A systematic method of summing the corrections to the renormalon residue arising from higher order renormalons is discussed.

1 Introduction

It is well known that the weak coupling expansion in field theory is not convergent but asymptotic. Its coefficients diverge factorially in the order of perturbation. One source of the divergence is renormalon. For example, when a Green’s function such as the Adler function $D(\alpha)$ is expanded as

$$D(\alpha) = \sum a_n \alpha^{n+1},$$ (1)

the infrared renormalon causes the coefficients to grow as

$$a_n \rightarrow K n! n^{-2\beta_1/\beta_0} (-2/\beta_0)^{-n},$$ for $n \rightarrow \infty,$ (2)

where $\beta_0, \beta_1$ are the first two coefficients of the $\beta$ function.

The constant $K$ is an all order quantity: it gets contribution not only from the well-known renormalon diagram, a chain of one loop bubbles, but also from infinitely many higher order renormalon diagrams. The purpose of this talk is to show summing these corrections is manageable, and that $K$ can be expressed in a calculable, convergent sequence.

2 Renormalons in QED

For our purpose it is convenient to consider the Adler function defined by

$$D(\alpha(q^2)) = \frac{\partial}{\partial q^2} \Pi(q^2)$$
$$= \text{const.} + \sum_{n=0}^{\infty} a_n (\mu^2/q^2) \alpha(\mu)^{n+1}$$ (3)

where $\Pi$ is the vacuum polarization function of the electromagnetic current and $\mu$ is the renormalization scale.

The large order behavior in Eq. 3 arises from an exchange of the all-order Gell-Mann-Low (GL) effective charge $a$ defined by

$$a(\alpha(\mu), t) = \frac{\alpha(\mu)}{1 + \Pi(\alpha(\mu), t)}$$ (4)

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where \( t = k^2/\mu^2 \). Therefore, as far as we are concerned with the infrared renormalon associated with the asymptotic behavior, we can write the Adler function as

\[
D(\alpha(\mu), \mu^2/q^2) = \int_0 f(k^2) a(k^2) \, dk^2
\]

(5)

where \( a(k^2) \) denotes the GL effective charge and

\[
f(k^2) = \frac{-e_f^2 k^2}{8\pi^3 q^4} \quad \text{for} \quad k^2 \to 0
\]

(6)

with \( e_f \) denoting the electric charge.

The divergence in Eq. (2) causes a singularity in Borel plane. The Borel transform of \( D(\alpha(\mu), \mu^2/q^2) \) is defined by

\[
D(\alpha(\mu), \mu^2/q^2) = \int_0^\infty \exp\left(-\frac{b}{\alpha(\mu)}\right) \tilde{D}(b) \, db.
\]

(7)

With the perturbative series in Eq. (1) the Borel transform becomes

\[
\tilde{D}(b) = \sum_{n=0}^\infty \frac{a_n}{n!} b^n.
\]

(8)

Substitution of the asymptotic form in Eq. (2) into Eq. (8) gives

\[
\tilde{D}(b) \to \frac{K(-2\beta_1/\beta_0^2)!}{(1 + \beta_0 b/2)^{1-2\beta_1/\beta_0^2}}
\]

(9)

in the neighborhood of the singularity at \( b = -2/\beta_0 \). Thus, to calculate \( K \) we need to evaluate the residue of the singularity.

Let us now consider the renormalization group equation for the GL effective charge:

\[
\left(\mu^2 \frac{\partial}{\partial \mu^2} + \beta(\alpha) \frac{\partial}{\partial \alpha}\right) a(\alpha(\mu), t) = 0.
\]

(10)

Solving the equation we may write the GL effective charge as

\[
a(k^2) = \frac{1}{A(k^2) + C(a(k^2))}
\]

(11)

where \( C(a) \) is a scheme independent function. The effective coupling \( A \) is defined by

\[
A(\alpha(\mu), t) = \frac{1}{-\beta_0 \left( \ln t + \int^{\alpha(\mu)} \frac{1}{\beta(\alpha)} \, d\alpha - \frac{p_1}{\beta_0} \right)}.
\]

(12)

where \( p_1 \) is the constant term in the one loop vacuum polarization function. The function \( C(a(k^2)) \) may be expanded in an asymptotic series as

\[
C(a(k^2)) = c_1 \ln(a(k^2)) + \sum_{i=2}^\infty c_i [a(k^2)]^{i-1}.
\]

(13)
The \( c_i \) can be easily determined in terms of the coefficients of the \( \beta \) function and the vacuum polarization function.

