Robin Heat Semigroup and HWI Inequality on Manifolds with Boundary*

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Abstract

Let \( M \) be a complete connected Riemannian manifold with boundary \( \partial M \), \( Q \) a bounded continuous function on \( \partial M \), and \( L = \Delta + Z \) for a \( C^1 \)-vector field \( Z \) on \( M \). By using the reflecting diffusion process generated by \( L \) and its local time on the boundary, a probabilistic formula is presented for the semigroup generated by \( L \) on \( M \) with Robin boundary condition \( \langle N, \nabla f \rangle + Qf = 0 \), where \( N \) is the inward unit normal vector field of \( \partial M \). As an application, the HWI inequality is established on manifolds with (nonconvex) boundary. In order to study this semigroup, Hsu’s gradient estimate and the corresponding Bismut’s derivative formula are established on a class of noncompact manifolds with boundary.

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1 Introduction

Let \( M \) be a \( d \)-dimensional connected complete Riemannian manifold with boundary \( \partial M \) and \( L = \Delta + Z \) for some \( C^1 \)-vector field \( Z \) such that

\[
\text{Ric} - \nabla Z \geq -K
\]

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holds on $M$ for some constant $K \in \mathbb{R}$. This curvature condition is well known by Bakry and Emery \[1\].

Let $X_t$ be the reflecting diffusion process generated by $L$ on $M$, and let $l_t$ be its local time on the boundary $\partial M$. Let $\tau$ be the first hitting time of $X_t$ to $\partial M$. It is well known that the following heat equation can be described by using the process $X_t$:

\begin{equation}
\partial_t u = Lu, \quad u(0, \cdot) = f,
\end{equation}

where $f \in \mathcal{B}_b(M)$. With Dirichlet boundary condition $u|_{\partial M} = 0$ the solution can be formulated as

$$u(t, x) = \mathbb{E}^x[f(X_t)1_{\{t<\tau\}}]$$

while under the Neumann boundary condition $Nu|_{\partial M} = 0$ one has

$$u(t, x) = \mathbb{E}^x f(X_t),$$

where $\mathbb{E}^x$ is the expectation taking for the process $X_t$ starting at $x$. In this paper we shall provide the corresponding probability formula for the solution under the Robin boundary condition (cf. \[4\], page 102):

\begin{equation}
Nf := \langle N, \nabla f \rangle = -Qf \quad \text{on } \partial M,
\end{equation}

where $Q \in C_b(\partial M)$ and $N$ is the inward unit normal vector field on $\partial M$. It turns out that under a reasonable assumption the solution to \eqref{eq:1.2} under condition \eqref{eq:1.3} can be formulated by

\begin{equation}
(u(t, x) = P_t^Q f(x) := \mathbb{E}^x \left\{ f(X_t) e^{\int_0^t Q(X_s)\,dl_s} \right\}, \quad t \geq 0, x \in M.
\end{equation}

As soon as $P_t^Q$ is well defined, the semigroup property follows immediately from the Markov property of the reflecting diffusion process $X_t$. To ensure the boundedness of $P_t^Q$ under the uniform norm, it is natural to ask the local time $l_t$ to be exponentially integrable. According to calculations from \[13\] (see also the proof of Lemma 2.1 below), for this we shall need the following assumption.

\[ \text{(A)} \quad \text{The boundary } \partial M \text{ has a bounded second fundamental form and a strictly positive injectivity radius, the sectional curvature of } M \text{ is bounded above, and there exists } r > 0 \text{ such that } Z \text{ is bounded on the } r\text{-neighborhood of } \partial M. \]

Let $\rho_{\partial M}$ be the Riemannian distance to the boundary. Then the $r$-neighborhood of $\partial M$ is $\partial_r M := \{ x \in M : \rho_{\partial M}(x) < r \}$, where $\rho_{\partial M}$ is the Riemannian distance to the
boundary $\partial M$. Next, the injectivity radius $i_{\partial M}$ of $\partial M$ is the largest number $r$ such that the exponential map

$$[0,r) \times \partial M \ni (s,x) \mapsto \exp[sN_x] \in \partial_r M$$

is diffeomorphic. In particular, $\rho_{\partial M}$ is smooth on $\partial r M$ for $r \leq i_{\partial M}$.

