Some Properties of the Arithmetic–Geometric Index

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Abstract: Recently, the arithmetic–geometric index (AG) was introduced, inspired by the well-known and studied geometric–arithmetic index (GA). In this work, we obtain new bounds on the arithmetic–geometric index, improving upon some already known bounds. In particular, we show families of graphs where such bounds are attained.

Keywords: arithmetic–geometric index; topological index; chemical graph theory

1. Introduction

In mathematical chemistry, a topological descriptor is a function that associates each molecular graph with a real value, and if it correlates well with some chemical property, it is called a topological index. Since Wiener’s work (see [1]), numerous topological indices have been defined and discussed, since the growing interest in their study is due to their several applications in chemistry, for example in QSPR/QSAR research (see [2–4]). For more information on other important applications of topological indices to specific problems in physics, computer science and environment science (see [5–7]). In particular, among the topological descriptors, the most studied from the mathematical point of view due to their practical scope are the so-called vertex-degree-based topological indices. Probably the most studied, with more than 500 papers, is the Randić index defined as

$$R(H) = \sum_{ij \in E(H)} \frac{1}{\sqrt{d_i d_j}},$$

where $ij$ denotes the edge of the graph $H$ and $d_i$ is the degree of the vertex $i$.

In [8,9], the variable Zagreb indices are defined as

$$M_1^\alpha(H) = \sum_{i \in V(H)} d_i^\alpha, \quad M_2^\alpha(H) = \sum_{ij \in E(H)} (d_i d_j)^\alpha,$$

with $\alpha \in \mathbb{R}$.

Note that for $\alpha = 2$, $\alpha = -1$, $\alpha = 3$, the index $M_1^\alpha$ is the first Zagreb index $M_1$, the inverse index $ID$, the forgotten index $F$, respectively; also for $\alpha = 1$, $\alpha = -1/2$, $\alpha = -1$, the index $M_2^\alpha$ is the second Zagreb index $M_2$, the Randić index $R$, the modified Zagreb index.

The general sum-connectivity index was defined in [10] as

$$\chi_\alpha(H) = \sum_{ij \in E(H)} (d_i + d_j)^\alpha.$$

Note that $\chi_{-1/2}$ is the sum-connectivity index, $2\chi_{-1}$ is the harmonic index $Har$, etc. The max–min rodeg index and min–max rodeg index were defined in [11] respectively as
was introduced in [22] as power, we used a datum for entropy (S) of octane isomers, and the results are compared

AG topological indices have shown better correlation with some physico–chemical properties from both theoretical and practical points of view are different. In some cases, the reciprocal indices are mathematically represented by an inverse relationship, their scope and results

GA reciprocal of the well-studied geometric–arithmetic index MG mm index reciprocal Randić index M1 Zagreb index example, the first Zagreb index AG and a discussion on the effect of deleting an edge from a graph on the arithmetic–geometric ties involving GA

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AG index of some kinds of trees was discussed in the papers [22,23]. Moreover, the AG index of graphene, which is the most conductive and effective material for electromagnetic interference shielding, was computed in [24]. The paper [25] studied the spectrum and energy of arithmetic–geometric matrix, in which the sum of all elements is equal to 2AG. Other bounds of the arithmetic–geometric energy of graphs appeared in [26,27]. The paper [28] studies optimal AG-graphs for several classes graphs, and it includes inequalities involving \( GA + AG \) and \( GA \cdot AG \). In [29–32], there are more bounds on the AG index and a discussion on the effect of deleting an edge from a graph on the arithmetic–geometric index. Motivated by these papers, we obtain new bounds of the AG index, improving upon some already known bounds. Furthermore, we show families of graphs where such bounds are attained. Some of these families are regular graphs, and we recall that some regular graphs play an important role in mathematical chemistry; for instance, Isaac graphs are well-known regular graphs that are isomorphic to hydrogen-suppressed molecular graphs [33].

Given a topological index \( I(H) = \sum_{ij \in E(H)} f(d_i, d_j) \), we can consider the reciprocal topological index defined as \( J(H) = \sum_{ij \in E(H)} 1/f(d_i, d_j) \). It is essential to point out that several important topological indices are associated with the above relationships. For example, the first Zagreb index \( M_1 \) and the first modified Zagreb index \( \mu M_1 \), the second Zagreb index \( M_2 \) and the second modified Zagreb index \( \mu M_2 \), the Randić index \( R \) and the reciprocal Randić index \( M_1^{1/2} \), the max–min rodeg index \( MM_{\text{MM}} \) and the min–max rodeg index \( mM_{\text{MM}} \), etc.

Inspired by these ideas, the arithmetic–geometric index \( AG \) was defined, which is the reciprocal of the well-studied geometric–arithmetic index \( GA \). Although these topological indices are mathematically represented by an inverse relationship, their scope and results from both theoretical and practical points of view are different. In some cases, the reciprocal topological indices have shown better correlation with some physico–chemical properties than their related indices. In the case of the AG index, in order to investigate its predictive power, we used a datum for entropy (S) of octane isomers, and the results are compared
with those obtained for the GA index, (see Figure 1). The correlation coefficient obtained for the AG index is $r_{AG} = -0.927$, while for the GA index, it is $r_{GA} = 0.912$, so the AG index, in this case, shows better predictive power than the GA index. However, when we used a datum for the boiling point of octane isomers, it turned out that the GA index showed better predictive power than the AG index. After this paper was accepted, Ref. [34] showed that both indices have the same predictive power for many kinds of graphs.

Figure 1. Graphs showing correlation between S and AG, S and GA respectively.

The arithmetic–geometric index was proposed recently and few important papers have been published on the subject. In this paper, we find several new mathematical properties (that cannot be obtained from the GA index), especially bounds that improve those already known.

Throughout this work, $H = (V(H), E(H))$ denotes a finite simple graph with at least an edge in each connected component of $H$. We denote by $m, n, \delta, \Delta$ the cardinality of the set of edges $E(H)$ and vertices $V(H)$, and the minimum and maximum degree of $H$, respectively.

2. Relationships between AG and Other Important Topological Indices

One can check that the following lemma holds:

Lemma 1. Let $f$ be the function $f(x, y) = \frac{x + y}{\sqrt[2]{xy}}$ defined on the rectangle $[a, b] \times [a, b]$ with $a > 0$. Then:

$$1 \leq f(x, y) \leq \frac{a + b}{2 \sqrt{ab}}.$$ 

The following inequalities for graphs $H$, follow from Lemma 1:

$$m \leq AG(G) \leq \frac{\Delta + \delta}{2 \sqrt{\Delta \delta}} m.$$  \hspace{1cm} (1)

The lower bound in (1) also follows from the inequalities $GA(H) \cdot AG(H) \geq m^2$ and $GA(H) \leq m$, see [15,16]. The upper bound in (1) appears in [31].

The following result shows the relationship between the AG index and the Randić index that correlates well with several physico–chemical properties. For this reason, it is one of the most studied indices, with innumerable applications in chemistry and pharmacology.

