Conflict between trajectories and density description: the statistical source of disagreement

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We study an idealized version of intermittent process leading the fluctuations of a stochastic dichotomous variable $\xi$. It consists of an overdamped and symmetric potential well with a cusp-like minimum. The right-hand and left-hand portions of the potential corresponds to $\xi = W$ and $\xi = -W$, respectively. When the particle reaches this minimum is injected back to a different and randomly chosen position, still within the potential well. We build up the corresponding Frobenius-Perron equation and we evaluate the correlation function of the stochastic variable $\xi$, called $\Phi_\xi(t)$. We assign to the potential well a form yielding $\Phi_\xi(t) = (T/(t + T))^\beta$, with $\beta > 0$. Thanks to the symmetry of the potential, there are no biases, and we limit ourselves to considering correlation functions with an even number of times, indicated for concision, by $\langle 12 \rangle$, $\langle 1234 \rangle$ and, more, in general, by $\langle 1...2n \rangle$. The adoption of a formal treatment, based on density, and thus of the operator driving the density time evolution, establishes a prescription for the evaluation of the correlation functions, yielding $\langle 12 \rangle = \langle 1234 \rangle = \langle 1 \rangle$. We study the same dynamic problem using trajectories, and we establish that the resulting two-time correlation function coincides with that afforded by the density picture, as it should. We then study the four-times correlation function and we prove that in the non-Poisson case it depart from the density prescription, namely, from $\langle 1234 \rangle = \langle 1 \rangle$. We conclude that this is the main reason why the two pictures yield two different diffusion processes, as noticed in an earlier work [M. Bologna, P. Grigolini, B.J. West, Chem. Phys. 284, (1-2) 115-128 (2002)].

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I. INTRODUCTION

The assumption of the equivalence between trajectories and density is one of the major tenets of modern physics. In fact, it is of fundamental importance in statistical mechanics, where the concept of density was originally born, as a consequence of the Gibbs perspective resting on the use of infinitely many copies of the same dynamic system \[1\]. The concept of density, under the form of statistical density matrix is also at the basis of the foundation of quantum mechanics and of decoherence theory \[2\] that enables us to interpret all the statistical processes occurring in nature as being compatible with the unitary time evolution of the whole Universe. As already pointed out by Joos years ago \[3\], the derivation of equation of motion for density, non diagonal in the position representation of the density matrix, does not lead to the unique result of letting trajectories emerge. On one side, this seems to make unnecessary to alter the quantum-mechanical law of evolution so that localized states for macro-objects naturally emerge \[4\] and, on the other side, generates the conviction that there is no more need for the postulate of wave-function collapse \[5\].

It comes to be, therefore, a great surprise that the authors of Ref. \[6\] found a conflict between trajectories and density picture to exist in a case of non-ordinary statistical mechanics. This has to do with the diffusion process generated by a fluctuating variable $\xi(t)$, and consequently with the very simple equation of motion

$$\frac{dx}{dt} = \xi(t).$$

The anomalous nature of the resulting diffusion process is generated by the fact that the correlation function of the stochastic variable is

$$\Phi_\xi(t) = \left(\frac{T}{t + T}\right)^\beta,$$

with

$$0 \leq \beta \leq 1.$$

Notice that the form of Eq.\,(2) is essential for the realization of anomalous diffusion. For this form of anomalous diffusion to be of Lévy kind, it is also fundamental to assume the variable $\xi$ to be dichotomous. In this paper, we assume, in fact, that $\xi$ can only have the values $W$ and $-W$. The Lévy nature of the resulting diffusion process can be established using either the Continuous Time Random Walk \[7\] or, as pointed out in Ref. \[8\], the Generalized Central Limit Theorem \[9\]. Both methods, however, are based on the direct use of trajectories, and...
consequently are not yet a rigorous derivation of Lévy statistics from within the Liouville-like approach made necessary by the adoption of density picture [10]. This ambitious task was addressed by the authors of Ref. 11, but it became clear later that the generalized diffusion equation built up by these authors becomes compatible with Lévy statistics through the adoption of a Markov approximation that forces it to depart significantly from the exact density time evolution. The form of this generalized diffusion equation is:

$$\frac{\partial}{\partial t} \sigma(x,t) = \langle \xi^2 \rangle \int_0^t \Phi_\varepsilon(t-t') \frac{\partial^2}{\partial x^2} \sigma(x,t') dt'. \quad (4)$$

