A link invariant from the symplectic geometry of nilpotent slices

Paul Seidel, Ivan Smith

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Abstract

We define an invariant of oriented links in $S^3$ using the symplectic geometry of certain spaces which arise naturally in Lie theory. More specifically, we present a knot as the closure of a braid, which in turn we view as a loop in configuration space. Fix an affine subspace $S_m$ of the Lie algebra $\mathfrak{sl}_2m(C)$ which is a transverse slice to the adjoint action at a nilpotent matrix with two equal Jordan blocks. The adjoint quotient map restricted to $S_m$ gives rise to a symplectic fibre bundle over configuration space. An inductive argument constructs a distinguished Lagrangian submanifold $L_{\mathbb{P}}\pm$ of a fibre $Y_{m,t_0}$ of this fibre bundle; we regard the braid $\beta$ as a symplectic automorphism of the fibre, and apply Lagrangian Floer cohomology to $L_{\mathbb{P}}\pm$ and $\beta(L_{\mathbb{P}}\pm)$ inside $Y_{m,t_0}$. The main theorem asserts that this group is invariant under the Markov moves, hence defines an oriented link invariant. We conjecture that this invariant coincides with Khovanov’s combinatorially defined link homology theory, after collapsing the bigrading of the latter to a single grading.

1 Introduction

From its first introduction in [14], the Jones polynomial has played a decisive role in knot theory. This invariant associates to an oriented link $\kappa \subset S^3$ a Laurent polynomial $V_\kappa(t^{1/2})$. It is completely characterised by $V_{\text{unknot}} = 1$ and a relation obtained from the Kauffman bracket calculus:

$$t^{-1/2}V_\bowtie + t^{3v/2}V_\bowtie + t^{-1}V_\bowtie = 0,$$

$$t^{3v/2}V_\bowtie + t^{1/2}V_\bowtie + tV_\bowtie = 0. \quad (1)$$

The two equations are for a positive and negative crossing, respectively. In the complement of the crossing under consideration, take the arc which ends at
the top left corner of the crossing. Then $v$ is the signed number of crossings between this arc and the other connected components of the complement (this term compensates for the non-local change of orientation that occurs in one of the two ways of resolving the crossing). (1) allows one to successively reduce the number of crossings, which in principle is suitable for algorithmic computation. However, despite arising in a farrago of contexts, the geometric meaning of the Jones polynomial has remained somewhat mysterious.

In [17] Section 7] Mikhail Khovanov introduced a categorified Jones polynomial, which is a bigraded abelian group $Kh^{*,*}(\kappa)$. $Kh^{i,j}(\text{unknot})$ is $\mathbb{Z}$ for $i = 0, j = \pm 1$, and zero otherwise. The role of equations (1) is played by long exact sequences

$$
\cdots \rightarrow Kh^{i,j}(\kappa) \rightarrow Kh^{i,j-1}(\kappa) \rightarrow Kh^{i-v,j-3v-2}(\kappa) \rightarrow \cdots
$$

and

$$
\cdots \rightarrow Kh^{i,j}(\kappa) \rightarrow Kh^{i+1,j}(\kappa) \rightarrow \cdots
$$

Starting from these, an obvious computation shows that the graded Euler characteristic $\chi_{\kappa}(q) = \sum_{i,j} (-1)^i q^j \dim(Kh^{i,j}(\kappa) \otimes \mathbb{Q})$ is, up to normalisation and change of variables, the Jones polynomial:

$$
V_{\kappa} = \left. \frac{\chi_{\kappa}(q)}{q + q^{-1}} \right|_{q = -t^{1/2}}
$$

$Kh^{*,*}$ is known to be a strictly stronger invariant than $V_{\kappa}$ [3]; by definition it remains algorithmically computable.

The groups $Kh^{*,*}$ are only the starting point for a very rich theory. First of all, one can vary their definition in many ways, giving rise to potentially useful additional invariants, such as Lee’s spectral sequence [25]. More spectacularly, they can be shown to fit into a topological quantum field theory for two-knots in four-space [18, 13, 16]. Very recently, Rasmussen [34] has used both properties to give a proof of Milnor’s conjecture on the slice genus of torus knots. This was previously accessible only via gauge theory (instanton invariants [23] originally, then Seiberg-Witten theory [31], and finally the Ozsváth-Szabó rebirth of the latter in terms of pseudoholomorphic curves [32]). Rasmussen’s argument is explicitly modelled on Ozsváth-Szabó theory, and in fact he conjectures the equality of a certain numerical invariant obtained from $Kh^{*,*}$ and its geometric counterpart [34, p. 2]. This use of Khovanov homology as a combinatorial substitute for gauge theory does not come as a complete surprise. The structural resemblance of the two theories, for instance looking at (2),(3) versus Floer’s
exact triangle, had been noticed for some time, and has found a concrete expression in the spectral sequence from [30], which goes from a variant of Khovanov homology to the Heegard Floer homology of the branched double cover. In that picture, Khovanov homology appears as a combinatorial approximation to Heegard Floer homology, and does not itself take on a geometric meaning.

The approach proposed in this paper is different, in that we give a tentative symplectic geometry description of Khovanov homology itself. The construction is fairly involved, but we can give a very rapid and superficial sketch at once: following Jones and others, we present an oriented link $\kappa$ as a closure of an $m$-stranded braid $b \in Br_m$. Adding another $m$ trivial strands gives $b \times 1^m \in Br_{2m}$, which can be represented as a loop $\beta$ in configuration space $Conf_{2m}(\mathbb{C})$ with respect to some base point $t_0$. We introduce a $4m$-dimensional noncompact symplectic manifold $M = Y_{m,t_0}$ (here and below, the more complicated notation is the one used in the body of the paper). This is the fibre at $t_0$ of a symplectic fibration over configuration space, whose monodromy along $\beta$ yields a symplectic automorphism $\phi = h_{\beta \text{ rez}}$. Our $M$ also contains a canonical (up to isotopy) Lagrangian submanifold $L = L_{\varphi_\pm}$ diffeomorphic to $(S^2)^m$. We apply Lagrangian Floer cohomology to these geometric data, and set

$$Kh_{\text{symp}}^*(\kappa) = HF^{*+m+w}(L, \phi(L)) = HF^{*+m+w}(L_{\varphi_\pm}, h_{\beta \text{ rez}}(L_{\varphi_\pm}))$$

where $m$ is as before, and $w$ is the writhe of the braid presentation (the number of positive minus the number of negative crossings).

(1) **Theorem:** Up to isomorphism of graded abelian groups, $Kh_{\text{symp}}^*(\kappa)$ depends only on the oriented link $\kappa$, and not on its presentation as a braid closure.

Reversing the orientation of all components leaves $Kh_{\text{symp}}^*$ unchanged, in particular we get an invariant of unoriented knots. The proposed relation with Khovanov homology is that our invariant should be obtained from it by collapsing the bigrading (actually, the sign of the $j$-grading should be reversed first, which is a simple change of conventions already applied in [18]).

(2) **Conjecture:**

$$Kh_{\text{symp}}^k(\kappa) \cong \bigoplus_{i-j=k} Kh^{i,j}(\kappa).$$

We admit at once that, at least for the time being, the symplectic theory does not come with a bigrading corresponding to the one in $Kh^{*,*}$. This prevents us from seeing the connection to the Jones polynomial geometrically, since the Euler characteristic only recovers the uninteresting specialization $V_\kappa(t^{1/2} = 1)$ which counts components of the link. Evidence for Conjecture 2 comes from various sources. The two sides have the same value for the unknot, and more generally they behave in the same way under adding an unlinked unknot com-
ponent. Another example is provided by the trefoil knot, whose $Kh^*_{\text{symp}}$ we will compute by direct geometric means. More speculatively, a generalization of \[38\] should yield the counterpart for $Kh_{\text{symp}}^*$ of the long exact sequences \[2, 3\] (but there are many details still to be carried out). Starting from this, one could follow the construction of the “hypercubes of crossing resolutions” of \[17\] in $Kh_{\text{symp}}^*$ and thereby obtain a spectral sequence which starts with $E_2 = Kh_{\text{symp}}$, and converges to $Kh_{\text{symp}}^*$ (this is precisely the approach used by Ozsváth-Szabó \[30\]). The conjectural vanishing of the higher order differentials in this sequence seems more difficult to explain at present, even though the fact that both theories are $\mathbb{Z}$-graded restricts the possibilities somewhat. A possible more fundamental explanation for Conjecture 2 would arise from a relation, on the derived level, between the Fukaya categories of $M = Y_{m,t}$ and differential graded modules over the arc algebras $H_m$ from \[18\]. This was one of Khovanov’s motivations when he proposed that these particular manifolds should be relevant for understanding $Kh_{\text{symp}}^*$ \[15\]. Because of its abstract homological algebra nature, this approach may seem far-fetched, but it has been successfully carried out in a toy model case \[19, 39\].

After this rather tentative discussion, we return to the concrete geometry underlying the definition of $Kh_{\text{symp}}^*$. The manifolds $M = Y_{m,t_0}$ can be described in elementary terms (as given by matrices of a certain special form, and with prescribed eigenvalues), but the proper framework for understanding them is provided by Lie theory. For any semisimple complex Lie algebra $\mathfrak{g}$, one can consider the adjoint quotient map $\chi : \mathfrak{g} \to \mathfrak{h}/W$. In the case of $\mathfrak{g} = \mathfrak{sl}_2$, which is the one relevant to us here, this associates to each matrix its characteristic polynomial. Thinking of $\mathfrak{h}/W$ as the set of unordered eigenvalues with multiplicities, we can identify an open dense subset $\mathfrak{h}^{*e}/W$ with the space $Conf_{2m}(\mathbb{C})$ of configurations having zero center of mass, and the restriction of $\chi$ to that subspace is a differentiable fibre bundle. We actually want to restrict $\chi$ to a so-called transverse slice, which is an affine subspace of $\mathfrak{g}$ intersecting all orbits of the natural adjoint $G$-action transversally. There is a well-known general construction of such slices $S^{JM} \subset \mathfrak{g}$, which starts with a nilpotent element of $\mathfrak{g}$ and invokes the Jacobson-Morozov theorem. The restrictions $\chi|_{S^{JM}} : S^{JM} \to \mathfrak{h}/W$ are still differentiable fibre bundles over the subset $\mathfrak{h}^{*e}/W$, and their topological monodromy has been used by Slodowy \[42\] and others to give an alternative construction of Springer’s Weyl group representations. For our particular slices $S_m$, we take the nilpotent $n^+ \in \mathfrak{g}$ which has two Jordan blocks of size $m$, and use a slight generalization of the Jacobson-Morozov construction (this is purely for technical reasons: $S_m$ is isomorphic to the corresponding Jacobson-Morozov slice $S^{JM}$, but is slightly easier to use). The fibre of $\chi|_{S_m}$ at the point $t_0$ (translated by a constant to put it into $Conf^0_{2m}(\mathbb{C})$, to be precise) is our $Y_{m,t_0}$.

Crucially, if we take a point $t \in \mathfrak{h}/W$ where two eigenvalues come together, the fibre of $\chi|_{S_m}$ over $t$ has a fibered $(A_1)$ singularity. By this we mean that the stratum of singular points is itself smooth, and that the normal structure of this stratum is that of an ordinary double point singularity $a^2 + b^2 + c^2 = 0$. 

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Moreover, up to symplectic isomorphism the singular stratum itself can be identified with a regular fibre of $\chi|_{S_{m-1}}$. This leads to an inductive scheme, where we construct Lagrangian submanifolds in $\mathcal{Y}_{m,t_0}$ by bringing the eigenvalues together in successive pairs. More precisely, each “crossingless matching” $\wp$ of the configuration $t_0$ in the plane gives rise to a Lagrangian submanifold $L_{\wp}$, and the $L = L_{\wp^\pm}$ which appeared above is just the example obtained from a certain standard choice of matching. To prove Theorem 1, one needs to check the invariance of our symplectic Floer cohomology group under the Markov moves which relate different braid presentations of the same link. For the most difficult (because it changes the number $m$ of strands) type II move this will follow from the observation that if $t$ is a point where three eigenvalues are brought together, the fibre of $\chi|_{S_m}$ at that point has a fibered $(A_2)$ type singularity. The main lesson to be learnt from this is that the data which enter into the definition of $Kh^*_\text{symp}$, and the properties which make it an oriented link invariant, are all derived from the basic geometry of the adjoint quotient, and general facts from symplectic geometry. To emphasize that, the paper alternates sections of general exposition with others more specifically tailored to our needs.

Alternatively, one can think of $\mathcal{Y}_{m,t_0}$ as a space of solutions of Nahm’s equations with certain boundary data (these are basically the equations for $\mathbb{R}^3$-invariant instantons on $\mathbb{R}^4$, with gauge group $SU_{2m}$, cf. [11] generalizing [22]). Via an ADHM transform $\mathcal{Y}_{m,t_0}$ can be viewed as a quiver variety, cf. [29, Theorem 8.4] (strictly this is partly conjectural, since the cited work does not quite cover our case). These descriptions show geometric properties of $\mathcal{Y}_{m,t_0}$ which may not be immediately apparent from the point of view taken here, and lead to several possible avenues for further development (other than the obvious question of trying to prove Conjecture 2). For instance, there is an involution on $\mathcal{Y}_{m,t_0}$ whose fixed point set is related to the Jacobian of the double cover of $\mathbb{CP}^1$ branched along the configuration of points $t_0 \in \mathbb{C}$, and that leads to a geometric interpretation of the relation between Khovanov and Ozsváth-Szabó theory. This will be discussed in detail in a sequel.

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## 2 Geometry of the adjoint quotient

This section gathers some background material from geometric Lie theory. All of it is essentially extracted from the textbooks [6, 5] and more advanced works [42, 11, 24, 36]. To make the exposition more focused and easier for non-specialists to read, we only deal with the Lie algebras $\mathfrak{g} = \mathfrak{sl}_n$. In this case, most proofs are within the reach of elementary linear algebra, and we have omitted some of them. However, each section ends with some brief remarks which outline the
situation for general semisimple $g$, and gives more references to the literature.

(A) Local slices

Let $G = SL_n(\mathbb{C})$, and $g = sl_n$ its Lie algebra. The adjoint quotient map $\chi$ associates to each matrix $x \in g$ its characteristic polynomial. Since that polynomial has $(n - 1)$ nontrivial coefficients, the adjoint quotient is a holomorphic map $\chi : g \rightarrow \mathbb{C}^{n-1}$. Equivalently, one can think of it as giving the eigenvalues of the matrix (with multiplicities). From that perspective, $\chi(x) \in \mathfrak{h}/W$, where $\mathfrak{h} \subset g$ is the subspace of diagonal matrices, and $W = S_n$ the permutation group acting on it. The connection between the two points of view is established by the elementary symmetric functions, which give a holomorphic isomorphism $\mathfrak{h}/W \cong \mathbb{C}^{n-1}$.

(3) Example: Take $g = sl_2$. The adjoint quotient is just the determinant, which for a suitable choice of coordinates on $g$ can be written as

$$\chi : \mathbb{C}^3 \rightarrow \mathbb{C}, \quad (a, b, c) \mapsto a^2 + b^2 + c^2.$$  

It has a single nondegenerate critical point at the origin, so the fibre $\chi^{-1}(0)$ has an ordinary double point singularity. Following common terminology, we will call these $(A_1)$ type critical point and singularity, respectively.

$G$ acts on $g$ by conjugation, $Ad(g)y = gyg^{-1}$. The corresponding infinitesimal action of $g$ on itself is $ad(x)y = [x, y]$. Either of these is usually called the adjoint action. $\chi$ is constant along $G$-orbits (and in fact can be thought of as the projection to the algebro-geometric quotient $g/G \cong \mathfrak{h}/W$).

(4) Example: Suppose that $x \in g$ has $n$ pairwise distinct eigenvalues. Since it is semisimple and $\chi$ is $G$-invariant, we may assume that $x \in \mathfrak{h}$. Moreover, since the eigenvalues are distinct, $x$ is a regular point of the projection $\mathfrak{h} \rightarrow \mathfrak{h}/W$. This implies that $x$ is a regular point of $\chi$.

Because of the $G$-symmetry, the local geometry of $\chi$ can be studied in terms of transverse slices, which we will now define. Fix $x \in g$. By definition, the tangent space to the adjoint orbit $Gx$ is

$$T_x(Gx) = ad(g)x = [x, g].$$

A local transverse slice to the orbit is simply a local complex submanifold $S \subset g$, $x \in S$, whose tangent space at $x$ is complementary to $T_x(Gx)$. Take such a slice, and in addition, let $K \subset G$ be a local submanifold containing the identity $e \in G$, such that $T_xK$ is complementary to the stabilizer $g_x = \{ y : [y, x] = 0 \}$. Then
we get a commutative diagram

\[
\begin{array}{c}
K \times S \xrightarrow{Ad|K \times S} \mathfrak{g} \\
\downarrow \text{projection} \quad \downarrow \\
S \xrightarrow{\chi|S} \mathfrak{h}/W
\end{array}
\] (4)

where the top map, \((g, y) \mapsto \text{Ad}(g)y\), is a local holomorphic isomorphism at \((e, y)\). In words, this means that \(\chi\) looks locally like \(\chi|S\) times a constant map in the remaining coordinates.

(5) **Lemma:**
(i) For all \(y \in S\) sufficiently close to \(x\), the intersection \(S \cap Gy\) is transverse at \(y\).
(ii) For all \(y \in S\) sufficiently close to \(x\), we have that \(y\) is a critical point of \(\chi|S\) iff it is a critical point of \(\chi\).
(iii) Any two local transverse slices at \(x\) are locally isomorphic, by an isomorphism which moves points only inside their \(G\)-orbits.

**Proof.** (i) is clear from the definition. (ii) follows from (4). For (iii), if \(S'\) is another slice, then the desired isomorphism is

\[
S' \xrightarrow{\text{inclusion}} \mathfrak{g} \xrightarrow{Ad|K \times S} K \times S \xrightarrow{\text{projection}} S. \quad \blacksquare
\] (5)

(6) **Example:** Let \(x\) be the nilpotent consisting of a single maximal Jordan block,

\[
x = \begin{pmatrix} 0 & 1 & \cdots & 1 \\ \vdots & \cdots & \cdots & \vdots \\ 0 & 1 \\ 1 & 0 \end{pmatrix}.
\] (6)

Then the space \(S\) of matrices

\[
y = \begin{pmatrix} 0 & 1 \\ y_{21} & 1 \\ \vdots & \vdots \\ y_{n-1,1} & 1 \\ y_{n1} & 0 \end{pmatrix}
\] (7)

is a slice at \(x\), and \(\chi|S = \text{id}_{\mathbb{C}^{n-1}}\). In view of Lemma(5)(ii) above, it follows that \(y\) is a regular point of \(\chi\) itself.

(7) **Example:** In \(\mathfrak{g} = \mathfrak{sl}_3\), the space \(S\) of matrices

\[
y = \begin{pmatrix} \alpha & 0 & 1 \\ \beta & -2\alpha & 0 \\ \delta & \gamma & \alpha \end{pmatrix}
\] (8)
is a slice at \( x = \{ \alpha = \beta = \gamma = \delta = 0 \} \). After changing coordinates to \( \alpha = \frac{1}{2} a, \beta = b, \gamma = -c, \delta = \frac{3}{2} a^2 - d \), the characteristic polynomial of \( y \) is \( t^3 - td + (a^3 - ad + bc) \), so one can write

\[
\chi|\mathcal{S} : \mathbb{C}^4 \to \mathbb{C}^2, \quad \chi(a, b, c, d) = (d, a^3 - ad + bc).
\] (9)

This map is known to singularity theorists as the miniversal unfolding of the \((A_2)\) type surface singularity \( a^3 + bc = 0 \).

Both examples above describe slices at nilpotent points. These are particularly important because the geometry of the general case can be reduced to the nilpotent one, as we will now explain. Consider an \( x \in \mathfrak{g} \) whose eigenvalues are \( (\mu_1, \ldots, \mu_n) \). For simplicity, assume that the first \( k \) eigenvalues are equal, and that there are no other coincidences between them. Write \( x \) as the sum of its semisimple and nilpotent parts, \( x = x_s + x_n \). The stabilizer \( \mathfrak{g}_{x_s} = \{ y : [y, x_s] = 0 \} \subset \mathfrak{g} \) is a Lie subalgebra of block diagonal matrices. To write this down explicitly, let \( E \) be the \( \mu_1 \)-eigenspace of \( x_s \), and \( L_{k+1}, \ldots, L_n \) the remaining eigenspaces (which are one-dimensional by assumption). Then \( \mathfrak{g}_{x_s} \) is the trace-free part of \( \mathfrak{gl}(E) \oplus \mathfrak{gl}(L_{k+1}) \cdots \oplus \mathfrak{gl}(L_n) \). One can further decompose this as

\[
\mathfrak{g}_{x_s} = \hat{\mathfrak{g}} \oplus \mathfrak{j}.
\] (10)

Here, the first factor is \( \hat{\mathfrak{g}} = \mathfrak{sl}(E) \), while the second one \( \mathfrak{j} \) is the center, consisting of the trace-free part of \( \{ \mathbb{C} \cdot 1 \subset \mathfrak{gl}(E) \} \oplus \mathfrak{gl}(L_{k+1}) \cdots \oplus \mathfrak{gl}(L_n) \). Note that \( \mathfrak{j} \) can be identified with \( \mathbb{C}^{n-k} \subset \mathbb{C}^{n-k+1} \) in an obvious way, without choosing bases for our eigenspaces. The nilpotent part \( x_n \) naturally lies in \( \mathfrak{g} \). Suppose that we are given a slice \( \hat{\mathcal{S}} \subset \hat{\mathfrak{g}} \) to \( x_n \), with respect to the adjoint action on that smaller Lie algebra. An explicit comparison of \( [x, \mathfrak{g}] \) with \( [x_n, \hat{\mathfrak{g}}] \) shows that

(8) Lemma: \( \mathcal{S} = x_s + \hat{\mathcal{S}} + \mathfrak{j} \subset \mathfrak{g} \) is a local transverse slice for the adjoint action at \( x \).

