A note on the Joint Spectral Radius

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Last two decades have been characterized by an increasing interest in the analysis of the maximal growth rate of long products generated by matrices belonging to a specific set/family. The maximal growth rate can be evaluated considering a generalization of the spectral radius of a single matrix to the case of a set of matrices.

This generalization can be formulated in many different ways, nevertheless in the commonly studied cases of bounded or finite families all the possible generalizations coincide in a unique value that is usually called joint spectral radius or simply spectral radius. The joint spectral radius, however, can prove to be hard to compute and can lead even to undecidable problems. We present in this paper all the possible generalizations of the spectral radius, their properties and the associated theoretical challenges.

From an historical point of view the first two generalizations of spectral radius, the so–called joint and common spectral radius, were introduced by Rota and Strang in the three pages paper “A note on the joint spectral radius” published in 1960 [62]. After that more than thirty years had to pass before a second paper was issued on this topic: in 1992 Daubechies and Lagarias [21] published “Sets of matrices all infinite products of which converge” introducing the generalized spectral radius, conjecturing it was equal to the joint spectral radius (this was proven immediately after by
Berger and Wang [2]) and presenting examples of applications. From then on there has been a rapidly increasing interest on this subject and the more years pass the more the number of mathematical branches and applications directly involved in the study of these quantities increases [3].

The study of infinite products convergence properties proves to be of primary interest in a variety of contexts:

Nonhomogeneous Markov chains, deterministic construction of functions and curves with self-similarities under changes in scale like the von Koch snowflake and the de Rham curves, two-scale refinement equations that arise in the construction of wavelets of compact support and in the dyadic interpolation schemes of Deslauriers and Dubuc [21, 61], the asymptotic behavior of the solutions of linear difference equations with variable coefficients [28, 29, 30], coordination of autonomous agents [41, 55, 24], hybrid systems with applications that range from intelligent traffic systems to industrial process control [11], the stability analysis of dynamical systems of autonomous differential equations [15], computer-aided geometric design in constructing parametrized curves and surfaces by subdivision or refinement algorithms [53, 16], the stability of asynchronous processes in control theory [67], the analysis of magnetic recording systems and in particular the study of the capacity of codes submitted to forbidden differences constraints [54, 6], probabilistic automata [57], the distribution of random power series and the asymptotic behavior of the Euler partition function [61], the logarithm of the joint spectral radius appears also in the context of discrete linear inclusions as the Lyapunov indicator [11, 35]. For a more extensive and detailed list of applications we refer the reader to the Gilbert Strang’s paper “The Joint Spectral Radius” [65] and to the doctoral theses by Jungers and Theys [42, 66].

The paper develops as following: in Section 1 we give notation and terminology used throughout this paper; Section 2 presents first a case of study associated with the asymptotic behavior analysis of the solutions of linear
difference equations with variable coefficients, further, it contains the definitions and properties of all the possible generalizations of spectral radius for a set of matrices, in particular the irreducibility, nondefectivity and finiteness properties are discussed.

1 Terminology, notation and basic properties

In this Section we provide notation, terminology, definitions and properties which are employed in this paper. We use the expression $\mathbb{N}_0$ meaning the set of natural numbers, included zero. All the matrices and vectors that we consider have real or complex entries. We denote the conjugate transpose of an $m$–by–$n$ matrix $X = [x_{ij}]$ by $X^* = [\bar{x}_{ji}]$, while the simple transpose as $X^T = [x_{ji}]$.

For $p \in [1, \infty)$, the $l^p$ norm of a vector $w \in \mathbb{C}^n$ is given by $\|w\|_p = \sqrt[p]{\sum_{i=1}^{n} |w[i]|^p}$

In particular:

\begin{itemize}
  \item $l^1$ – The sum norm $\|w\|_1 = \sum_{i} |w[i]|$
  \item $l^2$ – The Euclidean norm $\|w\|_2 = \sqrt{\sum_{i=1}^{n} |w[i]|^2} = \sqrt{w^*w}$
  \item $l^\infty$ – The max norm $\|w\|_\infty = \max_{j=1,...,n} |w[j]|$
\end{itemize}

If $A$ is a square matrix, its characteristic polynomial is $p_A(t) := \det(tI - A)$, where $\det$ stands for determinant [40] Section 0.3; the (complex) zeroes of $p_A(t)$ are the eigenvalues of $A$. A complex number $\lambda$ is an eigenvalue of $A$ if and only if there are nonzero vectors $x$ and $y$ such that $Ax = \lambda x$ and $y^*A = \lambda y^*$; $x$ is said to be an eigenvector (more specifically, a right eigenvector) of $A$ associated with $\lambda$ and $y$ is said to be a left eigenvector of $A$ associated with $\lambda$. The set of all the eigenvalues of $A$ is called the spectrum of $A$ and is denoted by $\sigma(A)$. The determinant of $A$, $\det A$, is equivalent to the product of all its eigenvalues. If the spectrum of $A$ does not contain 0 the matrix is said nonsingular (A nonsingular if and only if $\det A \neq 0$). The spectral
radius of $A$ is the nonnegative real number $\rho(A) = \max \{|\lambda| : \lambda \in \sigma(A)\}$. If $\lambda \in \sigma(A)$, its algebraic multiplicity is its multiplicity as a zero of $p_A(t)$; its geometric multiplicity is the maximum number of linearly independent eigenvectors associated with it. The geometric multiplicity of an eigenvalue is never greater than its algebraic multiplicity. An eigenvalue whose algebraic multiplicity is one is said to be simple. An eigenvalue $\lambda$ of $A$ is said to be semisimple if and only if $\text{rank}(A - \lambda I) = \text{rank}(A - \lambda I)^2$ i.e. $\lambda$ has the same geometric and algebraic multiplicity. If the geometric multiplicity and the algebraic multiplicity are equal for every eigenvalue, $A$ is said to be nondefective, otherwise is defective.

We let $e_1$ indicate the first column of the identity matrix $I$: $e_1 = [1 \ 0 \ \cdots \ 0]^T$. We let $e = [1 \ 1 \ \cdots \ 1]^T$ denote the all–ones vector. Whenever it is useful to indicate that an identity or zero matrix has a specific size, e.g., $r$–by–$r$, we write $I_r$ or $0_r$. Two vectors $x$ and $y$ of the same size are orthogonal if $x^*y = 0$. The orthogonal complement of a given set of vectors is the set (actually, a vector space) of all vectors that are orthogonal to every vector in the given set.

An $n$–by–$r$ matrix $X$ has orthonormal columns if $X^*X = I_r$. A square matrix $U$ is unitary if it has orthonormal columns, that is, if $U^*$ is the inverse of $U$.

A square matrix $A$ is a projection if $A^2 = A$.

A square matrix $A$ is row–stochastic if it has real nonnegative entries and $Ae = e$, which means that the sum of the entries in each row is 1; $A$ is column–stochastic if $A^T$ is row–stochastic. We say that $A$ is stochastic if it is either row–stochastic or column–stochastic.

The direct sum of $k$ given square matrices $X_1, \ldots, X_k$ is the block diagonal matrix

$$
\begin{bmatrix}
X_1 & \cdots & 0 \\
\vdots & \ddots & \vdots \\
0 & \cdots & X_k
\end{bmatrix} = X_1 \oplus \cdots \oplus X_k.
$$
The $k$–by–$k$ Jordan block with eigenvalue $\lambda$ is

$$J_k(\lambda) = \begin{bmatrix}
\lambda & 1 & 0 \\
\vdots & \ddots & \ddots \\
\vdots & & \ddots & 1 \\
& & & \lambda
\end{bmatrix}, \quad J_1(\lambda) = [\lambda].$$

Each square complex matrix $A$ is similar to a direct sum of Jordan blocks, which is unique up to permutation of the blocks; this direct sum is the Jordan canonical form of $A$. The algebraic multiplicity of $\lambda$ as an eigenvalue of $J_k(\lambda)$ is $k$; its geometric multiplicity is 1. If $\lambda$ is a semisimple eigenvalue of $A$ with multiplicity $m$, then the Jordan canonical form of $A$ is $\lambda I_m \oplus J$, in which $J$ is a direct sum of Jordan blocks with eigenvalues different from $\lambda$; if $\lambda$ is a simple eigenvalue, then $m = 1$ and the Jordan canonical form of $A$ is $[\lambda] \oplus J$. $A$ is diagonalizable, i.e. its Jordan canonical form is given by a diagonal matrix, if and only if is nondefective.

