On the moduli space of diffeomorphic algebraic surfaces

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Abstract. In this paper we show that the number of deformation types of complex structures on a fixed smooth oriented four-manifold can be arbitrarily large. The examples that we consider in this paper are locally simple abelian covers of rational surfaces. The proof involves the algebraic description of rational blow down, classical Brill-Noether theory and deformation theory of normal flat abelian covers.

0. Introduction

One of the main problems concerning the differential topology of algebraic surfaces leaving unsolved by the “Seiberg-Witten revolution” was to determine whether the differential type of a compact complex surface determines the deformation type. Two compact complex manifolds have the same deformation type if they are fibres of a proper smooth family over a connected base space (cf. [FrMo1]). Varieties of the same deformation type are also called deformation equivalent.

Here we restrict our attention to minimal surfaces of general type; in this case it is very useful to interpret the above question in terms of the moduli space of surfaces of general type $M$.

The space $M$ is a disjoint countable union of quasiprojective varieties $[Gi]$, the points of $M$ correspond to the isomorphism classes of minimal surfaces of general type and two surfaces belong to the same connected component if and only if they are deformation equivalent (cf. [Gi], [Ma4], [Ca2]).

The main result of this paper is the following

Theorem A. Let $X$ be a smooth oriented compact four-manifold and let $M_X$ be the moduli space of minimal surfaces of general type orientedly diffeomorphic to $X$.

Then for every $k > 0$ there exists $X$ as above with first Betti number $b_1(X) = 0$ such that $M_X$ has at least $k$ connected components.

By the above remark this result gives a strong negative answer to the def=diff? problem ([Ca5, §7], [FrMo1, Speculation of §3], [Ty], [Do]). Note that for compact complex surfaces with $b_1 = 0$ and Kodaira dimension $< 2$ the differential type determines the deformation type [Fri].
We recall that for an algebraic surface, $K^2$ and $\chi$ are topological invariants; by Gieseker theorem $\mathcal{M}_X$ is a quasiprojective variety and then it has a finite number of components.

Analsogs of theorem A for the moduli space of homeomorphic complex surfaces already exist in [Ca4], [Ma2], [Ma3] and rely over the very simple algebro-geometric criterion for homeomorphism given by Freedman’s theorem ([Fre, 1.5], [Ca1, 4.4], [Ma4, V.4]).

In view of some recent complexity results (especially [Ma3], [Ch]) and of the algebraic dependence of Donaldson and Seiberg-Witten invariants, theorem A is not completely surprising.

The starting idea of this paper is the more subtle end effective evidence to theorem A given by the existence of Kollár - Shepherd-Barron - Alexeev compactification of the moduli space of surfaces of general type ([KS], [Vie], [Ale]). For given positive integers $a, b$ let $M_{a,b}$ be the moduli space of minimal surfaces of general type $S$ with numerical invariants $K_S^2 = a, \chi(\mathcal{O}_S) = b$; it is then possible to embed $\mathcal{M}_{a,b}$ into a complete variety $\overline{\mathcal{M}}_{a,b}^{\text{sm}}$ which is a coarse moduli space for smoothable stable surfaces of general type [Vie, 8.39], [Ale], with numerical invariants $K^2 = a, \chi = b$.

As noted in [KS, 5.12] it may happen that nonhomeomorphic smooth surfaces belong to the same connected component of $\overline{\mathcal{M}}_{a,b}^{\text{sm}} = \bigsqcup \mathcal{M}_{a,b}$; the substantial reason of this “pathology” is the existence of certain normal semilognormal canonical surface singularities, each one of which admits at least two $\mathbb{Q}$-Gorenstein smoothings with nonhomemorphic Milnor fibres.

One can avoid this phenomenon by considering the moduli space $\mathcal{M}^T \subset \overline{\mathcal{M}}_{a,b}^{\text{sm}}$ of surfaces with at most quotient singularities. The smoothability condition implies that only some special quotient singularities, called of class $T$ (Definition 1.1), can appear in the surfaces represented by points of $\mathcal{M}^T$.

By a well known result [EV] $\mathcal{M}^T$ is open in $\overline{\mathcal{M}}_{a,b}^{\text{sm}}$ and it is possible to prove that smooth minimal surfaces belonging to the same connected component of $\mathcal{M}^T$ are diffeomorphic. Because of the extreme wilderness of the moduli space of surfaces of general type it is natural to suspect that the natural map $\pi_0(\mathcal{M}) \to \pi_0(\mathcal{M}^T)$ is not injective.

In practice one observe that the compactified moduli space $\overline{\mathcal{M}}_{a,b}^{\text{sm}}$ is a quite complicated object whose existence is not necessary to the proof of theorem A; for this reason throughout the rest of the paper we consider the simpler notion of deformation $T$-equivalence (Definition 1.4) instead of $\mathcal{M}^T$. Roughly speaking two smooth surfaces are deformation $T$-equivalent if they are fibres of a proper flat map $f : S \to Y$ such that $Y$ is connected, $S$ is $\mathbb{Q}$-Gorenstein and the fibres of $f$ are normal surfaces with at most quotient singularities.

The main result of section 1 (Theorem 1.5) says that deformation $T$-equivalence implies smooth equivalence. The proof of this fact follows essentially from the existence of certain well-defined surgery operations on the category of smooth four-manifolds called “rational blow-downs” and already considered in [FS].

The simplest singularity $X_0$ of class $T$ which is not a rational double point is the cone over the projectively normal rational curve of degree 4 in $\mathbb{P}^4$; this singularity can be easily described as
a bidouble cover $X_0 \to \mathbb{C}^2$ with branching divisors three generic lines passing through $0 \in \mathbb{C}^2$. Moreover every $\mathbb{Q}$-Gorenstein deformation of $X_0$ preserves the $(\mathbb{Z}/2)^2$-action and it is obtained by deforming the branching divisors (cf. 3.18).

This simple remark, together with our previous experience about abelian covers, motivates the use of $(\mathbb{Z}/2)^r$-covers in the construction of nontrivial examples of deformation T-equivalent surfaces. A short but fundamental role in this paper is covered by classical Brill-Noether theory. In fact the (not yet completely developed) machinery of abelian covers allows to produce components of the moduli space of surfaces by starting from components of parameter spaces of branching divisors. The Brill-Noether theory gives examples of disconnected spaces $M_{k,n}^0$ (see §2) of branching divisors of the same topological type.

The detailed construction of our examples is too complicated to be explained in this introduction; however we can give here a coarse idea of their structure. Given a polarization $H$ over $Q = \mathbb{P}^1 \times \mathbb{P}^1$, a zerodimensional subscheme (Cluster) $\xi \subset Q$ with ideal sheaf $I_\xi$ is called special if $I_\xi(H)$ is generated by global sections; we first take the blow up $S \to Q$ along a reduced special subscheme $\xi$ of finite length $n < H^2$, then we consider a suitable abelian covers with group $(\mathbb{Z}/2)^r$ of $S$; in the construction of the cover we must take as branching divisors $D_\sigma \subset S$ the exceptional curves of $S \to Q$, the strict transform of some sections of $I_\xi(H)$ and some other very ample divisors. For an appropriate choice of $H, n$ and $D_\sigma$ we get the desired examples.

Most part of the complexity of this construction is introduced for technical reasons (because the ramification of an abelian cover can be viewed as an obstruction to deformations and degenerations). Together with the working construction we also give (Example 3.20) some very simple examples of pairs of diffeomorphic surfaces that, at least in some case, we conjecture of different deformation type.

The paper consists of 5 sections. The first four are devoted to some preparatory material, most of which we consider of independent interest and susceptible of future applications (cf. example 3.15). Section 5 is completely devoted to the construction of explicit examples of arbitrarily large sets of deformation T-equivalent surfaces belonging to different connected components of the moduli space $M$.

**Notation.** All the varieties and schemes that we consider in this paper are defined over the field $\mathbb{C}$. For every variety $X$ we denote by $\Omega^1_X$, $\theta_X = \mathcal{H}om(\Omega^1_X, \mathcal{O}_X)$ the cotangent and tangent sheaves respectively.

A cluster on a variety $X$ is a zero-dimensional subscheme $\xi \subset X$, given a cluster $\xi$ we denote by $I_\xi \subset \mathcal{O}_X$ its ideal sheaf $a$ for a coherent sheaf $\mathcal{F}$ we sometimes write $\mathcal{F} - \xi = I_\xi \otimes \mathcal{F}$. We shall say that a cluster $\xi$ is $\mathcal{F}$-special if $\mathcal{F} - \xi$ is generated by global sections.

In the paper we shall use freely the following results of commutative algebra (cf. [Mat, §23]).

A) Let $G$ be a finite group acting on a local notherian $\mathbb{C}$-algebra $A$ and let $A^G \subset A$ be the invariant subalgebra. If $A$ is normal (resp. Cohen-Macaulay) then $A^G$ is normal (resp.
Cohen-Macaulay).

B) Let \((A, m) \to (B, n)\) be a flat morphism of local noetherian rings. Then \(B\) is Cohen-Macaulay (resp. Gorenstein) if and only if \(A\) and \(B/mB\) are both Cohen-Macaulay (resp.: Gorenstein). If \(A\) and \(B/mB\) are reduced (resp.: normal) then \(B\) is reduced (resp.: normal).

1. Deformation \(T\)-equivalence of algebraic surfaces.

In this section we introduce the notion of deformation \(T\)-equivalence of algebraic surfaces. It will be clear from the definition that deformation equivalence implies deformation \(T\)-equivalence, while the main result of this section will be the proof that deformation \(T\)-equivalence implies smooth equivalence.

**Definition 1.1.** ([KS, 3.7]) A normal surface singularity is of class \(T\) if it is a quotient singularity and admits a one-parameter \(\mathbb{Q}\)-Gorenstein smoothing.

We recall that a normal complex space is \(\mathbb{Q}\)-Gorenstein if it is Cohen-Macaulay and a multiple of the canonical divisor is Cartier.

A one-parameter \(\mathbb{Q}\)-Gorenstein smoothing of a normal singularity \((X_0, 0)\) is a deformation \((X, 0) \to (\mathbb{C}, 0)\) of \((X_0, 0)\) such that there exists a Stein representative (as in [Lo, 2.8]) \(X \to \Delta \subset \mathbb{C}\) which is \(\mathbb{Q}\)-Gorenstein and \(X_t\) is smooth for every \(t \neq 0\).

The singularities of class \(T\) and their smoothings are well understood (cf. e.g. [KS], [LW], [Wa2], [Ma7]). Before recalling the classification we recall a standard notation concerning cyclic singularities.

If \(p > 0, a, b\) are integers without common irreducible factors we shall call cyclic singularity of type \(\frac{1}{p}(a, b)\) the quotient singularity \(\mathbb{C}^2/\mu_p\), where the group \(\mu_p = \{\xi \in \mathbb{C}|\xi^p = 1\}\) acts on \(\mathbb{C}^2\) by the diagonal action \(\xi(u, v) = (\xi^a u, \xi^b v)\). It is well known (see e.g. [BPV, III.5]) that every cyclic quotient singularity is isomorphic to a singularity of type \(\frac{1}{p}(1, q)\) with \(p, q\) relatively prime.

**Proposition 1.2.** The singularities of class \(T\) are the following:

i) Smooth points.

ii) Rational double points.

iii) Cyclic singularities of type \(\frac{1}{dn^2}(1, dna - 1)\) for \(a, d, n > 0\) and \(a, n\) relatively prime.

**Proof.** This is a well known result, for a proof see e.g. [KS, 3.11], [Ma7].

\[\square\]

**Definition 1.3.** A surface of class \(T\) is a normal algebraic surfaces with at most singularities of class \(T\).

**Definition 1.4.** The deformation \(T\)-equivalence is the equivalence relation in the set of isomorphism classes of surfaces of class \(T\) generated by the following relation \(\sim:\)
Given two surfaces of class $T$, $S_1, S_2$ we set $S_1 \sim S_2$ if they are fibres of a proper flat analytic family $f: S \to C$ such that $C$ is a smooth irreducible curve, every fibre of $f$ is of class $T$ and $S$ is $\mathbb{Q}$-Gorenstein.

It is useful to point out that the property of being $\mathbb{Q}$-Gorenstein for a one-parameter normal flat family is stable under base change and then, in the notation of 1.4, if $B \to C$ is a nonconstant morphism of smooth curves then $S \times_C B$ is still $\mathbb{Q}$-Gorenstein.

**Theorem 1.5.** If two smooth surfaces $S_1, S_2$ are deformation $T$-equivalent then there exists an orientation preserving diffeomorphism $S_1 \simeq S_2$.

Before proving 1.5 we need to recall some results about diffeomorphism of lens spaces and classification of $\mathbb{Q}$-Gorenstein deformations of quotient singularities of class $T$.

For every oriented smooth manifold (possibly with boundary) $X$ we consider the following groups:

1) $Diff(X)$ the group of diffeomorphism $X \to X$.
2) $Diff^+(X) \subset Diff(X)$ the subgroup of orientation preserving diffeomorphism.
3) $Diff_0^+(X) \subset Diff^+(X)$ the subgroup of orientation preserving diffeomorphism which are isotopic to the identity.

Let $p, q$ be relatively prime positive integers and let $S^3 = \{(x_1, x_2) \in \mathbb{C}^2||x_1|^2 + |x_2|^2 = 1\}$ be the three-dimensional sphere. The lens space $L(p, q)$ is by definition the quotient $S^3/\mu_p$, where the action is given by $\xi(x_1, x_2) = (\xi x_1, \xi^q x_2)$. The action is free and orientation preserving; therefore $L(p, q)$ has a natural structure of oriented 3-manifold.

Let $\tau: \mathbb{C}^2 \to \mathbb{C}^2$ be the complex conjugation $\tau(x_1, x_2) = (\overline{x_1}, \overline{x_2})$, we have $\tau \xi = \xi^{-1} \tau$ for every $\xi \in \mu_p$ and therefore $\tau$ factors to an orientation preserving diffeomorphism $\tau \in Diff^+(L(p, q))$.

Let $\sigma: \mathbb{C}^2 \to \mathbb{C}^2$ be the involution defined by $\sigma(x_1, x_2) = (x_2, x_1)$. If $q^2 \equiv 1 \mod(p)$ then $\sigma \xi = \xi^q \sigma$ and therefore $\sigma$ induces a diffeomorphism of $\sigma \in Diff^+(L(p, q))$. It is notationally convenient to define $\sigma \in Diff^+(L(p, q))$ as the identity if $q^2 \not\equiv 1 \mod(p)$.

There are other “trivial” diffeomorphisms of lens spaces; these are given by choosing a pair $(\alpha, \beta) \in S^1 \times S^1$ and passing to the quotient the diffeomorphism $\varrho_{(\alpha, \beta)}(x_1, x_2) = (\alpha x_1, \beta x_2)$. It is immediate to observe that every diffeomorphism $\varrho_{(\alpha, \beta)}$ is isotopic to the identity. A simple computation also shows that if $q \equiv 1 \mod(p)$ (resp.: $q \equiv -1 \mod(p)$) then $\sigma$ (resp.: $\sigma \tau$) is conjugated to $\varrho_{(1, -1)}$ and then it is isotopic to the identity.

The result we need is

**Proposition 1.6.** (Bonahon) The group $\frac{Diff^+(L(p, q))}{Diff_0^+(L(p, q))}$ is generated by $\tau$ and $\sigma$.

**Proof.** This is exactly Proposition 2 in [Bon]. Actually a little calculation, left as exercise, is needed because Bonahon consider the lens space as a union of two solid tori $V_1, V_2 \simeq S^1 \times D^2$ and define the diffeomorphisms $\sigma, \tau$ in terms of toroidal coordinates on $V_1, V_2$.

$\square$
Note that if \( p = dn^2, q = dna - 1 \) with \( (a, n) = 1 \) then \( q^2 \equiv 1 \mod(p) \) if and only if \( a = 1, n \leq 2 \).

For reader convenience we recall the notion of link of an isolated singularity and of Milnor fibre of a smoothing. For more details and proofs we refer to [Mi], [Lo] and [Wa1]. For simplicity of exposition we consider here only the case of normal surface singularities.

Let \((X_0, 0)\) be a normal surface singularity (hence isolated, irreducible and Cohen-Macaulay) and let \(io:(X_0, 0) \to (\mathbb{C}^N, 0)\) be a closed embedding. A basic result [Mi, 2.9] asserts that there exists a positive real number \( r \) such that for every \( 0 < r' \leq r \) the sphere \( S_{r'} = \{ z \in \mathbb{C}^N | \| z \| = r' \} \) intersects transversally \( X_0 \). We shall call the oriented smooth compact 3-manifold \( L(X_0) = X_0 \cap S_{r'} \), \( 0 < r' << 1 \), the link of \((X_0, 0)\). It is not difficult to prove that different choices of the embedding \( i_0 \) and of \( r' \) give isotopic links inside \( X_0 - \{0\} \). More precisely it is proved in [Lo] that for every pair \( d_1, d_2:X_0 \to [0, +\infty) \) of real analytic functions such that \( d^{-1}_1(0) = d^{-1}_2(0) = \{0\} \) there exists \( \epsilon > 0 \) such that for every \( r_1 \leq \epsilon \), \( r_2 \leq \epsilon \) the loci \( d^{-1}_1(r_1) \) and \( d^{-1}_2(r_2) \) are isotopic smooth subvarieties of \( X_0 - \{0\} \).

Note that \( L(p, q) \) is the link of the cyclic singularity \( X_0 \) of type \( \frac{1}{p}(1, q) \); in fact it is sufficient to take as real analytic function \( d:X_0 \to [0, +\infty) \) the usual norm of \( \mathbb{C}^2 \) which is clearly \( \mu_p \) invariant.

Assume now that \( f:(X, 0) \to (\mathbb{C}, 0) \) is a one-parameter smoothing of \((X_0, 0)\) and let \( i:(X, 0) \to (\mathbb{C}^N \times \mathbb{C}, 0) \) be a closed embeddings extending \( i_0 \) such that \( f \) is the composition of \( i \) with the projection on the second factor. Let \( r > 0 \) as above, for every \( 0 < r' \leq r \) there exists a \( \delta > 0 \) such that \( X_t = f^{-1}(t) \) intersects transversally \( S_{r'} \) for every \( t \in \mathbb{C}, |t| \leq \delta \).

