A note on local $\omega$-consistency and reflection

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Abstract

In this note we answer a question of R. Kaye and H. Kotlarski regarding the relationship between the schemata of local $\omega$-consistency $\omega\text{-Con}(\text{PA})$ and local reflection $\text{Rfn}(\text{PA})$ for Peano arithmetic PA. We do this by investigating the properties of the schema $\omega\text{-Con}(T)$ and its partial variants, and characterizing it in terms of partial reflection principles for $T$ in the style of C. Smoryński’s characterization of the uniform version of partial $\omega$-consistency principles. We use this characterization along with the other properties of local $\omega$-consistency and reflection to establish the key fact that $\text{Con}(\text{PA} + \text{Rfn}(\text{PA}))$ is provable in $\text{PA} + \omega\text{-Con}(\text{PA})$, implying that $\omega\text{-Con}(\text{PA})$ does not follow from $\text{PA} + \text{Rfn}(\text{PA})$. We also introduce and discuss a modified version of the definable reflection principle, namely, the schema of uniform reflection with $\Sigma_n$-definable parameters.

Keywords: $\omega$-consistency, reflection principles, definable elements.

1 Introduction

A formal theory $T$ is said to be $\omega$-inconsistent if there is a formula $\varphi(x)$ such that the following two conditions hold simultaneously

(i) $T \vdash \exists x \varphi(x),$

(ii) $T \vdash \neg \varphi(n)$ for each natural number $n$,
where \( n \) is the \( n \)th numeral, i.e., the term \( 0'\ldots' \) with \( n \) successor symbols. If this is not the case, we say that \( T \) is \( \omega \)-consistent. This version of the consistency assertion was one of the conditions in the original formulation of Gödel’s first incompleteness theorem. Clearly, \( \omega \)-consistency implies (ordinary) consistency and is implied by the semantic notion of \textit{soundness} of \( T \), i.e., the assertion that every theorem of \( T \) is true in the standard model \( \mathbb{N} \).

In the contemporary formulations of Gödel’s incompleteness theorems a weaker condition of \( \Sigma_1 \)-\textit{soundness} (which, for sufficiently strong theories \( T \), is equivalent to the points (i) and (ii) above with \( \varphi(x) \) restricted to \( \Delta_0 \)-formulas) is often used instead of \( \omega \)-consistency.

On the syntactic level, various semantic notions of soundness are usually translated into the so-called \textit{reflection principles}. These principles typically express the following form of soundness

\[ T \vdash \varphi \implies \mathbb{N} \models \varphi, \]

with different conditions being imposed on the formula \( \varphi \) and on the notion of provability in \( T \).

Reflection principles were shown to be a convenient tool for the analysis of formal theories by demonstrating that many other principles, e.g., induction or different forms of consistency, can be expressed as some form of reflection and due to the \textit{unboundedness theorems} of G. Kreisel and A. Lévy [9], which allow to conclude that one formal theory \( T \) cannot be axiomatized over another formal theory \( U \) by the arithmetical sentences of a certain logical complexity, whenever \( T \) proves the corresponding form of reflection for \( U \). For a survey of the results on reflection principles, see C. Smoryński [13] and L. Beklemishev [4].

In particular, C. Smoryński [12] proved that the global \( \omega \)-consistency assertion (formulated as a single arithmetical sentence by quantifying over the formulas \( \varphi(x) \)) is equivalent to (the uniform) \( \Pi_3 \)-soundness of a theory \( T + \text{RFN}(T) \), where \( \text{RFN}(T) \) is the \textit{uniform reflection principle} for \( T \), formally asserting that for each formula \( \varphi(x) \) and a natural number \( n \) we have

\[ T \vdash \varphi(n) \implies \mathbb{N} \models \varphi(n). \]

See also [13, Theorem 4.2.5] for the refined version of this result, where the restricted variants of \textit{uniform} \( \omega \)-\textit{consistency} are characterized in the same manner.

The schema of local \( \omega \)-consistency and various kinds of reflection principles for \( \text{PA} \) were studied by R. Kaye and H. Kotlarski in [6] from the point of view of \textit{ACT}-extensions of models (extensions constructed by the means of the arithmetized completeness theorem). More specifically, the authors characterize these principles as the
first-order theories of the class of models of PA having ACT-extensions with certain model-theoretic properties.

The following questions regarding the relationship between local ω-consistency \(\omega\text{-}Con(PA)\) and local reflection \(\text{Rfn}(PA)\) were listed as open in [6].

**Problem 8.1.** Do there exists models \(M \models PA + \text{Rfn}(PA) + \neg \omega\text{-}Con(PA)\)?

**Problem 8.3.** Is it true that \(PA + \text{Rfn}(PA) \not\vdash \omega\text{-}Con\text{Th}(PA)\)?

**Problem 8.4.** Over PA, does \(\omega\text{-}Con(PA)\) imply \(\text{Con}(PA + \text{Rfn}(PA))\)?

For the formal definitions of the \(\omega\)-consistency and reflection principles mentioned in these questions see Section 2.

We answer all of these three questions positively by characterizing the schema \(\omega\text{-}Con(T)\) and its restricted variants in terms of partial reflection principles for \(T\) in the style of Smoryński (see Theorem 1). Using this characterization along with the other properties of local \(\omega\)-consistency and reflection schemata we prove the main result, Theorem 2, which shows that, in particular, \(PA + \omega\text{-}Con(PA)\) implies \(\text{Con}(PA + \text{Rfn}(PA))\), solving Problem 8.4 and, consequently, the other two problems as well. We also show that the schema \(\omega\text{-}Con\text{Th}(T)\) is equivalent to \(\text{RFN}(T)\) (see Theorem 3). Finally, we introduce a modified variant of the definable reflection principle considered in [6], namely, the schema of uniform reflection with \(\Sigma_n\)-definable parameters and show it to be equivalent to the corresponding form of local reflection. This fact is then used to give a short model-theoretic proof of the \(\Sigma_{n+2}\)-conservativity of uniform \(\Sigma_{n+1}\)-reflection over relativized local \(\Sigma_{n+1}\)-reflection (see Theorem 4).

