The mod 2 Steenrod and Dyer-Lashof algebras as quotients of a free algebra

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Abstract. A non-connected neither of finite type Hopf algebra $F_0$ is defined over $\mathbb{Z}/2\mathbb{Z}$ and its hom dual turns out to be a tensor product of polynomial algebras. Certain quotient Hopf algebras include the Steenrod and Dyer-Lashof algebras. This setting provides a map between the Steenrod coalgebra and a direct limit of Dyer-Lashof coalgebras.

Certain important algebras in algebraic topology admit the structure of a Hopf algebra and their duals turned out to be polynomial algebras. It is well known that the homology Hopf algebras $H_*(BU;\mathbb{Z})$ and $H_*(BO;\mathbb{Z}/p\mathbb{Z})$ enjoy a very special property: that they are self-dual, so that they are isomorphic to the cohomology Hopf algebras $H^*(BU;\mathbb{Z})$ and $H^*(BO;\mathbb{Z}/p\mathbb{Z})$. Following the original paper of Milnor and Moore [14] people have studied Hopf algebras, mainly connected and of finite type. Connected structure arise ubiquitously in algebraic topology. The homology of non-connected but grouplike homotopy associative H-space leads to the more general setting of component Hopf algebras. Most of the structure Theorems for connected Hopf algebras are best derived from the Poincaré-Birkoff-Witt Theorem. If one drops any one of the hypotheses, then one is likely to find counterexamples.

The purpose of this note is to connect two of the most well known Hopf algebras in algebraic topology. The first is the mod 2 Steenrod algebra and the second is the mod 2 Dyer-Lashof algebra. We re-derive Milnor’s and Madsen’s results. Milnor [13] showed that the hom dual of the mod 2 Steenrod algebra is a polynomial algebra and Madsen [10] proved that the hom dual of the mod 2 Dyer-Lashof algebra turned out to be a tensor product of polynomial algebras. The first is connected of finite type and the second is not. The corresponding proofs were not an easy task and their description as algebras were given in terms of generators. This phenomenon studied by certain authors, namely the notion of a Hopf algebra been copolynomial i.e. to have polynomial dual. It is the purpose of this note to provide a general setting including both the Steenrod (section 2) and Dyer-Lashof algebras (section 3). This setting also provides an induced map from the Steenrod coalgebra to a direct limit of Dyer-Lashof coalgebras (corollary 5).

A lot of research has been done concerning similarities between the Steenrod ([18]) and Araki-Kudo ([1]) Dyer-Lashof ([5]) algebras. Let us just recall a few of

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them. First of all May’s paper (12) gave a treatment of co-homology operations in an algebraic setting which included all the usual cases. In his setting \( P^0 \) (the zero operator) need not be the identity operator. He proved that the Araki-Kudo Dyer-Lashof and Steenrod operations are special cases of a single general construction. Bisson and Joyal (2) defined a \( Q \)-ring equipped with square operations satisfying the Adem-Cartan formulae. It turned out that the homology of the 0-space of the sphere spectrum is the free \( Q \)-ring on one generator and the dual of the Steenrod algebra is also a \( Q \)-ring provided the zero operator need not be the identity. Pengelley and Williams (16) constructed a bigraded algebra, called the Kudo-Araki-May algebra, \( K \) such that it is isomorphic to \( R \) as coalgebras. They also proved that there is an algebra bijection between \( A_2 \) and a certain inverse limit of subspaces of \( K \).

It shall also be mentioned that certain Hopf algebras admit the action of the opposite of the Steenrod algebra. The action of the Steenrod algebra on certain duals has been studied extensively because of important applications.

We thank Professor May for bringing to our attention many relevant works.

Let \( F_0 \) denote the free graded associative algebra with unit generated by \( \{ Q^i \mid i \geq 0 \} \) over \( \mathbb{F}_2 \) with degrees \( |Q^i| = i \). A typical element of \( F_0 \) is given by juxtaposition and is of the form

\[ Q^I = Q^{i_k} \ldots Q^{i_1} \]

associated with the sequence \( I = (i_k, \ldots, i_1) \), \( I \in \mathbb{N}^k \). Define the degree, length and excess of \( I \) by

\[ |I| = \sum_{t=1}^{k} i_t, \quad l(I) = k \quad \text{and} \quad e(I) = i_k - \left( \sum_{t=1}^{k-1} i_t \right) \]

The empty sequence representing the identity element is of degree zero, length zero, and of infinite excess. Let \( F_0^{(i)} \) denote the vector subspace of \( F_0 \) spanned by monomials of degree \( i \). Since \( F_0^{(i)} \) is an infinite dimensional vector space, \( F_0 \) is an object considered intractable. It turns out to be manageable using the length.

\( F_0 \) admits the structure of a Hopf algebra with coproduct defined on generators by

\[ \psi(Q^I) = \sum_{t=0}^{i} Q^{i-t} \otimes Q^t \]

and with augmentation defined on \( F_0^{(0)} = \mathbb{F}_2[Q^0] \) by \( \varepsilon(Q^0 \cdots Q^0) = 1 \). Here \( F_0^{(0)} \) is a polynomial algebra. Thus \( F_0 \) is not connected, neither of finite type. We prove that the dual of \( F_0 \), \( F_0^* \), is a tensor product of polynomial algebras (Theorem 1). This is a key result. Certain interesting quotients of \( F_0 \) including both the Steenrod and Dyer-Lashof algebras can be studied in this setting.

Let \( R[k] \) denote the subcoalgebra of the Dyer-Lashof Hopf algebra \( R \) consisting of fixed length \( k \) elements (see 6). A coalgebra map \( R[k] \rightarrow R[k+1] \) is defined respecting the opposite of the Steenrod algebra action and the family of these maps induce a direct limit \( \lim_{\rightarrow} R[k] \). Finally a coalgebra map between the Steenrod algebra and the previous limit is induced in corollary 6.

We extend our results to the \( p \) odd prime number case in \( 9 \). But instead of the Steenrod and Dyer-Lashof algebras one will consider the sub-Hopf algebras generated by the reduced powers \( P^n \) and \( Q^n \) respectively.
1. The free algebra

For each monomial $Q^I$ we denote its sequence $J$ by $I \langle Q^J \rangle$. The sum $I + J$ and difference $I - J$ of two sequences of length $k$ is defined termwise. $I - J$ is undefined, if an entry is less than zero. Then $e \left( I + J \right) = e(I) + e(J)$, $d(I + J) = d(I) + d(J)$.