In a block matrix, the symbol $\star$ denotes a block whose entries are not required to take particular values.

We consider $A^0 = I$. A matrix $B$ is said to be normal if $BB^* = B^*B$, unitary if $BB^* = B^*B = I$, Hermitian if $B = B^*$. Hermitian and unitary matrices are, by definition, normal matrices.

A proper subset of a set $\mathcal{A}$ is a set $\mathcal{B}$ that is strictly contained in $\mathcal{A}$. This is written as $\mathcal{B} \subsetneq \mathcal{A}$.

Besides the Jordan canonical form, we need to introduce an additional matrix factorization, the so–called singular value decomposition (in short svd): Given a square matrix $A \in \mathbb{C}^{n \times n}$ with rank $k \leq n$, there always exists a diagonal matrix $\Lambda \in \mathbb{R}^{n \times n}$ with nonnegative diagonal entries $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_k > \sigma_{k+1} = \cdots = \sigma_n = 0$ and two unitary matrices $U, V \in \mathbb{C}^{n \times n}$ such that $A = UV^*$, which is defined as the singular value decomposition of $A$. The matrix $\Lambda = \text{diag}(\sigma_1, \ldots, \sigma_n)$ is always uniquely determined and $\sigma_1^2 \geq \cdots \geq \sigma_n^2$ correspond to the eigenvalues of the Hermitian matrix $AA^*$. Values $\sigma_1, \ldots, \sigma_n$ are the so–called singular values of $A$. 

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The trace of an \( n \times n \)–matrix \( A \), denoted by \( \text{tr}(A) \), is given by the sum of the diagonal elements of \( A \), \( \text{tr}(A) = \sum_{i=0}^{n} a_{ii} \), and it is also equal to the sum of all the eigenvalues in the spectrum of \( A \), \( \text{tr}(A) = \sum_{\lambda \in \sigma(A)} \lambda \).

The spectral radius of a square matrix \( A \in \mathbb{C}^{n\times n} \) is defined as

\[
\rho(A) = \max \left\{ |\lambda| : \lambda \in \sigma(A) \right\}
\]

It is easy to prove that \( \rho(A^k) = (\rho(A))^k \) for every \( k \in \mathbb{N} \) and, thus, given a generic power \( k \) of the matrix \( A \), the value \( (\rho(A^k))^{1/k} \) is just equal to the spectral radius of the matrix.

It is possible to characterize the spectral radius using the trace of the matrix. Since \( \lambda^k \in \sigma(A^k) \) for every eigenvalue \( \lambda \in \sigma(A) \) and for every \( k \in \mathbb{N} \), it follows that

\[
|\text{tr}(A^k)|^{1/k} = \rho(A) \left| \sum_{\lambda \in \sigma(A)} \lambda^k / (\rho(A))^k \right|^{1/k}
\]

which converges to \( \rho(A) \) as \( k \to \infty \)

\[
\rho(A) = \lim_{k \to \infty} \left| \text{tr}(A^k) \right|^{1/k}.
\]

For a square matrix \( A \) and for \( p \in [1, \infty] \), \( \|A\|_p \) is the matrix norm induced by the corresponding \( p \)–vector norm. The induced matrix norms are sometimes defined as operator norms [40, Definition 5.6.3]. Among the induced matrix norms we will make use of the following

The maximum column–sum norm \( \|A\|_1 = \max_{\|x\|_1=1} \|Ax\|_1 = \max_j \sum_i |a_{i,j}| \)

The spectral norm \( \|A\|_2 = \max_{\|x\|_2=1} \|Ax\|_2 = \sigma_1(A) = \sqrt{\rho(AA^*)} \)

The maximum row–sum norm \( \|A\|_\infty = \max_{\|x\|_\infty=1} \|Ax\|_\infty = \max_j \sum_i |a_{i,j}| \)

Every induced matrix norm \( \| \cdot \|_* \) is submultiplicative i.e. \( \|AB\|_* \leq \|A\|_* \|B\|_* \) for every square matrix \( A \) and \( B \).
Another family of induced matrix norms are the \textit{ellipsoidal norms}. Let us consider an Hermitian positive definite matrix $P \succ 0$ (i.e. $P$ is a nonsingular Hermitian matrix such that $x^*Px > 0$ for all nonzero $x \in \mathbb{C}^n$ or, equivalently, $P$ is a Hermitian matrix such that all its eigenvalues are strictly positive). The \textit{vector ellipsoidal norm} is defined as

$$
\|x\|_P = \sqrt{x^*Px}.
$$

The corresponding induced matrix norm is given by

$$
\|A\|_P = \max_{\|x\|_P = 1} \|Ax\|_P = \max_{\sqrt{x^*Px} = 1} \sqrt{x^*A^*PAx}
$$

(4)

Recalling that \cite{40} Corollary 7.2.9 $P$ is positive definite if and only if there exists a nonsingular upper triangular matrix $T \in \mathbb{C}^{n \times n}$, with strictly positive diagonal entries, such that $P = T^*T$, which is defined as the \textit{Cholesky decomposition} of $P$, we can rewrite $\|A\|_P = \max_{\sqrt{x^*T^*Tx} = 1} \sqrt{x^*A^*T^*TAX} = \max_{\|Tx\|_2 = 1} \|TAX\|_2$ and if we rename $y = Tx$, $T$ by construction is nonsingular so $x = T^{-1}y$, we get

$$
\|A\|_P = \max_{\|y\|_2 = 1} \|TAT^{-1}y\|_2 = \|TAT^{-1}\|_2 = \sqrt{\rho(TAT^{-1}(TAT^{-1})^*)}
$$

(6)

Since $T$ is nonsingular and remembering that the spectrum of a matrix is invariant under \textit{similarity transformation}, two matrices $M$ and $T^{-1}MT$ have the same eigenvalues, counting multiplicity. So from (6) we obtain that

$$
\|A\|_P = \sqrt{\rho(TAP^{-1}A^*T^*)} = \sqrt{\rho(AP^{-1}A^*P)}
$$

(7)

Given a generic power $k$ of the matrix $A$, the value $\|A^k\|^{1/k}$ is defined as the \textit{normalized norm} of the matrix, in the sense that is normalized with respect to the length of the product.

Given the family $\mathcal{F} = \{A_i\}_{i \in \mathcal{I}}$ of complex square $n \times n$–matrices, with $\mathcal{I}$ a set of indices, $\mathcal{F}$ is defined \textit{bounded} if it does exist a constant $C < +\infty$ such that $\sup_{i \in \mathcal{I}} \|A_i\| \leq C$. While we define the set \textit{finite} if it is constituted by a finite number of matrices. Trivially a finite set is always bounded.
A matrix $A$ is said to be nondefective if and only if its Jordan canonical form is diagonal i.e. each eigenvalue of $A$ is semisimple or, equivalently, it has geometric multiplicity equal to algebraic multiplicity, otherwise $A$ is defined defective. In this paper we deal with a weaker condition of nondefectivity: a matrix $A$ is said to be weakly nondefective if and only if the eigenvalues of $A$ with modulus equal to the spectral radius, i.e. with maximal modulus, are semisimple, if it is not the case the matrix is defined weakly defective. Using the Jordan canonical form of $A$ it is easy to prove that, whenever $\rho(A) > 0$, defined $A^* = A/\rho(A)$, $A$ is weakly nondefective if and only if powers $(A^*)^k$ are bounded for every $k \geq 1$.