Write \( \hat{\chi} : \hat{\mathfrak{g}} \to \hat{\mathfrak{h}}/\hat{\mathcal{W}} \) for the adjoint quotient map on \( \hat{\mathfrak{g}} \). Take the isomorphism \( \hat{\mathcal{S}} \times \hat{\mathfrak{j}} \to \mathcal{S}, (y, z) \mapsto x_s + y + z \). On the bases, consider the map \( \hat{\mathfrak{h}}/\hat{\mathcal{W}} \times \mathfrak{j} \to \hat{\mathfrak{h}}/\mathfrak{W} \) which takes \( \lambda, \rho \in \hat{\mathfrak{h}} \times \mathfrak{j} \) and adds up the three collections of eigenvalues \( (\lambda_1, \ldots, \lambda_k, 0, \ldots, 0), (\mu_1, \ldots, \mu_1, \rho_{k+1}, \ldots, \rho_n) \) and \( (\mu_1, \ldots, \mu_n) \). This is a local isomorphism near the origin, since the subgroup of \( \mathcal{W} \) fixing \( (\mu_1, \ldots, \mu_n) \) is precisely \( \mathcal{W} \). Together, these maps fit into a commutative diagram

\[
\begin{array}{ccc}
\hat{\mathcal{S}} \times \mathfrak{j} & \xrightarrow{\text{isomorphism}} & \mathcal{S} \\
\chi|\hat{\mathcal{S}} \times \text{Id} \downarrow & & \downarrow \chi|\mathcal{S} \\
\hat{\mathfrak{h}}/\hat{\mathcal{W}} \times \mathfrak{j} & \xrightarrow{\text{local isomorphism}} & \hat{\mathfrak{h}}/\mathfrak{W}.
\end{array}
\] (11)

As promised, this means that the local structure of \( \chi|\mathcal{S} \) reduces to that of \( \chi|\hat{\mathcal{S}} \).
times the identity map in the remaining coordinates.

(9) Example: Suppose that $x$ as above, with eigenvalues $(\mu_1, \ldots, \mu_n)$, is itself semisimple, so that $x_1 = x$ and $x_n = 0$. Then $S = \hat{g} = \mathfrak{sl}(E)$, hence $S = g$. Now assume in addition that $k = 2$. By choosing coordinates on $\mathfrak{sl}(E)$ as in Example 3 and using $(11)$, we get the following local picture of $\chi|S$:

$$\chi(a, b, c, \mu_1, \ldots, \mu_{n-2}) = (a^2 + b^2 + c^2, \mu_1, \ldots, \mu_{n-2}).$$

(12)

To make this even simpler, take a disc $D \subset \mathfrak{h}/W$ corresponding to eigenvalues $(\mu_1 - \sqrt{\epsilon}, \mu_2 + \sqrt{\epsilon}, \mu_3, \ldots, \mu_n)$ for small $\epsilon$. By $(11)$, $\chi^{-1}(D) \cap S$ just singles out the $S$-factor of the slice. Hence, the restriction of $\chi$ to $\chi^{-1}(D) \cap S$ has an $(A_1)$ type critical point at $x$.

The case of a general $x$, where several eigenvalues may have nontrivial multiplicities $k_1, \ldots, k_r > 1$, is analogous. One splits $\hat{g}_x$ into $\hat{g} \cong \mathfrak{sl}_{k_1} \times \cdots \times \mathfrak{sl}_{k_r}$ and the center $\mathfrak{z} \cong \mathbb{C}^{n-k_1-\cdots-k_r-1}$. For each $j = 1, \ldots, r$, choose a slice $\hat{S}_j \subset \mathfrak{sl}_{k_j}$ at the corresponding component of $x_n$. By combining these with $\mathfrak{z}$, one again obtains a slice $S$ for $x$. The outcome is that $\chi|S$ looks like the product of the maps $\chi_j|\hat{S}_j$ for $j = 1, \ldots, k$, together with the identity map on $\mathfrak{z}$.

(10) Example: Suppose that $x \in \mathfrak{g}$ is a matrix which has a single Jordan block for each eigenvalue. This means that for each $j$, the $\mathfrak{sl}_{k_j}$-component of $x_n$ is as in Example 2. We know that these are regular points of the adjoint quotient maps $\hat{\chi}_j$, hence $x$ itself is a regular point of $\chi$ (these are in fact all the regular points).

(11) Example: Let $x \in \mathfrak{g}$ be a semisimple matrix with eigenvalues $(\mu_1, \ldots, \mu_n)$, where the first $2k$ form equal pairs $(\mu_1 = \mu_2, \mu_3 = \mu_4, \ldots, \mu_{2k-1} = \mu_{2k})$, and with no other coincidences. This is quite similar to Example 3, $S = \mathfrak{g}_x$, and each $\hat{S}_j$ is the whole of $\mathfrak{sl}_2$. Consider the polydisc $P \subset \mathfrak{h}/W$ formed by the sets of eigenvalues $(\mu_1 - \sqrt{\epsilon}, \mu_2 + \sqrt{\epsilon}, \ldots, \mu_{2k-1} - \sqrt{\epsilon}, \mu_{2k} + \sqrt{\epsilon}, \mu_{2k+1}, \ldots, \mu_n)$. Then, the restriction of $\chi$ to $\chi^{-1}(P) \cap S$ looks locally like the product of $k$ copies of the $(A_1)$ type map.

(12) Remarks: For a general semisimple Lie group $G$, with Lie algebra $\mathfrak{g}$, one defines the adjoint quotient as the projection $\chi: \mathfrak{g} \to \mathfrak{g}/G \cong \mathfrak{h}/W$, where $\mathfrak{h}$ is a Cartan subalgebra and $W$ the Weyl group (the isomorphism is Chevalley’s theorem $\mathbb{C}[\mathfrak{g}]^G = \mathbb{C}[\mathfrak{h}]^W$, see e.g. [3, Theorem 3.1.38]). The basic facts about slices, notably $(11)$ and Lemma 6, continue to hold, since they actually apply to general holomorphic $G$-actions and $G$-invariant maps. Examples 3 and 7 are special cases of a fundamental result of Brieskorn and Slodowy [4], which says that if $S$ is a slice to a subregular nilpotent element inside a simply-laced $\mathfrak{g}$, then $\chi|S$ is the miniversal unfolding of the corresponding simple (ADE type)
The general version of Lemma 8 looks as follows. Any $x \in \mathfrak{g}$ has a canonical decomposition $x = x_s + x_n$ into mutually commuting semisimple and nilpotent parts. Since $x_s$ is semisimple, $\mathfrak{g}_{x_s}$ is a reductive Lie algebra [Lemma 2.1.2], hence splits into its center $\mathfrak{z}$ and the semisimple derived Lie algebra $\hat{\mathfrak{g}} = [\mathfrak{g}_{x_s}, \mathfrak{g}_{x_s}]$, as in (10). Moreover, $ad(x_s)$ is a semisimple endomorphism of $\mathfrak{g}$, so $\mathfrak{g} = [x_s, \mathfrak{g}] \oplus \mathfrak{g}_{x_s}$. Finally, $ad(x_s), ad(x_n)$ are polynomials in $ad(x)$ with zero constant terms, hence $[x, \mathfrak{g}] = [x_s, \mathfrak{g}] + [x_n, \mathfrak{g}]$. Taking these three facts together, one finds that if $\hat{\mathfrak{S}} \subset \hat{\mathfrak{g}}$ is a $\hat{\mathfrak{g}}$-slice at the point $x_n$, then $\mathfrak{S} = x_s + \mathfrak{z} + \hat{\mathfrak{S}}$ is a $\mathfrak{g}$-slice at $x$. Restricting the adjoint quotient to this slice, one obtains a diagram like (11). For instance, $x$ is a regular point of $\chi$ iff $x_n \in \hat{\mathfrak{g}}$ is a regular nilpotent, which is the general version of Example 6.

(B) Homogeneous slices

Let $x \in \mathfrak{g}$ be nilpotent. The Jacobson-Morozov Lemma says that one can find a triple $(n^+, n^-, h)$ of elements of $\mathfrak{g}$, where $n^+ = x$, which satisfy

$$[h, n^+] = 2n^+, \quad [h, n^-] = -2n^-, \quad [n^+, n^-] = h. \quad (13)$$

There are many different choices of $(n^-, h)$ for a fixed $n^+$. However, Kostant’s uniqueness theorem says that any two are conjugate by an element of the stabilizer $G_{n^+} \subset G$.

The elements of a Jacobson-Morozov (JM) triple define a homomorphism $\mathfrak{sl}_2 \rightarrow \mathfrak{g}$, which in combination with the adjoint action makes $\mathfrak{g}$ into an $\mathfrak{sl}_2$-module. This allows one to apply elementary facts about $\mathfrak{sl}_2$-representations. For instance, $ad(h) \in End(\mathfrak{g})$ is necessarily semisimple and has integer eigenvalues. Hence, the vector field on $\mathfrak{g}$ defined by

$$K_y = -[h, y] + 2y \quad (14)$$

generates a linear $\mathbb{C}^*$-action $\lambda_r(y) = r^2 \exp(-log(r)h) \cdot y \cdot \exp(log(r)h)$. Via the adjoint quotient map, this is compatible with the $\mathbb{C}^*$-action on $\mathfrak{h}/\mathfrak{W}$ which multiplies all eigenvalues by $r^2$. By definition $K_x = 0$, so $x$ is a fixed point of $\lambda$. We define a homogeneous slice at $x$ to be an affine subspace $\mathfrak{S} \subset \mathfrak{g}$ invariant under $\lambda$, which is a local transverse slice for the adjoint action at $x$. There is actually a canonical choice of homogeneous slice, namely the JM slice $\mathfrak{S}^{JM} = x + \mathfrak{g}_{n^-}$. The fact that this is a slice, or equivalently that $\mathfrak{g} = [n^+, \mathfrak{g}] \oplus \mathfrak{g}_{n^-}$,
is another easy observation from \( \mathfrak{sl}_2 \)-representation theory.

(13) **Example:** If \( x \) is as in Example 6 one can take

\[
h = \begin{pmatrix} n-1 & n-3 & n-5 & \cdots & -n+1 \\ 0 & n-1 & 2(n-2) & 3(n-3) & \cdots & n-1 \end{pmatrix},
\]

\[
n^- = \begin{pmatrix} r^2 y_{11} & y_{12} & \cdots & r^{4n} y_{1n} \\ r^4 y_{21} & r^2 y_{22} & \cdots \\ \cdots & \cdots \\ r^{2n} y_{n1} & \cdots & r^2 y_{nn} \\ y_{n-1,1} & \cdots & y_{n,n-1} & r^2 y_{nn} \end{pmatrix}.
\]

The associated \( \mathbb{C}^* \)-action is

\[
\lambda_r : y \mapsto \begin{pmatrix} r^2 y_{11} & y_{12} & \cdots & r^{4n} y_{1n} \\ r^4 y_{21} & r^2 y_{22} & \cdots \\ \cdots & \cdots \\ r^{2n} y_{n1} & \cdots & r^2 y_{nn} \\ y_{n-1,1} & \cdots & y_{n,n-1} & r^2 y_{nn} \end{pmatrix}.
\]

By listing the eigenvalues of \( \text{ad}(h) \), one sees that that the \( \mathfrak{sl}_2 \)-module \( \mathfrak{g} \) breaks up into indecomposables of rank \( 3, 5, \ldots, 2n-1 \) (one each). The subspace \( \mathfrak{g}_{n^-} \) is therefore \( (n-1) \)-dimensional, which means that it is spanned by powers of \( n^- \), so \( S^{JM} = n^+ + (\mathbb{C} n^- \oplus \mathbb{C}(n^-)^2 \oplus \cdots) \). One should compare this with the slice considered in Example 6 which is homogeneous for the same choice of \( (h, n^-) \), but not a JM slice.

(14) **Example:** Take the nilpotent \( x \in \mathfrak{g} = \mathfrak{sl}_3 \) from Example 7. The slice constructed there is a JM slice, obtained by taking

\[
h = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}, \quad n^- = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.
\]

By Kostant’s theorem, any other JM triple with \( n^+ = x \) is the conjugate of this one by an element of \( G_{n^+} \), which is the group of matrices

\[
g = \begin{pmatrix} \delta & \epsilon & \kappa \\ \delta^{-2} & \tau \end{pmatrix}.
\]

Note that those \( g \) which are diagonal preserve \( n^- \), hence the entire triple. Thus, the space of all JM triples with \( n^+ = x \) becomes isomorphic to \( G_{n^+}/G_{n^+} \cap G_{n^-} \).
or equivalently, to the subgroup \( U \subset G_{n^+} \) of unipotent matrices. To interpret this in a more geometric way, note that \( x \) gives rise to a flag in \( \mathbb{C}^3 \),
\[
0 \subset F^1 = \text{im}(x) \subset F^2 = \ker(x) \subset \mathbb{C}^3.
\]

Given a JM triple, the eigenspaces of \( h \) yield a splitting of this flag into one-dimensional spaces. This is obvious for \((17)\), and follows for general triples by Kostant’s theorem. \( U \) acts simply transitively on the space of such splittings, and this proves that choices of splittings and JM triples correspond to each other bijectively.

We will now look at the general properties of homogeneous slices \( S \). Decompose \( g \) and \( T_xS \) into \( \text{ad}(h)\)-eigenspaces \( g^{(j)}, (T_xS)^{(j)}, j \in \mathbb{Z} \). Again appealing to basic facts about \( \mathfrak{sl}_2 \)-representations, we see that the map \( \text{ad}(x)^{(j)} : g^{(j)} \to g^{(j+2)} \) is injective for \( j < 0 \), and surjective for \( j > -2 \). By definition, \((T_xS)^{(j)}\) is a complementary subspace to the image of \( \text{ad}(x)^{(j-2)} \), hence is zero for all \( j > 0 \).

This implies that every homogeneous slice is necessarily contained in the subset
\[
\mathcal{T} = x + \bigoplus_{j < 2} g^{(j)}
\]
of those \( y \in g \) which satisfy \( \lim_{r \to 0} \lambda_r(y) = x \). In words, the \( \mathbb{C}^* \)-action shrinks the slice to the point \( x \). This immediately leads to an improved version of the first two parts of Lemma 5:

(15) Lemma: Let \( S \) be a homogeneous slice for \( x \). Then, (i) the intersection of \( S \) with any adjoint orbit is transverse. (ii) A point of \( S \) is a critical point of \( \chi \) iff it is a critical point of \( \chi|S \).

Define \( u = \bigoplus_{j < 0} g^{(j)} \), and let \( U = \text{exp}(u) \subset G \) be the corresponding subgroup. For any \( u \in u \), the vector field defined by \( L_y = [u, y] \) is tangent to \( \mathcal{T} \). Hence, the adjoint action of \( U \) preserves \( \mathcal{T} \), in particular we get a map
\[
\text{Ad}|(U \times S) : U \times S \to \mathcal{T}.
\]
The derivative of this at the point \((e, x)\) is \( u \oplus T_xS \to T_x\mathcal{T}, (u, y) \mapsto [u, x] + y \). In view of the observations made above, this is invertible. Equip \( U \times S \) with the \( \mathbb{C}^* \)-action \( r \cdot (g, y) = (\exp(-\log(r)h) \cdot g \cdot \exp(\log(r)h), \lambda_r(y)) \), and \( \mathcal{T} \) with \( \lambda \), so that the map between the two becomes equivariant. Since both actions contract the relevant spaces to the point \((e, x)\) respectively its image \( x \), it follows that (19) is a global \( \mathbb{C}^* \)-equivariant isomorphism.

(16) Lemma: Let \( S, S' \) be two homogeneous slices at \( x \), possibly defined using different JM triples. Then, there is an isomorphism \( S \to S' \) which is \( \mathbb{C}^* \)-equivariant, and which moves points only inside their adjoint orbits.
Proof: Suppose first that our two slices share the same underlying JM triple. From (19) we then get isomorphisms
\[ S \rightarrow T/U \leftarrow S', \] (20)
On the other hand, if we are considering Jacobson-Morozov slices associated to different choices of \((n^-, h)\), then the conjugating element provided by Kostant’s theorem directly yields the isomorphism in question. In both cases, all the desired properties are obvious; and by combining them, the general result follows.

Remark: For the Jacobson-Morozov Lemma in the setting of general semisimple \(g\), see [6, Theorem 3.3.1]. Entirely in parallel with the case \(g = \mathfrak{sl}_n\), this leads to a definition of the \(C^*\)-action \(\lambda\), and to the notion of homogeneous slices. Among these, Jacobson-Morozov slices are the most commonly used ones in the literature. The description of these slices as quotients [24], which uses \(h\) but not \(n^-\), is due to Kronheimer [24, Lemma 11].

The general version of Kostant’s theorem can be found e.g. in [6, Theorem 3.4.10]. The statement is actually a little better than the version given here, since it says that the conjugating elements can be taken to lie in a certain unipotent subgroup of \(G_{n+}\) (this is quite visible in Example 14, for instance).

Simultaneous resolution

Consider the open subset \(h^{reg}/W \subset \mathfrak{h}/W\) corresponding to \(n\)-tuples of pair-wise different eigenvalues. We will identify this with the subspace \(Conf^0_n(\mathbb{C}) \subset Conf^n_n(\mathbb{C})\) of point configurations with zero center of mass. Each \(t \in Conf^0_n(\mathbb{C})\) is a regular value of \(\chi\), by Example 4 (in fact, these are all the regular values). Therefore, the part of the adjoint quotient lying over \(Conf^0_n(\mathbb{C})\) is a submersion.

We need to show that it is in fact a differentiable fibre bundle. The technical difficulty is that the fibres are not compact, and we will resolve this by using Grothendieck’s simultaneous resolution and a suitable \(C^*\)-action.

Let \(\mathfrak{g}\) be the space of pairs \((x, F)\), where \(x \in \mathfrak{g}\) and \(F\) is a complete flag such that \(x(F^i) \subset F^i\) for all \(i\). Since the flag manifold is compact, projection \(\mathfrak{g} \rightarrow \mathfrak{g}\) is a proper map. Next, note that for \((x, F) \in \mathfrak{g}\), we can consider the endomorphism of each quotient \(F^i/F^{i-1}\) induced by \(x\), which is multiplication by some \(\tilde{t}_i \in \mathbb{C}\). The \(\tilde{t}_i\) are the eigenvalues of \(x\), with the correct multiplicities, and the flag \(F\) provides a preferred ordering of them. This means that the map \(\tilde{\chi} : \mathfrak{g} \rightarrow \mathfrak{h} \cong \mathfrak{h}^{reg}\)
\( \mathbb{C}^{n-1} \subset \mathbb{C}^n, \tilde{\chi}(x, F) = (\tilde{t}_1, \ldots, \tilde{t}_n), \) fits into a commutative diagram

\[
\begin{array}{ccc}
\tilde{\mathfrak{g}} & \longrightarrow & \mathfrak{g} \\
\tilde{\chi} \downarrow & & \downarrow \chi \\
\mathfrak{h} & \longrightarrow & \mathfrak{h}/W.
\end{array}
\]

(18) Lemma: (i) \( \tilde{\chi} \) is a submersion, so each fibre \( \tilde{\chi}^{-1}(\tilde{t}) \) is smooth. (ii) If \( t \) lies in \( \text{Conf}_n^0(\mathbb{C}) \) and \( \tilde{t} \in \mathfrak{h} \) is any point lying over it, then \( \tilde{\chi}^{-1}(\tilde{t}) \cong \chi^{-1}(t) \).

Proof: Both properties are elementary. For (i), since everything is invariant under the \( G \)-action by conjugation, we may assume that \( F \) is the standard flag, so \( x \) is a diagonal matrix. Then, by changing only the diagonal coefficients, we get a subspace of \( T_x\tilde{\mathfrak{g}} \) which projects isomorphically to \( \mathfrak{h} \). For (ii), note that if the eigenvalues are pairwise different, any ordering of them determines a unique compatible flag. ■

(19) Example: We wrote down the adjoint quotient map for \( \mathfrak{g} = \mathfrak{sl}_2 \) in Example 3. The simultaneous resolution is a classical algebroid-geometric construction: it consists of doing the base extension \( t^2 = a^2 + b^2 + c^2 \) and then taking a small resolution of that, which replaces the singular point with a \( \mathbb{C}\mathbb{P}^1 \).

(20) Lemma: \( \tilde{\chi} \) is naturally a differentiable fibre bundle.

Proof. The diagonal \( \mathbb{C}^* \)-action \( \rho \) on \( \mathfrak{g} \) obviously descends to an action on \( \mathfrak{h}/W \). Note that both actions have positive weights, hence contract the relevant spaces to a point (the origin). \( \rho \) also lifts to an action \( \tilde{\rho} \) on \( \tilde{\mathfrak{g}} \), which keeps the flags constant. Choose a hermitian inner product on \( \mathfrak{g} \), take the function

\[ \psi(y) = \frac{1}{2}||y||^2, \]

and pull it back to a function \( \tilde{\psi} \) on \( \tilde{\mathfrak{g}} \). By homogeneity, \( \tilde{\chi}^{-1}(0) \) intersects all the level sets \( \tilde{\psi}^{-1}(c), c > 0 \), transversally. It follows that there is a small ball \( B \subset \mathfrak{h} \) around the origin, such that for all \( \tilde{t} \in B, \tilde{\chi}^{-1}(\tilde{t}) \) intersects \( \psi^{-1}(1) \) transversally. Using the \( \mathbb{C}^* \)-action to rescale things, one sees that \( \tilde{\chi}^{-1}(\tilde{t}) \) intersects \( \tilde{\psi}^{-1}(c) \) transversally for all \( \tilde{t} \in B \) and \( c \geq 1 \). An obvious argument with the gradient flow of \( \tilde{\psi} \) on \( \tilde{\chi}^{-1}(\tilde{t}) \) shows that one can write that manifold as a union of a compact piece with boundary, which is \( \tilde{\chi}^{-1}(\tilde{t}) \cap \tilde{\psi}^{-1}(0; 1]) \), and an infinite cone over that boundary. This is true for all \( \tilde{t} \in B \), which means that \( \tilde{\chi}^{-1}(B) \cap \tilde{\psi}^{-1}(0; 1]) \) is a differentiable fibre bundle with compact fibres, and that the whole of \( \tilde{\chi}^{-1}(B) \) is obtained from it by attaching infinite cones to the fibre boundaries. Finally, using the \( \mathbb{C}^* \)-action once more, the fibre bundle structure can be extended from \( B \) to the whole of \( \mathfrak{h} \). ■
From this and Lemma 15(ii) it follows that \( \chi : \mathfrak{g} \to \mathfrak{h}/W \) itself, when restricted to \( Conf^n_n(\mathbb{C}) \), also becomes a differentiable fibre bundle.