Finally, to state our result, we introduce the following class of references functions:

$$\mathcal{D}_0 := \{ f \in C^\infty_0(M) : Nf + Qf = 0 \text{ on } \partial M \}.$$ 

**Theorem 1.1.** Assume (A) and (1.1) hold and let $Q \in C^b(\partial M)$.

1. $\{P_t^Q\}_{t \geq 0}$ is a positivity-preserving strong Feller semigroup of bounded linear operators on $\mathcal{B}(M)$, whose generator is $L$ with domain containing all functions $f \in C^2_0(M)$ such that $Nf + Qf = 0$ holds on $\partial M$.

2. Let $Z = \nabla V$ for some $V \in C^2(M)$ with $\mu(dx) := e^{V(x)}dx$ not necessarily finite. Then $\{P_t^Q\}_{t \geq 0}$ provides a bounded symmetric $C_0$-semigroup on $L^2(\mu)$. If in particular $Q \leq 0$, then $P_t^Q$ is sub-Markovian and the associated symmetric Dirichlet form $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ is the closure of $(\mathcal{E}, \mathcal{D}_0)$ with

$$\mathcal{E}(f,g) = \mu(\langle f,g \rangle) - \mu_0(Qfg), \quad f,g \in \mathcal{D}_0.$$ 

We note that a solution to the heat equation (1.2) under the Robin condition (1.3) can be represented by (1.4) provided it is bounded in $x \in M$. Indeed, by the boundary condition and the Itô formula, for fixed $t > 0$,

$$du(t-s, X_s) = dM_s + Nu(t-s, \cdot)(X_s) \, dl_s = dM_s - (Qu(t-s, \cdot))(X_s) \, dl_s$$

holds for some local martingale $M_s$ up to time $t$. So, $s \mapsto u(t-s, X_s) \exp[\int_0^s Q(X_r)dl_r]$ is a local martingale as well. Since $u$ is bounded and $l_t$ is exponential integrable due to Lemma 2.1 below, it is indeed a martingale. Thus, (1.4) holds.

Since the local time $l_t$ is not absolutely continuous in $t$, the semigroup $P_t^Q$ is essentially different from the well developed Schrödinger semigroup. According to Theorem 1.1 below $P_t^Q$ is generated by $L$ under the boundary condition $Nf + Qf = 0$, where $N$ is the inward unit normal vector field on $\partial M$. So, the formula (1.4) will be important in the study of this boundary value problem on $M$. In this paper we shall explain how can one apply this semigroup to the study of HWI inequality on manifolds with boundary. This inequality links three important quantities including the entropy, the energy and the Wasserstein distance (or the optimal transportation cost), and was found in [3, 2] on manifolds without boundary.

To study the HWI inequality, we consider the symmetric case that $Z = \nabla V$ for some $V \in C^2(M)$ such that $\mu(dx) = e^{V(x)}dx$ is a probability measure on $M$, where $dx$ is the Riemannian volume measure on $M$. Let $P_t$ be the semigroup of the reflecting diffusion
process generated by $L$ on $M$, which is then symmetric in $L^2(\mu)$. When $\partial M$ is convex (1.1) implies the following gradient estimate (cf. [9, 12]):

$$|\nabla P_t f| \leq e^{Kt} P_t |\nabla f|, \quad f \in C_b^1(M).$$

Combining this estimate and an argument of [2], we can easily obtain the following HWI inequality:

$$\mu(f^2 \log f^2) \leq 2 \sqrt{\mu(|\nabla f|^2)} W_2(f^2 \mu, \mu) + \frac{K}{2} W_2(f^2 \mu, \mu)^2, \quad \mu(f^2) = 1,$$

where $W_2$ is the $L^2$-Wasserstein distance induced by the Riemannian distance function $\rho$ on $M$. More precisely, for a probability measure $\nu$ on $M$ (note that we are using $\rho^2$ to replace $\frac{1}{2} \rho^2$ in [2])

$$W_2(\nu, \mu)^2 := \inf_{\pi \in \mathcal{C}(\nu, \mu)} \int_{M \times M} \rho(x, y)^2 \pi(dx, dy),$$

where $\mathcal{C}(\nu, \mu)$ is the class of all couplings of $\nu$ and $\mu$.

To see that $P_t^Q$ is important in the study of the HWI inequality on a nonconvex manifold, let us briefly introduce the main idea for the proof of (1.6) on a convex manifold using (1.5). Firstly, due to Bakry and Emery, (1.5) implies the semigroup log-Sobolev inequality

$$P_t(f^2 \log f^2) \leq \frac{2(e^{2Kt} - 1)}{K} P_t |\nabla f|^2.$$

Taking integration for both sides with respect to $\mu$ we arrive at

$$\mu(f^2 \log f^2) \leq \frac{2(e^{2Kt} - 1)}{K} \mu(|\nabla f|^2) + \mu((P_t f^2) \log P_t f^2)).$$

On the other hand, according to [2] Proof of Lemma 4.2 the gradient estimate (1.5) implies (again note that the $W_2^2$ here is twice of the one in [2])

$$\mu((P_t f^2) \log P_t f^2) \leq \frac{K e^{2Kt}}{2(e^{2Kt} - 1)} W_2(f^2 \mu, \mu)^2.$$

Combining this with (1.8) and minimizing in $t > 0$, one derives (1.6).

Now, what can we do for the nonconvex setting? According to [5], in this case the local time and the second fundamental form will be naturally involved in the upper bound.
of $|\nabla P_t f|$. Let the second fundamental form be bounded below by $-\sigma$ for some $\sigma \geq 0$, i.e.

\begin{equation}
\|X, X\| := -\langle \nabla_X N, X \rangle \geq -\sigma |X|^2, \quad X \in T\partial M.
\end{equation}

Recall that $N$ is the inward unit normal vector field on $\partial M$. According to [3, Theorem 5.1], if $M$ is compact and $V = 0$ then (1.1) and (1.10) imply

\begin{equation}
|\nabla P_t f|(x) \leq e^{Kt} \mathbb{E}^x \{|\nabla f|(X_t)e^{\sigma_l_t}\}, \quad x \in M, t \geq 0, f \in C^1_b(M),
\end{equation}

where $X_t$ is the reflecting $L$-diffusion process and $l_t$ is its local time on $\partial M$. In this paper we shall prove (1.11) for $Z \neq 0$ on noncompact manifolds under assumption (A), see Proposition 2.2 below.

Since the local time is unbounded, we are not able to derive from (1.11) the semigroup log-Sobolev inequality like (1.7). But Theorem 1.2 enables us to derive a log-Sobolev inequality of type (1.8) using (1.11), from which we can prove the following HWI inequality (1.12).

**Theorem 1.2.** Let $Z = \nabla V$ for some $V \in C^2(M)$ such that $\mu$ is a probability measure. Assume (A) and (1.1). Let $\| \geq -\sigma$ for some $\sigma \in \mathbb{R}$. Then

\begin{equation}
\eta_\lambda(s) := \sup_{x \in M} \mathbb{E}^x e^{\lambda s} < \infty, \quad s, \lambda \geq 0
\end{equation}

holds, and for any $t > 0$,

\begin{equation}
\mu(f^2 \log f^2) \leq 4 \left(\int_0^t e^{2Ks} \eta_\sigma(s) ds\right) \mu(|\nabla f|^2) + \frac{W_2(f^2 \mu, \mu)^2}{4 \int_0^t e^{-2Ks} \eta_\sigma(s)^{-1} ds}, \quad \mu(f^2) = 1.
\end{equation}

