Symmetric-periodic solutions for some types of generalized neutral equations

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Abstract The existence of symmetric-periodic outcomes for a class of fractional differential equations has been increasingly studied. Such study has used various methods such as fixed point theory, critical point theory, and approximation theory. In this work, we study the m-pseudo almost automorphic (m-PAAA) outcomes for a category of fractional neutral differential equations. To satisfy this aim, we introduce composition results under suitable conditions and employ them to establish some extant outcomes using interpolation theory mixed with fixed point technique. Examples are illustrated.

Keywords Fractional calculus • Fractional differential equations • Periodic symmetric solution

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Introduction

The symmetry in the field of differential equations is a transformation that preserves its domestic of results invariant. Symmetry study can be utilized to resolve some classes of ordinary, partial, fractional differential equations, though defining the symmetries can be computationally concentrated like other mathematical methods. The best method for symmetry is by finding the periodic solution of the differential equation.

In 1962, Bochner [1] introduced the concept of almost automorphy, which is an important generalization of almost periodicity. The concept of almost periodic functions was introduced by Bohr [2]. It was named as PAA functions because they originally presented themselves, in their work in differential geometry, as scalars and tensors on manifolds with (discrete) groups of automorphisms [3]. Later, PAAA function has become one of the most attractive topics in the qualitative theory of evolution equations, and there have been several interesting, natural and powerful generalizations of the classical PAAA functions [4–9]. Recently, Digana et al., studied the concept for different classes of ODF and PDE (see [10–14]). Xiao et al. [15] introduced the concept of PAAA functions for a natural and a significant extension of PAAA functions. Moreover, they proved that the space of PAAA functions is complete; so they solve a key fundamental problem on this issue and pave the road to further study the applications of PAAA functions. They investigated the existence of PAAA to

\[ u'(t) = \Lambda(t)u(t) + f(t) \]

and

\[ u'(t) = \Lambda(t)u(t) + f(t, u(t)) \]

in a Banach space. Chang and Luos [16] presented a composition theorem for m-PAAA function, which was proved under appropriate conditions. They applied this theorem to investigate whether the m-PAAA solutions exist in the neutral differential equation as follows:
\[ \frac{d}{dt}[u(t) + f(t, u(t))] = \Lambda u(t) + g(t, u(t)), \quad t \in \mathbb{R}. \]

Periodic motion is a very important and special phenomena not only in natural science, but also in social science, such as climate, food supplement, insecticide population and sustainable development. Periodic solutions are desired property in differential equations, constituting one of the most important research directions in the theory of differential equations. The existence of periodic solutions is often a desired property in dynamical systems, constituting one of the most important research directions in the theory of dynamical systems, with applications ranging from celestial mechanics to biology and finance. Fractional differential equations (FDEs) are the most important generalizations of the field of ODE [17–21]. Recent investigations in physics, engineering, biological sciences and other fields have demonstrated that the dynamics of many systems are described more accurately using FDEs, and that FDE with delay are often more realistic to describe natural phenomena than those without delay. Periodic solution fractional differential equations have been studied by many researchers. They studied periodic solutions of the equation (see [17, 18])

\[ D^\alpha u(t) + BD^{\beta} u(t) + f(t) = 0, \quad 0 \leq t \leq 2\pi, \]

where \( A \) and \( B \) are closed linear operators defined on a complex Banach space \( X \) with domains \( D(A) \) and \( D(B) \), respectively, \( 0 \leq \beta < \alpha \leq 2. \)

The aim of this paper is to study the existence of periodic solutions for the following FDE:

\[ D^\alpha\left(v(t) + \varphi(t, v(t))\right) = \Lambda v(t) + \vartheta(t, v(t)), \quad t \in \mathbb{R} \]  \hspace{1cm} (1)

\[ \forall \mu \in (0, 1], \] where \( \Lambda : dom(\Lambda) \subset \chi \rightarrow \chi \) is considered the operator of a hyperbolic analytic semigroup \( T(t)_{t \geq 0} \), and \( \varphi : \mathbb{R} \times \chi \rightarrow \chi \) are appropriate continuous functions; \( \chi \) refers to the appropriate interpolation space and \( D^\mu \) is the Riemann–Liouville fractional differential operator (R–L operator).

This paper is classified as follows. In “Setting”, we present some basic definitions, lemmas, and setting results which will be used in this study. In “Findings”, we introduce some existence results of almost-periodic and mild solutions of the fractional neutral differential equation. Examples are illustrated in the sequel.

\section*{Setting}

The researchers allocated this section to investigate some results required in the sequel. In this paper, the notations \( (\chi, \| \cdot \|) \) and \( (\mathcal{Y}, \| \cdot \|_Y) \) denote the two Banach spaces, whereas \( BC(\mathbb{R}, \chi) \) refers to the Banach space of all bounded continuous functions from \( \mathbb{R} \) to \( \chi \) qualified with the supremum norm \( \| \varphi \|_\infty = \sup_{t \in \mathbb{R}} \| \varphi(t) \| \). Let \( \chi_\lambda \) be a space mediated between \( dom(\Lambda) \) and \( \chi_\lambda B(\mathbb{R}, \chi_\lambda) \) for \( \lambda \in (0, 1) \) refers to Banach space of all bounded continuous functions \( \sigma : \mathbb{R} \rightarrow \chi_\lambda \) when supported with the \( \lambda - \sup \) norm:

\[ \| \sigma \|_{\lambda, \infty} := \sup_{t \in \mathbb{R}} \| \sigma(t) \|_\lambda \]

for \( \sigma \in BC(\mathbb{R}, \chi_\lambda) \). Throughout this paper, \( \varphi \) denotes the Lebesgue field of \( \mathbb{R} \) and \( \mathbb{N} \) the set of all positive numbers \( m \) on \( \varphi \) satisfying \( m(\mathbb{R}) = +\infty \) and \( m([a, b]) = +\infty \), for all \( a, b \in \mathbb{R} \).

