IDÈLIC CLASS FIELD THEORY FOR
3-MANIFOLDS AND VERY ADMISSIBLE LINKS

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Abstract. We study a topological analogue of idèlic class field theory for 3-manifolds. We start with introducing the notion of a very admissible link $K$ in a 3-manifold $M$ as an analogue of the set of primes of a number field and prove its existence. Next, we recall the basic analogies in arithmetic topology and introduce the notion of the universal $K$-branched cover, where the role of base points is also discussed. Then we introduce the notion of idèle for such a pair $(M,K)$ and establish the two main theorems: Firstly, we prove an analogue of the global reciprocity law of idèlic class field theory for $(M,K)$. Secondly, by introducing certain topologies on our idèle class group, called the standard topology and the norm topology, we show a topological analogue of the existence theorem for $(M,K)$. In addition, we discuss an analogue of the norm residue symbol for a 3-manifold, and finally calculate the Tate cohomology of our idèle class group in light of the comparison with the axiom of class field theory.

1. Introduction

This is a continuation of the previous paper [Ni14] by the first author. Following the analogies between 3-dimensional topology and number theory, the first author studied a topological analogue of idèlic class field theory for 3-manifolds, and showed, among other things, an analogue of Artin’s global reciprocity law over an integral homology 3-sphere. The purpose of this paper is to continue this work further, and show analogues of Artin’s global reciprocity law and the existence theorem of idèlic class field theory, over a general (closed, oriented, connected) 3-manifold.

Let $M$ be a closed, oriented, connected 3-manifold, equipped with a very admissible link $(knot set) K$, the notion introduced in [Ni14] as an analogue of the set of primes in a number ring. For this notion, we give a refined treatment in section 2.

For $(M,K)$, we also introduce the notion of a universal $K$-branched cover in section 3. It is an analogue of an algebraic closure of a number field. We fix it and restrict our argument to the branched covers which are obtained as its quotients. (It is equivalent to consider isomorphism classes of branched covers with basepoints.)

We then have the idèle group $I_{M,K}$, the principal idèle group $P_{M,K}$ and the idèle class group $C_{M,K} := I_{M,K}/P_{M,K}$ as defined in a functorial way in [Ni14]. Let $\text{Gal}(M,K)^{ab}$ denote the inverse limit of the Galois groups $\text{Gal}(X_L^{ab}/X_L)$, where $L$ runs through the finite links in $K$, and $X_L^{ab} \to X_L$ is the maximal abelian cover over the exterior of $L$ in $M$.

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The first main result of this paper is the following analogue of Artin’s global reciprocity law in idelic class field theory, which was first shown in [Nii14] for the case over an integral homology 3-sphere.

**Theorem 5.4** (The global reciprocity law for 3-manifolds). There is a canonical isomorphism called the global reciprocity map

\[ \rho_{M,K} : C_{M,K} \cong \text{Gal}(M,K)^{ab} \]

such that (i) for any finite abelian cover \( h : N \to M \) branched over a finite link in \( K \), \( \rho_{M,K} \) induces an isomorphism of finite abelian groups

\[ C_{M,K}/h_* (C_{N,h^{-1}(K)}) \cong \text{Gal}(h), \]

and (ii) \( \rho_{M,K} \) is compatible with the local theories.

In this paper, we also show an analogue of the existence theorem of idelic class field theory, which gives a bijective correspondence between finite abelian covers of \( M \) branched over finite links in \( K \) and certain subgroups of \( C_{M,K} \). For this purpose, we introduce certain topologies on \( C_{M,K} \), called the standard topology and the norm topology, following after the case of number fields ([KKS11], [Neu99]).

Now our existence theorem is stated as follows:

**Theorem 7.4** (The existence theorem). The following correspondence

\[ (h : N \to M) \mapsto h_* (C_{N,h^{-1}(K)}) \]

gives a bijection between the set of (isomorphism classes of) finite abelian covers of \( M \) branched over finite links \( L \) in \( K \) and the set of open subgroups of finite indices of \( C_{M,K} \) with respect to the standard topology. Moreover, the latter set coincides with the set of open subgroups of \( C_{M,K} \) with respect to the norm topology.

Here are contents of this paper. In section 2, we discuss very admissible links. We refine the definition in [Nii14], prove the existence, and give some remarks. In section 3, we briefly review the basic analogy between knots and primes, introduce the notion of a universal \( K \)-branched cover, and discuss the role of base points. In section 4, we recall the idelic class field theory for number fields, whose analogue will be discussed in the later sections. In section 5, following section 5 of [Nii14], we introduce the notions of idèle group \( I_{M,K} \) and idèle class group \( C_{M,K} \), together with the global reciprocity map \( \rho_{M,K} : C_{M,K} \to \text{Gal}(M,K)^{ab} \) for a 3-manifold \( M \) equipped with a very admissible link \( K \). Then we verify the global reciprocity law for 3-manifolds.

In section 6, we introduce the standard topology on the idèle class group \( C_{M,K} \), and prove the existence theorem for it. In section 7, we do the same for the norm topology. In section 8, we discuss the norm residue symbols for 3-manifolds. In section 9, we calculate the Tate cohomology of idèle class group and compare with the axiom of class field theory.

We note that idelic class field theory for 3-manifolds was firstly studied by A. Sikora ([Sik03], [Sik1], [Sikb]). Although our approach is slightly different from his, we were inspired by his works in the beginning of our study.

**Notation.** For a manifold \( X \), we simply denote by \( H_1(X) \) its 1st homology group with coefficients in \( \mathbb{Z} \). For a group \( G \) and its subgroup \( G_1 \), we write \( G_1 < G \). For
a branched cover \( h : N \to M \) and a link \( L \) in \( M \), \( h^{-1}(L) \) denotes the link in \( N \) defined by the preimage. When \( h \) is Galois (regular), \( \text{Gal}(h) = \text{Deck}(h) \) denotes the Galois group, that is, the group of covering transformations of \( N \) over \( M \).

Throughout this paper, a branched cover means one equipped with base points, unless otherwise mentioned.

2. Very admissible links

In the previous paper [Ni14], the notion of a very admissible knot set (link) \( K \) in a 3-manifold \( M \) was introduced, as an analogue object of the set of all the primes in a number field. However, the construction was not sufficient.

In this section, we refine the definitions, prove the existence of a very admissible link \( K \) in a 3-manifold \( M \), and give some remarks.

**Definition 2.1.** Let \( M \) be a closed, oriented, connected 3-manifold, and let \( \mathcal{K} \) be a finite or infinite link in \( M \) equipped with a tubular neighborhood. We say \( \mathcal{K} \) is an admissible link of \( M \) if the components of \( \mathcal{K} \) generate \( H_1(M) \). We say \( \mathcal{K} \) is a very admissible link of \( M \) if for any finite abelian cover \( h : N \to M \) branched over a finite link in \( \mathcal{K} \), the components of the link \( h^{-1}(K) \) generates \( H_1(N) \).

In the following, a link means a finite or infinite link equipped with a tubular neighborhood. For a link \( L \) in a 3-manifold \( M \), we denote by \( V_L = \bigcup_{K \subset L} V_K \) the tubular neighborhood, where \( K \) runs through the components of \( \mathcal{K} \), and \( V_K \) is a tubular neighborhood of \( K \). We denote the meridian by \( \mu_K \in H_1(\partial V_K) \), and fix an element \( \lambda_K \in H_1(\partial V_K) \) such that \( \mu_K \) and \( \lambda_K \) form a basis of \( H_1(\partial V_K) \).

In this paper, we call such \( \lambda_K \) a longitude of \( K \). For a branched cover \( h : N \to M \), for each component of \( h^{-1}(K) \) in \( N \), we fix a meridian and a longitude which are components of the preimages of those of \( K \).

**Lemma 2.2.** Let \( M \) be a closed, connected, oriented 3-manifold and \( L \) a link in \( M \). Then, there is a link \( \mathcal{L} \) in \( M \) such that \( L \subset \mathcal{L} \) and for any finite cover \( h : N \to M \) branched over a finite sublink of \( L \), \( H_1(N) \) is generated by the components of the preimage \( h^{-1}(\mathcal{L}) \).

**proof.** The set of all the finite branched covers of \( M \) branched over finite sublinks of \( L \) is countable, and can be written as \( \{ h_i : N_i \to M \}_{i \in \mathbb{N}} \), where \( h_0 = id_M \). Indeed, for a finite sublink \( L' \subset L \), each branched cover branched over \( L' \) can be obtained as the Fox completion of a cover of the exterior \( X_{L'} := M \setminus L' \). Each cover of \( X_{L'} \) corresponds to each subgroup of \( \pi_1(X_{L'}) \) by Galois theory. Each group \( \pi_1(X_{L'}) \) is finitely generated, and the set of its subgroups of finite indices is countable. Since the set of finite links of \( L \) is countable, so the set of branched covers.