Theorem 1. If $H$ is a graph with $m$ edges, minimum degree $\delta$ and maximum degree $\Delta$, then:

$$AG(H) \leq m + \frac{(\sqrt{\Delta} - \sqrt{\delta})^2}{2} R(H).$$

The equality in the bound is attained if and only if $H$ is regular or biregular.
Proof. Note that:

\[
\frac{d_i + d_j}{\sqrt{d_i d_j}} = 1 + \frac{1}{2} \left( \frac{\sqrt{d_i} - \sqrt{d_j}}{\sqrt{d_i d_j}} \right)^2,
\]

\[
AG(H) = m + \sum_{ij \in E(H)} \frac{1}{2 \sqrt{d_i d_j}} \left( \frac{\sqrt{d_i} - \sqrt{d_j}}{\sqrt{d_i d_j}} \right)^2.
\]

Since:

\[
\sum_{ij \in E(H)} \frac{1}{2 \sqrt{d_i d_j}} \left( \frac{\sqrt{d_i} - \sqrt{d_j}}{\sqrt{d_i d_j}} \right)^2 \leq \frac{1}{2 \sqrt{d_i d_j}} \left( \sqrt{\Delta} - \sqrt{\delta} \right)^2,
\]

we have:

\[
AG(H) \leq m + \frac{\left( \sqrt{\Delta} - \sqrt{\delta} \right)^2}{2} R(H).
\]

The bound is tight if and only if:

\[
\left( \sqrt{d_i} - \sqrt{d_j} \right)^2 = \left( \sqrt{\Delta} - \sqrt{\delta} \right)^2
\]

for every \( ij \in E(H) \), and this happens if and only if \( d_i = \Delta \) and \( d_j = \delta \), or vice versa, for every \( ij \in E(H) \), so \( H \) is regular if \( \Delta = \delta \) or is otherwise biregular. \( \square \)

The following theorem shows a relationship between the index \( AG \) and the index \( M_{2}^{-a} \), the second variable Zagreb index.

**Theorem 2.** If \( H \) is a graph with minimum degree \( \delta \) and maximum degree \( \Delta \), and \( a \in \mathbb{R} \), then:

\[
AG(H) \leq K_a M_{2}^{-a}(H),
\]

with:

\[
K_a := \begin{cases} 
\delta^2 a, & \text{if } a \leq -1/2, \\
\max \left\{ \delta^2 a, \frac{1}{2} (\delta + \Delta) (\Delta a^{-1/2}) \right\}, & \text{if } -1/2 < a \leq 0, \\
\max \left\{ \Delta^2 a, \frac{1}{2} (\delta + \Delta) (\Delta a^{-1/2}) \right\}, & \text{if } 0 < a < 1/2, \\
\Delta^2 a, & \text{if } a \geq 1/2. 
\end{cases}
\]

The equality in the bound is attained for some fixed \( a \notin (-1/2, 1/2) \) if and only if \( H \) is a regular graph.

Proof. Let us optimize the function \( g : [\delta, \Delta] \times [\delta, \Delta] \to (0, \infty) \) defined as

\[
g(x, y) = \frac{x+y}{(xy)^{a^{-1}}} = \frac{1}{2} (xy)^{a^{-1/2}} (x + y) = \frac{1}{2} \, x^{a+1/2} y^{a-1/2} + \frac{1}{2} \, x^{a-1/2} y^{a+1/2}.
\]

If \( a \geq 1/2 \), then \( a + 1/2 > a - 1/2 \geq 0 \) and \( g \) strictly increases in each variable. Thus:

\[
g(x, y) \leq g(\Delta, \Delta) = \Delta^{2a}
\]

and the bound is tight if and only if \( x = y = \Delta \). Therefore:

\[
AG(H) \leq \Delta^{2a} M_{2}^{-a}(H).
\]
Let us now consider the case $-1/2 \leq a < 1/2$. Since $g$ is a symmetric function, we can also assume that $x \leq y$. We have:

$$\frac{\partial g}{\partial x}(x, y) = \frac{1}{2} (1/2 + a)x^{a-1/2}y^{a-1/2} + \frac{1}{2} (a - 1/2)x^{a-3/2}y^{a+1/2} = \frac{1}{2} x^{a-3/2}y^{a-1/2}((1/2 + a)x + (a - 1/2)y),$$

and thus, $\frac{\partial g}{\partial y}(x, y) = \frac{1}{2} y^{a-3/2}x^{a-1/2}((1/2 + a)y + (a - 1/2)x)$.

Assume first that $0 < a < 1/2$. Thus, $a + 1/2 > 0$ and:

$$(1/2 + a)y + (a - 1/2)x \geq (1/2 + a)x + (a - 1/2)x = 2ax > 0$$

and thus, $\partial g / \partial y > 0$. Therefore, the maximum value of $g$ is attained on $\{\delta \leq x \leq \Delta, y = \Delta\}$. Since:

$$\frac{\partial g}{\partial x}(\Delta, \Delta) = \frac{1}{2} \Delta^{2a-2}((a + 1/2)\Delta + (a - 1/2)\Delta) = a\Delta^{2a-1} > 0,$$

and $\partial g / \partial x(\delta, \Delta) = 0$ at most once when $x \in [\delta, \Delta]$, we have:

$$\max_{x, y \in [\delta, \Delta]} g(x, y) = \max_{x \in [\delta, \Delta]} g(x, \Delta) = \max \{g(\delta, \Delta), g(\Delta, \Delta)\} = \max \left\{\frac{1}{2} (\Delta\delta)^{a-1/2}(\Delta + \delta), \Delta^{2a}\right\}.$$

Assume now that $-1/2 < a \leq 0$. We have $a + 1/2 > 0$ and:

$$(1/2 + a)x + (a - 1/2)y \leq (1/2 + a)x + (a - 1/2)y = 2ay \leq 0$$

and thus, $\partial g / \partial x \leq 0$. Therefore, the maximum value of $g$ is attained on $\{x = \delta, \delta \leq y \leq \Delta\}$. Since:

$$\frac{\partial g}{\partial y}(\Delta, \Delta) = \frac{1}{2} \Delta^{2a-2}((a + 1/2)\Delta + (a - 1/2)\Delta) = a\Delta^{2a-1} > 0,$$

and $\partial g / \partial y(\delta, \Delta) = 0$ at most once when $y \in [\delta, \Delta]$, we have:

$$\max_{x, y \in [\delta, \Delta]} g(x, y) = \max_{y \in [\delta, \Delta]} g(\delta, y) = \max \{g(\delta, \delta), g(\delta, \Delta)\} = \max \left\{\frac{1}{2} (\Delta\delta)^{a-1/2}(\Delta + \delta), \delta^{2a}\right\}.$$

Finally, assume that $a \leq -1/2$. Hence, $a - 1/2 < a + 1/2 \leq 0$ and $g$ strictly decreases in each variable. Thus:

$$g(x, y) \leq g(\delta, \delta) = \delta^{2a}$$

and the bound is tight if and only if $x = y = \delta$. Therefore:

$$AG(H) \leq \delta^{2a}M_2^{-a}(H).$$

The properties of the function $g$ give that the bound is tight for some fixed $a \geq 1/2$ (respectively, $a \leq -1/2$) if and only if $d_i = d_j = \Delta$ (respectively, $d_i = d_j = \delta$) for every $ij \in E(H)$, and this happens if and only if $H$ is a regular graph. \(\square\)

**Remark 1.** The proof of Theorem 2 allows us to obtain that:

$$C_\delta M_2^{-a}(H) \leq AG(H),$$
Theorem 3. If $H$ is a graph with minimum degree $\delta$, then:

\[ AG(H) \geq m = \sum_{ij \in E(H)} \frac{(d_i \delta)}{(d_i)^{1/2}} \geq \delta \sum_{ij \in E(H)} \frac{1}{(d_i)^{1/2}} = C_\delta M_2^{-1}(H). \]

Theorem 2 has the following result for the Randić, reciprocal Randić and modified Zagreb indices.

