Ricci flow on Kähler manifolds

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1 Introduction and main theorems

In the last two decades, the Ricci flow, introduced by R. Hamilton in [7], has been a subject of intense study. The Ricci flow provides an indispensable tool of deforming Riemannian metrics towards canonical metrics, such as Einstein ones. It is hoped that by deforming a metric to a canonical metric, one can further understand geometric and topological structures of underlying manifolds. For instance, it was proved [7] that any closed 3-manifold of positive Ricci curvature is diffeomorphic to a spherical space form. We refer the readers to [10] for more information.

If the underlying manifold is a Kähler manifold, the normalized Ricci flow in a canonical Kähler class preserves the Kähler class. It follows that the Ricci flow can be reduced to a fully nonlinear parabolic equation on almost pluri-subharmonic functions:

\[
\frac{\partial \varphi}{\partial t} = \log \det \left( \frac{(\omega + \partial \bar{\partial} \varphi)^n}{\omega^n} \right) + \varphi - h, 
\]

where \( \varphi \) is the evolved Kähler potential; and \( \omega \) is the fixed Kähler metric in the canonical Kähler class, while \( \text{Ric}(\omega) \) is the corresponding Ricci form and

\[
\text{Ric}(\omega) - \omega = \partial \bar{\partial} h, \quad \text{and} \quad \int_M (e^{h} - 1) \omega^n = 0.
\]

Usually, this reduced flow is called the Kähler Ricci flow. H.D. Cao [2] proved that the Kähler Ricci flow always has a global solution. He also proved that the solution converges to a Kähler-Einstein metric if the first Chern class of the underlying Kähler manifold is zero or negative. Consequently, he reproved the famous Calabi-Yau theorem [17]. On the other hand, if the first Chern class of the underlying Kähler manifold is positive, the solution of a Kähler Ricci flow may not converge to any Kähler-Einstein metric. This is because there are compact Kähler manifolds with positive first Chern class which do not admit any Kähler-Einstein metrics (cf. [8], [13]). A natural and challenging problem is whether or not the Kähler Ricci flow on a compact Kähler-Einstein manifold

1A Kähler class is canonical if the first Chern class is proportional to this Kähler class.
manifold converges to a Kähler-Einstein metric. It was proved by S. Bando [1] for 3-dimensional Kähler manifolds and by N. Mok [2] for higher dimensional Kähler manifolds that the positivity of bisectional curvature is preserved under the Kähler Ricci flow. A long standing problem in the study of the Ricci flow is whether or not the Kähler Ricci flow converges to a Kähler-Einstein metric if the initial metric has positive bisectional curvature? In view of the solution of the Frankel conjecture by S. Mori [3] and Siu-Yau [4], we suffice to study this problem on a Kähler manifold which is biholomorphic to $\mathbb{C}P^n$. Since $\mathbb{C}P^n$ admits a Kähler-Einstein metric, the above problem can be restated as follows: On a compact Kähler-Einstein manifold, does the Kähler Ricci flow converge to a Kähler-Einstein metric? In this note, we announce an affirmative solution to this problem.

**Theorem 1.1.** Let $M$ be a Kähler-Einstein manifold with positive scalar curvature. If the initial metric has nonnegative bisectional curvature and positive at least at one point, then the Kähler Ricci flow will converge exponentially fast to a Kähler-Einstein metric with constant bisectional curvature.

**Remark 1.2.** The above theorem in complex dimension 1 was proved first by Hamilton [5]. B. Chow [6] later showed that the assumption that the initial metric has positive curvature in $S^2$ can be removed since the scalar curvature will become positive after finite time anyway.

**Corollary 1.3.** The space of Kähler metrics with non-negative bisectional curvature is path-connected. Moreover, the space of metrics with non-negative curvature operator is also path-connected.

**Remark 1.4.** Using the same arguments, we can also prove the version of our main theorem for Kähler orbifolds.

**Remark 1.5.** What we really need is that the Ricci curvature is positive. Since the condition on Ricci may not be preserved under the Ricci flow, in order to have the positivity of the Ricci curvature, we will use the fact that the positivity of the bisectional curvature is preserved.

**Remark 1.6.** We need to assume the existence of Kähler-Einstein metric because we will use a nonlinear inequality from [16]. Such an inequality is nothing but the Moser-Trudinger-Onofri type inequality if the Kähler-Einstein manifold is the Riemann sphere.

