{2j}} \wedge \eta = \int_{\{u > -j\}} \omega^n_{u_j} \wedge \eta.
\]

It then follows that

\[
\int_{\{u_{2j} \leq -j\}} \omega^n_{u_{2j}} \wedge \eta = \int_{\{u_j \leq -j\}} \omega^n_{u_j} \wedge \eta = \int_{\{u \leq -j\}} \omega^n_{u_j} \wedge \eta.
\]

By definition of \( u \in \mathcal{E}(M, \xi, \omega^T) \), it follows that, as \( j \to \infty \),

\[
\int_{\{\psi \leq -j\}} \omega^n_{\psi_j} \wedge \eta \leq \int_{\{u \leq -j\}} \omega^n_{u_j} \wedge \eta \to 0.
\]

Hence \( \psi = v/2 \in \mathcal{E}(M, \xi, \omega^T) \). To show that \( v \in \mathcal{E}(M, \xi, \omega^T) \), we observe that \( \{v \leq -2j\} = \{\psi \leq -j\} \) and \( \omega_{\psi_j} \geq \omega_{v_{2j}}/2 \), hence

\[
\int_{\{v \leq -2j\}} \omega^n_{v_{2j}} \wedge \eta \leq 2^n \int_{\{v \leq -2j\}} \omega^n_{\psi_j} \wedge \eta \leq 2^n \int_{\{\psi \leq -j\}} \omega^n_{\psi_j} \wedge \eta.
\]

By letting \( j \to \infty \), we can then conclude that \( v \in \mathcal{E}(M, \xi, \omega^T) \). \( \Box \)

**Proposition 3.8** (Generalized Bedford–Taylor identity) For \( u \in \mathcal{E}(M, \xi, \omega^T) \), \( v \in \text{PSH}(M, \xi, \omega^T) \), then \( \max\{u, v\} \in \mathcal{E}(M, \xi, \omega^T) \) and

\[
\chi_{\{u > v\}} \omega^n_{\max(u, v)} \wedge \eta = \chi_{\{u > v\}} \omega^n_{u} \wedge \eta. \tag{3.17}
\]

**Proof** Our argument is identical to the Kähler setting; see [44, Corollary 1.7] and [30, Lemma 2.5]. Proposition 3.7 implies that \( w := \max\{u, v\} \in \mathcal{E}(M, \xi, \omega^T) \). Now observe that \( \max\{u_j, v_{j+1}\} = \max\{u, v, -j\} = w_j \). Since the cutoffs are bounded we have

\[
\chi_{\{u > v\}} \omega^n_{u_j} \wedge \eta = \chi_{\{u > v\}} \omega^n_{w_j} \wedge \eta. \tag{3.18}
\]

By 3.7, we know that \( \chi_{\{u > v\}} \omega^n_{u_j} \wedge \eta \to \chi_{\{u > v\}} \omega^n_{u} \wedge \eta \) and \( \chi_{\{u > v\}} \omega^n_{w_j} \wedge \eta \to \chi_{\{u > v\}} \omega^n_{w} \wedge \eta \) as \( j \to \infty \) (we also use the fact that \( u, w \in \mathcal{E}(M, \xi, \omega^T) \)). Since

\[
\{u > v\} \subset \{u_j > v_{j+1}\} \text{ and } \{u_j > v_{j+1}\}\{u > v\} \subset \{u \leq -j\},
\]

it follows that

\[
0 \leq (\chi_{\{u_j > v_{j+1}\}} - \chi_{\{u > v\}}) \omega^n_{u_j} \wedge \eta \leq \chi_{\{u \leq -j\}} \omega^n_{u_j} \wedge \eta \to 0.
\]

Similarly since

\[
\{u_j > v_{j+1}\}\{u > v\} \subset \{w \leq -j\}
\]

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we also obtain that
\[ 0 \leq (\chi_{[u_j>v_{j+1}]} - \chi_{[u>v_j]}^n\omega^n_{w_j} \wedge \eta \leq \chi_{[w\leq-j]}^n\omega^n_{w_j} \wedge \eta \rightarrow 0. \]

By taking limit in (3.18) together with the limit facts above, we get the desired result.

Next we introduce finite-energy class on Sasaki manifolds, following [44]. By considering Young weights \( \chi \in W_+^p \) (see [30, Chapter 1] for a short introduction to Young weights), one can introduce various finite-energy subclasses of \( \mathcal{E}(M, \xi, \omega^T) \),
\[ \mathcal{E}_\chi(M, \xi, \omega^T) := \{ u \in \mathcal{E}(M, \xi, \omega^T) : E_\chi(u) < \infty \}, \]
where \( E_\chi \) is the \( \chi \)-energy defined by
\[ E_\chi(u) := \int_M \chi(u)\omega^n_u \wedge \eta. \]

Of special importance are the weights \( \chi^p(t) = |t|^p/p \) and the associated classes \( \mathcal{E}_p(M, \xi, \omega^T) \). For these weights it is clear that \( \mathcal{E}_p(M, \xi, \omega^T) \subset \mathcal{E}_1(M, \xi, \omega^T) \) for \( p \geq 1 \). We will need the following straightforward fact:

**Proposition 3.9** For any \( u \in \mathcal{E}_1(M, \xi, \omega^T) \), \( u \) has Lelong number zero at every point.

**Proof** For similar results in Kähler case, see [44, Corollary 1.8]. This is straightforward. We can assume \( \sup u = 0 \). For \( u \in \mathcal{E}_1(M, \xi, \omega^T) \), we have
\[ \int_M (-u)\omega^n_u \wedge \eta < \infty. \]

We consider locally \((0,0) \in W_\alpha = (-\delta, \delta) \times V_\alpha\) in a foliation chart. Then we have
\[ 2\delta \int_{V_\alpha} (-u)\omega^n_u < \int_M (-u)\omega^n_u \wedge \eta < \infty. \]

This implies that \( u \) has Lelong number zero at \((0,0)\).

The following result implies that to test membership in \( \mathcal{E}_\chi(M, \xi, \omega^T) \) it is enough to test the finiteness condition \( E_\chi(u) < \infty \) on canonical cutoffs.

**Proposition 3.10** Suppose \( u \in \mathcal{E}(M, \xi, \omega^T) \) with canonical cutoffs \( \{ u_k \}_{k \in \mathbb{N}} \). If \( h : \mathbb{R}_+ \rightarrow \mathbb{R}_+ \) is continuous and increasing, then
\[ \int_M h(|u|)\omega^n_u \wedge \eta < \infty \iff \limsup_{k \to \infty} \int_M h(|u_k|)\omega^n_{u_k} \wedge \eta < \infty. \]

Moreover, if the above condition holds, then
\[ \int_M h(|u|)\omega^n_u \wedge \eta = \lim_{k \to \infty} \int_M h(|u_k|)\omega^n_{u_k} \wedge \eta. \]
Proof. Our proof is similar to the Kähler case, see [30, Proposition 2.6]. Without loss of generality we can assume that $u \leq 0$. If $\limsup_{k \to \infty} \int_M h(|u_k|) \omega^n_{u_k} \wedge \eta < \infty$, we obtain that the sequence of Radon measures $h(|u_k|) \omega^n_{u_k} \wedge \eta$ is weakly compact. Hence there exists a subsequence $h(|u_{k_j}|) \omega^n_{u_{k_j}} \wedge \eta$ converging weakly to a Radon measure $\mu$. Recall that $h(|u_{k_j}|)$ is an increasing sequence of lower semicontinuous functions converging to $h(|u|)$ and $\omega^n_{u_{k_j}} \wedge \eta \wedge \omega^n_u \wedge \eta$, this yields that $h(|u|) \omega^n_u \wedge \eta \leq \mu$ as measure. In particular $\int_M \omega^n_u \wedge \eta \leq \mu(M) < \infty$.

Now assume $\int_M h(|u|) \omega^n_u \wedge \eta < \infty$. If $\lim_{t \to +\infty} h(t) = +\infty$, we have

$$\lim_{k \to \infty} \int_{\{u \leq -k\}} h(|u|) \omega^n_u \wedge \eta = \lim_{l \to +\infty} \int_{\{h(|u|) > l\}} h(|u|) \omega^n_u \wedge \eta = 0.$$ 

It follows from Propositions 3.5, 3.8 and Definition 3.3 that

$$\int_{\{u \leq -k\}} \omega^n_{u_k} \wedge \eta = \int_{\{u \leq -k\}} \omega^n_u \wedge \eta.$$ 

Then by Propositions 3.5, 3.8 and Definition 3.3 again we have

$$\left| \int_M h(|u_k|) \omega^n_{u_k} \wedge \eta - \int_M h(|u|) \omega^n_u \wedge \eta \right| \leq \int_{\{u \leq -k\}} h(k) \omega^n_{u_k} \wedge \eta + \int_{\{u \leq -k\}} h(|u|) \omega^n_u \wedge \eta$$

$$= h(k) \int_{\{u \leq -k\}} \omega^n_u \wedge \eta + \int_{\{u \leq -k\}} h(|u|) \omega^n_u \wedge \eta$$

$$\leq 2 \int_{\{u \leq -k\}} h(|u|) \omega^n_u \wedge \eta.$$ 

It follows that $\int_M h(|u_k|) \omega^n_{u_k} \wedge \eta$ is bounded and $\int_M h(|u|) \omega^n_u \wedge \eta = \lim_{k \to \infty} \int_M h(|u_k|) \omega^n_{u_k} \wedge \eta$.

If $\lim_{t \to +\infty} h(t) = L < \infty$, it follows from Proposition 3.5 that $\int_M h(|u_k|) \omega^n_{u_k} \wedge \eta$ is bounded. Moreover for any $\epsilon > 0$ there exists $N > 0$ such that $0 < L - h(t) < \epsilon$ for all $t > N$. By Propositions 3.5, 3.8 and Definition 3.3 we have

$$\left| \int_M (L - h(|u_k|)) \omega^n_{u_k} \wedge \eta - \int_M (L - h(|u|)) \omega^n_u \wedge \eta \right| \leq \int_{\{u \leq -k\}} (L - h(|u_k|)) \omega^n_{u_k} \wedge \eta + \int_{\{u \leq -k\}} (L - h(|u|)) \omega^n_u \wedge \eta$$

$$\leq 2 \epsilon \text{Vol}(M)$$

for $k > N$. It yields that $\int_M h(|u|) \omega^n_u \wedge \eta = \lim_{k \to \infty} \int_M h(|u_k|) \omega^n_{u_k} \wedge \eta$. □

\[ \text{Springer} \]
With the proposition above, we can then prove the so-called fundamental estimate.

**Proposition 3.11** (Fundamental estimate) Suppose $\chi \in \mathcal{W}^+_p$ and $u, v \in \mathcal{E}_\chi(M, \xi, \omega^T)$ such that $u \leq v \leq 0$. Then

$$E_\chi(v) \leq (p + 1)^n E_\chi(u). \tag{3.19}$$

**Proof** The proof is similar to the Kähler case, see [44, Lemma 3.5]. First of all we assume that $u, v \in \text{PSH}(M, \xi, \omega^T) \cap L^\infty$. For $0 \leq j \leq n - 1$, we have

$$\int_M \chi(u) \omega^{j+1}_v \wedge \omega^{n-j-1}_u \wedge \eta = \int_M \chi(u) \omega^T \wedge \omega^j_v \wedge \omega^{n-j-1}_u \wedge \eta$$

$$+ \int_M \sqrt{-1} \chi(u) \partial_B \bar{\partial}_B v \wedge \omega^j_v \wedge \omega^{n-j-1}_u \wedge \eta.$$}

Recall that $\chi'(l) \leq 0$ for $l < 0$. Using integration by parts, we have

$$\int_M \chi(u) \omega^T \wedge \omega^j_v \wedge \omega^{n-j-1}_u \wedge \eta = \int_M \chi(u) \wedge \omega^j_v \wedge \omega^{n-j}_u \wedge \eta$$

$$- \int_M \sqrt{-1} \chi(u) \partial_B \bar{\partial}_B u \wedge \omega^j_v \wedge \omega^{n-j-1}_u \wedge \eta$$

$$= \int_M \chi(u) \wedge \omega^j_v \wedge \omega^{n-j}_u \wedge \eta$$

$$+ \int_M \sqrt{-1} \chi'(u) \partial_B \bar{\partial}_B u \wedge \omega^j_v \wedge \omega^{n-j-1}_u \wedge \eta$$

$$\leq \int_M \chi(u) \wedge \omega^j_v \wedge \omega^{n-j}_u \wedge \eta.$$}

Recall that $\chi'(l) \leq 0$ for $l < 0$ and $l \chi'(l) \leq p \chi(l)$ for $l \geq 0$. Using the integration by parts repeatedly, we have

$$\int_M \sqrt{-1} \chi(u) \partial_B \bar{\partial}_B v \wedge \omega^j_v \wedge \omega^{n-j-1}_u \wedge \eta$$

$$= \int_M \sqrt{-1} v \chi''(u) \partial_B \bar{\partial}_B u \wedge \omega^j_v \wedge \omega^{n-j-1}_u \wedge \eta$$

$$+ \int_M \sqrt{-1} v \chi'(u) \partial_B \bar{\partial}_B u \wedge \omega^j_v \wedge \omega^{n-j-1}_u \wedge \eta$$

$$\leq \int_M \sqrt{-1} v \chi'(u) \partial_B \bar{\partial}_B u \wedge \omega^j_v \wedge \omega^{n-j-1}_u \wedge \eta$$

$$\leq \int_M \chi'(u) \omega^j_v \wedge \omega^{n-j}_u \wedge \eta = \int_M |v| \chi'(|u|) \omega^j_v \wedge \omega^{n-j}_u \wedge \eta$$

$$\leq \int_M |u| \chi'(|u|) \omega^j_v \wedge \omega^{n-j}_u \wedge \eta \leq p \int_M \chi(|u|) \omega^j_v \wedge \omega^{n-j}_u \wedge \eta.$$
Combine the inequalities above we obtain
\[
\int_M \chi(u)\omega_v^{j+1} \wedge \omega_u^{n-j-1} \wedge \eta \leq (p+1) \int_M \chi(u)\omega_v^j \wedge \omega_u^{n-j} \wedge \eta.
\]
It follows that
\[
E_X(v) \leq \int_M \chi(u)\omega_v^n \wedge \eta \leq (p+1)^n E_X(u).
\]
In the general case \(u, v \in \mathcal{E}_X(M, \xi, \omega^T)\), we have \(E_X(v_k) \leq (p+1)^n E_X(u_k)\) for the canonical cutoffs \(u_k, v_k\) of \(u, v\). It follows from Proposition 3.10 that \(E_X(v) \leq (p+1)^n E_X(u)\). \(\square\)

As a direct consequence, we obtain the \textit{monotonicity property} for \(\mathcal{E}_X(M, \xi, \omega^T)\).

**Proposition 3.12** Suppose \(u \in \mathcal{E}_X(M, \xi, \omega^T)\) and \(v \in \text{PSH}(M, \xi, \omega^T)\). If \(u \leq v\), then \(v \in \mathcal{E}_X(M, \xi, \omega^T)\).

**Proof** Without loss of generality we can assume that \(u \leq v \leq 0\). The monotonicity property implies that \(v \in \mathcal{E}(M, \xi, \omega^T)\). We have \(u \leq v_k\) for the canonical cutoffs \(v_k\) of \(v\), then \(E_X(v_k) \leq (p+1)^n E_X(u)\) according to the Proposition 3.11. It follows from Proposition 3.10 that \(E_X(v) \leq (p+1)^n E_X(u)\) and \(v \in \mathcal{E}_X(M, \xi, \omega^T)\). \(\square\)

We also have the following,

**Proposition 3.13** Suppose \(u, v \in \mathcal{E}_X(M, \xi, \omega^T)\) for \(\chi \in \mathcal{W}_p^+.\) If \(u, v \leq 0\), then
\[
\int_M \chi(u)\omega_v^n \wedge \eta \leq p2^p (E_X(u) + E_X(v)).
\]

**Proof** For similar result is Kähler case, see [44, Proposition 3.6]. For \(\delta > 0\), we have \(\tilde{\chi}(t) = \chi(t) + \delta|t| \in \mathcal{W}_p^+.\) Assume that \(t > 0\), it is obvious \(\tilde{\chi}(t), \tilde{\chi}'(t) > 0\). Recall that \(\epsilon^p \tilde{\chi}(t) \leq \tilde{\chi}(\epsilon t)\) and \(t\tilde{\chi}'(t) \leq p\tilde{\chi}(t)\) for \(\tilde{\chi} \in \mathcal{W}_p^+\) and \(0 < \epsilon < 1\), hence we have \(\tilde{\chi}'(2t) \leq 2p\tilde{\chi}'(t)\). It follows from the convexity of the function \(\tilde{\chi}(t)\) that \(\frac{\tilde{\chi}(t)}{t} \leq \tilde{\chi}'(t)\). Then
\[
\frac{\tilde{\chi}'(2t)}{2} = 2t \frac{\tilde{\chi}'(2t)}{\tilde{\chi}(2t)} \frac{\tilde{\chi}(t)}{t} \leq p2^{p-1}\tilde{\chi}'(t).
\]
Then \(\delta \to 0\) implies that \(\tilde{\chi}'(2t) \leq p2^{p-1}\tilde{\chi}'(t)\) for \(t > 0\).

By Proposition 3.6 and \(\{|u| > 2t\} \subset \{u < v - t\} \cup \{v < -t\}\), we have
\[
\int_M \chi(u)\omega_v^n \wedge \eta = \int_0^\infty \chi'(t)\omega_v^n \wedge \eta |u| > t)dt
\leq p2^p \int_0^\infty \chi'(t)\omega_v^n \wedge \eta |u| > 2t)dt
\leq p2^p \left(\int_0^\infty \chi'(t)\omega_v^n \wedge \eta |u| < v - t)dt\right)
\]
\[ + \int_{0}^{\infty} \chi'(t) \omega^n_v \wedge \eta [v < -t] \, dt \]
\[ \leq p2^p \left( \int_{0}^{\infty} \chi'(t) \omega^n_u \wedge \eta [u < v - t] \, dt + E_X(v) \right) \]
\[ \leq p2^p \left( \int_{0}^{\infty} \chi'(t) \omega^n_u \wedge \eta [u < -t] \, dt + E_X(v) \right) \]
\[ = p2^p (E_X(u) + E_X(v)). \]

\[ \square \]

**Proposition 3.14** Suppose \( u \in \mathcal{E}_X(M, \xi, \omega^T), \chi \in \mathcal{W}^+_p \). Then there exists \( \tilde{\chi} \in \mathcal{W}^+_{2p+1} \) such that \( \chi(t) \leq \tilde{\chi}(t), \chi(t)/\tilde{\chi}(t) \to 0 \) as \( t \to \infty \) and \( u \in \mathcal{E}_{\tilde{\chi}}(M, \xi, \omega^T) \).

**Proof** This construction borrows from similar results in Kähler case, see [30, Lemma 2.10]. Take \( \chi_0 = \chi \), recall that \( \lim_{t \to \infty} \chi_0(t) = \infty \) and \( u \in \mathcal{E}_\chi(M, \xi, \omega^T) \), we have
\[ \lim_{t \to \infty} \int_{|u| > t} \chi(|u|) \omega^n_u \wedge \eta = \lim_{s \to \infty} \int_{\chi(u) > s} \chi(|u|) \omega^n_u \wedge \eta = 0. \]

Then one can choose \( t_1 > 0 \) such that \( \int_{|u| > t_1} \chi(|u|) \omega^n_u \wedge \eta < \frac{1}{2^2} \). We define \( \chi_1 : \mathbb{R}^+ \to \mathbb{R}^+ \) by the formula:
\[ \chi_1(t) = \begin{cases} \chi_0(t) & \text{if } t \leq t_1 \\ \chi_0(t_1) + 2(\chi_0(t) - \chi_0(t_1)) & \text{if } t > t_1. \end{cases} \]

Then it is easy to verify that

1. \( \chi_0(t) \leq \chi_1(t) \);
2. \( \lim_{t \to \infty} \frac{\chi_1(t)}{\chi_0(t)} = 2; \)
3. \( E_{\chi_1}(u) \leq E_{\chi_0}(u) + \frac{1}{2}; \)
4. \( \sup_{t > 0} \frac{|\chi_1'(t)|}{|\chi_0(t)|} \leq \sup_{t > 0} \frac{2|\chi_0'(t)|}{|\chi_0(t)|} < 2^p + 1; \)
5. \( \lim_{t \to \infty} \frac{t \chi_1'(t)}{\chi_1(t)} \leq p. \)

These properties imply that for \( t_2 > t_1 \) big enough, the function \( \chi_2 : \mathbb{R}^+ \to \mathbb{R}^+ \)
\[ \chi_2(t) = \begin{cases} \chi_1(t) & \text{if } t \leq t_2 \\ \chi_1(t_2) + 2(\chi_1(t) - \chi_1(t_2)) & \text{if } t > t_2 \end{cases} \]
satisfies

1. \( \chi_1(t) \leq \chi_2(t) \);
2. \( \lim_{t \to \infty} \frac{\chi_2(t)}{\chi_1(t)} = 2; \)
3. \( E_{\chi_2}(u) \leq E_{\chi_1}(u) + \frac{1}{2^r}; \)
Continuing the above construction we can obtain an increasing sequence \( \{ \chi_n \}_{n \in \mathbb{N}} \) and the limit weight \( \tilde{\chi}(t) = \lim_{k \to \infty} \chi_k(t) \) will satisfy the requirements of the proposition. \( \square \)

**Proposition 3.15** Assume that \( \{ \psi_k \}_{k \in \mathbb{N}}, \{ \phi_k \}_{k \in \mathbb{N}}, \{ v_k \}_{k \in \mathbb{N}} \subset \mathcal{E}_x(M, \xi, \omega^T) \) decrease (increase a. e) to \( \phi, \psi, v \in \mathcal{E}_x(M, \xi, \omega^T) \), respectively. Suppose

1. \( \psi_k \leq \phi_k \) and \( \psi_k \leq v_k \).
2. \( h : \mathbb{R} \to \mathbb{R} \) is continuous with \( \limsup_{|l| \to \infty} |h(l)|/\chi(l) \leq C \) for some \( C \geq 0 \).

Then we have the weak convergence of

\[
    h(\phi_k - \psi_k) \omega^n_{v_k} \wedge \eta \to h(\phi - \psi) \omega^n_v \wedge \eta.
\]

**Proof** For similar results in Kähler case, see [30, Proposition 2.11]. Without loss of generality one can assume all the functions \( \phi_k, \phi, \psi_k, \psi, v, v_k \) are negative. We will only prove the proposition for decreasing sequences, the case of increasing sequences can be proved similarly.

First of all we suppose that the functions involved are uniformly bounded below, that is, there exists \( L > 1 \) such that \(-L \leq \phi_k, \phi, \psi_k, \psi, v_k, v \leq 0 \). Given \( \epsilon > 0 \), it follows from Theorem 6.3 that there exists an open subset \( \tilde{O}_\epsilon \subset M \) such that \( \text{cap}(\tilde{O}_\epsilon) < \epsilon \) and \( \phi_k, \phi, \psi_k, \psi, v_k, v \) are continuous on \( M - \tilde{O}_\epsilon \). Then \( \phi_k \to \phi \) and \( \psi_k \to \psi \) uniformly on \( M - \tilde{O}_\epsilon \). Hence there exists \( N \) such that for \( k > N \) we have

\[
    |h(\phi_k - \psi_k) - h(\phi - \psi)| < \epsilon \quad \text{on} \quad M - \tilde{O}_\epsilon
\]

and the term

\[
    \int_M h(\phi_k - \psi_k) \omega^n_{v_k} \wedge \eta - \int_M h(\phi - \psi) \omega^n_v \wedge \eta
\]

is bounded by \( 2\epsilon L^n \max_{|l| \leq L} |h(l)| + \epsilon \text{Vol}(M) \). Hence

\[
    \int_M h(\phi_k - \psi_k) \omega^n_{v_k} \wedge \eta - \int_M h(\phi - \psi) \omega^n_v \wedge \eta \to 0. \quad (3.20)
\]

Given \( \epsilon > 0 \), it follows from Theorem 6.3 that there exists an open subset \( \tilde{O}_\epsilon \) such that \( \text{cap}(\tilde{O}_\epsilon) < \epsilon \) and \( \phi, \psi \) are continuous on \( M - \tilde{O}_\epsilon \). By the Tietze’s extension theorem, the function \( h(\phi - \psi) |_{M - \tilde{O}_\epsilon} \) can be extended to a continuous function \( \alpha \) on \( M \) bounded by \( \max_{|l| \leq L} |h(l)| \). By Proposition 3.1 we have \( \omega^n_{v_k} \wedge \eta \to \omega^n_v \wedge \eta \) weakly. Then there exists a constant \( N \) such that for \( k > N \) we have

\[
    |\int_M \alpha \omega^n_{v_k} \wedge \eta - \int_M \alpha \omega^n_v \wedge \eta| < \epsilon
\]

and the term

\[
    \int_M h(\phi - \psi) \omega^n_{v_k} \wedge \eta - \int_M h(\phi - \psi) \omega^n_v \wedge \eta
\]
we only have to show that Geometric Pluripotential Theory on Sasaki Manifolds

3.12. for $l \psi$ is bounded by $4 \epsilon L^n \max_{|l| \leq L} |h(l)| + \epsilon$. Hence

$$
\int_M h(\phi - \psi) \omega^n_{\nu_k} \wedge \eta - \int_M h(\phi - \psi) \omega^n_{\nu} \wedge \eta
$$

is bounded by $4 \epsilon L^n \max_{|l| \leq L} |h(l)| + \epsilon$. Hence

$$
\int_M h(\phi - \psi) \omega^n_{\nu_k} \wedge \eta - \int_M h(\phi - \psi) \omega^n_{\nu} \wedge \eta \to 0. \quad (3.21)
$$

It follows from (3.20) and (3.21) that $h(\phi_k - \psi_k) \omega^n_{\nu_k} \wedge \eta \to h(\phi - \psi) \omega^n_{\nu} \wedge \eta$.

Now consider the general case when $\phi_k, \phi, \psi_k, \psi, v_k, v$ are unbounded. Let $\phi_k^l, \phi^l, \psi_k^l, \psi^l, v_k^l, v^l$ be the canonical cutoffs of the corresponding potentials, then we only have to show that

$$
\int_M h(\phi_k - \psi_k) \omega^n_{\nu_k} \wedge \eta - \int_M h(\phi^l - \psi^l) \omega^n_{\nu^l} \wedge \eta \to 0 \quad (3.22)
$$

and

$$
\int_M h(\phi - \psi) \omega^n_{\nu} \wedge \eta - \int_M h(\phi^l - \psi^l) \omega^n_{\nu^l} \wedge \eta \to 0 \quad (3.23)
$$

as $l \to \infty$ uniformly with respect to $k$.

By Proposition 3.14 there exists $\tilde{\chi} \in \mathcal{W}_{2p+1}$ such that $\chi \leq \tilde{\chi}$, $\lim_{l \to \infty} \tilde{\chi}(t) = 0$ and $\psi \in \mathcal{E}_\tilde{\chi}(M, \xi, \omega^T)$. Then $\psi_k, \phi_k, \phi, v_k, v \in \mathcal{E}_\tilde{\chi}(M, \xi, \omega^T)$ according to Proposition 3.12.