Take a homogeneous slice \( S \) for a nilpotent \( x \), and let \( \tilde{S} \) be its preimage in \( \mathcal{g} \). Take \( y \in S \) and a preimage \( \tilde{y} \in \tilde{S} \). Because the adjoint action lifts to \( \mathcal{g} \), the space \( T_y(Gy) \) is contained in the image of the differential \( T_{\tilde{y}} \tilde{S} \to T_{\tilde{y}}S \). In view of Lemma 15(i), this has the following consequences: the projection \( \tilde{g} \to \mathcal{g} \) is transverse to \( S \); hence, \( \tilde{S} \subset \tilde{\mathcal{g}} \) is a smooth submanifold and transverse to all \( G \)-orbits; finally, \( \tilde{\chi} : \tilde{S} \to \mathfrak{h} \) is a submersion.

\begin{equation}
(21) \text{Lemma: } \tilde{\chi} : \tilde{S} \to \mathfrak{h} \text{ is naturally a differentiable fibre bundle.}
\end{equation}

\text{Proof: } \lambda \text{ lifts to a } \mathbb{C}^* \text{-action } \lambda \text{ on } \tilde{S}, \text{ which contracts that space to the compact subset lying over the point } x \in S. \text{ On the base spaces, the corresponding } \mathbb{C}^* \text{-action on } \mathfrak{h}/W \text{ obviously lifts to } \mathfrak{h}, \text{ and contracts that space to the origin. Let } \xi : \tilde{S} \to \mathbb{R} \text{ be an exhausting function which is } \lambda \text{-homogeneous, } \xi(\lambda(x)y) = r^{2\alpha}\xi(y) \text{ for some } \alpha > 0. \text{ After pulling this back to a function } \xi \text{ on } S, \text{ one finds that } \\
\xi^{-1}(c) \text{ intersects } \tilde{\chi}^{-1}(0) \cap \tilde{S} \text{ transversally for all } c > 0. \text{ The rest is as before.} \blacksquare
\]

As in Lemma 15 the map \( \tilde{\chi}^{-1}(t) \cap \tilde{S} \cong \chi^{-1}(t) \cap S \) is an isomorphism for all \( t \in Conf^n_n(\mathbb{C}) \). Hence, the restriction of \( \chi|S : S \to \mathfrak{h}/W \) to \( Conf^n_n(\mathbb{C}) \) is again a differentiable fibre bundle.

We will also need a variation on the idea of simultaneous resolution, involving partial flags. Fix some \( k < n \). Let \( \mathfrak{g}^{\text{mult}} \) be the space of pairs \( (x, E) \), where \( x \in \mathfrak{g} \) and \( E \subset \mathbb{C}^n \) is a \( k \)-dimensional subspace, such that \( x|E \) is some multiple of the identity map. \( \mathfrak{g}^{\text{mult}} \) is a smooth manifold (in fact a bundle over the Grassmannian with Lie algebra fibres). Correspondingly, let \( \mathfrak{h}^{\text{mult}} \subset \mathfrak{h} \) be the subspace of diagonal matrices whose first \( k \) entries coincide, and \( W^{\text{mult}} \subset W \) the subgroup of permutations which leave the first \( k \) entries fixed. The quotient is \( \mathfrak{h}^{\text{mult}}/W^{\text{mult}} \cong \mathbb{C} \times \mathbb{C}^{n-k-1}/S_{n-k} \cong \mathbb{C}^{n-k} \). There is a natural holomorphic map \( \chi^{\text{mult}} : \mathfrak{g}^{\text{mult}} \to \mathfrak{h}^{\text{mult}}/W^{\text{mult}} \), where the first entry of \( \chi^{\text{mult}}(x, E) \in \mathbb{C} \times \mathbb{C}^{n-k-1}/S_{n-k} \) is the (single, common) eigenvalue of \( x|E \). Now let \( \mathcal{g}^{\text{mult}} \) be the space of pairs \( (x, F) \), where \( x \in \mathfrak{g} \) and \( F = \{0 = F^0 \subset F^1 \subset F^{k+1} \subset \cdots \subset F^n = \mathbb{C}^n\} \) a partial flag, satisfying \( x(F^i) \subset F^{i+1} \) for all \( i \), and \( (x, F^k) \in \mathfrak{g}^{\text{mult}} \). This fits into a commutative diagram, where as usual the left \( \downarrow \) is a holomorphic submersion:

\begin{align*}
\tilde{\mathfrak{g}}^{\text{mult}} & \longrightarrow \mathfrak{g}^{\text{mult}} \quad h^{\text{mult}} \longrightarrow h^{\text{mult}}/W^{\text{mult}}. \\
\chi^{\text{mult}} & \downarrow \quad \chi^{\text{mult}} \downarrow \\
\tilde{h}^{\text{mult}} & \longrightarrow h^{\text{mult}}/W^{\text{mult}}.
\end{align*}
y ∈ \mathcal{S}, and \tilde{\mathcal{S}}^{\text{mult}} the corresponding subspace of \tilde{\mathfrak{g}}^{\text{mult}}. By restricting (21) one gets a diagram

\[
\begin{array}{ccc}
\tilde{\mathcal{S}}^{\text{mult}} & \longrightarrow & \mathcal{S}^{\text{mult}} \\
\chi^{\text{mult}} & \downarrow & \chi^{\text{mult}} \\
\mathfrak{h}^{\text{mult}} & \longrightarrow & \mathfrak{h}^{\text{mult}}/W^{\text{mult}}.
\end{array}
\]

(22) Remarks: For a general semisimple Lie algebra \mathfrak{g}, one defines \tilde{\mathfrak{g}} to be the space of pairs (x, b), where b ⊂ \mathfrak{g} is a Borel subalgebra containing x. This comes with a map \tilde{\chi} : \tilde{\mathfrak{g}} → \mathfrak{h}, which is a simultaneous resolution of the adjoint quotient map \chi (\text{Section 3.3 of}[4]) (the corresponding result for algebraic groups already appears in [4]). With that at hand, Lemma 21 generalizes easily (see [36] for another version of the same argument, and a more detailed study of the monodromy of the resulting fibration over \mathfrak{h}^{\text{reg}}/W).

The use of simultaneous resolution for Jacobson-Morozov slices also goes back to Brieskorn and Slodowy [4, 42], who looked at subregular nilpotent elements in simply-laced \mathfrak{g}. For those slices, one recovers the simultaneous resolutions of ADE singularities that had been previously discovered by Brieskorn by more elementary means; Example 19 is the simplest case.

3 \ ((m, m))-type nilpotent slices

We now focus on the particular slices \mathcal{S}_m ⊂ \mathfrak{sl}_{2m} relevant to our main construction. Running the basic idea of (4) and Lemma 15 in reverse, one finds that the geometry of \chi|\mathcal{S}_m is modelled on that of the whole adjoint quotient map \chi. In particular, in a process which goes back to Examples 3 and 7, we will see (A_1) and (A_2) singularities appearing. The other main point is that \mathcal{S}_{m-1} is embedded into \mathcal{S}_m in a natural way (Lemma 25); this will form the basis for several inductive arguments later on.

(A) Definition and first properties

From now on \( n = 2m \), which means that we will work with \( \mathfrak{g} = \mathfrak{sl}_{2m} \). Let \( x ∈ \mathfrak{g} \) be a nilpotent element with two Jordan blocks of size \( m \). We find it convenient to think of \( \mathbb{C}^{2m} = \mathbb{C}^2 ⊕ \cdots ⊕ \mathbb{C}^2 \), and then to write \( n^+ \) as in (0), but where the scalar entries 0, 1 are replaced by the corresponding 2x2 matrices (the zero matrix and the identity matrix in \( \mathfrak{gl}_2 \)) mapping the \( \mathbb{C}^2 \) summands to
each other. The equations (15) and (16), when re-interpreted in the same sense, then describe a JM triple \((n^+ = x, n^-, h)\) and the associated \(\mathbb{C}^\ast\)-action \(\lambda\) on \(g\).

Consider the affine space \(S_m = n^+ + \ker(z \mapsto zn^-)\), which consists of matrices of the form

\[
y = \begin{pmatrix}
y_{11} & 1 & & \\
y_{21} & 1 & & \\
& \ddots & \ddots & \\
y_{m-1,1} & & 1 & \\
y_{m1} & & & 0
\end{pmatrix}
\]

(23)

with \(y_{11} \in \mathfrak{s}l_2\), and \(y_{i1} \in \mathfrak{gl}_2\) for \(i > 1\). In parallel with Example (16), we have:

(23) Lemma: \(S_m\) is a homogeneous slice for the adjoint action at \(x\).

Proof: Suppose that \(z \in g\) is such that only the first two columns of \([n^+, z]\) are nonzero. Inspection of \([n^+, z]\) shows that \(z\) must be upper triangular when written in our usual block form, and from that one sees that \([n^+, z] = 0\). This shows that the tangent space to the adjoint orbit intersects the tangent space to our slice trivially. By looking at the eigenvalues of \(ad(h)\), one sees that the \(\mathfrak{s}l_2\)-module \(g\) splits into a direct sum of irreducible representations of dimensions \(1, 3, \ldots, 2m - 1\). More precisely, there are three trivial summands of dimension \(1\), and four summands of all the other dimensions. Thinking of the Jacobson-Morozov procedure, this means that a transverse slice must have dimension \(3 + 4(m - 1)\). This shows that the tangent space to our slice is indeed complementary to the orbit directions. \(\lambda\)-invariance is obvious from (16).

(24) Lemma: For any \(y \in S_m\) and any \(\mu \in \mathbb{C}\), projection to the first two coordinates yields an injective map \(\ker(\mu \cdot 1 - y) \rightarrow \mathbb{C}^2\). In particular, that eigenspace is at most two-dimensional.

Proof: Suppose that the contrary is true, which means that \(\ker(\mu \cdot 1 - y)\) has nonzero intersection with \(\{0\}^2 \times \mathbb{C}^{3m-2}\). Using the \(\mathbb{C}^\ast\)-action, one sees that the same holds for \(\ker(r^2\mu \cdot 1 - \lambda_r(y))\), and as \(r \rightarrow 0\), one obtains a nonzero element in \(\ker(n^+) \cap (\{0\}^2 \times \mathbb{C}^{2m-2})\), which is a contradiction.

(25) Lemma: The subspace of those \(y \in S_m\) such that \(\ker(y)\) is two-dimensional can be canonically identified with \(S_{m-1}\). This identification is compatible with the adjoint quotient map: if \(y\) has eigenvalues \((0, 0, \mu_3, \ldots, \mu_{2m})\), then the corresponding element \(\bar{y} \in S_{m-1}\) has eigenvalues \((\mu_3, \ldots, \mu_{2m})\).

Proof: In the previous proof, we saw that a vector in \(\ker(y)\) is uniquely determined by its first two entries. For there to be two linearly independent such vectors, it is necessary and sufficient that \(y_{m1} = 0\). The identification of this subspace with \(S_{m-1}\) is the straightforward one. One can see it as restriction of linear maps to the subspace \(\mathbb{C}^{2m-2} \times \{0\}^2 \supset im(y)\), and then the second
statement becomes obvious. ■

(26) Remark: By Lemma 16, $S_m$ is orbit-preservingly isomorphic to the Jacobson-Morozov slice at $n^+$, and correspondingly for $S_{m-1}$. As a consequence, Lemma 24 also holds for Jacobson-Morozov slices, but the isomorphism obtained in this way is no longer quite canonical (nor as simple as before), and we have not found a more direct construction. This is what makes $S_m$ more convenient for our purpose.

(B) Two eigenvalues coincide

Let $t \in \mathfrak{h}/W$ be a point corresponding to a collection of $2m$ pairwise different eigenvalues $(\mu_1, \ldots, \mu_{2m})$. By Example 4 and Lemma 15(ii), this is a regular value of $\chi|_{S_m}$, so the fibre $(\chi|_{S_m})^{-1}(t)$ is a smooth complex manifold of dimension $2m$.

Next, take the case where $\mu_1 = \mu_2$, with no other coincidences between the eigenvalues. Then $\chi^{-1}(t)$ is the union of two orbits: the regular orbit $O_{\text{reg}}$ (of matrices with an indecomposable Jordan block of size two for the eigenvalue $\mu_1$), which is open and dense in $\chi^{-1}(t)$; and the subregular orbit $O_{\text{sub}}$ (of matrices having two independent $\mu_1$-eigenvectors), which is closed. From Example 10 we know that elements of $O_{\text{reg}}$ are regular points of $\chi$, whereas those of $O_{\text{sub}}$ have singularities of $(A_1)$ type in transverse direction to the orbit. In view of Lemma 15, the intersections $O_{\text{reg}} \cap S_m$, $O_{\text{sub}} \cap S_m$ have the same properties with respect to the map $\chi|_{S_m}$. We need a more precise global version of the latter statement, in which one can see an entire neighbourhood of $O_{\text{sub}} \cap S_m$.

At every point $y \in O_{\text{sub}} \cap S_m$, choose a subspace of $T_yS_m$ which is complementary to $T_y(O_{\text{sub}} \cap S_m)$ and depends holomorphically on $y$. This splitting problem has a positive solution because $O_{\text{sub}} \cap S_m$ is affine, so that the relevant $\text{Ext}^1$ obstruction group is zero. Translate those subspaces by adding $y$, to obtain a family of affine subspaces $S'_y$ which form a tubular neighbourhood of $O_{\text{sub}} \cap S_m$ inside $S_m$. Because $S_m$ intersects $O_{\text{sub}}$ transversally, each $S'_y$ is also a local slice at $y$ for the adjoint action on $\mathfrak{g}$. On the other hand, $y$ is semisimple, so we are precisely in the situation discussed in Example 9 which means that we can construct another local slice at $y$ by setting $S_y = y + (\mathfrak{g} \oplus \mathfrak{z})$. Recall that $\mathfrak{g} = sl(E_y)$, where $E_y$ is the $\mu_1$-eigenspace of $y$. By Lemma 24 projection yields preferred isomorphisms $E_y \cong \mathbb{C}^2$ and hence $\mathfrak{g} \cong sl_2$. The remaining part $\mathfrak{z}$ can moreover be identified with $\mathbb{C}^{2m-2}$. By Lemma 5(iii), we can find a local isomorphism $S_y \cong S'_y$ which moves points only inside their adjoint orbits, hence relates $\chi|S_y$ to $\chi|S'_y$. Strictly speaking, this isomorphism requires a choice of local submanifold $K_y \subset G$ complementary to $G_y$. One can view this as another splitting problem, which can be solved in a way that depends holomorphically on $y$ for cohomological reasons. Alternatively, an elementary argument starting
from $g = [g, y] \oplus g_y$ shows that in this case, $K_y$ can be explicitly taken to be $\exp([g, y])$. In either way, the outcome is that we get a family of isomorphisms (defined locally near $y$) $S'_y \cong y + (\mathfrak{sl}_2 \oplus \mathbb{C}^{2m-2})$. Moreover, with respect to that trivialization, the adjoint quotient map becomes (22) on each slice. In particular, we obtain:

(27) **Lemma:** Let $D \subset \mathfrak{h}/W$ be a disc corresponding to eigenvalues $(\mu_1 - \sqrt{\epsilon}, \mu_2 + \sqrt{\epsilon}, \mu_3, \ldots, \mu_{2m})$ with $\epsilon$ small. Then there is a neighbourhood of $O^{\text{sub}} \cap S_m$ inside $\chi^{-1}(D) \cap S_m$, and an isomorphism of that with a neighbourhood of $(O^{\text{sub}} \cap S_m) \times \{0\}^3$ inside $(O^{\text{sub}} \cap S_m) \times \mathbb{C}^3$. This isomorphism fits into a commutative diagram

$$
\begin{array}{ccc}
\chi^{-1}(D) \cap S_m & \xrightarrow{\text{local \& defined near } O^{\text{sub}} \cap S_m} & (O^{\text{sub}} \cap S_m) \times \mathbb{C}^3 \\
\downarrow \chi & & \downarrow a^2 + b^2 + c^2 \\
D & \xrightarrow{a + b + c} & \mathbb{C}
\end{array}
$$

where $a, b, c$ are coordinates on $\mathbb{C}^3$. ■

This means that the function $\chi|_{\chi^{-1}(D) \cap S_m}$ is nondegenerate in transverse direction to the critical submanifold $O^{\text{sub}} \cap S_m$. This is the analogue of the Morse-Bott condition in real topology, and we will refer to it by saying that the critical submanifold is of fibered $(A_1)$ type (note that by using the explicit local slices given by Lemma 8, we have reached this conclusion without appealing to any Morse Lemma-type arguments). Our case has the additional feature that the normal data along the critical submanifold, consisting of the normal bundle and the nondegenerate quadratic form on its fibres, are trivial. It is instructive to compare it to the behaviour of the adjoint quotient map on $\chi^{-1}(D)$ (without the slice): $O^{\text{sub}}$ is still a critical submanifold of fibered $(A_1)$ type, but the normal data are no longer trivial.

As a variation on this theme, one can consider the case where $2r$ eigenvalues come together in pairs, as in Example 11. The relevant fibre $\chi^{-1}(t)$ consists of $2^r$ orbits, since the restriction of $x \in \chi^{-1}(t)$ to its generalized $\mu_{2j-1}$-eigenspace may be either semisimple or not, for each $j = 1, \ldots, r$. The smallest orbit $O^{\text{min}}$, which consists of those $x$ that are actually semisimple, is closed. A straightforward adaptation of the previous argument yields the following description of the local structure near $O^{\text{min}} \cap S_m$:

(28) **Lemma:** Let $P \subset \mathfrak{h}/W$ be a $k$-dimensional polydisc corresponding to eigenvalues $(\mu_1 - \sqrt{\epsilon_1}, \mu_2 + \sqrt{\epsilon_1}, \ldots, \mu_{2k-1} - \sqrt{\epsilon_k}, \mu_{2k} + \sqrt{\epsilon_k}, \mu_{2k+1}, \ldots, \mu_{2m})$ with the $\epsilon$’s small. Then there is a neighbourhood of $O^{\text{min}} \cap S_m$ inside $\chi^{-1}(P) \cap S_m$, and an isomorphism of that with a neighbourhood of $(O^{\text{min}} \cap S_m) \times \{0\}^{3k}$ inside.
\((O^{\text{min}} \cap S_m) \times \mathbb{C}^{3k}\). The isomorphism fits into a commutative diagram

\[
\begin{array}{ccc}
\chi^{-1}(P) \cap S_m & \xrightarrow{\text{local} \equiv \text{defined near } O^{\text{min}} \cap S_m} & (O^{\text{min}} \cap S_m) \times \mathbb{C}^{3k} \\
\downarrow & & \\
P & \xrightarrow{(a_1^2+b_1^2+c_1^2, \ldots, a_k^2+b_k^2+c_k^2)} & \mathbb{C}^r
\end{array}
\]

where \(a_j, b_j, c_j\) are coordinates on \(\mathbb{C}^{3k}\).

\[(C)\] Three eigenvalues coincide

Now take a point \(t \in \mathfrak{h}/W\) which corresponds to a set of eigenvalues \((\mu_1, \ldots, \mu_{2m})\) of which the first three coincide, \(\mu_1 = \mu_2 = \mu_3\), and which are otherwise pairwise distinct. The adjoint fibre \(\chi^{-1}(t)\) contains three orbits: the regular orbit \(O^{\text{reg}}\) has an indecomposable Jordan block of size 3 for the eigenvalue \(\mu_1\); the subregular orbit \(O^{\text{sub}}\) has two Jordan blocks of sizes 1, 2; and the minimal orbit consists of matrices with three independent \(\mu_1\)-eigenvectors. However, that last orbit does not intersect \(S_m\), due to Lemma 24. As a consequence, \(O^{\text{sub}} \cap S_m\) is closed in \(S_m\), which makes the situation fairly similar to the one we looked at before.

Take \(y \in O^{\text{sub}} \cap S_m\), and let \(E_y\) be the \(\mu_1\)-eigenspace of its semisimple part \(y_s\). As a first step, we want to choose a JM triple for \(y_n\) inside \(\mathfrak{sl}(E_y)\). As explained in Example 14, such triples correspond bijectively to splittings of the flag

\[
0 \subset F^1_y = (\mu_1 \cdot 1 - y)(E_y) \subset F^2_y = \ker(\mu_1 \cdot 1 - y) \subset E_y.
\]

Because \(O^{\text{sub}} \cap S_m\) is affine, one can use the vanishing of \(\text{Ext}^1\) to find such splittings which vary holomorphically with \(y\), and thence holomorphically varying JM slices \(\hat{S}_y \subset \mathfrak{g}\) for \(y\) itself, without any further choices. To find a better global picture, we appeal again to Kostant’s uniqueness theorem. This says that we can find an isomorphism

\[
E_y \cong \mathbb{C}^3
\]

such that the induced map \(\mathfrak{sl}(E_y) \cong \mathfrak{sl}_3\) identifies our JM triple with the one explicitly given in Example 14. The isomorphism is unique up to the action of the subgroup \(\mathbb{C}^* \times \mathbb{C}^* \subset GL_3\) consisting of diagonal matrices \(\text{diag}(\zeta, \tau, \zeta)\): namely, the isomorphism of Lie algebras is unique up to the subgroup of diagonal matrices in \(\mathbb{C}^* \times \mathbb{C}^* \subset GL_3\), and lifting that to an isomorphism of vector spaces adds the central \(\mathbb{C}^* \subset GL_3\). By construction, \(\text{diag}(\zeta, \zeta^{-1}, \zeta)\) takes \(F^2_y\) to \(\mathbb{C}^2 \times \{0\}\). Another isomorphism \(F^2_y \cong \mathbb{C}^2\) is given by Lemma 24, and we can constrain \(25\) by asking that the two maps have equal determinant. This reduces the ambiguity to a single \(\mathbb{C}^*\) factor \(\text{diag}(\zeta, \zeta^{-1}, \zeta)\), which means that \(24\) is completely determined.
by the induced identification \( F^1_y \to \mathbb{C} \times \{0\}^2 \). The adjoint action of the \( \mathbb{C}^* \) on the model slice, in the coordinates \( (a,b,c,d) \), is
\[
(a, b, c, d) \mapsto (a, \zeta^{-2}b, \zeta^2 c, d).
\]
(26)

With that in mind, the argument from Lemma 27 carries over up to some easy modifications, with the following outcome:

(29) Lemma: Let \( \mathcal{F} \to \mathcal{O}^{\text{sub}} \cap S_m \) be the line bundle whose fibres are the spaces \( F^1_y \) from (24). Consider the associated vector bundle
\[
(F \setminus 0) \times_{\mathbb{C}^*} \mathbb{C}^4 = \mathbb{C} \oplus F^{-2} \oplus F^2 \oplus \mathbb{C}
\]
(27)

with respect to the action from (26); here 0 denotes the zero-section and \( \mathbb{C} \) the trivial line bundle over \( \mathcal{O}^{\text{sub}} \cap S_m \). Let \( P \hookrightarrow \mathfrak{h}/\mathbb{W} \) be a small bidisc parametrized by \((d, z)\), corresponding to the set of eigenvalues
\[
(\mu_1 + \{ \text{all solutions of } \lambda^3 - d\lambda + z = 0 \}, \mu_4, \ldots, \mu_{2m}).
\]

There is a neighbourhood of \( \mathcal{O}^{\text{sub}} \cap S_m \) inside \( \chi^{-1}(P) \cap S_m \), and an isomorphism of that with a neighbourhood of the zero-section inside \( (F \setminus 0) \times_{\mathbb{C}^*} \mathbb{C}^4 \), which fits into a commutative diagram
\[
\begin{array}{ccc}
\chi^{-1}(P) \cap S_m & \xrightarrow{\text{local } \cong \text{ defined near } \mathcal{O}^{\text{sub}} \cap S_m} & (F \setminus 0) \times_{\mathbb{C}^*} \mathbb{C}^4 \\
\chi \downarrow & & p \downarrow \\
P & \xrightarrow{(d,z)} & \mathbb{C}^2
\end{array}
\]

where \( p \) is given by (19) on each \( \mathbb{C}^4 \) fibre.

