Robustly shadowable chain transitive sets and hyperbolicity

Mohammad Reza Bagherzad\textsuperscript{a} and Keonhee Lee\textsuperscript{b}

\textsuperscript{a}Department of Mathematics, Behin Andishan Rayyar Research Group, Tehran, Iran; \textsuperscript{b}Department of Mathematics, Chungnam National University, Daejeon, Korea

\section*{Abstract}
We say that a compact invariant set $\Lambda$ of a $C^1$-vector field $X$ on a compact boundaryless Riemannian manifold $M$ is robustly shadowable if it is locally maximal with respect to a neighbourhood $U$ of $\Lambda$, and there exists a $C^1$-neighbourhood $U$ of $X$ such that for any $Y \in U$, the continuation $\Lambda_Y$ of $\Lambda$ for $Y$ and $U$ is shadowable for $Y$. In this paper, we prove that any chain transitive set of a $C^1$-vector field on $M$ is hyperbolic if and only if it is robustly shadowable.

\section{Introduction}

The main goal of the study of differentiable dynamical systems is to understand the structure of the orbits of vector fields (or diffeomorphisms) on a compact boundaryless Riemannian manifold. To describe the dynamics on the underlying manifold, it is usual to use the dynamic properties on the tangent bundle such as hyperbolicity and dominated splitting. A fundamental problem in recent years is to study the influence of a robust dynamic property (i.e. property that holds for a given system and all $C^1$-nearby systems) on the behaviour of the tangent map on the tangent bundle (e.g. see \cite{4,7–9,11}).

Recently, several results dealing with the influence of a robust dynamics property of a $C^1$-vector field were appeared. For instance, Lee and Sakai \cite{7} proved that a non-singular vector field $X$ is robustly shadowable (i.e. $X$ and its $C^1$-nearby systems are shadowable) if and only if it satisfies both Axiom A and the strong transversality condition (i.e. it is structurally stable). Afterwards, Pilyugin and Tikhomirov \cite{11} gave a description of robustly shadowable oriented vector fields which are structurally stable. In particular, it is proved in \cite{8} that any robustly shadowable chain component $C_X(\gamma)$ of $X$ containing a hyperbolic periodic orbit $\gamma$ does not contain a hyperbolic singularity, and it is hyperbolic if $C_X(\gamma)$ has no non-hyperbolic singularity. Here we say that the chain component $C_X(\gamma)$ is \textit{robustly shadowable} if there is a $C^1$-neighbourhood $U$ of $X$ such that for any $Y \in U$, the continuation $C_Y(\gamma_Y)$ of $Y$ containing $\gamma_Y$ is shadowable for $Y$, where $\gamma_Y$ is the continuation of $\gamma$ with respect to $Y$. Very recently, Gan et al. \cite{4} showed that the set of all robustly shadowable oriented vector fields is contained in the set of vector fields with $\Omega$-stability. In this direction, the following question is still open: \textit{if the chain component $C_X(\gamma)$ of a $C^1$-vector field $X$ on...}
a compact boundaryless Riemannian manifold $M$ containing a hyperbolic periodic orbit $\gamma$ is robustly shadowable, then is it hyperbolic?

In this paper, we study the dynamics of robustly shadowable chain transitive sets. More precisely, we prove that any chain transitive set of a vector field $X$ is hyperbolic if and only if it is robustly shadowable. For this, we first show that if a compact invariant set $\Lambda \subset M$ is robustly shadowable then every singularity and periodic orbit in $\Lambda_Y$ are hyperbolic for $Y$, where $\Lambda_Y$ is the continuation of $\Lambda$ with respect to a $C^1$-nearby vector field $Y$. Moreover, we see that any robustly shadowable chain transitive set $\Lambda$ does not contain a singularity. Finally we show that $\Lambda$ admits a dominated splitting, and it is indeed a hyperbolic splitting. Note that Ribeiro [12] showed that $C^1$-generically, an isolated set $\Lambda$ is chain transitive and has the shadowing property if and only if it is a transitive hyperbolic set.

Similar results for diffeomorphisms were obtained in [14]. However, we cannot apply the reasoning developed therein to vector fields in general. For, since there may be singularities for vector fields, we cannot define the Poincaré maps of given vector fields in the whole space, and thus, we cannot apply the reasoning to the maps developed for diffeomorphisms. Moreover, a chain transitive set does not necessarily contain a periodic orbit. For a diffeomorphism $f$ of $M$, it is possible to prove the existence of a periodic point by assuming that $f$ is robustly shadowable. In fact, we see that if $f$ is robustly shadowable, then $f$ is in the $C^1$ interior of the set, $F^1(M)$, of diffeomorphisms whose periodic points are hyperbolic. From this, the existence of a periodic point is readily proved by the standard method based on the fact that for an integer $n > 0$, the map $f \mapsto P_n(f)$ is finite valued and continuous. Here, $P_n(f)$ is the set of periodic points of $f$ with period $n$. For vector fields, however, the above method does not work because, for a periodic orbit $\gamma$, the period $\tau \in \mathbb{R}$ of $\gamma$ is not an integer in general.

Now we round out the introduction with some notations, definitions and main theorem which we will use throughout the paper. Let $M$ be a compact boundaryless Riemannian manifold with dimension $n$. Denote by $\mathcal{X}^1(M)$ as the set of all $C^1$ vector fields of $M$ endowed with the $C^1$ topology. Then every $X \in \mathcal{X}^1(M)$ generates a $C^1$ flow $X_t : M \times \mathbb{R} \rightarrow M$, that is, a family of diffeomorphisms on $M$ such that $X_t \circ X_s = X_{t+s}$, for all $t, s \in \mathbb{R}$, $X_0 = \text{Id}$ and $\frac{\partial X_t}{\partial t}|_{t=0} = X(p)$ for any $p \in M$. Throughout the paper, for $X, Y, \ldots \in \mathcal{X}^1(M)$, we always denote the generated flows by $X_t$, $Y_t$, …, respectively. For $x \in M$, let us denote the orbit $\{X_t(x), t \in \mathbb{R}\}$ of the flow $X_t$ (or $X$) through $x$ by $\text{orb}(x, X_t)$, or $O(x)$ if no confusion is likely. We say that a point $x \in M$ is a singularity of $X$ if $X(x) = 0$; and an orbit $O(x)$ is closed (or periodic) if it is diffeomorphic to a circle $S^1$. Let $d$ be the distance induced from the Riemannian structure on $M$. A sequence $\{(x_i, t_i) : x_i \in M; t_i \geq 1; a < i < b\} (-\infty \leq a < b \leq \infty)$ is called a $\delta$-pseudo orbit (or a $\delta$-chain) of $X_t$ if for any $a < i < b - 1$,

$$d\left(X_{t_i}(x_i), X_{t_{i+1}}\right) < \delta.$$  

Roughly speaking, a pseudo orbit is composed by a set of segments of real orbits. We need the restriction $t_i \geq 1$ because without this, for any $\delta > 0$, all points $x, y \in M$ can be connected by a $\delta$-pseudo orbit.

Let $\text{Rep}$ be the set of all increasing homeomorphisms (called reparametrizations) $h : \mathbb{R} \rightarrow \mathbb{R}$ such that $h(0) = 0$. We say that a compact invariant set $\Lambda$ of $X_t$ is shadowable if for any $\varepsilon > 0$, there is $\delta > 0$ satisfying the following property: given any $\delta$-pseudo orbit $\{(x_i, t_i) : -\infty \leq i \leq \infty\}$ in $\Lambda$, there exist a point $y \in M$ and $h \in \text{Rep}$ such that for all $t \in \mathbb{R}$ we
have
\[ d(X_{h(t)}(y), x_0 * t) < \varepsilon, \]
where \( x_0 * t = X_{t-S_i}(x_i) \) for any \( t \in [S_i, S_{i+1}] \), and \( S_i \) is given by
\[
S_i = \begin{cases} 
\sum_{j=0}^{i-1} t_j & \text{for } i > 0, \\
0 & \text{for } i = 0, \\
-\sum_{j=1}^{i-1} t_j & \text{for } i < 0.
\end{cases}
\]

Note that the above concept of pseudo orbit is slightly different from that of pseudo orbit in [7,11]. However we point out here that a compact invariant set \( \Lambda \) is shadowable for \( X_t \) under the above definition if and only if it is shadowable for \( X_t \) under the definition in [7,11]. A point \( x \in M \) is called chain recurrent if for any \( \delta > 0 \), there exists a \( \delta \)-pseudo orbit \( \{(x_i, t_i) : 0 \leq i < n\} \) with \( n > 1 \) such that \( x_0 = x \) and \( d(X_{t_{n-1}}(x_{n-1}), x) < \delta \). The set of all chain recurrent points of \( X_t \) is called the chain recurrent set of \( X_t \), and is denoted by \( CR(X_t) \). For any \( x, y \in M \), we say that \( x \sim y \) if for any \( \delta > 0 \), there are a \( \delta \)-pseudo orbit \( \{(x_i, t_i) : 0 \leq i < n\} \) with \( n > 1 \) such that \( x_0 = x \) and \( d(X_{t_{n-1}}(x_{n-1}), y) < \delta \) and a \( \delta \)-pseudo orbit \( \{(x'_i, t'_i) : 0 \leq i < m\} \) with \( m > 1 \) such that \( x'_0 = y \) and \( d(X_{t'_{m-1}}(x'_{m-1}), x) < \delta \). It is easy to see that \( \sim \) gives an equivalence relation on the set \( CR(X_t) \). An equivalence class of \( \sim \) is called a chain component of \( X_t \) (or \( X \)). We say that a compact invariant set \( \Lambda \) of \( X_t \) is chain transitive if for any \( x, y \in \Lambda \) and any \( \delta > 0 \), there is a \( \delta \)-pseudo orbit \( \{(x_i, t_i) \in \Lambda \times \mathbb{R} : t_j \geq 1, 0 \leq i < n\} \) with \( n > 1 \) such that \( x_0 = x \) and \( d(X_{t_{n-1}}(x_{n-1}), y) < \delta \).

