Optimal Liquidation Trajectories for the Almgren-Chriss Model with Lévy Processes

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Overview

1. Introduction
2. Lévy Models and Optimal Liquidation Trajectories
3. Linear Approximation of Exponential Lévy Models
4. Properties of the Optimal Trajectories and the Temporary Impact Function
5. Conclusion and References
Introduction

The Almgren-Chriss model

Let $Y_t$ denote the number of shares the investor holds at time $t$, which is assumed to take the form $Y_t = Y_0 - \int_0^t \xi_u \, du$. Then the investor sells at prices $S_t$ given by

$$S_t = s + \sigma W_t + \alpha(Y_t - Y_0) - F(\xi_t), \quad t \geq 0.$$  

The temporary impact function $F$ is of the form $F(x) = \beta x^\gamma$, where typically $\gamma$ is around 0.6.

The objective for the investor is to maximize

$$\mathbb{E}[C_T^Y] - \lambda \text{Var}(C_T^Y),$$

where $C_T^Y$ is the cash position at time $T$ corresponding to the liquidation strategy $Y$.  

Introduction

This talk
We consider an infinite time-horizon where the investor sells at prices given by

\[ S_t = s + L_t + \alpha(Y_t - Y_0) - F(\xi_t), \quad t \geq 0, \]

where \( L_t \) is a Lévy process and \( F \) is a function satisfying certain properties.

The objective for the investor is to find a strategy \( Y \) which maximize

\[ \mathbb{E}[-\exp(-AC_Y^\infty)], \]

where \( A \) denotes the investor’s risk aversion and \( C_Y^\infty \) denotes the investor’s cash position at the end of time.
Initial Observations

Observation 1
In the Brownian motion case, the optimal solution to the problem of maximising the expected exponential utility of the cash position is equal to the solution to the problem of maximising the mean-variance criterion over deterministic strategies (the optimal liquidation trajectories coincide).

Observation 2
For small time-horizons, exponential Lévy models provide a good fit to observed stock price data (e.g. exponential variance-gamma or NIG models), and liquidation is normally completed within short time-horizons.

Observation 3
Models based on Brownian motion underestimate the probability of large market movements.
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Some Key Questions

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**Question 4**
Are there any natural relations between the temporary impact function $F$ and the probability law of the underlying asset?
Assumption

We assume that $L$ is a non-trivial Lévy process such that $L_1$ has finite second moment and

$$
\bar{\delta} = \inf \{ \delta < 0 \mid \mathbb{E}[e^{\delta L_1}] < \infty \} < 0.
$$

Such a Lévy process admits a decomposition

$$
L_t = \mu t + \sigma \mathcal{W}_t + \int_{\mathbb{R}} x \tilde{N}(t, dx), \quad t \geq 0,
$$

where $\mu \in \mathbb{R}$ and $\sigma \geq 0$ are constants, $\mathcal{W}$ is a standard Brownian motion, $\tilde{N}$ is a compensated Poisson random measure. We let $\nu$ denote the Lévy measure associated with $L$. 
Model Description

Admissible strategies

Given an initial share position $y$, the set $\mathcal{A}(y)$ of admissible strategies consists of adapted, absolutely continuous, non-increasing processes $Y$ of the form $Y_t = y - \int_0^t \xi_u du$, satisfying

$$
\int_0^\infty \|Y_t\|_{L^\infty(P)} \, dt < \infty \quad \text{if } \mu \neq 0,
$$

(1)

and

$$
\int_0^\infty \|Y_t\|^2_{L^\infty(P)} \, dt < \infty \quad \text{if } \mu = 0.
$$

(2)

We let $\mathcal{A}_D(y)$ denote the set of all deterministic strategies in $\mathcal{A}(y)$. 
Model Description

The investor sells his shares at prices given by

\[ S_t = s + L_t + \alpha(Y_t - Y_0) - F(\xi_t), \quad t \geq 0, \]

where \( \alpha \geq 0 \) and \( F : [0, \infty) \to [0, \infty) \) satisfies

(i) \( F \) is continuous and continuously differentiable on \( (0, \infty) \);

(ii) \( F(0) = 0 \);

(iii) the function \( x \mapsto xF(x) \) is strictly convex on \( [0, \infty) \);

(iv) \( \lim_{x \to 0}xF'(x) \) exists;

(v) the function \( x \mapsto x^2F'(x) \) is strictly increasing on \( [0, \infty) \), and tends to \( +\infty \) as \( x \to \infty \).
The Cash Position

