de Sitter extremal surfaces

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Abstract

We study extremal surfaces in de Sitter space in the Poincare slicing in the upper patch, anchored on spatial subregions at the future boundary $I^+$, restricted to constant boundary Euclidean time slices (focussing on strip subregions). We find real extremal surfaces of minimal area as the boundaries of past lightcone wedges of the subregions in question: these are null surfaces with vanishing area. We find also complex extremal surfaces as complex extrema of the area functional, and the area is not always real-valued. In $dS_4$ the area is real and has some structural resemblance with entanglement entropy in a dual $CFT_3$. There are parallels with analytic continuation from the Ryu-Takayanagi expressions for holographic entanglement entropy in $AdS$. We also discuss extremal surfaces in the $dS$ black brane and the de Sitter “bluewall” studied previously. The $dS_4$ black brane complex surfaces exhibit a real finite cutoff-independent extensive piece. In the bluewall geometry, there are real surfaces that go from one asymptotic universe to the other through the Cauchy horizons.
1 Introduction

de Sitter space is fascinating for many reasons, in particular for holographic explorations towards addressing questions of cosmology and time. In this regard, some versions of dS/CFT duality \cite{1, 2, 3} associate to de Sitter space a dual Euclidean CFT on the future timelike infinity $\mathcal{I}^+$ boundary (in the Poincare slicing). A concrete realization in the context of higher spin theories appears in \cite{4}. Further work on dS/CFT appears in e.g. \cite{5, 6, 7, 8, 9, 10, 11, 12, 13}.

In AdS/CFT, there has been considerable interest in understanding information theoretic notions in terms of geometric quantities via holography, in particular stemming from the Ryu-Takayanagi prescription \cite{14, 15} (see \cite{16, 17} for reviews) for calculating holographic entanglement entropy of a subsystem in the strongly coupled boundary field theory. This is the area of a bulk minimal surface (in Planck units) anchored at the subsystem interface and dipping inwards up to a certain maximal depth typically called the turning point. A different way to think about this appears in \cite{18}. More generally, these are extremal surfaces \cite{19}. In this light, one might speculate that the bulk subregion enclosed by the entangling surface and the boundary subsystem in some sense encodes bulk physics corresponding to that part of the boundary theory contained in the subsystem, although a detailed understanding of the hologram (and bulk locality) would seem more intricate.

It is interesting to consider these questions in the context of de Sitter space and dS/CFT. Assuming there is translation invariance with respect to a boundary Euclidean time direction, imagine constructing a subregion on a Euclidean time slice of the future boundary $\mathcal{I}^+$. Tracing out the complement of this subregion would lead to some loss of information and thereby give some associated entropy, which one might attribute to the subregion being entangled with the complement. In the bulk, intuition from the Ryu-Takayanagi prescription in AdS/CFT suggests that we study extremal surfaces in de Sitter space (in the Poincare slicing) on a constant boundary Euclidean time slice, defined as anchored on the subregion on the future (spacelike) boundary and dipping inwards (i.e. in the bulk time direction, towards the past). We find (sec. 2) that the bulk extremization problem exhibits some crucial sign differences from the AdS case. Focussing first on real surfaces, there are correspondingly some technical differences such as the absence of a natural turning point (where the surface stops dipping
inward). For sufficiently symmetric subregions such as strips (with an axis of symmetry), extremal surfaces can be defined as the union of two half-extremal-surfaces joined continuously but with a sharp cusp. Upon requiring that we choose minimal area, the extremal surfaces become null surfaces with zero area. In fact these are simply the boundaries of the past lightcone wedges of the subregion in question, and are thus the analogs of causal wedges associated with causal holographic information [20] (note that these bulk causal wedges and the corresponding causal holographic information in general do not coincide with the bulk entangling subregion, and entanglement entropy). This answer – past lightcone wedges – is well-defined for arbitrary boundary subregions, even without sufficiently high symmetry, and gives vanishing area.

It is interesting to look for complex saddle points of the extremization problem, motivated by considerations in $dS/CFT$. Focussing again on strip subregions, we indeed find these complex extremal surfaces: they exhibit “turning points” in the interior. They should be thought of as living in some auxiliary space, and are distinct from the bulk past lightcone wedges (which define real subregions in bulk $dS_4$). The area of these surfaces is in general not real-valued. In $dS_4$, we find that $x(\tau)$ parametrizing the strip width being real-valued suggests that the bulk time $\tau$ parametrizes a complex path $\tau = iT$. The area (in Planck units) of these surfaces is real-valued and has structural resemblance with entanglement entropy in a dual (non-unitary) 3-dim CFT, with central charge $C \sim -\frac{R^2}{G_4}$, with a leading area law divergence, which is negative, and subleading terms. There are parallels with analytic continuation from the Ryu-Takayanagi holographic entanglement expressions from $AdS$. In $dS_{d+1}$ with $d$ even, the nature of these complex surfaces is different and the area is pure imaginary. From the point of view of the dual Euclidean CFT living on the future boundary $I^+$, one might formally associate a density matrix w.r.t. boundary Euclidean time evolution and a reduced density matrix to the subregion obtained by tracing out the complement. It would be interesting to explore this further, perhaps in $dS/CFT$ as entanglement entropy in the dual Euclidean CFT.

We then discuss (sec. 3) an asymptotically de Sitter space [13] – the $dS$ black brane – where subleading normalizable metric components are turned on: in $dS_4/CFT_3$, they have the interpretation of saddle points representing the Euclidean CFT with uniform energy-momentum density expectation value. The corresponding extremal surfaces in the $dS_4$ black brane exhibit a finite cutoff-independent real-valued extensive piece (again negative) with some resemblance to a thermal entropy. Finally we discuss (real) extremal surfaces in the closely related $dS$ “bluewall” geometry, which are not obtained by analytic continuation: here there are real extremal surfaces which cross from one asymptotic universe to the other through the Cauchy horizons.
2 Extremal surfaces in de Sitter space

de Sitter space $dS_{d+1}$ in the Poincare slicing or planar coordinate foliation is given by the metric
\[ ds^2 = \frac{R_{dS}^2}{\tau^2}(-d\tau^2 + dw^2 + dx_i^2), \]
where half of the spacetime, e.g. the upper patch, has $I^+$ at $\tau = 0$ and a coordinate horizon at $\tau = -\infty$. This may be obtained by analytic continuation of a Poincare slicing of $AdS$,
\[ r \rightarrow -i\tau, \quad R_{AdS} \rightarrow -iR_{dS}, \quad t \rightarrow -iw, \]
where $w$ is akin to boundary Euclidean time, continued from time in $AdS$.

The dual Euclidean CFT is taken as living on the future $\tau = 0$ boundary $I^+$. We assume translation invariance with respect to a boundary Euclidean time direction, say $w$, and consider a subregion on a $w = \text{const}$ slice of $I^+$. One might imagine that tracing out the complement of this subregion then gives entropy in some sense stemming from the information lost. In the bulk, we study de Sitter extremal surfaces on the $w = \text{const}$ slice, analogous to the Ryu-Takayanagi prescription in $AdS/CFT$. Operationally these extremal surfaces begin at the interface of the subsystem (or subregion) and dip inwards (towards the past, in the bulk time direction). For simplicity, consider a strip on the $w = \text{const}$ surface (i.e. a constant boundary Euclidean time surface): this bulk $d$-dim subspace has metric
\[ ds^2 = \frac{R_{dS}^2}{\tau^2}\left(-d\tau^2 + \sum_{x_i \neq w} dx_i^2\right). \]
This is not a spacelike subspace in the bulk and it might seem that the extremal surfaces are timelike in general: however we will find that this is not the case.