The paper is organized as follows. Section 2 introduces the basic notions and notation used throughout this note. In Section 3 we prove the main results establishing the relationship between the local \(\omega\)-consistency schema and the reflection principles, and use these results to answer the questions posed above. In Section 4 we introduce and discuss the schema of uniform reflection with \(\Sigma_n\)-definable parameters.

## 2 Preliminaries

In this note we consider first-order theories in the language of arithmetic. As our basic theory we take Elementary arithmetic \(EA\) (sometimes denoted as \(I\Delta_0(\exp)\)), that is, the first-order theory formulated in the language \(0, (\cdot)', +, \times\) extended by the unary function symbol \(\exp\) for the exponentiation function \(2^x\). It has the standard defining axioms for these symbols and the induction schema for all \textit{elementary formulas} (we also call such formulas \textit{bounded}), i.e., formulas in the language with exponent containing only bounded quantifiers. We define classes \(\Sigma_0\) and \(\Pi_0\) to be
the classes of all elementary (bounded) formulas. After that the classes $\Sigma_n$ and $\Pi_n$ of arithmetical hierarchy are defined in a standard way for all $n \geq 0$.

If we allow induction for all arithmetical formulas, the resulting theory is Peano arithmetic denoted by PA. For a fixed class of arithmetical formulas $\Gamma$ the fragment of PA obtained by restricting the induction schema

$$\varphi(0) \land \forall x (\varphi(x) \rightarrow \varphi(x+1)) \rightarrow \forall x \varphi(x), \quad \varphi(x) \in \Gamma$$

to $\Gamma$-formulas without parameters is denoted by $\Gamma^-$ and called parameter free $\Gamma$-induction. If, in the schema above $\varphi(x)$, is allowed to contain parameters, then we obtain the usual $\Gamma$-induction schema and the corresponding theory is denoted by $\Gamma$. We also consider the schema $B\Sigma_1$ of $\Sigma_1$-collection

$$\forall x < z \exists y \varphi(x,y,a) \rightarrow \exists u \forall x < z \exists y < u \varphi(x,y,a), \quad \varphi(x,y,a) \in \Sigma_1.$$ 

For more details on these theories, see [7].

All the theories considered in this note are supposed to be recursively axiomatizable consistent extensions of $\text{EA}$. We assume that some standard arithmetization of syntax and the gödelnumbering of syntactic objects has been fixed. In particular, we write $\lceil \varphi \rceil$ for the (numeral of the) gödelnumber of $\varphi$. As usual, each theory $T$ is given to us by an elementary formula $\sigma_T(x)$, defining the set of axioms of $T$ in the standard model of arithmetic. The formula $\sigma_T(x)$ is used in the construction of the formula $\text{Prf}_T(y,x)$ representing the relation “$y$ codes a $T$-proof of the formula with gödelnumber $x$”. The standard provability predicate for $T$ is given by $\exists y \text{Prf}_T(y,x)$, and we denote this formula by $\Box_T(x)$. We often write $\Box_T \varphi$ instead of $\Box_T(\lceil \varphi \rceil)$ and use the notation $\Diamond_T \varphi$ for $\neg \Box_T \neg \varphi$. The sentence $\Diamond_T \top$ is the consistency assertion for $T$ and is also denoted by $\text{Con}(T)$.

The predicate $\Box_T$ satisfies Löb’s derivability conditions provably in $\text{EA}$ (cf. [4]):

1. If $T \vdash \varphi$, then $\text{EA} \vdash \Box_T \varphi$.
2. $\text{EA} \vdash \Box_T(\varphi \rightarrow \psi) \rightarrow (\Box_T \varphi \rightarrow \Box_T \psi)$.
3. $\text{EA} \vdash \Box_T \varphi \rightarrow \Box_T \Box_T \varphi$.

Point 3 follows from the general fact known as provable $\Sigma_1$-completeness:

$$\text{EA} \vdash \forall x_1, \ldots, x_m (\sigma(x_1, \ldots, x_m) \rightarrow \Box_T \sigma(\underline{x}_1, \ldots, \underline{x}_m)),$$

whenever $\sigma(x_1, \ldots, x_m)$ is a $\Sigma_1$-formula. Here the underline notation $\lceil \varphi(\underline{x}) \rceil$ stands for the elementarily definable term, representing the elementary function that maps
k to the Gödel number $\Gamma \varphi(k)$. In what follows we usually write just $\Box_T \varphi(x)$ instead of $\Box_T \varphi(\overline{x})$.

If two theories $T$ and $U$ have the same theorems, we say that they are *deductively equivalent* and denote this by $T \equiv U$. If they prove the same arithmetical sentences of complexity $\Gamma$, we write $T \equiv_T U$.

In this note we are mainly interested in the following three principles (or schemata) for a given arithmetical theory $T$:

- **Local $\omega$-consistency** $\omega-\text{Con}(T)$:
  \[ \forall x_1, \ldots, x_n \Box_T \varphi(x_1, \ldots, x_n) \rightarrow \Diamond_T \forall x_1, \ldots, x_n \varphi(x_1, \ldots, x_n), \]
  for each arithmetical formula $\varphi(x_1, \ldots, x_n)$;

- **Local reflection** $\text{Rfn}(T)$:
  \[ \Box_T \varphi \rightarrow \varphi, \]
  for each arithmetical sentence $\varphi$;

- **Uniform reflection** $\text{RFN}(T)$:
  \[ \forall x_1, \ldots, x_n \left( \Box_T \varphi(x_1, \ldots, x_n) \rightarrow \varphi(x_1, \ldots, x_n) \right), \]
  for each arithmetical formula $\varphi(x_1, \ldots, x_n)$.

For a more detailed analysis of the principles above we also consider their *partial* variants, which are obtained by imposing the restriction $\varphi \in \Gamma$, where $\Gamma$ is some class of arithmetical formulas (usually, $\Sigma_n$ or $\Pi_n$). Corresponding partial principles are denoted by $\omega-\text{Con}_\Gamma(T)$, $\text{Rfn}_\Gamma(T)$ and $\text{RFN}_\Gamma(T)$, respectively.

Note that, in the definitions of $\omega-\text{Con}(T)$ and $\text{RFN}(T)$ (and their partial analogues), by using sequence coding functions, we can, equivalently, restrict these schemata to the formulas $\varphi(x)$ with a single free variable $x$.

