**Abstract.** Although previous research has found several facts concerning chord lengths of regular polytopes, none of these investigations has considered whether any of these facts define relationships that might generalize to the chord lengths of all regular polytopes. Consequently, this paper explores whether four findings of previous studies—viz., the four facts relating to the sums and products of squared chord lengths of regular polygons inscribed in unit circles—can be generalized to all regular (n-dimensional) polytopes (inscribed in unit n-spheres). We show that (a) one of these four facts actually does generalize to all regular polytopes (of dimension \( n \geq 2 \)), (b) one generalizes to all regular polytopes except most simplices, (c) one generalizes only to the family of crosspolytopes, and (d) one generalizes only to the crosspolytopes and 24-cell. We also discover several corollaries (due to reciprocation) and some theorems specific to the three-dimensional regular polytopes along the way.

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1. **Introduction**

Several discoveries have been made concerning the chord lengths of certain types of regular polytopes (most often, the regular two-dimensional polytopes, i.e., regular polygons; e.g., see [6, 9, 13, 15]). However, no single rule has been found to apply to the chord lengths of all regular polytopes.

To remedy this situation, we consider whether any of the four facts discovered in one area of previous investigation (regarding the chord lengths of two-dimensional regular polytopes) can be generalized to apply to the chord lengths of all \( n \)-dimensional regular polytopes where \( n \geq 2 \). (In this paper, although the 0-dimensional polytope (i.e., a single point) and the 1-dimensional polytope (i.e., a line segment) are regular, we exclude them as being trivial.)
we will call an $n$-dimensional polytope an “$n$-polytope,” an $n$-dimensional sphere an “$n$-sphere,” etc.).

1.1. The Four Known Facts

First, recall that a chord of a regular 2-polytope (i.e., regular polygon) is a line segment whose two endpoints are vertices of the polygon.

Let $P$ be a regular polygon with $E$ edges that is inscribed in a unit circle (i.e., “2-sphere”).

1. The sum of the squared chord lengths of $P$ equals $E^2$.
2. The sum of the squared distinct chord lengths of $P$ equals:
   (a) $E$ (when $E$ is odd) \[9\]
   (b) some integer (when $E$ is even) \[9, pp. 490-491\].
3. The product of the squared chord lengths of $P$ equals $E^E$.
4. The product of the squared distinct chord lengths of $P$ equals:
   (a) $E$ (when $E$ is odd)
   (b) some integer (when $E$ is even) \[9, pp. 490-491\].

(Facts 1 and 3 are given in an unpublished 2013 article by S. Mustonen).

1.2. Overview of Procedure

To test the generalization of these four 2-polytope facts to all dimensions (greater than or equal to 2), we apply the following four steps:

1. Consider a regular $n$-polytope inscribed in a unit $n$-sphere ($n \geq 2$).
2. Compute both the sum and product of (a) the polytope’s square chord lengths and (b) the polytope’s squared distinct chord lengths.
3. Compare the results obtained in Step 2 with the values (which are usually some variant on $E$) given by the relevant 2-polytope fact.
4. If the compared values in Step 3 are different, compare the results obtained in Step 2 with other values related to the $n$-polytope (e.g., other $j$-face cardinalities).

The next section contains a review of some basic definitions and properties necessary to carrying out these steps. In the four remaining sections, we carry out these steps for each of the four facts in turn.

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2Mustonen’s unpublished article is titled “Lengths of edges and diagonals and sums of them in regular polygons as roots of algebraic equations” and can be accessed at http://www.survo.fi/papers/Roots2013.pdf as of 16 March 2019.

Although Mustonen states Facts 1 and 3 without mathematical proof, a proof of Fact 1 may be obtained by substituting $R = 1$ in Solution 6.73 of [12]. (An English version of [12], edited and translated by Dimitry Leites, is in preparation with the title, *Encyclopedia of Problems in Plane and Solid Geometry*).

Likewise, Fact 3 can be derived from a statement proven in [8, pp. 161-162], namely, that if $P$ is a regular polygon with $E$ edges inscribed in a unit circle, the product $p$ of the lengths of the chords of $P$ emanating from a given vertex $P$ equals $E$. Denoting the vertices of $P$ by $P_i$ for $i = 1, 2, ..., V$ and their associated products by $p_i$, we have $p_1p_2...p_V = p^V = E^V$. Since each chord’s length occurs twice in $p_1p_2...p_V$, this is also equal to the product of the squared chord lengths of $P$. Since, for regular polygons, $V = E$ (see [4]), we obtain Fact 3.
2. Preliminaries

In this paper, all \( n \)-polytopes being considered are finite convex regions of Euclidean \( n \)-space. For a full definition, see [3 pp.126-127].

An \( n \)-polytope \( P \) has one or more \( j \)-faces where \( j \in \{−1, 0, 1, 2, ..., n\} \) (cf. [10]). For \( j = 0, 1, ..., n−1 \), each of these \( j \)-faces is the \( j \)-dimensional intersection of \( P \) and \( H \) (where \( H \) is an \( (n−1) \)-plane such that \( P \) lies in one of the closed half-spaces determined by \( H \) and has a nonempty intersection with \( H \)) [10]. For \( j = n \), the sole \( j \)-face is just the \( n \)-polytope \( P \) itself [10], and, for \( j = −1 \), the sole \( j \)-face is the empty set [10]. For any \( n \)-polytope, the 0-faces are called vertices, the 1-faces are called edges, the \((n−2)\)-faces are called ridges, and the \((n−1)\)-faces are called facets [10].

An \( n \)-polytope is called regular if:

1. Whenever one of its \((j−2)\)-faces is incident with one of its \( j \)-faces, there are exactly two \((j−1)\)-faces that are incident with both the \((j−2)\)-face and the \( j \)-face (for \( j = 1, 2, ..., n \)), and

2. For a given \( j \in \{0, 1, 2, ..., n\} \), all of the \( j \)-faces are equidistant from a point \( O \) called the polytope’s center [2]. (That is, for each \( j \in \{0, 1, 2, ..., n\} \), there is an \( n \)-sphere centered at \( O \) that passes through all the \( j \)-faces’ centers). Without loss of generality, all polytopes will be considered as being centered at the origin.

One useful consequence of regularity is that, for a given \( j \), any of a polytope’s \( j \)-faces can be interchanged with any of the polytope’s other \( j \)-faces via one of its symmetries [7]. A second useful consequence of regularity is that, when a regular \( n \)-polytope is circumscribed about an \( n \)-sphere, \( n \geq 2 \), it may be reciprocated with respect to the \( n \)-sphere to form a new regular \( n \)-polytope inscribed inside the \( n \)-sphere (this maps the original polytope’s \((j−1)\)-faces to the new polytope’s \((n−j)\)-faces for \( j = 0, 1, ..., n \)) [3].

There are exactly five regular 3-polytopes, six regular 4-polytopes, and three regular \( n \)-polytopes in each dimension \( n \geq 5 \) [3]. Table 1 shows the number and shape of the \( j \)-faces of each of these polytopes.

| Polytope | 0-face | 1-faces | 2-face | 2-face | Shape of 2-Face |
|----------|--------|---------|--------|--------|----------------|
| tetrahedron | 4      | 6       | 4      |        | triangle       |
| octahedron | 6      | 12      | 8      |        | triangle       |
| cube      | 8      | 12      | 6      |        | square         |
| icosahedron | 12     | 30      | 20     |        | triangle       |
| dodecahedron | 20     | 30      | 12     |        | pentagon       |

Table 1: Cardinality and Shape of \( j \)-Faces of Regular Polytopes [3].
### 4-polytopes

| 4-polytopes | 0-face | 1-face | 2-face | 3-face | 2-face | 3-face |
|-------------|--------|--------|--------|--------|--------|--------|
| 5-cell      | 5      | 10     | 10     | 5      | triangle | tetrahedron |
| 16-cell     | 8      | 24     | 32     | 16     | triangle | tetrahedron |
| 8-cell      | 16     | 32     | 24     | 8      | square   | cube    |
| 24-cell     | 24     | 96     | 96     | 24     | triangle | octahedron |
| 600-cell    | 120    | 720    | 1200   | 600    | triangle | tetrahedron |
| 120-cell    | 600    | 1200   | 720    | 120    | pentagon | dodecahedron |

### n-polytopes

| n-polytopes | any j-face | any j-face |
|-------------|------------|------------|
| n-simplex   | \(^{n+1}\binom{j+1}{j}\) | \(^j\)-simplex |
| n-crosspolytope | \(2^{j+1}\binom{n}{j+1}\) | \(^j\)-simplex |
| n-cube      | \(2^{n-j}\binom{n}{j}\) | \(^j\)-cube |

\(^a\) The shape of a 0-face is a point; the shape of a 1-face is a line segment.

Lastly, a chord of a regular n-polytope is being defined, in this paper, as a line segment whose endpoints are vertices (i.e., 0-faces) of the polytope. Thus, the chords of a regular polytope consist of all of its edges and diagonals.

### 3. FACT 1: Sum of Squared Chords

The first 2-polytope fact stated that, for any regular 2-polytope inscribed in a unit 2-sphere, the sum of its squared chord lengths equals \(E^2\). To test generalization of this fact to all regular n-polytopes \((n \geq 2)\), we will make use of the following three lemmas and definition.

#### 3.1. Preliminaries

**Lemma 3.1.** Let \(\mathcal{P}\) be a regular n-polytope inscribed in a unit n-sphere, \(n \geq 2\), and let \(x\) denote the ratio of the n-sphere’s radius \(\rho R\) to the polytope’s half-edge length \(l\). Then the edge length \(e\) of \(\mathcal{P}\) is given by the formula \(e = \frac{2}{x}\).

