Tilted stable subordinators, Gamma time changes and Occupation
Time of rays by Bessel Spiders

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We exhibit, in the form of some identities in law, some connections between tilted stable subordinators, time-changed by independent Gamma processes and the occupation times of Bessel spiders, or their bridges. These identities in law are then explained thanks to excursion theory.

1 Introduction

1.1 Aim of this work

The genesis of this paper is our interest in a class of subordinators \((T_t, t \geq 0)\) which enjoy the following properties: (i) \((T_t, t \geq 0)\) is of GGC (Generalized Gamma convolution) type. That is, it has no drift and its Lévy measure is of the form

\[
\theta \frac{d\xi}{x} \mathbb{E}[e^{-xG}]
\]

for \(G\), a random variable such that \(\mathbb{E}[(\log(1/G))^+] < \infty\), and \(\theta > 0\); (ii) each marginal law of \(T_t\), for fixed \(t\), can be described explicitly. Of course, this second demand is not mathematically very precise but we are seeking descriptions such as:

\[T_t \sim X_t Y_t,\]

for fixed \(t\), where \(X_t\) and \(Y_t\) have classical (e.g. beta, gamma) distributions. [Throughout, we shall write: \(U \sim V\), to mean that \(U\) and \(V\) are identically distributed.] As we see above the law of the subordinator \((T_t)\), which can be written more precisely as \((T^{(0,G)}_t)\) depends both on \(G\) and \(\theta\). However, we note that \((T^{(0,G)}_t, t \geq 0)\) is equivalent (in law) to \((T^{(1,G)}_{\theta t}, t \geq 0)\).

Remark 1. The general class of GGC random variables has been extensively studied by Thorin (1977) and Bondesson (1992). These are infinitely divisible random variables and hence one can naturally construct subordinators from them. We note that in general the Lévy measure of such GGC subordinators may be defined as \(x^{-1} \int_0^\infty e^{-xy}U(dy)\), where \(U\) is a sigma-finite measure. That is to say it does not necessarily correspond to, or arise from, a random variable. The case where \(U(\infty) = \infty\), contains many important classes of random variables, including positive stable random variables of index \(0 < \alpha < 1\) and the GIG (Generalized Inverse Gaussian) class. See Bondesson (1992) for many properties and examples of GGC random variables.

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1.2 First examples

Here, we next give two examples of subordinators which satisfy the desired properties (i) and (ii) mentioned above. First let us take $\theta = 1/2$, then:

Example 1 $G := G_{1/2} \sim \beta_{(1/2,1/2)}$, where $\beta_{(a,b)}$ denotes a beta random variable, with parameters $(a,b)$.

Example 2 $G := 1/G_{1/2} \sim 1/\beta_{(1/2,1/2)}$.

We denote these subordinators by $\varepsilon_1$ and $\varepsilon_2$, respectively (where here $\varepsilon$ stands for example). Then, as we shall see in section 3, there are the equations:

\begin{align}
\mathbb{E}[\exp(-\lambda \varepsilon_1(t))] &= (\sqrt{1 + \lambda} - \sqrt{\lambda})^t = \frac{1}{(\sqrt{1 + \lambda} + \sqrt{\lambda})^t} \\
\text{and } \varepsilon_1(t) &\sim \gamma_{t/2}/\beta_{(1/2,(1+t)/2)}.
\end{align}

\begin{align}
\mathbb{E}[\exp(-\lambda \varepsilon_2(t))] &= \frac{2^t}{(1 + \sqrt{1 + \lambda})} \\
\text{and } \varepsilon_1(t) &\sim \gamma_{t/2}/\beta_{(1/2,(1+t)/2)}.
\end{align}

Remark 2. Interestingly, the processes $\varepsilon_1$ and $\varepsilon_2$, have appeared previously. $\varepsilon_1$ is a special case of the random variables recently studied in Bertoin, Fujita, Roynette, and Yor (2006) and Fujita and Yor (2006). The fact that $\varepsilon_1(t) \sim \gamma_{t/2}/\beta_{(1/2,(1+t)/2)}$, as indicated in (1.2), is demonstrated in Royquette and Yor (2006) where connections to Feller (1966) are also noted. The fact that $\varepsilon_2(t) \sim \gamma_{t/2}/\beta_{(1/2,(1+t)/2)}$, as indicated in (1.3), follows from a result of Cifarelli and Melilli (2000), concerning Dirichlet mean functionals, as discussed in James (2006).