If \( 0 < |t| << 1 \) the smooth oriented manifold with boundary \( F = X_t \cap \{ z \in \mathbb{C}^N | \| z \| \leq r' \} \) is called the Milnor fibre of the smoothing. Again its diffeomorphism class is independent from the choice of \( i, r' \) an \( t \) and by Ehresmann fibration theorem \( \partial F = L(X_0) \). If \((X_0, 0)\) is a complete intersection then the diffeomorphism type of \( F \) is also independent from the smoothing [Lo], in general \( F \) is not uniquely determined by \((X_0, 0)\).

By Lefschetz duality we have \( H^2(F, \mathbb{Z}) = H_2(F, \partial F, \mathbb{Z}) \) and the natural map \( H_2(F, \mathbb{Z}) \to H_2(F, \partial F, \mathbb{Z}) \) gives the intersection product \( q:H_2(F, \mathbb{Z}) \times H_2(F, \mathbb{Z}) \to \mathbb{Z} \).

It is also trivial to see that the Milnor fibre is invariant under base change and if the smoothing admits simultaneous resolution [Tya] then \( F \) is isomorphic to a neighbourhood of the exceptional curve of the minimal resolution of \( X_0 \).

**Proposition 1.7.** The Milnor fibre \( F \) of a \( \mathbb{Q} \)-Gorenstein smoothing of a cyclic singularity of class \( T \) is unique up to orientation preserving diffeomorphism. Moreover the restriction homomorphism \( Diff^+(F) \to Diff^+(\partial F) \) is surjective.

**Proof.** As \( \partial F \) is collared in \( F \) the restriction map \( Diff^+_0(F) \to Diff^+_0(\partial F) \) is surjective. As \( \partial F \) is a lens space, according to 1.6 it is sufficient to prove that \( \sigma, \tau \) lift to \( Diff^+(F) \).
Let $d, n, a$ be fixed positive integers with $(a, n) = 1$ and let $X_0$ be the cyclic quotient singularity of type $\frac{1}{dn^2}(1, dnn-1)$. If $\xi \in \mu_{dn^2}$ then $\xi$ acts on $\mathbb{C}^2$ by a linear transformation of determinant $\xi^{dnn}$ and then the subgroup $\mu_{dn} \subset \mu_{dn^2}$ acts on $\mathbb{C}^2$ by linear transformations preserving the holomorphic form $dx_1 \wedge dx_2$. In particular $X_0$ can be described as $Y_0/\mu_n$, where $Y_0 = \mathbb{C}^2/\mu_{dn}$ is the rational double point of type $A_{dn-1}$ (i.e. the cyclic singularity of type $\frac{1}{dn}(1, -1)$).

Setting $u = x_1^{dn}, v = x_2^{dn}$ and $y = x_1x_2$ we can describe $Y_0$ as the hypersurface singularity of $\mathbb{C}^3$ defined by the equation $uv - y^{dn} = 0$; in this coordinates $\mu_n$ acts by

$$\mu_n \ni \xi; (u, v, y) \mapsto (\xi u, \xi^{-1}v, \xi^a y)$$

Moreover the diffeomorphisms $\tau, \sigma$ of $L(Y_0)$ are induced by $\tau, \sigma \in Diff(\mathbb{C}^3)$

$$\tau(u, v, y) = (\overline{u}, \overline{v}, \overline{y}), \quad \sigma(u, v, y) = (v, u, y).$$

It is known [KS, 3.17], that every $\mathbb{Q}$-Gorenstein one-parameter deformation of $(X_0, 0)$ is isomorphic to the pull-back, via a suitable holomorphic germ map $(\mathbb{C}, 0) \to (\mathbb{C}^d, 0)$, of the deformation $\pi: Y/\mu_n \to \mathbb{C}^d$, where $t_0, ..., t_{d-1}$ are coordinates over $\mathbb{C}^d$. If $Y \subset \mathbb{C}^3 \times \mathbb{C}^d$ is the hypersurface of equation $uv - y^{dn} = \sum_{k=0}^{d-1} t_k y^k$ and $\mu_n$ acts as

$$\mu_n \ni \xi; (u, v, y, t_0, ..., t_{d-1}) \mapsto (\xi u, \xi^{-1}v, \xi^a y, t_0, ..., t_{d-1}).$$

Since $\mathbb{C}^d$ is locally irreducible, the basic theory of the discriminant tell us that all the Milnor fibres of $\mathbb{Q}$-Gorenstein smoothings of $X_0$ are orientedly diffeomorphic. It is therefore sufficient to prove that $\sigma, \tau$ extends to diffeomorphism of $Y_t/\mu_n$ for some $t \in \mathbb{C}^d$; this is immediate to check if $t = (t_0, 0, ..., 0)$ for $t_0$ a real number $0 < t_0 << 1$.

\[\square\]

**Example 1.8.** A simple calculation which we omit shows that the Milnor fibre of a $\mathbb{Q}$-Gorenstein smoothing of the cyclic singularity of type $\frac{1}{4}(1, 1)$ is the complement in $\mathbb{P}^2$ of a tubular neighbourhood of the conic of equation $uv - y^2 = 0$. The diffeomorphism $\tau, \sigma$ are induced by the diffeomorphism of $\mathbb{P}^2$

$$\tau(u, v, y) = (\overline{u}, \overline{v}, \overline{y}), \quad \sigma(u, v, y) = (v, u, y).$$

**Proof of 1.5.** We prove 1.5 by associating to every surface $S$ of class $T$ an oriented smooth 4-manifold $\hat{S}$ which is equal to $S$ when $S$ has no singular points; next we will prove that if $S_1 \sim S_2$, where $\sim$ is the relation introduced in 1.4, then $\hat{S}_1$ is diffeomorphic to $\hat{S}_2$.

As a first step we define for every surface of class $T$, $\hat{S} = \hat{S}'$ where $S' \to S$ is the minimal resolution of all rational double points of $S$.

Let $S$ be a surface with at most cyclic singularities of class $T$ at points $p_1, ..., p_n$. Choose closed embeddings $\phi_i: (S, p_i) \to (\mathbb{C}^N, 0), \ r > 0$ a sufficiently small real number and denote
study of the Kuranishi families of singularities of class $T_r > t$. If $F_i$ is the Milnor fibre of a \( \mathbb{Q} \)-Gorenstein smoothing of \((S, p_i)\) we define

\[
\hat{S} = (S - \cup_i V_i) \cup (\cup_i F_i)
\]

where the pasting is made by choosing for every \( i \) an orientation preserving diffeomorphism $\partial V_i \to \partial F_i$. By proposition 1.7 \( \hat{S} \) is a well defined smooth oriented four manifold.

By a simple topological argument it is sufficient to prove that if $f: S \to \Delta = \{ t \in \mathbb{C} \mid |t| < 1 \}$ is a \( \mathbb{Q} \)-Gorenstein deformation of a surface $S_0$ of class $T$ then $\hat{S_0} = \hat{S}_t$ for $|t| << 1$.

By using the simultaneous resolution of rational double points we can assume without loss of generality that $S_0$ has at most cyclic singularities at points $p_1, \ldots, p_n$.

\textbf{Step 1).} Assume first that $S_t$ is smooth for $t \neq 0$, $\phi_i: (S, p_i) \to (\mathbb{C}^{N_i} \times \Delta, 0)$ closed embeddings; then for $r > 0$ sufficiently small, $V_i = S_0 \cap \phi_i^{-1}(B_r)$, $F_i = S_t \cap \phi_i^{-1}(B_r)$, $t \neq 0$. By integration of vector fields it is easy to construct, for $0 < |t| << 1$, a diffeomorphism $(S_0 - \cup_i V_i) \to (S_t - \cup_i F_i)$ sending $\partial V_i$ into $\partial F_i$ and then by proposition 1.7

\[
\hat{S}_0 = (S_0 - \cup_i V_i) \cup (\cup_i F_i) = (S_t - \cup_i F_i) \cup (\cup_i F_i) = S_t.
\]

\textbf{Step 2).} Assume now $S_t$ possibly singular and let $p \in S_0$ be a singular point; by using the classification theorem of $\mathbb{Q}$-Gorenstein deformations of singularities of class $T$ (see the proof of 1.7) it is immediate to find a $\mathbb{Q}$-Gorenstein deformation $\tilde{f}: (\hat{S}, p) \to (\Delta_t \times \Delta_s, 0)$ such that $f$ is the pull-back of $\tilde{f}$ via the inclusion $\Delta_t \times \{0\} \subset \Delta_t \times \Delta_s$ and $S_{t,s}$ is smooth for $s \neq 0$.

Therefore $\hat{S}$ gives a local simultaneous $\mathbb{Q}$-Gorenstein smoothing of the fibres of $f$. Step 1) and a simple pasting argument concludes the proof.

\[ \square \]

\textbf{Remark.} Our first proof of 1.5 didn’t make use of proposition 1.7 but used a more careful study of the Kuranishi families of singularities of class $T$; however this approach (i.e. assuming Bonahon’s theorem) is considerably simpler and more elegant.

If $V \to S$ is the minimal resolution of a surface of class $T$ then, in the terminology of [FS], $\hat{S}$ is obtained by rationally blowing down $V$.

\section*{2. Special clusters on $\mathbb{P}^1 \times \mathbb{P}^1$ and related spaces.}

Let $Q = \mathbb{P}^1 \times \mathbb{P}^1$ be the quadric and let $H = \mathcal{O}_Q(a, b)$ be a fixed polarization over $Q$ with $a, b \geq 3$; we have $H^2 = 2ab$.

For every pair of integers $k \geq 0$, $0 \leq n \leq 2ab$, let $M^{a,b}_{k,n}$ be the subset of $|H|^k \times Q^n$ consisting of the elements $(C_1, \ldots, C_k, p_1, \ldots, p_n)$ such that:

\begin{itemize}
  \item[i)] the curves $C_i$’s are smooth.
  \item[ii)] For every $i \neq j$ $C_i$ intersect transversally $C_j$.
  \item[iii)] $p_i \in C_j$ for every $i, j$, i.e. the points $p_i$ are contained in the base locus of the linear system generated by $C_1, \ldots, C_k$.
\end{itemize}
iv) \( p_i \neq p_j \) if \( i \neq j \).

For simplicity of notation we shall write \( M_{k,n} \) instead of \( M_{k,n}^{a,b} \) whenever there is non ambiguity about the polarization.

The set \( M_{k,n} \) carries a natural structure of locally closed subscheme of \(|H|^k \times Q^n\). For \( l \leq k \), \( m \leq k \) we shall call natural projection \( M_{k,n} \to M_{l,m} \) the map

\[
(C_1, ..., C_k, p_1, ..., p_n) \to (C_1, ..., C_l, p_1, ..., p_m);
\]

it is clearly a regular morphism of schemes.

If \( Aut_0(Q) = PGL(2) \times PGL(2) \) denotes the group of biregular automorphisms of \( Q \) acting trivially over \( Pic(Q) \) then there exists a natural regular action of \( Aut_0(Q) \times \Sigma_k \times \Sigma_n \) over \( M_{k,n} \), where \( \Sigma_i \) is the symmetric group of permutations of \( i \) elements.

**Lemma 2.1.** The scheme \( M_{k,n} \) is connected, if \( k \leq 2 \) it is also smooth irreducible.

**Proof.** If \( k \leq 1 \) or \( n = 0 \) the lemma is trivially true. By the implicit function theorem the natural projection \( M_{2,n} \to M_{2,0} \) is an unramified covering, it is therefore sufficient to prove that there exists a fibre contained in a path-connected subset of \( M_{2,n} \).

Let \( C \in |H| \) be a fixed smooth curve, as \(|H| \) cuts over \( C \) a very ample linear system, by the general position theorem (see below) the fibre over \( C \) of the projection \( M_{2,n} \to M_{1,0} \) is connected.

If \( k \geq 3 \) let \( V \subset M_{k,n} \) be the subset of elements \((C_1, ..., C_k, p_1, ..., p_n)\) such that every curve \( C_i \) belong to the pencil generated by \( C_1, C_2 \). The natural projection \( V \to M_{2,n} \) is a smooth map with irreducible fibres, in particular \( V \) is also smooth irreducible. The conclusion follows from the fact that it is always possible to join by a path every point of \( M_{k,n} \) with a point of \( V \).

\[ \square \]

In the above proof we have used the following version of the general position theorem; for a proof we refer to [ACGH] p. 112.

**General position theorem.** Let \( C \subset \mathbb{P}^r \) be a smooth curve of degree \( d \). Then

\[
I = \{(p_1, ..., p_d, H) \in C^d \times (\mathbb{P}^r)^{\vee} \mid p_i \neq p_j, \{p_1, ..., p_d\} = H \cap C\}
\]

is smooth irreducible of dimension \( r \); hence for every \( s \leq d \) the image of the projection of \( I \to C^s \), \((p_1, ..., p_d, H) \to (p_1, ..., p_s)\) is irreducible too.

Let \( M_{k,n}^0 \subset M_{k,n} \) be the (possibly empty) open subset of elements \((C_1, ..., C_k, p_1, ..., p_n)\) such that the base locus of the linear system generated by \( C_1, ..., C_k \) is exactly \( \{p_1, ..., p_n\} \).

Consider now a positive integer \( 0 < c < \frac{1}{2}a \) and let \( L = \mathcal{O}_Q(a - c, b) \), \( F = \mathcal{O}_Q(c, 0) = H - L \). Let \( n = H \cdot L = b(2a - c) \) and consider for every \( k \geq 1 \)

\[
M_{k,L} = \{(C_1, ..., C_k, p_1, ..., p_n) \in M_{k,n} \mid \mathcal{O}_{C_1}(\sum p_i) = \mathcal{O}_{C_1}(L)\}
\]
Note that if $c \geq 2$ then the equality $\mathcal{O}_{C_1}(\sum p_i) = \mathcal{O}_{C_1}(L)$ does not imply that the divisor $p_1 + ... + p_n \subset C_1$ is cutted by a curve in the linear system $H^0(Q, L)$.

$M_{k,L}$ has a natural structure of closed subscheme of $M_{k,n}$, in order to show this, using the fact that $M_{k,L}$ is the fibred product of $M_{1,L} \to M_{1,n}$ and the natural projection $M_{k,n} \to M_{1,n}$, it is not restrictive to assume $k = 1$.

Let $\mathcal{C} \xrightarrow{\pi} M_{1,n}$ be the universal curve, $\mathcal{C} \subset Q \times M_{1,n}$, $q: \mathcal{C} \to Q$ the projection and let $p_i: M_{1,n} \to \mathcal{C}$ be the sections of marked points, then, as the fibres of $\pi$ are smooth irreducible curves, $M_{1,L}$ is naturally isomorphic to the first Fitting subscheme of the coherent sheaf $\pi_* \mathcal{O}_C(q^* L - \sum p_i)$.

We consider the Fitting subscheme instead of the support because we want that $M_{k,L}$ represent the corresponding functor from the opposite category of schemes over $\mathbb{C}$ to the category of sets.

**Lemma 2.2.** Let $C_1 \in |H|$ be a smooth curve, then the restriction map $H^0(Q, F) \to H^0(C_1, F)$ is an isomorphism. In particular if $(C_1, ..., C_k, p_1, ..., p_n) \in M_{k,L}$ then $\mathcal{O}_{C_1}(\sum p_i) = \mathcal{O}_{C_1}(L)$ for every $i = 1, ..., k$ and $M_{k,L}$ is stable under the action of $\text{Aut}_0(Q) \times \Sigma_k \times \Sigma_n$.

**Proof.** The first part follows immediately from the vanishing of $H^0(Q, -L)$ and $H^1(Q, -L)$. For the second part it is not restrictive to assume $k = 2$. Write $C_1 \cap C_2 = \sum p_i + D$ with $D \in |\mathcal{O}_{C_1}(F)|$ a reduced divisor, by the first part of the lemma there exists a curve $F_1 \in |F|$ such that $D = C_1 \cap F_1$. Since $D$ is reduced over $C_1$ of degree $cb$ we also have $D = C_2 \cap F_1$ and therefore $\mathcal{O}_{C_2}(\sum p_i) = \mathcal{O}_{C_2}(H - F) = \mathcal{O}_{C_2}(L)$.

□

**Proposition 2.3.** In the above notation the scheme $M_{k,L}$ is smooth irreducible of dimension $n + 2(a + b) - 1 + (k - 1)(c + 1)$. Moreover the natural projections $M_{k,L} \to M_{1,L} \to M_{1,0}$ are smooth morphisms.

**Proof.** The space $M_{1,0}$ is a Zariski open subset of $|H|$ and therefore it is smooth irreducible of dimension $ab + a + b$.

We have $H^1(Q, L) = H^2(Q, L - H) = 0$ and therefore for every smooth curve $C \in |H|$ we have $g(C) = (a - 1)(b - 1)$ and $H^1(C, L) = 0$.

By Riemann-Roch formula $b^0(C, L) = L \cdot C + 1 - g(C) = b(a - c) + a + b$. Let $C \xrightarrow{\pi} M_{1,0}$ be the universal curve, then $p_* q^* L$ is a locally free sheaf of rank $b(a - c) + a + b$. Now $M_{1,L}$ is an unramified covering of an open subset of the projectivized of $p_* q^* L$ and therefore it is smooth of dimension $n + 2(a + b) - 1$. The same proof shows also that the natural projection $M_{1,L} \to M_{1,0}$ is smooth and by the general position theorem its fibres are irreducible.

In order to conclude the proof we show that the fibres of the projection $M_{k,L} \xrightarrow{\pi} M_{1,L}$ are smooth irreducible of dimension $(k - 1)(c + 1)$.

First note that for every $m = (C, p_1, ..., p_n) \in M_{1,L}$ the complete linear system $|\mathcal{O}_C(H - \sum p_i)| = |\mathcal{O}_C(F)|$ is base point free of dimension $c$. From this it follows that the fibre of $\pi$ over $m$ is a nonempty open subset of $\mathbb{P}(H^0(H - p_1 - ... - p_n))^{k-1}$.