A compact invariant set \( \Lambda \) of \( X_t \) is called hyperbolic if there are constants \( C > 0 \) and \( \lambda > 0 \) such that the tangent flow \( DX_t : T_{\Lambda}M \to T_{\Lambda}M \) leaves a continuous invariant splitting \( T_{\Lambda}M = E^s \oplus \langle X \rangle \oplus E^u \) satisfying
\[
\|DX_t|_{E^s(x)}\| \leq Ce^{-\lambda t} \quad \text{and} \quad \|DX_t|_{E^u(x)}\| \leq Ce^{-\lambda t}
\]
for any \( x \in \Lambda \) and \( t > 0 \), where \( \langle X \rangle \) denotes the subspace generated by the vector field \( X \). For any hyperbolic closed orbit \( \gamma \), the sets
\[
W^s(\gamma) = \{x \in M : X_t(x) \to \gamma \text{ as } t \to \infty\} \quad \text{and} \quad W^u(\gamma) = \{x \in M : X_t(x) \to \gamma \text{ as } t \to -\infty\}
\]
are said to be the stable manifold and unstable manifold of \( \gamma \), respectively. We say that the dimension of the stable manifold \( W^s(\gamma) \) of \( \gamma \) is the index of \( \gamma \), and denoted by \( ind(\gamma) \).

The homoclinic class of \( X_t \) associated to \( \gamma \), denoted by \( H_X(\gamma) \), is defined as the closure of the transversal intersection of the stable and unstable manifolds of \( \gamma \), that is
\[
H_X(\gamma) = W^s(\gamma) \cap W^u(\gamma).
\]
By definition, we easily see that the set is closed and \( X_t \)-invariant. Let \( C_X(\gamma) \) be the chain component of \( X_t \) containing a hyperbolic periodic orbit \( \gamma \). Then we have \( H_X(\gamma) \subset C_X(\gamma) \), but the converse is not true in general. For two hyperbolic closed orbits \( \gamma_1 \) and \( \gamma_2 \) of \( X_t \), we
say $\gamma_1$ and $\gamma_2$ are homoclinically related, denoted by $\gamma_1 \sim \gamma_2$, if

$$W^s(\gamma_1) \cap W^u(\gamma_2) \neq \emptyset \text{ and } W^s(\gamma_2) \cap W^u(\gamma_1) \neq \emptyset.$$ 

By Birkhoff–Smale’s theorem (see [1]), we know that

$$H_X(\gamma) = \{ \gamma' : \gamma' \sim \gamma \}.$$

A point $x \in M$ is called non-wandering if for any neighbourhood $U$ of $x$, there is $t \geq 1$ such that $X_t(U) \cap U \neq \emptyset$. The set of all non-wandering points of $X_t$ is called the non-wandering set of $X_t$, denoted by $\Omega(X_t)$. Let $\text{Sing}(X)$ be the set of all singularities of $X$, and let $\text{PO}(X_t)$ be the set of all periodic orbits (which are not singularities) of $X_t$. Clearly we have

$$\text{Sing}(X) \cup \text{PO}(X_t) \subset \Omega(X_t) \subset CR(X_t).$$

We say that $X$ satisfies Axiom $A$ if $\text{PO}(X_t)$ is dense in $\Omega(X_t) \setminus \text{Sing}(X)$, and $\Omega(X_t)$ is hyperbolic for $X_t$. A point $y \in M$ is said to be an $\omega$ limit point of $x$ if there exists a sequence $t_i \to +\infty$ such that $X_{t_i}(x) \to y$. Denote the set of all omega limit points of $x$ by $\omega(x)$. We say that a compact invariant set $\Lambda$ of $X_t$ is transitive if there is $x \in \Lambda$ such that $\omega(x) = \Lambda$.

Let $\Lambda$ be a compact invariant set of $X_t$. For any $C^1$-close $Y$ to $X$ and a neighbourhood $U$ of $\Lambda$, the set

$$\Lambda_Y = \bigcap_{t \in \mathbb{R}} Y_t(U)$$

is called the continuation of $\Lambda$ for $Y$ and $U$. If there exists a neighbourhood $U$ of $\Lambda$ satisfying $\Lambda = \bigcap_{t \in \mathbb{R}} X_t(U)$, then we say that $\Lambda$ is locally maximal with respect to $U$, and $U$ is called an isolating block of $\Lambda$. Let $\gamma$ be a hyperbolic closed orbit of $X_t$. Then we know that there are a $C^1$ neighbourhood $U$ of $X$ and a neighbourhood $U$ of $\gamma$ such that for any $Y \in U$, there is a unique hyperbolic closed orbit $\gamma_Y$ in $U$ which is equal to the set $\bigcap_{t \in \mathbb{R}} Y_t(U)$. Note that every $\gamma_Y$ is locally maximal with respect to $U$. The chain component of $Y \in U$ containing the continuation $\gamma_Y$ will be denoted by $C_Y(\gamma_Y)$.

Now we give the definition of robust shadowability for invariant sets of vector fields.

**Definition 1.1:** We say that a compact invariant set $\Lambda$ of $X_t$ is robustly shadowable if it has an isolating block $U$, and there exists a $C^1$-neighbourhood $U$ of $X$ such that for any $Y \in U$, the continuation $\Lambda_Y$ for $Y$ and $U$ is shadowable for $Y_t$. Here $U$ is said to be an admissible neighbourhood of $X$ with respect to $\Lambda$.

In this paper, we prove the following main theorem.

**Main Theorem** For any $X \in \mathcal{X}^1(M)$, a compact invariant chain transitive set $\Lambda$ of $X_t$ is a hyperbolic basic set for $X_t$ (that is, locally maximal and transitive set) if and only if it is robustly shadowable.

**Corollary 1.2:** For a chain component $\Lambda$ of $X_t$, we have that $\Lambda$ is a hyperbolic homoclinic class (for some periodic orbit) if and only if it is robustly shadowable.
\textbf{Proof:} Suppose a chain component $\Lambda$ is robustly shadowable. Then we see that $\Lambda$ is a hyperbolic homoclinic class by applying the above main theorem and Lemma 4.1. The converse is clear. \hfill $\Box$

2. Linear Poincaré flows and quasi hyperbolic orbit arcs

Hereafter, we assume that the exponential map
\[ \exp_p : T_p M(1) \to M \]
is well defined for all $p \in M$, where $T_p M(r)$ denotes the $r$-ball $\{v \in T_p M : \|v\| \leq r\}$ in $T_p M$. For any regular point $x \in M$ (i.e. $X(x) \neq 0$), we let
\[ N_x = \text{span} X(x) \perp \subset T_x M, \]
and $N_x(r)$ the $r$-ball in $N_x$. Let $\hat{N}_{x,r} = \exp_x(N_x(r))$. Given any regular point $x \in M$ and $t \in \mathbb{R}$, we can take a constant $r > 0$ and a $C^1$ map $\tau : \hat{N}_{x,r} \to \mathbb{R}$ such that $\tau(x) = t$ and $X_{\tau(y)}(y) \in \hat{N}_{X(x),1}$ for any $y \in \hat{N}_{x,r}$. Now we define the \textit{Poincaré map}
\[ f_{x,t} : \hat{N}_{x,r} \to \hat{N}_{X(x),1}, \quad f_{x,t}(y) = X_{\tau(y)}(y) \]
for $y \in \hat{N}_{x,r}$. Let $M_X = \{x \in M : X(x) \neq 0\}$. Then it is easy to check that for any fixed $t$ there exists a continuous map $r_0 : M_X \to (0, 1)$ such that for any $x \in M_X$, the Poincaré map $f_{x,t} : \hat{N}_{x,r_0(x)} \to \hat{N}_{X(x),1}$ is well defined and the respective time function $\tau(y)$ satisfies $2t/3 < \tau(y) < 4t/3$ for $y \in \hat{N}_{x,r_0(x)}$.

