Benign ghosts in higher-derivative systems

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Abstract. A brief review of the physics of systems including higher derivatives in the Lagrangian is given. All such systems involve ghosts, i.e. the spectrum of the Hamiltonian is not bounded from below and the vacuum ground state is absent. Usually this leads to collapse and loss of unitarity. In certain special cases, this does not happen, however: ghosts are benign.

This happens, in particular, in exactly solvable higher-derivative theories, but exact solvability seems to be a sufficient but not a necessary condition for the benign nature of the ghosts.

We speculate that the Theory of Everything is a higher-derivative field theory, characterized by the presence of such benign ghosts and defined in a higher-dimensional bulk. Our Universe represents then a classical solution in this theory, having the form of a 3-brane embedded in the bulk.

1. Motivation

In the vast majority of problems in theoretical mechanics and theoretical physics, the Lagrangians depend on generalized coordinates and generalized velocities, but not on generalized accelerations or still higher derivatives of dynamic variables. In particular, this is true for the Lagrangian of the Standard Model.

However, having unravelled by ~1975 the underlying structure of matter at the scale $\sim 10^{-17}$ cm, the theorists met a major impasse: the attempts to construct quantum theory of gravity have not been successful so far. There are two main reasons for that.

(i) A direct attempt to quantize Einstein’s gravity gives a nonrenormalizable quantum theory.

Such a theory does not make sense not only because of the impossibility of perturbative calculations, but also due to impossibility to define path integrals: in a nonrenormalizable theory, the lattice path integral has no continuum limit.

(ii) In any gravity theory, time and spatial coordinates are intertwined so that the classical equations of motion do not represent a Cauchy problem — the evolution in independent flat time. As a result, causality is lost. Einstein’s equations admit strange Gödel solutions with closed time loops [1]. Thus, even the classical general relativity has fundamental difficulties whose resolution is not in sight. And things do not become more assuring when one tries to quantize it.

Today the prevailing opinion of the experts is that the Theory of Everything that includes quantum gravity is a superstring theory of some kind. However, the latter has its own serious difficulties.

(i) Its building is impressive and beautiful, but the lowest stories of this building are hid in mist. String theory is simply not formulated at the fundamental nonperturbative level.
More that 30 years have now passed since the superstring revolution of 1985, but string theory still cannot boast of phenomenological successes. It has not contributed much in our understanding of why the world we see is this and not that.

Bearing all this in mind, it is interesting to pursue as far as we can an alternative, more conservative line of reasoning, assuming that the TOE is not a string theory, but a conventional field theory living in flat space with universal flat time.

The first question that one should be able to answer in this approach is how to explain the observed fact that the space-time we live in is curved. And the only way to do so is to assume that our (3+1)-dimensional Universe represents a thin curved film embedded in a flat higher-dimensional bulk, like a 2-dimensional soap bubble is embedded in flat 3-dimensional space.

The fundamental TOE should be formulated in this bulk, and our Universe should represent a classical solution in this theory, a kind of Abrikosov string, but extended not in one, but in three spatial directions, and very thin in the transverse directions so that the latter cannot be perceived.

Then the question is how this higher-dimensional theory can look like. The problem is that we cannot write any Lagrangian, familiar from 4-dimensional physics, like

$$\mathcal{L} = -\frac{1}{2g^2} \text{Tr} \{ F_{\mu\nu} F_{\mu\nu} \}.$$  \hfill (1)

We cannot do that because at $D > 4$ the coupling constant $g$ acquires dimension, and the theory is no longer renormalizable.

What one can try to do is to add extra derivatives. For example, the 6-dimensional Lagrangian

$$\mathcal{L}^{D=6} = \alpha \text{Tr} \{ F_{\mu\nu} \Box F_{\mu\nu} \} + \beta \text{Tr} \{ F_{\mu\nu} F_{\nu\alpha} F_{\alpha\mu} \}$$  \hfill (2)

involves dimensionless coupling constants $\alpha, \beta$ and is renormalizable.

We thus arrive at a higher-derivative (HD) theory, and that was my own main motivation to study such theories — I tend to believe that the Holy Grail TOE is a HD theory living in a higher-dimensional bulk, with our Universe representing a curved 3-brane embedded there [2,3].

But we should say right now that all such theories have a common striking feature — they are ghost-ridden. This means that the quantum Hamiltonian of any such theory does not have a ground state and its spectrum involves the levels with arbitrarily low energies. Neither has this Hamiltonian a “sky state” — the spectrum also involves the levels with arbitrarily high energies.

For many years people thought that such theories are inherently sick and did not study them. Indeed, the absence of the vacuum ground state is a very strange and unusual feature. And in many cases such theories are sick, indeed. They involve a collapse leading to violation of unitarity. But it has become clear recently that the collapse and violation of unitarity is a feature of many, but not of all ghost-ridden theories. There are quantum mechanical and field theory systems with “benign ghosts” — their spectrum is bottomless and there is no vacuum state, but still the quantum problem is well posed, there is no collapse and the evolution operator is unitary.

Unfortunately, there has been much confusion in the literature devoted to this problem. But correct understanding of this issue is now being gradually achieved. We mostly follow in our talk the review [4].

1 Unfortunately, we have now absolutely no idea what this theory might be.
2. Pais-Uhlenbeck oscillator

The dynamics of HD systems was first studied by Ostrogradsky who developed back in 1848 the Hamiltonian method in classical mechanics independently of Hamilton and applied it to the ordinary and HD systems [5]. The inherent feature of the Ostrogradsky Hamiltonian is a term involving the linear dependence of one of the canonical momenta, so that the classical energy can acquire both arbitrary large positive and arbitrary large negative values [6].

The dynamics of a simplest quantum HD Hamiltonian was first studied in Ref. [7]. The classical Pais-Uhlenbeck Hamiltonian is derived from the Lagrangian

\[ L = \frac{1}{2} \left[ \dot{x}^2 - (\omega_1^2 + \omega_2^2)\dot{x}^2 + \omega_1^2 \omega_2^2 x^2 \right] \]  

and involves two pairs of canonical variables: \( x, p_x \) and \( v, p_v \). It reads

\[ H = p_x v + \frac{p_v^2}{2} + \frac{(\omega_1^2 + \omega_2^2)v^2}{2} - \frac{\omega_1^2 \omega_2^2 x^2}{2} \]  

(4)

One of the Hamilton equations of motion gives \( \partial H / \partial p_x = \dot{v} = \dot{x} \).

Pais and Uhlenbeck noticed that the spectrum of the quantum version of (4) (with \( p_x \rightarrow \hat{p}_x = -i \partial / \partial x \), \( p_v \rightarrow \hat{p}_v = -i \partial / \partial v \)) is not bounded neither from above, nor from below — the ground state of \( \hat{H} \) is absent. It was later understood that the absence of the ground state is an inherent feature of quantum HD systems [8].