\begin{definition} \[ [3] \] A continuous function \( \varphi : \mathbb{R} \rightarrow \chi \) referred to as automorphic in the case that every sequence \( (z_n)_{n \in \mathbb{N}} \) has a subsequence \( (z'_n)_{n \in \mathbb{N}} \subset (z_n)_{n \in \mathbb{N}} \) such that

\[ \lim_{n \to \infty} \| \varphi(t + z_n - z'_n) - \varphi(t) \| = 0. \]

Define

\[ \rho \Lambda \Lambda_0(\mathbb{R}, \chi) = \left\{ \Phi \in \mathbb{W}G(\mathbb{R}, \chi) : \lim_{s \to +\infty} \frac{1}{T - s} \int_s^T \| \varphi(\tau) \| d\tau = 0 \right\}. \]

Likely, \( \rho \Lambda \Lambda_0(\mathbb{R} \times \chi, \chi) \) is defined as the gathering of combined continuous functions \( \varphi : \mathbb{R} \times \chi \rightarrow \chi \) which belong to \( BC(\mathbb{R} \times \chi, \chi) \) and satisfy

\[ \lim_{T \to +\infty} \frac{1}{T - s} \int_s^T \| \varphi(\tau, x) \| d\tau = 0 \]

uniformly in a compact subset of \( \chi \).

\begin{definition} \[ [15] \] A continuous function \( \varphi : \mathbb{R} \rightarrow \chi \) (respectively, \( \mathbb{R} \times \chi \rightarrow \chi \)) refers to a periodic function when decomposed as \( \varphi = \vartheta + \Phi \), where \( \vartheta \in \Lambda \Lambda(\mathbb{R}, \chi) \) (respectively, \( \Lambda \Lambda(\mathbb{R} \times \chi, \chi) \)) and \( \Phi \in \rho \Lambda \Lambda_0(\mathbb{R}, \chi) \) (respectively, \( \rho \Lambda \Lambda_0(\mathbb{R} \times \chi, \chi) \)). Denote by \( \rho \Lambda \Lambda(\mathbb{R}, \chi) \) (respectively, \( \rho \Lambda \Lambda(\mathbb{R} \times \chi, \chi) \)) the set of all such functions.

\begin{definition} \[ [19] \] Let \( m \in \mathbb{N} \). A bounded continuous function \( \varphi : \mathbb{R} \rightarrow \chi \) is referred to as \( m \)-ergodic if \( \varphi \) is ergodic with respect to \( m \) (measure) i.e.

\[ \lim_{r \to +\infty} \frac{1}{m([-r,r])} \int_{-r}^r \| \varphi(t) \| dm(t) = 0. \]

The space of all such functions is denoted as \( \zeta(\mathbb{R}, \chi, m) \) and \( \zeta_\infty(\mathbb{R}, \chi, m) \) \( \| \cdot \|_\infty \) is a Banach space (see [19], Proposition 2.13]). The word ergodic (work) is employed to explain the dynamical system which has the same behavior averaged during the item time as averaged over the phase space.

\begin{definition} \[ [19] \] Let \( m \in \mathbb{N} \). A continuous function \( \varphi : \mathbb{R} \rightarrow \chi \) is stated to be \( m \)-\( \rho \Lambda \Lambda \) if \( \varphi \) comes in the form

\[ \varphi(t) = \vartheta(t) + \Phi(t), \quad t \in \mathbb{R}, \]

where \( \vartheta(t) \in \Lambda \Lambda(\mathbb{R}, \chi) \) and \( \Phi \in \rho \Lambda \Lambda_0(\mathbb{R}, \chi) \).

\]
φ = ˙θ + Φ, where ˙θ ∈ ΔΔ(ℵ, φ) and Φ ∈ ζ(ℵ, φ, m). So, all such functions have a space denoted by ρΔΔ(ℵ, φ, m). Most clearly, we have ΔΔ(ℵ, φ) ⊂ ρΔΔ(ℵ, φ, m) ⊂ BC(ℵ, φ).

Lemma 2.5 [20, Theorem 2.2.6] If φ : ℵ × ℵ → ℵ is PAA, and assume that φ(t, •) is uniformly continuous on each bounded subset κ ⊂ ℵ uniformly for t ∈ ℵ, that is for any ζ > 0, there exists S > 0 such that x, y ∈ κ and ||x − y|| < S imply that ||φ(t, x) − φ(t, y)|| < ζ for all t ∈ ℵ. Let Φ : ℵ → ℵ be PAA. Then the function F : ℵ → ℵ defined by F(t) = φ(t, Φ(t)) is PAA.

