A viral propagation model with nonlinear infection rate and free boundaries

Lei Li\textsuperscript{a}, Siyu Liu\textsuperscript{b,c}, Mingxin Wang\textsuperscript{a,2}

\textsuperscript{a} School of Mathematics, Harbin Institute of Technology, Harbin 150001, PR China.
\textsuperscript{b} School of Science and Engineering, The Chinese University of Hong Kong, Shenzhen 518172, China.
\textsuperscript{c} School of Mathematical Sciences, University of Science and Technology of China, Hefei 230026, China.

Abstract

In this paper we put forward a viral propagation model with nonlinear infection rate and free boundaries and investigate the dynamical properties. This model is composed of two ordinary differential equations and one partial differential equation, in which the spatial range of the first equation is the whole space $\mathbb{R}$, and the last two equations have free boundaries. As a new mathematical model, we prove the existence, uniqueness and uniform estimates of global solution, and provide the criteria for spreading and vanishing, and long time behavior of the solution components $u, v, w$. Comparing with the corresponding ordinary differential systems, the Basic Reproduction Number $R_0$ plays a different role. We find that when $R_0 \leq 1$, the virus cannot spread successfully; when $R_0 > 1$, the successful spread of virus depends on the initial value and varying parameters.

Keywords: viral propagation model, free boundaries, basic reproduction number, spreading-vanishing, long time behavior.

AMS Subject Classification (2100): 35K57, 35B40, 35R35, 92D30

1 Introduction

Background In order to clarify the pathogenesis of diseases and seek effective treatment measures, viral dynamics have been a hot research topic (cf. \cite{1,2}), which usually cannot be answered by biological experimental methods alone but require the help of mathematical models. For this reason, a simple model was introduced few decades ago by Nowak and Bangham \cite{3}. See also Nowak and May \cite{4}. The basic model of viral dynamics is the following set of differential equations

\begin{equation}
\begin{aligned}
    u' &= \theta - au - bw, \\
    v' &= bw - cv, \\
    w' &= kv - qw,
\end{aligned}
\end{equation}

where $u, v$ and $w$ represent the population of uninfected cells, infected cells and viruses, respectively; uninfected cells are produced at a constant rate $\theta$ and with death rate $au$; $cv$ is the death rate of infected cells; virus particles $w$ infect uninfected cells with rate $bw$, and meanwhile virus particles

\footnotesize
\textsuperscript{1}This work was supported by NSFC Grants 11771110, 11971128
\textsuperscript{2}Corresponding author. E-mail: mxwang@hit.edu.cn
are produced by infected cells with rate \( kv \) and have death rate \( qw \). It has been shown that if the Basic Reproduction Number \( R_0 = \theta \frac{kb}{acq} \) \(<\ 1\), then the system returns to the uninfected state \((\theta/a, 0, 0)\). If \( R_0 \> 1 \), then the system will converge to the unique positive equilibrium state \((\frac{qc}{kb}, \frac{\theta}{a} - \frac{qa}{kb}, \frac{\theta}{q} - \frac{a}{b})\). This indicates that in the initial stage of infection, if each infected cell infects less than one cell on average, then the infection cannot spread; if each infected cell infects \( R_0 \> 1 \) cells on average, the number of infected cells increases and the number of uninfected cells declines.

Mathematical Model. To investigate the impact of spatial dynamics on this model, Stancevic et al. \[5\] extended this model to include spatially random diffusion and spatially directed chemotaxis. Invoked by their ideas, we give the basic model assumptions as follows:

(i) A nonlinear infection rate can happen due to saturation at high virus concentration, where the infectious fraction is so high that exposure is very likely. Moreover, with the increase of the virus concentration the living environment for cells becomes worse and worse. Thus, it is reasonable for us to assume that the rate of infection for virus and the virion production rate for infected cells are both nonlinear. Here we use

\[
\begin{align*}
  f_1(u, w) &= \theta - au - \frac{buw}{1 + w}, \\
  f_2(u, v, w) &= \frac{buw}{1 + w} - cv, \\
  f_3(v, w) &= \frac{kv}{1 + w} - qw
\end{align*}
\]

instead of the three terms in the right hand side of (1.1).

(ii) We assume that the major spatial dispersal comes from the moving (diffusion) of viruses in vivo, while both the uninfected and infected cells are immobile (do not diffuse). So we add only a diffusion term to the differential equation of viruses;

(iii) Since the infected cells are caused by viruses, their distribution range is the same;

(iv) The distribution of viruses and infected cells is a local range, which is small relative to the distribution of uninfected cells, so we think that uninfected cells are distributed over the whole space. Such kind of assumptions have been used in the species invasion models (cf. \[6, 7, 8\] for example);

(v) Initially, viruses are distributed over a local range \( \Omega_0 \) (the initial habitat). They will spread from boundary to expand their habitat as a result of the spatial dispersal freely. That is, as time \( t \) increases, \( \Omega_0 \) will evolve into expanding region \( \Omega(t) \) with expanding front \( \partial \Omega(t) \). Initial function \( w_0(x) \), and as a result \( v_0(x) \), will evolve into positive functions \( w(t, x) \) and \( v(t, x) \) which vanish on the moving boundary \( \partial \Omega(t) \);

(vi) For simplicity, we restrict our problem to the one dimensional case. Based on the deduction of free boundary conditions given in \[9\], we have the following free boundary conditions

\[
\begin{align*}
  g'(t) &= -\mu w_x(t, g(t)), \\
  h'(t) &= -\beta w_x(t, h(t)).
\end{align*}
\]

All of these assumptions (i)-(vi) suggest the following model, which governs the spatial and
temporal evolution of viruses and cells, as well as free boundaries:

\[
\begin{align*}
    u_t &= f_1(u, w), & t > 0, & -\infty < x < \infty, \\
    v_t &= f_2(u, v, w), & t > 0, & g(t) < x < h(t), \\
    w_t - dw_{xx} &= f_3(v, w), & t > 0, & g(t) < x < h(t), \\
    v(t, x) &= w(t, x) = 0, & t > 0, & x \notin (g(t), h(t)), \\
    g'(t) &= -\mu w_x(t, g(t)), & h'(t) &= -\beta w_x(t, h(t)), & t \geq 0, \\
    u(0, x) &= u_0(x), & -\infty < x < \infty, \\
    v(0, x) &= v_0(x), & w(0, x) &= w_0(x), & -h_0 \leq x \leq h_0, \\
    h(0) &= -g(0) = h_0,
\end{align*}
\]  

(1.2)

where \( x = g(t) \) and \( x = h(t) \) are the moving boundaries to be determined together with \( u(t, x), v(t, x) \) and \( w(t, x) \); \( d, \theta, a, b, c, k, q, \mu, \beta, h_0 \) are positive constants.

Denote by \( C^{1-}(I) \) the space of Lipschitz continuous functions in \( I \). We assume that the initial functions \( u_0, v_0, w_0 \) satisfy

\[
\begin{align*}
    u_0 &\in C^{1-}(\mathbb{R}) \cap L^{\infty}(\mathbb{R}), & v_0 &\in C^{1-}([-h_0, h_0]), & w_0 &\in W^2_p((-h_0, h_0)), \\
    v_0(\pm h_0) &= w_0(\pm h_0) = 0, & w'(-h_0) &> 0, & w'(h_0) &< 0, \\
    u_0 > 0 &\text{ in } \mathbb{R}, & v_0, w_0 &> 0 \text{ in } (-h_0, h_0)
\end{align*}
\]

(1.3)

with \( p > 3 \). Denote by \( L_0 \) and \( L_* \) the Lipschitz constant of \( u_0 \) and \( v_0 \), respectively.

Partially degenerate reaction-diffusion systems, which mean that several diffusion coefficients are zeros, have been increasingly applied to epidemiology, population biology etc; see \[10\ \[11\], for example. Some researchers have introduced the Stefan type free boundary to the partially degenerate systems, please refer to \[12\ \[13\ \[14\ \[15\] and the references therein.

Aims and Main Results  This paper concerns with the dynamics of \[12\]. The global existence, uniqueness, regularity and uniform estimates in time \( t \) of solution are first studied. Then a spreading-vanishing dichotomy is established, i.e., either

(i) **Spreading** (virus persistence): the virus successfully infects the uninfected cells and spreads itself to the uninfected area in the sense that \( \lim h(t) = -\lim g(t) = \infty \), and

\[
\limsup_{t \to \infty} \|v(t, \cdot)\|_{C([g(t), h(t)])} > 0, \quad \limsup_{t \to \infty} \|w(t, \cdot)\|_{C([g(t), h(t)])} > 0.
\]

In addition, if \( \mathcal{R}_0 + \sqrt{\mathcal{R}_0} > b/a \), where \( \mathcal{R}_0 = \theta kb/(acq) \), then

\[
\begin{align*}
    u_\infty \leq \liminf_{t \to \infty} u(t, x) \leq \limsup_{t \to \infty} u(t, x) \leq \bar{u}_\infty, \\
    v_\infty \leq \liminf_{t \to \infty} v(t, x) \leq \limsup_{t \to \infty} v(t, x) \leq \bar{v}_\infty, \\
    w_\infty \leq \liminf_{t \to \infty} w(t, x) \leq \limsup_{t \to \infty} w(t, x) \leq \bar{w}_\infty
\end{align*}
\]

locally uniformly in \( \mathbb{R} \) for some positive constants \( u_\infty, \bar{u}_\infty, v_\infty, \bar{v}_\infty, w_\infty \) and \( \bar{w}_\infty \). Particularly, under a stronger assumption that \( b \leq 2a \), we will derive

\[
\begin{align*}
    \lim_{t \to \infty} u(t, x) &= u^*, \quad \lim_{t \to \infty} v(t, x) = v^*, \quad \lim_{t \to \infty} w(t, x) = w^* \text{ locally uniformly in } \mathbb{R},
\end{align*}
\]
where \((u^*, v^*, w^*)\) is the unique positive root of \((4.9)\); 

or 

(ii) \textbf{Vanishing} (virus dies out): the virus \(w\) and the infected cells \(v\) will vanish in a bounded area, i.e., \(-\infty < \lim_{t \to \infty} g(t) < \lim_{t \to \infty} h(t) < \infty\) and

\[
\lim_{t \to \infty} \|v(t, \cdot)\|_{C([g(t), h(t))]} = \lim_{t \to \infty} \|w(t, \cdot)\|_{C([g(t), h(t))]} = 0, \quad \lim_{t \to \infty} u = \theta/a \text{ uniformly in } \mathbb{R}.
\]

Moreover, \(\lim_{t \to \infty} h(t) - \lim_{t \to \infty} g(t) \leq \pi \sqrt{acd/(\theta kb - acq)}\) if \(R_0 > 1\).

As for the Basic Reproduction Number \(R_0 = \theta kb/(acq)\), in our results, we will show that it plays a different role, comparing with the corresponding ordinary differential systems. When \(R_0 \leq 1\), \textit{vanishing} always happens, that is, the virus cannot spread successfully. On the other hand, when \(R_0 > 1\), we have a criterion as follows: if the initial occupying area \([-h_0, h_0]\) is beyond a critical size, namely \(2h_0 \geq \pi \sqrt{acd/(\theta kb - acq)}\), then \textit{spreading} happens regardless of the moving parameter \(\mu, \beta\) and initial population density \((u_0, v_0, w_0)\). While \(2h_0 < \pi \sqrt{acd/(\theta kb - acq)}\), whether \textit{spreading} or \textit{vanishing} happens depends on the initial population density \((v_0, w_0)\) and the moving parameter \(\mu\) and \(\beta\).

The paper is organized as follows. Section 2 concerns with the existence, uniqueness and uniform estimates of global solution. In Section 3 we give some preliminaries which will be used later. In Section 4 we study the long time behavior of solution components \(u, v, w\), and in Section 5 we discuss the criteria for \textit{spreading} and \textit{vanishing}. At the last section, we give a brief discussion.

Before ending this section we mention that in recent years, more and more free boundary problems of reaction diffusion systems have been introduced to describe the dynamics of species after the pioneering work \cite{17}. Interested readers can refer to, except for the above cited papers, \cite{18, 19, 20, 21, 22} for competition models, \cite{23, 24, 25} for prey-predator models.

## 2 Existence, uniqueness and uniform estimates of solution of \((1.2)\)

In this section we prove the global existence and uniqueness of the solution to problem \((1.2)\). For convenience, we first introduce some notations. Denote

\[
A_1 = \max \{\|u_0\|_\infty, \theta/a\}, \quad B_1 = \|v_0\|_\infty + 1, \quad B_2 = \|w_0\|_\infty + 1,
\]

\[
\mathcal{A} = \{a, b, c, d, k, q, h_0, \mu, \beta, \alpha, A_1, B_1, B_2, \|w_0\|_{W^2_0((-h_0, h_0))}, w_0^\prime(\pm h_0)\},
\]

\[
\Pi_T = [0, T] \times \mathbb{R}, \quad \Delta_T = [0, T] \times [-1, 1], \quad D^T_{g,h} = \{0 \leq t \leq T, g(t) < x < h(t)\}.
\]

Let \(X\) be a Banach space and \(\varphi, \psi \in X\). Denote \(\|\varphi\|_X, \|\psi\|_X\) for simplicity.

