Eigenfunctions of a discrete elliptic integrable particle model with hyperoctahedral symmetry

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I. Preliminaries

Normalized Jacobi theta functions

Let

\[ [z]_1 = [z; \alpha, p]_1 = \frac{\vartheta_1(\frac{\alpha}{2} z; p)}{\sin(\frac{\alpha}{2}) \vartheta_1'(0; p)}, \quad [z]_r = [z; \alpha, p]_r = \frac{\vartheta_r(\frac{\alpha}{2} z; p)}{\vartheta_r'(0; p)} \ (r = 2, 3, 4), \]

with \( z \in \mathbb{C}, \ 0 < \alpha < 2\pi, \) and \( 0 < |p| < 1. \)

The normalization chosen is such that:

\[
\lim_{p \to 0} [z; \alpha, p]_1 = \frac{\sin(\frac{\alpha}{2} z)}{\sin(\frac{\alpha}{2})} \quad \text{and} \quad \lim_{\alpha \to 0} [z; \alpha, p]_1 = z.
\]

Hamiltonian (vD ’94)

\[
H = \sum_{1 \leq j \leq n} B_j(x) T_j + B_j(-x) T_j^{-1}
\]

with

\[
B_j(x) = \left( \prod_{1 \leq k \leq n} \frac{[x_j-x_k+g]_1}{[x_j-x_k]_1} \frac{[x_j+x_k+g]_1}{[x_j+x_k]_1} \right) \prod_{1 \leq r \leq 4} \frac{[x_j+gr]_r}{[x_j]_r}
\]

and \((T_j f)(x_1, \ldots, x_n) = f(x_1, \ldots, x_j-1, x_j+1, x_j+1, \ldots, x_n)\)
Features

- After a gauge trafo, $H$ recovers the quantization of Inozemtsev’s ‘89 Calogero model by scaling the step size to 0:

$$H \rightarrow \sum_{1 \leq j \leq n} \frac{\partial^2}{\partial x_j^2} - g(g-1) \sum_{1 \leq j \neq k \leq n} \left( \wp(x_j - x_k) + \wp(x_j + x_k) \right)$$

$$- \sum_{1 \leq j \leq n} \left( g_0(g_0 - 1)\wp(x_j) + g_1(g_1 - 1)\wp(x_j + \omega_1) + g_2(g_2 - 1)\wp(x_j + \omega_2) + g_3(g_3 - 1)\wp(x_j + \omega_1 + \omega_2) \right)$$

- At $p = 0$, $H$ degenerates to the Macdonald-Koornwinder difference operator.
- $H$ is quantum integrable (Komori & Hikami ’97).
- $H$ enjoys remarkable reflection symmetries in the parameter space (Ruijsenaars ’04).
- $H$ arises in the study of $BC_n$ elliptic hypergeometric integrals and biorthogonal functions, and connects to the Sklyanin algebra and the elliptic DAHA (Rains ’06, ’10, ’20).
- The quantum integrals of $H$ are generated by a quantum Lax matrix (Chalykh ’19).
- Some explicit eigenfunctions of $H$ are known (see e.g. Atai ’20, Atai & Noumi ’22, Ruijsenaars ’15, Spiridonov ’07).
- $H$ arises for $n = 1$ as a reduction of the Lax operator for Sakai’s elliptic difference Painlevé equation (Noumi, Ruijsenaars, & Yamada ’20).
- $H$ describes surface defects in compactifications of conformal matter theories on a punctured Riemann surface (see e.g. Nazzal, Nedelin, & Razamat ’21).
II. Lattice model on bounded partitions

Discretization

We restrict $H$ onto a lattice of shifted partitions:

$$\rho + \Lambda^{(n)}$$

where $\rho = (\rho_1, \ldots, \rho_n)$ with $\rho_j = (n - j)g + g_1$ and

$$\Lambda^{(n)} = \{\lambda \in \mathbb{Z}^n \mid \lambda_1 \geq \cdots \geq \lambda_n \geq 0\}.$$
Truncation at level $m$

Upon picking the real period of the form

$$\alpha = \frac{\pi}{m + (n-1)g + g_1 + g_2} \quad \text{with } m \in \mathbb{N}$$