Recall that there exists $L > 0$ such that $\chi(L) \geq 1$ and $|h(t)| \leq (C + 1) \chi(t)$ for $|t| > L$. Take $\tilde{C} = \max\{C + 1, \frac{\max_{0 \leq |l| \leq L} |h(l)|}{\chi(L)}\}$, then we have

$$
|h(l_1 - l_2)| \leq \tilde{C} \chi(l_2)
$$

for $l_2 \leq -L$ and $l_2 \leq l_1 \leq 0$. Using Propositions 3.8, 3.11, and 3.13, we have

$$
\left| \int_M h(\phi_k - \psi_k) \omega^n_{\nu_k} \wedge \eta - \int_M h(\phi^l_k - \psi^l_k) \omega^n_{\nu^l_k} \wedge \eta \right|
$$

$$
= \left| \int_{\{\psi_k \leq -l\}} h(\phi_k - \psi_k) \omega^n_{\nu_k} \wedge \eta - \int_{\{\psi_k \leq -l\}} h(\phi^l_k - \psi^l_k) \omega^n_{\nu^l_k} \wedge \eta \right|
$$

$$
\leq \int_{\{\psi_k \leq -l\}} |h(\phi_k - \psi_k)| \omega^n_{\nu_k} \wedge \eta + \int_{\{\psi_k \leq -l\}} |h(\phi^l_k - \psi^l_k)| \omega^n_{\nu^l_k} \wedge \eta
$$

$$
\leq \tilde{C} \int_{\{\psi_k \leq -l\}} \chi(\psi_k) \omega^n_{\nu_k} \wedge \eta + \int_{\{\psi_k \leq -l\}} \chi(\psi^l_k) \omega^n_{\nu^l_k} \wedge \eta
$$

$$
\leq \tilde{C} \sup_{s \leq -l} \frac{\chi(s)}{\tilde{\chi}(s)} \left( \int_{\{\psi_k \leq -l\}} \tilde{\chi}(\psi_k) \omega^n_{\nu_k} \wedge \eta + \int_{\{\psi_k \leq -l\}} \tilde{\chi}(\psi^l_k) \omega^n_{\nu^l_k} \wedge \eta \right)
$$
Proposition 3.16 Suppose \( \chi \in \mathcal{W}_p^+ \) and \( \{u_k\}_{k \in \mathbb{N}} \subset \mathcal{E}_X(M, \xi, \omega^T) \) is a decreasing sequence converging to \( u \in \text{PSH}(M, \xi, \omega^T) \). If \( \sup_k E_X(u_k) < \infty \), then

\[
E_X(u) = \lim_{k \to \infty} E_X(u_k).
\]

Proof For similar results in Kähler case, see [44, Proposition 5.6]. Without loss of generality, we assume that \( u_1 \leq 0 \). The canonical cutoffs \( u_k' = \max\{u_k, -l\} \) decreases to the canonical cutoff \( u' = \max\{u, -l\} \). As \(-l \leq u' \leq u_k' \leq 0\), Propositions 3.15 and 3.11 imply that

\[
E_X(u') = \lim_{k \to \infty} E_X(u_k') \leq (p + 1)^n \sup_k E_X(u_k).
\]
By Proposition 3.10, \( u \in \mathcal{E}_\chi(M, \xi, \omega^T) \). Applying the previous proposition in the case \( \psi_k = v_k = u_k, \phi_k = 0 \) gives that \( E_\chi(u) = \lim_{k \to \infty} E_\chi(u_k) \).

A very important notion in pluripotential theory is the **envelop construction**, which we shall describe below. In our setting on a compact Sasaki manifold, given a usc function \( f \in M \to [-\infty, \infty) \) such that \( f \) is invariant under the Reeb flow, we consider the envelop

\[
P(f) := \sup\{u \in \text{PSH}(M, \xi, \omega^T) \text{ such that } u \leq f\}.
\]

(3.24)

As in Kähler setting, we have the following

**Proposition 3.17** The envelop construction \( P(f) \in \text{PSH}(M, \xi, \omega^T) \).

**Proof** This statement is local in nature, hence we only need to argue in foliations charts \( W_\alpha = (-\delta, \delta) \times V_\alpha \), where \( V_\alpha \subset \mathbb{C}^n \) give a transverse holomorphic charts. Since \( P(f) \) is invariant under the Reeb flow, its usc regularization \( P(f)^* \) is invariant under the Reeb flow. Hence by \( P(f)^* \) is \( \omega^T_\alpha \)-psh on each \( V_\alpha \), see [12, Theorem 1.2.3 (viii)]. Since \( f \) is usc, hence \( P(f)^* \leq f^* = f \). Hence \( P(f)^* \) is a candidate in the definition of \( P(f) \), gives that \( P(f)^* \leq P(f) \). This implies that \( P(f) = P(f)^* \) and \( P(f) \in \text{PSH}(M, \xi, \omega^T) \).

We also introduce the notion **rooftop envelop**, for usc functions \( f_1, \ldots, f_n \) which are invariant under the Reeb flow,

\[
P(f_1, \ldots, f_n) := P(\min\{f_1, \ldots, f_n\}).
\]

We have the following,

**Theorem 3.1** Given \( f \in C^\infty_B(M) \), then we have the following estimate

\[
\|P(f)\|_{C^{1,1}} \leq C(M, \omega^T, g, \|f\|_{C^{1,1}}).
\]

Moreover, if \( u_1, \ldots, u_k \in \mathcal{H}_\Delta \), where we use the notation

\[
\mathcal{H}_\Delta = \{u \in \text{PSH}(M, \xi, \omega^T) : \|u\|_{C^{1,1}} < \infty\}
\]

then \( P(u_1, \ldots, u_k) \in \mathcal{H}_\Delta \).

We shall prove Theorem 3.1 in Appendix. The following result (for similar result in Kähler case, see [2, Corollary 9.2]) would be very essential for the rooftop envelop \( P(u_0, u_1) \):

**Lemma 3.2** For \( u_0, u_1 \in \mathcal{H}_\Delta \), then

\[
\omega^T_{P(u_0, u_1)} \wedge \eta = 0
\]

(3.25)

on the non-contact set \( \Gamma = \{P(u_0, u_1) < \min(u_0, u_1)\} \).
Proof First we assume $\xi$ is regular or quasiregular, then the proof follows similarly as in Kähler setting. We sketch the proof briefly. We consider the quotient Kähler manifold (orbifold) $(Z = M/\mathcal{F}_\xi, \omega_Z)$ such that $\omega^T = \pi^* \omega_Z$, where $\pi : M \to Z$ is the natural quotient map. Since $u_0, u_1$, and $P(u_0, u_1)$ are all basic functions, and they descend to $Z$ to define the functions on $Z$, which we still denote as $u_0, u_1$, and $P(u_0, u_1)$. We only need to show that $(\omega_Z + \sqrt{-1} \partial \bar{\partial} P(u_0, u_1))^n = 0$ on $\Gamma_Z := \{ z \in Z : P(u_0, u_1) < \min(u_0, u_1) \}$. Note that $\Gamma_Z = \pi(\Gamma)$. This simply follows from [2, Corollary 9.2].

Now we deal with the case when $\xi$ is irregular. We need to use a Type-I deformation to approximate $(M, \xi, \eta, g, \Phi)$, as in Theorem 6.1. Denote $T^k$ to be the torus in $\text{Aut}(\xi, \eta, g)$ with the Lie algebra $t$. Take $\rho_i \in t$ such that $\rho_i \to 0$ (convergence is smooth with respect to a fixed metric $g$). We can take $\rho_i$ such that $\xi_i = \xi + \rho_i$ is quasiregular. Consider the Type-I deformation $(M, \xi_i, \eta_i, g_i, \Phi_i)$ as in Definition 2.3. Given $u_0, u_1 \in \mathcal{H}_\Delta$ and we know that $P(u_0, u_1) \in \mathcal{H}_\Delta$ (see Theorem 3.1), by Lemma 6.1, there exists $\epsilon_i \to 0$ such that $(1 - \epsilon_i)u_0, (1 - \epsilon_i)u_1, (1 - \epsilon_i)P(u_0, u_1) \in \text{PSH}(M, \xi_i, \omega_i^T)$. Define

$$P_i = P_i((1 - \epsilon_i)u_0, (1 - \epsilon_i)u_1) = \sup\{ v \in \text{PSH}(M, \xi_i, \omega_i^T), v \leq (1 - \epsilon_i)u_0, (1 - \epsilon_i)u_1 \}. \tag{3.26}$$

Since $(1 - \epsilon_i)P(u_0, u_1) \in \text{PSH}(M, \xi_i, \omega_i^T)$ and $(1 - \epsilon_i)P(u_0, u_1) \leq (1 - \epsilon_i)u_0, (1 - \epsilon_i)u_1$, hence $(1 - \epsilon_i)P(u_0, u_1) \leq P_i$. On the other hand, we apply Lemma 6.1 and we know there exists $\epsilon_i \to 0$ such that $(1 - \epsilon_i)P_i \in \text{PSH}(M, \xi, \omega^T)$. It follows that

$$(1 - \epsilon_i)P_i \leq P(u_0, u_1) \leq P_i(1 - \epsilon_i)^{-1}. \tag{3.26}$$

By Theorem 3.1, we know that $|d\Phi dP_i|$ is uniformly bounded and hence $P_i \to P(u_0, u_1)$ in $C^{1,\alpha}$. For any compact subset $K \subset \Gamma = \{ P(u_0, u_1) < \min(u_0, u_1) \}$, we can choose $i$ sufficiently large, such that $P_i < \min\{(1 - \epsilon_i)u_0, (1 - \epsilon_i)u_1\}$. Since $\xi_i$ is quasiregular, by (3.26), we can then get that

$$\left( \omega_i^T + \frac{1}{2} d\Phi_i dP_i \right)^{\epsilon} \wedge \eta_i = 0, \text{ on } K. \tag{3.26}$$

Taking $i \to \infty$, by Lemma 6.2, we get that

$$\left( \omega^T + \frac{1}{2} d\Phi dP(u_0, u_1) \right)^{\epsilon} \wedge \eta = 0, \text{ on } K. \tag{3.26}$$

This completes the proof. \hfill \Box

As a consequence, we get a volume partition formula for $\omega^n_{P(u_0, u_1)} \wedge \eta$ as follows:

Lemma 3.3 For $u_0, u_1 \in \mathcal{H}_\Delta$, denote $\Lambda_{u_0} = \{ P(u_0, u_1) = u_0 \}$ and $\Lambda_{u_1} = \{ P(u_0, u_1) = u_1 \}$. Then we have the following

$$\omega^n_{P(u_0, u_1)} \wedge \eta = \chi_{\Lambda_{u_0}} \omega^n_{u_0} \wedge \eta + \chi_{\Lambda_{u_1}} \omega^n_{u_1} \wedge \eta. \tag{3.27}$$
Proof The proof is similar to the Kähler case, see [29, Proposition 2.2]. The previous lemma implies that the measure $\omega^n_{P(u_0,u_1)} \wedge \eta$ is supported on the set $\Lambda_{u_0} \cup \Lambda_{u_1}$. It follows from Theorem 3.1 that $P(u_0,u_1)$ has bounded Laplacian, hence all second partial derivatives of $P(u_0,u_1)$ are in $L^p(M)$ for all $p > 1$. Then all the second-order partial derivatives of $P(u_0,u_1)$ and $u_0$ coincide on $\Lambda_{u_0}$ almost everywhere, all the second-order partial derivatives of $P(u_0,u_1)$ and $u_1$ coincide on $\Lambda_{u_1}$ almost everywhere. Recall the definition of Monge–Ampere operators on psh functions belong to $W^{2,n}$, we can write:

$$\omega^n_{P(u_0,u_1)} \wedge \eta = \chi_{\Lambda_{u_0}} \omega^n_{u_0} \wedge \eta + \chi_{\Lambda_{u_1}} \omega^n_{u_1} \wedge \eta.$$

□

Lemma 3.4 Suppose $\chi \in W^+_p$ and $u_0, u_1 \in \mathcal{E}_\chi(M, \xi, \omega_T)$. Then $P(u_0,u_1) \in \mathcal{E}_\chi(M, \xi, \omega_T)$. If $u_0, u_1 \leq 0$, then the following estimates hold

$$E_\chi(P(u_0,u_1)) \leq (p + 1)^n (E_\chi(u_0) + E_\chi(u_1)). \quad (3.28)$$

Proof The proof is similar to the Kähler case, see [29, Lemma 3.4]. Without loss of generality we can assume $u_0, u_1 < 0$. It follows from Lemma 3.1 that there exist negative transverse Kähler potentials $u_0^k, u_1^k \in \mathcal{H}$ deceasing to $u_0, u_1$ respectively. By Theorem 3.1, the rooftop envelopes $P(u_0^k, u_1^k) \in \mathcal{H}_\Delta$ decreases to $P(u_0, u_1)$. And we have the following inequality by Lemma 3.3:

$$\omega^n_{P(u_0^k, u_1^k)} \wedge \eta \leq \chi_{\Lambda_{u_0}} \omega^n_{u_0} \wedge \eta + \chi_{\Lambda_{u_1}} \omega^n_{u_1} \wedge \eta.$$

Then

$$E_\chi(P(u_0^k, u_1^k)) = \int_M \chi(P(u_0^k, u_1^k)) \omega^n_{P(u_0^k, u_1^k)} \wedge \eta$$

$$\leq \int_{P(u_0^k, u_1^k) = u_0^k} \chi(u_0^k) \omega^n_{u_0^k} \wedge \eta + \int_{P(u_0^k, u_1^k) = u_1^k} \chi(u_1^k) \omega^n_{u_1^k} \wedge \eta$$

$$\leq E_\chi(u_0^k) + E_\chi(u_1^k)$$

$$\leq (p + 1)^n (E_\chi(u_0) + E_\chi(u_1)).$$

By Proposition 3.16 we have $P((u_0, u_1)) \in \mathcal{E}_\chi(M, \xi, \omega_T)$ and the required inequality holds. □

As a corollary we know that $\mathcal{E}_\chi(M, \xi, \omega_T)$ is convex.

Corollary 3.1 If $u_0, u_1 \in \mathcal{E}_\chi(M, \xi, \omega_T)$, then $tu_0 + (1 - t)u_1 \in \mathcal{E}_\chi(M, \xi, \omega_T)$ for any $t \in [0, 1]$.

Proof By the previous lemma we have $P(u_0, u_1) \in \mathcal{E}_\chi(M, \xi, \omega_T)$. Notice that $P(u_0, u_1) \leq tu_0 + (1 - t)u_1$ for $t \in [0, 1]$, then the monotonicity property of $\mathcal{E}_\chi(M, \xi, \omega_T)$ implies that $tu_0 + (1 - t)u_1 \in \mathcal{E}_\chi(M, \xi, \omega_T)$. □
To finish this subsection, we establish a domination principle which will be needed later.

**Lemma 3.5** Let \( U \subset M \) be a Borel set with \( (\omega^T)^n \wedge \eta(U) > 0 \) and \( u \in \mathcal{E}_1(M, \xi, \omega^T) \). Then there exists \( \varphi \in \mathcal{E}_1(M, \xi, \omega^T) \) with \( \varphi \leq u \) and \( \omega^n_{\varphi} \wedge \eta(U) > 0 \).

**Proof** The proof is similar to the Kähler case, see [30, Lemma 2.22]. Without loss of generality we can assume that \( u < 0 \). Then we choose a sequence \( u_k \in \mathcal{H} \) decreasing to \( u \) with \( u_k < 0 \). For a constant \( \tau > 0 \), we have \( \{P(u_k + \tau, 0) = u_k + \tau\} \subset \{u_k \leq -\tau\} \). It follows from Proposition 3.3 that

\[
\omega^n_{P(u_k+\tau,0)} \wedge \eta \leq \chi_{\{u_k \leq -\tau\}} \omega^n_{u_k} \wedge \eta + (\omega^T)^n \wedge \eta \leq -\frac{u_k}{\tau} \omega^n_{u_k} \wedge \eta + (\omega^T)^n \wedge \eta.
\]

The sequence \( P(u_k + \tau, 0) \in \mathcal{E}_1(M, \xi, \omega^T) \) decreases to \( P(u + \tau, 0) \in \mathcal{E}_1(M, \xi, \omega^T) \). It follows from Proposition 3.15 that

\[
\omega^n_{P(u+\tau,0)} \wedge \eta \leq -\frac{\mu}{\tau} \omega^n_u \wedge \eta + (\omega^T)^n \wedge \eta.
\]

Hence we have

\[
\omega^n_{P(u+\tau,0)} \wedge \eta(\mathcal{M} - U) \leq \frac{1}{\tau} \int_{\mathcal{M} - U} |u| \omega^n_u \wedge \eta + (\omega^T)^n \wedge \eta(\mathcal{M} - U)
\]

\[
\leq \frac{1}{\tau} \int_{\mathcal{M}} |u| \omega^n_u \wedge \eta + (\omega^T)^n \wedge \eta(\mathcal{M} - U).
\]

It follows from \( \omega^n_{P(u+\tau,0)} \wedge \eta(\mathcal{M}) = (\omega^T)^n \wedge \eta(\mathcal{M}) = \text{Vol}(\mathcal{M}) \) that

\[
\omega^n_{P(u+\tau,0)} \wedge \eta(U) \geq (\omega^T)^n \wedge \eta(U) = \frac{1}{\tau} \int_{\mathcal{M}} |u| \omega^n_u \wedge \eta
\]

and \( \omega^n_{P(u+\tau,0)} \wedge \eta(U) > 0 \) for \( \tau \) big enough. Then \( \varphi = P(u + \tau, 0) - \tau \) satisfies the requirements. \(\square\)

**Lemma 3.6** (The domination principle) If \( u, v \in \mathcal{E}_1(M, \xi, \omega^T) \) and \( u \leq v \) almost everywhere with respect to the measure \( \omega^n_u \wedge \eta \). Then \( u \leq v \).

**Proof** The proof is similar to the Kähler case, see [30, Proposition 2.21]. We only have to prove \( u \leq v \) almost everywhere with respect to \( (\omega^T)^n \wedge \eta \) for \( u, v < 0 \).

Suppose that \( (\omega^T)^n \wedge \eta(|u > v|) > 0 \). The previous lemma implies that there exists \( \varphi \in \mathcal{E}_1(M, \xi, \omega^T) \) with \( \varphi \leq u \) and \( \omega^n_\varphi \wedge \eta(|u > v|) > 0 \). It follows from Corollary 3.1 that \( t\varphi + (1 - t)u \in \mathcal{E}_1(M, \xi, \omega^T) \) for \( t \in [0, 1] \). Using the fact \( \omega^n_{t\varphi + (1 - t)u} \wedge \eta \geq t^n \omega^n_\varphi \wedge \eta \), the Comparison principle (3.15) and \( \{v < t\varphi + (1 - t)u\} \subset \{v < u\} \), we have

\[
t^n \int_{\{v < t\varphi + (1 - t)u\}} \omega^n_\varphi \wedge \eta \leq \int_{\{v < t\varphi + (1 - t)u\}} \omega^n_{t\varphi + (1 - t)u} \wedge \eta
\]
\[
\leq \int_{\{v < t\phi + (1-t)u\}} \omega^n_v \wedge \eta \\
\leq \int_{\{v < u\}} \omega^n_v \wedge \eta \\
= 0
\]

and \(\omega^n_{\phi} \wedge \eta(\{v < t\phi + (1-t)u\}) = 0\) for \(t \in (0, 1]\). Then

\[
\omega^n_{\phi} \wedge \eta(\{v < u\}) = \lim_{k \to \infty} \omega^n_{\phi} \wedge \eta\left(\left\{v < \frac{1}{k}\phi + (1 - \frac{1}{k})u\right\}\right) = 0.
\]

This leads to a contradiction. \(\square\)

### 3.2 The Space of Transverse Kähler Potentials and \((\mathcal{H}, d_2)\)

The Riemannian structure on \(\mathcal{H}\) has been studied extensively, notably by Guan–Zhang [42]. Guan–Zhang proved that for any two points \(\phi_1, \phi_2 \in \mathcal{H}\), there exists a unique \(C_{\overline{B}}^1\) geodesic which realizes the distance of \((\mathcal{H}, d_2)\) and \((\mathcal{H}, d_2)\) is a metric space. The Riemannian structure would play a very central role, as in Chen’s result [20] in Kähler setting.

We shall recall these results. For \(\psi_1, \psi_2 \in T_{\phi}\mathcal{H} = C_{\overline{B}}^\infty(M)\), define a \(L^2\) inner product on this tangent space

\[
(\psi_1, \psi_2)_{\phi} = \int_M \psi_1 \psi_2 d\mu_{\phi}
\]

and the length \(\|\psi\|_\phi\) of a vector \(\psi \in T_{\phi}\mathcal{H}\) is

\[
\|\psi\|_{2, \phi} = \left(\int_M \psi_1 \psi_2 d\mu_{\phi}\right)^{\frac{1}{2}},
\]

where we use the notation

\[
d\mu_{\phi} = \omega^n_{\phi} \wedge \eta_{\phi} = \omega^n_{\phi} \wedge \eta. \quad (3.29)
\]

For a smooth path \(\phi_t \in \mathcal{H}\), the length of the path is defined to be

\[
l(\phi_t) = \int_0^1 \|\dot{\phi}_t\|_{2, \phi_t} dt.
\]

This is a direct adaption of Mabuchi’s metric [53] on the space of Kähler potentials to Sasaki setting. The Levi-Civita connection \(\nabla\) is torsion free and satisfies

\[
\frac{d}{dt}(u_t, v_t)_{\phi_t} = (\nabla_{\dot{\phi}_t} u_t, v_t)_{\phi_t} + (u_t, \nabla_{\dot{\phi}_t} v_t)_{\phi_t}
\]
for any smooth vector fields \( u_t, v_t \) along the path \( \phi_t \) in \( \mathcal{H} \). Let \( u_t \in C^\infty_B(M) \) be smooth vector fields along a smooth curve \( \phi_t \) in \( \mathcal{H} \), then

\[
\nabla_{\dot{\phi}_t} u_t = \dot{u}_t - \frac{1}{4} < \nabla \dot{\phi}_t, \nabla u_t >_{\phi_t}.
\]

(3.30)

The geodesic equation can be written as

\[
\nabla_{\dot{\phi}_t} (\dot{\phi}_t) = \ddot{\phi}_t - \frac{1}{4} |\nabla \dot{\phi}_t|^2_{\phi_t} = 0.
\]

(3.31)

Given \( \phi_0, \phi_1 \in \mathcal{H} \), to solve the geodesic equation, Guan–Zhang \cite{42} introduced the following perturbation equation, for a path \( \phi_t : M \times [0, 1] \rightarrow \mathbb{R} \),

\[
\begin{cases}
\left( \ddot{\phi}_t - \frac{1}{4} |\nabla \dot{\phi}_t|^2_{\phi_t} \right) \omega^m_{\phi_t} \wedge \eta = \epsilon (\omega^T)_{\phi_t} \wedge \eta, M \times (0, 1) \\
\phi|_{t=0} = \phi_0 \\
\phi|_{t=1} = \phi_1.
\end{cases}
\]

(3.32)

Define a function \( \psi \) on \( M \times [1, 3/2] \), as a subset of the cone \( X \),

\[
\psi(\cdot, r) = \phi_t(\cdot) + 4 \log r, \quad t = 2r - 2.
\]

Set a \((1, 1)\) form by,

\[
\Omega_\psi = \omega_X + \frac{r^2}{2} \sqrt{-1} \left( \partial \bar{\partial} \psi - \frac{\partial \psi}{\partial r} \partial \bar{\partial} r \right).
\]

Guan–Zhang wrote an equivalent form of (3.32) in terms of a complex Monge–Ampère equation on \( \psi \) of the following form (with \( f = r^2, \epsilon \in (0, 1) \)),

\[
\begin{cases}
(\Omega_\psi)^{n+1} = \epsilon f (\omega_X)^{n+1}, M \times (1, \frac{3}{2}) \\
\psi|_{M \times \{r=1\}} = \phi_0, \psi|_{M \times \{r=3/2\}} = \psi_1 + 4 \log(3/2).
\end{cases}
\]

(3.33)

Guan–Zhang proved the following results regarding (3.33):

**Theorem 3.2** (Guan–Zhang) Fix a Sasaki structure \((M, \xi, \eta, g)\) on a compact manifold \(M\). For any positive basic function \( f \) and any two points \( \phi_0, \phi_1 \in \mathcal{H} \), there exists a unique smooth solution of \( \psi \) to (3.33), satisfying the following estimates: \( \psi \) is basic and there exists a constant \( C > 0 \), depending only on \( \|f\|^{\frac{5}{2}}_{C^2(M \times [1, \frac{3}{2}])}, \|\phi_0\|_{C^{2,1}}, \|\phi_1\|_{C^{2,1}} \) such that

\[
\|\psi\|_{C^1_w} := \|\psi\|_{C^1} + \sup |\Delta \psi| \leq C.
\]

(3.34)

Denote the corresponding solution of (3.32) by \( \phi^\epsilon_t \), then \( \phi^\epsilon_t \) is called a \( \epsilon \)-geodesic (smooth) connecting \( \phi_0, \phi_1 \) satisfying

\[
\|\phi^\epsilon_t\|_{C^1} + \sup (\ddot{\phi}^\epsilon + |\nabla \phi^\epsilon_t|g + \Delta_g \phi^\epsilon_t) \leq C.
\]

(3.35)
When \( \epsilon \to 0 \), there exists a unique (weak \( C^2_w \)) limit \( \phi_t \) of \( \phi^\epsilon_t : M \times [0, 1] \to \mathbb{R} \) connecting \( \phi_0, \phi_1 \) such that \( \Omega^{\phi^\epsilon_t + 4\log r} \) is positive. The later is equivalent to

\[
\omega_{\phi^\epsilon_t} > 0, \quad \ddot{\phi}^\epsilon_t - \frac{1}{4} |\nabla \dot{\phi}^\epsilon_t |_{\omega_{\phi^\epsilon_t}}^2 > 0.
\]

As a consequence, \( (\mathcal{H}, d_2) \) is a metric space.

**Remark 3.2** The constant \( 1/4 \) appears in the geodesic equation

\[
\ddot{\phi}_t - \frac{1}{4} |\nabla \dot{\phi}_t |_{\omega_{\phi_t}}^2 = 0.
\]

This constant is insignificant. In Kähler setting, some authors write the constant as \( 1/2 \) and some write as \( 1 \), depending on the gradient \( \nabla \) is interpreted as real or complex; they differ by a constant 2. The constant \( 1/4 \) appears in Sasaki setting in [42] since the authors use the real gradient and use the space of Sasaki potentials (transverse Kähler potentials) defined as

\[
\{ \phi : d\eta + \sqrt{-1} \partial B \bar{\partial} B \phi > 0 \}.
\]

In the following, we shall write the geodesic equation as

\[
\ddot{\phi}_t - |\nabla \dot{\phi}_t |_{\omega_{\phi_t}}^2 = 0,
\]

where we use complex gradient, and our choice space of transverse Kähler potentials is as

\[
\mathcal{H} = \{ \phi \in C^\infty_B (M) : \omega^T + \sqrt{-1} \partial B \bar{\partial} B \phi > 0 \}.
\]

To prove \( (\mathcal{H}, d_2) \) is a metric space, Guan–Zhang [42, Lemma 14, Proof of Theorem 2] proved the following triangle inequality,

**Lemma 3.7** (Guan–Zhang) Let \( \psi(s) : [0, 1] \to \mathcal{H} \) be a smooth curve, \( \phi \in \mathcal{H} \setminus \psi([0, 1]) \). Fix \( \epsilon \in (0, 1]. \) Let \( u^\epsilon \in C^\infty_B ([0, 1] \times [0, 1] \times M) \) be the function such that \( u^\epsilon_t (\cdot, s) \) is the \( \epsilon \)-geodesic connecting \( \phi \) and \( \psi_s \), for \( t \in [0, 1] \). Then the following estimate holds:

\[
l(u^\epsilon_t (\cdot, 0)) \leq l(\psi) + l(u^\epsilon_t (\cdot, 1)) + \epsilon C,
\]

where \( C = C(\phi, \psi, g) \) is a uniform constant, independent of \( \epsilon \).

There are several estimates which are not explicitly stated or not proved in [42]. We include these estimates below since we shall need them below. Regarding (3.32), first we have the following comparison principle,
Lemma 3.8  Suppose we have two solutions $\varphi, \phi$ with boundary datum $\varphi_0, \varphi_1$ and $\phi_0, \phi_1$, respectively,

$$
\left(\ddot{\phi}_t - |\nabla \dot{\phi}_t|_{\omega_{\phi_0}}^2\right) \omega_{\phi}^n \wedge \eta = \epsilon (\omega^T)^n \wedge \eta = \left(\ddot{\varphi}_t - |\nabla \dot{\varphi}_t|_{\omega_{\varphi_0}}^2\right) \omega_{\varphi}^n \wedge \eta,
$$

then we have the following

$$
\max |\phi - \varphi| \leq \max |\phi_0 - \varphi_0| + \max |\phi_1 - \varphi_1|.
$$

Proof  This is a standard comparison principle. We sketch the proof for completeness. Denote the operator

$$
F(D^2\phi) = \log \det \left( \frac{\ddot{\phi}}{(\nabla \dot{\phi})^T g_{ij} + \phi_{ij}} \right) - \log \det (g_{ij}^T) = \log \left( \ddot{\phi}_t - |\nabla \dot{\phi}_t|_{\omega_{\phi_0}}^2 \right) + \log \frac{\det(g_{ij}^T + \phi_{ij})}{\det(g_{ij}^T)}.
$$

The $\epsilon$-geodesic equation can be written as $F(D^2\phi) = \epsilon$. Now suppose $F(D^2\phi) = F(D^2\varphi) = \epsilon > 0$, then (3.38) holds. Otherwise suppose at some interior point

$$
\phi - \varphi > \max |\phi_0 - \varphi_0| + \max |\phi_1 - \varphi_1|.
$$

Hence $\phi - \varphi + at(1-t)$ obtains its maximum at an interior point $p$ for some $a > 0$. Denote $v = \phi + at(t-1)$. Then on one hand,

$$
F(D^2v) > F(D^2\phi) = \epsilon.
$$

On the other hand at $p$, $D^2v \leq D^2\varphi$. It follows from the concavity of $F$, we have at $p$,

$$
F(D^2v) - F(D^2\varphi) \leq \mathcal{L}_F(v - \varphi) \leq 0,
$$

where $\mathcal{L}_F$ is the linearized operator of $F$ at $v$. Contradiction.