2.1 Real extremal surfaces

Let us consider a strip subregion with width direction say $x$, the remaining $x_i$ being labelled $y_i$. A bulk surface on the $w = \text{const}$ slice bounding this subregion and dipping inward (towards the past) is bulk codim-2: its area functional in Planck units is
\[ S_{dS} = \frac{1}{4G_{d+1}} \int \prod_{i=1}^{d-2} \frac{R_{dS} dy_i}{\tau} \frac{R_{dS}}{\tau} \sqrt{d\tau^2 - dx^2} = \frac{R_{dS}^{d-1} V_{d-2}}{4G_{d+1}} \int \frac{d\tau}{\tau^{d-1}} \sqrt{1 - \left(\frac{dx}{d\tau}\right)^2}. \]
We consider extremizing the action to find extremal surfaces with minimal area, along the lines of the Ryu-Takayanagi prescription for entanglement entropy in $AdS$. The $S_{dS}$ extremization gives a conserved quantity ($\dot{x} \equiv \frac{dx}{d\tau}$)
\[ -\frac{\dot{x}}{\sqrt{1 - \dot{x}^2}} = B\tau^{d-1} \quad \Rightarrow \quad 1 - \dot{x}^2 = \frac{1}{1 + B^2\tau^{2d-2}} \quad \text{i.e.} \quad \dot{x}^2 = \frac{B^2\tau^{2d-2}}{1 + B^2\tau^{2d-2}}. \]
We see that \( \dot{x}^2 \to 0 \) near the boundary \( \tau \to 0 \). Assuming the conserved constant satisfies \( B^2 > 0 \) makes all the expressions real-valued and means \( \dot{x}^2 > 0 \), with \( \dot{x} \to 1 \) in the deep interior for large \( |\tau| \). For \( B^2 > 0 \), these are timelike surfaces. This gives the solution (upto boundary conditions) and corresponding area integral

\[
\dot{x}(\tau) = \pm \int \frac{B\tau^{d-1}d\tau}{\sqrt{1 + B^2\tau^{2d-2}}} \equiv \pm X(\tau), \quad S_{\text{ds}} = \frac{R_{dS}^{d-1}}{4G_{d+1}} V_{d-2} \int_\epsilon^{\tau_0} \frac{d\tau}{\tau^{d-1}} \frac{1}{\sqrt{1 + B^2\tau^{2d-2}}}. \tag{6}
\]

The main difference between this case and the minimal surface in \( AdS \) stems from \( B^2 > 0 \) implying that there is no smooth “turning point” where \( \ddot{x}^2 = \frac{B^2 + 2d - 2}{1 + B^2\tau^{2d-2}} \to \infty \). In fact \( B^2 > 0 \) means \( \dot{x}^2 \) is bounded, with \( 0 < \dot{x}^2 < 1 \). For any finite \( B^2 > 0 \), the extremal surface in this case begins to dip inwards from one boundary of the strip subregion and (rather than turning around as in \( AdS \)) continues indefinitely, eventually approaching \( \dot{x} \to \pm 1 \). With a view to associating a bulk subregion with the boundary subregion in question, let us artificially cut off the inward dipping surface at some interior location \( \tau = \tau_0 \), the bulk subregion then defined by the interior of the boundary strip subregion and the joined surface. So consider the half-extremal-surfaces,

\[
x_L(\tau) = X(\tau) - X(\tau_0) = -x_R(\tau), \quad x_L(0) = x_R(\tau_0), \quad x_L(0) = -\frac{l}{2} = -x_R(0) \quad \Rightarrow \quad \frac{l}{2} = X(\tau_0). \tag{7}
\]

This gives an extremal surface made of two half-extremal-surfaces joined continuously but with a sharp cusp at \( \tau_0 \) (see Figure 1). This defines the corresponding wedge-like bulk subregion, enclosed by this extremal surface and the boundary subregion. These conditions do not determine the parameters \( B, \tau_0 \) uniquely, given the subregion width \( l \). Varying \( B \) gives different extremal surfaces. By comparison, in the \( AdS \) case, the turning point \( \tau_0 = \frac{1}{B} \) is fixed by the global nature of the entangling surface as the location where \( \dot{x}^2 \to \infty \), the surface turning around.

To follow the Ryu-Takayanagi prescription, we would want to identify those extremal surfaces that have minimal area. From (6), we see that as \( B \) increases, the area \( S_{\text{ds}} \) decreases. Furthermore, (5) shows that as \( B \) increases, \( \dot{x}^2 \) increases and eventually approaches \( \dot{x}^2 \to 1 \) as \( B \to \infty \). In this limit, \( x(\tau) \to \pm \tau \) and \( S_{\text{ds}} \) appears to vanish. In fact this is a sensible result: in hindsight, it should have been obvious from (4) that minimal area arises when the extremal surface becomes null. This null extremal surface is in fact simply the boundary of the past bulk lightcone of the subregion.

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1. One might instead want to consider spacelike surfaces with \( \dot{x}^2 > 1 \) and therefore take, instead of (4), the area functional as \( S_{\text{ds}} = \frac{R_{dS}^{d-1}}{4G_{d+1}} V_{d-2} \int \frac{dx}{\tau} \sqrt{\left(\frac{dx}{d\tau}\right)^2 - 1} \). We will discuss this in the next subsection.

2. Note that surfaces with maximum area correspond to minimizing \( B \): this gives \( B = 0 \), which are disconnected surfaces \( x(\tau) = \text{const} \), with area \( S_{\text{ds}} \sim \frac{R_{dS}^{d-1}}{4G_{d+1}} V_{d-2} \int_\epsilon^{\tau_0} \frac{d\tau}{\tau} \) with a leading divergent piece.
Figure 1: Extremal surfaces in de Sitter made of two half-extremal surfaces joined continuously but with a sharp cusp at \( \tau_0 \). As \( B \) increases (till eventually \( B \gg \frac{1}{e^{\frac{1}{d-1}}} \)), the surface approaches \( x^2 \to 1 \) (figure on the right).

An alternative argument corroborating the above conclusion is the following. Physically, the shortest length (or time) scale here is \( \tau_{UV} = \epsilon \) so that in (4) when \( B e^{d-1} \gg 1 \) we can approximate \( x^2 \sim 1 \) and so \( x(\tau) \sim \pm \tau \) giving \( \frac{1}{2} \sim \tau_0 \). Thus one might estimate (6) as

\[
S_{dS} \sim \frac{R_{dS}^{d-1}}{4G_{d+1}} V_{d-2} \int_0^{\tau_0} \frac{d\tau}{\tau^{d-1}} + \frac{R_{dS}^{d-1}}{4G_{d+1}} V_{d-2} \int_0^{\tau_0} \frac{d\tau}{\tau^{d-1}} \left( \frac{1}{\sqrt{1 + B^2 \tau^{2d-2}}} - 1 \right),
\]

where the second integral can be seen to vanish as \( \tau \to 0 \). Now the first integral scales as \( \frac{V_{d-2}}{\epsilon^{d-1}} \) while the second integral can be expressed in terms of the hypergeometric function

\[
2F_1 \left( \frac{1}{2}, \frac{3-2d}{2} , \frac{3-2d}{2} - \frac{1}{B^2 \tau^{2d-2}} \right) - B^2 \tau_0^{2d-2}) \quad \text{(the extremal surface in (3) is itself expressed as \( x(\tau) = \pm \tau 2F_1 \left( \frac{1}{2}, \frac{3-2d}{2} , \frac{3-2d}{2} - \frac{1}{B^2 \tau^{2d-2}} \right) \) or \( \pm B \tau^d 2F_1 \left( \frac{1}{2}, \frac{d-2}{d-2} , \frac{3d-2}{2d-2} , -B^2 \tau^{2d-2} \right) \), using the integral representations of \( 2F_1 \)). As \( B^2 \) increases, this second integral is seen to scale as

\[-\frac{R_{dS}^{d-1}}{G_{d+1}} V_{d-2} B^{(d-2)/(d-1)} \]

Thus when \( B \sim \frac{1}{\epsilon^{d-1}} \) this cancels the earlier contribution and we again see the leading \( S_{dS} \) scaling to be vanishing.

In \( dS_3 \) (i.e. \( d = 2 \)), we obtain \( x(\tau) = \pm \int \frac{R_{dS} d\tau}{\sqrt{1 + B^2 \tau^2}} = \pm \frac{1}{B} \sqrt{1 + B^2 \tau^2} \) and the boundary conditions give \( \frac{1}{B} \left( \sqrt{1 + B^2 \tau_0^2} - 1 \right) = \frac{1}{2} \), the area integral becoming \( S_{dS} = \frac{R_{dS} d\tau}{4G_3} \int_0^{\tau_0} \frac{d\tau}{\sqrt{1 + B^2 \tau^2}} \). Analysing these vindicates the conclusions above.

It is worth noting that our construction of joining two half-extremal-surfaces appears invalid unless the subsystem has sufficiently high symmetry (in particular an axis of symmetry). Relatedly, one might look askance at the entire extremization procedure here: if we allow non-smooth surfaces (with cusps), one might wonder if more general surfaces need to be considered, e.g. a zigzag null surface formed by joining multiple partial surfaces with multiple cusps. This would be useful to systematise more rigorously. However the final answer, the past lightcone wedge, is well-defined for an arbitrary subregion, comprising two piecewise smooth extremal surfaces joined with just a single cusp (rather than multiple cusps). The past lightcone wedge boundary is however a complicated surface: it would be interesting to understand the shape dependence here. The resulting area is of course always zero for all these null surfaces, and does not reflect entanglement structure.

