Abstract. We study rank 3 stable bundles $E$ on $\mathbb{P}^3$ as extensions of a line bundle $L$ on a smooth surface $S \subset \mathbb{P}^3$ by $3 \oplus \mathcal{O}_{\mathbb{P}^3}(-\nu)$. In most cases, $S$ (the dependency locus of three sections of $E(\nu)$) lies in the Noether-Lefschetz locus. We give a detailed analysis when $S$ contains a line $L$ and $L$ is constructed from divisors of the form $aL + bC$ for $H = L + C$ a hyperplane section of $S$. We study the parameter space of this construction and compare it to the full (Gieseker-Maruyama) moduli space. We also analyze the situation when $L$ is a power of the hyperplane bundle.

The same approach is used to study rank 2 stable bundles on $\mathbb{P}^3$.

1. Introduction

The purpose of this paper is to begin a study of stable vector bundles of rank three on three dimensional projective space. Our approach is to express such a bundle $E$ (normalized so that $c_1 = 0$, $-1$ or $-2$) as an extension

$$\begin{align*}
0 \longrightarrow & \quad 3 \oplus \mathcal{O}_{\mathbb{P}^3}(-\nu) \quad \sigma \quad \longrightarrow \quad E \quad \longrightarrow \quad j_{S*}L \quad \longrightarrow \quad 0
\end{align*}$$

for $S \subset \mathbb{P}^3$ a smooth surface of degree $k = 3\nu + c_1$ and $L$ a line bundle on $S$, using Serre’s Theorem A and the Kleiman Transversality Theorem. We study $E$ through $S$, $L$, and the extension class $\tau$ of (1.1) which appears in the dual sequence

$$\begin{align*}
0 \longrightarrow & \quad E^* \quad \sigma^* \quad \longrightarrow \quad 3 \oplus \mathcal{O}_{\mathbb{P}^3}(\nu) \quad \tau \quad \longrightarrow \quad j_{S*}L^*(k) \quad \longrightarrow \quad 0.
\end{align*}$$

Chern class calculations show that, in most cases, $S$ must belong to the Noether-Lefschetz locus, that is, it must support a line bundle not equal to a power of the hyperplane bundle.

To produce examples, we reverse the above procedure and start with $c_1 \in \{0, -1, -2\}$, $\nu \in \mathbb{Z}_+$, a surface $S \subset \mathbb{P}^3$ of degree $k = 3\nu + c_1$, and a line bundle $L$ on $S$ and consider extensions (1.1). We make a detailed study of the case where the surface contains a line $L$ and the line bundles are constructed from divisors of the form $aL + bC$ for $L + C$ a hyperplane section of $S$ containing $L$, $C$ a curve of degree $k-1$, and $a, b \in \mathbb{Z}$ (Section 9). It is determined when the resulting coherent sheaf $E$ is locally free and (modulo one unresolved case) when it is stable (Theorem 4). We count the moduli of our construction (Proposition 5) by proving that the correspondence $(E, \sigma) \leftrightarrow (S, L, \tau)$ is 1-to-1. Then we estimate the
dimension of the component of the full moduli space containing $E$, $\mathcal{M}$ (Theorem 5). When the degree of $S$ is 2 or 3, $\dim \mathcal{M}$ is determined exactly and we can conclude, in many cases, that our examples form a subset $\mathcal{Y}$ of $\mathcal{M}$ of equal dimension and that $\mathcal{M}$ is smooth at $E$ (Theorem 4 and Theorem 7). For arbitrary $k$, we give a separate analysis of the special case where the line bundle $L$ is a power of the hyperplane bundle (Section 8) and show that the corresponding space of parameters $\mathcal{Y}$ is an open subscheme of $\mathcal{M}$. We address the general problem of putting a scheme structure on the parameter space $\mathcal{Y}$ in Section 6.

The examples we construct and study provide evidence for the general problem of determining the dimension of the moduli space of stable bundles when the base variety has dimension $\geq 3$.

For $E$ a rank $r$ stable bundle on a smooth projective variety $X$ and $\mathcal{M}$ the corresponding moduli space (see Section 2), $T_{\mathcal{M}}E \cong H^1(X; \text{End}_0 E)$ ($\text{End}_0 E$ is the bundle of trace-free endomorphisms of $E$). 

(1.3) $h^1(X; \text{End}_0 E) - h^2(X; \text{End}_0 E) \leq \dim_{\mathcal{E}} \mathcal{M} \leq h^1(X; \text{End}_0 E)$.

The expected dimension of $\mathcal{M}$ is defined by

(1.4) $ed(\mathcal{M}) \equiv h^1(X; \text{End}_0 E) - h^2(X; \text{End}_0 E)$.

When $X$ is a surface, Riemann-Roch calculates

(1.5) $ed(\mathcal{M}) \equiv 2rc_2(E) - (r - 1)c_1(E)^2 - (r^2 - 1)\chi(\mathcal{O}_E)$.

Also for the surface case, important work by Gieseker and Li ([8] and [9]), and O’Grady [22] implies that, for $c_2(E)$ large enough (with $c_1(E)$ fixed), $\mathcal{M}$ is irreducible, generically smooth, and of dimension $ed(\mathcal{M})$, and, on a Zariski open subset of $\mathcal{M}$, $h^2(X; \text{End}_0 E) = 0$.

When the base variety has dimension $\geq 3$, no results of this type have been proven. And there is no expression for $ed(\mathcal{M})$ in terms of chern classes (like (1.5))—because of the higher dimensional groups $h^i(X; \text{End}_0 E)$, $i \geq 3$. By varying the discrete parameters in the examples of Section 5, Section 8 and Section 9, one finds many bundles $E$ for which $\dim \mathcal{M}$ is much larger than $ed(\mathcal{M})$ and for arbitrarily large $c_2(E)$. For these examples, $h^2(X; \text{End}_0 E)$ is in fact much larger than $ed(\mathcal{M})$. One could ask whether the term ”expected dimension” should be applied to (1.3) when the base manifold has dimension three or greater. The problem remains: Understand $\dim \mathcal{M}$ for stable bundles over smooth varieties of dimension $\geq 3$.

The technical backbone of this paper’s theorems consists of the intersection properties of $L$ and $C$ on $S$ and results on the cohomology of the line bundles $\mathcal{O}_S(iL + jC)$ (Section 4).

The same methods are also applied to stable rank two bundles on $\mathbb{P}^3$ (Section 5 and Section 6). In general, the examples produced from surfaces containing a line seem to comprise a higher codimension subset of $\mathcal{M}$ than in the rank three case.

Our approach can also be used to discuss stable bundles $E \to X$ of various ranks on other smooth projective varieties $X$. This will be the subject of future papers.

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2. Preliminaries

By a stable bundle we shall mean Mumford-stable (or $\mu-$ stable), that is

**Definition 1.** Let $X$ be a smooth projective variety of dimension $n$, $\mathcal{O}_X(1)$ a very ample line bundle on $X$, and $H$ a corresponding hyperplane section of $X$. A coherent torsion-free rank $r$ sheaf $E$ on $X$ is called Gieseker-stable (resp. Gieseker-semistable) if, for any proper subsheaf $F \subset E$ of rank $r_1 < r$, $r_1^{-1}c_1(F) \cdot H^{n-1} < r^{-1}c_1(E) \cdot H^{n-1}$ (resp. $\leq$).

**Definition 2.** A coherent torsion-free rank $r$ sheaf $E$ on $X$ is called Gieseker-stable (resp. Gieseker-semistable) if, for any proper subsheaf $F \subset E$ of rank $r_1$, $r_1^{-1}\chi(E) < r^{-1}\chi(E)$ (resp. $\leq$) for $l \geq 0$, where $\chi(E)$ is the Hilbert polynomial of $E$.

There is a coarse moduli space ([3] page 153 and [10] page 38 and chapter 4]) for the Gieseker-semistable sheaves on $X$ with fixed Hilbert polynomial, a projective scheme $\mathcal{M}$ whose closed points correspond to the $S$-equivalence classes ([16] page 22)]of Gieseker-semistable sheaves on $X$. From the definitions and Riemann-Roch it follows that stable $\Rightarrow$ Gieseker-stable $\Rightarrow$ Gieseker-semistable $\Rightarrow$ semistable. The stable sheaves with fixed $\chi(E)$ form an open subset of $\mathcal{M}$.

The Riemann-Roch formula ([3] Append. A, sec.4] for a rank $r$ coherent sheaf $E$ on $\mathbb{P}^3$ is

$$\chi(\mathbb{P}^3; E) = r + \frac{11}{6}c_1 + (c_1^2 - c_2) + \frac{1}{6}(c_1^3 - 3c_1c_2 + 3c_3).$$

Here we have identified the Chern classes $c_i(E) = c_i$ with integers using the positive generator $\omega_0$ of $H^2(\mathbb{P}^3; \mathbb{Z})$ and the generators $\omega_i$ of $H^2(\mathbb{P}^3; \mathbb{Z})$ $i=0$ to 3.

The Grothendieck-Riemann-Roch formula for a closed embedding of smooth varieties $f : X \to Y$ and a coherent sheaf $E$ on $X$ ([5] Chapter 15] and [3] Append. A, sec.4] is

$$ch(f, E) = f_*[ch(E)td(N_{X/Y})^{-1}].$$

We make frequent use of the Kleiman Transversality Theorem ([17] and [13] Thm10.8]); Let $X$ be a homogeneous variety with group variety $G$ over an algebraically closed field $k$ of characteristic 0. Let $f : Y \to X$ and $\phi : Z \to X$ be morphisms of nonsingular varieties $Y,Z$ to $X$. For any $g \in G(k)$, let $Y^g$ be $Y$ with the morphism $g \circ f$ to $X$. Then there is a nonempty (Zariski)open subset $U \subset G$ such that for every $g \in U(k)$, $Y^g \times_X Z$ is nonsingular and either empty or of dimension exactly $\dim Y + \dim Z - \dim X$.

We use Kleiman Transversality in the following situation (see [10] page 212]). Let $Y$ be a smooth projective variety of dimension $n$ and $E$ a rank $r$ vector bundle on $Y$ which is globally generated. Set $H = H^0(Y; E)$. For $X$ the Grassmannian of $r$-dimensional quotient spaces of $H$, evaluation of sections defines a regular map $f : Y \to X$ such that, $E \cong f^* \mathcal{O}$ for $\mathcal{O}$ the tautological quotient bundle on $X$. For any $k$ sections of $E$, $\sigma_j$, $j=1$ to $k$, which generate a $k$-dimensional subspace $V$ of $H$, and $0 \leq l \leq k$, set $Y_l \equiv \{ y \in Y \mid \dim \text{span}\{\sigma_1(y), \ldots, \sigma_k(y)\} \leq l\}$. Define $Z_l \equiv \{ H/K \in X \mid \dim K \cap V \geq k - l \}$, a Schubert variety of codimension $(k - l)(r - l)$. $Z_l$ is smooth away from $Z_{l-1}$ and $Y_l = f^{-1}Z_l$. The group $G$ is $GL(dim H, \mathbb{C})$ acting on $X$. Now Kleiman Transversality implies that, for generic $\sigma_j$, $j=1$ to $k$, $Y_l$ is of codimension $(k - l)(r - l)$, empty if $(k - l)(r - l) > n,$
and the singular locus of $Y_i$ is of codimension $(k - l - 1)(r - l - 1)$, empty if $(k - l - 1)(r - l - 1) > n$.

3. Stable Bundles of Rank 2 on $\mathbb{P}^3$

Let $E \to \mathbb{P}^3$ be a rank 2 normalized bundle ($c_1 = 0$ or $-1$). For $\nu$ large enough, $E(\nu)$ is globally generated and a generic $\sigma = (\sigma_1, \sigma_2) \in \oplus H^0(\mathbb{P}^3; E(\nu))$ gives an exact sequence

$$E \to \mathbb{P}^3(\nu) \to E \to 0$$

(For fixed chern classes $\exists \nu_0 \in \mathbb{Z}_+$ so that this holds $\forall \nu \geq \nu_0$ and all semistable $E$, since this family is bounded [16 Thm 3.3.7] [24 Thm 1.1]). By Kleiman transversality, the generic $\sigma$ produces a degeneracy locus $S = Z_{\sigma_1 \wedge \sigma_2}$ which is a smooth hypersurface $S \to \mathbb{P}^3$ of degree $k = 2\nu + c_1$, a line bundle $L$ on $S$, and zero sets $Z_{\sigma_j}$, $j = 1, 2$ which are smooth curves of degree $c_2(E(2)) = c_2(E) + c_1\nu + \nu^2$. It follows that, though the $Z_{\sigma_j}$ need not be irreducible, their components are mutually disjoint. This gives a basepoint-free pencil of curves on $S$, $Z_{\tau_1 \sigma_1 + \tau_2 \sigma_2}$, and a regular map $S \to \mathbb{P}^1$. Therefore $S$ belongs to the Noether-Lefschetz locus, i.e. it supports a line bundle not equal to a power of the hyperplane bundle.

Applying $\text{Hom}_{\mathbb{P}^3}(\mathbb{P}^3, \mathbb{O}_{\mathbb{P}^3})$ to (3.1) gives

$$0 \to E^* \to \mathbb{P}^3(\nu) \to \mathbb{P}^3(\nu) \to \mathbb{P}^3(\nu) \to 0$$

for $\tau^\tau = (\tau_1, \tau_2) \in \oplus H^0(S; L^*(\nu + c_1))$. I explain why, after possibly multiplying $\tau_1$ and $\tau_2$ by the same non-zero constant,

$$\tau_1 = \sigma_2 S \wedge \tau_2 = -\sigma_1 S \wedge.$$ 

View $L$ as the quotient sheaf $E/\text{im}(\sigma_1 \oplus \sigma_2)$. For $g = (g_1, g_2) \in \oplus \mathbb{O}_{\mathbb{P}^3}(\nu)$, set $T(g) = (g_1 \sigma_2 - g_2 \sigma_1) |_S \wedge$ and apply to $[f] \in L$, $f \in E$ to get $(g_1 \sigma_2 - g_2 \sigma_1) \wedge [f] \in \mathbb{O}_S(\nu)$. Note that this is well-defined independent of $f \in [f]$ and applied to $g = (\sigma_1, \sigma_2)(\psi) = (\psi(\sigma_1), \psi(\sigma_2)) \wedge \psi \in E^*$ gives $(\psi(\sigma_1) \sigma_2 - \psi(\sigma_2) \sigma_1) |_S \wedge = c_2(\sigma_1 \wedge \sigma_2) |_S \wedge = 0$. It follows that $T = \tau_1 \oplus \tau_2$ up to non-zero constant multiple.

$Z_{\sigma_1} = Z_{\tau_2}$, $Z_{\sigma_2} = Z_{\tau_1}$, $Z_{\tau_1} \cdot Z_{\tau_2} = 0$, and so $Z_{\tau_1} \cdot Z_{\tau_2} = 0$ (which also follows from (3.2)). Therefore $c_1(L^*(\nu + c_1)) = (\nu + c_1) \omega_0 - c_1(L)$ gives

$$((\nu + c_1) \omega_0 - c_1(L))^2 = 0 \quad \text{intersection on } S \text{ i.e.}$$

$$(\nu + c_1)^2(2\nu + c_1) - 2(\nu + c_1) \omega_0 \cdot c_1(L) + c_1(L)^2 = 0.$$
Now assume that $E$ is stable. If $\mathcal{L}$ has the form $\mathcal{L} = \mathcal{O}_S(l)$, $h^0(\mathbb{P}^3; E) = 0$ implies $l < 0$ but then (3.3) gives $(\nu + c_1 - l)^2 = 0$ which is impossible. Therefore

For $E$ stable, $\mathcal{L} \neq \mathcal{O}_S(l)$ for any $l$.

Applying Grothendieck Riemann-Roch to $j_{S*}\mathcal{L}$ [3] gives

$$ (3.6) \quad c_2(E) = (\nu + c_1)^2 - \omega_0 \cdot c_1(\mathcal{L}) \quad (\text{intersection on } S) $$

and reproves (3.5) from the fact that $c_3(E) = 0$.

To construct some concrete bundles $E$, reverse the above procedure, begin with a given $\nu \in \mathbb{Z}_+$, a smooth $S \hookrightarrow \mathbb{P}^3$ of degree $k = 2\nu + c_1$ ($c_1 = 0$ or $-1$), and a line bundle $\mathcal{L}$ on $S$ and consider extensions

$$ (3.7) \quad 0 \longrightarrow \mathbb{O}_{\mathbb{P}^3}(-\nu) \longrightarrow E \longrightarrow j_{S*}\mathcal{L} \longrightarrow 0. $$

These are classified by $Ext^1(\mathbb{P}^3; j_{S*}\mathcal{L}, \mathbb{O}_{\mathbb{P}^3}(-\nu)) \cong \mathbb{H}^0(S; \mathcal{L}^*(\nu + c_1))$; we want to determine which extensions are locally free. Applying $\text{Hom}_{\mathcal{O}_{\mathbb{P}^3}}(, \mathcal{O}_{\mathbb{P}^3})$ to (3.7) gives

$$ (3.8) \quad 0 \longrightarrow E^* \longrightarrow \mathbb{O}_{\mathbb{P}^3}(\nu) \longrightarrow \mathcal{L}^*(k) \longrightarrow Ext^1_{\mathcal{O}_{\mathbb{P}^3}}(E, \mathcal{O}_{\mathbb{P}^3}) \longrightarrow 0. $$

$E$ is locally free iff $Ext^1_{\mathcal{O}_{\mathbb{P}^3}}(E, \mathcal{O}_{\mathbb{P}^3}) = 0$ iff $\mathcal{L}^*(\nu + c_1)$ is globally generated by $\tau = (\tau_1, \tau_2)$ which is the extension class mentioned above. It follows that

The generic extension (3.7) is locally free iff $\mathcal{L}^*(\nu + c_1)$ is globally generated (necessarily by two sections). In this case, $((\nu + c_1)\omega_0 - c_1(\mathcal{L}))^2 = 0$.