□
We are now able to prove the main result of this section

**Theorem 2.4.** In the notation above, for every $k \geq 3$, $M^0_{k,L} := M_{k,L} \cap M^0_{k,n}$ is a nonempty connected component of $M^0_{k,n}$.

**Proof.** By proposition 2.3 the space $M_{k,L}$ is smooth irreducible. Given generic curves $C \in |H|$, $D \in |L|$, $F_1, F_2 \in |F|$ and setting $\{p_1, ..., p_n\} = C \cap D$ we have by Bertini’s theorem that $(C_1, ..., C_k, p_1, ..., p_n) \in M^0_{k,L}$, where $C_1, ..., C_k$ are generic curves in the twodimensional linear system generated by $C, D + F_1$ and $D + F_2$.

The scheme $M^0_{k,L}$ is the restriction of $M_{k,L}$ to $M^0_{k,n}$ and it is therefore closed is the latter; by implicit function theorem it is sufficient to prove that for every $m = (C_1, ..., C_k, p_1, ..., p_n) \in M^0_{k,L}$ the natural map of Zariski tangent spaces $T_{m,M_{k,L}} \to T_{m,M_{k,n}}$ is surjective.

We have already proved that the composition $M_{k,L} \xrightarrow{i} M_{k,n} \xrightarrow{\alpha} M_{1,0}$ is smooth; we now prove that every morphism $\phi: T = Spec(\mathbb{C}[e]/(e^2)) \to (M_{k,n}, m)$ such that $\phi \alpha = 0$ can be lifted to $M_{k,L}$.

Note that for every line bundle $\mathcal{L}$ over $Q$, the line bundle $q^*\mathcal{L}$ is the unique extension of $\mathcal{L}$ to $Q \times T$, where $q: Q \times T \to Q$ is the projection.

The morphism $\phi$ represents $k$ smooth curves over $T$, $C_{T,1}, ..., C_{T,k}$ which are divisors of $q^*H$ together with $n$ sections $p_{T,i}: T \to \cap C_{T,j} \subset Q \times T$ such that their restriction to the closed point of $T$ give $m$; the condition $\alpha \phi = 0$ is equivalent to $C_{T,1} = C_1 \times T$.

For every $i = 2, ..., k$ let $D_i = C_1 \cap C_i - \sum p_j \in |\mathcal{O}_{C_1}(F)|$ and let $s_i \in H^0(Q, F)$ be a section whose restriction to $C_1$ gives the divisor $D_i$. The assumption $m \in M^0_{k,L}$ implies that the linear system $V \subset H^0(Q, F)$ generated by $s_2, ..., s_k$ has no fixed components and since $F^2 = 0$ it has no base points.

Denoting by $W \subset H^0(C_1, F)$ the isomorphic image of $V$ under the restriction map, we have a commutative diagram with vertical epimorphisms and horizontal cup products

$$
\begin{array}{ccc}
V \otimes H^0(Q, K_Q + L) & \xrightarrow{\beta} & H^0(Q, K_Q + C_1) \\
W \otimes H^0(C_1, K_{C_1} - F) & \xrightarrow{\mu_0} & H^0(C_1, K_{C_1})
\end{array}
$$

The map $\beta$ is surjective; in fact it is well known (and easy to prove) that, denoting $S_d \subset \mathbb{C}[x, y]$ the subspace of homogeneous polynomials of degree $d$, if $V \subset S_c$ is a base point free (over $\mathbb{P}^1$) vector subspace then $V \otimes S_{a-c} \to S_a$ is surjective for every $a \geq 2c - 1$. Moreover it is also well known that if $A, B$ are effective divisor over $Q$ then the cup product $H^0(Q, A) \otimes H^0(Q, B) \to H^0(Q, A + B)$ is surjective; in conclusion $\beta$ and then $\mu_0$ are surjective.

The map $\mu_0$ plays an important role in the description of the Zariski tangent space of the Brill-Noether varieties $W^r_d(C_1)$ and $G^r_d(C_1)$ (see [ACGH] for the definitions) at the points $\mathcal{O}_{C_1}(F)$ and $W$ respectively.

In particular by [ACGH, IV.4.1], the annihilator $Im(\mu_0) \subset H^1(\mathcal{O}_{C_1}) = H^0(K_{C_1})(\cap)$ classifies the first order extension of $\mathcal{O}_{C_1}(F)$ where the divisors $D_i$ extend. As $\mu_0$ is surjective it follows immediately that $\mathcal{O}_{C_1 \times T}(q^*H - \sum p_{T,i}) = \mathcal{O}_{C_1 \times T}(q^*F)$ and then $\phi$ can be lifted to $M_{k,L}$.
By symmetry the above results are still true if \( L = \mathcal{O}_Q(a, b - c') \) with \( 2c' < b \). The proof of 2.4 does not work if \( L = \mathcal{O}_Q(a - c, b - c') \), \( c, c' > 0 \), even in the case \( a >> b, c' \), because in general \( V \) is not base point free and the map \( \mu_0 \) cannot be surjective.

**Corollary 2.5.** Assume \( b = la \) for some integer \( l \geq 2 \) and \( n = la(2a - c) \), \( 0 < 2c < a \), \( k \geq 3 \). Then \( M^0_{k,n} \) contains at least two connected components which are stable under the action of \( \text{Auto}(Q) \times \Sigma_k \times \Sigma_n \).

**Proof.** Setting \( L_1 = \mathcal{O}_Q(a - c, b), \ L_2 = \mathcal{O}_Q(a, b - lc) \) we have by 2.3 \( M_{k,L1} \cap M_{k,L2} = \emptyset \) and then we can take \( M^0_{k,L1} \) and \( M^0_{k,L2} \) as connected components.

A simpler proof of the equality \( M_{k,L1} \cap M_{k,L2} = \emptyset \) which works also for \( k = 1 \) follows from 2.2. In fact if \( (C_1, ..., C_k, p_1, ..., p_n) \in M_{k,L1} \cap M_{k,L2} \) then

\[
h^0(\mathcal{O}_{C_i}(H - \sum p_i)) = \begin{cases} h^0(\mathcal{O}_{C_i}(H - L_1)) = c + 1 \\ h^0(\mathcal{O}_{C_i}(H - L_2)) = lc + 1 \end{cases}.
\]

By a careful reading of the proofs we see that all the above results concerning the local structure of the spaces \( M_{k,L} \) and \( M_{k,n} \) at a point \( (C_1, ..., C_k, p_1, ..., p_n) \) depends only on the properties of the linear system generated by the curves \( C_1, ..., C_k \). We can generalize these results by considering the “closure” \( \overline{M}_{k,n} \subset |H|^k \times \mathcal{Q}^n \) whose elements \( (C_1, ..., C_k, p_1, ..., p_n) \) satisfy the conditions:

i) the curves \( C_i \)'s are reduced without common components.

ii) \( p_i \in C_j \) for every \( i, j \), i.e. the points \( p_i \) are contained in the base locus of the linear system \( V \) generated by \( C_1, ..., C_k \).

iii) \( p_i \neq p_j \) if \( i \neq j \).

iv) The subscheme of base points of \( V \) is reduced, i.e. if \( q \) belongs to the intersection of \( C_1, ..., C_k \) then the local equations of the curves \( C_i \)'s at the point \( q \) generate the maximal ideal.

We define \( \overline{M}^0_{k,n} \subset \overline{M}_{k,n} \) as the open subscheme of points \( (C_1, ..., C_k, p_1, ..., p_n) \) such that \( \{p_1, ..., p_n\} = C_1 \cap ... \cap C_k \).

It is immediate to see, by using Bertini’s theorem, that if \( (C_1, ..., C_k, p_1, ..., p_n) \in \overline{M}^0_{k,n} \) (resp.: \( \overline{M}^0_{k,n} \)) then for generic curves \( D_1, ..., D_k \) in the linear system generated by the divisors \( C_i \) we have \( (D_1, ..., D_k, p_1, ..., p_n) \in M_{k,n} \) (resp.: \( M^0_{k,n} \)).

We then define \( \overline{M}_{k,L} \) as the set of \( (C_1, ..., C_k, p_1, ..., p_n) \in \overline{M}_{k,n} \) such that \( \mathcal{O}_D(\sum p_i) = \mathcal{O}_D(L) \) for some smooth curve \( D \) in the linear system generated by \( C_1, ..., C_k \). By Lemma 2.2 if \( \mathcal{O}_D(\sum p_i) = \mathcal{O}_D(L) \) for one smooth curve then the same is true for every smooth curve in the linear system \( < C_1, ..., C_k > \).
As above \( \overline{M}_{k,L} \) and \( \overline{M}^0_{k,L} = \overline{M}_{k,L} \cap \overline{M}^0_{k,n} \) are closed subschemes of \( \overline{M}_{k,n} \), \( \overline{M}^0_{k,n} \) respectively.

**Theorem 2.6.** \( \overline{M}_{k,L} \) is a smooth irreducible variety and if \( m = (C_1, ..., C_k, p_1, ..., p_n) \in \overline{M}^0_{k,L} \)
then \( \overline{M}_{k,L} \to \overline{M}_{k,n} \) is a local isomorphism at \( m \).

**Proof.** It is convenient to work with the \((\mathbb{C}^*)^k\)-principal bundles \( F_{k,n} \to M_{k,n} \), \( F_{k,L} \to M_{k,L} \), \( \overline{F}_{k,n} \to \overline{M}_{k,n} \), \( \overline{F}_{k,L} \to \overline{M}_{k,L} \), obtained by considering the sections of the line bundle \( H \) instead of the divisors.

Let \( m = (f_1, ..., f_k, p_1, ..., p_n) \in \overline{F}_{k,n} \) be a fixed point and let \( A = (a_{ij}) \in GL(k, \mathbb{C}) \) be a matrix, we then define \( Am = (g_1, ..., g_k, p_1, ..., p_n) \) where \( g_i = \sum a_{ij}f_j \).
If \( A \) is generic then \( Am \in F_{k,n} \) and \( A \) extends to an isomorphism of germs of complex spaces \( A : (\overline{F}_{k,n}, m) \to (F_{k,n}, Am) \); this germ isomorphism clearly preserves the subscheme \( \overline{F}_{k,L} \) and the theorem follows by the above results.

\[ \square \]

We also point out that \( M_{k,n} \) is Zariski open and dense in \( \overline{M}_{k,n} \). The motivation of our definition of \( \overline{M}_{k,n} \) and \( \overline{M}^0_{k,n} \) will be clear in the next sections. Moreover the inclusion \( M^0_{k,n} \subset \overline{M}^0_{k,n} \) induces a bijection between the sets \( \pi_0 \)'s of connected components.

Consider now a fixed \( m = (C_1, ..., C_k, p_1, ..., p_n) \in \overline{M}_{k,n} \) with \( k \geq 2 \) and let \( S = S_m \xrightarrow{\nu} Q \) be the blow up at the points \( p_1, ..., p_n \). If \( E_i \subset S \) is the \((-1\rangle\)-curve over \( p_i \) and \( C_i' = \nu^*C_i - \sum_j E_j \) then \( m \in \overline{M}^0_{k,n} \) if and only if \( \cap C_i' = \emptyset \).

**Lemma 2.7.** In the above notation \( h^0(\theta_S) = h^2(\theta_S) = 0 \), \( h^1(\theta_S) = 2n - 6 \).

**Proof.** The vanishing of \( H^2(\theta_S) \) is a general fact which holds for every smooth rational surface. Let \( E_i \subset S \) be the exceptional curve over the point \( p_i \), \( 1 = 1, .. n \). It is well known that there exists two exact sequences of sheaves

\[
0 \to \theta_S \xrightarrow{\nu^*} \theta_Q \to \oplus O_{E_i}(1) \to 0
\]

\[
0 \to \nu_*\theta_S \to \theta_Q \to \oplus T_{p_i,Q} \to 0
\]

and we also have the vanishing of the higher direct image sheaf \( R^1\nu_*\theta_S = 0 \).

It is therefore sufficient to prove that the natural restriction map \( H^0(Q, \theta_Q) \to \oplus T_{p_i,Q} \) is injective. Let \( \pi_i : Q \to \mathbb{P}^1 \), \( i = 1, 2 \) be the natural projections; as \( n > \max(2a, 2b) \) and the curves \( C_i \) have no common components, the points \( p_i \) lie in at least three different fibres of \( \pi_1 \), \( \pi_2 \). Since \( \theta_Q = O_Q(2, 0) \oplus O_Q(0, 2) \) the injectivity of the above restriction map is immediate.

\[ \square \]

By a well known formula we have an isomorphism \( Pic(S) = \nu^*Pic(Q) \oplus \mathbb{Z}E_1 \oplus ... \oplus \mathbb{Z}E_n \); from now on we shall write \( O_S(r,s) = \nu^*O_Q(r,s) \). By the Leray spectral sequence we get \( h^i(O_S(r,s)) = h^i(O_Q(r,s)) \).
Definition 2.8. We shall say that a line bundle $\mathcal{L} = \mathcal{O}_S(r, s) - \sum a_i E_i$ is combinatorially ample if $a_i \geq 2$ for every $i = 1, \ldots, n$ and $r, s > \sum (a_i + 1)$.

Note that if $\mathcal{L}$ is combinatorially ample then for every $D \in \text{Pic}(S)$ the line bundle $D + \alpha \mathcal{L}$ is combinatorially ample for $\alpha$ sufficiently large.

Proposition 2.9. Let $\mathcal{L} \in \text{Pic}(S)$ be a combinatorially ample line bundle, then:

i) $\mathcal{L}$ is ample and base point free.

ii) $H^1(\mathcal{L}) = H^1(-\mathcal{L}) = 0$.

iii) $H^0(\theta_S(-\mathcal{L})) = H^1(\theta_S(-\mathcal{L})) = 0$.

Proof. Consider a line bundle $\mathcal{L} = \mathcal{O}_S(r, s) - \sum a_i E_i$ with $a_i > 0$ for every $i$.

a) If $r, s \geq \sum a_i$ then $\mathcal{L}$ is effective and nef. In fact $\mathcal{L}^2 = 2rs - \sum a_i^2 > 0, \mathcal{L} \cdot E_i = a_i > 0$; moreover if $V \subset H^0(\mathcal{L})$ is the subspace generated by sections whose zero locus is contained in a finite union of fibres of the two projections $\pi_1, \pi_2: S \to \mathbb{P}^1$, then it is immediate to observe that the possible fixed part of $V$ is contained in the exceptional divisor $E = \cup E_i$. In particular $\mathcal{L} \cdot C \geq 0$ for every irreducible curve on $S$.

b) If $r, s > \sum a_i$ then $\mathcal{L}$ is ample. This follows by point a) and Nakai criterion.

c) If $\delta: X \to S$ is the blow up at a point with exceptional curve $E \subset X$ the same arguments of points a) and b) show that if $r, s \geq 3 + \sum a_i$ then $\delta^* \mathcal{L} - E$ is ample.

d) By adjunction formula $K_S = \mathcal{O}_S(-2, -2) + \sum E_i$ and by point b), if $\mathcal{L}$ is combinatorially ample then $\mathcal{L}, \mathcal{L} - K_S$ are ample. By Kodaira vanishing we have $H^1(-\mathcal{L}) = 0, H^1(\mathcal{L}) = H^1(K_S - \mathcal{L})^\vee = 0$.

The same arguments shows that $\delta^*(\mathcal{L} - K_S) - 2E$ is ample on $X$, in particular $H^1(X, K_X - \delta^* \mathcal{L} + E) = 0$ and $\mathcal{L}$ is base point free on $S$.

e) We have an exact sequence

$$0 \longrightarrow \theta_S(-\mathcal{L}) \longrightarrow (\mathcal{O}_S(2, 0) - \mathcal{L}) \oplus (\mathcal{O}_S(0, 2) - \mathcal{L}) \longrightarrow \mathcal{O}_{E_i}(1 - a_i) \longrightarrow 0.$$ 

By taking the associate cohomology exact sequence, if $\mathcal{L}$ is combinatorially ample then by the above steps we get $H^0(\theta_S(-\mathcal{L})) = H^1(\theta(-\mathcal{L})) = 0$. 

Let $\mathcal{S} \to \overline{\mathcal{M}}_{k,n} \times Q$ be the blow up of the sections $p_i: \overline{\mathcal{M}}_{k,n} \to Q$. It is easy to see, using the fact that the union of the images of $p_1, \ldots, p_n$ is a locally complete intersection subscheme of $\overline{\mathcal{M}}_{k,n} \times Q$, that the natural map $\mathcal{S} \to \overline{\mathcal{M}}_{k,n}$ is flat and commute with base change; in particular $p^{-1}(m) = S_m$ and if $E_i \subset \mathcal{S}$ is the exceptional divisor over the section $p_i$ then $E_i \cap S_m$ is exactly the $(-1)$ curve over the point $p_i$.

Denoting by $q: \mathcal{S} \to Q$ the natural projection, if $r, s > \sum (a_i + 1)$ then the line bundle $\mathcal{L} = q^* \mathcal{O}_Q(r, s) - \sum a_i E_i$ is combinatorially ample on every fibre of $p$ and $p_* \mathcal{L}$ is locally free.
3. \((\mathbb{Z}/2)^r\)-covers and their deformations

A quite useful machinery in the explicit construction of components of moduli spaces is the technique of Galois deformations of abelian covering. This technique has been extensively studied in [Ca1], [Ma2] in the case of \(\mathbb{Z}/2^2\)-covers, in [Ma3] for simple iterated double covers and in [Par], [FP] for general nonsingular abelian covers.

It is possible to prove that the main results of [FP] holds, with minor changes, also for flat, locally simple, normal abelian covering. Here, in order to avoid long and unnecessary computations we consider only the case of \(\mathbb{Z}/2^r\)-covers, leaving the possible generalizations elsewhere.

A) Basic theory of normal flat \(\mathbb{Z}/2^r\)-covers.

For readers convenience we recall the basic definition and results about \(\mathbb{Z}/2^r\)-covers of algebraic varieties; for the proofs we refer to the “standard” reference [Par].