From the point of view of bulk de Sitter alone, one could consider volume subregions in
the full $d$-dim boundary $I^+$ (at $\tau = 0$), i.e. not on the constant boundary Euclidean time slice. These would give codim-1 surfaces. For a strip subregion with width direction say $x$, the remaining $x_i$ being labelled $y_i$, analysing the area integral of the bulk surface for extremization gives

\[ S_{dS} \sim R^d_{dS} V_{d-1} \int \frac{d\tau}{\tau^d} \sqrt{1 - \left(\frac{dx}{d\tau}\right)^2}, \quad -\frac{\dot{x}}{\sqrt{1 - \dot{x}^2}} = B \tau^d, \]

\[ \Rightarrow \dot{x}^2 = \frac{B^2 \tau^{2d}}{1 + B^2 \tau^{2d}}, \quad S_{dS} = R^d_{dS} V_{d-1} \int \tau_i^0 \frac{d\tau}{\tau^d} \frac{1}{\sqrt{1 + B^2 \tau^{2d}}}. \tag{9} \]

This has volume scaling. Again $S_{dS}$ decreases with increasing $B$, with $\dot{x}^2 \to 1$: the resulting extremal surfaces are null surfaces defining the past lightcone wedges of the volume subregion, with vanishing area.

### 2.2 Complex extremal surfaces

For what follows, it is useful to recall the $dS/CFT$ correspondence, for de Sitter space in the Poincare slicing (1), obtained by analytic continuation (2) of Poincare AdS. A version of $dS/CFT$ (1,2,3) states that quantum gravity in de Sitter space is dual to a Euclidean CFT living on the boundary $I^+$ or $I^-$. More specifically, the CFT partition function with specified sources $\phi_{i0}(\vec{x})$ coupled to operators $O_i$ is identified with the wavefunctional of the bulk theory as a functional of the boundary values of the fields dual to $O_i$ given by $\phi_{i0}(\vec{x})$. In the semiclassical regime this becomes $Z_{CFT} = \Psi[\phi_{i0}(\vec{x})] \sim e^{i I_{cl} [\phi_{i0}]}$ where we need to impose regularity conditions on the past cosmological horizon $\tau \to -\infty$: e.g. scalar modes satisfy $\phi_k(\tau) \sim e^{ik\tau}$, which are Hartle-Hawking (or Bunch-Davies) initial conditions. Operationally, certain $dS/CFT$ observables can be obtained by analytic continuation (2) from $AdS$ (see e.g. [3], as well as [5]). The Bunch-Davies initial condition itself can be thought of as analytic continuation of regularity in the $AdS$ interior. Complex saddle points thus appear in $dS/CFT$. From this point of view, it is natural to ask if there are additional (perhaps complex) extrema of the area functional that could be considered in de Sitter space, with possible $dS/CFT$ interpretations.

With a view to considering spacelike surfaces with $\dot{x}^2 > 1$, let us take, instead of (4), the $dS_{d+1}$ area functional on a $w = \text{const}$ slice as

\[ S_{dS} = \frac{R^d_{dS}}{4G_{d+1}} \int \frac{d\tau}{\tau^{d-1}} \sqrt{\left(\frac{dx}{d\tau}\right)^2 - 1}, \quad \frac{\dot{x}}{\sqrt{\dot{x}^2 - 1}} = A \tau^{d-1}, \tag{10} \]

where we are considering strip subsystems with width along $x$. The second expression above is the conserved quantity obtained in the extremization. This is essentially the same as (5),

\[3\text{Recall that complex geodesics appeared in [21] in the context of the black hole interior. Complex extremal surfaces have recently appeared in a different context [22], as well as [23].} \]
but with $B^2 = -A^2 < 0$, and $A$ being real. One might ask if this can be interpreted as a real surface $\dot{x}^2 = \frac{A^2 \tau^{2d-2}}{1 - A^2 \tau^{2d-2}}$. However we need to require that the surface reaches the boundary $\tau \to 0$ from where it drops down (inward): near $\tau \to 0$, we find $\dot{x}^2 \sim -A^2 \tau^{2d-2}$, so that

$$\dot{x}^2 = \frac{-A^2 \tau^{2d-2}}{1 - A^2 \tau^{2d-2}},$$

this being a complex surface in some sense.

Let us focus now on $dS_4/CFT_3$ for concreteness, to understand this better. The extremal surface near $\tau \to 0$ in this case is $x(\tau) \sim \pm iA\tau^3 + x(0)$. We want $x(\tau)$ to be real-valued since it parametrizes a space direction in the dual $CFT_3$: this requires that $\tau$ takes imaginary values.

In more detail, near $\tau \to 0$, we have $x \to \pm \frac{i}{2}$ and the two ends of the surface are parametrized as $x_L(\tau) \sim -\frac{i}{2} + iA\tau^3$ and $x_R(\tau) \sim \frac{i}{2} - iA\tau^3$. For $x_{L,R}$ to be real-valued with $A$ real, we must have pure imaginary $\tau = iT$ with $T$ real, giving $x_L \sim -\frac{i}{2} + AT^3 \sim -x_R$: as $T$ increases, $x_L$ increases from $-\frac{i}{2}$ and $x_R$ decreases from $\frac{i}{2}$. The global structure of the surface shows a “turning point” at $\tau_* A^2 = 1$, where $\dot{x}^2 \to \infty$, very similar to the situation in $AdS$. From the point of view of the discussion in the previous subsection, the two half-extremal-surfaces $x_L, x_R$ in this case join smoothly at the turning point $\tau_*$ as in $AdS$, with $x_L(\tau_*) = 0 = x_R(\tau_*)$ and $\dot{x}_L, \dot{x}_R$ matching. This gives the width

$$\Delta x \frac{1}{2} = \frac{i}{2} = \frac{i}{\sqrt{1 - A^2 \tau^2}} \int_{0}^{\tau_*} \frac{A \tau^2}{\sqrt{1 - A^2 \tau^2}} d\tau \quad \Rightarrow \quad \tau_* \sim i \frac{l}{A}.$$  

The reality of $\Delta x = l$ with $A$ real again suggests that we parametrize the $\tau$-integral over the path $\tau = iT$ in a complex $\tau$-plane, we have then rescaled $T$ using $A$ to make the integration variable dimensionless (and $# = \int_{0}^{1} y^2 dy = \sqrt{\pi \Gamma(\frac{5}{4})}$). The turning point here is $\tau_* = \frac{i}{\sqrt{A}}$. The integral can be parametrized in terms of hypergeometric functions $_2F_1$. The extremal surface $x(\tau)$ with $\tau$ imaginary does not correspond to any real bulk subregion in $dS_4$ enclosed by the surface, but really lives in some auxiliary space. In a sense, the structure here is very much like analytic continuation of the $AdS_4$ expressions a la Ryu-Takayanagi: we will discuss this more below. From that point of view, since the analytic continuation [2] faithfully maps $AdS_4 \leftrightarrow dS_4$, this is a faithful map from the subsystem to the auxiliary bulk subregion. The area now becomes

$$S_{dS} = 2 \frac{R_{dS}^2}{4G_4} \frac{V_1}{V_{UV}} \int_{\tau_{UV}}^{\tau_*} \frac{d\tau}{\tau^2} \frac{1}{\sqrt{A^2 \tau^4 - 1}} = -i \frac{R_{dS}^2}{4G_4} \frac{V_1}{V_{UV}} \int_{\tau_{UV}}^{\tau_*} \frac{d\tau}{\tau^2} \frac{2}{\sqrt{1 - A^2 \tau^4}}.$$  

\[4\]Strictly speaking, this may be too restrictive. We have required $x(\tau)$ for all $\tau$ be real-valued: this means that each point on the surface directly maps to a corresponding real-valued spatial location within the strip in the dual $CFT_3$. One might instead think that one need only require the boundary value $x(0)$ be real, which would not restrict the $\tau$-path. This would suggest more general complex extremal surfaces defined over complex $\tau$-space, with the width $\Delta x$ required to be real-valued. See e.g. [23] for some discussions along these lines: I thank S. Fischetti for a discussion on this point.