$(g, g_r > 0)$, we implement a (Racah type) truncation that restricts $H$ to the finite lattice of bounded shifted partitions:

$$\rho + \Lambda^{(n,m)}$$

with

$$\Lambda^{(n,m)} = \{ \lambda \in \mathbb{Z}^n \mid m \geq \lambda_1 \geq \cdots \geq \lambda_n \geq 0 \}.$$
Hilbert space

Let $\ell^2(\Lambda^{(n,m)}, \Delta)$ be the $\binom{n+m}{n}$-dimensional space of functions $f : \Lambda^{(n,m)} \to \mathbb{C}$ endowed with the inner product:

$$\langle f, g \rangle_{\Delta} = \sum_{\lambda \in \Lambda^{(n,m)}} f_\lambda \overline{g}_\lambda \Delta_\lambda$$

$$\Delta_\lambda = \prod_{1 \leq j \leq n} \frac{[2\rho_j + 2\lambda_j]}{[2\rho_j]} \prod_{1 \leq r \leq 4} \frac{[\rho_j + g_r]_{r,\lambda_j}}{[\rho_j + 1 - g_r]_{r,\lambda_j}} \times \prod_{1 \leq j < k \leq n} \frac{[\rho_j \pm \rho_k + \lambda_j \pm \lambda_k]}{[\rho_j \pm \rho_k]} \frac{[\rho_j \pm \rho_k + g]_{1,\lambda_j \pm \lambda_k}}{[\rho_j \pm \rho_k + 1 - g]_{1,\lambda_j \pm \lambda_k}},$$

where $[z]_{r,t} = \prod_{0 \leq k < t} [z + k]_r$ (with $[z]_{r,0} = 1$).
Self-adjointness

Proposition

The action of $H$ in $\ell^2(\Lambda^{(n,m)}, \Delta)$:

$$(Hf)_\lambda = \sum_{1 \leq j \leq n, \varepsilon = \pm 1}^{\lambda + \varepsilon \epsilon_j \in \Lambda^{(n,m)}} B_{\lambda, \varepsilon_j} f_{\lambda + \varepsilon \epsilon_j}$$

with

$$B_{\lambda, \varepsilon_j} = \prod_{1 \leq r \leq 4} \frac{[\rho_j + \lambda_j + \varepsilon g_r]_r}{[\rho_j + \lambda_j]_r} \prod_{1 \leq k \leq n}^{k \neq j} \frac{[\rho_j \pm \rho_k + \lambda_j \pm \lambda_k + \varepsilon g]_1}{[\rho_j \pm \rho_k + \lambda_j \pm \lambda_k]_1}$$

is self-adjoint

$$\forall f, g \in \ell^2(\Lambda^{(n,m)}, \Delta) : \langle Hf, g \rangle_{\Delta} = \langle f, Hg \rangle_{\Delta}.$$
Spectrum

Proposition (Eigenvalues)

(i) The eigenvalues of the difference operator $H$ in $\ell^2(\Lambda^{(n,m)},\Delta)$ are given by real-analytic functions $E_\nu$, $\nu \in \Lambda^{(n,m)}$ in $p \in (-1,1)$ that specialize at $p = 0$ to

$$E_\nu|_{p=0} = 2 \sum_{1 \leq j \leq n} \cos \frac{\alpha}{2} (\hat{\rho}_j + \nu_j),$$

where $\hat{\rho}_j = (n - j)g + \frac{1}{2} (g_1 + g_2)$, $j = 1, \ldots, n$.

(ii) For generic coupling values, the eigenvalues $E_\nu$, $\nu \in \Lambda^{(n,m)}$ from part (i) are distinct as analytic functions of $p \in (-1,1)$.
**Eigenfunctions**

For generic values of the coupling parameters such that the discriminant

\[
\Delta(H) = \prod_{\mu, \nu \in \Lambda^{(n, m)}} (E_\nu - E_\mu) \neq 0
\]

(as an analytic function of \( p \in (-1, 1) \)), let

\[
h^{(\nu)} = \left( \prod_{\mu \in \Lambda^{(n, m)}} \frac{H - E_\mu}{E_\nu - E_\mu} \right) \chi
\]

with

\[
\chi_\lambda = \begin{cases} 
1, & \text{if } \lambda = 0, \\
0, & \text{if } \lambda \neq 0,
\end{cases} \quad (\lambda \in \Lambda^{(n, m)}).
\]
Theorem (Eigenfunctions)

The following statements hold for generic positive parameters subject to the level $m$ truncation condition.