It is notationally convenient to consider the group \(G = \mathbb{Z}/2^r\) as a vector space of dimension \(r\) over \(\mathbb{Z}/2\); there exists a natural isomorphism between the dual vector space \(G^\vee\) and the group of characters \(G^*\)

\[ G^\vee \ni \chi \leftrightarrow (-1)^\chi \in G^*. \]

Given any abelian group \(\Lambda\), a pair of maps \(D: G \to \Lambda, L: G^\vee \to \Lambda\) satisfies the cover condition if:

i) \(D_0 = L_0 = 0\).
ii) For every \(\chi, \eta \in G^\vee\),

\[ L_\chi + L_\eta = L_{\chi+\eta} + \sum_{\chi(\sigma) = \eta(\sigma) = 1} D_\sigma. \]

If \(i: \{0,1\} \to \mathbb{R}\) is the inclusion and \([\cdot]: \mathbb{R} \to \mathbb{Z}\) is the integral part, it is easy to see that a pair \((L,D) \in \Lambda^{G^\vee} \oplus \Lambda^{G}\) satisfies the cover condition if and only if \(D_0 = L_0 = 0\) and for every \(k \geq 0, \chi_1, \ldots, \chi_k \in G^\vee\) holds the equality

\[ L_{\chi_1} + \ldots + L_{\chi_k} = L_{\chi_1+\ldots+\chi_k} + \sum_{\sigma \in G} \left[ \frac{1}{2} \sum_{j=1}^{k} i(\chi_j(\sigma)) \right] D_\sigma. \]

As an application of this formula we see easily that, if \(\chi_1, \ldots, \chi_r\) is a basis of \(G^\vee\), \(D: G \to \Lambda\) is an application such that \(D_0 = 0\) and \(L_{\chi_1}, \ldots, L_{\chi_r} \in \Lambda\) satisfy the equations

\[ 2L_{\chi_i} = \sum_{\chi_i(\sigma) = 1} D_\sigma, \quad i = 1, \ldots, r \]

then there exists unique an extension \(L: G^\vee \to \Lambda\) such that \((L,D)\) satisfies the cover condition.

We denote by \(Z(G, \Lambda) \subset \Lambda^{G^\vee} \oplus \Lambda^{G}\) the subgroup consisting of pairs \((L,D)\) satisfying the cover condition.

Lemma 3.1. Let \(G\) be as above and let

\[ 0 \to K \to \tilde{\Lambda} \xrightarrow{\alpha} \Lambda \to C \]

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be an exact sequence of abelian groups with $K$ a $\mathbb{Q}$-vector space and $C$ torsion free.

Then for every $(L, D) \in Z(G, \Lambda)$ and every lifting $\tilde{D} : G \rightarrow \tilde{\Lambda}$ of $D$ there exists unique a lifting $(\tilde{L}, \tilde{D}) \in Z(G, \tilde{\Lambda})$ of $(L, D)$.

Proof. It is immediate to prove that if $\alpha(\tilde{\lambda}) = 2v$ then there exist unique $\tilde{v} \in \tilde{\Lambda}$ such that $\alpha(\tilde{v}) = v$ and $2\tilde{v} = \tilde{\lambda}$.

Let $\chi_1, ..., \chi_r$ be a basis of $G^\vee$, by the above remark there exists a bijection between the liftings of $L$ satisfying the cover conditions and the liftings of $L \chi_j$ satisfying the equations

$$2\tilde{L}_{\chi_i} = \sum_{\chi_i(\sigma) = 1} \tilde{D}_\sigma, \quad i = 1, ..., r.$$ 

It is very easy to find solutions of the cover condition for every group $\Lambda$. For every subspace $0 \neq H \subset G$ and every $v \in \Lambda$ we can define an elementary solution $(L, D)_{H,v} \in Z(G, \Lambda)$ in the following way (cf. [Par, Ex. 4.1]):

i) If $\dim H = 1$ we set

$$L_\chi = \begin{cases} 0 & \text{if } \chi \in H^\perp, \\ v & \text{if } \chi \notin H^\perp \end{cases}, \quad D_\sigma = \begin{cases} 2v & \text{if } \sigma \in H - \{0\}, \\ 0 & \text{otherwise} \end{cases}$$

ii) If $\dim H = d + 2 \geq 2$

$$L_\chi = \begin{cases} 0 & \text{if } \chi \in H^\perp, \\ 2^d v & \text{if } \chi \notin H^\perp \end{cases}, \quad D_\sigma = \begin{cases} v & \text{if } \sigma \in H - \{0\}, \\ 0 & \text{otherwise} \end{cases}$$

It is not difficult to see that if $\Lambda$ is finitely generated then the elements $(L, D)_{H,v}$ generate $Z(G, \Lambda)$; we don’t need this result.

Lemma 3.2. Assume the dimension of $G$ at least 4 and let $H \subset G$ a proper subspace. Let $\alpha : \Lambda \rightarrow \mathbb{Z}$ be a nonzero homomorphism of groups. Given any positive integer $N$ and any application $D : H \rightarrow \Lambda$ such that $D_0 = 0$, there exists an extension $(L, D) \in Z(G, \Lambda)$ such that:

a) $\alpha(L_\chi - D_\sigma) \geq N$ for every $\chi \in G^\vee - \{0\}, \sigma \in G$.

b) $\alpha(D_\sigma) \geq N$ for every $\sigma \in G - H$.

Proof. Let $q$ be a positive integer, $v \in \Lambda$ such that $\alpha(v) > 0$ and $\eta \in G - H$.

For every $\tau \in G$ let $(L^\tau, D^\tau) \in Z(G, \Lambda)$ be the following elementary solution:

i) If $\tau \in H$ then

$$L^\tau_\chi = \begin{cases} 0 & \text{if } \chi(\tau) = 0, \\ D_\tau & \text{otherwise} \end{cases}, \quad D^\tau_\sigma = \begin{cases} D_\tau & \text{if } \sigma = \tau, \eta, \tau + \eta, \\ 0 & \text{otherwise} \end{cases}$$

ii) If $\tau \notin H$ then

$$L^\tau_\chi = \begin{cases} 0 & \text{if } \chi(\tau) = 0, \\ qv & \text{if } \chi(\tau) = 1 \end{cases}, \quad D^\tau_\sigma = \begin{cases} 2qv & \text{if } \sigma = \tau, \\ 0 & \text{otherwise} \end{cases}$$
A simple computation shows that for \( q \gg 0 \) the sum \( \sum_{\tau \in G} (L^\tau, D^\tau) \) satisfies the required conditions.

Assume now that the group \( G \) acts faithfully on a normal irreducible complex algebraic variety \( X \) and let \( \pi: X \to Y \) be the projection to the quotient. We also make the assumption that the quotient map \( \pi \) is flat; this assumption is in general quite strong but it is always satisfied if \( Y \) is smooth.

For every \( \sigma \in G - \{0\} \) let \( Fix(\sigma) \subset X \) be the (possibly empty) closed subscheme of fixed points of the involution \( \sigma \) and let \( R_\sigma \subset Fix(\sigma) \) be its Weil divisorial part; by convention we set \( R_0 = \emptyset \).

It is well known and easy to see (cf. [Ca1]) that \( R_\sigma \) is reduced; as the group \( G \) is abelian every \( R_\sigma \) is \( G \)-stable and if \( \tau \neq \sigma \) then \( R_\sigma \cap R_\tau \) have no common components.

Let \( D_\sigma = \pi(R_\sigma) \subset Y \) with the reduced structure, \( D_\sigma \) is a Weil divisor which is not Cartier in general.

The map \( \pi \) is flat and then \( \pi_* \mathcal{O}_X \) is locally free and there exists a character decomposition

\[
\pi_* \mathcal{O}_X = \bigoplus_{\chi \in G^\vee} L^{-1}_\chi
\]

where by definition \( L^{-1}_\chi = \{ f \in \pi_* \mathcal{O}_X \mid \sigma f = (-1)^{\chi(\sigma)} f, \forall \sigma \in G \} \), note moreover that \( L_0^{-1} = \mathcal{O}_Y \) and, since \( \pi \) is a \( G \)-principal bundle over the generic point of \( Y \), \( L^{-1}_\chi \) is a locally free \( \mathcal{O}_Y \)-module of rank 1 for every \( \chi \in G^\vee \).

For every \( x \in X \) let \( I_x = \{ \sigma \mid x \in R_\sigma \} \), \( I = \bigcup I_x = \{ \sigma \mid R_\sigma \neq \emptyset \} \); in the terminology of [Par], \( \pi: X \to Y \) is called a \( (G, I) \)-cover. It is clear that \( I_x \) is contained in the stabilizer \( Stab_x \) of the point \( X \); we shall see that the condition \( \pi \) flat implies that \( I_x \) generates \( Stab_x \).

**Definition 3.3.** In the above notation the cover \( \pi: X \to Y \) is called:

a) Totally ramified if \( I \) generates \( G \).

b) Simple if \( I \) is a basis of \( G \).

c) Locally simple if \( I_x \) is a set of linearly independent vectors for every \( x \in X \).

For every pair \( \chi, \eta \in G^\vee \) let \( \beta_{\chi, \eta}: L^{-1}_\chi \otimes L^{-1}_\eta \to L^{-1}_{\chi+\eta} \) be the multiplication map, it is immediate to observe that the \( \beta_{\chi, \eta} \)'s induce a structure of commutative \( \mathcal{O}_Y \)-algebra on \( \oplus L^{-1}_\chi \) if and only if

\[
\beta_{\chi, \eta} = \beta_{\eta, \chi}, \quad \beta_{\chi, 0} = 1, \quad \beta_{\chi, \eta} \beta_{\chi+\eta, \mu} = \beta_{\chi, \mu} \beta_{\chi+\mu, \eta}
\]

(3.4)

for every \( \chi, \eta, \mu \in G^\vee \).

We can interpret \( \beta_{\chi, \eta} \) as a section of the line bundle \( L_\chi + L_\eta - L_{\chi+\eta} \) and the computation of [Par] shows that

\[
\text{div}(\beta_{\chi, \eta}) = \sum_{\chi(\sigma) = \eta(\sigma) = 1} D_\sigma
\]

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Therefore over the open set $U \subset Y$ of regular points the divisors $D_{\sigma}$ are Cartier and the pair $(L, D) \in \text{Pic}(U)^{G'} \oplus \text{Pic}(U)^{G}$ satisfies the cover condition. The pair $(L, D)$ is called the building data of the cover $\pi: X \to Y$.

**Corollary 3.5.** Let $\pi: X \to Y$ be a normal flat $G$-cover, $Z$ an irreducible normal variety and $f: Z \to Y$ be a morphism such that $f(Z) \not\subset \cup D_{\sigma}$ and the codimension in $Z$ of $f^{-1}(\text{Sing}(Y) \cup \pi(\text{Sing}(X)))$ is at least 2.

If the fibred product $X' = X \times_Y Z$ is normal then the divisors $f^*(D_{\sigma})$ are reduced without common components.

**Proof.** Denote by $\pi': X' = X \times_Y Z \to Z$ the pullback of the projection and by $f^*\beta_{\chi, \eta}$ the multiplication map of the $O_Z$-algebra $f^*\pi_*O_X = \pi'_*O_{X'}$, then we have

$$\text{div}(f^*\beta_{\chi, \eta}) = \sum_{\chi(\sigma) = \eta(\sigma) = 1} f^*D_{\sigma}$$

and the normality of $X'$ implies that these divisors are reduced.

$\square$

Note that if $\pi$ is locally simple then the divisors $D_{\sigma}$ are Cartier; in fact if $I_x = \{\sigma_1, \ldots, \sigma_s\}$ is a set of linearly independent vectors then there exists $\chi_1, \ldots, \chi_s \in G'$ such that $\chi_i(\sigma_j) = \delta_{ij}$ and therefore by the cover condition $2L_{\chi_i} = D_{\sigma_i}$ in some Zariski neighbourhood of $y = \pi(x)$.

Conversely, given a normal variety $Y$, line bundles $L_{\chi} \in \text{Pic}(Y)$ and reduced Weil divisors $D_{\sigma}$ without common components such that $(L, D)$ satisfies the cover condition on the Picard group of $Y - \text{Sing}(Y)$, there exist a normal, flat $G$-cover $\pi: X \to Y$ with building data exactly $(L, D)$. Moreover if $Y$ is complete then $\pi$ is uniquely determined up to $G$-isomorphism.

In fact let $U \subset Y$ be the open set of smooth points, $\pi: V \to Y$ be the total space of the vector bundle $\oplus L_{\chi}$, $w_{\chi} \in H^0(V, \pi^*L_{\chi})$ the tautological sections and $f_{\sigma} \in H^0(U, D_{\sigma})$ a section defining $D_{\sigma}$. For every $\chi, \eta \in G'$ let $\beta_{\chi, \eta} = \prod_{\chi(\sigma) = \eta(\sigma) = 1} f_{\sigma}$; as the variety $Y$ is normal, $\beta_{\chi, \eta}$ extends uniquely to a section of $L_{\chi} + L_{\eta} - L_{\chi + \eta}$. An elementary computation shows that the sections $\beta_{\chi, \eta}$ satisfy the equations (3.4).

We then define $X \subset V$ as the subvariety defined by the equations

$$F_{\chi, \eta} := w_{\chi}\omega_{\eta} - w_{\chi + \eta}\beta_{\chi, \eta} = 0, \quad \forall \chi, \eta \in G'.
$$

For the unicity of $\pi$ when $Y$ is complete follows from the following argument; another proof will be made later on this section.

Let $f_{\sigma}'$ be another set of equation of the divisors $D_{\sigma}$ and let $\beta_{\chi, \eta}'$ be the corresponding multiplication maps and $X' \subset V$ defined by the equations $w_{\chi}\omega_{\eta} - w_{\chi + \eta}\beta_{\chi, \eta}' = 0$. For every pair $\chi, \eta \in G'$ there exists an invertible function $\alpha_{\chi, \eta} \in H^0(O_{Y'})$ such that $\beta_{\chi, \eta}' = \alpha_{\chi, \eta}\beta_{\chi, \eta}'$. Assume now that for every $\chi \in G'$ there exists a square root of $\alpha_{\chi, \chi}$ in $H^0(O_{Y'})$, then there exist invertible functions $g_{\chi} \in H^0(O_{Y'})$ such that

$$g_{\chi}g_{\eta} = g_{\chi + \eta}\alpha_{\chi, \eta} \quad (3.6)$$

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for every \( \chi, \eta \in G' \); the isomorphism of the vector bundle \( V \) defined by \( w_{\chi} \to g_{\chi}w_{\chi} \) will induces an isomorphism \( X \simeq X' \).

We can choose the function \( g_\chi \) in the following way; let \( \chi_1, ..., \chi_r \) be a basis of \( G' \) and for every \( \chi = \sum a_i\chi_i \), \( a_i = 0,1 \), we define the length \( l(\chi) = \sum a_i \). Let \( g_{\chi_i} \) be a fixed square root of \( \alpha_{\chi_1,\chi_i} \); let \( \chi = \eta + \chi_i \) be a decomposition such that \( l(\chi) = l(\eta) + 1 \), by induction on \( l \) we can suppose \( g_\eta \) is defined, then we set \( g_\chi = g_\eta g_{\chi_i} \alpha_{\eta,\chi_i}^{-1} \); the functions \( g_\chi \) satisfy the cocycle condition (3.6).

**Proposition 3.7.** Let \( \pi: X \to Y \) be a normal flat \( G \)-cover with \( Y \) smooth. Then \( X \) is smooth if and only if \( \pi \) is locally simple, the divisors \( D_\sigma \) are smooth and intersects transversally.

**Proof.** [Par.3.1].

\( \Box \)

In the case \( Y \) smooth surface there exists simple formulas for the numerical invariants \( \chi(\mathcal{O}_X) \), \( K_X^2 \) in terms of the building data \((L,D)\). In fact by Hurwitz formula we have \( K_X = \pi^*K_Y + \sum R_\sigma \), \( \pi^*D_\sigma = 2R_\sigma \) and if \( g = \vert G \vert = 2^r \) then for every divisors \( A,B \) on \( Y \) we have

\[ \pi^*A \cdot \pi^*B = gA \cdot B, \quad R_\sigma \cdot \pi^*A = \frac{1}{2}gD_\sigma \cdot A, \quad R_\sigma \cdot R_\tau = \frac{1}{4}gD_\sigma \cdot D_\tau \]

Since \( \pi \) is finite \( H^i(\mathcal{O}_X) = \oplus H^i(Y, L_X^{-1}) \) and therefore we get

\[ K_X^2 = g(K_Y + \frac{1}{2} \sum D_\sigma)^2, \quad \chi(\mathcal{O}_X) = g\chi(\mathcal{O}_Y) + \frac{1}{2} \sum \chi(L_X \cdot (L_X + K_Y)) \]

A simple calculation shows that if the building data are sufficiently ample then the invariants \( K^2, \chi \) spread in the region \( 4\chi \leq K^2 \leq 8\chi \).

**Remark.** When we say that the building data \((L,D)\) is sufficiently ample we shall mean that there exists a basis \( \sigma_1, ..., \sigma_r \) of \( G \) such that the line bundles \( L_{\sigma_i}, \chi \neq 0 \) and \( \mathcal{O}_Y(D_{\sigma_i}) \), \( i = 1, ..., r \), are sufficiently ample.

By the computation of \([\text{Ma3}]\) it follows that every real number \( \beta \in [4,8] \) can be approximated by the ratios \( \frac{K_X^2}{\chi} \) of simple \( (\mathbb{Z}/2)^r \)-covers.

