Wild cyclic-by-tame extensions

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Abstract

Suppose $G$ is a semi-direct product of the form $\mathbb{Z}/p^n \rtimes \mathbb{Z}/m$ where $p$ is prime and $m$ is relatively prime to $p$. Suppose $K$ is a complete local field of characteristic $p > 0$ with algebraically closed residue field. The main result states necessary and sufficient conditions on the ramification filtrations that occur for wildly ramified $G$-Galois extensions of $K$. In addition, we prove that there exists a parameter space for $G$-Galois extensions of $K$ with given ramification filtration, and we calculate its dimension in terms of the ramification filtration. We provide explicit equations for wild cyclic extensions of $K$ of degree $p^3$.

Key words: Local field, Galois, ramification filtration

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1. Introduction

This paper is about wildly ramified Galois extensions of a complete local field $k((t))$ where $k$ is an algebraically closed field of characteristic $p > 0$. We prove that the lower jumps of the ramification filtration of a Galois extension of $k((t))$ with group $\mathbb{Z}/p^n \rtimes \mathbb{Z}/m$ are all congruent modulo $m$, Proposition 4.2. We also prove that one can dominate a given Galois extension having group $\mathbb{Z}/p^{n-1} \rtimes \mathbb{Z}/m$ by a Galois extension having group $\mathbb{Z}/p^n \rtimes \mathbb{Z}/m$, with control over the last jump in the ramification filtration, Proposition 5.1. Together with well-known results about ramification filtrations of Galois extensions with group $\mathbb{Z}/p^n$ [11], this yields (see Theorem 5.2):

\textbf{Theorem 1.1.} Let $G$ be a semi-direct product of the form $\mathbb{Z}/p^n \rtimes \mathbb{Z}/m$ where $p \nmid m$. Let $\sigma \in G$ have order $p^n$ and let $m' = |\text{Cent}_G(\sigma)|/p^n$. A sequence $u_1 \leq \cdots \leq u_m$ of rational numbers occurs as the set of positive breaks in the...
upper numbering of the ramification filtration of a $G$-Galois extension of $k((t))$ if and only if:

(a) $u_i \in \frac{1}{m} \mathbb{N}$ for $1 \leq i \leq n$;

(b) $\gcd(m, mu_1) = m'$;

(c) $p \nmid mu_1$ and, for $1 < i \leq n$, either $u_i = pu_{i-1}$ or both $u_i > pu_{i-1}$ and $p \nmid mu_i$;

(d) and $mu_i \equiv mu_1 \mod m$ for $1 \leq i \leq n$.

In the first author’s doctoral thesis, Theorem 1.1 yields restrictions on the stable reduction of certain branched covers of the projective line.

Our other main result, Theorem 5.6, states that, given a group $G$ and a ramification filtration $\eta$ satisfying conditions (a)-(d) as in Theorem 1.1 there exists a parameter space $M_\eta$ whose $k$-points are in natural bijection with isomorphism classes of $G$-Galois extensions of $k((t))$ having ramification filtration $\eta$. We calculate the dimension of $M_\eta$ in terms of the upper jumps of $\eta$.

Here is the paper’s outline: in Section 2 we introduce the framework of study, including ramification filtrations and field theory; Section 3 contains several structural descriptions of cyclic $p$-group extensions; in Section 4 we prove results about tame actions on cyclic extensions; and the main results on ramification filtrations and parameter spaces for $G$-Galois extensions appear in Section 5.

Our original motivation for this topic was to find explicit equations for $\mathbb{Z}/p^3$-Galois extensions of $k((t))$, see Section 4. Such equations are useful and are difficult to find in the literature. For example, in [5, II, Lemma 5.1], the authors use equations for $\mathbb{Z}/p^2$-Galois extensions in order to prove a case of Oort’s Conjecture, namely, that every $\mathbb{Z}/p^2$-Galois extension of $k((t))$ lifts to characteristic 0 [5, Thm. 2].

Similar results for elementary abelian $p$-group extensions are in [2].

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2. Framework of study

This section contains background on extensions of complete local fields and ramification filtrations and introduces the situation studied in this paper, in which the Galois group is a semi-direct product of the form $\mathbb{Z}/p^n \rtimes \mathbb{Z}/m$.

2.1. Extensions of complete local fields

Let $k$ be an algebraically closed field of characteristic $p > 0$. We fix a compatible system of roots of unity of $k$. In particular, this fixes an isomorphism $\mathbb{Z}/p \simeq \mathbb{F}_p$ and fixes a primitive $m$th root of unity $\zeta$ in $k$. Let $R$ be an equal characteristic complete discrete valuation ring with residue field $k$ and fraction field $K$. Then $R \simeq k[[t]]$ and $K \simeq k((t))$ for some uniformizing parameter $t$. 

Suppose $L/K$ is a separable Galois field extension with group $G$. Let $S$ be the integral closure of $R$ in $L$. Then $S/R$ is a Galois extension of rings with group $G$ which is totally ramified over the prime ideal $(t)$.

This type of field extension arises in the following context. Suppose $\phi : Y \to X$ is a Galois cover of smooth $k$-curves. Suppose $y \in Y$ is a ramified point with inertia group $G$. Consider the complete local rings $S = \mathcal{O}_{Y,y}$ and $R = \mathcal{O}_{X,\phi(y)}$. Then $S/R$ is a Galois extension of rings with group $G$ which is totally ramified over the unique valuation of $R$ as described in the preceding paragraph.

For a Galois extension $L/K$ as above, the group $G$ is a semi-direct product of the form $P \rtimes \mathbb{Z}/m$ where $P$ is a $p$-group and $p \nmid m$. Throughout the paper, we assume that the subgroup $P$ is cyclic.

### 2.2. Subgroups of a semi-direct product

Suppose $G$ is a semi-direct product of the form $P \rtimes \mathbb{Z}/m$ where $P \cong \mathbb{Z}/p^n$ and $p \nmid m$. Let $\sigma$ be a chosen generator of $P$. Let $c$ be a chosen element of order $m$ in $G$ and let $M = \langle c \rangle$. Let $m' = |\text{Cent}_G(\sigma)|/p^n$. In other words, $m' = \# \{ g \in M \mid g\sigma g^{-1} = \sigma \}$.

For $0 \leq i \leq n$, the element $\sigma_i := \sigma^{p^i}$ has order $p^{n-i}$ and $H_i := \langle \sigma_i \rangle$ is the unique subgroup of order $p^{n-i}$ in $G$. Then $\{ \text{id} \} = H_n \subset H_{n-1} \subset \cdots \subset H_0 = P$.

The semi-direct product is determined by the conjugation action of $M$ on $P$. Since $c\sigma c^{-1}$ also generates $P$, then $c\sigma c^{-1} = \sigma^{\alpha'}$ for some integer $\alpha'$ such that $1 \leq \alpha' < p^n$ and $p \nmid \alpha'$. The action of $c$ stabilizes $H_i$. Let $J_i := (H_{i-1}/H_i) \rtimes M$.

