THE ‘REGULARITY SINGULARITY’ AT POINTS OF GENERAL RELATIVISTIC SHOCK WAVE INTERACTION

MORITZ REINTJES

Abstract. We show that the regularity of the gravitational metric tensor in spherically symmetric spacetimes cannot be lifted from $C^{0,1}$ to $C^{1,1}$ by any $C^{1,1}$ coordinate transformation in a neighborhood of a point of shock wave interaction in General Relativity, without forcing the determinant of the metric tensor to vanish at the point of interaction. This is in contrast to Israel’s Theorem which states that such coordinate transformations always exist in a neighborhood of a point on a smooth single shock surface [10]. The results thus imply that points of shock wave interaction represent a new kind of spacetime singularity for perfect fluids, singularities that lie in physical spacetime, that can form from the evolution of smooth initial data, but at which the spacetime is not locally Minkowskian under any coordinate transformation. In particular, at such regularity singularities, delta function sources in the second derivatives of the gravitational metric tensor exist in all coordinate systems, but due to cancelation, the curvature tensor remains bounded (in $L^\infty$).

We announced the main results of this paper in [15], where we also sketched the main steps in some of the proofs. In this paper, those results together with their complete proofs and some additional propositions are presented.

Contents

1. Introduction 2
2. Preliminaries 6
3. A Point of Regular Shock Wave Interaction in SSC 12
4. Functions $C^{0,1}$ Across a Hypersurface 15
5. A Necessary and Sufficient Condition for Smoothing Metrics 19
6. Metric Smoothing on Single Shock Surfaces and a Constructive Proof of Israel’s Theorem in Spherical Symmetry 24
7. Shock Wave Interactions are Regularity Singularities in GR; Transformations in the $(t, r)$-Plane 30
8. Shock Wave Interactions are Regularity Singularities in GR; the Full Atlas 36
9. The Loss of Locally Inertial Frames 41
10. The Riemann Curvature Tensor is Bounded 41
11. Discussion 43
12. Conclusion 44
Appendix A. The Integrability Condition 45
Acknowledgments 46
References 46
1. Introduction

In 1915, seeking to conciliate Newtonian gravity with Special Relativity, Albert Einstein formulated the theory of General Relativity [3]. At its heart lies the (gravitational) metric tensor describing the geometry of spacetime and thus the gravitational field. A novel feature of General Relativity (GR) was the prediction of spacetime singularities, those are points where the gravitational metric suffers a lack of regularity. For instance, the singularity at the center of the Schwarzschild metric or at its Schwarzschild radius, where the metric tensor fails to be bounded. The first one is an example of a non-removable singularity which persist in every coordinate system, those singularities are usually characterized by a blow-up in the scalar curvature and they lie outside of physical spacetime. The apparent singularity at the Schwarzschild radius is an example of a removable singularity, i.e., there exist coordinates in which the metric is regular enough to be non-singular. A metric is non-singular if its components, their first and second derivatives and its curvature are bounded and if it is \( \text{locally inertial} \), that is, around any point \( p \) exist coordinates in which the gravitational metric at \( p \) is the Minkowski metric up to second order corrections. (The physical meaning of the metric being locally inertial is that an observer in freefall through a gravitational field should observe the physics of special relativity, except for the second order acceleration effects due to the gravitational field.) However, the metric tensor is governed by the Einstein equations, which are a system of PDE’s, so that the Einstein equations by themselves determine the smoothness of the gravitational metric tensor by the evolution they impose. Thus the condition on spacetime that it be non-singular cannot be assumed at the start, but must be determined by regularity theorems for the Einstein equations.

Whenever the sources of matter and energy are modeled by a perfect fluid, this issue becomes all the more interesting and intriguing. Then the Einstein equations imply the GR compressible Euler equations through the Bianchi identities, and the compressible Euler equations create shock waves out of smooth initial data whenever the flow is sufficiently compressive [2, 16, 17]. At a shock wave, the fluid density, pressure, velocity, and hence \( T \) are discontinuous, so the Einstein equations imply the curvature \( G \) must also become discontinuous at shocks. But discontinuous curvature by itself is not inconsistent with the assumption that spacetime be non-singular. For example, if the gravitational metric tensor were \( C^{1,1} \), (differentiable with first derivatives being Lipschitz continuous, \( C^{0,1} \)), then second derivatives of the metric are at worst discontinuous, the curvature is bounded (in \( L^\infty \)) and the metric has enough smoothness to be locally inertial, [18]. Furthermore, Israel’s theorem asserts that a metric \( C^{0,1} \) regular across a smooth single shock surface, is lifted to \( C^{1,1} \) by the \( C^{1,1} \) coordinate map to Gaussian normal coordinates, and this is again smooth enough for spacetime to be non-singular at each point. In [7], Groah and Temple set out a framework in which to address these issues rigorously by providing the first general existence theory for spherically symmetric shock wave solutions of

---

1 Also referred to as \( \text{locally Lorentzian} \) or \( \text{locally Minkowskian} \).
the Einstein-Euler equations allowing for arbitrary numbers of interacting shock waves of arbitrary strength. In coordinates where their analysis is feasible, Standard Schwarzschild Coordinates (SSC), (a general spherically symmetric metric can generically be transformed to SSC, c.f. [25]), the gravitational metric is only $C^{0,1}$ at shock waves, and it has remained an open problem as to whether the weak solutions constructed by Groah and Temple could be smoothed to $C^{1,1}$ by coordinate transformation, like the single shock surfaces addressed by Israel.

The resolution of the open problem of Groah and Temple is achieved in [15] and the present paper by proving that there do not exist $C^{1,1}$ coordinate transformations that can lift the regularity of a gravitational metric tensor from $C^{0,1}$ to $C^{1,1}$ at a point of shock wave interaction in a spherically symmetric spacetime, without forcing the determinant of the metric tensor to vanish at the point of interaction. Consequently, in contrast to Israel’s theorem for single shock surfaces, shock wave solutions cannot be continued as $C^{1,1}$ strong solutions of the Einstein equations beyond the first point of shock wave interaction. The results imply that points of shock wave interaction represent a new kind of singularity in General Relativity that can form from the evolution of smooth initial data, that lies within physical spacetime, but at which second order metric derivatives are distributional and spacetime is not locally inertial under any $C^{1,1}$ coordinate transformation. Due to cancelation, the Einstein tensor is sup-norm bounded and free of delta function sources at points of shock wave interaction. This result contrasts the common assumption about the metric being $C^{1,1}$ regular, for example, this is assumed in the singularity theorems of Hawking and Penrose, [9]. In this paper we present the complete proofs of the results announced in [15], where many proofs were omitted.

To state the main result precisely, let $g_{\mu\nu}$ denote a spherically symmetric spacetime metric in SSC, that is, the metric takes the form

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu = -A(t,r)dt^2 + B(t,r)dr^2 + r^2d\Omega^2,$$  \hspace{1cm} (1.1)

where either $t$ or $r$ can be taken to be timelike, and $d\Omega^2 = d\vartheta^2 + \sin^2(\vartheta)d\phi^2$ is the line element on the unit 2-sphere, c.f. [19]. In Section 3 we make precise the definition of a point of regular shock wave interaction in SSC. Essentially, this is a point where two shock waves enter or leave the point $p$ at distinct speeds, such that the metric is Lipschitz continuous across each shock, the Rankine Hugoniot (RH) jump conditions hold across the shocks, and the SSC Einstein equations hold strongly away from the shocks. The main result of this paper is the following theorem, c.f. [15]:

**Theorem 1.1.** Assume $p$ is a point of regular shock wave interaction in SSC, in the sense of Definition 3.1, for the SSC metric $g_{\mu\nu}$. Then there does not exist a $C^{1,1}$ regular coordinate transformation, defined in a neighborhood of $p$, such that the metric components are $C^1$ functions of the new coordinates and such that the metric has a nonzero determinant at $p$.

The proof of Theorem 1.1 is constructive in the sense that we characterize the Jacobians of $(t, r)$ coordinate transformations that could smooth the components
of the gravitational metric in a deleted neighborhood of a point $p$ of regular shock wave interaction, and then prove that any such Jacobian must have a vanishing determinant at $p$ itself. We then extend the result to $C^{1,1}$ transformations allowing for changes of angular variables as well.

We emphasize, when discontinuities in the fluid are present, $C^{1,1}$ coordinate transformations constitute the atlas of transformations capable of lifting the regularity of the metric one order (across a single shock surface), while still preserving the weak formulation of the Einstein equations, c.f. [18]. For this and other reasons, c.f. Section 11, is the atlas of $C^{1,1}$ coordinate transformation generic for addressing shock wave interactions in GR.

Our assumptions in Theorem 1.1 apply to the upper half ($t \geq 0$) and the lower half ($t \leq 0$) of a shock wave interaction (at $t = 0$) separately, general enough to include the case of two timelike (or spacelike) interacting shock waves of opposite families that cross at the point $p$, but also general enough to include the cases of two outgoing shock waves created by the focusing of compressive rarefaction waves, or two incoming shock waves of the same family that interact at $p$ to create an outgoing shock wave of the same family and an outgoing rarefaction wave of the opposite family, c.f. [17]. In particular, our framework is general enough to incorporate the shock wave interaction which was numerically simulated in [24].

Even though our research was motivated by shock wave solutions of the Einstein-Euler equations, we do not assume a perfect fluid as the matter source in Theorem 1.1, so that our main result is applicable to other matter models as well, provided $T^{\mu\nu}$ is bounded and satisfies the RH conditions, $[T^{\mu\nu}]n_\mu = 0$, on each of the shock curves.

Historically, the issue of the smoothness of the gravitational metric tensor across interfaces began with the matching of the interior Schwarzschild solution to the vacuum across an interface, followed by the celebrated work of Oppenheimer and Snyder who gave the first dynamical model of gravitational collapse by matching a pressureless fluid sphere to the Schwarzschild vacuum spacetime across a dynamical interface [13]. In [18], Smoller and Temple extended the Oppenheimer-Snyder model to nonzero pressure by matching the Friedmann metric to a static fluid sphere across a shock wave interface that modeled a blast wave in GR. In his celebrated 1966 paper [10], Israel gave the definitive conditions for regular matching of gravitational metrics at smooth interfaces, by showing that if the second fundamental form is continuous across a single smooth interface, then the RH conditions also hold, and Gaussian normal coordinates provide a locally inertial coordinate system at each point on the surface. In [7] Groah and Temple addressed these issues rigorously in the first general existence theory for shock wave solutions of the Einstein-Euler equations in spherically symmetric spacetimes. The results were extended to Gowdy spacetime in [11].

Although points of shock wave interaction are straightforward to construct for the relativistic compressible Euler equations in flat spacetime, we know of no rigorous construction of a point of regular shock wave interaction in GR. However, all evidence indicates points of shock wave interaction to exist, have
the structure we assume in SSC, and cannot be avoided in solutions consisting of, say, an outgoing spherical shock wave (the blast wave of an explosion) evolving inside an incoming spherical shock wave (the leading edge of an implosion). Namely, the existence theory of Temple and Groah [7] lends strong support to this claim, establishing existence of weak solutions of the Einstein-Euler equations in spherically symmetric spacetimes. The theory applies to arbitrary numbers of initial shock waves of arbitrary strength, existence is established beyond the point of shock wave interaction, and the regularity assumptions of our theorem are within the regularity class to which the Groah-Temple theory applies. Moreover, the recent work of Vogler and Temple gives a numerical simulation in which two shock waves emerge from a point of interaction where two compression waves focus into a discontinuity in density and velocity, and the numerics demonstrate that the structure of the emerging shock waves meet the assumptions of our theorem.

It is instructive at this point to clarify the difference between the essential $C^{0,1}$ singularities in the metric at points of shock wave interaction, and the essential $C^{0,1}$ singularities at surface layers like the “thin shells” introduced in Israel’s illuminating paper [10]. (See also [6].) On surface layers, the delta function sources in $T$ are the cause of the essential $C^{0,1}$ singularity in the metric $g$, because second derivatives of $g$ must have distributional sources and consequently $g$ cannot be $C^{1,1}$ in any regular coordinate system. For shock wave solutions of $G = \kappa T$ the issue is more delicate because $T$ is sup-norm bounded, so that the constraint of $G$ having delta function sources is removed and, at first sight, there is no clear obstacle to the existence of coordinate systems that smooth the metric to $C^{1,1}$. Israel’s theorem confirms there is no obstacle to $C^{1,1}$ smoothness in the special case of single shock surfaces, but the methods in [7] are only sufficient to prove existence of solutions in $C^{0,1}$, and the question as to whether there is an obstacle for more complicated solutions with interactions has remained unresolved until now.

The argument in [15] and the present paper resolves this issue by proving that at points of shock wave interaction, the Einstein-Euler equations in SSC generate an essential $C^{0,1}$ singularity in the metric that cannot be smoothed to $C^{1,1}$ by coordinate transformation, even though there are no delta function sources present in $T$ or $G$ to explain the $C^{0,1}$ singularity in the metric $g$.

We conclude that points of shock wave interaction are a new kind of regularity singularity in the gravitational field that are not generated by delta function sources in $T$, that can form from the evolution of smooth initial data, that correctly reflect the physics of the equations even though the spacetime is not locally Minkowskian under any local $C^{1,1}$ coordinate transformation, and where the metric tensor does not have sufficient regularity to satisfy the Einstein-Euler equations strongly in any coordinate system of the $C^{1,1}$ atlas. At such singularities, delta function sources in the second derivatives of the gravitational metric tensor exist in all coordinate systems of the $C^{1,1}$ atlas, but due to cancelation, the Riemann curvature tensor remains sup-norm bounded.
In Section 3, we set up our basic framework and define the notion of a point of regular shock wave interaction in SSC. We make precise what we mean by a function (or a metric) to be $C^{0,1}$ across a hypersurface and derive a canonical form for such functions in Section 4. In Section 5, we show that the property of the metric tensor being $C^{0,1}$ across a shock surface is covariant under $C^2$ coordinate transformations, but not under $C^{4,1}$ transformations. We use this lack of covariance to derive conditions on the Jacobians of general $C^{1,1}$ coordinate transformations (in principal) necessary and sufficient to lift the regularity of a metric tensor from $C^{0,1}$ to $C^1$ at points on a shock surface. This condition enables us to represent all such Jacobians in terms of the canonical form introduced in Section 4, unique up to addition of an $C^1$ function.

In Section 6, we give a new constructive proof of Israel’s theorem for spherically symmetric spacetimes, by showing that the Jacobians expressed in our canonical form do smooth the gravitational metric to $C^{1,1}$ at points on a single shock surface. The essential difficulty is to prove that the “gauge” freedom to add an arbitrary $C^1$-function to our canonical form is sufficient to satisfy the integrability condition, which is required to integrate the Jacobian to coordinates. This is possible within the $C^1$ gauge freedom if and only if the so-called Rankine-Hugoniot conditions and the Einstein equations hold at the shock interface.

The main step towards Theorem 1.1 is achieved in Section 7 where we prove that at a point of regular shock wave interaction in SSC there exist no coordinate transformations in the $(t,r)$-plane, (that is, transformations that keep the angular part fixed), that lift the metric regularity to $C^1$. The central method herein is that the $C^1$ gauge freedom in our canonical forms is insufficient to satisfy the integrability condition on the Jacobians, without forcing the determinant of the Jacobian to vanish at the point of interaction. In Section 8 we extend this result to the full atlas of spacetime, thereby proving Theorem 1.1. We prove the loss of locally inertial frames in Section 9 and show that the Riemann curvature tensor is bounded at points of shock wave interaction in Section 10.

2. Preliminaries

In this section we discuss the framework of General Relativity and introduce the coupled Einstein-Euler equations together with shock waves as solutions of the latter. In addition, we give a short review of Israel’s result [10, 18], introduce spherically symmetric spacetimes and discuss the Einstein equations in Standard Schwarzschild Coordinates regarding the metric regularity.

Spacetime is a four dimensional Lorentz manifold, that is, a manifold endowed with a metric tensor of signature $(-1,1,1,1)$, the so-called Lorentz metric. Requiring spacetime to be a manifold reflects Einstein’s original insight of general covariance [3], that is, all physical equations must be formulated as tensor equations (c.f. (2.2)). The signature of the metric induces a notion of causality [9, 20, 25].

For this paper, an essential feature of a manifold is that around each point $p \in M$ there exists an open neighborhood $N_p$ (called a patch) together with
a homeomorphism \( x = (x^0, \ldots, x^3) : \mathcal{N}_p \to \mathbb{R}^4 \), which defines coordinates on its image in \( \mathbb{R}^4 \). \( \mathcal{N}_p \) together with the homeomorphism \( x \) are called a chart. The collection of all such charts (covering the manifold) is called an atlas. If the intersection of two coordinate patches is nonempty, \( x \circ y^{-1} \) defines a mapping from an open set in \( \mathbb{R}^4 \) to another one (often referred to as a coordinate transformation). Given that all coordinate transformations in the atlas are \( C^k \) differentiable, the manifold \( M \) is called a \( C^k \)-manifold and its atlas a \( C^k \)-atlas. In General Relativity the manifold is usually assumed to be \( C^2 \), however, in order to address shock wave solutions of the Einstein Euler equations it is crucial to consider \( C^{1,1} \)-manifolds. In fact, lowering the regularity to \( C^{1,1} \) is the crucial step allowing for a smoothing of the metric in the presence of a single shock wave, (c.f. [5] and Theorem [6]).

Throughout this paper we use the Einstein summation convention, that is, we always sum over repeated upper and lower indices, e.g. \( v^\mu w_\mu = v^0 w_0 + \ldots + v^3 w_3 \) for \( \mu, \nu \in \{0, \ldots, 3\} \). Moreover, we use the type of index to indicate in which coordinates a tensor is expressed in, for instance, \( T^\mu_\nu \) denotes a (1,1)-tensor in coordinates \( x^\mu \) and \( T^\alpha_\beta \) denotes the same tensor in (different) coordinates \( x^\alpha \). Under a change of coordinates, tensors transform via contraction with the Jacobian \( J^\mu_\alpha \) of the coordinate transformation, which is given by

\[
J^\mu_\alpha = \frac{\partial x^\mu}{\partial x^\alpha}.
\]

(2.1)

For example, a vector transforms as \( v^\mu = J^\mu_\alpha v^\alpha \), a one-form as \( v_\mu = J^\alpha_\mu v_\alpha \) and a (1,1)-tensor as \( T^\mu_\nu = J^\mu_\alpha J^\beta_\nu T^\alpha_\beta \), where \( J^\alpha_\mu \) denotes the inverse of (2.1), c.f. [9] for some basics on tensors. The metric tensor transforms according to

\[
g^\mu_\nu = J^\alpha_\mu J^\beta_\nu g^\alpha_\beta,
\]

(2.2)

which is crucial for the methods in this paper. We denote with \( g^{ij} \) the inverse of the metric, defined via

\[
g^{\mu_\sigma} g_{\sigma_\nu} = \delta^\mu_\nu,
\]

(2.3)

in terms of the Kronecker-symbol \( \delta^\mu_\nu \). By convention, we raise and lower tensor indices with the metric, for example \( T^\mu_\nu = g_{\nu_\sigma} T^\sigma_\mu \).