Corollary 1. If $H$ is a graph with a maximum degree $\Delta$ and minimum degree $\delta$, then:

\[ AG(H) \leq \delta^{-2} M_2(H), \]
\[ AG(H) \leq \Delta R(H), \]
\[ AG(H) \leq \delta^{-1/2} M_2^{1/2}(H), \]
\[ AG(H) \leq \Delta^{1/2} M_2^{1/2}(H). \]

The following result shows a relationship between the $AG(H)$ index and the $\chi_b(H)$ index, which for different values of $b$ generalizes the indices $M_1$, $Har$, $\chi$ ($b = 1$, $b = -1$, $b = -1/2$, respectively).

Theorem 3. If $H$ is a graph with minimum degree $\delta$ and maximum degree $\Delta$, and $b \in \mathbb{R}$, then:

\[ AG(H) \leq B_b \chi_b(H), \]

with:

\[ B_b := \begin{cases} \max \left\{ \frac{1}{2} (\Delta \delta)^{-1/2} (\Delta + \delta)^{1-b}, (2\Delta)^{-b} \right\}, & \text{if } b < 0, \\ \max \left\{ \frac{1}{2} (\Delta \delta)^{-1/2} (\Delta + \delta)^{1-b}, (2\delta)^{-b} \right\}, & \text{if } 0 \leq b < 1/2, \\ (2\delta)^{-b}, & \text{if } b \geq 1/2. \end{cases} \]

The equality in the bound is attained for some fixed $b \geq 1/2$ if and only if $H$ is a regular graph.

Proof. For each $b < 1/2$, let us define:

\[ a = \frac{b}{2b - 2} \in \left( -\frac{1}{2}, \frac{1}{2} \right). \]

Let us consider the function: $g : [\delta, \Delta] \times [\delta, \Delta] \to (0, \infty)$ defined as

\[ g(x, y) = \frac{1}{2} (xy)^{a-1/2}(x + y) = \frac{1}{2} x^{a+1/2} y^{-1/2} + \frac{1}{2} x^{a-1/2} y^{a+1/2}. \]

Since $g$ is a symmetric function, we can assume $x \leq y$. We have:

\[ \frac{\partial g}{\partial x}(x, y) = \frac{1}{2} (1/2 + a) x^{a-1/2} y^{a-1/2} + \frac{1}{2} (a - 1/2) x^{a-3/2} y^{a+1/2} \]
\[ = \frac{1}{2} x^{a-3/2} y^{a-1/2} ((1/2 + a)x + (a - 1/2)y), \]
\[ \frac{\partial g}{\partial y}(x, y) = \frac{1}{2} y^{a-3/2} x^{a-1/2} ((1/2 + a)y + (a - 1/2)x). \]

Assume first that $0 < a < 1/2$. Thus, $1/2 + a > 0$ and:

\[ (1/2 + a)y + (a - 1/2)x \geq (1/2 + a)x + (a - 1/2)x = 2ax > 0 \]
and thus, \( \partial g / \partial y > 0 \). Therefore, the maximum value of \( g \) is attained on \( \{ \delta \leq x \leq \Delta, y = \Delta \} \).

Since:
\[
\frac{\partial g}{\partial x} (\Delta, \Delta) = \frac{1}{2} \Delta^{2a-2} ((a + 1/2)\Delta + (a - 1/2)\Delta) = a\Delta^{2a-1} > 0,
\]
and \( \partial g / \partial x (x, \Delta) = 0 \) at most once when \( x \in [\delta, \Delta] \), we have:
\[
\max_{x,y \in [\delta, \Delta]} g(x, y) = \max_{x \in [\delta, \Delta]} g(x, \Delta) = \max (g(\delta, \Delta), g(\Delta, \Delta))\]
\[
= \max \left\{ \frac{1}{2} (\Delta \delta)^{a-1/2}(\Delta + \delta), \Delta^{2a} \right\}.
\]

Assume now that \(-1/2 < a \leq 0 \). We have \( a + 1/2 > 0 \) and:
\[
(1/2 + a)x + (a - 1/2)y \leq (1/2 + a)y + (a - 1/2)y = 2ay \leq 0
\]
and thus, \( \partial g / \partial x \leq 0 \). Therefore, the maximum value of \( g \) is attained on \( \{ x = \delta, \delta \leq y \leq \Delta \} \).

Since:
\[
\frac{\partial g}{\partial y} (\Delta, \Delta) = \frac{1}{2} \Delta^{2a-2} ((a + 1/2)\Delta + (a - 1/2)\Delta) = a\Delta^{2a-1} > 0,
\]
and \( \partial g / \partial y (\delta, y) = 0 \) at most once when \( y \in [\delta, \Delta] \), we have:
\[
\max_{x,y \in [\delta, \Delta]} g(x, y) = \max_{y \in [\delta, \Delta]} g(\delta, y) = \max (g(\delta, \delta), g(\delta, \Delta))\]
\[
= \max \left\{ \frac{1}{2} (\Delta \delta)^{a-1/2}(\Delta + \delta), \delta^{2a} \right\}.
\]

Define:
\[
C_a := \begin{cases} 
\max \left\{ \delta^{2a}, \frac{1}{2}(\delta + \Delta)(\delta \Delta)^{a-1/2} \right\}, & \text{if } -1/2 < a \leq 0, \\
\max \left\{ \Delta^{2a}, \frac{1}{2}(\delta + \Delta)(\delta \Delta)^{a-1/2} \right\}, & \text{if } 0 < a < 1/2,
\end{cases}
\]
we have:
\[
(xy)^{a-1/2} (x + y) \leq 2C_a,
\]
\[
(xy)^{1/(2b-2)} (x + y) \leq 2C_a.
\]

Since \( b < 1/2 \), we have \( 1 - b > 0 \) and:
\[
\frac{1}{2} (xy)^{-1/2} (x + y)^{1-b} \leq \frac{1}{2} (2C_a)^{1-b}.
\]

If \( 0 \leq b < 1/2 \), then \(-1/2 < a \leq 0 \) and:
\[
\frac{1}{2} (2C_a)^{1-b} = \frac{1}{2} \left( 2 \max \left\{ \frac{1}{2} (\Delta \delta)^{a-1/2}(\Delta + \delta), \delta^{2a} \right\} \right)^{1-b}
\]
\[
= \frac{1}{2} \left( 2 \max \left\{ \frac{1}{2} (\Delta \delta)^{1/(2b-2)}(\Delta + \delta), \delta^{b/(b-1)} \right\} \right)^{1-b}
\]
\[
= \max \left\{ \frac{1}{2} (\Delta \delta)^{-1/2}(\Delta + \delta)^{1-b}, (2\delta)^{-b} \right\} = B_b.
\]

