GLOBAL WELL-POSEDNESS TO INCOMPRESSIBLE NON-INERTIAL QIAN-SHENG MODEL

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Abstract. In this paper we study the incompressible non-inertial Qian-Sheng model, which describes the hydrodynamics of nematic liquid crystals without inertial effect in the $Q$-tensor framework. Under some proper assumptions on the viscous coefficients, we prove the local well-posedness with large initial data and the global existence with small size of the initial data in the classical solutions regime.

1. Introduction.

1.1. Incompressible non-inertial Qian-Sheng model. In this paper, we aim at studying the incompressible non-inertial Qian-Sheng model over $(t, x) \in \mathbb{R}^+ \times \mathbb{R}^d$ with $d = 2, 3$:

\[
\begin{aligned}
\partial_t u + u \cdot \nabla u - \frac{\beta}{2} \Delta u + \nabla p &= \text{div}(\Sigma_1 + \Sigma_2 + \Sigma_3), \\
\text{div} u &= 0, \\
\mu_1 \dot{Q} &= L \Delta Q - a Q + b Q^2 - \frac{1}{4} |Q|^2 I_d - c Q |Q|^2 + \frac{\beta}{2} A + \mu_1 \{\Omega, Q\},
\end{aligned}
\]

where the notations $\Sigma_i$ ($i = 1, 2, 3$) are given as

\[
\begin{aligned}
\Sigma_1 &= -L \nabla Q \odot \nabla Q, \\
\Sigma_2 &= \beta_1 Q \text{tr}(QA) + \beta_2 A Q + \beta_3 Q A, \\
\Sigma_3 &= \frac{\omega}{2} (\dot{Q} - [\Omega, Q]) + \mu_1 [Q, (\dot{Q} - [\Omega, Q])],
\end{aligned}
\]

and $Q$ subjects to the constraint $Q \in S_0^{(d)}$ with the definition

\[
S_0^{(d)} := \{Q \in \mathbb{R}^d; Q_{ij} = Q_{ji}, \text{tr}(Q) = 0, i, j = 1, \ldots, d\}.
\]

Here $\dot{f} = (\partial_t + u \cdot \nabla)f$ represents a time material derivative and for any two $d \times d$ matrices $M, N$, we denote their commutator by $[M, N] := MN - NM$. We also denote the inner product on the space of matrices by $M : N = \text{tr}(MN)$ and $|M|$ stands for the Frobenius norm of the matrix, i.e. $|M| = \sqrt{M : M}$. Moreover,

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we employ the notations $A_{ij} := \frac{1}{d}(\partial_j u_i + \partial_i u_j)$ and $\Omega_{ij} := \frac{1}{d}(\partial_j u_i - \partial_i u_j)$ for $i, j = 1, \ldots, d$ to represent the rate-of-strain tensor and the vorticity tensor of the bulk velocity, respectively. The entries of $A$, $\Omega$, and $d$ denote the rate-of-strain tensor and the vorticity tensor of the bulk velocity, respectively. The entries of $\nabla Q \circ \nabla Q$ are defined as $(\nabla Q \circ \nabla Q)_{ij} = \sum_{k,l=1}^{d} \partial_k Q_{kl} \partial_l Q_{kl}$, where, for a scalar function $f$, we write $\partial_i f$ for $\frac{\partial f}{\partial x_i}$. The $I_d$ denotes the $d \times d$ identity matrix.

Furthermore, $L > 0$ is usually very small and is neglected on most mathematical literatures. $J$ is the inertial constant. In the physical experiment, the inertial constant $J$ is large enough compared to the other remaining viscosities, so that we can obtain the necessary energy dissipation.

Finally, the initial conditions of the system (1) are imposed on
\[ (u, Q)(x, t) \big|_{t=0} = (u^{in}(x), Q^{in}(x)), \]
which subject to the compatibilities $\text{div}\ u^{in} = 0$ and $Q^{in} \in S_0^{(d)}$.

1.2. Motivations. The model (1) describes the hydrodynamics of nematic liquid crystals \textit{without} inertial effect in the $Q$-tensor framework. At most tensorial models, this one provides an extension of the classical parabolic version Ericksen-Leslie model [6] without inertial effect, in particular capturing the biaxial alignment of the molecules, a feature not available in the classical Ericksen-Leslie model.

The corresponding model of (1) \textit{with} inertial effect is the following so-called incompressible inertial Qian-Sheng model proposed by Qian and Sheng in [14] (see also [1]):
\[
\begin{cases}
\partial_t u + u \cdot \nabla u + \nabla p - \frac{\beta_6}{2} \Delta u \\
\text{div} \left( - L \nabla Q \circ \nabla Q + \beta_1 Q \text{tr}(Q A) + \beta_6 A Q + \beta_6 Q A \right) \\
+ \text{div} \left( \frac{\beta_2}{2} (\hat{Q} - [\Omega, Q]) + \mu_1 (Q - \beta_2 [\Omega, Q]) \right), \\
\text{div}
\end{cases}
\]
\[
J \dot{Q} + \mu_1 \dot{Q} = L \Delta Q - a Q + b(Q^2 - \frac{1}{d} |Q|^2 I_d) - c |Q|^2 + \frac{\beta_2}{2} A + \mu_1 [\Omega, Q],
\]
where $J \dot{Q}$ is the inertial term. In the physical experiment, the inertial constant $J > 0$ is usually very small and is neglected on most mathematical literatures. From the mathematical aspects, the system (5) is a hyperbolic convection-diffusion system for matrix-valued function coupled an incompressible Navier-Stokes system, which models the evolution of the orientations of the nematic molecules over the flow. The inertial term $J \dot{Q}$ corresponding to the hyperbolic character of the equation describes the inertia effect of the orientation of the molecules. This feature is also presented in the Ericksen-Leslie model (see [6] or [4]) but it is usually neglected in the mathematical studies, see [7] for instance.

But, from a rigorously mathematical point of view, why can the inertial term $J \dot{Q}$ be ignored? We can positively answer the question provided that one is able to \textit{rigorously} justify the limit from the systems (5) to (1) as taking $J \to 0$. We notice that the problem mentioned above is a singular limit problem. Usually, there are two ways to deal with the singular limit problems in the classical solutions regime:

1. to obtain a uniform (in small parameter) bounds on the solutions to the original scaled singular equations and then to extract a convergent subsequence converging to the solutions of the target (limit) equations as the small parameter going to zero.
2. to obtain the solutions for the limiting equations and then to construct a sequence of special solutions of the original scaled singular equations for the small parameter, which is often named the Hilbert expansion method.

We remark that the first way is usually much harder than the second way, since the uniform bounds are very difficult to be sought in most situations. For the singular limit problem mentioned above, we plan to employ the second way rather than the first one, since the uniform (in \( J > 0 \)) energy bounds are almost not to be found. Consequently, the first thing is to construct the classical solutions of the limit system (1), which is what we do in this paper. Then, Luo and I will construct a sequence of special solutions of the system (5) around the solutions of (1) in another work, which is preparing.

1.3. Physical aspects. The velocity \( u \) of the centers of masses of the molecules satisfies an incompressible convection-diffusion fluid-type equation (here and in the following we use the Einstein summation convention, of summation over repeated indexes)

\[
\dot{u}_i = \partial_j (-p\delta_{ij} + \sigma_{ij} + \sigma'_{ij})
\]

with the divergence-free constraint

\[
\partial_k u_k = 0 ,
\]

where \( p \) is the pressure, \( \sigma \) is the distortion stress with the form

\[
\sigma_{ij} := -\frac{\partial \mathcal{F}[Q]}{\partial (\partial_i Q_{\alpha\beta})} \partial_j Q_{\alpha\beta}.
\]

Here \( \mathcal{F}[Q] \) is the simplest of the Landau-de Gennes free energy density

\[
\mathcal{F}[Q] := \frac{L}{2} |\nabla Q|^2 + \psi_B(Q),
\]

which models the spatial variations through the \( \frac{L}{2} |\nabla Q|^2 \) term with positive diffusion coefficient \( L > 0 \), and the nematic ordering enforced through the “bulk term” with the standard form \([9]\)

\[
\psi_B(Q) = \frac{a}{2} \text{tr}(Q^2) - \frac{b}{3} \text{tr}(Q^3) + \frac{c}{4} (\text{tr}(Q^2))^2.
\]

Moreover, the viscous stress \( \sigma' \) is defined as

\[
\sigma'_{ij} := \beta_1 Q_{ij} Q_{lk} A_{lk} + \beta_4 A_{ij} + \beta_5 Q_{jl} A_{li} + \beta_6 Q_{li} A_{lj} + \frac{1}{2} \mu_2 \mathcal{N}_{ij} + \mu_1 \{Q, \mathcal{N}\}_{ij},
\]

where \( \beta_1, \beta_4, \beta_5, \beta_6, \mu_1 \) and \( \mu_2 \) are viscosity coefficients subjecting to the relations (3) and \( \mathcal{N} \) represents the co-rotational time flux of \( Q \), whose \((i, j)\)-th component is defined as \([1]\)

\[
\mathcal{N}_{ij} = (\dot{Q} - [\Omega, Q])_{ij} = \dot{Q}_{ij} - \Omega_{il} Q_{lj} + Q_{il} \Omega_{lj}.
\]

We remark that the first equality of (3) is the well-known Parodi’s relation for the Leslie stress tensor, and the second condition in (3) is not always satisfied by physical materials however it is sometimes assumed in the physics literature in the more specialized form \( \beta_5 = \beta_6 = 0 \), see \([13]\) for instance.