In (2.1), the Jacobian is defined for a given change of coordinates. Vice versa, for a set of given functions \( J^\mu_\beta \) to be a Jacobian of a coordinate transformation it is necessary and sufficient to satisfy (see appendix A for details),

\[
J^\mu_\alpha,\beta = J^\mu_\beta,\alpha \quad \text{and} \quad \det (J^\mu_\beta) \neq 0,
\]

(2.4)

(2.5)

where \( f_\alpha = \frac{\partial f}{\partial x_\alpha} \) denotes partial differentiation with respect to coordinates \( x^\alpha \). (2.4) ensures \( J^\mu_\alpha \) to be integrable to coordinates \( x^\mu \), while (2.5) ensures the invertibility of the coordinate functions and that non-singular tensors remain non-singular under change of coordinates. We henceforth refer to (2.4) as the integrability condition.

We now discuss the Einstein equations that govern the gravitational field through coupling the spacetime curvature to the energy- and matter-content.
The Einstein equations read \[ G^{\mu\nu} = \kappa T^{\mu\nu} \] (2.6)

(in units where \( c = 1 \)), where \( \kappa := 8\pi G \) incorporates Newton’s gravitational constant \( G \) into the equation and

\[ G^{\mu\nu} = R^{\mu\nu} - \frac{1}{2} R g^{\mu\nu} + \Lambda g^{\mu\nu} \] (2.7)
is the Einstein tensor, \( \Lambda \) denotes the cosmological constant. (Our method applies regardless the choice of \( \Lambda \in \mathbb{R} \), since the term \( \Lambda g^{\mu\nu} \) gives a continuous contribution to the Einstein equations (2.6).) The Ricci tensor \( R^{\mu\nu} \) is defined to be the trace of the Riemann tensor \( R^{\mu\nu} = R_{\mu\nu}^{\sigma} \) and the trace of the Ricci tensor \( R = R^{\sigma}_{\sigma} \) is called the scalar curvature. The metric tensor \( g^{\mu\nu} \) enters the Einstein equations through the Christoffel symbols

\[ \Gamma_{\nu\rho}^{\mu} = \frac{1}{2} g_{\mu\sigma} \left( g_{\nu\sigma,\rho} + g_{\rho\sigma,\nu} - g_{\nu\rho,\sigma} \right) \] (2.8)
since the Riemann curvature tensor is given by

\[ R^{\mu}_{\nu\rho\sigma} = \Gamma^{\mu}_{\nu\rho,\sigma} - \Gamma^{\mu}_{\nu\sigma,\rho} + \Gamma^{\mu}_{\lambda\sigma} \Gamma_{\nu\rho}^{\lambda} - \Gamma^{\mu}_{\lambda\rho} \Gamma_{\nu\sigma}^{\lambda} \]

Taking into account the symmetry of the metric tensor \( g^{\mu\nu} = g_{\nu\mu} \), (and the Ricci tensor \( R^{\mu\nu} = R_{\nu\mu} \)), the Einstein equations form a set of 10 second order differential equations on \( g^{\mu\nu} \). For the Einstein tensor to be defined in a strong (almost everywhere) sense a \( C^{1,1} \) metric regularity is necessary, however, at the level of shock waves only Lipschitz continuity is guaranteed and one must introduce the Einstein tensor in a weak (distributional) sense, c.f. [18].

By construction, due to the Bianchi identities of the Riemann curvature tensor, the Einstein tensor is divergence free

\[ G^{\mu\nu}_{;\nu} = 0, \] (2.9)

where the semicolon in (2.9) denotes covariant differentiation, that is

\[ v^{\mu}_{;\nu} = v^{\mu}_{,\nu} + \Gamma^{\mu}_{\sigma\nu} v^{\sigma} \] (2.10)

for a vector field \( v^{\mu} \). Through (2.9), the Einstein equations ensure conservation of energy in the matter source, that is,

\[ T^{\mu\nu}_{;\nu} = 0, \] (2.11)

which was one of the guiding principles Einstein followed in the construction of the Einstein tensor [3]. In the case of a perfect fluid, that is, the energy momentum tensor is given by

\[ T^{\mu\nu} = (p + \rho) u^{\mu} u^{\nu} + pg^{\mu\nu}, \] (2.12)

(2.11) are the general relativistic Euler equations, with \( \rho \) being the density, \( p \) the pressure and \( u^{\mu} \) the tangent vector of the fluid flow normalized such that \( u^{\mu} u_{\mu} = -1 \), (see for example [20]). In a locally inertial frame around a point \( p \), that is, in coordinates where the metric satisfies (c.f. Definition 9.1)

\[ g_{\mu\nu}(p) = \eta_{\mu\nu} \]
and \( g_{\mu\nu,\sigma}(p) = 0 \) \( \forall \mu, \nu, \sigma \in \{0, \ldots, 3\} \), \( (2.13) \)

with \( \eta_{\mu\nu} \) the Minkowski metric, \( (2.11) \) reduce to the special relativistic Euler equations,

\[ T_{\mu\nu} = 0 \]

at the point \( p \). \( (2.6) \) together with \( (2.11) \), are the coupled \textit{Einstein Euler} equations. Having unknowns \( g_{\mu\nu}, \rho \) and \( u^\nu \), the coupled Einstein Euler equations are under-determined, however, prescribing an (barotropic) equation of state, \( p = p(\rho) \), the system closes.

The Euler equations \( (2.11) \) are a system of conservation laws, where shock waves form from smooth initial whenever the flow is sufficiently compressive,\cite{2, 12, 17} This makes the study of shock waves inevitable for perfect fluid sources, arguing the result of this paper being fundamental to General Relativity.

Being discontinuous, shock wave solutions satisfy the conservation law only in a weak sense. To obtain the weak formalism of a PDE, first multiply the equation with a smooth (“test”) function, then integrate the resulting equation and finally apply the divergence theorem to shift all derivatives onto the test function, c.f. \cite{18}. The weak form of the Euler equations reads

\[ \int_M T_{\mu\nu} \varphi^\nu d\mu_M = 0, \quad (2.14) \]

where \( \varphi \in C^\infty_0(M) \) is a test function and \( d\mu_M \) is the spacetime volume element. Across the surface of discontinuity \( \Sigma \) (the so-called shock surface) the solution satisfies the Rankine Hugoniot conditions (RH conditions),

\[ [T_{\mu\nu}] N^\nu = 0, \quad (2.15) \]

where \( N^\nu \) is normal to the hypersurface \( \Sigma \) and \( [u] := u_L - u_R \) for \( u_L \) and \( u_R \) denoting the left and right limit (to \( \Sigma \)) respectively. \( [u] \) is referred to as the “jump” in \( u \) across \( \Sigma \) and provides a measure for the strength of the shock. In fact, suppose \( T_{\mu\nu} \) is a strong solution everywhere away from the surface of discontinuity, then \( T_{\mu\nu} \) is a weak solution in a whole neighborhood of the surface if and only if the RH conditions \( (2.15) \) hold everywhere on the surface. Thus, one can bypass the weak formalism and instead work with the RH conditions and the strong solution away from the shock surface, as we do in this paper. Again, we point out that we do not specifically use perfect fluid sources in our method, it suffices to assume that \( T_{\mu\nu} \) satisfies the RH condition \( (2.15) \) on \( \Sigma \) and is continuous away from some \( \Sigma \), (c.f. Definition \( 3.1 \)).

For the Einstein tensor to be defined in a strong sense almost everywhere a \( C^{1,1} \) metric-regularity is required. However, at the level of shock waves only Lipschitz continuity of the matric is guaranteed, as we discuss below (see also \cite{7}). For this low regularity objects like the Einstein or Riemann curvature tensor can only be introduced in a weak (distributional) sense. Nevertheless, in his 1966 paper \cite{10} Israel proved that, at a single smooth shock surface, there exists coordinates where metric is \( C^{1,1} \) regular if and only if the energy momentum tensor is bounded almost everywhere. If any of the above holds, the Rankine
Hugoniot jump conditions are satisfied everywhere on the shock surface. Another part of Israel’s precise result is that the gravitational metric is smoothed to $C^{1,1}$ in Gaussian Normal Coordinates if and only if the second fundamental form of the metric is continuous across the surface. The latter is an invariant condition meaningful for metrics Lipschitz continuous across a hypersurface, and is often referred to in the literature as the junction condition, c.f. [25].

At the heart of Israel’s method lies the ad-hoc choice of Gaussian Normal Coordinates with respect to the shock surface $\Sigma$, that is, we first arrange by a smooth coordinate transformation that locally $\Sigma = \{ p \in M : x^a(p) = 0 \}$ and then exchange the $n-\text{th}$ coordinate function by arc-length of geodesics normal to $\Sigma$. In more detail, Gaussian Normal Coordinates are the mapping assigning a point $p \in M$ (sufficiently close to $\Sigma$) to a point $x^\alpha(p)$ in $\mathbb{R}^n$ as follows:

$$x^\alpha(p) = \left( s, x^{n-1}(q), ..., x^1(q) \right),$$  \hspace{1cm} (2.16)

where $s$ is the arc-length parameter of a geodesic curve $\gamma$ starting at the point $q \in \Sigma$ in the direction $\frac{\partial}{\partial x^n}$ normal to $\Sigma$, with $\gamma(s) = p$, and $x^\alpha(q) = x^\alpha(q)$ for all $\alpha = i \in \{1, ..., n-1\}$. Computing now the Einstein tensor in coordinates (2.16), one finds that each component of the resulting Einstein tensor contains only a single second order normal derivative, $g_{\alpha\beta,nn}$, while all other terms in the Einstein equations are in $L^\infty$ and thus $g_{\alpha\beta,nn} \in L^\infty$ as well. Since all other second order metric derivatives are bounded by assumption we conclude $g_{\alpha\beta} \in C^{1,1}$.

In Section 6 we give a new constructive proof of Israel’s result, based on the method we introduce in Section 5.

In the following we introduce and discuss spherically symmetric spacetimes. Many spacetimes of fundamental interest to General Relativity are spherically symmetric, for example, the Schwarzschild, the Oppenheimer Volkoff or the Friedmann Robertson Walker spacetimes, [9]. A spherically symmetric spacetime is a Lorentz manifold, allowing for three independent spacelike Killing vector field, such that the subspaces parameterized by the flow of the Killing vectors have a positive constant curvature [25]. We refer to those subspaces as the spaces of symmetry, which in suitable coordinates are given by a family of two spheres of smoothly varying radii. A vector field $X^\mu$ is called a Killing vector if it satisfies Killing’s equations

$$X_{\mu;\nu} + X_{\nu;\mu} = 0.$$  \hspace{1cm} (2.17)

Killing’s equation ensures that the flow of a solution $X^\mu$ is an isometry of spacetime, since a vector field $X^\mu$ solves (2.17) if and only if the Lie derivative of the metric in direction of $X^\mu$ vanishes [9 26 25],

$$L_{X}g = 0.$$  \hspace{1cm} (2.18)

In a spherically symmetric spacetime, assuming that one of the spaces of symmetry has constant scalar curvature $K = 1$, one can always introduce coordinates

\[2\text{Here we use that Lipschitz continuity of a function is equivalent for it to be in the Sobolev-space } W^{1,\infty}, \text{ containing all functions with first (weak) derivatives in } L^\infty. \text{ Both imply the function to be differentiable almost everywhere [4].}\]
\(\vartheta\) and \(\varphi\) such that the metric takes on the form
\[
ds^2 = -Adt^2 + Bdr^2 + 2Edtdr + Cd\Omega^2,
\]
where
\[
d\Omega^2 = d\varphi^2 + \sin^2(\vartheta)\,d\vartheta^2
\]
is the line element on the two-sphere and the metric coefficients \(A, B, C\) and \(E\) only depend on \(t\) and \(r\), [25]. This simplifies the metric significantly, as its original ten free components reduce to four and the ten Einstein equations reduce to four independent ones accordingly. The metric structure (2.19) is preserved under coordinate transformations in the \((t, r)\)-plane, that is, transformations which keep the angular variables \(\vartheta\) and \(\varphi\) fixed. Under generic conditions, one can simplify the metric further by introducing a new “radial” variable \(r' := \sqrt{C}\) and removing the off-diagonal element through an appropriate coordinate transformation in the \((t, r')\)-plane [25], denoting the resulting coordinates again by \(t\) and \(r\) the new metric reads
\[
ds^2 = -Adt^2 + Bdr^2 + r'^2d\Omega^2.
\]
(2.20)
Coordinates where the metric is given by (2.20) are called Standard Schwarzschild Coordinates (SSC). In SSC, the metric has only two free components and the Einstein equations simplify significantly:
\[
B_r + B \frac{B - 1}{r} = \kappa AB^2 r T^{00}
\]
(2.21)
\[
B_t = -\kappa AB^2 r T^{01}
\]
(2.22)
\[
A_r - A \frac{1 + B}{r} = \kappa AB^2 r T^{11}
\]
(2.23)
\[
B_{tt} - A_{rr} + \Phi = -2\kappa AB r^2 T^{22},
\]
(2.24)
with
\[
\Phi := -\frac{BA_t B_t}{2AB} - \frac{B_t^2}{2B} - \frac{A_r}{r} + \frac{AB_r}{rB} + \frac{A_r^2}{2A} + \frac{A_r B_r}{2B}.
\]
The first three Einstein equations in SSC are central in the method we develop and in the proof of Theorem 7.1.

Finally, we address the issue of the metric regularity in the presence of shock waves. A shock wave is a (weak) solution \(T^{\mu\nu}\) of the Euler equations (2.11), discontinuous across a timelike hypersurface \(\Sigma\) and smooth away from \(\Sigma\), where the discontinuity satisfies the RH conditions (2.15) across \(\Sigma\). Now, from the first three Einstein equations in SSC, (2.21)-(2.23), it is straightforward to read off that the metric cannot be any smoother than Lipschitz continuous if the matter source \(T^{\mu\nu} \in L^\infty\) is discontinuous. In this paper we henceforth assume that the gravitational metric in SSC is Lipschitz continuous, providing us a consistent framework to address shock waves in General Relativity, agreeing with various examples of solutions to the coupled Einstein Euler equations, for example, the

\[\text{footnote}{\phi}It is not clear to us if shock wave solutions with a lower metric regularity do exist.\]
solutions in [7] and [18]. Moreover, Lipschitz continuity arises naturally in the problem of matching two spacetimes across an interface [10].

As mentioned above, Israel proved the remarkable result that whenever a metric is Lipschitz continuous across a smooth single shock surface Σ and has an almost everywhere bounded Einstein tensor, then there always exists a coordinate transformation defined in a neighborhood of Σ smoothing the components of the gravitational metric to $C^{1,1}$. In spherically symmetric spacetimes, Smoller and Temple showed in [18] that in a neighborhood of a single radial shock surface there exists coordinates where the metric is $C^{1,1}$ regular if and only if the RH conditions (2.15) hold everywhere on the surface. Thus, for example, the metric across single radial shock surfaces can be no smoother than Lipschitz continuous in SSC, but can be smoothed to $C^{1,1}$ by coordinate transformations. However, it has remained an open problem whether or not such a theorem applies to the more complicated $C^{0,1}$-solutions containing shock wave interactions, for example, the SSC solutions proven to exist by Groah and Temple [7]. Our purpose here is to show that such solutions cannot be smoothed to $C^{1}$ in a neighborhood of a point of regular shock wave interaction in SSC, a notion we now make precise.

3. A Point of Regular Shock Wave Interaction in SSC

In this section we set up our basic framework and give the definition of a point of regular shock wave interaction in SSC, that is, a point $p$ where two radial shock waves enter or leave the point $p$ with distinct speeds. We henceforth restrict attention to radial shock waves, that is, shock surfaces $Σ$ that can (locally) be parameterized by

$$Σ(t, \vartheta, \phi) = (t, x(t), \vartheta, \phi),$$

and across which the stress-energy-momentum tensor $T$ is discontinuous. Said differently, for a fixed time $t$ the shock surface is a two sphere (of symmetry) with radius $x(t)$. Our subsequent methods apply to spacelike and timelike surfaces alike, (inside or outside a black hole, c.f. [19]), since we did not need to specify the signs of the metric coefficient, but without loss of generality and for ease of notation we henceforth restrict to timelike surfaces.

For radial hypersurfaces in SSC, the angular variables play a passive role, and the essential issue regarding smoothing the metric components lies within the atlas of $C^{1,1}$ coordinate transformations acting on the $(t, r)$-plane, i.e., angular coordinates are kept fixed. (In fact, for the proof of Theorem 6.1 and 7.1 we consider only the $(t, r)$-plane and standard methods suffice to extend the result to the full atlas, leading to Theorem 1.1) Therefore it suffices to work with the so-called shock curve $γ$, that is, the shock surface $Σ$ restricted to the $(t, r)$-plane,

$$γ(t) = (t, x(t)),$$

with normal 1-form

$$n_ν = (\dot{x}, -1).$$
For radial shock surfaces (3.1) in SSC, the RH conditions (2.15) take the simplified form

\[
\begin{align*}
[T^{00}] & \dot{x} = [T^{01}], \quad (3.4) \\
[T^{10}] & \dot{x} = [T^{11}]. \quad (3.5)
\end{align*}
\]

In Section 7 we prove the main result of this paper by establishing that SSC metrics Lipschitz continuous across two intersecting shock curves, cannot be smoothed to \( C^{1,1} \) by a \( C^{1,1} \) coordinate transformation. To establish our basic framework, suppose two timelike shock surfaces \( \Sigma_i \) are parameterized in SSC by

\[
\Sigma_i(t, \theta, \phi) = (t, x_i(t), \theta, \phi), \quad (3.6)
\]

for \( i = 1, 2, 3, 4 \), where \( \Sigma_1 \) and \( \Sigma_2 \) are defined for \( t \leq 0 \) and \( \Sigma_3 \) and \( \Sigma_4 \) are defined for \( t \geq 0 \) described in the \((t, r)\)-plane by,

\[
\gamma_i(t) = (t, x_i(t)), \quad (3.7)
\]

with normal 1-forms

\[
(n_i)_\nu = (\dot{x}_i, -1). \quad (3.8)
\]

Denoting with \([\cdot]_i\) the jump across the \( i \)-th shock curve the RH conditions read, in correspondence to (3.4)-(3.5),

\[
\begin{align*}
[T^{00}]_i & \dot{x}_i = [T^{01}]_i, \quad (3.9) \\
[T^{10}]_i & \dot{x}_i = [T^{11}]_i. \quad (3.10)
\end{align*}
\]

For the proof of our main result (Theorem 7.1) it suffices to restrict attention to the lower \((t < 0)\) or upper \((t > 0)\) part of a shock wave interaction that occurs at \( t = 0 \). That is, it suffices to impose conditions on either the lower or upper half plane (c.f. Figure 1)

\[
\mathbb{R}^2_- = \{(t, r) : t < 0\},
\]

or

\[
\mathbb{R}^2_+ = \{(t, r) : t > 0\},
\]

respectively, whichever half plane contains two shock waves that intersect at \( p \) with distinct speeds. Thus, without loss of generality, let \( t < 0 \) and let \( \gamma_i(t) = (t, x_i(t)), \quad (i = 1, 2), \) be two shock curves in the lower \((t, r)\)-plane intersecting in a point \((0, r_0)\), for \( r_0 > 0 \), that is,

\[
x_1(0) = r_0 = x_2(0). \quad (3.11)
\]

We are now prepared to give the definition of what we call a point of regular shock wave interaction in SSC. By this we mean a point \( p \) where two shock waves collide with distinct speeds, such that the metric is smooth away from the shock curves and Lipschitz continuous across each shock curve, allowing for a discontinuous \( T^{\mu \nu} \) and the RH condition to hold. Recall, we assume without loss of generality a lower shock wave interaction in \( \mathbb{R}^2_- \).

---

\(^4\)Shock waves typically change their speeds discontinuously at the point of interaction.

\(^5\)The intersection of the shock surfaces is a two sphere, but abusing language we refer to it as a point, consistent with us suppressing the (trivial) angular dependence in our method.
Figure 1. Example of intersecting shock curves.