To determine the spectrum of the PU Hamiltonian, it is convenient to perform the following quantum canonical transformation \(^3\) [10]:

\[ X_1 = \frac{1}{\omega_1} \hat{p}_x + \frac{\omega_2^2 v}{\sqrt{\omega_1^2 - \omega_2^2}}, \quad \hat{P}_1 = -i \frac{\partial}{\partial X_1} = \omega_1 \frac{\hat{p}_x + \omega_2^2 x}{\sqrt{\omega_1^2 - \omega_2^2}} \]

\[ X_2 = \frac{\hat{p}_v + \frac{\omega_2^2 x}{\sqrt{\omega_1^2 - \omega_2^2}}}{\sqrt{\omega_1^2 - \omega_2^2}}, \quad \hat{P}_2 = -i \frac{\partial}{\partial X_2} = \frac{\hat{p}_x + \frac{\omega_2^2 v}{\sqrt{\omega_1^2 - \omega_2^2}}}{\sqrt{\omega_1^2 - \omega_2^2}} \]  

(5)

(We assumed that \( \omega_1 > \omega_2 \). The case of equal frequencies will be treated a little later.) In these terms, the Hamiltonian reduces to a difference of two oscillator Hamiltonians:

\[ H = \frac{\hat{P}_1^2 + \omega_1^2 X_1^2}{2} - \frac{\hat{P}_2^2 + \omega_2^2 X_2^2}{2}. \]  

(6)

Its spectrum is

\[ E_{nm} = \left( n + \frac{1}{2} \right) \omega_1 - \left( m + \frac{1}{2} \right) \omega_2 \]  

(7)

with positive integer \( n, m \).

Consider the case \( \omega_1 = 2\omega_2 \). Then

\[ E_{nm} = \left( 2n - m + \frac{1}{2} \right) \omega_2. \]  

(8)

One can in principle get rid of the linear term in the Hamiltonian and thereby “exorcise the ghost” by imposing certain constraints that effectively reduce the dimensionality of the phase space [9]. But this just means that, if these constraints are explicitly resolved and one goes back to the Lagrangian formalism, higher derivatives disappear.

\(^3\) For a general theory of quantum canonical transformations see [11] and references therein. One can also first perform the canonical transformation at the classical level and quantize afterwards.
We see that the spectrum is discrete, not bounded and that each level is infinitely degenerate. The same properties hold for any rational ratio \( \omega_1/\omega_2 \).

An interesting situation arises for incommensurable frequencies. In that case, the spectrum is everywhere dense: for any energy \( E \) one can find an arbitrary close eigenvalue \( E_{nm} \). However, the spectrum is not continuous, but rather pure point: the wave functions of all eigenstates are normalizable.

In the degenerate case, \( \omega_1 = \omega_2 \equiv \omega \), the situation is somewhat more complicated. We cannot perform the canonical transformation (5) anymore and the Hamiltonian is not reduced to a difference of two oscillator Hamiltonians. The solution of the problem can be obtained by the following trick [7,12].

At the first step, we get rid of the potential terms \( \propto x^2 \) and \( \propto v^2 \) in the Hamiltonian

\[
\hat{H} = \hat{p}_x v + \frac{\hat{p}_v^2}{2} + \omega^2 v^2 - \frac{\omega^4 x^2}{2} \tag{9}
\]

by representing its eigenfunctions as

\[
\Psi(x,v) = e^{-i\omega^2 xv} \phi(x,v). \tag{10}
\]

The Hamiltonian acting on \( \phi(x,v) \) has the form

\[
\hat{H}_\phi = \frac{1}{2} \hat{p}_v^2 + v \hat{p}_x - \omega^2 x \hat{p}_v. \tag{11}
\]

At the second step, we perform the quantum canonical transformation

\[
x \to \frac{1}{\omega} x + \frac{1}{4\omega^2} \hat{p}_v, \quad \hat{p}_x \to \omega \hat{p}_x, \quad v \to v + \frac{1}{4\omega} \hat{p}_x, \quad \hat{p}_v \to \hat{p}_v. \tag{12}
\]

The transformation (12) is the superposition of the scale transformation \( x \to x/\omega, \hat{p}_x \to \omega \hat{p}_x \) and the unitary transformation

\[
\hat{O} \to \exp \left\{ \frac{i}{4\omega} \frac{\partial^2}{\partial x \partial v} \right\} \hat{O} \exp \left\{ -\frac{i}{4\omega} \frac{\partial^2}{\partial x \partial v} \right\}.
\]

It gives the new Hamiltonian

\[
\hat{H}'_\phi = \frac{\hat{p}_x^2 + \hat{p}_v^2}{4} + \frac{\omega^2}{4} (v \hat{p}_x - x \hat{p}_v) \tag{13}
\]

acting on the wave function \( \phi'(x,v) \) related to \( \phi(x,v) \) according to

\[
\phi(x,v) = \exp \left\{ -\frac{i}{4\omega} \frac{\partial^2}{\partial x \partial v} \right\} \phi'(\omega x, v). \tag{14}
\]

The first term in the Hamiltonian (13) describes free motion in the transformed \((x,v)\) plane, and the second term is proportional to the “angular momentum” operator

\[
\hat{l} = v \hat{p}_x - x \hat{p}_v. \tag{15}
\]

Expressed in the original variables, this operator reads

\[
\hat{l} = \frac{v \hat{p}_x - \omega}{2} x \hat{p}_v + \frac{1}{4\omega} \left( \hat{p}_v^2 - \frac{\hat{p}_x^2}{\omega^2} \right) + \frac{3\omega}{4} (v^2 - \omega^2 x^2). \tag{16}
\]
A dedicated reader may check that it commutes with the original Hamiltonian (9).

The eigenfunctions of (13) are the same as for the free Laplacian, but the energy is shifted by $l\omega$, where $l$ is the integer eigenvalue of (15). They are

$$\phi_{lk}^\prime(\omega x, v; t) \propto J_l(kr) e^{-it\theta} e^{-it(l\omega+k^2/4)},$$  \hspace{1cm} (17)

where $(r, \theta)$ are the polar coordinates in the plane $(\omega x, v)$.

The eigenfunctions of the original Hamiltonian are

$$\Psi_{lk}(x, v; t) \propto e^{-i\omega_1 x v} \exp\left\{i\frac{4}{4\omega_2^2}\frac{\partial^2}{\partial x \partial v}\right\} J_l\left(k\sqrt{v^2 + \omega_2^2 x^2}\right) \left(\frac{\omega_1 - iv}{\omega_1 + iv}\right)^{l/2} e^{-it(l\omega+k^2/4)} \hspace{1cm} (18)$$

They are not normalizable and describe a continuum spectrum rather than a pure point spectrum characteristic for a nondegenerate system with $\omega_1 \neq \omega_2$. Any real energy is admissible, and each energy level is infinitely degenerate: the eigenfunctions $\Psi_{l+1,k}$ and $\Psi_{l,\sqrt{k^2+4\omega_2^2}}$ have the same energy.

The difference in the quantum behaviour of the degenerate ($\omega_1 = \omega_2$) and non-degenerate ($\omega_1 \neq \omega_2$) systems matches the difference in their classical behaviour. The motion described by the Hamiltonian (6) is finite and that is why the spectrum is pure point. And in the degenerate case the situation is different. The equation of motion

$$\frac{d^4}{dt^4} + 2\omega_1^2 \frac{d^2}{dt^2} + \omega_4^2 x = 0.$$  \hspace{1cm} (19)

admits not only ordinary oscillatory solutions $x(t) \propto e^{i\omega t}$, but also the solutions

$$x(t) \propto te^{i\omega t},$$  \hspace{1cm} (20)

where the amplitude of oscillations grows with time. An infinite classical motion produces a continuum spectrum in the quantum problem.