**Lemma 2.1.** (i) The value of $\alpha'$ does not depend on the choice of generator of $P$;

(ii) The value of $\alpha'$ depends on the choice of generator of $M$ as follows; if $c_0 = c^\beta$ for some integer $\beta$, then $\alpha'_0 \equiv (\alpha')^\beta \mod p^n$.

**Proof.** (i) If $\tau = \sigma^\gamma$, then $c\tau c^{-1} = (c\sigma c^{-1})^\gamma = (\sigma^{\alpha'})^\gamma = \tau^{\alpha'}$.

(ii) By induction, $c^i \sigma c^{-i} = \sigma^{(\alpha')^i}$. Thus $c_0\sigma c_0^{-1} = \sigma^{\alpha'_0}$.

**Lemma 2.2.** The groups $J_i$ are canonically isomorphic for $1 \leq i \leq n$.

**Proof.** The groups $J_i$ are semi-direct products of the form $\mathbb{Z}/p \rtimes \mathbb{Z}/m$. Thus it suffices to show that the action of $c$ on the equivalence class of $\sigma_{i-1}$ modulo $\langle \sigma_i \rangle$ is the same for $1 \leq i \leq n$. Note that $c\sigma c^{-1} = \sigma^{\alpha'}$. Thus $c\sigma c^{-1} = \sigma_i^{\alpha'}$.

The residue of $\alpha'$ modulo $p$ can be identified with an element $\alpha \in (\mathbb{Z}/p)^*$ and thus with an element $\alpha \in \mathbb{F}_p^*$. Also $m/m'$ is the order of $\alpha$ in $\mathbb{F}_p^*$. 

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2.3. Towers of fields

Suppose $L/K$ is a separable Galois extension whose group $G$ is of the form $\mathbb{Z}/p^n \rtimes \mathbb{Z}/m$ with $p \nmid m$. We fix an identification of $\text{Aut}(L/K)$ with $G$ and indicate this by writing that $L/K$ is a $G$-Galois extension.

Consider the fixed fields $L_i = L^{H_i}$ and $K_i = L_i^{H_i \times M}$ for $0 \leq i \leq n$. So, $L_n = L$ and $K_0 = K$. Let $v_i$ be the natural valuation on $L_i$. Let $\Theta_i$ be the integral closure of $R$ in $L_i$. Then $L/L_i$ is an $H_i$-Galois extension and $L_i/L_0$ is a $P/H_i$-Galois extension. Also $L_i/K_{i-1}$ is a $J_i$-Galois extension. This yields a tower of fields:

$$
\begin{array}{cccccc}
L_0 & \overset{z/p}{\rightarrow} & L_1 & \overset{z/p}{\rightarrow} & \cdots & \overset{z/p}{\rightarrow} & L_{n-1} & \overset{z/p}{\rightarrow} & L \\
& & & & & & & \\
& & & & & & & K_0 & \overset{z/m}{\rightarrow} & K_1 & \overset{z/m}{\rightarrow} & \cdots & \overset{z/m}{\rightarrow} & K_{n-1} & \overset{z/m}{\rightarrow} & K_n \\
\end{array}
$$

By Kummer theory, there exists $x \in L_0$ such that $L_0 \simeq K[x]/(x^m - 1/t)$. After choosing $c \in G$ such that $c(x) = \zeta x$, one can determine the values of $\alpha'$ and $\alpha$ for the extension $L/K$.

2.4. Ramification filtrations

Here is a brief review of the theory of ramification filtrations from [12, IV]. Consider the natural valuation $v = v_n$ on $L$ and a uniformizing parameter $\pi \in L$. For $r \in \mathbb{N}$, let $I_r$ be the $r$th ramification group in the lower numbering for the extension $L/K$. In other words, $I_r$ is the normal subgroup of all $g \in G$ such that $v(g(\pi) - \pi) \geq r + 1$.

The ramification filtration is important because it determines the degree $\delta$ of the different of $S/R$. Namely, by [12, IV, Prop. 4], $\delta = \sum_{r \geq 0}(|I_r| - 1)$. If $\phi : Y \to X$ is a cover of smooth projective connected $k$-curves, the genus of $Y$ can be found using the Riemann-Hurwitz formula [6, IV, Cor. 2.4] and this formula relies on the degree of the different at each ramification point of $\phi$.

Let $g \in G$ with $g \neq 1$. The lower jump for $g$ is the non-negative integer $j$ so that $v_l(g(\pi) - \pi) = j + 1$. Then $g \in I_j$ and $g \notin I_{j+1}$. By [12, IV, Prop. 11], $p \nmid j$ for any positive lower jump $j$. If $|P| = p^n$, then there are $n$ positive indices $j_1 \leq \cdots \leq j_n$ at which there is a break in the ramification filtration in the lower numbering, which are called the lower jumps of $L/K$.

There is also a ramification filtration $I^u$ in the upper numbering. The upper jumps of $L/K$ are the positive breaks $u_1 \leq \cdots \leq u_n$ in the ramification filtration in the upper numbering. The lower numbering is stable for subextensions [12 IV, Prop. 2] and the upper numbering is stable for quotients [12 IV, Prop. 14]. Using Herbrand’s formula [12 IV, §3], one can translate between the two ramification filtrations: letting $j_0 = u_0 = 0$, then $u_i - u_{i-1} = (j_i - j_{i-1})/p^{i-1}m$ for $1 \leq i \leq n$. 
3. Wild cyclic extensions

In this section, we describe the equations and ramification filtration of the $\mathbb{Z}/p^n$-Galois subextension $L/L_0$. The material in this section is mostly known, but it is all necessary for later results in the paper.

3.1. Cyclic towers of Artin-Schreier extensions

Lemma 3.1. The $i$th lower jump $j_i$ of $L/K$ equals the lower jump of $L_i/L_{i-1}$.

Proof. The $i$th lower jump $j_i$ of $L/K$ is the lower jump of the automorphism $\sigma_{i-1}$. This is the same as the lower jump of $\sigma_{i-1}$ for the extension $L/L_{i-1}$ by \[12, IV, Prop. 2\]. Since this is the smallest lower jump for the extension $L/L_{i-1}$, it also equals the upper jump of $\sigma_{i-1}$ for $L/L_{i-1}$. By \[12, IV, Prop. 14\], this is then the same as the upper jump, and thus the lower jump, of $L_i/L_{i-1}$. \[
\]

3.2. Witt Vectors and $p$-power cyclic extensions

We recall some Witt vector theory. Let $\wp$ be the operation $\text{Fr} - \text{Id}$ on Witt vectors, where $\text{Fr}$ denotes Frobenius. An element $a$ of a field $F$ of characteristic $p$ is a $\wp$th power in $F$ if the polynomial $z^p - z - a$ has a root in $F$.