1.3 A general class of variables $G$

The fact that $\beta_{(1/2,1/2)}$ is the arcsine law, and that this law is that of the time spent in $\mathbb{R}^+$ (or $\mathbb{R}^-$) by real valued Brownian motion, led us to look for generalizations of the above examples, where the variable $G$ in (1.1) would be replaced by a variable distributed as the time spent in $\mathbb{R}^+$ by a symmetrized Bessel process and even to look for further generalizations. As shown in Barlow, Pitman and Yor (1989), the time spent in $\mathbb{R}^+$ by a symmetrized Bessel process with dimension $d = 2(1-\alpha)$, for $0 < \alpha < 1$, is distributed as:

$$
\frac{T_\alpha}{T_\alpha + T'_\alpha},
$$

where $T_\alpha$ and $T'_\alpha$ are two independent standard stable ($\alpha$) random variables, i.e. $\mathbb{E}[\exp(-\lambda T_\alpha)] = \exp(-\lambda^\alpha)$. More generally, we looked for a description of the subordinators satisfying (1.1) with

$$
G \overset{d}{=} \frac{\sum_{j=1}^N \nu_j T^{(j)}_\alpha}{\sum_{j=1}^N \nu_j T^{(j)}},
$$

where all the coefficients are assumed to be non-negative, and the $(T^{(j)}_\alpha, j = 1, 2, \ldots, N)$ are $N$ iid standard stable ($\alpha$) variables.

It turns out that the subordinator $T_\alpha$ associated with $G$ given in (1.4) may be obtained from

$$
G_{\mu} := G_{\mu}^{(\mu,\nu)} = \sum_{i=1}^N \mu_i T^{(i)}(u)
$$
where the \( (T^{(u)}_\alpha(u); u \geq 0) \) are constructed from \( N \) independent stable \((\alpha)\) subordinators, each of them being Esscher transformed with parameter \((\nu_i)\), that is,

\[
\mathbb{E}[\exp(-\lambda T^{(u)}_\alpha(u))] = \exp(-u[(\nu_i + \lambda) - \nu_i^\alpha])
\]

In fact, \((T_t)\) is constructed as \((\mathcal{C}_{\gamma_t}, t \geq 0)\) where \((\gamma_t, t \geq 0)\) is a standard Gamma process independent of \((\mathcal{C}_{u}, u \geq 0)\). We call the process \((\mathcal{C}_{\gamma_t}, t \geq 0)\) a (generalized) positive Linnik process with index \(\alpha\), and parameters \((\mu, \nu)\). The name we use comes from its link to the class of random variables expressible as \(L_{\alpha,t} = \gamma_t^{1/\alpha}T_\alpha\), defined for each fixed \(t > 0\), where \(\gamma_t\) and \(T_\alpha\) are independent. Naturally these random variables are equivalent in distribution to a positive stable \((\alpha)\) subordinator time changed by a gamma process \(\gamma_t\), for each fixed \(t\). The random variables, \(L_{\alpha,t}\), are known in the literature (see for instance Bondesson (1992, p.38) and Devroye (1990, 1996)) and are sometimes called generalized positive Linnik random variables. These random variables have a host of interesting properties and distributional representations. Huillet (2000) discusses some results related to the subordinator. James (2006) provides an extensive recent study of \(L_{\alpha,t}\) and its Esscher transformed version which connects with the work of Barlow, Pitman and Yor (1989), and our present exposition. In this sense, \(L_{\alpha,t}\) and its Esscher transformed version arise as special cases of \((\mathcal{C}_{\gamma_t}, t \geq 0)\). Moreover, \((\mathcal{C}_{\gamma_t}, t \geq 0)\), also contains an interesting subclass of the models discussed in the recent related work of Bertoin, Fujita, Roynette and Yor (2006). We shall show in Section 4 that we are able to describe precisely the one-dimensional marginals of the generalized positive Linnik process, \((\mathcal{C}_{\gamma_t}, t \geq 0)\), which therefore answers positively our demands (i) and (ii) made in (1.1). These one-dimensional marginals are closely related with the occupation times of the Bessel spiders living in webs constituted by \(N\) rays as described in Barlow, Pitman and Yor (1989). Once this was noticed, it became of interest to explain this coincidence. This is what we do in the last two sections of the paper.

1.4 Organization of the paper

We now describe the organization of the remainder of this paper: in Section 2, we recall notation and results about Bessel spiders and their occupation times of rays; in Section 3, we show that we can describe the one-dimensional marginals and the Lévy measures of the generalized Linnik process, in Section 4, we compare the results obtained in Section 2 and 3; in the final Section 5, we give an explanation, based on excursion theory, for the close relationship between the distributions we have encountered in Sections 2 and 3.

