A WEAK $L^2$ ESTIMATE FOR A MAXIMAL DYADIC SUM OPERATOR ON $\mathbb{R}^n$

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Abstract. Lacey and Thiele have recently obtained a new proof of Carleson’s theorem on almost everywhere convergence of Fourier series. This paper is a generalization of their techniques (known broadly as time-frequency analysis) to higher dimensions. In particular, a weak-type (2,2) estimate is derived for a maximal dyadic sum operator on $\mathbb{R}^n$, $n > 1$. As an application one obtains a new proof of Sjölin’s theorem on weak $L^2$ estimates for the maximal conjugated Calderón-Zygmund operator on $\mathbb{R}^n$.

1. Introduction

In 1966, Carleson [1] proved his celebrated theorem on almost everywhere convergence of Fourier series of square integrable functions on $\mathbb{R}$. This was followed by a new proof given by C. Fefferman [2] in 1973. The techniques used by C. Fefferman have become known as time-frequency analysis and have found wide application in harmonic analysis in recent years. In particular, Lacey and Thiele [3], [4] have refined and extended these ideas in their pioneering work on the bilinear Hilbert transform on $\mathbb{R}$. In 2000, they obtained a new proof of Carleson’s theorem [5] in which these techniques play a crucial role. These powerful techniques stem from interaction of extremely deep ideas which include delicate orthogonality estimates, combinatorics, and quasi-orthogonal decompositions well-localized in both time and space. It is the goal of this paper to extend the techniques of time-frequency analysis of [5] to higher dimensions.

The main result of this paper is a weak-type $L^2$ estimate for a maximal dyadic sum operator in $\mathbb{R}^n$, $n > 1$. In dimension one, this operator may be
thought of as a linearized and discretized version of the Carleson operator $\mathcal{C}$,

$$\mathcal{C}f(x) := \sup_N \left| \int_{-\infty}^{N} \hat{f}(\xi)e^{2\pi ix\xi} d\xi \right|.$$ 

The main point of Lacey and Thiele’s proof is to show that the discretized operator satisfies a weak-type $(2,2)$ estimate. This in turn implies a similar estimate for $\mathcal{C}$, which is a key ingredient in proving that the Fourier series of a square-integrable function on the circle converges almost everywhere.

We introduce a higher dimensional analogue of the linearized and discretized Carleson operator and adapt the methodology of Lacey and Thiele to prove that this operator maps $L^2(\mathbb{R}^n)$ to $L^{2,\infty}(\mathbb{R}^n)$. One of the distinguishing aspects of our proof is the introduction of an ordering of points in $\mathbb{R}^n$ which allows us to organize the higher dimensional rectangles and thus control the large sums that appear in the operator. Unlike the situation in dimension one, the mapping property above does not lead to an almost everywhere convergence result in higher dimensions. However, it gives as a corollary a result of Sjölin on the weak $L^2$ boundedness of the maximal conjugated Calderón-Zygmund operator on $\mathbb{R}^n$.

The proof is divided into seven sections. The first section explains the notation and terminology and gives the statement of the main theorem. The second section lists the main ingredients of the proof and the argument that binds them together. The subsequent four sections are devoted to proving the different lemmas needed in the main argument. The final section provides a new proof of Sjölin’s theorem as an application of our main result.

2. Main Theorem

Time-frequency analysis provides the crucial set of ideas in the recent progress made in the understanding of Carleson’s theorem. In this type of analysis one heavily uses the structure of dyadic intervals. A dyadic interval has the form $[m2^k, (m+1)2^k)$ where $k$ and $m$ are integers and $k$ is called the scale. A dyadic cube $I \subset \mathbb{R}^n$ is of the form

$$\prod_{j=1}^{n} I^j = \prod_{j=1}^{n} [m_j2^k, (m_j+1)2^k),$$

where $k$ and $m_j$ are integers for all $j = 1, 2, \cdots, n$. We easily see that the $n$-dimensional volume is given by $|I| = 2^{nk}$. Let $c(I) = (c(I^1), \cdots, c(I^n))$ denote the center of $I$, and for $a > 0$, $aI$ will denote the cube with the same center as $I$ and whose volume is $a^n|I|$.

Consider the time-frequency plane in $2n$ dimensional space with points $(x, \xi)$, where $x$ denotes the time coordinate in $\mathbb{R}^n$ and $\xi$ denotes the frequency coordinate in $\mathbb{R}^n$. A “rectangle” in the time-frequency plane is the cross product of a dyadic cube from the time plane and a dyadic cube from the
frequency plane. To be more precise, for a rectangle, \( p \), the projection onto the time plane will be denoted by \( I_p \), and its projection onto the frequency plane will be denoted by \( \omega_p \). We will denote by \( D \) the set of rectangles \( p = I_p \times \omega_p \) such that \( |I_p| |\omega_p| = 1 \). An element of \( D \) will be called a tile.

As mentioned earlier, it is important for the higher dimensional version of our time-frequency analysis to introduce an ordering in \( \mathbb{R}^n \) that will play a role analogous to the linear ordering on \( \mathbb{R} \). This is especially relevant in a certain selection scheme used in Section 5 in analogy with the work of Lacey and Thiele. Although the choice of ordering is not unique (we will mention an alternative in Section 5), we find it convenient to work with the lexicographical order defined as follows. Given \( \mathbf{a} = (a_1, a_2, \ldots, a_n), \mathbf{b} = (b_1, b_2, \ldots, b_n) \in \mathbb{R}^n \),

\[
\mathbf{a} < \mathbf{b} \iff \begin{cases} a_1 < b_1 \\ a_1 = b_1, a_2 < b_2 \\ \vdots \\ a_1 = b_1, a_2 = b_2, \ldots, a_{n-1} = b_{n-1}, a_n < b_n, \end{cases}
\]

where the right hand side above is to be read with Boolean “or” standard.

For a tile \( p \) with \( \omega_p = \omega_1^p \times \omega_2^p \times \ldots \times \omega_n^p \), we can divide each dyadic interval \( \omega_i^p \) into two parts. In other words, for \( j = 1, 2, \ldots, n \), we get

\[
\omega_i^p = (\omega_i^p \cap (-\infty, c(\omega_i^p)) \cup (\omega_i^p \cap [c(\omega_i^p), \infty))).
\]

Then \( \omega_p \) can be decomposed into \( 2^n \) subcubes formed from all combinations of cross products of these half intervals. We number these subcubes using the lexicographical order on the centers and denote the subcubes by \( \omega_p(i) \) for \( i = 1, 2, \ldots, 2^n \). A tile \( p \) is then the union of \( 2^n \) semi-tiles given by \( p(i) = I_p \times \omega_p(i) \) for \( i = 1, 2, \ldots, 2^n \).

Let us define translation, modulation, and dilation operators by

\[
T_\mathbf{y} f(x) := f(x - \mathbf{y}) \\
M_\mathbf{q} f(x) := f(x)e^{2\pi i \mathbf{q} \cdot \mathbf{x}} \\
D_\lambda f(x) := \lambda^{-n/4} f(\lambda^{-1} x), \lambda > 0.
\]

Note that if we set \( q = 2 \), these operators are isometries on \( L^2(\mathbb{R}^n) \). We fix a Schwartz function \( \phi \) such that \( \hat{\phi} \) is real, nonnegative, supported in the cube \([-1/10, 1/10]^n\) and equal to 1 on the cube \([-9/100, 9/100]^n\). For a tile \( p \in D \) and \( x \in \mathbb{R}^n \) we define

\[
\phi_p(x) = M_{c(\omega_p(1))} T_{c(I_p)} D_{|I_p|^{1/n}}^2 \phi(x).
\]

Using the following definition of the Fourier transform

\[
\hat{f}(\mathbf{\xi}) = \int_{\mathbb{R}^n} f(x)e^{-2\pi i \mathbf{x} \cdot \mathbf{\xi}} \, dx,
\]
one easily can see that
\begin{equation}
\hat{\phi}_p(\xi) = T_{c(\omega_p(1))} M_{-c(I_p)} D_{|\omega_p|^{1/n}} \hat{\phi}(\xi).
\end{equation}
Equation (2.1) tells us that for each \( p \) the function \( \phi_p \) is well localized in time with most of its mass in \( I_p \), while equation (2.2) tells us that \( \hat{\phi}_p \) is supported in \( \frac{1}{2}\omega_p(1) \). Note also that the \( \phi_p \) have the same \( L^2(\mathbb{R}^n) \) norm.

Let \( m \) be a multiplier in \( C^\infty(\mathbb{R}^n \setminus \{0\}) \) which is homogeneous of degree 0, and define
\[
\left( \psi_p^\zeta \right)^\sim(\xi) = m(\xi - \zeta) \hat{\phi}_p(\xi),
\]
where \( \zeta \) is contained in \( \omega_{p(r)} \) for some fixed \( r \in \{2, 3, \ldots, 2^n\} \). Note that we have the following fact for all \( \zeta \in \omega_{p(r)} \):
\[
|\psi_p^\zeta(x)| \leq C_\nu |I_p|^{-1/2} \left( 1 + \frac{|x - c(I_p)|}{|I_p|^{1/n}} \right)^{-\nu},
\]
where \( \nu \) is a large integer whose value may vary at different places in the proof. To see this fact we write
\[
\psi_p^\zeta(x) = \int_{\mathbb{R}^n} e^{2\pi ip \cdot \xi} \hat{\phi}_p(\xi) m(\xi - \zeta) \, d\xi
\]
\[
= \int_{\mathbb{R}^n} e^{2\pi ip \cdot \xi} T_{c(\omega_p(1))} M_{-c(I_p)} D_{|\omega_p|^{1/n}} \hat{\phi}(\xi) m(\xi - \zeta) \, d\xi
\]
\[
= e^{2\pi ic(I_p) \cdot \omega_p(1)} \int_{\mathbb{R}^n} e^{2\pi ip \cdot \xi} (x - c(I_p)) \cdot (x - c(I_p)) |I_p|^{-1/2} \hat{\phi}(\xi) \, d\xi
\]
\[
\times \int_{\mathbb{R}^n} e^{2\pi ip \cdot \xi - c(I_p)} |I_p|^{-1/2} \hat{\phi}_p(|I_p|^{1/n} (\xi - c(\omega_p(1)))) m(\xi - \zeta) \, d\xi.
\]
Making the change of variable
\[
\xi' = |I_p|^{1/n} (\xi - c(\omega_p(1)))
\]
in the above integral, we obtain
\[
\psi_p^\zeta(x) = e^{2\pi i c(\omega_p(1)) \cdot c(\omega_p(1))} \int_{\mathbb{R}^n} e^{2\pi i (\xi' - (x - c(I_p)) |I_p|^{-1/2} \hat{\phi}(\xi')}
\times \int_{\mathbb{R}^n} e^{2\pi i \xi' \cdot (x - c(I_p)) |I_p|^{-1/2} \hat{\phi}_p(|I_p|^{1/n} (\xi' - c(\omega_p(1))))} m(\xi' + |I_p|^{1/n} (c(\omega_p(1)) - \zeta)) \, d\xi'.
\]
where the second equality is obtained using the fact that \( m \) is homogeneous of degree 0. Since \( \zeta \in \omega_{p(r)} \) and \( \xi \in \frac{1}{2}\omega_p(1) \), we have
\[
|\zeta - \xi| \gtrsim |\omega_p|^{1/n},
\]
from which it follows that
\[
|\xi' + |I_p|^{1/n} (c(\omega_p(1)) - \zeta)| \gtrsim 1.
\]
Thus all the derivatives of \( m(\xi' + |P|^{|1/n|}(c(\omega_{p(1)}) - \xi)) \) are bounded. A standard integration by parts argument finishes the proof.

Using the following definition for the inner product

\[
\langle f, g \rangle = \int_{\mathbb{R}^n} f(x)g(x) \, dx,
\]
given \( \xi \in \mathbb{R}^n \) and \( f \in L^2(\mathbb{R}^n) \), we define an operator

\[
B^\xi_{\eta}f(\cdot) = \sum_{p \in D} \langle f, \phi_p \rangle \psi^\xi_{\eta}(\cdot)1_{\omega_p(r)}(\eta).
\]

**Theorem 1.** There exists a constant \( C \), depending only on dimension, so that for all \( f \in L^2(\mathbb{R}^n) \) and \( r \in \{2, 3, \ldots, 2^n\} \)

\[
\| \sup_{\zeta \in \mathbb{R}^n} |B^\xi_{\zeta}f| \|_{L^2, \infty(\mathbb{R}^n)} \leq C\|f\|_{L^2(\mathbb{R}^n)}.
\]

To prove the theorem, we will work with a linearized version of the operator. Consider a measurable function \( x \to N(x) = (N_1(x), N_2(x), \ldots, N_n(x)) \) from \( \mathbb{R}^n \) to \( \mathbb{R}^n \) and define a linear operator

\[
B^\xi_{N}(x) := B^\xi_{N(x)}(x) = \sum_{p \in D} \langle f, \phi_p \rangle \psi^\xi_{N(x)}(x)(1_{\omega_p(r)} \circ N)(x).
\]

To prove (2.3) it will suffice to show that there exists a constant \( C > 0 \) such that for all \( f \in L^2(\mathbb{R}^n) \)

\[
\sup_{N: R^n \to R^n} \|B^\xi_{N}f\|_{L^2, \infty(\mathbb{R}^n)} \leq C\|f\|_{L^2(\mathbb{R}^n)},
\]

where the supremum is taken over all measurable functions \( N \) on \( \mathbb{R}^n \).

By duality we will show that the adjoint operator

\[
g \to \sum_{p \in D} \langle ((1_{\omega_p(r)} \circ N)\psi^N_p, g) \phi_p
\]

maps \( L^{2,1}(\mathbb{R}^n) \) into \( L^2(\mathbb{R}^n) \) with bounds independent of the measurable function \( N \). Since \( L^{2,1}(\mathbb{R}^n) \) is a Lorentz space, it suffices to show that the dual operator maps \( L^{2,1}(\mathbb{R}^n) \cap \{1_E : E \subset \mathbb{R}^n, E \text{ measurable}, |E| < \infty\} \) into \( L^2(\mathbb{R}^n) \). Hence, we need to show

\[
\| \sum_{p \in D} \langle ((1_{\omega_p(r)} \circ N)\psi^N_p, 1_E) \phi_p \|_{L^2(\mathbb{R}^n)} \leq C|E|^{1/2}.
\]

By duality, (2.5) is equivalent to

\[
| \sum_{p \in D} \langle ((1_{\omega_p(r)} \circ N)\psi^N_p, 1_E) \phi_p, f \rangle | \leq C|E|^{1/2},
\]

for all Schwartz functions \( f \) with \( L^2 \) norm one. We will further restrict the sum to an arbitrary finite subset \( P \) of \( D \).
Now for all integers $j$ we have the identity
\[
\sum_{p \in \mathcal{P}} |\langle (1_{\omega_p(r)} \circ N)\psi_p^N, 1_E \rangle \phi_p, f| = 
2^{-\frac{m}{j}} \sum_{u \in p^{\prime}(j)} |\langle (1_{\omega_u(r)} \circ N_j)\psi_u^N, 1_{2^j \otimes E} \rangle \phi_u, 2^{-\frac{m}{j}} f(2^{-j}(\cdot))|,
\]
where for any set $A$ we define $2^j \otimes A = \{2^j y : y \in A\}$, $N_j(x) = 2^{-j}N(2^{-j}x)$, and $\mathcal{P}(j) = \{(2^j \otimes I_p) \times (2^{-j} \otimes \omega_p) : p \in \mathcal{P}\}$. By picking $j$ so that $1 \leq 2^{jn}|E| \leq 2$, we can absorb $|E|$ into the constant on the right hand side of (2.6). Finally we note that the left hand side of (2.6) can be rewritten so that the estimate we need to show now becomes
\[
(2.7) \quad \sum_{p \in \mathcal{P}} |\langle 1_{E \cap N^{-1}[\omega_p(r)]} \psi_p^N, \phi_p, f \rangle| \leq C,
\]
for all Schwartz functions $f$ with $L^2$ norm one, measurable functions $N$, measurable sets $E$ with $|E| \leq 1$, and all finite subsets $\mathcal{P}$ of $\mathcal{D}$. For the rest of the paper we fix $f$, $N$, and $E$ in this manner. By $N^{-1}[\omega_p(r)]$ we mean $\{x : N(x) \in \omega_p(r)\}$.