By \[7, p. 331, Ex. 50\], every Galois extension of $L_0 \cong k((x^{-1}))$ with group $\mathbb{Z}/p^n$ has Witt vector equations

$$(y_1^p, \ldots, y_n^p) = (y_1, \ldots, y_n) +' (x_1, \ldots, x_n). \quad (1)$$

where $x_i \in L_0$ for $1 \leq i \leq n$ such that $x_1$ is not a $\wp$th power in $L_0$ and where $+'$ denotes addition of Witt vectors: Moreover, there is a generator $\tau$ of $\mathbb{Z}/p^n$ such that the action of $\tau$ on Witt vectors is

$$\tau(y_1, \ldots, y_n) = (y_1, \ldots, y_n) +' (1, 0, \ldots, 0). \quad (2)$$

Modifying $(x_1, \ldots, x_n)$ by an element $w \in W^n(L_0)$, where $W^n$ is the $n$th truncation of the Witt vectors, changes the isomorphism class of the extension precisely when $w \not\in \wp(W^n(L_0))$. Thus, since $k$ is algebraically closed, one can choose $(x_1, \ldots, x_n)$ to be in standard form, i.e., $x_i \in k[x]$ and either $x_i = 0$ or $x_i$ has no exponent divisible by $p$.

To make (1) more explicit, for $0 \leq i \leq n-1$, let $W_i = \sum_{d=0}^i p^d x_1^d x_{i+1}^{i-d}$ be the $i$th Witt polynomial, \[12, II, §6\]. Define $S_i \in \mathbb{Z}[X_1, \ldots, X_{i+1}, Y_1, \ldots, Y_{i+1}]$ to be the unique formal polynomial such that

$$\begin{align*}
W_i(X_1, \ldots, X_{i+1}) + W_i(Y_1, \ldots, Y_{i+1}) &= \\
W_i(S_0(X_1, Y_1), S_1(X_1, X_2, Y_1, Y_2), \ldots, S_i(X_1, \ldots, X_{i+1}, Y_1, \ldots, Y_{i+1})).
\end{align*}$$

The indexing of these variables is shifted by one from that of \[12, II, §6\] in order to be more consistent with notation in this paper. By \[12, II, Thm. 6\], the $S_i$ are well defined and have integer coefficients.
Lemma 3.2. In \( \mathbb{Z}[X_1, \ldots, X_i, Y_1, \ldots, Y_i] \),

\[
S_{i-1}(X_1, \ldots, X_i, Y_1, \ldots, Y_i) = X_i + Y_i + \sum_{d=1}^{i-1} \frac{1}{p^d} (X_d^{p^d-1} + Y_d^{p^d-1} - S_d^{p^d-1})
\]

and the degree of every monomial of \( S_{i-1} \) is congruent to one modulo \( p - 1 \).

Proof. The equation follows from \( \sum_{d=0}^{i-1} p^d S_d^{p^d-1} = \sum_{d=0}^{i-1} p^d (X_d^{p^d-1} + Y_d^{p^d-1}) \) (see [11, Footnote 4]) and the statement about degrees from induction.

For \( 1 \leq i \leq n \), let \( \bar{S}_{i-1} \in \mathbb{F}_p[X_1, \ldots, X_i, Y_1, \ldots, Y_i] \) be the reduction of \( S_{i-1} \) modulo \( p \) and let \( f_i(Y_1, \ldots, Y_{i-1}, X_1, \ldots, X_i) = \bar{S}_{i-1} - Y_i \). Then \( f_i = X_i + g_i \), where \( g_i \in \mathbb{F}_p[X_1, \ldots, X_{i-1}, Y_1, \ldots, Y_{i-1}] \) is a polynomial whose terms each have degree congruent to one modulo \( p - 1 \). The meaning of \( \bar{S} \) is that a Galois extension with group \( \mathbb{Z}/p^n \) has equations \( y_i^p - y_i = f_i(y_1, \ldots, y_{i-1}, x_1, \ldots, x_i) \).

Lemma 3.3. Let \( L/L_0 \) be a \( \mathbb{Z}/p^n \)-Galois extension and \( \sigma \) a generator of \( \mathbb{Z}/p^n \). There exist \( x_i \in L_0 \) and \( y_i \in L \) for \( 1 \leq i \leq n \) such that \( L/L_0 \) is isomorphic to the \( \langle \sigma \rangle \)-Galois extension with Witt vector equations and Galois action

\[
(y_1^p, \ldots, y_n^p) = (y_1, \ldots, y_n) +^f (x_1, \ldots, x_n)
\]

\[
\sigma(y_1, \ldots, y_n) = (y_1, \ldots, y_n) +^f (1, 0, \ldots, 0).
\]

Furthermore, there is a unique choice for \((x_1, \ldots, x_n)\) in standard form.

Proof. There exist \( x_i \in L_0 \) and \( y_i \in L \) and a generator \( \tau \) of \( \mathbb{Z}/p^n \) such that \( L/L_0 \) has Witt vector equations \( \langle \bar{1} \rangle \) and Galois action \( \langle \bar{2} \rangle \). Now \( \sigma = \tau^b \) for some \( b \in (\mathbb{Z}/p^n)^* \). Then \( \sigma(y_1, \ldots, y_n) = (y_1, \ldots, y_n) +^f b(1, 0, \ldots, 0) \). Since \( b \) is invertible in \( \mathbb{Z}/p^n \cong W^n(\mathbb{Z}/p) \subset W^n(L_0) \), one can replace \((y_1, \ldots, y_n)\) and \((x_1, \ldots, x_n)\) with the Witt vectors \( \frac{1}{b}(y_1, \ldots, y_n) \) and \( \frac{1}{b}(x_1, \ldots, x_n) \). Since \( \text{Fr} \) is a ring homomorphism \([2, \text{p. 331, Ex. 48}]\), the extension \( L/L_0 \) still has Witt vector equations \( \langle \bar{1} \rangle \) and now \( \sigma(y_1, \ldots, y_n) = (y_1, \ldots, y_n) +^f (1, 0, \ldots, 0) \).

By a generalization of [8, Lemma 2.1.5], there is a unique choice of \((x_1, \ldots, x_n)\) in standard form compatible with the restriction on the Galois action.

3.3. Ramification filtrations for cyclic \( p \)-group extensions

The ramification filtration of a \( \mathbb{Z}/p^n \)-Galois extension is completely determined by either its lower or upper jumps, which in turn can be determined by the Witt vector equation.

Lemma 3.4. Let \( L/L_0 \) be a \( \mathbb{Z}/p^n \)-Galois extension with Witt vector \((x_1, \ldots, x_n)\) in standard form. Let \( u = \max\{-p^n-i v_0(x_i)\}_{i=1}^n \). Then \( u \) is the last upper jump of \( L/L_0 \).