Recall that a rank 2 bundle $E$ on $\mathbb{P}^3$ is stable iff $h^0(\mathcal{O}_{\mathbb{P}^3}; E) = 0$ and semistable ($c_1 = 0$ case) iff $h^0(\mathcal{O}_{\mathbb{P}^3}; E(-1)) = 0$ [23, pages 165–166]. This gives

$E$ of the form (3.4) is stable iff $\nu > 0$ and $h^0(S; \mathcal{L}) = 0$.

$E$ is semistable ($c_1 = 0$ case) iff $\nu \geq 0$ and $h^0(S; \mathcal{L}(-1)) = 0$.

Now $S$ must be chosen from the Noether-Lefschetz locus. The hypersurfaces of $\mathbb{P}^3$ of degree $k$ are parametrized by a $\mathbb{P}^{N_k}$ for $N_k = \binom{k+3}{3} - 1$. M. Noether stated and Lefschetz proved that there is a countable union of subvarieties $NL \subset \mathbb{P}^{N_k}$ such that $S \notin NL$ implies $Pic(S) \cong \mathbb{Z}$ is generated by $\mathcal{O}_S(1)$. See [12] for a modern proof and also [2] and [10] for interested properties, references, and questions about the Noether-Lefschetz locus. The component of $NL$ of smallest codimension $k-3$ (and the only such component) consists of the surfaces in $\mathbb{P}^3$ containing a line [11, 10, 26].

4. SURFACES IN $\mathbb{P}^3$ CONTAINING A LINE

Let $S \subset \mathbb{P}^3$ be a smooth degree $k$ surface ($k \geq 2$) containing a line $L$. Denote the pencil of hyperplane sections of $S$ containing $L$ by $H_t$ $t \in \mathbb{P}^1$ and let $H$ be a general hyperplane section (not containing $L$).
Lemma 1. For $H_t = L + C_t$, degree $C_t = k - 1$ and, using intersection on $S$,

\[
L^2 = 2 - k \\
C_t \cdot L = k - 1 \\
C_t^2 = 0.
\]

Furthermore, the generic $C_t$ is irreducible and smooth and the pencil $\{C_t\}$ is base point free and thus gives a regular map $S \xrightarrow{\pi} \mathbb{P}^1$.

Proof. The genus formula applied to $L$ gives $0 = 1 + \frac{1}{2}(L^2 + K_S \cdot L) = 1 + \frac{1}{2}(L^2 + k - 4)$ i.e. $L^2 = 2 - k$.

\[
H_t^2 = k = L^2 + 2L \cdot C_t + C_t^2 \\
2k - 2 = 2L \cdot C_t + C_t^2 \\
(4.1) \\
H_t \cdot C_t = k - 1 = L \cdot C_t + C_t^2
\]

Subtraction gives $L \cdot C_t = k - 1$ and so $C_t^2 = 0$. The base locus of $\{C_t\}$ is contained in $L$ and therefore is finite. Now $0 = C_t^2 = C_{t_1} \cdot C_{t_2} \geq 0$ implies that $\{C_t\}$ is base point free. Use $C$ to denote an arbitrary $C_t$ and consider

\[
(4.2) \\
0 \rightarrow O_S \rightarrow O_S(C) \rightarrow O_C(C) \rightarrow 0.
\]

Since $H^1(S; O_S(j)) = 0 \ \forall j$ (as follows from the cohomology sequence of $0 \rightarrow O_{\mathbb{P}^3}(j - k) \rightarrow O_{\mathbb{P}^3}(j) \rightarrow O_S(j) \rightarrow 0$), and $O_C(C) = O_C$, the cohomology sequence of $O_S$ gives

\[
0 \rightarrow H^0(S; O_S) \rightarrow H^0(S; O_S(C)) \rightarrow H^0(C; O_C) \rightarrow 0
\]

which shows that $h^0(S; O_S(C)) = 2$ and that $O_S(C)$ is globally generated. Bertini’s theorem implies that the generic $C_t$ is smooth. If $C_t$ had two distinct irreducible components, they must be disjoint by smoothness. But this is impossible because they are both contained in the same plane. Therefore $C_t$ is irreducible. \qed

Let $O_S(aL + bC)(j) \equiv O_S(aL + bC) \otimes O_S(j)$ for $a, b, j \in \mathbb{Z}$ and note that this is isomorphic to $O_S((a - b)L)(b + j)$ and to $O_S((b - a)C)(a + j)$. We will make frequent use of these isomorphisms and the

Lemma 2. For $a, b, j \geq 0$

i) $H^0(S; O_S(-aL)(j)) = 0$ iff $a > j$.

ii) $H^0(S; O_S(-bC)(j)) = 0$ iff $b > j$.

iii) $h^0(S; O_S(bC)) = b + 1$ and $O_S(bC)$ is globally generated.

iv) $H^1(S; O_S(-aL)(-j)) = 0$ iff $j > (a - 1)(k - 2) \text{ or } j = 0, a = 1 \text{ or } a = 0$.

v) $H^1(S; O_S(-bC)(-j)) = 0$ iff $j > 0$, or $j = 0, b = 1 \text{ or } b = 0$.

vi) For $j > k - 4$ and $b \geq 0$, $h^0(S; O_S(bC)(j)) = \binom{j + 3}{3} - \binom{j - k + 3}{3} + b\binom{j + 2}{2} - \binom{j - k + 3}{2}$.
Proof. For $\sigma \in H^0(S; \mathcal{O}_S(-aL)(-j))$ choose $C$ which is not an irreducible component of $Z_\sigma$. Then $0 \leq Z_\sigma \cdot C = (-aL + jH) \cdot C = -a(k-1) + j(k-1)$ so $a \leq j$. If $a \leq j$, $\mathcal{O}_S(-aL)(j)$ clearly has global sections so i) holds.

To prove ii) first note that $\mathcal{O}_S(-bC)(j)$ also clearly has sections if $b \leq j$. If $b > j$ suppose $\sigma \in H^0(S; \mathcal{O}_S(-bC)(j))$. If $L \not\subseteq Z_\sigma, 0 \leq L \cdot Z_\sigma = L \cdot (-bC + jH) = -b(k-1) + j < 0$, a contradiction. Therefore $L \subseteq Z_\sigma$ and so $\mathcal{O}_S(-bC - L)(j) = \mathcal{O}_S(-bL)(j-1)$ has a global section. Repeating this argument gives that $\mathcal{O}_S(-(b - j)C)$ has a non-zero global section, which is not true.

Note that $\mathcal{O}_S(bC) \cong \mathcal{O}_S(C)^{\oplus b}$ is globally generated because $\mathcal{O}_S(C)$ is. This and the cohomology sequence of

$$0 \rightarrow \mathcal{O}_S((j-1)C) \rightarrow \mathcal{O}_S(jC) \rightarrow \mathcal{O}_C(jC) \cong \mathcal{O}_C \rightarrow 0$$

gives

$$0 \rightarrow H^0(S; \mathcal{O}_S((j-1)C)) \rightarrow H^0(S; \mathcal{O}_S(jC)) \rightarrow H^0(C; \mathcal{O}_C) \rightarrow 0$$

and iii) follows by induction.

The group $H^1(S; \mathcal{O}_S(-j))$ vanishes for all $j$. Careful examination of the cohomology sequences

$$0 \rightarrow H^0(S; \mathcal{O}_S(-iL)(-j)) \rightarrow H^0(S; \mathcal{O}_S(-(i-1)L)(-j))$$
$$\quad \rightarrow H^0(L; \mathcal{O}_L((k-2)(i-1) - j)) \rightarrow H^1(S; \mathcal{O}_S(-iL)(-j))$$
$$\quad \rightarrow H^1(S; \mathcal{O}_S(-(i-1)L)(-j)) \rightarrow H^1(L; \mathcal{O}_L((k-2)(i-1) - j)) \ldots$$

for $1 \leq i \leq a$ shows that $H^1(S; \mathcal{O}_S(-aL)(-j)) = 0$ iff $j > (a-1)(k-2)$ or $j = 0, a = 0, 1$ , proving iv). The sequences

$$0 \rightarrow H^0(S; \mathcal{O}_S(-iC)(-j)) \rightarrow H^0(S; \mathcal{O}_S(-(i-1)C)(-j)) \rightarrow H^0(C; \mathcal{O}_C(-j))$$
$$\quad \rightarrow H^1(S; \mathcal{O}_S(-(i-1)C)(-j)) \rightarrow H^1(S; \mathcal{O}_S(-iC)(-j)) \rightarrow H^1(S; \mathcal{O}_S(-(i-1)C)(-j)) \rightarrow \ldots$$

for $1 \leq i \leq b$ imply that $H^1(S; \mathcal{O}_S(-bC)(-j)) = 0$ exactly when $j > 0, b \geq 0$ and $j = 0, b = 0, 1$ , proving v). The cohomology sequence of

$$0 \rightarrow \mathcal{O}_S(-j) \rightarrow \mathcal{O}_S(j) \rightarrow \mathcal{O}_S(j) \rightarrow 0$$

gives $h^0(S; \mathcal{O}_S(j)) = \binom{j+2}{2} - \binom{j-k+3}{3}$. Similarly, $h^0(C; \mathcal{O}_C(j)) = \binom{j-2}{2} - \binom{j-k+3}{2}$. The sequences

$$0 \rightarrow H^0(S; \mathcal{O}_S((i-1)C)(j)) \rightarrow H^0(S; \mathcal{O}_S(iC)(j)) \rightarrow H^0(C; \mathcal{O}_C(j))$$
$$\quad \rightarrow H^1(S; \mathcal{O}_S((i-1)C)(j)) \rightarrow \ldots$$

$1 \leq i \leq b$ and the vanishing $H^1(S; \mathcal{O}_S((i-1)C)(j)) \cong H^1(S; \mathcal{O}_S(-(i-1)C)(-j + k-4)) = 0$ for $j > k - 4$ (by part v) give

$$h^0(S; \mathcal{O}_S(bC)(j)) = h^0(S; \mathcal{O}_S(j)) + bh^0(C; \mathcal{O}_C(j))$$

and vi) follows. \qed
Lemma 3. If $a$ or $b < 0$, $h^0(S; \mathcal{O}_S(aL + bC)) = 0$. When $a, b \geq 0$:

For $b \geq a \geq k - 3$ or $a \geq b$ and $b(k - 1) - a(k - 2) \geq 0$,

$$h^0(S; \mathcal{O}_S(aL + bC)) = (k - 1)ab - \frac{(k - 2)a^2 - (k - 4)}{2}(a + (k - 1)b) + \binom{k - 1}{3} + 1.$$  

For $a \geq b$ and $j_0$ the largest integer between $0$ and $a - b$ such that $b - j_0(k - 2) \geq 0$,

$$h^0(S; \mathcal{O}_S(aL + bC)) = \binom{b + 3}{3} - \binom{b - k + 3}{3} + (b + 1)j_0 - (k - 2)\binom{j_0 + 1}{2}.$$  

For $b \geq a, a \leq k - 2$,

$$h^0(S; \mathcal{O}_S(aL + bC)) = \binom{a + 2}{2}b - \frac{2a}{3} + 1.$$  

Proof. For $a, b < 0$, it is clear that $h^0(S; \mathcal{O}_S(aL + bC)) = 0$. If $b \geq 0 > a$, $h^0(S; \mathcal{O}_S(aL + bC)) = h^0(S; \mathcal{O}_S((a - b)L)(b) = 0$ by Lemma 2. The case $a \geq 0 > b$ is handled in the same way.

Assume $b \geq a \geq k - 3$. Then $h^0(S; \mathcal{O}_S(aL + bC)) = h^0(S; \mathcal{O}_S((b - a)C)(a))$ and by Lemma 2, this is

$$\binom{a + 3}{3} - \binom{a - k + 3}{3} + (b - a)[\binom{a + 2}{2} - \binom{a - k + 3}{2}]$$

which is easily shown to equal our formula.

If $b \geq a$ and $a \leq k - 2$, we use the mapping $S \xrightarrow{\pi} \mathbb{P}^1$ and $H^0(S; \mathcal{O}_S((b - a)C)(a)) \cong H^0(\mathbb{P}^1; \pi_*\mathcal{O}_S((b - a)C)(a))$. To calculate the direct image sheaf, let the homogeneous coordinates of $\mathbb{P}^3 = PV$ be chosen so that the line $L$ is given by $x_2 = 0, x_3 = 0$. Set $W = \{\xi = \xi_2x_2 + \xi_3x_3 \in H^0(S; \mathcal{O}_S(1)) \cong V^*\}$, if $S \to \mathbb{P}^3$ is defined in by $g = 0$ then $g = x_2g_2 + x_3g_3$, for $g_2, g_3$ of degree $k - 1$. $S \cap H_\xi = L + C_\xi$

where $C_\xi$ is defined by $g_\xi = \xi_3g_3 - \xi_2g_2 = 0$. Then $S \to \mathbb{P}^W$ is given by $p(\xi) = \{\xi \in W | g_\xi(p) = 0\}$ and $C_\xi$ is the fiber over $\xi \in \mathbb{P}^1$. Let $C$ be a fixed fiber defined by $t = 0$, for a coordinate on $\mathbb{P}^1$. Then the isomorphism $H^0(C; \mathcal{O}_C((b - a)C)(a)) \cong H^0(C_\xi; \mathcal{O}_{C_\xi}(a))$ is given by $t^{b-a}s \to s$ for $s \in H^0(C_\xi; \mathcal{O}_{C_\xi}(a))$. For $Ann\xi \equiv \{x \in V | \xi(x) = 0\}$, $C_\xi$ is a curve in $\mathbb{P}^W$ of degree $k - 1$ and we have a restriction isomorphism $H^0(\mathbb{P}^Ann\xi; \mathcal{O}_{\mathbb{P}^Ann\xi}(a)) \cong H^0(C_\xi; \mathcal{O}_{C_\xi}(a))$ when $a \leq k - 2$.

Writing $V^* = U \oplus W$ for $U = span\{x_0, x_1\}$,

$$H^0(\mathbb{P}^Ann\xi; \mathcal{O}_{\mathbb{P}^Ann\xi}(a)) \cong \text{Sym}^a(V^*/C_\xi)$$

$$\cong \bigoplus_{i=0}^a \text{Sym}^i U \otimes \text{Sym}^{a-i}(W/C_\xi)$$

which gives

$$\pi_*\mathcal{O}_S((b - a)C)(a) \cong \mathcal{O}_{\mathbb{P}^1}(b - a) \otimes \bigoplus_{i=0}^a \text{Sym}^i U \otimes \mathcal{O}_{\mathbb{P}^1}(a - i)$$

$$\cong \bigoplus_{i=0}^a \mathcal{O}_{\mathbb{P}^1}(b - i)^{\oplus i+1}.$$
Therefore
\[ h^0(S; \mathcal{O}_S((b-a)C)(a)) = \sum_{i=0}^{a} (i+1)(b-i+1) = \left( \frac{a+2}{2} \right)[b - \frac{2a}{3} + 1]. \]

Now assume \( a \geq b \) and \( b(k-1) - a(k-2) \geq 0 \). Then \( h^0(S; \mathcal{O}_S(aL + bC)) = h^0(S; \mathcal{O}_S((a-b)L)(b)) \) and consider the cohomology of the sequences
\[ 0 \rightarrow \mathcal{O}_S((j-1)L)(b) \rightarrow \mathcal{O}_S(jL)(b) \rightarrow \mathcal{O}_L(b - j(k-2)) \rightarrow 0 \]
for \( 1 \leq j \leq a-b \). Since \( b(k-1) - a(k-2) \geq 0 \), \( b-j(k-2) \geq 0 \) for all \( j \) and \( h^1(S; \mathcal{O}_S((j-1)L)(b)) = h^1(S; \mathcal{O}_S((1-j)L)(k-4-b)) = 0 \) by Lemma \( 2 \) because \( b-k+4 > (k-2)(a-b-2) \) is \( b(k-1) - a(k-2) + k > 0 \). This gives
\[ h^0(S; \mathcal{O}_S((a-b)L)(b)) = \left( \frac{b+3}{3} \right) - \left( \frac{b-k+3}{3} \right) + (a-b)(b+1) - (k-2) \left( \frac{a-b+1}{2} \right) \]
which is equivalent to our formula.

