ON THE SPLITTING CONJECTURE IN THE HYBRID MODEL FOR THE RIEMANN ZETA FUNCTION

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Abstract. We show that the splitting conjecture in the hybrid model of Gonek–Hughes–Keating holds to order on the Riemann hypothesis. Our results are valid in a larger range of the parameter $X$ which mediates between the partial Euler and Hadamard products. We also show that the asymptotic splitting conjecture holds for this larger range of $X$ in the cases of the second and fourth moments.

1. Introduction

The moments of the Riemann zeta function have been the subject of several conjectural methods in recent years. Since the second and fourth moments of Hardy–Littlewood [23] and Ingham [33], it is only relatively recently that a full conjecture for all moments was given. This began with the work of Keating–Snaith [34] who used the now famous connection with random matrix theory to conjecture that for real $k > -1/2$,

$$\frac{1}{T} \int_T^{2T} |\zeta(\frac{1}{2} + it)|^{2k} dt \sim a(k)g(k)(\log T)^{k^2}$$

where

$$a(k) = \prod_p \left(1 - \frac{1}{p}\right)^{k^2} \sum_{m \geq 0} \frac{d_k(p^m)^2}{p^m}$$

and

$$(2) \quad g(k) = \frac{G(k+1)^2}{G(2k+1)}$$

where $G$ is the Barnes $G$-function. This was preceded by conjectures for the 6th and 8th moments due to Conrey–Ghosh [14] and Conrey–Gonek [15], respectively, using number theoretic methods. The Keating–Snaith conjecture has since been derived with various different approaches [13, 18, 22].

A drawback of Keating and Snaith’s method was that the arithmetic factor $a(k)$ had to be incorporated in an ad-hoc way since there was no input from primes in their random matrix theory model. This was remedied in the method of Gonek–Hughes–Keating (G–H–K) [22] which forms the main focus of this paper.
The first step of G–H–K’s method was to express the zeta function as the product of partial Euler and Hadamard products. Precisely, Theorem 1 of [22] states that for \( 2 \leq X \leq t^{1/3} \) and large \( t \),

\[
\zeta\left(\frac{1}{2} + it\right) = P_X\left(\frac{1}{2} + it\right)Z_X\left(\frac{1}{2} + it\right)
\left(1 + O\left(\frac{1}{\log X}\right)\right)
\]

where

\[
P_X(s) = \exp\left(\sum_{n \leq X} \frac{\Lambda(n)}{n^s \log n}\right), \quad Z_X(s) = \exp\left(-\sum_\rho U((s - \rho) \log X)\right)
\]

and

\[
U(z) = \int_0^\infty u(x)E_1(z \log x)dx
\]

with \( E_1(z) = \int_0^\infty e^{-w} dw/w \) and \( u(x) \) a smooth, non-negative function of mass 1 with support in \([e^{1-1/X}, e]\). To give a rough idea of these objects, note that from the support conditions on \( u \) we have \( U(z) \approx E_1(z) \). This has mass concentrated in the region \( z \ll 1 \) where we have the approximation \( E_1(z) \approx -\gamma - \log z \). Thus, roughly speaking, \( Z_X(s) \approx \prod_{|\Im(s) - \Im(\rho)| \leq 1/\log X} ((s - \rho)e^\gamma \log X). \) Also, from the definition of the von Mangoldt function and the Taylor series for the logarithm we find \( P_X(s) \approx \prod_{p \leq X}(1 - p^{-s} - 1) \). Therefore, we can indeed view \( P_X(s) \) and \( Z_X(s) \) as partial Euler and Hadamard products.

G–H–K then proceeded to compute the moments of the Euler product, showing that for \( X \ll (\log T)^{2-\epsilon} \),

\[
\frac{1}{T} \int_T^{2T} |P_X\left(\frac{1}{2} + it\right)|^{2k} dt \sim a(k)(e^\gamma \log X)^{k^2}, \quad k \in \mathbb{R}.
\]

They conjectured with random matrix theory that

\[
\frac{1}{T} \int_T^{2T} |Z_X\left(\frac{1}{2} + it\right)|^{2k} dt \sim g(k)\left(\frac{\log T}{e^\gamma \log X}\right)^{k^2}, \quad k > -1/2
\]

and then proved this in the cases \( k = 1, 2 \) for \( X \ll (\log T)^{2-\epsilon} \). In order to recover the Keating–Snaith conjecture they assumed that the moments of the product of \( P_X \) and \( Z_X \) should split as the product of moments.

**Conjecture 1** (Splitting conjecture, [22]). Let \( X, T \to \infty \) with \( X \ll (\log T)^{2-\epsilon} \). Then for fixed \( k > -1/2 \) we have

\[
\frac{1}{T} \int_T^{2T} |P_X\left(\frac{1}{2} + it\right)Z_X\left(\frac{1}{2} + it\right)|^{2k} dt \sim \frac{1}{T} \int_T^{2T} |P_X\left(\frac{1}{2} + it\right)|^{2k} dt \cdot \frac{1}{T} \int_T^{2T} |Z_X\left(\frac{1}{2} + it\right)|^{2k} dt.
\]
Their reasoning behind this conjecture was that since \( P_X \) and \( Z_X \) oscillate at different scales (\( 1/\log X \) vs. \( 1/\log T \)), their contributions should act independently and hence the moment should split to leading order. They verified this in the cases \( k = 1, 2 \) for \( X \ll (\log T)^{2-\epsilon} \). The methodology of the hybrid model has since been used in various different settings to acquire conjectures for all sorts of \( L \)-functions \([1, 8, 9, 10, 11, 19, 25]\). In all cases, an equivalent version of the splitting conjecture plays a key role.

In this paper we prove that the splitting conjecture holds to order on the Riemann hypothesis (RH). Furthermore, we can extend the range of \( X \) past \((\log T)^{2-\epsilon}\).

**Theorem 1.** Assume RH. Let \( \epsilon, k > 0 \) be fixed and suppose \( X, T \to \infty \) with \( X \leq (\log T)^{\theta_k - \epsilon} \) where \( \theta_k = 2 \sqrt{1 + 1/2|k|} \). Then

\[
\frac{1}{T} \int_T^{2T} |P_X(\frac{1}{2} + it)Z_X(\frac{1}{2} + it)|^{2k} dt \sim \frac{1}{T} \int_T^{2T} |P_X(\frac{1}{2} + it)|^{2k} dt \cdot \frac{1}{T} \int_T^{2T} |Z_X(\frac{1}{2} + it)|^{2k} dt.
\]

As mentioned, this holds in a range of \( X \) larger than originally conjectured. We can also extend the range of \( X \) in the asymptotic results \((4)\) and \((5)\), both unconditionally and on RH. This gives the following.

**Theorem 2.** The Splitting conjecture holds for \( k = 1, 2 \) in the range

\[
X \leq \frac{1}{10^4}(\log T)^2(\log_2 T)^2.
\]

Assuming RH, we may take

\[
X \leq \begin{cases} 
(\log T)^{\sqrt{6} - \epsilon} & \text{when } k = 1, \\
(\log T)^{\sqrt{5} - \epsilon} & \text{when } k = 2.
\end{cases}
\]

Our proofs utilise the recent developments in the theory of moments of \( L \)-functions due to Soundararajan [43], Harper [24] and Radziwiłł–Soundararajan [38]. These techniques were originally geared for upper bounds although they can be brought to bear on lower bounds too [27]. We highlight three main ideas.

The first is an innovation of Soundarajan [43]. This was to note that \( \log |\zeta(\frac{1}{2} + it)| \) can be bounded from above by a sum over primes alone since the zeros contribute negatively to this quantity (see Lemma 6 below and c.f. formula (3)). With this, \( |\zeta(\frac{1}{2} + it)| \) can be bounded from above by an Euler product of flexible length.

The second idea can be found in a paper of Radziwiłł [37] and features heavily in the later works of Harper [24] and Radziwiłł–Soundararajan [38]. It allows one to compute moments of Euler products provided one can restrict to a certain subset of \([T, 2T]\). For the purposes of this discussion we consider the example

\[
\exp \left( \sum_{p \leq Y} p^{-1/2-it} \right)
\]
with \( Y = T^{1/(\log \log T)^2} \). On the face of it, this is a very long Dirichlet polynomial. However, if we can restrict \( t \) to a subset of \([T, 2T]\) on which \(|\sum_{p \leq Y} p^{-1/2-it}| \leq V\) for a given \( V \), then we can truncate the exponential series effectively using the fact that

\[
(6) \quad e^z \sim \sum_{j=0}^{10V} \frac{z^j}{j!}
\]

for \(|z| \leq V\) and large \( V \). The choice of \( V \) is naturally dictated by the variance: setting \( V = \sum_{p \leq Y} p^{-1} \sim \log \log T\) we get a Dirichlet polynomial of length \( Y^{10V} = T^{10/\log \log T} \). This is now short and so the mean square is easily computed. Also, the exceptional set in this case is of small measure.

The final main input in the arguments of Harper and Radziwill–Soundararajan allows one to push the length of the prime sum up to \( Y = T^{\theta} \), for some fixed \( \theta > 0 \). This involves breaking the sum it into subsums of progressively smaller variance. A similar splitting has appeared in the work of Brun on the pure sieve (see Hooley’s refinement \([29]\)).

This circle of ideas has been used in a wide variety of different contexts recently. These include; short interval maxima of the Riemann zeta function \([2, 3, 4]\), unconditional bounds for the moments of zeta and \( L \)-functions \([21, 26, 27]\), value distribution of \( L \)-functions \([16, 30, 39]\), sign changes in Fourier coefficients of modular forms \([35]\), non-vanishing of central values of \( L \)-functions \([17]\) and equidistribution of lattice points on the sphere \([32]\). In our case, we use these ideas to prove the following.

**Proposition 1.** Let \( \epsilon > 0 \) and \( k \in \mathbb{R} \) be fixed. Suppose \( X \leq \eta_k (\log T)^2 (\log_2 T)^2 \) with \( \eta_k = \frac{1}{16k^2} - \epsilon \). Then

\[
\frac{1}{T} \int_T^{2T} |P_X\left(\frac{1}{2}+it\right)|^{2k} dt \sim a(k)(e^\gamma \log X)^{k^2}
\]

where \( a(k) \) is given by (1). Assuming RH, this holds for \( X \leq (\log T)^{\theta_k - \epsilon} \) with \( \theta_k = 2\sqrt{1+1/2k} \).

**Proposition 2.** Suppose \( X \leq \frac{1}{10^2}(\log T)^2 (\log_2 T)^2 \). Then for \( k = 1, 2 \) we have

\[
\frac{1}{T} \int_T^{2T} |Z_X\left(\frac{1}{2}+it\right)|^{2k} dt \sim g(k)\left(\frac{\log T}{e^\gamma \log X}\right)^{k^2}
\]

where \( g(k) \) is given by (2). Assuming RH we may take \( X \leq (\log T)^{\sqrt{5}-\epsilon} \) when \( k = 1 \) and \( X \leq (\log T)^{\sqrt{7}-\epsilon} \) when \( k = 2 \).
Proposition 3. Assume RH and let $\epsilon, k > 0$ be fixed. Suppose $X \leq (\log T)^{\theta_k - \epsilon}$ with $\theta_k$ as above. Then

$$\frac{1}{T} \int_T^{2T} |Z_X(\frac{1}{2} + it)|^{2k} dt \asymp \left( \frac{\log T}{\log X} \right)^{k^2}.$$  

Remark. The lower bound in Proposition 3 can be made unconditional provided $X \leq \eta_k(\log T)^2(\log_2 T)^2$. We say more on this in section 7.

Since $\int_T^{2T} |\zeta(\frac{1}{2} + it)|^{2k} dt$ is $\ll T(\log T)^{k^2}$ on RH [24] and $\gg T(\log T)^{k^2}$ unconditionally [27], Theorem 1 follows from Propositions 1 and 3 when combined with (3). Likewise, Theorem 2 follows on combining Propositions 1 and 2 and (3).

Using the case of $P_X$ as an example, we describe how the range of $X$ can be increased past $(\log T)^{2 - \epsilon}$. First of all, note that since

$$\sum_{n \leq X} \Lambda(n) n^{1/2 + it} \log n \ll (1 + o(1)) X^{1/2} \log X,$$

we can approximate $P_X(1/2 + it)^k$ with a Dirichlet polynomial of length $X^{20|k|X^{1/2}/\log X}$ by using (6) to truncate the exponential. If $X \ll (\log T)^{2 - \epsilon}$ then this is $T^{o(1)}$ and so we have a short Dirichlet polynomial. Note this holds for all $t \in \mathbb{T}$ since the bound (7) is pointwise. G–H–K computed a Dirichlet polynomial approximation in a slightly different way, although in order for it to be short they required the same bound on $X$, perhaps unsurprisingly.

If $X$ is larger, then in order to have a short Dirichlet polynomial we must restrict to a subset of $[T, 2T]$ and in this case we need good bounds on the exceptional set. Typically, one would expect Gaussian bounds of the shape

$$\frac{1}{T} \mu \left( \left\{ t \in [T, 2T] : |\Re \sum_{n \leq X} \frac{\Lambda(n)}{n^{1/2 + it} \log n} | \geq V \right\} \right) \ll \exp \left( - \frac{V^2}{\log \log T} \right)$$

in a wide range of $V$. In practice we are limited to $V \ll \sqrt{(\log T)(\log_2 T)/\log X}$ which may be much smaller than the maximum $2X^{1/2}/\log X$. For the remaining range of $V$ one must settle for weaker bounds. For example, in [43] it is shown that the tails of $\log |\zeta(1/2 + it)|$ can be bounded by $e^{-V \log V}$ when $V \gg \log_2 T \log_3 T$. We can show that the tails of our sum satisfy the same bound in the range $\log_2 T \log_3 T \leq V \leq 2X^{1/2}/\log X$ provided $X \ll (\log T)^2$. However, for our purposes the weaker bound of $e^{-AV}$ with large $A$ is sufficient and this affords us slightly more room in the size of $X$.

Another avenue for improvement is to reduce the trivial bound in (7). This becomes a manageable task under RH and thus we are able to make further gains in the size of $X$ under this assumption. We shall prove the following.
Theorem 3. Assume RH. Then for large \( t \in [T, 2T] \) and \( 2(\log T)^2 \leq X \leq T \) we have

\[
\left| \sum_{n \leq X} \Lambda(n) \frac{\Lambda(n)}{n^{1/2+it} \log n} \right| \leq \left( \frac{1}{2} + o(1) \right) \left( \log \left( \frac{X^{1/2}}{\log T} \right) + 4 \log \log X \right) \frac{\log T}{\log \log T}.
\]

For the imaginary part we can replace the factor \( \frac{1}{2} + o(1) \) by \( \frac{1}{\pi} + o(1) \).

The factor of \( \frac{1}{2} + o(1) \) here is related to the function \( S(t) \) and can be read as \( 2c \) where \( c \) is a permissible constant in the bound \( S(t) \leq (c + o(1)) \log t / \log \log t \). The current best is due to Carneiro–Chandee–Milinovich [12] who give \( c = 1/4 \). Our approach to Theorem 3 is to relate the sum with \( S(t) \) via contour integrals and then input these bounds. We have not made attempts to further optimise this argument but it would be interesting to see if one could use the extremal function machinery of Carneiro et al. in a more direct way.

Regarding further improvements in the size of \( X \), if the conjectural bound \( S(t) \ll \sqrt{\log t \log_2 t} \) of Farmer–Gonek–Hughes [20] holds, then one could take \( X \leq \exp(C'(\log t)^{1/4}) \). Also, assuming that the bounds for the exceptional set in (8), or some minor variant of this, hold in the full range of \( V \) for a given \( X \), then our arguments can reproduce Theorems 1 and 2 for \( X \) as large as \( T^{1/C} \log \log T \). This supports the view of G–H–K that the splitting conjecture may hold as long as \( X = o(T) \).

The paper is organised as follows. We first prove Theorem 3 in section 2 and then the asymptotic results of Propositions 1 and 2 in sections 3 and 4, respectively. In section 5 we describe some tools for later use. Then in section 6 we prove the upper bound of Proposition 3 and in sections 7 and 8 we prove the lower bound in the cases \( 0 \leq k \leq 1 \) and \( k \geq 1 \), respectively.