To derive an explicit HWI inequality, we shall estimate $\eta_\sigma$ as in [13] by using the Itô formula for $\phi \circ \rho_{\partial M}(X_t)$ with a specific choice of $\phi$ (see Lemma 2.1 below). From this we obtain the following consequence of Theorem 1.1 immediately. Let $\text{Sect}_M$ be the sectional curvature of $M$, and let

\begin{equation}
\delta_r(Z) := \sup_{\partial M} \langle Z, \nabla \rho_{\partial M} \rangle, \quad r > 0.
\end{equation}

**Corollary 1.3.** Let $Z = \nabla V$ for some $V \in C^2(M)$ such that $\mu$ is a probability measure. Assume (A) and (1.1). Let $r_0, \sigma, k, > 0$ be such that $\delta_{r_0}(Z) < \infty, -\sigma \leq \| \leq \gamma$ and $\text{Sect}_M \leq k$. For any

\begin{equation}
0 < r \leq \min \left\{i_{\partial M}, \frac{1}{\sqrt{k}} \arcsin \left(\frac{\sqrt{k}}{\sqrt{k + \gamma^2}}\right) \right\},
\end{equation}

5
and for $K_r := K + \frac{\sigma_d}{r} + \sigma \delta_r(Z) + 4\sigma^2$, the HWI inequality

$$
\mu(f^2 \log f^2) \leq 2e^{2\lambda d/r} \sqrt{\mu(|\nabla f|^2)} W_2(f^2 \mu, \mu) + \frac{K_r e^{2\lambda d/r}}{2} W^2(f^2 \mu, \mu), \quad \mu(f^2) = 1
$$

holds.

As preparations, in the next section we shall confirm the exponential integrability of $l_t$ and establish (1.11) on noncompact manifolds. The above two theorems are then proved in Sections 3 and 4 respectively. To prove the strong Feller property of $P_t^Q$ and for further applications in the literature, the Bismut type formula for $P_t$ on manifolds with boundary is addressed in Appendix at the end of the paper.

2 Exponential estimate and Hsu’s gradient estimate

As explained in Section 1, to ensure that $P_t^Q$ is well defined, we first study the exponential integrability of the local time.

**Lemma 2.1.** Let $r_0 > 0$ be such that $\delta_{r_0}(Z) < \infty$ and let $k, \gamma$ be in Corollary 1.3. Then

$$
\sup_{x \in M} E_x e^{\lambda t} \leq \exp \left[ \frac{\lambda dr}{2} + \left( \frac{\lambda d}{r} + \lambda \delta_r(Z) + 2\lambda^2 \right) t \right], \quad t \geq 0, \lambda \geq 0
$$

holds for any

$$
0 < r \leq \min \left\{ i \partial M, r_0, \frac{1}{\sqrt{k}} \arcsin \left( \frac{\sqrt{k}}{\sqrt{k} + \gamma^2} \right) \right\}.
$$

**Proof.** Let

$$
h(s) = \cos \left( \sqrt{k} s \right) - \frac{\gamma}{\sqrt{k}} \sin \left( \sqrt{k} s \right), \quad s \geq 0.
$$

Then $h$ is the unique solution to the equation

$$
h'' + kh = 0, \quad h(0) = 1, h'(0) = -\gamma.
$$

By the Laplacian comparison theorem for $\rho_{\partial M}$ (cf. [7, Theorem 0.3] or [14]),

$$
\Delta \rho_{\partial M} \geq \frac{(d-1)h'}{h}(\rho_{\partial M}), \quad \rho_{\partial M} < i_{\partial M} \wedge h^{-1}(0).
$$