3. Main Argument

We now set up some tools that we will use throughout the rest of the paper. Define a partial order $<$ on the set of tiles $\mathcal{D}$ by setting
\[
p < p' \iff I_p \subset I_{p'} \quad \text{and} \quad \omega_p \subset \omega_{p'},\n\]
We have the property that if two tiles $p, p' \in \mathcal{D}$ intersect, then either $p < p'$ or $p' < p$. To see this, observe that dyadic cubes have the property that if two of them intersect, then one is contained in the other. This extends from the same property for dyadic intervals in dimension one. Now, suppose two tiles $p$ and $p'$ in $\mathcal{D}$ intersect, and without loss of generality let $|I_p| \leq |I_{p'}|$. Then $p$ and $p'$ intersect in both the time and frequency components, i.e. $I_p \cap I_{p'} \neq \emptyset$, $\omega_p \cap \omega_{p'} \neq \emptyset$. From size considerations, one obtains that $I_p \subset I_{p'}$ and $\omega_{p'} \subset \omega_p$, hence $p < p'$. A consequence of this property is that for a finite set of tiles $\mathcal{P}$, all maximal elements of $\mathcal{P}$ under $<$ must be disjoint sets.

A finite set of tiles $\mathcal{T}$ is called a tree if there exists a tile $t \in \mathcal{D}$ such that $p < t$ for all $p \in \mathcal{T}$. We call $t$ the top of the tree $\mathcal{T}$ and denote it by $p_T = I_T \times \omega_T$. Note that the top is unique but not necessarily an element of the tree. Another useful observation is that any finite set of tiles $\mathcal{P}$ can be written as a union of trees. Consider all maximal elements of $\mathcal{P}$ under $\subset$. Then a nonmaximal element $p \in \mathcal{P}$ must be less than, under “$\subset$”, some maximal element $t \in \mathcal{P}$ which places $p$ in the tree with top $t$. For $i \in \{1, 2, \ldots, 2^n\}$, we call a tree an $i$-tree, denoted by $\mathcal{T}^i$, if $\omega_{\mathcal{T}^i} \subset \omega_{p(i)}$ for all $p \in \mathcal{T}^i$. Observe that any tree can be written as the disjoint union of $i$-trees. Also for fixed $i_0$, and $p, p' \in \mathcal{T}^{i_0}$,
the subcubes $\omega_{p(i)}$ and $\omega_{p'(i)}$ are pairwise disjoint and disjoint from $\omega_{T(i)}$ for all $i \in \{1, 2, \ldots, 2^n\} \setminus \{i_0\}$.

For $p \in \mathbf{D}$, define the mass of $\{p\}$ as

$$M(\{p\}) = \sup_{p < u} \int_{E \cap N^{-1}[\omega_u]} \frac{|I_u|^{-1}}{(1 + \frac{|x-c(I_u)|}{|I_u|^{1/n}})^{10n}} dx.$$ 

We can then define the mass of a finite set of tiles $\mathbf{P}$ to be

$$M(\mathbf{P}) = \sup_{p \in \mathbf{P}} M(\{p\}).$$

Note that the mass of any set of tiles is at most one since by a change of variables

$$M(\mathbf{P}) \leq \int_{\mathbb{R}^n} \frac{1}{(1 + |x|)^{10n}} dx \leq 1.$$ 

The energy, depending on $r$, of a finite set of tiles $\mathbf{P}$ is defined

$$E(\mathbf{P}) = \sup_{T_r \in \mathbf{P}} \left( |I_{T_r}|^{-1} \sum_{p \in T_r} |\langle f, \phi_p \rangle|^2 \right)^{1/2}.$$ 

Recall that $r \in \{2, 3, \ldots, 2^n\}$ is fixed and $f$ is a fixed Schwartz function of $L^2(\mathbb{R}^n)$ norm one. The following three lemmata will provide the main steps in proving the theorem, and their proofs will be shown in the next four sections of the paper.

**Lemma 1.** There exists a constant $C_1$ such that for any finite set of tiles $\mathbf{P}$ there is a subset $\mathbf{P}'$ of $\mathbf{P}$ such that

$$(3.1) \quad M(\mathbf{P} \setminus \mathbf{P}') \leq \frac{1}{4} M(\mathbf{P})$$

and $\mathbf{P}'$ is the union of trees $T_j$ satisfying

$$(3.2) \quad \sum_j |I_{T_j}| \leq \frac{C_1}{M(\mathbf{P})}.$$ 

**Lemma 2.** There exists a constant $C_2$ such that for any finite set of tiles $\mathbf{P}$ there is a subset $\mathbf{P}''$ of $\mathbf{P}$ such that

$$(3.3) \quad E(\mathbf{P} \setminus \mathbf{P}'') \leq \frac{1}{2} E(\mathbf{P})$$

and $\mathbf{P}''$ is the union of trees $T_j$ satisfying

$$(3.4) \quad \sum_j |I_{T_j}| \leq \frac{C_2}{E(\mathbf{P})^2}.$$ 

**Lemma 3.** (The Tree Inequality) There exists a constant $C_3$ such that for all trees $T$

$$(3.5) \quad \sum_{p \in T} |\langle 1_{E \cap N^{-1}[\omega_{p(r)}]}, \psi_{N}^p(\phi_p, f) \rangle| \leq C_3 |I_T| E(T) M(T).$$
We will now prove (2.7), and hence Theorem 1, assuming the three lemmata. In the argument below set
\[ C_0 = C_1 + C_2. \]

Given a finite set of tiles \( P \), find a very large integer \( m_0 \) such that \( E(P) \leq 2^{m_0 n} \) and \( M(P) \leq 2^{2m_0 n} \). We construct by decreasing induction a sequence of pairwise disjoint sets \( P_{m_0}, P_{m_0-1}, P_{m_0-2}, P_{m_0-3}, \ldots \) such that
\[
\bigcup_{j=-\infty}^{m_0} P_j = P
\]
and such that the following properties are satisfied
1. \( E(P_j) \leq 2^{(j+1)n} \) for all \( j \leq m_0 \).
2. \( M(P_j) \leq 2^{(2j+2)n} \) for all \( j \leq m_0 \).
3. \( E(P \setminus (P_{m_0} \cup \cdots \cup P_j)) \leq 2^{jn} \) for all \( j \leq m_0 \).
4. \( M(P \setminus (P_{m_0} \cup \cdots \cup P_j)) \leq 2^{2jn} \) for all \( j \leq m_0 \).
5. \( P_j \) is a union of trees \( T_{jk} \) such that \( \sum_k |I_{T_{jk}}| \leq C_0 2^{-2jn} \) for all \( j \leq m_0 \).

Assume momentarily that we have constructed a sequence \( P_j \) as above. Then to obtain estimate (2.7) we use (1), (2), (5), the observation that the mass is always bounded by 1, and Lemma 3 to obtain
\[
\sum_{s \in P} \left| \langle 1_{E \cap N^{-1}[\omega_p(r)], \psi_p^N} \rangle(f, \phi_p) \right| \\
\leq \sum_j \sum_{p \in P_j} \left| \langle 1_{E \cap N^{-1}[\omega_p(r)], \psi_p^N} \rangle(f, \phi_p) \right| \\
\leq \sum_j \sum_k \sum_{p \in T_{jk}} \left| \langle 1_{E \cap N^{-1}[\omega_p(r)], \psi_p^N} \rangle(f, \phi_p) \right| \\
\leq C_3 \sum_j \sum_k |I_{T_{jk}}| E(T_{jk}) M(T_{jk}) \\
\leq C_3 \sum_j \sum_k |I_{T_{jk}}| 2^{(j+1)n} \min(1, 2^{(2j+2)n}) \\
\leq C_3 \sum_j C_0 2^{-2jn} 2^{(j+1)n} \min(1, 2^{(2j+2)n}) \\
\leq 2^{3n} C_0 C_3 \sum_j \min(2^n, 2^{-jn}) \leq C_n.
\]

This proves estimate (2.7).

It remains to construct a sequence of disjoint sets \( P_j \) satisfying (1)-(5). We start our induction at \( j = m_0 \) by setting \( P_{m_0} = \emptyset \). Then (1), (2), and (5) are
clearly satisfied, while
\[ E(P \setminus P_{m_0}) = E(P) \leq 2^{m_0 n} \]
\[ \mathcal{M}(P \setminus P_{m_0}) = \mathcal{M}(P) \leq 2^{2m_0 n}, \]
hence (3) and (4) are also satisfied for \( P_{m_0} \).

Suppose we have selected pairwise disjoint sets \( P_{m_0}, P_{m_0 - 1}, \ldots, P_m \) for some \( m < m_0 \) such that (1)-(5) are satisfied for all \( j \in \{m_0, m_0 - 1, \ldots, m\} \). We will construct a set of tiles \( P_{m-1} \) disjoint from all the sets already constructed such that (1)-(5) are satisfied for all \( j = m - 1 \). This procedure is given by decreasing induction. We will need to consider the following four cases.

**Case 1.** \( E(P \setminus (P_{m_0} \cup \cdots \cup P_m)) \leq 2^{(m-1)n} \) and \( \mathcal{M}(P \setminus (P_{m_0} \cup \cdots \cup P_m)) \leq 2^{2(m-1)n} \).

In this case set \( P_{m-1} = \emptyset \) and observe that (1)-(5) trivially hold.

**Case 2.** \( E(P \setminus (P_{m_0} \cup \cdots \cup P_m)) > 2^{(m-1)n} \) and \( \mathcal{M}(P \setminus (P_{m_0} \cup \cdots \cup P_m)) \leq 2^{2(m-1)n} \).

Use Lemma 2 to find a subset \( P_{m-1} \) of \( P \setminus (P_{m_0} \cup \cdots \cup P_m) \) such that
\[ (3.6) \quad E(P \setminus (P_{m_0} \cup \cdots \cup P_m \cup P_{m-1})) \leq \frac{1}{2} E(P \setminus (P_{m_0} \cup \cdots \cup P_m)) \leq \frac{1}{2} 2^{mn} \]
and \( P_{m-1} \) is a union of trees (whose set of tops we denote by \( P_{m-1}^* \)) such that
\[ (3.7) \quad \sum_{t \in P_{m-1}^*} |I_t| \leq C_2 E(P \setminus (P_{m_0} \cup \cdots \cup P_m))^{-2} \leq C_2 2^{-2(m-1)n}. \]

Then \( 3.6 \) gives (3) and \( 3.7 \) gives (5) for \( j = m - 1 \). Since
\[ E(P_{m-1}) \leq E(P \setminus (P_{m_0} \cup \cdots \cup P_m)) \leq 2^{mn} = 2^{(m-1)+1)n}, \]
estimate (1) is satisfied for \( j = m - 1 \). Also by our induction hypothesis we have
\[ \mathcal{M}(P \setminus (P_{m_0} \cup \cdots \cup P_m)) \leq \mathcal{M}(P \setminus (P_{m_0} \cup \cdots \cup P_m)) \leq 2^{2(m-1)n}, \]
hence (4) is satisfied for \( j = m - 1 \). Finally
\[ P_{m-1} \subset P \setminus (P_{m_0} \cup \cdots \cup P_m) \]
and hence its mass is at most the mass of the latter which is trivially bounded by \( 2^{2(m-1)+2)n} \), thus (2) is also satisfied for \( j = m - 1 \).

**Case 3.** \( E(P \setminus (P_{m_0} \cup \cdots \cup P_m)) \leq 2^{(m-1)n} \) and \( \mathcal{M}(P \setminus (P_{m_0} \cup \cdots \cup P_m)) \geq 2^{(m-1)n} \).
In this case we repeat the argument in case 2 with the roles of the mass and energy reversed. Precisely, use Lemma 1 to find a subset $P_{m-1}$ of the set $P \setminus (P_{m_0} \cup \cdots \cup P_m)$ such that
\begin{equation}
\mathcal{M}(P \setminus (P_{m_0} \cup \cdots \cup P_m)) \leq \frac{1}{4} \mathcal{M}(P \setminus (P_{m_0} \cup \cdots \cup P_m)) \leq \frac{1}{4} 2^{2mn}
\end{equation}
and $P_{m-1}$ is a union of trees (whose set of tops we denote by $P_{m-1}^*$) such that
\begin{equation}
\sum_{t \in P_{m-1}^*} |I_t| \leq C_1 \mathcal{M}(P \setminus (P_{m_0} \cup \cdots \cup P_m))^{-1} \leq C_1 2^{-2(m-1)n}.
\end{equation}
Then (3.8) gives (4) and (3.9) gives (5) for $j = m-1$. By induction we have
\begin{equation}
\mathcal{M}(P_{m-1}) \leq \mathcal{M}(P \setminus (P_{m_0} \cup \cdots \cup P_m)) \leq 2^{2mn} = 2^{(2(m-1)+2)n},
\end{equation}
thus (2) is satisfied for $j = m-1$. Finally (1) and (3) follow from the inclusion $P_{m-1} \subset P \setminus (P_{m_0} \cup \cdots \cup P_m)$ and the assumption $\mathcal{E}(P \setminus (P_{m_0} \cup \cdots \cup P_m)) \leq 2^{(m-1)n}$. This concludes the proof of (1)-(5) for $j = m-1$.

**Case 4.** $\mathcal{E}(P \setminus (P_{m_0} \cup \cdots \cup P_m)) > 2^{(m-1)n}$ and $\mathcal{M}(P \setminus (P_{m_0} \cup \cdots \cup P_m)) > 2^{2(m-1)n}$.