In [6] the authors have also considered the following schema $\omega-\text{Con}^\text{Th}(T)$ (note that we use slightly different notation), which is a strengthening of $\omega-\text{Con}(T)$,

- **Local $\omega$-consistency of the theory of the model** $\omega-\text{Con}^\text{Th}(T)$:
  \[ \sigma \land \forall x \Box_T (\sigma \rightarrow \varphi(x)) \rightarrow \Diamond_T (\sigma \land \forall x \varphi(x)), \]
  for each arithmetical formula $\varphi(x)$ with a single free variable $x$ and arithmetical sentence $\sigma$.  

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It is shown that \(\omega^{-\text{Con}}(\mathsf{PA})\) implies \(\text{Rfn}(\mathsf{PA})\) (take \(\varphi(x)\) to be \(x = x\)) and is implied by \(\text{RFN}(\mathsf{PA})\).

In what follows, when arguing about the principles introduced above, we freely use their dual forms without specifically mentioning it, e.g., \(\text{Rfn}_{\Sigma_n}(T)\) is equivalent to \(\pi \land \Diamond_T \pi\),

where \(\pi\) is a \(\Pi_n\)-sentence, and \(\omega^{-\text{Con}}_{\Sigma_n}(T)\) is equivalent to

\[\square_T \exists x \pi(x) \land \exists x \Diamond_T \pi(x),\]

where \(\pi(x)\) is \(\Pi_n\)-formula.

Throughout this note we often use the following principle known as the small reflection (formalized in \(\mathsf{EA}\)). We include a proof of this important principle for the sake of completeness.

**Proposition 2.1.** For each formula \(\varphi(x)\)

\[\mathsf{EA} \vdash \forall x, y \Diamond_T (\text{Prf}_T(y, \lceil \varphi(x) \rceil) \rightarrow \varphi(x)).\]

**Proof.** By weakening \(\text{Prf}_T\) to \(\Box_T\), we get

\[\mathsf{EA} \vdash \text{Prf}_T(y, \lceil \varphi(x) \rceil) \rightarrow \Box_T \varphi(x),\]

\[\quad \rightarrow \Box_T (\text{Prf}_T(y, \lceil \varphi(x) \rceil) \rightarrow \varphi(x)).\]

On the other hand, by provable \(\Sigma_1\)-completeness

\[\mathsf{EA} \vdash \lnot \text{Prf}_T(y, \lceil \varphi(x) \rceil) \rightarrow \Box_T (\lnot \text{Prf}_T(y, \lceil \varphi(x) \rceil))\]

\[\quad \rightarrow \Box_T (\text{Prf}_T(y, \lceil \varphi(x) \rceil) \rightarrow \varphi(x)),\]

whence the result follows. \(\square\)

It is known that for each \(n > 0\) there exists an arithmetical \(\Pi_n\)-formula \(\text{True}_{\Pi_n}(x)\) (known as a truth definition for \(\Pi_n\)-formulas) such that

\[\mathsf{EA} \vdash \forall x_1, \ldots, x_m (\varphi(x_1, \ldots, x_m) \leftrightarrow \text{True}_{\Pi_n}(\lceil \varphi(x_1, \ldots, x_m) \rceil)),\]

for every \(\Pi_n\)-formula \(\varphi(x_1, \ldots, x_m)\), and this fact itself is formalizable in \(\mathsf{EA}\), so

\[\mathsf{EA} \vdash \forall \pi \in \Pi_n \Box_T (\pi \leftrightarrow \text{True}_{\Pi_n}(\pi)).\]

Using these properties, it is not hard to check that over \(\mathsf{EA}\), the schema \(\text{RFN}_{\Sigma_n}(T)\) is equivalent to its universal instance with \(\varphi(x)\) taken to be \(\lnot \text{True}_{\Pi_n}(x)\).
3 Local ω-consistency and reflection

In this section we study the partial local ω-consistency principles ω-Con_{Σ_n}(T). We start by proving several useful properties and obtaining the characterization of these schemata in terms of partial reflection principles, analogous to that of Smoryński (Theorem 1). The main result of this section is Theorem 2 which allows us to answer all the questions posed in the introduction. In addition, we prove that the schema ω-Con^Th(T) is actually equivalent to RFN(T) (Theorem 3).

It is known that partial uniform reflection principles satisfy the equivalence RFN_{Π_{n+1}}(T) ≡ RFN_{Σ_n}(T) for each n ≥ 0 (see [4, Lemma 2.4]). We prove the same result for partial local ω-consistency principles.

Proposition 3.1. Over EA, ω-Con_{Π_{n+1}}(T) ≡ ω-Con_{Σ_n}(T) for each n ≥ 0.

Proof. Fix an arbitrary Π_{n+1}-formula ∀y φ(x, y) with a single free variable x (see the remark above about formulas with several variables), where φ(x, y) ∈ Σ_n. We derive

EA + ω-Con_{Σ_n}(T) ⊢ ∀x □_T ∀y φ(x, y) → ∀x, y □_T φ(x, y)
→ □_T ∀x, y φ(x, y),

where the last implication follows from the corresponding axiom of ω-Con_{Σ_n}(T). □

The next proposition shows how much reflection is sufficient to derive ω-Con_{Σ_n}(T).

Proposition 3.2. EA + RFN_{Σ_n}(T) + Rfn_{Σ_{n+1}}(T) ⊢ ω-Con_{Σ_n}(T) for each n ≥ 0.

Proof. Fix an arbitrary Σ_n-formula φ(x). We have

EA + RFN_{Σ_n}(T) + Rfn_{Σ_{n+1}}(T) ⊢ ∀x □_T φ(x) → ∀x φ(x)
→ □_T ∀x φ(x).

□

Corollary 3.3. For each n ≥ 0,

(i) EA + RFN_{Σ_{n+1}}(T) ⊢ ω-Con_{Σ_n}(T).

(ii) EA + RFN(T) ⊢ ω-Con(T).

The following theorem provides the characterization of local ω-consistency principles in terms of partial reflection principles for T. It can be seen as a direct analogue of a theorem due to Smoryński [13, Theorem 4.2.5], where the corresponding results about uniform ω-consistency assertions are obtained.
Theorem 1. Over EA,

(i) $\omega\text{-Con}_{\Sigma_n}(T) \equiv \text{Rfn}_{\Sigma_n}(T + \text{RFN}_{\Sigma_n}(T))$ for each $n > 0$,

(ii) $\omega\text{-Con}_{\Sigma_0}(T) \equiv \text{Rfn}_{\Sigma_1}(T)$.