One can actually be more precise about the estimate (3.35) [and (3.34)]. For simplicity, we state the result for (3.32).

Lemma 3.9  The $\epsilon$ geodesic $\phi^\epsilon_t$ connecting $\phi_0, \phi_1 \in \mathcal{H}$ satisfies the following estimate,

$$
\max |\dot{\phi}_t^\epsilon| \leq \max |\phi_1 - \phi_0| + C \max |\nabla (\phi_1 - \phi_0)|_g^2 + \epsilon,
$$

where $C$ depends only on $\phi_0, \phi_1$. Moreover, we have

$$
|\nabla \phi_t^\epsilon|_g + \sup \Delta_g \phi^\epsilon \leq C(\|\phi_0\|_{C^1}, \|\phi_1\|_{C^1}, \sup \Delta_g \phi_0, \Delta_g \phi_1, g).
$$

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Proof The first estimate follows from \( \ddot{\phi}_t^\epsilon > 0 \) and the following \( C^0 \) estimate (3.41), which can be proved similarly using the concavity of \( F \). First there exists \( a > 0 \) such that

\[
\dot{a}t(t - 1) + (1 - t)\phi_0 + t\phi_1 \leq \phi_t^\epsilon \leq (1 - t)\phi_0 + t\phi_1. \tag{3.41}
\]

The right-hand side is a direction consequence of \( \ddot{\phi}_t^\epsilon > 0 \), while the left-hand side can be argued as follows. Denote \( U^a = \dot{a}t(t - 1) + (1 - t)\phi_0 + t\phi_1; \) we know \( \phi_t^\epsilon \) agrees with \( U^a \) on the boundary. Hence if \( \phi_t^\epsilon < U^a \), then \( \phi_t^\epsilon - U^a \) takes its minimum at some interior point \( p \). At \( p \), we know \( D^2\phi_t^\epsilon \geq D^2 U_a^a \). By concavity of \( F \), we get

\[
0 \leq L_{\phi_t^\epsilon}(\phi_t^\epsilon - U^a) \leq F(D^2\phi_t^\epsilon) - F(D^2 U^a).
\]

That is \( F(D^2 U^a) \leq \log \epsilon \). This is a contradiction when \( a > 0 \) is sufficiently large. Indeed, a direct computation shows that if \( a \geq C \max |\nabla (\phi_1 - \phi_0)|^2 + \epsilon \), then \( F(D^2 U^a) > \log \epsilon \). Hence for such choice of \( a \), (3.41) holds. By convexity in \( t \) direction, we know that

\[
\phi_t^\epsilon(\cdot, 0) \leq \phi_t^\epsilon(\cdot, 0) \leq \phi_t^\epsilon(\cdot, 1) \leq \phi_t^\epsilon(\cdot, 1).
\]

It is evident to show that

\[
- a + \phi_1 - \phi_0 \leq \phi_t^\epsilon(\cdot, 0) \leq \phi_1 - \phi_0 \leq \phi_t^\epsilon(\cdot, 1) \leq a + \phi_1 - \phi_0.
\]

Hence (3.39) follows. The gradient estimate \( |\nabla \phi_t^\epsilon| \) is given by [42, Proposition 2]. The estimate on \( \Delta_g \phi_t^\epsilon \), depending only on \( \phi_0, \phi_1 \) up to second-order derivative, was proved for Kähler setting by the first named author [47, Theorem 1.1] (for \( \epsilon = 0 \), it was proved earlier in [8] using pluripotential theory). The method in [47] is to deal with Eq. (3.32) directly, and it can be carried over to prove the interior estimate of \( \Delta_g \phi_t^\epsilon \) word by word (since in Sasaki setting, this estimate only involves transverse Kähler structure and basic functions). For completeness, we sketch the proof. Denote \( \phi_t^\epsilon = \phi \) for simplicity. We write the equations as

\[
\log(\dot{\phi} - |\nabla \phi|_\phi^2) + \log \det (g_{i\bar{j}}^T + \phi_{i\bar{j}}) = \log \epsilon + \log \det (g_{i\bar{j}}^T), \tag{3.42}
\]

using the transverse Kähler metric \( g_{i\bar{j}}^T \). For any basic function \( h \), we denote

\[
Dh = \Delta_\phi h + \frac{h_{t\bar{t}} + g_{i\bar{j}}^T g_{\phi}^T h_{i\bar{j}} \phi_{t\bar{t}} \phi_{i\bar{j}}}{\phi_{t\bar{t}} - |\nabla \phi_{t\bar{t}}|^2_\phi} - \frac{g_{i\bar{j}}^T (h_{t\bar{t}} \phi_{t\bar{t}} + h_{i\bar{j}} \phi_{t\bar{t}})}{\phi_{t\bar{t}} - |\nabla \phi_{t\bar{t}}|^2_\phi}, \tag{3.43}
\]
where \((g_{i\bar{j}}^\phi)^{-1}\) is the inverse of the transverse Kähler metric \((g_T^i\bar{j} + \phi_i\bar{j})\). Proceeding exactly as in the computation in the Kähler setting (see [47, (2.4)–(2.19)]), we compute

\[
D(\log(n + \Delta \phi) - C\phi + t^2) > g_{i\bar{j}}^{\phi} g_T^{i\bar{j}} + (\dot{\phi} - |\nabla\dot{\phi}|_\phi^2)^{-1} - (n + 1)C, \quad (3.44)
\]

where \(C\) depends only on the background transverse Kähler metric \(g_T^i\bar{j}\) and \(n\). Hence we have

\[
D(\log(n + \Delta \phi) - C\phi + t^2) > (n + \Delta \phi + \dot{\phi} - |\nabla\dot{\phi}|_\phi^2)^{\frac{1}{n}} \epsilon^{-\frac{1}{n}} - (n + 1)C, \quad (3.45)
\]

where we have used the elementary inequality

\[
\left(\sum_{i=0}^{n} a_i^{-1}\right)^n \geq \frac{\sum_{i=0}^{n} a_i}{\prod_{i=0}^{n} a_i}.
\]

Hence it follows that either \(\log(n + \Delta \phi) - C\phi + t^2\) achieves its maximum on the boundary, or at an interior maximum point \(P\),

\[
(n + \Delta \phi + \dot{\phi} - |\nabla\dot{\phi}|_\phi^2)^{\frac{1}{n}} \epsilon^{-\frac{1}{n}} - (n + 1)C \leq D(\log(n + \Delta \phi) - C\phi + t^2)(P) \leq 0.
\]

This gives the desired bound

\[
-n < \Delta \phi^\epsilon \leq C(\|\phi_0\|_{C^1}, \|\phi_1\|_{C^1}, \Delta g\phi_0, \Delta g\phi_1, g).
\]

\[\square\]

By taking \(\epsilon \to 0\), we have the following,

**Lemma 3.10** Suppose \(\phi\) is the weak geodesic connecting \(\phi_0, \phi_1 \in \mathcal{H}\), then for some positive constant \(C = C(M, g, \|\phi_0\|_{C^2}, \|\phi_1\|_{C^2})\), we have

\[
|\dot{\phi}| \leq \max |\phi_1 - \phi_0| + C \max |\nabla\phi_1 - \nabla\phi_0|_g^2
\]

As a consequence, when \(\phi_0 \to \phi_1\) in \(\mathcal{H}\), then \(d_2(\phi_0, \phi_1) \to 0\).

**Remark 3.3** One can get a much sharper estimate,

\[
|\dot{\phi}| \leq \max |\phi_1 - \phi_0|
\]

using the uniqueness and comparison for the generalized solutions of complex Monge–Ampere in the sense of Bedford–Taylor, see [30, Lemma 3.5] for Kähler setting. We shall prove this sharper version below.

Using Lemmas 3.7 and 3.10, it follows that the distance function \(d_2(\phi_0, \phi_1)\) is realized by the weak geodesic \(\phi\) connecting \(\phi_0, \phi_1\). In particular,
Lemma 3.11  Given \( \phi_0, \phi_1 \in \mathcal{H} \), we have,

\[
d_2(\phi_0, \phi_1) = \| \dot{\phi} \|_{2, \phi_t}, \forall t \in [0, 1]
\]  \quad (3.46)

Proof  Let \( \phi^\epsilon \) be the \( \epsilon \) geodesic connecting \( \phi_0, \phi_1 \). Then we compute

\[
\frac{d}{dt} \int_M |\dot{\phi}^\epsilon|^2 (\omega_{\phi^\epsilon})^n \wedge \eta = 2 \int_M \dot{\phi}^\epsilon (\dddot{\phi}^\epsilon - |\nabla \dot{\phi}^\epsilon|_{\phi^\epsilon}) (\omega_{\phi^\epsilon})^n \wedge \eta.
\]  \quad (3.47)

Since \( |\dot{\phi}^\epsilon| \) is uniformly bounded, letting \( \epsilon \to 0 \), we get that

\[
\frac{d}{dt} \int_M |\dot{\phi}|^2 (\omega_{\phi_t})^n \wedge \eta = 0.
\]

This proves (3.46). In particular if \( \phi_0 \neq \phi_1 \), \( \dot{\phi}_t \) is not identically zero for any \( t \). Moreover, if \( \epsilon \) is small enough, depending on \( \phi_0 \neq \phi_1 \), then \( \dot{\phi}^\epsilon \) is not identically zero for any \( t \in [0, 1] \). This follows from (3.47) and it is easy to see that \( \int_M |\dot{\phi}^\epsilon|^2 (\omega_{\phi^\epsilon})^n \wedge \eta \) has a positive lower bound for any \( t \) (say \( l(\phi^\epsilon)/2 \)), if \( \epsilon \) is sufficiently small. \( \Box \)

We also have the following

Theorem 3.3  (Guan–Zhang, Theorem 2). For \( u, v, w \in \mathcal{H} \),

\[
d_2(u, w) \leq d_2(u, v) + d_2(v, w).
\]

3.3 The Orlicz–Finsler Geometry on Sasaki Manifolds

The Orlicz–Finsler geometry on the space of Kähler potentials was introduced by Darvas [28] and it has played an important role in problems regarding csck and Calabi’s extremal metric in Kähler geometry. In particular, the Finsler metric \( d_1 \) will play an important role and it is used to define the properness of \( K \)-energy. In this section, we discuss the Orlicz–Finsler geometry on Sasaki manifolds. We prove the following theorem, which is the counterpart of Darvas’s [28, Theorem 1] in Sasaki setting.

Theorem 3.4  If \( \chi \in \mathcal{W}^+_p, p \geq 1 \), then \( (\mathcal{H}, d_\chi) \) is a metric space and for any \( u_0, u_1 \in \mathcal{H} \), the \( C^{1, \bar{1}}_B \) geodesic \( t \to u_t \) connecting \( u_0, u_1 \) satisfies

\[
d_\chi(u_0, u_1) = \| \dot{u}_t \| \chi_{u_t}, t \in [0, 1].
\]  \quad (3.48)

Theorem 3.4 is the generalization for \( d_2 \) to general Young weights. This important result in Darvas’s theory says that, the same \( C^{1, \bar{1}}_B \) geodesic (with respect to \( d_2 \)) is “length minimizing” for all \( d_\chi \) metric structures and this holds in Sasaki setting. The proof of Theorem 3.4 pretty much follows Darvas’s proof [30, Theorem 3.4], with minor modifications adapted to Sasaki setting. The main point is that only transverse
Kähler structure is involved, and hence this is essentially the same as in Kähler setting. We include the details for completeness.

Following Darvas (see [30, Chapter 3]), we define the Orlicz–Finsler length of \( v \in T_u \mathcal{H} = C^\infty_B(M) \) for any weight \( \chi \in \mathcal{W}^+_p \):

\[
\|v\|_{\chi,u} = \inf \left\{ r > 0 : \frac{1}{\text{Vol}(M)} \int_M \chi\left(\frac{v}{r}\right) \omega^n_u \wedge d\eta \leq \chi(1) \right\}.
\] (3.49)

For simplicity, we shall assume \( \text{Vol}(M) = 1 \) in this section. Given a smooth curve \( \gamma : t \in [0, 1] \to \mathcal{H} \), its length is computed by the formula

\[
l_{\chi}(\gamma_t) = \int_0^1 \|\dot{\gamma}_t\|_{\chi,\gamma_t} \, dt.
\] (3.50)

Furthermore, the distance \( d_{\chi}(u_0, u_1) \) between \( u_0, u_1 \in \mathcal{H} \) is defined to be

\[
d_{\chi}(u_0, u_1) = \inf \{ l_{\chi}(\gamma_t) : \gamma_t \text{ is a smooth curve with } \gamma_0 = u_0, \gamma_1 = u_1 \}.
\] (3.51)

First we have the following,

**Proposition 3.18** Suppose \( \chi \in \mathcal{W}^+_p \cap C^\infty(\mathbb{R}) \). For a smooth curve \( u_t (t \in [0, 1]) \) in \( \mathcal{H} \) and a vector field \( f_t \in C^\infty_B(M) \) along this curve with \( f_t \not\equiv 0 \), we have

\[
\frac{d}{dt} \|f_t\|_{\chi,u_t} = \frac{\int_M \chi'\left(\frac{f_t}{\|f_t\|_{\chi,u_t}}\right) \nabla f_t d\mu_{u_t}}{\int_M \chi'\left(\frac{f_t}{\|f_t\|_{\chi,u_t}}\right) \frac{f_t}{\|f_t\|_{\chi,u_t}} d\mu_{u_t}}.
\] (3.52)

**Proof** This works as in [28, Proposition 3.1] word by word. We skip the details. \( \square \)

**Lemma 3.12** Suppose \( \chi \in \mathcal{W}^+_p \cap C^\infty(\mathbb{R}) \) and \( u_0, u_1 \in \mathcal{H}, u_0 \neq u_1 \). Then the \( \epsilon \)-geodesics \( [0, 1] \ni t \to u_t^\epsilon \in \mathcal{H} \) connecting \( u_0, u_1 \) satisfies the following estimate:

\[
\int_M \chi(\dot{u}_t^\epsilon) \omega^n_{u_t^\epsilon} \wedge \eta \geq \max \left( \int_M \chi(\min(u_1 - u_0, 0)) \omega^n_{u_0} \wedge \eta, \right. \\
\left. \int_M \chi(\min(u_0 - u_1, 0)) \omega^n_{u_1} \wedge \eta \right) - \epsilon C
\] (3.53)

for all \( t \in [0, 1] \), where \( C := C(\chi, \|u_0\|_{C^2(M)}, \|u_1\|_{C^2(M)}) \).

**Proof** This follows exactly as in Kähler setting [30, Lemma 3.8], by a direct computation and the convexity of \( \chi \). \( \square \)

**Lemma 3.13** Suppose \( \chi \in \mathcal{W}^+_p \cap C^\infty(\mathbb{R}) \) and \( u_0, u_1 \in \mathcal{H}, u_0 \neq u_1 \). Then there exists a constant \( \epsilon_0 \) that depends on \( u_0, u_1 \) such that for all \( \epsilon \in (0, \epsilon_0] \) the \( \epsilon \)-geodesic \( [0, 1] \ni t \to u_t^\epsilon \in \mathcal{H} \) connecting \( u_0, u_1 \) satisfies:
Choose \( \epsilon \)

**Proposition 3.18.**

**Proposition 3.19.**

Suppose \( t \in [0, 1] \), if \( \epsilon \) exists such that for any \( \epsilon \in (0, \epsilon_0] \), given \( u_0 \neq u_1 \), see (3.46). Then the results follows from Proposition 3.18.

We have the following, similar to (3.46) (for \( d_2 \)),

**Proposition 3.19.**

Suppose \( \chi \in \mathcal{W}^+_p \cap C^\infty (\mathbb{R}) \) and \( u_0, u_1 \in \mathcal{H} \), \( u_0 \neq u_1 \). Then there exists \( \epsilon_0 > 0 \) such that for any \( \epsilon \in (0, \epsilon_0] \) the \( \epsilon \)-geodesic \([0, 1] \ni t \rightarrow u^\epsilon_t \in \mathcal{H} \) connecting \( u_0, u_1 \) satisfies

\[(i) \ |\dot{u}^\epsilon_t|_{\mathcal{X}, u^\epsilon_t} > R_0, t \in [0, 1];
(ii) \ |d/dt| |\dot{u}^\epsilon_t|_{\mathcal{X}, u^\epsilon_t} | \leq \epsilon R_1, t \in [0, 1],
\]

where \( \epsilon_0, R_0, R_1 \) depend on upper bounds for \( |u_0|_{C^2(M)}, |u_1|_{C^2(M)} \) and lower bounds for \( ||\chi(u_1 - u_0)||_{L^1((\omega^T)\mu \wedge \eta)} \), \( \omega^T_{u_0} \wedge \eta_{u_0} \) and \( \omega^T_{u_1} \wedge \eta_{u_1} \).

**Proof.**

(i) Recall Eq. (1.11) in [30]

\[
||f||_{\mathcal{X}, \mu} \geq \min \left\{ \frac{\int_{\Omega} \chi(f) d\mu}{\chi(1)}, \left( \frac{\int_{\Omega} \chi(f) d\mu}{\chi(1)} \right) ^{1/\beta} \right\}
\]

and Lemma 3.12, the estimate in (i) follows immediately.

(ii) Choose \( \epsilon_0 \) small so that Lemma 3.13 applies. Recall the Young identity

\[
\chi(a) + \chi^*(\chi'(a)) = a \chi'(a), a, b \in \mathbb{R}, \chi'(a) \in \partial \chi(a)
\]

Then we have

\[
\frac{d}{dt} |\dot{u}^\epsilon_t|_{\mathcal{X}, u^\epsilon_t} = \epsilon \frac{\int_{\mathcal{M}} \chi' \left( \frac{\dot{u}^\epsilon_t}{||\dot{u}^\epsilon_t||_{\mathcal{X}, u^\epsilon_t}} \right) (\omega^T)^n \wedge \eta}{\int_{\mathcal{M}} \chi' \left( \frac{\dot{u}^\epsilon_t}{||\dot{u}^\epsilon_t||_{\mathcal{X}, u^\epsilon_t}} \right) (\omega^T)^n \wedge \eta} - \epsilon \frac{\int_{\mathcal{M}} \chi' \left( \frac{\dot{u}^\epsilon_t}{||\dot{u}^\epsilon_t||_{\mathcal{X}, u^\epsilon_t}} \right) (\omega^T)^n \wedge \eta}{\chi(1)} + \frac{\int_{\mathcal{M}} \chi^* \left( \chi' \left( \frac{\dot{u}^\epsilon_t}{||\dot{u}^\epsilon_t||_{\mathcal{X}, u^\epsilon_t}} \right) \right) (\omega^T)^n \wedge \eta}{\chi(1)}
\]

\[
\leq \frac{\epsilon}{\chi(1)} \left| \int_{\mathcal{M}} \chi' \left( \frac{\dot{u}^\epsilon_t}{||\dot{u}^\epsilon_t||_{\mathcal{X}, u^\epsilon_t}} \right) (\omega^T)^n \wedge \eta \right| .
\]

(3.55)

\( \Box \) Springer
Then the estimates (ii) follows from (i) and the fact that \( \dot{u}^\epsilon_t \) is uniformly bounded in terms of \( ||u_0||_{C^2(M)} \), \( ||u_1||_{C^2(M)} \).

\[ \square \]

**Remark 3.4** The estimate (i) in Proposition 3.19 holds for general weights \( \chi \in \mathcal{W}_p^+ \). Recall that \( \dot{u}^\epsilon_t \) is uniformly bounded in terms of \( ||u_0||_{C^2(M)} \), \( ||u_1||_{C^2(M)} \). We can choose smooth weights \( \chi_k \in \mathcal{W}^+_{pk} \cap C^\infty(\mathbb{R}) \) which approximate \( \chi \) uniformly on compact subsets of \( \mathbb{R} \). Moreover we have \( \lim_{k \to \infty} ||\dot{u}^\epsilon_t||_{\chi_k,u_t^\epsilon} = ||\dot{u}^\epsilon_t||_{\chi,u_t^\epsilon} \) [30, Section 1]. It follows that the estimates (i) hold for \( \chi \).

Next we are ready to prove the triangle inequality, as in Lemma 3.7 for \( d_2 \) and [28, Proposition 3.4] in Kähler setting.

**Proposition 3.20** Suppose \( \chi \in \mathcal{W}_p^+ \cap C^\infty(\mathbb{R}) \), \( \psi_s \in \mathcal{H} \) is a smooth curve, \( \phi \in \mathcal{H} \setminus \psi([0, 1]) \), and \( \epsilon > 0 \). \( u^\epsilon \in C^\infty([0, 1] \times [0, 1] \times M) \) is the smooth function for which \( t \to u^\epsilon_t(s, \cdot) = u^\epsilon(t, s, \cdot) \) is the \( \epsilon \)-geodesic connecting \( \phi \) and \( \psi_s \). Then there exists \( \epsilon_0(\phi, \psi) > 0 \) such that for any \( \epsilon \in (0, \epsilon_0) \) the following holds:

\[ l_\chi(u^\epsilon_t(\cdot, 0)) \leq l_\chi(\psi_s) + l_\chi(u^\epsilon_t(\cdot, 1)) + \epsilon R \]

for some \( R(\phi, \psi, \chi, \epsilon_0) > 0 \) independent of \( \epsilon \).

**Proof** Fix \( s \in [0, 1] \). By Propositions 3.18 and 3.19, there exists a constant \( \epsilon_0(\phi, \psi) > 0 \) such that for \( \epsilon \in (0, \epsilon_0) \)

\[
\frac{d}{ds} l_\chi(u_t(\cdot, s)) = \int_0^1 \frac{d}{ds} ||\dot{u}(t, s, \cdot)||_{\chi,u(t,s,\cdot)} dt \\
= \int_0^1 \frac{\int_M \chi' \left( \frac{\dot{u}}{||\dot{u}||_{\chi,u}} \right) \nabla_{\frac{\dot{u}}{||\dot{u}||_{\chi,u}}} \dot{u} d\mu_{u_t}}{\int_M \chi' \left( \frac{\dot{u}}{||\dot{u}||_{\chi,u}} \right) ||\dot{u}||_{\chi,u} d\mu_{u_t}} dt \\
= \int_0^1 \frac{\int_M \chi' \left( \frac{\dot{u}}{||\dot{u}||_{\chi,u}} \right) \nabla_{\frac{\dot{u}}{||\dot{u}||_{\chi,u}}} \dot{u} d\mu_{u_t}}{\chi(1) + \int_M \chi' \left( \frac{\dot{u}}{||\dot{u}||_{\chi,u}} \right) \left( \frac{\dot{u}}{||\dot{u}||_{\chi,u}} \right) d\mu_{u_t}} dt \\
= \int_0^1 \frac{\int_M \chi' \left( \frac{\dot{u}}{||\dot{u}||_{\chi,u}} \right) \frac{d}{ds} d\mu_{u_t}}{\chi(1) + \int_M \chi' \left( \frac{\dot{u}}{||\dot{u}||_{\chi,u}} \right) \left( \frac{\dot{u}}{||\dot{u}||_{\chi,u}} \right) d\mu_{u_t}} dt \\
= \int_0^1 \frac{d}{ds} \int_M \chi' \left( \frac{\dot{u}}{||\dot{u}||_{\chi,u}} \right) \nabla_{\frac{\dot{u}}{||\dot{u}||_{\chi,u}}} \left( \frac{\dot{u}}{||\dot{u}||_{\chi,u}} \right) d\mu_{u_t} \\
= \frac{d}{dt} \int_0^1 \frac{d}{ds} \int_M \chi' \left( \frac{\dot{u}}{||\dot{u}||_{\chi,u}} \right) \nabla_{\frac{\dot{u}}{||\dot{u}||_{\chi,u}}} \left( \frac{\dot{u}}{||\dot{u}||_{\chi,u}} \right) d\mu_{u_t} dt.
\]

Moreover we have

\[
\nabla_{\frac{\dot{u}}{||\dot{u}||_{\chi,u}}} \left( \frac{\dot{u}}{||\dot{u}||_{\chi,u}} \right) d\mu_{u_t} = \chi'' \left( \frac{\dot{u}}{||\dot{u}||_{\chi,u}} \right) \left( \frac{\nabla_{\frac{\dot{u}}{||\dot{u}||_{\chi,u}}} \dot{u}}{||\dot{u}||_{\chi,u}} \right) - \frac{\nabla_{\frac{\dot{u}}{||\dot{u}||_{\chi,u}}} \dot{u}}{||\dot{u}||_{\chi,u}} \frac{d}{dt} ||\dot{u}||_{\chi,u} d\mu_{u_t},
\]

It follows from Proposition 3.19 that \( ||\dot{u}||_{\chi,u} \) is uniformly bounded away from zero and both \( \nabla_{\frac{\dot{u}}{||\dot{u}||_{\chi,u}}} d\mu_{u_t} \) and \( \frac{d}{dt} ||\dot{u}||_{\chi,u} \) are uniformly bounded by the form \( \epsilon R \), where \( R \)
is uniformly bounded. Moreover $\dot{u}$, $\frac{du}{ds}$ are uniformly bounded independent of $\epsilon$ [42, Lemma 14]. Hence

$$
\frac{d}{ds} l_\chi(u_t(\cdot, s)) = \int_0^1 \frac{d}{dr} \left( \chi(1) + \int_M \chi^\ast \left( \chi' \left( \frac{\dot{u}}{||\dot{u}||_{\chi,u}} \right) \right) d\mu_{u_t} \right) dr + \epsilon R",

where $R$ is uniform bounded independent of $\epsilon$.

Recall that $\chi^\ast (\chi'(l)) = l$ for $l \in \mathbb{R}$, the expression

$$
\frac{d}{dr} \left( \chi(1) + \int_M \chi^\ast \left( \chi' \left( \frac{\dot{u}}{||\dot{u}||_{\chi,u}} \right) \right) d\mu_{u_t} \right)
$$

is a term of type $\epsilon R$. Hence we can write

$$
\frac{d}{ds} l_\chi(u_t(\cdot, s)) = \int_0^1 \frac{d}{dr} \left( \chi(1) + \int_M \chi^\ast \left( \chi' \left( \frac{\dot{u}}{||\dot{u}||_{\chi,u}} \right) \right) d\mu_{u_t} \right) dr + \epsilon R
$$

where the last line follows from the Young inequality

$$
\chi(a) + \chi^\ast(b) \geq ab, a, b \in \mathbb{R}.
$$

The integration of the above inequality with respect to $s \in [0, 1]$ yields the desired inequality. \hfill \Box

Now we are ready to prove Theorem 3.4. Certainly the proof follows closely Darvas’s result in Kähler setting [28, Section 3].