If \( a \geq b \) but \( b-j(k-2) < 0 \) for some \( 1 \leq j < a-b \), the above argument is easily adjusted to give our result. \( \square \)

5. Examples of Rank 2 Bundles

Let \( S \in \mathbb{P}^3 \) be a smooth surface of degree \( k = 2\nu + c_1 \) containing a line \( L \), \( \nu \in \mathbb{Z}, c_1 = 0 \) or \(-1\), and \( L \) a line bundle on \( S \) determined by \( L = \mathcal{O}_S(-aL - bC)(\nu + c_1) \) where \( a, b \in \mathbb{Z} \). This gives \( L^* (\nu + c_1) = \mathcal{O}_S(aL + bC) \). As in section 3, we examine the rank 2 extensions
\[ \begin{align*}
0 & \rightarrow \mathbb{P}^3(-\nu)_{\sigma_{12}} \rightarrow E_{\tau_{1\tau_2}} \rightarrow j_{S*}\mathcal{O}_S(-aL - bC)(\nu + c_1) \rightarrow 0
\end{align*} \]
and determine which divisors \( aL + bC \) have the property that the generic extension (5.1) is a stable bundle. Recall that in the dual sequence, in the case that \( E \) is locally free,
\[ \begin{align*}
0 & \rightarrow E^*_{\tau_{1\tau_2}} \rightarrow \mathbb{P}^3(\nu)_{\sigma_{12}} \rightarrow j_{S*}\mathcal{O}_S(aL + bC)(\nu) \rightarrow 0
\end{align*} \]
\( \tau \in \mathbb{P}^3 H^0(S; \mathcal{O}_S(aL + bC)) \) is the extension class of (5.1).

**Theorem 1.** The generic extensions of the form (5.1) with \( D = aL + bC \) are stable rank 2 bundles in exactly the following cases:\( \text{recall } k = 2\nu + c_1 \)

(1) For \( k = 2(\nu = 1, c_1 = 0) \), \( a = 0 \) and \( b \geq 2 \) or vice versa. Here \( S \) is a smooth quadric \( Q \). Using the bidegree notation for line bundles on \( Q \), either \( L = \mathcal{O}_Q(1,1-b) \) for \( b \geq 2 \) and \( L^* (\nu + c_1) = \mathcal{O}_Q(0,b) \) or \( L = \mathcal{O}_Q(1-b,1) \) and \( L^* (\nu + c_1) = \mathcal{O}_Q(b,0) \). \( c_2(E) = b-1 \).
(2) For $k \geq 3$, $a = 0$ and $b > \nu + c_1$. $L^*(\nu + c_1) = O_S(bC)$ and $L = O_S(-bC)(\nu + c_1)$. $c_2(E) = b(2\nu + c_1 - 1) - \nu(\nu + c_1) = b(k - 1) - (k^2 - c_1^2)/4$.

Note that, if $k = 2$ in statement (2), statement (1) results. No other values of $a$ and $b$ produce stable bundles.

Proof. From Section 3 we know that $k=1$ can not occur and that the generic extension $E$ is a stable bundle iff $O_S(D)$ is globally generated, $D^2 = 0$, and $h^0(S; \mathcal{L}) = 0$. $D^2 = 0$ gives $a^2(2 - k) + 2ab(k - 1) = 0$ and so

\[(5.3) \quad a = 0 \text{ or } 2b(k - 1) - a(k - 2) = 0.\]

Since $O_L(D) = O_L(aL^2 + bC \cdot L) = O_L(b(k - 1) - a(k - 2))$ is globally generated,

\[(5.4) \quad b(k - 1) - a(k - 2) \geq 0.\]

Since $O_C(D)$ is globally generated, $0 \leq \deg O_C(D) = C \cdot (aL + bC)$, i.e.

\[(5.5) \quad a(k - 1) \geq 0.\]

If $k=2$, then $\nu = 1$, $c_1 = 0$, $S=Q$, and $L$ and $C$ are lines from the two pencils of lines on $Q$. The equations give that either $a$ or $b = 0$ and the other is non-negative. We can assume $a = 0$. Then $\mathcal{L} \cong O_Q(1, 1 - b)$ and $h^0(Q; \mathcal{L}) = 0$ implies $b \geq 2$. Now $c_2(E) = b - 1$ follows from (5.6).

If $k \geq 3$, $a = 0$ because otherwise (5.3) and (5.4) give $b(k - 1) \geq a(k - 2) = 2b(k - 1)$ and so $0 \geq b(k - 1)$ i.e. $b \leq 0$. Now (5.4) and (5.5) give $a = 0 = b$. But then $\mathcal{L} = O_S(\nu + c_1)$ has non-zero global sections. Now $\mathcal{L} = O_S(-bC)(\nu + c_1)$ will have $h^0 = 0$ iff $b > \nu + c_1$ by Lemma 2 ii.\qed
Proposition 1. Let $E \to \mathbb{P}^3$ be a rank 2 stable bundle of the type constructed in Theorem 4. Then

i) For $l \geq -c_1 - 4$, $H^3(O_{\mathbb{P}^3}; E(l)) = 0$.

ii) For $l > \nu - 4$, $H^2(O_{\mathbb{P}^3}; E(l)) = 0$.

iii) For $l > b(k - 1) - \nu - c_1 - 2$, $H^1(O_{\mathbb{P}^3}; E(l)) = 0$.

iv) $E(l)$ is globally generated iff $l \geq b(k - 1) - \nu - c_1$.

v) The line $L \subset S$ is a jumping line of $E$ of jump size $m = b(k - 1) - \nu$ i.e. $E_L \cong O_L(m) \oplus O_L(-m + c_1)$.

Proof. $H^3(O_{\mathbb{P}^3}; E(l)) \cong H^0(O_{\mathbb{P}^3}; E(-l - c_1 - 4)) = 0$ for $-l - c_1 - 4 \leq 0$ because $E$ is stable so i) holds. The cohomology sequence of

$$0 \to 2 \oplus O_{\mathbb{P}^3}(l - \nu) \to E(l) \to j_* O_S(-bC)(\nu + c_1 + l) \to 0$$

gives $H^2(O_{\mathbb{P}^3}; E(l)) \cong H^2(S; O_S(-bC)(\nu + c_1 + l))$ when $l > \nu - 4$. $H^2(S; O_S(-bC)(\nu + c_1 + l)) \cong H^0(S; O_S(bC)(\nu - c_1 - l - k - 4))^* \cong H^0(S; O_S(-bL)(\nu - 4 - l + b))^* = 0$ if $l > \nu - 4$ by Lemma 2. The sequence also gives $H^1(O_{\mathbb{P}^3}; E(l)) \cong H^1(S; O_S(-bC)(\nu + c_1 + l)) \cong H^1(S; O_S(bC)(\nu - 4 - l))^* \cong H^1(S; O_S(-bL)(b + \nu - 4 - l))^* = 0$ for $l > b(k - 1) - \nu - c_1 - 2$ by Lemma 2. This proves ii). From

$$0 \to H^0(\mathbb{P}^3; O_{\mathbb{P}^3}(l - \nu)) \to H^0(\mathbb{P}^3; E(l)) \to H^0(S; O_S(-bC)(\nu + c_1 + l)) \to 0$$

one sees that $E(l)$ is globally generated iff 1) $l \geq \nu$ and 2) $O_S(-bC)(\nu + c_1 + l) \cong O_S(bL)(\nu + c_1 + l - b)$ is globally generated. A necessary condition for 2) is $i \equiv \nu + c_1 + l - b \geq 0$. (Lemma 2). The cohomology sequences of

$$0 \to O_S((j - 1)L)(i) \to O_S(jL)(i) \to O_L(jL)(i) \cong O_L(i - (k - 2)j) \to 0$$

for $j = 1$ to b show that $i - (k - 2)b \geq 0$ is also necessary. It is also sufficient because $H^1(S; O_S((j - 1)L)(i)) \cong H^1(S; O_S(-(j - 1)L)(i - k - 4))^* = 0$ for $i - (k - 2) > (b - 2)(k - 2)$, that is, $i > b(k - 2) - b$ by Lemma 2. Thus $l \geq b(k - 1) - \nu - c_1$ is necessary and sufficient for 2). Note that $b(k - 1) - \nu - c_1 \geq (\nu + c_1 + 1)(2\nu + c_1 - 1) - \nu - c_1 = 2\nu^2 + 3c_1\nu - 1 - 2c_1 \geq \nu$ for $k \geq 2$. Therefore $l \geq b(k - 1) - \nu - c_1$ is a necessary and sufficient condition for $E(l)$ to be globally generated.

To examine $L$ as a jumping line of $E$ express $E_L = O_L(m) \oplus O_L(-m + c_1)$ for some $m \geq 0$ and restrict $E_L$ to $L$ to get

$$0 \to im(\sigma_{1L} \oplus \sigma_{2L}) \to O_L(m) \oplus O_L(-m + c_1) \to O_L(\nu + c_1 - b(k - 1)) \to 0.$$

Because $\nu + c_1 - b(k - 1) < 0$, it is clear that $-m + c_1 = \nu + c_1 - b(k - 1)$ which gives the result. □

To make some observations about moduli, let $\mathcal{M}$ be the moduli space of $S$-equivalence classes of semi-stable rank two sheaves on $\mathbb{P}^3$ with fixed chern classes $c_1 = 0$ or $-1, c_2,$ and $c_3 = 0$, a projective scheme containing the stable rank two bundles as an open subset. For $E$ a rank two stable bundle, the Zariski tangent space of $\mathcal{M}$ at $E$ is

$$TM_E \cong H^1(\mathbb{P}^3; \text{End}(E))$$
and one knows that

\[(5.7) \quad h^1(\mathbb{P}^3; \mathcal{E}nd(E)) \geq \dim EM \geq h^1(\mathbb{P}^3; \mathcal{E}nd(E)) - h^2(\mathbb{P}^3; \mathcal{E}nd(E))\]

and \(h^2(\mathbb{P}^3; \mathcal{E}nd(E)) = 0\) implies that \(M\) is smooth at \(E\) [10, Sect. 4.5]. From Riemann-Roch,

\[(5.8) \quad h^1(\mathbb{P}^3; \mathcal{E}nd(E)) - h^2(\mathbb{P}^3; \mathcal{E}nd(E)) = 8c_2(E) + 2c_1 - 3.\]

We want to count the parameters of our construction. Note that the basic sequences \((5.1)\) and \((5.2)\) or, more generally, \((3.1)\) and \((3.2)\) are dual to one another. Also note that, when sequences \((5.1)\) and \((5.2)\) or, more generally, \((3.1)\) and \((3.2)\) are dual to one another. Also when \(k \geq 3\), the isomorphism class of \(L = \mathcal{O}_S(-b\ell)(\nu + c_1) \cong \mathcal{O}_S(b\ell)(\nu + c_1 - b)\) is determined by the line \(L\) because two lines on \(S\) (or integer multiples of lines) can not be linearly equivalent (or even homologically equivalent): if \(L, L' \subset S\) are homologically equivalent, \(L^2 = L \cdot L' \geq 0\); but we know \(L^2 = -(k - 2) < 0\). Also when \(k \geq 3\), \(S\) can contain only a finite number of lines. To see this, let \(G\) be the grassmannian of lines and \(S\) the universal sub-bundle over \(G\). Then the degree \(k\) polynomial \(g\) defining \(S\) can be viewed as a global section of \(\text{Sym}^k(S^*)\) whose zeroes are the lines contained in \(S\). The zero set of \(g\) is either finite or of positive dimension. In the latter case, since \(\text{Pic}(S)\) is discrete, there are linearly equivalent lines on \(S\), a contradiction.

For fixed \(c_1, \nu, a = 0\), and \(b\), \((S, L, \tau)\) defines \((E, \sigma)\) and the function \((S, L, \tau) \rightarrow (E, \sigma)\) is injective but not a priori surjective, as we explain. From Theorem [11] \(E\) has the form

\[(5.9) \quad 0 \longrightarrow \oplus \mathcal{O}_{\mathbb{P}^3}(\nu) \longrightarrow E \longrightarrow j_{S*}\mathcal{O}_S(-b\ell)(\nu + c_1) \longrightarrow 0.\]

Choose a different \(\bar{\sigma} \in \oplus H^0(\mathbb{P}^3; E(\nu))\); this produces another sequence

\[(5.10) \quad 0 \longrightarrow \oplus \mathcal{O}_{\mathbb{P}^3}(\nu) \longrightarrow E \longrightarrow j_{\bar{S}*}\bar{\ell} \longrightarrow 0\]

for \(\bar{S}\) another smooth surface of degree \(k\) and \(\bar{\ell}\) a line bundle on \(\bar{S}\). Does \(\bar{S}\) contain a line and, if so, is \(\bar{\ell}\) of the form \(\mathcal{O}_S(-b\ell)(\nu + c_1)\)? We show, somewhat surprisingly, that the answer to both questions is affirmative. Note that these considerations are relevant only when \(b \leq k\) because \(b > k\) implies that \(h^0(\mathbb{P}^3; E(\nu)) = 2\) and so \(\bar{\sigma}\) differs from \(\sigma\) by a basis change.

The cohomology sequences of \((5.7)\) and \((5.10)\) and Lemma [2] imply that \(h^0(\bar{S}; \bar{\ell}(b-\nu - c_1) = h^0(S; \mathcal{O}_S(b\ell)) = 1\) Therefore \(\bar{\ell}(b-\nu - c_1) \cong \mathcal{O}_S(D)\) for \(D\) effective. The chern class formulas \((3.5)\) and \((3.5)\) imply

\[(5.11) \quad \deg \bar{D} = \omega_0 \cdot c_1(\mathcal{O}_S(\bar{D}))
= \omega_0 \cdot c_1(\mathcal{O}_S(b\ell))
= H \cdot bL
= b\]

and
\[c_1(\mathcal{L})^2 = c_1(O_S(-bC)(\nu + c_1))^2\]
\[\left(\bar{D} - (b - \nu - c_1)H\right)^2 = ((\nu + c_1)H - bC)^2\]
\[\bar{D}^2 = -(k - 2)b^2.\]

Express \(\bar{D} = \sum_i m_i Y_i\) for \(Y_i\) irreducible curves and \(m_i \in \mathbb{Z}^+\). The genus formula gives
\[g_i = 1 + \frac{1}{2}(Y_i^2 + (k - 4)\deg Y_i)\]
\[Y_i^2 \geq -(k - 2)\deg Y_i.\]

Now (5.12) implies
\[-(k - 2)b^2 = \sum_i m_i^2 Y_i^2 + 2 \sum_{i<j} m_i m_j Y_i \cdot Y_j\]
\[\geq -(k - 2) \sum_i m_i^2 \deg Y_i.\]

Using \(\sum_i m_i \deg Y_i = b\),
\[\sum_i m_i^2 \deg Y_i - b^2 = \sum_i m_i^2 \deg Y_i - b \sum_i m_i \deg Y_i\]
\[= \sum_i m_i \deg Y_i (m_i - b)\]
\[\leq 0\]
with equality if and only if there is only one term in the sum, \(b = m_1\), and \(\deg Y_1 = 1\).

But (5.13) and (5.14) show that equality must hold and so \(\bar{D} = bL\) for \(L\) a line on \(\bar{S}\). This gives \(\mathcal{L} \cong O_S(-bC)(\nu + c_1)\) for \(H = L + C\) a hyperplane section of \(\bar{S}\). We have proven that, for \(k \geq 3\), there is a 1-to-1 correspondence

\[(S, L, \tau) \leftrightarrow (E, \sigma).\]

Note that a linear change in \((\tau_1, \tau_2)\) produces an isomorphic \(E\) and a corresponding linear change in \((\sigma_1, \sigma_2)\). Similarly, a linear change in \((\sigma_1, \sigma_2)\), does not change \(S\) or \(\mathcal{L}\) and produces a linear change is \((\tau_1, \tau_2)\). Let \(G_2 \equiv G_2(H^0(\mathbb{P}^3; E(\nu)))\) and \(G^*_2 \equiv G_2H^0(S; \mathcal{L}^*(\nu + c_1))\) be grassmannians. For \([\sigma] \in G_2\) and \([\tau] \in G^*_2\), our 1-to-1 correspondence can be refined to:

\[(S, L, [\tau]) \leftrightarrow (E, [\sigma]).\]

When \(k=2\), \(S\) is a smooth quadric \(Q\) with two linear equivalence classes of lines, \(\pm\). In this case the 1-to-1 correspondence is \((E, \sigma) \leftrightarrow (Q, \pm, \tau)\).

Denote by \(\mathcal{Y}\) the subset of \(\mathcal{M}\) consisting of isomorphism classes of stable bundles of the form (5.1). Define \(\dim \mathcal{Y}\) as the number of independent parameters determining \(E\) (see Proposition below). In general we expect \(\dim E\mathcal{Y} < \dim E\mathcal{M}\). In Section 6 we will discuss what conditions imply that \(\mathcal{Y}\) has a natural scheme structure and that \(\mathcal{Y} \hookrightarrow \mathcal{M}\) is a regular map.
Proposition 2. Let $E \rightarrow \mathbb{P}^3$ be a rank 2 stable bundle of the type constructed in Theorem 1. Let $\mathcal{Y} \subset \mathcal{M}$ be the set of these bundles. Then

$$
dim \mathcal{Y} = \begin{cases} 
\left(\frac{k+3}{3}\right) + 2b - k & \text{when } b > k \geq 3 \\
21 & \text{when } k = 3, b = 3 \\
11 & \text{when } k = 3, b = 2 \\
2b + 7 & \text{when } k = 2, b \geq 3 \\
5 & \text{when } k = 2, b = 2 \\
\left(\frac{k+3}{3}\right) + 2b - k - 2\left(\frac{k-b+3}{3}\right) & \text{when } b \leq k, k \geq 4.
\end{cases}
$$

Proof. From the 1-to-1 correspondence (5.15),

$$
dim \mathcal{Y} = \dim \{S\} + \dim \{\tau\} - \dim \{\sigma\}
$$

(by deriving an upper bound for $\dim \{S, L\}$ = $\dim \{S\}$ when $k \geq 3$ since $S$ contains at most a finite number of lines.