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2. Bounds for prime sums: Proof of Theorem 3

We first give a lemma which relates our prime sum to \( S(t) \). This shares some similarities with previous convolution formulas from the literature [42, 45].

Lemma 1. Assume RH. For large \( t \in [T, 2T] \) and \( 2 \leq X \leq T, Y \leq T/2 \), we have

\[
\sum_{n \leq X} \Lambda(n) \frac{\Lambda(n)}{n^{1/2+it} \log n} = \int_{t-Y}^{t+Y} S(y) \frac{1 - X^{-i(t-y)}}{t-y} dy + E(X, Y, T)
\]

\[1\)In fact, anything of the form \( e^{-AV} \) with large \( A \) would be sufficient.
where
\[ E(X, Y, T) \ll \frac{X^{1/2}(\log X + \frac{\log T}{\log X})}{Y} + \frac{\log T}{\log_2 T} \left( \frac{1}{Y^{1/2}} + \frac{X^{C/\log_2 T}}{Y} \mathbb{1}_{X>(\log T)^A} \right) \]

and \( A \) is a large constant.

**Proof.** By Perron’s formula (Lemma 3.19, [44]) we have
\[ \sum_{n \leq X} \frac{\Lambda(n)}{n^{1/2+it} \log n} = \frac{1}{2\pi i} \int_{1/2+1/\log X-iY}^{1/2+1/\log X+iY} \log \zeta(z + \frac{1}{2} + it) X^z \frac{dz}{z} + O\left(\frac{X^{1/2} \log X}{Y}\right). \]

We shift the contour to the line with real part \( \Re(z) = 1/\log X \). Restricting \( Y \ll T/2 \) we don’t encounter any poles. In the region \( 1/2 + C/\log \log \tau \leq \sigma \leq 1 \), \( \tau \gg 1 \), we have
\[ \log \zeta(\sigma + it) \ll \frac{(\log \tau)^{2-2\sigma} \log_3 \tau}{\log_2 \tau}. \]

This follows from (14.14.5) of [44], the Phragmen–Lindelöf principle and the bound \( \log \zeta(\sigma + it) \ll \log_3 \tau, \sigma \geq 1 \) (the latter can be deduced from the proof of Theorem 14.8 of [44]). From Theorem 14.14 (B) of [44] we have
\[ \log \zeta(\sigma + it) \ll \frac{\log \tau}{\log_2 \tau} \log \left( \frac{2}{(\sigma - 1/2) \log_2 \tau} \right), \quad \frac{1}{2} < \sigma \leq \frac{1}{2} + C/\log_2 \tau. \]

Therefore, the horizontal contours contribute
\[ \ll X^{C/\log_2 T} \log \left( \frac{\log X}{\log_2 T} \right) \log T \mathbb{1}_{X>(\log T)^A} + \frac{X^{1/2} \log_3 T}{Y \log_2 T \log(\frac{X^{1/2} \log X}{T^{2)}}, 1) + \frac{X^{1/2} \log_3 T}{Y \log X}. \]

By formula (14.10.5) of [44] (see also (14.12.4) there) we have
\[ \log \zeta(z + \frac{1}{2} + it) = i \int_{t/2-Y}^{2t+Y} \frac{S(y)}{z + i(t-y)} dy + O\left(\frac{\log T}{T}\right). \]

This implies
\[ \ll \frac{1}{2\pi i} \int_{1/\log X-iY}^{1/\log X+iY} \log \zeta(z + \frac{1}{2} + it) X^z \frac{dz}{z} = i \int_{t/2-Y}^{2t+Y} S(y) I(t-y) dy + O\left(\frac{(\log T)^2}{T}\right). \]

where
\[ I(t-y) = \frac{1}{2\pi i} \int_{1/\log X-iY}^{1/\log X+iY} \frac{X^z}{z(z + i(t-y))} dz. \]

A trivial estimate gives
\[ I(t-y) \ll \int_{-Y}^{Y} \frac{dx}{(1/\log X + |x|)(1/\log X + |x + t - y|)} \ll \log X. \]
On the other hand, shifting the contour to the left we find

\[(12) \quad I(t-y) = \frac{1 - X^{-i(t-y)}}{i(t-y)} \mathbb{1}_{|t-y| \leq Y} + \frac{1}{i(t-y)} \mathbb{1}_{|t-y| > Y} + O\left(\frac{1}{Y|Y - |t-y|| \log X}\right).\]

For a given \(\delta > 0\) to be chosen, we find by (11) that

\[
\int_{|t-y| \leq \delta} S(y)I(t-y)dy \ll \delta \log X \log T / \log_2 T
\]

since \(S(\tau) \ll \log \tau / \log_2 \tau\). In the integral over the remaining region, the error term of (12) contributes

\[
\ll \frac{\log T}{\delta Y \log X \log_2 T} + \frac{\log Y \log T}{Y \log X \log_2 T}
\]

after considering the regions \(\delta < |Y - |t-y|| \leq 1\) and \(1 < |Y - |t-y||\) separately. Therefore, on choosing \(\delta = 1/(Y^{1/2} \log X)\) we find that the integral on the right of (10) is

\[
i \int_{t/2 - Y \leq y \leq 2t + Y} S(y) \left(\frac{1 - X^{-i(t-y)}}{i(t-y)} \mathbb{1}_{|t-y| \leq Y} + \frac{1}{i(t-y)} \mathbb{1}_{|t-y| > Y}\right)dy + O\left(\frac{\log T}{Y^{1/2} \log_2 T}\right).
\]

Integrating by parts along with the bound \(S_1(\tau) \ll \log \tau / (\log_2 \tau)^2\), we find that the second term in this integral is \(\ll \log T / (Y \log_2 T)^2\) which can be absorbed into the error term immediately above. The range of integration of the first term can be extended to \(|t-y| \leq Y\) at the cost of an error \(\ll \log T / (Y \log_2 T)\). Combining this in (10) along with the error terms of (9) the result follows.

\[\square\]

**Proof of Theorem 3.** We apply Lemma 1 with

\[Y = \frac{X^{1/2}(\log X)^{2+\epsilon}}{\log T}.\]

With this choice we have \(E(X, Y, T) = o(\log T / \log \log T)\) since \(\log X \gg \log_2 T\). Therefore, on taking real parts in Lemma 1 we find

\[\Re \sum_{n \leq X} \frac{\Lambda(n)}{n^{1/2+it} \log n} = \int_{-Y}^{Y} S(t + y) \frac{1 - \cos(y \log X)}{y} dy + o(\log T / \log_2 T).\]

From [12] we have

\[|S(\tau)| \leq (1 + o(1)) \frac{\log \tau}{4 \log \log \tau}.\]
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for large $\tau$ and so

$$\left| \Re \sum_{n \leq X} \frac{\Lambda(n)}{n^{1/2+it} \log n} \right| \leq (1 + o(1)) \frac{\log T}{4 \log \log T} \int_{-Y}^{Y} \frac{1 - \cos y}{|y|} dy.$$ 

The integral here is

$$2 \int_{-\pi}^{Y \log X} \frac{1 - \cos y}{y} dy + O(1) \leq 2 \sum_{n=1}^{Y \log X} \frac{1}{\pi n} \int_{\pi n}^{\pi(n+1)} (1 - \cos y) dy + O(1)$$

$$= 2 \log(Y \log X) + O(1).$$

Thus, we acquire

$$\left| \Re \sum_{n \leq X} \frac{\Lambda(n)}{n^{1/2+it} \log n} \right| \leq (1 + o(1)) \frac{\log(Y \log X) \log T}{2 \log \log T}$$

and the result follows on inputting our choice of $Y$.

In the case of the imaginary part, we acquire the integral

$$\int_{-\pi}^{\pi} \frac{1 - \cos y}{y} dy$$

which, on following the same argument, is $\leq (4/\pi) \log(Y \log X) + O(1)$. $\square$

From the proof we see that the factor $\log(X^{1/2} / \log T)$ comes from the divergent integral $\int_{-\pi}^{\pi} |1 - X^{-iy}| / |y| dy$. One may then wonder if smoothing would help here, that is, if the problem could be modified so that we consider a smoothed sum instead; say

$$\sum_{n \leq X} \frac{\Lambda(n)}{n^{1/2+it} \log n} \left( 1 - \frac{\log n}{\log X} \right).$$

The imaginary part responds well to this procedure and the above proof recovers the formulas of Selberg [42] and Tsang [45] which have the convergent integrand $\sin^2(y \log X) / y^2$. Unfortunately, for the real part the integrand is again of the form $\approx 1/|y|$ owing to large negative values of $\Re \log \zeta(1/2 + it)$ (cf. Lemma 5 of [45]).

3. MOMENTS OF THE EULER PRODUCT: PROOF OF PROPOSITION 1

Recall that

$$P_X(s) = \exp \left( \sum_{n \leq X} \frac{\Lambda(n)}{n^s \log n} \right).$$

Here and throughout the paper we consider the following subsets of $[T, 2T]$ on which the sum in the exponential attains typical values. For $V \geq 0$ set

$$S_R(V) = \left\{ t \in [T, 2T] : \left| \Re \sum_{n \leq X} \frac{\Lambda(n)}{n^{1/2+it} \log n} \right| \leq V \right\},$$
and define the set relating to the imaginary part $S_\Im(V)$ similarly. Let
\[ S(V) = S_R(V) \cap S_\Im(V) \]
and denote
\[ V_0 = \log_2 T \log_3 T, \quad S = S(V_0). \]
Define the complementary sets by
\[ E_R(V) = S_\Re(V), \quad E_\Im(V) = S_\Im(V), \quad E(V) = S(V), \quad E = S \]
where $A = [T, 2T] \setminus A$ for a given set $A$. Also, let
\[ V_{\text{max}} = \max_{t \in [T, 2T]} \left( |\Re \sum_{n \leq X} \frac{\Lambda(n)}{n^{1/2+it} \log n}|, |\Im \sum_{n \leq X} \frac{\Lambda(n)}{n^{1/2+it} \log n}| \right). \]
Note that unconditionally $V_{\text{max}} \leq (2+o(1))X^{1/2}/\log X$ and that on the Riemann hypothesis $V_{\text{max}} \leq (1/2 + o(1))(\log(X^{1/2}/\log T) + O(\log_2 X)) \log T/\log_2 T$ by Theorem 3. The reason we work with the real and imaginary parts (as opposed to working with the modulus directly) is so that we have slightly better conditional bounds for $V_{\text{max}}$ whilst still maintaining control over the modulus (which is important for Lemma 5 below). This gives better exponents for our logarithms in the conditional results but entails slightly more work.

To the estimate the measure of the complementary sets we use the following.

**Lemma 2 ([43]).** Let $T$ be large and let $2 \leq x \leq T$. Let $m$ be a natural number such that $x^m \leq T/\log T$. Then for any complex numbers $a(p)$ we have
\[ \frac{1}{T} \int_T^{2T} \left| \sum_{p \leq x} \frac{a(p)}{p^{1/2+it}} \right|^{2m} dt \ll m! \left( \sum_{p \leq x} \left| \frac{a(p)}{p} \right|^2 \right)^m. \]

**Lemma 3.** Let $\epsilon, \kappa > 0$. Then for $X \leq \eta_\kappa (\log T)^2(\log_2 T)^2$ with $\eta_\kappa = \frac{1}{16\kappa^2} - \epsilon$ we have
\[ \mu(E_R(V)) \ll T e^{-(2\kappa+o(1))V}, \quad V_0 \leq V \leq V_{\text{max}} \]
where $\mu$ denotes Lebesgue measure. Assuming RH, the same bound holds provided $X \leq (\log T)^{\theta_\kappa - \epsilon}$ where
\[ \theta_\kappa = 2 \sqrt{1 + \frac{1}{2\kappa}}. \]

The same results hold for $E_\Im(V)$ also.

**Proof.** We first prove the unconditional result. Write
\[ \sum_{n \leq X} \frac{\Lambda(n)}{n^{1/2+it} \log n} = \sum_{p \leq X} \frac{1}{p^{1/2+it}} + \sum_{p \leq \sqrt{X}} \frac{1}{2p^{1+2it}} + O(1). \]
Then from Jensen’s inequality in the form \((a + b + c)^{2m} \leq 3^{2m-1}(a^{2m} + b^{2m} + c^{2m})\) with \(m \geq 1\), we have

\[
\mu(\mathcal{E}_R(V)) \leq \frac{3^{2m-1}}{V^{2m}} \left( \int_T^{2T} \left| \sum_{p \leq X} \frac{1}{p^{1/2+it}} \right|^{2m} \, dt + \int_T^{2T} \left| \sum_{p \leq \sqrt{X}} \frac{1}{2p^{1/2+2it}} \right|^{2m} \, dt + O(TC^{2m}) \right).
\]

By Lemma 2 the right hand side is

\[
\ll T^{3^{2m}m!} \left( \sum_{p \leq X} \frac{1}{p} \right)^m \ll T^{m^{1/2}} \left( \frac{9m \log \log X}{eV^2} \right)^m
\]

provided \(m \leq (\log T - \log_2 T)/\log X\). Choosing

\[
m = \frac{2\kappa V}{\log V}
\]

the bound \(\ll Te^{-(2\kappa+o(1))V}\) follows since \(\log \log X \leq V^{o(1)}\). Note that our choice of \(m\) is legal since

\[
m \leq \frac{2\kappa V_{\text{max}}}{\log V_{\text{max}}} \sim \frac{8\kappa X^{1/2}}{(\log X)^2} \leq \frac{4\kappa \eta^{1/2} \log T}{\log X} \leq (1 - \epsilon) \frac{\log T}{\log X}
\]

where \(\ll\) means \(\leq\) times a constant of the form \(1 + o(1)\).

Assuming RH, then on writing \(\theta_\kappa = 2 + \nu_\kappa\) we find

\[
V_{\text{max}} \leq \left( \frac{1}{2} + o(1) \right)(\log(\frac{X^{1/2}}{\log T}) + 4 \log \log X) \frac{\log T}{\log \log T} \leq \frac{\nu_\kappa - \epsilon}{4} \log T
\]

by Theorem 3. Choosing \(m\) as before gives the desired bound for \(\mu(\mathcal{E}_R(V))\) and, again, our choice of \(m\) is legal since in this case we have

\[
m \leq \frac{\kappa (\nu_\kappa - \epsilon) \log T}{2 \log \log T}
\]

which is \(\leq (\log T - \log_2 T)/\log X\) provided

\[
2 + \nu_\kappa \leq \frac{2}{\kappa \nu_\kappa} \iff \nu_\kappa \leq \frac{1}{2 \kappa} \sqrt{1 + \frac{1}{2\kappa} - 2}.
\]

\[
\square
\]

**Lemma 4.** Let \(\epsilon > 0\), \(\nu \in \mathbb{R}\) and suppose \(X \leq \eta_\nu (\log T)^2 (\log_2 T)^2\) with \(\eta_\nu = \frac{1}{16e^2} - \epsilon\).