**Definition 1.** We define a total ordering on the set of finite sequences as follows.

a) $I < J$, if $l(I) < l(J)$.
b) If $I = (i_k, \ldots, i_1)$ let $I_t = (i_t, \ldots, i_1)$ for $1 \leq t \leq k$. For sequences of the same length we define $I < J$, if $e(I_t) < e(J_t)$ for the smallest $t$ such that $e(I_t) \neq e(J_t)$.

Observe that $e(I_t) = e(J_t)$ for all $t$ implies that $I = J$ provided $l(I) = l(J)$.

**Lemma 1.** $\mathcal{F}_0$ is a component coalgebra with respect to length and if $\pi\mathcal{F}_0 = \{Q^I \mid \psi(Q^I) = Q^I \otimes Q^J\}$, then $\pi\mathcal{F}_0$ is the free monoid generated by $Q^0$. Moreover, the component $\mathcal{F}_0[k]$ of $(Q^0)^k$ is the subcoalgebra of $\mathcal{F}_0$ spanned by

$$\{Q^I \mid l(I) = k\}.$$  

Thus

$$\mathcal{F}_0 = \bigoplus_{k \geq 0} \mathcal{F}_0[k] \text{ with } \mathcal{F}_0[0] = \mathbb{F}_2.$$  

Moreover, $\mathcal{F}_0[k] \otimes \mathcal{F}_0[l] \rightarrow \mathcal{F}_0[k + l]$.

We shall determine the primitive elements of $\mathcal{F}_0[k]$. Define

$$x_{k,0} = \underbrace{Q^0 \cdots Q^0}_k \text{ for } 1 \leq k;$$

$$x_{k,i} = \underbrace{Q^0 \cdots Q^0}_{k-i}Q^1Q^0 \cdots Q^0 \text{ for } 1 \leq i \leq k.$$  

**Lemma 2.** $x_{k,i}$ is a primitive monomial for $k \geq 1$ and $k \geq i \geq 1$. Moreover, \{x_{k,i} \mid 1 \leq i \leq k\} is a basis for the primitive elements of $\mathcal{F}_0[k]$.

**Proof.** Proceed by induction on $k$. The case $k = 1$ is trivial. Let $I = (i_k, J)$ with $J = (i_{k-1}, \ldots, i_1)$. There are two cases to be considered: $i_k = 0$ and $i_k = 1$. If $i_k = 0$, then $Q^I$ is a primitive element. Let $i_k = 1$ and $J \neq (0, \ldots, 0)$. Then $\psi(Q^I)$ contains the summand $Q^1Q^{i_0} \cdots Q^i$. The claim follows. \qed

To describe the hom dual of $\mathcal{F}_0$ we proceed by analogy with Madsen’s computations of the dual of the Dyer-Lashof algebras. The computation of $\mathcal{F}_0[k]^*$ is based on multiplication of the duals of monomials.

Let $I[k]$ be the basis of monomials of $\mathcal{F}_0[k]$.

**Lemma 3.** Define $\phi : \mathbb{N}^k \rightarrow I[k]$ by $\phi(n_k, \ldots, n_1) = \bigoplus_{k} n_i I(x_{k,i})$. Then $\phi$ is an isomorphism of sets.
Since \( \mathcal{F}_0 \) is a coassociative, cocommutative coalgebra, \( \mathcal{F}_0^* = \prod \mathcal{F}_0 [k]^* \) is an algebra. We define elements of \( \mathcal{F}_0 [k]^* \) by

\[
y_{k,0} = \left( (Q^0)^k \right)^*, \quad k \geq 0; \quad y_{k,i} = (x_{k,i})^*, \quad 1 \leq i \leq k.
\]

Now \( y_{k,0} \) is the identity element in \( \mathcal{F}_0 [k]^* \) and \( \prod y_{k,0} \) is the identity element in \( \mathcal{F}_0^* \). The augmentation of \( \mathcal{F}_0^* \) is given by

\[
\varepsilon \prod a_k y_{k,0} = a_0.
\]

\( \mathcal{F}_0^* \) is not a coalgebra and \( \mathcal{F}_0 [k]^* \cdot \mathcal{F}_0 [l]^* = 0 \), for \( k \neq l \).

**Theorem 1.** \( \mathcal{F}_0 [k]^* \) is a polynomial algebra on \( \{y_{k,i} \mid 1 \leq i \leq k\} \).

**Proof.** Let \( \langle \cdot , \cdot \rangle : \mathcal{F}_0 [k]^* \otimes \mathcal{F}_0 [k] \rightarrow \mathbb{F}_2 \) be the Kronecker product and \( \Phi_k : \mathcal{F}_0 [k] \rightarrow \mathcal{F}_0 [k + 1] \) be the map given by \( \Phi_k (Q^I) = Q^0 Q^I \). \( \Phi \) is a degree preserving map of coalgebras such that

\[
\Phi (x_{k, i}) = x_{k + 1, i + 1} \quad \text{and} \quad \Phi^* (y_{k + 1, i + 1}) = y_{k, i} \quad \text{for} \quad i \leq k \quad \text{and} \quad \Phi^* (y_{k + 1, 1}) = 0.
\]

We compare the polynomial algebra on \( \{y_{k,i} \mid 1 \leq i \leq k\} \) with \( \mathcal{F}_0 [k]^* \).

Let \( \Lambda = (\lambda_1, \cdots, \lambda_k) \in \mathbb{N}^k \) and

\[
\xi^\Lambda = \prod y_{k,i}^\lambda.
\]

Let \( Q^\Lambda (I) = Q^{\lambda_1} \cdots Q^{\lambda_k} \) and \( \sum \lambda_i = \lambda \). Here \( \Lambda (I) = \sum \lambda_i I (x_{k,i}) \). We claim that

\[
\langle \xi^\Lambda, Q^J \rangle \equiv 1 \text{ mod } 2 \implies J \geq \Lambda (I).
\]

Let \( \psi (\lambda) : \mathcal{F}_0 [k] \rightarrow (\mathcal{F}_0 [k])^\Lambda \) be the iterated coproduct. Then

\[
\psi (\lambda) (Q^I) = \sum Q^{J_1} \otimes \cdots \otimes Q^{J_k} \quad \text{such that} \quad \sum J_t = J.
\]

Let \( J = (\mu_k, \cdots, \mu_1) \) with \( \sum \mu_i = \lambda \).

\[
\langle \xi^\Lambda, Q^J \rangle \equiv 1 \text{ mod } 2 \implies \mu_1 \geq \lambda_1 \text{ and } J \geq \Lambda (I).
\]

Next we consider the case \( \mu_1 = \lambda_1 \). Let \( \Lambda' = (\lambda_k, \cdots, \lambda_2, 0) \) and \( J' = J - \lambda_1 I (x_{k,1}) \).

