ROHLIN FLOWS ON VON NEUMANN ALGEBRAS

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Abstract. We will introduce the Rohlin property for flows on von Neumann algebras and classify them up to strong cocycle conjugacy. This result provides alternative approaches to some preceding results such as Kawahigashi’s classification of flows on the injective type II₁ factor, the classification of injective type III factors due to Connes, Krieger and Haagerup and the non-fullness of type III₀ factors. Several concrete examples are also studied.

1. Introduction

In this paper, we study flows on von Neumann algebras. Our purpose is to classify highly outer flows called Rohlin flows.

A flow, that is, a one-parameter automorphism group, appears in many scenes in the theory of operator algebras, and it has attracted attention among operator algebraists. We have known some examples of classification of non-periodic flows on injective factors. In [19], Haagerup has solved the Connes’ bicentralizer problem for injective type III₁ factors. As an important consequence, the uniqueness of the injective type III₁ factor follows. In other words, trace scaling flows on the injective type II₁ factor are (cocycle) conjugate to one another if their Connes-Takesaki modules are equal. In the type II₁ setting, Kawahigashi has studied several kinds of flows on the injective type II₁ factor [30, 31, 32, 33]. Among them, he has obtained the classification of flows on the injective type II₁ factor such that they have the full Connes spectrum and fix a Cartan subalgebra.

We can expect that these examples may possess some sort of right “outerness”, and consequently they are classifiable. Thus it is a natural attempt to give a comprehensive method of classifying flows on von Neumann algebras. In classification of group actions, “outerness”, which, to be precise, includes the central freeness, is considered as an essentially important notion. In this point, the usual pointwise outerness is known to be not so sufficiently strong that we can classify flows up to cocycle conjugacy. Indeed, Kawahigashi has found a family of non-cocycle conjugate outer flows on the injective factor of type II₁ [33]. Thus it is conceivable that both pointwise outerness and pointwise central non-triviality are not right notions of “outerness” for flows.

One formulation of “outerness” is to observe how non-trivially a given group is acting on a central sequence algebra. This is the case for actions of discrete amenable groups [6, 26, 29, 32] or duals of compact groups [17, 19]. A flow, however, causes a serious problem concerning discontinuity on a central sequence algebra.
algebra $\mathcal{M}_\omega$. One prescription of that is to focus on the much smaller subalgebra $\mathcal{M}_{\omega,\alpha}$, which consists of $(\alpha, \omega)$-equicontinuous sequences (see Definition 3.4). Then the Rohlin property, which has been introduced by Kishimoto to flows on $C^*$-algebras [37] and later by Kawamuro to flows on finite von Neumann algebras [35], can be a candidate of “outerness”. This property means that we can find out a unitary eigenvector in $\mathcal{M}_{\omega,\alpha}$ with the eigenvalue $p$ for any $p \in \mathbb{R}$.

Assuming the Rohlin property, we will prove the following main theorem of this paper (Theorem 5.14).

**Theorem 1.** Let $\alpha, \beta$ be Rohlin flows on a von Neumann algebra with separable predual. Then $\alpha$ and $\beta$ are strongly cocycle conjugate if and only if $\alpha_t \beta_{-t}$ is approximately inner for all $t \in \mathbb{R}$.

We emphasize that either the factoriality or the injectivity are not required in our assumption. For injective factors, we obtain the following result in terms of the Connes-Takesaki module (Corollary 5.15).

**Corollary 2.** Let $\alpha, \beta$ be Rohlin flows on an injective factor. Then $\alpha$ and $\beta$ are strongly cocycle conjugate if and only if $\text{mod}(\alpha_t) = \text{mod}(\beta_t)$ for all $t \in \mathbb{R}$.

It turns out that if a flow $\alpha$ on the injective type II$_1$ factor fixes a Cartan subalgebra and the Connes spectrum $\Gamma(\alpha)$ equals $\mathbb{R}$, then $\alpha$ has the Rohlin property. Thus Theorem 1 implies the following Kawahigashi’s result (Theorem 6.4).

**Theorem 3** (Kawahigashi). Let $\alpha$ be a flow on the injective type II$_1$ factor $\mathcal{M}$. If $\alpha$ pointwise fixes a Cartan subalgebra of $\mathcal{M}$ and $\Gamma(\alpha) = \mathbb{R}$, then $\alpha$ is cocycle conjugate to a product type flow, and absorbs any product type flows. Thus such action $\alpha$ is unique up to cocycle conjugacy.

Thanks to works due to Connes and Haagerup, a modular automorphism group on any injective factor is an approximately inner flow, and hence the dual flow has the Rohlin property (Theorem 4.11, Proposition 4.19). Then Theorem 1 implies the following result (Theorem 6.17).

**Theorem 4** (Connes, Haagerup, Krieger). Let $\mathcal{M}_1$ and $\mathcal{M}_2$ be injective factors of type III. Then they are isomorphic if and only if their flows of weights are isomorphic.

This paper is organized as follows. In Section 2, the basic notions such as the core of a von Neumann algebra and an ultraproduct von Neumann algebra are reviewed.

In Section 3, to a Borel map $\alpha: \mathbb{R} \to \text{Aut}(\mathcal{M})$, we introduce the notion of $(\alpha, \omega)$-equicontinuity and the $(\alpha, \omega)$-equicontinuous parts $\mathcal{M}_\alpha^\omega$ and $\mathcal{M}_{\omega,\alpha}$ of $\mathcal{M}$, respectively.

In Section 4, the Rohlin property and the invariant approximate innerness are introduced. We show they are dual notions to each other.

Section 5 is devoted to proving the main classification result. We first prove the 2-cohomology vanishing for Borel cocycle actions of $\mathbb{R}$ with Rohlin property. We next obtain the approximate vanishing of the 1-cohomology of a Rohlin flow. We
show that by disintegration, it suffices to prove the main theorem for centrally ergodic flows. Then the Bratteli-Elliott-Evans-Kishimoto intertwining argument achieves strong cocycle conjugacy.

In Section 6, we apply the main result to give alternative proofs of some known results: Kawahigashi’s results about flows on the injective type $\text{II}_1$ factor, the classification of injective type III factors (assuming Haagerup’s work on a bicentralizer) and the non-fullness of an arbitrary type III$_0$ factor, more precisely, the approximate innerness of a modular automorphism group. We also discuss results obtained by Hui and Aoi-Yamanouchi in [1, 22]. Some concrete examples of Rohlin flows are given. In particular, we will classify product type flows and quasi-free flows coming from a Cuntz algebra up to cocycle conjugacy.

In Section 7, we will give a characterization of the Rohlin property which states that a flow $\alpha$ on a factor $M$ has the Rohlin property if and only if $\alpha$ is faithful on $M_{\omega,\alpha}$.

In Section 8, we will pose a plausible conjecture on a characterization of the Rohlin property. Some unsolved problems are also mentioned.

We will close this paper with appendix in Section 9, where basic results on measure theory and a disintegration of automorphisms are studied. Also, with some assumptions on a factor, we will show that the condition of Theorem 1 derives an approximation of $\alpha_t$ by $\text{Ad} \nu(t) \circ \beta_t$ with $\nu$ being a continuous unitary path.

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2. Preliminary

Throughout this paper, we mainly treat a von Neumann algebra with separable predual unless otherwise noted.

2.1. Notation. Let $\mathcal{M}$ be a not necessarily separable von Neumann algebra. Let us denote by $\mathcal{M}^U$, $\mathcal{M}^P$, $\mathcal{M}^{Pl}$ and $\mathcal{M}_1$ the set of unitaries, projections, partial isometries and contractions in $\mathcal{M}$, respectively. The center of $\mathcal{M}$ is denoted by $Z(\mathcal{M})$. The set of faithful normal semifinite weights is denoted by $W(\mathcal{M})$.

For $\alpha \in \text{Aut}(\mathcal{M})$, $a \in \mathcal{M}$ and $\varphi \in \mathcal{M}_\tau$, let $\alpha(\varphi)$, $a \varphi$, $\varphi a$, $[a, \varphi] \in \mathcal{M}_\tau$ be

\[
\alpha(\varphi) := \varphi \circ \alpha^{-1}, \quad a \varphi(x) := \varphi(ax), \quad a \varphi(x) := \varphi(xa), \quad [a, \varphi] := a \varphi - \varphi a,
\]

respectively. For $a \in \mathcal{M}$ and $\varphi \in (\mathcal{M}_\tau)_+$, we define the following seminorms:

\[
\|a\|_\varphi := \varphi(a^* a)^{1/2}, \quad \|a\|_\varphi^\sharp := 2^{-1/2}(\varphi(a^* a) + \varphi(aa^*))^{1/2}.
\]

In this paper, $\{\mathcal{H}, J, \mathcal{P}\}$ denotes the standard Hilbert space of $\mathcal{M}$ (see [17] for the notations). We regard $\mathcal{H}$ as an $\mathcal{M}$-$\mathcal{M}$-bimodule as follows:

\[
x \xi y := x J y^* J \xi, \quad x, y \in \mathcal{M}, \quad \xi \in \mathcal{H}.
\]

For $\alpha \in \text{Aut}(\mathcal{M})$, there uniquely exists a unitary $U(\alpha)$ on $\mathcal{H}$ such that $\alpha(x) = \text{Ad} U(\alpha)(x)$ for $x \in \mathcal{M}$, $J U(\alpha) = U(\alpha) J$ and $U(\alpha) \mathcal{P} = \mathcal{P}$. We use the notation as
\[ \alpha(\xi) := U(\alpha)\xi. \] Then we have \[ \alpha(x\xi y) = \alpha(x)\alpha(\xi)\alpha(y). \] Since \( U(\Ad u) = uJuJ \) for \( u \in \mathcal{M}^U \), we have \( \Ad u(\xi) = u\xi u^* \).

We equip \( \Aut(\mathcal{M}) \) with the \( u \)-topology as usual. Namely, a net \( \alpha_\lambda \in \Aut(\mathcal{M}) \) converges to \( \alpha \in \Aut(\mathcal{M}) \) if \( \alpha_\lambda(\varphi) \to \alpha(\varphi) \) for all \( \varphi \in \mathcal{M}_r \). Then the map \( \Aut(\mathcal{M}) \ni \alpha \mapsto U(\alpha) \) is strongly continuous. If \( \mathcal{M} \) is separable, that is, \( \mathcal{M}_r \) is norm separable, then \( \Aut(\mathcal{M}) \) is a Polish group.

Let us denote by \( \Int(\mathcal{M}) \) the set of inner automorphisms. An automorphism which belongs to the closure \( \overline{\Int(\mathcal{M})} \) of \( \Int(\mathcal{M}) \) is said to be \textit{approximately inner}.

Throughout this paper, we always equip \( \mathbb{R}^n \) with the usual Lebesgue measure.

### 2.2. Actions and cocycle actions.

In this paper, we mean by a \textit{flow} a one-parameter automorphism group on a von Neumann algebra, that is, a group homomorphism \( \alpha : \mathbb{R} \to \Aut(\mathcal{M}) \) with the following continuity:

\[
\lim_{t \to 0} \|\alpha_t(\varphi) - \varphi\| = 0 \quad \text{for all } \varphi \in \mathcal{M}_r,
\]

or equivalently,

\[
\lim_{t \to 0} \|\alpha_t(\xi) - \xi\| = 0 \quad \text{for all } \xi \in \mathcal{H}.
\]

By \( \mathcal{M}^\alpha \), we denote the fixed point algebra of \( \alpha \). We say that \( \alpha \) is \textit{ergodic} if \( \mathcal{M}^\alpha = \mathbb{C} \), and \textit{centrally ergodic} if \( Z(\mathcal{M})^\alpha = \mathbb{C} \).

A flow \( \alpha \) is said to be \textit{inner} if \( \alpha_t \in \Int(\mathcal{M}) \) for all \( t \in \mathbb{R} \), and \textit{outer} if \( \alpha_t \notin \Int(\mathcal{M}) \) for all \( t \in \mathbb{R} \setminus \{0\} \). Thanks to \cite{27}, Theorem 0.1 or \cite{50} Theorem 5, if \( \mathcal{M} \) is separable, then an inner flow \( \alpha \) is implemented by a one-parameter unitary group \( u : \mathbb{R} \to \mathcal{M}^U \). See also Corollary \ref{corollary-inner-flow}.

An \( \alpha \)-\textit{cocycle} means a strongly continuous unitary path \( v \) in \( \mathcal{M} \) such that \( v(s)\alpha_s(v(t)) = v(s + t) \). The perturbed flow is defined by \( \alpha^\nu_t := \Ad v(t) \circ \alpha_t \).

Let \( \alpha \) and \( \beta \) be flows on von Neumann algebras \( \mathcal{M} \) and \( \mathcal{N} \), respectively. They are said to be

- \textit{conjugate} if there exists an isomorphism \( \theta : \mathcal{N} \to \mathcal{M} \) such that \( \alpha_t = \theta \circ \beta_t \circ \theta^{-1} \). We write \( \alpha \approx \beta \);
- \textit{cocycle conjugate} if there exist an isomorphism \( \theta : \mathcal{N} \to \mathcal{M} \) and an \( \alpha \)-cocycle \( v \) such that \( \alpha^\nu_t = \theta \circ \beta_t \circ \theta^{-1} \). We write \( \alpha \sim \beta \);
- \textit{stably conjugate} if \( \alpha \otimes \id_{\mathcal{B}(\mathcal{N})} \) and \( \beta \otimes \id_{\mathcal{B}(\mathcal{N})} \) are cocycle conjugate.

When \( \mathcal{M} = \mathcal{N} \), \( \alpha \) and \( \beta \) are said to be \textit{strongly cocycle conjugate} if there exist \( \theta \in \overline{\Int(\mathcal{M})} \) and an \( \alpha \)-cocycle \( v \) such that \( \alpha^\nu_t = \theta \circ \beta_t \circ \theta^{-1} \).

A \textit{Borel cocycle action} means a pair \( (\alpha, c) \) of Borel maps \( \alpha : \mathbb{R} \to \Aut(\mathcal{M}) \) and \( c : \mathbb{R}^2 \to \mathcal{M}^U \) such that for all \( r, s, t \in \mathbb{R}, c(s, 0) = 1 = c(0, s), \alpha_0 = \id \) and

\[
\alpha_s \circ \alpha_t = \Ad c(s, t) \circ \alpha_{s+t},
\]

\[
c(r, s)c(r + s, t) = \alpha_r(c(s, t))c(r, s + t).
\]

The perturbation of \( (\alpha, c) \) by a Borel unitary path \( v : \mathbb{R} \to \mathcal{M}^U \) is the Borel cocycle action \( (\alpha^\nu, c^\nu) \) defined by

\[
\alpha^\nu_t := \Ad v(t) \circ \alpha_t, \quad c^\nu(s, t) := v(s)\alpha_s(v(t))c(s, t)v(s + t)^* \quad \text{for all } s, t \in \mathbb{R}.
\]
As is well-known, if $\mathcal{M}$ is properly infinite, then any 2-cocycle is a coboundary. However, the solution presented below is always “big” even if a given 2-cocycle is close to 1.

**Lemma 2.1.** Let $\mathcal{M}$ be a properly infinite von Neumann algebra and $(\alpha, c)$ a Borel cocycle action of $\mathbb{R}$ on $\mathcal{M}$. Then there exists a Borel unitary path $u(t) \in \mathcal{M}$ such that $u(t)\alpha_s(u(s))c(t, s)u(t + s)^* = 1$ for all $(t, s) \in \mathbb{R}^2$.

**Proof.** Let $H$ be a separable infinite dimensional Hilbert space. Regard $B(H)$ as a von Neumann subalgebra of $\mathcal{M}$ such that $B(H)' \cap \mathcal{M}$ is properly infinite. Let $\{e_{ij}\}_{i,j=1}^\infty$ be a system of matrix units of $B(H)$ such that $\sum_i e_{ii} = 1$ and $e_{11}$ is minimal in $B(H)$. Take an isometry $v$ with $vv^* = e_{11}$. Set $w(t) := \sum_i e_{ii} v\alpha_t(v^* e_{11})$. It is easy to see $w(t)$ is a Borel unitary path, and $w(t)\alpha_s(e_{ij})w(t)^* = e_{ij}$.

Hence we may and do assume that $(\alpha, c)$ is of the form $(\beta \otimes \text{id}, d \otimes 1)$ on $\mathcal{M} = N \otimes B(L^2(\mathbb{R}))$ for a von Neumann algebra $N \subset B(K)$ and a Hilbert space $K$. As given in the proof of [56, Proposition 2.1.3], the following $u(t)$ does the job:

$$(u(t)\xi)(s) = d(t, s)\xi(t + s) \quad \text{for all } \xi \in K \otimes L^2(\mathbb{R}), \; s, t \in \mathbb{R}.$$

$\square$

**Remark 2.2.** In the proof above, it turns out that the unitary path $w$ is in fact an $\alpha$-cocycle when $\alpha$ is a flow. Thus if $\alpha$ is a flow on a properly infinite von Neumann algebra $\mathcal{M}$, then $\alpha \sim \alpha \otimes \text{id}_{B(H)}$. Indeed,

$$\alpha \sim \beta \otimes \text{id}_{B(H)} \approx \beta \otimes \text{id}_{B(H)} \otimes \text{id}_{B(H)} \sim \alpha \otimes \text{id}_{B(H)}.$$

Hence the stable conjugacy implies the cocycle conjugacy if $\mathcal{M}$ is properly infinite. When $\mathcal{M}$ is finite, this is not true in general (see [33, Theorem 2.9]).

Let $\alpha$ be a flow on $\mathcal{M}$. We define $\pi_\alpha(x), \lambda^\alpha(t) \in \mathcal{M} \otimes B(L^2(\mathbb{R}))$ for $x \in \mathcal{M}$ and $t \in \mathbb{R}$ as follows:

$$(\pi_\alpha(x)\xi)(s) = \alpha_{-s}(x)\xi(s), \quad (\lambda^\alpha(t)\xi)(s) = \xi(s - t) \quad \text{for } \xi \in \mathcal{H} \otimes L^2(\mathbb{R}), \; s \in \mathbb{R}.$$ Then the crossed product $\mathcal{M} \rtimes_\alpha \mathbb{R}$ is the von Neumann algebra generated by $\pi_\alpha(\mathcal{M})$ and $\lambda^\alpha(\mathbb{R})$. Note that $\lambda^\alpha(t) = 1 \otimes \lambda(t)$, where $\lambda(t)$ denotes the left regular representation. Let us denote by $\rho(t)$ the right regular representation.

The dual flow $\hat{\alpha}$ on $\mathcal{M} \rtimes_\alpha \mathbb{R}$ is defined as

$$\hat{\alpha}_p(\pi_\alpha(x)) = \pi_\alpha(x), \quad \hat{\alpha}_p(\lambda^\alpha(t)) = e^{-ip\lambda^\alpha(t)} \quad \text{for } x \in \mathcal{M}, \; p, t \in \mathbb{R}.$$

2.3. **Core and canonical extension.** The core $\tilde{\mathcal{M}}$ of a von Neumann algebra $\mathcal{M}$ is introduced in [16], and that is generated by a copy of $\mathcal{M}$ and a one-parameter unitary group $\{\varphi^\alpha t\}_{t \in \mathbb{R}}, \varphi \in W(\mathcal{M})$. Their relations are described as follows: for $x \in \mathcal{M}, \; t \in \mathbb{R}$ and $\varphi, \psi \in W(\mathcal{M})$,

$$\lambda^{\varphi^\alpha}\tau_\alpha^\varphi(x)\lambda^{\varphi^\alpha}(t), \quad \lambda^\varphi(t) = [D\varphi : D\psi], \lambda^\psi(t).$$

Then the core $\tilde{\mathcal{M}}$ is naturally isomorphic to $\mathcal{M} \rtimes_{\sigma^\varphi} \mathbb{R}$.

The restriction of the dual flow $\theta$ in $\sigma^\varphi$ on $Z(\tilde{\mathcal{M}})$ is called the (smooth) flow of weights of $\mathcal{M}$ [10]. Note that $Z(\tilde{\mathcal{M}})^\theta = Z(\mathcal{M})$. 


It is known that the flow of weights is a complete invariant for isomorphic classes among injective type III factors. We will present a proof of this fact in Theorem \[\text{6.17}\] as an application of our classification of Rohlin flows.

Let \( \mathcal{N} \) be another von Neumann algebra. Any isomorphism \( \pi \) from \( M \) onto \( \mathcal{N} \) extends to the isomorphism \( \tilde{\pi} : M \to \tilde{\mathcal{N}} \) such that for \( x \in M \) and \( t \in \mathbb{R} \),

\[
\tilde{\pi}(x) := \pi(x), \quad \tilde{\pi}(\lambda^\varphi(t)) := \lambda^{\pi(\varphi)}(t),
\]

where \( \varphi \in W(M) \) and \( \pi(\varphi) := \varphi \circ \pi^{-1} \). We call \( \tilde{\pi} \) the canonical extension of \( \pi \) (see [16, Theorem 2.4] and \[21\, \text{Proposition 12.1}\]). Let \( \theta^M \) and \( \theta^N \) be the dual flows on \( \tilde{M} \) and \( \tilde{\mathcal{N}} \). Then \( \tilde{\pi} \) intertwines them, that is, \( \theta^N \circ \tilde{\pi} = \tilde{\pi} \circ \theta^M \). The restriction \( \tilde{\pi}|_{Z(\tilde{M})} : Z(\tilde{M}) \to Z(\tilde{\mathcal{N}}) \) is called the Connes-Takesaki module of \( \pi \) \([\text{10}]\).

When \( \mathcal{N} = M \), we note that the canonical extension \( \text{Aut}(M) \ni \alpha \to \tilde{\alpha} \in \text{Aut}(\tilde{M}) \) is a continuous group homomorphism.

Let \( G \) be a locally compact group and \( \alpha : G \to \text{Aut}(M) \) an action. For \( \varphi \in W(M) \), we denote by \( \tilde{\varphi} \) the dual weight on \( M \rtimes_{\alpha} G \). Then we have

\[
\sigma^\tilde{\varphi}_t(\pi_\alpha(x)) = \pi_\alpha(\sigma^\varphi_t(x)), \quad \sigma^\tilde{\alpha}_t(\lambda^\varphi(g)) = \delta_G(g)^{it}\lambda^\alpha(g)\pi_\alpha([D\varphi \circ \alpha_g : D\varphi]|_t), \quad (2.1)
\]

where \( \delta_G \) denotes the modular function of \( G \). See [18, Theorem 3.2].

We introduce the action \( \tilde{\alpha} : G \to \text{Aut}(\tilde{M}) \) defined by \( \tilde{\alpha}_g := \tilde{\alpha}_g \circ \theta_{\log \delta_G(g)} \). Then the core of \( M \rtimes_{\alpha} G \) is canonically isomorphic to \( \tilde{M} \rtimes_{\tilde{\alpha}} G \) as shown below.

**Lemma 2.3.** One has the isomorphism \( \rho : (M \rtimes_{\alpha} G) \to \tilde{M} \rtimes_{\tilde{\alpha}} G \) such that

- \( \rho(\pi_\alpha(x)) = \pi_\alpha(x) \) for all \( x \in M \);
- \( \rho(\lambda^\alpha(g)) = \lambda^\alpha(g) \) for all \( g \in G \);
- \( \rho(\lambda^\tilde{\alpha}(t)) = \pi_\alpha(\lambda^\varphi(t)) \) for all \( \varphi \in W(M) \) and \( t \in \mathbb{R} \).

In particular, when \( G \) is abelian, we have \( \rho \circ \tilde{\alpha}_p = \tilde{\alpha}_p \circ \rho \) for all \( p \in \hat{G} \).

**Proof.** Let \( \mathcal{N} := M \rtimes_{\alpha} G \) and \( \varphi \in W(M) \). Then we have the canonical isomorphisms \( \Xi_{\tilde{\varphi}} : \tilde{N} \to M \rtimes_{\alpha \varphi} \mathbb{R} \) and \( \Lambda_{\tilde{\varphi}} : \tilde{M} \rtimes_{\tilde{\varphi}} G \to (M \rtimes_{\alpha \varphi} \mathbb{R}) \rtimes_{\alpha} G \). Let \( U : L^2(G \times \mathbb{R}) \to L^2(\mathbb{R} \times G) \) be the flip unitary. Set the unitary \( V \in M \otimes L^\infty(\mathbb{R} \times G) \) defined by \( V(t, g) := \delta_G(g)^{it}[D\varphi : D\varphi \circ \alpha_g]|_{-t} \).

We will show \( \rho := \Lambda_{\tilde{\varphi}}^{-1} \circ \text{Ad} V(1 \otimes U) \circ \Xi_{\tilde{\varphi}} \) is the well-defined isomorphism from \( \tilde{N} \) onto \( M \rtimes_{\alpha} \mathbb{R} \) satisfying the required conditions. Let \( x \in M \). Then \( \Xi_{\tilde{\varphi}}(\pi_\alpha(x)) = \pi_{\alpha \varphi}(\pi_\alpha(x)) \). By \( (2.1) \), we have \( \text{Ad} V(1 \otimes U)(\Xi_{\tilde{\varphi}}(\pi_\alpha(x))) = \pi_\alpha(\pi_{\alpha \varphi}(x)) \). Thus we get \( \rho(\pi_\alpha(x)) = \pi_\alpha(x) \). For \( g \in G \), we have \( \Xi_{\tilde{\varphi}}(\lambda^\alpha(g)) = \pi_{\alpha \varphi}(\lambda^\alpha(g)) \). Recall that \( \sigma^\tilde{\alpha}_t(\lambda^\varphi(g)) = \lambda^\alpha(g)\pi_\alpha(V_{t,g}^*) \). Thus for \( \xi, \eta \in \mathcal{H} \otimes L^2(\mathbb{R} \times G) \), we have

\[
\langle \text{Ad} V(1 \otimes U)(\Xi_{\tilde{\varphi}}(\lambda^\alpha(g)))\xi, \eta \rangle \\
= \int_{\mathbb{R}} \int_G \langle V_{t,h} \cdot (\sigma^\tilde{\alpha}_t(\lambda^\alpha(g))V^*_{t,g})\xi(t, h), \eta(t, h) \rangle \\
= \int_{\mathbb{R}} \int_G \langle V_{t,h} \cdot (\lambda^\alpha(g)\pi_\alpha(V_{t,g}^*)V^*_{t,g})\xi(t, h), \eta(t, h) \rangle \\
= \int_{\mathbb{R}} \int_G \langle V_{t,h} \alpha_{h^{-1}}(V^*_{t,g})V^*_{t,g}^{-1}h\xi(t, g^{-1}h), \eta(t, h) \rangle.
\]
\[= \int dt \int dh \langle \xi(t, g^{-1}h), \eta(t, h) \rangle = \langle \Lambda_\varphi(\lambda^\alpha(g)) \xi, \eta \rangle.\]

Thus \(\rho(\lambda^\alpha(g)) = \lambda^\alpha(g)\).

Since \(\Xi_\varphi(\lambda^\alpha(t)) = \lambda^\sigma(t)\) and \(\alpha_g^{-1}(\lambda^\sigma(t)) = \delta_G(g)^{it} \pi_\sigma([D\varphi \circ \alpha_g : D\varphi]_t) \lambda^\sigma(t)\), we have

\[\pi_\sigma(\lambda^\sigma(t)) = \pi_\sigma(V(-t, \cdot)^*) \lambda^\sigma(t).\]

Then

\[V\lambda^\sigma(t)V^*\pi_\sigma(\lambda^\sigma(t)^*) = V\lambda^\sigma(t)V^*\pi_\sigma(V(-t, \cdot)).\]

For \((s, g) \in \mathbb{R} \times G\), we have

\[(V\lambda^\sigma(t)V^*\lambda^\sigma(t)^*)(s, g) = V(s, g)V(-t + s, g)^*\]
\[= \delta_G(g)^{i[s][D\varphi : D\varphi \circ \alpha_g]_{s}} \cdot \delta_G(g)^{i[t - s][D\varphi : D\varphi \circ \alpha_g]_{t - s}}\]
\[= \delta_G(g)^{i[s][\sigma^\circ s]([D\varphi \circ \alpha_g : D\varphi]_t)}\]
\[= \pi_\sigma(V(-t, \cdot)^*)(s, g),\]

and \(V\lambda^\sigma(t)V^*\pi_\sigma(\lambda^\sigma(t)^*) = 1.\)

**Remark 2.4.** The previous lemma shows that \(\tilde{M}\) is regarded as a von Neumann subalgebra of \((M \rtimes_\alpha G)^\sim\). This is generalized as follows. Let \(N \subset M\) be an inclusion of von Neumann algebras. When there exists an operator valued weighted \(T\) from \(M\) onto \(N\), we can regard \(\tilde{N}\) as a von Neumann subalgebra of \(\tilde{M}\) in such a way that \(\lambda^\alpha(t) = \lambda^{\varphi \circ T}(t)\) for \(\varphi \in W(N)\) and \(t \in \mathbb{R}\). Note that this identification depends on the choice of \(T\). If we take \(T\) as the canonical operator valued weighted \(T_\alpha : M \rtimes_\alpha G \rightarrow \pi_\alpha(M)\), which is given by \(T_\alpha(x) = \int_G \hat{\alpha}(p) dp\) when \(G\) is abelian, then the associated map is nothing but \(\pi_\alpha : \tilde{M} \rightarrow \tilde{M} \rtimes_\alpha G\).

### 2.4. Ultraproduct von Neumann algebras

Our standard reference is [52 Chapter 5]. Let \(M\) be a von Neumann algebra. We denote by \(\ell^\infty(M)\) the C*-algebra of norm bounded sequences in \(M\). Let \(\omega\) be a free ultrafilter over \(\mathbb{N}\).

An element \((x^\nu)_\nu\) of \(\ell^\infty(M)\) is said to be

- **trivial** if \(x^\nu \rightarrow 0\) as \(\nu \rightarrow \infty\) in the strong* topology;
- **\(\omega\)-trivial** if \(x^\nu \rightarrow 0\) as \(\nu \rightarrow \omega\) in the strong* topology;
- **central** if for all \(\varphi \in M_*\), \(\| [\varphi, x^\nu] \|_{M_*} \rightarrow 0\) as \(\nu \rightarrow \infty\);
- **\(\omega\)-central** if for all \(\varphi \in M_*\), \(\| [\varphi, x^\nu] \|_{M_*} \rightarrow 0\) as \(\nu \rightarrow \omega\).

Let \(\mathcal{I}_\omega(M)\) and \(\mathcal{C}_\omega(M)\) be the collections of \(\omega\)-trivial and \(\omega\)-central sequences in \(M\), respectively, which are unital C*-subalgebras of \(\ell^\infty(M)\). Let \(\mathcal{N}_\omega(M)\) be the normalizer of \(\mathcal{I}_\omega(M)\) in \(\ell^\infty(M)\). Then \(\mathcal{I}_\omega(M) \subset \mathcal{C}_\omega(M) \subset \mathcal{N}_\omega(M)\). We often simply write \(\mathcal{I}_\omega, \mathcal{C}_\omega\) and \(\mathcal{N}_\omega\) for them unless otherwise confused.

The quotient C*-algebras \(M^\omega := \mathcal{N}_\omega / \mathcal{I}_\omega\) and \(M_\omega := \mathcal{C}_\omega / \mathcal{I}_\omega\) are in fact von Neumann algebras. We call them ultraproduct von Neumann algebras. The quotient map from \(\mathcal{N}_\omega\) onto \(M^\omega\) is denoted by \(\pi_\omega\). Each \(x \in M\) is mapped to the constant sequence \((x, x, \ldots) \in \mathcal{M}_\omega\). Then \(M\) is regarded as a von Neumann subalgebra of \(M^\omega\).
Let $\tau^\omega: M^\omega \to M$ be the map defined by $\tau^\omega(\pi_\omega((x^\nu)_\nu)) := \lim_{\nu \to \omega} x^\nu$, where the limit is taken in the $\sigma$-weak topology in $M$. Then $\tau^\omega$ is a faithful normal conditional expectation. For $\varphi \in M_*$, we denote by $\varphi^\omega$ the functional $\varphi \circ \tau^\omega$. Any element $a \in M_\omega$ commutes with $\varphi^\omega$, that is, $\varphi^\omega a = a \varphi^\omega$.

If $M$ is a factor, then $\tau^\omega$ gives a faithful normal tracial state on $M_\omega$. We often denote the trace by $\tau_\omega$. Note that in this case, $\varphi^\omega = \varphi(1)\tau_\omega$ on $M_\omega$ for all $\varphi \in M_*$.

Each $\alpha \in \text{Aut}(M)$ extends to the automorphism $\alpha^\omega \in \text{Aut}(M^\omega)$ by putting $\alpha^\omega(\pi_\omega((x^\nu)_\nu)) = \pi_\omega((\alpha(x^\nu))_\nu)$. Then $\alpha^\omega(M_\omega) = M_\omega$. We often simply write $\alpha$ for $\alpha^\omega$.

When $\alpha^\omega$ is trivial on $M_\omega$, $\alpha$ is said to be centrally trivial. Denote by $\text{Cnt}(M)$ the set of centrally trivial automorphisms that is a Borel subgroup of $\text{Aut}(M)$ as shown in Lemma 9.7.

In this paper, the compactness of a subset of $\mathcal{H}$ or $M_*$ means the norm compactness.

**Lemma 2.5.** If $(x^\nu)_\nu \in \mathcal{F}_\omega$ and $\Psi \subset \mathcal{H}$ is compact, then $\sup_{\eta \in \Psi} (\|x^\nu \eta\| + \|\eta x^\nu\|)$ converges to 0 as $\nu \to \omega$.

**Proof.** Take $C > 0$ with $C > \sup_{\nu} \|x^\nu\|$. Let $\epsilon > 0$ and take $\eta_1, \ldots, \eta_n \in \Psi$ so that any $\eta \in \Psi$ has some $\eta_i$ with $\|\eta - \eta_i\| < \epsilon/4C$. Using such $\eta_i$, we have

$$
\|x^\nu \eta\| + \|\eta x^\nu\| \leq \|x^\nu (\eta - \eta_i)\| + \|x^\nu \eta_i\| + \|\eta_i x^\nu\| + \|((\eta_i - \eta)x^\nu)\|
$$

$$
\leq C\epsilon/4C + \|x^\nu \eta_i\| + \|\eta_i x^\nu\| + C\epsilon/4C
$$

$$
\leq \epsilon/2 + \max_i (\|x^\nu \eta_i\| + \|\eta_i x^\nu\|).
$$

Thus

$$
\sup_{\eta \in \Psi} (\|x^\nu \eta\| + \|\eta x^\nu\|) \leq \epsilon/2 + \max_i (\|x^\nu \eta_i\| + \|\eta_i x^\nu\|) \quad \text{for all } \nu \in \mathbb{N}.
$$

If $\nu$ is sufficiently close to $\omega$, the second term in the right hand side becomes less than $\epsilon/2$. Hence we obtain $\lim_{\nu \to \omega} \sup_{\eta \in \Psi} (\|x^\nu \eta\| + \|\eta x^\nu\|) \leq \epsilon$. \hfill $\Box$

In a similar way, we can prove the following.

**Lemma 2.6.** Let $(x^\nu)_\nu \in \mathcal{F}_\omega$. Then for any compact set $\Psi \subset \mathcal{H}$, $\sup_{\eta \in \Psi} \|[x^\nu, \eta]\|$ converges to 0 as $\nu \to \omega$.

**Lemma 2.7.** Let $(x^\nu)_\nu \in \ell^\infty(M)$ and $\xi \in \mathcal{H}$ a cyclic and separating vector for $M$. Then the following statements are equivalent:

1. $(x^\nu)_\nu \in \mathcal{N}_\omega$;
2. For any $\epsilon > 0$ and compact set $\Psi \subset \mathcal{H}$, there exist $\delta > 0$ and $W \in \omega$ such that if $y \in M_1$ and $\|y\| + \|\xi y\| < \delta$, then $\sup_{\eta \in \Psi} (\|x^\nu y \eta\| + \|\eta x^\nu y\|) < \epsilon$, and $\sup_{\eta \in \Psi} (\|\eta x^\nu \eta\| + \|\eta x^\nu\|) < \epsilon$ for all $\nu \in W$.

**Proof.** (1) $\Rightarrow$ (2). Suppose on the contrary that there exist $\epsilon > 0$ and a compact set $\Psi \subset \mathcal{H}$ such that for any $n \in \mathbb{N}$, there exists $y_n \in M_1$ with $\|y_n\| + \|\xi y_n\| < 1/n$, but the following set belongs to $\omega$:

$$
A_n := \left\{ \nu \in \mathbb{N} \mid \sup_{\eta \in \Psi} (\|x^\nu y_n \eta\| + \|\eta x^\nu y_n\|) + \sup_{\eta \in \Psi} (\|y_n x^\nu \eta\| + \|\eta y_n x^\nu\|) \geq \epsilon \right\}.
$$

\hfill 9
Let $W_0 := \mathbb{N}$ and $W_n := A_1 \cap \cdots \cap A_n \cap \{n, \infty\}$ for $n \geq 1$. We may and do assume that $W_n \supseteq W_{n+1}$. For $\nu \in W_n \setminus W_{n+1}$, we set $z^{\nu} := y_n$. Then $(z^{\nu})_\nu \in \mathcal{T}_\omega$, and $(x^{\nu}z^{\nu})_\nu, (z^{\nu}x^{\nu})_\nu \in \mathcal{T}_\omega$ since $(x^{\nu})_\nu \in \mathcal{M}_\omega$. Nevertheless, we have
\[
\sup_{\eta \in \Psi}(\|x^{\nu}z^\nu\eta\| + \|\eta x^{\nu}z^\nu\|) + \sup_{\eta \in \Psi}(\|z^\nu x^{\nu}\eta\| + \|\eta z^\nu x^{\nu}\|) \geq \varepsilon \quad \text{for all } \nu \in \mathbb{N},
\]
which is a contradiction to Lemma 2.5.

(2)$\Rightarrow$(1). This implication is trivial. \qed

The following result is probably well-known for experts (see [5, Lemma 2.11] for example), but we give a proof for readers’ convenience.

**Lemma 2.8.** Let $\mathcal{M}$ be a separable von Neumann algebra and $\Omega$ a separable type I factor. Put $N := \mathcal{M} \otimes \Omega$. Then $N^\omega \cong M^\omega \otimes \Omega$ and $N^\omega \cong M^\omega \otimes \mathbb{C}$, naturally.

**Proof.** Let us use the notations $\mathcal{T}_\omega(N), \mathcal{M}_\omega(N), \mathcal{T}_\omega(M)$ and $\mathcal{M}_\omega(M)$ to distinguish $\mathcal{T}_\omega$ and $\mathcal{M}_\omega$ of $N$ and $M$. We prove the following claim.

**Claim.** $(x^{\nu})_\nu \in \mathcal{M}_\omega(M)$ if and only if $(x^{\nu} \otimes 1)_\nu \in \mathcal{M}_\omega(N)$.

**Proof of Claim.** The “if” part is trivial. We show the “only if” part. Suppose that $(x^{\nu})_\nu \in \mathcal{M}_\omega(M)$. Let $(y^{\nu})_\nu \in \mathcal{T}_\omega(N)$. Let $\theta_1 \in (\mathcal{M}_\omega)_+$ and $\theta_2 \in (\mathcal{O}_\omega)_+$ be faithful states. Then $\|y^{\nu}(x^{\nu} \otimes 1)\|_{\theta_1 \otimes \theta_2} = \theta_1((x^{\nu})^*(id \otimes \theta_2)((y^{\nu})^*y^{\nu})x^{\nu})$. Since $(id \otimes \theta_2)((y^{\nu})^*y^{\nu})^{1/2} \in \mathcal{T}_\omega(M)$, it turns out that $\|y^{\nu}(x^{\nu} \otimes 1)\|_{\theta_1 \otimes \theta_2} \rightarrow 0$ as $\nu \rightarrow \omega$. Thus $y^{\nu}(x^{\nu} \otimes 1) \rightarrow 0$ strongly as $\nu \rightarrow \omega$. In a similar way, we can show that $y^{\nu}(x^{\nu} \otimes 1)$ and $(x^{\nu} \otimes 1)y^{\nu}$ converges to 0 in the strong* topology as $\nu \rightarrow \omega$. Thus $(x^{\nu} \otimes 1)_\nu \in \mathcal{M}_\omega(N)$. \qed

Let us consider the inclusion $\mathcal{Q} \subset N^\omega$. Since $\mathcal{Q}$ is a type I factor, we have the tensor product decomposition $N^\omega = (\mathcal{Q} \cap N^\omega) \vee \mathcal{Q} \cong (\mathcal{Q} \cap N^\omega) \otimes \mathcal{Q}$. Let $p$ be a minimal projection of $\mathcal{Q}$. Let $x \in \mathcal{Q} \cap N^\omega$ and $(x^{\nu})_\nu$ its representing sequence. Take $x^{\nu} \in \mathcal{M}$ with $(1 \otimes p)a^{\nu}(1 \otimes p) = x^{\nu} \otimes p$. Then $x(1 \otimes p) = (1 \otimes p)x(1 \otimes p) = \pi_{\omega}(x^{\nu} \otimes \nu)$. Since $(x^{\nu} \otimes \nu)_\nu \in \mathcal{M}_\omega(N), (x^{\nu})_\nu \in \mathcal{M}_\omega(M)$.

By the claim above, we can consider the element $\pi_{\omega}(x^{\nu} \otimes 1)_\nu \in N^\omega$. Hence we have $x(1 \otimes p) = \pi_{\omega}(x^{\nu} \otimes 1)_\nu(1 \otimes p)$. Since the normal *-homomorphism $\mathcal{Q} \cap N^\omega \ni y \mapsto y(1 \otimes p) \in (\mathcal{Q} \cap N^\omega)_p$ is faithful, we have $x = \pi_{\omega}(x^{\nu} \otimes 1)_\nu$.

Thus we obtain the natural *-homomorphism $\Phi: \mathcal{Q} \cap N^\omega \rightarrow M^\omega$ defined by $\Phi(x) = \pi_{\omega}(x^{\nu})_\nu$. The faithfulness of $\Phi$ is trivial. The claim above implies the surjectivity of $\Phi$. Hence $\Phi$ is an isomorphism, and we obtain an isomorphism $\Psi: N^\omega \rightarrow M^\omega \otimes \mathcal{Q}$.

Since $N^\omega \subset Q \cap N^\omega$, $\Phi$ maps $N^\omega$ into $M^\omega \otimes \mathcal{Q}$. Then it is immediately verified that the image is precisely equal to $M^\omega \otimes \mathcal{Q}$.

We verify the naturality of $\Psi$ as follows. Let $\{e_{ij}\}_{i,j \in I}$ be a system of matrix units of $\mathcal{Q}$ such that $e_{ii}$ are minimal projections and $\sum_i e_{ii} = 1$. Let $x = \pi_{\omega}(x^{\nu})_\nu \in N^\omega$. For each $\nu$, we have the decomposition $x^{\nu} = \sum_{i,j} x^{\nu}_{ij} \otimes e_{ij}$ with $x^{\nu}_{ij} \in \mathcal{M}$. It is easy to see that $(x^{\nu}_{ij})_\nu \in \mathcal{M}_\omega(M)$ for all $i, j \in I$. 

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Then $\Psi((x_{ij}^\nu \otimes e_{ij})_{ij}) = \pi_{ij}(x_{ij}^\nu) \otimes e_{ij}$. For a diagonal finite rank projection $q \in Q$, we obtain

$$\Psi((1 \otimes q)x(1 \otimes q)) = (1 \otimes q) \cdot (\sum_{i,j} \pi_{ij}(x_{ij}^\nu) \otimes e_{ij}) \cdot (1 \otimes q)$$

Since $\|\Psi((1 \otimes q)x(1 \otimes q))\| \leq \|x\|$ for any $q$, the operator $\sum_{i,j} \pi_{ij}(x_{ij}^\nu) \otimes e_{ij}$ is norm bounded. Hence letting $q \to 1$, we have

$$\Psi(x) = \sum_{i,j} \pi_{ij}(x_{ij}^\nu) \otimes e_{ij}.$$ 

\[ \square \]

2.5. Ultraproduct of reduced von Neumann algebras. Let $M$ be a von Neumann algebra. For a projection $p \in M$, we denote by $M_p$ the reduced von Neumann algebra.

**Lemma 2.9.** The following properties hold:

1. $\mathcal{T}_\omega(M_p) = \{(px^\nu)_\nu \mid (x^\nu)_\nu \in \mathcal{T}_\omega(M)\} \subset \mathcal{T}_\omega(M)$;
2. $\mathcal{N}_\omega(M_p) = \{(px^\nu)_\nu \mid (x^\nu)_\nu \in \mathcal{N}_\omega(M)\} \subset \mathcal{N}_\omega(M)$;
3. $\mathcal{C}_\omega(M_p) = \{(px^\nu)_\nu \mid (x^\nu)_\nu \in \mathcal{C}_\omega(M)\}$.

**Proof.** (1). It is trivial.

(2). It suffices to show that the inclusion $\mathcal{N}_\omega(M_p) \subset \mathcal{N}_\omega(M)$ because the others are clear. Let $(x^\nu)_\nu \in \mathcal{N}_\omega(M_p)$ and $(y^\nu)_\nu \in \mathcal{N}_\omega(M)$. Let $\xi \in H$. Since $p|y|^2p \to 0$ in the strong* topology as $\nu \to \omega$, we obtain

$$\|y^\nu x^\nu \xi\|^2 = \langle p|y|^2px^\nu\xi, x^\nu\xi \rangle \to 0.$$ 

Hence $(x^\nu)_\nu \in \mathcal{N}_\omega$.

(3). It is clear that the right hand side is contained in the left. We will show the converse inclusion. Let $(x^\nu)_\nu \in \mathcal{C}_\omega(M_p)$ and $\varphi \in M_\ast$. In the following, we assume that $\|x^\nu\| \leq 1$. Put $q := 1 - p$. For any $y \in M_q$, we have

$$[[\varphi, x^\nu + y] = [p\varphi p, x^\nu] + q\varphi x^\nu - x^\nu \varphi q + [q\varphi q, y] + p\varphi y - y\varphi p.$$ 

Thus it suffices to show that there exists $(y^\nu)_\nu \in \mathcal{C}_\omega(M_q)$ such that

$$\lim_{\nu \to \omega} \|q\varphi x^\nu - y^\nu \varphi p\| = 0, \quad \lim_{\nu \to \omega} \|x^\nu \varphi q - p\varphi y^\nu\| = 0 \quad \text{for all } \varphi \in M_\ast. \quad (2.2)$$

If this is the case, then indeed we obtain $(x^\nu + y^\nu)_\nu \in \mathcal{C}_\omega(M)$ and $x^\nu = p(x^\nu + y^\nu)p$.

Take a maximal orthogonal family of projections $\{q_i\}_{i \in I}$ such that $q_i \leq q$ and $q_i \leq p$. We let $r := q - \sum i q_i$. Then there exist $z_1, z_2 \in \mathcal{Z}(M)^p$ such that $z_1 + z_2 = 1$ and $rz_1 \leq p z_1 r z_2 \leq p z_2$. By maximality, we deduce $pz_1 = 0$ and $rz_2 = 0$. Hence $p z_2 = p$. Since $z_2 x^\nu = x^\nu z_2$, we get $q\varphi x^\nu = q\varphi z_2 x^\nu = qz_2 \varphi x^\nu$, and likewise, $x^\nu \varphi q = x^\nu \varphi q z_2$ for all $\nu$ and $\varphi$. Hence we may and do assume that $z_2 = 1$, that is, $r = 0$. Then $q = \sum q_i$.

For each $i$, take $v_i \in M^{p_i}$ such that $q_i = v_i^* v_i$ and $v_i v_i^* \leq p$. Set $y^\nu := \sum_i v_i^* x^\nu v_i$. Then $\|y^\nu\| \leq \|x^\nu\| \leq 1$, and $(y^\nu)_\nu \in \ell^\infty(M_q)$. We will check that $(y^\nu)_\nu \in \mathcal{C}_\omega(M_q)$.
Let $\varepsilon > 0$ and $\varphi \in \mathcal{M}$ a state. Take a finite subset $J \subset I$ such that $\varphi(q - q_J) < \varepsilon$ with $q_J := \sum_{i \in J} q_i$. Then

$$
\|q\varphi q - q_J \varphi q_J\| \leq \|q_J \varphi \cdot (q - q_J)\| + \|(q - q_J) \varphi q\|
\leq 2 \varphi(q - q_J)^{1/2} < 2\varepsilon^{1/2}.
$$

Using this, we obtain

$$
\|y'q\varphi q - q\varphi y''\| \leq 2\|y''\|\|q\varphi q - q_J \varphi q_J\| + \|y''q_J \varphi q_J - q_J \varphi q_J y''\|
\leq 4\varepsilon^{1/2} + \sum_{i,j \in J} \|v_i^* x' v_i \varphi q_j - q_J \varphi v_j^* x' v_i\|
= 4\varepsilon^{1/2} + \sum_{i,j \in J} \|v_i^* [x', v_i \varphi v_j^*] v_j\|
\leq 4\varepsilon^{1/2} + \sum_{i,j \in J} \|v_i^* [x', v_i \varphi v_j^*]\|.
$$

Since $\|v_i^* [x', v_i \varphi v_j^*]\| \to 0$ as $\nu \to \omega$, we have $\|y''q\varphi q - q\varphi y''\| < 5\varepsilon^{1/2}$ for $\nu$ being close to $\omega$. Thus $(y''_\nu)_{\nu} \in \mathcal{C}_\omega(\mathcal{M}_q)$.

Now we will check (2.2). Using $\varphi(q - q_J) < \varepsilon$, we have

$$
\|q\varphi x'' - y'' \varphi p\| \leq \|q\varphi x'' - q_J \varphi x''\| + \|y''(q_J - q) \varphi p\| + \sum_{i \in J} \|q_i \varphi x'' - v_i^* x' v_i \varphi p\|
\leq 2\varepsilon^{1/2} + \sum_{i \in J} \|v_i^* [v_i \varphi p, x'']\|.
$$

Hence for $\nu$ being close to $\omega$, we have $\|q\varphi x'' - y'' \varphi p\| < 3\varepsilon^{1/2}$. Similarly, $\|x'' \varphi q - p \varphi y''\| < 3\varepsilon^{1/2}$ Thus (2.2) holds.

The previous lemma implies the following result.

**Proposition 2.10.** Let $\mathcal{M}$ be a von Neumann algebra and $p \in \mathcal{M}^P$. Then the ultraproduct von Neumann algebras $(\mathcal{M}_p)^\omega$ and $(\mathcal{M}_p)_\omega$ are realized in $\mathcal{M}^\omega$ as follows:

$$(\mathcal{M}_p)^\omega = (\mathcal{M}^\omega)_p, \quad (\mathcal{M}_p)_\omega = (\mathcal{M}_\omega)_p.$$
continuously by the given flow. However, the continuity is insufficient in lifting a
continuous or Borel path from \( M^\omega \) to \( M \). As a result, we have to think of a much
smaller von Neumann subalgebra in \( M^\omega \) that is called the \((\alpha, \omega)\)-equicontinuous
part (see Definition 3.1).

3.1. \( \omega \)-equicontinuity.

**Definition 3.1.** Let \((\Omega, d)\) be a metric space and \( \{x^\nu: \Omega \to M\}_{\nu \in \mathbb{N}} \) a family of
maps. We will say that \( \{x^\nu\}_\nu \) is \( \omega \)-equicontinuous if for any \( \varepsilon > 0 \) and finite set
\( \Phi \subset \mathcal{H} \), there exist \( \delta > 0 \) and \( W \in \omega \) such that for all \( s, t \in \Omega \) with \( d(s, t) < \delta \),
\( \nu \in W \) and \( \xi \in \Phi \), we have

\[
\| (x^\nu(s) - x^\nu(t))\xi \| < \varepsilon, \quad \| \xi (x^\nu(s) - x^\nu(t)) \| < \varepsilon.
\]

Several statements in this paper can be replaced with normal functionals for
vectors in a standard Hilbert space. We should note that the \( \omega \)-equicontinuity
does not necessarily require the continuity of each \( x^\nu \).

**Lemma 3.2.** Let \((\Omega, d)\) be a metric space and \( \{x^\nu: \Omega \to M\}_\nu \) a family of uni-
formly bounded maps, that is, \( \sup_{t \in \Omega, \nu \in \mathbb{N}} \| x^\nu(t) \| < \infty \). Then the following state-
ments are equivalent:

(1) \( \{x^\nu: \Omega \to M\}_\nu \) is \( \omega \)-equicontinuous;

(2) For any \( \varepsilon > 0 \) and compact set \( \Psi \subset \mathcal{H} \), there exist \( \delta > 0 \) and \( W \in \omega \) such
that for all \( s, t \in \Omega \) with \( d(s, t) < \delta \), \( \nu \in W \) and \( \xi \in \Psi \), we have

\[
\| (x^\nu(s) - x^\nu(t))\xi \| < \varepsilon, \quad \| \xi (x^\nu(s) - x^\nu(t)) \| < \varepsilon.
\]

(3) Let \( \xi_0 \in \mathcal{H} \) be a cyclic and separating vector for \( M \). For any \( \varepsilon > 0 \), there
exist \( \delta > 0 \) and \( W \in \omega \) such that for all \( s, t \in \Omega \) with \( d(s, t) < \delta \) and
\( \nu \in W \), we have

\[
\| (x^\nu(s) - x^\nu(t))\xi_0 \| < \varepsilon, \quad \| \xi_0 (x^\nu(s) - x^\nu(t)) \| < \varepsilon.
\]

**Proof.** (1)\( \Rightarrow \) (2). Take \( C > 0 \) with \( C \geq \sup_{t, \nu} \| x(t)^\nu \| \). Let \( \Psi \) and \( \varepsilon \) be given.
Choose \( \{\xi_i\}_{i=1}^N \in \Psi \) such that for any \( \xi \in \Psi \), there exists \( \xi_i \) such that \( \| \xi - \xi_i \| < \varepsilon/4C \).
Using the \( \omega \)-equicontinuity of \( x^\nu \), we can take \( \delta > 0 \) and \( W \in \omega \) such that for
any \( s, t \in \Omega \) with \( d(s, t) < \delta \), \( \nu \in W \) and \( i = 1, \ldots, N \), we have

\[
\| (x^\nu(s) - x^\nu(t))\xi_i \| < \varepsilon/2, \quad \| \xi_i (x^\nu(s) - x^\nu(t)) \| < \varepsilon/2.
\]

Then it is clear that these \( \delta \) and \( W \) are desired ones.

(2)\( \Rightarrow \) (3). This implication is trivial.

(3)\( \Rightarrow \) (1). Let \( \varepsilon > 0 \) and \( \Phi := \{\xi_i \mid i = 1, \ldots, N\} \subset \mathcal{H} \). Take \( C > 0 \) as \( C \geq \sup_{t, \nu} \| x^\nu(t) \| \).
Let \( a_i, b_i \in M \) such that \( \| \xi_i - \xi_0 a_i \| < \varepsilon/4C \) and \( \| \xi_i - b_i \xi_0 \| < \varepsilon/4C \).
Set \( M := \max\{\|a_i\|, \|b_i\|, 1 \mid i = 1, \ldots, N\} \). By our assumption, there exist \( \delta > 0 \)
and \( W \in \omega \) such that for all \( s, t \in \Omega \) with \( d(s, t) < \delta \) and \( \nu \in W \), we have

\[
\| (x^\nu(s) - x^\nu(t))\xi_0 \| < \varepsilon/2M, \quad \| \xi_0 (x^\nu(s) - x^\nu(t)) \| < \varepsilon/2M.
\]

Then

\[
\| (x^\nu(s) - x^\nu(t))\xi_i \| \leq \| (x^\nu(s) - x^\nu(t))(\xi_i - \xi_0 a_i) \| + \| (x^\nu(s) - x^\nu(t))\xi_0 a_i \|
\leq 2C \cdot \varepsilon/4C + M \cdot \varepsilon/2M = \varepsilon.
\]
Similarly, we obtain \( \|\xi_i(x^\nu(s) - x^\nu(t))\| < \varepsilon \). Thus \( \{x^\nu\} \) is \( \omega \)-equicontinuous. \( \square \)

The following result is frequently used in this paper.

**Lemma 3.3.** Let \((\Omega, d)\) be a metric space and \(E \subset \Omega\) a relatively compact set. Let \( \{f^\nu: \Omega \to \mathbb{C}\} \) be a family of functions. Suppose that \( \{f^\nu: E \to \mathbb{C}\} \) is \( \omega \)-equicontinuous. Then the convergence \( \lim_{\nu \to \omega} f^\nu(t) \) is uniform on \( E \).

**Proof.** Put \( F(t) := \lim_{\nu \to \omega} f^\nu(t) \) for \( t \in E \). For \( \varepsilon > 0 \), take \( \delta > 0 \) and \( W_1 \subset \omega \) such that for all \( s, t \in E \) with \( d(s, t) < \delta \) and \( \nu \in W_1 \), we have \( |f^\nu(s) - f^\nu(t)| < \varepsilon \). Letting \( \nu \to \omega \), we obtain \( |F(s) - F(t)| \leq \varepsilon \) which shows the uniform continuity of \( F \) on \( E \).

Let us keep \( \varepsilon, \delta \) and \( W_1 \) introduced above. Since \( E \) is relatively compact, there exists a finite set \( E_0 \subset E \) such that \( E \subset \bigcup_{t \in E_0} B(t, \delta) \), where \( B(t, \delta) := \{s \in \Omega \mid d(s, t) < \delta\} \). Then there exists \( W_2 \subset \omega \) such that for all \( s \in E_0 \) and \( \nu \in W_2 \), we have \( |f^\nu(s) - F(s)| < \varepsilon \). Let \( t \in E \), and take \( s \in E_0 \) with \( d(s, t) < \delta \). Then for all \( \nu \in W_1 \cap W_2 \), we have
\[ |f^\nu(t) - F(t)| \leq |f^\nu(t) - f^\nu(s)| + |f^\nu(s) - F(s)| + |F(s) - F(t)| < 3\varepsilon. \]

Thus we are done. \( \square \)

### 3.2 Borel maps and flows.

**Definition 3.4.** Let \( \alpha: \mathbb{R} \to \text{Aut}(\mathcal{M}) \) be a Borel map. An element \( (x^\nu)\) \( \in \ell^\infty(\mathcal{M}) \) is said to be \((\alpha, \omega)\)-equicontinuous if for any Borel set \( E \subset \mathbb{R} \) with \( \mu(E) < \infty \) and \( \varepsilon > 0 \), there exists a compact \( K \subset E \) such that

- \( \alpha|_K \) is continuous;
- \( \mu(E \setminus K) < \varepsilon \);
- the family \( \{K \ni t \mapsto \alpha_t(x^\nu) \in \mathcal{M}\} \) is \( \omega \)-equicontinuous.

We denote by \( \mathcal{E}_{\alpha}^\omega \) the set of \((\alpha, \omega)\)-equicontinuous sequences.

In the definition above, \( \mu \) denotes the Lebesgue measure on \( \mathbb{R} \) with \( \mu([0, 1]) = 1 \). Lemma 3.1 implies that for any Borel set \( E \subset \mathbb{R} \), we can take a compact \( K \subset E \) satisfying the first and second conditions above. The reason why we must consider the third is to make the stability holds with respect to a perturbation of a cocycle action (Lemma 3.8). To flows, the following characterization is useful.

**Proposition 3.5.** Let \( \alpha \) be a flow on \( \mathcal{M} \) and \( (x^\nu) \in \ell^\infty(\mathcal{M}) \). Let \( \xi \in \mathcal{H} \) be a cyclic and separating vector for \( \mathcal{M} \). Then the following statements are equivalent:

1. \((x^\nu)\) is \((\alpha, \omega)\)-equicontinuous;
2. For any \( \varepsilon > 0 \), there exist \( \delta > 0 \) and \( W \in \omega \) such that if \( |t| < \delta \) and \( \nu \in W \), we have
\[ \|\alpha_t(x^\nu) - x^\nu\| + \|\xi(x^\nu) - x^\nu\| < \varepsilon. \]
3. For any \( \varepsilon > 0 \) and compact set \( \Psi \subset \mathcal{H} \), there exist \( \delta > 0 \) and \( W \in \omega \) such that if \( |t| < \delta \) and \( \nu \in W \), we have
\[ \sup_{\eta \in \Psi}(\|\alpha_t(x^\nu) - x^\nu\| + \|\eta(x^\nu) - x^\nu\|) < \varepsilon. \]
(4) For any $T > 0$, $\{[-T, T] \ni t \mapsto \alpha_t(x^\nu) \in M_\nu\}$ is $\omega$-equicontinuous.

Proof. (4)$\Rightarrow$(1) is trivial.