Then

\[
\frac{1}{T} \int_\epsilon \left| \mathcal{P}_X(\frac{1}{2} + it) \right|^{2\nu} \, dt \ll e^{-\delta \nu_0}
\]

for some \(\delta > 0\) dependent on \(\epsilon\). Assuming RH, this holds provided \(X \leq (\log T)^{\theta_\nu - \epsilon}\) where \(\theta_\nu\) is given by (13).
Proof. Since $A \cup B = A \cup (\overline{A} \cap B)$ we may write
\begin{equation}
\mathcal{E} = \mathcal{E}_{\mathbb{R}}(V_0) \cup \mathcal{E}_{\mathbb{I}}(V_0) = \mathcal{E}_{\mathbb{R}}(V_0) \cup (\mathcal{S}_{\mathbb{R}}(V_0) \cap \mathcal{E}_{\mathbb{I}}(V_0)).
\end{equation}
The integral over $\mathcal{E}_{\mathbb{R}}(V_0)$ is
\begin{equation}
\leq \frac{1}{T} \int_{\mathcal{E}_{\mathbb{R}}(V_0)} \exp \left(2|\nu| \cdot \Re \left( \sum_{n \leq X} \frac{\Lambda(n)}{n^{1/2+it} \log n} \right) \right) dt
\end{equation}
\begin{equation}
= \frac{1}{T} e^{2|\nu| V_0} \mu(\mathcal{E}_{\mathbb{R}}(V_0)) + \frac{2|\nu|}{T} \int_{V_0}^{\nu_{\max}} e^{2|\nu| V} \mu(\mathcal{E}_{\mathbb{R}}(V)) dV.
\end{equation}
Since $\eta_\nu \leq 1/16(|\nu| + \delta/2)^2 - \epsilon/2$ for some $\delta > 0$ dependent on $\epsilon$, Lemma 3 gives $\mu(\mathcal{E}_{\mathbb{R}}(V)) \ll T e^{-2|\nu|+\delta V}$. Hence the quantity in (16) is $\ll e^{-\delta V_0}$. Similarly, the integral over $\mathcal{S}_{\mathbb{R}}(V_0) \cap \mathcal{E}_{\mathbb{I}}(V_0)$ is
\begin{equation}
\leq \frac{1}{T} e^{2|\nu| V_0} \mu(\mathcal{E}_{\mathbb{I}}(V_0)) \ll e^{-\delta V_0}.
\end{equation}
The result then follows by the union bound. Likewise, $\theta_\nu - \epsilon \leq \theta_{\nu+\delta/2} - \epsilon/2$ and so the same argument gives the conditional case. \qed

We now show that $P_X(1/2 + it)^k$ can be approximated by a short Dirichlet polynomial provided $t \in \mathcal{S}$. To do this we note that for $|z| \leq Z$ we have
\begin{equation}
\left| e^z - \sum_{j=0}^{10Z} \frac{z^j}{j!} \right| \leq e^{-10Z}
\end{equation}
by a Taylor expansion and Stirling’s formula.

**Lemma 5.** Suppose $t \in \mathcal{S}$. Let $k \in \mathbb{R}$ and define
\begin{equation}
W_0 = W_0(k, T) = 20|k| V_0.
\end{equation}
Then as $X \to \infty$,
\begin{equation}
P_X(\frac{1}{2} + it)^k = \left(1 + O(e^{-19|k| V_0})\right) D(t, k)
\end{equation}
where
\begin{equation}
D(t, k) = \sum_{n \in S(X)} \frac{\alpha_k(n)}{n^{1/2+it}}
\end{equation}
with $S(X) = \{ n \in \mathbb{N} : p|n \implies p \leq X \}$ and where the coefficients $\alpha_k(n)$ satisfy the following properties:
\begin{itemize}
  \item $\alpha_k(n)$ is supported on integers $n \leq X^{W_0}$ and $|\alpha_k(n)| \leq d_{|k|}(n)$.
\end{itemize}
THE SPLITTING CONJECTURE

• If \( \Omega(n) \leq W_0 \) then \( \alpha_k(n) = \beta_k(n) \) where \( \beta_k(n) \) is a multiplicative function satisfying

\[
\beta_k(n) = d_k(n)
\]

if \( p^m | n \implies p^m \leq X \) and \( |\beta_k(n)| \leq d_{|k|}(n) \) in general.

Proof. Since \( t \in S \) we have

\[
\left| \sum_{\ell \leq X} \frac{\Lambda(\ell)}{\ell^{1/2 + it} \log \ell} \right| \leq 2 \log_2 T \log_3 T,
\]

and so by (17) we acquire

\[
P_X(\frac{1}{2} + it)^k = (1 + O(e^{-19|k|W_0})) \sum_{j=0}^{W_0} \frac{k^j}{j!} \left( \sum_{\ell \leq X} \frac{\Lambda(\ell)}{\ell^{1/2 + it} \log \ell} \right)^j.
\]

Writing the sum on the right as the Dirichlet polynomial \( D(t, k) \) it remains to deduce

the properties of the coefficients \( \alpha_k(n) \).

Clearly, this is a Dirichlet polynomial of length \( X^{W_0} \) over the \( X \)-smooth numbers \( S(X) \). For the remaining properties, first note that we may write

\[
\exp \left( k \sum_{\ell \leq X} \frac{\Lambda(\ell)}{\ell^s \log \ell} \right) = \prod_{p \leq X} \exp \left( -k \log(1 - p^{-s}) - k \sum_{m: p^m > X} \frac{1}{mp^ms} \right).
\]

Note also that after performing a Taylor expansion of the left hand side and collecting like terms for \( n^s \), the coefficients are a sum of positive terms if \( k > 0 \), whilst they can be bounded from above by the same sum but involving \( |k| \) if \( k < 0 \). From these observations it is clear that \( |\alpha_k(n)| \leq d_{|k|}(n) \) since the right hand side is the generating function for the divisor functions \( d_{|k|}(n) \) with some terms removed.

Moreover, if we form the product on the right hand side of (21) into a series

\[
\sum_{n \in S(X)} \beta_k(n)n^{-s},
\]

then we see that the coefficients are multiplicative and satisfy \( \beta_k(n) = d_k(n) \) if \( p^m | n \implies p^m \leq X \). Since the highest power \( j \) in (20) is \( W_0 \) we see that, certainly, if \( \Omega(n) \leq W_0 \) then \( \alpha_k(n) = \beta_k(n) \).

Proof of Proposition 1. Write

\[
\frac{1}{T} \int_T^{2T} |P_X(\frac{1}{2} + it)|^{2k} dt = \frac{1}{T} \int_S |P_X(\frac{1}{2} + it)|^{2k} dt + \frac{1}{T} \int_E |P_X(\frac{1}{2} + it)|^{2k} dt.
\]

By Lemma 4, the second integral here is \( o(1) \). For the first integral, Lemma 5 gives

\[
\frac{1}{T} \int_S |P_X(\frac{1}{2} + it)|^{2k} dt \sim \frac{1}{T} \int_S |D(t, k)|^2 dt = \frac{1}{T} \int_T^{2T} |D(t, k)|^2 dt + O \left( \frac{1}{T} \int_E |D(t, k)|^2 dt \right).
\]
By the Cauchy–Schwarz inequality the error term here is
\[(22) \ll \left( \frac{\mu(\mathcal{E})}{T} \right)^{1/2} \left( \frac{1}{T} \int_T^{2T} |D(t, k)|^4 dt \right)^{1/2}.
\]

Since \(D(t, k)\) is of length \(X^{W_0} \leq e^{C(\log_2 T)^2 \log_3 T}\), the Montgomery–Vaughan mean value Theorem (see (39) below) and the coefficient bounds of Lemma 5 give
\[
\frac{1}{T} \int_T^{2T} |D(t, k)|^4 dt = (1 + o(1)) \sum_{n_1n_2=n_3n_4 \atop n_j \in S(X)} \frac{\alpha_k(n_1)\alpha_k(n_2)\alpha_k(n_3)\alpha_k(n_4)}{(n_1n_2n_3n_4)^{1/2}}
\]
\[
\ll \prod_{p \in X} \left( 1 + \frac{4k^2}{p} + O(p^{-2}) \right) \ll (\log X)^{4k^2}.
\]

More generally, for fixed \(m \in \mathbb{N}\) we have
\[(24) \frac{1}{T} \int_T^{2T} |D(t, k)|^{2m} dt \ll (\log X)^{m^2k^2}.
\]

Therefore, by Lemma 3 the expression in (22) is
\[
\ll e^{-|k|(\log_2 T)(\log_3 T)}(\log X)^{2k^2} = o(1)
\]
and hence we may concentrate on the integral of \(|D(t, k)|^2\) over the full set \([T, 2T]\).

Applying the Montgomery–Vaughan mean value theorem again gives
\[
\frac{1}{T} \int_T^{2T} |D(t, k)|^2 dt = (1 + o(1)) \sum_{n \in S(X)} \frac{\alpha_k(n)^2}{n}.
\]

Since \(|\alpha_k(n)| \leq d_{|k|}(n)\), the sum over terms with \(\Omega(n) > W_0\) is, for any \(1 < r < 2\),
\[(25) \sum_{n \in S(X) \atop \Omega(n) > W_0} \frac{\alpha_k(n)^2}{n} \ll r^{-W_0} \sum_{n \in S(X)} \frac{d_{|k|}(n)^2 r^{\Omega(n)}}{n} \ll r^{-W_0}(\log X)^{rk^2} = o(1)
\]
where in the first inequality we have applied Rankin’s trick in the form \(r^{\Omega(n) - W_0} \geq 1\).

Since \(\alpha_k(n) = \beta_k(n)\) if \(\Omega(n) \leq W_0\) the main term is
\[
\sum_{n \in S(X) \atop \Omega(n) \leq W_0} \frac{\alpha_k(n)^2}{n} = \sum_{n \in S(X)} \frac{\beta_k(n)^2}{n} + O\left( \sum_{n \in S(X) \atop \Omega(n) > W_0} \frac{|\beta_k(n)|^2}{n} \right).
\]
The bound $|\beta_k(n)| \leq d_k(n)$ and the same analysis as in (25) shows that this error term is $o(1)$. From the properties of $\beta_k(n)$ we get

$$\sum_{n \in S(X)} \frac{\beta_k(n)^2}{n} = \prod_{p \leq X} \left( \sum_{m : p^m \leq X \atop m > X} \frac{d_k(p^m)^2}{p^m} + \sum_{m : p^m > X} \frac{\beta_k(p^m)^2}{p^m} \right)$$

$$= \prod_{p \leq X} \sum_{m \geq 0} \frac{d_k(p^m)^2}{p^m} \prod_{p \leq X} \left( 1 + O\left( \sum_{m : p^m > X} \frac{d_k(p^m)^2}{p^m} \right) \right).$$

We split the second product at $\sqrt{X}$ and apply the bound $d_k(n) \ll n^\epsilon$ to find that it is

$$\prod_{p \leq \sqrt{X}} \left( 1 + O\left( \frac{1}{X^{1-\epsilon}} \right) \right) \prod_{\sqrt{X} < p \leq X} \left( 1 + O\left( \frac{1}{p^{2-\epsilon}} \right) \right) = 1 + O(X^{-1/2+\epsilon})$$

by the prime number theorem. Then by Mertens’ theorem we have

$$\prod_{p \leq X} \sum_{m \geq 0} \frac{d_k(p^m)^2}{p^m} \sim a(k)(e^\gamma \log X)^{k^2}$$

since $a(k)$ is an absolutely convergent product.

4. ASYMPTOTICS FOR THE 2ND AND 4TH MOMENTS OF THE HADAMARD PRODUCT: PROOF OF PROPOSITION 2

From (3) we have

$$Z_X(\frac{1}{2} + it) = \zeta(\frac{1}{2} + it)P_X(\frac{1}{2} + it)^{-1}(1 + O(1/\log X))$$

and thus it suffices to consider the second and fourth moment of the object on the right. Our aim is to first replace $P_X$ by its Dirichlet polynomial approximation and then apply formulas for the twisted second and fourth moments of the zeta function.

4.1. The second moment. As before, we decompose the integral as

$$\frac{1}{T} \int_T^{2T} |Z_X(\frac{1}{2} + it)|^2 dt = \frac{1}{T} \int_S |Z_X(\frac{1}{2} + it)|^2 dt + \frac{1}{T} \int_{E} |Z_X(\frac{1}{2} + it)|^2 dt.$$

Working unconditionally first, we apply the Cauchy–Schwarz inequality to the integral over $E$ to find it is

$$\ll (\log T)^2 \left( \frac{1}{T} \int_E |P_X(\frac{1}{2} + it)^{-1}|^4 dt \right)^{1/2}$$

using Ingham’s asymptotic for the fourth moment. Since $X \leq \frac{1}{10^4}(\log T)^2(\log_2 T)^2$ and $1/10^4 < \eta_2$ we find that this is $\ll (\log T)^2 e^{-\delta_{\psi}} = o(1)$ by Lemma 4.
If we assume RH we can apply Hölder's inequality in the form
\[
\frac{1}{T} \int_E |Z_X(\frac{1}{2} + it)|^2 dt \ll \left( \frac{1}{T} \int_T^{2T} |\zeta(\frac{1}{2} + it)|^{2(1+\epsilon)} dt \right)^{\frac{\epsilon}{1+\epsilon}} \left( \frac{1}{T} \int_E |P_X(\frac{1}{2} + it)|^{-2(1+\epsilon)} dt \right)^{\frac{1}{1+\epsilon}}
\] (28)
for some \( \epsilon > 0 \). The first term on the right is \( \ll (\log T)^{4(1+\epsilon)} \) by Harper’s \cite{24} conditional bound \( \int_T^{2T} |\zeta(1/2 + it)|^{2k} \ll T(\log T)^k \). Since \( X \leq (\log T)^{\sqrt{6} - \epsilon'} \) and \( \sqrt{6} - \epsilon' \leq \theta_{1+\epsilon} - \epsilon'/2 \) on choosing \( \epsilon \) small enough, the second term on the right is \( \ll e^{-4\epsilon_0} \) by Lemma 4. Therefore, the quantity in (28) is \( o(1) \).

By Lemma 5 we have
\[
\frac{1}{T} \int_S |Z_X(\frac{1}{2} + it)|^2 dt \sim \frac{1}{T} \int_S |\zeta(\frac{1}{2} + it)|^2 |D(t, -1)|^2 dt
\]
which we write as
\[
\frac{1}{T} \int_T^{2T} |\zeta(\frac{1}{2} + it)|^2 |D(t, -1)|^2 dt + O\left( \frac{1}{T} \int_E |\zeta(\frac{1}{2} + it)|^2 |D(t, -1)|^2 dt \right).
\]
Applying the Cauchy–Schwarz inequality twice along with Ingham’s fourth moment bound, Lemma 3 and (24) we find that the error term here is \( o(1) \).

It remains to show that
\[
I_1 := \frac{1}{T} \int_T^{2T} |\zeta(\frac{1}{2} + it)|^2 |D(t, -1)|^2 dt \sim \frac{\log T}{e^\gamma \log X}.
\]
The mean square of the zeta function times an arbitrary Dirichlet polynomial has been computed before e.g. see \cite{5, 7}. From there we see that
\[
I_1 = \sum_{m,n \in S(X)} \frac{\alpha_{-1}(m)\alpha_{-1}(n)(m,n)}{mn} \log \left( \frac{BT(m,n)^2}{mn} \right) + o(1)
\]
for some constant \( B \). On applying the bound \( \alpha_{-1}(n) \ll d_1(n) \ll 1 \) and following the argument in the proof of Theorem 3, pg. 530, of \cite{22} we easily see that
\[
\sum_{m,n \in S(X)} \frac{\alpha_{-1}(m)\alpha_{-1}(n)(m,n)}{mn} \log \left( \frac{B(m,n)^2}{mn} \right) \ll (\log X)^{10}
\]
and thus we can consider the remaining sum.

As in the previous section, we first estimate the sum over integers for which \( \Omega(m), \Omega(n) > W_0 \) (which equals \( 10 \log_2 T \log_3 T \) in this case). Applying the bound
\( \alpha_1(n) \ll 1 \) leads to an error of size
\[
\ll \log T \sum_{m,n \in S(X), \Omega(m) > W_0} \frac{(m,n)}{mn} \ll (\log T)^{-W_0} \sum_{m,n \in S(X)} \frac{(m,n)^{-\Omega(m)}}{mn}
\]
for any \( 1 < r < 2 \). A short computation shows this last sum is
\[
\prod_{p \leq X} \left( 1 + (2r + 1)p^{-1} + O(p^{-2}) \right) \ll (\log X)^{2r+1}
\]
and so the terms with \( \Omega(m), \Omega(n) > W_0 \) contribute an error of size \( o(1) \).

In the main term we replace \( \alpha_1(n) \) with \( \beta_1(n) \) and then re-extend the sum to include those integers for which \( \Omega(m), \Omega(n) > W_0 \). By the bounds \( \beta_1(n) \ll d_1(n) \ll 1 \), the same argument shows that this introduces an error of \( o(1) \). Thus,
\[
I_1 = \log T \sum_{m,n \in S(X)} \frac{\beta_1(m)\beta_1(n)m,n}{mn} + O((\log X)^{10}).
\]

By symmetry and the properties of \( \beta_k(n) \) we find that the sum is
\[
\prod_{p \leq X} \left( \sum_{m,n \geq 0} \frac{\mu(p^m)\mu(p^n)}{p^{m+n-\min(m,n)}} + O\left( \sum_{m,n \geq 0} \frac{1}{p^{m+n-\min(m,n)}} \right) \right)
\]
\[
= \prod_{p \leq X} \left( 1 - \frac{1}{p} \right) \prod_{p \leq X} \left( 1 + O\left( \sum_{m,n \geq 0} \frac{1}{p^{m+n-\min(m,n)}} \right) \right).
\]

To estimate the second product we note that the sum in the error term is \( \ll \sum_{m > X}(m + 1)p^{-m} \ll (\log X)p^{-\lfloor \log X / \log p \rfloor} \) and then split the product at \( \sqrt{X} \), as before. In this way we find it is \( 1 + O(X^{-1/2+\epsilon}) \) and therefore by Mertens’ Theorem
\[
I_1 \sim \frac{\log T}{e^\gamma \log X}
\]
as desired.