Then \( \langle \xi^\Lambda, Q^{J'} \rangle \equiv 1 \text{ mod } 2 \) and

\[
Q^{J'} = \Phi (Q^{J'}) \quad \text{such that} \quad \sum_{i=2}^k \mu_i I_{k-1,t-1} = J'.
\]

Let \( \Lambda'' = (\lambda_k, \cdots, \lambda_2) \), then

\[
\langle \xi^\Lambda, \Phi (Q^{J'}) \rangle \equiv 1 \text{ mod } 2 \implies \langle \Phi^* (\xi^{\Lambda'}), Q^{J''} \rangle \equiv 1 \text{ mod } 2 \text{ and }
\]

\[
\langle \xi^\Lambda, Q^{J''} \rangle \equiv 1 \text{ mod } 2.
\]

As before \( \mu_2 \geq \lambda_2 \). Continuing in this fashion \( J \geq \Lambda (I) \).

Now, \( \mathbb{F}_2 [y_{k,i} \mid 1 \leq i \leq k] \) is contained in \( \mathcal{F}_0 [k]^* \), but the two vector spaces have the same dimension in each degree. \[\square\]
Corollary 1. $F_0^* = \lim \otimes F_0[k]^* = \prod F_0[k]^*$ which is not a Hopf algebra but it admits a coproduct on its generators:

$$\psi y_{k,i} = \sum_{t=1}^{i-1} y_{k-t,0} \otimes y_{t,i} + \sum_{t=1}^{k-1} y_{k-t,i-t} \otimes y_{t,0}.$$ 

A map

$$(1.1) \varphi_k : F_0[k] \to F_0[k+1]$$

is defined by $\varphi_k Q^I = Q^I Q^0$. It is a coalgebra injection map and the direct limit, $\lim \to F_0[k]$, is defined under these maps.

Definition 2. Let $A$ denote the mod 2 Steenrod algebra. We define a left action of the Steenrod algebra $A$ on $F_0$ as follows

- $Sq^a Q^b = \left(\frac{b-a}{a}\right)Q^{b-a}$ for $b \geq 0$;
- $Sq^a (Q^0 r) = Q^0 Sq^a(r)$;
- $Sq^a (Q^b r) = \sum_t \left(\frac{b-a}{a-2t}\right)Q^{b-a+t} Sq^t(r)$ for $b > 0$.

Here $Q^j = 0$, if $0 > j$.

Compare with the Nishida relations \[15\]. The previous action of $A$ on $F_0$ is called the action of the opposite of the Steenrod algebra and is abbreviated by $A^{op}$.

The next lemma follows.

Lemma 4. $F_0[k]$ is closed under the $A^{op}$ action and $F_0$ is an unstable coalgebra over $A^{op}$.

On the hom dual $F_0[k]^*$ we have.

Lemma 5. $Sq^a y_{k,i}^b = \binom{b}{a} y_{k,i}^{b+a}$.

Proof. It follows from the action of $A^{op}$ on $x_{k,i}$ and the fact that $y_{k,i} = (x_{k,i})^*$. $\square$

We wish to examine two relatively different directions: one obtained by the relation $Q^0 = 1$ and the second by eliminating elements of negative excess.

$F_0$

$$F := F_0/\langle Q^0 - 1 \rangle \quad \quad U := F_0/\langle \text{negative excess} \rangle$$

The first provides the Steenrod algebra $A_2$ and the second the Dyer-Lashof algebra $R$.

2. The direction to the Steenrod algebra

Let $F$ be the quotient of $F_0$ by the ideal generated by $Q^0 - 1$. In other words $F$ is the free associative algebra with unit generated by $\{Q^i \mid i \geq 1\}$ over $\mathbb{F}_2$ with degrees $|Q^i| = i$. $F$ is a quotient Hopf algebra of $F_0$ of finite type.

Definition 3. Let $F(k)$ be the vector subspace generated by all monomials $Q^S$ such that $l(S) \leq k$. 
Now $\mathcal{F}(k) \subseteq \mathcal{F}(k + 1)$ and $\mathcal{F} = \lim\to \mathcal{F}(k)$.

**Lemma 6.** $\mathcal{F}(k)$ is a coalgebra closed under the $A^{op}$ action and $\mathcal{F}$ is an unstable coalgebra over $A^{op}$.

We note that $\mathcal{F}$ is isomorphic with the mod 2 homology of the space $\Omega \Sigma CP^\infty$ and it has been studied in relation with quasi-symmetric functions ([11]).

There exists an obvious coalgebra epimorphism

$$\mathcal{F}_0[k] \twoheadrightarrow \mathcal{F}(k).$$

The next proposition follows.

**Proposition 1.** There exists an injection of $A$-maps induced by the quotient map above:

$$\mathcal{F}(k)^* \hookrightarrow \mathcal{F}[k]^*.$$ 

As a corollary to our result we get a result proved by Crossley.

**Corollary 2.** ([11]) $\mathcal{F}^*$ is a polynomial algebra.

The proof of Crossley uses the Borel-Hopf theorem. It follows that $\mathcal{F}$ is a copolynomial algebra. Moreover, Crossley following work of Gelfand et all ([6]) computed the number of generators in each degree of the algebra $\mathcal{F}^*$.

Let $A_2$ be the quotient of $\mathcal{F}$ by the ideal $I(A)$ generated by the elements known as the **Adem relations** in cohomology

$$Q^i Q^j \in [i/2] \left( \frac{j - k - 1}{i - 2k} \right) Q^{i+j-k}Q^k, \text{ } i \text{ and } j > 0 \text{ such that } i < 2j.$$ 

We recall that a monomial $Q^S = Q^{s_k}...Q^{s_1}$ is called admissible in $A_2$, if $s_k \geq 2s_{k-1}$, $\cdots$, $s_2 \geq 2s_1$.