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In principle, we could assign ±i in the second expression, as a choice of the branch of the square root: the choice of the minus sign leads to an appropriate coefficient as we see below. The integral itself is just as in AdS, giving

\[ S_{dS} = -i \frac{R_{dS}^2}{2G_4} V_1 \left( \frac{1}{\gamma_{UV}} - c_3 \frac{1}{\tau_x} \right) = -\frac{R_{dS}^2}{2G_4} V_1 \left( \frac{1}{\epsilon} - c_3 \frac{1}{\tau} \right) \sim C V_1 \left( \frac{1}{\epsilon} - c_3 \frac{1}{\tau} \right), \quad (14) \]

where \( c_3 = 2\pi (\frac{\Gamma(\frac{1}{4})}{\Gamma(\frac{3}{4})})^2 \) is a constant as in AdS, stemming the finite cutoff-independent part of the integral. Note that here we have used the relation \( \tau_{UV} = i\epsilon \) for the ultraviolet cutoff in the dual Euclidean field theory, suggested by previous investigations\footnote{See e.g. \cite{3, 5, 4, 11}, which discuss this (in some cases implicitly). Heuristically, we expect that evolution in the bulk direction is encoded by renormalization group flow in the dual field theory: see e.g. \cite{24, 25, 26, 27, 28, 29} and more recently e.g. \cite{30, 31} for discussions on this in the AdS context. In the present dS case, the bulk description is time evolution \( i \delta_{\tau} \Psi = \mathcal{H} \Psi \), with \( \mathcal{H} \) being an evolution operator. Through dS/CFT, this becomes \( \delta Z_{CFT} \tau = \mathcal{O} Z_{CFT} \), if \( \tau_{UV} = i\epsilon \), with \( \mathcal{O} \) an appropriate operator generating RG flow. If we view \( \epsilon \) as a floating RG parameter, this again suggests the path \( \tau = iT \) in complex \( \tau \)-space for a dS/CFT interpretation.} in dS/CFT. Also we have rewritten the last expression in the last line in terms of the central charge \( C \sim -\frac{R_{dS}^2}{G_4} \). The first term resembles an area law divergence \cite{32, 33}, proportional to the area of the interface between the subregion and the environment, in units of the ultraviolet cutoff. It is also proportional to the central charge which represents the number of degrees of freedom in the dual CFT: in this case, \( \mathcal{C} < 0 \) reflecting the fact that the CFT is non-unitary. The second term is a finite cutoff independent piece. Whether the expression \( (14) \) should really be thought of physically as holographic entanglement entropy in dS/CFT\textsubscript{3} is less clear, the dual CFT being non-unitary with central charge \( C = -\frac{R_{dS}^2}{G_4} < 0 \).

In some sense, \( -S_{dS} \) appears to resemble entanglement entropy in AdS/CFT, sharing various features including subadditivity. For instance, the quantity \( I_{dS}[A, B] = S_{dS}[A] + S_{dS}[B] - S_{dS}[A \cup B] \) for two disjoint subsystems \( A, B \), exhibits various properties of holographic mutual information in AdS including an analog of the disentangling transition in the classical gravity approximation \cite{34}, but with some crucial differences. For strip subregions that are sufficiently nearby but disjoint, \( I_{dS}[A, B] \) is nonzero: e.g. using \( (14) \) for a single strip, we obtain for two parallel strips of equal width \( l \) and separation \( x \),

\[ I_{dS}[A, B] = S_{dS}[A] + S_{dS}[B] - S_{dS}[A \cup B] \sim -\frac{R_{dS}^2}{G_4} c_3 V_1 \left( -\frac{2}{l} + \frac{1}{x} + \frac{1}{2l + x} \right). \quad (15) \]

\( S_{dS}[A \cup B] \) arises from the area of the connected surface between \( A, B \) as \( S(2l + x) + S(x) \). This is similar to the structure of holographic mutual information for strips in AdS\textsubscript{4}, e.g. the UV divergent pieces cancel, with a cutoff-independent divergence \( \mathcal{O} \frac{1}{x} \) as the subregions collide. The striking difference is that \( I_{dS}[A, B] \leq 0 \), rather than positive definite, following
from the fact that \( C = -\frac{R_{ds}^2}{G_{d+1}} < 0 \). Thus \( I_{ds}[A, B] \) is large and negative when the subregions are nearby, then increases as the separation \( \frac{x}{T} \) increases, and eventually approaches zero as \( \frac{x}{T} \to \frac{\sqrt{6}-1}{2} \sim 0.62 \). Beyond this critical value, \( I_{ds}[A, B] \) vanishes identically and the two subregions are disconnected. This disentangling transition in the classical gravity approximation arises from the transition between the connected and disconnected surface for \( A \cup B \). What we are seeing is that \( S_{ds}[A] + S_{ds}[B] \leq S_{ds}[A \cup B] \), i.e. \( -S_{ds} \) satisfies strong subadditivity for disjoint parallel strips \( A, B \). By comparison, using the real lightcone wedge surfaces in the previous subsection, we see that disjoint boundary subregions are always causally disconnected and thus uncorrelated for any nonzero separation. Correlation functions are nonzero: the disentangling transition above is in the classical gravity approximation, and we expect subleading terms in a large-N-like expansion of \( I_{ds}[A, B] \) (see e.g. [35] in the AdS context).

We now discuss \( dS_{d+1} \) for even \( d \) (in particular \( dS_3, dS_5 \)) where the nature of these extremal surfaces is different. We would like to retain the relation \( \tau_{UV} = i\epsilon \) as following quite generally in \( dS/CFT \) from time evolution mapping to renormalization group flow. This suggests we parametrize the bulk time parameter \( \tau \) along a complex path \( \tau = iT \) as in \( dS_4 \). However now with \( A^2 > 0 \) the surface (11) near \( \tau \to 0 \) gives \( \dot{x} \sim \pm iA\tau^{d-1} \), i.e. \( x(\tau) \sim \pm iA\tau^d \). Thus \( x \) cannot be made real-valued for any even \( d \) in this manner. A way out is to take the parameter \( A^2 \to -A^2 \): the surface equation now is the same as (5) but with the bulk time parametrized as \( \tau = iT \). The expressions (10), (11) then give

\[
\dot{x}^2 = \frac{A^2\tau^{2d-2}}{1 + A^2\tau^{2d-2}}, \quad S_{dS} = -i\frac{R_{ds}^{d-1}}{4G_{d+1}} V_{d-2} \int_{\tau_{UV}}^{\tau_{s}} \frac{d\tau}{\tau^{d-1}} \frac{2}{\sqrt{1 + A^2\tau^{2d-2}}} \int_{\tau_{s}}^{T_s} \frac{dT}{T^{d-1}} \frac{1}{\sqrt{1 + (-1)^{d-2}A^2T^{2d-2}}} . \tag{16}
\]

For even \( d \), the \((-1)^{d-1}\) gives rise to a “turning point” at \( T_s^{2d-2}A^2 = 1 \): the width now scales as \( l \sim T_s \sim -i\tau_s \). The integral is as in \( AdS \), giving

\[
S_{dS} \sim i^{1-d}\frac{R_{ds}^{d-1}}{2G_{d+1}} V_{d-2} \left( \frac{1}{\epsilon^{d-2}} - c_{d} \frac{1}{l^{d-2}} \right) . \tag{17}
\]

The leading divergence \( S_{dS}^{\text{div}} \sim i^{1-d}\frac{R_{ds}^{d-1}}{2G_{d+1}} V_{d-2} \) resembling an area law, appears independent of the shape of the subregion, expanding (10) and assuming that \( \dot{x} \) is small near the boundary \( \tau_{UV} \). Unlike \( dS_4 \), note that \( S_{dS} \) in \( dS_{d+1} \) with even \( d \) is not real-valued, in particular for \( dS_3, dS_5 \). For instance, in \( dS_3 \), we obtain from (16)

\[
\tau = iT , \quad x(\tau) \sim \pm \frac{1}{A} \sqrt{1 + A^2\tau^2} , \quad S_{dS} \sim -i\frac{R_{ds}}{G_3} \log \frac{\tau_s}{\tau_{UV}} = -i\frac{R_{ds}}{G_3} \log \frac{l}{\epsilon} . \tag{18}
\]

Note that \( x(\tau) \) appears real, although the parametrization is \( \tau = iT \).
It is interesting to recall the Ryu-Takayanagi expression for entanglement entropy for an (infinitely long) strip-shaped subsystem with width along the $x$-direction, given as the area of the corresponding minimal surface in the bulk $AdS_{d+1}$ geometry (with radius $R$),

$$ S_{AdS}[R, x(r), r] = \frac{R^{d-1}}{4G_{d+1}} V_{d-2} \int \frac{dr}{r^{d-1}} \sqrt{1 + \left( \frac{dx}{dr} \right)^2}, \quad (x')^2 = \frac{A^2 r^{2d-2}}{1 - A^2 r^{2d-2}}, \quad (19) $$

where the conserved quantity $A$ in the extremization is related to the turning point as $r_*^{d-1} = \frac{1}{A}$. Noting that $dS_{d+1}$ in Poincare slicing (3) is just the analytic continuation of the corresponding $t = constant$ spatial slice in $AdS_{d+1}$, obtained by (2), i.e. $r \rightarrow -i\tau, \ t \rightarrow -iw, \ R \rightarrow -iR_{ds}$, let us carry out this analytic continuation on the Ryu-Takayanagi expression. Indeed we see that $S_{ds}$ in (4) appears very much like the analytic continuation of $S_{AdS}[x(r), r]$, with the various factors of $i$ conspiring to leave a single $i$ behind, i.e.