In the statement we have used \( d \) both as a coordinate on \( \mathbb{C}^4 \) and on \( P \), but that should not be troublesome since the first component of \( p \) maps one identically to the other. Note also that the second component \( a^3 - ad + bc \) makes sense as a holomorphic function on (27) because \( b \) and \( c \) are sections of inverse line bundles.

(D) A partial Grothendieck resolution

Let’s apply the construction from (22), with \( k = 2 \) and \( n = 2m > 2 \), to our slices \( S_m \). Denote the resulting spaces by \( S_{m, \text{mult}} \), \( \tilde{S}_{m, \text{mult}} \). Consider the open subset \( \mathfrak{h}^{\text{mult, reg}} \subset \mathfrak{h}^{\text{mult}} \) of those \( (\mu_1, \mu_2 = \mu_1, \mu_3, \ldots, \mu_{2m}) \) where the \( (\mu_3, \ldots, \mu_{2m}) \) are pairwise disjoint among themselves (but any of them may agree with \( \mu_1 = \mu_2 \)). It is clear from the definitions that if \( (y, E) \in S_{m, \text{mult}} \) is such that \( \chi^{\text{mult}}(y, E) \in \mathfrak{h}^{\text{mult, reg}}/\mathbb{W}^{\text{mult}} \), then a choice of preimage \( (y, F) \in \tilde{S}_{m, \text{mult}} \) is the same as an ordering of the eigenvalues \( \mu_3, \ldots, \mu_{2m} \). In other words, the restriction of (22) to
\[ \mathfrak{h}^{\text{mult, reg}}/\mathcal{W}^{\text{mult}} \] is a pullback diagram. As a consequence, the restriction of

\[ \chi^{\text{mult}}|_{S_m} : S_m^{\text{mult}} \to \mathfrak{h}^{\text{mult}}/\mathcal{W}^{\text{mult}} \]  \hspace{1cm} (28)

to \( \mathfrak{h}^{\text{mult, reg}}/\mathcal{W}^{\text{mult}} \) becomes a fibre bundle. Fix some \((\mu_1, \ldots, \mu_{2m}) \in \mathfrak{h}^{\text{mult, reg}}\), let \( t^{\text{mult}} \) be its image in \( \mathfrak{h}^{\text{mult, reg}}/\mathcal{W}^{\text{mult}} \), and \( t \) its image in \( \mathfrak{h}/\mathcal{W} \). By definition, the fibre of \( (28) \) over \( t^{\text{mult}} \) is the set of pairs \((y, E)\), where \( y \in S_m \cap \chi^{-1}(t) \) is such that \( y|E \) is a multiple of the identity. That multiple must be the unique multiple eigenvalue \( \mu_1 \), and then necessarily \( E = \ker(\mu_1 \cdot 1 - y) \) by Lemma \[24\]. In other words, the fibre of \( (28) \) can be identified with the subspace of \( \chi^{-1}(t) \cap S_m \) consisting of those \( y \) which have two independent \( \mu_1 \)-eigenvalues.

From the results in Sections (B) and (C), we see that this subspace is just the set of singular points of \( \chi^{-1}(t) \cap S_m \), which we will denote by \( \mathcal{C}_{m,t} \) from now on.

The upshot of this discussion is that the \( \mathcal{C}_{m,t} \) form a differentiable fibre bundle

\[ \mathcal{C}_m \to \mathfrak{h}^{\text{mult, reg}}/\mathcal{W}^{\text{mult}}. \]  \hspace{1cm} (29)

4 Parallel transport, Floer cohomology

This section deals with the necessary Kähler and symplectic geometry, largely in the context of Stein fibre bundles. The main objectives are the definition of relative vanishing cycles, and two technical statements about Floer cohomology groups in situations where the Lagrangian submanifolds concerned are constructed as such cycles (Lemma \[42\] and Lemma \[44\]). Our main technical trick involves deforming Kähler forms to make them agree with the standard forms on certain subsets where we have preferred holomorphic coordinates. One has to worry whether the resulting families of Lagrangian submanifolds remain inside those subsets, and addressing that requires some technical estimates of parallel transport vector fields.

(A) Parallel transport

Let \( \pi : Y \to T \) be a holomorphic map between complex manifolds, which is a submersion with fibres \( Y_t \). Suppose that \( Y \) carries a Kähler metric, and equip the fibres with the induced metrics, which in particular makes them into symplectic manifolds. Take a path \( \gamma : [0; 1] \to T \) on the base. The parallel transport vector field \( H_\gamma \) is a vector field on the pullback \( \gamma^*Y \to [0; 1] \): it consists of the unique sections of \( TY|_{Y_\gamma(s)} \) which project to \( \gamma(s) \), and which are orthogonal to the tangent space along the fibres. In the case where \( T = \mathbb{C} \), one can write explicitly

\[ H_\gamma = \frac{\nabla_\gamma \pi}{\|\nabla_\pi\|^2}\gamma(s). \]  \hspace{1cm} (30)
When $\pi$ is proper, integrating $H_\gamma$ yields a symplectic isomorphism between fibres, the parallel transport map $h_\gamma : Y_\gamma(0) \to Y_\gamma(1)$. If properness fails, the integral lines may not exist for all times. In some cases, explicit estimates of $H_\gamma$ may allow one to show that $h_\gamma$ is still defined everywhere, or at least on a subset which is sufficiently large to contain the geometric objects (Lagrangian submanifolds, in our applications) that one wants to apply parallel transport to. Alternatively, one can try to modify the vector field, so as to make the domain of definition larger. We will now explain a basic argument of the second kind, for the case when the fibres $Y_t$ are Stein manifolds with finite topology (to simplify the description, we impose slightly sharper technical conditions than strictly necessary).

Suppose that there is a proper bounded below function $\psi : Y \to \mathbb{R}$ such that

- $-dd^c\psi$ is the given Kähler form on $Y$.
- Outside a compact subset of $Y$, $||\nabla \psi||^2 \leq \rho \psi$ for some $\rho > 0$.
- The fibrewise critical set of $\psi$, consisting of those points where $d\psi|_\text{ker}(D\pi)$ is zero, maps properly to $T$.

Fix $t \in T$. The function $\psi_t = \psi|_{Y_t}$ is proper and has a compact set of critical points. Let $Z_t$ be its gradient vector field. We have $Z_t \cdot \psi_t = ||\nabla \psi_t||^2 \leq ||\nabla \psi||^2 \leq \rho \psi_t$ outside a compact subset, which ensures that the flow of $Z_t$ is defined for all times. In symplectic geometry terms, this is a Liouville vector field, so $Y_t$ is a manifold with an infinite convex contact-type cone. As before, let $\gamma$ be a path in $T$. Because of the properness condition on the fibrewise critical point set, we can choose $c > 0$ in such a way that the critical values of $\psi$ on the fibres $Y_{\gamma(s)}$ all lie in $[0; c)$. Using that and the properness of $\psi$, one can find a $\sigma > 0$ such that the modified parallel transport vector field

$$\tilde{H}_\gamma = H_\gamma - \sigma Z_{\gamma(s)}$$

satisfies $d\psi(\tilde{H}_\gamma) < 0$ at all points $y \in Y_{\gamma(s)}$ with $\psi(y) = c$. The integral lines of that vector field necessarily stay within $\psi^{-1}([0; c])$, hence give rise to a well-defined map $Y_{\gamma(0)} \cap \psi^{-1}([0; c]) \to Y_{\gamma(1)}$. This is only conformally symplectic, but one can repair that by composing with the time $\sigma$ map of the Liouville flow on $Y_{\gamma(1)}$. The result is a symplectic embedding

$$h^{\text{resc}}_\gamma : Y_{\gamma(0)} \cap \psi^{-1}([0; c]) \to Y_{\gamma(1)}$$

called rescaled symplectic parallel transport (to make the distinction clear, we will sometimes refer to the maps $h_\gamma$ obtained by simply integrating horizontal vector fields as naive parallel transport). It is independent of the choice of $\sigma$ up to isotopy within the class of symplectic embeddings. One can take $c$ arbitrarily large, and thereby define $h^{\text{resc}}_\gamma$ on arbitrarily big compact subsets of $Y_{\gamma(0)}$. Passing from some value of $c$ to a larger one yields a map whose restriction
to the smaller domain is isotopic to the previous one. As a consequence, the image $h_{\gamma}(L)$ of a closed Lagrangian submanifold $L \subset Y_{\gamma(0)}$ is well-defined up to Lagrangian isotopy, which is the most important fact for our purpose.

(30) Remark: Under the same assumptions, one can construct rescaled parallel transport maps which are defined on the whole fibre and are symplectic isomorphisms, modelled at infinity on contactomorphisms (this means that they commute with the Liouville flows outside compact subsets). The first step is Gray’s stability theorem, which shows that the hypersurfaces $\psi^{-1}(c) \cap Y_{\gamma(s)}$ for various $s$ are isomorphic as contact manifolds. Hence, the cone-like ends of the fibres are symplectically isomorphic. One modifies the parallel transport vector field to be compatible with these isomorphisms outside a compact subset; for details see [17, Section 6].

(B) Relative vanishing cycles

We now consider the local Kähler geometry around a fibered $(A_1)$ type critical set. Since our main intended application is provided by Lemma 27, we will concentrate on the case where the normal data to the critical point set are trivial (for a partial loosening of this restriction, see Remark 33). Therefore, take any complex manifold $X$ and consider

$$\pi : Y = X \times \mathbb{C}^3 \longrightarrow \mathbb{C}, \quad \pi(x, a, b, c) = a^2 + b^2 + c^2.$$ (31)

The critical point set is $\text{Crit}(\pi) = \{a = b = c = 0\}$, hence can be identified with $X$. We equip $Y$ with any Kähler metric, the fibres $Y_t$ with the induced metrics, and $X$ with the restriction of the metric to $\text{Crit}(\pi)$. The symplectic form is denoted by $\Omega \in \Omega^2(Y)$. Since the second derivative of $\pi$ in transverse direction to $X$ is nondegenerate, the real part $\text{re}(\pi)$ is a Morse-Bott function. Define its stable manifold $W \subset Y$ to be the set of points $y$ such that the flow line of $-\nabla \text{re}(\pi)$ starting at $y$ exists for all times $s \geq 0$, and converges to a critical point in the limit $s \to \infty$. Obviously $X$ itself is contained in $W$. Moreover, since the negative gradient flow of $\text{re}(\pi)$ is also the Hamiltonian vector field of $\text{im}(\pi)$, it leaves $\text{im}(\pi)$ invariant, hence $W$ lies inside $\pi^{-1}(\mathbb{R}_{\geq 0})$.

(31) Lemma: (i) $W \subset Y$ is a local real submanifold of codimension 3, and its tangent space along $\text{Crit}(\pi)$ is $TX \times \mathbb{R}^3$. (ii) The map $l : W \to X$ which assigns to a point its limit under the negative gradient flow is a smooth submersion. (iii) $\Omega|W$ is equal to the pullback of $\Omega|X$ under $l$.

The first two statements are standard Morse-Bott theory. One possible approach, carried out in detail in [2, Appendix A], goes roughly as follows. One first shows that the convergence of gradient flow lines towards critical points happens with exponential speed. Then, defining $W$ as a subspace of the Banach
manifold of all paths in $Y$ converging exponentially towards a critical point, one finds that it is smooth by using the implicit function theorem. The dimension can be computed from the index of a suitably linearized problem, and the map $l$ is smooth by construction. As for (iii), the gradient flow is symplectic, and if we restrict it to $W$, then the limit of its derivative as $s \to \infty$ gives $Dl$.

\textbf{(32) Lemma:} Let $K \subset X$ be a compact Lagrangian submanifold. Then for sufficiently small $t > 0$, $L_t = l^{-1}(K) \cap Y_t$ is a Lagrangian submanifold of $Y_t$ diffeomorphic to $K \times S^2$.

\textbf{Proof:} $\pi|W: W \to \mathbb{R}$ is a function with a Morse-Bott type nondegenerate minimum along $X \subset W$. The same holds if we restrict it to the submanifold $l^{-1}(K)$ of points whose limit lies in $K$. The Morse-Bott Lemma, together with the fact that the normal bundle of $X$ in $W$ is trivial, imply that the sets $L_t$ for small $t$ are trivial $S^2$-bundles over $K$. The Lagrangian property follows immediately from the previous Lemma. \hfill \Box

We call $L_t$ the relative vanishing cycle associated to $K$. Of course, by multiplying $\pi$ with some constant in $S^1$, one can define stable manifolds which lie over other half-lines in $\mathbb{C}$, and relative vanishing cycles $L_t \subset Y_t$ for all sufficiently small $t \in \mathbb{C}^*$. There is also an equivalent formulation in terms of parallel transport. Take the path $\gamma: [0; 1] \to \mathbb{C}$, $\gamma(s) = (1-s)t$, which runs straight into the critical value. Then $L_t$ is the set of those $y \in Y_t$ such that the (naive) parallel transport maps $h_{\gamma|[0;s]}$ are well-defined near $y$ for all $s < 1$, and such that as $s \to 1$, $h_{\gamma|[0;s]}(y)$ converges to a point of $K$. The two definitions are equivalent essentially because for $t > 0$, $H_\gamma$ and $-\nabla \text{re}(\pi)$ agree up to a positive scalar factor, see (30).

\textbf{(33) Remark:} A variant of this geometry is where one has a holomorphic line bundle $\mathcal{L} \to X$ and looks at $Y = \mathbb{C} \oplus \mathbb{L}^{-1} \oplus \mathcal{L}$ with the function $\pi: Y \to \mathbb{C}$, $\pi(a,b,c) = a^2 + bc$ where $(a,b,c)$ are the fibre coordinates. The construction of relative vanishing cycles $L_t \subset Y_t$ from $K \subset X$ goes through as before, the only difference being that topologically $L_t$ is a possibly nontrivial $S^2$-bundle over $K$, in fact the projectivization $P(\mathcal{L} \oplus \mathbb{C})|K$.

To supplement the previous discussion, and (more importantly) as a warmup exercise for (C) below, we will now explicitly estimate $H_\gamma$ and thereby bound the position of the relative vanishing cycles. Fix a relatively compact open subset $U \subset X$ and a ball $B \subset \mathbb{C}^3$ around the origin, and set $V = U \times B \subset Y$.

\textbf{(34) Lemma:} There is a constant $\nu > 0$ such that on $V$,

$$||\nabla \pi||^2 \geq \nu^{-1} |\pi|.$$
Proof: This would hold everywhere, with \( \nu = 1/4 \), if our metric was the product of some metric on \( X \) and the standard metric on \( \mathbb{C}^3 \). Since \( V \) is relatively compact, the statement is independent of the choice of metric. \( \blacksquare \)

Now take a compact Lagrangian submanifold \( K \subset U \subset X \). Consider it as lying in the critical set of \( \pi \), and denote by \( \delta > 0 \) its distance from \( \partial V \) with respect to the given metric. We want to show that the relative vanishing cycle \( L_t \) is well-defined and lies in \( V \) for all

\[
0 < |t| < (1/100)\nu^{-1}\delta^2.
\]

Assume that \( t > 0 \), and think in terms of parallel transport along \( \gamma(s) = s \). From the definition (30) and the Lemma above, one sees that inside \( V \), the horizontal vector field on \( Y_{\gamma_t} \) is bounded by

\[
||H_{\gamma_t}|| \leq \nu^{1/2}s^{-1/2}.
\]

Suppose that we have a flow line of this vector field defined for \( s \in (0; t) \), and which converges to a point of \( K \) as \( s \to 0 \). Supposing that \( t \) satisfies (32), then by integrating (33) one finds that the whole flow line lies at distance at most \( 2\nu^{1/2}s^{-1/2} < \delta/2 \) from \( K \), hence it extends to \( s = t \). With that in mind, the well-definedness of the vanishing cycles and the fact that they lie in \( V \) is clear.

A similar estimate shows that if we take \( t \) as in (32) and consider the circle \( \gamma_t : [0; 2\pi] \to \mathbb{C}^* \), \( \gamma_t(s) = t \exp(is) \), then parallel transport \( h_{\gamma_t}(y) \) is well-defined and lies in \( V \) for all \( y \in L_t \). A priori we now have two Lagrangian submanifolds in \( Y_t \), namely the relative vanishing cycle \( L_t \) and its monodromy image \( h_{\gamma_t}(L_t) \), however:

\[\text{(35) Lemma: } L_t \text{ is Lagrangian isotopic to } h_{\gamma_t}(L_t) \text{ inside } V \cap Y_t.\]

It may be helpful to first consider the case when the metric on \( Y \) is the product of some Kähler metric on \( X \) and the standard metric on \( \mathbb{C}^3 \). Then the relative vanishing cycles are

\[
L_t = K \times \sqrt{t}S^2 \subset Y_t
\]

where \( \sqrt{t}S^2 \subset \sqrt{t}\mathbb{R}^3 \subset \mathbb{C}^3 \). The monodromy is \( \text{id}_X \) times the standard Picard-Lefschetz (Dehn twist) monodromy, see [38] for an explicit computation, and the Lemma is trivially true, since \( h_{\gamma_t}(L_t) = L_t \). For general metrics one argues as follows. The estimates made above show that for all \( \tau = \gamma_t(s) \) the relative vanishing cycle \( L_\tau \) is well-defined, and so is the (naive) parallel transport along \( \gamma_t|[s; 2\pi] \) at least near \( L_\tau \). Clearly \( h_{\gamma_t|[s; 2\pi]}(L_\tau) \) is a family of Lagrangian submanifolds connecting \( h_{\gamma_t}(L_t) \) with \( L_t \).

Guided by Lemma 28, we also want to look at the situation where the construction of vanishing cycles can be iterated. Namely take \( Y = X \times \mathbb{C}^{3k} \), with the function

\[
\pi : Y \to \mathbb{C}^k, \quad \pi(x, a_1, b_1, c_1, \ldots) = (a_1^2 + b_1^2 + c_1^2, \ldots, a_k^2 + b_k^2 + c_k^2)
\]
and a compact Lagrangian submanifold $K$ of $X = X \times \{0\}^{3k} \subset Y$. One starts with the first component of $\pi$, restricted suitably to a function

$$\pi_1 : X \times \mathbb{C}^3 \times \{0\}^{3k-3} \to \mathbb{C}.$$ 

This yields a relative vanishing cycle $L_{t_1} \in \pi_1^{-1}(t_1)$ for small $t_1 \neq 0$. Fix some value of that parameter, and consider the next component

$$\pi_2 : \pi^{-1}\{t_1\} \times \mathbb{C}^{k-1} \cap (X \times \mathbb{C}^6 \times \{0\}^{3k-6}) \to \mathbb{C}.$$ 

This has $\pi_1^{-1}(t_1)$ as its critical locus, and by writing down things explicitly one sees that the relative vanishing cycle construction can be applied to $L_{t_1}$ yielding an $L_{t_1,t_2} \in \pi_2^{-1}(t_2)$. By repeating this one finally obtains an iterated relative vanishing cycle, which is a Lagrangian submanifold $L_{t_1,\ldots,t_k} \subset Y_{t_1,\ldots,t_k}$ diffeomorphic to $K \times (S^2)^k$. A priori this may appear to work only for $0 < |t_k| \ll |t_{k-1}| \ll |t_{k-2}| \ll \cdots \ll |t_1|$, but an inspection of the relevant parallel transport vector fields shows that there is a uniform bound for all coordinates, meaning that there is a $\sigma > 0$ such that $L_{t_1,\ldots,t_k}$ is defined whenever $0 < |t_j| < \sigma$ for all $j$. With this in mind, it makes sense to state:

\[(36) \text{Lemma:} \quad \text{If one changes the order in which the components of } \pi \text{ are used to construct the iterated vanishing cycle, the outcome is the same up to Lagrangian isotopy, at least as long as all the } |t_j| \text{ are sufficiently small.}\]

\[\text{Proof:} \quad \text{Let } \Omega = \Omega^{(0)} \text{ be the given Kähler form on } Y. \text{ By restricting it to } X \times \{0\}^{3k} \text{ and taking the sum of that and the standard form on } \mathbb{C}^{3k}, \text{ one gets another Kähler form } \Omega^{(1)}. \text{ The statement of the Lemma would be trivial for } \Omega^{(1)}, \text{ since the corresponding iterated vanishing cycles are simply}\]

$$L_{t_1,\ldots,t_k}^{(1)} = K \times \sqrt{t_1}S^2 \times \cdots \times \sqrt{t_k}S^2.$$  

\[(35)\]

We will now use a Moser Lemma argument. Take the family $\Omega^{(r)}$, $0 \leq r \leq 1$, of Kähler forms which interpolate linearly between the two previously mentioned ones. By integrating radially away from $X$, one can write $\Omega^{(1)} - \Omega^{(0)} = d\Theta$ for some one-form $\Theta$ such that $\Theta_y = 0$ for each $y = (x,0,\ldots,0) \in X \times \{0\}^{3k}$. Take a relatively compact open subset $V \subset Y$ which contains $K$, and choose sufficiently small $t_1,\ldots,t_k \neq 0$. Then the iterated vanishing cycle $L_{t_1,\ldots,t_k}^{(r)}$ is well-defined for all $r$, and moreover, the Moser vector fields constructed from $\Theta|_{Y_{t_1,\ldots,t_k}}$ integrate to give a family of symplectic embeddings

$$\phi_{t_1,\ldots,t_k}^{(r)} : (V \cap \pi^{-1}(t_1,\ldots,t_k), \Omega^{(r)}) \to (\pi^{-1}(t_1,\ldots,t_k), \Omega^{(0)}).$$

From this one gets a Lagrangian isotopy from $L_{t_1,\ldots,t_k}$ to $\phi_{t_1,\ldots,t_k}^{(1)}(L_{t_1,\ldots,t_k}^{(1)})$. The same can be done for the iterated vanishing cycles constructed using a different ordering of the components of $\pi$, and since the endpoints of the two isotopies are the same by (36), the result follows. \[\]
(C) Fibered $(A_2)$ singularities

Basically the same strategy can be applied to the geometric situation which appears in Lemma 29. However, since that is somewhat more complicated, we prefer to first explain the argument in the simplest example, which is just the map $\pi = p : Y = \mathbb{C}^4 \to \mathbb{C}^2$ from [3]. We write $(d, z)$ for the coordinates on the base $\mathbb{C}^2$. The critical point set is $\text{Crit}(\pi) = \{b = c = 0, d = 3a^2\}$, and its image is the cusp curve $\Sigma = \{4d^3 = 27z^2\}$. We want to view $d$ as an auxiliary parameter, so we consider the restrictions of $\pi$ to $Y_d = C^3 \times \{d\}$ as a family of functions $\pi_d : Y_d \to \mathbb{C} = \mathbb{C}$, writing $Y_d(z) = \pi_d^{-1}(d, z) = \pi_d^{-1}(z)$ for their fibres. The critical point set can then be written as $\text{Crit}(\pi_d) = \{b = c = 0, a = \pm \sqrt{d/3}\}$. For $d \neq 0$ this consists of two nondegenerate critical points, denoted by $\text{Crit}(\pi_d)_{\pm}$, which project to the critical values $\zeta_d^\pm = \pm \sqrt{4d^3/27}$. Later on, we will mostly consider the case when $d > 0$, and then the convention is that $\zeta_d^+ > 0$. For $d = 0$ the two critical points coalesce into a single more degenerate one, which is of course exactly how singularity theorists came to study $\pi$.