Lemma 2.6 [19, Theorem 4.1] Let m ∈ ℵ and φ ∈ ρΔΔ(ℵ, φ, m) be such that φ = ˙θ + Φ, where ˙θ ∈ ΔΔ(ℵ, φ) and Φ ∈ ζ(ℵ, φ, m). If ρΔΔ(ℵ, φ, m) is translation invariant, then
\[ \{ ˙φ(t) : t ∈ ℵ \} \subset \{ φ(t) : t ∈ ℵ \} \]
(the closure of the range of φ).

Definition 2.8 The Riemann–Liouville fractional integral is defined as follows:
\[ I^α u(t) = \frac{1}{Γ(α)} \int_0^t (t − z)^{α−1} u(z) dz, \]
where Γ denotes the gamma function (see [22, 23]).

Definition 2.9 The Riemann–Liouville fractional derivative is defined as follows:
\[ D^α u(t) = \frac{1}{Γ(1−α)} \frac{d}{dt} \int_0^t (t − z)^{−α} u(z) dz, \quad 0 < t < ∞. \]

Definition 2.10 [8, Definition 2.5] A semi group \((T(t))_{t≥0}\) on ℵ is stated to be hyperbolic if there is a projection η and constants η, S > 0 such that each T(t) commutes with η. Kerη is invariant with respect to T(t), T(t) : ImQ → ImQ is invertible and for every x ∈ ℵ
\[ \| T(t)ρx \| ≤ Mg^{−S}\| x \|, \quad \text{fort} ≥ 0; \]
\[ \| T(t)Qx \| ≤ Mg^{−S}\| x \|, \quad \text{fort} ≤ 0, \]
where \( Q := I − ρ \) and, for t < 0, \( T(t) = (−t)^{−1}. \)

Definition 2.11 [14] A linear operator \( Λ : dom(Λ) ⊂ ℵ → ℵ \) (not necessarily densely defined) is referred to as sectorial if the following hold: there exist constants \( m ∈ ℵ, \underline{θ} ∈ (π, 0) \) and M > 0 such that \( p(Λ) ⊂ \underline{z}_{0,13} := \{ x ∈ ℵ : x = \underline{z}, |arg(x − \underline{z})| < 0 \}, \)
\[ \| Φ(x, Λ) \| ≤ \frac{2M}{|x − \underline{z}|^S}, \quad x ∈ \underline{z}_{0,13}. \]

Definition 2.12 [8, Definition 2.7] Let \( 0 ≤ λ ≤ 1 \). A Banach space \( Y \) such that \( Z → Y → ℵ \) refers to the class \( J_λ \) between \( Z \) and \( Y \) if there is a constant \( c > 0 \) such that
\[ \| x \|_Y ≤ c\| x \|^{1−λ}_Z(\| x \|_Z). \]
In this case, we write \( Y ∈ J_λ((X), Z) \).

Definition 2.13 [8, Definition 2.8] Let \( Λ : dom(Λ) ⊂ ℵ → ℵ \) be a sectorial operator. A Banach space \((ℵ_λ, \| • \|_λ, \| • \|_Z)\) \( λ ∈ (0, 1) \) is said to be an intermediate space between \( ℵ \) and \( dom(Λ) \) if \( ℵ_λ ∈ J_λ \).

For the problem (1), we list the following assumptions:
\[ \text{(H1)} \quad \text{If } 0 ≤ λ ≤ θ < 1, \text{ then we let } k_1 \text{ be the bound of the embedding } ℵ_θ → ℵ, \text{ that is} \]
\[ \| v \| ≤ k_1 \| v \|_θ \text{ for } v ∈ ℵ_θ. \]
\[ \text{(H2)} \quad \text{Let } 0 ≤ λ < θ < 1, \text{ and the function } φ : ℵ × ℵ → ℵ_θ \text{ belongs to } \rhoΔΔ(ℵ, φ_θ, m), \text{ while } ˙θ : ℵ × ℵ → ℵ_θ \text{ belongs to } \rhoΔΔ(ℵ, φ, m). \text{ Moreover, the functions } φ, ˙θ \text{ are uniformly Lipschitz in rotation to the second following argument: there exist } K > 0 \text{ such that} \]
\[ \| φ(t, v) − φ(t, v) \|_θ ≤ K\| v − v \| \]
and
\[ \| ˙φ(t, v) − ˙φ(t, v) \| ≤ K\| v − v \| \]
for all \( v, v ∈ ℵ \) and \( t ∈ ℵ \).

Findings
In the present section, a composition theorem is proved for \( m-\text{PAA} \) functions under appropriate conditions. Then, we apply this composition theorem to obtain some results regarding Eq. (1).
Theorem 3.1 Let \( m \in \mathbb{R} \) and \( \varphi = \partial + \vec{n} \in \rho \Lambda(\mathbb{R} \times \mathbb{R}, \mathbb{R}, m) \). Suppose that

(\text{H3}) \quad \varphi(t, \chi) \text{ is uniformly continuous on any bounded subset } \kappa \subset \mathbb{R} \text{ uniformly in } t \in \mathbb{R};

(\text{H4}) \quad \partial(t, \chi) \text{ is uniformly continuous on any bounded subset } \kappa \subset \mathbb{R} \text{ uniformly in } t \in \mathbb{R}.

If \( \Phi \in \rho \Lambda(\mathbb{R} \times \mathbb{R}, \mathbb{R}, m) \) then \( F(\cdot) := \Phi(\cdot, \Phi(\cdot)) \in \rho \Lambda(\mathbb{R} \times \mathbb{R}, \mathbb{R}, m) \).