4.2. The fourth moment: Initial clearing. Not surprisingly, the fourth moment requires more work in both the initial stages and the arithmetic computations. Our aim is to show that
\[
I_2 := \frac{1}{T} \int_T^{2T} \left| \zeta' \left( \frac{1}{2} + it \right) \right|^4 |P_X(\frac{1}{2} + it)|^{-4} dt \sim \frac{1}{12} \left( \frac{\log T}{e^\gamma \log X} \right)^4.
\]

In this subsection the goal is to replace \( P_X(1/2 + it)^{-2} \) by \( D(t, -2) \).
Splitting the integral as in (27) we see that our first task is to bound

\[(29) \quad \frac{1}{T} \int_{\mathcal{E}} |\zeta(\frac{1}{2} + it)|^4 |P_X(\frac{1}{2} + it)|^{-4} dt.\]

On RH we can deal with this by applying Hölder’s inequality as in (28). Following the same argument and using the fact that $\sqrt{5} - \epsilon' \leq \theta_{2(1+\epsilon)} - \epsilon'/2$ shows that this is $o(1)$.

To bound this unconditionally requires more work. First note that since $E \subset \mathcal{E}' := \{ t \in [T, 2T] : \left| \sum_{n \leq X} \Lambda(n) \frac{1}{n^{1/2+it} \log n} \right| \geq V_0 \}$

we can upper bound by the integral over $\mathcal{E}'$. Let $V_j = e^j V_0$ and define $\mathcal{J}$ to be the maximal $j$ such that $V_j \leq V_{\text{max}}$. Let

\[\mathcal{E}_j = \{ t \in [T, 2T] : V_j \leq \left| \sum_{n \leq X} \Lambda(n) \frac{1}{n^{1/2+it} \log n} \right| \leq V_{j+1} \}\]

so that

\[\mathcal{E}' = \bigcup_{j=0}^{\mathcal{J}} \mathcal{E}_j.\]

Then

\[(30) \quad \frac{1}{T} \int_{\mathcal{E}_j} |\zeta(\frac{1}{2} + it)|^4 |P_X(\frac{1}{2} + it)|^{-4} dt \leq \sum_{j=0}^{\mathcal{J}} e^{V_{j+1}} \frac{1}{T} \int_{\mathcal{E}_j} |\zeta(\frac{1}{2} + it)|^4 dt \]

\[\leq \sum_{j=0}^{\mathcal{J}} e^{4V_{j+1} - 2r_j} \frac{1}{T} \int_T^{2T} |\zeta(\frac{1}{2} + it)|^4 \left| \sum_{n \leq X} \Lambda(n) \frac{1}{n^{1/2+it} \log n} \right|^{2r_j} dt \]

for any given integer $r_j \geq 0$. The combinatorics are simplified if we focus on the prime sums so we apply Jensen’s inequality in the form

\[\left| \sum_{n \leq X} \frac{\Lambda(n)}{n^{1/2+it} \log n} \right|^{2r_j} \leq 9^{r_j} \left( \left| \sum_{p \leq X} \frac{1}{p^{1/2+it}} \right|^{2r_j} + \left| \sum_{p \leq \sqrt{X}} \frac{1}{2p^{1+2it}} \right|^{2r_j} + O(C^{2r_j}) \right).\]

It will be clear after the computations that the first sum here gives the dominant contribution and so we focus on this. Note that by the multinomial theorem,

\[\left( \sum_{p \leq X} \frac{1}{p^{1/2+it}} \right)^{r_j} = r_j! \sum_{n \leq X^{r_j}} \frac{\mathfrak{g}(n)}{n^{1/2+it}} \]

\[\mathfrak{g}(n) = \prod_{\Omega(n) = r_j} \frac{1}{p^{1/2+it}}.\]
where \( g(n) \) is the multiplicative function satisfying \( g(p^\alpha) = 1/\alpha! \). Accordingly, (30) is

\[
(31) \quad \ll \sum_{j=0}^{J} e^{4V_{j+1}}g_{j}V_{j}^{-2\alpha_{j}} \cdot \frac{1}{T} \int_{T}^{2T} |\zeta(\frac{1}{2} + it)|^4 r_{j}! \sum_{\substack{n \leq X^{r_{j}} \\ \Omega(n) = r_{j}}} \frac{g(n)}{n^{1/2 + it}} dt
\]

The twisted fourth moment of the zeta function has been computed before [6, 31] and has been applied in similar situations [26]. Provided \( X^{r_{j}} \leq T^{1/4 - \epsilon} \) i.e.

\[
(32) \quad r_{j} \leq (1/4 - \epsilon) \frac{\log T}{\log X},
\]

Proposition 4 and formula (8) of [26] (see section 6 there) give

\[
\frac{1}{T} \int_{T}^{2T} |\zeta(\frac{1}{2} + it)|^4 r_{j}! \sum_{\substack{n \leq X^{r_{j}} \\ \Omega(n) = r_{j}}} \frac{g(n)}{n^{1/2 + it}}^2 dt \ll (\log T)^{4}g_{j} \sum_{\substack{m, n \leq X^{r_{j}} \\ \Omega(m) = \Omega(n) = r_{j}}} r_{j}! g(n)g(m) [n, m].
\]

Following the arguments of [26] which lead to formula (9) there, we find that this is

\[
(33) \quad \ll (\log T)^{4}g_{j}r_{j}! \left( \sum_{p \leq X} \frac{1}{p} \right) r_{j} \exp \left( \sum_{p \leq X} \frac{1}{p} \right) \ll (\log T)^{5}r_{j}^{1/2} \left( \frac{9r_{j}\log \log X}{e} \right)^{r_{j}}.
\]

We choose

\[
r_{j} = \frac{12V_{j}}{\log V_{j}} \leq \frac{12V_{\max}}{\log V_{\max}} \lesssim \frac{48X^{1/2}}{(\log X)^{2}} \lesssim \frac{24 \log T}{100 \log X} \quad \forall j,
\]

since \( X \leq \frac{1}{10^{4}}(\log T)^{2}(\log_{2} T)^{2} \). Clearly, this satisfies (32). Then applying (33) in (31) we find that

\[
\frac{1}{T} \int_{T}^{2T} |\zeta(\frac{1}{2} + it)|^4 |P_{X}(\frac{1}{2} + it)|^{-4} dt
\]

\[
\ll (\log T)^{5} \sum_{j=0}^{J} e^{4V_{j+1}} \left( \frac{C \log X}{V_{j} \log V_{j}} \right)^{12V_{j}/\log V_{j}}
\]

\[
\ll (\log T)^{5} \sum_{j=0}^{J} e^{4V_{j+1} - (12 - o(1))V_{j}} \ll (\log T)^{5} \sum_{j=0}^{J} e^{-e^{V_{0}(12-4\epsilon + o(1))}} = o(1).
\]

We have therefore arrived at

\[
I_{2} = (1 + o(1)) \frac{1}{T} \int_{S} |\zeta(\frac{1}{2} + it)|^4 |D(t, -2)|^2 dt + o(1)
\]
on applying Lemma 5 in the integral over $S$. After extending to the full range of integration $[T, 2T]$ it remains to estimate

$$\frac{1}{T} \int_S |\zeta(\frac{1}{2} + it)|^4 |D(t, -2)|^2 dt.$$  

However, from the definition of $D(t, k)$ we have

$$|D(t, k)| \leq \sum_{j=0}^{W_0} \left| \frac{k^j}{j!} \right| \sum_{n \leq X} \frac{\Lambda(n)}{n^{1/2+it} \log n} \leq \exp \left( \left| k \sum_{n \leq X} \frac{\Lambda(n)}{n^{1/2+it} \log n} \right| \right).$$

and so we can apply the same argument as above to acquire the bound $o(1)$ for this integral. Thus we have

$$I_2 \sim J_2$$

where

$$J_2 := \frac{1}{T} \int_T^{2T} |\zeta(\frac{1}{2} + it)|^4 |D(t, -2)|^2 dt.$$  

4.3. The fourth moment: Arithmetic computations. In this section our aim is to show that

$$J_2 \sim \frac{1}{12} \left( \frac{\log T}{e^\gamma \log X} \right)^4.$$  

The formulas for the twisted fourth moment of the zeta function given in the literature [6, 31] apply to smoothed integrals and accordingly we must first smooth $J_2$. Let $\Phi_-, \Phi_+$ be smooth approximations of compact support satisfying

(34) $$\Phi_-(t) \leq 1_{t \in [T, 2T]} \leq \Phi_+(t)$$

with derivatives $\Phi_{\pm}^{(j)}(t) \ll T^\epsilon$. For example, we may take $\Phi_-$ to be compactly supported on $[1, 2]$ and equal to 1 on the interval $[1 + T^{-\epsilon}, 2 - T^{-\epsilon}]$ with smooth, monotonic decay to zero at each endpoint. Then, on letting $\Phi$ be either $\Phi_-$ or $\Phi_+$ we consider the smoothed integral

$$J_{2, \Phi} := \frac{1}{T} \int_{\mathbb{R}} \Phi \left( \frac{t}{T} \right) |\zeta(\frac{1}{2} + it)|^4 |D(t, -2)|^2 dt.$$  

We note that the error incurred from these approximations will be $\ll T^{1-\epsilon}$ which is tolerable given the asymptotic we seek.

Theorem 4 (Theorem 1.2 of [6]). Let $\Phi$ be as above and $\alpha_1, \alpha_2, \alpha_3, \alpha_4 \ll 1/\log T$. Let $\Xi$ be the subgroup of $S_4$ consisting of the identity, those permutations which swap just one element of $\{1, 2\}$ with $\{3, 4\}$ and the permutation satisfying $\tau(1) = 3, \tau(2) = 4$.  

Then for any Dirichlet polynomial \( \sum_{n \leq y} a(n)n^{-s} \) satisfying \( y \leq T^{1/4-\varepsilon} \) and \( a(n) \ll n^\varepsilon \) we have

\[
\int_{\mathbb{R}} \zeta(\frac{1}{2} + \alpha + it)\zeta(\frac{1}{2} + \alpha + it)\zeta(\frac{1}{2} - \alpha - it)\zeta(\frac{1}{2} - \alpha - it) \left| \sum_{n \leq y} \frac{a(n)}{m^{1/2+it}} \right|^2 \Phi\left( \frac{t}{T} \right) dt = \sum_{m_1, m_2 \leq y} a(m_1)a(m_2) \int_{\mathbb{R}} \Phi\left( \frac{t}{T} \right) \sum_{\tau \in \Xi} \left( \frac{t}{2\pi} \right)^{\sum_{j=1}^{N} \alpha_{\tau(j)} - \alpha_j} Z_{\tau(\alpha_1), \tau(\alpha_2), \tau(\alpha_3), \tau(\alpha_4), m_1, m_2} dt + O(T^{1-\varepsilon})
\]

where

\[
Z_{\alpha_1, \alpha_2, \alpha_3, \alpha_4, m_1, m_2} = \sum_{m_1n_1n_2=m_2n_3n_4} \frac{1}{(m_1m_2)^{1/2}n_1^{1/2+\alpha_1}n_2^{1/2+\alpha_2}n_3^{1/2-\alpha_3}n_4^{1/2-\alpha_4}} V\left( \frac{n_1n_2n_3n_4}{t^2} \right)
\]

and

\[
V(x) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{G(s)}{s} (4\pi^2 x)^{-s} ds, \quad c > 0
\]

with \( G(s) \) an even function of rapid decay in vertical strips satisfying \( G(0) = 1 \).

**Remark.** We remark that the choice of function \( G(s) \) is flexible and it can be prescribed to have zeros at linear combinations of the shifts. This is fairly typical and is used to cancel unnecessary poles later on. We will take \( G(s) = Q_\alpha(s) \exp(s^2) \) where \( Q_\alpha(s) \) is an even polynomial which is 1 at \( s = 0 \) and zero at \( 2s = \alpha_3 - \alpha_1, \alpha_4 - \alpha_1, \alpha_3 - \alpha_2, \alpha_4 - \alpha_2 \). Note these conditions on \( Q \) imply that for fixed \( \Re(s) \),

\[
G(s) \ll (\log T)^4 e^{-\Im(s)^2}
\]

since \( \alpha_j \ll 1/\log T \).

Let us compute term corresponding to the identity: \( \tau = \id \). Denote this by

\[
\mathcal{K} = \mathcal{K}_\alpha(t, X) := \sum_{m_1, m_2 \in S(X)} \alpha_{-2}(m_1)\alpha_{-2}(m_2)Z_{\alpha_1, \alpha_2, \alpha_3, \alpha_4, m_1, m_2}
\]

\[
= \sum_{m_1n_1n_2=m_2n_3n_4} \frac{\alpha_{-2}(m_1)\alpha_{-2}(m_2)}{(m_1m_2)^{1/2}n_1^{1/2+\alpha_1}n_2^{1/2+\alpha_2}n_3^{1/2-\alpha_3}n_4^{1/2-\alpha_4}} V\left( \frac{n_1n_2n_3n_4}{t^2} \right).
\]

By shifting the contour in \( V \) to the either the left or right depending on whether \( x \ll 1 \) or \( x \gg 1 \), respectively, we find that \( V(x) \ll (1 + |x|)^{-A} \) for any \( A > 0 \). Accordingly, on applying the bounds \( \alpha_k(m) \ll m^\varepsilon \cdot 1_{m \leq T^\gamma} \) we may restrict the above sum to those \( n_j \) satisfying \( n_1n_2n_3n_4 \ll t^{2+\varepsilon} \ll T^{2+\varepsilon} \) at the cost of an error of size
o(1). Then the contribution from those $m_1$ with $\Omega(m_1) > W_0$ is for $1 < r < 2$
\[ \ll r^{-W_0} \sum_{m \leq T^{2+\varepsilon}} \frac{r^{\Omega(m)} d_4(m)^2}{m} \ll r^{-W_0} (\log T)^{16r} = o(1) \]
where for the first inequality we have applied Rankin’s trick in the form $r^{\Omega(m)} - W_0 \geq 1$ along with the bound $\alpha_{-2}(n) \ll d(n)$. The same bound holds for the sum over $\Omega(m_2) > W_0$. Then on replacing $\alpha_{-2}(m)$ with $\beta_{-2}(m)$ and re-extending the sums (which by the same arguments incurs an error of $o(1)$) we have
\[ \mathcal{K} = \sum_{m_1, n_2 = m_2 n_3 n_4} \frac{\beta_{-2}(m_1) \beta_{-2}(m_2)}{(m_1 m_2)^{1/2 n_1^{1/2 + \alpha_1} n_2^{1/2 + \alpha_2} n_3^{1/2 - \alpha_3} n_4^{1/2 - \alpha_4}}} V \left( \frac{n_1 n_2 n_3 n_4}{t^2} \right) + o(1). \]
Unfolding the integral for $V(x)$ and pushing the sum through we find
\[ \mathcal{K} = \frac{1}{2 \pi i} \int_{-i \infty}^{c+i \infty} \mathcal{F}_{\alpha, X}(s) \frac{G(s)}{s} \left( \frac{t}{2\pi} \right)^{2s} ds + o(1) \]
where
\[ \mathcal{F}_{\alpha, X}(s) = \sum_{m_1, n_2 = m_2 n_3 n_4} \frac{\beta_{-2}(m_1) \beta_{-2}(m_2)}{(m_1 m_2)^{1/2 n_1^{1/2 + \alpha_1 + s} n_2^{1/2 + \alpha_2 + s} n_3^{1/2 - \alpha_3 + s} n_4^{1/2 - \alpha_4 + s}}} = \sum_{m_1, n_2 = m_2 n_3 n_4} \frac{\beta_{-2}(m_1) \beta_{-2}(m_2) \sigma_{\alpha_1, \alpha_2}(n_1) \sigma_{-\alpha_3, -\alpha_4}(n_2)}{(m_1 m_2)^{1/2 (n_1 n_2)^{1/2 + s}}} \]
with $\sigma_{u,v}(n) = \sum_{d_1 d_2 = n} d_1^{-u} d_2^{-v}$. Expressing this as an Euler product we have
\[ \mathcal{F}_{\alpha, X}(s) = \mathcal{A}_\alpha(s) \mathcal{G}_{\alpha, X}(s) \]
where
\[ \mathcal{A}_\alpha(s) = \frac{\zeta(1 + \alpha_1 - \alpha_3 + 2s) \zeta(1 + \alpha_1 - \alpha_4 + 2s) \zeta(1 + \alpha_2 - \alpha_3 + 2s) \zeta(1 + \alpha_2 - \alpha_4 + 2s)}{\zeta(2 + \alpha_1 + \alpha_2 - \alpha_3 - \alpha_4 + 4s)} \]
and
\[ \mathcal{G}_{\alpha, X}(s) = \prod_{p \in \mathcal{X}} \sum_{m_1 + n_3 = m_2 + n_2} \frac{\beta_{-2}(p^{m_1}) \beta_{-2}(p^{m_2}) \sigma_{\alpha_1, \alpha_2}(p^{n_1}) \sigma_{-\alpha_3, -\alpha_4}(p^{n_2})}{p^{(m_1 + m_2) + \frac{1}{2}(s)(n_1 + n_2)}} / \sum_{n \geq 0} \frac{\sigma_{\alpha_1, \alpha_2}(p^n) \sigma_{-\alpha_3, -\alpha_4}(p^n)}{p^{n(1 + 2s)}}. \]
Shifting the line of integration in (36) to \( \Re(s) = -1 / \log X \) we pick up a simple pole only at \( s = 0 \) (the poles of \( A_\alpha(s) \) being cancelled by the zeros of \( G(s) \)). Since \( \beta_{-2}(n) \ll d(n) \) and \( \sigma_{\alpha_i,\alpha_j}(p^n) \ll p^n/\log T d(p^n) \) we find that on the new line of integration

\[
G_\alpha(X)(s) \ll (\log X)^{O(1)}.
\]