We construct an inclusion sequence of links \( \{ L_i \}_i \) as follows: First, we put \( L_{-1} = L \). Next, for \( i \in \mathbb{N} \), let \( L_{i-1} \) be given. Since \( N_i \) is compact, \( H_1(N_i) \) is finitely generated, and we can take a link \( L_i \) in \( N_i \) such that it includes \( h_{i-1}^{-1}(L_{i-1}) \), its components generates \( H_1(N_i) \), and its image \( h_i(L_i) \) is again a link in \( M \). We put \( L_i = h_i(L_i) \). Then, the union \( \mathcal{L} := \bigcup_i L_i \) satisfies the expected condition. \( \square \)

**Theorem 2.3.** Let \( M \) be a closed, connected, oriented 3-manifold, and \( L \) a link in \( M \). Then, there is a very admissible link \( K \) which includes \( L \). Moreover, we can remove the condition “abelian” in the definition.
proof. We construct an inclusion sequence of links \( \{K_i\} \) as follows: First, we put \( K_1 = L \). Next, for \( i \in \mathbb{N} \), let \( K_{i-1} \) be given, and let \( K_i \) be a link obtained from \( K_{i-1} \) by the above lemma. Then the union \( K := \bigcup K_i \) is a very admissible link, and the condition of abelian is removed. \( \square \)

Links \( L \) and \( K \) in the lemma and theorem above may be taken smaller than in the constructions. It may be interesting to ask whether they can be finite.

Let \( M = S^3 \). The unknot is very admissible link. If \( L \) is the trefoil, by taking branched 2-cover, we see that \( K_1 \) is greater than \( L \). We expect that \( K \) has to be infinite.

Next, let \( M \) be a 3-manifold, and \( L \) a minimum admissible link (\( L \) can be empty). For an integral homology 3-sphere \( M \), we have \( K = L = \phi \). For a lens space \( M = L(p, 1) \) or \( M = S^2 \times S^1 \), we can take a knot (simple loop) \( K = L = K \).

In the latter sections of this paper, we assume that a very admissible link \( K \) is an infinite link. However, our argument are applicable for finite \( K \) also.

Remark 2.4 (variants). (1) We will discuss a weaker condition on an admissible link in the end of section 5, Remark 5.9.

(2) Let \( L \) be an infinite link such that any (ambient isotopy class of) finite link in \( M \) is contained in \( L \). There exists such a link. Indeed, since the classes of finite links are countable, by putting links side by side in \( S^3 = \mathbb{R}^3 \cup \{\infty\} \), we obtain such a link \( L \), with one limit point at \( \infty \). By using the metric of \( \mathbb{R}^3 \), we can take a tubular neighborhood of \( L \). If we start the construction form such an infinite link, then we obtain a special very admissible link \( K \), which controls all the branched covers of \( M \) branched over any finite links in \( M \).

(3) We could remove the condition of “abelian” on the covers in the definition and the constructions. In addition, we can replace every “the components generates \( H_1 \)” by “the components with paths to the base point generates \( \pi_1 \)”.

Strictly speaking, a very admissible link is an analogue of the set of finite primes. According to \([Mor12]\), counterparts of infinite primes are ends of 3-manifolds. F. Hajir also studies cusps of hyperbolic 3-manifolds as analogues of infinite primes of number fields (\([Haj12]\)). In this paper, however, since we deal with closed manifolds, the counterpart of the set of infinite primes is empty.

3. THE UNIVERSAL \( K \)-BRANCHED COVER

In this section, we briefly review the analogies between knots and primes. Then, for a 3-manifold \( M \) equipped with an infinite (very admissible) link \( K \), we introduce the notion of the universal \( K \)-branched cover, which is an analogue of an algebraic closure of a number field. We also discuss the role of base points.

The analogies between knots and primes has been studied systematically by B. Mazur (\([Maz64]\)), M. Kapranov (\([Kap95]\)), A. Reznikov (\([Rez97]\), \([Rez00]\)), M. Morishita (\([Mor02]\), \([Mor10]\), \([Mor12]\)), A. Sikora (\([Sik03]\)) and others, and their research is called arithmetic topology. Here is a basic dictionary of the analogies.
For a number field \( k \), let \( \mathcal{O}_k \) denote the ring of integer.

| 3-manifold \( M \) | number ring \( \text{Spec} \mathcal{O}_k \) |
|---------------------|----------------------------------|
| knot \( K : S^1 \hookrightarrow M \) | prime \( \mathfrak{p} : \text{Spec}(\mathcal{O}_k) \rightarrow \text{Spec} \mathcal{O}_k \) |
| link \( L = \{K_1, \ldots, K_r\} \) | set of primes \( S = \{p_1, \ldots, p_r\} \) |
| (branched) cover \( h : N \rightarrow M \) | (ramified) extension \( F/k \) |
| fundamental group \( \pi_1(M) \) | \( \text{étale fundamental group} \ \pi_1^\text{\text{ét}}(\text{Spec} \mathcal{O}_k) \) |
| 1st homology group \( H_1(M) \) | ideal class group \( \text{Cl}_k \) |

There is also an analogy between the Hurewicz isomorphism and the Artin reciprocity in unramified class field theory:

\[
H_1(M) \cong \text{Gal}(M^{ab}/M) \cong \pi_1(M)^{ab} \quad \text{Cl}_k \cong \text{Gal}(k^{ab}/k) \cong \pi_1^{\text{et}}(\text{Spec} \mathcal{O}_k)^{ab}
\]

Here, \( M^{ab} \rightarrow M \) and \( k^{ab}/k \) denote the maximal abelian cover and the maximal unramified abelian extension respectively. Moreover, there are branched Galois theories in a parallel manner, where the fundamental groups of the exteriors of knots and primes dominate the branched covers and the ramified extensions respectively. Our project of idélic class field theory for 3-manifolds aims to pursue the research of analogies in this line. For more analogies, we also consult \cite{Uck, Ulke, MTTU} and \cite{Nii14}.

In the following, we discuss an analogue of an algebraic closure of a number field. If we say branched covers, unless otherwise mentioned, we consider branched covers endowed with base points, that is, we fix base points in all spaces that are compatible with covering maps. For a space \( X \), we denote by \( b_X \) the base point.

First, we recall the notion of an isomorphism of branched covers. For covers \( h : N \rightarrow M \) and \( h' : N' \rightarrow M \) branched over \( L \), we say they are isomorphic (as branched covers endowed with base points) and denote by \( h \cong h' \) if there is a (unique) homeomorphism \( f : (N, b_N) \xrightarrow{\sim} (N', b_{N'}) \) such that \( h = h' \circ f \). Let \( \bar{h} : Y_L \rightarrow X_L \) and \( \bar{h}' : Y_{L'} \rightarrow X_L \) denote the restrictions to the exteriors. Then, \( h \cong h' \) is equivalent to that \( \bar{h}(\pi_1(Y_L, b_{Y_L})) = \bar{h}'(\pi_1(Y_{L'}, b_{Y_{L'}})) \in \pi_1(X_L, b_{X_L}) \).

Such notion is extended to the class of branched pro-covers, which are objects obtained as inverse limits of finite branched covers.

Next, we introduce an analogue notion of an algebraic closure of a number field.

For a finite link \( L \) in a 3-manifold, a branched pro-cover \( h_L : \tilde{M}_L \rightarrow M \) is a universal \( L \)-branched cover of \( M \) if it satisfies a certain universality: \( h_L : \tilde{M}_L \rightarrow M \) is a minimal object such that any finite cover of \( M \) branched over \( L \) factor through it. It is unique up to the canonical isomorphisms, and it can be obtained by Fox completion of a universal cover of the exterior \( h_L : \tilde{X}_L \rightarrow X_L \). (Note that Fox completion is defined for a spread of locally connected \( T_1 \)-spaces in general. (\cite{Fox57}))

Now, let \( M \) be a 3-manifold equipped with an infinite (very admissible) link \( K \). A branched pro-cover \( h_K : \tilde{M}_K \rightarrow M \) is a universal \( K \)-branched cover of \( M \) if it satisfies a certain universality: \( h_K : \tilde{M}_K \rightarrow M \) is a minimal object such that any finite cover of \( M \) branched over a finite link \( L \) in \( K \) factor through it.
It can be obtained as the inverse limit of a family of universal $L$-branched covers, as follows: For each finite link $L$ in $K$, let $h_L : \tilde{M}_L \to M$ be a universal $L$-branched cover of $M$. By the universality, for each $L \subset L'$, we have a unique map $f_{L,L'} : \tilde{M}_{L'} \to \tilde{M}_L$ such that $h_{L'} = h_L \circ f_{L,L'}$. Thus $\{h_L\}_{L \subset K}$ forms an inverse system. By putting $\tilde{M}_K = \varprojlim \tilde{M}_L$, we obtained a universal $K$-branched cover $h_K : \tilde{M}_K \to M$ as the composite of the natural map $\tilde{M}_K \to \tilde{M}_L$ and $h_L$.

