Noncommutative Riemann Conditions

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Abstract

In this paper we study the holomorphic bundles over a noncommutative complex torus. We define a noncommutative abelian variety as a kind of deformation of abelian variety and we show that for a restricted deformation parameter, one can define a noncommutative abelian variety. Also, along the cohomological deformation, we discuss the noncommutative analogue of usual Riemann conditions. This will be done by using the real cohomologies instead of the rational ones.
1 Introduction

The noncommutative tori is known to be the most accessible examples of noncommutative geometry developed by A. Connes [1]. It also provides the best example in applications of noncommutative geometry to string/M-theory which was initiated in [2]. Analogously the geometry and gauge theory of noncommutative torus has been explicitly studied in many papers, such as [2], [3], [4], [5]. Rather recently, complex geometry of noncommutative torus was developed by A. Schwarz [6] and some detailed analysis were made in [7] for the two-dimensional case (see also [8] for four-dimensional ones). It provided a basic result for subsequent papers [9], [10] and [11] which study the Kontevisch’s homological mirror symmetry conjecture [12] on the two-dimensional noncommutative tori. One of our main underlying motivations is to study the mirror symmetry on the higher dimensional cases and the goal of this paper is to set up a very first step for that study.

Our main interest in this paper is to compare complex geometries of both noncommutative and ordinary complex torus, focused on the four-dimensional case. As discussed in [13], the ordinary Riemann conditions are to determine whether a complex torus is an abelian variety or not. We show that the definition of holomorphic structures given in [6] is a noncommutative analogue of Riemann conditions, which determines the existence of theta vectors. Since noncommutative torus is defined as a deformation quantization of an ordinary torus [14], we may regard a holomorphic bundle over a noncommutative complex torus as a deformation of a bundle over a commutative complex torus. Compared with other examples of noncommutative geometry, the Chern character of noncommutative torus is easily understood because it takes the values in the real cohomological group of the commutative torus [4]. The Chern character can be obtained from the ordinary Chern character via deformation parameter. Our deformed bundle will be given in terms of constant curvature connections such that the curvature matrix reflects the Chern character deformation. Such a bundle equipped with a constant curvature connection is easily seen as a deformation of a projectively flat bundle over a complex torus. From this point of view, we will define a noncommutative abelian variety as a deformation of abelian variety and we find that such a noncommutative abelian variety can be defined for a restricted deformation parameter. We will also study noncommutative analogue of ordinary Riemann conditions given in [13]. This will be done by using the real cohomologies instead of the rational ones, considering the usual Riemann conditions are obtained by comparing the rational structure and the complex structure of the cohomology of a torus.

This paper is organized as follows. In Section 2, we review a general concepts on noncommutative complex tori defined in [6] and we define a deformed bundle using the Chern character deformation. By regarding an abelian variety as a pair of complex torus and a line bundle on it, we define a noncommutative abelian variety, which is a deformation of abelian variety. In Section 3, we first find an explicit solutions for the
theta vectors on a four-dimensional noncommutative complex torus. We also discuss
the analogue the definition of holomorphic bundles in two different ways depending on
the complex structures to define a noncommutative complex torus. We conclude in
Section 4.

2 Preliminaries

Let $T^d$, $d = 2g$, be a $d$-dimensional noncommutative torus. It is generated by $d$-unitaries
$U_1, \cdots, U_d$ such that

$$U_i U_j = \exp(2\pi i \theta_{ij}) U_j U_i,$$

where $\theta = (\theta_{ij})$ is a real $d \times d$ skew-symmetric matrix. The relation (1) defines the
presentation of the involutive algebra

$$A^d_\theta = \left\{ \sum_{(n_1, \cdots, n_d) \in \mathbb{Z}^d} a_{n_1} U_1^{n_1} \cdots U_d^{n_d} \mid a_{n_1} \cdots n_d \in \mathcal{S}(\mathbb{Z}^d) \right\},$$

where the coefficient function $(n_1, \cdots, n_d) \mapsto a_{n_1} \cdots n_d$ rapidly decays at infinity. By
definition, the algebra $A^d_\theta$ is the algebra of smooth functions on $T^d_\theta$ and the bundles
over $T^d_\theta$ correspond to finitely generated projective (left) $A^d_\theta$-modules.

The ordinary torus $T^d$ acts on the algebra $A^d_\theta$ (cf. [4]) and the infinitesimal form of
the action of $T^d$ on $A^d_\theta$ defines a Lie algebra homomorphism

$$\delta : L \longrightarrow \text{Der}(A^d_\theta),$$

where $L \cong \mathbb{R}^d$ is the Lie algebra of $T^d$ and $\text{Der}(A^d_\theta)$ denotes the Lie algebra of derivations of $A^d_\theta$. Thus for $x \in L$,

$$\delta_x(uv) = \delta_x(u)v + u\delta_x(v), \quad u, v \in A^d_\theta.$$

For a basis $\{\lambda_i\}_{i=1, \cdots, 2g}$ for $L$, set $\delta_{\lambda_i} := \delta_i$. Then

$$\delta_j(U_j) = 2\pi i U_j, \quad \delta_i(U_j) = 0 \quad \text{for } i \neq j,$$

or

$$\delta_j \left( \sum_{n=(n_1, n_2, \cdots, n_d) \in \mathbb{Z}^d} a_n U_1^{n_1} \cdots U_d^{n_d} \right) = \sum_{n \in \mathbb{Z}^d} 2\pi i n_j a_n U_1^{n_1} \cdots U_d^{n_d}.$$

Note that the Lie algebra $\text{Der}(A^d_\theta)$ plays the role of tangent bundle to $T^d_\theta$.

Since $T^d$ is commutative, the cohomology group $H^*(T^d, \mathbb{R})$ can be identified with
the exterior algebra $\wedge^* L^*$, where $L^*$ is the dual vector space of $L$. Let $(x_1, \cdots, x_{2g})$
be the dual real coordinates on \( L \) and \( dx_1, \cdots, dx_{2g} \) the corresponding 1-forms on \( T^d \).