If \( b < 0 \), then \( 0 < a < 1/2 \) and:
\[
\frac{1}{2} (2C_a)^{1-b} = \frac{1}{2} \left( 2 \max \left\{ \frac{1}{2} (\Delta \delta)^{a-1/2}(\Delta + \delta), \Delta^{2a} \right\} \right)^{1-b}
\]
\[
= \frac{1}{2} \left( 2 \max \left\{ \frac{1}{2} (\Delta \delta)^{1/(2b-2)}(\Delta + \delta), \Delta^{b/(b-1)} \right\} \right)^{1-b}
\]
\[
= \max \left\{ \frac{1}{2} (\Delta \delta)^{-1/2}(\Delta + \delta)^{1-b}, (2\Delta)^{-b} \right\} = B_b.
\]
If \( b \geq 1/2 \), then the function \( A : [\delta, \Delta] \times [\delta, \Delta] \to \mathbb{R} \) defined as
\[
A(x, y) = 2\sqrt{xy} (x + y)^{b-1}
\]
satisfies:
\[
\frac{\partial A}{\partial x}(x, y) = x^{-1/2}y^{1/2}(x + y)^{b-1} + 2x^{1/2}y^{1/2}(b - 1)(x + y)^{b-2}
\]
\[
= x^{-1/2}y^{1/2}(x + y)^{b-2}(x + y + (2b - 2)x)^{b-2}
\]
\[
= x^{-1/2}y^{1/2}(x + y)^{b-2}((2b - 1)x + y)^{b-2}
\]
\[
\geq x^{-1/2}y^{1/2-(3/2)}(x + y)^{b-2} > 0,
\]
\[
\frac{\partial A}{\partial y}(x, y) = y^{-1/2}x^{1/2}(x + y)^{b-2}((2b - 1)y + x)^{b-2}
\]
\[
\geq y^{-1/2}x^{b-3/2}(x + y)^{b-2} > 0.
\]
Thus, \( A \) is a strictly increasing function in each variable and thus:
\[
2\sqrt{xy} (x + y)^{b-1} = A(x, y) \geq A(\delta, \delta) = (2\delta)^b,
\]
with equality if and only if \( x = y = \delta \). Hence:
\[
(2\delta)^b \frac{x + y}{2\sqrt{xy}} \leq (x + y)^b \quad \forall x, y \in [\delta, \Delta],
\]
\[
\frac{d_i + d_j}{2\sqrt{d_i d_j}} \leq (2\delta)^{-b}(d_i + d_j)^b \quad \forall ij \in E(H),
\]
\[
AG(H) \leq B_b \chi_s(H),
\]
and the equality in this last inequality is attained if and only if \( d_i = d_j = \delta \) for every \( ij \in E(H) \), i.e., \( H \) is a regular graph. \( \square \)

**Remark 2.** The proof of Theorem 3 allows us to obtain that:
\[
A_b \chi_s(H) \leq AG(H),
\]
with:
\[
A_b := \begin{cases} 
(2\delta)^{-b}, & \text{if } b < 0, \\
(2\Delta)^{-b}, & \text{if } b \geq 0.
\end{cases}
\]

However, this inequality is direct, since:
\[
AG(H) \geq m = \sum_{ij \in E(H)} (d_i + d_j)^b \geq A_b \sum_{ij \in E(H)} (d_i + d_j)^b = A_b \chi_s(H).
\]

Theorem 3 has the following consequence for the first Zagreb, harmonic and sum-connectivity indices.

**Corollary 2.** Let \( H \) be a graph with minimum degree \( \delta \) and maximum degree \( \Delta \). Then:
\[
AG(H) \leq \frac{1}{2\delta} M_1(H),
\]
\[
AG(H) \leq \frac{1}{2} \max \left\{ \frac{1}{2} (\Delta \delta)^{-1/2}(\Delta + \delta)^2, 2\Delta \right\} Har(H),
\]
\[
AG(H) \leq \max \left\{ \frac{1}{2} (\Delta \delta)^{-1/2}(\Delta + \delta)^{3/2}, (2\Delta)^{1/2} \right\} \chi(H).
\]

The following result relates \( AG \) and \( SDD \) indices.
Theorem 4. Let $H$ be a graph with $m$ edges, minimum degree $\delta$ and maximum degree $\Delta$. Then:

\[
\sqrt{\frac{\Delta \delta}{\Delta + \delta}} \left( \frac{1}{2} SDD(H) + m \right) \leq AG(H) \leq \frac{1}{4} SDD(H) + \frac{1}{2} m.
\]

The equality in the lower bound is attained if $H$ is a regular or biregular graph. The equality in the upper bound is attained if and only if each connected component of $H$ is a regular graph.

Proof. Lemma 1 gives:

\[
\frac{\Delta + \delta}{\sqrt{\Delta \delta}} AG(H) = \sum_{ij \in E(H)} \frac{\Delta + \delta}{\sqrt{\Delta \delta}} \frac{d_i + d_j}{2 \sqrt{d_i d_j}} \geq \frac{1}{2} \sum_{ij \in E(H)} \left( \frac{d_i + d_j}{\sqrt{d_i d_j}} \right)^2
\]

\[
= \frac{1}{2} \sum_{ij \in E(H)} \left( \sqrt{\frac{d_i}{d_j}} + \sqrt{\frac{d_j}{d_i}} \right)^2 = \frac{1}{2} \sum_{ij \in E(H)} \left( \frac{d_i}{d_j} + \frac{d_j}{d_i} \right) + \frac{1}{2} \sum_{ij \in E(H)} \frac{2}{1} = \frac{1}{2} SDD(H) + m.
\]

If $H$ is a regular or biregular graph, then:

\[
\frac{\sqrt{\Delta \delta}}{\Delta + \delta} \left( \frac{1}{2} SDD(H) + m \right) = \frac{1}{2} \frac{\sqrt{\Delta \delta}}{\Delta + \delta} \left( \left( \frac{\Delta}{\Delta + \delta} \right) m + 2m \right)
\]

\[
= \frac{1}{2} \frac{\sqrt{\Delta \delta}}{\Delta + \delta} (\Delta^2 + \delta^2 + 2\Delta \delta) m = \frac{\Delta + \delta}{2 \sqrt{\Delta \delta}} m = AG(H).
\]

Lemma 1 gives:

\[
AG(H) = \sum_{ij \in E(H)} \frac{d_i + d_j}{2 \sqrt{d_i d_j}} \leq \frac{1}{4} \sum_{ij \in E(H)} \left( \frac{d_i + d_j}{\sqrt{d_i d_j}} \right)^2
\]

\[
= \frac{1}{4} \sum_{ij \in E(H)} \left( \frac{d_i}{d_j} + \frac{d_j}{d_i} \right)^2 = \frac{1}{4} \sum_{ij \in E(H)} \left( \frac{d_i}{d_j} + \frac{d_j}{d_i} \right) + \frac{1}{4} \sum_{ij \in E(H)} \frac{2}{1} = \frac{1}{4} SDD(H) + \frac{1}{2} m.
\]

If the equality in this bound is attained, then Lemma 1 gives $d_i = d_j$ for every $ij \in E(H)$ and so, each connected component of $H$ is a regular graph.