\textbf{Theorem 2.1.} (Local solution) \textit{For any given} \(\alpha \in (0, 1)\) \textit{and} \(p > 3/(1 - \alpha)\), \textit{there exists a} \(T > 0\) \textit{such that the problem} \((1.2)\) \textit{has a unique local solution} \((u, v, w, g, h) \in C^{1, 1}_\alpha(\Pi_T) \times C^{1, 1}_\alpha(D^T_{g,h}) \times W^{1, 2}_p(D^T_{g,h}) \times [C^{1+\frac{\alpha}{2}}([0, T])]^2\). \textit{Moreover,}

\[
\begin{align*}
u_0 > 0 & \text{ in } \Pi_T; \quad v, w > 0 \text{ in } D^T_{g,h}; \quad g'(t) < 0, \quad h'(t) > 0 \text{ in } [0, T],
\end{align*}
\]

\(u \in C^{1, 1}_\alpha(D^T_{g,h})\) \textit{means that} \(u\) \textit{is continuously differentiable in} \(t \in [0, T]\) \textit{and is Lipschitz continuous in} \(x \in [g(t), h(t)]\) \textit{for all} \(t \in [0, T]\).
Proof. Invoked by the proof of [13, Theorem 2.1] and [14, Theorem 1.1], we divide the proof into several steps. Unless otherwise specified in the proof, positive constants $C_i$ depend only on $A$.

**Step 1:** Given $T > 0$, we say $u \in C^1_x(\Pi_T)$ if there is a constant $L_u(T)$ such that

$$|u(t, x_1) - u(t, x_2)| \leq L_u(T)|x_1 - x_2|, \quad \forall x_1, x_2 \in \mathbb{R}, \ 0 < t \leq T.$$

For $s > 0$, define

$$X^s_{u_0} = \{ \phi \in C(\Pi_s): \phi(0, x) = u_0(x), \ 0 \leq \phi \leq A_1 \}.$$

For any given $u \in X^1_{u_0} \cap C^1_x(\Pi_1)$ we consider the following problem

$$\begin{align*}
&v_t = f_2(u(t, x), v, w), \quad t > 0, \ g(t) < x < h(t), \\
&w_t - dw_{xx} = f_3(v, w), \quad t > 0, \ g(t) < x < h(t), \\
&v(t, x) = w(t, x) = 0, \quad t > 0, \ x \notin (g(t), h(t)), \\
&g'(t) = -\mu w_x(t, g(t)), \ h'(t) = -\beta w_x(t, h(t)) \quad t \geq 0, \\
&v(0, x) = v_0(x), w(0, x) = w_0(x), \quad |x| \leq h_0, \\
&h(0) = -g(0) = h_0.
\end{align*}$$

By [14, Theorem 1.1], we know that for some $0 < T \ll 1$, [24] has a unique solution $(v, w, g, h) \in C^{1,1-}(D^T_{g,h}) \times C^{1,1+\alpha}(D^T_{g,h}) \times [C^{1+\frac{\alpha}{2}}([0, T])]^2$. Moreover,

$$\begin{align*}
&\|w\|_{W^{1,2}(D^T_{g,h})} + \|w_x\|_{C(D^T_{g,h})} + \|g, h\|_{C^{1+\frac{\alpha}{2}}([0, T])} \leq M,
\end{align*}$$

where $M$ depends only on $A$.

**Step 2:** For the function $w(t, x)$ obtained in Step 1, we consider the following parameterized ODE problem,

$$\begin{align*}
&\tilde{u}_t = f_1(\tilde{u}, w(t, x)), \quad (t, x) \in (0, T] \times \mathbb{R}, \\
&\tilde{u}(0; x) = u_0(x) > 0, \quad x \in \mathbb{R}.
\end{align*}$$

By the standard ODE theory, [23] has a unique solution $\tilde{u} \in C^{1,1-}(\Pi_T)$ and $0 < \tilde{u} \leq A_1$.

Now we estimate the Lipschitz constant of $\tilde{u}$ in $x$. Since it can be easily derived from (2.2) that $|w(t, x_1) - w(t, x_2)| \leq M|x_1 - x_2|$ for any given $(t, x_1), (t, x_2) \in \Pi_T$, we have

$$\begin{align*}
|\tilde{u}(t, x_1) - \tilde{u}(t, x_2)| &= \left| \int_0^t \tilde{u}_t(s, x_1) - \tilde{u}_t(s, x_2) ds + u_0(x_1) - u_0(x_2) \right| \\
&\leq \int_0^t |\tilde{u}_t(s, x_1) - \tilde{u}_t(s, x_2)| ds + \int_0^t |f_1(\tilde{u}(s, x_1), w(s, x_1)) - f_1(\tilde{u}(s, x_2), w(s, x_2))| ds \\
&\leq \int_0^t (a + b)|\tilde{u}(s, x_1) - \tilde{u}(s, x_2)| ds + (bA_1TM + L_0)|x_1 - x_2|.
\end{align*}$$
Then noticing $0 < T \leq 1$ and making use of the Gronwall inequality, we obtain

$$|\tilde{u}(t, x_1) - \tilde{u}(t, x_2)| \leq (bA_1M + L_0)e^{a+b}|x_1 - x_2|.$$  

This shows that $L_{\tilde{u}} = (bA_1M + L_0)e^{a+b}$ is the Lipschitz constant of $\tilde{u}$. Define

$$\mathcal{Y}^T_{u_0} = \{ \phi \in C(\Pi_T) : \phi(0, x) = u_0(x), 0 \leq \phi \leq A_1, |\phi(t, x) - \phi(t, y)| \leq L_{\tilde{u}}|x - y| \}.$$  

Obviously, $\mathcal{Y}^T_{u_0}$ is complete with the metric $d(\phi_1, \phi_2) = \sup_{(t, x) \in \Pi_T} |\phi_1(t, x) - \phi_2(t, x)|$. The above analysis allows us to define a map $\mathcal{F}(u) = \tilde{u}$, and $\mathcal{F}$ maps $\mathcal{Y}^T_{u_0}$ into itself.

**Step 3:** We are in the position to prove that $\mathcal{F}$ is a contraction mapping in $\mathcal{Y}^T_{u_0}$ for sufficiently small $T$. In fact, for $i = 1, 2$, let $v_i, w_i, g_i, h_i$ be the unique solution of (2.1) with $u = u_i$. By arguing as in the proof of [14, Theorem 1.1], we can show that there exists a constant $L_v$, which only depends on $A$, such that for any given $(t, x_1), (t, x_2) \in \mathcal{Y}^T_{g_i, h_i}$,

$$|v_i(t, x_1) - v_i(t, x_2)| \leq L_v|x_1 - x_2|.$$  

(2.4)

Denote $U = u_1 - u_2, \tilde{U} = \tilde{u}_1 - \tilde{u}_2, V = v_1 - v_2$ and $W = w_1 - w_2$. Since $\tilde{u}_i$ satisfy

$$\begin{cases}
\tilde{u}_i,t = f_1(\tilde{u}_i, w_i), \quad (t, x) \in (0, T] \times \mathbb{R}, \\
\tilde{u}_i(0, x) = u_0(x) > 0, \quad x \in \mathbb{R},
\end{cases}$$

it follows that, for any $(t, x) \in \Pi_T$,

$$|\tilde{U}(t, x)| \leq \int_0^t (a + b)|\tilde{U}(s, x)|ds + bA_1T\|W\|_{L^\infty(\Pi_T)}.$$  

By virtue of the Gronwall inequality again, it yields

$$|\tilde{U}(t, x)| \leq bA_1e^{a+b}T\|W\|_{L^\infty(\Pi_T)}. $$  

(2.5)

The following arguments are devoted to an estimate of $\|W\|_{L^\infty(\Pi_T)}$. Evidently, $w_i$ satisfy

$$\begin{cases}
w_{i,t} - dw_{i,xx} = f_3(v_i, w_i), \quad 0 < t \leq T, \quad g_i(t) < x < h_i(t), \\
w_i(t, x) = 0, \quad 0 < t \leq T, \quad x \notin (g_i(t), h_i(t)), \\
w_i(0, x) = w_0(x), \quad |x| \leq h_0.
\end{cases}$$

We straighten the boundaries and define

$$x_i(t, y) = \frac{(h_i(t) - g_i(t))y + h_i(t) + g_i(t)}{2}, \quad z_i(t, y) = w_i(t, x_i(t, y)), \quad r_i(t, y) = v_i(t, x_i(t, y)).$$

For simplicity, we introduce the following notations $\xi = \xi_1 - \xi_2, \zeta = \zeta_1 - \zeta_2, z = z_1 - z_2, r = r_1 - r_2, h = h_1 - h_2, g = g_1 - g_2$, where

$$\begin{align*}
\xi_i(t) &= \frac{4}{(h_i(t) - g_i(t))^2}, \\
\zeta_i(t, y) &= \frac{h_i'(t) + g_i'(t)}{h_i(t) - g_i(t)} + \frac{(h_i'(t) - g_i'(t))y}{h_i(t) - g_i(t)}.
\end{align*}$$

Then $z$ satisfies

$$\begin{cases}
z_t - d\xi_1z_{yy} - \xi_1z_y = d\xi z_{2,yy} + \zeta z_{2,y} + \frac{kr}{1 + z_1} - \frac{kr_2z}{(1 + z_1)(1 + z_2)} - qz, \quad 0 < t \leq T, \quad |y| < 1, \\
z(t, \pm 1) = 0, \\
z(0, y) = 0,
\end{cases}$$

$$0 \leq t \leq T,$$

$$|y| \leq 1.$$
By the $L^p$ estimates for parabolic equations, we see
\[
\|z\|_{W^{1,2}_p(\Delta_T)} \leq C_1 (\|g\|_{C^1([0,T])} + \|r\|_{C(\Delta_T)}).
\]

We now estimate $\|r\|_{C(\Delta_T)}$. For any given $(t, y) \in \Delta_T$, it follows that
\[
|r(t, y)| = |v_1(t, x_1(t, y)) - v_2(t, x_2(t, y))| \leq |V(t, x_1(t, y))| + |v_2(t, x_1(t, y)) - v_2(t, x_2(t, y))|.
\]
It follows from the inequality (2.4) that
\[
|v_2(t, x_1(t, y)) - v_2(t, x_2(t, y))| \leq C_2 \|g, h\|_{C^1([0,T])}.
\]

Additionally, we can prove the following inequality:
\[
|V(t, x_1(t, y))| \leq C_3 (\|g, h\|_{C^1([0,T])} + T\|U, W\|_{L^\infty(\Pi_T)}).
\]
(2.6)

Its proof will be put in the next step on account of the length. Thus we have
\[
\|r\|_{C(\Delta_T)} \leq C_4 (\|g, h\|_{C^1([0,T])} + T\|U, W\|_{L^\infty(\Pi_T)}).
\]

Then it follows that
\[
\|z\|_{W^{1,2}_p(\Delta_T)} \leq C_5 (\|g, h\|_{C^1([0,T])} + T\|U, W\|_{L^\infty(\Pi_T)}).
\]

By utilizing the similar methods in Step 2 of [26, Theorem 2.1] and the embedding theorem:
\[
[z]_{C^0(\Delta_T)} \leq C \|z\|_{W^{1,2}_p(\Delta_T)}
\]
for some positive constant $C$ independent of $T^{-1}$ ([27, Theorem 1.1]), we can show that
\[
\|W\|_{L^\infty(\Pi_T)} \leq C_6 \left( T\|W\|_{L^\infty(\Pi_T)} + \|T\|_{L^\infty(\Pi_T)} \right).
\]

Hence
\[
\|W\|_{L^\infty(\Pi_T)} \leq 2C_6 \|U\|_{L^\infty(\Pi_T)} \quad \text{if} \quad 0 < T \ll 1.
\]

This combined with (2.5) arrives at
\[
\|\hat{U}\|_{L^\infty(\Pi_T)} \leq C_7 T\|U\|_{L^\infty(\Pi_T)} \leq \frac{1}{2} T\|U\|_{L^\infty(\Pi_T)} \quad \text{if} \quad 0 < T \ll 1.
\]

As a consequence, $\mathcal{F}$ is a contraction mapping and there exists a unique local solution $(u, v, w, g, h)$. Moreover, the desired properties of the local solution can be obtained from the above arguments.

**Step 4:** In this step, we are going to tackle the estimate (2.6), which will be divided into several cases. By the definition of $x_1(t, y)$, it is easy to see that $g_1(t) \leq x_1(t, y) \leq h_1(t)$. We denote $x_1 = x_1(t, y)$ for simplicity.