The main purpose of this paper is to identify the crucial property that is responsible for the conflict between trajectories and density discovered by the authors of Ref. 12. However, to reach this important result in a satisfactory manner it is necessary to complete the program of the authors of Ref. 12. These authors proposed a Frobenius-Perron operator to describe the dynamics of the fluctuation process responsible for anomalous diffusion. We have to prove that this Frobenius-Perron operator yields the same correlation function as the trajectories produced by the same dynamics. This is usually taken for granted, but in this case, where the equivalence between the two pictures is questioned, it is of fundamental importance to prove that density and trajectory approach yield the same two-time correlation function. This is useful to identify the true source of breakdown. As already pointed out by the authors of Ref. 12 (see also 13), once the Frobenius-Perron equation is established, we have to work with it without any more direct recourse to the concept of trajectories. Thus, we proceed as follows. We devote Section 2 to the derivation of the Frobenius-Perron operator. In Section 3 we derive the correlation function of Eq. (4) using trajectory. In Section 4 we derive the same correlation function from the Frobenius-Perron operator. As earlier stated, this is a crucial result, confirming that the source of discrepancy between trajectory and density has to be found somewhere else. In Section 5 we derive Eq. (4) and in Section 6 we identify the true source of the trajectories-density conflict. It is caused by the four-times correlation function, and plausibly, of higher order. We devote Section 7 to concluding remarks.

II. AN IDEALIZED MODEL OF INTERMITTENT DYNAMICS AND ITS FROBENIUS-PERRON EQUATION

As done in Ref. 1, let us consider the following dynamical system. Let us consider a variable $y$ moving within the interval $I = [0, 2]$, within a sort of overdamped potential with a cusp-like minimum located at $y = 1$. If the initial condition of the particle is $y(0) > 1$, the particle moves from the right to the left towards the potential minimum. If the initial condition is $y(0) < 1$, then the motion of the particle towards the potential minimum takes place from the left to the right. When the particle reaches the potential bottom, it is injected back to an initial condition in the interval $I$. This initial condition, different from $y = 1$, is chosen in a random manner with uniform probability. We thus realize a mixture of randomness and slow deterministic dynamics. The left and right portions of the potential $V(y)$ correspond to the laminar regions of turbulent dynamics, while randomness is concentrated at $y = 1$. In other words, this is an idealization of the map used by Zumofen and Klafter 4, which does not affect the long-time dynamics of the process, yielding only the benefit of a neat distinction between random and deterministic dynamics.

The form of the Frobenius-Perron operator is the following

$$\frac{\partial}{\partial t} p(y, t) = -\frac{\partial}{\partial y} (\lambda p(y, t)) + C(t), \quad (5)$$

where, as we shall see,

$$C(t) \propto p(1, t), \quad (6)$$

which implies the random injection into any point $y$ of the interval $I$, with equal probability, due to the fact that $C(t)$ is independent of $y$. Eq. (5) must fulfill the following physical conditions

$$\frac{d}{dt} \int_{y=[0,2]} p(y, t) dy = \int_{y=[0,2]} \frac{\partial}{\partial t} p(y, t) dy = 0. \quad (7)$$

The condition (7) means that the evolution operator preserves the mass of the distribution that without loss of generality is assumed to be given by

$$\int_{[0,2]} p(y, t) dy = 1. \quad (8)$$

To ensure that plugging Eq. (5) into Eq. (6) yields the natural condition of Eq. (5) and, at the same time, the probability conservation of Eq. (5), we are forced to adopt the following condition for $y$,

$$y = \lambda [\Theta(1-y)+\Theta(-1)(2-y)] + \frac{\Delta_{\lambda_\text{random}}(t)}{\tau_{\text{random}}} \delta(y-1). \quad (9)$$

The function $\Theta(x)$ is the ordinary Heaviside step function, $\Delta_{\lambda}(t)$ is a random function of time that can get any value of the interval $[-1, 1]$, and $\tau_{\text{random}}$ is the injection time that must fulfill the condition of being infinitely smaller than the time of sojourn in one of the laminar phases. Eq. (5), as well as Eq. (6), is the combination of two processes, one deterministic and regular, the motion within the laminar phase, and the other, expressed by the second term on the right hand side of Eq. (6), totally random.