Then

\[
H^*(\mathbb{T}^d, \mathbb{R}) = \mathbb{R}\{dx_1\}_1, \quad H^*(\mathbb{T}^d, \mathbb{Z}) = \mathbb{Z}\{dx_1\}_1. \tag{2}
\]

A complex structure on \( L \) can be given in the form

\[
L \oplus iL \cong L^{1,0} \oplus L^{0,1} = V \oplus \bar{V}, \quad V \cong \mathbb{C}^g.
\]

Thus we may consider \( T^d \) as a complex torus \( T^d = V/\Lambda \), where \( \Lambda \) is a lattice of maximal rank \( 2g \) which is defined by the kernel of the exponential map \( \exp : V \rightarrow \mathbb{T}^d \). Let \( \lambda_1, \cdots, \lambda_{2g} \) be an integral basis for \( \Lambda \), which will also be a basis for the real vector space \( L \) and let \( e_1, \cdots, e_g \) be a complex basis for \( V \). We take the period matrix of \( \Lambda \subset V \) to be a \( g \times d \) complex matrix \( \mathcal{U} = (\mathcal{U}_{\alpha i}) \) such that

\[
\lambda_i = \sum_{\alpha} \mathcal{U}_{\alpha i} e_{\alpha}.
\]

Let \( z = (z_1, \cdots, z_g) \) be Euclidean coordinates on \( V \) and let \( \{dz_1, \cdots, dz_g\} \) and \( \{d\bar{z}_1, \cdots, d\bar{z}_g\} \) be corresponding 1-forms on \( \mathbb{T}^d \). Then

\[
dz_\alpha = \sum_i \mathcal{U}_{\alpha i} dx_i, \quad d\bar{z}_\alpha = \sum_i \overline{\mathcal{U}_{\alpha i}} dx_i
\]

so that the matrix \( \begin{pmatrix} \mathcal{U} \\ \overline{\mathcal{U}} \end{pmatrix} \) gives the change of basis from \( \{dx_i\} \) to \( \{dz_\alpha, d\bar{z}_\alpha\} \). We can also identify

\[
H^*(\mathbb{T}^d, \mathbb{C}) = \wedge^* V^* \otimes \wedge^* \overline{V^*} = \mathbb{C}\{dz_I \wedge d\bar{z}_J\}_{1,1}.
\]

In the case of complex tori, the Kodaira’s embedding theorem is that the complex torus \( T^d = V/\Lambda \) is an abelian variety if and only if there is a closed, positive \((1, 1)\)-form \( \omega \) whose cohomology class \([\omega]\) is rational, so that \([\omega] \in H^{1,1}(T^d) \cap H^2(T^d, \mathbb{Z})\). We will understand an abelian variety as a pair \((T^d, \mathcal{L})\), where \( T^d \) is a complex torus and \( \mathcal{L} \) is a line bundle whose first Chern class is in \([\omega] \in H^{1,1}(T^d) \cap H^2(T^d, \mathbb{Z})\). In terms of the period matrix \( \mathcal{U} \), the Riemann condition II in [13] is stated as follows: The complex torus \( T^d = V/\Lambda \) is an abelian variety if and only if there exists an integral, skew-symmetric matrix \( q \) such that

\[
\mathcal{U}q^{-1}\overline{\mathcal{U}} = 0, \quad -i\partial\overline{\partial}q^{-1}\overline{\mathcal{U}} > 0.
\]
For any integral 2-form on $\mathbb{T}^d$, we can find an integral basis $\lambda_1, \cdots, \lambda_{2g}$ for $\Lambda$ such that
\[
\omega = \sum_{i=1}^{g} m_i dx_i \wedge dx_{g+i}, \quad m_i \in \mathbb{Z}.
\]
By taking a basis for $V$ the vectors
\[
e_\alpha = m_\alpha^{-1} \lambda_\alpha, \quad \alpha = 1, \cdots, g
\]
the period matrix $\mathcal{U}$ will be of the form
\[
\mathcal{U} = (M \ Z) = \begin{pmatrix}
m_1 & 0 & \cdots & 0 & Z_{11} & \cdots & Z_{1g} \\
0 & m_2 & \cdots & 0 & \cdot & \cdots & \cdot \\
\cdot & \cdot & \cdots & \cdot & \cdot & \cdots & \cdot \\
0 & \cdots & 0 & m_g & Z_{g1} & \cdots & Z_{gg}
\end{pmatrix}.
\]
The Riemann condition II implies the Riemann condition III in [13]: the complex torus $\mathbb{T}^d$ is an abelian variety if and only if the $g \times g$ complex matrix $Z$ is symmetric and $\text{Im} \ Z$ is positive definite.

The period matrix $\mathcal{U} = (\mathcal{U}_{\alpha \beta})$ of $\Lambda$ in $V$ gives the change of basis $\{dx_i\}$ to $\{dz_\alpha\}$. On the other hand the Lie algebra of derivations of $A^d_g$ is spanned by the derivations $\delta \lambda_1, \cdots, \delta \lambda_d$ and since the derivations $\delta \lambda_\alpha$ correspond to vector fields $\frac{\partial}{\partial x_i}$, we will have to use dual-change-of-basis matrix $\left(\Omega^t \ \Omega\right)$ of $\mathcal{U}$ which gives the change of basis from $\{\frac{\partial}{\partial z_\alpha}\}$ to $\{\frac{\partial}{\partial z_\alpha}, \frac{\partial}{\partial \bar{z}_\alpha}\}$. Those two matrices are related by
\[
\left(\frac{\mathcal{U}}{\Omega}\right) \cdot \left(\frac{\Omega}{\Omega^t}\right) = \text{Id}_d
\]
so that we can identify the matrix $\left(\Omega^t \ \Omega\right)$ with the matrix which changes the basis from $\{dz_\alpha, d\bar{z}_\alpha\}$ to $\{dx_i\}$. Now the Riemann condition I in [13] is given in terms of dual period matrix $\Omega$: The complex torus $\mathbb{T}^d = V/\Lambda$ is an abelian variety if and only if there exists an integral, skew-symmetric matrix $q$ such that
\[
\Omega q \Omega^t = 0, \quad -i \Omega q \overline{\Omega^t} > 0.
\]

