3-uniform hypergraphs: modular decomposition and realization by tournaments

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Abstract

Let $H$ be a 3-uniform hypergraph. A tournament $T$ defined on $V(T) = V(H)$ is a realization of $H$ if the edges of $H$ are exactly the 3-element subsets of $V(T)$ that induce 3-cycles. We characterize the 3-uniform hypergraphs that admit realizations by using a suitable modular decomposition.

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1 Introduction

Let $H$ be a 3-uniform hypergraph. A tournament $T$, with the same vertex set as $H$, is a realization of $H$ if the edges of $H$ are exactly the 3-element subsets of the vertex set of $T$ that induce 3-cycles. The aim of the paper is to characterize the 3-uniform hypergraphs that admit realizations (see [2, Problem 1]). This characterization is comparable to that of the comparability graphs, that is, the graphs admitting a transitive orientation (see [10]).

In Section 2, we recall some of the classic results on modular decomposition of tournaments.

In the section below, we introduce a new notion of module for hypergraphs. We introduce also the notion of a modular covering, which generalizes the notion of a partitive family. In Subsection 3.1, we show that the set of the modules of a hypergraph induces a modular covering. In Subsection 3.2, we consider the

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notion of a strong module, which is the usual strengthening of the notion of a module (for instance, see Subsection 2.1 for tournaments). We establish the analogue of Gallai’s modular decomposition theorem for hypergraphs.

Let $H$ be a realizable 3-uniform hypergraph. Clearly, the modules of the realizations of $H$ are modules of $H$ as well, but the converse is false. Consider a realization $T$ of $H$. In Section 4, we characterize the modules of $H$ that are not modules of $T$. We deduce that a realizable 3-uniform hypergraph and its realizations share the same strong modules. Using Gallai’s modular decomposition theorem, we prove that a realizable 3-uniform hypergraph is prime (i.e. all its modules are trivial) if and only if each of its realizations is prime too. We have similar results when we consider a comparability graph and its transitive orientations (for instance, see [11, Theorem 3] and [11, Corollary 1]).

In Section 5, by using the modular decomposition tree, we demonstrate that a 3-uniform hypergraph is realizable if and only if all its prime, 3-uniform and induced subhypergraphs are realizable. We pursue by characterizing the prime and 3-uniform hypergraphs that are realizable. Hence [2, Problem 1] is solved.

At present, we formalize our presentation. We consider only finite structures. A hypergraph $H$ is defined by a vertex set $V(H)$ and an edge set $E(H)$, where $E(H) \subseteq 2^{V(H)} \setminus \{\emptyset\}$. In the sequel, we consider only hypergraphs $H$ such that $E(H) \subseteq 2^{V(H)} \setminus (\emptyset \cup \{\{v\} : v \in V(H)\})$.

Given $k \geq 2$, a hypergraph $H$ is $k$-uniform if

$$E(H) \subseteq \binom{V(H)}{k}.$$

A hypergraph $H$ is empty if $E(H) = \emptyset$. Let $H$ be a hypergraph. With each $W \subseteq V(H)$, we associate the subhypergraph $H[W]$ of $H$ induced by $W$, which is defined by $V(H[W]) = W$ and $E(H[W]) = \{e \in E(H) : e \subseteq W\}$.

**Definition 1.** Let $H$ be a hypergraph. A subset $M$ of $V(H)$ is a module of $H$ if for each $e \in E(H)$ such that $e \cap M \neq \emptyset$ and $e \setminus M \neq \emptyset$, there exists $m \in M$ such that $e \cap M = \{m\}$ and for every $n \in M$, we have $(e \setminus \{m\}) \cup \{n\} \in E(H)$.

**Notation 2.** Given a hypergraph $H$, the set of the modules of $H$ is denoted by $\mathcal{M}(H)$. For instance, if $H$ is an empty hypergraph, then $\mathcal{M}(H) = 2^{V(H)}$.

We study the set of the modules of a hypergraph. Let $S$ be a set. A family $\mathcal{F}$ of subsets of $S$ is a partitive family [3, Definition 6] on $S$ if it satisfies the following assertions.
• $\emptyset \in \mathcal{F}$, $S \in \mathcal{F}$, and for every $x \in S$, $\{x\} \in \mathcal{F}$.

• For any $M, N \in \mathcal{F}$, $M \cap N \in \mathcal{F}$.

• For any $M, N \in \mathcal{F}$, if $M \cap N \neq \emptyset$, $M \setminus N \neq \emptyset$ and $N \setminus M \neq \emptyset$, then $M \cup N \in \mathcal{F}$ and $(M \setminus N) \cup (N \setminus M) \in \mathcal{F}$.

We generalize the notion of a partitive family as follows.

**Definition 3.** Let $S$ be a set. A modular covering of $S$ is a function $\mathfrak{M}$ which associates with each $W \subseteq S$ a set $\mathfrak{M}(W)$ of subsets of $W$, and which satisfies the following assertions.

(A1) For each $W \subseteq S$, $\mathfrak{M}(W)$ is a partitive family on $W$.

(A2) For any $W, W' \subseteq S$, if $W \subseteq W'$, then

$$\{ M' \cap W : M' \in \mathfrak{M}(W') \} \subseteq \mathfrak{M}(W).$$

(A3) For any $W, W' \subseteq S$, if $W \subseteq W'$ and $W \in \mathfrak{M}(W')$, then

$$\{ M' \in \mathfrak{M}(W') : M' \subseteq W \} = \mathfrak{M}(W).$$

(A4) Let $W, W' \subseteq S$ such that $W \subseteq W'$. For any $M \in \mathfrak{M}(W)$ and $M' \in \mathfrak{M}(W')$, if $M \cap M' = \emptyset$ and $M' \cap W \neq \emptyset$, then $M \in \mathfrak{M}(W \cup M')$.

(A5) Let $W, W' \subseteq S$ such that $W \subseteq W'$. For any $M \in \mathfrak{M}(W)$ and $M' \in \mathfrak{M}(W')$, if $M \cap M' \neq \emptyset$, then $M \cup M' \in \mathfrak{M}(W \cup M')$.

We obtain the following result.

**Proposition 4.** *Given a hypergraph $H$, the function defined on $2^{V(H)}$, which maps each $W \subseteq V(H)$ to $\mathfrak{M}(H[W])$, is a modular covering of $V(H)$.*

Let $H$ be a hypergraph. By Proposition 4, $\emptyset, V(H)$ and $\{v\}$, where $v \in V(H)$, are modules of $H$, called trivial. A hypergraph $H$ is indecomposable if all its modules are trivial, otherwise it is decomposable. A hypergraph $H$ is prime if it is indecomposable with $v(H) \geq 3$.

To state Gallai’s modular decomposition theorem, we need to define the quotient of a hypergraph by a modular partition (see Section 2).

**Definition 5.** Let $H$ be a hypergraph. A partition $P$ of $V(H)$ is a modular partition of $H$ if $P \subseteq \mathcal{M}(H)$. Given a modular partition $P$ of $H$, the quotient $H/P$ of $H$ by $P$ is defined on $V(H)/P = P$ as follows. For $E \subseteq P$, $E \in E(H/P)$ if $|E| \geq 2$, and there exists $e \in E(H)$ such that $E = \{ X \in P : X \cap e \neq \emptyset \}$.

As for tournaments, we introduce the following strengthening of the notion of a module. Let $H$ be a hypergraph. A module $M$ of $H$ is strong if for every module $N$ of $H$, we have

if $M \cap N \neq \emptyset$, then $M \subseteq N$ or $N \subseteq M$. 

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Notation 6. We denote by $\Pi(H)$ the set of proper strong modules of $H$ that are maximal under inclusion. Clearly, $\Pi(H)$ is a modular partition of $H$ when $v(H) \geq 2$.

Gallai’s modular decomposition theorem for hypergraphs follows. It is the analogue of Theorem 16.

**Theorem 7.** Given a hypergraph $H$ with $v(H) \geq 2$, $H/\Pi(H)$ is an empty hypergraph, a prime hypergraph or a complete graph (i.e. $E(H/\Pi(H)) = \binom{\Pi(H)}{2}$).

A realization of a 3-uniform hypergraph is defined as follows. To begin, we associate with each tournament a 3-uniform hypergraph in the following way.

**Definition 8.** The 3-cycle is the tournament $C_3 = (\{0, 1, 2\}, \{01, 12, 20\})$. Given a tournament $T$, the $C_3$-structure of $T$ is the 3-uniform hypergraph $C_3(T)$ defined on $V(C_3(T)) = V(T)$ by

$$E(C_3(T)) = \{X \subseteq V(T) : T[X] is isomorphic to C_3\}$$

(see [2]).

**Definition 9.** Given a 3-uniform hypergraph $H$, a tournament $T$, with $V(T) = V(H)$, realizes $H$ if $H = C_3(T)$. We say also that $T$ is a realization of $H$.

Whereas a realizable 3-uniform hypergraph and its realizations do not have the same modules, they share the same strong modules.

**Theorem 10.** Consider a realizable 3-uniform hypergraph $H$. Given a realization $T$ of $H$, $H$ and $T$ share the same strong modules.

The next result follows from Theorems 7 and 10.

**Theorem 11.** Consider a realizable 3-uniform hypergraph $H$. For a realization $T$ of $H$, we have $H$ is prime if and only if $T$ is prime.

Lastly, we characterize the realizable 3-uniform hypergraphs. To begin, we establish the following theorem by using the modular decomposition tree.

**Theorem 12.** Given a 3-uniform hypergraph $H$, $H$ is realizable if and only if for every $W \subseteq V(H)$ such that $H[W]$ is prime, $H[W]$ is realizable.

We conclude by characterizing the prime and 3-uniform hypergraphs that are realizable (see Theorems 53 and 55).

## 2 Background on tournaments

A tournament is a linear order if it does not contain $C_3$ as a subtournament. Given $n \geq 2$, the usual linear order on $\{0, \ldots, n - 1\}$ is the tournament $L_n = (\{0, \ldots, n - 1\}, \{pq : 0 \leq p < q \leq n - 1\})$. With each tournament $T$, associate its dual $T^*$ defined on $V(T^*) = V(T)$ by $A(T^*) = \{vw : vw \in A(T)\}$. 
2.1 Modular decomposition of tournaments

Let $T$ be a tournament. A subset $M$ of $V(T)$ is a module \cite{14} of $T$ provided that for any $x, y \in M$ and $v \in V(T)$, if $xv, vy \in A(T)$, then $v \in M$. Note that the notions of a module and of an interval coincide for linear orders.

**Notation 13.** Given a tournament $T$, the set of the modules of $T$ is denoted by $\mathcal{M}(T)$.

We study the set of the modules of a tournament. We need the following weakening of the notion of a partitive family. Given a set $S$, a family $\mathcal{F}$ of subsets of $S$ is a weakly partitive family on $S$ if it satisfies the following assertions.

- $\emptyset \in \mathcal{F}$, $S \in \mathcal{F}$, and for every $x \in S$, $\{x\} \in \mathcal{F}$.
- For any $M, N \in \mathcal{F}$, $M \cap N \in \mathcal{F}$.
- For any $M, N \in \mathcal{F}$, if $M \cap N \neq \emptyset$, then $M \cup N \in \mathcal{F}$.
- For any $M, N \in \mathcal{F}$, if $M \setminus N \neq \emptyset$, then $N \setminus M \in \mathcal{F}$.

The set of the modules of a tournament is a weakly partitive family (for instance, see \cite{5}). We generalize the notion of a weakly partitive family as follows.

**Definition 14.** Let $S$ be a set. A weak modular covering of $S$ is a function $\mathfrak{M}$ which associates with each $W \subseteq S$ a set $\mathfrak{M}(W)$ of subsets of $W$, and which satisfies Assertions (A2),..., (A5) (see Definition 3), and the following assertion. For each $W \subseteq S$, $\mathfrak{M}(W)$ is a weakly partitive family on $W$.

Since the proof of the next proposition is easy and long, we omit it.

**Proposition 15.** Given a tournament $T$, the function defined on $2^{V(H)}$, which maps each $W \subseteq V(H)$ to $\mathfrak{M}(T[W])$, is a weak modular covering of $V(T)$.

Let $T$ be a tournament. By Proposition 15, $\emptyset, V(T)$ and $\{v\}$, where $v \in V(T)$, are modules of $T$, called trivial. A tournament is indecomposable if all its modules are trivial, otherwise it is decomposable. A tournament $T$ is prime if it is indecomposable with $v(T) \geq 3$.

We define the quotient of a tournament by considering a partition of its vertex set in modules. Precisely, let $T$ be a tournament. A partition $P$ of $V(T)$ is a modular partition of $T$ if $P \in \mathcal{M}(T)$. With each modular partition $P$ of $T$, associate the quotient $T/P$ of $T$ by $P$ defined on $V(T/P) = P$ as follows. Given $X, Y \in P$ such that $X \neq Y$, $XY \in A(T/P)$ if $xy \in A(T)$, where $x \in X$ and $y \in Y$.

We need the following strengthening of the notion of module to obtain an uniform decomposition theorem. Given a tournament $T$, a subset $X$ of $V(T)$ is a strong module \cite{9,12} of $T$ provided that $X$ is a module of $T$ and for every module $M$ of $T$, if $X \cap M \neq \emptyset$, then $X \subseteq M$ or $M \subseteq X$. With each tournament $T$, with $v(T) \geq 2$, associate the set $\Pi(T)$ of the maximal strong module of $T$ under the inclusion amongst all the proper and strong modules of $T$. Gallai’s modular decomposition theorem follows.
Theorem 16 (Gallai [9, 12]). Given a tournament $T$ such that $v(T) \geq 2$, $\Pi(T)$ is a modular partition of $T$, and $T/\Pi(T)$ is a linear order or a prime tournament.