For the calculations that we shall make in Section 4, it is convenient to split the density $p(y, t)$ into symmetric and antisymmetric part with respect to $y = 1$,

$$p(y, t) = p_S(y, t) + p_A(y, t). \quad (10)$$
Of course we always have that:

\[
\int_{[0,2]} p_A(y,t) dy = 0
\]
\[
\int_{[0,2]} p_S(y,t) dy = 1.
\]  
(11)

The Frobenius-Perron equation of Eq. (3), with \( y \) given by Eq. (4), expressed in terms of the symmetric and antisymmetric part, yields the following two equations

\[
\frac{\partial}{\partial t} p_S(y,t) = -\lambda \Theta(1-y) \frac{\partial}{\partial y} (y^2 p_S(y,t)) + \lambda \Theta(y-1) \frac{\partial}{\partial y} [(2-y)^2 p_S(y,t)] + C(t)
\]

and

\[
\frac{\partial}{\partial t} p_A(y,t) = -\lambda \Theta(1-y) \frac{\partial}{\partial y} [(2-y)^2 p_A(y,t)].
\]  
(12)

Note that Eq. (12) yields as an equilibrium state the invariant distribution \( p_0(y) \), whose explicit expression is easily obtained from Eq. (12) to be

\[
p_0(y) = \frac{2-z}{2} \left[ \Theta(1-y) \frac{\partial}{\partial y} \frac{\partial}{\partial y} p_A(y,t) \right] + \lambda \Theta(1-y) \frac{\partial}{\partial y} [(2-y)^2 p_A(y,t)].
\]  
(13)

The reader can easily check this to be the solution of Eq. (12), by setting \( C(t) = \lambda p_0(1) \), \( p_S(y,t) = p_0(y) \), and Eq. (14) to determine \( p_0(1) \) and \( p_0(y) \).

We denote this equilibrium state with the quantum-like symbol \( |p_0\rangle \). We refer ourselves to a notation where the corresponding left state, \( \langle p_0 | \) is nothing but the constant \( 1 \). We write the correlation function \( \Phi_\xi(t) \) under the quantum-like form

\[
W^2 \Phi_\xi(t) = \langle \xi(t) \xi(0) \rangle = \langle p_0 | \xi \exp(\hat{\Gamma} t) \xi | p_0 \rangle,
\]  
(15)

where \( W^2 = \langle \xi \xi \rangle \). We note that when equilibrium is reached, Eq. (15) leaves room only for the state \( |p_0\rangle \). All the other excited states would imply an out of equilibrium condition. Note that the diffusion process under study in this paper is stationary. This means that the correlation function is evaluated in the correspondence of the equilibrium condition \( |p_0\rangle \). The consequence of this is significant. In fact the application of the operator \( \hat{\xi} \) to \( |p_0\rangle \) generates an antisymmetric state. The time evolution of this excited state is driven by the operator \( \hat{\Gamma} \) defined by Eq. (13). This means that the time evolution takes place in the antisymmetric space. When we apply to this time dependent state the operator \( \hat{\xi} \) we obtain again a symmetric state. However, the only symmetric state available is \( |\tilde{p}_0\rangle \). This explains the form of Eq. (15). If we apply the same argument to the four-times correlation function we obtain, in the time ordered case \( t_1 \leq t_2 \leq t_3 \leq t_4 \),

\[
\langle \xi(t_1) \xi(t_2) \xi(t_3) \xi(t_4) \rangle = \langle \xi(t_2-t_1) \xi(t_3-t_4) \xi(t_2-t_1) \xi(t_3-t_4) \rangle.
\]  
(16)

This is the four-times case of a more general property that, using a concise notation, with an evident meaning, we write as

\[
\langle 12...2n \rangle = \langle 12 \rangle \langle (n-1)2n \rangle.
\]  
(17)

The authors of Ref. [5] proved that the property of Eq. (17) yields the generalized diffusion equation of Eq. (3). This confirms that Eq. (3) is a legitimate and rigorous consequence of the Frobenius-Perron picture.

III. THE CORRELATION FUNCTION OF THE DICHOTOMOUS FLUCTUATION FROM THE TRAJECTORY PICTURE

Let us focus our attention on Eq. (9) and consider the initial condition \( y_0 \in [0,1] \). Then, as it is straightforward to prove, the solution for \( y < 1 \) is:

\[
y(t) = y_0 (1 - \lambda (z-1) y_0 z^{-1} t^{-1/z-1}).
\]  
(18)

Using Eq. (18) and setting \( y(T) = 1 \), we find the time at which the trajectory reaches the point \( y = 1 \). The solution is:

\[
T = T(y_0) = \frac{1 - y_0 z^{-1}}{\lambda(z-1) y_0^{1/z-1}}.
\]  
(19)