This is the most difficult case since it involves elements from both of the previous cases. We start by using Lemma 1 to find a subset $P_{m-1}^*$ of the set $P \setminus (P_{m_0} \cup \cdots \cup P_m)$ such that
\begin{equation}
\mathcal{M}(P \setminus (P_{m_0} \cup \cdots \cup P_m \cup P_{m-1}^*)) \leq \frac{1}{4} \mathcal{M}(P \setminus (P_{m_0} \cup \cdots \cup P_m)) \leq \frac{1}{4} 2^{2mn}
\end{equation}
and $P_{m-1}^*$ is a union of trees (whose set of tops we denote by $(P_{m-1}^*)^*$) such that
\begin{equation}
\sum_{t \in (P_{m-1}^*)^*} |I_t| \leq C_1 \mathcal{M}(P \setminus (P_{m_0} \cup \cdots \cup P_m))^{-1} \leq C_1 2^{-2(m-1)n}.
\end{equation}

We now consider the following two subcases of case 4.

**Subcase 4(a).** $\mathcal{E}(P \setminus (P_{m_0} \cup \cdots \cup P_m \cup P_{m-1}^*)) \leq 2^{(m-1)n}$

In this subcase, we set $P_{m-1} = P_{m-1}^*$. Then (3) is automatically satisfied for $j = m-1$ and also (5) is satisfied in view of (3.11). By the inductive hypothesis we have $\mathcal{E}(P \setminus (P_{m_0} \cup \cdots \cup P_m)) \leq 2^{mn} = 2^{(m-1)+1)n}$ and also $\mathcal{M}(P \setminus (P_{m_0} \cup \cdots \cup P_m)) \leq 2^{2mn} = 2^{2(m-1)+2)n}$. Since $P_{m-1}$ is contained in $P \setminus (P_{m_0} \cup \cdots \cup P_m)$ the same estimates hold for $\mathcal{E}(P_{m-1})$ and $\mathcal{M}(P_{m-1})$, thus (1) and (2) also hold for $j = m-1$. Finally (4) for $j = m-1$ follows from (3.8) since $P_{m-1}^* = P_{m-1}$.

**Subcase 4(b).** $\mathcal{E}(P \setminus (P_{m_0} \cup \cdots \cup P_m \cup P_{m-1}^*)) > 2^{(m-1)n}$
Here we use Lemma 4 one more time to find a subset $P''_{m-1}$ of the set $P \setminus (P_{m_0} \cup \cdots \cup P_m \cup P'_{m-1})$ such that

$$E(P \setminus (P_{m_0} \cup \cdots \cup P_m \cup P'_{m-1} \cup P''_{m-1})) \leq \frac{1}{2}E(P \setminus (P_{m_0} \cup \cdots \cup P_m \cup P'_{m-1}))$$

(3.12)

and $P''_{m-1}$ is a union of trees (whose set of tops we denote by $(P''_{m-1})^*$) such that

$$\sum_{t \in (P''_{m-1})^*} |I_t| \leq C_2E(P \setminus (P_{m_0} \cup \cdots \cup P_m \cup P'_{m-1})))^{-2} \leq C_22^{2(m-1)n}.$$ 

(3.13)

We set $P_{m-1} = P'_{m-1} \cup P''_{m-1}$ and we observe that $P_{m-1}$ is disjoint from all the previously selected $P_j$’s. Since by the induction hypothesis the last term in (3.12) is bounded by $\frac{1}{2}E(P \setminus (P_{m_0} \cup \cdots \cup P_m)) \leq \frac{1}{2}2^{mn}$, the first term in (3.13) is also bounded by $2^{(m-1)n}$, thus (3) holds for $j = m-1$. Likewise, since

$$E(P_{m-1}) \leq E(P \setminus (P_{m_0} \cup \cdots \cup P_m)) \leq 2^{mn} = 2^{((m-1)+1)n}$$

$$\mathcal{M}(P_{m-1}) \leq \mathcal{M}(P \setminus (P_{m_0} \cup \cdots \cup P_m)) \leq 2^{2mn} = 2^{2(m-1)+2)n},$$

(1) and (2) are satisfied for $j = m-1$. Since

$$\mathcal{M}(P \setminus (P_{m_0} \cup \cdots \cup P_m \cup P'_{m-1})) \leq \mathcal{M}(P \setminus (P_{m_0} \cup \cdots \cup P_m \cup P'_{m-1}))$$

implies that (4) is satisfied for $j = m-1$. Now each of $P'_{m-1}$ and $P''_{m-1}$ is given as a union of trees, thus the same is true for $P_{m-1}$. The set of tops of all of these trees, call it $(P_{m-1})^*$, is contained in the union of the set of tops of the trees in $P'_{m-1}$ and the trees in $P''_{m-1}$, i.e in $(P'_{m-1})^* \cup (P''_{m-1})^*$. This implies that

$$\sum_{t \in (P_{m-1})^*} |I_t| \leq \sum_{t \in (P'_{m-1})^*} |I_t| + \sum_{t \in (P''_{m-1})^*} |I_t| \leq (C_1 + C_2)2^{-2(m-1)n} = C_02^{-2(m-1)n}$$

in view of (3.11) and (3.13). This proves (5) for $j = m-1$ and concludes the inductive step $j = m-1$. The construction of the $P_j$’s is now complete.

4. PROOF OF LEMMA 1

Given a finite set of tiles $P$, set $\mu = \mathcal{M}(P)$ and define

$$P' = \{ p \in P : \mathcal{M}(\{ p \}) > \frac{1}{4}\mu \}.$$
Clearly $\mathcal{M}(\mathbf{P} \setminus \mathbf{P}') \leq \frac{1}{2} \mu$, thus it remains to show that $\mathbf{P}'$ satisfies (3.2). By definition of the mass, for each $p \in \mathbf{P}'$ there is a tile $u(p) = u \in \mathcal{D}$ such that

\begin{equation}
\int_{E \cap N^{-1}[\omega_u]} \frac{|I_u|^{-1}}{\left(1 + \frac{|x - c(I_u)|}{|I_u|^{1/n}}\right)^{10n}} dx > \frac{\mu}{4}.
\end{equation}

Set $U = \{u(p) : p \in \mathbf{P}'\}$, and let $U_{\text{max}}$ be the subset of $U$ containing all maximal elements of $U$ under the partial order on tiles. As observed earlier, the tiles in $U$ can be grouped into trees with tops in $U_{\text{max}}$. Now $U$ is not necessarily a subset of $\mathbf{P}'$, but each $u \in U$ is associated to a $p \in \mathbf{P}'$ as described above. In particular, if $p$ is a maximal element in $\mathbf{P}'$, then there exists a $u \in U$ with $p < u$ such that (3.2) holds. If this $u$ is not in $U_{\text{max}}$ then there exists $u' \in U_{\text{max}}$ with $u < u'$. We must then have $u'$ associated to another $p' \in \mathbf{P}'$ which implies, by maximality of $p$, that $p' < p$. Hence for each maximal element $p \in \mathbf{P}'$ there exists a unique element $u \in U_{\text{max}}$ with $p < u$, and there is at most one such maximal element for each $u \in U_{\text{max}}$. Therefore, we will show

\begin{equation}
\sum_{u \in U_{\text{max}}} |I_u| \leq C_1 \mu^{-1},
\end{equation}

which implies (3.2). Now we will rewrite (4.2) as

$$\frac{2^n - 1}{2n+2} \mu \sum_{k=0}^\infty 2^{-kn} < \sum_{k=0}^\infty \int_{E \cap N^{-1}[\omega_u] \cap (2^k I_u \setminus 2^{k-1} I_u)} \frac{|I_u|^{-1}}{\left(1 + \frac{|x - c(I_u)|}{|I_u|^{1/n}}\right)^{10n}} dx,$$

where we set $\frac{1}{2} I_u = \emptyset$. This estimate holds for all $u \in U$, so in particular for every $u \in U_{\text{max}}$ there exists a $k \geq 0$ such that

$$\frac{2^n - 1}{2n+2} \mu |I_u| 2^{-kn} < \int_{E \cap N^{-1}[\omega_u] \cap (2^k I_u \setminus 2^{k-1} I_u)} \frac{1}{\left(1 + \frac{|x - c(I_u)|}{|I_u|^{1/n}}\right)^{10n}} dx \leq \frac{|E \cap N^{-1}[\omega_u] \cap (2^k I_u \setminus 2^{k-1} I_u)|}{C(\sqrt{n})^{10n} 2^{10kn}},$$

where the second inequality above follows from the fact that

$$|I_u|^{-1/n} |x - c(I_u)| \sim \sqrt{n} 2^k$$

for $x \in (2^k I_u \setminus 2^{k-1} I_u)$. Here and throughout the paper $C$ denotes a constant depending only on dimension and whose value may change at different places in the proof. Now we define for $k \geq 0$

$$U_k = \{ u \in U_{\text{max}} : C \mu |I_u| 2^{kn} < |E \cap N^{-1}[\omega_u] \cap 2^k I_u| \}.$$
Since \( U_{\text{max}} = \bigcup_{k=0}^{\infty} U_k \), if we show that
\[
(4.3) \quad \sum_{u \in U_k} |I_u| \leq C2^{n-8kn}\mu^{-1},
\]
summing over \( k \in \mathbb{Z}^+ \) gives us estimate (4.2).

We now concentrate on showing estimate (4.3). Fix \( k \geq 0 \) and select an element \( v_0 \in U_k \) so that \( |I_{v_0}| \) is largest possible. Then select an element \( v_1 \in U_k \setminus \{v_0\} \) such that the enlarged rectangle \((2^k I_{v_1}) \times \omega_{v_1}\) is disjoint from \((2^k I_{v_0}) \times \omega_{v_0}\) and \( |I_{v_1}| \) is largest possible. Continuing by induction, at the \( j \)-th step we select an element \( v_j \in U_k \setminus \{v_0, \ldots, v_{j-1}\} \) so that \((2^k I_{v_j}) \times \omega_{v_j}\) is disjoint from the enlarged rectangles of previously selected tiles and \( |I_{v_j}| \) is largest possible. Since we have a finite set of tiles, this process will terminate, and we will have the set of selected tiles in \( U_k \), which we will call \( V_k \).

Next we make some key observations about the tiles. First, note that elements of \( U_k \) are maximal in \( U \) and therefore disjoint. Second, for any \( u \in U_k \) there exits a selected tile \( v \in V_k \) with \( |I_u| \leq |I_v| \) and such that the enlarged rectangles of \( u \) and \( v \) intersect. We will associate \( u \) to this \( v \). Third, if \( u \) and \( u' \) are both associated to the same \( v \), then \( I_u \) and \( I_{u'} \) are disjoint. Indeed, \((2^k I_u) \times \omega_u \) intersects \((2^k I_v) \times \omega_v \) which means \( 2^n |I_u| \wedge 2^n I_v \neq \emptyset \) and \( \omega_u \cap \omega_v \neq \emptyset \). This implies, together with the fact that \( |I_u| \leq |I_v| \), that \( \omega_u \supset \omega_v \). Similarly \( \omega_{u'} \supset \omega_v \). Therefore, one of \( \omega_u \) and \( \omega_{u'} \) contains the other. But \( u \) and \( u' \) are disjoint, thus \( I_u \) is disjoint from \( I_{u'} \). Finally, all tiles \( u \in U_k \) associated to a particular \( v \in V_k \) satisfy \( I_u \subset 2^{n+2} I_v \).

From the observations above and the definition of \( U_k \), we have
\[
\sum_{u \in U_k} |I_u| \leq \sum_{v \in V_k} \sum_{u \in U_k \text{ assoc } v} |I_u| \\
= \sum_{v \in V_k} \left| \bigcup_{u \in U_k \text{ assoc } v} I_u \right| \leq \sum_{v \in V_k} 2^{(k+2)n} |I_v| \\
\leq C \mu^{-1-2^{-9kn}} 2^{(k+2)n} \sum_{v \in V_k} |E \cap \mathbb{N}^{-1} [\omega_v] \cap 2^k I_v| \\
\leq C 2^{2n} \mu^{-1-2^{-8kn}} |E| \leq C 2^{2n} \mu^{-1-2^{-8kn}},
\]
where we have used that for \( v \in V_k \), the enlarged rectangles are disjoint, and therefore so are the subsets \( E \cap \mathbb{N}^{-1} [\omega_v] \cap 2^k I_v \) of \( E \).

5. Proof of Lemma 2

We begin by fixing a finite set of tiles \( P \) and \( r = 2^n \). This choice of \( r \) ensures that \( \eta < \xi \) in the lexicographical order for all \( \eta \in \omega_{p(1)} \) and \( \xi \in \omega_{p(r)} \). For \( r \neq 2^n \) the proof goes through by a suitable permutation of the coordinates of \( \mathbb{R}^n \) which changes the coordinate that takes precedence in the lexicographical order. Here we note that we can be less precise by taking any linear functional...
that separates $\omega_p^{(1)}$ and $\omega_p^{(r)}$ in any given cube $\omega_p$. Then we let $\eta < \xi$ if $L(\eta) < L(\xi)$. In particular, we can take $L$ to be the projection onto the appropriate axis so that the usual linear ordering on $\mathbb{R}$ is relevant. Let $\epsilon$ denote $\mathcal{E}(P)$. Define for $T'$ a $2^n$-tree

$$\Delta(T') = \left( |I_{T'}|^{-1} \sum_{p \in T'} |\langle f, \phi_p \rangle|^2 \right)^{1/2}.$$ 

Consider all $2^n$-trees $T'$ contained in $P$ which satisfy

$$\Delta(T') \geq \frac{1}{2} \epsilon. \quad (5.1)$$

Among these select a $2^n$-tree $T'_1$ such that $c(\omega_{T'_1})$ is minimal in the lexicographical order. Let $T_1$ be the set of all $p \in P$ such that $p < p_{T'_1} = p_{T_1}$. In other words $T_1$ is the maximal tree containing $T'_1$ with the same top as $T'_1$. Now consider all $2^n$-trees contained in $P \setminus T_1$ and select a $2^n$-tree $T_2$, such that $c(\omega_{T'_2})$ is minimal. Let $T_2$ be the set of all $p \in P$ such that $p < p_{T'_2} = p_{T_2}$. Continue inductively to obtain a finite sequence of pairwise disjoint $2^n$-trees $T'_1, T'_2, \ldots, T'_q$ and pairwise disjoint trees $T_1, T_2, \ldots, T_q$ where $p_{T'_j} = p_{T_j}, T'_j \subset T_j$, and the $T'_j$ satisfy (5.1). Let

$$P'' = \bigcup_{j=1}^q T_j,$$

then clearly

$$\mathcal{E}(P \setminus P'') \leq \frac{1}{2} \epsilon.$$ 

Thus we need to show that $P''$ satisfies condition (3.4) of Lemma 2, i.e.

$$\sum_{j=1}^q |I_{T_j}| \leq \frac{C_2}{\epsilon^2}. \quad (5.2)$$

Since the trees $T'_j$ satisfy (5.1) and $\|f\|_{L^2(\mathbb{R}^n)} = 1,

$$\frac{1}{4} \epsilon^2 \sum_{j} |I_{T'_j}| \leq \sum_{j} \sum_{p \in T'_j} |\langle f, \phi_p \rangle|^2$$

$$= \sum_{j} \sum_{p \in T'_j} \langle f, \phi_p \rangle \overline{\langle f, \phi_p \rangle}$$

$$= \langle f, \sum_{j} \sum_{p \in T'_j} \langle f, \phi_p \rangle \phi_p \rangle$$

$$\leq \left\| \sum_{j} \sum_{p \in T'_j} \langle f, \phi_p \rangle \phi_p \right\|_{L^2(\mathbb{R}^n)}.$$ 

(5.3)
Letting $U = \bigcup_j T'_j$, we will show that

\[
\left\| \sum_{p \in U} \langle f, \phi_p \rangle \phi_p \right\|_{L^2(\mathbb{R}^n)} \leq C \left( c^2 \sum_j |I_{T_j}| \right)^{1/2},
\]

which, together with (5.3), will give us (5.2). The square of the left hand side of (5.4) can be estimated by

\[
\sum_{p, u \in U} \frac{|\langle f, \phi_p \rangle \langle f, \phi_u \rangle| + 2 \sum_{p, u \in U, \omega_p \subset \omega_u^{(1)}} |\langle f, \phi_p \rangle \langle f, \phi_u \rangle \langle \phi_p, \phi_u \rangle|}{\omega_p \cap \omega_u^{(1)}}.
\]

Here we have used that $\langle \phi_p, \phi_u \rangle = 0$ unless $\omega_p^{(1)}$ intersects $\omega_u^{(1)}$ which implies that either $\omega_p = \omega_u$ or $\omega_u^{(1)}$ contains $\omega_p$ or $\omega_p^{(1)}$ contains $\omega_u$. We are then able to utilize the symmetry in $p$ and $u$ to combine the off diagonal terms.