Now define an \( N \) th order GL effective charge:

\[
a^{(N)}(k^2) = \frac{1}{A(k^2) + c_1 \ln(a^{(N)}(k^2)) + \sum_{2}^{N} c_i a^{(N)}(k^2)^{i-1}}
\]

and introduce a modified Borel transform

\[
a^{(N)}(\alpha(\mu), t) = \int_{0}^{\infty} \exp\left(-\frac{b}{A(t)}\right) \tilde{a}^{(N)}(b) \, db.
\]

Substituting Eq. 15 into Eq. 5, we can write \( D^{(N)}(\alpha(\mu), \mu^2/q^2) \), which is defined by replacing \( a(k^2) \) in Eq. 5 with \( a^{(N)}(k^2) \), as

\[
D^{(N)}(\alpha(\mu), \mu^2/q^2) = \int \exp \left[ b \beta_0 \int \alpha(\mu) \frac{d\alpha}{\beta(\alpha)} \right] \left\{ e^{-bp_1 \tilde{f}(b)} \tilde{a}^{(N)}(b) \right\} \, db,
\]

where

\[
\tilde{f}(b) = \int_{0}^{M} f(t) \exp (b\beta_0 \ln t) \, dt.
\]

The first IR renormalon singularity arises from the IR divergence in the integral in Eq. 17. Substituting Eq. 8 into Eq. 17

\[
\tilde{f}(b) = \int_{0}^{M} f(t) e^{b\beta_0 \ln t} \, dt
= -\frac{e_2^2 \mu^4}{8\pi^2 q^4} \int_{0}^{M} t e^{b\beta_0 \ln t} \, dt
= -\frac{e_2^2 \mu^4}{8\pi^2 q^4} \frac{1}{2 + b\beta_0} \left( 1 + (2 + b\beta_0) \ln M + \cdots \right),
\]

where \( M \) is an arbitrary UV cutoff. Notice that the leading renormalon singularity is cutoff independent.

In the renormalization scheme in which the \( \beta \) function is given by

\[
\beta(\alpha) = \frac{\beta_0 \alpha^2}{1 - \lambda \alpha}
\]

with \( \lambda = \beta_1/\beta_0 \), the Eq. 19 becomes the modified Borel transform introduced by Brown, Yaffe, and Zhai. Then using the relation between the ordinary Borel transform and the modified one, we have the ordinary Borel transform

\[
\tilde{D}^{(N)}(b) \rightarrow -\frac{e_2^2 \mu^4}{16\pi^2 q^4} e^{-bp_1 e^{-\lambda_0 \ln b_0} \tilde{a}^{(N)}(b_0)} \frac{(-2\lambda/\beta_0)!}{(1 + \frac{1}{2} b\beta_0)^{1-2\lambda/\beta_0}}
\]

where \( b_0 = -2/\beta_0 \).
Using Eq. 14 the Borel transform of the effective charge can be obtained without difficulty. The final expression is given by

\[
b_{bc_1} \tilde{a}^{(N)}(b) = \frac{b_{bc_1}}{2\pi i} \int e^{b y} \sum_{k=0}^{\infty} \frac{(-b)^k}{k!} \sum_{l=k}^{N-1} \frac{h_{Nkl}}{y^l} \sum_{i=0}^{N} \tilde{c}_i y^{b_{c_1} - i - 1} \, d y
\]

\[
= \sum_{k=0}^{\infty} \sum_{l=k}^{N-1} \sum_{i=0}^{N} \frac{(-1)^k h_{Nkl} \tilde{c}_i}{k! \Gamma(l + i + 1 - b_{c_1})} y^{k + l + i}
\]