Note that the function \( \hat{\xi}(t) \) is determined by \( \hat{\xi}(t) = \exp(\hat{\Gamma} t) \). Thus, from Eq. (15) we obtain:

\[
\hat{\xi}(t) W = [\Theta(1-y_0) \Theta(T(y_0) - t) - \Theta(y_0 - 1) \Theta(T(2-y_0) - t)]
\]

\[
\times \left[ \Theta \left( \sum_{k=0}^{i+1} k \tau_k - t \right) - \Theta \left( \sum_{k=0}^{i} k \tau_k - t \right) \right],
\]  
(20)

where the times of sojourn in the laminar phases \( \tau_i \)s read

\[
\tau_0 = T(y_0) = \frac{1 - y_0 z^{-1}}{\lambda(z-1) y_0^{1/z-1}} \Theta(1-y_0)
\]

\[
+ \frac{1 - (2-y_0) z^{-1}}{\lambda(z-1)(2-y_0)^{1/z-1}} \Theta(y_0 - 1)
\]

\[
\tau_{i \geq 1} = \frac{1 - [\Delta y(\tau_i)] z^{-1}}{\lambda(z-1)[\Delta y(\tau_i) z^{-1}] \Theta(-\Delta y(\tau_i))}
\]

\[
+ \frac{1 - [\Delta y(\tau_i)] z^{-1}}{\lambda(z-1)[1 - \Delta y(\tau_i)] z^{-1} \Theta(\Delta y(\tau_i))}.
\]  
(21)
Then, we have:
\[
\frac{\langle \xi(t)\xi(0) \rangle}{W^2} = \langle \Theta(1 - y_0)\Theta(T(y_0) - t) + \Theta(y_0 - 1)\Theta(T(2 - y_0) - t) \rangle + \sum_{i=0}^{\infty} \text{sign}(y_0 - 1)\text{sign} \left[ \Delta_y \left( \sum_{k=0}^{i} \tau_k \right) \right] \\
\times \left[ \Theta \left( \sum_{k=0}^{i+1} \tau_k - t \right) - \Theta \left( \sum_{k=0}^{i} \tau_k - t \right) \right].
\]

As pointed out in Section 2, the calculation of the correlation function rests on averaging on the invariant distribution of Eq. (14). As a consequence of this averaging the second term in (22) vanishes. In fact, the quantity to average is antisymmetric, whereas the statistical weight is symmetric.

It is possible to write the surviving term as:
\[
\frac{\langle \xi(t)\xi(0) \rangle}{W^2} = (2 - z) \int_0^1 \Theta \left( \frac{1 - y^{z-1}}{(z-1)y^{z-1} - 1} - t \right) \frac{1}{y^{z-1}}dy \\
= (2 - z) \int_0^{(1+\lambda(z-1)t)^{-1}(z-1)} y^{-z+1}dy \\
= (1 + \lambda(z - 1)t)^{-2(z-1)/(z-1)} \\
\equiv (1 + \lambda(z - 1)t)^{-\beta},
\]
with
\[
\beta = \frac{2 - z}{z - 1}.
\]

Since we focus our attention on $0 < \beta < 1$, we have to consider $3/2 < z < 2$. Note that the region $1 < z < 3/2$ does not produce evident signs of deviation from ordinary statistics. However, as we shall see in Section 5, an exact agreement between density and trajectory is recovered only at $z = 1$, when the correlation function becomes identical to the exponential function $\exp(-\lambda t)$. Note also that Eq. (23) becomes identical to Eq. (2) after setting the condition
\[
\lambda(z - 1) = \frac{1}{T}.
\]

IV. THE CORRELATION FUNCTION OF THE DICHTOMOUS FLUCTUATION FROM THE DENSITY PICTURE

The result of the preceding section is reassuring, since it proves that the intermittent model we are using generates the wanted inverse power law form for the correlation function of the dichotomous variable $\xi(t)$. However, it is based on the adoption of trajectories. In this Section we get an even more important result, this being the fact that the same correlation function is derived from the use of the Frobenius-Perron equation of Eq. (5), with no direct use of trajectories.

To fix the ideas, let us consider the following system: a particle in the interval $[0,1]$ moves towards $y = 1$ following the prescription $\dot{y} = \lambda y^z$ and when it reaches $y = 1$ it is injected backwards at a random position in the interval. The evolution equation obeyed by the densities defined on this interval is the same as Eq. (3), with $C(t) = \lambda p(1,t)$. This dynamic problem was already addressed in Refs. [11, 12], and solved using the method of characteristics detailed in Ref. [13]. Let us remind the reader that the solution afforded by the method of characteristics, in this case yields:
\[
p(y,t) = \int_0^t \frac{p(1,\xi)}{g_z((t - \xi)y^{z-1})} d\xi \\
+ p \left( 1 - \frac{y}{g_z(y^{z-1}t)}, 0 \right) \frac{1}{g_z(y^{z-1}t)},
\]
where
\[
g_z(x) \equiv (1 + \lambda(z - 1)x)^{z/(z-1)}.
\]