Proof. (i) Fix a $\Pi^1_n$-formula $\pi(x)$ with a single free variable $x$. Using the axiom $\forall x \ (\pi(x) \rightarrow \Box_T \pi(x))$ of $\text{RFN}_{\Sigma_n}(T)$ under $\Box_{T+\text{RFN}_{\Sigma_n}(T)}$ we derive

$$\text{EA} + \text{Rfn}_{\Sigma_2}(T + \text{RFN}_{\Sigma_n}(T)) \vdash \Box_T \exists x \pi(x) \rightarrow \Box_{T+\text{RFN}_{\Sigma_n}(T)} \exists x \pi(x)$$

$$\rightarrow \Box_{T+\text{RFN}_{\Sigma_n}(T)} \exists x \Box_T \pi(x)$$

$$\rightarrow \exists x \Box_T \pi(x),$$

where the last implication uses local $\Sigma_2$-reflection. Now, we prove that for each formula $\forall x \sigma(x)$, where $\sigma(x)$ is a $\Sigma_1$-formula, we have

$$\text{EA} + \omega\text{-Con}_{\Sigma_n}(T) \vdash \forall x \sigma(x) \rightarrow \Box_{T+\text{RFN}_{\Sigma_n}(T)} \forall x \sigma(x).$$

Assuming $\varphi(x)$ to be a $\Sigma_n$-formula, we use provable $\Sigma_1$-completeness and small reflection principle as follows

$$\text{EA} + \omega\text{-Con}_{\Sigma_n}(T) \vdash \forall x \varphi(x) \rightarrow \forall x \Box_T \varphi(x)$$

$$\rightarrow \forall x, y, z \ (\varphi(x) \rightarrow \varphi(y))$$

$$\rightarrow \Box_T (\forall x, y, z \ (\varphi(x) \rightarrow \varphi(y)))$$

$$\rightarrow \Box_T (\forall x, y, z \ (\varphi(x) \rightarrow \varphi(y)))$$

$$\rightarrow \Box_T (\forall x, y, z \ (\varphi(x) \rightarrow \varphi(y)))$$

where the third implication uses local $\omega$-consistency for the $\Sigma_n$-formula under $\Box_T$ in the second line. The final formula is equivalent to $\Box_{T+\text{RFN}_{\Sigma_n}(T)} \forall x \sigma(x)$, if we take $\varphi(y)$ to be $\neg \text{True}_{\Pi_n}(y)$ for $n > 0$.

(ii) For $\Sigma_0$-formula $\delta(x)$ we have

$$\text{EA} + \omega\text{-Con}_{\Sigma_0}(T) \vdash \forall x \delta(x) \rightarrow \forall x \Box_T \delta(x)$$

$$\rightarrow \Box_T \forall x \delta(x).$$

Conversely, we argue by contraposition

$$\text{EA} + \text{Rfn}_{\Sigma_1}(T) \vdash \Box_T \exists x \neg \delta(x) \land \forall x \Box_T \delta(x)$$

$$\rightarrow \exists x \Box_T \neg \delta(x) \land \forall x \Box_T \delta(x)$$

$$\rightarrow \exists x \Box_T (\neg \delta(x) \land \delta(x))$$

$$\rightarrow \Box_T \bot$$

$$\rightarrow \bot,$$
where the first and the last implications use $\text{Rfn}_{\Sigma_1}(T)$. As a result, we get

$$\text{EA} + \text{Rfn}_{\Sigma_1}(T) \vdash \forall x \square_T \delta(x) \rightarrow \Diamond_T \forall x \delta(x).$$

□

**Corollary 3.4.** Over EA, $\omega\text{-Con}(T) \equiv \bigcup_{n<\omega} \text{Rfn}_{\Sigma_2}(T + \text{RFN}_{\Sigma_n}(T))$.

The above characterization together with Corollary 3.3 yields the following

**Proposition 3.5.** $\text{EA} + \omega\text{-Con}_{\Sigma_{n+1}}(T) \vdash \text{Con}(T + \omega\text{-Con}_{\Sigma_n}(T))$ for each $n \geq 0$.

**Proof.** By Theorem 1 we have

$$\text{EA} + \omega\text{-Con}_{\Sigma_{n+1}}(T) \vdash \text{Rfn}_{\Sigma_2}(T + \text{RFN}_{\Sigma_{n+1}}(T)),$n

whence, in particular, $\text{EA} + \omega\text{-Con}_{\Sigma_{n+1}}(T) \vdash \text{Con}(T + \text{RFN}_{\Sigma_{n+1}}(T))$.

Point (i) of Corollary 3.3 can be formalized in EA and strengthened to get

$$\text{EA} \vdash \forall \varphi \left( \square_{T + \omega\text{-Con}_{\Sigma_n}(T)} \varphi \rightarrow \square_{T + \text{RFN}_{\Sigma_{n+1}}(T)} \varphi \right).$$

(1)

To see this, observe that the proof of Corollary 3.3 goes by showing that each axiom of $T + \omega\text{-Con}_{\Sigma_n}(T)$ is provable in $T + \text{RFN}_{\Sigma_{n+1}}(T)$, and this fact is formalizable in EA. Using $\Sigma_1$-collection schema (see [3, Proposition 5.1] for the analogous argument), we conclude that each theorem of $T + \omega\text{-Con}_{\Sigma_n}(T)$ is provable in $T + \text{RFN}_{\Sigma_{n+1}}(T)$, i.e.,

$$\text{EA} + B\Sigma_1 \vdash \forall \varphi \left( \square_{T + \omega\text{-Con}_{\Sigma_n}(T)} \varphi \rightarrow \square_{T + \text{RFN}_{\Sigma_{n+1}}(T)} \varphi \right),$$

Now, using $\Pi_2$-conservativity of $\text{EA} + B\Sigma_1$ over EA (see [3] Corollary 4.1), this yields [11]. Taking $\varphi$ to be $\bot$ in [11] we get

$$\text{EA} \vdash \text{Con}(T + \text{RFN}_{\Sigma_{n+1}}(T)) \rightarrow \text{Con}(T + \omega\text{-Con}_{\Sigma_n}(T)),$n

whence $\text{EA} + \omega\text{-Con}_{\Sigma_{n+1}}(T) \vdash \text{Con}(T + \omega\text{-Con}_{\Sigma_n}(T))$, as required. □