(1)$\Rightarrow$(2). Suppose that $(x^\nu)_\nu$ is $(\alpha, \omega)$-equicontinuous. For $E := [0, 1]$, there exists a compact set $K \subset E$ such that $\mu(K) \geq 1/2$, $\alpha|_K$ is continuous, and $\{K \ni t \mapsto \alpha_t(x^\nu) \in M_\nu\}$ is $\omega$-equicontinuous. Since $\mu(K) > 0$, we can find $\delta > 0$ with $(-\delta, \delta) \subset K - K$.

Set $\Psi := \{\alpha_s(\xi) \mid s \in K\}$ that is compact. Then by Lemma 3.2 for any $\varepsilon > 0$, there exist $\delta' > 0$ and $W \in \omega$ such that for all $s, t \in K$ with $|s - t| < \delta'$ and $\nu \in W$, we have

$$\sup_{\eta \in \Psi} (\|\alpha_s(x^\nu) - \alpha_t(x^\nu)\|_\eta + \|\eta(\alpha_s(x^\nu) - \alpha_t(x^\nu))\|) < \varepsilon.$$ 

Then for $s, t \in K$ with $|s - t| < \delta'$ and $\nu \in W$, we have

$$\|(\alpha_s(x^\nu) - x^\nu)\| + \|\xi(\alpha_s(x^\nu) - x^\nu)\|$$

$$= \|(\alpha_s(x^\nu) - \alpha_t(x^\nu))\alpha_t(\xi)\| + \|\alpha_t(\xi)(\alpha_s(x^\nu) - \alpha_t(x^\nu))\|$$

$$\leq \sup_{\eta \in \Psi} (\|\alpha_s(x^\nu) - \alpha_t(x^\nu)\|_\eta + \|\eta(\alpha_s(x^\nu) - \alpha_t(x^\nu))\|)$$

$$< \varepsilon.$$ 

Hence if $|t| < \min(\delta, \delta')$ and $\nu \in W$, then

$$\|(\alpha_t(x^\nu) - x^\nu)\| + \|\xi(\alpha_t(x^\nu) - x^\nu)\| < \varepsilon.$$ 

(2)$\Rightarrow$(3). By compactness, it suffices to prove (3) for a finite $\Psi$. We may and do assume that $\|x^\nu\| \leq 1/2$ for all $\nu \in \mathbb{N}$. Let $\varepsilon > 0$. Take $\delta > 0$ so that if $a \in M$ satisfies $\|a\| \leq 1$ and $\|\alpha\xi\| + \|\xi a\| < \delta$, then $\sup_{\eta \in \Psi} (\|a\| + \|\eta a\|) < \varepsilon$. By (2), there exist $\delta' > 0$ and $W \in \omega$ such that if $|t| \leq \delta'$ and $\nu \in W$, then

$$\|(\alpha_t(x^\nu) - x^\nu)\| + \|\xi(\alpha_t(x^\nu) - x^\nu)\| < \delta.$$ 

Hence it implies

$$\sup_{\eta \in \Psi} (\|\alpha_t(x^\nu) - x^\nu\|_\eta + \|\eta(\alpha_t(x^\nu) - x^\nu)\|) < \varepsilon.$$ 

(3)$\Rightarrow$(4). Let $\varepsilon > 0$, $T > 0$ and $\Psi := \{\alpha_t(\xi) \mid |t| \leq T\}$. By (3), there exist $\delta > 0$ and $W \in \omega$ such that for all $t$ with $|t| < \delta$ and $\nu \in W$, 

$$\sup_{\eta \in \Psi} (\|\alpha_t(x^\nu) - x^\nu\|_\eta + \|\eta(\alpha_t(x^\nu) - x^\nu)\|) < \varepsilon.$$ 

This implies the following:

$$\|(\alpha_t+\varepsilon(s\nu) - \alpha_s(x^\nu))\| + \|\xi(\alpha_t+\varepsilon(s\nu) - \alpha_s(x^\nu))\| < \varepsilon \text{ if } |t| < \delta, |s| \leq T, \nu \in W.$$ 

Hence we are done.\hfill$\square$

Lemma 3.6. Let $(\Omega, d)$ be a compact metric space, $\{x^\nu : \Omega \rightarrow M\}_\nu$ and $\{y^\nu : \Omega \rightarrow M\}_\nu$ families of maps. Suppose that the following conditions hold:

- They are uniformly bounded and $\omega$-equicontinuous;
- For each $t \in \Omega$, $(x(t)^\nu)_\nu$ and $(y(t)^\nu)_\nu$ belong to $\mathcal{M} (\nu)$. 

Then the family of their multiplications $\{x^\nu y^\nu : \Omega \rightarrow M\}_\nu$ is also $\omega$-equicontinuous.
Proof. We may and do assume that $\|x(t)^\nu\|, \|y(t)^\nu\| \leq 1$ for all $t \in \Omega$ and $\nu \in \mathbb{N}$. Let $\varepsilon > 0$ and $\xi \in \mathcal{H}$ a cyclic separating vector for $\mathcal{M}$. Then there exists $\delta > 0$ and $W_1 \in \omega$ such that if $s, t \in \Omega$ satisfies $d(s, t) < \delta$ and $\nu \in W_1$, then

$$\|(y(s)^\nu - y(t)^\nu)\xi\| < \varepsilon. \quad (3.1)$$

Since $\Omega$ is compact, there exists a finite subset $F \subset \Omega$ such that each $s \in \Omega$ has $t \in F$ with $d(s, t) < \delta$. Then by Lemma 2.7 we can take $\delta' > 0$ and $W_2 \in \omega$ such that if $a \in \mathcal{M}$ with $\|a\| \leq 2$ satisfies $\|a\xi\| + \|\xi a\| \leq \delta'$, then $\|ay(s)^\nu\xi\| < \varepsilon$ for $s \in F$.

By $\omega$-equicontinuity of $\{x^\nu\}_\nu$, we take $\delta'' > 0$ and $W_3 \in \omega$ such that if $s, t \in \Omega$ satisfies $d(s, t) < \delta''$ and $\nu \in W_2$, then

$$\|(x(s)^\nu - x(t)^\nu)\xi\| + \|\xi(x(s)^\nu - x(t)^\nu)\| < \delta'.$$

This implies

$$\|(x(s)^\nu - x(t)^\nu)y(t_0)^\nu\xi\| < \varepsilon \quad \text{for all } t_0 \in F. \quad (3.2)$$

Let $s, t \in \Omega$ with $d(s, t) < \min(\delta, \delta'')$ and $\nu \in W_1 \cap W_2 \cap W_3$. Take $t_0 \in F$ with $d(t, t_0) < \delta$. Then we have

$$\begin{align*}
\|(x(s)^\nu y(s)^\nu - x(t)^\nu y(t)^\nu)\xi\| \\
\leq \|(x(s)^\nu (y(s)^\nu - y(t)^\nu)\xi\| + \|(x(s)^\nu y(t)^\nu - y(t_0)^\nu)\xi\| \\
+ \||(x(s)^\nu - x(t)^\nu)y(t_0)^\nu\xi\| + \|(x(t)^\nu y(t_0)^\nu - y(t)^\nu)\xi\| \\
\leq \|(y(s)^\nu - y(t)^\nu)\xi\| + \|(y(t)^\nu - y(t_0)^\nu)\xi\| \\
+ \|(x(s)^\nu - x(t)^\nu)y(t_0)^\nu\xi\| + \|(y(t_0)^\nu - y(t)^\nu)\xi\| \\
< \varepsilon + \varepsilon + \varepsilon + 4\varepsilon \quad \text{by (3.1), (3.2).}
\end{align*}$$

Likewise, we can show that there exist $\delta''' > 0$ and $W_4 \in \omega$ such that if $s, t \in \Omega$ satisfies $d(s, t) < \delta'''$ and $\nu \in W_4$, then

$$\|\xi(x(s)^\nu y(s)^\nu - x(t)^\nu y(t)^\nu)\| < 4\varepsilon.$$

Hence we are done. \qed

Lemma 3.7. Let $\alpha: \mathbb{R} \to \text{Aut}(\mathcal{M})$ be a Borel map. Then the following hold:

1. If $(x^\nu)_\nu \in \mathcal{E}_\alpha$ and $(y^\nu)_\nu \in \ell^\infty(\mathcal{M})$ satisfy $(x^\nu - y^\nu)_\nu \in \mathcal{T}_\omega$, then $(y^\nu)_\nu \in \mathcal{E}_\alpha$;
2. $\mathcal{E}_\alpha$ contains $\mathcal{T}_\omega$;
3. $\mathcal{E}_\alpha \cap \mathcal{M}_\omega$ is a $C^*$-subalgebra of $\ell^\infty(\mathcal{M})$.

Proof. (1). Let $E$ be a Borel set and $K$ a compact set in $E$ such that $\alpha|_K$ is continuous and $\{t \mapsto \alpha_t(x^\nu)\}_\nu$ is $\omega$-equicontinuous. Then for $\varepsilon > 0$ and a finite set $\Phi \subset \mathcal{H}$, there exist $\delta > 0$ and $W \in \omega$ such that for $s, t \in K$ with $|s - t| < \delta$, $\nu \in W$ and $\xi \in \Phi$, we have

$$\|/(\alpha_s(x^\nu) - \alpha_t(x^\nu))\xi\| + \|\xi(\alpha_s(x^\nu) - \alpha_t(x^\nu))\| < \varepsilon/2.$$ 

Set $\Psi := \{\alpha_s^{-1}(\xi) \mid \xi \in \Phi, s \in K\}$ that is a compact subset of $\mathcal{H}$. We let $s^\nu := \sup_{\eta \in \Psi} \|(x^\nu - y^\nu)\eta\| + \|\eta(x^\nu - y^\nu)\|$. By Lemma 2.5, we have $\lim_{\nu \to \omega} s^\nu = 0$. Thus we may and do assume that $s^\nu < \varepsilon/4$ for $\nu \in W$.\]
Then for $\xi \in \Phi$, $s, t \in K$ with $|s-t| < \delta$ and $\nu \in W$, we have
\[
\| (\alpha_s(y') - \alpha_t(y')) \xi \| + \| (\alpha_s(y') - \alpha_t(y')) \xi \|
\leq \| (\alpha_s(y') - \alpha_t(y')) \xi \| + \| (\alpha_s(x') - \alpha_t(x')) \xi \| + \| (\alpha_t(x') - \alpha_t(y')) \xi \|
\leq \| (\alpha_s(y') - \alpha_t(y')) \xi \| + \| (\alpha_s(x') - \alpha_t(x')) \xi \| + \| (\alpha_t(x') - \alpha_t(y')) \xi \|
\leq \| (\alpha_s(y') - \alpha_t(y')) \xi \| + \| (\alpha_s(x') - \alpha_t(x')) \xi \| + \| (\alpha_t(x') - \alpha_t(y')) \xi \|
\leq \| (\alpha_s(y') - \alpha_t(y')) \xi \| + \| (\alpha_s(x') - \alpha_t(x')) \xi \| + \| (\alpha_t(x') - \alpha_t(y')) \xi \|
\leq 2s' + \varepsilon/2 < \varepsilon/2 + \varepsilon/2 = \varepsilon.
\]

Hence $(y')_\nu$ is $(\alpha, \omega)$-equicontinuous.

(2). Let $K \subset \mathbb{R}$ be a compact set on which $\alpha$ is continuous. Then $\Psi := \{ \alpha^{-1}_s(\xi) \mid s \in K \}$ is compact in $H$. Thus if $(x')_\nu \in \mathcal{H}$, then $s' := \sup_{\eta \in \Psi} (\|x'\eta\| + \|x'\eta\|) \to 0$ as $\nu \to \omega$ by Lemma 2.7. Then the statement is clear because of the inequalities $\| (\alpha_s(x') - \alpha_t(x')) \xi \| \leq 2s'$ and $\| (\alpha_s(x') - \alpha_t(x')) \xi \| \leq 2s'$.

(3). It is easy to see that $\mathcal{E}_\alpha^\omega$ is a norm closed operator system in $\ell^\infty(M)$. We show that $\mathcal{E}_\alpha^\omega \cap \mathcal{N}_\omega$ is closed under multiplication.

Let $E \subset \mathbb{R}$ be a Borel set with $0 < \mu(E) < \infty$ and $0 < \kappa < 1/2$. Let $(x')_\nu, (y')_\nu \in \mathcal{E}_\alpha^\omega \cap \mathcal{N}_\omega$. Take a compact set $K_0 \subset E$ such that $\mu(E \setminus K_0) < \kappa$, $\alpha|_K$ is continuous and the maps $\{ K_0 \ni t \mapsto \alpha_t(x'_\nu) \}, \{ K_0 \ni t \mapsto \alpha_t(y'_\nu) \}$ are $\omega$-equicontinuous. Hence $\{ K_0 \ni t \mapsto \alpha_t(x'_\nu y'_\nu) \}$ is $\omega$-equicontinuous by the previous lemma.

**Lemma 3.8.** Let $(\alpha, c)$ be a Borel cocycle action of $\mathbb{R}$. Then the following statements hold:

1. $\mathcal{E}_\alpha^\omega \cap \mathcal{N}_\omega$ is $\alpha$-invariant;
2. Let $(x')_\nu, (y')_\nu$ be the perturbation by a Borel unitary path $v: \mathbb{R} \to M^u$. Then $\mathcal{E}_\alpha^\omega \cap \mathcal{N}_\omega = \mathcal{E}_\alpha^\omega \cap \mathcal{N}_\omega$.

**Proof.** (1). Let $E \subset \mathbb{R}$ be a Borel set with $0 < \mu(E) < \infty$. Let $\varepsilon, \kappa > 0$ and $(x')_\nu \in \mathcal{E}_\alpha^\omega \cap \mathcal{N}_\omega$. We may and do assume $\|x'\| \leq 1$ for all $\nu \in \mathbb{N}$. Fix $s \in \mathbb{R}$. Then we can take a compact set $K_1 \subset E + s$ such that $\mu((E + s) \setminus K_1) < \kappa$, $\alpha|_{K_1}$ is continuous, and $\{ K_1 \ni t \mapsto \alpha_t(x'_\nu) \}$ is $\omega$-equicontinuous.

Next, we take a compact set $K_2 \subset E$ such that $\mu(E \setminus K_2) < \kappa$, and the map $K_2 \ni t \mapsto c(t, s)$ is continuous. Set $K := (K_1 - s) \cap K_2$, which satisfies $K \subset E$ and $\mu(E \setminus K) < 2\kappa$. Let $\xi \in \mathcal{H}$ be a cyclic and separating vector for $M$. We set the following compact set

$$
\Psi := \{ c(t, s)^* \xi \mid t \in K \} \cup \{ \alpha^{-1}_t(c(t, s)^* \xi) \mid t \in K \}.
$$

Take $\delta > 0$ and $W_1 \in \omega$ such that for all $t, t' \in K_1$ with $|t - t'| < \delta$ and $\nu \in W_1$, we have

$$
\sup_{\eta \in \Psi} \| \eta(\alpha_t(x'_\nu) - \alpha_{t'}(x'_\nu)) \| < \varepsilon, \quad \sup_{\eta \in \Psi} \| (\alpha_t(x'_\nu) - \alpha_{t'}(x'_\nu)) \eta \| < \varepsilon.
$$

(3.3)

By Lemma 2.7, there exist $\delta' > 0$ and $W_2 \in \omega$ such that if $a \in M$ with $\|a\| \leq 2$ and $\|\alpha \xi\| + \|\xi a\| < \delta'$, then $\sup_{\eta \in \Psi} (\|x'a \eta\| + \|x' a \| < \varepsilon$, and $\sup_{\eta \in \Psi} (\|x' a \eta\| + \|x' a \| < \varepsilon$ for all $\nu \in W_2$.
Take $\delta'' > 0$ so that if $t, t' \in K$ with $|t - t'| < \delta''$, then

$$
\| (c(t, s) - c(t, s)) x \| + \| (c(t, s) - c(t, s)) \| < \varepsilon,
$$

(3.4)

Then for $t, t' \in K$ with $|t - t'| < \min(\delta, \delta'')$ and $\nu \in W_1 \cap W_2$, we have

$$
\sup_{\eta \in \Psi} \| \alpha_{t, s}^{-1}(c(t, s) - c(t, s)) x^\nu \eta \| < \varepsilon,
$$

(3.5)

and

$$
\| (\alpha_t(\alpha_s(x^\nu)) - \alpha_{t'}(\alpha_s(x^\nu))) \| \\
= \| (c(t, s) \alpha_{t+s}(x^\nu) c(t, s)^* - c(t', s) \alpha_{t'+s}(x^\nu) c(t', s)^*) \| \\
= \| c(t, s) \alpha_{t+s}(x^\nu) c(t, s)^* - c(t', s) \| + \| c(t, s) \alpha_{t+s}(x^\nu) - \alpha_{t'+s}(x^\nu) \| c(t', s)^* \| \\
+ \| c(t, s) - c(t', s) \| \alpha_{t+s}(x^\nu) c(t', s)^* \| \\
\leq \| (c(t, s)^* - c(t', s)^*) x^\nu \| + \sup_{\eta \in \Psi} \| (\alpha_{t+s}(x^\nu) - \alpha_{t'+s}(x^\nu)) \| \\
\leq \| (c(t, s)^* - c(t', s)^*) x^\nu \| + \sup_{\eta \in \Psi} \| \alpha_{t+s}(c(t, s) - c(t', s)) x^\nu \eta \| \\
< \varepsilon + \varepsilon + \sup_{\eta \in \Psi} \| \alpha_{t+s}(c(t, s) - c(t', s)) x^\nu \eta \| \\
< \varepsilon \text{ by (3.5)}.
$$

We can obtain a similar estimate for $\| (\alpha_t(\alpha_s(x^\nu)) - \alpha_{t'}(\alpha_s(x^\nu))) \|$. Therefore, $(\alpha_s(x^\nu))_{\nu} \in E^\omega_{\alpha}$. (2). Let $(x^\nu)_{\nu} \in E^\omega_{\alpha} \cap \mathcal{M}_\omega$. Let $E \subset \mathbb{R}$ be a Borel set with $0 < \mu(E) < \infty$. Take a compact set $K \subset E$ such that

- $\mu(E \setminus K) < \kappa$;
- $\alpha, \nu$ are continuous on $K$;
- $\{K \ni t \mapsto \alpha_t(x^\nu)\}_{\nu}$ is $\omega$-equicontinuous.

Then $\{K \ni t \mapsto v_{t} \alpha_t(x^\nu) v_{t}^*\}_{\nu}$ is $\omega$-equicontinuous by Lemma 3.6

In the following, we generalize the $\omega$-equicontinuous part of $\mathcal{M}_\omega$ introduced in Definition 2.2 to a Borel map.

**Definition 3.9.** Let $\alpha : \mathbb{R} \to \text{Aut}(\mathcal{M})$ be a Borel map. We let $\mathcal{M}_\omega^\alpha$ be the quotient $C^*$-algebra $(E^\omega_{\alpha} \cap \mathcal{M}_\omega)/\mathcal{I}_\omega$, and $\mathcal{M}_{\omega, \omega} : = \mathcal{M}_\omega^\alpha \cap \mathcal{M}_\omega$. We call them the $(\alpha, \omega)$-equicontinuous parts of $\mathcal{M}_\omega$ and $\mathcal{M}_\omega$, respectively.

**Lemma 3.10.** The $C^*$-subalgebras $\mathcal{M}_\omega^\alpha$ and $\mathcal{M}_{\omega, \omega}$ are von Neumann subalgebras of $\mathcal{M}_\omega$ and $\mathcal{M}_\omega$, respectively.

**Proof.** We show the unit ball of $\mathcal{M}_\omega^\alpha$ is strongly closed in $\mathcal{M}_\omega$. Suppose a sequence $X_n \in (\mathcal{M}_\omega^\alpha)_1$ strongly converges to $X \in (\mathcal{M}_\omega)_1$ as $n \to \infty$. Let $0 < \kappa < 1/2$. Let $E$ be a Borel set with $0 < \mu(E) < \infty$ and $K_0 \subset \mathbb{R}$ a compact set such that $\mu(E \setminus K_0) < \kappa$ and $\alpha : K_0 \to \text{Aut}(\mathcal{M})$ is continuous. Let $\varphi \in \mathcal{M}_\omega$ be a faithful state. Recall the fact that for any $\phi \in \mathcal{M}_\omega$, the function $K_0 \ni t \mapsto \alpha_t(\phi) \in (\mathcal{M}_\omega)_s$ is continuous since $\alpha_t(\phi^\omega) = \alpha_t(\phi)^\omega$. Thus $\Psi := \{\alpha_t^{-1}(\phi^\omega) \mid t \in K_0\}$ is compact in $(\mathcal{M}_\omega)_s$, and we have $\sup_{t \in K_0} \| X_n - X \|_{\alpha_t^{-1}(\phi^\omega)}^2 \to 0$ as $n \to \infty$. 


Let $\varepsilon > 0$. Then we can find $n_0$ such that $\sup_{t \in K_0} \|X_{n_0} - X\|^2_{\alpha_t^{-1}(\varphi)} < \varepsilon/3$. Fix representing sequences of $X_n$ and $X$, $(x'_n)_\nu$ and $(x''_\nu)_\nu$ with $\|x'_n\|, \|x''_\nu\| \leq 1$, respectively. Then again by compactness of $\Psi$, there exists $W_1 \in \omega$ such that
\[
\|x'_\nu - x''_\nu\|^2_{\alpha_t^{-1}(\varphi)} < \varepsilon/3 \quad \text{for all } t \in K, \nu \in W_1.
\] Since $X_{n_0} \in M^\omega$, there exist a compact set $K_1 \subset K_0$, $0 < \delta < 1$ and $W_2 \in \omega$ such that $\mu(K_0 \setminus K_1) < \kappa$, and
\[
\|\alpha_s(x'_{n_0}) - \alpha_t(x'_{n_0})\|_{\varphi} < \varepsilon/3 \quad \text{for all } s, t \in K_1, |s - t| < \delta, \nu \in W_2.
\] Thus for $s, t \in K_1$ with $|s - t| < \delta$ and $\nu \in W_1 \cap W_2$, we have $\|\alpha_s(x''_\nu) - \alpha_t(x''_\nu)\|_{\varphi} < \varepsilon$. This shows that $X \in M^\omega$ since $\mu(E \setminus K_1) < 2\kappa$. Hence $M^\omega$ is a von Neumann algebra, and so is $M_{\omega, \alpha} = M^\omega \cap M^\omega$. 

We should note that $M \subset M^\omega$ and $M_{\omega, \alpha} \subset M^\omega \cap M^\omega$.

Suppose that an flow $\alpha$ fixes $p \in M^P$. Denote the reduced flow by $\alpha^p$. It is trivial that $\delta^\omega_{\alpha^p} \subset \delta^\omega_{\alpha}$. By Lemma 2.9 we obtain the following result.

**Corollary 3.11.** Let $M$ be a von Neumann algebra and $\alpha$ a flow on $M$. Suppose that $p \in M^P$ is fixed by $\alpha$. Then the $(\alpha, \omega)$-equicontinuous parts of $(M_p)^\omega$ and $(M_p)_{\omega, \alpha}$ are described as follows:

\[
(M_p)^\omega_{\alpha^p} = (M^\omega_{\alpha^p})_p, \quad (M_p)_{\omega, \alpha^p} = (M_{\omega, \alpha^p})_p.
\]

The following result is a direct consequence of Lemma 3.8 and this shows that the $(\alpha, \omega)$-equicontinuous parts $M^\omega_{\alpha}$ and $M_{\omega, \alpha}$ are invariant under perturbation.

**Lemma 3.12.** If $(\alpha, c)$ be a Borel cocycle action of $\mathbb{R}$ on a von Neumann algebra $M$, and $(\beta, d)$ is a perturbation of $(\alpha, c)$ by a Borel unitary path. Then $M^\omega_{\alpha} = M^\omega_{\beta}$ and $M_{\omega, \alpha} = M_{\omega, \beta}$.

### 3.3. Flows on $M^\omega_{\alpha}$ or $M_{\omega, \alpha}$

**Lemma 3.13.** Let $M$ be a von Neumann algebra. The following statements hold:

1. If $\alpha$ is a flow on $M$, then so is $\alpha$ on $M^\omega_{\alpha}$;
2. If $(\alpha, c)$ is a Borel cocycle action of $\mathbb{R}$ on $M$, then $\alpha$ is a flow on $M_{\omega, \alpha}$.

**Proof.** (1) Let $\varphi \in M^\omega_{\alpha}$ be a faithful state. Since $\varphi$ is faithful, $\{a\varphi \mid a \in M^\omega_{\alpha}\}$ is dense in $(M^\omega_{\alpha})_\varphi$, the predual of $M^\omega_{\alpha}$. Then
\[
\|\alpha_t(a\varphi) - a\varphi\|_{(M^\omega_{\alpha})_\varphi} \leq \|\alpha_t(a)(\alpha_t(\varphi) - \varphi)\|_{(M^\omega_{\alpha})_\varphi} + \|(\alpha_t(a) - a)\varphi\|_{(M^\omega_{\alpha})_\varphi} \leq \|a\|\|\alpha_t(\varphi) - \varphi\|_{M^\omega_{\alpha}} + \|\alpha_t(a) - a\|_{\varphi}.
\]
If $t \to 0$, the last two terms converge to 0 because $\mathbb{R} \ni t \mapsto \alpha_t(a)$ is strongly continuous for all $a \in M^\omega_{\alpha}$ by Proposition 3.5.

(2) Note that $\alpha_s\alpha_t = \alpha_{s+t}$ on $Z(M)$. Since the group homomorphism $\alpha: \mathbb{R} \to \text{Aut}(Z(M))$ is the composition of the Borel map $\alpha: \mathbb{R} \to \text{Aut}(M)$ and the restriction $\text{Aut}(M) \to \text{Aut}(Z(M))$, which is continuous, $\alpha$ is a Borel homomorphism that is in fact continuous because $\text{Aut}(Z(M))$ is Polish.
Let $\varphi \in M_*$ be a faithful normal state and $E : M \to Z(M)$ be the conditional expectation such that $\varphi \circ E = \varphi$. We put $\chi := \varphi|_{Z(M)}$. As in (1), $\{a\varphi^\omega \mid a \in M_{\omega,\alpha}\}$ is dense in $(M_{\omega,\alpha})_*$. Using $\varphi^\omega = \chi \circ \tau^\omega$ on $M_\omega$, we have
\[
\|\alpha_t(\varphi^\omega) - a\varphi^\omega\|_{(M_{\omega,\alpha})_*} \leq \|a\|\|\alpha_t(\chi) - \chi\|_{Z(M)_*} + \|\alpha_t(a) - a\|_{\varphi^\omega}.
\]
Since the first term in the right hand side converges to 0 as $t \to 0$, it suffices to show $\|\alpha_t(a) - a\|_{\varphi^\omega} \to 0$ as $t \to 0$.

Let $a \in M_{\omega,\alpha}$ and $(a^\nu)_\nu$ a representing sequence. Since $\alpha$ is a flow on $Z(M)$, the set $\{\alpha_r(\chi) \circ E \mid r \in [0, 1]\}$ is compact in $M_*$. Then for any $\varepsilon > 0$, there exist a compact set $K \subset [0, 1]$ with $\mu(K) > 0$, $\delta > 0$ and $W \in \omega$ such that for $s, t \in K$ with $|s - t| < \delta$ and $\nu \in W$, we have $\|\alpha_s(a^\nu) - \alpha_t(a^\nu)\|_{\alpha_r(\chi) \circ E} < \varepsilon$ for all $r \in [0, 1]$. Letting $\nu \to \omega$, we have $\|\alpha_s(a) - \alpha_t(a)\|_{\alpha_r(\chi) \circ \tau^\omega} \leq \varepsilon$. Then
\[
\|\alpha_{s-t}(a) - a\|_{\varphi^\omega} = \|\alpha_s(a) - \alpha_t(a)\|_{\alpha_r(\chi) \circ \tau^\omega} \leq \varepsilon \quad \text{for all} \quad s, t \in K, \quad |s - t| < \delta
\]
because $\alpha_s \alpha_t = \alpha_{s+t}$ on $M_{\omega,\alpha}$. Since the set $K - K$ contains an open neighborhood of 0, there exists $\delta' > 0$ such that if $|t| < \delta'$, then $\|\alpha_t(a) - a\|_{\varphi^\omega} \leq \varepsilon$. Therefore, $\alpha$ is a flow on $M_{\omega,\alpha}$.

Let $\alpha$ be a flow on a von Neumann algebra $M$. For $f \in L^1(\mathbb{R})$ and $x \in M$, we let $\alpha_f(x) := \int_\mathbb{R} f(t) \alpha_t(x) \, dt$. The following result provides us with a method of creating elements which belongs to $M^\omega_\alpha$ though those may be trivial sequences.

**Lemma 3.14.** Let $(x^\nu)_\nu \in \ell^\infty(M)$ and $f \in L^1(\mathbb{R})$. If $\alpha$ is a flow, then the following statements hold:

1. $(\alpha_f(x^\nu))_\nu \in \mathcal{E}^\omega_{\alpha}$;
2. If $(x^\nu)_\nu \in \mathcal{E}^\omega_{\alpha} \cap \mathcal{N}_\omega$, then $(\alpha_f(x^\nu))_\nu \in \mathcal{E}^\omega_{\alpha} \cap \mathcal{N}_\omega$;
3. If $(x^\nu)_\nu \in \mathcal{E}_\omega$, then $(\alpha_f(x^\nu))_\nu \in \mathcal{E}_\omega$.

**Proof.** (1). Observe that $\alpha_t(\alpha_f(x^\nu)) - \alpha_f(x^\nu) = \alpha_{\lambda_t f - f}(f)$, where $(\lambda_t f)(s) = f(s-t)$. Let $C := \sup_\nu \|x^\nu\|$. Then
\[
\|\alpha_t(\alpha_f(x^\nu)) - \alpha_f(x^\nu)\| \leq C\|\lambda_t f - f\|_1 \quad \text{for all} \quad \nu \in \mathbb{N}.
\]
Hence $(\alpha_f(x^\nu))_\nu \in \mathcal{E}^\omega_{\alpha}$ by Lemma 3.13.

(2). Suppose that $(x^\nu)_\nu \in \mathcal{E}^\omega_{\alpha} \cap \mathcal{N}_\omega$. Let $(y^\nu)_\nu \in \mathcal{F}_\omega$. Then
\[
y^\nu \alpha_f(x^\nu) = \int_{-\infty}^{\infty} f(t)y^\nu \alpha_t(x^\nu) \, dt \quad \text{for all} \quad \nu \in \mathbb{N}.
\]
By Lemma 3.13, $y^\nu \alpha_t(x^\nu) \to 0$ compact uniformly in the strong topology as $\nu \to \omega$. This implies that $y^\nu \alpha_f(x^\nu) \to 0$. Hence $(\alpha_f(x^\nu))_\nu \in \mathcal{N}_\omega$.

(3). Suppose that $(x^\nu)_\nu$ is $\omega$-central. For $\varepsilon > 0$, take $T > 0$ such that $\|f - f1_{[-T,T]}\|_1 < \varepsilon$. By Lemma 2.7, there exists $W \in \omega$ such that if $\nu \in W$, then
sup_{t \in [-T,T]} \| x' \alpha_{-t}(\xi) - \alpha_{-t}(\xi)x' \| < \varepsilon. \) Then for any \( \xi \in \mathcal{H}, \)

\[
\| \alpha_f(x')\xi - \xi \alpha_f(x') \| \leq \int_{-T}^{T} |f(t)| \| \alpha_t(x')\xi - \xi \alpha_t(x') \| \, dt
\]

\[
\leq \int_{-T}^{T} |f(t)| \| x' \alpha_{-t}(\xi) - \alpha_{-t}(\xi)x' \| \, dt
\]

\[
+ \int_{[-T,T]^c} |f(t)| \| \alpha_t(x')\xi - \xi \alpha_t(x') \| \, dt
\]

\[
\leq \| f \|_1 \varepsilon + 2C \| \xi \| \varepsilon.
\]

Hence \((\alpha_f(x'))_\nu \) is \( \omega \)-central.

\[\cdash\]

**Lemma 3.15.** Let \( x = \pi_\omega((x')_\nu) \in M_\omega^N \) and \( f \in L^1(\mathbb{R}) \). If \( \alpha \) is a flow, then \( \alpha_f(x) = \pi_\omega((\alpha_f(x')_\nu)) \).

**Proof.** Put \( a := \pi_\omega((\alpha_f(x')_\nu)) \) that belongs to \( M_\omega^N \) by the previous lemma. It suffices to show that \( \tau^\omega(ay) = \tau^\omega(\alpha_f(x)y) \) for all \( y \in M_\omega^N \), where \( \alpha_f(x) \) is well-defined by Lemma 3.3. On the one hand, we have

\[
\varphi(\alpha_f(x')y') = \int_{-\infty}^{\infty} f(t)\varphi(\alpha_t(x')y') \, dt \quad \text{for all } \varphi \in M_*, \ \nu \in \mathbb{N}.
\]

By Lemma 3.3 \( \varphi(\alpha_t(x')y') \to \varphi(\tau^\omega(\alpha_t(x)y)) \) compact uniformly as \( \nu \to \omega \).

Hence

\[
\varphi(\tau^\omega(ay)) = \lim_{\nu \to \omega} \varphi(\alpha_f(x')y') = \int_{-\infty}^{\infty} f(t)\varphi(\tau^\omega(\alpha_t(x)y)) \, dt.
\]

On the other hand, the normality of the conditional expectation \( \tau^\omega: M_\alpha^\omega \to \mathcal{M} \) and the continuity of \( \alpha \) on \( M_\omega^N \) implies that \( \int_{-\infty}^{\infty} f(t)\tau^\omega(\alpha_t(x)y) \, dt = \tau^\omega(\alpha_f(x)y) \).

Hence \( \varphi(\tau^\omega(ay)) = \varphi(\tau^\omega(\alpha_f(x)y)) \) for any \( \varphi \in M_* \), and we have \( \tau^\omega(ay) = \tau^\omega(\alpha_f(x)y) \). \(\cdash\)

### 3.4. Connes spectrum of \( \alpha_\omega \)

We show the fast reindexation trick is applicable to our interesting case. Namely, we will construct a reindexation map in the \((\alpha, \omega)\)-equicontinuous part \( M_\alpha^\omega \). Our proof is almost in parallel with [52, Lemma 5.3], but we should be careful of a construction of that because a reindexation map constructed in [52, Lemma 5.3] may not send given elements into \( M_\omega^N \) nor commute with \( \alpha_t \) for all \( t \in \mathbb{R} \).

**Lemma 3.16** (Fast reindexation trick). Let \( \alpha \) be a flow on a von Neumann algebra \( \mathcal{M} \), and \( F \subset M^\omega \) and \( N \subset M_\omega^N \) separable von Neumann subalgebras. Suppose that \( N \) is \( \alpha \)-invariant. Then there exists a faithful normal \( * \)-homomorphism \( \Phi: N \to M_\alpha^\omega \) with the following properties:

1. \( \Phi = \text{id} \) on \( N \cap \mathcal{M} \);
2. \( \Phi(N \cap M_\omega, \alpha) \subset F' \cap M_\omega, \alpha \);
3. \( \tau^\omega(\Phi(a)x) = \tau^\omega(a)\tau^\omega(x) \) for all \( a \in N, \ x \in F \);
4. \( \alpha_t \circ \Phi = \Phi \circ \alpha_t \) on \( N \) for all \( t \in \mathbb{R} \).

We call such \( \Phi \) a fast reindexation map.
Proof. Let us introduce the same notations as the proof of [52, Lemma 5.3]. We may suppose that $M \subset N$. For $n \in N$, we take finite subsets $N_n \subset N_{n+1}$ of $N$, $F_n \subset F_{n+1}$ of $F$, $M_n \subset M_{n+1}$ of $M$, and $B_n \subset B_{n+1}$ of $\mathcal{B} := \{\alpha_t \mid t \in \mathbb{Q}\}$ such that

- $\tilde{N} := \bigcup_n N_n$ is a unital $*$-algebra over $\mathbb{Q} + i\mathbb{Q}$, weakly dense in $N$;
- $\tilde{N}$ is globally invariant by $\mathcal{B}$;
- $\tilde{N} \cap M$ is weakly dense in $M$;
- $\tilde{N} \cap M_{\omega,\alpha}$ is weakly dense in $N \cap M_{\omega,\alpha}$;
- $\tilde{F} := \bigcup_n F_n$ is weakly dense in $F$;
- $\bigcup_n M_n$ is norm dense in $M$;
- $\mathcal{B} = \bigcup_n B_n$.

For each $x \in F \cup N$, we choose a representing sequence $(x^\nu)_\nu$ such that for all $\nu \in N$ and $\lambda \in \mathbb{C}$, we have $\|x^\nu\| \leq \|x\|$, $(x^\nu)^* = (x^\nu)^*$, $(\lambda x^\nu)_\lambda$ is weakly dense in $N$, and $(x^\nu)_\nu$ is constant if $x \in M$.

Let $\phi \in M_\alpha$ be a faithful state. For each $x \in M_\alpha^n$ and $n \in N$, we find $\delta_n(x) > \delta_{n+1}(x) > 0$ and a neighborhood $W_n(x) \supseteq W_{n+1}(x)$ of $\omega$ in $N$ such that for all $y \in M_\omega$ with $\|y\| < \delta_n(x)$, we have $\|x^\nu y\|_\phi^2 + \|yx^\nu\|_\phi^2 < 1/n$ for $\nu \in W_n(x)$.

For $n \in N$ and $\varepsilon > 0$, take $\gamma_{n,\varepsilon} > 0$ such that the following set belongs to $\omega$:

$$E_{n,\varepsilon} := \{\nu \in N \mid \|\alpha_t(x^\nu) - x^\nu\|_\phi^2 < \varepsilon, |t| \leq \gamma_{n,\varepsilon}, x \in N_n\}$$

For $n \geq 1$, we choose $p(n) \in N$ such that $p(n) \geq n$ and

- $\{x \in N_n \mid x \in W_n(x) \cap \bigcap_{m=1}^n E_{m,1/m}\}$;
- $\|x^{p(n)}y^{p(n)} - (xy)^{p(n)}\|_\phi^2 < 1/n$ for $x, y \in N_n$;
- $\|x^{p(n)}, a^n\|_\phi^2 < 1/n$ for $x \in N_n \cap M_{\omega,\alpha}$, $a \in F_n$;
- $\|\psi(a^nx^{p(n)}) - \psi(a^nx^{\tau(x)})\|_\phi^2 < 1/n$ for $x \in N_n$, $a \in F_n$, $\psi \in M_\alpha$;
- $\|\beta(x^{p(n)}) - (\beta^x)^{p(n)}\|_\phi^2 < 1/n$ for $x \in N_n$, $\beta \in B_n$.

Letting $\Phi(x) = \pi_\omega((x^{p(n)}))$ for $x \in \tilde{N}$, we obtain a faithful normal $*$-homomorphism $\Phi: N \to M_\alpha$ which satisfies (1), (2) and (3), and commutes with $\alpha_t$ for $t \in \mathbb{Q}$. We will check that $\Phi(N)$ is contained in the $(\alpha, \omega)$-equicontinuous part. Let $\varepsilon > 0$ and $x \in N_m$. Take a large $m_\varepsilon \in N$ such that $1/m_\varepsilon < \varepsilon$ and $m \leq m_\varepsilon$. Then

$$\{n \in N \mid n \geq m_\varepsilon, \|\alpha_t(x^{p(n)}) - x^{p(n)}\|_\phi^2 < \varepsilon, |t| \leq \gamma_{m_\varepsilon,1/m_\varepsilon}\} = [m_\varepsilon, \infty) \cap N.$$

Indeed, let $n \geq m_\varepsilon$. Then $p(n) \in E_{m_\varepsilon,1/m_\varepsilon}$. It turns out that $\|\alpha_t(x^{p(n)}) - x^{p(n)}\|_\phi^2 < 1/m_\varepsilon < \varepsilon$ for all $|t| \leq \gamma_{m_\varepsilon,1/m_\varepsilon}$ since $x \in N_{m_\varepsilon}$. This implies that $(x^{p(n)})_n$ is $(\alpha, \omega)$-equicontinuous, and $\Phi(x) \in M_\alpha$ for $x \in N_m$. Since $\Phi$ is normal, we see that $\Phi$ maps $N$ into $M_\alpha$.

Then the commutativity $\Phi \circ \alpha_t = \alpha_t \circ \Phi$ holds for all $t \in \mathbb{R}$ since $\alpha$ is a flow on $M_\alpha$ by Lemma 3.13.

Lemma 3.17. Let $\alpha$ be a flow on a von Neumann algebra $M$. Then the following statements hold:

1. $\Gamma(\alpha|_{M_{\omega,\alpha}}) \subset \Gamma(\alpha)$;
(2) If $\alpha$ is centrally ergodic, then $\text{Sp}(\alpha|_{M_{\omega,\alpha}}) = \Gamma(\alpha|_{M_{\omega,\alpha}})$. In particular, $\text{Sp}(\alpha|_{M_{\omega,\alpha}})$ is the annihilator group of $\ker(\alpha|_{M_{\omega,\alpha}})$.

Proof. (1). By Lemma 3.14 and 3.15, if $f \in L^1(\mathbb{R})$ satisfies $\alpha_f = 0$ on $M$, then $\alpha_f = 0$ on $M_{\omega,\alpha}$. Hence $\text{Sp}(\alpha|_{M_{\omega,\alpha}}) \subset \text{Sp}(\alpha)$. Applying this observation to $\alpha^e$ with a projection $e \in M^e$, we have $\text{Sp}(\alpha^e|_{M_{\omega,\alpha}^e}) \subset \text{Sp}(\alpha^e)$. By Corollary 3.11 we have the natural identification $(M_{\omega,\alpha})_e = (M_{\alpha})_{\omega,\alpha}$. Thus $\text{Sp}(\alpha|_{(M_{\omega,\alpha})_e}) \subset \text{Sp}(\alpha^e)$.

Let $z$ be the central support projection of $e$ in $M_{\omega,\alpha}$. Then $z$ is fixed by $\alpha$, and the map $(M_{\omega,\alpha})_z \ni x \mapsto xe \in (M_{\omega,\alpha})_e$ is an isomorphism. Obviously, this intertwines the flows coming from $\alpha$. Hence,

$$\Gamma(\alpha|_{M_{\omega,\alpha}}) \subset \text{Sp}(\alpha|_{(M_{\omega,\alpha})_e}) = \text{Sp}(\alpha|_{(M_{\omega,\alpha})_e}) \subset \text{Sp}(\alpha^e).$$

Since $e$ is arbitrary, we have $\Gamma(\alpha|_{M_{\omega,\alpha}}) \subset \Gamma(\alpha)$.

(2). Let $p \in \text{Sp}(\alpha|_{M_{\omega,\alpha}})$, $\varepsilon > 0$ and $T > 0$ be given. Then there exists a non-zero $x \in M_{\omega,\alpha}$ such that $\|\alpha_t(x) - e^{ipt}x\| < \varepsilon\|x\|$ for all $t \in [-T, T]$. Let $f \in (M_{\omega,\alpha})^\alpha$ be a non-zero projection, $N := \{\alpha_t(x) \mid t \in \mathbb{R}\}''$ and $F = \{f\}''$. Take a fast reindexation map $\Phi: N \to F' \cap M_{\alpha}$ as in the previous lemma. Since $\alpha$ is centrally ergodic, $\tau^\alpha(f) \in Z(M)^\alpha = \mathbb{C}$. This implies $\|\Phi(a)f\|_2 = \|a\|_2\|f\|_2$ for all $a \in N$. Hence the $*$-homomorphism $N \ni a \mapsto \Phi(a)f$ is faithful, and we have $\|\Phi(a)f\| = \|a\|$. Thus for $t \in [-T, T]$, we obtain

$$\|\alpha_t(\Phi(x)f) - e^{ipt}\Phi(x)f\| = \|\Phi(\alpha_t(x) - e^{ipt}x)f\|$$

$$< \varepsilon\|x\| = \varepsilon\|\Phi(x)f\|.$$

This means $p \in \text{Sp}(\alpha^e|_{M_{\omega,\alpha}})$. Therefore $p \in \Gamma(\alpha|_{M_{\omega,\alpha}})$.

In particular, if $\alpha$ is a flow on a factor $M$ with $\Gamma(\alpha) = \{0\}$, then $\Gamma(\alpha|_{M_{\omega,\alpha}}) = \{0\}$, that is, $\alpha = \text{id}$ on $M_{\omega,\alpha}$. We do not know whether the converse holds or not for injective factors.

Proposition 3.18. Let $\alpha$ be a centrally ergodic flow on a von Neumann algebra $M$. If $\Gamma(\alpha) = \{0\}$ and 0 is isolated in $\text{Sp}(\alpha)$, then any element in $M_{\omega,\alpha}$ is represented by a sequence in $M^\alpha$.

Proof. Let $x = \pi_\omega((x')_\nu) \in M_{\omega,\alpha}$. By the previous lemma, $\alpha_t(x) = x$ for all $x \in \mathbb{R}$. Since 0 is isolated in $\text{Sp}(\alpha)$, there exists a non-negative $f \in L^1(\mathbb{R})$ such that $\alpha_f$ gives a faithful normal conditional expectation from $M$ onto $M^\alpha$. By Lemma 3.15 we have $x = \alpha_f(x) = \pi_\omega((\alpha_f(x'))_\nu)$. \hfill $\Box$

3.5. Lift of Borel unitary path. In this subsection, we solve the problem concerning a lift of a Borel unitary path $U: \mathbb{R} \to M_2^\alpha$ in Lemma 3.21. A Borel path $U: \mathbb{R} \to M_2^\alpha$ means that $\{U(t) \mid t \in \mathbb{R}\}$ generates a separable von Neumann subalgebra, and $U$ is a Borel map into it.

Lemma 3.19. Let $M$ be a von Neumann algebra, $\phi \in M_*$ a state and $u \in M^U$. Then $\|e^{t\text{Log}(u)} - 1\|_\phi \leq \sqrt{2}\|u - 1\|_{\phi}^{1/2}$ for $|t| \leq 1$, where $\text{Log} e^{ix} = ix$ for $-\pi \leq x < \pi$. 

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Then we obtain
\[ v_t = \int_{-\pi}^{\pi} e^{i\lambda} dE(\lambda). \]

Hence using the triangle inequality.

Proof. Let \( u = \int_{-\pi}^{\pi} e^{i\lambda} dE(\lambda) \) be the spectral decomposition on the torus \( \mathbb{R}/2\pi \mathbb{Z} = [-\pi, \pi) \), and \( \varepsilon := \|u - 1\|_\phi \leq 2 \). Then we have
\[
\varepsilon^2 = \int_{-\pi}^{\pi} |e^{i\lambda} - 1|^2 d\phi(E(\lambda)) = \int_{-\pi}^{\pi} 4\sin^2(\lambda/2) d\phi(E(\lambda)).
\]

Thus if we set \( A_\varepsilon := \{ \lambda \in [-\pi, \pi) \mid \sin(\lambda/2) \geq \varepsilon^{1/2} \} \), then we have \( \phi(E(A_\varepsilon)) \leq \varepsilon/4 \). Using \( \log(u) = \int_{-\pi}^{\pi} i\lambda dE(\lambda) \) and \( e^{i\theta \log(u)} = \int_{-\pi}^{\pi} e^{i\theta \lambda} dE(\lambda) \), we have
\[
\|e^{i\theta \log(u)} - 1\|_\phi^2 = \int_{-\pi}^{\pi} |e^{i\theta \lambda} - 1|^2 d\phi(E(\lambda)) = \int_{-\pi}^{\pi} 4\sin^2(\theta\lambda/2) d\phi(E(\lambda))
\]
\[
= \int_{A_\varepsilon} 4\sin^2(\theta\lambda/2) d\phi(E(\lambda)) + \int_{A_\varepsilon^c} 4\sin^2(\theta\lambda/2) d\phi(E(\lambda))
\]
\[
\leq 4\phi(E(A_\varepsilon)) + \int_{A_\varepsilon^c} 4\sin^2(\lambda/2) d\phi(E(\lambda))
\]
\[
\leq 4\phi(E(A_\varepsilon)) + \int_{-\pi}^{\pi} 4\sin^2(\lambda/2) d\phi(E(\lambda)) = 4\varphi(E(A_\varepsilon)) + \varepsilon^2.
\]

If \( \varepsilon \leq 1 \), then \( 4\varphi(E(A_\varepsilon)) \leq \varepsilon \). If \( \varepsilon > 1 \), then \( A_\varepsilon = \emptyset \), and \( \varphi(E(A_\varepsilon)) = 0 \). As a result, we obtain \( \|e^{i\theta \log(u)} - 1\|_\phi^2 \leq 2\varepsilon \) in both cases.

Lemma 3.20. Let \( t_1, \ldots, t_n \in \mathbb{R} \) with \( t_1 < t_2 < \cdots < t_n \). Suppose that unitaries \( u_1, \ldots, u_n \in \mathcal{M} \) are given. If \( \varepsilon > 0 \) and a faithful state \( \phi \in \mathcal{M}_\phi \), we have \( \|u_j - u_{j+1}\|_\phi^2 < \varepsilon \) for \( j = 1, \ldots, n-1 \), then there exists a continuous unitary path \( u: [t_1, t_n] \to \mathcal{M} \) such that \( u(t_j) = u_j \) for all \( j \), and \( \|u(t) - u(t_j)\|^2 < \sqrt{2}\varepsilon^{1/2} \) for \( t \in [t_j, t_{j+1}] \). If moreover, we have \( \|u_i - u_j\|^2 < \varepsilon \) for all \( i, j \), then \( \|u(s) - u(t)\|^2 < 4\varepsilon^{1/2} \) for all \( s, t \in [t_1, t_n] \).

Proof. Let \( 1 \leq j \leq n - 1 \). Set \( v_j(\theta) := u_j \exp(\theta \log(u_j^* u_{j+1})) \) for \( \theta \in [0, 1] \). The previous lemma implies the following:
\[
\|v_j(\theta) - u_j\|_\phi = \|\exp(\theta \log(u_j^* u_{j+1})) - 1\|_\phi \leq \sqrt{2}\|u_j - u_{j+1}\|_\phi^{1/2},
\]
and
\[
\|v_j(\theta)^* - u_j^*\|_\phi = \|e^{i\theta \log(u_j^* u_{j+1})} u_j^* - u_j^*\|_\phi = \|u_j^* e^{i\theta \log(u_j u_{j+1}^*)} - u_j^*\|_\phi \leq \sqrt{2}\|u_j^* - u_{j+1}^*\|_\phi^{1/2}.
\]

Hence
\[
\|v_j(\theta) - u_j\|_\phi^2 \leq \sqrt{2}(\|u_j - u_{j+1}\|_\phi^2)^{1/2} < \sqrt{2}\varepsilon^{1/2}.
\]

Then \( v_j: [0, 1] \to \mathcal{M}^U \) is strongly continuous and \( v_j(0) = u_j \) and \( v(1) = u_{j+1} \). By connecting \( v_j \)'s, we have a desired path \( u(t) \). The last statement is verified by using the triangle inequality. \( \square \)
Lemma 3.21 (Lift of Borel unitary path). Let \( \alpha : \mathbb{R} \to \text{Aut}(\mathcal{M}) \) be a Borel map. Let \( U : \mathbb{R} \to \mathcal{M}_\alpha^w \) be a Borel unitary path. Then for any Borel set \( E \subset \mathbb{R} \) with \( 0 < \mu(E) < \infty \) and \( \varepsilon > 0 \), there exist a compact set \( K \subset E \) and a sequence \( (u(t)\nu)_\nu \) for \( t \in E \) such that

- \( \pi_\omega((u(t)\nu)_\nu) = U(t) \) for almost every \( t \in E \), and the equality holds for all \( t \in K \);
- \( \mu(E \setminus K) < \varepsilon \);
- For all \( \nu \in \mathbb{N} \), the map \( E \ni t \mapsto u(t)\nu \) is Borel, and the map \( K \ni t \mapsto u(t)\nu \) is strongly continuous;
- the family \( \{ K \ni t \mapsto u(t)\nu \in \mathcal{M}_\nu \} \) is \( \omega \)-equicontinuous.

Proof. By Lemma 9.11, we have a compact set \( K \subset E \) such that \( \mu(E \setminus K) < \varepsilon \) and \( U \) is continuous on \( K \). Continuing this process, we get a mutually disjoint series of compact sets \( K = K_0, K_1, \ldots \subset E \) such that \( \mu(E \setminus \bigcup_j K_j) = 0 \) and \( U \) is continuous on each \( K_j \). By lifting piecewise, we see that it suffices to show the existence of a continuous lift for \( U : K \to \mathcal{M}_\alpha^w \).

We may and do assume that \( K \subset [0, 1] \) by changing the variable of \( U(t) \). Let \( \varphi \in \mathcal{M}_\star \) be a faithful state. For each \( t \in K \), we choose a representing unitary sequence \( (\tilde{U}(t)\nu)_\nu \) of \( U(t) \). Then for each \( k \in \mathbb{N} \), we can construct by induction \( N_k \in \mathbb{N} \) \((N_0 := 1)\), \( F_k \in \omega \) \((F_0 := \emptyset)\) and a finite set \( A_k \subset K \) \((A_0 := \emptyset)\) with the following properties:

- If \( s, t \in K \) satisfies \( |s - t| \leq 1/N_k \), then \( \| U(s) - U(t) \|_{\varphi^w}^2 < 1/2k \);
- \( N_k > N_{k-1} \) and \( 2N_k + 1/(2N_{k-1}) < 1/N_{k-1} \);
- \([k, \infty) \supset F_{k-1} \supseteq F_k \);
- \( A_k := \{a^k_j, b^k_j\}_{j=0}^{N_k-1} \cup A_{k-1} \), where
  \[ a^k_j := \min[j/N_k, (j+1)/N_k] \cap K, \quad b^k_j := \max[j/N_k, (j+1)/N_k] \cap K; \]
- If \( s, t \in A_k \) and \( \nu \in F_k \), then
  \[
  \| \tilde{U}(s)\nu - \tilde{U}(t)\nu \|_{\varphi^w}^2 \leq \| U(s) - U(t) \|_{\varphi^w}^2 + 1/2k. \tag{3.6}
  \]

Note that \( \Delta^k_j := [j/N_k, (j+1)/N_k] \cap K \) may be empty, and \( a_j, b_j \) are not defined in this case. Since \( |s - t| \leq 1/N_k \) for \( s, t \in A_k \cap [a^k_j, b^k_j] \), we have

\[
\| \tilde{U}(s)\nu - \tilde{U}(t)\nu \|_{\varphi^w}^2 \leq \| U(s) - U(t) \|_{\varphi^w}^2 + 1/2k < 1/k \quad \text{for all } \nu \in F_k.
\]

Applying Lemma 3.20 to \( A_k \cap [a^k_j, b^k_j] \) for each \( j \), \( \tilde{U}(t)\nu \) and \( \varepsilon := 1/k \), we obtain a continuous unitary path \( U(t)^{k,\nu} \) on \( \bigcup_j [a^k_j, b^k_j] \) such that \( U(t)^{k,\nu} = \tilde{U}(t)\nu \) for all \( t \in \bigcup_j A_k \cap [a^k_j, b^k_j] \) and

\[
\| U(s)^{k,\nu} - U(t)^{k,\nu} \|_{\varphi^w}^2 \leq 4/k^{1/2} \quad \text{for all } s, t \in [a^k_j, b^k_j], \ \nu \in F_k. \tag{3.7}
\]

Put \( u(t)^{\nu} := U(t)^{k,\nu} \) for \( \nu \in F_k \setminus F_{k+1} \) and \( t \in K \). We show \( \{ K \ni t \mapsto u(t)^{\nu} \}_\nu \) is \( \omega \)-equicontinuous. Let \( s, t \in K \) with \( |s - t| < 1/2N_k \) and \( \nu \in F_k \). Take \( m \geq k \) with \( \nu \in F_m \setminus F_{m+1} \). Let \( s_0, t_0 \in A_m \) be the nearest points from \( s, t \), respectively.
Then we have

\[ \|u(s)\nu - u(t)\nu\|^2 \]
\[ \leq \|U(s)^{\nu} - U(s_0)^{\nu}\|^2 + \|\tilde{U}(s_0)^{\nu} - \tilde{U}(t_0)^{\nu}\|^2 + \|U(t_0)^{\nu} - U(t)^{\nu}\|^2 \]
\[ \leq 4/m^{1/2} + \|\tilde{U}(s_0)^{\nu} - \tilde{U}(t_0)^{\nu}\|^2 + 4/m^{1/2} \quad \text{by (3.7)} \]
\[ \leq 8/k^{1/2} + \|U(s_0) - U(t_0)\|^2 \quad \text{by (3.6)}. \]

Since

\[ |s_0 - t_0| \leq |s_0 - s| + |s - t| + |t - t_0| \leq 1/N_m + 1/(2N_k) + 1/N_m \leq 1/N_k, \]

we have \( \|U(s_0) - U(t_0)\|^2 \nu < 1/k \), and

\[ \|u(s)\nu - u(t)\nu\|^2 \leq 8/k^{1/2} + 1/k + 1/k \leq 10/k^{1/2}. \]

Thus \( \{K \ni t \mapsto u(t)\nu\} \nu \) is \( \omega \)-equicontinuous, and the function \( K \ni t \mapsto \pi_\omega((u(t)\nu)_\nu) \in \mathcal{M} \) is continuous. Since \( u(t)\nu = \tilde{U}(t)\nu \) for all \( t \in A_k \) and \( \nu \in F_k \),

\[ \pi_\omega((u(t)\nu)_\nu) = U(t) \quad \text{for all } t \in \bigcup_k A_k. \]

It is clear that \( \bigcup_k A_k \) is dense in \( K \), and we have \( \pi_\omega((u(t)\nu)_\nu) = U(t) \) for all \( t \in K \).

We close this section with the following three lemmas.

**Lemma 3.22.** Let \( K_1, K_2 \subset \mathbb{R} \) be compact sets. Let \( \alpha : \mathbb{R} \to \text{Aut}(\mathcal{M}) \) be a Borel map and \( \{w^\nu : K_2 \to \mathcal{M}\}_\nu \) a family of continuous maps. Suppose that

• \( \alpha \) is continuous on \( K_1 \);
• \( \{K_1 \ni s \mapsto \alpha_s(w(t)\nu)\}_\nu \) is \( \omega \)-equicontinuous for each \( t \in K_2 \);
• \( \{K_2 \ni t \mapsto w(t)\nu\}_\nu \) is \( \omega \)-equicontinuous.

Then \( \{K_1 \times K_2 \ni (s, t) \mapsto \alpha_s(w(t)\nu)\}_\nu \) is \( \omega \)-equicontinuous.

**Proof.** Let \( \varepsilon > 0 \) and \( \xi \in \mathcal{H} \) a cyclic and separating vector for \( \mathcal{M} \). Set \( \Psi := \{\alpha_s^{-1}(\xi) \mid s \in K_1\} \) that is a compact set. By Lemma 3.2 there exist \( \delta > 0 \) and \( W_1 \in \omega \) such that for all \( t, t' \in K_2 \) with \( |t - t'| < \delta \), \( \nu \in W_1 \) and \( \eta \in \Psi \), we have

\[ \|(w(t)\nu - w(t')\nu)\eta\| < \varepsilon, \quad \|\eta(w(t)\nu - w(t')\nu)\| < \varepsilon. \quad (3.8) \]

Take \( \{t_1, \ldots, t_N\} \) in \( K_2 \) such that each \( t \in K_2 \) has \( t_i \) with \( |t - t_i| < \delta \). By the second condition, there exist \( \delta' > 0 \) and \( W_2 \in \omega \) such that for all \( s, s' \in K_1 \) with \( |s - s'| < \delta' \), \( \nu \in W_2 \) and \( i = 1, \ldots, N \), we have

\[ \|\alpha_s(w(t_i)\nu) - \alpha_{s'}(w(t_i)\nu)\| < \varepsilon, \quad \|\xi(\alpha_s(w(t_i)\nu) - \alpha_{s'}(w(t_i)\nu))\| < \varepsilon. \quad (3.9) \]
Now let $s, s' \in K_1$ and $t, t' \in K_2$ with $|s - s'| < \delta'$ and $|t - t'| < \delta$. Take $t_i$ such that $|t' - t_i| < \delta$. Then for $\nu \in W_1 \cap W_2$, we obtain

$$\|(\alpha_s(w(t)^\nu) - \alpha_{s'}(w(t')^\nu))\xi\| \leq \|(\alpha_s(w(t)^\nu) - \alpha_s(w(t_i)^\nu))\xi\| + \|(\alpha_s(w(t_i)^\nu) - \alpha_{s'}(w(t_i)^\nu))\xi\| + \|(\alpha_{s'}(w(t_i)^\nu) - \alpha_{s'}(w(t')^\nu))\xi\|$$

$$\leq \|(w(t)^\nu - w(t_i)^\nu)\alpha_s^{-1}(\xi)\| + \|(w(t_i)^\nu - w(t_i)^\nu)\alpha_s^{-1}(\xi)\| + \varepsilon \quad \text{by (3.9)}$$

$$+ \|(w(t_i)^\nu - w(t')^\nu)\alpha_s^{-1}(\xi)\|$$

$$\leq 4\varepsilon \quad \text{by (3.8).}$$

Similarly, we obtain

$$\|\xi(\alpha_s(w(t)^\nu) - \alpha_{s'}(w(t')^\nu))\| < 4\varepsilon.$$  

Hence we are done. \hfill \Box

**Lemma 3.23.** Let $\alpha : \mathbb{R} \to \text{Aut}(\mathcal{M})$ be a Borel map and $C \subset \mathbb{R}$ a compact set. Suppose that $\{C \ni t \mapsto x(t)^\nu \in \mathcal{M}_\nu\}$ is $\omega$-equicontinuous and $(x(t)^\nu)_{\nu}$ is $\delta_{\nu}^{\omega}$ for all $t \in C$. Then for all $\kappa > 0$ and Borel set $E \subset \mathbb{R}$ with $0 < \mu(E) < \infty$, there exists a compact set $L \subset E$ such that

- $\mu(E \setminus L) < \kappa$;
- $\alpha$ is continuous on $L$;
- $\{L \ni s \mapsto \alpha_s(x(t)^\nu)\}_{\nu}$ is $\omega$-equicontinuous for all $t \in C$.