**Proof.** From the hypotheses, we have \(x = \frac{\rho R}{l}\). To find the edge length \(e\) (which is equal to \(2l\)), we first solve for \(2l\) and then replace \(\rho R\) with 1 (since the n-sphere has a radius of 1). We obtain: \(e = 2l = \frac{2\rho R}{x} = \frac{2}{x}\). \(\Box\)

**Remark.** To use Lemma 3.1 in our proofs, we will substitute values (or formulas) for \(x\) obtained from \(\text{[3]}\) for select polytopes. For the regular 3- and 4-polytopes, we will substitute the values of \(x\) that are listed in the tables of “regular polyhedra in ordinary space” and “regular polytopes in four dimensions” in \(\text{[3]}\) (i.e., Table II-ii on pp. 292-293) and are also given in the cells marked with an asterisk in our Table 2. For the regular \(n\)-simplices,

\(^3\) The values of \(x\) are referred to as the values of \(\frac{\rho R}{l}\) in \(\text{[3]}\).
n-crosspolytopes, and n-hypercubes, we will first substitute $j = 0$ into the formulas for $\frac{2^R}{r}$ (the ratio of the “radius of n-sphere intersecting each j-face’s center” to “half-edge length”) listed in the table of “regular polytopes in n dimensions” in [3] (i.e., Table Iiii on pp. 294-295) to obtain formulas for $\frac{2^R}{r}$ (i.e., formulas for $x$) that we can then substitute into Lemma 3.1 to obtain the edge length $e$ for each of these three $n$-polytopes. Each of these steps are shown in the last three rows on Table 2.

| Polytope       | $\frac{2^R}{r}$ | $\frac{a^R}{r}$ ($= x$) | $e$   |
|----------------|-----------------|--------------------------|-------|
| 3-polytopes$^a$|                 |                          |       |
| cube           | NA              | $\sqrt{3}^*\frac{2}{\sqrt{3}}$ |       |
| icosahedron    | NA              | $\sqrt{\tau}\sqrt{5}^*\frac{2}{\sqrt{\tau\sqrt{5}}}$ |       |
| dodecahedron   | NA              | $\tau\sqrt{3}^*\frac{2}{\tau\sqrt{3}}$ |       |
| 4-polytope     | 24-cell         | NA                       |       |
|                |                 | $2^*1$                    |       |
| n-polytopes$^b$|                 |                          |       |
| n-simplex      | $\sqrt{\frac{2}{j+1} - \frac{2}{n+1}}$ | $\sqrt{2 - \frac{2}{n+1}}\frac{2}{\sqrt{2 - \frac{2}{n+1}}}$ |       |
| n-crosspolytope| $\sqrt{\frac{2}{j+1}}$ | $\sqrt{2}$ | $\sqrt{2}$ |       |
| n-cube         | $\sqrt{n-j}$    | $\sqrt{n}$               |       |
|                |                 | $\frac{2}{\sqrt{n}}$   |       |

$^a$ In these rows (and throughout the paper), $\tau$ denotes the golden ratio $\frac{1+\sqrt{5}}{2}$.

$^b$ Although Coxeter [3] only states that his formulas for $\frac{2^R}{r}$ are valid for $n \geq 5$ (see his Table Iiii), it is evident from his development of these formulas (see pp. 133-134, 158-159) that they are, in fact, valid for $n \geq 2$.

**Lemma 3.2.** Let $P$ be a regular polytope with $V$ vertices, and let $m_i$ be the number of chords of length $d_i$ emanating from a given vertex of $P$. Then the total number of chords of length $d_i$ is given by $N_i = \frac{m_i V}{2}$.

**Proof.** Let $P_1, P_2, \ldots, P_V$ be the vertices of $P$. Without loss of generality, consider the vertex $P_1$. If vertex $P_1$ has $m_i$ chords of length $d_i$ emanating from it, then, by the mappings induced by regularity, each of the $V$ vertices of the polytope also has $m_i$ chords of length $d_i$ emanating from it. However, since the chord $P_j P_j$ is equal to the chord $P_j P_j$ for all distinct $j, j' \in \{1, 2, \ldots, V\}$, this yields a total of only $\frac{m_i V}{2}$ chords of length $d_i$. □

**Lemma 3.3.** Let $P$ be a regular polytope with $V$ vertices and $N$ total chords. Then $N = \frac{V(V-1)}{2}$.

**Proof.** Since each chord in $P$ corresponds to a single pair of distinct vertices, the total number of chords $N$ is equal to the total number of pairs of distinct vertices, i.e., $N = \binom{V}{2} = \frac{V(V-1)}{2}$. □
A useful property of centrally-symmetric polytopes is that, for each vertex $P$, there is a unique vertex $P'$ lying diametrically directly opposite to it (cf. [3]). All regular $n$-polytopes, $n \geq 2$, are centrally symmetric except for the odd-edged polygons and simplices [3].

3.2. Generalizing Fact 1

**Theorem 3.4.** Let $\mathcal{P}$ be any regular $n$-polytope inscribed in a unit $n$-sphere (where $n \geq 2$). Then

$$\sum_{i=1}^{N} c_i^2 = V^2$$

where $c_i$ is the length of each $i^{th}$ chord of $\mathcal{P}$, $N$ is the total number of chords, and $V$ is the total number of vertices.

**Proof.** A regular $n$-polytope, where $n \geq 2$, is either centrally symmetric or not centrally symmetric (in the latter case, it is either an odd-edged polygon or an $n$-simplex). Thus, we consider three cases: (a) $\mathcal{P}$ is centrally symmetric, (b) $\mathcal{P}$ is an odd-edged polygon, or (c) $\mathcal{P}$ is an $n$-simplex.

**Case 1 (\(\mathcal{P}\) is centrally symmetric).** Let $k$ be the number of distinct chords lengths of $\mathcal{P}$. Without loss of generality, chose a vertex $P_0$ of the polytope. Let $P_i$ for $i = 1, 2, ..., k$ be $k$ other vertices of the polytope such that $P_0$ and $P_i$ form the endpoints of $k$ chords each having a distinct length $d_i$ and $d_1 < d_2 < \cdots < d_k$. We now find a formula (or value) for each of these $k$ distinct lengths.

Observe that—except in the case where both $P_0$ and $P_i$ are collinear with the center $O$ of the unit $n$-sphere—each chord $P_0P_i$ is the side of a triangle $\triangle P_0OP_i$. Moreover, the other two sides of the triangle ($OP_0$ and $OP_i$) form radii of the $n$-sphere and so have length 1. Applying the Law of Cosines, we have $d_i^2 = 1^2 + 1^2 - 2(1)(1)\cos \theta_i = 2 - 2\cos \theta_i$ where $\theta_i$ is the angle between the two unit-long sides of the triangle.

When both $P_0$ and $P_i$ are collinear with $O$, the chord $P_0P_i$ is a diameter of the unit $n$-sphere and so has length 2. Since a diameter is the longest possible chord of an $n$-sphere, $d_k = 2$. Furthermore, only one diameter can emanate from $P_0$, so Lemma 3.2 reveals that $\mathcal{P}$ has a total of $\frac{1V}{2}$ chords of length $d_k$.

Letting $N_i$ be the number of chords of length $d_i$ ($i = 1, 2, ..., k$), we have

$$\sum_{i=1}^{N} c_i^2 = \sum_{i=1}^{k} N_i d_i^2 = \sum_{i=1}^{k-1} N_i (2 - 2\cos \theta_i) + N_k d_k^2 =$$

$$2 \sum_{i=1}^{k-1} N_i - 2 \sum_{i=1}^{k-1} N_i \cos \theta_i + \frac{V}{2} (2^2) = 2 \left(\sum_{i=1}^{k} N_i - \frac{V}{2}\right) - 2 \sum_{i=1}^{k-1} N_i \cos \theta_i + 2V.$$
Since $\sum_{i=1}^{k} N_i$ must equal the total number of chords (i.e., $N$) and since $N = \frac{V(V-1)}{2}$ by Lemma 3.3, we have:

$$\sum_{i=1}^{N} c_i^2 = 2 \left[ \frac{V(V-1)}{2} - \frac{V}{2} \right] - 2 \sum_{i=1}^{k-1} N_i \cos \theta_i + 2V = V^2 - 2 \sum_{i=1}^{k-1} N_i \cos \theta_i.$$

Next, we show that $\sum_{i=1}^{k-1} N_i \cos \theta_i = 0$. Since $P$ is centrally symmetric, each vertex $P_i$ has an opposite vertex $P'_i$ for $i = 0, 1, \ldots, k$. Thus, for $i \in \{1, 2, \ldots, k-1\}$, $P_0P_k$ and $P_iP'_i$ intersect at $O$ to form vertical angles $\angle P_0OP_i$ and $\angle P_kOP'_i$. Hence, these two angles are congruent. Furthermore, observe that $\angle P_0OP_i'$ is the supplement of $\angle P_0OP_i$ (which has measure $\theta_i$). Thus, $\angle P_0OP_i'$ has measure $\pi - \theta_i$. However, the line segment $P_0P_i'$ (lying opposite to the angle $\angle P_0OP_i'$) is a side of the triangle $\triangle P_0OP_i'$ (whose other sides, $OP_0$ and $OP_i'$, are radii of length 1) and, moreover, must be the same length as one of the original $P_0P_i$ for $i = 1, 2, \ldots, k-1$ (because these $k-1$ chords are of all possible distinct chord lengths—except for that of the unique diameter emanating from $P_0$—and we know $P_0P_i' \neq P_0P_k$). Hence, by side-side-side, $\triangle P_0OP_i'$ is congruent to one of the “original triangles $\triangle P_0OP_i$.” Thus, the “original $P_0P_i'$” (i.e., the one that is the same length as $P_0P_i'$) must lie opposite to one of the “original angles $\angle P_0OP_i$,” which must have the same measure as $\angle P_0OP_i'$ (i.e., $\pi - \theta_i$). Thus, we have shown that, for each angle of measure $\theta_i$ contained in the set $\{\angle P_0OP_i : i = 1, 2, \ldots, k-1\}$, there is an angle (not necessarily distinct from the angle of measure $\theta_i$) that is also contained in this set and has measure $\pi - \theta_i$. Moreover, since we have shown that, for every chord of length $d_i$ (corresponding to an angle of measure $\theta_i$) that emanates from $P_0$, there is a chord of length $d'_i$ (corresponding to an angle of measure $\pi - \theta_i$) that also emanates from $P_0$. Lemma 3.2 tells us that there must be the same number $N_i$ of chords of length $d_i$ as chords of length $d'_i$. Call the latter number $N_i'$.