**Definition 3.1.** Let \( r_0 > 0 \), and let \( g_{\mu\nu} \) be an SSC metric in \( C^{0,1} \left( \mathcal{N} \cap \overline{\mathbb{R}^2} \right) \) where \( \mathcal{N} \subset \mathbb{R}^2 \) is a neighborhood of a point \( p = (0, r_0) \) of intersection of two timelike shock curves \( \gamma_i(t) = (t, x_i(t)) \in \mathbb{R}^2, t \in (-\epsilon, 0) \). Assume the shock speeds \( \dot{x}_i(0) = \lim_{t \to 0} \dot{x}_i(t) \) exist and are distinct, \( \dot{x}_1(0) \neq \dot{x}_2(0) \), and let \( \mathcal{N} \) denote the neighborhood consisting of all points in \( \mathcal{N} \cap \overline{\mathbb{R}^2} \) not in the closure of the two intersecting curves \( \gamma_i(t) \). Then we say that \( p \) is a point of regular shock wave interaction in SSC if:

(i) The pair \( (g, T) \) is a strong solution of the SSC Einstein equations \([\text{2.21}-\text{2.24}]\) in \( \mathcal{N} \), with \( T^{\mu\nu} \in C^0(\mathcal{N}) \) and \( g_{\mu\nu} \in C^2(\mathcal{N}) \).

(ii) The limits of \( T^{\mu\nu} \) and of metric derivatives \( g_{\mu\nu,\sigma} \) exist on both sides of each shock curve \( \gamma_i(t) \) for all \( t \in (-\epsilon, 0) \).

(iii) The jumps in the metric derivatives \([g_{\mu\nu,\sigma}]_i(t)\) are \( C^1 \) function with respect to \( t \) for \( i = 1, 2 \) and for \( t \in (-\epsilon, 0) \).

(iv) The limits

\[
\lim_{t \to 0}[g_{\mu\nu,\sigma}]_i(t) = [g_{\mu\nu,\sigma}]_i(0)
\]

exist for \( i = 1, 2 \).

(v) The metric \( g \) is continuous across each shock curve \( \gamma_i(t) \) separately, but no better than Lipschitz continuous in the sense that, for each \( i \) there exists \( \mu, \nu \) such that

\[
[g_{\mu\nu,\sigma}]_i(n_i)^\sigma \neq 0
\]
at each point on $\gamma_i$, $t \in (-\epsilon, 0)$ and
\[
\lim_{t \to 0} [g_{\mu\nu,\sigma}])_{\gamma_i}(n_i)^\sigma \neq 0.
\]

(vi) The stress tensor $T$ is bounded on $\mathcal{N} \cap \mathbb{R}^2_+$ and satisfies the RH conditions
\[
[T^{\nu\sigma}]_{\gamma_i} (n_i)_{\sigma} = 0
\]

at each point on $\gamma_i(t)$, $i = 1, 2$, $t \in (-\epsilon, 0)$, and the limits of these jumps exist up to $p$ as $t \to 0$.

The structure assumed in Definition 3.1 reflects the regularity of shock wave solutions of the coupled Einstein-Euler equations consistent with the theory in [7] and confirmed by the numerical simulation in [24]. If one wants to include the structure of single general relativistic shock waves, for instance, as in [18, 19, 20, 21, 22, 23], we expect Definition 3.1 to be the most natural way of introducing shock wave interactions. In more detail, (i)-(iii) and (vi) are straightforward generalizations from either the flat case or the case of a single relativistic shock wave. Note, we do not require $T^{\mu\nu}$ to match up continuously at $p$. Condition (iv) prohibits blow-up phenomena in the matter sources and condition (v) ensures the shock to persist up to the interaction.

Our method of proof works for timelike and spacelike shock surfaces alike, but breaks down for null hypersurfaces. Also, our method fails in the unphysical case $\dot{x}_1(0) = \dot{x}_2(0)$, due to (7.19).

Note, we do not explicitly assume a perfect fluid source in Definition 3.1 only an energy momentum tensor $T^{\mu\nu}$ that satisfies the Rankine Hugoniot jump condition. Furthermore, the structure we defined is general enough to include all shock wave interaction described in [17]. In more detail, it includes the case of two interacting shock waves of opposite families that cross at the point $p$, but also general enough to include the cases of two outgoing shock waves created by the focusing of compressive rarefaction waves (as simulated in [24]), or two incoming shock waves resulting in one outgoing shock wave and an outgoing rarefaction wave.

Even though the Groah-Temple theory [7] establishes existence of $C^{0,1}$ shock waves before and after interaction, and the work of Vogler numerically simulates the detailed structure of the metric at a point of shock wave interaction, the mathematical theory still lacks a complete proof that rigorously establishes the detailed structure of shock wave interactions summarized in Definition 3.1. Such a proof would be very interesting, and remains to be done.

4. Functions $C^{0,1}$ Across a Hypersurface

In this section we first define a function (or a metric) being $C^{0,1}$ across a hypersurface. We then study the relation of a metric being $C^{0,1}$ across a hypersurface to the Rankine Hugoniot jump condition through the Einstein equations and derive a set of three equations central to our methods in Sections 5 - 7. Finally we derive a canonical form functions $C^{0,1}$ across a hypersurface can be represented in, which is fundamental to the proof of our main theorem.
Definition 4.1. Let $\Sigma$ be a smooth hypersurface in some open set $\mathcal{N} \subset \mathbb{R}^d$ with a normal vector-field nowhere lightlike. We call a function $f \in C^{0,1}(\mathcal{N})$ “Lipschitz continuous across $\Sigma$”, (or $C^{0,1}$ across $\Sigma$), if $f$ is smooth in $\mathcal{N}\setminus \Sigma$ and limits of derivatives of $f$ to $\Sigma$ exist on each side of $\Sigma$ separately and are smooth functions. We call a metric $g_{\mu\nu}$ Lipschitz continuous across $\Sigma$ in coordinates $x^\mu$ if all metric components are $C^{0,1}$ across $\Sigma$.

For us, “smooth” means enough continuous derivatives so that regularity is not an issue. Usually $f \in C^2(\mathcal{N}\setminus \Sigma)$ with left-/right-limits of first and second order derivatives existing suffices.

The main point of the above definition is that we assume smoothness of $f$ away and tangential to the hypersurface $\Sigma$, but allow for the normal derivative of $f$ to be discontinuous, that is,

$$[f,\sigma] n^\sigma \neq 0,$$  \hspace{1cm} (4.1)

where $n^\sigma$ is normal to $\Sigma$ with respect to some (Lorentz-) metric $g_{\mu\nu}$ defined on $\mathcal{N}$. Moreover, the continuity of $f$ across $\Sigma$ implies the continuity of all derivatives of $f$ tangent to $\Sigma$, i.e.,

$$[f,\sigma] v^\sigma = 0,$$  \hspace{1cm} (4.2)

for all $v^\sigma$ tangent to $\Sigma$.

In the following we clarify the implications of Definition 4.1, particularly (4.2), together with the Einstein equations on the RH conditions (3.4), (3.5). For this, consider a spherically symmetric spacetime metric in SSC (1.1) and assume the first three Einstein equations (2.21)-(2.23) hold and the stress tensor $T$ is discontinuous across a smooth radial shock surface, described in the $(t,r)$-plane by $\gamma(t)$ as in (3.1)-(3.3). To this end, condition (4.2) applied to each metric component $g_{\mu\nu}$ in SSC, c.f. (2.20), reads

$$[B_r] = -\dot{x}[B_t],$$  \hspace{1cm} (4.3)

$$[A_t] = -\dot{x}[A_r].$$  \hspace{1cm} (4.4)

On the other hand, the first three Einstein equations in SSC (2.21)-(2.23) imply

$$[B_r] = \kappa AB^2 r[T^00],$$  \hspace{1cm} (4.5)

$$[B_t] = -\kappa AB^2 r[T^01],$$  \hspace{1cm} (4.6)

$$[A_r] = \kappa AB^2 r[T^{11}].$$  \hspace{1cm} (4.7)

Now, using the jumps in Einstein equations (4.5)-(4.7), we find that (4.3) is equivalent to the first RH condition (3.4), while the second condition (4.4) is independent of equations (4.5)-(4.7), because $A_t$ does not appear in the first order SSC equations (2.21)-(2.23). The result, then, is that in addition to the assumption that the metric be $C^{0,1}$ across the shock surface in SSC, the RH conditions (3.4) and (3.5) together with the Einstein equations (4.5)-(4.7), yield only one additional condition over and above (4.3) and (4.4), namely,

$$[A_r] = -\dot{x}[B_t].$$  \hspace{1cm} (4.8)

This observation is consistent with Lemma 9, page 286, of [18], where only one jump condition need to be imposed to meet the full RH relations.
The RH conditions together with the Einstein equations will enter our method in Section 5 only through equations (4.8), (4.3) and (4.4). In particular, our proofs of Theorems 6.1 and 7.1 below will rely on the Einstein equation and the RH condition only in the sense of (4.8), (4.3) and (4.4).

The following lemma provides a canonical form for any function \( f \) Lipschitz continuous across a single shock curve \( \gamma \) in the \((t,r)\)-plane, under the assumption that the vector \( n^\mu \), normal to \( \gamma \), is obtained by raising the index in (3.3) with respect to a Lorentzian metric \( C_{\mu\nu} \) across \( \gamma \).

**Lemma 4.2.** Suppose \( f \) is \( C^{0,1} \) across a smooth curve \( \gamma(t) = (t,x(t)) \) in the sense of Definition 4.1, \( t \in (-\epsilon,\epsilon) \), in an open subset \( N \) of \( \mathbb{R}^2 \). Then there exists a function \( \Phi \in C^1(N) \) such that

\[
f(t,r) = \frac{1}{2} \varphi(t) |x(t) - r| + \Phi(t,r),
\]

if and only if

\[
\varphi(t) = \frac{[f_{,\mu}]n^\mu}{n^\sigma n_{\sigma}} \in C^1(-\epsilon,\epsilon),
\]

where \( n_\mu(t) = (\dot{x}(t),-1) \) is a 1-form normal to \( v^\mu(t) = \dot{v}^\mu(t) \) and indices are raised and lowered by a Lorentzian metric \( g_{\mu\nu} \) which is \( C^{0,1} \) across \( \gamma \).

**Proof.** We first prove the explicit expression for \( \varphi \) in (4.10). Suppose there exists \( \Phi \in C^1(N) \) satisfying (4.9), defining \( X(t,r) := x(t) - r \) this implies that

\[
[f_{,\mu}]n^\mu = \frac{1}{2} \varphi[H(X)]X_{,\mu}n^\mu \quad \text{with} \quad H(X) := \begin{cases} -1 & \text{if } X < 0 \\ +1 & \text{if } X > 0 \end{cases},
\]

where we use \( \frac{d}{dX} |X| = H(X) \) and \( [\Phi_{,\mu}] = 0 \). Since \( [H(X)] = 2 \) and \( X_{,\mu}n^\mu = n_\mu n^\mu \) by (4.10), we conclude that

\[
[f_{,\mu}]n^\mu = \varphi n_\mu n^\mu.
\]

Solving the above equation for \( \varphi \) we obtain the expression claimed in (4.10). The \( C^1 \) regularity of \( \varphi \) follows from our incoming assumption on the metric and \( f \).

We now prove the opposite direction. Suppose \( \varphi \) as defined in (4.10) is \( C^1 \) regular. To show the existence of \( \Phi \in C^1(N) \) define

\[
\Phi = f - \frac{1}{2} \varphi |X|,
\]

then (4.9) holds and it remains to prove the \( C^1 \) regularity of \( \Phi \). It suffices to prove

\[
[\Phi_{,\mu}]n^\mu = 0 = [\Phi_{,\mu}]v^\mu,
\]

since \( \Phi \in C^1(N \setminus \gamma) \) follows immediately from (4.13) and the \( C^1 \) regularity of \( f \) and \( \varphi \) away from \( \gamma \). By assumption \( f \) satisfies (4.2), so that

\[
[\Phi_{,\mu}]v^\mu = -\varphi X_{,\mu}v^\mu.
\]

The right hand side vanishes since \( v^\mu(t) = \frac{T(1,\dot{x}(t))}{|T(1,\dot{x}(t))|} \) and thus \( [\Phi_{,\mu}]v^\mu = 0 \). Finally, \( \varphi \) defined in (4.10) together with

\[
X_{,\mu}n^\mu = n_\mu n^\mu
\]
show that
\[ [\Phi_{\mu}] n^\mu = \varphi n_\mu n^\mu - \varphi X_\mu n^\mu = 0. \]
This completes the proof. \( \square \)

In words, the canonical form (4.9) separates off the \( C^{0,1} \) kink of \( f \) across \( \gamma \) into the function \( |x(t) - r| \), from its more regular \( C^1 \) behavior incorporated into the functions \( \varphi \), giving the strength of the jump, and \( \Phi \), which encodes the remaining \( C^1 \) behavior of \( f \).

In Section 7 we need a canonical form analogous to (4.9) for two shock curves, which is provided by the next corollary. To this end, suppose timelike shock surfaces described in the \((t,r)\)-plane by, \( \gamma_i(t) = (t,x_i(t)) \), such that (3.6) - (3.8) applies. To cover the generic case of shock wave interaction, we assume each \( \gamma_i(t) \) is \( C^2 \) away from \( t = 0 \) with the lower/upper-limit of the tangent vectors existing up to \( t = 0 \). For our main results (Theorem 7.1 and 1.1) it suffices to consider the upper \((t > 0)\) or lower part \((t < 0)\) of a shock wave interaction (at \( t = 0 \)) separately, whichever part contains two shock waves that interact with distinct speeds. In the following we restrict without loss of generality to the lower part of a shock wave interactions, that is, to \( \mathbb{R}_-^2 = \{(t,r) : t < 0\} \).

**Corollary 4.3.** Let \( \gamma_i(t) = (t,x_i(t)) \) be two smooth curves defined on \( I = (-\epsilon,0) \), for some \( \epsilon > 0 \), such that the limits \( \lim_{t \to 0^-} \gamma_i(t) = (0,r_0) \) and \( \dot{x}_i(0) = \lim_{t \to 0^-} \dot{x}_i(t) \) both exist for \( i = 1, 2 \). Let \( \mathcal{N} \) be an open neighborhood of \( p = (0,r_0) \) in \( \mathbb{R}^2 \) and suppose \( f \) defined on \( \mathcal{N} \cap \mathbb{R}_-^2 \) is in \( C^{0,1} \) across each curve \( \gamma_i((-\epsilon,0)) \). Then there exists a function \( \Phi \in C^1(\mathcal{N} \cap \mathbb{R}_-^2) \), such that

\[
 f(t,r) = \frac{1}{2} \sum_{i=1,2} \varphi_i(t) |x_i(t) - r| + \Phi(t,r), \tag{4.14}
\]

for all \((t,r) \in \mathcal{N} \cap \mathbb{R}_-^2\), if and only if

\[
 \varphi_i(t) = \frac{[f_{,\mu}](n_i)^\mu}{(n_i)^\mu(n_i)_\mu} \in C^{0,1}(I), \tag{4.15}
\]

where \( (n_i)_\mu(t) = (\dot{x}_i(t),-1) \) is a 1-form normal to \( u_\mu(t) = \dot{\gamma}_i^\mu(t) \), \((i = 1, 2)\), and indices are raised and lowered by a Lorentzian metric \( C^{0,1} \) across \( \gamma \).

**Proof.** The Corollary follows by the same arguments as in the proof of Lemma 4.2 on each of the curves \( \gamma_i \), \((i = 1, 2)\), but defining

\[
 \Phi := f - \frac{1}{2} \sum_{i=1,2} \varphi_i |X_i|, \tag{4.16}
\]

for \( X_i(t,r) := x_i(t) - r \), in place of (4.13) and using that \( H(X_i) \) is discontinuous across \( \gamma_i \), but \([H(X_i)]_l = 0 \) for \( l \neq i \). (In particular, \( f \) meets (4.2) across each of the curves \( \gamma_i \).) \( \square \)

In Section 7 we use the canonical form (4.14) to characterize all Jacobians in the \((t,r)\)-plane, that could possibly lift the metric regularity from \( C^{0,1} \) to \( C^{1,1} \) on a deleted neighborhood of the point \( p \) of shock wave interaction, unique up
to addition of a function $\Phi \in C^1$. It is precisely the $C^1$ regularity across $\gamma$ that forces the determinant of those Jacobians to vanish at $p$ when taking the limit, resulting in a singular metric at $p$.

5. A Necessary and Sufficient Condition for Smoothing Metrics

In this section we derive a point-wise condition on the Jacobians of a coordinate transformation, necessary and sufficient for the Jacobian to lift the metric regularity from $C^{0,1}$ to $C^{1,1}$ in a neighborhood of a point on a single shock surface $\Sigma$. This condition is the starting point for our strategy and lies at the heart of the proofs in Section 6 and 7. We begin with the covariant transformation law

$$g_{\alpha\beta} = J^{\mu}_{\alpha} J^{\nu}_{\beta} g_{\mu\nu},$$

(5.1)

for the metric components at a point on a hypersurface $\Sigma$ for a general $C^{1,1}$ coordinate transformation $x^\mu \to x^\alpha$, where, as customary, the indices indicate the coordinate system. Let $J^{\mu}_{\alpha}$ denote the Jacobian of the transformation, that is,

$$J^{\mu}_{\alpha} = \frac{\partial x^\mu}{\partial x^\alpha}.$$ 

Now, assume the metric components $g_{\mu\nu}$ are only Lipschitz continuous across $\Sigma$, with respect to coordinates $x^\mu$. Then, differentiating (5.1) with respect to $\frac{\partial}{\partial x^\gamma}$ and taking the jump across $\Sigma$ we obtain

$$\left[ g_{\alpha\beta,\gamma} \right] = J^{\mu}_{\alpha} J^{\nu}_{\beta} \left[ g_{\mu\nu,\gamma} \right] + g_{\mu\nu} J^{\nu}_{\beta} \left[ J^{\mu}_{\alpha,\gamma} \right] + g_{\mu\nu} J^{\mu}_{\alpha} \left[ J^{\nu}_{\beta,\gamma} \right] + J^{\mu}_{\alpha} J^{\nu}_{\beta} \left[ g_{\mu\nu,\gamma} \right],$$

(5.2)

where $[f] = f_L - f_R$ denotes the jump in the quantity $f$ across the shock surface $\Sigma$, c.f. (2.15). Since both $g_{\mu\nu}$ and $J^{\mu}_{\alpha}$ are $C^{0,1}$ across $\Sigma$, the jumps are only on the derivative-terms. Now, $g_{\alpha\beta}$ is in $C^{1,1}$ if and only if

$$\left[ g_{\alpha\beta,\gamma} \right] = 0$$

(5.3)

for every $\alpha, \beta, \gamma \in \{0, ..., 3\}$, while (5.2) implies that (5.3) holds if and only if

$$\left[ J^{\mu}_{\alpha,\gamma} \right], J^{\nu}_{\beta} g_{\mu\nu} + \left[ J^{\nu}_{\beta,\gamma} \right] J^{\mu}_{\alpha} g_{\mu\nu} + J^{\mu}_{\alpha} J^{\nu}_{\beta} \left[ g_{\mu\nu,\gamma} \right] = 0.$$ 

(5.4)

(5.4) is a necessary and sufficient condition for smoothing the metric from $C^{0,1}$ to $C^{1,1}$ by the means of a coordinate transformation, (c.f. Corollary 5.2).