**Proof.** Take an increasing sequence of finite sets $C_1 \subset C_2 \subset \cdots C$ such that their union is dense in $C$. Then for each $n \in \mathbb{N}$, we can find a compact set $L_n \subset E$ such that

- $\mu(E \setminus L_n) < \kappa/2^{n+1}$;
- $\alpha$ is continuous on $L_n$;
- $\{L_n \ni s \mapsto \alpha_s(x(t)^\nu)\}_{\nu}$ is $\omega$-equicontinuous for all $t \in C_n$.

Set $L := \cap_n L_n$. Then $\mu(E \setminus L) \leq \sum_n \kappa/2^{n+1} < \kappa$, and $\alpha$ is continuous on $L$.

We will check the third condition. Let $\xi \in \mathcal{H}$ be a cyclic and separating vector. Let $\varepsilon > 0$ and $\Psi := \{\alpha_s^{-1}(\xi) | s \in L\}$ that is compact. Then there exist $\delta > 0$ and $W \in \omega$ such that if $t, t' \in C$ with $|t' - t| < \delta$ and $\nu \in W$, then

$$\|(x(t)^\nu - x(t')^\nu)\xi\| + \|\xi(x(t)^\nu - x(t')^\nu)\| < \varepsilon \quad \text{for all } \xi \in \Psi. \quad (3.10)$$

Fix $t \in C$ and take $t_0 \in C_n$ with $|t - t_0| < \delta$. Then by $(\alpha, \omega)$-equicontinuity, we have $\delta' > 0$ and $W' \in \omega$ such that if $s, s' \in L$ with $|s - s'| < \delta'$ and $\nu \in W'$, then

$$\|(\alpha_s(x(t_0)^\nu) - \alpha_{s'}(x(t_0)^\nu))\xi\| + \|\xi(\alpha_s(x(t_0)^\nu) - \alpha_{s'}(x(t_0)^\nu))\| < \varepsilon. \quad (3.11)$$
Then for all $s, s' \in L$ with $|s - s'| < \delta'$ and $\nu \in W \cap W'$,

\[
\| (\alpha_s(x(t)) - \alpha_{s'}(x(t))) \| \leq \| (\alpha_s(x(t)) - \alpha_s(x(t_0))) \| \\
+ \| (\alpha_s(x(t_0)) - \alpha_{s'}(x(t_0))) \| \\
+ \| (\alpha_{s'}(x(t_0)) - \alpha_{s'}(x(t))) \| \\
< \| (x(t) - x(t_0)) \alpha_s^{-1}(\xi) \| \\
+ \varepsilon \quad \text{by (3.11)} \\
+ \| (x(t_0) - x(t)) \alpha_{s'}^{-1}(\xi) \|
\]

In a similar way, we obtain $\| (\alpha_s(x(t)) - \alpha_{s'}(x(t))) \| < 3\varepsilon$. Hence $\{ L \ni s \mapsto \alpha_s(x(t)) \} \nu$ is $\omega$-equicontinuous.

**Lemma 3.24.** Let $(\alpha, c)$ be a Borel cocycle action of $\mathbb{R}$ on $\mathcal{M}$. Suppose that $U: \mathbb{R} \to \mathcal{M}_\omega$ is a Borel unitary path. Then for any $T > 0$, $\delta > 0$ with $0 < \delta < 1$ and finite set $\Phi \subset \mathcal{M}_\delta^+$, there exist a compact set $K \subset [-T, T] \times [-T, T]$ and a lift $(u(t))_\nu$ of $U$ as in Lemma 3.21 such that

- $\mu(K) \geq 4T^2(1 - \delta)$;
- $\{ K \ni (t, s) \mapsto u(t)^{\alpha_t}(u(s)^{\nu}) c(t, s)(u(t+s)^{\nu})^* \} \nu$ is $\omega$-equicontinuous;
- The following limit is the uniform convergence on $K$ for all $\varphi \in \Phi$:

\[
\lim_{\nu \to \omega} \| u(t)^{\nu} \alpha_t(u(s)^{\nu}) c(t, s)(u(t+s)^{\nu})^* - 1 \|_\varphi = \| U(t) \alpha_t(U(s)) c(t, s) U(t+s)^* - 1 \|_\varphi.
\]

**Proof.** Let $\eta := \delta/6$, $k \in \mathbb{N}$. Take a compact set $C \subset [-2T, 2T]$ for $U(t)$ as in Lemma 3.21, that is, $\mu(C) \geq 4T(1 - \eta)$, $\mu(C \cap [-T, T]) \geq 2T(1 - \eta)$ and $\{ C \ni t \mapsto u(t)^{\nu} \} \nu$ is $\omega$-equicontinuous.

By the previous lemma, we have a compact subset $L \subset [-T, T]$ such that $\mu(L) \geq 2T(1 - \eta)$, $L \ni t \mapsto \alpha_t \in \text{Aut}(\mathcal{M})$ is continuous and $\{ L \ni t \mapsto \alpha_t(u(s)^{\nu}) \} \nu$ is $\omega$-equicontinuous for all $s \in C$. Then $\mu(C \cap L) \geq 2T(1 - 2\eta)$, and the family $\{ L \times C \ni (t, s) \mapsto \alpha_t(u(s)^{\nu}) \} \nu$ is $\omega$-equicontinuous by Lemma 3.22.

Next we consider the Borel map $[-T, T]^2 \ni (t, s) \mapsto c(t, s) \in \mathcal{M}^U$. Take a compact subset $M \subset [-T, T]^2$ such that $\mu(M) \geq 4T^2(1 - \eta)$, and $c$ is continuous on $M$ as before.

Note that the map $C \times C \ni (t, s) \mapsto u(t+s)^{\nu}$ may not be $\omega$-equicontinuous. Let $f(t, s) = t + s$ on $[-T, T]^2$ and set the compact set $N := f^{-1}(C)$. Then $\{ N \ni (t, s) \mapsto u(t+s)^{\nu} \} \nu$ is $\omega$-equicontinuous, and we have

\[
\mu(N^c \cap [-T, T]^2) = \int_{-T}^T dt \int_{-T}^T ds \mathbf{1}_{f([-T, T]^2) \cap N^c} (t, s) \\
= \int_{-T}^T \mu((C^c \cap [-2T, 2T] - t) \cap [-T, T]) dt \\
\leq \int_{-T}^T \mu(C^c \cap [-2T, 2T]) dt \\
= 2T \mu(C^c \cap [-2T, 2T]) \leq 8T^2 \eta.
\]
Now we set the compact subset $K$ in $[-T, T]^2$ as follows:

$$K := ((C \cap L) \times C) \cap M \cap N.$$ 

Then

$$\mu(K^c \cap [-T, T]^2) = \mu ((((C \cap L) \times C) \cap [-T, T]^2)^c \cup M^c \cup N^c)$$

$$\leq 4T^2 - \mu ((C \cap L) \times (C \cap [-T, T])) + 4T^2\eta + 8T^2\eta$$

$$= 4T^2 - \mu(C \cap L)\mu(C \cap [-T, T]) + 12T^2\eta$$

$$\leq 4T^2 - 2T(1 - 2\eta) \cdot 2T(1 - \eta) + 12T^2\eta$$

$$= 4T^2\eta(6 - 2\eta) < 24T^2\eta = 4T^2\delta.$$ 

Then $\{K \ni (t, s) \mapsto u(t)^*\alpha_t(u(s)^*)c(t, s)(u(t + s)^*)v\} \omega$ is $\omega$-equicontinuous by Lemma 3.6, and we have the uniform convergence stated in Lemma 3.3. □

4. Rohlin flows

4.1. Rohlin flows. In [37], Kishimoto has introduced the notion of the Rohlin property for flows on $C^*$-algebras. This property has been defined also for finite von Neumann algebras by Kawamuro [35]. Following their works, we will introduce the Rohlin property for a Borel cocycle action.

**Definition 4.1.** Let $(\alpha, c)$ be a Borel cocycle action of $\mathbb{R}$ on a separable von Neumann algebra $\mathcal{M}$. We will say that $\alpha$ has the **Rohlin property** if for any $p \in \mathbb{R}$, there exists a unitary $v \in \mathcal{M}_{\omega, \alpha}$ such that $\alpha_t(v) = e^{ipt}v$ for all $t \in \mathbb{R}$.

A flow $\alpha$ with Rohlin property is simply called a Rohlin flow. We call the unitary $v$ in the above a Rohlin unitary for $p \in \mathbb{R}$. By definition, $\alpha_t$ is centrally non-trivial if $t \neq 0$. Therefore, any full factor does not admit a Rohlin flow. Several examples are investigated in Section 6.

Lemma 3.17 implies the following result.

**Lemma 4.2.** If $\alpha$ is a Rohlin flow on a factor, then $\Gamma(\alpha) = \mathbb{R}$.

Thus it is natural to ask if an outer flow with full Connes spectrum on the injective type $\Pi_1$ factor has the Rohlin property or not. This problem has been open so far. See Section 8 for related problems.

We remark that there does not exist a strongly continuous path $\mathbb{R} \ni p \mapsto w_p \in \mathcal{M}_{\omega, \alpha}$ such that $\alpha_t(w_p) = e^{ipt}w_p$ when $\mathcal{M}$ is a factor. Indeed, $\tau_\omega$ gives an $\alpha$-invariant inner product on $\mathcal{M}_{\omega, \alpha}$, and $\{w_p\}_p$ is an orthonormal system. In particular, this spans a non-separable Hilbert space.

Lemma 3.12 implies the stability of the Rohlin property under cocycle perturbation.

**Lemma 4.3.** If a Borel cocycle action of $\mathbb{R}$ on a von Neumann algebra has the Rohlin property, then so does its any perturbation.

The following result states a sequence-version of the definition of the Rohlin property.
Lemma 4.4. Let $\alpha$ be a flow on a von Neumann algebra $\mathcal{M}$. Then the following statements hold:

1. $\alpha$ has the Rohlin property;
2. For any $p \in \mathbb{R}$, there exists a unitary central sequence $(u^n)_o$ such that $\alpha_t(u^n) - e^{ipt}u^n \rightarrow 0$ compact uniformly in the strong topology as $\nu \rightarrow \infty$;
3. For any $p \in \mathbb{R}$, there exists a unitary central sequence $(v^n)_o$ such that for each $t \in \mathbb{R}$, one has $\alpha_t(v^n) - e^{ipt}v^n \rightarrow 0$ in the strong topology as $\nu \rightarrow \infty$.

Proof. (1)$\Rightarrow$(2). Let $p \in \mathbb{R}$. Take a unitary $v \in \mathcal{M}_{\omega,\alpha}$ with $\alpha_t(v) = e^{ipt}v$. Let $(v^n)_o$ be a unitary representing sequence of $v$.

Take a compact set $K \subset \mathbb{R}$ with $\mu(K) > 0$ such that $K = -K$, $\alpha|_K$ is continuous and $\{\alpha_t(v^n)\}_o$ is $\omega$-equicontinuous.

Let $\xi \in \mathcal{H}$ be a cyclic and separating vector for $\mathcal{M}$. Then it turns out that $\sup_{s,t \in K} \|\alpha_t(v^n) - e^{ipt}v^n\|\alpha_s(\xi)\| \rightarrow 0$ as $\nu \rightarrow \omega$ by Lemma 3.3 and the compactness of $\{\alpha_s(\xi) \mid s \in K\}$. By taking an appropriate subsequence, we may and do assume that $(v^n)_o$ is central, and $\sup_{s,t \in K} \|\alpha_t(v^n) - e^{ipt}v^n\|\alpha_s(\xi)\| \rightarrow 0$ as $\nu \rightarrow \infty$. Let $s, t \in K$. Then

$$\|\alpha_{s-t}(v^n) - e^{ip(s-t)}v^n\| = \|\alpha_s(\alpha_{s-t}(v^n)) - e^{-ipt}v^n\| = \|\alpha_t(v^n) - e^{ipt}v^n\|\alpha_s(\xi)\| + \|\alpha_t(v^n) - e^{ipt}v^n\|\alpha_s(\xi)\| = \|\alpha_{s-t}(v^n) - e^{-ipt}v^n\|\alpha_s(\xi)\| + \|\alpha_t(v^n) - e^{ipt}v^n\|\alpha_s(\xi)\|.$$ 

Hence

$$\lim_{\nu \rightarrow \infty} \sup_{s,t \in K} \|\alpha_{s-t}(v^n) - e^{ip(s-t)}v^n\| = 0.$$ 

Let $\delta > 0$ with $(-\delta, \delta) \subset K - K$. Then we have $\sup_{|t| \leq \delta} \|\alpha_t(v^n) - e^{ipt}v^n\| \rightarrow 0$ as $\nu \rightarrow \infty$. From this fact, we can deduce that the uniform convergence on any compact sets.

(2)$\Rightarrow$(1). This implication is trivial.

(3)$\Rightarrow$(1). Let $p \in \mathbb{R}$, and take such a sequence $(v^n)_o$. Let $\xi \in \mathcal{H}$. Then for each $t \in \mathbb{R}$, we obtain

$$\|\xi(\alpha_t(v^n) - e^{ipt}v^n)\| \leq \|\xi(\alpha_t(v^n))\| + \|\alpha_t(v^n) - e^{ipt}v^n\| + \|v^n, \xi\| = \|\alpha_{s-t}(\xi)\| + \|\alpha_t(v^n) - e^{ipt}v^n\| + \|v^n, \xi\|,$$

which converges to 0 since $(v^n)_o$ is central. Thus we have the strong* convergence $\alpha_t(v^n) - e^{ipt}v^n \rightarrow 0$ as $\nu \rightarrow \infty$. Let $f(t) = e^{-ipt}1_{[0,1]}(t) \in L^1(\mathbb{R})$. Then we have

$$\|\alpha_f(v^n) - v^n\| \leq \int_0^1 \|\alpha_t(v^n) - e^{ipt}v^n\| dt,$$

which converges to 0 as $\nu \rightarrow \infty$ by the dominated convergence theorem. Likewise, we obtain $\|\xi(\alpha_f(v^n) - v^n)\| \rightarrow 0$ as $\nu \rightarrow \infty$. Thus $(v^n)_o$ belongs to $\mathcal{E}_b \cap \mathcal{E}_\omega$ by Lemma 3.17 and 3.14. Hence $v := \pi_{\omega}(v^n)_o \in \mathcal{M}_{\omega,\alpha}$ satisfies $\alpha_t(v) = e^{ipt}v$. \hfill $\square$

4.2. Invariant approximate innerness. We investigate a relation between the Rohlin property and the invariant approximate innerness.
Definition 4.5. Let $\alpha$ be a flow on a von Neumann algebra $M$. We will say that $\alpha$ is \textit{invariantly approximately inner} if for any $T \in \mathbb{R}$, there exists a sequence of unitaries $(w^{\nu})_\nu$ in $M$ such that

- $\alpha_T = \lim_{\nu \to \infty} \text{Ad} w^{\nu}$ in $\text{Aut}(M)$;
- $\|((\alpha_t(w^{\nu}) - w^{\nu})\xi) + \|\xi(\alpha_t(w^{\nu}) - w^{\nu})\| \to 0$ compact uniformly for $t \in \mathbb{R}$ as $\nu \to \infty$ for all $\xi \in H$.

Lemma 4.6. Let $\alpha$ be a flow on a von Neumann algebra $M$. Then the following statements are equivalent:

1. $\alpha$ is invariantly approximately inner;
2. For any $T \in \mathbb{R}$, there exists a sequence of unitaries $(w^{\nu})_\nu$ in $M$ such that
   - $\alpha_T = \lim_{\nu \to \infty} \text{Ad} w^{\nu}$ in $\text{Aut}(M)$;
   - $\|((\alpha_t(w^{\nu}) - w^{\nu})\xi) + \|\xi(\alpha_t(w^{\nu}) - w^{\nu})\| \to 0$ for each $t \in \mathbb{R}$ and $\xi \in H$ as $\nu \to \infty$.
3. For any $T \in \mathbb{R}$, there exists a unitary $w \in M^\omega_\alpha$ such that
   - $\alpha_T(x) = wxw^*$ for all $x \in M$;
   - $\alpha_t(w) = w$ for all $t \in \mathbb{R}$;

Proof. (1)$\Rightarrow$(2). This implication is trivial.
(2)$\Rightarrow$(3). Take such a sequence $(w^{\nu})_\nu$. Then as in the proof of Lemma 4.4 we can show that $(w^{\nu})_\nu \in \mathcal{E}^\omega_\alpha$. Since $\text{Ad} w^{\nu} \to \alpha_T$ in $\text{Aut}(M)$, $(w^{\nu})_\nu$ normalizes $\mathcal{I}$. Thus we can consider a unitary $w := \pi_\omega((w^{\nu})_\nu)$ in $M^\omega_\alpha$ which satisfies the required properties.
(3)$\Rightarrow$(1). We suppose that the conditions of (2) are fulfilled. Let $(w^{\nu})_\nu$ be a unitary representing sequence of $w$. Let $T > 0$, $\varepsilon > 0$ and $\Phi \subset H$ a finite set. By $(\alpha, \omega)$-equicontinuity, there exist $N \in \mathbb{N}$ and $W_1 \in \omega$ such that if $s,t \in [-T,T]$ satisfies $|s-t| < T/N$ and $\nu \in W_1$, then

$$\|((\alpha_s(w^{\nu}) - \alpha_t(w^{\nu}))\xi)\| < \varepsilon, \quad \|\xi(\alpha_s(w^{\nu}) - \alpha_t(w^{\nu}))\| < \varepsilon \quad \text{for all } \xi \in \Phi.$$ 

Put $t_j := jT/N$, $j = -N, \ldots, N$. Since $\alpha_t(w) = w$, there exists $W_2 \in \omega$ such that if $\nu \in W_2$, then

$$\|((\alpha_{t_j}(w^{\nu}) - w^{\nu})\xi)\| < \varepsilon, \quad \|\xi(\alpha_{t_j}(w^{\nu}) - w^{\nu})\| < \varepsilon \quad \text{for all } j = -N, \ldots, N, \xi \in \Phi.$$ 

Let $t \in [-T,T]$ and take $t_j$ with $|t-t_j| < T/N$. If $\nu \in W_1 \cap W_2$, then

$$\|((\alpha_t(w^{\nu}) - w^{\nu})\xi)\| \leq \|((\alpha_{t_j}(w^{\nu}) - \alpha_t(w^{\nu}))\xi\| + \|((\alpha_{t_j}(w^{\nu}) - w^{\nu}))\xi\| < 2\varepsilon.$$ 

Likewise, we obtain $\|\xi(\alpha_t(w^{\nu}) - w^{\nu})\| < 2\varepsilon$ for $\xi \in \Phi$, $t \in [-T,T]$ and $\nu \in W_1 \cap W_2$. Then an appropriate subsequence of $w^{\nu}$ satisfies the condition of Definition 4.5. \hfill \Box

Lemma 4.7. The invariant approximate innerness is stable under cocycle perturbation.

Proof. Let $\alpha$ be an invariantly approximately inner flow on a von Neumann algebra $M$. Let $\nu$ be an $\alpha$-cocycle. For $T \in \mathbb{R}$, take a unitary $w \in M^\omega_\alpha$ such that
Lemma 4.8. Let $\alpha$ be a flow on a von Neumann algebra $M$ and $\nu \in \mathbb{R}$. Suppose that there exists a unitary central sequence $(\nu^t)_\nu$ in $M$ such that for each $t \in \mathbb{R}$,
$$\lim_{\nu \to \infty} (\alpha_t(\nu^t) - e^{i\nu t} \nu^t) = 0$$
in the strong topology. Then $\hat{\alpha}_p = \lim_{\nu \to \infty} \text{Ad} \pi_\alpha(\nu^t)$ in $\text{Aut}(M \rtimes_\alpha \mathbb{R})$.

Proof. Set $N := M \rtimes_\alpha \mathbb{R}$ and $e_{-\nu} \in C_b(\mathbb{R})$ defined by $e_{-\nu}(t) := e^{-i\nu t}$. Then $\pi_\alpha(\nu^t) - \nu^t \otimes e_{-\nu} \to 0$ in the strong topology in $M \otimes B(L^2(\mathbb{R}))$. Indeed, let $\xi \in H$ and $f \in L^2(\mathbb{R})$. Then
$$||\pi_\alpha(\nu^t) - \nu^t \otimes e_{-\nu}(\xi \otimes f)||^2 = \int_{\mathbb{R}} |f(t)|^2 ||(\alpha_{-t}^{(\nu^t)}(\nu^t) - e^{-i\nu t} \nu^t)\xi||^2 dt,$$
which converges to 0 by the dominated convergence theorem. Thus for $\phi \in M_\pi$ and $\psi \in B(L^2(\mathbb{R}))$, we have
$$||\pi_\alpha(\nu^t)(\phi \otimes \psi)\pi_\alpha((\nu^t)^*) - \nu^t \phi(\nu^t)^* \otimes e_{-\nu} \psi \pi_{\nu^t}|| \to 0.$$ Since $(\nu^t)_\nu$ is central, we have
$$||\pi_\alpha(\nu^t)(\phi \otimes \psi)\pi_\alpha((\nu^t)^*) - \phi \otimes e_{-\nu} \psi \pi_{\nu^t}|| \to 0.$$ This means $\text{Ad} \pi_\alpha(\nu^t) \to \text{Ad}(1 \otimes e_{-\nu})$ in $\text{Aut}(M \otimes B(L^2(\mathbb{R})))$. Since $\hat{\alpha}_p = \text{Ad}(1 \otimes e_{-\nu})$ on $N$, we have $\text{Ad} \pi_\alpha(\nu^t) \to \hat{\alpha}_p$ in $\text{Aut}(N)$. \qed

Remark 4.9. In the proof above, we have used the following fact. Let $N \subset M$ be an inclusion of von Neumann algebras. Denote by $\text{Aut}(M, N)$ the set of automorphisms $\alpha$ on $M$ such that $\alpha(N) = N$. It is fairly easy to see that $\text{Aut}(M, N)$ is a closed subgroup of $\text{Aut}(M)$ with respect to the $u$-topology. Then the map $\text{Aut}(M, N) \ni \alpha \mapsto \alpha|_N \in \text{Aut}(N)$ is continuous.

Indeed, let $\alpha, \beta \in \text{Aut}(M, N)$. Take $\varphi \in N_\pi$ and its normal extension $\psi \in M_\pi$. Then trivially, $||\alpha(\varphi) - \beta(\varphi)||_N \leq ||\alpha(\psi) - \beta(\psi)||_M$. This shows the continuity.

We recall the modular conjugation of $M \rtimes_\alpha \mathbb{R}$ introduced in [18, Lemma 2.8]:
$$J(\xi)(s) = J_{\alpha_s}(\xi(-s)) \quad \text{for} \ \xi \in H \otimes L^2(\mathbb{R}).$$

Lemma 4.10. Let $\alpha$ be a flow on a von Neumann algebra $M$ and $\nu \in \mathbb{R}$. Suppose that there exists a sequence of unitaries $(\nu^t)_\nu$ in $M$ such that $\alpha_p = \lim_{\nu \to \infty} \text{Ad} \nu^t$ in $\text{Aut}(N)$ and $\alpha_t(\nu^t) - \nu^t \to 0$ as $\nu \to \infty$ in the strong* topology for each $t \in \mathbb{R}$. Then the sequence $(\nu^t)_\nu$ defined by $\nu^t := \lambda^\alpha(p)^* \pi_\alpha(\nu^t)$ is central in $M \rtimes_\alpha \mathbb{R}$, and belongs to $\mathcal{E}_\alpha^\nu$. In particular, one has $\hat{\alpha}(v) = e^{i\nu t} v$ for $t \in \mathbb{R}$ putting $v := \pi_\alpha((\nu^t)_\nu).$

Proof. We will check that $(\nu^t)_\nu$ is central. As in the proof of the previous lemma, we can show that $\pi_\alpha(\nu^t) - \nu^t \otimes 1 \to 0$ as $\nu \to \infty$ in the strong* topology. Then for all $\eta \in H$ and $f \in L^2(\mathbb{R})$, we have
$$\lim_{\nu \to \infty} ||\nu^t(\eta \otimes f) - (\eta \otimes f)\nu^t|| = \lim_{\nu \to \infty} ||\lambda^\alpha(p)^*(\nu^t \eta \otimes f) - (\eta \otimes f)\lambda^\alpha(p)^* \pi_\alpha(\nu^t)||$$
The right hand side equals 0. Indeed,
\[
\|\lambda^\alpha(p)\ast(w\eta \otimes f) - (\eta \otimes f)\lambda^\alpha(p)\ast\pi_\alpha(w\eta)\|
= \|w\eta \otimes \lambda(p)\ast f - \tilde{J}\pi_\alpha(w\eta)\ast\lambda^\alpha(p)\tilde{J}(\eta \otimes f)\|
= \|w\eta \otimes \lambda(p)\ast f - \alpha_\rho(\eta)w\eta \otimes \rho(p)f\|
= \|w\eta - \alpha_\rho(\eta)w\eta\|\|f\|,
\]
where we have used \(\tilde{J}\lambda^\alpha(p)\tilde{J} = U_\alpha(p) \otimes \rho(p)\) and \(\tilde{J}\pi_\alpha(x)\tilde{J} = Jx^*J \otimes 1\) for all \(p \in \mathbb{R}\) and \(x \in \mathcal{M}\). Hence \((v\nu)_\nu\) is central. Since \(\hat{\alpha}_t(v\nu) = e^{it}v\nu\) for all \(\nu \in \mathbb{N}\), \((v\nu)_\nu\) belongs to \(\mathcal{E}_0\). Thus \(v = \pi_\omega((v\nu)_\nu) \in (\mathcal{M} \rtimes_\alpha \mathbb{R})_{\omega,\hat{\alpha}}\) and \(\hat{\alpha}_t(v) = e^{it}v\). □

The following result is the von Neumann algebra version of [39, Theorem 1.3]. This states that the Rohlin property and the invariantly approximate innerness are mutually dual notions. See [25, Lemma 3.8] for the corresponding result in the case of finite group actions on C*-algebras.

**Theorem 4.11.** Let \(\alpha\) be a flow on a von Neumann algebra \(\mathcal{M}\). Then the following statements hold:

1. \(\alpha\) has the Rohlin property if and only if \(\hat{\alpha}\) is invariantly approximately inner;
2. \(\alpha\) is invariantly approximately inner if and only if \(\hat{\alpha}\) has the Rohlin property.

**Proof.** (1). Set \(\mathcal{N} := \mathcal{M} \rtimes_\alpha \mathbb{R}\). Suppose that \(\alpha\) is a Rohlin flow. Then Lemma 4.8 shows that \(\hat{\alpha}\) is invariantly approximately inner because \(\hat{\alpha}\) fixes \(\pi_\alpha(\mathcal{M})\).

Suppose that \(\hat{\alpha}\) is invariantly approximately inner. By the previous lemma, the dual flow of \(\hat{\alpha}\) has the Rohlin property, and so does the flow \(\alpha \otimes \text{Ad} \rho\) on \(\mathcal{M} \otimes B(L^2(\mathbb{R}))\) by Takesaki duality [58]. Lemma 2.8 implies that \(\alpha\) has the Rohlin property.

(2). If a flow \(\alpha\) on \(\mathcal{M}\) is invariantly approximately inner, then the dual flow \(\hat{\alpha}\) has the Rohlin property by the previous lemma. Conversely, suppose that \(\hat{\alpha}\) is a Rohlin flow. Then by Takesaki duality and Lemma 4.7, \(\beta := \alpha \otimes \text{id}_{B(\ell^2)}\) on \(\mathcal{N} := \mathcal{M} \otimes B(\ell^2)\) is invariantly approximately inner. Let \(T \in \mathbb{R}\). Then by Lemma 4.6, there exists a unitary \(w \in \mathcal{N}_0^\omega\) such that \(wx = \beta_T(x)w\) and \(\beta_t(w) = w\) for all \(x \in \mathcal{N}\) and \(t \in \mathbb{R}\). By the description of \(\mathcal{N}_0^\omega\) in Lemma 2.8, we get the natural isomorphism \(\mathcal{N}_0^\omega \cong \mathcal{M}_0^\omega \otimes B(\ell^2)\). In fact, it turns out that the isomorphism maps \(\mathcal{N}_0^\omega\) onto \(\mathcal{M}_0^\omega \otimes B(\ell^2)\). Hence \(w\) is regarded as an element in \(\mathcal{M}_0^\omega \otimes B(\ell^2)\), and we have \(wx = (\alpha_T \otimes \text{id})(x)w\) for any \(x \in \mathcal{M} \otimes B(\ell^2)\). Then \(w\) commutes with \(1 \otimes y\) for any \(y \in B(\ell^2)\), and \(w \in \mathcal{M}_0^\omega \otimes \mathbb{C}\). This shows the invariantly approximate innerness of \(\alpha\). □

**Remark 4.12.** Let \(\alpha\) be a flow on a von Neumann algebra \(\mathcal{M}\). Then the following statements hold:

1. If \(\alpha\) has the Rohlin property, then so does \(\tilde{\alpha}\);
2. If \(\alpha\) is invariantly approximately inner, then so is \(\tilde{\alpha}\).
The first one follows from the inclusion $\mathcal{M}_\omega \subset \tilde{\mathcal{M}}_\omega$ (see the proof of \cite[Lemma 4.11]{48}). The second is directly proved.

We obtain the following useful corollaries of the previous theorem.

**Corollary 4.13.** If $\alpha$ is a Rohlin flow on a von Neumann algebra $\mathcal{M}$, then
\[
\pi_\alpha(M)' \cap (M \rtimes_\alpha \mathbb{R}) = \pi(Z(M)),
\]
\[
\pi_\alpha(M)' \cap (\tilde{\mathcal{M}} \rtimes_{\tilde{\alpha}} \mathbb{R}) = \pi_{\tilde{\alpha}}(Z(\tilde{\mathcal{M}})).
\]

In particular, $Z(\tilde{\mathcal{M}} \rtimes_{\tilde{\alpha}} \mathbb{R}) = Z(\tilde{\mathcal{M}})\tilde{\alpha}$.

**Proof.** By the previous theorem, $\tilde{\alpha}$ is invariantly approximately inner. In fact, by Lemma 4.8 each $\tilde{\alpha}_T$ is approximated by $\text{Ad}\, \pi_\alpha(w^\nu)$ with $w^\nu \in M_\nu$. Thus $\tilde{\alpha}$ fixes $\pi_\alpha(M)' \cap (M \rtimes_\alpha \mathbb{R})$, and we get the first equality. The second equality is proved similarly. \hfill \Box

Hence if $\mathcal{M}$ is a type $\mathbb{III}_1$ factor, then so is $M \rtimes_\alpha \mathbb{R}$.

**Corollary 4.14.** If $\alpha$ is an invariantly approximately inner on a von Neumann algebra $\mathcal{M}$, then
\[
\pi_\alpha(M)' \cap (M \rtimes_\alpha \mathbb{R}) = \pi(Z(M \rtimes_\alpha \mathbb{R})),
\]
\[
\pi_{\tilde{\alpha}}(\tilde{\mathcal{M}})' \cap (\tilde{\mathcal{M}} \rtimes_{\tilde{\alpha}} \mathbb{R}) = \pi_{\tilde{\alpha}}(Z(\tilde{\mathcal{M}} \rtimes_{\tilde{\alpha}} \mathbb{R})).
\]

In particular, $Z(\tilde{\mathcal{M}} \rtimes_{\tilde{\alpha}} \mathbb{R})\tilde{\alpha} = Z(\tilde{\mathcal{M}})$.

**Proof.** By Theorem 4.11, $\tilde{\alpha}$ has the Rohlin property. Then we get the result by employing the previous result and the following mirroring (see \cite[Lemma 5.7]{24} for its proof):
\[
\tilde{J}(\pi_\alpha(M)' \cap (M \rtimes_\alpha \mathbb{R}))\tilde{J} = (M \rtimes_\alpha \mathbb{R})' \cap (M \otimes B(L^2(\mathbb{R}))).
\]

\hfill \Box

Therefore, if $\alpha$ is a Rohlin flow on $\mathcal{M}$ and invariantly approximately inner, then the inclusion $\pi_\alpha(M) \subset M \rtimes_\alpha \mathbb{R}$ has the common flow of weights. In particular, $M \rtimes_\alpha \mathbb{R}$ is of the same type as $\mathcal{M}$ when $\mathcal{M}$ is a factor. This assumption corresponds to the central freeness and the approximate innerness for discrete group actions on a factor.

**Corollary 4.15.** Let $\alpha$ be a Rohlin flow on a von Neumann algebra $\mathcal{M}$. Suppose that $\alpha$ is centrally ergodic. Then
\[
\text{Sp}_d(\alpha|_{Z(\mathcal{M})}) = \{p \in \mathbb{R} \mid \tilde{\alpha}_p \in \text{Int}(M \rtimes_\alpha \mathbb{R})\}.
\]

**Proof.** If $\tilde{\alpha}_p = \text{Ad}\, u$ for some unitary $u \in M \rtimes_\alpha \mathbb{R}$, then $u \in \pi_\alpha(M)' \cap (M \rtimes_\alpha \mathbb{R}) = \pi_\alpha(Z(\mathcal{M}))$. Then putting $u = \pi_\alpha(v), v \in Z(\mathcal{M})$, we have
\[
\pi_\alpha(\alpha_t(v)) = \lambda_t^\alpha u(\lambda_t^\alpha)^* = \lambda_t^\alpha \tilde{\alpha}_p(\lambda_t^\alpha)^* u = e^{ipt}\pi_\alpha(v).
\]

Hence $p \in \text{Sp}_d(\alpha|_{Z(\mathcal{M})})$.

Suppose conversely that $p \in \text{Sp}_d(\alpha|_{Z(\mathcal{M})})$. By polar decomposition, there exists a non-zero partial isometry $v \in Z(\mathcal{M})$ such that $\alpha_t(v) = e^{ipt} v$ for $t \in \mathbb{R}$. The central ergodicity implies that $v$ is in fact a unitary. Then $\tilde{\alpha}_p = \text{Ad}\, \pi_\alpha(v)$. \hfill \Box
A classification of invariably approximately inner flows will be treated in §6.1. A typical example of an invariably approximately inner flow not of infinite tensor product type comes from a modular flow, or more generally, an extended modular flow as introduced below.

**Definition 4.16.** We will say that a flow $\beta$ on a von Neumann algebra $N$ is **extended modular** when $\beta_t$ is an extended modular automorphism for each $t \in \mathbb{R}$, that is, $\tilde{\beta}_t \in \text{Int}(\tilde{N})$.

The definition above is slightly different from that of [10, Proposition IV.2.1]. However, it is essential to consider the canonical extension in what follows, and we adopt the definition above (see also [20, Proposition 5.4] and [24, Definition 3.1]).

**Lemma 4.17.** Let $\beta$ be an extended modular flow on a von Neumann algebra $N$ and $T \in \mathbb{R}$. Suppose that there exists a unitary $(v^\nu)_\nu$ in $\ell^\infty(N)$ such that $\beta_T = \lim_{\nu \to \infty} \text{Ad } v^\nu$. Then $\beta_t(v^\nu) - v^\nu$ converges to 0 compact uniformly in the strong topology as $\nu \to \infty$.

**Proof.** The canonical extension $\tilde{\beta}$ is inner. Thanks to the result due to Kallman and Moore as mentioned in [22.2] we can take a one-parameter unitary group $w_t \in \tilde{N}$ such that $\tilde{\beta}_t = \text{Ad } w_t$.

Let $\mathcal{K}$ be the standard Hilbert space of $\tilde{N}$. We regard $\mathcal{K}$ as an $\tilde{N}$-$\tilde{N}$ bimodule as usual. Let $\xi \in \mathcal{K}$, $S > 0$ and $\Psi := \{w^*_t \xi \mid t \in [-S, S]\}$. Then we have $\sup_{\eta \in \Psi} \|v^\nu\eta - \tilde{\beta}_T(\eta)v^\nu\| \to 0$ since $\tilde{\beta}_T = \lim_{\nu \to \infty} \text{Ad } v^\nu$ in $\text{Aut}(\tilde{N})$, and $\Psi$ is compact. Thus,

$$
\|(\beta_t(v^\nu) - v^\nu)\xi\| = \|(w_tw^\nu w^*_t - v^\nu)\xi\|
\leq \|w_tw^\nu w^*_t - w_t\tilde{\beta}_T(w^*_t)\nu\| + \|w_t\tilde{\beta}_T(w^*_t)\nu - v^\nu\xi\|
\leq \sup_{\eta \in \Psi} \|v^\nu\eta - \tilde{\beta}_T(\eta)v^\nu\| + \|\tilde{\beta}_T(\xi) - v^\nu\xi\|,
$$

where we have used $\tilde{\beta}_T(w_t) = w_T w_t w^*_T = w_{T+t-T} = w_t$. The last terms are converging to 0 as $\nu \to \infty$. Hence we have $\|\beta_t(v^\nu) - v^\nu\xi\| \to 0$ uniformly on $[-S, S]$ as $\nu \to \infty$. Similarly, $\|\xi(\beta_t(v^\nu) - v^\nu)\| \to 0$ uniformly on $[-S, S]$ as $\nu \to \infty$.

**Remark 4.18.** For a modular automorphism group, the previous lemma is shown without the use of the canonical extension. Indeed, let us assume that a faithful state $\varphi \in N_+$ and $T \in \mathbb{R}$ satisfy $\sigma_T^\varphi = \lim_{\nu \to \infty} \text{Ad } v^\nu$ in $N$ as above. Using $\sigma_T^\varphi(\varphi) = \varphi$, we have $\|v^\nu, \varphi\| = \|\text{Ad } v^\nu(\varphi) - \varphi\| \to 0$ as $\nu \to \infty$. Thus by [3, Lemma 2.7], $\|\sigma_T^\varphi(v^\nu) - v^\nu\|_{\varphi} \to 0$ compact uniformly as $\nu \to \infty$.

By Lemma 4.10, Theorem 4.11 and Lemma 4.17 we have the following result.
Proposition 4.19. Let $\beta$ be an extended modular flow on a von Neumann algebra $\mathcal{N}$. If $\beta$ is pointwise approximately inner, the dual flow $\alpha := \hat{\beta}$ has the Rohlin property.

Let $\alpha, \beta$ be as above. We show that the Connes-Takesaki module flow of $\alpha$ is faithful. Denote by $\mathcal{M}$ the crossed product $\mathcal{N} \rtimes_{\beta} \mathbb{R}$. Then $\tilde{\mathcal{M}} = \tilde{\mathcal{N}} \rtimes_{\beta} \mathbb{R}$ and $\tilde{\alpha} = \tilde{\beta}$ by Lemma 2.3. Since $\tilde{\beta}$ is implemented by a one-parameter unitary group as mentioned before, we have an isomorphism $\{\tilde{\mathcal{M}}, \theta, \tilde{\alpha}\} \cong \{\tilde{\mathcal{N}} \otimes L(\mathbb{R}), \theta, \text{id} \circ \text{Ad} e_{-t}\}$, where $e_{-t} \in C_b(\mathbb{R})$ is $e_{-t}(s) := e^{-ist}$. By simple calculation, we have $c_s(t) \in Z(\mathcal{N})$ satisfying
\[
\theta_s(a \otimes \lambda(t)) = (\theta_s(a) \otimes 1)(c_s(t) \otimes \lambda(t)).
\]
Since $c_s(t + t') = c_s(t)c_s(t')$, we have a positive operator $K_s$ affiliated with $Z(\mathcal{N})$ such that $c_s(t) = K_s^t$.

Then by the isomorphism $\pi : L(\mathbb{R}) \to L^\infty(\mathbb{R}_+^\ast)$ with $\pi(\lambda(t))(h) = h^t$, we have $\{\tilde{\mathcal{M}}, \theta, \tilde{\alpha}\} \cong \{\tilde{\mathcal{N}} \otimes L^\infty(\mathbb{R}_+^\ast), \theta, \text{id} \circ \text{Ad} \lambda(t)\}$, where $(\lambda(t)\xi)(s) = \xi(e^{-ts})$ for $t, s > 0$. In particular, $\text{mod}(\alpha_t)$ is the translation on $\mathbb{R}_+^\ast := \{h \in \mathbb{R} \mid h > 0\}$.

If we regard $K_s$ as the function $K_s : X_N \to \mathbb{R}_+^\ast$, we have the following for all $x \in X_N$ and $h > 0$:
\[
\theta_s(1 \otimes f)(x, h) = f(K_s(x)h)
\]
Hence the flow space $X_M$ is naturally isomorphic to $X_N \times \mathbb{R}_+^\ast$. Let $F^M$ and $F^N$ be the flow of weights of $\mathcal{M}$ and $\mathcal{N}$, respectively. Then we have $F^M_s(x, h) = (F^N_s x, K_s(x)h)$. Summarizing the discussion above, we have the following result.

Theorem 4.20. Let $\beta$ be an extended modular flow on $\mathcal{N}$ and $\alpha := \hat{\beta}$ be the dual flow on $\mathcal{M} := \mathcal{N} \rtimes_{\beta} \mathbb{R}$. Then there exists an $\mathbb{R}_+^\ast$-valued $F^N$-cocycle $K : X_N \times \mathbb{R} \to \mathbb{R}_+^\ast$ such that

1. $X_M = X_N \times \mathbb{R}_+^\ast$, $F^M_s(x, h) = (F^N_s x, K_s(x, s)h)$ for all $s \in \mathbb{R}$, $x \in X_M$ and $h > 0$;
2. $\text{mod}(\alpha_t)(x, h) = (x, e^{-th})$ for all $t \in \mathbb{R}$, $x \in X_M$ and $h > 0$.

5. Classification of Rohlin flows

In this section, we will prove our main theorem (Theorem 5.14) of this paper.

5.1. Rohlin projection and averaging technique. The classification of general Rohlin flows will be reduced to that of centrally ergodic Rohlin flows (see the proof Theorem 5.14). Hence let us assume that $(\alpha, c)$ is a Borel cocycle action of $\mathbb{R}$ on a von Neumann algebra $\mathcal{M}$ with the following properties:

- Rohlin property;
- $Z(\mathcal{M})^\alpha = \mathbb{C}$ and $\text{Sp}_d(\alpha|_{Z(\mathcal{M})}) \neq \mathbb{R}$.

The case that $\text{Sp}_d(\alpha|_{Z(\mathcal{M})}) = \mathbb{R}$ will be treated separately in the proof of Lemma 5.12. Let us put $H_\alpha := \text{Sp}_d(\alpha|_{Z(\mathcal{M})})$ that is a Borel subgroup of $\mathbb{R}$. The following result is probably well-known to experts, but we present a proof for readers’ convenience.

Lemma 5.1. For any $\varepsilon > 0$, there exists $p > 0$ such that $p < \varepsilon$ and $Zp \cap H_\alpha = \{0\}$. 

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Proof. Let \( E := [0, 1] \subset \mathbb{R} \), \( E_0 := \{ t \in E \mid \text{Z} t \cap H_\alpha = \{0\} \} \) and
\[
E_k := \{ t \in E \mid \text{Z} t \notin H_\alpha, \ell = 1, \ldots, k - 1, k t \in H_\alpha \}, \quad k \in \mathbb{N}.
\]
Since \( E_0 = \bigcap_{t=1}^{\infty} (1/\ell) H_\alpha \bigcup \{0\} \), and \( E_k = (1/k) H_\alpha \cap \bigcap_{t=1}^{k-1} (1/\ell) H_\alpha \), \( E_k \)'s are Borel sets. Hence \( 1 = \mu(E) = \sum_{k=0}^{\infty} \mu(E_k) \). Suppose that \( \mu(E_k) > 0 \) for some \( k > 0 \). Then there exists \( \delta > 0 \) such that \( (-\delta, \delta) \subset E_k - E_k \). Since \( k E_k \subset H_\alpha \), we have \( (-k\delta, k\delta) \subset k E_k - k E_k \subset H_\alpha \). This forces \( H_\alpha \) to be \( \mathbb{R} \), which is a contradiction. Thus \( \mu(E_0) = 1 \), and we are done.

The above lemma states that an arbitrarily large \( S > 0 \) can be chosen in such a way that \((2\pi/S)\mathbb{Z} \cap H_\alpha = 0 \). Let \( v \in M_{\omega, \alpha} \) be a Rohlin unitary for \(-2\pi/S\), that is, \( \alpha_t(v) = e^{-2\pi it/S} v \). Then \( \alpha \) is a flow on \( W^*(v) \) with period \( S \). By the equality \( \tau^\omega \circ \alpha_t = \alpha_t \circ \tau^\omega \), we have \( \tau^\omega(v^n) \in Z(\mathcal{M}) \) satisfies \( \alpha_t(\tau^\omega(v^n)) = e^{-2\pi it/S} \tau^\omega(v^n) \) for \( n \in \mathbb{Z} \), which yields, however, \( \tau^\omega(v^n) = 0 \) if \( n \neq 0 \) because \(-2\pi it/S \notin H_\alpha \). Hence \( \tau^\omega \) is a faithful normal state on \( W^*(v) \).

Let \( v = \int_0^{2\pi} e^{i\lambda} dE(\lambda) \) be the spectral decomposition on \( T = [0, 2\pi) \). By easy calculation, we have \( \alpha_t(dE(\lambda)) = dE(\lambda + 2\pi t/S) \). We set \( e(\lambda) := E(2\pi \lambda/S) \) for \( \lambda \in [0, S) \). Then \( \alpha_t(dE(\lambda)) = dE(\lambda + t) \) and \( v = \int_0^S e^{2\pi i\lambda/S} de(\lambda) \). Thus \( dt\tau^\omega(e(\cdot)) \) coincides with the Haar measure on the torus \([0, S) = \mathbb{R}/\mathbb{Z} \), that is, the normalized Lebesgue measure. Therefore, for \( f \in L^\infty([0, S)) \), we can define \( f(v) = \int_0^S f(t) de(\lambda) \). Then \( L^\infty([0, S)) \ni f \mapsto f(v) \in W^*(v) \) is an isomorphism.

Following \[35, 67\], we introduce a normal \(*\)-homomorphism \( \Theta : M \otimes L^\infty([0, S)) \rightarrow M^\omega_\alpha \) such that \( \Theta(a \otimes f) = a f(v) \).

**Lemma 5.2.** Let \( S > 0 \) with \((2\pi/S)\mathbb{Z} \cap H_\alpha = \{0\} \). Then there exists an isomorphism \( \Theta : M \otimes L^\infty([0, S)) \rightarrow M \vee W^*(v) \) such that
- \( \Theta(a \otimes f) = a f(v) \) for all \( a \in M \) and \( f \in L^\infty([0, S)) \);
- \( \alpha_t \circ \Theta = \alpha_t \otimes \gamma_t \), where \([0, S) \) is regarded as a circle \( \mathbb{R}/\mathbb{Z} \), and \( \gamma_t \) denotes the rotation by \( t \) on \([0, S) \);
- \( \tau^\omega \circ \Theta = \text{id}_M \otimes \mu \), where \( \mu \) denotes the integration by the normalized Lebesgue measure.

**Proof.** Let \( z(\lambda) := e^{2\pi i\lambda/S} \) for \( \lambda \in [0, S) \). Then we have
\[
\varphi^\omega(a v^n) = \varphi(a \tau^\omega(v^n)) = \delta_{n,0} \varphi(a) = (\varphi \otimes \mu)(a \otimes z^n).
\]
Since \( \{a \otimes z^n \mid n \in \mathbb{Z}\} \) and \( \{a v^n \mid n \in \mathbb{Z}\} \) span strongly dense \(*\)-algebras in \( M \otimes L^\infty([0, S)) \) and \( M \vee W^*(v) \), respectively, we have such \( \Theta \).

The map \( \Theta \) plays a role of Shapiro’s lemma, that is, \( \Theta(a), a \in M \otimes L^\infty([0, S)) \), can be regarded as the average of \( a(s) \) along with the Rohlin tower \( e(s) \). We may write \( \Theta(a) \) in a formal manner as
\[
\Theta(a) = \int_0^S a(s) de(s).
\]
From the previous lemma, for any \( \varphi \in M^+_\alpha \), we obtain the following equality:
\[
\|\Theta(w)\|_{2, \varphi}^2 = \frac{1}{S} \int_0^S \|w(s)\|_{\varphi}^2 ds.
\]
Lemma 5.3. Let \((\alpha, c)\) be a Borel cocycle action of \(\mathbb{R}\) as before. Let \(S > 0\) with \((2\pi/S)\mathbb{Z} \cap H_\alpha = \{0\}\). Let \(u: [-T, T] \times [0, S] \to \mathcal{M}^U\) be a Borel map, \(\Phi \subset \mathcal{M}_+\) a finite set and \(e(\lambda) \in \mathcal{M}_{\alpha, \alpha}\) a Rohlin projection over [0, S). Set \(w(t) := \Theta(u(t, \cdot))\), which is a Borel unitary path in \(\mathcal{M}^\alpha\). Then for any \(\varepsilon > 0\) with

\[
\frac{1}{2ST} \int_{-T}^{T} dt \int_{0}^{S} ds \|u(t, s, \varphi)\| < \varepsilon \quad \text{for all } \varphi \in \Phi,
\]

there exist \(W \in \omega\) and a lift \((w(t)^\nu)_\nu\) of \(w(t)\) as in Lemma 5.21 with respect to \(E := [-T, T]\) such that

\[
\frac{1}{2T} \int_{-T}^{T} \|w(t)^\nu, \varphi\| dt < 3\varepsilon \quad \text{for all } \varphi \in \Phi, \; \nu \in W.
\]

Proof. Since \(\|u, \varphi\| = \|u, \varphi^*\|\) for a unitary \(u\), we may and do assume that \(\Phi^* = \Phi\). Note that \([-T, T] \ni t \mapsto u(t, \cdot) \in \mathcal{M} \otimes L^\infty(\mathbb{T})\) is a Borel unitary path. Hence so is \(t \mapsto \Theta(u(t, \cdot)) \in \mathcal{M}_\nu\). Fix \(0 < \delta < 1\) so that for all \(\varphi \in \Phi\),

\[
\delta^2 + 4\delta\|\varphi\| < \delta^{1/2}, \; (\delta + \varepsilon)/(1 - \delta^{1/2}) + 2\delta^{1/2}\|\varphi\| < \delta^{1/4} + 2\varepsilon, \; 5\delta^{1/4} + 2\delta\|\varphi\| < \varepsilon/2. \tag{5.2}
\]

Since \(u(t, s)\) is Borel, there exists a compact set \(K \subset [-T, T] \times [0, S]\) such that \(\mu(K) \geq 2ST(1 - \delta)\) and \(u\) is continuous on \(K\).

Let \(N \in \mathbb{N}\), and for \(i = -N, \ldots, N - 1\) and \(j = 0, \ldots, N - 1\), we set

\[
I_i := \{t \in \mathbb{R} \mid iT/N \leq t < (i + 1)T/N\},
\]

\[
J_j := \{s \in \mathbb{R} \mid jS/N \leq s < (j + 1)S/N\},
\]

\[
\Delta_{i,j} := I_i \times J_j.
\]

Fix a large \(N\) so that for all \((t, s), (t', s') \in \Delta_{i,j} \cap K\) and \(\varphi \in \Phi\), we have

\[
\|u(t, s) - u(t', s')\|_\varphi^2 < \delta, \quad \|\varphi, u(t, s) - u(t', s')\| < \delta. \tag{5.3}
\]

If \(\Delta_{i,j} \cap K \neq \emptyset\), we fix an element \(k_{i,j} \in \Delta_{i,j} \cap K\). If empty, we put \(k_{i,j} := (iT/N, jS/N)\). We set the following unitary in \(\mathcal{M}\):

\[
u_0(t, s) := \sum_{i,j} u(k_{i,j}) 1_{\Delta_{i,j}}(t, s), \quad (t, s) \in [-T, T] \times [0, S).
\]

Then

\[
\|u(t, s) - \nu_0(t, s)\|_\varphi^2 < \delta \quad \text{for all } (t, s) \in K, \; \varphi \in \Phi. \tag{5.4}
\]

Let \(V(t) := \Theta(\nu_0(t, \cdot)) \in \mathcal{M}_\nu\). Then

\[
V(t) = \sum_{i,j} u(k_{i,j}) 1_{I_i}(t)c(J_j).
\]

We estimate \(\|w(t) - V(t)\|_\varphi^2\) as follows. Put \(K_0 := \text{pr}(K) \subset [-T, T]\), where \(\text{pr}\) denotes the projection \((x, y) \mapsto x\). Then \(K_0^c \times [0, S] \subset K^c\), where \(K_0^c\) and \(K^c\) are the complements in \([-T, T]\) and \([-T, T] \times [0, S]\), respectively. Hence
\( \mu(K^c_{0}) \cdot S \leq 2ST\delta \), and \( \mu(K_0) \geq 2T(1 - \delta) \). For \( t \in [-T, T] \), we set \( K_t := \{ s \in [0, S] \mid (t, s) \in K \} \). Then for \( t \in [-T, T] \), we have
\[
\|w(t) - V(t)\|^2_{\mid \varphi \mid} = \|\Theta(u(t, \cdot)) - \Theta(u_0(t, \cdot))\|^2_{\mid \varphi \mid}
\]
\[
= \frac{1}{S} \int_0^S \|u(t, s) - u_0(t, s)\|^2 ds \quad \text{by (5.1)}
\]
\[
= \frac{1}{S} \int_{K_t} \|u(t, s) - u_0(t, s)\|^2 ds
\]
\[
+ \frac{1}{S} \int_{K_t^c} \|u(t, s) - u_0(t, s)\|^2 ds
\]
\[
\leq \delta^2 + 4\mu(K^c_{0})\|\varphi\|/S \quad \text{by (5.4)}.
\]
Note that
\[
\int_{-T}^T \mu(K^c_{t}) dt = \int_{-T}^T (S - \mu(K_t)) dt = 2ST - \mu(K) \leq 2ST\delta.
\]
Then by (5.2),
\[
\int_{-T}^T \|w(t) - V(t)\|^2_{\mid \varphi \mid} dt \leq (2\delta^2 + 8\delta\|\varphi\|)T \leq 2T\delta^{1/2} \quad \text{for all } \varphi \in \Phi.
\]
Let \( C \) be a compact set in \([-T, T]\) as in Lemma 3.21 with respect to \( w(t) \) such that \( \mu(C) \geq 2T(1 - \delta) \). By the inequality above, we get
\[
\int_C \|w(t) - V(t)\|^2_{\mid \varphi \mid} dt \leq 2T\delta^{1/2} \quad \text{for all } \varphi \in \Phi. \quad (5.5)
\]
Put \( u_{i,j} := u(k_{i,j}) \). For \( (t, s) \in K \), we have
\[
\sum_{i,j} ||[\varphi, u_{i,j}]||1_{\Delta_{i,j} \cap K}(t, s) \leq ||[\varphi, u(t, s)]|| + \sum_{i,j} ||[\varphi, u_{i,j} - u(t, s)]||1_{\Delta_{i,j} \cap K}(t, s)
\]
\[
\leq ||[\varphi, u(t, s)]|| + \delta \quad \text{by (5.3)}. \quad (5.6)
\]
Integrating them by \( (t, s) \in K \), we have
\[
\sum_{i,j} ||[\varphi, u_{i,j}]||\mu(\Delta_{i,j} \cap K) \leq \int_K ||[\varphi, u(t, s)]|| dt ds + 2ST\delta
\]
\[
\leq \int_{-T}^T dt \int_0^S ds \|[\varphi, u(t, s)]|| + 2ST\delta
\]
\[
\leq 2ST\varepsilon + 2ST\delta = 2ST(\varepsilon + \delta). \quad (5.7)
\]
Note that \( \sum_{i,j} \mu(\Delta_{i,j} \cap K^c) = \mu(K^c) < 2ST\delta \). Set
\[
L := \{ (i, j) \mid \mu(\Delta_{i,j} \cap K^c) < ST\delta^{1/2}/N^2 \}.
\]
Then \( |L^c|/2N^2 < \delta^{1/2} \) by the Chebyshev inequality, and for \( (i, j) \in L \),
\[
\mu(\Delta_{i,j} \cap K) = \mu(\Delta_{i,j}) - \mu(\Delta_{i,j} \cap K^c) > ST/N^2 - ST\delta^{1/2}/N^2
\]
\[
= (1 - \delta^{1/2})ST/N^2.
\]
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Thus by (5.7), we have

$$\sum_{(i,j) \in L} ||[\varphi, u_{i,j}]||/2N^2 \leq (\delta + \varepsilon)/(1 - \delta^{1/2}).$$

By definition of $L$, we obtain

$$\sum_{(i,j) \in L^c} ||[\varphi, u_{i,j}]||/2N^2 \leq 2||\varphi||/2N^2 < 2\delta^{1/2}||\varphi||.$$

Hence by (5.2), we get the following inequality for $\varphi \in \Phi$:

$$\sum_{i,j} ||[\varphi, u_{i,j}]||/2N^2 \leq (\delta + \varepsilon)/(1 - \delta^{1/2}) + 2\delta^{1/2}||\varphi|| < \delta^{1/4} + 2\varepsilon. \quad (5.8)$$

Let $(E_j^\nu)_\nu$ be a representing sequence of $E_j := e(J_j)$ consisting of projections with $\sum_j E_j^\nu = 1$ for each $\nu$. We set $V(t)^\nu := \sum_{i,j} 1_{I_i}(t)u_{i,j}E_j^\nu$. On $C \cap I_i$, which may be non-compact, $\{t \mapsto w(t)^\nu\}$ and $\{t \mapsto V(t)^\nu\}$ are $\omega$-equicontinuous since $V(t)^\nu$ is constant. Thus by Lemma 3.3, we have

$$\lim_{\nu \to \omega} ||w(t)^\nu - V(t)^\nu||_{\varphi}^2 = ||w(t) - V(t)||_{\varphi}^2$$

is a uniform convergence on $C \cap I_i$. We set

$$\Phi' := \{[\varphi, u_{i,j}] | \varphi \in \Phi, -N \leq i \leq N - 1, 0 \leq j \leq N - 1\}.$$

Note that $\tau^\nu(E_j) = 1/N$ by Lemma 5.2. Using (5.5) and the above uniform convergence, we can find $W \in \omega$ such that

$$\int_C ||w(t)^\nu - V(t)^\nu||_{\varphi}^2 \, dt < 3T\delta^{1/2} \quad \text{for all } \varphi \in \Phi, \nu \in W; \quad (5.9)$$

$$||[\varphi, E_j^\nu]|| < \varepsilon/6N \quad \text{for all } \varphi \in \Phi \cup \Phi', \ 0 \leq j \leq N - 1, \nu \in W,$n

$$||\psi||/N - \psi(E_j^\nu) < \varepsilon/6N \quad \text{for all } \psi \in \Phi', \ 0 \leq j \leq N - 1, \nu \in W.$$

Then for all $\varphi \in \Phi, i, j$ and $\nu \in W$,

$$||[\varphi, u_{i,j}E_j^\nu]|| \leq ||[\varphi, u_{i,j}]E_j^\nu|| + ||u_{i,j}[\varphi, E_j^\nu]|| \leq ||[\varphi, u_{i,j}]E_j^\nu|| + \varepsilon/6N.$$

Putting $\psi := [\varphi, u_{i,j}]$ and $\psi = u|\psi|$, the polar decomposition, we obtain the following estimate for $x \in M$ with $||x|| \leq 1$ and $\nu \in W$:

$$||[\varphi, u_{i,j}E_j^\nu](x)|| = ||\psi|(E_j^\nu xu) - ||\psi|(E_j^\nu xuE_j^\nu) - ||\psi|(E_j^\nu xu)| + ||\psi|(E_j^\nu xuE_j^\nu)|$$

$$\leq ||[\varphi, E_j^\nu]|x|| + ||\psi|(E_j^\nu)^{1/2}|x|||\psi|(E_j^\nu)^{1/2}$$

$$< \varepsilon/6N + ||\psi|(E_j^\nu) < \varepsilon/3N + ||\psi||/N.$$

Then for all $\varphi \in \Phi, i, j$ and $\nu \in W$, we obtain

$$||[\varphi, u_{i,j}E_j^\nu]|| < \varepsilon/2N + ||[\varphi, u_{i,j}]||/N.$$
Hence for $\varphi \in \Phi$ and $\nu \in W$,
\[
\int_{-T}^{T} \|[\varphi, V(t)^\nu]\| \, dt \leq \sum_{i,j} \int_{-T}^{T} \|[\varphi, u_{i,j}E_j^\nu]\|1_{I_i}(t) \, dt
\leq \sum_{i,j} (\varepsilon/2N + \|[\varphi, u_{i,j}]\|/N) \cdot (T/N)
\leq \varepsilon T + 2(\delta^{1/4} + 2\varepsilon)T \quad \text{by } (5.8)
\leq 2T(\delta^{1/4} + 5\varepsilon/2).
\]

(5.10)

We estimate $\int_C \|[\varphi, w(t)^\nu - V(t)^\nu]\| \, dt$ as follows:
\[
\int_C \|[\varphi, w(t)^\nu - V(t)^\nu]\| \, dt
\leq \int_C \|[w(t)^\nu]^* - (V(t)^\nu)^*\| \, dt + \int_C \|w(t)^\nu - V(t)^\nu\| \nu^* \, dt
\leq \mu(C)^{1/2} \left( \int_C \|[w(t)^\nu]^* - (V(t)^\nu)^*\| \nu^* \, dt \right)^{1/2}
+ \mu(C)^{1/2} \left( \int_C \|w(t)^\nu - V(t)^\nu\|^2 \nu^* \, dt \right)^{1/2}
\leq \mu(C)^{1/2} \cdot (2 \cdot 3T\delta^{1/2})^{1/2} + \mu(C)^{1/2} \cdot (2 \cdot 3T\delta^{1/2})^{1/2}
\leq 7T\delta^{1/4}.
\]

(5.11)

Then for $\nu \in W$,
\[
\int_C \|[\varphi, w(t)^\nu]\| \, dt \leq \int_C \|[\varphi, w(t)^\nu - V(t)^\nu]\| \, dt + \int_C \|[\varphi, V(t)^\nu]\| \, dt
\leq 2T(5\delta^{1/4} + 5\varepsilon/2) \quad \text{by } (5.10, 5.11),
\]

and
\[
\int_{-T}^{T} \|[\varphi, w(t)^\nu]\| \, dt \leq \int_C \|[\varphi, w(t)^\nu]\| \, dt + \int_{C^c} \|[\varphi, w(t)^\nu]\| \, dt
\leq 2T(5\delta^{1/4} + 5\varepsilon/2) + 2\|\varphi\|\mu(C^c)
\leq 2T(5\delta^{1/4} + 5\varepsilon/2) + 2\|\varphi\| \cdot 2T\delta
= 2T(5\delta^{1/4} + 2\delta\|\varphi\| + 5\varepsilon/2) < 6T\varepsilon \quad \text{by } (5.2).
\]

5.2. 2-cohomology vanishing. Let $(\alpha, c)$ be a Borel cocycle action of $\mathbb{R}$ on a von Neumann algebra $M$ as in the previous subsection, that is, it has the Rohlin property and the ergodicity on $Z(M)$ such that $\alpha$ on $Z(M)$ is not conjugate to the translation on $L^\infty(\mathbb{R})$. We will show that the 2-cocycle $c$ can be perturbed to be close to 1. Let $0 < \delta < 1$, $T > 0$ and a finite set $\Phi \subset M^*_\alpha$. Take $S > T$.
such that \((2\pi / S)\mathbb{Z} \cap H_\alpha = \{0\}\) and
\[
4T^{1/2} / S^{1/2} < \delta / 24T^2.
\] (5.12)

Let \(e(\lambda)\) be a Rohlin projection over \([0, S]\). We put \(U(t) := \Theta(c(t, \cdot - t)^*)\), where \(\tilde{c}\) denotes the periodization of \(c\) with respect to the second variable, that is, \(\tilde{c}(x, y) := c(x, y)\) for \(y \in [0, S]\), and \(\tilde{c}(x, y + S) = \tilde{c}(x, y)\) for \(y \in \mathbb{R}\).

