Comment on “Generic rotating regular black holes in general relativity coupled to non-linear electrodynamics”

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We show that there is an inconsistency in the class of solutions obtained in Phys. Rev. D 95, 084037 (2017). This inconsistency is due to the approximaterelation between lagrangian density and its derivative for Non-Linear Electrodynamics. We present an algorithm to obtain new classes of solutions.

PACS numbers: 04.50.Kd, 04.70.Bw

I. DEMONSTRATION

The action of General Relativity coupled to Non-linear Electrodynamics (NED) is given by

\[ S = \int dx^4 \sqrt{-g} \left[ R + 8\pi G \mathcal{L}_{\text{NED}}(F) \right], \]

where \( R \) is the scalar curvature and \( F = F_{\mu\nu}F^{\mu\nu} \), with \( F_{\mu\nu} = \partial_{\mu}A_{\nu} - \partial_{\nu}A_{\mu} \) and \( A_{\mu} \) the four-potential. The functional variation of the action in relation to metric and the four-potential \( A_{\mu} \) results in

\[ R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = 8\pi GT_{\mu\nu}, \]

\[ \nabla_{\mu} [F^{\mu\nu}\mathcal{L}_F] = 0, \]

where

\[ T_{\mu\nu} = g_{\mu\nu}\mathcal{L}_{\text{NED}} - \mathcal{L}_F F_{\mu}^{\alpha}F_{\nu\alpha}, \]

\[ \mathcal{L}_F = \frac{\partial \mathcal{L}_{\text{NED}}}{\partial F}. \]

Constructing a regular metric with axial symmetry through NEWMAN-JANIS algorithm we have \[ 1 \]

\[ ds^2 = \left( 1 - \frac{2\rho r}{\Sigma} \right) dt^2 - \frac{\Sigma}{\Delta} dr^2 + 2\alpha \sin^2 \theta \frac{2\rho r}{\Sigma} dt d\phi - \Sigma d\theta - \sin^2 \theta \frac{(r^2 + a^2)^2 - a^2 \Delta \sin^2 \theta}{\Sigma} d\phi^2. \]

with \( \Sigma = r^2 + a^2 \cos^2 \theta \), \( \Delta = r^2 - 2r\rho + a^2 \) and \( \rho(r) \) the mass function, where \( \lim_{r \to +\infty} \rho(r) = M_{\text{ADM}} \) with \( M_{\text{ADM}} \) the ADM mass.

For a space-time with axial symmetry, the Maxwell tensor obeys the following equation \[ 2 \]
\[ [L_{K^\alpha} F_{\mu\nu}] \equiv 0, \]

where \( L \) represents the Lie’s derivative and \( K^\alpha \) the Killing vectors of axial symmetry. Through this symmetry we have the following non-zero components for the Maxwell tensor \[ 2 \]
\( F_{01}(t,r,\theta), F_{02}(t,r,\theta), F_{13}(t,r,\theta) \) and \( F_{23}(t,r,\theta) \). As the solution must be stationary and coming from a NEWMAN-JANIS algorithm, the following constraint can be made for the four-potential components \[ 2 \]

\[ A_\mu = Q a \cos \theta \frac{\theta_\mu}{\Sigma} \delta_\mu^0 - Q \frac{(r^2 + a^2) \cos \theta}{\Sigma} \delta_\mu^3. \]

\[ 7 \]

\[ 8 \]

\[ 9 \]
Now, according to [1], calculating the Maxwell tensor, the scalar $F$ is given by
\[ F = \frac{Q^2}{4\Sigma^4}[a^4(3 - \cos 4\theta) + 4(6a^2r^2 + 2r^4 + a^2(a^2 - 6r^2)\cos 2\theta)]. \tag{8} \]

Again according to [1], using (6) and (7) in (2), solving the equations in terms to relationship [4] and [5].

from NEWMAN-JANIS algorithm, whose Lagrangean density and its derivative are (9) and (10), with mass function
we can explicitly verify that the consistency equation (12) is not satisfied as in (11). Therefore any solution coming
published solutions in Non-Linear Electrodynamics.

of the electromagnetic field. Moreover, there is well-accepted fact that similar notion can be related to many other
Ref. [1] can be considered as an approximate solution with the comparatively small incorrectness in the behaviour
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possible solutions, since it gives the correct results in non rotating limit as in Ref. [5]. Furthermore, the numerical

The central question here is that solving like equations of motion (2), the Lagrangian density and its derivative will
not necessarily be related as in (5). This should be imposed so that the solution is consistent with NED. Let’s now
check this property. We can rewrite (5) as
\[ \mathcal{L}_F = \left[ \frac{\partial \mathcal{L}_{NED}}{\partial r} \left( \frac{\partial F}{\partial r} \right)^{-1} + \frac{\partial \mathcal{L}_{NED}}{\partial \theta} \left( \frac{\partial F}{\partial \theta} \right)^{-1} \right] = 0. \tag{11} \]