(i) $H$ is diagonalized in the Hilbert space $\ell^2(\Lambda^{(n,m)}, \Delta)$ by an orthonormal basis of eigenfunctions $f^{(\nu)}$, $\nu \in \Lambda^{(n,m)}$ that depend analytically on $p \in (-1, 1)$ such that $Hf^{(\nu)} = E^{\nu}f^{(\nu)}$ (with $E^{\nu}$ as before).

(ii) For generic coupling parameters such that $\Delta(H) \neq 0$, one has that

$$Hh^{(\nu)} = E^{\nu}h^{(\nu)}$$

and

$$\langle h^{(\nu)}, h^{(\tilde{\nu})} \rangle_{\Delta} = \begin{cases} h^{(\nu)}_0 & \text{if } \nu = \tilde{\nu}, \\ 0 & \text{if } \nu \neq \tilde{\nu}. \end{cases}$$

Moreover, the functions $h^{(\nu)} \in \ell^2(\Lambda^{(n,m)}, \Delta)$ coincide with the orthonormal eigenfunctions in part (i) up to normalization:

$$h^{(\nu)} = \frac{f^{(\nu)}_0}{f^{(\nu)}} f^{(\nu)}.$$
(iii) At $p = 0$ the eigenbasis is given explicitly by

$$h^{(\nu)}_{\lambda}|_{p=0} = \frac{c_{\lambda, q}}{N_{\nu, q}} P_{\lambda}(q^{\hat{\rho} + \nu}; q, t, a, b, c, d),$$

where

$$c_{\lambda, q} = \prod_{1 \leq j \leq n} [\rho_j]_r, q, \lambda_j \prod_{1 \leq j \leq k \leq n} [\rho_j \pm \rho_k]_r, q, \lambda_j \prod_{r=1,2} \prod_{1 \leq j \leq n} [\rho_j]_r, q, \lambda_j \prod_{1 \leq j \leq k \leq n} [\rho_j \pm \rho_k + g]_r, q, \lambda_j \pm \lambda_k$$

and $P_{\lambda}$ denotes the monic Macdonald-Koornwinder polynomial with parameters

$$q = e^{i\alpha}, \quad t = q^g, \quad a = q^{(g_1 + g_2)/2}, \quad b = -q^{(g_1 + g_2)/2}, \quad c = q^{(g_1 - g_2 + 1)/2}, \quad d = -q^{(g_1 - g_2 + 1)/2}.$$

Here

$$N_{\nu, q} = \sum_{\lambda \in \Lambda(n, m)} c_{\lambda, q}^2 P_{\lambda}^2(q^{\hat{\rho} + \nu}; q, t, a, b, c, d) \Delta_{\lambda, q},$$

$$\Delta_{\lambda, q} = \prod_{1 \leq j \leq n} [2\rho_j + 2\lambda_j]_1, q \prod_{r=1,2} [\rho_j]_r, q, \lambda_j \prod_{1 \leq j \leq k \leq n} [\rho_j \pm \rho_k + \lambda_j \pm \lambda_k]_1, q [\rho_j \pm \rho_k + g]_1, q, \lambda_j \pm \lambda_k,$$

$$[z]_1, q = \frac{\sin(\frac{\alpha}{2}z)}{\sin(\frac{\alpha}{2})} = \frac{q^{\frac{z}{2}} - q^{-\frac{z}{2}}}{q^\frac{1}{2} - q^{-\frac{1}{2}}}, \quad [z]_2, q = \cos(\frac{\alpha}{2}z) = \frac{q^{\frac{z}{2}} + q^{-\frac{z}{2}}}{2}.$$
Level \( m \) truncated difference Heun equation

For \( \lambda = 0, \ldots, m \):

\[
\begin{align*}
f_{\lambda+1} \prod_{1 \leq r \leq 4} \frac{[\lambda+g_1+g_r]_r}{[\lambda+g_1]_r} + f_{\lambda-1} \prod_{1 \leq r \leq 4} \frac{[\lambda+g_1-g_r]_r}{[\lambda+g_1]_r} &= E f_{\lambda} \\
\end{align*}
\]

(with \( \alpha = \frac{\pi}{m+g_1+g_2} \) and \( g_r > 0 \)).