\footnote{In [8], Hamilton proved that the positivity of the curvature operator is preserved under Ricci flow in any compact manifold.}
2 Outline of Proof

The standard methods in the Ricci flow involve the pointwise estimate of curvature by using its evolution equation, the blow-up analysis and the classification of possible singularity type. Unfortunately, there are only a few examples of proving convergence of the Ricci flow to a canonical one (cf. [7] [8] and [11] et al). One of the main reasons is that it is very hard to detect geometric information from a singular model arisen from a blow-up analysis.

In order to overcome this difficulty, we define a set of new functionals $E_k(\phi) (k = 0, 1, \cdots, n)$. The leading term of $E_k (k = 0, 1, \cdots, n)$ is

$$
\int_M \left( \ln \det \left( \frac{\omega_{\phi}^n}{\omega^n} \right) - h_\omega \right) \sum_{i=0}^k Ric(\omega_{\phi})^i \wedge \omega^{k-i} \wedge \omega_{\phi}^{n-k}.
$$

Here $\omega_{\phi} = \omega + \partial \bar{\partial} \phi$ is the Kähler metric determined by the Kähler potential $\phi$. The Euler-Lagrange equation for the functional $E_k (k = 0, 1, \cdots, n)$ is

$$
\Delta_{\omega_{\phi}} \left( \frac{Ric(\omega_{\phi})^k \wedge \omega_{\phi}^{n-k}}{\omega_{\phi}^n} \right) - \frac{n-k}{k+1} \left( \frac{Ric(\omega_{\phi})^{k+1} \wedge \omega_{\phi}^{n-k-1}}{\omega_{\phi}^n} \right) = - \frac{n-k}{k+1} \int_M \omega_{\phi}^n.
$$

If $k = 0$, this leads to the usual constant scalar curvature metric equation. In general, critical metrics may not have constant scalar curvature.

The derivative of the functional $E_k$ over a one parameter family of automorphisms gives rise to a holomorphic invariant $F_k (k = 0, 1, 2, \cdots, n)$. One shall note that these functionals $E_k$ and holomorphic invariants $F_k$ are defined in any Kähler class of any Kähler manifold. However, if the Kähler class is canonical on a Kähler-Einstein manifold, then these invariants vanish simultaneously. It follows that on a Kähler-Einstein manifold, these functionals are invariant under action of automorphisms. This important property enables us to modify the flow by automorphisms so that we can apply the fully nonlinear inequality of Tian [16]. In fact, this inequality is a generalized version of the Moser-Trudinger-Onfri inequality. Using the inequality, we can derive a uniform lower bound of the evolved volume form along the modified Kähler Ricci flow; Consequently, we can derive a uniform lower bound of these functionals along the Kähler Ricci flow.

Furthermore, these functionals essentially decrease along the Kähler Ricci flow. Since the functional $E_k$ has a uniform lower bound over the entire Kähler Ricci flow, by computing its derivative, we have

$$
\int_0^\infty \int_M R(\omega_{\phi})(Ric(\omega_{\phi})^k - \omega_{\phi}^k) \wedge \omega^{n-k} \, dt < C
$$

In $S^2$, we need to require a function to be perpendicular to the first eigenspace of the standard metric in $S^2$ in order to get better estimates. We need to do exactly the same here: we need to adjust the Kähler Ricci flow by automorphisms so that the evolved Kähler potential is always perpendicular to the first eigenspace of a fixed Kähler-Einstein metric.
for some uniform constant $C$. In particular when $k = 1$, we obtain

$$\int_{t=0}^{\infty} \int_M (R(\omega_\varphi) - r)^2 \omega_\varphi^n \, dt < C,$$

where $r$ is the average of the scalar curvature — a constant depending only on the Kähler class.

This means that $\int_M (R(\omega_\varphi) - r)^2 \omega_\varphi^n$ is small for almost all the time. In complex dimension 2, combining this integral estimate with Cao’s Harnack inequality for the scalar curvature and a generalized version of Klingenberg estimate on injective radius, we can prove that the scalar curvature is uniformly bounded from above over the entire Kähler Ricci flow. Once the scalar curvature is uniformly bounded, we then follow the standard arguments in the theory of parabolic equations to deduce the exponential convergence to a Kähler-Einstein metric for this flow. Therefore, we prove Theorem 1 in complex dimension 2.

For high dimensional Kähler manifolds, it is more complicated to control the scalar curvature and we need to use some new ingredients from the theory of pluri-subharmonic functions.

The detailed proof of these results will appear elsewhere.

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