Global U(1) R-symmetry and Conformal Invariance of (0,2) Models

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We derive a condition under which (0,2) linear sigma models possess a “left-moving” conformal stress tensor in $\overline{Q}_+^+$ cohomology (i.e. which leaves invariant the “right-moving” ground states) even away from their critical points. At the classical level this enforces quasihomogeneity of the superpotential terms. The persistence of this structure at the quantum level on the worldsheet is obstructed by an anomaly unless the charges and superpotential degrees satisfy a condition which is equivalent to the condition for the cancellation of the anomaly in a particular “right-moving” U(1) R-symmetry.

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1. Introduction

The perturbative conditions for conformal invariance of nonlinear sigma models have long been studied. Of particular relevance for string theory are N=2 superconformal theories. Recently it has been noticed (see [1] and [2] and references therein) that (0,2) string vacua can be studied by computing renormalization group invariant quantities in simpler (0,2) models that are not conformally invariant but may flow in the infrared to conformal fixed points. The models that are useful in this respect are linear sigma models, in general coupled to gauge fields.

If one is to study (0,2) string vacua by studying linear sigma models in the same universality class, one would like criteria for knowing which linear sigma models are likely to have conformal fixed points in the infrared. For instance, consider the $CP^n$ model – a well-known (2,2) model which we will think of as a (0,2) model, ignoring the left-moving supersymmetry. One can certainly find a simple linear sigma model with gauge fields that is equivalent to the $CP^n$ model in the infrared. One would not expect this model to flow in the infrared to a non-trivial conformal field theory. We would like to understand what restrictions on the gauge charges and superpotential interactions in linear sigma models are necessary to exclude models such as the $CP^n$ model which do not describe superstring vacua.

The global right-moving supercharges of a (0,2) or (2,2) model are operators $\bar{Q}_+$ and $Q_+$ that obey $\bar{Q}_+^2 = Q_+^2 = 0$, $\{\bar{Q}_+, Q_+\} = P_+$; in particular, if we think of $\bar{Q}_+$ as a cohomology operator, then the cohomology is the space of right-moving ground states (states of $P_+ = 0$). One of the main observations of [3] was that at the classical level a (2,2) Landau-Ginzburg theory (with a quasi-homogeneous superpotential) has a left-moving superconformal algebra even away from criticality. Indeed, one can find operators commuting with $\bar{Q}_+$, and generating by operator products the left-moving $N=2$ algebra (modulo terms of the form $\{\bar{Q}_+, \ldots\}$) directly in the non-critical Landau-Ginzburg theory. It is plausible to expect that the $\bar{Q}_+$-trivial error terms $\{\bar{Q}_+, \ldots\}$ vanish in the infrared limit and to interpret the left-moving $N=2$ algebra that one finds away from criticality as a precursor of the conjectured superconformal fixed point.

We want to carry out here a similar program for (0,2) models. A model possessing a (0,2) superconformal fixed point will at that fixed point have a left-moving stress-tensor $T_{--}$ which satisfies the conformal algebra and commutes with the right-moving global supersymmetry charges. We will determine which linear models possess such a left-moving
conformal symmetry away from criticality at the level of $\overline{Q}_+$ cohomology. In the process, we will also generalize some of the previous results concerning (2,2) models.

Finding a left-moving conformal symmetry at the level of $\overline{Q}_+$ cohomology in the non-critical model is neither necessary nor sufficient to ensure the existence of a conformally invariant infrared fixed point. It is obviously not sufficient, a priori. It is also not necessary in general, since if a conformal limit exists, the left-moving stress tensor that commutes with $\overline{Q}_+$ might appear only at the fixed point. (The stress tensor would have to appear paired with another new state in the $\overline{Q}_+$ cohomology with right-moving $U(1)$ charge differing by one and the same left-moving quantum numbers. In many instances the $\overline{Q}_+$ cohomology of the conformal theory has no suitable states; in that case the existence of the left-moving conformal symmetry mod $\{\overline{Q}_+\}$ in the non-critical theory is indeed necessary.) Nevertheless, we think that the occurrence of the off-shell conformal symmetry is a very interesting hint of the existence of the fixed point.

In section 2 we will explain the classical condition that arises from this requirement. The classical condition is not enough: even if the left-moving conformal symmetry arises classically, there may be a quantum anomaly. In section 3 we compute quantum anomaly in $[\overline{Q}_+, T_{--}]$ and relate this to the anomaly in a particular right-moving R-symmetry.