Returning to $\sum_{i=1}^{k-1} N_i \cos \theta_i$, this shows that, if the angle $\angle P_0OP_i$ of measure $\theta_i$ and its supplement (the one contained in the set $\{\angle P_0OP_i : i = 1, 2, \ldots, k\}$) are distinct angles, we have $N_i \cos \theta_i + N_i' \cos (\pi - \theta_i) = N_i \cos \theta_i + N_i \cos (\pi - \theta_i) = N_i (\cos \theta_i - \cos \theta_i) = 0$ and these two angles contribute nothing to $\sum_{i=1}^{k-1} N_i \cos \theta_i$. If the angle of measure $\theta_i$ and its supplement are actually the same angle, then $\theta_i = \frac{\pi}{2}$ and $N_i \cos \left(\frac{\pi}{2}\right) = 0$. Thus, this angle also contributes nothing to $\sum_{i=1}^{k-1} N_i \cos \theta_i$. Therefore, we see that $\sum_{i=1}^{k-1} N_i \cos \theta_i = 0$ and $\sum_{i=1}^{N} c_i^2 = V^2$.

**Case 2 ($P$ is an odd-edged polygon).** Since $P$ is a regular polygon, Fact 1 tells us that $\sum_{i=1}^{N} c_i^2 = E^2$. Since the number of vertices, $V$, of an $E$-edged polygon is equal to $E$ [4], we have: $\sum_{i=1}^{N} c_i^2 = V^2$.

**Case 3 ($P$ is an $n$-simplex).** A simplex has no diagonals (since all of its vertices are connected by edges; cf. [3]), thus, all of the chords of $P$ are edges. An edge is a 1-face, so the number of edges $E$ is obtained by letting $j = 1$ in
the cardinality formula for the $j$-faces of $n$-simplices given in Table 1:

$$E = \binom{n+1}{j+1} = \frac{(n+1)n}{2}.$$  

The edge length $e$ of an $n$-simplex is $2\left(2 - \frac{2}{n+1}\right)^{-1/2}$ (see Table 2). We have:

$$\sum_{i=1}^{N} c_i^2 = E e^2 = \frac{(n+1)n}{2} \left(\frac{4}{2 - \frac{2}{n+1}}\right) = (n+1)^2.$$  

Since the vertices are 0-faces, the number of vertices $V$ of $P$ can be seen to be $n+1$ from Table 1. Therefore, $\sum_{i=1}^{N} c_i^2 = (n+1)^2 = V^2$. \qed

Corollary 3.5. Let $Q$ be any regular $n$-polytope circumscribed about a unit $n$-sphere (where $n \geq 2$). Then $\sum_{i=1}^{N} s_i^2 = F^2$ where $s_i$ is the length of each $i$th line segment whose endpoints are centers of facets of $Q$, $N$ is the number of these line segments, and $F$ is the number of facets of $Q$.

Proof. The centers of the facets of $Q$ are also the vertices of a reciprocal regular $n$-polytope $P$ of $Q$ with respect to the unit $n$-sphere about which $Q$ is circumscribed (cf. [3]). Since the vertices of $P$ are the centers of the facets $Q$, the chords of $P$ are the line segments whose endpoints are the centers of facets of $Q$. Hence, the lengths of these line segments ($s_i$ for $i = 1, 2, \ldots, N$) are also the lengths of the chords of $P$. Thus, by Theorem 3.4, we have $\sum_{i=1}^{N} s_i^2 = V^2$ where $V$ is the number of vertices of $P$. Moreover, since every facet of $Q$ corresponds to a vertex of $P$ (and vice versa), the number of facets $F$ of $Q$ is the same as the number of vertices $V$ of $P$. Therefore, we have $\sum_{i=1}^{N} s_i^2 = F^2$ for $Q$. \qed

3.3. Discussion

Theorem 3.4 shows that Fact 1 does, in fact, apply to all regular $n$-polytopes where $n \geq 2$—if the “number of edges” ($E$) mentioned in Fact 1 is reinterpreted as “number of vertices” ($V$). This reinterpretation is valid since, for 2-polytopes, $E = V$.

4. FACT 2: Sum of Squared Distinct Chords

The second 2-polytope fact stated that, for any regular 2-polytope inscribed in a unit 2-sphere, the sum of its squared distinct chord lengths equals (a) $E$ (when $E$ is odd) and (b) some integer (when $E$ is even). To test generalization of this fact to all regular $n$-polytopes ($n \geq 2$), we use the following lemmas.

4.1. Preliminaries

Lemma 4.1. Let $P$ be a regular 2-polytope with $E$ edges and $k$ chords of distinct lengths. If $E$ is odd, then $E = 2k + 1$. 
Proof. Let \( O \) be the center of \( P \), let \( P_0 \) be a vertex of \( P \), and let \( P_i \) for \( i = 1, 2, ..., k \) be \( k \) other vertices of \( P \) such that \( P_0 \) and \( P_i \) form the endpoints of \( k \) chords each having a distinct length \( d_i \). Since \( E \) is odd, the line \( l \) that passes through \( O \) and \( P_0 \) is an axis of symmetry of \( P \) (cf. [1]). Furthermore, only one of the vertices of \( P \) lies on \( l \) (i.e., \( P_0 \); cf. [1]). If we reflect \( P \) across the line \( l \), we see that, for each \( P_i \) where \( i \neq 0 \), there is another vertex, call it \( P_i' \), that is the same distance \( d_i \) from \( P_0 \) (because it is now in the same location as \( P_i \) had been before reflecting \( P \) and \( P_0 \) has not moved). Since there are no other (non-identity) symmetries of \( P \) that hold \( P_0 \) constant, this shows that these two vertices, \( P_i \) and \( P_i' \), are the only two vertices that are a distance of \( d_i \) away from \( P_0 \) for each \( i \in \{1, 2, ..., k\} \). Thus, \( P \) has \( 2k \) vertices besides \( P_0 \) and the total number of vertices \( V \) of \( P \) is \( 2k + 1 \). Since \( E = V \) for 2-polytopes, we have \( E = 2k + 1 \). \( \square \)

Lemma 4.2. Let \( P \) be a regular \( n \)-polytope, \( n \geq 2 \), that is not a simplex of dimension \( n \geq 3 \). If \( P \) has an odd number of edges \( E \), then \( n = 2 \).

Proof. From Table 1, we see that all the regular 3- and 4-polytopes have even \( E \). Likewise, substituting \( j = 1 \) into the \( j \)-face cardinality formulas for the \( n \)-crosspolytopes and \( n \)-cubes in Table 1 yields \( E = 2^2 \binom{n}{2} = 2n(n - 1) \) and \( E = 2^{n-1} \binom{n}{1} = 2^{n-1}n \), respectively, both of which are even. Since \( P \) is not a simplex of dimension \( n \geq 3 \) by assumption, this means \( P \) is not a polytope of dimension \( n \geq 3 \). Therefore, \( n = 2 \). \( \square \)

4.2. Generalizing Fact 2

Theorem 4.3. Let \( P \) be a regular \( n \)-polytope inscribed in a unit \( n \)-sphere (\( n \geq 2 \)). Let \( P \) have \( V \) vertices, \( E \) edges, and \( k \) chords of distinct lengths, and let \( d_i \) denote the \( i \)th distinct chord length of \( P \).

1. If \( P \) is a simplex of dimension \( n \geq 3 \), then \( \sum_{i=1}^{k} d_i^2 \) is a non-integral rational number.
2. If \( P \) is not a simplex of dimension \( n \geq 3 \), then \( \sum_{i=1}^{k} d_i^2 \) is an integer. Specifically:
   (a) If \( E \) is odd, this integer is \( 2k + 1 \).
   (b) If \( E \) is even, this integer is \( 2k + 2 \).

Proof. We prove Part 1, Part 2a, and then Part 2b.

Part 1. Let \( P \) be a regular \( n \)-simplex inscribed in a unit \( n \)-sphere where \( n \geq 3 \). Recall that a simplex has only one type of chord (i.e., its edges), all of which have length \( e = 2 \left( 2 - \frac{2}{n+1} \right)^{-1/2} \). Thus, we have

\[
\sum_{i=1}^{k} d_i^2 = e^2 = \frac{4}{2 - \frac{2}{n+1}} = \frac{2(n+1)}{n}.
\]

Observe that both the numerator and denominator of \( \frac{2(n+1)}{n} \) are integers (and that \( n \neq 0 \)). Thus, \( \frac{2(n+1)}{n} \) is a rational number. Moreover, since \( n \geq 3 \), \( n \) does not divide 2. On the other hand, since \( n \) and \( n + 1 \) are consecutive
integers, they are relatively prime. Hence, since $n \neq 1$, $n$ does not divide $n + 1$. We conclude that $\frac{2(n+1)}{n}$ is not an integer but is a rational number.

