Slow $m = 1$ instabilities of softened gravity Keplerian discs

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ABSTRACT

We present the simplest model that permits a largely analytical exploration of the $m = 1$ counter-rotating instability in a ‘hot’ nearly Keplerian disc of collisionless self-gravitating matter. The model consists of a two-component softened gravity disc, whose linear modes are analysed using the Wentzel–Kramers–Brillouin approximation. The modes are slow in the sense that their (complex) frequency is smaller than the Keplerian orbital frequency by a factor which is of order the ratio of the disc mass to the mass of the central object. Very simple analytical expressions are derived for the precession frequencies and growth rates of local modes; it is shown that a nearly Keplerian disc must be unrealistically hot to avoid an overstability. Global modes are constructed for the case of zero net rotation.

Key words: instabilities – stellar dynamics – celestial mechanics – galaxies: nuclei.

1 INTRODUCTION

Galactic nuclei are thought to harbour supermassive black holes and dense clusters of stars, whose structural and kinematical properties appear to be correlated with global galaxy properties (Gebhardt et al. 1996, 2000; Ferrarese & Merritt 2000). The imprint of galaxy formation is expected to be recorded in the nature of stellar orbits. A remarkable case is that of our nearest large neighbouring galaxy M31, whose centre has a double-peaked distribution of stars (Light, Danielson & Schwarzschild 1974; Lauer et al. 1993, 1998; Kormendy & Bender 1999). Tremaine (1995) proposed that the off-centred peak marks the region, in a disc of stars, where lie the apoapses of many eccentric orbits. Self-gravitating models of such an eccentric disc have been proposed (Bacon et al. 2001; Salow & Statler 2001; Sambhus & Sridhar 2002). Of particular interest to the present investigation is the model proposed in Sambhus & Sridhar (2002), because it included a few per cent of stars on retrograde (i.e. counter-rotating) orbits. Here, it was proposed that the lopsidedness of the nuclear disc of M31 could have been excited by the presence of the retrograde stars, which were accreted to the centre of the galaxy in the form of a globular cluster that spiralled in due to dynamical friction. This proposal was motivated by the work of Touma (2002), which suggested that even a small fraction of mass in retrograde orbits could excite a linear lopsided instability.

Counter-rotating streams of matter in a self-gravitating disc are known to be unstable to lopsided modes (Zang & Hohl 1978; Araki 1987; Sawamura 1988; Merritt & Stiavelli 1990; Palmer & Papaloizou 1990; Sellwood & Merritt 1994; Lovelace, Jore & Haynes 1997). The dynamics of galactic nuclei involve nearly Keplerian systems of stars or other collisionless matter (Rauch & Tremaine 1996; Sridhar & Touma 1999; Sridhar, Syer & Touma 1999; Tremaine 2001; Touma, Tremaine & Kazandjian 2009). Touma (2002) considered a softened gravity version of Laplace–Lagrange theory of planetary motions, and showed that a nearly Keplerian axissymmetric disc is linearly unstable to a $m = 1$ mode when even a small fraction of the disc mass is counter-rotating. Softened gravity was introduced by Miller (1971) to simplify the analysis of the dynamics of stellar systems. In this form of interaction, the Newtonian $1/d^2$ gravitational potential is replaced by $1/\sqrt{d^2 + b^2}$, where $b > 0$ is called the softening length. In the context of waves in discs, it is well known that the softening length mimics the epicyclic radius of stars on nearly circular orbits. Therefore, a disc composed of cold collisionless matter interacting via softened gravity provides a surrogate for a ‘hot’ collisionless disc.

The goal of this paper is to formulate and analyse the counter-rotating instability in the simplest model of a ‘hot’ nearly Keplerian disc of collisionless self-gravitating matter. To this end we make the following choices: (i) the discs are assumed to be made of matter whose self-interaction is through softened gravity; (ii) a Wentzel–Kramers–Brillouin (WKB) analysis is made of the linearized equations governing the perturbations. The unperturbed two-component nearly Keplerian disc is introduced in Section 2 and the apse precession rate is defined. The equations governing the linearized perturbations and relevant potential theory for softened gravity is given in Section 3. The local (or WKB) approximation and dispersion relation for local modes is derived in Section 4. This is used to discuss stability, instability and overstability. Section 5 considers the construction of global modes for the case of a Kuzmin disc with equal masses in the counter-rotating components, and we conclude in Section 6.
2 THE UNPERTURBED TWO-COMPONENT DISC