To find the quantity $\langle \xi(t)\xi(0) \rangle$ using only density, we have to solve Eqs. (12) and (13), which are equations of the same form as that yielding Eq. (24). For this reason we adopt the method of characteristics again. To do the calculation in this case, it is convenient to adopt a frame symmetric respect to $y = 1$. Then, let us define $Y = y - 1$. Using the new variable and Eq. (23), we find for Eqs. (12) and (13) the following solution
\[
ps(Y,t) = \int_0^t \frac{ps(0,\xi)}{g_z((t - \xi)(1 - |Y|)^{z-1})} d\xi \\
+ ps \left( 1 - \frac{1 - |Y|}{g_z((1 - |Y|)^{z-1}t)}, 0 \right) \times \\
\times \frac{1}{g_z((1 - |Y|)^{z-1}t)}
\]
and
\[
p_A(Y,t) = p_A \left( \text{sign}(Y) \left[ 1 - \frac{1 - |Y|}{g_z((1 - |Y|)^{z-1}t))}, 0 \right) \times \\
\times \frac{1}{g_z((1 - |Y|)^{z-1}t)}
\]
Then, the solution consists of two terms:

- the former is an even term and is responsible for the long-time limit of the distribution evolution;
- the latter is an odd term and it disappears in the long-time limit. Let us point out why this feature is beneficial: Eq. (13) does not contain the injection term $C(t)$ and the equilibrium distribution function is given by the density distribution of Eq. (14), which is an even function, no matter what the symmetry of the initial distribution function is.
As pointed out in Sections 2 (see Eq. (13) and Eq. (16)), the correlation function of the operator $\xi$ is determined by the time independent operator driving the time evolution of the anti-symmetric distribution. As proved in Section 5, this property is of fundamental importance for a rigorous derivation of the generalized diffusion equation of Eq. (4). In fact, this property means that the anti-symmetric space is the irrelevant space, to be traced out for the derivation of this important diffusion equation. The direct calculation of this correlation function yields:

$$\langle \xi(t)\xi(0) \rangle = (2 - z) \int_0^1 \frac{1}{(1 - \xi)^{z-1}} |_{\xi = 1 - \frac{1}{g_z((1 - Y)^{z-1})}} \times \frac{1}{g_z((1 - Y)^{z-1})} dY. \quad (30)$$

The integral (30) is exactly solvable and leads to:

$$\langle \xi(t)\xi(0) \rangle = (1 + \lambda(z-1)t)^{-(2-z)/(z-1)} \quad (31)$$

which is the same result as that found in Section 3, using the trajectories. In a similar fashion, it is possible to calculate $(Y(t)Y(0))$: its temporal behavior is a power law with the same exponent as that of Eq. (31).

V. THE GENERALIZED DIFFUSION EQUATION REVISITED

In this section we shall derive again the generalized diffusion equation of Eq. (4) in a way that is more closely connected with the formalism of Sections 2 to 4. We define

$$|p_1\rangle = \frac{\xi}{W}|p_0\rangle. \quad (32)$$

This is an antisymmetric state. Consequently its time evolution is driven by the operator $\hat{\Gamma}$ defined by Eq. (13). We now assume (17) that the operator $\hat{\Gamma}$ has a complete basis set of eigenvectors with a non-necessarily discrete spectrum, even if we adopt for simplicity a discrete notation, setting aside the easy problem of moving from the discrete to the continuous condition. Let us assume

$$\hat{\Gamma}|\mu\rangle = \gamma_\mu|\mu\rangle \quad (33)$$

and

$$\langle \hat{\mu}|\Gamma^+ = \langle \hat{\mu}|\gamma_\mu^*. \quad (34)$$

so that the biorthonormal condition

$$\langle \hat{\mu}|\mu'\rangle = \delta_{\mu,\mu'} \quad (35)$$

applies. All this allows us to write the correlation function $\Phi_\xi(t)$ under the following attractive form

$$\Phi_\xi(t) = \langle p_1|\exp(\hat{\Gamma}t)|p_1\rangle = \sum_{\mu} \langle \hat{p}_1|\mu\rangle \langle \hat{\mu}|p_1\rangle \exp(\gamma_\mu t) \quad (36)$$