(5.2) also holds for Jacobians that are just Lipschitz continuous, possibly having discontinuous derivatives away from $\Sigma$, provided one defines the jump $[\cdot]$ along some (arbitrarily) prescribed curve, that is, define

$$[u] = \lim_{t \to 0^+} u \circ c(t) - \lim_{t \to 0^-} u \circ c(t)$$

for some curve $c$ transversal to $\Sigma$ such that $c(0) \in \Sigma$.

Note, if the coordinate transformation is $C^2$, the jumps in $J^{\mu}_{\alpha,\beta}$ vanish, and (5.4) reduces to

$$0 = J^{\mu}_{\alpha} J^{\nu}_{\beta} \left[ g_{\mu\nu,\gamma} \right],$$

which is tensorial. Now, since the Jacobians are non-singular, we find that it is precisely the lack of covariance in (5.2) for $C^{1,1}$ transformations, providing the necessary degrees of freedom, (namely $[J^{\mu}_{\alpha,\gamma}]$), making it possible for a Lipschitz
metric to be smoothed by coordinate transformation at points on a single shock surface. This illustrates, there is no hope of lifting the metric regularity within a $C^2$ atlas.

Let’s consider (5.4) as an inhomogeneous linear system with unknowns $[J^\mu_\alpha,\gamma]$, interpreting the $J^\mu_\alpha$-factors as free parameters. A solution of (5.4) alone does not ensure the existence of a Jacobian, since (5.4) does not yet ensure the existence of $C^{0,1}$ functions $J^\mu_\alpha$ that take on the values $[J^\mu_\alpha,\gamma]$ and satisfy the integrability condition (2.4), necessary for integrating $J^\mu_\alpha$ to coordinates. It is therefore crucial to impose in addition to (5.4) an appropriate integrability condition, namely

$$[J^\mu_\alpha,\beta] = [J^\mu_\beta,\alpha].$$

In the following we solve the linear system obtained from (5.4) and (5.5) for $[J^\mu_\alpha,\sigma]$, subject to Lemma 5.1. To simplify (5.4), we subsequently restrict to spherically symmetric spacetimes and assume that $g_{\mu\nu}$ denotes the metric in Standard Schwarzschild coordinates (2.20). To this end, suppose we are given a single radial shock surface $\Sigma$ in SSC locally parameterized by

$$\Sigma(t, \theta, \phi) = (t, x(t), \theta, \phi),$$

(5.6)
described in the $(t, r)$-plane by the corresponding shock curve

$$\gamma(t) = (t, x(t)).$$

(5.7)

For such a hypersurface in SSC, the angular variables play a passive role, and the essential issue regarding smoothing the metric components by $C^{1,1}$ coordinate transformations, lies in the atlas of $(t, r)$-coordinate transformations. Thus we restrict to the atlas of $(t, r)$-coordinate transformations, which keep the SSC angular coordinates fixed, c.f. (2.20). Then

$$(J^\mu_\alpha) = \begin{pmatrix} J^t_0 & J^t_1 & 0 & 0 \\ J^r_0 & J^r_1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

(5.8)

with the coefficients just depending on the SSC $t$ and $r$, which implies

$$[J^\mu_\alpha,\beta] = 0 \quad \text{whenever } \mu \in \{\varphi, \vartheta\} \text{ or } \alpha \in \{2, 3\} \text{ or } \beta \in \{2, 3\}.$$  

(5.9)

Now, (5.4) and (5.5) form a linear inhomogeneous $8 \times 8$ system in eight unknowns $[J^\mu_\alpha,\gamma]$. The following lemma shows that this system is uniquely solvable.

**Lemma 5.1.** Consider a metric in SSC,

$$g_{\mu\nu} dx^\mu dx^\nu = -A(t, r) dt^2 + B(t, r) dr^2 + r^2 d\Omega^2,$$

(5.10)

let $\Sigma$ denote a single radial hypersurface (5.6) across which $g_{\mu\nu}$ is Lipschitz continuous and assume (5.9). Then there exists a unique solution $[J^\mu_\alpha,\sigma]$ of the system (5.4) together with (5.5). The solution is given by:

$$[J^\mu_0,\varphi] = -\frac{1}{2} \left( \frac{[A_t]}{A} J^\sigma_0 + \frac{[A_r]}{A} J^\sigma_r \right); \quad [J^\mu_t,\varphi] = -\frac{1}{2} \left( \frac{[A_t]}{A} J^\sigma_t + \frac{[B_t]}{A} J^\sigma_r \right);$$

$^8$Indices $\mu$, $\nu$ and $\sigma$ always refer to SSC, if nothing else is mentioned.
where \( T \) with unknowns (5.10) drop out and (5.4) reduces to the following 6 × 6 major simplification of (5.9) is that the angular coefficients of the metric in SSC solve (5.4) for the metric in SSC (5.10) and Jacobians in the \((t,r)\)-plane. Thus only considered 6 unknowns in (5.14). We first setup the 8 linear system formed by (5.4) and (5.5) for the metric in SSC (5.10) and the case of Jacobians in the \((t,r)\)-plane. A \( J_{a,\beta} \) denotes indices whenever they appear on the Jacobian \( J_{a,\beta} \).

Proof. We first setup the 8 × 8 linear system formed by (5.4) and (5.5) for the metric in SSC (5.10) and Jacobians in the \((t,r)\)-plane. A major simplification of (5.9) is that the angular coefficients of the metric in SSC (5.10) drop out and (5.4) reduces to the following 6 × 6 linear system:

\[
A\vec{v} = \vec{w} \tag{5.13}
\]

with unknowns

\[
\vec{v} = T([J_{0,0}^t], [J_{0,1}^t], [J_{1,0}^t], [J_{1,1}^t], [J_{0,0}^r], [J_{0,1}^r], [J_{1,0}^r], [J_{1,1}^r]), \tag{5.14}
\]

where \( T \) denotes the transpose. The right hand side in (5.13) is given by

\[
\vec{w} = T(w_1, w_2, w_3, w_4, w_5, w_6) = T(w_{000}, w_{010}, w_{110}, w_{001}, w_{011}, w_{111}), \tag{5.15}
\]

for

\[
w_{a,\beta,\gamma} = J_{a,\alpha}^0 J_{\beta,\beta}^0 [A_{\alpha,\gamma}] - J_{a,\alpha}^1 J_{\beta,\beta}^1 [B_{\alpha,\gamma}]. \tag{5.16}
\]

The matrix \( A \) reads

\[
A = \begin{pmatrix}
2a & 0 & 0 & 2d & 0 & 0 \\
0 & c & a & 0 & b & d \\
0 & 2c & 0 & 0 & 2b & 0 \\
0 & 2a & 0 & 0 & 2d & 0 \\
0 & c & a & 0 & b & d \\
0 & 0 & 2c & 0 & 0 & 2b
\end{pmatrix} \tag{5.17}
\]

with \( a := -J_{0}^0 A , \ b := J_{1}^1 B , \ c := -J_{0}^0 A \) and \( d := J_{0}^1 B \). Observe the we already incorporated the integrability condition (5.12) into the linear system (5.13) and thus only considered 6 unknowns in (5.14). (5.13) has a unique solution given by,

\[
[J_{0,0}^t] = \frac{2w_2d - bw_1 - dw_4}{2AB|J|}. \tag{5.18}
\]
\[ [J^t_{0,1}] = \frac{dw_3 - bw_4}{2AB|J|}; \]
\[ [J^t_{1,1}] = \frac{dw_6 + bw_3 - 2bw_5}{2AB|J|}; \]
\[ [J^r_{0,0}] = \frac{aw_4 - w_2a + cw_1}{2AB|J|}; \]
\[ [J^r_{0,1}] = \frac{cw_4 - aw_3}{2AB|J|}; \]
\[ [J^r_{1,1}] = \frac{2cw_5 - aw_6 - cw_3}{2AB|J|}. \] (5.18)

where \(|J| = \det(J^\mu_\alpha) = J^0_0J^1_1 - J^0_1J^1_0\) denotes the determinant of the Jacobian.

It remains to change coordinates to SSC on the left hand side of (5.18), in order to recover the expressions in (5.11). From the chain rule and the continuity of the inverse Jacobian \(J^\sigma_{\gamma}\) we get

\[ [J^\mu_{\alpha,\nu}] = [J^\mu_{\alpha,\gamma}]J^\nu_{\gamma}. \]

and (for \(\sigma = t, r\))

\[ [g_{\mu\nu,\gamma}] = [g_{\mu\nu,\sigma}]J^\sigma_{\gamma}. \]

Using the definition of \(w_i\) in (5.16) together with (5.18) and the jumps in the integrability condition (5.5), a straightforward (but lengthy) calculation leads to (5.11).

In order to exemplify the procedure we outline the calculation leading to the expression for \([J^t_{0,t}]\) in (5.11):

\[ [J^t_{0,t}] = [J^t_{0,0}]J^0_t + [J^t_{0,1}]J^1_t = [J^t_{0,0}]\frac{J^t_1}{|J|} - [J^t_{0,1}]\frac{J^1_t}{|J|}, \] (5.19)

where we used Cramer’s rule in the last equality. Now, substituting (5.18) together with the definition of \(w_i\) in (5.16) and the integrability condition (5.5) a straightforward calculation leads to

\[ [J^t_{0,t}] = -\frac{1}{2} \left( [A_t]J^t_0 + [A_r]J^r_0 \right), \]

as claimed in (5.11). Performing similar computations for the remaining expression finishes the proof. \(\square\)

By Lemma 5.1 (5.11) is a necessary and sufficient condition for \([g_{\alpha\beta,\gamma}] = 0\), since it solves (5.4) and (5.5) uniquely. Putting (5.11) in words, the Jacobian must mirror the metric regularity, compensating for all discontinuous first order metric derivatives, to ensure \([g_{\alpha\beta,\gamma}] = 0\). In fact, assuming the \([J^\mu_{\alpha,\sigma}]\)-terms to come from an actual Jacobian of a coordinate transformation, \(J^\mu_{\alpha, (5.11)}\) is

9We want to consider the shock curve \(\gamma\) in the coordinates we start in (SSC) and thus express the condition (5.11) in terms of the given coordinates \(x^\mu\).

10This is why we expect a \(C^{1,1}\) atlas to be generic for possibly lifting the metric regularity from \(C^{0,1}\) to \(C^{1,1}\).
necessary and sufficient for raising the metric regularity to $C^{1,1}$ at a point on a single shock surface, as is proven in the next Corollary 5.2.

**Corollary 5.2.** Let $p$ be a point on a single smooth shock curve $\gamma$, and let $g_{\mu\nu}$ be the metric in SSC, $C^{0,1}$ across $\gamma$ in the sense of Definition 4.1. Let $J_{\alpha}^{\mu}$ be the Jacobian of an actual coordinate transformation defined on a neighborhood $\mathcal{N}$ of $p$ and assume $J_{\alpha}^{\mu}$ is $C^{0,1}$ across $\gamma$. Then the metric in the new coordinates, $g_{\alpha\beta}$, is in $C^{1,1}(\mathcal{N})$ if and only if $J_{\alpha}^{\mu}$ satisfies (5.11).

**Proof.** We first prove that $g_{\alpha\beta} \in C^{1,1}(\mathcal{N})$ implies (5.4). Suppose there exist coordinates $x^{\alpha}$ such that the metric in the new coordinates $g_{\alpha\beta}$ is $C^{1,1}$ regular, then

$$[g_{\alpha\beta\gamma}] = 0 \quad \forall \alpha, \beta, \gamma \in \{0, ..., 3\}.$$ 

This directly implies (5.4) and since $J_{\alpha}^{\mu}$ are the Jacobians of an actual coordinate transformation they satisfy the integrability condition (5.5) as well. By Lemma 5.1 the jumps in the derivatives of the Jacobian $[J_{\alpha\gamma}^{\mu}]$ then satisfy (5.11).

We now prove the opposite direction. Suppose the Jacobians $J_{\alpha}^{\mu}$ satisfy (5.11), by Lemma 5.1 they then meet the smoothing condition (5.4), which implies all metric derivatives $g_{\alpha\beta\gamma}$ to match continuously across the shock curve $\gamma$, that is, $[g_{\alpha\beta\gamma}] = 0$ for all $\alpha, \beta, \gamma \in \{0, ..., 3\}$. Since $g_{\mu\nu}$ and $J_{\alpha}^{\mu}$ are assumed to be smooth away from $\gamma$, it follows that $g_{\alpha\beta} \in C^{1}(\mathcal{N})$. Moreover, the existence of the limit towards $\gamma$ of all first and second order metric derivatives is postulated, (c.f. Definition 4.1), which then implies $g_{\alpha\beta} \in C^{1,1}(\mathcal{N})$. Namely, for any $q \in \Sigma$ one can bound

$$|g_{\alpha\beta\gamma}(t, r) - g_{\alpha\beta\gamma}(q)| < c |(t, r) - (q^0, q^1)|,$$

(5.20)

by a finite constant $c$, for instance choose

$$c = \sum_{\alpha, \beta, \gamma, \delta = 0, 1, 2, 3} \left( \sup_{q' \in \Sigma'} |(g_{\alpha\beta\gamma\delta})_L (q')| + \sup_{q' \in \Sigma'} |(g_{\alpha\beta\gamma\delta})_R (q')| \right) + \sum_{\alpha, \beta, \gamma} \sup_{(t, r) \in \mathcal{N}'} |g_{\alpha\beta\gamma}(t, r)|,$$

for $\mathcal{N}' \subset \mathcal{N}$ a compact neighborhood of $q$ and $\Sigma' = \Sigma \cap \mathcal{N}'$. □

Lemma 5.1 and Corollary 5.2 establish the remarkable result that there is no algebraic obstruction to lifting the metric regularity. Namely, the smoothing condition (5.4) has the solution (5.11), and its solvability is neither connected to the RH conditions (3.4)-(3.5) nor the Einstein equations. However, by [18, 10], the RH conditions must be imposed for coordinate transformations smoothing out the metric to exist. The point here is, that to prove the existence of coordinate transformations lifting the regularity of SSC metrics to $C^{1,1}$ at $p \in \Sigma$, one must prove that there exists a set of functions $J_{\alpha}^{\mu}$ defined in a neighborhood of $p$, such that (5.11) holds at $p$, and such that the integrability condition (2.4), necessary for $J_{\alpha}^{\mu}$ to be the Jacobian of a coordinate transformation, holds in a neighborhood of $p$. In Section 6, we give an alternative proof of Israel’s Theorem by showing that such $J_{\alpha}^{\mu}$ exist in a neighborhood of a point $p$ on a single shock surface if and only if the Einstein equations and the RH condition hold. Such
a Jacobian fails to exist if two radial shock surfaces intersect, as we prove in Section 7.

6. Metric Smoothing on Single Shock Surfaces and a Constructive Proof of Israel’s Theorem in Spherical Symmetry

We have shown in Corollary 5.2 that (5.11) is a necessary and sufficient condition on \( [J^\mu_{\alpha,\sigma}] \) for lifting the SSC metric regularity to \( C^{1,1} \) in a neighborhood of a shock curve, provided the value \( [J^\mu_{\alpha,\sigma}] \) comes from an actual Jacobian \( J^\mu_{\alpha} \). We now address the issue of how to obtain such Jacobians, that is, we study how to construct a set of functions \( J^\mu_{\alpha} \) satisfying (5.11) on the shock curve, while the integrability condition (2.4) holds in a neighborhood of the curve. For a single shock surfaces, this can be done if and only if the RH condition hold, as recorded in the following theorem.

**Theorem 6.1.** (Israel’s Theorem) Suppose \( g_{\mu\nu} \) is an SSC metric that is \( C^{0,1} \) across a radial shock surface \( \Sigma \) in the sense of Definition 4.1, such that it solves the Einstein equations (2.21) - (2.24) strongly away from \( \Sigma \). Assume \( T^{\mu\nu} \) is smooth away from \( \Sigma \) and let \( p \) be a point on \( \Sigma \). Then the following is equivalent:

(i) There exists a coordinate transformation of the \( (t,r) \)-plane, defined in a neighborhood \( N \) of \( p \) and with its Jacobian \( J^\mu_{\alpha} \) being \( C^{0,1} \) across \( \Sigma \), such that the transformed metric components \( g_{\alpha\beta} \) are \( C^{1,1} \) functions of the new coordinates.

(ii) The RH conditions (3.4) - (3.5) hold on \( \Sigma \) in a neighborhood of \( p \).

Theorem 6.1 is the spherically symmetric version of results first proven by Israel [10], see also [18]. In the remainder of this section we develop a new method to prove Theorem 6.1 which also allows us to address shock interactions in Section 7, finally leading to our main result, Theorem 1.1. The first step is to construct Jacobians acting on the \( (t,r) \)-plane that satisfy the smoothing condition (5.11) on the shock curve, the condition guaranteeing \( [g_{\alpha\beta,\gamma}] = 0 \). For a single shock curves, both the RH conditions and the Einstein equations (in the form of (4.8)) are necessary and sufficient for such functions \( J^\mu_{\alpha} \) to exist, c.f. the discussion in the end of Section 5, as presented in the following lemma.

**Lemma 6.2.** Let \( p \) be a point on a single shock curve \( \gamma \) across which the SSC metric \( g_{\mu\nu} \) is Lipschitz continuous in the sense of Definition 4.1. Then there exists a set of functions \( J^\mu_{\alpha} \) defined on \( N \) being \( C^{0,1} \) across \( \Sigma \) and satisfying the smoothing condition (5.11) on \( \gamma \cap N \) if and only if (4.8) holds on \( \gamma \cap N \). Furthermore, all \( J^\mu_{\alpha} \) that are \( C^{0,1} \) across \( \Sigma \) and satisfy (5.11) on \( \gamma \cap N \) are given by

\[
\begin{align*}
J^0_0(t,r) &= \frac{[A_r] \phi(t) + [B_t] \omega(t)}{4A \circ \gamma(t)} |x(t) - r| + \Phi(t,r) \\
J^0_1(t,r) &= \frac{[A_r] \nu(t) + [B_t] \zeta(t)}{4A \circ \gamma(t)} |x(t) - r| + N(t,r) \\
J^0_2(t,r) &= \frac{[B_t] \phi(t) + [B_r] \omega(t)}{4B \circ \gamma(t)} |x(t) - r| + \Omega(t,r)
\end{align*}
\]
\[ J_\alpha'(t,r) = \frac{[B_t]\nu(t) + [B_r]\zeta(t)}{4B \circ \gamma(t)} |x(t) - r| + Z(t,r), \] (6.1)

for arbitrary functions \( \Phi, \Omega, Z, N \in C^1(\mathcal{N}) \) and

\[
\phi = \Phi \circ \gamma, \quad \omega = \Omega \circ \gamma, \quad \nu = N \circ \gamma, \quad \zeta = Z \circ \gamma. \quad (6.2)
\]

**Proof.** Suppose there exists a set of functions \( J_\alpha^\mu \) satisfying (5.11) and being \( C^{0,1} \) across \( \Sigma \), which implies

\[
[J_\alpha^\mu]\neq -\dot{x}[J_\alpha^\mu] \quad (6.3)
\]

for all \( \mu \in \{t, r\} \) and \( \alpha \in \{0, 1\} \). Combining (6.3) for the special case \( \mu = t \) and \( \alpha = 0 \) with the right hand side in (5.11) leads to

\[
-\frac{1}{2} \left( \frac{[A_t]}{A} J_0^t + \frac{[A_r]}{A} J_0^r \right) = \frac{\dot{x}}{2} \left( \frac{[A_t]}{A} J_0^t + \frac{[B_t]}{A} J_0^r \right). \]

Using now the jump relations for the metric tensor, (4.3) - (4.4), finally gives \([A_t] = -\dot{x}[B_t]\), that is, (4.8).