We now define a noncommutative complex torus using the matrix $\Omega = (\omega_{\alpha \beta})$. Let
\[
\delta_{\Omega, \alpha} = \sum_i \omega_{\alpha i} \delta_i, \quad \alpha = 1, \cdots, g.
\]
Then
\[
\delta_{\Omega, \alpha} \left( \sum_{n \in \mathbb{Z}^d} a_{n_1, \ldots, n_d} U_1^{n_1} U_2^{n_2} \cdots U_d^{n_d} \right) = \sum_i \omega_{\alpha i} \delta_i \left( \sum_{n \in \mathbb{Z}^d} a_{n_1, \ldots, n_d} U_1^{n_1} U_2^{n_2} \cdots U_d^{n_d} \right) = \sum_{n \in \mathbb{Z}^d} 2\pi i \left( \sum_i \omega_{\alpha i} n_i \right) a_{n_1, \ldots, n_d} U_1^{n_1} U_2^{n_2} \cdots U_d^{n_d}.
\]

The noncommutative torus \( T^d_\theta \) equipped with a complex structure \( \Omega \) is called a non-commutative complex torus and is denoted by \( T^d_\theta, \Omega \) and the algebra of smooth functions on \( T^d_\theta, \Omega \) is denoted by \( A^d_\theta, \Omega \).

Let \( E \) be a finitely generated projective left \( A^d_\theta \)-module. A connection \( \nabla \) on \( E \) is a linear map
\[
\nabla : E \rightarrow E \otimes L^*
\]
such that, for \( x \in L \),
\[
\nabla_x (u \cdot e) = u \cdot \nabla_x e + \delta_x (u) \cdot e, \quad e \in E, \quad u \in A^d_\theta.
\]
For the integral basis \( \{ \lambda_i \} \) for \( L \), we set \( \nabla_{\lambda_i} = \nabla_i \). The curvature of the connection \( \nabla \) is constant if
\[
[\nabla_i, \nabla_j] = 2\pi i F_{ij} \cdot 1, \quad F_{ij} \in \mathbb{R}.
\]
A complex (holomorphic) structure on a right \( A^d_\theta \)-module \( E \) (compatible with the complex structure on \( A^d_\theta, \Omega \)) is a collection of \( \mathbb{C} \)-linear operators \( \nabla_1, \cdots, \nabla_g \) on \( E \) satisfying

1. \( \nabla_\alpha(u \cdot e) = u \cdot \nabla_\alpha e + \delta_{\Omega, \alpha}(u) \cdot e \).
2. \( [\nabla_\alpha, \nabla_\beta] = 0 \).

A projective left \( A^d_\theta \)-module \( E \) equipped with holomorphic structure is called a holomorphic bundle. A vector \( \phi \in E \) is called a holomorphic vector if \( \nabla \phi = 0 \). We will see in the next section that the condition 2 above assures the existence of holomorphic vectors in \( E \).

Since \( T^d_\theta \) is a deformation quantization of the commutative torus \( T^d \), a bundle over \( T^d_\theta \) may be understood as a deformation of a bundle over \( T^d \). In fact, the vector bundles over \( T^d \) are classified by the \( K \)-theory of \( T^d \) and there is a natural isomorphism \( \text{ch} : K^*(T^d) \rightarrow H^*(T^d, \mathbb{Z}) \otimes \mathbb{Q} \). Similarly, finitely generated projective \( A^d_\theta \)-modules are classified by \( K_0(A^d_\theta) \) and the Chern character takes values in \( H^*(T^d, \mathbb{R}) \). The targets of the both Chern characters are related by the deformation parameter \( \theta \in \Lambda^2 L^* \). The relation is summarized by the following diagram:

\[
\begin{array}{ccc}
K^0(T^d) & \xrightarrow{\text{ch}} & H^{\text{even}}(T^d, \mathbb{Z}) \otimes \mathbb{Q} \\
 & e^{i\theta} \downarrow & \\
K_0(A^d_\theta) & \xrightarrow{\text{Ch}} & H^{\text{even}}(T^d, \mathbb{R})
\end{array}
\]
where $i(\theta)$ denotes the contraction with 2-vector $\theta$. Thus, for a given vector bundle $E$ over $\mathbb{T}^d$, one can construct an $A_d^\theta$-module $\mathcal{E}$ such that
\[
\text{Ch}(\mathcal{E}) = e^{i(\theta)} \text{ch}(E)
\]
and such a module $\mathcal{E}$ will be called a deformation of $E$ when the zeroth component of $\text{Ch}(\mathcal{E})$ is strictly positive. We note that such a deformation of a bundle may not be unique. Now, in terms of cohomology, the deformation of $\mathbb{T}^d$ to $\mathbb{T}^d_\theta$ is related with the deformation of $H^*(\mathbb{T}^d, \mathbb{Z})$ to $H^*(\mathbb{T}^d, \mathbb{R})$ and the condition $[\nabla_\alpha, \nabla_\beta] = 0$ can be understood as the existence of a cohomology class in $H^{1,1}(\mathbb{T}^d) \cap H^2(\mathbb{T}^d, \mathbb{R})$. Finally, for an abelian variety $(\mathbb{T}^d, L)$, we can construct a pair $(\mathbb{T}^d_\theta, \Omega, L)$, where $L$ is a deformation of the line bundle $L$. Such a pair $(\mathbb{T}^d_\theta, \Omega, L)$ will be referred as a noncommutative abelian variety.