Orthogonality weights

\[
\Delta_{\lambda} = \prod_{1 \leq j \leq n} \frac{[2g_1+2\lambda]_1}{[2g_1]_1} \prod_{1 \leq r \leq 4} \frac{[g_1+g_r]_{r,\lambda}}{[g_1+1-g_r]_{r,\lambda}}
\]
Elliptic Racah polynomials

For \( k = 0, 1, 2, \ldots \) we define the elliptic Racah polynomials:

\[
P_k(E) = \det \begin{bmatrix}
    E & -b_0^+ & 0 & \cdots & 0 \\
    -b_1^- & E & \ddots & \ddots & \\
    0 & -b_2^- & \ddots & -b_{k-3}^+ & 0 \\
    \vdots & \ddots & \ddots & E & -b_{k-2}^+ \\
    0 & \cdots & 0 & -b_{k-1}^- & E
\end{bmatrix}
\]

(so \( P_0(E) = 1 \)) with

\[
b_k^+ = \prod_{1 \leq r \leq 4} \frac{[g_1+g_r+k]_r}{[g_1+k]_r} \quad \text{and} \quad b_k^- = \prod_{1 \leq r \leq 4} \frac{[g_1-g_r+k]_r}{[g_1+k]_r}.
\]

Eigenvalues

The eigenvalues of \( H \) are given by the roots

\[
E_0 > E_1 > \cdots > E_m
\]

of \( P_{m+1}(E) \).
Eigenfunctions

\[ h_\lambda^{(\nu)} = \frac{c_\lambda}{N_\nu} P_\lambda(E_\nu) \quad (0 \leq \lambda, \nu \leq m) \]

with

\[ c_\lambda = \prod_{1 \leq r \leq 4} \frac{[g_1]_{r,\lambda}}{[g_{1+r}]_{r,\lambda}} \]

and

\[ N_\nu = \sum_{0 \leq \lambda \leq m} c_\lambda^2 P_\lambda^2(E_\nu) \Delta_\lambda = c_m^2 \Delta_m P_m(E_\nu) \prod_{0 \leq j \leq m} (E_\nu - E_j). \]
For $g=1$ the gauge transformation

$$H \rightarrow \tilde{H} = V_\lambda H V_\lambda^{-1} \quad \text{with} \quad V_\lambda = \prod_{1 \leq j < k \leq n} \frac{[\rho_j \pm \rho_k + \lambda_j \pm \lambda_k]}{[\rho_j \pm \rho_k]}$$

transforms the particle model to $n$ free fermions on the open lattice \{0, 1, 2, \ldots, m + n - 1\} placed in an external field of elliptic Racah type:

$$\left(\tilde{H}\tilde{f}\right)_\lambda = \sum_{1 \leq j \leq n, \lambda + e_j \in \Lambda^{(n, m)}} b_{n-j+\lambda}^+ \tilde{f}_\lambda + e_j + \sum_{1 \leq j \leq n, \lambda - e_j \in \Lambda^{(n, m)}} b_{n-j+\lambda}^- \tilde{f}_\lambda - e_j$$

(where $\tilde{f}_\lambda = V_\lambda f_\lambda$).
Let $g=1$, $\alpha = \frac{\pi}{m+n-1+g_1+g_2}$ ($gr > 0$) and let

$$E_0 > E_1 > \cdots > E_{n+m-1}$$

denote the roots of $P_{n+m}(E)$.