For proving the opposite direction it suffices to show that all \( t \) and \( r \) derivatives of \( J_\alpha^\mu \), defined in the above ansatz (6.1), satisfy (5.11) for all \( \mu \in \{t, r\} \) and \( \alpha \in \{0, 1\} \). Since (6.2) implies the identities

\[
\phi = J_0^t \circ \gamma, \quad \nu = J_0^t \circ \gamma, \quad \omega = J_0^t \circ \gamma, \quad \zeta = J_0^t \circ \gamma, \quad (6.4)
\]

and since the \( C^1 \) regularity of \( \Phi, \Omega, Z \) and \( N \) implies

\[
[U_r] = 0 = [U_t], \quad \text{for} \quad U = \Phi, \Omega, Z \text{ or } N, \quad (6.5)
\]

(4.3), (4.4) and (4.8) immediately show that the Jacobian ansatz (6.1), satisfies (5.11). This proves the existence of functions \( J_\alpha^\mu \) satisfying (5.11).

For the supplement, applying Corollary 4.2 confirms that all functions satisfying (5.11) can be written in the canonical form (6.1).

To complete the proof of Israel’s Theorem, it remains to study under what condition the functions \( J_\alpha^\mu \) defined in (6.1) satisfy the integrability condition (2.4) in a neighborhood of the shock, ensuring that \( J_\alpha^\mu \) is integrable to coordinates. This is accomplished in the following two lemmas. The first lemma gives an equivalent form of the integrability condition (2.4), adapted to the free functions \( \Phi, \Omega, Z \) and \( N \) of the Jacobian ansatz (6.1). The main step in its proof is to write (2.4) in SSC.

**Lemma 6.3.** There exist coordinates \( x^\alpha \) such that \( J_\alpha^\mu \) defined in (6.1) satisfy the integrability condition (2.4) if and only if the functions \( \Phi, \Omega, N \) and \( Z \), in (6.1), satisfy the following system of PDE’s:

\[
(\dot{\alpha}|X| + \Phi_\ell) (\beta|X| + N) + \Phi_\alpha (\epsilon|X| + Z) - (\alpha|X| + \Phi) \left( \dot{\beta}|X| + N_t \right) = 0,
\]

\[
-N_r (\delta|X| + \Omega) + fH(X) = 0, \quad (6.6)
\]

\[
(\dot{\delta}|X| + \Omega_t) (\beta|X| + N) + \Omega_r (\epsilon|X| + Z) - (\dot{\epsilon}|X| + Z_t) (\alpha|X| + \Phi) = 0,
\]

\[
-Z_r (\delta|X| + \Omega) + hH(X) = 0 \quad (6.7)
\]
where \( X(t, r) = x(t) - r \), \( H(X) \) denotes the Heaviside step function,

\[
\alpha(t) = \frac{[A_r] \phi(t) + [B_t] \omega(t)}{4A \circ \gamma(t)};
\beta(t) = \frac{[A_r] \nu(t) + [B_t] \zeta(t)}{4A \circ \gamma(t)};
\delta(t) = \frac{[B_t] \phi(t) + [B_r] \omega(t)}{4B \circ \gamma(t)};
\epsilon(t) = \frac{[B_t] \nu(t) + [B_r] \zeta(t)}{4B \circ \gamma(t)};
\]

(6.8)

and

\[
f = (\beta \delta - \alpha \epsilon) |X| + \alpha \dot{x}N - \beta \dot{x} \Phi + \beta \Omega - \alpha Z
\]

\[
h = (\beta \delta - \alpha \epsilon) \dot{x}|X| + \delta \dot{x}N - \epsilon \dot{x} \Phi + \epsilon \Omega - \delta Z.
\]

(6.9)

**Proof.** Suppose there exist coordinates \( x^\alpha \), such that \( J_\alpha^\mu \) satisfies the integrability condition (2.4), that is

\[
J_\alpha^\mu, \beta = J_\beta^\mu, \alpha.
\]

(6.10)

From the chain rule we find that (6.10) implies

\[
J_\alpha^\mu, \nu J_\nu^\beta = J_\beta^\mu, \nu J_\nu^\alpha,
\]

(6.11)

(where \( x^\nu \) denote SSC). Substituting the Jacobian defined in (6.1) into (6.11) proves that \( \Phi, \Omega, N \) and \( Z \) satisfy (6.6)-(6.7).

We now prove the opposite direction. Observe that, in light of (6.1), the integrability condition (6.11) is in fact equivalent to (6.6)-(6.7). It remains to prove that (6.11) implies the existence of coordinates \( x^\nu \) such that (6.10) holds. For this, we show that (6.11) implies the integrability condition on the inverse Jacobian \( J_\alpha^\mu \), that is,

\[
J_\mu^\alpha = \frac{\partial x^\mu}{\partial x^\alpha}.
\]

(6.13)

By the chain rule, (6.13) gives the point-wise (linear algebraic) inverse of \( J_\alpha^\mu \). Moreover, from (6.13), the \( J_\alpha^\mu \) satisfy the integrability condition (6.10) by the commutativity of partial derivatives. We conclude that Lemma 6.3 is proven, once we have shown that (6.11) imply (6.12).

We now prove this implication. Suppose (6.11) holds, which is equivalent to

\[
J_1^1 J_{0,r}^1 - J_0^1 J_{1,r}^1 = J_0^1 J_{1,t}^1 - J_1^1 J_{0,t}^1
\]

(6.14)

\[
J_1^r J_{0,r}^r - J_0^r J_{1,r}^r = J_0^r J_{1,t}^r - J_1^r J_{0,t}^r.
\]

(6.15)
A direct computation, using (6.14) - (6.15), verifies (6.12). In more detail, the
linear algebraic inverse $J^\alpha_\mu$ of $J^\mu_\alpha$ is given by
\[
\begin{pmatrix}
J^0_t & J^0_r \\
J^1_t & J^1_r
\end{pmatrix} = \frac{1}{|J|} \begin{pmatrix}
J_1^r & -J_1^t \\
-J_0^r & J_0^t
\end{pmatrix},
\] (6.16)
where $|J|$ denotes the determinant of $J^\mu_\alpha$. From (6.16), a straightforward com-
putation gives
\[
J^0_{t,r} = \frac{J^1_t}{|J|^2} (J^0_t J^r_1 - J^0_r J^r_1) - \frac{J^1_r}{|J|^2} (J^1_t J^r_0 - J^0_r J^r_1),
\] (6.18)
where (6.16) enters the last equality. A similar computation verifies the remain-
ing equation in (6.12), $J^1_{t,r} = J^1_{r,t}$. This shows that (6.11) implies (6.12) and
completes the proof.

The proof of Israel’s Theorem is complete once we prove the existence of $C^1$
regular solutions $\Phi, \Omega, N$ and $Z$ of (6.6) and (6.7). For this, it suffices to choose
$N$ and $Z$ arbitrarily, then (6.6) - (6.7) reduce to a system of 2 linear first order
PDE’s for the unknown functions $\Phi$ and $\Omega$. Thus (6.6) - (6.7) can be solved along
characteristic lines and the only obstacle to solutions $\Phi$ and $\Omega$ with the
necessary $C^1$ regularity is the presence of the (discontinuous) Heaviside function
$H(X)$ in (6.6) - (6.7). However, the coefficients of $H(X)$, $f$ and $h$, vanish precisely when
the RH jump conditions hold on $\gamma$, as stated in the next lemma.

Lemma 6.4. Assume the SSC metric $g_{\mu \nu}$ is $C^{0,1}$ across $\gamma$, in the sense of Def-
inition 4.1, and solves the first three Einstein equations strongly away from $\gamma$. Let $f$ and $h$
be the functions defined in (6.9). Then,
\[
f \circ \gamma = 0 = h \circ \gamma
\] (6.17)
if and only if the RH conditions (3.4) - (3.5) hold on $\gamma$.

Proof. The first terms of $f \circ \gamma$ and $h \circ \gamma$ in (6.9) drop out, since $X \circ \gamma = 0$. Now,
using (6.2) gives
\[
\begin{align*}
f \circ \gamma &= \alpha \dot{x} \nu - \beta \dot{x} \phi + \beta \omega - \alpha \zeta \\
h \circ \gamma &= \delta \dot{x} \nu - \epsilon \dot{x} \phi + \epsilon \omega - \delta \zeta
\end{align*}
\] (6.18)
In (6.18), replace $\alpha$ and $\beta$ by their definition, (6.8), then a straightforward com-
putation shows that
\[
f \circ \gamma = 0
\] is equivalent to
\[
([A_t] + \dot{x}[B_t]) (\phi \zeta - \nu \omega) = 0.
\] (6.19)
Now, using
\[
(\phi \zeta - \nu \omega) = \det (J^\mu_\alpha \circ \gamma) \neq 0,
\] (6.20)
we conclude that $f \circ \gamma = 0$ if and only if \([4.8]\) holds, which is equivalent to the second RH condition \([3.5]\).

Similarly, replacing \(\delta\) and \(\epsilon\) in \([6.18]\) by their definition, \([6.8]\), a straightforward computation shows the equivalence of

\[
h \circ \gamma = 0
\]

and

\[
([B_t] + \dot{x}[B_r]) (\phi \zeta - \nu \omega) = 0. \quad (6.21)
\]

Now, using again \([6.20]\), it follows that $h \circ \gamma = 0$ if and only if \([4.4]\) holds,\(^{11}\) which is equivalent to the first RH condition \([3.4]\). This completes the proof. \(\square\)

In summary, Lemma 6.4 asserts that $\Phi$, $\Omega \in C^1(N)$ if and only if the Rankine Hugoniot conditions \([3.4]-[3.5]\) hold. We are now in the position to prove Israel’s Theorem:

**Proof.** (of Theorem 6.1) Assume there exist coordinates $x^\alpha$ such that $g_{\alpha\beta}$ is in $C^{1,1}$. We now prove that the RH condition \([3.4]-[3.5]\) hold as a consequence. Recall that the first RH condition is implied by the continuity of the metric $g_{\mu\nu}$, see the discussion following \([4.3]\). It remains to verify \([4.8]\). The $C^{1,1}$ regularity of $g_{\alpha\beta}$ implies that the Jacobian $J^\mu_\alpha$ satisfies \([5.11]\) on $\gamma$. Thus, by Lemma 6.2, the Jacobians are of the canonical form \([6.1]\) with $\Phi$, $\Omega$, $N$ and $Z$ being $C^1$ regular, hence satisfying \([6.5]\). Moreover, the functions $\Phi$, $\Omega$, $N$ and $Z$ satisfy the integrability condition. Taking the jumps of the integrability condition \([6.6]-[6.7]\) and using \([6.5]\) implies $f \circ \gamma = 0$ and $h \circ \gamma = 0$. By Lemma 6.4 we conclude that \([4.8]\) holds and therefore also the second RH condition \([3.5]\). This proves (ii) following from (i).

We now show that (ii) implies (i). Suppose the RH conditions \([3.4]-[3.5]\) and thus \([4.8]\) hold on $\gamma \cap N$. By Lemma 6.2, the functions $J^\mu_\alpha$ defined in \([6.1]\) satisfy the smoothing condition \([5.11]\) and thus lift the metric regularity from $C^{0,1}$ to $C^{1,1}$. By Lemma 6.3 to ensure the existence of coordinates $x^\alpha$ such that $J^\mu_\alpha = \frac{\partial x^\mu}{\partial x^\alpha}$, we need to prove the existence of $C^1$ regular functions $\Phi$, $\Omega$, $N$ and $Z$, that satisfy the integrability condition \([6.6]\). For this, without loss of generality, we choose $N$ and $Z$ arbitrarily and solve the resulting linear system of PDEs, \([6.6]\), for $\Phi$ and $\Omega$.

In more detail, choose $N$ and $Z$ to be smooth functions on $N$ subject to the condition

\[
\zeta \neq \dot{x}\nu \quad (6.22)
\]

on the shock curve, where $\nu = N \circ \gamma \neq 0$ and $\zeta = Z \circ \gamma \neq 0$. Due to \([6.22]\), the characteristic curves of \([6.6]\) are transversal to the shock curve,\(^{12}\) $\gamma(t) = (t, x(t))$. This allows us to prescribe initial data on the shock curve, i.e., to prescribe $\Phi \circ \gamma = \phi$ and $\Omega \circ \gamma = \omega$, then the derivative-terms in \([6.6]\), $\dot{\alpha}$ and $\dot{\delta}$, are given functions. Now, \([6.6]\) is a linear symmetric hyperbolic system of first order

\(^{11}\)Due to the metric regularity, \([4.4]\) always holds, c.f. Section 4.

\(^{12}\)This holds, since $(\nu, \zeta)$ is tangent to the characteristic curve on $\gamma$, c.f. [11] for an introduction to the method of characteristics.
PDEs for $\Phi$ and $\Omega$. Choosing smooth initial data, the method of characteristics in [11], (chapter 2.5; exchanging $(t,x)$ by $(r,t)$), shows existence of a solution $(\Phi,\Omega)$ on some open neighborhood of $\gamma$ as follows: Integrating (6.6) along its characteristic curves, the method in [11] yields a $C^0$ solution to the resulting integral equation, (even for the discontinuous coefficients of lower order terms in (6.6)-(6.7)). Once we prove the $C^1$ regularity of this solution it is clear that it also solves the original PDE (6.6).

We now prove the $C^1$ regularity of the solution $(\Phi,\Omega)$. Away from the shock curve $\gamma$ (and tangential to $\gamma$) all coefficients in (6.6) are smooth, thus we can follow the method in [11] proving smoothness of $(\Phi,\Omega)$ away from the shock curve. In particular, choosing initial data, $N$ and $Z$ such that the limit of first and second derivatives exists on each side of $\gamma$, then $\Phi$ and $\Omega$ are $C^{0,1}$ across $\gamma$. It remains to prove a $C^1$ regularity across the shock curve, that is, (6.5). For this, the vanishing of $f$ and $h$ in (6.17), the continuity of $(\Phi,\Omega)$ and the integrability condition (6.6) imply that derivatives of $(\Phi,\Omega)$ along the characteristic curves are continuous. That is,

$$\begin{align*}
[\Phi_t]\nu + [\Phi_r]\zeta &= 0 \\
[\Omega_t]\nu + [\Omega_r]\zeta &= 0.
\end{align*}$$

(6.23)

(Note, $(\nu, \zeta)$ is tangent to the characteristic curve of (6.6) at $\gamma$.) On the other hand, $(\Phi, \Omega)$ being $C^{0,1}$ across $\gamma$ implies (c.f. (4.2))

$$\begin{align*}
[\Phi_t] + [\Phi_r] x &= 0, \\
[\Omega_t] + [\Omega_r] x &= 0.
\end{align*}$$

(6.24)

Now, since (6.22) ensures that the shock curve is non-characteristic, (6.23) and (6.24) are independent condition, which then yield the desired $C^1$ regularity across $\gamma$, (6.5). We conclude that $(\Phi,\Omega)$ are $C^1$ regular on some neighborhood $\mathcal{N}$ and smooth on $\mathcal{N}\setminus\gamma$. Therefore, $J^\mu_\alpha$ is $C^{0,1}$ across the shock curve.

Moreover, we can choose initial data and the free functions, $\phi, ..., \zeta$, such that $\det(J) \neq 0$ everywhere on the shock curve and by continuity of the solution $\det(J)$ is non-vanishing in some neighborhood of $\gamma$.

In summary, assuming the RH conditions on $\gamma$, the Jacobian $J^\mu_\alpha$ is $C^{0,1}$ across the shock, satisfies the smoothing condition (5.11) on $\gamma$ and satisfies the integrability condition on some neighborhood $\mathcal{N}$. Therefore, by Corollary 5.2, $J^\mu_\alpha$ maps the SSC metric, $g_{\mu\nu}$, to a $C^{1,1}$ regular metric $g_{\alpha\beta}$. This completes the proof of Theorem 6.

In fact, in Theorem 6 one could replace the assumption of $J^\mu_\alpha$ being $C^{0,1}$ across $\Sigma$ by the assumption that the coordinate transformation is $C^{1,1}$ regular. Namely, if there exist a $C^{1,1}$ regular coordinate transformation smoothing the SSC metric to $C^{1,1}$, one can assume without loss of generality that the Jacobians are $C^{0,1}$ across $\Sigma$, since the smoothing condition (5.1) is a point-wise condition on the shock surface. Note, (5.1) is also valid for Lipschitz continuous Jacobians not smooth away from the shock surface, provided one defines the jump $[\cdot]$ in terms of some curve transversal to $\Sigma$, c.f. Section 5. Moreover, proving that (i) implies (ii) is not affected, since the Jacobian constructed is $C^{0,1}$ across $\gamma$ anyways.
7. Shock Wave Interactions are Regularity Singularities in GR; Transformations in the \((t,r)\)-Plane

The main step in the proof of Theorem 1.1 is to show the result for the smaller atlas of coordinate transformations in the \((t,r)\)-plane first. That is, to prove there do not exist \(C^{1,1}\) coordinate transformations of the \((t,r)\)-plane (i.e., keeping the SSC angular variables fixed) lifting the regularity of the SSC metric \(g_{\mu\nu}\) from \(C^{0,1}\) to \(C^{1,1}\) in a neighborhood of a point \(p\) of regular shock wave interaction in SSC. We then prove in Section 8 that no such transformation can exist within the full \(C^{1,1}\) atlas that transforms all four variables of spacetime, including the SSC angular variables, leading to Theorem 1.1. We formulate the main step for lower shock wave interactions in \(\mathbb{R}^2_-\) in the following theorem, which is the topic of this section. A corresponding result applies to upper shock wave interactions in \(\mathbb{R}^2_+\), as well as to shock wave interactions in \(\mathbb{R}^2_1\), as well as to shock wave interactions in a neighborhood of \(p\).

**Theorem 7.1.** Suppose that \(p\) is a point of regular shock wave interaction in SSC for the SSC metric \(g_{\mu\nu}\), in the sense of Definition 3.1. Then there does not exist a \(C^{1,1}\) coordinate transformation \(x^\alpha \circ (x^\mu)^{-1}\) of the \((t,r)\)-plane, defined on \(N \cap \mathbb{R}^2_-\) for a neighborhood \(N\) of \(p\) in \(\mathbb{R}^2\), such that the metric components \(g_{\alpha\beta}\) are \(C^1\) functions of the coordinates \(x^\alpha\) in \(N \cap \mathbb{R}^2_-\) and such that the metric has a non-vanishing determinant at \(p\), (that is, such that \(\lim_{q \to p} \det (g_{\alpha\beta}(q)) \neq 0\)).

Theorem 1.1 implies the non-existence of coordinates on a neighborhood \(N\) of \(p\) in \(\mathbb{R}^2\). In the remainder of this section we present the proof of Theorem 7.1 which mirrors the constructive proof of Theorem 6.1 in that it uses the extension of our Jacobian ansatz (6.1) to the case of two interacting shock waves. But now, to prove non-existence, we must show the ansatz is general enough to include all \(C^{0,1}\) Jacobians that could possibly lift the regularity of the metric. We conclude the proof by showing that this canonical form is inconsistent with the assumption that \(\det (g_{\alpha\beta}) \neq 0\) at \(p\), by using the continuity of the Jacobians up to \(p\).