We estimate $|\langle f, \phi_p \rangle|$ and $|\langle f, \phi_u \rangle|$ by the larger one and bound the first term in (5.5) by

\[
\sum_{p \in U} |\langle f, \phi_p \rangle|^2 \sum_{u \in U, \omega_p \cap \omega_u^{(1)}} |\langle \phi_p, \phi_u \rangle| \leq C \sum_{p \in U} |\langle f, \phi_p \rangle|^2 \sum_{u \in U, \omega_p \subset \omega_u^{(1)}} C \frac{\min \left( \frac{|I_p|}{|I_u|}, \frac{|I_u|}{|I_p|} \right)^{1/2}}{\left( 1 + \frac{|x - c(I_p)|}{\max(|I_p|,|I_u|)^{1/n}} \right)^{10n}} \int_{I_u} \frac{1}{|I_p|} \left( 1 + \frac{|x - c(I_p)|}{|I_p|^{1/n}} \right)^{-10n} dx \leq C \sum_{p \in U} |\langle f, \phi_p \rangle|^2 = C \sum_j \sum_{p, u \in U, \omega_p \subset \omega_u^{(1)}} |I_{T_j}|^{-1} |\langle f, \phi_p \rangle|^2 \leq C \sum_j |I_{T_j}| \epsilon^2,
\]

where we have used that for $p \in U$, the $I_u$ for which $\omega_p = \omega_u$ are pairwise disjoint.

Using Cauchy-Schwarz, the second term in (5.5) can be estimated by

\[
2 \sum_j \sum_{p \in T'_j} \left| \langle f, \phi_p \rangle \right| \sum_{u \in U, \omega_p \subset \omega_u^{(1)}} \left| \langle f, \phi_u \rangle \right| |\langle \phi_p, \phi_u \rangle|.
\]
\[
\leq 2 \sum_j \left\{ \sum_{p \in T'_j} |\langle f, \phi_p \rangle|^2 \right\}^{1/2} \left\{ \sum_{p \in T'_j} \left( \sum_{u \in U} \left| \langle f, \phi_u \rangle \right| \langle \phi_p, \phi_u \rangle \right) \right\}^{1/2} \\
\leq 2 \sum_j |T'_j|^{1/2} \Delta(T'_j) \left\{ \sum_{p \in T'_j} \left( \sum_{u \in U} \left| \langle f, \phi_u \rangle \right| \langle \phi_p, \phi_u \rangle \right) \right\}^{2} \\
\leq 2c |T'_j|^{1/2} \left\{ \sum_{p \in T'_j} \left( \sum_{u \in U} \left| \langle f, \phi_u \rangle \right| \langle \phi_p, \phi_u \rangle \right) \right\}^{2}.
\]

To complete the proof, we need to show that the expression inside the curly brackets is bounded by \( C \epsilon^2 |T'_j| \). Since for a single tile \( u \)
\[
\mathcal{E}(\{u\}) = \langle |I_u|^{-1} \langle f, \phi_u \rangle \rangle^{1/2} = |I_u|^{-1/2} |\langle f, \phi_u \rangle| \leq \epsilon,
\]
we get that
\[
\sum_{p \in T'_j} \left( \sum_{u \in U} \left| \langle f, \phi_u \rangle \right| \langle \phi_p, \phi_u \rangle \right)^2 \leq \epsilon^2 \sum_{p \in T'_j} \left( \sum_{u \in U} |I_u|^{1/2} |\langle \phi_p, \phi_u \rangle| \right)^2.
\]

Thus we now need to show that
\[
(5.6) \sum_{p \in T'_j} \left( \sum_{u \in U} |\langle I_u \rangle^{1/2} |\langle \phi_p, \phi_u \rangle| \right)^2 \leq C |T'_j|.
\]

To prove this, we will need the following lemma.

**Lemma 4.** Let \( p \in T'_j \) and \( u \in T'_k \). Then if \( \omega_p \subset \omega_u(1) \), we have \( I_u \cap I_{T'_j} = \emptyset \). If \( u \in T'_k, v \in T'_l, u \neq v, \omega_p \subset \omega_u(1) \), and \( \omega_p \subset \omega_u(2^n) \) for some fixed \( p \in T'_j \), then \( I_u \cap I_v = \emptyset \).

**Proof.** Since \( \omega_p \subset \omega_u(1) \) and \( T'_j \) and \( T'_k \) are \( 2^n \)-trees, \( T'_j \) and \( T'_k \) are not the same tree. Otherwise \( \omega_p \subset \omega_u(2^n) \). We know that \( \omega_T \subset \omega_p \subset \omega_u(1) \), which implies that \( c(\omega_T) \) is contained in \( \omega_u(1) \). We also have \( \omega_{T'_j} \subset \omega_u(2^n) \), which implies that \( c(\omega_{T'_j}) \) is contained in \( \omega_u(2^n) \). Therefore \( c(\omega_{T'_j}) \leq c(\omega_{T'_k}) \) in the lexicographical order which means that \( T'_j \) was chosen before \( T'_k \) in the original selection process. Now suppose \( I_u \cap I_{T'_j} \neq \emptyset \). Then either \( I_u \subset I_{T'_j} \) or \( I_u \supset I_{T'_j} \), however \( \omega_{T'_j} \subset \omega_u \) implies that \( I_u \subset I_{T'_j} \). Thus we have \( \omega_{T'_j} \subset \omega_u \) and \( I_u \subset I_{T'_j} \) which says that \( u \) belongs to the tree \( T'_j \). However, \( u \in T'_k \) and thus was chosen from \( P \setminus T'_j \), which gives a contradiction. Thus \( I_u \cap I_{T'_j} = \emptyset \).
Next suppose that \( u \in T_k^v, v \in T'_k, u \neq v \), and \( \omega_p \subset (\omega_u(1) \cap \omega_v(1)) \) for some fixed \( p \in T'_k \). We have three cases to consider: (a) \( \omega_u \subset \omega_v(1) \) which means \( I_u \cap I_{T_k} = \emptyset \) and thus \( I_u \cap I_u = \emptyset \), (b) \( \omega_v \subset \omega_u(1) \) which means \( I_u \cap I_{T_k} = \emptyset \) and thus \( I_v \cap I_u = \emptyset \), and (c) \( \omega_v = \omega_u \) which tells us \( |I_u| = |I_v| \), thus \( I_u \) and \( I_v \) are disjoint since \( u \) and \( v \) don’t coincide.

We now return to estimate (5.6). Observe that Lemma 4 tells us that for the tiles \( u \in U \) appearing in the interior sum of (5.6), the \( I_u \) are pairwise disjoint and contained in \((I_{T_k})^c\). Thus we have

\[
\sum_{p \in T'_k} \left( \sum_{u \in \mathbb{U}} \frac{|I_u|^{1/2}}{\omega_p \subset \omega_u(1)} |\phi_p, \phi_u| \right)^2 \\
\leq C \sum_{p \in T'_k} \left( \sum_{u \in \mathbb{U}} \frac{|I_u|^{1/2}}{\omega_p \subset \omega_u(1)} \right)^2 \int_{I_u} \frac{|I_p|^{-1}}{1 + \frac{\|e^{-c(I_p)}\|^{1/\|p\|_n}}{10n}} d\mathbf{x} \\
\leq C \sum_{p \in T'_k} |I_p|^\circ \left( \int_{(I_{T_k})^c} \frac{|I_p|^{-1}}{1 + \frac{\|e^{-c(I_p)}\|^{1/\|p\|_n}}{10n}} d\mathbf{x} \right)^2 \\
\leq C \sum_{p \in T'_k} |I_p| \left( \int_{(I_{T_k})^c} \frac{|I_p|^{-1}}{1 + \frac{\|e^{-c(I_p)}\|^{1/\|p\|_n}}{10n}} d\mathbf{x} \right)^2 \\
\leq C \sum_{k=0}^\infty 2^{-kn} |I_{T_k}| \sum_{p \in T'_k, |I_p| = 2^{-k}|I_{T_k}|} \left( \int_{(I_{T_k})^c} \frac{|I_p|^{-1}}{1 + \frac{\|e^{-c(I_p)}\|^{1/\|p\|_n}}{10n}} d\mathbf{x} \right)^2.
\]

The proof of Lemma 3 will be complete if we can show that

\[
\sum_{p \in T'_k, |I_p| = 2^{-kn}|I_{T_k}|} \int_{(I_{T_k})^c} \frac{|I_p|^{-1}}{1 + \frac{\|e^{-c(I_p)}\|^{1/\|p\|_n}}{10n}} d\mathbf{x} \lesssim 2^{k(n-1)},
\]

thus allowing the sum in \( k \) to converge. Throughout the paper, \( A \lesssim B \) means that \( A \) is less than or equal to \( B \) up to a constant depending only on dimension.
The first observation we have is that

$$\int_{(I_T)^c} \frac{\|I_p\|^{-1}}{\left(1 + \frac{|x-c(I_p)|}{|I_p|^{1/n}}\right)^{10n}} \, dx \lesssim \left(\frac{\text{dist}((I_T)^c, I_p)}{|I_p|^{1/n}}\right)^{-9n}.$$

To see this, note that by a change of variables, it suffices to let the center of $I_T$ be at the origin. Also note that we have the inequality

$$\left(1 + \frac{|x - c(I_p)|}{|I_p|^{1/n}}\right)^{10n} \geq \prod_{i=1}^{n} \left(1 + \frac{|x_i - c(I_p)|}{|I_p|^{1/n}}\right)^{10}.$$

Therefore, the integral above is bounded by a constant times

$$\prod_{i=1}^{n} \left(\int_{x_i > \frac{1}{4}|I_T|} \frac{|I_p|^{-1/n}}{\left(1 + \frac{|x_i - c(I_p)|}{|I_p|^{1/n}}\right)^{10}} \, dx_i\right)$$

$$\lesssim \prod_{i=1}^{n} \left(\text{dist}((I_T)^c, I_p)\right)^{-9} \lesssim \left(\frac{\text{dist}((I_T)^c, I_p)}{|I_p|^{1/n}}\right)^{-9n},$$

where we have used that $|x_i - c(I_p)| \geq \text{dist}((I_T)^c, I_p)$ for all $i = 1, 2, \ldots, n$. Now we need to sum over $p$ for a fixed scale $k$. Consider an $n-1$ dimensional face of $I_T$, and fix a cube $I_p$ whose face is contained in the face of $I_T$. We allow the remaining coordinate to vary and sum over those $I_p$ in this “column”. In a fixed column, the distances from $(I_T)^c$ to each $I_p$ sum as additive multiples of $|I_p|^{1/n}$. For each face, there are $2^{k(n-1)}$ such columns. Thus

$$\sum_{p \in T} \left(\frac{\text{dist}((I_T)^c, I_p)}{|I_p|^{1/n}}\right)^{-9n} \lesssim 2^{k(n-1)} \times (\# \text{ of faces}) \sum_{m=0}^{\infty} \frac{1}{m^{9n}}$$

$$\lesssim 2^{k(n-1)}.$$

6. Proof of Lemma 3 - The Tree Inequality

Let $J$ be the collection of all maximal dyadic cubes $J$ such that $3J$ does not contain any $I_p$ with $p \in \mathbf{T}$. Then $J$ is a partition of $\mathbb{R}^n$.

