REALIZING HOMOLOGY CLASSES UP TO COBORDISM

MARK GRANT, ANDRÁS SZÜCS, AND TAMÁS TERPAI

Abstract. It is known that neither immersions nor maps with a fixed finite set of multisingularities are enough to realize all mod 2 homology classes in manifolds. In this paper we define the notion of realizing a homology class up to cobordism; it is shown that for realization in this weaker sense immersions are sufficient, but maps with a fixed finite set of multisingularities are still insufficient.

1. Introduction

In 1949 Steenrod [4] posed the following question: given a homology class \( h \) of a space \( X \), does there exist a closed manifold \( V \) and a continuous map \( f : V \to X \) such that \( f_*[V] = h \), where \([V]\) is the fundamental class of \( V \)? Thom’s famous result answers the question affirmatively if \( h \) is a \( \mathbb{Z}_2 \)-homology class, and shows that for integral homology the answer in general is negative. It is a natural further question whether \( f \) can be chosen to be “nice” if \( X \) itself is a smooth manifold. For example, can it be always an embedding or an immersion? If not, then can \( f \) be chosen to have only mild singularities?

For embeddings Thom himself gave some necessary and sufficient conditions. From these conditions it is not hard to deduce that there are \( \mathbb{Z}_2 \)-homology classes of codimension 2 not realizable by embeddings in some manifolds.

In [5] it was shown that for any \( k > 1 \) there is a manifold \( M \) (of dimension approximately \( 4k \)) and a cohomology class \( \alpha \in H^k(M; \mathbb{Z}_2) \) such that the Poincaré dual of \( \alpha \) cannot be realized by an immersion. Moreover it was shown there that for any \( k > 1 \) singular maps of finite complexity (see Section 3 for the precise definition) are insufficient to realize all codimension \( k \) homology classes in manifolds.

Therefore in order to obtain positive answers it is natural to relax the notion of “realization of a homology class”. The relaxed version we use will be “realization up to cobordism”. For this purpose we define the cobordism group of pairs \((M^n, \alpha)\) where \( M^n \) is a closed smooth \( n \)-manifold and \( \alpha \in H^k(M; \mathbb{Z}_2) \) for a fixed \( k \).

**Definition:** Given two pairs \((M^n, \alpha)\) and \((N^n, \beta)\) we say that they are cobordant if there is a pair \((W^{n+1}, \gamma)\) such that \( W^{n+1} \) is a compact \((n + 1)\)-manifold with boundary \( \partial W = M \sqcup N \) and \( \gamma \in H^k(W; \mathbb{Z}_2) \) is a cohomology class such that \( \gamma|_M = \alpha \) and \( \gamma|_N = \beta \).

**Remark:** The obtained group of pairs is clearly isomorphic to \( \mathbb{M}_n(K(\mathbb{Z}_2, k)) \), the \( n \)th bordism group of the Eilenberg-MacLane space \( K(\mathbb{Z}, k) \).

**Definition:** Let \( \mathcal{F} \) be a class of smooth maps (for example, embeddings, immersions, or singular maps of some given complexity). We say that a pair \((M, \alpha)\)
Realization by immersions

**Theorem 1.** Any pair \((M, \alpha)\) is realizable by immersions up to cobordism.

For conciseness, (co)homology coefficients \(\mathbb{Z}_2\) will be omitted and \(K\) will stand for \(K(\mathbb{Z}_2, k)\).

In what follows, \(MO(k)\) denotes as usual the Thom space of the universal vector bundle over \(BO(k)\), and for any space \(X\) we denote by \(\Gamma X\) the space \(\Omega^\infty S^\infty X = \lim_{N \to \infty} \Omega^N S^N X\). Recall that \(\Gamma MO(k)\) is the classifying space of codimension \(k\) immersions, in particular, the group of cobordism classes of codimension \(k > 0\) immersions into a fixed closed manifold \(P\) (where cobordisms are codimension \(k\) immersions into \(P \times [0,1]\)) is isomorphic to the group of homotopy classes \([P, \Gamma MO(k)]\).

It is well-known that \(\Gamma MO(k)\) is stably equivalent to a bouquet that contains \(MO(k)\) (i.e. there is a space \(Y\) such that \(\Gamma MO(k) \cong MO(k) \vee Y\)). Hence \(H^*(MO(k))\) embeds naturally into \(H^*(\Gamma MO(k))\). In particular the Thom class \(u_k \in H^k(MO(k))\) can be considered (uniquely, since \(Y\) is known to be \(2k-1\)-connected) as a cohomology class of \(\Gamma MO(k)\). Denote by \(u\) the corresponding map into \(K\), that is, \(u : \Gamma MO(k) \to K\) has the property that \(u^*(u_k) = u_k\), where \(u_k \in H^k(K)\) is the fundamental class.