To implement these ideas, the main step is to show that the canonical form (4.14) of Corollary 4.3 can be applied to the Jacobians \(J^\mu_i\) in the presence of a shock wave interaction. The result is recorded in the following lemma:

**Lemma 7.2.** Let \(p\) be a point of regular shock wave interaction in SSC in the sense of Definition 3.1, corresponding to the SSC metric \(g_{\mu\nu}\) defined on \(N \cap \mathbb{R}^2_-\). Then the following is equivalent:

(i) There exists a set of functions \(J^\mu_i\), defined on \(N \cap \mathbb{R}^2_-\), which is \(C^{0,1}\) across \(\gamma_i \cap N\) and satisfies the smoothing condition (5.11) on \(\gamma_i \cap N\), for \(i = 1, 2\).

(ii) The RH condition (4.8) holds on each shock curve \(\gamma_i \cap N\), for \(i = 1, 2\).

Moreover, if (i) or (ii) is valid, then the functions \(J^\mu_i\) assume the canonical form

\[
J^0_i(t, r) = \sum_i \alpha_i(t) |x_i(t) - r| + \Phi(t, r),
\]

\[
J^1_i(t, r) = \sum_i \beta_i(t) |x_i(t) - r| + N(t, r),
\]
\[ J_0^\alpha(t, r) = \sum_i \delta_i(t) |x_i(t) - r| + \Omega(t, r), \]
\[ J_1^\alpha(t, r) = \sum_i \epsilon_i(t) |x_i(t) - r| + Z(t, r), \]
(7.1)

where \( \Phi, \Omega, Z, N \in C^1(\mathcal{N} \cap \mathbb{R}^2) \) and
\[
\alpha_i(t) = \frac{[A_r]_i \phi_i(t) + [B_r]_i \omega_i(t)}{4A \circ \gamma_i(t)},
\]
\[
\beta_i(t) = \frac{[A_r]_i \nu_i(t) + [B_r]_i \zeta_i(t)}{4A \circ \gamma_i(t)},
\]
\[
\delta_i(t) = \frac{[B_r]_i \phi_i(t) + [B_r]_i \omega_i(t)}{4B \circ \gamma_i(t)},
\]
\[
\epsilon_i(t) = \frac{[B_r]_i \nu_i(t) + [B_r]_i \zeta_i(t)}{4B \circ \gamma_i(t)},
\]
(7.2)

with
\[
\phi_i = \Phi \circ \gamma_i, \quad \omega_i = \Omega \circ \gamma_i, \quad \zeta_i = Z \circ \gamma_i, \quad \nu_i = N \circ \gamma_i.
\]
(7.3)

**Proof.** The proof is analogous to the proof of the single shock version in Lemma 6.2. In more detail, suppose that there exist functions \( J_\alpha^\mu \) that are \( C^{0,1} \) across each \( \gamma_i \) and meet the smoothing condition (5.11) on each \( \gamma_i \), for \( i = 1, 2 \). \( J_\alpha^\mu \) being \( C^{0,1} \) across \( \gamma_i \) implies (c.f. (4.2) and (6.3))
\[
[J_{\alpha,r}]_i = -\dot{\gamma}_i [J_{\alpha,r}]_i \quad \text{for} \quad i = 1, 2.
\]
(7.4)

Substituting into (7.4) the expressions for \( [J_{\alpha,\nu}]_i \) from the smoothing condition (5.11) (with respect to \( \gamma_i \)) and using that \( g_{\alpha \nu} \) is \( C^{0,1} \) across \( \gamma_i \), in the sense of (4.3)-(4.4), finally yields (4.8) on \( \gamma_i \). Now, (4.3) together with (4.8) imply the RH conditions on \( \gamma_i \) for \( i = 1, 2 \).

We now prove the implication from (ii) to (i). Suppose that the RH conditions hold on each \( \gamma_i, i = 1, 2 \), then (4.3), (4.4) and (4.8) hold on each shock curve, due to \( g_{\alpha \nu} \) being \( C^{0,1} \) across each \( \gamma_i \). Now, a straightforward computation using (4.3), (4.4) and (4.8) to eliminate \( \dot{x}_i \) shows that the Jacobian ansatz defined in (7.1) satisfies the smoothing condition (5.11) on each shock curve \( \gamma_i \). This proves the existence of functions satisfying (5.11).

From Corollary 4.3 we conclude that if functions \( J_\alpha^\mu \) as in (i) exist, then there exist functions \( \Phi, \Omega, Z, N \in C^1(\mathcal{N}) \), such that the \( J_\alpha^\mu \) can be written as in (7.1). This completes the proof. \( \square \)

We now have a canonical form for all functions \( J_\alpha^\mu \) that meet the necessary and sufficient condition for \([g_{\alpha \beta \gamma}] = 0, \ \text{(5.11)}\). However, for \( J_\alpha^\mu \) to be proper Jacobians, integrable to a coordinate system, we must use the free functions \( \Phi, \Omega, Z \) and \( N \) to meet the integrability condition (2.4). In the spirit of Lemma 6.3, the following lemma gives a form of the integrability condition tailored to our Jacobian ansatz (7.1). To prove Theorem 7.1, the precise form of the integrability condition is not relevant, except the structure of the coefficients of
the (discontinuous) $H(X)$-terms, $f$ and $h$. The main difference to (6.6) is the appearance of additional mixed terms in $f$ and $h$, recorded in (7.6).

**Lemma 7.3.** For the ansatz (7.1) the integrability condition (2.4) is of the form

$$\tilde{G} (\varphi_t, \varphi_r, \varphi, \dot{\varphi}_i; t, r) + \sum_{i=1,2} F_i H(X_i) = 0, \tag{7.5}$$

where $X_i(t, r) = x_i(t) - r$ and $H(.)$ denotes the Heaviside function, i.e., $H(X) = -1$ for $X < 0$ and $H(X) = 1$ for $X > 0$. $\varphi = T (\Phi, \Omega, Z, N)$ and $\dot{\varphi}_i = \frac{d}{dt} (\varphi \circ \gamma_i) (t)$. $\tilde{G}$ is a (2-D) vector valued function continuously depending on its arguments and $F_i = T (f_i, h_i)$, where $f_i$ and $h_i$ are real valued functions continuously depending on $\varphi$, $t$ and $r$, with their restriction to the shock curves given by

$$f_i \circ \gamma_i = (\alpha_i \dot{x}_i \nu_i - \beta_i \dot{x}_i \phi_i + \beta_i \omega_i - \alpha_i \zeta_i) + (\alpha_i \dot{x}_i \beta_i - \beta_i \dot{x}_i \alpha_i + \beta_i \delta_i - \alpha_i \epsilon_i) |x_i(.) - x_l(.)|,$$

$$h_i \circ \gamma_i = (\delta_i \dot{x}_i \nu_i - \epsilon_i \dot{x}_i \phi_i + \epsilon_i \omega_i - \delta_i \zeta_i) + (\delta_i \dot{x}_i \beta_i - \epsilon_i \dot{x}_i \alpha_i + \epsilon_i \delta_i - \delta_i \epsilon_i) |x_i(.) - x_l(.)|, \tag{7.6}$$

for $i = 1, 2$ and $l \neq i$.

**Proof.** The proof is similar to the one of Lemma 6.3 in that the equivalence of the integrability condition (2.4) and the integrability condition in SSC (6.11), that is,

$$J^\mu_{\alpha,x} J^\nu_{\beta} = J^\mu_{\beta,\nu} J^\nu_{\alpha},$$

follows by exactly the same arguments. (In fact, this equivalence holds independently of the Jacobian ansatz (6.1).) Substituting now the Jacobian ansatz for the intersecting shock curves (7.1) into (6.11) and separating continuous and discontinuous terms (i.e. terms containing $H(X)$) leads to (7.5). □

For proving Theorem 7.1, the point of interest in (7.5) lies in the additional mixed terms appearing in the coefficients $f_i$ and $h_i$ of the $H(X)$-terms. In contrast to the single shock case (6.6), those mixed terms do not vanish on the shock curves $\gamma_i$ by the RH condition alone. But, as a consequence of the $C^1$ regularity of $\Phi, \Omega, Z$ and $N$, $f_i$ and $h_i$ must vanish on the shock curves, which imposes an additional constraint. Taking the limit of this constraint to the point $p$ of shock wave interaction finally yields the condition that $\det (J^\mu_\alpha)$ vanishes at $p$, which immediately implies $\det (g_{\alpha \beta}) \bigg|_p = 0$.

The main step in the proof of Theorem 7.1 is recorded in Lemma 7.4. To appreciate its significance, let’s first compute the determinant of the Jacobian at the point of shock interaction. From (7.3) together with (7.1), we find

$$\det (J^\mu_\alpha \circ \gamma_i(t)) = (J^1_0 J^2_1 - J^1_1 J^2_0) \bigg|_{\gamma_i(t)} = \phi_i(t) \zeta_i(t) - \nu_i(t) \omega_i(t) \tag{7.7}$$

and taking the limit to the point of shock wave interaction gives

$$\lim_{t \to 0} \det (J^\mu_\alpha \circ \gamma_i(t)) = \phi_0 \zeta_0 - \nu_0 \omega_0, \tag{7.8}$$
where we use the notation
\[ \phi_0 = \lim_{t \to 0^+} \phi_i(t), \quad \omega_0 = \lim_{t \to 0^+} \omega_i(t), \quad \zeta_0 = \lim_{t \to 0^+} \zeta_i(t) \quad \text{and} \quad \nu_0 = \lim_{t \to 0^+} \nu_i(t). \] (7.9)

Note, that the limit in (7.9) is independent of the shock curve \( \gamma_i \), \( i = 1, 2 \), since \( J^\mu_{\alpha} \) is assumed to be continuous and since the limits
\[ \lim_{t \to 0^+} [g_{\mu\nu,\sigma}]_i(t) = [g_{\mu\nu,\sigma}]_i(0) \]
are assumed to exist, c.f. Definition 3.1. From (7.8), the importance of the next lemma for the proof of Theorem 7.1 is obvious.

Lemma 7.4. Let \( p \in \mathcal{N} \) be a point of regular shock wave interaction in SSC in the sense of Definition 3.1. Assume the functions \( J^\mu_{\alpha} \) defined in (7.1) satisfy the integrability condition
\[ J^\mu_{\alpha,\beta} = J^\mu_{\beta,\alpha} \] (7.10)
on \( \mathcal{N} \cap \mathbb{R}^2 \), then the following equation holds at \( t = 0 \),
\[ \frac{1}{4B} \left( \frac{\dot{x}_1 \dot{x}_2}{A} + \frac{1}{B} \right) [B_r]_1 [B_r]_2 (\dot{x}_1 - \dot{x}_2) (\phi_0 \zeta_0 - \nu_0 \omega_0) = 0. \] (7.11)
(The metric coefficients \( A \) and \( B \) in (7.11) are evaluated at \( p = (0, r_0) \) and \( \phi_0, ..., \omega_0 \) are defined in (7.9).)

Proof. Suppose the functions \( J^\mu_{\alpha} \) defined in our Jacobian ansatz (7.1), solve the integrability condition (7.10). Then, by Lemma 7.3, the functions \( \Phi, \Omega, Z \) and \( N \) satisfy the integrability condition in SSC (7.5). Taking the jump of (7.5) across \( \gamma_i \) gives
\[ \left[ \vec{G}(\varphi_t, \varphi_r, \varphi, \phi_i, t, r) \right]_i + \sum_{j=1,2} [F_j H(X_j)]_i = 0. \] (7.12)
Using \( \Phi, \Omega, Z, N \in C^1(\mathcal{N} \cap \overline{\mathbb{R}^2}) \) and using that \( \vec{G} \) and \( F_j \) depend continuously on their arguments, (7.12) reduces to
\[ \sum_{j=1,2} F_j [H(X_j)]_i = 0. \]
Using further that \( [H(X_j)]_i = 2\delta_{ij} \), we get
\[ f \circ \gamma_i(t) = 0 = h \circ \gamma_i(t), \]
for all \( t \in (-\epsilon, 0) \) and for \( i = 1, 2 \).

In the following we compute the limit of the equation \( h \circ \gamma_i(t) = 0 \) as \( t \) approaches 0, which directly leads to (7.11) and completes the proof. Without loss of generality we choose \( i = 1 \) (and thus \( l = 2 \) in (7.6)). Using (4.3) with respect to \( \gamma_1 \), which holds since the metric is \( C^{0,1} \) across \( \gamma_1 \), the first term in
\footnote{Note, one can prove that \( h \circ \gamma_i(t) = 0 \) implies \( f \circ \gamma_i(t) = 0 \). The inverse implication holds, provided \( \dot{x}_i(t) \neq 0 \). However, computing the limit of \( f \circ \gamma_i(t) = 0 \) as \( t \) approaches 0 leads to (7.11) multiplied with \( \dot{x}_i(0) \), which does not suffice to prove Theorem 7.1 unless we assume \( \dot{x}_i(0) \neq 0 \).}
vanishes on $\gamma_1$ by the same arguments as in the proof of Lemma 6.4. We obtain
\[ h_1 \circ \gamma_1 = (\delta_1 \dot{x}_1 \beta_2 - \epsilon_1 \dot{x}_1 \alpha_2 + \epsilon_1 \delta_2 - \delta_1 \epsilon_2) |x_1(\cdot) - x_2(\cdot)|. \tag{7.13} \]
For $t \neq 0$, the initial assumption $\dot{x}_1(0) \neq \dot{x}_2(0)$ implies $|x_1(t) - x_2(t)| \neq 0$ for all $t \neq 0$ sufficiently close to 0. Thus $h \circ \gamma_1(t) = 0$ is equivalent to
\[ \delta_1 \dot{x}_1 \beta_2 - \epsilon_1 \dot{x}_1 \alpha_2 + \epsilon_1 \delta_2 - \delta_1 \epsilon_2 = 0, \tag{7.14} \]
sufficiently close to $t = 0$. By continuity, (7.14) holds at $t = 0$ as well.

We next compute the limit of (7.14) as $t$ approaches 0 from below. From the definition of $\alpha_i$ and $\delta_i$ in (7.2), together with the RH condition in the form (4.3) and (4.8), we get the identity
\[ \alpha_i = -\dot{x}_i \frac{B}{A} \delta_i, \tag{7.15} \]
for $i = 1, 2$, where $A$ and $B$ are evaluated at $p = (0, r_0)$. Now, from the definition of $\epsilon_i$ and $\beta_i$ in (7.2), as well as (7.15) and the continuity of the Jacobian at $p$, we conclude that (7.14) at $t = 0$ is equivalent to
\[ \begin{align*}
&\left( \frac{\dot{x}_1 [A_i]_2}{A} - \frac{[B_i]_2}{B} \right) \delta_1 - \left( \frac{\dot{x}_2 [A_i]_2}{A} - \frac{[B_i]_2}{B} \right) \delta_2 \right) \nu_0 \\
&+ \left( \frac{\dot{x}_1 [A_i]_2}{A} - \frac{[B_i]_2}{B} \right) \delta_1 - \left( \frac{\dot{x}_2 [A_i]_2}{A} - \frac{[B_i]_2}{B} \right) \delta_2 \right) \zeta_0 = 0. \tag{7.16}
\end{align*} \]
Using (4.3) and (4.8) to eliminate $[B_i]_i$ and $[A_r]_i$ we find that the coefficients to $\delta_1$ and $\delta_2$ in (7.16) are related as follows:
\[ \begin{align*}
&\dot{x}_i \frac{[B_i]_i}{A} - \frac{[B_i]_i}{B} = -\left( \frac{\dot{x}_1 \dot{x}_2}{A} + \frac{1}{B} \right) [B_i]_i, \\
&\dot{x}_i \frac{[A_r]_i}{A} - \frac{[B_i]_i}{B} = \dot{x}_i \left( \frac{\dot{x}_1 \dot{x}_2}{A} + \frac{1}{B} \right) [B_i]_i, \tag{7.17}
\end{align*} \]
for $i = 1, 2$ and $l \neq i$. Substituting (7.17) back into (7.16) gives
\[ \left( \frac{\dot{x}_1 \dot{x}_2}{A} + \frac{1}{B} \right) \left( (\dot{x}_2 [B_i]_2 \delta_1 - \dot{x}_1 [B_i]_1 \delta_2) \nu_0 + (-[B_i]_2 \delta_1 + [B_i]_1 \delta_2) \zeta_0 \right) = 0. \tag{7.18} \]
The definition of $\delta_i$ in (7.2) gives, at $t = 0$,
\[ \delta_i(0) = \frac{[B_i]_i \phi_0 + [B_i]_i \omega_0}{4B}, \]
for $i = 1, 2$. Substituting the above equation into (7.18) proves that (7.14) is equivalent to (7.11). This completes the proof. \(\square\)

**Proof. (of Theorem 7.1)** Assume there exists a $C^{1,1}$-coordinate transformation in the $(t, r)$-plane mapping SSC to coordinates $x^\alpha$ such that $g_{\alpha \beta}$ is in $C^1(\mathcal{N} \cap \mathbb{R}^2)$ for some neighborhood $\mathcal{N}$ of $p$. Thus the Jacobian of the coordinate transformation satisfies the smoothing condition (5.4). Since (5.4) is a point-wise condition on the shock cures, also valid for Lipschitz continuous Jacobians not smooth away
from the shocks, we assume without loss of generality that the Jacobians are $C^{0,1}$ across each $\gamma_i$, $(i = 1, 2)$.

Now, by Lemma 5.1, the Jacobian $J^\mu_\alpha$ of the coordinate transformation satisfies the smoothing condition (5.11) on each of the shock curves $\gamma_i$, $(i = 1, 2)$. Consequently, applying Lemma 7.2, the Jacobians $J^\mu_\alpha$ can be written in terms of the canonical form (7.1) with $C^1$ regular functions $\Phi, \Omega, N$ and $Z$.

By assumption, the $J^\mu_\alpha$ are integrable to coordinates and thus satisfy the integrability condition (2.4), (c.f. Appendix A). This together with $J^\mu_\alpha$ being of the canonical form (7.1), allows us to apply Lemma 7.4, which yields that (7.11) holds at the point $p$ of shock wave interaction, that is,

$$\frac{1}{4B} \left( \frac{\dot{x}_1 \dot{x}_2}{A} + \frac{1}{B} \right) [B_r]_1 [B_r]_2 (\dot{x}_1 - \dot{x}_2) (\phi_0 \zeta_0 - \nu_0 \omega_0) = 0. \quad (7.19)$$

To finish the proof of Theorem 7.1, observe that the first three terms in (7.19) are nonzero by our assumption that shock curves are non-null, and have distinct speeds at $t = 0$. In more detail, by our assumptions in definition 3.1

$$(\dot{x}_1 - \dot{x}_2) \neq 0,$$

while

$$\frac{\dot{x}_i \dot{x}_l}{A} + \frac{1}{B} \neq 0$$

for time-like shock curves, since such curves satisfy $B \dot{x}_j^2 < A$. Finally,

$$[B_r]_i \neq 0,$$

since otherwise $[T^{\mu\nu}] = 0$ for all $\mu, \nu = 0, 1$, due to the Einstein equations (4.5)-(4.7), in contradiction to our assumptions in Definition 3.1. We conclude

$$\phi_0 \zeta_0 - \nu_0 \omega_0 = 0 \quad (7.20)$$

and using the explicit expression for the Jacobian determinant at the point $p$ of shock interaction, (7.7)-(7.8), we get

$$\det J^\mu_\alpha (p) = (\phi_0 \zeta_0 - \nu_0 \omega_0) = 0. \quad (7.21)$$

Now, the covariant transformation law (2.2), that is, $g_{\alpha\beta} = J^\mu_\alpha J^\nu_\beta g_{\mu\nu}$, immediately implies

$$\det g_{\alpha\beta} (p) = (\det J^\mu_\alpha (p))^2 \det g_{\mu\nu} (p) = 0.$$

This completes the proof of Theorem 7.1. \Box

We remark that at first there appears to be more than enough freedom to choose the free functions $\Phi, \Omega, Z, N$ of the canonical form to arrange for the discontinuous term in the integrability condition to vanish. This together with the fact that the derivatives of $J^\mu_\alpha$ are uniquely solvable in condition (5.11), lead us to believe until the very end that one could construct coordinates in which $g_{\alpha\beta}$ was $C^{1,1}$. But at the very last step, taking the limit of the integrability constraints to the limit of shock wave interaction $p$, we find that the crucial $C^1$ regularity of $\Phi, \Omega, Z, N$, expressing that $[g_{\alpha\beta,\gamma}]$ vanishes at shocks, has the effect of freezing out all the freedom in $\Phi, \Omega, Z, N$, thereby forcing the determinant of
the Jacobian to vanish at \( p \). The answer was not apparent until the very last step, and thus we find the result remarkable, and most surprising! \[15\]

8. Shock Wave Interactions are Regularity Singularities in GR; the Full Atlas

In Theorem \[7.1\] we proved that \( C^{1,1} \) coordinate transformations of the \((t, r)\)-plane, capable of mapping a \( C^{0,1} \) regular SSC metric \( g_{\mu \nu} \) over to a metric \( g_{\alpha \beta} \) in \( C^{1,1} \), do not exist. In this section, we extend the result to the full atlas of coordinate transformations, that is, transformations depending on all four spacetime coordinates, thereby proving Theorem \[1.1\].