**B)** The geometry of locally simple covers.

Let \( \pi: X \to Y \) be a normal flat simple \((G,I)\)-cover with \( I = \{\sigma_1, ..., \sigma_r\} \) basis of \( G = (\mathbb{Z}/2)^r \) and \( Y \) complete irreducible. Let \( \chi_1, ..., \chi_r \in G' \) be the dual basis of \( \sigma_1, ..., \sigma_r \) and let \( f_i \) be an equation defining \( D_{\sigma_i} \). In the above notation \( X \) is defined, up to isomorphism, in \( V \) by the equations

\[
\begin{align*}
w_{\chi_i}^2 &= f_i & \text{for } i = 1, ..., r \\
\sum a_i \chi_i &= \prod w_{\chi_i}^{a_i} & \text{for } a_i = 0,1
\end{align*}
\]

An immediate simplification of this equations gives \( X \) as a complete intersection in the total space of \( L_{\chi_1} \oplus ... \oplus L_{\chi_r} \) defined by the \( r \) equations \( w_{\chi_i}^2 = f_i \). Note moreover that the divisor
$R_{\sigma_i}$ is defined by the equation $w_{\chi_i} = 0$ and then it is a Cartier divisors in $X$. From this description follows immediately the projection formula

$$\pi_*O_{R_{\sigma}} = \bigoplus_{\chi(\sigma) = 0} O_{D_{\chi}}(-L_{\chi}).$$

Note that our definition of simple cover is compatible with the, apparently different, notions of simple covers given in [Ma3], [Ca3], [Par].

If $\pi$ is locally simple then for every $y \in Y$ there exists a neighbourhood $U \subset Y$ such that $X \times_Y U$ is isomorphic to the cover defined by equations $w_i^2 = f_i$, $i = 1, ..., r$. In particular a normal flat $G$-cover is unramified at $x$ if and only if $I_x = \emptyset$; more generally if $H \subset G$ is the subspace generated by $I_x$ then $X/H \rightarrow Y$ if still flat and ramified in codimension $\geq 2$ at the point $x$, by the above remark $X/H \rightarrow Y$ is unramified at $x$ and then $H = \text{Stab}_x$.

A more important consequence is that if $\pi: X \rightarrow Y$ is locally simple then $X$ is a local complete intersection in $V$ and the divisors $R_{\sigma}$ are Cartier.

Let $\pi: X \rightarrow Y$ be a locally simple $G$-cover, we have seen that for every $\sigma$, $R_{\sigma}$ is a $G$-stable Cartier divisors, in particular the $G$-action on $X$ can be naturally lifted, in the sense of [Mu2, p.110-111], to $G$-actions on the coherent sheaves $O_X(-R_{\sigma})$, $O_{R_{\sigma}}(-R_{\sigma})$.

**Theorem 3.8.** For every locally simple normal flat $(\mathbb{Z}/2)^r$-cover $\pi: X \rightarrow Y$ there exists a $G$-equivariant exact sequence of sheaves

$$0 \rightarrow \pi^*\Omega_Y^1 \rightarrow \Omega_X^1 \rightarrow \bigoplus_{\sigma} O_{R_{\sigma}}(-R_{\sigma}) \rightarrow 0.$$

**Proof.** Assume first $\pi$ simple, then there exist line bundles $L_1, ..., L_r$ over $Y$ such that $X$ is the complete intersection subvariety of $W = L_1 \oplus ... \oplus L_r$ defined by the equations $F_i = w_i^2 - f_i = 0$, where $D_i = \{f_i = 0\} \subset Y$ are the branching divisors of $\pi$ and $w_i \in H^0(W, \pi^*L_i)$ are the tautological sections.

The equations $F_i \in H^0(W, \pi^*D_i)$ are $G$-invariant, in particular the conormal bundle of $X$ in $W$ is $G$-isomorphic to the pull back $\pi^*(\oplus\mathcal{O}_Y(-D_i)) = \oplus\mathcal{O}_X(-2R_i)$. On the other hand there exists a natural $G$-isomorphism $\Omega^1_{W/Y} = \oplus\pi^*L_i^{-1}$. Tensoring the exact sequence of sheaves over $W$

$$0 \rightarrow \pi^*\Omega_Y^1 \rightarrow \Omega_W^1 \rightarrow \Omega_{W/Y}^1 \rightarrow 0$$

by $\mathcal{O}_X$ we get

$$\text{Tor}_1^W(\mathcal{O}_X, \Omega_{W/Y}^1) = 0 \rightarrow \pi^*\Omega_Y^1 \rightarrow \Omega_W^1 \otimes \mathcal{O}_X \rightarrow \Omega_{W/Y}^1 \otimes \mathcal{O}_X \rightarrow 0.$$

We have moreover the exact sequence

$$0 \rightarrow \bigoplus_{i} \mathcal{O}_X(-2R_i) \rightarrow \Omega_{W/Y}^1 \otimes \mathcal{O}_X \rightarrow \Omega_{X/Y}^1 \rightarrow 0,$$
where \( d(F_i) = d(w_i^2 - f_i) = w_i dw_i \in \mathcal{O}_X(-R_i) = \pi^*L_i^{-1} \). This gives a natural \( G \)-isomorphism
\[
\Omega^1_{X/Y} = \bigoplus_i \frac{\mathcal{O}_X(-R_i)}{w_i \mathcal{O}_X(-2R_i)} = \bigoplus_i \mathcal{O}_{R_i}(-R_i).
\]
The injectivity of \( \pi^* \Omega^1_Y \to \Omega^1_X \) follows by considering the snake exact sequence of the commutative diagram
\[
\begin{array}{cccccc}
0 & \to & \oplus \mathcal{O}_X(-\pi^*D_i) & \to & \Omega^1_{W} \otimes \mathcal{O}_X & \to & \Omega^1_X & \to & 0 \\
& & \| & & \downarrow & & \downarrow & & \\
0 & \to & \oplus \mathcal{O}_X(-\pi^*D_i) & \to & \Omega^1_{W/Y} \otimes \mathcal{O}_X & \to & \Omega^1_{X/Y} & \to & 0
\end{array}
\]
This proves the theorem for simple covers.

In the general case, for every \( \sigma \in G \) let \( Z_\sigma = X/\sigma \) and \( \pi_\sigma : X \to Z_\sigma \) be the projection to the quotient. As \( R_\sigma \) is Cartier, by [Ma3] 3.1, the map \( \pi_\sigma \) is flat in a neighbourhood \( U \) of \( R_\sigma \) and by the computation in the simple case there exists a natural isomorphism \( \Omega^1_{U/Z_\sigma} = \mathcal{O}_{R_\sigma}(-R_\sigma) \) and morphisms
\[
\Omega^1_{X/Y} \to \Omega^1_{X/Z_\sigma} \to \mathcal{O}_{R_\sigma}(-R_\sigma)
\]
Taking the direct sum of \( \mathcal{O}_{R_\sigma}(-R_\sigma) \) over all \( \sigma \in G \) we get a morphism \( \Omega^1_{X/Y} \to \oplus \mathcal{O}_{R_\sigma}(-R_\sigma) \) and the computation made in the simple case shows that it is a \( G \)-isomorphism.

\[ \square \]

**Lemma 3.9.** Let \( \pi : X \to Y \) be a normal flat locally simple \( G \)-cover, then for every \( \sigma \in G \)

\[
\text{Ext}^1_X(\mathcal{O}_{R_\sigma}(-R_\sigma), \mathcal{O}_X) = \bigoplus_{\chi(\sigma) = 0} H^{i-1}(\mathcal{O}_{D_\sigma}(D_\sigma - L_\chi))
\]

**Proof.** By applying the functor \( \mathcal{H}om_X(\_ , \mathcal{O}_X) \) to the exact sequence
\[
0 \to \mathcal{O}_X(-2R_\sigma) \to \mathcal{O}_X(-R_\sigma) \to \mathcal{O}_{R_\sigma}(-R_\sigma) \to 0
\]
we get
\[
\mathcal{E}xt^1_X(\mathcal{O}_{R_\sigma}(-R_\sigma), \mathcal{O}_X) = \begin{cases} \mathcal{O}_{R_\sigma}(2R_\sigma) & i = 1 \\ 0 & i \neq 1 \end{cases}
\]
and by the Ext spectral sequence \( \text{Ext}^1_X(\mathcal{O}_{R_\sigma}(-R_\sigma), \mathcal{O}_X) = H^{i-1}(\pi_* \mathcal{O}_{R_\sigma}(2R_\sigma)) \).

A local computation shows that \( 2R_\sigma = \pi^*D_\sigma \), \( \pi_* \mathcal{O}_{R_\sigma} = \bigoplus_{\chi(\sigma) = 0} \mathcal{O}_{D_\sigma}(-L_\chi) \) and then
\[
\pi_* \mathcal{O}_{R_\sigma} = \bigoplus_{\chi(\sigma) = 0} \mathcal{O}_{D_\sigma}(-L_\chi).
\]

\[ \square \]

**Corollary 3.10.** If \( \pi : X \to Y \) is normal, flat, locally simple \( G \)-cover such that \( H^1(Y, L^{-1}_\chi) = 0, \text{Ext}^1_Y(\Omega^1_Y, L^{-1}_\chi) = 0, H^0(Y, D_\sigma - L_\chi) = 0 \) for every \( \chi \neq 0 \) and every \( \sigma \in \chi^{-1} \).
Then $G$ acts trivially on $\text{Ext}^1_X(\Omega^1_X, \mathcal{O}_X)$.

Proof. Let $\chi \neq 0$ be a fixed character, taking the $\chi$-equivariant part of the long $\text{Ext}_X(\cdot, \mathcal{O}_X)$ sequence associated to

$$0 \longrightarrow \pi^*\Omega^1_Y \longrightarrow \Omega^1_X \longrightarrow \oplus_{\sigma} \mathcal{O}_{R_{\sigma}}(-R_{\sigma}) \longrightarrow 0$$

we get

$$\bigoplus_{\sigma \in \chi^{-1}} H^0(\mathcal{O}_{D_{\sigma}}(D_{\sigma} - L_{\chi})) \longrightarrow \text{Ext}^1_X(\Omega^1_X, \mathcal{O}_X)^{\chi} \longrightarrow \text{Ext}^1_X(\pi^*\Omega^1_Y, \mathcal{O}_X)^{\chi}$$

The left side lies in the middle of $H^0(Y, D_{\sigma} - L_{\chi})$ and $H^1(Y, -L_{\chi})$ and then it is equal to 0. As $\pi$ is finite flat there exists a natural isomorphism (cf. [Ma3] 2.1) $\text{Ext}^1_X(\pi^*\Omega^1_Y, \mathcal{O}_X) = \text{Ext}^1_Y(\Omega^1_Y, \pi_*\mathcal{O}_X)$ and therefore $\text{Ext}^1_X(\pi^*\Omega^1_Y, \mathcal{O}_X)^{\chi} = \text{Ext}^1_Y(\Omega^1_Y, L^{-1}_X) = 0$.

Note that if $Y$ is Gorenstein of dimension $n \geq 2$ then by Serre duality

$$\text{Ext}^1_Y(\Omega^1_Y, L^{-1}_X) = \text{Ext}^1_Y(\Omega^1_Y \otimes L_X \otimes K_Y, K_Y) = H^{n-1}(\Omega^1_Y \otimes L_X \otimes K_Y)^{\vee}$$

and this space is 0 whenever $L_X$ is sufficiently ample.

C) Galois deformations of $(\mathbb{Z}/2)^r$-covers.

Let $\pi: X \rightarrow Y$ be a normal flat $(\mathbb{Z}/2)^r$-cover with $Y$ complete irreducible variety and building data $L_X$, $D_{\sigma}$. Here we want to study the deformations of $\pi$ obtained by “moving” the branching divisors. We assume for simplicity that $Y$ is smooth and $H^0(Y, \theta_Y) = H^0(X, \theta_X) = 0$; although these assumption are not strictly necessary for the main results of this section, they are sufficient for our application and allow simpler proofs. We also note that in the case $X$ smooth our results are contained in [FP].

Let $\text{Art}$ be the (small) category of local Artinian $\mathbb{C}$-algebras and denote by

$$\text{Def}_X, \text{Def}_Y, \text{Def}_{\pi}: \text{Art} \rightarrow \text{Set}$$

the functors of deformations of $X$, $Y$, $\pi$ respectively. All of them satisfy Schlessinger conditions H1, H2, H3 of [Sch] and linearity condition L of [FaMa]; therefore they admit a hull and a good obstruction theory.

Since $X, Y$ are normal we have

$$T^i\text{Def}_X = \text{Ext}^i_X(\Omega^1_X, \mathcal{O}_X), \quad T^i\text{Def}_Y = \text{Ext}^i_Y(\Omega^1_Y, \mathcal{O}_Y), \quad i = 1, 2$$

where as usual we denote by $T^1$ the tangent space and by $T^2$ the obstruction space arising from the cotangent complex [Fle].

Let $\text{Def}_{(Y,D_{\sigma})}: \text{Art} \rightarrow \text{Set}$ be the functor of deformations of the closed inclusions $D_{\sigma} \rightarrow Y$; more precisely for $A \in \text{Art}$, $\text{Def}_{(Y,D_{\sigma})}(A)$ is the set of isomorphism classes of:
1) a deformation of $Y$, $Y_A \to \text{Spec}(A)$.
2) for every $\sigma \in G$, a closed $A$-flat embedding $D_{A,\sigma} \subset Y_A$ extending $D_{\sigma}$ (the subvarieties $D_{A,\sigma}$ will be automatically Cartier divisors).

Note that $Def_{f(Y,D_{\sigma})}$ is prorepresented by the fibred product of the relative Hilbert scheme of the Kuranishi family of $Y$.

**Lemma 3.11.** In the above set up let $(Y_A, D_{A,\sigma}) \in Def_{f(Y,D_{\sigma})}(A)$; then for every $\chi$ there exists a unique extension $L_{A,\chi} \in \text{Pic}(Y_A)$ of $L_\chi$ such that $(L_{A,\chi}, D_{A,\sigma})$ satisfies the cover condition.

**Proof.** Let $0 \to \mathcal{C} \to A \to B \to 0$ be a small extension in $\text{Art}$, then we have an exact sequence of sheaves

$$0 \to \mathcal{O}_Y \to \mathcal{O}_{Y_A}^* \to \mathcal{O}_{Y_B}^* \to 0$$

Since $Y$ is complete, reduced and irreducible the pullback map $B^* = (B - m_B) \to H^0(\mathcal{O}_{Y_B}^*)$ is surjective and then we have an exact sequence of abelian groups

$$0 \to H^1(\mathcal{O}_Y) \to \text{Pic}(Y_A) \to \text{Pic}(Y_B) \to H^2(\mathcal{O}_Y).$$

The conclusion follows by lemma 3.1 and induction on the length of $A$.

\[\square\]

**Proposition 3.12.** There exists a natural transformation of functors

$$Def_{f(Y,D_{\sigma})} \to Def_\pi$$

commuting with the projections $Def_\pi \to Def_Y$, $Def_{f(Y,D_{\sigma})} \to Def_Y$.

**Proof.** Let $f_\sigma \in H^0(Y, D_\sigma)$ be equations of the divisors $D_\sigma$, then $X$ is $G$-isomorphic to the subvariety of $V = \oplus L_\chi$ defined by

$$F_{\chi,\eta} = w_\chi w_\eta - w_{\chi+\eta} \prod_{\chi(\sigma) = \eta(\sigma) = 1} f_\sigma = 0, \quad \chi, \eta \in G^\vee.$$ 

Consider now $A \in \text{Art}$ and $(Y_A, D_{A,\sigma}) \in Def_{f(Y,D_{\sigma})}(A)$. By 3.11 there exist unique line bundles $L_{A,\chi} \in \text{Pic}(Y_A)$ such that $(L_{A,\chi}, D_{A,\sigma})$ satisfies the cover condition.

Denoting $\pi_A: V_A = \oplus L_{A,\chi} \to Y_A$ we define $X_A \subset V_A$ by the equations

$$F_{A,\chi,\eta} = w_\chi w_\eta - w_{\chi+\eta} \prod_{\chi(\sigma) = \eta(\sigma) = 1} f_{A,\sigma} = 0, \quad \chi, \eta \in G^\vee.$$ 

where $f_{A,\sigma} \in H^0(Y_A, D_{A,\sigma})$ is an equation of $D_{A,\sigma}$ extending $f_\sigma$. We also define $\pi_A: X_A \to Y_A$ as the natural projection.

We first note that $\pi_A$ is flat and then $X_A$ is flat over $A$. In fact by construction $\pi_A_\ast \mathcal{O}_{X_A} = \oplus L_{A,\chi}^{-1}$ which is locally free. It is clear that our construction commutes with morphisms in $\text{Art}$.
and then remains only to prove that the isomorphism deformation class of $\pi_A$ does not depend on the choice of $f_{A,\sigma}$. More precisely, we also need to prove the independence from the choice of the isomorphisms of line bundles $L_{A,\chi} \otimes L_{A,\eta} \otimes L_{A,\chi+\eta}^{-1} \simeq O(\sum_{\chi(\sigma)=\eta(\sigma)=1} D_{A,\sigma})$, but the effect over $\pi_A: X_A \rightarrow Y_A$ of a change of isomorphism is the same of a change of the equations $f_{A,\sigma}$.

Let $f'_{A,\sigma}$ be another set of equations, as $Y$ is complete reduced and irreducible, for every $\sigma$ there exists an invertible element $a_\sigma \in A$ such that $f'_{A,\sigma} = a_\sigma f_{A,\sigma}$. Let $\pi'_A: X'_A \rightarrow Y_A$ be the cover defined by $f'_{A,\sigma}$.

The maps $\pi_A, \pi'_A$ are uniquely determined by the isomorphism classes of the $O_{Y_A}$-algebras

$$A = \oplus L_{A,\chi}^{-1}, \quad A' = \oplus L_{A,\chi}^{-1}$$

with the multiplication maps induced respectively by the sections

$$\beta_{A,\chi,\eta} = \prod_{\chi(\sigma)=\eta(\sigma)=1} f_{A,\sigma}, \quad \beta'_{A,\chi,\eta} = \prod_{\chi(\sigma)=\eta(\sigma)=1} f'_{A,\sigma}$$

The sections $\beta, \beta'$ differ by multiplication by an invertible element of $A$.