Proof. This follows from [4, Thm. 1.1]; see also [13, Prop. 4.2(1)].
Lemma 3.5. A sequence of positive integers \( w_1 \leq \cdots \leq w_n \) occurs as the set of upper jumps of a \( \mathbb{Z}/p^n \)-Galois extension of \( L_0 \) if and only if \( p \nmid w_1 \) and, for \( 1 < i \leq n \), either \( w_i = pw_{i-1} \) or both \( w_i > pw_{i-1} \) and \( p \nmid w_i \).

Proof. The result, originally found in [11], follows from Lemma 3.4; see also [9, Lemma 19].

The following lemma will be used to compare the upper jumps of the \( G \)-Galois extension \( L/K \) and the \( \mathbb{Z}/p^n \)-Galois extension \( L/L_0 \).

Lemma 3.6. Suppose \( L/K \) has upper jumps \( u_1 \leq \cdots \leq u_n \). Then \( L/L_0 \) has upper jumps \( w_1 \leq \cdots \leq w_n \) where \( w_i = mu_i \) for \( 1 \leq i \leq n \).

Proof. By [12, IV, Prop. 2], the lower jumps of \( L/L_0 \) equal the lower jumps \( j_1 \leq \cdots \leq j_n \) of \( L/K \). Herbrand’s formula [12, IV, §3] implies that \( u_i - u_{i-1} = (j_i - j_{i-1})/p^{i-1}m \) and that \( w_i - w_{i-1} = (j_i - j_{i-1})/p^{i-1} \) for \( 1 \leq i \leq n \).

4. Cyclic-by-tame extensions

Suppose \( L/K \) is a separable \( G \)-Galois field extension as in Sections 2.2-3.1. In this section, we find necessary conditions on the ramification filtrations and equations arising from the \( \mathbb{Z}/m \)-Galois action on \( L \).

4.1. The case of Galois extensions with group \( \mathbb{Z}/p \times \mathbb{Z}/m \)

Lemma 4.1. Consider the \( J_1 \)-Galois extension \( L_1/K \) with equations \( x^m = 1/t \) and \( y_1^p - y_1 = x_1 \) and Galois action \( c(x) = \zeta x \) and \( \sigma(y_1) = y_1 + 1 \).

(i) The lower jump \( j \) of \( L_1/L_0 \) satisfies \( m' = \gcd(m, j) \).

(ii) Also \( m | j(p - 1) \). In particular, \( j \equiv jp^r \mod m \) for any \( r \in \mathbb{N} \).

(iii) Also \( c(y_1) = \alpha^{-1}y_1 = \zeta^jy_1 \).

Proof. (i) This follows from [12, IV, Prop. 9], see also [8, Lemma 1.4.1(iv)].

(ii) The conjugation action of \( \mathbb{Z}/m \) on \( \mathbb{Z}/p \) gives a homomorphism \( \nu : \mathbb{Z}/m \to \text{Aut}(\mathbb{Z}/p) \). By definition, \( \text{Im}(\nu) \) has order \( m/m' \) and \( \text{Ker}(\nu) = \langle c^{m/m'} \rangle \). Thus \( m|m'(p - 1) \). By part (i), \( m' = \gcd(m, j) \), so \( m | j(p - 1) \).

(iii) [8, Lemma 1.4.1(ii)-iii)].

4.2. A congruence condition on the ramification filtration

Proposition 4.2. (i) The lower jumps in the ramification filtration of the \( P \)-Galois extension \( L/L_0 \) are all congruent modulo \( m \).

(ii) The upper jumps in the ramification filtration of the \( P \)-Galois extension \( L/L_0 \) are all congruent modulo \( m \).
Proof. (i) The ith lower jump of \(L/L_0\) is \(j_i\) by [12, IV, Prop. 2]. Let \(\pi\) be a uniformizer of \(\Theta_n\) and let \(u = c(\pi)/\pi \in \Theta_n^\times\). Then \(u\) equals \(\theta_0(c) \in k^\times\) in the notation of [12, IV, Prop. 7]. The order of \(u\) is \(m\) by [12, IV, Prop. 7]. By the proof of Lemma 4.2, \(c\sigma_{i-1}c^{-1} = \sigma_{i-1}^{\alpha'}\) for \(1 \leq i \leq n\). Since \(\sigma_{i-1}\) generates \(H_{i-1}/H_i = I_{j_i}/I_{j_{i-1}}, [12, IV, Prop. 9]\) shows that \(\theta_j(\sigma_{i-1}^{\alpha }) = u^i\theta_j(\sigma_{i-1})\) for \(1 \leq i \leq n\). Thus \(u^i = \alpha \in k^\times\) for \(1 \leq i \leq n\) and so \(j_1 \equiv \cdots \equiv j_n \mod m\).

(ii) Let \(w_1 \leq \cdots \leq w_n\) be the upper jumps of the \(P\)-Galois extension \(L/L_0\). Since \(P\) is abelian, the Hasse-Arf Theorem implies that \(w_i \in \mathbb{N}\). By Herbrand’s formula, \(w_i - w_{i-1} = (j_i - j_{i-1})/p^{i-1}\). Thus \(w_i - w_{i-1} \equiv 0 \mod m\) by part (i).

\(\square\)

Class field theory approach: If \(k\) is instead a finite field, here is a different proof of Proposition 4.2 which uses class field theory.

Second proof of Proposition 4.2 The \(G\)-Galois extension \(L/K\) dominates the \(\langle c \rangle\)-Galois extension \(L_0/K\) where \(L_0 \simeq k[[x^{-1}]]\), \(x^m = 1/t\), and \(c(x) = \zeta x\). Let \(L/L_0\) be the \(P\)-Galois subextension, which has upper jumps \(w_1 \leq \cdots \leq w_n\), where \(w_i = mw_i\) by Lemma 3.6. Thus the upper ramification group \(I^d\) of \(L/L_0\) equals \(H_i\) if \(w_i < \ell \leq w_{i+1}\).

Let \(Q = (x^{-1})\) be the maximal ideal of \(k[[x^{-1}]]\). Consider the ring groups \(U^d = 1 + Q^d\) of \(k[[x^{-1}]]\). By [12, IV, Prop. 6], \(U^d/U^{d+1}\) is canonically isomorphic to \(Q^d/Q^{d+1}\). Now, \(Q^d\) carries a natural \(\langle c \rangle\)-module structure where \(c((x^{-1})^d) = \zeta_m^{-d}(x^{-1})^d\). Thus \(U^d/U^{d+1}\) carries a natural structure as a \(\langle c \rangle\)-module, and this structure depends on the congruence class of \(d\) modulo \(m\).