Therefore, on combining this with the bound for \( G(s) \) given in (35) we see the integral over the new line is bounded by

\[
\ll t^{-2/\log X} (\log T)^{O(1)} = o(1)
\]

since \( t \asymp T \). Hence

\[
K_\alpha = A_\alpha(0) G_\alpha(X)(0) + o(1).
\]

We have satisfactorily computed the contribution from a single \( Z \) term and thus it remains to find the combinatorial sum of these which appears in Theorem 4. Using the results of [13] we can express this sum as a multiple contour integral. Precisely, Lemma 2.5.1 there gives

\[
\sum_{\tau \in \mathcal{Z}} \left( \frac{t}{2\pi} \right)^{\sum_{j=1}^4 \alpha_{\tau(j)} - \alpha_j} K_{\tau(\alpha)}
= \frac{1}{4(2\pi i)^4} \int_{|z_j| = 3/\log T} A_{z_1,z_2,z_3,z_4}(0) G_{z_1,z_2,z_3,z_4,X}(0) \Delta(z_1,z_2,z_3,z_4)^2 \prod_{i,j=1}^4 (z_i - \alpha_j)
\times \left( \frac{t}{2\pi} \right)^{\sum_{j=1}^4 (z_{j+2} - z_j)/2} \, dz + o(1)
\]

where \( \Delta(z) \) denotes the vandermonde determinant. A short calculation shows that

\[
\frac{\partial}{\partial z_j} G_{z_1,z_2,z_3,z_4,X}(0) \bigg|_{z=0} \ll G_{0,X}(0) \sum_{p \leq X} \frac{\log p}{p} \ll G_{0,X}(0) \log X
\]

and hence we acquire the Taylor expansion

\[
G_{z_1,z_2,z_3,z_4,X}(0) = G_{0,X}(0) \left( 1 + O \left( \log X \sum_{j=1}^4 |z_j| \right) \right)
\]

whilst from the Laurent expansion of the zeta function we get

\[
A_{z_1,z_2,z_3,z_4,X}(0) = \frac{1}{\zeta(2)} \prod_{i,j=1}^2 \frac{1}{(z_i - z_{j+2})} \left( 1 + O \left( \sum_{j=1}^4 |z_j| \right) \right).
\]
On setting the shifts $\alpha_j$ equal to zero, substituting $z_j \mapsto z_j/\frac{1}{2}\log(t/2\pi)$, and applying these expansions we find

$$J_{2,\Phi} = \int_{\mathbb{R}} \Phi\left(\frac{t}{T}\right) \left(c_4 \log^4(t/2\pi) \cdot \frac{G_{\Phi, X}(0)}{\zeta(2)} \left(1 + O\left(\frac{\log X}{\log T}\right)\right)\right) dt + o(T)$$

where

$$c_4 = \frac{1}{4 \cdot 2^4 (2\pi i)^4} \int_{|z_j|=3/4 \atop 1 \leq j \leq 4} \Delta(z_1, z_2, z_3, z_4)^2 \prod_{i, j=1}^{4} (z_i - z_{j+2}) \prod_{j=1}^{4} \frac{dz_j}{z_j^4}.$$ 

From section 2.7 of [13] we know that $c_4 = g(2) = 1/12$ where $g(k)$ is given by (2). Furthermore,

$$G_{\Phi, X}(0) = \prod_{p \leq X} \sum_{m_1, n_1, m_2, n_2} \frac{\beta_2(p^{n_1}) \beta_2(p^{n_2}) d(p^{n_1}) d(p^{n_2})}{p^{\frac{1}{2}(m_1+m_2+n_1+n_2)}} / \sum_{n \geq 0} d(p^n)^2 p^n.$$

The denominator here is

$$\prod_{p \leq X} \left( \sum_{n \geq 0} \frac{d(p^n)^2}{p^n} \right)^{-1} = \prod_{p \leq X} \frac{(1 - p^{-1})^4}{1 - p^{-2}} \sim \frac{\zeta(2)}{(e^\gamma \log X)^4},$$

whereas the numerator is

$$G_{\Phi, X}(0) = \prod_{p \leq X} \left(1 + O\left( \sum_{m_1, n_1, m_2, n_2} d(p^{m_1}) d(p^{m_2}) d(p^{n_1}) d(p^{n_2}) / p^{\frac{1}{2}(m_1+m_2+n_1+n_2)} \right) \right)$$

since $\beta_2(p^m) = d_2(p^m)$ for $p^m \leq X$ and $\beta_2(p^m) \ll d(p^m)$ in general. Then since $n_1 \geq 0$, the sum in the error is $\ll \sum_{m: p^m > X} d_4(p^m)^2 / p^m$ after forming the convolution. Therefore, we can apply the same argument which gave (26) to find that the numerator is $1 + O(X^{-1/2+\epsilon})$. Consequently, we have

$$J_{2,\Phi} \sim \frac{\Phi(0)}{12} \cdot T \left(\frac{\log T}{e^\gamma \log X}\right)^4 = \frac{T}{12} \left(\frac{\log T}{e^\gamma \log X}\right)^4 + O(T^{1-\epsilon})$$

and so the result follows by (34).

5. SOME USEFUL TOOLS

In this short section we describe some tools which will come in handy when proving Proposition 3. The first relates to the exponential truncation of a more general prime sum and will be used extensively throughout.
Given a general Dirichlet polynomial of the form \( D(s) = \sum_{p \leq Y} a(p)p^{-s}, \) suppose \( t \in [T, 2T] \) is such that \( |kD(s)| \leq Z. \) Then by (17) we have

\[
|e^{kD(s)} - \sum_{0 \leq j \leq 10Z} \frac{(kD(s))^j}{j!}| \leq e^{-10Z}.
\]

By the multinomial theorem the truncated exponential series can be written as

\[
\sum_{0 \leq j \leq 10Z} \frac{1}{j!} \left( k \sum_{p \leq Y} \frac{a(p)}{p^s} \right)^j = \sum_{\Omega(n) \leq 10Z} \frac{k^{\Omega(n)}a(n)g(n)}{n^s}
\]

where we recall \( g \) is the multiplicative function such that \( g(p^\alpha) = 1/\alpha! \), and \( a(n) \) is the completely multiplicative extension of \( a(p) \). Observe this is a Dirichlet polynomial of length \( Y^{10Z}. \)

Our remaining observations relate to mean values of Dirichlet polynomials. We first state the mean value theorem of Montgomery and Vaughan \([36]\) which gives for any complex coefficients \( a(n) \),

\[
\frac{1}{T} \int_T^{2T} \left| \sum_{n \in N} \frac{a(n)}{n^{it}} \right|^2 dt = (1 + O(N/T)) \sum_{n \in N} |a(n)|^2.
\]

Suppose we are given \( R \) Dirichlet polynomials

\[
A_j(s) = \sum_{n \in S_j} a_j(n)n^{-s},
\]

where the \( \prod_{j=1}^R n_j \leq N = o(T) \) for all \( n_j \in S_j \) i.e. the product of the \( A_j(s) \) is short. Then the Montgomery–Vaughan mean value theorem readily implies

\[
\frac{1}{T} \int_T^{2T} \prod_{j=1}^R |A_j(it)|^2 dt \sim \sum_{n \in N} \left| \sum_{n=n_1 \cdots n_R, \ n_j \in S_j} a_1(n_1) \cdots a_R(n_R) \right|^2
\]

\[
= \sum_{n=n_1 \cdots n_{R+1} \cdots n_{2R}, \ n_j \in S_j} a_1(n_1) \cdots a_R(n_R)a_1(n_{R+1}) \cdots a_R(n_{2R}).
\]

Suppose in addition that for any \( j_1, j_2 \) with \( j_1 \neq j_2 \) the elements of \( S_{j_1} \) are all coprime to the elements of \( S_{j_2} \). Then there is at most one way to write \( n = \prod_{j=1}^R n_j \) with
\[ n_j \in \mathcal{S}_j \text{ and thus several applications of the mean value theorem imply} \]
\[ \frac{1}{T} \int_T^{2T} \prod_{j=1}^R |A_j(it)|^2 \, dt = (1 + O(NT^{-1})) \sum_{n \leq N} \left| \sum_{n_j = n_{j_1} \ldots n_{j_R} \in \mathcal{S}_j} \prod_{j=1}^R a_j(n_j) \right|^2 \]
\[ = (1 + O(NT^{-1})) \prod_{j=1}^R \left( \sum_{n_j \in \mathcal{S}_j} |a_j(n_j)|^2 \right) \]
\[ = (1 + O(NT^{-1}))^{1-R} \prod_{j=1}^R \left( \frac{1}{T} \int_T^{2T} |A_j(it)|^2 \, dt \right). \]

(41)

We now move on to proving the upper bound of Proposition 3.

6. The upper bound of Proposition 3

6.1. Initial cleaning. In this section we are required to show that on RH,
\[ \frac{1}{T} \int_T^{2T} |Z_X(\frac{1}{2} + it)|^{2k} \, dt \ll \left( \frac{\log T}{\log X} \right)^{k^2}. \]
On assuming RH, it is a simple task to replace \(|P_X(1/2 + it)|^{-2k}\) by \(|D(t, -k)|^2\) on the left hand side. Indeed; from Harper’s [24] conditional bound \(\int_T^{2T} |\zeta(1/2 + it)|^{2k} \ll T(\log T)^{k^2}\), the bound for the moments of \(D(t, k)\) in \((24)\), Lemmas 3, 4, 5, and the usual arguments involving the decomposition \([T, 2T] = S \cup E\) along with Hölder’s inequality we have
\[ \frac{1}{T} \int_T^{2T} |Z_X(\frac{1}{2} + it)|^{2k} \, dt \sim \frac{1}{T} \int_T^{2T} |\zeta(\frac{1}{2} + it)|^{2k} |D(t, -k)|^2 \, dt \]
for all \(k \geq 0\) on RH.

To bound the right hand side we use an upper bound for \(\zeta(1/2 + it)\) which incorporates the recent developments of Harper [24] on moments of the zeta function, although we present the result more in the style of Radziwiłł–Soundararajan [38] (see the key inequality of section 3 there). Such a treatment is similar to that of [35].

6.2. An upper bound for \(\zeta(\frac{1}{2} + it)\). We start with a proposition of Soundararajan in a mildly adapted form of Harper.

**Lemma 6.** Assume RH. Let \(t \in [T, 2T]\) be large and suppose \(4 \leq x \leq T^2\). Then
\[ \log |\zeta(\frac{1}{2} + it)| \leq \Re \sum_{p \leq x} \frac{1}{p^{1/2 + 1/\log x + it}} \log(x/p) + \Re \sum_{p \leq \min(\sqrt{T}, \log T)} \frac{1}{2p^{1+2it}} + \frac{\log T}{\log x} + O(1). \]

*Proof.* This is Proposition 1 of [24]. \(\square\)
For the splitting of the prime sums we denote
\[ T_{-1} = A, \quad T_j = T^{(\log e^j)^2} \]
where \( A > 0 \) is fixed and \( j = 0, \ldots, J \) with \( J \) the maximal integer such that \( e^j / (\log \log T)^2 \leq 1/10^{12} \) (so that \( J \approx \log \log \log T \)). In this section we take \( A = 1 \) although later we need to take it sufficiently large. Let
\[ \theta_j = \frac{e^j}{(\log \log T)^2}, \quad \ell_j = \theta_j^{-3/4} \]
so that \( T_j = T^{\theta_j} \) for \( 0 \leq j \leq J \). Now, write
\[ w_j(p) = \frac{1}{p^{1/\theta_j} \log T} \frac{\log(T_j/p)}{\log T_j} \]
and
\[ \mathcal{P}_{i,j}(t) := \sum_{T_{i-1} < p \leq T_i} \frac{w_j(p)}{p^{1/2 + it}}. \]
With this notation the first sum over primes in Lemma 6 with \( x = T_j \) can be written as
\[ \sum_{p \leq T_j} \frac{w_j(p)}{p^{1/2 + it}} = \sum_{i=0}^{j} \mathcal{P}_{i,j}(t). \]

Also, write
\[ \mathcal{N}_{i,j}(t, k) = \sum_{p \mid \Omega(n) \leq 10^\ell_i \text{ and } n^{1/2 + it}} \frac{k^{\Omega(n)} w_j(n) g(n)}{n^{1/2 + it}} \]
and note that on the set of \( t \in [T, 2T] \) such that \( |k \mathcal{P}_{i,j}(t)| \leq \ell_i \) we have
\[ \exp(2k \Re \mathcal{P}_{i,j}(t)) = (1 + O(e^{-9^i})) \left| \mathcal{N}_{i,j}(t, k) \right|^2 \]
by (37) and (38). Accordingly, if \( t \) is such that \( |k \mathcal{P}_{i,j}(t)| \leq \ell_i \) for all \( 0 \leq i \leq j \) then on applying (43) we have
\[ \exp \left( 2k \Re \sum_{p \leq T_j} \frac{w_j(p)}{p^{1/2 + it}} \right) \ll \prod_{i=0}^{j} \left| \mathcal{N}_{i,j}(t, k) \right|^2 \]
since \( \sum_{i=0}^{j} e^{-9^i} \) is a rapidly converging series. Note this is a Dirichlet polynomial of length
\[ \prod_{i=0}^{J} T_i^{10^\ell_i} = T^{10 \sum_{i=0}^{j} \theta_j^{i/4}} \leq T^{20e^{J/4}/(\log \log T)^{1/2}} \leq T^{1/50}. \]
We can now state an upper bound for the zeta function in terms of these short Dirichlet polynomials.