2. The Classical Condition

We follow the conventions in [1] so that our action is

$$S = \int d^2y d^2\theta \left\{ \frac{1}{8e^2} Tr \overline{\Upsilon} \Upsilon - \frac{i}{2} \Phi_i(D_0 - D_1) \Phi_i - \frac{1}{2} \Lambda_a \Lambda_{-a} \right\}$$

$$+ \left( \int d^2y d\theta^+ \left\{ \frac{t}{4} \Upsilon_{\theta=0} - \frac{1}{\sqrt{2}} \Lambda_{-a} J^a_{\sigma=0} \right\} + h.c. \right)$$

(2.1)

Here we have $N_i$ chiral multiplets $\Phi_i = \phi_i + \sqrt{2} \theta^+ \psi_i + i \theta^+ \bar{\Upsilon}^+(D_0 + D_1) \phi_i$ with gauge charges $Q_i$, $N_a$ fermionic multiplets $\Lambda_a = \lambda_{-a} - \sqrt{2} \theta^+ G_a - i \theta^+ \bar{\Upsilon}^+(D_0 + D_1) \lambda_{-a} - \sqrt{2} \theta^+ E_a$ with gauge charges $Q_a$, and a $U(1)$ gauge multiplet with fermionic field strength $\Upsilon = -\chi_+ + \theta^+(v_{01} + i D) + i \theta^+ \bar{\Upsilon}^+(D_0 + D_1) \chi_-$. $E_a$ and $J^a$ are holomorphic functions of the $\Phi_i$. The fermionic multiplets satisfy $\overline{D}_+ \Lambda_{-a} = \sqrt{2} E_a(\Phi_i)$, and $E_a J^a = 0$. Here all derivatives are gauge-covariant derivatives.
Consider the following candidate stress tensor (a slight generalization of that in [4] and [3]):

\[
T_{--} = \left( i \frac{-i}{2e^2} \Upsilon (D_0 - D_1) \overline{\Upsilon} + 2 (D_0 - D_1) \Phi_i (D_0 - D_1) \overline{\Phi}_i \\
+ i [\Lambda_{-a} (D_0 - D_1) \overline{\Lambda}_{-a} - (D_0 - D_1) \Lambda_{-a} \overline{\Lambda}_{-a}] \right) \\
- \sum_i \alpha_i (D_0 - D_1) [\Phi_i (D_0 - D_1) \overline{\Phi}_i] + i \sum_a \alpha_a (D_0 - D_1) (\Lambda_{-a} \overline{\Lambda}_{-a}) \bigg|_{\theta = \overline{\theta} = 0}
\]

(2.2)

As discussed in [3], one can study $\overline{Q}_+$ cohomology by studying $\overline{D}_+$ cohomology since the two operators are conjugate:

\[
\overline{Q}_+ = \exp[-2i\theta^+\overline{\theta}^+] \overline{D}_+ \exp[2i\theta^+ \theta^+]
\]

Using the equations of motion

\[
\frac{1}{4e^2} \partial_- \overline{D}_+ \Upsilon = i Q_i \Phi_i (D_0 - D_1) \overline{\Phi}_i + Q_a \overline{\Lambda}_{-a} \Lambda_{-a}
\]

(2.3)

\[
\overline{D}_+ (D_0 - D_1) \Phi = -i \sqrt{2} (\Lambda_{-a} \frac{\partial J_a}{\partial \Phi_i} - \overline{\Lambda}_{-a} \frac{\partial E_a}{\partial \overline{\Phi}_i})
\]

(2.4)

\[
\overline{D}_+ \overline{\Lambda}_{-a} = \sqrt{2} J_a
\]

(2.5)

and the constraint

\[
\overline{D}_+ \Lambda_{-a} = \sqrt{2} E_a
\]

(2.6)

we find that $\overline{D}_+ T_{--} = 0$ classically provided that the following quasihomogeneity conditions on $E_a$ and $J_a$ are satisfied:

\[
\alpha_a J^a + \sum_i \alpha_i \Phi_i \frac{\partial J^a}{\partial \Phi_i} = J^a
\]

(2.7)

and

\[
-\alpha_a E_a + \sum_i \alpha_i \Phi_i \frac{\partial E_a}{\partial \Phi_i} = E_a
\]

(2.8)

These conditions reduce to the usual quasihomogeneity condition on the superpotential in the (2,2) case. $T_{--}$ is not $\overline{D}_+ (...)$, so $T_{--}$ is a nontrivial element of $\overline{D}_+$ cohomology.

We would like to understand the operator algebra satisfied by $T_{--}$ as an element of the chiral algebra of the (0,2) model, i.e. in $\overline{D}_+$ cohomology. One finds that the superpotential interactions do not contribute to the singularities in the OPE. To see this, note
that since $\phi$ has mass dimension zero, the superpotential terms $|E_a|^2 + |J_a|^2$ must come with a dimensionful coupling constant $\mu^2$. Then (with $x^2 = x_+ x_-$) the first superpotential corrections to the OPE $T_-(x)T_-(0)$ would be of the form $\mu^2 x^2, \mu^2 x^2 O_-, \mu^2 x^2 \partial_+ O'_-$, for some operators $O_-$ and $O'_-$, and so would vanish as $x_+ \to 0$; i.e. for Lorentz invariance $\mu^2$ would always be accompanied by $x^2$.