2 On the occupation times of rays by a Bessel spider

In this section, we simply recall the definition of a Bessel spider, with index \(\alpha \in (0, 1)\), and whose state space is \(E\), the union of \(N\) half lines, or rays, \(I_1, \ldots, I_N\), originating from a common point which we denote as \(0\). In fact, these random processes are indexed by both \(\alpha \in (0, 1)\), and \(p = (p_1, \ldots, p_N)\), a generic probability on \(\{1, 2, \ldots, N\}\). Now, we may introduce \(\mathcal{S}(\alpha, p) = (S_t, t \geq 0)\), a Bessel spider with index \(\alpha\), and probability \(p\), which lives on \(E\), as, informally, a process \((S_t, t \geq 0)\) which behaves like a BES\((\alpha)\) process on each of the rays, \(I_1, \ldots, I_N\), and which, when coming back to \(0\), rides off in all directions at once, and chooses its ray \(I_i\), with probability \(p_i\). Of course, this is only a heuristic description as the point \(0\) is regular for itself (with respect to the Markov process \((S_t)\)), but rigorous constructions have been provided by Barlow, Pitman and Yor (1989), with the help in particular of excursion theory.

The case \(\alpha = 1/2\) is particularly interesting, as the \((1/2, p)\) spider behaves like a Brownian motion in each of its rays; We also note that for \(\alpha = 1/2\), and \(N = 2\), \(p = (p_1, p_2)\), \(\mathcal{S}(1/2, p)\) may be identified with the skew Brownian motion of, say, parameter \(p_1\) (or \(p_2\)).
Theorem 2
1) For any \( S \) a reflecting Brownian motion, and \( (S, t \geq 0) \) admits a local time at 0, \( (L_t, t \geq 0) \) such that \( (|S|^{2a} - L_t, t \geq 0) \) is a martingale; we refer the reader to Donati-Martin, Roynette, Vallois, and Yor (2005) for a discussion of different choices of this local time found in the literature. In the following theorem we describe the joint laws of \( \{A_t^{(i)} = \int_0^t \delta s(S_s \in I_i); 1 \leq i \leq N; L_t\} \), for fixed \( t \), as well as the sequence \( \{a_t^{(i)}; i \leq N, \ell_1\} \), the same quantity, but now considered for the standard spider’s bridge.

Theorem 1
1) There is the identity in law
\[
\{A_t^{(i)}; 1 \leq i \leq N\} \overset{\text{law}}{=} \left\{ \frac{p^{1/\alpha}_i T_{(i)}}{\sum_j p^{1/\alpha}_j T^{(j)}}, 1 \leq N; \frac{1}{\sum_j p^{1/\alpha}_j T_{(j)}} \right\},
\]
where \( E[\exp(-\lambda T_{(j)})] = \exp(-\lambda^\alpha) \).

2) The law of \( \{a_t^{(i)}; i \leq N, t_1^{1/\alpha}\} \) is absolutely continuous w.r.t that of \( \{(A_t^{(i)}; i \leq N); L_t^{1/\alpha}\} \) with density \( \Gamma(1 - \alpha)L_1 \). This may be written as: \( \forall f \geq 0 \), Borel, \( \gamma_t \): \( [0, 1]^N \times R_+ \to R_+ \),
\[
E[f((a^{(i)}, i \leq N); t_1^{1/\alpha})] = E[f\left(\frac{p^{1/\alpha}_i T_{(i)}}{\sum_j p^{1/\alpha}_j T^{(j)}}; 1 \leq N; \frac{1}{\sum_j p^{1/\alpha}_j T_{(j)}}\right) \Gamma(1 + \alpha) \sum_j p^{1/\alpha}_j T^{(j)}]\].

In the next section, we shall see, in a different context, similar quantities—in particular, the ratios: \( T_{(i)}/\sum_j p^{1/\alpha}_j T^{(j)} \) appear.

3 An identity about multivariate positive generalized Linnik processes

In this section, we consider
\[
\mathcal{E}_\alpha^{(\nu)}(u) = (T_{(\nu)}^{(\alpha)}(u); 1 \leq i \leq N), u \geq 0,
\]
an \( R_+^N \)-valued process, whose components are independent \( \nu \)-Esscher transforms of stable(\( \alpha \))-subordinators. Let, furthermore, \( (\gamma_t, t \geq 0) \) denote a standard gamma process independent of \( \mathcal{E}_\alpha^{(\nu)} \).

The next theorem offers a description of the joint law of \( \{(\mathcal{E}_\alpha^{(\nu)}(\gamma_t), \gamma_t)\} \), in terms of a sequence of iid stable-(\( \alpha \)) variables, \( (T_{(\nu)}^{(\alpha)}, i \leq N) \), and an independent gamma \( (\alpha \sigma) \) variable.