It is obvious that $A_2$ is isomorphic with the Steenrod algebra as Hopf algebras ([18]).

We admit that it is unorthodox to use the symbols $Q^i$ for the Steenrod algebra, but we would like to distinguishing between the action of the opposite of the Steenrod algebra on these algebras.

**Remark 1.** $A_2 = \mathcal{F}/I(A)$ is not closed under the $A^{op}$ action.

We examine the action of $Sq^2$ on $Q^3Q^2$ before and after applying the Adem relations.

$$Sq^2 Q^3 Q^2 = \left( \frac{3 - 2}{2 - 2 \cdot 1} \right) Q^{3-2+1}sq^1 Q^2 = Q^2 Q^1 \text{ but}$$

$$Q^3 Q^2 = 0, \text{ because of Adem relations.}$$

According to the Adem relations, $Q^1 Q^2 = Q^3 Q^0 = Q^3$. Hence after applying the iterated coproduct on a monomial $Q^I$ of length $k$, it may be the case that a summand turns out to be of length less than $k$ after applying the Adem relations. So we define $A_2(k)$ to be the vector space spanned by admissible monomials of length at most $k$. Thus $A_2(k) \hookrightarrow A_2(k + 1)$ and $A_2 = \lim\to A_2(k)$. Moreover, we have the following obvious coalgebra quotient maps:

$$\mathcal{F}_0[k] \twoheadrightarrow \mathcal{F}(k) \twoheadrightarrow A_2(k) \text{ and } \lim\to \mathcal{F}_0[k] \twoheadrightarrow \mathcal{F} \twoheadrightarrow A_2.$$
Following Milnor we describe the primitive monomials of positive degree for the convenience of the reader.

**Theorem 2.** a) A Milnor basis for primitive elements of \( A_2 \) is given by
\[
\left\{ Q^{S_{n+1,n+1}} = Q^{2^n} \cdots Q^2 Q^1 \mid 0 \leq n \right\}.
\]

b) Let \( MA_2 (k) \) be the subset of \( A_2 (k) \) consisting of all admissible monomials on \( A_2 (k) \). Then the map \( f : \mathbb{N}^k \to MA_2 (k) \) given by
\[
f (n_1, \cdots, n_k) = Q^{\sum_{i} S_i, i}
\]
is an isomorphism of sets. Moreover, \( f^{-1} : MA_2 (k) \to \mathbb{N}^k \) is given by
\[
f^{-1} (Q^{s_k} \cdots Q^{s_1}) = (n_1, \cdots, n_k) \text{ such that } n_{k-t} = s_{t+1} - 2s_t - \cdots - 2^t s_1.
\]
Here \( 0 \leq t \leq k - 1 \).

c) If \( Q^S \) is not admissible in \( A_2 (k) \) and \( Q^S = \sum a_J Q^J \) after applying the Adem relations, then \( a_J \equiv 1 \mod 2 \) implies \( J < S \).

**Proof.** This is a bookkeeping application. \( \square \)

As a corollary to theorem 1 we obtain Milnor’s result that the Steenrod algebra is copolynomial.

**Theorem 3.** \( A^* \) is a polynomial algebra and a set of generators is given by
\[
\left\{ \xi_n := (Q^{S_{n+1,n+1}})^* \mid 0 \leq n \right\}.
\]

A special case of a copolynomial algebra is a bipolynomial algebra which is a graded connected bicommutative Hopf algebra such that both it and its dual are polynomial algebras. Husemoller ([7]) studied a universal construction of this kind of algebras and Ravenel and Wilson ([17]) proved that being bipolynomial of finite type over \( \mathbb{F}_2 \) determines the Hopf algebra structure.

For completeness we quote Ravenel and Wilson’s result.

**Theorem 4.** ([17]) \( P \) is bipolynomial isomorphic to its dual as Hopf algebras.

All the above have applications on the \( Q \) spectrum for Brown-Peterson cohomology.

Next, let \( P \) denote the polynomial algebra \( \mathbb{F}_2 [Q^i \mid 1 \leq i] \) given as a quotient of \( \mathcal{F} \) by the commutative relation. As a corollary to our main Theorem we obtain that \( P \) is a bipolynomial Hopf algebra.

A basis for the primitive elements for \( P \) using Newton’s polynomials and an idea described by May ([3]) can be found in Wellington’s book ([19]). We note that Madsen ([10]) used that set to provide a basis for the primitive elements of \( H_* (Q_0 S^0) \).

We also note that \( P \) is isomorphic to a Hopf subalgebra of \( H_* (Q_0 S^0) \) and the action of \( A^{op} \) on its primitive elements is known ([19]).
3. The direction to the Dyer-Lashof algebra

Next we proceed to the other direction, namely eliminating elements of negative excess.

**Definition 4.** Let $\mathcal{U}$ be the quotient of $\mathcal{F}_0$ by the ideal generated by elements of negative excess.

$\mathcal{U}$ is a quotient Hopf algebra of $\mathcal{F}_0$ not connected neither of finite type. As before $\mathcal{U}[k]$ stands for the subspace generated by monomials of fixed length $k$.

We recall that each admissible sequence admits a unique decomposition with respect to the primitive monomial elements, therefore we repeat the appropriate results from [8] for the convenience of the reader.

**Definition 5.** Let $t$ be a natural number such that $1 \leq t \leq k$.

Define $J_{k,t} = \left(\begin{array}{c} 2t-1, \ldots, 2, 1, 1, 0, \ldots, 0 \\ t \end{array}\right)_{k-t}$. The degree of $J_{k,t}$ is $2t$.

**Proposition 2.** [8] Let $QJ$ be an admissible monomial of length $k$ and of positive degree. Then there exist natural numbers $(a_1, a_1, \cdots, a_k)$ such that $J$ can be written uniquely as $J = \sum_{i=1}^{k} a_i J_{k,i}$.

**Lemma 7.** $\mathcal{U}[k]$ is closed under the $A^{op}$ action and $\mathcal{U}$ is an unstable coalgebra over $A^{op}$.

**Proof.** The excess condition must be checked. Let $b < c$, then $Sq^a Q^b Q^c = \sum (a-2t) \binom{t}{c-t} Q^{b-a+t} Q^{c-t}$.

Restrictions should be taken into account. $2t \leq a$ implies $b + 2t < c + a$, so we get $b + 2t < c + a$. Finally, $b - a + t < c - t$. \hfill $\square$

The opposite Steenrod coalgebra epimorphism $\mathcal{F}_0 [k] \twoheadrightarrow \mathcal{U} [k]$ induces the following corollary.