Recall:

**Theorem 1.1** Suppose that \( p \) is a point of regular shock wave interaction in SSC, in the sense of Definition \[3.1\] for the SSC metric \( g_{\mu \nu} \). Then there does not exist a \( C^{1,1} \) regular coordinate transformation \( x^\alpha \circ (x^\mu)^{-1} \), defined on a neighborhood \( N \) of \( p \), such that the metric components \( g_{\alpha \beta} \) are \( C^1 \) functions of the coordinates \( x^\alpha \) and such that the metric has a nonzero determinant at \( p \).

The method of proof is as follows, we assume for contradiction the existence of coordinates in which the metric is \( C^1 \) regular, in general the metric does have the full ten components and is not of the box diagonal form \[2.19\]. However, in spherically symmetric spacetimes one can change to coordinates, such that the resulting metric is of box diagonal form \[2.19\], is \( C^1 \) regular and is related to \( g_{\mu \nu} \) by a transformation in the \((t, r)\)-plane. This contradicts Theorem \[7.1\].

For constructing these coordinates we partially follow \[25\], (chapter 13). Showing that this coordinate transformation preserves the \( C^1 \) metric regularity is our own contribution.

Before we begin with the proof of Theorem \[1.1\] we need to study the regularity of a Killing vector-field of a given \( C^1 \) metric:

**Lemma 8.1.** Let \( X^j \) be a Killing vector-field of a metric tensor \( g_{ij} \). If in some coordinates \( x^j \) the metric is in \( C^1 \) and the Killing vector-field \( X^j \) is \( C^{0,1} \) across a smooth (timelike) hypersurface \( \Sigma \), then, in those coordinates, \( X^j \) is also \( C^1 \) regular.

**Proof.** Lipschitz continuity across \( \Sigma \) (c.f. Definition \[4.1\]) implies \( X_{i,j} \) to be possibly discontinuous across \( \Sigma \) but smooth elsewhere. Denoting with \([\cdot]\) the jump across \( \Sigma \), continuity of \( X_i \) implies (c.f. \[4.2\])

\[
[X_{i,j}]v^j = 0 \tag{8.1}
\]

for any smooth vector field \( v^j \) tangent to \( \Sigma \).

On the other hand, \( X^j \) being a Killing vector-field implies that its dual vector, \( X_j \), satisfies Killing’s equations:

\[
X_{i,j} + X_{j,i} = 2\Gamma^k_{ij}X_k, \tag{8.2}
\]

where \( \Gamma^k_{ij} \) denote the Christoffel symbols of \( g_{ij} \). Taking the jump across \( \Sigma \) of Killing’s equations \[8.2\] and using the continuity of the Christoffel symbols \( \Gamma^k_{ij} \),
leads to

\[ [X_{i,j}] + [X_{j,i}] = 0. \quad (8.3) \]

Define \( A_{i,j} = \delta^{il}X_{l,j} \), where \( \delta^{il} \) denotes Kronecker’s symbol. By (8.3), \( A \) is (point-wise) an antisymmetric matrix. Thus, \( A \) is diagonalizable and has a vanishing trace, \( \text{tr}(A) = 0 \). Now, by (8.1), \( A \) has a three dimensional nullspace, so that \( \text{tr}(A) = 0 \) implies all eigenvalues being zero and therefore

\[ [X_{i,j}] = 0 \quad \text{for all } i, j \in \{0, \ldots, 3\}. \quad (8.4) \]

This proves that \( X^i = g^{ij}X_j \) is in \( C^1 \), completing the proof of Lemma (8.1). \( \square \)

**Proof. (of Theorem 1.1)**

Assume for contradiction there exist coordinates \( \tilde{x}^j \), such that the transformed metric \( \tilde{g}_{ij} = \frac{\partial x^\mu}{\partial \tilde{x}^i} \frac{\partial x^\nu}{\partial \tilde{x}^j} g_{\mu\nu} \) is \( C^1 \) regular and has a non-vanishing determinant at \( p \). In general \( \tilde{g}_{ij} \) is not of the box diagonal form (2.19). We now construct a coordinate transformation taking \( \tilde{g}_{ij} \) over to a metric of box diagonal form, maintaining the \( C^1 \) metric regularity, in contradiction to Theorem 7.1.

For this, observe that the Killing vectors \( \frac{\partial}{\partial x^i} \) of the SSC metric are smooth \(^{14} \) and are \( C^{0,1} \) regular in any other coordinate system, which can be reached within the atlas of \( C^{1,1} \) coordinate transformations. Now, due to the smoothing conditions \(^{5.1} \), a Jacobian lifting the metric regularity to \( C^1 \) is in \( C^{0,1} \setminus C^1 \) on the shock surfaces, while its differentiability elsewhere is negligible as long that it doesn’t lower the metric regularity. Thus, we assume without loss of generality that the Jacobians are smooth off the shock surfaces and consequently the Killing vectors are \( C^{0,1} \) across each of the shock surfaces, excluding their intersection.

Now, since the metric \( \tilde{g}_{ij} \) is in \( C^1 \), Lemma 8.1 immediately shows that those Killing vectors are \( C^1 \) regular on a deleted neighborhood of \( p \), by which we refer to a neighborhood of \( p \) not containing the intersection of the shock surfaces. Given two point-wise linearly independent \( C^1 \) regular Killing vectors with a vanishing Lie-bracket one can choose two \( C^1 \) vectors, such that those four vectors are integrable to coordinates \(^{16} \) where all Killing vectors satisfy (possibly after reordering)

\[ X^j = 0 \quad \text{for } j = 0, 1, \quad (8.5) \]

while the \( C^1 \) metric regularity is maintained on a deleted neighborhood. We denote these coordinates again with \( \tilde{x}^j \).

We now construct new coordinates, \( x^j \), such that the resulting metric satisfies

\[ g_{al} = 0, \quad (8.6) \]

\(^{14}\)For ease of notation we subsequently refer to vector fields as vectors.

\(^{15}\)In more detail, there a three smooth generating Killing vectors in SSC and for our purposes it suffices to restrict consideration to smooth linear combination of those.

\(^{16}\)A set of vectors is integrable to a coordinate system if and only if their Lie-brackets mutually vanish. Now, the condition of vanishing Lie-brackets ties together the regularity of the vectors, so that one can always stay within the same class of smoothness.
for $a \in \{0, 1\}$ and $l \in \{2, 3\}$. For ease of notation, we subsequently split up indices as $a, b \in \{0, 1\}$ and $k, l, m, n \in \{2, 3\}$. Now, introduce the coordinates $x^j$ implicitly as
\[
\tilde{x}^a = x^a \quad \text{for} \quad a \in \{0, 1\},
\tilde{x}^l = \tilde{x}^l(x) \quad \text{for} \quad l \in \{2, 3\},
\] (8.7)
such that $\tilde{x}^l(x)$ satisfies, (with summation over $k = 2, 3$ only),
\[
\frac{\partial \tilde{x}^k}{\partial x^a} \tilde{g}_{mk}(\tilde{x}) + \tilde{g}_{ma}(\tilde{x}) = 0,
\] (8.8)
with initial data $x^l_0 = \tilde{x}^l(x_0)$ for some $x^l_0$ in our coordinate patch. From the covariant transformation law, (2.2), we find the metric in the new coordinates, $g_{ij}$, satisfying
\[
g_{lk} = \frac{\partial \tilde{x}^l}{\partial x^a} \left( \frac{\partial \tilde{x}^m}{\partial x^a} \tilde{g}_{lm} + \tilde{g}_{la} \right)
\]
and using (8.8) we immediately find (8.6) to hold true. Moreover, (8.5) is maintained due to the second equation in (8.7).

To see that the $C^1$ metric regularity is preserved, write (8.8) in its equivalent form
\[
\frac{\partial \tilde{x}^l}{\partial x^a} = -\hat{g}^{lk} \hat{g}_{ka},
\] (8.9)
where $\hat{g}^{lk}$ denotes the $2 - D$ reciprocal of $\tilde{g}_{lk}$, i.e., $\hat{g}^{lk} \tilde{g}_{lm} = \delta^k_m$ restricting summation to $l \in \{2, 3\}$. Since the right hand side in (8.9) is in $C^1$, (2.2) implies the $C^1$ metric regularity being maintained.

We now prove existence of solutions to (8.8), for which it suffices to show that the right hand side of (8.9) allows for commuting derivatives, that is,
\[
\frac{\partial}{\partial x^b} \left( \frac{\partial F^k_a}{\partial x^a} \right) = \frac{\partial}{\partial x^a} \left( \frac{\partial F^k_b}{\partial x^b} \right),
\] (8.10)
where we define $F^k_a = \hat{g}^{lk} \hat{g}_{la}$. From (8.7), the chain rule and (8.9), we find (8.10) to be equivalent to
\[
F^k_a \partial_{x^b} F_b^l - F^l_a \partial_{x^b} F_b^k = \partial_{x^b} F^k_{b,a} - F^l_{b,a}.
\] (8.11)
We now use Killing’s equation to show the validity of (8.11). Using (2.8), we write Killing’s equations (8.2) in its equivalent form (summation over $\sigma \in \{0,\ldots, 3\}$)
\[
X^\sigma \hat{\tilde{g}}_{\sigma j} + X^\sigma \hat{\tilde{g}}_{\sigma i} + X^\sigma \hat{\tilde{g}}_{ij, \sigma} = 0.
\] (8.12)
For indices $i = l$ and $j = k$, contracting (8.12) with $\hat{g}^{ln} \hat{g}^{km}$ and using $\hat{g}^{mn} = -\hat{g}^{ln} \hat{g}^{km} \hat{g}_{lk, \sigma}$ gives
\[
X^m \hat{g}^{ln} = X^k \hat{g}^{mn} - X^n \hat{g}^{lm}.
\] (8.13)
For indices $i = k$ and $j = a$, contracting (8.12) with $\hat{g}^{kl}$ gives
\[
X^l \hat{g}_{k, a} = -X^m \hat{g}^{kl} \hat{g}_{ma} - X^k \hat{g}^{ml} \hat{g}_{ka, m}.
\] (8.14)
Substituting (8.13) into (8.14) and recalling $F^k_a = \hat{g}^{kl} \hat{g}_{la}$ leads to
\[
X^l \hat{g}_{k, a} = X^l \hat{g}_{k, a}.
\] (8.15)
Taking $\partial_b$ of (8.15), substituting again (8.14) and using $X^{\ell}_{\alpha\beta} - X^{\ell}_{\beta\alpha} = 0$ we obtain

$$0 = X^{\ell}_{k} \left( F_{b,m}^{\ell} F_a^{m} - F_{b,a}^{k} - F_{a,m}^{k} F_{b}^{m} + F_{a,b}^{k} \right) + X^{k} \left( F_{b,km}^{l} F_{a}^{m} - F_{a,km}^{l} F_{b}^{m} - F_{a,kb}^{l} + F_{b,ka}^{l} + F_{a,b}^{m} F_{b,k}^{l} - F_{b,a}^{m} F_{b,b}^{l} \right).$$

(8.16)

Observe that at each point $q$ and for any antisymmetric 2-form $A_{ik}$ one can find a Killing vector with $X^{k}(q) = 0$ and $g_{il} X^{l}_{k}(q) = A_{ik}$. Therefore, since (8.16) holds for all Killing vectors, (8.16) implies (8.11). The existence of a solution to (8.9) now follows with standard ODE-methods. (Namely, for fixed $l$ and $x^{b}$ (8.9) is an ODE in $x^{a}$, defining $\tilde{x}(x) = \tilde{x}(x^{a}, x^{b})$ to be its solution for all $x^{b}$ with $b \neq a$ and for appropriate initial data, (8.10) immediately shows $\tilde{x}(x)$ to satisfy the remaining equation in (8.9) (for fixed $l$).)

In this and the next two paragraphs we show that $g_{ij}$, in the coordinates $x^{i}$ constructed above, is of box-diagonal form up to a change of $(x^{2}, x^{3})$-coordinates. Recall that indices are split as $a, b \in \{0, 1\}$ and $k, l, m, n \in \{2, 3\}$. Using (8.5) as well as (8.6), we find the $(a, b)$-components of Killing's equations (8.1) to imply

$$X^{k} g_{ab,k} = 0.$$  

(8.17)

Since this holds for all Killing vectors we conclude

$$g_{ab,k} = 0,$$  

(8.18)

the $(a, b)$-components of the metric are independent of $x^{2}$ and $x^{3}$.

Denote with $\bar{g}_{ij}$ the induced metric on one of the spaces of symmetry, which we consider fixed and henceforth refer to as the space of symmetry. Then, on the spaces of symmetry the metric agrees with the induced metric up to a factor, that is,

$$g_{lk}(x^{0}, x^{1}, x^{2}, x^{3}) = C(x^{0}, x^{1}) \bar{g}_{lk}(x^{2}, x^{3}),$$  

(8.19)

for some real valued positive function $C$ and $k, l \in \{2, 3\}$, as we show in the remainder of this paragraph. Let $q$ be an arbitrary point lying in our coordinate patch. Suppose $X^{k}$ is a Killing vector satisfying

$$X^{k}(q) = 0.$$  

(8.20)

Now, at $q$, (8.12) gives

$$\left. \left( X^{k,i} \bar{g}_{kj} + X^{k,j} \bar{g}_{ki} \right) \right|_{q} = 0.$$

(8.21)

Defining $\bar{X}_{m,l}(q) = \left( X^{k} \bar{g}_{km} \right)_{,l}(q) = X^{k,l}(q) \bar{g}_{km}(q)$ and $B_{j}^{m} = \bar{g}^{mk} g_{kj}$, (8.20) is equivalent to

$$\bar{X}_{m,l}(q) \left( \delta^{l}_{i} B_{j}^{m} + \delta^{m}_{j} B_{i}^{l} \right) \bigg|_{p} = 0.$$  

(8.22)

Thus, since $\bar{X}_{m,l}(q)$ is antisymmetric due to (8.20), the expression in the brackets must be symmetric in $m$ and $l$, that is,

$$\left( \delta^{l}_{i} B_{j}^{m} + \delta^{m}_{j} B_{i}^{l} \right) \bigg|_{p} = \left( \delta^{m}_{i} B_{j}^{l} + \delta^{l}_{j} B_{i}^{m} \right) \bigg|_{p}.$$  

(8.23)
Contracting now with $\bar{g}_i^j$ gives
\[ \bar{g}^{mk} g_{kj} = \frac{1}{2} \bar{g}^{m}{}_{ik} g_{ki}, \]
which is equivalent to
\[ g_{lj} = C \bar{g}_{lj}, \quad (8.23) \]
for $C = \frac{1}{2} \bar{g}_{ik} g_{ki}$, and since $q$ was arbitrary this holds on the whole coordinate patch. Now, $C$ only depends on coordinates $x^0$ and $x^1$, since substituting $(8.23)$ into $(8.12)$ and using that any arbitrary Killing vector $X^k$ of $g$ is also a Killing vector of $\bar{g}$, together gives for all $X^k$
\[ X^k C_{,k} = 0, \quad (8.24) \]
from which we conclude $C_{,l} = 0$ for $l \in \{2, 3\}$. This proves $(8.18)$.

Now, $(8.6)$, $(8.17)$ and $(8.18)$ together show that, in the coordinates $x^i$ constructed in the beginning, the metric is given by
\[ ds^2 = g_{ab}(x^0, x^1) dx^a dx^b + C(x^0, x^1) \bar{g}_{lk}(x^2, x^3) dx^l dx^k, \quad (8.25) \]
where summation over $a, b$ runs from 0 to 1, summation over $l, k$ runs from 2 to 3, $C$ is some $C^1$ function and $\bar{g}_{ij}(u)$ denotes the induced metric on the space of symmetry.

Now, assume without loss of generality\footnote{By assumption of spherical symmetry, the spaces of symmetry have constant curvature, so that, without loss of generality, we can choose the space with $K = 1$.} the space of symmetry has constant curvature $K = 1$, then the induced metric $\bar{g}_{ij} dx^i dx^j$ can be taken over to the line element $d\Omega^2 = d\vartheta^2 + \sin^2(\vartheta) d\varphi^2$ of the SSC metric we started in by a coordinate transformation on the spheres of symmetry alone, (c.f. [25], chapter 13.2). Setting $\bar{t} = x^0$ and $\bar{r} = x^1$, the resulting metric,
\[ ds^2 = -A(t, r) dt^2 + B(t, r) d\bar{r}^2 + 2E(t, r) d\bar{t} d\bar{r} + C(t, r) d\Omega^2, \quad (8.26) \]
is again in $C^1$ and has the same induced metric on the spheres of symmetry than the original SSC metric, $g_{\mu\nu}$. Consequently $(8.26)$ can be taken over to $g_{\mu\nu}$ by a transformation in the $(t, r)$-plane, since by our incoming assumption $g_{ij}$ and the SSC metric are related by some coordinate transformation. This contradicts Theorem 7.1 since the metric $(8.26)$ is in $C^1$ on a deleted neighborhood of $p$ and has a non-vanishing determinant at $p$, which proves of Theorem 1.1.

We close this chapter by stating a straightforward consequence of Theorem 1.1, asserting that at points of regular shock wave interaction the Einstein equations can only hold in the weak sense.

**Corollary 8.2.** Suppose that $p$ is a point of regular shock wave interaction in SSC, in the sense of Definition 3.1, for the SSC metric $g_{\mu\nu}$. Then there do not exist coordinates that can be reached from SSC by a $C^{1,1}$ coordinate transformation, such that the metric is a $C^{1,1}$ function of the new coordinates, has a non-vanishing determinant and solves the Einstein equations point-wise almost everywhere in a neighborhood of $p$.  \[ \square \]
9. The Loss of Locally Inertial Frames

In this section we prove the non-existence of locally inertial frames around a point of regular shock wave interaction. This is in strong contrast to the situation of a single shock wave, where such coordinates exist, c.f. [13]. We first clarify what we mean by a locally inertial frame.