**Part 2a.** Let $P$ be an odd-edged regular $n$-polytope (inscribed in a unit $n$-sphere) of dimension $n \geq 2$ that is not a simplex of dimension $n \geq 3$. Then, by Lemma 4.2, $P$ is 2-dimensional. Hence, by Fact 2, we have $\sum_{i=1}^{k} d_i^2 = E$. Since $E$ is odd, $\sum_{i=1}^{k} d_i^2 = E = 2k + 1$ by Lemma 4.1.

**Part 2b.** Let $P$ be an even-edged regular $n$-polytope (inscribed in a unit $n$-sphere) of dimension $n \geq 2$ that is not a simplex of dimension $n \geq 3$. Thus, $P$ is neither an odd-edged polygon nor a simplex of dimension $n \geq 2$. Hence, $P$ is centrally symmetric.

Choose a vertex $P_0$ of $P$ and let $P_1$ for $i = 1, 2, \ldots, k$ be $k$ other vertices of the polytope such that $P_0$ and $P_1$ form the endpoints of $k$ different chords each having a distinct length $d_i$ and $d_1 < d_2 < \cdots < d_k$ (just as in Case 3 of the proof of Theorem 3.4). From the proof of Theorem 3.4, we know that $d_i^2 = 2 - 2 \cos \theta_i$ for $i = 1, 2, \ldots, k$ (where $\theta_i$ is the measure of the angle $\angle P_0P_i$) and $d_k^2 = 2^2$. Therefore, we have

$$\sum_{i=1}^{k} d_i^2 = \sum_{i=1}^{k-1} (2 - 2 \cos \theta_i) + d_k^2 = 2(k - 1) - 2 \sum_{i=1}^{k-1} \cos \theta_i + 4.$$

Next, we show that $\sum_{i=1}^{k-1} \cos \theta_i = 0$. Recall from the proof of Theorem 3.4 that, for each angle of measure $\theta_i$ contained in the set $\{ \angle P_0P_i : i = 1, 2, \ldots, k \}$, there is an angle (not necessarily distinct from the angle of measure $\theta_i$) that is supplementary to the angle of measure $\theta_i$ and is also contained in the set. Thus, if the angle of measure $\theta_i$ and its supplement are distinct angles, it follows that $\cos \theta_i + \cos (\pi - \theta_i) = \cos \theta_i - \cos \theta_i = 0$ and these two angles contribute nothing to $\sum_{i=1}^{k-1} \cos \theta_i$. If the angle of measure $\theta_i$ and its supplement are actually the same angle, then $\theta_i = \frac{\pi}{2}$ and $\cos \left( \frac{\pi}{2} \right) = 0$.

Therefore, we see that $\sum_{i=1}^{k-1} \cos \theta_i = 0$ and $\sum_{i=1}^{N} d_i^2 = 2k + 2$. \hfill $\Box$

**Corollary 4.4.** Let $Q$ be a regular $n$-polytope circumscribed about a unit $n$-sphere ($n \geq 2$) having $E$ edges and $F$ facets. Let $t_i$ for $i = 1, 2, \ldots, k$ be the distinct lengths of the line segments whose endpoints are centers of facets.

1. If $Q$ is a simplex of dimension $n \geq 3$, then $\sum_{i=1}^{k} t_i^2$ is a non-integral rational number.
2. If $Q$ is not a simplex of dimension $n \geq 3$, then $\sum_{i=1}^{k} t_i^2$ is an integer. Specifically:
   (a) If $E$ is odd, this integer is $2k + 1$.
   (b) If $E$ is even, this integer is $2k + 2$.

**Proof.** This corollary follows by much the same reasoning as Corollary 3.5. Note that the reciprocal of a regular simplex of dimension $n \geq 3$ is a regular simplex of dimension $n \geq 3$, the reciprocal of an odd-edged regular polygon is an odd-edged regular polygon, and the reciprocal of a centrally-symmetric regular polytope is a centrally-symmetric regular polytope (cf. 3). \hfill $\Box$
4.2.1. Special Cases (the regular 3-polytopes).

Theorem 4.5. Let $P$ be a regular 3-polytope inscribed in a unit 3-sphere with $V$ vertices and $k$ distinct chords. Let $d_i$ be the $i^{th}$ distinct chord length of $P$.

1. For the self-dual tetrahedron, $\sum_{i=1}^{k} d_i^2 \in \mathbb{Q}$.
2. For the dual pair of octahedron and cube, $\sum_{i=1}^{k} d_i^2 = V$.
3. For the dual pair of icosahedron and dodecahedron, $\sum_{i=1}^{k} d_i^2 = 2k + 2$.

Proof. Parts 1 and 3 follow directly from Parts 1 and 2c of Theorem 4.3, respectively (since the icosahedron and dodecahedron have an even number of edges). For Part 2, recall that the regular octahedron and cube have an even number of edges. Hence, for each of them, Part 2c of Theorem 4.3 guarantees that $\sum_{i=1}^{k} d_i^2 = 2k + 2$. We determine $k$ for the octahedron and cube.

Octahedron. The octahedron is just the regular 3-crosspolytope, so consider a regular $n$-crosspolytope of edge length $e$ inscribed in a unit $n$-sphere (where $n \geq 2$). It can be constructed by (a) creating a Cartesian cross ($n$ mutually orthogonal lines through a point $O$; see [3]), (b) finding the points on each of these lines that are all a distance $e\sqrt{0.5}$ away from the point $O$, and (c) connecting all pairs of these points that do not lie on the same line of the Cartesian cross by line segments (cf. the construction in [14]).

We now have a regular $n$-crosspolytope whose vertices are the $2n$ points found in part “b” and whose edges are all the line segments constructed in part “c.” (These edges do have length $e$, as can be checked by applying the Pythagorean Theorem to a triangle whose vertices consist of any one of the pairs of points mentioned in part “c” and the point $O$).

Now we determine $k$. From part “c” of our construction, it is readily apparent that any vertex $P$ of the crosspolytope is connected to all other vertices by edges except for the vertex $P'$ lying on the same “Cartesian cross” line as $P$. Thus, the chords of the crosspolytope come in two types: (a) edges and (b) inner diagonals $PP'$ that are portions of the “Cartesian cross” lines. Thus, $k = 2$ and $\sum_{i=1}^{k} d_i^2 = 2k + 2 = 6$, which is the number of vertices.

Cube. The cube is the 3-cube, so consider an $n$-cube. An $n$-cube’s $j$-faces are $j$-cubes (see Table 1) and its chords thus come in $n$ types: (a) edges and (b) $n - 1$ types of diagonals (each of which is the “longest diagonal” of a $j$-face, where $2 \leq j \leq n$, of the $n$-cube). Thus, $k = n$ for a regular $n$-cube and $k = 3$ for the 3-cube. Therefore, $\sum_{i=1}^{k} d_i^2 = 2k + 2 = 8$, which is the number of vertices. □

Corollary 4.6. Let $Q$ be a regular 3-polytope circumscribed about a unit 3-sphere with $F$ facets. Let $t_i$, for $i = 1, 2, \ldots, k$, be the distinct lengths of the line segments whose endpoints are centers of facets of $Q$. Then

1. For the self-dual tetrahedron, $\sum_{i=1}^{k} t_i^2 \in \mathbb{Q}$.
2. For the dual pair of octahedron and cube, $\sum_{i=1}^{k} t_i^2 = F$.
3. For the dual pair of icosahedron and dodecahedron, $\sum_{i=1}^{k} t_i^2 = 2k + 2$. 


4.3. Discussion
Theorem 4.3 (combined with Lemma 4.2) shows that Fact 2 does, in fact, apply to all regular $n$-polytopes (where $n \geq 2$) except most $n$-simplices. Moreover, this theorem shows that the “$E$” in Fact 2 can be re-characterized as $2k + 1$ (where $k$ is the number of distinct chords) and specifies the “some integer” in Fact 2 as $2k + 2$. Thus, these two integers are remarkably similar.

5. FACT 3: Product of Squared Chords
The third 2-polytope fact stated that, for any regular 2-polytope inscribed in a unit 2-sphere, the product of its squared chord lengths equals $E^E$. We test generalization of this fact to (a) the regular $n$-crosspolytopes (where $n \geq 2$), (b) the regular 24-cell, and (c) the other regular polytopes.

5.1. Preliminaries
No additional information is need to test generalization of Fact 3 to the $n$-crosspolytopes. However, in testing generalization to the 24-cell, we will make use of the following definition, and, in testing generalization to the other regular polytopes, we will make use the cardinalities in Table 8.

Definition. A section of a regular $n$-polytope $P$ is a nonempty intersection of $P$ and an $(n - 1)$-plane $H$ (cf. [3]). This intersection is a $j$-polytope for some $j \in \{0, 1, ..., n - 1\}$ (cf. [3]).