Proof Let \( \varphi = \partial + \vec{n} \) with \( \partial \in \Lambda(\mathbb{R} \times \mathbb{R}, \mathbb{R}), \vec{n} \in \zeta(\mathbb{R} \times \mathbb{R}, \mathbb{R}, m) \) and \( \Phi = v + \vec{n} \), with \( v \in \Lambda(\mathbb{R} \times \mathbb{R}, \mathbb{R}) \) and \( v \in \zeta(\mathbb{R} \times \mathbb{R}, m) \). Define a function \( f \) as follows:

\[
\begin{align*}
f(t) &= \varphi(t, \Phi(t)) - \varphi(t, v(t)) \quad = \varphi(t, v(t)) + \varphi(t, \Phi(t)) - \varphi(t, v(t)) + \vec{n}(t, v(t)).
\end{align*}
\]

Let us restate \( G(t) = \varphi(t, v(t)), \varphi(t, \Phi(t)) - \varphi(t, v(t)), H(t) = \vec{n}(t, v(t)). \)

Therefore, we obtain \( f(t) = G(t) + \varphi(t) + H(t) \). By Lemma 2.5, we conclude that \( G(t) \in \Lambda(\mathbb{R} \times \mathbb{R}, \mathbb{R}) \) and obviously \( \varphi(t) \in BC(\mathbb{R}, \mathbb{R}) \). We proceed to show that \( \varphi(t) \in \zeta(\mathbb{R}, \mathbb{R}, m) \). It suffices to prove that

\[
\lim_{r \to \infty} \frac{1}{m([-r, r])} \int_{[-r, r]} \| \varphi(t) \| dm(t) = 0.
\]

By Lemma 2.6, \( v(\mathbb{R}) \subset \Phi(\mathbb{R}) \), which is a bounded set. From the hypothesis \( (\text{H3}) \) with \( \kappa = \Phi(\mathbb{R}) \) yields that for each \( \cdot \in \mathbb{R} \), \( \| \Phi - v \| \leq S \), \( \| \varphi(t, \Phi(t)) - \varphi(t, v(t)) \| \leq S \).

From the following set: \( \Lambda_{r, s} = \{ t \in [-r, r] : \| \varphi(t) \| > r \} \), we get

\[
\Lambda_{r, s} = \Lambda_{r, s} \Phi(t, \Phi(t)) - \varphi(t, u(t)) \subset \Lambda_{r, s} \Phi(t) - \varphi(t, v(t)) = \Lambda_{r, s} v.
\]

Therefore, the following inequality carries:

\[
\frac{m\{ t \in [-r, r] : \| \varphi(t, \Phi(t)) - \varphi(t, v(t)) \| > S \}}{m([-r, r])} \leq \frac{m\{ t \in [-r, r] : \| \Phi(t) - v(t) \| > S \}}{m([-r, r])}.
\]

Since \( \Phi(t) = v(t) + v(t) \) and \( \Phi \in \zeta(\mathbb{R} \times \mathbb{R}, m) \), Lemma 2.7 states that for the above-mentioned \( S \), we have

\[
\lim_{r \to \infty} \frac{m\{ t \in [-r, r] : \| \varphi(t) - v(t) \| > r \}}{m([-r, r])} = 0,
\]

and then we get

\[
\lim_{r \to \infty} \frac{m\{ t \in [-r, r] : \| H(t) \| > r \}}{m([-r, r])} = 0.
\]

Again, in view of Lemma 2.7 and Eq. (8), we attain \( \varphi(t) \in \zeta(\mathbb{R} \times \mathbb{R}, m) \). Finally, we have to prove \( H(t) = \vec{n}(t, v(t)) \in \zeta(\mathbb{R} \times \mathbb{R}, m) \). Since \( v \) is continuous on \( \mathbb{R} \) as \( \Lambda(\mathbb{R} \times \mathbb{R}, \mathbb{R}) \), the set \( v([-r, r]) \) can be taken as compact. Therefore, the function \( \partial \in \Lambda(\mathbb{R} \times \mathbb{R}, \mathbb{R}) \), and \( \partial \) is uniformly continuous on \( [-r, r] \). Then, \( (\text{H3}) \) implies that \( \vec{n}(t, \chi) \) is uniformly continuous with \( X \in \zeta(\mathbb{R} \times \mathbb{R}, m) \) uniformly in \( t \in \mathbb{R} \). Thus for any \( \zeta > 0 \), a constant \( S > 0 \) exists such that for \( X_1, X_2 \in v([-r, r]) \) with \( \| X_1 - X_2 \| < S \), we have

\[
\| H(t, X_1) - H(t, X_2) \| < \frac{\zeta}{2}, \quad \forall t \in [-r, r].
\]