**Case 1:** $x_1 \notin (g_2(t), h_2(t))$. In this case $v_2(t, x_1) = 0$, and either $g_1(t) \leq x_1 \leq g_2(t)$ or $h_2(t) \leq x_1 \leq h_1(t)$. We only deal with the former case. Hence
\[
|V(t, x_1)| = |v_1(t, x_1) - v_1(t, g_1(t))| \leq L_v |x_1 - g_1(t)|
\leq L_v |g_2(t) - g_1(t)| \leq L_v \|g\|_{C^1([0,T])}.
\]

**Case 2:** $x_1 \in (g_2(t), h_2(t))$ and either $x_1 > h_0$ or $x_1 < -h_0$. We deal with only the case $x_1 > h_0$. Then we can uniquely find $0 < t_{x_1}, t'_{x_1} \leq t$ such that $h_1(t_{x_1}) = x_1$ and $h_2(t'_{x_1}) =
conclude that there exists $A$ for some $0 < \varepsilon < 1$. Without loss of generality, we assume $t'_{x_1} > t_{x_1}$. Then $h_1(t'_{x_1}) > h_1(t_{x_1}) = \varphi_1 = h_2(t'_{x_1})$, $x_1 \in (g_1(s), h_1(s))$ for all $t'_{x_1} < s < t$ and $x_1 \in (g_1(t'_{x_1}), h_1(t'_{x_1})) \setminus (g_2(t'_{x_1}), h_2(t'_{x_1}))$. Hence,

$$|V(t'_{x_1}, x_1)| = v_1(t'_{x_1}, x_1) \leq L_v\|g, h\|_{C([0, T])}$$

by the conclusion of Case 1. Integrating the differential equation of $v_i$ from $t'_{x_1}$ to $s(t'_{x_1} < s \leq t)$ we obtain

$$v_1(s, x_1) = v_1(t'_{x_1}, x_1) + \int_{t'_{x_1}}^{s} f_2(u, v, w, g, h) \bigg|_{x=x_1} d\tau,$$

$$v_2(s, x_1) = \int_{t'_{x_1}}^{s} f_2(u, v, w, g, h) \bigg|_{x=x_1} d\tau.$$

It then follows that

$$|V(s, x_1)| \leq v_1(t'_{x_1}, x_1) + \int_{t'_{x_1}}^{s} \left| \frac{bu_1w_1}{1+w_1} - \frac{bu_2w_2}{1+w_2} + c(v_2 - v_1) \right| d\tau \leq L_v\|g, h\|_{C([0, T])} + TC_8(\|V\|_{C(t', x_1)}),$$

where $C_8 = \max\{c, bA_1, b\}$. It follows from that

$$|V(t, x_1)| \leq C_{10} (\|g, h\|_{C([0, T])} + T\|U, W\|_{L^\infty(\Pi_T)}).$$

if $T > 0$ is sufficiently small.

**Case 3:** $x_1 \in (g_2(t), h_2(t))$ and $x_1 \in [-h_0, h_0]$. In this case we can derive

$$|V(t, x_1)| \leq C_{11} T\|U, W\|_{L^\infty(\Pi_T)}$$

by using similar methods. Since it is actually much simpler, we omit the details. In conclusion, we have proved the estimate (2.6). \hfill \Box

**Theorem 2.2.** (Global solution) The problem (1.2) has a unique global solution $(u, v, w, g, h)$, and there exist four positive constants $A_i$, $i = 1, 2, 3, 4$, such that

$$(u, v, w, g, h) \in C^{1, -1}(\Pi_\infty) \times C^{1, -1}(\Pi_\infty) \times W^{1, 2} p(D_{g, h}) \times [C^{1+\frac{\theta}{2}}([0, \infty]])^2,$$

$$0 < u \leq A_1 \text{ in } \Pi_\infty; \quad 0 < v \leq A_2; \quad 0 < w \leq A_3 \text{ in } D_{g, h}^\infty; \quad 0 < -g'(t), h'(t) \leq A_4 \text{ in } [0, \infty),$$

where $A_1 = \max\{\|u_0\|_{\infty}, \theta/A\}$.

**Proof.** It follows from Theorem 2.1 that the problem (1.2) has a unique local solution $(u, v, w, g, h)$ for some $0 < T \ll 1$ and $g'(t) > 0, h'(t) > 0$ for $0 \leq t \leq T$.

It is easy to show that $0 < u \leq A_1$ in $\Pi_T$. Recalling the equations of $(v, w)$ we can readily conclude that there exists $A_2, A_3 > 0$ such that $0 < v \leq A_2, 0 < w \leq A_3$ in $D_{g, h}^T$. Making use of the similar arguments in the proof of Lemma 2.1, we can show that there exists constant $A_4 > 0$, which only depends on the initial data, such that $0 < -g'(t), h'(t) \leq A_4$ in $[0, T]$.

With above estimates, we can extend the local solution uniquely to the global solution, and

$$(u, v, w, g, h) \in C^{1, -1}(\Pi_\infty) \times C^{1, -1}(\overline{D_{g, h}}^\infty) \times W^{1, 2} p(D_{g, h}^\infty) \times [C^{1+\frac{\theta}{2}}([0, \infty]])^2;$$

see [27 Corollary 1.1] for the details. It follows from the standard parabolic regularity theory that $(u, v, w, g, h)$ is the unique classical solution of (1.2). Combining $v(t, x) = 0$ for $x \notin (g(t), h(t))$ and the equation satisfied by $v$, we easily derive that $v \in C^{1, -1}(\Pi_\infty)$. The proof is ended. \hfill \Box
Since $g'(t) < 0$, $h'(t) > 0$, there exist $g_\infty \in [-\infty, 0)$ and $h_\infty \in (0, \infty]$ such that

$$\lim_{t \to \infty} g(t) = g_\infty, \quad \lim_{t \to \infty} h(t) = h_\infty.$$ 

The case $h_\infty = -g_\infty = \infty$ is called Spreading, and the case $h_\infty - g_\infty < \infty$ is called Vanishing.

**Theorem 2.3.** (Uniform estimates) Let $(u, v, w, g, h)$ be the unique global solution of (1.2). Then there exists a constant $C > 0$ such that

$$\|w(t, \cdot)\|_{C^1([g(t), h(t)])} \leq C, \quad \|g', h'\|_{C^{\alpha/2}([1, \infty))} \leq C, \quad \forall \ t \geq 1.$$ 

(2.7)

**Proof.** Remember $0 \leq v \leq A_2$, $0 \leq w \leq A_3$. The estimates (2.7) can be proved by using analogous methods in [29, Theorem 2.1] for the case $h_\infty - g_\infty < \infty$ and [25, Theorem 2.2] for the case $h_\infty - g_\infty = \infty$. We omit the details here. $\square$

### 3 Preliminaries

In this section, we will show some preliminaries which are crucial in the later parts. First we will investigate an eigenvalue problem and analyze the properties of its principal eigenvalue which will pave the ground for later discussion. It is well known that the eigenvalue problem

$$\begin{cases}
  d\hat{\phi}_{xx} + a_{11} \hat{\phi} = \rho \hat{\phi}, & l_1 < x < l_2, \\
  \hat{\phi}(l_i) = 0, & i = 1, 2
\end{cases}$$

has a principal eigenpair $(\rho_1, \hat{\phi}_1)$, where

$$\rho_1 = a_{11} - \frac{d\pi^2}{(l_2 - l_1)^2}, \quad \hat{\phi}_1(x) = \cos \frac{\pi(2x - l_2 - l_1)}{2(l_2 - l_1)}.$$

Now we consider the following eigenvalue problem

$$\begin{cases}
  d\phi_{xx} + a_{11}\phi + a_{12}\psi = \lambda \phi, & l_1 < x < l_2, \\
  a_{21}\phi + a_{22}\psi = \lambda \psi, & l_1 < x < l_2, \\
  \phi(l_i) = 0, & i = 1, 2
\end{cases}$$

(3.1)

with $a_{12}, a_{21} > 0$ and $a_{11}, a_{22} < 0$. It is clear that if $(\lambda, (\phi, \psi))$ is an eigenpair of (3.1), then $\lambda \neq a_{22}$ and $\psi(l_i) = 0, i = 1, 2$. Define

$$\mathcal{L} = \begin{pmatrix}
  d & a_{11} & a_{12} \\
  a_{21} & a_{22}
\end{pmatrix},$$

and choose the domain of $\mathcal{L}$:

$$\mathcal{D}(\mathcal{L}) = \{(\phi, \psi) \in H^2(l_1, l_2) \times L^2(l_1, l_2) : \phi(l_i) = 0, i = 1, 2\}.$$ 

Similarly to the proof of [15 Theorem 3.1], we can prove the following results by means of [30 Theorem 2.3, Remark 2.2]. The details are omitted here.
Theorem 3.1. Let $\sigma(\mathcal{L})$ be the spectral set of $\mathcal{L}$ and $s(\mathcal{L}) := \sup\{\text{Re}\lambda : \lambda \in \sigma(\mathcal{L})\}$. Then the following properties hold true:

(i) $s(\mathcal{L})$ is the principal eigenvalue of (3.1) with positive eigenvectors $(\phi_1, \psi_1)$;

(ii) $s(\mathcal{L}) = \frac{1}{2} [\rho_1 + a_{22} + \sqrt{(\rho_1 - a_{22})^2 + 4a_{12}a_{21}}]$ and has the same sign with $\rho_1 - a_{12}a_{21}/a_{22}$;

(iii) $s(\mathcal{L})$ is strictly monotone increasing in the length of the interval $(l_1, l_2)$ and strictly monotone decreasing in $d$.

By Theorem 3.1 we can easily deduce the following result.

Corollary 3.2. Define $\Gamma = a_{11} - \frac{a_{12}a_{21}}{a_{22}}$. Let $\lambda_1$ be the principal eigenvalue of the problem (3.1). Then the following properties are valid:

(i) If $\Gamma \leq 0$, then $\lambda_1 < 0$ for any $d > 0$ and $(l_1, l_2)$;

(ii) If $\Gamma > 0$, we fix the domain $(l_1, l_2)$ and let $d^*(l_1, l_2) = \Gamma(l_2 - l_1)^2 \pi^{-2}$. Then $\lambda_1 > 0$ when $0 < d < d^*(l_1, l_2)$, $\lambda_1 = 0$ when $d = d^*(l_1, l_2)$, and $\lambda_1 < 0$ when $d > d^*(l_1, l_2)$;

(iii) If $\Gamma > 0$, we fix $d > 0$ and set $L^*(d) = \pi \sqrt{d/\Gamma}$. Then $\lambda_1 > 0$ when $l_2 - l_1 > L^*(d)$, $\lambda_1 = 0$ when $l_2 - l_1 = L^*(d)$ and $\lambda_1 < 0$ when $l_2 - l_1 < L^*(d)$.

Let $\lambda_1$ be the principal eigenvalue of (3.1), that is, two components of the corresponding eigenfunction are both positive or negative. Then we have

$$\lambda_1 > a_{22}, \quad \rho_1 = \lambda_1 - a_{12}a_{21}/(\lambda_1 - a_{22}).$$

Thus by the uniqueness of $\rho_1$ we easily derive the uniqueness of the principal eigenvalue of (3.1).

Let $(\mu_1, u_1)$ be the first eigenpair of $-\Delta$ with homogeneous Dirichlet boundary condition on $(l_1, l_2)$ and $(\lambda_1, (\phi_1, \psi_1))$ be the principal eigenpair of the problem (3.1). The direct calculation yields

$$\begin{pmatrix} -d\mu_1 + a_{11} & a_{12} \\ a_{12} & a_{22} \end{pmatrix} \begin{pmatrix} \langle \phi_1, u_1 \rangle \\ \langle \psi_1, u_1 \rangle \end{pmatrix} = \lambda_1 \begin{pmatrix} \langle \phi_1, u_1 \rangle \\ \langle \psi_1, u_1 \rangle \end{pmatrix}, \quad (3.2)$$

where $\langle \cdot, \cdot \rangle$ denotes the inner product in $L^2((l_1, l_2))$.

The following lemma will play an important role in the study of long time behaviors of $(u, v, w)$ when $h_{\infty} - g_{\infty} < \infty$.