We consider a disc of mass $M_d$ orbiting a central mass $M$. We specialize to the Keplerian case, $M_d \ll M$. Therefore the force on the disc material is mostly Newtonian, giving rise to a near equality between the frequencies of azimuthal and radial oscillations. Test particle orbits may be thought of as osculating Keplerian ellipses, whose apsides precess due to the self-gravity of the disc, at rates that are smaller than the orbital frequency by a factor $\sim = M_d/M \ll 1$. Other forces could also be responsible for the evolution of the disc over similar slow time-scales. We consider a cold collisionless disc, composed of particles orbiting on two counter-rotating streams. The disc particles interact with each other through softened gravity. The central mass and the disc attract each other through (unsoftened) Newtonian gravity. Our notation closely follows Tremaine (2001).

We employ polar coordinates $(r, \phi)$ in the disc plane and place the central mass at the origin. The unperturbed components are axisymmetric with surface densities $\Sigma_d^+ (r) \geq 0$ and $\Sigma_d^- (r) \geq 0$, and circular velocities $v^+_d = \pm r \Omega_d(r) \epsilon_d$, respectively. The angular frequency, $\Omega_d (r) > 0$, is determined by the total gravitational potential:

$$\Phi_d (r) = -\frac{GM_d}{r} + \Phi_d(r),$$

where $\Phi_d$ is the self-gravity of the disc determined by the total surface density $\Sigma_d (r) = \Sigma_d^+ + \Sigma_d^-$; it is $O(\epsilon)$ compared to $GM/r$. Test particles on nearly circular prograde orbits have azimuthal and radial frequencies, $\Omega > 0$ and $\kappa > 0$; particles on nearly circular retrograde orbits have frequencies, $-\Omega < 0$ and $-\kappa < 0$. The frequencies are given by

$$\Omega^2 (r) = \frac{GM}{r^3} + \frac{1}{r} \frac{d \Phi_d}{dr},$$

$$\kappa^2 (r) = \frac{GM}{r^3} + \frac{d^2 \Phi_d}{dr^2} + \frac{3}{r} \frac{d \Phi_d}{dr}.$$  

(2)

(3)

The precession rate of the apsides of a nearly circular orbit of angular frequency $\pm \Omega (r)$ is given by $\pm \kappa$, where

$$\dot{\phi} (r) = \Omega - \kappa = -\frac{1}{2 \Omega (r)} \left( \frac{d^2 \Phi_d}{dr^2} + \frac{2}{r} \frac{d \Phi_d}{dr} \right) + O(\kappa^2).$$

(4)

The above can be generalized to include a centrally symmetric external potential, $\Phi_e (r)$; assuming that $\Phi_e$ is also $O(\epsilon)$ just like $\Phi_d$, all that needs to be done is to replace $\Phi_d$ by $(\Phi_d + \Phi_e)$ in equations (1)–(4).

3 LINEAR RESPONSE

Let $\psi^+ = u^+_a (r, t) \epsilon_d + v^+_a (r, t) \epsilon_d$ and $\psi^- = u^-_a (r, t) \epsilon_d + v^-_a (r, t) \epsilon_d$ be infinitesimal perturbations to $\psi^+_d$ and $\psi^-_d$. They satisfy the linearized Euler and continuity equations, appropriate to a cold disc:

$$\frac{\partial \psi^+}{\partial t} + (\psi^+ \cdot \nabla) \psi^+ + (\psi^+ \cdot \nabla) \psi^- = -\nabla \Phi_a,$$  

$$\frac{\partial \psi^-}{\partial t} + (\psi^- \cdot \nabla) \psi^- + (\psi^- \cdot \nabla) \psi^+ = 0,$$

(5)