**Proof** First we show that for $u_0, u_1 \in \mathcal{H}$ and the weak $C^{1,T}_B$-geodesic $u_t$ connecting $u_0, u_1$

$$
d_{\chi}(u_0, u_1) = l_\chi(u_t). \tag{3.57}
$$

We assume $u_0 \neq u_1$. Recall that, by Guan–Zhang [42], $\epsilon$-geodesics $u^\epsilon_t$ connecting $u_0, u_1$ converge to the weak $C^{1,T}_B$ geodesic $u_t$ in $C^{1,\alpha}$. Hence $\dot{u}^\epsilon_t$ converges uniformly to $\dot{u}_t$. \hfill \Box

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Recall that $\hat{u}_t^\epsilon$ is uniformly bounded in terms of $||u_0||_{C^2(M)}$, $||u_1||_{C^2(M)}$. Combine with the remark after Proposition 3.19, there exist constants $0 < C_1 < C_2$ such that for sufficiently small $\epsilon > 0$

$$C_1 \leq ||\hat{u}_t^\epsilon||_{X,u_t^\epsilon} \leq C_2.$$ 

Take a cluster point $N$ of $\{||\hat{u}_t^\epsilon||_{X,u_t^\epsilon}\}_{\epsilon > 0}$, after taking a subsequence, we can assume that $||\hat{u}_t^\epsilon||_{X,u_t^\epsilon} \to N$ as $\epsilon \to 0$. Then $\frac{\hat{u}_t^\epsilon}{||\hat{u}_t^\epsilon||_{X,u_t^\epsilon}}$ converges to $\frac{u_t}{N}$ uniformly. Moreover, we have $\omega^\alpha_{u_t^\epsilon} \wedge \eta$ converges to $\omega^\alpha_{u_t} \wedge \eta$ weakly.

Recall $||f||_{X,\mu} = \alpha > 0$ if and only if $\int_A X(\frac{f}{\alpha})d\mu = \chi(1)[30, \text{Section } 1]$. We have

$$\chi(1) = \int_M X\left(\frac{\hat{u}_t^\epsilon}{||\hat{u}_t^\epsilon||_{X,u_t^\epsilon}}\right) \omega^\alpha_{u_t^\epsilon} \wedge \eta \to \int_M X\left(\frac{u_t}{N}\right) \omega^\alpha_{u_t} \wedge \eta$$

and $N = ||\hat{u}_t||_{X,u_t}$. Hence $||\hat{u}_t||_{X,u_t}$ is the only possible cluster point of $\{||\hat{u}_t^\epsilon||_{X,u_t^\epsilon}\}_{\epsilon > 0}$. It means that

$$||\hat{u}_t^\epsilon||_{X,u_t^\epsilon} \to ||\hat{u}_t||_{X,u_t}$$

as $\epsilon \to 0$. Then by the dominated convergence theorem we have

$$\lim_{\epsilon \to 0} l_X(u_t^\epsilon) = l_X(u_t) \quad (3.58)$$

and $d_X(u_0, u_1) \leq l_X(u_t)$.

Then Eq. (3.57) follows if we can prove

$$l_X(\phi_t) \geq l_X(u_t) \quad (3.59)$$

for all smooth curves $\phi_t$ in $\mathcal{H}$ connecting $u_0, u_1$.

First we consider the case $\chi \in \mathcal{W}_p^+ \cap C^\infty(\mathbb{R})$. We can assume that $u_1 \notin \phi([0, 1])$ and take $h \in [0, 1)$. Applying Proposition 3.20 to the case $\phi = u_1$ and $\psi_s = \phi|[0, h]$, letting $\epsilon \to 0$, we can obtain

$$l_X(u_t) \leq l_X(\phi_t|[0, h]) + l_X(w_t^h),$$

where $u_t$ is the $C_{\bar{B}}^{1,1}$ geodesic connecting $u_1, u_0$, and $w_t^h$ is the $C_{\bar{B}}^{1,1}$ geodesic connecting $u_1, \phi_t$. By Lemma 3.9, $l_X(w_t^h) \to 0$ as $h \to 1$. Hence $l_X(\phi_t) \geq l_X(u_t)$.

For the general weight $\chi \in \mathcal{W}_p^+$, we need to do approximation as in [28, Proposition 2.4]. There exists sequence $\chi_k \in \mathcal{W}_{pk}^+ \cap C^\infty(\mathbb{R})$ such that $\chi_k$ converges to $\chi$ uniformly on compact subsets. Then we have

$$\int_0^1 ||\phi_t||_{\chi_k,\phi_t}dt = l_{\chi_k}(\phi_t) = l_{\chi_k}(u_t) = \int_0^1 ||\hat{u}_t||_{\chi_k,u_t}dt.$$
and \( ||\dot{\phi}_t||_{\chi,\phi_t} \rightarrow ||\dot{\phi}_t||_{\chi,\phi_t}, ||\dot{u}_t||_{\chi,u_t} \rightarrow ||\dot{u}_t||_{\chi,u_t} \) [30, Section 1]. Moreover, \( \dot{u}_t, \dot{\phi}_t \) are uniformly bounded. By the dominated convergence theorem, \( l_{\chi}(\phi_t) \geq l_{\chi}(u_t) \). This completes the proof of 3.57.

Recall \( l_{\chi}(u_t) = \int_0^1 ||\dot{u}_t||_{\chi,u_t}dt \) and by Lemma 3.14, we have

\[
d_{\chi}(u_0, u_1) = ||\dot{u}_0||_{\chi,u_0} = ||\dot{u}_1||_{\chi,u_1}.
\]

Suppose \( u_0 \neq u_1 \in \mathcal{H} \), take \( \epsilon \rightarrow 0 \) in the estimate Lemma 3.12 we obtain \( \dot{u}_0 \neq 0 \) and \( d_{\chi}(u_0, u_1) = ||\dot{u}_0||_{\chi,u_0} > 0 \). This implies that \( (\mathcal{H}, d_{\chi}) \) is a metric space. \( \square \)

**Lemma 3.14** Let \( u_t \) be the weak \( C^1_{\beta,1} \) geodesic connecting \( u_0, u_1 \). Then for any \( \chi \in \mathcal{W}_p^+ \) and \( t_0, t_1 \in [0, 1] \), the following holds

\[
d_{\chi}(u_0, u_1) = ||\dot{u}_0||_{\chi,u_0} = ||\dot{u}_1||_{\chi,u_1}.
\]

**Proof** It had been shown that for \( \epsilon \)-geodesics \( u^\epsilon_t \) joining \( u_0, u_1 \), we have

\[
||\dot{u}^\epsilon_{t_0}||_{\chi,u^\epsilon_{t_0}} \rightarrow ||\dot{u}_{t_0}||_{\chi,u_{t_0}}, ||\dot{u}^\epsilon_{t_1}||_{\chi,u^\epsilon_{t_1}} \rightarrow ||\dot{u}_{t_1}||_{\chi,u_{t_1}}
\]

as \( \epsilon \rightarrow 0 \). Proposition 3.19 implies that

\[
||\dot{u}_{t_0}||_{\chi,u_{t_0}} - ||\dot{u}_{t_1}||_{\chi,u_{t_1}} \leq |t_0 - t_1|\epsilon R_1.
\]

Then taking \( \epsilon \rightarrow 0 \) we have \( ||\dot{u}_{t_0}||_{\chi,u_{t_0}} = ||\dot{u}_{t_1}||_{\chi,u_{t_1}}. \) \( \square \)

Finally, we have the following triangle inequality,

**Lemma 3.15** For \( u, v, w \in \mathcal{H}, \chi \in \mathcal{W}_p^+, p \geq 1, \)

\[
d_{\chi}(u, w) \leq d_{\chi}(u, v) + d_{\chi}(v, w).
\]

### 4 The Metric Space \( (E_p(M, \xi, \omega^T), d_p) \)

In this section, we prove Theorem 2. We shall follow the Kähler setting closely as in [28, Section 4], but we shall only consider \( d_p \) distance. Given \( u_0, u_1 \in E_p(M, \xi, \omega^T), p \geq 1 \), by Lemma 3.1 there exists decreasing sequences \( \{u_0^k\}_{k \in \mathbb{N}}, \{u_1^k\}_{k \in \mathbb{N}} \subset \mathcal{H} \) such that \( u_0^k \searrow u_0 \) and \( u_1^k \searrow u_1 \). We shall prove that the following formula for distance \( d_p \) is well defined,

\[
d_p(u_0, u_1) = \lim_{k \to \infty} d_p(u_0^k, u_1^k)
\]

and the definition in (4.1) coincides with (3.51) (we only consider \( \chi(l) = |l|^p / p \)). We will prove that

**Theorem 4.1** \( (E_p(M, \xi, \omega^T), d_p) \) is a complete geodesic metric space extending \( (\mathcal{H}, d_p) \).
We start with the notion of generalized solution of complex Monge–Ampere in the sense of Bedford–Taylor in Sasaki setting, which was considered by van Coevering in [58, Section 2.4]. Let $S = [0, 1] \times S^1$ be the cylinder and $N = M \times S$. Then $N$ is a manifold of dimension $2n + 3$ with boundary and $N$ has a transverse holomorphic structure, simply the product structure of transverse holomorphic structure on $M$, and holomorphic structure on $S$. A path $\phi : [0, 1] \to C^\infty_M$ corresponds to an $S^1$-invariant function $\Phi_w$ on $N$. If $\phi_\epsilon$ is a smooth path in $\mathcal{H}$ then a direct computation gives

$$
(\pi^* \omega^T + \sqrt{-1} \partial_B \bar{\partial}_B \Phi)^{n+1} = c_m \left( |\nabla \phi_\epsilon|^2_\omega \right) (\omega^T_{\phi_\epsilon})^n \wedge dw \wedge d\bar{w}.
$$

(4.2)

Note that this choice of complexification [see van Coevering (4.2)] is different with the choice of Guan–Zhang (3.33). It seems that (4.2) would be more natural to discuss weak solutions. By (4.2), a smooth geodesic then corresponds to a solution of homogeneous complex Monge–Ampere for basic function $\Phi : N \to \mathbb{R}$,

$$
(\pi^* \omega^T + \sqrt{-1} \partial_B \bar{\partial}_B \Phi)^{n+1} \wedge \eta = 0.
$$

We define a weak geodesic between $u_0, u_1 \in \text{PSH}(M, \xi, \omega^T) \cap L^\infty$ as follows, for $\Phi(\cdot, w) = \Phi(\cdot, t) \in \text{PSH}(N^\circ, \xi, \pi^* \omega^T) \cap L^\infty, (t = \text{Re}(w))$, it satisfies

$$
\begin{cases}
(\pi^* \omega^T + \sqrt{-1} \partial_B \bar{\partial}_B \Phi)^{n+1} \wedge \eta = 0 \\
\lim_{t \to 0} \Phi(\cdot, t) = u_0, \lim_{t \to 1} \Phi(\cdot, t) = u_1.
\end{cases}
$$

(4.3)

We have the following strong maximum principle, see [58, Theorem 2.5.3], [11, Theorem 21], and [30, Theorem 3.2].

**Lemma 4.1** Let $u, v \in \text{PSH}(N^\circ, \xi, \pi^* \omega^T) \cap L^\infty(N)$. Suppose that

$$
(\pi^* \omega^T + \sqrt{-1} \partial_B \bar{\partial}_B u)^{n+1} \wedge \eta \leq (\pi^* \omega^T + \sqrt{-1} \partial_B \bar{\partial}_B v)^{n+1} \wedge \eta
$$

and $\lim_{x \to \partial N} (u - v)(x) \geq 0$, then $u \geq v$ on $N$.

**Proof** Our proof is similar to Kähler case, see [30, Theorem 3.2]. Fix $\epsilon > 0$ and $v_\epsilon := \max(u, v - \epsilon) \in \text{PSH}(N^\circ, \xi, \omega^T) \cap L^\infty$. Then $v_\epsilon = u$ near the boundary $\partial N = M \times S^1 \times \{t = 0\} \cup M \times S^1 \times \{t = 1\}$. Hence it is enough to show that $u = v_\epsilon$ on $N$.

We write $N = M \times S$ and $\omega_u = \pi^* \omega^T + \ddc_B u$, etc. Note that on each foliation chart $W_\alpha = (-\delta, \delta) \times V_\alpha$ of $M$, we have the following inequality on $V_\alpha \times S$ for complex Monge–Ampere measure [12, Theorem 2.2.10]

$$
\omega^{n+1}_{v_\epsilon} \geq \chi_{[u \geq v - \epsilon]} \cap V_\alpha \omega^{n+1}_u + \chi_{[u < v - \epsilon]} \cap V_\alpha \omega^{n+1}_v \geq \omega^{n+1}_u.
$$

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It follows that on $N$, we have
\[
\omega_{v_\epsilon}^{n+1} \wedge \eta \geq \omega_u^{n+1} \wedge \eta.
\]
Then we have the following
\[
0 \leq \int_N (v_\epsilon - u) (\omega_{v_\epsilon}^{n+1} - \omega_u^{n+1}) \wedge \eta. \tag{4.4}
\]
Using integration by parts, we obtain that
\[
\int_N d(u - v_\epsilon) \wedge d^c_B (u - v_\epsilon) \wedge \omega_u^k \wedge \omega_{v_\epsilon}^{n-k} \wedge \eta = 0, \quad 0 \leq k \leq n.
\]
By an induction argument as in [30, Theorem 3.2], we can prove that
\[
\int_N d(u - v_\epsilon) \wedge d^c_B (u - v_\epsilon) \wedge \omega_u^k \wedge (\pi^* \omega^T)^{n-k} \wedge \eta = 0, \quad 0 \leq k \leq n.
\]
For $k = n$, this shows that
\[
\int_{M \times S} d(u - v_\epsilon) \wedge d^c_B (u - v_\epsilon) \wedge (\pi^* \omega^T)^n \wedge \eta = 0.
\]
Writing $\rho = u - v_\epsilon$, this reads
\[
\int_{M \times S} |\partial_t \rho|^2 dt \wedge ds \wedge (\pi^* \omega^T)^n \wedge \eta = 0.
\]
Hence $\partial_t \rho = 0$. Since $\rho = 0$ near the boundary $\partial N = M \times S^1 \times \{t = 0\} \cup M \times S^1 \times \{t = 1\}$, this shows that $\rho = 0$. It completes the proof. \qed

**Remark 4.1** One can certainly formulate a general version of comparison principle as in [30, Theorem 3.2]. But one would need certainly a (transverse) Kähler form. Note that $\pi^* \omega^T$ is not transverse Kähler (it is zero along $S$-direction). Here we use the product structure of $N$.

With this maximum principle for bounded TPSH, we have the following,

**Lemma 4.2** Given $u_0, u_1 \in \mathcal{H}$, let $u_t : [0, 1] \to \mathcal{H}$ be the unique $C^{1,1}_B$ geodesic connecting $u_0, u_1$. Then we have the following,
\[
\|\dot{u}_t\|_{C^0} \leq \|u_0 - u_1\|_{C^0}, \quad \forall t \in [0, 1].
\]

**Proof** Note that this gives a much sharper estimate than Lemma 3.10. The proof follows the Kähler setting [30, Lemma 3.5]. Denote $C = \max |u_0 - u_1|$. By the convexity of $u$ in $t$-variable, we know that
\[
\dot{u}_0 \leq \dot{u}_t \leq \dot{u}_1.
\]
Note that \( v_t = u_0 - C t \) is a smooth geodesic connecting \( u_0 \) and \( u_0 - C \). Hence its complexification gives a solution to (4.3). By Lemma 4.1, we know that \( v_t \leq u_t \), for \( t \in [0, 1] \), since \( u_0 - C \leq u_1 \). It follows that \( -C \leq \dot{u}_0 \). Similarly, one can prove that \( \dot{u}_1 \leq C \), by considering \( \tilde{v}_t = u_0 + C t \).

**Remark 4.2** The upper envelop construction was used to construct bounded weak geodesic segment in Kähler setting by Berndtsson [9], where he proved that Lemma 4.2 holds for \( u_0, u_1 \in \text{PSH}(M, \omega) \) (when \( (M, \omega) \) is Kähler). A direct adaption to Sasaki setting using Lemma 4.1 would lead to an extension of Berndtsson’s result to Sasaki setting.

In general, \( \Phi(\cdot, w) \in \text{PSH}(N^\circ, \xi, \pi^* \omega^T) \) will be called weak subgeodesic, if \( \Phi(\cdot, t) = \Phi(\cdot, \text{Re}(w)) \), \( t = \text{Re}(w) \). For \( u_0, u_1 \in \text{PSH}(M, \xi, \omega^T) \), we define

\[
\Phi(\cdot, t) \in \text{PSH}(N^\circ, \xi, \pi^* \omega^T), \quad \lim_{t \to 0,1} \Phi(\cdot, t) \leq u_{0,1}.
\]

(4.5)

We have the following:

**Proposition 4.1** \( u \in \text{PSH}(N^\circ, \xi, \pi^* \omega^T) \). Denote \( u_t = u(\cdot, t) \). We refer \( t \to u_t \) to the weak geodesic segment connecting \( u_0, u_1 \).

**Proof** Note that \( u^* \) is basic, and \( u^* \in \text{PSH}(N^\circ, \xi, \pi^* \omega^T) \). Since \( \Phi \) is convex in \( t \) direction, it follows that \( \Phi(\cdot, t) \leq (1-t)u_0 + tu_1 \). Hence \( u_t \leq (1-t)u_0 + tu_1 \). It follows that

\[
u^* \leq (1-t)u_0 + tu_1.
\]

In other words, \( u^* \leq u \) by definition. It follows that \( u^* = u \). □

**Proposition 4.2** If \( u_0, u_1 \in \text{PSH}(M, \xi, \omega^T) \cap L^\infty \), \( u \) is defined by (4.5) and \( u_t = u(\cdot, t) \) is the weak geodesic. Let \( C \) be a constant \( \geq ||u_1 - u_0||_{L^\infty} \).

1. We have

\[
\max(u_0 - Ct, u_1 - C(1-t)) \leq u_t \leq (1-t)u_0 + tu_1.
\]

(4.6)

2. \( u_t \in \text{PSH}(M, \xi, \omega^T) \cap L^\infty \) and \( u \) is the unique solution of (4.3).

3. \( u_t \) is uniformly Lipschitz continuous with respect to \( t \):

\[
|u_t - u_s| \leq C|s - t|.
\]

for \( s, t \in [0, 1] \).

4. The derivatives \( \dot{u}_0, \dot{u}_1 \) exists and

\[
|\dot{u}_0| \leq C, \quad |\dot{u}_1| \leq C.
\]
Proof. (1) It is obvious that $u_0 - Ct, u_1 - C(1 - t)$ are weak subgeodesics. It follows from the definition of $u_t$ (4.5) that

$$\max (u_0 - Ct, u_1 - C(1 - t)) \leq u_t.$$ 

The other half of the inequality comes from the convexity of $u_t$ with respect to $t$. 

(2) By the inequality (4.6) we have $u_t \in \text{PSH}(M, \xi, \omega^T) \cap L^\infty$ and \( \lim_{t \to 0, 1} u_t = u_{0, 1} \).

Then $u \in \text{PSH}(N^\circ, \xi, \pi^*\omega^T) \cap L^\infty$. Using the classical Perron-Bremmerman argument, we have $(\pi^*\omega^T + \sqrt{-1} \partial_B \overline{\partial} u)^{n+1} \wedge \eta = 0$. Hence $u$ is a solution of (4.3). The uniqueness of the solution of (4.3) follows from the strong maximum principle.

(3) If one of $s, t$ equals to $0$ or $1$, the required inequality is a direct consequence of (4.6). If $0 < s < t < 1$, by the convexity of $u_t$ with respect to $t$ we have

$$\frac{t - s}{s} (u_s - u_0) \leq u_t - u_s \leq \frac{t - s}{1 - s} (u_1 - u_s)$$

and the inequality follows from the case $t = 0, 1$ we have proved.

(4) By the convexity of $u_t$, we have

$$\frac{u_{t_1} - u_0}{t_1} \leq \frac{u_{t_2} - u_0}{t_2}$$

for $0 < t_1 < t_2$. These quantities are uniformly bounded by $C$. Hence $\dot{u}_0$ exists and $|\dot{u}_0| \leq C$. The case of $u_1$ follows by a similar argument. 

\[\square\]

Remark 4.3 If $u_0, u_1 \in \mathcal{H}$, the weak geodesic $u_t$ coincides with the $C^{1, \bar{1}}_B$ geodesic.

The weak geodesic $u_t$ connecting $u_0, u_1 \in \text{PSH}(M, \xi, \omega^T)$ has the advantage of admitting some homogeneous structures and offering a new interpretation of rooftop envelope. Moreover, it is closed for class $\mathcal{E}_p(M, \xi, \omega^T)$: the weak geodesic $u_t$ connecting $u_0, u_1 \in \mathcal{E}_p(M, \xi, \omega^T)$ stays in the same class. It is called the finite-energy geodesic in $\mathcal{E}_p(M, \xi, \omega^T)$.

Proposition 4.3 Let $u^k_0, u^k_1 \in \text{PSH}(M, \xi, \omega^T)$ be sequences decreasing to $u_0, u_1 \in \text{PSH}(M, \xi, \omega^T)$, respectively. Suppose that $u^r_t, u_t \in \text{PSH}(M, \xi, \omega^T)$ be the weak geodesic connecting $u^k_0, u^k_1$ and $u_0, u_1$, respectively. Then

(1) $u^r_t$ decreases to $u_t$ for $t \in [0, 1]$;

(2) For any $t_1, t_2 \in [0, 1], [0, 1] \supset t \to u_{(1-t)t_1 + t t_2} \in \text{PSH}(M, \xi, \omega^T)$ is the weak geodesic connecting $u_{t_1}$ and $u_{t_2}$.

Proof. (1) By the definition of $u^r_t$ (4.5) it is obvious that $\{u^r_t\}_{k \in \mathbb{N}}$ is decreasing and $v_t = \lim_{k \to \infty} u^r_t \in \text{PSH}(M, \xi, \omega^T)$. Again by the definition of $u^r_t, u_t$ (4.5) we have $u^r_t \geq u_t$, hence $v_t \geq u_t$.

Recall that $u^r_t$ is convex with respect to $t$. Then $u^r_t \leq (1 - t)u^k_0 + tu^k_1$ and $v_t \leq (1 - t)u_0 + tu_1$. It follows from the definition of $u_t$ (4.5) that $v_t \leq u_t$.

Consequently the sequence $\{u^r_t\}_{k \in \mathbb{N}}$ decreases to $u_t$ for $t \in [0, 1]$.
(2) Recall that \( u_0, u_1 \) are the decreasing limits of their canonical cutoffs, it follows from part (1) that we only have to prove the proposition for \( u_0, u_1 \) in \( L^\infty \). \( v_t = u_{(1-t)t_1 + t_2} \) is a path connecting \( u_{t_1}, u_{t_2} \). By Proposition 4.2 we have \( \lim_{t \to 0,1} v_t = u_{t_1, t_2} \) and \( \Phi(\cdot, t) = v_t \) is a solution of Eq. \((4.3)\) with initial data \( u_{t_1}, u_{t_2} \). Then it follows from Proposition 4.2(2) that \( v_t = u_{(1-t)t_1 + t_2} \) is the weak geodesic connecting \( u_{t_1}, u_{t_2} \).

\[ \square \]

**Lemma 4.3** Suppose \( u_0, u_1 \in PSH(M, \xi, \omega^T) \) and \( t \to u_t \) is the weak geodesic segment connecting \( u_0, u_1 \).

1. For any \( \tau \in \mathbb{R} \) we have
   \[
   \inf_{t \in (0,1)} (u_t - t\tau) = P(u_0, u_1 - \tau), \tau \in \mathbb{R}. \tag{4.7}
   \]

2. If \( u_0, u_1 \in PSH(M, \xi, \omega^T) \cap L^\infty \), then
   \[
   \{ \hat{u}_0 \geq \tau \} = \{ P(u_0, u_1 - \tau) = u_0 \}. \tag{4.8}
   \]

3. If \( u_0, u_1 \in \mathcal{E}_p(M, \xi, \omega^T) \), then \( u_t \in \mathcal{E}_p(M, \xi, \omega^T) \) for \( t \in [0, 1] \).

**Proof** (1) First note that \( t \to v_t = u_t - t\tau \) is the weak geodesic connecting \( u_0, u_1 - \tau \), hence the proof can be reduced to the particular case \( \tau = 0 \).

By definition \( P(u_0, u_1) \leq u_0, u_1 \). As a result, the constant weak subgeodesic \( t \to h_t := P(u_0, u_1) \) is a candidate for definition of \( u_t \), hence \( h_t \leq u_t, t \in [0, 1] \).

It follows that \( P(u_0, u_1) \leq \inf_{t \in [0,1]} u_t \).

For the other direction, we use Kiselman minimum principle [33, Chapter I, Theorem 7.5], which asserts that \( w := \inf_{t \in [0,1]} u_t \in PSH(M, \xi, \omega^T) \) (note that \( u_t \) is a genuine plurisubharmonic function on foliation charts, for each \( t \) and \( u_t \) is convex in \( t \) variable; hence Kiselman minimum principle applies, as in Kähler setting).

Note that \( u_t \leq (1 - t)u_0 + tu_1 \), it follows that \( w \) is a candidate for \( P(u_0, u_1) \) and hence \( w \leq P(u_0, u_1) \). This completes the proof.

(2) For \( x \in M \) we have \( P(u_0, u_1 - \tau)(x) = u_0(x) \) if and only if \( \inf_{t \in [0,1]} (u_t(x) - t\tau) = u_0(x) \). By the convexity of \( u_t \) in the \( t \) variable, it is equivalent to \( \hat{u}_0(x) \geq \tau \).

(3) By Lemma 3.4, we have \( P(u_0, u_1) \in \mathcal{E}_p(M, \xi, \omega^T) \). Notice that \( P(u_0, u_1) \leq u_0, u_1 \). It follows from (1) that \( P(u_0, u_1) \leq u_t \). By Proposition 3.11 we have \( u_t \in \mathcal{E}_p(M, \xi, \omega^T) \) for \( t \in [0, 1] \).

\[ \square \]

Now we prove Theorem 4.1, through a series of propositions and lemmas, following [28, Section 4] (and in particular [30, Section 3]).

First of all, the \( d_p \) distance between comparable smooth potentials behaves well.

**Lemma 4.4** Suppose \( u, v \in \mathcal{H} \) with \( u \leq v \). We have

\[
\max \left\{ \frac{1}{2n+p} \int_M |u - v|^p \omega_u^n \wedge \eta, \int_M |u - v|^p \omega_v^n \wedge \eta \right\} \leq d_p(u, v)^p \leq \int_M |u - v|^p \omega_v^n \wedge \eta. \tag{4.9}
\]
**Proof** Let \( w_t : [0, 1] \to \mathcal{H} \) be the \( C^{1,1}_B \) geodesic connecting \( u \) and \( v \). By Theorem 3.4, we have
\[
d_p(u, v)^p = \int_M |\dot{w}_0|^p \omega_u^n \wedge \eta = \int_M |\dot{w}_1|^p \omega_v^n \wedge \eta. \tag{4.10}
\]
By Lemma 4.1, we have \( u \leq w_t \) given \( u \leq v \). Since \( w_t \) is convex in \( t \), it follows that
\[
0 \leq \dot{w}_0 \leq v - u \leq \dot{w}_1. \tag{4.11}
\]
It then follows that, by (4.10) and (4.11),
\[
\int_M |u - v|^p \omega_u^n \wedge \eta \leq d_p(u, v)^p \leq \int_M |v - u|^p \omega_v^n \wedge \eta. \tag{4.12}
\]
Next we use \( \omega_u^n \wedge \eta \leq 2^n \omega_{u+v}^n \wedge \eta \) to obtain that
\[
2^{-n} \int_M |u - v|^p \omega_u^n \wedge \eta \leq \int_M |v - u|^p \omega_u^n \wedge \eta.
\]
We write the right-hand side above as follows and apply (4.12) for \( u \leq (u + v)/2 \) to obtain
\[
2^{-p} \int_M |u - v|^p \omega_{u+v}^n \wedge \eta = \int_M \left| u - \frac{u + v}{2} \right|^p \omega_{u+v}^n \wedge \eta \leq d_p\left(u, \frac{u + v}{2}\right)^p.
\]
The lemma below implies that \( d_p(u, (u + v)/2) \leq d_p(u, v) \), completing the proof. \( \square \)

Lemma 4.5 Suppose \( u, v, w \in \mathcal{H} \) and \( u \leq v \leq w \). Then we have
\[
d_p(u, v) \leq d_p(u, w), \ d_p(v, w) \leq d_p(u, w).
\]

**Proof** Let \( \alpha_t, \beta_t \) be the \( C^{1,1}_B \) geodesic segments connecting \( u, v \) and \( u, w \), respectively. Since \( u \leq v \leq w \), by Lemma 4.1 we have \( u \leq \alpha_t \leq v \) and \( u \leq \beta_t \leq w \); moreover, \( \alpha_t \leq \beta_t \). Since \( \alpha_0 = \beta_0 \), this gives that \( 0 \leq \dot{\alpha}_0 \leq \dot{\beta}_0 \). Theorem 3.4 then implies that \( d_p(u, v) \leq d_p(u, w) \). Similarly we can prove \( d_p(v, w) \leq d_p(u, w) \). \( \square \)

Next we prove that the distance formula (4.1) is well defined and extends the original definition (3.51).