Lemma 3.3. Let $m(t, x)$ be a bounded function, $d, C, \mu$ and $\eta_0$ be positive constants, and constant $x_0 < \eta_0$. Let $\eta \in C^1([0, \infty))$, $w \in W^{1,2}_p(D_T)$ and $w_0 \in W^2_p((x_0, \eta_0))$ for any $T > 0$ with some $p > 1$, and $w_x \in C(D)$, where $D_T = \{(t, x) : 0 < t < T, x_0 < x < \eta(t)\}$, $D = \{(t, x) : 0 \leq t_{\infty}, x_0 < x \leq \eta(t)\}$. Assume that $(w, \eta)$ satisfies

$$\begin{cases}
    w_t - dw_{xx} + m(t, x)w_x \geq -Cw, & t > 0, \quad x_0 < x < \eta(t), \\
    w \geq 0, & t > 0, \quad x = x_0, \\
    w = 0, \quad \eta'(t) \geq -\mu w_x, & t > 0, \quad x = \eta(t), \\
    w(0, x) = w_0(x) \geq, & x \in (x_0, \eta_0), \\
    \eta(0) = \eta_0,
\end{cases}$$

$$10$$
and \( \lim_{t \to \infty} \eta(t) = \eta_{\infty} < \infty, \lim_{t \to \infty} \eta'(t) = 0, \)

\[
\|w(t, \cdot)\|_{C^1([x_0, \eta(t)])} \leq M, \quad \forall t \geq 1
\]

for some constant \( M > 0. \) Then \( \lim_{t \to \infty} \max_{x_0 \leq x \leq \eta(t)} w(t, x) = 0. \)

**Proof.** When \( x_0 = 0 \) and \( m(t, x) = 0, \) this lemma is exactly Proposition 2 in [31]; when \( x_0 = 0 \) and \( m(t, x) = \gamma \) is a constant, this lemma is exactly Lemma 3.1 in [32]. For our present case, by the maximum principle we have \( w(t, x) > 0 \) for \( t > 0 \) and \( x_0 < x < \eta(t). \) If we follow the proof of [21] Theorem 2.2] word by word we can prove this lemma. We will leave out the details because the advection term and boundary condition at \( x = x_0 \) do not influence the availability of the argument in [21] Theorem 2.2]. \( \square \)

**Lemma 3.4.** (Comparison principle) Let \( T > 0, \) \( \bar{g}, \bar{h} \in C^1([0, T]) \) and \( \bar{g} < \bar{h} \) in \([0, T]. \) Let \( \bar{u} \in C^{1,0}([0, T] \times \mathbb{R}), \) \( \bar{v} \in C^{1,0}((\Omega), \bar{w} \in C(\Omega) \cap C^{1,2}(\Omega) \) with \( \Omega = \{ 0 < t \leq T, \bar{g}(t) < x < \bar{h}(t) \}. \) Assume that \((\bar{u}, \bar{v}, \bar{w}, \bar{g}, \bar{h})\) satisfies

\[
\begin{align*}
\bar{u}_t &\geq \theta - a\bar{u}, & t > 0, \quad -\infty < x < \infty, \\
\bar{v}_t &\geq f_2(\bar{u}, \bar{v}, \bar{w}), & t > 0, \quad \bar{g}(t) < x < \bar{h}(t), \\
\bar{w}_t - d\bar{w}_{xx} &\geq f_3(\bar{v}, \bar{w}), & t > 0, \quad \bar{g}(t) < x < \bar{h}(t), \\
\bar{v}(t, x) &\equiv \bar{w}(t, x) = 0, & t > 0, \quad x = \bar{g}(t), \bar{h}(t), \\
\bar{g}'(t) &\leq -\mu \bar{w}_x(t, \bar{g}(t)), & t \geq 0, \\
\bar{h}'(t) &\geq -\beta \bar{w}_x(t, \bar{h}(t)), & t \geq 0, \\
\bar{v}(0, x), \bar{w}(0, x) &\geq 0, & \bar{g}(0) \leq x \leq \bar{h}(0).
\end{align*}
\]

If \( \bar{g}(0) \leq -h_0, \bar{h}(0) \geq h_0, u_0(x) \leq \bar{u}(0, x) \) in \( \mathbb{R}, \) and \( v_0(x) \leq \bar{v}(0, x), w_0(x) \leq \bar{w}(0, x) \) on \([-h_0, h_0]. \)

Then the solution \((u, v, w, g, h)\) of \( (1.2) \) satisfies

\[
g \geq \bar{g}, \ h \leq \bar{h} \text{ on } [0, T]; \quad u \leq \bar{u} \text{ on } [0, T] \times \mathbb{R}; \quad v \leq \bar{v}, \ w \leq \bar{w} \text{ on } D^{T}_{g,h},
\]

where \( D^{T}_{g,h} \) is defined as in the beginning of Section 2.

**Proof.** Take \( 0 < \rho < 1 \) and let \((u_\rho, v_\rho, w_\rho, g_\rho, h_\rho)\) be the corresponding unique solution of \((1.2)\) with \((h_0, v_0, w_0)\) replaced by \((\rho h_0, v_0, w_0, \rho), \) where \( v_{0,\rho}(x), w_{0,\rho}(x) \) satisfy \( (1.3) \) with \( h_0 \) replaced by \( \rho h_0, \) and satisfy

\[
0 < v_{0,\rho}(x) \leq v_0(x), \ 0 < w_{0,\rho}(x) \leq w_0(x) \quad \text{on } (-\rho h_0, \rho h_0),
\]

as well as

\[
\lim_{\rho \to 1} v_{0,\rho}(\rho x) = v_0(x) \quad \text{in } W^1_\infty((-h_0, h_0)), \quad \lim_{\rho \to 1} w_{0,\rho}(\rho x) = w_0(x) \quad \text{in } W^2_\rho((-h_0, h_0)).
\]

11
By a simple comparison consideration, we have $u_\rho \leq \bar{u}$ on $[0,T] \times \mathbb{R}$. Thus $(v_\rho, w_\rho)$ satisfies

$$
\begin{align*}
& v_{\rho,t} \leq f_2(\bar{u}, v_\rho, w_\rho), & t > 0, & g_\rho(t) < x < h_\rho(t), \\
& w_{\rho,t} - dw_{\rho,xx} = f_3(v_\rho, w_\rho), & t > 0, & g_\rho(t) < x < h_\rho(t), \\
& v_\rho(t, x) = w_\rho(t, x) = 0, & t > 0, & x \notin (g_\rho(t), h_\rho(t)), \\
& g'_\rho(t) = -\mu w_{\rho,x}(t, g_\rho(t)), & h'_\rho(t) = -\beta w_{\rho,x}(t, h_\rho(t)), & t \geq 0, \\
& v_\rho(0, x) = v_{0,\rho}(x), & w_\rho(0, x) = w_{0,\rho}(x), & -\rho h_0 \leq x \leq \rho h_0, \\
& h_\rho(0) = -g_\rho(0) = \rho h_0.
\end{align*}
$$

Similarly to [17, Lemma 3.5], by use of the indirect arguments and strong maximum principle we can show that $g_\rho(t) > \bar{g}(t), h_\rho(t) < \bar{h}(t)$ for $0 \leq t \leq T$. Thus $v_\rho(t, x) < \bar{v}(t, x), w_\rho(t, x) < \bar{w}(t, x)$ for $0 < t \leq T$ and $g_\rho(t) \leq x \leq h_\rho(t)$ by the standard comparison principle. Letting $\rho \to 1$ and using the continuous dependence of solutions on parameters we have $(u_\rho, v_\rho, w_\rho, g_\rho, h_\rho) \to (u, v, w, g, h)$. The details are omitted.

\section{Long time behavior of $(u, v, w)$}

This section concerns with the long time behavior of $(u, v, w)$. We first study the vanishing case $(h_\infty - g_\infty < \infty)$.

\begin{theorem}
Let $(u, v, w, g, h)$ be the unique global solution of \eqref{1.2}. If $h_\infty - g_\infty < \infty$, then

$$
\lim_{t \to \infty} \|w(t, \cdot)\|_{C([g(t), h(t)])} = \lim_{t \to \infty} \|v(t, \cdot)\|_{C([g(t), h(t)])} = 0,
$$

$$
\lim_{t \to \infty} u(t, x) = \theta/a \quad \text{uniformly in } \mathbb{R}.
$$
\end{theorem}

\begin{proof}
By the second estimate in \eqref{2.7}, we see that both $g'(t)$ and $h'(t)$ are uniformly continuous in $[1, \infty)$. Since $h_\infty - g_\infty < \infty$, it is easy to deduce $\lim_{t \to \infty} g'(t) = \lim_{t \to \infty} h'(t) = 0$. Then, using the first estimate of \eqref{2.7}, Lemma 3.3 in $[0, h(t))$ and a similar version of Lemma 3.3 in $(g(t), 0]$, one can arrive at $\lim_{t \to \infty} \|w(t, \cdot)\|_{C([g(t), h(t)])} = 0$. For any $\varepsilon > 0$, there exists $T > 0$ such that $bA_1 w(t, x)/(1 + w(t, x)) \leq \varepsilon$ for $t \geq T$ and $x \in \mathbb{R}$. Thus $v$ satisfies

$$
\begin{align*}
& v_t \leq \varepsilon - cv, & t \geq T, & g(t) < x < h(t), \\
& v(t, g(t)) = v(t, h(t)) = 0, & t \geq T, \\
& v(T, x) \geq 0.
\end{align*}
$$

By the comparison principle, we have $\limsup_{t \to \infty} \|v(t, \cdot)\|_{C([g(t), h(t)])} \leq \varepsilon/c$. The arbitrariness of $\varepsilon$ implies $\lim_{t \to \infty} \|v(t, \cdot)\|_{C([g(t), h(t)])} = 0$. Similarly, we can easily deduce that

$$
\limsup_{t \to \infty} u(t, x) \leq \theta/a \quad \text{uniformly in } \mathbb{R}. \tag{4.1}
$$

On the other hand, for any $\varepsilon_1 > 0$, there exists $T_1 > 0$ such that $w(t, x)/(1 + w(t, x)) \leq \varepsilon_1$ for $t \geq T_1$ and $x \in \mathbb{R}$. So $u$ satisfies

$$
\begin{align*}
& u_t \geq \theta - (a + b\varepsilon_1)u, & t \geq T_1, & x \in \mathbb{R}, \\
& u(T_1, x) > 0, & x \in \mathbb{R}.
\end{align*}
$$

12
Let \( u \) be the unique solution of the problem
\[
\begin{align*}
  u_t &= \theta - (a + b \varepsilon_1) u, \quad t \geq T_1, \\
  u(T_1) &= 0.
\end{align*}
\]
By using comparison principle and the fact that \( \lim_{t \to \infty} u(t) = \theta/(a + b \varepsilon_1) \), we have that \( \liminf_{t \to \infty} u(t, \cdot) \geq \theta/(a + b \varepsilon_1) \) uniformly in \( \mathbb{R} \). Due to the arbitrariness of \( \varepsilon_1 \) and (4.1), we derive the desired result.

In the following we study the spreading case \( (h_\infty - g_\infty = \infty) \). To get the accurate limits of the solution components \( u, v, w \) of (1.2), we first give a proposition which concerns the existence, uniqueness and asymptotic behavior of positive solution of a boundary value problem.

**Proposition 4.2.** Let \( m, l \) be positive constants and consider the following problem
\[
\begin{align*}
  f_2(m, v, w) &= 0, \quad -l < x < l, \\
  -dw_{xx} &= f_3(v, w), \quad -l < x < l, \\
  w(x) &= 0, \quad x = \pm l. 
\end{align*}
\]  

(i) Let \( \lambda_1 \) be the principal eigenvalue of
\[
\begin{align*}
  d\phi_{xx} - q\phi + k\psi &= \lambda \phi, \quad -l < x < l, \\
  bm\phi - c\psi &= \lambda \psi, \quad -l < x < l, \\
  \phi(\pm l) &= 0. 
\end{align*}
\]  

Then (4.2) has a positive solution if and only if \( \lambda_1 > 0 \). Moreover, the positive solution of (4.2) is unique when it exists.

(ii) To stress the dependence on \( l \), we denote the unique positive solution of (4.2) by \((v_l, w_l)\). Then \((v_l, w_l)\) is nondecreasing in \( l \), and converges to \((\hat{v}, \hat{w})\) locally uniformly in \( \mathbb{R} \) as \( l \to \infty \), where \((\hat{v}, \hat{w})\) is the unique positive root of
\[
\begin{align*}
  f_2(m, v, w) &= 0, \\
  f_3(v, w) &= 0. 
\end{align*}
\]  

Proof. (i) Clearly, the problem (4.2) is equivalent to
\[
\begin{align*}
  -dw_{xx} + qw &= \frac{kbmw}{c(1 + w)^2}, \quad v = \frac{bmw}{c(1 + w)}, \quad -l < x < l, \\
  w(x) &= 0, \quad x = \pm l.
\end{align*}
\]  

For clarity of exposition, we will always use the problem (4.2) in later discussion.