In view of Gödel’s second incompleteness theorem, Proposition 3.5 and Theorem [11] imply that the hierarchy of partial local $\omega$-consistency principles $\omega\text{-Con}_{\Sigma_n}(T)$ does not collapse if and only if $T + \omega\text{-Con}(T)$ is consistent. We also have the following

**Proposition 3.6.** $T + \omega\text{-Con}(T)$ is equiconsistent with $T + \text{RFN}(T)$.
Proof. By Corollary 3.3 the consistency of $T + \text{RFN}(T)$ implies that of $T + \omega\text{-Con}(T)$. Now, assume $T + \text{RFN}(T)$ is inconsistent. Then the theory $T + \text{RFN}_{\Sigma_n}(T)$ is inconsistent for some $n \geq 0$, whence, by $\Sigma_1$-completeness, $\text{EA} \vdash \neg \text{Con}(T + \text{RFN}_{\Sigma_n}(T))$, but also by Theorem 1 $\text{EA} + \omega\text{-Con}(T) \vdash \text{Con}(T + \text{RFN}_{\Sigma_n}(T))$, so $\text{EA} + \omega\text{-Con}(T)$ is inconsistent.

The following theorem is another consequence of the characterization of $\omega\text{-Con}(T)$.

**Theorem 2.** $\text{EA} + \omega\text{-Con}_{\Sigma_1}(T) \vdash \text{Con}(T + \text{Rfn}(T))$.

Proof. Lemma 3.5 from [8] for $n = 0$ (which is essentially a formalization of the fact that $\text{Rfn}(T)$ is contained in $T$ together with all true $\Pi_1$-sentences) implies that

$$\text{EA} \vdash \text{Con}(T + \text{RFN}_{\Sigma_1}(T)) \to \text{Con}(T + \text{Rfn}(T)),$$

and by point (i) of Theorem 1 we have $\text{EA} + \omega\text{-Con}_{\Sigma_1}(T) \vdash \text{Con}(T + \text{RFN}_{\Sigma_1}(T))$, whence the result follows.

In particular, $\text{PA} + \omega\text{-Con}_{\Sigma_1}(\text{PA}) \vdash \text{Con}(\text{PA} + \text{Rfn}(\text{PA}))$, which solves Problem 8.4 and, consequently, Problems 8.1 and 8.3 as well, because then

$$\text{PA} + \text{Rfn}(\text{PA}) \not\vdash \omega\text{-Con}(\text{PA}),$$

by Gödel’s second incompleteness theorem, whence, certainly,

$$\text{PA} + \text{Rfn}(\text{PA}) \not\vdash \omega\text{-Con}^{\text{Th}}(\text{PA}).$$

Recall from Section 2 that $\omega\text{-Con}^{\text{Th}}(\text{PA})$ is implied by $\text{RFN}(\text{PA})$. We prove that these two schemata are equivalent for an arbitrary theory $T$, which provides an alternative solution to Problem 8.3.

**Theorem 3.** Over $\text{EA}$, $\omega\text{-Con}^{\text{Th}}(T) \equiv \text{RFN}(T)$.

Proof. To derive $\omega\text{-Con}^{\text{Th}}(T)$ from $\text{RFN}(T)$ argue as follows

$$\text{EA} + \text{RFN}(T) \vdash \sigma \land \forall x \Box_T(\sigma \to \varphi(x)) \to \sigma \land \forall x (\sigma \to \varphi(x)) \to \sigma \land \forall x \varphi(x) \to \Box_T(\sigma \land \forall x \varphi(x)),$$

where the first and the last implications use $\text{RFN}(T)$.
To show the converse fix a formula $\varphi(x)$ with a single free variable $x$ and let $\sigma$ be the sentence $\neg \forall x \ (\square_T \varphi(x) \to \varphi(x))$, i.e., the negation of the corresponding instance of $\text{RFN}(T)$. We derive $\neg \sigma$ using contraposition and small reflection

$$\text{EA} + \omega\text{-Con}^\text{Th}(T) \vdash \sigma \to [\sigma \land \forall x, y \ (\Prf_T(y, \varphi(\overline{x})) \to \varphi(\overline{x})]$$

$$\to [\sigma \land \forall y \ (\Prf_T(y, \varphi(\overline{x})) \to \varphi(\overline{x})])$$

$$\to \Diamond_T (\sigma \land \neg \sigma)$$

whence $\text{EA} + \omega\text{-Con}^\text{Th}(T) \vdash \square_T \top \to \neg \sigma$ and $\text{EA} + \omega\text{-Con}^\text{Th}(T) \vdash \neg \sigma$, so

$$\text{EA} + \omega\text{-Con}^\text{Th}(T) \vdash \text{RFN}(T).$$

\[\square_T \top \to \neg \sigma\]

\[\Diamond_T (\sigma \land \neg \sigma)\]

\[\Diamond_T \bot,\]

In particular, $\text{PA} + \text{Rfn}(\text{PA}) \not\vdash \text{RFN}(\text{PA})$, since $\text{PA} + \text{Rfn}(\text{PA}) \not\vdash \text{RFN}(\text{PA})$, which provides an alternative solution to Problem 8.3.

To formulate these corollaries for an arbitrary theory $T$ in place of $\text{PA}$ we recall the following definition. A theory $T$ is said to have \textit{infinite characteristic}, if the theory $T^\omega$ is consistent, where

$$T_0 := T, \ T_{n+1} := T_n + \text{Con}(T_n), \ T_\omega := \bigcup_{n<\omega} T_n.$$ 

The following two facts are known about this notion (see Corollary 2.38 and Corollary 2.35 in [4]):

1. $T + \text{Rfn}(T) \not\vdash \text{RFN}(T)$ for a theory $T$ of infinite characteristic.

2. $T + \text{Rfn}(T)$ is equiconsistent with $T_\omega$.