(6)

where $\Phi_a (r, t)$ is the perturbing potential. We write the variables $(\Sigma^+_a, u^+_a, v^+_a, \Phi_a)$ in the form $X^+_a (r, \phi, t) = X^+_a (r, \exp [m \phi - \omega t])$. Substituting these in equations (5) and (6), straightforward manipulations give

$$u^{\pm}_a = -i \frac{D^m}{m} \left[ \pm \frac{\kappa^2}{2\Omega} \frac{d}{dr} \mp \frac{2m \Omega - \omega}{\kappa} \right] \Phi^m_a,$$

$$v^{\pm}_a = \frac{1}{D^m} \left[ \pm \frac{\kappa^2}{2\Omega} \frac{d}{dr} \mp \frac{2m \Omega - \omega}{\kappa} \right] \Phi^m_a,$$

$$\Phi^m_a = -2\pi G \frac{\exp (-|k|b)}{|k|} \left( \Sigma^+_a + \Sigma^-_a \right).$$

(7)

(8)

(9)

(10)

(11)

Equations (7)–(10) give the linear responses of the surface densities and velocities of the two components, to a specified perturbing potential, $\Phi_a$.

For a self-consistent response, the perturbing potential, $\Phi^m_a (r)$, depends only on the total surface density, $(\Sigma^+_a + \Sigma^-_a)$; in fact this is the only coupling between the two counter-rotating components. The Poisson integral is

$$\Phi^m_a (r) = \int_0^r d'r' P_m (r, r') \left[ \Sigma^+_a (r') + \Sigma^-_a (r') \right].$$

The unperturbed disc potential, $\Phi_d$, is related to the unperturbed surface density, $\Sigma_d = (\Sigma^+_d + \Sigma^-_d)$, through equation (11), when $m = 0$. The kernel,

$$P_m (r, r') = - \frac{\pi G}{r^2} B_m^0 (\alpha, \beta) + \frac{\pi G}{r^2} \frac{r}{r'} \left( \delta_{n1} + \delta_{n-1} \right),$$

includes direct and indirect contributions. Here $r_\prec = \min (r, r')$, $r_\succ = \max (r, r')$, $\alpha = r_\prec / r_\succ$, $\beta = b / r_\succ$, and

$$B_m^0 (\alpha, \beta) = 2 \pi \int_0^\infty d\theta \cos m \theta \sin \frac{\pi}{2} \left( 1 - 2 \alpha \cos \theta + \alpha^2 + \beta^2 \right)^{m/2}$$

(12)

(13)

is a generalization of the Laplace coefficients, introduced by Touma (2002); in the limit of no softening, $B_m^0 (\alpha, 0) = b_{m/2} (\alpha)$, the Laplace coefficients familiar from celestial mechanics (Murray & Dermott 1999). Equations (7)–(13) determine the self-consistent, linear modes of axisymmetric discs, whose counter-rotating components interact through softened gravity.

4 THE LOCAL APPROXIMATION

A perturbation, $X^+_m (r) = |X^+_m (r)| \exp [i \int dr ' k (r')]$, is referred to as tightly wound, if the radial wavenumber is large in the sense $|k (r')| \gg |m|$. To leading order in $|m| / k (r')$, the WKB approximations to the linear responses of equations (7)–(9) are

$$u^m_a (r) = \frac{(\pm m \Omega - \omega)}{D^m} k \Phi^m_a,$$

$$v^m_a (r) = \pm i \frac{\kappa^2}{2\Omega D^m} k \Phi^m_a,$$

$$\Sigma^m_a (r) = - \frac{\Sigma^+_a + \Sigma^-_a}{D^m} k^2 \Phi^m_a.$$

(14)

(15)

(16)

The responses are singular at radii, where $D^m = 0$. For a self-consistent $\Phi^m_a$, the potential theory of the previous section simplifies in the WKB limit (Miller 1971),

$$\Phi^m_a = -2\pi G \frac{\exp (-|k|b)}{|k|} \left( \Sigma^+_a + \Sigma^-_a \right).$$

(17)

Substituting for $\Sigma^+_a$ and $\Sigma^-_a$ from equation (16), and eliminating $\Phi^m_a$, gives the WKB dispersion relation

$$D^m D_a^m = 2\pi G |k| \exp (-|k|b) \left[ D^+_a \Sigma^+_a + D^-_a \Sigma^-_a \right].$$