Thus,

$$
(2.1) \quad L \rho_{\partial M} \geq \frac{(d-1)h'}{h}(\rho_{\partial M}) - \delta_r(Z), \quad \rho_{\partial M} \leq r.
$$
Now, let
\[
\alpha = (1 - h(r))^{1-d} \int_0^r (h(s) - h(r))^{d-1} ds,
\]
\[
\psi(s) = \frac{1}{\alpha} \int_0^s (h(t) - h(r))^{1-d} dt \int_{t \wedge r}^r (h(u) - h(r))^{d-1} du, \quad s \geq 0.
\]
We have \( \psi(0) = 0 \), \( 0 \leq \psi' \leq \psi'(0) = 1 \). Moreover, as observed in [13, Proof of Theorem 1.1],
\[
\alpha \geq \frac{r}{d}, \quad \psi(\infty) = \psi(r) \leq \frac{r^2}{2\alpha} \leq \frac{dr}{2}.
\]
Combining this with (2.1) we obtain (note that \( \psi'(s) = 0 \) for \( s \geq r \))
\[
L\psi \circ \rho_{\partial M} = \psi' \circ \rho_{\partial M} L\rho_{\partial M} + \psi'' \circ \rho_{\partial M} \geq -\frac{1}{\alpha} - \delta_r(Z) \geq -\frac{d}{r} - \delta_r(Z).
\]
On the other hand, since \( \psi'(0) = 1 \), by the Itô formula we have
\[
d\psi \circ \rho_{\partial M}(X_t) = \sqrt{2} \psi' \circ \rho_{\partial M}(X_t) dB_t + L\psi \circ \rho_{\partial M}(X_t) dt + dl_t,
\]
where \( B_t \) is the one-dimensional Brownian motion. Then it follows from (2.2) and (2.3) that (note that \( |\psi'| \leq 1 \))
\[
\mathbb{E} e^{\lambda t} = \mathbb{E} \exp \left[ \lambda \psi \circ \rho_{\partial M}(X_t) + \left( \frac{d\lambda}{r} + \lambda \delta_r(Z) \right) t \right] - \sqrt{2} \mathbb{E} \int_0^t \psi' \circ \rho_{\partial M}(X_s) dB_s
\]
\[
\leq \exp \left[ \frac{1}{2} \lambda dr + \left( \frac{d\lambda}{r} + \lambda \delta_r(Z) \right) t \right] \mathbb{E} \exp \left[ 4\lambda^2 \int_0^t (\psi' \circ \rho_{\partial M}(X_s))^2 ds \right]^{1/2}
\]
\[
\leq \exp \left[ \frac{1}{2} \lambda dr + \left( \frac{d\lambda}{r} + \lambda \delta_r(Z) + 2\lambda^2 \right) t \right].
\]
This Lemma ensures the boundedness of \( P_t^Q \) under the uniform norm. Next, we intend to prove (1.11) under assumption (A), which is known by [5] for compact \( M \) and \( Z = 0 \).

**Proposition 2.2.** Assume that (A). Let \( \kappa_1, \kappa_2 \in C_b(M) \) be such that
\[
\text{Ric} - \nabla Z \geq -\kappa_1, \quad \mathbb{I} \geq -\kappa_2
\]
hold on $M$ and $\partial M$ respectively. Then

\[(2.6) \quad |\nabla P_tf|(x) \leq \mathbb{E}^x \left\{ |\nabla f|(X_t) \exp \left[ \int_0^t \kappa_1(X_s)ds + \int_0^t \kappa_2(X_s)dl_s \right] \right\} \]

holds for all $f \in C^1_b(M), t > 0, x \in M$.

We first provide a simple proof of (2.6) under a further condition that $|\nabla P_tf|$ is bounded on $[0, T] \times M$ for any $T > 0$, then drop this assumption by an approximation argument. Since this condition is trivial for compact $M$, our proof below is much shorter than that in [5].