**Lemma 7.** Assume RH. Then either

$$|k\mathcal{P}_{0,j}(t)| > \ell_0$$

for some $0 \leq j \leq J$ or

$$|\zeta(\frac{1}{2} + it)|^{2k} \ll \left( \prod_{i=0}^{j} |\mathcal{N}_{i,j}(t, k)|^{2} + \sum_{0 \leq j \leq J-1 \atop j+1 \leq l \leq J} \exp \left( \frac{2k}{\ell_j} \right) \left( \frac{|k\mathcal{P}_{j+1,l}(t)|}{\ell_{j+1}} \right)^{2s_j} \prod_{i=0}^{j} |\mathcal{N}_{i,j}(t, k)|^{2} \right) \times |\mathcal{M}(t, k)|^{2}$$

for any positive integers $s_j$ where

$$\mathcal{M}(t, k) = \sum_{\Omega(n) \leq 10k(\log_2 T)^2 \atop p|n \Rightarrow p \leq \log T} \frac{(k/2)^{\Omega(n)} g(n)}{n^{1+2it}}.$$

**Proof.** Suppose $|k\mathcal{P}_{0,j}(t)| < \ell_0$. For $0 \leq j \leq J - 1$ let

$$S(j) = \left\{ t \in [T, 2T] : |k\mathcal{P}_{i,l}(t)| \leq \ell_i \quad \forall 1 \leq i \leq j, \quad \forall j \leq l \leq J; \right\} \quad |k\mathcal{P}_{j+1,l}(t)| > \ell_{j+1} \text{ for some } j + 1 \leq l \leq J \text{ for some}$$

and

$$S(J) = \left\{ t \in [T, 2T] : |k\mathcal{P}_{i,l}(t)| \leq \ell_i \quad \forall 1 \leq i \leq J \right\}.$$

Then since $[T, 2T] = \cup_{j=0}^{J} S(j)$, for $t \in [T, 2T]$ we have

$$|\zeta(\frac{1}{2} + it)|^{2k} \leq \mathbb{1}_{t \in S(J)} \cdot |\zeta(\frac{1}{2} + it)|^{2k} + \sum_{0 \leq j \leq J-1 \atop j+1 \leq l \leq J} \mathbb{1}_{t \in S_{i}(j)} \cdot |\zeta(\frac{1}{2} + it)|^{2k}$$

where

$$S_{i}(j) = \left\{ t \in [T, 2T] : \begin{array}{l} |k\mathcal{P}_{i,l}(t)| \leq \ell_i \quad \forall 1 \leq i \leq j, \quad \forall j \leq l \leq J; \quad |k\mathcal{P}_{j+1,l}(t)| > \ell_{j+1} \quad \forall j + 1 \leq l \leq J; \end{array} \right\}.$$

We apply Lemma 6 to each zeta function on the right hand side of (47). If $t \in S_{i}(j)$ then we take $x = T_j$ to give

$$|\zeta(\frac{1}{2} + it)|^{2k} \ll \exp \left( 2k \Re \sum_{p \leq T_j} \frac{\omega_j(p)}{p^{1/2} + it} + 2k \Re \sum_{p \leq \log T} \frac{1}{2p^{1/2} + 2it} + \frac{2k}{\ell_j} \right).$$
For the first sum over primes in the exponential we apply (45). For the second sum we note that, since 
\[ \left| \sum_{p \leq T} \frac{1}{p^{1/2+it}} \right| \leq 2 \log_3 T \leq (\log T)^2, \]
we have
\[ \exp \left( 2k \Re \sum_{p \leq T} \frac{1}{2p^{1/2+it}} \right) = (1 + O(e^{-9k(\log T)^2})) \left| \sum_{\Omega(n) \leq 10k(\log \log T)^2} \frac{(k/2)^{\Omega(n)} g(n)}{n^{1+2it}} \right|^2 \]
by (37) and (38). This is \( \ll |M(t, k)|^2 \). Finally, to capture the small size of the set, we multiply by
\[ \left( \frac{|k P_{j+1,l}(t)|}{\ell_{j+1}} \right)^{2s_j} > 1. \]
If \( t \in S(J) \) then we omit this last step since of course there is no such \( P_{j+1,l}(t) \).

6.3. Proof of the upper bound in Proposition 3. From (42) we are required to show that
\[ \frac{1}{T} \int_T^{2T} |\zeta(1/2+it)| D(t, -k) |D(t, k)| dt \ll \left( \frac{\log T}{\log X} \right)^{k^2}. \]
To apply Lemma 7 we must consider two cases; that where \( |k P_{0,j}(t)| > \ell_0 \) and otherwise. We consider the former case first since this is simpler.

So, let \( E \subset [T, 2T] \) be the subset on which \( |k P_{0,j}(t)| > \ell_0 \), that is, when
\[ \left| \sum_{p \leq T^{1/(\log \log T)^2}} \frac{w_j(p)}{p^{1/2+it}} \right| > \frac{(\log \log T)^{3/2}}{k}. \]
By Chebyshev’s inequality and Lemma 2 we have
\[ \mu(E) \ll \left( \frac{k^2}{(\log \log T)^3} \right)^m \int_T^{2T} \left| \sum_{p \leq T^{1/(\log \log T)^2}} \frac{w_j(p)}{p^{1/2+it}} \right|^{2m} dt \ll T^m! \left( \frac{k^2}{(\log \log T)^3} \right)^m \left( \sum_{p \leq T^{1/(\log \log T)^2}} \frac{1}{p^m} \right)^m \]
provided \( m \leq (1 - o(1))(\log \log T)^2 \) where in the last line we have used \( |w_j(p)| \leq 1 \) for all \( j \). By Stirling’s formula and Mertens’ theorem this is
\[ (48) \quad \mu(E) \ll T^m \left( \frac{k^2 m}{e(\log \log T)^2} \right)^m \leq T e^{-c(\log \log T)^2} \]
for some \( c > 1 \) on choosing \( m = \lfloor \min(1, k^2)(\log \log T)^2 \rfloor \). Therefore by Hölder’s inequality, Harper’s bound for the moments of zeta and (24) we have
\[ \frac{1}{T} \int_E |\zeta(1/2+it)|^{2k} |D(t, -k)|^2 dt \ll e^{-c(\log \log T)^2} (\log T)^{O(1)} = o(1). \]
We may now consider the second case where \(|k\mathcal{P}_{0,j}(t)| > \ell_0\) and accordingly concentrate on the integral

\[
\frac{1}{T} \int_{[T,2T]} |\zeta(\frac{1}{2} + it)|^{2k} |D(t,-k)|^2 dt.
\]

By the second part of Lemma 7 this is

\[
\ll \frac{1}{T} \int_T^{2T} \left( \prod_{i=0}^j |\mathcal{N}_{i,j}(t,k)|^2 + \sum_{0 \leq j \leq J-1 \atop j+1 \leq l \leq J} \exp \left( \frac{2k}{\theta_j} \left( \frac{|k\mathcal{P}_{j+1,l}(t)|}{\ell_{j+1}} \right)^{2s_j} \prod_{i=0}^j |\mathcal{N}_{i,j}(t,k)|^2 \right) \right)
\]

\times |\mathcal{M}(t,k)|^2 |D(t,-k)|^2 dt.
\]

(49)

To compute the resultant integrals we apply the following lemma.

**Lemma 8.** For \(0 \leq s_j \leq 1/(10\theta_j)\) we have

\[
\frac{1}{T} \int_T^{2T} \left| D(t,-k) \right|^2 |\mathcal{M}(t,k)|^2 |\mathcal{P}_{j+1,l}(t)|^{2s_j} \prod_{i=0}^j |\mathcal{N}_{i,j}(t,k)|^2 dt \ll s_j! P_{s_j} \left( \frac{\log T}{\log X} \right)^{k^2}
\]

where

\[
P_{j+1} = \sum_{T_j < p \leq T_{j+1}} \frac{1}{p}.
\]

**Proof.** We write the integrand as a multiple sum. First off, by the multinominal theorem we have

\[
\mathcal{P}_{j+1,l}(t)^{s_j} = \left( \sum_{T_j < p \leq T_{j+1}} \frac{w_l(p)}{p^{1/2+it}} \right)^{s_j} = s_j! \sum_{\Omega(n) = s_j} \frac{w_l(n)g(n)}{n^{1/2+it}}.
\]

Therefore

\[
D(t,-k)\mathcal{M}(t,k)\mathcal{P}_{j+1,l}(t)^{s_j} \prod_{i=0}^j \mathcal{N}_{i,j}(t,k)
\]

\[
= s_j! \sum_{n \in S(X)} \frac{\alpha_{-k}(n)}{n^{1/2+it}} \sum_{\Omega(n) \leq 10^k (\log_2 T)^2 \atop p | n} \frac{(k/2)^{\Omega(n)}g(n)}{n^{1/2+it}} \sum_{\Omega(n) = s_j} \frac{w_l(n)g(n)}{n^{1/2+it}} \times \prod_{i=0}^j \left( \sum_{\Omega(n) \leq 10\ell_i} \frac{k^{\Omega(n)}w_j(n)g(n)}{n^{1/2+it}} \right).
\]
Since \( X \leq T_0 \) we may group together all the sums over \( T_0 \)-smooth numbers as a single sum and write the above as

\[
s_j! \sum_n \frac{\gamma(n)}{n^{1/2 + it}} \sum_{\Omega(n)=s_j} \frac{w_1(n)g(n)}{n^{1/2 + it}} \prod_{i=1}^j \left( \sum_{\Omega(n) \leq 10\ell_i} \frac{k^{\Omega(n)}w_j(n)g(n)}{n^{1/2 + it}} \right)
\]

for some coefficients \( \gamma(n) \) where the product is empty if \( j = 0 \). Since this is a short Dirichlet polynomial we find by (41) that

\[
(50) \quad \frac{1}{T} \int_T^{2T} |D(t, -k)|^2 |M(t, k)|^2 |P_{j+1,l}(t)|^2 s_j \prod_{i=0}^j |N_{i,j}(t, k)|^2 dt
\]

\[
\ll (s_j!)^2 \sum_n \frac{\gamma(n)^2}{n} \sum_{\Omega(n)=s_j} \frac{w_1(n)^2g(n)^2}{n} \prod_{i=1}^j \left( \sum_{\Omega(n) \leq 10\ell_i} \frac{k^{\Omega(n)}w_j(n)^2g(n)^2}{n} \right).
\]

Now, by (40) we explicitly have

\[
\sum_n \frac{\gamma(n)^2}{n} = \sum' \frac{\alpha_0(n_1)\alpha_0(n_4)k^{\Omega(n_2n_5)}(k/2)^{\Omega(n_3n_6)}W(n)}{(n_1n_2n_4n_5)^{1/2}n_3n_6}
\]

where

\[
W(u) = w_j(n_2)w_j(n_5)g(n_2)g(n_3)g(n_5)g(n_6)
\]

and the \( t \) in the sum denotes that \( n_1, n_3 \in S(X) \) and

\[
\begin{align*}
\Omega(n_2), \Omega(n_5) &\leq 10\ell_0 & \Omega(n_3), \Omega(n_6) &\leq 10k(\log_2 T)^2 \\
p | n_2, n_5 &\implies p < T_0 & p | n_3, n_6 &\implies p \leq \log T.
\end{align*}
\]

We first estimate the terms with \( \Omega(n_1), \Omega(n_4) > W_0 \). By the usual arguments, for \( 1 < r < 2 \) these terms are bounded by,

\[
\ll r^{-W_0} \sum_{n_1n_2n_3^2 = n_4n_5n_6^2} \frac{r^{\Omega(n_1)}d_k(n_1)d_k(n_4)k^{\Omega(n_2n_3n_5n_6)}W(n)}{(n_1n_2n_4n_5)^{1/2}n_3n_6} 
\]

\[
=r^{-W_0} \prod_{p \leq T} \left( 1 + \frac{(2r + 2)k^2}{p} + O(p^{-2}) \right) \ll e^{-W_0(\log T)^{6k^2}} = o(1).
\]
We then replace $\alpha - k(n)$ with $\beta - k(n)$ in the remaining sum. The usual arguments also allow us to remove the restrictions on all the $\Omega(n_j)$ at the cost of an error of size $o(1)$. Expressing the resultant sum as an Euler product we get

$$\sum_n \frac{\gamma(n)^2}{n} = \prod_{p \leq X} \left(1 + O(p^{-2})\right) \prod_{X < p < T_0} \left(1 + \frac{k^2}{p} + O(p^{-2})\right) + o(1)$$

$$\times \left(\frac{\log T_0}{\log X}\right)^{k^2}$$

since $\beta - k(n)$ is supported on $X$-smooth numbers where it satisfies $\beta - k(p) = -k$ and $\beta - k(p^m) \ll d_k(p^m)$ for $m \geq 2$.

The second sum in (50) is

$$(s_j!)^2 \sum_{\Omega(n) = s_j} \frac{w_j(n)^2 g(n)^2}{n} \leq s_j! \sum_{p | n \implies T_j \leq p < T_{j+1}} \frac{s_j! g(n)}{n} = s_j! \left(\sum_{T_j \leq p < T_{j+1}} \frac{1}{p}\right)^{s_j},$$

since $g(n) \leq 1$, whilst the third sum is

$$\prod_{i=1}^{j} \sum_{\Omega(n) \leq 10^i} \frac{k^{2\Omega(n)} w_j(n)^2 g(n)^2}{n} \ll \prod_{i=1}^{j} \prod_{T_{i-1} \leq p < T_i} \left(1 + \frac{k^2}{p} + O(p^{-2})\right)$$

$$\times \left(\frac{\log T_i}{\log T_0}\right)^{k^2}.$$

Combining these estimates gives the result. \qed

Completion of proof of upper bound in Proposition 3. Applying Lemma 8 in (49) gives an upper bound of the form

$$\left(\frac{\log T_j}{\log X}\right)^{k^2} + \sum_{0 \leq j \leq J-1} \exp \left(\frac{2k}{\theta_j}\right) s_j! \left(\frac{k P_{j+1}}{\ell_{j+1}^2}\right)^{s_j} \left(\frac{\log T_j}{\log X}\right)^{k^2}.$$

On noting that

$$J - j \ll \log(1/\theta_j), \quad P_{j+1} = \log \left(\frac{\log T_{j+1}}{\log T_j}\right) + o(1) \leq 2, \quad \ell_{j+1}^2 = \theta_j^{-3/2},$$

...
and setting \( s_j = 1/(10\theta_j) \), then by Stirling’s formula this is

\[
\ll \left( \frac{\log T}{\log X} \right)^{k^2} \left( 1 + \sum_{0 < j < J-1} \log \left( \frac{1}{\theta_j} \right) \theta_j^{-1/10\theta_j} c_{1/\theta_j} \theta_j^{3/20\theta_j} \right)
\]

\[
\ll \left( \frac{\log T}{\log X} \right)^{k^2} \left( 1 + \sum_{0 < j < J-1} e^{-C\theta_j^{-1} \log(1/\theta_j)} \right)
\]

for some constants \( c, C > 0 \). Since this last series is bounded the result follows. \( \Box \)

### 7. Lower Bound for \( 0 \leq k \leq 1 \)

We keep a similar setup to the previous section but with a few minor changes. There is no longer any need for the weights \( w_j(p) \) so we can simplify our notation and let

\[
\mathcal{N}_i(t, k) = \sum_{\Omega(n) \leq 10 \ell_i, \quad p | n \implies T_{i-1} < p \leq T_i} \frac{k_{\Omega(n)} \mathcal{g}(n)}{n^{1/2 + it}}
\]

where the \( T_i \) and \( \ell_i \) are as before for \( 0 \leq i \leq J \). As remarked earlier, in this section we take \( T_{-1} = A \) with \( A \) sufficiently large to be chosen later. We then form the product

\[
(51) \quad \mathcal{N}(t, k) := \prod_{i=0}^{J} \mathcal{N}_i(t, k) = \sum_{n \leq Y} \frac{\gamma_k(n)}{n^{1/2 + it}}
\]

for some coefficients \( \gamma_k(n) \) where from (46) we have \( Y = T^{1/50} \). We can think of this as an approximation to \( \zeta(1/2 + it)^k \); It possesses several nice features akin to an Euler product whilst also being a short Dirichlet polynomial.