By [12, XV.2, Cor. 3 & pg. 229], there is a reciprocity isomorphism \(\omega : L_0^\times/NL^* \rightarrow P\) and thus there are isomorphisms \(\omega_n : U^d/(U^{d+1}NU_L^\psi(w)) \rightarrow I^d/I^{d+1}\). Here \(N : L \rightarrow L_0\) is the norm map and \(\psi\) is Herbrand’s function. In particular, taking \(d = w_i\), then \(U^{w_i}/(U^{w_i+1}NU_L^{\psi(w_i)}) = H_{i-1}/H_i\).

Now \(H_{i-1}/H_i\) has a \(\langle c \rangle\)-module structure and this \(\langle c \rangle\)-module structure is independent of \(i\) by Lemma 2.2. After pulling back by \(\omega\), this implies that the \(\langle c \rangle\)-module structure of \(U^{w_i}/(U^{w_i+1}NU_L^{\psi(w_i)})\) and thus of \(U^{w_i}\) is independent of \(i\). Thus \(\zeta_m^{-w_i}\) is independent of \(i\) and so \(w_i \equiv w_1 \mod m\).

The lower jumps are also congruent modulo \(m\) by Herbrand’s formula. \(\square\)

At this point, one can prove that the conditions in Theorem 1.1 are necessary; we will postpone this until Section 5.2.

4.3. Actions and isomorphisms

This section contains two results that will be needed in Section 5

Proposition 4.3. Suppose \(L_0 \simeq k[x]/(x^m - 1/t)\) and \(c(x) = \zeta x\). Suppose \(L/L_0\) is a \(P\)-Galois extension with Witt vector equations (7), Galois action (2), and first lower jump \(j\) such that \(\zeta^j = \alpha^{-1}\). Then \(L/K\) is a \(G\)-Galois extension if and only if \(c(x_i) = \zeta^j x_i\) and \(c(y_i) = \zeta^j y_i\) for \(1 \leq i \leq n\).
Proof. Suppose $L/K$ is a $G$-Galois extension. Then $L_1/K$ is a $J_1$-Galois extension. By Lemma 4.3 (iii), $c(y_1)/y_1 = \alpha^{-1} = \zeta^j$. Since $y_1^n - y_1 = x_1$, this implies that $c(x_1) = \zeta^j x_1$. As an inductive hypothesis, suppose that $c(x_i) = \zeta^j x_i$ and $c(y_i) = \zeta^j y_i$ for $1 \leq i \leq n - 1$.

Now $L_n/K_{n-1}$ is a $J_n$-Galois extension of local fields and $J_n$ and $J_1$ are canonically isomorphic by Lemma 4.4. In other words, the value of $\alpha$ for $\text{Aut}(L_n/K_{n-1})$ is the same as for $\text{Aut}(L_1/K)$. By Kummer theory, there exists a uniformizer $\pi_{n-1}$ of $L_{n-1}$ such that $c$ acts on $\pi_{n-1}$ via multiplication by some $\gamma \in \mu_n$. Then $L_n/K_{n-1}$ satisfies the hypotheses of Lemma 4.1 with $1/\pi_{n-1}$, $y_n$, $J_n$, and $\gamma^{-1}$ replacing $x$, $y_1$, $j$, and $\zeta$ respectively. Applying Lemma 4.1 (iii) to $L_n/K_{n-1}$ implies that $c(y_n)/y_n = \gamma^{-1} = \zeta^{-1}$.

The equation for $L_n/L_{n-1}$ is $y_n^p - y_n = x_n + g_n$ where the terms of the polynomial $g_n \in \mathbb{F}_p[x_1, \ldots, x_{n-1}, y_1, \ldots, y_{n-1}]$ each have degree congruent to one modulo $p - 1$. By the inductive hypothesis and Lemma 4.1 (ii), $c$ scales $g_n$ by $\zeta$. Thus $c$ scales both $y_n^p - y_n - x_n$ and $y_n$ by $\zeta^j$, which implies $c(x_n) = \zeta^j x_n$.

Conversely, suppose $c(x_i) = \zeta^j x_i$ and $c(y_i) = \zeta^j y_i$ for $1 \leq i \leq n$. The proof that $L/K$ is $G$-Galois proceeds by induction on $n$; the case $n = 1$ can be computed explicitly, see e.g. [4] Lemma 1.4.1. As an inductive hypothesis, suppose that $L_{n-1}/K$ is a $G/H_{n-1}$-Galois extension. To finish, it suffices to show that the action of $c$ extends to an automorphism of $L_n$, i.e., that $c$ stabilizes the equation $y_n^p - y_n = f_n$ for $L_n/L_{n-1}$. By Lemmas 3.2 and 4.1 (ii), the action of $c$ scales every term of this equation by $\zeta^j$.

Lemma 4.4. Suppose $L/K$ is a $G$-Galois extension as in Section 2.3.

(i) There is a Witt vector $(x_1, \ldots, x_n)$ in standard form for the subextension $L/L_0$ and it is uniquely determined up to multiplication by $\mu_m/m'$.

(ii) There are $\varphi(m)/\varphi(m/m')$ different non-isomorphic $G$-Galois structures on the field extension $L/K$ such that the action of $\sigma$ on $L$ is as in (3).

Proof. For part (i), by Lemma 4.3 for fixed $x$, there is a uniquely determined Witt vector $(x_1, \ldots, x_n)$ in standard form for the subextension $L/L_0$. Now $x$ is determined up to multiplication by $\zeta^d$, for $d \in \mathbb{Z}$. By Proposition 4.3, every monomial in $x$ has degree congruent to $j$ mod $m$. Replacing $x$ with $\zeta^d x$ scales $x_i$ by $\zeta^{dj}$. The values of $\zeta^{dj}$ range over $\mu_m/m'$ by Lemma 4.1 (i).

For part (ii), a $G$-Galois structure on $L/K$ satisfying the requirement for $\sigma$ is determined by an isomorphism $\iota: G \to \text{Aut}(L/K)$ such that $\iota(\sigma)(y_1, \ldots, y_n) = (y_1, \ldots, y_n)^{-1}(1, 0, \ldots, 0)$. If $h \in \text{Aut}(L/K)$, then the map $h : L \to L$ yields an isomorphism of $G$-Galois extensions $L/K \to L/K$, the first with structure morphism $\iota$ and the second with structure morphism $h \iota h^{-1}$. Thus, modifying $\iota$ by an inner automorphism yields an isomorphic $G$-Galois structure on $L/K$. So the number of isomorphism classes of $G$-Galois structures with this requirement on $\sigma$ is given by the number of elements of $\text{Aut}(G)$ fixing $\sigma$, divided by the number of $\text{Inn}(G)$ fixing $\sigma$.