Alternatively, we may use the universal property of the functor \(\Gamma\) that is as follows ([2, p. 39.], [6, pp. 42–43]): for any map \(f : X \to Y\) from a compactly generated Hausdorff space \(X\) to an infinite loop space \(Y\) there is a homotopically unique extension \(f : \Gamma X \to Y\) that is an infinite loop map. Applying this property to \(u_k\) yields the map \(u\).

For any \(P\) the map \(u^P : [P, \Gamma MO(k)] \to [P, K]\) induced by \(u\) associates to (a cobordism class of) an immersion the Poincaré dual of the homology class represented by the immersion.

This shows that Theorem 1 has the following equivalent reformulation:

**Theorem 1’.** The map \(u : \Gamma MO(k) \to K\) induces an epimorphism of the bordism groups in any dimension. That is, for any \(n\)

\[
\mu_n : \mathfrak{B}_n(\Gamma MO(k)) \to \mathfrak{B}_n(K)
\]

is onto.

**Proof:** It is well-known ([4]) that there is an isomorphism \(H_*(X; \mathbb{Z}_2) \otimes \mathfrak{B}_* \to \mathfrak{B}_*(X)\), natural in \(X\), defined by taking a representative \(\{\hat{\alpha} : M_\alpha \to X\} \in \mathfrak{B}_*(X)\) for all elements \(\alpha\) of a basis of \(H_*(X)\) and mapping \(\sum_j \alpha_j \otimes [N_j] \to \sum_j (\hat{\alpha}_j \circ pr_j : M_{\alpha_j} \times N_j \to X)\), where \(pr_j : M_{\alpha_j} \times N_j \to M_{\alpha_j}\) is the projection. Hence a map induces epimorphism of the (unoriented) bordism groups if and only if it does so in the \(\mathbb{Z}_2\)-homology groups.

For later use, recall that for any space \(X\) the ring \(H_*(\Gamma X)\) is a polynomial ring (multiplication being the Pontryagin product) in variables \(x_\lambda, y_{l,\lambda}\), where \(\{x_\lambda\}_\lambda\) is
a homogeneous basis of $H_\ast(X)$ and $y_{1,\lambda}$ are further variables defined using Kudo-Araki operations as $y_{1,\lambda} = Q^I x_\lambda$ (their precise description will be unimportant in our argument).

In order to show that

$$u_\ast : H_\ast(\Gamma MO(k)) \to H_\ast(K)$$

is onto it is enough to show that the composition

$$\varphi : \overline{H}_\ast(MO(k)) \xrightarrow{(u_\ast)_\ast} \overline{H}_\ast(K) \xrightarrow{p} Q(H_\ast(K)) = \overline{H}_\ast(K)/\mu (\overline{H}_\ast(K) \otimes \overline{H}_\ast(K))$$

is onto, where $\mu : H_\ast(K) \otimes H_\ast(K) \to H_\ast(K)$ is the multiplication map and $p$ is the natural projection onto the quotient group of indecomposables. Indeed, assume that $\varphi$ is onto and for all $j$ choose elements in $H_j(K)$ such that they form a (linear) basis in $\overline{H}_j(K)/\mu (\overline{H}_j(K) \otimes \overline{H}_j(K))$. It is easy to see by induction on $j$ that the chosen elements generate $\overline{H}_\ast(K)$ multiplicatively and hence the subring of $H_\ast(\Gamma MO(k))$ generated by the preimages of these elements is mapped onto the entire $H_\ast(K)$ (here we use that $u_\ast$ is a ring homomorphism, since $u$ is an infinite loop map).

Hence to prove Theorem 1 we have to show that $\varphi : H_\ast(MO(k)) \to QH_\ast(K)$ is onto. This is equivalent to the dual homomorphism $\varphi^\ast$ being injective. By [7, Proposition 3.10], the dual of $QH_\ast(K)$ is $PH^\ast(K)$, the submodule of primitive elements of the Hopf algebra $H^\ast(K)$. This latter group is known to be

$$PH^\ast(K) = \mathbb{Z}_2 \left\langle Sq^I \iota_k : I \text{ admissible of excess } e(I) \leq k \right\rangle,$$

the vector space over $\mathbb{Z}_2$ freely generated by the $Sq^I \iota_k$ (see eg. [1, p. 23.]). The dual of $H_\ast(MO(k))$ is $H^\ast(MO(k))$ and can be identified with the ideal generated by $w_k$ in $\mathbb{Z}_2[w_1, \ldots, w_k]$ ($w_k$ corresponds to the Thom class $u_k$). The map $\varphi^\ast$ maps $\iota_k$ to $w_k$ and then to $w_k$, and commutes with the action of the Steenrod algebra, allowing to calculate the image of $\varphi^\ast$.