Theorem 16 is deduced from the following two results. We use the following notation.

Notation 17. Let $P$ be a partition of a set $S$. For $W \subseteq S$, $W/P$ denotes the subset $\{X \in P : X \cap W \neq \emptyset\}$ of $P$. For $Q \subseteq P$, set

$$\cup Q = \bigcup_{X \in Q} X.$$  

Proposition 18. Given a modular partition $P$ of a tournament $T$, the following two assertions hold.

1. If $M$ is a strong module of $T$, then $M/P$ is a strong module of $T/P$.

2. Suppose that all the elements of $P$ are strong modules of $T$. If $M$ is a strong module of $T/P$, then $\cup M$ is a strong module of $T$.

Theorem 19. Given a tournament $T$, all the strong modules of $T$ are trivial if and only if $T$ is a linear order or a prime tournament.

Definition 20. Given a tournament $T$, the set of the nonempty strong modules of $T$ is denoted by $\mathcal{D}(T)$. Clearly, $\mathcal{D}(T)$ endowed with inclusion is a tree called the modular decomposition tree of $T$.

Let $T$ be a tournament. The next proposition allows us to obtain all the elements of $\mathcal{D}(T)$ by using successively Theorem 16 from $V(T)$ to the singletons.

Proposition 21 (Ehrenfeucht et al. [5]). Given a tournament $T$, consider a strong module $M$ of $T$. For every $N \subseteq M$, the following two assertions are equivalent

1. $N$ is a strong module of $T$;

2. $N$ is a strong module of $T[M]$.

We use the analogue of Proposition 21 for hypergraphs (see Proposition 44) to prove Proposition 46.

2.2 Critical tournaments

Definition 22. Given a prime tournament $T$, a vertex $v$ of $T$ is critical if $T - v$ is decomposable. A prime tournament is critical if all its vertices are critical.

Schmerl and Trotter [13] characterized the critical tournaments. They obtained the tournaments $T_{2n+1}$, $U_{2n+1}$ and $W_{2n+1}$ defined on $\{0, \ldots, 2n\}$, where $n \geq 1$, as follows.
• The tournament $T_{2n+1}$ is obtained from $L_{2n+1}$ by reversing all the arcs between even and odd vertices (see Figure 1).

• The tournament $U_{2n+1}$ is obtained from $L_{2n+1}$ by reversing all the arcs between even vertices (see Figure 2).

• The tournament $W_{2n+1}$ is obtained from $L_{2n+1}$ by reversing all the arcs between $2n$ and the even elements of $\{0, \ldots, 2n-1\}$ (see Figure 3).

**Theorem 23** (Schmerl and Trotter [13]). Given a tournament $\tau$, with $v(\tau) \geq 5$, $\tau$ is critical if and only if $v(\tau)$ is odd, and $\tau$ is isomorphic to $T_{v(\tau)}$, $U_{v(\tau)}$ or $W_{v(\tau)}$.

### 2.3 The $C_3$-structure of a tournament

The $C_3$-structure of a tournament (see Definition 8) is clearly a 3-uniform hypergraph. The main theorem of [2] follows. It plays an important role in Section 5.

**Theorem 24** (Boussaïri et al. [2]). Let $T$ be a prime tournament. For every tournament $T'$, if $C_3(T') = C_3(T)$, then $T' = T$ or $T^*$.
3 Modular decomposition of hypergraphs

Definition 1 is not the usual definition of a module of a hypergraph. The usual definition follows.

**Definition 25.** Let $H$ be a hypergraph. A subset $M$ of $V(H)$ is a module of $H$ if for any $e, f \subseteq V(H)$ such that $|e| = |f|$, $e \setminus M = f \setminus M$, and $e \setminus M \neq \emptyset$, we have $e \in E(H)$ if and only if $f \in E(H)$.

**Remark 26.** Given a hypergraph $H$, a module of $H$ in the sense of Definition 1 is a module in the sense of Definition 25. The converse is not true. Given $n \geq 3$, consider the 3-uniform hypergraph $H$ defined by $V(H) = \{0, \ldots, n-1\}$ and $E(H) = \{0p: 2 \leq p \leq n-1\}$. In the sense of Definition 25, $\{0, 1\}$ is a module of $H$ whereas it is not a module of $H$ in the sense of Definition 1.

Let $H$ be a realizable 3-uniform hypergraph. Consider a realization $T$ of $H$. Given $e \in E(H)$, all the modules of $T[e]$ are trivial. To handle close modular decompositions for $H$ and $T$, we try to find a definition of a module of $H$ for which all the modules of $H[e]$ are trivial as well. This is the case with Definition 1 and not with Definition 25. Moreover, note that, with Definition 25, $H$ and $T$ do not share the same strong modules, but but they do with Definition 1 (see Theorem 10). Indeed, consider the 3-uniform hypergraph $H$ defined on $\{0, \ldots, n-1\}$ in Remark 26. In the sense of Definition 25, $\{0, 1\}$ is a strong module of $H$. Now, consider the tournament $T$ obtained from $L_n$ by reversing all the arcs between 0 and $p \in \{2, \ldots, n-1\}$. Clearly, $T$ realizes $H$. Since $T[\{0, 1, 2\}]$ is a 3-cycle, $\{0, 1\}$ is not a module of $T$, so it is not a strong module.

3.1 Modular covering

The purpose of the subsection is to establish Proposition 4. To begin, we show that the set of the modules of a hypergraph is a partitive family (see Proposition 30). We need the next three lemmas.

**Lemma 27.** Let $H$ be a hypergraph. For any $M, N \in \mathcal{M}(H)$, we have $M \cap N \in \mathcal{M}(H)$.
Proof. Consider \( M, N \in \mathcal{M}(H) \). To show that \( M \cap N \in \mathcal{M}(H) \), consider \( e \in E(H) \) such that \( e \cap (M \cap N) \neq \emptyset \) and \( e \setminus (M \cap N) \neq \emptyset \). Since \( e \cap (M \cap N) \neq \emptyset \), assume for instance that \( e \setminus M \neq \emptyset \). Since \( M \) is a module of \( H \) and \( e \cap M \neq \emptyset \), there exists \( m \in M \) such that \( e \cap M = \{m\} \). Since \( e \cap (M \cap N) \neq \emptyset \), we obtain \( e \cap (M \cap N) = \{m\} \).

Let \( n \in M \cap N \). Since \( M \) is a module of \( H \), \( (e \setminus \{m\}) \cup \{n\} \in E(H) \).

Lemma 28. Let \( H \) be a hypergraph. For any \( M, N \in \mathcal{M}(H) \), if \( M \cap N \neq \emptyset \), then \( M \cup N \in \mathcal{M}(H) \).

Proof. Consider \( M, N \in \mathcal{M}(H) \) such that \( M \cap N \neq \emptyset \). To show that \( M \cup N \in \mathcal{M}(H) \), consider \( e \in E(H) \) such that \( e \cap (M \cup N) \neq \emptyset \) and \( e \setminus (M \cup N) \neq \emptyset \). Since \( e \cap (M \cup N) \neq \emptyset \), assume for instance that \( e \cap M \neq \emptyset \). Clearly \( e \setminus M \neq \emptyset \) because \( e \setminus (M \cup N) \neq \emptyset \). Since \( M \) is a module of \( H \), there exists \( m \in M \) such that \( e \cap M = \{m\} \), and

\[
(e \setminus \{m\}) \cup \{n\} \in E(H) \text{ for every } n \in M. \tag{1}
\]

Consider \( n \in M \cap N \). By (1), \( (e \setminus \{m\}) \cup \{n\} \in E(H) \). Set

\[
f = (e \setminus \{m\}) \cup \{n\}.
\]

Clearly \( n \in f \cap N \). Furthermore, consider \( p \in e \setminus (M \cup N) \). Since \( m \in M \), we have \( p \neq m \), and hence \( p \in f \setminus N \). Since \( N \) is a module of \( H \), we obtain \( f \cap N = \{n\} \) and

\[
(f \setminus \{n\}) \cup \{n'\} \in E(H) \text{ for every } n' \in N. \tag{2}
\]

Since \( (f \setminus \{n\}) \cup \{n'\} = (e \setminus \{m\}) \cup \{n'\} \) for every \( n' \in N \), it follows from (2) that

\[
(e \setminus \{m\}) \cup \{n'\} \in E(H) \text{ for every } n' \in N. \tag{3}
\]

Therefore, it follows from (1) and (3) that

\[
(e \setminus \{m\}) \cup \{n\} \in E(H) \text{ for every } n \in M \cup N.
\]

Moreover, since \( f \cap N = \{n\} \), we have

\[
e \cap N = (\{m\} \cup (e \setminus \{m\})) \cap N
= (\{m\} \cup (f \setminus \{n\})) \cap N
= \{m\} \cap N,
\]

and hence \( e \cap N \subseteq \{m\} \). Since \( e \cap M = \{m\} \), we obtain \( e \cap (M \cup N) = \{m\} \). Consequently, \( M \cup N \) is a module of \( H \).

Lemma 29. Let \( H \) be a hypergraph. For any \( M, N \in \mathcal{M}(H) \), if \( M \setminus N \neq \emptyset \), then \( N \setminus M \in \mathcal{M}(H) \).
Proposition 30. Given a hypergraph $H$, $\mathcal{M}(H)$ is a partitive family on $V(H)$.
Proof. It is easy to verify that $\emptyset \in \mathcal{M}(H)$, $V(H) \in \mathcal{M}(H)$, and for every $v \in V(H)$, $\{v\} \in \mathcal{M}(H)$. Therefore, it follows from Lemmas 21, 28 and 29 that \( \mathcal{M}(H) \) is a weakly partitive family on $V(H)$. To prove that \( \mathcal{M}(H) \) is a partitive family on $V(H)$, consider any $M,N \in \mathcal{M}(H)$ such that $M \cap N \neq \emptyset$, $N \setminus M \neq \emptyset$ and $M \cap N \neq \emptyset$. We have to show that $(M \setminus N) \cup (N \setminus M) \in \mathcal{M}(H)$. Hence consider $e \in E(H)$ such that $e \cap ((M \setminus N) \cup (N \setminus M)) \neq \emptyset$ and $e \setminus ((M \setminus N) \cup (N \setminus M)) \neq \emptyset$. Since $e \cap ((M \setminus N) \cup (N \setminus M)) \neq \emptyset$, assume for instance that $e \cap (M \setminus N) \neq \emptyset$. Clearly $e \setminus (M \setminus N) \neq \emptyset$ because $e \setminus ((M \setminus N) \cup (N \setminus M)) \neq \emptyset$. Since $N \setminus M \neq \emptyset$, it follows from Lemma 29 that $M \setminus N$ is a module of $H$. Thus, there exists $m \in M \setminus N$ such that $e \cap (M \setminus N) = \{m\}$. We distinguish the following two cases.

1. Suppose that $e \subseteq M$. Since $e \setminus ((M \setminus N) \cup (N \setminus M)) \neq \emptyset$, $e \cap (M \cap N) \neq \emptyset$. Therefore $e \cap N \neq \emptyset$. Furthermore, since $e \cap (M \setminus N) \neq \emptyset$, we have $e \setminus N \neq \emptyset$. Since $N$ is a module of $H$, there exists $n \in N$ such that $e \cap N = \{n\}$. Since $e \cap (M \cap N) \neq \emptyset$, we get $e \cap (M \cap N) = \{n\}$. Since $e \subseteq M$ and $e \cap (M \setminus N) = \{m\}$, we obtain $e = mn$. It follows that

$$e \cap ((M \setminus N) \cup (N \setminus M)) = \{m\}. \tag{7}$$

Let $p \in (M \setminus N) \cup (N \setminus M)$. We have to show that

$$(e \setminus \{m\}) \cup \{p\} = np \in E(H). \tag{8}$$

Recall that $M \setminus N$ is a module of $H$. Consequently (5) holds whenever $p \in M \setminus N$. Suppose that $p \in N \setminus M$. Since $N$ is a module of $H$ and $mn \in E(H)$, we get $mp \in E(H)$. Now, since $M$ is a module of $H$ and $mp \in E(H)$, we obtain $np \in E(H)$. It follows that (5) holds for each $p \in (M \setminus N) \cup (N \setminus M)$. Lastly, it follows from (7) that there exists $m \in M \setminus N$ such that

$$\left\{ \begin{array}{l}
e \cap ((M \setminus N) \cup (N \setminus M)) = \{m\} \\
and \\
for each \ p \in (M \setminus N) \cup (N \setminus M), \ (e \setminus \{m\}) \cup \{p\} \in E(H). \end{array} \right.$$

2. Suppose that $e \setminus M \neq \emptyset$. Since $e \setminus (M \setminus N) = \{m\}$, $m \in e \cap M$. Since $M$ is a module of $H$, there exists $m' \in M$ such that $e \cap M = \{m'\}$. Since $e \setminus (M \setminus N) = \{m\}$, we have $m = m'$, and hence

$$e \cap (M \setminus N) = e \cap M = \{m\}. \tag{9}$$

It follows that $e \cap (M \cap N) = \emptyset$. Since $e \setminus ((M \setminus N) \cup (N \setminus M)) \neq \emptyset$, we obtain

$$e \setminus (M \cup N) \neq \emptyset.$$ 

Since $M \cap N \neq \emptyset$, it follows from Lemma 28 that $M \cup N$ is a module of $H$. Therefore, there exists $p \in M \cup N$ such that $e \cap (M \cup N) = \{p\}$, and
for every \( q \in M \cup N \), \((e \setminus \{p\}) \cup \{q\} \in E(H)\). Since \( e \cap M = \{m\} \), we get \( p = m \). Thus, \( e \cap (M \cup N) = \{m\} \), and hence
\[
e \cap ((M \setminus N) \cup (N \setminus M)) = \{m\}.
\]
(9)
Since \( p = m \), we have \((e \setminus \{m\}) \cup \{q\} \in E(H)\) for every \( q \in M \cup N \). It follows that
\[
(e \setminus \{m\}) \cup \{q\} \in E(H)
\]
for every \( q \in (M \setminus N) \cup (N \setminus M) \), where \( \{m\} = e \cap ((M \setminus N) \cup (N \setminus M)) \) by (9).
\[\square\]

To prove Proposition 4, we need the next four lemmas.