Theorem 2
1) For any \( F: R_+^N \to R_+ \), and \( g: R_+ \to R_+ \), Borel, one has:
\[
E[F((\mathcal{E}_\alpha^{(\nu)}(\gamma_t))g(\gamma_t))] = \frac{\Gamma(1 + \alpha t)}{\Gamma(1 + t)} E\left[ F \left( \frac{\gamma_t}{\nu \cdot T_{(\alpha)}} \right) g\left( \frac{\gamma_t}{\nu \cdot T_{(\alpha)}}^\alpha \exp\left( \frac{\gamma_t}{\nu \cdot T_{(\alpha)}} \right)^{\alpha(\sigma - 1)} \right) \right]
\]
where \( \sigma = \sum_{i=1}^N \nu_i \), and \( \nu \cdot T_{(\alpha)} = \sum_{j=1}^N \nu_j T_{(\alpha)}^{(j)} \). In particular for \( \sigma = 1 \), the formula simplifies as:
\[
E[F((\mathcal{E}_\alpha^{(\nu)}(\gamma_t))g(\gamma_t))] = \frac{\Gamma(1 + \alpha t)}{\Gamma(1 + t)} E\left[ F \left( \frac{\gamma_t}{\nu \cdot T_{(\alpha)}} \right) g\left( \frac{\gamma_t}{\nu \cdot T_{(\alpha)}}^\alpha \right) \frac{1}{\nu \cdot T_{(\alpha)}^{\alpha}} \right].
\]

2) If \( \mathcal{L}(dx_1, \ldots, dx_N) \) denotes the Lévy measure of the \( N \)-dimensional Lévy process \( \{(\mathcal{E}_\alpha^{(\nu)}(\gamma_t), t \geq 0) \) then for any \( F: R_+^N \to R_+ \) continuous, with compact support in \( (0, \infty)^N \):
\[
\langle \mathcal{L}, F \rangle = \lim_{t \to 0} \frac{1}{t} E[F(\mathcal{E}_\alpha^{(\nu)}(\gamma_t))] = \alpha \int_0^\infty \frac{du}{u} e^{-u} E[F(u \cdot \nu \cdot T_{(\alpha)})].
\]
Remark 3. Note that the last formula follows from

$$\mathbb{E}[F(X_t)] = \mathbb{E}[\int_0^t ds L F(X_s)],$$

where \((X_t)\) is a Lévy process, starting at 0, \(F\) a regular function, equal to 0 on a neighborhood of 0, and \(L\) is the infinitesimal generator of \(X\). If \(X\) has no drift, and no Brownian component, then

$$LF(0) = (\mathcal{L}, F).$$

Corollary 1 In the case \(\sigma = 1\), the variables \(\nu \cdot \mathcal{C}(\nu)(\gamma_t)/\gamma_t^{1/\alpha}\) are independent and satisfy:

(i) \(\nu \cdot \mathcal{C}(\nu)(\gamma_t) \overset{d}{=} \gamma_t^{1/\alpha}\).

(ii)

$$\mathbb{E} \left[ H \left( \frac{1}{\gamma_t^{1/\alpha}} \mathcal{C}(\nu)(\gamma_t) \right) \right] = \frac{\Gamma(1 + \alpha t)}{\Gamma(1 + t)} \mathbb{E} \left[ H(T_{\alpha}) \frac{1}{(\nu \cdot T_{\alpha})^{\alpha}} \right],$$

for any positive measurable function \(H : \mathbb{R}_+^N \rightarrow \mathbb{R}_+\), or equivalently:

$$\frac{1}{\gamma_t^{1/\alpha}} \mathcal{C}(\nu)(\gamma_t) \overset{d}{=} \mathcal{C}(\nu^{1/\alpha})(1),$$

where on the RHS of (3.4), the law of the variable appearing there is the mixture, with respect to the law of \((\nu^{1/\alpha}_t)\) of the laws of \(\mathcal{C}(\nu)(1)\), i.e. the \(\mu\)-Esscher transform of \(\mathcal{C}(\nu)(1)\). \(\gamma_t\) is assumed to be independent of the process \((\mathcal{C}(\mu)(\gamma_t), \mu \geq 0)\).

