Nonlocal solitons supported by non-parity-time-symmetric complex potentials

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Abstract

We report on the existence and stability of fundamental and out-of-phase dipole solitons in nonlocal focusing Kerr media supported by one-dimensional non-parity-time (PT)-symmetric complex potentials. These fundamental and dipole solitons bifurcate from different discrete eigenvalues in the linear spectra. Below the phase transition of the non-PT-symmetric complex potentials, these solitons are stable in the low power region. While above the phase transition, they are stable in the moderate power region. The eigenvalues in linear-stability spectra of solitons appear as conjugation pairs \((\delta, \delta^*)\). The transverse power flow and the nonlinear contribution to refractive index are asymmetric functions. Moreover, the degree of nonlocality can also influence the stability of these solitons.

1. Introduction

The concept of parity-time (PT) symmetry comes from quantum mechanics, where a Hamiltonian with a complex potential can also have completely real spectrum [1, 2]. The one-dimensional (1D) PT-symmetric complex potential satisfies the condition: \(V(x) = V^*(-x)\). Here, the superscript * is the complex conjugation and \(x\) is the normalized transverse coordinate, respectively. In the last decades, the studies of the PT symmetry [3–10] and solitons in PT-symmetric potentials [11–23] have attracted much attention.

The non-PT-symmetric complex potentials which follow the relation \(V(x) \neq V^*(-x)\) can also have entirely real spectra [24–28] and support continuous families of solitons [29, 30]. The method that can construct non-PT-symmetric complex potentials with completely real spectra was introduced [27]. Above phase transition, the unique feature of spectra of new classes of non-PT-symmetric complex potentials was also reported [28]. Continuous families of fundamental solitons can be stable in the 1D non-PT-symmetric single-hump complex potential [29]. The reasons why continuous families of solitons can exist in non-PT-symmetric complex potentials were explained [30]. The non-PT-symmetric double-hump complex potential with focusing and defocusing Kerr nonlinearities can also support stable continuous families of solitons. Moreover, the form \(V(x) = g(x) + ig_0(x)\) (This potential was introduced by Wadati [31], the constant-intensity wave solutions and their modulation instability in such potential have also been investigated theoretically [32]) is proved to be the only 1D non-PT-symmetric complex potential to support continuous families of solitons [33]. Stable continuous families of fundamental solitons can also exist in \(\chi^{(2)}\) media with non-PT-symmetric complex potentials [34]. The eigenvalues of linear-stability spectra of solitons in non-PT-symmetric complex potentials with local focusing Kerr nonlinearity appear in quartets \((\lambda, -\lambda, \lambda^*, -\lambda^*)\) [35]. Moreover, below the phase transition of the non-PT-symmetric complex potentials, fundamental solitons are stable in their existence domain and dipole solitons are stable in the low power region. Above the phase transition, fundamental solitons are stable in the high power region but are unstable in the low power region. Recently, vector solitons in 1D non-PT-symmetric complex potentials were reported [36]. We found that the vector solitons with the first component is fundamental mode and the second component is dipole mode can be stable below and above the phase transition.
Fundamental [37–40] and dipole solitons [37] in PT-symmetric optical lattices (periodic potentials) with nonlocal nonlinearity were investigated. However, these nonlocal solitons in non-PT-symmetric complex potentials have not been studied yet. In this article, we demonstrate that continuous families of nonlocal fundamental and dipole solitons can be stable in 1D non-PT-symmetric complex potentials. They are stable in the low power region below the phase transition and are stable in the moderate power region above the phase transition. The eigenvalues in linear-stability spectra of these solitons appear in conjugation pairs \((\delta, \delta^*)\).

Moreover, the shapes of the nonlinear contribution to refractive index and the transverse power flow exhibit the asymmetry and the degree of nonlocality can affect the stability of these nonlocal solitons.

2. Model

In nonlocal focusing Kerr media, the normalized nonlinear Schrödinger-like equations to describe beam propagation in 1D non-PT-symmetric complex potentials are [29, 30, 37, 39]

\[
\begin{align*}
\frac{\partial U}{\partial z} + \frac{\partial^2 U}{\partial x^2} + V(x)U + nU &= 0, \quad (1a) \\
 d \frac{\partial^2 n}{\partial x^2} - n + |U|^2 &= 0. \quad (1b)
\end{align*}
\]

where \(U\) is the complex light field amplitude, \(z\) is the longitudinal coordinate, \(d\) is the degree of nonlocality, and \(n\) is the nonlinear contribution to refractive index. The non-PT-symmetric complex potentials is represented by \(V(x)\) and the form is [27, 35]

\[V(x) = g^2(x) + 2g(x) + ig_x(x).\] (2)

Here \(c\) is a real constant and \(g(x)\) is a real function. We choose

\[g(x) = \tanh^2(x + 2.6) - \tanh(x - 2.6).\] (3)

The linear spectra of non-PT-symmetric complex potentials can be obtained by using the Fourier collocation method [41]. For \(c = 0.25, c = 0,\) and \(c = -0.25,\) the linear spectra of the non-PT-symmetric complex potentials are depicted in figures 1(d)–(f), respectively. The critical threshold is \(c_{th} = -0.2205.\) When \(c \leq c_{th},\) a phase transition will occur and the spectrum becomes partially complex. For \(c > c_{th},\) the spectrum is entirely real.