3 Holomorphic structures on $\mathbb{T}^d_\theta$

In this section, we apply the general discussion, given in Section 2, to the four dimensional case. We will construct a bundle over $\mathbb{T}^d_\theta$ which is a deformation of a projectively flat bundle over $\mathbb{T}^4$ and we will find an explicit formula of holomorphic vectors in the deformed bundle. Finally, we discuss how Riemann conditions correspond to the existence of holomorphic vectors.

Let $E$ be a projectively flat $U(n)$ bundle on the complex torus $\mathbb{T}^4 = V/\Lambda$, here we follow the same notations as in Section 2 adapted to the four dimensional case. The bundle $E$ carries a Hermitian connection $\nabla^E$ whose curvature form is given by
\[
R_{\nabla^E} = \lambda \cdot \text{Id}_E
\]
where $\lambda$ is a complex 2-form on $\mathbb{T}^4$ and $\text{Id}_E$ denotes the identity endomorphism of $E$.

The first and the second Chern classes are given as follows:
\[
c_1(E) = \frac{i}{2\pi} \text{Tr } R_{\nabla^E} = \frac{i}{2\pi} \lambda \cdot \text{rank } E = \frac{i}{2\pi} \lambda n
\]
\[
c_2(E) = -\frac{1}{8\pi^2} (\text{Tr } (R_{\nabla^E} \wedge R_{\nabla^E}) - \text{Tr } R_{\nabla^E} \wedge \text{Tr } R_{\nabla^E}) = -\frac{1}{4\pi^2} \frac{n(n-1)}{2} \lambda^2
\]
\[
\frac{1}{2} c_1^2(E) - c_2(E) = -\frac{1}{8\pi^2} n\lambda^2.
\]

Note that the Chern character of $E$ is given by
\[
\text{ch}(E) = \text{rank}(E) + c_1(E) + \left(\frac{1}{2} c_1^2(E) - c_2(E)\right) \in H^{\text{even}}(\mathbb{T}^4, \mathbb{Q}).
\]
The first Chern class $c_1(E)$ is an integral, invariant 2-form on $\mathbb{T}^4$ and we can find an integral basis $\lambda_1, \lambda_2, \lambda_3, \lambda_4$ for $\Lambda$ such that
\[
c_1(E) = \frac{1}{2} (m_1 dx_{13} + m_2 dx_{24}),
\]
where \( dx_{\mu\nu} = dx_{\mu} \wedge dx_{\nu} \). Then we have

\[
\text{ch}(E) = n + \frac{1}{2}(m_1 dx_{13} + m_2 dx_{24}) + \frac{m_1 m_2}{n} dx_{1234}.
\]

We take a basis for the complex vector space \( V \) the vectors

\[
e_\alpha = m^{-1}_\alpha \lambda_\alpha, \quad \alpha = 1, 2.
\]

Then the period matrix \( \mathcal{U} \) for \( \Lambda \) in \( V \) will be of the form

\[
\mathcal{U} = \begin{pmatrix}
m_1 & 0 & Z_{11} & Z_{12} \\
0 & m_2 & Z_{21} & Z_{22}
\end{pmatrix} := (\Delta_m \ Z)
\]

and by a suitable basis change, we can take the dual period matrix \( \Omega \) as in the following form

\[
\Omega = \begin{pmatrix}
\frac{1}{m_1} & 0 & Z_{11} & Z_{12} \\
0 & \frac{1}{m_2} & Z_{21} & Z_{22}
\end{pmatrix} := (\Delta^{-1}_m \ Z),
\]

where \( Z = -\mathcal{Z} \) (cf. [13]). Let us denote the corresponding integral, skew-symmetric \( 4 \times 4 \) matrix \( q \) by

\[
q = \begin{pmatrix}
0 & \Delta_m \\
-\Delta_m & 0
\end{pmatrix}.
\]

On the other hand, the bundle \( E \) can be deformed to a projective module \( \mathcal{E} \) over the noncommutative torus \( T^4_{\theta} \) whose Chern character is given in the following form

\[
\text{Ch}(\mathcal{E}) = e^{i(\theta)} \text{ch}(E) \in H^{\text{even}}(T^4, \mathbb{R}),
\]

where \( i(\theta) \) denotes the contraction with \( \theta \in \wedge^2 L^* \). More explicitly, we have

\[
\text{Ch}(\mathcal{E}) = (n + \frac{1}{2} \text{Tr}(q\theta) + \frac{m_1 m_2}{n} \text{Pf}(\theta)) + \frac{1}{2}(q + \star \theta \frac{m_1 m_2}{n})_{ij} dx_{ij} + \frac{m_1 m_2}{n} dx_{1234},
\]

where \( \text{Pf}(\theta) = \theta_{13}\theta_{24} - \theta_{12}\theta_{34} - \theta_{14}\theta_{23} \) is the Paffian of \( \theta \). If there is a constant curvature connection in \( \mathcal{E} \), then the corresponding curvature up to a factor \( 2\pi i \) can be identified with matrix

\[
F = 2\pi i \frac{q + \star \theta \frac{m_1 m_2}{n} \text{Pf}(\theta)}{n + \frac{1}{2} \text{Tr}(q\theta) + \frac{m_1 m_2}{n} \text{Pf}(\theta)} = 2\pi i \frac{q + \star \theta \frac{m_1 m_2}{n}}{\dim \mathcal{E}}.
\]