Let $t_0$ be fixed. At each $x \in M_X$, one can consider a \textit{flow box chart} $(\hat{U}_{x,t_0,\delta}, F_{x,t_0})$ at $x$ such that
\[ \hat{U}_{x,t_0,\delta} = \{tX(x) + y : 0 \leq t \leq t_0, y \in N_x(\delta)\} \subset T_x M, \]
where $F_{x,t_0} : \hat{U}_{x,t_0,\delta} \to M$ is defined by $F_{x,t_0}(tX(x) + y) = X_t(\exp_x y)$. Then it is well known that if $X_t(x) \neq x$ for any $t \in (0, t_0]$, then there is $\delta > 0$ such that $F_{x,t_0} : \hat{U}_{x,t_0,\delta} \to M$ is an embedding.

For $\epsilon > 0$ and $r > 0$, let $N'_\epsilon(\hat{N}_{x,r})$ be the set of all diffeomorphisms $\phi : \hat{N}_{x,r} \to \hat{N}_{x,r}$ such that
\[ \text{supp}(\phi) \subset \hat{N}_{x,\epsilon/2} \text{ and } d_{C^1}(\phi, id) < \epsilon. \]

Here $d_{C^1}$ is the usual $C^1$ metric, $id$ denotes the identity map and the $\text{supp}(\phi)$ is the closure of the set of points where it differs from $id$.

\textbf{Proposition 2.1:} Let $X \in \mathcal{X}^1(M)$, and let $\mathcal{U} \subset \mathcal{X}^1(M)$ be a neighbourhood of $X$. For any constant $t_0 > 0$, there are a constant $\epsilon > 0$ and a $C^1$-neighbourhood $\mathcal{V}$ of $X$ such that for any $Y \in \mathcal{V}$, there exists a continuous map $r : M_Y \to (0, 1)$ satisfying the following property: for any $x \in M_Y$ satisfying $Y_t(x) \neq x$ for $0 < t \leq 2t_0$ and any $\phi \in N'_\epsilon(\hat{N}_{x,r(x)})$, there is $Z \in \mathcal{U}$ such that $Y(z) = Z(z)$ for all $z \in M \setminus F_x(\hat{U}_x)$ and $Z_t(y) = Y_t(\phi(y))$ for any $y \in \hat{N}_{x,r(x)}$ and $2t_0/3 < t < 4t_0/3$, where $F_x(\hat{U}_x)$ is the flow box of $Y$ at $x$. 


Proof: See [13, p. 293–295]. \qed

Remark 2.2: In the above proposition, it is easy to see that if \( \phi(x) = x \), then \( f_{x, t_0} \circ \phi \) is the Poincaré map of \( Z \), where \( f_{x, t_0} : \hat{N}_{x, r(x)} \to \hat{N}_{X_0(x), 1} \) is the Poincaré map of \( Y \).

For the study of stability conjecture (see [5]) posed by Palis and Smale, Liao [10] introduced the notion of linear Poincaré flow for a \( C^1 \)-vector field as follows. Let \( \mathcal{N} = \bigcup_{x \in M_X} N_x \) be the normal bundle based on \( M_X \). Then we can introduce a flow (which is called a linear Poincaré flow for \( X \))

\[
\Psi_t : \mathcal{N} \to \mathcal{N}, \quad \Psi_t|_{N_x} = \pi_{N_x} \circ D_x X_t|_{N_x},
\]

where \( \pi_{N_x} : T_x M \to N_x \) is the natural projection along the direction of \( X(x) \), and \( D_x X_t \) is the derivative map of \( X_t \). Then we can see that

\[
\Psi_t|_{N_x} = D_x f_{x, t} \quad \text{and} \quad f_{x, t} \circ \exp_x = \exp_{X_t(x)} \circ \Psi_t.
\]

Using Proposition 2.1, we can prove the following lemma which has the same philosophy with the Franks’ Lemma for diffeomorphisms. One can find another proof for the lemma in [2].

Lemma 2.3: Let \( \mathcal{U} \) be a \( C^1 \) neighbourhood of \( X \in \mathcal{X}^1(M) \). For any \( T > 0 \), there exists a constant \( \eta > 0 \) such that for any tubular neighbourhood \( U \) of an orbit arc \( \gamma = X_{[0, T]}(x) \) of \( X_t \) and for any \( \eta \)-perturbation \( F \) of the linear Poincaré flow \( \Psi_T|_{N_x} \), there exists a vector field \( Y \in \mathcal{U} \) such that the linear Poincaré flow \( \Psi_T|_{N_x} \) associated to \( Y \) coincides with \( F \), and \( Y \) coincides with \( X \) outside \( U \) and along \( X_{[-t_1, t_2]}(x) \), where \( t_1 = \min \{ t > 0, X_{-t}(x) \in \partial U \} \) and \( t_2 = \min \{ t > 0, X_t(x) \in \partial U \} \).

We introduce the notions of dominated splitting and hyperbolic splitting for linear Poincaré flows as follows.

Definition 2.4: Let \( \Lambda \) be an invariant set of \( X_t \) which contains no singularity. We call a \( \Psi_t \)-invariant splitting \( \mathcal{N}_\Lambda = \Delta^s \oplus \Delta^u \) as an \( l \)-dominated splitting (or \( \Lambda \) admits an \( l \)-dominated splitting) if

\[
\left\| \Psi_t|_{\Delta^s(x)} \right\| \cdot \left\| \Psi_{-t}|_{\Delta^u(X_t(x))} \right\| \leq \frac{1}{2}
\]

for any \( x \in \Lambda \) and any \( t \geq l \), where \( l > 0 \) is a constant. Moreover, if \( \dim(\Delta^s) \) is constant for all \( x \in \Lambda \), then we say that the splitting is a homogeneous dominated splitting. Furthermore, a \( \Psi_t \)-invariant splitting \( \mathcal{N}_\Lambda = \Delta^s \oplus \Delta^u \) is said to be a hyperbolic splitting if there exist \( C > 0 \) and \( \lambda \in (0, 1) \) such that

\[
\left\| \Psi_t|_{\Delta^s(x)} \right\| \leq C \lambda^t \quad \text{and} \quad \left\| \Psi_{-t}|_{\Delta^u(x)} \right\| \leq C \lambda^t
\]

for any \( x \in \Lambda \) and \( t > 0 \).

The following proposition which is crucial to prove the hyperbolicity of invariant sets was proved by Doering and Liao [3,10]. For a detailed proof, see Proposition 1.1 in [3].
Proposition 2.5: Let $\Lambda \subset M$ be a compact invariant set of $X$, such that $\Lambda \cap \text{Sing}(X) = \emptyset$. Then $\Lambda$ is hyperbolic for $X$, if and only if the linear Poincaré flow $\Psi_t$ restricted on $\Lambda$ has a hyperbolic splitting $\mathcal{N}_\Lambda = \Delta^s \oplus \Delta^u$.

Proposition 2.6: Let $\Lambda$ be a locally maximal set of $X$, with an isolating block $U$. Suppose that $X$ has a $C^1$-neighbourhood $\mathcal{U}$ such that for any $Y \in \mathcal{U}$, every periodic orbit and singularity of $Y$ in $U$ are hyperbolic. Then $X$ has a neighbourhood $\tilde{U}$, together with two uniform constants $\tilde{\eta} > 0$ and $\tilde{T} > 1$ such that for any $Y \in \tilde{U}$,

1. whenever $x$ is a point on a periodic orbit of $Y$, in $U$ and $\tilde{T} \leq t < \infty$, then
$$\frac{1}{t} \left[ \log m(\Psi_t^Y |_{\Delta^s_t}) - \log \|\Psi_t^Y |_{\Delta^u_t}\| \right] \geq 2\tilde{\eta};$$

2. whenever $P$ is a periodic orbit of $Y$ in $U$ with period $T$, $x \in P$, and whenever an integer $m \geq 1$ and a partition $0 = t_0 < t_1 < \cdots < t_i = mT$ of $[0, mT]$ are given that satisfy

$$t_k - t_{k-1} \geq \tilde{T}, \quad k = 1, 2, \ldots, l,$$

then
$$\frac{1}{mT} \sum_{k=0}^{l-1} \log m(\Psi_{t_{k+1}-t_k}^Y |_{\Delta^s_{t_{k+1}}(x)}) \leq -\tilde{\eta},$$

and
$$\frac{1}{mT} \sum_{k=0}^{l-1} \log m(\Psi_{t_{k+1}-t_k}^Y |_{\Delta^u_{t_{k+1}}(x)}) \geq \tilde{\eta}.$$

Proof: See Theorem 2.6 in [9].