An automorphism $\gamma$ of $G$ which fixes $\sigma$ is determined by $\gamma(c)$. Also $\gamma(c)$ must have order $m$ and have the same conjugation action as $c$ on $\sigma$, as determined by
Lemma 2.1(ii). When $G$ is abelian, then $\alpha' = 1$ and there are $\varphi(m')$ choices for $\gamma(c)$. This yields the count $\varphi(m)/\varphi(m/m')$ since $m' = m$ and since $\text{Inn}(G)$ is trivial. If $G$ is non-abelian, then the image of $\gamma(c)$ in $M$ must have order $m$ and be congruent to $c$ modulo $\langle cm/m' \rangle = \ker(\nu)$. There are $p^m \varphi(m)/\varphi(m/m')$ choices for $\gamma(c)$. This yields the desired count, since there are $p^m$ inner automorphisms of $G$ which fix $\sigma$, namely conjugation by powers of $\sigma$. \hfill $\square$

5. Main results

Let $G$ be a semi-direct product of the form $\mathbb{Z}/p^n \rtimes \mathbb{Z}/m$. This section contains three results: first we prove that one can dominate a given Galois extension having group $G/H$ by a Galois extension having group $\mathbb{Z}/p^n \rtimes \mathbb{Z}/m$, with control over the last upper jump; second, we give necessary and sufficient conditions for the ramification filtration of a $G$-Galois extension; third, we define a parameter space for $G$-Galois extensions of $K$ with given ramification filtration $\eta$ and calculate its dimension in terms of the upper jumps.

5.1. A wild embedding problem

We prove that one can embed a given Galois extension having group $\mathbb{Z}/p^{n-1} \times \mathbb{Z}/m$ by a Galois extension having group $\mathbb{Z}/p^n \rtimes \mathbb{Z}/m$, with control over the last upper jump. See [3, 24.42] for an earlier version of this result, in which $m = 1$ and there is no control over the upper jump. Recall that $G/H_{n-1}$ is a semi-direct product of the form $\mathbb{Z}/p^{n-1} \times \mathbb{Z}/m$.

**Proposition 5.1.** Suppose $L_{n-1}/K$ is a $G/H_{n-1}$-Galois extension with upper jumps $u_1 \leq \cdots \leq u_{n-1}$. Let $u_n \in \frac{1}{m} \mathbb{N}$ be such that either $u_n = pu_{n-1}$ or both $u_n > pu_{n-1}$ and $p \not| mu_n$. Suppose also that $mu_n \equiv mu_1 \mod m$. Then there exists a $G$-Galois extension $L_n/K$ with upper jumps $u_1 \leq \cdots \leq u_n$ that dominates $L_{n-1}/K$.

**Proof.** Without loss of generality, one can suppose $L_0 \simeq K[x]/(x^m - 1/t)$ and $c(x) = \zeta x$. The $\mathbb{Z}/p^{n-1}$-Galois extension $L_{n-1}/L_0$ has upper jumps $mu_1 \leq \cdots \leq mu_{n-1}$ by Lemma 3.6. By Section 3.2, $L_{n-1}/L_0$ is given by a Witt vector equation $(y_1^p, \ldots, y_{n-1}^p) = (y_1, \ldots, y_{n-1}) + (x_1, \ldots, x_{n-1})$ for some $x_i \in L_0$, such that $x_1$ is not a $p$th power in $L_0$. Furthermore, one can choose $(x_1, \ldots, x_{n-1})$ to be in standard form. In particular, if $x_i \neq 0$, then $p \not| v_0(x_i)$.

By Proposition 4.3 if $1 \leq i \leq n - 1$, then $c(x_i) = \zeta^j x_i$ and $c(y_i) = \zeta^j y_i$, where $j = mu_1$. By Lemma 3.4, $mu_{n-1} = \max\{-p^{n-1}v_0(x_i)\}_{i=1}^{n-1}$.

If $u_n \neq pu_{n-1}$, let $x_n = x^{mu_n}$. In this case, $-v_0(x_n) = mu_n$. If $u_n = pu_{n-1}$, let $x_n = 0$. In this case, $-v_0(x_n) = -\infty < pu_{n-1}$. In both cases, $(x_1, \ldots, x_n)$ is a Witt vector in standard form. Then the Witt vector equation $(y_1^p, \ldots, y_n^p) = (y_1, \ldots, y_n) + (x_1, \ldots, x_n)$ yields a $P$-Galois extension $L_n/L_0$ dominating $L_{n-1}/L_0$, with upper jumps $mu_1 \leq \cdots \leq mu_n$ by Lemma 3.4 (i.e., [4, Thm. 1.1]).

By the definition of $x_n$, then $c(x_n) = \zeta^j x_n$. Let $c(y_n) = \zeta^j y_n$. By Proposition 4.3, $L_n/K$ is a $G$-Galois extension dominating $L_{n-1}/K$, and it has upper jumps $u_1 \leq \cdots \leq u_n$ by Lemma 3.6. \hfill $\square$
5.2. Conditions on the ramification filtration

The ramification filtration of a Galois extension with group \( G \) of the form \( \mathbb{Z}/p^n \times \mathbb{Z}/m \) is completely determined by either its lower or upper jumps. Here are the statement and proof of Theorem 5.1 giving necessary and sufficient conditions on the ramification filtrations of \( G \)-Galois extensions of \( K \).

**Theorem 5.2.** Let \( G \) be a semi-direct product of the form \( \mathbb{Z}/p^n \times \mathbb{Z}/m \) where \( p \nmid m \). Let \( \sigma \in G \) have order \( p^n \) and let \( m' = |\text{Cent}_G(\sigma)|/p^n \). A sequence \( u_1 \leq \cdots \leq u_n \) of rational numbers occurs as the set of positive breaks in the upper numbering of the ramification filtration of a \( G \)-Galois extension of \( K(t) \) if and only if:

(a) \( u_i \in \frac{1}{m} \mathbb{N} \) for \( 1 \leq i \leq n \);

(b) \( \gcd(m, mu_1) = m' \);

(c) \( p \nmid mu_1 \) and, for \( 1 < i \leq n \), either \( u_i = pu_{i-1} \) or both \( u_i > pu_{i-1} \) and \( p \nmid mu_i \);

(d) and \( mu_i \equiv mu_1 \mod m \) for \( 1 \leq i \leq n \).

**Proof.** Conditions (a)-(d) are necessary: let \( u_1 \leq \cdots \leq u_n \) be the set of upper jumps of a \( G \)-Galois extension of \( K((t)) \). The upper jumps of the \( \mathbb{Z}/p^n \)-subextension \( L/L_0 \) are \( w_1 \leq \cdots \leq w_n \) where \( w_i = mu_i \) by Lemma 3.6. Condition (a) follows since \( w_i \in \mathbb{N} \) by the Hasse-Artin Theorem. Condition (b) follows from Lemma 3.5. Condition (c) is due to [11], see Lemma 3.5. Condition (d) follows from Proposition 4.2(ii).