where

\[
\tilde{c}_i = \begin{cases} 
1 & \text{for } i = 0 \\
c_1 & \text{for } i = 1 \\
(i - 1) c_i & \text{for } i \geq 2,
\end{cases}
\]

and

\[
h_{Nkl} = k! \sum_{\{n_i\}} \frac{\prod_{i=1}^{N-1} c_{i+1}^{n_i}}{\prod_{i=1}^{N-1} n_i!}
\]

with the set \(\{n_i\}\) of nonnegative integers satisfying

\[
\sum_{i=1}^{N-1} n_i \cdot i = l, \quad \sum_{i=1}^{N-1} n_i = k.
\]

To find the renormalon residue of \(\tilde{D}^{(N)}(b)\), we have to evaluate \(b_{bc_1} \tilde{a}^{(N)}(b)\) at the first IR renormalon position, \(b_0 = -2/\beta_0\). If we directly substitute \(b\) in Eq. 21 with \(b_0\), the resulting large order behavior does not have a finite limit for \(N \to \infty\). The reason for this is that \(\tilde{a}(b)\) is singular at the UV and IR renormalon positions, and its radius of convergence when it is expanded as in Eq. 21 is given by the position at \(b = 1/\beta_0\) of the first UV renormalon, which is the closest renormalon to the origin in the Borel plane. Therefore we cannot substitute \(b\) with \(b_0\) in Eq. 21 to correctly evaluate the Borel transform at the first IR renormalon.

This problem can be avoided by introducing an analytic transform of the Borel plane so that the closest renormalon to the origin in the new complex plane is the first IR renormalon. Because the singularity of \(\tilde{a}(b)\) at the IR renormalon is such that it is finite but has a divergent derivative, we can then express the residue as a convergent series.

For this purpose, we can take any analytic transform that puts the IR renormalon as the closest singularity to the origin. Here we take:

\[
z = \frac{\beta_0 b_{c_1}}{1 - \beta_0 b_{c_1}}.
\]

In the \(z\) - plane, the closest singularity to the origin is the first IR renormalon at

\[
z_0 = -\frac{2}{3},
\]

and all the UV renormalons are pushed beyond \(z = -1\) on the real axis.
Table 1: The first four elements of the sequence for the first IR renormalon residue in QED

| $N_f$ | $\kappa_0$ | $\kappa_1$ | $\kappa_2$ | $\kappa_3$ |
|-------|-----------|------------|------------|------------|
| 1     | 1         | 1.63       | 0.71       | -1.53      |
| 2     | 1         | 1.32       | 1.31       | 1.25       |
| 3     | 1         | 1.21       | 1.31       | 1.41       |
| 4     | 1         | 1.16       | 1.27       | 1.39       |
| 5     | 1         | 1.13       | 1.24       | 1.34       |
| 100   | 1         | 1.00       | 1.02       | 1.02       |

To find $b^{bc_1}\tilde{a}^{(N)}(b)$ at the first IR renormalon, we have to substitute $b$ in Eq. 21 with the inverse of Eq. 25 and expand it in Taylor series at $z = 0$ to order $N$, and evaluate it at $z = z_0$. Then the Borel transform of GL effective charge at the first IR renormalon is given by

$$\kappa_N = b^{bc_1}\tilde{a}^{(N)}(b)\bigg|_{b=\lambda b_0} = \sum_{M=0}^{N} q_M z_0^M,$$  (27)

where it is straightforward to express $q_M$ in terms of $c_i$.

Finally, the coefficient $K$ in Eq. 2 is given by

$$K = \lim_{N \to \infty} K_N$$  (28)

where

$$K_N = -\frac{e^2 g^4}{16\pi^3 q^3} e^{-b_0 p_1} K_N.$$

The first four elements of the sequence is given in Table 1 for different number of fermion flavors.