By the compactness of the set \( v([-r, r]) \), we conclude that there is an existence of finite balls \( \Theta_K \) with \( \partial \in \zeta(\mathbb{R} \times \mathbb{R}, m) \). Finally, we have to prove \( H(t) = \vec{n}(t, v(t)) \in \zeta(\mathbb{R} \times \mathbb{R}, m) \). Since \( v \) is continuous on \( \mathbb{R} \) as \( \Lambda(\mathbb{R} \times \mathbb{R}, \mathbb{R}) \), the set \( v([-r, r]) \) can be taken as compact. Therefore, the function \( \partial \in \Lambda(\mathbb{R} \times \mathbb{R}, \mathbb{R}) \), and \( \partial \) is uniformly continuous on \( [-r, r] \). Then, \( (\text{H3}) \) implies that \( \vec{n}(t, \chi) \) is uniformly continuous with \( X \in \zeta(\mathbb{R} \times \mathbb{R}, m) \) uniformly in \( t \in \mathbb{R} \). Thus for any \( \zeta > 0 \), a constant \( S > 0 \) exists such that for \( X_1, X_2 \in v([-r, r]) \) with \( \| X_1 - X_2 \| < S \), we have

\[
\| H(t, X_1) - H(t, X_2) \| < \frac{\zeta}{2}, \quad \forall t \in [-r, r].
\]
\( H(t) = \mathcal{H}(t, v(t)) \in \zeta(\mathcal{R}, \chi, m). \)

This ends the proof. \( \square \)

Throughout the remaining parts of this paper, it is proposed that there is existence of two real numbers \( \lambda, \delta \) such that \( 0 < \lambda < \delta < 1 \) and \( 2\delta > \lambda + 1 \). Moreover, we define the following fractional operators \( \gamma^\mu_{\alpha}, \gamma^\mu_{\beta}, \gamma^\mu_{\gamma}, \gamma^\mu_{\delta} \) by

\[
\begin{align*}
\gamma^\mu_{\alpha}(v)(t) &= \int_0^t N^\mu_{\alpha}(t-\zeta)\rho \phi(\zeta, v(\zeta))d\zeta, \\
\gamma^\mu_{\beta}(v)(t) &= \int_0^t \frac{N^\mu_{\beta}(t-\zeta)Q}{\Gamma(\mu)} \phi(\zeta, v(\zeta))d\zeta, \\
\gamma^\mu_{\gamma}(v)(t) &= \int_0^t \frac{N^\mu_{\gamma}(t-\zeta)\rho}{\Gamma(\mu)} \theta(\zeta, v(\zeta))d\zeta, \\
\gamma^\mu_{\delta}(v)(t) &= \int_0^t \frac{N^\mu_{\delta}(t-\zeta)Q}{\Gamma(\mu)} \psi(\zeta, v(\zeta))d\zeta,
\end{align*}
\]

where \( \mu \in (0, 1) \) and \( T(t)_{t \geq 0} \) is the analytic semigroup. It is clear that, if \( T^\mu = (t-\zeta)^{\mu-1} \), then we obtain the RL-integral operator.

**Lemma 3.2** Let \( m \in \mathbb{N}, \; v \in \rho \Delta \Lambda(\mathcal{R}, \chi, m) \) and \((H1)-(H2)\) hold. Then,

\[
\gamma^\mu_{\alpha}, \gamma^\mu_{\beta}, \gamma^\mu_{\gamma}, \gamma^\mu_{\delta} : \rho \Delta \Lambda(\mathcal{R}, \chi, m) \rightarrow \rho \Delta \Lambda(\mathcal{R}, \chi, m).
\]

**Proof** Let \( v \in \rho \Delta \Lambda(\mathcal{R}, \chi, m) \). Putting \( \mathcal{H}(t) = \vartheta(t, v(t)) \) and by Theorem 3.1, it indicates that \( \mathcal{H} \in \rho \Delta \Lambda(\mathcal{R}, \chi, m) \) for each \( v \in \rho \Delta \Lambda(\mathcal{R}, \chi, m) \). Setting \( \mathcal{H} = \Phi + \zeta \), where \( \Phi \in \Delta \Lambda(\mathcal{R}, \chi) \) and \( \zeta \in \zeta(\mathcal{R}, \chi) \). Therefore, \( \gamma^\mu_{\alpha}(v) \) can be read as

\[
\gamma^\mu_{\alpha}(v)(t) = \int_0^t \frac{T^\mu_{\alpha}(t-\zeta)\rho}{\Gamma(\mu)} \Phi(\zeta) d\zeta + \int_0^t \frac{T^\mu_{\alpha}(t-\zeta)\rho}{\Gamma(\mu)} \zeta(\zeta) d\zeta.
\]

Let

\[
\phi(t) = \int_0^t \frac{T^\mu_{\alpha}(t-\zeta)\rho}{\Gamma(\mu)} \Phi(\zeta) d\zeta
\]

and

\[
\psi(t) = \int_0^t \frac{T^\mu_{\alpha}(t-\zeta)\rho}{\Gamma(\mu)} \zeta(\zeta) d\zeta,
\]

for each \( t \in \mathcal{R} \). It can be realized that \( \phi \in \Delta \Lambda(\mathcal{R}, \chi) \). Consider a sequence \( (z_\eta)_{\eta \in \mathbb{N}} \), then there is a subsequence \( (z_\eta)_{\eta \in \mathbb{N}} \) such that

\[
\lim_{\eta, n \to \infty} ||\Phi(t + z_\eta - z_n) - \Phi(t)|| = 0. \tag{6}
\]