When $k = 2$, i.e. $S$ is a smooth quadric $Q$, we can calculate directly that $h^0(Q; \mathcal{O}_Q(0, b)) = b + 1$ and $h^0(Q; \mathcal{O}_Q(2, 2 - b)) = 3$ when $b = 2$ and 0 when $b \geq 3$ which gives the result in this case.

When $k \geq 3$, using Lemma 2ii,

$$
dim \mathcal{Y} = \left(\frac{k+3}{3}\right) - k + 2b - 2h^0(S; \mathcal{O}_S(-bC)(k)).
$$

For $b > k$, $h^0(S; \mathcal{O}_S(-bC)(k)) = 0$ by Lemma 2i. For $b \leq k$, use $h^0(S; \mathcal{O}_S(-bC)(k)) = h^0(S; \mathcal{O}_S(hL)(k - b))$ and the sequences

$$
0 \rightarrow \mathcal{O}_S(jL)(k - b) \rightarrow \mathcal{O}_S(jL)(k - b) \rightarrow \mathcal{O}_L(k - b - (k - 2)j) \rightarrow 0
$$

for $1 \leq j \leq b$. For $j \geq 2$ or $j = 1$ and either $k \geq 4$ or $b \geq 3$, $k - b - (k - 2)j < 0$ and so the cohomology sequence gives $h^0(S; \mathcal{O}_S(bL)(k - b)) = h^0(S; \mathcal{O}_S(k - b)) = \left(\frac{k-b+3}{3}\right)$. The remaining case, $k=3$, $b=2$, yields $h^0(\mathcal{O}_S(2L)(1)) = 5$. This gives our formula when $k \geq 3$.

For the bundles of Theorem 1, the formula (5.14) becomes

$$
h^1(\mathbb{P}^3; \mathcal{E}nd(E)) - h^2(\mathbb{P}^3; \mathcal{E}nd(E)) = 8b(k - 1) - 2k^2 - 3.
$$

Therefore by choosing $b$ large compared to $k^2$, one gets $\dim \mathcal{E} \mathcal{M}$ much smaller that $\dim \mathcal{E} \mathcal{M}$ but, choosing $b = k + 1$, one gets, for large $k$, $\dim \mathcal{E} \mathcal{Y} >$ the expected dimension of $\mathcal{M}$ at $E$.

We now obtain an upper bound for $h^1(\mathbb{P}^3; \mathcal{E}nd(E))$ by deriving an upper bound for $h^2(\mathbb{P}^3; \mathcal{E}nd(E))$. When $k=2$ or 3, this will give $\dim \mathcal{E} \mathcal{M}$ exactly. To set up the framework for these calculations, write out (5.11) and (5.2) in this case,

$$
0 \rightarrow 2\mathcal{O}_{\mathbb{P}^3}(-\nu) \rightarrow E \rightarrow j_{\ast} \mathcal{O}_S(-bC)(\nu + c_1) \rightarrow 0
$$
(5.18) \[ 0 \to E^* \to \oplus O_{\mathbb{P}^3}(\nu) \to j^* O_S(bC)(\nu) \to 0. \]

Tensor (5.18) with \( E \) to get

(5.19) \[ 0 \to \text{End}(E) \to \oplus E(\nu) \to j^* E_S(bC)(\nu) \to 0. \]

Tensoring (5.17) with \( O_S \) and calculating \( O_S \oplus O_{\mathbb{P}^3} O_S(-bC)(\nu+c_1) \cong O_S(-bC)(\nu+c_1) \) and \( \text{Tor}_1^{O_S}(O_S, O_S(-bC)(\nu+c_1)) = O_S(-bC)(-\nu) \) yields

\[ 0 \to O_S(-bC)(-\nu) \to \oplus O_S(-\nu) \to E_S \to O_S(-bC)(\nu+c_1) \to 0 \]

which can be written as

(5.20) \[ 0 \to O_S \to \oplus O_S(bC) \to E_S(bC)(\nu) \to O_S(k) \to 0 \]

which can be broken up into two short exact sequences

(5.21) \[ 0 \to O_S \to \oplus O_S(bC) \to K \to 0 \]

The cohomology sequence of (5.19) and Proposition 1ii yield

(5.22) \[ 0 \to H^0(\mathbb{P}^3; \text{End}(E)) \to \oplus H^0(\mathbb{P}^3; E(\nu)) \to H^0(S; E_S(bC)(\nu)) \to \]

\[ H^1(\mathbb{P}^3; \text{End}(E)) \to \oplus H^1(\mathbb{P}^3; E(\nu)) \to H^1(S; E_S(bC)(\nu)) \to H^2(\mathbb{P}^3; \text{End}(E)) \to 0 \]

and therefore \( h^2(\mathbb{P}^3; \text{End}(E)) \leq h^1(S; E_S(bC)(\nu)). \) From the cohomology of the second sequence in (5.21), \( h^1(S; E_S(bC)(\nu)) \leq h^1(S; K) \) and from the first sequence, \( h^1(S; K) \leq 2 h^1(S; O_S(bC)) + h^2(S; O_S). \) Note that \( h^2(S; O_S) = h^0(S; O_S(k-4)) = \binom{k-1}{3} = 0 \) for \( k=2,3 \) and \( h^1(S; O_S(bC)) = h^1(S; O_S(-bC)(k-4)) = 0 \) for \( k=2,3 \) by Lemma 2i. Therefore by 5.7 and 5.10.

**Theorem 2.** Let \( E \) be a stable rank 2 bundle as constructed in Theorem 1. For \( k=2 \) or 3, \( H^2(\mathbb{P}^3; \text{End}(E)) = 0 \) and so the moduli space \( \mathcal{M} \) containing \( E \) is smooth at \( E \) and \( T \mathcal{M} = H^1(\mathbb{P}^3; \text{End}(E)). \) Its dimension is

\[
\dim \mathcal{M} = \begin{cases} 
8b - 11 & k = 2, b \geq 2 \\
16b - 21 & k = 3, b \geq 2.
\end{cases}
\]

For \( k \geq 4, b > k-4 \) the inequalities above only give an estimate for \( h^1(\mathbb{P}^3; \text{End}(E)) \) and thus \( \dim \mathcal{M}. \) Riemann-Roch calculates \( \chi(S; O_S(bC)) = 1 + \binom{k-1}{3} - (k-4)(k-1)b/2 \) and, since \( h^0(S; O_S(bC)) = b + 1 \) and \( h^2(S; O_S(bC)) = h^3(S; O_S(-bC))(k-4) = 0 \) (using \( b > k-4 \) and Lemma 2ii), \( h^1(S; O_S(bC)) = (k-4)(k-1)b/2 - (k-1)b + b. \) Putting all this together,

(5.23) \[ h^2(\mathbb{P}^3; \text{End}(E)) \leq (k^2 - 5k + 6)b - \binom{k-1}{3}. \]
Therefore by (5.7) and (5.16), for $k \geq 4$, $b \geq k - 4$,

\[(5.24)\]

\[8(k - 1)b - 2k^2 - 3 \leq \dim E \mathcal{M} \leq 8(k - 1)b - 2k^2 - 3 + (k^2 - 5k + 6)b - \left(\frac{k - 1}{3}\right)\]

\[= (k^2 + 3k - 2)b - (k^3 + 5k^2 + 11k + 12)/6.\]

This shows that, for fixed $k$ and large $b$, the codimension of $\mathcal{Y}$ in $\mathcal{M}$ is at least of order $(8k - 10)b$.

Returning to the $k = 2, 3$ cases and comparing $\dim \mathcal{Y}$ with $\dim E \mathcal{M}$ shows that equality holds only when $k = 2, b = 2, 3$ and $k = 3, b = 2$. When $k = 2, b = 2$ then $c_1(E) = 0, c_2(E) = 1$ and these are the null-correlation bundles classified by Barth [14] and Wever [27]. The moduli space of these stable bundles is isomorphic to $\mathbb{P}^5 - \mathcal{G}(1,3)$ where $\mathcal{G}$ is the grassmannian of lines in $\mathbb{P}^3$ [14] page 266. When $k = 2, b = 3$ then $c_1(E) = 0, c_2(E) = 2$ and these stable bundles were classified and studied in detail by Hartshorne [14]. The moduli space of these bundles is smooth, irreducible, and of dimension 13. When $k = 3, b = 2$ then $c_1(E) = -1, c_2(E) = 2$ and these bundles were analyzed by Hartshorne and Sols [15]. The moduli space of these stable bundles is smooth, irreducible, and rational of dimension 11.

6. Scheme Structures Related to the Parameter Space

Recall that $\mathcal{Y}$ denotes the set of isomorphism classes of stable rank two bundles of the form

\[(6.1)\]

\[0 \longrightarrow \sigma_1 \oplus \sigma_2 \longrightarrow E \longrightarrow j_S \mathcal{O}_S(bL)(\nu + c_1 - b) \longrightarrow 0\]

where the extension class $\tau = (\tau_1, \tau_2) \in \text{Ext}^1(S, \mathcal{O}_S(bC))$ appears as a homomorphism in the dual sequence

\[(6.2)\]

\[0 \longrightarrow E^* \longrightarrow \sigma_1 \oplus \sigma_2 \longrightarrow j_S \mathcal{O}_S(-bL)(\nu + b) \longrightarrow 0.\]

Note that we are using the isomorphism $\mathcal{O}_S(bC) \cong \mathcal{O}_S(-bL)(b)$. Here $\nu$, $b$, and $c_1$ are fixed and $S$, $L$, $\tau$, $E$, and $\sigma$ vary. We have shown that the sequences give a 1-to-1 correspondence (see Section 5)

\[(6.3)\]

\[(E, \sigma_1, \sigma_2) \longleftrightarrow (S, L, \tau_1, \tau_2)\]

which can be refined to

\[(6.4)\]

\[(E, [\sigma_1, \sigma_2]) \longleftrightarrow (S, L, [\tau_1, \tau_2]).\]

We would like to show that $\mathcal{Y}$ has a natural scheme structure and that there is a regular map $\mathcal{Y} \rightarrow \mathcal{M}$ into the full moduli space but this seems to be the case only under certain circumstances. To discuss the situation we use auxiliary parameter spaces.
\[ \mathcal{Y}_2 \equiv \{(S, L, \tau)\} \]

(where S is a smooth surface of degree \( k = 2\nu + c_1 \) containing the line L and \( \tau \) globally generates \( \mathcal{O}_S(bC) \)) and

\[ \mathcal{Y}_1 \equiv \{(S, L, [\tau])\}. \]

By (6.4) there is an bijective function from \( \mathcal{Y}_1 \) to the set \( \{(E, [\sigma])\} \). We have \( \mathcal{Y} = \{E\} \) and the obvious projection functions

\[ \mathcal{Y}_2 \xrightarrow{\eta_2} \mathcal{Y}_1 \xrightarrow{\eta_1} \mathcal{Y}. \]

We will show that \( \mathcal{Y}_2 \) and \( \mathcal{Y}_1 \) have natural scheme structures and regular maps into \( M \). Then we will point out some situations in which these results descend to \( \mathcal{Y} \).

Let \( P \) be the projective space of surfaces of degree \( k \) and \( G \) the grassmannian of lines in \( \mathbb{P}^3 \). Define \( Z \equiv \{(S, L, p) \mid L \subset S, p \in S\} \), \( W \equiv \{(S, L) \mid L \subset S\} \), and \( \pi : Z \to W \) the projection. \( Z \) and \( W \) are clearly projective varieties. Let \( Z_0 \subset Z \) and \( W_0 \subset W \) to be the Zariski open subsets defined by requiring that \( S \) is smooth. We define a line bundle \( \mathcal{F} \) on \( Z_0 \) such that, for all \( (S, L) \in W_0 \), \( \mathcal{F}|_{\pi^{-1}(S, L)} \cong \mathcal{O}_S(-bL)(b) \). Actually we define \( \mathcal{F}' \) such that \( \mathcal{F}'|_{\pi^{-1}(S, L)} \cong \mathcal{O}_S(-bL) \) and then set \( \mathcal{F} \equiv \mathcal{F}' \otimes \pi_3^* \mathcal{O}_{\mathbb{P}^3}(b) \).

For \( (S, L, p) \in Z_0 \) such that \( p \notin L \), set \( \mathcal{F}'_{(S, L, p)} \equiv \mathcal{O}_{Z_0, (S, L, p)} \). If \( p \in L \), we proceed as follows. Let \( S \) be defined by \( g(x) = 0 \) so that \( S = [g] \in P \) and let \( L \) be given by the two linear equations \( l_1 = 0 \) and \( l_2 = 0 \) so that \( L = [l_1 \land l_2] \in G \). For each \( (S, L) \in W_0 \), \( g = g_1 + g_2 \) for \( g_1 \) and \( g_2 \) of degree \( k - 1 \). Since \( S \) is smooth, at least one of \( g_1 \) and \( g_2 \) does not vanish in a Zariski open neighborhood in \( S \) of the given point \( p \in L \). Assume \( g_1 \) never vanishes. On this neighborhood,

\[ l_1 = -\frac{l_2 g_2}{g_1} \]

and so the pencil of hyperplane sections of \( S \) containing \( L \), \( \{H_t\} \), which are defined by \( t_1 l_1 + t_2 l_2 = 0 \), can be expressed as

\[ \frac{l_2}{g_1} (-t_1 g_2 + t_2 g_1) = 0. \]

The local equations for \( L \) and \( C_t \) on \( S \) are therefore \( l_2 = 0 \) and \( -t_1 g_2 + t_2 g_1 = 0 \) respectively.

As \( (S, L, x) \) varies in an open neighborhood \( U \) of a point \((S_0, L_0, p_0)\) of \( Z_0 \), we need to demonstrate that \( l_2(x) \) above can be chosen as a regular function of \((S, L, x)\). This requires knowing that \( g_1(x) \) has no zeroes on \( U \) and so it is sufficient to show that \( g_1 \) is a regular function of \((S, L, x)\). By a coordinate change we can assume that \( L_0 \) is defined by \( x_2 = 0 \) and \( x_3 = 0 \) and so, for \((S, L, x) \in U \),

\[ l_1 = x_2 + l_1'(x_0, x_1) \]
\[ l_2 = x_3 + l_2'(x_0, x_1). \]

Expanding \( g(x) = g(x_0, x_1, l_1 - l_1', x_3) \) gives \( g = l_1 g_1 + g_2 \) for \( g_2 \) not involving \( x_2 \) in fact
\[ \hat{g}_2 = g(x_0, x_1, -l'_1, x_3) \]
\[ g_1 = \frac{g(x) - \hat{g}_2(x)}{l_1}. \]

This shows that \( g_1 \) and \( \hat{g}_2 \) are regular functions on \( U \subset Z_0 \). Expanding \( \hat{g}_2(x_0, x_1, x_3) = \hat{g}_2(x_0, x_1, l_2 - l'_2) \) gives

\[ \hat{g}_2(x_0, x_1, -l'_2) = 0 \]
\[ \frac{\hat{g}_2(x_0, x_1, x_3)}{l_2} = g_2 \]

and shows that \( g_2 \) is also a regular function.

If \( p \in L \), define \( \mathcal{F}'_{(S,L,p)} = \{ f = l_1^h \mid h \text{ is regular on } Z_0 \text{ at } (S,L,p) \} \). It is clear that our two definitions of \( \mathcal{F}' \) patch together and give a line bundle in the form of a subsheaf of the sheaf of total quotient rings on \( Z_0 \) \cite[page 144]{13}. It follows from the definition that, for all \( (S,L,\tau) \in W_0 \), \( \mathcal{F}_{\pi^{-1}(S,L)} \cong \mathcal{O}_S(-bL)(b) \).

Since \( h^0 \equiv h^0(S;\mathcal{O}_S(-bL)(b)) = h^0(S;\mathcal{O}_S(bC)) \) is constant in \( (S,L), F \equiv \pi, \mathcal{F} \) is a vector bundle of rank \( h^0 \) on \( W_0 \). The parameter space \( \mathcal{Y}_2 \) is the Zariski open subset of \( F^\oplus 2 \) consisting of \( \tau = (\tau_1, \tau_2) \in H^0(S;\mathcal{O}_S(-bL)(b))^\oplus 2 \) such that \( \tau_1 \) and \( \tau_2 \) generate \( \mathcal{O}_S(-bL)(b) \). This defines the structure of \( \mathcal{Y}_2 \) as a variety and hence as a scheme \cite[Chapter 2, Proposition 2.6]{13}.