The evolution of the order tensor \( Q \) without inertial effect is governed by

\[
h_{ij} + h'_{ij} - \lambda \dot{\mathcal{N}}_{ij} - \varepsilon_{ijk} \lambda_k = 0 ,
\]

where \( \varepsilon_{ijk} \) is the Levi-Civita symbol, \( h \) is the elastic molecular with form

\[
h_{ij} := -\frac{\partial \mathcal{F}[Q]}{\partial Q_{ij}} + \partial_k \left( \frac{\partial \mathcal{F}[Q]}{\partial (\partial_k Q_{ij})} \right).
\]
and the viscous molecular field $h'$ is defined as

$$h'_{ij} := \frac{1}{2} \tilde{\mu}_2 A_{ij} - \mu_1 N_{ij}.$$  

Moreover, the $\lambda$, $\lambda_k$ are Lagrange multipliers with forms $\lambda = -b|Q|^2 I_d$ and $\lambda_k = 0$, respectively, enforcing the tracelessness and symmetry of the tensor. For the viscosity $\tilde{\mu}_2$, there are two different choices:

1. $\tilde{\mu}_2 = \mu_2$ in the paper [14], which can be derived from the Onsager theorem [3] with entropy source.
2. $\tilde{\mu}_2 = -\mu_2$, for details see [12, 13].

Furthermore, due to the coefficient $\beta_4$ corresponds to the standard Newtonian stress tensor and $\mu_1$ is the molecular viscosity coefficient, we can assume $\beta_4 > 0$ and $\mu_1 > 0$.

1.4. Historical remarks. The mathematical study of the incompressible inertial Qian-Sheng model of liquid crystal flow was initialized by F. De Anna and A. Zarnescu [1], which was the first work to deal with a second order material derivative. They derived the energy law and proved the local well-posedness for bounded initial data and global well-posedness under the assumptions that the initial data are small in suitable norm and the coefficients satisfy some further damping property. In [1], they also provided an example of twist-wave solutions, which are solutions of the coupled system for which the flow vanishes for all times. Furthermore, for the inviscid version of the inertial Qian-Sheng model, in [2], Feireisl et al. proved a global existence of the dissipative solution which is inspired from that of incompressible Euler equation defined by P.-L. Lions [8].

There are many works on the $Q$-tensor model without inertial effect but being different with the Qian-Sheng model. For instance, M. Paicu and A. Zarnescu [11] proved the global weak solutions of the non-inertial Beris-Edwards system, a $Q$-tensor model. For more $Q$-tensor models of liquid crystals, readers can be referred to the works [5, 10, 15, 16] and the related references therein, for instance.

1.5. Notations and main results. For the sake of convenience, we first introduce some notations throughout this paper. We denote by $A \lesssim B$ if there exists a constant $C > 0$, such that $A \leq CB$. For convenience, we also denote $L^p := L^p(\mathbb{R}^d)$ by the standard $L^p$ space for all $p \in [1,+\infty]$. For $p = 2$, we use the notation $\langle \cdot, \cdot \rangle$ to represent the inner product on the Hilbert space $L^2$.

For any multi-index $k = (k_1, k_2, \cdots, k_d)$ in $\mathbb{N}^d$, we denote the $k$-th partial derivative operator by

$$\partial^k = \partial_{x_1}^{k_1} \partial_{x_2}^{k_2} \cdots \partial_{x_d}^{k_d}.$$  

We employ the notation $k \leq k'$ to represent that every component of $k \in \mathbb{N}^d$ is not greater than that of $k' \in \mathbb{N}^d$. Moreover, $k < k'$ means that $k \leq k'$ and $|k| < |k'|$, where $|k| = k_1 + k_2 + \cdots + k_d \in \mathbb{N}$. We now define the following two Sobolev spaces $H^s$ and $\dot{H}^s$ endowed with the norms

$$\|f\|_{H^s} = \left( \sum_{|k|=0}^s \|\partial^k f\|_{L^2}^2 \right)^{\frac{1}{2}} , \quad \|f\|_{\dot{H}^s} = \left( \sum_{|k|=1}^s \|\partial^k f\|_{L^2}^2 \right)^{\frac{1}{2}},$$

respectively.

We now define the following energy $E(t)$:

$$E(t) = \|u\|_{H^s}^2 + L\|\nabla Q\|_{H^s}^2 + a\|Q\|_{H^s}^2. \quad (6)$$
In particular, the initial energy is
\[ E^{in} = \|u^{in}\|_{H^s}^2 + L\|\nabla Q^{in}\|_{H^s}^2 + a\|Q^{in}\|_{H^s}^2. \]

We next state our main theorems as follows.

**Theorem 1.1** (Local well-posedness with large initial data). Let \( s > \frac{d}{2} + 1 \) (\( d = 2, 3 \)) be an integer. We assume that the coefficients satisfy (3) and
\[ L > 0, \beta_1 \geq 0, \beta_4 > 0, \mu_1 > 0, \mu_2 = \mu_2 = 0, a > 0 \]
or
\[ L > 0, \beta_1 \geq 0, \beta_4 > 0, \mu_1 > 0, \bar{\mu}_2 = -\mu_2 \neq 0, a > 0, \]
where the constants \( \beta = \tilde{\beta}(\mu_1, \mu_2) \) is an explicitly computable coefficient. If \( E^{in} < +\infty \), then there are \( T, C_0 > 0 \), depending only on the initial data, the all coefficients, \( s \) and \( d \), such that the Cauchy problem (1)-(4) admits a unique solution \((u, Q)\) satisfying
\[ u \in L^\infty(0, T; H^s), \quad Q \in L^\infty(0, T; H^{s+1}), \quad \nabla u, \dot{Q} \in L^2(0, T; H^s) \]
and obeying the following energy bound
\[
\sup_{t \in [0, T]} \left( \|u\|_{H^s}^2 + L\|\nabla Q\|_{H^s}^2 + a\|Q\|_{H^s}^2 \right) + \int_0^T \|\nabla u\|_{H^s}^2 dt

+ \sum_{|k|=0}^s \int_0^T \|Q : \partial^k A\|_{L^2}^2 dt + \sum_{|k|=0}^s \int_0^T \|\partial^k \dot{Q} - [\partial^k \Omega, Q]\|_{L^2}^2 dt \leq C_0.
\]

**Theorem 1.2** (Global well-posedness with small initial data). Under the same assumptions as in Theorem 1.1, if we further assume
\[ E^{in} = \|u^{in}\|_{H^s}^2 + L\|\nabla Q^{in}\|_{H^s}^2 + a\|Q^{in}\|_{H^s}^2 \leq \epsilon_0 \]
for some small \( \epsilon_0 > 0 \), depending on the all coefficients, \( s \) and \( d \), then the Cauchy problem (1)-(4) admits a unique global solution \((u, Q)\) satisfying
\[ u \in L^\infty(\mathbb{R}^+; H^s), \quad Q \in L^\infty(\mathbb{R}^+, H^{s+1}), \quad \nabla u, \dot{Q} \in L^2(\mathbb{R}^+; H^s), \]
and subjecting to the following inequality
\[
\sup_{t \geq 0} \left( \|u\|_{H^s}^2 + L\|\nabla Q\|_{H^s}^2 + a\|Q\|_{H^s}^2 \right) + \int_0^\infty \|\nabla u\|_{H^s}^2 dt

+ \sum_{|k|=0}^s \int_0^\infty \|\partial^k \dot{Q} - [\partial^k \Omega, Q]\|_{L^2}^2 dt \leq C_1 E^{in}
\]
for some \( C_1 > 0 \), depending only on the all coefficients, \( s \) and \( d \).

1.6. **Ideas and sketch of proofs.** One of the important assumptions in the above theorems is that the coefficient \( a > 0 \). This captures a regime of physical interest but unfortunately not the most interesting physical regime (which would be for \( a \leq 0 \), meaning the “deep nematic” regime, see [9]). Technically the assumption \( a > 0 \) provides an additional damping effect \( a\dot{Q} \).

Two another important assumptions are \( \bar{\mu}_2 = \mu_2 = 0 \) or \( \beta_4 > \bar{\beta} > 0 \) under \( \bar{\mu} = -\mu_2 \neq 0 \). For the relation (11) in Section 2, namely
\[ H_2 - \mu_1\|\dot{Q}\|_{L^2}^2 = -\beta_1\|Q : A\|_{L^2}^2 - \mu_1\|\dot{Q} - [\Omega, Q]\|_{L^2}^2

+ \mu_2\langle A, [\Omega, Q] \rangle + \frac{1}{2}(\bar{\mu}_2 - \mu_2)\langle A, \dot{Q} \rangle, \]
we need to control the unsigned quantities $\mu_2\langle A, [\Omega, Q]\rangle$ and $\frac{1}{2}(\tilde{\mu}_2 - \mu_2)\langle A, \dot{Q}\rangle$.

Under the setting $\tilde{\mu}_2 = \mu_2$, what we need to dominate is the term $\mu_2\langle A, [\Omega, Q]\rangle$. However, corresponding to the low-regularity norms, the energy and energy dissipative rate are $\|u\|_{L^2}^2 + \|\nabla Q\|_{L^2}^2 + a\|Q\|_{L^2}$ and $\frac{1}{2}\beta_3\|\nabla u\|_{L^2}^2 + \mu_1\|\dot{Q} - [\Omega, Q]\|_{L^2}^2$, respectively, which can not control the term $\mu_2\langle A, [\Omega, Q]\rangle$ with $\mu_2 \neq 0$. In other words, the energy of the system is not dissipative. In this sense, we assume $\tilde{\mu}_2 = \mu_2 = 0$.