**Corollary 3.** $\mathcal{U}[k]^*$ is a polynomial algebra closed under the Steenrod algebra action.

As in the diagram (2.1), the quotient maps above induce the map $\lim \mathcal{F}_0 [k] \twoheadrightarrow \lim \mathcal{U} [k]$.

The Dyer-Lashof algebra $\mathcal{R}$ is given as the quotient of $\mathcal{U}$ modulo the ideal generated by the Adem relations in homology:

$Q^r Q^s - \sum_{t>0} \binom{t-s-1}{2t-r} Q^{r+s-t} Q^t$ such that $r > 2s$.

**Definition 6.** Let $k \geq 1$, $R[k]$ is the sub $A$-coalgebra spanned by admissible monomials of fixed length $k$ and non-negative excess.

$\{Q^{(I,\varepsilon)} \mid (I,\varepsilon) \text{ admissible, } l(I) = k \text{ and } e(I,\varepsilon) \geq 0 \}$. 
It is a non-negatively graded Hopf algebra and also a component coalgebra $R = \bigoplus_{k \geq 0} R[k]$ with respect to the length.

We recall that each admissible sequence admits a unique decomposition with respect to the primitive monomial elements. We repeat the appropriate results from \[10\].

**Definition 7.** Let $t$ be a natural number such that $0 \leq t \leq k - 1$ and the term $2^{-1}$ is omitted in the following expressions. Define $I_{k,t} = \{2^{k-1} - 2^{t-1}, \ldots, 2^{k-t} - 1, 2^{k-t-1}, \ldots, 1\}$. The degree of $I_{k,t}$ is $2^k - 2^t$.

**Proposition 3.** \[10\] Let $Q^I$ be an admissible monomial of length $k$ and of positive degree. Then there exist natural numbers $(a_0, a_1, \cdots, a_{k-1})$ such that $I$ can be written uniquely as

$$I = \sum_{t=0}^{k-1} a_t I_{k,t}.$$  

It is known that $R[k]$ respects the action of the opposite of the Steenrod algebra.

**Lemma 8.** \[10\] The Dyer-Lashof algebra $\mathcal{R}$ is an unstable coalgebra over $\mathcal{A}^{op}$. Moreover, $\mathcal{R}[k]$ is closed under the $\mathcal{A}^{op}$ action.

**Proof.** The proof uses the fact that $H_*(Q^0 S^0)$ is an unstable coalgebra over $\mathcal{A}^{op}$ and the evaluation map $e : \mathcal{R} \rightarrow H_*(Q^0 S^0)$ given by $e(Q^I) = Q^I i_0.$ Here $i_0$ is the fundamental class of $H_0(S^0) \rightarrow H_*(Q^0 S^0)$.

The opposite Steenrod coalgebra epimorphisms $\mathcal{F}_0[k] \rightarrow \mathcal{U}[k] \rightarrow \mathcal{R}[k]$ induce the following corollary.

**Corollary 4.** $\mathcal{R}[k]^*$ is a polynomial algebra closed under the Steenrod algebra action.

Let us examine the induced map $\varphi_k : \mathcal{R}[k] \rightarrow \mathcal{R}[k+1]$ given by $\varphi_k Q^I = Q^I Q^0$ (see \[11\]). It is not an injection any longer because of the Adem relations, neither an onto map. For example $\varphi_1 Q^1 = Q^1 Q^0 = 0$ and there is no $Q^I$ such that $\varphi_1 Q^I = Q^2 Q^1$.

**Proposition 4.** The map $\varphi_k : \mathcal{R}[k] \rightarrow \mathcal{R}[k+1]$ given by $\varphi_k Q^I = Q^I Q^0$ is an $\mathcal{A}^{op}$-coalgebra map for $k \geq 1$. Moreover,

i) if $I \notin 2N^k$ and $I$ is admissible, then $\varphi_k Q^I = 0$;  

ii) Let $I = \sum_{i=0}^{k-1} 2a_i I_{i+1}$, then $\varphi_k Q^I = Q^J$ where $J = \sum_{i=0}^{k-1} a_i I_{k+1,i+1}$.

**Proof.** We start with length 2 sequences in order to demonstrate our method. Let $I = (2a_0 + a_1, a_0 + a_1)$. There are three cases to be considered:

i) $a_0 + a_1 \equiv 1 \mod 2$;  
ii) $a_0 + a_1 \equiv 0 \mod 2$ and $a_0 \equiv 1 \mod 2$; and  
iii) $a_0, a_1 \equiv 0 \mod 2$.

i) $Q^{a_0+a_1} Q^0 = \sum (2t-(a_0+a_1)) Q^{a_0+a_1-t} Q^t$. We must also consider the restrictions:
$a_0 + a_1 - t \geq t$ and $2t - (a_0 + a_1) \geq 0$. Therefore $2t = a_0 + a_1$, which contradicts our assumption. Hence,

$$Q^tQ^0 = 0.$$  

ii) $Q^{a_0+a_1}Q^0 = \sum (2t-(a_0+a_1))Q^{a_0+a_1-t}Q^t$. By the argument above, we have:

$$Q^{a_0+a_1}Q^0 = \frac{Q^{a_0+a_1}}{2} \cdot \frac{Q^{a_0+a_1}}{2}.$$  

Now we consider the monomial $Q^{2a_0+a_1}Q^{a_0+a_1}Q^{a_0+a_1}Q^{a_0+a_1}$ and in particular the term $Q^{2a_0+a_1}Q^{a_0+a_1}$.

$$Q^{2a_0+a_1}Q^{a_0+a_1}Q^{a_0+a_1} = \sum \left( t - \frac{a_0 + a_1}{2} \right)Q^{2a_0+a_1}Q^{a_0+a_1}Q^{a_0+a_1}.$$  

We shall also take into consideration the restrictions:

$$2t - (2a_0 + a_1) \geq 0$$  

and

$$2a_0 + a_1 + \frac{a_0 + a_1}{2} - t \geq t + \frac{a_0 + a_1}{2}.$$  

Therefore $2t = 2a_0 + a_1$ which contradicts our assumption.

iii) In this case we consider the monomial

$$Q^{2a_0+a_1}Q^{a_0+a_1}Q^{a_0+a_1}.$$

Applying the Adem relations we get

$$Q^{2a_0+a_1}Q^{a_0+a_1}Q^{a_0+a_1} = Q^{2a_0+a_1}Q^{a_0+a_1}Q^{a_0+a_1}.$$  

For the general case let $I = \sum_{k-1}^{k-1} a_iI_k = (ik, \cdots, i_1)$, we call

$$i_k = b_{k-t}$$  

such that $b_j = 2b_{j+1} - a_{j+1}$ for $j \leq k - 2$ and $b_{k-1} = \sum a_i$.