(18)
which is a quartic equation in $\omega$. All the coefficients of the various powers of $\omega$ being real, if $\omega$ is a solution, so is its complex conjugate, $\omega^*$. It should be noted that we have not made any assumptions about the Keplerian nature of the disc. Therefore, the dispersion relation of equation (18) is valid for a non-Keplerian disc as well, with or without the central mass. In general, $\omega$ will not be small, compared to either $\Omega$ or $\kappa$, so the dispersion relation is not restricted to slow perturbations. When the counter-rotating component is absent (i.e. $\Sigma^+_c = 0$), equation (18) reduces to the WKB dispersion relation, \[ \omega^2 = \kappa^2 - 2\pi G \Sigma d |k| \exp (-|k| b), \] which is identical to the dispersion relation for a disc without counter-rotating components. It is straightforward to prove [as problem (6–5) of Binney & Tremaine (2008)] invites the reader to] that the disc is stable to short-wavelength axisymmetric perturbations, if \[ b > b_0 = \frac{2\pi G \Sigma d}{\kappa^2 e}. \] (20)

4.1 Slow $m = 1$ perturbations

When the azimuthal wavenumber is $m = 1$, the near equality between $\Omega$ and $\kappa$ – see equations (2) and (3) – enables slow modes, for which $|\omega/\Omega| \sim O(\epsilon) \ll 1$. It is convenient to define \[ \Sigma^+_d = (1 - \eta) \Sigma d \quad \text{and} \quad \Sigma^-_d = \eta \Sigma d, \] (21) where $0 \leq \eta(r) \leq 1$ is the local mass fraction in the counter-rotating component. Note that, since the form of $\eta(r)$ is quite arbitrary, the prograde and retrograde discs can have completely different radial profiles and total mass. We also introduce a local frequency, \[ S(r, |k|) = \frac{\pi G \Sigma d(r)}{\Omega(r)} |k| \exp (-|k| b), \] (22) whose maximum value at any $r$, \[ S_{\text{max}}(r) = \frac{\pi G \Sigma d}{e \Omega(r)} = \frac{\kappa^2 b_0}{2 \Omega} \simeq \frac{\Omega b_0}{2 \eta}, \] (23) is attained for $|k| = 1/b$. Here $b_0$ is the minimum softening length, defined in equation (20), that ensures local stability to axisymmetric perturbations.

From equations (4) and (10), we have \[ D^+_1 = 2\Omega (\omega - \sigma) + O(e^2), \] (24) \[ D^-_1 = -2\Omega (\omega + \sigma) + O(e^2). \] (25) When these are substituted in equation (18), a little rearrangement provides the dispersion relation for slow, $m = 1$ perturbations: \[ \omega^2 - S (1 - 2\eta) \omega - \sigma (\sigma + S) = 0, \] (26) whose solution is \[ \omega = \frac{S}{2} (1 - 2\eta) \pm \frac{1}{2} \sqrt{S^2 (1 - 2\eta)^2 + 4\sigma (\sigma + S)}. \] (27) Equations (26) and (27) are invariant under $(\eta, \omega) \rightarrow (1 - \eta, -\omega)$, because this operation is equivalent to interchanging the meaning of ‘prograde’ and ‘retrograde’. It is convenient to first consider two special cases.

\begin{itemize}
\item \textbf{(i) No counter-rotation, $\eta = 0$.} When $\eta = 0$, equation (27) admits the two roots, $\omega = \sigma + S$, and $\omega = -\sigma$. The former root corresponds to equation (14) of Tremaine (2001), and implies that the disc is locally stable to all $m = 1$ perturbations. However, $\omega = -\sigma$ is a spurious solution, arising from multiplication by $D^+_1$ in the derivation of equation (18). Henceforth we assume that $\eta \neq 0$.
\item \textbf{(ii) Equal counter-rotation, $\eta = 1/2$.} When there is equal mass (locally) in the prograde and retrograde components, the two roots of equation (27) are $\omega = \pm \sqrt{\sigma} (\sigma + S)$. If $\sigma$ happens to be positive, then $\omega$ is real, and the disc is locally stable. However, $\sigma < 0$ for most continuous discs, hence $\omega$ can be either real or purely imaginary; there is no local overstability. The criteria for (in)stability are discussed below, along with the case of general $\eta$.
\end{itemize}