Under the setting $\tilde{\mu}_2 = -\mu_2 \neq 0$, the quantity $\mu_2\langle A, [\Omega, Q]\rangle + \frac{1}{2}(\tilde{\mu}_2 - \mu_2)\langle A, \dot{Q}\rangle = -\mu_2\langle A, \dot{Q} - [\Omega, Q]\rangle$ can be bounded by

$$|\mu_2\|\nabla u\|_{L^2}\|\dot{Q} - [\Omega, Q]\|_{L^2} \leq \frac{1}{2}\mu_1\|\dot{Q} - [\Omega, Q]\|_{L^2}^2 + \frac{\mu_2^2}{2\mu_1}\|\nabla u\|_{L^2}^2.$$ 

Thus, it can be absorbed by the energy dissipative rate provided that $\frac{1}{2}\beta_4 > \frac{1}{2}\overline{\beta} = \frac{\mu_2^2}{2\mu_1} > 0$. We also emphasize that the key cancellation $(10)$, i.e., $L\langle \Delta Q, u \cdot \nabla Q \rangle + \langle \text{div}\Sigma_1, u \rangle = 0$ are employed in justifying the local well-posedness with large initial data.

When verifying the global existence under small size of initial data, we need to additionally utilize the molecular viscous effect $L\Delta Q$ and the molecular damping effect $aQ$.

1.7. Organization of this paper. In the next section, we derive a priori estimate of the incompressible Qian-Sheng model. In Section 3, based on the a priori estimate, we prove the local well-posedness with large initial data in which we employ mollifier method. Finally, in Section 4, we prove the global classical solution to the system $(1)$ with small initial data by utilizing the additional dissipative and damping effects.

2. A priori estimates. In this section, we derive the a priori estimate of the system $(1)$. We first define the energy dissipative rate $D(t)$: if $\mu_2 = \tilde{\mu}_2 = 0$, i.e., $(7)$,

$$D(t) = \frac{1}{2}\beta_4\|\nabla u\|_{H^s}^2 + \beta_1 \sum_{|k|=0}^s \|Q : \partial^k A\|_{L^2}^2 + \mu_1 \sum_{|k|=0}^s \|\partial^k \dot{Q} - [\partial^k \Omega, Q]\|_{L^2}^2,$$

and if $\mu_2 = \tilde{\mu}_2 \neq 0$, i.e., $(8)$,

$$D(t) = \frac{1}{2}(\beta_4 - \overline{\beta})\|\nabla u\|_{H^s}^2 + \overline{\beta} \sum_{|k|=0}^s \|Q : \partial^k A\|_{L^2}^2 + \frac{\beta_1}{\sqrt{2}} \sum_{|k|=0}^s \|\partial^k \dot{Q} - [\partial^k \Omega, Q]\|_{L^2}^2,$$

where $\overline{\beta} = \frac{\mu_2^2}{2\mu_1} > 0$.

Lemma 2.1. Let $s > \frac{d}{2} + 1$ ($d = 2, 3$) be any fixed integer and $(u, Q)$ be a sufficiently smooth solution to system $(1)$-$(4)$ on $[0, T]$ for some $T > 0$. Then there is a constant $C > 0$, depending only on the all coefficients, $s$ and $d$, such that for all $t \in [0, T]$, we have

$$\frac{d}{dt}E(t) + D(t) \leq CE(t)(E^{\frac{1}{2}}(t) + E(t) + E^2(t)),$$

where the energy functional $E(t)$ is defined in $(6)$.

Proof. We first deduce the $L^2$-estimate of the system of $(1)$. Taking $L^2$-inner product with $u$ in the first equation of $(1)$, one obtains

$$\frac{1}{2}\frac{d}{dt}\|u\|_{L^2}^2 + \frac{1}{2}\beta_4\|\nabla u\|_{L^2}^2 = \langle \text{div}\Sigma_1, u \rangle + \langle \text{div}\Sigma_2, u \rangle + \langle \text{div}\Sigma_3, u \rangle.$$
From taking $L^2$-inner product with $\dot{Q}$ in the third equation of (1), we derive

$$\frac{1}{2} \frac{d}{dt} (L\|\nabla Q\|^2_{L^2} + a\|Q\|^2_{L^2}) + \mu_1 \|\dot{Q}\|^2_{L^2}$$

$$= L(\Delta Q, u \cdot \nabla Q) + b((Q^2 - \frac{1}{4} \text{tr}(QQ)I_d), \dot{Q}) - c(Q\text{tr}(QQ), \dot{Q})$$

$$+ \frac{\epsilon_2}{2} (A, \dot{Q}) + \mu_1 ([\Omega, Q], \dot{Q}).$$

Hence, we obtain

$$\frac{1}{2} \frac{d}{dt} (\|u\|^2_{L^2} + L\|\nabla Q\|^2_{L^2} + a\|Q\|^2_{L^2}) + \mu_1 \|\dot{Q}\|^2_{L^2} + \frac{1}{2} \beta_4 \|\nabla u\|^2_{L^2}$$

$$= \langle \text{div} \Sigma_1, u \rangle + L(\Delta Q, u \cdot \nabla Q)$$

$$\begin{aligned}
&+ \langle \text{div} \Sigma_2, u \rangle + \langle \text{div} \Sigma_3, u \rangle + \frac{\epsilon_2}{2} (A, \dot{Q}) + \mu_1 ([\Omega, Q], \dot{Q}) \\
&+ b((Q^2 - \frac{1}{4} \text{tr}(QQ)I_d), \dot{Q}) - c(Q\text{tr}(QQ), \dot{Q})
\end{aligned}
$$

(9)

Now we compute the terms $H_1$, $H_2$ and $H_3$ on the right-hand side of the equality (9). Simple calculation tells us that

$$H_1 = 0,$$

(10)

which is the key cancellation.

In order to deal with the quantity $H_2$, we will repeatedly use that $\nabla u = A + \Omega$ and the fact that $\text{tr}(BC) = B : C$ is null for any symmetric matrix $B$ and skew-adjoint matrix $C$ (i.e., $C_{ij} = -C_{ji}$). We first derive

$$-\beta_5 (Q\text{tr}(QA), \nabla u) = -\beta_1 \int_{\mathbb{R}^d} Q_{ij}Q_{kl}A_{kl}(A_{ij} + \Omega_{ij})dx$$

$$= -\beta_1 \int_{\mathbb{R}^d} Q_{ij}Q_{kl}A_{kl}A_{ij}dx - \beta_1 \int_{\mathbb{R}^d} Q_{ij}Q_{kl}A_{kl}\Omega_{ij}dx$$

$$= -\beta_1 \int_{\mathbb{R}^d} (Q : A)^2 dx - \beta_1 \int_{\mathbb{R}^d} (Q : \Omega)(Q : A)dx$$

$$= -\beta_1 \int_{\mathbb{R}^d} (Q : A)^2 dx,$$

where we also use the fact $Q : \Omega = 0$. We then have

$$-\beta_5 (AQ, \nabla u) = -\beta_5 (AQ, \nabla u) + \frac{\epsilon_2}{2} ([\Omega, Q], \nabla u)$$

$$= -\beta_5 \int_{\mathbb{R}^d} \text{tr}(QA(A + \Omega))dx - \beta_6 \int_{\mathbb{R}^d} \text{tr}(AQ(A + \Omega))dx + \frac{\epsilon_2}{2} \int_{\mathbb{R}^d} \text{tr}([\Omega, Q](A + \Omega))dx$$

$$= -\beta_5 \int_{\mathbb{R}^d} \text{tr}((QA + AQ)(A + \Omega))dx - (\beta_6 - \beta_5) \int_{\mathbb{R}^d} \text{tr}(AQ(A + \Omega))dx$$

$$+ \frac{\epsilon_2}{2} \int_{\mathbb{R}^d} \text{tr}([\Omega, Q](A + \Omega))dx$$

$$= -\beta_5 \int_{\mathbb{R}^d} \text{tr}((QA + AQ)A)dx - (\beta_6 - \beta_5) \int_{\mathbb{R}^d} \text{tr}(AQ(A + \Omega))dx$$

$$+ \frac{\epsilon_2}{2} \int_{\mathbb{R}^d} \text{tr}([\Omega, Q](A + \Omega))dx$$