For the universal $K$-branched cover, the inverse limit $\pi_1(X_K)$ of the fundamental groups of exteriors $\pi_1(X_L)$ ($L \subset K$) acts on it in a natural way. The finite branched covers of $M$ obtained as quotients of $h_K$ by subgroups of $\pi_1(X_K)$ form a complete system of representatives of the isomorphism classes of covers of $M$ branched over links in $K$.

Therefore, in the latter section of this paper, if we take $(M, K)$, we silently fix a universal $K$-branched cover, call it “the” universal $K$-branched cover, and restrict our argument to the branched subcovers obtained as its quotients.

Finally, we discuss an analogue of a base point. The following facts explain the role of base points in branched covers:

**Proposition 3.1.** (1) For $(M, K)$, we fix a universal $K$-branched cover $h_K$. Then, for a branched cover $h : N \to M$ whose base point is forgotten, taking a branched pro-cover $f : \tilde{M}_K \to N$ such that $h \circ f = h_K$ is equivalent to fixing a base point in $N$ such that $h(b_N) = b_M$.

(2) Let $h : N \to M$ be a branched cover. Then, a base point of a universal $K$-branched cover $h_K$ defines a branched pro-cover $f : \tilde{M}_K \to N$ such that $h_K = h \circ f$.

An analogue of a base point in a 3-manifold is a geometric point of a number field. Let $\Omega$ be a sufficiently large field which includes $\mathbb{Q}$, for instance, $\Omega = \mathbb{C}$. Then, for a number field $k$, choosing a geometric point $x : \text{Spec} \Omega \to \text{Spec} O_k$ is equivalent to choosing an inclusion $k \to \Omega$. Moreover, choosing base points in a cover $h : N \to M$ which are compatible with the covering map is an analogue of choosing inclusion $k \subset F \to \Omega$ for an extension $F/k$. For an algebraic closure $\overline{k}/k$ and an extension $F/k$ of a number field $k$, we have following facts:

**Proposition 3.2.** (1) If we fix $\overline{k}/k$ in $\Omega$, taking an inclusion $F \hookrightarrow \overline{k}$ is equivalent to taking an inclusion $F \hookrightarrow \Omega$.

(2) For an extension $F/k$ in $\Omega$, an inclusion $\overline{k} \hookrightarrow \Omega$ defines $F \hookrightarrow \overline{k}$.

In addition, we have $\text{Spec} O_k = \{\text{finite primes}\} \cup \text{Spec} k$, and $(\text{Spec} k)(\Omega) = \{\Omega\text{-rational points of Spec } k\} := \text{Hom}(\text{Spec } \Omega, \text{Spec } k) \cong \text{Hom}(k, \Omega)$. Accordingly, choosing a geometric point (an injection) $k \to \Omega$ is an analogue of choosing a base point in the exterior of $K$ in $M$. If $k/\mathbb{Q}$ is Galois, we have a non canonical isomorphism $\{\text{the choices of a geometric point of } k\} = \text{Hom}(k, \Omega) \cong \text{Gal}(k/\mathbb{Q})$. This map depends on the fact that an inclusion of $\mathbb{Q}$ into a field is unique. In order to state an analogue for $(M, K)$, we need to fix an analogue of $k/\mathbb{Q}$. If we fix a Galois branched cover $h_M : M \to S^3$ whose base point is forgotten, an infinite link $\overline{K}$ in $S^3$ such that $h^{-1}(\overline{K}) = K$, and a base point $b_0$ in $S^3$, then we have a non-canonical map $\{\text{the choices of base points in } M\} \cong \text{Gal}(h_M)$. 
Thereby, we obtained the following dictionary:

| 3-manifold with very admissible link \((M, K)\) | number ring \(\text{Spec} \mathcal{O}_k\) |
|------------------------------------------------|-----------------------------------------|
| universal \(K\)-branched cover \(h_K : M_K \to M\) | algebraic closure \(\overline{k}/k\) |
| base point \(b_M : \{\text{pt}\} \to M\) | geometric point \(x : \text{Spec} \Omega \to \text{Spec} \mathcal{O}_k\) |

In this paper, since we consider only regular (Galois) covers, we can forget base points. Then weaker equivalence classes of branched covers should be considered.

### 4. Idèlic Class Field Theory for Number Fields

In this section, we briefly review the idèlic class field theory for number fields, whose topological analogues will be studied in later sections. We consult [KKST1] and [Neu99] as basic references for this section.

#### 4.1. Local theory

We firstly review the local theory. Let \(k\) be a number field, that is, a finite extension of the rationals \(\mathbb{Q}\), and let \(p \subset \mathcal{O}_k\) be a prime ideal of its integer ring. Then, for a local field \(k_p\), we have the following commutative diagram of splitting exact sequences.

\[
\begin{array}{ccccccccc}
0 & \to & \mathcal{O}_p^\times & \to & k_p^\times & \to & \mathbb{Z} & \to & 0 \\
& & v_p & \downarrow & \rho_p & & & & \\
0 & \to & \text{Gal}(k_p^{ab}/k_p^{ur}) & \to & \text{Gal}(k_p^{ab}/k_p) & \to & \text{Gal}(k_p^{ur}/k_p) & \to & 0
\end{array}
\]

Here, \(\mathcal{O}_p^\times\) is the local unit group, \(v_p\) is the valuation, \(k_p^{ab}/k_p\) is the maximal abelian extension, and \(k_p^{ur}/k_p\) is the maximal unramified abelian extension. The map \(\rho_p\) is called the local reciprocity homomorphism, which is a canonical injective homomorphism with dense image, and controls all the abelian extensions of the local field \(k_p\). In the lower line, \(I_p^{ab} = \text{Gal}(k_p^{ab}/k_p^{ur})\) is the abelian quotient of the inertia group, and we have \(\text{Gal}(k_p^{ur}/k_p) \cong \text{Gal}(\overline{\mathbb{F}}_p/\mathbb{F}_p) \cong \hat{\mathbb{Z}}\).

The theory of a local field is rather complicated. There are non-abelian extensions, and there are notions of wild and tame for ramifications. For the tame quotients, we have an exact sequence

\[
1 \to I_p^{ur} = \langle \tau \rangle \to \pi_1^p(\text{Spec}(k_p)) = \langle \tau, \sigma | \tau^{q-1}[\tau, \sigma] \rangle \to \text{Gal}(\overline{\mathbb{F}}_p/\mathbb{F}_p) = \langle \sigma \rangle \to 1,
\]

where \(\tau\) and \(\sigma\) are called the monodromy and the Frobenius, respectively.

#### 4.2. Definitions

Next, we review the global theory. Let \(k\) be a number field. We define the idèle group \(I_k\) of \(k\) by the following restricted product of \(k_p^\times\) with respect to the local unit groups \(U_p\) over all primes \(p\) of \(k\):

\[
I_k := \prod_p k_p^\times = \left\{ (a_p)_{p} \in \prod_{p\text{ prime}} k_p^\times \mid v_p(a_p) = 0 \text{ for almost all finite primes } p \right\}.
\]

Since we have \(v_p(a) = 0\) for \(a \in k^\times\) and for almost all finite primes \(p\), \(k^\times\) is embedded into \(I_k\) diagonally. We define the principal idèle group \(P_k\) of \(k\) by the image of \(k^\times\) in \(I_k\), and the idèlle class group of \(k\) by the quotient \(C_k := I_k/P_k\).

Then, the homomorphism to the ideal group \(\varphi : I_k \to I(k)\); \((a_p)_p \mapsto \prod_p p^{v_p(a_p)}\) induces an isomorphism

\[
I_k/(U_k \cdot P_k) \cong Cl_k,
\]
where \( U_k = \text{Ker} \varphi = \prod_p U_p \) denotes the unit idèle group and \( Cl_k \) denotes the ideal class group of \( k \).

### 4.3. Standard topology.

The idèle class group \( C_k \) equips the standard topology, which is the quotient topology of the restricted product topology on the idèle group \( I_k \) of the local topologies, defined as follows.