Let \( \gamma = (\gamma_{ij}) = \psi - \theta \), where

\[
\psi = (\psi_{ij}) = \begin{pmatrix}
0 & 0 & \frac{n}{m_1} & 0 \\
0 & 0 & 0 & \frac{n}{m_2} \\
-\frac{n}{m_1} & 0 & 0 & 0 \\
0 & -\frac{n}{m_2} & 0 & 0
\end{pmatrix} = -nq^{-1}.
\]
Then
\[
Pf(\gamma) = Pf(\psi - \theta) = (\frac{n}{m_1} - \theta_{13})(\frac{n}{m_2} - \theta_{24}) - \theta_{12}\theta_{34} - \theta_{14}\theta_{23}
\]
\[
= \frac{n}{m_1m_2} \left( n + \frac{1}{2}Tr(q\theta) + \frac{m_1m_2}{n}Pf(\theta) \right)
\]
and since \(\gamma_{ij} = \psi_{ij} - \theta_{ij}\), we have
\[
\gamma^{-1} = \frac{1}{Pf(\gamma)} \begin{pmatrix}
0 & \gamma_{34} & -\gamma_{24} & \gamma_{23} \\
-\gamma_{34} & 0 & \gamma_{14} & -\gamma_{13} \\
\gamma_{24} & -\gamma_{14} & 0 & \gamma_{12} \\
-\gamma_{23} & \gamma_{13} & -\gamma_{12} & 0
\end{pmatrix} = -\frac{q + \theta_{m_1m_2}}{n + \frac{1}{2}Tr(q\theta) + \frac{m_1m_2}{n}Pf(\theta)}.
\]
Thus
\[
F = \frac{2\pi}{i} \gamma^{-1}.
\]

Note that if, for \(\theta\) not rational, the zeroth component of the Chern character is strictly positive, then the gauge bundle \(E\) belongs to the positive cone of \(K_0(A_4)\) and it can be written as a direct sum of the form \(S(\mathbb{R}^2 \times F)\) ([4]), where \(F\) is a finite abelian group. Thus by assuming \(\text{dim} \ E > 0\), let \(E = S(\mathbb{R}^2 \times F)\). The projective flat bundle \(E\) over \(\mathbb{T}^4\) can be specified by operators \(W_1, W_2, W_3, W_4\) acting on \(C(\mathbb{Z}_{m_1} \times \mathbb{Z}_{m_2}) = \mathbb{C}^{m_1} \otimes \mathbb{C}^{m_2}\):
\[
W_1f(k_1, k_2) = f(k_1 - n, k_2) \quad W_2f(k_1, k_2) = f(k_1, k_2 - n)
\]
\[
W_3f(k_1, k_2) = \exp(-2\pi i \frac{k_1}{m_1})f(k_1, k_2) \quad W_4f(k_1, k_2) = \exp(-2\pi i \frac{k_2}{m_2})f(k_1, k_2).
\]
The operators obey the commutation relation
\[
W_iW_j = \exp(2\pi i \gamma_{ij})W_jW_i.
\]
Let us consider an embedding of \(\Lambda\) into \(\mathbb{R}^2 \times (\mathbb{R}^2)^*\) in the sense of [4]. Such an embedding map can be represented by a non-singular matrix \(T = (T_{ij})\) such that
\[
(\wedge^2 T^*)(e_3 \wedge e_2 + e_4 \wedge e_1) = -\gamma \in \wedge^2 \mathbb{R}^2.
\]
Associated to the embedding \(T\), the left module action on \(S(\mathbb{R}^2)\) is defined by the Heisenberg representation of \(\Lambda\) in \(S(\mathbb{R}^2)\):
\[
(V_i f)(r, s) := (V_i f)(r, s) = \exp[2\pi i (r T_{3i} + s T_{4i})] f(r + T_{1i}, s + T_{2i}),
\]
which obeys the commutation relations:
\[
V_i V_j = \exp(-2\pi i \gamma_{ij}) V_j V_i.
\]
and 
\[ \gamma_{ij} = \begin{vmatrix} T_{1i} & T_{1j} \\ T_{3i} & T_{3j} \end{vmatrix} + \begin{vmatrix} T_{2i} & T_{2j} \\ T_{4i} & T_{4j} \end{vmatrix}. \]

Let us consider the operator 
\[ U_i = V_i \otimes W_i \] 
acting on \( E = \mathcal{S}(\mathbb{R}^2 \times \mathbb{Z}_{m_1} \times \mathbb{Z}_{m_2}) \). The operators \( U_i \)'s hold the commutation relation 
\[ U_i U_j = \exp(2\pi i \theta_{ij}) U_j U_i \]
and hence it defines a left \( A_4^\theta \)-module action on \( E \). Analogously, we may define a constant curvature connection on \( E \) as follows: for \((r,s) \in \mathbb{R}^2\),
\[ (\nabla_i f)(r,s) = 2\pi i r A_{i1} f(r,s) + 2\pi i s A_{i2} f(r,s) - A_{i3} \frac{\partial f}{\partial r}(r,s) - A_{i4} \frac{\partial f}{\partial s}(r,s). \] (4)

It is easy to compute the curvature of \( \nabla \) and it takes of the form
\[ F_{ij} = [\nabla_i, \nabla_j] = 2\pi i \begin{vmatrix} A_{i1} & A_{i3} \\ A_{j1} & A_{j3} \end{vmatrix} + \begin{vmatrix} A_{i2} & A_{i4} \\ A_{j2} & A_{j4} \end{vmatrix}. \]
Thus if \( A = (A_{ij}) = (T^{-1})^* \), then we get
\[ F_{ij} = -\frac{2\pi i}{\text{Pf}(\gamma)} * \gamma_{ij} \]
and hence the curvature matrix of \( \nabla \) is identified with \(-2\pi i \gamma^{-1} \).

Let us consider the noncommutative complex torus \( T_4^\theta \), where the complex structure on \( T_4^\theta \) is specified by the dual period matrix \( \Omega = (\Delta^{-1}_m \ Z) \) of \( \Lambda \) in \( V \). The compatible holomorphic structure on the deformed bundle or the finitely generated projective left \( A_4^\theta \)-module \( E \) is determined by the following defining equation
\[ [\nabla_\alpha, \nabla_\beta] = 0, \quad \alpha, \beta = 1, 2 \] (5)
where \( \nabla = \Omega \nabla \), and hence
\[
\begin{pmatrix} \nabla_1 \\ \nabla_2 \end{pmatrix} = \begin{pmatrix} \frac{1}{m_1} & 0 \\ 0 & \frac{1}{m_2} \end{pmatrix} \begin{pmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{pmatrix} \begin{pmatrix} \nabla_1 \\ \nabla_2 \end{pmatrix} = \begin{pmatrix} \frac{1}{m_1} \nabla_1 + Z_{11} \nabla_3 + Z_{12} \nabla_4 \\ \frac{1}{m_2} \nabla_2 + Z_{21} \nabla_3 + Z_{22} \nabla_4 \end{pmatrix}.
\]
Since the curvature matrix \( F \) of \( \nabla \) is \(-2\pi i \gamma^{-1} \), the defining equation (5) is equivalent to the equation
\[ \Omega \gamma^{-1} \Omega^t = 0. \]
Also we have a useful defining equation
\[
[\nabla_\alpha, \nabla_\beta] = \left[ \frac{1}{m_1} \nabla_1 + Z_{11} \nabla_3 + Z_{12} \nabla_4 + \frac{1}{m_2} \nabla_2 + Z_{21} \nabla_3 + Z_{22} \nabla_4 \right] = 0
\]