If each connected component of $H$ is a regular graph, then:

\[
\frac{1}{4} SDD(H) + \frac{1}{2} m = \frac{1}{4} (2m + 2m) = m = AG(H).
\]

It is easy to check that $SDD(H) \geq 2m$ and thus, Theorem 4 has the following consequence.

Corollary 3. Let $H$ be a graph. Then:

\[
AG(H) \leq \frac{1}{2} SDD(H).
\]

The inequality in Corollary 3 appears in [30, Theorem 10] for connected graphs. (Note that the definition of $SDD$ in [30] is slightly different.) Our argument gives it for general graphs, and Theorem 4 improves this inequality.
We present here elementary relations between $AG$, $M_{sde}$ and $M_{sde}$ indices.

**Proposition 1.** If $H$ is a graph, then:

$$AG(H) = \frac{1}{2} M_{sde}(H) + \frac{1}{2} m M_{sde}(H), \quad AG(H) \leq M_{sde}(H).$$

The equality in the bound is attained if and only if each connected component of $H$ is a regular graph.

**Proof.** We have:

$$AG(H) = \sum_{ij \in E(H)} \frac{d_i + d_j}{2 \sqrt{d_i d_j}} = \frac{1}{2} \sum_{ij \in E(H)} \left( \sqrt{\frac{d_i}{d_j}} + \sqrt{\frac{d_j}{d_i}} \right) = \frac{1}{2} \sum_{ij \in E(H)} \left( \frac{\max\{d_i, d_j\}}{\min\{d_i, d_j\}} + \frac{\min\{d_i, d_j\}}{\max\{d_i, d_j\}} \right) = \frac{1}{2} M_{sde}(H) + \frac{1}{2} m M_{sde}(H).$$

In addition:

$$AG(H) = \sum_{ij \in E(H)} \frac{1}{2} \left( \sqrt{\frac{d_i}{d_j}} + \sqrt{\frac{d_j}{d_i}} \right) \leq \sum_{ij \in E(H)} \frac{\max\{d_i, d_j\}}{\min\{d_i, d_j\}} = M_{sde}(H).$$

The bound is tight if and only if:

$$\sqrt{\frac{d_i}{d_j}} = \sqrt{\frac{d_j}{d_i}} = \frac{\max\{d_i, d_j\}}{\min\{d_i, d_j\}}$$

for every $ij \in E(H)$, i.e., $d_i = d_j$ for every $ij \in E(H)$, and this happens if and only if each connected component of $H$ is a regular graph. \hfill \Box

### 3. A General Bound of the $AG$ Index

In this section, we find and show optimal inequalities, which do not involve other topological indices, for the topological index $AG$ as a function of graph invariants such as the number of edges and the minimum and maximum degree.

We will need the following definitions. Given a graph $H$ with maximum degree $\Delta$ and minimum degree $\delta < \Delta - 1$, we denote by $a_0, a_1, a_2$, the cardinality of the subsets of edges

- $A_0 = \{ij \in E(H) : d_i = \delta, d_j = \Delta\}$,
- $A_1 = \{ij \in E(H) : d_i = \delta, \delta < d_i < \Delta\}$,
- $A_2 = \{ij \in E(H) : d_i = \Delta, \delta < d_j < \Delta\}$,

respectively.

**Theorem 5.** Let $H$ be a graph with maximum degree $\Delta$, minimum degree $\delta < \Delta - 1$ and $m$ edges. Then:

$$AG(H) \leq \frac{\Delta + \delta}{2\sqrt{\Delta \delta}} m - a_1 \left( \frac{\Delta + \delta}{2\sqrt{\Delta \delta}} - \frac{\Delta + \delta - 1}{2\sqrt{\Delta - 1} \delta} \right) - a_2 \left( \frac{\Delta + \delta}{2\sqrt{\Delta \delta}} - \frac{\Delta + \delta + 1}{2\sqrt{\Delta (\delta + 1)}} \right) + m \left( \frac{\delta + \Delta}{2\sqrt{\delta \Delta}} - 1 \right),$$

$$AG(H) \geq m + a_0 \left( \frac{\delta + \Delta}{2\sqrt{\delta \Delta}} - 1 \right) + a_1 \left( \frac{1 + 2\delta}{2\sqrt{\delta + 1} \delta} - 1 \right) + a_2 \left( \frac{2\Delta - 1}{2\sqrt{\Delta - 1} \Delta} - 1 \right).$$
Proof. Let us consider the function \( g(t) = \frac{t + \sqrt{t^2 - 4}}{2} \) on the interval \((0, \infty)\). We have \( g'(t) = \frac{\sqrt{t^2 - 4}}{2t} \), therefore \( g'(t) < 0 \) for \( t \in (0, 1) \) and \( g'(t) > 0 \) for \( t \in (1, \infty) \). Then, \( g \) decreases on \((0, 1]\) and \( g \) increases on \([1, \infty)\).

From the above argument, it follows that the function:

\[
\frac{\delta + d_j}{\sqrt{\delta d_j}} = \mathcal{G}\left(\left(\frac{d_j}{\delta}\right)^{1/2}\right)
\]

is increasing in \( d_j \in (\delta, \Delta) \) and thus:

\[
\frac{\delta + (\delta + 1)}{2\sqrt{\delta(\delta + 1)}} \leq \frac{\delta + d_j}{\sqrt{\delta d_j}} \leq \frac{\delta + \Delta - 1}{2\sqrt{\delta(\Delta - 1)}},
\]

for every \( ij \in A_1\).

In a similar way, the function:

\[
\frac{\Delta + d_j}{\sqrt{\Delta d_j}} = \mathcal{G}\left(\left(\frac{d_j}{\Delta}\right)^{1/2}\right)
\]

is decreasing in \( d_j \in (\delta, \Delta) \) and thus:

\[
\frac{\Delta + (\Delta - 1)}{2\sqrt{\Delta(\Delta - 1)}} \leq \frac{\Delta + d_j}{\sqrt{\Delta d_j}} \leq \frac{\Delta + \delta + 1}{2\sqrt{\Delta(\delta + 1)}},
\]

for every \( ij \in A_2\).