Applying geometric invariant theory to the quotient

\[ \mathcal{Y}_2 \xrightarrow{\tau_2} \mathcal{Y}_1 \]

by the reductive group \( GL(2, \mathbb{C}) \) gives an induced scheme structure to \( \mathcal{Y}_1 \). More precisely, let \( \tilde{U}_1 \) be an affine open subset of \( W_0 \) and let \( q : F^\oplus 2 \to W_0 \) be the bundle projection. Then for \( \tilde{U}_1 \) small enough, \( \tilde{U}_1 \equiv q^{-1}(U_1) \equiv U_1 \times \mathbb{C}^{h^0} \) is also affine and these sets cover \( F^\oplus 2 \). Let \( \tilde{U}_1 \) be defined as the orbit space of \( \tilde{U}_1 \) under the action of \( GL(2, \mathbb{C}) \). By \cite[Theorem 6.3.1]{13}, \( \tilde{U}_1 \) has the structure of an affine scheme and these structures for different \( \tilde{U}_1 \) patch together to give a scheme structure to the orbit space of the \( GL(2, \mathbb{C}) \)-action on \( F^\oplus 2 \). Because the equations on \( F^\oplus 2 \) defining \( \mathcal{Y}_2 \) as a Zariski open subset are clearly \( GL(2, \mathbb{C}) \)-invariant, they determine \( \mathcal{Y}_1 \) as a Zariski open subset of the orbit space of the \( GL(2, \mathbb{C}) \)-action on \( F^\oplus 2 \).

There is a natural regular map \( \mathcal{Y}_2 \to \mathcal{M} \) defined by using the universal property of \( \mathcal{M} \) as follows. We will define a family of stable rank 2 vector bundles on \( \mathbb{P}^3 \) parameterized by \( \mathcal{Y}_2 \), that is, a coherent sheaf \( \mathcal{E} \) on \( \mathcal{Y}_2 \times \mathbb{P}^3 \) such that, for every \( (S,L,\tau) \in \mathcal{Y}_2, E \equiv \mathcal{E}_{|(S,L,\tau)\times\mathbb{P}^3} \) is given by \cite[6.1]{13}. Since these restrictions have the same Hilbert polynomial, \( \mathcal{E} \) is flat over \( \mathcal{Y}_2 \). This defines a unique regular map \( \mathcal{Y}_2 \xrightarrow{\mathcal{E}} \mathcal{M} \) sending closed points of \( \mathcal{Y}_2 \) to closed points of \( \mathcal{M} \).

We construct \( \mathcal{E} \) by first defining a coherent sheaf \( \mathcal{F}_2 \) on \( \mathcal{Y}_2 \times \mathbb{P}^3 \) and a sheaf mapping \( \pi_2^*\mathcal{O}_{\mathbb{P}^3}(\nu)^\oplus 2 \xrightarrow{\phi} \mathcal{F}_2 \) such that, for each \( (S,L,\tau) \in \mathcal{Y}_2, \) the restriction of \( \phi \) to \( (S,L,\tau) \times \mathbb{P}^3 \) is given by \cite[6.2]{13}. Then \( \mathcal{E} \) is defined as the dual of the kernel of \( \phi \). The construction of \( \mathcal{F}_2 \) and \( \phi \) is very similar to that of \( \mathcal{F} \) above and so is left to the reader.

Because \( \mathcal{Y}_2 \xrightarrow{\mathcal{E}} \mathcal{M} \) is constant on the fibers of \( \mathcal{Y}_2 \xrightarrow{\tau_2} \mathcal{Y}_1 \), it induces a regular map \( \mathcal{Y}_1 \xrightarrow{\tau_1} \mathcal{M} \).
Note that the above development is much simpler when \( k = 2 \). Denoting the two types of lines on a smooth quadric \( Q \) by \( \pm \), the parameter schemes have the form \( \mathcal{Y}_2 = \{(Q, \pm, \tau)\} \) and \( \mathcal{Y}_1 = \{(Q, \pm, [\tau])\} \).

When \( \max(a, b) > k \), \([\sigma]\) is unique, \( \mathcal{Y}_1 = \mathcal{Y} \), and so we have a regular map of schemes \( \mathcal{Y} \to \mathcal{M} \) injective on closed points. When \( \max(a, b) \leq k \), the fibers of \( \mathcal{Y}_1 \to \mathcal{Y} \) are open subsets of the grassmannian of two dimensional subspaces of \( H^0(\mathbb{P}^3; E(\nu)) \). It is not clear that the scheme structure of \( \mathcal{Y}_1 \) descends to \( \mathcal{Y} \).

The arguments and results of this section apply equally well to the examples of stable rank 3 bundles studied in Section 9.

7. Stable Bundles of Rank 3 on \( \mathbb{P}^3 \)

Let \( E \) be a rank 3 normalized bundle \((c_1 = 0, -1, \text{ or } -2)\) on \( \mathbb{P}^3 \). For \( \nu \) large enough, \( E(\nu) \) is globally generated and, using Kleiman transversality, the generic \( \sigma = (\sigma_1, \sigma_2, \sigma_3) \in \oplus H^0(\mathbb{P}^3; E(\nu)) \) produces

\[
(7.1) \quad o \longrightarrow 3 \oplus \mathcal{O}_{\mathbb{P}^3}(-\nu) \stackrel{\sigma}{\longrightarrow} E \longrightarrow j_{S*} \mathcal{L} \longrightarrow 0
\]

for

1. \( S \equiv Z_{\sigma_1 \wedge \sigma_2 \wedge \sigma_3} \subset \mathbb{P}^3 \) a smooth hypersurface of degree \( k = 3\nu + c_1 \) and \( \mathcal{L} \) a line bundle on \( S \).
2. \( Z_{\sigma_i} \equiv \mathbb{P}(c_3(E(\nu)) = c_3(E) + \nu c_2(E) + \nu^2 c_1 + \nu^3 \) smooth points.
3. \( Z_{\sigma_i \wedge \sigma_j} \equiv i < j \) smooth curves of degree \( c_3(E(\nu)) = c_3(E) + \nu c_2(E) + \nu^2 c_1 + 3\nu^2 \).

**Proposition 3.** For \( V \equiv \text{span}\{\sigma_1, \sigma_2, \sigma_3\} \subset H^0(\mathbb{P}^3; E(\nu)) \) generic as above, the two-dimensional linear system of curves \( Y = Z_{s_1 \wedge s_2} \) for \( s_1 \wedge s_2 \in \wedge^2 V \) on the surface \( S \) satisfies

i) The curves \( Y \) are connected and the generic \( Y \) is smooth.
ii) \( Y^2 = c_3(E(\nu)) \) (intersection on \( S \))
iii) \( \text{genus}(Y) = 1 + 1/2(c_3(E(\nu)) + c_2(E(\nu))(k - 4)) \)

**Proof.** Set \( P \equiv PV \equiv \mathbb{P}^2 \) and \( Z \equiv \{(x, s) \in \mathbb{P}^3 \times P \mid s(x) = 0\} \). For \((x, s) \in Z, x \in S \) and the smoothness of \( S \) implies that \( x \) determines \( s \) (the subspace of \( V \) that vanishes at \( x \) is one-dimensional). It follows that \( Z \equiv S \). In

\[
(7.2) \quad Z \xrightarrow{\pi_1} S \\
\pi_2 \quad \downarrow \quad P
\]

the curves \( Y \) are \( \pi_2^{-1}(L) \) for the lines \( L \subset P, Y \) connected follows from the Fulton-Hansen connectedness theorem 4. Note that \( Z_{\sigma_1 \wedge \sigma_2} \cap Z_{\sigma_2 \wedge \sigma_3} = Z_{\sigma_2} \) (it is obvious, \( \subset \) results from the fact that \( S \) is smooth). An easy local coordinate argument shows that \( Z_{\sigma_1 \wedge \sigma_2} \) and \( Z_{\sigma_2 \wedge \sigma_3} \) meet transversely on \( S \) at each point of \( Z_{\sigma_2} \). Therefore \( Y^2 = Z_{\sigma_1 \wedge \sigma_2} \cdot Z_{\sigma_2 \wedge \sigma_3} = c_3(E(\nu)) \). The genus formula for \( Y \) now follows from the usual genus formula for the curve \( Y \) on the surface \( S \) and \( K_S = (k - 4)\omega_0 \). \( \square \)
This proposition is a special case of a more general result [25].

Applying $\text{Hom}_{\mathcal{O}_{\mathbb{P}^3}}(\mathcal{O}_{\mathbb{P}^3}, \mathcal{O}_{\mathbb{P}^3})$ to \((7.4)\) gives

\[
\begin{array}{cccccccc}
0 & \longrightarrow & E^* & \longrightarrow & \mathcal{O}_{\mathbb{P}^3}(\nu) & \longrightarrow & \mathcal{O}_{\mathbb{P}^3}(\nu) & \longrightarrow & 0 \\
& & \longrightarrow & \tau & \longrightarrow & \mathcal{O}_{\mathbb{P}^3}(\nu) & \longrightarrow & \mathcal{O}_{\mathbb{P}^3}(\nu) & \longrightarrow & 0 \\
\end{array}
\]

for $\tau = (\tau_1, \tau_2, \tau_3) \in \mathbb{Z}^3$. Arguing as in the rank 2 case

\[
\begin{array}{cccccccc}
(7.4) & \tau_1 = (\sigma_j \wedge \sigma_k)_S & \wedge & \text{for } (ijk) \text{ an even permutation of } (123). \\
\end{array}
\]

Applying Grothendieck Riemann-Roch to $j_{S*} L$ yields

\[
(7.5) \quad c_2(E) = 3\nu^2 + 3\nu c_1 + c_2 - c_1(L) \cdot \omega_0
\]

Applying $\text{Hom}_{\mathcal{O}_{\mathbb{P}^3}}(\mathcal{O}_{\mathbb{P}^3}, \mathcal{O}_{\mathbb{P}^3})$ to \((7.6)\) gives

\[
(7.6) \quad c_3(E) = (2\nu + c_1)^3 - (3\nu + 2c_1)c_1(L) \cdot \omega_0 + c_1(L)^2 \quad \text{(intersection on } S). \\
\]

Now assume $E$ is stable. If $L$ has the form $\mathcal{O}_S(l)$, $h^0(\mathbb{P}^3; E) = 0$ implies $l < 0$. Then \((7.5)\) implies

For a fixed stable bundle $E$ and $\nu$ large enough, any representation of $E$ of the form \((7.1)\) implies that $L \not\cong \mathcal{O}_S(l)$ for any $l$ and therefore $S$ belongs to the Noether-Lefschetz locus.

As in the rank 2 case we want to construct some specific rank 3 stable bundles so reverse the above development, begin with a given $\nu \in \mathbb{Z}_+$, a smooth surface $S \subset \mathbb{P}^3$ of degree $k = 3\nu + c_1$, a line bundle $L$ on $S$ and consider extensions

\[
(7.7) \quad 0 \longrightarrow \mathcal{O}_{\mathbb{P}^3}(\nu) \longrightarrow E \longrightarrow j_{S*} L \longrightarrow 0.
\]

They are classified by $\tau \in \mathbb{Z}^3 \oplus H^0(S; \mathcal{L}^*(2\nu + c_1))$. Applying $\text{Hom}_{\mathcal{O}_{\mathbb{P}^3}}(\mathcal{O}_{\mathbb{P}^3}, \mathcal{O}_{\mathbb{P}^3})$ to \((7.7)\),

\[
(7.8) \quad 0 \longrightarrow \mathcal{O}_{\mathbb{P}^3}(\nu) \longrightarrow \mathcal{O}_{\mathbb{P}^3}(\nu) \longrightarrow \mathcal{O}_{\mathbb{P}^3}(\nu) \longrightarrow 0.
\]

They are classified by $\tau \in \mathbb{Z}^3 \oplus H^0(S; \mathcal{L}^*(2\nu + c_1))$. Applying $\text{Hom}_{\mathcal{O}_{\mathbb{P}^3}}(\mathcal{O}_{\mathbb{P}^3}, \mathcal{O}_{\mathbb{P}^3})$ to \((7.7)\),

\[
(7.8) \quad 0 \longrightarrow \mathcal{O}_{\mathbb{P}^3}(\nu) \longrightarrow \mathcal{O}_{\mathbb{P}^3}(\nu) \longrightarrow \mathcal{O}_{\mathbb{P}^3}(\nu) \longrightarrow 0.
\]

So $E$ is locally free iff $\mathcal{L}^*(2\nu + c_1)$ is globally generated by $\tau$. Therefore

The generic extension \((7.7)\) is locally free iff $\mathcal{L}^*(2\nu + c_1)$ is globally generated (necessarily by three sections).

A rank 3 reflexive sheaf $E$ on $\mathbb{P}^3$ is stable iff $h^0(\mathbb{P}^3; E) = 0$ and $h^0(\mathbb{P}^3; E^*) = 0$ $(c_1 = 0)$, $h^0(\mathbb{P}^3; E^*(-1)) = 0$ $(c_1 = -1, -2)$. When $c_1 = 0$, $E$ is semistable iff $h^0(\mathbb{P}^3; E(-1)) = 0$ and $h^0(\mathbb{P}^3; E^*(-1)) = 0$ [25] page 167]. Therefore \((7.7)\) and \((7.8)\) imply

The bundle $E$ of the form \((7.7)\) is stable iff the following two conditions hold:

A) $\nu \geq 1$ and $h^0(S; L) = 0$
B) $\mathcal{O}_{\mathbb{P}^3}(\nu) \longrightarrow H^0(S; \mathcal{L}^*(k))$ is injective $(c_1 = 0)$

$\mathcal{O}_{\mathbb{P}^3}(\nu - 1) \longrightarrow H^0(S; \mathcal{L}^*(k - 1))$ is injective $(c_1 = -1, -2)$.\]
E is semistable \((c_1 = 0)\) iff the following two conditions hold:

\[ A' \quad \nu \geq 0 \text{ and } h^0(S; L(-1)) = 0 \]

\[ B' \quad H^0(\mathbb{P}^3; O_{\mathbb{P}^3}(\nu - 1)) \xrightarrow{\tau} H^0(S; L^*(k - 1)) \text{ is injective.} \]

8. Examples of Rank 3 Bundles I

First we take \( \mathcal{L} = O_S(-l) \) and examine rank 3 bundles \( E \) of the form

\[
\begin{array}{c}
0 \xrightarrow{\frac{3}{\sigma}} O_{\mathbb{P}^3}(-\nu) \xrightarrow{\sigma} E \xrightarrow{\tau} j_S S^* O_S(-l) \xrightarrow{\tau} 0
\end{array}
\]

for \( \nu, l \in \mathbb{Z}_+ \) and show that they are stable for generic \( \sigma \).

The dual sequence is

\[
\begin{array}{c}
0 \xrightarrow{\tau} E^* \xrightarrow{\sigma} O_{\mathbb{P}^3}(\nu) \xrightarrow{\tau} j_S S^* O_S(l + k) \xrightarrow{\tau} 0.
\end{array}
\]

For \( E \) to be locally free, \( \tau_1, \tau_2, \tau_3 \) must globally generate \( O_S(l + 2\nu + c_1) \). We can identify the \( \tau_i \) with homogeneous polynomials of degree \( l + 2\nu + c_1 \) with no simultaneous zeroes on \( S \). If \( g \) is the degree \( k \) homogeneous polynomial that defines \( S \), \( (8.3) \) can be expressed as

\[
\begin{array}{c}
0 \xrightarrow{\tau} E^* \xrightarrow{\sigma} O_{\mathbb{P}^3}(\nu) \oplus O_{\mathbb{P}^3}(l) \xrightarrow{\tau \oplus g} O_{\mathbb{P}^3}(l + k) \xrightarrow{\tau} 0.
\end{array}
\]

and the condition is that \( \tau_1, \tau_2, \tau_3, g \) have no common zeroes on \( \mathbb{P}^3 \), which holds for generic \( \tau \). To verify stability condition B) from Section 7, in the \( c_1 = 0 \) case, let \( \Gamma_i \equiv \text{the homogeneous polynomials of degree } l \)

\[
\begin{array}{c}
\Gamma_i \xrightarrow{\tau \oplus g} \Gamma_i + k.
\end{array}
\]

By [3, Lemma 3.1], a pre-Koszul complex graded adaptation of the Koszul complex, if \( \psi = (\psi_1, \psi_2, \psi_3, \psi_4) \) is in the kernel, \( \psi_i = \sum_j B_{ij} \tau_j \) for \( \tau_4 = g \) and \( B = (B_{ij}) \) is a skew-symmetric matrix of homogeneous polynomials with

\[
\text{deg} B_{ij} = \begin{cases} 
 l + k - 2(\nu + c_1) & \text{for } 1 \leq i, j \leq 3 \\
 l + k - (\nu + c_1) - k & \text{for } 1 \leq i \leq 3, j = 4 \text{ or vice versa.} 
\end{cases}
\]

In both cases these degrees are negative meaning \( B = 0 \) and \( \psi = 0 \). E is therefore stable. The stability condition for the cases \( c_1 = -1 \) or \( -2 \) is checked in the same way. Taking the dual of \( (8.3) \),

\[
\begin{array}{c}
0 \xrightarrow{\sigma} O_{\mathbb{P}^3}(-l - k) \xrightarrow{(\tau, \sigma)} O_{\mathbb{P}^3}(-\nu) \oplus O_{\mathbb{P}^3}(-l) \xrightarrow{\sigma \oplus h} E \xrightarrow{\tau} 0.
\end{array}
\]

where \( \sigma \in O_{\mathbb{P}^3}(\mathbb{P}^3; E(\nu)) \) and \( h \in H^0(\mathbb{P}^3; E(l)) \). Note that we may drop the condition that \( S \) be smooth, require only that \( \tau \) and \( g \) have no common zeroes, and define the stable bundle directly by \( (8.4) \). For \( \nu, l, \) and \( c_1 \) fixed, this gives a one-to-one correspondence

\[
(\tau, g) \longleftrightarrow (E, \sigma, h).
\]
Define \( \mathcal{Y} \) to be the set of stable rank 3 bundles of the form \( \mathcal{E} \). We will show that \( \mathcal{Y} \) has a natural scheme structure, that the inclusion \( \mathcal{Y} \hookrightarrow \mathcal{M} \) is a regular map, and that \( \dim \mathcal{Y} = \dim \mathcal{M} \). To begin, define

\[
\mathcal{Y}_0 \equiv \{ (\tau, g) \in \Gamma_{l+3}^{\mathbb{P}^3} \times \Gamma_k \mid \tau_1, \tau_2, \tau_3, g \text{ have no common zeroes on } \mathbb{P}^3 \}.
\]

\( \mathcal{Y}_0 \) is a Zariski open subset of an affine space and there is a coherent sheaf \( \mathcal{E} \) on \( \mathcal{Y}_0 \times \mathbb{P}^3 \), flat over \( \mathcal{Y}_0 \), such that, for each \( (\tau, g) \in \mathcal{Y}_0 \), \( \mathcal{E} = \mathcal{E}_{(\tau, g) \times \mathbb{P}^3} \) is given by \( \mathcal{E} \). By the universal property of \( \mathcal{M} \), there is a unique regular map \( f_0 : \mathcal{Y}_0 \to \mathcal{M} \) sending the closed points of \( \mathcal{Y}_0 \) to closed points of \( \mathcal{M} \).