For an admissible liquidation strategy \( Y \in \mathcal{A}(y) \), the investor’s cash position at the end of time is

\[
C^Y_\infty = c - \int_0^\infty S_t \, dY_t
= c + sy - \frac{1}{2} \alpha y^2 + \int_0^\infty Y_t - dL_t - \int_0^\infty \xi_t F(\xi_t) \, dt.
\]

Also, recall that the investor’s optimisation problem is to find a \( Y^* \in \mathcal{A}(y) \), which maximise

\[
\mathbb{E}[- \exp(-AC^Y_\infty)].
\]
Problem Simplification

It turns out that in order to solve the optimal liquidation problem, it is sufficient to solve (the derivation is similar to that of Schied, Schöneborn and Tehranchi (2010))

\[
V(y) = \inf_{Y \in A_D(y)} \int_0^\infty \left\{ \kappa_A(Y_t) + A \xi_t F(\xi_t) \right\} dt,
\]

where

\[
\kappa_A(y) = \ln \mathbb{E}[\exp(-A y L_1)].
\]

Moreover, if \( \mu > 0 \), then there are no admissible optimal solutions. Hence from now on we assume that the asset has a drift \( \mu \leq 0 \).
The Optimal Liquidation Trajectory

Let $G : [0, \infty) \to [0, \infty)$ be the inverse function of $x \mapsto x^2 F'(x)$, and define

$$
\tau = \int_0^y \frac{1}{G(\frac{\kappa_A(u)}{A})} \, du, \quad y \in [0, \bar{\delta}_A).
$$

Let $Y^*$ be the unique process satisfying

$$
\int_{Y_t^*}^y \frac{1}{G(\frac{\kappa_A(u)}{A})} \, du = t, \quad \text{if } t \leq \tau, \quad \text{and} \quad Y_t^* = 0, \quad \text{if } t > \tau.
$$

Then $Y^* \in A_D(y)$ is the optimal liquidation strategy, and the associated speed $\xi^*$ is given by

$$
\xi_t^* = G\left(\frac{\kappa_A(Y_t^*)}{A}\right), \quad 0 \leq t < \tau.
$$
Approximation of Exponential Models

Stock price data are typically calibrated to exponential Lévy models, so we want to derive a linear approximation of such models. Consider the exponential model

\[ \tilde{S}_t = s \exp(\tilde{L}_t) + I_t, \]

where \( I_t = \alpha(Y_t - Y_0) + F(\xi_t) \) is the price impact at time \( t \), and \( \tilde{L} \) is a Lévy process with characteristics \((\tilde{\mu}, \tilde{\sigma}, \tilde{\nu})\) satisfying

\[ d\tilde{\nu}(x) = \tilde{f}(x) \, dx \quad \text{and} \quad \int_{|z| \geq 1} e^{2z} \tilde{\nu}(dz) < \infty. \]

We want to choose a Lévy process \( L \) such that \( S \) given by

\[ S_t = s + L_t + I_t \]

is the linear approximation of \( \tilde{S} \).
Approximation of Exponential Models

Solution
Define a measure $\nu$ by

$$\nu(dx) = \frac{1}{1 + x} \tilde{f}\left(\ln(1 + x)\right) dx, \quad x > -1, x \neq 0,$$

and set

$$m = \tilde{\mu} + \frac{\tilde{\sigma}^2}{2} + \int_{\mathbb{R}} (e^z - 1 - z 1_{\{|z| < 1\}}) \tilde{\nu}(dz).$$

Let $L$ be a Lévy process with characteristics $(m, \tilde{\sigma}, \nu)$. Then

$$S_t = s + sL_t + l_t$$

is the linear approximation of $\tilde{S}$. 
Example; Variance-Gamma

Consider the case where $\tilde{L}$ is the variance-gamma process with parameters $(\theta, \rho, \eta)$. Then $\tilde{\sigma} = 0$, and $\tilde{\nu}$ has a density

$$f(z) = \frac{1}{\eta |z|} e^{Cz - D|z|}, \quad z \in \mathbb{R},$$

where

$$C = \frac{\theta}{\rho^2} \quad \text{and} \quad D = \sqrt{\frac{\theta^2 + \frac{2\rho^2}{\eta}}{\rho^2}}.$$

Our assumptions are satisfied if $D - C > 2$, and the cumulant generating function $\tilde{\kappa}$ admits the expression

$$\tilde{\kappa}(x) = -\frac{1}{\eta} \ln \left( 1 - \frac{x^2 \rho^2 \eta}{2} - \theta \eta x \right).$$
Example; Variance-Gamma