From now on, for the sake of simplicity and to be coherent with the literature on the spectral radius of sets, we use the expressions strongly nondefective and strongly defective in place of nondefective and defective, whereas we make use of the words nondefective and defective meaning weakly nondefective and weakly defective.

Let us now recall basic relations between spectral radius and matrix norms:

1.1 Theorem ([40, Theorem 5.6.9]). If $\| \cdot \|$ is any matrix norm on $\mathbb{C}^{n \times n}$ and if $A \in \mathbb{C}^{n \times n}$, then $\rho(A) \leq \| A \|$.

Furthermore

1.2 Lemma ([40, Lemma 5.6.10]). Let $A \in \mathbb{C}^{n \times n}$, for every $\varepsilon > 0$ there is a matrix norm $\| \cdot \|_\varepsilon$ such that

$$\rho(A) \leq \| A \|_\varepsilon \leq \rho(A) + \varepsilon$$  \hspace{1cm} (8)

The spectral radius of $A$ is not itself a matrix or vector norm, but if we let $\varepsilon \to 0$ in (8) we have that $\rho(A)$ is the greatest lower bound for the values of all matrix norms of $A$

$$\rho(A) = \inf_{\| \cdot \| \in \mathcal{N}} \| A \|$$ \hspace{1cm} (9)
where \( N \) denotes the set of all possible induced matrix norms (the so-called operator norms).

Spectral radius allows to characterize convergent matrices, i.e. those matrices whose successive powers tends to zero:

**1.3 Theorem** ([40, Theorem 5.6.12]). Let \( A \in \mathbb{C}^{n \times n} \), then \( \lim_{k \to \infty} A^k = 0 \iff \rho(A) < 1 \)

As a Corollary of the previous Theorem we have the so-called Gelfand’s formula:

**1.4 Corollary** ([40 Corollary 5.6.14]). Let \( \| \cdot \| \) be any matrix norm on \( \mathbb{C}^{n \times n} \), then

\[
\rho(A) = \lim_{k \to \infty} \| A^k \|^{1/k} \quad \text{for all} \quad A \in \mathbb{C}^{n \times n}
\]

The Gelfand’s formula gives us two information:

- the spectral radius of \( A \) represent the asymptotic growth rate of the normalized norm of \( A^k \): \( \| A^k \|^{1/k} \sim \rho(A) \) as \( k \to \infty \)
- the normalized norm \( \| A^k \|^{1/k} \) can be used to approximate the spectral radius and in the limit for \( k \to \infty \) the two quantities coincide.

Given a row–stochastic matrix \( A \) its maximum row–sum matrix norm is equal 1 by definition of row–stochasticity. By Theorem 1.1, choosing as matrix norm the maximum row–sum, we have that for every stochastic matrix \( A \)

\[
\rho(A) \leq \| A \|_\infty = 1
\]

The row–stochasticity of \( A \) can be formulated also as

\[
Ae = e
\]

with \( e \) the all–ones vector and \( \lambda = 1 \) the eigenvalue of \( A \) associated with the right eigenvector \( x = e \). So we have that \( \rho(A) = 1 \). Remembering that the
set of stochastic matrices is closed under matrix multiplication, we observe that the very same result can be proved also using the Gelfand formula: choosing as matrix norm the maximum row–sum we have that \( \|A^k\|_\infty^{1/k} = 1 \) for every integer \( k \geq 1 \).

In the following we generalize all these notions to the case of a family of matrices.

For a systematic discussion of a broad range of matrix analysis issues, see [40].

2 Framework

2.1 A case of study

Given a stable discrete time system we want to analyze its robustness with respect to perturbations not a priori quantifiable.

Let us consider the system

\[
x(k+1) = A_0x(k), \quad k \in \mathbb{N}_0.
\]  

(13)

with \( x(0) \in \mathbb{C}^n \) and \( A_0 \in \mathbb{C}^{n \times n} \) such that the system is asymptotically stable, i.e. \( \rho(A_0) < 1 \) (ref Theorem 1.3). We consider the perturbed system given by time–varying perturbations

\[
x(k+1) = \left( A_0 + \sum_{i=1}^{p} \delta_i(k)A_i \right) x(k), \quad k \in \mathbb{N}_0.
\]  

(14)

The matrices \( \{A_i\}_{i=1}^p \) are known, but the perturbations \( \{\delta_i(k)\}_{i=1}^p \) are not. The perturbations may depend on incomplete modeling, neglect of dynamics or measurement uncertainty. We are interested to know if a stability result for the theoretical model (13) holds also for the real system (14).

The perturbed system (14) can be regarded as a first order system of difference equations with variable coefficients

\[
x(k+1) = Y_kx(k), \quad k \in \mathbb{N}_0.
\]  

(15)
where $x(0) \in \mathbb{C}^n$ and $Y_{ik} \in \mathbb{C}^{n \times n}$ is an element of the following family

$$
\mathcal{F}_\alpha = \left\{ A_0 + \sum_{i=1}^{p} \delta_i A_i \left| \| \delta \| \leq \alpha \right. \right\}
$$

(16)

where $\delta = (\delta_1 \delta_2 \cdots \delta_p)^T$ and the bound on the uncertainties is known. This kind of problems arise in several contexts such as when applying numerical methods to non-autonomous systems of differential equations.

From a point of view of robustness or worst case analysis the goal is to determine the largest uncertainty level $\alpha^*$ such that for every $\alpha < \alpha^*$ the system remains stable (see e.g. [70]).

If the sequence of matrices $Y_{ik}$ is known, for $k \geq 0$, then the solution of (15) is given by

$$
x(k+1) = P_k x(0), \quad \text{with} \quad P_k = \prod_{j=1}^{k} Y_{ij}, \quad k \geq 1
$$

(17)

where asymptotic stability may be studied directly (although this is not an easy task in general). Nevertheless we want to study the case where the sequence $\{Y_{ik}\}_{k \geq 1}$ is not known a priori and may be whatever.

2.1 Definition (Uniform asymptotic stability – u.a.s.). We say that (15) is uniformly asymptotically stable if

$$
\lim_{k \to \infty} x(k) = 0
$$

(18)

for any initial $x(0)$ and any sequence $\{Y_{ik}\}_{k \geq 1}$ of elements in $\mathcal{F}_\alpha$.

It is easy to prove that Definition 2.1 is equivalent of requiring that any possible left product $Y_{ik} \cdot Y_{ik-1} \cdot \cdots \cdot Y_{i1}$ of matrices from $\mathcal{F}_\alpha$ vanishes as $k \to \infty$.

We observe that in the context of the discrete linear inclusions some authors refer to the uniform asymptotic stability as absolute asymptotic stability [35] [66].

For the single matrix case we have that u.a.s. holds if and only if the spectral radius of the matrix is strictly less than one, while for the general
case of a family of matrices $\mathcal{F}$ we are driven to the problem of computing the joint spectral radius of $\mathcal{F}$. The intrinsic difficulty in exploiting this quantity is due to the non–commutativity of matrix multiplication.