**Theorem (Diagonalization for $g=1$)**

(i) The eigenvalues of $H$ become in terms of the elliptic Racah roots:

$$E_\nu = \sum_{1 \leq j \leq n} E_{n-j+\nu_j} \quad (\nu \in \Lambda^{(n,m)}).$$

(ii) The eigenfunctions are given by *Schur polynomials of elliptic Racah type*:

$$h_\lambda^{(\nu)} = \frac{c_{\lambda\nu}}{n_\nu} s_\lambda^{(\nu)} \quad (\lambda, \nu \in \Lambda^{(n,m)})$$

with

$$s_\lambda^{(\nu)} = \frac{A_\lambda^{(\nu)}}{A_0^{(\nu)}}, \quad A_\lambda^{(\nu)} = \det [P_{n-i+\lambda_i}(E_{n-j+\nu_j})]_{1 \leq i,j \leq n}.$$

Here the normalizations are governed by

$$c_\lambda = \frac{1}{V_\lambda} \prod_{1 \leq j \leq n} \frac{[\rho_j]_{r,\lambda_j}}{[\rho_j + gr]_{r,\lambda_j}}$$

and

$$N_\nu = \sum_{\lambda \in \Lambda^{(n,m)}} c_\lambda^2 (s_\lambda^{(\nu)})^2 \Delta_\lambda = \frac{1}{(A_0^{(\nu)})^2} \prod_{1 \leq j \leq n} \frac{N_{n-j+\nu_j}}{\Delta_{n-j} c_{n-j}^2}. $$
VI. The level $m = 1$ case: one-column partitions

The lattice $\Lambda^{(n,1)}$ consists of one-column partitions

$$(1^k) = \underbrace{1, \ldots, 1}_k, 0, \ldots, 0 \quad (0 \leq k \leq n).$$

The level $m = 1$ eigenvalue equation for $H$ becomes of triangular form:

$$B_{-k} f_{(1^k-1)} + B_{k+1} f_{(1^{k+1})} = Ef_{(1^k)} \quad (0 \leq k \leq n),$$

with

$$B_{-k} = B_{(1^k),-k} = \frac{[2\rho_{k+1}+2, \rho_0 - \rho_k, \rho_k + \rho_{n+1} + 1, 1]}{[\rho_1 + \rho_k + 2, \rho_k + \rho_{k+1} + 1, \rho_k - \rho_{n+1}, 1]} \prod_{1 \leq r \leq 4} \frac{[\rho_{k+1} - gr]_r}{[\rho_{k+1}]_r},$$

and

$$B_k = B_{(1^k),k+1} = \frac{[\rho_0 + \rho_{k+1} + 1, 2\rho_{k+1}, \rho_{k+1} - \rho_{n+1}, 1]}{[\rho_k + \rho_{k+1} + 1, \rho_1 - \rho_{k+1} + 1, \rho_{k+1} + \rho_{n+1}, 1]} \prod_{1 \leq r \leq 4} \frac{[\rho_{k+1} + gr]_r}{[\rho_{k+1}]_r}.$$
**Upshot:** at level $m = 1$ the diagonalization can again be performed by means of polynomials

$$P_{(1^k)}(E) = \det \begin{bmatrix} E & -B_1 & 0 & \cdots & 0 \\ -B_{-1} & E & \ddots & \vdots \\ 0 & -B_{-2} & \ddots & -B_{k-2} & 0 \\ \vdots & \vdots & \ddots & E & -B_{k-1} \\ 0 & \cdots & 0 & -B_{-k+1} & E \end{bmatrix},$$

$k = 0, \ldots, n + 1$. 
Theorem (Diagonalization at level $m=1$)

(i) The eigenvalues of $H$ are given by the simple roots of $P_{(1n+1)}(E)$:

$$E_{(10)} > E_{(11)} > \cdots > E_{(1n)}.$$ 

(ii) The eigenfunctions $h^{(1l)}$, $0 \leq l \leq n$ of $H$ are given by

$$h^{(1l)}_{(1k)} = \frac{c^{(1k)}}{N(1l)} P_{(1k)}(E_{(1l)}) \quad (0 \leq k, l \leq n)$$

with

$$N(1l) = \sum_{0 \leq k \leq n} c^{2}_{(1k)} P_{(1k)}^{2}(E_{(1l)}) \Delta_{(1k)}$$

$$= c^{2}_{(1n)} \Delta_{(1n)} P_{(1n)}(E_{(1l)}) \prod_{0 \leq j \leq n} \left( E_{(1l)} - E_{(1j)} \right),$$

where

$$c^{(1k)} = \prod_{1 \leq j \leq k} B_{j}^{-1} \quad \text{and} \quad \Delta_{(1k)} = \prod_{1 \leq j \leq k} B_{j} B_{j-1}^{-1}.$$
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Thank You!