We now find a holomorphic vector in \( E \), which is a solution for the equation
\[
\nabla \phi (r, s) = 0,
\]

By the definition of \( \nabla \) and (4), we have
\[
\nabla_\alpha f(r, s) = 2\pi i r \Omega^\alpha \cdot A_1 f(r, s) + 2\pi i s \Omega^\alpha \cdot A_2 f(r, s) - \Omega^\alpha \cdot A_3 \frac{\partial f}{\partial r}(r, s) - \Omega^\alpha \cdot A_4 \frac{\partial f}{\partial s}(r, s)
\]

where \( \Omega^\alpha \) denotes the \( \alpha \)-th row of \( \Omega \) and \( A_i \) is the \( i \)-th column of the matrix \( A \).

The solution of the equation (7) should be of the form
\[
\phi(r, s) = \exp[\pi i (r \ s) H \begin{pmatrix} r \\ s \end{pmatrix}],
\]

where \( H = \begin{pmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{pmatrix} \) is a complex \( 2 \times 2 \) matrix. If \( H \) is not symmetric, then the equation (7) does not have a solution, and \( \phi \in S(\mathbb{R}^2) \) only when \( \text{Im} \ H > 0 \). Thus we assume that \( H \) is symmetric and \( \text{Im} \ H > 0 \). Then
\[
\phi(r, s) = \exp[\pi i (r^2 H_{11} + s^2 H_{22} + 2rs H_{12})]
\]

and hence
\[
\frac{\partial \phi}{\partial r}(r, s) = 2\pi i (r H_{11} + s H_{12}) \phi(r, s)
\]
\[
\frac{\partial \phi}{\partial s}(r, s) = 2\pi i (r H_{12} + s H_{22}) \phi(r, s)
\]

Applying (8) to the equation (7), we have four equations with three unknowns \( H_{11} \), \( H_{12} \), and \( H_{22} \):
\[
(\Omega^1 \cdot A_3) H_{11} + (\Omega^1 \cdot A_4) H_{12} = \Omega^1 \cdot A_1
\]
\[
(\Omega^1 \cdot A_3) H_{12} + (\Omega^1 \cdot A_4) H_{22} = \Omega^1 \cdot A_2
\]
\[
(\Omega^2 \cdot A_3) H_{11} + (\Omega^2 \cdot A_4) H_{12} = \Omega^2 \cdot A_1
\]
\[
(\Omega^2 \cdot A_3) H_{12} + (\Omega^2 \cdot A_4) H_{22} = \Omega^2 \cdot A_2
\]
Such a linear system of equations will have a solution if
\[
\begin{vmatrix}
\Omega^1 \cdot A_3 & 0 & \Omega^1 \cdot A_4 & \Omega^1 \cdot A_1 \\
0 & \Omega^1 \cdot A_4 & \Omega^1 \cdot A_3 & \Omega^1 \cdot A_2 \\
\Omega^2 \cdot A_3 & 0 & \Omega^2 \cdot A_4 & \Omega^2 \cdot A_1 \\
0 & \Omega^2 \cdot A_4 & \Omega^2 \cdot A_3 & \Omega^2 \cdot A_2
\end{vmatrix} = 0.
\]
In the above, the determinant is computed to
\[
\begin{align*}
&\{(\Omega^1 \cdot A_1)(\Omega^2 \cdot A_3) - (\Omega^2 \cdot A_1)(\Omega^1 \cdot A_3) + (\Omega^1 \cdot A_2)(\Omega^2 \cdot A_4) - (\Omega^2 \cdot A_2)(\Omega^1 \cdot A_4)\} \\
&\times\{(\Omega^1 \cdot A_3)(\Omega^2 \cdot A_4) - (\Omega^2 \cdot A_3)(\Omega^1 \cdot A_4)\}.
\end{align*}
\]
We first compute the following:
\[
(\Omega^1 \cdot A_i)(\Omega^2 \cdot A_j) - (\Omega^2 \cdot A_i)(\Omega^1 \cdot A_j) = \frac{1}{m_1 m_2} \begin{vmatrix} A_{1i} & A_{1j} \\ A_{2i} & A_{2j} \end{vmatrix} - \frac{1}{m_2} Z_{11} \begin{vmatrix} A_{2i} & A_{2j} \\ A_{3i} & A_{3j} \end{vmatrix} - \frac{1}{m_1} Z_{21} \begin{vmatrix} A_{3i} & A_{3j} \\ A_{4i} & A_{4j} \end{vmatrix} + \det Z \begin{vmatrix} A_{3i} & A_{3j} \\ A_{4i} & A_{4j} \end{vmatrix}.
\]
In particular, if either \( m_1 = m_2 \) or \( Z \) is symmetric, we can write (9) in the following form:
\[
(\Omega^1 \cdot A_i)(\Omega^2 \cdot A_j) - (\Omega^2 \cdot A_i)(\Omega^1 \cdot A_j) = \begin{vmatrix} -Z_{11} & -Z_{12} & A_{1i} & A_{1j} \\ -Z_{21} & -Z_{22} & A_{2i} & A_{2j} \\ 0 & 0 & A_{3i} & A_{3j} \\ \frac{1}{m_1} & \frac{1}{m_2} & A_{4i} & A_{4j} \end{vmatrix}.
\]
Using the identity (9), we now have
\[
\{ (\Omega^1 \cdot A_1)(\Omega^2 \cdot A_3) - (\Omega^2 \cdot A_1)(\Omega^1 \cdot A_3) + (\Omega^1 \cdot A_2)(\Omega^2 \cdot A_4) - (\Omega^2 \cdot A_2)(\Omega^1 \cdot A_4) \} \\
= \frac{1}{m_1 m_2} \gamma_{34} + \det Z \gamma_{12} - \frac{1}{m_2} Z_{11} \gamma_{14} + \frac{1}{m_2} Z_{12} \gamma_{13} - \frac{1}{m_1} Z_{21} \gamma_{24} - \frac{1}{m_1} Z_{22} \gamma_{23} \\
= 0 \quad \text{by (3)}.
\]
Thus the equation (7) has a solution. As a conclusion, the condition \([\nabla_\alpha, \nabla_\beta] = 0\) implies that holomorphic vectors in \( \mathcal{E} \) do exist. By assuming \((\Omega^1 \cdot A_3)(\Omega^2 \cdot A_4) - (\Omega^2 \cdot A_3)(\Omega^1 \cdot A_4) \neq 0\)
we have the solution for the equation (7) as follows:
\[ H_{11} = \frac{(\Omega^1 \cdot A_1)(\Omega^2 \cdot A_4) - (\Omega^2 \cdot A_1)(\Omega^1 \cdot A_4)}{(\Omega^1 \cdot A_3)(\Omega^2 \cdot A_4) - (\Omega^2 \cdot A_3)(\Omega^1 \cdot A_4)} \]