### 2.2 Definitions and properties

#### 2.2.1 Definitions

From now on we consider always complex square $n \times n$–matrices and sub-multiplicative norms if not differently specified. Let $\mathcal{F} = \{A_i\}_{i \in \mathcal{I}}$ be a family of matrices, $\mathcal{I}$ being a set of indices. For each $k = 1, 2, \ldots$, consider the set $\mathcal{P}_k(\mathcal{F})$ of all possible products of length $k$ whose factors are elements of $\mathcal{F}$, that is $\mathcal{P}_k(\mathcal{F}) = \{A_{i_1} \cdots A_{i_k} | i_1, \ldots, i_k \in \mathcal{I}\}$, and set

$$\mathcal{P}(\mathcal{F}) = \bigcup_{k \geq 1} \mathcal{P}_k(\mathcal{F})$$

(19)

to be the multiplicative semigroup associated with $\mathcal{F}$. While, defined $\mathcal{P}_0(\mathcal{F}) := I$, we have

$$\mathcal{P}^*(\mathcal{F}) = \bigcup_{k \geq 0} \mathcal{P}_k(\mathcal{F})$$

(20)

the multiplicative monoid associated with $\mathcal{F}$.

We present four different generalizations of the concept of spectral radius of a single matrix to the case of a family of matrices $\mathcal{F}$.

The first generalization is due to Rota and Strang, in the seminal paper [62] published in 1960 they presented the generalization of the notion of spectral radius as limit of the normalized norm of a single matrix:

#### 2.2 Definition (Joint Spectral Radius – jsr).

If $\| \cdot \|$ is any matrix norm on $\mathbb{C}^{n \times n}$, consider $\hat{\rho}_k(\mathcal{F}) := \sup_{P \in \mathcal{P}_k(\mathcal{F})} \|P\|^{1/k}$, $k \in \mathbb{N}$ i.e. the supremum among the normalized norms of all products in $\mathcal{P}_k(\mathcal{F})$, and define the joint spectral radius of $\mathcal{F}$ as

$$\text{jsr}(\mathcal{F}) = \hat{\rho}(\mathcal{F}) = \lim_{k \to \infty} \hat{\rho}_k(\mathcal{F})$$

(21)
The joint spectral radius does not depend on the matrix norm chosen thanks to the equivalence between matrix norms in finite dimensional spaces.

We observe that in the discrete linear inclusions literature the logarithm of the joint spectral radius is sometimes called Lyapunov indicator [1].

In 1992 Daubechies and Lagarias [21] introduced the generalized spectral radius as a generalization of the \( \limsup \) over all the spectral radii \( \rho(A^k)^{1/k} \), \( k \geq 1 \), which are, trivially, always equal to \( \rho(A) \).

2.3 Definition (Generalized Spectral Radius – gsr). Let \( \overline{\rho}(\cdot) \) denote the spectral radius of an \( n \times n \)–matrix, consider \( \overline{\rho}_k(\mathcal{F}) := \sup_{P \in \mathcal{P}_k(\mathcal{F})} \rho(P)^{1/k}, \ k \in \mathbb{N} \) i.e. the supremum among the spectral radii of all products in \( \mathcal{P}_k(\mathcal{F}) \) normalized taking their \( k \)–th root, and define the generalized spectral radius of \( \mathcal{F} \) as

\[
gsr(\mathcal{F}) = \overline{\rho}(\mathcal{F}) = \limsup_{k \to \infty} \overline{\rho}_k(\mathcal{F}) \tag{22}
\]

For this two definitions it has been proved by Daubechies and Lagarias [21] [22] the following

2.4 Proposition (Four members inequality). For any set of matrices \( \mathcal{F} \) and any \( k \geq 1 \)

\[
\overline{\rho}_k(\mathcal{F}) \leq \overline{\rho}(\mathcal{F}) \leq gsr(\mathcal{F}) \leq jsr(\mathcal{F}) = \hat{\rho}(\mathcal{F}) \leq \hat{\rho}_k(\mathcal{F}) \tag{23}
\]

independently of the submultiplicative norm used to define \( \hat{\rho}_k(\mathcal{F}) \).

As a consequence of this we have that:

\[
\hat{\rho}(\mathcal{F}) = \inf_{k \geq 1} \hat{\rho}_k(\mathcal{F}) \tag{24}
\]

\[
\overline{\rho}(\mathcal{F}) = \sup_{k \geq 1} \overline{\rho}_k(\mathcal{F}) \tag{25}
\]

For the first equality see also [42], Lemma 1.2; for the second one, since \( \rho(M^k) = \rho(M)^k \) for every \( k \in \mathbb{N} \) and considering that by definition of \( \limsup \)

\[
\limsup_{k \to \infty} \overline{\rho}_k(\mathcal{F}) = \inf_{k \geq 1} \sup_{n \geq k} \overline{\rho}_n(\mathcal{F}),
\]

13
if it does exist a finite product \( P \in \mathcal{P}_r(\mathcal{F}), r \in \mathbb{N} \), such that \( \rho(P)^{1/r} = \overline{\rho}(\mathcal{F}) \), then, for every \( m \in \mathbb{N} \), \( \rho(P^m)^{1/mr} = \overline{\rho}(\mathcal{F}) \) and, thus, \( \sup_{n \geq k} \overline{\rho}_n(\mathcal{F}) = \overline{\rho}(\mathcal{F}) \) for every \( k \in \mathbb{N} \). This last equality is valid also if it does not exists such a finite product, in fact in this case the \( \sup \) is achieved only for \( n \to \infty \). So in both cases it results \( \inf_{k \geq 1} \sup_{n \geq k} \overline{\rho}_n(\mathcal{F}) = \sup_{k \geq 1} \overline{\rho}_k(\mathcal{F}) \), i.e. equation (25) holds true.

A third definition has been introduced by Chen and Zhou in 2000 \([17]\) and is based on a generalization of the formula associating the spectral radius of a matrix with its trace:

**2.5 Definition** (Mutual Spectral Radius – msr). Let \( \text{tr}(P) \) be the trace of the product \( P \in \mathcal{P}_k(\mathcal{F}) \) then \( \sup_{P \in \mathcal{P}_k(\mathcal{F})} |\text{tr}(P)| \) is the maximal absolute value among all the traces of the products of length \( k \). Define the mutual spectral radius of \( \mathcal{F} \) as

\[
msr(\mathcal{F}) = \limsup_{k \to \infty} \sup_{P \in \mathcal{P}_k(\mathcal{F})} |\text{tr}(P)|^{1/k}
\]  

(26)

We present now the last characterization of the spectral radius of a family of matrices. For bounded sets (ref Section 1) it is possible to generalize the concept, express in equation (9), of spectral radius as the \( \inf \) over the set of all possible induced matrix norms of \( A \).

**2.6 Definition** (Common Spectral Radius – csr). Given a norm \( \| \cdot \| \) on the vector space \( \mathbb{C}^n \) and the corresponding induced matrix norm, we define

\[
\| \mathcal{F} \| := \sup_{i \in \mathcal{I}} \| A_i \|
\]

(27)

where we assume that the family \( \mathcal{F} \) is bounded. We define the common spectral radius of \( \mathcal{F} \) (see \([62]\) and \([23]\)) as

\[
csr(\mathcal{F}) = \overline{\rho}(\mathcal{F}) = \inf_{\| \cdot \| \in \mathcal{N}} \| \mathcal{F} \|
\]

(28)

where \( \mathcal{N} \) denotes the set of all possible induced matrix norms.
This definition was first introduced by Rota and Strang in 1960 \cite{62} and re-introduced 35 years later by Elsner \cite{23}.

In the case of bounded sets, it is possible to prove that the four characterizations we presented coincide.

**2.7 Theorem** (The Complete Spectral Radius Theorem). *For a bounded family $\mathcal{F}$ the following equalities hold true*

$$g_{sr}(\mathcal{F}) = j_{sr}(\mathcal{F}) = c_{sr}(\mathcal{F}) = m_{sr}(\mathcal{F})$$

(29)

The equality of $g_{sr}$ and $j_{sr}$ was conjectured by Daubechies and Lagarias and it was proven by Berger and Wang \cite{2}, Elsner \cite{23}, Chen and Zhou \cite{17}, Shih et al. \cite{64}. For the equality of $c_{sr}$ and $j_{sr}$ we refer the reader to the seminal work of Rota and Strang \cite{62} or again \cite{23}. Chen and Zhou \cite{17} proved the last equality.