Let \( (\tau, g) \in \mathcal{Y}_0 \) determine \( \mathcal{E} \) and the sequence \( \mathcal{E} \) and \( \bar{(\bar{\tau}, \bar{g})} \) the sequence for \( \mathcal{Y}_0 \). An easy argument, using \( H^1(\mathbb{P}^3; \mathcal{O}_{\mathbb{P}^3}(j)) = 0 \) for all \( j \) and the fact that elements of \( \mathcal{Y}_0 \) have no common zeroes, shows that \( \mathcal{E} \cong \bar{\mathcal{E}} \) if and only if the isomorphism extends to an isomorphism of sequences

\[
\begin{array}{ccccccccc}
0 & \longrightarrow & \mathcal{O}_{\mathbb{P}^3}(-l - k) & \stackrel{(\tau, g)}{\longrightarrow} & \mathcal{O}_{\mathbb{P}^3}(-\nu) \oplus \mathcal{O}_{\mathbb{P}^3}(-l) & \stackrel{\sigma \oplus h}{\longrightarrow} & E & \longrightarrow & 0 \\
\downarrow & & \downarrow \text{id} & & \downarrow \psi & & \downarrow & & \downarrow \\
0 & \longrightarrow & \mathcal{O}_{\mathbb{P}^3}(-l - k) & \stackrel{(\bar{\tau}, \bar{g})}{\longrightarrow} & \mathcal{O}_{\mathbb{P}^3}(-\nu) \oplus \mathcal{O}_{\mathbb{P}^3}(-l) & \stackrel{\bar{\sigma} \oplus h}{\longrightarrow} & \bar{E} & \longrightarrow & 0
\end{array}
\]

where

\[
\psi = \begin{pmatrix}
A & v(x) \\
w^t(x) & b
\end{pmatrix}
\]

for \( A \in GL(3; \mathbb{C}), b \in \mathbb{C}^*, v(x) \in \Gamma_{l+3}^{\mathbb{P}^3}, \) and \( w(x) \in \Gamma_{\nu-\nu}^{\mathbb{P}^3} \). The homomorphisms \( \psi \) form a Lie group \( H \) whose dimension equals \( 10 + 3\binom{l-\nu+3}{3} \) if \( l \neq \nu \) and \( 16 \) if \( l = \nu \). We have shown that \( \mathcal{Y} = \mathcal{Y}_0/H \). Using the fact that the isomorphisms of \( \mathcal{E} \) are scalar multiples of the identity and that this multiple is fixed by requiring that the left vertical homomorphism above is the identity, we can compute

\[
(8.6) \quad \dim \mathcal{Y} = \dim \{ \tau \} + \dim \{ g \} - \dim H
\]

\[
= 3\binom{l + k - \nu + 3}{3} + \binom{k + 3}{3} - \begin{cases}
10 + 3\binom{l-\nu+3}{3} & \text{for } l \neq \nu \\
16 & \text{for } l = \nu.
\end{cases}
\]

To identify the scheme structure of \( \mathcal{Y} \) as a Zariski open subset of a projective scheme \( \bar{\mathcal{Y}} \), consider the three cases \( l > \nu, l = \nu, \) and \( l < \nu \). When \( l > \nu \), the action of \( H \) is given by \( \bar{\tau} = A\tau + gv \) and \( \bar{g} = bg \). The space of orbits \( \bar{\mathcal{Y}} \) is therefore the grassmann bundle \( G_3(W) \) where \( W \) is the vector bundle on \( \mathbb{P}^3 \) defined by

\[
0 \longrightarrow \mathcal{O}\mathcal{P} \Gamma_k(-1) \otimes \Gamma_{l-\nu} \longrightarrow \mathcal{O}\mathcal{P} \Gamma_k \otimes \Gamma_{k+l-\nu} \longrightarrow W \longrightarrow 0
\]

For \( l = \nu, H = GL(4; \mathbb{C}) \) and \( \bar{\mathcal{Y}} = G_4(\Gamma_l) \). Finally for \( l < \nu, \bar{\tau} = A\tau \) and \( \bar{g} = w^t\tau + bg \) and this gives \( \bar{\mathcal{Y}} = \text{Proj}(\text{Sym} \mathcal{F}) \) for \( \mathcal{F} \) the coherent sheaf over \( G_3(\Gamma_{k+l-\nu}) \) defined by

\[
\mathcal{K} \xrightarrow{\phi} \mathcal{O}_{G_3(\Gamma_{k+l-\nu})} \otimes \Gamma_k \longrightarrow \mathcal{F} \longrightarrow 0
\]
where $\mathcal{K}$ is the vector bundle with fiber $\Gamma_{\nu-l} \otimes \mathbb{C}^3$ associated with the principal frame bundle of the tautological sub-bundle on $G_3(\Gamma_{k+l-\nu})$ by the group action $B(\tau, w) = (B\tau, B^{-1}\tau)$ for $B \in GL(3; \mathbb{C})$ and where $\phi(\tau, w) = w^t \tau$. $\mathcal{Y}$ is a projective scheme [13, Chapter II, Proposition 7.10].

It is clear that $f_0$ induces a regular map $f: \mathcal{Y} \to M$.

We now compute $h_1^1(\mathbb{P}^3; \text{End}(E))$. The cohomology sequence of (8.4) implies $H^1(\mathbb{P}^3; E(j)) = 0$ for all $j$ and $H^2(\mathbb{P}^3; E(k+l)) = 0$. Tensoring (8.3) with $E$ gives

$$0 \to \text{End}(E) \to E(\nu) \otimes 3 \oplus E(l) \to E(l+k) \to 0$$

and therefore

$$0 \to H^0(\mathbb{P}^3; \text{End}(E)) \to H^0(\mathbb{P}^3; E(\nu)) \otimes 3 \oplus H^0(\mathbb{P}^3; E(l)) \to H^0(\mathbb{P}^3; E(k+l)) \to H^1(\mathbb{P}^3; \text{End}(E)) \to 0$$

(8.7)

$$H^2(\mathbb{P}^3; \text{End}(E)) \cong H^2(\mathbb{P}^3; E(\nu)) \otimes 3 \oplus H^2(\mathbb{P}^3; E(l)).$$

It follows from (8.4) or (8.4) that

$$h^0(\mathbb{P}^3; E(\nu)) = \begin{cases} 3 & l > \nu \\ 3 + (\nu-l+1)/3 & l \leq \nu \end{cases}$$

$$h^0(\mathbb{P}^3; E(l)) = \begin{cases} 1 + 3(l-\nu+3)/3 & l \geq \nu \\ 1 & l < \nu \end{cases}$$

$$h^0(\mathbb{P}^3; E(k+l)) = 3 \left( \binom{l+k-\nu+3}{3} + \binom{k+3}{3} - 1 \right)$$

and so (8.6) and (8.7) imply

**Theorem 3.** Let $E$ be a stable rank 3 bundle on $\mathbb{P}^3$ of the form

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^3}(-l-k) \longrightarrow \mathcal{O}_{\mathbb{P}^3}(-\nu) \otimes \mathbb{C}^3 \oplus \mathcal{O}_{\mathbb{P}^3}(-l) \longrightarrow E \longrightarrow 0.$$

If $\mathcal{Y} \to M$ is the set of these bundles, then

$$h^1(\mathbb{P}^3; \text{End}(E)) = \dim \mathcal{Y} = \dim E \cdot M$$

$$= 3 \left( \binom{l+k-\nu+3}{3} + \binom{k+3}{3} - \begin{cases} 10 + 3(l-\nu+3)/3 & \text{for } l \neq \nu \\ 3 & \text{for } l = \nu \end{cases} \right)$$

$\mathcal{Y}$ is an open subscheme of $M$ and $M$ is smooth at $E$.

The second Chern class of these bundles is (from (8.3))

$$c_2(E) = 3\nu^2 + 3\nu c_1 + c_1^2 + lk = \frac{1}{3}[k^2 + c_1 k + c_1^2] + lk.$$
From Riemann-Roch one gets

\[(8.10) \quad h^1(\mathbb{P}^3; \mathcal{E}nd(E)) - h^2(\mathbb{P}^3; \mathcal{E}nd(E)) = 12c_2(E) - 4c_1^2 - 8.\]

Combining the above two equations with (8.9) gives

\[h^2(\mathbb{P}^3; \mathcal{E}nd(E)) = 3\left(\frac{l+k-\nu+3}{3}\right) + \left(\frac{k+3}{3}\right) - 4k[k + c_1 + 3l] - \begin{cases} 2 + 3\left(\frac{|l-\nu|+3}{3}\right) & \text{if } l \neq \nu, \\ 8 & \text{if } l = \nu. \end{cases}\]

Returning to the scheme structure of \(\mathcal{Y}\) and the regular map \(\mathcal{Y} \rightarrow \mathcal{M}\), it is tempting to suppose that the closure of \(\mathcal{Y}\) in \(\mathcal{M}\) is the projective scheme \(\mathcal{Y}\) given above. This is not the case unless \(k = l = \nu = 1\) in which case \(\mathcal{Y}\) is a point and \(E \cong T\mathbb{P}^3(-2)\). When \(\nu > 1\), there are points \([\tau, g] \in \mathcal{Y} \smallsetminus \mathcal{Y}\) which correspond to reflexive sheaves of rank three which are unstable. More precisely, let \(Z \subset \mathbb{P}^3\) be the subscheme defined by the vanishing of \((\tau, g)\) and define \(E\) by \(\mathbb{Q}_{\mathcal{O}_{\mathbb{P}^3}(\tau, \mathcal{O}_{\mathbb{P}^3})}\). For \(\text{codim}Z \geq 2\), \(E\) is torsion free, locally free on \(\mathbb{P}^3 \smallsetminus Z\). Apply \(\text{Hom}_{\mathcal{O}_{\mathbb{P}^3}}(\mathcal{I}, \mathcal{O}_{\mathbb{P}^3})\) to get

\[
\begin{CD}
0 @>>> E^* @>>> \mathcal{O}_{\mathbb{P}^3}(\nu) \oplus \mathcal{O}_{\mathbb{P}^3}(l) @>>> \mathcal{I}(l + k) @>>> 0
\end{CD}
\]

for \(\mathcal{I}\) the ideal sheaf of \(Z\). \(E^*\) is reflexive (the dual of any coherent sheaf is reflexive) and the sequence exhibits \(E^*\) as a second syzygy sheaf. If \(Z\) is a 0-dimensional locally complete intersection, taking the dual again shows that \(E\) is reflexive. Now take \(\tau = (x_0^{k+i-\nu}x_3, x_1^{k+i-\nu-1}x_3, x_0^{k+i-\nu-1}x_2, x_1^{k+i-\nu-1}x_2)\) and \(g \equiv \sum_{i=0}^3 x_i^k\). Then \(Z\) is a zero-dimensional locally complete intersection in \(\mathbb{P}^3\). Define \(f \equiv (x_2, 0, -x_3, 0)\), a section of \(\mathcal{O}_{\mathbb{P}^3}(1) \oplus \mathcal{O}_{\mathbb{P}^3}(l - \nu + 1)\). Because \(\hat{f}\) is in the kernel of \(\tau \oplus g\), it defines a section \(f\) of \(E^*(-(\nu - 1))\). This implies that \(E\) is unstable when \(\nu > 1\).

The family \(\mathbb{Q}_{\mathcal{O}_{\mathbb{P}^3}(\tau)}\) also contains \(E\) which fail to be torsion free: if \(l - \nu \geq 1\) and \(\tau = (x_1^{l-\nu}g, x_2^{l-\nu}g, x_3^{l-\nu}g)\) then \(E\) has torsion. When \(l = \nu\) and \(k \geq 2\), it is also easy to construct \(E\) with torsion.

9. Examples of Rank Three Bundles II

We now construct examples from surfaces \(S \hookrightarrow \mathbb{P}^3\) containing a line \(L\), choosing a line bundle on \(S\) of the form \(\mathcal{L} = \mathcal{O}_S(-aL - bC)(2\nu + c_1)\) for \(a, b \in \mathbb{Z}\). Then \(\mathcal{L}^*(2\nu + c_1) = \mathcal{O}_S(aL + bC)\). We analyze rank 3 extensions

\[(9.1) \quad 0 \rightarrow \mathcal{O}_{\mathbb{P}^3}(-\nu) \rightarrow E \rightarrow j_{S*}\mathcal{O}_S(-aL - bC)(2\nu + c_1) \rightarrow 0\]

and determine the divisors \(aL + bC\) for which the generic extension is a stable bundle. The dual sequence is

\[(9.2) \quad 0 \rightarrow E^* \rightarrow \mathcal{O}_{\mathbb{P}^3}(\nu) \rightarrow j_{S*}\mathcal{O}_S(aL + bC)(\nu) \rightarrow 0\]
and \( \tau \in \bigoplus H^0(S; \mathcal{O}_S(aL + bC)) \) is the extension class.

**Theorem 4.** The generic extension of the form (9.2) is a stable rank 3 bundle in the following cases: (recall \( k = 3\nu + c_1 \))

1. \( k = 1, \nu = 1, c_1 = -2, S = \) hyperplane \( H \). There is no curve \( C, a > 0 \), and \( \mathcal{L} = \mathcal{O}_H(-a) \). \( E \) is a special case of (8.4), \( c_2(E) = a+1 \) and \( c_3(E) = a^2 - a \).

2. \( k = 2, \nu = 1, c_1 = -1, S = \) smooth quadric \( Q \), and \( L \) and \( C \) belong to the two pencils of lines on \( Q \). Using the bidegree notation for line bundles on \( Q \), \( \mathcal{L}^*(2\nu + c_1) = \mathcal{O}_Q(a, b), \mathcal{L} = \mathcal{O}_Q(1 - a, 1 - b) \) for \( a, b \geq 0, \max(a, b) \geq 2 \). \( c_2(E) = a + b - 1 \) and \( c_3(E) = 2ab - a - b + 1 \).

3. \( k \geq 3 \) and \( a > b \geq \frac{(k-2)}{(k-1)}a > 0 \) (which implies \( a \geq k - 1, b \geq k - 2 \)) —except for the case \( k=3, a = 2, b = 1 \). \( c_2(E) = a + b(k - 1) - [k^2 - c_1^2]/3 \) and \( c_3(E) = 2ab(k-1) - a^2(k-2) - a(k-1)b(k-c_1)/3 + (k - c_1)^2(2k+c_1))/27 \).

4. \( k \geq 3, b \geq a > \nu/2(c_1 = 0), (\nu - 1)/2(c_1 = -1, -2), \) and \( b > 2\nu + c_1 \). The chern classes of \( E \) are as in case 3.

When \( k \geq 3, b \geq a, b > 2\nu + c_1 \) but \( a \leq \nu/2(c_1 = 0), (\nu - 1)/2(c_1 = -1, -2) \) the generic extension is locally free but it is not known if it is stable. No other values of \( a \) and \( b \) produce stable bundles.