$$ S_{AdS}[R, x(r), r] \rightarrow -i \frac{R_{ds}^{d-1}}{4G_{d+1}} V_{d-2} \int \frac{d\tau}{\tau^{d-1}} \sqrt{1 - \left( \frac{dx}{d\tau} \right)^2} = S_{ds}[R_{ds}, x(\tau), \tau]. \quad (20) $$

On the analytic continuation of the extremization itself, we obtain

$$ -\dot{x}^2 = \frac{(-1)^{2d-2} A^2 \tau^{2d-2}}{1 - (-1)^{2d-2} A^2 \tau^{2d-2}}, \quad i.e. \quad \dot{x}^2 = \frac{(-1)^{d-1} A^2 \tau^{2d-2}}{1 - (-1)^{d-1} A^2 \tau^{2d-2}}, $$

$$ S_{ds} = -i \frac{R_{ds}^{d-1}}{4G_{d+1}} V_{d-2} \int \frac{d\tau}{\tau^{d-1}} \frac{1}{\sqrt{1 - (-1)^{d-1} A^2 \tau^{2d-2}}}. \quad (21) $$

This expression corroborates the minus sign in (16) and (13), (14). The analytic continuation essentially recovers our earlier calculations in $dS_4$ and $dS_{d+1}$ for even $d$. For instance, in $dS_5$ (i.e. $d = 4$), we obtain

$$ \dot{x}^2 = \frac{A^2 \tau^{6}}{1 + A^2 \tau^{6}}, \quad S_{ds} = -i \frac{R_{ds}^{3}}{4G_5} V_2 \int \frac{d\tau}{\tau^3 \sqrt{1 + A^2 \tau^{6}}}. \quad (22) $$

With real $A$, this is as such a real extremal surface as in the previous subsection: taking $A$ large minimises the area and we obtain the null surfaces earlier with vanishing area representing the past lightcone wedge of the subregion. However parametrizing as $\tau = iT$, there is a turning point at $\tau_* = \frac{i}{A^2}$, and a corresponding complex surface and corresponding area given by (16). The area in (22) then becomes $S_{ds} \sim -i \frac{R_{ds}^{3}}{4G_5} V_2 (\tau^4 \tau^{6})$. The extra $i$ can be thought of as arising from the odd powers of $R_{ds}$ under the analytic continuation from $AdS_5$. It is interesting to note that for even $d$ (in particular, $dS_3$ and $dS_5$), the expression $S_{ds}$ obtained by analytic continuation of the Ryu-Takayanagi entanglement prescription appears similar to (5), (6), parametrizing $\tau$ as real-valued, giving real null extremal surfaces with vanishing area. There are complex surfaces here also with $\tau = iT$, and the corresponding area $S_{ds}$ is imaginary.

To summarize, we have studied bulk de Sitter codim-2 extremal surfaces. Real extremal surfaces are the boundaries of the past lightcone wedges of the boundary subregions, with
vanishing area. Complex extremal surfaces have some structural resemblance with entanglement entropy in a dual CFT. In $dS_i/CFT_3$, the area is real-valued and negative: in this sense, these complex surfaces have lower area, suggesting that they are the preferred minimal surfaces. Our calculations here have been done for a strip subregion but it would appear that generalizations to other subregion shapes will exhibit similar features. For instance, the spherical subregion extremal surface presumably exhibits a logarithmic term with with interesting coefficient.

It is worth noting that this analysis of bulk extremal surfaces is different from studies of entanglement entropy of bulk fields in de Sitter space e.g. [36, 37, 38, 39].

3 Extremal surfaces in the $dS$ black brane

We now study extremal surfaces in the asymptotically $dS$ spacetime studied in [13], i.e.

$$ds^2 = \frac{R_{AdS}^2}{\tau^2} \left( -{\frac{d \tau^2}{1 + \alpha \tau_0^d \tau^d}} + (1 + \alpha \tau_0^d \tau^d)dw^2 + \sum_{i=1}^{d-1} dx_i^2 \right),$$

(23)

with $\alpha$ a complex phase and $\tau_0$ is some real parameter of dimension length inverse. An analog of regularity in the interior for an asymptotically $AdS$ solution is obtained here by a Wick rotation $\tau = il$ and demanding that the resulting spacetime (thought of as a saddle point in a path integral) in the interior approaches flat Euclidean space in the $(l, w)$-plane with no conical singularity. This makes the $w$-coordinate angular with fixed periodicity (and $l$ is a radial coordinate), giving $\alpha = -(d-i)^d$, $l \geq \tau_0$, $w \simeq w + \frac{4\pi}{(d-1)\tau_0}$. Thus the spacetime (23) is a complex metric which satisfies Einstein’s equation with a positive cosmological constant $R_{MN} = \frac{d}{R_{AdS}^2}g_{MN}$, $\Lambda = \frac{d(d-1)}{2R_{AdS}^2}$. This resulting metric satisfying regularity is equivalent to one obtained by analytically continuing the Euclidean $AdS$ black brane

$$ds^2 = \frac{R_{AdS}^2}{r^2} \left( \frac{dr^2}{1 - r_0^d r^d} + (1 - r_0^d r^d)d\theta^2 + \sum_{i=1}^{d-1} dx_i^2 \right),$$

(24)

where $\theta \sim \theta + \frac{4\pi}{(d-1)\tau_0}$, to the asymptotically de Sitter spacetime (23) using (2) and we identify $r_0 \equiv \tau_0$, giving the phase $\frac{-1}{(d-1)i}$. The regularity criterion is simply the analog of regularity of the $EAdS$ black brane. The condition $l \geq \tau_0$ is equivalent to the radial coordinate having the range $r \geq r_0$. We see that “normalizable” metric pieces are turned on in (23). We then expect a nonzero expectation value for the energy-momentum tensor here, as in the $AdS$ context [40, 41, 28, 29]. In the present case [13], we have $T_{ij} = \frac{2}{\sqrt{h}} \frac{\delta Z_{CFT}}{\delta h^{ij}} = \frac{2i}{\sqrt{h}} \frac{\delta \Psi}{\delta h^{ij}} \propto iR_{AdS}^{l-1}g^{(d)}_{ij}$, where $g^{(d)}_{ij}$ is the coefficient of the normalizable $\tau^{d-2}$ term in the Fefferman-Graham expansion of the metric (23). This definition of $T_{ij}$ is natural for a CFT with partition function $Z_{CFT}$, equated with $\Psi$: thus, most notably, the $i$ arising from $\Psi$, the wavefunction of the universe, implies that the energy-momentum tensor is real only if $g^{(d)}_{ij}$ is pure imaginary. In effect, this $dS/CFT$
energy-momentum tensor can be thought of as the analytic continuation of the EAdS one. The spacetime \([23]\) for \(dS_4/CFT_3\) gives real \(T_{ij}\), with \(T_{uu} = -\frac{R_{dS}^d}{G_4} \tau^3_0 \) with \(T_{ww} + (d-1)T_{ii} = 0\).