We observe that the first equality is the generalization of the Gelfand’s formula (Corollary 1.4) to the case of a family of matrices.

Another observation is that even though the joint and generalized spectral radius can be defined also for unbounded families the first equality does not hold in general. Consider for example the unbounded family:

$$\mathcal{F} = \left\{ \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \ldots, \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}, \ldots \right\}$$

For this family since every product of the two matrices is upper triangular with ones in the main diagonal it is evident that $p(\mathcal{F}) = 1$ and obviously $\hat{p}(\mathcal{F}) = +\infty$ since the family is unbounded (see \cite{66} for details and \cite{21} for another example).

We observe also that Gurvits in \cite{35} give a counterexample to the first equality in the case of two operators in an infinite dimensional Hilbert space.

From now on and if not differently specified we will always consider bounded sets of matrices. Theorem 2.7 implies that we can simply refer to the spectral radius $\rho(\mathcal{F})$ of the family of matrices $\mathcal{F}$. 

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2.8 Definition (Trajectory). Given a family $\mathcal{F} = \{A_i\}_{i \in \mathcal{I}}$, we define, for an arbitrary nonzero vector $x \in \mathbb{C}^n$, the trajectory

$$\mathcal{T}[\mathcal{F}, x] = \{Px \mid P \in \mathcal{P}(\mathcal{F})\}$$

(30)

as the set of vectors obtained by applying all the products $P$ in the multiplicative semigroup $\mathcal{P}(\mathcal{F})$ to the vector $x$.

2.9 Definition (Discrete linear inclusion). The discrete linear inclusion is the set of all the trajectories associated with all the possible vectors in $\mathbb{C}^n$. This set is denoted by $DLI(\mathcal{F})$.

2.2.2 Properties

We resume now properties valid for the spectral radius of a bounded set of matrices $\mathcal{F} = \{A_i\}_{i \in \mathcal{I}}$

1. Multiplication by a scalar: For any set $\mathcal{F}$ and for any number $\alpha \in \mathbb{C}$

$$\rho(\alpha \mathcal{F}) = |\alpha| \rho(\mathcal{F})$$

(31)

2. Continuity: The joint spectral radius is continuous in its entries as established by Heil and Strang [38]. Wirth has proved [70] that the joint spectral radius is even locally Lipschitz continuous on the space of compact irreducible sets of matrices (an explicit formula for the related Lipschitz constant has been evaluated by Kozyakin [47]).

3. Powers of the family: For any set $\mathcal{F}$ and for any $k \geq 1$

$$\rho(\mathcal{F}^k) \leq \rho^k(\mathcal{F})$$

4. Invariance under similarity: The spectral radius of the family is invariant under similarity transformation, so for any set of matrices $\mathcal{F}$, and any invertible matrix $T$

$$\rho(\mathcal{F}) = \rho(T \mathcal{F} T^{-1})$$

(32)
This because to any product $A_1 \cdots A_k \in \mathcal{P}_k(\mathcal{F})$ corresponds a product $T \cdot A_1 \cdots A_k \cdot T^{-1} \in \mathcal{P}_k(T\mathcal{F}T^{-1})$ with equal spectral radius.

5. **Conjugate or transposed family:** The conjugate or transposed family (family obtained taking the conjugate/transpose of every matrix in the original set) has the same spectral radius as the original one [27, Lemma 5.1]

$$\rho(\mathcal{F}^*) = \rho(\mathcal{F}) \quad \rho(\mathcal{F}^T) = \rho(\mathcal{F})$$ (33)

6. **Block triangular matrices:** Given a family of block upper triangular matrices

$$\mathcal{F} = \left\{ \begin{pmatrix} A_i & B_i \\ 0 & C_i \end{pmatrix} \right\}_{i \in \mathcal{I}}$$

we have that

$$\rho(\mathcal{F}) = \max \left\{ \rho(\{A_i\}_{i \in \mathcal{I}}), \rho(\{C_i\}_{i \in \mathcal{I}}) \right\}.$$ (34)

This is due to the closure, with respect to the multiplication, of block upper triangularity [2, Lemma II (c)]. Clearly the same holds for lower triangular matrices. This result generalizes to the case of more than two blocks on the diagonal.

7. **Closure and convex hull:** The closure and the convex hull of a set have the same spectral radius of the original set

$$\rho(\text{conv}\mathcal{F}) = \rho(\text{cl}\mathcal{F}) = \rho(\mathcal{F})$$ (35)

This result was first obtained by Barabanov in 1988 [1]. An alternative proof, given by Theys in [66, page 17], is based on the common spectral radius definition (28) and the property

$$\sup_{A_i \in \mathcal{F}} \|A_i\| = \sup_{A_i \in \text{conv}\mathcal{F}} \|A_i\| = \sup_{A_i \in \text{cl}\mathcal{F}} \|A_i\|. \quad (36)$$
8. **Uniform asymptotic stability characterization** [2, Theorem I (b)]:

For any bounded set of matrices \( \mathcal{F} \) and for any \( k \geq 1 \), all matrix products \( P \in \mathcal{P}_k(\mathcal{F}) \) converge to the zero matrix as \( k \to \infty \), i.e. \( \mathcal{F} \) is uniformly asymptotically stable (ref page 11), if and only if \( \rho(\mathcal{F}) < 1 \).

In other words the spectral radius of the family of matrices \( \mathcal{F} \) gives information about the uniform asymptotic stability of the associated dynamical system \( \text{DLI}(\mathcal{F}) \), defined on page 16.

9. **Product boundedness** [2, Theorem I (a)]: Given a bounded set of matrices \( \mathcal{F} \), if products \( P \in \mathcal{P}_k(\mathcal{F}), k \in \mathbb{N} \), converge as \( k \to \infty \). Then, the multiplicative monoid \( \mathcal{P}^*(\mathcal{F}) \) defined in (20) is bounded and \( \rho(\mathcal{F}) \leq 1 \).

The opposite implication is not true in general:

Given a defective family with \( \rho(\mathcal{F}) = 1 \), products \( P \in \mathcal{P}_k(\mathcal{F}), k \in \mathbb{N} \), explode for \( k \to \infty \) by Definition 2.11.

We return on this aspect in 13.

10. **Special cases**:

1. Recalling that the set of stochastic matrices is closed under matrix multiplication and that every stochastic matrix has spectral radius equal 1 (ref Section 1), if the matrices in \( \mathcal{F} \) are all stochastic then the spectral radius of the family is exactly 1.

2. If the matrices in \( \mathcal{F} \) are all upper–triangular, if they can be simultaneously upper–triangularized, if all the matrices in \( \mathcal{F} \) commutes or, more in general, if the Lie algebra associated with the set of matrices is solvable (commutative families are Abelian Lie algebras which are always solvable), if they are all symmetric or, more in general, if they are all normal or, finally, if they can be simultaneously normalized, then

\[
\rho(\mathcal{F}) = \max_{A_i \in \mathcal{F}} \{ \rho(A_i) \}
\]  

(37)
For more details see \cite{35, 25, 20, 66, 42}.