Lemma 5.4. In the above setting, there exist \(W \in \omega\) and a lift \((u(t)^\nu)_\nu\) of \(U(t)\) as in Lemma 3.24 such that for all \(\varphi \in \Phi\),
\[
\int_{-T}^{T} dt \int_{-T}^{T} ds \|u(t)^\nu \alpha_t(u(s)^\nu)c(t, s)(u(t+s)^\nu)^* - 1\|_\varphi^2 < \delta \quad \text{for all } \nu \in W.
\]

If \(\varepsilon > 0\) satisfies
\[
\frac{1}{2ST} \int_{-T}^{T} dt \int_{0}^{S} ds \|[c(t, s), \varphi]\| < \varepsilon \quad \text{for all } \varphi \in \Phi,
\]
then one can take \(W\) so that
\[
\frac{1}{2T} \int_{-T}^{T} ||[u(t)^\nu, \varphi]\| dt < 3\varepsilon \quad \text{for all } \varphi \in \Phi, \nu \in W.
\]

Proof. By Lemma 5.2, we have \(\alpha_t(U(s)) = \Theta(\alpha_t(\tilde{c}(s, \cdot - t - s)^*))\). Let \(-T \leq s, t \leq T.\) When \(t + s \geq 0,\) then
\[
\tilde{c}(x, \lambda - t - s) = c(x, \lambda - t - s)1_{[t+s,S]}(\lambda) + c(x, \lambda - t - s + S)1_{[0,t+s]}(\lambda).
\]

Thus we have
\[
U(t)\alpha_t(U(s))c(t, s)U(t+s)^* \\
= \Theta(\tilde{c}(t, \cdot - t)^* \alpha_t(\tilde{c}(s, \cdot - t - s)^*)c(t, s)\tilde{c}(t+s, \cdot - t - s)) \\
= \Theta(c(t, \cdot - t)^* \alpha_t(c(s, \cdot - t - s)^*)c(t, s)c(t+s, \cdot - t - s)1_{[t+s,S]}(\cdot)) \\
+ \Theta(\tilde{c}(t, \cdot - t)^* \alpha_t(\tilde{c}(s, \cdot - t - s)^*)c(t, s)\tilde{c}(t+s, \cdot - t - s))1_{[0,t+s]}(\cdot)) \\
= \Theta(1_{[t+s,S]}(\cdot)) \\
+ \Theta(\tilde{c}(t, \cdot - t)^* \alpha_t(\tilde{c}(s, \cdot - t - s)^*)c(t, s)\tilde{c}(t+s, \cdot - t - s))1_{[0,t+s]}(\cdot)).
\]

Then for \(\varphi \in \Phi_+\) and \(t, s\) with \(t + s \geq 0:\)
\[
\|U(t)\alpha_t(U(s))c(t, s)U(t+s)^* - 1\|_\varphi^2 \\
\leq \|1_{[t+s,S]}(\cdot) - 1\|_{\varphi \otimes \mu}^2 + \|1_{[0,t+s]}(\cdot)\|_{\varphi \otimes \mu}^2 \\
= 2\|1_{[0,t+s]}(\cdot)\|_\varphi^2 \\
= 2\|\varphi\|_{L^2(\mathbb{R})}^2 + \|1_{[0,t+s]}(\cdot)\|_{\varphi \otimes \mu}^2 \\
\leq 2\sqrt{2}\|\varphi\|_{L^2(\mathbb{R})}^2 / \|1_{[0,t+s]}(\cdot)\|_{\varphi \otimes \mu}^2 < \delta / 24T^2 \quad \text{by (5.12)}.
\]

The same inequality also holds when \(t + s \leq 0.\) Hence for all \(t, s \in [-T, T]\) and \(\varphi \in \Phi,\)
\[
\|U(t)\alpha_t(U(s))c(t, s)U(t+s)^* - 1\|_\varphi^2 < \delta / 24T^2. \tag{5.13}
\]
Then by Lemma 3.24, there exist a compact subset $K \subset [-T,T]^2$ and a lift $(u(t)^\nu)_\nu$ of $U(t)$ such that $\mu(K) \geq 4T^2(1 - \delta/24T^2)$, and for all $\varphi \in \Phi$, we have the following uniform convergence on $K$ as $\nu \to \omega$:

$$\|u(t)^\nu a_t(u(s)^\nu)c(t,s)(u(t+s)^\nu)^*-1\|_\varphi^2 \to \|u(t)a_t(U(s))c(t,s)U(t+s)^*-1\|_\varphi^2.$$  

By (5.13), there exists $W \in \omega$ such that if $(t,s) \in K$, $\nu \in W$ and $\varphi \in \Phi$, then

$$\|u(t)^\nu a_t(u(s)^\nu)c(t,s)(u(t+s)^\nu)^*-1\|_\varphi^2 < \delta/24T^2.$$  

If $\nu \in W$ and $\varphi \in \Phi$, then we obtain

$$\int_{-T}^T dt \int_{-T}^T ds \|u(t)^\nu a_t(u(s)^\nu)c(t,s)(u(t+s)^\nu)^*-1\|_\varphi^2$$

$$= \int_{K} dtds \|u(t)^\nu a_t(u(s)^\nu)c(t,s)(u(t+s)^\nu)^*-1\|_\varphi^2$$

$$+ \int_{K^c} dtds \|u(t)^\nu a_t(u(s)^\nu)c(t,s)(u(t+s)^\nu)^*-1\|_\varphi^2$$

$$\leq \mu(K)\delta/24T^2 + 2\|\varphi\|^{1/2}\mu(K^c)$$

$$\leq 4T^2(1 + 2\|\varphi\|^{1/2})\delta/24T^2 < \delta/2 \text{ by (5.12).}$$

Next if we have $(1/2ST) \int_{-T}^T dt \int_0^S ds \|[\varphi,c(t,s)]\| < \varepsilon$ for all $\varphi \in \Phi$, then

$$\frac{1}{2ST} \int_{-T}^T dt \int_0^S ds \|[\varphi,\hat{c}(t,s-t)]\| = \frac{1}{2ST} \int_{-T}^T dt \int_0^S ds \|[\varphi,c(t,s)]\| < \varepsilon,$$

and we can apply Lemma 5.3 to $U(t) = \Theta(\hat{c}(t,\cdot-t))$. Then we have the following for $\nu$ close to $\omega$:

$$\frac{1}{2T} \int_{-T}^T \|[\varphi,u(t)^\nu]\|_\varphi dt < 3\varepsilon \text{ for all } \varphi \in \Phi.$$

\[\square\]

Let us take a decreasing sequence $\{\varepsilon_n\}_n$, increasing sequences $\{T_n\}_n$ and $\{S_n\}_n$ such that $0 < \varepsilon_n < 1/n$, $T_n, S_n > n$, $(2\pi/S_n)^2 \mathbb{Z} \cap H_\alpha = \{0\}$, and

$$T_n + S_n < T_{n+1}, \quad \sum_{k=n+1}^\infty \sqrt{44T_k}\varepsilon_k < \varepsilon_n, \quad 4T_{n+1}^{1/2}S_{n+1}^{1/2} < \varepsilon_{n+1}/24T_{n+1}^2. \quad (5.14)$$

The last inequality satisfies (5.12) for $\delta = \varepsilon_{n+1}$. For a finite set $\Phi \subset \mathcal{M}_+$, we define

$$d(\Phi) := \max \{1 \cup \{\|\varphi\| \mid \varphi \in \Phi\}. \text{ Theorem 5.5 (2-cohomology vanishing). Let } (\alpha,c) \text{ be a Borel cocycle action of } \mathbb{R} \text{ on a von Neumann algebra } \mathcal{M}. \text{ Suppose that } (\alpha,c) \text{ has the Rohlin property, and } \alpha \text{ is an ergodic flow on } Z(\mathcal{M}) \text{ that is not conjugate to the translation on } L^\infty(\mathbb{R}). \text{ Then the following statements hold:}

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(1) The 2-cocycle $c$ is a coboundary, that is, there exists a Borel unitary path $v$ in $\mathcal{M}$ such that

$$v(t)\alpha_t(v(s))c(t,s)v(t+s)^* = 1$$

for almost every $(t, s) \in \mathbb{R}^2$;

(2) If for some $n \geq 2$ and a finite set $\Phi \subset (\mathcal{M}_*)_+$, one has

$$\int_{-T_n}^{T_n} dt \int_{-T_n}^{T_n} ds \|c(t,s) - 1\|_\varphi^2 \leq \varepsilon_{n+1}$$

for all $\varphi \in \Phi$,

then one can choose $v(t)$ in (1) such that

$$\int_{-T_n}^{T_n} \|v(t) - 1\|_\varphi^2 dt < \varepsilon_{n-1}d(\Phi)^{1/2}$$

for all $\varphi \in \Phi$;

(3) If for some $n \geq 2$ and a finite set $\Phi \subset \mathcal{M}_*$, one has

$$\int_{-T_n}^{T_n} dt \int_{0}^{T_n} ds \|\{c(t,s), \varphi\}\| < \varepsilon$$

for all $\varphi \in \Phi$,

then one can take $v$ in (1) satisfying

$$\int_{-T_n}^{T_n} \|\{v(t), \varphi\}\| dt \leq (3\varepsilon_n + 3\varepsilon)d(\Phi)$$

for all $\varphi \in \Phi$.

**Proof.** We may assume that $\Phi$ is contained in the unit ball of $\mathcal{M}_*$.

(1), (2). First we assume that $\mathcal{M}$ is finite. Let $\tau \in \mathcal{M}_*$ be a faithful tracial state. Let $I_n := [-T_n, T_n]$ and $J_n := I_n \times I_n$.

Employing Lemma 5.4, we have a Borel unitary path $v_n(t)$ such that with $\alpha^n_t := \text{Ad} v_n(t) \circ \alpha_t$ and $c_n(t,s) := v_n(t)\alpha_t(v_n(s))c(t,s)v_n(t+s)^*$, we get

$$\int_{J_n} \|c_n(t,s) - 1\|_2 dt ds < \varepsilon_{n+1},$$

(5.15)

where $\|\cdot\|_2 = \|\cdot\|_\varphi = \|\cdot\|_\varphi^2$. It suffices to prove (1) and (2) for $\alpha^n$. Then $(\alpha^n, c^n)$ has the Rohlin property by Lemma 4.3. Again by Lemma 5.4 there exists a Borel path $u(t)$ such that

$$\int_{J_{n+2}} \|u(t)\alpha^n_t(u(s))c_n(t,s)u(t+s)^* - 1\|_2 dt ds < \varepsilon_{n+2}.$$ 

By (5.15), we have

$$\int_{J_{n+1}} \|u(t)\alpha^n_t(u(s))u(t+s)^* - 1\|_2 dt ds < 2\varepsilon_{n+1}.$$ 

(5.16)

Let $e(\lambda) \in \mathcal{M}_{\omega,\alpha}$ be a Rohlin projection over $[0, S_n)$ with respect to $\alpha$. Then $e(\lambda)$ is also a Rohlin projection for $\alpha^n$ by Lemma 4.3. Set $W := \Theta(\tilde{u}(\cdot)) \in \mathcal{M}_e^n$, where $\tilde{u}$ is the periodization of $u$ with period $S_n$. By Lemma 5.2 $\alpha^n_t(W) = \Theta(\alpha^n_t(\tilde{u}(\cdot - t)))$. For $0 \leq t \leq T_n$, we have

$$\tilde{u}(\lambda - t) = u(\lambda - t)1_{[t,S_n)}(\lambda) + u(\lambda - t + S_n)1_{[0,t)}(\lambda).$$
Then
\[ W^* u(t) \alpha^n_t(W) = \Theta(\bar{u}(\cdot) u(t) \alpha^n_t(\bar{u}(\cdot - t))) \]
\[ = \Theta(u(\cdot)^* u(t) \alpha^n_t(u(\cdot - t)) 1_{[t, S_n]}(\cdot)) \]
\[ + \Theta(u(\cdot)^* u(t) \alpha^n_t(u(\cdot - t + S_n)) 1_{[0, t]}(\cdot)). \]

Thus
\[
\int_0^{T_n} \| W^* u(t) \alpha^n_t(W) - 1 \|^2_2 dt \\
\leq 2 \int_0^{T_n} \| \Theta((u(\cdot)^* u(t) \alpha^n_t(u(\cdot - t) - 1) 1_{[t, S_n]}(\cdot))\|^2_2 dt \\
+ 2 \int_0^{T_n} \| \Theta(u(\cdot)^* u(t) \alpha^n_t(u(\cdot - t + S_n) - 1) 1_{[0, t]}(\cdot))\|^2_2 dt \\
\leq \frac{2}{S_n} \int_0^{T_n} dt \int_t^{S_n} ds \| u(s)^* u(t) \alpha^n_t(u(s - t)) - 1 \|^2_2 dt ds \\
+ \frac{2}{S_n} \int_0^{T_n} 4t dt \\
\leq \frac{2}{S_n} \int_{J_{n+1}} \| u(t + s)^* u(t) \alpha^n_t(u(s)) - 1 \|^2_2 dt ds \\
+ 4T_n^2 / S_n \\
\leq 8\varepsilon_{n+1}/S_n + 4T_n^2 / S_n < 9\varepsilon_n \quad \text{by (5.14), (5.16).}
\]

Similarly, we have the same inequality as the above for the integration over \([-T_n, 0]\). Hence
\[
\int_{-T_n}^{T_n} \| W^* u(t) \alpha^n_t(W) - 1 \|^2_2 dt < 18\varepsilon_n.
\]

Let \((w^\nu)\) be a representing unitary sequence of \(W \in \mathcal{M}_\alpha\). Take a compact set \(K \subset [-T_n, T_n]\) with \(\mu(K) > 2T_n(1 - \varepsilon_n / 2T_n)\), such that \(\alpha, \nu_n\) and \(u\) are continuous on \(K\), and moreover, the family \(\{ K \ni t \mapsto \alpha^n_t(w^\nu) \}\) is \(\omega\)-equicontinuous. Then we have the following estimate by Lemma 3.3
\[
\lim_{\nu \to \omega} \int_K \| (w^\nu)^* u(t) \alpha^n_t(w^\nu) - 1 \|^2_2 dt = \int_K \| W^* u(t) \alpha^n_t(W) - 1 \|^2_2 dt < 18\varepsilon_n.
\]

Hence
\[
\lim_{\nu \to \omega} \int_{-T_n}^{T_n} \| (w^\nu)^* u(t) \alpha^n_t(w^\nu) - 1 \|^2_2 dt \\
< 18\varepsilon_n + \lim_{\nu \to \omega} \int_{K^c \cap [-T_n, T_n]} \| (w^\nu)^* u(t) \alpha^n_t(w^\nu) - 1 \|^2_2 dt \\
\leq 18\varepsilon_n + 4\mu(K^c \cap [-T_n, T_n]) \\
< 18\varepsilon_n + 4\varepsilon_n = 22\varepsilon_n.
\]
Thus for some $\nu \in \mathbb{N}$, we obtain
\[
\int_{-T_n}^{T_n} \| (w^\nu)^* u(t) \alpha_t^n(w^\nu) - 1 \|^2 dt < 22\varepsilon_n.
\]
We set $v_{n+1}(t) := (w^\nu)^* u(t) \alpha_t^n(w^\nu)$ for $t \in \mathbb{R}$. Then
\[
\int_{-T_n}^{T_n} \| v_{n+1}(t) - 1 \|^2 dt \leq (2T_n)^{1/2} \left( \int_{-T_n}^{T_n} \| v_{n+1}(t) - 1 \|^2 dt \right)^{1/2}
< (44T_n\varepsilon_n)^{1/2},
\]
and
\[
\int_{J_{n+2}} \| v_{n+1}(t) \alpha_t^n(\nu_{n+1}(s)) c^n(t, s) v_{n+1}(t + s)^* - 1 \|^2 dt ds < \varepsilon_{n+2}.
\]
Let $(\alpha^{n+1}, c_{n+1})$ be the perturbation of $(\alpha^n, c_n)$ by $v_{n+1}$.

Repeating the above process, we obtain a family of Borel cocycle actions $(\alpha^k, c_k)$ and Borel unitary paths $v_k$, $k = n, n+1, \ldots$ such that $(\alpha^{k+1}, c_{k+1})$ is the perturbation of $(\alpha^k, c_k)$ by $v_{k+1}$, and for $k \geq n+1$,
\[
\int_{I_{k-1}} \| v_k(t) - 1 \|^2 dt < (44T_{k-1}\varepsilon_{k-1})^{1/2}, \quad \int_{J_{k+1}} \| c_k(t, s) - 1 \|^2 dt ds < \varepsilon_{k+1}.
\]

Then a subsequence of $v_k(t)v_{k-1}(t) \cdots v_{n+1}(t)$ strongly converges to a Borel path $v(t)$ almost everywhere on $\mathbb{R}$, and we have $v(t)\alpha_t^n(v(s)) c(t, s) v(t + s)^* = 1$ almost everywhere on $\mathbb{R}^2$. On the norm $\| v(t) - 1 \|^2$, we have
\[
\int_{-T_n}^{T_n} \| v(t) - 1 \|^2 dt = \limsup_{k \to \infty} \int_{-T_n}^{T_n} \| v_k(t)v_{k-1}(t) \cdots v_{n+1}(t) - 1 \|^2 dt
\leq \sum_{k=n}^{\infty} \sqrt{44T_k}\varepsilon_k < \varepsilon_{n-1} \quad \text{by (5.14)}.
\]

Next we consider the case that $M$ is properly infinite. By Lemma 5.4. we perturb $(\alpha, c)$ to $(\alpha^n, c^n)$ so that
\[
\int_{J_{n+1}} \| c^n(t, s) - 1 \|^2 dt ds < \varepsilon_{n+1} \quad \varphi \in \Phi.
\]
By Lemma 2.1. we can take a Borel map $u(t)$ as
\[
u(t)\alpha_t^n(u(s)) c^n(t, s) u(t + s)^* = 1 \quad \text{for all } (t, s) \in \mathbb{R}^2.
\]
Thus we have
\[
\int_{J_{n+1}} \| u(t + s)^* u(t) \alpha_t^n(u(s)) - 1 \|^2 dt ds < \varepsilon_n, \quad |t|, |s| \leq T_{n+1}, \quad \varphi \in \Phi.
\]

A computation as given in the finite case shows that for $W := \Theta(\tilde{u})$, we have
\[
\int_{-T_n}^{T_n} \| W^* u(t) \alpha_t^n(W) - 1 \|^2 dt < 18\varepsilon_n \quad \varphi \in \Phi.
\]

Then we can prove (1) and (2) in a similar way to the above.
(3). We may assume that $\Phi^* = \Phi$. By Lemma 5.4, we find a Borel unitary path $w$ such that

$$\int_{-T_{n+1}}^{T_{n+1}} ds \ |w(t)\alpha_t(w(s))c(t, s)w(t + s)^*- 1|^2 \leq \varepsilon_{n+1},$$

and $\int_{-T_n}^{T_n} ||[\varphi, w(t)]|| \ dt < 3\varepsilon$ for all $\varphi \in \Phi$. Let $(\alpha', c')$ be the perturbation of $(\alpha, c)$ by $w$. Then $(\alpha', c')$ is a Borel cocycle action with Rohlin property. By (2), there exists a Borel unitary path $v$ such that $v(t)\alpha'_t(v(s))c'(t, s)v(t + s)^* = 1$ almost everywhere on $\mathbb{R}^2$, and $\int_{-T_n}^{T_n} ||v(t) - 1||_{[\varphi]}^2 \ dt < \varepsilon_{n-1}$ for all $\varphi \in \Phi$. Thus

$$\int_{-T_n}^{T_n} ||[\varphi, v(t)]|| \ dt = \int_{-T_n}^{T_n} ||[\varphi, v(t) - 1]|| \ dt$$

$$
\leq \int_{-T_n}^{T_n} ||v(t)^* - 1||_{[\varphi]} \ dt + \int_{-T_n}^{T_n} ||v(t) - 1||_{[\varphi]^*} \ dt
$$

$$
< 2\sqrt{2}\varepsilon_{n-1}.
$$

Then we obtain

$$\int_{-T_n}^{T_n} ||[\varphi, v(t)w(t)]|| \ dt \leq \int_{-T_n}^{T_n} ||[\varphi, v(t)]|| \ dt + \int_{-T_n}^{T_n} ||[\varphi, w(t)]|| \ dt
$$

$$
< 2\sqrt{2}\varepsilon_{n-1} + 3\varepsilon.
$$

\[\square\]

5.3. Approximation by cocycle perturbation. Let $\alpha, \beta$ be flows on a von Neumann algebra $\mathcal{M}$ with $\alpha_t\beta_t^{-1} \in \text{Int}(\mathcal{M})$. Then we can approximate $\beta_t$ by a perturbation of $\alpha_t$ for finite $t$’s. We would like to connect these unitaries by a continuous path, but we have not solved this problem. We can do for an ITPFI factor with a lacunary product state (see Proposition 9.15). Instead, we connect those by a Borel unitary path.

**Lemma 5.6.** For any $T > 0, \varepsilon > 0$ and a compact set $\Phi \subset \mathcal{M}_*$, there exists a Borel unitary path $\{u(t)\}_{|t| \leq T}$ such that

$$||\text{Ad} \ u(t) \circ \alpha_t(\varphi) - \beta_t(\varphi)|| < \varepsilon \quad \text{for all} \ \varphi \in \Phi, \ t \in [-T, T].$$

**Proof.** Since $\Phi$ is compact, we can take a finite set $\Phi_0 \subset \Phi$ such that each $\varphi \in \Phi$ has $\varphi_0 \in \Phi_0$ such that $||\varphi - \varphi_0|| < \varepsilon/4$. Choose a large $N \in \mathbb{N}$ so that

$$||\alpha_t(\varphi) - \varphi|| < \varepsilon/6, \quad ||\beta_t(\varphi) - \varphi|| < \varepsilon/6 \quad \text{for all} \ \varphi \in \Phi_0, \ |t| \leq T/N.$$

For each $t_j := jT/N, j = -N, \ldots, N$, we can take a unitary $u_j$ such that

$$||\text{Ad} \ u_j \circ \alpha_{t_j}(\varphi) - \beta_{t_j}(\varphi)|| < \varepsilon/6 \quad \text{for all} \ \varphi \in \Phi_0, \ j = -N, \ldots, N.$$
We put a unitary \( u(t) := \sum_{j=-N}^{N-1} u_j(t_{j+1}, t_j) + u_N(t_N) \). Then for \( t \in [t_j, t_{j+1}) \), \( j = -N, \ldots, N-1 \) and \( \varphi \in \Phi_0 \), we have

\[
\| \text{Ad} u(t) \circ \alpha_t(\varphi) - \beta_t(\varphi) \| = \| \text{Ad} u_j \circ \alpha_t(\varphi) - \beta_t(\varphi) \|
\]

\[
\leq \| \text{Ad} u_j(\alpha(t) - \alpha(t_j)) \| + \| \text{Ad} u_j(\beta_j(\varphi)) \|
\]

\[
+ \| \beta_j(\varphi) - \beta_t(\varphi) \|
\]

\[
< \frac{\varepsilon}{6} + \frac{\varepsilon}{6} + \frac{\varepsilon}{6} = \frac{\varepsilon}{2}.
\]

For \( \varphi \in \Phi \), take \( \varphi_0 \in \Phi_0 \) such that \( \| \varphi - \varphi_0 \| < \frac{\varepsilon}{4} \). Then

\[
\| \text{Ad} u(t) \circ \alpha_t(\varphi) - \beta_t(\varphi) \| \leq \| \text{Ad} u(t) (\alpha_t(\varphi) - \alpha_t(\varphi_0)) \|
\]

\[
+ \| \text{Ad} u(t) (\beta_t(\varphi_0) - \beta_t(\varphi)) \| + \| \beta_t(\varphi) - \beta_t(\varphi) \|
\]

\[
\leq \| \varphi - \varphi_0 \| + \frac{\varepsilon}{2} + \| \varphi - \varphi_0 \| < \varepsilon.
\]

\[
\square
\]

**Lemma 5.7.** For any \( T > 0 \), \( \varepsilon > 0 \) and a finite set \( \Phi \subset \mathcal{M}_* \), there exists a Borel\nunitary path \( \{u(t)\}_{|t| \leq 2T} \) such that

\[
\| \text{Ad} u(t) \circ \alpha_t(\varphi) - \beta_t(\varphi) \| < \varepsilon \quad \text{for all } \varphi \in \Phi, \ t \in [-2T, 2T]
\]

\[
\|[u(t) \alpha_t(u(s)) u(t+s)]^*, \varphi]\| < \varepsilon \quad \text{for all } \varphi \in \Phi, \ t, s \in [-T, T].
\]

**Proof.** Let \( \Psi := \{ \beta_t(\varphi) \mid \varphi \in \Phi, |t| \leq 2T \} \), which is a compact set by continuity
of \( \beta \). By the previous lemma, we can take a Borel unitary path \( u(t) \) such that

\[
\| \text{Ad} u(s) \circ \alpha_s(\psi) - \beta_s(\psi) \| < \frac{\varepsilon}{3} \quad \text{for all } \psi \in \Psi, s \in [-2T, 2T].
\]

Then for \( t, s \in [-T, T] \) and \( \psi \in \Psi \), we have

\[
\| \text{Ad} u(t) \alpha_t(u(s)) \circ \alpha_{t+s}(\psi) - \beta_{t+s}(\psi) \|
\]

\[
\leq \| \text{Ad} u(t) \circ \alpha_t(\beta(s)) - \beta_t(\beta(s)) \|
\]

\[
+ \| \alpha_t(\beta(s)) - \beta_t(\beta(s)) \|
\]

\[
= \| \text{Ad} u(s) \circ \alpha_s(\psi) - \beta_s(\psi) \| + \| \text{Ad} u(t) \circ \alpha_t(\beta(s)) - \beta_t(\beta(s)) \|
\]

\[
< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = 2\varepsilon/3.
\]

Together with \( \| \alpha_{t+s}(\psi) - \text{Ad} u(t+s)^* \beta_{t+s}(\psi) \| < \frac{\varepsilon}{3} \), we have

\[
\| \text{Ad} u(t) \alpha_t(u(s)) u(t+s)^* \beta_{t+s}(\psi) - \beta_{t+s}(\psi) \| < \varepsilon \quad \text{for all } \psi \in \Psi, t, s \in [-T, T].
\]

Since \( \beta_{t-s}(\varphi) \in \Psi \) for \( \varphi \in \Phi \), we are done.

\[
\square
\]

**Lemma 5.8.** Suppose that \( \alpha \) is a centrally ergodic Rohlin flow on \( \mathcal{M} \) such that
\( \text{Sp}_d(\alpha|_{\mathcal{Z}(\mathcal{M})}) \neq \mathbb{R} \). Then for any \( T > 0 \), \( \varepsilon > 0 \) and finite set \( \Phi \subset \mathcal{M}_* \), there exists an \( \alpha \)-cocycle \( u \) for \( \alpha \) such that

\[
\int_{-T}^{T} \| \text{Ad} u(t) \circ \alpha_t(\varphi) - \beta_t(\varphi) \| \ dt < \varepsilon \quad \text{for all } \varphi \in \Phi.
\]
Proof. We may and do assume that $\Phi$ is contained in the unit ball of $M_*$. Let $\eta > 0$ be such that $8\eta^{1/4} < 1$ and $2\eta + 8\eta^{1/4} < \varepsilon$. Take a large $N \in \mathbb{N}$ so that

$$\|\alpha_t(\varphi) - \varphi\| < \eta, \quad \|\beta_t(\varphi) - \varphi\| < \eta/T \quad \text{for all } \varphi \in \Phi, \ |t| \leq T/N. \tag{5.17}$$

Set

$$A(N,T) := \{t_j \mid j = -N, \ldots, N\}, \quad t_j := jT/N, \tag{5.18}$$

and

$$\Psi := \{\beta_t(\varphi) \mid \varphi \in \Phi, \ t \in A(N,T)\}.$$ 

Recall (5.14) and fix $n \in \mathbb{N}$ with $T < T_n$ and $3\varepsilon_{n-1} < \eta/(4N + 2)$. By Lemma 5.7 there exists a Borel unitary path $v(t)$ such that

$$\|\text{Ad} \ v(t) \circ \alpha_t(\psi) - \beta_t(\psi)\| \ dt < \eta/T_{n+1} \quad \text{for all } \psi \in \Psi, \ |t|, |s| \leq 2T_{n+1},$$

$$\||v(t)\alpha_t(v(s))v(s + t)^*\psi|| < \eta/(8N + 4)T_nT_{n+1} \quad \text{for all } \psi \in \Psi, \ |t|, |s| \leq T_{n+1}.$$

Set $\gamma_t := \text{Ad} \ v(t) \circ \alpha_t$, and $c(t,s) := v(t)\alpha_t(v(s))v(t+s)^*$. Then $(\gamma, c)$ is a Borel cocycle action. By Theorem 5.3 there exists a Borel unitary path $w$ such that $w(t)\gamma_t(w(s))c(t,s)w(t+s)^* = 1$ almost everywhere on $\mathbb{R}^2$, and

$$\int_{-T_n}^{T_n} \||[\beta_{t_j}(\varphi), w(t)]\| \ dt < 2\eta/(2N + 1) \quad \text{for all } j = -N, \ldots, N, \ \varphi \in \Phi.$$

It is known that there exists a Borel $\alpha$-cocycle $u(t)$ that coincides with $w(t)v(t)$ almost everywhere on $\mathbb{R}$ (see [10] Remark III.1.9]). Moreover, any Borel $\alpha$-cocycle is automatically strongly continuous (see the remark after the proof). We set

$$B_{\varphi,j} := \{t \in [-T_n, T_n] \mid \|[\beta_{t_j}(\varphi), w(t)]\| < \eta^{1/2}/T_n\}, \quad j = -N, \ldots, N, \ \varphi \in \Phi.$$

By the Chebyshev inequality, we have $\mu(B_{\varphi,j}^c) < \eta^{1/2}/(2N + 1)$, where $B_{\varphi,j}^c = [-T_n, T_n] \setminus B_{\varphi,j}$. We let $B_{\varphi} := \bigcap_{j=-N}^{N} B_{\varphi,j}$. Then $\mu(B_{\varphi}^c) \leq \sum_j \mu(B_{\varphi,j}^c) < \eta^{1/2}$. Thus for all $\varphi \in \Phi$,

$$\int_{t_j}^{t_{j+1}} \|[\beta_{t_j}(\varphi), w(t)]\| \ dt = \int_{B_{\varphi} \cap [t_j, t_{j+1}]} \|[\beta_{t_j}(\varphi), w(t)]\| \ dt \quad \text{for all } \varphi \in \Phi, \ j = -N, \ldots, N.$$

By the Chebyshev inequality, we have $\mu(B_{\varphi,j}^c) < \eta^{1/2}/(2N + 1)$, where $B_{\varphi,j}^c = [-T_n, T_n] \setminus B_{\varphi,j}$. We let $B_{\varphi} := \bigcap_{j=-N}^{N} B_{\varphi,j}$. Then $\mu(B_{\varphi}^c) \leq \sum_j \mu(B_{\varphi,j}^c) < \eta^{1/2}$. Thus for all $\varphi \in \Phi$,

$$\int_{t_j}^{t_{j+1}} \|[\beta_{t_j}(\varphi), w(t)]\| \ dt < \eta^{1/2}T/NT_n + 2\mu(B_{\varphi} \cap [t_j, t_{j+1}]), \tag{5.19}$$
Therefore, for $\varphi \in \Phi$, we have
\[
\int_{-T}^{T} \|[\beta_t(\varphi), w(t)]\| \, dt = \sum_{j=-N}^{N-1} \int_{t_j}^{t_{j+1}} \|[\beta_t(\varphi), w(t)]\| \, dt
\]
\[
\leq \sum_{j=-N}^{N-1} \int_{t_j}^{t_{j+1}} \|[\beta_t(\varphi) - \beta_{t_j}(\varphi), w(t)]\| \, dt
\]
\[
+ \sum_{j=-N}^{N-1} \int_{t_j}^{t_{j+1}} \|[\beta_{t_j}(\varphi), w(t)]\| \, dt
\]
\[
< \sum_{j=-N}^{N-1} \int_{t_j}^{t_{j+1}} 2\|[\beta_t(\varphi) - \beta_{t_j}(\varphi)]\| \, dt
\]
\[
+ \sum_{j=-N}^{N-1} \left( \eta^{1/2} T/NT_n + 2\mu(B^c_{\varphi} \cap [t_j, t_{j+1}]) \right) \text{ by (5.19)}
\]
\[
< 2N \times 2\eta/N + 2\eta^{1/2} + 2\mu(B^c_{\varphi}) \text{ by (5.17)}
\]
\[
\leq 4\eta + 2\eta^{1/2} + 2\eta^{1/2} < 8\eta^{1/2}.
\]

By the inequality above, we obtain
\[
\int_{-T}^{T} \| Ad u(t) \circ \alpha_t(\varphi) - \beta_t(\varphi) \| \, dt = \int_{-T}^{T} \| Ad w(t)v(t) \circ \alpha_t(\varphi) - \beta_t(\varphi) \| \, dt
\]
\[
\leq \int_{-T}^{T} \| Ad w(t)(Ad v(t) \circ \alpha_t(\varphi) - \beta_t(\varphi)) \| \, dt
\]
\[
+ \int_{-T}^{T} \| Ad w(t) \circ \beta_t(\varphi) - \beta_t(\varphi) \| \, dt
\]
\[
< 2\eta + \int_{-T}^{T} \|[w(t), \beta_t(\varphi)]\| \, dt
\]
\[
< 2\eta + 8\eta^{1/2} < \varepsilon.
\]

\[\square\]

**Remark 5.9.** Let $\alpha$ be a flow on a separable von Neumann algebra $\mathcal{M}$, and $v$ a Borel $\alpha$-cocycle in $\mathcal{M}$. Then $v$ is strongly continuous. Indeed, in the crossed product $\mathcal{M} \rtimes_{\alpha} \mathbb{R}$, we have the Borel one-parameter unitary group $v(t)\lambda^\alpha(t)$. Since $(\mathcal{M} \rtimes_{\alpha} \mathbb{R})^U$ is Polish, $v(t)\lambda^\alpha(t)$ is continuous, and so is $v(t)$.

**Lemma 5.10.** Let $\alpha$ be a flow on a von Neumann algebra $\mathcal{M}$ and $u$ an $\alpha$-cocycle. Then for all $t \in \mathbb{R}$ and $\varphi \in \mathcal{M}_*$, one has
\[
\|[u(t), \varphi]\| = \|\alpha_{-t}(\varphi) - Ad u(-t) \circ \alpha_{-t}(\varphi)\|.
\]
Proof. Since $\text{Ad} u(t) \circ \alpha_t \circ \text{Ad} u(-t) \circ \alpha_{-t} = \text{id}$, we have
\[
\|[u(t), \varphi]\| = \|\text{Ad} u(t)(\varphi) - \varphi\|
\]
\[
= \|\text{Ad} u(t) \circ \alpha_t \circ \alpha_{-t}(\varphi) - \text{Ad} u(t) \circ \alpha_t \circ \text{Ad} u(-t) \circ \alpha_{-t}(\varphi)\|
\]
\[
= \|\alpha_{-t}(\varphi) - \text{Ad} u(-t) \circ \alpha_{-t}(\varphi)\|.
\]
\[
\square
\]

5.4. Approximate vanishing of 1-cohomology.

**Theorem 5.11.** Let $\alpha$ be a Rohlin flow on a von Neumann algebra $M$. Suppose that $\alpha$ is centrally ergodic and $\text{Sp}_d(\alpha|Z(M)) \neq \mathbb{R}$. Let $\epsilon, \delta, T > 0$ and $\Phi \subset M_*$ a finite set. Take $S > 0$ such that $T\|\varphi\|/S < \epsilon^2/4$ for all $\varphi \in \Phi$ and $(2\pi/S)\mathbb{Z} \cap \text{Sp}_d(\alpha|Z(M)) = \{0\}$. Then for any $\alpha$-cocycle $u$ with
\[
\frac{1}{S} \int_0^S \|[u(t), \varphi]\| \, dt < \delta \quad \text{for all } \varphi \in \Phi,
\]
there exists a unitary $w \in M$ such that
\[
\|[w, \varphi]\| < 3\delta \quad \text{for all } \varphi \in \Phi,
\]
\[
\|[\varphi \cdot (u(t)\alpha_t(w)w^* - 1)]\| < \epsilon, \quad \|(u(t)\alpha_t(w)w^* - 1) \cdot \varphi\| < \epsilon \quad \text{for all } |t| \leq T, \varphi \in \Phi.
\]

**Proof.** We may and do assume that $\Phi^* = \Phi$. Let $e(\lambda) \in M_{\omega, \alpha}$ be a Rohlin projection over $[0, S)$. Put $W := \Theta(\tilde{u}) \in U(M_{\alpha}^\omega)$, where $\tilde{u}$ is the periodization of $u$ with period $S$. Then it is trivial that
\[
\tilde{u}(\lambda - t) = u(\lambda - t)1_{[t,S]}(\lambda) + u(\lambda - t + S)1_{[0,t]}(\lambda) \quad \text{for all } \lambda \in [0, S).
\]

For $0 \leq t \leq T$, we have
\[
u(t)\alpha_t(W)W^* = \Theta(u(t)\alpha_t(\tilde{u}(\cdot - t)))\tilde{u}(\cdot)^*
\]
\[
\Theta(u(t)\alpha_t(\tilde{u}(\cdot - t))u(\cdot)^*1_{[t,S]}(\cdot)) + \Theta(u(t)\alpha_t(u(\cdot - t + S))u(\cdot)^*1_{[0,t]}(\cdot))
\]
\[
\Theta(1_{[t,S]}(\cdot)) + \Theta(u(t)\alpha_t(u(\cdot - t + S))u(\cdot)^*1_{[0,t]}(\cdot)).
\]

Hence for all $\varphi \in \Phi$, we have
\[
\|[u(t)\alpha_t(W)W^* - 1]\|^2_{[\varphi]} \
\quad \leq \|[\Theta(1_{[t,S]}(\cdot)) - 1]\|^2_{[\varphi]}
\]
\[
+ \|[\Theta(u(t)\alpha_t(u(\cdot - t + S))u(\cdot)^*1_{[0,t]}(\cdot))\|_{[\varphi]}^2
\]
\[
= 2\|1_{[0,t]}(\cdot)\|_{[\varphi]}^2
\]
\[
\leq 2t^{1/2}\|\varphi\|^{1/2}/S^{1/2}.
\]

The same estimate as the above is valid for $-T \leq t \leq 0$. Thus if $|t| \leq T$, we have
\[
\|[u(t)\alpha_t(W)W^* - 1]\|^2_{[\varphi]} \leq 2t^{1/2}\|\varphi\|^{1/2}/S^{1/2} < \epsilon \quad \text{for all } \varphi \in \Phi.
\]

Take a representing sequence $(w^\nu)_\nu$ of $W \in M_{\alpha}^\omega$. By Lemma 3.3 we have the following uniform convergence on $[-T, T]$
\[
\lim_{n \to \omega} \|[u(t)\alpha_t(w^\nu)(w^\nu)^* - 1]\|^2_{[\varphi]} = \|[u(t)\alpha_t(W)W^* - 1]\|^2_{[\varphi]}.
\]
Hence if \( \nu \in \mathbb{N} \) is close to \( \omega \), then for all \( t \in [-T, T] \),
\[
\| \varphi \cdot (u(t)\alpha_t(w^\nu)(w^\nu)^* - 1) \| < \varepsilon, \quad \| (u(t)\alpha_t(w^\nu)(w^\nu)^* - 1) \cdot \varphi \| < \varepsilon.
\]
Applying Lemma 5.3 to \( u(t, s) := u(s) \), we have \( \| [w^\nu, \varphi] \| < 3\delta \) for \( \nu \) being close to \( \omega \).

5.5. Proof of the main theorem. We will prove our main theorem for centrally ergodic Rohlin flows by using the Bratteli-Elliott-Evans-Kishimoto intertwining argument [15].

Lemma 5.12. Let \( \alpha \) and \( \beta \) be Rohlin flows on a von Neumann algebra \( M \). Suppose that \( \alpha \) is centrally ergodic. Then \( \alpha \) and \( \beta \) are strongly cocycle conjugate if and only if \( \alpha_t \beta_{-t} \in \text{Int}(M) \) for all \( t \in \mathbb{R} \).

Proof. The “only if” part is trivial. We will prove the “if” part. Assume \( \alpha_t \beta_{-t} \in \text{Int}(M) \) for all \( t \in \mathbb{R} \).

Case 1. \( \text{Sp}_d(\alpha|_{Z(M)}) = \mathbb{R} \).

In this case, the covariant system \( \{L^\infty(\mathbb{R}), \text{Ad} \lambda \} \) embeds into \( \{Z(M), \alpha\} \). Since \( \alpha \) is centrally ergodic, this embedding is surjective. (See Remark 5.13) Note that \( \alpha = \beta \) on \( Z(M) \) since \( \alpha_t \beta_{-t} \in \text{Int}(M) \). By duality theorem, we obtain the following decompositions:
\[
M = M^\alpha \vee Z(M) \cong M^\alpha \otimes Z(M).
\]
(5.20)
Note that \( Z(M^\alpha) \subset Z(M) \), and \( Z(M^\alpha) = \mathbb{C} = Z(M^\beta) \), that is, the fixed point algebra \( M^\alpha \) is a factor.

The ergodic flow \( \alpha = \beta \) on \( Z(M) \) is identified with the translation on \( L^\infty(\mathbb{R}) \), which produces the groupoid \( \mathcal{G} := \mathbb{R} \times \mathbb{R} \). Applying [61, Corollary XIII.3.29] to \( \mathcal{G} \) and \( \alpha, \beta: \mathcal{G} \to G \) with \( G := \text{Int}(M^\alpha) \) and \( H := \text{Int}(M^\alpha) \), we can find \( \theta \in \text{Aut}(M) \) and a Borel unitary path \( u: \mathbb{R} \to M \) such that \( \theta = \int_\mathbb{R} \theta_x dx \) with \( \theta_x \in \text{Int}(M^\alpha) \) and
\[
\text{Ad } u(t) \circ \alpha_t = \theta \circ \beta_t \circ \theta^{-1}.
\]
We should note that the statement of [61, Corollary XIII.3.29] is concerned with a properly ergodic flow, but the proof is also applicable to \( \mathbb{R} \times \mathbb{R} \) by setting a base space \( Z \) with natural transformation and a ceiling function \( r = 1 \).

By Theorem 9.3, \( \theta \) is approximately inner. If we use [61, Proposition XIII.3.34], then it turns out that we may arrange \( u(t) \) to an \( \alpha \)-cocycle. In our case, it is directly checked as follows. Since \( \beta \) is a flow, the cocycle action \( (\text{Ad } u(t) \circ \alpha_t, c(t, s)) \) must be a flow, where we have put \( c(t, s) = u(t)\alpha_t(u(s))u(t + s)^* \) which belongs to \( Z(M) \). By the conjugacy \( \{M, \alpha\} \cong \{M^\alpha \otimes L^\infty(\mathbb{R}), \text{id} \otimes \text{Ad } \lambda\} \), \( c \) is regarded as an \( L^\infty(\mathbb{R}) \)-valued 2-cocycle with respect to the translation. Then \( c \) is a coboundary by [13, Proposition A.2]. Hence we may assume that \( v \) is an \( \alpha \)-cocycle, and we are done.

Case 2. \( \text{Sp}_d(\alpha|_{Z(M)}) \neq \mathbb{R} \).

Take \( \varepsilon_n, T_n > 0 \) and \( S_n > 0 \) which satisfy (5.13). Let us denote \( H_\alpha := \text{Sp}_d(\alpha|_{Z(M)}) \) as before. We should note that the choice of \( S_n \) depends on \( \alpha \),
that is, \((2\pi/S_n)\mathbb{Z} \cap H_\alpha = \{0\}\). In what follows, we introduce a sequence of flows \(\gamma^{(m)}\). They are cocycle perturbations of \(\alpha\) and \(\beta\), and \(H_\alpha = H_\beta = H_{\gamma^{(m)}}\). Hence \((2\pi/S_n)\mathbb{Z} \cap H_{\gamma^{(m)}} = \{0\}\), and we can apply the preceding results to \(\gamma^{(m)}\).

Now for \(N \in \mathbb{N}\) and \(T > 0\), let \(A(N, T)\) be as defined in (5.18). Let \(\{\psi_i\}_{i=1}^\infty\) be a dense countable set of the unit ball of \(M_*\), and set \(\Phi_n = \{\psi_i\}_{i=0}^n\) with a faithful state \(\varphi_0 \in M_*\). Set \(\hat{\Phi}_n := \Phi_0, \hat{\Phi}_1 := \Phi_1, \gamma_1^{(-1)} := \beta_t\) and \(\gamma_1^{(0)} := \alpha_t\).

By Lemma 5.8 there exists a \(\gamma^{(-1)}\)-cocycle \(u^1(t)\) such that

\[
\int_{-T_1}^{T_2} \| \text{Ad} u^1(t) \circ \gamma_t^{(-1)}(\varphi) - \gamma_t^{(0)}(\varphi) \| dt < \varepsilon_1 \quad \text{for all } \varphi \in \hat{\Phi}_1.
\]

Set \(\gamma_t^{(1)} := \text{Ad} u^1(t) \circ \gamma_t^{(-1)}, w_1 := 1, \hat{\varphi}_t(t) := 1, v^1(t) := u^1(t) =: \hat{v}^1(t)\) and \(\theta_{-1} = \theta_0 = \theta_1 = \text{id}\). Choose \(M_1 \in \mathbb{N}\) such that \(\| (\hat{v}^1(t) - \hat{v}^1(s)) \varphi \| < \varepsilon_1\) and \(\| \varphi \cdot (\hat{v}^1(t) - \hat{v}^1(s)) \| < \varepsilon_1\) for \(t, s \in [-T_1, T_1]\) with \(|t - s| \leq T_1/M_1\) and \(\varphi \in \hat{\Phi}_0\).

We will inductively construct a flow \(\gamma^{(n)}\), an automorphism \(\theta_n \in \text{Int}(M)\), a \(\gamma^{(n-2)}\)-cocycle \(u^n(t)\), unitary paths \(v^n(t), \hat{v}^n(t)\), a unitary \(w_n \in M^U\), a natural number \(M_n \in \mathbb{N}\), and a finite set \(\hat{\Phi}_n < M\) satisfying the following conditions:

\[
\begin{align*}
(1.1) & \quad \hat{\Phi}_n = \Phi_n \cup \{ \omega(t) \mid t \in A(M_k, T_k), 1 \leq k \leq n - 1 \}; \\
(2.2) & \quad \gamma_t^{(n)} = \text{Ad} u^n(t) \circ \gamma_t^{(n-1)}; \\
(3.3) & \quad v^n(t) = u^n(t) \gamma_t^{(n-2)}(w_n^*) w_n^*; \\
(4.4) & \quad \theta_n = \text{Ad} w_n \circ \theta_{n-2}; \\
(5.5) & \quad \int_{-T_n+1}^{T_n+1} \| \gamma_t^{(n)}(\varphi) - \gamma_t^{(n-1)}(\varphi) \| dt < \varepsilon_n \quad \text{for } \varphi \in \hat{\Phi}_n; \\
(6.6) & \quad \| (\hat{v}^n(t) - 1) \varphi \| < \varepsilon_n, \quad \| (\hat{v}^n(t) - 1) \| < \varepsilon_n, \quad |t| \leq T_n, \quad \varphi \in \hat{\Phi}_n; \\
(7.7) & \quad \| w_n \varphi \| < 3\varepsilon_n, \quad \varphi \in \hat{\Phi}_n; \\
(8.8) & \quad \| \hat{v}^n(t) - \hat{v}^n(s) \| \varphi \| < \varepsilon_n \quad \text{and } \| \varphi \cdot (\hat{v}^n(t) - \hat{v}^n(s)) \| < \varepsilon_n \quad \text{for } t, s \in [-T_n, T_n] \quad \text{with } |t - s| \leq T_n/M_n \quad \text{and } \varphi \in \hat{\Phi}_n.
\end{align*}
\]

Suppose we have constructed them up to the \(n\)-th step. Define \(\hat{\Phi}_{n+1}\) as the condition (n+1.1). Employing Lemma 5.8, we take a \(\gamma^{(n-1)}\)-cocycle \(u^{n+1}(t)\) such that

\[
\int_{-T_{n+2}}^{T_{n+2}} \| \text{Ad} u^{n+1}(t) \circ \gamma_t^{(n-1)}(\varphi) - \gamma_t^{(n)}(\varphi) \| dt < \varepsilon_{n+1} \quad \text{for } \varphi \in \hat{\Phi}_n \cup \hat{\Phi}_{n+1}.
\]

Combining this with (n.5), we have

\[
\int_{-T_{n+1}}^{T_{n+1}} \| \text{Ad} u^{n+1}(t) \circ \gamma_t^{(n-1)}(\varphi) - \gamma_t^{(n-1)}(\varphi) \| dt < 2\varepsilon_n \quad \text{for } \varphi \in \hat{\Phi}_n.
\]

By Lemma 5.10

\[
\int_{-T_{n+1}}^{T_{n+1}} \| [u^{n+1}(t), \varphi] \| < 2\varepsilon_n \quad \text{for } \varphi \in \hat{\Phi}_n.
\]

Using \(T_n/S_n < \varepsilon_n/4\), \(S_n < T_{n+1}\) and Theorem 5.11 we get a unitary \(w_{n+1} \in M\) such that

\[
\| [w_{n+1}, \varphi] \| < 6\varepsilon_n/S_n < 3\varepsilon_n \quad \text{for } \varphi \in \hat{\Phi}_n,
\]

\[
\| \varphi \cdot (u^{n+1}(t) \gamma_t^{(n-1)}(w_{n+1}) w_{n+1}^* - 1) \| < \varepsilon_n \quad \text{if } |t| \leq T_n, \quad \varphi \in \hat{\Phi}_n.
\]
and

\[ \| (u^{n+1}(t) \gamma_t^{(n-1)} (w_{n+1}) w_{n+1}^* - 1) \varphi \|_2 < \varepsilon_n \quad \text{if } |t| \leq T_n, \quad \varphi \in \Phi_n. \]

Set \( \gamma_t^{(n+1)} = \text{Ad} u^{n+1}(t) \circ \gamma_t^{(n-1)}, \theta_{n+1} = \text{Ad} w_{n+1} \circ \theta_{n-1} \) and

\[ v_t^{n+1} = u^{n+1}(t) \gamma_t^{(n-1)} (w_{n+1}) w_{n+1}^*, \quad \hat{v}^{n+1} = v^{n+1}(t) w_{n+1} \hat{v}^{n-1}(t) w_{n+1}^*. \]

Then the conditions from \((n + 1, 2)\) to \((n + 1.7)\) are satisfied. Choose \( M_{n+1} \in \mathbb{N} \) as in \((n + 1.8)\) and the induction procedure is done.

We first show the convergence of \( \{ \theta_{2n} \}_n \) and \( \{ \theta_{2n+1} \}_n \). By \((n + 2.7)\),

\[ \| \theta_{n+2}(\varphi) - \theta_n(\varphi) \| = \|[w_{n+2}, \theta_n(\varphi)]\|_2 < 4\varepsilon_{n+1} \]

and

\[ \| \theta_{n+2}^{-1}(\varphi) - \theta_n^{-1}(\varphi) \| = \|[w_{n+2}, \varphi]\|_2 < 4\varepsilon_{n+1} \quad \text{for all } \varphi \in \Phi_{n+1}. \]

Thus the limits \( \lim_{n \to \infty} \theta_{2n} = \sigma_0 \) and \( \lim_{n \to \infty} \theta_{2n+1} = \sigma_1 \) exist.

We next show \( \hat{v}^{2n}(t) \) converges compact uniformly. If \(|t| \leq T_{n+2}\), then

\[
\| \varphi_0 \cdot (\hat{v}^{n+2}(t) - \hat{v}^n(t)) \|
= \| \varphi_0 \cdot (w_{n+2}(t) w_{n+2} \hat{v}^n(t) w_{n+2}^* - \hat{v}^n(t)) \|
\leq \| \varphi_0 \cdot (w_{n+2}(t) - 1) \| + \| \varphi_0 \cdot (w_{n+2} \hat{v}^n(t) w_{n+2}^* - \hat{v}^n(t)) \|
< \varepsilon_{n+2} + \| \varphi_0 \cdot (w_{n+2} \hat{v}^n(t) w_{n+2}^* - \hat{v}^n(t)) \| \quad \text{by } (n + 2.6). \quad (5.21)
\]

For \(|t| \leq T_n\), take \( t_0 \in \mathcal{A}(M_n, T_n) \) so that \( 0 \leq |t - t_0| < T_n/M_n \). Note \( \varphi_0 \hat{v}^n(t_0) \in \Phi_{n+1} \). By \((n,8)\),

\[ \| \varphi_0 \cdot (\hat{v}^n(t) - \hat{v}^n(t_0)) \| < \varepsilon_n. \]

Hence the second term of \((5.21)\) can be estimated as follows;

\[
\| \varphi_0 \cdot (w_{n+2} \hat{v}^n(t) w_{n+2}^* - \hat{v}^n(t)) \|
\leq \|[w_{n+2}, \varphi_0]\| + \|[w_{n+2} \varphi_0 \hat{v}^n(t) - \varphi_0 \hat{v}^n(t)] w_{n+2}\|
\leq 3\varepsilon_{n+1} + 2\varepsilon_n + \|[w_{n+2} \varphi_0 \hat{v}^n(t_0) - \varphi_0 \hat{v}^n(t_0)] w_{n+2}\| \quad \text{by } (n + 2.7), \ (n,8)
< 3\varepsilon_{n+1} + 2\varepsilon_n + 3\varepsilon_{n+1} < 8\varepsilon_n \quad \text{by } (n + 2.7).
\]

Thus we get

\[ \| \varphi_0 \cdot (\hat{v}^{n+2}(t) - \hat{v}^n(t)) \| < 9\varepsilon_n \quad \text{if } |t| \leq T_n. \]

We estimate \( \| (\hat{v}^{n+2}(t) - \hat{v}^n(t)) \varphi_0 \| \) as follows:

\[
\| (\hat{v}^{n+2}(t) - \hat{v}^n(t)) \varphi_0 \|
= \| (w_{n+2} \hat{v}^n(t) w_{n+2}^* - \hat{v}^n(t)) \varphi_0 \|
\leq \| (w_{n+2} \hat{v}^n(t) - 1) w_{n+2} \varphi_0 \| + \| (w_{n+2} \hat{v}^n(t) w_{n+2}^* - \hat{v}^n(t)) \varphi_0 \|.
\]

In the same way as above, we can show \( \| (w_{n+2} \hat{v}^n(t) w_{n+2}^* - \hat{v}^n(t)) \varphi_0 \| < 8\varepsilon_n \). The first term is estimated as follows. We take \( t_0 \) as above. Since \( \hat{v}^n(t_0) \varphi_0 \in \Phi_{n+1}, \)
we have
\[
\| (v_{n+2}^n(t) - 1) w_{n+2} \hat{v}^n(t) w_{n+2}^* \varphi_0 \| \\
\leq 2 \| w_{n+2}^* \varphi_0 \| + \| (v_{n+2}^n(t) - 1) w_{n+2} \hat{v}^n(t) \varphi_0 \| \\
\leq 6 \varepsilon_{n+1} + \| (v_{n+2}^n(t) - 1) w_{n+2} (\hat{v}^n(t) - \hat{v}^n(t_0)) \varphi_0 \| \\
+ \| (v_{n+2}^n(t) - 1) [w_{n+2}, \hat{v}^n(t_0)] \varphi_0 \| + \| (v_{n+2}^n(t) - 1) \hat{v}^n(t_0) \varphi_0 w_{n+2} \| \\
\leq 6 \varepsilon_{n+1} + 2 \varepsilon_n + 3 \varepsilon_{n+1} + \varepsilon_{n+2} \\
< 12 \varepsilon_{n+1}.
\]
Hence we have
\[
\| (\hat{v}^{n+2}(t) - \hat{v}^n(t)) \cdot \varphi_0 \| < 20 \varepsilon_n \quad \text{if } |t| \leq T_n.
\]
Thus \( \hat{v}^{2n}(t) \) converges compact uniformly in the strong* topology. In the same way, so does \( \hat{v}^{2n+1}(t) \). Put \( \hat{v}^0(t) := \lim_{n \to \infty} \hat{v}^{2n}(t) \) and \( \hat{v}^1(t) := \lim_{n \to \infty} \hat{v}^{2n+1}(t) \), which are cocycles of \( \sigma_0 \circ \alpha_t \circ \sigma_0^{-1} \) and \( \sigma_1 \circ \beta_t \circ \sigma_1^{-1} \), respectively. By (n.5), we have
\[
\Ad \hat{v}^0(t) \circ \sigma_0 \circ \alpha_t \circ \sigma_0^{-1} = \Ad \hat{v}^1(t) \circ \sigma_1 \circ \beta_t \circ \sigma_1^{-1} \quad \text{for all } t \in \mathbb{R}.
\]
Therefore, \( \alpha \) and \( \beta \) are strongly cocycle conjugate. \( \square \)

**Remark 5.13.** Let \( \mathcal{P} \) be a von Neumann algebra and \( \alpha \) a flow on \( \mathcal{P} \). Suppose that the covariant system \( \{ L^\infty(\mathbb{R}), \Ad \lambda \} \) is embedded into \( \{ Z(\mathcal{P}), \alpha \} \). Then any \( \alpha \)-cocycle is a coboundary as checked below.