\[ H_{12} = H_{21} = \frac{(\Omega^1 \cdot A_3)(\Omega^2 \cdot A_1) - (\Omega^2 \cdot A_3)(\Omega^1 \cdot A_1)}{(\Omega^1 \cdot A_3)(\Omega^2 \cdot A_4) - (\Omega^2 \cdot A_3)(\Omega^1 \cdot A_4)} \]

\[ H_{22} = \frac{(\Omega^1 \cdot A_2)(\Omega^2 \cdot A_3) - (\Omega^2 \cdot A_2)(\Omega^1 \cdot A_3)}{(\Omega^1 \cdot A_3)(\Omega^2 \cdot A_4) - (\Omega^2 \cdot A_3)(\Omega^1 \cdot A_4)} \]

We have shown that the defining equation (6) assures the existence of holomorphic vectors in a holomorphic bundle over \( T^4_{\theta,\Omega} \). In what follows, we discuss more consequences of the equation (6).

First note that in terms of the rational matrix \( \psi = -nq^{-1} \) defined in (3) we can state the the ordinary Riemann condition \( I \) as follows: The commutative complex torus \( T^4 = V/\Lambda \) is an abelian variety if and only if

\[ \Omega \psi^{-1} \Omega^t = 0, \quad i\Omega \psi^{-1} \Omega^t > 0 \]  

(10)

On the other hand, the equation (6) is equivalent to

\[ \Omega \gamma^{-1} \Omega^t = \Omega(\psi - \theta)^{-1} \Omega^t = 0 \]  

(11)

and the equation determines a holomorphic structure on a bundle over \( T^4_{\theta,\Omega} \). As discussed in Section 2, the deformation of commutative torus \( T^4 \) to a noncommutative torus \( T^4_{\theta} \) corresponds to the cohomological deformation from \( H^*(T^4, \mathbb{Z}) \) to \( H^*(T^4, \mathbb{R}) = \wedge^* L^* \). Since the construction of the equation (11) reflects the cohomological deformation, we might call the equation (11) as the \textit{noncommutative Riemann condition} under the condition that the corresponding solution to the equation \( \nabla \phi = 0 \) is in \( \mathcal{S}(\mathbb{R}^2) \).

Topologically, any commutative torus \( T^4 \) can be deformed to a noncommutative torus \( T^4_{\theta} \) with the deformation parameter \( \theta \in \wedge^2 L^* \). Along with this, one can ask a following question: For any deformation parameter \( \theta \), can every abelian variety be deformed to a noncommutative abelian variety? More precisely, for an abelian variety \( T^4 = V/\Lambda \) equipped with a line bundle \( L \) such that \( c_1(L) = m_1 dx_{13} + m_2 dx_{24} \), the question is to construct a holomorphic bundle \( \mathcal{L} \) over \( T^4_{\theta} \) which is a deformation of \( L \) in the sense of the definition given in Section 2. In what follows, we discuss that the answer is negative. Let us consider a simple example where the deformation parameter is given in the following form:

\[
\theta = \begin{pmatrix}
0 & 0 & \theta_{13} & 0 \\
0 & 0 & 0 & \theta_{24} \\
-\theta_{13} & 0 & 0 & 0 \\
0 & -\theta_{24} & 0 & 0
\end{pmatrix}
\]
and thus

\[ \gamma = \begin{pmatrix} 0 & 0 & \gamma_{13} & 0 \\ -\gamma_{13} & 0 & 0 & \gamma_{24} \\ 0 & -\gamma_{24} & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & \frac{1}{m_1} - \theta_{13} & 0 \\ 0 & 0 & 0 & \frac{1}{m_2} - \theta_{24} \\ -\frac{1}{m_1} + \theta_{13} & 0 & 0 & 0 \\ 0 & 0 & -\frac{1}{m_2} + \theta_{24} & 0 & 0 \end{pmatrix}. \]

In this case, the defining equation or the noncommutative Riemann condition is

\[ [\nabla_1, \nabla_2] = \frac{1}{m_2} Z_{12} \gamma_{13} - \frac{1}{m_1} Z_{21} \gamma_{24} = 0. \]  \hspace{1cm} (12)