Every $O_{Y_A}$-isomorphism from $A$ to $A'$ is determined by taking for every $\chi \in G^\vee$ an invertible element $g_\chi \in \text{Hom}(L_{A,\chi}, L_{A,\chi}) = A$ such that $g_0 = 1$ and

$$\beta'_{A,\chi,\eta} = \frac{g_\chi g_\eta}{g_{\chi+\eta}} \beta_{A,\chi,\eta}, \quad \forall \chi, \eta \in G^\vee$$

Given a section $s: \text{Spec}(A) \rightarrow Y_A$ it is clearly sufficient to check these conditions over the image of $s$, or equivalently to prove that $s^*A$ is isomorphic to $s^*A'$. This last condition is trivially satisfied if the image of $s$ does not intersect the branching divisors $D_{A,\sigma}$, in fact every principal $G$-bundle over a fat point is trivial.

Let $\eta: Df_{(Y,D_\sigma)} \rightarrow Df_X$ be the composition of the morphisms defined in 3.8 and the natural projection $Df_x \rightarrow Df_X$. Let moreover $Def^G_X \subset Df_X$ be the subfunctor of deformations of $X$ for which there exists an extension of the $G$-action. Using the fact that $H^0(X, \theta_X) = 0$ it is easy to see that $Def^G_X$ is a functor with good deformation theory, i.e. satisfies the Schlessinger conditions, and it is prorepresented by the $G$-invariant subgerm of the Kuranishi base space of $X$. By Cartan’s lemma $Def^G_X = Df_X$ if and only if $G$ acts trivially on $T^1_X = \text{Ext}^1_X(\Omega^1_X, O_X)$.

**Theorem 3.13.** In the above notation, if $\pi: X \rightarrow Y$ is locally simple then the natural transformation $\eta$ induces an isomorphism $Def_{(Y,D_\sigma)} \simeq Def^G_X$.

**Proof.** We prove the theorem by constructing explicitly an inverse natural transformation $\theta: Def^G_X \rightarrow Def_{(Y,D_\sigma)}$. The first step is to construct a natural transformation $\mu: Def^G_X \rightarrow Df_Y$ by “taking the quotients”.

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For every $A \in \text{Art}$ it is useful to consider every deformation $X_A \in \text{Def}^G_X(A)$ as a sheaf $\mathcal{O}_A$ of flat $A$-algebras over $X$ together with a lifting of the $G$-action. As $G$ is abelian there exists a character decomposition

$$\pi_*\mathcal{O}_A = \bigoplus_{\chi \in G^\vee} (\pi_*\mathcal{O}_A)^\chi$$

Since $\pi$ is a finite affine morphism, the sheaf $\pi_*\mathcal{O}_A$ is still flat over $A$ and then every direct summand $(\pi_*\mathcal{O}_A)^\chi$ is $A$-flat. In particular $Y_A := (Y, (\pi_*\mathcal{O}_A)^0) \to \text{Spec}(A)$ is a flat morphism.

If $A \to B$ is a morphism in $\text{Art}$ and $\mathcal{O}_B = \mathcal{O}_A \otimes_A B$ we have $\pi_*\mathcal{O}_B = (\pi_*\mathcal{O}_A) \otimes_A B$; moreover for every character $\chi$ we have $(\pi_*\mathcal{O}_A)^\chi \otimes_A B \subset \pi_*\mathcal{O}_B)^\chi$. Putting together the above relations we get that $(\pi_*\mathcal{O}_A)^\chi \otimes_A B = \pi_*\mathcal{O}_B)^\chi$; in particular $Y_A$ is a deformation of $Y$ and the induced application $\mu: \text{Def}^G_Y \to \text{Def}^G_Y$ commute with base change, i.e. it is a morphism of functors.

Note also that $(\pi_*\mathcal{O}_A)^\chi$ is a deformation of $L^{-1}_\chi$ and therefore it is a line bundle over $Y_A$.

Let $x \in X$ be a point, $y = \pi(x)$ and $\sigma_1, ..., \sigma_r \in G$ a basis extending $L_x$; denote by $\chi_1, ..., \chi_r$ the dual basis. Over a sufficiently small affine open neighbourhood $U$ of $y$ the cover $\pi: X \to Y$ is isomorphic to $\pi_1: X_U \to U$ where $X_U \subset U \times \mathbb{C}_w^r$ is defined by equations

$$w_i^2 = f_{\sigma_i}, \quad i = 1, ..., r$$

and $\pi_1$ is the projection on the first factor. The $G$-action over $X_U$ is induced by the $G$-action on the affine variety $U \times \mathbb{C}^r$ defined by $\sigma(w_i) = (-1)^{\chi_i(\sigma)}w_i$.

Let $(X, \mathcal{O}_A) \in \text{Def}^G_X(A)$ and let $U_A = (U, (\pi_*\mathcal{O}_A)^0)$, it is well known [Ar] that $(X_U, \mathcal{O}_A) \to U_A$ can be described as an embedded deformation inside $U_A \times \mathbb{C}^r$, moreover as $G$ is finite we can choose the embedded deformation $G$-stable.

$X_U$ is a complete intersection in $U \times \mathbb{C}^r$ and then every $G$-stable deformation over $A$ is given by the equations

$$w_i^2 = f_{A, \sigma_i}, \quad i = 1, ..., r$$

with $f_{A, \sigma_i}$ a lifting of $f_{\sigma}$ over $U_A$.

The regular function $f_{A, \sigma}$ defines a Cartier divisor $D_{A, \sigma}$ over $U_A$ which is $A$-flat. It is clear that these local constructions of $D_{A, \sigma}$ patch together over $Y_A$ and commute with base change.

We have therefore constructed a natural transformation $\theta: \text{Def}^G_X \to \text{Def}^{(Y, D_\sigma)}$. By construction it is immediate to observe that $\theta \eta$ is the identity on $\text{Def}^{(Y, D_\sigma)}$.

The same proof of 3.12 shows that the morphism $\theta$ is injective and then $\eta \theta$ is the identity on $\text{Def}^G_X$.

\[ \square \]

The above proof has the advantage of being quite elementary but it seems very hard to extend it to natural deformations ([FP] def. 3.2). For this reason we sketch here a shorter but more technical proof of 3.13 which involves the notion of relative obstructions and the standard criterion of smoothness [FaMa].
Let $\nu: \text{Def}_{(Y,D_\sigma)} \to \text{Def}_Y$ be the natural projection, it is easy to compute the tangent and obstruction space of $\nu$; they are

$$T^i\nu = \bigoplus_{\sigma} H^{i-1}(\mathcal{O}_{D_\sigma}(D_\sigma)), \quad i = 1,2.$$ 

This gives an exact sequence

$$0 \to \bigoplus_{\sigma} H^0(\mathcal{O}_{D_\sigma}(D_\sigma)) \to T^1\text{Def}_{(Y,D_\sigma)} \to \text{Ext}^1_Y(\Omega^1_Y, \mathcal{O}_Y) \to \bigoplus_{\sigma} H^1(\mathcal{O}_{D_\sigma}(D_\sigma)) \to T^2\text{Def}_{(Y,D_\sigma)} \to \text{Ext}^2_Y(\Omega^1_Y, \mathcal{O}_Y)$$

On the other hand, taking the long $\text{Ext}_X(-, \mathcal{O}_X)$ sequence of

$$0 \to \pi^*\Omega^1_Y \to \Omega^1_X \oplus_{\sigma}\mathcal{O}_{R_\sigma}(-R_\sigma) \to 0$$

we get

$$\bigoplus_{\chi} \text{Ext}^0_Y(\Omega^1_Y, L^{-1}_\chi) \to \bigoplus_{\sigma, \chi(\sigma) = 0} H^0(\mathcal{O}_{D_\sigma}(D_\sigma - L_\chi)) \to T^1\text{Def}_X \to \bigoplus_{\chi} \text{Ext}^1_Y(\Omega^1_Y, L^{-1}_\chi) \to \bigoplus_{\sigma, \chi(\sigma) = 0} H^1(\mathcal{O}_{D_\sigma}(D_\sigma - L_\chi)) \to T^2\text{Def}_X \to \bigoplus_{\chi} \text{Ext}^2_Y(\Omega^1_Y, L^{-1}_\chi)$$

If $\text{Ext}^1_Y(\Omega^1_Y, L^{-1}_\chi) = 0$, $H^0(\mathcal{O}_{D_\sigma}(D_\sigma - L_\chi)) = 0$ for every $\chi \neq 0$ and $\sigma \in \chi^\perp$, then a simple diagram chasing involving the above exact sequences shows that the natural maps

$$T^i\text{Def}_{(Y,D_\sigma)} \to T^i\text{Def}_X$$

are surjective for $i = 1$ and injective for $i = 2$. By the standard smoothness criterion the morphism $\eta$ is smooth. If in addition $H^0(\theta_Y) = \text{Ext}^0_Y(\Omega^1_Y, \mathcal{O}_Y) = 0$ then also $H^0(\theta_X) = 0$ and $\text{Def}_X$ is prorepresentable. By diagram chasing $\eta$ induces an isomorphism on tangent spaces and then it is an isomorphism.

For future reference we collect the main results in the following corollary

**Corollary 3.14.** Let $\pi: X \to Y$ be a normal flat locally simple $(\mathbb{Z}/2)^r$-cover with building data $(L_\chi, D_\sigma)$ such that:

1. $Y$ is smooth of dimension $\geq 2$, $H^0(\theta_Y) = 0$.
2. $\text{Ext}^1_Y(\Omega^1_Y, L^{-1}_\chi) = H^1(L^{-1}_\chi) = 0$ for every $\chi \neq 0$.
3. $H^0(Y, D_\sigma - L_\chi) = 0$ for every $\chi \neq 0$, $\sigma \in \chi^\perp$.

Then $H^0(\theta_X) = 0$ and the natural morphism $\eta: \text{Def}_{(Y,D_\sigma)} \to \text{Def}_X$ is an isomorphism of prorepresentable functors.

The functor $\text{Def}_{(Y,D_\sigma)}$ corresponds essentially to the functor $D\text{gal}_X$ of Galois deformations introduced in [FP]; we shall say that the family of Galois deformations is complete if $\eta$ is smooth.
Let $H \subset G$ be a proper subspace and let $D_\sigma$, $\sigma \in H - \{0\}$, be reduced effective divisors in the smooth variety $Y$ without common components. The same argument of lemma 3.2, with $\Lambda = \text{Pic}(Y)$ and $v$ ample, shows that if the dimension of $G$ is at least 4 there exists sufficiently ample and general divisors $D_\sigma$, $\sigma \in G - H$, and sufficiently ample line bundles $L_\chi$ such that the pair $(L, D)$ satisfies the cover condition. Moreover if $\dim Y \geq 2$ and $H^0(\theta_Y) = 0$ we can choose $(L, D)$ which satisfies the hypothesis of corollary 3.14. In particular if $X \to Y$ is the $G$-cover with building data $(L, D)$ then the family of Galois deformations of $X$ is complete.

**Example 3.15.** The corollary 3.14 provides a techniques for constructing irreducible components of the moduli space of surfaces of general type whose members have singular canonical model.

Consider for example the Pascal configuration in $\mathbb{P}^2$, i.e. a smooth conic $C$ and an inscribed hexagon $E = L_1 + L_2 + \ldots + L_6$, $L_i$ line. This configuration is uniquely determined by $C$ and the six vertices $p_i = C \cap L_i \cap L_{i+1}$, $(L_7 = L_1)$, $i = 1, \ldots, 6$. Assume $p_1, \ldots, p_6 \in C$ in general position and let $q_i = L_1 \cap L_4$, $q_2 = L_2 \cap L_5$, $q_3 = L_3 \cap L_6$. Pascal theorem asserts that the $q_i$ are collinear end then the generic quadric passing through $q_i$ has an ordinary double point (node).

Let $Y \to \mathbb{P}^2$ be the blow up of $p_1, \ldots, p_6, q_1, q_2, q_3$. By Kodaira stability theorem the 9 exceptional curves of $\nu$ are stable under deformations, in particular it is possible to choose a very ample line bundle $\mathcal{O}_Y(1) \in \text{Pic}(Y)$ which extends to every deformation of $Y$.

Let $\sigma_1, \ldots, \sigma_8 \in (\mathbb{Z}/2)^8$ be a basis and let $D: H = (\mathbb{Z}/2)^8 \to \text{Div}(Y)$ defined in the following way:

\begin{align*}
D_{\sigma_i} &= \text{strict transform of } L_i, \ i = 1, \ldots, 6. \\
D_{\sigma_7} &= \text{strict transform of } C. \\
D_{\sigma_8} &= \text{strict transform of a generic quadric passing through } q_1, q_2, q_3. \\
D_{\sigma} &= 0 \text{ if } \sigma \neq \sigma_1.
\end{align*}

It is immediate to see that every deformation of the data $(Y, D_\sigma)$ preserves the Pascal configuration; in particular if $(Y_A, D_{A, \sigma}) \in \text{Def}_{(Y, D_\sigma)}(A)$ then the flat family of curves $D_{A, \sigma} \to \text{Spec}(A)$ is locally trivial.

Let now $D: G = (\mathbb{Z}/2)^9 \to \text{Div}(Y)$, $L: G \to \text{Pic}(Y)$ an extension of $D: H \to \text{Div}(Y)$ such that for every $\sigma \in G - H$ the divisor $D_\sigma$ is a generic element of the linear system $|\mathcal{O}_Y(l)|$, $l >> 0$; the pair $(L, D)$ satisfies the cover conditions and the hypothesis of Corollary 3.14.

Let $\pi: X \to Y$ be the $G$-cover with building data $(L, D)$, it is easy to see that $X$ is a normal surface with exactly $2^8$ simple nodes as singularities and ample canonical bundle. By 3.14 every deformation of $X$ is Galois and then contains at least $2^8$ nodes.

Examples of irreducible components of the moduli space of surfaces of general type whose general member is a regular surface with singular canonical model were constructed in [Ca7] by using a different method.
The use of Galois deformations allow to give a simple proof of the following

**Proposition 3.16.** Let $\pi: X \to Y$ be a normal flat $(\mathbb{Z}/2)^r$-cover with $Y$ smooth; then $X$ is $\mathbb{Q}$-Gorenstein of index $\leq 2$. Moreover the following three conditions are equivalent:

a) $\pi$ is locally simple.

b) $X$ is locally complete intersection.

c) $X$ is Gorenstein.

**Proof.** As $Y$ is smooth and $\pi$ is flat, by a well known result in commutative algebra [Mat.,§23], the variety $X$ is Cohen-Macaulay (resp.: Gorenstein) if and only if the fibres of $\pi$ are Cohen-Macaulay (resp.: Gorenstein). The fibres, and hence $X$, are Cohen-Macaulay because they have dimension 0. By Hurwitz formula $2K_X = f^*K_Y + \sum D_\sigma$; this proves that $X$ is $\mathbb{Q}$-Gorenstein of index $\leq 2$.

We have already proved that a) $\Rightarrow$ b), while b) $\Rightarrow$ c) is a general fact of commutative algebra. We now prove c) $\Rightarrow$ a). If $r = 1$ there is nothing to prove; we begin by considering the case $r = 2$.

Every fibre is defined in $\mathbb{C}^3$ by the six equations

\[
\begin{align*}
  w_i^2 &= f_j f_k \\
  w_j w_k &= w_i f_i.
\end{align*}
\]

If $X$ is not locally simple at a point $x$, then the fibre containing $x$ has $f_1 = f_2 = f_3 = 0$ and then it is isomorphic to $\text{Spec}(\mathbb{C}[w_1, w_2, w_3]/(w_1, w_2, w_3)^2)$ and then it is not Gorenstein because the socle has dimension 3.

As the property of being Gorenstein is local the same conclusion holds if $r \geq 2$ but the stabilizer of every point $x \in X$ is a subspace of dimension $\leq 2$.

In general assume $r \geq 3$ and $\pi$ not locally simple at $x$; it is not restrictive to shrink $Y$ to a sufficiently small affine neighbourhood $U$ of $y = \pi(x)$ such that $D_\sigma \cap U$ are principal divisors. Let $\sigma, \tau \in G$ such that $\sigma, \tau, \sigma + \tau \in I_x$. Since the property of being Gorenstein is stable under small deformations, we can deform the divisors $D_\sigma$ by moving away from $y$ the divisors $D_\alpha$ with $\alpha \neq \sigma, \tau, \sigma + \tau$. After this deformation $I_x = \{\sigma, \tau, \sigma + \tau\}$ the stabilizer of $x$ has dimension 2 and the above computation gives a contradiction.

\[\square\]

**Corollary 3.17.** Let $\pi: X \to Y$ be a $(\mathbb{Z}/2)^r$-cover of algebraic surfaces. If $Y$ is smooth and $X$ has at most rational double points as singularities then $\pi$ is locally simple.

**Proof.** Immediate from Proposition 3.12.

\[\square\]

**Example 3.18.** Let $(X_0, 0)$ be the cyclic singularity of type $\frac{1}{4}(1,1)$, it is well known [Rie] that the semiuniversal deformation of $(X_0, 0)$ contains two irreducible components: the Artin component and the $\mathbb{Q}$-Gorenstein component.
The $\mathbb{Q}$-Gorenstein component is the one-parameter smoothing $\pi: \mathcal{X} = \mathcal{Y}/\mu_2 \to \mathbb{C}_t$ where $\mathcal{Y} \subset \mathbb{C}^3 \times \mathbb{C}_t$ is defined by the equation $uv = t + y^2$ and the group $\mu_2$ is generated by the involution $(u, v, y) \to (-u, -v, -y)$.

The group $G = (\mathbb{Z}/2) \times (\mathbb{Z}/2)$ generated by the involutions

$$\sigma(u, v, y, t) = (v, u, y, t), \quad \rho(u, v, y, t) = (u, v, -y, t)$$

acts on $\mathcal{X}$ preserving fibres. A set of generator of the $\mathbb{C}$-algebra $O_{\mathcal{Y}} = O_{\mathcal{X}}$ which are eigen-vectors for the $G$-action is given by

$$t, \ x = uv = y^2 + t, \ z = u^2 + v^2, \ w_1 = uy + vy, \ w_2 = u^2 - v^2, \ w_3 = uy - vy$$

It is trivial to see that $\mathcal{X}/G$ is smooth and $\mathcal{X}$ is isomorphic to the subvariety of $\mathbb{C}^5 \times \mathbb{C}_t$ defined by the equations

$$w_i^2 = f_j f_k, \quad w_i w_j = w_k f_k, \quad \{i, j, k\} = \{1, 2, 3\}$$

where $f_1 = z - 2x$, $f_2 = x + t$, $f_3 = z + 2x$.