The sign of the discriminant of equation (27), \[ D = S^2 (1 - 2\eta)^2 + 4\sigma (\sigma + S), \] (28) determines whether $\omega$ is real or complex. If $\sigma > 0$, then $D > 0$, hence $\omega$ is real. However, $\sigma < 0$ for most continuous discs, and it is straightforward to determine that $D < 0$ if $S$ lies in the range of values, $0 < S_- < S < S_+$, where \[ S_+ = \frac{2|\sigma|}{(1 - 2\eta)^2} \left[ 1 \pm 2\sqrt{\eta(1 - \eta)} \right]. \] (29) However, we noted earlier that the maximum value that $S$ can take is $S_{\text{max}}$, given by equation (23). Hence, at a specified $r$, $D$ is positive for all $k$, if $S_+ > S_{\text{max}}$. Therefore, the disc is stable to all short-wavelength $m = 1$ perturbations, if \[ b > b_1 \equiv \frac{\pi G \Sigma d}{\epsilon \Omega |\sigma|} \left[ \frac{(1 - 2\eta)^2}{2 - 4\sqrt{\eta(1 - \eta)}} \right]. \] (30)

The $\eta$-dependent factor in equation (30) does not vanish for any value of $\eta$. In fact, as $\eta \rightarrow 1/2$, the term in $[\ldots]$ approaches unity, giving the stability criterion for the case $\eta = 1/2$, as may be verified independently. Since $(|\sigma|/\Omega) \sim O(\epsilon) \ll 1$, the $b$ required for local stability, according to equation (30), equals $b_0$ (which is the minimum softening required for local axisymmetric stability) multiplied by a large factor of order $1/e$. To the extent softening mimics ‘heat’ (more precisely, the epicyclic radius) in collisionless discs, this criterion suggests that a Keplerian disc would have to be very hot indeed, to be able to avoid a local instability to $m = 1$ perturbations. Hence one is led to consider overstable perturbations.

Overstability occurs when $D < 0$. We write $\omega = \Omega_p \pm i\Gamma$, where $\Omega_p$ is the pattern speed of the $m = 1$ perturbation, and $\Gamma > 0$ is the growth rate. From equation (27), \[ \Omega_p = \frac{S}{2} (1 - 2\eta), \] (31) \[ \Gamma = \frac{1}{2} \sqrt{|S^2 (1 - 2\eta)^2 + 4\sigma (\sigma + S)|}. \] (32)
assume that both the prograde and retrograde discs have identical radial profiles and masses. It is useful to take the axisymmetric unperturbed disc to be a Kuzmin disc because (i) the Kuzmin disc is centrally concentrated and is hence a plausible candidate for being a quite generic case; (ii) the surface density, the self-gravitational potential and the precession rate all have explicit analytical forms; (iii) the slow modes of the Kuzmin disc were studied in Tremaine (2001) for the case of no counter-rotation.

The surface density of the Kuzmin disc is given by

$$\Sigma_r(r) = \frac{aM_d}{2\pi(r^2 + a^2)^{3/2}},$$

where $M_d$ is the disc mass and $a$ is the central concentration parameter. The precession rate due to the Kuzmin disc is given by

$$\sigma = \frac{3GM_d a^2}{2\Omega(r)(r^2 + a^2)^{3/2}},$$

where the rotational frequency is given by the Keplerian flow due to the central mass $M$ as $\Omega(r) = \sqrt{GM/r^3}$. Let us first consider the stability of the disc under axisymmetric perturbations. In Section 4 we derived the minimum value of the softening parameter $b_0$ that ensures local stability. In the slow mode limit $\kappa(r) = \Omega(r)$ to the zeroth approximation. Therefore, we find

$$b_0 = \frac{2\pi G \Sigma_d}{\kappa^2 e} = \frac{aM_d}{eM} \left( \frac{r}{r^2 + a^2} \right)^{3/2}.$$  

(35)

The largest value of $b_0 = aM_d/eM$, therefore, the smallest softening parameter $b$ that ensures stability everywhere satisfies $b/a > M_d/eM$. Let us define a parameter $R = b/a$ and another parameter $\mu$ through $M_d/M_0 = \mu R$. In terms of these parameters the previous inequality gives $\mu < e$. Substituting $\eta = 1/2$ in equation (27), we get

$$S = \frac{\omega^3 - \sigma^2}{\sigma}.$$  

(36)