We firstly consider on \( \mathcal{O}_k \times \prod_p \mathcal{O}_p \) the relative topology of the local norm topology of \( k \times \prod_p \mathcal{O}_p \), and re-define the local topology on \( k \times \prod_p \mathcal{O}_p \) as the unique topology such that the inclusion \( \mathcal{O}_k \times \prod_p \mathcal{O}_p \hookrightarrow k \times \prod_p \mathcal{O}_p \) is open and continuous. (For this local topologies, \( I_k \) is the restricted products with respect to the family of open sets \( \{ \mathcal{O}_p \subset \mathcal{O}_p \}_{p} \).)

Next, for each finite set of primes \( T \) which includes all the infinite primes, we consider the product topology on \( G(T) = \prod_{p \in T} k_p \times \prod_{p \not\in T} \mathcal{O}_p \).

Then, we define the standard topology on \( I_k \) so that each subgroup \( H < I_k \) is open if and only if \( H \cap G(T) \) is open for every \( T \).

This standard topology on \( C_k \) differs from the one defined as the quotient topology of the relative topology of the product topology on \( I_k < \prod \mathcal{O}_p \), and it is finer than the latter.

### 4.4. Norm topology.

For a finite abelian extension \( F/k \), the norm map \( N_{F/k} : C_F \to C_k \) is defined as follows.

Let \( p \) be a prime of \( k \) and \( F_p^\times := \prod_{q \mid p} F_q^\times \). Each \( \alpha_p \in F_p^\times \) defines a \( k_p \)-linear automorphism \( \alpha_p : F_p^\times \to F_p^\times : x \mapsto \alpha_p x \), and the norm of \( \alpha_p \) is defined by \( N_{F_p/k_p}(\alpha_p) = \det(\alpha_p) \). It induces a homomorphism \( N_{F_p/k_p} : F_p^\times \to k_p^\times \), and the norm homomorphism \( N_{F/k} : I_F \to I_k \) on the idèle groups. Since \( N_{F/k} \) sends the principal idèles to principal idèles, it also induces the norm homomorphism \( N_{F/k} : C_F \to C_k \) on the idèle class groups.

For a number field \( k \), the idèle class group \( C_k \) equips the norm topology, so that it is a topological group, and the family of \( N_{F/k}(C_F) \) is a fundamental system of neighborhoods of 0, where \( F/k \) runs through all the finite abelian extensions of \( k \).

**Proposition 4.1.** A subgroup \( H \) of \( C_k \) is open and of finite index with respect to the standard topology if and only if it is open with respect to the norm topology.

### 4.5. Global theory.

Here is a main theorem of idèle class field theory for number fields (cf. [Neu99], §6, Theorem 6.1): 

**Theorem 4.2** (Idèle class field theory for number fields). Let \( k \) be a number field and let \( k^{ab} \) denote the maximal abelian extension of \( k \) which are fixed in \( \mathbb{C} \).

1. (Artin’s global reciprocity law.) There is a canonical surjective homomorphism, called the global reciprocity map,

\[
\rho_k : C_k \to \text{Gal}(k^{ab}/k)
\]

which has the following properties:

- For any finite abelian extension \( F/k \) in \( \mathbb{C} \), \( \rho_k \) induces an isomorphism

\[
C_k/\text{N}_F/k(C_F) \cong \text{Gal}(F/k).
\]
(ii) For each prime \( p \) of \( k \), we have the following commutative diagram

\[
\begin{array}{c}
\overset{\rho_p}{\longleftarrow} & \overset{\iota_p}{\longrightarrow} & \\
\overset{k_p^\times \rho_p}{\downarrow} & \downarrow & \downarrow \\
C_k & \longrightarrow & \Gal(k_{ab}^p/k_p) \\
\end{array}
\]

where \( \iota_p \) is the map induced by the natural inclusion \( k_p^\times \to I_k \).

(2) (The existence theorem.) The correspondence

\[
F \mapsto N = N_{F/k}(C_F)
\]

gives a bijection between the set of finite abelian extensions \( F/k \) in \( \mathbb{C} \) and the set of open subgroups \( N \) of \( C_k \) with respect to the standard topology. Moreover, the latter set coincides with the set of open subgroups of \( C_k \) with respect to the norm topology.

In the proof, we use the norm residue symbol \( (\cdot, F/k) : C_k \to \Gal(F/k) \). For this map, we have \( \operatorname{Ker}(\cdot, F/k) = N_{F/k}(C_F) \).

### 5. The global reciprocity law

In this section, we recall the local theory and the id`elic class field theory for 3-manifolds, and present the global reciprocity law over a 3-manifold equipped with a very admissible link. This extends the main result of the previous paper [Nii14] to the general cases.

#### 5.1. Local theory. Let \( K \) be a knot in its tubular neighborhood \( V_K \). In our context, the local theory for 3-manifolds is nothing but the Galois theory for the covers of \( \partial V_K \), which is dominated by an abelian group \( \pi_1(\partial V_K) = \langle \mu_K, \lambda_K | [\mu_K, \lambda_K] \rangle \cong H_1(\partial V_K) \cong \mathbb{Z}^2 \). (In a sense, the tame case of a local field is a “quantized” version of this case.) For each manifold \( X \), let \( \Gal(\widetilde{X}/X) \) denote the Galois group of the universal cover. We have the following commutative diagram of exact sequences.

\[
\begin{array}{cccccccc}
0 & \longrightarrow & \langle \mu_K \rangle & \longrightarrow & H_1(\partial V_K) & \overset{v_K}{\longrightarrow} & H_1(V_K) = \langle \lambda_K \rangle & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow \text{Hurewicz} & & \downarrow & & \\
0 & \longrightarrow & \Gal(\partial V_K/\partial \widetilde{V}_K) & \longrightarrow & \Gal(\widetilde{V}_K/\partial V_K) & \longrightarrow & \Gal(\widetilde{V}_K/V_K) & \longrightarrow & 0
\end{array}
\]

By an isomorphism \( \partial_* : H_2(V_K, \partial V_K) = \langle [D_K] \rangle \overset{\cong}{\to} \langle \mu_K \rangle \), classes of meridian disks \( D_K \) can be seen as analogues of local units. The map \( v_K \) is the one induced by the natural injection \( \partial V_K \to V_K \), which is an analogue of the valuation map \( v_p \) in number theory. The vertical maps are the Hurewicz isomorphisms. In the lower line, \( I_K := \Gal(\partial \widetilde{V}_K/\partial V_K) \) is the inertia group, and we have \( \Gal(\widetilde{V}_K/V_K) \cong \Gal(\overline{K}/K) \cong \mathbb{Z} \).

#### 5.2. Definitions. Let \( M \) be a closed, connected, oriented 3-manifold. Let \( K \) be a link in \( M \) with finite or infinite components which equips a tubular neighborhood \( V_K = \cup_{K \subset K} V_K \). For a sublink \( L \) of \( K \), let \( V_L \subset V_K \) denote its tubular neighborhood.
Definition 5.1 (idèle group). We define the idèle group of \((M, K)\) by the restricted product of \(H_1(\partial V_K)\) with respect to the subgroups \(\{\langle \mu_K \rangle \}_{K \subset K} = \{\ker(v_K)\}_{K \subset K}:\)

\[
I_{M,K} := \prod_{K \subset K} H_1(\partial V_K) = \left\{ (a_K)_K \in \prod_{K \subset K} H_1(\partial V_K) \mid v_K(a_K) = 0 \text{ for almost all } K \right\}.
\]

For a finite link \(L\) in \(K\), let \(\text{Gal}(X^\text{ab}_L/X_L)\) denote the Galois group of the maximal abelian cover over its exterior. Then, the set of all these Galois groups \(\{\text{Gal}(X^\text{ab}_L/X_L) \mid L \subset K\}\) forms an inverse system, indexed by \(L \subset K\) with the natural order defined by the inclusions. We define the abelian group of \((M, K)\) to be its inverse limit: \(\text{Gal}(M, K)^\text{ab} := \lim_{\leftarrow L} \text{Gal}(X^\text{ab}_L/X_L)\).