**Proof.** The conditions from Section 4 for \( E \) to be locally free and stable are applied to \( \mathcal{L} = \mathcal{O}_S(-D)(2\nu + c_1) \) and \( \mathcal{L}^*(2\nu + c_1) = \mathcal{O}_S(D) \) for \( D = aL + bC \). Case (1) follows from the analysis of the bundles (8.1). If \( \mathcal{O}_S(aL + bC) \) is globally generated, \( \mathcal{O}_L(aL + bC) = \mathcal{O}_L(b(k-1) - a(k-2)) \) is globally generated and so

\[
(9.3) \quad b(k-1) - a(k-2) \geq 0
\]

and \( \mathcal{O}_C(aL + bC) \) is globally generated and of degree \( C \cdot (aL + bC) = (k-1)a \) so

\[
(9.4) \quad a(k-1) \geq 0.
\]

The condition that \( h^0(\mathcal{L}) = 0 \) is, expressing

\[
\mathcal{L} \cong \mathcal{O}_S(-(a-b)L)(2\nu + c_1 - b) \quad \text{for} \quad a \geq b
\]

\[
\cong \mathcal{O}_S(-(b-a)C)(2\nu + c_1 - a) \quad \text{for} \quad b \geq a
\]

and applying Lemma 2 equivalent to

\[
(9.5) \quad \max(a, b) > 2\nu + c_1.
\]

For \( k \geq 3 \) and \( a > b \) (case 3), \( 1 \leq a - b \leq a - \frac{k^2}{k-1} = \frac{a}{k-1} \), and so \( a \geq k - 1 \) and \( b \geq k - 2 \). Condition (9.3) is satisfied except for the case \( k=3, a=2, b=1 \). To show that \( \mathcal{O}_S(aL+bC) \) is globally generated first note that \( \mathcal{O}_S(aL+bC) = \mathcal{O}_S((a-b)L)(b) \) is clearly globally generated on \( S \setminus L \). For \( 1 \leq j \leq a - b \), consider the sequences

\[
(9.6) \quad 0 \rightarrow \mathcal{O}_S((j-1)L)(b) \rightarrow \mathcal{O}_S(jL)(b) \rightarrow \mathcal{O}_L(b - (k-2)j) \rightarrow 0
\]

and note that \( b - (k-2)j \geq b - (k-2)(a-b) = -(k-2)a + (k-1)b \geq 0 \) so that \( \mathcal{O}_L(b - (k-2)j) \) is globally generated. The cohomology sequences now show that \( \mathcal{O}_S(aL+bC) \) is globally generated because \( H^1(S; \mathcal{O}_S((j-1)L)(b) \cong \)
and this yields our result. The injective so that we again get sequence and the fact that reduces to 2.

(9.7) \[ 0 \rightarrow \wedge^3 V \otimes \mathcal{O}_S(-2aL - 2bC)(\nu) \rightarrow \wedge^2 V \otimes \mathcal{O}_S(-aL - bC)(\nu) \rightarrow V \otimes \mathcal{O}_S(\nu) \rightarrow \mathcal{O}_S(aL + bC)(\nu) \rightarrow 0. \]

Break this up into two short exact sequences

(9.8) \[ 0 \rightarrow \wedge^3 V \otimes \mathcal{O}_S(-2aL - 2bC)(\nu) \rightarrow \wedge^2 V \otimes \mathcal{O}_S(-aL - bC)(\nu) \rightarrow K \rightarrow 0 \]

(9.9) \[ 0 \rightarrow K \rightarrow V \otimes \mathcal{O}_S(\nu) \rightarrow \mathcal{O}_S(aL + bC)(\nu) \rightarrow 0. \]

Considering the second sequence, we must show that \( H^0(S; K) = 0 \). By the first sequence and the fact that \( \mathcal{O}_S(-aL - bC)(\nu) \cong \mathcal{O}_S(-a - bL)(\nu - b) \) has no global sections (Lemma 2v), it is enough to prove that \( H^1(S; \mathcal{O}_S(-2aL - 2bC)(\nu)) \cong H^1(S; \mathcal{O}_S(-2(a - b)L)(\nu - 2b)) = 0 \). Except in the cases \( k = 3, a = 2l, b = l, \) for \( l \geq 2, \) this follows from \( 2b - \nu > (k - 2)(2(a - b) - 1) \) reduces to \( 2[b(k - 1) - a(k - 2)] + 2\nu - 2 > 0 \) which holds since \( b(k - 1) - a(k - 2) \geq 0 \) and \( 2\nu - 2 > 0 \) and both equalities hold if \( \nu = 1, k = 3, \) and \( a = 2l, b = l \) for \( l \in \mathbb{Z}^+ \).

When \( k = 3, a = 2l, b = l \) for \( l \geq 2, \) \( h^1(S; \mathcal{O}_S(-2aL - 2bC)(\nu)) = h^1(S; \mathcal{O}_S(-4lL - 2lC)(1)) = 1 \) but \( H^1(S; \mathcal{O}_S(-4lL - 2lC)(1)) \rightarrow \mathcal{O}_S(-2lL - lC)(1) \rightarrow \mathcal{O}_S(-2lL - lC)(1) \rightarrow \mathcal{O}_S(-lL)(1 - l) \) and consider

\[
0 \rightarrow \mathcal{O}_S(-lL)(1 - l) \rightarrow \mathcal{O}_S(-lL)(1 - l) \rightarrow \mathcal{O}_L \rightarrow 0
\]

and note that the homomorphism \( \psi \) induced by \( \tau \) is given by a non-zero \( v \in \mathbb{C}^3 \).

The cohomology ladder and Lemma 2 for \( l \geq 2 \) gives

\[
H^0(L; \mathcal{O}_L) \cong \mathbb{C} \rightarrow \mathcal{O}_S(-lL)(1 - l) \rightarrow \mathcal{O}_L \rightarrow 0
\]

and this yields our result. The \( c_1 = -1, -2 \) case follows in the same way.

Now assume \( k \geq 3 \) and \( b \geq a \) (case 4.). Then \( 3.11 \) requires \( b > 2\nu + c_1 \) and \( 3.14 \) implies \( a \geq 0 \). Note that \( \mathcal{O}_S(aL + bC) \cong \mathcal{O}_S((b-a)C)(a) \cong \mathcal{O}_S(a) \otimes \mathcal{O}_S((b-a)C) \) is
Theorem 4 for 1))

globally generated because both \(O_S(a)\) and \(O_S((b-a)C)\) are (Lemma 2 ii). To verify that \(a > \nu/2\) (when \(c_1 = 0\)) implies that \(\nabla^3 H^0(\mathbb{P}^3; O_{\mathbb{P}^3}(\nu)) \rightarrow H^{0}(S; O_S(aL + bC)(\nu))\) is injective and that \(a > (\nu-1)/2\) (when \(c_1 = -1, -2\)) implies \(\nabla^3 H^0(\mathbb{P}^3; O_{\mathbb{P}^3}(\nu-1)) \rightarrow H^{0}(S; O_S(aL + bC)(\nu-1))\) is injective, proceed as in case 3. In the \(c_1 = 0\) situation, this reduces to knowing that \(H^1(S; O_S(2(b-a)C)(\nu - 2\nu)) = 0\). By Lemma 2, this holds if \(a > \nu/2\) or \(a = \nu/2 = b\). The second case cannot occur since \(b > 2\nu \geq \nu/2\). The \(c_1 = -1, -2\) case is similar.

In case 2, \(S = Q \cong \mathbb{P}^1 \times \mathbb{P}^1\), the argument follows the same pattern and is left to the reader.

**Proposition 4.** Let \(E\) be a rank 3 stable bundle on \(\mathbb{P}^3\) of the type constructed in Theorem 4 for \(k \geq 3\). Then

1. For \(c_1 = 0\), \(l \geq -4\), \(H^3(\mathbb{P}^3; E(l)) = 0\). For \(c_1 = -1, -2\), \(l \geq -3\), \(H^3(\mathbb{P}^3; E(l)) = 0\).
2. For \(l > \nu - 4\), \(H^2(\mathbb{P}^3; E(l)) = 0\) iff \(l > \min(a, b) + \nu - 4\).
3. \(H^1(\mathbb{P}^3; E(l)) = 0\) in exactly the following cases: For \(b = a\), \(l > b + \nu - 4 + (k - 2)[b - a - 1]\). For \(b = a + 1\), \(l = b + \nu - 4\). For \(a > b\), \(l > a + \nu - 4\). For \(l = a + \nu - 4\), \(a = b + 1\).
4. For \(a \geq b\), \(E(l)\) is globally generated iff \(l \geq \max(a - k + \nu, \nu)\). For \(b > a\), \(E(l)\) is globally generated iff \(l \geq b(k - 1) - a(k - 2) - k + \nu\).

**Proof.** This is very similar to the proof of Proposition 11 and so is left to the reader.

To count moduli, we proceed as in the previous examples. Fix \(\nu\), \(c_1\), \(a\), and \(b\) and consider the dual defining sequences for \(E\)

\[
\begin{array}{cccccc}
0 & \rightarrow & \nabla^3 O_{\mathbb{P}^3}(-\nu) & \rightarrow & E & \rightarrow & j_{S*}O_S(-aL - bC)(2\nu + c_1) & \rightarrow & 0 \\
\sigma & & & & \tau & & & & \\
(9.10)
\end{array}
\]

\[
\begin{array}{cccccc}
0 & \rightarrow & E^* & \rightarrow & \nabla^3 O_{\mathbb{P}^3}(\nu) & \rightarrow & j_{S*}O_S(aL + bC)(\nu) & \rightarrow & 0 \\
\sigma' & & & & \tau & & & & \\
(9.11)
\end{array}
\]

Here \(\sigma \in \nabla^3 H^0(\mathbb{P}^3; E(\nu))\) and \(\tau \in \nabla^3 H^0(S; O_S(aL + bC))\). The sequences imply that the function \((S, L, \tau) \rightarrow (E, \sigma)\) is injective but not a priori surjective (when \(b \leq k\)), as explained in Section 5. For \(E\) of the form (9.10), choose a different \(\tilde{\sigma} \in \nabla^3 H^0(\mathbb{P}^3; E(\nu))\); this gives another sequence

\[
\begin{array}{cccccc}
0 & \rightarrow & \nabla^3 O_{\mathbb{P}^3}(-\nu) & \rightarrow & E & \rightarrow & j_{S*}\tilde{L} & \rightarrow & 0 \\
\sigma & & & & \tau & & & & \\
(9.12)
\end{array}
\]

and we must show that the surface \(\tilde{S}\) contains a line \(\tilde{L}\) and that \(\tilde{L}\) has the form \(O_{\tilde{S}}(-aL - bC)(2\nu + c_1)\). This is obvious when \(S\) and \(\tilde{S}\) are quadrics \((k = 2)\) so assume \(k \geq 3\).

Consider case 4 of Theorem 4 b \(\geq a\), \(a \geq \nu/2\) (for \(c_1 = 0\)), \(\geq (\nu - 1)\) (for \(c_1 = -1, -2\)), \(b > 2\nu + c_1\). Arguing exactly as in Section 5 one finds \(\tilde{L}(b - 2\nu - c_1) \cong O_{\tilde{S}}(\tilde{D})\) for \(\tilde{D}\) effective, \(\deg \tilde{D} = b - a\), \(\tilde{D}^2 = -(k - 2)(b - a)^2\), and finally that \(D = (b - a)\tilde{L}\) for \(\tilde{L} \subset \tilde{S}\) a line.
Now consider case 3 of Theorem \[ a > b, b(k - 1) - a(k - 2) \geq 0. \] Using \( b \leq k \), there are only two possibilities: \( a = k, b = k - 1 \) and \( a = k - 1, b = k - 2 \). Again proceeding as in Section \[ 9 \] \( \deg \bar{D} = k - 1 \) and \( \bar{D}^2 = 0 \).

When \( a = k, b = k - 1 \), the cohomology sequences of \[ 11 \] and \[ 12 \] and Lemma \[ 2 \] imply \( h^0(\bar{S}; L(\nu)) = h^0(S; \mathcal{O}_S(s)) = 2 \). Therefore \( L(\nu) \cong \mathcal{O}_S(D) \) for \( D \) effective. Since \( \mathcal{O}_S(C) \) is globally generated, so is \( E(\nu) \) and therefore \( \mathcal{O}_S(D) \). So \( |\bar{D}| \) has no base locus and, by Bertini, we can assume \( \bar{D} \) is smooth. \( \bar{D} = \sum_i Y_i \) for the \( Y_i \) smooth disjoint curves and \( Y_i^2 = 0 \). The cohomology sequence of

\[
0 \rightarrow \mathcal{O}_{\bar{S}} \rightarrow \mathcal{O}_{\bar{S}}(\bar{D}) \rightarrow \bigoplus_{i=1}^r \mathcal{O}_{Y_i} \rightarrow 0
\]

and \( h^0(\bar{S}; \mathcal{O}_{\bar{S}}(\bar{D})) = 2 \) imply \( r = 1 \). So \( \bar{D} \) is an irreducible smooth curve of degree \( k - 1 \). To show that \( \bar{D} \) is contained in a plane, we show that, for \( \Gamma \) denoting the homogeneous polynomials in \( x \) of degree 1, the restriction \( \Gamma \rightarrow H^0(\bar{D}; \mathcal{O}_D(1)) \) has non-trivial kernel. This follows from the cohomology sequence of

\[
0 \rightarrow \mathcal{O}_{\bar{S}}(-\bar{D})(1) \rightarrow \mathcal{O}_{\bar{S}}(1) \rightarrow \mathcal{O}_D(1) \rightarrow 0
\]

and \( h^0(\bar{S}; \mathcal{O}_{\bar{S}}(-\bar{D}))(1) = h^0(S; \mathcal{O}_S(-C)(1)) = h^0(S; \mathcal{O}_S(L)) = 1 \). So \( \bar{S} \) has a hyperplane section \( H = \bar{D} + L \) for \( L \) a line. This gives \( \mathcal{L} \cong \mathcal{O}_{\bar{S}}(-kL - (k-1)\bar{D})(2\nu + c_1) \).

When \( a = k - 1, b = k - 2 \), the argument is the same except that we must work harder to show that \( |\bar{D}| \) has no base locus. The sequences \[ 11 \] and \[ 12 \] show that \( L(\nu - 1) \cong \mathcal{O}_S(\bar{D}) \) for \( \bar{D} \) effective and that \( h^0(\bar{S}; \mathcal{O}_{\bar{D}}(\bar{D})) = 2 \). Note that \( \mathcal{O}_{\bar{S}}(\bar{D})(1) \) is globally generated but that \( \mathcal{O}_{\bar{S}}(\bar{D}) \) is not. Therefore to prove that \( |\bar{D}| \) has no base locus it is enough to show that the bilinear multiplication map

\[
m : \Gamma_1 \times H^0(\bar{S}; \mathcal{O}_{\bar{S}}(\bar{D})) \rightarrow H^0(\bar{S}; \mathcal{O}_{\bar{S}}(\bar{D})(1))
\]

is surjective. Since \( h^0(\bar{S}; \mathcal{O}_{\bar{S}}(\bar{D})(1)) = 7 \), we must show that the dimension of the kernel of \( m \) is \( \leq 1 \). For \( V \cong \mathbb{C}^4 \) defined by \( \mathbb{P}^3 \equiv \mathbb{P}V \) and \( x \) the homogeneous coordinates on \( \mathbb{P}^3 \), consider

\[
(9.13) \quad 0 \rightarrow K_1 \rightarrow \mathcal{O}_{\bar{S}}(\bar{D}) \otimes V \rightarrow \mathcal{O}_{\bar{S}}(\bar{D})(1) \rightarrow 0.
\]

and note that \( \text{Ker} \ m \cong H^0(\bar{S}; \mathcal{K}_1) \). Extend \[ 9.13 \] to a Koszul sequence which breaks up into three short exact sequences, \[ 9.13 \] and

\[
0 \rightarrow \mathcal{K}_2 \rightarrow \mathcal{O}_{\bar{S}}(\bar{D})(-1) \otimes \wedge^2 V \rightarrow \mathcal{K}_1 \rightarrow 0
\]

\[
0 \rightarrow \mathcal{O}_{\bar{S}}(\bar{D})(-3) \otimes \wedge^4 V \rightarrow \mathcal{O}_{\bar{S}}(\bar{D})(-2) \otimes \wedge^3 V \rightarrow \mathcal{K}_2 \rightarrow 0.
\]

\( h^i(\bar{S}; \mathcal{O}_{\bar{S}}(\bar{D})(-j)) = h^i(S; \mathcal{O}_S(C)(-j)) = h^i(S; \mathcal{O}_S(-L)(-j - 1))) = 0 \) for \( i = 0, 1 \) and \( j = 1, 2, 3 \). by Lemma \[ 2 \]. Therefore \( \dim \ \text{Ker} \ m \) equals the dimension of the kernel of

\[
H^2(\bar{S}; \mathcal{O}_{\bar{S}}(\bar{D})(-3)) \otimes \wedge^4 V \xrightarrow{\sim} H^2(\bar{S}; \mathcal{O}_{\bar{S}}(\bar{D})(-2)) \otimes \wedge^3 V
\]

which, by Serre duality and \[ 9.10 \], equals the dimension of the cokernel of

\[
(9.14) \quad H^0(S; \mathcal{O}_S(-C)(k - 2)) \otimes \wedge^3 V \xrightarrow{x \wedge} H^0(S; \mathcal{O}_S(-C)(k - 1)) \otimes \wedge^4 V.
\]
Using $\mathcal{O}_S(-C)(k - 2) \cong \mathcal{O}_S(L)(k - 3)$ and the sequence

$$0 \rightarrow \mathcal{O}_S(k - 3) \rightarrow \mathcal{O}_S(L)(k - 3) \rightarrow \mathcal{O}_L(-1) \rightarrow 0$$

we see that $\mathcal{O}_S(-C)(k - 2)$ is not globally generated because all its sections vanish on $L$. Similarly, $\mathcal{O}_S(-C)(k - 3) \cong \mathcal{O}_S(L)(k - 2)$ is globally generated and we can take a basis for $H^0(S; \mathcal{O}_S(L)(k - 2))$ of the form $s_0, s_1, \ldots, s_n$ where $s_0$ is non-vanishing on $L$ and the $s_i$ are zero on $L$ for $i = 1$ to $n$. Let $\xi = 0$ define $L$. Then the sequence shows that $s \in H^0(S; \mathcal{O}_S(L)(k - 3))$ has the form $s = \xi P$ for $P \in \Gamma_{k-3}$. Similarly, $\tilde{s} \in \text{span}\{s_1, \ldots, s_n\}$ has the form $\tilde{s} = \xi \tilde{P}$ for $\tilde{P} \in \Gamma_{k-2}$. This shows that the cokernel of (9.14) has dimension 1.