Let $(a_0, \cdots, a_{k-1}) \notin 2\mathbb{N}^k$ and there exists an $a_{k-t_0} \equiv 1 \mod 2$ with $t_0$ the smallest among the appropriate indices. Therefore $a_{k-t_0+s} \equiv 0 \mod 2$ for $s > 0$.

If $b_{k-1} = i_1 \equiv 1 \mod 2$, then $Q^iQ^0 = 0$ by the Adem relations and the restrictions.

If $b_{k-1} = i_1 \equiv 0 \mod 2$, then $Q^iQ^0 = Q^{\frac{i}{2}}Q^{\frac{i}{2}}$.

Let $b_{k-1} = i_1 \equiv 0 \mod 2$ and $a_{k-t_0} \equiv 1 \mod 2$ as above. In this case, we consider the following monomial:

$$Q^{b_0} \cdots Q^{b_{k-t_0-1}}Q^{b_{k-t_0}}Q^{b_{k-t_0+1}+b_{k-1}}Q^{b_{k-t_0}} \cdots Q^{b_{k-1}}.$$  

Now $b_{k-t_0-1} \equiv 1 \mod 2$ and the Adem relations imply

$$Q^{b_{k-t_0-1}}Q^{b_{k-t_0+1}+b_{k-1}}Q^{b_{k-t_0}} =$$

$$\sum \left( t - \frac{b_{k-t_0} + b_{k-1}}{2} \right)Q^{b_{k-t_0-1}+b_{k-t_0+1}+b_{k-1}} - tQ^s.$$  

Because of the restrictions $t = \frac{b_{k-t_0+1}}{2}$, which contradicts our assumption.

If we had $b_{k-t_0-1} \equiv 0 \mod 2$, then we would have $t = \frac{b_{k-t_0-1}}{2}$ and

$$Q^{b_{k-t_0-1}+b_{k-t_0+1}+b_{k-1}} - tQ^s = Q^{b_{k-t_0-1}+b_{k-t_0+1}+b_{k-1}}Q^{b_{k-t_0}}.$$
We proceed to the Steenrod algebra action by considering the following diagram.

\[
\begin{array}{ccc}
S\overline{q}_s^{2^m} & Q^{J'} & \xrightarrow{\varphi_s} \\
\downarrow & \downarrow & \\
Q^{2I_{k,i}} & Q^J & \xrightarrow{\varphi_s} Q^{2I_{k+1,i+1}} \\
\end{array}
\]

Madsen proved that \( S\overline{q}_s^{2^m} Q^{J'} = Q^{I_{k+1,i+1}} \) if and only if \( J = I_{k+1,i+1} + I_{k+1,k} \) for \( i + 1 \leq m = k \). We prove the analogue statement for \( Q^{2I_{k,i}} \).

**Claim:** \( S\overline{q}_s^{2^m} Q^{J'} = Q^{2I_{k,i}} \) if and only if

\[
J' = \begin{cases} 
2I_{k,i-1} & \text{for } m = i < k; \\
2I_{k,i} + 2I_{k,k-1} & \text{for } i + 1 < m = k; \\
2I_{k,k-1} & \text{for } i + 1 = m = k.
\end{cases}
\]

**Proof of claim.** Here \( m \leq k \). First we consider what expression \( J' \) admits by degree arguments.

\[
\left| J' \right| = 2^{k+1} - 2^{i+1} + 2^m = \sum_{0}^{m-1} a_i |I_{k,t}| = \sum_{0}^{m-1} a_i 2^{k} - 2^t.
\]

There are three cases to consider. In all cases induction is applied.

a) \( m = i \) and \( J' = 2I_{k,i-1} \).

b) \( m = k \), \( i < k-1 \) and \( J' = 2I_{k,i} + 2I_{k,k-1} \).

c) \( m = k-1 \) and \( J' = 2I_{k,i} + I_{k,k-1} \).

a) We recall that \( 2I_{k,i} = (2^k - 2^i, \ldots, 2^{k+1-i} - 2, 2^{k-i}, \ldots, 2) \) and

\[
I_{k,k-1} = (2^{k-2}, \ldots, 2, 1, 1).
\]

From the action of the opposite of the Steenrod algebra \( (2) \) we get

\[
S\overline{q}_s^{2^i} Q^{2^k-2^{i-1}} = \sum \left( \frac{2^k - 2^{i-1} - 2^t}{2^i - 2t} \right) Q^{2^k-2^{i-1} - 2^t} S\overline{q}_s^{2^i}.
\]

For \( t = 2^{i-1} \) we get \( Q^{2^k-2^i} S\overline{q}_s^{2^i} \). If \( t < 2^{i-1} \), then \( 2^i - 2^t \) contains a summand \( 2^s \) such that \( s < i - 1 \) and \( 2^k - 2^{i-1} - 2^t \) does not. Therefore \( (2^k-2^{i-1} - 2^t) \equiv 0 \mod 2 \).

Inductively we get

\[
S\overline{q}_s^{2^i} Q^{2I_{k,i-1}} = Q^{(2^k-2^i, \ldots, 2^{k+1-i-2})} S\overline{q}_s^{2^i} Q^{(2^{k+1-i}, \ldots, 2)}.
\]

Now \( (2^{k+1-i} - 2) \equiv 1 \) for \( t = 0 \) or \( 1 \). For \( t = 0 \) we get the expected element

\[
S\overline{q}_s^{2^i} Q^{2I_{k,i-1}} = Q^{2I_{k,i}}.
\]

For \( t = 1 \) we get

\[
Q^{(2^k-2^i, \ldots, 2^{k+1-i-2})} Q^{2^k-1} S\overline{q}_s^{2^i} Q^{2^{k-1,i}} Q^{(2^{k-1,i}, \ldots, 2)} =
Q^{(2^k-2^i, \ldots, 2^{k+1-i-2})} Q^{2^k-1} Q^{2^{k-1,i}} Q^{(2^{k-1,i}, \ldots, 2)}
\]

and the last monomial has excess zero.