Finally, we need to show that the set $\{ Sq^I (w_k) : I \text{ admissible with } e(I) \leq k \}$ is linearly independent in the ideal $(w_k) \subset \mathbb{Z}_2[w_1, \ldots, w_k]$. This is the immediate consequence of [8, Remark 2.4] that shows that the Steenrod algebra acts freely unstably on $w_k$ in $H^\ast(BO(k))$, and this finishes the proof of Theorem 1.\hfill $\Box$

3. Non-realizability up to cobordism by singular maps of finite complexity

Recall some definitions from singularity theory that are necessary for the formulation of Theorem [2]

DEFINITION: Fix a natural number $k \geq 1$ and consider equivalence classes of germs $\eta : (\mathbb{R}^{n-k}, 0) \to (\mathbb{R}^n, 0)$, $n \geq k$, up to left-right equivalence and stabilization, that is, we consider $\eta$ to be equivalent to $\eta \times id_{\mathbb{R}} : (\mathbb{R}^{n-k+1}, 0) \to (\mathbb{R}^{n+1}, 0)$. An equivalence class is called a (codimension $k$) local singularity (even if its rank is maximal).

DEFINITION: A multisingularity is a finite multiset (set with elements equipped with multiplicities) of local singularities.

DEFINITION: Let $f : M \to N$ be a smooth map such that for any $y \in N$ the preimage $f^{-1}(y)$ is a finite set. For $y \in N$ and $f^{-1}(y) = \{x_1, \ldots, x_m\}$ let $[f_{x_j}]$ denote the local singularity class of the germ $f$ at $x_j$. The multiset $\{[f_{x_1}], \ldots, [f_{x_m}]\}$ is called the multisingularity of $f$ at $y$.\hfill
Definition: Let \( \tau \) be a set of multisingularities. The map \( f \) is said to be a \( \tau \)-map if its multisingularity at any point \( y \in N \) belongs to \( \tau \).

Theorem 2. Let \( \tau \) be any finite set of multisingularities of codimension \( k > 1 \) stable maps and let \( \mathcal{F} \) be the class of \( \tau \)-maps. Then the class \( \mathcal{F} \) is insufficient for realizing up to cobordism all codimension \( k \) homology classes in manifolds. That is, for any \( k > 1 \) there is a pair \((M, \alpha)\) with \( M \) a smooth manifold and \( \alpha \in H^k(M) \) such that \((M, \alpha)\) is not \( \mathcal{F} \)-realizable up to cobordism.

Proof: The proof given in [5, Theorem 1.3.] for non-realizability of homologies by \( \tau \)-maps also proves the stronger statement of Theorem 2. In that proof there was a classifying space \( X_\tau \) for \( \tau \)-maps (analogously to \( \Gamma MO(k) \) being the classifying space for immersions). \( X_\tau \) has a single nonzero element in its first nontrivial (reduced) cohomology group, \( H^k(X_\tau) \), which can be called the Thom class \( u_\tau : X_\tau \to K \). If any pair \((M, \alpha)\) could be realizable up to cobordism by \( \tau \)-maps, then the map \( u_\tau \) would induce an epimorphism \((u_\tau)_* : \mathcal{N}_*(X_\tau) \to \mathcal{N}_*(K)\) between the unoriented bordism groups or, equivalently, between the homology groups (using the same argument as in the proof of Theorem 1). But [5] shows that for any sufficiently high dimension \( j \) (under the assumption that \( k > 1 \)) we have \( \dim_{\mathbb{Z}_2} H_j(X_\tau) < \dim_{\mathbb{Z}_2} H_j(K) \), hence \((u_\tau)_* : H_j(X_\tau) \to H_j(K)\) cannot be surjective. \( \square \)

Remark: In particular, embeddings or immersions with self-intersection multiplicity bounded by a fixed number are insufficient for realizing all homology classes in manifolds even up to cobordism.

References

[1] A. Clement: Integral Cohomology of Finite Postnikov Towers, Thèse de doctorat, Université de Lausanne, 2002
[2] F.R. Cohen, T.J. Lada, J.P. May: The homology of iterated loop spaces, Lecture Notes in Mathematics 533, Springer, 1976
[3] P.E. Conner, E.E. Floyd: Differentiable periodic maps, Bull. Amer. Math. Soc. 68 (2) (1962), 76–86.
[4] S. Eilenberg: Problems in topology, Ann. Math. 50 (1949), 246–260.
[5] M. Grant, A. Szűcs: On realising homology classes by maps of restricted complexity, Bull. London Math. Soc. 45 (2) (2013), 329–340.
[6] J.P. May: The geometry of iterated loop spaces, Lecture Notes in Mathematics 271, Springer, 1972
[7] J.W. Milnor, J.C. Moore: On the structure of Hopf algebras, Annals of Math. 81 (2) (1965), 211–264.
[8] D.J. Pengelley, F. Williams: Global structure of the mod two symmetric algebra, \( H^*(BO; F_2) \), over the Steenrod Algebra, Alg. Geom. Topol. 3 (2003), 1119–1139.