Corollary 2 Let \(\mu_i \geq 0, \nu_i > 0; 1 \leq i \leq N\). The generalized positive Linnik process

$$\mathcal{C}(\mu, \nu) \equiv \sum_{i=1}^N \mu_i T_{\alpha}(\nu_i)(\gamma_t) \equiv \mu \cdot \mathcal{C}(\nu)(\gamma_t)$$

is a GGC subordinator with Lévy measure:

$$\alpha x^{-1} \mathbb{E} \left[ \exp \left( -x \frac{\nu \cdot T_{\alpha}}{(\mu \cdot T_{\alpha})} \right) \right] dx.$$
(then, changing variables \( x = y^\alpha \)),
\[
\frac{\alpha}{\Gamma(t)} \int_0^\infty dy y^{\alpha-1} (y^\alpha)^{t-1} e^{y^\alpha (\sigma-1)} E[F(y^\alpha, y^\alpha) \exp(-y(\nu \cdot T_\alpha))] \]

(then changing variables: \( y = z/(\nu \cdot T_\alpha) \)),
\[
\frac{\alpha \Gamma(\alpha t)}{\Gamma(t)} \frac{\Gamma(\alpha)}{\Gamma(t)} E[F(\gamma_{\alpha t}/\nu \cdot T_\alpha, (\gamma_{\alpha t}/\nu \cdot T_\alpha)^\alpha \exp((\gamma_{\alpha t}/\nu \cdot T_\alpha)^\alpha (\sigma - 1)))]
\]

which yields the desired result.

Remark 4 We also note that the preceding argument yields, but we do not give the details, the following 1-parameter extension of formula (3.1)
\[
E[F((\mathcal{C}(\nu) \gamma_{1/2}) g(\gamma_t)) = \Gamma(1 + \alpha t) \frac{\Gamma(1) \Gamma(1 + t)}{\Gamma(1 + t)} E\left[ m^t F\left( (\gamma_{\alpha t}/\nu \cdot T_\alpha) \frac{\gamma_{\alpha t}}{\nu \cdot T_\alpha} \right) g(m(\gamma_{\alpha t}/\nu \cdot T_\alpha)^\alpha \exp((\gamma_{\alpha t}/\nu \cdot T_\alpha)^\alpha (\sigma - m)) \right]
\]

3.1 Examples

We now illustrate Theorem 2 and Corollary 2 in the particular case \( N = 2 \) and \( \alpha = 1/2 \), for which the law of \( T_{1/2} \) is explicit (recall \( E[e^{-\lambda T_{1/2}}] = e^{-\sqrt{\lambda}} \)). We shall also assume \( \sqrt{\nu_1} + \sqrt{\nu_2} = 1 \).

Example 1. Set \( \nu_1 = 1, \nu_2 = 0, \mu_1 = \mu_2 = 1; E[\exp(-\lambda T_t)] = (\sqrt{1+\lambda} - \sqrt{\lambda})^t \).

Proposition 3.1 (i) The Lévy measure of the process \((T_t)\) is
\[
\frac{1}{2} \frac{du}{u} E[\exp(-u\beta_{1/2,1/2})]
\]

(ii) The distribution of the variable \( T_t \) is that of
\[
\frac{\gamma_{t/2}}{\beta_{1/2,1/2}}.
\]

Example 2. Set \( \nu_1 = \nu_2 = 1/4, \mu_1 = 1/4, \mu_2 = 0; E[\exp(-\lambda T_t)] = (2/(1 + \sqrt{1+\lambda}))^t \).

Proposition 3.2 (i) The Lévy measure of the process \((T_t)\) is
\[
\frac{1}{2} \frac{du}{u} E[\exp(-u/\beta_{1/2,1/2})]
\]

(ii) The distribution of the variable \( T_t \) is that of
\[
\frac{\gamma_{t/2}}{\beta_{1/2,1/2}}.
\]

Example 3 (This is the general case for \( \alpha = 1/2, N = 2 \): Set \( \sqrt{\nu_1} + \sqrt{\nu_2} = 1, \mu_1, \mu_2 \geq 0 \).)

Proposition 3.3 (i) The Lévy measure of the process \((T_t)\) is
\[
\frac{1}{2} \frac{du}{u} E[\exp(-u/(\mu_1 \mathcal{T}_{1/2} + \nu_2 \mathcal{T}_{1/2}))]
\]
(ii) The law of the variable $T_t$ satisfies: $\forall f \geq 0$

$$
E[f(T_t)] = C_1 \left[ f \left( \frac{\mu_1 T_t^{(1)} + \mu_2 T_t^{(2)}}{\nu_1 T_t^{(1)} + \nu_2 T_t^{(2)}} \right) \frac{1}{(\nu_1 T_t^{(1)} + \nu_2 T_t^{(2)})^{1/2}} \right]
$$

for some universal constant $C_1$, and $\gamma$, $\gamma'$ are independent gamma processes.

Rather than proving Propositions 3.1, 3.2, 3.3 in that order, we shall first prove Proposition 3.3, and then see that the results of Proposition 3.1 and 3.2 follow. In fact, we only need to prove the point (ii) in each of the Propositions.