Conditions (a)-(d) are sufficient: recall that \( G \) has generators \( \sigma \) (of order \( p^n \)) and \( c \) (of order \( m \)) and \( \sigma \sigma^{-1} = \sigma' \) for some integer \( \alpha' \) such that \( 1 \leq \alpha' < p^n \) and \( p \nmid \alpha' \). Let \( \alpha \in \mathbb{F}_p^* \simeq (\mathbb{Z}/p)^* \) be such that \( \alpha \equiv \alpha' \mod p \). Let \( j = mu_1 \). By condition (b), \( \zeta^j \) has order \( m/m' \) in \( k^* \). Likewise, \( \alpha^{-1} \) has order \( m/m' \) in \( k^* \). Thus there exists an integer \( \beta \) such that \( \zeta^{\beta j} = \alpha^{-1} \).

Consider the \( (c) \)-Galois extension \( L_0/K \) with equation \( x^m = 1/t \) and Galois action \( c(x) = \zeta^j x \). Let \( x_1 \in x^k[k[[x^{-m}]]]^* \). Consider the \( \mathbb{Z}/p \)-Galois extension \( L_1/L \) with equation \( y^p - y = x_1 \) and Galois action \( \sigma(y_1) = y_1 + 1 \). By [8, Lemma 1.4.1], \( L_1/K \) is a \( J_1 \)-Galois extension. It has lower jump \( j \) and thus upper jump \( u_1 \). By conditions (a), (c), (d), and Proposition 5.1 there exists a \( G \)-Galois extension \( L/K \) dominating \( L_1/K \) with upper jumps \( u_1 \leq \cdots \leq u_n \).

**Corollary 5.3.** Let \( G \) be a semi-direct product of the form \( \mathbb{Z}/p^n \times \mathbb{Z}/m \) where \( p \nmid m \). Suppose \( \eta \) is a ramification filtration of \( G \) satisfying conditions (a)-(d). Let \( f \) be the order of \( p \) modulo \( m/m' \) and let \( q = p^f \). Then there exists a \( G \)-Galois extension \( L/K \) with ramification filtration \( \eta \) which is defined over \( \mathbb{F}_q \).

**Proof.** It suffices to produce a \( G \)-Galois extension \( L/K \) whose equations and Galois action have coefficients in \( \mathbb{F}_q \). Note that \( \zeta^{\beta j} \) has order \( m/m' \) in \( k^* \). By the definition of \( f \), the field \( \mathbb{F}_{p^f} \) contains the \( (m/m') \)th roots of unity, and thus contains \( \zeta^{\beta j} \). The case \( n = 1 \) follows by direct computation with the equation...
$y_1^p - y_1 = x_1^{mp_1}$, see \[\text{Lemma 1.4.1}\]. The result then proceeds by induction on $n$. For the inductive step, one produces an equation for the extension $L/L_{n-1}$ using Proposition 5.1. In the proof of that result, recall that $x_n \in F_p[x]$ by definition. Thus the equation has coefficients in $F_p$ by Lemma \[\text{3.2}\]. The Galois action is defined over $F_q$ by \[\text{1.2}\] and Proposition 4.3.

5.3. Parameter space for $G$-Galois extensions

Given a sequence $u_1 \leq \cdots \leq u_n$ satisfying conditions (a)-(d), let $\eta$ be the ramification filtration of $G$ having upper jumps $u_1 \leq \cdots \leq u_n$. By Theorem 5.2, there exists a $G$-Galois extension of $k((t))$ with ramification filtration $\eta$. We prove there is a scheme $M_\eta$ such that there is a natural bijection between the $k$-points of $M_\eta$ and isomorphism classes of $G$-Galois extensions of $k((t))$ with ramification filtration $\eta$. We calculate the dimension of $M_\eta$ in terms of the sequence $u_1 \leq \cdots \leq u_n$.

**Notation 5.4.** Given positive integers $w$ and $m$, let

$$\epsilon_p(w, m) = \# \{e \in \mathbb{Z} \mid 1 \leq e \leq w, e \equiv w \mod m, p \nmid e \}.$$

**Lemma 5.5.** Let $\delta_p(w, m) = 1$ if $w \equiv ap \mod m$ for some $1 \leq a \leq r$, where $r$ is the remainder when $\lfloor w/p \rfloor$ is divided by $m$, and $\delta_p(w, m) = 0$ otherwise. Then $\epsilon_p(w, m) = \lfloor w/m \rfloor - \lfloor w/mp \rfloor - \delta_p(w, m)$.

**Proof.** The number of integers $e$ such that $1 \leq e \leq w$ and $e \equiv w \mod m$ is $\lfloor w/m \rfloor$. To count the number of these which are divisible by $p$, consider the set $A = \{p, 2p, \ldots, \lfloor w/p \rfloor \}$. Then $A$ contains at least $\lfloor w/p \rfloor / m = \lfloor w/mp \rfloor$ elements $e$ such that $e \equiv w \mod m$. Let $r$ be the remainder when $\lfloor w/p \rfloor$ is divided by $m$. Then $A$ contains one additional element $e \equiv w \mod m$ if and only if an element of $\{p, 2p, \ldots, rp\}$ is congruent to $w$ modulo $m$. The formula holds since $\delta_p(w, m) = 1$ precisely in this case.

Given a positive integer $N$, the root of unity $\zeta_{m/m'}$ acts on the affine variety $\mathbb{A}_N$ via multiplication on each coordinate. Let $\mathbb{A}_N/\mu_{m/m'}$ denote the quotient.

**Theorem 5.6.** Let $G$ be a semi-direct product of the form $\mathbb{Z}/p^n \rtimes \mathbb{Z}/m$ where $p \nmid m$. Let $u_1 \leq \cdots \leq u_n$ be a sequence satisfying conditions (a)-(d) and $\eta$ be the ramification filtration of $G$ with upper jumps $u_1 \leq \cdots \leq u_n$. Let $N_\eta = \sum_{i=1}^n \epsilon_p(mu_i, m)$. Then there is an open subscheme $U_\eta \subset \mathbb{A}_N/\mu_{m/m'}$ and a finite étale map $\pi : M_\eta \to U_\eta$ of degree $\varphi(m)/\varphi(m/m')$ such that the $k$-points of $M_\eta$ are in natural bijection with isomorphism classes of $G$-Galois extensions of $k((t))$ with ramification filtration $\eta$.

It is clear that $\dim(M_\eta) = N_\eta$ depends only on $p, m, u_1, \ldots, u_n$.

**Proof.** By Lemma 4.4 it suffices to show that the collection of Witt vectors $(x_1, \ldots, x_n)$ in standard form, which, as in Proposition 4.3, yield $G$-Galois extensions $L/K$ with ramification invariants $u_1 \leq \cdots \leq u_n$, is in natural bijection with the $k$-points of an open subscheme of $\mathbb{A}_N$.  