One can observe that both in the non-degenerate and in the degenerate case both the classical and the quantum dynamics of the system are quite benign. The wave functions are known explicitly. The Hamiltonian is Hermitian and the quantum evolution is unitary. For example, the evolution operator for the system (6) is simply a product of the evolution operators for the individual oscillators:

$$K(X_1', X_2'; X_1, X_2; t) = \sum_{n=0}^{\infty} \psi_n^*(X_1') \psi_n(X_1) e^{-i\omega_1 t(n+1/2)} \sum_{m=0}^{\infty} \psi_m^*(X_2') \psi_m(X_2) e^{i\omega_2 t(m+1/2)} \hspace{1cm} (21)$$

where $\psi_n(X)$ are the standard oscillator eigenfunctions.

One can meet in the literature a statement that the Pais-Uhlenbeck oscillator with degenerate frequencies is not unitary due to the presence of Jordan blocks [13]. But this is not correct. Infinite-dimensional Jordan blocks appear in this problem if one tries to describe the motion in terms of the oscillator wave functions, as if the motion were finite. And their appearance is simply a manifestation of the fact that the spectrum in this case is continuous, and not of the violation of unitarity. We refer the reader to Refs. [14], [15], [4], where we clarify this point.

The Minkowskian path integral for the evolution operator (21) can be expressed either in the Lagrangian form

$$\sim \int \prod_t dx(t) \exp\left\{i \int dt L(\dot{x}, \ddot{x}, x)\right\} \hspace{1cm} (22)$$
or in the Hamiltonian form
\[
\sim \int \prod_t dx(t) dv(t) dp_x(t) dp_v(t) \exp \left\{ i \int dt \left[ p_v \dot{v} + p_x \dot{x} - H(p_v, p_x; v, x) \right] \right\} .
\] (23)

Indeed, substituting in (23) the Hamiltonian (4) and integrating over \( \prod_t dp_x(t) \), we obtain the factor
\[
\prod_t \delta[v(t) - \dot{x}(t)] .
\] (24)

Performing the integral over \( \prod_t dv(t) \) with this factor and doing the Gaussian integral over \( \prod_t dp_v(t) \), we reproduce (22) with the Lagrangian (3).

In ordinary quantum theories including a well-defined vacuum state we can perform the Wick rotation \( t \to -i\tau \) and define an Euclidean path integral. This does not work, however, for the integral (23). The contribution to this integral coming from the first term in the Hamiltonian (4)
\[
\prod_{\tau} \int_{-\infty}^{\infty} dp_x(\tau) \exp \left\{ \int d\tau p_x(\tau) \left[ \frac{dx(\tau)}{d\tau} - v(\tau) \right] \right\} 
\] (25)
diverges and does not give anything like (24). (The latter circumstance means that the variable \( v(\tau) \) can no longer be interpreted as the velocity.) The fact that the Euclidean path integral diverges agrees well with that the evolution operator (21) is not defined at imaginary time — at \( t = -i\tau \) the second series diverges with a vengeance.

Another way of going into Euclidean space was suggested in [16]. Instead of performing an analytic continuation of the Euclidean path integral (23), one could try to do it at the Lagrangian level, capitalizing on the fact that the Euclidean counterpart of the Lagrangian (3) is positive definite. Trading \( t \) for \( i\tau \) in (22) \(^4\), one obtains a converging path integral.

However, if one tries to take the expression for the Euclidean evolution operator thus derived and goes back to real Minkowsky time, one obtains something which does not describe a unitary evolution with a Hermitian Hamiltonian [16–18]. It is thus better not to go to Euclidean space whatsoever, but keep the time real all the time.

One can sometimes meet a statement that the only way to define a Minkowski path integral is by an analytic continuation from Euclidean space. If true, that would mean that, given the fact that the Euclidean path integral is not defined, the Minkowski path integral is not defined either. But it is not so. The fact that Minkowski path integrals can be defined independently of Euclidean ones is especially clearly seen in the semi-classical limit where the Minkowski evolution operator can be evaluated as \(^5\)
\[
\langle x' t' | x t \rangle \approx \Delta \exp \{ iS_{cl} \} ,
\] (26)

where \( S_{cl} \) is the action on the corresponding classical trajectory (which is assumed to be large in the units of \( \hbar \)) and \( \Delta \) is a well defined Gaussian integral. The corrections to \( \Delta \) in any perturbative order can also be evaluated. Finally, Minkowski path integrals can be calculated numerically using the technique of Ref. [20].

\(^4\) The standard Wick rotation \( t \to -i\tau \) works if one changes the sign in the definition of \( L \).

\(^5\) See also Eq. (3.62) in the book [19] that expresses the Minkowskian path integral for the evolution operator in the problem with a linearly rising potential \( V(x) = -fx \). (Obviosly, the spectrum of this problem does not have a bottom.)
3. Philological digression

We have proven that the nondegenerate quantum higher-derivative systems have no ground state. How should one call this phenomenon? This question is actually not so irrelevant. Experience shows that an inexact, not carefully chosen name may excite false associations and lead eventually to wrong claims.

Traditionally, one says that such bottomless systems involve ghosts. Let us explain where this name came from and what it means. By “ghosts” one usually has in mind the states with negative norm. Negative norm means negative probability and production of such states means violation of unitarity. Such ghosts appear e.g. in gauge theories. These are scalar photons in QED, and scalar gluons and Faddeev-Popov ghosts in non-Abelian theories.

In the framework of the Gupta-Bleuler quantization procedure for QED, one introduces the creation and annihilation operators $a_\mu(k)\,$ and $a_\mu(k)^\dagger\,$ for all four photon polarizations, which satisfy the following commutation relations:

$$[a_\mu(k),a_\nu^\dagger(q)] = -\eta_{\mu\nu}\delta(k-q).$$

(27)

The commutator $[a_0, a_0^\dagger]$ is then negative. That means that the scalar photon state has a negative norm. Indeed, introduce the Fock vacuum $|\Phi\rangle$ whose norm is positive. We then have

$$\left|\left\langle a_0^\dagger\Phi\right|\right|^2 = \langle\Phi|a_0a_0^\dagger|\Phi\rangle = \langle\Phi|[a_0, a_0^\dagger]|\Phi\rangle = -\langle\Phi|\Phi\rangle.$$  

(28)

In gauge theories, such ghosts are harmless: one can define a reduced physical space involving only the physical states with positive norm (transverse photons or transverse gluons) and show that the ghosts states are not created in collisions of physical particles.\(^6\)

Let us go back to the Pais-Uhlenbeck oscillator. Consider the non-degenerate case, which is more simple. Introducing the creation and annihilation operators in the usual way, we may write the quantum Hamiltonian as

$$H = \omega_1a_1a_1^\dagger - \omega_2a_2a_2^\dagger + C.$$  

(29)

The excitations of the second term give a tower of states with decreasing energies all the way down to $-\infty$. Suppose, however, that we do not like these negative energies so much that we want to get rid of them by any cost. Then one can formally define a state $|\Phi\rangle$ which is annihilated by $a_2^\dagger$ rather than by $a_2$. This state is strange and unhandy: its wave function is\(^{[22]}\)

$$\Phi(X_2) \propto \exp\left\{\frac{\omega_2}{2}X_2^2\right\}$$

(30)

so that the state is not normalizable. Still, formally, the state $|\Phi\rangle$ has the lowest energy. We can even bring it to zero or to any other finite value by choosing an infinite positive constant $C$ in (29). Then the operator $a_2$ acting on $|\Phi\rangle$ would increase its energy, and we can call it a creation operator $b^\dagger$. And $a_2^\dagger$ is now interpreted as the annihilation operator $b$. The price one has to pay for that is that the commutator $[b, b^\dagger]$ is now negative and the states describing excitations above the “vacuum” $|\Phi\rangle$ are ghost states, they have negative norm.