Moreover, we have

\[
\begin{align*}
\phi(t + z_\eta - z_n) - \phi(t) &= \int_{-\infty}^{z_\eta - z_n} T^\mu_{\alpha}(t + z_\eta - z_n - \zeta)\rho \Phi(\zeta) d\zeta \\
&\quad - \int_{-\infty}^t \frac{T^\mu_{\alpha}(t - \zeta)\rho}{\Gamma(\mu)} \Phi(\zeta) d\zeta \\
&\quad - \int_{-\infty}^t T^\mu_{\alpha}(t - \zeta)\rho \Phi(\zeta) d\zeta \\
&\quad - \int_{-\infty}^0 \frac{T^\mu_{\alpha}(\zeta)\rho}{\Gamma(\mu)} (\Phi(\zeta + t + z_\eta - z_n) \\
&\quad - \Phi(\zeta + t)) d\zeta.
\end{align*}
\]

Then, we get

\[
||\phi(t + z_\eta - z_n) - \phi(t)||_\lambda \\
\leq \int_{-\infty}^0 \left( \frac{T^\mu_{\alpha}(\zeta)\rho}{\Gamma(\mu)} (\Phi(\zeta + t + z_\eta - z_n) - \Phi(\zeta + t)) \right) d\zeta.
\]

Hence, by (4) and the fact that \( ||T^\mu||_\lambda \leq ||T||_\lambda \), we conclude

\[
||\phi(t + z_\eta - z_n) - \phi(t)||_\lambda \\
\leq \int_{-\infty}^0 \frac{M(\zeta)}{\Gamma(\mu)} \|\Phi(\zeta + t + z_\eta - z_n) - \Phi(\zeta + t)\| d\zeta.
\]

The outcome obtains from (10) and the Lebesgue dominated convergence theorem. Lastly, we aim to show that \( \psi(t) \in \zeta(\mathcal{R}, \chi, m) \). A computation yields

\[
\frac{1}{m([-r, r])} \int_{[-r, r]} \|\psi(t)\|_\lambda dm(t)
\]

\[
= \frac{1}{m([-r, r])} \int_{[-r, r]} \left( \frac{1}{\Gamma(\mu)} \int_{-\infty}^t \frac{T^\mu_{\alpha}(t-\zeta)\rho}{\Gamma(\mu)} \zeta(\zeta) d\zeta \right) dm(t)
\]

\[
\leq \frac{1}{m([-r, r])} \int_{[-r, r]} \left( \frac{1}{\Gamma(\mu)} \int_{-\infty}^t \frac{T^\mu_{\alpha}(t-\zeta)\rho}{\Gamma(\mu)} \zeta(\zeta) d\zeta \right) dm(t)
\]

\[
\leq \frac{1}{m([-r, r])} \int_{[-r, r]} \left( \frac{1}{\Gamma(\mu)} \int_{-\infty}^t \frac{M(\zeta)}{\Gamma(\mu)} \|\zeta(\zeta)\| dm(t) \right) dm(t)
\]

\[
\lesssim \frac{M(\zeta)}{\Gamma(\mu)} \int_{-\infty}^t \left( \frac{1}{m([-r, r])} \int_{[-r, r]} \|\zeta(\zeta)\| dm(t) \right) dm(t)
\]

In fact, the space \( \zeta(\mathcal{R}, \chi, m) \) is invariant (preserved by some function); it shows that \( t - \zeta(t - \zeta) \) belongs to \( \zeta(\mathcal{R}, \chi, m) \) for each \( \zeta \in \mathcal{R} \) and hence

\[
\lim_{t \to \infty} \frac{1}{m([-r, r])} \int_{[-r, r]} \|\zeta(t - \zeta)\| dm(t) = 0.
\]

Consequently, by utilizing the Lebesgue dominated convergence theorem, we have

\[
\lim_{t \to \infty} \frac{M(\zeta)}{\Gamma(\mu)} \int_{-\infty}^t \left( \frac{1}{m([-r, r])} \int_{[-r, r]} \|\zeta(t - \zeta)\| dm(t) \right) d\zeta = 0,
\]

similarly, by applying (5) to \( \gamma^\mu_{\delta}(v) \). This completes the proof. \( \square \)
Lemma 3.3 Let \( m \in \mathbb{N} \), and \( v \in \rho \Lambda \Lambda(\mathbb{R}, \chi, m) \). If (H1)–(H2) are satisfied, then
\[
\gamma^\mu_1, \gamma^\mu_2 : \rho \Lambda \Lambda(\mathbb{R}, \chi, m) \to \rho \Lambda \Lambda(\mathbb{R}, \chi, m).
\]

Proof. Let \( v \in \rho \Lambda \Lambda(\mathbb{R}, \chi, m) \) and \( \mathfrak{h}(t) = \phi(t, v(t)) \). Then in view of Theorem 3.1, it implies that \( \mathfrak{h} \in \rho \Lambda \Lambda(\mathbb{R}, \chi, m) \) whenever \( v \in \rho \Lambda \Lambda(\mathbb{R}, \chi, m) \). In particular,

\[
\| \mathfrak{h} \|_{\delta} = \sup_{t \in \mathbb{R}} \| \phi(t, v(t)) \|_{\delta} < \infty.
\]