3. If $\mathcal{F} = \{A, A^*\}$ then $\rho(\mathcal{F}) = \rho(AA^*)^{1/2} = \sigma_1(A)$ i.e. the largest singular value of $A$. In fact \cite[Proposition 6.20]{66} using the four members inequality \cite{23} for $k = 2$ we have

$$\rho(AA^*)^{1/2} = \sigma_1(A) = \|AA^*\|_2^{1/2}$$ \hspace{1cm} (38)

4. \cite[Theorem 4]{56 and 26}. Consider the family $\mathcal{F} = \{A, B\}$ with

$$A := \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad B := \begin{pmatrix} a & -b \\ -c & d \end{pmatrix} \quad a, b, c, d \in \mathbb{R}$$

The joint spectral radius of the family $\mathcal{F}$ is given by

$$\rho(\mathcal{F}) = \begin{cases} \rho(A) = \rho(B) & \text{if } bc \geq 0 \\ \sqrt{\rho(AB)} & \text{if } bc < 0 \end{cases}$$

5. \cite[Theorem 5]{56 and 26}. Consider the family $\mathcal{F} = \{A, B\}$ with

$$A := \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad B := \begin{pmatrix} d & c \\ b & a \end{pmatrix} \quad a, b, c, d \in \mathbb{R}$$

The joint spectral radius of the family $\mathcal{F}$ is given by

$$\rho(\mathcal{F}) = \begin{cases} \rho(A) = \rho(B) & \text{if } |a - d| \geq |b - c| \\ \sqrt{\rho(AB)} & \text{if } |a - d| < |b - c| \end{cases}$$

6. Let $|\mathcal{F}|$ be the family of matrices obtained from $\mathcal{F}$ as follows:

$$A = [a_{ij}] \in \mathcal{F} \quad \rightarrow \quad |A| = [|a_{ij}|] \in |\mathcal{F}|.$$  

Then

$$\rho(|\mathcal{F}|) \geq \rho(\mathcal{F})$$  \hspace{1cm} (39)

From the previous result and the four members inequality \cite{23} we have that

$$\overline{\rho}_k(\mathcal{F}) \leq \rho(\mathcal{F}) \leq \rho(|\mathcal{F}|)$$
So if $P \in \mathcal{P}_k(\mathcal{F})$, $k \in \mathbb{N}$, is such that $\rho(P)^{1/k} = \rho(|\mathcal{F}|)$, then

$$\rho(\mathcal{F}) = \rho(P)^{1/k}.$$ 

**11. Non–algebraicity:** Any set composed of $k$ real $n \times n$–matrices can be seen as a point in the space $\mathbb{R}^{kn^2}$. Therefore, a subset of $\mathbb{R}^{kn^2}$ is a set of $k$–tuples of $n \times n$–matrices. Given a subset of $\mathbb{R}^{kn^2}$ this is defined *semi–algebraic* if it is a finite union of sets that can be expressed by a finite list of polynomial equalities and inequalities. Kozyakin [45] has shown that, for all $k, n \geq 2$, the set of points $x \in \mathbb{R}^{kn^2}$ such that $\rho(x) < 1$ is not semi–algebraic and, for all $k, n \geq 2$, the set of points $x \in \mathbb{R}^{kn^2}$ corresponding to a bounded semigroup $\mathcal{P}(x)$ is not semi–algebraic (the original paper by Kozyakin contains a flaw and the correction has been published by the same author only in Russian. For a corrected version in English we refer the reader to the Doctoral work of Theys [66, Section 4.2]). In practice in the general case, given a discrete linear inclusion $\text{DLI}(\mathcal{F})$, there is no procedure involving a finite number of operations that allows to decide whether $\text{DLI}(\mathcal{F})$ is uniformly asymptotically stable or not i.e. the uniform asymptotic stability of $\text{DLI}(\mathcal{F})$ is in general hard to determine.

**12. NP–hardness:** In [68] Tsitsiklis and Blondel proved that, given a set of two matrices $\mathcal{F}$ and unless $P = \text{NP}$, the spectral radius $\rho(\mathcal{F})$ is not polynomial–time approximable. This holds true even if all nonzero entries of the two matrices are constrained to be equal. Let us recall that the function $\rho(\mathcal{F})$ is *polynomial–time approximable* if there exists an algorithm $\rho^*(\mathcal{F}, \varepsilon)$, which, for every rational number $\varepsilon > 0$ and every set of matrices $\mathcal{F}$ with $\rho(\mathcal{F}) > 0$, returns an approximation of $\rho(\mathcal{F})$ with a relative error of at most $\varepsilon$ (i.e. such that $|\rho^* - \rho| \leq \varepsilon \rho$) in time polynomial in the bit size of $\mathcal{F}$ and $\varepsilon$ (if $\varepsilon = p/q$, with $p$ and $q$ relatively prime numbers, its *bit size* is equal to $\log(pq)$); however, there are algorithms which are polynomial either in the bit size of $\mathcal{F}$
or in $\varepsilon$. We conclude that the computation of the spectral radius of a set of matrices is in general \textit{NP–hard} and, consequently, it is NP–hard to decide the stability of all products of a set of matrices (for a survey of NP–hardness and undecidability we refer the reader to [12]). We observe here that Gurvits in [36] provides a polynomial–time algorithm for the case of binary matrices.

13. \textbf{Undecidability:} A decision problem is a problem which output is binary and can be interpreted as “yes” or “not”. For instance the problem of deciding whether an integer matrix is nonsingular is a decision problem. Since the nonsingularity can be checked, for example, by computing the determinant of the matrix and comparing it to zero it is a \textit{decidable problem}, i.e. a problem for which there exists an algorithm that always halts with the right answer. But there are also problems for which this kind of algorithm does not exist, these are \textit{undecidable problems}.

Given a set of matrices $\mathcal{F}$:

- The problem of determining if the semigroup $\mathcal{P}(\mathcal{F})$ is bounded is undecidable
- The problem of determining if $\rho(\mathcal{F}) \leq 1$ is undecidable

These two results, which remain true even if $\mathcal{F}$ contains only rational entries [13, 4], teach us that does not exist any algorithm allowing to compute the spectral radius of a generic set $\mathcal{F}$ in finite time.

It is still unknown if it does exist in the generic case an algorithm that, given a finite set of matrices $\mathcal{F}$, decides whether $\rho(\mathcal{F}) < 1$. Such an algorithm would allow to check the uniform asymptotic stability of the dynamical system ruled by the generic set $\mathcal{F}$. In the following we discuss the relation between this kind of algorithm and the so–called finiteness property.
The actual computation of $\rho(\mathcal{F})$ is an important problem in several applications, as we mentioned in the introduction of the present paper. According to the previous properties of non-algebraicity, NP-hardness and undecidability the problem appears quite difficult in general.

However, this is not reason enough for declaring the problem intractable and refraining from further research. As we discover in the next subsection the existence of an s.m.p. for the family (i.e. a product in the semigroup $\mathcal{P}(\mathcal{F})$ with particular properties) allows in the general case to evaluate exactly the spectral radius of a family making use of the Definition 2.6 as an actual computational tool. In order to do this we need the inf in equation (28) to be a min, but this is always true for irreducible families.

### 2.3 Irreducibility, nondefectivity and finiteness property

When the inf in (28) is a min we say that the family $\mathcal{F} = \{A_i\}_{i\in\mathcal{I}}$ admits an extremal norm.

#### 2.10 Definition (Extremal norm).

A norm $\|\cdot\|_*$ satisfying the condition

$$\rho(\mathcal{F}) = \|\mathcal{F}\|_* := \sup_{i\in\mathcal{I}} \|A_i\|_*$$

is said to be extremal for the family $\mathcal{F}$ (for an extended discussion see [71]).

Equivalently a norm $\|\cdot\|_*$ is called extremal for a given set $\mathcal{F} = \{A_i\}_{i\in\mathcal{I}}$ if it satisfies $\|A_i\|_* \leq \rho(\mathcal{F})$ for every $i \in \mathcal{I}$.

From Proposition 2.4 it is clear that, for a given norm, this inequality cannot be strict simultaneously for all the matrices in the set.