The \(w\)-coordinate is naturally interpreted as Euclidean time from the structure of the energy-momentum tensor: so let us now consider a strip subregion on a \(w = \text{const} \) surface in \([23]\). The area functional (in Planck units) of a bulk surface bounding this strip and dipping inwards is

\[
S_{dS} = -i \frac{R_{dS}^{d-1}}{G_{d+1}} V_{d-2} \int \frac{d\tau}{\tau^{d-1}} \sqrt{\frac{1}{1 + \alpha r^d_0 \tau^d}} - \left(\frac{dx}{d\tau}\right)^2,
\]

(25)
defined so that for \(\tau_0 = 0\), this reduces to our de Sitter discussion in sec. 2.2. For the \(dS_4\) brane (i.e. \(d = 3\)), we obtain for the extremization,

\[
\Delta x = \int_0^{\tau_*} \frac{i A \tau^2 d\tau}{\sqrt{(1 - i \tau^3_0 \tau^3)(1 - A^2 \tau^4)}}, \quad S = -i \frac{V_1 R_{dS}^2}{4G_4} \int_{\tau_{UV}}^{\tau_*} \frac{d\tau}{\tau^2} \sqrt{(1 - i \tau^3_0 \tau^3)(1 - A^2 \tau^4)}.
\]

(26)

Now for small width \(l\), this is essentially similar to the previous discussion on pure \(dS_4\) and we have \(\frac{\Delta x}{2} \equiv \delta \sim \tau_* = \frac{i}{\sqrt{A}}\), where \(A\) is real. In particular, the width \(\Delta x\) being real-valued suggests that \(\tau\) parametrizes a complex path \(\tau = iT\) with \(T\) real. As \(l\) increases however, the other denominator approaches a zero also, with \(\tau \to \frac{i}{\tau_0}\). In this limit, we thus have \(\tau_* \to \frac{i}{\sqrt{A}} \sim \frac{i}{\tau_0}\) and large \(l \sim -i\tau_*\), obtained from the double zero as

\[
\Delta x = \int_0^{\tau_*} \frac{i A \tau^2 d\tau}{\sqrt{(1 - i \tau^3_0 \tau^3)(1 - A^2 \tau^4)}} \int_0^{T_*} \frac{i A \tau^3 d\tau}{\sqrt{(1 - i \tau^3_0 \tau^3)(1 - A^2 \tau^4)}}.
\]

(27)

Note that reality of the width \(\Delta x\) implies now that the range of \(T\) is restricted as \(T \leq \frac{1}{\tau_0}\) i.e. asymptotically \(\tau \to \frac{1}{\tau_0}\). This is similar to the fact that in the \(AdS\) black brane, static minimal surfaces in the IR limit (large subsystem width) wrap the horizon but do not penetrate beyond.

Now the area integral exhibits a cutoff-independent piece which can be estimated from the contribution in the deep interior where \(\tau \to \tau_*\): the contribution to the integral near the double zero thus scales as \(i\Delta x\) giving

\[
S_{\text{fin}} \sim -i \frac{V_1 R_{dS}^2}{G_4} \frac{1}{\tau_*^2}(il) = -\frac{R_{dS}^2}{G_4} \tau_0^2 V_1 l \equiv CT_0^2 V_1 l,
\]

(28)

which resembles an extensive thermal entropy in a 3-dim CFT with central charge \(C \sim -\frac{R_{dS}^2}{G_4}\) at temperature \(T_0 \equiv \tau_0\). Note that \(S_{\text{fin}} < 0\).

We recall that the entanglement entropy area functional for the \(AdS_{d+1}\) black brane from the Ryu-Takayanagi prescription is \(S = \frac{V_{d-3} R_{d-1}}{4G_{d+1}} \int_{r_{\text{min}}}^{r_{\text{max}}} \sqrt{(\partial_r x)^2 + \frac{1}{1 - r_0^d r^d}}\), giving

\[
(x')^2 = \frac{A^2 r^{2d-2}}{(1 - r_0^d r^d)(1 - A^2 r^{2d-2})}, \quad S = \frac{V_{d-3} R_{d-1}}{4G_{d+1}} \int_{r_{\text{min}}}^{r_{\text{max}}} \frac{2}{r^{d-1} \sqrt{(1 - r_0^d r^d)(1 - A^2 r^{2d-2})}}.
\]

(29)
Under the analytic continuation, we obtain

\[
\dot{x}^2 = \frac{-(1)^{d-1}A^2\tau^{2d-2}}{(1 - (-i)^d\tau^d)(1 - (-1)^{d-1}A^2\tau^{2d-2})},
\]

\[
S = \frac{V_{d-2}R_{dS}^{d-1}}{4G_{d+1}} \int_{\tau_{UV}}^{\tau} \frac{-i d\tau}{\tau^{d-1}} \frac{2}{\sqrt{(1 - (-i)^d\tau^d)(1 - (-1)^{d-1}A^2\tau^{2d-2})}}. \tag{30}
\]

For generic dimension \(d\), we see that \(S\) is not real, as in the earlier discussion with \(\tau_0 = 0\).

### 3.1 The de Sitter bluewall

We now explore metrics of the form (23), but with the parameter \(\alpha = -1\) here\(^6\), i.e.

\[
ds^2 = \frac{R_{dS}^2}{\tau^2} \left( -\frac{d\tau^2}{1 - \tau_0^d \tau^d} + (1 - \tau_0^d \tau^d)dw^2 + dx_i^2 \right) \equiv \frac{R_{dS}^2}{\tau^2} \left( -\frac{d\tau^2}{f(\tau)} + f(\tau)dw^2 + dx_i^2 \right). \tag{31}
\]

The \(w\)-coordinate here has the range \(-\infty \leq w \leq \infty\). This spacetime\(^7\) has a Penrose diagram shown in Figure 2 which resembles that of the \(AdS\) black brane rotated by \(\frac{\pi}{2}\): there are two asymptotically \(dS\) universes (for \(\tau \lesssim \frac{1}{\tau_0}\)), and timelike singularities cloaked by Cauchy horizons at \(\tau = \frac{1}{\tau_0}\), which “cross” at a bifurcation region. The Penrose diagram has many similarities with the interior of the Reissner-Nordstrom black hole (or wormhole). Late time infalling observers near the Cauchy horizon see incoming light rays from early times as highly blueshifted, essentially stemming from light rays “crowding” near the Cauchy horizon, suggesting an instability. It is unclear if this spacetime has any interpretation in \(dS/CFT\): nevertheless, formally, one finds the energy-momentum \(T_{ij}\) to be imaginary in \(dS_4/CFT_3\), perhaps reflecting the blueshift instability here. Here we will simply look for bulk codim-2 extremal surfaces, lying either on a \(w = \text{const}\) slice or an \(x = \text{const}\) slice (from a bulk point of view alone, either could be taken as Euclidean time slices), restricting to real surfaces which may also be timelike.

The area functional for a surface in (31) bounding a subregion on a \(x = \text{const}\) slice of \(\mathcal{I}^+\), and wrapping the other \(x_i \neq x\), is

\[
S = \frac{R_{dS}^{d-1}}{4G_{d+1}} V_{d-2} \int \frac{d\tau}{\tau^{d-1}} \sqrt{\frac{1}{f(\tau)} - f(\tau) \left(\frac{dw}{d\tau}\right)^2}, \quad f(\tau) = 1 - \tau_0^d \tau^d. \tag{32}
\]

This does not correspond to any analytic continuation from the Ryu-Takayanagi formula for the \(AdS_4\) black brane, so we analyse this directly focussing on the \(dS_4\) bluewall. Along our earlier discussions in sec. 2, we find real extremal surfaces corresponding to

\[
\dot{w}^2 = \frac{1}{(1 - \tau_0^3 \tau^3)^2} \frac{B^2 \tau^4}{1 - \tau_0^3 \tau^3 + B^2 \tau^4}, \quad S = \frac{V_1 R_{dS}^2}{4G_4} \int \frac{d\tau}{\tau^{d-1}} \frac{1}{\sqrt{1 - \tau_0^3 \tau^3 + B^2 \tau^4}}, \tag{33}
\]

\(^6\)The metric (23) with \(\alpha = +(-i)^d\) is similar to the \(dS\) black brane, except with \(T_{ij}\) of the opposite sign, while \(\alpha = +1\) gives a real spacetime with a spacelike singularity at \(\tau \to \infty\).
where the constant $B$ arises from a conserved quantity in the extremization. The first equation describing the surface can be rewritten as

$$
\frac{d\tau}{d\tau_*} \equiv \frac{d\tau}{1 - \tau_0^4 \tau^3}, \quad \left(\frac{dw}{d\tau_*}\right)^2 \equiv (w')^2 = \frac{B^2 \tau^4}{1 - \tau_0^4 \tau^3 + B^2 \tau^4}, \quad (34)
$$

where we are using $\tau_*$ here for the “tortoise” coordinate in this bluewall geometry [13], analogous to the Schwarzschild tortoise coordinate $r_*$. Parametrized thus, we see as in the $dS_4$ case that increasing $B$ decreases the area, as long as we restrict the surface to lie within the future asymptotic universe $I$, i.e. $f(\tau) > 0$. As $B^2 \to \infty$, these extremal surfaces become null with $(w')^2 = 1$, corresponding to the past lightcone wedges of the boundary subregion, and have vanishing area. Thus extremal surfaces for a given subregion at $I^+$ can be constructed as in $dS_4$ (Figure 1) by joining two half-extremal surfaces: this is the blue wedge in region $I$ in Figure 2 (the half-surface when not cut off continues as a null surface through the Cauchy horizon into region $III$, represented by the dotted extension of the blue line). As the subregion grows in size, this blue wedge approaches and eventually wraps the future Cauchy horizons.