Let $\Lambda \subset M_X$ be a closed invariant set of $X$, that has a continuous $\Psi_t$-invariant splitting $\mathcal{N}_\Lambda = \Delta^s \oplus \Delta^u$ with $\dim \Delta^s = p$, $1 \leq p \leq \dim M - 2$. For two real numbers $T > 0$ and $\eta > 0$, an orbit arc $(x, t) = X_{[0, t]}(x)$ will be called $(\eta, T, p)$-quasi hyperbolic orbit arc of $X$ with respect to the splitting $\Delta^s \oplus \Delta^u$ if $[0, t]$ has a partition

$$0 = T_0 < T_1 < \cdots < T_1 = t$$

such that $T \leq T_i - T_{i-1} < 2T, i = 1, 2, \ldots, l$, and the following three conditions are satisfied:

$$\frac{1}{T_k} \sum_{j=1}^{k} \log m(\Psi_{T_j - T_{j-1}}^Y |_{\Delta^s(X_{T_{j-1}})(x)}) \leq -\eta,$$

$$\frac{1}{T_i - T_{k-1}} \sum_{j=k}^{l} \log m(\Psi_{T_j - T_{j-1}}^Y |_{\Delta^u(X_{T_{j-1}})(x)}) \geq \eta,$$

$$\log m(\Psi_{T_k - T_{k-1}}^Y |_{\Delta^s(X_{T_{k-1}})(x)}) - \log m(\Psi_{T_k - T_{k-1}}^Y |_{\Delta^u(X_{T_{k-1}})(x)}) \leq -2\eta,$$

for $k = 1, 2, \ldots, l$. 

\[\square\]
Liao [10] proved the following shadowing lemma which says that any quasi hyperbolic orbit arc with close enough end points can be shadowed by a hyperbolic periodic orbit. Very recently, Han and Wen [6] generalized the Liao’s result for a sequence of quasi-hyperbolic strings for flows.

**Proposition 2.7:** Let \( \Lambda \) be a compact invariant set of \( X_t \) without singularities. Assume that there exists a continuous invariant splitting \( N_{\Lambda} = \Delta^s \oplus \Delta^u \) with \( \dim \Delta^s = p \), \( 1 \leq p \leq \dim M - 2 \). Then for any \( \eta > 0 \), \( T > 0 \), and \( \varepsilon > 0 \), there exists \( \delta > 0 \) such that if \( (x, \tau) \) is an \( (\eta, T, p) \)-quasi hyperbolic orbit arc of \( X_t \) with respect to the splitting \( \Delta^s \oplus \Delta^u \) and \( d(X_\tau(x), x) < \delta \) then there exists a hyperbolic periodic point \( y \in M \) and an orientation preserving homeomorphism \( g : [0, \tau] \to \mathbb{R} \) with \( g(0) = 0 \) such that \( d(X_{g(t)}(y), X_t(x)) < \varepsilon \) for any \( t \in [0, \tau] \) and \( X_{g(\tau)}(y) = y \).

### 3. From robust shadowing to dominated splitting

In this section, we prove that if a nontrivial chain transitive subset \( \Lambda \) of \( X_t \) is robustly shadowable, then it admits a dominated splitting. First, we show that any continuation \( \Lambda_Y \) of \( \Lambda \) does not contain both a non-hyperbolic singularity and a non-hyperbolic periodic orbit. Next, we show that \( \Lambda \) does not contain a singularity. Finally, we prove that \( \Lambda \) admits a dominated splitting.

**Lemma 3.1:** Let \( \Lambda \) be a chain transitive set of \( X_t \). If \( \Lambda \) is robustly shadowable, then it is transitive.

**Proof:** The proof is straightforward. □

Using the perturbation technique developed by Pugh and Robinson [13], Pilyugin and Tikhomirov [11] showed that if \( M \) is robustly shadowable for \( X_t \) then there is a \( C^1 \)-neighbourhood \( U \) of \( X \) such that for any \( Y \in U \), every critical element of \( Y_t \) is hyperbolic. In the Proposition 3.2, we prove that any continuation \( \Lambda_Y \) of a robustly shadowable chain transitive set \( \Lambda \) does not contain both a non-hyperbolic singularity and a non-hyperbolic periodic orbit.

**Proposition 3.2:** Let \( \Lambda \) be a robustly shadowable set of \( X_t \). Then there exists a \( C^1 \)-neighbourhood \( U \) of \( X \) such that for any \( Y \in U \), every singularity and periodic orbit of \( Y_t \) in \( \Lambda_Y \) are hyperbolic for \( Y_t \).

**Proof:** Suppose \( \Lambda \) is a robustly shadowable set of \( X_t \). Then there exist a \( C^1 \)-neighbourhood \( U \) of \( X \) and a neighbourhood \( U_{\Lambda} \) of \( \Lambda \) such that for any \( Y \in U \), the continuation \( \Lambda_Y = \cap_{t \in \mathbb{R}} Y_t(U) \) is shadowable for \( Y_t \).

**Case 1:** Suppose that there is \( Y \in U \) such that \( \Lambda_Y \) contains a non-hyperbolic singularity \( \sigma \). By using the Taylor’s theorem, we may assume that in a neighbourhood of \( \sigma \) the dynamical system induced by \( Y \) is expressed by the following differential equation:

\[
\dot{x} = Ax + K(x),
\]

where \( A \in M_{n \times n}(\mathbb{R}) \) and \( K : \mathbb{R}^n \to \mathbb{R}^n \) is a continuous map satisfying

\[
\lim_{x \to 0} \frac{K(x)}{\|x\|^2} = 0.
\]
Since $\sigma$ is not hyperbolic, there is an eigenvalue $\lambda$ of $A$ with zero real part. First we assume that $\lambda = 0$. By changing coordinate, if necessary, we may assume that there is a $n \times n$-matrix $D$ close enough to $A$ such that

$$D = \begin{bmatrix} 0 & 0 \\ 0 & B \end{bmatrix},$$

where $B$ is a $(n - 1) \times (n - 1)$-matrix with real entries. We represent the coordinates of a point $x$ in a neighbourhood of $\sigma$ by $x = (y, z)$ with respect to $D$. Let $\varepsilon > 0$, and choose a real valued $C^\infty$ bump function $\beta : \mathbb{R} \to \mathbb{R}$ that satisfies the following conditions:

$$\begin{cases} 
\beta(x) \subset [0, 1] & \text{for } x \in \mathbb{R}, \\
\beta(x) = 0 & \text{for } |x| \geq \varepsilon, \\
\beta(x) = 1 & \text{for } |x| \leq \frac{\varepsilon}{4}, \\
0 \leq \beta'(x) < \frac{2}{\varepsilon} & \text{for } x \in \mathbb{R}.
\end{cases}$$

Define $\rho : \mathbb{R}^n \to \mathbb{R}$ by $\rho(x) = \beta(\|x\|)$. By taking $\varepsilon$ small enough, one can see that the vector field $Z$ obtained from the following differential equation:

$$\dot{x} = Dx + (1 - \rho(x))K(x)$$

is $C^1$-close to $Y$. Moreover, we have $B_{\varepsilon/4}(\sigma) \subset U$. Consequently we see that $Z \in U$, $\sigma \in \text{Sing}(Z) \cap \Lambda_Z$ and $\Lambda_Z$ is shadowable for $Z$. Since $\rho(x) = 1$ for $\|x\| < \frac{\varepsilon}{4}$, in the $\frac{\varepsilon}{4}$-neighbourhood of $\sigma$, the differential equation associated to $Z$ is given by

$$\begin{cases} 
\dot{y} = 0 \\
\dot{z} = Bz
\end{cases}.$$ 

By considering coordinates represented in Equation (1), for any $x = (y, z) \in B_{\varepsilon/4}(\sigma)$, we have

$$Z_t(x) = Z_t(y, z) = (y, \exp(Bt)z).$$

This implies that if $|y| \leq \frac{\varepsilon}{4}$ then $(y, 0) \in \text{Sing}(Z) \cap U$, and so $(y, 0) : |y| < \frac{\varepsilon}{4} \subset \Lambda_Z$. Let $\delta > 0$ be a corresponding constant from the definition of shadowing of $\Lambda_Y$ for $\frac{\varepsilon}{8}$. Choose $\alpha_0 = 0 < \alpha_1 < \cdots < \alpha_n = \frac{\varepsilon}{2}$ such that $|\alpha_i - \alpha_{i-1}| < \delta$ for $i = 1, \ldots, n$. Let

$$x_i = (y_i, z_i) \text{ and } t_i = 1 \text{ for } i = 1, \ldots, n.$$ 

Clearly $\{(x_i, t_i) \mid i = 0, \ldots, n\}$ is a finite $\delta$-pseudo orbit of $Z_t$ in $\Lambda_Z$. Since $x_0$ and $x_n$ are singularities, we can put

$$x_i = x_0, \ t_i = 1 \text{ for } i \leq 0; \text{ and } x_i = x_n, \ t_i = 1 \text{ for } i > n.$$
Then \( \{ (x_i, t_i) \mid i \in \mathbb{Z} \} \) is a \( \delta \)-pseudo orbit of \( Z_t \) in \( \Lambda_Z \). Since \( \Lambda_Z \) is shadowable, there are \( (y, z) \in M \) and a reparametrization \( h \) such that

\[
d(X_{h(t)}(y, z), x_0 * t) < \frac{\varepsilon}{8}
\]

for all \( t \in \mathbb{R} \). This implies \( O(y) \subset B_{\frac{\varepsilon}{8}}(0) \). Since the intersections of planes formulated by \( \{ (y, z) \mid y = c \} \) with \( B_{\frac{\varepsilon}{8}}(0) \) are invariant (\( c \) is a constant), there is \( c_0 \in (-\frac{\varepsilon}{8}, \frac{\varepsilon}{8}) \) such that \( O(y) \subset \{ (y, z) \mid y = c_0 \} \). Without loss of generality, we may assume \( c_0 = 0 \). Then we get a contradiction since \( d(X_t(y, z), x_1) \geq \frac{\varepsilon}{4} \) for all \( t \in \mathbb{R} \).