Equip $Y$ with some Kähler form $\Omega$. For any $d > 0$ and $0 < \epsilon \ll d$ there is a natural Lagrangian two-sphere

$$L_{d, \epsilon} \subset Y_{d, \zeta_d^- + \epsilon},$$

namely the vanishing cycle of $\pi_d : Y_d \to \mathbb{C}$ associated to the critical value $\zeta_d^-$ (this is the classical vanishing cycle construction, which is the special case $X = \text{point}$ of the discussion in [B] above). Take the path $\gamma_{d, \epsilon}$ in $\mathbb{C} \setminus \{\zeta_d^\pm\}$ which runs from $\zeta_d^- + \epsilon$ to 0 along the real axis, then makes a positive full circle around $\zeta_d^+$, and finally goes back to its starting point along the real axis, see Figure 1. We want to look at the image of the vanishing cycle by parallel transport, which is another Lagrangian two-sphere

$$h_{\gamma_{d, \epsilon}}(L_{d, \epsilon}) \subset Y_{d, \zeta_d^- + \epsilon}.$$  

(37) Lemma: On each $B_d = B \cap Y_d$ one has

$$||\nabla \pi_d||^2 \geq \nu^{-1}|d|^{1/2} \min \left( |\pi_d - \zeta_d^+|, |\pi_d - \zeta_d^-| \right)$$

where $\nu > 0$ is a constant independent of $d$.

Proof: As when proving Lemma 34, we may assume that the Kähler form is standard. By using the $S^1$-action $(x, a, b, c, d) \mapsto (x, r^d a, r^d b, r^d c, r^d d)$, we may also assume that $d > 0$. A simple computation shows that

$$|\pi_d - \zeta_d^\pm| \leq |b|^2 + |c|^2 + |a| \mp \sqrt{d/3} \cdot |a \pm 2\sqrt{d/3}|.$$
On the other hand, 
\[ \|\nabla \pi_d\|^2 = 4|b|^2 + 4|c|^2 + 9|a - \sqrt{d/3}|^2 \cdot |a + \sqrt{d/3}|^2. \]
For \( re(a) \geq 0 \), \( |a + \sqrt{d/3}| \geq \frac{1}{2}|a + 2\sqrt{d/3}| \) and \( |a + \sqrt{d/3}| \geq \sqrt{d/3} \), so
\[
\|\nabla \pi_d\|^2 \geq \sqrt{d} \left( \frac{4}{\sqrt{d}} |b|^2 + \frac{4}{\sqrt{d}} |c|^2 + \frac{9}{2\sqrt{3}} |a - \sqrt{d/3}|^2 \cdot |a + 2\sqrt{d/3}| \right)
\geq \nu^{-1} |d|^{1/2} |\pi_d - \zeta_d^+|
\]
where \( \nu \) is \( \geq 2\sqrt{3}/9 \) and is also an upper bound for \( \sqrt{d}/4 \) on \( B \). The other part of the minimum in \( \ref{eq:43} \) takes care of the case \( re(a) \leq 0 \). ■

For sufficiently small \( d > 0 \), the critical points of \( \pi_d \) will lie close to 0, and so will the sphere \( L_{d,\epsilon} \) for \( \epsilon \ll d \). Using Lemma \( \ref{lem:37} \), we can now estimate the length of any flow line of the parallel transport vector field along \( \gamma_{d,\epsilon} \), as long as that line remains inside \( B \). For the straight pieces from \( \zeta_d^- + \epsilon \) to the origin and back, one gets
\[
\int \|\nabla \pi_d\|^{-1} \leq 2 \int_0^{\zeta_d^+} \nu^{1/2} d^{-1/4} s^{-1/2} ds \leq 100 \nu^{1/2} d^{1/2},
\]
and for the circle around \( \zeta_d^+ \),
\[
\int \|\nabla \pi_d\|^{-1} \leq \int_0^{2\pi \zeta_d^+} \nu^{1/2} d^{-1/4} (\zeta_d^+)^{-1/2} ds \leq 100 \nu^{1/2} d^{1/2}.
\]
Arguing as in our discussion of vanishing cycles, we arrive at the desired conclusion: for \( 0 < \epsilon \ll d \) small, parallel transport \( h_{\gamma_{d,\epsilon}} \) is well-defined near \( L_{d,\epsilon} \), and moreover the image \( \ref{eq:43} \) still lies in \( B \).
To obtain a more concrete picture, assume momentarily that $\Omega$ is the standard Kähler form on $\mathbb{C}^4$. Borrowing from [19, 40], we consider projection to the $a$-coordinate, $q_{d,z} : Y_{d,z} \to \mathbb{C}$. The fibres of this are affine quadrics, three of which are singular, corresponding to the solutions of

$$a^3 - ad - z = 0.$$  \hfill (39)

The $S^1$ part of the $\mathbb{C}^*$-action from [20] is a Hamiltonian circle action which is fibrewise with respect to $q_{d,z}$, and whose moment map is $\mu(a, b, c, d) = |c|^2 - |b|^2$. The intersection

$$C_{d,z,a} = \mu^{-1}(0) \cap q_{d,z}^{-1}(a) = \{(b, c) : |b|^2 = |c|^2, bc = -a^3 + ad + z\}$$  \hfill (40)

is a circle if $a$ is a regular value, and shrinks to a point for the singular values. To any embedded path $\alpha : [0; 1] \to \mathbb{C}$ such that $\alpha(r)^3 - \alpha(r)d - z$ vanishes exactly for $r = 0, 1$ one can associate an embedded smooth Lagrangian sphere in $Y_{d,z}$,

$$\Lambda_\alpha = \bigcup_{r=0}^1 C_{d,z,a(r)}.$$  \hfill (41)

Suppose that $d > 0$ and $z = \zeta_d^2 + \epsilon$ for $0 < \epsilon \ll d$, so that (39) has three real solutions, of which the rightmost two are close to $a = \sqrt{d/3}$.

**Figure 2:**

**Lemma:** Assuming that the Kähler form on $Y$ is standard, the vanishing cycle (36) and its monodromy image (37) are the Lagrangian spheres (41) associated to paths in $\mathbb{C}$ which are isotopic to $\alpha$ and $t_\beta(\alpha)$, respectively. Here $\alpha$ and $\beta$ are as in Figure 2 and $t_\beta$ denotes the positive half-twist around $\beta$.

**Proof:** The parallel transport vector fields on $\pi_d : Y_d \to \mathbb{C}$ are invariant with respect to the $S^1$-action (26), and $d\mu$ vanishes on them. Since the critical points are fixed points of the action and lie in $\mu^{-1}(0)$, it follows that all vanishing cycles are $S^1$-invariant and also lie in $\mu^{-1}(0)$. One sees easily that any Lagrangian sphere in $Y_{d,z}$ with these properties is necessarily of the form $\Lambda_\alpha$ for some path $\alpha$ as described above. Concerning $L_{d,\epsilon}$, one knows in addition that it must lie close to the critical point $\text{Crit}(\pi_d)^-$ which has $a = \sqrt{d/3}$, hence the corresponding $\alpha$ must stay close to that value in $\mathbb{C}$, which determines its isotopy class uniquely. The same argument as before proves that (37) is of the form $\Lambda_\alpha$.
for some path $\alpha'$. As one moves $z$ along $\gamma_{d,\epsilon}$, the two leftmost solutions of (39) get exchanged, and more precisely perform a positive half-twist around each other, moving along a circle. The path $\alpha'$ is isotopic to the image of $\alpha$ under the resulting monodromy map of the three-pointed plane, which is precisely $t_\beta$ (for more details on this last step see [19, Lemma 6.15]).

We now turn to the realistic situation. Let $T \to X$ be a holomorphic line bundle over some complex manifold, $Y = (T \setminus 0) \times \mathbb{C}$, and $\pi : Y \to \mathbb{C}^2$ the map which is equal to (34) on each $\mathbb{C}^4$ fibre. $Y_d, \pi_d$ and $Y_d,\zeta$ are defined in analogy with the notation above. We equip $Y$ with an arbitrary Kähler form $\Omega$, and restrict that to $X$ by identifying the latter space with the zero-section of $Y$.

Let $K, K'$ be two closed Lagrangian submanifolds of $X$. The first step in the construction, which was trivial in the previously considered case $X = \text{point}$, goes as follows. Normalize the cusp curve of critical values by the map $n : \mathbb{C} \to \Sigma$, $n(w) = (3w^2, 2w^3)$. The pullback of $\text{Crit}(\pi) \to \Sigma$ is the projection

$$n^*\text{Crit}(\pi) \cong X \times \mathbb{C} \to \mathbb{C},$$

and while the pullback of $\Omega$ is no longer positive, it is still nondegenerate on each fibre, which is sufficient to define symplectic parallel transport. Starting with our original Lagrangian submanifolds, which lie in the fibre over zero of (42), we get two smooth families of Lagrangian submanifolds in the fibres nearby. Changing back to the original parameter, and supposing that $d > 0$ is sufficiently close to zero, one now has Lagrangian submanifolds

$$K_d, K'_d \subset \text{Crit}(\pi_d)^-$$

which as $d \to 0$ converge to $K, K'$ respectively. We take the associated relative vanishing cycles for the map $\pi_d$, which are Lagrangian submanifolds

$$L_{d,\epsilon}, L'_{d,\epsilon} \subset Y_{d,\zeta_d^+ + \epsilon}$$

for $0 < \epsilon \ll d$ (this is actually the variant vanishing cycle construction from Remark 43 with line bundle $L = T^2$, so 43 could be nontrivial $S^2$-bundles over $K, K'$; however, that won’t happen in our application, see Remark 43 below).

Take a relatively compact open subset $U \subset X$ containing $K, K'$, and an open subset of $V \subset Y$ which, with respect to some metric on the vector bundle $Y \to X$, is the unit ball bundle over $U$. For $0 < \epsilon \ll d$, $43$ will lie inside $V$, and one has the same estimates for $\nabla \pi$ on $V$ as $37$ (they can be derived for instance by covering $\bar{U}$ with finitely many open subsets over which $T$ is trivial, and applying the previous argument to each of them). Hence, parallel transport along the path $\gamma_{d,\epsilon}$ is well-defined near $L'_{d,\epsilon}$, and yields another Lagrangian submanifold

$$L''_{d,\epsilon} = h_{\gamma_{d,\epsilon}}(L'_{d,\epsilon}) \subset Y_{d,\zeta_d^+ + \epsilon} \cap V.$$
Background: At this point, we need to recall some general facts about symplectic associated bundles. Let \((X, \omega)\) be a symplectic manifold, and \(\mathcal{L} \to X\) a complex line bundle with a hermitian metric and compatible connection. Let \(P \subset \mathcal{L}\) be the unit circle bundle, and \(\alpha \in \Omega^1(P)\) the connection one-form. Our normalization is that if \(R\) is the rotational vector field on \(P\) (whose orbits are 2\(\pi\)-periodic), then \(\alpha(R) \equiv 1\). Let \((M, \eta)\) be any symplectic manifold with a Hamiltonian circle action, whose Killing field is \(Z\) and whose moment map is \(\mu\), so \(i_Z \eta = -d\mu\). The associated symplectic fibre bundle is

\[
Y = P \times_{S^1} M \to X.
\]

Take the two-form \(\Omega = \omega + \eta + d(\alpha \mu)\) on \(P \times M\), where \(\omega\) is pulled back via \(P \to X\). This satisfies \(\Omega((R, -Z), \cdot) = 0\), hence it descends in a unique way to a two-form on \(Y\), also called \(\Omega\). Take \(V_1, V_2 \in T_x X\) and lift them in the unique way to horizontal tangent vectors \(V_1^e, V_2^e\) on the circle bundle, so \(\alpha(V_1^e) = 0\). Take also \(W_1, W_2 \in T_m M\), and project \((V_1^e, W_k)\) to tangent vectors in the quotient \(Y\). Then

\[
\Omega((V_1^e, W_1), (V_2^e, W_2)) = \eta(V_1, V_2) + \omega(W_1, W_2) + \mu(m)d\alpha(V_1^e, V_2^e).
\]

This shows that \(\Omega\) is nondegenerate, hence a symplectic form, in a neighbourhood of \(P \times_{S^1} \mu^{-1}(0) \subset Y\). Moreover, if \(K\) is a Lagrangian submanifold of \(X\), and \(L\) an \(S^1\)-invariant Lagrangian submanifold of \(M\) such that \(\mu|L \equiv 0\), then the associated \(L\)-bundle over \(K\),

\[
\Lambda = (P|K) \times_{S^1} L,
\]

is a Lagrangian submanifold of \((Y, \Omega)\).

Now assume that \(X\) and \(M\) are Kähler, \(\mathcal{L}\) is a holomorphic line bundle and the connection is compatible with this structure, and the circle action on \(M\) is part of a holomorphic \(\mathbb{C}^*\)-action \(\rho\). One can then identify \(Y\) with the holomorphic associated bundle \((\mathcal{L} \setminus 0) \times_{\mathbb{C}^*} M\), and \(\Omega\) is Kähler near \(P \times_{S^1} \mu^{-1}(0)\). To see this, take a nowhere vanishing holomorphic section \(e\) of \(\mathcal{L}\) over some open subset \(U\). This defines a holomorphic trivialization of \((\mathcal{L} \setminus 0) \times_{\mathbb{C}^*} M\) over \(U\). Its normalized form \(e/\|e\|\) gives a corresponding trivialization of \(P \times_{S^1} M\), with respect to which the connection one-form is \(\alpha = -d\log \|e\|\). The difference between the two trivializations is the map

\[
U \times M \to U \times M, \quad (x, m) \mapsto (x, \rho_{\exp(h(x))}(m)),
\]

involving the radial part of the \(\mathbb{C}^*\)-action and \(h = -\log \|e\|\). The pullback of \(\Omega\) by \(\mathbb{H}\) is \(\eta + \rho^*_{\exp(h)}\omega - dh \wedge \rho^*_{\exp(h)}(d^c \mu) + d^c h \wedge \rho^*_{\exp(h)}d\mu - dd^c h \cdot \rho^*_{\exp(h)}\mu\), which is obviously of type \((1, 1)\).

Returning to our discussion: starting with a Kähler form on \(X\), a hermitian metric on \(\mathcal{F}\) and the standard Kähler form on \(\mathbb{C}^4\), we construct an associated form \(\Omega\) on \(Y = \mathcal{F} \times_{\mathbb{C}^*} \mathbb{C}^4\) which is Kähler at least in a neighbourhood of the set.
Given \( (d, z) \in \mathbb{C}^2 \setminus \Sigma \), a Lagrangian submanifold \( K \subset X \), and a path \( \alpha \) in \( \mathbb{C} \) of the same kind as before, one can define a Lagrangian submanifold

\[
\Lambda_{d, z, K, \alpha} \subset Y_{d, z}
\]

by starting with the Lagrangian sphere \( \text{11} \) and applying the construction \( \text{45} \). Under the map \( Y_{d, z} \to X \), this is an \( S^2 \)-bundle over \( K \). The following result is the generalization of Lemma \( \text{38} \) and has the same proof:

\[\text{(40) Lemma: } \text{Suppose that we use the Kähler form obtained from a Kähler form on } X, \text{ a hermitian metric on } F, \text{ and the standard Kähler form on } \mathbb{C}^4. \text{ Let } K, K' \subset X \text{ be closed Lagrangian submanifolds, and consider the Lagrangian submanifolds from } \text{43}, \text{44}. \text{ Then}
\]

\[
L_d, \epsilon = \Lambda_{d, \zeta^2 + \epsilon, K, \alpha}, \quad L''_d, \epsilon = \Lambda_{d, \zeta^2 + \epsilon, K', \alpha'}
\]

for paths \( \alpha \) isotopic to the one from Figure \( \text{2} \) and \( \alpha' \) isotopic to \( t_\beta(\alpha) \). \( \blacksquare \)

\[\text{(D) Floer cohomology: background} \]

Let \( (M, \omega) \) be a Kähler manifold such that \( \omega \) is exact and the underlying complex structure is Stein, meaning that there is an exhausting plurisubharmonic function \( \psi \). We assume that \( c_1(M) = 0 \) and \( H^1(M) = 0 \). Let \( L, L' \) be closed connected Lagrangian submanifolds of \( M \) with \( H_1(L) = H_1(L') = 0 \) and \( w_2(L) = w_2(L') = 0 \). Then there is a well-defined Floer cohomology group

\[
HF(L, L') = H(CF(L, L'), d_J)
\]

which is a finitely generated, relatively graded abelian group. One can of course replace \( \mathbb{Z} \) with any other abelian coefficient group, and the universal coefficient theorem holds as usual. Relatively graded means that there is a \( \mathbb{Z} \)-grading which is unique up to an overall constant shift. We recall briefly the definition, which is essentially Floer’s original one \( \text{8} \) except for the orientations of moduli spaces, which come from \( \text{11} \). First, after a small Lagrangian perturbation, one may assume that the intersection \( L \cap L' \) is transverse. One then defines the Floer chain complex to be the abelian group

\[
CF(L, L') = \bigoplus_{x \in L \cap L'} O_x
\]

where \( O_x \) is the orientation group of \( x \). Formally, this is an abelian group canonically associated to \( x \) and generated by two elements – labelled the possible “coherent orientations” of \( x \) – with the relation that the sum of these orientations is zero. Hence, \( O_x \cong \mathbb{Z} \) but not canonically so. The association to \( x \) of a well-defined orientation group proceeds essentially as in the case of Hamiltonian
Floer cohomology [9], but there is a “family index anomaly” due to which the consistency of the definition requires Spin structures on $L$ and $L'$ [11]. Of course, our standing topological assumptions on $L$ and $L'$ imply that such structures exist and are unique. Similarly, one can associate to any pair of intersection points $x, y$ a relative Maslov index $\Delta \mu(x, y) \in \mathbb{Z}$, which satisfies $\Delta \mu(x, y) + \Delta \mu(y, z) = \Delta \mu(x, z)$ and establishes the relative grading. The differential $d_J$ is defined by considering solutions of Floer’s equation

\[
\begin{cases}
  u : \mathbb{R} \times [0; 1] \to M, \\
  u(s, 0) \in L, \quad u(s, 1) \in L', \\
  \partial_s u + J_t(u) \partial_t u = 0, \\
  \lim_{s \to \pm \infty} u(s, \cdot) = x_\pm
\end{cases}
\]

where $J = (J_t)_{0 \leq t \leq 1}$ is a generic smooth family of $\omega$-compatible almost complex structures, which all agree with the given complex structure outside a compact subset, and $x_\pm \in L \cap L'$. More precisely, $d_J(x_+) = \sum_{x_-} n_{x_+, x_-} x_-$ where $n_{x_+, x_-}$ counts isolated solutions $u$ of (48) (mod translation in the $s$-variable), with a sign that can be canonically encoded as an isomorphism $\delta_u : O_{x_+} \cong O_{x_-}$.

The geometric assumptions set out at the beginning of the section enter in two crucial ways (one elementary, and one going back to Floer [8]). First, the fact that our almost complex structures are standard at infinity, together with the Stein property of $M$, allows one to apply the maximum principle to $\psi \circ u$; as a consequence, all solutions of (48) remain within a fixed compact subset of $M$. Secondly, the exactness of $\omega$ and the fact that $H^1(L) = H^1(L') = 0$ ensure that there is a well-defined action functional on the space of paths from $L$ to $L'$, which implies that the moduli spaces of (48) have well-behaved compactifications (bounded energy, no bubbling). The other assumptions are of lesser importance, though of some relevance given Conjecture [2]. Existence of spin structures on $L$ and $L'$ enables one to define Floer theory with integral rather than mod 2 coefficients; vanishing of $c_1(M)$ and $H^1(M)$ are respectively relevant to the existence and uniqueness of a $\mathbb{Z}$-grading in Floer cohomology, as discussed in Section [1A].