Personally, I find this construction rather awkward. It is much better to talk about negative energies, keeping the norm positive. But, historically, people thought in these terms and that is why they (wrongly) believed for a long time that higher derivatives necessarily entail the violation of unitarity.

R. Woodard suggested to call this phenomenon “Ostrogradskian instability” without the reference to ghosts. This name has, however, its own drawbacks. First of all, Ostrogradsky

\(^6\) This is explained in many textbooks including my own book [21].
knew nothing about quantum Hamiltonians and their spectra. The observation of the absence of the ground states in HD quantum systems belongs not to him, but to Pais and Uhlenbeck.

In addition, the word “instability” invokes wrong images. Having heard it, a physicist imagines a ball on the top of the hill and thinks of the exponential growth of deviations from the equilibrium. But for certain HD systems, there is no such growth. The spectrum has no bottom, but there is no instability — neither at the classical nor at the quantum level.

Thus, we prefer not to use this word and not invent anything new, but rather to stick to the traditional and more familiar to most people name “ghost”, not invoking, however, the negative metric description. For us, a ghost system is by definition a system where the Hamiltonian does not have a ground state but involves oscillatory-type excitations with arbitrarily negative energies. However, all the states of this Hamiltonian have positive norm. We will call the excitations with negative energies ghosts, but will distinguish the systems with benign ghosts, where the quantum problem is well defined, as it is for the Pais-Uhlenbeck oscillator, and the systems with malignant ghosts involving a collapse and loss of unitarity. The latter systems are more common. They will be discussed in Sect. 5. And some examples of the benign ghost systems will be given in Sects. 6, 7.

The last comment concerns the paper of C. Bender and P. Mannheim [23] who suggested to bust ghosts by replacing the original Hilbert space spanned over the eigenfunctions $\Psi(x, v)$ of the Hamiltonian (4) or, which is equivalent for different frequencies, the Hilbert space spanned by the oscillator wave functions of the Hamiltonian (6) by another Hilbert space involving the functions depending on real $X_1$ and imaginary $X_2 = iY_2$ and normalized in that region. Then the wave function (30) becomes a good normalizable vacuum state, $\int_{-\infty}^{\infty} |\Phi|^2 dY_2 < \infty$, and all other eigenstates have positive energies (and positive norm).

Personally, I do not quite see a point of doing so. One should understand that this complexification brings us to another quantum problem having little to do with the original one. No ghosts in the former does not mean no ghosts in the latter.

4. Including interactions

We discussed so far only mechanical systems, but it is easy to write down a field-theory generalization of (3). We can write

$$L = \frac{1}{2} \left[ (\Box \phi)^2 - (M_1^2 + M_2^2)(\partial_\mu \phi)^2 + M_1^2 M_2^2 \phi^2 \right]. \tag{31}$$

When $M_1 \neq M_2$, one can go over to the Hamiltonian, perform an appropriate canonical transformation and bring it again in the Lagrangian form to obtain

$$L = \frac{1}{2} \left[ (\partial_\mu \Phi_1)^2 - M_1^2 \Phi_1^2 \right] - \frac{1}{2} \left[ (\partial_\mu \Phi_2)^2 - M_2^2 \Phi_2^2 \right]. \tag{32}$$

This Lagrangian includes an ordinary scalar field $\Phi_1$ and a ghost field (i.e., in our terminology, a field whose quanta carry negative energies) $\Phi_2$. We can expand each field in Fourier modes, in which case the Lagrangian is split into an infinite number of noninteracting sectors with a definite momentum $p$. In each such sector, the dynamics is described by the Hamiltonian (6) with $\omega_{1,2}^2 = p^2 + M_{1,2}^2$. In free theory, the negative sign of the second term in (6) or (32) is irrelevant. The classical equations of motion and their solutions do not depend on this sign. The spectra of the quantum Hamiltonians are different for different signs, but in the absence of interactions, energy does not really play a dynamical role and serves only for bookkeeping.

A trouble may set in, however, if interactions are included. Heuristically, one could expect in this case an instability due to copious production of ghosts. Let us see whether an interactive system has or not this instability.
We modify the Lagrangian (3) by adding there nonlinear terms. Consider for example the system \[24\]
\[
L = \frac{1}{2} \left[ \dddot{x}^2 - 2\omega^2 \dot{x}^2 + \omega^4 x^2 \right] - \frac{1}{4} \alpha x^4 .
\] (33)
The classical equations of motion read
\[
\left( \frac{d^2}{dt^2} + \omega^2 \right) x - \alpha x^3 = 0 .
\] (34)
The classical trajectories depend on four initial conditions. There is an obvious stationary point
\[
x(0) = \dot{x}(0) = \ddot{x}(0) = \lambda(0) = 0.
\] (35)
The behaviour of the system at the vicinity of this point depend on the sign of \(\alpha\). If \(\alpha < 0\), the Ostrogradsky Hamiltonian acquires an extra negative contribution to the energy and all the trajectories other than \(x(t) = 0\) are unstable — they run away to infinity in finite time. This instability is even worse that Lyapunov’s exponential growth of perturbations. We are dealing here with a collapse: the classical dynamical problem is not well posed beyond a certain time horizon.

The situation is better for positive \(\alpha\). The stationary point (35) lies in the center of an “island of stability” — the trajectories with initial conditions at its vicinity do not go astray, but exhibit an oscillatory behaviour. However, this island has a shore. When the deviations of initial conditions from (35) are large enough, the trajectory collapses. To chart this shore, one should perform a numerical study. Assume for simplicity that \(\dot{x}(0), \ddot{x}(0)\) and \(\lambda(0)\) stay zero, and \(x(0) = c > 0\). Then a critical value \(c_{\text{crit}}\) can be found, above which the trajectories go astray and run to infinity, but the system only exhibits benign oscillations around zero when \(c < c_{\text{crit}}\). This value is roughly \(c_{\text{crit}} \approx 0.3 \omega^2 / \sqrt{\alpha}\) (the dependence on \(\omega\) and \(\alpha\) follows, of course, from simple scaling arguments). For illustration, we plotted in Fig.1 the solution to the equations of motion (34) for \(\omega = \alpha = 1\) and \(x(0)\) just above \(c_{\text{crit}}\). After some quasiharmonic oscillations, the trajectory finally goes astray and runs to infinity. And for \(x(0) < c_{\text{crit}}\), it keeps oscillating forever.

The same behaviour is seen if three other initial conditions are shifted from zero. The island has a finite area (or rather a finite phase space volume).