Suppose that \( \lambda = ib \) for some non-zero \( b \in \mathbb{R} \). By the same techniques as above, we can construct a vector field \( Z \) which is \( C^1 \)-close to \( Y \) and in a neighbourhood of \( \sigma \), the differential equation associated to \( Z \) is given by

\[
\dot{x} = Ax = \begin{bmatrix} C & 0 \\ 0 & B \end{bmatrix} \begin{bmatrix} y \\ z \end{bmatrix},
\]

where \( C = \begin{bmatrix} \cos(b) & \sin(b) \\ -\sin(b) & \cos(b) \end{bmatrix} \). By considering the coordinates obtained from Equation (2) in the \( \frac{\varepsilon}{4} \) neighbourhood of \( \sigma \), we can see that every point \( x = (y_1, y_2, 0) \) is periodic. Since the intersections of cylinders formulated by \( \{ (y_1, y_2, z) \mid y_1^2 + y_2^2 = c, c \in \mathbb{R} \} \) and \( B_{\frac{\varepsilon}{8}}(\sigma) \) are invariant, we can derive a contradiction by using the same techniques as above.

**Case 2:** Suppose that there is \( Y \in \mathcal{U} \) such that \( \Lambda_Y \) contains a non-hyperbolic periodic orbit \( \gamma \). Let \( p \in \gamma \), and denote the period of \( \gamma \) by \( \pi(p) \). Then the linear Poincaré map \( \Psi_{\pi(p)}: N_p \to N_p \) has an eigenvalue of modulus 1. Hence, we can find a linear map \( P: N_p \to N_p \) arbitrarily close to \( \Psi_{\pi(p)} \) that has an eigenvalue \( \lambda \) of modulus 1, the multiplicity of \( \lambda \) is 1, and \( \lambda \) is a root of unity (i.e. \( \lambda^n = 1 \) for some \( n \in \mathbb{N} \)). Using Lemma 2.3, we may assume that \( \Psi_{\pi(p)} = P \). By changing the coordinates in \( N_p \), if necessary, we may assume that

\[
\Psi_{\pi(p)} = \begin{bmatrix} C & 0 \\ 0 & B \end{bmatrix}
\]

and \( C = \lambda w \) for some \( (w, 0) \in N_p \), where \( C \) is a \( 1 \times 1 \) (or \( 2 \times 2 \))-matrix. Choose \( r > 0 \) such that \( \hat{N}_r \subset U \) and the Poincaré map \( f_{p,\pi(p)}: \hat{N}_{x,r} \to \hat{N}_{p,1} \) is well defined. Since \( f_{p,\pi(p)} \) is a \( C^1 \) map, using the same techniques as in Case 1, we can find a map

\[
g_{p,\pi(p)}: \hat{N}_{x,r} \to \hat{N}_{p,1}
\]

which is arbitrarily \( C^1 \)-close to \( f_{p,\pi(p)} \) and \( \exp_p^{-1} \circ g \circ \exp_p \mid_{\hat{N}_{x,z}} = \Psi_{\pi(p)} \mid_{\hat{N}_{x,z}} \). By Proposition 2.1, we may assume that \( f_{p,\pi(p)} = g \).

By the tubular flow theorem for closed orbits in Section 2.5.2 in [1], we can find constants \( s, \delta_0, l > 0 \) such that if \( x \in \hat{N}_p \cap B_s(p), y \in M \) and \( \varepsilon \in (0, \delta_0) \) then \( d(x, y) < \varepsilon \) implies \( y = Y_{\gamma'}(y') \), for some \( y' \in \hat{N}_p \) and \( |t'|, d(y', x) < ls \). Let \( \delta > 0 \) be a corresponding constant for \( \varepsilon < \min\{\delta_0, \frac{s}{4l} \} \) obtained from the shadowing property of \( \Lambda_Y \). Let \( v \) be a scalar multiplication of \( w \) which obtained in Equation (3) satisfying \( \|v\| = s \). To make a \( \delta \)-pseudo orbit,
fix $N > 0$ and define

$$x_i = \begin{cases} 
  p & i \leq 0, \\
  \exp_p\left(\frac{i}{N}C^0, 0\right) & 0 \leq i \leq N - 1, \\
  \exp_p(C^0, 0) & i \geq N,
\end{cases}$$

and $t_i = \tau(x_i)$, where $\tau$ is the first return map. Then we get

$$d(X_{t_i}(x_i), x_{i+1}) \leq \frac{i}{N}C^1 - \frac{i + 1}{N}C^1, \quad i \leq l \leq \frac{i + 1}{N}C^1, \quad i \geq N,$$

for sufficient large $N$. Since $\lambda^n = \lambda^n v = v$, we see that each $\{x_i\}$ is periodic and $O(x_i) \subset U$ for all $i \in \mathbb{Z}$. Consequently, we get $x_i \in \Lambda_Y$ for all $i \in \mathbb{Z}$. Since $\Lambda_Y$ satisfies the shadowing property, there are $x \in M$ and $h \in \text{Rep}$ such that

$$O(x) \subset B_{\varepsilon}(\gamma) \text{ and } d(X_{h(t)}(x), x_0 \ast t) < \varepsilon.$$

for all $t \in \mathbb{R}$. Hence, there are $t_1, t_2 \in \mathbb{R}$ such that

$$d(Y_{t_1}(x), p) < \varepsilon \text{ and } d(Y_{t_2}(x), x_N) < \varepsilon.$$

By the above fact, we can choose $t_1'$ and $t_2'$ in $\mathbb{R}$ such that

$$d(Y_{t_1'}(x), p) < \varepsilon < \frac{s}{4}, \quad d(Y_{t_2'}(x), x_N) < \varepsilon < \frac{s}{4}, \text{ and } Y_{t_1'}(x), Y_{t_2'}(x) \in \hat{N}_p. \quad (4)$$

Suppose that

$$Y_{t_1'}(x) = \exp_p(v_1, w_1) \text{ and } Y_{t_2'}(x) = \exp_p(v_2, w_2).$$

Then Equation (4) implies that

$$\| (v_1, w_1) \| < \frac{s}{4} \text{ and } \| (v_2, w_2) - (C^N v, 0) \| < \frac{s}{4}. \quad (5)$$

Moreover, we see that $(v_1, w_1)$ and $(v_2, w_2)$ belongs to the same orbit of $\Psi_{\pi(p)}$. Hence, without loss of generality, we may assume that there is $j \in \mathbb{N}$ such that $v_1 = C^j v_2$. Consequently, we get

$$\| v_1 \| = \| C^j v_2 \| = \| v_2 \|.$$

But Equation (5) implies that

$$\| v_1 \| < \frac{s}{4} \text{ and } \| v_2 - C^N v \| < \frac{s}{4}.$$

On the other hand, we have $\| C^n v \| = \| v \| = s$, and so the contradiction completes the proof of our proposition. \qed
Recently, Gan et al. [4] showed that if $M$ is robustly shadowable for $X \in \mathcal{X}^1(M)$, then there is no singularity $\sigma \in \text{Sing}(X)$ exhibiting homoclinic connection. Here the homoclinic connection is the closure of a orbit of a regular point which is contained in both the stable and the unstable manifolds of $\sigma$.

Proposition 3.3: Let $\Lambda$ be a nontrivial chain transitive set of $X$. If $\Lambda$ is robustly shadowable, then it does not contain a singularity of $X$.

Proof: Let $U$ be an isolating block of $\Lambda$, and suppose $U$ contains a singularity $\sigma$. By Proposition 3.2, it must be hyperbolic.