(41) REMARK: Even if the Kähler form is defined only on some open subset $U \subset M$ which is holomorphically weakly convex (meaning that holomorphic discs cannot touch $\partial U$ from the inside, unless they are completely contained in $\partial U$), one can still define Floer cohomology groups for Lagrangian submanifolds $L, L' \subset U$ satisfying the conditions set out above. The reason is that to achieve transversality of the moduli spaces, it is sufficient to consider almost complex structures $J_t$ which agree with the given complex structure outside any given neighbourhood $U'$ of $L \cup L'$. By taking a $U'$ whose closure is inside $U$ and applying weak convexity, one sees that all solutions of (48) remain in $U$, so that their energy can be estimated by the symplectic area and hence the action functional. The most obvious application of this is to sublevel sets $U = \{ \psi < C \}$.
The most important aspect of Floer cohomology is its strong invariance properties. A Hamiltonian isotopy of \( M \) is a path \((g_t)\) of symplectomorphisms starting at \( g_0 = \text{id} \) and defined by the flow of the vector field associated to some smooth time-dependent function \( H_t \), for which \( H : M \times I \to \mathbb{R} \) has compact support. Floer [8] proved invariance of \( HF^*(L, L') \) under Hamiltonian isotopies of either \( L \) or \( L' \). Now let \( \{L_t = \phi_t(L)\}_{t \in [0,1]} \) be an arbitrary Lagrangian isotopy of a closed Lagrangian submanifold \( L \) in an exact symplectic manifold \((M, \omega, \theta)\), where \( d\theta = \omega \). The isotopy is said to be exact if \( [\theta]_{L_t} \in H^1(L_t) \cong H^1(L) \) is constant. Note that this makes sense, in that there is a canonical identification \( H^1(L_t) \cong H^1(L_0) \) for all \( t \). By explicitly writing down and integrating the appropriate vector fields, one sees that such an isotopy can be embedded in a global ambient Hamiltonian isotopy of \( M \); this is analogous to the characterisation of Hamiltonian symplectomorphisms as those having zero flux, cf. [28, Chapter 10]. That is, there is some \( (g_t) \) with \( g_t(L) = L_t \). Under our standing assumptions \( H^1(L) = H^1(L') = 0 \) exactness is automatic, and it follows that the Floer cohomology \( HF^*(L, L') \) is invariant under arbitrary Lagrangian isotopies.

For a second invariance property, suppose that we have an isotopy \((\omega_s)\) of symplectic forms on \( M \), together with closed submanifolds \( L, L' \) which are \( \omega_s \)-Lagrangian for every \( s \). Suppose as usual that \( H_1(L) = H_1(L') = 0 \) and that the other geometric assumptions required for well-definition of Floer cohomology hold. In particular, suppose that all the \( \omega_s \) are Kähler forms making \( M \) geometrically bounded at infinity, for instance making \( M \) Stein. Then we claim \( HF^*(L, L') \) is independent of the particular symplectic form \( \omega \) (given that the Floer differential counts solutions to an equation defined without explicit mention of \( \omega \) this is perhaps not as surprising as it first seems). The result is proved using a parametrised version of the Floer equation [48], as in Floer’s original [8]. The main technical difficulties stem from the parameter values where birth-death processes occur for the intersection points of the Lagrangian submanifolds; a careful treatment of these issues has been given in [28]. Note that the discussion in [26] analyses bifurcations occurring in rather general “one-parameter homotopies of Floer data”, and applies equally to a parametrized Floer equation in which the almost complex structures are compatible with a smoothly varying family of symplectic forms. Another approach would be to combine parametrized moduli spaces with the continuation map technique, noting the energy bounds required for compactness of spaces of solutions to the continuation map equation carry over essentially as usual to this case. A closely related but more difficult statement – proving invariance of symplectic homology of Stein manifolds under continuous variation of the symplectic form – was proved by Viterbo in [44].

Combining the two statements, if one has a holomorphic submersion \( Y \to T \) whose fibres are Stein, a path \( \gamma : [0;1] \to T \) and families \( L_r, L'_r \) of closed Lagrangian submanifolds in the fibres \( Y_{\gamma(r)} \), with the required additional conditions to make \( HF^*(L_r, L'_r) \) well-defined, then it is the same for all \( r \). A helpful, albeit informal, general principle is thus that Floer cohomology is invariant un-
der smooth deformation of all geometric objects involved, as long as one remains within the class where it is well-defined.

(E) Floer cohomology: computations

We will need two simple Floer cohomology computations for the geometric situations studied in Sections 3(B) and (C). They have the flavour of a “Künneth formula” and “Thom isomorphism” respectively. The first computation takes place in the following context:

- $Y$ is a complex manifold with a holomorphic function $\pi : Y \to \mathbb{C}$. We have a complex submanifold $X \subset Y$ and an isomorphism between a neighbourhood of that submanifold and a neighbourhood of $X \times \{0\}^3 \subset X \times \mathbb{C}^3$, such that the following diagram commutes:

$$
\begin{array}{ccc}
Y & \xrightarrow{\text{local \, \, \, \, \, \, \, \, defined \, near}} & X \times \mathbb{C}^3 \\
\downarrow \pi & & \downarrow \begin{pmatrix} a^2 + b^2 + c^2 \end{pmatrix} \\
\mathbb{C} & \xrightarrow{\text{(49)}} & \mathbb{C}
\end{array}
$$

- $Y$ is Stein, carries an exact Kähler form $\Omega$, and satisfies $c_1(Y) = 0$ (which implies that $c_1(Y_t)$ and $c_1(X)$ also vanish). Moreover, $H^1(Y_t) = 0$ for small $t \neq 0$, and $H^1(X) = 0$.

We equip $X$ and the smooth fibres $Y_t$ with the restrictions of $\Omega$. Let $K, K'$ be closed Lagrangian submanifolds of $X$ which have the properties necessary to define $HF(K, K')$, and consider for sufficiently small $t \neq 0$ the associated relative vanishing cycles $L_t, L'_t \subset Y_t$. Since these are products of $K, K'$ with $S^2$, their Floer cohomology $HF(L_t, L'_t)$ is again well-defined, and is independent of $t$ by the basic invariance principle discussed above.

(42) Lemma: $HF(L_t, L'_t) \cong HF(K, K') \otimes H^*(S^2)$, where $H^*(S^2)$ carries its standard grading.

Proof: One can find finitely many holomorphic functions $\phi_1, \ldots, \phi_l : Y \to \mathbb{C}$ whose common vanishing set is $X$ (this is a general result about complex submanifolds of Stein manifolds [11]; however, note that all our applications will be in the affine algebraic context where the counterpart is trivial, so we are appealing to it only to keep the current exposition general). Take a sublevel set $U = \{\psi(y) < C\} \cap X \subset X$ which contains both $K$ and $K'$, and consider the open subset $V = \{\psi(y) < C, \; |\phi_1(y)| < \delta, \ldots, |\phi_l(y)| < \delta\}$ for some $\delta > 0$. Since $U$ is relatively compact, one can make $\delta$ sufficiently small so as to ensure that $V$ is contained in the neighbourhood of $X$ where the isomorphism [11] is
defined. By taking $t$ small, one can achieve that the relative vanishing cycles $L_t, L'_t$ lie in $Y_t \cap V$. By definition $V$ is holomorphically weakly convex, so for a suitable choice of almost complex structure, the definition of $HF(L_t, L'_t)$ is local, meaning that all solutions of (48) stay inside $Y_t \cap V$.

On $V$ there is another Kähler form $\Omega^{(1)}$, which is obtained by taking the product of $\Omega|_X$ and the standard form on $\mathbb{C}^3$, and pulling that back by (49). We can consider the linear family of Kähler forms $\Omega^{(s)}$ interpolating between $\Omega^{(0)} = \Omega|_V$ and $\Omega^{(1)}$. For each $s \in [0; 1]$ there are relative vanishing cycles constructed from $K, K'$ using $\Omega^{(s)}$. These will be well-defined inside $Y_t \cap V$ for sufficiently small $t \neq 0$ (one can see this explicitly by estimating the parallel transport vector field, using Lemma 34), and the Floer cohomology is local in the same sense as before. It follows that $HF(L_t, L'_t)$ is isomorphic to the Floer cohomology of the corresponding vanishing cycles for $\Omega^{(1)}$. But in the coordinates given by the isomorphism (49), these cycles are simply $K \times \sqrt{t}S^2$ and $K' \times \sqrt{t}S^2$, compare (34). At this point, the Künneth isomorphism in Floer cohomology (which has the same essentially trivial proof as in ordinary Morse theory) finishes the argument.

The second situation we are concerned with is as follows:

- $Y$ is a complex manifold with a holomorphic map $\pi : Y \to \mathbb{C}^2$. We have a complex submanifold $X \subset Y$ equipped with a holomorphic line bundle $\mathcal{F}$, and an isomorphism between a neighbourhood of that submanifold and a neighbourhood of the zero-section inside $(\mathcal{F} \setminus 0) \times_{\mathbb{C}^*} \mathbb{C}^4$, where the associated bundle is formed with respect to (26). This should fit into a commutative diagram

$$
\begin{array}{ccc}
Y & \xrightarrow{\text{local } \mathcal{F} \text{ defined near } X} & (\mathcal{F} \setminus 0) \times_{\mathbb{C}^*} \mathbb{C}^4 \\
\pi \downarrow & & \downarrow p \\
\mathbb{C}^2 & \xrightarrow{\text{ }} & \mathbb{C}^2,
\end{array}
$$

(50)

where $p$ is given on each fibre by (9).

- $Y$ is Stein, carries an exact Kähler form $\Omega$, and satisfies $c_1(Y) = 0$ (which implies that $c_1(Y_{d,z})$, for a regular value $(d, z)$ of $\pi$, and $c_1(X)$ also vanish). We also require $H^1(Y_{d,z}) = 0$ for small regular values $(d, z)$, and $H^1(X) = 0$. Finally, $\mathcal{F}$ should be a subbundle of the trivial $\mathbb{C}^2$-bundle over $X$.

Take closed Lagrangian submanifolds $K, K' \subset X$, satisfying the conditions above so in particular $HF(K, K')$ is well-defined. The construction in (C) above associates to these new Lagrangian submanifolds as in (38), (44) inside the fibre.
\[ Y_{d, \zeta - \epsilon} \text{ for } 0 < \epsilon \ll d. \]

**Remark:** The last-made assumption in the list above implies that there is a short exact sequence of vector bundles \( 0 \to \mathcal{F} \to \mathbb{C}^2 \to \mathcal{F}^{-1} \to 0. \) Because \( X \) is Stein, it follows that \( \mathcal{F} \oplus \mathcal{F}^{-1} \) is trivial, and therefore so is the vector bundle
\[
(\mathcal{F} \setminus 0) \times_{\mathbb{C}} \mathbb{C}^4 = \mathbb{C} \oplus \mathcal{F}^{-2} \oplus \mathcal{F}^2 \oplus \mathbb{C} = (\mathcal{F} \oplus \mathcal{F}^{-1}) \otimes \mathbb{C}^2.
\]

Another consequence is that \( P(\mathcal{F}^2 \oplus \mathbb{C}) \) is trivial, which by construction means that the Lagrangian submanifolds we constructed will be diffeomorphic to \( K \times S^2, K' \times S^2 \) respectively. In particular they again satisfy \( w_2 = 0, \) so the Floer cohomology \( HF(L_{d, \epsilon}, L'_{d, \epsilon}) \) is well-defined.

**Lemma:** \( HF(L_{d, \epsilon}, L''_{d, \epsilon}) \cong HF(K, K'). \)

**Proof:** Suppose that \( K, K' \) intersect transversally. The first steps of the proof are the same as in Lemma 42: one can achieve that the Lagrangian submanifolds concerned lie inside the set where the local isomorphism (50) is defined, and moreover Floer cohomology can be localized to that subset. This means that from now on, our computations will all take place in the fibre bundle
\[ (\mathcal{F} \setminus 0) \times_{\mathbb{C}} \mathbb{C}^4 \to X. \] (51)

Moreover, one can replace the given Kähler form \( \Omega \) by an \( \Omega^{(1)} \) which is constructed from \( \Omega|_X, \) a hermitian metric on \( \mathcal{F}, \) and the standard form on \( \mathbb{C}^4 \) (\( \Omega^{(1)} \) is strictly speaking defined only in a neighbourhood of the zero-section, but all our arguments will take place inside that neighbourhood). The Lagrangian submanifolds can then be described explicitly as in Lemma 40, in fact after a suitable Lagrangian isotopy we may assume that

\[
L_{d, \epsilon} = \Lambda_{d, \zeta - \epsilon, K, \alpha}, \quad L''_{d, \epsilon} = \Lambda_{d, \zeta - \epsilon, K', \beta(\alpha)}.
\] (52)

where the paths \( \alpha, \beta(\alpha) \) are as in Figure 2, as opposed to merely lying in the same isotopy class. Note that the two paths intersect only in one endpoint, which is the rightmost solution \( a_+ \) of \( a^3 - ad + bc = \zeta - \epsilon. \) By construction this implies that in the fibre over each point \( x \in K \cap K', \) there is a unique (and transverse) intersection point \( y \in L_{d, \epsilon} \cap L''_{d, \epsilon}, \) given by \( b = c = 0 \) and \( a = a_+. \)

Consider solutions \( u \) of Floer’s equation for the Lagrangian submanifolds inside using the standard complex structure as \( J. \) Under projection to the \( a \)-variable, \( u \) gets mapped to a finite energy holomorphic map \( \mathbb{R} \times [0; 1] \to \mathbb{C} \) whose boundary lies on the paths \( \alpha, \beta(\alpha). \) This is necessarily constant equal to \( a_+, \) and that implies that \( u \) lies in the subset of where \( a = a_+, \) and because the only points of our Lagrangian submanifolds which satisfy \( a = a_+ \) also have \( b = c = 0, \) we find that the \( (b, c) \) components of \( u(s, t) \) vanish for \( t = 0, 1. \) The \( b \)
component, for instance, can be viewed as a holomorphic section of the pullback bundle
\[ u^*F^{-2} \to \mathbb{R} \times [0; 1] \]
which is zero along \( \mathbb{R} \times \{0; 1\} \), so unique continuation shows that it is identically zero; similarly for the \( c \)-component. This sets up a bijective correspondence between solutions of Floer’s equation in our fibre bundle (51) with boundary conditions (52), and those of the corresponding equation in \( X \) with boundary condition \( K, K' \) (the correspondence is given by projection in one direction, and conversely by lifting to the submanifold \( X_+ \subset Y \) defined by \( a = a_+, b = c = 0 \)).

This would be a way of proving our result if we could assume that the standard complex structure was regular in the sense of Floer theory for the pair \( (K, K') \).

That is not a realistic assumption, but the way to repair the argument is a standard exercise in Floer theory, so we will only sketch it (one could even claim that this step is trivial, at the price of relying on somewhat complicated virtual perturbation theory).

Fix small neighbourhoods \( U_1 \subset \bar{U}_1 \subset U_2 \) of \( X_+ \) inside (51). On the base \( X \), choose \( \bar{J} = (\bar{J}_t) \) which is a small compactly supported perturbation of the standard complex structure, making the solutions of Floer’s equation for \( (K, K') \) regular. From \( \bar{J} \) and the given connection on \( F \) one gets an induced \( t \)-dependent almost complex structure on (51) which is compatible with our symplectic form.

We choose a \( J = (J_t) \) which agrees with this induced almost complex structure on \( U_1 \), and is equal to the standard complex structure outside \( U_2 \). This can be done in such a way that \( J \) is still everywhere a small perturbation of the standard complex structure, and a Gromov compactness argument will tell us that all solutions of Floer’s equation for this \( J \) are contained in \( U_1 \), so that for all practical purposes this is the almost complex structure induced from \( \bar{J} \) and the connection. As a consequence, we still have all the properties used in the argument above \( (J_t \)-holomorphicity of the projection to the \( a \)-variable, the fact that the \( b \) component can be viewed as a holomorphic section of (53) and similarly for \( c \), and \( (J_t, \bar{J}_t) \)-holomorphicity of projection to the base \( X \)), but now with the added benefit of regularity.

\[ \blacksquare \]

5 Symplectic geometry of \( \chi|\mathcal{S}_m \)

With the symplectic techniques at hand, we now return to the specifics of \( \chi|\mathcal{S}_m \). In particular we define the Lagrangian submanifolds \( L_\varphi \), the monodromy maps \( h^\text{gau}_{\varphi} \), and therefore \( Kh_{\text{sym}} \). Its invariance under the Markov moves (and its behaviour under adding unlinked unknotted components) is a swift consequence of the preceding material. As in the previous section we deal with Floer cohomology as a relatively graded group only, but this will be remedied in the following section.
(A) Open braids and parallel transport maps

Take the affine transverse slice $S_m$ from Section 3 inside $g = sl_2$. Recall that points $t \in Conf_{2m}(C) \subset h/W$, which correspond to pairwise different eigenvalues $(\mu_1, \ldots, \mu_{2m})$ with

$$\mu_1 + \cdots + \mu_{2m} = 0,$$ (54)

are regular values of $\chi|S_m$. We denote by $Y_{m,t} = (\chi|S_m)^{-1}(t)$ the fibre over $t$. Occasionally we will extend this notation to points of the whole configuration space $Conf_{2m}(C)$, with the understanding that in that case, $Y_{m,t}$ is really the fibre over the normalized configuration $(\mu_1 - (\sum_k \mu_k)/2m, \ldots, \mu_{2m} - (\sum_k \mu_k)/2m)$.

(45) Lemma: $H^1(Y_{m,t}) = 0$.

This can be read off from the literature. Because of the simultaneous resolution, $Y_{m,t}$ is diffeomorphic to the fibre over 0 of $\tilde{S}_m \rightarrow h$. By [41, p. 50] that fibre deformation retracts onto its “compact core”, which is the preimage of the nilpotent $x = n^+$ under $\tilde{S}_m \rightarrow S_m$, or equivalently by definition the set of flags stabilized by $n^+$. The inclusion of that set into the full flag variety induces a surjective map on cohomology [41, p. 60], and of course that variety has $H^1 = 0$. Alternatively one can follow [15] and appeal to the presentation of the entire algebra $H^* (Y_{m,t})$ given in [7].

When it comes to choosing a Kähler metric on $S_m$, we are guided by the requirements of the (rescaled) parallel transport construction and the proof of Lemma 21. Fix some real number $\alpha > m$. For each $i = 2, 4, \ldots, 2m$ take the functions $\xi_i(z) = |z|^{\alpha/i}$ on $C$. By applying $\xi_i$ to each coordinate of $S_m$ on which $\lambda$ acts with weight $i$, which in terms of (23) means each entry of $y_{ii}$, and summing up all these terms, one gets a proper $C^1$-function $\xi$ on $S_m$. Now find compactly supported functions $\eta_k$ on $C$ such that $\psi_i = \eta_i + \xi_i$ is $C^\infty$, and add the $\psi_i$ in the same way as before to form another function $\psi$ on $S_m$.

(46) Lemma: $\psi$ is asymptotically homogeneous for the radial part of the $C^*$-action $\lambda$, in the sense that

$$\lim_{r \rightarrow \infty} \frac{\psi \circ \lambda_r}{r^{2\alpha}} = \xi$$

where the convergence is uniform in $C^1$-sense.

Proof: As $r \rightarrow \infty$, the rescaled functions $\eta_i(r^i z)$ on $C$ are supported on progressively smaller neighbourhoods of the origin. Their $C^0$-norms are of course uniformly bounded, and their derivatives grow like $r^i$. Since $i \leq 2m < 2\alpha$, the limit $\eta_i(r^i z)/r^{2\alpha}$ goes to zero uniformly. ■
Let $\tilde{\xi}, \tilde{\psi}$ be the lifts of $\xi, \psi$ to the simultaneous resolution $\tilde{S}_m$.

(47) Lemma: The union of the critical points of $\tilde{\psi}$ on the fibres of $\tilde{\chi}|\tilde{S}_m$ forms a subset of $\tilde{S}_m$ which projects properly to $h$.

Proof: Suppose on the contrary that there is a sequence of points $\tilde{y}_j \in \tilde{S}_m$ which are critical points in the respective fibres of $\tilde{\chi}|\tilde{S}_m$, such that $\tilde{y}_j$ goes to infinity but $\tilde{\chi}(\tilde{y}_j)$ remains bounded. After rescaling with a suitable sequence $\lambda_j$ and applying Lemma 46, one obtains a limiting point $\tilde{y} \in \tilde{S}_m$ whose projection to $S_m$ lies on the unit sphere for the obvious identification $S_m \cong \mathbb{C}^{4m-1}$, such that $\tilde{\chi}(\tilde{y}) = 0$, and which is a critical point for $\tilde{\xi}$ on $\tilde{\chi}^{-1}(0) \cap \tilde{S}_m$. But that is impossible due to the homogeneity of $\xi$. ■

We will assume from now on that the $\eta_k$ have been chosen in such a way that $-\dd^c \psi_k > 0$ everywhere, and equip $S_m$ with the metric defined by the Kähler form $\Omega = -\dd^c \psi$.

(48) Lemma: Outside a compact subset, we have an inequality $||\nabla \psi||^2 < \rho \psi$ for some $\rho > 0$.

Proof: An explicit computation shows that $||\nabla \psi_k||^2 \leq c + \rho \psi$ for some $c, \rho > 0$. Since our metric is the product of Kähler metrics on each coordinate, we can add this up and suitably adjust the constants, to get $||\nabla \psi||^2 \leq c + (\rho/2)\psi$. But $(\rho/2)\psi > c$ outside a compact subset. ■

In view of the two Lemmas above, the argument from Section 4(A) shows that the family $\chi^{-1}((\text{Conf}_{2m}(\mathbb{C})) \cap S_m \to \text{Conf}_{2m}(\mathbb{C})$ has well-defined rescaled parallel transport maps (defined on arbitrarily large compact subsets of the fibres, or even on the entire fibres if one is willing to take the slightly more complicated route indicated in Remark 30). If $\beta$ is a piecewise smooth path $[0; 1] \to \text{Conf}_{2m}(\mathbb{C})$, the associated rescaled parallel transport is denoted by

$$h_{\beta}^{\text{res}} : Y_{m,\beta(0)} \to Y_{m,\beta(1)}.$$  \hspace{1cm} (55)

As before, we extend this notation to arbitrary open braids, which are paths in $\text{Conf}_{2m}(\mathbb{C})$, with the understanding that one translates each $\beta(s)$ by an $s$-dependent amount so that (54) is again satisfied.