For a knot \(K\) and a finite link \(L\) in \(K\), the natural maps induced by the inclusion \(\partial V_K \hookrightarrow X_L\) and the Hurewicz maps commute.

\[
\begin{array}{ccc}
H_1(\partial V_K) & \xrightarrow{\cong} & \text{Gal}(\partial V_K/\partial V_K) \\
\downarrow & & \downarrow \\
H_1(X_L) & \xrightarrow{\cong} & \text{Gal}(X^\text{ab}_L/X_L)
\end{array}
\]

Let \(\rho_{K,L} : H_1(\partial V_K) \to \text{Gal}(X^\text{ab}_L/X_L)\) denote their composite, and let \(\rho_L : I_{M,K} \to \text{Gal}(X^\text{ab}_L/X_L) : (a_K)_K \mapsto \sum_{K \subset K} \rho_{K,L}(a_K)\) be the map defined by the summation of \(\rho_{K,L}\) over \(K\) in \(K\). This sum makes sense, because it is actually a finite sum for each \((a_K)_K \in I_{M,K}\), by the definition of the restricted product. Since \((\rho_L)_L\) is compatible with the inverse system, the following homomorphism is induced:

\[
\rho_{M,K} : I_{M,K} \to \text{Gal}(M, K)^\text{ab}.
\]

If \(K\) is an admissible link, this map is surjective.

Definition 5.2 (Principal idèle group, idèle class group). We define the principal idèle group by the kernel of the reciprocity map \(P_{M,K} := \ker \rho_{M,K}\), and the idèle class group by the the quotient \(C_{M,K} := I_{M,K}/P_{M,K}\).

Remark 5.3. The principal idèle group \(P_{M,K}\) is the image of certain subgroup of \(\lim_{\leftarrow L \subset K} H_2(X_L, \partial X_L)\), \(X_L := M \setminus \text{Int}(V_L)\), and can be realized as the boundaries of certain surfaces in the exterior. This fact justifies our definition in the context of arithmetic topology.

For a finite abelian cover \(h : N \to M\) branched over a finite link \(L\) in \(K\), a link \(h^{-1}(K)\) in \(N\) is defined by its preimage. Then, on each object, the covering map \(h\) induces the norm maps on each objects \(h_* : I_{N,h^{-1}(K)} \to I_{M,K}\), \(h_* : P_{N,h^{-1}(K)} \to P_{M,K}\) and \(h_* : C_{N,h^{-1}(K)} \to C_{M,K}\). They satisfy the transitivity (functoriality) in a natural way.

5.3. The global reciprocity law. Here is the first part of the idèle global class field theory for 3-manifolds and admissible links, which is the counter part of Theorem 3.1 (1).

Theorem 5.4 (The global reciprocity law for 3-manifolds). Let \(M\) be a closed, connected, oriented 3-manifold equipped with a very admissible link \(K\). Then, there
is a canonical isomorphism called the global reciprocity map
\[ \rho_{M,K} : C_{M,K} \xrightarrow{\cong} \text{Gal}(M,K)^{ab} \]
which satisfies the following properties:
(i) For any finite abelian cover \( h : N \to M \) branched over a finite link \( L \) in \( K \), \( \rho_M \) induces an isomorphism
\[ C_{M,K}/h_*(C_{N,h^{-1}(K)}) \cong \text{Gal}(h). \]
(ii) For each knot \( K \) in \( K \), we have the following commutative diagram:
\[
\begin{array}{ccc}
H_1(\partial V_K) & \xrightarrow{\cong} & \text{Gal}(\partial V_K/\partial V_K) \\
\downarrow & & \downarrow \\
C_{M,K} & \xrightarrow{\rho_{M,K}} & \text{Gal}(M,K)^{ab} ,
\end{array}
\]
where the vertical maps are induced by the natural inclusions.

**Remark 5.5.** This theorem generalizes the main result of [Ni14]. We have removed the assumption that \( M \) is an integral homology 3-sphere, that is, \( H_1(M) = 0 \).

Since \( H_1(M) \) is an analogue of \( Cl_k \) and \( Cl_k \) is always finite, it may be interesting to restrict the class of \( M \) to the rational homology 3-spheres, that is, \( H_1(M) \) is finite. There are many results on the class number \( \#Cl_k \) via idèle theory, and our theory may enable us to consider their analogues.

The compatibility with the local theories (ii) follows from the definition of the map. We will give the proof of (i) in the following.

**Definition 5.6.** We define the unit idèle group of \( (M,K) \) by the meridian group
\[ U_{M,K} := \{ (ak)K \in I_{M,K} \mid v_K(ak) = 0 \text{ in } H_1(V_K) \} , \]
that is, a subgroup of the “infinite linear combinations” \( \sum_{K \subseteq \mathbb{K}} m_K h_K \) \((m_K \in \mathbb{Z})\) of the meridians of \( K \) with \( \mathbb{Z} \)-coefficients.

**Lemma 5.7** (An improvement of [Ni14] Proposition 5.7). Let \( M \) be a closed, connected, oriented 3-manifold equipped with an admissible link \( K \), and \( L \) be a finite link in \( K \). We write \( U_{M,K} = U_L \oplus U_{\text{non}L} \), where \( U_L \) is the subgroup generated by the meridians of \( L \), and \( U_{\text{non}L} := \ker(\text{pr}_L : U_{M,K} \to U_L) \). Then we have \( I_{M,K}/(P_{M,K} + U_{M,K}) \cong H_1(L) \).

Especially, if we put \( L = \phi \), we have \( I_{M,K}/(P_{M,K} + U_{M,K}) \cong H_1(M) \). Moreover, if \( M \) is an integral homology 3-sphere, we have \( I_{M,K} = P_{M,K} + U_{M,K} \).

**Remark 5.8.** The assumption in [Ni14] Proposition 5.7 can be paraphrased as follows: \( H_1(M) \) is torsion free, and the knots \( K \) in \( K \) with non-trivial images in \( H_1(M) \) forms its free basis. However, these are removed.

In the proofs, we abbreviate \( M,K \) by \( M \), and \( N,h^{-1}(K) \) by \( N \) for simplicity.

**Proof.** For a map \( \varphi_L : I_M \to H_1(X_L) \), we prove \( \ker \varphi_L = P_M + U_{\text{non}L} \). Consider the composite \( \varphi_L : I_M \to I_M/P_M = C_M \cong \text{Gal}(M,K)^{ab} \cong \varprojlim H_1(X_L') \to H_1(X_L) \). For each \( L \subseteq L' \subseteq K \), it factorizes as \( \varphi_L : I_M \to H_1(X_L') \xrightarrow{\text{pr}} H_1(X_L) \).

For the meridian \( \mu_K \) of \( K \) in \( I_M \), the Mayer–Vietoris exact sequence proves \( N_L := \ker(\text{pr} : H_1(X_L') \to H_1(X_L)) = \langle \varphi_{L'}(\mu_K) \rangle \in L' \setminus L \). Hence \( \ker(I_M/P_M \to \)
$H_1(X_L) = U_{\text{non}L} \mod P_M \cong \varprojlim_{L} N_L = \ker(\lim_{L} H_1(X_L) \to H_1(X_L))$, and therefore $\ker \varphi_L = P_M + U_{\text{non}L}$. \hfill $\square$

**proof of Theorem 5.4 (i).** Since there are isomorphisms

$$C_M/h_* (C_N) \cong (I_M/P_M)/h_* (I_N/P_N) \cong I_M/(P_M + h_*(I_N)),$$

we consider the natural surjection $\varphi' : I_M \to H_1(X_L) \to H_1(X_L)/h_*(H_1(Y_L))$.

Since $\mathcal{K}$ is very admissible, there is a surjection $I_N \to H_1(Y_L)$, and hence a surjection $h_*(I_N) \to h_*(H_1(Y_L))$. Then, there is the following commutative diagram.

$$\begin{array}{ccc}
I_M & \xrightarrow{\varphi_L} & H_1(X_L) \\
\downarrow & & \downarrow \\
I_M & \xrightarrow{\varphi} & H_1(X_L) \\
\end{array}$$

Since $\ker \varphi_L = P_M + U_{\text{non}L} < P_M + h_*(I_N)$, we have $\ker \varphi' = \ker \varphi_L + h_*(I_N) = P_M + h_*(I_N)$, and hence $I_M/(P_M + h_*(I_N)) \cong H_1(X_L)/h_*(H_1(Y_L)) \cong \text{Gal}(h)$. \hfill $\square$

**Remark 5.9.** In the above proof, we used the assumption that “$\mathcal{K}$ is very admissible” only to say that $I_{N,h^{-1}(K)} \to h_*(H_1(Y_L))$ is surjective. Therefore, we can replace the assumption on $\mathcal{K}$ by a weaker (and necessary) one: “for any $h : N \to M$, the natural map $I_{N,h^{-1}(K)} \to h_*(H_1(N))$ is surjective”.

Especially, if $M$ is an integral homology 3-sphere, that is, if $H_1(M) = 0$, we can remove the assumption on the link $K$ in the global reciprocity law (Theorem 5.4).