**Lemma 2.3.** Assume that $f \in C^1_b(M)$ such that $|\nabla P_tf|$ is bounded on $[0, T] \times M$ for any $T > 0$. Then (2.6) holds.

**Proof.** For any $\varepsilon > 0$, let

$$\zeta_s = \sqrt{\varepsilon + |\nabla P_{t-s}f|^2}(X_s), \quad s \leq t.$$

By the Itô formula we have

$$d\zeta_s = dM_s + \frac{L|\nabla P_{t-s}f|^2 - 2(\nabla LP_{t-s}f, \nabla P_{t-s}f)}{2\varepsilon + |\nabla P_{t-s}f|^2}(X_s)ds$$

$$- \frac{|\nabla|\nabla P_{t-s}f|^2|^2}{4(\varepsilon + |\nabla P_{t-s}f|^2)^{3/2}}(X_s)ds + \frac{N|\nabla P_{t-s}f|^2}{2\varepsilon + |\nabla P_{t-s}f|^2}(X_s)dl_s, \quad s \leq t,$$

where $M_s$ is a local martingale. Combining this with (2.5) and (see [8, (1.14)])

$$L|\nabla u|^2 - 2(\nabla Lu, \nabla u) \geq -2\kappa_1|\nabla u|^2 + \frac{|\nabla|\nabla u|^2|^2}{2|\nabla u|^2},$$

we obtain

$$d\zeta_s \geq dM_s - \frac{\kappa_1|\nabla P_{t-s}f|^2}{\varepsilon + |\nabla P_{t-s}f|^2}(X_s)\zeta_s ds - \frac{\kappa_2|\nabla P_{t-s}f|^2}{\varepsilon + |\nabla P_{t-s}f|^2}(X_s)\zeta_s dl_s, \quad s \leq t.$$

Since $\zeta_s$ is bounded on $[0, t], \kappa_1$ and $\kappa_2$ are bounded, and by Lemma 2.1 $\mathbb{E}e^{\lambda t} < \infty$ for all $\lambda > 0$, this implies that

$$[0, t] \ni s \mapsto \zeta_s \exp \left[ \int_0^s \frac{\kappa_1|\nabla P_{t-r}f|^2}{\varepsilon + |\nabla P_{t-r}f|^2}(X_r)dr + \int_0^s \frac{\kappa_2|\nabla P_{t-r}f|^2}{\varepsilon + |\nabla P_{t-r}f|^2}(X_r)dl_r \right]$$

is a submartingale for any $\varepsilon > 0$. Letting $\varepsilon \downarrow 0$ we conclude that

$$[0, t] \ni s \mapsto |\nabla P_{t-s}f|(X_s) \exp \left[ \int_0^s \kappa_1(X_r)dr + \int_0^s \kappa_2(X_r)dl_r \right]$$

is a submartingale as well. This completes the proof. \qed
By Lemma 2.3 to prove Proposition 2.2 it suffices to confirm the boundedness of \(|\nabla P \cdot f|\) on \([0, T] \times M\) for \(f \in C^1(M)\). Below we first consider \(f \in C^\infty(M)\) satisfying the Neumann boundary condition.

Lemma 2.4. Assume (A). If (1.1) holds then for any \(T > 0\) and \(f \in C^\infty(M)\) such that \(Nf|_{\partial M} = 0\), \(|\nabla P \cdot f|\) is bounded on \([0, T] \times M\).

Proof. We shall take a conformal change of metric as in [14] to make the boundary convex, so that the known estimates for the convex case can be applied. As explained on page 1436 in [14], under assumption (A) there exists \(\phi \in C^\infty(M)\) and a constant \(R > 1\) such that \(1 \leq \phi \leq R, |\nabla \phi| \leq R, N \log \phi|_{\partial M} \geq \sigma\), and \(\nabla \phi = 0\) outside \(\partial \, r\, M\). Since \(\bar{I} \geq -\sigma\), by [14, Lemma 2.1] \(\partial M\) is convex under the new metric \(\langle \cdot, \cdot \rangle = \bar{\phi} \cdot \cdot \cdot \).