Taking (8), (9) and (10), the consistency equation (11) becomes
\[ \mathcal{L}_F - \left[ \frac{\partial \mathcal{L}_{NED}}{\partial r} \left( \frac{\partial F}{\partial r} \right)^{-1} + \frac{\partial \mathcal{L}_{NED}}{\partial \theta} \left( \frac{\partial F}{\partial \theta} \right)^{-1} \right] = \frac{1}{4Q^2} \left\{ 2 \left[ 2(r^2 - a^2 \cos^2 \theta)\rho' - r(r^2 + a^2 \cos^2 \theta)\rho'' \right] + \right. \\
+ 16r^4 \left[ (6a^2 - 4r^2 + 6a^2 \cos 2\theta)\rho' + r(a^2 + 2r^2 + a^2 \cos 2\theta)\rho'' \right] + \\
- \left[ -2(a^2 - 2r^2 + a^2 \cos 2\theta)(5a^2 + 2r^2 + 5a^2 \cos 2\theta)\rho' - \frac{1}{2}r(a^2 + 2r^2 + a^2 \cos 2\theta)((33a^4 + 84r^2 + 44a^4 \cos 2\theta + \\
+ 11a^4 \cos 4\theta)\rho'' + 8a^2r \cos 2\theta(a^2 + 2r^2 + a^2 \cos 2\theta)\rho''') \right] \left[a^2(4r^2 + 2(a^2 - 4r^2) \cos 2\theta + a^2 \cos 4\theta) \right]^{-1} \right\}. \tag{12} \]

Now using the model of mass function (30) on [1]
\[ \rho(r) = M + \frac{\alpha^{-1}q^2r^\mu}{(\nu^\mu + \nu^\rho)r^\mu}, \tag{13} \]

we can explicitly verify that the consistency equation (12) is not satisfied as in (11). Therefore any solution coming
from NEWMAN-JANIS algorithm, whose Lagrangean density and its derivative are (9) and (10), with mass function
give by (13) is inconsistent. Some spherically symmetric models of regular black holes have consistently used this
relationship [4] and [5].

Despite, the discussed inconsistency in Eq. (11), one could see that the derived solution in Ref. [1] is one of the most
possible solutions, since it gives the correct results in non rotating limit as in Ref. [2]. Furthermore, the numerical
calculations show that the difference between two parts of Eq. (11) is comparably small. The solution obtained in
Ref. [1] can be considered as an approximate solution with the comparatively small incorrectness in the behaviour
of the electromagnetic field. Moreover, there is well-accepted fact that similar notion can be related to many other
published solutions in Non-Linear Electrodymanics.

We conclude this section by emphasizing that the class of rotating regular black holes solutions obtained in [1] does
not fulfil the whole set of field equations in (2) and (3), which will be better explained in the next section.

II. POSSIBILITY TO NEW SOLUTIONS

The NEWMAN-JANIS algorithm can present some pathology [7] in some cases, but its generalization does not appear
to present these pathologies [8]. The authors to [1] has used the generalization of this algorithm. The
algorithm itself seems to show no inconsistency, but the way in which it was used by the authors of [1], it was not
correct. Let’s comment this better now.

We will start this section by explaining why the authors of [1] did not obtain an exact solution of the theory coming
from the action [1] with axial symmetry. First, they did not begin by integrating the modified Maxwell equation to
NED, thus suggesting a solution to the potential in equation (25) de [1]. Second, they attempted to solve Einstein’s
equations $G_{\mu\nu} = T_{\mu\nu}$, which are five equations with only three arbitrary functions $\{\rho(r), \mathcal{L}, \mathcal{L}_F\}$. This eventually left some of the equations of motion (Einstein and Maxwell modified) not satisfied.

Now let’s point out a way to obtain exact solutions for this system. First we must integrate the modified Maxwell equations

\[
\nabla_{\mu} \left[ \mathcal{L}_F F^{\mu\nu} \right] = 0, \nabla_{\mu} \ast F^{\mu\nu} = 0, \nabla_{[\mu} F_{\alpha\beta]} = 0, \ast F^{\mu\nu} = \frac{1}{2} \eta^{\mu\nu\alpha\beta} F_{\alpha\beta}, \eta^{0123} = \frac{-1}{\sqrt{-g}}.
\]

The non-zero Maxwell tensor components are $\{F_{10}, F_{20}, F_{13}, F_{23}\}$, but two are dependent on the others, remaining only $\{F_{10}, F_{20}\}$. These Maxwell tensor components may perhaps be integrated to obtain a solution that depends on the Lagrangian and its derivative $\mathcal{L}_F$. We could then insert them into the Einstein equations projected with the tetrads $G_{(a)}^{(b)} = T_{(a)}^{(b)}$, where we have only two equations for the case $T_{(2)}^{(2)} = T_{(3)}^{(3)}$. We can now solve the two equations to obtain $\mathcal{L}_F$, remember that there is a constraint equation $\mathcal{L}_F = \partial \mathcal{L} / \partial F$, which if soluble, then we have a new class of exact solutions.

The algorithm specified above to obtain new classes of solutions is not simple and it may be that the equation $\mathcal{L}_F = \partial \mathcal{L} / \partial F$ is highly non-linear in the mass function and does not present solution by known methods. So this should be a future work separate from this comment.

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