Subset-Sum Representations of Domination Polynomials

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Abstract The domination polynomial \(D(G, x)\) is the ordinary generating function for the dominating sets of an undirected graph \(G = (V, E)\) with respect to their cardinality. We consider in this paper representations of \(D(G, x)\) as a sum over subsets of the edge and vertex set of \(G\). One of our main results is a representation of \(D(G, x)\) as a sum ranging over spanning bipartite subgraphs of \(G\). Let \(d(G)\) be the number of dominating sets of \(G\). We call a graph \(G\) conformal if all of its components are of even order. Let \(\text{Con}(G)\) be the set of all vertex-induced conformal subgraphs of \(G\) and let \(k(G)\) be the number of components of \(G\). We show that

\[
d(G) = \sum_{H \in \text{Con}(G)} 2^{k(H)}.
\]

Keywords Domination polynomial · Dominating set · Graph polynomial

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1 Introduction

Let $G = (V, E)$ be an undirected graph. All graphs considered in this paper are assumed to be finite and simple. The closed neighborhood $N_G[v]$ of a vertex $v \in V$ is the set consisting of $v$ and all its neighbor vertices in $G$. For any subset $W \subseteq V$, we denote by $N_G[W]$ the closed neighborhood of $W$ in $G$, that is

$$N_G[W] = \bigcup_{v \in W} N_G[v].$$

If the graph is clear from the context, then we write $N[v]$ and $N[W]$ instead of $N_G[v]$ and $N_G[W]$, respectively. A dominating set of $G$ is a vertex subset $W \subseteq V$ such that $N[W] = V$. Let $W \subseteq V$ be a given vertex subset of the graph $G = (V, E)$. We denote by $\partial(W)$ the set of all edges of $G$ that have exactly one of their end vertices in $W$, that is

$$\partial(W) = \{ \{u, v\} \in E \mid u \in W, v \in V \setminus W \}.$$

The edges of $\partial(W)$ link vertices of $W$ with vertices of $V \setminus W$. Whether a given set $W$ is a dominating set of $G$ depends neither on edges lying completely inside $W$ nor on edges that have no end vertex in $W$, which gives the following statement.

**Proposition 1** Let $G = (V, E)$ be a graph, $W \subseteq V$, and $F \subseteq E$. Then $W$ is a dominating set of $(V, F)$ if and only $W$ is dominating in $(V, F \cap \partial(W))$, i.e.

$$N_{(V, F)}[W] = V \iff N_{(V, F \cap \partial(W))}[W] = V.$$

**Definition 1** Let $G = (V, E)$ be an undirected graph and $d_k(G)$ the number of dominating sets of cardinality $k$ in $G$ for $k = 0, \ldots, n = |V|$. The domination polynomial of $G$ is

$$D(G, x) = \sum_{k=0}^{n} d_k(G)x^k.$$

We denote by $d(G)$ the number of dominating sets of $G$. Consequently, we find $d(G) = D(G, 1)$.

The domination polynomial of a graph has been introduced by Arocha and Llano in [7]. More recently it has been investigated with respect to special graphs, zeros, and applications in network reliability, see [1–3,5,6,9].

The domination polynomial can also be represented as a sum over vertex subsets of $G$,

$$D(G, x) = \sum_{U \subseteq V \atop N[U] = V} x^{|U|}.$$