A similar island of stability was observed in \([25]\) for the model Hamiltonian \[7\]
\[
H = \frac{1}{2} \left[ p_x^2 - (p_y^2 + y^2) + x^2 y^2 \right] ,
\] (36)
where a free particle is coupled to the ghost oscillator. If we pose the initial conditions
\[
x(0) = y(0) = a, \quad p_x(0) = p_y(0) = 0 ,
\] (37)
the system undergoes benign oscillations at the vicinity of the origin if \(a \lesssim 0.59\) and goes astray, running to the infinity, for larger deviations.

What happens in the quantum case? One cannot give a quite definite answer to this question without a special numerical study, but certain heuristic arguments that the quantum problem is also malignant and collapsing in this case will be given below.

Classical and quantum collapse are well known for certain special ordinary (not HD) systems. The simplest such system describes the 3-dimensional motion of a particle with the Hamiltonian
\[
H = \frac{\hat{p}^2}{2m} - \frac{\kappa}{r^2} .
\] (38)
Figure 1. Oscillating and collapsing

Figure 2. Falling on the center for the Hamiltonian (38) with \( m = 1 \) and \( \kappa = .05 \). The energy is slightly negative. The particles with positive energies escape to infinity.

Classically, for certain initial conditions, the particle falls to the center in a finite time. An example of such a trajectory is given in Fig. 2. The quantum dynamics of this system depends on the value of \( \kappa \). If \( m\kappa < 1/8 \) (where \( m \) is the mass of the particle), the ground state exists and unitarity is preserved. If \( m\kappa > 1/8 \), the spectrum is not bounded from below and, which is worse, the quantum problem cannot be well posed until the singularity at the origin is removed. For example, one can pose \( V(r) = -\kappa/r^2 \) for \( r > a \) and \( V(r) = -\kappa/a^2 \) for \( r \leq a \). The spectrum then depends on \( a \) [26–28]. Without such a cutoff, the probability “leaks” into the singularity.

\footnote{The authors of this paper were interested in the toy Hamiltonian (36) in association with a certain cosmological problem.}
and unitarity is violated. And for \( m\kappa < 1/8 \) quantum fluctuations cope successfully with the attractive force of the potential and prevent the system from collapsing.

Going back to the system (33), a variational analysis shows that, in contrast to the system (38) with small \( \kappa \), the ground state in the spectrum is absent [14]. We see that for some HD systems involving collapsing classical trajectories, their ghosts are malignant and unitarity is violated by the same mechanism as for the strong attractive potential (38). On the other hand, one can also conjecture that

*If the classical dynamics of the system is benign, its quantum dynamics is also benign, irrespectively of whether the spectrum has or does not have a bottom.*

5. Benign ghosts: classical and quantum mechanics

In some special cases, a nontrivial interacting ghost system may be completely benign: all the classical trajectories (and not only the trajectories restricted to a limited region of phase space) are stable and behave well at all times. Capitalizing on the conjectured property formulated above, the corresponding quantum problems are also well posed — a special study of this complicated question is not required!

5.1. Exactly solvable models

5.1.1. A simplest nontrivial model

The first example of such a system was found in [29] \(^8\). The Hamiltonian involves two pairs of the dynamic variables, \((x,p)\) and \((D,P)\), and reads

\[
H = pP + DV'(x),
\]

where \(V(x)\) is an arbitrary smooth even function. The simplest nontrivial case is

\[
V(x) = \frac{\omega^2 x^2}{2} + \frac{\lambda x^4}{4}, \quad \lambda > 0.
\]

The kinetic part of the Hamiltonian (39) is not positive definite, and, obviously, the spectrum of its quantum counterpart has no bottom. If one introduces the variables

\[
X_{1,2} = \sqrt{\frac{\omega}{2}} x \pm \frac{1}{\sqrt{2\omega}} D, \quad P_{1,2} = \frac{1}{\sqrt{2\omega}} p \pm \frac{\sqrt{\omega}}{2} P,
\]

the Hamiltonian (39) acquires the form

\[
H = \frac{P_1^2 + \omega^2 X_1^2}{2} - \frac{P_2^2 + \omega^2 X_2^2}{2} + \frac{\lambda}{4\omega} (X_1 - X_2)(X_1 + X_2)^3.
\]

In other words, this is the Hamiltonian (6) with degenerate frequencies, where an extra quartic interaction of a special form is added.

A nice distinguishing feature of the system (39) is its integrability. Indeed, it involves besides \(H\) another integral of motion:

\[
N = \frac{P^2}{2} + V(x).
\]

The Poisson bracket \(\{H,N\}\) vanishes. This allows one to find the solution analytically.

The classical equations of motion are

\[
\ddot{x} + V'(x) = 0, \quad \ddot{D} + V''(x)D = 0.
\]

\(^8\) It was found by studying a certain higher-derivative supersymmetric system, but we will not explore here the supersymmetric aspects of this problem.
The first equation is especially simple. It describes oscillations in the quartic potential (40). The solutions are the elliptic functions whose parameters depend on the integral of motion $N$:

$$x(t) = x_0 \text{cn}[\Omega(t - t_0), k]$$  \hspace{1cm} (45)$$

where

$$\alpha = \frac{\omega^4}{\lambda N}, \quad \Omega = \left[\lambda N(4 + \alpha)\right]^{1/4}, \quad k^2 \equiv m = \frac{1}{2} \left[1 - \sqrt{\frac{\alpha}{4 + \alpha}}\right],$$

$$x_0 = \left(\frac{N}{\lambda}\right)^{1/4} \sqrt{4 + \alpha - \sqrt{\alpha}}. \hspace{1cm} (46)$$

Here $k$ is the parameter of the Jacobi elliptic functions [30] and $\text{cn}(\tau)$ is a periodic function with the period

$$4K = 4 \int_0^{\pi/2} \frac{d\theta}{\sqrt{1 - k^2 \sin^2 \theta}}. \hspace{1cm} (47)$$

The equation for $D$ represents an elliptic variety of the Mathieu equation. Generically, it is not the simplest kind of equation, but in our case the solutions can be found in an explicit form. One of the solutions is

$$D_1(t) \propto \dot{x}(t) \propto \text{sn}[\Omega t, k] \text{dn}[\Omega t, k]$$  \hspace{1cm} (48)$$

(we have chosen $t_0 = 0$). The second solution can be found from the condition that the time derivative of the Wronskian $W = D_1 D_2 - D_2 D_1$ vanishes. We find

$$D_2(t) \propto \dot{x}(t) \int_t^{\tau} \frac{dt'}{x'^2(t')} \propto \text{sn}[\Omega t, k] \text{dn}[\Omega t, k] \int_t^\tau \frac{dt'}{\text{sn}^2[\Omega t', k] \text{dn}^2[\Omega t', k]}.$$  \hspace{1cm} (49)$$

Two independent solutions (48) and (49) exhibit oscillatory behaviour with constant and linearly rising amplitude, correspondingly (see Fig. 3). This linear growth has the same nature as in (20) and does not lead to any inconsistency.