Since:

\[
1 \leq \frac{d_i + d_j}{\sqrt{d_i d_j}} \leq \frac{\Delta + \delta}{2\sqrt{\Delta \delta}}
\]

for every \( ij \in E(H) \), we have:

\[
AG(H) = \sum_{ij \in E(H) \setminus A_0 \cup A_1 \cup A_2} \frac{d_i + d_j}{\sqrt{d_i d_j}} + \sum_{ij \in A_0} \frac{d_i + d_j}{\sqrt{d_i d_j}} + \sum_{ij \in A_1} \frac{\Delta + \delta}{2\sqrt{\Delta \delta}} + \sum_{ij \in A_2} \frac{\Delta + d_i}{\sqrt{\Delta d_i}}.
\]

therefore:

\[
AG(H) \geq m - a_0 - a_1 - a_2 + a_0 \frac{\Delta + \delta}{2\sqrt{\Delta \delta}} + a_1 \frac{2\delta + 1}{2\sqrt{\delta(\delta + 1)}} + a_2 \frac{2\Delta - 1}{2\sqrt{\Delta(\Delta - 1)}},
\]

and:

\[
AG(H) \leq (m - a_0 - a_1 - a_2) \frac{\Delta + \delta}{2\sqrt{\Delta \delta}} + a_0 \frac{\Delta + \delta}{2\sqrt{\Delta \delta}} + a_1 \frac{\Delta + \delta - 1}{2\sqrt{\Delta(\Delta - 1)}} + a_2 \frac{\Delta + \delta + 1}{2\sqrt{\Delta(\delta + 1)}}
\]

\[
= \frac{\Delta + \delta}{2\sqrt{\Delta \delta}} m - a_1 \left( \frac{\Delta + \delta}{2\sqrt{\Delta \delta}} - \frac{\Delta + \delta - 1}{2\sqrt{\Delta(\Delta - 1)}} \right) - a_2 \left( \frac{\Delta + \delta}{2\sqrt{\Delta \delta}} - \frac{\Delta + \delta + 1}{2\sqrt{\Delta(\delta + 1)}} \right).
\]

Lemma 2. If \( \nu(t) = \frac{1+t}{2\sqrt{t}} \), then:

(1) \( \nu(t) \leq \frac{1}{8}(1-t)^2 + 1 \) for every \( t \in [1, \infty) \),

(2) \( \nu(t) \geq \frac{1}{16}(1-t)^2 + 1 \) for every \( t \in (0, 1.945] \).
**Proof.** We have for every $s \in [1, \infty)$ and $t = s^2 \in [1, \infty)$:

$$(s - 1)^3(s^2 + 3s + 4) \geq 0,$$

$$s^5 - 2s^3 - 4s^2 + 9s - 4 \geq 0,$$

$$4(s^2 + 1) \leq s(8 + (1 - s^2)^2),$$

$$v(t) = \frac{t + 1}{2\sqrt{t}} \leq \frac{1}{8}(1-t)^2 + 1.$$

Let $s_1 = 1.39485...$ be the unique real solution of $s^3 + 2s^2 + s - 8 = 0$ in the interval $(0, \infty)$. We have for every $s \in [0,s_1]$ and $t = s^2 \in (0,1.945) \subset (0,s_1^2)$:

$$(s - 1)^2(s^3 + 2s^2 + s - 8) \leq 0,$$

$$s^5 - 2s^3 - 8s^2 + 17s - 8 \leq 0,$$

$$8(s^2 + 1) \geq s(16 + (1 - s^2)^2),$$

$$v(t) = \frac{1+t}{2\sqrt{t}} \geq \frac{1}{16}(1-t)^2 + 1.$$

\[\square\]

**Proposition 2.** Let $H$ be a graph with maximum degree $\Delta$, minimum degree $\delta < \Delta - 1$ and $m$ edges.

1. If $\delta$ is an even integer, then:

$$AG(H) \geq m + \min \left\{ \frac{\Delta + \delta}{2\sqrt{\Delta \delta}} + \frac{2\delta + 1}{2\sqrt{\delta(\delta + 1)}} - 2, \frac{2\delta + 1}{\sqrt{\delta(\delta + 1)}} + \frac{2\Delta - 1}{2\sqrt{(\Delta - 1)(\Delta - 1)}} - 3 \right\}.$$

2. If $\Delta$ is an even integer, then:

$$AG(H) \geq m + \min \left\{ \frac{\Delta + \delta}{2\sqrt{\Delta \delta}} + \frac{2\Delta - 1}{2\sqrt{\Delta(\Delta - 1)}} - 2, \frac{2\delta + 1}{2\sqrt{\delta(\delta + 1)}} + \frac{2\Delta - 1}{\sqrt{(\Delta - 1)(\Delta - 1)}} - 3 \right\}.$$

3. If $\delta$ and $\Delta$ are even integers, then:

$$AG(H) \geq m + \frac{2\delta + 1}{\sqrt{\delta(\delta + 1)}} + \frac{2\Delta - 1}{\sqrt{\Delta(\Delta - 1)}} - 4.$$

**Proof.** Assume first that $\delta$ is an even integer.

Let $H_1$ be the subgraph of $H$ induced by the $n_1$ vertices with degree $\delta$ in $V(H)$, and denote by $m_1$ the cardinality of the set of edges of $H_1$. Handshaking Lemma gives $n_1\delta - \alpha_0 - \alpha_1 = 2m_1$. Since $\delta$ is an even integer, $\alpha_0 + \alpha_1$ is also an even integer; since each component of $H$ is a connected graph, we have $\alpha_0 + \alpha_1 \geq 1$ and so, $\alpha_0 + \alpha_1 \geq 2$.

If $\alpha_0 \geq 2$, then Theorem 5 gives:

$$AG(H) \geq m + \alpha_0 \left( \frac{\delta + \Delta}{2\sqrt{\Delta \delta}} - 1 \right) + \alpha_1 \left( \frac{1 + 2\delta}{2\sqrt{(\delta + 1)\delta}} - 1 \right) + \alpha_2 \left( \frac{2\Delta - 1}{2\sqrt{(\Delta - 1)\Delta}} - 1 \right)$$

$$\geq m + \frac{\delta + \Delta}{\sqrt{\delta \Delta}} - 2.$$
If $\alpha_0 = 1$, then $\alpha_1 \geq 1$ and Theorem 5 gives:

$$AG(H) \geq m + \alpha_0 \left( \frac{\delta + \Delta}{2\sqrt{\delta \Delta}} - 1 \right) + \alpha_1 \left( \frac{1 + 2\delta}{2\sqrt{\delta + 1)} - 1 \right) + \alpha_2 \left( \frac{2\Delta - 1}{2\sqrt{\Delta - 1)} - 1 \right)$$

$$\geq m + \frac{\Delta + \delta}{2\sqrt{\Delta \delta}} + \frac{2\delta + 1}{2\sqrt{\delta + 1)} - 2.$$  

If $\alpha_0 = 0$, then $\alpha_1 \geq 2$ and $\alpha_2 \geq 1$, and Theorem 5 gives:

$$AG(H) \geq m + \alpha_0 \left( \frac{\delta + \Delta}{2\sqrt{\delta \Delta}} - 1 \right) + \alpha_1 \left( \frac{1 + 2\delta}{2\sqrt{\delta + 1)} - 1 \right) + \alpha_2 \left( \frac{2\Delta - 1}{2\sqrt{\Delta - 1)} - 1 \right)$$

$$\geq m + 2\delta + 1 \frac{\Delta - 1}{2\sqrt{\Delta(\Delta - 1)} - 3.}$$

Since Lemma 1 gives:

$$\frac{\Delta + \delta}{2\sqrt{\Delta \delta}} \geq \frac{2\delta + 1}{2\sqrt{\delta + 1)}},$$

we have:

$$AG(H) \geq m + \min \left\{ \left. \frac{\delta + \Delta}{2\sqrt{\delta \Delta}} + \frac{2\delta + 1}{2\sqrt{\delta + 1)} - 2 \right\} \right. \left. - \frac{2\delta + 1}{2\sqrt{\delta + 1)} + \frac{2\Delta - 1}{2\sqrt{\Delta(\Delta - 1)} - 3 \right\}.$$  

Assume now that $\Delta$ is an even integer. Let $H_2$ be the subgraph of $H$ induced by the $n_2$ vertices with a degree $\Delta$ in $V(H)$, and denote by $m_2$ the cardinality of the set of edges of $H_2$. Handshaking Lemma gives $n_2\Delta - \alpha_0 - \alpha_2 = 2m_2$. Since $\Delta$ is an even integer, $\alpha_0 + \alpha_2$ is also an even integer; since each component of $H$ is a connected graph, we have $\alpha_0 + \alpha_2 \geq 1$ and thus, $\alpha_0 + \alpha_2 \geq 2$.