First we show that there is $z \in W^s(\sigma) \cap W^u(\sigma)$ such that

$$\Gamma := \{\sigma\} \cup O(z) \subset \Lambda.$$ 

Choose $x \in \Lambda \\setminus\{\sigma\}$, and let $\eta > 0$ be a constant to ensure that the local stable manifold $W^s_0(\sigma)$ and the local unstable manifold $W^u_0(\sigma)$ of $\sigma$ are embedded sub-manifolds of $M$. Take $\delta_0 > 0$ satisfying $\bigcup_{y \in \Lambda} B_{\delta_0}(y) \subset U$. Let $\delta > 0$ be a corresponding constant for $\varepsilon = \min\left(\frac{\eta}{2}, \frac{\delta_0}{2}, \frac{d(\sigma, x)}{2}\right)$ obtained from the shadowability of $\Lambda$. Since $\Lambda$ is transitive, there are two finite $\delta$-pseudo orbits in $\Lambda$

$$\{(x_i', t_i') \mid t_i' \geq 1, i = 1, \ldots, n\} \text{ and } \{(x_i'', t_i'') \mid t_i'' \geq 1, i = 1, \ldots, m\}$$

such that $x_0' = x_m'' = \sigma$, and $x_n' = x_0'' = x$. Define an infinite $\delta$-pseudo orbit in $\Lambda$ as follows:

$$(x_i, t_i) = \begin{cases} 
(\sigma, 1) & i < 0, \\
(x_i', t_i') & 0 \leq i < n, \\
(x_i'', t_i'') & n \leq i < n + m, \\
(\sigma, 1) & i \geq n + m.
\end{cases}$$

Then there are $z \in M$ and $h \in \text{Rep}$ such that

$$d(X_{h(t)}(z), x_0 \ast t) < \varepsilon$$

for all $t \in \mathbb{R}$. This implies that there is $T > 0$ such that

$$d(X_t(z), \sigma) < \eta \text{ for all } t > T \text{ and } t < -T.$$ 

By our construction, we see that $z \in W^s(\sigma) \cap W^u(\sigma)$. Hence, we have

$$\sup_{t \in \mathbb{R}, y \in \Lambda} (d(X_t(z), y)) < \varepsilon < \frac{\delta_0}{2}.$$ 

This implies that $O(z) \subset U$ and $z \in \Lambda$.

Second, we show that there is $x \in W^s(\sigma) \cap W^u(\sigma)$ such that

$$\Gamma' := \{\sigma\} \cup O(x) \subset \Lambda \text{ and } x \notin O(z).$$
Let $\varepsilon > 0$ be such that $\bigcup_{x \in \Gamma} B_{\varepsilon}(x) \subset U$, and let $\delta$ be a corresponding constant for $\varepsilon$ obtained from the shadowing property of $\Lambda$. Since $z \in W^s(\sigma) \cap W^u(\sigma)$, there is $m \in \mathbb{N}$ such that

$$d(X_n(z), \sigma) < \frac{\delta}{2} \quad \text{and} \quad d(X_{-n}(z), \sigma) < \frac{\delta}{2}$$

for all $n \geq m$. Consider a $\delta$-pseudo orbit in $\Lambda$

$$(x_i, t_i) = \begin{cases} (X_i(z), 1) & \text{for } i \leq m, \\ (X_{i-2m}(z), 1) & \text{for } i > m. \end{cases}$$

Then there are $x \in M$ and $h \in \text{Rep}$ such that $d(X_{h(t)}(x), x_0^* t) < \varepsilon$. We also easily check that $x \in W^s(\sigma) \cap W^u(\sigma)$, $O(x) \subset U$ and $x \not\in O(z)$.

This implies that $\dim E^s = \dim W^s(\sigma) = k \geq 2$. By applying Lemma 3.5 in [4], we can assume that there is a dominated splitting $E^s = E^c \oplus E^{ss}$ such that $\dim E^c = 1$. We also perturb $\Gamma$ and $\Gamma'$ to make sure that

$$(\Gamma \cup \Gamma') \cap W^{ss}(\sigma) = \{\sigma\},$$

where $W^{ss}(\sigma)$ is the strong stable sub-manifold of $M$ tangent to $E^{ss}$. Furthermore, we may perturb that in a neighbourhood $V$ of $\sigma$, the dynamic induced by $X$ is expressed by the following differential equation:

$$\begin{bmatrix} \dot{x}^c \\ \dot{x}^{ss} \\ \dot{x}^u \end{bmatrix} = A \begin{bmatrix} x^c \\ x^{ss} \\ x^u \end{bmatrix} = \begin{bmatrix} B_1 & 0 & 0 \\ 0 & B_2 & 0 \\ 0 & 0 & C \end{bmatrix} \begin{bmatrix} x^c \\ x^{ss} \\ x^u \end{bmatrix},$$

(6)

where $B_1$, $B_2$ and $C$ are preserving the splitting $E^c \oplus E^{ss} \oplus E^u$. Here, the eigenvalues of $B_2$ and $C$ have negative and positive real parts, respectively, and the spectrum of $B_1 = \{\lambda_1\}$. For more details on these perturbations, see [4]. Since the dynamic on $V$ is induced by the differential equation $\dot{x} = Ax$, we can express every point $y$ in $V$ by $y = (y^c, y^{ss}, y^u)$ based on the coordinates obtained from $E^c \oplus E^{ss} \oplus E^u$. Then we get

$$X_t(y) = (X_t(y^c), X_t(y^{ss}), X_t(y^u)) = (e^{B_1 t} y^c, e^{B_2 t} y^{ss}, e^{C t} y^u).$$

(7)

Next, we are going to get some useful properties for $\Gamma \cup \Gamma'$ that helps us to complete the proof. Choose $x \in \Gamma'$ and $z_1, z_2 \in \Gamma$ satisfying

$$x, z_1 \in W^s(\sigma), \quad z_2 \in W^u(\sigma), \quad O^+(x) \cup O^+(z_1) \subset V, \quad \text{and} \quad O^-(z_2) \subset V.$$ 

Fix $r > 0$ and let $y \in \hat{N}_{x,r}$. Assume that there exists $t > 0$ such that

$$X_t(y) \in \hat{N}_{x_2,r} \text{ and } X_{[0,t]} \subset V.$$
For any \( y \in \hat{N}_{x,r} \), denote by \( \tau(y) \) the minimum of \( t \) with the above property (if such a \( t \) exists). Define a map \( P_r \) by

\[
P_r : \text{Dom}(P_r) \subset \hat{N}_{x,r} \rightarrow \hat{N}_{z_2,r}, \quad P_r(y) = X_{\tau(y)}(y).
\]

We show that there is \( r_0 > 0 \) such that \( \text{Dom}(P_r) \neq \emptyset \) for any \( r \in (0, r_0] \). Fix \( r_0 > 0 \) such that

\[
\bigcup_{t \geq 0} \{B_{r_0}(X_t(x)) \cup B_{r_0}(X_{-t}(z_2))\} \subset V.
\]

Let \( r \in (0, r_0] \), and take \( r_1 \in (0, r] \) such that \( d(X_t(y), x) < r_1 \) for \( y \in M \) and \( t > 0 \). Then there is \( t' \in [0, t] \) such that

\[
X_{t'}(y) \in \hat{N}_{x,r} \quad \text{and} \quad X_{[t', t]}(y) \subset B_r(x).
\]

If \( d(X_t(y), z_2) < r_1 \), then there is \( t' \in [t, \infty) \) such that

\[
X_{t'}(y) \in \hat{N}_{z_2,r} \quad \text{and} \quad X_{[t', t]}(y) \subset B_r(z_2).
\]

Let \( \delta > 0 \) be a corresponding constant for \( \frac{r_1}{2} \) obtained from the shadowing property of \( \Lambda \). Let \( m \in \mathbb{N} \) be such that

\[
d(X_m(x), \sigma) \leq \frac{\delta}{2} \quad \text{and} \quad d(X_{-m}(z_2), \sigma) < \frac{\delta}{2}.
\]

Consider the following \( \delta \)-pseudo orbit:

\[
(x_i, t_i) = \begin{cases} 
(x, 1) & i \leq m, \\
(X_{-m}(z_2), 1) & i \geq m + 1.
\end{cases}
\] (8)

Then there are \( y \in M \) and \( h \in \text{Rep} \) such that

\[
d(X_{h(t)}(y), x_0 \ast t) < \frac{r_1}{2}.
\]

This implies that there are \( 0 \leq t_1 < t_2 < t_3 < \infty \) such that

\[
d(X_{[h(t_1), h(t_3)]}(y), X_{[0,m]}(x)) < \frac{r_1}{2} \quad \text{and} \quad d(X_{[h(t_2), h(t_3)]}(y), X_{[-m,0]}(z_2)) < \frac{r_1}{2}.
\]

Hence, we have \( X_{[h(t_1), h(t_3)]}(y) \subset V \). Let \( t'_1, t'_2 \) be constants corresponding to \( h(t_1), h(t_3) \), respectively, obtained from the same way we get \( r_1 \). Then we get

\[
X_{t'_1}(y) \in \hat{N}_{x,r}, \quad X_{t'_2}(y) \in \hat{N}_{z_2,r}, \quad \text{and} \quad X_{[t'_1, t'_2]}(y) \subset \exp_{\sigma}(T_{h}M(1)).
\]

Consequently, we have \( y \in \text{Dom}(P_r) \) and so \( \text{Dom}(P_r) \neq \emptyset \).