Thanks to [44, Theorem 1], it turns out that \( \alpha \) is a dual flow. Indeed, \( e_s \in L^\infty(\mathbb{R}) \) satisfies \( \Ad \lambda_t(e_s) = e^{-ist} e_s \), where \( e_s(x) = e^{isx} \). Let \( \pi : L^\infty(\mathbb{R}) \to Z(\mathcal{P}) \) be an equivariant normal *-homomorphism. Put \( w(s) := e_s \). Since \( w(s) \) belongs to \( Z(\mathcal{P}) \), we obtain
\[
\mathcal{P} = \mathcal{P}^\alpha \vee \{ w(\mathbb{R}) \}'' \cong \mathcal{P}^\alpha \rtimes_{\Ad w} \mathbb{R} \cong \mathcal{P}^\alpha \otimes L^\infty(\mathbb{R}).
\]
Then \( \alpha \) is conjugate to \( \id \otimes \Ad \lambda \). Thus \( \alpha \) is stable (see the proof of [60, Theorem XII.1.11]).

Now we will prove the main theorem for general Rohlin flows.

**Theorem 5.14.** Let \( \alpha \) and \( \beta \) be Rohlin flows on a von Neumann algebra \( \mathcal{M} \). Then \( \alpha \) and \( \beta \) are strongly cocycle conjugate if and only if \( \alpha_t \beta_{-t} \in \text{Int}(\mathcal{M}) \) for all \( t \in \mathbb{R} \).

**Proof.** We only prove the “if” part. The assumption implies that \( \alpha = \beta \) on \( Z(\mathcal{M}) \). Let \((X, \mu)\) be a measure theoretic spectrum of \( Z(\mathcal{M})^\alpha = Z(\mathcal{M})^\beta \). Then we obtain the following disintegrations:
\[
\mathcal{M} = \int_X \mathcal{M}_x \, d\mu(x), \quad \alpha^x_t = \int_X \alpha^x_t \, d\mu(x), \quad \beta^x_t = \int_X \beta^x_t \, d\mu(x).
\]
Note that \( \alpha^x \) and \( \beta^x \) are centrally ergodic flows on \( \mathcal{M}_x \) for almost every \( x \).

**Claim 1.** For almost every \( x \in X \), \( \alpha^x_t \beta^x_{-t} \in \text{Int}(\mathcal{M}_x) \) for all \( t \in \mathbb{R} \).
**Proof of Claim 1.** Employing Theorem 9.4, we deduce that for each \( t \in \mathbb{R} \), \( \alpha_t^x \beta_{-t}^x \in \text{Int}(M_x) \) for almost every \( x \in X \). Thus by usual measure theoretic discussion, it turns out that for almost every \( x \in X \), \( \alpha_t^x \beta_{-t}^x \in \text{Aut}(M_x) \) is continuous, we see that for almost every \( x \in X \), \( \alpha_t^x \beta_{-t}^x \in \text{Int}(M_x) \) for all \( t \in \mathbb{R} \). \( \square \)

**Claim 2.** For almost every \( x \in X \), \( \alpha^x \) and \( \beta^x \) are Rohlin flows.

**Proof of Claim 2.** Let \( p \in \mathbb{R} \). Employing Lemma 4.4, we take a central sequence \((v^x)_\nu \) in \( M \) such that \( v^x \in M^U \) and \( \alpha_t(v^x) - e^{ipt} v^x \to 0 \) compact uniformly in the strong topology as \( \nu \to \infty \). By Lemma 9.8, a subsequence of \((v^x)_\nu \) is central for almost every \( x \in X \). Hence we may and do assume that \((v^x)_\nu \) is central for almost every \( x \in X \).

Let \( \{\varphi^k\}_k \) be a norm dense sequence in \( M_x \). Then

\[
\left\| (\alpha_t(v^x) - e^{ipt} v^x) \varphi^k \right\| = \int_X \left\| (\alpha_t^x(v^x) - e^{ipt} v^x) \varphi^k \right\| d\mu(x),
\]

\[
\left\| \varphi^k \cdot (\alpha_t(v^x) - e^{ipt} v^x) \right\| = \int_X \left\| \varphi^k \cdot (\alpha_t^x(v^x) - e^{ipt} v^x) \right\| d\mu(x),
\]

As the discussion in the proof of Lemma 9.8, we may and do assume that for each \( t \in \mathbb{R} \), \( \alpha_t^x(v^x) - e^{ipt} v^x \) converges to 0 in the strong topology as \( \nu \to \infty \) for almost every \( x \in X \).

Let \( f(t) = e^{-ipt}1_{[0,1]}(t) \). Then for all \( \nu \in \mathbb{N} \) and \( x \in X \), we have

\[
\left\| (\alpha_f(v^x) - v^x) \varphi^k \right\| \leq \int_0^1 \left\| (\alpha_t^x(v^x) - e^{ipt} v^x) \varphi^k \right\| dt.
\]

Hence

\[
\int_X \left\| (\alpha_f(v^x) - v^x) \varphi^k \right\| d\mu(x) \leq \int_0^1 \int_X d\mu(x) \left\| (\alpha_t^x(v^x) - e^{ipt} v^x) \varphi^k \right\| dt = \int_0^1 \left\| (\alpha_t(v^x) - e^{ipt} v^x) \varphi^k \right\| dt,
\]

which converges to 0 as \( \nu \to \infty \). Similarly we have \( \int_X \left\| \varphi^k \cdot (\alpha_f(v^x) - v^x) \right\| \to 0 \). Again by taking a subsequence if necessary, we may and do assume that \( (\alpha_f(v^x) - v^x)_\nu \) is a trivial sequence for almost every \( x \in X \).

By Lemma 3.14 \((v^x)_\nu \) is a Rohlin unitary for \( p \in \mathbb{R} \) for almost every \( x \in X \). \( \square \)

Combining the above claims, Lemma 5.12 and Theorem 9.13, we see that \( \alpha \) and \( \beta \) are strongly cocycle conjugate.

Thanks to [34, Theorem 1 (i)] and Theorem 9.11 we know \( \text{Int}(M) = \ker(\text{mod}) \) when \( M \) is injective. Hence we obtain the following corollary.

**Corollary 5.15.** Let \( M \) be an injective von Neumann algebra and \( \alpha, \beta \) Rohlin flows on \( M \). Then \( \alpha \) and \( \beta \) are strongly cocycle conjugate if and only if \( \text{mod}(\alpha_t) = \text{mod}(\beta_t) \) for all \( t \in \mathbb{R} \).

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Corollary 5.16. If $\mathcal{M}$ is an injective factor of type $\text{II}_1$ or $\text{III}_1$, then any Rohlin flow is cocycle conjugate to $\text{id}_M \otimes \alpha^0$, where $\alpha^0$ is a (unique) Rohlin flow on the injective factor of type $\text{II}_1$.

Proof. Since $\text{Aut}(\mathcal{M}) = \overline{\text{Int}}(\mathcal{M})$, any Rohlin flows are cocycle conjugate. □

Corollary 5.17. Let $\alpha$ be a flow on the type $\text{II}_1$ injective factor $\mathcal{M}$. Then the following statements are equivalent:

1. $\alpha$ has the Rohlin property;
2. $\alpha$ is invariantly approximately inner and $\Gamma(\alpha) = \mathbb{R}$.

In this case, $\alpha$ is stably self-dual, that is, $\hat{\alpha} \sim \alpha \otimes \text{id}_{B(\ell^2)}$.

Proof. (1)$\Rightarrow$(2). The $\alpha$ is cocycle conjugate to a product type action as we will see in Example 6.13 in the next section. Thus $\alpha$ is invariantly approximately inner by Lemma 4.7.

(2)$\Rightarrow$(1). Suppose that the condition (2) holds. Then the dual flow $\hat{\alpha}$ has the Rohlin property by Theorem 4.11. Since $\hat{\alpha}$ preserves the trace on the type $\text{II}_\infty$ factor $\mathcal{N} := \mathcal{M} \rtimes_\alpha \mathbb{R}$, mod($\hat{\alpha}$) is trivial. Hence $\hat{\alpha}$ is cocycle conjugate to $\sigma \otimes \text{id}$ on $\mathcal{M} \otimes B(\ell^2)$, where $\sigma$ is the a product type action given in Example 6.13. Thus the bidual flow $\hat{\hat{\alpha}}$ is conjugate to $\hat{\sigma} \otimes \text{id}$.

By Takesaki duality, $\hat{\hat{\alpha}} \sim \alpha \otimes \text{id}$ on $\mathcal{M} \otimes B(\ell^2)$. Hence $\alpha \otimes \text{id} \sim \hat{\sigma} \otimes \text{id}$ on $\mathcal{M} \otimes B(\ell^2)$. Since $\sigma$ is invariantly approximately inner, $\hat{\sigma}$ has the Rohlin property by Theorem 4.11. Thus $\alpha$ has the Rohlin property by Lemma 2.8. Since both $\alpha \otimes \text{id}$ and $\hat{\alpha}$ are trace preserving Rohlin flows, they are cocycle conjugate. □

Since the tensor product flow of an arbitrary flow and a Rohlin flow has the Rohlin property, we see the following result.

Corollary 5.18. Let $\alpha$ and $\beta$ be flows on an injective factor $\mathcal{M}$ such that $\text{mod}(\alpha_t) = \text{mod}(\beta_t)$ for all $t \in \mathbb{R}$. Let $\sigma$ be a Rohlin flow on the injective factor $\mathcal{N}$ of type $\text{II}_1$. Then $\alpha \otimes \sigma$ is strongly cocycle conjugate to $\beta \otimes \sigma$.

6. APPLICATIONS

In this section, we discuss several applications of our classification result. Among them, we present a new proof of Kawahigashi’s work on flows on the injective type $\text{II}_1$ factor [30, 31, 32] and also that of classification of injective type III factors.

6.1. Classification of invariantly approximately inner flows. Let $\alpha, \beta$ be flows on an injective factor $\mathcal{M}$. Let $\mathcal{N}_1 := \mathcal{M} \rtimes_\alpha \mathbb{R}$ and $\mathcal{N}_2 := \mathcal{M} \rtimes_\beta \mathbb{R}$. We denote by $\{Z(\mathcal{N}_1), \theta^1, \text{mod}(\hat{\alpha})\}$ and $\{Z(\mathcal{N}_2), \theta^2, \text{mod}(\hat{\beta})\}$ the triples of the flow spaces, the flows of weights and the Connes-Takesaki modules for $\hat{\alpha}$ and $\hat{\beta}$, respectively.

Theorem 6.1. Let $\alpha, \beta$ be flows on an injective factor $\mathcal{M}$. Suppose that they are invariantly approximately inner. Then the following statements hold:

1. Two flows $\alpha$ and $\beta$ are stably conjugate if and only if the types (I, II and III) of $\mathcal{M} \rtimes_\alpha \mathbb{R}$ and $\mathcal{M} \rtimes_\beta \mathbb{R}$ are same, and the $\mathbb{R}^2$-actions $\theta^1_t \circ \text{mod}(\hat{\alpha}_t)$ and $\theta^2_t \circ \text{mod}(\hat{\beta}_t)$ on the corresponding flow spaces are conjugate:
(2) In (1), if one of the following conditions holds, then $\alpha$ and $\beta$ are cocycle conjugate:

- $\mathcal{M}$ is infinite;
- $\mathcal{M} \rtimes_\alpha \mathbb{R}$ and $\mathcal{M} \rtimes_\beta \mathbb{R}$ are not of type I.

Proof. (1). The “only if” part is clear. We will show the “if” part. Recall that the dual flows $\hat{\alpha}$ and $\hat{\beta}$ have the Rohlin property by Theorem 4.11. Since $\mathcal{M}$ is a factor, the dual flows are centrally ergodic. In particular, $\mathcal{N}_1$ and $\mathcal{N}_2$ are von Neumann algebras of type I, II, II$_{\infty}$ or III. To show the stable conjugacy, we may and do assume that $\mathcal{N}_1$ and $\mathcal{N}_2$ are properly infinite by considering $\alpha \otimes \text{id}_{B(\ell^2)}$ and $\beta \otimes \text{id}_{B(\ell^2)}$. Then their core von Neumann algebras are isomorphic.

Let $\Theta : Z(\tilde{\mathcal{N}_1}) \to Z(\tilde{\mathcal{N}_2})$ be an isomorphism which conjugates $\theta_1^t \circ \text{mod}(\hat{\alpha}_t)$ and $\theta_2^t \circ \text{mod}(\hat{\beta}_t)$. Then by Lemma 6.19 which will be proved later, there exist an isomorphism $\rho : \mathcal{N}_1 \to \mathcal{N}_2$ and a self-adjoint operator $h$ affiliated with $Z(\mathcal{N}_2)$ such that $\Theta = \theta_h^2 \circ \text{mod}(\rho)$. (See (6.4) for the definition of $\theta_h$.)

We set $\gamma_t := \rho \circ \alpha_t \circ \rho^{-1}$. Then

$$\text{mod}(\gamma_t) = \text{mod}(\rho) \circ \text{mod}(\hat{\alpha}_t) \circ \text{mod}(\rho)^{-1}$$

$$= \theta_{-h}^2 \circ \Theta \circ \text{mod}(\hat{\alpha}_t) \circ \Theta^{-1} \circ \theta_h^2$$

$$= \theta_{-h}^2 \circ \Theta \circ \text{mod}(\hat{\alpha}_t) \circ \Theta^{-1} \circ \theta_h^2$$

$$= \theta_{-h}^2 \circ \text{mod}(\hat{\beta}_t) \circ \theta_h^2$$

$$= \text{mod}(\hat{\beta}_t).$$

By Corollary 5.15, it turns out that $\gamma \sim \hat{\beta}$. Hence $\hat{\alpha} \sim \hat{\beta}$, and $\alpha \otimes \text{id}_{B(\ell^2)} \sim \beta \otimes \text{id}_{B(\ell^2)}$, by Takesaki duality.

(2). If $\mathcal{M}$ is infinite, then $\alpha \sim \beta$ as usual. Hence we suppose that $\mathcal{M}$ is of type II$_1$. Then $\mathcal{N}_1$ and $\mathcal{N}_2$ are of type II by our assumption. Since $\alpha$ and $\beta$ are stably conjugate, there exist an $\alpha \otimes \text{id}_{B(\ell^2)}$-cocycle $w$ and $\theta \in \text{Aut}(\mathcal{M} \otimes B(\ell^2))$ such that

$$\text{Ad} \, w_t \circ (\alpha_t \otimes \text{id}_{B(\ell^2)}) = \theta \circ (\beta_t \otimes \text{id}_{B(\ell^2)}) \circ \theta^{-1}.$$ 

It turns out from (1) that $\alpha$ and $\alpha \otimes \text{id}_\mathcal{M}$ are also stably conjugate. Thus there exist an $\alpha \otimes \text{id}_{B(\ell^2)}$-cocycle $v$ and an isomorphism $\theta_\alpha : \mathcal{M} \otimes \mathcal{M} \otimes B(\ell^2) \to \mathcal{M} \otimes B(\ell^2)$ such that

$$\text{Ad} \, v_t \circ (\alpha_t \otimes \text{id}_{B(\ell^2)}) = \theta_\alpha \circ (\alpha_t \otimes \text{id}_\mathcal{M} \otimes \text{id}_{B(\ell^2)}) \circ \theta^{-1}_\alpha.$$ 

$$\text{Ad} \, \theta^{-1}_\alpha (v^*_t) \circ (\alpha_t \otimes \text{id}_\mathcal{M} \otimes \text{id}_{B(\ell^2)}) = \theta^{-1}_\alpha \circ (\alpha_t \otimes \text{id}_{B(\ell^2)}) \circ \theta_\alpha.$$ 

We set

$$\theta_1 := \theta_\alpha \circ (\text{id}_\mathcal{M} \otimes \theta^{-1}) \circ \theta^{-1}_\alpha \circ \theta, \quad v'_t := \theta_1 (\theta^{-1}_\alpha (v^*_t)), \quad w'_t := \theta_1 (\theta^{-1}_1 (w_t)).$$
The $\theta_t$ is an automorphism on $M \otimes B(\ell^2)$ which satisfies $\text{mod}(\theta_t) = 1$. This is verified by computing its module with respect to a tracial weight. Then
\[
\theta_t \circ (\beta_t \otimes \text{id}_{B(\ell^2)}) \circ \theta_t^{-1} = \text{Ad} w'_t \circ \theta_t \circ (\text{id}_M \otimes \theta^{-1}) \circ \theta_t^{-1} \circ (\alpha_t \otimes \text{id}_{B(\ell^2)}) \circ \theta_t \circ (\text{id}_M \otimes \theta) \circ \theta_t^{-1}
\]
\[
= \text{Ad} (w'_t v'_t) \circ \theta_t \circ (\text{id}_M \otimes \theta^{-1}) \circ (\alpha_t \otimes \text{id}_M \otimes \text{id}_{B(\ell^2)}) \circ (\text{id}_M \otimes \theta) \circ \theta_t^{-1}
\]
\[
= \text{Ad} (w'_t v'_t) \circ \theta_t \circ (\alpha_t \otimes \text{id}_M \otimes \text{id}_{B(\ell^2)}) \circ \theta_t^{-1}
\]
\[
= \text{Ad} (w'_t v'_t) \circ (\alpha_t \otimes \text{id}_{B(\ell^2)}).
\]

Hence we may and do assume that $w$ and $\theta$ satisfy
\[
\text{Ad} w \circ (\alpha \otimes \text{id}_{B(\ell^2)}) = \theta \circ (\beta \otimes \text{id}_{B(\ell^2)}) \circ \theta^{-1}, \quad \text{mod}(\theta) = 1.
\]

By the latter condition, we can take a unitary $u \in M \otimes B(\ell^2)$ and $\sigma \in \text{Aut}(M)$ such that $\text{Ad} u \circ \theta = \sigma \otimes \text{id}_{B(\ell^2)}$. By setting $u_t := uw_t(\alpha_t \otimes \text{id}_{B(\ell^2)})(u^*)$, we have
\[
\text{Ad} u_t \circ (\alpha_t \otimes \text{id}_{B(\ell^2)}) = (\sigma \circ \beta_t \circ \sigma^{-1}) \otimes \text{id}_{B(\ell^2)}.
\]

From the above relation, $u_t \in M \otimes \mathbb{C}$ is clear, and we obtain
\[
\text{Ad} u_t \circ \alpha_t = \sigma \circ \beta_t \circ \sigma^{-1}.
\]

Therefore $\alpha$ and $\beta$ are cocycle conjugate. \qed

**Lemma 6.2.** Let $\theta$ be an automorphism on a von Neumann algebra $N$ and $\alpha$ an automorphism on the crossed product $N \rtimes_{\theta} \mathbb{Z}$. Suppose that $\theta$ is ergodic and faithful on $Z(N)$, and $\alpha = \text{id}$ on $N$. Then there exists a sequence of unitaries $(v_n)_n$ in $Z(N)$ such that $\alpha = \lim_{n \to \infty} \text{Ad} v_n$ in $\text{Aut}(M)$.

**Proof.** Put $\mathcal{A} := Z(N)$ and $U := \lambda^\theta(1)$, the implementing unitary of $M := N \rtimes_{\theta} \mathbb{Z}$. Since $Z(M)^\theta = \mathcal{A}^\theta = \mathbb{C}$, $\hat{\theta}$ is a centrally ergodic action of $T$. By [51, Corollary VI.1.3], we have $\mathcal{A}' \cap M = N$. In particular, $Z(M) = \mathcal{A}^\theta = \mathbb{C}$.

Then $c(m) := \alpha(U^m)U^{*m}$ belongs to $N' \cap M = \mathcal{A}$, which is a $\theta$-cocycle. We will show that $c$ is approximated by a coboundary.

Let $\varphi \in \mathcal{N}$ be a faithful state and $\hat{\varphi}$ the dual state on $M$. Note that $\hat{\varphi} \circ \alpha = \hat{\varphi}$ since $\alpha$ fixes $\mathcal{N}$. Let $n \in \mathbb{N}$ and $\varepsilon_n := 1/2n^2(2n + 1)$. Take $\delta_n > 0$ so that if $x \in \mathcal{A}_1$ satisfies $|x|_{\varphi} < \delta_n$, then $|\theta^k(x)|_{\varphi} < \varepsilon_n$ with $|k| \leq n$. Next take $N_n \in \mathbb{N}$ such that $12/N_n < \delta_n$.

By [51, Lemma 10], there exists a partition of unity $\{f\} \cup \{e_i\}_{i=0}^{N_n}$ in $\mathcal{A}$ such that $\theta(e_i) = e_{i+1}$, $i = 0, \ldots, N_n - 1$, $|f|_{\varphi} < 1/N_n$, $|e_0|_{\varphi} < 1/N_n$ and $|e_{N_n}|_{\varphi} < 2/N_n$. Using the following inequalities:
\[
\theta(e_{N_n}) \leq e_0 + f \leq \theta(e_{N_n}) + \theta(f) + f, \quad \theta(f) = f + e_0 - \theta(e_{N_n}),
\]
we get
\[
|\theta(e_{N_n})|_{\varphi} < 2/N_n, \quad |\theta(f)|_{\varphi} < 4/N_n, \quad |\theta^{-1}(e_0)|_{\varphi} < 3/N_n, \quad |\theta^{-1}(f)|_{\varphi} < 6/N_n.
\]
Set $v_n := f + \sum_{i=0}^{N_n} c(i)e_i \in \mathcal{A}$. Then $v_n$ is a unitary, and
\[
c(1)\theta(v_n) - v_n = c(1)\theta(f) + \sum_{i=0}^{N_n} c(1)\theta(c(i))\theta(e_i) - v_n
\]
\[
= c(1)\theta(f) + \sum_{i=0}^{N_n} (c(1+i))\theta(e_i) - v_n
\]
\[
= c(1)\theta(f) + \sum_{i=1}^{N_n} c(i)e_i + c(1+N_n)\theta(e_{N_n}) - v_n
\]
\[
= c(1)\theta(f) - f - c(0)e_0 + c(1+N_n)\theta(e_{N_n}).
\]

Hence we have
\[
|c(1)\theta(v_n) - v_n|_{\varphi} \leq |\theta(f)|_{\varphi} + |f|_{\varphi} + |e_0|_{\varphi} + |\theta(e_{N_n})|_{\varphi}
\]
\[
< 4/N_n + 1/N_n + 1/N_n + 2/N_n
\]
\[
= 8/N_n < \delta_n,
\]
and by $c(-1) = \theta^{-1}(c(1)^*)$,
\[
|c(-1)\theta^{-1}(v_n) - v_n|_{\varphi} = |c(1)\theta(v_n) - v_n|_{\theta(\varphi)}
\]
\[
\leq |\theta(f)|_{\theta(\varphi)} + |f|_{\theta(\varphi)} + |e_0|_{\theta(\varphi)} + |\theta(e_{N_n})|_{\theta(\varphi)}
\]
\[
= |f|_{\varphi} + |\theta^{-1}(f)|_{\varphi} + |\theta^{-1}(e_0)|_{\varphi} + |e_{N_n}|_{\varphi}
\]
\[
< 1/N_n + 6/N_n + 3/N_n + 2/N_n
\]
\[
= 12/N_n < \delta_n.
\]

Therefore, if $|k| \leq n$, then
\[
|\theta^k(c(1)\theta(v_n) - v_n)|_{\varphi} < \varepsilon_n,
\]
\[
|\theta^k(c(-1)\theta^{-1}(v_n) - v_n)|_{\varphi} < \varepsilon_n.
\]

We will prove the following inequality by induction:
\[
|c(k)\theta^k(v_n) - v_n|_{\varphi} < |k|\varepsilon_n \quad \text{if } |k| \leq n.
\]

We first consider when $k > 0$. Suppose that we have proved the inequality above for $k-1$. Using the cocycle identity $c(k) = c(k-1)\theta^{k-1}(c(1))$, we obtain the following:
\[
|c(k)\theta^k(v_n) - v_n|_{\varphi} = |c(k-1)\theta^{k-1}(c(1)\theta(v_n)) - v_n|_{\varphi}
\]
\[
\leq |\theta^{-1}(c(1)\theta(v_n) - v_n)|_{\varphi} + |c(k-1)\theta^{k-1}(v_n) - v_n|_{\varphi}
\]
\[
< \varepsilon_n + |c(k-1)\theta^{k-1}(v_n) - v_n|_{\varphi} \quad \text{by (6.1)}
\]
\[
< k\varepsilon_n.
\]

Next we consider when $k < 0$. Suppose that we have proved the inequality above for $k+1$. Using the cocycle identity $c(k) = c(k+1)\theta^{k+1}(c(-1))$, we obtain
the following:

\[ |c(k)\theta^k(v_n) - v_n|_\varphi = |c(k + 1)\theta^{k+1}(c(-1)\theta^{-1}(v_n)) - v_n|_\varphi \]
\[ \leq |\theta^{k+1}(c(-1)\theta^{-1}(v_n) - v_n)|_\varphi + |c(k + 1)\theta^{k+1}(v_n) - v_n|_\varphi \]
\[ < \varepsilon_n + |c(k + 1)\theta^{k+1}(v_n) - v_n|_\varphi \quad \text{by (6.2)} \]
\[ \leq -k\varepsilon_n. \]

Hence (6.3) holds.

We will prove that \( \alpha = \lim_n \text{Ad} v_n \) in \( \text{Aut}(M) \). Since \( \hat{\varphi} \circ \alpha = \hat{\varphi} \) and \( v_n \in M_{\hat{\varphi}} \), it suffices to show that \( \alpha(x) = \lim_n \text{Ad} v_n(x) \) in the strong topology for \( x = \sum_{|k| \leq \ell} x_k U^k \) with \( x_k \in N \). When \( n \geq \ell \), we obtain

\[ \|\alpha(x) - v_n x v_n^*\|_\varphi^2 = \left\| \sum_{|k| \leq \ell} x_k (c(k) - v_n \theta^k(v_n^*)) U^k \right\|_\varphi^2 \]
\[ < \sum_{|k| \leq \ell} \|x_k (c(k) - v_n \theta^k(v_n^*))\|_\varphi^2 \]
\[ \leq \sum_{|k| \leq \ell} \|x\|_2^2 \cdot 2|c(k) - v_n \theta^k(v_n^*)|_\varphi \]
\[ < \|x\|_2^2 \cdot 2(2\ell + 1)\varepsilon_n \quad \text{by (6.3)} \]
\[ \leq \|x\|_2^2 / n. \]

Thus \( \alpha(x) = \lim_n v_n x v_n^* \), and we are done. \( \square \)

**Lemma 6.3.** Let \( \alpha \) be a flow on an injective factor \( M \). Suppose that \( \alpha \) fixes a Cartan subalgebra \( A \) of \( M \). Then \( \alpha \) is invariantly approximately inner.

**Proof.** By [9, Theorem 10], we may and do assume that there exists a \( \mathbb{Z} \)-action \( \theta \) on \( A \) such that \( M = A \rtimes_{\theta} \mathbb{Z} \). The factoriality of \( M \) implies the ergodicity and the faithfulness of \( \theta \). Thus by the previous lemma, \( \alpha \) is invariantly approximately inner. \( \square \)

In particular, Kawahigashi’s example [33, Theorem 1.4] is invariantly approximately inner.

If a flow \( \alpha \) fixes a Cartan subalgebra of an injective type II\(_1\) factor and \( \Gamma(\alpha) = \mathbb{R} \), then \( \alpha \) has the Rohlin property by Corollary 5.17 and the previous lemma. This means the uniqueness of \( \alpha \). As a result, we have proved the following main result of [32]. See Example 6.13 for a product type flow with the Rohlin property.

**Theorem 6.4** (Kawahigashi). Let \( \alpha \) be a flow on the injective type II\(_1\) factor \( M \). If \( \alpha \) fixes a Cartan subalgebra of \( M \) and \( \Gamma(\alpha) = \mathbb{R} \), then \( \alpha \) is cocycle conjugate to a product type flow, and absorbs any product type actions. Thus such an action \( \alpha \) is unique up to cocycle conjugacy.

In [1, Proposition 6.5, Theorem 6.6], Aoi and Yamanouchi have generalized the above Kawahigashi’s result to the case of an action of a locally compact group on injective factors by groupoid method. We will prove their result for flows making use of a classification of Rohlin flows.
Corollary 6.5 (Aoi-Yamanouchi). Let \( \alpha, \beta \) be flows on an injective injective factor \( \mathcal{M} \). Suppose that they fix a Cartan subalgebra of \( \mathcal{M} \), and both \( \mathcal{M} \rtimes_\alpha \mathbb{R} \) and \( \mathcal{M} \rtimes_\beta \mathbb{R} \) are not of type I. Then \( \alpha \) and \( \beta \) are cocycle conjugate if and only if the \( \mathbb{R}^2 \)-actions \( \theta^1_\alpha \circ \text{mod}(\tilde{\alpha}) \) and \( \theta^2_\beta \circ \text{mod}(\tilde{\beta}) \) on the corresponding flow spaces are conjugate.

Proof. By Lemma 6.3, \( \alpha \) and \( \beta \) are invariantly approximately inner. When \( N_1 \) is of type III, so is \( N_2 \) since their flow spaces are isomorphic. Hence when \( N_1 \) is of type II, so is \( N_2 \). By Theorem 6.1, \( \alpha \sim \beta \). 

Remark 6.6. If \( N := \mathcal{M} \rtimes_\alpha \mathbb{R} \) is of type I, then \( \Gamma(\alpha) = \{0\} \). This fact is verified as follows. Take a type I subfactor \( \mathcal{P} \) in \( N \) such that \( \mathcal{P} \cap N = Z(N) \). Hence we have the tensor product decomposition \( N = \mathcal{P} \otimes Z(N) \). Let \( \gamma \) be the restriction of \( \tilde{\alpha} \) on \( Z(N) \). Then \( \gamma \) is an ergodic flow. Since \( (\text{id} \otimes \gamma_{-t}) \circ \tilde{\alpha} = \text{id} \) on \( Z(N) \), \( (\tilde{\alpha} \otimes \gamma_{-t}) \circ \tilde{\alpha} \) is inner. Take a measurable unitary map \( \mathbb{R} \ni t \to U_t \in N \) such that \( \tilde{\alpha} = \text{Ad} U_t \circ (\text{id} \otimes \gamma) \). Then \( c(t,s) := (\text{id} \otimes \gamma_t)(U_s^* U_t^* U_{t+s}) \) is a 2-cocycle of \( \gamma \) which belongs to \( \mathbb{C} \otimes Z(N) \). Thanks to [10] Proposition A.2, \( c \) is a coboundary. Hence we may and do assume that \( U \) is an \( (\text{id} \otimes \gamma) \)-cocycle. Then \( \mathcal{P} \otimes (Z(N) \rtimes_\gamma \mathbb{R}) \cong N \rtimes_\alpha \mathbb{R} \cong \mathcal{M} \otimes B(L^2(\mathbb{R})) \) that is a factor. In particular, \( \gamma \) is faithful, that is, \( \{0\} = \ker \gamma = \Gamma(\alpha) \).

By the remark above, we have interest in a classification of flows with trivial Connes spectrum.

Lemma 6.7. Let \( \mathcal{M} \) be a factor and \( \alpha \) a flow on \( \mathcal{M} \). If \( \Gamma(\alpha) = \{0\} \), then \( \alpha \) is invariantly approximately inner.

Proof. When \( \alpha \) is inner, \( \alpha \) is implemented by one-parameter unitary group as usual. Thus the statement is trivial.

Suppose that \( \alpha \) is not inner. Let \( N := \mathcal{M} \otimes B(L^2) \) and \( \tilde{\alpha} := \alpha \otimes \text{id}_{B(L^2)} \). By the same discussion as [4] §5.3, there exist a perturbation \( \sigma \) of \( \tilde{\alpha} \) and \( \mu > 0 \) such that \( 0 \in \text{Sp}(\sigma) \) is isolated and \( N = N^\sigma \rtimes_\gamma \mathbb{Z} \) whose implementing unitary \( \lambda^\gamma(1) \) satisfies \( \text{Sp}_\sigma(\lambda^\gamma(1)) \subset [\mu, \infty) \).

Claim. The \( Z(N^\sigma) \) is non-atomic, and \( \gamma \) is a faithful ergodic action on \( Z(N^\sigma) \).

Proof of Claim. The factoriality of \( N \) implies the central ergodicity of \( \gamma \). Note that \( Z(N^\sigma) \neq \mathbb{C} \). If so, then \( \Gamma(\sigma) = \text{Sp}(\sigma) \supset \text{Sp}_\sigma(\lambda^\gamma(1)) \neq \{0\} \). This is a contradiction. Assume that \( Z(N^\sigma) \) were atomic. Take a minimal projection \( e \in Z(N^\sigma) \). By [4] Lemme 2.3.3, \( \{0\} = \Gamma(\sigma) = \Gamma(\sigma^e) \). However, \( (N_e)^{\sigma^e} = (N^\sigma)_e \) is a factor, and \( \Gamma(\sigma^e) = \text{Sp}(\sigma^e) \). Hence \( \sigma^e \) is trivial. Since \( N \) is a factor, \( \sigma \) must be inner by [4] Lemme 1.5.2, and this is a contradiction. Therefore, \( Z(N^\sigma) \) is non-atomic.

Suppose that \( n > 0 \) is the period of \( \gamma \) on \( Z(N^\sigma) \). The ergodicity of \( \gamma \) implies that \( Z(N^\sigma) \cong (\mathbb{Z}/n\mathbb{Z})_{\infty} \), which is atomic. Hence \( \gamma \) must be faithful on \( N^\sigma \). 

By Lemma 6.2, we can deduce the invariantly approximate innerness of \( \sigma \) and that of \( \alpha \).
Remark 6.8. We can also prove the previous lemma in the following way. The condition $\Gamma(\alpha) = \{0\}$ means that $\hat{\alpha}$ is faithful on $Z(M \rtimes_\alpha \mathbb{R})$. If we use Theorem 6.21, then it turns out that $\hat{\alpha}$ has the Rohlin property. Thus $\alpha$ is invariantly approximately inner by Theorem 4.11.

By Theorem 6.11 and the previous lemma, we obtain the following result due to Kawahigashi for type II$_1$ case [30] Theorem 1.4 and Hui for type III case [22] Theorem 1.3.

Corollary 6.9. Let $\alpha, \beta$ be flows on the injective factor $M$. Suppose that $\Gamma(\alpha) = \{0\} = \Gamma(\beta)$. Then the following statements hold:

1. The $\alpha$ is stably conjugate to $\beta$ if and only if the types (I, II and III) of $M \rtimes_\alpha \mathbb{R}$ and $M \rtimes_\beta \mathbb{R}$ are same, and the $\mathbb{R}^2$-actions $\theta^t_\alpha \circ \text{mod}(\hat{\alpha}_t)$ and $\theta^t_\beta \circ \text{mod}(\hat{\beta}_t)$ are conjugate as before;
2. In (1), if one of the following conditions hold, then $\alpha$ and $\beta$ are cocycle conjugate:
   - $M$ is infinite;
   - $M \rtimes_\alpha \mathbb{R}$ and $M \rtimes_\beta \mathbb{R}$ are not of type I.

Example 6.10 (Kawahigashi). The difference between the cocycle conjugacy and the stable conjugacy occurs only when $M$ is of type II$_1$. We let $\alpha^{(k)}$ be the flow on the injective factor of type II$_1$ defined by

$$\alpha^{(k)}_{t} = \bigotimes_{n=1}^{\infty} \text{Ad} \begin{pmatrix} 1 & 0 \\ 0 & e^{2\pi i 3^n + kt} \end{pmatrix}. $$

They are mutually stably conjugate, but not cocycle conjugate. See [33] Theorem 2.9. In particular, their crossed products are of type I.

6.2. Examples of Rohlin flows on the injective factor of type II$_1$. We recall the notion of minimality (cf. [31]).

Definition 6.11. Let $\alpha$ be an action of a locally compact group $G$ on a factor $M$. We will say that $\alpha$ on $M$ is minimal if $\alpha$ is faithful and $(M^\alpha)' \cap M = \mathbb{C}$.

Theorem 6.12. Let $\alpha$ be an almost periodic and minimal flow on the injective type II$_1$ factor $M$. Then $\alpha$ has the Rohlin property.

Proof. Set $M(p) := \{x \in M \mid \alpha_t(x) = e^{ipt}x\}$, $p \in \mathbb{R}$, and $H := \text{Sp}_d(\alpha) = \{p \in \mathbb{R} \mid M(p) \neq \{0\}\}$. As shown in [62] Lemma 2.4, Proposition 7.3, the eigenspace $M(p)$ contains a unitary, and $H$ is a dense countable subgroup of $\mathbb{R}$. Take a unitary $v(p) \in M(p)$, and set $\gamma_p := \text{Ad} v(p)|_{M^\alpha}$, $c(p,q) := v(p)v(q)v(p+q)^*$. Then $(\gamma, c)$ is a free cocycle action on $M^\alpha$, where we regard $H$ as a discrete group. Since $(M^\alpha)' \cap M = \mathbb{C}$, $M^\alpha$ is an injective subfactor of type II$_1$. By the 2-cohomology vanishing theorem [52] Theorem 7.6, we may assume $v(p)$ is a unitary representation of $H$, and $\gamma$ is a free action of $H$ on $M^\alpha$. By [52] Lemma 9.2, there exists $\{u(p)_n\}_{n=1}^{\infty} \subset U(M^\alpha)$ for each $p \in H$ such that for all $p, q \in H$,

$$\gamma_p = \lim_{n \to \infty} \text{Ad} u(p)_n, \quad \lim_{n \to \infty} \|\gamma_p(u(q)_n) - u(q)_n\|_2 = 0.$$
In fact, we may assume \( \lim_{n \to \infty} \| u(p)_n u(q)_n - u(p + q)_n \|_2 = 0 \) holds, but this is unnecessary in this proof.

Fix \( p \in H \), and set \( w_n := u(p)_n^* v(p) \). Then trivially \( \alpha_t(w_n) = e^{ipt} w_n \), and \((w_n)_n\) is a central sequence as verified below. For \( x \in \mathcal{M}^\alpha \) and \( q \in H \), we have

\[
\| w_n x v(q) - x v(q) w_n \|_2 = \| u(p)_n^* v(p) x v(q) - x v(q) u(p)_n^* v(p) \|_2 \\
= \| u(p)_n^* \gamma_{q} v(p + q) - x \gamma_q u(p)_n^* v(p + q) \|_2 \\
= \| \gamma_{p}(x) - u(p)_n x \gamma_q u(p)_n^* \|_2 \\
\leq \| \gamma_{p}(x) - u(p)_n x u(p)_n^* \|_2 + \| x (\gamma_q u(p)_n) - u(p)_n^* \|_2 ,
\]

and the last terms converge to 0 by the choice of \( \{u(p)_n\} \). Since the linear span of \( \{\mathcal{M}^\alpha v(q)\}_{q \in H} \) is a strongly dense *-subalgebra of \( \mathcal{M} \), \((w_n)_n\) is central. Thus \( \pi_\omega((w_n)_n) \in \mathcal{M}_{\omega,\alpha} \) is a Rohlin unitary for \( p \in H \).

We next show the existence of a Rohlin unitary for an arbitrary \( p \in \mathbb{R} \). Take a strongly dense countable set \( \{x_j\}_{j=1}^\infty \subset \mathcal{M}_1 \).

For any \( n > 0 \), take \( q_n \in H \) such that \( |e^{ipt} - e^{iq_n t}| < 1/n, \ |t| \leq n \). Let \( w_n \in \mathcal{M} \) be a unitary such that

\[
\alpha_t(w_n) = e^{iq_n t} w_n , \quad \|[w_n, x_j]\|_2 < 1/n, \quad 1 \leq j \leq n.
\]

Then for \( |t| \leq n \),

\[
\| \alpha_t(w_n) - e^{ipt} w_n \|_2 < 1/n.
\]

Thus \( \pi_\omega((w_n)_n) \) is a Rohlin unitary for \( p \).

**Example 6.13.** Let \( \mu, \nu \in \mathbb{R} \setminus \{0\} \) be such that \( \mu/\nu \notin \mathbb{Q} \). Let \( \sigma \) be the flow on the injective type \( \mathrm{II}_1 \) factor \( \mathcal{M} \) as defined below

\[
\sigma_t := \bigotimes_{n=1}^\infty \text{Ad} \begin{pmatrix} 1 & 0 & 0 \\ 0 & e^{i\mu t} & 0 \\ 0 & 0 & e^{i\nu t} \end{pmatrix}.
\]

It is straightforward to check that \( \sigma \) satisfies the condition of Theorem 6.4 or Theorem 6.12. Indeed, \( \text{Sp}_d(\sigma) = \mu \mathbb{Z} + \nu \mathbb{Z} \). Hence \( \sigma \) has the Rohlin property. We can also show this fact from Corollary 7.3 without using the classification result of discrete amenable group actions due to Ocneanu.

By uniqueness of a Rohlin flow on the injective type \( \mathrm{II}_1 \) factor, any Rohlin flow is cocycle conjugate to \( \sigma \). We will study product type flows as above in more general setting in [6.5].

**Example 6.14.** Let \( \theta \in \mathbb{R} \setminus \mathbb{Q} \) and \( A_\theta \) be the irrational rotation \( C^* \)-algebra generated by \( u \) and \( v \) satisfying \( uv = e^{2\pi i \theta} vu \). For \( \mu, \nu \in \mathbb{R} \setminus \{0\} \), we introduce the ergodic flow \( \alpha \) on \( A_\theta \) defined by \( \alpha_t(u) = e^{i\mu t} u \) and \( \alpha_t(v) = e^{i\nu t} v \). If \( \mu/\nu \notin \mathbb{Q} \) and \( \mu/\nu \notin \text{GL}(2, \mathbb{Q}) \theta \), where each \( g \in \text{GL}(2, \mathbb{Q}) \) acts on \( \mathbb{R} \) as the linear fractional transformation, then \( \alpha \) is a Rohlin flow in the \( C^* \)-sense by [37], Proposition 2.5].

We lift \( \alpha \) to the weak closure of \( A_\theta \) with respect to the unique trace. Then we obtain a Rohlin flow on the injective type \( \mathrm{II}_1 \) factor. Hence Theorem 5.13 implies that the flow \( \sigma \) introduced in Example 6.14 and \( \alpha \) are mutually cocycle conjugate. This fact is originally proved by Kawahigashi [31], Theorem 16].
This example can be generalized to a higher-dimensional noncommutative torus [37, Proposition 2.5].

**Remark 6.15.** Let $A$ be a C*-algebra and $\pi: A \to B(H)$ be a representation. A central sequence $(x^\nu)_\nu$ in $A$ needs not to be central in $\pi(A)''$ in the von Neumann algebra sense. Indeed, let $\varphi \in A^*$ be a state and $\pi_\varphi: A \to B(H_\varphi)$ be the GNS representation. Suppose that the normal extension of $\varphi$ on $M := \pi_\varphi(A)''$ is faithful. Then $\{\varphi \pi_\varphi(a) \mid a \in A\}$ is a dense subspace of $M_*$. Let $(x^\nu)_\nu$ be a central sequence in $A$. Then we have

$$\|\varphi \pi_\varphi(a), \pi_\varphi(x^\nu)\|_{M_*} \leq \|a\| \|\varphi, \pi_\varphi(x^\nu)\|_{M_*} + \|[a, x^\nu]\|.$$  

By the Kaplansky density theorem, we have $\|\varphi(x^\nu)\|_{M_*} = \|\varphi(x^\nu)\|_{A^*}$. Therefore, $(\pi_\varphi(x^\nu))_\nu$ is central in $M$ if and only if $\lim_{\nu \to \infty} \|\varphi(x^\nu)\|_{A^*} = 0$.

### 6.3. The classification of injective factors of type III.

We recall the following fundamental result. A sketch of a proof is given in order that we can understand the outline.

**Theorem 6.16 (Connes, Haagerup, Krieger).** Let $M$ be an injective von Neumann algebra and $\varphi$ a faithful normal state on $M$. Then $\sigma_\varphi^t \in \text{Int}(M)$ for all $t \in \mathbb{R}$.

**Proof.** By Theorem 9.4 and the disintegration of $\sigma_\varphi$ as (9.2), we may and do assume that $M$ is a factor. The semifinite case is trivial.

If $M$ is of type III$_\lambda$ with $0 < \lambda < 1$, then there exists an automorphism $\theta$ on the injective type II$_1$ factor $N$ such that $M = N \rtimes_\theta \mathbb{Z}$ and $\tau \circ \theta = \lambda \tau$. Then by [6, Lemma 5, Theorem 1.2.5], $\theta$ has the Rohlin property as a $\mathbb{Z}$-action, and the dual action $\hat{\theta}_t = \sigma_\varphi^t$ is invariantly approximately inner as a torus action. This fact is proved in a similar way to Theorem 4.11.

By [5, Proposition 3.9], a modular automorphism is approximately inner for any factor of type III$_0$. We will prove this fact by using Rohlin flows in Corollary 6.24.

If $M$ is of type III$_1$, then the asymptotic bicentralizer of any faithful normal state is trivial [19, Corollary 2.4]. As is proved in [3, Theorem IV.1], the semidiscreteness implies the approximate innerness of a modular automorphism. □

Assuming this theorem, we present the classification of injective type III von Neumann algebras from a viewpoint of Rohlin property.

**Theorem 6.17 (Connes, Haagerup, Krieger).** Let $M_1$ and $M_2$ be injective von Neumann algebras of type III. Then they are isomorphic if and only if their flows of weights are isomorphic.

**Proof.** The “only if” part is clear, and we will show the “if” part. Let $M_1$ and $M_2$ as above. Let $\varphi_1$ and $\varphi_2$ be faithful normal states on $M_1$ and $M_2$, respectively. By our assumption and the uniqueness of the injective type II$_\infty$ factor, we may regard $\tilde{M}_1 = M_2$, and the dual flows $\tilde{\theta}_1 := \tilde{\sigma}_{\varphi_1}$ and $\tilde{\theta}_2 := \tilde{\sigma}_{\varphi_2}$ are equal on $Z(\tilde{M}_1)$.

By standard prescription (see the proof of Lemma 6.19), we may and do assume
that $\theta^1$ and $\theta^2$ are scaling the same trace $\tau$ as $\tau \circ \theta^1_t = e^{-t}\tau = \tau \circ \theta^2_t$. Thus $\text{mod}(\theta^1_t) = \text{mod}(\theta^2_t)$.

By Proposition 4.19 and the previous theorem, it turns out that $\theta^1$ and $\theta^2$ have the Rohlin property. Thus by Corollary 5.15, they are cocycle conjugate. In particular, $M_1 \rtimes_{\theta^1} \mathbb{R} \cong M_2 \rtimes_{\theta^2} \mathbb{R}$. Hence $M_1 \cong M_2$ by Takesaki duality. □

Let us denote by $R_0$, $R_{0,1}$ and $R_\infty$ the injective factors of type II$_1$, II$_\infty$ and III$_1$. Let us focus on the injective type III$_1$ factor $R_\infty$. The core of $R_\infty$ is isomorphic to $R_{0,1}$. The dual flow $\theta$ scales the trace $\tau$ as $\tau \circ \theta_t = e^{-t}\tau$. As we have seen, $\theta$ has the Rohlin property. We summarize this fact as follows.

**Theorem 6.18.** A trace scaling flow on $R_{0,1}$ has the Rohlin property.

**Proof.** Let $\theta$ be a trace scaling flow on $R_{0,1}$ and $\tau$ a faithful normal tracial weight on $R_{0,1}$. Then there exists a non-zero $p \in \mathbb{R}$ such that $\tau \circ \theta_t = e^{-pt}\tau$ for $t \in \mathbb{R}$. Using $\theta'_t := \theta_{t/p}$ if $p \neq 1$, we may and do assume $\tau \circ \theta_t = e^{-t}\tau$. We have known that $\theta$ has the Rohlin property. □

Let us refine Theorem 6.17 for later use (see Theorem 6.28). Let $M$ be a von Neumann algebra. For a self-adjoint operator $h$ affiliated with $Z(M)$, we define $\theta_h \in \text{Aut}(M)$ by

$$\theta_h(x) = x, \quad \theta_h(\lambda^\varphi(t)) = e^{-ih}\lambda^\varphi(t) \quad \text{for } x \in M, \ \varphi \in W(M), \ t \in \mathbb{R}. \quad (6.4)$$

**Lemma 6.19.** Let $P_1$ and $P_2$ be injective von Neumann algebras which are one of type I, II$_1$, II$_\infty$ or III. Let $\{Z(P_1), \theta^1\}$ and $\{Z(P_2), \theta^2\}$ be the flow of weights of $P_1$ and $P_2$, respectively. Suppose that $P_1 \cong P_2$ as von Neumann algebras and there exists an isomorphism $\Theta: Z(P_1) \to Z(P_2)$ with $\Theta \circ \theta^1_t = \theta^2_t \circ \Theta$ on $Z(P_1)$ for all $t \in \mathbb{R}$. Then there exist an isomorphism $\rho: P_1 \to P_2$ and a self-adjoint operator $h$ that is affiliated with $Z(P_2)$ such that $\Theta = \theta^2_h \circ \text{mod}(\rho) = \text{mod}(\rho) \circ \theta_{\rho^{-1}(h)}^2$.

**Proof.** By the assumption of the injectivity, there exist a semifinite injective factor $Q$, an abelian von Neumann algebra $A$ and isomorphisms $\pi_j: P_j \to Q \otimes A$ with $j = 1, 2$. We put $\psi^1_j := \pi_j \circ \theta^1_t \circ \pi_j^{-1}$. By Proposition 4.19 and Theorem 6.16 they are Rohlin flows.

Then $\Theta_0 := \pi_2 \circ \Theta \circ \pi_1^{-1}|_A$ is an automorphism on $A$ such that $\Theta_0 \circ \psi^1_t = \psi^2_t \circ \Theta_0$ on $A$. By replacing with $\pi_1$ with $(\text{id}_Q \otimes \Theta_0) \circ \pi_1$, we may and do assume that $\psi^1_t = \psi^2_t$ on $A$ and $\pi_2^{-1} \circ \pi_1|_{Z(P_1)} = \Theta$.

**Claim.** There exists $\gamma \in \text{Aut}(Q \otimes A)$ such that $\gamma|_A = \text{id}_A$ and $\gamma \circ \psi^1_t \circ \gamma^{-1} = \psi^2_t$ modulo $\text{Int}(Q \otimes A)$.

**Proof of Claim.** When $P_1$ is of type I, so is $Q$. Since $\psi^1_t \psi^2_{-t}|_A = \text{id}_A$ and $\text{Aut}(Q) = \text{Int}(Q)$, we can deduce $\psi^1_t \psi^2_{-t} \in \text{Int}(Q \otimes A)$ by Theorem 9.3.

When $P_1$ is of type II$_1$, so is $Q$. Since $\text{Aut}(Q) = \text{Int}(Q)$, $\psi^1_t \psi^2_{-t} \in \text{Int}(Q \otimes A)$ by Theorem 9.3.

When $P_2$ is of type II$_\infty$ or III, $Q$ is the injective factor of type II$_\infty$. Let $\tau$ be a faithful normal semifinite tracial weight on $Q$. Take $\varphi_j \in W(A)$ with $(\tau \otimes \varphi_j) \circ \psi^1_t = e^{-t}(\tau \otimes \varphi_j)$ for $j = 1, 2$. 

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Realize $\mathcal{A}$ as $L^\infty(X,\mu_2)$ where $\varphi_2(a) = \int_X a(x)\,d\mu_2(x)$. Let $h(x)$ be a positive measurable function such that $\varphi_1(a) = \int_X h(x)a(x)\,d\mu_2(x)$. Take $\gamma \in \text{Aut}(Q \otimes \mathcal{A})$ of the form $\gamma = \int_X \gamma_x\,d\mu_2(x)$ such that $\tau \circ \gamma_x = h(x)\tau$. Then $(\tau \otimes \varphi_2) \circ \gamma = \tau \otimes \varphi_1$, and

$$(\tau \otimes \varphi_2) \circ \gamma \circ \psi^1 \circ \gamma^{-1} = e^{-t}(\tau \otimes \varphi_2).$$

Since $\gamma \circ \psi^1 \circ \gamma^{-1} = \psi^2$ on $\mathcal{A}$, we obtain $\text{mod}(\gamma \circ \psi^1 \circ \gamma^{-1}) = \text{mod}(\psi^2)$. By Theorem 9.11, $\gamma \circ \psi^1 \circ \gamma^{-1} \circ \psi^2 \in \text{Int}(Q \otimes \mathcal{A})$. □

Replacing $\pi_1$ with $\gamma \circ \pi_1$, we may assume that $\psi^1 = \psi^2$ modulo $\text{Int}(Q \otimes \mathcal{A})$. Then by Theorem 5.15 there exist $\gamma' \in \text{Int}(Q \otimes \mathcal{A})$ and a $\psi^2$-cocycle $w$ such that $\gamma' \circ \psi^1 \circ \gamma'^{-1} = \text{Ad} w \circ \psi^2$. Then $\pi_2^{-1} \circ \gamma' \circ \pi_1 = \Theta$ on $Z(\mathcal{P}_1)$. Thus we can extend $\Theta$ to the isomorphism from $\overline{\mathcal{P}_1}$ onto $\overline{\mathcal{P}_2}$ by putting $\Theta := \pi_2^{-1} \circ \gamma' \circ \pi_1$ on $\overline{\mathcal{P}_1}$.

Put $v_t := \pi_2^{-1}(w_t)$. Then $v$ is a $\theta^2$-cocycle and we obtain $\Theta \circ \theta^1_t \circ \Theta^{-1} = \text{Ad} v_t \circ \theta^2_t$.

By stability of $\theta^2$ [10, Theorem III.5.1], there exists a unitary $w \in \overline{\mathcal{P}_2}$ such that $v_t = w^* \theta^2_t(w)$. Set $\Theta' := \text{Ad} w \circ \Theta$. Then $\Theta' = \Theta$ on $Z(\mathcal{P}_1)$ and $\Theta' \circ \theta^1_t \circ (\Theta')^{-1} = \theta^2_t$ for $t \in \mathbb{R}$. Thus we may and do assume that $\Theta \circ \theta^1_t \circ \Theta^{-1} = \theta^2_t$. This implies that $\Theta(\mathcal{P}_1) = \mathcal{P}_2$. Put $\rho := \Theta|_{\mathcal{P}_1}$.

Let $\varphi \in W(\mathcal{P}_1)$. Then $u_t := \Theta(\lambda^\varphi(t))\lambda^{\rho(\varphi)}(t)^*$ belongs to $\overline{\mathcal{P}_2} = \mathcal{P}_2$, and

$$u_t \sigma^\rho(\varphi)(u_s) = \Theta(\lambda^\varphi(t))\lambda^{\rho(\varphi)}(t)^* \cdot \lambda^{\rho(\varphi)}(t)\Theta(\lambda^\varphi(s))\lambda^{\rho(\varphi)}(s)^* \lambda^{\rho(\varphi)}(t)^*$$

$$= u_{t+s}.$$ 

Hence there exists $\psi \in W(\mathcal{P}_2)$ such that $u_t = [D\psi : D\rho(\varphi)]_t$. For $x \in \mathcal{P}_2$, and $t \in \mathbb{R}$, we have

$$\sigma^\varphi_t(x) = u_t \sigma^\rho(\varphi)(x)u^*_t$$

$$= \Theta(\lambda^\varphi(t))x\Theta(\lambda^\varphi(t)^*) = \rho(\sigma^\varphi_t(\rho^{-1}(x)))$$

$$= \sigma^{\rho(\varphi)}(x).$$

Thus $u_t \in Z(\mathcal{P}_2)$. Let $h$ be a positive operator affiliated with $Z(\mathcal{P}_2)$ such that $u_t = e^{ith}$. The $u$ does not depend on $\varphi$. Indeed, for another $\chi \in \mathcal{P}_1$, we have

$$\Theta(\lambda^\chi(t)) = \Theta([D\chi : D\varphi]_t, \lambda^\varphi(t))$$

$$= [D\rho(\chi) : D\rho(\varphi)]_t \cdot u_t \lambda^{\rho(\varphi)}(t)$$

$$= u_t \lambda^{\rho(\chi)}(t).$$

This shows $\Theta = \theta^2_h \circ \rho$. □

6.4. Non-fullness of type $\text{III}_0$ factors. In [2], it is shown that a modular automorphism group on any type $\text{III}_0$ factor $\mathcal{M}$ is approximately inner. This implies the non-fullness of an arbitrary type $\text{III}_0$ factors. If we apply Lemma 6.7 to the discrete decomposition of $\mathcal{M}$, then the invariant approximate innerness of $\sigma^\varphi$ is immediately obtained. We will present another approach to that by showing that any non-periodic ergodic flow on a commutative von Neumann algebra has the Rohlin property.

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Let \((X, \mu, \mathcal{F}_t)\) be a non-singular properly ergodic flow with \(\mu(X) = 1\). We can assume that \((X, \mu, \mathcal{F}_t)\) is given by a flow built under a ceiling function with a base space \((Y, \nu, \mathcal{B})\) and a positive function \(r(y)\) on \(Y\). We represent \(x \in X\) as \((\pi(x), h(x)) \in Y \times \mathbb{R}\) with \(0 \leq h(x) < r(\pi(x))\). We can assume that \(0 < R_1 \leq r(y) < R_2\) for some \(R_1, R_2 > 0\). (In fact, we can assume that \(r(y)\) takes only two values. See \([41, 55]\).) Let \(\mathbb{Z} \ltimes Y\) be the groupoid whose multiplication rule is given by \((n, S^m y)(m, y) := (n + m, y)\). For \((n, y) \in \mathbb{Z} \ltimes Y\), we set

\[
T(n, y) := \begin{cases} 
\sum_{k=0}^{n-1} r(S^k y) & \text{if } n \geq 1, \\
0 & \text{if } n = 0, \\
-T(-n, S^n y) & \text{if } n \leq -1.
\end{cases}
\]

Then \(T: \mathbb{Z} \ltimes Y \to \mathbb{R}\) is a homomorphism.

Next for \((t, x) \in \mathbb{R} \ltimes X\), we define \(N(t, x) = m\) if \(T(m, \pi(x)) \leq t + h(x) < T(m + 1, \pi(x))\), where we note that \(\lim_{n \to \infty} T(n, x) = +\infty\) since \(r(y) \geq R_1 > 0\). It turns out that \(N: \mathbb{R} \ltimes X \to \mathbb{Z}\) is a homomorphism. The maps \(T\) and \(N\) are related to each other as follows:

\[
S^{N(t,x)} \pi(x) = \pi(\mathcal{F}_t x),
\]

\[
t = T(N(t, x), \pi(x)) + h(\mathcal{F}_t x) - h(x) \quad \text{for all } (t, x) \in \mathbb{R} \ltimes X.
\]  

(6.5)

Define a homomorphism \(p: \mathbb{R} \ltimes X \to \mathbb{Z} \ltimes Y\) by \(p(t, x) = (N(t, x), \pi(x))\). This induces the group homomorphism \(p^*: Z^1(\mathbb{Z} \ltimes Y) \to Z^1(\mathbb{R} \ltimes X)\) by \(p^*(c) := c \circ p\), where each \(Z^1(\cdot)\) denotes the set of \(\mathbb{T}\)-valued cocycles. Let us denote by \(B^1(\cdot)\) the set of coboundaries. Then \(p^*(B^1(\mathbb{Z} \ltimes Y)) \subset B^1(\mathbb{R} \ltimes X)\). Hence the following group homomorphism is well-defined:

\[
p^*: H^1(\mathbb{Z} \ltimes X) \to H^1(\mathbb{R} \ltimes X).
\]  

(6.6)

In fact, \(p^*\) is an isomorphism though we do not use this fact in what follows. See \([10\text{, Proposition A.2}]\) or \([57\text{, Theorem 3.1}]\) for its proof.