If $\alpha_0 \geq 2$, then Theorem 5 gives:

$$AG(H) \geq m + \alpha_0 \left( \frac{\delta + \Delta}{2\sqrt{\delta \Delta}} - 1 \right) + \alpha_1 \left( \frac{1 + 2\delta}{2\sqrt{\delta + 1)} - 1 \right) + \alpha_2 \left( \frac{2\Delta - 1}{2\sqrt{\Delta - 1)} - 1 \right)$$

$$\geq m + \frac{\Delta + \delta}{2\sqrt{\Delta \delta}} - 2.$$  

If $\alpha_0 = 1$, then $\alpha_2 \geq 1$ and Theorem 5 gives:

$$AG(H) \geq m + \alpha_0 \left( \frac{\delta + \Delta}{2\sqrt{\delta \Delta}} - 1 \right) + \alpha_1 \left( \frac{1 + 2\delta}{2\sqrt{\delta + 1)} - 1 \right) + \alpha_2 \left( \frac{2\Delta - 1}{2\sqrt{\Delta - 1)} - 1 \right)$$

$$\geq m + \frac{\Delta + \delta}{2\sqrt{\Delta \delta}} + \frac{2\Delta - 1}{2\sqrt{\Delta(\Delta - 1)} - 2.$$  

If $\alpha_0 = 0$, then $\alpha_2 \geq 2$ and $\alpha_1 \geq 1$, and Theorem 5 gives:

$$AG(H) \geq m + \alpha_0 \left( \frac{\delta + \Delta}{2\sqrt{\delta \Delta}} - 1 \right) + \alpha_1 \left( \frac{1 + 2\delta}{2\sqrt{\delta + 1)} - 1 \right) + \alpha_2 \left( \frac{2\Delta - 1}{2\sqrt{\Delta - 1)} - 1 \right)$$

$$\geq m + \frac{1 + 2\delta}{2\sqrt{\delta + 1)} + \frac{2\Delta - 1}{\sqrt{\Delta(\Delta - 1)} - 3.$$  

Since:

$$\frac{\Delta + \delta}{2\sqrt{\Delta \delta}} \geq \frac{2\Delta - 1}{2\sqrt{\Delta(\Delta - 1)}},$$
we have:

\[ AG(H) \geq m + \min \left\{ \frac{\Delta + \delta}{2 \sqrt{\delta \Delta}} + \frac{2\Delta - 1}{2 \sqrt{\Delta(\Delta - 1)}} - 2, \frac{2\delta + 1}{2 \sqrt{(\delta + 1)\delta}} + \frac{2\Delta - 1}{2 \sqrt{(\Delta - 1)\Delta}} - 3 \right\}. \]

Finally, assume that \( \delta \) and \( \Delta \) are even integers. The previous arguments give \( a_0 + a_1 \geq 2 \) and \( a_0 + a_2 \geq 2 \).

If \( a_0 \geq 2 \), then Theorem 5 gives:

\[
AG(H) \geq m + a_0 \left( \frac{\delta + \Delta}{2 \sqrt{\delta \Delta}} - 1 \right) + a_1 \left( \frac{1 + 2\delta}{2 \sqrt{(\delta + 1)\delta}} - 1 \right) + a_2 \left( \frac{2\Delta - 1}{2 \sqrt{(\Delta - 1)\Delta}} - 1 \right)
\]

\[
\geq m + \frac{\Delta + \delta}{\sqrt{\delta \Delta}} - 2.
\]

If \( a_0 = 1 \), then \( a_1, a_2 \geq 1 \) and Theorem 5 gives:

\[
AG(H) \geq m + a_0 \left( \frac{\delta + \Delta}{2 \sqrt{\delta \Delta}} - 1 \right) + a_1 \left( \frac{1 + 2\delta}{2 \sqrt{(\delta + 1)\delta}} - 1 \right) + a_2 \left( \frac{2\Delta - 1}{2 \sqrt{(\Delta - 1)\Delta}} - 1 \right)
\]

\[
\geq m + \frac{\Delta + \delta}{2 \sqrt{\Delta \delta}} + \frac{2\delta + 1}{2 \sqrt{(\delta + 1)\delta}} + \frac{2\Delta - 1}{2 \sqrt{\Delta(\Delta - 1)}} - 3.
\]

If \( a_0 = 0 \), then \( a_1, a_2 \geq 2 \), and Theorem 5 gives:

\[
AG(H) \geq m + a_0 \left( \frac{\Delta + \delta}{2 \sqrt{\Delta \delta}} - 1 \right) + a_1 \left( \frac{2\delta + 1}{2 \sqrt{(\delta + 1)\delta}} - 1 \right) + a_2 \left( \frac{2\Delta - 1}{2 \sqrt{(\Delta - 1)\Delta}} - 1 \right)
\]

\[
\geq m + \frac{2\delta + 1}{\sqrt{(\delta + 1)\delta}} + \frac{2\Delta - 1}{\sqrt{(\Delta - 1)\Delta}} - 4.
\]

We claim now:

\[
1 + \frac{\delta + \Delta}{2 \sqrt{\delta \Delta}} \geq \frac{1 + 2\delta}{2 \sqrt{(\delta + 1)\delta}} + \frac{2\Delta - 1}{2 \sqrt{(\Delta - 1)\Delta}}.
\]

Assuming that this inequality holds, we have:

\[
m + \frac{\delta + \Delta}{\sqrt{\delta \Delta}} - 2 \geq m + \frac{\delta + \Delta}{2 \sqrt{\delta \Delta}} + \frac{1 + 2\delta}{2 \sqrt{(\delta + 1)\delta}} + \frac{2\Delta - 1}{2 \sqrt{(\Delta - 1)\Delta}} - 3,
\]

\[
m + \frac{\Delta + \delta}{2 \sqrt{\delta \Delta}} + \frac{2\delta + 1}{2 \sqrt{(\delta + 1)\delta}} + \frac{2\Delta - 1}{2 \sqrt{(\Delta - 1)\Delta}} - 3 \geq m + \frac{1 + 2\delta}{\sqrt{(\delta + 1)\delta}} + \frac{2\Delta - 1}{\sqrt{(\Delta - 1)\Delta}} - 4,
\]

and we conclude:

\[
AG(H) \geq m + \frac{1 + 2\delta}{\sqrt{(\delta + 1)\delta}} + \frac{2\Delta - 1}{\sqrt{(\Delta - 1)\Delta}} - 4.
\]