Let \(\Delta', \nabla', \text{Ric}'\) be corresponding to the new metric. By [14, Lemma 2.2] \(L' := \phi^2 L = \Delta' + (d - 2) \phi \nabla \phi + \phi^2 Z =: \Delta' + Z'\).

As in [15] we shall now calculate the curvature tensor \(\text{Ric}' - \nabla' Z'\) under the new metric. By [14, (9)], for any unit vector \(U \in T M\), \(U' := \phi U\) is unit under the new metric, and the corresponding Ricci curvature satisfies

\[
(2.8) \quad \text{Ric}'(U', U') \geq \phi^2 \text{Ric}(U, U) + \phi \Delta \phi - (d - 3)|\nabla \phi|^2 - 2(U \phi)^2 + (d - 2) \phi \text{Hess}_\phi(U, U).
\]

Noting that

\[
\nabla'_X Y = \nabla X Y - \langle X, \nabla \log \phi \rangle Y - \langle Y, \nabla \log \phi \rangle X + \langle X, Y \rangle \nabla \log \phi, \quad X, Y \in T M,
\]

we have

\[
\langle \nabla'_{U'} Z', U' \rangle = \langle \nabla U Z', U \rangle - \langle Z', \nabla \log \phi \rangle = \phi^2 \langle \nabla U Z, U \rangle + (U \phi)^2 \langle Z, U \rangle + (d - 2)(U \phi)^2 + (d - 2) \phi \text{Hess}_\phi(U, U) - \langle Z', \nabla \log \phi \rangle.
\]

Combining this with (2.8), (1.1), \(\|Z\|_r < \infty\) and the properties of \(\phi\) mentioned above, we find a constant \(K' \geq 0\) such that

\[
\text{Ric}'(U', U') - \langle \nabla'_{U'} Z', U' \rangle \geq -K', \quad \langle U', U' \rangle = 1.
\]

For any \(x, y \in M\), let \((X'_t, Y'_t)\) be the coupling by parallel displacement of the reflecting diffusion processes generated by \(L'\) with \((X'_0, Y'_0) = (x, y)\). Let \(\rho'\) be the Riemannian distance induced by \(\langle \cdot, \cdot \rangle'.\) Since \((M, \langle \cdot, \cdot \rangle')\) is convex, we have (see [12, (3.2)])
\[ \rho'(X'_t, Y'_t) \leq e^{tK't} \rho(x, y), \quad t \geq 0. \]

Since \( 1 \leq \phi \leq R \), we have \( R^{-1} \rho \leq \rho' \leq \rho \) so that

(2.9) \[ \rho(X'_t, Y'_t) \leq Re^{K't} \rho(x, y), \quad t \geq 0. \]

To derive the gradient estimate of \( P_t \), we shall make time changes

\[ \xi_x(t) = \int_0^t \phi^2(X'_s)ds, \quad \xi_y(t) = \int_0^t \phi^2(Y'_s)ds. \]

Since \( L' = \phi^2 L \), we see that \( X_t := X'_{\xi_x(t)} \) and \( Y_t := Y'_{\xi_y(t)} \) are generated by \( L \) with reflecting boundary. Again by \( 1 \leq \phi \leq R \) we have

\[ R^{-2}t \leq \xi_x^{-1}(t), \xi_y^{-1}(t) \leq t, \quad t \geq 0. \]

Combining this with \( |\nabla \phi| \leq R, 1 \leq \phi \leq R \) and (2.9) we arrive at

\[ |\xi_x^{-1}(t) - \xi_y^{-1}(t)| \leq \int_{\xi_x^{-1}(t) \wedge \xi_y^{-1}(t)}^t \phi^2 (Y'_s)ds = |\xi_y \circ \xi_y^{-1}(t) - \xi_y \circ \xi_x^{-1}(t)| \]