**Proofs of points (ii):**

(a) From Theorem 2, we know:

$$
E[f(T_t)] = \frac{\Gamma(1+\frac{\alpha}{2})}{\Gamma(1+t)} \left[ f \left( \frac{\mu_1 T_t^{(1)} + \mu_2 T_t^{(2)}}{\nu_1 T_t^{(1)} + \nu_2 T_t^{(2)}} \right) \frac{1}{(\nu_1 T_t^{(1)} + \nu_2 T_t^{(2)})^{1/2}} \right].
$$

Since $T_t^{(i)} \overset{d}{=} c/\gamma_t/2$ for $i = 1, 2$, we get, with obvious notation:

$$
E[f(T_t)] = C_1 \left[ f \left( \frac{\mu_1 \hat{T}_t^{(1)} + \mu_2 \hat{T}_t^{(2)}}{\nu_1 \hat{T}_t^{(1)} + \nu_2 \hat{T}_t^{(2)}} \right) \left( \frac{\hat{\gamma}_1^{\prime} T_t^{(1)} + \nu_2 \hat{\gamma}_2^{\prime} T_t^{(2)}}{\nu_1 \hat{T}_t^{(1)} + \nu_2 \hat{T}_t^{(2)}} \right)^{1/2} \right],
$$

which simplifies to (3.6).

(b) In the case of Proposition 3.2, we have: $\nu_1 = \nu_2$, and then $\hat{\gamma}_1^{\prime}(1+t)/2 + \hat{\gamma}_1(1+t)/2$ is a beta$(1/2, (1+t)/2)$ random variable, independent from $(\hat{\gamma}_1^{\prime}(1+t)/2 + \hat{\gamma}_1(1+t)/2)$.

(c) In the case of Proposition 3.1, the formula (3.6) in point (ii), Proposition 3.3., simplifies as:

$$
E[f(T_t)] = C_1 \left[ f \left( \hat{\gamma}_1(1+t)/2 + \hat{\gamma}_1(1+t)/2 \right) \frac{1}{(\hat{\gamma}_1^{\prime}(1+t))^{1/2}} \right].
$$

It might also be of interest to discuss the case $N = 2$ and general $0 < \alpha < 1$ in the same vein, because the law of $(T_\alpha/T_\alpha')$ is simple, as shown by Lamperti (1958). Let us denote by $R_\alpha$ this ratio.

Then we have the following:

**Proposition 3.4** ($N = 2, 0 < \alpha < 1, \nu_1^2 + \nu_2^2 = 1$).

(i) The Lévy measure of the process is:

$$
\hat{\alpha} \frac{du}{u} \left[ \exp \left( -\frac{\nu_1 R_\alpha + \nu_2}{\mu_1 R_\alpha + \mu_2} \right) \right].
$$

(ii) The law of $T_t$ satisfies $\forall f \geq 0$,

$$
E[f(T_t)] = C_1 \left[ f \left( \frac{\nu_1 R_\alpha + \nu_2}{\nu_1 R_\alpha + \nu_2} \right) \frac{1}{(\nu_1 R_\alpha + \nu_2)^{\alpha t}} \frac{1}{(T_\alpha')^{\alpha t}} \right],
$$

where, on the RHS, $\gamma_t/2$ is independent from the pair $(T_\alpha, T_\alpha')$ and $R_\alpha = T_\alpha/T_\alpha'$. 
3.2 Looking for a simplification of Proposition 3.4

Using a conditional expectation argument we can replace \((T'_{\alpha})^{-\alpha t}\) appearing in (3.7) with

\[
h_{\alpha t}(R_{\alpha}) := \mathbb{E}[(T'_{\alpha})^{-\alpha t} | R_{\alpha}].
\]

Naturally this expression has utility only if it has a tractable form. Here we obtain such an expression by using Kanter’s (1975) explicit representation of the density of a positive stable random variable, in conjunction with Lamperti’s expression for the density of \(R_{\alpha}\). This form of the stable density is apparently not well-known. First let \(f_{\alpha}\) denote the density of a unilateral stable-(\(\alpha\)) density. Then generically the conditional density of \(T'_{\alpha}|R_{\alpha} = r\) is given by

\[
f_{T'_{\alpha}|R_{\alpha}}(s|r) = \frac{s f_{\alpha}(rs) f_{\alpha}(s)}{f_{R_{\alpha}}(r)}
\]

Now setting

\[
K_{\alpha}(u) = \left(\frac{\sin(\pi \alpha u)}{\sin(\pi u)}\right)^{-\frac{1}{\alpha}} \left(\frac{\sin((1-\alpha)\pi u)}{\sin(\pi \alpha u)}\right),
\]

it follows from Kanter (1975)[see also Devroye (1996)] that

\[
f_{\alpha}(s) = \frac{\alpha}{1-\alpha} s^{-1/(1-\alpha)} \int_0^1 e^{-s^{-\frac{\alpha}{1-\alpha}}} K_{\alpha}(u) K_{\alpha}(u) du.
\]