The eigenvalues and eigenfunctions of the quantum Hamiltonian can also be found explicitly in this case [29]. Most of the states belong to the continuum spectrum (note again the similarity with the situation for the free PU oscillator with degenerate frequencies discussed in Sect.3). There are, however, not one, but many bands. The bands with positive energies start from $E = \omega$, $E = 2\omega$, etc, and extend upwards, while the bands with negative energies start from $E = -\omega$, $E = -2\omega$, etc, and extend downwards. The energies depend on $N$:

$$E_n(N) = \frac{\pi n}{2K(k)} \left[\lambda N \left(4 + \frac{\omega^4}{\lambda N}\right)\right]^{1/4} \hspace{1cm} (50)$$

with $n = \pm 1, \pm 2, \ldots$ and $K(k)$ defined in (47). This means that the levels with $E \in (\omega, 2\omega)$ and $E \in (-2\omega, -\omega)$ are not degenerate, the levels with $E \in (2\omega, 3\omega)$ and $E \in (-3\omega, -2\omega)$ are doubly degenerate, the levels with $E \in (3\omega, 4\omega)$ and $E \in (-4\omega, -3\omega)$ are 3-fold degenerate, etc. The wave functions are

$$\Psi_n(x, D) \propto \frac{1}{\sqrt{N - V(x)}} \exp\left\{iD \sqrt{2[N - V(x)]} + \frac{iE_n(N)}{\sqrt{2}} \int_x^y \frac{dy}{\sqrt{N - V(y)}}\right\}. \hspace{1cm} (51)$$
Figure 3. A typical behaviour of $D(t)$, as follows from the solution of (44).

If setting $n = 0$ in Eq. (50) and substituting this in (51), we obtain an infinity of zero-energy states with the wave functions

$$
\Psi_{0N}(x, D) \propto \frac{1}{\sqrt{N - V(x)}} \exp \left\{ iD \sqrt{2[|N - V(x)|]} \right\} .
$$

(52)

The functions (52) are not normalizable, but we can in fact choose the basis with normalizable eigenfunctions. Indeed, one can show that any function

$$
\Psi_0^{(g)}(x, D) = \int_{-\infty}^{\infty} g \left( \frac{P^2}{2} + V(x) \right) e^{iPD} dP
$$

(53)
is a solution to the Schrödinger equation $\hat{H}\Psi = 0$. The solutions (52) are obtained from (53) by setting $g(N) = \delta(N - N_0)$. But we can also choose $g_k(N) = N^k e^{-N}$ giving the normalizable functions $\Psi_{0k}$. The full spectrum of the quantum Hamiltonian is represented in Fig. 4. It is Hermitian and the evolution operator is unitary.

5.1.2. Ghosts in the Toda chain

There are many other exactly solvable systems with benign ghosts [31]. A large class of such systems is related to ordinary ghost-free nonlinear exactly solvable systems.

Consider as an example the closed Toda chain with three particles. The Hamiltonian is

$$H = \frac{1}{2}(p_1^2 + p_2^2 + p_3^2) + V_{12} + V_{23} + V_{31},$$

(54)

where $V_{12} = e^{q_1 - q_2}$, etc. Besides the energy, the system involves an obvious integral of motion

$$P = p_1 + p_2 + p_3,$$

as well as the less obvious cubic invariant

$$I = \frac{1}{3}(p_1^3 + p_2^3 + p_3^3) + p_1(V_{12} + V_{31}) + p_2(V_{12} + V_{23}) + p_3(V_{23} + V_{31}).$$

(55)

The Poisson brackets $\{H, I\}$, $\{H, P\}$ and $\{I, P\}$ vanish. For the system with three degrees of freedom, we have three integrals of motion that are in involution. The system is exactly solvable.

In contrast to the open Toda chain, where the potential represents the sum of only two terms $V_{12} + V_{23}$, the potential in (54) keeps the three particles together. If we impose the requirement $q_1 + q_2 + q_3 = 0$ (the center of mass does not move), the motion is finite, with the classical trajectories representing complicated nonlinear oscillations. In the quantum problem [32], we
impose the constraint $\hat{P}\Psi(q_1, q_2, q_3) = 0$, in which case the spectrum of $\hat{H}$ is discrete. The eigenfunctions of (54) are simultaneously the eigenfunctions of $\hat{I}$.  

Now note that $\hat{I}$ is cubic in momenta and its eigenvalues can be both positive and negative. The following simple property holds:

**Proposition 1.** For each state with energy $E$ and positive eigenvalue of $\hat{I}$, there exists a state with the same energy and the negative eigenvalue of $\hat{I}$.

**Proof.** The operator $\hat{H}$ is invariant under the following discrete symmetry: 

$$\hat{P} : \quad q_1 \leftrightarrow -q_2, \quad q_3 \rightarrow -q_3. \quad (56)$$

The Hamiltonian commutes with $\hat{P}$, so that, if $\Psi$ is an eigenfunction of $\hat{H}$, $\hat{P}\Psi$ is also an eigenfunction of $\hat{H}$ with the same energy. But $\hat{I}$ anticommutes with $\hat{P}$. It follows that, if $\hat{I}\Psi = \lambda\Psi$, then 

$$\hat{I}\hat{P}\Psi = -\hat{P}\hat{I}\Psi = -\lambda\hat{P}\Psi.$$ 

\[ \square \]

In other words, the spectrum of $\hat{I}$ is symmetric under reflection $\lambda \rightarrow -\lambda$. To prove that this spectrum extends indefinitely down to $-\infty$ (and extends indefinitely up to $\infty$), it is sufficient to consider the classical trajectories with large values of $|\hat{I}|$. The highly excited eigenstates of $\hat{H}$ and $\hat{I}$ are related to these trajectories by WKB correspondence. But it is obvious that both the energy and $|\hat{I}|$ can be arbitrary large. Simply take the initial conditions with $q_j(0) = 0$ and $p_2(0) = p_3(0) = -p_3(0)/2 = M \rightarrow \infty$.

There is only one step to go. We call $\hat{I}$ rather that $\hat{H}$ the Hamiltonian. The new Hamiltonian does not have a ground state and is hence ghost-ridden. On the other hand, the spectral problem for $\hat{I}$ is quite well defined. There is no collapse and no loss of unitarity.

A nice feature of the Hamiltonian $\hat{I}$, compared to the quantum counterpart of (39) is its discrete spectrum.

For the quantum Toda problem, it does not matter which operator, $\hat{H}$ or $\hat{I}$, is called the Hamiltonian: they have the same spectrum. But the classical dynamics associates with $\hat{H}$ and $\hat{I}$ are different. For some reason, the scholars have not been so much interested in the latter. I only was able to find in the literature the Hamilton equations of motion associated with the cubic invariant for the open Toda chain [33]. In our case, the Hamilton equations of motion for $\hat{I}$ read

$$\dot{q}_1 = p_1^2 + V_{12} + V_{31}, \quad \dot{p}_1 = (p_1 + p_3)V_{31} - (p_1 + p_2)V_{12},$$

$$\dot{q}_2 = p_2^2 + V_{23} + V_{12}, \quad \dot{p}_2 = (p_2 + p_1)V_{12} - (p_2 + p_3)V_{23},$$

$$\dot{q}_3 = p_3^2 + V_{31} + V_{23}, \quad \dot{p}_3 = (p_3 + p_2)V_{23} - (p_3 + p_1)V_{31}, \quad (57)$$

to be compared with the standard equations of motion

$$\ddot{q}_1 = \dddot{p}_1 = V_{31} - V_{12}, \quad \ddot{q}_2 = \dddot{p}_2 = V_{12} - V_{23}, \quad \ddot{q}_3 = \dddot{p}_3 = V_{23} - V_{31}. \quad (58)$$

To feel the difference, compare the classical trajectories associated with $\hat{H}$ and $\hat{I}$ (see Fig. 2).