Consider the following set:

\[
L = \{ (y^e, y^{\bar{h}}, y^w) \mid y^{\bar{h}} = 0 \} \subset \hat{N}_{z_2,r}.
\]
We will show that for any $\varepsilon > 0$ there is $r > 0$ satisfying

$$P_r(\hat{N}_{x,r}) \subset C_\varepsilon := \{(u, w) \in N_{z_2} \mid u \in L, \ w \in L^\perp, \ |w| \leq \varepsilon \ |u|\}. \tag{9}$$

Let $y \in \text{Dom}(P_r) \cap \hat{N}_{x,r}$. Since $P_r(y) \in \hat{N}_{z_2,r}$, we have

$$0 < \|z^u_2\| - r \leq \|P_r(y)^u\|$$

for sufficiently small $r > 0$. Using Equation (7), we get

$$\|P_r(y)^u\| \leq e^{C\tau(y)} \|y^u\|. \tag{10}$$

Hence $\tau(y) \to +\infty$ as $\|y^u\| \to 0$. On the other hand, we have

$$\frac{\|P_r(y)^s\|}{\|P_r(y)^c\|} \leq e^{b_2\|\tau(y)\|} \frac{\|y^s\|}{\|e_{s/c}(y)\|}. \tag{10}$$

Since $x^c \neq 0$, we get $y^f \not\to 0$ as $y \to x$. In addition, because $E^s \oplus E^u$ is a dominated splitting, the right side of Equation (10) tends to be zero as $\tau(y) \to +\infty$, and Equation (9) is proved.

Next, we perturb $X$ so that if $z_1 = X_{t'}(z_2)$ then $\Psi_t(L) \cap \Delta^s = \emptyset$, where $\Delta^s = N_{z_1} \cap T_{z_1} W^s(\sigma)$. If $\Psi_{t'}(L) \not\subset \Delta^s$ we have nothing to prove. Otherwise, let $u \in N_{z_1}$ be such that $u \not\in \Delta^s$. Fix $\alpha > 0$, and denote $u_\alpha = \alpha u + (1 - \alpha)v$, where $\Psi_{t'}(L) = \text{Span}\{v\}$. Then there is a linear map $H_\alpha : N_{z_1} \to N_{z_1}$ such that

$$H_\alpha(v) = u_\alpha \text{ and } \|H_\alpha\| \to 1 \text{ as } \alpha \to 0.$$ 

Define a map

$$\Psi' : N_{z_2} \to N_{z_1}, \ \Psi'(u) = H_\alpha \circ \Psi_{t'}(u).$$

Choose $\alpha > 0$ so small that we can use Lemma 2.3, and replace $\Psi_{t'}$ with $\Psi'$. Then we get

$$\Psi'(L) \cap \Delta^s = \text{Span}\{u_\alpha\} \cap \Delta^s = \{0\}.$$ 

Since the Poincaré map $f_{z_1,t} : \hat{N}_{z_2,r} \to \hat{N}_{z_1,r}$ is continuous, there is $\varepsilon > 0$ such that

$$f_{z_1,t}(C_r) \cap W^s(\sigma) \cap \hat{N}_{z_1,r} = \{z_1\},$$

where $C_r$ is defined in Equation (9). Let $r > 0$ be such that $r$ satisfies Equation (9) for $\varepsilon$, and let $\delta > 0$ be a corresponding constant for $\varepsilon' = \min\{r, \varepsilon, \eta\}$ obtained from the shadowing property of $\Lambda$. Consider the $\delta$-pseudo orbit (8) we constructed in the above. Then there are $y \in M$ and $h \in \text{Rep}$ such that

$$d(X_{h(t)}(y), x_0 \ast t) < \varepsilon'.$$
This implies that there are constants $0 < t_1 < t_2 < t_3 < t_4$ satisfying
\[
d(X_{h(t_1)}(y), x) < \varepsilon', \quad d(X_{h(t_1), h(t_2)}(y), X_{[0,m]}(x)) < \varepsilon', \\
d(X_{h(t_2), h(t_3)}(y), X_{[-m,0]}(z_2)) < \varepsilon', \quad d(X_{h(t_3), h(t_4)}(y), X_{[0,t]}(z_2)) < \varepsilon', \quad \text{and} \\
d(X_{h(t_4), \infty})(y), X_{[0,\infty]}(z_1)) < \varepsilon'.
\]

Without loss of generality, we may assume that
\[
X_{h(t_1)}(y) \in \hat{N}_{x,r}, \quad X_{t_1}(y) \in \hat{N}_{z_2,r}, \quad \text{and} \quad X_{h(t_4)}(y) \in \hat{N}_{z_1, r}.
\]

This means that $X_{t_1}(y) = P_t(X_{h(t_1)}(y))$, and so we have $X_{h(t_3)}(y) \in C_\varepsilon$. Consequently, we get
\[
X_{h(t_4)} \notin W^s(\sigma) \cap \hat{N}_{z_1, r}.
\]

This is a contradiction to the fact that
\[
d(X_{h(t_4), \infty})(y), X_{[0,\infty]}(z_1)) < \eta,
\]
and so completes the proof. \hfill \Box

**Proposition 3.4:** Let $\Lambda$ be a chain transitive set. If $\Lambda$ is robustly shadowable, then it admits a homogeneous dominated splitting for $\Psi_t$.

**Proof:** If $\Lambda$ is a periodic orbit, then it admits a dominated splitting for $\Psi_t$ by Proposition 3.2. Hence, we suppose $\Lambda$ is not a periodic orbit, and take a point $x \in \Lambda$ be such that $\omega(x) = \Lambda$. By applying the Pugh's closing lemma (see [13]), we can select a sequence $\{Y^n\}_{n \in \mathbb{N}} \subset \mathcal{U}$ converging to $X$ such that each $Y^n$ has a periodic point $p_n$ converging to $x$; and for each $t > 0$, the sequence $\phi_n: [0, t] \to M$ given by $\phi_n(s) = Y^n(p_n)$ converges to $\phi: [0, t] \to M$, $\phi(s) = X_s(x)$. Note that here $O(p_n)$ is hyperbolic for $Y^n$ for every $n$. Moreover, we can see that the period of $p_n$ tends to $\infty$ as $n \to \infty$. By applying Proposition 2.6, we can take $l > 0$ such that the linear Poincaré flow of $Y_n$ over $O(p_n)$ admits an $l$-dominated splitting. By taking a subsequence, if necessary, we may assume that there is $k \in \mathbb{N}$ such that $\text{ind}(p_n) = k$ for all $n \in \mathbb{N}$.

Let $\{x_k\}$ be a sequence in $\Lambda$ converging to $x$, and let $E(x_k)$ be an $m$-dimensional subspace of $T_{x_k}M$. We say that $E(x_k)$ converges to $E(x)$ if, for each $k$, there is a basis $\{e^1_k, \ldots, e^m_k\}$ of $E(x_k)$ and a basis $\{e^1, \ldots, e^m\}$ of $E(x)$ such that $e^i_k \to e^i$ for each $i = 1, \ldots, m$.

Put
\[
\lim_{n \to \infty} E^s_n(p_n) = \Delta^s(x) \quad \text{and} \quad \lim_{n \to \infty} E^u_n(p_n) = \Delta^u(x).
\]

For each $t > 0$, we denote by
\[
\lim_{n \to \infty} E^s_n(Y^n_t(p_n)) = \Delta^s_n(X_t(x)) \quad \text{and} \quad \lim_{n \to \infty} E^u_n(Y^n_t(p_n)) = \Delta^u_n(X_t(x)),
\]
where \( T^n_{r}(p_n) M = E^s_n(Y^n(p_n)) \oplus E^u_n(Y^n(p_n)) \). Then we have

\[
\Delta^s(X_t(x)) = \lim_{n \to \infty} \Delta^s_n(Y^n(p_n)) = \lim_{n \to \infty} \Psi^n_t(\Delta^s_n(p_n)) = \Psi_t(\Delta^s(X_t(x))), \quad \text{and}
\]

\[
\Delta^u(X_t(x)) = \lim_{n \to \infty} \Delta^u_n(Y^n(p_n)) = \lim_{n \to \infty} \Psi^n_t(\Delta^u_n(p_n)) = \Psi_t(\Delta^u(X_t(x))),
\]

where \( \Psi^n_t \) is the linear Poincaré flow for \( Y^n \). This means that the splitting \( \Delta^s(x) \oplus \Delta^u(x) \) is \( \Psi_t \) invariant, and we have \( \mathcal{N}_x = \Delta^s(x) \oplus \Delta^u(x) \). If \( t \) is sufficiently large, then we can see that

\[
\| \Psi_t \cdot \Delta^s(x) \\| = \lim_{n \to \infty} \| \Psi_t^n \cdot \Delta^s_n(x) \\| \leq \frac{1}{2}.
\]

This means that the orbit \( O(x) \) admits a dominated splitting for \( \Psi_t \), and so \( \Lambda = \overline{O(x)} \) also has a dominated splitting for \( \Psi_t \).

\[\square\]

4. From dominated splitting to hyperbolicity

**Lemma 4.1:** If a chain transitive set \( \Lambda \) of \( X_t \) is robustly shadowable, then it admits a hyperbolic periodic orbit.