One might imagine that there are timelike surfaces which are not restricted to just region $I$ but instead start on $I^+$ in $I$ and cross over to $II$ ending on the past boundary $I^-$. These can be found with the parameter $B^2 > 0$ being finite. In this case, we see that $(w')^2 \to 0$ as $\tau \to 0$ and $(w')^2 \to 1$ as $\tau \to \frac{1}{\tau_0}$ near the horizon in $I$. Now after the surface crosses the future Cauchy horizon, we have $f(\tau) < 0$ in $IV$. Requiring that $(w')^2$ in (34) satisfies $(w')^2 \geq 0$ corresponding to real surfaces, it is possible to see (e.g. by plotting as a function of $\tau_0 \tau$) that the parameter $B^2$ is bounded below by a critical value. There is a family of such surfaces: we will isolate one “critical” surface for a particular value of $B$, in what follows. Drawing analogies with the study of the phase transition found in [42] (although the physical context there is different), we note that $(w')^2 \to \infty$ when the denominator in (34) approaches a double zero (with $y \equiv \tau_0 \tau$), i.e.

$$
1 - y_c^3 + \frac{B^2}{\tau_0^4} y_c^4 = 0, \quad -3y_c^2 + 4\frac{B^2}{\tau_0^4} y_c^3 = 0 \quad \Rightarrow \quad \frac{B^2}{\tau_0^4} = \frac{3}{4} 4^{1/3}, \quad y_c = 4^{1/3}. \quad (35)
$$

This corresponds to $\tau_c = 4^{1/3} \tau_0 \sim \frac{1.6}{\tau_0}$ which is just a little inside the Cauchy horizon in region $IV$. Note that $(w')^2 \big|_{\tau_c} \to \infty$ here means this curve is normal to the $w = \text{const}$ line here (these are

![Figure 2: de Sitter “bluewall” Penrose diagram and some extremal surfaces with at least one end anchored at $I^+$. The blue wedge is null, while the red timelike surface goes from $I^+$ to $I^-$.](image-url)
straight spacelike lines passing through the bifurcation point and hitting the singularity in \( IV \), or equivalently tangent to the \( \tau = \text{const} \) curve at \( \tau_c \) in \( IV \). The corresponding surface from \( \tau = 0 \) to \( \tau = \tau_c \) can be drawn as a curve in the \((\tau, w)\)-plane: it can be joined smoothly at \( \tau_c \) with a corresponding curve from \( I^- \), resulting approximately in the red curve in Figure 2. This surface crosses the upper and lower Cauchy horizons at \( \tau = \frac{1}{\tau_0}, w = +\infty \) and \( \tau = \frac{1}{\tau_0}, w = -\infty \). The area of this surface has a leading divergence \( S \sim R_2^2 \frac{V_1}{G_4} \). Near the double zero, \( \Delta w \) acquires a large contribution and we can estimate \( S \sim R_2^2 \frac{V_1}{G_4} \Delta w \). This surface is vaguely reminiscent of the extremal surface in [43] which goes from one timelike boundary to the other: since the \( dS \) bluewall metric itself is related to the AdS-Schwarzschild black brane by flipping minus signs, it is perhaps not surprising that there exists a similar surface here (but timelike), albeit with no obvious corresponding interpretation. In light of ER=EPR [44], it is amusing to speculate that the subregion here corresponds to copies on both \( I^\pm \) possibly “entangled”, in some sense, thinking of the bluewall geometry as a “timelike wormhole” with the bifurcation region being the Einstein Rosen bridge. Note however that strictly speaking, all timelike geodesics go from \( I^- \) to \( I^+ \) (unlike a shortcut in spacetime) either through the bifurcation region or through the Cauchy horizons, subject to the blueshift instability [13].

With a \( w = \text{const} \) slice, real extremal surfaces likewise have

\[
\dot{x}^2 = \frac{B^2 \tau^4}{(1 - \tau_0^3 \tau^3)(1 + B^2 \tau^4)}, \quad S = \frac{V_1 R_2^2}{4 G_4} \int_{\tau_U}^{\tau_V} d\tau \frac{2}{\tau^2 \sqrt{(1 - \tau_0^3 \tau^3)(1 + B^2 \tau^4)}}.
\]

(36)

For \( B^2 \to \infty \), these are again null extremal surfaces \( \dot{x}^2 = \frac{1}{1 - \tau_0^3 \tau^3} \) with vanishing area. These surfaces all lie on a \( w = \text{const} \) slice (thin black straight line from \( I^+ \) to \( I^- \) in Figure 2).

4 Discussion

We have considered extremal surfaces in bulk de Sitter space (in the Poincare slicing) on constant boundary Euclidean time slices bounding subregions at future timelike infinity, motivated by the Ryu-Takayanagi prescription for entanglement entropy in \( AdS/CFT \). Stemming from certain crucial sign differences, we have seen real extremal surfaces which are essentially the boundaries of the past lightcone wedges of the subregion: these are null surfaces with vanishing area. We have also seen complex extremal surfaces which do not always have real-valued area: this has parallels with analytically continuing from the Ryu-Takayanagi formula in \( AdS \). In \( dS_4 \), the area is real-valued, negative and has some structural resemblance with entanglement entropy in a dual \( CFT_3 \). We have also studied extremal surfaces in the \( dS \) black brane (where there is a finite cutoff-independent extensive piece), and the related \( dS \) bluewall spacetime. It is worth mentioning that there may exist other extrema of the area functional: for instance, we
have required that $x(\tau)$ parametrizing the strip width be real-valued, which suggests the path $\tau = iT$ in complex $\tau$-space. This appears consistent with possible $dS/CFT$ interpretations and also corroborates with our discussion of the $dS$ black brane. However this may be restrictive and more general complex extremal surfaces may be relevant in complex $\tau$-space (see e.g. [23]). It may be interesting to understand if the analysis of [18] can be applied in this case to obtain insights into extremal surfaces.

While this analysis of bulk extremal surfaces could be regarded as simply a study of certain kinds of probes of asymptotically de Sitter spaces, it cannot pinpoint whether the corresponding area is expected to have a physical interpretation as entanglement entropy in the dual CFT, and if it should be real-valued, negative/positive and so on, although the result in $dS_4/CFT_3$ does appear so. It is tempting to study this in light of the higher spin $dS/CFT$ duality of [4]. However the presence of massless higher spin fields might suggest that extremal surfaces which are geometric gravitational objects are not accurate (see e.g. [45, 46] which study entanglement entropy from Wilson lines in higher spin AdS holography). Nevertheless it is interesting to ask if these extremal surfaces have significance in some approximation where the higher spin symmetry is not exact. In this case, it would be interesting to explore the physical interpretation here more directly from a Euclidean $CFT_3$ point of view. One way to think about entanglement entropy in field theory (lattice models) is in terms of the eigenvalues of a correlation matrix and a corresponding von Neumann entropy (see e.g. [17] and more recently [18]). In that context, a simple model of a massless scalar field with wrong sign kinetic terms might suggest that the correlation matrix squared $C^2$ is related to that for an ordinary massless scalar field by a minus sign, so that $C$-eigenvalues $\lambda_k$ become $i\lambda_k$. Then the associated von Neumann entropy is in general not real-valued: it would be interesting to understand this better.

Acknowledgements: It is a pleasure to thank the participants of the “Entanglement from Gravity” Discussion meeting, ICTS, Bangalore, in particular S. Banerjee, G. Mandal and R. Myers for helpful comments. I also thank the organizers of this workshop and the Indian Strings Meeting 2014, Puri, for hospitality as this was being completed. This work is partially supported by a grant to CMI from Infosys Foundation.

References

[1] A. Strominger, “The dS / CFT correspondence,” JHEP 0110, 034 (2001) [hep-th/0106113].
[2] E. Witten, “Quantum gravity in de Sitter space,” [hep-th/0106109].
[3] J. M. Maldacena, “Non-Gaussian features of primordial fluctuations in single field inflationary models,” JHEP 0305, 013 (2003), [astro-ph/0210603].
[4] D. Anninos, T. Hartman and A. Strominger, “Higher Spin Realization of the dS/CFT Correspondence,” arXiv:1108.5735 [hep-th].

[5] D. Harlow and D. Stanford, “Operator Dictionaries and Wave Functions in AdS/CFT and dS/CFT,” arXiv:1104.2621 [hep-th].