Of particular interest to us are the sections known as simplified sections. These are the sections (a) that are formed when the $(n - 1)$-plane $H$ is orthogonal to a line $l$ that passes through the $n$-polytope’s center $O$ and a given, fixed vertex $P_0$ and (b) that form $j$-polytopes whose vertices are all vertices of the original $n$-polytope $P$. For these sections, which we will denote by $S_i$ for $i = 0, 1, 2, ..., k$, the distances between the fixed vertex $P_0$ and each one of the vertices in a given section ($S_i$ for some particular $i$) are always the same length (for a regular polytope; cf. [3]). Hence, if we let $P_i$ be one of the vertices in the section $S_i$ of a given regular polytope, the lengths of the chords $P_0P_i$ and $P_0P_j$ are distinct if $i \neq j$.

5.2. Generalizing Fact 3 to Crosspolytopes
Theorem 5.1. Let $P$ be a regular $n$-crosspolytope with $V$ vertices and $F$ facets inscribed in a unit $n$-sphere (where $n \geq 2$). Then
\[
\prod_{i=1}^{N} c_i^2 = F^V
\]
where $N$ is the total number of chords of $P$ and $c_i$ is the length of each $i^{th}$ chord.

Proof. Recall from the proof of Theorem 4.5 that the chords of an $n$-crosspolytope $(n \geq 2)$ come in two types: (a) edges and (b) inner diagonals. Let the lengths of the latter be denoted by $d$. 
Table 3. Number of “Vertices Incident to Any Edge” (\(\nu\)) and Number of “Edges Incident to Any Vertex” (\(\varepsilon\)) for the Regular 3-Polytopes \([3]\).

| 3-Polytope         | \(\nu\) | \(\varepsilon\) |
|--------------------|--------|------------------|
| tetrahedron        | 2      | 3                |
| octahedron         | 2      | 4                |
| cube               | 2      | 3                |
| icosahedron        | 2      | 5                |
| dodecahedron       | 2      | 3                |

An edge is a 1-face, so the number of edges \(E\), given by Table 1, is

\[E = 2^{j+1} \binom{n}{j+1} = 2^2 \binom{n}{2} = 2n(n-1).\]

The number of inner diagonals, \(N_d\), is the total number of chords (i.e., \(N\)) minus the number of edges. By Lemma 3.3 we have:

\[N_d = \frac{V(V-1)}{2} - E.\]

Recall from the proof of Theorem 4.5 that an \(n\)-crosspolytope has \(2n\) vertices \((n \geq 2)\). Thus, substituting the values for \(V\) and \(E\) yields

\[N_d = \frac{(2n)(2n-1)}{2} - 2n(n-1) = n.\]

Now that we know the number of chords of length \(e\) and length \(d\), we want to calculate these lengths. From Table 2 we see \(e = 2^{1/2}\). From the proof of Theorem 4.5, we see \(d\) is twice the distance from the polytope’s center \(O\) to any one of its vertices and that this distance is \(e\sqrt{0.5}\). Thus, \(d = 2e\sqrt{0.5} = 2\).

Therefore, we have:

\[
\prod_{i=1}^{N} c_i^2 = (e^2)^E(d^2)^{N_d} = 2^{2n(n-1)}(2)^{2n} = (2^n)^{2n} = (2^n)^V. 
\]

From Table 1 the number of facets is \(F = 2^n \binom{n}{2} = 2^n\). We conclude that

\[
\prod_{i=1}^{N} c_i^2 = (2^n)^V = F^V. 
\]

**Corollary 5.2.** Let \(Q\) be an \(n\)-cube with \(V\) vertices and \(F\) facets that circumscribes a unit \(n\)-sphere \((n \geq 2)\). Then \(\prod_{i=1}^{N} s_i^2 = V^F\) where \(s_i\) is the length of each \(i\)th line segment whose endpoints are centers of facets of \(Q\) and \(N\) is the total number of these line segments.

**Proof.** The reciprocal of a regular \(n\)-crosspolytope is an \(n\)-cube \([3]\). \(\square\)
5.3. Generalizing Fact 3 to the 24-cell

**Theorem 5.3.** Let $\mathcal{P}$ be a regular 24-cell with $E$ edges and $R$ ridges inscribed in a unit 4-sphere. Then

$$
\prod_{i=1}^{N} c_i^2 = 6^E = 6^R
$$

where $c_i$ is the length of each $i^{th}$ chord of $\mathcal{P}$ and $N$ is the total number of chords.

**Proof.** According to Table 2, the edge length of $\mathcal{P}$ is 1. To find its other distinct chord lengths, we consider the distance $d'_i$ from a given, fixed vertex $P_0$ of an arbitrary 24-cell to some vertex $P_i$ in its simplified section $S_i$ (where $i = 1, 2, ..., k$). (Note that—since these $k$ distances are distinct—they will actually be the polytope’s distinct chord lengths).

In Table V(i) in [3], there is a list of the values of $a_i$, which is the distance $d'_i$ divided by the edge length $e$ of an arbitrary 24-cell (so $d'_i = a_i e$).

Thus, to obtain the distinct chord lengths $d_i$ for $\mathcal{P}$, we multiply each of the values of $a_i$ by the edge length of $\mathcal{P}$ (i.e., by 1). These values are listed in the second column of Table 4 (For convenience’ sake, the values of $d_2^i$ are listed in the third column of Table 4).

To find the total number of chords of each distinct length, we will use the total number of vertices $V_i$ in each simplified section $S_i$ as listed in Table V(i) of [3]. Obviously, the number of chords $m_i$ of a given length $d_i$ emanating from $P_0$ is the same as $V_i$ (see the fourth column of Table 4). Thus, the total number of chords of length $d_i$ is $m_iV_i$ by Lemma 3.2. The total number of vertices $V$ of a 24-cell is 24 (see Table 1). Substituting the values of $m_i$ and $V$ into $m_iV_i/2$ yields the total number of chords of length $d_i$ for $i = 1, 2, ..., k$ (as listed in the last column of Table 4).

| $S_i$ | $a_i (= d'_i = d_i)$ | $d_i^2$ | # of vertices ($m_i$) | # of chords of length $d_i$ |
|-------|---------------------|--------|----------------------|-----------------------------|
| $S_1$ | 1                   | 1      | 8                    | 96                          |
| $S_2$ | $2^{1/2}$           | 2      | 6                    | 72                          |
| $S_3$ | $3^{1/2}$           | 3      | 8                    | 96                          |
| $S_4$ | 2                   | 4      | 1                    | 12                          |

Therefore, we have $\prod_{i=1}^{k} c_i^2 = (1)^{96} (2)^{72} (3)^{96} (4)^{12} = 6^{96}$. Table 1 reveals that, for the 24-cell, the number of edges $E$ and the number of ridges $R$ are both equal to 96. We conclude: $\prod_{i=1}^{k} c_i^2 = 6^E = 6^R$. □

\(^4\)Instead of “$a_i$,” Coxeter [3] actually uses “$a$.”
5.4. Generalizing Fact 3 to Other Regular Polytopes

**Theorem 5.4.** Let \( \mathcal{P} \) be a regular 3-polytope with \( E \) edges and \( V \) vertices inscribed in a unit 3-sphere. Then

\[
\prod_{i=1}^{N} c_i^2 = \nu^a / \varepsilon^b
\]

where \( c_i \) is the length of each \( i \)-th chord of \( \mathcal{P} \), \( \nu \) is the number of vertices incident to any edge, \( \varepsilon \) is the number of edges incident to any vertex, and \( a, b \) are positive integers such that \( E \) divides \( b \) and

1. for the self-dual tetrahedron,
   \[ a \equiv 0 \pmod{E} \text{ and } a \equiv E \pmod{V} \]
2. for the dual pair of octahedron and cube,
   \[ a \equiv V \pmod{E} \text{ and } a \equiv E \pmod{V} \]
3. for the dual pair of icosahedron and dodecahedron,
   \[ a \equiv V \pmod{E} \text{ and } a \equiv 0 \pmod{V} \]

In addition, the number \( \varepsilon \) in the equation above may be replaced by

\[
\frac{E}{(E, V)} = \frac{[E, V]}{V}
\]

where \( (E, V) \) is the greatest common divisor of \( E \) and \( V \) and \( [E, V] \) is the least common multiple of \( E \) and \( V \).

**Proof.** We consider each of these five cases in turn: (a) the tetrahedron, (b) the octahedron, (c) the cube, (d) the icosahedron, and (e) the dodecahedron.

**Tetrahedron.** Recall from the proof of Theorem 3.4 that a regular \( n \)-simplex inscribed in a unit \( n \)-sphere has \( \frac{(n+1)n}{2} \) edges of length

\[
2 \left( 2 - \frac{2}{n+1} \right)^{-1/2}
\]

and no diagonals. Substituting \( n = 3 \) into these expressions shows that, for a regular 3-simplex (i.e., the tetrahedron) inscribed in a unit 3-sphere, \( E = 6 \) and the edge length \( e \) is \( \frac{2\sqrt{6}}{3} \). Therefore, we have

\[
\prod_{i=1}^{N} c_i^2 = e^E = \left( \frac{2\sqrt{6}}{3} \right)^2 = \frac{218}{3^6}. \tag{5.1}
\]

Observe that we have 2 as the numerator’s base and 3 as the denominator’s base (which are \( \nu \) and \( \varepsilon \) for the tetrahedron, respectively), as desired. Next, we consider the exponents. Recall that, for a tetrahedron, \( E = 6 \) and \( V = 4 \). For the denominator’s exponent, observe that \( E \) divides 6. For the numerator’s exponent, observe that (a) \( 18 \equiv 6 \equiv E \pmod{E} \) and (b) \( 18 \equiv 2 \equiv 6 \equiv E \pmod{V} \) (as desired). Finally, the identity \( EV = (E, V) \cdot [E, V] \) implies

\[
\frac{E}{(E, V)} = \frac{[E, V]}{V}. \]

Substituting into the left-hand side of this equation yields

\[
\frac{6}{(6, 4)} = 3, \text{ which is the denominator’s base (as desired)}. \]
Octahedron. Recall from the proof of Theorem 5.1 that the chords of a regular $n$-crosspolytope inscribed in a unit $n$-sphere consist of $2n(n - 1)$ edges of length $2^{1/2}$ and $n$ inner diagonals of length 2. Substituting $n = 3$ into these expressions shows that a regular 3-crosspolytope (i.e., the octahedron) inscribed in a unit 3-sphere has 12 edges of length $2^{1/2}$ (which we denote by $e$) and three inner diagonals of length 2 (which we denote by $d$). Therefore:

$$\prod_{i=1}^{N} c_i^2 = (e^2)^{12} (d^2)^3 = (2)^{12} (2^2)^3 = 2^{18} = \frac{2^{18+24q}}{2^{24q}} = \frac{2^{18+24q}}{4^{12q}} \quad (5.2)$$

where $q$ is some integer.