For \(c_1, c_2 \in Z^1(\mathbb{R} \ltimes X)\) and \(d_1, d_2 \in Z^1(\mathbb{Z} \ltimes Y)\), we set the following metrics:

\[
\rho_X(c_1, c_2) := \max_{0 \leq t < R_1} \int_X |c(t, x) - c'(t, x)| \, d\mu(x),
\]

\[
\rho_Y(d_1, d_2) := \int_Y |d_1(1, y) - d_2(1, y)| \, d\nu(y).
\]

**Lemma 6.20.** For \(d_1, d_2 \in Z^1(\mathbb{Z} \ltimes Y)\), we have

\[
\rho_X(p^*(d_1), p^*(d_2)) \leq R_2 \rho_Y(d_1, d_2).
\]

In particular, \(p^*\) is continuous.

**Proof.** Note that \(N(t, x) \in \{0, 1\}\) for \(0 \leq t < R_1\). Indeed, since \(0 \leq h(x) < r(\pi(x))\), we obtain

\[
t \leq h(x) + t < r(\pi(x)) + t < r(\pi(x)) + r(S\pi(x)).
\]

If \(h(x) + t < r(\pi(x))\), then \(N(t, x) = 0\). If \(r(\pi(x)) \leq h(x) + t\), then \(N(t, x) = 1\).

Fix \(t\) with \(0 \leq t < R_1\). Let

\[
X_1 := \{x \in X \mid 0 \leq h(x) < r(x) - t\}, \quad X_2 := \{x \in X \mid r(x) - t \leq h(x) < r(x)\}.
\]

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Then for \( x_1 \in X_1 \) and \( x_2 \in X_2 \), we have \( N(t, x_1) = 0 \) and \( N(t, x_2) = 1 \). Hence \( p^*(d_1)(t, x_1) = 1 = p^*(d_2)(t, x_1) \). Then
\[
\int_X |p^*(d_1)(t, x) - p^*(d_2)(t, x)| \, d\mu(x) = \int_{X_2} |p^*(d_1)(t, x) - p^*(d_2)(t, x)| \, d\mu(x)
= \int_{X_2} |d_1(1, \pi(x)) - d_2(1, \pi(x))| \, d\mu(x)
\leq R_2 \int_Y |d_1(1, y) - d_2(1, y)| \, d\nu(y).
\]

\[\Box\]

**Theorem 6.21.** Let \( \mathcal{A} \) be an abelian von Neumann algebra and \( \alpha \) a non-periodic ergodic flow on \( \mathcal{A} \). Then \( \alpha \) has the Rohlin property.

*Proof.* If \( \text{Sp}_d(\alpha) = \mathbb{R} \), then \( \alpha \) is conjugate to the translation on \( L^\infty(\mathbb{R}) \). Thus the function \( t \mapsto e^{ipt} \) does the job.

We consider when \( \text{Sp}_d(\alpha) \neq \mathbb{R} \). Let \( (X, \mu, \mathcal{F}) \) be a point realization of \( \alpha \) with \( \mu(X) = 1 \). Note that \( \mathcal{F} \) is properly ergodic since \( \alpha \) is non-periodic. Then we represent \((X, \mu, \mathcal{F})\) as a flow built under a ceiling function \( r \) with a base space \((Y, \nu, S)\) as before. We let \( \theta(f)(y) := f(S^{-1}y) \) for \( f \in L^\infty(Y) \).

Let \( p \in \mathbb{R} \). For \((t, x) \in \mathbb{R} \times X\) and \((n, y) \in \mathbb{Z} \times Y\), we set \( c(t, x) := e^{ipt} \) and \( d(n, y) := e^{ipT(n, y)} \) which are cocycles of \( \mathbb{R} \times X \) and \( \mathbb{Z} \times Y \), respectively. Then for \((t, x) \in \mathbb{R} \times X\), we have
\[
p^*(d)(t, x) = d(N(t, x), \pi(x)) = e^{ipT(N(t, x), \pi(x))}
= e^{ip(t-h(Tx)+hx)} \quad \text{by (6.5)}
= c(t, x)e^{-ip(h(Tx))}e^{ip(x)}.
\]

This means \( p^*(d) = c \) in \( H^1(\mathbb{R} \times X) \).

We let \( u_n(y) := d(-n, y) \) for \( n \in \mathbb{Z} \) and \( y \in Y \). Then \( u : \mathbb{Z} \rightarrow L^\infty(Y)^U \) is a \( \theta \)-cocycle. By the proof of Lemma 6.2, for any \( \varepsilon > 0 \), there exists \( v \in L^\infty(Y) \) such that \( |u_k - v^*\theta^k(v)|_{\nu} < \varepsilon \) for \( k \) with \( |k| \leq 1 \).

Let \( v(y) \) be a bounded measurable function representing \( v \in L^\infty(Y) \). We set a coboundary \( \partial v \) on \( \mathbb{Z} \times Y \) defined by \( \partial v(n, y) := v(S^n y) v(y)^* \). Then
\[
\rho_Y(d, \partial v) = |u_{-1} - v^*\theta^{-1}(v)|_{\nu} < \varepsilon.
\]

By the previous lemma, we obtain
\[
\rho_X(p^*(d), p^*(\partial v)) \leq R_2\varepsilon.
\]

Therefore, we have proved that \( c \) is approximated by a coboundary. More precisely, for any \( n \in \mathbb{N} \), there exists a measurable function \( w_n : X \rightarrow \mathbb{T} \) such that
\[
\rho_X(c, \partial w_n) = \max_{0 \leq t < R_1} \int_X |e^{ipt} - w_n(x)^* \alpha_t(w_n)(x)| \, d\mu(x) < 1/n.
\]

Regarding \( w_n \) as a unitary in \( L^\infty(X, \mu) \), we obtain
\[
|\alpha_t(w_n) - e^{ipt} w_n|_{\mu} < 1/n \quad \text{if } 0 \leq t < R_1.
\]
Then $w := \pi_\omega((w_n)_n) \in \mathcal{A}_\omega$ satisfies $\alpha_t(w) = e^{ipt}w$ for all $t \in \mathbb{R}$. We will check that $(w_n)_n$ is $(\alpha, \omega)$-equicontinuous. Let $f(t) := R_1^{-1}e^{-ipt}1_{[0, R_1]}(t)$ and $w'_n := \alpha_f(w_n)$. Then

$$|\alpha_f(w_n) - w_n|_\mu \leq R_1^{-1} \int_0^{R_1} |\alpha_t(w_n) - e^{ipt}w_n|_\mu \, dt < 1/n\lambda_1.$$  

Thus $\alpha_f(w_n) - w_n \to 0$ in the strong* topology as $n \to \infty$. By Lemma 3.14, $w \in \mathcal{A}_{\alpha, \omega}$. \hfill \Box

**Remark 6.22.** By Theorem 5.11 any $\alpha$-cocycle is approximated by a coboundary. To see this fact, we can avoid the Shapiro type argument for the flow $\alpha$ when we use the fact that $p^*$ defined in (6.6) is isomorphism. The surjectivity of $p^*$ implies that any $\alpha$-cocycle $c$ may be assumed to be of the form $p^*(d)$ with $d$ a $\theta$-cocycle. As we have seen $d$ is approximated by a coboundary, and so is $c$.

**Corollary 6.23.** Let $\mathcal{N}$ be a von Neumann algebra and $\alpha$ a flow on $\mathcal{N}$. If $\alpha$ is non-periodic and ergodic on $\mathbb{Z}(\mathcal{N})$, then $\alpha$ has the Rohlin property.

Since the dual flow of a Rohlin flow is invariantly approximately inner by Theorem 4.11, we get the following result due to Connes [5].

**Corollary 6.24 (Connes).** Let $\mathcal{M}$ be a type III$_0$ factor. Then any modular automorphism group of $\mathcal{M}$ is approximately inner, and hence $\mathcal{M}$ is not a full factor.

### 6.5. Product type flows

We will generalize Example 6.13 to a factor of type III. Let $\lambda = (\lambda_1, \ldots, \lambda_m)$ with all $\lambda_j > 0$ and $\mu = (\mu_1, \ldots, \mu_m) \in \mathbb{R}^m$. We consider the following flow:

$$\mathcal{M}_\lambda := \bigotimes_{k=1}^{\infty} (M_{m+1}(\mathbb{C}), \phi_\lambda), \quad \phi_\lambda := \frac{1}{1 + \lambda_1 + \cdots + \lambda_m} \text{Tr} \cdot \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & \lambda_1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_m \end{pmatrix},$$  

$$\alpha_{\lambda, \mu}^t := \bigotimes_{k=1}^{\infty} \text{Ad} \left( \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & e^{\mu_1 t} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & e^{\mu_m t} \end{pmatrix} \right), \quad t \in \mathbb{R}.$$  

Let $\phi_\lambda = \bigotimes_{k=1}^{\infty} \phi_\lambda$. In what follows, we simply write $\alpha = \alpha_{\lambda, \mu}$, $\mathcal{M} = \mathcal{M}_\lambda$ and $\phi = \phi_\lambda$. It is trivial that $\alpha$ is invariantly approximately inner, and the dual flow has the Rohlin property. This fact will enable us to classify $\alpha$ in terms of $\lambda$ and $\mu$. Note that $(\mathcal{M}_\varphi^\alpha)' \cap \mathcal{M} = \mathbb{C}$ because the infinite symmetric group $\mathfrak{S}_\infty$ is represented into $(\mathcal{M}_\varphi^\alpha)'$ canonically.

The flow $\alpha$ fixes the diagonal Cartan subalgebra of $\mathcal{M}$, and we have already classified such flows in Theorem 6.1 for injective infinite factors. However, we can easily compute the flow of weights of $\mathcal{M} \rtimes_{\alpha} \mathbb{R}$ because of the minimality of $\alpha$, and we will classify $\alpha$ without use of the results obtained in 6.1.

By Lemma 2.3, we have the following identification:

$$\tilde{\mathcal{N}} = \tilde{\mathcal{M}} \rtimes_{\bar{\alpha}} \mathbb{R} = (\mathcal{M} \rtimes_{\sigma^\varphi} \mathbb{R}) \rtimes_{\bar{\alpha}} \mathbb{R}.$$
Note that $\tilde{\alpha}_t(\lambda^x(s)) = \lambda^x(s)$. Since $(M,\sigma)^{\alpha} \subset M$ has the trivial relative commutant, we have $\tilde{M}' \cap \tilde{N} \subset \mathbb{C} \otimes \{\lambda^x(\mathbb{R})\}'' \otimes \{\lambda^x(\mathbb{R})\}''$. From the picture of the product type flows, we can observe that $M$ is strongly densely spanned by eigenvectors for both $\sigma^x$ and $\alpha$. Take a non-zero element $x_j \in M$ such that $\sigma^x_j(\alpha_j(x_j)) = e^{i(\log \lambda_j s + \mu_j t)}x_j$. Then we have
\[
\pi_{\tilde{\alpha}}(\pi_{\sigma^x}(x_j)) = x_j \otimes e_{-\log \lambda_j} \otimes e_{-\mu_j},
\]
where $e_s(t) := e^{ist}$ for $s, t \in \mathbb{R}$. Hence we obtain the natural identification
\[
Z(\tilde{N}) = \tilde{M}' \cap \tilde{N} = (L(\mathbb{R}) \otimes L(\mathbb{R})) \cap \{e_{-\log \lambda_j} \otimes e_{-\mu_j} \mid j = 1, \ldots, m\}', \quad (6.7)
\]
where the flow of weights $\theta_s$ and the dual flow $\tilde{\alpha}_s$ are given by the restrictions of $\text{Ad} e_{-s} \otimes 1$ and $1 \otimes \text{Ad} e_{-t}$. By the Fourier transform, we have the isomorphism $L(\mathbb{R}) \otimes L(\mathbb{R}) \rightarrow L^\infty(\mathbb{R}) \otimes L^\infty(\mathbb{R})$ such that $\lambda^x(s) \otimes \lambda^x(t) \mapsto e_s \otimes e_t$. Then we have
\[
\tilde{M} \cap (\tilde{M} \rtimes G) = Z(\tilde{N}) = (L^\infty(\mathbb{R}) \otimes L^\infty(\mathbb{R})) \cap \{\lambda(\log \lambda_j) \otimes \lambda(\mu_j)\}'.
\]
The $\theta_i$ and $\tilde{\alpha}_s$ are transformed to $\text{Ad} \lambda(t) \otimes 1$ and $1 \otimes \text{Ad} \lambda(s)$, respectively. Note that $Z(\tilde{M}) = (\tilde{M}' \cap (\tilde{M} \rtimes G))^\hat{\alpha}$, and
\[
Z(\tilde{M}) \cong L^\infty(\mathbb{R}) \cap \{\lambda(\log \lambda_j)\}'.
\]
For $s_1, \ldots, s_k \in \mathbb{R}^2$, we denote by $\langle s_1, \ldots, s_k \rangle$ the closed subgroup generated by them. We put
\[
G_{\lambda,\mu} := \langle (\log \lambda_j, \mu_j), \mid j = 1, \ldots, m \rangle.
\]
Put $pr_1(x, y) := x$ and $pr_2(x, y) := y$. Then we have
\[
\overline{pr}_1(G_{\lambda,\mu}) = \Gamma(\sigma^x), \quad \overline{pr}_2(G_{\lambda,\mu}) = \Gamma(\alpha^x),
\]
where $\overline{pr}_j(G_{\lambda,\mu})$ denotes the closure of $pr_j(G_{\lambda,\mu})$.

**Theorem 6.25.** Let $M, \alpha := \alpha^x$ and $N := M \rtimes_{\alpha} \mathbb{R}$ as before. Let $(X_N, F^N)$ be the flow of weights of $N$. Then the following holds:

1. One has the identification $L^\infty(X_N) = L^\infty(\mathbb{R}^2)^G_{\lambda,\mu}$ that is the fixed point algebra of the translation action of $G_{\lambda,\mu}$ on $\mathbb{R}^2$.
2. The flow of weights $F^N$ is given by
\[
f(F^N_t(r, s)) = f(r + t, s) \quad \text{for } f \in L^\infty(X_N), \ r, s, t \in \mathbb{R};
\]
3. The Connes-Takesaki module of $\tilde{\alpha}_t$ is given by
\[
(\text{mod}(\tilde{\alpha}_t)f)(r, s) = f(r, s - t) \quad \text{for } f \in L^\infty(X_N), \ r, s, t \in \mathbb{R}.
\]

Thus we obtain the following characterization of the factoriality and the type of $N$. Remark 6.6 states that if $N$ were of type I, then $\Gamma(\alpha) = \{0\}$. Hence $\mu_j = 0$ for all $j$, and $N \cong M \otimes L^\infty(\mathbb{R})$ that is not of type I, and this is a contradiction. Thus $N$ must be a von Neumann algebra of type $\Pi_1$, $\Pi_\infty$ or III. When $N$ is a factor, its type is either $\Pi_\infty$ or III.

**Corollary 6.26.** The following statements hold:

1. $M \rtimes_{\alpha^x} \mathbb{R}$ is a factor if and only if $\overline{pr}_2(G_{\lambda,\mu}) = \mathbb{R}$;

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(2) $M_\lambda \times_{\alpha, \mu} \mathbb{R}$ is a factor of
- type II$_\infty$ if and only if $G_{\lambda, \mu} \cong \mathbb{R}$ and $G_{\lambda, \mu} \neq \mathbb{R} \times \{0\}$;
- type III$_1$ if and only if $G_{\lambda, \mu} = \mathbb{R}^2$;
- type III$_0$ if and only if $G_{\lambda, \mu} \cong \mathbb{Z}^2$ and $\overline{\text{pr}}_2(G_{\lambda, \mu}) = \mathbb{R}$;
- type III$_\rho$, $0 < \rho < 1$ if and only if $G_{\lambda, \mu} \cong \mathbb{R} \oplus \mathbb{Z}$ and $G_{\lambda, \mu} \neq \mathbb{R} \times \mathbb{Z}$.

Proof. We let $N := M_\lambda \times_{\alpha, \mu} \mathbb{R}$.

(1). This is because $\overline{\text{pr}}_2(G_{\lambda, \mu}) = \Gamma(\alpha)$, or $Z(N) = Z(\tilde{N})^\theta$.

(2). Recall that any closed subgroup in $\mathbb{R}^2$ is isomorphic to one of the following:

\[ \mathbb{R}^2, \mathbb{R} \times \mathbb{Z}, \mathbb{R}, \mathbb{Z}^2, \mathbb{Z}, 0. \]

The factoriality of $N$ excludes the cases $G_{\lambda, \mu} \cong \mathbb{Z}, 0$.

We know that $N$ is semifinite if and only if the flow of weights is the translation on $\mathbb{R}$, that is, $G_{\lambda, \mu} \cong \mathbb{R}$.

Since $N$ is a factor of type III$_1$ if and only if $L^\infty(X_N) = \mathbb{C}$, we have $G_{\lambda, \mu} = \mathbb{R}^2$.

Let us consider the case that $G_{\lambda, \mu} \cong \mathbb{Z}^2$. Then we can take a parallelogram as a fundamental domain of the action of $G_{\lambda, \mu}$ on $\mathbb{R}^2$. Then the flow of weights of $N$ must be non-periodic because of the ergodicity. Thus $N$ is of type III$_1$ if $N$ is a factor.

When $G_{\lambda, \mu} \cong \mathbb{R} \oplus \mathbb{Z}$, a fundamental domain is a segment. By the factoriality, we have $\overline{\text{pr}}_2(G_{\lambda, \mu}) = \mathbb{R}$, and $G_{\lambda, \mu} \neq \mathbb{R} \times \mathbb{Z}$. Then it is easy to see that the flow of weights of $N$ is periodic, that is, $N$ is of type III$_\rho$ with $0 < \rho < 1$. □

Example 6.27. We consider the case of $m = 2$. Then $G_{\lambda, \mu}$ is isomorphic to one of $\mathbb{R}$, $\mathbb{Z}$ and $\mathbb{Z}^2$. Hence if $N$ is a factor, then $N$ must be of type II$_\infty$ or III$_0$. The former comes from the modular flow $\sigma^a_{t\lambda}$ for some $a \in \mathbb{R}$. Let us consider the latter. Namely, $\mu_1, \mu_2 \neq 0$ and $\mu_1/\mu_2 \notin \mathbb{Q}$, and the vectors $v_1 := (\log \lambda_1, \mu_1)$ and $v_2 := (\log \lambda_2, \mu_2)$ are linearly independent. With this assumption, we obtain $(\lambda_1, \lambda_2) \neq (1, 1)$, that is, $M_\lambda$ is of type III. Then $G_{\lambda, \mu} = \mathbb{Z}v_1 + \mathbb{Z}v_2$ and $\overline{\text{pr}}_2(G_{\lambda, \mu}) = \mathbb{R}$. Denote by $\beta$ the dual flow of $\alpha^\lambda_{\mu}$.

We let $S^1_t(r, s) := (r + t, s)$ and $S^2_t(r, s) := (r, s + t)$ for $r, s, t \in \mathbb{R}$. Then $L^\infty(\mathbb{R})^{G_{\lambda, \mu}}$ is nothing but the fixed point algebra of $L^\infty(\mathbb{R})$ with respect to the transformations $S^1_{\log \lambda_1} S^2_{\mu_1}$, $j = 1, 2$. Recall that the flow of weights of $N$ and the Connes-Takesaki module of $\beta$ are given by $S^1_t$ and $S^2_t$ on $L^\infty(\mathbb{R})^{G_{\lambda, \mu}}$, respectively.

We pull back the flows $S^1$ and $S^2$ through the linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$, where

\[ T := \begin{pmatrix} \log \lambda_1 & \log \lambda_2 \\ \mu_1 & \mu_2 \end{pmatrix}. \]

Then we have

\[ T^{-1} \circ S^1_t \circ T \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix} + tT^{-1} \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad T^{-1} \circ S^2_t \circ T \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix} + tT^{-1} \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \]

Note that

\[ T^{-1} \circ S^1_{\log \lambda_1} S^2_{\mu_1} \circ T \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad T^{-1} \circ S^1_{\log \lambda_2} S^2_{\mu_2} \circ T \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \]
Thus the triple \((X_N, \mathcal{F}_t^N, \text{mod}(\beta_t))\) is conjugate to the Kronecker flow on \([0,1]^2\). More precisely, with this identification, we have

\[
\mathcal{F}_t^N \left( \begin{array}{c} x \\ y \end{array} \right) = \left( \begin{array}{c} x \\ y \end{array} \right) + \frac{t}{\Delta_{\lambda,\mu}} \left( \begin{array}{c} \mu_2 \\ -\mu_1 \end{array} \right), \quad \text{mod}(\beta_t) \left( \begin{array}{c} x \\ y \end{array} \right) = \left( \begin{array}{c} x \\ y \end{array} \right) + \frac{t}{\Delta_{\lambda,\mu}} \left( -\log \lambda_2, \log \lambda_1 \right),
\]

where \(\Delta_{\lambda,\mu} := \det(T) = \mu_2 \log \lambda_1 - \mu_1 \log \lambda_2\). This means that the vectors \(\lambda\) and \(\mu\) determine the directions of \(\text{mod}(\beta_t)\) and \(\mathcal{F}_t^N\), respectively. In particular, \(\text{mod}(\beta_t)\) is non-periodic if and only if \(\log \lambda_1 Z + \log \lambda_2 Z\) is dense in \(\mathbb{R}\), that is, \(M_\lambda\) is of type \(\text{III}_1\).

We will prove in Lemma 6.31 that the modular part of \(\alpha\) is equal to \(\text{pr}_2(G^1_{\lambda,\mu})\), where we denote by \(G^1_{\lambda,\mu}\) the annihilator group of \(G_{\lambda,\mu}\). Hence \((x, y) \in G^1_{\lambda,\mu}\) if and only if \((x, y) T \in (2\pi Z, 2\pi Z)\). This implies that \(\text{pr}_2(G^1_{\lambda,\mu}) = (2\pi/\Delta_{\lambda,\mu})(\log \lambda_1 Z + \log \lambda_2 Z)\). This is not equal to \(\mathbb{R}\). Hence \(\alpha\) is not extended modular.

Now we will prove that the closed subgroup \(G_{\lambda,\mu}\) is a complete invariant of the cocycle conjugacy class of \(\alpha_{\lambda,\mu}\).

**Theorem 6.28.** Let \(m, n \in \mathbb{N}\). Let \(\lambda \in \mathbb{R}^m\) and \(\rho \in \mathbb{R}^n\) be such that \(\lambda_j, \rho_k > 0\) for all \(j, k\). Let \(\mu \in \mathbb{R}^m\) and \(\nu \in \mathbb{R}^n\). Consider the flows \(\alpha_{\lambda,\mu}\) and \(\alpha_{\rho,\nu}\) on \(M_\lambda\) and \(M_\rho\), respectively. Then they are cocycle conjugate if and only if \(G_{\lambda,\mu} = G_{\rho,\nu}\).

**Proof.** Let us simply write \(\alpha_{\mu} := \alpha_{\lambda,\mu}\) and \(\alpha_{\nu} := \alpha_{\rho,\nu}\). Let \(\beta_{\mu} := \hat{\alpha}_{\lambda,\mu}\) and \(\beta_{\nu} := \hat{\alpha}_{\rho,\nu}\). We put \(\mathcal{P}_1 := M_\lambda \times \alpha_{\mu} \mathbb{R}\) and \(\mathcal{P}_2 := M_\rho \times \alpha_{\nu} \mathbb{R}\). Recall that we have the isomorphism \(Z(\mathcal{P}_1) \cong L^\infty(\mathbb{R}^2)^{G_{\lambda,\mu}}\) and \(Z(\mathcal{P}_2) \cong L^\infty(\mathbb{R}^2)^{G_{\rho,\nu}}\).

Suppose that \(\alpha_{\mu}\) and \(\alpha_{\nu}\) are cocycle conjugate flows, that is, there exist an \(\alpha_{\nu}\)-cocycle \(v\) and an isomorphism \(\phi: M_\lambda \to M_\rho\) such that \(\text{Ad } v(t) \circ \alpha_{\nu} = \phi \circ \alpha_{\mu} \circ \phi^{-1}\). Then \(\phi\) extends to the isomorphism \(\mathcal{P}_1 \to \mathcal{P}_2\) such that \(\phi(\pi_{\alpha_{\mu}}(x)) = \pi_{\alpha_{\nu}}(\phi(x))\) and \(\phi(\lambda_{\mu}(t)) = \pi_{\alpha_{\nu}}(v(t)) \lambda_{\nu}(t)\) for \(x \in M_\lambda\) and \(t \in \mathbb{R}\). Then \(\beta_{\nu} = \phi \circ \beta_{\mu} \circ \phi^{-1}\).

Let \(\tilde{\phi}: \mathcal{P}_1 \to \mathcal{P}_2\) be the canonical extension of \(\phi\). Let \(\theta^1\) and \(\theta^2\) be the dual flows on \(\mathcal{P}_1\) and \(\mathcal{P}_2\), respectively. Then \(\theta^1 = \tilde{\phi} \circ \theta^1 \circ \tilde{\phi}^{-1}\) and \(\theta^2 = \tilde{\phi} \circ \theta^2 \circ \tilde{\phi}^{-1}\). Hence we have an isomorphism \(L^\infty(\mathbb{R}^2)^{G_{\lambda,\mu}} \cong L^\infty(\mathbb{R}^2)^{G_{\rho,\nu}}\) preserving the translation of \(\mathbb{R}^2\). Thus \(G_{\lambda,\mu} = G_{\rho,\nu}\).

Next suppose that \(G_{\lambda,\mu} = G_{\rho,\nu}\). Then we have an isomorphism \(\sigma: Z(\mathcal{P}_1) \to Z(\mathcal{P}_2)\) preserving their flows of weights and the Connes-Takesaki modules of \(\beta_{\mu}\) and \(\beta_{\nu}\). Applying \(\text{pr}_j\) to \(G_{\lambda,\mu} = G_{\rho,\nu}\), we obtain \(\Gamma(\sigma^{\nu}) = \Gamma(\sigma^{\mu})\) and \(\Gamma(\alpha_{\mu}) = \Gamma(\alpha_{\nu})\). In particular, \(M_\lambda\) is semifinite if and only if \(\lambda_j = 1\) for all \(j\). In this case, \(\rho_j = 1\) for all \(j\), and \(M_\lambda \cong M_\rho\) is of type \(\text{II}_1\). When \(M_\lambda\) is of type \(\text{III}\), then \(M_\rho\) is the same type as \(M_\lambda\). Since they are not of type \(\text{III}_0\), \(M_\lambda \cong M_\rho\).

**Case 1.** \(\Gamma(\alpha_{\mu}) = \mathbb{R}\).

In this case, \(\Gamma(\alpha_{\nu}) = \mathbb{R}^2\). Thus \(\mathcal{P}_1\) and \(\mathcal{P}_2\) are factors. By Corollary 6.26 (2), \(\tilde{\mathcal{P}}_1\) and \(\tilde{\mathcal{P}}_2\) must be a von Neumann algebra of type \(\text{II}_\infty\). Hence \(\tilde{\mathcal{P}}_1 \cong \tilde{\mathcal{P}}_2\).

By Lemma 6.19 there exist \(s \in \mathbb{R}\) and an isomorphism \(\phi: \mathcal{P}_1 \to \mathcal{P}_2\) such that \(\sigma = \text{mod}(\phi) \circ \theta^1_s\).
Consider the flow \( \gamma_t := \phi \circ \beta_t^\mu \circ \phi^{-1} \) on \( \mathcal{P}_2 \). Then they satisfy
\[
\text{mod}(\gamma_t) = \text{mod}(\phi) \circ \text{mod}(\beta_t^\mu) \circ \text{mod}(\phi)^{-1}
\]
\[
= \sigma \circ \theta_{-\lambda}^1 \circ \text{mod}(\beta_t^\mu) \circ \theta_\lambda^1 \circ \sigma^{-1}
\]
\[
= \sigma \circ \text{mod}(\beta_t^\mu) \circ \sigma^{-1}
\]
\[
= \text{mod}(\beta_t^\nu).
\]

Since they have the Rohlin property, \( \gamma \sim \beta^\nu \) by Corollary 5.15. Hence by Takesaki duality, \( \alpha^\mu \otimes \text{id}_{B(L^2(\mathbb{R}))} \sim \alpha^\nu \otimes \text{id}_{B(L^2(\mathbb{R}))} \).

**Case 2.** \( \Gamma(\alpha^\mu) \neq \mathbb{R} \).

When \( \Gamma(\alpha^\mu) = \{0\} \), \( \mu_j = 0 = \nu_j \) for all \( j \). Thus there is nothing to prove because \( \alpha_t^\mu = \text{id}_{\mathcal{M}_\lambda} \) and \( \alpha_t^\nu = \text{id}_{\mathcal{M}_\mu} \).

We consider the case that \( \Gamma(\alpha^\mu) = \mathbb{Z} \) for some \( p > 0 \). Then \( \alpha^\mu \) and \( \alpha^\nu \) have the period \( T := 2\pi/p \). When we regard them as the actions of the torus \( \mathbb{R}/TZ \), we denote them by \( \gamma^1 \) and \( \gamma^2 \), respectively. Note that they are minimal.

Let \( \Omega_1 := \mathcal{M}_\lambda \rtimes_{\gamma_t^1} \mathbb{R}/TZ \) and \( \Omega_2 := \mathcal{M}_\mu \rtimes_{\gamma_t^2} \mathbb{R}/TZ \). Let \( \delta^1 \) and \( \delta^2 \) be the dual \( p\mathbb{Z} \)-actions of \( \gamma^1 \) and \( \gamma^2 \), respectively. By a similar computation to (6.7), we obtain the following natural isomorphism:
\[
Z(\widetilde{\Omega}_1) \cong L^\infty(\mathbb{R} \times p\mathbb{Z})^{G_{\lambda, \mu}}, \quad Z(\widetilde{\Omega}_2) \cong L^\infty(\mathbb{R} \times p\mathbb{Z})^{G_{\mu, \nu}},
\]
where the flows of weights act on the first coordinate, and the Connes-Takesaki modules of \( \delta^1 \) and \( \delta^2 \) do on the second. Define an isomorphism \( \sigma : \quad Z(\widetilde{\Omega}_1) \to Z(\widetilde{\Omega}_2) \) by this expression.

Since \( \widetilde{\Omega}_1 \) and \( \widetilde{\Omega}_2 \) must be of type \( \text{II}_\infty \), there exist an isomorphism \( \phi : \Omega_1 \to \Omega_2 \) and \( s \in \mathbb{R} \) such that \( \sigma = \text{mod}(\phi) \circ \theta_{s}^1 \) by Lemma 6.19 where \( \theta_s^1 \) denotes the flow of weights of \( \Omega_1 \). Putting \( \delta' := \phi \circ \delta^1 \circ \phi^{-1} \), we obtain \( \text{mod}(\delta') = \text{mod}(\delta^2) \).

We compute the modular invariants of \( \delta^1 \) and \( \delta^2 \). Suppose that \( \tilde{\delta}_n^1 = \text{Ad} u \) for some \( n \in \mathbb{Z} \) and \( u \in \widetilde{\Omega}_1^U \). Since \( \tilde{\delta}^1 \) fixes \( \widetilde{\mathcal{M}}_\lambda \) and \( \widetilde{\mathcal{M}}_\lambda \cap \widetilde{\Omega}_1 = Z(\widetilde{\Omega}_1) \), we must have \( u \in Z(\widetilde{\Omega}_1) \), that is, \( \tilde{\delta}_n^1 = \text{id}_{\widetilde{\Omega}_1} \). Thus \( n = 0 \). Likewise, it turns out that the modular part of \( \delta^2 \) is trivial. Hence \( \delta^1 \) and \( \delta^2 \) are centrally free.

Therefore, \( \delta^1 \sim \delta^2 \) as \( p\mathbb{Z} \)-actions by 20 Theorem 6.1 or 34 Theorem 20 (see 16 Theorem 3.1 for a simple proof). Thus \( \gamma^1 \otimes \text{id}_{B(\ell^2)} \sim \gamma^2 \otimes \text{id}_{B(\ell^2)} \) by Takesaki duality. This implies that \( \alpha^\mu \otimes \text{id}_{B(\ell^2)} \sim \alpha^\nu \otimes \text{id}_{B(\ell^2)} \) as \( \mathbb{R} \)-actions.

Hence in either case, \( \alpha^\mu \) is stably conjugate to \( \alpha^\nu \). If \( \mathcal{M} \) is infinite, this implies \( \alpha^\mu \sim \alpha^\nu \) as shown in Remark 2.2. When \( \mathcal{M}_\lambda \cong \mathcal{M}_\mu \) is finite, \( \{0\} = \Gamma(\sigma^{\nu}) = \text{Sp}(\sigma^{\mu}) \). Thus \( \lambda_j = 1 \) for all \( j \). Likewise, \( \rho_k = 1 \) for all \( k \). Then the condition \( G_{\lambda, \mu} = G_{\rho, \nu} \) implies that \( H := \langle \mu_j \mid j \rangle = \langle \nu_j \mid j \rangle \).

When \( H = T \mathbb{Z} \) for some \( T > 0 \), \( \alpha^\mu \) and \( \alpha^\nu \) have the period \( 2\pi/T \). Then they are regarded as minimal actions of the torus \( \mathbb{R}/(2\pi/T) \mathbb{Z} \) on the injective type \( \text{II}_1 \) factor. Hence \( \alpha^\mu \) is conjugate to \( \alpha^\nu \) by the uniqueness of a minimal action of the torus on the injective type \( \text{II}_1 \) factor.

When \( H = \mathbb{R} \), the both actions have the Rohlin property by Theorem 6.12 or 6.32. Therefore, they are cocycle conjugate by Corollary 5.16. \( \square \)
Next we describe the modular part \( \Lambda(\alpha^{\lambda \mu}) \) of \( \alpha := \alpha^{\lambda \mu} \) on \( \mathcal{M} := \mathcal{M}_\lambda \). By Lemma 6.7, \( \Lambda(\alpha^{\lambda \mu}) \) is a Borel subgroup of \( \mathbb{R} \) which consists of elements \( t \in \mathbb{R} \) such that \( \alpha_t \) are inner.

**Lemma 6.29.** One has \( \Lambda(\alpha) = \text{Sp}_d(\hat{\alpha}|_{Z(\tilde{\mathcal{M}} \ltimes \tilde{\alpha} \mathbb{R})}) \).

**Proof.** Let \( p \in \Lambda(\alpha) \). Take a unitary \( u_p \in \tilde{\mathcal{M}} \) such that \( \tilde{\alpha}_p = \text{Ad}(u_p) \). Then \( v_p := \pi_{\tilde{\alpha}}(u_p^* \lambda^{\tilde{\alpha}}(p)) \) belongs to \( \tilde{\mathcal{M}}' \cap (\tilde{\mathcal{M}} \ltimes \tilde{\alpha} \mathbb{R}) = Z(\tilde{\mathcal{M}} \ltimes \tilde{\alpha} \mathbb{R}) \), and \( \tilde{\alpha}_t(v_p) = e^{-ipt}v_p \). Thus \(-p\), and hence, \( p \) belong to \( \text{Sp}_d(\hat{\alpha}|_{Z(\tilde{\mathcal{M}} \ltimes \tilde{\alpha} \mathbb{R})}) \).

Suppose that \(-p \in \text{Sp}_d(\hat{\alpha}|_{Z(\tilde{\mathcal{M}} \ltimes \tilde{\alpha} \mathbb{R})}) \). Take a non-zero \( z \in Z(\tilde{\mathcal{M}} \ltimes \tilde{\alpha} \mathbb{R}) \) such that \( \tilde{\alpha}_t(z) = e^{-ipt}z \). We put \( a := z^* \lambda^{\tilde{\alpha}}(p) \). Then \( a \) is fixed by \( \tilde{\alpha} \), and \( a \in \pi_{\tilde{\alpha}}(\tilde{\mathcal{M}}) \). Let \( b \) be such that \( a = \pi_{\tilde{\alpha}}(b) \). For \( x \in \tilde{\mathcal{M}} \), we have \( bx = \tilde{\alpha}_p(x)b \). Then by ergodicity of \( \theta \) on \( Z(\tilde{\mathcal{M}}) \), we can take a unitary \( u \in \tilde{\mathcal{M}} \) such that \( ux = \tilde{\alpha}_p(x)u \) for all \( x \in \tilde{\mathcal{M}} \) (see the proof of [24, Proposition 3.4]). Hence \( p \in \Lambda(\alpha) \). \( \square \)

Let \( p \in \Lambda(\alpha) \) and take \( u_p, v_p \) as given in the proof above. Since \( Z(\tilde{\mathcal{M}} \ltimes \tilde{\alpha} \mathbb{R}) \) is included in \( \mathbb{C} \otimes L(\mathbb{R}) \otimes L(\mathbb{R}) \), there exists \( w_p \in L(\mathbb{R}) = \{ \lambda^{\tilde{\alpha}}(p) \}'' \) such that \( v_p = 1 \otimes w_p \otimes \lambda^{\tilde{\alpha}}(p) = \pi_{\tilde{\alpha}}(1 \otimes w_p) \lambda^{\tilde{\alpha}}(p) \). Thus \( u_p = 1 \otimes w_p^* \). In particular, \( \tilde{\alpha}_t(u_p) = u_p \).

Next we compute the modular invariant of \( u_p \). By (6.7), \( v_p \) commutes with \( 1 \otimes e(\log \lambda_j) \otimes e(\mu_j) \) for all \( j \). Hence \( \theta_{\log \lambda_j}(w_p) = e^{ip_{\mu_j}}w_p \).

Then we obtain the character \( \langle \log \lambda_j \mid j \rangle = \Gamma(\sigma^c) \ni t \mapsto \theta_t(w_p)w_p^* \in \mathbb{T} \). Thus there exists \( q \) such that \( e^{iq \log \lambda_j} = e^{ip_{\mu_j}} \) for all \( j \).

**Lemma 6.30.** An element \( p \in \mathbb{R} \) belongs to \( \Lambda(\alpha) \) if and only if there exists \( q \in \mathbb{R} \) such that \( e^{iq \log \lambda_j} = e^{ip_{\mu_j}} \) for all \( j = 1, \ldots, m \).

**Proof.** The “only if” part has been shown. We show the “if” part. Suppose that \( q \) satisfies \( e^{iq \log \lambda_j} = e^{ip_{\mu_j}} \) for all \( j = 1, \ldots, m \). Put \( u := \pi_{\tilde{\alpha}}(\lambda^c(q^*)^c \lambda^{\tilde{\alpha}}(p)) \). Then \( u \) is fixed by the action of \( G_{\lambda \mu} \), and it turns out that \( u \in Z(\mathcal{M} \ltimes \tilde{\alpha} \mathbb{R}) \). Thus we are done from the previous lemma. \( \square \)

We let \( G_{\lambda \mu}^\perp \) be the annihilator group of \( G_{\lambda \mu} \) with respect to the pairing of \( \mathbb{R}^2 \), that is, \( (a, b) \in G_{\lambda \mu}^\perp \) if and only if \( e^{i(ax+by)} = 1 \) for all \( (a, b) \in G_{\lambda \mu} \). Then the following lemma is an immediate consequence of the previous result.

**Lemma 6.31.** The modular part \( \Lambda(\alpha) \) coincides with \( \text{pr}_2(G_{\lambda \mu}^\perp) \).

Now we will characterize when \( \mu \) has the Rohlin property.

**Theorem 6.32.** Let \( \mathcal{M} := \mathcal{M}_\lambda \) and \( \alpha := \alpha^{\lambda \mu} \) as before. Then the following statements are equivalent:

1. The flow \( \alpha \) has the Rohlin property;
2. \( \tilde{\mathcal{M}}' \cap (\mathcal{M} \ltimes \tilde{\alpha} \mathbb{R}) = Z(\tilde{\mathcal{M}}) \);
3. \( \Gamma(\alpha) = \mathbb{R} \) and \( \alpha_t \notin \text{Cnt}((\mathcal{M}) \) for all \( t \neq 0 \);
Remark 6.33. □ and Corollary 5.17.

Lemma 6.34. Let $\tau$ be a minimal action of $\mathbb{T}$. Let $\gamma$ be the faithful normal conditional expectation. Suppose that $M\rtimes\gamma$ has the faithful normal tracial state $\tau$. Let $\varphi := \tau \circ E$. Then the following statements hold:

1. There exists $\lambda = (\lambda_1, \ldots, \lambda_m)$ such that $\lambda_j > 0$ and $\sigma_t^{\varphi} = \gamma(\exp^{i \log \lambda_1 t}, \ldots, \exp^{i \log \lambda_m t})$;
2. $\gamma(z_1, \ldots, z_m) \sim \gamma_{z_1}^{\lambda_1} \otimes \cdots \otimes \gamma_{z_m}^{\lambda_m}$, where the $\mathbb{T}$-action $\gamma_{z_j}^{\lambda_j}$ is defined as

$$M_{\lambda_j} := \bigotimes_{n=1}^{\infty} (M_2(\mathbb{C}), \phi_{j})''$$

$$\phi_{\lambda_j} := \frac{1}{1 + \lambda_j} \text{Tr} \begin{pmatrix} 1 & 0 \\ 0 & \lambda_j \end{pmatrix},$$

$$\gamma_{z_j}^{\lambda_j} = \bigotimes_{n=1}^{\infty} \text{Ad} \begin{pmatrix} 1 & 0 \\ 0 & z \end{pmatrix}.$$

Proof. (1). See [24], Proposition 5.2 (5).

(2). Let $N := M \rtimes_{\gamma} \mathbb{T}^m$. Then $\sigma_t^{\varphi}$ is implemented by $\lambda^{\gamma}((\exp^{i \log \lambda_1 t}, \ldots, \exp^{i \log \lambda_m t}))$. Thus $N$ is the injective factor of type $II_1$. Let $h$ be a positive operator affiliated with $N$ such that

$$h^{it} = \lambda^{\gamma}((\exp^{i \log \lambda_1 t}, \ldots, \exp^{i \log \lambda_m t})).$$
Then $\tau := \hat{\gamma}_{h^{-1}}$ is a faithful normal tracial weight on $\mathcal{N}$. We compute the module of $\hat{\gamma}$ as follows. Let $(k_1, \ldots, k_m) \in \mathbb{Z}^m$. By definition of the dual action, we get
\[
\hat{\gamma}(k_1, \ldots, k_m)(h^t) = e^{-i(k_1 \log \lambda_1 + \cdots + k_m \log \lambda_m) t} h^t,
\]
and
\[
\hat{\gamma}(k_1, \ldots, k_m)(h) = e^{-i(k_1 \log \lambda_1 + \cdots + k_m \log \lambda_m) \tau} = \lambda_1^{-k_1} \cdots \lambda_m^{-k_m} \tau.
\]
Hence
\[
\tau \circ \hat{\gamma}(k_1, \ldots, k_m) = \hat{\gamma}(-k_1, \ldots, -k_m)(h^{-1}) = e^{i(k_1 \log \lambda_1 + \cdots + k_m \log \lambda_m) \tau} = \lambda_1^{k_1} \cdots \lambda_m^{k_m} \tau.
\]
Thanks to [52, Theorem 2.9], we have the cocycle conjugacy as the $\mathbb{Z}^m$-actions:
\[
\hat{\gamma}(k_1, \ldots, k_m) \sim \theta_1^{k_1} \otimes \cdots \otimes \theta_m^{k_m}, \quad (k_1, \ldots, k_m) \in \mathbb{Z}^m,
\]
where $\theta_j$ is an aperiodic automorphism on the injective factor $\mathcal{R}_{0,1}$ such that $\tau_j \circ \theta_j = \lambda_j^{-1} \tau_j$ for a faithful semifinite normal trace $\tau_j$ on $\mathcal{R}_{0,1}$. By Takesaki duality, $\hat{\gamma} \sim \gamma \otimes \id_{B(\mathbb{F})}$. Thus we have the following conjugacy:
\[
\gamma(z_1, \ldots, z_m) \otimes \id_{B(\mathbb{F})} \approx \hat{\theta}_{z_1} \otimes \cdots \otimes \hat{\theta}_{z_m}, \quad (z_1, \ldots, z_m) \in \mathbb{T}^m.
\]
First we consider $j$ such that $\lambda_j \neq 1$. Let $\hat{\gamma}_j$ be the dual weight of $\tau_j$ on $\mathcal{R}_{0,1} \rtimes \theta_j \mathbb{Z}$. Then we have $(\hat{\theta}_j)^{\lambda_j \mu_j} = \sigma_t^{\lambda_j \mu_j}$, which is cocycle conjugate to $\sigma_t^{\lambda_j \mu_j}$ because $\mathcal{R}_{0,1} \rtimes \theta_j \mathbb{Z}$ is isomorphic to $\mathcal{M}_{\lambda_j}$. Thus $(\hat{\theta}_j)^{e^{it}} \sim \sigma_t^{\lambda_j \mu_j} = \gamma^{\lambda_j \mu_j}$.

Next we suppose that $\lambda_j = 1$. In this case, $\theta_j$ comes from an aperiodic automorphism on $\mathcal{R}_0$ that is unique up to cocycle conjugacy. Thus $\hat{\theta}_j \sim \beta \otimes \id_{B(\mathbb{F})}$, where $\beta$ is a minimal action of $\mathbb{T}$ on $\mathcal{R}_0$. By uniqueness of $\beta$, $\beta_z$ is conjugate to $\gamma_z^{\lambda_j}$. Hence $(\hat{\theta}_j)_z \sim \gamma_z^{\lambda_j} \otimes \id_{B(\mathbb{F})}$. Therefore,
\[
\gamma(z_1, \ldots, z_m) \otimes \id_{B(\mathbb{F})} \sim \gamma_{z_1}^{\lambda_j} \otimes \cdots \otimes \gamma_{z_m}^{\lambda_j} \otimes \id_{B(\mathbb{F})}. \quad (6.8)
\]
We will remove $\id_{B(\mathbb{F})}$ as follows. Recall that $\Gamma(\sigma_{\mathbb{F}}) = \Sp(\sigma_{\mathbb{F}}) = \langle \log \lambda_j \mid j \rangle$. Hence $\mathcal{M}$ is infinite, then $\lambda_j \neq 1$ for some $j$. By Remark 2.2 (to locally compact abelian groups), $\gamma \sim \gamma \otimes \id_{B(\mathbb{F})}$ and $\gamma_z^{\lambda_j} \sim \gamma_z^{\lambda_j} \otimes \id_{B(\mathbb{F})}$. Thus we are done.

Let us consider the case that $\mathcal{M}$ is finite, that is, $\lambda_j = 1$ for all $j$. Then $\gamma(z_1, \ldots, z_m)$ and $\gamma_{z_1}^{\lambda_j} \otimes \cdots \otimes \gamma_{z_m}^{\lambda_j}$ are minimal actions of $\mathbb{T}^m$ on $\mathcal{R}_0$, and they are conjugate. □

Let $\mathbf{\lambda}, \mathbf{\mu} \in \mathbb{R}^m$ as before. Regarding $\lambda_j, \mu_j \in \mathbb{R}^1$, we obtain the the flow $\alpha_{\lambda_j, \mu_j}$ as follows:

\[
\mathcal{M}_{\lambda_j} := \bigotimes_{n=1}^{\infty} (M_2(\mathbb{C}), \phi_{\lambda_j}^{n}), \quad \phi_{\lambda_j} := \frac{1}{1 + \lambda_j} \Tr \left( \begin{pmatrix} 1 & 0 \\ 0 & \lambda_j \end{pmatrix} \right),
\]

\[
\alpha_{\lambda_j, \mu_j} = \bigotimes_{n=1}^{\infty} \Ad \left( \begin{pmatrix} 1 & 0 \\ 0 & e^{i\mu_j t} \end{pmatrix} \right). \quad (77)
\]
Let us consider the gauge action $\gamma$ of $T^m$ on $M_\lambda$:

$$\gamma_z : = \bigotimes_{k=1}^{\infty} \text{Ad} \left( \begin{array}{cccc} 1 & 0 & \cdots & 0 \\ 0 & z_1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & z_m \end{array} \right), \quad (z_1, \ldots, z_m) \in T^m. \quad (6.9)$$

Then

$$\alpha^\lambda_\mu = \gamma(e^{i \lambda_1 \rho_1}, \ldots, e^{i \lambda_m \rho_m}), \quad \sigma^\phi_\lambda = \gamma(e^{i \log \lambda_1}, \ldots, e^{i \log \lambda_m}).$$

Employing Lemma 6.34, we obtain the following result.

**Theorem 6.35.** Let $\lambda, \mu$ be as before. Then $\alpha^\lambda_\mu \sim \alpha^\lambda_\mu \otimes \cdots \otimes \alpha^\lambda_\mu \otimes \mu$.

This implies the following result.

**Corollary 6.36.** Let $\lambda, \mu \in \mathbb{R}^m$ and $\rho, \nu \in \mathbb{R}^n$ with $\lambda_j > 0$ and $\rho_k > 0$ for all $j = 1, \ldots, m$ and $k = 1, \ldots, n$. Then we have $\alpha^{\lambda \otimes \mu \otimes \nu} \sim \alpha^\lambda_\mu \otimes \alpha^\rho_\nu$.

**Remark 6.37.** If we put $n = 1$, $\rho = 1$ and $\nu = 0$ in the above corollary, we have $\alpha^{\lambda \otimes \mu \otimes 0} \sim \alpha^\lambda_\mu \otimes \text{id}_{\mathbb{R}_0}$. Since $G_{\lambda \otimes \mu \otimes 0} = G_{\lambda, \mu}$, $\alpha^{\lambda \otimes \mu \otimes 0} \sim \alpha^\lambda_\mu$ by Theorem 6.28. Thus we have $\alpha^{\lambda \otimes \mu} \sim \alpha^\lambda_\mu \otimes \text{id}_{\mathbb{R}_0}$.

As an application of Theorem 6.32 and 6.35, we will give an example of Rohlin flows on the injective type III$_1$ factor. Let $\mathcal{P}$ be the injective type III$_1$ factor and $\varphi_1, \varphi_2 \in W(\mathcal{P})$. Let $\mathcal{M} := \mathcal{P} \otimes \mathcal{P}$. We study when the flow $\sigma^\varphi_{\mu t} \otimes \sigma^\varphi_{\nu t}$ on $\mathcal{M}$ has the Rohlin property for given $\mu$ and $\nu$.

Thanks to the Connes cocycle and the uniqueness of an injective type III$_1$ factor, we may and do assume that the both $\varphi_1$ and $\varphi_2$ are equal to the following product state $\chi = \phi_\lambda \otimes \phi_\rho$, where $\lambda, \rho$ satisfy $0 < \lambda, \rho < 1$, $\log \lambda / \log \rho \notin \mathbb{Q}$ and

$$\phi_\lambda := \bigotimes_{n=1}^{\infty} \frac{1}{1 + \lambda} \text{Tr} \cdot \left( \begin{array}{cc} 1 & 0 \\ 0 & \lambda \end{array} \right), \quad \phi_\rho := \bigotimes_{n=1}^{\infty} \frac{1}{1 + \rho} \text{Tr} \cdot \left( \begin{array}{cc} 1 & 0 \\ 0 & \rho \end{array} \right).$$

Then trivially we have

$$\sigma^\varphi_{\mu t} \otimes \sigma^\varphi_{\nu t} \sim \sigma^{\phi_\lambda}_{\mu t} \otimes \sigma^{\phi_\lambda}_{\nu t} \otimes \sigma^{\phi_\rho}_{\mu t} \otimes \sigma^{\phi_\rho}_{\nu t}.$$  

Thus letting

$$\lambda = (\lambda, \lambda, \rho, \rho), \quad \mu = (\mu \log \lambda, \nu \log \lambda, \mu \log \rho, \nu \log \rho),$$

we have $\sigma^\varphi_{\mu t} \otimes \sigma^\varphi_{\nu t} \sim \alpha^\lambda_\mu$. Then $G_{\lambda, \mu}$ is the closure of

$$(\mathbb{Z} \log \lambda + \mathbb{Z} \log \rho)(1, \mu) + (\mathbb{Z} \log \lambda + \mathbb{Z} \log \rho)(1, \nu).$$

Since $\log \lambda / \log \rho$ is irrational, $G_{\lambda, \mu}$ is the closure of $\mathbb{R}(1, \mu) + \mathbb{R}(1, \nu)$. Thus if $\mu \neq \nu$, then $G_{\lambda, \mu} = \mathbb{R}^2$, which is also equivalent to say $\{0\} \times \mathbb{R} \subset G_{\lambda, \mu}$.

**Proposition 6.38.** The $\sigma^\varphi_{\mu t} \otimes \sigma^\varphi_{\nu t}$ has the Rohlin property if and only if $\mu \neq \nu$. In this case, $\sigma^\varphi_{\mu t} \otimes \sigma^\varphi_{\nu t} \sim \alpha^0_{\nu} \otimes \text{id}_{\mathbb{R}_0}$, where $\alpha^0$ is a (unique) Rohlin flow on $\mathbb{R}_0$.

Therefore, for any $\varphi \in W(\mathbb{R}_0)$, $\sigma^\varphi_{\mu t} \otimes \text{id}_{\mathbb{R}_0}$ is a Rohlin flow unless $\mu = 0$ though this sounds a little strange since the modular flow is centrally trivial.
6.6. Quasi-free flows on Cuntz algebras. We recall basic facts on a Cuntz algebra and a quasi-free flow. Our standard references are [11, 14, 23].

Let \( 2 \leq n < \infty \) and \( \mathcal{O}_n \) the Cuntz algebra generated by isometries \( s_1, \ldots, s_n \) satisfying \( \sum_j s_j s_j^* = 1 \). Then \( \mathcal{O}_n \) admits the canonical action of the unitary group \( U(n) \), that is, each unitary \( u = (u_{ij})_{i,j} \in U(n) \) maps the generator \( s_k \) to \( \sum_j u_{jk} s_j \). We embed \( \mathbb{T}^n \) into \( U(n) \) diagonally. Denote by \( \gamma \) the action of \( \mathbb{T}^n \) on \( \mathcal{O}_n \), that is,

\[
\gamma(z_1, \ldots, z_n)(s_j) = z_j s_j, \quad j = 1, \ldots, n.
\]

We regard \( \mathbb{T} \) as a closed subgroup of \( \mathbb{T}^n \) via the map \( z \mapsto (z, \ldots, z) \). Denote by \( \mathcal{O}_{U(n)} \) and \( \mathcal{F}^n \) the fixed point algebras by \( U(n) \) and \( \mathbb{T} \), respectively. Let us denote by \( A_n \) the fixed point algebra \( (\mathcal{F}^n)\gamma \). Then it is trivial that

\[
\mathcal{O}_{U(n)} \subset A_n = \mathcal{O}_{\mathbb{T}}^\gamma \subset \mathcal{F}^n \subset \mathcal{O}_n.
\]

It is known that \( \mathcal{F}^n \) is canonically isomorphic to \( \bigotimes_n M_n(\mathbb{C}) \). Let us denote by \( F \) the conditional expectation from \( \mathcal{O}_n \) onto \( \mathcal{F}^n \) given by averaging the action of \( \mathbb{T} \). For \( \mu := (\mu_1, \ldots, \mu_n) \in \mathbb{R}^n \), we introduce the quasi-free flow \( \alpha^\mu \) as follows:

\[
\alpha^\mu(s_j) = e^{i\mu_j t} s_j, \quad j = 1, \ldots, n.
\]

Then we have \( \mathcal{O}_{U(n)} \subset \mathcal{O}_n^\mu \subset \mathcal{F}^n \). Put \( 1 := (1, \ldots, 1) \). Then \( \alpha^1 \) is nothing but the action of \( \mathbb{T} \) as stated above. Thus \( F(x) = \int_0^{2\pi} \alpha^1_t(x) dt \) for \( x \in \mathcal{O}_n \). Note that the restriction of \( \gamma(1, z_1, \ldots, z_{n-1}) \) on \( \mathcal{F}^n \) for \( z_j \in \mathbb{T} \) is of the form defined in (6.9).

By [14, Proposition 2.2] (and also [53, Theorem 2] in the case of \( \mu_j = 1 \)), \( \alpha^\mu \) has a KMS state if and only if \( \mu_i \mu_j > 0 \) for all \( i, j \). In fact, a KMS state \( \varphi^\mu \) and an inverse temperature \( \beta \in \mathbb{R} \) are unique, and given by

\[
\varphi^\mu = \psi^\mu \circ F, \quad \sum_{j=1}^n e^{-\beta \mu_j} = 1, \tag{6.10}
\]

where

\[
\psi^\mu := \bigotimes_{k=1}^\infty \text{Tr} \cdot \left( \begin{array}{ccccc} e^{-\beta \mu_1} & 0 & \cdots & 0 \\ 0 & e^{-\beta \mu_2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & e^{-\beta \mu_n} \end{array} \right).
\]

Let \( \pi_\mu : \mathcal{O}_n \to B(H_\mu) \) be the GNS representation with respect to the KMS state \( \varphi^\mu \). We simply write \( M \) and \( N \) for \( \pi_\mu(\mathcal{O}_n)'' \) and \( \pi_\mu(A_n)'' \), respectively.

The modular automorphism of \( \varphi^\mu \) is given by \( \sigma^\varphi_\mu = \alpha^{-\beta}\mu^\mu \). Hence we have \( \pi_\mu(\mathcal{O}_{U(n)})'' \subset N \subset M_{\varphi^\mu} \). Thanks to [23, Proposition 4.5], we have \( \pi_\mu(\mathcal{O}_{U(n)})'' \cap M = \mathbb{C} \). In particular, \( M \) is a type III factor, which is of type III_\lambda (0 < \lambda < 1) if \( \mu_i / \mu_j \in \mathbb{Q} \) for all \( i, j \), and of type III_1 otherwise [23, Theorem 4.7].

Since the \( \mathbb{T}^n \)-action \( \gamma \) preserves \( \varphi^\mu \), it extends to \( M \). Thus so does \( \alpha^\omega \) for any \( \omega \in \mathbb{R}^n \). Note \( N = M^\gamma \) and the following formulae:

\[
\alpha^\omega_t = \gamma(e^{i\omega_1 t}, \ldots, e^{i\omega_n t}), \quad \sigma^\varphi_\mu = \gamma(e^{-i\beta \mu_1 t}, \ldots, e^{-i\beta \mu_n t}).
\]
Since \( \pi_\mu(\mathcal{O}_{U(n)})' \cap M = \mathbb{C} \), \( \gamma \) is a minimal action of \( \mathbb{T}^n \) on an injective factor \( M \). Applying Lemma 6.34 to \( \varphi := \varphi_\mu \), we have

\[
\gamma(z_1, \ldots, z_n) \sim \gamma_1^{\lambda_1} \otimes \cdots \otimes \gamma_n^{\lambda_n},
\]

where \( \lambda_j := e^{-\beta \mu_j} < 1 \). Recall \( \alpha^t \omega \) defined in the previous subsection. Then by Theorem 6.35, we obtain

\[
\alpha_t^\omega = \gamma(e^{\omega_1}, \ldots, e^{\omega_n}) \sim \gamma_1^{\lambda_1} \otimes \cdots \otimes \gamma_n^{\lambda_n} = \alpha_t^{\lambda_1 \omega_1} \otimes \cdots \otimes \alpha_t^{\lambda_n \omega_n} \sim \alpha_t^\omega.
\]

Lemma 4.7 implies that \( \alpha^\omega \) is invariantly approximately inner.

We also get the following results by Corollary 6.26, Theorem 6.28 and Theorem 6.32 putting

\[
H_{\mu, \omega} := \langle (-\beta \mu_j, \omega_j) \mid j = 1, \ldots, n \rangle.
\]

Note that \( \beta \) depends on \( \mu \) as (6.10).

**Theorem 6.39.** Let \( \mu \in \mathbb{R}^m \) and \( \nu \in \mathbb{R}^n \) with \( \mu_i \mu_j > 0 \) for all \( i, j \) and \( \nu_k \nu_\ell > 0 \) for all \( k, \ell \). Let \( \omega \in \mathbb{R}^m \) and \( \eta \in \mathbb{R}^n \). Then the flows \( \alpha^\omega \) on \( \pi_\mu(\mathcal{O}_n)'' \) and \( \alpha^\eta \) on \( \pi_\nu(\mathcal{O}_n)'' \) are cocycle conjugate if and only if \( H_{\mu, \omega} = H_{\nu, \eta} \).