We can write the left hand side of (3.5) as follows, where the terms $\alpha_p$ are phase factors of modulus 1 which make up for the absolute value signs in (3.5):

$$\left\| \sum_{p \in \mathbf{T}} \alpha_p \langle f, \phi_p \rangle \psi_p^{N_1 E_{2p}} \right\|_1 \leq K_1 + K_2$$
A WEAK $L^2$ ESTIMATE FOR A MAXIMAL DYADIC SUM OPERATOR ON $\mathbb{R}^n$

where

\begin{equation}
E_{2p} := E \cap N^{-1}[\omega_p(2^n)],
\end{equation}

\begin{equation}
K_1 := \sum_{J \in \mathcal{J}} \sum_{p \in \mathcal{T} : |I_p| \leq 2^n |J|} \| \langle f, \phi_p \rangle \psi_p^N 1_{E_{2p}} \|_{L^1(J)},
\end{equation}

\begin{equation}
K_2 := \sum_{J \in \mathcal{J}} \left\| \sum_{p \in \mathcal{T} : |I_p| > 2^n |J|} \alpha_p(f, \phi_p) \psi_p^N 1_{E_{2p}} \right\|_{L^1(J)}.
\end{equation}

Let

$$
\epsilon = \mathcal{E}(\mathcal{T}) \quad \text{and} \quad \mu = \mathcal{M}(\mathcal{T}).
$$

We begin with $K_1$. For every $p \in \mathcal{T}$, $\{p\}$ is a $2^n$-tree contained in $\mathcal{T}$, and therefore

$$
|\langle f, \phi_p \rangle| \leq \epsilon |I_p|^\frac{1}{2}.
$$

\begin{align*}
K_1 & \leq C \epsilon \sum_{J \in \mathcal{J}} \sum_{p \in \mathcal{T} : |I_p| \leq 2^n |J|} |I_p| \int_{J \cap E \cap N^{-1}[\omega_p]} \frac{|I_p|^{-1}}{\left(1 + \frac{|x - c(I_p)|}{|I_p|^\frac{1}{2}}\right)^{20n}} \, dx \\
& \leq C \epsilon \mu \sum_{J \in \mathcal{J}} \sum_{p \in \mathcal{T} : |I_p| \leq 2^n |J|} |I_p| \sup_{x \in J} \left(1 + \frac{|x - c(I_p)|}{|I_p|^\frac{1}{2}}\right)^{-10n} \\
& \leq C \epsilon \mu \sum_{J \in \mathcal{J}} \sum_{k : 2^{kn} \leq 2^n |J|} 2^{kn} \sum_{p \in \mathcal{T} : |I_p| = 2^{kn}} \left(1 + \frac{\text{dist}(J, I_p)}{2^k}\right)^{-10n},
\end{align*}

where we have used that for $x \in J$,

$$
|x - c(I_p)| \geq \text{dist}(J, c(I_p)) \geq \text{dist}(J, I_p) + \frac{2^k}{2},
$$

hence

$$
\frac{|x - c(I_p)|}{2^k} \geq \frac{\text{dist}(J, I_p)}{2^k} \geq \frac{\text{dist}(J, I_p)}{2^k} + \frac{1}{2}.
$$

For all $p \in \mathcal{T}$ with $|I_p| = 2^{kn}$, the $I_p$ are pairwise disjoint and contained in $I_T$. Therefore $\text{dist}(J, I_p) \geq \text{dist}(J, I_T)$ and $|I_T|^{-\frac{1}{n}} \leq 2^{-k}$, which gives

\begin{equation}
\left(1 + \frac{\text{dist}(J, I_p)}{2^k}\right)^{-5n} \leq \left(1 + \frac{\text{dist}(J, I_T)}{|I_T|^\frac{1}{n}}\right)^{-5n}.
\end{equation}

We will treat the remaining powers with the following lemma.
Lemma 5. For $J \in \mathcal{J}$ such that $2^{kn} \leq 2^n |J|$, 
\[
\sum_{p \in \mathcal{T}, |I_p|=2^{kn}} \left(1 + \frac{\text{dist}(J, I_p)}{2^k}\right)^{-5n} \leq C(n),
\]
where $C(n)$ is independent of $J, k$ and $\mathcal{T}$.

Proof. We first observe that $\text{dist}(J, I_p)$ and $\text{dist}(c(J), I_p)$ are of comparable size. The inequality $\text{dist}(J, I_p) \leq \text{dist}(c(J), I_p)$ is clear. To see the other inequality, note that $|I_p| \leq 2^n |J| = 2|J|$ implies that $I_p$ is disjoint from $3J$, since $3J$ does not contain any $I_p$. Thus we have 
\[
\text{dist}(I_p, c(J)) \leq \text{dist}(I_p, J) + \text{dist}(\partial J, c(J)) \leq \text{dist}(I_p, J) + \frac{\sqrt{n}}{2} \text{dist}(I_p, J) \left(1 + \frac{\sqrt{n}}{2}\right) \text{dist}(I_p, J).
\]
Hence it suffices to replace $\text{dist}(J, I_p)$ by $\text{dist}(c(J), I_p)$. Let $x_0 = c(J)$ and decompose $\mathbb{R}^n$ as follows:
\[
\mathbb{R}^n = \bigcup_{m=1}^{\infty} O_m,
\]
where
\[
O_1 := B(x_0, 3\sqrt{n}2^k),
\]
\[
O_m := B(x_0, 3m\sqrt{n}2^k) \setminus B(x_0, 3(m-1)\sqrt{n}2^k).
\]
Let
\[
S_1 := \{p \in \mathcal{T} : |I_p| = 2^{kn}, I_p \cap B(x_0, 3\sqrt{n}2^k) \neq \emptyset\},
\]
\[
S_m := \{p \in \mathcal{T} : |I_p| = 2^{kn}, I_p \cap O_m \neq \emptyset, I_p \cap \bigcup_{i=1}^{m-1} O_i = \emptyset\}, \quad m \geq 2.
\]
Since the diameter of $I_p$ is $\sqrt{n}2^k$, $I_p$ will not intersect three annuli, so each $p$ in the sum is contained in exactly one $S_m$. In order to estimate the number of tiles in $S_m$, we consider the volume of the corresponding annulus $O_m$. Now, 
\[
\text{volume}(O_m) = (3\sqrt{n})^{n}2^{kn} (m^n - (m-1)^n) = C_n 2^{kn} m^{n-1}.
\]
Since the $I_p$-s are disjoint, there are $C_n m^{n-1}$ cubes $I_p$ of size $2^{kn}$ in the set $S_m$. Also for $p \in S_m$,
\[
3(m-1)\sqrt{n}2^k \leq \text{dist}(x_0, I_p) \leq 3m\sqrt{n}2^k.
\]
Thus,
\[
\sum_{p \in \mathcal{T}, |I_p|=2^{kn}} \frac{1}{\left(1 + \frac{\text{dist}(x_0, I_p)}{2^k}\right)^{5n}} \leq \sum_{m=1}^{\infty} \frac{m^{n-1}}{(1 + m)^{5n}} \leq \sum_{m=1}^{\infty} \frac{1}{m^6} < \infty.
\]
Using \((6.4)\) and the lemma, we have that \(K_1\) is bounded by

\[
C \epsilon \mu \sum_{J \in \mathcal{J}} \sum_{k_n = -\infty}^{\log_2 2^n |J|} 2^{|J|} \left(1 + \frac{\text{dist}(J, I_T)}{|I_T|^{1/3}}\right)^{-5n}
\]

\[
\leq C \epsilon \mu \sum_{J \in \mathcal{J}} |J| \left(1 + \frac{\text{dist}(J, I_T)}{|I_T|^{1/3}}\right)^{-5n}
\]

\[
\leq C \epsilon \mu \sum_{J \in \mathcal{J}} \int_J \left(1 + \frac{|x - c(I_T)|}{|I_T|^{1/3}}\right)^{-5n} dx
\]

\[
\leq C \epsilon \mu |I_T|.
\]

This completes the estimate of \(K_1\).

Now we consider \(K_2\) defined by \((6.3)\). We can assume that the summation runs only over those \(J \in \mathcal{J}\) for which there exists a \(p \in \mathcal{T}\) with \(2^n |J| < |I_p|\). Then we have \(J \subset 3I_T\) and \(2^n |J| < |I_T|\) for all \(J\) occurring in the sum.

Let us fix a dyadic cube \(J \in \mathcal{J}\) and observe that the set

\[
G_J = J \cap \bigcup_{p \in \mathcal{T} : |I_p| > 2^n |J|} E_{2p}
\]

has measure at most \(C \mu |J|\). To see this, let \(J'\) be the unique dyadic cube which contains \(J\) and \(|J'| = 2^n |J| < |I_T|\). By maximality of \(J\), \(3J'\) contains \(I_{p_0}\) for some \(p_0 \in \mathcal{T}\). There are two cases to consider. Case (a): \(I_{p_0}\) is the dyadic cube that is formed from taking the unique double of each side of \(J'\) which is also dyadic. In this case \(|I_{p_0}| = 2^n |J'|\) and we set \(p_0 = p' < I_T \times \omega_T\).

Case (b): \(I_{p_0}\) is contained in one of the dyadic cubes of size \(|J'|\) contained in \(3J'\). Since \(|J'| = 2^n |J| < |I_T|\), the dyadic cube which contains \(I_{p_0}\) is contained in \(I_T\). In this case there exists a tile \(p'\) with \(|I_{p'}| = |J'|\) so that \(I_{p_0} \subset I_{p'} \subset I_T\).

In both cases we have a tile \(p'\) such that \(p_0 < p' < I_T \times \omega_T\) and \(|\omega_{p'}|\) is either \(2^{-n}|J|^{-1}\) or \(2^{-2n}|J|^{-1}\). We claim that

\[
\bigcup_{p \in \mathcal{T} : |I_p| > 2^n |J|} E_{2p} \subset E \cap N^{-1} [\omega_{p'}].
\]

To see this, let us choose \(p \in \mathcal{T}\) such that \(|I_p| > 2^n |J|\). Then \(|\omega_p| < 2^{-n}|J|^{-1}\), which means \(|\omega_p| < |\omega_{p'}|\). But \(\omega_T \subset \omega_p \cap \omega_{p'}\), which leads us to conclude \(\omega_p \subset \omega_{p'}\). Recalling that

\[
E_{2p} = \{x : N(x) \in \omega_p(2^n)\} \cap E
\]
now completes the proof of the claim.

The above claim implies that \( G_J \subset J' \cap E \cap N^{-1}[\omega_p] \). Therefore,

\[
|G_J| \leq |J' \cap E \cap N^{-1}[\omega_p]| = \int_{E \cap N^{-1}[\omega_p]} 1_{J'}(x) \, dx.
\]

Since

\[
1_{J'}(x) \leq C \left( 1 + \frac{|x - c(I_p)|}{|I_p|^\frac{1}{n}} \right)^{-\nu},
\]

and mass(\( \{p\} \)) \( \leq \mu \), we get

\[
|G_J| \leq C\mu |J|.
\]

Let \( T_2 \) be the \( 2^n \)-tree of all \( p \in T \) such that \( \omega_{T(2^n)} \subset \omega_{p(2^n)} \) and let \( T_1 = T \setminus T_2 \). Define for \( j = 1, 2 \),

\[
F_j \omega := \sum_{p \in T_j : |I_p| > 2^n |J|} 1_{E_p} \psi_p^N 1_{E_{2p}}.
\]

First we consider \( F_1 \omega \). We have

\[
|F_1 \omega(x)| \leq \sum_{p \in T_1 : |I_p| > 2^n |J|} |(f, \phi_p)| |\psi_p^N(x)| 1_{E_p} 1_{E_{2p}}(x)
\]

\[
\leq C \epsilon \sum_{p \in T_1 : |I_p| > 2^n |J|} \left( 1 + \frac{|x - c(I_p)|}{|I_p|^\frac{1}{n}} \right)^{-\nu} 1_{E_p} 1_{E_{2p}}(x).
\]

We will sum the expression on the right hand side of the above inequality in two steps. First let us construct

\[
\mathcal{I} := \{ \omega : \text{there exists} \ p \in T_1 \ \text{such that} \ |I_p| > 2^n |J| \ \text{and} \ \omega_{p(2^n)} = \omega \},
\]

and \( \mathcal{P}_\omega := \{ p \in T_1 : |I_p| > 2^n |J|, \omega_{p(2^n)} = \omega \}, \) for \( \omega \in \mathcal{I} \).

This means that the sum estimating \( F_1 \omega \) may be written as

\[
\sum_{\omega \in \mathcal{I}} \sum_{p \in \mathcal{P}_\omega} \left( 1 + \frac{|x - c(I_p)|}{|I_p|^\frac{1}{n}} \right)^{-\nu} 1_{E_\omega} 1_{E_{2p}}(x),
\]

where

\[
E_\omega := E \cap \{ x : N(x) \in \omega \}.
\]

Now note that for \( p \in T_1 \), the semitiles \( I_p \times \omega_{p(2^n)} \) are disjoint. In particular, for \( p, p' \in \mathcal{P}_\omega, p \neq p' \), one has \( I_p \cap I_{p'} = \emptyset \). Therefore,

\[
\sum_{p \in \mathcal{P}_\omega} \left( 1 + \frac{|x - c(I_p)|}{|I_p|^\frac{1}{n}} \right)^{-\nu} \leq C
\]

The proof of this fact is similar to that of Lemma 5. This implies that

\[
|F_1 \omega(x)| \leq C \epsilon \sum_{\omega \in \mathcal{I}} 1_{E_\omega}(x) = C \epsilon 1_{E_\omega}(x).
\]
Here we have used the fact that the \( \omega \)'s in \( J \) are disjoint. This yields

\[
\| F_{1J}(x) \|_{L^1(J)} \leq C \epsilon \int \sum_{\omega \in \mathcal{J}_e} 1 \ dx = C \epsilon |G_J| \leq C \epsilon |J|,
\]

This estimate, summed over the disjoint \( J \subset 3I_T \) yields the desired bound.

To complete the proof of (3.1) we estimate \( F_{2J}(x) \). Fix \( x \) and assume that \( F_{2J}(x) \) is not zero. Since the cubes \( \omega_{p(2^n)} \) with \( p \in T_2 \) are all nested and \( E_{2p} = \{ x : N(x) \in \omega_{p(2^n)} \} \cap E_1 \), there is a largest cube \( \omega^+ \) of the form \( \omega_p \) with \( p \in T_2 \), \( x \in E_{2p} \) and \( |I_p| > 2^n |J| \). Similarly there is a smallest cube which we call \( \omega^- \) satisfying the above properties. Let us define \( \omega^- = \omega_{s(2^n)} \). Then \( x \in E_{2p} \) for some \( p \in T \) with \( |I_p| > 2^n |J| \) if and only if \( |\omega^-| < |\omega_p| \leq |\omega^+| \). Fix \( \xi_0 \in \omega_T \). We can now write \( F_{2J}(x) \) as

\[
F_{2J}(x) = \sum_{p \in T_2} \alpha_p(f, \phi_p) \psi_p^N(x),
\]

which may be decomposed as

\[
\sum_{p \in T_2} \alpha_p(f, \phi_p) \left( e^{2\pi i \xi_0 \cdot \cdot} K(\cdot) \ast \phi_p(\cdot) \right)(x)
\]

\[
+ \left( \left( e^{2\pi i N(x) \cdot \cdot} - e^{2\pi i \xi_0 \cdot \cdot} \right) K(\cdot) \right) \ast \sum_{p \in T_2} \alpha_p(f, \phi_p) \phi_p(\cdot)(x)
\]

\[
= \sum_{p \in T_2} \alpha_p(f, \phi_p) \left( \psi_p^0 \ast \left( M_{c(\omega_+)} D_{\frac{1}{2}|\omega_+| \frac{1}{2}} \phi - M_{c(\omega_-)} D_{\frac{1}{2}|\omega_-| \frac{1}{2}} \phi \right) \right)(x)
\]

\[
+ \left\{ \left( e^{2\pi i N(x) \cdot \cdot} - e^{2\pi i \xi_0 \cdot \cdot} \right) K(\cdot) \right\} \ast \sum_{p \in T_2} \alpha_p(f, \phi_p) \left( \phi_p \ast \left( M_{c(\omega_+)} D_{\frac{1}{2}|\omega_+| \frac{1}{2}} \phi - M_{c(\omega_-)} D_{\frac{1}{2}|\omega_-| \frac{1}{2}} \phi \right) \right)(x).
\]

We claim that the last equality follows from the geometry of the supports of the Fourier transforms of the two convolving functions. More specifically, \( \phi_p \) is supported on \( \frac{1}{3} \omega_{p(1)} \) while

\[
\left( M_{c(\omega_{\pm})} D_{\frac{1}{2}|\omega_{\pm}| \frac{1}{2}} \phi \right)(\xi) = \begin{cases} 1 & \text{if } \xi \in \omega_{\pm} \\ 0 & \text{if } \xi \notin \left( 1 + \frac{1}{3} \right) \omega_{\pm}. \end{cases}
\]

Therefore

\[
\left( M_{c(\omega_+)} D_{\frac{1}{2}|\omega_+| \frac{1}{2}} \phi - M_{c(\omega_-)} D_{\frac{1}{2}|\omega_-| \frac{1}{2}} \phi \right)(\xi) = \begin{cases} 1 & \text{if } \xi \in \omega_{\pm}. \end{cases}
\]
\[
\begin{cases}
1 & \text{if } \xi \in \omega_+ \setminus (1 + \frac{1}{5}) \omega_- \\
0 & \text{if } \xi \in \omega_- \cup \left((1 + \frac{1}{5}) \omega_+\right)^c.
\end{cases}
\]

For those \( p \in T_2 \) such that \(|\omega_-| < |\omega_p| \leq |\omega_+| \), we have
\[
\frac{1}{5} \omega_p(1) \subset \omega_+ \setminus \left(1 + \frac{1}{5}\right) \omega_-. 
\]
Conversely, for \( p \in T_2 \) such that \(|\omega_p| > |\omega_+| \) or \(|\omega_p| \leq |\omega_-| \), we have
\[
\frac{1}{5} \omega_p(1) \subset \omega_- \cup \left((1 + \frac{1}{5}) \omega_+\right)^c.
\]
This concludes the proof of the claim.