Therefore we can compute the OPE $T_-(x)T_-(y)$ using the free propagators

$$< \phi(x)\phi(y)> = \log(x-y)^2,$$

$$< \nu_\mu(x)\nu_\nu(y)> = e^{2\eta_{\mu\nu}} \log(x-y)^2,$$

and

$$< \chi_a(x)\chi_a(y)> = \frac{1}{(x_--y_-)} = < \chi_a(x)\chi_a(y) > \frac{1}{4e^2}.$$

We find that this stress tensor satisfies the conformal algebra

$$T_-(x)T_-(y) \sim \frac{c/2}{(x_- - y_-)^4} + \frac{2T_-(y)}{(x_- - y_-)^2} + \frac{\partial T_-(y)}{x_- - y_-} \quad (2.9)$$

with

$$c = 3 \sum_i (1 - 2\alpha_i) + (N_a - N_i) + \sum_i 3\alpha_i^2 - \sum_a 3\alpha_a^2 - \sum_g 2 \quad (2.10)$$

where the last sum is over the generators of the gauge group.

Note that if we shift the $\alpha_i$ by $Q_i$ and $\alpha_a$ by $Q_a$, the quasihomogeneity conditions are still satisfied by virtue of gauge invariance but the central charge shifts by an amount proportional to $\sum_i \alpha_i Q_i - \sum_a \alpha_a Q_a - \sum_i Q_i$, appearing to yield a family of conformal stress tensors. We will see in the next section that this situation will be avoided by the quantum anomaly.

3. The Anomaly in $[\overline{Q}_+, T_-]$

In order to determine whether we can maintain $[\overline{Q}_+, T_-]=0$ at the wordsheet “quantum” level, we compute the following time-ordered product:

$$0 = \int d^2x \partial_\mu T(S^\mu(x)T_-(y)\overline{W}(z)) = T([\overline{Q}_+, T_-(y)]\overline{W}(z)) + T(\{\overline{Q}_+, \overline{W}(z)\}T_-(y)) \quad (3.1)$$

where $S^\mu$ is the supersymmetry current whose charge is $\overline{Q}_+$. For this formula to be useful, $\overline{W}$ should be an operator whose commutator with $\overline{Q}_+$ is known; in that case, the above
formula gives information about $[Q_+, T_-]$. To ensure that the commutator of $\mathcal{W}$ with $Q_+$ is known, we will take $\mathcal{W}$ to be one of the elementary fermion fields of the model.

If we take $\mathcal{W}_1 = \partial_- \Upsilon \big|_{\theta = 0 = \bar{\sigma}} = (-2i) \partial_- \chi_-$ then using the equation of motion

$$\frac{1}{4e^2} \partial_+ \Upsilon = iQ_i \Phi_i (D_0 - D_1) \Phi_i + Q_a \lambda_{-a} \lambda_{-a}$$

we obtain

$$\{ Q_+, \mathcal{W}_1 \} = (-2i)(4e^2) [iQ_i \Phi_i (D_0 - D_1) \Phi_i + Q_a \lambda_{-a} \lambda_{-a}]$$

For the most singular terms in $T(\{ Q_+, \mathcal{W}_1(z) \} T_-(y))$ we can again use the free propagators since the only possible contribution from the superpotential terms would be proportional to $\frac{\mu^2 x^2}{x^2}$, which would vanish as $x_+ \to 0$. Computing using the free propagators, (2.2), and (3.3), we have the most singular terms

$$\frac{1}{(2i)(4e^2)} T(\{ Q_+, \mathcal{W}_1(z) \} T_-(y)) \sim T(2\partial_- \phi_i \partial_- \phi_i(y) iQ_i \phi_i \partial_- \phi_i(z))$$

$$- T(\alpha_i \partial_- (\phi_i \partial_- \phi)(y) iQ_i \phi_i \partial_- \phi_i(z))$$

$$+ T(i\alpha_a \partial_- (\lambda_{-a} \lambda_{-a})(y) Q_a \lambda_{-a} \lambda_{-a}(z))$$

$$\sim \frac{2i}{(y_- - z_-)^3} (\sum_i Q_i - \sum_i Q_i \alpha_i + \sum_a Q_a \alpha_a)$$

$$+ \text{less singular}$$

from which we deduce

$$[Q_+, T_-] = 2i\partial_- \chi_- (\sum_i Q_i - \sum_i Q_i \alpha_i + \sum_a Q_a \alpha_a)$$