If the complex torus \( T^4 = V/\Lambda \) is an abelian variety with the integral skew-symmetric matrix \( q, n = 1 \), then the complex matrix \( Z = \begin{pmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{pmatrix} \) is symmetric and the equation (12) becomes

\[ Z_{12} \left( \frac{\gamma_{13}}{m_2} - \frac{\gamma_{24}}{m_1} \right) = 0. \]

Since \( Z_{12} = Z_{21} \) is arbitrary, we have

\[ \frac{\gamma_{13}}{m_2} = \frac{\gamma_{24}}{m_1} \quad \text{or} \quad \frac{\theta_{13}}{m_2} = \frac{\theta_{24}}{m_1}. \]  \hspace{1cm} (13)

Thus we see that the deformation parameter \( \theta \) should be restricted. In other words, for an arbitrary \( \theta \), one cannot have a noncommutative abelian variety \( (T^4_{\theta, \Omega}, \mathcal{L}) \) which is a deformation of \( (T^4, L) \) and \( (T^4_{\theta, \Omega}, \mathcal{L}) \) is defined only when \( \theta \) holds the equation (13).

As discussed in Section 2, the deformation of \( T^4 \) to \( T^4_{\theta} \) corresponds to the cohomological deformation \( H^*(T^4, Z) \) to \( H^*(T^4, \mathbb{R}) \). On a commutative complex torus \( T^4 \), the Riemann conditions are obtained by comparing the rational structure and the complex structure on \( H^*(T^4) \). Thus, associated to the noncommutative Riemann conditions, it is natural to consider the relation between the real structure and the complex structure on \( H^*(T^4) \). Note that the cohomological deformation can be seen in the curvature matrices. Let us consider the constant curvature connection \( \nabla \) given in (11) whose curvature matrix is \( \gamma^{-1} \). If we assume the Euclidean metric for \( L = \mathbb{R}^4 \), we can use the orthogonal transformation to change matrix \( \gamma \) into the standard form:

\[ \tilde{\gamma} = \begin{pmatrix} 0 & 0 & \gamma_1 & 0 \\ 0 & 0 & 0 & \gamma_2 \\ -\gamma_1 & 0 & 0 & 0 \\ 0 & -\gamma_2 & 0 & 0 \end{pmatrix} = P \gamma^{-1} P^t, \quad P \in O(4). \]

Note that the connection \( \nabla \) is unitary equivalent to the connection \( \tilde{\nabla} \) whose curvature is \( \tilde{\gamma}^{-1} \). In terms of cohomologies, for any invariant, real 2-form

\[ \sum_{i=1}^{4} \gamma_{ij} dx_i \wedge dx_j, \]
one can find an orthonormal basis $\xi_1, \cdots, \xi_4$ for $L$ such that

$$\omega = \frac{1}{2}(\gamma_1^{-1}dx_{13} + \gamma_2^{-1}dx_{24}).$$

We take a basis for $V$ the vectors

$$\zeta_\alpha = \gamma_\alpha^{-1}\xi_\alpha, \quad \alpha = 1, 2.$$

Then the corresponding complex structure will be of the form

$$\tilde{\Omega} = \begin{pmatrix} \gamma_1 & 0 & \bar{Z}_{11} & \bar{Z}_{12} \\ 0 & \gamma_2 & \bar{Z}_{21} & \bar{Z}_{22} \end{pmatrix}.$$

We use the complex structure to define noncommutative complex torus $T_{\theta, \tilde{\Omega}}^4$ and a holomorphic structure on a bundle over $T_{\theta}^4$ equipped with the constant curvature connection $\tilde{\nabla}$. By the same argument as given above, we have the defining equation (5) in the following form

$$\tilde{\Omega}\gamma^{-1}\tilde{\Omega}^t = 0.$$

Since

$$\begin{pmatrix} \gamma_1 & 0 & \bar{Z}_{11} & \bar{Z}_{12} \\ 0 & \gamma_2 & \bar{Z}_{21} & \bar{Z}_{22} \end{pmatrix} = \begin{pmatrix} 0 & 0 & -\gamma_1^{-1} & 0 \\ \gamma_1^{-1} & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \gamma_1 & 0 \\ 0 & \gamma_2 \end{pmatrix} \begin{pmatrix} \bar{Z}_{11} & \bar{Z}_{12} \\ \bar{Z}_{21} & \bar{Z}_{22} \end{pmatrix},$$

we see that the matrix $\tilde{Z} = \begin{pmatrix} \bar{Z}_{11} & \bar{Z}_{12} \\ \bar{Z}_{21} & \bar{Z}_{22} \end{pmatrix}$ is symmetric. In this case, we can find a holomorphic vector in a simple form:

$$\phi(r, s) = \exp[\pi i (\gamma_1 r \quad \gamma_2 s) \bar{Z} \begin{pmatrix} \gamma_1 r \\ \gamma_2 s \end{pmatrix}].$$

This can be obtained from the form of solution obtained above by taking suitable entries of the matrix $A$ as given in [8]. Also, we see that the solution is in $S(\mathbb{R}^2)$ only when the matrix $\text{Im} \bar{Z}$ is negative definite. In [6], this kind of solution has been obtained in case of $\gamma_1 = \gamma_2 = 1$ and hence it is easily seen from our solution how deformation parameter is involved in holomorphic vectors.
4 Conclusion

In this paper, we found an explicit solution for holomorphic connections on a four dimensional noncommutative complex torus. We also related the definition of a holomorphic structure for a bundle over a noncommutative complex torus to the ordinary Riemann conditions. In doing so, we defined a deformation of abelian variety along the Chern character deformation and we found that deformation parameter should be restricted to define a noncommutative abelian variety. Finally, we studied a noncommutative variation of Kodaira’s embedding theorem. By using the real cohomology class instead integral ones, we obtain a noncommutative version of Riemann conditions. We expect all these results should be a first step to the study of the mirror symmetry for noncommutative complex torus or noncommutative abelian varieties. We will come to this in forthcoming paper.

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