That is \(T'_{\alpha} \overset{d}{=} \left(K_{\alpha}(U)/e\right)^{(1-\alpha)/\alpha}\), where \(U\) is a Uniform\([0,1]\) random variable independent of \(e\), which is exponential\((1)\). Additionally from Lamperti (1958)[see, also Chaumont and Yor (2003, p. 116)] we obtain,

\[
f_{R_{\alpha}}(r) = \frac{\sin(\pi \alpha)}{\pi} r^{2\alpha - 1} + 2r^{\alpha} \cos(\pi \alpha) + 1 \quad \text{for } r > 0.
\]

Combining these points we arrive at the following result

**Proposition 3.5**  

(i) The conditional density of \(T'_{\alpha}|R_{\alpha} = r\) is expressible as

\[
\frac{\pi r^{-\alpha(2-\alpha)\frac{1}{1-\alpha}}}{\sin(\pi \alpha)} \left(\frac{\alpha}{1-\alpha}\right) s^{-\frac{2}{1-\alpha} + 1} \int_0^1 \int_0^1 e^{-s^{-\frac{\alpha}{1-\alpha}}} C_{\alpha}(u_1, u_2) r K_{\alpha}(u_1) K_{\alpha}(u_2) du_1 du_2,
\]

where \(C_{\alpha}(u_1, u_2) = [r^{-\frac{\alpha}{1-\alpha}} K_{\alpha}(u_1) + K_{\alpha}(u_2)]\)

(ii) It follows that

\[
h_{\alpha t}(R_{\alpha}) := \mathbb{E}[(T'_{\alpha})^{-\alpha t} | R_{\alpha}] = \frac{\pi(R_{\alpha})^{-\frac{\alpha(2-\alpha)}{1-\alpha}}}{\sin(\pi \alpha)} \left(\frac{\alpha \Gamma(1(1-\alpha) + 2)}{1-\alpha}\right) D_{\alpha,\alpha t}(R_{\alpha})
\]

where

\[
D_{\alpha,\alpha t}(R_{\alpha}) = \int_0^1 \int_0^1 [C_{\alpha}(u_1, u_2) R_{\alpha}]^{-\frac{1(1-\alpha) + 2}{1-\alpha}} K_{\alpha}(u_1) K_{\alpha}(u_2) du_1 du_2.
\]

This leads to an alternative characterization of \((T_{\alpha})\) in proposition 3.4 as follows;

**Proposition 3.6** The distribution of the variable \(T_{\alpha}\) in Proposition 3.4 with \(N = 2\), \(0 < \alpha < 1\), \(\nu_{1}^2 + \nu_{2}^2 = 1\), satisfies \(\forall f \geq 0\),

\[
\mathbb{E}[f(T_{\alpha})] = C_{\alpha} f\left(\frac{\mu_1 R_{\alpha} + \mu_2}{\nu_1 R_{\alpha} + \nu_2}\right) h_{\alpha t}(R_{\alpha}) (\nu_1 R_{\alpha} + \nu_2)^{\alpha}.
\]

where \(h_{\alpha t}(R_{\alpha})\) is given by (3.8).
4 A comparison of Theorems 1 and 2

This comparison is only partial, as here we shall use the hypothesis: \( \sigma = \sum_{i=1}^{N} \nu_i = 1 \), so that we can associate with the Esscher sequence \((\nu_i)_{i \leq N}\) the probabilities \((p_i)_{i \leq N}\) defined by \( p_i = \nu_i \). Then a simple cross-inspection of Theorems 1 and 2 shows that, taking \( t = 1 \),

\[
\{(\nu_i \xi^{(\nu_i)}_\alpha)(i; i \leq N), \xi \} \overset{d}{=} \{(\gamma_\alpha a^{(i)}_{1}; i \leq N), (\gamma_\alpha)^a f_1 \},
\]

where, on the LHS, \( \xi \) is an independent \( \exp(1) \) variable, and on the RHS, \( \gamma_\alpha \) is independent of the Spider’s Bridge. We shall now transform the identity in law (4.1), or rather its RHS, by using the two following facts:

(i) if \( g = \sup\{s < 1 : S_s = 0\} \), then \( g \overset{d}{=} \beta(\alpha, 1 - \alpha) \),

(ii) \( \gamma_\alpha = \beta(\alpha, 1 - \alpha) \) where \( \beta(\alpha, 1 - \alpha) \) denotes a beta variable with parameters \((\alpha, 1 - \alpha)\), which is independent from \( \xi \).