Now we substitute for $\sigma$ and recall that we are looking for global unstable modes. Therefore $\omega^2 = \Gamma^2$, and

$$a|k| \exp(-|k|b) = \frac{4}{3} \frac{M_d}{M_0} \frac{2\Gamma^2}{\Omega a^2} \left( \frac{x_1^2 + 1}{x_2} \right)^2 + \frac{3}{1 + x_1^2},$$

where $x_* = r/a$. By defining $k_* = a|k|$ and $\Gamma_* = \sqrt{4/3}\Gamma/\mu\Omega(a)$, we finally obtain

$$R|k_*| \exp(-|k_*|) = \Phi(x_*; \Gamma_*; R),$$

(37)

$$\Phi(x_*; \Gamma_*; R) = \frac{\Gamma_*^2}{R} \left( \frac{x_1^2 + 1}{x_2} \right)^2 + \frac{3R}{1 + x_2},$$

(38)

5.1 Numerical results

Global unstable modes are determined by numerically solving equation (38) for $k_*$ for a given value of $\Gamma_*$ and $R$ and applying a quantization condition to obtain growth rate. Note that the right-hand side of equation (38) blows up at $x_* = 0$ and $x_* \to \infty$; however, it is bounded from below and has a minimum at $x_* \approx 1$, the exact value depending on the precise value of $\Gamma_*$ and $R$. Since the left-hand side of this equation has a maximum value equal to unity at $k_* = 1$, the equation does not admit any solutions if $\min[\Phi(x_*; \Gamma_*; R)] > 1$. However, for $\min[\Phi(x_*; \Gamma_*; R)] < 1$, the equation admits two real roots, one each on either side of $k_* = 1$. We denote the roots with $k_* < 1$ as the long-wavelength branch and the one with $k_* > 1$ as the short-wavelength branch.

As noted above, the right-hand side of equation (38) is unbounded from above and blows up at small and large values of $x_*$. It is clear that the real roots exist only for a finite range of the radial coordinate, $x_1 < x_* < x_2$, where both $x_1$ and $x_2$ depend on the parameters $\Gamma_*$ and $R$ in a complicated manner. We shall assume that at these points the wave is reflected, and we therefore impose the Bohr–Sommerfeld quantization condition

$$\int_{x_1}^{x_2} k_* dx_* = \left( n + \frac{1}{2} \right) \pi,$$

(39)

where $\int k_* dx_*$ corresponds to smaller values of the growth rate. The two
The slow modes explored in Tremaine (2001) or Saini, Gulati & Sridhar (2009) can have wavelengths comparable to the size of the disc. If this were true in the present case, it would be pertinent to discuss whether the WKB approximation is applicable. However, it is not clear that this happens in our case. The short-wavelength branch has modes with quite short wavelengths, so let us consider the long-wavelength branch of our slow modes. As Fig. 4 shows, the modes with the greatest growth rates also have the smallest number of nodes. However, the distance between the turning points for these modes also decreases, so that the mode wavelength can be much smaller than the size of the disc. In this case WKB can be expected to be valid. However, this expectation should be verified by the construction of eigenfunctions to the full problem, which requires the solution of a linear integrodifferential equation.

6 CONCLUSIONS

The principal aim of this work is to present the simplest model that permits a largely analytical exploration of the \( m = 1 \) counter-rotating instability in a ‘hot’ nearly Keplerian disc of collisionless self-gravitating matter. To this end we have considered a two-component softened gravity disc, and performed a linearized WKB analysis of both local and global modes. We derive an analytical expression for local WKB waves for arbitrary \( m \), which turns out to be quartic in the frequency \( \omega \). Specializing to \( m = 1 \), we show that \( \omega \) is smaller than the (Keplerian) orbital frequency by the small quantity \( \varepsilon = M_d/M \) (the ratio of the disc mass to the mass of the central object); in other words, the \( m = 1 \) modes are slow modes. The dispersion relation now reduces to a quadratic equation in \( \omega \). Hence the criteria for stability, instability and overstability can be readily derived in simple analytical forms. For a one-component disc (which does not have any counter-rotation), the \( m = 1 \) modes are stable, consistent with the results of Tremaine (2001). Equal mass in the two counter-rotating components corresponds to the case of no net rotation. In this case we find that the local modes are purely unstable (i.e. not overstable), consistent with Araki (1987), Palmer & Papaloizou (1990), Sellwood & Merritt (1994), Lovelace et al. (1997), Touma (2002) and Tremaine (2005). However, the general case of arbitrary mass ratio in the two counter-rotating components corresponds to overstability; note that overstability is already evident in the results of Touma (2002). Then we show analytically that the discs must be unrealistically hot to avoid an overstability.

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