Observe that we have 2 as the numerator’s base and 4 as the denominator’s base (which are $\nu$ and $\varepsilon$ for the octahedron, respectively), as desired. Next, we consider the exponents. Recall that $E = 12$ and $V = 6$ for the octahedron. For the denominator’s exponent, observe that $E$ divides $12q$ (as desired). For the numerator’s exponent, observe that (a) $18 + 24q \equiv 0 \equiv 12 \equiv E$ (mod $V$) and (b) $18 + 24q \equiv 0 \equiv V$ (mod $E$) (as desired). Finally, $\frac{12}{(12,6)} = 2$, which is the denominator’s base (as desired).

Cube. First, we determine the distinct chord lengths of a cube inscribed in a unit 3-sphere. Since a cube has eight vertices, we know there must be seven chords (not necessarily distinct) emanating from a given vertex $P$. From Table 3 we see that there are three edges emanating from $P$ (three of the seven chords). Table 2 in reveals that the cube’s edges have length $3^{1/2}$.

For the next distinct chord length, observe that three of the cube’s square faces meet at $P$ so that each pair of faces shares an edge emanating from $P$ (cf. [3]). Since a square has four vertices, each of these three faces has one vertex that is not joined to $P$ by an edge. Thus, each face has one diagonal emanating from $P$ and each of these three diagonals forms an (outer) diagonal of the cube. Using Pythagorean’s Theorem on the right triangle formed by one of these diagonals and two adjacent edges shows that these outer diagonals have length $\sqrt{(3^{1/2})^2 + (3^{1/2})^2} = 2^{1/2} \sqrt{6}$.

We have now determined the length of six of the seven chords emanating from $P$. For the seventh, recall that a cube is centrally-symmetric and, hence, $P$ has a vertex $P'$ diametrically opposite to it. Hence, the chord $PP'$ is a diameter. Thus, it has length 2 (and is the seventh chord).

Next, we determine the total number of chords of these distinct lengths. By Lemma 3.2 viz., $N_i = \frac{m_i V}{2}$, the cube has $\frac{3(8)}{2} = 12$ chords of length $3^{-1/2}$, $\frac{3(8)}{2} = 12$ chords of length $2\sqrt{6}$, and $\frac{1(8)}{2} = 4$ chords of length 2.

Therefore, we have

$$\prod_{i=1}^{N} c_i^2 = \left( \left( 3^{-1/2} \right)^2 \right)^{12} \left( \left( \frac{2\sqrt{6}}{3} \right)^2 \right)^{12} \left( (2)^2 \right)^4 = \frac{2^{68}}{3^{24}}. \quad (5.3)$$

Observe that 2 is the numerator’s base and 3 is the denominator’s base ($\nu$ and $\varepsilon$ for the cube, respectively), as desired. Next, we consider the exponents.
Recall that $E = 12$ and $V = 8$ for the cube. For the denominator’s exponent, observe that $E$ divides 24 (as desired). For the numerator’s exponent, observe that (a) $68 \equiv 12 \equiv E \pmod{V}$ and (b) $68 \equiv 8 \equiv V \pmod{E}$ (as desired). Finally, $\frac{12}{112.87} = 3$, which is the denominator’s base.

**Icosahedron.** The edge length of a regular icosahedron inscribed in a unit 3-sphere is $5^{-1/4}\tau^{-1/2}2$ (see Table [2]). The coordinates for a regular icosahedron of edge length 2 are given in [3]. To find the distinct chord lengths of a regular icosahedron of edge length $5^{-1/4}\tau^{-1/2}2$, we:

1. Select the coordinates of an arbitrary vertex (of a regular icosahedron of edge length 2).
2. Compute the distance between this vertex and the other vertices (of the regular icosahedron of edge length 2).
3. Divide each one of the $k$ distinct distances found in step 3 by the edge length 2.
4. Multiply each result in step 3 by the new edge length $5^{-1/4}\tau^{-1/2}2$.

Comparing with [3, p. 238, para. 3] shows the validity of steps 3 and 4.

We choose the vertex $\langle 0, \tau, 1 \rangle$ and compute the distance between it and the other vertices. To do this, we make repeated use of three identities, namely, $\tau^2 = \tau + 1$, $\tau^{-1} = \tau - 1$, and $\tau^{-2} = -\tau + 2$ (given in [5]):

\[
\begin{align*}
\langle 0, \tau, 1 \rangle & - \langle 0, \tau, -1 \rangle = \|\langle 0, 0, 1 \rangle\| = 2 \\
\langle 0, \tau, 1 \rangle & - \langle 0, -\tau, 1 \rangle = \|\langle 0, -2\tau, 0 \rangle\| = 2\tau \\
\langle 0, \tau, 1 \rangle & - \langle 0, -\tau, -1 \rangle = \|\langle 0, 2\tau, 2 \rangle\| = 2\sqrt{\tau + 2} \\
\langle 0, \tau, 1 \rangle & - \langle 1, 0, \tau \rangle = \|\langle -1, \tau, 1 - \tau \rangle\| = 2 \\
\langle 0, \tau, 1 \rangle & - \langle -1, 0, -\tau \rangle = \|\langle -1, \tau, 1 + \tau \rangle\| = 2\tau \\
\langle 0, \tau, 1 \rangle & - \langle -1, 0, -\tau \rangle = \|\langle 1, \tau, 1 + \tau \rangle\| = 2 \\
\langle 0, \tau, 1 \rangle & - \langle -\tau, 1, 0 \rangle = \|\langle -\tau, \tau - 1, 1 \rangle\| = 2 \\
\langle 0, \tau, 1 \rangle & - \langle -\tau, 1, 0 \rangle = \|\langle -\tau, \tau - 1, 1 \rangle\| = 2 \\
\langle 0, \tau, 1 \rangle & - \langle -\tau, -1, 0 \rangle = \|\langle -\tau, \tau + 1, 1 \rangle\| = 2\tau \\
\langle 0, \tau, 1 \rangle & - \langle -\tau, -1, 0 \rangle = \|\langle \tau, \tau - 1, 1 \rangle\| = 2 \\
\langle 0, \tau, 1 \rangle & - \langle -\tau, -1, 0 \rangle = \|\langle \tau, \tau + 1, 1 \rangle\| = 2\tau
\end{align*}
\]

Thus, there are 3 distinct chord lengths: 2, 2$\tau$, and $2\sqrt{\tau + 2}$. Dividing these by 2 yields 1, $\tau$, and $\sqrt{\tau + 2}$, and multiplying them by $5^{-1/4}\tau^{-1/2}2$, $5^{-1/4}\tau^{-1/2}2$, and $5^{-1/4}\tau^{-1/2}2\sqrt{\tau + 2}$, it can be shown that the last of these equals 2.

From our computations, we see that, emanating from the vertex $\langle 0, \tau, 1 \rangle$, there are 5 chords of length 2, 5 chords of length $2\tau$, and one chord of length $2\sqrt{\tau + 2}$. Thus, for the regular icosahedron of edge length $5^{-1/4}\tau^{-1/2}2$, there are 5 chords of length $5^{-1/4}\tau^{-1/2}2$, 5 chords of length $5^{-1/4}\tau^{-1/2}2$, and one chord of length 2. Since a regular icosahedron has 12 vertices, Lemma [3,2] viz., $N_i = \frac{m_i V}{2}$, shows that there are a total of 30 chords of length $5^{-1/4}\tau^{-1/2}2$, 30 chords of length $5^{-1/4}\tau^{-1/2}2$, and six chords
of length 2. Therefore, we have
\[
\prod_{i=1}^{N} c_i^2 = \left(\frac{5^{-1/4} \tau^{-1/2}}{2}\right)^{30} \left(\frac{5^{-1/4} \tau^{1/2}}{2}\right)^{30} \left(\frac{2}{2}\right)^6 = \frac{2^{132}}{5^{30}}. \tag{5.4}
\]

Observe that 2 is the numerator’s base and 5 is the denominator’s base (ν and ε for the icosahedron, respectively). Next, we consider the exponents. Recall that \( E = 30 \) and \( V = 12 \) for the icosahedron. For the denominator’s exponent, observe that \( E \) divides 30 (as desired). For the numerator’s exponent, observe that (a) 132 = 12(11) = V(11) and therefore \( 132 \equiv V \pmod{V} \) and that (b) \( 132 = 30(4) + 12 = E(4) + V \) and therefore \( 132 \equiv V \pmod{E} \) as desired. Finally, \( \frac{30}{(30,12)} = 5 \), which is the denominator’s base.