We are now in a position to derive the generalized diffusion equation of Eq. (4) from a Liouville-like equation. It is evident that the Liouville-like equation of the whole system, corresponding to the Frobenius-Perron treatment of the system of variables $\xi$ and $y$ of the preceding sections, is given by

$$\frac{\partial}{\partial t}\Pi(x, \xi, y, t) = \left(-\xi \frac{\partial}{\partial x} + \Gamma\right)\Pi(x, \xi, y, t). \quad (37)$$

Let us adopt the quantum-like formalism and let us set

$$\rho_\mu \equiv \langle \hat{\mu}|\Pi(t)\rangle \quad (38)$$

and

$$\sigma(x, t) \equiv \rho_0(x, t) \equiv \langle \hat{\rho}_0|\Pi(t)\rangle. \quad (39)$$

Thus, by multiplying Eq. (17) by $\langle \hat{\rho}_0|$ and $\langle \hat{\mu}|$, we get

$$\frac{\partial}{\partial t}\sigma(x, t) = -W \sum_\mu \langle \hat{\rho}_1|\mu\rangle \frac{\partial}{\partial x}\rho_\mu(x, t) \quad (40)$$

and

$$\frac{\partial}{\partial t}\rho_\mu(x, t) = -W \frac{\partial}{\partial x}\sigma(x, t) + \gamma_\mu \rho_\mu(x, t) \quad (41)$$

respectively. The solution of Eq. (41) is

$$\rho_\mu(x, t) = -W \int_0^t \exp(\gamma_\mu(t - t')) \frac{\partial}{\partial x}\sigma(x, t') dt'. \quad (42)$$

By plugging Eq. (42) into Eq. (40) we derive the fundamental equation of Eq. (4). It is easy to prove (8) that this equation yields a hierarchy of moments $\langle x^{2n}(t)\rangle$, with $n = 1, 2, ..., \infty$, which coincides with the hierarchy that would be generated by a fluctuation $\xi(t)$ with the correlation functions obeying the prescription of Eq. (17). In the next section we shall see that this conflicts with the prescriptions generated by the adoption of trajectories. The arguments of Section 6 are limited to the case of four-times correlation function. However, we think that this is enough to identify the source of the conflict between trajectories and density.

VI. FOUR-TIMES CORRELATION FUNCTION

In this section we show that the four-times correlation function stemming from probabilistic arguments and trajectories do not fulfill Eq. (16). We shall see that we can write the four-times correlation function in terms of a different combination of two-times correlation function. In this way we can connect higher-order correlation function to the waiting time distribution $\psi(t)$ previously defined. Note that the waiting time distribution and the correlation function are linked via

$$\Phi_\xi(t_2 - t_1) = \frac{1}{T} \int_{t_2 - t_1}^{\infty} [T - (t_2 - t_1)] \psi(T) dT. \quad (43)$$
This means that the correlation function can be viewed as the probability of finding $t_1$ and $t_2$ in the same laminar region. In fact, the occurrence of one or more transitions, between $t_1$ and $t_2$, would make the correlation function vanish due to the averaging on uncorrelated fluctuations, either positive or negative. Using similar arguments the four-times correlation function can be viewed as the probability that both $t_1$ and $t_2$ belong to the same laminar region. We shall also use the notation $p(ij)$ as the probability that $t_i$ and $t_j$ belong to the same laminar region. We shall also use the notation $p(\bar{i}j)$ as the probability that at least one transition occurs between $t_i$ and $t_j$. With the usual Bayesian notation we indicate with $p(A,B)$ the joint probability of events $A$ and $B$ and with $p(A|B)$ the conditional probability of $A$ given $B$ (thus implying $p(A,B) \equiv p(B)p(A|B)$). It is easy to cast the four-times correlation function in our notation as (with $W^2 = 1$):

$$
\langle \xi(t_1)\xi(t_2)\xi(t_3)\xi(t_4) \rangle = p(12,34)
$$

$$
= p(14) + p(23)p(12|23)p(34|23)
$$

$$
= p(14) + \frac{(p(12) - p(12,23))(p(34) - p(34,23))}{1 - p(23)}
$$

$$
= \Phi_\xi(t_4 - t_1)
$$

$$
+ \frac{\Phi_\xi(t_2 - t_1) - \Phi_\xi(t_3 - t_1)}{1 - \Phi_\xi(t_3 - t_2)},
$$

where we used the general property:

$$
p(A|B) = \frac{p(A) - p(A,B)}{1 - p(B)}
$$

and the fact that in our problem

$$
p(ij,jk) = p(ik).
$$

The leading order of Eq. (44) is the first term, namely $\Phi_\xi(t_4 - t_1)$. This yields, for the fourth moment