The initial conditions for both trajectories are identical and the conserved values of $H$ and $I$ are the same. The patterns of the two trajectories are similar—oscillations with beats. But for the equations of motion based on $I$, the period of beats is smaller.  

9 The operator $\hat{I}$ is defined by Eq. (55) where all the momenta $\hat{p}_j$ stay consistently on the left or consistently on the right: the two orderings give the same result.

10 It is also invariant under cyclic permutations of $q_{1,2,3}$, but this is irrelevant for our purposes.

11 There is no wonder that the classical trajectories associated with $H$ and $I$ are different. The phase space has four essential coordinates: $p_{1,2,3}, q_{1,2}$. The fixed values of two integrals of motion define a two-parametric surface in this space. Both trajectories lie on this surface, but they need not coincide.
5.2. Not integrable models

For the ghosts to be benign, it is not necessary that the system is exactly solvable. Many systems with benign ghosts that are not exactly soluble are known.

Well, a reservation is of order here. In contrast to regular periodic trajectories of exactly solvable integrable models, the trajectories of not integrable models exhibit stochastic behaviour. We can solve in this case the equations of motion only numerically and can never be sure that a surprise does not await us beyond the horizon of our calculations. Still, these numerical studies, some samples of which will be presented below, give us strong indications that there are no such surprises.

All models considered in this subsection represent certain deformations of the exactly solvable models presented above. In Ref. [29] a model with the Hamiltonian

$$H = pP + DV'(x) - \frac{\gamma}{2}(D^2 + P^2)$$

was considered. This model is not integrable and can only be studied numerically. This numerical study exhibits a benign behaviour of classical trajectories. The latter are even more handy than for the model (39) — if $\gamma$ is not too large, the linear growth of the amplitudes of the oscillations is absent.\footnote{A numerical analysis shows that, if $\gamma$ exceeds some limit that depends on the values of $\omega$, $\lambda$ and of the initial conditions, the amplitude starts to grow again, and this growth is exponential. Still, we did not observe a collapse and the solution exists at all the times.} One observes instead a finite motion with beats (Fig. 6).

Alternatively, one can add a nonlinear term in the Hamiltonian (6). One has found out that for many such deformations, the trajectories still behave in a benign way, not exhibiting collapse. This happens for the Hamiltonian \footnote{A numerical analysis shows that, if $\gamma$ exceeds some limit that depends on the values of $\omega$, $\lambda$ and of the initial conditions, the amplitude starts to grow again, and this growth is exponential. Still, we did not observe a collapse and the solution exists at all the times.}

$$H = \frac{P_1^2 + \omega_1^2 X_1^2}{2} - \frac{P_2^2 + \omega_2^2 X_2^2}{2} + \kappa(X_1 - X_2)(X_1 + X_2)^3;$$

\footnote{A numerical analysis shows that, if $\gamma$ exceeds some limit that depends on the values of $\omega$, $\lambda$ and of the initial conditions, the amplitude starts to grow again, and this growth is exponential. Still, we did not observe a collapse and the solution exists at all the times.}
Figure 6. A typical trajectory for the Hamiltonian (59).

for the Hamiltonian [34,35]

$$\begin{align*}
    H &= \frac{P_1^2 + \omega_1^2 X_1^2}{2} - \frac{P_2^2 + \omega_2^2 X_2^2}{2} + \lambda \sin^4(X_1 + X_2) \\
    &\text{and for the Hamiltonian [22]} \\
    H &= \frac{P_1^2 + \omega_1^2 X_1^2}{2} - \frac{P_2^2 + \omega_2^2 X_2^2}{2} + \lambda(X_1^4 - X_2^4) + \mu X_1^2 X_2^2, \quad \lambda \gg \mu.
\end{align*}$$

(61) (62)

One can also perturb the cubic Toda invariant (55) by adding to it an oscillator potential:

$$\tilde{I} = I + \alpha \left[ (q_1 - q_2)^2 + (q_1 - q_3)^2 + (q_2 - q_3)^2 \right].$$

(63)

The Poisson bracket \{\tilde{I}, p_1 + p_2 + p_3\} still vanish, but the bracket \{\tilde{I}, H\} does not. Thus, \(H\) is not an integral of motion anymore and the system is not integrable.

The trajectories can be found numerically. The trajectory for the Hamiltonian (63) with \(\alpha = 1\) and the same initial conditions as in Fig. 5 is shown in Fig. 7. We observe quasiperiodic oscillations (the regularity of trajectories seen in Fig. 5 is now lost, which confirms the fact that the system is no longer integrable). There is no collapse.

6. Benign ghosts: field theory

6.1. A nonintegrable model

The first example of a field theory enjoying benign ghosts was constructed in [36]. It is a straightforward generalization of (39). The Lagrangian of the model is

$$\mathcal{L} = \partial_\mu \phi \partial^\mu D - DV'(\phi),$$

(64)
where $\mu = 0, 1$ (the model is 2-dimensional) and
\[
V(\phi) = \frac{\omega^2 \phi^2}{2} + \frac{\lambda \phi^4}{4}, \quad \lambda > 0.
\]  
(65)

The model has two integrals of motion: the energy
\[
E = \int dx \left[ \dot{\phi} \dot{D} + \partial_x \phi \partial_x D + D \phi (\omega^2 + \lambda \phi^2) \right],
\]  
which can be both positive and negative, and the positive definite
\[
N = \int dx \left\{ \frac{1}{2} \left[ \dot{\phi}^2 + (\partial_x \phi)^2 \right] + \frac{\omega^2 \phi^2}{2} + \frac{\lambda \phi^4}{4} \right\}.
\]  
(66)

With only two integrals of motion for the infinite number of degrees of freedom, the model is not integrable and can only be solved numerically. We did so in the case of only one spatial dimension. The equations of motion are
\[
\Box \phi + \omega^2 \phi + \lambda \phi^3 = 0
\]
\[
\Box D + D (\omega^2 + 3 \lambda \phi^2) = 0.
\]  
(68)

We see that $\phi(x, t)$ satisfies a nonlinear wave equation. The solutions to this equation cannot grow — a growth is incompatible with the conservation of $N$. The amplitude of the oscillations of the field $D(x, t)$ can grow with time, however.

We played with different values of the parameters $\omega, \lambda$ and with different initial conditions and never found a collapse, only at worst a linear growth of $D(x, t)$ with time. A typical behaviour is shown in Fig. 8, where the dispersion $d(t) = \sqrt{\langle D^2 \rangle_x}$ is plotted as a function of time. This particular graph corresponds to the choice of the parameters $\omega = \lambda = 1$, to the length of the spatial box $L = 20$ (periodic boundary conditions in the spatial direction were imposed), while the initial conditions at $t = 0$ were chosen as follows:
\[
\phi(x, 0) = 5e^{-x^2}, \quad D(x, 0) = \cos(\pi x / 20), \quad \dot{\phi}(x, 0) = \dot{D}(x, 0) = 0.
\]  
(69)

The dispersion undergoes stochastic fluctuations, as is natural for a non-integrable nonlinear system. However, on the average, $d(t)$ grows linearly with time, similar to Fig. 3, and there is no trace of collapse.
Figure 8. Dispersion $d = \sqrt\langle D^2 \rangle$ as a function of time with the initial conditions (69).