**Theorem 6.40.** Let \( \alpha^\omega \) be the quasi-free flow on \( M := \pi_\mu(\mathcal{O}_n)'' \) as before. Then the following conditions are equivalent:

1. \( \alpha^\omega \) has the Rohlin property;
2. \( \mathcal{M}' \cap \mathcal{M} \times_{\alpha^\omega} \mathbb{R} = \mathcal{Z}(\mathcal{M}) \);
3. \( \Gamma(\alpha^\omega) = \mathbb{R} \) and \( \alpha^\omega_t \notin \text{Cnt}(\mathcal{M}) \);
4. \( \{1\} \times \mathbb{R} \subset H_{\mu, \omega} \).

Moreover, \( \mathcal{M} \times_{\alpha^\omega} \mathbb{R} \) is an injective type III factor of the same type as \( \mathcal{M} \).

When \( n = 2 \), \( H_{\mu, \omega} \) never fulfills the fourth condition above. Thus any quasi-free flow on \( \pi_\mu(\mathcal{O}_2)'' \) does not have the Rohlin property.

**Example 6.41.** Put \( n = 3 \) and \( \mu_1 = \mu_2 = \mu_3 = 1 \). Then the inverse temperature \( \beta \) equals \( \log 3 \). Put \( \omega_1 = 1 \), \( \omega_2 = 2 \) and \( \omega_3 = \sqrt{2} \). The group \( G_{\mu, \omega} \) contains \((0, 1) = (-\beta \mu_2, \omega_2) - (-\beta \mu_1, \omega_1) \) and \((0, \sqrt{2}) = (-\beta \mu_3, \omega_3) - (-\beta \mu_1, \omega_1) \). Thus \( \alpha^\omega \) on \( \pi_\mu(\mathcal{O}_3)'' \) has the Rohlin property.

So far, we have obtained the classification of \( \alpha^\omega \) by using product type flows. Let us prove the invariant approximate innerness of \( \alpha^\omega \) in another way which is motivated by Kishimoto’s results [38 40]. In those works, he has shown, in the \( C^* \)-algebra level, that \( \alpha^\omega \) has the Rohlin property as a flow on \( \mathcal{O}_n \) if and only if the semigroup generated by \( \omega_j \), \( j = 1, \ldots, n \), is dense in \( \mathbb{R} \). In his proof, a sequence of endomorphisms \( \phi_k: \mathcal{O}_n \to \mathcal{O}_n \) plays a crucial role. Those are based on the 1-cocycle property of the shift automorphism on \( \bigotimes \mathcal{M}_n(\mathbb{C}) \), which is proved by deeply understanding its gauge invariant \( C^* \)-subalgebra.

However, it turns out not so involved to prove that the 1-cocycle property in the von Neumann algebra level. For this, we should understand how the shift endomorphism \( \sigma \) on \( \mathcal{F}^n = \bigotimes \mathcal{M}_n(\mathbb{C}) \) acts on \( \mathcal{N} \), and furthermore, the ultraproduct von Neumann algebras \( \mathcal{N}^\omega \) and \( \mathcal{N}_\omega \) (one should not confuse the vector \( \omega \) with a free ultrafilter \( \omega \)).

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By definition of $\sigma$, we have $\psi^\mu \circ \sigma = \psi^\mu$ and $\sigma(\gamma_z(x)) = \gamma_z(\sigma(x))$ for $z \in \mathbb{T}^n$ and $x \in \mathcal{F}^n$. Thus $\sigma$ extends to $\mathcal{R} := \pi_\mu(\mathcal{F}^n)''$, and $\sigma(\mathcal{N}) \subset \mathcal{N} \subset \mathcal{R}$.

Let us denote by $\mathcal{R}' \cap \mathcal{N}^\omega$ the subalgebra of $\mathcal{N}^\omega$ which consists of $\pi_\omega((x')_\nu) \in \mathcal{N}^\omega$ such that $\|yx^\nu - x^\nu y\|_{\psi_\mu} \to 0$ as $\nu \to \infty$ for all $y \in \mathcal{R}$. Since there exists a faithful normal conditional expectation from $\mathcal{R}$ onto $\mathcal{N}$ by averaging $\gamma$, we obtain $\mathcal{N}^\omega \subset \mathcal{R}'$, and $\mathcal{R}' \cap \mathcal{N}^\omega$ is a von Neumann subalgebra of $\mathcal{N}^\omega$. Moreover, $\psi^\mu$ is a trace on $\mathcal{N}$, and $\mathcal{R}' \cap \mathcal{N}^\omega \subset \mathcal{N}^\omega$.

**Lemma 6.42.** The $\mathcal{R}' \cap \mathcal{N}^\omega$ is a type II$_1$ factor.

**Proof.** Let $x$ be a non-zero element in $\mathcal{R}' \cap \mathcal{N}^\omega$ with $\tau_\omega(x) = 0$. We will show that $x$ is not central. We can take a representing sequence $(x')_\nu$ of $x$ and $W_m \in \omega$ for $m \in \mathbb{N}$ such that $(m, \infty) \supset W_m \supset W_{m+1}$, $x^\nu \in \bigotimes_{k=m}^{\infty} M_n(\mathbb{C})''$ for $\nu \in W_m$ and $\psi^\mu(x') = 0$.

Since $\psi^\mu$ is the tracial state on the type II$_1$ factor $\mathcal{N}$, there exists a sequence of unitaries $(u')_\nu$ in $\mathcal{N}$ such that $u^\nu \in \bigotimes_{k=m}^{\infty} M_n(\mathbb{C})''$ and $\|u^\nu, x^\nu\|_2 \geq \|x^\nu\|_2 / 2$. Putting $u := \pi_\omega((u')_\nu)$, we have $u \in \mathcal{R}' \cap \mathcal{N}^\omega$ and $ux \neq xu$. $\square$

**Lemma 6.43.** The $\sigma$ is an aperiodic automorphism on $\mathcal{R}' \cap \mathcal{N}^\omega$.

**Proof.** Let $\phi(x) := n^{-1} \sum_j s_j^* x s_j$ for $x \in \mathcal{M}$. Then $\phi \circ \sigma = \text{id}_{\mathcal{M}}$. Since $\phi(\mathcal{N}) \subset \mathcal{N}$, $\phi$ extends to $\mathcal{N}^\omega$ (see [17, Lemma 3.2]). Using $\mathcal{R} = \{s_i s_j^*\}_{i,j}'' \vee \sigma(\mathcal{R})$, we obtain $\sigma(\mathcal{R}' \cap \mathcal{N}^\omega) \subset \mathcal{R}' \cap \mathcal{N}^\omega$. Since $\phi(axb) = \phi(\sigma(a)x\sigma(b))$ for all $a, b \in \mathcal{M}$, we have $\phi(\mathcal{R}' \cap \mathcal{N}^\omega) \subset \mathcal{R}' \cap \mathcal{N}^\omega$. Take any $x = \pi_\omega((x')_\nu)$ and $y = \pi_\omega((y')_\nu)$ in $\mathcal{R}' \cap \mathcal{N}^\omega$.

Then in the strong* topology, we have

$$\phi(x') \phi(x') = \frac{1}{n^2} \sum_{i,j} s_i^* x y s_i^* s_j^* y s_j \to \phi(x x'), \ \nu \to \omega.$$ 

Hence $\phi(xy) = \phi(y)$, that is, $\phi$ is a faithful endomorphism on $\mathcal{R}' \cap \mathcal{N}^\omega$. Since $\phi \circ \sigma = \text{id}_{\mathcal{R}' \cap \mathcal{N}^\omega}$, $\phi$ is a surjection, and an automorphism. Thus so is $\sigma = \phi^{-1}$.

Suppose that for some non-zero element $a \in \mathcal{R}' \cap \mathcal{N}^\omega$ and $k \in \mathbb{N}$, we have $ax = \sigma^k(x)a$ for all $x \in \mathcal{R}' \cap \mathcal{N}^\omega$. Let us denote by $\{e_{i,j}^\nu\}_{i,j}$ a system of $n \times n$-matrix units in the $\nu$-th matrix algebra $M_n(\mathbb{C}) \subset \mathcal{F}^n$. We let $x := \pi_\omega((e_{11}^\nu)_\nu)$. Then $x \in \mathcal{R}' \cap \mathcal{N}^\omega$ and

$$\|\sigma^k(x) - x\|_2^2 = \lim_{\nu \to \omega} \left(\psi^\mu(e_{11}^k + e_{11}^\nu) - 2\psi^\mu(e_{11}^k e_{11}^\nu)\right) = 2e^{-\beta \mu_1}(1 - e^{-\beta \mu_1}) > 0. \quad (6.11)$$

Since $\mathcal{R}' \cap \mathcal{N}^\omega \subset \mathcal{R}_\omega$, we have a fast reindexation map $\Phi: \{\sigma^\ell(x)\}_\ell \to \{a\}' \cap \mathcal{R}_\omega$ such that $\Phi(\sigma(y)) = \sigma(\Phi(y))$ and $\tau^\omega(\Phi(\sigma(y))aa^*) = \tau^\omega(y)\tau^\omega(aa^*)$ for all $y \in \{\sigma^\ell(x)\}_\ell$. By constructing method of $\Phi$, it turns out that $\Phi(x) \in \mathcal{R}' \cap \mathcal{N}^\omega$. On the one hand, we have

$$\tau^\omega ((\Phi(x) - \sigma^k(\Phi(x)))^2 aa^*) = \tau^\omega ((\Phi(x) - \sigma^k(\Phi(x))) \cdot (\Phi(x) - \sigma^k(\Phi(x))) aa^*)$$

$$= \tau^\omega ((\Phi(x) - \sigma^k(\Phi(x))) \cdot (\Phi(x) - \sigma^k(\Phi(x))) a) a^*) = 0.$$
On the other hand,

\[ \tau^\omega \left( (\Phi(x) - \sigma^k(\Phi(x)))^2aa^* \right) = \tau^\omega \left( (|x - \sigma^k(x)|^2)aa^* \right) 
\]

\[ = \|x - \sigma^k(x)\|^2 \tau^\omega(aa^*) \]

\[ > 0 \quad \text{by (6.11).} \]

This is a contradiction. \qed

Thanks to [6, Theorem 1.2.5], we can prove the following 1-cohomology almost vanishing by Shapiro’s lemma (see Lemma 6.2).

**Lemma 6.44.** For any unitary \( v \in \mathbb{N} \) and \( \varepsilon > 0 \), there exists a unitary \( N \) such that \( \|v - w\sigma(w^*)\|_2 < \varepsilon \), where \( \| \cdot \|_2 = \| \cdot \|_{\psi^\omega} \).

This is the von Neumann algebra version of [40, Theorem 1.1 (1)]. We apply this not to the permutation unitary \( \sum_{i,j} s_is_js^*_i s^*_j \) but to \( u_t := \sum_j e^{i\omega_j} s_js^*_j \in N \).

**Lemma 6.45.** The flow \( \alpha^\omega \) is invariantly approximately inner on \( \mathcal{M} \).

*Proof.* Fix \( T \in \mathbb{R} \). By the previous lemma, we obtain a sequence of unitaries \( (w_k)_k \) in \( N \) such that \( \|u_T - w_k\sigma(w_k)^*\|_2 \to 0 \) as \( k \to \infty \). Then we have

\[ \alpha^\omega_T(s_j) = u Ts_j = \lim_{k \to \infty} w_k s_j w_k^* \]

in the strong* topology. Thus for all \( x \in \mathcal{O}_n \), we have \( \alpha^\omega_T(x) = \lim_{k \to \infty} w_k x w_k^* \).

Let \( a \in \mathcal{O}_n \). Since \( w_k \in \mathcal{N} \subset \mathcal{M}_\omega \), \( w_k(\varphi^\mu a)w_k^* = \varphi^\mu w_k aw_k^* \), which converges to \( \varphi^\mu \alpha^\omega_T(a) = \alpha^\omega_T(\varphi^\mu a) \) in the norm topology of \( \mathcal{M}_\omega \). Hence \( \lim_{k \to \infty} \text{Ad } w_k = \alpha^\omega_T \) in \( \text{Aut}(\mathcal{M}) \). Then the statement is clear because \( w_k \in \mathcal{N} \subset \mathcal{M}_\alpha^\omega \). \qed

With this fact, we can proceed to compute the flow of weights of \( \mathcal{M} \rtimes_\alpha^\omega \mathbb{R} \) as shown in the previous subsection, and obtain Theorem 6.39 and 6.40.

**Remark 6.46.** The pointwise approximate innerness of \( \alpha^\omega \) is easily verified. Indeed, if \( \mathcal{M} \) is of type III_1, then it is trivial because \( \text{Aut}(\mathcal{M}) = \text{Int}(\mathcal{M}) \). When \( \mathcal{M} \) is of type III_\lambda with \( 0 < \lambda < 1 \), then \( \varphi^\mu \) is the periodic state which is invariant under \( \alpha^\omega \). Thus we have \( \text{mod}(\alpha^\omega_T) = \text{id} \) for all \( t \in \mathbb{R} \).

**Remark 6.47.** If we apply the 1-cohomology almost vanishing to the permutation \( \sum_{i,j} s_is_js^*_i s^*_j \), then by the same argument as that of [40], we can show that there exist endomorphisms \( \{\rho_k\}_{k \in \mathbb{N}} \) on \( \mathcal{M} \) with the following conditions:

1. There exists a unitary \( u_k \in \mathcal{N} \) such that \( \rho_k(s_j) = u_k s_j \) for all \( j \);
2. \( \rho_k \circ \gamma_z = \gamma_z \circ \rho_k \) for all \( z \in \mathbb{T}^n \);
3. \( \varphi^\mu \circ \rho_k = \varphi^\mu \);
4. \( \lim_{k \to \infty} [\rho_k(x), y] = 0 \) in the strong* topology for all \( x, y \in \mathcal{M} \).

On the last condition, \( (\rho_k(x))_k \) is central in \( \mathcal{M} \) if and only if \( x \in \mathcal{M}_\varphi \). Indeed, since \( \rho_k(\mathcal{M}) \subset \mathcal{M} \) and \( \varphi^\mu \circ \rho_k = \varphi^\mu \), we have \( \|[\rho_k(x), \varphi^\mu]|| \leq ||[x, \varphi^\mu]|| \) for all \( k \).
7. A characterization of Rohlin property

The main purpose of this section is to show Theorem 7.8 which states that a flow \(\alpha\) on a factor \(M\) has the Rohlin property if and only if \(\alpha\) faithfully acts on \(M_{\omega,\alpha}\).

We begin with the following lemma.

**Lemma 7.1.** Let \(p \in \text{Sp}(\alpha|_{M_{\omega,\alpha}}), \) and \(\mu_1, \ldots, \mu_n\) be finite Borel measures on \(\mathbb{R}\). Then for any \(\varepsilon > 0\), there exists a non-zero partial isometry \(v \in M_{\omega,\alpha}\) such that

\[
\int_{\mathbb{R}} \left\| \alpha_t(v) - e^{i\mu t}v \right\|_2 d\mu_j(t) < \varepsilon \|v\|_2 \sum_{j=1}^m \mu_j(\mathbb{R}).
\]

**Proof.** Fix a small positive number \(\eta\) such that \(\eta < \min(\varepsilon^2, 1/400)\). Let \(\mu := \sum_{j=1}^n \mu_j\) and \(\nu := \mu(\mathbb{R})^{-1}\mu\). Note that \(\nu\) is a regular Borel measure with \(\nu(\mathbb{R}) = 1\).

Take \(R > 0\) so that \(\nu([-R, R]) \leq \eta/4\).

Since \(\alpha\) preserves the tracial state \(\tau_\omega\) on \(M_{\omega,\alpha}\), there exists a non-zero element \(x \in M_{\omega,\alpha}\) such that

\[
\|\alpha_t(x) - e^{i\mu t}x\|_2 \leq \eta \|x\|_2/2 \quad \text{for all} \quad t \in [-R, R].
\]

Then

\[
\int_{\mathbb{R}} \left\| \alpha_t(x) - e^{i\mu t}x \right\|_2 d\nu(t) = \int_{-R}^R \left\| \alpha_t(x) - e^{i\mu t}x \right\|_2 d\nu(t)
\]

\[
+ \int_{R([-R, R])} \left\| \alpha_t(x) - e^{i\mu t}x \right\|_2 d\nu(t)
\]

\[
\leq \eta \|x\|_2/2 \cdot \nu([-R, R]) + 2 \|x\|_2 \cdot \nu([-R, R])
\]

\[
\leq \eta \|x\|_2.
\]

Since \(M_{\omega,\alpha}\) is finite, we can take a unitary \(w \in M_{\omega,\alpha}\) with \(x = w|x|\). Put \(y_t := e^{-i\mu t}w^*\alpha_t(x)\). Then we have

\[
\|y_t - x\|_2 = \|y_t - |x|\|_2 = \|\alpha_t(x) - e^{i\mu t}x\|_2.
\]

By the Powers-Størmer inequality, we have

\[
\|\alpha_t(|x|) - |x|\|_2^2 \leq 2\|y_t - |x||_2\|_2 x_2.
\]

Thus

\[
\int_{\mathbb{R}} \left\| \alpha_t(|x|) - |x| \right\|_2^2 d\nu(t) \leq \int_{\mathbb{R}} 2\|y_t - |x||_2\|_2 x_2 d\nu(t)
\]

\[
\leq 2\eta \|x\|_2.
\]

Next we have

\[
\int_{\mathbb{R}} \|y_t - y_t\|_2^2 d\nu(t) \leq \int_{\mathbb{R}} 2\|y_t - |x||_2^2 d\nu(t) + \int_{\mathbb{R}} 2\||x| - \alpha_t(|x|)||_2^2 d\nu(t)
\]

\[
\leq 4\|x\|_2 \cdot \eta \|x\|_2 + 2 \cdot 2\eta \|x\|_2^2
\]

\[
= 8\eta \|x\|_2^2.
\]
Using Fubini’s theorem and [4, Lemma 1.2.5, 1.2.6], we have
\[
\int_{\mathbb{R}^+} \left( \int_{\mathbb{R}} \| u_{\sqrt{\sigma}}(y_t) - u_{\sqrt{\sigma}}(|x|) \|_2^2 \, d\nu(t) \right) \, da = \int_{\mathbb{R}} \| y_t - |y_t| \|_2^2 \, d\nu(t) \leq 8\eta \| x \|_2^2,
\]
and
\[
\int_{\mathbb{R}^+} \left( \int_{\mathbb{R}} \| u_{\sqrt{\sigma}}(|y_t|) - u_{\sqrt{\sigma}}(|x|) \|_2^2 \, d\nu(t) \right) \, da \leq \int_{\mathbb{R}} \| y_t - |x| \|_2^2 \, d\nu(t) + |x|_2 \, d\nu(t)
\leq 2\eta \| x \|_2^2.
\]
Thus we obtain
\[
\int_{\mathbb{R}^+} \left( \int_{\mathbb{R}} \| u_{\sqrt{\sigma}}(y_t) - u_{\sqrt{\sigma}}(|x|) \|_2^2 \, d\nu(t) \right) \, da \leq 2(8\eta + 2\eta) \| x \|_2^2 = 20\eta \| x \|_2^2.
\]
Now let \( G(a) = \| u_{\sqrt{\sigma}}(|x|) \|_2^2 \). Then \( \int_{\mathbb{R}^+} G(a) \, da = \| x \|_2^2 \). We let
\[
A := \left\{ b > 0 \left| \int_{\mathbb{R}} \| u_{\sqrt{\sigma}}(y_t) - u_{\sqrt{\sigma}}(|x|) \|_2^2 \, d\nu(t) > \eta^{1/2} \| u_{\sqrt{\sigma}}(|x|) \|_2^2 \right. \right\}.
\]
Then
\[
\int_A G(a) \, da < \eta^{-1/2} \cdot 20\eta \| x \|_2^2 = 20\eta^{1/2} \| x \|_2^2.
\]
Since the measure \( G(a) \, da \) is normal, we can take an open set \( U \subset \mathbb{R}^+ \) such that \( 0 \in U \), \( A \subset U \) and \( \int_{U \cap \mathbb{R}^+} G(a) \, da < 20\eta^{1/2} \| x \|_2^2 \). Take the smallest \( b > 0 \) such that \( b \in U^c \) satisfies \((0, b) \subset U \). Then
\[
\int_{(0,b)} G(a) \, da \leq \int_U G(a) \, da < 20\eta^{1/2} \| x \|_2^2 < \| x \|_2^2.
\]
Hence \( b < \infty \). Then we have
\[
\int_{\mathbb{R}} \| u_{\sqrt{\sigma}}(y_t) - u_{\sqrt{\sigma}}(|x|) \|_2^2 \, d\nu(t) \leq \eta^{1/2} \| u_{\sqrt{\sigma}}(|x|) \|_2^2,
\]
and
\[
\| x - u_{\sqrt{\sigma}}(x) |x| \|_2^2 = \tau(|x|^2(1 - E_{\theta}(|x|^2))) \leq \int_0^b \tau(E_{\theta}(|x|^2)) \, ds
\]
\[
= \int_0^b G(s) \, ds < 20\eta^{1/2} \| x \|_2^2
\]
\[
< \| x \|_2^2.
\]
Hence \( v := u_{\sqrt{\sigma}}(x) \) is a non-zero partial isometry. Trivially, \( u_{\sqrt{\sigma}}(y_t) = \alpha_t(v) \), and we are done.

We consider the standard Hilbert space \( H \otimes L^2(\mathbb{R}) \) of the crossed product \( M \rtimes_\alpha \mathbb{R} \). Recall that for \( x \in M \), the right action of \( \pi_\alpha(x) \) on this Hilbert space is nothing but \( Jx^*J \otimes 1 \). Hence
\[
(\zeta \otimes f) \pi_\alpha(x) = \zeta x \otimes f \quad \text{for all } \zeta \in H, \ f \in L^2(\mathbb{R}).
\]
Note that the one-parameter unitary group associated with \( \hat{\alpha} \) is \( 1 \otimes e_\cdot \), that is,
\[
\hat{\alpha}_p(\xi) = (1 \otimes e_-p)\xi \quad \text{for all } p \in \mathbb{R}, \; \xi \in H \otimes L^2(\mathbb{R}).
\]

**Lemma 7.2.** Let \( p \in \text{Sp}(\alpha|_{M_\omega,p}) \) and \( f \in M^p \). For any \( \varepsilon > 0 \) and \( \xi_1, \ldots, \xi_n \in \pi_\alpha(f)(H \otimes L^2(\mathbb{R}))\pi_\alpha(f) \), there exists a non-zero \( x \in M_f \) such that
\[
\|\pi_\alpha(x)\xi_j - \hat{\alpha}_p(\xi_j)\pi_\alpha(x)\|^2 \leq \varepsilon \sum_{j=1}^n \|\pi_\alpha(x)\xi_j\|^2 \quad \text{for all } j = 1, \ldots, n.
\]

**Proof.** Note that for \( \xi \in H \otimes L^2(\mathbb{R}) \), the functionals \( M \ni x \mapsto \langle \pi_\alpha(x)\xi, \xi \rangle \) are normal. Hence for \( x = \pi_\omega((x^\nu)_\nu) \in M_\omega \), we obtain
\[
\lim_{\nu \to \omega} \|\pi_\alpha(x^\nu)\xi\| = \|x \|_2 \|\xi\| = \lim_{\nu \to \omega} \| (x^\nu \otimes 1)\xi \|. \quad (7.1)
\]

Let \( p, \varepsilon \) and \( \xi_k \) be given as in the statement above. Take \( \delta > 0 \) and \( R > 0 \) with \( 85\delta^2 + 42\delta(1 - \delta)^{-2} < \varepsilon \) and \( \|\xi_j - \eta_j\| < \delta\|\xi_j\| \), where we have put \( \eta_j := (1 \otimes 1|_{-R,R})\xi_j \). By Lemma 7.1 there exists a non-zero partial isometry \( v \in M_{\omega,\alpha} \) such that
\[
\int_{-R}^R \|\alpha_t(v) - e^{ipt}v\|_2\|\eta_j(-t)\|^2 dt < \delta\|v\|_2 \sum_{j=1}^n \|\eta_j\|^2.
\]
From the inequality \( \|\alpha_t(v) - e^{ipt}v\|_2 \leq 2\|v\|_2 \), we obtain
\[
\int_{-R}^R \|\alpha_t(v) - e^{ipt}v\|_2^2\|\eta_j(-t)\|^2 dt < 2\delta\|v\|_2^2 \sum_{j=1}^n \|\eta_j\|^2. \quad (7.2)
\]

Take a representing sequence \( (v^\nu)_\nu \) of \( v \) such that each \( v^\nu \) is a non-zero partial isometry. To proceed a proof, we need the following claim.

**Claim.** For each \( j = 1, \ldots, n \), the following holds:
\[
\lim_{\nu \to \omega} \|\pi_\alpha(v^\nu)\eta_j - (v^\nu \otimes e_-p)\eta_j\|^2 < 21\delta\|v\|_2^2 \sum_{j=1}^n \|\eta_j\|^2. \quad (7.3)
\]

**Proof of Claim.** Let \( \kappa > 0 \) so that \( 8\kappa^2 < \delta \). We fix \( j \). Take \( \zeta_1, \ldots, \zeta_m \in H \) and continuous functions \( f_1, \ldots, f_m \) such that \( \text{supp} f_k \subset [-R,R] \) for all \( k \), and \( \eta_j^o := \sum_{k=1}^m \zeta_k \otimes f_k \) satisfies \( \|\eta_j - \eta_j^o\| < \kappa\|\eta_j\| \). Then
\[
\|\pi_\alpha(v^\nu)\eta_j^o - (v^\nu \otimes e_-p)\eta_j^o\|^2 = \int_{-R}^R \|\alpha_{-t}(v^\nu) - e^{-ipt}v^\nu\|_2\|\eta_j(t)\|^2 dt
\]
\[
= \int_{-R}^R \| \sum_{k=1}^m f_k(t)(\alpha_{-t}(v^\nu) - e^{-ipt}v^\nu)\zeta_k \|^2 dt.
\]
Note that \( (v^\nu)_\nu \) is \( (\alpha, \omega) \)-equicontinuous. Hence the following functions
\[
\| \sum_{k=1}^m f_k(t)(\alpha_{-t}(v^\nu) - e^{-ipt}v^\nu)\zeta_k \|^2 = \sum_{k, \ell=1}^m \frac{f_k(t)f_\ell(t)}{85} \|\alpha_{-t}(v^\nu) - e^{-ipt}v^\nu\|_2^2 \zeta_k, \zeta_\ell \]
converge to
\[ \sum_{k,\ell=1}^m f_k(t) f_\ell(t) \| \alpha_{-\ell}(v) - e^{-ipt} v \|_2^2 \| \zeta_k, \zeta_\ell \| = \| \alpha_{-\ell}(v) - e^{-ipt} v \|_2^2 \| \eta_j^\alpha(t) \|^2. \]

uniformly on \([-R, R]\) as \(\nu \to \omega\). Thus we have
\[
\lim_{\nu \to \omega} \| \pi_\alpha(v^\nu) \eta_j^\alpha - (v^\nu \otimes e_{-p}) \eta_j^\alpha \|^2 = \int_{-R}^R \| \alpha_{-\ell}(v) - e^{-ipt} v \|_2^2 \| \eta_j^\alpha(t) \|^2 \, dt \\
\leq \int_{-R}^R 2 \| \alpha_{-\ell}(v) - e^{-ipt} v \|_2^2 \| \eta_j(t) \|^2 \, dt \\
+ \int_{-R}^R 2 \| \alpha_{-\ell}(v) - e^{-ipt} v \|_2^2 \| \eta_j(t) \|^2 \, dt \\
\leq 8 \| v \|_2^2 \| \eta_j^\alpha - \eta_j \|^2 + 4 \delta \| v \|_2^2 \sum_{j=1}^n \| \eta_j \|^2 \quad \text{by (7.2)} \\
\leq (8\kappa^2 + 4\delta) \| v \|_2^2 \sum_{j=1}^n \| \eta_j \|^2 \\
< 5\delta \| v \|_2^2 \sum_{j=1}^n \| \eta_j \|^2. \tag{7.4}
\]

From (7.1), (7.4) and the following inequality:
\[ \| \pi_\alpha(v^\nu) \eta_j - (v^\nu \otimes e_{-p}) \eta_j \|^2 \leq 4 \| \pi_\alpha(v^\nu) (\eta_j - \eta_j^\alpha) \|^2 + 4 \| (v^\nu \otimes e_{-p}) (\eta_j - \eta_j^\alpha) \|^2 \\
+ 4 \| \pi_\alpha(v^\nu) \eta_j^\alpha - (v^\nu \otimes e_{-p}) \eta_j^\alpha \|^2, \]

we obtain
\[
\lim_{\nu \to \omega} \| \pi_\alpha(v^\nu) \eta_j - (v^\nu \otimes e_{-p}) \eta_j \|^2 \leq 8 \| v \|_2^2 \| \eta_j^\alpha - \eta_j \|^2 + 5\delta \| v \|_2^2 \sum_{j=1}^n \| \eta_j \|^2 \\
\leq (8\kappa^2 + 20\delta) \| v \|_2^2 \sum_{j=1}^n \| \eta_j \|^2 \\
< 21\delta \| v \|_2^2 \sum_{j=1}^n \| \eta_j \|^2.
\]

Hence Claim follows. \(\square\)

In the following inequality:
\[ \| \pi_\alpha(v^\nu) \xi_j - \hat{\alpha}_p(\xi_j) \pi_\alpha(v^\nu) \| \leq \| \pi_\alpha(v^\nu) (\xi_j - \eta_j) \| + \| \hat{\alpha}_p(\eta_j - \xi_j) \pi_\alpha(v^\nu) \| \\
+ \| \pi_\alpha(v^\nu) \eta_j - \hat{\alpha}_p(\eta_j) \pi_\alpha(v^\nu) \|, \]

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we let $\nu \to \omega$ and then
\[
\lim_{\nu \to \omega} \| \pi_{\alpha}(v') \xi_j - \hat{\alpha}_p(\xi_j) \pi_{\alpha}(v') \| \leq 2\|v\|_2 \|\xi_j - \eta_j\| + \lim_{\nu \to \omega} \| \pi_{\alpha}(v') \eta_j - \hat{\alpha}_p(\eta_j) \pi_{\alpha}(v') \|
\]
\[
< 2\delta \|v\|_2 \|\xi_j\| + \lim_{\nu \to \omega} \| \pi_{\alpha}(v') \eta_j - \hat{\alpha}_p(\eta_j) \pi_{\alpha}(v') \|.
\]
(7.5)

On the last term, we have $\hat{\alpha}_p(\eta_j) \pi_{\alpha}(v') = (1 \otimes e_{-p}) \eta_j (v' \otimes 1)$, where $\eta_j (v' \otimes 1)$ means that $(J(v')^* J \otimes 1) \eta_j$. Thus
\[
\| \pi_{\alpha}(v') \eta_j - \hat{\alpha}_p(\eta_j) \pi_{\alpha}(v') \| \leq \| \pi_{\alpha}(v') \eta_j - (v' \otimes e_{-p}) \eta_j \| + \|[v' \otimes 1, \eta_j]\|.
\]
Since $(v')_\nu$ is $\omega$-central, we have
\[
\lim_{\nu \to \omega} \| \pi_{\alpha}(v') \eta_j - \hat{\alpha}_p(\eta_j) \pi_{\alpha}(v') \|^2 \leq 21\delta \|v\|^2 \sum_{j=1}^n \|\eta_j\|^2 \quad \text{by (7.3)}
\]
\[
< 21\delta (1 - \delta)^{-2} \|v\|^2 \sum_{j=1}^n \|\xi_j\|^2.
\]
(7.6)

By (7.5) and (7.6), we get
\[
\lim_{\nu \to \omega} \| \pi_{\alpha}(v') \xi_j - \hat{\alpha}_p(\xi_j) \pi_{\alpha}(v') \|^2 \leq (8\delta^2 + 42\delta (1 - \delta)^{-2}) \|v\|^2 \sum_{j=1}^n \|\xi_j\|^2
\]
\[
< \varepsilon \|v\|^2 \sum_{j=1}^n \|\xi_j\|^2
\]
\[
= \lim_{\nu \to \omega} \varepsilon \sum_{j=1}^n \| \pi_{\alpha}(v') \xi_j \|^2.
\]

Since $[v', f] \to 0$ as $\nu \to \omega$ in the strong* topology and $\xi_j = \pi_{\alpha}(f) \xi_j \pi_{\alpha}(f)$, we have
\[
\lim_{\nu \to \omega} \| \pi_{\alpha}(f v' f) \xi_j - \hat{\alpha}_p(\xi_j) \pi_{\alpha}(f v' f) \|^2 < \lim_{\nu \to \omega} \varepsilon \sum_{j=1}^n \| \pi_{\alpha}(f v' f) \xi_j \|^2.
\]

Note that $f v f = f v \neq 0$ in $M^\omega$. Hence $x := f v' f$ does the job for a sufficiently large $\nu$.

\[\text{Lemma 7.3. Let $\mathcal{N}$ be a von Neumann algebra and $L^2(\mathcal{N})$ its standard Hilbert space. Then for $x, y \in \mathcal{N}$ and $\xi, \eta \in L^2(\mathcal{N})$, one has}
\]
\[
\int_0^\infty \| u_{\sqrt{\sigma}}(x) \xi - \eta u_{\sqrt{\sigma}}(y) \|^2 \, da
\]
\[
\leq 4(\|x \xi - \eta y\| + \|x^* \eta - \xi y^*\|)(\|x \xi\| + \|x^* \eta\| + \|\xi y^*\| + \|\eta y\|).
\]

\[\text{Proof. Let}
\]
\[
H := \begin{pmatrix} 0 & x^* \\ x & 0 \end{pmatrix}, \quad K := \begin{pmatrix} 0 & y^* \\ y & 0 \end{pmatrix}, \quad \zeta := \begin{pmatrix} \xi & 0 \\ 0 & \eta \end{pmatrix}.
\]
Then we have
\[ u_{\sqrt{\alpha}}(H) = \begin{pmatrix} 0 & u_{\sqrt{\alpha}}(x) \\ u_{\sqrt{\alpha}}(x)^* \\ 0 \end{pmatrix}, \quad u_{\sqrt{\alpha}}(K) = \begin{pmatrix} 0 & u_{\sqrt{\alpha}}(y) \\ u_{\sqrt{\alpha}}(y)^* \\ 0 \end{pmatrix}, \]
and
\[ H\zeta - \zeta K = \begin{pmatrix} 0 & x^*\eta - \xi y^* \\ x\xi - \eta y \\ 0 \end{pmatrix}. \]

**Claim.**
\[ \int_0^\infty \| u_{\sqrt{\alpha}}(H) \zeta - \zeta u_{\sqrt{\alpha}}(K) \|^2 \, da \leq 4 \| H\zeta - \zeta K \| (\| H\zeta \| + \| \zeta K \|). \]

**Proof of Claim.** Applying the proof of [60, Proposition IX.1.22] to
\[ H' := \begin{pmatrix} H & 0 \\ 0 & K \end{pmatrix}, \quad \zeta' := \begin{pmatrix} \zeta \\ 0 \end{pmatrix}, \]
we obtain
\[ \int_0^\infty \| u_{\sqrt{\alpha}}(H'), \zeta' \|^2 \, da \leq 4 \| [H', \zeta'] \| (\| H'\zeta' \|^2 + \| \zeta'H' \|^2)^{1/2}. \]

This proves the claim. \( \square \)

By the claim above, we have
\[ \int_0^\infty \| u_{\sqrt{\alpha}}(x) \xi - \eta u_{\sqrt{\alpha}}(y) \|^2 + \| u_{\sqrt{\alpha}}(x)^*\eta - \xi u_{\sqrt{\alpha}}(y)^* \|^2 \, da \]
\[ = \int_0^\infty \| u_{\sqrt{\alpha}}(H) \zeta - \zeta u_{\sqrt{\alpha}}(K) \|^2 \, da \]
\[ \leq 4 \| H\zeta - \zeta K \| (\| H\zeta \| + \| \zeta K \|) \]
\[ \leq 4(\| x\xi - \eta y \| + \| x^*\eta - \xi y^* \|)(\| x\xi \| + \| x^*\eta \| + \| \xi y^* \| + \| \eta y \|). \]

**Lemma 7.4.** Let \( p \in \text{Sp}(\alpha|_{\mathcal{M}_{w,\alpha}}) \) and \( f \in \mathcal{M}^p \). For any \( \varepsilon > 0 \) and \( \xi_1, \ldots, \xi_n \in \pi_\alpha(f)(H \otimes L^2(\mathbb{R})) \pi_\alpha(f) \) with \( \xi_j = \tilde{J}\xi_j \), there exists a non-zero partial isometry \( v \in \mathcal{M} \) such that \( v = f v f \) and
\[ \| \pi_\alpha(v)\xi_j - \hat{\alpha}_p(\xi_j)\pi_\alpha(v) \|^2 \leq \varepsilon \sum_{j=1}^n \| \pi_\alpha(v)\xi_j \|^2 \quad \text{for all} \quad j = 1, \ldots, n. \]

**Proof.** Let \( \delta > 0 \) with \( 32\delta^{1/2} + 16\delta \leq \varepsilon \). By Lemma 7.2, we have a non-zero \( x \in \mathcal{M}_f \) such that
\[ \| \pi_\alpha(x)\xi_j - \hat{\alpha}_p(\xi_j)\pi_\alpha(x) \|^2 \leq \delta \sum_{j=1}^n \| \pi_\alpha(x)\xi_j \|^2 \quad \text{for all} \quad j = 1, \ldots, n. \]
Employing the previous lemma, we obtain
\[
\int_{0}^{\infty} \| \pi_{\alpha}(u_{\sqrt{\alpha}}(x))\xi_{j} - \hat{\alpha}_{p}(\xi_{j})\pi_{\alpha}(u_{\sqrt{\alpha}}(x)) \|^2 \, da \\
\leq 4(\| \pi_{\alpha}(x)\xi_{j} - \hat{\alpha}_{p}(\xi_{j})\pi_{\alpha}(x)\| + \| \pi_{\alpha}(x^{*})\hat{\alpha}_{p}(\xi_{j}) - \xi_{j}\pi_{\alpha}(x^{*})\|) \\
\cdot (\| \pi_{\alpha}(x)\xi_{j}\| + \| \pi_{\alpha}(x^{*})\hat{\alpha}_{p}(\xi_{j})\| + \|\xi_{j}\pi_{\alpha}(x^{*})\| + \|\hat{\alpha}_{p}(\xi_{j})\pi_{\alpha}(x)\|) \\
\leq 8\| \pi_{\alpha}(x)\xi_{j} - \hat{\alpha}_{p}(\xi_{j})\pi_{\alpha}(x)\| \cdot 2(\| \pi_{\alpha}(x)\xi_{j}\| + \|\hat{\alpha}_{p}(\xi_{j})\pi_{\alpha}(x)\|) \\
\leq 16\| \pi_{\alpha}(x)\xi_{j} - \hat{\alpha}_{p}(\xi_{j})\pi_{\alpha}(x)\| \\
\cdot (2\| \pi_{\alpha}(x)\xi_{j}\| + \| \pi_{\alpha}(x)\xi_{j} - \hat{\alpha}_{p}(\xi_{j})\pi_{\alpha}(x)\|) \\
\leq 32\delta^{1/2} \sum_{j=1}^{n} \| \pi_{\alpha}(x)\xi_{j}\|^2 + 16\delta \sum_{j=1}^{n} \| \pi_{\alpha}(x)\xi_{j}\|^2 \\
\leq \varepsilon \| \pi_{\alpha}(x)\xi_{j}\|^2 = \varepsilon \int_{0}^{\infty} \| \pi_{\alpha}(u_{\sqrt{\alpha}}(x))\xi_{j}\|^2 \, da.
\]
Thus for some \( a > 0 \), \( v := u_{\sqrt{\alpha}}(x) \) does the job. \( \Box \)

We can show the following lemma in the same way as in [8, Lemma III.3].

Lemma 7.5. Let \( p \in \text{Sp}(\alpha_{|M_{\omega},\alpha}) \) and \( f \in \mathcal{M}^{P} \). For any \( \varepsilon > 0 \) and \( \xi_{1}, \ldots, \xi_{n} \in \pi_{\alpha}(f)(H \otimes L^{2}(\mathbb{R}))\pi_{\alpha}(f) \) with \( \tilde{J}\xi_{j} = \xi_{j} \), there exist a non-zero projection \( E \in \mathcal{M}_{f} \) and a non-zero \( x \in \mathcal{M}_{f} \) such that

1. \( \|x\| \leq 1, \ x = ExE; \)
2. \( \sum_{j=1}^{n} \| \pi_{\alpha}(x)\xi_{j}\|^2 \geq \frac{1}{12800} \sum_{j=1}^{n} \| \pi_{\alpha}(E)\xi_{j}\|^2 ; \)
3. \( \| \pi_{\alpha}(E),\xi_{k}\|^{2} \leq \varepsilon^{2} \sum_{j=1}^{n} \| \pi_{\alpha}(E)\xi_{j}\|^{2}, \quad 1 \leq k \leq n; \)
4. \( \| \pi_{\alpha}(x)\xi_{k} - \hat{\alpha}_{p}(\xi_{k})\pi_{\alpha}(x)\|^{2} \leq \varepsilon^{2} \sum_{j=1}^{n} \| \pi_{\alpha}(x)\xi_{j}\|^{2}, \quad 1 \leq k \leq n. \)

Let \( \mathcal{P}_{N}^{2} \) be the natural cone of \( N := \mathcal{M} \rtimes_{\alpha} \mathbb{R} \) in the standard Hilbert space \( H \otimes L^{2}(\mathbb{R}) \).

Lemma 7.6. Let \( p \in \text{Sp}(\alpha_{|M_{\omega},\alpha}) \) and \( \xi_{0} \) be a cyclic and separating unit vector for \( \mathcal{M} \rtimes_{\alpha} \mathbb{R} \). Then for any \( \varepsilon > 0 \) and \( \xi_{1}, \ldots, \xi_{n} \in \mathcal{P}_{N}^{2}, \) there exists a non-zero \( x \in \mathcal{M} \) such that

- \( \|x\| \leq 1; \)
- \( \| \pi_{\alpha}(x)\xi_{0}\|^{2} \geq 1/102400; \)
- \( \| \pi_{\alpha}(x)\xi_{k} - \hat{\alpha}_{p}(\xi_{k})\pi_{\alpha}(x)\| \leq \varepsilon \sum_{j=1}^{n} \| \xi_{j}\|^{2} \) for all \( k = 1, \ldots, n. \)

Proof. We prove this lemma in a similar way to that of [8, Lemma III.4]. Put \( d = 12800^{-1} \). Let \( \mathcal{F} \) be the set of all \( (n+2) \)-tuples \( (x, E, \beta_{1}, \ldots, \beta_{n}) \) in \( \mathcal{M} \times \mathcal{M}^{P} \times (H \otimes L^{2}(\mathbb{R}))^{n} \) such that

1. \( \|x\| \leq 1, \ x = ExE; \)
2. \( \pi_{\alpha}(E)\beta_{j} = \beta_{j}, \eta_{j} := \xi_{j} - \beta_{j} - \tilde{J}\beta_{j} \in \mathcal{P}_{N}^{2} \) and \( [\pi_{\alpha}(E),\eta_{j}] = 0, \ 1 \leq j \leq n; \)
3. \( \| \beta_{k}\|^{2} \leq \varepsilon^{2} \sum_{j=1}^{n} \| E\xi_{j}\|^{2}; \)
4. \( \sum_{j=1}^{n} \| \pi_{\alpha}(x)\eta_{j}\|^{2} \geq d/2 \sum_{j=1}^{n} \| \pi_{\alpha}(E)\eta_{j}\|^{2}; \)
5. \( \| \pi_{\alpha}(x)\eta_{k} - \hat{\alpha}_{p}(\eta_{k})\pi_{\alpha}(x)\|^{2} \leq \varepsilon^{2} \sum_{j=1}^{n} \| \pi_{\alpha}(x)\xi_{j}\|^{2}, \quad 1 \leq k \leq n. \)
We will equip \( \mathcal{J} \) with a partial order as \((x, E, \beta_1, \ldots, \beta_n) \leq (x', E', \beta'_1, \ldots, \beta'_n)\) if

- \( E \leq E' \);
- \( x = x' E \);
- \( \pi_a(E)\beta'_j = \beta_j, \ 1 \leq j \leq n; \)
- \( \|\beta'_k - \beta_k\| \leq \varepsilon^2 \sum_{j=1}^{n} \|\pi_a(E' - E)\xi_j\|^2, \ 1 \leq k \leq n. \)

Take a maximal element \((x, E, \beta_1, \ldots, \beta_n)\). It suffices to show that \( E = 1 \). Indeed, suppose that we have proved \( E = 1 \). Let \( \varepsilon > 0 \) and \( \xi_1, \ldots, \xi_n \in \mathcal{P}_N^2 \) with \( \|\xi_j\| = 1 \). Put \( \zeta_j = \xi_0 \) for \( j = 1, \ldots, n^2 \) and \( \zeta_{n^2+k} = \xi_k \) for \( k = 1, \ldots, n \). Then for \( \varepsilon \) and \( \{\zeta_j\}_{j=1}^{n^2+n} \), we have non-zero \( x \in \mathcal{M}_1 \) such that

\[
\sum_{j=1}^{n^2+n} \|\pi_a(x)\zeta_j\|^2 \geq d/4 \sum_{j=1}^{n^2+n} \|\zeta_j\|^2 = (n^2 + n)d/4,
\]

\[
\|\pi_a(x)\zeta_k - \hat{\alpha}_p(\zeta_k)\pi_a(x)\|^2 \leq \varepsilon^2 \sum_{j=1}^{n} \|\pi_a(x)\zeta_j\|^2.
\]

Readers are referred to the proof of [45, Lemma B.1] for a detail. Then we have

\[
\|\pi_a(x)\zeta_0\|^2 \geq (\frac{n^2 + n}{4} + d - n) / n^2 = d/4 - (1 - d/4)/n.
\]

If we take a sufficiently large \( n \), then we have \( \|\pi_a(x)\zeta_0\|^2 \geq d/8. \)

Suppose on the contrary that \( f := 1 - E \) is not equal to 0. Then \( \pi_a(f)\beta_j = 0 = (J\beta_j)\pi_a(f) \). Hence \( \pi_a(f)\eta_j\pi_a(f) = \pi_a(f)\xi_j\pi_a(f) \). Take a non-zero projection \( F \in \mathcal{M}_f \) and a non-zero \( y \in \mathcal{M}_F \) which satisfy the conditions in the previous lemma for \( \pi_a(f)\xi_1\pi_a(f), \ldots, \pi_a(f)\xi_n\pi_a(f) \).

We set

\[
x' := x + y, \quad E' := E + F, \quad \beta'_j := \beta_j + \pi_a(F)\eta_j\pi_a(f - F).
\]

We will check that \((x', E', \beta'_1, \ldots, \beta'_n)\) belongs to \( \mathcal{J} \). The condition (1) is trivial. On the condition (2), we put \( \eta'_j := \xi_j - \beta'_j - J\beta'_j \). Then

\[
\eta'_j = \eta_j - \pi_a(F)\eta_j\pi_a(f - F) - \pi_a(f - F)\eta_j\pi_a(F) - \pi_a(E)\eta_j\pi_a(E) + \pi_a(F)\eta_j\pi_a(F) + \pi_a(f - F)\eta_j\pi_a(f - F)
\]

\[
= \pi_a(F)\eta_j\pi_a(F) + \pi_a(1 - F)\eta_j\pi_a(1 - F) \in \mathcal{P}_N^2,
\]

and \([\pi_a(E'), \eta'_j] = 0\).
Next, the condition (3) is verified as follows:

\[
\|\beta_k\|^2 = \|\beta_k\|^2 + \|\pi_\alpha(F)\eta_j\pi_\alpha(f - F)\|^2
\]

\[
\leq \varepsilon^2 \sum_{j=1}^{n} \|\pi_\alpha(E)\xi_j\|^2 + \|\pi_\alpha(F)[\pi_\alpha(f)\eta_j\pi_\alpha(f), \pi_\alpha(F)]\|^2
\]

\[
\leq \varepsilon^2 \sum_{j=1}^{n} \|\pi_\alpha(E)\xi_j\|^2 + \varepsilon^2 \sum_{j=1}^{n} \|\pi_\alpha(F)\pi_\alpha(f)\pi_\alpha\|^2
\]

\[
\leq \varepsilon^2 \sum_{j=1}^{n} \|\pi_\alpha(E')\xi_j\|^2.
\]

We will verify the condition (4). Since \(x = xE\) and \([\pi_\alpha(E), \eta_j] = 0\), we have

\[
\pi_\alpha(x')\eta_j' = \pi_\alpha(x + y)(\pi_\alpha(F)\eta_j\pi_\alpha(F) + \pi_\alpha(1 - F)\eta_j\pi_\alpha(1 - F))
\]

\[
= \pi_\alpha(x)\eta_j\pi_\alpha(1 - F) + \pi_\alpha(y)\eta_j\pi_\alpha(F)
\]

\[
= \pi_\alpha(x)\eta_j + \pi_\alpha(y)\eta_j\pi_\alpha(F),
\]

and

\[
\hat{\alpha}_p(\eta_j')\pi_\alpha(x') = \pi_\alpha(1 - F)\hat{\alpha}_p(\eta_j)\pi_\alpha(x) + \pi_\alpha(F)\hat{\alpha}_p(\eta_j)\pi_\alpha(y)
\]

\[
= \hat{\alpha}_p(\eta_j)\pi_\alpha(x) + \pi_\alpha(F)\hat{\alpha}_p(\eta_j)\pi_\alpha(y).
\]

Since

\[
\|\pi_\alpha(y)\eta_j\pi_\alpha(f)\|^2 = \|\pi_\alpha(y)\eta_j\pi_\alpha(F)\|^2 + \|\pi_\alpha(y)\eta_j\pi_\alpha(f - F)\|^2
\]

\[
= \|\pi_\alpha(y)\eta_j\pi_\alpha(F)\|^2 + \|\pi_\alpha(y)[\pi_\alpha(f)\eta_j\pi_\alpha(f), \pi_\alpha(F)]\|^2
\]

\[
\leq \|\pi_\alpha(y)\eta_j\pi_\alpha(F)\|^2 + \varepsilon^2 \sum_{j=1}^{n} \|\pi_\alpha(F)\pi_\alpha(f)\pi_\alpha\|^2,
\]

we have

\[
\|\pi_\alpha(y)\eta_j\pi_\alpha(F)\|^2 \geq \|\pi_\alpha(y)\eta_j\pi_\alpha(f)\|^2 - \varepsilon^2 \sum_{j=1}^{n} \|\pi_\alpha(F)\pi_\alpha(f)\eta_j\pi_\alpha(f)\|^2.
\]
Then it follows that
\[
\|\pi_{\alpha}(x')\eta_j'\|^2 = \|\pi_{\alpha}(x)\eta_j\|^2 + \|\pi_{\alpha}(y)\eta_j\pi_{\alpha}(F)\|^2
\]
\[
\geq d/2 \sum_{j=1}^n \|\pi_{\alpha}(E)\eta_j\|^2 + \|\pi_{\alpha}(y)\eta_j\pi_{\alpha}(F)\|^2
\]
\[
-\varepsilon^2 \sum_{j=1}^n \|\pi_{\alpha}(F)\pi_{\alpha}(f)\eta_j\pi_{\alpha}(F)\|^2
\]
\[
\geq d/2 \sum_{j=1}^n \|\pi_{\alpha}(E)\eta_j\|^2 + (d - \varepsilon^2) \sum_{j=1}^n \|\pi_{\alpha}(F)\pi_{\alpha}(f)\eta_j\pi_{\alpha}(F)\|^2
\]
\[
\geq d/2 \sum_{j=1}^n \|\pi_{\alpha}(E)\pi_{\alpha}(1 - F)\eta_j\pi_{\alpha}(1 - F)\|^2
\]
\[
+ d/2 \sum_{j=1}^n \|\pi_{\alpha}(F)\eta_j\pi_{\alpha}(F)\|^2
\]
\[
= d/2 \sum_{j=1}^n \|\pi_{\alpha}(E)\pi_{\alpha}(1 - F)\eta_j\pi_{\alpha}(1 - F) + \pi_{\alpha}(F)\eta_j\pi_{\alpha}(F)\|^2
\]
\[
= d/2 \sum_{j=1}^n \|\pi_{\alpha}(E + F)(\pi_{\alpha}(1 - F)\eta_j\pi_{\alpha}(1 - F) + \pi_{\alpha}(F)\eta_j\pi_{\alpha}(F))\|^2
\]
\[
= d/2 \sum_{j=1}^n \|\pi_{\alpha}(E')\eta_j'\|^2,
\]
which shows (4).

Using \(\pi_{\alpha}(F)\beta_j = 0 = (\tilde{J}\beta_j)\pi_{\alpha}(F)\) in (7.7) and (7.8), we can verify (5) as follows:

\[
\|\pi_{\alpha}(x')\eta_k' - \tilde{\alpha}_p(\eta_k')\pi_{\alpha}(x')\|^2
\]
\[
= \|\tilde{\alpha}_p(\eta_k)\pi_{\alpha}(x) - \tilde{\alpha}_p(\eta_k)\pi_{\alpha}(x)\|^2
\]
\[
+ \|\pi_{\alpha}(y)\eta_k\pi_{\alpha}(F) - \pi_{\alpha}(F)\tilde{\alpha}_p(\eta_k)\pi_{\alpha}(y)\|^2
\]
\[
\leq \|\tilde{\alpha}_p(\eta_k)\pi_{\alpha}(x) - \tilde{\alpha}_p(\eta_k)\pi_{\alpha}(x)\|^2
\]
\[
+ \|\pi_{\alpha}(y)\cdot\pi_{\alpha}(f)\xi_k\pi_{\alpha}(f) - \pi_{\alpha}(f)\tilde{\alpha}_p(\xi_k)\pi_{\alpha}(f)\cdot\pi_{\alpha}(y)\|^2
\]
\[
\leq \varepsilon^2 \sum_{j=1}^n \|\pi_{\alpha}(x)\xi_j\|^2 + \varepsilon^2 \sum_{j=1}^n \|\pi_{\alpha}(y)\xi_j\|^2
\]
\[
\leq \varepsilon^2 \sum_{j=1}^n \|\pi_{\alpha}(x')\xi_j\|^2.
\]

Therefore, \((x', E', \beta'_1, \ldots, \beta'_n)\) belongs to \(\mathcal{J}\), and is strictly larger than \((x, E, \beta_1, \ldots, \beta_n)\). This is a contradiction. Hence \(E = 1\). \(\Box\)
Thus we have proved there exists a sequence \((x_n)_n\) in \(\mathcal{M}_1\) which does not converge to 0 in the strong* topology, and
\[
\lim_{n \to \infty} \|\pi_\alpha(x_n)\xi - \hat{\alpha}_p(\xi)\pi_\alpha(x_n)\| = 0 \quad \text{for all } \xi \in H \otimes L^2(\mathbb{R}).
\]
Note that \((x_n)_n\) is a central sequence in \(\mathcal{M}\). Indeed, let \(\xi \in H \otimes L^2(\mathbb{R})\) and set \(\varphi(a) := \langle \pi_\alpha(a)\xi, \xi \rangle\) for \(a \in \mathcal{M}\). Then for \(y \in \mathcal{M}_1\),
\[
[x_n, \varphi](y) = \langle \pi_\alpha(yx_n)\xi, \xi \rangle - \langle \pi_\alpha(x_ny)\xi, \xi \rangle
= \langle \pi_\alpha(y) - \pi_\alpha(x_ny)\xi, \xi \rangle + \langle \pi_\alpha(x_n)\xi - \hat{\alpha}_p(\xi)\pi_\alpha(x_n)\xi, \xi \rangle
= \langle \pi_\alpha(y)\xi - \hat{\alpha}_p(\xi)\pi_\alpha(x_n)\xi, \xi \rangle
\]
holds, since
\[
\langle \pi_\alpha(y)\hat{\alpha}_p(\xi), \pi_\alpha(x_n)\hat{\alpha}_p(\xi) \rangle = \langle \pi_\alpha(x_ny)\hat{\alpha}_p(\xi), \hat{\alpha}_p(\xi) \rangle = \langle \hat{\alpha}_p(\pi_\alpha(x_ny)\xi), \hat{\alpha}_p(\xi) \rangle
= \langle \pi_\alpha(x_ny)\xi, \xi \rangle.
\]
Thus
\[
\| [x_n, \varphi] \|_{\mathcal{M}_*} \leq \| \pi_\alpha(x_n)\xi - \hat{\alpha}_p(\xi)\pi_\alpha(x_n)\| + \| \xi \pi_\alpha(x_n^*) - \pi_\alpha(x_n^*)\hat{\alpha}_p(\xi) \|,
\]
and the right hand side converges to 0 as \(n \to \infty\).

**Theorem 7.7.** Suppose that \(p \in \text{Sp}(\alpha|_{\mathcal{M}_{\omega,\alpha}})\). Then there exists a unitary central sequence \((u_n)_n\) in \(\mathcal{M}\) such that \(\hat{\alpha}_p = \lim_{n \to \infty} \text{Ad} \pi_\alpha(u_n)\).

**Proof.** Set \(Q := M_2(\mathbb{C}) \otimes \mathcal{M}\). Let \(A_\omega\) be the set of all \((X_n)_n\) in \(\ell^\infty(Q)\) such that \([(\text{id} \otimes \pi_\alpha)(X_n), \varphi \otimes \hat{\alpha}_p(\varphi)] \to 0\) as \(n \to \omega\) for all \(\varphi \in \mathcal{N}_\ast\). Then \(P_\omega := A_\omega / J_\omega(Q)\) is a von Neumann algebra. By the remark above, \(P_\omega \subset M_2(\mathbb{C}) \otimes \mathcal{M}_\omega\) naturally.

We will show that \(e := e_{11} \otimes 1\) is equivalent to \(f := e_{22} \otimes 1\) in \(P_\omega\). Let \(z\) be a central element of \(P_\omega\). Clearly, \(z = e_{11} \otimes a + e_{22} \otimes b\) for some \(a, b \in \mathcal{M}_\omega\). Take \((x_n)_n\) as above. Then \((x_n)_n\) defines a non-zero element \(x\) of \(\mathcal{M}_\omega\) and \(e_{21} \otimes x \in eP_\omega f\). Then \([z, e_{21} \otimes x] = 0\) implies \(xa = bx\). Since \(x\) is central, we may assume that \(xa = ax\), and moreover \(\|(a-b)x\|_2 = \|a-b\|_2\|x\|_2\) by the fast reindexation trick. Thus we have \(a = b\). Hence \(Z(P_\omega) \subset \mathbb{C} \otimes Z(\mathcal{M}_\omega)\). This implies that \(e \sim f\). □

By Lemma 4.10 we obtain the following result.

**Theorem 7.8.** Let \(\alpha\) be a flow on a factor \(\mathcal{M}\). Then \(p \in \text{Sp}(\alpha|_{\mathcal{M}_{\omega,\alpha}})\) if and only if there exists a unitary \(v \in \mathcal{M}_{\omega,\alpha}\) such that \(\alpha_t(v) = e^{ipt}v\) for all \(t \in \mathbb{R}\). In particular, the following statements are equivalent:

1. \(\alpha\) is faithful on \(\mathcal{M}_{\omega,\alpha}\);

(2) $\text{Sp}(\alpha_{|M_{\omega,\alpha}}) = \Gamma(\alpha_{|M_{\omega,\alpha}}) = \mathbb{R}$;
(3) $\alpha$ is a Rohlin flow on $M$.

This result immediately implies the following.

**Corollary 7.9.** Let $\alpha$ be a faithful flow on a finite factor. Then the product type flow $\bigotimes_{n=1}^{\infty} \alpha_t$ has the Rohlin property.

8. **Concluding remarks and Problems**

In this paper, we have studied Rohlin flows on von Neumann algebras. We can generalize the Rohlin property for actions of a locally compact abelian group by using the dual group. Hence it is natural attempt to extend our work more general. In studying this, one needs to think of the map $\Theta$ that is introduced in Lemma 5.2

**Problem 8.1.** Classify actions with the Rohlin property of a locally compact abelian group on von Neumann algebras.

Our main theorem is applicable to non-McDuff factors. When its central sequence algebra is non-trivial and commutative, there is a chance that they have a Rohlin flow.

**Problem 8.2.** Characterize a factor admitting a Rohlin flow.

We should notice that the classification of “outer actions” of $\mathbb{R}$ on injective factors has not yet been completely finished. Indeed, we do not know a characterization of the Rohlin property without using a central sequence algebra. In the light of classification results for amenable discrete or compact group actions obtained so far, we will pose the following plausible conjecture.