The first fact together with Theorem \[\square_T \top \to \neg \sigma\] yields the following

\textbf{Corollary 3.7}. \textit{For each theory $T$ of infinite characteristic}

$$T + \text{Rfn}(T) \not\vdash \omega\text{-Con}^\text{Th}(T).$$

The second fact together with Theorem \[\square_T \top \to \neg \sigma\] and G"{o}del's second incompleteness theorem gives a stronger result.
Corollary 3.8. For each theory $T$ of infinite characteristic

$$T + Rfn(T) \not\vdash \omega-\text{Con}_{\Sigma_1}(T).$$

Our characterizations also yield the following unboundedness results for the local $\omega$-consistency principles (see Corollary 2.22 (i) and Corollary 2.17 (ii) in [4]).

Corollary 3.9. For each $n > 0$,

(1) $\omega-\text{Con}_{\Sigma_n}(T)$ (and, consequently, $\omega-\text{Con}(T)$) is not contained in any consistent r.e. extension of $T + Rfn_{\Sigma_n}(T)$ of arithmetical complexity $\Pi_2$.

(2) $\omega-\text{Con}^T_{\Sigma_n}(T)$ is not contained in any consistent extension of $T$ of bounded arithmetical complexity.

4 Reflection with definable parameters

In [6] the authors have also introduced an intermediate reflection schema (we formulate it in a dual form, which is actually used in the proofs there) called

- *definable reflection* $\text{DRfn}(T)$

$$\forall x (\varphi(x) \land \forall y (\varphi(y) \rightarrow y = x) \rightarrow \Diamond_T [\varphi(x) \land \forall y (\varphi(y) \rightarrow y = x)],$$

for each arithmetical formula $\varphi(x)$ with a single free variable $x$.

It was shown that $\text{DRfn}(PA)$ is actually equivalent to $\text{RFN}(PA)$ using indicators. Let us include a proof of this fact without using indicators (note the change of the base theory to $PA$, since we use induction).

**Proposition 4.1.** Over $PA$, $\text{DRfn}(T) \equiv \text{RFN}(T)$.

**Proof.** Clearly, $EA + Rfn(T) \vdash DRfn(T)$. Fix a formula $\varphi(x)$ and let

$$\psi(x, y) := \text{Prf}_T(y, \langle \varphi(x) \rangle) \rightarrow \varphi(x).$$

We show that $PA + DRfn(T) \vdash \forall x, y \psi(x, y)$. Arguing informally in $PA$, assume $\exists x, y \neg \psi(x, y)$. By the least element principle $\exists z \delta(z)$, where $\delta(z)$ asserts that $z$ codes the least pair $\langle x, y \rangle$ satisfying $\neg \psi(x, y)$, so

$$PA + DRfn(T) \vdash \exists x, y \neg \psi(x, y) \rightarrow \exists z \delta(z)$$

$$\rightarrow \exists z (\delta(z) \land \forall u (\delta(u) \rightarrow u = z))$$
$$\rightarrow \exists z \Diamond_T (\delta(z) \land \forall u (\delta(u) \rightarrow u = z)), $$
where the last implication uses $\text{DRfn}(T)$. Since $\delta(z)$ implies $\neg \psi((z)_0, (z)_1)$ we get

$$\text{PA} + \text{DRfn}(T) \vdash \exists x, y \neg \psi(x, y) \rightarrow \exists z \diamond_T \neg \psi((z)_0, (z)_1),$$

which contradicts the small reflection principle $\text{PA} \vdash \forall z \Box_T \psi((z)_0, (z)_1)$, whence

$$\text{PA} + \text{DRfn}(T) \vdash \exists x, y \neg \psi(x, y) \rightarrow \bot,$$

and $\text{PA} + \text{DRfn}(T) \vdash \forall x, y \psi(x, y)$, which shows that $\text{PA} + \text{DRfn}(T) \vdash \text{RFN}(T)$.

In this section we consider a refined version of the definable reflection principle, namely, uniform reflection with $\Sigma^n$-definable parameters, and use it to give a model-theoretic proof of the $\Sigma_{n+2}$-conservativity of uniform reflection over relativized local reflection (Theorem 4). We also discuss a relationship between these principles and the schemata of induction with definable parameters introduced by A. Cordón-Franco et al. in [5].

Recall the uniform $\Sigma^k$-reflection principle $\text{RFN}_{\Sigma^k}(T)$:

$$\forall x \left( \square_T \varphi(x) \rightarrow \varphi(x) \right),$$

where $\varphi(x)$ is a $\Sigma^k$-formula. If we require the variable $x$ above to range only over the standard elements (numerals), then we get the schema that is equivalent to local reflection $\text{Rfn}_{\Sigma^k}(T)$. We investigate the question: can we expand the range of $x$ to some nonstandard elements while still obtaining the equivalent schema?

Formally, we define the following schema of

- uniform $\Sigma^k$-reflection with $\Sigma^n$-definable parameters $\text{RFN}_{\Sigma^k}^n(T)$:

$$\forall x \left( \text{Def}_\delta(x) \rightarrow (\square_T \varphi(x) \rightarrow \varphi(x)) \right),$$

for each $\Sigma^n$-formula $\delta(x)$ and $\Sigma^k$-formula $\varphi(x)$, where

$$\text{Def}_\delta(x) := \delta(x) \land \forall y, z (\delta(y) \land \delta(z) \rightarrow y = z)$$

is the formula asserting that $x$ is the unique element satisfying $\delta(x)$.

We aim at proving that these reflection principles are equivalent to their local counterparts. To cover the case $n > 1$ we need to introduce the notion of $n$-provability and corresponding local reflection principles. The following formula

$$[n]_T \varphi := \exists \pi \left( \text{True}_{\Pi^n}(\pi) \land \Box_T (\pi \rightarrow \varphi) \right),$$

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defines the predicate of \( n \)-provability, i.e., usual provability in \( T \) together with all true \( \Pi_n \)-sentences taken as additional axioms. The predicate \([n]_T\) satisfies the same derivability conditions as \( \Box_T \) and is provably \( \Sigma_{n+1} \)-complete (see [1]).

The relativized local reflection principles \( \text{Rfn}_n^\alpha(T) \) are defined analogously to \( \text{Rfn}_T(T) \) but with \( n \)-provability predicate \([n]_T\) instead of the usual provability \( \Box_T \).