It is important for us to observe that the singularity $X_0$ is the bidouble cover of $\mathbb{C}^2$ branched over three lines passing through $0$ and that every $\mathbb{Q}$-Gorenstein smoothing is obtained as a Galois deformation by moving a line away from $0$.

**Example 3.19.** Let $m = (C_1, ..., C_k, p_1, ..., p_n) \in \mathcal{M}_{k,n}$ and denote by $\nu: S_m \to \mathbb{P}^1 \times \mathbb{P}^1$ the blow up of $p_1, ..., p_n$, $E_i = \nu^{-1}(p_i)$ and $D_i = \nu^* C_i - \sum_j E_j$.

There is a natural morphism of functors of Artin rings $\phi: (\mathcal{M}_{k,n}, m) \to \text{Def}_{S_m, D_i}$ and by Kodaira stability theorem $\phi$ is a smooth morphism.

**Example 3.20.** In the notation of example 3.19 assume $k = 3$ and let $\pi: X_m \to S_m$ be the $(\mathbb{Z}/2)^2 = \{0, \alpha_1, \alpha_2, \alpha_3\}$-cover with building data $D_{\alpha_i} = D_i$, $i = 1, 2, 3$ and $L_X = O_{S_m}(D_1)$, $\chi \neq 0$.

In view of 3.7, 3.17 and 3.18 we see easily that:

i) if $m \in M^0_{3,n}$ then $X_m$ is smooth.

ii) if $m \in M_{3,n}$ then $X_m$ has at most cyclic singularities of type $1/4(1,1)$.

iii) if $X$ has at most rational double points as singularities then $m \in \mathcal{M}^0_{3,n}$.

This partially motivates the definitions of section 2. It is an interesting question to decide whether two surfaces $X_m$, $X_l$ are deformation equivalent whenever $m, l$ belongs to different connected components of $M^0_{3,n}$. We suspect that the answer is no in general but a proof of this fact seems inaccessible at the moment.

4. $(\mathbb{Z}/2)^r$-actions on rational double points and their smoothings
In this section we continue the investigation of [Ca3], [Ma3] concerning the quotients of rational double points by commuting involutions: in particular we are interested to simplicity and smoothability of the actions.

We first introduce some terminology. Assume the group $G = (\mathbb{Z}/2)^r$ acts faithfully on a normal surface singularity $(X, x)$ and let $\pi: (X, x) \to (Y, y)$ be the quotient map.

For every $\sigma \in G$ let $R_\sigma \subset X$ be the germ of the divisorial part of the fixed locus of $\sigma$. As in the global case, every irreducible curve germ through $X$ may belong to at most one $R_\sigma$. We denote by $D_\sigma$ the image of $R_\sigma$ with the reduced structure and $I_x = \{\sigma | R_\sigma \neq 0\}$.

**Definition 4.1.** The $G$-action is called *almost simple* if $I_x$ is a set of linearly independent vectors. It is called *simple* if it is almost simple and the quotient map $\pi$ is flat.

The computations of §3 show that if the $G$-action is simple then the divisors $R_\sigma$, $D_\sigma$ are principal and $I_x$ is a basis. In particular if $Y$ is smooth then the $G$-action is simple if and only if it is almost simple.

After [Ca3] we know that there exist exactly 13 conjugacy classes of $(\mathbb{Z}/2)^r$-actions on rational double points. We shall give this list in the next three tables.

**Table 4.2.** Equations of RDP’s in $\mathbb{C}^3$.

|   | Equation                                                      |
|---|----------------------------------------------------------------|
| $E_8$ | $z^2 + x^3 + y^5 = 0$                                      |
| $E_7$ | $z^2 + x(y^3 + x^2) = 0$                                  |
| $E_6$ | $z^2 + x^3 + y^4 = 0$                                      |
| $D_n$, $n \geq 3$ | $z^2 + x(y^2 + x^{n-2}) = 0$                        |
| $A_n$, $n \geq 0$ | $z^2 + x^2 + y^{n+1} = 0 \text{ or } uv + y^{n+1} = 0$ |

By a little abuse of terminology, we consider the smooth germ $(\mathbb{C}^2, 0)$ as the rational double point $A_0$ and $D_3 = A_3$; clearly $x, z$ are local coordinates on $\mathbb{C}^2$ at 0. Moreover we have $u = z + ix$, $v = z - ix$.

**Table 4.3.** ([Ca3]) The six canonical forms of involutions acting on the RDP’s.

|   | Transformation                                      |
|---|----------------------------------------------------|
| a) | $(x, y, z) \to (x, -y, z)$                        |
| b) | $(x, y, z) \to (x, -y, -z)$                        |
| c) | $(u, v, y) \to (-u, v, -y)$                       |
| d) | $(x, y, z) \to (-x, y, -z)$                        |
| e) | $(u, v, y) \to (-u, -v, -y)$                      |
| f) | $(x, y, z) \to (x, y, -z)$                        |
Table 4.4. Conjugacy classes of \((\mathbb{Z}/2)^r\)-actions on rational double points.

| \(r\) | \(X\) | basis | \(Y\) | \(|I_x|\) | simple? | smoothable? |
|---|---|---|---|---|---|---|
| 1 | 1 | \(E_6, D_n, A_{2n+1}\) | \(a\) | \(A_2, A_1, A_{n+1}\) | 1 | yes | yes |
| 2 | 1 | \(E_6, D_n, A_{2n+1}\) | \(b\) | \(E_7, D_{2n-2}, D_{n+3}\) | 0 | no | no |
| 3 | 1 | \(A_{2n}\) | \(c\) | \(B_n\) | 1 | no | yes |
| 4 | 1 | \(A_n\) | \(d\) | \(A_{2n+1}\) | 0 | no | no |
| 5 | 1 | \(A_{2n-1}\) | \(e\) | \(Y_n\) | 0 | no | yes |
| 6 | 1 | all RDP’s | \(f\) | \(A_0\) | 1 | yes | yes |
| 7 | 2 | \(E_6, D_n, A_{2n+1}\) | \(a, f\) | \(A_0\) | 2 | yes | yes |
| 8 | 2 | \(A_n\) | \(d, f\) | \(A_0\) | 2 | yes | yes |
| 9 | 2 | \(A_{2n+1}\) | \(a, d\) | \(A_{2n+1}\) | 1 | no | no |
| 10 | 2 | \(A_n\) | \(e, f\) | \(A_1\) | 1 | no | no |
| 11 | 2 | \(A_{2n+1}\) | \(b, d\) | \(D_{2n+4}\) | 0 | no | no |
| 12 | 2 | \(A_{2n+2}\) | \(c, d\) | \(A_{2n+2}\) | 2 | no | yes |
| 13 | 3 | \(A_{2n+1}\) | \(a, d, f\) | \(A_0\) | 3 | yes | yes |

The above Table 4.4 requires some explanations.

By definition \(B_n\) is the quotient of the rational double point \(A_{2n}\) by the involution \(c\); the computation of [Ca3] shows that \(B_n\) is the cyclic singularity of type \(\frac{1}{2n+1}(1, 2n-1)\) and by the Hirzebruch-Jung expansion as a continued fraction ([B-P-V] III.5) its Dynkin diagram is

\[
\begin{array}{cccccccc}
-2 & -2 & -2 & \cdots & -2 & -3 \\
\end{array}
\] \(\text{for } n \geq 2 \text{ vertices}\) \(\tag{4.5}\)

If \(Z\) is the fundamental cycle then \(Z^2 = -3\) and then \(B_n\) is a rational triple point; by Riemenschneider results ([Rie, Satz 10 and Satz 12]) every deformation of \(B_n\) admits simultaneous resolution. In particular if \(F\) is the Milnor fibre of a smoothing of \(B_n\) then \(H_2(F, \mathbb{Z})\) is torsion free of rank \(n\) with the intersection form induced by the Dynkin diagram (4.5).

By definition \(Y_n\) is the quotient of the rational double point \(A_{2n-1}\) by the involution \(e\). A simple computation ([Ca3, p. 92], [Ma1], [Ma7]) shows that \(Y_{n+1}\) is the cyclic quotient singularity of type \(\frac{1}{4n}(1, 2n-1)\) and its Dynkin diagram is

\[
\begin{array}{cccccccc}
-3 & -2 & -2 & \cdots & -2 & -3 \\
\end{array}
\] \(\text{for } n \geq 2 \text{ vertices}\) \(\tag{4.6}\)

The fundamental cycle has selfintersection \(Z^2 = -4\) and then \(Y_n\) is a rational quadruple point. Again by Riemenschneider results ([Rie, Satz 13]) the base space of the semiuniversal
deformations of $Y_n$ is the transversal union of two smooth germs parametrizing respectively deformations admitting simultaneous resolution and $\mathbb{Q}$-Gorenstein deformations.

If $F$ is the Milnor fibre of a $\mathbb{Q}$-Gorenstein smoothing of $Y_n$ then it is proved in ([Ma1, Prop. 13]) that the torsion subgroup of $H^2(F, \mathbb{Z})$ is isomorphic to $(\mathbb{Z}/2)$ and it is generated by the canonical class.

The first 4 columns of 4.4, namely $r$, $X$, $Y$ and the basis of $G$; come directly from [Ca3]. By $|I_x|$ we denote the cardinality of the set $I_x$: a case by case computation shows that all the above actions are almost simple.

The sixth column tell us if the action is simple, this is obtained by direct inspection. The last column tell us if the action is smoothable.

For reader convenience we recall the notion of smoothable action introduced in [Ma3].

**Definition 4.6.** Let $G$ be a finite group acting on a normal twodimensional singularity $(X, 0)$. The action is called smoothable if there exists a smoothing $(X, 0) \to (\mathbb{C}, 0)$ of $(X, 0)$ such that the $G$-action extends to $X$ preserving fibres and the quotient $(X/G, 0) \to (\mathbb{C}, 0)$ is a smoothing of the quotient $Y = X/G$. The Milnor fibre of $X/G \to \mathbb{C}$ is called an admissible Milnor fibre of the $G$-action.

**Remark. 4.7.** Not every Milnor fibre of singularity $X/G$ is admissible, for example if $(X, 0)$ is a rational double point and $G$ is a cyclic group acting freely on $X - 0$ then the admissible Milnor fibres are exactly the ones coming from $\mathbb{Q}$-Gorenstein smoothing.

In the definition 4.6, the proof that $Y = X/G$ is the central fibre of the smoothing $X/G$ is non trivial and rests on the Serre criterion [Mat, 23.8] and the fact that Cohen-Macaulay complex spaces are preserved under finite group actions.

The answer yes in the last column of 4.4 is given by writing down explicit smoothings, while the answer no is obtained by using the same methods of [Ma3] (see below).

Note that every simple action realize $X$ as a simple cover of $Y$ and then a generic Galois deformation of the cover gives a smoothing of the action (recall that every rational twodimensional singularity is smoothable, [Wa3, 1.3]).

In the following example we describe some smoothings of non simple actions (i.e. lines 3,5 and 12 of Table 3).

**Example 4.8.** The singularity $X \subset \mathbb{C}^3 \times \mathbb{C}$ defined by the equation $uv + y^{2n+1} + ty = 0$ defines a smoothing of the rational double point $A_{2n}$; the involution $c$) extends to $X$ and its quotient is a smoothing of $B_n$.

The negative answer to smoothability in lines 2,4 is proved in [Ma3, 4.6], the negative answer in lines 9,10,11 is proved in a similar way; here we illustrate the idea by showing that the action 9 is not smoothable.

The rational double point $X$ of type $A_{2n-1}$, $n > 0$, is defined in $\mathbb{C}^3$ by the equation $z^2 + x^2 + y^{2n} = 0$ and the action 9 is given by the involutions $a, d, e = ad$, $G = \{1, a, d, e\}$. 

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Since $G$ is finite, every $G$-stable smoothing of $X$ over $(\mathbb{C}, 0)$ is defined in $(\mathbb{C}^3 \times \mathbb{C}, 0)$ by an equation

$$F_t(x, y, z) = z^2 + x^2 + y^{2n} + t\phi(x, y, z, t) = 0$$

with $\phi$ a $G$-invariant function; see [Ma3, 4.5] for a proof of this fact.

In particular the power series expansion of $\phi$ cannot contain linear terms in $x, y, z$. Since for $t \neq 0$ sufficiently small the fibre $X_t = \{F_t = 0\}$ is smooth, we must have necessarily $\phi(0, 0, 0, t) \neq 0$ and the intersection of $X$ with the line $\{x = z = 0\} = \text{Fix}(d)$ is the germs of plane curve of equation $y^{2n} + t\phi(0, y, 0, t) = 0$. In particular, for $0 < |t| << 1$ there exists a point $x = (0, \xi, 0, t) \in \text{Fix}(d) \cap X_t$ with $\xi \neq 0$; hence $x$ is an isolated fixed point for the involution $d$ and $x \notin \text{Fix}(a) \cup \text{Fix}(e)$. This implies that the image of $x$ in the quotient $X_t/G$ is a singular point.

In the following three lemmas we deduce, in the style of [Ca3], some topological obstructions to degenerations.

**Lemma 4.9.** Let $V \to \mathbb{C}^2$ be the blow-up at a point and let $F_n$ be the Milnor fibre of a smoothing of a singularity of type $A_n, D_n, E_n, B_n$, $n > 0$.

Then cannot exist any open embedding $F \subset V$.

*Proof.* Assume that $F_n$ is an open subset of $V$, then there is a homomorphism of groups $i_*: H_2(F_n, \mathbb{Z}) \to H_2(V, \mathbb{Z})$ which preserves the intersection products. We have $H_2(V, \mathbb{Z}) = Ze$, where $e$ is the class of the exceptional curve, $e^2 = -1$. By simultaneous resolution the intersection form on $H_2(F_n, \mathbb{Z})$ is negative definite of rank $n$; this forces $n = 1$. But $H_2(F_1, \mathbb{Z}) = Ze$ with $e^2 = -2$ for smoothing of $A_1$ and $e^2 = -3$ for smoothing of $B_1$; in both cases $i_*$ cannot be an isometry.

□

**Lemma 4.10.** Let $F$ be the Milnor fibre of a smoothing of $Y_n$, then cannot exist any open embedding $F \subset \mathbb{C}^2$.

*Proof.* There are two different cases:

1) $F$ is the fibre of a smoothing admitting simultaneous resolution. In this case the intersection form on $H_2(F, \mathbb{Z})$ is negative definite and cannot exist any isometry $H_2(F, \mathbb{Z}) \to H_2(\mathbb{C}^2, \mathbb{Z}) = 0$.

2) $F$ is the fibre of a $\mathbb{Q}$-Gorenstein smoothing. As $F$ is open in $\mathbb{C}^2$ the (trivial) canonical class restricts to the canonical class of $F$ which is non trivial; this gives a contradiction.

□

**Lemma 4.11.** Let $F$ be the Milnor fibre of a smoothing of a rational double point of type $A_n$, $n \geq 2$ and let $V \to \mathbb{C} \times \mathbb{P}^1$ be the blow up at two points. Then cannot exist any open embedding $F \subset V$.

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We have seen that there exists sufficiently ample building data satisfying the above conditions, moreover by the computation of section 2 it follows easily that the triple $(S, L, D)$.

\[ \div[S, L, D] = \mathbb{Z}^3. \]

Since the hypothesis of corollary 3.14.

\section{A global construction}

Let $k > 0$ be a fixed natural number and consider integers $a_1, \ldots, a_k, b_1, \ldots, b_k \geq 3$, $n_1 < n_2 < \ldots < n_k$, $a_i \neq b_i$ such that $0 \leq n_i \leq 2a_ib_i$ and denote $M_{3,n_i} = \mathbb{M}_{3,n_i}^{a_i, b_i}$. Let $\mathcal{M} \subseteq \mathbb{M}_{a_1,b_1} \times \ldots \times \mathbb{M}_{a_k,b_k}$ be the open subscheme of $k$-uples $((C^i_1, p^i_1), \ldots, (C^i_k, p^i_k))$ such that $p^i_j \neq p^h_j$ for $(i, j) \neq (h, l)$ and the curves $C^i_j$ are without common components.

Define also

\[ \mathcal{M}^0 = \mathcal{M} \cap \prod_i \mathbb{M}_{3,n_i}^{0}. \]

Let $\nu: S \to \mathcal{M} \times Q$ be the blow up of the sections $p^i_j: \mathcal{M} \to \mathcal{M} \times Q$, $i = 1, \ldots, k$, $j = 1, \ldots, n_i$ and denote by $p: S \to \mathcal{M}$, $q = (q_1, q_2): S \to \mathbb{P}^1 \times \mathbb{P}^1 = Q$ the natural projections.

Denote by $E^i_j \subseteq S$ the exceptional Cartier divisor over the sections $p^i_j$. We have seen that the coherent sheaf $p_*(q^* \mathcal{O}_Q(r, s) - \sum a^i_j E^i_j)$ is locally free in the following cases:

i) $r, s \geq 0$ and $a^i_j = 0$ for every $i, j$.

ii) $a^i_j \geq 2$ for every $i, j$ and $r, s > \sum (a^i_j + 1)$.

Moreover in the above cases i), ii) the restriction of $q^* \mathcal{O}_Q(r, s) - \sum a^i_j E^i_j$ to every fibre of $p$ gives a base point free linear system.

For every $i = 1, \ldots, k$ define $G^i_1 = (\mathbb{Z}/2)^2 = \{0, \alpha^i_1, \alpha^i_2, \alpha^i_3\}$, $G^i_2 = (\mathbb{Z}/2)^{n_i}$ with basis $\epsilon^i_1, \ldots, \epsilon^i_{n_i}$.

Define moreover $G' = G^1_1 \oplus G^2_2 \oplus \ldots \oplus G^k_1 \oplus G^k_2 \oplus (\mathbb{Z}/2)^4$ with $\tau_1, \tau_2, \eta_1, \eta_2$ a basis of $(\mathbb{Z}/2)^4$ and $G = G' \oplus (\mathbb{Z}/2)$.