Lemma 4.6 Given \( u_0, u_1 \in \mathcal{E}_p(M, \xi, \omega^T) \), the limit (4.1) is finite and independent of the approximating sequences \( u_k^0, u_k^1 \in \mathcal{H} \).

**Proof** First we show that given \( u \in \mathcal{E}_p(M, \xi, \omega^T) \) and a sequence \( \{u_k\}_{k \in \mathbb{N}} \subset \mathcal{H} \) decreasing to \( u \). We have \( d_p(u_l, u_k) \to 0 \) as \( l, k \to \infty \). We can assume that \( l \leq k \) and hence \( u_k \leq u_l \). Then Lemma 4.4 implies that
\[
d_p(u_l, u_k)^p \leq \int_M |u_l - u_k|^p \omega_u^n \wedge \eta.
\]
Clearly, we have $u - u_l \leq u_k - u_l \leq 0$ and $u - u_l, u_k - u_l \in \mathcal{E}_p(M, \xi, \omega_{u_l})$. Hence applying Proposition 3.11 for the class $\mathcal{E}_p(M, \xi, \omega_{u_l})$, we obtain that

$$d_p(u_l, u_k)^p \leq \int_M |u_l - u_k|^p \omega_{u_k}^n \wedge \eta \leq (p + 1)^n \int_M |u - u_l|^p \omega_{u_l}^n \wedge \eta. \tag{4.13}$$

As $u_l$ decreases to $u \in \mathcal{E}_p(M, \xi, \omega^T)$, the monotone convergence theorem implies that the right-hand side above converges to zero as $l \to \infty$, hence $d_p(u_l, u_k) \to 0$ as $l, k \to \infty$.

Now by Lemma 3.15, we know that

$$|d_p(u_{l0}^l, u_1^l) - d_p(u_{k0}^l, u_1^k)| \leq d_p(u_{l0}^l, u_{k0}^l) + d_p(u_{10}^l, u_{11}^k) \to 0, l, k \to \infty.$$  

Hence this proved that the limit (4.1) is convergent and finite.

Next we show that the limit is independent of the choice of approximating sequences. Let $v_{l0}^l, v_{l1}^l$ be other approximating sequences. Certainly we can assume the sequences are strictly decreasing, by adding small constants if necessary. Fix $k$ and consider the sequence $\{\max\{u_{0k}^{k+1}, v_{0l}^l\} \}_{j \in \mathbb{N}}$ decreases pointwise to $u_{0k}^{k+1}$. By Dini’s lemma, the convergence is uniform (for fixed $k$) and hence we can choose $j_k$ sufficiently large such that $v_{0j}^l < u_{0k}^l, j \geq j_k$. Repeating the argument we can assume $v_{1j}^l < u_{1k}^l$, for $j \geq j_k$. By triangle inequality again, we have

$$|d_p(v_{l0}^l, v_{l1}^l) - d_p(u_{k0}^l, u_1^l)| \leq d_p(v_{l0}^l, u_{k0}^l) + d_p(u_{10}^l, u_{11}^k), j \geq j_k.$$  

By (4.13) we know that if $k$ is sufficiently large, $d_p(v_{l0}^l, u_{k0}^l) + d_p(u_{10}^l, u_{11}^k)$ is sufficiently small. Hence the distance $d_p(u_0, u_1)$ is independent of the choice of approximating sequence.

For $u_0, u_1 \in \mathcal{H}$, we can approximate $u_0, u_1$ by constant sequences. The previous lemma indicates that the distance (4.1) on $\mathcal{E}_p(M, \xi, \omega^T)$ is an extension of the distance (3.51) on $\mathcal{H}$ for weight $\chi(l) = |l|^p$.

For $u_0, u_1 \in \mathcal{E}_p(M, \xi, \omega^T)$, we choose a decreasing sequence $\{u_{0k}^k\}_{k \in \mathbb{N}}, \{u_{1k}^k\}_{k \in \mathbb{N}} \subset \mathcal{H}$ such that $u_{0k}^k \searrow u_0, u_{1k}^k \searrow u_1$. We connect $u_{0k}^k, u_{1k}^k$ by the unique $C^{1,1}_B$ geodesic segment $u_k^l$. By Lemma 4.1, it follows that $u_k^l$ decreases in $k$. Hence the limit $\lim_{k \to \infty} u_k^l$ exists. Using Dini’s lemma as above, one can show that the limit does not depend on the choice of approximating sequence. By Proposition 4.3 and the remark before it, the limit indeed coincides with the weak geodesic $u_t$ connecting $u_0, u_1$:

$$u_t = \lim_{k \to \infty} u_k^l.$$  

**Lemma 4.7** For $u_0, u_1 \in \mathcal{E}_p(M, \xi, \omega^T)$, the weak geodesic $u_t$ connecting them is a $d_p$-geodesic in the sense that

$$d_p(u_{t1}, u_{t2}) = |t_1 - t_2|d_p(u_0, u_1).$$
for \( t_1, t_2 \in [0, 1] \).

**Proof** Let \( \{u^k_0\}_{k \in \mathbb{N}}, \{u^1_1\}_{k \in \mathbb{N}} \subseteq \mathcal{H} \) be sequences strictly decreasing to \( u_0, u_1 \), respectively, and \( u^k_l \in \mathcal{H}_\triangle \) the unique \( C^1_B \) geodesic connecting \( u^k_0, u^k_1 \). By Theorem 3.4, we have

\[
d_p(u_0, u_1)^2 = \lim_{k \to \infty} d_p(u^k_0, u^k_1)^2 = \lim_{k \to \infty} \int_M |\dot{u}^k_0|^p \omega^0_{\omega^k_0} \wedge \eta. \tag{4.14}
\]

For \( l \in (0, 1) \), Lemma 4.1 implies that \( u^k_l \) strictly decreases to \( u_l \). Then one can choose a sequence \( \{u^k_l\}_{k \in \mathbb{N}} \subseteq \mathcal{H} \) such that

1. \( u^k_{l+1} \leq u^k_l \);
2. For the \( C^1_B \) geodesic \( v^k_l \) connecting \( u^k_0 \) and \( u^k_l \) with \( v^k_0 = u^k_0, v^k_1 = u^k_l \), we have

\[
\left| \int_M |v^k_0|^p \omega^0_{\omega^k_0} \wedge \eta - I^p \int_M |\dot{u}^k_0|^p \omega^0_{\omega^k_0} \wedge \eta \right| \leq \frac{1}{k}.
\]

In fact there exists a sequence \( \{\varphi^j\}_{j \in \mathbb{N}} \subseteq \mathcal{H} \) decreasing to \( u^k_l \). By Dini’s lemma, \( \varphi^j \) converges to \( u^k_l \) uniformly. It follows from Lemma 4.8 and Proposition 4.3 that for \( j \) big enough, \( u^k_l = \varphi^j \) will satisfy our requirements. By Theorems 3.4 and (4.14),

\[
d_p(u_0, u_l) = l d_p(u_0, u_1) \text{ for } l \in [0, 1].
\]

Without loss of generality, we assume that \( 0 \leq t_1 \leq t_2 \leq 1 \). By Proposition 4.3, \( h_t = u(t(1-t)) \) is the weak geodesic connecting \( u_{t_2} \) and \( u_0 \). By following from the results above, we have

\[
d_p(u_{t_2}, u_{t_1}) = \left(1 - \frac{t_1}{t_2}\right) d_p(u_{t_2}, u_0) = (t_2 - t_1) d_p(u_1, u_0).
\]

This completes the proof. \( \square \)

**Lemma 4.8** Suppose \( u_0, u_1 \in PSH(M, \xi, \omega^T) \cap L^\infty \). Let \( \{u^k_l\}_{k \in \mathbb{N}} \subseteq PSH(M, \xi, \omega^T) \cap L^\infty \) be a sequence decreasing to \( u_1 \) and \( u_l, u^k_l \in PSH(M, \xi, \omega^T) \cap L^\infty \) the weak geodesic connecting \( u_0, u_1 \) and \( u_0, u^k_1 \), respectively. Then

\[
\lim_{k \to \infty} \int_M |\dot{u}^k_0|^p \omega^0_{\omega^k_0} \wedge \eta = \int_M |\dot{u}_0|^p \omega^0_{\omega^0_0} \wedge \eta.
\]

**Proof** Denote by \( C = \max(\|u_1\| - u_0\|_{L^\infty}, \|u_1 - u_0\|_{L^\infty}) \). It follows Proposition 4.2 that \( \|\dot{u}_0\|_{L^\infty} \leq C, \|\dot{u}^k_0\|_{L^\infty} \leq C \). By Proposition 4.3 the sequence \( \{u^k_l\}_{k \in \mathbb{N}} \) decreases to \( u_l \) hence the sequence \( \{\dot{u}^k_0\}_{k \in \mathbb{N}} \) is decreasing with \( \dot{u}^k_0 \geq \dot{u}_0 \).

Moreover, we have \( \dot{u}_0 \) decreases to \( \dot{u}_0 \). If this is not true, we can find \( x_0 \in M, a \in \mathbb{R} \) such that \( \dot{u}_0^k > a > \dot{u}_0 \). Then there exists \( 0 < t_0 < 1 \) such that \( u^k_t(x_0) > u_t + a t > u_t(x_0) \) for \( t \in [0, t_0] \). It contradicts with the fact that \( u^k_t \) decreases to \( u_t \).
Then the lemma follows from Lebesgue’s dominated convergence theorem. □

Pythagorean formula about $d_p$ distance involves that rooftop envelope plays an essential role in Darvas’s results [28,29] and we have a similar formula in Sasaki setting:

**Theorem 4.2** (Pythagorean formula) Given $u_0, u_1 \in \mathcal{E}_p(M, \xi, \omega^T)$, we have $P(u_0, u_1) \in \mathcal{E}_p(M, \xi, \omega^T)$ and

$$d_p(u_0, u_1)^p = d(u_0, P(u_0, u_1))^p + d_p(u_1, P(u_0, u_1))^p.$$  \hspace{1cm} (4.15)

**Proof** First we prove the formula for $u_0, u_1 \in \mathcal{H}$. It follows from Theorem 3.1 that $P(u_0, u_1) \in \mathcal{H}_\Delta$. Let $u_t$ be the $C^{1,1}$ geodesic connecting $u_0, u_1$. Let $v_t$ be the weak geodesic connecting $P(u_0, u_1), u_1$. It follows from Lemma 4.1 that $P(u_0, u_1) \leq v_t$ for $t \in [0, 1]$. Hence we have $\dot{v}_0 \geq 0$. By Lemmas 4.9, Lemma 4.3, the definition of rooftop envelope, and Lemma 3.3, we have

$$d_p(P(u_0, u_1), u_1)^p = \int_M |\dot{v}_0|^p \omega_{P(u_0, u_1)}^n \wedge \eta$$

$$= \int_{[\dot{v}_0 > 0]} |\dot{v}_0|^p \omega_{P(u_0, u_1)}^n \wedge \eta$$

$$= p \int_0^{\infty} s^{p-1} \omega_{P(u_0, u_1)}^n \wedge \eta(\{\dot{v}_0 \geq s\}) ds$$

$$= p \int_0^{\infty} s^{p-1} \omega_{P(u_0, u_1)}^n \wedge \eta(\{P(P(u_0, u_1), u_1 - s) \})) ds$$

$$= P(u_0, u_1)) ds$$

$$= p \int_0^{\infty} s^{p-1} \omega_{P(u_0, u_1)}^n \wedge \eta(\{P(u_0, u_1 - s) = P(u_0, u_1)\}) ds$$

$$= p \int_0^{\infty} s^{p-1} \omega_{u_0}^n \wedge \eta(\{P(u_0, u_1 - s) = u_0\}) ds$$

$$= p \int_0^{\infty} s^{p-1} \omega_{u_0}^n \wedge \eta(\{P(u_0, u_1 - s) = u_0\}) ds$$

$$= p \int_0^{\infty} s^{p-1} \omega_{u_0}^n \wedge \eta(\{\dot{u}_0 \geq s\}) ds$$

$$= \int_{[\dot{u}_0 > 0]} |\dot{u}_0|^p \omega_{u_0}^n \wedge \eta.$$ 

By a similar argument we also have

$$d_p(u_0, P(u_0, u_1))^p = \int_{[\dot{u}_0 < 0]} |\dot{u}_0|^p \omega_{u_0}^n \wedge \eta.$$ 

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Now using Theorem 3.4 we have

$$d_p(u_0, u_1)^p = \int_M |\dot{u}_0|^p \omega_{u_0}^n \wedge \eta$$

$$= \int_{\{\dot{u}_0 > 0\}} |\dot{u}_0|^p \omega_{u_0}^n \wedge \eta + \int_{\{\dot{u}_0 < 0\}} |\dot{u}_0|^p \omega_{u_0}^n \wedge \eta$$

$$= d_p(u_0, P(u_0, u_1))^p + d_p(P(u_0, u_1), u_1)^p$$

and the Pythagorean formula follows from Lemma 4.11.

For the general case we can choose sequences \{u_0^k\}_{k \in \mathbb{N}}, \{u_1^k\}_{k \in \mathbb{N}} \subset \mathcal{H} decreases to \(u_0, u_1\), respectively. Then the sequence \(P(u_0^k, u_1^k) \in \mathcal{H}_\Delta\) decreases to \(P(u_0, u_1)\) and the Pythagorean formula follows from Lemma 4.11.

\[\square\]

**Lemma 4.9** Let \(u_t\) be the weak geodesic connecting \(u_0, u_1 \in \mathcal{H}_\Lambda\). Then the following holds:

$$d_p(u_0, u_1)^p = \int_M |\dot{u}_0|^p \omega_{u_0}^P \wedge \eta = \int_M |\dot{u}_1|^p \omega_{u_1}^P \wedge \eta.$$ 

**Proof** \(u_t = u_{1-t}\) is the weak geodesic connecting \(u_1, u_0\). By Lemma 4.3, we have

\[
\{P(u_0 + s, u_1) < u_1\} = M - \{P(u_0 + s, u_1) = u_1\} = M - \{\dot{v}_0 \geq -s\} = \{\dot{u}_1 > s\}.
\]

Recall that \(\omega_{u_1}^n \wedge \eta\) has total finite measure \(\text{Vol}(M)\), hence except for a countably many \(s \in \mathbb{R}\) we have \(\omega_{u_1}^n \wedge \eta(\{u_0 = u_1 - s\}) = 0\) and \(\omega_{u_1}^n \wedge \eta(\{\dot{u}_1 \geq s\}) = \omega_{u_1}^n \wedge \eta(\{\dot{u}_1 > s\})\).

For such real number \(s\), it follows from Lemma 3.3 that

\[
\omega_{P(u_0, u_1 - s)}^n \wedge \eta = \chi_{\{P(u_0, u_1 - s) = u_0\}} \omega_{u_0}^n \wedge \eta + \chi_{\{P(u_0, u_1 - s) = u_1 - s\}} \omega_{u_1}^n \wedge \eta
\]

and

\[
\text{Vol}(M) = \omega_{u_0}^n \wedge \eta(\{P(u_0, u_1 - s) = u_0\}) + \omega_{u_1}^n \wedge \eta(\{P(u_0, u_1 - s) = u_1 - s\}).
\]

It follows from Lemma 4.3, the definition of rooftop envelope, that

\[
\int_{\{\dot{u}_0 > 0\}} |\dot{u}_0|^p \omega_{u_0}^n \wedge \eta = p \int_0^\infty s^{p-1} \omega_{u_0}^n \wedge \eta(\{\dot{u}_0 \geq s\}) ds
\]

\[
= p \int_0^\infty s^{p-1} \omega_{u_0}^n \wedge \eta(\{P(u_0, u_1 - s) = u_0\}) ds
\]

\[
= p \int_0^\infty s^{p-1} (\text{Vol}(M) - \omega_{u_1}^n \wedge \eta(\{P(u_0, u_1 - s) = u_1 - s\})) ds
\]

\[
= p \int_0^\infty s^{p-1} \omega_{u_1}^n \wedge \eta(\{P(u_0, u_1 - s) < u_1 - s\}) ds
\]

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\[= p \int_0^\infty s^{p-1} \omega_{u_1}^n \wedge \eta(\{P(u_0 + s, u_1) < u_1\}) ds\]
\[= p \int_0^\infty s^{p-1} \omega_{u_1}^n \wedge \eta(\{\dot{u}_1 > s\}) ds\]
\[= p \int_0^\infty s^{p-1} \omega_{u_1}^n \wedge \eta(\{\dot{u}_1 \geq s\}) ds\]
\[= \int_{\{\dot{u}_1 > 0\}} |\dot{u}_1|^p \omega_{u_1}^n \wedge \eta.\]

A similar argument gives that
\[\int_{\{\dot{u}_0 < 0\}} |\dot{u}_0|^p \omega_{u_0}^n \wedge \eta = \int_{\{\dot{u}_1 < 0\}} |\dot{u}_1|^p \omega_{u_1}^n \wedge \eta.\]

It follows that
\[\int_M |\dot{u}_0|^p \omega_{u_0}^n \wedge \eta = \int_M |\dot{u}_1|^p \omega_{u_1}^n \wedge \eta.\]

Now choose sequence \(\{u_k^0\}_{k \in \mathbb{N}}, \{u_k^1\}_{k \in \mathbb{N}} \subset \mathcal{H}\) decreasing to \(u_0, u_1\), respectively. Let \(u_{k}^{l}\), \(u_{l}\) be the \(C^{1, \bar{1}}_B\) geodesic connecting \(u_0^k, u_1^k\) and \(u_0, u_1\), respectively. Let \(u_k^k\) be the \(C^{1, \bar{1}}_B\) geodesic connecting \(u_0^k, u_1^k\). It follows from Lemmas 4.11, 4.8 and the above results that
\[d_p(u_0^k, u_1^k)^p = \lim_{l \to \infty} d_p(u_k^l, u_1^k)^p = \lim_{l \to \infty} \int_M |\dot{u}_0^k|^p \omega_{u_0^k}^n \wedge \eta = \int_M |\dot{u}_0^k|^p \omega_{u_0^k}^n \wedge \eta\]
\[= \int_M |\dot{u}_1|^p \omega_{u_1}^n \wedge \eta.\]

Then using Lemmas 4.11, 4.8, and Proposition 4.3, we have
\[d_p(u_0, u_1)^p = \lim_{k \to \infty} d_p(u_0^k, u_1^k)^p = \lim_{k \to \infty} \int_M |\dot{u}_0^k|^p \omega_{u_0^k}^n \wedge \eta = \int_M |\dot{u}_1|^p \omega_{u_1}^n \wedge \eta.\]

This completes the proof. 

\[\square\]

**Lemma 4.10** Assume that \(u, v \in \mathcal{E}_p(M, \xi, \omega^T)\) with \(u \leq v\). Then we have
\[\max \left( \frac{1}{2^{n+p}} \int_M |v - u|^p \omega_u^n \wedge \eta, \int_M |u - v|^p \omega_u^n \wedge \eta \right) \leq d_p(u, v)^p \leq \int_M |v - u|^p \omega_u^n \wedge \eta.\]

**Proof** First we can choose \(u_k, w_k \in \mathcal{H}\) strictly decreasing to \(u, v\), respectively. Then \(\max(u_k, w_k) \in \text{PSH}(M, \xi, \omega^T)\) are continuous and strictly decreases to \(v\). By Dini’s lemma there exists \(v_k \in \mathcal{H}\) such that \(\max(u_{k-1}, v_{k-1}) \geq v_k \geq \max(u_k, v_k)\). Then \(v_k\) decreases to \(v\) and \(u_k \leq v_k\). It follows from Lemma 4.4 that
\[\square\]
\[
\max \left( \frac{1}{2n+p} \int_M |v_k - u_k|^p \omega_{ub}^n \wedge \eta, \int_M |u_k - v_k|^p \omega_{ub}^n \wedge \eta \right) \leq d_p(u_k, v_k)^p
\]

\[
\leq \int_M |v_k - u_k|^p \omega_{ub}^n \wedge \eta.
\]

By Proposition 3.15, the required inequality follows as \( k \to \infty \). \( \square \)

**Lemma 4.11** If the sequence \( \{u_k\}_{k \in \mathbb{N}} \), \( \{v_k\}_{k \in \mathbb{N}} \subset \mathcal{E}_p(M, \xi, \omega^T) \) decreases (increases) to \( u, v \in \mathcal{E}_p(M, \xi, \omega^T) \), respectively, then \( d_p(u_k, v_k) \to d_p(u, v) \) as \( k \to \infty \). In particular, \( d_p(u_k, u) \to 0 \).

**Proof** If the sequence \( \{u_k\}_{k \in \mathbb{N}} \) is decreasing, using the triangle inequality and Lemma 4.10, we have

\[
|d_p(u_k, v_k) - d_p(u, v)| \leq d_p(u_k, u) + d_p(v, v_k)
\]

\[
\leq \left( \int_M |u_k - u|^p \omega_{ub}^n \wedge \eta \right)^{\frac{1}{p}} + \left( \int_M |v_k - v|^p \omega_{ub}^n \wedge \eta \right)^{\frac{1}{p}}
\]

and the lemma follows from Lemma 3.15.

If the sequence \( \{u_k\}_{k \in \mathbb{N}} \) is increasing, using the triangle inequality and Lemma 4.10, we have

\[
|d_p(u_k, v_k) - d_p(u, v)| \leq d_p(u_k, u) + d_p(v, v_k)
\]

\[
\leq \left( \int_M |u_k - u|^p \omega_{ub}^n \wedge \eta \right)^{\frac{1}{p}} + \left( \int_M |v_k - v|^p \omega_{ub}^n \wedge \eta \right)^{\frac{1}{p}}
\]

and the lemma follows from Lemma 3.15. \( \square \)

Next we proceed to prove that \((\mathcal{E}_p(M, \xi, \omega^T), d_p)\) is a complete metric space.

**Lemma 4.12** Suppose \( u_0, u_1 \in \mathcal{E}_p(M, \xi, \omega^T) \). Then we have

\[
d_p\left(u_0, \frac{u_0 + u_1}{2}\right)^p \leq C d_p(u_0, u_1)^p.
\]

**Proof** It is obvious that \( P(u_0, u_1) \leq P(u_0, \frac{u_0 + u_1}{2}) \leq u_0 \) and \( P(u_0, u_1) \leq P(u_0, \frac{u_0 + u_1}{2}) \leq \frac{u_0 + u_1}{2} \). By the Pythagorean theorem 4.2, Lemmas 4.5, and 4.10, we have

\[
d_p\left(u_0, \frac{u_0 + u_1}{2}\right)^p = d_p\left(u_0, P\left(u_0, \frac{u_0 + u_1}{2}\right)\right)^p
\]

\[
+ d_p\left(\frac{u_0 + u_1}{2}, P\left(u_0, \frac{u_0 + u_1}{2}\right)\right)^p
\]

\[
\leq d_p(u_0, P(u_0, u_1))^p + d_p\left(\frac{u_0 + u_1}{2}, P(u_0, u_1)\right)^p
\]
For any $u_0, u_1 \in \mathcal{E}_p(M, \xi, \omega^T)$, we have

$$C^{-1}d_p(u_0, u_1)^p \leq \int_M |u_0 - u_1|^p (\omega^n_{u_0} \wedge \eta + \omega^n_{u_1} \wedge \eta) \leq Cd_p(u_0, u_1)^p. \quad (4.16)$$

**Proof** Using the triangle inequality, arithmetic–geometric mean inequality, and Lemma 4.10, we have:

$$d_p(u_0, u_1)^p \leq (d_p(u_0, \max(u_0, u_1)) + d_p(u_1, \max(u_0, u_1)))^p \leq 2^{p-1}(d_p(u_0, \max(u_0, u_1))^p + d_p(u_1, \max(u_0, u_1))^p) \leq 2^{p-1}\left(\int_M |u_0 - \max(u_0, u_1)|^p \omega^n_{u_0} \wedge \eta + \int_M |u_1 - \max(u_0, u_1)|^p \omega^n_{u_1} \wedge \eta\right) = 2^{p-1} \left(\int_{u_0 < u_1} |u_0 - u_1|^p \omega^n_{u_0} \wedge \eta + \int_{u_1 < u_0} |u_1 - u_0|^p \omega^n_{u_1} \wedge \eta\right) \leq 2^{p-1}\int_M |u_0 - u_1|^p (\omega^n_{u_0} \wedge \eta + \omega^n_{u_1} \wedge \eta).$$

By the previous lemma, the Pythagorean formula, and Lemma 4.10, there exists a constant $C$ such that

$$C d_p(u_0, u_1)^p \geq d_p \left(u_0, \frac{u_0 + u_1}{2}\right)^p \geq d_p(u_0, P \left(u_0, \frac{u_0 + u_1}{2}\right)^p \geq \int_M |u_0 - P \left(u_0, \frac{u_0 + u_1}{2}\right)|^p \omega^n_{u_0} \wedge \eta.$$
Recall that $\omega^n_{u_0} \wedge \eta \leq 2^n \omega^n_{u_0+u_1} \wedge \eta$. Similarly, we also have:

$$C_d_p(u_0, u_1)^p \geq d_p\left(\frac{u_0 + u_1}{2}, P\left(\frac{u_0 + u_1}{2}\right)\right)^p$$

$$\geq \int_M \left|\frac{u_0 + u_1}{2} - P\left(\frac{u_0 + u_1}{2}\right)\right|^p \omega^n_{u_0+u_1} \wedge \eta$$

$$\geq \frac{1}{2^n} \int_M \left|\frac{u_0 + u_1}{2} - P\left(\frac{u_0 + u_1}{2}\right)\right|^p \omega^n_{u_0} \wedge \eta.$$ 

Hence by the Holder inequality, we have:

$$(2^n + 1)C_d_p(u_0, u_1)^p \geq \int_M \left|u_0 - P\left(\frac{u_0 + u_1}{2}\right)\right|^p$$

$$+ \left|\frac{u_0 + u_1}{2} - P\left(\frac{u_0 + u_1}{2}\right)\right|^p \omega^n_{u_0} \wedge \eta$$

$$\geq \frac{1}{2^{2p-1}} \int_M |u_0 - u_1|^p \omega^n_{u_0} \wedge \eta.$$ 

By symmetry of $u_0, u_1$, we also have:

$$(2^n + 1)C_d_p(u_0, u_1)^p \geq \frac{1}{2^{2p-1}} \int_M |u_0 - u_1|^p \omega^n_{u_1} \wedge \eta.$$ 

Adding the last two inequalities, we obtain:

$$2^{p+1}(2^n + 1)C_d_p(u_0, u_1)^p \geq \int_M |u_0 - u_1|^p (\omega^n_{u_0} \wedge \eta + \omega^n_{u_1} \wedge \eta).$$ 

This completes the proof.  

\[\square\]

**Lemma 4.13** Let $\{u_k\}_{k \in \mathbb{N}} \subset E_p(M, \xi, \omega^T)$ be a $d_p$-bounded sequence decreasing (increasing) to $u$. Then $u \in E(M, \xi, \omega^T)$ and $d_p(u_k, u) \to 0$.

**Proof** If $\{u_k\}_{k \in \mathbb{N}}$ is decreasing, we can assume that $u_k < 0$. It follows from Lemma 4.10 that

$$\text{max}\left(\frac{1}{2^{n+p}} \int_M |u_k|^p \omega^n_{u_k} \wedge \eta, \int_M |u_k|^p (\omega^T)^n \wedge \eta\right) \leq d_p(u_k, 0)^p$$

are uniformly bounded. $\int_M |u_k|^p (\omega^T)^n \wedge \eta$ is uniformly bounded; the monotone convergence theorem and the dominated convergence theorem imply that $u_k \to u$ in $L^1$ and $u \in \text{PSH}(M, \xi, \omega^T)$. $E_p(u_k) = \int_M |u_k|^p \omega^n_{u_k} \wedge \eta$ is uniformly bounded; it follows from Proposition 3.16 and Lemma 4.11 that $u \in E_p(M, \xi, \omega^T)$ and $d_p(u_k, u) \to 0$.  