If (4.2) has a positive solution \((v, w)\), it is easy to show that \( q < \lambda_1(q) < kbm/c \), where \( \lambda_1(q) \) is the principal eigenvalue of
\[
\begin{align*}
  -d\phi_{xx} + q\phi &= \lambda \phi, \quad -l < x < l, \\
  \phi(x) &= 0, \quad x = \pm l. 
\end{align*}
\]  

Moreover, since \( \lambda_1(q) < kbm/c \), and the function \( bmk/(c + x) - x \) is decreasing in \( x > -c \), we can show that there exists the unique \( \lambda^* > 0 \) such that \( bmk/(c + \lambda^*) - \lambda^* = \lambda_1(q) \). Substituting this into (4.5), one can easily see that \( \lambda^* \) is the principal eigenvalue of (4.3), that is, \( \lambda^* = \lambda_1 > 0 \).
If $\lambda_1 > 0$, by the standard upper and lower solution methods we can show that (4.2) has at least one positive solution. Thanks to the structure of nonlinear terms of (4.2), the uniqueness is easily derived.

(ii) It follows from the above analysis and Corollary 3.2 that for large $l$, (4.2) has a unique positive solution $(v_l, w_l)$ provided that $kbm > qc$. A comparison argument (Lemma 2.1 in [16]) shows that $(v_l, w_l)$ is nondecreasing in $l$, and there exists $C > 0$ such that $v_l, w_l < C$ for all large $l$. Making use of the standard elliptic regularity theory, we have that $(v_l, w_l) \to (\tilde{v}, \tilde{w})$ in $C^2_{\text{loc}}(\mathbb{R})$, where $(\tilde{v}, \tilde{w})$ is a positive solution of

\[
\begin{cases}
  f_2(m, v, w) = 0, & -\infty < x < \infty, \\
  -dw_{xx} = f_3(v, w), & -\infty < x < \infty.
\end{cases}
\]

Obviously, $\tilde{w}$ satisfies

\[ -d\tilde{w}_{xx} = \frac{kbm}{c(1 + \tilde{w})^2} \tilde{w} - q\tilde{w}, \quad -\infty < x < \infty. \tag{4.6} \]

Since $\frac{kbm}{c(1 + w)^2} - q$ is decreasing in $w > 0$, the possible positive solution of (4.6) is a unique positive root of $kbm = qc(1 + w)^2$. Thus $\tilde{w} = \tilde{w}$, and consequently $\tilde{v} = \tilde{v}$. The proof is finished. \qed

**Theorem 4.3.** Suppose that $h_\infty = -g_\infty = \infty$. If $R_0 + \sqrt{R_0} > b/a$, then there are six positive constants $u_\infty, \tilde{u}_\infty, v_\infty, \tilde{v}_\infty, w_\infty$ and $\tilde{w}_\infty$ such that

\[
\begin{cases}
  u_\infty \leq \liminf_{t \to \infty} u(t, x) \leq \limsup_{t \to \infty} u(t, x) \leq \tilde{u}_\infty, \\
  v_\infty \leq \liminf_{t \to \infty} v(t, x) \leq \limsup_{t \to \infty} v(t, x) \leq \tilde{v}_\infty, \\
  w_\infty \leq \liminf_{t \to \infty} w(t, x) \leq \limsup_{t \to \infty} w(t, x) \leq \tilde{w}_\infty
\end{cases} \tag{4.7}
\]

locally uniformly in $\mathbb{R}$. Particularly, if we assume $b \leq 2a$, then

\[
\lim_{t \to \infty} u(t, x) = u^*, \quad \lim_{t \to \infty} v(t, x) = v^*, \quad \lim_{t \to \infty} w(t, x) = w^* \quad \text{locally uniformly in } \mathbb{R}, \tag{4.8}
\]

where $(u^*, v^*, w^*)$ is a unique positive root of

\[ f_1(u, w) = 0, \quad f_2(u, v, w) = 0, \quad f_3(v, w) = 0. \tag{4.9} \]

**Proof.** The condition $h_\infty - g_\infty = \infty$ implies $R_0 > 1$ (cf. Theorem 5.1). One can easily see that (4.9) has a unique positive root $(u^*, v^*, w^*)$. The following proof is actually an iterative process, and the idea comes from [28, 33].

**Step 1:** Clearly,

\[
\limsup_{t \to \infty} u(t, x) \leq \theta/a =: \bar{u}_1 \text{ uniformly in } \mathbb{R}.
\]

Then for any $\varepsilon > 0$, there exists $T > 0$ such that $u(t, x) \leq \theta/a + \varepsilon$ with $t \geq T$ and $x \in \mathbb{R}$. Thus $(v, w)$ satisfies

\[
\begin{cases}
  v_t \leq f_2(\theta/a + \varepsilon, v, w), & t > T, \quad g(t) < x < h(t), \\
  w_t - dw_{xx} = f_3(v, w), & t > T, \quad g(t) < x < h(t), \\
  v(t, x) = 0, \quad w(t, x) = 0, & t > T, \quad x = g(t) \text{ or } h(t), \\
  v(T, x) \geq 0, \quad w(T, x) \geq 0, & g(T) \leq x \leq h(T).
\end{cases}
\]
Consider the ODEs problem

\[
\begin{cases}
\dot{v}_t = f_2(\bar{u}_1 + \varepsilon, \bar{v}, \bar{w}), & \bar{w}_t = f_3(\bar{v}, \bar{w}), \quad t > T, \\
\bar{v}(T) = A_2, & \bar{w}(T) = A_3.
\end{cases}
\] (4.10)

Since \( R_0 > 1 \), the problem (4.10) has a unique positive equilibrium \((\bar{v}_1^\varepsilon, \bar{w}_1^\varepsilon)\) which is globally asymptotically stable. By a simple comparison consideration, we have \( v(t, x) \leq \bar{v}(t, x) \) and \( w(t, x) \leq \bar{w}(t, x) \) for \( t \geq T \) and \( x \in \mathbb{R} \). And so, \( \limsup_{t \to \infty} v(t, x) \leq \bar{v}_1^\varepsilon \) and \( \limsup_{t \to \infty} w(t, x) \leq \bar{w}_1^\varepsilon \) uniformly in \( \mathbb{R} \).

By the arbitrariness of \( \varepsilon \), we have

\[
\limsup_{t \to \infty} v(t, x) \leq \bar{v}_1, \quad \limsup_{t \to \infty} w(t, x) \leq \bar{w}_1 \quad \text{uniformly in } \mathbb{R},
\]

where \((\bar{v}_1, \bar{w}_1)\) is a unique positive root of the algebraic system (4.14) with \( m \) replaced by \( \bar{u}_1 \).

**Step 2:** For small \( \varepsilon > 0 \), there exists \( T > 0 \) such that \( w(t, x) \leq \bar{w}_1 + \varepsilon \) for \( t \geq T \) and \( x \in \mathbb{R} \).

Hence \( u \) satisfies

\[
\begin{cases}
\dot{u}_t \geq f_1(u, \bar{w}_1 + \varepsilon), \quad t > T, \quad x \in \mathbb{R}, \\
u(T, x) > 0, \quad x \in \mathbb{R}.
\end{cases}
\]

Using the comparison argument with the solution having initial value 0 we can deduce that

\[
\liminf_{t \to \infty} u(t, x) \geq \frac{\theta(1 + \bar{w}_1)}{a + a\bar{w}_1 + b\bar{w}_1} =: u_1 \quad \text{uniformly in } \mathbb{R}.
\]

Direct calculation shows that \( \bar{w}_1 = \sqrt{R_0} - 1 \) and

\[
kbu_1 > qc \quad \text{if and only if} \quad R_0 + \sqrt{R_0} > b/a.
\]

By our assumptions, we have \( kbu_1 > qc \), and then \( kb(u_1 - \varepsilon) > qc \) for small \( \varepsilon > 0 \).

Recall Proposition 4.2. For any large \( l \), let \((v_l, w_l)\) and \((v_1^\varepsilon, w_1^\varepsilon)\) be the unique positive solutions of (4.2) and (4.4) with \( m \) replaced by \( u_1 - \varepsilon \), respectively. Then \((v_l(x), w_l(x)) \rightarrow (v_1^\varepsilon(x), w_1^\varepsilon(x))\) locally uniformly in \( \mathbb{R} \) as \( l \rightarrow \infty \). For any given \( N \gg 1 \) and \( 0 < \sigma \ll 1 \), there exists a large \( l > N \) such that \( v_l(x) > v_1^\varepsilon - \sigma/2 \) and \( w_l(x) > w_1^\varepsilon - \sigma/2 \) for \( x \in [-N, N] \).

For such a fixed \( l > N \), let \((\lambda_1, (\phi, \psi))\) be the principal eigenpair of (4.3) with \( m \) replaced by \( u_1 - \varepsilon \). We can verify that for small \( \delta > 0 \), \((\delta \phi, \delta \psi)\) is a lower solution of (4.2) with \( m \) replaced by \( u_1 - \varepsilon \) (see the proof of Theorem 5.2 for details). Moreover, we may choose \( T \gg 1 \), \( 0 < \delta \ll 1 \) such that \([-l, l] \subseteq (g(t), h(t))\) for \( t \geq T \), \( u(t, x) \geq u_1 - \varepsilon \) on \([T, \infty) \times [-l, l]\), and \( \delta \psi(x) \leq v(T, x), \delta \phi(x) \leq w(T, x) \) on \([-l, l]\). Hence \((v, w)\) satisfies

\[
\begin{cases}
v_t \geq f_2(u_1 - \varepsilon, v, w), & t > T, \quad -l < x < l, \\
w_t - dw_{xx} = f_3(v, w), & t > T, \quad -l < x < l, \\
w(t, x) > 0, & t > T, \quad x = \pm l, \\
v(T, x) \geq \delta \psi(x), \ w(T, x) \geq \delta \phi(x), & -l \leq x \leq l.
\end{cases}
\]
Let \((\tilde{v}, \tilde{w})\) be a unique positive solution of the following problem

\[
\begin{cases}
\tilde{v}_t = f_2(\tilde{v} - b, \tilde{v}, \tilde{w}), & t > T, \ -l < x < l, \\
\tilde{w}_t - d\tilde{w}_{xx} = f_3(\tilde{v}, \tilde{w}), & t > T, \ -l < x < l, \\
\tilde{w}(t, x) = 0, & t > T, \ x = \pm l, \\
\tilde{v}(T, x) = \delta \psi(x), \quad \tilde{w}(T, x) = \delta \phi(x), & -l \leq x \leq l.
\end{cases}
\]

Then \(\tilde{v}\) and \(\tilde{w}\) are nondecreasing in \(t\). By the standard parabolic regularity we have \(\lim_{t \to \infty} (\tilde{v}(t, x), \tilde{w}(t, x)) = (v_1(x), w_1(x))\) uniformly in \([-l, l]\). There exists \(T_1 > T\) such that \(\tilde{v}(t, x) \geq v_1(x) - \sigma/2, \ \tilde{w}(t, x) \geq w_1(x) - \sigma/2\) for \(t > T_1\) and \(x \in [-l, l]\). Furthermore, by the comparison principle, \(v(t, x) \geq \tilde{v}(t, x), w(t, x) \geq \tilde{w}(t, x)\) for \(t > T\) and \(x \in [-l, l]\). So we have

\[
v(t, x) \geq \frac{v_1}{1} - \sigma, \quad w(t, x) \geq \frac{w_1}{1} - \sigma \quad \text{for} \quad t > T_1, \ |x| \leq N.
\]

These estimates combined with the arbitrariness of \(\varepsilon, \sigma\) and \(N\) yield

\[
\liminf_{t \to \infty} v(t, x) \geq v_1, \quad \liminf_{t \to \infty} w(t, x) \geq w_1 \quad \text{locally uniformly in} \quad \mathbb{R},
\]

where \((v_1, w_1)\) is a unique positive root of (4.4) with \(m\) replaced by \(\mathbb{u}_1\).

**Step 3:** For any given \(N > 0\) and \(0 < \varepsilon \ll 1\), there exists \(T > 0\) such that \(w(t, x) \geq w_1 - \varepsilon\) for \(t > T\) and \(-N \leq x \leq N\). So we have

\[
\begin{cases}
u_t \leq f_1(u, \frac{w_1}{1} - \varepsilon), & t > T, \ -N \leq x \leq N, \\
u(T, x) > 0, & -N \leq x \leq N.
\end{cases}
\]

Comparing with the following ODE problem

\[
\bar{u}_t = f_1(\bar{u}, \frac{w_1}{1} - \varepsilon), \quad t > T; \quad \bar{u}(T) = A_1,
\]

we can show that \(u(t, x) \leq \bar{u}(t)\) for \(t \geq T\) and \(-N \leq x \leq N\). Similarly to the preceding arguments, we have

\[
\limsup_{t \to \infty} u(t, x) \leq \frac{\theta(1 + \bar{u}_1)}{a + a\bar{w}_1 + b\bar{w}_1} =: \bar{u}_2 \quad \text{locally uniformly in} \quad \mathbb{R},
\]

and \(\bar{u}_2 \geq u_1\). Moreover, the direct calculation yields \(kb\bar{u}_2 >qc\).