Given a bounded family $\mathcal{F} = \{A_i\}_{i\in\mathcal{I}}$ of $n \times n$–matrices with $\rho(\mathcal{F}) > 0$, the normalized family is given by

$$\mathcal{F}^* = \{A_i/\rho(\mathcal{F})\}_{i\in\mathcal{I}}$$

(40)

with spectral radius $\rho(\mathcal{F}^*) = 1$ and $\mathcal{P}(\mathcal{F}^*)$ is the associated multiplicative semigroup (ref equation (19) on page 12).
The definition of (weakly) defective matrix, given in section 1, extends to bounded families of matrices as follows:

2.11 Definition (Defective and Nondefective Families). A bounded family $\mathcal{F}$ of $n \times n$–matrices is said to be defective if the corresponding normalized family $\mathcal{F}^*$ is such that the associated semigroup $\mathcal{P}(\mathcal{F}^*)$ is an unbounded set of matrices. Otherwise, if either $\rho(\mathcal{F}) = 0$ or $\rho(\mathcal{F}) > 0$ with $\mathcal{P}(\mathcal{F}^*)$ bounded, then the family $\mathcal{F}$ is said to be nondefective.

The following result can be found, for example, in [62] and [2]:

2.12 Proposition. A bounded family $\mathcal{F}$ of $n \times n$–matrices admits an extremal norm $\| \cdot \|_*$ if and only if it is nondefective.

As previously mentioned we want to make use of Definition 2.6 as an actual computational tool for the spectral radius $\rho(\mathcal{F})$. To do this we need to ensure that the family admits an extremal norm i.e. we have to check the defectivity or nondefectivity of the set $\mathcal{F}$.

Strictly connected to defectivity of a family there is the concept of reducibility.

2.13 Definition (Reducible and Irreducible families). A bounded family $\mathcal{F} = \{A_i\}_{i \in I}$ of $n \times n$–matrices is said to be reducible if there exist a nonsingular $n \times n$–matrix $M$ and two integers $n_1, n_2 \geq 1$, $n_1 + n_2 = n$, such that

$$M^{-1}A_iM = \begin{pmatrix} A_{i}^{(11)} & A_{i}^{(12)} \\ O & A_{i}^{(22)} \end{pmatrix}$$

for all $i \in \mathcal{I}$ (41)

where the blocks $A_{i}^{(11)}$, $A_{i}^{(12)}$, $A_{i}^{(22)}$ are $n_1 \times n_1$–, $n_1 \times n_2$– and $n_2 \times n_2$–matrices, respectively. On the contrary, if a family $\mathcal{F}$ is not reducible, then it is said to be irreducible.

Irreducibility means that only the trivial subspaces $0$ and $\mathbb{C}^n$ are invariant under all the matrices of the family $\mathcal{F}$. Otherwise $\mathcal{F}$ is called reducible.
The concept of irreducibility was introduced in the joint spectral radius theory by Barabanov in [1], where he named irreducible families *nonsingular sets*.

We observe that some authors refer to reducibility as decomposability (irreducibility as non–decomposability) in order to avoid confusion with the notion of reducibility commonly used in linear algebra [40, Definition 6.2.21].

An immediate consequence of irreducibility of $F$ is that $\rho(F) > 0$, in fact, in this case the semigroup $P(F)$ is irreducible and, therefore, does not consist of nilpotent elements, by the Levitzky Theorem [49]. So we can always normalize an irreducible set of matrices $F$ by $\rho(F)$ obtaining a set with generalized spectral radius equal to 1.

Another consequence of irreducibility of a family is stated in the next Theorem and its Corollary, which follow easily from the Barabanov’s construction of extremal norms for irreducible families of matrices [1].

**2.14 Theorem (23, Lemma 4).** If a bounded family $F$ of $n \times n$–matrices is defective, then it is reducible.

and, therefore,

**2.15 Corollary (11 23 59).** If a bounded family of matrices is irreducible then it is nondefective, i.e. it does exist an extremal norm for the family.

In Figure 1 it is represented the space $B$ of bounded families of matrices in $\mathbb{C}^{n \times n}$. This space can be split into the set of the reducible families $R$ and its complement $I_R$, the set of the irreducible ones. Families of matrices can be nondefective or defective: the set $D$ of the defective families is a proper subset of $R$ i.e. $D \subseteq R$. In fact Theorem 2.14 implies that a defective family is always reducible, but the opposite implication is not necessarily true. For example, for $n \geq 2$ all single families $\mathcal{F} = \{A\}$ are clearly reducible as the Jordan canonical form proves, but not necessarily defective. The set of nondefective families $N_D$, the complement of $D$ in $B$, is denoted by grey dots.
Figure 1: Space of bounded families of matrices $\mathcal{B}$: the set of defective families is denoted by $\mathcal{D}$, while its complement, highlighted by dots, is the set of nondefective families $\mathcal{N}_D$.

About the dimension of set $\mathcal{D}$ and $\mathcal{R}$ Maesumi [52] proposed the following conjecture

2.16 Conjecture. Reducible (decomposable) matrix sets form a set of measure zero in the corresponding space of matrices. Defective matrix sets form a set of measure zero within the set of reducible matrices.

In [18] we delve further this analysis especially explaining how reducible families can be handled.

We add just that Brayton and Tong in [15] give an alternative sufficient–condition for nondefectiveness. They prove that, considered each matrix $P$ in the semigroup $\mathcal{P}(\mathcal{F})$ and the associated similarity matrix $S_P$ that reduce $P$ into its Jordan form, if every $S_P$ has columns linearly independent uniformly on all $P \in \mathcal{P}(\mathcal{F})$, then $\mathcal{F}$ is nondefective. This alternative sufficient–condition represents the generalization of the concept of strongly nondefectiveness to the case of sets of matrices, in fact for a single matrix $A$ strongly nondefectiveness is equivalent to semisimplicity of all the eigenvalues in the spectrum of $A$ or equivalently to diagonalizability of $A$ (ref pages 25
4 and 8. Clearly strongly nondefectiveness implies nondefectiveness, but checking this sufficient–condition is not feasible in practice.

As previously mentioned there are not known algorithms for deciding uniform asymptotic stability of a generic set of matrices and it is unknown if this problem is algorithmically decidable in general. We have also seen that uniform asymptotic stability of the set $\mathcal{F}$ is equivalent to $\rho(\mathcal{F}) < 1$. In order to check if $\rho(\mathcal{F}) < 1$ for finite families we may think of using the four members inequality (23)

$$\rho_k(\mathcal{F}) \leq \rho(\mathcal{F}) \leq \hat{\rho}_k(\mathcal{F}) \quad \text{for all } k \geq 1$$

The procedure could be the following [21]:

1. We evaluate

$$\rho_k(\mathcal{F}) := \max_{P \in \mathcal{P}_k(\mathcal{F})} \rho(P)^{1/k} \quad \text{and} \quad \hat{\rho}_k(\mathcal{F}) := \max_{P \in \mathcal{P}_k(\mathcal{F})} \|P\|^{1/k}$$

for increasing values of $k \geq 1$.

2. As soon as $\hat{\rho}_k < 1$ or $\rho_k \geq 1$ we stop and declare the set uniform asymptotic stable or unstable respectively.

We observe that this procedure always stops after finitely many steps unless $\rho = 1$ and $\rho_k < 1$ for all $k \geq 1$, but this never occurs for families that, satisfying the finiteness property, have an s.m.p.

**2.17 Definition** (Finiteness property and s.m.p.). A finite family $\mathcal{F}$ of $n \times n$–matrices has the finiteness property if there exists, for some $k \geq 1$, a product $\overline{P} \in \mathcal{P}_k(\mathcal{F})$ such that

$$\rho(\overline{P}) = \rho(\mathcal{F})^k.$$

The product $\overline{P}$ is said to be a spectrum–maximizing product or s.m.p. for $\mathcal{F}$. Some authors refer to optimal product instead of s.m.p., see for instance [43, 52].