$$= -2\beta_5 \int_{\mathbb{R}^d} \text{tr}(AQAdx) - (\beta_6 - \beta_5) \int_{\mathbb{R}^d} \text{tr}(AQAdx) - (\beta_6 - \beta_5) \int_{\mathbb{R}^d} \text{tr}(AQAdx)$$
\[
+ \frac{\mu_2^2}{2} \int_{\mathbb{R}^d} \text{tr}(A[\Omega, Q])dx
\]
\[
- (\beta_5 + \beta_6) \int_{\mathbb{R}^d} \text{tr}(AQA)dx + \mu_2 \int_{\mathbb{R}^d} \text{tr}(A[\Omega, Q])dx = \mu_2 \int_{\mathbb{R}^d} \text{tr}(A[\Omega, Q])dx ,
\]
where we use the coefficients relations \(\beta_6 + \beta_5 = 0, \beta_6 - \beta_5 = \mu_2\) and
\[
(\beta_6 - \beta_5) \int_{\mathbb{R}^d} \text{tr}(AQ\Omega)dx = -(\beta_6 - \beta_5) \int_{\mathbb{R}^d} \text{tr}(AQ\Omega)dx .
\]
Finally, we calculate
\[
\mu_1([\Omega, Q, \dot{Q}] - \mu_1 ([Q, \dot{Q}], \nabla u)
\]
\[
= \mu_1 \int_{\mathbb{R}^d} (\Omega_{ik} Q_{kj} - Q_{ik} \Omega_{kj})\dot{Q}_{ji} dx - \mu_1 \int_{\mathbb{R}^d} (Q_{ik} \dot{Q}_{kj} - Q_{ik} \dot{Q}_{kj})\partial_j u_i dx
\]
\[
= \mu_1 \int_{\mathbb{R}^d} \text{tr}\{([\Omega, Q] - Q[\Omega, Q])(A + \Omega)\} dx + \mu_1 \int_{\mathbb{R}^d} \text{tr}\{([\Omega, Q] - Q[\Omega, Q])A\} dx
\]
\[
= \mu_1 \int_{\mathbb{R}^d} \text{tr}\{([\Omega, Q] - Q[\Omega, Q])\Omega_{ji\}} dx + \mu_1 \int_{\mathbb{R}^d} \text{tr}\{([\Omega, Q] - Q[\Omega, Q])\Omega_{ji\}} dx
\]
\[
= 2\mu_1 \int_{\mathbb{R}^d} \text{tr}\{([\Omega, Q] - Q[\Omega, Q])\} dx ,
\]
and
\[
\mu_1 ([Q, \Omega, Q], \nabla u) = \mu_1 \int_{\mathbb{R}^d} (Q_{ik} [\Omega, Q]_{kj} - [\Omega, Q]_{ik} Q_{kj})\partial_j u_i dx
\]
\[
= \mu_1 \int_{\mathbb{R}^d} \text{tr}\{([\Omega, Q]Q - Q[\Omega, Q])\} dx
\]
\[
= \mu_1 \int_{\mathbb{R}^d} \text{tr}\{([\Omega, Q]Q - Q[\Omega, Q])\} dx
\]
\[
= \mu_1 \int_{\mathbb{R}^d} \text{tr}\{([\Omega, Q]Q - Q[\Omega, Q])\} dx
\]
\[
= - \mu_1 \| [\Omega, Q]\|^2_{L^2} .
\]
Collecting the all above equalities deduces to
\[
H_2 - \mu_1 \| \dot{Q}\|^2_{L^2} = - \beta_1 \| Q \cdot A\|^2_{L^2} - \mu_1 \| \dot{Q} - [\Omega, Q]\|^2_{L^2}
\]
\[
+ \mu_2 \langle A, [\Omega, Q]\rangle + \frac{1}{2}(\mu_2 - \mu_2)\langle A, \dot{Q}\rangle .
\]
If \(\hat{\mu}_2 = \mu_2 = 0\), we immediately obtain
\[
H_2 - \mu_1 \| \dot{Q}\|^2_{L^2} = - \beta_1 \| Q \cdot A\|^2_{L^2} - \mu_1 \| \dot{Q} - [\Omega, Q]\|^2_{L^2} ,
\]
while if \(\hat{\mu}_2 = -\mu_2 \neq 0\), we have
\[
- \frac{\mu_2}{2} \langle \dot{Q}, \nabla u\rangle + \frac{\mu_2}{2} \langle A, \dot{Q}\rangle = - \frac{\mu_2}{2} \langle \dot{Q}, A\rangle + \frac{\mu_2}{2} \langle A, \dot{Q}\rangle = -\mu_2 (\dot{Q}, A) ,
\]
which means that
\[
H_2 - \mu_1 \| \dot{Q}\|^2_{L^2} = - \beta_1 \| Q \cdot A\|^2_{L^2} - \mu_1 \| \dot{Q} - [\Omega, Q]\|^2_{L^2} - \mu_2 \langle A, \dot{Q} [\Omega, Q]\rangle .
\]
For the term \(H_3\), from utilizing the Hölder inequality and Sobolev embedding, we can easily infer that
\[
H_3 \lesssim \| \nabla Q\|^2_{L^2} \| \dot{Q}\|_{L^2} + \| \nabla Q\|^3_{L^2} \| \dot{Q}\|_{L^2} .
\]
Summarizing all the previous estimates, we get
\[
\frac{1}{2} a(t) \left(\|u\|_{L^2}^2 + L\|\nabla Q\|_{L^2}^2 + a\|Q\|_{L^2}^2 + \frac{\beta_2}{2} \|\nabla u\|_{L^2}^2 \right) + \beta_1 \|Q : A\|_{L^2}^2 + \mu_1 \|\tilde{Q} - [\Omega, Q]\|_{L^2}^2 + \mu_2 \langle A, \tilde{Q} - [\Omega, Q]\rangle 
\leq \|\nabla Q\|_{L^2}\|\tilde{Q}\|_{L^2} + \|\nabla Q\|_{L^2}^2 \|\tilde{Q}\|_{L^2} \tag{12}
\]
for the case \(\tilde{\mu}_2 = -\mu_2 \neq 0\), while for the case \(\tilde{\mu}_2 = \mu_2 = 0\), the last term in the left-hand side of the inequality (12) is zero.

We next turn to calculate the higher order derivatives. First, acting the derivative operator \(\partial^k(1 \leq |k| \leq s)\) on the first equation of (1), and taking \(L^2\)-inner product with \(\partial^k u\) reduce to
\[
\frac{1}{2} \frac{d}{dt} \|\partial^k u\|_{L^2}^2 + \frac{\beta_4}{2} \|\nabla \partial^k u\|_{L^2}^2 = -\langle \partial^k(u \cdot \nabla u), \partial^k u \rangle + \langle \partial^k \text{div}(\Sigma_1 + \Sigma_2 + \Sigma_3), \partial^k u \rangle. \tag{13}
\]
Second, applying the derivative operator \(\partial^k\) for all multi-index \(k \in \mathbb{N}^d\) with \(1 \leq |k| \leq s\) to the third equation of (1), and taking \(L^2\)-inner product with \(\partial^k \tilde{Q}\) yield
\[
\frac{1}{2} \frac{d}{dt}(L\|\nabla \partial^k Q\|_{L^2}^2 + a\|\partial^k Q\|_{L^2}^2) + \mu_1 \|\partial^k \tilde{Q}\|_{L^2}^2 
= L\langle \Delta \partial^k Q, \partial^k(u \cdot \nabla Q) \rangle - a\langle \partial^k Q, \partial^k(u \cdot \nabla Q) \rangle + b\langle \partial^k (Q^2 - \frac{1}{2}\text{tr}(QQ))I_d, \partial^k \tilde{Q}\rangle 
- c\langle \partial^k (Q\text{tr}(QQ)), \partial^k \tilde{Q}\rangle + \frac{\beta_2}{2} \langle \partial^k A, \partial^k \tilde{Q}\rangle + \mu_1 \langle \partial^k([\Omega, Q]), \partial^k \tilde{Q}\rangle. \tag{14}
\]
Then, from combining the above equalities (13) and (14), we deduce that
\[
\frac{1}{2} \frac{d}{dt}(\|\partial^k u\|_{L^2}^2 + L\|\nabla \partial^k Q\|_{L^2}^2 + a\|\partial^k Q\|_{L^2}^2 + \frac{\beta_4}{2} \|\nabla \partial^k u\|_{L^2}^2 + \mu_1 \|\partial^k \tilde{Q}\|_{L^2}^2) 
= -\langle \partial^k(u \cdot \nabla u), \partial^k u \rangle - a\langle \partial^k Q, \partial^k(u \cdot \nabla Q) \rangle \bigg\|_{I_1} 
+ \langle \partial^k \text{div}\Sigma_1, \partial^k u \rangle + L\langle \Delta \partial^k Q, \partial^k(u \cdot \nabla Q) \rangle \bigg\|_{I_2} 
+ \langle \partial^k \text{div}\Sigma_2, \partial^k u \rangle + \langle \partial^k \text{div}\Sigma_3, \partial^k u \rangle + \frac{\beta_2}{2} \langle \partial^k A, \partial^k \tilde{Q}\rangle + \mu_1 \langle \partial^k([\Omega, Q]), \partial^k \tilde{Q}\rangle \bigg\|_{I_3} 
+ b\langle \partial^k (Q^2 - \frac{1}{2}\text{tr}(QQ))I_d, \partial^k \tilde{Q}\rangle - c\langle \partial^k (Q\text{tr}(QQ)), \partial^k \tilde{Q}\rangle \bigg\|_{I_4}. \tag{15}
\]
We now turn to deal with the terms \(I_i\) \((1 \leq i \leq 4)\). We take advantage of the Hölder inequality, the Sobolev embedding inequality and the fact \(\text{div} u = 0\) to get that
\[
I_1 \lesssim \sum_{m_1 + m_2 = k, 1 \leq |m_1| \leq k-1} \|\partial^{m_1} u\|_{L^4}\|\nabla \partial^{m_2} u\|_{L^4}\|\partial^k u\|_{L^2}^2 + \|\partial^k u\|_{L^2} \|\nabla u\|_{L^\infty} \|\partial^k u\|_{L^2} \tag{16}
\]
\[
\lesssim \|\nabla u\|_{H^s} \|u\|_{H^s}^2 + \|\nabla Q\|_{H^s} \|u\|_{H^s} \|Q\|_{H^{s}}. 
\]
As to $I_2$, it is easy to deduce that