Given a point $m = ((C^i_1, p^i_1), \ldots, (C^i_k, p^i_k)) \in \mathcal{M}$ we want to construct a family of normal flat $G$-covers over the fibre $S_m = p^{-1}(m)$, the building data $L: G' \to \text{Pic}(S_m)$, $D: G \to \text{Div}(S_m)$ of this cover must satisfy the following conditions:

i) $L_\chi - D_\sigma$ is combinatorially ample for every $\chi \neq 0, \sigma \in G$.

ii) $D_\alpha^i_j = q^* C^i_j - \sum_h E^i_h$ for $j = 1, 2, 3$, $i = 1, \ldots, k$.

iii) $D_\tau_j = E^i_j$, $D_\eta$, a fibre of $q_1$, $D_\eta$, a fibre of $q_2$. If $\sigma \in G'$ is not one of the previous cases then $D_\sigma = 0$.

iv) If $\sigma \in G - G'$ then $D_\sigma$ is combinatorially ample.

We have seen that there exists sufficiently ample building data satisfying the above conditions, moreover by the computation of section 2 it follows easily that the triple $(S, L, D)$ satisfies the hypothesis of corollary 3.14.

Since $\text{Pic}(S_m)$ is torsion free the map $L: G' \to \text{Pic}(S_m)$ is uniquely determined by $D: G \to \text{Div}(S_m)$. For a fixed $m$ and $L$ the set of building data $(L, D)$ as above is then parametrized.
by a product $U_m$ of projective spaces; by base change the union of all $U_m$, with fixed $L$ is an algebraic bundle $s:U \to \overline{M}$.

**Lemma 5.1.** Let $U^0 \subset U$ (resp.: $U^{00}$) be the subset of triples $(m, D, L)$ such that the divisors $D_\sigma$ are reduced without common components and the associated $G$-cover $X \to S_m$ has at most rational double points as singularities (resp.: $X$ is smooth).

Then $U^0, U^{00}$ are Zariski open in $U$.

Moreover $s(U^{00}) \subset s(U^0) \subset \overline{M}^0$.

**Proof.** For every $u = (s(u), L_u, D_u) \in U$ denote $S_u = S \times \overline{M} \{u\}$. By construction there exist (tautological) maps $D_U: G \to \text{Div}(S \times \overline{M} U)$, $L_U: G' \to \text{Pic}(S \times \overline{M} U)$ such that their restriction to $S_u$ are exactly $D_u, L_u$; define $\tilde{U} \subset U$ as the Zariski open subset consisting of points $u$ such that the divisors $D_{u, \sigma}$ are reduced without common components and then they are the branching divisors of a normal flat $G$-cover $X_u \to S_u$.

There exists a Zariski open covering $\tilde{U} = \cup V_i$ such that the maps $D_U, L_U$ satisfies the cover conditions on $S \times \overline{M} V_i$ and therefore there exist flat $G$-covers $X_{V_i} \to S \times \overline{M} V_i$, which by construction contains all the Galois deformations of $X_u \to S_u$ for every $u \in \tilde{U}$.

By the stability property of the classes of smooth and rational double points under deformations it follows that $V_i \cap U^0$ and $V_i \cap U^{00}$ are Zariski open subsets of $V_i$ and this proves that $U^0, U^{00}$ are Zariski open in $U$.

For $u \in U^0$ let $X_u \to S_u = S \times \overline{M} \{u\}$ be the associated $G$-cover, by Corollary 3.17 it is locally simple, hence for every $i = 1, ..., k$, $D_{a_1} \cap D_{a_2} \cap D_{a_3} = \emptyset$ and this is equivalent to the fact that $s(u) \in \overline{M}^0$.

\[ \square \]

If $\mathcal{M}$ denotes the moduli space of surfaces of general type the above proof shows that the natural map $\phi: U^0 \to \mathcal{M}$, $\phi(u) = [X_u]$ is a regular morphism of varieties. Note that the map $\phi$ is invariant under the natural action of $\text{Aut}_0(Q)$ over $U^0$; it is also useful to remark that $U^0$ and $U^{00}$ are stable under the action of $\text{Aut}_0(Q) \times \prod_{i=1}^{k} (\Sigma_3 \times \Sigma_{n_i})$ but in general $\phi$ is not invariant under this extended action.

**Proposition 5.2.** The morphism $\phi$ is open.

**Proof.** Immediate from the fact that, as the building data are choosed satisfying the hypothesis of 3.14, the Galois deformations are complete.

\[ \square \]

Next we want to prove that $\phi(V)$ is closed in $\mathcal{M}$ for every irreducible component $V$ of $U^0$; since $U^{00}$ is dense in $U^0$ it is sufficient to prove that there exists an open covering $U^{00} = \cup V_i$ such that $\phi(V_i \cap V) \subset \phi(V)$ for every $i$.

We choose the covering $\{V_i\}$ such that there exist global $G$-covers $\mathcal{X}_{V_i} \to S \times \overline{M} V_i$, by the valuative criterion the closure of $\phi(V)$ follows from the following
Theorem 5.3. Let $U^{00} = \cup V_i$ be as above and let $\Delta$ be a smooth affine curve, $0 \in \Delta$ and let $f: \mathcal{X} \to \Delta$ be a proper flat family of irreducible surfaces such that:

i) $X_t = f^{-1}(t)$ is smooth for $t \neq 0$ and $X_0$ has at most rational double points as singularities.

ii) $X_t$ has ample canonical bundle for every $t \in \Delta$.

iii) Let $\mathcal{X} = \mathcal{X} \times X_0$, $\Delta^* = \Delta - \{0\}$, then there exists a regular morphism $\eta: \Delta^* \to V_i \subset U^{00}$ such that $\mathcal{X}^* \to \Delta^*$ is isomorphic to the pull-back of $\mathcal{X}_i \to V_i$ under $\eta$.

Then the map $\eta: \Delta^* \to U^{00}$ can be extended, up to the action of $\text{Aut}_0(Q)$ on $U^{00}$, to a regular morphism $\eta: \Delta \to U^{00}$.

Proof. As the fibres of $f$ have at most rational double points and ample canonical bundle the hypothesis of the semicontinuity theorem [FP, 4.4] are satisfied and the $G$-action over $\mathcal{X}^*$ can be extended to a regular $G$-action on $\mathcal{X}$; moreover the restriction of this action to $X_0$ is faithful.

Let $\pi: \mathcal{X} \to \mathcal{Y}$ be the projection to the quotient and let $g: \mathcal{Y} = \mathcal{X}/G \to \Delta$ be the factorization of $f$. Denote by $Y_t = g^{-1}(t)$; note that if $Y_t$ is smooth or $X_t/G$ is smooth and $Y_t$ is normal then the natural morphism $X_t/G \to Y_t$ is an isomorphism of varieties.

Claim 1. If $x \in X_t$ is a smooth point then $g: \mathcal{Y} \to \Delta$ is smooth at $\pi(x)$, in particular $\mathcal{Y}$ is normal with isolated singularities.

We can check the smoothness of $g$ from the analytic point of view. Let $G_x \subset G$ be the stabilizer of $x$, according to [Ma3, 4.5] there exists a $G_x$-isomorphism of analytic germs $(\mathcal{X}, x) = (X_t, x) \times (\Delta, t)$ where $G_x$ acts trivially on $\Delta$. Then $(\mathcal{Y}, \pi(x)) = (X_t/G_x, x) \times (\Delta, t)$, using the fact that $Y_t$ is smooth for every $t \neq 0$ it follows that $(X_t/G_x, x)$ must be smooth. Since $\mathcal{X}$ is normal also $\mathcal{Y}$ is normal.

Claim 2. $g: \mathcal{Y} \to \Delta$ is smooth and $\pi: \mathcal{X} \to \mathcal{Y}$ is flat.

By Claim 1 if $g$ is not smooth at a point $y = \pi(x)$ then $x$ must be a singular point of $X_0$. As above let $G_x$ be the stabilizer of $x$; then $G_x$ acts on the smoothing $(\mathcal{X}, x) \to (\Delta, 0)$ preserving fibres and its quotient is the smoothing $(\mathcal{Y}, y) \to (\Delta, 0)$. Therefore the $G_x$-action on the rational double point $(X_0, x)$ is smoothable.

For every $\sigma \in G$ let $D_\sigma \subset \mathcal{Y}$ be the reduced branching divisor of the action of $\sigma$ on $\mathcal{X}$. If $F \subset Y_t$ is the Milnor fibre of $(\mathcal{Y}, y) \to (\Delta, 0)$ then for $t$ sufficiently near to $0$ we have $F \subset Y_t - (\cup_{\sigma \in I_t} D_\sigma)$ and then, recalling that among the divisors $D_\sigma \cap Y_t$ there are fibres of the two projections $q_1, q_2: Y_t \to \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^1$, we see that $F$ is biholomorphic to an open subset of $V$, where

a) $V = \mathbb{C}^2$ if $|I_x| = 0$, or

b) $V$ = the blow up of $\mathbb{C}^2$ at a point if $|I_x| = 1$, or

c) $V$ = the blow up of $\mathbb{C} \times \mathbb{P}^1$ at two points if $|I_x| = 2$.

By using table 4.4 and the topological obstructions 4.9, 4.10 and 4.11 we deduce that $Y_0$ must be smooth at $y$. 

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By 3.5 the divisors $D_\sigma \cap Y_i$ are reduced without common components and the same argument used in Claim 1 shows that for every $t$ every branching divisor of $X_t \to Y_t$ has the form $D_\sigma \cap Y_t$. Since $D_{\tau_1} \cdot D_{\tau_2} \cdot Y_t = 0$, we have $D_{\tau_1} \cap D_{\tau_2} = \emptyset$ and, possibly shrinking $\Delta$, $D_{\tau_1}$ is linearly equivalent to $D_{\tau_2}$ and then they define a morphism $\nu: Y \to \mathbb{P}^1$. The same argument applies to the divisors $D_{\alpha_1}, D_{\alpha_2}$ and then we have a morphism $\nu: Y \to Q \times \Delta$ such that for every $t \neq 0$ the restriction $\nu: Y \to Q$ is the blow up at $n$ distinct points.

Claim 3. $D_{\alpha_j} \cap Y_0$ is a smooth rational curve for every $i = 1, \ldots, k, \; j = 1, \ldots, n_i$.

Let $(i, j)$ be fixed pair as above and let $E = D_{\alpha_j} \cap Y_0$.

As $D_{\alpha_j}$ is reduced irreducible and the local rings of $\Delta$ are discrete valuation rings the map $g: D_{\alpha_j} \to \Delta$ is flat and then $E \subset Y_0$ is a reduced connected divisor of arithmetic genus $p_a(E) = 0$. It is then easy to see that if $E = \bigcup_i E_i \subset Y_0$ is the decomposition in irreducible components, then every irreducible component $E_i$ is smooth rational, $E_i \cdot E_j \leq 1$ for every $i \neq j$ and the dual intersection graph is a tree; a trivial calculation shows that $E^2 = \sum_i (E_i^2 + 2) - 2$.

The one-dimensional variety $\nu(D_{\alpha_j})$ is irreducible and then $\nu$ contracts $E$; by Mumford theorem [Mu1] every component $E_i$ has negative selfintersection. If $E_i^2 \leq -2$ for every $i$ then we get a contradiction by using the above formula $-1 = E^2 = \sum_i (E_i^2 + 2) - 2$. Let $E_0 \subset E$ be a component with $E_0^2 = -1$, then by Kodaira stability theorem [Ko], [Ho], there exist an analytic neighbourhood $0 \in \Delta' \subset \Delta$ and a smooth closed submanifold $E \subset g^{-1}(\Delta')$ such that $\nu \cap Y_0 = E_0$ and $g: E \to \Delta'$ is a $\mathbb{P}^1$ bundle. As $\nu$ contracts $E_0$ then it contracts all the fibres of $g: E \to \Delta'$ and therefore $E \subset D_{\alpha_j}^i$ for some $h, l$. Using the fact that the divisors $D_{\alpha_j}^i \cap Y_0$ have no common components we get $E \subset D_{\alpha_j}^i$; both divisors are irreducible and then $E = D_{\alpha_j}^i$.

From Claim 3 it follows that $\nu: Y_0 \to Q$ is the blow up at $n$ distinct points $p_1, \ldots, p_n$. Moreover $D_{\alpha_j}^i \cap Y_0$ is a reduced effective divisor linearly equivalent to $O_{Y_0}(a_i, b_i) - \sum E_j$ and $D_{\alpha_j}^i \cap D_{\alpha_j}^i \cap D_{\alpha_j}^i = \emptyset$; this clearly implies that

$$(\ldots, \nu(D_{\alpha_j} \cap Y_0), \nu(D_{\alpha_j} \cap Y_0), \nu(D_{\alpha_j} \cap Y_0), p_1^i, \ldots, p_n^i, \ldots) \in \overline{\mathcal{M}}^0$$

From this we get the required morphism $\eta: \Delta \to \mathcal{U}^0$.

☐

**Corollary 5.4.** Let $V$ be an irreducible (resp.: connected) component of $U^0$. Then $\phi(V)$ is an irreducible (resp.: connected) component of the moduli space $\mathcal{M}$.

**Proof.** It is an immediate consequence of the fact that $\phi$ is an open map and that $\phi(V)$ is closed in $\mathcal{M}$.

☐

There exists a natural diagonal action of $\text{Aut}_0(Q)^k$ on the spaces $\prod M_{3,n_1}$ and $\prod M^0_{3,n_1}$.

**Lemma 5.5.** Let $U_T \subset \widehat{U}$ be the open subset of points $u$ such that $X_u$ is a surface of class $T$. 37
Then the open subsets \( s(U_T) \cap \prod M_{3,n_i} \subset \prod M_{3,n_i} \), \( s(U^{00}) \cap \prod M_{3,n_i} \subset \prod M^0_{3,n_i} \) intersect every \( Aut_0(Q)^k \)-orbit. In particular \( s(U_T) \cap \prod M_{3,n_i} \) is connected.

**Proof.** Let \( m = (.., (C^1_j, p^1_i), ..) \) \( \in \prod M_{3,n_i} \), for a generic choice of \( \phi = (\phi_1, ..., \phi_k) \in Aut_0(Q)^k \), the curves
\[
\phi_1(C^1_1 + C^1_2 + C^1_3), ..., \phi_k(C^k_1 + C^k_2 + C^k_3)
\]
intersects transversally. This implies that there exists \( u \in U_T \) such that \( s(u) = \phi(m) \) and every point of \( X_u \) is either smooth or cyclic singular of type \( \frac{1}{4}(1,1) \). The same argument shows that if \( m \in \prod M^0_{3,n_i} \) then for generic \( u \in U_{\phi(m)} \) the surface \( X_u \) is smooth.

Since \( \prod M_{3,n_i} \) is connected and \( Aut_0(Q)^k \) is irreducible it is an easy exercise to deduce that \( s(U_T) \cap \prod M_{3,n_i} \) is connected.

As a corollary of 5.5 we get the following

**Proposition 5.6.** If \( u, v \in U^{00} \) then \( X_u, X_v \) are deformation \( T \)-equivalent and hence diffeomorphic.

**Proof.** It is an immediate consequence of 5.5 and the inclusion \( s(U^{00}) \subset s(U_T) \cap \prod M_{3,n_i} \).

It remains to estimate from below the number of connected components of \( \phi(U^{00}) \).

**Proposition 5.7.** If the building data \((L,D)\) are sufficiently ample then there exists an open dense subset \( U' \subset U^{00} \) such that if \( u, v \in U' \) and \( \phi(u) = \phi(v) \) then \( s(u) = h s(v) \) for some \( h \in Aut_0(Q) \times \prod (\Sigma_3 \times \Sigma_{n_i}) \). In particular the number of connected components of \( \phi(U^{00}) \) is greater or equal than the number of connected components of \( \prod M^0_{3,n_i} \) which are stable by the action of \( Aut_0(Q) \times \prod (\Sigma_3 \times \Sigma_{n_i}) \).

**Proof.** According to [FP.4.4], if \((L,D)\) are sufficiently ample there exists an open dense subset \( U' \subset U^{00} \) such that for every \( u \in U' \), \( Aut(X_u) = G \), i.e. every biregular automorphism of \( X_u \) is an automorphism of the cover \( X_u \to S_u \). (The same result can be obtained as a straightforward generalization of the main theorem of [Ma4]).

The point \( \phi(u) \in \mathcal{M} \) determines \( X_u \) up to biregular isomorphism; if \( X \simeq X_u, u \in U' \) then there exists a commutative diagram with horizontal isomorphisms
\[
\begin{array}{ccc}
X & \longrightarrow & X_u \\
\pi \downarrow & & \downarrow \\
X/\text{Aut}(X) & \longrightarrow & S_u
\end{array}
\]

The exceptional curves \( f^*(E'_i) \) are exactly the branching divisors \( D_\alpha \) of the \( G \)-cover \( \pi \) such that \( D_\alpha^2 = -1 \). By considering the branching divisors \( D_\alpha \) such that \( D_\alpha \cdot f^*(E'_i) = 1 \) for some \( i, j \), and using the fact that \( n_1 < n_2 < ... < n_k \) it is easy to recover from the isomorphism class of \( X_u \) the orbit of \( s(u) \) under the action of \( Aut_0(Q) \times \prod (\Sigma_3 \times \Sigma_{n_i}) \).
This proves the first part of the proposition. The second part follows immediately from the fact that $s(U')$ intersects every connected component of $\prod M_{g,n_i}^0$, and the fact that if $U_1, U_2 \subset U$ are connected components then either $\phi(U_1) \cap \phi(U_2) = \emptyset$ or $\phi(U_1) = \phi(U_2)$.

\[\square\]

**Corollary 5.8.** In the above situation for suitable values of $a_i, b_i, n_i$ the image $\phi(U^0) \subset \mathcal{M}$ is the union of at least $2^k$ irreducible components.

**Proof.** Immediate consequence of 5.7 and 2.5.

\[\square\]

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