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An s.m.p. is said minimal if it is not a power of another s.m.p. of $F$.

Any eigenvector $x \neq 0$ of $P$ related to an eigenvalue $\lambda$ with $|\lambda| = \rho(P)$ is said to be a leading eigenvector of $F$.

From the previous definition is evident that uniform asymptotic stability is algorithmically decidable for finite sets of matrices that have the finiteness property.

Lagarias and Wang in 1995 [48] conjectured that the finiteness property was valid for all finite families of real matrices (the so-called finiteness conjecture). Unfortunately this conjecture does not hold true: Bousch and Mairesse [14] and later other authors [10, 46] presented non-constructive counterproofs. In particular in [10] Blondel et al. proved that for the parametric family

$$\mathcal{F}_\alpha = \{A, \alpha B\} = \left\{ \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \alpha \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \right\} \quad \text{with} \quad \alpha \in [0,1]$$

there exist uncountably many values of the parameter $\alpha$ for which $\mathcal{F}_\alpha$ does not satisfy the finiteness conjecture. They were unable to find a single explicit value of $\alpha$ and they conjectured that the set of values $\alpha \in [0,1]$ for which the finiteness conjecture is not satisfied is of measure zero. Recently Hare et al. [37], using combinatorial ideas and ergodic theory, have been able to approximate, up to a desired precision, an explicit value $\alpha$ such that $\mathcal{F}_\alpha$ does not satisfy the finiteness conjecture. The question if there exist families of matrices with rational entries that violate the conjecture remains still open. Based on all the numerical experiments developed in the last years and the results previously mentioned a new conjecture has been introduced:

2.18 Conjecture ([10, 52, Blondel et al. and Maesumi]). The finiteness property is true a.e. in the space of finite families of complex square matrices, i.e. the set of families of matrices for which the finiteness property is not true has measure zero in the space of finite families.
If this conjecture is true then it suggests us to track s.m.p.’s candidates out and validate them with some procedure in order to find the spectral radius of the family. In [18] we explain how to perform the validation step using particular extremal norms for the given set.

The idea behind this last conjecture is that the NP–hardness, non–algebraicity and undecidability results are due to certain rare and extreme cases and that in the generic case the evaluation of the spectral radius, while could be computationally intensive, is possible. About the computational complexity we remind an example, given by Berger and Wang [2, Example 2.1], of a set of two $2 \times 2$–matrices with minimal s.m.p. of length $k \geq 1$ with $k$ arbitrarily large:

$$
F = \left\{ \alpha^k \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \alpha^{-1} \begin{pmatrix} \cos \frac{\pi}{2k} & \sin \frac{\pi}{2k} \\ -\sin \frac{\pi}{2k} & \cos \frac{\pi}{2k} \end{pmatrix} \right\} \quad \text{with} \quad 1 < \alpha < \left( \cos \frac{\pi}{2k} \right)^{-1}
$$

They prove that $\rho(F) = 1$, $\rho_j(F) < 1$ for $j \leq k$ and $\rho_{k+1}(F) = 1$.

We recall that Blondel and Tsitsiklis in [13] proved also that the effective finiteness conjecture is false:

2.19 Conjecture (Effective finiteness conjecture). For any finite set $\mathcal{F}$ of square matrices with rational entries there exists an effectively computable natural number $t(\mathcal{F})$ such that $\rho_{t(\mathcal{F})}(\mathcal{F}) = \rho(\mathcal{F})$

The falseness of this conjecture implies that, given a family of matrices with rational entries which admits a spectrum–maximizing product, the length of the s.m.p. can be arbitrary long and consequently the computation of the spectral radius can become a tough problem. Nevertheless for random families this product appears to be, luckily, quite short in general.

The finiteness property is known to hold in many cases:

- when the matrices in $\mathcal{F}$ are all simultaneously upper–triangularizable, or they can be simultaneously normalized, or the Lie algebra associated with the set $\mathcal{F}$ is solvable. In these cases, in fact, the spectral radius
is simply equal to the maximum of the spectral radii of the matrices (Property 10, special case 2 on page 18);

- when a finite set of real matrices admits an extremal piecewise analytic norm in $\mathbb{R}^n$. A piecewise analytic norm is any norm on $\mathbb{R}^n$ whose unit ball $B$ has a boundary which is contained in the zero set of a holomorphic function $f(z)$, i.e. complex differentiable at every point in its domain, defined on a connected open set $\Omega \subset \mathbb{C}^n$ containing 0, which has $f(0) \neq 0$ (Lagarias and Wang [48]);

- when a finite set of real matrices admits an extremal piecewise algebraic norm in $\mathbb{R}^n$. A piecewise algebraic norm is one whose boundary is contained in the zero set of a polynomial $p(z) \in \mathbb{R}[z_1, \ldots, z_n]$, which has $p(0) \neq 0$. This is the case when the unit ball of a norm is a polytope [18], or an ellipsoid (ref page 7), or the $l^p$ norm for rational $p$, with $1 \leq p \leq \infty$ (Lagarias and Wang in [18] extended the result proved by Gurvits in [35] for real polytope extremal norms to the general case of piecewise algebraic norms in $\mathbb{R}^n$);

- when a finite set of matrices admits a complex polytope extremal norm. This it has been proved by Guglielmi, Wirth and Zennaro in [27, Theorem 5.1] extending to the complex case the results by Gurvits [35] and Lagarias and Wang [18]. We come back to polytope norms in [18].

For other classes of sets of matrices the finiteness property has been only conjectured to be true, an example is the class of sets of matrices with rational entries. Indeed the proof of the finiteness property for sets of rational matrices would be satisfactory for practical applications: the matrices that one handles or enters in a computer are rational–valued.

Recently Blondel and Jungers [43] have proved the following Theorem:

**2.20 Theorem** ([43 Theorem 4]).

1. The finiteness property holds for all sets of nonnegative rational matrices if and only if it holds for all pairs of binary matrices.
2. The finiteness property holds for all sets of rational matrices if and only if it holds for all pairs of matrices with entries in \{-1,0,+1\}.

They proposed, consequently, the following conjecture

**2.21 Conjecture** ([6, 43 Blondel, Jungers and Protasov]). *Pairs of binary matrices have the finiteness property.*

If this conjecture is correct then, by Theorem 2.20 nonnegative rational matrices also satisfy the finiteness property and, thus, the question $\rho(\mathcal{F}) < 1$ becomes decidable for sets of matrices with nonnegative rational entries. From a decidability point of view this last result would be somewhat surprising since it is known that the closely related question $\rho(\mathcal{F}) \leq 1$ is known to be no algorithmically decidable for such sets of matrices (ref Property 13 on page 21). Blondel and Jungers [43] proved that pairs of $2 \times 2$ binary–matrices satisfy the finiteness property and observed that the length of the s.m.p.'s is always very short. This result is promising even though a generalization to the case of $n \times n$–matrices seems quite difficult due to the falseness of the effective finiteness conjecture 2.19 which implies that the length of the s.m.p.'s for families of $n \times n$–matrices can become extremely long.

A more general version of the previous Conjecture is the following

**2.22 Conjecture** ([6, 43 Blondel, Jungers and Protasov]). *The finiteness property holds for pairs of matrices with entries in \{-1,0,+1\} (the so–called sign–matrices).*

This last would imply, by Theorem 2.20 that the finiteness property holds for all sets of rational matrices. In [18] we prove analytically the finiteness property for pairs of $2 \times 2$ sign–matrices, i.e. matrices with entries in \{-1,0,+1\}. 

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