For any fixed \(0 < \varepsilon \ll 1\), we take \(K > \max\left\{ A_3, \frac{kb(\bar{u}_2 + \varepsilon)}{qc} \right\}\) and consider the problem

\[
\begin{cases}
-dw_{xx} = \frac{kb(\bar{u}_2 + \varepsilon)w}{c(1 + w)^2} - qw, & -l < x < l, \\
w(\pm l) = K.
\end{cases}
\]

Clearly, \(kb(\bar{u}_2 + \varepsilon) > qc\). By the standard method we can show that (4.11) has a unique positive solution \(w^l\) for large \(l\). Moreover, \(0 < w^l \leq K\). The comparison principle gives that \(w^l\) is non-increasing in \(l\) and \(w^l \geq w_1\). In the same way as the proof of Proposition 4.2(ii) we can derive \(\lim_{l \to \infty} w^l(x) = \bar{w}_2^l\) locally uniformly in \(\mathbb{R}\), where \(\bar{w}_2^l\) is a unique positive root of \(kb(\bar{u}_2 + \varepsilon) = qc(1 + w)^2\). Take

\[
v^l(x) = \frac{b(\bar{u}_2 + \varepsilon)w^l(x)}{c(1 + w^l(x))}, \quad \bar{v}^l_2 = \frac{b(\bar{u}_2 + \varepsilon)\bar{w}_2^l}{c(1 + \bar{w}_2^l)}.
\]
Then \( \lim_{l \to \infty} v^l(x) = \bar{v}_2 \) locally uniformly in \( \mathbb{R} \), and \((\bar{v}_2, \bar{w}_2)\) is a unique positive root of \([4.4]\) in there \( m \) is replaced by \( \bar{u}_2 + \varepsilon \).

For any given \( N \gg 1 \) and \( 0 < \sigma \ll 1 \), there exists a large \( l > N \) such that \( v^l(x) \leq \bar{v}_2 + \sigma \) and \( w^l(x) \leq \bar{w}_2 + \sigma \) for \( -N \leq x \leq N \). Moreover, there exists \( T > 0 \) such that \( u(t, x) \leq \bar{u}_2 + \varepsilon \) for \( (t, x) \in [T, \infty) \times [-l, l] \), and \( h(T) > l, g(T) < -l \). Thanks to the equation of \( v \) and \( K > A_3 \), we can find \( T_1 > T \) such that \( v(t, x) \leq \frac{\beta(\bar{u}_2 + \varepsilon)K}{\alpha(1+K)} := A_2^* \) on \([T_1, \infty) \times [-l, l] \). Therefore, \((v, w)\) satisfies

\[
\begin{align*}
    v_t &\leq f_2(\bar{u}_2 + \varepsilon, v, w), & t > T_1, \quad -l < x < l, \\
    w_t - dw_{xx} &= f_3(v, w), & t > T_1, \quad -l < x < l, \\
    w(t, x) &\leq K, & t > T_1, \quad x = \pm l, \\
    v(T_1, x) &\leq A_2^*, & w(T_1, x) \leq K, \quad -l \leq x \leq l.
\end{align*}
\]

Let \((\bar{v}, \bar{w})\) be a unique positive solution of the problem

\[
\begin{align*}
    \bar{v}_t &\leq f_2(\bar{u}_2 + \varepsilon, \bar{v}, \bar{w}), & t > T_1, \quad -l < x < l, \\
    \bar{w}_t - d\bar{w}_{xx} &= f_3(\bar{v}, \bar{w}), & t > T_1, \quad -l < x < l, \\
    \bar{w}(t, x) &= K, & t > T_1, \quad x = \pm l, \\
    \bar{v}(T_1, x) &= A_2^*, \quad \bar{w}(T_1, x) = K, \quad -l \leq x \leq l.
\end{align*}
\]

Then we can deduce that \((\bar{v}(t, x), \bar{w}(t, x)) \to (v^l(x), w^l(x))\) uniformly in \([-l, l]\) as \( t \to \infty \). Thus there exists \( T_2 > T_1 \) such that \( v(t, x) \leq \bar{v}(t, x) \leq v^l(x) + \sigma, w(t, x) \leq \bar{w}(t, x) \leq w^l(x) + \sigma \) for \( t > T_2 \) and \( x \in [-l, l] \).

A comparison consideration yields that \( v(t, x) \leq \bar{v}(t, x) \) and \( w(t, x) \leq \bar{w}(t, x) \) on \([T_1, \infty) \times [-l, l]\). Recalling our previous conclusion we immediately derive that

\[
v(t, x) \leq \bar{v}_2 + 2\sigma, \quad w(t, x) \leq \bar{w}_2 + 2\sigma, \quad t > T_2, -N \leq x \leq N.
\]

The arbitrariness of \( \varepsilon, \sigma \) and \( N \) implies

\[
\limsup_{t \to \infty} v(t, x) \leq \bar{v}_2, \quad \limsup_{t \to \infty} w(t, x) \leq \bar{w}_2 \quad \text{locally uniformly in} \quad \mathbb{R},
\]

where \((\bar{v}_2, \bar{w}_2)\) is a unique positive root of the equations \([4.4]\) with \( m \) replaced by \( \bar{u}_2 \).

We may argue as in Step 2 to conclude that

\[
\liminf_{t \to \infty} u(t, x) \geq \underline{u}_2, \quad \liminf_{t \to \infty} v(t, x) \geq \underline{v}_2, \quad \liminf_{t \to \infty} w(t, x) \geq \underline{w}_2 \quad \text{locally uniformly in} \quad \mathbb{R},
\]

where \( \underline{u}_2 = \frac{\theta(1+\bar{u}_2)}{\underline{a} + \underline{u}_2 + \bar{w}_2} \), and \((\underline{v}_2, \underline{w}_2)\) is a unique positive root of \([4.4]\) with \( m \) replaced by \( \underline{u}_2 \).

**Step 4:** According to the above arguments we have

\[
\underline{u}_1 < \underline{u}_2 < \bar{u}_2 < \bar{u}_1, \quad \underline{u}_1 < \underline{v}_2 < \bar{v}_2 < \bar{v}_1, \quad \underline{w}_1 < \underline{w}_2 < \bar{w}_2 < \bar{w}_1.
\]

Repeating the above procedures we can find six sequences \( \{\underline{u}_n\}, \{\bar{u}_n\}, \{\underline{v}_n\}, \{\bar{v}_n\}, \{\underline{w}_n\}, \{\bar{w}_n\} \) satisfying

\[
\underline{u}_1 < \underline{u}_2 < \cdots < \underline{u}_n < \cdots < \bar{u}_n < \cdots < \bar{u}_2 < \bar{u}_1,
\]

17
\[ v_1 < v_2 < \cdots < v_n < \cdots < \bar{v}_n < \cdots < \bar{v}_2 < \bar{v}_1, \]
\[ w_1 < w_2 < \cdots < w_n < \cdots < \bar{w}_n < \cdots < \bar{w}_2 < \bar{w}_1, \]

so that
\[
\begin{align*}
    u_n & \leq \liminf_{t \to \infty} u(t, x) \leq \limsup_{t \to \infty} u(t, x) \leq \bar{u}_n, \\
v_n & \leq \liminf_{t \to \infty} v(t, x) \leq \limsup_{t \to \infty} v(t, x) \leq \bar{v}_n, \\
w_n & \leq \liminf_{t \to \infty} w(t, x) \leq \limsup_{t \to \infty} w(t, x) \leq \bar{w}_n
\end{align*}
\]

locally uniformly in \( \mathbb{R} \). The limits of the above six sequences are well defined, and denoted by \( u_\infty, \bar{u}_\infty, v_\infty, \bar{v}_\infty, w_\infty \) and \( \bar{w}_\infty \) respectively. It is clear that (4.7) holds.

Now we assume \( b \leq 2a \) and prove (4.8). By the careful calculations one can obtain
\[
\begin{align*}
    \bar{u}_1 = \frac{\theta}{a}, \quad b\bar{u}_n \bar{w}_n \leq \frac{b\bar{u}_n \bar{w}_n}{1 + \bar{w}_n} = c\bar{v}_n, \quad \frac{k\bar{v}_n}{1 + \bar{w}_n} = q\bar{w}_n, \quad \frac{k\bar{v}_n}{1 + \bar{w}_n} = q\bar{w}_n, \quad \bar{u}_{n+1} = \frac{\theta(1 + \bar{w}_n)}{a + a\bar{w}_n + bw_n}.
\end{align*}
\]

Consequently, \( u_\infty, \bar{u}_\infty, v_\infty, \bar{v}_\infty, w_\infty, \bar{w}_\infty \) satisfy
\[
\begin{align*}
    \frac{b\bar{u}_\infty \bar{w}_\infty}{1 + \bar{w}_\infty} = c\bar{v}_\infty, \quad \frac{k\bar{v}_\infty}{1 + \bar{w}_\infty} = q\bar{w}_\infty, \quad \frac{\theta(1 + \bar{w}_\infty)}{a + a\bar{w}_\infty + bw_\infty}, \quad \frac{k\bar{v}_\infty}{1 + \bar{w}_\infty} = q\bar{w}_\infty, \quad \frac{\theta(1 + \bar{w}_\infty)}{a + a\bar{w}_\infty + bw_\infty}.
\end{align*}
\]

Using our assumptions \( R_0 > 1 \) and \( b/a \leq 2 \), we can derive after a series of calculations that
\[
\begin{align*}
    u_\infty = \bar{u}_\infty = u^*, \quad v_\infty = \bar{v}_\infty = v^*, \quad w_\infty = \bar{w}_\infty = w^*.
\end{align*}
\]

Thus (4.8) holds and the proof is ended. \( \square \)

5 Criteria for spreading and vanishing

In this section we study the criteria governing spreading \( (h_\infty - g_\infty = \infty) \) and vanishing \( (h_\infty - g_\infty < \infty) \). In the following, we divide our discussion into two cases based on the Basic Reproduction Number \( R_0 = \theta kb/(acq) \). For convenience, we denote \( \gamma = \max \{ \mu, \beta \} \).

5.1 The case \( R_0 \leq 1 \)

Theorem 5.1. Let \( (u, v, w, g, h) \) be the unique solution of (1.2). If \( R_0 \leq 1 \), then \( h_\infty - g_\infty < \infty \).

Proof. By a simple comparison argument, we have
\[
u(t, x) \leq \theta/a + \|u_0\|e^{-at} =: \hat{u}(t) \quad \text{for } t \geq 0, \ x \in \mathbb{R}.
\]

Hence \( v \) satisfies
\[
\begin{align*}
    v_t & \leq f_2(\hat{u}(t), v, w), \quad t > 0, \ g(t) < x < h(t), \\
v(t, g(t)) & = v(t, h(t)) = 0, \quad t > 0, \\
v(0, x) & = v_0(x), \quad |x| \leq h_0.
\end{align*}
\]
Notice that $R_0 = \theta kb/acq \leq 1$. It follows that by simple calculations

$$
\frac{d}{dt} \int_{g(t)}^{h(t)} (cw + kv) dx = \int_{g(t)}^{h(t)} (cw_t + kv_t) dx
$$

$$
\leq \int_{g(t)}^{h(t)} [cdw_{xx} + kb\|u_0\|_\infty e^{-at}w + (k\theta b/a - qc)w] dx
$$

$$
\leq \int_{g(t)}^{h(t)} (cdw_{xx} + kb\|u_0\|_\infty e^{-at}w) dx
$$

$$
\leq -cd\gamma^{-1}(h'(t) - g'(t)) + kbA_3\|u_0\|_\infty e^{-at}(h(t) - g(t)).
$$

Set

$$
f(t) = \int_{g(t)}^{h(t)} (cw + kv) dx, \quad \ell(t) = h(t) - g(t), \quad \varphi(t) = kbA_3\|u_0\|_\infty e^{-at}.
$$

Then we have

$$
cd\ell'(t) \leq -\gamma f'(t) + \gamma \varphi(t)\ell(t).
$$

Integrating the above differential inequality from 0 to $t$ yields

$$
\ell(t) \leq \ell(0) + \gamma (cd)^{-1}f(0) + \gamma (cd)^{-1} \int_0^t \varphi(s)\ell(s) ds.
$$

By virtue of the Gronwall inequality,

$$
\ell(t) \leq [\ell(0) + \gamma (cd)^{-1}f(0)] \exp \left\{ \gamma (cd)^{-1} \int_0^t \varphi(s) ds \right\} < \infty.
$$

Thus, $h_\infty - g_\infty < \infty$. \hfill \Box

### 5.2 The case $R_0 > 1$

In this subsection, we always assume that $R_0 > 1$, and consider $d$, $h_0$, $\mu$ and $\beta$ as varying parameters to depict the criteria for spreading and vanishing.