The case b) is similar. We proceed to case c).

c) \( 2I_{k,i} + I_{k,k-1} = (2^k - 2^i + 2^{k-2}, \ldots, 2^{k+1-i} - 2 + 2^{k-i}, 2^{k-i} + 2^{k-1-i}, \ldots, 2^2 + 1, 2 + 1) \).

\[
S\overline{q}_s^{2^k-1} Q^{2^{k+2}} Q^{2I_{k-1,i-1} + I_{k-1,k-2}} =
\]
\[
\sum \left( \frac{2^{k-1} - 2^i + 2^{k-2}}{2^{k-1} - 2t} \right) Q^{2^{k-1} - 2^i + 2^{k-2} + 4} Sq^i Q^{2t} Q^{2l_i-1, i-1 + l_{k-1} - 2}.
\]

The case \( t < 2^{k-2} \) is eliminated because of negative excess:
\[
2^{k-1} - 2^i + 2^{k-2} + 2t < 2^{k-1} + 2^{k-2}.
\]

Therefore only the case \( t = 2^{k-2} \) remains and this case ends up at
\[
Sq^i Q^{2l_i-1, i-1 + l_{k-1} - 2} = Q^{2l_i-1, i}.
\]

Finally, \( Sq^i Q^{2l_i-1, i} = Q^{2l_i, i-1} \neq Q^{2l_i, i} \).

The following corollary is just an application of the last proposition.

**Corollary 5.** Let \( I = \sum_{i=0}^{k-1} a_i I_{k,i} \) such that \( a_i \equiv 0 \mod 2^n \) for all \( i \), then
\[
\varphi_{k+n-1} \cdots \varphi_k \left( Q^I Q^0 \cdots Q^0 \right) = Q^J
\]
where \( J = \sum_{i=0}^{k-1} b_i I_{k+n,i+1} \) and \( a_i = 2^n b_i \) for all \( i \).

We end this section by summarizing the results above in the following two diagrams.

4. A map between the Steenrod and Dyer-Lashof coalgebras

We conclude this note by filling the diagram above discussing the induced map between \( A_2 (k) \) and \( R [k] \).

Let \( \phi_k : U [k] \to A_2 (k) \) be the obvious map given by
\[
\phi_k \left( Q^I Q^0 \cdots Q^0 \right) = Q^I.
\]

This is not an injection map. For example \( \phi_2 \left( Q^I Q^1 \right) = 0. \) But it is a coalgebra epimorphism:
Let \( Q^I \in A_2 (k) \) with \( l (I) = m \leq k \) and \( I \) an admissible sequence. Then, if \( I = (i_m, \cdots, i_1) \), \( i_t \geq 2i_{t-1} \) for all \( t \), \( \phi_k \left( Q^I Q^0 \cdots Q^0 \right) = Q^I. \)
We extend the map \( \phi_k \) from \( \mathcal{A}_2 (k) \) to \( \mathcal{R} [k] \) in the obvious way:

\[
\pi_k : \mathcal{A}_2 (k) \to \mathcal{R} [k] \text{ given by } \pi_k(Q^I) = \left( Q^I Q^n \cdots Q^1 \right) \text{ for } n = k - l (I)
\]

and \( \pi_k(Q^I) = Q^I \) for \( k = l (I) \).

We must note that the last diagram is not a commutative diagram under the given maps. For example: \( Q^1 Q^1 \in U [k] \cap \mathcal{R} [k] \) but

\[
\pi_2 \phi_2(Q^1 Q^1) = 0.
\]

Nevertheless, the map \( \pi_k \) is an epimorphism of coalgebras.

**Theorem 5.** The map \( \pi_k : \mathcal{A}_2 (k) \to \mathcal{R} [k] \) is an epimorphism of coalgebras.

**Proof.** Let \( Q^I \in \mathcal{R}' [k] \) such that \( I = \sum_{t=0}^{k-1} a_t K_{k,t} \) and \( \sum_{t=0}^{k-1} a_t > 0 \). We shall define an element \( K \in \mathcal{A}_2 (k) \) such that \( \pi_k(K) = Q^I \).

Let \( Q^I(t) \in \mathcal{A}_2 (k) \) such that

\[
J(I) = \sum_{t=0}^{k-1} 2^t a_t K_{k-t,0}.
\]

Namely, if \( J(I) = (b_k, \ldots, b_1) \), then \( b_{t+1} = 2^t (a_0 + \cdots + a_t) \) for \( 0 \leq t \leq k - 1 \). Let us also define \( J_t(I) = (b_k, \ldots, b_{t+1}) \) for \( 0 \leq t \leq k - 1 \).

Let us first consider the case \( I = (2a_0 + a_1, a_0 + a_1) \). In this case \( J(I) = (2(a_0 + a_1), a_0) \). We start with \( Q^2(a_0 + a_1) Q^{a_0} \) and apply the Adem relations in the Dyer-Lashof algebra.

\[
Q^2(a_0 + a_1) Q^{a_0} \xrightarrow{\text{Adem}} \sum \left( \frac{t - a_0 - 1}{2t - 2(a_0 + a_1)} \right) Q^{(2t+1)a_0 + 2a_1 - t} Q^t.
\]

Here we shall take into account the restrictions:

\[
(2 + 1) a_0 + 2a_1 \geq 2t.
\]

For \( t = a_0 + a_1 \), we have the required term

\[
Q^{2a_0 + a_1} Q^{a_0 + a_1}.
\]

For \( t = a_0 + a_1 + s \) such that \( 2s \leq a_0 \) and for \( \left(\frac{s + a_1 - 1}{2s}\right) \neq 0 \text{ mod } 2 \), we have the non-required term

\[
Q^{2a_0 + a_1 - s} Q^{a_0 + a_1 + s}.
\]

In other words the monomial \( Q^{J_2} Q^{2a_0 + a_1 - s} Q^{a_0 + a_1 + s} \) appears and it will be eliminated by adding an appropriate term \( Q^{J(1)} \) to \( Q^{J(I)} \), where

\[
J^{(1)} = (a_0 + a_1 + s, a_0 - s).
\]

This is because the term \( Q^{a_0 + a_1 + s} Q^{2a_0 - 2s} \) will provide \( Q^{2a_0 + a_1 - s} Q^{a_0 + a_1 + s} \) after applying the Adem relations. We start with the smallest \( s \) such that

\[
\left(\frac{s + a_1 - 1}{2s}\right) \neq 0 \text{ mod } 2
\]

and apply the Adem relations. Neither of the terms \( Q^{a_0 + a_1 + s} Q^{a_0 - 2s} \) will provide \( Q^{2a_0 + a_1} Q^{a_0 + a_1} \).
It is also important to note that \( J^{(1)} < J (I) \) and this property will be present in our procedure. In each step we define and use a new sequence less than the previous ones so that the non-required term is eliminated.