We assume that the stationary soliton solutions of equation (1) are the form of \(U(x, z) = q(x)e^{i\mu z},\) where \(q\) is a complex function and \(\mu\) is the real propagation constant. By substituting above relation into equation (1), we can get

![Figure 1. (a)–(c) Are the non-PT-symmetric complex potentials for \(c = 0.25, c = 0,\) and \(c = -0.25,\) respectively. The blue and red lines are real and imaginary parts, respectively. (d)–(f) are the corresponding linear spectra.](image-url)
Equation (4) can be numerically solved by a method that is developed from the modified squared-operator iteration method [42]. In addition, equation (4) can also be numerically solved by the spectral renormalization method [43]. We define the power of a soliton as 

\[ P = \int_{-\infty}^{\infty} |q|^2 \, dx. \]  

To confirm the stability of these nonlocal solitons, we perform stability analyses for them. The perturbations \( f(x) \) and \( t(x) \) are added into solitons [16]:

\[ U(x, z) = e^{i\delta z}[q(x) + f(x)e^{i\xi} + t^*e^{-i\xi}], \]

where \( \delta \) is the growth rate and \( |f|, |t| \ll |q| \). By taking equation (5) into equation (1) and linearizing, we can also get coupled eigenvalue equations

\[
\begin{align*}
\delta f &= i \left[ -\mu + \frac{\partial^2}{\partial x^2} + V + n \right] f + \Delta n q,
\delta t &= i \left[ -\mu + \frac{\partial^2}{\partial x^2} - V^* - n \right] t - \Delta n q^*.
\end{align*}
\]

Here \( n = \int_{-\infty}^{\infty} h(x-k)|q(k)|^2 \, dk \), \( h(x) = 1/(2d^{1/2}) \exp (|x|/d^{1/2}) \), and \( \Delta n = \int_{-\infty}^{\infty} h(x-k)|q^*(k)f(k) + q(k)t(k)| \, dk \).
Equation (6) can be solved numerically. If there are complex eigenvalues with $\text{Re}(\delta) > 0$, solitons are linearly unstable; otherwise, solitons are stable.

3. Numerical results

For $c = 0.25$, there are four discrete eigenvalues in the linear spectrum, as shown in figure 1(d). The largest and second largest eigenvalues are $\lambda_1 = 4.4742$ and $\lambda_2 = 3.2437$, respectively. Fundamental solitons can bifurcate from these. These fundamental solitons are stable in the low power region. With an increase of the degree of nonlocality $d$, the stability region shrinks. The stability domains of fundamental solitons for $d = 1$ and $\mu = 6$, the stable propagation of the perturbed soliton (the direct simulations of equation (1a) are added random noises with 5% soliton amplitude) is depicted in figure 2(f). By solving equation (6), the linear-stability spectra of solitons can be obtained. From figure 2(e), all the real parts of the eigenvalues in the linear-stability spectrum of the soliton are less than or equal to 0. It confirms that the soliton is stable. We also introduce the parameter $S = (i/2)(qq^* - q^*q)$ associated with the transverse power flow density [11]. As indicated in figure 2(d), $S$ is negative everywhere, the power flow from gain toward loss regions in one direction (from right to left). As the unstable case, figure 2(g) shows the profiles of the soliton and refractive index for $d = 5$ and $\mu = 6$. The corresponding unstable propagation of the perturbed soliton is shown in figure 2(i). The linear-stability spectrum of the soliton also demonstrates that the soliton is unstable. There are eigenvalues with real part greater than 0, as exhibited in figure 2(h).