**Theorem 5.2.** Let $(u, v, w, g, h)$ be the solution of (1.2). If $h_\infty - g_\infty < \infty$, then we have

$$
h_\infty - g_\infty \leq \pi \sqrt{acd/(kb\theta - acq)} =: \Lambda.
$$

This implies that if $h_0 \geq \Lambda/2$, then $h_\infty - g_\infty = \infty$. Moreover if $h_\infty - g_\infty = \infty$, then

$$
\limsup_{t \to \infty} \|v(t, \cdot)\|_{C([g(t), h(t)])} > 0, \quad \limsup_{t \to \infty} \|w(t, \cdot)\|_{C([g(t), h(t)])} > 0. \quad (5.1)
$$

**Proof.** Due to Theorem 4.1 and $h_\infty - g_\infty < \infty$, we see that $\lim_{t \to \infty} u(t, \cdot) = \theta/a$ uniformly in $\mathbb{R}$, and

$$
\lim_{t \to \infty} \|v(t, \cdot)\|_{C([g(t), h(t)])} = 0, \quad \lim_{t \to \infty} \|w(t, \cdot)\|_{C([g(t), h(t)])} = 0. \quad (5.2)
$$

Arguing indirectly, if $h_\infty - g_\infty > \Lambda$, then there exists $T > 0$ such that for any small $\varepsilon > 0$ satisfying $\frac{kb}{c}(\frac{\theta}{a} - \varepsilon) - q > 0$, we have

$$
u(t, x) > \theta/a - \varepsilon, \quad \forall \ t \geq T, \ x \in \mathbb{R};
$$

$$
h(t) - g(t) > \pi \sqrt{d} [kb(\theta/a - \varepsilon)/c - q]^{-1/2} =: \Lambda_\varepsilon, \quad \forall \ t \geq T.
$$
Then for any \([l_1, l_2] \subseteq (g(T), h(T))\) and \(l_2 - l_1 > \Lambda_\varepsilon\), we have
\[
\begin{cases}
  v_t \geq f_2(\theta/a - \varepsilon, v, w), & t > T, \ l_1 < x < l_2, \\
  w_t - dw_{xx} = f_3(v, w), & t > T, \ l_1 < x < l_2, \\
  w(t, x) > 0, & t > T, \ x = l_i, \ i = 1, 2, \\
  v(T, x) > 0, \ w(T, x) > 0, \ l_1 \leq x \leq l_2.
\end{cases}
\]
(5.3)

Consider the following eigenvalue problem
\[
\begin{cases}
  d\phi_{xx} - q\phi + k\psi = \lambda\phi, & l_1 < x < l_2, \\
  b(\theta/a - \varepsilon)\phi - c\psi = \lambda\psi, & l_1 < x < l_2, \\
  \phi(l_1) = 0, \ & i = 1, 2.
\end{cases}
\]
(5.4)

Denote the principal eigenpair of (5.4) by \((\lambda_1, (\phi, \psi))\) with \(\max_{x \in [l_1, l_2]} |\phi(x)| = 1\). It follows from Corollary 3.2 that \(\lambda_1 > 0\) due to \(l_2 - l_1 > \Lambda_\varepsilon\). Let
\[
\begin{align*}
  v(x) &= \delta\psi(x), \\
  w(x) &= \delta\phi(x)
\end{align*}
\]
with \(\delta > 0\) to be determined later.

We claim that there exists \(\delta_0 > 0\) sufficiently small such that for any \(0 < \delta < \delta_0\), we have
\[
\begin{cases}
  0 < f_2(\theta/a - \varepsilon, v, w), & l_1 < x < l_2, \\
  - dw_{xx} \leq f_3(v, w), & l_1 < x < l_2.
\end{cases}
\]
(5.5)

In fact, we have
\[
f_2(\theta/a - \varepsilon, v, w) = \delta\frac{b(\theta/a - \varepsilon)\phi}{1 + \delta\phi} - c\delta\psi = \delta\left(\frac{c + \lambda_1}{1 + \delta\phi} - \frac{c}{1 + \delta\phi} + \frac{c}{1 + \delta\phi} - c\right)\psi > 0
\]
provided that \(\delta > 0\) is small. The proof of the second inequality of (5.5) can be done in a similar manner.

Furthermore, one can choose small \(\delta > 0\) such that \(v(T, x) \geq \delta\psi(x)\) and \(w(T, x) \geq \delta\phi(x)\) for \(x \in [l_1, l_2]\). Then \((v, w)\) satisfies
\[
\begin{cases}
  v_t \leq f_2(\theta/a - \varepsilon, v, w), & t > T, \ l_1 < x < l_2, \\
  w_t - dw_{xx} \leq f_3(v, w), & t > T, \ l_1 < x < l_2, \\
  v(t, x) = 0, \ w(t, x) = 0, & t > T, \ x = l_i, \ i = 1, 2, \\
  v(T, x) \leq v(T, x), \ w(T, x) \leq w(T, x), \ l_1 \leq x \leq l_2.
\end{cases}
\]

By virtue of the comparison principle,
\[
v(t, x) \geq v(x), \ w_t(x) \geq w(x), \ t > T, \ l_1 \leq x \leq l_2.
\]
(5.6)

This is a contradiction with (5.2).

We now assume \(h_\infty - g_\infty = \infty\) and prove (5.1). By the comparison principle, it is easy to see that \(\lim_{t \to \infty} \|v(t, \cdot)\|_{C([g(T), h(t)])} = 0\) if and only if \(\lim_{t \to \infty} \|w(t, \cdot)\|_{C([g(T), h(t)])} = 0\). Hence if we assume that one of the two limits in (5.1) does not hold, we can similarly obtain \(\lim_{t \to \infty} \|v(t, \cdot)\|_{C([g(T), h(t)])} = \theta/a\) uniformly in \(\mathbb{R}\). By \(h_\infty - g_\infty = \infty\), we can derive an analogous contradiction as above. The proof is ended. \(\square\)
Obviously, \(h_0 \geq \Lambda/2\) is equivalent to \(d \leq 4h_0^2q(R_0 - 1)\pi^{-2} =: D\). So the above result suggests that when \(R_0 > 1\), the larger initial habitat \([-h_0, h_0]\) or the lower dispersal rate \(d\) of the virus is, the more possibility of successful spreading is observed.

**Theorem 5.3.** If \(h_\infty - g_\infty = \infty\), then \(h_\infty = \infty\) and \(g_\infty = -\infty\).

**Proof.** By way of contradiction, we assume that \(h_\infty < \infty\) and \(g_\infty = -\infty\). If we take \(L > \Lambda + 2\), where \(\Lambda\) is defined in Theorem 5.2, then there exists \(T_0 > 0\) such that \(g(T_0) < -L\). Then \(w\) satisfies

\[
\begin{align*}
     w_t - dw_{xx} &= f_3(v, w), & t > T_0, & -L < x < h(t), \\
     w(t, -L) &> 0, & w(t, h(t)) = 0, & t > T_0, \\
     h' &= -\beta w_x(t, h(t)), & t > T_0, \\
     w(T_0, x) &\geq 0, & -L \leq x \leq h(T_0).
\end{align*}
\]

As \(h_\infty < \infty\), using the second estimate in (2.7) we have \(\lim_{t \to \infty} h'(t) = 0\). Then, using the first estimate in (2.7) and Lemma 5.3, one can arrive at

\[
\lim_{t \to \infty} \|w(t, \cdot)\|_{C([-L, h(t)])} = 0.
\] (5.7)

Then we may argue as in the proof of Theorem 4.1 with minor modifications to derive that

\[
\lim_{t \to \infty} \max_{t \in [1 - L, h(T_0)]} v(t, \cdot) = 0, \quad \lim_{t \to \infty} \max_{t \in [1 - L, h(T_0)]} u(t, \cdot) = \theta/a.
\]

There exists \(T > T_0\) such that \(u(t, x) \geq \theta/a - \varepsilon\) for \((t, x) \in [T, \infty) \times [1 - L, h(T_0)]\). Let \(\varepsilon > 0\) be small enough satisfying \(L - 1 > \Lambda_{\varepsilon}\), where \(\Lambda_{\varepsilon}\) is defined as in Theorem 5.2. Choose an interval \([l_1, l_2] \subset (1 - L, h(T_0))\) with \(l_2 - l_1 > \Lambda_{\varepsilon}\). Then \((v, w)\) satisfies (5.3). As in the proof of Theorem 5.2, we can conclude that (5.6) holds. This is a contradiction with (5.7).

Analogously, we can prove that the case with \(h_\infty = \infty\) and \(g_\infty > -\infty\) also does not hold. Therefore, we must have \(h_\infty = \infty\) and \(g_\infty = -\infty\). \(\square\)

The following result implies that although the initial habitat is small or the dispersal rate is fast, the spreading also can occur if the expanding rate \(\mu\) or \(\beta\) is appropriately large. By using similar method in the proof of Lemma 3.2] with some modifications, we can prove the following lemma.

**Lemma 5.4.** If \(h_0 < \Lambda/2\) (or \(d > D\)), then there exists \(\mu^0 > 0\) (resp. \(\beta^0 > 0\)) such that if \(\mu \geq \mu^0\) (resp. \(\beta \geq \beta^0\)), then \(h_\infty - g_\infty = \infty\).

The above lemma also indicates that if \(\gamma = \max\{\mu, \beta\} \geq \max\{\mu^0, \beta^0\}\), then \(h_\infty - g_\infty = \infty\). Instinctively, we deem that if \(R_0 > 1\), \(h_0 < \Lambda/2\) (or \(d > D\)), \(\mu\) and \(\beta\) both are small, then the vanishing will happen. The lemma listed below supports our belief.

**Lemma 5.5.** Assume \(h_0 < \Lambda/2\) (or \(d > D\)). Then there exists \(\mu_0 > 0\) such that when \(\gamma = \max\{\mu, \beta\} \leq \mu_0\), we must have \(h_\infty - g_\infty < \infty\).
Proof. Let \( \hat{u} \) be the unique solution of the problem
\[
\hat{u}_t = \theta - a\hat{u}, \quad t > 0; \quad \hat{u}(0) = \max \{\|u_0\|_\infty, \theta/a\}.
\]
Then \( \hat{u}(t) \geq \theta/a \) and \( \lim_{t \to \infty} \hat{u}(t) = \theta/a \). By the comparison principle, \( u(t, x) \leq \hat{u}(t) \) in \( [0, \infty) \times \mathbb{R} \).
For any fixed \( h_0 < l < \Lambda/2 \), we consider the following eigenvalue problem
\[
\begin{cases}
-\frac{\pi^2}{(2l)^2} \varphi - q\varphi + k\psi = \lambda \varphi, & -l < x < l, \\
( b\theta/a ) \varphi - c\psi = \lambda \psi, & -l < x < l, \\
\phi(\pm l) = 0.
\end{cases}
\]
In view of Corollary 3.2 the principal eigenvalue \( \lambda_1 < 0 \) since \( 2l < \Lambda \). Moreover, by (3.2), there exists a positive constant \( \hat{\phi} \) such that
\[
-d\frac{\pi^2}{(2l)^2} \hat{\phi} - q\hat{\phi} + k = \lambda_1 \hat{\phi}, \quad \frac{b\theta}{a} \hat{\phi} - c = \lambda_1. \tag{5.8}
\]
Define
\[
f(t) = M \exp \left\{ \int_0^t \left[ \hat{\phi} b(\hat{u}(s) - \theta/a) + \lambda_1 \right] ds \right\}, \quad r(t) = \left( h_0^2 + \gamma \pi \hat{\phi} \right)^{1/2}, \quad \hat{v}(t, x) = f(t) \cos \frac{\pi x}{2r(t)}, \quad \hat{w}(t, x) = \hat{\phi} f(t) \cos \frac{\pi x}{2r(t)}, \quad -r(t) \leq x \leq r(t),
\]
where \( \gamma = \max \{\mu, \beta\} \), and \( M > 0 \) is taken large so that
\[
v_0(x) \leq M \cos \frac{\pi x}{2h_0} = \hat{v}(t, 0), \quad w_0(x) \leq \hat{\phi} M \cos \frac{\pi x}{2h_0} = \hat{w}(t, 0) \text{ in } [-h_0, h_0].
\]
As \( \lim_{t \to \infty} \hat{u}(t) = \theta/a \) and \( \lambda_1 < 0 \), it follows that \( \int_0^\infty f(s) ds < \infty \). Clearly, \( r'(t) > 0 \) for \( t \geq 0 \). Set
\[
\mu_0 = \frac{l^2 - h_0^2}{\pi \hat{\phi} \int_0^\infty f(s) ds}.
\]
Then \( r(t) < l \) for \( t \geq 0 \) provided \( 0 < \gamma \leq \mu_0 \).
Using (5.8) and \( \hat{u}(t) \geq \theta/a, r(t) < l \) for all \( t \geq 0 \), by a series of calculations we have
\[
\begin{align*}
\hat{v}_t - f_2(\hat{u}, \hat{v}, \hat{w}) &\geq f(t) \cos \frac{\pi x}{2r(t)} \left( \lambda_1 + c - \frac{\theta}{a} \hat{\phi} \right) = 0, \\
\hat{w}_t - dw_{xx} - f_3(\hat{v}, \hat{w}) &\geq f(t) \cos \frac{\pi x}{2r(t)} \left[ \hat{\phi} f(t) \cos \frac{\pi x}{2r(t)} \left( \hat{\phi} b \left( \hat{u} - \frac{\theta}{a} \right) + d \left( \frac{\pi}{2r(t)} \right)^2 \right) + \lambda_1 \hat{\phi} + q\hat{\phi} - k \right] \\
&= \hat{\phi} f(t) \cos \frac{\pi x}{2r(t)} \left[ \hat{\phi} b \left( \hat{u} - \frac{\theta}{a} \right) + d \left( \frac{\pi}{2r(t)} \right)^2 \right] \geq 0,
\end{align*}
\]
for \( t > 0 \) and \( -r(t) < x < r(t) \). And we easily see
\[
-r'(t) = -\gamma \hat{w}_x(t, -r(t)), \quad r'(t) = -\gamma \hat{w}_x(t, r(t)).
\]
Thus for any $0 < \gamma \leq \mu_0$, $(\hat{u}, \hat{v}, \hat{w}, -r, r)$ satisfies $r(0) = h_0$ and