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The proof is by induction on $n$. For the case $n = 1$, Lemma 3.3 shows that $x_1 \in k[x]$ must have degree $mu_1$. By Proposition 4.3, the extension $L_1/K$ is $J_1$-Galois if and only if $c(x_1) = \zeta^{mu_1}x_1$, in other words, if and only if all exponents of $x_1$ are congruent to $mu_1$ modulo $m$. Since $x_1$ is in standard form, it has no exponents with degree divisible by $p$. Thus the number of possible exponents is $\epsilon = \epsilon_p(mu_1, m)$. Since the leading coefficient of $x_1$ is nonzero, the choice of $x_n$ is equivalent to the choice of a $k$-point in an open subscheme of $\mathbb{A}^e$. (See also [8, Proposition 2.2.6]).

Now, suppose that $(x_1, \ldots, x_{n-1})$ is a Witt vector in standard form, which yields a $G/H_{n-1}$-Galois extension $L_{n-1}/K$ with upper jumps $u_1 \leq \cdots \leq u_{n-1}$. Let $\epsilon = \epsilon_p(mu_n, m)$. It suffices to show that Witt vectors $(x_1, \ldots, x_n)$ in standard form which yield an extension $L/K$ dominating $L_{n-1}/K$ with upper jumps $u_1 \leq \cdots \leq u_n$ are in natural bijection with the $k$-points of an open subscheme $U_n \subset \mathbb{A}^e$.

The Witt vector $(x_1, \ldots, x_n)$ for the extension $L/K$ is determined by the choice of $x_n \in k[x]$ in standard form. By Proposition 4.3, the extension $L/K$ is $G$-Galois if and only if $c(x_n) = \zeta^{mu_n}x_n$, in other words, if and only if all exponents of $x_n$ are congruent to $mu_n$ modulo $m$. Recall that $mu_1 \equiv mu_n \mod m$ by Proposition 1.2.

By Lemma 3.3 the extension $L/K$ has upper jump $u_n$ if and only if $\deg(x_n) = -v_0(x_n) \leq mu_n$, where equality must hold if $u_n > pu_{n-1}$. Thus, an exponent $e$ appearing in $x_n$ satisfies $0 \leq e \leq mu_n$, and $e \equiv mu_n \mod m$, and $p \nmid e$. The number of these exponents is $\epsilon = \epsilon_p(mu_n, m)$. The leading coefficient of $x_n$ must be non-zero when $u_n > pu_{n-1}$. The choice of $x_n$ is thus equivalent to the choice of a $k$-point in an open subscheme of $\mathbb{A}^e$.

\begin{remark}
Consider the contravariant functor $F_\eta$ from the category of schemes to sets, which associates to a scheme $B$ the set of $G$-Galois extensions of $O_B((t))$ whose geometric fibres have ramification filtration $\eta$. The scheme $M_\eta$ does not represent $F_\eta$ on the category of $k$-schemes because there are non-constant $G$-Galois covers defined over a base scheme $B$, which become constant after pullback by a finite morphism $B' \to B$. The scheme $M_\eta$ is a fine moduli space for $F_\eta$ on a category where such morphisms are trivialized; see [8, Thm. 2.2.10] for the case $n = 1$.
\end{remark}

\begin{remark}
In [11, Prop. 4.1.1], the authors calculate the dimension of the tangent space of the versal deformation space of a $\mathbb{Z}/p^n$-Galois extension in terms of its ramification filtration. Theorem 5.6 is less technical than their result and it is not clear how to compare them directly.
\end{remark}

\section{Equations for $\mathbb{Z}/p^3$-Galois extensions}

It is well-known that the methods of Section 3.2 can be used to find equations for $\mathbb{Z}/p^n$-extensions [10], but the equations themselves are difficult to find in the literature. Here are formulae for the general $\mathbb{Z}/p^3$-Galois extension of $K$.
Example 6.1. Suppose $L/K$ is a $\mathbb{Z}/p^3$-Galois extension of $K \cong k((t))$. Then there exist $x_1, x_2, x_3 \in K$ so that $L/K$ is isomorphic to the following extension:

\begin{align*}
y_1^p - y_1 &= x_1; \\
y_2^p - y_2 &= \frac{x_1 + y_1^p - (x_1 + y_1)^p}{p} + x_2; \\
y_3^p - y_3 &= \frac{x_1^2 + y_1^2 - (x_1 + y_1)^2}{p^2} + \frac{x_2^p + y_2^p - (x_2 + y_2 + \frac{x_1^2 + y_1^2 - (x_1 + y_1)^2}{p})^p}{p} + x_3.
\end{align*}

A generator $\sigma$ of the Galois group can be chosen so that its action is given by:

\begin{align*}
\sigma(y_1) &= y_1 + 1; \\
\sigma(y_2) &= y_2 + \frac{y_1^p + 1 - (y_1 + 1)^p}{p}; \\
\sigma(y_3) &= y_3 + \frac{y_1^2 + 1 - (y_1 + 1)^2}{p^2} + \frac{y_2^p - (y_2 + \frac{y_1^2 + 1 - (y_1 + 1)^2}{p})^p}{p}. 
\end{align*}

The integral coefficients in Example 6.1 can be considered to be in $\mathbb{F}_p \subseteq k$.

Proof. For the equations, it suffices to recursively compute $f_i = \overline{S}_{i-1} - y_i$ for $1 \leq i \leq 3$, starting with $S_0(x_1, y_1) = x_1 + y_1$ and $S_1(x_1, x_2, y_1, y_2) = x_2 + y_2 + (x_1^p + y_1^p - (x_1 + y_1)^p)/p$. The Galois action is given by $\sigma(y_i) = y_i + f_i$, where $f_i = f_i(y_1, \ldots, y_{i-1}, 1, 0, \ldots, 0)$. To see this, note that $y_i^p = y_i + f_i$ and (10) imply that $(y_1 + f_1, \ldots, y_n + f_n) = (y_1, \ldots, y_n) + (x_1, \ldots, x_n)$. Substituting $(1, 0, \ldots, 0)$ for $(x_1, \ldots, x_n)$ yields $(y_1 + f_1, \ldots, y_n + f_n) = (y_1, \ldots, y_n) + (1, 0, \ldots, 0)$, which equals $\sigma(y_1, \ldots, y_n)$ by Lemma 3.3. \hfill \square

Example 6.2. When $p = 2$ and $x = t^{-j}$, here are equations for a $\mathbb{Z}/8$-Galois extension of $k((t))$, which is defined over $\mathbb{F}_2$ and has upper jumps $j, 2j, 4j$:

\begin{align*}
y^2 - y &= x; \\
z^2 - z &= xy; \\
w^2 - w &= x^3 y + y^3 x + xyz.
\end{align*}

The Galois action is given by $y \mapsto y + 1$, $z \mapsto z + y$, and $w \mapsto w + y^3 + y + yz$.

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