Now, we write \( \mathfrak{h} = \Phi + \Psi \), where \( \Phi \in \Lambda \Lambda(\mathbb{R}, \chi, m) \), \( \Psi \in \zeta(\mathbb{R}, \chi, m) \), that is, \( \gamma^\mu_2 \mathfrak{h} = \Phi + \Psi \) where
\[
E\Phi(t) := \int_{-\infty}^{t} \frac{\Lambda T^\mu(t-z)\rho}{\Gamma(\mu)} \Phi(z)dz,
\]
\[
E\Psi(t) := \int_{-\infty}^{t} \frac{\Lambda T^\mu(t-z)\rho}{\Gamma(\mu)} \Psi(z)dz.
\]

First, we need to show that \( E\Phi(t) \in \Lambda \Lambda(\mathbb{R}, \chi, m) \). Consider a sequence \((\zeta_{\nu})_{\nu \in \mathbb{N}}\) in \( t \in \mathbb{R} \), since \( \Phi(t) \in \Lambda \Lambda(\mathbb{R}, \chi, m) \), a subsequence \((\zeta_{\nu})_{\nu \in \mathbb{N}}\) exists such that
\[
\lim_{\nu, \mu \to \infty} \| \Phi(t + \zeta_{\nu} - \zeta_{\mu}) - \Phi(t) \|_{\delta} = 0.
\]

In addition, since
\[
E\Phi(t + \zeta_{\nu} - \zeta_{\mu}) - E\Phi(t) = \int_{-\infty}^{t} \frac{\Lambda T^\mu(t + \zeta_{\nu} - \zeta_{\mu} - z)\rho}{\Gamma(\mu)} \Phi(z)dz - \int_{-\infty}^{t} \frac{\Lambda T^\mu(t - z)\rho}{\Gamma(\mu)} \Phi(z)dz
\]
\[
= \int_{-\infty}^{t} \frac{\Lambda T^\mu(-z)\rho}{\Gamma(\mu)} \Phi(z + t + \zeta_{\nu} - \zeta_{\mu} - \zeta_{t}) - \Phi(z + t)dz.
\]

Then, a computation implies
\[
\| E\Phi(t + \zeta_{\nu} - \zeta_{\mu}) - E\Phi(t) \|_{\delta} \leq \int_{-\infty}^{t} \left| \frac{\Lambda T^\mu(-z)\rho}{\Gamma(\mu)} \Phi(z + t + \zeta_{\nu} - \zeta_{\mu} - \zeta_{t}) - \Phi(z + t)dz \right|.
\]

Hence, by (6) and the fact that \( \| A(T^\mu) \| \leq \| A(T) \| \), we receive
\[
\| E\Phi(t + \zeta_{\nu} - \zeta_{\mu}) - E\Phi(t) \|_{\delta} \leq \int_{-\infty}^{t} \frac{Cz^{\mu-1} - C}{\Gamma(\mu)} \| \Phi(z + t + \zeta_{\nu} - \zeta_{\mu} - \zeta_{t}) - \Phi(z + t) \|_{\delta} dz.
\]

The result comes from Eq. (11) and the Lebesgue’s dominated theorem. Finally, we reveal that \( E\Psi(t) \in \zeta(\mathbb{R}, \chi, m) \). We have
\[
\begin{align*}
\frac{1}{m([-r, t])} \int_{[-r, t]} \| \mathcal{E}(t) \| dm(t) \\
= \frac{1}{m([-r, t])} \int_{[-r, t]} \left\| \frac{\Lambda T^\mu(t-z)\rho}{\Gamma(\mu)} \Psi(z)dz \right\| dm(t) \\
\leq \frac{1}{m([-r, t])} \int_{[-r, t]} \left\| \frac{\Lambda T^\mu(t-z)\rho}{\Gamma(\mu)} \Psi(z)dz \right\| dm(t) \\
\leq \frac{1}{m([-r, t])} \int_{[-r, t]} \left( \frac{c(\zeta_{\nu} - \zeta_{\mu})^{\mu-1} C}{m([-r, t])} \right) \left\| \Psi(z)dz \right\| dm(t)
\end{align*}
\]

Therefore, we obtain
\[
\lim_{t \to \infty} \frac{1}{m([-r, t])} \int_{[-r, t]} \| \Psi(t - \zeta) \|_{\delta} dm(t) = 0
\]

as
\[
\zeta \to \Psi(t - \zeta) \in \zeta(\mathbb{R}, \chi, m)
\]

for every \( \zeta \in \mathbb{R} \). The proof is completed by applying the Lebesgue’s dominated convergence theorem and similarly for \( \gamma^\mu_2 \).

\[\square\]

\textbf{m-PAAM outcomes}

The rest of this section is conducted to find the existence of \( m \)-PAAM mild solutions (\( m \)-PAAM) of Eq. (1). Recently, Ibrahim et al. studied the mild solution of a class of FDE, by utilizing the fractional resolvent concept (see [24, 25]).

\textbf{Definition 3.4} Let \( \lambda \in (0, 1) \). A bounded continuous function \( v : \mathbb{R} \to \chi_\lambda \) is stated to be a mild solution to (1) indicate that the function \( \zeta \to \Lambda T^\mu(t-\zeta)\phi(z, v(z)) \) is integrable on \([-\infty, t] \), \( \zeta \to \frac{\Lambda T^\mu(t-\zeta)}{\Gamma(\mu)} Q \phi(z, v(z)) \) is integrable on \((t, \infty) \) and
\[
v(t) = -\phi(t, v(t)) - \int_{-\infty}^{t} \frac{\Lambda T^\mu(t-\zeta)}{\Gamma(\mu)} \phi(z, v(z))dz + \int_{t}^{\infty} \frac{\Lambda T^\mu(t-\zeta)}{\Gamma(\mu)} Q \phi(z, v(z))dz - \int_{t}^{\infty} \frac{\Lambda T^\mu(t-\zeta)}{\Gamma(\mu)} \phi(z, v(z))dz
\]

for each \( t \in \mathbb{R} \).