Thus, it suffices to prove the claim.

\[
1 + \frac{\delta + \Delta}{2 \sqrt{\delta \Delta}} \geq \frac{2\delta + 1}{2 \sqrt{(\delta + 1)\Delta}} + \frac{2\Delta - 1}{2 \sqrt{(\Delta - 1)\Delta}},
\]

\[
1 + v\left(\frac{\Delta}{\delta}\right) \geq v\left(\frac{1 + \delta}{\delta}\right) + v\left(\frac{2\Delta - 1}{\Delta - 1}\right).
\]
where \( v(t) = \frac{1 + t}{2\sqrt{t}} \) is the function in Lemma 2. Since \( v \) is an increasing function in \([1, \infty)\) and \( \Delta \geq \delta + 2 \), we have:

\[
v\left(\frac{2 + \delta}{\delta}\right) \leq v\left(\frac{\Delta}{\delta}\right), \quad v\left(\frac{\Delta}{\Delta - 1}\right) \leq v\left(\frac{2 + \delta}{1 + \delta}\right).
\]

Hence, it suffices to show:

\[
1 + v\left(\frac{2 + \delta}{\delta}\right) \geq v\left(\frac{1 + \delta}{3}\right) + v\left(\frac{2 + \delta}{1 + \delta}\right),
\]

for every \( \delta \geq 1 \).

Note that (2) holds for \( \delta = 1, 2 \). Let us prove that it holds for \( \delta \geq 3 \). Note that:

\[
\frac{1}{8} \left(\frac{1}{\delta + 1}\right)^2 \leq \frac{1}{8} \frac{1}{\delta^2}, \quad 2 + \frac{1}{8} \frac{1}{\delta^2} + \frac{1}{8} \left(\frac{1}{\delta + 1}\right)^2 \leq 2 + \frac{1}{4} \frac{1}{\delta^2}.
\]

Since \( \delta \geq 3 \), we have:

\[
\frac{2 + \delta}{\delta} \leq \frac{5}{3} < 1.945, \quad \frac{1 + \delta}{\delta} > 1, \quad \frac{2 + \delta}{1 + \delta} > 1.
\]

Thus, Lemma 2 gives:

\[
v\left(\frac{1 + \delta}{\delta}\right) + v\left(\frac{2 + \delta}{1 + \delta}\right) \leq 1 + \frac{1}{8} \left(\frac{1 + \delta}{\delta} - 1\right)^2 + 1 + \frac{1}{8} \left(\frac{2 + \delta}{1 + \delta} - 1\right)^2
\]

\[
= 2 + \frac{1}{8} \frac{1}{\delta^2} + \frac{1}{8} \left(\frac{1}{\delta + 1}\right)^2 \leq 2 + \frac{1}{4} \frac{1}{\delta^2}
\]

\[
= 1 + 1 + \frac{1}{16} \left(1 - \frac{2 + \delta}{\delta}\right)^2 \leq 1 + v\left(\frac{2 + \delta}{\delta}\right).
\]

These inequalities give (2) for \( \delta \geq 3 \), and the proof is finished. \( \square \)

Finally, we show that the bound in Proposition 2 (3) is tight: let us consider the complete graphs \( K_5 \) and \( K_3 \), and fix \( u_1, u_2 \in V(K_5) \) and \( v_1, v_2 \in V(K_3) \). Denote by \( K_5^* \) the graph obtained from \( K_5 \) by deleting the edge \( u_1u_2 \). Let \( \Gamma \) be the graph with \( V(\Gamma) = V(K_5^*) \cup V(K_3) \) and \( E(\Gamma) = E(K_5^*) \cup E(K_3) \cup \{u_1v_1\} \cup \{u_2v_2\} \). Thus, \( \Gamma \) has a maximum degree \( \Delta = 4 \), minimum degree \( \delta = 2 \), \( \alpha_0 = 0 \), \( \alpha_1 = 2 \), \( \alpha_2 = 2 \); in addition, if \( ij \notin A_0 \cup A_1 \cup A_2 \), then \( d_i = d_j \), if \( ij \in A_1 \), then \( \{d_i, d_j\} = \{\delta, \delta + 1\} \), and if \( ij \in A_2 \), then \( \{d_i, d_j\} = \{\Delta, \Delta - 1\} \). Then, we have:

\[
AG(\Gamma) = m + \frac{2\delta + 1}{\sqrt{\delta(\delta + 1)}} + \frac{2\Delta - 1}{\sqrt{\Delta(\Delta - 1)}} - 4.
\]

4. Conclusions

Topological indices have become a useful tool for the study of theoretical and practical problems in different areas of science. An important line of research associated with topological indices is that of determining optimal bounds and relations between known topological indices—particularly to obtain bounds for the topological indices associated with the invariant parameters of a graph.

Ref. [35] proves that many upper bounds of \( GA \) are not useful, and shows the importance of obtaining upper bounds of \( GA \) that are less than \( m \). In a similar way, it is important to find lower bounds of \( AG \) greater than \( m \).

With this aim, we obtain in this paper several new lower bounds of \( AG \), which are greater than \( m \) for graphs with a maximum degree \( \Delta \) and minimum degree \( \delta < \Delta - 1 \):
1. If $\delta$ is an even integer, then:

$$AG(H) \geq m + \min \left\{ \frac{\Delta + \delta}{2\sqrt{\Delta \delta}} + \frac{2\delta + 1}{2\sqrt{\delta(\delta + 1)}} - 2, \frac{2\delta + 1}{2\sqrt{\delta(\delta + 1)}} + \frac{2\Delta - 1}{2\sqrt{\Delta(\Delta - 1)}} - 3 \right\}.$$ 

2. If $\Delta$ is an even integer, then:

$$AG(H) \geq m + \min \left\{ \frac{\Delta + \delta}{2\sqrt{\Delta \delta}} + \frac{2\Delta - 1}{2\sqrt{\Delta(\Delta - 1)}} - 2, \frac{2\delta + 1}{2\sqrt{\delta(\delta + 1)}} + \frac{2\Delta - 1}{2\sqrt{\Delta(\Delta - 1)}} - 3 \right\}.$$ 

3. If $\delta$ and $\Delta$ are even integers, then:

$$AG(H) \geq m + \frac{2\delta + 1}{\sqrt{\delta(\delta + 1)}} + \frac{2\Delta - 1}{\sqrt{\Delta(\Delta - 1)}} - 4.$$ 

We obtain several inequalities relating $AG$ with other topological indices, as

$$\frac{\sqrt{\Delta \delta}}{\Delta + \delta} \left( \frac{1}{2} SDD(H) + m \right) \leq AG(H) \leq \frac{1}{4} SDD(H) + \frac{1}{2} m.$$ 

This result improves the following bound already known in the literature:

$$AG(H) \leq \frac{1}{2} SDD(H).$$ 

Moreover, we find families of graphs where the bounds are attained.

Furthermore, we show that at least for entropy, the $AG$ index has better predictive power than $GA$, while for other physicochemical properties, the $GA$ index has better predictive power than $AG$.

We think that it would be interesting to obtain for the geometric–arithmetic index some results similar to those included in this work for the $AG$ index.

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