Definition 9.1. Let \( p \) be a point in a Lorentz manifold and let \( x^j \) be a coordinate system defined on a neighborhood \( \mathcal{N} \) of \( p \). We call \( x^j \) a locally inertial frame around \( p \) if the metric \( g_{ij} \) in those coordinates satisfies:

(i) \( g_{ij}(p) = \eta_{ij} \), where \( \eta_{ij} = \text{diag}(-1, 1, 1, 1) \) denotes the Minkowski metric,
(ii) \( g_{ij,l}(p) = 0 \) for all \( i, j, l \in \{0, ..., 3\} \),
(iii) \( g_{ij,kl} \) are in \( L^\infty \) for every compact region contained in \( \mathcal{N} \), for all \( i, j, k, l \in \{0, ..., 3\} \).

We refer to a metric \( g_{ij} \), that satisfies (i)-(iii), as a locally Minkowskian (or locally flat or locally inertial) metric around \( p \).

Condition (iii) ensures that, in General Relativity, gravitational effects induce second order correction to tensorial equation. That is, the difference of an equation in Minkowski space and in a (curved) spacetime is a correction term second order in the metric derivatives. (In most literature the metric is assumed to be smooth, thus implying condition (iii) of Definition 9.1.) Now, by Theorem 1.1, there exist distributional and thus unbounded second order derivatives of the metric, so that the following Corollary is a straightforward consequence.

Corollary 9.2. Let \( p \) be a point of regular shock wave interaction in SSC in the sense of Definition 3.1, then there does not exist a \( C^{1,1} \) coordinate transformation such that the resulting metric \( g_{ij} \) is locally Minkowskian around \( p \).

Proof. Assume for contradiction that there exist a locally inertial frame \( x^i \) around the point \( p \) of regular shock wave interaction. Then the metric in coordinates \( x^i \) satisfies \( g_{ij}(p) = \eta_{ij} \) and thus has a non-vanishing determinant.

By Theorem 1.1 the metric \( g_{ij} \) is not in \( C^1 \). Now, there exist indices \( i, j, l \in \{0, ..., 3\} \) such that \( g_{ij,l} \) is discontinuous and thus there exist indices \( i, j, l, k \in \{0, ..., 3\} \) such that \( g_{ij,kl} \) is distributional and therefore not in \( L^\infty \) for any neighborhood of \( p \). This completes the proof. \( \square \)

Corollary 9.2 proves that the gravitational metric cannot be locally Minkowskian at points of regular shock wave interaction in SSC since unbounded second order metric derivatives occur. However, it is presently not clear if there exist coordinates in which the metric satisfies condition (i)-(ii) of Definition 9.1 or how close the metric can be to locally Minkowski.

10. The Riemann Curvature Tensor is Bounded

In this section we prove that a regularity singularity is not a naked singularity, since the scalar-curvature, the Riemann curvature tensor and all scalar quantities incorporating it are bounded.
To begin with, recall that the scalar curvature is the trace of the Ricci tensor, that is,

\[ R = R^\mu_\mu. \]  

(10.1)

Taking the trace of the Einstein tensor \((2.7)\) gives

\[ G^\mu_\mu = R^\mu_\mu - \frac{1}{2} \delta^\mu_\mu R = -R, \]

and taking the trace of the Einstein equations \((2.6)\) we find

\[ R = -T^\mu_\mu. \]

(10.3)

Assuming \(T^\mu_\nu \in L^\infty\), \((10.3)\) implies \(R\) to be in \(L^\infty\) as well. We conclude that the scalar curvature stays bounded at points of regular shock wave interactions in SSC.

However, even though the scalar curvature is bounded, there could be other curvature scalars blowing up at points of shock wave interaction, curvature scalars of the form

\[ c = C^\mu_\nu^\rho_\sigma R^\mu_\nu^\rho_\sigma, \]

where \(R^\mu_\nu^\rho_\sigma\) denotes the Riemann curvature tensor and \(C^\mu_\nu^\rho_\sigma\) is some tensor with components in \(L^\infty\), (with respect to some coordinate system). The next lemma states that the Riemann curvature tensor and any such scalar is bounded for \(T^\mu_\nu \in L^\infty\).

**Lemma 10.1.** Assume the energy momentum tensor is bounded almost everywhere, that is, \(T^\mu_\nu \in L^\infty\). Let \(g_\mu_\nu\) be a \(C^{0,1}\) regular metric solving the Einstein equations \((2.6)\) weakly. Then the Riemann curvature tensor and any curvature scalar \(c\), as defined in \((10.4)\), are in \(L^\infty\).

**Proof.** Since \(C^\mu_\nu^\rho_\sigma\), in \((10.4)\), is in \(L^\infty\), it suffices to prove that all components of the Riemann curvature tensor are in \(L^\infty\).

To begin with, we compute the scalar curvature in SSC (using MAPLE):

\[ R = \frac{A_{rr} - B_{tt}}{AB} + \frac{A_r B_t - A_t B_r}{2A^2 B} + \frac{B_t^2 - A_r B_r}{2AB^2} + \frac{2A_r}{rAB} - \frac{2B_t}{rB^2} + \frac{2 - 2B}{r^2 B}. \]

(10.5)

For a Lipschitz continuous metric the first order metric derivatives exist almost everywhere, while the second order derivatives in \((10.5)\) have to be taken in a weak sense, that is, \(A_{rr}\) and \(B_{tt}\) are functions in \(L^\infty\) with the defining property

\[ \int (A_{rr} - B_{tt}) \varphi = -\int (A_r \varphi_r - B_t \varphi_t), \]

(10.6)

for all \(\varphi \in C_0^\infty\), with \(C_0^\infty\) denoting the set of smooth functions with compact support. By \((10.3)\), we conclude that \(R \in L^\infty\), thus \((10.5)\) together with the metric being Lipschitz continuous implies

\[ A_{rr} - B_{tt} \in L^\infty, \]

(10.7)

where \(A_{rr}\) and \(B_{tt}\) again refer to weak second order derivatives.

For a metric in \(C^{0,1}\), all components of the Riemann curvature tensor not containing second order metric derivatives lie in \(L^\infty\). In SSC, a straightforward
computation shows the only components (up to the symmetries of $R_{\mu\nu\rho\sigma}$) containing second order metric derivatives to be

$$R_{0101} = \frac{A_{rr} - B_{tt}}{2} + \frac{A_t B_t - A_r^2}{4A} + \frac{B_t^2 - A_r B_r}{4B}, \quad (10.8)$$

where we interpret second order derivatives again in the weak sense. From (10.8) and (10.7), we conclude that $R_{0101}$ lies in $L^\infty$. Finally, since all components of $R_{\mu\nu\rho\sigma}$ are in $L^\infty$ with respect to SSC, the Riemann curvature tensor is in $L^\infty$ in any coordinates that can be reached from SSC within a $C^{1,1}$ atlas. This completes the proof. \qed

By Lemma [10.1] even though delta function sources in the second derivatives of the gravitational metric tensor exist in all coordinate systems of the $C^{1,1}$ atlas, the Riemann curvature tensor and all curvature scalars remain uniformly bounded at regularity singularities due to cancelation. This further clarifies the difference between the essential $C^{0,1}$ singularities in the metric at points of shock wave interaction, and the essential $C^{0,1}$ singularities at surface layers like the “thin shells” introduced in Israel’s paper [10]. Namely, on surface layers, the delta function sources in $T$ are the cause of the essential $C^{0,1}$ singularity in the metric $g$, because second derivatives of $g$ must have distributional sources and consequently $g$ cannot be $C^{1,1}$ in any regular coordinate system. For shock wave solutions of $G = \kappa T$ the issue is more delicate because the constraint that $G$ have delta function sources is removed and, at first sight, there is no clear obstacle to the existence of coordinate systems that smooth the metric to $C^{1,1}$. In light of this, the existence of a non-removable $C^{0,1}$-singularity at points of shock wave interaction is rather subtle and surprising.

11. Discussion

In this section we discuss our choice of atlas, the role of spherical symmetry for the result of Theorem 1.1 and the possibility of physical effects caused by a regularity singularity.

Theorem 1.1 shows that there is a non-removable lack of $C^1$ regularity within the atlas of $C^{1,1}$ regular coordinate transformations. We believe this proves points of regular shock wave interaction to be essential spacetime singularities, since we expect that one can not extend the atlas to an even lower regularity and achieve a metric-smoothing within the larger atlas. Namely, a Jacobian which is Hölder but not Lipschitz continuous cannot meet the (necessary) smoothing condition [5.4] because it fails to provide enough free parameters $[\gamma_{\alpha,\beta}]$, said differently, a Hölder continuous Jacobian does not “mirror” the $C^{0,1}$ metric regularity appropriately. Moreover, we believe that lowering the atlas regularity to anything less smooth than $C^1$ would conflict with the weak formulation of the Einstein equations, since the metric regularity is too low to shift enough derivatives onto test functions, due to the quasi-linear structure of the equations.

Points of regular shock wave interaction are not hidden from observation by an event horizon. This opens up the opportunity of direct measurements of effects
caused by a regularity singularity. Predicting what such an effect could be is an open problem. Supernovae are cosmic events where we believe shock wave interactions to take place, namely, the collision of the outer shell of the collapsing star with the surface of the (cooled off) core.

The assumption of spherical symmetry enters our methods in Sections 3 - 7 extensively. However, we believe that removing our symmetry assumption does not alter our results, since the smoothing condition (5.4) is valid for metrics without any symmetries and we expect that (5.4) together with (5.5) are still uniquely solvable. But extending our method to the case without any symmetries is far from being straightforward.

Introducing (Navier Stokes type) viscosity terms into the perfect fluid source of the Einstein equation causes a smoothing of the discontinuities in the fluid variables smearing out the shock profiles. As a result the metric tensor is smooth and locally inertial frames exist. However, steep gradients in the fluid sources persist and we still expect to recover large second order metric derivatives that blow up as the viscosity tends to zero. From this, we conclude that the distributional second order metric derivatives could be of physical significance, but the loss of locally inertial frames is rather a mathematical curiosity.

12. Conclusion

Our results show that points of shock wave interaction give rise to a new kind of singularity which is different from the well known singularities of General Relativity. The famous examples of singularities are either non-removable singularities beyond physical spacetime, (for example the center of the Schwarzschild and Kerr metrics, and the Big Bang singularity in cosmology where the curvature cannot be bounded), or else they are removable in the sense that they can be transformed to locally inertial points of a regular spacetime under coordinate transformation, (for example, the apparent singularity at the Schwarzschild radius and any apparent singularity at smooth shock surfaces that become regularized by Israel’s Theorem, [10, 18]). In contrast, points of shock wave interaction are non-removable $C^{0,1}$ singularities that propagate in physically meaningful spacetimes, such that the curvature is uniformly bounded, but the spacetime is essentially not locally inertial at the singularity and second order metric derivatives are distributional. For this reason we call these regularity singularities.

Since the gravitation metric tensor is not locally inertial at points of shock wave interaction, it begs the question as to whether there are general relativistic gravitational effects at points of shock wave interaction that cannot be predicted from the compressible Euler equations in special relativity alone. Indeed, even if there are dissipativity terms, like those of the Navier Stokes Equations [18] which regularize the gravitational metric at points of shock wave interaction, our results assert that the steep gradients in the derivative of the metric tensor at small viscosity cannot be removed uniformly while keeping the metric determinant

\[18\] The issue of how to incorporate a relativistic viscosity that meets the speed of light bound is problematic. [20].
uniformly bounded away from zero, so one would expect the general relativistic effects at points of shock wave interaction to persist. Moreover, points of regular shock wave interaction are not hidden from observation by an event horizon, which opens up the opportunity of direct measurement of effects that could resemble unbounded scalar curvature. We thus wonder whether shock wave interactions might provide a physical regime where new general relativistic effects might be observed.

Appendix A. The Integrability Condition

In this section we review the equivalence of the existence of an integration factor and the integrability condition (2.4) on \( \mathbb{R}^2 \). Recall that the Jacobian is defined as

\[
J_\mu^\alpha = \frac{\partial x^\alpha}{\partial x^\mu},
\]

provided we are given a coordinate transformation from \( x^\mu \) to \( x^\alpha \), where the indices \( \alpha = 0, 1 \) and \( \mu = t, r \) label a set of four functions.

**Lemma A.1.** Let \( \Omega \) be an open set in \( \mathbb{R}^2 \) with coordinates \( x^\nu = (t, r) \). Suppose we are given a set of functions \( J_\mu^\alpha(x^\nu) \) in \( C^{0,1}(\Omega) \), \( C^1 \) away from some curve \( \gamma(t) = (t, x(t)) \), satisfying \( \det(J_\mu^\alpha) \neq 0 \). Then the following is equivalent:

(i) There exist locally invertible functions \( x^\alpha(t, r) \in C^{1,1}(\Omega) \), for \( \alpha = 0, 1 \), such that \( \frac{\partial x^\alpha}{\partial x^\mu} = J_\mu^\alpha \).

(ii) The set of functions \( J_\mu^\alpha \in C^{0,1}(\Omega) \) satisfy the integrability condition

\[
J_\mu^\alpha,\nu = J_\nu^\mu,\alpha.
\]

**Proof.** The implication from (i) to (ii) is trivial, since (weak) partial derivatives commute. We now prove that (ii) implies (i). Without loss of generality, we assume that \( \Omega \) is the square region \( (a, b)^2 \subset \mathbb{R}^2 \). For \( (t, r) \in \Omega \), introduce

\[
x^\alpha(t, r) = \int_a^r J_r^\alpha(t, x)dx + \int_a^t J_t^\alpha(t, \tau)d\tau,
\]

then

\[
\frac{\partial x^\alpha}{\partial r} = J_r^\alpha,
\]

follows immediately. Furthermore, using (A.2) we get

\[
\frac{\partial x^\alpha}{\partial t}(t, r) = \int_a^r J_{r,t}^\alpha(t, x)dx + J_t^\alpha(t, a) = \int_a^{x(t)} J_{t,x}^\alpha dx + \int_{x(t)}^r J_{t,r}^\alpha dx + J_t^\alpha(t, a),
\]

where \( x(t) \in (a, r) \) is the point of discontinuity of \( J_{t,r}^\alpha \). We apply the fundamental theorem of calculus to each of the above integrals separately and obtain

\[
\frac{\partial x^\alpha}{\partial t}(t, r) = J_t^\alpha(t, r).
\]
Moreover, the Inverse Function Theorem yields that the function $x^\alpha$ is bijective on some open set, since

$$\det \left( \frac{\partial x^\mu}{\partial x^\alpha} \right) = \det (J^\mu_\alpha) \neq 0,$$

in $\Omega$. □

**Acknowledgments**

I am grateful to Blake Temple for supervising my dissertation and proposing to work on the question whether one can smooth the gravitational metric at points of shock wave interaction for my thesis. I was partially supported by Blake Temple’s NSF Grant, where the problem was first proposed.

I am thankful to Felix Finster for his support during the last years, and for hosting me at the Mathematics Department of the University Regensburg during Fall 2010. I was partially funded by Felix Finster’s DFG Grant.

I also thank Lars Andersson for inviting me to the Max-Planck-Institute for Gravitational Physics and for their funding from October to December 2012.

I am grateful to Joel Smoller for his support.

**References**

[1] A. P. Barnes, P. G. Lefloch, B. G. Schmidt and J. M. Stewart, “The Glimm scheme for perfect fluids on plane-symmetric Gowdy spacetimes”, Class. Quantum Grav. 21 (2004), 5043.

[2] D. Christodoulou, *The Formation of Shocks in 3-Dimensional Fluids*, E.M.S. Monographs in Mathematics, 2007, ISBN: 978-3-03719-031-9.

[3] A. Einstein, *Die Feldgleichungen der Gravitation*, Preuss. Akad. Wiss., Berlin, Sitzber. 1915b, pp. 844-847. 314(1970), pp. 529-548.

[4] L. C. Evans, *Partial Differential Equations*, Vol 3A, Berkeley Mathematics Lecture Notes, 1994.

[5] H. Freistühler and M. Raoofi, “Stability of perfect-fluid shock waves in special and general relativity”, Class. Quantum Grav. 24, (2007), 4439 - 4455.

[6] R. Geroch and J. Traschen, *Strings and other distributional sources in general relativity*, Phys. Rev. D, 36, No. 4, August 1987.

[7] J. Groah and B. Temple, *Shock-Wave Solutions of the Einstein Equations with Perfect Fluid Sources: Existence and Consistency by a Locally Inertial Glimm Scheme*, Memoirs AMS, Vol. 172, Number 813, November 2004, ISSN 0065-9266.

[8] J. Groah and B. Temple, “A shock wave formulation of the Einstein equation”, Methods and Applications of Analysis, 7, No. 4, (2000), pp. 793-812.

[9] S.W. Hawking and G.F.R. Ellis, *The Large Scale Structure of Spacetime*, Cambridge University Press, 1973.

[10] W. Israel, “Singular hypersurfaces and thin shells in general relativity”, Il Nuovo Cimento, Vol. XLIV B, N. 1, 1966, pp. 1-14.

[11] F. John, *Partial Differential Equations*, Applied Mathematical Sciences 1, Springer-Verlag, New York, 1981.

[12] P. Lax, “Hyperbolic systems of conservation laws, II”, Comm. Pure Appl. Math., 10(1957), pp. 537-566.

[13] J. R. Oppenheimer and J. R. Snyder, “On continued gravitational contraction”, Phys. Rev., 56 (1939), pp. 455-459.
[14] M. Reintjes, “Shock Wave Interactions in General Relativity and the Emergence of Regularity Singularities”, Dissertation, University of California - Davis, September 2011.

[15] M. Reintjes and B. Temple, “Points of general relativistic shock wave interaction are ‘regularity singularities’ where space-time is not locally flat”, Proc. R. Soc. A, 2012.

[16] A. Rendall and F. Stahl, “Shock Waves in Plane Symmetric Spacetimes”, Communications in Partial Differential Equations, 33(11), 2020-2039.

[17] J. Smoller, *Shock Waves and Reaction Diffusion Equations*, Springer-Verlag, 1983.

[18] J. Smoller and B. Temple, “Shock wave Solutions of the Einstein equations: The Oppenheimer-Snyder model of gravitational collapse extended to the case of non-zero pressure”, Archive Rational Mechanics and Analysis, 128 (1994), pp. 249-297, Springer-Verlag 1994.

[19] J. Smoller and B. Temple, “Shock wave cosmology inside a black hole”, PNAS, Vol. 100, no. 20, 2003, pp. 11216-11218.

[20] J. Smoller and B. Temple, “Global Solutions of the Relativistic Euler Equations”, Commun. Math. Phys. 156, 67-99 (1993).

[21] J. Smoller and B. Temple, “Cosmology with a Shock-Wave”, Commun. Math. Phys. 210, 275-308 (2000).

[22] J. Smoller and B. Temple, “Cosmology, Black Holes and Shock Waves Beyond the Hubble Length”, Methods Appl. Anal. Volume 11, Number 1 (2004), 077-132

[23] J. Smoller and B. Temple, “Astrophysical shock wave solutions of the Einstein equations”, Phys. Rev. D, Vol. 51, No. 6, 1995, pp. 2733-2743.

[24] Z. Vogler and B. Temple, “Simulation of General Relativistic Shock Waves by a Locally Inertial Godunov Method featuring Dynamical Time Dilation”, Davis Preprint.

[25] S. Weinberg, *Gravitation and Cosmology*, John Wiley & Sons, New York, 1972.

[26] R. Wald, *General Relativity*.

Max-Planck-Institute for Gravitational Physics, Albert Einstein Institute, Am Mühlenberg 1, 14476 Golm, Germany

Mathematics Department, University Regensburg, Universitätsstraße 31, 93040 Regensburg, Germany

E-mail address: moritzreintjes@googlemail.com