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If \( \{u_k\}_{k \in \mathbb{N}} \) is increasing, it follows from Theorem 4.3 that there exists a constant \( C \) such that
\[
\int_M |u_k|^p (\omega_{u_k}^n \wedge \eta + (\omega^T)^n \wedge \eta) \leq C d_p(u_k, 0)^p
\]
is uniformly bounded. By Propositions 3.3 and 3.4, we have \( u_k \to u \) in \( L^1 \) and \( u \in \text{PSH}(M, \xi, \omega^T) \). By Proposition 3.16 and Lemma 4.11, we have \( u \in \mathcal{E}_p(M, \xi, \omega^T) \) and \( d_p(u_k, u) \to 0 \).

**Proposition 4.4** Given \( u_0, u_1, v \in \mathcal{E}_p(M, \xi, \omega^T) \),
\[
d_p(P(u_0, v), P(u_1, v)) \leq d_p(u_0, u_1).
\]

**Proof** By Theorem 3.1 and Lemma 4.11, we only have to prove the inequality for \( u_0, u_1, v \in \mathcal{H}_\Delta \). In this case, \( P(u_0, v), P(u_1, v) \in \mathcal{H}_\Delta \) according to Theorem 3.1.

First we assume that \( u_0 \leq u_1 \). Let \( u_t, v_t \) be the \( C^{1,1}_B \) geodesic connecting \( u_0, u_1 \) and \( P(u_0, v), P(u_1, v) \), respectively. Then \( P(u_0, v) \leq P(u_1, v) \leq u_0 \). \( P(u_1, v) \) and Proposition 4.1 imply that \( P(u_0, v) \leq v_t \leq u \). Hence for \( x \in \{P(u_0, v) = v\} \), \( v_t(x) \) is independent of \( t \) and \( \dot{v}_0(x) = 0 \). Then we have
\[
\int_{\{P(u_0,v)=v\}} |\dot{v}_0|^p \omega_v^n \wedge \eta = 0.
\]

Then it follows from Lemmas 4.9 and 3.3 that
\[
d_p(P(u_0, v), P(u_1, v))^p = \int_M |\dot{v}_0|^p \omega_{P(u_0,v)}^n \wedge \eta
\]
\[
\leq \int_{\{P(u_0,v)=v\}} |\dot{v}_0|^p \omega_{u_0}^n \wedge \eta + \int_{\{P(u_0,v)=v\}} |\dot{v}_0|^p \omega_v^n \wedge \eta
\]
\[
\leq \int_{\{P(u_0,v)=v\}} |\dot{u}_0|^p \omega_{u_0}^n \wedge \eta
\]
\[
\leq \int_M |\dot{u}_0|^p \omega_{u_0}^n \wedge \eta
\]
\[
= d_p(u_0, u_1)^p.
\]

For the general case, using the Pythagorean formula we have
\[
d_p(P(u_0, v), P(u_1, v))^p = d_p(P(u_0, v), P(u_0, u_1))^p
\]
\[= d_p(P(u_0, v), P(P(u_0, u_1), v))^p + d_p(P(u_1, v), P(P(u_0, u_1), v))^p \]
\[\leq d_p(u_0, P(u_0, u_1))^p + d_p(u_1, P(u_0, u_1))^p = d_p(u_0, u_1)^p.\]

This completes the proof. □

**Proposition 4.5** \((\mathcal{E}_p(M, \xi, \omega^T), d_p)\) is a complete metric space.

**Proof** First we show that \((\mathcal{E}_p(M, \xi, \omega^T), d_p)\) is a metric space. The symmetry of \(d_p\) is obvious and the triangle inequality is inherited from the triangle inequality for smooth potentials. We only have to check the non-degeneracy of \(d_p\). Suppose \(w_1, w_2 \in \mathcal{E}_p(M, \xi, \omega^T)\) and \(d_p(w_1, w_2) = 0\). It follows from the Pythagorean formula that \(d_p(w_1, P(w_1, w_2)) = 0\) and \(d_p(P(w_1, w_2), w_2) = 0\). Then Lemma 4.10 implies that \(w_1 = P(w_1, w_2) = w_2\) with respect to the measure \(\omega^n\) \(\eta\). Then the domination principle Lemma 3.6 implies that \(w_1 \leq P(w_1, w_2)\) and \(w_2 \leq P(W_1, w_2)\). It follows that \(w_1 = P(w_1, w_2) = w_2\). Hence \((\mathcal{E}_p(M, \xi, \omega^T), d_p)\) is a metric space.

Next we show that the metric space \((\mathcal{E}_p(M, \xi, \omega^T), d_p)\) is complete. Suppose \(\{u_k\}_{k \in \mathbb{N}} \subseteq \mathcal{E}_p(M, \xi, \omega^T)\) is a \(d_p\) Cauchy sequence. We will prove that there exists \(u \in \mathcal{E}_p(M, \xi, \omega^T)\) such that \(d_p(u_k, u) \to 0\).

Without loss of generality we can assume that

\[d_p(u_k, u_{k+1}) \leq \frac{1}{2^k}\]

for \(k \in \mathbb{N}\). Denote by \(u_k^l = P(u_k, u_{k+1}, \ldots, u_{k+l})\) for \(k, l \in \mathbb{N}\) and \(u_k^0 = u_k\). It follows from the definition of rooftop envelope and Proposition 4.4 that

\[d_p(u_k^l, u_k^{l+1}) = d_p(P(u_k^l, u_{k+l}), P(u_k^l, u_{k+l+1})) \leq d_p(u_{k+l}, u_{k+l+1}) \leq \frac{1}{2^{k+l}}\]

and the sequence \(\{u_k^l\}_{l \in \mathbb{N}} \subseteq \mathcal{E}_p(M, \xi, \omega^T)\) is \(d_p\) bounded and decreasing. According to Lemma 4.13, \(\tilde{u}_k = \lim_{l \to \infty} u_k^l \in \mathcal{E}_p(M, \xi, \omega^T)\) and \(d_p(u_k^l, \tilde{u}_k) \to 0\) as \(l \to \infty\). Moreover, \(u_k^{l+1} \leq u_k^{l+1}\) implies that \(\tilde{u}_k \leq \tilde{u}_{k+1}\) and \(\{\tilde{u}_k\}_{k \in \mathbb{N}}\) is an increasing sequence in \(\mathcal{E}_p(M, \xi, \omega^T)\).

It follows from Lemma 4.11, the definition of rooftop envelope, and Proposition 4.4 that

\[d_p(\tilde{u}_k, \tilde{u}_{k+1}) = \lim_{l \to \infty} d_p(u_k^{l+1}, u_k^l) = \lim_{l \to \infty} d_p(P(u_k^{l+1}, u_k), P(u_k^{l+1}, u_k^{l+1})) \leq \lim_{l \to \infty} d_p(u_k, u_{k+1}) \leq \frac{1}{2^k}\]
and the sequence \( \{ \tilde{u}_k \} \) is \( d_p \) bounded and increasing. By Lemma 4.13, \( u = \lim_{k \to \infty} \tilde{u}_k \in \mathcal{E}_p(M, \xi, \omega^T) \) and \( \lim_{k \to \infty} d_p(\tilde{u}_k, u) = 0 \). Moreover, by Proposition 4.4 we have

\[
d_p(u_k^l, u_k) = d_p(P(u_k, u_{k+1}^{l-1}), P(u_k, u_k)) \leq d_p(u_{k+1}^{l-1}, u_k) \leq d_p(u_{k+1}^{l-1}, u_{k+1}) + d_p(u_k, u_{k+1})
\]

and

\[
d_p(u_k^l, u_k) \leq d_p(u_k^0, u_{k+l}) + \sum_{j=1}^{l} d_p(u_{k+j-1}, u_{k+j}) = \sum_{j=1}^{l} d_p(u_{k+j-1}, u_{k+j}).
\]

It follows from Lemma 4.11 that

\[
d_p(\tilde{u}_k, u_k) \leq \sum_{j=1}^{\infty} \frac{1}{2k+j-1} = \frac{1}{2k-1}.
\]

By the triangle inequality

\[
d_p(u_k, u) \leq d_p(\tilde{u}_k, u_k) + d_p(\tilde{u}_k, u),
\]

we have \( d_p(u_k, u) \to 0 \). This completes the proof.

To end this section, we remark that Theorem 4.1 follows from Lemmas 4.6, 4.7, and Proposition 4.5. Our main Theorem 2 follows from Theorem 4.1, Lemmas 4.7, 4.3(3), and 4.12, and Theorem 4.3.

\section{5 Sasaki-Extremal Metric}

We give a brief discussion of existence of Sasaki-extremal metric and properness of modified \( K \)-energy. Calabi’s extremal metric was extended to Sasaki setting by Boyer–Galicki–Simanca [16]. A Sasaki metric is called Sasaki-extremal if its transverse Kähler metric is extremal in the sense of Calabi [17]. As in Kähler setting, given a priori estimates [49] and the pluripotential theory developed in the paper, we have the following:

\textbf{Theorem 5.1} A compact Sasaki manifold \( (M, \xi, \eta, g) \) admits a Sasaki-extremal metric in the transverse Kähler class \( [\omega^T] \) if and only if the modified \( K \)-energy is reduced proper.

We recall some basic notions [16,17,35,36,52]. We use the group \( \text{Aut}_0(\xi, J) \) to denote the subgroup of diffeomorphism group of \( M \) which preserves both \( \xi \) and transverse holomorphic structure. Its Lie algebra is the Lie algebra of all Hamiltonian holomorphic vector fields in the sense of [37, Definition 4.4].
First one can define Sasaki–Futaki invariant as follows, given $X \in \text{aut}$, the Lie algebra of $\text{Aut}_0(\xi, J)$,

$$\mathcal{F}_X(\omega^T) = \int_M X(f)\omega_T^n \wedge \eta, \tag{5.1}$$

where $f$ is the potential of transverse scalar curvature,

$$\Delta f = R^T - R.$$

The first step is certainly to verify that (5.1) does not depend on a particular choice of transverse Kähler form in $[\omega^T]$ (see [16, Proposition 5.1]). We are interested in the reduced part $\mathfrak{h}_0$ of $\text{aut}$, which consists of Hamiltonian holomorphic vector fields such that $\eta(Y)$ has non-empty zero. When $(M, \xi, \eta, g)$ is a Sasaki-extremal metric, then similar as in Calabi’s decomposition, we have [16, Theorem 4.8] the decomposition

$$\mathfrak{h} = a \oplus \mathfrak{h}_0,$$

where $a$ consists of parallel vector fields of the transverse Kähler metric $g^T$. Moreover, the reduced part $\mathfrak{h}_0$ has the decomposition

$$\mathfrak{h}_0 = \mathfrak{z}_0 \oplus J\mathfrak{z}_0 \oplus (\oplus_{\lambda > 0} \mathfrak{h}^\lambda),$$

where $\mathfrak{z}_0 = \text{aut}(\xi, \eta, g)/\{\xi\}$ and

$$\mathfrak{h}^\lambda = \{Y \in \mathfrak{h} : \mathcal{L}_X Y = \lambda Y, X = (\bar{\partial}R)\#, \}$$

where $X := (\bar{\partial}R)\#$ is the dual vector and it is the extremal vector field in $\mathfrak{h}_0$. In general, we can define Futaki–Mabuchi bilinear form [36] on $\mathfrak{h}_0$ as in Kähler setting (in Sasaki setting this is well defined on $\text{aut}$ since every Hamiltonian vector field has a potential, simply given by $\eta(Y)$; for example, $\xi$ has potential 1). Given $Y, Z \in \text{aut}$, define

$$B(Y, Z) = \int_M \eta(Y)\eta(Z)(\omega^T)^n \wedge \eta. \tag{5.2}$$

It is straightforward to check that (5.2) remains unchanged if $\eta \to \eta + d^*\phi$ for $\phi \in \mathcal{H}$. If we restrict us on the real Hamiltonian holomorphic vector fields such that $\eta(Y)$ is real, then there exists a unique vector field $V$ such that

$$\mathcal{F}_{\text{Re}(Y)} = B(\text{Re}(Y), V). \tag{5.3}$$

We call such $V$ and its corresponding $X = V - \sqrt{-1}J V$ the extremal vector field. As in Kähler setting, for $J V$-invariant metrics in $\mathcal{H}$, we define the modified $\mathcal{K}$-energy [41,56] as

$$\delta\mathcal{K}_V = -\int_M \delta\phi(R\phi - R - \eta_\phi(V))\omega_\phi^n \wedge \eta. \tag{5.4}$$

Let $\text{Aut}_0(\xi, J, V)$ be the subgroup of $\text{Aut}_0(\xi, J)$ which commutes with the flow of $J V$. 

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Proposition 5.1 The $K_V$ energy is invariant under the action of $\text{Aut}_0(\xi, J, V)$

Proof The proof is similar to Kähler setting [48, Lemma 2.1] and it follows in a tautologic way from Futaki invariant and definition of extremal vector field through Futaki–Mabuchi bilinear form. We fix a background transverse Kähler structure $\omega^T$ such that it is $JV$ invariant. For $\sigma \in \text{Aut}_0(\xi, J, V)$, let $\sigma_t$ be one parameter subgroup generated by the flow of $Y_R := \text{Re}(Y)$ for some $Y \in \text{aut}$. Since $Y$ commutes with $V$, hence $\sigma_t^* \omega_0$ is invariant with respect to $JV$ if $\omega_0 \in [\omega^T]$ is invariant. We compute

$$\frac{d}{dt} K(\sigma_t^* \omega_0) = - \int_M \sigma_t^* (\eta_0(\text{Re}(Y))(R_0 - R - \eta_0(V)) \omega_0^n \wedge \eta_0)$$

$$= - \int_M \eta_0(Y_R)(R_0 - R) \omega_0^n \wedge \eta_0 + \int_M \eta_0(Y_R) \eta_0(V) \omega_0^n \wedge \eta_0.$$

The right-hand side is zero by (5.3).

We define the distance $d_1$ modulo the group action $G_0 := \text{Aut}_0(\xi, J, V)$. Fix a compact subgroup $K$ of $G_0$ such that $K$ contains the flow of $JV$ (and $\xi$ of course). Denote

$$G_0^K = \{ \phi \in H_0, \phi \text{ is invariant under the flow of } K \}.$$

Note that $G_0$ acts on $H_0$ through $\omega_\phi \to \sigma^* \omega_\phi = \omega^T + \sqrt{-1} \partial \bar{\partial} \sigma(\phi)$. Given any $\phi, \psi \in H_0$, we can consider the distance modulo $G_0$ as follows [26]

$$d_{1,G_0}(\phi, \psi) = \inf_{\sigma_1, \sigma_2 \in G_0} d_1(\sigma_1[\phi], \sigma_2[\psi]) = \inf_{\sigma \in G_0} d_1(\phi, \sigma[\psi]).$$

Definition 5.1 We say $K_V$ is reduced proper for $K$-invariant metrics with respect to $d_1$, if the following conditions hold

1. $K_V$ is bounded below over $H^K$.
2. There exists constant $C, D > 0$ such that for $\phi \in H^K$

$$K_V(\phi) \geq C d_{1,G_0}(0, \phi) - D.$$

To prove Theorem 5.1, we proceed exactly as in [48], to consider the modified Chen’s continuity path [21], for a $K$-invariant transverse Kähler metric $\omega^T$,

$$t(R_\phi - R - \eta_\phi(V)) + (1 - t)(\Lambda_{\omega_\phi} \omega^T - n) = 0. \quad (5.5)$$

Given a priori estimates as in [49] and the pluripotential theory on Sasaki manifolds developed in this paper, we can then follow [48,49] to prove Theorem 5.1. Since the argument is almost identical, we only sketch the process and skip the details.

1. The openness of (5.5) is proved similarly [48, Theorem 3.4]; note that we assume transverse Kähler metrics and potentials are $K$-invariant.

\end{document}
(2) For $0 < t < 1$, $\mathcal{K}_V$ bounded below over $\mathcal{H}^K$ implies that the distance $d(0, \phi_t)$ is uniformly bounded by a constant in the order $C((1-t)^{-1} + 1)$, where $\phi_t$ is the solution of (5.5) at $t$. This together with the fact that $\phi_t$ minimizes $t \mathcal{K}_V + (1-t)$, gives the uniform upper bound of entropy of $H(\phi_t)$ (depending on $(1-t)^{-1}$). Hence estimates in [49, Theorem 2] apply to get the solution for any $t < 1$.

(3) Choose an increasing sequence $t_i \to 1$; first using the properness assumption, we can assume that there are $\sigma_i \in G$ such that $\psi_i := \sigma_i \phi_{t_i}$ ($\omega_{\psi_i} = \sigma_i^* \omega_{\phi_{t_i}}$) satisfies that $d(0, \psi_i)$ is uniformly bounded above. Then $\psi_i$ satisfies a scalar curvature-type equation

$$\omega_{\psi_i}^n = e^{F_i} (\omega^T)^n$$

$$\Delta_{\psi_i} F_i = h_i + \text{tr}_{\psi_i} \left( \text{Ric}(\omega^T) - \frac{1}{t_i} \omega_i \right),$$

where $h_i$ is uniformly bounded and $\omega_i = \sigma_i^* (\omega^T)$. One can use [49, Theorem 3] and arguments as in [48, Theorem 3.5] to conclude the convergence of $\psi_i, F_i$ to a smooth Sasaki-extremal structure.

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Appendix

Approximation Through Type-I Deformation and Regularity of Rooftop Envelop

Using Type-I deformation, we can obtain the following approximation of irregular Sasaki structure $(M, \xi, \eta, g)$, which would be important for us; see [55] and in particular [13, Theorem 7.1.10] for the approximation. Suppose $\xi$ is irregular, then the Reeb flow generates an isometry in $\text{Aut}(M, \xi, \eta, g)$. Let $T^k \subset \text{Aut}(M, \xi, \eta, g)$ ($k \geq 2$) be the torus generated by $\xi$ and denote $t$ to be its Lie algebra. We can then choose $\rho_i \to 0$, $\rho_i \in t$ such that $\xi_i = \xi + \rho_i$ is quasiregular. Define

$$\eta_i = \frac{\eta}{1 + \eta(\rho_i)}, \quad \Phi_i = \Phi - \frac{1}{1 + \eta(\rho_i)} \Phi \rho_i \otimes \eta, \quad \omega_i^T = \frac{1}{2} d\eta_i, \quad \Phi_i = \eta_i \otimes \eta_i + \omega_i^T \otimes \Phi_i,$$

where $\Phi$ is the $(1, 1)$ tensor field defined on the contact bundle $\mathcal{D} = \text{Ker}(\eta)$. We recall the following:

**Theorem 6.1** (Approximation of irregular Sasaki structure) Let $(M, \xi, \eta, g)$ be an irregular Sasaki structure on a compact manifold $M$. Then we can choose $\rho_i \to 0$ such that $\xi_i$ is quasiregular and (6.1) define a quasiregular Sasaki structure which is invariant under the action of $T^k$, the torus generated by $\xi$ in $\text{Aut}(M, \xi, \eta, g)$. 

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Lemma 6.1  Let \((M, \xi, \eta, g)\) be a Sasaki structure on a compact manifold \(M\). Consider a torus \(T \subset \text{Aut}(M, \xi, \eta, g)\) and \(\xi_i \in \mathfrak{t}\). Choose \(\xi_i = \xi + \rho_i\) for \(\rho_i\) sufficiently small. Consider two Sasaki structures \((\xi, \eta, \Phi, g) \leftrightarrow (\xi_i, \eta_i, \Phi_i, g_j)\) via Type-I deformation. Then we have the following. Suppose \(u\) is \(T\) invariant and \(u \in \text{PSH}(M, \xi, \omega^T)\) with \(|d\Phi du| \leq C_0\). Then for \(\rho_i\) sufficiently small, there exists positive constant \(\epsilon_i \to 0\) (as \(\rho_i \to 0\)) such that,
\[
(1 - \epsilon_i)u \in \text{PSH}(M, \xi_i, \omega^T_i). \tag{6.2}
\]
Similarly, suppose \(|d\Phi_i du| \leq C_0\) and \(u \in \text{PSH}(M, \xi_i, \omega^T_i)\), then there exists positive constant \(\epsilon_i \to 0\) as \(i \to \infty\), such that
\[
(1 - \epsilon_i)u \in \text{PSH}(M, \xi, \omega^T). \tag{6.3}
\]

Proof Since \(u\) is \(T^k\)-invariant, hence \(u\) is a basic function with respect to both \(\xi\) and \(\xi_i\). We write
\[
\omega_i^T + \sqrt{-1} \partial_B \bar{\partial}_B u = \omega_i^T + \frac{1}{2} d\Phi_i d\omega.
\]
Using (6.1), we compute
\[
\omega_i^T + \frac{1}{2} d\Phi_i d\omega = \frac{\omega_i^T}{1 + \eta(\rho_i)} + \eta \wedge d \left( \frac{1 - du(\Phi \rho_i)}{1 + \eta(\rho_i)} \right) + \frac{1}{2} d\Phi_i du + 2\omega_i^T \frac{du(\Phi \rho_i)}{1 + \eta(\rho_i)}
\]
\[
= \frac{1 + 2du(\Phi \rho_i)}{1 + \eta(\rho_i)} \omega_i^T + \frac{1}{2} d\Phi_i du + \eta \wedge d \left( \frac{1 - du(\Phi \rho_i)}{1 + \eta(\rho_i)} \right)
\]
\[
= \omega_i^T + \frac{1}{2} d\Phi_i du + \left( \frac{1 + 2du(\Phi \rho_i)}{1 + \eta(\rho_i)} - 1 \right) \omega_i^T + \eta \wedge d \left( \frac{1 - du(\Phi \rho_i)}{1 + \eta(\rho_i)} \right). \tag{6.4}
\]
If \(|d\Phi du| \leq C_0\), then (6.4) implies that \(|d\Phi_i du| \leq C_1\) (vice versa). Moreover, when \(\rho_i \to 0\),
\[
\frac{1 + 2du(\Phi \rho_i)}{1 + \eta(\rho_i)} \to 1, \quad d \left( \frac{1 - du(\Phi \rho_i)}{1 + \eta(\rho_i)} \right) \to 0.
\]
We can then choose \(\epsilon_i \to 0\) as \(\rho_i \to 0\), such that
\[
\omega_i^T + \frac{1}{2} d\Phi_i d(1 - \epsilon_i)) \geq 0.
\]
This proves (6.2). Note that given the relation of \(\Phi\) and \(\Phi_i\), then \(|d\Phi du| \leq C_0\) implies that \(|d\Phi_i du|\) is uniformly bounded (we suppose \(\rho_i\) is uniformly small in smooth topology). Interchanging \(\xi\) and \(\xi_i\), this proves (6.3). \(\square\)

Remark 6.1 Note that the complex structure on the cone remains unchanged under Type-I deformation [50, Lemma 2.2]. The transverse holomorphic structure is changed since the foliation is changed, due to the change of Reeb vector foliation; on the
other hand, the contact bundle $\mathcal{D}$ remains unchanged. Note that $(\mathcal{D}, \Phi)$ and $(\mathcal{D}, \Phi_i)$ can be identified to transverse holomorphic tangent bundle $T^{1,0}(\mathcal{F}_\xi)$ and $T^{1,0}(\mathcal{F}_{\xi_i})$ (the foliations are different). Since the term $\eta \wedge d \left( \frac{1-du(\Phi_0)}{1+\eta(\rho_i)} \right)$ vanishes on $\mathcal{D}$ and

$\left( \frac{1+2du(\Phi_0)}{1+\eta(\rho_i)} - 1 \right) \omega^T$ involves with only $du$, hence the above statement holds if we only assume that $|du|$ is uniformly bounded. Since we shall not need this, we skip the argument. However, it seems that assumption like $|du| \leq C$ is necessary and we are not able to extend this to PSH$(M, \xi, \omega^T)$.

As mentioned above, we fix a torus $T \subset \text{Aut}(N, \xi, \eta, g)$ and consider $\rho_i \in t$ sufficiently small. Let $\xi_i = \xi + \rho_i$ and let $(\xi_i, \eta_i, g_i, \Phi_i)$ be the Type-I deformation of $(\xi, \eta, g, \Phi)$.

**Lemma 6.2** Let $\rho_i \to 0$. Suppose a sequence of $T$-invariant functions $u_i \in \text{PSH}(M, \xi_i, \omega_i^T)$ with $|d \Phi du_i|_{\omega_i^T} \leq C_0$ converges to $u \in \text{PSH}(M, \xi, \omega^T)$. Then $|d \Phi du|_{\omega^T} \leq C_0$ and we have the following weak convergence of the measure

$$\left( \omega_i^T + \frac{1}{2} d \Phi du_i \right)^n \wedge \eta_i \to \left( \omega^T + \frac{1}{2} d \Phi du \right)^n \wedge \eta.$$

**Proof** By (6.4) and $|d \Phi du_i|_{\omega_i^T} \leq C_0$, $\omega_i^T + \frac{1}{2} d \Phi du_i$ and $\omega^T + \frac{1}{2} d \Phi du$ differ by a term with small $L^\infty$ norm, hence we only need to prove that

$$\left( \omega^T + \frac{1}{2} d \Phi du \right)^n \wedge \eta \to \left( \omega^T + \frac{1}{2} d \Phi du \right)^n \wedge \eta.$$

Note that $\eta_i = \eta/(1 + \eta(\rho_i))$ converges smoothly to $\eta$, then the above follows from the weak convergence of $(\omega^T + \frac{1}{2} d \Phi du)^n \wedge \eta$. \qed

Next we give a proof of Theorem 3.1 in Sasaki setting, regarding the regularity of envelop construction.

**Lemma 6.3** Assume $\beta > 0$ and $u, v \in \text{PSH}(M, \xi, \omega^T) \cap L^\infty$. If

$$(\omega_v^T)^n \wedge \eta \geq e^{\beta(v-f)} (\omega^T)^n \wedge \eta, \quad (\omega_u^T)^n \wedge \eta \leq e^{\beta(u-f)} (\omega^T)^n \wedge \eta$$

then $v \leq u$.

**Proof** By the comparison principle (3.6)

$$\int_{[u<v]} (\omega_v^T)^n \wedge \eta \leq \int_{[u<v]} (\omega_u^T)^n \wedge \eta.$$

Then we have

$$\int_{[u<v]} e^{\beta(v-f)} (\omega_v^T)^n \wedge \eta \leq \int_{[u<v]} e^{\beta(u-f)} (\omega_v^T)^n \wedge \eta.$$

\[\text{ Springer}\]
It follows that \( \{ u < v \} \) has zero Lebesgue measure and \( v \leq u \) almost everywhere with respect to Lebesgue measure. Moreover, we have \( v \leq u \) everywhere on \( M \) since they are \( \omega^T \)-plurisubharmonic.

Using the same computation to the transverse complex Monge–Ampere equation (as in the complex Monge–Ampere equation \([57, \text{p. 99}]\)), we can obtain the following Laplacian estimate.

**Lemma 6.4** Suppose \( u \in \mathcal{H} \) is a solution for the equation

\[
(\omega^T_u)^n \wedge \eta = e^g (\omega^T)^n \wedge \eta.
\]

Then

\[
\Delta \omega_u^T \log \text{Tr} \omega^T_u \omega^T_u \geq \frac{\Delta \omega^T g}{\text{Tr} \omega^T_u \omega^T_u} - 2B \text{Tr} \omega^T_u \omega^T_u
\]

where \( B > 0 \) is a constant which depends on \( \omega \).

**Theorem 6.2** Given \( f \in C^\infty_b(M) \), then we have the following estimate

\[
\| P(f) \|_{C^1,1} \leq C(M, \omega^T, g, \| f \|_{C^1,1}).
\]

Moreover, if \( u_1, \ldots, u_k \in \mathcal{H}_\Delta \), where we use the notation

\[
\mathcal{H}_\Delta = \{ u \in PSH(M, \xi, \omega^T) : \| u \|_{C^1,1} < \infty \}
\]

then \( P(u_1, \ldots, u_k) \in \mathcal{H}_\Delta \).

**Proof** The first result was proved by Berman–Demailly [8] in Kähler setting. Since all quantities are basic and only transverse Kähler structure is involved, the argument as in Kähler setting has a direct adaption; see [30, Theorem A.7] for details in Kähler setting.