\[ I_2 = - \langle \partial \partial^{k} Q_{\alpha \beta}, \partial_{j} u_{i} \partial_{j} \partial^{k} Q_{\alpha \beta} \rangle - \sum_{m_1 + m_2 = k, 1 \leq |m_2| \leq k-1} \langle \Delta \partial^{m_1} Q_{\alpha \beta} \partial_{i} \partial^{m_2} Q_{\alpha \beta}, \partial^{k} u_{i} \rangle \\
- \sum_{m_1 + m_2 = k, 1 \leq |m_2| \leq k-1} \langle \partial^{m_1} u_{i} \partial_{i} \partial^{m_2} Q_{\alpha \beta} + \partial_{j} \partial^{m_1} u_{i} \partial_{i} \partial^{m_2} Q_{\alpha \beta}, \partial_{j} \partial^{k} Q_{\alpha \beta} \rangle \\
- \langle \Delta Q_{\alpha \beta} \partial_{j} \partial^{k} Q_{\alpha \beta}, \partial^{k} u_{i} \rangle \\
\lesssim \| \nabla u \|_{L^\infty} \| \nabla \partial^{k} Q \|^2_{L^2} + \sum_{m_1 + m_2 = k, 1 \leq |m_2| \leq k-1} \| \Delta \partial^{m_1} Q \|_{L^2} \| \nabla \partial^{m_2} Q \|_{L^4} \| \partial^{k} u \|_{L^4} \\
+ \sum_{m_1 + m_2 = k, 1 \leq |m_2| \leq k-1} (\| \partial^{m_1} u \|_{L^\infty} \| \Delta \partial^{m_2} Q \|_{L^2} + \| \nabla \partial^{m_1} Q \|_{L^4} \| \nabla \partial^{m_2} u \|_{L^4}) \| \nabla \partial^{k} Q \|_{L^2} \\
+ \| \Delta Q \|_{L^4} \| \partial^{k} u \|_{L^4} \| \nabla \partial^{k} Q \|_{L^2} \\
\lesssim \| \nabla Q \|_{H^2} \| \nabla u \|_{H^k}, \]  

(17)

from using the Hölder inequality and the Sobolev embedding theory.

We next estimate $I_3$. We observe that it can give us some dissipative structures, which is similar as the derivations of the $L^2$-estimate. More precisely, we can divide $I_3$ into two parts $I_3^c$ and $I_3^m$:

\[ I_3 = I_3^c + I_3^m, \]

where

\[ I_3^c = - \beta_1 \langle Q \text{tr}(\partial^{k} A), \nabla \partial^{k} u \rangle - \beta_5 \langle \partial^{k} A Q, \nabla \partial^{k} u \rangle - \beta_6 \langle Q \partial^{k} A, \nabla \partial^{k} u \rangle \\
+ \frac{\alpha_2}{2} \langle \partial^{k} \Omega, Q \rangle, \nabla \partial^{k} u \rangle - \frac{\alpha_2}{2} \langle \partial^{k} \hat{Q}, \nabla \partial^{k} u \rangle + \frac{\alpha_2}{2} \langle \partial^{k} A, \partial^{k} \hat{Q} \rangle \\
- \mu_1 \langle Q, \partial^{k} \hat{Q} \rangle, \nabla \partial^{k} u \rangle + \mu_1 \langle \partial^{k} \Omega, Q \rangle, \partial^{k} \hat{Q} \rangle - \mu_1 \langle Q, [\partial^{k} \Omega, Q], \nabla \partial^{k} u \rangle, \]

and

\[ I_3^m = \beta_1 \sum_{m_1 + m_2 + m_3 = k, 0 \leq |m_3| \leq k-1} \langle \partial^{m_3} Q \text{tr}(\partial^{m_2} Q \partial^{m_3} A), \nabla \partial^{k} u \rangle \\
- \beta_5 \sum_{m_1 + m_2 = k, 0 \leq |m_1| \leq k-1} \langle \partial^{m_1} A \partial^{m_2} Q, \nabla \partial^{k} u \rangle - \beta_6 \sum_{m_1 + m_2 = k, 0 \leq |m_2| \leq k-1} \langle \partial^{m_1} Q \partial^{m_2} A, \nabla \partial^{k} u \rangle \\
+ \frac{\mu_2}{2} \sum_{m_1 + m_2 = k, 0 \leq |m_1| \leq k-1} \langle \partial^{m_1} \Omega, \partial^{m_2} Q \rangle, \nabla \partial^{k} u \rangle \\
- \mu_1 \sum_{m_1 + m_2 = k, 0 \leq |m_2| \leq k-1} \langle \partial^{m_1} Q, \partial^{m_2} \hat{Q} \rangle, \nabla \partial^{k} u \rangle + \mu_1 \sum_{m_1 + m_2 = k, 0 \leq |m_1| \leq k-1} \langle \partial^{m_1} \Omega, \partial^{m_2} Q \rangle, \partial^{k} \hat{Q} \rangle \]
\[ + \mu_1 \sum_{m_1 + m_2 + m_3 = k, 0 \leq |m_2| \leq k-1} \langle \partial^{m_1} Q, [\partial^{m_2} \Omega, \partial^{m_3} Q], \nabla \partial^k u \rangle. \]

From the similar derivations of the \( L^2 \)-estimates, one can deduce that if \( \tilde{\mu}_2 = \mu_2 = 0 \),

\[ I^c_3 = -\beta_1 \| Q : \partial^k A \|_{L^2}^2 + 2\mu_1 \langle [\partial^k \Omega, Q], \partial^k \dot{Q} \rangle - \mu_1 \| [\partial^k \Omega, Q] \|_{L^2}, \]

and if \( \tilde{\mu}_2 = \mu_2 \neq 0 \),

\[ I^c_5 = -\beta_1 \| Q : \partial^k A \|_{L^2}^2 - \mu_2 \langle [\partial^k \Omega, Q], \partial^k \dot{Q} \rangle \]

\[ + 2\mu_1 \| [\partial^k \Omega, Q], \partial^k \dot{Q} \| - \mu_1 \| [\partial^k \Omega, Q] \|_{L^2}. \]

We next estimate the term \( I^m_3 \) one by one, which can be dominated by the free energy or/and dissipation rate. Thanks to the Hölder inequality and the Sobolev embedding theory, one has

\[ I^m_{31} \lesssim \| \partial^k Q \|_{L^2} \| Q \|_{L^2} \| \nabla u \|_{L^2} \| \nabla \partial^k u \|_{L^2} \]

\[ + \sum_{m_1 + m_2 + m_3 = k, 0 \leq |m_2| \leq k-1} \| \partial^{m_1} Q \|_{L^\infty} \| \partial^{m_2} Q \|_{L^\infty} \| \nabla \partial^{m_3} u \|_{L^2} \| \nabla \partial^k u \|_{L^2} \]

\[ \lesssim (\| Q \|_{H^s} + \| \nabla Q \|_{H^s}) \| \nabla Q \|_{H^s} \| u \|_{H^s} \| \nabla u \|_{H^s}, \]

and

\[ I^m_{32} \lesssim (\| \nabla u \|_{L^s} \| \partial^k Q \|_{L^s} + \sum_{m_1 + m_2 = k, 0 \leq |m_2| \leq k-1} \| \nabla \partial^{m_1} u \|_{L^s} \| \partial^{m_2} Q \|_{L^\infty} ) \| \nabla \partial^k u \|_{L^2} \]

\[ \lesssim \| \nabla Q \|_{H^s} \| u \|_{H^s} \| \nabla u \|_{H^s}, \]

and

\[ I^m_{33} \lesssim (\| \nabla u \|_{L^s} \| \partial^k Q \|_{L^s} + \sum_{m_1 + m_2 = k, 0 \leq |m_2| \leq k-1} \| \nabla \partial^{m_1} u \|_{L^s} \| \partial^{m_2} Q \|_{L^\infty} ) \| \nabla \partial^k u \|_{L^2} \]

\[ \lesssim \| \nabla Q \|_{H^s} \| u \|_{H^s} \| \nabla u \|_{H^s}, \]

and

\[ I^m_{34} \lesssim (\| \nabla u \|_{L^s} \| \partial^k Q \|_{L^s} + \sum_{m_1 + m_2 = k, 0 \leq |m_2| \leq k-1} \| \nabla \partial^{m_1} u \|_{L^s} \| \partial^{m_2} Q \|_{L^\infty} ) \| \nabla \partial^k u \|_{L^2} \]

\[ \lesssim \| \nabla Q \|_{H^s} \| u \|_{H^s} \| \nabla u \|_{H^s}, \]

and

\[ I^m_{35} \lesssim (\| \nabla \partial^k Q \|_{L^2} \| \dot{Q} \|_{L^\infty} + \| \partial^k Q \|_{L^s} \| \nabla \dot{Q} \|_{L^s} + \| \nabla \partial Q \|_{L^s} \| \partial^{k-1} \dot{Q} \|_{L^s} \]

\[ + \| \partial^k Q \|_{L^\infty} \| \nabla \partial^{k-1} \dot{Q} \|_{L^2} ) \| \partial^k u \|_{L^2} \]

\[ + \sum_{m_1 + m_2 = k, 0 \leq |m_2| \leq k-2} (\| \nabla \partial^{m_1} Q \|_{L^s} \| \partial^{m_2} \dot{Q} \|_{L^s} + \| \partial^{m_1} \dot{Q} \|_{L^s} \| \nabla \partial^{m_2} \dot{Q} \|_{L^s} ) \| \partial^k u \|_{L^2} \]