6.2. Integrable models

There are many exactly solvable $(1 + 1)$ systems, which are characterized by an infinite number of integrals of motion. Each of the latter can be chosen as a Hamiltonian, and we obtain thereby a set of higher-derivative exactly solvable $(1 + 1)$-dimensional field theories. Take as an example the Sine-Gordon model. Its equation of motion reads

$$\phi_{tt} - \phi_{xx} = -\sin \phi.$$  \hspace{1cm} (70)

The conventional energy of the system,

$$E = \int_{-\infty}^{\infty} dx \left[ \frac{1}{2} (\phi_t^2 + \phi_x^2) + (1 - \cos \phi) \right]$$  \hspace{1cm} (71)

is conserved, and this gives the standard Hamiltonian.

But there is an infinity of other integrals of motion. The next in complexity after (71) is [37]

$$E_4 = \int dx \left[ \frac{1}{4} (\phi_t - \phi_x)^4 - (\phi_{tt} - 2\phi_{tx} + \phi_{xx})^2 - (\phi_t - \phi_x)^2 \cos \phi \right].$$  \hspace{1cm} (72)

This expression involves higher derivatives, is not bounded from below, and the Hamiltonian $H_4$ is ghost-ridden. To the best of my knowledge, the corresponding dynamical field equations were never studied (not speaking of quantum dynamics!), but such a study would be very interesting.

7. Discussion

Our main message is that the presence of ghosts (unboundness of the spectrum of the Hamiltonian) does not necessarily mean a collapse and disaster — there are nontrivial interacting ghost-ridden systems where this does not happen. In spite of the presence of ghosts, HD theories should be treated seriously, one should not throw them immediately to a garbage basket, as it was habitually done for many years. Instead, they should be interrogated one by one: the question whether a HD theory is benign or not (involves or does not involve the collapse) is a nontrivial dynamical question.

Unfortunately, we do not know today a lot of such benign ghost theories, and our dream to find a HD theory which might play the role of the TOE is far from being fulfilled. The
only known benign field-theory systems that we analyzed live in one spatial and one temporal dimension, while the TOE that includes gravity should be multidimensional, $D > 4$.

In Sect. 6, we discussed only the classical dynamics of the HD field theory models, but a hard question is what is their quantum dynamics? Ordinary field theories may be studied perturbatively. We single out the free Hamiltonian $H_0$, find its spectrum and treat nonlinear terms as perturbation. If the spectrum of $H_0$ includes a vacuum state and particle excitations above it, one may define the scattering matrix whose elements represent transition amplitudes between the particle asymptotic states. These transition amplitudes can in many cases be found perturbatively.

Here this approach does not work. Look at the system (64), (65). If the interaction is switched off, each level involves an infinite degeneracy: an “ordinary particle” (an excitation of the positive-energy oscillator) has the same energy as the state with 2 ordinary particles and one ghost, 3 particles and 2 ghosts, etc. And when the interaction is present ($\lambda \neq 0$), the spectrum is radically reshuffled. If we single out a particular Fourier mode and suppress artificially its interaction with the other modes, the Hamiltonian in this sector looks similar to (42) and its spectrum includes continuum bands as in Fig. 4. It is rather clear that even if $\lambda$ is small, the spectrum of the full Hamiltonian for the system (64) has little to do with the spectrum of the free system. 13 If trying to describe it in terms of particles, these particles (both the ordinary particles and the ghosts) must have a continuum spectrum of masses. 14

Without any doubt, all this looks very strange and unusual. But strange and unusual does not mean self-contradictory and inconsistent. And one can be almost sure that the TOE, whatever it is, must be strange and unusual. After all, people tried many less strange and more usual candidates for the role of the TOE, including strings, but these efforts have not been successful so far.

We have just recalled confining theories, where the spectra of the free Hamiltonian and of the full Hamiltonian are rather different. Bearing this in mind, the scenario of confinement of ghosts was discussed in the literature. The main idea is that even if the ghosts are present in the bare Hamiltonian, they may disappear from the physical spectrum due to confinement [41, 42]. This conjecture was inspired by the hypothesis [41, 43, 44] that the fundamental gravity action is not the Einstein-Hilbert action, but the conformally invariant Weyl HD action

$$S_{\text{Weyl}} \sim \frac{1}{f^2} \int \sqrt{-g} d^4 x C_{\mu\nu\lambda\sigma} C^{\mu\nu\lambda\sigma}, \quad (73)$$

where

$$C_{\mu\nu\lambda\sigma} = R_{\mu\nu\lambda\sigma} - \frac{1}{2} (g_{\mu\lambda} R_{\nu\sigma} - g_{\mu\sigma} R_{\nu\lambda} - g_{\nu\lambda} R_{\mu\sigma} + g_{\nu\sigma} R_{\mu\lambda}) + \frac{1}{6} (g_{\mu\lambda} g_{\nu\sigma} - g_{\mu\sigma} g_{\nu\lambda}) \quad (74)$$

is the Weyl tensor, or a supersymmetric generalization of this action. Conformal gravity and conformal supergravities are asymptotically free [45], so that classical (super)conformal symmetry is broken there by quantum effects. Like in QCD, this brings about a dimensional parameter — the scale where the coupling constant $f$ becomes large. It is natural to assume that this scale is of order of the Planck mass $M_p$. The EH term in the effective low-energy action along with the cosmological constant term are induced due to quantum effects by Sakharov’s mechanism [46, 47]. 15

13 Also in ordinary QCD-like theories without ghosts but with confinement the spectrum of the full theory has nothing to do with the spectrum of $H_0$. Still, in those cases (i) there is a kinematical region where the perturbative spectrum and perturbative scattering amplitudes make sense; (ii) even though the spectra of excitations for the full Hamiltonian $H$ and for the free Hamiltonian $H_0$ are very different, the ground vacuum state is well defined in both cases and the physical spectrum involves only ordinary particles carrying positive energies. It is difficult to calculate the hadron scattering amplitudes, but they can be defined and measured.

14 One can recall at this point supermembranes, where the mass spectrum is also continuous [38, 39].

15 Nobody has an idea today how to get rid of the huge cosmological constant, which appears in all approaches. This is a separate very hard question.
The propagator of the metric following from (73) behaves as
\[ D(k) \sim \frac{f^2}{k^4} , \] (75)
which brings about the confining static potential
\[ V(r) \propto f^2 \int \frac{d^3k}{k^4} e^{ikr} \propto f^2 r . \] (76)

One may thus suppose that this potential confines the ghost degrees of freedom and only ordinary excitations are left in the physical spectrum.

However,
- We now do not think that any gravity theory, regardless of whether it is the conventional Einstein’s gravity or a HD gravity, is a part of the TOE. We argued in the Introduction that the TOE should rather be a HD theory living in a flat higher-dimensional bulk, while the gravity is an effective theory living on the 3-brane, which is our Universe.
- As was mentioned above, the physical spectrum can be very different from the spectrum of the free Hamiltonian, and many degrees of freedom that are seen in the latter may indeed disappear from the spectrum — be confined. But a general theorem proven in [4,8] and valid for any nondegenerate HD system dictates that the physical spectrum of its Hamiltonian does not have a ground state and must involve ghosts. Thus, confinement or any other mechanism can drastically affect the spectrum of a HD Hamiltonian, but it does not allow one to get rid of all the ghosts.

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