The out-of-phase dipole solitons can bifurcate from $\lambda_2$. They are also stable in the low power region. On the contrary, the stability domain of dipole solitons widens with the increase of the degree of nonlocality. The stability domains of dipole solitons for $d = 1$, $d = 3$, and $d = 5$ are $3.244 \leq \mu \leq 3.344$, $3.244 \leq \mu \leq 3.364$, and $3.244 \leq \mu \leq 3.384$, respectively. As a stable example, figure 3(i) depicts the stable propagation of the perturbed...
dipole soliton with \( d = 5 \) and \( \mu = 3.3 \). In local focusing Kerr media, the eigenvalues of linear-stability of solitons in non-PT-symmetric complex potentials appear in quartets [35]. Much different with that, these eigenvalues in linear-stability spectra of solitons in non-PT-symmetric complex potentials with nonlocal focusing Kerr nonlinearity appear as conjugation pairs, which are clearly shown in figures 2(e), (b), 3(e), and (h). In PT-symmetric optical lattices [11, 37–40], the refractive index and the transverse power flow are even functions. However, as exhibited in figures 2(c), (d), 3(c), and (d), these are asymmetric in non-PT-symmetric complex potentials.

For \( \epsilon = 0 \), the linear spectrum is shown in figure 1(c). There are three discrete eigenvalues and the largest and second largest eigenvalues are \( \lambda_1 = 3.4929 \) and \( \lambda_2 = 2.3089 \). The fundamental and out-of-phase dipole solitons also bifurcate from \( \lambda_1 \) and \( \lambda_2 \), respectively. Both the fundamental and dipole solitons are stable in the low power regions. When the degree of nonlocality increases, the stability region of the fundamental solitons shrinks but the stability domain of dipole solitons widens. The existence and stability of the fundamental and dipole solitons are similar to \( \epsilon = 0.25 \).

Above phase transition, fundamental solitons are unstable in the low power region but are stable in the high power region in non-PT-symmetric complex potentials with local focusing Kerr nonlinearity [35]. However, nonlocal fundamental solitons are stable in the moderate power region but are unstable in the low and high power regions in non-PT-symmetric complex potentials. For \( \epsilon = -0.25 \), there are two discrete eigenvalues \( \lambda_1 = 2.5118 \) and \( \lambda_2 = 1.3755 \) in the linear spectrum, as shown in figure 1(f). Nonlocal fundamental solitons can also bifurcate from \( \lambda_1 \). The stability domain also shrinks with the increase of the degree of nonlocality. The three stability regions for \( d = 1 \), \( d = 3 \), and \( d = 5 \) are \( 2.969 \leq \mu \leq 4.180 \), \( 2.991 \leq \mu \leq 3.598 \), and \( 2.998 \leq \mu \leq 3.185 \), respectively. As two stable cases for above phase transition (\( \epsilon = -0.25 \)), figures 4(f) and (i) depict the stable propagations of the two perturbed solitons for \( d = 3 (\mu = 3.25) \) and \( d = 5 (\mu = 3.05) \), respectively.

For \( \epsilon = -0.25 \), nonlocal dipole solitons cannot be stable with \( d = 1 \), \( d = 3 \), and \( d = 5 \). When increase \( \epsilon \) to \( -0.221 \) (still above phase transition), dipole solitons can also stable in the moderate power region. There are also two discrete eigenvalues in the linear spectrum of the non-PT-symmetric complex potentials with \( \epsilon = -0.221 \).
They are $\lambda_1 = 2.6256$ and $\lambda_2 = 1.4839$. Nonlocal dipole solitons can also bifurcate from $\lambda_2$. For $d = 1, d = 3$, and $d = 5$, the stability domains are $1.486 \leq \mu \leq 1.511$, $1.487 \leq \mu \leq 1.570$, and $1.487 \leq \mu \leq 1.585$, respectively. In the three stability regions, figures 5(c), (f) and (i) show the stable propagations of three perturbed dipole solitons.

This non-PT-symmetric complex potentials (equations (2) and (3)) cannot support continuous families of solitons in the nonlocal PT-symmetric nonlinear Schrödinger equation (NLSE) [44]. Whether there are continuous families of solitons in this nonlocal PT-symmetric NLSE with other non-PT-symmetric complex potentials or not needs further study.

For a PT-symmetric single-hump potential, the profiles of the fundamental and out-of-phase dipole solitons, the refractive index, and the transverse power flow are all symmetric. However, they are all asymmetric in this non-PT-symmetric complex potential. Moreover, the complex eigenvalues in linear-stability spectra of solitons in a PT-symmetric single-hump potential cannot appear as conjugation pairs $(\delta,\delta^*)$. This article shows.

### 4. Conclusions

In conclusion, we have investigated the existence and stability of nonlocal fundamental and out-of-phase dipole solitons in 1D non-PT-symmetric complex potentials. Below the phase transition, fundamental and dipole solitons are stable in the low power region. While above the phase transition, these solitons are stable in the moderate power region. The eigenvalues in the linear-stability spectra of these solitons appear in conjugation pairs $(\delta,\delta^*)$. The nonlinear contribution to refractive index and transverse power flow show asymmetry in the non-PT-symmetric complex potentials. Moreover, the degree of nonlocality can also affect the stability of these solitons.
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