We acquire our lower bound by applying Hölder’s inequality, the form of which will depend on whether \( 0 < k \leq 1 \) or \( k \geq 1 \). The latter case is somewhat simpler so we give details for the case \( 0 < k \leq 1 \) first. By Hölder’s inequality, we have

\[
\left| \frac{1}{T} \int_{S} \zeta(\frac{1}{2} + it) \mathcal{N}(t, k - 1) \mathcal{N}(t, k) |D(t, -k)|^2 dt \right|
\]

\[
\ll \left( \frac{1}{T} \int_{S} |\zeta(\frac{1}{2} + it)|^{2k} |D(t, -k)|^2 dt \right)^{\frac{1}{k}} \left( \frac{1}{T} \int_{T}^{2T} |\zeta(\frac{1}{2} + it)\mathcal{N}(t, k - 1)|^2 |D(t, -k)|^2 dt \right)^{\frac{1-k}{k}}
\]

\[
\times \left( \frac{1}{T} \int_{T}^{2T} |\mathcal{N}(t, k)|^{\frac{k}{2}} |\mathcal{N}(t, k - 1)|^2 |D(t, -k)|^2 dt \right)^{\frac{k}{2}}.
\]

Since \( D(t, -k) \sim P_X(1/2 + it)^{-k} \) for \( t \in S \) the first integral on the right hand side is \( \ll \frac{1}{T} \int_{T}^{2T} |Z_X(1/2 + it)|^{2k} dt \).
Remark. Note also that in this argument we can modify the definition of $D$ to be

$$D(t, k) = \sum_{p|n \Rightarrow A < p \leq X} \frac{\alpha_k(n)}{n^{1/2+\epsilon} \log n}$$

since the removal of the $A$-smooth numbers from the sum in

$$\exp \left( k \sum_{n \leq X} \frac{\Lambda(n)}{n^{1/2+\epsilon} \log n} \right)$$

leads to a bounded multiplicative factor which can be absorbed into the $\ll$ sign. Also, to save space in the future we may absorb the condition $p|n \Rightarrow A < p \leq X$ into the coefficients $\alpha_k(n)$.

The lower bound in the case $0 < k \leq 1$ now follows from the subsequent Propositions.

**Proposition 4.** Suppose $X \leq \eta_k(\log T)^2(\log_2 T)^2$. Then

$$\frac{1}{T} \int \zeta(\frac{1}{2} + it)N(t, k-1)\overline{N(t, k)} |D(t, -k)|^2 dt \geq C \left( \frac{\log T}{\log X} \right)^{k^2}$$

for some $C > 0$. Assuming RH we may take $X \leq (\log T)^{\theta_k - \epsilon}$.

**Proposition 5.** For $X \leq T^{1/(\log_2 T)^2}$ we have

$$\frac{1}{T} \int_T^{2T} |\zeta(\frac{1}{2} + it)N(t, k-1)|^2 |D(t, -k)|^2 dt \ll \left( \frac{\log T}{\log X} \right)^{k^2}.$$

**Proposition 6.** For $X \leq T^{1/(\log_2 T)^2}$ we have

$$\frac{1}{T} \int_T^{2T} |N(t, k)|^2 |N(t, k-1)|^2 |D(t, -k)|^2 dt \ll \left( \frac{\log T}{\log X} \right)^{k^2}.$$

Note that our argument works unconditionally provided $X \leq \eta_k(\log T)^2(\log_2 T)^2$ as claimed in the introduction.

**7.1. Proof of Proposition 4.** Our first job is to extend the range of integration to the full set $[T, 2T]$. By the usual argument involving Hölder’s inequality the integral over $\mathcal{E}$ is $o(1)$. Indeed, from the conditions on $X$ and Lemma 3 we have $T^{-1} \mu(\mathcal{E}) \ll e^{-2|k|V_0}$ which is enough to kill any power of $\log T$. We also have the second moment bound for the zeta function, and by (24) the $m$th moment of $D(t, -k)$ is also $(\log T)^{O(1)}$. The only new ingredient required is a moment bound for $N(t, k)$.
but by the Montgomery–Vaughan mean value theorem this is, for \( m \leq 50 \),
\[
\frac{1}{T} \int_{T}^{2T} |N(t, k)|^{2m} dt \ll \sum_{n_1 \ldots n_m = n_{m+1} \ldots n_{2m}, n_j \leq T^{1/50}} \frac{\gamma_k(n_1) \ldots \gamma_k(n_{2m})}{(n_1 \ldots n_{2m})^{1/2}}
\]
\[
= \prod_{p \leq T} \left( 1 + \frac{m^2 k^2}{p} + O(p^{-2}) \right) \ll (\log T)^{m^2 k^2}.
\]
Therefore,
\[
\frac{1}{T} \int_{S} \zeta(\frac{1}{2} + it)N(t, k - 1)N(t, k)|D(t, -k)|^2 dt = I_3 + o(1)
\]
where
\[
I_3 = \frac{1}{T} \int_{T}^{2T} \zeta(\frac{1}{2} + it)N(t, k - 1)N(t, k)|D(t, -k)|^2 dt.
\]
To lower bound \( I_3 \) we apply the approximation
\[
\zeta(\frac{1}{2} + it) = \sum_{n \leq T} \frac{1}{n^{1/2 + it}} + O\left(\frac{1}{T^{1/2}}\right).
\]
The other terms of the integrand satisfy pointwise bounds of the form
\[
N(t, k) \ll Y^{1/2 + \epsilon} \ll T^{1/100 + \epsilon}, \quad \sum_{n} \frac{\alpha_k(n)}{n^{1/2 + it}} \ll X^{W_0(1/2 + \epsilon)} \ll T^{1/100},
\]
since \( k^{O(n)} \) has average order \( (\log n)^{k-1} \) and \( \alpha_k(n) \ll d_k(n) \ll n^{\epsilon} \). We then see that the error term in the approximation for zeta leads to an error of size \( o(1) \).

Therefore
\[
I_3 = \frac{1}{T} \int_{T}^{2T} \sum_{n_1 \leq T, n_2, n_3 \leq Y} \frac{\gamma_{k-1}(n_2) \gamma_k(n_3) \alpha_k(n_4) \alpha_k(n_5)}{(n_1 n_2 n_3 n_4 n_5)^{1/2}} \left( \frac{n_3 n_5}{n_1 n_2 n_4} \right)^{it} dt + o(1).
\]
By direct integration, the off-diagonal terms for which \( n_1 n_2 n_4 \neq n_3 n_5 \) lead to an error of size
\[
\frac{Y X^{W_0} T}{T} \sum_{n_1 \leq T, n_2, n_3 \leq Y} \frac{|\gamma_{k-1}(n)| |\gamma_k(n)| \left( \sum_{n} \frac{\alpha_k(n)}{n^{1/2}} \right)^2}{(n_1 n_2 n_3)^{1/2}} \ll \frac{Y^{2+\epsilon} X^{W_0(2+\epsilon)}}{T^{1/2}} = o(1)
\]
since \( |\log(n_3 n_5/n_1 n_2 n_4)| \gg 1/(Y X^{W_0}) \). Accordingly,
\[
I_3 = \sum_{n_1 n_2 n_4 = n_3 n_5, n_1 \leq T, n_2, n_3 \leq Y} \frac{\gamma_{k-1}(n_2) \gamma_k(n_3) \alpha_k(n_4) \alpha_k(n_5)}{(n_1 n_2 n_3 n_4 n_5)^{1/2}} + o(1).
\]
Since \( n_3 n_5 \leq Y X^{W_0} \leq T \) we may remove the condition \( n_1 \leq T \).
Now, since $\alpha_{-k}(n)$ is supported on prime powers $p^m$ with $A < p \leq X$ we may write our sum as

$$\sum_{n_1 n_2 n_4 = n_3 n_5} = \sum_{p \mid n_j \implies A < p \leq X} \sum_{n_1 n_2 = n_3} \cdot \sum_{p \mid n_j \implies X < p \leq T_0} \cdot \prod_{i=1}^{j} \sum_{p \mid n_i \implies T_{i-1} < p \leq T_i}$$

and note that we have essentially taken $n_4 = n_5 = 1$ in the second two sums on the right. Unfolding the coefficients using (51) gives the first sum as

$$\sum_{n_1 n_2 n_4 = n_3 n_5} \frac{(k-1)^{\Omega(n_2) k^{\Omega(n_3)}} g(n_2) g(n_3) \alpha_{-k}(n_4) \alpha_{-k}(n_5)}{(n_1 n_2 n_3 n_4 n_5)^{1/2}}$$

The usual arguments now allow us to replace $\alpha_{-k}(n_j)$ with $\beta_{-k}(n_j)$ and remove the restrictions on $\Omega(n_2), \Omega(n_3)$ at the cost of a term of size $o(1)$. We then express the resultant sum as an Euler product. Since $\beta_{-k}(p) = -k$ we find that the leading terms cancel and that (53) is

$$\prod_{A < p \leq X} \left(1 + O_k(p^{-2})\right) + o(1).$$

On taking $A$ sufficiently large we can guarantee that the term $O_k(p^{-2})$ is always $< 1$ in modulus and hence the above product is $\geq c$ for some constant $c > 0$.

With a similar computation the second sum in (52) is

$$\sum_{p \mid n_2 n_3 \implies X < p \leq T_0} \frac{(k-1)^{\Omega(n_2) k^{\Omega(n_3)}} g(n_2) g(n_3)}{(n_1 n_2 n_3)^{1/2}} + O(e^{-10\delta_0 (\log T_0 / \log X) O(1)})$$

$$= \prod_{X < p \leq T_0} \left(1 + \frac{k^2}{p} + O(p^{-2})\right) + o(1) \geq c \left(\frac{\log T_0}{\log X}\right)^{k^2}.$$

For the sums inside the product in (52), we must be a little more careful. The sums in question are given by

$$\sum_{n_1 n_2 n_3} \frac{(k-1)^{\Omega(n_2) k^{\Omega(n_3)}} g(n_2) g(n_3)}{(n_1 n_2 n_3)^{1/2}}.$$
Since \(0 < k \leq 1\) and \(d(n) \leq 2^{\Omega(n)} \leq e^{\Omega(n)}\), the error incurred from dropping the condition on \(\Omega(n_2)\) is, in absolute value,

\[
\leq e^{-10\ell_i} \sum_{p|n_2, n_3 \Rightarrow T_{i-1} < p \leq T_i} \frac{e^{\Omega(n_2)} g(n_2) g(n_3)}{(n_1 n_2 n_3)^{1/2}} \leq e^{-10\ell_i} \sum_{p|n \Rightarrow T_{i-1} < p \leq T_i} \frac{e^{2\Omega(n)} g(n)}{n}
\]

\[
= \exp \left( -10\ell_i + e^2 \sum_{T_{i-1} < p \leq T_i} \frac{1}{p} \right) \leq \exp \left( -10\ell_i + e^2 \log \left( \frac{\log T_i}{\log T_{i-1}} \right) + o(1) \right) \leq e^{-9\ell_i}
\]

since \(\log \left( \frac{\log T_i}{\log T_{i-1}} \right) \leq 2\) and \(\ell_i \geq 10^9\). Doing the same for \(n_3\) gives an error of the same size and hence the sum in (56) is

\[
\geq \prod_{T_{i-1} < p \leq T_i} \left( 1 + \frac{k^2}{p} + O(p^{-2}) \right) - e^{-8\ell_i}
\]

(57)

\[
\geq (1 - e^{-7\ell_i}) \prod_{T_{i-1} < p \leq T_i} \left( 1 + \frac{k^2}{p} + O(p^{-2}) \right).
\]

Hence, combining (54), (55) and (57) we find that

\[
I_3 \geq C \left( \frac{\log T_0}{\log X} \right)^{k^2} \prod_{i=1}^J \left( 1 - e^{-7\ell_i} \right) \prod_{T_{i-1} < p \leq T_i} \left( 1 + \frac{k^2}{p} + O(p^{-2}) \right) \geq C \left( \frac{\log T}{\log X} \right)^{k^2}
\]

since, again, \(\sum_{i=1}^J e^{-7\ell_i}\) is a rapidly converging series. This completes the proof of Proposition 4.

### 7.2. Proof of Proposition 5

We are required to show

\[
I_4 := \frac{1}{T} \int_T^{2T} |\zeta(1/2 + it)| N(t, k - 1)^2 |D(t, -k)|^2 dt \ll \left( \frac{\log T}{\log X} \right)^{k^2}.
\]

From the conditions on \(X\) given in the statement of the proposition, \(D(t, -k)\) is a Dirichlet polynomial of length \(X^{W_0} = T^{o(1)}\) which is still short. Thus, we can apply the results of [5, 7] after combining the two Dirichlet polynomials into a single sum. This gives

\[
I_2 = \sum_{m,n} \frac{h_k(m) h_k(n)(m,n)}{mn} \log \left( \frac{BT(m,n)^2}{mn} \right) + o(1)
\]

for some constant \(B\) where

\[
h_k(n) = \sum_{n_1 n_2 = n} \gamma_{k-1}(n_1) \alpha_{-k}(n_2).
\]
As in \[27\], we write
\[
\log \left( \frac{BT(m,n)^2}{mn} \right) = \frac{1}{2\pi i} \int_{|z|=1/\log T} \left( \frac{BT(m,n)^2}{mn} \right)^z \frac{dz}{z^2},
\]
so that the main term in \(I_2\) becomes
\[
\frac{1}{2\pi i} \int_{|z|=1/\log T} \sum_{m,n} \frac{h_k(m)h_k(n)(m,n)(BT(m,n)^2)^z}{mn} \frac{dz}{z^2}.\]
Then after a trivial estimate we get
\[
I_4 \ll (\log T)^2 \max_{|z|=1/\log T} \left| \sum_{m,n} \frac{h_k(m)h_k(n)(m,n)^{1+2z}}{(mn)^{1+z}} \right|
\]
and thus we are required to show this last sum is \(\ll \frac{1}{\log X}(\log T/\log X)^{k^2-1}\).
Unfolding the coefficients, the above sum becomes
\[
\sum_{m_1,m_2,n_1,n_2} \frac{\gamma_{k-1}(m_1)\gamma_{k-1}(n_1)\alpha_{k-1}(m_2)\alpha_{k-1}(n_2)(m_1m_2,n_1n_2)^{1+2z}}{(m_1m_2n_1n_2)^{1+z}}.
\]
Estimating the terms with \(\Omega(m_2), \Omega(n_2) \geq W_0\) in the usual way we may replace \(\alpha_{k-1}(n)\) by \(\beta_{k-1}(n)\) and then re-extend the sums at the cost of \(o(1)\). Then by multiplicativity we can express the resultant sum as
\[
\prod_{A<p \leq X} \left( 1 + \sum_{0 \leq j \leq k-1} \frac{(m_1+n_1)(-k)^{m_2+n_2}(p^{m_3+n_2}p)^{1+2z}}{p^{(m_1+m_2+n_1+n_2)(1+z)}} + O(p^{-2}) \right)
\]
where, again, we have taken \(m_2 = n_2 = 1\) in the second two sums since the functions \(\beta_k(n)\) are supported on \(X\)-smooth numbers. As usual, we drop the conditions on \(\Omega(m_j), \Omega(n_j)\) in these sums. Considering, for the moment, the first sum without these conditions we get
\[
\prod_{A<p \leq X} \left( 1 + \sum_{0 \leq j \leq k-1} \frac{(k-1)^{m_1+n_1}(-k)^{m_2+n_2}(p^{m_3+n_2}p)^{1+2z}}{p^{(m_1+m_2+n_1+n_2)(1+z)}} + O(p^{-2}) \right)
\]
\[
= \prod_{A<p \leq X} \left( 1 + \frac{2(k-1) - 2k}{p^{k+1}} + \frac{(k-1)^2 + k^2 - 2k(k-1)}{p} + O(p^{-2}) \right)
\]
\[
= \prod_{A<p \leq X} \left( 1 - \frac{1}{p} + O(p^{-2}) + O\left( \frac{1}{\log T} \log p \right) \right) \ll \frac{1}{\log X}
\]
where we have used $\beta_{-k}(n) \ll d_k(n)$ and $|z| \leq 1/\log T$ along with the bound
\[
\frac{1}{\log T} \sum_{p \leq X} (\log p)/p \ll 1.
\]
As usual, for the error term we apply Rankin’s trick along
with similar Euler product computations to give an error of size
\[
\ll e^{-10\ell_0 + O(\log \log X)}
\]
which is $o(1/\log X)$.