3 Renormalons in QCD

In QCD, there is unfortunately no satisfactory definition of renormalization scheme and scale invariant effective charge that may be used in the diagrammatic study of renormalon. However, if we are only interested in the calculation of the residue, the definition of the effective charge is not required. Indeed the calculation is cunningly simple; it only requires the strength of the renormalon singularity and the perturbative calculation of $D(\alpha)$. For this reason, the method described in the following can be equally well applied to the calculation of the UV renormalon residue.

Consider the Borel transform of the current correlation function in QCD. The renormalon singularity of $\tilde{D}(b)$ in QCD

$$\tilde{D}(b) \approx \frac{\tilde{D}}{(1 - b/b_0)^{1+\lambda b_0}}$$  (30)

gives the large order behavior

$$a_n \approx \frac{\tilde{D}}{(\lambda b_0)^n n! n^{\lambda b_0} b_0^{-n}}.$$  (31)
To calculate the residue $\hat{D}$, consider a function

$$R(b) = \bar{D}(b) \left(1 - b/b_0\right)^{1+\lambda b_0}.$$ \hspace{1cm} (32)

Then because of Eq. (30), we have

$$\hat{D} = R(b_0).$$ \hspace{1cm} (33)

To avoid the first UV renormalon, we introduce a new variable $z$, as we did in QED, which is defined by

$$z = -\frac{\beta_0 b}{1 - \beta_0 b}.$$ \hspace{1cm} (34)

with its inverse

$$b = -\frac{1}{\beta_0} \left( \frac{z}{1 - z} \right).$$ \hspace{1cm} (35)

In the $z-$plane, the IR renormalon at

$$z_0 = \frac{2}{3}$$ \hspace{1cm} (36)

is the closest singularity to the origin, and so the radius of convergence of the Taylor series of $\bar{D}(b(z))$ at $z = 0$ is given by the first IR renormalon.

Now $\hat{D}$ can be expressed in a convergent series form

$$\hat{D} = \bar{D}(b) \left(1 - b/b_0\right)^{1+\lambda b_0} \bigg|_{b=b_0} = \left( \sum_{n=0}^{\infty} \frac{a_n}{n!} [b(z)]^n \right) \left(1 - b(z)/b_0\right)^{1+\lambda b_0} \bigg|_{z=z_0} = \sum_{n=0}^{\infty} r_n z_0^n,$$ \hspace{1cm} (37)

where it is straightforward to find $r_n$ in terms of the perturbative coefficients $a_n$. Note that the series is convergent even if $R(b(z))$ is not analytic at $z = z_0$, because then the radius of convergence of the series is given by $z = z_0$, and $R(b(z_0))$ is finite.

Using the perturbative calculation of the current correlation function, and $D(\alpha)$, to three loop, we have

$$R(b(z)) = \frac{3 \sum_f Q_f^2}{16 \pi^3} \left[1.333 - 0.748 z - 0.311 z^2 + O(z^3)\right]$$ \hspace{1cm} (38)

for $N_f = 3$. This is in the renormalization scheme in which the one-loop renormalization point is same as that of $\overline{MS}$ scheme, and the $\beta$ function is given in the form in Eq. (17).

Evaluating this series at the renormalon position at $z = z_0$ we have

$$K_1 = \frac{1.333}{(\lambda b_0)!} = 0.946$$

$$K_2 = \frac{(1.323 - 0.748 z_0)}{(\lambda b_0)!} = 0.592$$

$$K_3 = \frac{(1.323 - 0.748 z_0 - 0.311 z_0^2)}{(\lambda b_0)!} = 0.494$$ \hspace{1cm} (39)
Table 2: The first three elements of the sequence for the large order behavior in QCD

|   | $N_f = 1$ | $N_f = 2$ | $N_f = 3$ | $N_f = 4$ | $N_f = 5$ |
|---|-----------|-----------|-----------|-----------|-----------|
| $K_1$ | .881      | .904      | .946      | 1.018     | 1.132     |
| $K_2$ | .521      | .546      | .592      | .674      | .813      |
| $K_3$ | .592      | .549      | .494      | .411      | .307      |

For several other flavor numbers we give $K_n$ in Table 2.

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