**Dodecahedron.** The edge length of a regular dodecahedron inscribed in a unit 3-sphere is \( 3^{-1/2} \tau^{-1/2} \) (see Table 2). The coordinates for a regular dodecahedron of edge length \( 2 \tau^{-1} \) are given in [3]. To find the distinct chord lengths of a regular dodecahedron of edge length \( 3^{-1/2} \tau^{-1/2} \), we apply the analogous four steps to the four steps listed in the “Icosahedron case” (replacing \( 5^{-1/4} \tau^{-1/2} \) with \( 3^{-1/2} \tau^{-1/2} \) and 2 with \( 2 \tau^{-1} \)).

We choose the vertex \( \langle 0, \tau^{-1}, \tau \rangle \) (which we will denote by \( P_0 \)) and compute the distance between this vertex and the other vertices:

\[
\begin{align*}
P_0 - \langle 0, \tau^{-1}, -\tau \rangle &= \|\langle 0, 0, 2\tau \rangle\| = 2\tau \\
P_0 - \langle 0, -\tau^{-1}, \tau \rangle &= \|\langle 0, 2\tau^{-1}, 0 \rangle\| = 2\tau^{-1} \\
P_0 - \langle 0, -\tau^{-1}, -\tau \rangle &= \|\langle 0, 2\tau^{-1}, 2\tau \rangle\| = 2\sqrt{3} \\
P_0 - \langle \tau, 0, -\tau^{-1} \rangle &= \|\langle -\tau, \tau^{-1}, \tau - \tau^{-1} \rangle\| = 2 \\
P_0 - \langle -\tau, 0, \tau^{-1} \rangle &= \|\langle \tau, \tau^{-1}, -\tau + \tau^{-1} \rangle\| = 2\sqrt{2} \\
P_0 - \langle -\tau, 0, -\tau^{-1} \rangle &= \|\langle \tau, \tau^{-1}, -\tau + \tau^{-1} \rangle\| = 2\sqrt{2} \\
P_0 - \langle -\tau^{-1}, \tau, 0 \rangle &= \|\langle -\tau^{-1}, \tau^{-1} - \tau, \tau \rangle\| = 2 \\
P_0 - \langle -\tau^{-1}, -\tau, 0 \rangle &= \|\langle -\tau^{-1}, \tau^{-1} + \tau, \tau \rangle\| = 2\sqrt{2} \\
P_0 - \langle 1, 1, 1 \rangle &= \|\langle -1, \tau^{-1} - 1, \tau - 1 \rangle\| = 2\tau^{-1} \\
P_0 - \langle 1, 1, -1 \rangle &= \|\langle -1, \tau^{-1} - 1, \tau + 1 \rangle\| = 2\sqrt{2} \\
P_0 - \langle 1, -1, 1 \rangle &= \|\langle -1, \tau^{-1} + 1, \tau - 1 \rangle\| = 2 \\
P_0 - \langle -1, 1, 1 \rangle &= \|\langle 1, \tau^{-1} - 1, \tau - 1 \rangle\| = 2\tau^{-1} \\
P_0 - \langle -1, -1, 1 \rangle &= \|\langle 1, \tau^{-1} + 1, \tau + 1 \rangle\| = 2\tau \\
P_0 - \langle -1, -1, -1 \rangle &= \|\langle 1, \tau^{-1} - 1, \tau + 1 \rangle\| = 2\sqrt{2} \\
P_0 - \langle 1, -1, -1 \rangle &= \|\langle 1, -1, \tau + 1 \rangle\| = 2\sqrt{2} \\
P_0 - \langle -1, -1, -1 \rangle &= \|\langle 1, \tau^{-1} + 1, \tau + 1 \rangle\| = 2\tau \\
P_0 - \langle 1, 1, 1 \rangle &= \|\langle 1, \tau^{-1} - 1, \tau + 1 \rangle\| = 2\sqrt{2} \\
P_0 - \langle 1, -1, 0 \rangle &= \|\langle 1, \tau^{-1} + 1, \tau - 1 \rangle\| = 2
\end{align*}
\]

Thus, there are 5 distinct chord lengths: \( 2\tau, 2\tau^{-1}, 2\sqrt{3}, 2, \) and \( 2\sqrt{2} \). Dividing these by \( 2\tau^{-1} \) yields \( \tau^2, 1, \tau\sqrt{3}, \tau, \) and \( \tau\sqrt{2} \), and multiplying them by \( 3^{-1/2} \tau^{-1/2} \) yields \( 3^{-1/2}\tau^2, 3^{-1/2}\tau^{-1/2} \), \( 2, 3^{-1/2}\tau^2 \), and \( 3^{-1/2}\tau\sqrt{2} \).

From our computations, we see that, emanating from the vertex \( \langle 0, \tau^{-1}, \tau \rangle \), there are three chords of length \( 2\tau \), three chords of length
2\tau^{-1}, one chord of length \(2\sqrt{3}\), six chords of length 2, and six chords of length \(2\sqrt{2}\). Thus, for the regular dodecahedron of edge length \(3^{-1/2}\tau^{-1}2\), there are three chords of length \(3^{-1/2}\tau 2\), three chords of length \(3^{-1/2}\tau^{-1}2\), one chord of length 2, six chords of length \(3^{-1/2}2\), and six chords of length \(3^{-1/2}2\sqrt{2}\). Since a regular dodecahedron has 20 vertices, Lemma 3.2, viz., \(N_i = \frac{m_iV}{2}\), shows that there are a total of 30 chords of length \(3^{-1/2}\tau 2\), 30 chords of length \(3^{-1/2}\tau^{-1}2\), 10 chord of length 2, 60 chords of length \(3^{-1/2}2\), and 60 chords of length \(3^{-1/2}2\sqrt{2}\). Therefore, we have

\[
\prod_{i=1}^{N} c_i^2 = \left( \left( \frac{\tau}{\sqrt{3}} \right)^2 \right)^{30} \left( \left( \frac{2}{\sqrt{3}} \right)^2 \right)^{30} (2^2)^{10} \left( \left( \frac{2}{\sqrt{3}} \right)^2 \right)^{60} \left( \left( \frac{2}{\sqrt{3}} \right)^2 \right)^{60} = 2^{\frac{2440}{3^{180}}}.
\]

Observe that 2 is the numerator’s base and 3 is the denominator’s base (\(\nu\) and \(\varepsilon\) for the dodecahedron, respectively). Next, we consider the exponents. Recall that \(E = 30\) and \(V = 20\) for the dodecahedron. For the denominator’s exponent, observe that \(E\) divides 180 as desired. For the numerator’s exponent, observe that (a) 440 = 20(22) = \(V(22)\) and therefore 440 \(\equiv V \pmod{V}\) and that (b) 440 = 30(14) + 20 = \(E(14) + V\) and therefore 440 \(\equiv V \pmod{E}\) as desired. Finally, \(\frac{30}{(30,20)} = 3\), which is the denominator’s base. □

5.5. Discussion

Theorem 5.1 shows that Fact 3 does, in fact, apply to at least one family of regular \(n\)-polytopes (where \(n \geq 2\))—i.e., the family of regular \(n\)-crosspolytopes—if the “number of edges” \((E)\) in the base is replaced by “number of facets” \((F)\) and the “number of edges” \((E)\) in the exponent is replaced by “number of vertices” \((V)\). These replacements are valid since, for 2-polytopes, the facets are, in fact, the edges and \(E = V\).

Theorem 5.3 shows that Fact 3 does not generalize to the 24-cell but does show that, for the 24-cell (inscribed in a unit 4-sphere), the product of the squared chord lengths is a “very nice” integer \((6^E)\), which is equal to \(6^{22}\).

Finally, our proof of Theorem 5.4 shows that Fact 3 does not generalize to the regular 3-simplex, 3-cube, icosahedron, or dodecahedron and therefore cannot generalize to all regular \(n\)-simplices or to all \(n\)-cubes (where \(n \geq 2\)) and probably does not generalize to the remaining regular \(n\)-simplices or \(n\)-cubes (i.e., those of dimension \(n > 3\)) or to the 600-cell (i.e., the “hypericosahedron”) or the 120-cell (i.e., the “hyperdodecahedron”).

6. FACT 4: Product of Squared Distinct Chords

The fourth 2-polytope fact stated that, for any regular 2-polytope inscribed in a unit 2-sphere, the product of its squared distinct chord lengths equals (a) \(E\) (when \(E\) is odd) and (b) some integer (when \(E\) is even). We test generalization of this fact to (a) the regular \(n\)-crosspolytopes \((n \geq 2)\), (b) the 24-cell, and (c) the other regular polytopes. We need no additional information.
6.1. Generalizing Fact 4 to Crosspolytopes

Theorem 6.1. Let $P$ be a regular $n$-crosspolytope inscribed in a unit $n$-sphere ($n \geq 2$) with $k$ distinct chords. Let $d_i$ be the $i^{th}$ distinct chord length. Then $\prod_{i=1}^{k} d_i^2 = 8$.

Proof. Recall from the proof of Theorem 5.1 that the chords of a regular $n$-crosspolytope inscribed in a unit $n$-sphere ($n \geq 2$) consist of edges of length $2^{1/2}$ and inner diagonals of length 2. Hence, $\prod_{i=1}^{N} d_i^2 = (2^{1/2})^2 (2)^2 = 8$. □

Corollary 6.2. Let $Q$ be a regular $n$-cube circumscribed about a unit $n$-sphere (where $n \geq 2$). Let $t_i$, for $i = 1, 2, ..., k$, be the distinct lengths of the line segments whose endpoints are centers of facets. Then $\prod_{i=1}^{k} t_i^2 = 8$.