\[ \lesssim (\| Q \|_{H^s} + \| \nabla Q \|_{H^s} ) \| \dot{Q} \|_{H^s} \| u \|_{H^s}, \]
and

\[ I_{37}^m \lesssim (\|\nabla u\|_{L^4} \|\partial^k Q\|_{L^4} + \sum_{m_1 + m_2 = k, \ 1 \leq |m_1| \leq k-1} \|\nabla \partial^{m_1} u\|_{L^2} \|\partial^{m_2} Q\|_{L^\infty}) \|\partial^k \hat{Q}\|_{L^2} \]

\[ \lesssim \|\nabla Q\|_{H^r} \|u\|_{H^s} \|\hat{Q}\|_{H^r}, \]

and

\[ I_{37}^m \lesssim \|\partial^k Q\|_{L^6} \|\nabla u\|_{L^6} \|\nabla \partial^k u\|_{L^2} \]

\[ + \sum_{m_1 + m_2 + m_3 = k, \ 1 \leq |m_3| \leq k-1} \|\partial^{m_1} Q\|_{L^\infty} \|\partial^{m_2} Q\|_{L^\infty} \|\nabla \partial^{m_3} u\|_{L^2} \|\nabla \partial^k u\|_{L^2} \]

\[ \lesssim (\|Q\|_{H^r} + \|\nabla Q\|_{H^r}) \|\nabla Q\|_{H^r} \|u\|_{H^r} \|\nabla u\|_{H^r}. \]

We thereby have the following estimates: if \( \tilde{\mu}_2 = \mu_2 = 0 \), there is a constant \( C > 0 \) such that

\[ I_3 \leq -\beta_1 \|Q : \partial^k A\|_{L^2}^2 + 2\mu_1 (\|\partial^k Q\|_{L^2}^2 + \mu_1 (\|\partial^k Q\|_{L^2}^2 + CI'), \]  \( \tag{18} \)

and if \( \tilde{\mu}_2 = -\mu_2 \neq 0 \), the terms \( I_3 \) can be estimated as

\[ I_3 \leq -\beta_1 \|Q : \partial^k A\|_{L^2}^2 - \mu_2 (\|\partial^k Q\|_{L^2}^2 + \|\partial^k Q\|_{L^2}^2 + \partial^k \hat{Q}) - C I', \]

\[ + 2\mu_1 (\|\partial^k Q\|_{L^2}^2 + \|\partial^k Q\|_{L^2}^2 + \partial^k \hat{Q}) - \mu_1 (\|\partial^k Q\|_{L^2}^2 + CI'), \]  \( \tag{19} \)

where

\[ I' = (\|\nabla Q\|_{H^r} + \|Q\|_{H^s} + \|\nabla Q\|_{H^r}^2) \|u\|_{H^r} \|\nabla u\|_{H^r} \]

\[ + (\|Q\|_{H^s} + \|\nabla Q\|_{H^r}) \|u\|_{H^r} \|\hat{Q}\|_{H^r}. \]

It remains to estimate the term \( I_4 \). One can easily calculate that

\[ b(\partial^k (Q^2 - \frac{1}{4} \text{tr}(QQ) I_d), \partial^k \hat{Q}) \lesssim \sum_{m_1 + m_2 + m_3 = k} \|\partial^{m_1} Q\|_{L^4} \|\partial^{m_2} Q\|_{L^4} \|\partial^k \hat{Q}\|_{L^2} \]

\[ \lesssim \|\nabla Q\|_{H^r}^2 \|\hat{Q}\|_{H^r}, \]

and we similarly deduce that

\[ -c(\partial^k (Q \text{tr}(QQ)), \partial^k \hat{Q}) \]

\[ \lesssim \sum_{m_1 + m_2 + m_3 = k} \|\partial^{m_1} Q\|_{L^4} \|\partial^{m_2} Q\|_{L^4} \|\partial^k \hat{Q}\|_{L^2} \]

\[ \lesssim \|\nabla Q\|_{H^r}^3 \|\hat{Q}\|_{H^r}. \]

As a result, we have

\[ I_4 \lesssim \|\nabla Q\|_{H^r}^2 \|\hat{Q}\|_{H^r} + \|\nabla Q\|_{H^r}^3 \|\hat{Q}\|_{H^r}. \]  \( \tag{20} \)

For the case \( \mu_2 = 0 \) i.e. (7), from substituting the estimates (16), (17), (18), (20) into (15), summing up for all \( 1 \leq |k| \leq s \), and combining the result with \( L^2\)-estimate (12), we infer that

\[ \frac{1}{2} \|u\|_{H^r}^2 + L \|\nabla Q\|_{H^r}^2 + a \|Q\|_{H^s}^2 \]

\[ + \beta_1 \sum_{|k|=0}^{s} \|Q : \partial^k A\|_{L^2}^2 + \mu_1 \sum_{|k|=0}^{s} \|\partial^k \hat{Q} - [\partial^k \Omega, Q]\|_{L^2}^2 \]

\[ \lesssim \|\nabla Q\|_{H^r} \|u\|_{H^r} \|Q\|_{H^r} + \|\nabla Q\|_{H^r}^2 \|\nabla u\|_{H^r} \]

\[ + (\|u\|_{H^r} + \|\nabla Q\|_{H^r} + \|Q\|_{H^s} + \|\nabla Q\|_{H^r}) \|\hat{Q}\|_{H^r} \]

\[ + (\|Q\|_{H^s} + \|\nabla Q\|_{H^r}) \|u\|_{H^r} \|\hat{Q}\|_{H^r}. \]  \( \tag{21} \)
If $\tilde{\mu}_2 = -\mu_2 \neq 0$, i.e. (8), we can estimate that

$$
-\mu_2 (\partial^k A, (\partial^k \tilde{Q} - [\partial^k \Omega, Q]) \leq \frac{1}{2} \bar{\beta} \|\nabla u\|_{L^2}^2 + \frac{\mu_1}{2} \|\partial^k \tilde{Q} - [\partial^k \Omega, Q]\|_{L^2}^2,
$$

(22)

where $\bar{\beta} = \frac{\mu_2}{\mu_1} > 0$. From substituting the estimates (16), (17), (19), (20) into (15), summing up for all $1 \leq |k| \leq s$ and combining the $L^2$-estimate (12) and the bound (22), we derive an inequality similar to (21), in which we replace the last three terms in the left-hand side of (21) by

$$
\frac{1}{2} (\beta_4 - \bar{\beta}) \|\nabla u\|_{H^s}^2 + \beta_1 \sum_{|k|=0}^s \|Q : \partial^k A\|_{L^2}^2 + \frac{\mu_1}{2} \sum_{|k|=0}^s \|\partial^k \tilde{Q} - [\partial^k \Omega, Q]\|_{L^2}^2.
$$

(23)

Therefore, the proof of Lemma 2.1 is finished.

\[\square\]

3. Local well-posedness with large initial data. In this section, we will prove the local well-posedness of (1)-(4) with large initial data, namely, prove Theorem 1.1. We employ mollifier method to achieve our goal.

**Proof of Theorem 1.1.** We first define the mollifying operator

$$
\mathcal{J}_\varepsilon f := \mathcal{F}^{-1} \left( 1_{|\xi| \leq \frac{1}{\varepsilon}} \mathcal{F}(f) \right),
$$

where the symbol $\mathcal{F}$ is the Fourier transform operator and $\mathcal{F}^{-1}$ is its inverse transform. It is easy to verify that the mollifier operator has the property $\mathcal{J}_\varepsilon^2 = \mathcal{J}_\varepsilon$.

Then the approximate system is constructed as follows:

$$
\begin{align*}
\partial_t u^\varepsilon + \mathcal{P}(\mathcal{J}_\varepsilon u^\varepsilon : \mathcal{J}_\varepsilon \nabla u^\varepsilon) - \frac{1}{2} \beta_4 \Delta \mathcal{J}_\varepsilon u^\varepsilon &= -L \nabla \cdot \mathcal{P} \left\{ \mathcal{J}_\varepsilon (\nabla \mathcal{J}_\varepsilon Q^\varepsilon \circ \nabla \mathcal{J}_\varepsilon Q^\varepsilon) \right\} \\
+ \nabla \cdot \mathcal{P} \left\{ \beta_1 \mathcal{J}_\varepsilon (\mathcal{J}_\varepsilon Q^\varepsilon \nabla \mathcal{J}_\varepsilon, \mathcal{J}_\varepsilon Q^\varepsilon \nabla \mathcal{J}_\varepsilon, \mathcal{J}_\varepsilon A^\varepsilon) \right\} + \beta_5 \mathcal{J}_\varepsilon (\mathcal{J}_\varepsilon A^\varepsilon \mathcal{J}_\varepsilon Q^\varepsilon) + \beta_6 \mathcal{J}_\varepsilon (\mathcal{J}_\varepsilon Q^\varepsilon \mathcal{J}_\varepsilon A^\varepsilon) \\
+ \nabla \cdot \mathcal{P} \left\{ \frac{\mu_1}{2} (\mathcal{J}_\varepsilon \mathcal{Q}^\varepsilon - \mathcal{J}_\varepsilon [\mathcal{J}_\varepsilon \Omega^\varepsilon, \mathcal{J}_\varepsilon Q^\varepsilon]) + \mu_1 \mathcal{J}_\varepsilon (\mathcal{J}_\varepsilon Q^\varepsilon - [\mathcal{J}_\varepsilon \Omega^\varepsilon, \mathcal{J}_\varepsilon Q^\varepsilon]) \right\},
\end{align*}
$$

(24)

where

$$
\Omega^\varepsilon := \frac{1}{2} (\nabla u^\varepsilon - \nabla^T u^\varepsilon),
$$

$$
A^\varepsilon := \frac{1}{2} (\nabla u^\varepsilon + \nabla^T u^\varepsilon),
$$

$$
\hat{Q}^\varepsilon := \partial_t Q^\varepsilon + \mathcal{J}_\varepsilon (\mathcal{J}_\varepsilon u^\varepsilon \cdot \nabla \mathcal{J}_\varepsilon Q^\varepsilon),
$$

and $\mathcal{P}$ denotes the Leray projector onto divergence-free vector fields.