We describe our method by starting with \( J (I) \).

Let us first recall the coefficient involved in the Adem relations \( (t-s-1) \) for \( Q'Q^r \) such that \( r > 2s \). Second, \( b_t+1 \equiv 0 \bmod{2^t} \) for \( t > 0 \). We start with the sequence \( J (I) \) and apply the relations between the first and the second element from the left and proceed to the pair consisting of the new just defined element and the next one. We continue in this fashion up to the pair where there is no relation. Each time a relation is applied in the sequence \( J^{(s)} \) a new sequence is defined abbreviated by \( J^{(s+1)} \) starting with \( J (I) = J^{(0)} \). Each time a relation is applied between \( b_t^{(s)} \) and \( b_{t-1}^{(s)} \) we first consider only the case \( t = \frac{k(s)}{2} \) and call

\[
b_t^{(s+1)} = b_t^{(s)} + b_{t-1}^{(s)} - \frac{b_t^{(s)}}{2} \quad \text{and} \quad b_{t-1}^{(s+1)} = \frac{b_t^{(s)}}{2}.
\]

Therefore a new sequence is defined \( J^{(s+1)} = \) \( \left( b_{k+1}^{(s)}, \ldots, b_{t+1}^{(s)}, b_{t}^{(s+1)}, b_{t-1}^{(s+1)} \right) \), \( b_{t-1}^{(s+1)} = b_t^{(s)} \). Following the pattern above we get

\[
J^{(k-1)} = \left( b_1^{(k-1)}, \ldots, b_k^{(k-1)} \right).
\]

with \( b_1^{(k-1)} = \sum_{t=0}^{k-1} a_t \) and \( b_t^{(k-1)} = 2^{i-1} \sum_{t=0}^{k-1} a_t - 2^{i-2} \sum_{t=0}^{i-1} a_t \) for \( k \geq i > 1 \).

According to our procedure

\[
b_1^{(k-1-k-2)} = \sum_{t=0}^{k-1} a_t, \quad b_2^{(k-1-k-2)} = 2 \sum_{t=0}^{k-1} a_t - a_{k-1} \quad \text{and} \quad b_t^{(k-1-k-2)} = 2^{i-1} \sum_{t=0}^{k-1} a_t - 2^{i-2} \sum_{t=0}^{i-1} a_t \quad \text{for} \quad k \geq i > 2.
\]

We note that there is no relation between \( b_2^{(k-1+k-2)} \) and \( b_1^{(k-1+k-2)} = b_2^{(k-1)} \). After \( k(k-1) \) steps we have \( J^{(k(k-1))} = I \). To finish off we also have to consider terms coming from the Adem relations with corresponding \( t \) such that \( t \geq \frac{k}{2} \). Our method is described after we demonstrate an example on the method above.

Example for \( k = 3 \).

\[
I = a_0 I_{3,0} + a_1 I_{3,1} + a_2 I_{3,2} \quad \text{and} \quad J = \left( 2^{2} \sum_{0}^{2} a_t, 2^{1} \sum_{0}^{1} a_t, a_0 \right).
\]

\[
J^{(1)} = \left( 2^{2} \sum_{0}^{2} a_t - 2 a_2, 2 \sum_{0}^{2} a_t, a_0 \right), \quad J^{(2)} = \left( 2^{2} \sum_{0}^{2} a_t - 2 a_2, 2 \sum_{0}^{2} a_t - \sum_{1}^{2} a_t, 2 \sum_{0}^{2} a_t \right)
\]

and \( J^{(3)} = \left( 2^{2} \sum_{0}^{2} a_t - 2 a_2 - a_1, 2 \sum_{0}^{2} a_t - \sum_{1}^{2} a_t, 2 \sum_{0}^{2} a_t \right) \).

Given \( Q^l \in \mathcal{R}' [k] \) such that \( I = \sum_{t=0}^{k-1} a_t I_{k,t} \) and \( \sum_{t=0}^{k-1} a_t > 0 \) a sequence \( J (I) \) defined above such that \( Q^{J (I)} \in \mathcal{A}_2 (k) \) and \( J (I) = \sum_{t=0}^{k-1} 2^t a_t I_{k-t,0} \). The Adem relations are applied on \( c_t Q^{J (I)} \):

\[
Q^{J (I)} \overset{Adem}{=} Q^l + \sum a_t (I) Q^{J (I)}.
\]
Here $Q^{J(I)}$ is an admissible monomial and $I_t(I) > I$ because of the Adem relations. Next we consider $t_0$ such that $I_{t_0}(I) < I_t(I)$ for $t \neq t_0$. A new sequence $J(I_{t_0}(I))$ is defined as above such that $Q^{J(I_{t_0}(I))} \in \mathcal{A}_2(k)$ and

$$Q^{J(I_{t_0}(I))} \overset{\text{Adem}}{=} Q^{I_{t_0}(I)} + \sum c_s (I_{t_0}) Q^{I_s(I_{t_0})}.$$ 

Here again $Q^{J(I_{t_0})}$ is an admissible monomial and $I_r(I_{t_0}) > I_{t_0}(I)$. Moreover, $J(I_{t_0}(I)) < J(I)$. Now we take

$$Q^{J(I)} + Q^{J(I_{t_0}(I))} \overset{\text{Adem}}{=} Q^I + \sum b_r Q^{I_r(I_{t_0})}$$

and consider $r_0$ such that $I_{r_0}(I,I_{t_0}) < I_r(I,I_{t_0})$ for $r \neq r_0$.

Proceeding in this fashion an element $K \in \mathcal{A}_2(k)$ will be defined in finite steps such that $\pi_k(K) = Q^I$.

**Corollary 6.** There exists an induced coalgebra map $\pi : \mathcal{A}_2 \to \lim \mathcal{R}[k]$.

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