**Lemma 31.** Given a hypergraph \( H \), consider subsets \( W \) and \( W' \) of \( V(H) \). If \( W \subseteq W' \), then \( \{M' \cap W : M' \in \mathcal{M}(H[W'])\} \subseteq \mathcal{M}(H[W]) \) (see Definition 8, Assertion (A2)).

**Proof.** Let \( M' \) be a module of \( H[W'] \). To show that \( M' \cap W \) is a module of \( H[W] \), consider \( e \in E(H[W]) \) such that \( e \cap (M' \cap W) \neq \emptyset \) and \( e \setminus (M' \cap W) \neq \emptyset \). We obtain \( e \in E(H[W']) \) and \( e \cap M' \neq \emptyset \). Since \( e \setminus (M' \cap W) \neq \emptyset \) and \( e \subseteq W \), we get \( e \setminus M' \neq \emptyset \). Since \( M' \) is a module of \( H[W'] \), there exists \( m' \in M' \) such that \( e \cap M' = \{m'\} \), and \((e \setminus \{m'\}) \cup \{n'\} \in E(H[W']) \) for each \( n' \in M' \). Let \( n' \in M' \cap W \). Since \( e \subseteq W \), \((e \setminus \{m'\}) \cup \{n'\} \subseteq W \). Hence \((e \setminus \{m'\}) \cup \{n'\} \in E(H[W])\) because \((e \setminus \{m'\}) \cup \{n'\} \in E(H[W'])\). Moreover, since \( e \cap (M' \cap W) \neq \emptyset \) and \( e \cap M' = \{m'\} \), we obtain \( e \cap (M' \cap W) = \{m'\} \).
\[\square\]

**Lemma 32.** Given a hypergraph \( H \), consider subsets \( W \) and \( W' \) of \( V(H) \) such that \( W \subseteq W' \). If \( W \in \mathcal{M}(H[W']) \), then \( \{M' \in \mathcal{M}(H[W']) : M' \subseteq W\} = \mathcal{M}(H[W]) \) (see Definition 8, Assertion (A3)).

**Proof.** By Lemma 31, \( \{M' \in \mathcal{M}(H[W']) : M' \subseteq W\} \subseteq \mathcal{M}(H[W]) \). Conversely, consider a module \( M \) of \( H[W] \). To prove that \( M \) is a module of \( H[W'] \), consider \( e \in E(H[W']) \) such that \( e \cap M \neq \emptyset \) and \( e \setminus M \neq \emptyset \). We distinguish the following two cases.

1. Suppose that \( e \subseteq W \). We obtain \( e \in E(H[W]) \). Since \( M \) is a module of \( H[W] \), there exists \( m \in M \) such that \( e \cap M = \{m\} \), and for each \( n \in M \), we have \((e \setminus \{m\}) \cup \{n\} \subseteq E(H[W]) \). Hence \((e \setminus \{m\}) \cup \{n\} \in E(H[W])\).

2. Suppose that \( e \setminus W \neq \emptyset \). Clearly, \( e \cap W \neq \emptyset \) because \( e \cap M \neq \emptyset \). Since \( W \) is a module of \( H[W'] \), there exists \( w \in W \) such that \( e \cap W = \{w\} \). Furthermore,
\[
\text{for each } w' \in W, (e \setminus \{w\}) \cup \{w'\} \in E(H[W']).
\]
(10)
Since \( e \cap M \neq \emptyset \), we get \( e \cap M = \{w\} \). Clearly, it follows from (10) that \((e \setminus \{w\}) \cup \{w'\} \in E(H[W'])\) for each \( w' \in M \).
\[\square\]
Lemma 33. Given a hypergraph $H$, consider subsets $W$ and $W'$ of $V(H)$ such that $W \subseteq W'$. For any $M \in \mathcal{M}(H[W])$ and $M' \in \mathcal{M}(H[W'])$, if $M \cap M' = \emptyset$ and $M' \cap W \neq \emptyset$, then $M \in \mathcal{M}(H[W \cup M'])$ (see Definition 3 Assertion A4).

Proof. Consider a module $M$ of $H[W]$ and a module $M'$ of $H[W']$ such that $M \cap M' = \emptyset$ and $M' \cap W \neq \emptyset$. We have to show that $M$ is a module of $H[W \cup M']$. Hence consider $e \in E(H[W \cup M'])$ such that $e \cap M \neq \emptyset$ and $e \setminus M \neq \emptyset$. We distinguish the following two cases.

1. Suppose that $e \subseteq W$. We obtain $e \in E(H[W])$. Since $M$ is a module of $H[W]$, there exists $m \in M$ such that $e \cap M = \{m\}$, and for each $n \in M$, we have $(e \setminus \{m\}) \cup \{n\} \in E(H[W])$. Hence $(e \setminus \{m\}) \cup \{n\} \in E(H[W \cup M'])$.

2. Suppose that $e \setminus W \neq \emptyset$. We obtain $e \cap (M' \setminus W) = \emptyset$. Since $e \cap M \neq \emptyset$, we have $e \setminus M' = \emptyset$. Since $M'$ is a module of $H[W']$, there exists $m' \in M'$ such that $e \cap M' = \{m'\}$, and for each $m' \in M'$, $(e \setminus \{m'\}) \cup \{m'\} \in E(H[W'])$. (11)

Since $e \cap (M' \setminus W) = \emptyset$ and $e \cap M' = \{m'\}$, we get $e \cap (M' \setminus W) = \{m'\}$. Let $w' \in W \cap M'$. Set

$$f = (e \setminus \{m'\}) \cup \{w'\}.$$

By (11), $f \in E(H[W'])$. Furthermore, since $e \cap (M' \setminus W) = \{m'\}$, we obtain $f \subseteq W$, and hence $f \in E(H[W])$. Since $e \cap M \neq \emptyset$, we have $f \cap M \neq \emptyset$. Moreover, $w' \in f \setminus M$ because $w' \in W \cap M'$ and $M \cap M' = \emptyset$. Since $M$ is a module of $H[W]$, there exists $m \in M$ such that $f \cap M = \{m\}$. Since $f = (e \setminus \{m'\}) \cup \{w'\}$, with $m', w' \notin M$, we get $e \cap M = f \cap M$, so $e \cap M = \{m\}$.

Lastly, consider $n \in M$. We have to verify that

$$(e \setminus \{m\}) \cup \{n\} \in E(H[W'])$$. (12)

Set

$$g_n = (f \setminus \{m\}) \cup \{n\}.$$

Since $M$ is a module of $H[W]$ such that $f \cap M = \{m\}$ and $w' \notin f \setminus \{m\}$, $g_n \in E(H[W])$. Hence $g_n \in E(H[W'])$. Since $n \in g_n \cap M$ and $M \cap M' = \emptyset$, $n \in g_n \setminus M'$. Clearly, $w' \notin M'$ because $w' \notin W \cap M'$. Furthermore, $w' \notin f$ because $f = (e \setminus \{m'\}) \cup \{w'\}$. Since $g_n = (f \setminus \{m\}) \cup \{n\}$, $m \in M$ and $M \cap M' = \emptyset$, we have $w' \notin g_n$. It follows that $w' \notin g_n \cap M'$. Since $M'$ is a module of $H[W']$, we have $g_n \cap M' = \{w'\}$ and $(g_n \setminus \{w'\}) \cup \{m'\} \in E(H[W'])$. We have

$$(g_n \setminus \{w'\}) \cup \{m'\} = (((f \setminus \{m\}) \cup \{n\}) \setminus \{w'\}) \cup \{m'\}$$

$$= (f \setminus \{m, w'\}) \cup \{m', n\}$$

$$= (((e \setminus \{m'\}) \cup \{w'\}) \setminus \{m, w'\}) \cup \{m', n\}$$

$$= (e \setminus \{m, m', w'\}) \cup \{m', n, w'\}$$

$$= (e \setminus \{m\}) \cup \{n\}.$$
Consider a module $M$. Let $e$ be such that $e \not\subseteq M$. Suppose that $e \cap M' \neq \emptyset$, then $M \cup M' \in \mathcal{M}(H[W \cup M'])$ (see Definition 3, Assertion (A5)).

**Lemma 34.** Given a hypergraph $H$, consider subsets $W$ and $W'$ of $V(H)$ such that $W \subseteq W'$. For any $M \in \mathcal{M}(H[W])$ and $M' \in \mathcal{M}(H[W'])$, if $M \cap M' \neq \emptyset$, then $M \cup M' \in \mathcal{M}(H[W \cup M'])$.

**Proof.** Consider a module $M$ of $H[W]$ and a module $M'$ of $H[W']$ such that $M \cap M' \neq \emptyset$. We have to prove that $M \cup M'$ is a module of $H[W \cup M']$. Hence consider $e \in E(H[W \cup M'])$ such that $e \cap (M \cup M') \neq \emptyset$ and $e \not\subseteq (M \cup M') \neq \emptyset$. Let $m \in M \cap M'$. We distinguish the following two cases.

1. Suppose that $e \cap M' \neq \emptyset$. Clearly $e \in E(H[W'])$. Moreover, $e \not\subseteq M'$ because $e \not\subseteq (M \cup M') \neq \emptyset$. Since $M'$ is a module of $H[W']$, there exists $m' \in M'$ such that $e \cap M' = \{m', n\}$, and $(e \setminus \{m'\}) \cup \{n\} \subseteq E(H[W'])$ for every $n' \in M'$. Hence, for every $n' \in M'$, we have

$$e \setminus \{m'\} \cup \{n\} \subseteq E(H[W \cup M']).$$

In particular, $(e \setminus \{m'\}) \cup \{m\} \subseteq E(H[W \cup M'])$. Set

$$f = (e \setminus \{m'\}) \cup \{m\}.$$  

Since $e \cap M' = \{m'\}$, we obtain $f \cap M' = \{m\}$. Hence $m \not\in f \cap M$. It follows that $f \subseteq E(H[W])$ because $e \subseteq E(H[W \cup M'])$. Clearly $e \not\subseteq M$ because $e \not\subseteq (M \cup M') \neq \emptyset$. Since $M$ is a module of $H[W]$, there exists $n \in M$ such that $f \cap M = \{n\}$, and $(f \setminus \{n\}) \cup \{p\} \subseteq E(H[W])$ for every $p \in M$. Since $m \not\in f \cap M$, we get $m = n$. Therefore, $f \cap M = f \cap M' = \{m\}$. It follows that $f \cap (M \cup M') = \{m\}$, so

$$e \cap (M \cup M') = \{m'\}.$$ 

By (15), it remains to show that $(e \setminus \{m'\}) \cup \{n\} \subseteq E(H[W \cup M'])$ for each $n \in M$. Let $n \in M$. Recall that $f \cap (M \cup M') = \{m\}$ and $e \cap (M \cup M') = \{m'\}$. Thus $e \setminus (M \cup M') = f \setminus (M \cup M')$. Hence $f \setminus (M \cup M') \neq \emptyset$ because $e \not\subseteq (M \cup M') \neq \emptyset$. It follows that $f \setminus M \neq \emptyset$. Recall that $f \subseteq E(H[W])$. Since $M$ is a module of $H[W]$, we obtain $(f \setminus \{m\}) \cup \{n\} \subseteq E(H[W])$. We have

$$(f \setminus \{m\}) \cup \{n\} = ((e \setminus \{m'\}) \cup \{m\}) \cup \{n\} = (e \setminus \{m'\}) \cup \{n\}.$$  

Therefore $(e \setminus \{m'\}) \cup \{n\} \subseteq E(H[W])$, so $(e \setminus \{m'\}) \cup \{n\} \subseteq E(H[W \cup M'])$.

2. Suppose that $e \cap M' = \emptyset$. We get $e \subseteq E(H[W])$. Clearly $e \not\subseteq M$ because $e \not\subseteq (M \cup M') \neq \emptyset$. Furthermore, since $e \cap (M \cup M') \neq \emptyset$ and $e \cap M' = \emptyset$, we obtain $e \cap (M \cup M') \neq \emptyset$. Since $M$ is a module of $H[W]$, there exists $q \in M$ such that

$$e \cap M = \{q\}$$.
and
\[ \text{for every } r \in M, \ (e \setminus \{q\}) \cup \{r\} \in E(H[W]). \]  
(15)

Since \( e \cap M' = \emptyset \), it follows from (14) that \( q \in M \setminus M' \) and
\[ e \cap (M \cup M') = \{q\}. \]  
(16)

By (15), \( (e \setminus \{q\}) \cup \{m\} \in E(H[W]) \). Set \( e' = (e \setminus \{q\}) \cup \{m\} \).

Clearly, \( m \in e' \cap M' \). Moreover, since \( e \cap (M \cup M') = \{q\} \), we obtain
\[
\begin{align*}
\{e' \cap (M \cup M') = \{m\} \\
\text{and} \ \\
(e \setminus (M \cup M') = e' \setminus (M \cup M').
\end{align*}
\]

Therefore \( e' \cap (M \cup M') \neq \emptyset \), and \( e' \setminus (M \cup M') \neq \emptyset \) because \( e \setminus (M \cup M') \neq \emptyset \).