We will also need a version of this discussion for the critical point set fibration (29). The base space $\mathfrak{h}^{\text{main, res}}/W^{\text{main}}$ of that can be identified with $\text{Conf}_{2m-2}(\mathbb{C})$ by forgetting the first two eigenvalues. The total space comes with a natural map $\mathcal{C}_m \to S_m$ which is an embedding on each fibre, and we pull back the Kähler form by it. As explained in Section 3(D), the arguments from Lemma 21 can be easily adapted to show that (29) is a differentiable fibre bundle. A combination of these arguments and the ones used above proves that there are
well-defined parallel transport maps

\[ h^\text{sec}_{\tilde{\beta}} : \mathcal{C}_{m,\tilde{\beta}(0)} \to \mathcal{C}_{m,\tilde{\beta}(1)} \quad (56) \]

for any path \( \tilde{\beta} \) in \( \text{Conf}_{2m-2}(\mathbb{C}) \). Lemma 22 says that the fibre \( \mathcal{C}_{m,t} \) over a point \( t \in \text{Conf}_{2m-2}(\mathbb{C}) \subset \text{Conf}_{2m-2}(\mathbb{C}) \) can be identified with the corresponding space \( \mathcal{Y}_{m-1,t} \). This is compatible with our choice of symplectic forms (provided one takes the same \( \alpha \) and functions \( \psi_k \) for both \( m \) and \( m - 1 \), see Lemma 53 below). Hence, if \( \tilde{\beta} \) lies in \( \text{Conf}_{2m-2}(\mathbb{C}) \) then the parallel transport \( (56) \) is the same as the corresponding map \( (56) \) for \( m - 1 \). Note that even though there is no canonical isomorphism \( \mathcal{C}_{m,t} \cong \mathcal{Y}_{m-1,t} \) for general \( t \in \text{Conf}_{2m-2}(\mathbb{C}) \), one can partially remedy this by moving \( t \) into the subset \( \text{Conf}^0_{2m}(\mathbb{C}) \) by translation, and then combining the isomorphism defined there with parallel transport \( (56) \) along the family of translated configurations to get back to the original fibre.

(B) Lagrangian submanifolds from matchings

Take \( t = (\mu_1, \ldots, \mu_{2m}) \in \text{Conf}_{2m}(\mathbb{C}) \). A crossingless matching \( \varphi \) with endpoints \( t \) is a collection of \( m \) disjoint embedded unoriented arcs \( (\delta_1, \ldots, \delta_m) \) in \( \mathbb{C} \) which join together the points of \( t \) in pairs. For the moment, we include an ordering of the arcs as part of the data (although that will be dropped at some point later on), and order the configuration correspondingly, so that \( \delta_k \) has endpoints \( \mu_{2k-1}, \mu_{2k} \). We will associate to each such \( \varphi \) a Lagrangian submanifold

\[ L_{\varphi} \subset \mathcal{Y}_{m,t} \quad (57) \]

which is diffeomorphic to \( (S^2)^m \) and unique up to Lagrangian isotopy. Choose a path \( [0; 1] \to \text{Conf}_{2m}(\mathbb{C}) \) starting at \( t \) which moves the points as follows: the endpoints of \( \delta_2, \ldots, \delta_m \) remain fixed, and the two endpoints of \( \delta_1 \) move towards each other along that arc, colliding in the limit \( s \to 1 \). For simplicity we assume that the arc is a straight line near its midpoint, and that the colliding points move towards each other with the same speed for \( s \) close to 1. After translating to meet the normalization condition \( (54) \) at all times, we get a path \( \gamma : [0; 1] \to h/W \) such that the point \( \gamma(1) \) corresponds to a collection of eigenvalues \( (\mu_1', \ldots, \mu_{2m}') \) where \( \mu_1' = \mu_2' \), and \( \mu_k' = \mu_k + \mu_1/(m-1) - \mu_1'/((m-1) \) for \( k \geq 3 \). Note that all eigenvalues except the first two are pairwise distinct.

For \( m = 1 \) the construction is straightforward. \( \chi : S^1 \to h/W = \mathbb{C} \) has a single nondegenerate critical point in the fibre over \( \gamma(1) = 0 \), hence in the nearby fibres \( \gamma(1-s) \) for small \( s \) we have an associated vanishing cycle, which is a Lagrangian two-sphere. We then use reverse parallel transport along \( \gamma|[0; 1-s] \) to move this back to the fibre \( \mathcal{Y}_{1,t} \), which gives us \( (57) \). In the general case, one proceeds by induction on \( m \). Let \( \tilde{\varphi} \) be the crossingless matching obtained from \( \varphi \) by removing the component \( \delta_1 \), and \( t \in \text{Conf}_{2m-2}(\mathbb{C}) \) its endpoints. By assumption, there is a well-defined Lagrangian submanifold \( L_{\tilde{\varphi}} \in \mathcal{Y}_{m-1,t} \).
Lemma 25 says that one can identify $Y_{m-1,i}$ with the fibre of (29) over the point $(\mu_3 + \mu_1 / (m-1), \ldots, \mu_{2m} + \mu_1 / (m-1)) \in Conf_{2m-2}(C) \cong h^{\text{mult},\text{reg}} \backslash W^{\text{mult}}$. Using the parallel transport maps (56) over a path which translates this configuration, one can move the Lagrangian submanifold to the fibre of (29) over $(\mu_3', \ldots, \mu_{2m}')$, which by our discussion in Section 3(D) is the singular locus of $Y_{m,\gamma(1)}$. The local model from Lemma 27 shows that one can apply the relative vanishing cycle construction, which yields a Lagrangian submanifold in the fibre $Y_{m,\gamma(1-s)}$ for small $s$. As before, reverse parallel transport is then used to move this to the original fibre, which gives rise to (57). Topologically, the relative vanishing cycle procedure takes the product of a given Lagrangian submanifold with $S^2$, hence the outcome is diffeomorphic to $(S^2)^m$ as claimed. While the construction involves many choices, none of them carries any nontrivial topology (the space of possible choices in each step is path-connected, and indeed weakly contractible), so that the outcome is well-defined up to Lagrangian isotopy.

There is another property which follows directly from the definition. Namely, suppose that we have an open braid $\beta : [0; 1] \rightarrow Conf_{2m}(C)$ and a smooth family $\wp(s)$ of crossingless matchings with endpoints $\beta(s)$. Then one can construct the $L_{\wp(s)} \subset Y_{m,\beta(s)}$ in such a way that they depend smoothly on $s$. By using parallel transport over $\beta([s; 1])$ to carry them into a common fibre, one finds that there is a Lagrangian isotopy

$$L_{\wp(1)} \simeq h^{\text{ord}}_{\beta}(L_{\wp(0)}).$$

As a particular obvious special case, a smooth family of crossingless matchings with the same endpoints leads to a family of isotopic Lagrangian submanifolds.

The reverse of the previous statement is false: non-isotopic crossingless matchings can also sometimes lead to isotopic Lagrangian submanifolds. Let $\wp, \wp'$ be crossingless matchings with the same endpoints, which are related to each other as in Figure 3: we choose an embedded path joining the first and second arc of

![Figure 3](image-url)
and avoiding all other components (shown dashed in the picture), and then define \( \wp' \) by passing the second component over the first as indicated by that path.

\[\text{(49) Lemma: } L_\wp, L_\wp' \text{ are Lagrangian isotopic.}\]

\[\text{Proof: } \text{In view of (58) we may assume that the endpoints } t = (\mu_1, \ldots, \mu_{2m}) \text{ of } \wp \text{ satisfy } \text{(61)} \text{ as well as } \mu_1 = -\mu_2, \text{ and that the point } 0 \in \mathbb{C} \text{ lies on } \delta_1. \text{ We can then choose } \gamma \text{ so that the point } \gamma(1) \text{ corresponds to } (0, 0, \mu_3, \ldots, \mu_{2m}), \text{ and the construction of } L_\wp \text{ simplifies slightly in that the step involving (56) becomes trivial. Namely, one considers } L_\bar{\wp} \in \mathcal{Y}_{m-1, \bar{t}} \text{ where } \bar{t} = (\mu_3, \ldots, \mu_{2m}), \text{ identifies the latter space with the singular set of } \mathcal{Y}_{m, \gamma(1)}, \text{ takes the associated relative vanishing cycle in } \mathcal{Y}_{m, \gamma(1-s)}, \text{ and then carries it back to } \mathcal{Y}_{m, \bar{t}} \text{ by parallel transport. The definition of } L_\wp' \text{ is the same except that we start with } L_\bar{\wp}'. \text{ But } \bar{\wp} \text{ and } \bar{\wp}' \text{ are isotopic as crossingless matchings with fixed endpoints, hence } L_\bar{\wp} \simeq L_\bar{\wp}', \text{ and that carries over to the associated relative vanishing cycles.} \]

\[\text{(50) Lemma: Up to Lagrangian isotopy, } L_\wp \text{ is independent of the ordering of the components of } \wp.\]

\[\text{Proof: } \text{Because of the recursive nature of the definition, we only need to show that exchanging } \delta_1 \text{ and } \delta_2 \text{ does not affect the Lagrangian submanifold. In view of (62) we may suppose that } \delta_1 \text{ is a straight short line segment } [-\sqrt{e_1}; \sqrt{e_1}] \subset \mathbb{C}, \text{ and similarly } \delta_2 = [\lambda - \sqrt{e_2}; \lambda + \sqrt{e_2}], \text{ for small } e_1, e_2 \neq 0 \text{ (in fact, we will only see in the course of the argument what the precise bounds are, but that is not a problem). By taking the paths short, we remove the need for using parallel transport (55) in the definition of } L_\wp', \text{ at least for the last two steps in the recursive procedure. What remains of these steps is the following: one starts with an already defined Lagrangian submanifold inside the singular point set of } \mathcal{Y}_{m-1, \bar{t}_1}, \text{ where } \bar{t}_1 = (\lambda, \lambda, \mu_5, \ldots, \mu_{2m}). \text{ The relative vanishing cycle procedure associates to this a Lagrangian submanifold inside a nearby smooth fibre } \mathcal{Y}_{m-1, \bar{t}}, \text{ such as } \bar{t} = (\mu_3, \ldots, \mu_{2m}) \text{ if } e_2 \text{ has been chosen sufficiently small. } \mathcal{Y}_{m-1, \bar{t}} \text{ can in turn be identified with the singular point set of } \mathcal{Y}_{m, t_1} \text{ for } t_1 = (0, 0, \mu_3, \ldots, \mu_{2m}), \text{ and forming the relative vanishing cycle again gives a Lagrangian submanifold in the nearby fibre } \mathcal{Y}_{m, t}, \text{ which is } L_\wp'. \text{ We will now reformulate this as follows. Let } w : P \hookrightarrow h/W \text{ be a small embedded bidisc, so that } w(z_1, z_2) \text{ corresponds to the set of eigenvalues } (-\sqrt{e_1}, \sqrt{e_1}, \lambda - \sqrt{e_2}, \lambda + \sqrt{e_2}, \mu_5, \ldots, \mu_{2m}). \text{ Using Lemma 25 one can identify the singular set of } \mathcal{Y}_{m-1, \bar{t}_1} \text{ with the intersection } \mathcal{Y}_{m, w(0,0)} \cap O^{\min}; \text{ here } O^{\min} \subset g \text{ is the orbit consisting of matrices where both the kernel and the } \lambda \text{-eigenspace are two-dimensional. Our construction starts with a Lagrangian submanifold inside this intersection, forms the relative vanishing cycle inside the critical set of } \mathcal{Y}_{m, w(0,e_2)},\]

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and then takes the relative vanishing cycle of that inside the whole smooth fibre \( \mathcal{Y}_{m, w(e_1, e_2)} \). The local structure of \( \mathcal{S}_m \cap \chi^{-1}(w(P)) \) near \( \mathcal{O}_{\min} \) is described by Lemma 28 and our iteration of the vanishing cycle procedure is precisely that discussed at the end of Section 4(B). Lemma 36 allows one to reverse the order in such a procedure, which in our case corresponds exactly to exchanging \( \delta_1 \) and \( \delta_2 \). ■

We may therefore drop the ordering of the components in the definition of a crossingless matching.

(51) Remark: As the reader may have noticed, the construction of Lagrangian submanifolds also goes through if one starts with a matching with ordered components which may intersect each other (of course, the endpoints must still be distinct, and in addition disjoint from the interiors of the arcs). However, the result is not really more general: by the same argument as in the proof of Lemma 49 one can slide the intersections of the components off each other to make the matching into a crossingless one, and the Lagrangian submanifold will not be affected.

(C) Definition of the invariant

Fix a \( t_0 \in \text{Conf}_{2m}(\mathbb{C}) \) which is a configuration of points on the real line. We denote by \( \varphi_+, \varphi_- \) the crossingless matchings with endpoints \( t_0 \) which consist of a family of concentric arcs in the upper, respectively lower, half plane (Figure 4). These are unique up to isotopy, hence so are the associated submanifolds \( L_{\varphi_+}, L_{\varphi_-} \). A repeated application of Lemma 49 shows that \( L_{\varphi_+}, L_{\varphi_-} \) are actually isotopic, so we will usually just write \( L_{\varphi_{\pm}} \) instead. Next, take an oriented link \( \kappa \) presented as a braid closure with \( 2m \) strands (Figure 5), with the left side of
the diagram being \( b \in Br_m \). Via the standard inclusion \( Br_m \times Br_m \hookrightarrow Br_{2m} \) we turn this into a \( 2m \)-stranded braid \( b \times 1^m \), and represent that by a loop \( \beta : [0; 1] \rightarrow Conf_{2m}(\mathbb{C}) \) starting and ending at \( t_0 \). As already explained in the Introduction, we set

\[
\begin{align*}
\text{Figure 5:} \\
\end{align*}
\]

\((52)\) \text{Definition:}

\[
Kh_{\text{symp}}(\kappa) = HF(L_{\wp^\pm}, h_{\beta}^\text{resc}(L_{\wp^\pm})).
\tag{59}
\]

The Floer cohomology is taken inside \( \mathcal{Y}_{m,t_0} \) which is certainly Stein, with the exact Kähler form \( \Omega = -dd^c \psi \). Since \( \mathcal{Y}_{m,t_0} \) is a regular fibre of a holomorphic map \( S_m \rightarrow \mathfrak{h}/W \) between affine spaces, its Chern classes are zero, and moreover \( H^1(\mathcal{Y}_{m,t_0}) = 0 \) by Lemma \( \text{[lem:chern]} \). Finally, \( L_{\wp^\pm} \) is diffeomorphic to \((S^2)^m\), hence has \( H_1 = 0 \) and is spin, so the Floer cohomology group above really is well-defined. Explicitly, the choices made in constructing \( L_{\wp^\pm} \) — including the choice of representative \( \beta \) for \( b \times 1^m \) and of the choices entering into the definition of \( h_{\beta}^\text{resc} \) — affect \( L_{\wp^\pm} \) up to Lagrangian isotopy, and any such isotopy is exact.

\((53)\) \text{Lemma:} \text{ The Floer cohomology group } \text{[def:kc]} \text{ is independent of the choices made in the definition of the Kähler form } \Omega. \text{ The statement is independence of the Kähler form only within the very restricted class that we are considering, which means independence of the choice of } \alpha \text{ and of compactly supported functions } \eta_k. \text{ But one can linearly interpolate between any two such forms, and get corresponding smooth families of Lagrangian sub-}
manifolds to use in (59), so the result follows from the invariance properties of Floer cohomology described in Section 4(D) (proved by bifurcation analysis).

The fact that (59) is an oriented link invariant will follow from its invariance under Markov moves. The type I move replaces the braid $b$ by $s_k b s_k$ where $s_1, \ldots, s_{m-1}$ are the standard generators of $Br_m$ (if one sees braids as diffeomorphisms of the punctured disc, the $s_k$ are positive half-twists; it is an unfortunate consequence of the standard convention that these are represented by braids with a negative crossing). We will use the same notation for the generators of $Br_{2m}$, and choose representatives $\sigma_k$ for them which are loops in $(Conf_{2m}(\mathbb{C}), t_0)$.

(54) Lemma: Up to Lagrangian isotopy, $L_{\varphi_{\pm}}$ is invariant under parallel transport along $\sigma_{2m-k} \circ \sigma_k$.

Proof: We use (58). Moving the crossingless matching $\varphi_+$ smoothly, so that its endpoints follow $\sigma_{2m-k} \circ \sigma_k$, yields another crossingless matching shown on the right in Figure 6. That is clearly obtained from $\varphi_+$ by an operation as in Figure 3, so the associated Lagrangian submanifold is isotopic to $L_{\varphi_{\pm}}$ because of Lemma 49.

Figure 6:

(55) Proposition: Up to isomorphism of relatively graded abelian groups, the Floer cohomology (59) is invariant under type I Markov moves.

Proof: By symplectomorphism invariance of Floer cohomology, and the previous Lemma, we have

$$HF(L_{\varphi_{\pm}}, h_{\sigma_{2m-k}^{-1}} h_{\sigma_k}^{-1} h_{\sigma_k} h_{\sigma_{2m-k}} (L_{\varphi_{\pm}}))$$

$$\cong HF(h_{\sigma_k} (L_{\varphi_{\pm}}), h_{\sigma_k}^{-1} h_{\sigma_k} h_{\sigma_k} h_{\sigma_k}^{-1} h_{\sigma_k} (L_{\varphi_{\pm}}))$$

$$\cong HF(h_{\sigma_{2m-k}} (L_{\varphi_{\pm}}), h_{\sigma_k}^{-1} h_{\sigma_k} h_{\sigma_k} h_{\sigma_k}^{-1} h_{\sigma_k} (L_{\varphi_{\pm}}))$$

$$\cong HF(L_{\varphi_{\pm}}, h_{\sigma_{2m-k}^{-1}} h_{\sigma_k}^{-1} h_{\sigma_k} h_{\sigma_{2m-k}} (L_{\varphi_{\pm}}))$$
and since $s_{2m-k}$ and $b \times 1^m$ commute in $Br_{2m}$, this is

$$\cong HF(L_{\tilde{\varphi}_\pm}, h_{\tilde{\beta}}^{\text{res}} h_{\tilde{\varphi}_\pm}^{\text{res}} (L_{\tilde{\varphi}_\pm}))$$

$$\cong HF(L_{\tilde{\varphi}_\pm}, h_{\tilde{\varphi}_\pm}^{\text{res}}).$$

\[\Box\]

(D) Markov II

Before going on to the remaining Markov move we want to deal with a different property of our Floer groups, whose proof is simpler but somewhat analogous, hence can serve as a warmup exercise. Namely, suppose that our oriented link has an unknotted and unlinked component, which appears in the braid presentation as shown in Figure 7. This means that $b = \bar{b} \times 1$, where $\bar{b} \in Br_{m-1}$ and we embed that into $Br_m$ by considering the leftmost $m-1$ strands.

![Figure 7](image)

For simplicity, assume that $t_0 = (\mu_1, \ldots, \mu_{2m}) \in Conf_{2m}^0(\mathbb{C})$, and that the middle two points of the configuration are $\mu_m, \mu_{m+1} = \pm \sqrt{e}$ for some small $e > 0$. Define $t_1 \in \mathfrak{g}/W$ by replacing these two with $(0,0)$. Similarly we define $\bar{t}_0 \in Conf_{2m-2}(\mathbb{C})$ by deleting the same points from the configuration, and get a crossingless matching $\bar{\varphi}_\pm$ with endpoints $\bar{t}_0$ by removing the corresponding component from $\varphi_\pm$. We choose a representative $\beta$ of $b$ in which the points $\pm \sqrt{e}$ remain fixed, so that by deleting these points one gets a representative $\beta$ of $\bar{b}$.

Let’s start by restating part of the definition of $h_{\tilde{\beta}}^{\text{res}} (L_{\tilde{\varphi}_\pm})$. One starts with $L_{\tilde{\varphi}_\pm} \in Y_{m-1,t_0}$, identifies the latter space with the set of singular points of
\( y_{m,t_0} \) using Lemma \( \ref{lem:25} \) and then uses the relative vanishing cycle construction to obtain a Lagrangian submanifold in the nearby fibre \( y_{m,t_0} \), which by definition is just \( L_{\varphi_z} \). One then applies the monodromy along \( \beta \) to that submanifold. Our first claim is that the order of the two last steps can be inverted, in the sense that if one applies monodromy along \( \beta \) to \( L_{\varphi_z} \) and then takes the associated vanishing cycle of the result, an isotopic Lagrangian submanifold is obtained. The reason is that one can interpolate continuously between the two processes, by starting with \( L_{\varphi_z} \) and applying the monodromy along \( \beta \) to that. The outcome of our discussion is that the pair of Lagrangian submanifolds \( (L_{\varphi_z}, h^{\text{resc}}_{\beta}(L_{\varphi_z})) \) in \( y_{m,t_0} \) is obtained by taking the pair \( (L_{\varphi_z}, h^{\text{resc}}_{\beta}(L_{\varphi_z})) \) inside \( y_{m-1,t_0} \) and applying the relative vanishing cycle construction to both.

We want to apply Lemma \( \ref{lem:42} \) to this situation. The total space \( Y \) will be \( \chi^{-1}(D) \cap S_m \), where \( D \subset \mathfrak{h}/W \) is a small disc corresponding to eigenvalues \( (\mu_1, \ldots, \mu_{m-1}, -\sqrt{z}, \sqrt{z}, \mu_{m+2}, \ldots, \mu_{2m}) \). \( X \subset Y \) is the subset of matrices which have a two-dimensional kernel. The local structure around \( X \) is described by Lemma \( \ref{lem:27} \) and the other assumptions of Lemma \( \ref{lem:42} \) are satisfied for obvious reasons, so

\[
HF(L_{\varphi_z}, h^{\text{resc}}_{\beta}(L_{\varphi_z})) \cong HF(L_{\varphi_z}, h^{\text{resc}}_{\beta}(L_{\varphi_z})) \otimes H^*(S^2).
\] (60)

By definition, the first factor on the right is the Floer group associated to the braid presentation obtained from our original one by removing the unknotted component.