**Proposition 4.2.** For each \( k > n \geq 0 \) we have \( \text{EA} + \text{Rfn}_n^\alpha(T) \vdash \text{RFN}_{\Sigma_k}^{K^{n+1}}(T) \).

**Proof.** Fix some \( \Sigma_{n+1} \)-formula \( \delta(x) \) and \( \Sigma_k \)-formula \( \varphi(x) \). We will derive the corresponding axiom of \( \text{RFN}_{\Sigma_k}^{K^{n+1}}(T) \). Using provable \( \Sigma_{n+1} \)-completeness and an instance of \( \text{Rfn}_n^\alpha(\Sigma_k)(T) \) for the \( \Sigma_k \)-sentence \( \exists u (\delta(u) \land \varphi(u)) \) (we use \( n < k \) here) we derive

\[
\text{EA} + \text{Rfn}_n^\alpha(\Sigma_k)(T) \vdash \text{Def}_\delta(x) \land \Box_T \varphi(x) \rightarrow [n]_T \delta(x) \land \Box_T \varphi(x) \\
\rightarrow [n]_T (\delta(x) \land \varphi(x)) \\
\rightarrow [n]_T \exists u (\delta(u) \land \varphi(u)) \\
\rightarrow \exists u (\delta(u) \land \varphi(u)).
\]

Since \( \text{EA} \vdash \text{Def}_\delta(x) \land \delta(u) \rightarrow x = u \), we get

\[
\text{EA} + \text{Rfn}_n^\alpha(\Sigma_k)(T) \vdash \text{Def}_\delta(x) \land \delta(u) \land \varphi(u) \rightarrow (x = u) \land \varphi(u) \\
\rightarrow \varphi(x),
\]

whence \( \text{EA} + \text{Rfn}_n^\alpha(\Sigma_k)(T) \vdash (\text{Def}_\delta(x) \land \exists u (\delta(u) \land \varphi(u))) \rightarrow \varphi(x) \), and so

\[
\text{EA} + \text{Rfn}_n^\alpha(\Sigma_k)(T) \vdash \text{Def}_\delta(x) \rightarrow (\Box_T \varphi(x) \rightarrow \varphi(x)),
\]

as required. \( \square \)

In particular, the local \( \Sigma_k \)-reflection is equivalent to the uniform \( \Sigma_k \)-reflection with \( \Sigma_1 \)-definable parameters.

**Corollary 4.3.** For each \( k > 0 \) over \( \text{EA} \), \( \text{Rfn}_{\Sigma_k}(T) \equiv \text{RFN}_{\Sigma_k}^{K^1}(T) \).

**Proof.** By Proposition 4.2 it suffices to show that \( \text{EA} + \text{RFN}_{\Sigma_k}^{K^1}(T) \vdash \Box_T \varphi \rightarrow \varphi \), where \( \varphi \) is a \( \Sigma_k \)-sentence. Define \( \delta(x) \) to be the formula \( (x = 0) \) and consider the corresponding axiom of \( \text{RFN}_{\Sigma_k}^{K^1}(T) \). Note that \( \text{EA} \vdash \text{Def}_\delta(0) \), whence, by instantiating the axiom with \( 0 \), we obtain \( \Box_T \varphi \rightarrow \varphi \). \( \square \)

Let us recall several notions related to the models of arithmetic (for more details, see [7]). Given a model \( M \) and a natural number \( n \) we denote by \( K^n(M) \) the
substructure of \( M \) consisting of all \( \Sigma_n \)-definable elements without parameters. Given a substructure \( N \subseteq M \), we say that \( N \) is a \( \Sigma_n \)-elementary substructure (denoted \( N \prec \Sigma_n M \)) if and only if for each \( \Sigma_n \)-formula \( \sigma(x_1, \ldots, x_m) \) and \( a_1, \ldots, a_m \in N \)

\[ N \models \sigma(a_1, \ldots, a_m) \iff M \models \sigma(a_1, \ldots, a_m). \]

The models \( K^{n+1}(M) \) possess the following useful property (see Remark (i) after Theorem 2.1 in [7]).

**Lemma 4.4.** For each \( n \geq 0 \), \( K^{n+1}(M) \prec \Sigma_n M \), whenever \( M \models I \Sigma_n \).

We have the following generalization of Corollary 4.3.

**Proposition 4.5.** For each \( k > n \geq 0 \) over \( EA + I \Sigma_n \), \( \text{Rfn}^n_{\Sigma_k}(T) \equiv \text{RFN}^n_{\Sigma_k}(T) \).

**Proof.** By Proposition 4.2, \( EA + \text{Rfn}^n_{\Sigma_k}(T) \vdash \text{RFN}^n_{\Sigma_k}(T) \), so we only prove the converse implication (over \( EA + I \Sigma_n \)). Fix some \( \Sigma_k \)-sentence \( \sigma \) and a model \( M \) of \( EA + I \Sigma_n \). By Lemma 4.4 we have \( K^{n+1}(M) \prec \Sigma_n M \), implying \( K^{n+1}(M) \models [n]_T \sigma \), since \( [n]_T \sigma \) is a \( \Sigma_n \)-sentence. It follows that there exists a \( \pi \in K^{n+1}(M) \) such that

\[ K^{n+1}(M) \models \text{True}_{\Pi_n}(\pi) \land \Box_T(\pi \rightarrow \sigma). \]

Using \( K^{n+1}(M) \prec \Sigma_n M \) again, we get \( M \models \text{True}_{\Pi_n}(\pi) \land \Box_T(\pi \rightarrow \sigma) \), whence

\[ M \models \text{True}_{\Pi_n}(\pi) \land \Box_T(\text{True}_{\Pi_n}(\pi) \rightarrow \sigma), \] (2)

since \( EA \models \forall \pi \in \Pi_n \Box_T(\pi \leftrightarrow \text{True}_{\Pi_n}(\pi)) \) and \( M \models EA \). Let \( \delta(x) \) be a \( \Sigma_{n+1} \)-formula such that \( M \models \text{Def}_{\delta}(\pi) \). We have

\[ M \models \text{Def}_{\delta}(\pi) \land \Box_T(\text{True}_{\Pi_n}(\pi) \rightarrow \sigma). \]

Now, since \( \text{True}_{\Pi_n}(\pi) \rightarrow \sigma \) is a \( \Sigma_k \)-formula, we can use \( \text{RFN}^n_{\Sigma_k}(T) \) in \( M \) to obtain

\[ M \models \text{True}_{\Pi_n}(\pi) \rightarrow \sigma, \]

whence \( M \models \sigma \), as required, since \( M \models \text{True}_{\Pi_n}(\pi) \) by (2).