\textbf{Theorem 3.5} Let \( m \in \mathbb{N} \). Under the assumptions (H1) and (H2), Eq. (1) admits a unique \( m \)-PAAM solution for some constants \( K > 0 \).

Proof. Consider the fractional integral operator \( \Lambda : \rho \Lambda \Lambda(\mathbb{R}, \chi, m) \to \rho \Lambda \Lambda(\mathbb{R}, \chi, m) \) such that
Now, for $\gamma_1^m$, we conclude

$$\|\gamma_1^m(t) - \gamma_1^m(s)\| \leq \int_0^t \left( \int_{-\infty}^0 \left| \frac{\tau^\mu(t-\tau)\rho}{\Gamma(\mu)} \right| d\tau \right) dt \leq \int_0^t \left( \int_{-\infty}^0 \left| \frac{\tau^\mu(t-\tau)\rho}{\Gamma(\mu)} \right| d\tau \right) dt \leq \int_0^t \left( \int_{-\infty}^0 \left| \frac{\tau^\mu(t-\tau)\rho}{\Gamma(\mu)} \right| d\tau \right) dt \leq C \|\gamma_1^m\|_{1,\infty}.$$}

Joining the above inequalities yields $\|\gamma - \gamma\|_{1,\infty} \leq K\Xi$, where

$$\Xi := \frac{k_1 + k_2 + k_3 + k_4}{\Gamma(\mu)}.$$ 

Therefore, if $K < \Xi^{-1}$, then in view of the Banach fixed point theorem, Eq. (1) has a unique solution, which clearly is the only $m$-PAAM solution. This completes the proof.

**Example 3.6** Consider the equation

$$D^\mu \left( (1+t)v(t) \right) = \left( (1+t)v(t) \right), \quad t \in [0,1], \mu \in (0,1].$$

Let $\mu = 0.15$. It is clear that $\phi(t,v(t)) = \vartheta(t,v(t)) = tv(t)$. Thus, they are Lipschitz with $K = 1$, $t \in [0,1]$. Moreover, $\kappa_1 = \kappa_2 = \kappa_3 = \kappa_4 = 2.7/2$, this implies that $\frac{k_1 + k_2 + k_3 + k_4}{\Gamma(0.15)} = 4.811 \times 0.874 \to \Xi^{-1} = 1.14 > K = 1$, $k_1 = 1$ such that $\lambda = 1/2$ and $\|v\| \leq k_1\|v\|_1$. In addition, the functions $\varphi$ and $\vartheta$ are bounded and uniformly continuous for all $t \in [0,1]$. Hence, all the conditions of Theorem 3.5 are achieved; therefore, Eq. (12) has a unique periodic solution. Note that if $\mu = 0.5$, we obtain $\|v_{n+1} - v_n\| = 3.0 > \Xi^{-1} = 0.3 < K = 1$, then Theorem 3.5 is achieved. From the above computation, Eq. (12) has a unique periodic solution when $0 < \mu < 0.4$.

**Example 3.7** Consider the equation

$$D^\mu \left( (1/4 + \sin t)v(t) \right) = \left( (1/2 + \cos t)v(t) \right), \quad t \in [0,2\pi], \mu \in (0,1].$$

Obviously, $\phi(t,v(t)) = \sin tv(t)$, $\vartheta(t,v(t)) = \cos tv(t)$. Thus, they are Lipschitz with $K = \max_{t \in [0,2\pi]} \{ |\sin(t)|, |\cos(t)| \} = 1$, $\kappa_1 = \kappa_2 = \kappa_3 = \kappa_4 = 2.7/4$; this implies that $\frac{k_1 + k_2 + k_3 + k_4}{\Gamma(0.15)} = 0.4 \to \Xi^{-1} = 2.5 > K = 1$. Moreover, $k_1 = 1/2$ such that $\lambda = 1/4$ and $\|v\| \leq k_1\|v\|_1$. In addition, the functions $\varphi$ and $\vartheta$ are bounded and uniformly continuous for all $t \in [0,2\pi]$. Hence, all the conditions of Theorem 3.5 are achieved; therefore, Eq. (13) admits a unique periodic solution. If $\mu = 0.5$, then we have $\Xi^{-1} = 0.66 < K = 1$ and hence Theorem 3.5 is true when $0 < \mu < 0.4$.

**Conclusions**

In the current study, we suggested symmetry of a class of fractional differential equations (the fractional calculus depends on the RL fractional operators) by utilizing its periodic solutions. We studied a special class of FDEs. This class is a generalization of the neutral equation. We
proved a composition theorem for m-PABA functions under appropriate conditions. Our technique is based on interpolation theory and Banach’s fixed point theorem. Therefore, the solution, in this case, is unique. Moreover, we investigated the mild solution, for such a class by illustrating a new fractional resolvent concept. This functional is constructed to keep the periodicity of the solution and consequently its symmetry.

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**Compliance with ethical standards**

**Conflict of interest** The authors declare that they have no competing interests

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