The expression for \( F_{2J} \) may therefore be written as
\[
(6.5) \quad \left( G_1 * \left( M_{c(\omega_+)} D_{\frac{1}{2}|\omega_+|} \phi - M_{c(\omega_-)} D_{\frac{1}{2}\omega_-} \phi \right) \right)(x) + \left[ \left( e^{2\pi i N(x) \cdot (\cdot)} - e^{2\pi i \xi_0 \cdot (\cdot)} \right) K(\cdot) * G_2(\cdot) \right. \\
\left. \quad \ast \left( M_{c(\omega_+)} D_{\frac{1}{2}|\omega_+|} \phi - M_{c(\omega_-)} D_{\frac{1}{2}\omega_-} \phi \right) \right](x),
\]
where
\[
G_1(x) = \sum_{p \in T_2} \alpha_p(f, \phi_p) \psi_\xi^p(x) \\
= \sum_{p \in T_2} \alpha_p(f, \phi_p) \left( e^{2\pi i \xi_0 \cdot (\cdot)} K(\cdot) * \phi_p(\cdot) \right)(x),
\]
\[
G_2(x) = \sum_{p \in T_2} \alpha_p(f, \phi_p) \phi_p.
\]

Claim 1.
\[
(6.6) \quad |F_{2J}(x)| \leq C \left( \sup_{J \subset I} \frac{1}{|I|} \int_I |G_1(z)| \, dz + \sup_{J \subset I} \frac{1}{|I|} \int_I |G_2(z)| \, dz \right),
\]
where the suprema above are taken over all cubes \( I \) containing \( J \).

The proof of the claim is given in the next section. One should recognize the claim as a slight variant of the classical inequality
\[
T^* \tilde{f} \lesssim M(T \tilde{f}) + M(\tilde{f}),
\]
where
\[
Tg = \left( e^{2\pi i \xi_0 \cdot (\cdot)} K(\cdot) \right) * g, \quad \tilde{f} = G_2(x),
\]
\( T^* \) is the maximal operator corresponding to \( T \), and \( M \) denotes the Hardy-Littlewood maximal function. In the rest of this section we show how the proof of Theorem 1 may be completed using (6.6).
We observe that the right hand side of the above expression is constant on $J$ and that $F_2,1,J$ is supported on $G_J$, which is of measure less than or equal to $C|J|$. Hence,

$$\sum_{J \in \mathcal{J}} ||F_{2,J}||_{L^1(J)} \leq C \mu \sum_{J \in \mathcal{J}, J \subset 3I_T} |J| \left( \sup_{J \subset I} \frac{1}{|J|} \int_I |G_1(z)| \, dz + \sup_{J \subset I} \frac{1}{|J|} \int_I |G_2(z)| \, dz \right)$$

$$\leq C \mu \left( \left\| M \left( \sum_{p \in T_2} \alpha_p \langle f, \phi_p \rangle \psi_{\xi_p} \right) \right\|_{L^1(3I_T)} \right) \left( \left\| M \left( \sum_{p \in T_2} \alpha_p \langle f, \phi_p \rangle \phi_p \right) \right\|_{L^1(3I_T)} \right)$$

$$\leq C \mu |I_T|^{1/2} \left( \sum_{p \in T_2} \alpha_p \langle f, \phi_p \rangle \psi_{\xi_p} \right)_{L^2(\mathbb{R}^n)} + \sum_{p \in T_2} \alpha_p \langle f, \phi_p \rangle \phi_p \right)_{L^2(\mathbb{R}^n)} \right).$$

Here we have used the $L^2$ boundedness of the Hardy-Littlewood maximal function $M$. We would now like to show that $||G_1||_{L^2}$ and $||G_2||_{L^2}$ are bounded above by a constant multiple of $\epsilon |I_T|^{1/2}$.

$$||G_1||_{L^2}^2 = \sum_{p,p' \in T_2} \alpha_p \alpha_p' \langle f, \phi_p \rangle \langle f, \phi_{p'} \rangle \langle \psi_{\xi_p}, \psi_{\xi_{p'}} \rangle$$

$$= \sum_{p,p' \in T_2} \alpha_p \alpha_p' \langle f, \phi_p \rangle \langle f, \phi_{p'} \rangle \langle (m(\cdot - \xi_0)\phi_p, m(\cdot - \xi_0)\phi_{p'}) \rangle$$

$$+ \sum_{p,p' \in T_2, \omega_p = \omega_{p'}} \alpha_p \alpha_p' \langle f, \phi_p \rangle \langle f, \phi_{p'} \rangle \langle \psi_{\xi_p}, \psi_{\xi_{p'}} \rangle.$$ Similarly,

$$||G_2||_{L^2}^2 = \sum_{p,p' \in T_2} \alpha_p \alpha_{p'} \langle f, \phi_p \rangle \langle f, \phi_{p'} \rangle \langle \phi_p, \phi_{p'} \rangle$$

$$= \sum_{p,p' \in T_2, \omega_p \neq \omega_{p'}} \alpha_p \alpha_{p'} \langle f, \phi_p \rangle \langle f, \phi_{p'} \rangle \langle \phi_p, \phi_{p'} \rangle$$

$$+ \sum_{p,p' \in T_2, \omega_p = \omega_{p'}} \alpha_p \alpha_{p'} \langle f, \phi_p \rangle \langle f, \phi_{p'} \rangle \langle \phi_p, \phi_{p'} \rangle.$$ But $\langle \phi_p, \phi_{p'} \rangle = 0$ for $p,p' \in T_2, \omega_p \neq \omega_{p'}$, since $\phi_p$ and $\phi_{p'}$ have disjoint supports in this case. Therefore we only need to consider the second sum in the expressions for $||G_1||_{L^2}^2$ and $||G_2||_{L^2}^2$. Our pointwise bounds imply that $\langle \phi_p, \phi_{p'} \rangle, \langle \psi_{\xi_p}, \psi_{\xi_{p'}} \rangle \lesssim \epsilon$.
so it is enough to estimate the right hand side above for $p, p'$ satisfying $\omega_p = \omega_{p'}$. Upon simplification this reduces to

$$|I_p|^{-1} \int_{\mathbb{R}^n} \left(1 + \frac{|x - c(I_p)|}{|I_p|^\frac{1}{p}}\right)^{-\nu} \left(1 + \frac{|x - c(I_{p'})|}{|I_{p'}|^\frac{1}{p}}\right)^{-\nu} \, dx,$$

With this estimate, the proof that $|||G_1|||_{L^2}$ and $|||G_2|||_{L^2}$ are less than $C\varepsilon |I_T|^\frac{1}{2}$ is similar to an argument outlined in the proof of Lemma 2. One needs to follow the proof of the estimate of the first term of (6.5) to complete the proof of Lemma 3, given the claim.

7. Proof of Claim 1

Let us estimate the first term in the expression (6.3). We denote by $\tilde{c}$ any one of the two constants $\frac{1}{6}|\omega_+|^{-\frac{1}{2}}$ or $\frac{1}{6}|\omega_-|^{-\frac{1}{2}}$. By translation invariance, let $I_-$ and $I_+$ be the unique dyadic cubes of the form

$$I_{\pm} = \prod_{j=1}^n [0, |\omega_{\pm}|^{-\frac{1}{p}}].$$

For a dyadic cube $I = \prod_{j=1}^n [0, 2^k]$ and $r = (r_1, r_2, \cdots, r_n) \in \mathbb{Z}^n$, $I + r$ will denote the unique dyadic cube of the form

$$[r_1 2^k, (r_1 + 1)2^k] \times \cdots \times [r_n 2^k, (r_n + 1)2^k].$$

Now,

$$\int |G_1(y)| \tilde{c}^{-n} \left| \phi \left(\frac{x - y}{\tilde{c}}\right) \right| \, dy$$

$$\leq \tilde{c}^{-n} \int_{|x - y| \leq \tilde{c}} |G_1(y)| \, dy + \tilde{c}^{-n} \sum_{j \geq 1} 2^{-j\nu} \int_{|x - y| \sim 2^j} |G_1(y)| \, dy$$

$$\leq \tilde{c}^{-n} \int_{|x - y| \leq \tilde{c}} |G_1(y)| \, dy + \sum_{j \geq 1} 2^{-j} \frac{1}{(\tilde{c}2^j)^n} \int_{|x - y| \leq \tilde{c}2^j} |G_1(y)| \, dy$$

$$\leq \sup_{I: J \subset I} \frac{1}{|I|} \int_I |G_1(z)| \, dz.$$
Here we have used the fact that since \( x \in J \) and \( |J| < |I_+| < |I_-| \), we have \( J \subset \{ y : |x_i - y_i| \leq \epsilon \text{ for all } i, 1 \leq i \leq n \} \).

We denote by \( B \) the second term in (5.7).

\[
B = \int_{(y,z) \in \mathbb{R}^n \times \mathbb{R}^n} \left( e^{2\pi i N(x) y} - e^{2\pi i \xi_0 y} \right) K(y) \\
\quad \times \left[ e^{2\pi i c(z-y) \left( \frac{1}{6} |\omega_+|^{-\frac{1}{n}} \right)} - e^{2\pi i c(z-y) \left( \frac{1}{6} |\omega_-|^{-\frac{1}{n}} \right)} \right] G_2(x-z) \, dy \, dz.
\]

To estimate \( B \) we write it as

\[
B = B_1 - B_2 + B_3 + B_4,
\]

where

\[
B_1 := \int_{x \in I_+} \int_{y \in \mathbb{R}^n} \left( e^{2\pi i N(x) y} - e^{2\pi i \xi_0 y} \right) K(y) e^{2\pi i c(z-y) \left( \frac{1}{6} |\omega_+|^{-\frac{1}{n}} \right)} G_2(x-z) \, dy \, dz,
\]

\[
B_2 := \int_{x \in I_-} \int_{y \in \mathbb{R}^n} \left( e^{2\pi i N(x) y} - e^{2\pi i \xi_0 y} \right) K(y) e^{2\pi i c(z-y) \left( \frac{1}{6} |\omega_-|^{-\frac{1}{n}} \right)} G_2(x-z) \, dy \, dz,
\]

\[
B_3 := \sum_{r \in \mathbb{Z}^n \setminus \{0\}} \int_{x \in I_- + r} \int_{y \in \mathbb{R}^n} \left( e^{2\pi i N(x) y} - e^{2\pi i \xi_0 y} \right) K(y) \\
\quad \times \left[ e^{2\pi i c(z-y) \left( \frac{1}{6} |\omega_+|^{-\frac{1}{n}} \right)} - e^{2\pi i c(z-y) \left( \frac{1}{6} |\omega_-|^{-\frac{1}{n}} \right)} \right] G_2(x-z) \, dy \, dz,
\]

and

\[
B_4 := \int_{x \in I_- \setminus I_+} \int_{y \in \mathbb{R}^n} \left( e^{2\pi i N(x) y} - e^{2\pi i \xi_0 y} \right) K(y) e^{2\pi i c(z-y) \left( \frac{1}{6} |\omega_+|^{-\frac{1}{n}} \right)} G_2(x-z) \, dy \, dz.
\]
We have the following bound for $B_1$:

$$|B_1| \lesssim \left[ \int_{z \in I_+} |\omega_+| \int_{y \in I_+} |\omega_+| \sup_{m \in \mathbb{Z}^n \setminus \{0\}} \int_{y \in I_+ + m} \int_{z \in I_+} \int_{y \in I_+ + m} \left| e^{2\pi i \epsilon \mathbf{x} \cdot \mathbf{y}} - e^{2\pi i \epsilon \mathbf{x}_0 \cdot \mathbf{y}} \right| K(y) \left| |\omega_+| \phi \left( \frac{z - y}{|\omega_+|^{\frac{1}{2}}} \right) \right| d\mathbf{y} |G_2(\mathbf{x} - \mathbf{z})| d\mathbf{z} \right]$$

$$\lesssim \int_{z \in I_+} |\omega_+| \int_{y \in I_+} |\mathbf{N}(\mathbf{x}) - \mathbf{x}_0||y|^{1-\nu} d\mathbf{y} |G_2(\mathbf{x} - \mathbf{z})| d\mathbf{z}$$

$$\lesssim |\omega_+| |\omega_-|^{\frac{1}{2}} |I_+|^{\frac{1}{2}} \int_{z \in I_+} |G_2(\mathbf{x} - \mathbf{z})| d\mathbf{z}$$

$$\lesssim \sup_{I : J \subset I} \frac{1}{|I|} \int_I |G_2(\mathbf{z})| d\mathbf{z}$$

for large $\nu$. The treatment for $B_2$ is similar.