It follows from the first case above applied with \( e' \) that
\[ \text{for every } s \in M \cup M', \ (e' \setminus \{m\}) \cup \{s\} \in E(H[W \cup M']). \]  
(17)

Recall that \( e \cap (M \cup M') = \{q\} \) by (16). Consequently, we have to show that \( (e \setminus \{q\}) \cup \{s\} \in E(H[W \cup M']) \) for every \( s \in M \cup M' \). Let \( s \in M \cup M' \). We have
\[
(e' \setminus \{m\}) \cup \{s\} = ((e \setminus \{q\}) \cup \{m\}) \setminus \{m\} \cup \{s\} = (e \setminus \{q\}) \cup \{s\}.
\]

It follows from (17) that \( (e \setminus \{q\}) \cup \{s\} \in E(H[W \cup M']). \)

Now, we can prove Proposition 4.

**Proof of Proposition 4.** For Assertion (A1) (see Definition 3), consider \( W \subseteq V(H) \). By Proposition 30, \( \mathcal{M}(H[W]) \) is a partitive family on \( W \). Furthermore, it follows from Lemmas 31, 32, 33, and 34 that Assertions (A2), (A3), (A4), and (A5) hold.

### 3.2 Gallai’s decomposition

The purpose of the subsection is to demonstrate Theorem 7. We use the following definition.

**Definition 35.** Let \( P \) be a partition of a set \( S \). Consider \( Q \subseteq P \). A subset \( W \) of \( S \) is a transverse of \( Q \) if \( W \subseteq \cup Q \) and \( |W \cap X| = 1 \) for each \( X \in Q \).

The next remark makes clearer Definition 5.
Remark 36. Consider a modular partition $P$ of a hypergraph $H$. Let $e \in E(H)$ such that $|e|/P \geq 2$ (see Notation 17). Given $X \in e/P$, we have $e \cap X \neq \emptyset$, and $e \setminus X \neq \emptyset$ because $|e|/P \geq 2$. Since $X$ is a module of $H$, we obtain $|e \cap X| = 1$. Therefore, $e$ is a transverse of $e/P$. Moreover, since each element of $e/P$ is a module of $H$, we obtain that each transverse of $e/P$ is an edge of $H$.

Given $E \subseteq P$ such that $|E| \geq 2$, it follows that $E \in E(H/P)$ if and only if every transverse of $E$ is an edge of $H$.

Lastly, consider a transverse $t$ of $P$. The function $\theta_t$ from $t$ to $P$, which maps each $x \in t$ to the unique element of $P$ containing $x$, is an isomorphism from $H[t]$ onto $H/P$.

In the next proposition, we study the links between the modules of a hypergraph with those of its quotients.

Proposition 37. Given a modular partition $P$ of a hypergraph $H$, the following two assertions hold

1. if $M$ is a module of $H$, then $M/P$ is a module of $H/P$ (see Notation 17);
2. if $M$ is a module of $H/P$, then $\cup M$ is a module of $H$.

Proof. For the first assertion, consider a module $M$ of $H$. Consider a transverse $t$ of $P$ such that

$$\text{for each } X \in M/P, \ t \cap X \in M. \tag{18}$$

By Lemma 31, $M \cap t$ is a module of $H[t]$. Since $\theta_t$ is an isomorphism from $H[t]$ onto $H/P$ (see Remark 36),

$$\theta_t(M \cap t), \text{ that is, } M/P$$

is a module of $H/P$.

For the second assertion, consider a module $M$ of $H/P$. Let $t$ be any transverse of $P$. Since $\theta_t$ is an isomorphism from $H[t]$ onto $H/P$, $(\theta_t)^{-1}(M)$ is a module of $H[t]$. Set

$$\mu = (\theta_t)^{-1}(M).$$

Denote the elements of $M$ by $X_0, \ldots, X_m$. We verify by induction on $i \in \{0, \ldots, m\}$ that $\mu \cup (X_0 \cup \ldots \cup X_i)$ is a module of $H[t \cup (X_0 \cup \ldots \cup X_i)]$. It follows from Lemma 31 that $\mu \cup X_0$ is a module of $H[t \cup X_0]$. Given $0 \leq i < m$, suppose that $\mu \cup (X_0 \cup \ldots \cup X_i)$ is a module of $H[t \cup (X_0 \cup \ldots \cup X_i)]$. Similarly, it follows from Lemma 31 that $\mu \cup (X_0 \cup \ldots \cup X_i \cup X_{i+1})$ is a module of $H[t \cup (X_0 \cup \ldots \cup X_{i+1})]$. By induction, we obtain that $\mu \cup (X_0 \cup \ldots \cup X_m)$ is a module of $H[t \cup (X_0 \cup \ldots \cup X_m)]$.

Observe that

$$\mu \cup (X_0 \cup \ldots \cup X_m) = \cup M.$$  

Lastly, denote the elements of $P \setminus M$ by $Y_0, \ldots, Y_n$. Using Lemma 31, we show by induction on $0 \leq j \leq n$ that $(\cup M)$ is a module of $H[t \cup (X_0 \cup \ldots \cup X_m) \cup (Y_0 \cup \ldots \cup Y_j)]$. Consequently, we obtain that $(\cup M)$ is a module of $H[t \cup (X_0 \cup \ldots \cup X_m) \cup (Y_0 \cup \ldots \cup Y_n)]$, that is, $H$.  

\[ \square \]
The next proposition is similar to Proposition 37 but it is devoted to strong modules. It is the analogue of Proposition 18 for hypergraphs.

**Proposition 38.** Given a modular partition $P$ of a hypergraph $H$, the following two assertions hold:

1. If $M$ is a strong module of $H$, then $M/P$ is a strong module of $H/P$ (see Notation 17).
2. Suppose that all the elements of $P$ are strong modules of $H$. If $M$ is a strong module of $H/P$, then $\cup M$ is a strong module of $H$.

**Proof.** For the first assertion, consider a strong module $M$ of $H$. By the first assertion of Proposition 37, $M/P$ is a module of $H/P$. To show that $M/P$ is strong, consider a module $\mathcal{M}$ of $H/P$ such that $(M/P) \cap \mathcal{M} \neq \emptyset$. By the second assertion of Proposition 37, $\cup \mathcal{M}$ is a module of $H$. Furthermore, since $(M/P) \cap \mathcal{M} \neq \emptyset$, there exists $X \in (M/P) \cap \mathcal{M}$. We get $X \cap M \neq \emptyset$ and $X \subseteq \cup \mathcal{M}$. Therefore, $M \cap (\cup \mathcal{M}) \neq \emptyset$. Since $M$ is a strong module of $H$, we obtain $\cup M \subseteq M$ or $M \subseteq \cup \mathcal{M}$. In the first instance, we get $\mathcal{M} \subseteq M/P$, and, in the second one, we get $M/P \subseteq \mathcal{M}$.

For the second assertion, suppose that all the elements of $P$ are strong modules of $H$. Consider a strong module $\mathcal{M}$ of $H/P$. To begin, we make two observations. First, if $\mathcal{M} = \emptyset$, then $\cup \mathcal{M} = \emptyset$, and hence $\cup \mathcal{M}$ is a strong module of $H$. Second, if $|\mathcal{M}| = 1$, then $\cup \mathcal{M} \in P$, and hence $\cup \mathcal{M}$ is a strong module of $H$ because all the elements of $P$ are. Now, suppose that

$$|\mathcal{M}| \geq 2. \quad (19)$$

By the second assertion of Proposition 37, $\cup \mathcal{M}$ is a module of $H$. To show that $\cup \mathcal{M}$ is strong, consider a module $M$ of $H$ such that $M \cap (\cup \mathcal{M}) \neq \emptyset$. Let $x \in M \cap (\cup \mathcal{M})$. Denote by $X$ the unique element of $P$ containing $x$. We get $X \in (M/P) \cap \mathcal{M}$. Since $\mathcal{M}$ is a strong module of $H/P$, we obtain $M/P \subseteq \mathcal{M}$ or $\mathcal{M} \subseteq M/P$. In the first instance, we obtain $\cup(M/P) \subseteq \cup \mathcal{M}$, so we have $M \subseteq \cup(M/P) \subseteq \cup \mathcal{M}$. Lastly, suppose $\mathcal{M} \subseteq M/P$. It follows from (19) that

$$|M/P| \geq 2.$$ 

Let $Y \subseteq M/P$. We have $Y \cap M \neq \emptyset$. Since $|M/P| \geq 2$, we have $M \setminus Y \neq \emptyset$. Since $Y$ is a strong module of $P$, we obtain $Y \subseteq M$. It follows that $M = \cup(M/P)$. Since $\mathcal{M} \subseteq M/P$, we obtain $\cup \mathcal{M} \subseteq \cup(M/P)$, and hence $\cup \mathcal{M} \subseteq M$. \[\square\]

**Remark 39.** We use the characterization of disconnected hypergraphs in terms of a quotient (see Lemma 41 below) to prove the analogue of Theorem 19 (see Theorem 42 below). Recall that a hypergraph $H$ is connected if for distinct $v, w \in V(H)$, there exist a sequence $(e_0, \ldots, e_n)$ of edges of $H$, where $n \geq 0$, satisfying $v \in e_0$, $w \in e_n$, and (when $n \geq 1$) $e_i \cap e_{i+1} \neq \emptyset$ for every $0 \leq i \leq n - 1$. Given a hypergraph $H$, a maximal connected subhypergraph of $H$ is called a component of $H$.
Notation 40. Given a hypergraph $H$, the set of the components of $H$ is denoted by $\mathcal{C}(H)$.

Let $H$ be a hypergraph. For each component $C$ of $H$, $V(C)$ is a module of $H$. Thus, $\{V(C) : C \in \mathcal{C}(H)\}$ is a modular partition of $H$. Furthermore, for each component $C$ of $H$, $V(C)$ is a strong module of $H$. We conclude the remark with the following result.

**Lemma 41.** Given a hypergraph $H$ with $v(H) \geq 2$, the following assertions are equivalent

1. $H$ is disconnected;
2. $H$ admits a modular bipartition $P$ such that $|P| \geq 2$ and $H/P$ is empty;
3. $\Pi(H) = \{V(C) : C \in \mathcal{C}(H)\}$, $|\Pi(H)| \geq 2$, and $H/\Pi(H)$ is empty.

Let $H$ be a hypergraph such that $v(H) \geq 2$. Because of the maximality of the elements of $\Pi(H)$ (see Notation 40), it follows from the second assertion of Proposition 38 that all the strong modules of $H/\Pi(H)$ are trivial. To prove Theorem 42 we establish the following result, which is the analogue of Theorem 19.

**Theorem 42.** Given a hypergraph $H$, all the strong modules of $H$ are trivial if and only if $H$ is an empty hypergraph, a prime hypergraph or a complete graph.

**Proof.** Clearly, if $H$ is an empty hypergraph, a prime hypergraph or a complete graph, then all the strong modules of $H$ are trivial.

To demonstrate the converse, we prove the following. Given a hypergraph $H$, if all the strong modules of $H$ are trivial, and $H$ is decomposable, then $H$ is an empty hypergraph or a complete graph.

To begin, we show that $H$ admits a modular bipartition. Since $H$ is decomposable, we can consider a maximal nontrivial module $M$ of $H$ under inclusion. Since $M$ is a nontrivial module of $H$, $M$ is not strong. Consequently, there exists a module $N$ of $H$ such that $M \cap N \neq \emptyset$, $M \setminus N \neq \emptyset$ and $N \setminus M \neq \emptyset$. Since $M \cap N \neq \emptyset$, $M \cup N$ is a module of $H$ by Lemma 28. Clearly, $M \subseteq M \cup N$ because $N \setminus M \neq \emptyset$. Since $M$ is a maximal nontrivial module of $H$, $M \cup N$ is a trivial module of $H$, so $M \cup N = V(H)$. Since $M \cap N \neq \emptyset$, $N \setminus M$ is a module of $H$ by Lemma 29. But, $N \setminus M = V(H) \setminus M$ because $M \cup N = V(H)$. It follows that $\{M, V(H) \setminus M\}$ is a modular bipartition of $H$.

We have $H/\{M, V(H) \setminus M\}$ is an empty hypergraph or a complete graph. We distinguish the following two cases.

1. Suppose that $H/\{M, V(H) \setminus M\}$ is an empty hypergraph. We prove that $H$ is an empty hypergraph. By Lemma 41 $H$ is disconnected. Let $C \in \mathcal{C}(H)$. As recalled in Remark 39, $V(C)$ is a strong module of $H$. By hypothesis, $V(C)$ is trivial. Since $H$ is disconnected, $V(C) \nsubseteq V(H)$. It follows that $v(C) = 1$. Therefore, $H$ is isomorphic to $H/\{V(C) : C \in \mathcal{C}(H)\}$. It follows from Lemma 41 that $H$ is empty.
2. Suppose that $H/\{M, V(H) \setminus M\}$ is a complete graph. We prove that $H$ is a complete graph. Consider the graph $H^c$ defined on $V(H)$ by

$$E(H^c) = (E(H) \setminus \binom{V(H)}{2}) \cup \binom{V(H)}{2} \setminus E(H). \quad (20)$$

It is easy to verify that $H$ and $H^c$ share the same modules. Therefore, they share the same strong modules. Consequently, all the strong modules of $H^c$ are trivial, $H^c$ is decomposable, and $\{M, V(H) \setminus M\}$ is a modular bipartition of $H$. Since $H/\{M, V(H) \setminus M\}$ is a complete graph, $H^c/\{M, V(H) \setminus M\}$ is empty. It follows from the first case that $H^c$ is empty. Hence $E(H^c) = \emptyset$, and it follows from (20) that $E(H) = \binom{V(H)}{2}$.