By ODE theory, we know that there is a maximal $T_\varepsilon > 0$ such that the approximate system (24) has a unique solution $u^\varepsilon \in C([0, T_\varepsilon); H^k(\mathbb{R}^d))$ and $Q^\varepsilon \in C([0, T_\varepsilon); H^{k+1}(\mathbb{R}^d))$. Notice that the fact $\mathcal{J}_\varepsilon^2 = \mathcal{J}_\varepsilon$, we know $(\mathcal{J}_\varepsilon u^\varepsilon, \mathcal{J}_\varepsilon Q^\varepsilon)$ is also a solution to the approximate system (24). Then by the uniqueness of the solution we know that $(\mathcal{J}_\varepsilon u^\varepsilon, \mathcal{J}_\varepsilon Q^\varepsilon) = (u^\varepsilon, Q^\varepsilon)$. Therefore, the solution $(u^\varepsilon, Q^\varepsilon)$ also solves
the following system

\[
\begin{align*}
\partial_t u^\epsilon + \mathcal{P} \mathcal{J}_\epsilon (u^\epsilon \cdot \nabla u^\epsilon) - \frac{1}{2} \beta_4 \Delta u^\epsilon &= -L \nabla \cdot \left\{ \mathcal{J}_\epsilon (\nabla Q^\epsilon \otimes \nabla Q^\epsilon) \right\} \\
+ \nabla \cdot \mathcal{P} \left\{ \beta_1 \mathcal{J}_\epsilon (Q^\epsilon \text{tr}(Q^\epsilon A^\epsilon)) + \beta_0 \mathcal{J}_\epsilon (A^\epsilon Q^\epsilon) + \beta_0 \mathcal{J}_\epsilon (Q^\epsilon A^\epsilon) \right\} \\
+ \nabla \cdot \mathcal{P} \left\{ \frac{\mu_1}{2} (\dot{Q}^\epsilon - \mathcal{J}_\epsilon (\nabla Q^\epsilon)) + \mu_1 \mathcal{J}_\epsilon (\dot{Q}^\epsilon - (\nabla Q^\epsilon)) \right\}, \\
\text{div} u^\epsilon &= 0, \\
\mu_1 \dot{Q}^\epsilon &= L \Delta Q^\epsilon - a Q^\epsilon + b \mathcal{J}_\epsilon (Q^\epsilon Q^\epsilon) - \text{tr} \left\{ \mathcal{J}_\epsilon (Q^\epsilon Q^\epsilon) \right\} \frac{1}{\beta_2} \\
&- c \mathcal{J}_\epsilon \left\{ Q^\epsilon \text{tr}(Q^\epsilon Q^\epsilon) \right\} + \frac{\mu_2}{2} A^\epsilon + \mu_1 \mathcal{J}_\epsilon [\Omega^\epsilon, Q^\epsilon], \\
(u^\epsilon, Q^\epsilon)(x, t)|_{t=0} &= (\mathcal{J}_\epsilon u^{in}, \mathcal{J}_\epsilon Q^{in})(x).
\end{align*}
\]

From employing the almost same arguments in deriving the a priori estimates in Section 2, we obtain the energy estimate of the approximate system (25) as following:

\[
\frac{d}{dt} E_\epsilon(t) + D_\epsilon(t) \lesssim E_\epsilon(t) \left( E_\epsilon^2(t) + E_\epsilon(t) + E_\epsilon^2(t) \right)
\]

for all \( t \in [0, T_\epsilon) \), where the approximate energy functionals \( E_\epsilon(t) \) and the approximate dissipative rate \( D_\epsilon(t) \) is

\[
D_\epsilon(t) = \frac{1}{2} \beta_4 \| \nabla u^\epsilon \|_{H^s}^2 + \beta_1 \sum_{|k|=0}^s \| Q^\epsilon : \partial^k A^\epsilon \|_{L^2}^2 + \mu_1 \sum_{|k|=0}^s \| \partial^k \dot{Q}^\epsilon - \partial^k \Omega^\epsilon, Q^\epsilon \|_{L^2}^2
\]

for the case \( \bar{\mu}_2 = \mu_2 = 0 \), i.e., (7), and

\[
D_\epsilon(t) = \frac{1}{2} (\beta_2 - \beta) \| \nabla u^\epsilon \|_{H^s}^2 + \beta_1 \sum_{|k|=0}^s \| Q^\epsilon : \partial^k A^\epsilon \|_{L^2}^2
\]

\[
+ \mu_1 \sum_{|k|=0}^s \| \partial^k \dot{Q}^\epsilon - \partial^k \Omega^\epsilon, Q^\epsilon \|_{L^2}^2
\]

for the case \( \bar{\mu}_2 \neq \mu_2 \neq 0 \), i.e., (8), where the constant \( \beta = \frac{\mu_2}{\bar{\mu}_2} > 0 \).

Then by the continuity arguments imply that there is a \( T_\epsilon > 0 \), depending only on all the coefficients, \( s \) and \( d \), such that the energy bound

\[
\sup_{t \in [0, T]} E_\epsilon(t) + \int_0^T D_\epsilon(t) dt \leq C_0
\]

holds uniformly in \( \epsilon > 0 \), where the positive constant \( C_0 \) depends only on all the coefficients, \( s \), \( d \), and initial data \( E^{in} \). Then by compactness arguments, we get functions \( (u, Q) \in \mathbb{R}^d \times S^d_0 \) satisfying \( u \in L^\infty(0, T; H^s) \cap L^2(0, T; H^{s+1}) \) and \( Q \in L^\infty(0, T; H^{s+1}) \), which solves the incompressible non-inertial Qian-Sheng model (1) with initial data (4). Moreover, \( (u, Q) \) satisfies the bound

\[
\sup_{t \in [0, T]} E(t) + \int_0^T D(t) dt \leq C_0.
\]

Then the proof of Theorem 1.1 is finished. \( \square \)
4. Global well-posedness with small initial data. In this section, we will prove the global classical solution to the system (1)-(4) with small initial data. We note that the $H^s$-estimante (21) or (23) does not have enough dissipation to prove the global solution with small initial data. To overcome this difficulty, we shall seek some dissipation of the $Q$. We first introduce the following energy functional $\mathcal{E}_\eta$

$$\mathcal{E}_\eta = \|u\|_{H^s}^2 + L\|\nabla Q\|_{H^s}^2 + (a + \mu_1 \eta)\|Q\|_{H^s}^2,$$

and the dissipative rate $\mathcal{D}_\eta$

$$\mathcal{D}_\eta = \beta_1 \sum_{|k|=0}^s \|Q : \partial^k A\|_{L^2}^2 + \mu_1 \sum_{|k|=0}^s \|\partial^k \dot{Q} - [\partial^k \Omega, Q]\|_{L^2}^2$$

$$+ L\eta \|\nabla Q\|_{H^s}^2 + 2\eta \|Q\|_{H^s}^2 + \frac{1}{2} \beta_4 \|\nabla u\|_{H^s}^2,$$

for the case $\beta_2 = \mu_2 = 0$, or

$$\mathcal{D}_\eta = \beta_1 \sum_{|k|=0}^s \|Q : \partial^k A\|_{L^2}^2 + \mu_1 \sum_{|k|=0}^s \|\partial^k \dot{Q} - [\partial^k \Omega, Q]\|_{L^2}^2$$

$$+ L\eta \|\nabla Q\|_{H^s}^2 + 2\eta \|Q\|_{H^s}^2 + \frac{1}{4}(\beta_4 - \beta) \|\nabla u\|_{H^s}^2,$$

for the case $\beta_2 = - \mu_2 \neq 0$, where $\eta \in (0, \frac{1}{2}]$ to be determined and $\beta = \frac{\mu_4}{\mu_1} > 0$.

Then we derive the following Lemma in order to prove the global existence.

**Lemma 4.1.** Assume that $(u, Q)$ is the solution to the system (1) on $[0, T]$ constructed in Theorem 1.1. Then there exists a constant $\eta_0$, depending only on the all coefficients, $s$ and $d$, such that

$$\frac{1}{2} \frac{d}{dt} \mathcal{E}_\eta + \mathcal{D}_\eta \leq CD_\eta \sum_{l=1}^2 \mathcal{E}_\eta^l$$

holds for all $0 < \eta \leq \eta_0$ and $t \in [0, T]$, where the positive constant $C$ depends only on the all coefficients, $s$ and $d$.