$$
\langle x^4(t) \rangle = 8 \int_0^{t_4} dt_4 \int_0^{t_4} dt_3 \int_0^{t_3} dt_2 \int_0^{t_2} dt_1 \langle \xi(t_1)\xi(t_2)\xi(t_3)\xi(t_4) \rangle,
$$

and, if the correlation function has a slow decay as in $\xi(t) \propto t^{4-\beta}$, notice that if condition (44) held true, we would have $\langle x^4(t) \rangle \propto t^{4-2\beta}$. Generalizing the 4-times treatment to the $2n$-times, the trajectory prescription yields

$$
\langle x^{2n}(t) \rangle \propto t^{2n-\beta},
$$

as recently shown in Ref. [17]. On the other hand, the factorization of the even correlation functions, which is exact in the density treatment, leads to $\langle x^{2n}(t) \rangle \propto t^{2n(1-\beta/2)}$, fulfilling (44). This latter property reflects the fact that the solution of the generalized diffusion equation (4) has a solution which rescales with time, while its trajectory counterpart does not obey an exact rescaling (7).

Remarkably, in the exponential case, with some tedious algebra it is straightforward to show that

$$
\langle \xi(t_1)\xi(t_2)\xi(t_3)\xi(t_4) \rangle = \langle \xi(t_1)\xi(t_2) \rangle \langle \xi(t_3)\xi(t_4) \rangle, \quad \text{(49)}
$$

which coincides with prescription (16). With boring but straightforward arguments it is possible to prove that Eq. (49) yields the more general condition of Eq. (17). In conclusion, in the exponential case the agreement between density and trajectories is complete.

VII. CONCLUDING REMARKS

The discrepancy between trajectories and density is a surprising result. The first report of this disconcerting property was given in Ref. [8]. However, in that paper the derivation of the generalized diffusion equation, Eq. (4), from a Frobenius-Perron picture was not as accurate as in the present paper, thereby leaving room to the criticism that this equation is not compatible with a rigorous Frobenius-Perron picture. This is the reason why Sections 2, 3 and 4 have been devoted to a careful discussion of the Frobenius-Perron picture. Of remarkable interest is the content of Section 4 which derives the crucial correlation function of Eq. (4) without using the concept of trajectory. There is no doubt, therefore, that the density picture adopted is the proper representation of the process under study.

The importance of the Gibbs picture for statistical mechanics is well known, and it is well known that the ergodic properties have been introduced to replace the average over infinitely many copies of the same system with a more realistic average in time [18]. However, there is no guarantee that the two pictures are equivalent, in general. Thus, to a first sight the reader might think that the breakdown of the equivalence between trajectories and density rests on the failure of the ergodic condition, a relatively trivial property. The effect discovered by the authors of Ref. [6] does not have anything to do with the breakdown of the ergodic condition. If we focus our attention on the variable $y$ and on the Frobenius-Perron picture of Section 2, we do not find any disagreement between trajectories and density, the results of Sections 3 and 4 lead us to expect that the numerical study of a bunch of trajectory yields for the time evolution of a non equilibrium probability distribution the same time evolution as that prescribed by the Frobenius-Perron operator. On top of that the variable $y$ is ergodic and the generalized diffusion equation of Eq. (4) is based on the correct assumption that the Gibbs averages are equal to the time average on time of a single trajectory. This is a correct assumption. The reason of the failure of Eq. (4)
to recover the results of numerical calculations based on trajectories is much deeper and subtler than the failure of the ergodic condition. This reason has to do with the fact that the higher-order correlation functions are expressed in terms of the two-time correlation functions in two distinct ways, the trajectory and the density way. We have shown in details that the four-times correlation function compatible with the generalized diffusion equation of Eq. (6) is the product of two second-order (two-times) correlation functions. In the non-exponential case, the four-times correlation function generated by the trajectories breaks this condition. Extending the kind of arguments used in Section 6 to correlation functions of higher order involves much more complicated calculations, but the ballistic contribution always shows up, and the formulas can be expressed in terms of pairs of two-time correlation functions, although by means of expressions of increasing length. In conclusion, even if for simplicity it is not done here, it is possible to prove that in the case of exponential relaxation, the factorization condition is fulfilled by the correlation functions of any order. This means that the exponential case is the only one where the equivalence between trajectory and density is complete.