6.2. Generalizing Fact 4 to the 24-cell

Theorem 6.3. Let $P$ be a 24-cell that is inscribed in a unit 4-sphere and has $V$ vertices, $F$ facets, and $k$ distinct chord lengths. Let $d_i$ denote the $i^{th}$ distinct chord length of $P$. Then

$$\prod_{i=1}^{k} d_i^2 = F = V.$$

Proof. Recall from the proof of Theorem 5.3 that a regular 24-cell inscribed in a unit 4-sphere has four distinct chord lengths: 1, $2^{1/2}$, $3^{1/2}$, and 2. Squaring these yields 1, 2, 3, and 4. Thus, $\prod_{i=1}^{k} d_i^2 = 4! = 24$. Table reveals that the 24-cell has 24 vertices and 24 facets. Therefore: $\prod_{i=1}^{k} d_i^2 = F = V$. □

Corollary 6.4. Let $Q$ be a 24-cell circumscribed about a unit 4-sphere with $V$ vertices and $F$ facets. Let $t_i$, for $i = 1, 2, ..., k$, be the distinct lengths of the line segments whose endpoints are centers of facets. Then $\prod_{i=1}^{k} t_i^2 = V = F$.

Proof. Note that a regular 24-cell is self-reciprocal. □

6.3. Generalizing Fact 4 to Other Regular Polytopes

Theorem 6.5. Let $P$ be a regular 3-polytope with $E$ edges and $V$ vertices inscribed in a unit 3-sphere, and let $a$ and $b$ be the exponents in Theorem 5.4. Then

$$\prod_{i=1}^{k} d_i^2 = \frac{\nu^c}{\varepsilon^d}$$

where $d_i$ is the length of each $i^{th}$ distinct chord of $P$, $\nu$ is the number of vertices incident to any edge, $\varepsilon$ is the number of edges incident to any vertex, and $c, d \in \mathbb{Z}$ such that

1. for the self-dual tetrahedron, $b = dE$,
2. for the cube and icosahedron, $b = dE$, and
3. for the octahedron and dodecahedron, $a = cV m$ (where $m = 1$ or 2, respectively).
In addition, the number $\varepsilon$ in the equation above may be replaced by either

$$\frac{E}{(E,V)} = \frac{|E,V|}{V}.$$ 

Proof. We consider each of these five cases in turn: (a) the tetrahedron, (b) the cube, (c) the icosahedron, (d) the octahedron, and (e) the dodecahedron.

**Tetrahedron.** Recall that a regular tetrahedron inscribed in a unit 3-sphere has one type of chord: edges of length $\frac{2\sqrt{6}}{3}$. Therefore, we have

$$\prod_{i=1}^{k} d_i^2 = \left(\frac{2\sqrt{6}}{3}\right)^2 = \frac{2^3}{3^2}. \tag{6.1}$$

Observe that 2 is the numerator’s base and 3 is the denominator’s base ($\nu$ and $\varepsilon$, respectively), as desired. For the denominator’s exponent, recall from the proof of Theorem 5.4 that $E = 6$ and $b = 6$ for the tetrahedron. Observe that $d = 1 = \frac{6}{6} = \frac{b}{E}$, so $b = dE$ (as desired). Finally, recall from the proof of Theorem 5.4 that, for the tetrahedron, $\frac{E}{(E,V)} = \frac{|E,V|}{V} = 3$, which is the denominator’s base.

**Cube.** Recall from the proof of Theorem 5.4 that a cube inscribed in a unit 3-sphere has three types of chords: edges of length $3^{-1/2}$, outer diagonals of length $\frac{2\sqrt{6}}{3}$, and inner diagonals of length 2. Therefore, we have

$$\prod_{i=1}^{k} d_i^2 = \left(3^{-1/2}\right)^2 \left(\frac{2\sqrt{6}}{3}\right)^2 (2)^2 = \frac{2^7}{3^2}. \tag{6.2}$$

Observe that the numerator’s base is 2 and the denominator’s base is 3 ($\nu$ and $\varepsilon$, respectively). For the denominator’s exponent, recall from the proof of Theorem 5.4 that $E = 12$ and $b = 24$ for the cube. Observe that $d = 2 = \frac{24}{12} = \frac{b}{E}$, so $b = dE$. Finally, recall from the proof of Theorem 5.4 that, for the cube, $\frac{E}{(E,V)} = \frac{|E,V|}{V} = 3$, which is the denominator’s base.

**Icosahedron.** Recall that a regular icosahedron inscribed in a unit 3-sphere has three distinct chord lengths: $\frac{2\sqrt{5}}{3}$, $\frac{2\sqrt{2} \sqrt{\tau}}{3}$, and 2. Therefore:

$$\prod_{i=1}^{k} d_i^2 = \left(\frac{2\sqrt{5}}{3}\right)^2 \left(\frac{2\sqrt{2} \sqrt{\tau}}{3}\right)^2 (2)^2 = \frac{2^6}{5}. \tag{6.3}$$

Observe that the numerator’s base is 2 and the denominator’s base is 5 ($\nu$ and $\varepsilon$, respectively). For the denominator’s exponent, recall from the proof of Theorem 5.4 that $E = 30$ and $b = 30$ for the icosahedron. Observe that $d = 1 = \frac{30}{30} = \frac{b}{E}$, so $b = dE$ (as desired). Finally, recall from the proof of Theorem 5.4 that, for the icosahedron, $\frac{E}{(E,V)} = \frac{|E,V|}{V} = 5$, which is the denominator’s base.

**Octahedron.** Recall the proof of Theorem 5.4 that a regular octahedron inscribed in a unit 3-sphere has two types of chords: edges of length $\sqrt{2}$ and
inner diagonals of length 2. Therefore, we have

$$\prod_{i=1}^{k} d_i^2 = \left(\sqrt{2}\right)^2 (2)^2 = \frac{2^{3+4q}}{24q} = \frac{2^{3+4q}}{4^{2q}}. \quad (6.4)$$

where $q$ is the same as in the proof of Theorem 5.4. Observe that the numerator’s base is 2 and the denominator’s base is 4 ($\nu$ and $\varepsilon$, respectively). For the numerator’s exponent, recall from the proof of Theorem 5.4 that $V = 6$ and $a = 18 + 24q$ for the octahedron. Observe that $c = 3 + 4q = \frac{18+24q}{6} = \frac{a}{V}$, so $a = cV \cdot 1 = cVm$. Finally, recall from the proof of Theorem 5.4 that, for the octahedron, $\frac{E}{(E,V)} = \frac{[E,V]}{V} = 2$, which is the base of the denominator in $\frac{2^{3+4q}}{2^{4q}}$.

**Dodecahedron.** Recall that a regular dodecahedron inscribed in a unit 3-sphere has five distinct chord lengths: $2\sqrt[3]{3}$, $\frac{2\sqrt[3]{3}}{\tau}$, $2$, $2\sqrt[3]{3}$, and $2\sqrt{2}\sqrt[3]{3}$. Therefore:

$$\prod_{i=1}^{k} d_i^2 = \left(\frac{2\tau}{\sqrt[3]{3}}\right)^2 \left(\frac{2}{\sqrt[3]{3}\tau}\right)^2 \left(\frac{2}{\sqrt[3]{3}}\right)^2 \left(\frac{2\sqrt{2}}{\sqrt[3]{3}}\right)^2 = \frac{2^{11}}{3^4}. \quad (6.5)$$

Observe that the numerator’s base is 2 and the denominator’s base is 3 ($\nu$ and $\varepsilon$, respectively). For the numerator’s exponent, recall from the proof of Theorem 5.4 that $V = 20$ and $a = 440$ for the dodecahedron. Observe that $c = 11 = \frac{440}{40} = \frac{a}{2}$, so $a = cV \cdot 2 = cVm$. Finally, recall from the proof of Theorem 5.4 that, for the dodecahedron, $\frac{E}{(E,V)} = \frac{[E,V]}{V} = 3$, which is the base of the denominator’s base. □

6.4. Discussion

Theorem 6.1 shows that Fact 4 does, in fact, apply to the regular $n$-crosspolytopes (where $n \geq 2$) since they have even $E$ and, thus, fall under the “even $E$” case of Fact 4.

Theorem 6.3 shows that Fact 4 also applies to the 24-cell (which has even $E$) and shows, for the 24-cell, that the “some integer” is the “number of facets” ($F$) or, equivalently, the “number of vertices” ($V$).

Finally, our proof of Theorem 6.5 shows that Fact 4 does not generalize to the regular 3-simplex, 3-cube, icosahedron, or dodecahedron and therefore cannot generalize to all regular $n$-simplices or all $n$-cubes ($n \geq 2$) and probably does not generalize to the remaining regular $n$-simplices or $n$-cubes (i.e., those of dimension $n > 3$) or to the 600-cell (i.e., the “hypericosahedron”) or the 120-cell (i.e., the “hyperdodecahedron”).

7. Conclusion

We have succeeded in finding a single rule that applies to the chord lengths of all regular $n$-polytopes, $n \geq 2$ (namely, the rule given by Theorem 3.4). Facts 2-4 generalized to several types of regular polytopes. Given that the literature contained little information regarding the chord lengths of regular
n-polytopes, $n \geq 3$, it is clear that this investigation has taken the study of chord regular polytopes’ chord lengths to a new level (and higher dimensions).

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