For each \( \beta > 0 \), consider the equation

\[
\left( \omega^T_{u_\beta} \right)^n \wedge \eta = e^{\beta (u_\beta - f)} (\omega^T)^n \wedge \eta.
\]

This reads locally

\[
\frac{\det (g^T_{ij} + u_\beta_{ij})}{\det (g^T_{ij})} = e^{\beta (u_\beta - f)}.
\]

The transverse version of Aubin–Yau theorem implies that there exists a unique solution \( u_\beta \) for any \( \beta > 0 \) and a smooth function \( f \). The unique solution \( u_\beta \) satisfies the following:

\[
\| u_\beta - P(f) \|_{C^0} \to 0, \beta \to \infty
\]

\( \square \) Springer
and there exists $\beta_0 > 0$ and a uniform constant $C$ such that $\beta \geq \beta_0$,

$$-n < \Delta^{\omega_T} u_\beta \leq C. \quad (6.7)$$

To prove (6.6), we choose $x_0 \in M$ such that $u_\beta - f$ obtains its maximum at $x_0$. Combining with Eq. (6.5), we have

$$\sqrt{-1} \partial_B \bar{\partial}_B (f - u_\beta) \geq 0$$

and

$$u_\beta - f = \frac{1}{\beta} \log \frac{(\omega_{u_\beta})^n \wedge \eta}{(\omega_f)^n \wedge \eta} \leq \frac{1}{\beta} \log \frac{(\omega_f)^n \wedge \eta}{(\omega_f)^n \wedge \eta}$$

at $x_0$. It follows that

$$u_\beta - f \leq \frac{C}{\beta} \quad (6.8)$$

On the other hand, we choose $v \in \mathcal{H}$ and $L > 0$ such that

$$\omega_v^T \geq L \omega^T \text{ and } v \leq f.$$ 

One can choose $\beta_1 > 2$ such that $\epsilon = \frac{2n \log \beta}{\beta} < 1$ for all $\beta \geq \beta_1$. Take $\beta_2 = \max\{\frac{1}{L}, \beta_1\}$, then for any $\beta \geq \beta_2$, we have

$$0 < \delta, \epsilon < 1 \text{ and } e^{-\beta \epsilon} \leq \delta^n L^n$$

where $\delta = \frac{1}{\beta}$. It follows that

$$u_{\delta, \epsilon} := (1 - \delta) P(f) + \delta v - \epsilon \leq f - \epsilon$$

and

$$(\omega_{u_{\delta, \epsilon}})^n \wedge \eta \geq \delta^n (\omega_v^T)^n \geq \delta^n L^n (\omega_f^T)^n \wedge \eta \geq e^{-\beta \epsilon} (\omega_f)^n \wedge \eta \geq e^{\beta(u_{\delta, \epsilon} - f)} (\omega_f)^n \wedge \eta.$$ 

By Eq. (6.5) and Lemma (6.3), we have

$$u_{\delta, \epsilon} = \left(1 - \frac{1}{\beta}\right) P(f) + \frac{v}{\beta} - \frac{2n \log \beta}{\beta} \leq u_\beta.$$
\[ P(f) \leq \frac{\beta}{\beta - 1} u_\beta + \frac{2n \log \beta}{\beta - 1} - \frac{1}{\beta - 1} \inf_M v. \]

Combined with Eq. ((6.8)) we can derive that
\[
\frac{1}{\beta} \inf_M v - \frac{1}{\beta} \sup_M f - \frac{2n \log \beta}{\beta} \leq u_\beta - P(f) \leq \frac{C}{\beta}
\] (6.9)

for \( \beta \geq \beta_2 \). Then (6.6) follows immediately.

It is standard to deduce the lower bound in (6.7) from the fact
\[ \omega^T B \omega - \partial_B \bar{B} u_\beta \geq 0. \]

By Eq. ((6.5)) and Lemma 6.4, we have
\[ \quad 2B \Tr_{\omega_\beta} \omega^T \omega_\beta^T + \Delta_{\omega_\beta} \log \Tr_{\omega_\beta} \omega_\beta^T \geq \beta \frac{\Delta_{\omega_\beta} (u_\beta - f)}{\Tr_{\omega_\beta} \omega_\beta^T}. \]

It follows that
\[ 2nB + \Delta_{\omega_\beta} (\log \Tr_{\omega_\beta} \omega_\beta^T - Bu_\beta) \geq \beta \frac{\Tr_{\omega_\beta} \omega_\beta^T - 2n - \Delta_{\omega_\beta} f}{\Tr_{\omega_\beta} \omega_\beta^T} \]

and
\[ \beta \Tr_{\omega_\beta} \omega_\beta^T e^{-Bu_\beta} \leq \beta (2n + \Delta_{\omega_\beta} f) e^{-Bu_\beta} \]
\[ + [2nB + \Delta_{\omega_\beta} \log \Tr_{\omega_\beta} \omega_\beta^T e^{-Bu_\beta}] \Tr_{\omega_\beta} \omega_\beta^T e^{-Bu_\beta}. \]

Assume that \( \Tr_{\omega_\beta} \omega_\beta^T e^{-Bu_\beta} \) obtains its maximum \( s \) at \( x_1 \in M \) and \( C_1 = \sup_M (2n + \Delta_{\omega_\beta} f) \), then we have
\[ \beta s \leq \beta C_1 e^{-Bu_\beta(x_1)} + 2nB s. \]

By the inequality (6.9) and \( P(f) \leq f, u_\beta \) is uniformly bounded. Hence we obtain an upper bound for \( \Tr_{\omega_\beta} \omega_\beta^T e^{-Bu_\beta} \) if \( \beta \geq \beta_0 = \max\{3nB, \beta_2\} \). We conclude that \( \Delta_{\omega_\beta} u_\beta \leq C \) for \( \beta \geq \beta_0 \).

The first statement follows from (6.6) and (6.7).

For the second statement, first note that we only need to show that if \( u_0, u_1 \in \mathcal{H}_\Delta \), then \( P(u_0, u_1) \in \mathcal{H}_\Delta \). Let \( u_t \) be the geodesic segment connecting \( u_0, u_1 \), then by Lemma 3.9, we know that \( u_t \in \mathcal{H}_\Delta \) (see [8] and [47] for Kähler setting). Now we have already known \( P(u_0, u_1) = \inf_{t \in [0, 1]} u_t \), then by [31, Proposition 4.4] (applied to each foliation chart), \( \Delta u_t \) is uniformly bounded. This shows that \( P(u_0, u_1) \in \mathcal{H}_\Delta \). \( \Box \)
More generally, one can obtain results as in [31] that \( P(f_1, \ldots, f_n) \in C^{1,1}_B \) given \( f_1, \ldots, f_n \in C^{1,1}_B \). The point is that given two functions \( f_1, f_2, h = \min\{f_1, f_2\} \) satisfy \( \Delta h \leq \max\{\Delta f_1, \Delta f_2\} \) in viscosity sense, writing \( h = \frac{f_1 + f_2}{2} - \frac{|f_1 - f_2|}{2} \). The argument as in [30, Theorem A.7] applies using the maximum principle in viscosity sense. Since we do not need this, we shall skip the details.

**Complex Monge–Ampere Operator and Intrinsic Capacity on Compact Sasaki Manifolds**

We discuss briefly the Bedford–Taylor theory on Sasaki manifolds. For details for complex Monge–Ampere operator, see Bedford–Taylor [2]. We also extend intrinsic Monge–Ampere capacity to Sasaki setting, see [43] for Kähler setting.

Given a Sasaki structure, there is a splitting of tangent bundle \( TM = L\xi \otimes \mathcal{D} \), where \( \mathcal{D} = \text{Ker}(\eta) \), with \( \Phi : \mathcal{D} \to \mathcal{D} \) inducing a splitting \( \mathcal{D} \otimes \mathbb{C} = \mathcal{D}^{1,0} \oplus \mathcal{D}^{0,1} \). Hence the subbundle \( \Lambda^2p(\mathcal{D}^*) \) of \( \Lambda^{2p}M \) is well defined and \( \Phi \) induces a splitting to give bidegree of forms in \( \Lambda^2p(\mathcal{D}^*) \). Note that we have the following,

\[
\Lambda^2p(\mathcal{D}^*) = \{ \theta : \theta \in \Lambda^{2p}M, t_\xi \theta = 0 \}.
\]

We do not assume that \( \theta \in \Lambda^2p(\mathcal{D}^*) \) is basic. That is, the coefficients of \( \theta \) might not be invariant under the Reeb flow. A simple observation shows that if \( \theta \in \Lambda^2p(\mathcal{D}^*) \), then \( \theta \) is basic if it is closed, \( d\theta = 0 \) (since \( t_\xi \theta = 0 \)). Hence a closed \( 2p \)-form in \( \Lambda^2p(\mathcal{D}^*) \) is basic and can be regarded as a transverse closed \( 2p \)-form, defined as in [58]. In general, \( d\Lambda^2p(\mathcal{D}^*) \) is not in \( \Lambda^{2p+1}(\mathcal{D}^*) \).

Next, we give a very brief discussion of transverse positive closed currents of bidegree of \( (p, p) \) on \( M \), \( 0 \leq p \leq n \); see [58] for similar treatment. We simply treat them as closed differential forms of bidegree \( (p, p) \) in \( \Lambda^{2p}(\mathcal{D}^*) \) with measurable coefficients which are invariant under the Reeb flow. Its total variation is controlled by

\[
\|T\| := \int_M T \wedge (\omega^T)^{n-p} \wedge \eta.
\]

Given \( \phi \in \text{PSH}(M, \xi, \omega^T) \), we write \( \phi \in L^1(T) \) if \( \phi \) is integrable with respect to the measure \( T \wedge (\omega^T)^{n-p} \wedge \eta \). In this case, the current \( \phi T \) is well defined and we write

\[
\omega_\phi \wedge T := \omega^T \wedge T + dd^*_B(\phi T)
\]

\[
\omega_\phi \wedge T \wedge (\omega^T)^{n-p-1} \wedge \eta = T \wedge (\omega^T)^{n-p} \wedge \eta + dd^*_B(\phi T) \wedge (\omega^T)^{n-p-1} \wedge \eta.
\]

The positivity is a local notion and we simply think \( T \) as a positive closed \( (p, p) \)-form on each foliation chart. Hence \( \omega_\phi \wedge T \) is also a transverse closed positive \( (p+1, p+1) \) form. Note that we think transverse positive closed currents of bidegree of \( (p, p) \)-type as a linear functional on \( \Lambda^{n-p,n-p}(\mathcal{D}^*) \), hence the test forms are of bidegree \( (n - p, n - p) \). A main point is that test forms are not restricted to basic forms. In
other words, given such a current $T$ and $\gamma \in \Lambda^{n-p, n-p}(D^*)$, we have the following pairing:

$$\gamma \rightarrow \int_M \gamma \wedge T \wedge \eta.$$  

When $\phi \in \text{PSH}(M, \xi, \omega^T) \cap L^\infty$, it follows that $\phi \in L^1(T)$ for any transverse positive closed current $T$ of bidegree $(p, p)$ and hence one can define inductively $\omega^k_\phi \wedge (\omega^T)^{n-k}$; in particular, this leads to the definition of transverse complex Monge–Ampere operator $\omega^p_\phi$ of bidegree $(n, n)$. Moreover, the cocycle condition on transverse holomorphic structure ensures that $\omega^k_\phi \wedge (\omega^T)^{n-k}$ is well defined on $M$. In particular, $\omega^p_\phi \wedge \eta$ defines a positive Borel measure on $M$.

It is more convenient to consider this construction locally in foliations charts $W_\alpha = (-\delta, \delta) \times V_\alpha$. By taking test forms $\gamma \in \Lambda^{n-p, n-p}(D^*)$ with compact support, we can consider $T \wedge \eta$ on a foliation chart for a transverse positive closed $(p, p)$ current $T$. In particular, this give a local description of the complex Monge–Ampere measures $\omega^k_\phi \wedge (\omega^T)^{n-k} \wedge \eta$. By taking test functions $f$ supported in a foliation chart, the measure $\omega^k_\phi \wedge (\omega^T)^{n-k} \wedge \eta$ for each $k$ is regarded as the product measure $\omega^k_\phi \wedge (\omega^T)^{n-k} \wedge dx$ on $W_\alpha$, where $\xi = \partial_x$ is the Reeb direction. Note that $\omega^k_\phi \wedge (\omega^T)^{n-k}$ is defined on $V_\alpha$ as the usual way in Kähler setting, and the cocycle condition on transverse holomorphic structure ensures that $\omega^k_\phi \wedge (\omega^T)^{n-k}$ is well defined as a transverse positive closed current of bidegree $(n, n)$. On each foliation chart, we have $\omega^k_\phi \wedge (\omega^T)^{n-k} \wedge \eta = \omega^k_\phi \wedge (\omega^T)^{n-k} \wedge dx$ as a product measure. This coincides with the local description given by van Coevering [58, Section 2].

Moreover, when $u, v \in \text{PSH}(M, \xi, \omega^T) \cap L^\infty$, $du \wedge d^c_B v \wedge T$ can also be defined, where $T$ is a transverse closed positive current of bidegree $(n-1, n-1)$. By the polarization formula we only need to define $du \wedge d^c_B u \wedge T$. By adding a positive constant if necessary, we assume $u \geq 0$. Then we define

$$du \wedge d^c_B u \wedge T := \frac{1}{2} dd^c_B(u^2) \wedge T - udd^c_B u \wedge T. \quad (6.10)$$

In particular, $du \wedge d^c_B u \wedge T$ is positive if $T$ is a transverse closed positive current of bidegree $(n-1, n-1)$. We can then define $du \wedge d^c_B u \wedge T \wedge \eta$ as a positive Borel measure. Using the polarization formula, we have the following Cauchy–Schwarz inequality, for $u, v \in \text{PSH}(M, \xi, \omega^T) \cap L^\infty$,

$$\left| \int_M du \wedge d^c_B v \wedge T \wedge \eta \right|^2 \leq \left( \int_M du \wedge d^c_B u \wedge T \wedge \eta \right) \left( \int_M dv \wedge d^c_B v \wedge T \wedge \eta \right). \quad (6.11)$$

We also record the following Stokes’ theorem in Sasaki setting, and its proof follows the Bedford–Taylor theory as in Kähler setting via approximation (Lemma 3.1); see [58, Theorem 2.3.1, Proposition 2.3.2].
Lemma 6.5  Let $u, v, \phi \in PSH(M, \xi, \omega^T) \cap L^\infty$, then for each $0 \leq k \leq n - 1$, we have

$$\int_M u \ddc_B v \wedge \omega^k_\phi \wedge (\omega^T)^{n-k-1} \wedge \eta = \int_M v \ddc_B u \wedge \omega^k_\phi \wedge (\omega^T)^{n-k-1} \wedge \eta = - \int_M du \wedge d\ddc_B v \wedge \omega^k_\phi \wedge (\omega^T)^{n-k-1} \wedge \eta.$$  

(6.12)

We record a basic inequality in Sasaki setting, usually referred to Chern–Levine–Nirenberg inequality.

Proposition 6.1 (Chern–Levine–Nirenberg inequalities) Let $T$ be a positive closed current of bidegree $(p, p)$ on $M$ and $\phi \in PSH(M, \xi, \omega^T) \cap L^\infty$. Then $\|\omega_\phi \wedge T\| = \|T\|$. Moreover, if $\psi \in PSH(M, \xi, \omega^T) \cap L^1(T)$, then $\psi \in L^1(\omega_\phi \wedge T)$ and

$$\|\psi\|_{L^1(T \wedge \omega_\phi)} \leq \|\psi\|_{L^1(T)} + (2 \max\{\sup \psi, 0\} + \sup \phi - \inf \phi) \|T\|.  \quad (6.13)$$

Proof By Stokes’ theorem, we have $\int_M \ddc_B (\phi T) \wedge (\omega^T)^{n-p-1} \wedge \eta = 0$, hence

$$\|\omega_\phi \wedge T\| = \int_M \omega^T \wedge T \wedge (\omega^T)^{n-p-1} \wedge \eta = \|T\|.$$

To prove (6.13), we first assume $\psi \leq 0, \phi \geq 0$. By assumption, $\psi \in L^1(T)$, then

$$\|\psi\|_{L^1(T \wedge \omega_\phi)} := \int_M -\psi T \wedge \omega_\phi \wedge (\omega^T)^{n-p-1} \wedge \eta = \|\psi\|_{L^1(T)} + \int_M -\psi \ddc_B (\phi T) \wedge (\omega^T)^{n-p-1} \wedge \eta.$$

By Stokes’ theorem, we compute

$$\int_M -\psi \ddc_B (\phi T) \wedge (\omega^T)^{n-p-1} \wedge \eta = \int_M \ddc_B (-\psi) \wedge \phi T \wedge (\omega^T)^{n-p-1} \wedge \eta \leq \int_M \phi T \wedge (\omega^T)^{n-p} \wedge \eta \leq \sup M \phi \int_M T \wedge (\omega^T)^{n-p} \wedge \eta = (\sup M \phi) \|T\|.$$

Now suppose $\sup \psi > 0$. Replacing $\phi$ by $\phi - \inf \phi$, we compute

$$\|\psi\|_{L^1(T \wedge \omega_\phi)} \leq \int_M (2 \sup \psi - \psi) T \wedge \omega_\phi \wedge (\omega^T)^{n-p-1} \wedge \eta.$$

The same argument as above leads to (6.13) for the general case. \qed
For a Borel subset $E$ on a Sasaki manifold $(M, \xi, \omega^T)$, we define the capacity as

$$\text{cap}_{\omega^T}(E) := \sup \left\{ \int_E \omega^n \phi \wedge \eta : \phi \in \text{PSH}(M, \xi, \omega^T), 0 \leq \phi \leq 1 \right\}.$$ 

It is obvious that $\text{cap}_{\omega^T}(\cup_{k=1}^{\infty} E_k) \leq \sum_{k=1}^{\infty} \text{cap}_{\omega^T}(E_k)$ for a sequence of Borel sets $E_k$. We have the following:

**Proposition 6.2** Let $\phi \in \text{PSH}(M, \xi, \omega^T)$ with $0 \leq \phi \leq 1$ and $\psi \in \text{PSH}(M, \xi, \omega^T)$ such that $\psi \leq 0$. Then

$$\int_M -\psi \omega^n \wedge \eta \leq \int_M (-\psi)(\omega^T)^n \wedge \eta + n \int_M (\omega^T)^n \wedge \eta. \quad (6.14)$$

**Proof** We only need to prove (6.14) for canonical cutoffs $\psi_k = \max\{\psi, -k\}$ ($-\psi_k$ increases to $-\psi$ and we can apply monotone convergence theorem). We have the following:

$$\int_M -\psi_k \omega^n \wedge \eta = \int_M -\psi_k \omega^{n-1} \wedge (\omega^T + \sqrt{-1} \partial B \bar{\partial} B \phi) \wedge \eta$$

$$= \int_M -\psi_k \omega^{n-1} \wedge \omega^T \wedge \eta + \int_M -\psi_k \omega^{n-1} \wedge \sqrt{-1} \partial B \bar{\partial} B \phi \wedge \eta$$

$$= \int_M -\psi_k \omega^{n-1} \wedge \omega^T \wedge \eta + \int_M \phi \omega^{n-1} \wedge (-\sqrt{-1} \partial B \bar{\partial} B \psi_k) \wedge \eta$$

$$\leq \int_M -\psi_k \omega^{n-1} \wedge \omega^T \wedge \eta + \int_M (\omega^T)^n-1 \wedge \omega^T \wedge \eta$$

$$\leq \int_M -\psi_k \omega^{n-1} \wedge \omega^T \wedge \eta + \int_M (\omega^T)^n \wedge \eta.$$

We can then proceed inductively to obtain (6.14). Note that the argument above is a special case of (6.13). \qed

**Proposition 6.3** Suppose that $u \in \text{PSH}(M, \xi, \omega^T)$ and $u \leq 0$. Then for $t > 0$ we have

$$\text{cap}_{\omega^T}(|u < -t|) \leq \frac{1}{t} \left( \int_M (-u)(\omega^T)^n \wedge \eta + n \int_M (\omega^T)^n \wedge \eta \right).$$

**Proof** This is a direct consequence of Proposition 6.2. Denote $K_t = \{u < -t\}$, then

$$\int_{K_t} \omega^n \wedge \eta \leq \frac{1}{t} \int_M -\psi \omega^n \wedge \eta$$

$$\leq \frac{1}{t} \left( \int_M -\psi(\omega^T)^n \wedge \eta + n \int_M (\omega^T)^n \wedge \eta \right).$$

\qed
Proposition 6.4 Suppose that $u_k, u \in \text{PSH}(M, \xi, \omega^T) \cap L^\infty$ and $u_k$ decreases to $u$. Then for $\delta > 0$ we have

$$\text{cap}_{\omega^T}(\{u_k > u + \delta\}) \to 0, \ k \to \infty.$$\

Proof This proceeds exactly the same as in [43, Proposition 3.7]. We sketch the argument briefly. We assume $\text{Vol}(M) = 1$ for simplicity. Fix $\delta > 0$ and $\phi \in \text{PSH}(M, \xi, \omega^T)$ such that $0 \leq \phi \leq 1$. We have

$$\int_{\{u_k > u + \delta\}} \omega^n \wedge \eta \leq \delta^{-1} \int_M (u_k - u) \omega^n \wedge \eta.$$\

By Stokes’ theorem, we write

$$\int_M (u_k - u) \omega^n \wedge \eta = \int_M (u_k - u) \wedge \omega^T \wedge \omega^{n-1} \wedge \eta$$

$$+ \int_M (u_k - u) \wedge \dd c_B \phi \wedge \omega^{n-1} \wedge \eta$$

$$= \int_M (u_k - u) \wedge \omega^T \wedge \omega^{n-1} \wedge \eta$$

$$- \int_M d(u_k - u) \wedge d^c_B \phi \wedge \omega^{n-1} \wedge \eta.$$

By the Cauchy–Schwarz inequality, setting $f_k = u_k - u \geq 0$,

$$\left| \int_M d(u_k - u) \wedge d^c_B \phi \wedge \omega^{n-1} \wedge \eta \right|^2 \leq \int_M d f_k \wedge d^c_B f_k \wedge \omega^{n-1} \wedge \eta$$

$$\wedge \eta \int_M d \phi \wedge d^c_B \phi \wedge \omega^{n-1} \wedge \eta.$$\

We compute

$$\int_M d \phi \wedge d^c_B \phi \wedge \omega^{n-1} \wedge \eta = \int_M \phi(-\dd c_B \phi) \wedge \omega^{n-1} \wedge \eta$$

$$\leq \int_M \phi \omega^T \wedge \omega^{n-1} \wedge \eta \leq 1.$$\

Similarly, we compute

$$\int_M d f_k \wedge d^c_B f_k \wedge \omega^{n-1} \wedge \eta = \int_M f_k (\dd c_B u - \dd c_B u_k) \wedge \omega^{n-1} \wedge \eta$$

$$\leq \int_M f_k \omega_u \wedge \omega^{n-1} \wedge \eta.$$
Combining all these together gives
\[
\int_M (u_k - u) \omega^n \wedge \eta \leq \int_M (u_k - u) \wedge \omega^T \wedge \omega^{n-1}_\phi \wedge \eta + \left( \int_M (u_k - u) \omega_u \wedge \omega^{n-1}_\phi \wedge \eta \right)^{1/2}.
\]

Suppose \( u_k - u \leq c_0 \) for a fixed positive constant \( c_0 \geq 1 \). Then we have
\[
\int_M (u_k - u) \omega^n \wedge \eta \leq \sqrt{c_0} \left( \int_M (u_k - u) \wedge \omega^T \wedge \omega^{n-1}_\phi \wedge \eta \right)^{1/2} + \left( \int_M (u_k - u) \omega_u \wedge \omega^{n-1}_\phi \wedge \eta \right)^{1/2}.
\]

Hence we have
\[
\int_M (u_k - u) \omega^n \wedge \eta \leq \sqrt{2c_0} \left( \int_M (u_k - u) \wedge (\omega^T + \omega_u) \wedge \omega^{n-1}_\phi \wedge \eta \right)^{1/2}.
\]

We can proceed inductively by replacing \( \omega_\phi \) by \( \omega^T + \omega_u \) to obtain
\[
\int_M (u_k - u) \omega^n \wedge \eta \leq (\sqrt{2c_0}) \left( \int_M (u_k - u) \wedge (\omega^T + \omega_u)^n \wedge \eta \right)^{1/2^n}.
\]

The dominated convergence theorem implies the right-hand side goes to zero, independent of \( \phi \). This completes the proof.

As a consequence, we have the following:

**Theorem 6.3** Let \( \varphi \in \text{PSH}(M, \xi, \omega^T) \), then for any \( \epsilon > 0 \) there exists an open subset \( O_\epsilon \subset M \) such that \( \text{cap}_{\omega^T}(O_\epsilon) < \epsilon \) and \( \varphi \) is continuous on \( M - O_\epsilon \).

**Proof** By Proposition 6.3 there exists \( t_0 > 0 \) such that \( \text{cap}_{\omega^T}(O_0) < \frac{\epsilon}{2} \) for the open subset \( O_0 = \{ u < -t_0 \} \). Take the cutoff \( u_{t_0} = \max\{ u, -t_0 \} \in \text{PSH}(M, \xi, \omega^T) \), then there exists a sequence \( u_k \in \mathcal{H} \) decreasing to \( u \). By Proposition 6.4, we can choose a subsequence \( u_{k,j} \) such that \( \text{cap}_{\omega^T}(O_j) < \frac{\epsilon}{2^{j+1}} \) for the open subset \( O_j = \{ u_{k,j} > u + \frac{1}{j} \} \). Then for the open subset \( O_\epsilon = \bigcup_{j=0}^{\infty} O_j \) we have \( \text{cap}_{\omega^T}(O) < \epsilon \). Moreover \( u_{K,j} \) converges uniformly to \( u \) on \( M - O_\epsilon \), hence \( u \) is continuous on \( M - O_\epsilon \).

**Remark 6.2** The discussions above are taken from Kähler setting [43, Section 3]. Note that in (6.13) it is necessary to replace sup \( \psi \) by max\{sup \( \psi \), 0\} (similarly one needs to replace sup\( _X \psi \) by max\{sup\( _X \psi \), 0\} in [43, Proposition 3.1]).

We also need the following uniqueness in Sasaki setting, see [44, Theorem 3.3].
Theorem 6.4 Suppose \( u, v \in \mathcal{E}_1(M, \xi, \omega^T) \) such that
\[
\omega^n_u \wedge \eta = \omega^n_v \wedge \eta,
\]
then \( u - v = \text{const.} \)

Proof This follows exactly as in [44, Theorem 3.3] and we sketch the argument. The first step is that for \( u \in \mathcal{E}_1(M, \xi, \omega^T) \) and its canonical cutoffs \( u_j = \max\{u, -j\} \), then \( \nabla u_j \in L^2(d\mu_g) \) and has uniformly bounded \( L^2 \) norm (see [44, Proposition 3.2]).

We can assume that \( u \leq 0 \) and hence \( u_j \leq 0 \). Then for \( \phi \in \text{PSH}(M, \xi, \omega^T) \cap L^\infty \) such that \( \phi \leq 0 \), we know that, for any basic positive closed of \((n-1, n-1)\) type.

\[
\int_M (-\phi) \omega \wedge T = \int_M (-\phi)(\omega \phi - dd_c B \phi) \wedge T = \int_M (-\phi)\omega \phi \wedge T \]
\[
+ \int_M d\Phi \wedge d^c_B \phi \wedge T \leq \int_M (-\phi)\omega \phi \wedge T.
\]

An inductive argument applies to \( T = \omega^k \wedge (\omega^T)^{n-k-1} \), we get that
\[
0 \leq \int_M d\Phi \wedge d^c_B \phi \wedge T \leq \int_M (-\phi)\omega \phi \wedge \eta.
\]

(6.15)

Taking \( \phi = u_j \) in (6.15) and noting that the right-hand side is uniformly bounded, we get \( \nabla u_j \) is uniformly bounded in \( L^2(d\mu_g) \), hence \( \nabla u \in L^2(d\mu_g) \).