plus possibly other terms whose OPEs with $\mathcal{W}_1 = (2i) \partial_- \chi_-$ are nonsingular. Considering other $\mathcal{W}$'s linear in the fundamental fields yields no further divergent contributions to $T(\{ Q_+, \mathcal{W} \} T_-)$ at the one-loop level. The contribution (3.4) is proportional to $q^3/q^2$ in momentum space (where $q$ is the momentum carried by $T_-$) and cannot be cancelled by any local counterterm. We have found that the existence of $T_-$ requires

$$\sum_i Q_i - \sum_i Q_i \alpha_i + \sum_a Q_a \alpha_a = 0$$

(3.6)
This condition (3.6) for the cancellation of this anomaly in $[Q_+, T_-]$ is the same as the condition for the cancellation of the anomaly in the following right-moving R-symmetry:

\[\begin{align*}
\psi_{+i} &\rightarrow e^{i\epsilon(1-\alpha_i)}\psi_{+i} \\
\chi_- &\rightarrow e^{-i\epsilon}\chi_- \\
\lambda_{-a} &\rightarrow e^{-i\epsilon\alpha_a}\lambda_{-a} \\
\phi_i &\rightarrow e^{-i\epsilon\alpha_i}\phi_i
\end{align*}\]  

(3.7)

Note that it is not sufficient to have some right-moving R-symmetry; we need this particular one. The R-symmetry that arises in this way has the following property: in case of a (0,2) linear sigma model that happens to be a (2,2) model, it commutes with the left-moving supersymmetry.

The above condition rules out the $CP^n$ model. At the classical level, the $CP^n$ model is a (2,2) model with left- and right-moving R symmetry (and can be derived from a linear sigma model with those properties). At the quantum level, the axial R symmetry is anomalous but the vector symmetry survives; let us call it $V$. If one ignores the left-moving supersymmetry, then $V$ transforms the right-moving supersymmetries as an R symmetry, so one could view the $CP^n$ model as a (0,2) model with a right-moving R symmetry. However, the condition for left-moving conformal invariance is not merely that there should be an anomaly-free right-moving R symmetry, but that the particular symmetry in (3.7) should be anomaly-free; this is not so for the $CP^n$ model.

In the (2,2) case the above condition for cancellation of the anomaly reduces to the condition $\sum_i Q_i = 0$, which together with gauge invariance was shown in [1] to reproduce the Calabi-Yau condition in the appropriate limit ($r=\text{Re}(t)\gg 0$).

Distler and Kachru recently analyzed the string theories arising from (0,2) linear sigma models in their Landau-Ginzburg phases [2]. They not only imposed the condition derived here for the right-moving R-symmetry but also insisted on a non-anomalous left-moving $U(1)$ current $J$. In view of [2] one might wonder about the condition for the chirality of $J$, which fills out the spacetime gauge group and plays the important role of defining the GSO projection and orbifold twisting.

For

\[J = (1 - \tilde{\alpha}_a)\Lambda_{-a}\overline{\Lambda}_{-a} - i\tilde{\alpha}_i\Phi_i\overline{D_+}\Phi_i\]  

(3.8)
a similar calculation to the one presented above reveals that \([\overline{Q}_+, J]\) vanishes at the classical level given the quasihomogeneity conditions

\[
\tilde{\alpha}_a J^a + \sum_i \tilde{\alpha}_i \Phi_i \frac{\partial J^a}{\partial \Phi_i} = J^a
\]  

(3.9)

and

\[
-\tilde{\alpha}_a E_a + \sum_i \tilde{\alpha}_i \Phi_i \frac{\partial E_a}{\partial \Phi_i} = -E_a
\]  

(3.10)

That is, in order to impose the existence of the left \(U(1)\) as well as the left stress tensor, one needs to find constants \(\alpha_a, \alpha_i, \tilde{\alpha}_a\) and \(\tilde{\alpha}_i\) and polynomials \(E_a\) and \(J_a\) to satisfy (2.7), (2.8), (3.9), and (3.10). The cases considered in \([2]\) had \(E_a = 0\) and satisfied these conditions with \(\tilde{\alpha}_a = \alpha_a\) and \(\tilde{\alpha}_i = \alpha_i\). In the gauged (2,2) case discussed in \([1]\) one satisfies these conditions with \(\tilde{\alpha}_\Sigma = -\alpha_\Sigma\) for the gauge field strength \(\Sigma\) and for all other fields \(\tilde{\alpha} = \alpha\). At the quantum level we obtain the condition \(\sum_a Q_a - \sum_a \tilde{\alpha}_a Q_a + \sum_i \tilde{\alpha}_i Q_i = 0\). This is just the standard condition for the symmetry generated by \(J\) to be free of gauge anomalies – in contrast to the left-moving conformal symmetry where the anomaly in \(\{\overline{Q}_+, \ldots\}\) gave new information.

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