For the second sum we get, with a similar argument,
\[
\sum_{p|m_1n_1 \Rightarrow X < p \leq T_0} \frac{\gamma_{k-1}(m_1)\gamma_{k-1}(n_1)(m_1, n_1)^{1+2z}}{(m_1n_1)^{1+z}}
= \prod_{X < p \leq T_0} \left(1 + \frac{k^2 - 1}{p} + O(p^{-2})\right) + O(e^{-10\ell_0 + O(\log \log T_0/\log X)})
\]
which is $\ll (\log T_0/\log X)^{k^2 - 1}$. Finally, for the product of sums we have
\[
\prod_{i=1}^J \sum_{p|m_1n_1 \Rightarrow T_{i-1} < p \leq T_i} \frac{\gamma_{k-1}(m_1)\gamma_{k-1}(n_1)(m_1, n_1)^{1+2z}}{(m_1n_1)^{1+z}}
= \prod_{i=1}^J \left[ \prod_{T_{i-1} < p \leq T_i} \left(1 + \frac{k^2 - 1}{p} + O(p^{-2})\right) + O(e^{-10\ell_i + O(\log \log T_i/\log T_{i-1})})\right]
\]
since $\log T_i/\log T_{i-1} \leq 2$. This last term is $\ll (\log T_0/\log X)^{k^2 - 1}$ and so combining these
bounds in (58) we get
\[
I_4 \ll \log T \cdot \frac{1}{\log X} \cdot \left(\frac{\log T_0}{\log X}\right)^{k^2 - 1} \cdot \left(\frac{\log T}{\log T_0}\right)^{k^2 - 1} \ll \left(\frac{\log T}{\log X}\right)^{k^2}
\]
which completes the proof of Proposition 5.

7.3. Proof of Proposition 6. We begin with the following lemma from [27].

Lemma 9. Let
\[
\mathcal{P}_j(t) := \sum_{T_{j-1} < p \leq T_j} \frac{1}{p^{1/2 + \sigma}}.
\]
Then for $0 \leq j \leq J$
\[
|\mathcal{N}_j(t, k-1)\mathcal{N}_j(t, k)^{1/2}|^2 \leq |\mathcal{N}_j(t, k)|^2 (1 + O(e^{-9\ell_j})) + O(Q_j(t)),
\]

where the implied constants are absolute, and

\[ Q_j(t) = \left( \frac{e|P_j(t)|}{10\ell_j} \right)^{20\ell_j} \sum_{r=0}^{10\ell_j/k} \left( \frac{2e|P_j(t)|}{r+1} \right)^{2r} . \]

**Proof.** This is essentially Lemma 1 of [27]. Our sequence \( T_j \) is defined slightly differently but one can check that this makes no difference to the end result. \( \square \)

**Lemma 10.** With the above notation

\[ \frac{1}{T} \int_T^{2T} Q_0(t)|D(t, -k)|^2 dt \ll e^{-10\ell_0} (\log X)^{2k^2} \]

and for \( 1 \leq j \leq J \)

\[ \frac{1}{T} \int_T^{2T} Q_j(t) dt \ll e^{-10\ell_j} . \]

**Proof.** We prove the first bound since this is new, the second bound follows similarly (and is essentially Lemma 2 of [27]). Let \( L = 10\ell_0 \). From the definition of \( Q_0(t) \) we have

\[ \frac{1}{T} \int_T^{2T} Q_0(t)|D(t, -k)|^2 dt = \left( \frac{12L}{L} \right)^{2L} \sum_{r=0}^{L/k} \left( \frac{2e}{r+1} \right)^{2r} \cdot \frac{1}{T} \int_T^{2T} |P_0(t)|^{2L+2r} |D(t, -k)|^2 dt . \]

By the Cauchy–Schwarz inequality and (23) the integral is

\[ \ll (\log X)^{2k^2} \left( \frac{1}{T} \int_T^{2T} |P_0(t)|^{2L+2r} |D(t, -k)|^2 dt \right)^{1/2} . \]

Since

\[ P_0(t)^{2L+2r} = (2L + 2r)! \sum_{\Omega(n) = 2L+2r} \frac{g(n)}{n^{1/2+ir}} , \]

the Montgomery–Vaughan mean value theorem gives that our original integral is

\[ \ll (\log X)^{2k^2} \left( \frac{12L}{L} \right)^{2L} \sum_{r=0}^{L/k} \left( \frac{2e}{r+1} \right)^{2r} \cdot (2L + 2r)! \sum_{\Omega(n) = 2L+2r} \frac{g(n)^2}{n} \right)^{1/2} . \]

(59)
since $g(n)^2 \leq g(n)$. Letting $P = \sum_{p \leq T} p^{-1}$ we find by Stirling’s formula that the summand is
\[ \ll (2/e)^L (8e)^r (r + 1)^{-2r} (L + r)^{L + r + 1/4} P^{L + r} \]
which is maximised at the solution of $r^2 = 8P(L + r)(1 + O(1/r))$. Since $P \leq 2 \log \log T = o(L)$ this solution $r = r_0$ satisfies $2\sqrt{2}(PL)^{1/2} \leq r_0 \leq 3(PL)^{1/2}$. Therefore, (59) is
\[ \ll (\log X)^{2k^2} \left( \frac{C}{L} \right)^{2L} \frac{L}{k} \cdot (3L!)^{1/2} P^{L + r_0} r_0^{-2r_0} \]
since $2r_0 \leq L$. This is then
\[ \ll (\log X)^{2k^2} L^{-L/2 + o(1)} \ll e^{-10\ell_0} \]
and the result follows. \qed

By Lemma 9 and (41) we find that
\[
I_5 := \frac{1}{T} \int_T^{2T} |N(t, k)|^2 |N(t, k - 1)|^2 |D(t, -k)|^2 dt \\
\ll \frac{1}{T} \int_T^{2T} \left( |N_0(t, k)|^2 (1 + O(e^{-10\ell_0})) + O(Q_0(t)) \right) |D(t, -k)|^2 dt \\
\times \prod_{j=1}^{J+1} \frac{1}{T} \int_T^{2T} \left( |N_j(t, k)|^2 (1 + O(e^{-10\ell_j})) + O(Q_j(t)) \right) dt
\]
since $J \asymp \log \log \log T$. By Lemma 10 we have
\[
\prod_{j=1}^{J} \frac{1}{T} \int_T^{2T} \left( |N_j(t, k)|^2 (1 + O(e^{-10\ell_j})) + O(Q_j(t)) \right) dt \\
= \prod_{j=1}^{J} \left( 1 + O(e^{-10\ell_j}) \right) \sum_{\Omega(n) \leq 10\ell_j} \frac{k^{2\Omega(n)} g(n)^2}{n} + O(e^{-10\ell_j})
\]
(60)
\[
\ll \prod_{j=1}^{J} \prod_{T - 1 < p \leq T_i} \left( 1 + \frac{k^2}{p} + O(p^{-2}) \right) \ll \left( \frac{\log T}{\log T_0} \right)^{k^2}.
\]
By Lemma 10 again we find
\[
\frac{1}{T} \int_T^{2T} \left( |N_0(t, k)|^2 (1 + O(e^{-10\ell_0})) + O(Q_0(t)) \right) |D(t, -k)|^2 dt \\
= (1 + O(e^{-10\ell_0})) \frac{1}{T} \int_T^{2T} |N_0(t, k)|^2 \left| \sum_n \frac{\alpha_k(n)}{n^{1/2 + it}} \right|^2 dt + o(1).
\]
This last integral is
\[
\sum_{m_1 n_1 = m_2 n_2 \atop \Omega(m_j) \leq 1060, p | m_j \Rightarrow A < p \leq T_0} \frac{k^{\Omega(m_1) + \Omega(m_2)} g(m_1) g(m_2) \alpha_k(n_1) \alpha_k(n_2)}{(m_1 m_2 n_1 n_2)^{1/2}} + o(1).
\]

The usual arguments allow us to remove the conditions on \(\Omega(m_j)\) and replace \(\alpha_k(n)\) by \(\beta_k(n)\) at the cost of \(o(1)\). We then find that the resultant sum is
\[
\prod_{A < p \leq X} \left(1 + O(p^{-2})\right) \prod_{X < p \leq T_0} \left(1 + \frac{k^2}{p} + O(p^{-2})\right) \ll \left(\frac{\log T_0}{\log X}\right)^{k^2}
\]
since the leading terms cancel over \(A < p \leq X\). Combining this with (60) gives
\[
I_5 \ll \left(\frac{\log T}{\log X}\right)^{k^2}
\]
thus completing the proof of Proposition 6.

8. The lower bound of Proposition 3 for \(k \geq 1\)

The lower bound for \(k \geq 1\) is similar to the case \(0 \leq k \leq 1\), if not a little simpler. In this case we take
\[
T_{-1} = Bk^2
\]
for some \(B > 0\) to be chosen and we alter the definition of \(J\) slightly so that it is the maximal integer such that \(e^j / (\log \log T)^2 \leq 1 / (10^{12} k^4)\). This implies that \(\ell_j = 10^9 k^3\) for all \(j \leq J\).

We perform Hölder’s inequality in the form
\[
|T \int_{S} \zeta(\frac{1}{2} + it) N(t, k - 1) \overline{N(t, k)} | D(t, -k) |^2 dt |
\ll \left( \int_{T} |Z_X(\frac{1}{2} + it)|^{2k} dt \right)^{1/k} \left( \int_{T} |N(t, k - 1) \overline{N(t, k)} |^{2k} |D(t, -k)|^2 dt \right)^{1/2k}
\]
where again we have used Lemma 5. The integral on the left can be dealt with rather similarly to Proposition 4 although the change in parameters requires some modifications. We detail these alterations first before dealing with the second integral on the right.
8.1. **Modifying the proof of Proposition 4.** We see that we can arrive at (52) in exactly the same way. When dealing with (53), the errors incurred from dropping the conditions on $\Omega(n_2)$ etc. are now

$$\ll e^{-10\ell_i}(\log X)^{Ck^2}$$

which of course is still $o(1)$ since $k$ is fixed. Note that on writing (53) as a single sum $\sum f(n)/n$, the coefficients are supported on $X$-smooth numbers and satisfy the bounds $|f(n)| \leq k^{2\Omega(n)}d_3(n)^2 \leq (3k)^{2\Omega(n)}$. Then we find that the equivalent of (53) is

$$\prod_{Bk^2 < p \leq X} \left( 1 + O\left( \frac{k^4}{p^2} \right) \right) + o(1) \geq C$$

for some $C$ on taking $B$ large enough. In a similar way we find that the equivalent of (55) is

$$\prod_{X < p < T_0} \left( 1 + \frac{k^2}{p} + O\left( \frac{k^4}{p^2} \right) \right) + o(1) \geq C \left( \frac{\log T_0}{\log X} \right)^{k^2}.$$

It remains to deal with the term

$$(62) \prod_{i=1}^{J} \sum_{n_1n_2n_3 \leq T_i, \Omega(n_2),\Omega(n_3) \leq 10\ell_i} \frac{(k-1)^{\Omega(n_2)}k^{\Omega(n_3)}g(n_2)g(n_3)}{(n_1n_2n_3)^{1/2}}$$

in the current context. The error from removing the condition on $\Omega(n_2)$ in the sums is

$$\leq e^{-10\ell_i} \sum_{p|n_2, n_3 \rightarrow T_i-1 < p < T_i} \frac{(ek)^{\Omega(n_2)}k^{\Omega(n_3)}g(n_2)g(n_3)}{(n_1n_2n_3)^{1/2}}$$

$$\leq e^{-10\ell_i} \sum_{p|n \rightarrow T_i-1 < p < T_i} \frac{(ek)^{2\Omega(n)}g(n)}{n} = \exp \left( -10\ell_i + e^2k^2 \sum_{T_i-1 < p < T_i} \frac{1}{p} \right)$$

$$\leq \exp \left( -10\ell_i + e^2k^2 \log\left( \frac{\log T_i}{\log T_{i-1}} \right) + o(1) \right) \leq e^{-9\ell_i}$$

where we have used $d(n) \leq 2^{\Omega(n)} \leq e^{\Omega(n)}$. Then (62) is

$$\geq \prod_{i=1}^{J} \left( \prod_{T_i-1 < p < T_i} \left( 1 + \frac{k^2}{p} + O\left( \frac{k^4}{p^2} \right) \right) - e^{-8\ell_i} \right)$$

$$\geq \prod_{i=1}^{J} \left( 1 - e^{-7\ell_i} \right) \prod_{T_i-1 < p \leq T_i} \left( 1 + \frac{k^2}{p} + O\left( \frac{k^4}{p^2} \right) \right) \geq C \left( \frac{\log T_i}{\log T_0} \right)^{k^2}.$$
Combining these we get the desired bound

\[(63) \quad \frac{1}{T} \int_S \zeta(\frac{1}{2} + it)N(t, k - 1)\overline{N(t, k)}|D(t, -k)|^2 dt \geq C \left( \frac{\log T}{\log X} \right)^{k^2}.\]

8.2. The remaining integral. We let

\[I_6 = \frac{1}{T} \int_T^{2T} |N(t, k - 1)N(t, k)|^{\frac{2k}{2k-1}}|D(t, -k)|^2 dt.\]

Then Proposition 3 in the case \(k \geq 1\) will follow from Hölder’s inequality (61) and (63) if we can show that

\[I_6 \ll \left( \frac{\log T}{\log X} \right)^{k^2}.\]

Lemma 11. For \(0 \leq i \leq J\) we have

\[|N_i(t, k - 1)N_i(t, k)|^{\frac{2k}{2k-1}} \leq (1 + O(e^{-9\ell_i}))|N_i(t, k)|^2 + \left( \frac{ek|P_i(t)|}{10\ell_i} \right)^{40\ell_i}.\]

Proof. If \(k|P_i(t)| \leq 10\ell_i\) then by (37) we have

\[|N_i(t, k - 1)N_i(t, k)|^{\frac{2k}{2k-1}} = (1 + O(e^{-9\ell_i})) \left| \exp \left( (k - 1)P_i(t) + kP_i(t) \right) \right|^{\frac{2k}{2k-1}} \]

\[= (1 + O(e^{-9\ell_i})) \left| \exp \left( kP_i(t) \right) \right|^2 \]

\[= (1 + O(e^{-9\ell_i}))|N_i(t, k)|^2.\]

If \(k|P_i(t)| > 10\ell_i\) then

\[|N_i(t, k - 1)| \leq \sum_{r=0}^{10\ell_i} \frac{(k - 1)|P_i(t)|^r}{r!} \leq (k|P_i(t)|)^{10\ell_i} \sum_{r=0}^{10\ell_i} (10\ell_i)^{r-10\ell_i} \frac{1}{r!} \]

\[\leq \left( \frac{ek|P_i(t)|}{10\ell_i} \right)^{10\ell_i}.\]

The same bound holds for \(|N_i(t, k)|\) and hence the result follows since \(2k/(2k - 1) \leq 2\). \(\square\)

Lemma 12. We have

\[\frac{1}{T} \int_T^{2T} \left( \frac{ek|P_0(t)|}{10\ell_0} \right)^{40\ell_0} |D(t, -k)|^2 dt \ll e^{-10\ell_0}(\log X)^{2k^2}\]

and for \(1 \leq i \leq J\).

\[\frac{1}{T} \int_T^{2T} \left( \frac{ek|P_i(t)|}{10\ell_i} \right)^{40\ell_i} dt \ll e^{-10\ell_i}.\]
Proof. We prove the first formula since the second follows similarly. By the Cauchy–Schwarz inequality and (23) we have
\[
\frac{1}{T} \int_T^{2T} |P_0(t)|^{4\ell_0} |D(t, -k)|^2 dt \ll (\log X)^{2k^2} \left( \frac{1}{T} \int_T^{2T} |P_0(t)|^{4\ell_0} dt \right)^{1/2}.
\]
Letting \( L = 10\ell_0 = 10(\log \log T)^{3/2} \) the last integral is
\[
\ll (4L)^2 \sum_{\Omega(n)=4L, p|n} \frac{g(n)^2}{n} \leq (4L)! \left( \sum_{T-1 < p < T_0} \frac{1}{p} \right)^{4L} \leq (4L)! (\log \log T)^{4L}
\]
using \( g(n)^2 \leq g(n) \). Thus, the original integral is
\[
\ll (\log X)^{2k^2} \left( \frac{e^k}{L} \right)^{4L} (4L)!^{1/2} \left( \frac{L}{10} \right)^{4L/3} \ll (\log X)^{2k^2} L^{-2L/3} e^L
\]
by Stirling’s formula. The result follows.

As in subsection 7.3 we now combine Lemmas 11 and 12 along with (41) and the usual calculations to give
\[
I_6 \ll \left( \frac{\log T}{\log X} \right)^{k^2}.
\]
This completes the proof of Proposition 3 in the case \( k \geq 1 \).

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