\[
\begin{align*}
\dot{u}_t &= \theta - a\hat{u}, & t > 0, \\
\dot{v}_t &\geq f_2(\hat{u}, \hat{v}, \hat{w}), & t > 0, \quad -r(t) < x < r(t), \\
\dot{w}_t - d\dot{w}_{xx} &\geq f_3(\hat{v}, \hat{w}), & t > 0, \quad -r(t) < x < r(t), \\
\hat{v}(t, \pm r(t)) &= \hat{w}(t, \pm r(t)) = 0, & t > 0, \\
-r'(t) &\leq -\mu \hat{w}_x(t, -r(t)), & t \geq 0, \\
\hat{u}(0) &\geq u_0(x), \quad \hat{w}(0, x) \geq w_0(x), & -\infty < x < \infty, \\
\hat{v}(0, x) &\geq v_0(x), \quad \hat{w}(0, x) \geq w_0(x), & |x| \leq h_0.
\end{align*}
\]

By the comparison principle (Lemma 3.4), $-r(t) \leq g(t), h(t) \leq r(t)$ for $t \geq 0$. As a result, we have

\[g_\infty \geq -\lim_{t \to \infty} r(t) \geq -l, \quad h_\infty \leq \lim_{t \to \infty} r(t) \leq l,\]

which implies $h_\infty - g_\infty < \infty$. This completes the proof.

According to the above proof, we can see that $\mu_0$ is independent of $v_0$ and $w_0$ and strictly decreasing in $M$. Thus for any given $\mu$ and $\beta$, there exists $M > 0$ sufficiently small such that $\gamma \leq \mu_0$. Meanwhile, if both $v_0$ and $w_0$ are small enough such that for such $M$

\[v_0(x) \leq M \cos \frac{\pi x}{2h_0}, \quad w_0(x) \leq \tilde{\phi} M \cos \frac{\pi x}{2h_0}, \quad \forall \ x \in [-h_0, h_0],\]

we still can derive $h_\infty - g_\infty < \infty$ by the above arguments. Hence we have the following conclusion.

**Remark 5.6.** Assume $h_0 < \Lambda/2 \ (d > D)$, and that $(v_0, w_0)$ satisfies (1.3). Then vanishing happens if both $v_0$ and $w_0$ are small enough.

Combining the above two lemmas, by the similar arguments in [28 Theorem 5.2] we can show the following conclusion.

**Theorem 5.7.** If $h_0 < \Lambda/2 \ (d > D)$. There exists $0 < \mu_* \leq \mu^*$ such that $h_\infty - g_\infty < \infty$ if $\gamma \leq \mu_*$ or $\gamma = \mu^*$, and $h_\infty - g_\infty = \infty$ if $\gamma > \mu^*$.

## 6 Discussion

In this paper we proposed a viral propagation model with nonlinear infection rate and free boundaries and investigated the dynamical properties. This model is composed of two ordinary differential equations and one partial differential equation, in which the spatial range of the first equation is the whole space $\mathbb{R}$, and the last two equations have free boundaries. As a new mathematical model, we have proved the existence, uniqueness and uniform estimates of global solution, and provided the criteria for spreading and vanishing, and long time behavior of the solution components $u, v, w$.

Comparing with the corresponding ordinary differential systems, the **Basic Reproduction Number** $R_0 = \theta kb/(acq)$ plays a different role:
(i) For the corresponding ordinary differential systems, by the Lyapunov function method we can prove that if $R_0 < 1$ then the infection can not spread successfully, while if $R_0 > 1$ then the infection will spread successfully. When $R_0 = 1$ the dynamical property is not clear;

(ii) For our present model, the results indicate that when $R_0 \leq 1$, the virus cannot spread successfully; when $R_0 > 1$, the successful spread of virus depends on the initial value and varying parameters. If the initial occupying area $[-h_0, h_0]$ is beyond a critical size, namely $2h_0 \geq \pi \sqrt{acd/(bk\theta - acq)}$, then spreading happens regardless of the moving parameter $\mu$, $\beta$ and initial population density $(u_0, v_0, w_0)$. While $2h_0 < \pi \sqrt{acd/(bk\theta - acq)}$, whether spreading or vanishing happens depends on the initial population density $(v_0, w_0)$ and the moving parameter $\mu$ and $\beta$.

From a biological point of view, our model and results seem closer to the reality. On the other hand, our model shows more complex and precise dynamical properties from a mathematical point of view.

Before ending this paper, we mention that for the corresponding Cauchy problem:

\[
\begin{align*}
  u_t - d_1 \Delta u &= f_1(u, w), \\
  v_t - d_2 \Delta v &= f_2(u, v, w), \\
  w_t - d_3 \Delta w &= f_3(v, w), \\
  u(0, x) &= u_0(x), \\  v(0, x) &= v_0(x), \\  w(0, x) &= w_0(x)
\end{align*}
\]

we guess that the Basic Reproduction Number is still $R_0 = \theta kb/(acq)$ and it plays the same role as in the corresponding ODEs.

Acknowledgments: The authors would like to thank the reviewers for their helpful comments and suggestions that significantly improve the initial version of this paper.

References

[1] X. P. Wei, S. K. Ghosn, M. E. Taylor, et al., Viral dynamics in human immunodeficiency virus type 1 infection, Nature, 373 (1995), 117-122.

[2] A. S. Perelson, A. U. Neumann, M. Markowitz, et al., HIV-1 dynamics in vivo: virion clearance rate, infected cell life-span, and viral generation time, Science, 271 (1996), 1582-1586.

[3] M. A. Nowak, C. R. M. Bangham, Population dynamics of immune responses to persistent viruses, Science, 272 (1996), 74-79.

[4] M. A. Nowak, R. M. May, Virus dynamics: mathematical principles of immunology and virology, Oxford University Press, Oxford, UK, 2000.

[5] O. Stancevic, C. N. Angstmann, J. M. Murray, et al., Turing patterns from dynamics of early HIV infection, Bull Math Biol., 75 (2013), 774-795.

[6] Y. H. Du, Z. G. Lin, The diffusive competition model with a free boundary: Invasion of a superior or inferior competitor, Discrete Cont. Dyn. Syst. B., 19 (2014), 3105-3132.

[7] J. F. Zhao, M. X. Wang, A free boundary problem of a predator-prey model with higher dimension and heterogeneous environment, Nonlinear Anal.: RWA, 16 (2014), 250-263.
[8] Y. G. Zhao, M. X. Wang, Free boundary problems for the diffusive competition system in higher dimension with sign-changing coefficients, IMA J. Appl. Math., 81 (2016), 255-280.

[9] G. Bunting, Y. H. Du, K. Krakowski, Spreading speed revisited: Analysis of a free boundary model, Networks and Heterogeneous Media, 42 (7) (2012), 583-603.

[10] V. Capasso, L. Maddalena, Convergence to equilibrium states for a reaction-diffusion system modelling the spatial spread of a class of bacterial and viral diseases, J. Math. Biol., 13 (1981), 173-184.

[11] M. A. Lewis, G. Schmitz, Biological invasion of an organism with separate mobile and stationary states: Modelling and analysis, Forma, 11 (1996), 1-25.

[12] I. Ahn, S. Baek, Z. G. Lin, The spreading fronts of an infective environment in a man-environment-man epidemic model, Appl. Math. Modelling, 40 (2016), 7082-7101.

[13] S. Y. Liu, H. M. Huang, M. X. Wang, A free boundary problem for a prey-predator model with degenerate diffusion and predator-stage structure, Discrete Cont. Dyn. Syst.-B, 2020. Doi: 10.3934/dcdsb.2019245.

[14] S. Y. Liu, M. X. Wang, Existence and uniqueness of solution of free boundary problem with partially degenerate diffusion, Nonlinear Anal.: RWA, 54 (2020) 103097.

[15] J. Wang, J. F. Cao, The spreading frontiers in partially degenerate reaction-diffusion systems, Nonlinear Analysis, 122 (2015), 215-238.

[16] Y. H. Du, L. Ma, Logistic type equations on $\mathbb{R}^N$ by a squeezing method involving boundary blow-up solutions, J. London Math. Soc., 64 (2001), 107-124.

[17] Y. H. Du, Z. G. Lin, Spreading-Vanishing dichotomy in the diffusive logistic model with a free boundary, SIAM J. Math. Anal., 42 (2010), 377-405.

[18] Y. H. Du, M. X. Wang, M. L. Zhou, Semi-wave and spreading speed for the diffusive competition model with a free boundary, J. Math. Pures Appl., 107 (2017), 253-287.

[19] J. S. Guo, C. H. Wu, On a free boundary problem for a two-species weak competition system, J. Dyn. Diff. Equat., 24 (2012), 873-895.

[20] J. S. Guo and C. H. Wu, Dynamics for a two-species competition-diffusion model with two free boundaries, Nonlinearity, 28 (2015), 1-27.

[21] M. X. Wang, J. F. Zhao, Free boundary problems for a Lotka-Volterra competition system, J. Dyn. Diff. Equat., 26 (2014), 655-672.

[22] M. X. Wang and Y. Zhang, Note on a two-species competition-diffusion model with two free boundaries, Nonlinear Anal: TMA, 159 (2017), 458-467.

[23] J. P. Wang, M. X. Wang, The diffusive Beddington-DeAngelis predator-prey model with nonlinear prey-taxis and free boundary, Math. Meth. Appl. Sci., 41 (2018), 6741-6762.

[24] M. X. Wang, On some free boundary problems of the prey-predator model, J. Differential Equations, 256 (2014), 3365-3394.

[25] M. X. Wang, Y. Zhang, Dynamics for a diffusive prey-predator model with different free boundaries, J. Differential Equations, 264 (2018), 3527-3558.

[26] L. Li, W. J. Sheng, M. X. Wang, Systems with nonlocal vs. local diffusions and free boundaries, J. Math. Anal. Appl., 483 (2) (2020) 123646.
[27] M. X. Wang, *Existence and uniqueness of solutions of free boundary problems in heterogeneous environments*. Discrete Cont. Dyn. Syst.-B, 24 (2) (2019), 415-421.

[28] M. X. Wang, J. F. Zhao, *A free boundary problem for the predator-prey model with double free boundaries*, J. Dyn. Diff. Equat., 29 (3) (2017), 957-979.

[29] M. X. Wang, *A diffusive logistic equation with a free boundary and sign-changing coefficient in time-periodic environment*, J. Funct. Anal., 270 (2016), 483-508.

[30] W. D. Wang, X. Q. Zhao, *Basic reproduction numbers for reaction-diffusion epidemic models*, SIAM J. Appl. Dyn. Syst., 11 (2012), 1652-1673.

[31] M. X. Wang, Q. Y. Zhang, *Dynamics for the diffusive leslie-gower model with double free boundaries*, Discrete Cont. Dyn. Syst., 38 (5) (2018), 2591-2607.

[32] Q. Y. Zhang, M. X. Wang, *Dynamics for the diffusive mutualist model with advection and different free boundaries*, J. Math. Anal. Appl., 474 (2) (2019), 1512-1535.

[33] M. X. Wang, Y. Zhang, *The time-periodic diffusive competition models with a free boundary and sign-changing growth rates*, Z. Angew. Math. Phys., 67 (5) (2016), 132.

[34] M. X. Wang, Y. Zhang, *Two kinds of free boundary problems for the diffusive prey-predator model*, Nonlinear Anal.: RWA, 24 (2015), 73-82.