Next we estimate $B_3$. Note that it suffices to consider $|r| \geq 2$. The case $|r| < 2$ follows with a similar argument as in the treatment of $B_1$.

$$B_3 = \sum_{r \in \mathbb{Z}^n \setminus \{0\}} \int_{z \in I_+ + r} \int_{\xi \in \mathbb{R}^n} (m(\mathbf{x} - \mathbf{N}(\mathbf{x})) - m(\mathbf{x} - \mathbf{x}_0)) e^{2\pi i \mathbf{x} \cdot \mathbf{\xi}}$$

$$\times \left[ \hat{\phi} \left( \frac{1}{6} |\omega_+|^{-\frac{1}{2}} (\mathbf{\xi} - c(\mathbf{\omega}_+)) \right) - \hat{\phi} \left( \frac{1}{6} |\omega_-|^{-\frac{1}{2}} (\mathbf{\xi} - c(\mathbf{\omega}_-)) \right) \right] d\mathbf{\xi} |G_2(\mathbf{x} - \mathbf{z})| d\mathbf{z}.$$
For simplicity, let us consider only those terms where \((L_\xi)^N\) is applied to either one of the terms

\[ m(\xi - N(x)) - m(\xi - \xi_0)) \text{ or } \left( \hat{\phi} \left( \frac{1}{6} |\omega_+|^{-\frac{1}{5}} (\xi - c(\omega_+)) \right) - \hat{\phi} \left( \frac{1}{6} |\omega_-|^{-\frac{1}{5}} (\xi - c(\omega_-)) \right) \right). \]

The analysis for the other terms is similar. First let us look at the inner integral

\[
(7.1) \quad \int_{\xi \in \mathbb{R}^n} e^{2\pi i x \cdot \xi} L_\xi^N [m(\xi - N(x)) - m(\xi - \xi_0)]) \times \left[ \left( \hat{\phi} \left( \frac{1}{6} |\omega_+|^{-\frac{1}{5}} (\xi - c(\omega_+)) \right) - \hat{\phi} \left( \frac{1}{6} |\omega_-|^{-\frac{1}{5}} (\xi - c(\omega_-)) \right) \right] d\xi.
\]

We observe that

\[ |L_\xi^N [(m(\xi - N(x)) - m(\xi - \xi_0))]| \lesssim (\xi - N(x))^{-2N} + |\xi - \xi_0|^{-2N}, \]

and that the integrand is supported on \((1 + \frac{1}{2})_\omega \setminus \omega_\cdot\). Also, given \(\omega_+\) and \(\omega_\cdot\), there exists a unique sequence of nested intervals

\[ \omega_\cdot \subset \omega_{p_1} \subset \omega_{p_2} \subset \cdots \subset \omega_{p_{M-1}} = \omega_+, \quad |\omega_{p_{i+1}}| = 2^n |\omega_{p_i}|. \]

It is not difficult to see that if \(\xi \in (1 + \frac{1}{2})_\omega \setminus \omega_{p_i}\) and \(N(x), \xi_0 \in \omega_\cdot \subset \omega_{p_i}\), then

\[ |N(x) - \xi|, |\xi - \xi_0| \gtrsim |\omega_{p_i}|^{\frac{1}{5}}, \quad 1 \leq i \leq M - 1. \]

This implies that the expression in (7.1) is bounded by a constant multiple of

\[ \sum_{i=1}^{M-1} |\omega_{p_i}|^{-\frac{2N}{5}} \lesssim |\omega_\cdot|^{-\frac{2N}{5}}. \]

Next we consider the inner integral

\[
(7.2) \quad \int_{\xi \in \mathbb{R}^n} e^{2\pi i x \cdot \xi} [(m(\xi - N(x)) - m(\xi - \xi_0))]
\]

\[
\times L_\xi^N \left[ \left( \hat{\phi} \left( \frac{1}{6} |\omega_+|^{-\frac{1}{5}} (\xi - c(\omega_+)) \right) - \hat{\phi} \left( \frac{1}{6} |\omega_-|^{-\frac{1}{5}} (\xi - c(\omega_-)) \right) \right] d\xi
\]

\[
= \int_{\xi \in \mathbb{R}^n} e^{2\pi i x \cdot \xi} [(m(\xi - N(x)) - m(\xi - \xi_0))]
\]

\[
\times \left[ |\omega_+|^{-\frac{2N}{5}} \left( \frac{1}{6} \right)^{2N} (L_\xi N) \hat{\phi} \left( \frac{1}{6} |\omega_+|^{-\frac{1}{5}} (\xi - c(\omega_+)) \right) \right.
\]

\[
- |\omega_-|^{-\frac{2N}{5}} \left( \frac{1}{6} \right)^{2N} (L_\xi N) \hat{\phi} \left( \frac{1}{6} |\omega_-|^{-\frac{1}{5}} (\xi - c(\omega_-)) \right) \] \] \] \] d\xi.
\]
Therefore the expression in (7.3) is bounded above by a constant multiple of 
\[ |\omega_+|^{-2N} + |\omega_-|^{-2N} \lesssim |\omega_-|^{-2N}. \]

All the other terms originating from the integration by parts yield the same bound. Also note that for \( \mathbf{z} \in I_- + r \), we have \( |\mathbf{z}|^{-2N} \sim (|r||I_-|)^{-2N} \) since \( |r| \geq 2 \). Therefore choosing \( N \) large enough, we obtain 
\[
|B_3| \lesssim \sum_{r \in \mathbb{Z}^n \setminus \{0\}} \left| \sum_{z \in I_- + r} |\mathbf{z}|^{-2N} |\omega_-|^{-2N} |G_2(x - \mathbf{z})| \right| dz 
\[
\lesssim \sum_{r \in \mathbb{Z}^n \setminus \{0\}} |\omega_-|^{-2N} \left( |r||I_-|^\frac{1}{N} \right)^{-2N} \int_{\mathbf{z} \in I_- + r} |G_2(x - \mathbf{z})| \right| dz 
\[
\lesssim \sum_{r \in \mathbb{Z}^n \setminus \{0\}} |r|^{-2N} \frac{1}{(|r||I_-|^\frac{1}{N})^2} \int_{\mathbf{z} \in C|r|I_-} |G_2(x - \mathbf{z})| \right| dz 
\[
\lesssim \sup_{I: J \subseteq I} \frac{1}{|I|} \int_{I} |G_2(\mathbf{z})| \right| dz,
\]

since \( J \subset C|r|I_- \).

It therefore remains to analyze \( B_4 \). Since \( \mathbf{z} \in I_- \setminus I_+ \), there exists 
\[
\mathbf{r} \in \mathbb{Z}^n \setminus \{0\}, 1 \leq |\mathbf{r}| \lesssim \left( \frac{|\omega_+|}{|\omega_-|} \right)^{\frac{1}{N}}
\]
such that \( \mathbf{z} \in I_+ + \mathbf{r} \). Let us first analyze the integral in \( y \). Now,
\[
\left| \int_{y \in \mathbb{R}^n} \left( e^{2\pi i N(x) \cdot y} - e^{2\pi i \xi_0 \cdot y} \right) K(y) e^{2\pi i \mathbf{z} \cdot (\mathbf{z} - y)} \right| \cdot \left( \frac{1}{6 |\omega_+|^{\frac{1}{N}}} \right)^n \phi \left( \frac{\mathbf{z} - \mathbf{y}}{\frac{1}{6} |\omega_+|^{\frac{1}{N}}} \right) dy 
\[
\lesssim \sum_{m \in \mathbb{Z}^n} \left| \int_{y \in I_+ + m} \left( e^{2\pi i N(x) \cdot y} - e^{2\pi i \xi_0 \cdot y} \right) K(y) e^{2\pi i \mathbf{z} \cdot (\mathbf{z} - y)} \right| \cdot \left( \frac{1}{6 |\omega_+|^{\frac{1}{N}}} \right)^n \phi \left( \frac{\mathbf{z} - \mathbf{y}}{\frac{1}{6} |\omega_+|^{\frac{1}{N}}} \right) dy 
\[
\lesssim \int_{y \in I_+} \left| \omega_- \right|^{\frac{1}{N}} |y|^{1-n} |\omega_+| |\mathbf{r}|^{-v} dy + 
\[
\left( \sum_{m \neq 0: |m - r| \leq \frac{2\pi}{N}} + \sum_{m \neq 0: |m - r| > \frac{2\pi}{N}} \right) \left( |m||I_+|^\frac{1}{N} \right)^{1-n} \left| \omega_- \right|^{\frac{1}{N}} \min \left( 1, |m - r|^{-v} \right) 
\[
\lesssim \left| \omega_- \right|^{\frac{1}{N}} |\omega_+||I_+|^\frac{1}{N} |\mathbf{r}|^{-v} + \left( |r||I_+|^\frac{1}{N} \right)^{1-n} |\omega_-|^{\frac{1}{N}}.
Therefore, 

\[ g \leq (\|z\|^{n-1} - \|\omega\|^{n-1}) \] 

Therefore, 

\[ |B_{\gamma}| \leq |\omega_{\gamma}| \int_{|I_+|^{\frac{n}{n+1}}} |z|^{1-n} |G_2(x - z)| dz \leq g \ast |G_2|(x), \]

where 

\[ g(z) = h(|z|) = |\omega_{\gamma}| |z|^{1-n} 1_{|I_+|^{\frac{n}{n+1}}} 1_{|I_+|^{\frac{n}{n+1}}} \](z). 

We observe that g ∈ L^1, ||g||_1 ≤ C and g is radially decreasing. Let us approximate g from below by \( g_{\gamma} \) defined as follows,

\[
g_{\gamma}(x) = \begin{cases} 0 & \text{if } |x| \leq |I_+|^\frac{n}{n+1} \\ h \left(|I_+|^\frac{n}{n+1} + k\gamma\right) & \text{if } (k-1)\gamma + |I_+|^\frac{n}{n+1} < |x| \leq k\gamma + |I_+|^\frac{n}{n+1}, \\ 0 & \text{if } |x| > |I_+|^\frac{n}{n+1} \end{cases}
\]

where \( k_0 \gamma = |I_+|^{\frac{n}{n+1}} - |I_+|^\frac{n}{n+1} \).

We can write \( g_{\gamma} \) as

\[
g_{\gamma} = -h \left(|I_+|^\frac{n}{n+1} + \gamma\right) 1_{B(0;|I_+|^\frac{n}{n+1})} + \sum_{k=1}^{k_0} \left(h \left(|I_+|^\frac{n}{n+1} + k\gamma\right) - h \left(|I_+|^\frac{n}{n+1} + (k+1)\gamma\right)\right) 1_{B(0;|I_+|^\frac{n}{n+1} + k\gamma)}. 
\]

Therefore,

\[
|g_{\gamma} \ast |G_2| | \leq h(|I_+|^\frac{n}{n+1} + \gamma) |G_2| * 1_{B(0;|I_+|^\frac{n}{n+1})} + \sum_{k} \left(h \left(|I_+|^\frac{n}{n+1} + k\gamma\right) - h \left(|I_+|^\frac{n}{n+1} + (k+1)\gamma\right)\right) |G_2| * 1_{B(0;|I_+|^\frac{n}{n+1} + k\gamma)},
\]

which in turn is bounded by

\[
\left( \sup_{I; J \subseteq I} \frac{1}{|I|} \int_{I} |G_2(z)| dz \right) \times \left[ h \left(|I_+|^\frac{n}{n+1} + \gamma\right) |B(0;|I_+|^\frac{n}{n+1})| \right] + \sum_{k} \left(h \left(|I_+|^\frac{n}{n+1} + k\gamma\right) - h \left(|I_+|^\frac{n}{n+1} + (k+1)\gamma\right)\right) |B(0;|I_+|^\frac{n}{n+1} + k\gamma)| \]

\[
\leq \left( \sup_{I; J \subseteq I} \frac{1}{|I|} \int_{I} |G_2(z)| dz \right) ||\hat{g}_{\gamma}||_1. 
\]
Here $\tilde{g}_\gamma$ is given by
\[
\tilde{g}_\gamma = h \left( |I_+|^\frac{1}{n} + \gamma \right) 1_{B(0;|I_+|^\frac{1}{n} \gamma)} + \sum_k \left( h \left( |I_+|^\frac{1}{n} + k\gamma \right) - h \left( |I_+|^\frac{1}{n} + (k+1)\gamma \right) \right) 1_{B(0;|I_+|^\frac{1}{n} + k\gamma)}.
\]
In other words,
\[
\tilde{g}_\gamma(x) = \begin{cases} 
2h \left( |I_+|^\frac{1}{n} + \gamma \right) & \text{if } 0 \leq |x| \leq |I_+|^\frac{1}{n} \\
h \left( |I_+|^\frac{1}{n} + k\gamma \right) & \text{if } |I_+|^\frac{1}{n} + (k-1)\gamma < |x| < |I_+|^\frac{1}{n} + k\gamma \\
0 & \text{if } |x| > |I_-|^\frac{1}{n}
\end{cases}
\]
Therefore,
\[
\|\tilde{g}_\gamma\|_1 \lesssim \|g\|_1 + |\omega_-|^\frac{1}{n} \int_{|y| \leq |I_+|^\frac{1}{n}} |y|^{-n} \, dy \lesssim 1,
\]
since $|\omega_-| < |\omega_+|$. Thus,
\[
g_\gamma \ast |G_2|(x) \lesssim \sup_{I : J \subseteq I} \frac{1}{|I|} \int_I |G_2(z)| \, dz.
\]
Letting $\gamma \to 0$ and applying the dominated convergence theorem now yields the desired bound for $B_4$.

8. An Application

As an immediate application of the weak $L^2$ mapping property of the maximal dyadic sum operator, we obtain a new proof of Sjölin’s theorem \[ on a weak-type $(2,2)$ estimate for the maximal conjugated Calderón-Zygmund operator on $\mathbb{R}^n$, $n > 1$.

**Theorem 2.** Let
\[
K(x) = \Omega \left( \frac{x}{|x|} \right) |x|^{-n}
\]
be a Calderón-Zygmund kernel in $\mathbb{R}^n$ with $\Omega \in C^\infty(S^{n-1})$. Let
\[
Bf = f \ast K,
\]
and
\[
Cf(x) = \sup_{\xi \in \mathbb{R}^n} \left| e^{2\pi i \xi \cdot x} B e^{-2\pi i \xi \cdot (-)} f \right|(x).
\]
Then,
\[
\|Cf\|_{L^2,\infty} \leq C \|f\|_{L^2},
\]
with a constant $C$ independent of $f$. 
The proof of Theorem 2 again follows techniques similar to those used by Lacey and Thiele in proving Carleson’s theorem on almost everywhere convergence of Fourier series. Following [3], we introduce the operators

\[ A_N f := \sum_{p \in \mathcal{D}} (f, \phi_p) \phi_1 \omega_p(2^n)(\eta) \]

\[ Af := \lim_{N \to \infty} \frac{1}{K_N} \int_{K_N \times [0,1]} M_{-\eta} T_{-y} D_{2^{-n}}^2 A_2 A_2^* T_y M_\eta f \, dy \, d\eta \, d\kappa, \]

where \( K_N \) is any increasing sequence of rectangles filling out \( \mathbb{R}^n \times \mathbb{R}^n \). For any Schwartz function \( f \) and any \( x \in \mathbb{R}^n \), the limit representing \( Af(x) \) exists by the argument given by Lacey and Thiele.

Note that by rotation invariance, it is enough to prove Theorem 2 when the multiplier is supported on a nonempty open cone in \( \mathbb{R}^n \).

**Lemma 6.** There exists a nonempty open cone \( \tilde{K}_0 \) with vertex at the origin,

\[ \tilde{K}_0 \subset \{ \xi = (\xi_1, \xi_2, \cdots, \xi_n); \xi_i \leq 0 \text{ for all } i \}, \]

such that for all \( \xi \in \tilde{K}_0 \),

\[ (Af)(\xi) = c \hat{f}(\xi), \]

where \( c \) is a constant independent of \( f \).