**Proof of Theorem 7.** For a contradiction, suppose that $H/\Pi(H)$ admits a non-trivial strong module $\mathcal{S}$. By the second assertion of Proposition 38, $\cup \mathcal{S}$ is a strong module of $H$. Given $X \in \mathcal{S}$, we obtain $X \subseteq \cup \mathcal{S} \subseteq V(H)$, which contradicts the maximality of $X$. Consequently, all the strong modules of $H/\Pi(H)$ are trivial. To conclude, it suffices to apply Theorem 12 to $H/\Pi(H)$.

**Definition 43.** Let $H$ be a hypergraph. As for tournaments (see Definition 20), the set of the nonempty strong modules of $H$ is denoted by $\mathcal{D}(H)$. Clearly, $\mathcal{D}(H)$ endowed with inclusion is a tree called the *modular decomposition tree* of $H$. For convenience, set

$$\mathcal{D}_{\geq 2}(H) = \{X \in \mathcal{D}(H) : |X| \geq 2\}.$$

Moreover, we associate with each $X \in \mathcal{D}_{\geq 2}(H)$, the label $\varepsilon_H(X)$ defined as follows

$$\varepsilon_H(X) = \begin{cases} \triangle & \text{if } H[X]/\Pi(H[X]) \text{ is prime,} \\ \bigcirc & \text{if } H[X]/\Pi(H[X]) \text{ is empty} \\ \bullet & \text{if } H[X]/\Pi(H[X]) \text{ is a complete graph.} \end{cases}$$

To conclude, we prove the analogue of Proposition 21 for hypergraphs.

**Proposition 44.** Given a hypergraph $H$, consider a strong module $M$ of $H$. For every $N \subseteq M$, the following two assertions are equivalent

1. $N$ is a strong module of $H$;
2. $N$ is a strong module of $H[M]$.

**Proof.** Let $N$ be a subset of $M$. To begin, suppose that $N$ is a strong module of $H$. Since $N$ is a module of $H$, $N$ is a module of $H[M]$ by Lemma 31. To show that $N$ is a strong module of $H[M]$, consider a module $X$ of $H[M]$ such that $N \cap X \neq \emptyset$. Since $M$ is a module of $H$, $X$ is a module of $H$ by Lemma 32. Since $N$ is a strong module of $H$, we obtain $N \subseteq X$ or $X \subseteq N$. 

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Conversely, suppose that \( N \) is a strong module of \( H[M] \). Since \( M \) is a module of \( H \), \( N \) is a module of \( H \) by Lemma 32. To show that \( N \) is a strong module of \( H \), consider a module \( X \) of \( H \) such that \( N \cap X \neq \emptyset \). We have \( M \cap X \neq \emptyset \) because \( N \subseteq M \). Since \( M \) is a strong module of \( H \), we obtain \( M \subseteq X \) or \( X \subseteq M \).

In the first instance, we get \( N \subseteq M \subseteq X \). Hence, suppose that \( X \subseteq M \). By Lemma 31, \( X \) is a module of \( H[M] \). Since \( N \) is a strong module of \( H[M] \) and \( N \cap X \neq \emptyset \), we obtain \( N \subseteq X \) or \( X \subseteq N \).

\[ \square \]

### 4 Realization and decomposability

Consider a realizable 3-uniform hypergraph. Let \( T \) be a realization of \( H \). A module of \( T \) is clearly a module of \( H \), but the converse is false. Nevertheless, we have the following result (see Proposition 46). We need the following notation.

**Notation 45.** Let \( H \) be a 3-uniform hypergraph. For \( W \subseteq V(H) \) such that \( W \neq \emptyset \), \( \widetilde{W}^H \) denotes the intersection of the strong modules of \( H \) containing \( W \). Note that \( \widetilde{W}^H \) is the smallest strong module of \( H \) containing \( W \).

**Proposition 46.** Let \( H \) be a realizable 3-uniform hypergraph. Consider a realization \( T \) of \( H \). Let \( M \) be a module of \( H \). Suppose that \( M \) is not a module of \( T \), and set

\[ \neg \tau M = \{ v \in V(H) \setminus M : M \text{ is not a module of } T[M \cup \{v\}] \}. \]

The following four assertions hold

1. \( M \cup (\neg \tau M) \) is a module of \( T \);
2. \( M \) is not a strong module of \( H \);
3. \( M \cup (\neg \tau M) \subseteq \widetilde{M}^H \);
4. \( \varepsilon_H(\widetilde{M}^H) = \emptyset \) and \( |\Pi(H[\widetilde{M}^H])| \geq 3 \).

**Proof.** Since \( M \) is not a module of \( T \), we have \( \neg \tau M \neq \emptyset \). Let \( v \in \neg \tau M \). Since \( M \) is not a module of \( T[M \cup \{v\}] \), we obtain

\[ \begin{cases} N_T^{-}(v) \cap M \neq \emptyset \\ N_T^{+}(v) \cap M \neq \emptyset \end{cases} \]  

(21)

Furthermore, consider \( v^- \in N_T^{-}(v) \cap M \) and \( v^+ \in N_T^{+}(v) \cap M \). Since \( M \) is a module of \( H \), \( v^- v^+ \notin E(H) \). Hence \( v^- vv^+ \notin E(C_3(T)) \). Since \( v^- v, vv^+ \in A(T) \), we get \( v^- v^+ \in A(T) \). Therefore, for each \( v \in \neg \tau M \), we have

\[ \text{for } v^- \in N_T^{-}(v) \cap M \text{ and } v^+ \in N_T^{+}(v) \cap M, v^- v^+ \in A(T). \]  

(22)

Now, consider \( v, w \in \neg \tau M \) such that \( vw \in A(T) \). Let \( v^- \in N_T^{-}(v) \cap M \). Suppose for a contradiction that \( v^- \in N_T^{+}(w) \cap M \). We get \( v^- vw \in E(C_3(T)), \)
and hence $v^-vw \in E(H)$. Since $M$ is a module of $H$, we obtain $\mu vw \in E(H)$ for every $\mu \in M$. Thus, since $vw \in A(T)$, $\mu v \in A(T)$ for every $\mu \in M$. Therefore, $M \subseteq N_T(v)$, so $N_T(v) \cap M = \emptyset$, which contradicts (21). It follows that for $v, w \in \gamma_T M$, we have

$$\text{if } vw \in A(T), \text{ then } N_T(v) \cap M \subseteq N_T(w) \cap M. \quad (23)$$

For the first assertion, set

$$M^- = \{v \in V(H) \setminus M : vm \in A(T) \text{ for every } m \in M\}$$

and

$$M^+ = \{v \in V(H) \setminus M : mv \in A(T) \text{ for every } m \in M\}.$$  

Note that $\{M^-, M, \gamma_T M, M^+\}$ is a partition of $V(H)$. Let $m^- \in M^-$ and $v \in \gamma_T M$. By (21), there exist $v^- \in N_T(v) \cap M$ and $v^+ \in N_T(v) \cap M$. Suppose for a contradiction that $vm^-v^- \in A(T)$. We get $vm^-v^- \in E(C_3(T))$. Hence $vm^-v^- \in E(H)$. Since $m^-v^+, v^-v^+ \in A(T)$, we have $vm^-v^- \notin E(C_3(T))$. Thus $vm^-v^- \notin E(H)$, which contradicts the fact that $M$ is a module of $H$. It follows that $m^-v \in A(T)$ for any $m^- \in M^-$ and $v \in \gamma_T M$. Similarly, $vm^+ \in A(T)$ for any $m^+ \in M^+$ and $v \in \gamma_T M$. It follows that $M \cup (\gamma_T M)$ is a module of $T$.

For the second assertion, consider $v \in (\gamma_T M)$. Set

$$N_v = (N_T(v) \cap M) \cup \{w \in (\gamma_T M) : N_T(w) \cap M \subseteq N_T(v) \cap M\}. \quad (24)$$

We show that $N_v$ is a module of $T$. If $m^- \in M^-$, then $m^-n \in A(T)$ for every $n \in N_v$ because $M \cup (\gamma_T M)$ is a module of $T$. Similarly, if $m^+ \in M^+$, then $nm^+ \in A(T)$ for every $n \in N_v$. Now, consider $m \in M \setminus N_T(v)$. We get $m \in M \setminus N_T(v)$. Therefore, we have $m \in M \setminus N_T(v) \cap M$ for every $w' \in \{w \in (\gamma_T M) : N_T(w) \cap M \subseteq N_T(v) \cap M\}$. Thus, $m \in N_T(v) \cap M$ for every $w' \in \{w \in (\gamma_T M) : N_T(w) \cap M \subseteq N_T(v) \cap M\}$. Since $m \in N_T(v) \cap M$, it follows from (22) that $v^-m \in A(T)$ for every $v^- \in N_T(v) \cap M$. Furthermore, since $m \in N_T(v) \cap M$ for every $w' \in \{w \in (\gamma_T M) : N_T(w) \cap M \subseteq N_T(v) \cap M\}$, we have $w'm \in A(T)$ for every $w' \in \{w \in (\gamma_T M) : N_T(w) \cap M \subseteq N_T(v) \cap M\}$. Therefore, we obtain $nm \in A(T)$ for every $n \in N_v$. Lastly, consider $u \in (\gamma_T M) \setminus N_v$. We get $u \in (\gamma_T M)$ and $N_T(u) \cap M \not\subseteq N_T(v) \cap M$. It follows from (23) that $vu \in A(T)$. By (24), again, we have $N_T(v) \cap M \not\subseteq N_T(u) \cap M$. Thus $v^-u \in A(T)$ for each $v^- \in N_T(v) \cap M$. Let $w' \in \{w \in (\gamma_T M) : N_T(w) \cap M \subseteq N_T(v) \cap M\}$. We get $N_T(w') \cap M \not\subseteq N_T(v) \cap M$. It follows from (23) that $w'u \in A(T)$. Consequently, $N_v$ is a module of $T$ for each $v \in (\gamma_T M)$. Hence,

$$N_v \text{ is a module of } H \text{ for each } v \in (\gamma_T M). \quad (25)$$

(We use (24) to prove the third assertion below.) Let $v \in (\gamma_T M)$. Clearly, $v \in N_v \setminus M$. Moreover, it follows from (21) that there exist $v^- \in N_T(v) \cap M$ and $v^+ \in N_T(v) \cap M$. We get $v^- \in M \cap N_T(v)$ and $v^+ \in M \setminus N_T(v)$. Since $N_v$ is a module of $H$, $M$ is not a strong module of $H$.

For the third assertion, consider $v \in (\gamma_T M)$. As previously proved, $N_v$ is a module of $H$. Furthermore, by considering $v^- \in N_T(v) \cap M$ and $v^+ \in N_T(v) \cap M$,
we obtain $M \cap N_v \neq \emptyset$ and $M \setminus N_v \neq \emptyset$. Hence $\overline{M} \cap N_v \neq \emptyset$ and $\overline{M} \setminus N_v \neq \emptyset$. Since $\overline{M}$ is a strong module of $H$, we get $N_v \subseteq \overline{M}$. Thus $v \in \overline{M}$ for every $v \in (\gamma_T M)$. Therefore $M \cup (\gamma_T M) \subseteq \overline{M}$.

For the fourth assertion, we prove that for each $v \in (\gamma_T M)$,

$$P_v = \{N_T(v) \cap M, N_T^+(v) \cap M, \gamma_T M\}$$

is a modular partition of $H[M \cup (\gamma_T M)]$. Let $v \in (\gamma_T M)$. By (22), $N_T(v) \cap M$ and $N_T^+(v) \cap M$ are modules of $T[M]$. Thus, $N_T(v) \cap M$ and $N_T^+(v) \cap M$ are modules of $H[M]$. Since $M$ is a module of $H$, it follows from Lemma 32 that $N_T(v) \cap M$ and $N_T^+(v) \cap M$ are modules of $H$. By Lemma 31, $N_T(v) \cap M$ and $N_T^+(v) \cap M$ are modules of $H[M \cup (\gamma_T M)]$. Now, we prove that $\gamma_T M$ is a module of $H[M \cup (\gamma_T M)]$. It suffices to prove that there exists no $e \in E[H[M \cup (\gamma_T M)]]$ such that $e \cap (\gamma_T M) \neq \emptyset$ and $e \cap M \neq \emptyset$. Indeed, suppose to the contrary that there exists $e \in E[H[M \cup (\gamma_T M)]]$ such that $e \cap (\gamma_T M) \neq \emptyset$ and $e \cap M \neq \emptyset$. Since $M$ is a module of $H$, we get $|e \cap M| = 1$ and $|e \cap (\gamma_T M)| = 2$. Therefore, there exist $v, w \in e \cap (\gamma_T M)$ and $m \in e \cap M$ such that $vw, wn \in E(H)$. By replacing $v$ by $w$ if necessary, we can assume that $vw \in A(T)$. Since $H = C_3(T)$, we obtain $vw, wm, mw \in A(T)$, which contradicts (23). Therefore, $\gamma_T M$ is a module of $H[M \cup (\gamma_T M)]$. Consequently, $P_v = \{N_T(v) \cap M, N_T^+(v) \cap M, \gamma_T M\}$ is a modular partition of $H[M \cup (\gamma_T M)]$. Furthermore, given $v \in (\gamma_T M)$, consider $v^{-} = N_T(v) \cap M$ and $v^{+} = N_T^+(v) \cap M$. It follows from (22) that $v^{-} v^{+} v \notin E(C_3(T))$, and hence $v^{-} v^{+} v \notin E(H)$. Consequently,

$$H[M \cup (\gamma_T M)]/P_v \text{ is empty.} \quad (26)$$

Since $M \cup (\gamma_T M)$ is a module of $T$ by the first assertion above, $M \cup (\gamma_T M)$ is a module of $H$. By Lemma 31, $M \cup (\gamma_T M)$ is a module of $H[\overline{M}]$. Given $v \in (\gamma_T M)$, it follows from Lemma 32 that each element of $P_v$ is a module of $H[\overline{M}]$.