### 6. The standard topology and the existence theorem 1/2

In this section, we introduce the standard topology on the idèle class group of a 3-manifold, and prove the existence theorem 1/2.

Let $M$ be a closed, connected, oriented 3-manifold equipped with a very admissible link $\mathcal{K}$. For each group $\pi_1(\partial V_K) \cong H_1(\partial V_K) = \langle \mu_K \rangle \oplus \langle \lambda_K \rangle \cong \mathbb{Z}^\oplus 2$ of the boundary of tubular neighborhood of $K$, we will define an analogue of the local topology of $k^\times_F$. Here $\mu_K$ and $\lambda_K$ denote the fixed meridian and longitude of $K$ respectively. We first consider the **local norm topology** on $H_1(\partial V_K)$, whose open subgroups corresponds to the finite abelian covers of $\partial V_K$. This topology is equal to the **Krull topology**, whose open subgroups are the subgroups of finite indices. Then we consider the relative topology on the local inertia group $\langle \mu_K \rangle < H_1(\partial V_K)$, and re-define the **local topology** on $H_1(\partial V_K)$ as the unique topology such that the inclusion $\iota : \langle \mu_K \rangle \to H_1(\partial V_K)$ is open and continuous. For this topology, under the identification $\mathbb{Z} \cong \langle \mu_K \rangle \to H_1(\partial V_K) = \langle \mu_K \rangle \oplus \langle \lambda_K \rangle \cong \mathbb{Z} \oplus \mathbb{Z}$, the open subgroup of $H_1(\partial V_K)$ has the form $\langle (a,b),(0,c) \rangle$ with some $a,b,c \in \mathbb{Z}$, $a \neq 0$. Then, the local existence theorem is stated as the 1-1 correspondence between the open subgroups of finite indices and the finite abelian covers.

With this local topology, $I_{M,K}$ is the restricted product with respect to the open subgroups $\mathcal{U}_K = \langle \mu_K \rangle < H_1(\partial V_K)$, and $I_{M,K}$ equips the **restricted product topology** as follows. For each finite link $L$ in $\mathcal{K}$, let $G(L) := \prod_{K \subset L} H_1(\partial V_K) \times \prod_{K \subset L} \langle \mu_K \rangle$, and consider the product topology on $G(L)$. Then a subgroup $H < I_{M,K}$ is open if and only if $H \cap G(L)$ is open for every $L$.
Definition 6.1. The restricted product topology on $I_{M,K}$ is finer than the subspace topology of the product topology of the local topologies, but both define the same quotient topology on $C_{M,K}$. We call it the standard topology of $C_{M,K}$.

proof. Note that there is a factorization $I_{M,K}/P_{M,K} = C_{M,K} \cong \lim \left\{ H_1(X_L) \to H_1(X) \mid \text{for each finite link } L \right\}$ for any $\lambda \in \pi_1(M)$, where $H_1(X)$ denotes the first homology group of the manifold $X$. Then, for each finite link $L$ in $K$, $H_1(X_L)$ is generated by the images of $\mu_K(K \subset L)$ and $\lambda_K(K \subset L_0)$. Therefore, for each knot $K' \not\subset L_0$ in $K$, the image of $\lambda_{K'}$ in $C_{M,K}$ is of finite index. □

Proposition 6.2. Let $C_{M,K}$ be endowed with the standard topology. If $M$ is a rational homology 3-sphere, an open subgroup of $C_{M,K}$ is of finite index.

proof. The assumption on $M$ means $H_1(M)$ is finite, and $(\oplus_{K \subset L_0} \mathbb{Z})$ falls into the torsion part of $C_{M,K}$ up to the meridians. Then by the definition of the restricted product topology and the forms of the open sets in $I'$, an open subgroup of $C_{M,K}$ is of finite index. □

Theorem 6.3 (The existence theorem 1/2). Let $C_{M,K}$ be endowed with the standard topology. Each open subgroup of $C_{M,K}$ of finite index corresponds to each (isomorphism class of) finite abelian cover of $M$ branched over a finite link in $K$.

proof. Let $V$ be an open subgroup of $C_{M,K}$, and let $\tilde{V}$ denote the preimage of $V$ in $I_{M,K}$. Then, $\tilde{V}$ is also the preimage of an open subgroup of $I'$. $\tilde{V}$ contains $\text{Ker}(\text{Id} : I_{M,K} \to I = \prod_{K \subset L} H_1(\partial V_K))$ for some finite link $L$ in $K$. Therefore, the image of $V$ under the projection $C_{M,K} \to H_1(X_L)$ is of finite index, and the ordinary Galois theory for branched covers gives a finite abelian cover branched over $L$ in $K$ which corresponds to $V$. The converse is clear. □

7. THE NORM TOPOLOGY AND THE EXISTENCE THEOREM

In this section, we introduce the norm topology on the idèle class group, and present the existence theorem.

Let $M$ be a closed, oriented, connected 3-manifold equipped with a very admissible link $K$ as before. In the proofs, we use the abbreviations $C_M = C_{M,K}$ and $C_N = C_{N,h^{-1}(K)}$ for a branched cover $h : N \to M$.

Definition 7.1. We define the norm topology on $C_M$ to be the topology of topological group generated by the family $V := \left\{ h_*(C_{N,h^{-1}(K)}) \right\}$, where $h : N \to M$ runs through all the finite abelian covers of $M$ branched over finite links in $K$.

Lemma 7.2. $V$ is a fundamental system of neighborhoods of 0.

proof. For any $V_1, V_2 \in V$, it is suffice to prove $\exists V_3 \in V$ such that $V_3 \subset V_1 \cap V_2$. However, we will prove $V_3 := V_1 \cap V_2 \in V$. 

Let \( h_i : N_i \to M \) be a finite abelian cover branched over \( L_i \) in \( K \) for \( i = 1, 2 \). Let \( L := L_1 \cup L_2 \), and let \( G_L := \text{Gal}(X_L^{ab}/X_L) \) denote the Galois group of the maximal abelian cover over the exterior \( X_L = M \setminus \text{Int}(V_L) \). Then, if a cover \( h : N \to M \) is unbranched outside \( L \), the map \( \varphi_L : C_M \to \text{Gal}(h) \) factors through the natural map \( \varphi_L : C_M \to G_L \).

Let \( G_i := \text{Ker}(G_L \to \text{Gal}(h_i)) < G_L \) for \( i = 1, 2 \), and let \( G_3 := G_1 \cap G_2 \). Since \( G_3 \) is also a subgroup of \( G_L \) of finite index, the ordinary Galois theory for branched covers gives a cover \( h_3 : N_3 \to M \) such that \( G_3 = \text{Ker}(G_L \to \text{Gal}(h_3)) \). (This cover \( h_3 \) should be called the “composition cover” of \( h_1 \) and \( h_2 \), because it is an analogue of the composition field \( k_1k_2 \) of \( k_1 \) and \( k_2 \) in number theory.)

Now, Theorem 5.4 (the global reciprocity law) implies \( h_{is}(C_{N_i}) = \varphi_L^{-1}(G_i) \) for \( i = 1, 2, 3 \), and therefore \( h_{is}(C_{N_3}) = \varphi_L^{-1}(G_3) = \varphi_L^{-1}(G_1 \cap G_2) = \varphi_L^{-1}(G_i) \cap \varphi_L^{-1}(G_2) = h_{is}(C_{N_1}) \cap h_{is}(C_{N_2}). □

**Proposition 7.3.** Let \( C_{M,K} \) be endowed with the norm topology. A subgroup \( V \) of \( C_{M,K} \) is open if and only if it is closed and of finite index.

**proof.** Let \( V \) be an open subgroup of \( C_M \). The coset decomposition of \( C_M \) by \( V \) proves that \( V \) is closed. Lemma 7.2 gives a finite abelian branched cover \( h : N \to M \) such that \( h_{is}(C_N) < V \). Then Theorem 5.4 implies \( h_{is}(C_N) : V(C_M) = (h_{is}(C_N) : C_M) = \# \text{Gal}(h) \), and hence \( V \) is of finite index.

The converse is also clear by the coset decomposition. □

Now we present the existence theorem for 3-manifolds with respect to both the standard topology and the norm topology, which is the counterpart of Theorem 4.2 (2).

**Theorem 7.4 (The existence theorem).** Let \( M \) be a closed, connected, oriented 3-manifold equipped with a very admissible link \( K \). Then the correspondence

\[
(h : N \to M) \mapsto h_{is}(C_{N,h^{-1}(K)})
\]

gives a bijection between the set of (isomorphism classes of) finite abelian covers of \( M \) branched over finite links \( L \) in \( K \) and the set of open subgroups of finite indices of \( C_{M,K} \) with respect to the standard topology. Moreover, the latter set coincides with the set of open subgroups of \( C_{M,K} \) with respect to the norm topology.