Thus the identity in law (4.1) may be rephrased as:

\[
\{(\nu_i \xi^{(\nu_i)}_\alpha)(i; i \leq N), \xi \} \overset{d}{=} \{(A^{(i)}_{\beta \xi}; i \leq N), L_\xi \},
\]

where \( g_t = \sup\{s < t : S_s = 0\} \), which is equivalent to:

\[
\tau_\ell = \inf\{t : L_t > \ell\}.
\]

In the next section, we give an explanation of (4.3).

5 An explanation of (4.3)

We shall now show that (4.3) follows from the conjunction of the next lemma and again the scaling property of a Bessel spider.

5.1 Lemma

**Lemma 5.1** Let \((X_t, t \geq 0)\) denote a nice Markov process taking values in a general space \( E \); let \( 0 \) belong to \( E \), such that \( 0 \) is regular for itself (with respect to \( X \)), and denote by \((L_t, t \geq 0)\) a choice of its local time at \( 0 \); let \((\tau_\ell, \ell \geq 0)\) denote the inverse local time, i.e.

\[
\tau_\ell = \inf\{t : L_t > \ell\}.
\]

It is a subordinator with associated Bernstein function, \( \psi(\theta) \):

\[
\mathbb{E}[e^{-\theta \tau_\ell}] = e^{-\ell \psi(\theta)}.
\]

Then, if \( e_\theta \) denotes an exponential \((\theta)\) random variable independent from \((X_t, t \geq 0)\), and \((A_t, t \geq 0)\) is a continuous additive functional, one has

(i) \( L_\theta \) is exponentially distributed with parameter \( \psi(\theta) \);

(ii) The conditional formula holds:

\[
\mathbb{E}[e^{-A_{\theta \xi}|L_\theta = \ell}] = \frac{E[e^{-A_{\tau_\ell} - \theta \tau_\ell}]}{E[e^{-\theta \tau_\ell}]}.
\]
This lemma is quite classical (it may be obtained from excursion theory), let us admit it for a moment and derive (4.3) from it. For this purpose, we take: \( A_t = \sum_{j=1}^{N} \lambda_j A_t^{(j)}, \lambda_j \geq 0 \), and we write (ii) in the lemma as follows:

\[
E[\exp(-\sum_{j} \lambda_j A^{(j)}_{L_{e_0}})|L_{e_0} = l] = E[\exp(-\sum_{j} (\lambda_j + \theta) A_t^{(j)})]\exp(l\theta^\alpha)
\]

(since: \( \tau_t = \sum_{j} A_t^{(j)} \))

\[
= \exp\left(-l \sum_{j} \nu_j ((\lambda_j + \theta)^\alpha - \theta^\alpha)\right)
\]

(since the Ito excursion measure \( \nu \) associated with the local time \( (L_t) \) satisfies \( \nu = \sum_{j} \nu_j \nu_j \), where \( \nu_j \) is \( \nu \) restricted to excursions on \( I_j \))

\[
= E \left[ \exp\left(-\sum_{j} \lambda_j \nu_j T_{\alpha}^{(\nu_j \theta)}(l)\right)\right]
\]

and it now suffices to take \( \theta = 1 \).

### 5.2 A global view of the lemma

In fact, the lemma is only a part of the more general statement.

**Proposition 5.1** (We use the same notation as in the Lemma). The following holds:

(i) The pre-\( e_0 \) process \( \{X_u, u \leq g_{e_0}\} \) and the post-\( e_0 \) process \( \{X_{g_{e_0} + v}, v \leq e_0 - g_{e_0}\} \) where \( d_t = \inf\{s > t : X_s = 0\} \), are independent;

(ii) The law of \( \{X_u, u \leq g_{e_0}\} \) may be described as follows:

(a) \( L_{e_0} \equiv L_{g_{e_0}} \) is \( \exp(\psi(\theta)) \) distributed;

(b) \( \{(X_u, u \leq g_{e_0})|L_{e_0} = l\} \) is distributed as: \( \{X_u, u \leq \tau_\ell\} \) under the probability \( \exp(-\theta \tau_\ell + l\psi(\theta)) \cdot P \)

(iii) The law of the process \( \{X_{g_{e_0} + v}, v \leq e_0 - g_{e_0}\} \) is

\[
\frac{1}{\psi(\theta)}(1 - \exp(-\theta V(\varepsilon)))\nu(d\varepsilon)
\]

where \( V(\varepsilon) \) denotes the lifetime of the generic excursion \( \varepsilon \).

For details of the proof and further references, the reader may consult Salminen, Vallois and Yor (2006).

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