Let $v \in (\gamma_T M)$. For a contradiction, suppose that there exist $Y \in P_v$ and $X \in \Pi(H[\overline{M}])$ such that $Y \not\subseteq X$. We get $X \cap (M \cup (\gamma_T M)) \neq \emptyset$. Since $M \cup (\gamma_T M)$ is a module of $H[\overline{M}]$ and $X$ is a strong module of $H[\overline{M}]$, we have $M \cup (\gamma_T M) \subseteq X$ or $X \not\subseteq M \cup (\gamma_T M)$. Furthermore, since $X$ is a strong module of $H[\overline{M}]$ and $\overline{M}$ is a strong module of $H$, it follows from Proposition 44 that $X$ is a strong module of $H$. Since $X \not\subseteq \overline{M}$, it follows from the minimality of $\overline{M}$ that we do not have $M \cup (\gamma_T M) \subseteq X$. Therefore, $X \not\subseteq M \cup (\gamma_T M)$. Let $x \in X \setminus Y$. We have $x \in (M \cup (\gamma_T M)) \setminus Y$. Denote by $Y'$ the unique element of $P_v \setminus \{Y\}$ such that $x \in Y'$. Also, denote by $Z$ the unique element of $P_v \setminus \{Y, Y'\}$. We get $X \cap Y' \neq \emptyset$ and $Y \not\subseteq X \setminus Y'$. Since $X$ is a strong module of $H[\overline{M}]$, we get $Y' \subseteq X$. Since $X \not\subseteq M \cup (\gamma_T M)$, we obtain $X \cap Z = \emptyset$. Thus $X = Y \cup Y'$. Since $H[M \cup (\gamma_T M)]/P_v$ is empty by (26), $\{Y, Z\}$ is a module of $H[M \cup (\gamma_T M)]/P_v$. By the second assertion of Proposition 37, $Y \cup Z$ is a module of $H[M \cup (\gamma_T M)]$. As previously seen, $M \cup (\gamma_T M)$ is a module of $H[\overline{M}]$. By Lemma 32, $Y \cup Z$ is a module of $H[\overline{M}]$, which contradicts the fact that $X$ is a strong module of $H[\overline{M}]$. Consequently,

for any $Y \in P_v$ and $X \in \Pi(H[\overline{M}])$, we do not have $Y \not\subseteq X$. \quad (27)
Let $Y \in P_v$. Set

$$Q_Y = \{X \in \Pi(H[\tilde{M}^H]) : X \cap Y \neq \emptyset\}.$$ 

For every $X \in Q_Y$, we have $Y \not\subseteq X$ or $X \not\subseteq Y$ because $X$ is a strong module of $H[\tilde{M}^H]$. By (27), we have $X \subseteq Y$. It follows that

$$\text{for each } Y \in P_v, \text{ we have } Y = \cup Q_Y.$$ 

Therefore, $|\Pi(H[\tilde{M}^H])| \geq |P_v|$, that is,

$$|\Pi(H[\tilde{M}^H])| \geq 3.$$ 

Finally, we prove that $H[\tilde{M}^H]/\Pi(H[\tilde{M}^H])$ is empty. Suppose that $M \cup \langle \gamma_T M \rangle \not\subseteq \tilde{M}^H$, and set

$$Q_{M \cup \langle \gamma_T M \rangle} = \{X \in \Pi(H[\tilde{M}^H]) : X \cap (M \cup \langle \gamma_T M \rangle) \neq \emptyset\}.$$ 

Since $M \cup \langle \gamma_T M \rangle$ is a module of $H[\tilde{M}^H]$, it follows from the first assertion of Proposition 37 that $Q_{M \cup \langle \gamma_T M \rangle}$ is a module of $H[\tilde{M}^H]/\Pi(H[\tilde{M}^H])$. Moreover, it follows from (28) that $|Q_{M \cup \langle \gamma_T M \rangle}| \geq 3$. Since each element of $\Pi(H[\tilde{M}^H])$ is a strong element of $\tilde{M}^H$, we get $M \cup \langle \gamma_T M \rangle = \cup Q_{M \cup \langle \gamma_T M \rangle}$. Since $M \cup \langle \gamma_T M \rangle \not\subseteq \tilde{M}^H$, we obtain that $Q_{M \cup \langle \gamma_T M \rangle}$ is a nontrivial module of $H[\tilde{M}^H]/\Pi(H[\tilde{M}^H])$. Hence $H[\tilde{M}^H]/\Pi(H[\tilde{M}^H])$ is decomposable. It follows from Theorem 4 that $H[\tilde{M}^H]/\Pi(H[\tilde{M}^H])$ is empty. Lastly, suppose that $M \cup \langle \gamma_T M \rangle = \tilde{M}^H$. Suppose also that there exists $Y \in P_v$ such that $|Q_Y| \geq 2$. As previously, we obtain that $Q_Y$ is a nontrivial module of $H[\tilde{M}^H]/\Pi(H[\tilde{M}^H])$, and hence $H[\tilde{M}^H]/\Pi(H[\tilde{M}^H])$ is empty. Therefore, suppose that $|Q_Y| = 1$ for every $Y \in P_v$. By (28), $\Pi(H[\tilde{M}^H]) = P_v$. Hence $H[\tilde{M}^H]/\Pi(H[\tilde{M}^H])$ is empty by (20).

The next result is an easy consequence of Proposition 47.

**Corollary 47.** Consider a realizable 3-uniform hypergraph $H$, and a realization $T$ of $H$. The following two assertions are equivalent

- $H$ and $T$ share the same modules;
- for each strong module $X$ of $H$ such that $|X| \geq 2$, we have

$$\text{if } \varepsilon_H(X) = \emptyset, \text{ then } |\Pi(H[X])| = 2.$$ 

**Proof.** To begin, suppose that $H$ and $T$ do not share the same modules. There exists a module $M$ of $H$, which is not a module of $T$. By the last assertion of Proposition 47, we obtain $\varepsilon_H(\tilde{M}^H) = \emptyset$ and $|\Pi(H[\tilde{M}^H])| \geq 3$.

Conversely, suppose that there exists a strong module $X$ of $H$, with $|X| \geq 2$, such that $\varepsilon_H(X) = \emptyset$ and $|\Pi(H[X])| \geq 3$. It follows from the second assertion of Proposition 47 that $X$ is a module of $T$. Observe that $T[X]$ realizes $H[X]$. Let $Y \in \Pi(H[X])$. Since $Y$ is a strong module of $H[X]$, it follows from the second assertion of Proposition 47 applied to $H[X]$
$T[X]$ that $Y$ is a module of $T[X]$. Thus, $\Pi(H[X])$ is a modular partition of $T[X]$. Since $H[X]/\Pi(H[X])$ is empty, $T[X]/\Pi(H[X])$ is a linear order. Denote by $Y_{\text{min}}$ the smallest element of $T[X]/\Pi(H[X])$. Similarly, denote by $Y_{\text{max}}$ the largest element of $T[X]/\Pi(H[X])$. Since $H[X]/\Pi(H[X])$ is empty, $(Y_{\text{min}}, Y_{\text{max}})$ is a module of $H[X]/\Pi(H[X])$. By the second assertion of Proposition 57, $Y_{\text{min}} \cup Y_{\text{max}}$ is a module of $H[X]$. Since $X$ is a module of $H$, it follows from Lemma 32 that $Y_{\text{min}} \cup Y_{\text{max}}$ is a module of $H$. Lastly, since $|\Pi(H[X])| \geq 3$, there exists $Y \in \Pi(H[X]) \setminus \{Y_{\text{min}}, Y_{\text{max}}\}$. Since $Y_{\text{min}}$ is the smallest element of $T[X]/\Pi(H[X])$ and $Y_{\text{max}}$ is the largest one, we obtain $\max Y_{\text{min}} Y, Y_{\text{max}} \in A(T[X]/\Pi(H[X]))$. Therefore, for $y_{\text{min}} \in Y_{\text{min}}$, $y \in Y$ and $y_{\text{max}} \in Y_{\text{max}}$, we have $y_{\text{min}} y, y_{\text{max}} \in A(T[X])$, and hence $y_{\text{min}} y, y_{\text{max}} \in A(T)$. Consequently, $Y_{\text{min}} \cup Y_{\text{max}}$ is not a module of $T$. \hfill \Box

Now, we prove Theorem 10 by using Proposition 46 and the following lemma.

**Lemma 48.** Consider a realizable 3-uniform hypergraph $H$. Given a realization $T$ of $H$, all the strong modules of $H$ are strong modules of $T$.

**Proof.** Consider a strong module $M$ of $H$. By the second assertion of Proposition 46, $M$ is a module of $T$. Let $N$ be a module of $T$ such that $M \cap N \neq \emptyset$. Since $N$ is a module of $T$, $N$ is a module of $H$. Furthermore, since $M$ is a strong module of $H$, we obtain $M \subseteq N$ or $N \subseteq M$. Therefore, $M$ is a strong module of $T$. \hfill \Box

**Proof of Theorem 10.** By Lemma 48, all the strong modules of $H$ are strong modules of $T$.

Conversely, consider a strong module $M$ of $T$. Since $M$ is a module of $T$, $M$ is a module of $H$. Let $N$ be a module of $H$ such that $M \cap N \neq \emptyset$. If $N$ is a module of $T$, then $M \subseteq N$ or $N \subseteq M$ because $M$ is a strong module of $T$. Hence suppose that $N$ is not a module of $T$. By the last assertion of Proposition 46

$$
\begin{align*}
&\left\{ H[\tilde{N}^H] / \Pi(H[\tilde{N}^H]) \right\} \text{ is empty} \\
&\text{and} \\
&|\Pi(H[\tilde{N}^H])| \geq 3.
\end{align*}
$$

(29)

Since $M \cap N \neq \emptyset$, $M \cap \tilde{N}^H \neq \emptyset$. Since $\tilde{N}^H$ is a strong module of $H$, we get $\tilde{N}^H \subseteq M$ or $M \nsubseteq \tilde{N}^H$. Clearly, if $\tilde{N}^H \subseteq M$, then $N \subseteq M$. Thus, suppose that

$M \nsubseteq \tilde{N}^H$.

We prove that $M \subseteq N$. By Lemma 48, $\tilde{N}^H$ is a strong module of $T$. Since $M$ is a strong module of $T$, it follows from Proposition 44 that $M$ is a strong module of $T[\tilde{N}^H]$. For each $X \in \Pi(H[\tilde{N}^H])$, $X$ is a strong module of $T[\tilde{N}^H]$ by Lemma 48. Therefore, $\Pi(H[\tilde{N}^H])$ is a modular partition of $T[\tilde{N}^H]$. Clearly, $T[\tilde{N}^H] / \Pi(H[\tilde{N}^H])$ is a realization of $H[\tilde{N}^H] / \Pi(H[\tilde{N}^H])$. Set

$$Q_M = \{ X \in \Pi(H[\tilde{N}^H]) : M \cap X \neq \emptyset \}.$$  

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By the first assertion of Proposition 18, \( Q_M \) is a strong module of \( T[\tilde{N}^H]/\Pi(H[\tilde{N}^H]) \). Since \( H[\tilde{N}^H]/\Pi(H[\tilde{N}^H]) \) is empty by (20), \( T[\tilde{N}^H]/\Pi(H[\tilde{N}^H]) \) is a linear order. By Theorem 19, \( Q_M \) is a trivial module of \( T[\tilde{N}^H]/\Pi(H[\tilde{N}^H]) \).

For a contradiction, suppose that \( Q_M = \Pi(H[\tilde{N}^H]) \). Since \( \tilde{N}^H \) is a strong module of \( H \), we get \( \tilde{N}^H \subseteq \tilde{N}^H \), which contradicts \( \tilde{N}^H \subset \tilde{N}^H \). It follows that \(|Q_M| = 1\). Hence there exists \( X_M \in \Pi(H[\tilde{N}^H]) \) such that

\[
M \subseteq X_M.
\]

Since \( N \) is not a module of \( T \), it follows from the second assertion of Proposition 46 that \( N \) is not a strong module of \( H \). Thus \( N \subseteq \tilde{N}^H \). We obtain

\[
Q_N = \{ X \in \Pi(H[\tilde{N}^H]) : N \cap X \neq \emptyset \}.
\]

Since \( \tilde{N}^H \) is a strong module of \( H \), it follows from Proposition 44 that each element of \( \Pi(H[\tilde{N}^H]) \) is a strong module of \( H \). It follows from the minimality of \( \tilde{N}^H \) that \(|Q_N| \geq 2\). Since each element of \( \Pi(H[\tilde{N}^H]) \) is a strong module of \( H \), we obtain

\[
N = \cup Q_N.
\]

Since \( M \cap N \neq \emptyset \), we get \( X_M \in Q_N \). We obtain \( M \subseteq X_M \subseteq N \).

Lastly, we establish Theorem 11 by using Theorems 7 and 10.

**Proof of Theorem 11.** Suppose that \( H \) is prime. Since all the modules of \( T \) are modules of \( H \), \( T \) is prime.