**Proof.**

\[ (Af)(\xi) = \lim_{N \to \infty} \frac{1}{|K_N|} \int_{K_N \times [0,1]} (M_{-\eta} T_{-y} D_{2^{-n}}^2 A_2 A_2^* T_y M_\eta f)(\xi) \, dy \, d\eta \, d\kappa \]

\[ = \lim_{N \to \infty} \frac{1}{|K_N|} \int_{K_N \times [0,1]} \sum_{p \in \mathcal{D}} (f, M_{-\eta} T_{-y} D_{2^{-n}}^2 A_2 \phi_p)(M_{-\eta} T_{-y} D_{2^{-n}}^2 \phi_p)(\xi) \, dy \, d\eta \, d\kappa \]

\[ = \lim_{N \to \infty} \frac{1}{|K_N|} \int_{K_N \times [0,1]} \sum_{p \in \mathcal{D}} \int_{\mathbb{R}^n} \hat{f}(\xi') 2^{-\kappa n} \phi_p (2^{-\kappa} (\xi' + \eta)) e^{-2\pi i y \cdot (\xi' + \eta)} \, d\xi' \]

\[ \times 2^{-\frac{\alpha n}{2}} \phi_p (2^{-\kappa} (\xi + \eta)) e^{2\pi i y \cdot (\xi + \eta)} \omega_p(2^n) (2^{-\kappa} \eta) \, dy \, d\eta \, d\kappa \]

\[ = \lim_{N \to \infty} \frac{1}{|K_N|} \int_{K_N \times [0,1]} \sum_{p \in \mathcal{D}} \int_{\mathbb{R}^n} \hat{f}(\xi') 2^{-\kappa n + \alpha n} \phi_p \left( 2^{\alpha - \kappa} (\xi' + \eta) - \left( 1 + \frac{1}{4} \right) \right) \]

\[ \times e^{-2\pi i (m + \frac{1}{2}) (2^{-\kappa} (\xi' + \eta) - (1 + \frac{1}{4}))} \phi \left( 2^{\alpha - \kappa} (\xi + \eta) - \left( 1 + \frac{1}{4} \right) \right) \]

\[ \times e^{2\pi i (m + \frac{1}{2}) (2^{-\kappa} (\xi + \eta) - (1 + \frac{1}{4}))} \omega_p(2^n) (2^{-\kappa} \eta) \, dy \, d\eta \, d\xi', \]
where we have expressed $I_p$ and $\omega_p$ as

$$I_p = \prod_{i=1}^{n} [m_i2^\alpha, (m_i + 1)2^\alpha), \quad \omega_p = \prod_{i=1}^{n} [l_i2^{-\alpha}, (l_i + 1)2^{-\alpha})$$

with $m = (m_1, m_2, \ldots, m_n)$, $l = (l_1, l_2, \ldots, l_n) \in \mathbb{Z}^n$.

Interpreting the sum

$$\sum_{m \in \mathbb{Z}^n} e^{2\pi i (m + \frac{1}{4}) 2^{\alpha} (\xi - \xi')};$$

in the sense of distributions, we find that

$$\text{(8.1)} \quad (Af)(\xi) = c \lim_{N \to \infty} \frac{1}{|K_N|} \int_{K_N \times [0,1]} \sum_{l \in \mathbb{Z}^n, \alpha \in \mathbb{Z}} \hat{f}(\xi) \times |\phi \left(2^{\alpha} (\xi + \eta) - (1 + \frac{1}{4})\right)|^2 1_{\omega_p(2^{\alpha})} d\eta d\kappa.$$

Now, $\hat{\phi} \left(2^{\alpha}(\xi + \eta) - (1 + \frac{1}{4})\right)$ is supported on

$$\left\{ (\xi, \eta) : -\frac{1}{10} \leq 2^{\alpha} (\xi_i + \eta_i) - (l_i + \frac{1}{4}) \leq \frac{1}{10} \text{ for all } i, 1 \leq i \leq n \right\} =$$

$$\left\{ (\xi, \eta) : (l_i + \frac{3}{20}) 2^{\kappa - \alpha} \leq \xi_i + \eta_i \leq (l_i + \frac{7}{20}) 2^{\kappa - \alpha} \text{ for all } i, 1 \leq i \leq n \right\}.$$

Also,

$$1_{\omega_p(2^{\alpha})} \neq 0 \Leftrightarrow \left(l_i + \frac{1}{2}\right) 2^{-\alpha} \leq 2^{\kappa - \alpha} \eta_i \leq (l_i + 1) 2^{-\alpha} \Leftrightarrow \left(l_i + \frac{1}{2}\right) 2^{\kappa - \alpha} \leq \eta_i \leq (l_i + 1) 2^{\kappa - \alpha}.$$

Therefore, the integrand in the right hand side of (8.1) is supported in

$$\bigcup_{\alpha \in \mathbb{Z}} \left\{ -\frac{17}{20} 2^{\kappa - \alpha} \leq \xi_i \leq -\frac{3}{20} 2^{\kappa - \alpha} \text{ for all } i, 1 \leq i \leq n \right\}.$$

Moreover, if

$$\text{(8.2)} \quad -\frac{13}{20} 2^{\kappa - \alpha} \leq \xi_i \leq -\frac{7}{20} 2^{\kappa - \alpha} \text{ for all } i, 1 \leq i \leq n,$$

then

$$\left\{ \eta : (l_i + \frac{3}{20}) 2^{\kappa - \alpha} \leq \xi_i + \eta_i \leq (l_i + \frac{7}{20}) 2^{\kappa - \alpha} \text{ for all } i, 1 \leq i \leq n \right\} \subset \left\{ \eta : (l_i + \frac{1}{2}) 2^{\kappa - \alpha} \leq \eta_i \leq (l_i + 1) 2^{\kappa - \alpha} \right\}.$$
Note that there exists a nonempty open cone $\tilde{K}_0$ with vertex at the origin such that for all $\xi \in \tilde{K}_0$, there exist $\alpha \in \mathbb{Z}$ and $\kappa \in [0, 1]$ satisfying (8.2). In the sequel we work with such a cone.

For $\xi \in \tilde{K}_0$ and choosing $K_N = [-N, N]^n \times [-N, N]^n$, we have

$$(Af)^\wedge (\xi) = c \hat{f}(\xi) \lim_{N \to \infty} \frac{1}{|K_N|} \int_{K_N \times [0, 1]} \sum_{l \in \mathbb{Z}^n, \alpha \in \mathbb{Z}} \left| \hat{\phi} \left( 2^{\alpha-\kappa}(\xi + \eta) - \left(1 + \frac{1}{4}\right) \right) \right|^2 dy \, d\eta \, d\kappa$$

$$= c \hat{f}(\xi) \lim_{N \to \infty} \frac{1}{N^n} \int_{[\eta, \kappa) \in [-N, N]^n \times [0, 1]} \sum_{l \in \mathbb{Z}^n, \alpha \in \mathbb{Z}} \left| \hat{\phi} \left( 2^{\alpha-\kappa}(\xi + \eta) - \left(1 + \frac{1}{4}\right) \right) \right|^2 d\eta \, d\kappa$$

$$= c \hat{f}(\xi) \lim_{N \to \infty} N^{-n} \int_0^1 \sum_{l \in \mathbb{Z}^n, \alpha \in \mathbb{Z}} \left| \hat{\phi} \left( 2^{\alpha-\kappa}(\xi + \eta) - \left(1 + \frac{1}{4}\right) \right) \right|^2 d\eta$$

$$= c \hat{f}(\xi) \lim_{N \to \infty} N^{-n} \int_0^1 \sum_{\alpha \in \mathbb{Z}} (2^{\alpha-\kappa})^n \, d\kappa$$

$$= c \hat{f}(\xi) \int_0^1 \sum_{\alpha \in \mathbb{Z}} 1 \, d\kappa$$

$$= c \hat{f}(\xi).$$

Now, let $m$ be the multiplier associated with the Calderón-Zygmund kernel $K$. Then $m \in C^\infty(\mathbb{R}^n \setminus \{0\})$ and is homogeneous of degree 0. Suppose further, without loss of generality, that $m$ is supported on the cone $\tilde{K}_0$ described earlier. We may reduce the problem to this case via a partition of unity and by invoking rotation invariance.

Recalling that

$$Bf = f * K,$$

and

$$Cf(x) = \sup_{\eta} |BM_{\eta}B_{-\eta}f|(x),$$
we define, for $\zeta \in \mathbb{R}^n$, 
\[ B\zeta f := \sum_{p \in D} \langle f, \phi_p \rangle \psi_p \omega_{p(2^n)}(\zeta), \]

where 
\[ \left( \psi_p \right) \omega(\xi) = m(\xi - \zeta) \hat{\phi}_p(\xi). \]

Using Lemma 6, it is not hard to see that 
\[ Bf := \lim_{N \to \infty} \frac{1}{|K_N|} \int_{K_N \times [0,1]} M_{-\eta} T_{-\eta} D^2_{2^{-\kappa}} B_{2^{-\kappa}} D^2_{2^{-\kappa}} T_y M_\eta f \, d\eta \, d\kappa. \]

In fact,
\[ (Bf)^\wedge(\xi) = \lim_{N \to \infty} \frac{1}{|K_N|} \int_{K_N \times [0,1]} \sum_{p \in D} \langle f, M_{-\eta} T_{-\eta} D^2_{2^{-\kappa}} \phi_p \rangle \times \left( M_{-\eta} T_{-\eta} D^2_{2^{-\kappa}} \psi^2_{2^{-\kappa}} \right) \left( 1_{\omega_{p(2^n)}}(2^{-\kappa}) \right) \, d\eta \, d\kappa. \]

Since 
\[ T_{-\eta} M_\eta D^2_{2^{-\kappa}} \left[ m(\xi - \eta 2^{-\kappa}) \hat{\phi}_p(\xi) \right] = m(\xi) T_{-\eta} M_\eta D^2_{2^{-\kappa}} \hat{\phi}_p(\xi), \]
we get that 
\[ (Bf)^\wedge(\xi) = m(\xi)(Af)^\wedge(\xi) = c \, m(\xi) \hat{f}(\xi), \]
where the last equality follows from the claim and the fact that supp $m \subset K_0$.

Now, for $f \in C_0^\infty(\mathbb{R}^n)$ and $\zeta \in \mathbb{R}^n$
\[ M_\zeta B M_{-\zeta} f(x) \]

\[ = \lim_{l \to \infty} \frac{1}{|K|} \int_{K \times [0,1]} M_{\zeta} M_{-\eta} T_{-\eta} D^2_{2^{-\kappa}} B_{2^{-\kappa}} D^2_{2^{-\kappa}} T_y M_\eta M_{-\xi} f(x) \, d\eta \, d\kappa \]

\[ = \lim_{l \to \infty} \frac{1}{|K|} \int_{K \times [0,1]} M_{\zeta} M_{-\eta} T_{-\eta} D^2_{2^{-\kappa}} B_{2^{-\kappa}} D^2_{2^{-\kappa}} T_y M_\eta M_{-\xi} f(x) \, d\eta \, d\kappa \]

\[ = \lim_{l \to \infty} \frac{1}{|K|} \int_{K \times [0,1]} M_{-\eta} T_{-\eta} D^2_{2^{-\kappa}} B_{2^{-\kappa}}(\eta^* + \xi) D^2_{2^{-\kappa}} T_y M_\eta f(x) \, d\eta \, d\kappa \]

\[ = \lim_{l \to \infty} \frac{1}{|K|} \int_{K \times [0,1]} M_{-\eta} T_{-\eta} D^2_{2^{-\kappa}} B_{2^{-\kappa}}(\eta^* + \xi) D^2_{2^{-\kappa}} T_y M_\eta f(x) \, d\eta \, d\kappa, \]
where $K^\zeta := \zeta + K_i$. The last equality follows from the fact that for $f \in C_0^\infty(\mathbb{R}^n)$ and any fixed $\zeta$, the integrand gets arbitrarily small on the domain $K^\zeta \Delta K^i$ as $l \to \infty$. The details are left to the interested reader.

Therefore,

$$
\sup_{\zeta} |M_{\zeta} B M_{-\zeta} f(x)| \leq \lim_{l \to \infty} \frac{1}{|K_l|} \int_{K_l \times [0,1]} \left| M_{-\eta'} T_{-y} D_{2^{-\kappa}}^2 \left( \sup_{\zeta} B_\zeta \right) D_{2^\kappa}^2 T_y M_{\eta'} f(x) \right| \, dy \, d\eta' \, d\kappa.
$$

We recall the following fact about the weak $L^2$ norm: there exist universal constants $C_1, C_2$ such that

$$
C_1 \sup_E \frac{|\{f, 1_E\}|}{|E|^{1/2}} \leq \|f\|_{L^{2,\infty}} \leq C_2 \sup_E \frac{|\{f, 1_E\}|}{|E|^{1/2}},
$$

where the supremum is over all measurable sets $E$ with finite Lebesgue measure. This implies

$$
\left\| \sup_{\zeta} |M_{\zeta} B M_{-\zeta} f| \right\|_{L^{2,\infty}} \leq \sup_E \frac{1}{|E|^{1/2}} \int_E \left[ \lim_{l \to \infty} \frac{1}{|K_l|} \int_{K_l \times [0,1]} \left| M_{-\eta'} T_{-y} D_{2^{-\kappa}}^2 \left( \sup_{\zeta} B_\zeta \right) D_{2^\kappa}^2 T_y M_{\eta'} f(x) \right| \, dy \, d\eta' \, d\kappa \right] \, dx
$$

$$
\leq \lim_{l \to \infty} \frac{1}{|K_l|} \int_{K_l \times [0,1]} \left[ \sup_E \frac{1}{|E|^{1/2}} \int_E \left| M_{-\eta'} T_{-y} D_{2^{-\kappa}}^2 \left( \sup_{\zeta} B_\zeta \right) D_{2^\kappa}^2 T_y M_{\eta'} f(x) \right| \, dy \, d\eta' \, d\kappa \right] \, dx
$$

$$
\leq \lim_{l \to \infty} \frac{1}{|K_l|} \int_{K_l \times [0,1]} \left\| M_{-\eta'} T_{-y} D_{2^{-\kappa}}^2 \left( \sup_{\zeta} B_\zeta \right) D_{2^\kappa}^2 T_y M_{\eta'} f \right\|_{L^{2,\infty}} \, dy \, d\eta' \, d\kappa
$$

since the weak $L^2$ norm is invariant under the translation, dilation and modulation operators defined in Section 2. The same invariance properties also hold true for the $L^2$ norm. In order to prove the weak $L^2$ bound for the Carleson operator, it therefore suffices to show that

$$
\| \sup_{\zeta} B_\zeta f \|_{L^{2,\infty}} \leq C \|f\|_{L^2},
$$

which is the conclusion of Theorem 1.
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