**proof.** The former part is done by Theorem 6.3. We prove the theorem for the norm topology. For a finite abelian cover \( h : N \to M \) branched over a finite link in \( K \), the isomorphism \( C_M/h_{is}(C_N) \cong \text{Gal}(h) \) in Theorem 5.4 (the global reciprocity law) gives the following bijections.

\[
\{C' | h_{is}(C_N) < C' < C_M\} \leftrightarrow \{H | H < C_M/h_{is}(C_N) \cong \text{Gal}(h)\} \leftrightarrow \{\text{subcovers of } h\}
\]

(Injectivity) For covers \( h_1 \) and \( h_2 \), this bijections proves that \( h_{1*}(C_{N_1}) < h_{2*}(C_{N_2}) \iff h_2 \) is a subcover of \( h_1 \), and hence \( h_{1*}(C_{N_1}) = h_{2*}(C_{N_2}) \iff h_2 = h_1 \).

(Surjectivity) For an open subgroup \( C' < C_M \), Lemma 7.2 gives a cover \( h : N \to M \) such that \( h_{is}(C_N) < C' \), and then the above bijection gives a cover \( h' \) which corresponds to \( C' \).

**Corollary 7.5.** If \( M \) is a rational homology 3-sphere, the standard topology and the norm topology on \( C_{M,K} \) coincide.

**proof.** By Proposition 6.2, it follows immediately from the existence theorem. □
8. The norm residue symbols

In this section, we introduce the norm residue symbol for 3-manifolds, as an analogue of the norm residue symbol for number fields. We also explain that they generalize the linking number \( \text{lk}(K_1, K_2) \) and the Legendre symbol \( \left( \frac{p}{q} \right) \).

**Definition 8.1.** For a finite abelian extension \( F/k \), the norm residue symbol \( (\ , F/k) : C_k \to \text{Gal}(F/k) \) is defined as the composite of \( \rho_k : C_k \to \text{Gal}(k^{ab}/k) \) and \( \text{Gal}(k^{ab}/k) \to \text{Gal}(F/k) \). For this map, we have \( \text{Ker}(\ , F/k) = N_{F/k}(C_F) \).

The relation with Legendre’s quadratic residue symbol can be seen as follows: Let \( p \) and \( q \) be distinct primes in \( k = \mathbb{Q} \), and let \( F = \mathbb{Q}(\sqrt{q}) \) be the quadratic extension of \( \mathbb{Q} \) ramified at \( q \). Then \([\text{KKS11}] \) Lemma 5.19 states the following equivalences:

\[
\left( \frac{q}{p} \right) = 1 \iff (p) = p_1p_2 \text{ with two primes } p_1, p_2 \text{ in } \mathcal{O}_F \text{ (decomposed)},
\]

\[
\left( \frac{q}{p} \right) = -1 \iff (p) \text{ is a prime in } \mathcal{O}_F \text{ (inert)}.
\]

On the other hand, under the identification \( \text{Gal}(F/k) \cong \{ \pm 1 \} \), there are the following equivalences:

\[
((p), F/k) = 1 \iff (p) \in N_{F/k}(C_{F/k}) \iff (p) \text{ is decomposed in } F/k.
\]

Therefore, we have \( \left( \frac{q}{p} \right) = ((p), \mathbb{Q}(\sqrt{q})/\mathbb{Q}) \in \{ \pm 1 \} \).

**Definition 8.2.** Let \( M \) be a 3-manifold equipped with a very admissible knot set \( \mathcal{K} \). For a finite abelian cover \( h : N \to M \) branched over a finite link \( L \) in \( \mathcal{K} \), the norm residue symbol \( (\ , h) : C_{M,K} \to \text{Gal}(h) \) is defined as the composite of \( \rho_{M,K} : C_{M,K} \to \text{Gal}(M^{ab}/M) \) and \( \text{Gal}(M^{ab}/M) \to \text{Gal}(h) \). For this map, we have \( \text{Ker}(\ , h) = h_*(C_{N,h^{-1}(K)}) \).

The relation with the linking number can be seen as follows: Let \( h_2 : N \to M \) be the double cover of \( M = S^3 \) branched over a knot \( K_2 \) in two component link \( K_1 \sqcup K_2 \). We identify \( \text{Gal}(h_2) \cong \mathbb{Z}/2\mathbb{Z} \). Then, for a longitude \( \lambda_1 \) of \( K_1 \) in \( C_{M,K} \), we have \( (\lambda_1, h_2) = \text{lk}(K_1, K_2) \mod 2 \). Moreover, there are the following equivalences:

\[
(\lambda_1, h_2) = 0 \iff h^{-1}(K_1) = K'_1 \sqcup K''_1 \text{ with knots } K'_1, K''_1 \text{ in } N \text{ (decomposed)},
\]

\[
(\lambda_1, h_2) = 1 \iff h^{-1}(K_1) = \widetilde{K}_1 \text{ is a knot in } N \text{ (inert)}.
\]

Thus, we have obtained an extension of the dictionary of analogies.

| linking number \( \text{lk}(K_1, K_2) \mod 2 \) | Legendre symbol \( \left( \frac{p}{q} \right) \) | norm residue symbol \( (\ , F/k) \) |
|----------------------------------------|-----------------|-----------------|

Let \( p \) and \( q \) be distinct odd primes and \( q^* := (-1)^{\frac{q-1}{2}}q \). Then by using \([\text{KKS11}] \) Lemma 5.19, the quadratic reciprocity law \( \left( \frac{q^*}{p} \right) = \left( \frac{p}{q} \right) \) can be obtained. On the other hand, for knots \( K_1 \) and \( K_2 \), we have \( \text{lk}(K_1, K_2) = \text{lk}(K_2, K_1) \). For this analogy, we also consult \([\text{Mor12}] \) Chapter 4 and 5.

In the proof of Artin’s global reciprocity law for number fields (Theorem \([4.2]\)), the norm residue symbol plays an essential role (\([\text{Neu99}] \)). By using the norm
residue symbol for 3-manifolds, we can also give a parallel proof for the global reciprocity law for 3-manifolds (Theorem 5.4), although it becomes a little more complicated-looking than our proof in this paper.

9. AXIOM OF CLASS FIELD THEORY

Finally, we calculate the Tate cohomology of idèle class group, and compare the result with the axiom of class field theory.

When a finite cyclic group $G = \langle \sigma \rangle$ of order $n$ acts on a module $A$, by definition, the Tate cohomology is calculated as

$$\hat{H}^0(G, A) = A^G / \operatorname{Nr} A, \quad \hat{H}^1(G, A) \cong \operatorname{Ker}(\operatorname{Nr})/(1 - \sigma)A.$$  

Here, $\operatorname{Nr} = \sum_{0 \leq i \leq n-1} \sigma^i$ denotes the norm map (in a general sense), and the isomorphism class of $\hat{H}^1(G, A)$ depends only on $i \mod 2$.

Now the class field axiom for number fields is stated:

**Theorem 9.1** (The axiom of class field theory for number fields [Neu99]). Let $F/k$ be a cyclic extension of number fields with degree $n$, and $G = \operatorname{Gal}(F/k)$. Then the idèle class group $C_F$ satisfies

$$\hat{H}^0(G, C_F) \cong \mathbb{Z}/n\mathbb{Z}, \quad \hat{H}^1(G, C_F) = 1.$$  

In addition, we have

**Proposition 9.2** ([Neu99]). If $F/k$ is an unramified cyclic extension, the unit idèle group $U_F$ satisfies $\hat{H}^i(G, U_F) = 1$ for all $i \in \mathbb{Z}$.

In the case of number fields, the axiom ensures the Artin reciprocity law. Therefore, the following question is natural:

**Question 9.3** (The axiom of class field theory for 3-manifolds). Let $M$ be an oriented, connected, closed 3-manifold equipped with a very admissible link $\mathcal{K}$, $h : N \to M$ a cyclic branched cover of degree $n$, and $G = \operatorname{Gal}(h)$. Do the following hold? (i) $\hat{H}^0(G, C_{N,h^{-1}(K)}) \cong \mathbb{Z}/n\mathbb{Z}$, (ii) $\hat{H}^1(G, C_{N,h^{-1}(K)}) = 0$.

In the following, we will check that the axiom does not hold in general. Let $p$ be a prime number and suppose $n = p$. For each $A$, we write $\hat{H}^0(G, A) = \hat{H}^0(A)$ and $A_{N,h^{-1}(K)} = A_N$ for simplicity. For each $G$-module $A$, we write $\hat{H}^1(G, A) = \hat{H}^1(A)$ for simplicity. For each $A = I, P$, and $C$, we use the abbreviation $A_{N,h^{-1}(K)} = A_N$.