Conversely, suppose that \( T \) is prime. Hence, all the strong modules of \( T \) are trivial. By Theorem 11, all the strong modules of \( H \) are trivial. We obtain

\[
\Pi(H) = \{ \{v\} : v \in V(H) \}.
\]

Thus, \( H \) is isomorphic to \( H/\Pi(H) \). It follows from Theorem 7 that \( H \) is an empty hypergraph, a prime hypergraph or a complete graph. Since \( T \) is prime, we have \( E(C_3(T)) \neq \emptyset \). Since \( E(C_3(T)) = E(H) \), there exists \( e \in E(H) \) such that \(|e| = 3\). Therefore, \( H \) is not an empty hypergraph, and \( H \) is not a graph. It follows that \( H \) is prime.

\[
\square
\]

### 5 Realizability of 3-uniform hypergraphs

The next proposition is useful to construct realizations from the modular decomposition tree of a realizable 3-uniform hypergraph. We need the following notation and remark.

**Notation 49.** Let \( H \) be a 3-uniform hypergraph. We denote by \( \mathcal{R}(H) \) the set of the realizations of \( H \).
Remark 50. Let $H$ be a realizable 3-uniform hypergraph. Consider $T \in \mathcal{R}(H)$. It follows from Theorem 10 that

$$\mathcal{D}(H) = \mathcal{D}(T).$$

By the same, for each $X \in \mathcal{D}_2(H)$, we have

$$\Pi(H[X]) = \Pi(T[X]).$$

Therefore, for each $X \in \mathcal{D}_2(H)$, $T[X]/\Pi(T[X])$ realizes $H[X]/\Pi(H[X])$, that is,

$$T[X]/\Pi(T[X]) \in \mathcal{R}(H[X]/\Pi(H[X])).$$

Set

$$\mathcal{R}_2(H) = \bigcup_{X \in \mathcal{D}_2(H)} \mathcal{R}(H[X]/\Pi(H[X])).$$

We denote by $\delta_H(T)$ the function

$$\begin{align*}
\mathcal{D}_2(H) & \rightarrow \mathcal{R}_2(H) \\
Y & \mapsto T[Y]/\Pi(T[Y]).
\end{align*}$$

Lastly, we denote by $\mathcal{D}(H)$ the set of the functions $f$ from $\mathcal{D}_2(H)$ to $\mathcal{R}_2(H)$ satisfying $f(Y) \in \mathcal{R}(H[Y]/\Pi(H[Y]))$ for each $Y \in \mathcal{D}_2(H)$. Under this notation, we obtain the function

$$\delta_H : \mathcal{R}(H) \rightarrow \mathcal{D}(H)$$

$$T \mapsto \delta_H(T).$$

Proposition 51. For a 3-uniform hypergraph, $\delta_H$ is a bijection.

Proof. To begin, we show that $\delta_H$ is injective. Let $T$ and $T'$ be distinct realizations of $H$. There exist distinct $v, w \in V(H)$ such that $vw \in A(T)$ and $vw \in A(T')$. Consider $Z_v, Z_w \in \Pi(H[\{v, w\}^H])$ (see Notation 13) such that $v \in Z_v$ and $w \in Z_w$. Since $\{v, w\}^H$ is the smallest strong module of $H$ containing $\{v, w\}$, we obtain $Z_v \neq Z_w$. It follows from Theorem 10 that $\Pi(H[\{v, w\}^H]) = \Pi(T[\{v, w\}^H])$ and $\Pi(H[\{v, w\}^H]) = \Pi(T'[\{v, w\}^H])$. Since $vw \in A(T)$ and $vw \in A(T')$, we obtain

$$
\begin{align*}
Z_v Z_w & \in A(T[\{v, w\}^H]/\Pi(T[\{v, w\}^H])) \\
Z_w Z_v & \in A(T'[\{v, w\}^H]/\Pi(T'[\{v, w\}^H])).
\end{align*}
$$

Consequently, $\delta_H(T)(\{v, w\}^H) \neq \delta_H(T')(\{v, w\}^H)$. Thus, $\delta_H(T) \neq \delta_H(T')$.

Now, we prove that $\delta_H$ is surjective. Consider $f \in \mathcal{D}(H)$, that is, $f$ is a function from $\mathcal{D}_2(H)$ to $\mathcal{R}_2(H)$ satisfying $f(Y) \in \mathcal{R}(H[Y]/\Pi(H[Y]))$ for each $Y \in \mathcal{D}_2(H)$. We construct $T \in \mathcal{R}(H)$ such that $\delta_H(T) = f$ in the following
manner. Consider distinct vertices $v$ and $w$ of $H$. Clearly, $\{v, w\}^H$ is a strong module of $H$ such that $|\{v, w\}^H| \geq 2$. There exist $Z_v, Z_w \in \Pi(H[\{v, w\}^H])$ such that $v \in Z_v$ and $w \in Z_w$. Since $\{v, w\}^H$ is the smallest strong module of $H$ containing $v$ and $w$, we obtain $Z_v \neq Z_w$. Set

$$
\begin{aligned}
&\text{if } Z_v Z_w \in A(f(\{v, w\}^H)) , \\
&\text{and } \\
&\text{if } Z_w Z_v \in A(f(\{v, w\}^H)).
\end{aligned}
$$

(30)

We obtain a tournament $T$ defined on $V(H)$.

Lastly, we verify that $T$ realizes $H$. First, consider distinct vertices $u, v, w$ of $H$ such that $uvw \in E(H)$. There exist $Z_u, Z_v, Z_w \in \Pi(H[\{u, v, w\}^H])$ such that $u \in Z_u$, $v \in Z_v$, and $w \in Z_w$. For a contradiction, suppose that $Z_u = Z_v$. Since $Z_u$ is a module of $H$ and $uvw \in E(H)$, we get $w \in Z_u$. Thus, $Z_u = Z_v = Z_w$, which contradicts the fact that $\{u, v, w\}^H$ is the smallest strong module of $H$ containing $u, v$, and $w$. It follows that $Z_u \neq Z_v$. Similarly, we have $Z_u \neq Z_w$ and $Z_v \neq Z_w$. It follows that $Z_u Z_v, Z_u Z_w \in E(H[\{u, v, w\}^H]/\Pi(H[\{u, v, w\}^H]))$. Since $f(\{u, v, w\}^H)$ realizes $H[\{u, v, w\}^H]/\Pi(H[\{u, v, w\}^H])$, we obtain $Z_u Z_v, Z_u Z_w, Z_w Z_u \in A(\{u, v, w\}^H)$. By exchanging $u$ and $v$ if necessary, assume that

$$Z_u Z_v, Z_v Z_w, Z_w Z_u \in A(f(\{u, v, w\}^H)).$$

Since $Z_u \neq Z_v$, we obtain $\{u, v\}^H = \{u, v, w\}^H$. Similarly, we have $\{u, w\}^H = \{u, v, w\}^H$ and $\{v, w\}^H = \{u, v, w\}^H$. It follows from (30) that $uv, uv, wu \in A(T)$. Hence, $T[\{u, v, w\}]$ is a 3-cycle.

Conversely, consider distinct vertices $u, v, w$ of $T$ such that $T[\{u, v, w\}]$ is a 3-cycle. There exist $Z_u, Z_v, Z_w \in \Pi(H[\{u, v, w\}^H])$ such that $u \in Z_u$, $v \in Z_v$, and $w \in Z_w$. For a contradiction, suppose that $Z_u = Z_v$. Since $\{u, v, w\}^H$ is the smallest strong module of $H$ containing $u, v$, and $w$, we obtain $Z_u \neq Z_w$. Therefore, we have $\{u, w\}^H = \{u, v, w\}^H$ and $\{v, w\}^H = \{u, v, w\}^H$. For instance, assume that $Z_u Z_w \in A(f(\{u, v, w\}^H))$. It follows from (30) that $uw, vw \in A(T)$, which contradicts the fact that $T[\{u, v, w\}]$ is a 3-cycle. Consequently, $Z_u \neq Z_v$. It follows that $\{u, v\}^H = \{u, v, w\}^H$. Similarly, we have $\{u, w\}^H = \{u, v, w\}^H$ and $\{v, w\}^H = \{u, v, w\}^H$. For instance, assume that $uw, vw, wu \in A(T)$. It follows from (30) that $Z_u Z_v, Z_u Z_w, Z_w Z_u \in A(f(\{u, v, w\}^H))$. Since $f(\{u, v, w\}^H)$ realizes $H[\{u, v, w\}^H]/\Pi(H[\{u, v, w\}^H])$, we obtain $Z_u Z_v, Z_u Z_w \in E(H[\{u, v, w\}^H]/\Pi(H[\{u, v, w\}^H]))$. It follows that $uw \in E(H[\{u, v, w\}^H])$, and hence $uww \in E(H)$. 

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Consequently, $T \in \mathcal{R}(H)$. Let $X \in \mathcal{D}_2(H)$. As seen at the beginning of Remark 50, we have $\Pi(H[X]) = \Pi(T[X])$, and

$$T[X] / \Pi(T[X]) \in \mathcal{R}(H[X] / \Pi(H[X])).$$

Consider distinct elements $Y$ and $Z$ of $\Pi(H[X])$. For instance, assume that $YZ \in A(T[X] / \Pi(T[X]))$. Let $v \in Y$ and $w \in Z$. We obtain $vw \in A(T)$. Moreover, we have $\{v, w\}^H = X$ because $Y, Z \in \Pi(H[X])$ and $Y \neq Z$. It follows from (30) that $YZ \in A(f(X))$. Therefore,

$$T[X] / \Pi(T[X]) = f(X).$$

(31)

Since (31) holds for every $X \in \mathcal{D}_2(H)$, we have $\delta_H(T) = f$. \hfill $\square$

Theorem 12 is an easy consequence of Proposition 51.

Proof of Theorem 12. Clearly, if $H$ is realizable, then $H[W]$ is also for every $W \in V(H)$. Conversely, suppose that $H[W]$ is realizable for every $W \in V(H)$ such that $H[W]$ is prime. We define an element $f$ of $\mathcal{R}(H)$ as follows. Consider $Y \in \mathcal{D}_2(H)$. By Theorem 7, $H[Y] / \Pi(H[Y])$ is empty or prime. First, suppose that $H[Y] / \Pi(H[Y])$ is empty. We choose for $f(Y)$ any linear order defined on $\Pi(H[Y])$. Clearly, $f(Y) \in \mathcal{R}(H[Y] / \Pi(H[Y]))$. Second, suppose that $H[Y] / \Pi(H[Y])$ is prime. Consider a transverse $W$ of $\Pi(H[Y])$ (see Definition 55). The function

$$\theta_W : W \rightarrow \Pi(H[Y]),$$

$$\theta_W : w \rightarrow \Pi(H[Y]),$$

is an isomorphism from $H[W]$ onto $H[Y] / \Pi(H[Y])$. Thus, $H[W]$ is prime. By hypothesis, $H[W]$ admits a realization $T_W$. We choose for $f(Y)$ the unique tournament defined on $\Pi(H[Y])$ such that $\theta_W$ is an isomorphism from $T_W$ onto $f(Y)$. Clearly, $f(Y) \in \mathcal{R}(H[Y] / \Pi(H[Y]))$.

By Proposition 51, $(\delta_H)^{-1}(f)$ is a realization of $H$. \hfill $\square$

Theorem 12 leads us to study the realization of prime and 3-uniform hypergraphs. We need to introduce the analogue of Definition 22 for 3-uniform hypergraphs.

Definition 52. Given a prime and 3-uniform hypergraph $H$, a vertex $v$ of $H$ is critical if $H - v$ is decomposable. A prime and 3-uniform hypergraph is critical if all its vertices are critical.

For critical and 3-uniform hypergraphs, we obtain the following characterization, which is an immediate consequence of Theorems 11 and 28.

Theorem 53. Given a critical and 3-uniform hypergraph $H$, $H$ is realizable if and only if $v(H)$ is odd, and $H$ is isomorphic to $C_3(T_{v(H)})$, $C_3(U_{v(H)})$ or $C_3(W_{v(H)})$. 

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We pursue with the characterization of non critical, prime and 3-uniform hypergraphs that are realizable. We need the following notation.

**Notation 54.** Let $H$ be a 3-uniform hypergraph. Consider a vertex $x$ of $H$. Set 
$$V_x = V(H) \setminus \{x\}.$$ 
We denote by $G_x$ the graph defined on $V_x$ as follows. Given distinct elements $v$ and $w$ of $V_x$,
$$vw \in E(G_x) \text{ if } xvw \in E(H) \text{ (note that the graph } G_x \text{ is used in [8]).}$$

Also, we denote by $I_x$ the set of the isolated vertices of $G_x$. Lastly, suppose that $H - x$ admits a realization $T_x$. Consider a bipartition $P$ of $V_x \setminus I_x$. Denote one element of $P$ by $X^-\{x\}$, and the other one by $X^+\{x\}$. Now, denote by $Y^-$ the set of $v \in I_x$ such that there exists a sequence $v_0, \ldots, v_n$ satisfying
- $v_0 \in X^-$;
- $v_n = v$;
- $v_1, \ldots, v_n \in I_x$;
- for $i = 0, \ldots, n - 1$, $v_i v_{i+1} \in A(T_x)$.

Dually, denote by $Y^+$ the set of $v \in I_x$ such that there exists a sequence $v_0, \ldots, v_n$ satisfying
- $v_0 \in X^+$;
- $v_n = v$;
- $v_1, \ldots, v_n \in I_x$;
- for $i = 0, \ldots, n - 1$, $v_i v_{i+1} \in A(T_x)$.

**Theorem 55.** Let $H$ be a non critical, prime, and 3-uniform hypergraph. Consider a vertex $x$ of $H$ such that $H - x$ is prime. Suppose that $H - x$ admits a realization $T_x$. Then, $H$ is realizable if and only if the following two assertions hold.