The basic setup for the type \( II^+ \) Markov move is that one starts with \( \tilde{b} \in Br_{m-1} \), and then adds a single strand plus a half-twist of that strand with its neighbour, \( b = s_{m-1}(\tilde{b} \times 1) \in Br_m \). Suppose that \( m \geq 3 \), and assume that the base point \( t_0 = (\mu_1, \ldots, \mu_{2m}) \) lies in \( Conf_{2m}^0(\mathbb{C}) \), with the \( \mu_k \) ordered in the obvious way on the real line, and with \( \mu_{m-1}, \mu_m, \mu_{m+1} \) small and satisfying \( \mu_{m-1} + \mu_m + \mu_{m+1} = 0 \) (we will constrain the choice of these during the course of the argument, which is not a problem). Let \( t_0 \in Conf_{2m-2}^0(\mathbb{C}) \) be the configuration \( (\mu_1, \ldots, \mu_{m-2}, 0, \mu_{m+2}, \ldots, \mu_{2m}) \). We choose loops \( \beta \) and \( \beta' \) in \( Conf_{2m}^0(\mathbb{C}) \) based at \( t_0 \), representing \( b \times 1^m \) and \( \tilde{b} \times 1^{m+1} = s_{m-1}^{-1}(b \times 1^m) \) respectively, and similarly a loop \( \tilde{\beta} \) in \( Conf_{2m-2}^0(\mathbb{C}) \) based at \( \tilde{t}_0 \) which represents \( \tilde{b} \times 1^{m-1} \) (Figure 8).

As in Lemma \( \ref{lem:28} \) we consider the embedding \( w : P \to \mathfrak{h}/W \) of a small bidisc, so that \( w(d, z) \) corresponds to eigenvalues \( (\mu_1, \ldots, \mu_{m-2}, \text{all solutions of } \lambda^3 - d\lambda + z = 0, \mu_{m+2}, \ldots, \mu_{2m}) \). Start with the pair

\[
K = L_{\varphi_z}, \quad K' = h^{\text{resc}}_{\beta}(L_{\varphi_z})
\] (61)

of Lagrangian submanifolds in \( y_{m-1,t_0} \), and identify \( y_{m-1,t_0} \) itself with the singular point set of \( (\chi|S_m)^{-1}w(0,0) \). The key to the argument is Lemma \( \ref{lem:28} \) which
Figure 8:

Figure 8 describes the local structure of \((\chi | S_m)^{-1} w(P)\) near that subset, since that allows us to apply the results of the discussion from Section 4(C); we will use terms from that discussion freely from now on. By using parallel transport in the family of singular sets over the cusp curve in \(P\) (which corresponds to sets of eigenvalues where two coincide), one moves both of \((61)\) to Lagrangian submanifolds \(K_d, K'_d\) in the singular set of a nearby fibre \((\chi | S_m)^{-1} w(d, \zeta_d)\). Here \(d > 0\) is small and \(\zeta_d\) is the negative solution of \(4d^3 + 27z^2 = 0\). Then, by the relative vanishing cycle procedure applied to the family of fibres \((\chi | S_m)^{-1} w(d, z)\) with fixed \(d\), one gets associated Lagrangian submanifolds \(L_{d, \epsilon}, L'_{d, \epsilon}\) in \((\chi | S_m)^{-1} w(d, \zeta_d + \epsilon)\) for \(0 < \epsilon \ll d\). It is no problem to assume that our base point \(t_0\) was in fact chosen so that \(\mu_{d-1}, \mu_d, \mu_{d+1}\) are the solutions of \(\lambda^3 - d\lambda + (\zeta_d + \epsilon) = 0\), and then the Lagrangian submanifolds which we constructed lie precisely in \(Y_{m, t_0}\). The same argument as in the proof of \((60)\), inverting the order of parallel transport and relative vanishing cycle procedures, allows us to identify the outcome up to Lagrangian isotopy:

\[
L_{d, \epsilon} \simeq L_{\psi^\pm}, \quad L'_{d, \epsilon} \simeq h_{\beta^\prime}^{\psi^\pm}(L_{\psi^\pm}).
\] (62)

The second Lagrangian submanifold in \((62)\) is not yet quite the one which would appear in the formula \((59)\) for the link diagram from Figure 8. What we are missing is the generator \(s_{k-1}\), which corresponds to moving \(\mu_{m-1}, \mu_m\) around each other in a positive half-circle. Reversing an argument made in Section...
we find that in terms of coordinates on $P$ this can be achieved by fixing $d$ and moving $z$ along the loop $\gamma_{d,\epsilon}$ from Figure 1. We therefore obtain the following modified version of (62): if $L''_{d,\epsilon}$ is defined as in (44) then

$$L_{d,\epsilon} \simeq L_{\psi_{\pm}}, \quad L''_{d,\epsilon} \simeq h_{\text{resc}}^{\beta}(L_{\psi_{\pm}}).$$

(63)

Applying Lemma 44, whose assumptions are easily verified due to Lemma 29, one finds that $HF(L_{\psi_{\pm}}, h_{\text{resc}}^{\beta}(L_{\psi_{\pm}}))$ is isomorphic to $HF(K, K')$, which by definition is the Floer group associated to the link presentation with $2m-2$ strands and braid $b$. Up to now, we have excluded the lowest strand case $m=2$, since then the conditions imposed above on $t$ are impossible to satisfy without violating (54). Concretely, this means that one has to bring the three eigenvalues $\mu_{m-1}, \mu_m, \mu_{m+1}$ together at some nonzero point. That adds a small and entirely harmless intermediate step to the proof, which is the use of parallel transport for a suitable family $C_1, \gamma(s)$ to bring that point back to zero, and there to make the identification with $Y_{1,t_0}$. With that taken into account, we have shown:

(56) Proposition: Up to isomorphism of relatively graded abelian groups, the Floer cohomology is invariant under type $II^+$ Markov moves. ■

The discussion above can be easily adapted to the Markov move of type $II^-$, where $s_{m-1}$ occurs instead of $s_{m-1}$. The geometry of this situation is very similar to the previous one, the difference being that the path $\gamma_{d,\epsilon}$ has to be taken with reversed orientation, and consequently that $t_{\beta}(\alpha)$ in Figure 2 should be replaced by $t_{\beta}^{-1}(\alpha)$. This still intersects $\alpha$ only in the rightmost endpoint, so the proof of Lemma 44 goes through exactly as before. We omit the details. By combining this with Proposition 55 and 56, one gets that $Kh_{\text{symp}}(\kappa)$ (as a relatively graded group) is an invariant of the oriented link $\kappa$.

6 Miscellany

This section lists some concluding observations. First we explain how to equip $Kh_{\text{symp}}$ with a suitable absolute grading, which rounds off the construction of the invariant in the form presented in the Introduction. Secondly, we see why orientation reversal of all components leaves it unchanged. Finally, we compute it for the trefoil knot. Since these are somewhat peripheral topics (even though they have some relevance in view of Conjecture 2), we will give less details than usual.
(A) Gradings

Let $M$ be a Stein manifold with an exact Kähler form, satisfying $c_1(M) = 0$ and $H^1(M) = 0$. Pick a differentiable trivialization of the canonical bundle, which is a nowhere zero complex volume form $\eta_M$. Any Lagrangian submanifold $L \subset M$ comes with a canonical circle-valued squared phase function $\alpha_L : L \to S^1$, defined by

$$\alpha_L(x) = \frac{\eta_M(\xi_1, \ldots, \xi_n)^2}{|\eta_M(\xi_1, \ldots, \xi_n)|^2}$$

for any orthonormal basis $\xi_1, \ldots, \xi_n$ of $TL_x$. By definition a grading of $L$ is a lift of this to a real-valued function $\tilde{\alpha}_L$, let’s say for concreteness $\exp(2\pi i \tilde{\alpha}_L) = \alpha_L$. This is always possible in the context we used for defining Floer cohomology, since $H^1(L) = 0$ was part of the assumptions. For a pair of Lagrangian submanifolds $L_0, L_1$ equipped with gradings, the relative grading of Floer cohomology can be improved to an absolute one [20, 37]. If we denote by $L \mapsto L[1]$ the process which subtracts the constant 1 from the grading, then

$$HF^*(L_0, L_1[1]) = HF^*(L_0[-1], L_1) = HF^{*+1}(L_0, L_1).$$

It may seem that this theory depends on the choice of $\eta_M$, but in fact all that matters is its homotopy class as smooth trivialization, which is unique since $H^1(M) = 0$.

In our application, we start by choosing arbitrary trivializations $\eta_{S_m}$ and $\eta_{h/W}$ on the total space and base space of $\chi|S_m$. There is an induced family of trivializations of the canonical bundles of the regular fibres, characterized by

$$\eta_{y_{m,t}} \wedge \chi^*\eta_{h/W} = \eta_{S_m} \text{ on } y_{m,t}.$$ (66)

Choose a grading for $L_{\phi^+} \subset y_{m,t_0}$. Given a path $\beta: [0; 1] \to \text{Conf}_{2m}(\mathbb{C})$ starting at $t_0$, one can continue the given grading uniquely to a smooth family of gradings of the images $h_{\beta|[0;3]}^\text{resc}(L_{\phi^+})$, in particular the monodromy images which appear in (59) carry induced gradings, so the Floer cohomology group in that definition is now absolutely graded. Shifting the original choice of grading affects both Lagrangian submanifolds involved in the same way, and the effect on Floer cohomology cancels out due to (65). We take this absolutely graded group and apply a final shift to the grading, which depends on the number of strands $m$ and writhe $w$ of the braid presentation, thus arriving at the final definition:

$$Kh^*(\kappa) = HF^{*+m+w}(L_{\phi^+}; h_{\beta}^\text{resc}(L_{\phi^+})).$$ (67)

The isomorphisms in the proof of Proposition 55 are compatible with the absolute gradings, and the writhe does not change since we add one positive and one negative crossing, so (67) is invariant under Markov $I$.

We next look at the role of absolute gradings in (60). The basic situation in the
the set of eigenvalues $(\mu_1, \ldots, \mu_m, -\sqrt{z}, \sqrt{z}, \mu_{m+2}, \ldots, \mu_{2m})$; and $y_{m-1, \bar{t}_0}$ is identified with the singular set of $Y_m$. We can assume that on the subset of $\chi^{-1}(w(D)) \cap S_m$ which is the domain of the $\equiv$ in (68), $\eta_{m,w}$ is the wedge product of a previously defined $\eta_{m-1, \bar{t}_0}$, the standard form $da \wedge db \wedge dc$ on $\mathbb{C}^3$, and the form $d\mu_1 \wedge \cdots \wedge d\mu_{m-1} \wedge d\mu_{m+2} \wedge \cdots \wedge d\mu_{2m}$. Similarly, we may assume that on the image of $w$, $\eta_{b \cap W} = dz \wedge d\mu_1 \wedge d\mu_{m-1} \wedge d\mu_{m+2} \wedge \cdots \wedge d\mu_{2m}$ where $z$ is the parameter of $D$. This is because we have complete freedom in the choice of these volume forms, so we can prescribe them arbitrarily on any subset with zero first Betti number (this condition is imposed to ensure extendibility to the whole space).

As in the discussion preceding (60), we start with the Lagrangian submanifolds $K = L_{\varphi_+}$ and $K' = h_{\text{sym}}(L_{\varphi_+})$ inside $y_{m-1, \bar{t}_0}$, which have already been equipped with gradings following the prescription given above, and then take the associated relative vanishing cycles $L_z, L'_z$ in $y_{m,w(z)}$ for some small $z \neq 0$. For this we may use the local isomorphism (68) and a Kähler form which is the product of the given one on $y_{m-1, \bar{t}_0}$ and the standard form on $\mathbb{C}^3$, since that is how the Floer cohomology computation in Lemma 102 is carried out anyway.

Then $L_z = K \times \sqrt{z}S^2$, $L'_z = K' \times \sqrt{z}S^2$, and a straightforward computation of the relevant phases shows that

$$\alpha_{L_z} = \alpha_K \cdot \frac{z}{|z|}, \quad \alpha_{L'_z} = \alpha_{K'} \cdot \frac{z}{|z|}.$$  

To equip the relative vanishing cycles with gradings, what one has to do is therefore to choose a branch of $\text{arg}(z)$. The main thrust of the proof of (60) is that these vanishing cycles are related to $L_{\varphi_+}$ and $h_{\text{sym}}(L_{\varphi_+})$, respectively. Inspection of the argument shows that in order to make this relation work on the level of Lagrangian submanifolds equipped with gradings, the same branch of $\text{arg}(z)$ has to be used for both $L_z$ and $L'_z$. In that case, the Künneth formula from Lemma 102 holds as an isomorphism of absolutely graded groups where $H^*(S^2)$ carries its natural grading. Taking into account the additional shift that comes from the number of strands, we find that:

(57) **PROPOSITION:** Under disjoint sum with an unlinked unknot, $Kh^*_\text{sym}(L \sqcup U) \cong Kh^*_\text{sym}(L) \otimes H^{*+1}(S^2)$.  

The role of the grading in Markov $II^+$ is fundamentally very similar, with an additional contribution to the phase function coming from the $S^2$ factor added.
when taking relative vanishing cycles, and a corresponding correction term to the degree of intersection points. This correction will be the same in all cases, so it is enough to look at the toy model example studied at the beginning of Section 4(C). Hence, let \( Y_d, \zeta \) be the fibre of the map \( p : \mathbb{C}^4 \to \mathbb{C}^2 \) from (9) for some \( 0 < \epsilon \ll d \). We consider the Lagrangian spheres \( L_d, \epsilon \) and \( h_\gamma d, \epsilon \) from (36) and (37), equipping the first one with an arbitrary grading and the second one with the induced grading, coming from the fact that it is a monodromy image of the first. We take the standard Kähler form, and then apply a Lagrangian isotopy if necessary, so that following Lemma 38 our Lagrangian spheres are \( \Lambda_\alpha \) and \( \Lambda_t, (\alpha) \) respectively. As shown in the proof of that Lemma, the monodromy corresponds to the half-twist around \( \beta \) in the base, so the gradings we have chosen will have the property that they are approximately the same at the unique intersection point, which we call \( q \) (assuming that as represented in Figure 2, the angle between \( \alpha \) and \( t, (\alpha) \) at the common endpoint is small). Near \( q \) one can locally write \( \Lambda_\alpha = \text{graph}(dh) \subset T^* \Lambda_{t, (\alpha)} \), where \( h \) has a nondegenerate local minimum at \( q \). In view of the fact about the gradings mentioned above, standard properties of the Maslov index imply that the Maslov index of \( q \) reduces to the Morse index of \( (D^2 h)_q \), which is 0.

The same argument works for Markov II except that the second path is now \( t, (\alpha) \), which runs to the left of \( \alpha \), and so the function \( h \) has a local maximum, leading to a Maslov index of 2. These are the desired correction terms, and so the graded versions of the isomorphism arising from Lemma 44 and its analogue are as follows:

**(58) Lemma:** Take \( \bar{b} \in Br_{m-1} \) and set \( b^\pm = s^\pm_{m-1}(\bar{b} \times 1) \in Br_m \). Let \( \bar{\beta} \) be a path in Conf\(_{2m-2}(\mathbb{C}) \) representing \( \bar{b} \times 1 \), and similarly \( \beta^\pm \) paths in Conf\(_{2m}(\mathbb{C}) \) representing \( b^\pm \times 1 \). Then there are isomorphisms of graded abelian groups,

\[
HF^*(L_{\bar{\nu}^\pm}, h_{\bar{\beta}^*}^{\text{rec}}(L_{\bar{\nu}^\pm})) \cong HF^*(L_{\bar{\nu}^\pm}, h_{\beta^*}^{\text{rec}}(L_{\bar{\nu}^\pm})),
\]

\[
HF^*(L_{\nu^\pm}, h_{\beta^*}^{\text{rec}}(L_{\nu^\pm})) \cong HF^*_{-2}(L_{\nu^\pm}, h_{\beta^*}^{\text{rec}}(L_{\nu^\pm})).
\]

**■**

Inspection of (67) shows that these precisely cancel out against the changes in \( m + w \). This proves the invariance under Markov II of \( Kh^*_{\text{symp}} \) as a graded group, and thereby completes our proof of Theorem 1.

**B Orientation-reversal**

Complex conjugation \( c(y) = \bar{y} \) acts on \( S_m \), and induces the obvious map, also denoted by \( c \), on the base of the adjoint quotient map \( \mathfrak{h}/W \). In particular,
if \( t \in \text{Conf}_{2m}(\mathbb{C}) \) consists of real eigenvalues, like our base point \( t_0 \), we have an induced involution of \( \mathfrak{y}_{m,t} \) which reverses the sign of the Kähler form. As an obvious consequence of its behaviour on \( \mathfrak{y}/W \), and the definition of parallel transport and of the Lagrangian submanifolds associated to crossingless matchings, we have

\[
c \circ h_{\beta}^{\text{resc}} \circ c = h_{c(\beta)}^{\text{resc}};
\]

\[
c(L_\psi) = L_{c(\psi)}.
\]

(69)

In particular, since \( c(\psi_{\pm}) = \psi_{\mp} \), our basic Lagrangian submanifold \( L_{\psi_{\pm}} \) is invariant under this involution up to isotopy. Since \( c \) is antisymplectic, it induces isomorphisms on Floer cohomology groups which exchange the two factors involved,

\[
\text{HF}(c(L_0), c(L_1)) \cong \text{HF}(L_1, L_0).
\]

(70)

A choice of grading for \( L_k \) induces a grading for \( c(L_k) \), and with that in mind (70) becomes an isomorphism of graded groups. This is just the fact that complex conjugation acts on the first cohomology of the Grassmannian of Lagrangian subspaces in a symplectic vector space by multiplication by \(-1\); suitably unwound (and coupled with the definition of the absolute grading [37]), this implies that \( c(L_k[-1]) = (c(L_k))[1] \), in other words increasing the absolute Maslov index of an intersection point of the \( L_k \) decreases the index of the point viewed as an intersection of the \( c(L_k) \). By combining this with (69) and symplectomorphism invariance of Floer cohomology, one finds that

\[
\text{HF}^*(L_{\psi_{\pm}}, h_{c(\beta)}^{\text{resc}}(L_{\psi_{\mp}})) \\
\cong \text{HF}^*((c \circ h_{\beta}^{\text{resc}})(L_{\psi_{\pm}}), c(L_{\psi_{\pm}})) \\
\cong \text{HF}^*(L_{\psi_{\pm}}, (h_{c(\beta)}^{\text{resc}})^{-1}(L_{\psi_{\mp}})).
\]

If \( \beta \) represents \( b \times 1^m \) for some \( b \in Br_m \), then \( c(\beta)^{-1} \) represents \( c(b) \times 1^m \), where the braid \( c(b) \) is obtained from \( b \) by the antiautomorphism of \( Br_m \) which inverts the order of the letters in a word with respect to the standard presentation. If \( b \) gives a braid presentation for an oriented link \( \kappa \), then \( c(b) \) corresponds to the same link with the orientation of all components reversed. Since both presentations have the same number of strands and the same writhe, the computation above shows:

(59) **Proposition:** Up to isomorphism of graded groups, \( \text{Kh}_{\text{symp}}^*(\kappa) \) remains unchanged if we reverse the orientation of all components of \( \kappa \).  ■

(C) **The trefoil**

We now look at the left-handed trefoil knot \( \kappa \) (more precisely, the knot coming from the braid closure with \( b = s_1^3 \in Br_2 \)). The first part of the proof is to reduce things to an open subset of \( \mathfrak{y}_{2,t_0} \) where one has nice holomorphic coordinates,
and then deform the Kähler form to a more standard one. This runs entirely parallel to the corresponding argument for Markov II, so we will omit it and simply state the outcome.

Let \( X \subset \mathbb{C}^3 \) be the quadric \( u^2 + v^2 + w^2 = z \) for some \( z \neq 0 \). Choose some \( \sqrt{z} \), and consider the line bundle \( \mathcal{F} \to X \) whose fibre is the \( i\sqrt{z} \)-eigenspace of the matrix
\[
\begin{pmatrix}
  iu & v + iw \\
  -v + iw & -iu
\end{pmatrix}.
\]

Inside \( \mathbb{C} \oplus \mathcal{F}^{-2} \oplus \mathcal{F}^2 \to X \) with fibre coordinates \((a, b, c)\), consider the sub-fibre bundle \( Y \) defined by \( a^3 - ad + bc = \zeta_d + \epsilon \) for some small \( 0 < \epsilon \ll d \), and where \( \zeta_d \) is as in Section 4(C). We construct a Kähler form on \( Y \) (or more precisely on an open subset which is sufficiently large for our purpose) by combining the standard form on the base and fibre, and a hermitian metric on \( \mathcal{F} \), as set out in Background 39. Define a Lagrangian submanifold \( L \subset Y \) by taking \( \sqrt{z} S^2 \subset X \) on the base, and fibrewise over it the Lagrangian sphere \( \Lambda^\alpha \) from (41) in the fibres; and another Lagrangian \( L' \subset Y \) in the same way using \( \alpha' = t_{3/\beta} \alpha \) instead, see Figure 9 (the basic notation is carried over from Figure 2).

\[ \alpha' = t_{3/\beta} \alpha \]

![Figure 9:](image)

(60) **Lemma:** \( L \cap L' \cong S^2 \sqcup \mathbb{R}P^3 \).

**Proof:** The paths \( \alpha \) and \( \alpha' \) intersect in one endpoint and one interior point, and the corresponding intersections of \( \Lambda^\alpha \) and \( \Lambda'^\alpha \) consist of a single point and a circle \( \text{[14]} \) respectively. This takes place in each fibre over \((u, v, w) \in \sqrt{z}S^2 \), leading to a total intersection which is the disjoint union of a copy of the \( S^2 \) and a circle bundle over it. The degree of the circle bundle equals the multiplicity of the \( S^1 \)-action \( [20] \) on the circle \( [14] \), which is \( \pm 2 \). \( \blacksquare \)

The intersection \( L \cap L' \) is clean in the sense of \[33\], so one has a Morse-Bott type long exact sequence
\[
\cdots \rightarrow H^{*-2}(\mathbb{R}P^3) \rightarrow HF^*(L, L') \rightarrow H^*(S^2) \xrightarrow{\partial} H^{*-1}(\mathbb{R}P^3) \cdots
\]

(71)

We have given the gradings and resulting Maslov indices in this sequence without proof, but the nontrivial contribution to them comes from the geometry in the fibres of \( Y \to X \), and can be read off from \[19\] Lemma 6.18. The differential \( \partial \)
is necessarily zero, and taking into account the shift factor \( m + w = 2 - 3 = -1 \), we have

(61) **Proposition:** \( K_h^* \text{symp}(\kappa) \cong H^{*-1}(S^2) \oplus H^{*-3}(\mathbb{RP}^3) \). ■

This agrees with the computation of [17, Section 7] after collapsing the bigrading according to the prescription of Conjecture 2.

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