[6] G. S. Ng and A. Strominger, “State/Operator Correspondence in Higher-Spin dS/CFT,” Class. Quant. Grav. 30, 104002 (2013) arXiv:1204.1057 [hep-th].

[7] D. Anninos, “De Sitter Musings,” Int. J. Mod. Phys. A 27, 1230013 (2012) arXiv:1205.3855 [hep-th].

[8] D. Das, S. R. Das, A. Jevicki and Q. Ye, “Bi-local Construction of Sp(2N)/dS Higher Spin Correspondence,” JHEP 1301, 107 (2013) arXiv:1205.5776 [hep-th].

[9] D. Anninos, F. Denef and D. Harlow, “The Wave Function of Vasiliev’s Universe - A Few Slices Thereof,” Phys. Rev. D 88, 084049 (2013) arXiv:1207.5517 [hep-th].

[10] D. Anninos, F. Denef, G. Konstantinidis and E. Shaghoulian, “Higher Spin de Sitter Holography from Functional Determinants,” arXiv:1305.6321 [hep-th].

[11] D. Das, S. R. Das and G. Mandal, “Double Trace Flows and Holographic RG in dS/CFT correspondence,” arXiv:1306.0330 [hep-th].

[12] S. Banerjee, A. Belin, S. Hellerman, A. Lepage-Jutier, A. Maloney, J. J. Radievi and S. Shenker, “Topology of Future Infinity in dS/CFT,” JHEP 1311, 026 (2013) arXiv:1306.6629 [hep-th].

[13] D. Das, S. R. Das and K. Narayan, “dS/CFT at uniform energy density and a de Sitter 'bluewall','’ JHEP 1404, 116 (2014) arXiv:1312.1625 [hep-th].

[14] S. Ryu and T. Takayanagi, “Holographic derivation of entanglement entropy from AdS/CFT,” Phys. Rev. Lett. 96, 181602 (2006) hep-th/0603001.

[15] S. Ryu and T. Takayanagi, “Aspects of Holographic Entanglement Entropy,” JHEP 0608, 045 (2006) hep-th/0605073.

[16] T. Nishioka, S. Ryu and T. Takayanagi, “Holographic Entanglement Entropy: An Overview,” J. Phys. A 42 (2009) 504008;

[17] T. Takayanagi, “Entanglement Entropy from a Holographic Viewpoint,” Class. Quant. Grav. 29 (2012) 153001 arXiv:1204.2450 [gr-qc].

[18] A. Lewkowycz and J. Maldacena, “Generalized gravitational entropy,” JHEP 1308, 090 (2013) arXiv:1304.4926 [hep-th].

[19] V. E. Hubeny, M. Rangamani and T. Takayanagi, “A Covariant holographic entanglement entropy proposal,” JHEP 0707 (2007) 062 arXiv:0705.0016 [hep-th].

[20] V. E. Hubeny and M. Rangamani, “Causal Holographic Information,” JHEP 1206, 114 (2012) arXiv:1204.1698 [hep-th].
[21] L. Fidkowski, V. Hubeny, M. Kleban and S. Shenker, “The Black hole singularity in AdS / CFT,” JHEP 0402, 014 (2004) [hep-th/0306170].

[22] S. Fischetti and D. Marolf, “Complex Entangling Surfaces for AdS and Lifshitz Black Holes?,” Class. Quant. Grav. 31, no. 21, 214005 (2014) [arXiv:1407.2900 [hep-th]].

[23] S. Fischetti, D. Marolf and A. Wall, “A paucity of bulk entangling surfaces: AdS wormholes with de Sitter interiors,” [arXiv:1409.6754 [hep-th]].

[24] E. T. Akhmedov, “A Remark on the AdS / CFT correspondence and the renormalization group flow,” Phys. Lett. B 442, 152 (1998) [hep-th/9806217].

[25] E. Alvarez and C. Gomez, “Geometric holography, the renormalization group and the c theorem,” Nucl. Phys. B 541, 441 (1999) [hep-th/9807226].

[26] D. Z. Freedman, S. S. Gubser, K. Pilch and N. P. Warner, “Renormalization group flows from holography supersymmetry and a c theorem,” Adv. Theor. Math. Phys. 3, 363 (1999) [hep-th/9904017].

[27] J. de Boer, E. P. Verlinde and H. L. Verlinde, “On the holographic renormalization group,” JHEP 0008, 003 (2000) [hep-th/9912012].

[28] S. de Haro, S. N. Solodukhin and K. Skenderis, “Holographic reconstruction of spacetime and renormalization in the AdS/CFT correspondence,” Commun. Math. Phys. 217, 595 (2001) [arXiv:hep-th/0002230].

[29] K. Skenderis, “Lecture notes on holographic renormalization,” Class. Quant. Grav. 19, 5849 (2002) [arXiv:hep-th/0209067].

[30] I. Heemskerk and J. Polchinski, “Holographic and Wilsonian Renormalization Groups,” JHEP 1106, 031 (2011) [arXiv:1010.1264 [hep-th]].

[31] T. Faulkner, H. Liu and M. Rangamani, “Integrating out geometry: Holographic Wilsonian RG and the membrane paradigm,” JHEP 1108, 051 (2011) [arXiv:1010.4039 [hep-th]].

[32] L. Bombelli, R. K. Koul, J. Lee and R. D. Sorkin, “A Quantum Source of Entropy for Black Holes,” Phys. Rev. D 34 (1986) 373.

[33] M. Srednicki, “Entropy and area,” Phys. Rev. Lett. 71 (1993) 666 [hep-th/9303048].

[34] M. Headrick, “Entanglement Renyi entropies in holographic theories,” Phys. Rev. D 82, 126010 (2010) [arXiv:1006.0047 [hep-th]].

[35] T. Faulkner, A. Lewkowycz and J. Maldacena, “Quantum corrections to holographic entanglement entropy,” JHEP 1311, 074 (2013) [arXiv:1307.2892].

[36] J. Maldacena and G. L. Pimentel, “Entanglement entropy in de Sitter space,” JHEP 1302, 038 (2013) [arXiv:1210.7244 [hep-th]].

[37] S. Kanno, J. Murugan, J. P. Shock and J. Soda, “Entanglement entropy of α-vacua in de Sitter space,” JHEP 1407, 072 (2014) [arXiv:1404.6815 [hep-th]].
[38] N. Iizuka, T. Noumi and N. Ogawa, “Entanglement Entropy of de Sitter Space $\alpha$-Vacua,” [arXiv:1404.7487 [hep-th]].

[39] W. Fischler, S. Kundu and J. F. Pedraza, “Entanglement and out-of-equilibrium dynamics in holographic models of de Sitter QFTs,” JHEP 1407, 021 (2014) [arXiv:1311.5519 [hep-th]].

[40] V. Balasubramanian and P. Kraus, “A stress tensor for anti-de Sitter gravity,” Commun. Math. Phys. 208, 413 (1999) [arXiv:hep-th/9902121].

[41] R. C. Myers, “Stress tensors and Casimir energies in the AdS/CFT correspondence,” Phys. Rev. D 60, 046002 (1999) [arXiv:hep-th/9903203].

[42] K. Narayan, T. Takayanagi and S. P. Trivedi, “AdS plane waves and entanglement entropy,” JHEP 1304, 051 (2013) [arXiv:1212.4328 [hep-th]].

[43] T. Hartman and J. Maldacena, “Time Evolution of Entanglement Entropy from Black Hole Interiors,” JHEP 1305, 014 (2013) [arXiv:1303.1080 [hep-th]].

[44] J. Maldacena and L. Susskind, “Cool horizons for entangled black holes,” Fortsch. Phys. 61, 781 (2013) [arXiv:1306.0533 [hep-th]].

[45] J. de Boer and J. I. Jottar, “Entanglement Entropy and Higher Spin Holography in AdS$_3$,” JHEP 1404, 089 (2014) [arXiv:1306.4347 [hep-th]].

[46] M. Ammon, A. Castro and N. Iqbal, “Wilson Lines and Entanglement Entropy in Higher Spin Gravity,” JHEP 1310, 110 (2013) [arXiv:1306.4338 [hep-th]].

[47] I. Peschel and V. Eisler, “Reduced density matrices and entanglement entropy in free lattice models”, J.Phys.,A42,504003 (2009), [arXiv:0906.1663 [cond-mat]].

[48] C. P. Herzog and M. Spillane, “Tracing Through Scalar Entanglement,” Phys. Rev. D 87, 025012 (2013) [arXiv:1209.6368 [hep-th]].