The Lévy measure for the process $L$ appearing in the expression for the linear approximation of $\tilde{S}$ in the VG case is

$$
\nu(dx) = \begin{cases} 
\frac{-1}{\eta \ln(1+x)} (1 + x)^{C+D-1} \, dx, & x \in (-1, 0), \\
\frac{1}{\eta \ln(1+x)} (1 + x)^{C-D-1} \, dx, & x \in (0, \infty).
\end{cases}
$$

The corresponding $\kappa_A$ function is

$$
\kappa^\text{vg}_A(u) = -Amu + \int_{-1}^{\infty} \left( e^{-Au x} - 1 + Au x \right) \nu(dx),
$$

where $m = \tilde{\kappa}(1)$. 
Examples; Power-Law Temporary Impact

In this case, the temporary impact function takes the form

\[ F(x) = \beta x^\gamma. \]

For our numerical examples, we choose \( \gamma = 0.6 \) and \( \beta = 4.7 \times 10^{-5} \) (which we believe are reasonable in view of Almgren et. al. (2005) when we work with a daily volatility of roughly 0.02, take daily volume to be \( 2 \times 10^6 \) and choose an initial stock price of \( s = 100 \)). The function \( G \) then takes the form

\[ G(x) = \left( \frac{x}{\beta \gamma} \right)^{\frac{1}{\gamma+1}}, \quad x \geq 0, \]

and the optimal liquidation trajectory is given by

\[ \xi_t^* = \left( \frac{\kappa_{sA}(Y_t^*)}{A \beta \gamma} \right)^{\frac{1}{\gamma+1}}. \]
Example; Power-Law Temporary Impact

We choose a risk aversion of $A = 2$, and recall that the unaffected price process is given by $100 \times \exp(\tilde{L}_t)$.

Exponential VG case

We choose $\tilde{L}$ to be a VG Lévy process with parameters $\theta = -0.002$, $\rho = 0.02$ and $\eta = 0.6$, which are typical values for daily stock price data.

Exponential BM case

In this case we choose $\tilde{L}_t = \mu t + \sigma W_t$, where $W$ denotes a standard BM. We let the parameter $\mu$ and $\sigma$ be given by $
mu + \frac{\sigma^2}{2} = \kappa(1)$ and $2\mu + \frac{4\sigma^2}{2} = \kappa(2)$, in which case the VG and the BM models have the same mean and variance.

Initial position

We assume that the investor initially holds $y = 2 \times 10^5$ number of shares, which is 10% of daily volume.
If the investor follows the optimal liquidation strategy, it takes roughly $2 \times 10^{-4}$ days until the time the investor has liquidated 99% of his shares.
If the investor follows the optimal liquidation strategy, it takes less than $2 \times 10^{-10^{8263}}$ days until the time he has liquidated 99% of his shares.

So essentially, the investor liquidates 99% of his shares instantly.

Even if we take $\gamma = 1000$, the time it takes the investor to liquidate 99% of a position of $2 \times 10^5$ number of shares would be less than $2 \times 10^{-168}$. 
Optimal Liquidation Trajectory VG-Case

The previous example was an extreme case with risk aversion $A = 2$. Almgren typically assumes a risk aversion in the region $A = 10^4$ to $A = 10^5$.

For risk aversions in the region $A = 10^5$, the BM and the VG models will produce liquidation strategies that are comparable.
An Equivalence Relation

Consider the VG model with temporary impact function $F_{vg}$ and the BM model with temporary impact function $F_{bm}$. Then the optimal liquidation trajectories for the two models coincide iff

$$G_{vg}\left(\frac{\kappa_{sA}^{vg}(Y_t^*)}{A}\right) = \xi_t^* = G_{bm}\left(\frac{\kappa_{sA}^{bm}(Y_t^*)}{A}\right), \quad t \geq 0,$$

where $G_{vg}$ is the inverse function of $x \mapsto x^2 F'_{vg}(x)$ and $G_{bm}$ is the inverse function of $x \mapsto x^2 F'_{bm}(x)$. We can solve for $F_{vg}$ to obtain

$$F_{vg}(x) = \int_0^x \frac{1}{Az^2} \kappa_{sA}^{vg}\left((\kappa_{sA}^{bm})^{-1}(Az^2 F'_{bm}(z))\right) \, dz.$$

If $F_{bm}(x) \sim x^\gamma$ for $x$ small, then $F_{vg}(x) \sim x^\gamma$ for $x$ small.
Temporary Impact; VG case
Conclusion

- We obtain semi-explicit solutions for the optimal liquidation trajectories when the risk is modelled by a Lévy process.
- We obtain an explicit expression for the connection between the temporary impact function for the Lévy model and the temporary impact function for the BM model, such that the optimal liquidation trajectories are identical.
- There might be a connection between the distribution of the returns and the temporary impact function, but this would require further investigations.
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Thank You