We have demonstrated the 1-to-1 correspondences

$$(E, \sigma) \leftrightarrow (S, L, \tau)$$

for $k \geq 3$ and

$$(E, \sigma) \leftrightarrow (Q, \pm, \tau) \quad \text{for } k = 2.$$ 

**Proposition 5.** Let $\mathcal{Y}$ be the set of isomorphism classes of stable rank 3 bundles $E$ as in Theorem 4. For $b \geq a \geq k - 3$,

$$\dim \mathcal{Y} = 3(k - 1)ab - \frac{3(k - 2)}{2}a^2 - \frac{3(k - 4)}{2}(a + (k - 1)b)$$

$$+ 3\binom{k - 1}{3} + \binom{k + 3}{3} - \text{sup}(k - 3, 0) - 7$$

$$+ (\text{when } b \leq k) \begin{cases} -3\binom{k-b+3}{3} & \text{if } k \geq 3 \\ 3a - 9 & \text{if } k = 2. \end{cases}$$

For $a > b$,

$$\dim \mathcal{Y} = 3(k - 1)ab - \frac{3(k - 2)}{2}a^2 - \frac{3(k - 4)}{2}(a + (k - 1)b)$$

$$+ 3\binom{k - 1}{3} + \binom{k + 3}{3} - \text{sup}(k - 3, 0) - 7$$

$$+ (\text{when } a \leq k) \begin{cases} -6 & \text{if } a = k \\ -21 & \text{if } a = k - 1. \end{cases}$$

For $b \geq a$, $a \leq k - 2$,

$$\dim \mathcal{Y} = \binom{a + 2}{2} |3b - 2a + 3| + \binom{k + 3}{3} - \text{sup}(k - 3, 0) - 10$$

$$+ (\text{when } b \leq k) \begin{cases} -3\binom{k-b+3}{3} & \text{if } k \geq 3 \\ -9 & \text{if } k = 2. \end{cases}$$

**Proof.** Counting parameters from the 1-to-1 correspondences and using $\dim \{(S, L)\} = \dim \{S\}$ when $k \geq 3$, we get
\[ \dim \mathcal{Y} = \dim \{ S \} + \dim \{ \tau \} - \dim \{ \sigma \} \]
\[ = \left( k + \frac{3}{3} \right) - \text{sup}(k - 3, 0) - 10 + 3h^0(S; \mathcal{O}_S(\alpha L + bC)) \]
\[ - 3h^0(S; \mathcal{O}_S(-\alpha L - bC)(k)). \]

When \( \max(a, b) > k \), \( h^0(S; \mathcal{O}_S(-\alpha L - bC)(k)) = 0 \) and \( h^0(S; \mathcal{O}_S(\alpha L + bC)) \) is given by Lemma 3. When \( \max(a, b) \leq k \), Lemma 3 can also be applied to \( h^0(S; \mathcal{O}_S(-\alpha L - bC)(k)) = h^0(S; \mathcal{O}_S((k-a)L-(k-b)C)), \) using the restrictions on \( a \) and \( b \) imposed by Theorem 4. This gives our result. □

From (9.16) \[ 0 \longrightarrow 3 \oplus \mathcal{O}_{\mathbb{P}^3} \overset{\sigma}{\longrightarrow} E(\nu) \longrightarrow j_{S*}\mathcal{O}_S(-\alpha L - bC)(k) \longrightarrow 0 \]

it follows that

(9.17) \[ H^j(\mathbb{P}^3; E(\nu)) \cong H^j(S; \mathcal{O}_S(-\alpha L - bC)(k)) \quad j = 1, 2, 3 \]

(9.18) \[ h^0(\mathbb{P}^3; E(\nu)) = 3 + h^0(S; \mathcal{O}_S(-\alpha L - bC)(k)) \]
\[ = 3 \quad \text{if } \max(a, b) > k \]
\[ \text{(by Lemma 2 and ii).} \]

To establish a framework in which we can try to calculate or estimate the dimension of the Zariski tangent space of \( \mathcal{M} \) at \( E \) we argue as in Section 5 to obtain

(9.19) \[ 0 \longrightarrow \mathcal{E}nd(E) \longrightarrow 3 \oplus E(\nu) \longrightarrow j_{S*}E_S(\alpha L + bC)(\nu) \longrightarrow 0 \]

(9.20) \[ 0 \longrightarrow \mathcal{O}_S \longrightarrow 3 \oplus \mathcal{O}_S(\alpha L + bC) \longrightarrow \mathcal{K} \longrightarrow 0 \]
\[ 0 \longrightarrow \mathcal{K} \longrightarrow E_S(\alpha L + bC)(\nu) \longrightarrow \mathcal{O}_S(k) \longrightarrow 0 \]

(9.21) \[ 0 \longrightarrow H^0(\mathbb{P}^3; \mathcal{E}nd(E)) \longrightarrow 3 \oplus H^0(\mathbb{P}^3; E(\nu)) \longrightarrow H^0(S; E_S(\alpha L + bC)(\nu)) \]
\[ \longrightarrow H^1(\mathbb{P}^3; \mathcal{E}nd(E)) \longrightarrow 3 \oplus H^1(\mathbb{P}^3; E(\nu)) \longrightarrow H^1(S; E_S(\alpha L + bC)(\nu)) \]
\[ \longrightarrow H^2(\mathbb{P}^3; \mathcal{E}nd(E)) \longrightarrow 3 \oplus H^2(\mathbb{P}^3; E(\nu)) \longrightarrow H^2(S; E_S(\alpha L + bC)(\nu)) \longrightarrow 0. \]

The main result of the following calculations will be to make an effective comparison of \( \dim T\mathcal{M}_E = h^1(\mathbb{P}^3; \mathcal{E}nd(E)) \) and \( \dim \mathcal{Y} \) when \( k=2 \) or \( 3 \) and an estimation of the codimension of \( \mathcal{Y} \) in \( \mathcal{M} \) at \( E \) when \( k \geq 4 \). Using Lemma 2 we get
(9.22)
\[ h^1(S; \mathcal{O}_S(aL + bC)) = 0 \quad \text{in case 3.} \]
\[ = 0 \quad \text{in case 4. if } a > k - 4 \text{ or } a = k - 4, b = a, a + 1. \]

(9.23)
\[ h^2(S; \mathcal{O}_S(aL + bC)) = 0 \quad \text{iff } \max(a, b) > k - 4. \]

From the cohomology sequences of (9.20) it follows that

(9.24)
\[ h^2(S; E_S(aL + bC)(\nu)) = 0 \quad \text{for } \max(a, b) > k - 4 \]
\[ h^1(S; E_S(aL + bC)(\nu)) = \dim(\ker\delta) \]
\[ = 0 \quad \text{for } k = 2, 3 \]
\[ h^0(S; E_S(aL + bC)(\nu)) = \left(\frac{k + 3}{3}\right) - 2 + 3h^0(S; \mathcal{O}_S(aL + bC)) - \dim(\ker\delta) \]

where \( \delta \) is the connecting homomorphism \( H^0(S; \mathcal{O}_S(k)) \to H^1(S; K). \) When \( k = 2, 3, \delta = 0 \) because \( H^1(S; K) = 0. \) Starting from (9.24), routine calculations using Lemma \( \text{[4]} \) give

(9.25)
\[ h^2(P^3; E(\nu)) = h^2(S; \mathcal{O}_S(-aL - bC)(k)) \]
\[ = h^0(S; \mathcal{O}_S(aL + bC)(-4)) \]
\[ = h^0(S; \mathcal{O}_S((a - 4)L + (b - 4)C)) \quad \text{(see Lemma \( \text{[4]} \) when } \min(a, b) \geq 4) \]
\[ = 0 \quad \text{if } \min(a, b) < 4 \]

and

(9.26)
\[ h^1(P^3; E(\nu)) = h^1(S; \mathcal{O}_S(-aL - bC)(k)) \]
\[ = 0 \quad \text{in case 3 of Theorem \( \text{[4]} \) if } b \geq k \text{ and } (k - 1)b - (k - 2)a > 2 \text{ or } \]
\[ a = k + 1, b = k \text{ and in case 4 of Theorem \( \text{[4]} \) if } a > k \]
\[ \text{or } a = k, b = k, k + 1. \]

**Theorem 5.** Let \( S \hookrightarrow \mathbb{P}^3 \) be a smooth surface of degree \( k \) that contains a line \( L \) and let \( E \) be a stable rank 3 bundle on \( \mathbb{P}^3 \) of the form

\[
0 \longrightarrow \mathcal{O}_{\mathbb{P}^3}(-\nu) \longrightarrow E \longrightarrow j_* s^* \mathcal{O}_S(-aL - bC)(2\nu + c_1) \longrightarrow 0
\]
as in Theorem \( \text{[4]} \). Assume in case 3 that \( b \geq k \) and \( b(k - 1) - a(k - 2) > 2 \) or that \( a = k + 1, b = k \) and in case 4 that \( a > k \) or \( a = k, b = k + 1. \) Then

(9.27)
\[ h^1(P^3; \text{End } E) = \left(\frac{k + 3}{3}\right) - 10 + 3h^0(S; \mathcal{O}_S(aL + bC)) - \dim(\ker\delta). \]
For $\mathcal{Y} \subset \mathcal{M}$ the set of these bundles, $\mathcal{Y}$ is a subscheme of $\mathcal{M}$ of codimension no larger than $\max(k - 3, 0) - \dim(\im \delta) \leq \max(k - 3, 0)$.

**Proof.** By (9.26), our hypothesis insures that $H^1(\mathbb{P}^3; E(\nu)) = 0$ and that $H^0(S; \mathcal{O}_S(aL - bC)(k)) = 0$. Now the sequence (9.21) gives

$$h^1(\mathbb{P}^3; \mathcal{E}nd E) = h^0(S; \mathcal{E}nd E) + 1$$

and the formula follows from (9.24) and (9.18). Comparing this with $\dim \mathcal{Y}$ (given by (9.15)) and using $\dim_E \mathcal{M} \leq h^1(\mathbb{P}^3; \mathcal{E}nd E)$ gives the codimension estimate. $\mathcal{Y}$ is a subscheme of $\mathcal{M}$ from arguments parallel to those in Section 6. □

We now assume $k = 2$ or 3 and obtain much more precise information. It will be convenient to calculate $h^1(\mathbb{P}^3; \mathcal{E}nd E)$ via

$$h^1(\mathbb{P}^3; \mathcal{E}nd E) = 12c_2(E) - 4c_1 - 8$$

and the formula follows from (9.21) and (9.24). From Riemann-Roch (8.10) and the chern class formula of Theorem 4,

$$h^1(\mathbb{P}^3; \mathcal{E}nd E) - h^2(\mathbb{P}^3; \mathcal{E}nd E) = 12c_2(E) - 4c_1 - 8$$

(9.28)

From (9.21), (9.24), and (9.25),

$$h^2(\mathbb{P}^3; \mathcal{E}nd E) = 3h^0(S; \mathcal{O}_S((a - 4)L + (b - 4)C))$$

(see Lemma 8 when $\min(a, b) \geq 4$)

$$= 0 \text{ if } \min(a, b) < 4$$

for $k = 2$ or 3.

(9.29)

Our conclusions are summarized in the following two theorems.

**Theorem 6.** Let $Q \hookrightarrow \mathbb{P}^3$ be a smooth quadric and let $E$ be a stable rank 3 bundle on $\mathbb{P}^3$ of the form

$$0 \rightarrow \mathcal{O}_{\mathbb{P}^3}(-1) \rightarrow E \rightarrow j_Q^* \mathcal{O}_Q(1 - a, 1 - b) \rightarrow 0$$

with $a, b \geq 0$, $\max(a, b) \geq 2$ as in Theorem 7. Let $\mathcal{Y} \subset \mathcal{M}$ be the set of these bundles. Then

$$\dim \mathcal{Y} = \begin{cases} 3(a + 1)(b + 1) & \text{for } \max(a, b) \geq 3 \\ 12a + 12b - 24 & \text{for } \max(a, b) \leq 3 \end{cases}$$

$$h^1(\mathbb{P}^3; \mathcal{E}nd E) = \begin{cases} 3(a + 1)(b + 1) & \text{for } \min(a, b) \geq 3 \\ 12a + 12b - 24 & \text{for } \min(a, b) \leq 3 \end{cases}$$

$$h^2(\mathbb{P}^3; \mathcal{E}nd E) = \begin{cases} 3(a - 3)(b - 3) & \text{for } \min(a, b) \geq 3 \\ 0 & \text{for } \min(a, b) \leq 3 \end{cases}$$

In all cases $\dim_E \mathcal{M} = h^1(\mathbb{P}^3; \mathcal{E}nd E) = \dim T \mathcal{M}_E$ and $\mathcal{M}$ is smooth at $E$. $\dim E \mathcal{M} = \dim \mathcal{Y}$ in all cases except when $\max(a, b) \geq 4$ and $\min(a, b) \leq 2$ both hold.
Proof. For \( k = 2 \), Lemma \( 3 \) reduces to the elementary result

\[
0 = \bigoplus_{Q} \mathcal{O}_{Q}(i, j) = \begin{cases} 
(i + 1)(j + 1) & \text{if } i, j \geq 0 \\
0 & \text{if } i \text{ or } j < 0.
\end{cases}
\]

This along with (9.15), (9.28), and (9.29) give our three formulas. In every case, either \( h^1(\mathbb{P}^3; \mathcal{E}nd E) = \dim \mathcal{Y} \) or \( h^2(\mathbb{P}^3; \mathcal{E}nd E) = 0 \).

\[\square\]

**Theorem 7.** Let \( S_3 \looparrowright \mathbb{P}^3 \) be a smooth cubic and let \( E \) be a stable rank 3 bundle on \( \mathbb{P}^3 \) of the form

\[
0 \rightarrow 3 \oplus \mathcal{O}_{\mathbb{P}^3}(-1) \rightarrow E \rightarrow j_{S_3*} \mathcal{O}_{S_3}(-aL - bC)(2) \rightarrow 0
\]

as in Theorem 4. Let \( \mathcal{Y} \subset \mathcal{M} \) be the set of these bundles. Then

\[
\dim \mathcal{Y} = \begin{cases} 
6ab + 3b - 3\binom{a}{3} + 13 & \text{for } \max(a, b) \geq 4 \\
12a + 24b - 44 & \text{for } \max(a, b) \leq 3, a \geq b \\
52 & b = 3, a = 2 \\
37 & b = 3, a = 1
\end{cases}
\]

\[
h^2(\mathbb{P}^3; \mathcal{E}nd E) = \begin{cases} 
0 & \text{for } \min(a, b) \leq 3 \\
3[2ab - \frac{1}{2}a^2 - \frac{7}{2}a - 7b + 19] & \text{if } \min(a, b) \geq 4 \text{ and } b \geq a \text{ or } a > b \geq 2 + a/2 \text{ both hold} \\
3[2b^2 - 14b + 25] & \text{if } \min(a, b) \geq 4 \text{ and } b < 2 + a/2 \text{ both hold}
\end{cases}
\]

\[
h^1(\mathbb{P}^3; \mathcal{E}nd E) = \begin{cases} 
12a + 24b - 44 & \text{for } \min(a, b) \leq 3 \\
6ab + 3b - 3\binom{a}{3} + 13 & \text{if } \min(a, b) \geq 4 \text{ and } b \geq a \text{ or } a > b \geq 2 + a/2 \text{ both hold} \\
12a + 6b^2 - 18b + 31 & \text{if } \min(a, b) \geq 4 \text{ and } b < 2 + a/2 \text{ both hold}
\end{cases}
\]

[Note that in the case where \( \min(a, b) \geq 4 \) and \( b < 2 + a/2 \text{ both hold} \), the requirement \( 2b - a \geq 0 \) from Theorem 4 gives that \( b = a/2 \) or \( a/2 + 1 \) when \( a \) is even and \( b = (a + 1)/2 \) or \( (a + 1)/2 + 1 \) when \( a \) is odd.]

\[\dim \mathcal{Y} = \dim \mathcal{M} = h^1(\mathbb{P}^3; \mathcal{E}nd E) \] and \( \mathcal{M} \) is smooth at \( E \) when \( \min(a, b) \geq 4 \) and \( b \geq a \text{ or } a > b \geq 2 + a/2 \text{ both hold} \), when \( \max(a, b) \leq 3, a \geq b, \) when \( b = 3, a = 2, \) and when \( \min(a, b) \geq 4, b = a/2 + 3/2 \) for \( a \) odd. When \( \min(a, b) \leq 3, \mathcal{M} \) is smooth at \( E \).

Proof. This follows from Lemma 3, Proposition 5, (9.28), and (9.29). \( \square \)
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