**Conjecture 8.3.** Let $\alpha$ be a flow on an injective factor $M$. Then the following conditions are equivalent:

1. $\alpha$ has the Rohlin property;
2. $\pi_{\widetilde{\alpha}}(\widetilde{M})' \cap (\widetilde{M} \rtimes_{\widetilde{\alpha}} \mathbb{R}) = \pi_{\widetilde{\alpha}}(Z(\widetilde{M}))$.

We have seen the implication (1) $\Rightarrow$ (2) holds for a general von Neumann algebra in Corollary 4.13.

**Example 8.4.** Let $\alpha$ be a flow on an injective type III$_0$ factor $M$ such that $\alpha$ is ergodically and faithfully acting on the space of the flow of weights. Let $N = M \rtimes_{\alpha} \mathbb{R}$. Then $\widetilde{N} = \widetilde{M} \rtimes_{\widetilde{\alpha}} \mathbb{R}$ canonically. Since the action $\widetilde{\alpha}$ on $Z(\widetilde{M})$ is faithful, we have $\pi_{\widetilde{\alpha}}(Z(\widetilde{M}))' \cap \widetilde{N} = \pi_{\widetilde{\alpha}}(\widetilde{M})$ by [51 Corollary VI.1.3]. Hence $\alpha$ satisfies (2) in Conjecture 8.3. The dual flow of an extended modular flow has such property (see Theorem 4.20).

Thus it must be interesting to consider the weak version of Conjecture 8.3.

**Problem 8.5.** Suppose that $\alpha$ is a flow on an injective type III$_0$ factor such that $\text{mod}(\alpha)$ is faithful and ergodic. Then does $\alpha$ have the Rohlin property?
Readers are referred to [24] for a faithful action of a compact group on a flow space. The following problem is related with Lemma 3.17.

**Problem 8.6.** Assume that \( M \) is an injective factor and \( \alpha \) is a pointwise centrally non-trivial flow. Then \( \Gamma(\alpha|_{M_\omega,\alpha}) = \Gamma(\alpha) \)?

Since we always have \( \Gamma(\alpha|_{M_\omega,\alpha}) \subset \Gamma(\alpha) \) by Lemma 3.17, what we must do is to study the case when \( \Gamma(\alpha) \neq \{0\} \). If this problem is affirmatively solved, the following condition, which looks much weaker than (2) in the above conjecture, could imply the Rohlin property of \( \alpha \) by Theorem 7.8:

(3) \( \alpha \) is pointwise centrally non-trivial and \( \Gamma(\alpha) = \mathbb{R} \).

Indeed in Theorem 6.32 and 6.40, we have shown that the three conditions mentioned above are equivalent for product type flows and quasi-free flows induced from Cuntz algebras. We should note that if \( \alpha \) is an action of a compact abelian group, then the above condition (3) indeed implies the Rohlin property.

**Problem 8.7.** Let \( \alpha \) be a flow on the injective type II\(_1\) factor \( \mathcal{R}_0 \). Suppose that \( \alpha \otimes \text{id}_{\mathcal{R}_0} \) has the Rohlin property. Then does \( \alpha \) also have?

This problem is a direct consequence of the above conjecture. Proposition 6.38 shows that the assumption on the type is necessary.

In Theorem 6.18, we have proved that a trace scaling flow on the injective type II\(_\infty\) factor has the Rohlin property by using Connes-Haagerup theory. Another proof of that theorem will yield the uniqueness of the injective type III\(_1\) factor. Hence we have interest in the following problem.

**Problem 8.8.** Prove Theorem 6.18 without using the fact that \( \sigma_t^\alpha \in \text{Int}(\mathcal{M}) \) for an injective type III\(_1\) factor.

9. **Appendix**

9.1. **Basic measure theoretic results.** We recall the following elementary result of measure theory.

**Lemma 9.1.** Let \( X \) be a Polish space and \( f: \mathbb{R}^n \to X \) be a Borel map. Let \( E \subset \mathbb{R}^n \) be a Borel set with \( 0 < \mu(E) < \infty \), where \( \mu \) denotes the Lebesgue measure on \( \mathbb{R}^n \). Then the following statements hold:

1. For any \( \varepsilon > 0 \), there exists a compact set \( K \subset E \) such that \( \mu(E \setminus K) < \varepsilon \) and \( f \) is continuous on \( K \);
2. Let \( d \) be a complete metric on the Polish space \( X \). For any \( \varepsilon_1, \varepsilon_2 > 0 \), there exist disjoint Borel sets \( A_0, A_1, \ldots, A_N \subset E \) and \( x_1, \ldots, x_N \in X \) such that \( \sum_{j=0}^N A_j = E \), \( \mu(A_0) < \varepsilon_1 \), \( d(f(t), x_j) < \varepsilon_2 \) for all \( t \in A_j \), \( j = 1, \ldots, N \).

**Proof.** (1). This is shown by using Lusin’s theorem. See [35] Theorem 17.12

(2). Let \( K \) be as above with \( \varepsilon = \varepsilon_1 \). Since \( f|_K \) is uniformly continuous, for any \( \varepsilon_2 > 0 \), there exists a Borel partition \( \{A_j\}_{j=1}^N \) of \( K \) such that any \( s, t \in A_j \) satisfy \( d(f(s), f(t)) < \varepsilon_2 \). Put \( A_0 := E \setminus K \), and take an element \( x_j \in f(A_j) \) for each \( j \). Then these \( x_j \) have the required property. \( \square \)
Updating the previous lemma, we have the following result.

**Lemma 9.2.** Let $\mathcal{M}$ be a separable finite von Neumann algebra with a faithful normal tracial state $\tau$. Let us write $\|x\|_2 := \tau(x^*x)^{1/2}$ as usual. Let $w: [0, 1] \rightarrow \mathcal{M}^U$ be a Borel map. Then for any $\varepsilon_1, \varepsilon_2 > 0$, there exist disjoint Borel sets $A_0, A_1, \ldots, A_N$ of $[0, 1]$ and $u_1, \ldots, u_N \in \mathcal{M}^U$ such that $\mu(A_0) \leq \varepsilon_1$, $\|w_t - u_j\|_2 < \varepsilon_2$ for all $t \in A_j$, $j = 1, \ldots, N$.

**9.2. Disintegration of automorphisms.** Let $(X, \mathcal{B})$ be a standard Borel space and $\mu$ a $\sigma$-finite Borel measure. Recall the following basic result.

**Lemma 9.3.** Let $f: X \rightarrow \mathbb{R}$ be a $\mu$-measurable function. Then there exists a Borel $\mu$-null set $N \subset X$ such that the restriction $f: X \setminus N \rightarrow \mathbb{R}$ is Borel.

**Proof.** Take a sequence $\{q_n\}_n$ whose union equals $Q$. Then $A_n := \{x \mid f(x) > q_n\}$ is $\mu$-measurable. Since $A_n$ is $\mu$-measurable, we can take Borel sets $B_n, B'_n \subset X$ such that $B_n \subset A_n \subset B'_n$ and $\mu(B'_n \setminus B_n) = 0$. We put $N_n := B'_n \setminus B_n$. Then we have $A_n \setminus N_n = B_n$ that is Borel. We let $N := \bigcup_{k=1}^{\infty} N_n$. Then

$$\{x \in X \mid x \notin N, f(x) > q_n\} = A_n \cap N_n \cap \bigcap_{k \neq n} N_k,$$

and this set is Borel. Since $Q$ is dense in $\mathbb{R}$, $f: X \setminus N \rightarrow \mathbb{R}$ is Borel.

Let $\{H_x\}_{x \in X}$ and $\{\mathcal{M}_x\}_{x \in X}$ be measurable fields of separable Hilbert spaces and separable von Neumann algebras, respectively, such that $\mathcal{M}_x \subset B(H_x)$. Let $\{\alpha_x\}_{x \in X}$ be a measurable field of automorphisms with $\alpha_x \in \text{Aut}(\mathcal{M}_x)$. We set

$$\mathcal{M} := \int_X \mathcal{M}_x d\mu(x), \quad H := \int_X H_x d\mu(x), \quad \alpha := \int_X \alpha_x d\mu(x).$$

**Theorem 9.4.** Let $\alpha$ and $\alpha^x$ be as above. Then $\alpha \in \overline{\text{Int}}(\mathcal{M})$ if and only if $\alpha_x \in \overline{\text{Int}}(\mathcal{M}_x)$ for almost every $x \in X$.

**Proof.** We may and do assume that $H_x = H_0$ with $H_0 \cong \ell^2$ for all $x \in X$, and $\mu(X) < \infty$. Let $\text{vN}(H_0)$ be the set of von Neumann algebras on $H_0$. We equip $\text{vN}(H_0)$ with the Effros Borel structure as usual [12]. Then we have a $\mu$-measurable map $X \ni x \mapsto \mathcal{M}_x \in \text{vN}(H_0)$. Using the previous lemma and choice functions of $\text{vN}(H_0)$, we may and do assume that the maps $x \mapsto \mathcal{M}_x$, $x \mapsto \mathcal{M}_x^*$ and $x \mapsto \alpha_x$ are Borel.

Suppose that $\alpha \in \overline{\text{Int}}(\mathcal{M})$. Then there exists a sequence of unitaries $\{v^\nu\}_\nu$ in $\mathcal{M}$ such that $\alpha = \lim_{\nu} \text{Ad} v^\nu$ in the $u$-topology. Then we obtain

$$\|\alpha(\varphi) - v^\nu \varphi v^{\nu^*}\| = \int_X \|\alpha_x(\varphi_x) - v^\nu_x \varphi_x v^{\nu_x^*}\|_{\mathcal{M}_x}, d\mu(x) \quad \text{for all } \varphi \in \mathcal{M}_*.$$

Thus there exists a subsequence $\{v^{\nu_k}\}_k$ such that for all $\varphi \in \mathcal{M}_*$, we have $\|\alpha_x(\varphi_x) - v^{\nu_k}_x \varphi_x v^{\nu_k_x^*}\| \rightarrow 0$ as $k \rightarrow \infty$ for almost all $x$. Hence $\alpha_x \in \overline{\text{Int}}(\mathcal{M}_x)$ for almost all $x$ since $\{\varphi_x \mid \varphi \in \mathcal{M}_*\}$ is dense in $(\mathcal{M}_x)$, for almost every $x$.

Suppose conversely that $\alpha_x \in \overline{\text{Int}}(\mathcal{M}_x)$ for almost every $x$. For an integrable Borel map $X \ni x \mapsto \varphi_x \in B(H_0)_*$, we set

$$F_\varphi: X \times B(H_0)^U \ni (x, v) \mapsto \|\alpha_x(\varphi_x|_{\mathcal{M}_x}) - v\varphi_x v^*|_{\mathcal{M}_x}\|_{\mathcal{M}_x}, \in \mathbb{R},$$

and this set is Borel. Since $Q$ is dense in $\mathbb{R}$, $f: X \setminus N \rightarrow \mathbb{R}$ is Borel.
where $B(H_0)^U$ denotes the unitary group that is Polish with respect to the strong* topology. The function $F_\varphi$ is Borel. Indeed, let $\{a_x\}_x$ be a Borel operator field such that $a_x \in M_x$. Then

$$ (a_x(\varphi_x|M_x) - v\varphi_x v^*)(a_x) = \varphi(\alpha_x^{-1}(a_x)) - v\varphi_x v^*(a_x). \quad (9.1) $$

It is trivial that $x \mapsto \varphi_x(\alpha_x^{-1}(a_x))$ is Borel. Since the maps $B(H_0)^U \times B(H_0)_* \ni (v, \varphi) \mapsto v\varphi v^* \in B(H_0)_*$ and the coupling $B(H_0)_* \times B(H_0)_* \to \mathbb{C}$ are both continuous, the second term in (9.1) is Borel. Hence $F_\varphi$ is Borel.

Let $b^j: v\Sigma\mathcal{N}(H_0) \to B(H_0)_1$ be a Borel choice function such that $\{b^j(N)\}_{j=1}^\infty$ is strongly dense in $\mathcal{N}_1$ for all $N \in v\Sigma\mathcal{N}(H_0)$. We let $b^j_x := b^j(M_x')$ that is a Borel function from $X$ into $B(H_0)_1$. Let $\{\varphi^k\}_{k=1}^\infty$ be a norm dense set in $L^1_{B(H_0)}(X, \mu)$. We set the function

$$ G_n: X \times B(H_0)^U \ni (x, v) \mapsto \sup_{1 \leq j, \varphi \leq n} \|[v, b^j_x]|\varphi_x\| \in \mathbb{R}. $$

Then $G_n$ is Borel since the left multiplication $B(H_0) \times B(H_0)_* \to B(H_0)_*$ is continuous. For $m, n \in \mathbb{N}$, we set the following Borel subset:

$$ Z_m := \bigcap_{k=1}^m F_{\varphi}^{-1}([0, 1/m]) \cap \bigcap_{n=1}^\infty G_n^{-1}\{0\}. $$

Note that

$$ Z_m = \left\{(x, v) \in X \times B(H_0)^U \mid \sup_{1 \leq k \leq m} \|\alpha_x(\varphi^k_x|M_x) - v\varphi^k_x v^*|M_x\|_{(M_x)_*} \leq 1/m, \ v \in M_x \right\}. $$

Let $\text{pr}_1: X \times B(H_0)^U \to X$ be the projection. By approximate innerness of $\alpha_x$, it turns out that $\text{pr}_1|Z_m: Z_m \to X$ is surjective. Thanks to the measurable cross section theorem (see [3, Theorem 3.2.4] or [59, Theorem A.16]), we have a $\mu$-measurable map $f: X \to Z_m$ such that $\text{pr}_1 \circ f = \text{id}_X$.

Let $\text{pr}_2: X \times B(H_0)^U \to B(H_0)^U$ be the projection. Since $\text{pr}_2$ is Borel, $\text{pr}_2 \circ f: X \to B(H_0)^U$ is $\mu$-measurable. We set $v_x := \text{pr}_2(f(x)) \in M_x$ and

$$ v := \int_X v_x \, d\mu(x) \in M. $$

Then for all $k = 1, \ldots, m$,

$$ \|\alpha(\varphi^k) - v\varphi^k v^*\|_{M_*} = \int_X \|\alpha_x(\varphi^k_x|M_x) - v_x\varphi^k_x v^*|M_x\|_{(M_x)_*} \, d\mu(x) \leq \mu(X)/m. $$

This means that $\alpha \in \overline{\text{Int}(M)}$. \hfill $\square$

In the proof above, we have implicitly proved the following result which has been proved by Lance [43, Theorem 3.4].

**Theorem 9.5.** Let $\alpha = \int_X \alpha^x \, d\mu(x)$ be an automorphism on $M$ as before. Then $\alpha \in \text{Int}(M)$ if and only if $\alpha_x \in \text{Int}(M_x)$ for almost every $x$.

Next we study centrally trivial automorphisms.
Lemma 9.6. Let us fix a faithful normal state $\psi$ on $M$. An automorphism $\alpha$ on $M$ is centrally trivial if and only if for any $\varepsilon > 0$, there exist $\delta > 0$ and a finite set $F \subset M_*$ such that if $a \in M_1$ satisfies $\norm{[a, \varphi]} < \delta$ for all $\varphi \in F$, then $\norm{\alpha(a) - a}_{[\psi]}^2 < \varepsilon$.

Lemma 9.7. The subgroup $\operatorname{Cnt}(M)$ is Borel in $\operatorname{Aut}(M)$.

Proof. Note that if $\alpha \in \operatorname{Cnt}(M)$ and $(a^\nu)_\nu$ is central, then $\norm{\alpha(a^\nu) - a^\nu}_{[\psi]}^2 \to 0$ as $\nu \to \infty$. Let $\{F_m\}_m$ be an increasing sequence of finite subsets in $M_*$ such that their union is norm dense in $M_*$. Let $\{a_j\}_j$ be a strongly dense sequence in $M_1$. For $m \in \mathbb{N}$, we let $J_m := \{j \in \mathbb{N} \mid \norm{[a_j, \varphi]} < 1/m, \varphi \in F_m\}$.

Then the previous lemma implies that $\alpha \in \operatorname{Aut}(M)$ is centrally trivial if and only if

$$\inf_{m \in \mathbb{N}} \sup_{j \in J_m} \norm{\alpha(a_j) - a_j}_{[\psi]}^2 = 0.$$ 

Since $\norm{\alpha(a_j) - a_j}_{[\psi]}^2$ is continuous with respect to $\alpha \in \operatorname{Aut}(M)$, the function $\alpha \mapsto \inf_m \sup_{j \in J_m} \norm{\alpha(a_j) - a_j}_{[\psi]}^2$ is Borel. In particular, $\operatorname{Cnt}(M)$ is a Borel subset in $\operatorname{Aut}(M)$. □

Lemma 9.8. If a sequence $a^\nu := \int_X a^\nu_x d\mu(t)$ is central in $M$, then a subsequence $(a^\nu_m)_m$ is central in $M_x$ for almost every $x \in X$.

Proof. Let $\{\varphi^k\}_{k \in \mathbb{N}}$ be a dense sequence of $M_*$. Then $(a^\nu)_\nu$ is central if and only if

$$\norm{[a^\nu, \varphi^k]} = \int_X \norm{[a^\nu_x, \varphi^k_x]} d\mu(x) \to 0 \quad \text{as } \nu \to \infty$$

for all $k$. This means $\norm{[a^\nu, \varphi^k]} \to 0$ in $L^1(X, \mu)$. Hence we are done. □

Theorem 9.9. Let $\alpha = \int_X a_x d\mu(x)$ be an automorphism on $M$ as before. Then $\alpha \in \operatorname{Cnt}(M)$ if and only if $\alpha_x \in \operatorname{Cnt}(M_x)$ for almost every $x \in X$.

Proof. We may and do assume that all relevant maps such as $x \mapsto M_x$ and $x \mapsto \alpha_x$ are Borel as before. By replacing $\mu$ if necessary, $\mu$ is assumed to be finite. Let $\varphi_0 \in B(H_0)_1$ be a faithful state and $\psi := \varphi_0 \otimes \mu$.

Suppose that $\alpha_x \in \operatorname{Cnt}(M_x)$ for almost every $x \in X$. If $\alpha$ were not centrally trivial, there exist $\varepsilon_0 > 0$ and a central sequence $(a^\nu)_\nu$ in $M$ such that $\inf_{\nu} \norm{\alpha(a^\nu) - a^\nu}_{[\psi]}^2 \geq \varepsilon_0$. This implies for all $\nu$,

$$\int_X \norm{\alpha_x(a^\nu_x) - a^\nu_x}_{[\psi_0]}^2 d\mu(t) \geq \varepsilon_0^2.$$ 

By the previous lemma, a subsequence $(a^\nu_m)_m$ is central for almost every $x \in X$. Hence $\norm{\alpha_x(a^\nu_m) - a^\nu_m}_{[\psi_0]}^2 \to 0$ as $m \to \infty$. Then by the dominated convergence theorem, the left hand side above converges to 0, and this is a contradiction.

Suppose conversely $\alpha \in \operatorname{Cnt}(M)$. We let $a^\mu : X \to B(H_0)_1$ be a choice function such that $\{a^\mu_x\}_x$ is strongly dense in $(M_x)_1$. By discarding $\mu$-null sets, we may take a norm dense sequence $\{\varphi^k\}_k$ in $L_B(H_0)_1, (X, \mu)$ such that $\{\varphi^k_x|_{M_x}\}_k$ is norm dense in $(M_x)_1$ for all $x \in X$. We set

$$A_{j,m} := \{x \in X \mid \norm{[a^\mu_x, \varphi^k_x]}_{(M_x)_1} < 1/m, \quad k = 1, \ldots, m\},$$
which is a Borel subset of $X$ since the map $B(H_0) \times \nu N(H_0) \ni (\varphi, \mathcal{M}) \mapsto \|\varphi|_\mathcal{M}\|$ is Borel. Then

$$\{ x \in X \mid \alpha_x \in Cnt(M_x) \} = \left\{ x \in X \mid \inf_{m \in \mathbb{N}} \sup_{j \in \mathbb{N}} 1_{A_{j,m}}(x) \|\alpha_x(a^j_x) - a^j_x\|_{\mathcal{V}_0} = 0 \right\}.$$ 

Since $x \mapsto \|\alpha_x(a^j_x) - a^j_x\|_{\mathcal{V}_0}$ is Borel, the set $\{ x \in X \mid \alpha_x \notin Cnt(M_x) \}$ is Borel.

Suppose that $N := \{ x \in X \mid \alpha_x \notin Cnt(M_x) \}$ satisfied $\mu(N) > 0$. Since the positive function $g: N \ni x \mapsto \inf_{m} \sup_{j} 1_{A_{j,m}}(x) \|\alpha_x(a^j_x) - a^j_x\|_{\mathcal{V}_0}$ is Borel, there exists $\varepsilon_1 > 0$ such that $N_1 := \{ x \in N \mid g(x) > \varepsilon_1 \}$ satisfies $\mu(N_1) > 0$. Thus for all $m \in \mathbb{N}$, we obtain $j_m \in \mathbb{N}$ such that

$$\mu(\bigcup_{j=1}^{j_m} \{ x \in A_{j,m} \mid \|\alpha_x(a^j_x) - a^j_x\|_{\mathcal{V}_0} \geq \varepsilon_1 \}) = \mu(N_1)/2.$$ 

Let $B_{j,m} := \{ x \in A_{j,m} \mid \|\alpha_x(a^j_x) - a^j_x\|_{\mathcal{V}_0} \geq \varepsilon_1 \}$ and $X_m := B_{1,m} \cup \cdots \cup B_{j_m,m}$. We set $c^m_x$ as follows:

$$c^m_x := \begin{cases} a^j_x & \text{if } x \in B_{j,m} \cap B_{j-1,m} \cap \cdots \cap B_{1,m} \\
0 & \text{if } x \in X \setminus X_m. \end{cases}$$ 

Then $\|[c^m_x, \varphi^k_x]|_{(M_x)_*} \leq 1/m$ for all $x \in X$ and $k = 1, \ldots, m$, and $\|\alpha_x(c^m_x) - c^m_x\|_{\mathcal{V}_0} \geq \varepsilon_1$ for all $x \in X_m$. Put $c^m := \int_X c^m_x \, d\mu(x)$. Then we obtain

$$\|[c^m, \varphi^k]|_{(M_x)_*} = \int_X \|[c^m_x, \varphi^k_x]|_{(M_x)_*} \, d\mu(x) \leq \mu(X)/m \quad \text{for all } k = 1, \ldots, m,$$

and

$$\|\alpha(c^m) - c^m\|_{\mathcal{V}}^2 = \int_{X_m} \|\alpha_x(c^m_x) - c^m_x\|_{\mathcal{V}_0}^2 \, d\mu(t) \geq \varepsilon_1^2 \mu(X_m) \geq \varepsilon_1^2 \mu(N_1)/2.$$ 

Then $(c^m)_m$ is central, but $\liminf_m \|\alpha(c^m) - c^m\|_{\mathcal{V}}^2 \geq \varepsilon_1^2 \mu(N_1)/2$. This is a contradiction. $\square$

Let $\mathcal{M}$ be a von Neumann algebra and $\varphi$ a faithful normal state on $\mathcal{M}$. Then we obtain the central decompositions of $\mathcal{M}$ and $\varphi$ as follows:

$$\mathcal{M} = \int_X \mathcal{M}_x \, d\mu(x), \quad \varphi = \int_X \varphi_x \, d\mu(x),$$

where $Z(\mathcal{M})$ is identified with $L^\infty(X, \mu)$.

We may and do assume that all $\mathcal{M}_x$ are von Neumann subalgebras acting on a common Hilbert space $H_0$ as before. Thus $\mathcal{M}$ acts on $H := L^2(X, \mu) \otimes H_0$.

Let $N_x := \mathcal{M}_x \times_{\sigma_{\mathcal{M}_x}} \mathbb{R}$ and $K_x := H_0 \otimes L^2(\mathbb{R})$. We will show that $\{N_x, K_x\}_x$ is a measurable field of von Neumann algebras with respect to $\int_X K_x \, d\mu(x) = L^2(X) \otimes H_0 \otimes L^2(\mathbb{R})$.

Let $a^j: X \to B(H_0)_1$ be a $\mu$-measurable choice function such that $\{a^j_x\}_x \in 1$ is strongly dense in $(\mathcal{M}_x)_1$ for almost every $x$. Then $x \mapsto \pi_{\sigma_{\mathcal{M}_x}}(a^j_x)$ is $\mu$-measurable.
Indeed, let $\xi, \eta \in \int_X K_x d\mu(x)$. Then

$$\langle \pi_{\sigma \varphi_x} (a_x^j) \xi_x, \eta_x \rangle = \int_\mathbb{R} \langle \sigma_{\varphi_t}(a_x^j) \xi_x(t), \eta_x(t) \rangle d\mu(t).$$

Since $(x,t) \mapsto \langle \sigma_{\varphi_t}(a_x^j) \xi_x(t), \eta_x(t) \rangle$ is $\mu$-measurable, $x \mapsto \langle \pi_{\sigma \varphi_x} (a_x^j) \xi_x, \eta_x \rangle$ is $\mu$-measurable by Fubini’s theorem.

Note that $\{\pi_{\sigma \varphi_x} (a_x^j)\}_j$ is strongly dense in $\pi_{\sigma \varphi_x} (\mathcal{M}_x)_1$. Each $\mathcal{N}_x$ contains the left regular representation $\lambda_{\varphi(t)} = 1 \otimes \lambda(t)$. Thus $\{\mathcal{N}_x, K_x\}$ is a $\mu$-measurable field. Let $\mathcal{N}$ be the disintegration of $\mathcal{N}_x$, that is,

$$\mathcal{N} := \int_X \mathcal{N}_x d\mu(x),$$

which acts on the Hilbert space $L^2(X, \mu) \otimes H_0 \otimes L^2(\mathbb{R})$. Using

$$\sigma_i^\varphi = \int_X \sigma_i^{\varphi(t)} d\mu(t),$$

we obtain

$$\int_X \pi_{\sigma \varphi_x} (a_x^j) d\mu(x) = \pi_{\sigma \varphi}(a^j).$$

It is trivial that $\int_X \lambda_{\varphi(t)} d\mu(x) = \lambda_{\varphi(t)}$. Thus $\mathcal{N}$ is nothing but $\mathcal{M} \rtimes_{\sigma \varphi} \mathbb{R}$. Summarizing this discussion, we obtain the following result.

**Lemma 9.10.** In the above setting, one has the following natural identification:

$$\mathcal{M} \rtimes_{\sigma \varphi} \mathbb{R} = \int_X \mathcal{M}_x \rtimes_{\sigma \varphi_x} \mathbb{R} d\mu(x).$$

Let $\alpha \in \text{Aut}(\mathcal{M})$ which fixes any element of $Z(\mathcal{M})$. Then $\alpha$ is described as

$$\alpha = \int_X \alpha_x d\mu(x).$$

By the previous lemma, we have

$$\tilde{\alpha} = \int_X \tilde{\alpha}_x d\mu(x), \quad \text{mod}(\alpha) = \int_X \text{mod}(\alpha_x) d\mu(x).$$

Combining Theorem 9.21, 9.35, 9.40 and Kawahigashi-Sutherland-Takesaki’s result [34, Theorem 1], we obtain the following.

**Theorem 9.11.** Let $\mathcal{M}$ be an injective von Neumann algebra with separable predual. Then the following statements hold:

1. $\overline{\text{Int}(\mathcal{M})} = \ker(\text{mod})$;
2. $\text{Cnt}(\mathcal{M}) = \{\alpha \in \text{Aut}(\mathcal{M}) \mid \tilde{\alpha} \in \text{Int}(\tilde{\mathcal{M}})\}$.

We need the following lemma.

**Lemma 9.12.** Let $\mathcal{M}$ be a von Neumann algebra and $\theta \in \text{Aut}(\mathcal{M})$. Then $\theta \in \overline{\text{Int}(\mathcal{M})}$ if and only if for any $\varepsilon > 0$ and a finite $\Phi \subset \mathcal{M}_1^+$, there exists $a \in \mathcal{M}_1$ such that

$$\|\theta(\varphi) - a \varphi a^*\| < \varepsilon, \quad \|a^*a - 1\|_{\varphi} + \|aa^* - 1\|_{\varphi} < \varepsilon.$$
Proof. By assumption, we obtain a sequence \((a_n)_n\) in \(M_1\) such that for all positive \(\varphi \in M_*\),
\[
\theta(\varphi) = \lim_{n \to \infty} a_n \varphi a_n^*, \quad \lim_{n \to \infty} \|a_n^* a_n - 1\|_\varphi + \|a_n a_n^* - 1\|_\varphi = 0.
\]
This also implies that \(\lim_n a_n^* a_n = \theta^{-1}(\varphi)\).

It turns out that \((a_n)_n\) belongs to \(\mathcal{M}_\omega(M)\). Indeed, let \((x_n)_n \in \mathcal{F}_\omega(M)\) with \(\sup_n \|x_n\| \leq 1\). Then we have
\[
\|x_n a_n \varphi\| \leq \|x_n a_n \varphi \cdot (1 - a_n^* a_n)\| + \|x_n a_n \varphi a_n^* a_n\|
\leq \|\varphi \cdot (1 - a_n^* a_n)\| + \|x_n (a_n \varphi a_n^* - \theta(\varphi)) a_n\| + \|x_n \theta(\varphi) a_n\|
\leq \|1 - a_n^* a_n\| \varphi + \|a_n \varphi a_n^* - \theta(\varphi)\| + \|x_n \theta(\varphi)\|,
\]
and
\[
\|\varphi a_n x_n\| \leq \|(1 - a_n a_n^*) \varphi a_n x_n\| + \|a_n a_n^* \varphi a_n x_n\|
\leq \|1 - a_n a_n^*\| \varphi + \|a_n (a_n^* \varphi a_n - \theta(\varphi)) x_n\| + \|a_n \theta^{-1}(\varphi) x_n\|
\leq \|1 - a_n a_n^*\| \varphi + \|a_n^* \varphi a_n - \theta(\varphi)\| + \|a_n \theta^{-1}(\varphi) x_n\|.
\]
Thus \(\|x_n a_n \varphi\| \to 0\) and \(\|\varphi a_n x_n\| \to 0\) as \(n \to \infty\). Hence \((a_n)_n\) normalizes \(\mathcal{F}_\omega(M)\).

We let \(u := \pi_\omega((a_n)_n) \in M^\omega\) that is a unitary. Take a unitary representing sequence \((u_n)_n\) of \(u\). This satisfies \(u_n - a_n \to 0\) in the strong* topology as \(n \to \omega\), and we are done.

Let \(M\) be a von Neumann algebra and \(A\) a von Neumann subalgebra of \(Z(M)\). Let \((X, \mu)\) be the measure theoretic spectrum of \(A\). Let \(\mathcal{K}\) be the standard Hilbert space of \(M\). Then we have the following disintegrations putting \(A = L^\infty(X, \mu)\) as usual:
\[
M = \int_X M_x d\mu(x), \quad \mathcal{H} = \int_X \mathcal{H}_x d\mu(x).
\]
We may assume that \(\dim \mathcal{H}_x\) is constant in what follows. Let \(\mathcal{K}\) be a Hilbert space with \(\dim \mathcal{K} = \dim \mathcal{H}_x\) for all \(x\). Then \(\mathcal{H}_x\) is regarded as a constant field \(\mathcal{K}_x\), and we obtain the natural identification
\[
\mathcal{H} = \int_X \mathcal{K} d\mu(x) = L^2(X, \mu) \otimes \mathcal{K}.
\]
Note that any automorphism \(\theta\) on \(M_x\) is implemented by a unitary on \(\mathcal{K}\).

Now let \(\alpha\) and \(\beta\) be actions of a locally compact group \(G\) on \(M\) which are fixing \(A\). Then they are written as follows:
\[
\alpha_t = \int_X \alpha_t^x d\mu(x), \quad \beta_t = \int_X \beta_t^x d\mu(x) \quad \text{for all } t \in G.
\]

**Theorem 9.13.** Let \(\alpha\), \(\beta\) be as above. Then the following statements hold:

1. They are cocycle conjugate if and only if \(\alpha^x\) and \(\beta^x\) are for almost every \(x \in X\);
2. They are strongly cocycle conjugate if and only if \(\alpha^x\) and \(\beta^x\) are for almost every \(x \in X\).
Proof. (1). It is useful to consider those actions in terms of a Kac algebra [13]. Namely, \( \alpha, \beta \) are regarded as the faithful normal *-homomorphisms \( \alpha, \beta : M \to M \otimes L^\infty(G) \) by putting \( (\alpha(a)\xi)(t) = \alpha_t(a)\xi(t), (\beta(a)\xi)(t) = \beta_t(a)\xi(t) \) for all \( \xi \in H \otimes L^2(G) \) and \( t \in G \). Then we obtain

\[
(\alpha \otimes \text{id}) \circ \alpha = (\text{id} \otimes \delta) \circ \alpha, \quad (\beta \otimes \text{id}) \circ \beta = (\text{id} \otimes \delta) \circ \beta,
\]

where the coproduct \( \delta : L^\infty(G) \to L^\infty(G) \otimes L^\infty(G) \) is defined by \( \delta(f)(r, s) := f(rs) \) for \( f \in L^\infty(G) \) and \( r, s \in G \).

Take unitary representations \( U, V : G \to B(H) \) such that \( \alpha_t = \text{Ad} U_t \) and \( \beta_t = \text{Ad} V_t \) on \( M \). Regarding \( U, V \in B(H) \otimes L^\infty(G) \), we have

\[
(\text{id} \otimes \delta)(U) = U_{12}U_{13}, \quad (\text{id} \otimes \delta)(V) = V_{12}V_{13},
\]

and

\[
\alpha(a) = U(a \otimes 1)U^\ast, \quad \beta(a) = V(a \otimes 1)V^\ast \quad \text{for all} \quad a \in M.
\]

Since \( \mathcal{A} \) is fixed by \( \alpha \) and \( \beta \), \( U \) and \( V \) are diagonalizable, that is,

\[
U = \int_X U^x d\mu(x), \quad V = \int_X V^x d\mu(x),
\]

where \( U^x, V^x \in B(H) \otimes L^\infty(G) \) and we have used the following identification:

\[
H \otimes L^2(G) = \int_X H \otimes L^2(G) d\mu(x).
\]

Then \( U^x \) and \( V^x \) implement \( \alpha^x \) and \( \beta^x \), respectively, for almost every \( x \). Note that a unitary \( v \in B(H) \otimes L^\infty(G) \) is an \( \alpha^x \)-cocycle if and only if \( v \in M_x \otimes L^\infty(G) \) and \( vU^x \) is a unitary representation, that is, it satisfies

\[
(\text{id} \otimes \delta)(vU^x) = (vU^x)_{12}(vU^x)_{13}.
\]

By Lemma 9.3, we may and do assume that all the relevant measurable maps in what follows are in fact Borel. Take Borel maps \( a^j : X \to (M_x)_1 \) and \( b^k : X \to (M'_x)_1 \) for \( j, k \in \mathbb{N} \).

Let \( \{\xi_i\}_{i \in \mathbb{N}} \) be a dense sequence of \( K \otimes L^2(G) \). Let \( Y_m \) be the subset of \( X \times (B(H) \otimes L^\infty(G))^U \times B(H)^U \) which consists of elements \((x, v, w)\) such that for \( i, j, k = 1, \ldots, m \),

\[
\|v, b_j^k \otimes 1\|\xi_i\| < 1/m, \quad (\text{id} \otimes \delta)(vU^x) = (vU^x)_{12}(vU^x)_{13},
\]

\[
\|[w a_j^k w^\ast, b_j^k]\|\| \| w^\ast a_j^k w^\ast, b_j^k \|\xi_i\| < 1/m, \quad \| (v\alpha^x(a_j^k)v^\ast - (w \otimes 1)U^x(w^\ast a_j^k w \otimes 1)(U^x)^*(w^\ast \otimes 1)) \|\xi_i\| < 1/n.
\]

We can show that \( Y_m \) is Borel as before. Thus \( Y := \bigcap_m Y_m \) is Borel. Then \( (x, v, w) \in Y \) if and only if \( v \in M_x \otimes L^\infty(G) \), \( wM_xw^\ast = M_x \) and

\[
(\text{id} \otimes \delta)(v) = (v \otimes 1)(\alpha^x \otimes \text{id})(v), \quad \text{Ad} v \circ \alpha^x = (\theta \otimes \text{id}) \circ \beta^x \circ \theta^{-1},
\]

where we have put \( \theta := \text{Ad} w|_{M_x} \). By our assumption, \( Y \) is non-empty, and we get the Borel projection \( \text{pr}_X : Y \to X \).
Then there exists a measurable cross section \( s : X \to Y \) with \( \text{pr}_X \circ s = \text{id}_X \). We let \( s(x) = (x, v^x, w^x) \). Hence \( \{v^x\}_x \) and \( \{w^x\}_x \) are measurable, and we set
\[
v := \int_X v^x \, d\mu(x) \in M \otimes L^\infty(G), \quad w := \int_X w^x \, d\mu(x) \in \bigoplus_X B(K) \, d\mu(x).
\]
Put \( \theta := \text{Ad} \, w|_M \) and we obtain
\[
(\text{id} \otimes \delta)(v) = (v \otimes 1)(\alpha \otimes \text{id})(v), \quad \text{Ad} \, v \circ \alpha = (\theta \otimes \text{id}) \circ \beta \circ \theta^{-1}.
\]
Thus we are done.

(2). We first show that the following set is Borel:
\[
Z := \{(x,w) \in X \times B(\mathcal{K})^U \mid \text{Ad} \, w|_{M_x} \in \overline{\text{Int}}(M_x)\}.
\]
Take a norm dense sequence \( \{\varphi^k\}_k \) in \( L^1_{B(\mathcal{K})}(X,\mu) \) such that \( \{\varphi^k|_{M_x}\}_k \) is norm dense in \( M_x \) for almost every \( x \). For \( m \in \mathbb{N} \), we define \( Z_m \subseteq X \times B(\mathcal{K})^U \) which consists of elements \( (x,w) \) such that there exists \( \ell \in \mathbb{N} \) satisfying the following conditions for all \( i,j,k \):

\[
\|\{wa^i_xw^*,b^j_x\} \xi_i\| + \|\{w^*a^i_xw,b^j_x\} \xi_i\| < 1/m,
\]
\[
\|w\varphi^i_xw^*|_{M_x} - a^\ell_x\varphi^i_x(a^\ell_x)^*|_{M_x}\| < 1/m,
\]
\[
\|(a^\ell_x)^*a^\ell_x - 1\} \xi_i\| + \|(a^\ell_x(a^\ell_x)^* - 1)\} \xi_i\| < 1/m.
\]

Then \( Z_m \) is Borel.

We will show that \( Z = \bigcap_m Z_m \). Let \( (x,w) \in \bigcap_m Z_m \). Then \( \theta = \text{Ad} \, w|_{M_x} \in \text{Aut}(M_x) \), and for any \( \varepsilon > 0 \) and a finite set \( \Phi \subseteq (M_x)^*_+ \), there exists an element \( a \in (M_x)_1 \) such that
\[
\|\theta(\varphi) - a\varphi a^*\| < \varepsilon,
\]
\[
\|a^*a - 1\|_{\varphi} + \|aa^* - 1\|_{\varphi} < \varepsilon \quad \text{for all } \varphi \in \Phi.
\]
This implies that \( \theta \) is approximately inner by Lemma \ref{approx}. Hence \( (x,w) \in Z \).

Suppose conversely that \( (x,w) \in Z \). Put \( \theta := \text{Ad} \, w|_{M_x} \in \overline{\text{Int}}(M_x) \). Then for any \( \varepsilon > 0 \) and a finite \( \Phi \subseteq (M_x)^*_+ \), there exists a unitary \( u \in M_x \) such that \( \|\theta(\varphi) - u\varphi u^*\| < \varepsilon \) for \( \varphi \in \Phi \). We can take a subsequence \( \{a^k_x\}_n \) which converging to \( u \) in the strong* topology. Thus \( (x,w) \in \bigcap_m Z_m \).

Therefore \( Z = \bigcap_m Z_m \), which is Borel. We modify \( Y \) defined above as follows:
\[
Y' := Y \cap \{(x,v,w) \mid (x,w) \in Z\}.
\]
By our assumption, \( Y' \) is non-empty, and we obtain the projection \( \text{pr}_X : Y' \to X \). Then we are done in a similar way to the above. \( \square \)

As an application, we obtain the following result due to Kallman and Moore. See \[27\, \text{Theorem 0.1}\] or \[50\, \text{Theorem 5}\].

**Corollary 9.14** (Kallman, Moore). *Any inner flow on a separable von Neumann algebra is implemented by a one-parameter unitary group.*
Proof. Let \( \alpha \) be such a flow on a von Neumann algebra \( \mathcal{M} \). Since \( \alpha \) fixes \( Z(\mathcal{M}) \), we obtain the central decomposition \( \alpha_t = \int_{\mathcal{X}} \alpha_t^x \, d\mu(x) \). Then \( \alpha^x \) is an inner flow on a factor \( \mathcal{M}_x \) for almost every \( x \). Since a \( \mathbb{T} \)-valued 2-cocycle of \( \mathbb{R} \) is a coboundary, \( \alpha^x \) is cocycle conjugate to the trivial flow \( \text{id} \). The previous result implies \( \alpha \sim \text{id} \).

9.3. Perturbation by continuous unitary path. Let \( \varphi \) be a normal state on a von Neumann algebra \( \mathcal{M} \). We need the following basic inequalities:

\[
\|x\varphi\| \leq \|x\|_{\varphi}, \quad \|\varphi x\| \leq \|x^*\|_{\varphi}, \quad \|x\varphi\| + \|\varphi x\| \leq \|x\|_{\varphi} + \|x^*\|_{\varphi} \leq 2\|x\|_{\varphi}.
\]

\[
\|x\|_{\varphi}^2 \leq \|x\varphi\|\|x\|, \quad \|x^*\|_{\varphi}^2 \leq \|\varphi x\|\|x\|, \quad \|x\|_{\varphi}^2 \leq (\|x\|_{\varphi} + \|\varphi x\|) \|x\|/2.
\]

For \( \psi \in \mathcal{M}_x \), let \( \psi = w_t|\psi| \) and \( \psi = |\psi^*|w_r \) be the left and right polar decompositions, respectively. Then

\[
\|\psi x\| = \|\psi x\| \leq \|x\psi^*\|, \quad \|\psi x\| = \|x\psi^*\| \leq \|x\|_{\psi^*},
\]

In what follows, we assume that \( \mathcal{M} = \bigotimes_{k=1}^{\infty} (L_k, \rho_k) \) and \( \varphi_0 := \bigotimes_{k=1}^{\infty} \rho_k \) is lacunary, i.e., 1 is isolated in \( \text{Sp}(\Delta_{\varphi_0}) \), where \( L_k \) is a finite dimensional type I factor, and \( \rho_k \) is a faithful normal state on \( L_k \). Denote \( \hat{L}_k := L_1 \otimes \cdots \otimes L_k \). Note that any injective type II and III \( x \) factors with \( 0 < \lambda < 1 \) have such form (see [2]). Then we can strengthen Lemma 9.6 as follows.

**Proposition 9.15.** Let \( \mathcal{M} \) be an ITPFI factor as above. Let \( \alpha \) and \( \beta \) be flow on \( \mathcal{M} \) with \( \text{mod}(\alpha_t) = \text{mod}(\beta_t) \) for all \( t \in \mathbb{R} \). Then for any \( T > 0 \), \( \varepsilon > 0 \) and a finite set \( \Phi \subset \mathcal{M}_x \), there exists a continuous unitary path \( \{u(t)\}_{|t| \leq T} \) such that

\[
|\text{Ad} u(t) \circ \alpha_t(\varphi) - \beta_t(\varphi)| < \varepsilon, \quad \text{for all } \varphi \in \Phi, \quad t \in [-T, T].
\]

We first recall some results proved in [5].

**Lemma 9.16 ([5] Proposition 3.2, Lemma 2.7).** The following statements hold:

1. There exists a universal constant \( C_0 > 0 \) such that for any von Neumann algebra \( \mathcal{M} \) and its faithful normal state \( \varphi \),

\[
\|\sigma_t^\varphi(x) - x\|_{\varphi}^2 < C_0(1 + |t|)\|x, \varphi\||\!\|x, \varphi\||^{1/2}, \quad \text{for all } t \in \mathbb{R}.
\]

2. Let \( \mathcal{M} \) be a von Neumann algebra and \( \varphi \in \mathcal{M}_x \) a faithful lacunary state. Let \( E_{\varphi} \) be the \( \varphi \)-preserving conditional expectation from \( \mathcal{M} \) onto \( \mathcal{M}_x \). Then there exists a constant \( C_{\varphi} \), which depends only on \( \varphi \), such that for all \( x \in \mathcal{M} \),

\[
\|E_{\varphi}(x) - x\|_{\varphi}^2 < C_{\varphi}\|x, \varphi\||\!\|x, \varphi\||^{1/2}.
\]

**Proof.** (1). See [5]. (2). Choose a positive \( f \in L^1(\mathbb{R}) \) as follows;

\[
\int_{\mathbb{R}} f(t) \, dt = 1, \quad \int_{\mathbb{R}} |t||f(t)| \, dt < \infty, \quad \text{supp}(\hat{f}) \cap \text{Sp}(\Delta_{\varphi}) = \{1\},
\]

where \( \hat{f}(\lambda) = \int_{\mathbb{R}} \lambda^t f(t) \, dt \). Then \( \sigma_t^\varphi(x) = E_{\varphi}(x) \). Set \( C_{\varphi} := \int_{\mathbb{R}} C_0(1 + |t|)|f(t)| \, dt \) and we are done. \( \square \)

The following result can be found in [26] Lemma 3.2.1.
Lemma 9.17. Let $\mathcal{M}$ be a finite von Neumann algebra with a normal tracial state $\tau$, and $a \in \mathcal{M}$ such that $\|a^*a - 1\|_\tau < \delta$. Then there exists a unitary $v \in \mathcal{M}$ such that $\|v - a\|_\tau < (3 + \|a\|)\delta$.

Lemma 9.18. Let $\Phi \subset \mathcal{M}_*$ be a finite set, and assume $v \in U(\mathcal{M})$ satisfies
$$\|\varphi \cdot (v - 1)\| < \varepsilon, \| (v - 1) \varphi \| < \varepsilon, \quad \varphi \in \Phi.$$ Then there exists a continuous unitary path $v(t)$, $0 \leq t \leq 1$, such that $v(0) = 1$, $v(1) = v$ and
$$\|\varphi \cdot (v(t) - 1)\| < \sqrt{2\varepsilon}, \| (v(t) - 1) \varphi \| < \sqrt{2\varepsilon}, \quad \varphi \in \Phi.$$ Proof. Let $v = \int_{-\pi}^{\pi} e^{i\lambda} \, d\lambda$ be the spectral decomposition of $v$. We set $v(t) := \int_{-\pi}^{\pi} e^{it\lambda} \, d\lambda$. Then for all $\varphi \in \Phi$, we have
$$\int_{-\pi}^{\pi} |e^{-i\lambda} - 1|^2 \, d|\varphi|(e) = \|v^*-1\|_{|\varphi|}^2 \leq \|\varphi \cdot (v - 1)\|_v \leq 2\varepsilon.$$ Thus
$$\|v(t)^*-1\|_{|\varphi|}^2 = \int_{-\pi}^{\pi} |e^{-it\lambda} - 1|^2 \, d|\varphi|(e) \leq \int_{-\pi}^{\pi} |e^{-i\lambda} - 1|^2 \, d|\varphi|(e) \leq 2\varepsilon$$ holds for $0 \leq t \leq 1$. Hence
$$\|\varphi \cdot (v(t) - 1)\| \leq \|\varphi \cdot (v - 1)\| \leq \|v(t)^*-1\|_{|\varphi|} \leq \sqrt{2\varepsilon}.$$ If we replace $|\varphi|$ with $|\varphi^*|$ above, then we get
$$\|(v(t) - 1) \varphi\| \leq \sqrt{2\varepsilon}.$$ \hfill $\square$

Lemma 9.19. Let $\alpha$ and $\beta$ be as in Proposition 9.12. For any $k \in \mathbb{N}$, a finite set $F \subset \hat{L}_k$, $T > 0$, and $\varepsilon > 0$, there exists a continuous unitary path $\{u(t)\}_{|t| \leq T}$ such that
$$\|\text{Ad} \, u(t) \circ \alpha_t(\varphi_0 a) - \beta_t(\varphi_0 a)\| < \varepsilon, \quad a \in F, \quad t \in [-T, T].$$

Proof. Let $m := \left(\dim \hat{L}_k\right)^{1/2}$ and $\{e_{ij}\}_{1 \leq i,j \leq m}$ a system of matrix units of $\hat{L}_k$. We may assume that $\{1\} \cup \{e_{ij}\}_{i,j} \subset F \subset M_1$. Set
$$\Phi_0 := \{\varphi_0 a \mid a \in F\}, \quad \Phi := \{a\varphi_0 \mid a \in F\} \cup \{\varphi_0 a \mid a \in F\}.$$ Let $\varepsilon' > 0$. Fix $N \in \mathbb{N}$ with $N \geq m$ so that if $|t| \leq T/N$ and $\varphi \in \Phi$,
$$\|\alpha_t(\varphi) - \beta_t(\varphi)\| < \varepsilon'/4m, \quad \|\alpha_t(\varphi) - \varphi\| < \varepsilon'/4m, \quad \|\beta_t(\varphi) - \varphi\| < \varepsilon'/4m. \quad (9.3)$$ Set $t_0 := T/N$. Since $\alpha_t \beta_t^{-1} \in \text{Int}(\mathcal{M})$ for all $t \in \mathbb{R}$, we can take an $\alpha_0$-cocycle $\{w_n\}_{n \in \mathbb{Z}}$ such that
$$\|\text{Ad} \, w_n \circ \alpha_{nt_0}(\varphi) - \beta_{nt_0}(\varphi)\| \leq \varepsilon'/4m, \quad |n| \leq N, \quad \varphi \in \Phi. \quad (9.4)$$ Set $w := w_1, B := \alpha_{t_0}(\hat{L}_k), \psi_0 := \alpha_{t_0}(\varphi_0), \Psi := \alpha_{t_0}(\Phi)$ and $f_{ij} := \alpha_{t_0}(e_{ij})$. We will find a continuous unitary path $w(t)$, $0 \leq t \leq t_0$ so that $w(t)$ connects 1 and
$w$, and $\text{Ad} w(t) \circ \alpha_t$ approximates $\beta_t$ on $F$. Note $\text{Sp}(\Delta_{\psi_0}) = \text{Sp}(\Delta_{\varphi_0})$. Thus we can assume $C_{\psi_0} = C_{\varphi_0}$ in Lemma 9.16. Using (9.4), we have

\[ \|\text{Ad} w(\psi) - \beta_{t_0} \alpha_{t_0}^{-1}(\psi)\| \leq \varepsilon'/4m, \quad \psi \in \Psi. \]

Thus for $\psi = \alpha_{t_0}(\varphi) \in \Psi$, we have

\[
\|[w, \psi]\| = \|\text{Ad} w(\psi) - \psi\|
\leq \|\text{Ad} w(\psi) - \beta_{t_0} \alpha_{t_0}^{-1}(\psi)\| + \|\beta_{t_0}(\varphi) - \alpha_{t_0}(\varphi)\|
\leq \varepsilon'/2m. \quad (9.5)
\]

Since $\psi_0, \psi_0 f_{ij} \in \Psi$, we have

\[
\|\psi_0 \cdot (wf_{ij} - f_{ij}w)\| \leq \|\psi_0 wf_{ij} - w\psi_0 f_{ij}\| + \|[w, \psi_0 f_{ij}]\|
\leq \|\psi_0 w - w\psi_0\| + \|[w, \psi_0 f_{ij}]\|
\leq \varepsilon'/m.
\]

In the same way, we have $\|(wf_{ij} - f_{ij}w)\psi_0\| < \varepsilon'/m$.

Let $E(x) = m^{-1} \sum_{i,j} f_{ij} x f_{ji}$. Then $E$ is a conditional expectation from $\mathcal{M}$ onto $B' \cap \mathcal{M}$. Set $c := E(w)$, and then

\[
\|\psi_0 \cdot (c - w)\| \leq \frac{1}{m} \sum_{i,j} \|\psi_0 \cdot (f_{ij} w f_{ji} - w f_{ij} f_{ji})\| \leq \frac{1}{m} \sum_{i,j} \|\psi_0 [w, f_{ij}]\| < \varepsilon'.
\]

We also have $\|(c - w)\psi_0\| < \varepsilon'$. Using (9.5), we obtain

\[
\|[c, \psi_0]\| \leq \|[w, \psi_0]\| + \|\psi_0 \cdot (c - w)\| + \|(c - w)\psi_0\| < 3\varepsilon'.
\]

Since $\|w - c\|_{\psi_0} < \varepsilon'$ and $\|(w - c)^*\|_{\psi_0} < \varepsilon'$, we have

\[
\|w - c\|^2_{\psi_0} < \varepsilon'. \quad (9.6)
\]

Set $d := E_{\psi_0}(c)$. Note that $d \in (B' \cap \mathcal{M})_{\psi_0}$, and $\text{Sp}(\Delta_{\psi_0})_{B' \cap \mathcal{M}} \subset \text{Sp}(\Delta_{\varphi_0})$. Thus we can assume $C_{\psi_0} |_{B' \cap \mathcal{M}} = C_{\varphi_0}$ by the definition of $C_{\varphi_0}$ in Lemma 9.16. By the lemma,

\[
\|d - c\|^2_{\psi_0} \leq C_{\varphi_0} \|[c, \psi_0]\|^2 \leq C_{\varphi_0} \sqrt{3\varepsilon'}. \quad (9.6)
\]

Hence by (9.6), we get

\[
\|w - d\|^2_{\psi_0} < \sqrt{3\varepsilon'}(1 + C_{\varphi_0}).
\]

We should note that $[x, \psi] = 0$ for $x \in (B' \cap \mathcal{M})_{\psi_0}$ and $\psi \in \Psi$ since $\Psi = \{b\psi_0, \psi_0 b \mid b \in \alpha_{t_0}(F)\}$ and $\alpha_{t_0}(F) \subset \alpha_{t_0}(\tilde{L}_k) = B$. We have

\[
\|d^* d - 1\|_{\psi_0} \leq \|d^* d - w^* d\|_{\psi_0} + \|w^* d - w^* w\|_{\psi_0}
\leq \|d^* - w^*\|_{\psi_0} + \|d - w\|_{\psi_0}
\leq 2\sqrt{3\varepsilon'}(1 + C_{\varphi_0}).
\]

Thus by Lemma 9.17, we can take a unitary $v \in (B' \cap \mathcal{M})_{\psi_0}$ such that

\[
\|d - v\|_{\psi_0} = \|d - v\|^2_{\psi_0} < 8\sqrt{3\varepsilon'}(1 + C_{\varphi_0}).
Then setting \( \varepsilon'' := \varepsilon' + 8\sqrt{3\varepsilon'(1 + C_{\varphi_0})} \), we obtain
\[
\|w - v\|_{\psi_0} < \varepsilon'' .  \tag{9.7}
\]

Let \( v(t) \in (B' \cap M)_{\psi_0} \), \( t_0/2 \leq t \leq t_0 \), be a continuous path of unitaries with \( v(t_0/2) = v \) and \( v(t_0) = 1 \). Set \( u(t) := wv(t)^* \). Note that \( (B' \cap M)_{\psi_0} \subset M_{\psi_0} \). Since \([v(t), \psi]\) = 0 for all \( \psi \in \Psi \), for all \( \varphi \in \Phi \), we have
\[
\|\text{Ad} u(t) \circ \alpha_{t_0}(\varphi) - \beta_{t_0}(\varphi)\| = \|\text{Ad} w \circ \alpha_{t_0}(\varphi) - \beta_{t_0}(\varphi)\| < \varepsilon'/4m.  \tag{9.8}
\]

We will find a path which connects 1 and \( wv^* \). For \( b \in \alpha_{t_0}(F) \), we obtain
\[
\|([\psi_0 b, wv] - \psi_0 v)\| = \|\psi_0 bw - \psi_0 v\| 
\leq \|[\psi_0 b, w]\| + \|[w\psi_0 b - \psi_0 v]\| 
\leq \|[\psi_0 b, w]\| + \|[w\psi_0 - \psi_0 v]\| 
\leq \|[\psi_0 b, w]\| + \|[w, \psi_0]\| + \|\psi_0 w - \psi_0 v\| 
\leq \varepsilon' + 2\varepsilon'' \quad \text{by (9.4), (9.5), (9.7),}
\]
and by (9.7),
\[
\|(wv^* - 1)\psi_0 b\| = \|(w - v)\psi_0 v\| \leq \|(w - v)\psi_0\| \leq 2\varepsilon''.
\]
Hence by Lemma 9.18 there exists a continuous unitary path \( u(t), 0 \leq t \leq t_0/2 \) such that \( u(0) = 1, u(t_0/2) = wv^* \) and
\[
\|(u(t) - 1)\psi_0 a\| \leq \sqrt{2(\varepsilon' + 2\varepsilon'')} \quad \text{and} \quad \|(u(t) - 1)\psi_0 a\| \cdot (\psi_0 a) \cdot (u(t) - 1)\| \leq \sqrt{2(\varepsilon' + 2\varepsilon'')}, \quad a \in \alpha_{t_0}(F).
\]

Summarizing the inequalities (9.3) and (9.8), we get a continuous unitary path \( u(t), 0 \leq t \leq t_0 \), such that \( u(0) = 1, u(t_0) = w \), and
\[
\|\text{Ad} u(t) \circ \alpha_{t_0}(\varphi) - \beta_{t_0}(\varphi)\| < 2\sqrt{2(\varepsilon' + 2\varepsilon'')} + \varepsilon', \quad \varphi \in \Phi_0, 0 \leq t \leq t_0.
\]
Then we have
\[
\|\text{Ad} u(t) \circ \alpha_t(\varphi) - \beta_t(\varphi)\| < 2\sqrt{2(\varepsilon' + 2\varepsilon'')} + 2\varepsilon', \quad \varphi \in \Phi_0, 0 \leq t \leq t_0.
\]
For \( t \in [-T, T] \), let \( t = nt_0 + s, 0 \leq s < t_0 \), and define \( u(t) := wn_0\alpha_{nt_0}(u(s)) \). Then \( u(t) \) is a continuous unitary path and for \( \varphi \in \Phi_0 \),
\[
\|\text{Ad} u(t) \circ \alpha_t(\varphi) - \beta_t(\varphi)\| 
= \|\text{Ad} w_n \circ \alpha_{nt_0} \circ \text{Ad} u(s) \circ \alpha_s(\varphi) - \beta_{nt_0} \circ \beta_s(\varphi)\| 
\leq \|\text{Ad} w_n \circ \alpha_{nt_0} \circ \text{Ad} u(s) \circ \alpha_s(\varphi) - \text{Ad} w_n \circ \alpha_{nt_0} \circ \beta_s(\varphi)\| 
+ \|\text{Ad} w_n \circ \alpha_{nt_0} \circ \beta_s(\varphi) - \beta_{nt_0} \circ \beta_s(\varphi)\| 
\leq \|\text{Ad} u(s) \circ \alpha_s(\varphi) - \beta_s(\varphi)\| + \varepsilon'/m + \|\text{Ad} w_n \circ \alpha_{nt_0}(\varphi) - \beta_{nt_0}(\varphi)\| 
< 2\sqrt{2(\varepsilon' + 2\varepsilon'')} + 3\varepsilon' \quad \text{by (9.4).}
\]
For any given \( \varepsilon > 0 \), we choose \( \varepsilon' \) as \( 2\sqrt{2(\varepsilon' + 2\varepsilon'')} + 3\varepsilon' < \varepsilon \), and we get the conclusion. \( \square \)
Proof of Proposition 9.15. Let \( \Phi := \{ \psi_i \}_{i=1}^n \) be a finite set of \( M^* \). Since the set \( \{ \varphi a \mid a \in \bigcup_{k=1}^\infty \hat{L}_k \} \) is dense in \( M^* \), there exist \( k > 0 \) and \( \{ a_i \}_{i=1}^n \subset \hat{L}_k \) such that \( \| \psi_i - \varphi a_i \| < \varepsilon/3 \). By Lemma 9.19 there exists a continuous path of unitaries \( u(t), |t| \leq T \), such that
\[
\| \text{Ad} u(t) \circ \alpha_t(\varphi a_i) - \beta_t(\varphi a_i) \| < \varepsilon/3, \quad 1 \leq i \leq n, \quad |t| \leq T.
\]
Then we obtain
\[
\| \text{Ad} u(t) \circ \alpha_t(\psi_i) - \beta_t(\psi_i) \| < \varepsilon, \quad 1 \leq i \leq n, \quad |t| \leq T.
\]
\[\square\]

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