We assume that \( u, v \leq -1 \) and \( \text{Vol}(M) = 1 \). Set \( f = (u - v)/2 \) and \( h = (u + v)/2 \). We need to establish that \( \nabla f = 0 \) by showing that
\[
\int_M d\Phi \wedge d^c_B B \wedge \omega \wedge \eta = 0.
\]

If we assume \( u, v \) are bounded, then we have
\[
\int_M d\Phi \wedge d^c_B B \wedge \omega \wedge \eta \leq \sum \int_M d\Phi \wedge d^c_B B \wedge \omega \wedge \omega^k \wedge \omega^{n-k-1} \wedge \eta = - \int_M \frac{1}{2} (\omega^n_u - \omega^n_v) \wedge \eta,
\]

(6.16)

where we use the fact that \( dd^c_B f = (\omega_u - \omega_v)/2 \). We shall also establish the following a priori bound, when \( u, v \) are bounded,
\[
\int_M d\Phi \wedge d^c_B B \wedge (\omega^T)^{n-1} \wedge \eta \leq 3^\eta \left( \int_M d\Phi \wedge d^c_B B \wedge \omega^k \wedge \eta \right)^{1/2^{n-1}}.
\]

(6.17)

We apply (6.16) and (6.17) to the canonical cutoffs \( u_j, v_j \) (writing \( f_j, h_j \) correspondingly and using Proposition 3.15),
\[
\lim \int_M d\Phi \wedge d^c_B B \wedge (\omega^T)^{n-1} \wedge \eta = 0.
\]

We can then conclude that
\[
\int_M d\Phi \wedge d^c_B B \wedge (\omega^T)^{n-1} \wedge \eta = 0.
\]
This implies that \( u - v \) is a constant. To establish (6.17), we need several observations as follows. First observe that for \( l = n - 2, \ldots, 0 \),

\[
\int_M (-h) \omega_h^{2+l} \wedge (\omega^T)^{n-2-l} \wedge \eta \leq \int_M (-h)(\omega^T)^n \wedge \eta \leq 1,
\]

where the last inequality follows from \( -h \leq 1 \) and the normalization of the volume. We can then apply the following inequality inductively for \( T = \omega_h^l \wedge (\omega^T)^{n-l-1} \) such that

\[
\int_M d f \wedge d^c_B f \wedge \omega^T \wedge T \wedge \eta \leq 3 \left( \int_M d f \wedge d^c_B f \wedge \omega_h \wedge T \wedge \eta \right)^{1/2}, \tag{6.18}
\]

which proves (6.17). Now we establish (6.18). We write

\[
d f \wedge d^c_B f \wedge \omega^T = d f \wedge d^c_B f \wedge \omega_h - d f \wedge d^c_B f \wedge d d^c_B h
\]

hence we obtain, integrating by parts,

\[
\int_M d f \wedge d^c_B f \wedge \omega^T \wedge T \wedge \eta = \int_M d f \wedge d^c_B f \wedge \omega_h \wedge T \wedge \eta \\
+ \int_M d f \wedge d^c_B h \wedge \frac{\omega_u - \omega_v}{2} \wedge T \wedge \eta.
\]

By Cauchy–Schwartz inequality, we have

\[
\left| \int_M d f \wedge d^c_B h \wedge \omega \wedge T \wedge \eta \right|^2 \leq 4 \int_M d f \wedge d^c_B f \wedge \omega_h \wedge T \wedge \eta \int_M d h \wedge d^c_B h \wedge \omega_h \wedge T \wedge \eta.
\]

We can get a similar control

\[
\left| \int_M d f \wedge d^c_B h \wedge \omega \wedge T \wedge \eta \right|^2 \leq 4 \int_M d f \wedge d^c_B f \wedge \omega_h \wedge T \wedge \eta \int_M d h \wedge d^c_B h \wedge \omega_h \wedge T \wedge \eta.
\]

Clearly, we have the following \( (h \leq 0, S = \omega_h^l \wedge (\omega^T)^{n-l-2}) \)

\[
\int_M d h \wedge d^c_B h \wedge \omega_h \wedge S \wedge \eta \leq \int_M (-h)\omega_h^2 \wedge S \wedge \eta \leq 1.
\]

Combining these estimate altogether we conclude that,

\[
\int_M d f \wedge d^c_B f \wedge \omega^T \wedge S \wedge \eta \leq \int_M d f \wedge d^c_B f \wedge \omega_h \wedge T \wedge \eta \\
+ 2 \left( \int_M d f \wedge d^c_B f \wedge \omega_h \wedge T \wedge \eta \right)^{1/2}.
\]
The last observation is that
\[
\int_M df \wedge d_B f \wedge \omega_h \wedge S \wedge \eta = \frac{1}{4} \int_M (u - v)(\omega_v - \omega_u) \wedge \omega_h \wedge S \wedge \eta \\
\leq \int_M (-h)^2 \omega_h \wedge S \wedge \eta \leq 1.
\]

This completes the proof of (6.18) by combining two inequalities above. \(\square\)

**Functionals in Finite-Energy Class \(E_1\) and Compactness**

We discuss briefly well-known functionals in Kähler geometry and their properties over finite-energy class \(E_1\), see [30, Section 3.8]. The energy functionals include Monge–Ampere energy \(I\) and Aubin’s \(I\)-functional on \(E_1\), see [1,4,5,30] for Kähler setting. These results have a direct adaptation in Sasaki setting. Recall Aubin’s \(I\)-functional in Sasaki setting, for \(u, v \in \mathcal{H}\)

\[
I(u, v) := I(\omega_u, \omega_v) = \frac{1}{n!} \int_M (v - u)(\omega_u^n - \omega_v^n) \wedge \eta. \quad (6.19)
\]

We also recall the \(J\)-functional

\[
J(u, v) := J(\omega_u, \omega_v) = \frac{1}{n!} \int_M (v - u)\omega_u^n \wedge \eta - \|_{\omega_u}(v), \quad (6.20)
\]

where the \(\|_{\omega_u}(v)\)-functional is given by

\[
\|_{\omega_u}(v) = \frac{1}{(n + 1)!} \int_M (v - u) \sum_{k=0}^{n} \omega_u^k \wedge \omega_v^{n-k} \wedge \eta. \quad (6.21)
\]

We define the \(\|\)-functional (with the base \(\omega^T\)) on \(\mathcal{H}\),

\[
\|_{\omega^T}(u) = \frac{1}{(n + 1)!} \int_M u \sum_{k=0}^{n} \omega_u^k \wedge \omega_T^{n-k} \wedge \eta. \quad (6.22)
\]

The \(\|\)-functional is also called the Monge–Ampère energy, since if \(t \to v_t \in \mathcal{H}\) is smooth, then we have (as in Kähler setting),

\[
\frac{d}{dt} \|_{v_t} = \frac{1}{n!} \int_M \dot{v}_t \omega_v^n \wedge \eta. \quad (6.23)
\]

We mention that \(I\) is symmetric with respect to \(u, v\) but \(J\) is not. \(I, J\) are both defined on the metric level, independent of the choice of normalization of potentials \(u, v\); while
\(\mathbb{I}_{\omega_u}(v)\) depends on the normalization of \(u, v\). When \(u, v\) are bounded, then Bedford–Taylor theory allows to integrate by parts and the \(I\)-functional takes the formula
\[
I(\omega_u, \omega_v) = \frac{1}{(n+1)!} \sum_{j=0}^{n-1} \int_M d(u - v) \wedge d^c_B (u - v) \wedge \omega_u^j \wedge \omega_v^{n-1-j} \wedge \eta. \quad (6.24)
\]
Hence it is non-negative.

We need more information about \(\mathbb{I}\)-functional, see [30, Section 3.7] for Kähler setting. These properties in Sasaki setting follow in a rather straightforward way given pluripotential theory extended to Sasaki setting. We include these facts here for completeness.

**Proposition 6.5** Given \(u, v \in \text{PSH}(M, \xi, \omega_T) \cap L^\infty\), the following cocycle condition holds
\[
\mathbb{I}(u) - \mathbb{I}(v) = \frac{1}{(n+1)!} \sum_{k=0}^{n} \int_M (u - v) \omega_u^k \wedge \omega_v^{n-k} \wedge \eta = \mathbb{I}_{\omega_u}(v). \quad (6.25)
\]
Moreover, we have \(\mathbb{I}(u)\) is concave in \(u\) in the sense that,
\[
\frac{1}{n!} \int_M (u - v) \omega_u^n \wedge \eta \leq \mathbb{I}(u) - \mathbb{I}(v) \leq \frac{1}{n!} \int_M (u - v) \omega_v^n \wedge \eta. \quad (6.26)
\]
As a direct consequence, if \(u, v \in \text{PSH}(M, \xi, \omega_T) \cap L^\infty\) such that \(u \geq v\). Then \(\mathbb{I}(u) \geq \mathbb{I}(v)\).

**Proof** This follows almost identical as in [30, Proposition 3.8], given the pluripotential theory established in Sasaki setting in the paper. We sketch the argument. When \(u, v \in \mathcal{H}\), this follows exactly the same as in Kähler setting, by taking \(h_t = (1-t)u + tv\) and then use (6.23) to compute directly. When \(u, v \in \text{PSH}(M, \xi, \omega_T) \cap L^\infty\), we then use \(u_k, v_k \in \mathcal{H}\) decreasing to \(u, v\) (Lemma 3.1), respectively. Using Bedford–Taylor’s theorem in Sasaki setting [58, Theorem 2.3.1], we proceed exactly as in Kähler setting to conclude that \(\mathbb{I}(u_k) \to \mathbb{I}(u)\), etc. For the estimate (6.26), we compute
\[
\int_M (u - v) \omega_u^k \wedge \omega_v^{n-k} \wedge \eta = \int_M (u - v) \omega_u^{k-1} \wedge \omega_v^{n-k+1} \wedge \eta
\]
\[
+ \int_M (u - v) \sqrt{-1} \partial \bar{\partial} (u - v) \wedge \omega_u^{k-1} \wedge \omega_v^{n-k} \wedge \eta
\]
\[
= \int_M (u - v) \omega_u^{k-1} \wedge \omega_v^{n-k+1} \wedge \eta
\]
\[
- \int_M \sqrt{-1} \partial (u - v) \wedge \bar{\partial} (u - v) \wedge \omega_u^{k-1} \wedge \omega_v^{n-k} \wedge \eta
\]
\[
\leq \int_M (u - v) \omega_u^{k-1} \wedge \omega_v^{n-k+1} \wedge \eta.
\]
Using the estimate inductively for the terms in (6.25) leads to (6.26). Clearly, $I(u)$ is concave in $u$ given (6.26).

The monotonicity property allows to define $I(u)$ for $u \in \text{PSH}(M, \xi, \omega^T)$ through the limit process, using the canonical cutoffs $u_k = \max\{u, -k\}$

$$I(u) = \lim_{k \to \infty} I(\max\{u, -k\}).$$

Though the above limit is well defined, it may equal $-\infty$. It turns out $I(u)$ is finite exactly on $E_1(M, \xi, \omega^T)$. We record some further properties of $I(u)$ for $u \in E_1(M, \xi, \omega^T)$. The proofs are almost identical and we shall skip the details, see [30, Propositions 3.40, 3.42, 3.43; Lemma 3.41].

**Proposition 6.6** Let $u \in \text{PSH}(M, \xi, \omega^T)$. Then $-\infty < I(u)$ if and only if $u \in E_1(M, \xi, \omega^T)$. Moreover,

$$|I(u_0) - I(u_1)| \leq d_1(u_0, u_1), \quad u_0, u_1 \in E_1(M, \xi, \omega^T).$$

(6.27)

**Proposition 6.7** Suppose $u_0, u_1 \in E_1(M, \xi, \omega^T)$ and $t \to u_t$ is the finite-energy geodesic connecting $u_0, u_1$. Then $t \to I(u_t)$ is linear in $t$. We also have the following distance formula:

$$d_1(u_0, u_1) = I(u_0) + I(u_1) - 2I(P(u_0, u_1)).$$

In particular, $d_1(u_0, u_1) = I(u_0) - I(u_1)$ if $u_0 \geq u_1$.

We have the following (see [30, Lemma 3.47])

**Lemma 6.6** Suppose $u, u^j, v, v^j \in E_1(M, \xi, \omega^T)$ and $u^j \searrow u$ and $v^j \searrow v$. Then the following hold:

$$I(u, v) = I(u, \max\{u, v\}) + I(\max\{u, v\}, v).$$

(6.28)

Moreover, $\lim_{j \to \infty} I(u^j, v^j) = I(u, v)$.

**Proof** By Proposition 3.8, we have

$$\chi_{\{v > u\}} \omega^\alpha_{\max\{u, v\}} \wedge \eta = \chi_{\{v > u\}} \omega^\alpha_v \wedge \eta.$$

Hence it follows that

$$I(u, \max\{u, v\}) = \frac{1}{(n+1)!} \int_{\{v > u\}} (u - v)(\omega^\alpha_v - \omega^\alpha_u) \wedge \eta.$$

Interchange $u \leftrightarrow v$, we get $I(v, \max\{u, v\}) = \int_{\{u > v\}} (u - v)(\omega^\alpha_v - \omega^\alpha_u) \wedge \eta$. This proves (6.28). We write

$$I(u^j, v^j) = I(u^j, \max\{u^j, v^j\}) + I(v^j, \max\{u^j, v^j\}).$$
Since \( u^j, v^j \leq \max\{u^j, v^j\} \), we can apply Proposition 3.15 to conclude \( I(u^j, \max\{u^j, v^j\}) \rightarrow I(u, \max\{u, v\}) \) and \( I(v^j, \max\{u^j, v^j\}) \rightarrow I(v, \max\{u, v\}) \), using the formula (6.19). This completes the proof.

We have the following well-known inequalities:

**Proposition 6.8** For \( u, v \in \text{PSH}(M, \xi, \omega^T) \cap L^\infty \), we have

\[
\frac{1}{n+1} I(u, v) \leq J(u, v) \leq \frac{n}{n+1} I(u, v).
\]

Moreover, \( J(u, v) \) is convex in \( v \) since \( I_{\omega^T}(v) \) is concave in \( v \).

**Proof** This is well known, by direct computation [38, Proposition 4.2.1] for \( u, v \in H \).

A direct approximation argument using Lemma 3.1 shows that this can be generalized for \( u, v \in \text{PSH}(M, \xi, \omega^T) \cap L^\infty \).

The functionals \( (I, J, \|\|) \) are well defined for \( u, v \in E_1(M, \xi, \omega^T) \) [see Proposition (3.16)]. Note that (6.26) and Proposition 6.8 both hold in \( E_1(M, \xi, \omega^T) \) (see [4] for Kähler setting). This follows by an approximation argument applying Proposition 3.15. Next we prove the following, as a direct adaption of [5, Theorem 1.8],

**Lemma 6.7** There exists a positive \( C = C(n) \) such that for \( u, v, w \in E_1(M, \xi, \omega^T) \), then

\[
I(u, v) \leq C(I(u, w) + I(v, w)).
\]  

(6.29)

**Proof** With Lemma 6.6, we only need to argue (6.29) holds for bounded potentials, with \( u, v, w \) replaced by canonical cutoffs \( u_k, v_k, w_k \). The proof follows exactly as in [5, Theorem 1.8, Lemma 1.9], and we include the proof for completeness. For \( u, v, \psi \in \text{PSH}(M, \xi, \omega^T) \cap L^\infty \), set

\[
\|d(u - v)\|_{\psi} := \left( \int_M d(u - v) \wedge d\psi(u - v) \wedge \omega^{n-1} \wedge \eta \right)^{\frac{1}{2}}.
\]

Using (6.24), it is straightforward to see that

\[
\|d(u - v)\|_{\phi}^{2} \leq I(u, v) \leq 2^{n-1} \|d(u - v)\|_{\phi}^{2}.
\]  

(6.30)

We need the following, there exists a constant \( C = C(n) \) for \( u, v, \psi \in \text{PSHM}, \xi, \omega^T \cap L^\infty \), we have the following (see [5, Lemma 1.9]),

\[
\|d(u - v)\|_{\psi}^{2} \leq C I(u, v)^{1/2^{n-1}} \left( I(u, \psi)^{1-1/2^{n-1}} + I(v, \psi)^{1-1/2^{n-1}} \right).
\]  

(6.31)

With (6.31) we prove (6.29). Taking \( \phi = \frac{u + v}{2} \), the triangle inequality gives,

\[
\|d(u - v)\|_{\phi} \leq \|d(u - w)\|_{\phi} + \|d(v - w)\|_{\phi}.
\]
Using (6.30) and (6.31), we have

\[ I(u, v) \leq 2^{n-1} \|d(u - v)\|_\phi^2 \leq C(\|d(u - w)\|_\phi^2 + \|d(v - w)\|_\phi^2) \]
\[ \leq CI(u, w)^{1/2n-1} \left(I(u, \phi)^{1-1/2n-1} + I(w, \phi)^{1-1/2n-1}\right) \]
\[ + CI(v, w)^{1/2n-1} \left(I(v, \phi)^{1-1/2n-1} + I(w, \phi)^{1-1/2n-1}\right). \]

By Proposition 6.8, we have

\[ I(u, \phi) \leq nI(u, v), I(v, \phi) \leq nI(v, u), I(w, \phi) \leq n(I(w, u) + I(w, v)). \]

It follows that

\[ I(u, v) \leq C \left(I(u, w)^{1/2n-1} + I(v, w)^{1/2n-1}\right) \left(I(u, \phi)^{1-1/2n-1} + I(w, \phi)^{1-1/2n-1}\right). \]

We assume \( I(u, v) \geq \max\{I(u, w), I(v, w)\} \) (otherwise we are done). Hence it follows

\[ I(u, v)^{1/2n-1} \leq C \left(I(u, w)^{1/2n-1} + I(v, w)^{1/2n-1}\right). \]

This is sufficient to prove that

\[ I(u, v) \leq C(I(u, w) + I(v, w)). \]

Now we establish (6.31) (see [5, Lemma 1.9]). First observe that

\[ \|d(u - v)\|_\psi \leq \|d(u - \psi)\|_\psi + \|d(v - \psi)\|_\psi \leq I(u, \psi)^{1/2} + I(v, \psi)^{1/2}. \]

Hence we have

\[ \|d(u - v)\|_\psi^2 \leq 2(I(u, \psi) + I(v, \psi)). \]

Hence if \( I(u, v) \geq I(u, \psi) + I(v, \psi) \), clearly we have

\[ \|d(u - v)\|_\psi^2 \leq 2(I(u, \psi) + I(v, \psi)) \]
\[ \leq CI(u, v)^{1/2n-1} \left(I(u, \psi)^{1-1/2n-1} + I(v, \psi)^{1-1/2n-1}\right). \quad (6.32) \]

Now we suppose \( I(u, v) \leq I(u, \psi) + I(v, \psi) \). Taking \( \phi = \frac{u + v}{2} \), we consider

\[ b_p := \int_M d(u - v) \wedge d_B(u - v) \wedge \omega_\psi^p \wedge \omega_{\phi}^{n-p-1} \wedge \eta. \]
By \((6.30)\), \(b_0 \leq I(u, v)\) and \(b_{n - 1} = \|d(u - v)\|_\psi^2\). We claim that, \(p = 0, \ldots, n - 2\),

\[
b_{p+1} \leq b_p + 4\sqrt{b_p I(\psi, \phi)}.
\] (6.33)

We compute

\[
b_{p+1} - b_p = \int_M d(u - v) \wedge d_B^c(u - v) \wedge dd_B^c(\psi - \phi) \omega_p^\psi \wedge \omega_{\psi}^{n-p-2} \wedge \eta
\]

\[
= -\int_M d(u - v) \wedge dd_B^c(u - v) \wedge d_B^c(\psi - \phi) \omega_p^\psi \wedge \omega_{\psi}^{n-p-2} \wedge \eta
\]

\[
= -\int_M d(u - v) \wedge (\omega_u - \omega_v) \wedge d_B^c(\psi - \phi) \omega_p^\psi \wedge \omega_{\psi}^{n-p-2} \wedge \eta.
\]

Using Cauchy–Schwarz inequality, we compute

\[
\left| \int_M d(u - v) \wedge \omega_u \wedge d(\psi - \phi) \omega_p^\psi \wedge \omega_{\psi}^{n-p-2} \wedge \eta \right|
\]

\[
\leq \left( \int_M d(u - v) \wedge d_B^c(u - v) \wedge \omega_u \wedge \omega_p^\psi \wedge \omega_{\psi}^{n-p-2} \wedge \eta \right)^{1/2}
\]

\[
\times \left( \int_M d(\psi - \phi) \wedge d_B^c(\psi - \phi) \wedge \omega_u \wedge \omega_p^\psi \wedge \omega_{\psi}^{n-p-2} \wedge \eta \right)^{1/2}
\]

\[
\leq 2\sqrt{b_p I(\psi, \phi)},
\]

where we have used that \(\omega_u \leq 2\omega_p\) and \((6.24)\). We can get the same estimate for

\[
\left| \int_M d(u - v) \wedge \omega_v \wedge d(\psi - \phi) \omega_p^\psi \wedge \omega_{\psi}^{n-p-2} \wedge \eta \right|.
\]

This establishes \((6.33)\). By Proposition 6.8, we know that

\[
I(\psi, \phi) \leq (n + 1)J(\psi, \phi) \leq \frac{n}{2}(I(\psi, u) + I(\psi, v)).
\]

Denote \(a = (I(\psi, u) + I(\psi, v))\). We write \((6.33)\) as

\[
b_{p+1} \leq b_p + 4\sqrt{b_p a}, \quad p = 0, \ldots, n - 2.
\]

Note that \(b_0 = I(u, v) \leq a\), hence it is evident that \(b_p \leq Ca\). Hence it follows that, for \(p = 0, \ldots, n - 2\),

\[
b_{p+1} \leq C\sqrt{b_p a}.
\]

A direct computation gives that,

\[
b_{n-1} \leq Cb_0^{1/2^{n-1}}a^{1 - \frac{1}{2^{n-1}}}.
\]
This completes the proof. □

More generally, we have the following [30, Proposition 3.48]

**Proposition 6.9** Suppose $C > 0$ and $\phi, \psi, u, v \in \mathcal{E}_1(M, \xi, \omega^T)$ satisfies

\[-C \leq I(\phi), I(\psi), I(u), \sup M \psi, \sup M u, \sup M v \leq C.\]

Then there exists a continuous function $f_C : \mathbb{R}^+ \to \mathbb{R}^+$ depending only on $C$ with $f_C(0) = 0$ such that

\[
\left| \int_M \phi (\omega^n_u - \omega^n_v) \wedge \eta \right| \leq f_C(I(u, v)) \\
\left| \int_M (u - v)(\omega^n_\psi - \omega^n_\phi) \wedge \eta \right| \leq f_C(I(u, v)).
\] (6.34)

**Proof** The proof is similar in philosophy as Lemma 6.7 and follows almost identically as in Kähler setting, see [30, Proposition 3.48]. Hence we skip the details. □

As a consequence, we have the following [30, Theorem 3.46]:

**Theorem 6.5** Suppose $u_k, u \in \mathcal{E}_1(M, \xi, \omega^T)$. Then the following holds:

1. $d_1(u_k, u) \to 0$ if and only if $\int_M |u_k - u| \omega^n_u \wedge \eta \to 0$ and $\mathbb{I}(u_k) \to \mathbb{I}(u)$.
2. If $d_1(u_k, u) \to 0$, then $\omega^n_{u_k} \wedge \eta \to \omega^n_u \wedge \eta$ weakly and $\int_M |u - u_k| \omega^n_u \wedge \eta \to 0$ for $v \in \mathcal{E}_1(M, \xi, \omega^T)$.

**Proof** If $d_1(u_k, u) \to 0$, then Propositions 6.6 and 6.9 imply (1) and (2). For the reverse direction in (1), it follows almost identically as in Kähler setting, see [30, Proposition 3.52], using Proposition 6.9 and approximation argument. We sketch the process. First we have

\[
\int_M u_k \omega^n_u \wedge \eta \to \int_M u \omega^n_u \wedge \eta.
\]

And then one argues that

\[
I(u, u_k) \leq (n + 1) \left( \mathbb{I}(u_k) - \mathbb{I}(u) - \int_M (u - u_k) \omega^n_u \wedge \eta \right)
\]

Hence this shows that $I(u, u_k) \to 0$. Using Proposition 6.9 and Lemma 6.6, one can then show

\[
\int_M |u_k - u| \omega^n_u \wedge \eta, \int_M |u_k - u| \omega^n_{u_k} \wedge \eta \to 0, k \to \infty.
\]

This gives the desired convergence $d_1(u_k, u) \to 0$. □
As an application of results established above, we have the following compactness result in Sasaki setting, following [30, Theorem 4.45].

**Theorem 6.6** Let \( u_j \in \mathcal{E}_1(M, \xi, \omega^T) \) be a \( d_1 \)-bounded sequence for which the entropy

\[
\sup_j H(u_j) < \infty.
\]

Then \( \{u_j\} \) contains a \( d_1 \)-convergence sequence.

**Proof** We sketch the proof for completeness; for details see [30, Theorem 4.45]. First \( d_1 \) bounded implies that \( I \) and \( \sup u \) are both bounded. Together with Proposition 3.4, this implies that \( d_1 \) bounded set is precompact in \( L^1 \). That is, there exists \( u \in \mathcal{E}_1(M, \xi, \omega^T) \) such that after passing by subsequence,

\[
\int_M |u_k - u| (\omega^T)^n \wedge \eta \to 0.
\]

Moreover, we have (see [30, Proposition 4.14, Corollary 4.15])

\[
\limsup I(u_k) \leq I(u).
\]

Since all elements in \( \mathcal{E}_1(M, \xi, \omega^T) \) have zero Lelong number, we apply Zeriahi’s uniform version of the famous Skoda integrability theorem [59] (we apply Zeriahi’s theorem in each foliation chart) to obtain: for any \( p \geq 1 \), there exists \( C = C(p) \) such that

\[
\int_M e^{-pu_j}(\omega^T)^n \wedge \eta \leq C.
\]

Since \( \sup u_j \leq C \), we have

\[
\int_M e^{pu_j}(\omega^T)^n \wedge \eta \leq C.
\]

Now we need to use the assumption that \( H(u_j) \) is uniformly bounded above. We proceed as in the proof of [30, Theorem 4.45] to conclude

\[
\int_M |u_j - u| \omega^n_{u_j} \wedge \eta \to 0.
\]

By Proposition 6.26 (which holds for \( \mathcal{E}_1 \)), we can then conclude that \( \liminf I(u_j) \geq I(u) \). This gives \( I(u_j) = I(u) \). Hence \( d_1(u_j, u) \to 0 \), as a consequence of Theorem 6.5.

Finally we have the extension of \( \mathcal{K} \)-energy, see [7, Theorem 1.2] for Kähler setting.
Theorem 6.7 The $K$-energy can be extended to a functional $K : E^1 → \mathbb{R} \cup \{+∞\}$. Such a $K$-energy in $E^1$ is the greatest $d_1$-lsc extension of $K$-energy on $\mathcal{H}$. Moreover, $K$-energy is convex along the finite-energy geodesics of $E^1$.

Proof As in Kähler setting [19], we can write the $K$-energy as the following:

$$K(φ) = H(φ) + \mathbb{J}_{ω^T, − Ric}(φ),$$

where $H(φ)$ is the entropy part and $\mathbb{J}$ is the entropy part, taking the formula, respectively,

$$H(φ) = \int_M \log \frac{ω^n}{ω^n T} ∧ η d v_φ,$$

$$\mathbb{J}_{− Ric}(φ) = \frac{n R}{(n + 1)!} \int_M φ \sum_{k=0}^{n} ω^k T ∧ ω^{n−k} φ ∧ η − \frac{1}{n!} \int_M φ \sum_{k=0}^{n−1} Ric ∧ ω^k T ∧ ω^{n−1−k} φ ∧ η.$$

As a direct consequence of this formula, $K(φ)$ is well defined for $φ ∈ H/Δ^1$. More importantly, for $φ_0, φ_1 ∈ H$, and $φ_t ∈ H/Δ^1$ being the geodesic connecting $φ_0, φ_1$, $K(φ_t)$ is convex with respect to $t ∈ [0, 1]$.

Now we extend $H(φ)$ and $\mathbb{J}_{− Ric}$ to $E^1$ separately. As in [7], the extension of $\mathbb{J}_{− Ric}$ to $E^1$ is $d_1$-continuous, while since $d_1(uk, u) → 0$ implies that $ω^n_{uk} ∧ η → ω^n φ ∧ η$ weakly (Theorem 6.5), this implies that the extension of $φ → H(φ)$ to $E^1$ is $d_1$ lsc. Moreover, by [49, Lemma 5.4], the extension of $K$ is the greatest lsc extension. In the end, the convexity of the extended $K$-energy along the finite-energy geodesic segments follows exactly as in [7, Theorem 4.7].

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