**Proof.** For all $0 \leq |k| \leq s$, from acting $\partial^k$ on the third equation of the system (1) and taking $L^2$-inner product with $\partial^k Q$, we deduce that

$$\frac{\mu_1}{2} \frac{d}{dt} \|\partial^k Q\|_{L^2}^2 + L \|\nabla \partial^k Q\|_{L^2}^2 + a \|\partial^k Q\|_{L^2}^2$$

$$= - \mu_1 \langle \partial^k (u \cdot \nabla Q), \partial^k Q \rangle + b \langle \partial^k (Q^2 - \frac{1}{4} \text{tr}(QQ) I_d), \partial^k Q \rangle$$

$$- c \langle \partial^k (Q \text{tr}(QQ)), \partial^k Q \rangle + \frac{\beta_4}{2} \langle \partial^k A, \partial^k Q \rangle + \mu_1 \langle \partial^k \Omega, Q \rangle, \partial^k Q \rangle. \tag{26}$$

We now estimate the terms in the right-hand side of (26) term by term. It is easy to derive that

$$- \langle \partial^k (u \cdot \nabla Q), \partial^k Q \rangle = \sum_{m_1 + m_2 = k, \atop |m_1| \geq 1} \langle \partial^{m_1} u \nabla \partial^{m_2} Q, \partial^k Q \rangle$$

$$\lesssim \sum_{m_1 + m_2 = k, \atop |m_1| \geq 1} \|\partial^{m_1} u\|_{L^4} \|\nabla \partial^{m_2} Q\|_{L^4} \|\partial^k Q\|_{L^4} \lesssim \|\nabla u\|_{H^s} \|\nabla Q\|_{H^s} \|Q\|_{H^s}, \tag{27}$$

where we utilize the Hölder inequality and the fact $\text{div} u = 0.$
For the term $b(\partial^k Q^2 - \frac{1}{3} \text{tr}(QQ) I_d, \partial^k Q)$, we estimate that
\[
b(\partial^k Q^2 - \frac{1}{3} \text{tr}(QQ) I_d, \partial^k Q) \
\lesssim \sum_{m_1 + m_2 = k} ||\partial^{m_1} Q||_{L^2} ||\partial^{m_2} Q||_{L^2} ||\partial^k Q||_{L^2} \lesssim ||\nabla Q||_{H^s}^2 ||Q||_{H^s}.
\]
(28)

From using Hölder inequality and the Sobolev embedding theory, one easily deduces that
\[
- c(\partial^k (Q \text{tr}(QQ)), \partial^k Q) \
\lesssim \sum_{m_1 + m_2 + m_3 = k} ||\partial^{m_1} Q||_{L^2} ||\partial^{m_2} Q||_{L^2} ||\partial^{m_3} Q||_{L^2} ||\partial^k Q||_{L^2} \lesssim ||\nabla Q||_{H^s}^3 ||Q||_{H^s}.
\]
(29)

and
\[
\mu_2 (\partial^k A, \partial^k Q) \lesssim |\tilde{\mu}_2||\nabla u||_{H^s} ||Q||_{H^s}.
\]
(30)

and
\[
\mu_1 (\partial^k ([\Omega, Q]), \partial^k Q) \
\lesssim ||\partial^k \Omega||_{L^2} ||Q||_{L^2} ||\partial^k Q||_{L^2} + \sum_{m_1 + m_2 = k, m_3 \geq 1} ||\partial^{m_1} \Omega||_{L^2} ||\partial^{m_2} Q||_{L^2} ||\partial^k Q||_{L^2} \lesssim ||\nabla u||_{H^s} ||Q||_{H^s} + ||\nabla Q||_{H^s}.
\]
(31)

Noticing that
\[
|\tilde{\mu}_2||\nabla u||_{H^s} ||Q||_{H^s} \lesssim \frac{1}{2} ||Q||_{H^s}^2 + C\tilde{\mu}_2^2 ||\nabla u||_{H^s}^2.
\]
(32)

and plugging the inequalities (27), (28), (29), (30), (31) and (32) into (26), we have
\[
\frac{\mu_1}{2} \frac{d}{dt} ||\partial^k Q||_{H^s}^2 + L||\nabla \partial^k Q||_{H^s}^2 + \frac{\mu_1}{4} ||\partial^k Q||_{H^s}^2 - C\tilde{\mu}_2^2 ||\nabla u||_{H^s}^2 \
\lesssim (||\nabla Q||_{H^s}^2 + ||\nabla Q||_{H^s}) ||Q||_{H^s} + ||\nabla u||_{H^s} ||Q||_{H^s} + ||\nabla Q||_{H^s}).
\]
(33)

From choosing $\eta_0 = \frac{1}{2} \min \{ \frac{\beta_4}{2C\eta_2}, \frac{\beta_4 - \beta}{2C\eta_2}, 1 \} \in (0, \frac{1}{2})$, multiplying the inequality (33) by $\eta \in (0, \eta_0]$, and adding them to the inequality (21), we derive that
\[
\frac{1}{2} \frac{d}{dt} (||u||_{H^s}^2 + L||\nabla Q||_{H^s}^2 + (a + \mu_1 \eta)||Q||_{H^s}^2) + \beta_1 \sum_{|k|=0}^s ||Q : \partial^k A||_{L^2}^2 \
+ (\frac{1}{2} \beta_4 - C\tilde{\mu}_2^2 \eta)||\nabla u||_{H^s}^2 + \mu_1 \sum_{|k|=0}^s ||\partial^k \tilde{Q} - [\partial^k \Omega, Q]||_{L^2}^2 \lesssim C \Delta \eta \sum_{l=1}^2 \mathcal{E}_l^2 (t)
\]
(34)

for the case $\tilde{\mu}_2 = \mu_2 = 0$. If $\tilde{\mu}_2 = -\mu_2 \neq 0$, we can gain an inequality similar to (34), just replacing the last five terms on the left-hand side of the inequality (34) by
\[
\beta_1 \sum_{|k|=0}^s ||Q : \partial^k A||_{L^2}^2 + (\frac{1}{2} (\beta_4 - \tilde{\beta}) - C\tilde{\mu}_2^2 \eta)||\nabla u||_{H^s}^2 \
+ \frac{1}{2} \mu_1 \sum_{|k|=0}^s ||\partial^k \tilde{Q} - [\partial^k \Omega, Q]||_{L^2}^2 + L\eta ||\nabla Q||_{H^s}^2 + \frac{3a}{2} \eta ||Q||_{H^s}^2,
\]
where the constant $C$ depends on the all coefficients, $s$ and $d$. We thereby complete the proof of Lemma 4.1.

**Proof the Theorem 1.2: Global well-posedness.** At the end of this paper, we use the continuity arguments to prove the global-in-time solution. It is obvious that

$$C_2 E(t) \leq \mathcal{E}_\eta(t) \leq C^2 E(t)$$

for some positive constants $C_2, C^2 > 0$, and

$$\mathcal{D}_\eta(t) \geq \frac{1}{4} \beta_4 \| \nabla u \|_{H}^2,$$

for the case $\mu_2 = \mu_2 = 0$, or

$$\mathcal{D}_\eta(t) \geq \frac{1}{4} (\beta_4 - \bar{\beta}) \| \nabla u \|_{H}^2,$$

for the case $\mu_2 = -\mu_2 \neq 0$. Moreover, we have $C_2 E^{in} \leq \mathcal{E}_\eta(0) \leq C^2 E^{in}$.

We then define

$$T^* = \sup \{ \tau > 0 ; \sup_{t \in [0,\tau]} C \sum_{l=1}^{2} \mathcal{E}_\eta^2(t) \leq \frac{1}{2} \} \geq 0,$$

where the constant $C > 0$ is given in Lemma 4.1. We then choose the sufficient small positive number $\epsilon_0 = \frac{1}{C^2} \min \{ 1, \frac{1}{16C} \} > 0$. If the initial energy $E^{in} \leq \epsilon_0$, we can deduce that

$$C \sum_{l=1}^{2} \mathcal{E}_\eta^2(0) \leq \frac{1}{2} < \frac{1}{2}.$$

Then, from taking advantage of the continuity of the energy functional $\mathcal{E}_\eta(t)$, one derives that $T^* > 0$. Thus,

$$\frac{1}{2} \frac{d}{dt} \mathcal{E}_\eta(t) + [1 - C \sum_{l=1}^{2} \mathcal{E}_\eta^2(t)] \mathcal{D}_\eta(t) \leq 0$$

holds for all $t \in [0, T^*)$, which also implies that we have $\mathcal{E}_\eta(t) \leq \mathcal{E}_\eta(0) \leq C^2 E^{in}$ for all $t \in [0, T^*)$. Then we can derive that $\sup_{t \in [0, T^*)} \{ C \sum_{l=1}^{2} \mathcal{E}_\eta^2(t) \} \leq \frac{1}{4}$.

Based on the above analysis, we claim that $T^* = +\infty$. Otherwise, the continuity of the energy $\mathcal{E}_\eta(t)$ implies that there exists a sufficiently small positive $\epsilon > 0$ such that

$$\sup_{t \in [0, T^* + \epsilon]} \{ C \sum_{l=1}^{2} \mathcal{E}_\eta^2(t) \} \leq \frac{3}{8} < \frac{1}{2},$$

which contradicts to the definition of $T^*$. As a consequence, we get

$$\sup_{t \geq 0} \mathcal{E}_\eta(t) + \int_{0}^{\infty} \mathcal{D}_\eta(t) dt \leq C_1 E^{in},$$

where the positive constant $C_1$ depends only on the all coefficients, $s$ and $d$. Thus we complete the proof of Theorem 1.2.

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