What might be the consequence of this result? In the introduction we mentioned the possible consequences on the foundation of quantum mechanics. One might be tempted to conclude that the foundation of Lévy processes on the basis of classical trajectories rules out the possibility of deriving Lévy diffusion from a quantum mechanical treatment, in conflict with the conclusions of several papers that suggest the opposite to be true \cite{19,20,21,22}. We have to point out that all these papers studied the kicked rotor, and with this dynamical system the condition of ballistic motion for extended times is realized by a trajectory sticking to the border between chaotic sea and acceleration islands. This sticking is actually due to the fact that the classical trajectories keeps exploring fractal regions of decreasing size. The arguments used to justify the birth of anomalous diffusion rest on classical trajectories, and the quantum breakdown of this condition of anomalous diffusion is due to the broadening of the quantum wave function that prevents it from exploring border regions of even infinitesimally small size. It is plausible that the realization of Lévy statistics cannot be complete, and that the breakdown of anomalous diffusion takes place in the very moment when tunneling processes are activated. All this seems to prove that the papers of Refs. \cite{19,21,22} cannot be used to argue against the conclusion of this paper, and cannot be used to support them either.

We think that the conclusions of this paper might be of interest for quantum mechanics at a different level. The experiment on resonant fluorescence on single atoms have revealed experimentally the occurrence of quantum jumps \cite{23}. Nevertheless the discovery that the Bloch equations consists of two contributions, one compatible with the unitary nature of quantum mechanics, and one equivalent to the von Neumann measurement postulate, makes it plausible to reach the conclusion that, as a relevant example, the reader can find in the introduction of the book of Ref. \cite{24}. This is that the hypothesis of spontaneous collapses is unnecessary. The contraction on the irrelevant degrees of freedom would be enough to mimic properties that are naively interpreted as a manifestation of collapses.

This view is generally accepted, but it is challenged by new discoveries. We note that the unraveling process compatible with the occurrence of quantum jumps rests its foundation on the connection between random walker trajectories and master equation, established years ago by the authors of Ref. \cite{25}. These authors proved that an exponential waiting time distribution implies a Markovian master equation. Using the more modern jargon of de-coherence literature \cite{26} we can say that the exponential waiting time distribution is compatible with the quantum mechanical master equations that have the Lindblad structure. However, recent experimental studies on semiconductor quantum dots \cite{26,27} reveal intermittent signals corresponding to a bright state emitting many photons and to a dark state with no photon emission. The waiting time distributions of these two states turn out to exhibit inverse power law properties for several decades. There are already conjectures that this strong deviation from an exponential behavior might be due to the existence of slow modulation of barrier widths or heights \cite{23}.

It is worth noticing that the authors of Ref. \cite{26} used this modulation hypothesis to discuss an ideal physical problem that it is very similar to the blinking semiconductors. These authors studied the case where a given Pauli matrix $σ_x$ is forced by the environment to yield a symbolic sequence of numbers, corresponding to its two eigenvalues, a sequence that can be thought of as corresponding to that of a blinking semiconductor, with, for instance, the eigenvalue 1 standing for light, and the eigenvalue $-1$ for darkness. This fluctuating dipole drives the motion of another 1/2 spin system, serving a detection purpose. Actually, this second spin system is essentially equivalent to the Kubo stochastic dipole \cite{30} used by the authors of Ref. \cite{21}.

The authors of Ref. \cite{29} found a surprising effect. They studied the relaxation properties of the detecting dipole driven by the intermittent signal, with a modulation assumption about the origin of its non-Poisson nature that fits the conjectures made to explain the properties of blinking quantum dots \cite{28}. The waiting time distribution was assumed to have a diverging second moment, a property shared indeed by the blinking quantum dots \cite{28}. The calculation of these spectroscopy properties was done in two distinct ways. The former was compatible with the density prescription, with the only constraint that the fluctuation correlation function has to coincide with that of the symbolic sequence realized by the environmental induced process. The latter calculation was done, on the contrary, noticing that the spectroscopy property can also be expressed through a
characteristic function. The explicit analytical form of this characteristic function is determined by the central limit theorem, either normal, in the case of ordinary statistical mechanics, or generalized, in the case of interest, that is, the case of waiting time distributions with a diverging second moment. This latter way of doing the calculation yielded a result conflicting with the former, with the detecting dipole correlation function decaying as an exponential rather than as an inverse power law. We think that the present article affords a satisfactory explanation of why the two methods adopted yield different results. This is so because only the exponential relaxation make the correlation function stemming from the density picture identical to those produced by trajectories. A deviation from the Poissonian condition seems to have dramatic consequences. We hope that this paper might serve the purpose of triggering further investigation on this important issue.

VIII. ACKNOWLEDGMENTS

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