(M1) There exists a bipartition $\{X^-, X^+\}$ of $V_x \setminus I_x$ satisfying
- for each component $C$ of $G_x$, with $v(C) \geq 2$, $C$ is bipartite with bipartition $\{X^- \cap V(C), X^+ \cap V(C)\}$;
- for $v^- \in X^-$ and $v^+ \in X^+$, we have 
  $$v^- v^+ \in E(G_x) \text{ if and only if } v^- v^+ \in A(T_x). \quad (32)$$

(M2) We have $Y^- \cap Y^+ = \emptyset$ and $Y^- \cup Y^+ = I_x$. Furthermore, for $x^- \in X^-$, $x^+ \in X^+$, $y^- \in Y^-$ and $y^+ \in Y^+$, we have $y^+ x^-, x^+ y^-, y^+ y^- \in A(T_x)$. 

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Moreover, if $H$ is realizable, then there exists a unique realization $T$ of $H$ such that $T - x = T_x$. Precisely, suppose that there exists a realization $T$ of $H$ such that $T - x = T_x$. For $x^- \in X^-$, $x^+ \in X^+$, $y^- \in Y^-$ and $y^+ \in Y^+$, we have $xx^-, x^+x, xy^-, y^+x \in A(T)$.

**Proof.** To begin, suppose that $H$ admits a realization $T$. Clearly, $T - x$ is a realization of $H - x$. Since $T_x$ is a realization of $H - x$, we have $C_3(T_x) = C_3(T - x)$. Since $H - x$ is prime, it follows from Theorem 11 that $T - x$ is prime as well. Since $C_3(T_x) = C_3(T - x)$, it follows from Theorem 24 that $T_x = T - x$ or $(T - x)^*$. By exchanging $T$ and $T^*$ if necessary, we can assume that

$$T_x = T - x.$$  

We show that $G_x$ is bipartite. Consider a sequence $(v_0, \ldots, v_{2n})$ of distinct elements of $V_x$, where $n \geq 2$, such that $v_i v_{i+1} \in E(G_x)$ for every $0 \leq i \leq 2n - 1$. For instance, assume that $v_0 v_1 \in A(T_x)$. Hence $v_0 v_1 \in A(T)$. Since $v_0 v_1 \in E(G_x)$, we have $xv_0 v_1 \in E(H)$. Since $T$ is a realization of $H$, with $v_0 v_1 \in A(T)$, we obtain $xv_0, v_1 x \in A(T)$. Since $v_1 v_2 \in E(G_x)$, we have $xv_1 v_2 \in E(H)$. Since $T$ is a realization of $H$, with $v_1 x \in A(T)$, we obtain $xv_2, v_1 v_2 \in A(T)$. By continuing this process, we obtain

$$\begin{align*}
xv_0, xv_2, \ldots, xv_{2n} &\in A(T) \\
\text{and} & \\
v_1, \ldots, v_{2n-1} &\in A(T).
\end{align*}$$

Thus $xv_0, xv_{2n} \in A(T)$, and hence $T\{x, v_0, v_{2n}\}$ is not a 3-cycle. It follows that $xv_0 v_{2n} \notin E(H)$, so $xv_0 v_{2n} \notin E(G_x)$. Therefore, $G_x$ does not contain odd cycles.

For Assertion (M1), consider a component $C$ of $G_x$ such that $v(C) \geq 2$. Consider distinct vertices $c_0, c_1, c_2$ of $C$ such that $c_0 c_1, c_1 c_2 \in E(G_x)$. We show that

$$\begin{align*}
c_0 c_1, c_2 c_1 &\in A(T_x) \\
or & \\
c_1 c_0, c_1 c_2 &\in A(T_x),
\end{align*}$$

(33)

Otherwise, suppose that $c_0 c_1, c_1 c_2 \in A(T_x)$. Since $c_0 c_1, c_1 c_2 \in E(G_x)$, $T\{x, c_0, c_1\}$ is a 3-cycle. Hence, $x c_0, c_1 x \in A(T_x)$ because $c_0 c_1, c_1 c_2 \in A(T_x)$. We obtain $c_1 c_2, c_1 x \in A(T_x)$. Thus, $T\{x, c_1, c_2\}$ is not a 3-cycle, which contradicts $c_1 c_2 \in E(G_x)$. It follows that (33) holds. Now, denote by $V(C)^-$ the set of the vertices $c^-$ of $C$ such that there exists $c^+ \in V(C)$ satisfying $c^- c^+ \in E(G_x)$ and $c^- c^+ \in A(T_x)$. Dually, denote by $V(C)^+$ the set of the vertices $c^+$ of $C$ such that there exists $c^- \in V(C)$ satisfying $c^+ c^- \in E(G_x)$ and $c^+ c^- \in A(T_x)$. Since $C$ is a component of $G_x$, we have $V(C) = V(C)^- \cup V(C)^+$. Moreover, it follows from (33) that

$$V(C)^- \cap V(C)^+ = \emptyset.$$  

Also, it follows from the definition of $V(C)^-$ and $V(C)^+$ that $V(C)^-$ and $V(C)^+$ are stable subsets of $C$. Therefore, $C$ is bipartite with bipartition

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\{V(C)^{-}, V(C)^{+}\}. Set \\
\[X^{-} = \bigcup_{C \in \varepsilon(G_x)} V(C)^{-} \quad \text{and} \quad X^{+} = \bigcup_{C \in \varepsilon(G_x)} V(C)^{+} \] (see Notation [40]).

Clearly, \(\{X^{-}, X^{+}\}\) is a bipartition of \(V_x \setminus I_x\). Consider again a component \(C\) of \(G_x\) such that \(v(C) \geq 2\). Since \(V(C)^{-} = X^{-} \cap V(C)\) and \(V(C)^{+} = X^{+} \cap V(C)\), \(C\) is bipartite with bipartition \(\{X^{-} \cap V(C), X^{+} \cap V(C)\}\). To prove that [32] holds, consider \(v^{-} \in X^{-}\) and \(v^{+} \in X^{+}\). First, suppose that \(v^{-}v^{+} \in E(G_x)\). Hence, \(v^{-}\) and \(v^{+}\) belong to the same component of \(G_x\). Denote it by \(C\). We obtain \(v^{-} \in V(C)^{-}\) and \(v^{+} \in V(C)^{+}\). By definition of \(V(C)^{-}\), there exists \(c^{-} \in V(C)\) such that \(v^{-}c^{-} \in E(G_x)\) and \(v^{-}c^{+} \in A(T_x)\). Since \(v^{-}v^{+} \in E(G_x)\), it follows from [33] that \(v^{-}v^{+} \in A(T_x)\). Second, suppose that \(v^{-}v^{+} \in A(T_x)\). Since \(v^{-} \in V(C)^{-}\), there exists \(c^{-} \in V(C)\) such that \(v^{-}c^{-} \in E(G_x)\) and \(v^{-}c^{+} \in A(T_x)\). It follows that \(xv^{-}, c^{-}x \in A(T)\). Similarly, since \(v^{+} \in V(C)^{+}\), there exists \(c^{+} \in V(C)\) such that \(xv^{-}, v^{+}x \in A(T)\). We obtain \(xv^{-}, v^{+}v^{-}, v^{+}x \in A(T)\). Thus, \(T[\{x, v^{-}, v^{+}\}]\) is a 3-cycle, so \(v^{-}v^{+} \in E(G_x)\).

For Assertion (M2), consider \(x^{-} \in X^{-}\). Denote by \(C\) the component of \(G_x\) such that \(x^{-} \in V(C)\). We have \(x^{-} \in V(C)^{-}\). Therefore, there exists \(c^{-} \in V(C)\) satisfying \(x^{-}c^{-} \in E(G_x)\) and \(x^{-}c^{+} \in A(T_x)\). Since \(T[\{x, x^{-}, c^{+}\}]\) is a 3-cycle and \(x^{-}c^{+} \in A(T)\), we get \(xx^{-} \in A(T)\). Hence,

\[
xx^{-} \in A(T) \quad \text{for every} \quad x^{-} \in X^{-}. \tag{34}
\]

Dually, we have

\[
x^{+}x \in A(T) \quad \text{for every} \quad x^{+} \in X^{+}. \tag{35}
\]

Now, consider \(y^{-} \in Y^{-}\). There exists a sequence \(v_0, \ldots, v_n\) satisfying \(v_0 \in X^{-}\), \(v_n = y^{-}\), \(v_1, \ldots, v_n \in I_x\), and \(v_i, v_{i+1} \in A(T_x)\) for \(i = 0, \ldots, n - 1\). We show that \(xx_i \in A(T_x)\) by induction on \(i = 0, \ldots, n\). By [34], this is the case when \(i = 0\). Consider \(i \in \{0, \ldots, n - 1\}\) and suppose that \(xx_i \in A(T_x)\). Since \(v_{i+1} \in I_x\), \(v_i, v_{i+1} \notin E(G_x)\). Thus \(T[\{x, v_i, v_{i+1}\}]\) is a linear order. Since \(xv_i, v_i, v_{i+1} \in A(T_x)\), we obtain \(xv_{i+1} \in A(T_x)\). It follows that

\[
xy^{-} \in A(T) \tag{36}
\]

for every \(y^{-} \in Y^{-}\). Dually, we have

\[
y^{+}x \in A(T) \tag{37}
\]

for every \(y^{+} \in Y^{+}\). It follows from [36] and [37] that \(Y^{-} \cap Y^{+} = \emptyset\). By definition of \(Y^{-}\) and \(Y^{+}\), \(Y^{-} \subseteq I_x\) and \(Y^{+} \subseteq I_x\). Set

\[
W = I_x \setminus (Y^{-} \cup Y^{+}).
\]

Let \(w \in W\). Since \(w \notin Y^{-}\), we have \(wz^{-} \in A(T_x)\) for every \(z^{-} \in X^{-} \cup Y^{-}\). Therefore,

\[
wz^{-} \in A(T_x) \tag{38}
\]
for \( w \in W \) and \( z^- \in X^- \cup Y^- \). Dually,
\[
z^+ w \in A(T_x)
\] (39)
for \( w \in W \) and \( z^+ \in X^+ \cup Y^+ \). It follows from (34), (35), (36), (37), (38) and (39) that \( \{x\} \cup W \) is a module of \( T \). Since \( H \) is prime, it follows from Theorem 11 that \( T \) is prime as well. Therefore, \( W = \emptyset \), so
\[
Y^- \cup Y^+ = I_x.
\]

It follows also from (34), (35), (36) and (37) that \( T \) is the unique realization of \( H \) such that \( T - x = T_x \). To conclude, consider \( x^- \in X^- \), \( x^+ \in X^+ \), \( y^- \in Y^- \) and \( y^+ \in Y^+ \). Since \( y^- \notin Y^+ \), \( x^+ y^- \notin A(T_x) \). Dually, we have \( y^+ x^- \notin A(T_x) \). It follows from (36) and (37) that \( y^+ x, xy^- \notin A(T_x) \). Since \( y^-, y^+ \notin I_x \), \( y^-y^+ \notin E(G_x) \). Thus, \( T[\{x, y^-, y^+\}] \) is a linear order. Consequently, we have \( y^-y^+ \notin A(T_x) \).

Conversely, suppose that Assertions (M1) and (M2) hold. Let \( T \) be the tournament defined on \( V(H) \) by
\[
\begin{align*}
T - x &= T_x, \\
&\text{for every } z^- \in X^- \cup Y^-, \ xz^- \in A(T), \\
&\text{and} \\
&\text{for every } z^+ \in X^+ \cup Y^+, \ z^+x \in A(T).
\end{align*}
\] (40)

We verify that \( T \) is a realization of \( H \). Since \( T_x \) realizes \( H) - x \), it suffices to verify that for distinct \( v, w \in V_x \), \( vw \in E(G_x) \) if and only if \( T[\{x, v, w\}] \) is a 3-cycle. Hence, consider distinct \( v, w \in V_x \). First, suppose that \( vw \in E(G_x) \). Denote by \( C \) the component of \( G_x \) containing \( v \) and \( w \). Since Assertion (M1) holds, \( C \) is bipartite with bipartition \( \{X^- \cap V(C), X^+ \cap V(C)\} \). By exchanging \( v \) and \( w \) if necessary, we can assume that \( v \in X^- \cap V(C) \) and \( w \in X^+ \cap V(C) \). It follows from (32) that \( vw \in A(T_x) \). Furthermore, it follows from (10) that \( xv \in A(T) \) and \( wx \in A(T) \). Therefore, \( T[\{x, v, w\}] \) is a 3-cycle. Second, suppose that \( T[\{x, v, w\}] \) is a 3-cycle. By exchanging \( v \) and \( w \) if necessary, we can assume that \( vw, wx, xv \in A(T) \). It follows from (10) that \( v \in X^- \cup Y^- \) and \( w \in X^+ \cup Y^+ \).

Moreover, since Assertion (M2) holds and \( vw \in A(T_x) \), we obtain \( v \in X^- \) and \( w \in X^+ \). It follows from (32) that \( vw \in E(G_x) \). \( \square \)

We conclude by counting the number of realizations of a realizable 3-uniform hypergraph. This counting is an immediate consequence of Proposition 51. We need the following notation.

**Notation 56.** Let \( H \) be a 3-uniform hypergraph. Set
\[
\begin{align*}
\mathcal{D}_\triangle(H) &= \{X \in \mathcal{D}_2(H) : \varepsilon_H(X) = \triangle\} \text{ (see Definition 13)} \\
\text{and} \\
\mathcal{D}_\circ(H) &= \{X \in \mathcal{D}_2(H) : \varepsilon_H(X) = \circ\}.
\end{align*}
\]

**Corollary 57.** For a realizable 3-uniform hypergraph, we have
\[
|\mathcal{D}(H)| = 2^{\left|\mathcal{D}_\triangle(H)\right|} \times \prod_{X \in \mathcal{D}_\circ(H)} |\Pi(H[X])|!.
\]

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