**Proof:** Let \( \Delta^s \oplus \Delta^u \) be the \( l \)-dominated splitting of \( (T_{r}, M, \Psi_t|_{\mathcal{N}^s}) \) obtained in Proposition 3.4. By using Lemma 3.4 in [5] and Theorem 2.6 we may assume that \( \dim(\Delta^s) \leq \dim M - 2 \). Denote by

\[
\alpha = \min\{\| \Psi_t \cdot \mathcal{N}^s \| : z \in \Lambda, t \in [-3, 3]\}.
\]

For any \( \varepsilon > 0 \), choose \( \varepsilon' \in (0, \frac{\varepsilon}{2}) \), \( \delta' > 0 \), and \( Y \in U \) having a periodic point \( p \) such that

\[
\begin{align*}
\log(s + \varepsilon') &\leq \log(s) + \varepsilon, \quad \forall s \in \left[ \frac{e}{2}, \infty \right), \\
\log\left( \frac{1}{\varepsilon - \varepsilon'} \right) &\geq \log\left( \frac{1}{\varepsilon} \right) - \varepsilon, \quad \forall s \in \left[ \frac{e}{2}, \infty \right), \\
\| \Psi_t \cdot \Delta^u(z) \| - \| \Psi_t \cdot \Delta^s(y) \| &< \varepsilon', \quad \forall t \in [-3, 3], \ d(z, y) < \delta', \ z \in \Lambda, \ y \in O(p), \\
\delta_H(O(p), \Lambda) &< \delta',
\end{align*}
\]

(11)

where \( \Psi \) and \( \Psi' \) are linear Poincaré flows of \( X \) and \( Y \), respectively. Since \( p \) is a hyperbolic periodic point of \( Y_t \), there are \( C > 0 \) and \( \lambda \in (0, 1) \) such that

\[
\| \Psi_t \cdot \Delta^u(y) \| \leq C \lambda^t \text{ and } \| \Psi_t \cdot \Delta^s(y) \| \leq C \lambda^t
\]

for all \( t \geq 0 \) and \( y \in O(p) \). Denote by \( C' = \max\{C, C^{-1}\} \), and let \( \delta \) be a constant as in Proposition 2.7 for the triple \( (\varepsilon, T, \eta) = (\varepsilon, 1, -(\log(\varepsilon') + \varepsilon)) \). Because \( x \) is a non-wandering point, there is \( t' > 0 \) such that \( d(X_{t'}(x), x) < \delta \). Let \( T_0, \ldots, T_m \in \mathbb{R} \) be such that

\[
0 = T_0 < T_1 < T_2 < \cdots < T_m = t'
\]
is a partition for \([0, t']\) with \(T_{i+1} - T_i \in [1, 2]\). Let \(p_0, \ldots, p_m \in O(p)\) be such that
\[
d(p_j, XR_j(x)) < \delta' \quad \text{for} \quad j = 0, \ldots, m.
\]

We show that \(XR_{[0,t']} (x)\) is an \((\varepsilon, T, \eta)\)-quasi hyperbolic arc. By using Equation (11), we have
\[
\frac{1}{T_k} \sum_{j=1}^{k} \log \| \psi_{T_{j-1}} |_{\Delta^u(X_{T_{j-1}}(x))} \| \leq \frac{1}{T_k} \sum_{j=1}^{k} \log(\| \psi_{T_{j-1}} |_{\Delta^u(p_j)} \| + \varepsilon')
\]
\[
\leq \frac{1}{T_k} \sum_{j=1}^{k} (\log(\| \psi_{T_{j-1}} |_{\Delta^u(p_j)} \|) + \varepsilon) \leq \frac{1}{T_k} \sum_{j=1}^{k} \log(C T_{j-1}^T) + \frac{k}{T_k} \varepsilon
\]
\[
\leq \frac{1}{T_k} \sum_{j=1}^{k} \log(C T_{j-1}^T) + \frac{k}{T_k} \varepsilon \leq \log(C) + \varepsilon = -\eta.
\]

For the first and second inequality, we used the properties in Equation (11); for the third inequality, we used the hyperbolicity of \(O(p)\); and for the fourth and fifth inequality, we used the property \(T_{j} - T_{j-1} \geq 1\).

On the other hand, we have
\[
m(\psi_{j}' |_{\Delta^u_T(\gamma)}) = \frac{1}{\| \psi_{j}' |_{\Delta^u_T(\gamma)} \|} \geq C^{-1} \lambda^{-i} \geq C^{-1} \lambda^{-i}.
\]

Hence, we get
\[
\frac{1}{T_m - T_{k-1}} \sum_{j=k}^{m} \log m(\psi_{T_{j-1}} |_{\Delta^u(X_{T_{j-1}}(x))})
\]
\[
= \frac{1}{T_m - T_{k-1}} \sum_{j=1}^{k} \log \left( \frac{1}{\| \psi_{T_{j-1}} |_{\Delta^u(X_{T_{j-1}}(x))} \|} \right)
\]
\[
\geq \frac{1}{T_m - T_{k-1}} \sum_{j=1}^{k} \log \left( \frac{1}{\| \psi_{T_{j-1}} |_{\Delta^u(p_j)} \| - \varepsilon'} \right)
\]
\[
\geq \frac{1}{T_m - T_{k-1}} \sum_{j=1}^{k} \left( \log(\| \psi_{T_{j-1}} |_{\Delta^u(p_j)} \|) - \varepsilon) \right)
\]
\[
\geq \frac{1}{T_m - T_{k-1}} \sum_{j=k}^{m} \left( (T_{j-1} - T_{j}) (\log(C) + \log(\lambda)) \right) - \frac{m - k + 1}{T_m - T_{k-1}} \varepsilon
\]
\[
\geq -\log(C) - \varepsilon - \log(\lambda) \geq -(\log(C) + \varepsilon) = \eta.
\]
Similarly, we obtain

\[
\log \| \Psi_{T_k - T_{k-1}} \| \Delta^u(x_{T_k-1}(x)) \| - \log m \left( \Psi_{T_k - T_{k-1}} \| \Delta^u(x_{T_k-1}(x)) \right) \\
\leq \log(C') + (T_k - T_{k-1})\log(\lambda) + \varepsilon - \left( -\log(C') + (-T_k + T_{k-1})\log(\lambda) - \varepsilon \right) \\
= 2\log (C') + 2\varepsilon + 2(T_k - T_{k-1})\log(\lambda) \\
\leq 2\log (C') + 2\varepsilon = -2\eta,
\]

for all \( k \in \{1, \ldots, m \} \). Consequently, we can see that \( \Lambda \) contains a hyperbolic periodic orbit by Proposition 2.7.

In Section 5 of [8], the authors showed that if a chain component \( C_X(\gamma) \) satisfies a dominated splitting but not hyperbolic, then there exists a \( C^1 \) -perturbation \( Y \) of \( X \) such that the chain component \( C_Y(\gamma' \gamma) \) contains a periodic orbit which does not satisfy the assumptions of Proposition 2.6. To show this, they used several lemmas and propositions. Here, we can apply the same techniques in [8] for the case of non-trivial locally maximal chain transitive sets to get the similar results, and finally obtain the following proposition.

**Proposition 4.2:** Let \( \gamma \) be a hyperbolic periodic orbit, and \( \Lambda \) be a robustly shadowable basic set of \( X_t \), with an isolating block \( U \), such that \( \gamma \subset \Lambda \subset C_X(\gamma) \). If \( \Lambda \) satisfies a homogeneous \( l \)-dominated splitting \( \Delta^l \oplus \Delta^u \), and \( \Delta^l \) is not contracting for \( \Psi_t \), then for any constants \( T > 0 \) and \( \eta > 0 \), there exists a hyperbolic periodic point \( z \in U \) such that

1. \( O(z) \subset U \),
2. there are positive numbers \( 0 = t_0 < t_1 < \cdots < t_n \) with \( T < t_{k+1} - t_k \), \( k = 0, \ldots, n - 1 \) such that
\[
\frac{1}{t_n} \sum_{i=0}^{n-1} \log \| \Psi_{t_{i+1} - t_i} \|_{\Delta^l_{X_i}(x)} \| > -\eta.
\]
3. \( X_{t_n}(z) = z \).

**Proof:** See Proposition 5.11 in [8].

**End of proof of main theorem:** Let \( \Lambda \) be a chain transitive set, and suppose it is robustly shadowable. Then \( \Lambda \) contains a hyperbolic periodic orbit, say \( \gamma \), by Lemma 4.1. Since \( \Lambda \) is transitive, we see that \( \Lambda \subset C_X(\gamma) \) and also \( \Lambda \subset H_X(\gamma) \). Since \( \Lambda \) is compact and the periodic points are dense in \( \Lambda \), we may assume that for any \( T > 0 \), there is a periodic point \( p \) in \( \Lambda \) whose period is bigger than \( T \). Let \( \tilde{T}, \eta > 0 \) be constants as in Proposition 2.6. If \( \Delta^l \) is not contracting for \( \Psi_t \), by applying Proposition 4.2, we reach a contradiction. Similarly, we can show that \( \Delta^u \) is expanding. So the dominated splitting \( \mathcal{N}_\Lambda = \Delta^l \oplus \Delta^u \) is in fact a hyperbolic splitting for \( \Psi_t \). Consequently, we can see that \( \Lambda \) is hyperbolic for \( X_t \) by applying Proposition 2.5.

The converse is clear by the robust property of hyperbolic sets and the shadowability of the hyperbolic sets, and so completes the proof of our main theorem. \( \square \)
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