Now we can give a relatively short model-theoretic proof of the following consequence of the so-called reduction property (see [3, Proposition 4.6]).
Theorem 4. If $U$ is a $\Pi_{n+2}$-axiomatized extension of $\text{EA}$, then
$$U + \text{RFN}_{\Sigma_{n+1}}(T) \equiv \Sigma_{n+2} U + \text{Rfn}^n_{\Sigma_{n+1}}(T).$$

Proof. The inclusion $\supseteq$ is clear from the definitions of the schemata and $[n]_T$, so we only prove the converse. Assume $U + \text{RFN}_{\Sigma_{n+1}}(T) \models \sigma$, where $\sigma$ is a $\Sigma_{n+2}$-sentence. Fix an arbitrary model $M \models U + \text{RFN}_{\Sigma_{n+1}}(T)$. We will show that $M \models \sigma$. By Theorem 1 (i) in [2], $\text{EA} + \text{Rfn}^n_{\Sigma_{n+1}}(\text{EA}) \models I\Sigma_n^-$, whence, certainly, $U + \text{RFn}^n_{\Sigma_{n+1}}(T) \models I\Sigma^-$, and so, in particular, $M \models \text{EA} + I\Sigma^-$. Lemma 4.2 then implies $K^{n+1}(M) \prec_{\Sigma_{n+1}} M$.

We have $K^{n+1}(M) \models U$, since $M \models U$ and $U$ is a $\Pi_{n+2}$-extension (the truth of $\Pi_{n+2}$-sentences is preserved downwards by the relation $\prec_{\Sigma_{n+1}}$). We will show that $K^{n+1}(M) \models \text{RFN}_{\Sigma_{n+1}}(T)$. In this case $K^{n+1}(M) \models U + \text{RFN}_{\Sigma_{n+1}}(T)$, whence, by the assumption, $K^{n+1}(M) \models \sigma$, so $M \models \sigma$, as required, since $\sigma$ is a $\Sigma_{n+2}$-sentence (the truth of $\Sigma_{n+2}$-sentences is preserved upwards by the relation $\prec_{\Sigma_{n+1}}$).

The rest of the proof is close to that of Proposition 4.5. Aiming for a contradiction, assume $K^{n+1}(M) \not\models \text{RFN}_{\Sigma_{n+1}}(T)$, i.e., there exists a $\Sigma_{n+1}$-formula $\varphi(x)$ and an element $a \in K^{n+1}(M)$ with $K^{n+1}(M) \models \square_T \varphi(a) \land \lnot \varphi(a)$, which, using $K^{n+1}(M) \prec_{\Sigma_{n+1}} M$, implies
$$M \models \square_T \varphi(a) \land \lnot \varphi(a). \tag{3}$$

Let $\delta(x)$ be a $\Sigma_{n+1}$-formula such that $M \models \text{Def}_\delta(a)$. Since $M \models \text{EA} + \text{Rfn}^n_{\Sigma_{n+1}}(T)$, Proposition 4.2 implies that $M \models \text{RFN}_{\Sigma_{n+1}}^{K^{n+1}}(T)$ and, in particular,
$$M \models \text{Def}_\delta(a) \rightarrow (\square_T \varphi(a) \rightarrow \varphi(a)),$$

since $\varphi(x)$ is a $\Sigma_{n+1}$-formula. Together with (3) this yields $M \models \varphi(a)$, which contradicts $M \models \lnot \varphi(a)$ in (3). 

Let us also note the following relationship between Proposition 4.5 and the following proposition proved in [5, Proposition 4.1] (we use slightly different notation).

Proposition 4.6. For each $n \geq 0$ over $\text{EA} + I\Sigma^-$,
$$\Pi^n \equiv I(\Sigma_{n+1}^-, K^{n+1}) \equiv I(\Sigma_{n+1}, I^{n+1}, K^{n+1}_1).$$

Here $I(\Sigma_{n+1}^-, K^{n+1})$ is the local variant of $\Sigma_{n+1}$-induction schema, where the conclusion of the induction axiom is relativized to $\Sigma_{n+1}$-definable elements (for the formal definitions of this schema and $I(\Sigma_{n+1}, I^{n+1}, K^{n+1}_1)$, see [5]). Thus, Proposition 4.5 can be seen as an analogue of Proposition 4.6 but for the reflection principles instead of the induction schemata.
The connection between the two propositions is based on the following fact (see [2, Theorem 1 (ii)] for \(n > 0\) and [5, Theorem 5.2] for \(n = 0\), for each \(n \geq 0\)
\[
\Pi_{n+1}^- \equiv \text{EA} + \text{Rfn}_{\Sigma_{n+2}}^n (\text{EA}).
\]
In view of this result, Propositions 4.5 and 4.6 we have the following

**Corollary 4.7.** For each \(n \geq 0\) over \(\text{EA} + \Sigma_n^-\),
\[
\text{RFN}_{\Sigma_n+2}^{K_{n+1}} (\text{EA}) \equiv I(\Sigma_{n+1}^-, K^{n+1}) \equiv I(\Sigma_{n+1}, I^{n+1}, K_1^{n+1}).
\]

This may be contrasted with the famous result by D. Leivant [10] and H. Ono [11] (cf. [4, Theorem 7]), that \(\text{EA} + \text{RFN}_{\Sigma_{n+2}} (\text{EA}) \equiv \Sigma_{n+1}^-\) for each \(n \geq 0\), and a result by L. Beklemishev (see [2, Theorem 1 (i)]), that \(\text{EA} + \text{Rfn}_{\Sigma_{n+2}}^{n+1} (\text{EA}) \equiv \Sigma_{n+1}^-\) for each \(n \geq 0\). In our case we also have the equivalence between certain forms of \(\Sigma_{n+2}^-\)-reflection for \(\text{EA}\) and \(\Sigma_{n+1}^-\)-induction, namely, for the versions of reflection and induction restricted to \(\Sigma_{n+1}^-\)-definable elements.

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