The exact sequence $0 \to P_N \to I_N \to C_N \to 0$ yields the long exact sequence of the Tate cohomologies.

A natural map called the transfer $h^1 : C_*(M) \to C_*(N)$ is defined by taking a connected component $\tilde{c}_1$ of the preimage of an open simplex $c$ and sending $c \mapsto \sum_{g \in G} g\tilde{c}_1$. (It is also defined by using the language of Serre-fibration on the exterior of branch locus $L$, and by extending to whole $M$ so that it is compatible with the boundary map $\partial : C_{i+1} \to C_i$.) This map induces the transfers on $Z_*$, $B_*$, and hence on $H_*$. The local behavior of the transfer map is as follows:

**Proposition 9.4.** Let $K$ be a knot in $\mathcal{K}$ and let $\tilde{K} = h^{-1}(K)$.

1. If $K$ is branched, then $h^1 : H_1(\partial V_K) \to H_1(\partial V_{\tilde{K}})$ maps $\mu_K \mapsto \mu_{\tilde{K}}, \lambda_K \mapsto p\lambda_{\tilde{K}}$.
2. If $K$ is inert, then $h^1 : H_1(\partial V_K) \to H_1(\partial V_{\tilde{K}})$ maps $\mu_K \mapsto p\mu_{\tilde{K}}, \lambda_K \mapsto \lambda_{\tilde{K}}$.
3. If $K$ is decomposed, say $\tilde{K} = K_1 \cup \ldots \cup K_p$, then $h^1 : H_1(\partial V_K) \to H_1(\partial V_{\tilde{K}})$ maps $\mu_K \mapsto \sum_i \mu_{K_i}, \lambda_K \mapsto \sum_i \lambda_{K_i}$.  


The transfers are also induced on the idèle groups, the principal idèle groups, and the idèle class groups naturally, and satisfies \( \text{Nr} = h^* \circ h_* \) on each of them.

**Proposition 9.5.** The idèle group satisfies \( \hat{H}^1(G, I_{N,h^{-1}(K)}) = 0 \). However, (ii) \( \hat{H}^1(G, C_{N,h^{-1}(K)}) = 0 \) does not hold in general.

**proof.** Let \( K_{BI} \) and \( K_D \) denote the sublinks of \( h^{-1}(K) \) which consist of the branched or inert components and the decomposed components respectively, and put \( I_i := \prod_{K \subset K_i} H_1(\partial V_K) \) for \( i = BI, D \). Then \( I_N = I_{BI} \oplus I_D \) as \( G \)-modules.

The former part \( I_{BI} \) is point-wise fixed by \( G \), and \( \text{Nr} \) acts on it as multiplication by \( p \). Since \( I_{BI} \) is torsion-free, \( \text{Ker} \text{Nr} | I_{BI} = 0 \), and hence \( \hat{H}^1(I_{BI}) = 0 \). The latter part satisfies \( I_D = (\text{meridians}) \oplus (\text{longitudes}) \cong \mathbb{Z}[G/N] \oplus \mathbb{Z}[G^\oplus/N] \), and it is the inverse limit of a surjective system of free \( \mathbb{Z}[G] \)-modules. Since \( G \) is finite, we obtain \( \hat{H}^1(I_D) = 0 \). Therefore \( \hat{H}^1(I_N) = 0 \).

Then, the exact sequence \( 0 \to \hat{H}^{-1}(I_N) \to \hat{H}^{-1}(C_N) \to \hat{H}^0(P_N) \to \hat{H}^0(I_N) \) yields \( \hat{H}^1(C_N) \cong \hat{H}^{-1}(C_N) \cong \text{Ker} | \hat{H}^0(P_N) \to \hat{H}^0(I_N)) = \text{Ker}(P^G_N/\text{Nr}(P_N) \to I^G_N/\text{Nr}(I_N)) \).

Here, \( P^G_N/\text{Nr}(P_N) \to I^G_N/\text{Nr}(I_N) \) is not injective in general. Indeed, if \( K \) is an inert knot in \( h^{-1}(K) \) and \( \lambda_K \) represents a \( p \)-torsion element in \( H_1(N \setminus h^{-1}(K)) \), then \( \lambda_K \neq 0 \) in \( P^G_N/\text{Nr}(P_N) \setminus \lambda_K = p\lambda_K = 0 \), hence \( \lambda_K \) corresponds to a non-trivial element in \( \hat{H}^1(C_N) \). Such a case exists; consider a lens space, for instance.

Let \( d_1 \) and \( d_2 \) denote the numbers of branched components and inert components of \( h^{-1}(K) \) respectively, where \( d_2 \) can be infinite, and let \( K_{BI} \) be as in the proof above.

**Proposition 9.6.** The idèle group satisfies \( \hat{H}^0(G, I_{N,h^{-1}(K)}) \cong \prod_{K' \subset K} \mathbb{Z}/p\mathbb{Z}^2 \), and \( I^G_{N,h^{-1}(K)}/h^1(I_{M,K}) \cong (\mathbb{Z}/p\mathbb{Z})^{d_1+d_2} \). Moreover, (i) \( \hat{H}^0(G, C_{N,h^{-1}(K)}) \cong \mathbb{Z}/p\mathbb{Z} \) does not hold in general.

**proof.** Recall the standard isomorphism \( I_N = \prod_{K' \subset K \setminus h^{-1}(K)} H_1(\partial V_{K'}) \cong \prod_{K'} \mathbb{Z}^2 \) defined by the meridians and the fixed longitudes. Then \( \hat{H}^1(I_N) \cong \prod_{K' \subset K} \mathbb{Z}/p\mathbb{Z}^2 \) is clear. By proposition \( \exists \mathbb{Z}/p\mathbb{Z}^2 \to h^1(I_{M,K}) \cong (\mathbb{Z}/p\mathbb{Z})^{d_1} \oplus (\mathbb{Z}/p\mathbb{Z})^{d_2} \) is also clear, where \( (\mathbb{Z}/p\mathbb{Z})^{d_1} \) and \( (\mathbb{Z}/p\mathbb{Z})^{d_2} \) correspond to the longitudes of the branched part and the meridians of the inert part respectively. Indeed, the transfer \( h^1 : I_{M,K} \to I^G_N \) is bijective on the decomposed part, the longitudes of the inert part and the meridians of the branched part.

Meanwhile, the maps \( I^G_N \to C^G_N \) and \( h^1(I_{M,K}) \to h^1(C_{M,K}) \) yield a natural map \( I^G_N/h^1(I_{M,K}) \to C^G_N/h^1(C_{M,K}) \). In general, this map is clearly non-zero, and especially \( C^G_N/h^1(C_{M,K}) = 0 \) does not hold; Take the meridian of an inert knot \( K \) in \( h^{-1}(K) \) which is non-trivial in \( H_1(N \setminus K) \), for instance.

Now we have an exact sequence \( 0 \to C_{M,h^*(C_N)} \to C^G_N/h^1(C_{M,K}) \to C^G_N/h^1(C_{M,K}) \to 0 \), an equation \( \hat{H}^0(C_{N,K}) = C^G_N/h^1(C_{M,K}) \), and the isomorphism \( G = \text{Gal}(h) \cong C_{M,K} \) by theorem \( \text{Id} \) (the global reciprocity law). Thus, \( \hat{H}^0(C_{N,K}) \) differs from \( G \cong \mathbb{Z}/p\mathbb{Z} \) by the term \( C^G_N/h^1(C_{M,K}) \).

Here is another way of calculation: an isomorphism \( C_{M,K} \cong \varprojlim_{L \subset K} H_1(M \setminus L) \) deduces the calculation of \( \hat{H}^1(C_{N,h^{-1}(K)}) \) to the case of a finite link \( L \) and the ordinary exact sequence \( 0 \to B_1(X_L) \to Z_1(X_L) \to H_1(X_L) \to 0 \).

In addition, the unit idèle group satisfies the following:
Proposition 9.7. The Tate cohomology of the unit idèle group counts the number of branched or inert components: \( \hat{H}^0(U_{N,h^{-1}(K)}) \cong (\mathbb{Z}/p\mathbb{Z})^{d_1+d_2}, \hat{H}^1(U_{N,h^{-1}(K)}) = 0. \)

Especially, if \( h \) is unbranched, it counts the number \( d_2 \) of inert components.

Thereby, we have checked the counter part of the axiom of class field theory. It is interesting that the main theorems of idèle class field theory hold nevertheless.

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