(2.10)

\[ = |\xi_x \circ \xi_x^{-1}(t) - \xi_y \circ \xi_x^{-1}(t)| \leq \int_0^{\xi_x^{-1}(t)} |\phi^2 (X'_s) - \phi^2 (Y'_s)|ds \]

\[ \leq 2R^2 \rho(x, y) \int_0^t e^{K's}ds \leq 2te^{K't}R^2 \rho(x, y). \]

Therefore,

(2.11)

\[ |P_t f(x) - P_t f(y)| = |\mathbb{E}\{f(X'_{\xi_x^{-1}(t)}(t)) - f(Y'_{\xi_y^{-1}(t)}(t))\}| \]

\[ \leq \mathbb{E}|f(X'_{\xi_x^{-1}(t)}(t)) - f(Y'_{\xi_y^{-1}(t)}(t))| + |\mathbb{E}\{f(X'_{\xi_x^{-1}(t)}(t)) - f(X'_{\xi_y^{-1}(t)}(t))\}| =: I_1 + I_2. \]

By (2.9) and \( \xi_y^{-1}(t) \leq t \) we obtain

(2.12)

\[ I_1 \leq \|\nabla f\|_{\infty} e^{K't} R \rho(x, y). \]

Moreover, since \( f \in C^\infty_0(M) \) with \( Nf|_{\partial M} = 0 \), it follows from the Itô formula and (2.10) that

\[ I_2 \leq \mathbb{E}\left[ \int_{\xi_x^{-1}(t) \wedge \xi_y^{-1}(t)} L' f(X'_s)ds \right] \leq \|L' f\|_{\infty} \mathbb{E}|\xi_x^{-1}(t) - \xi_y^{-1}(t)| \leq c_1 te^{K't} \rho(x, y). \]
holds for some constant $c_1 > 0$. Combining this with (2.11) and (2.12) we conclude that
\[ \|\nabla P_t f\|_\infty \leq c_2 (1 + t) e^{K't}, \quad t \geq 0 \]
for some constant $c_2 > 0$. \qed

**Proof of Proposition 2.2.** Let $f \in C^1_b(M)$. By Lemma 2.3 we only have to prove the boundedness of $|\nabla P_t f|$ on $[0, T] \times M$.

(a) Let $f \in C^\infty_0(M)$. In this case there exist a sequence of functions $\{f_n\}_{n \geq 1} \subset C^\infty_0(M)$ such that $Nf_n|_{\partial M} = 0$, $f_n \to f$ uniformly as $n \to \infty$, and $\|\nabla f_n\|_\infty \leq 1 + \|\nabla f\|_\infty$ holds for any $n \geq 1$, see e.g. [11]. By Lemmas 2.3 and 2.4, (2.6) holds for $f_n$ in place of $f$ so that Lemma 2.1 implies
\[ |P_t f_n(x) - P_t f_n(y)| \rho(x, y) \leq C, \quad t \leq T, n \geq 1, x \neq y \]
for some constant $C > 0$. Letting first $n \to 0$ then $y \to x$, we conclude that $|\nabla P_t f|$ is bounded on $[0, T] \times M$.

(b) Let $f \in C^\infty_b(M)$. Let $\{g_n\}_{n \geq 1} \subset C^\infty_b(M)$ be such that $0 \leq g_n \leq 1$, $|\nabla g_n| \leq 2$ and $g_n \uparrow 1$ as $n \uparrow \infty$. By (a) and Lemma 2.3 we may apply (2.6) to $g_n f$ in place of $f$ such that Lemma 2.1 implies
\[ |P_t (g_n f)(x) - P_t (g_n f)(y)| \rho(x, y) \leq C, \quad t \leq T, n \geq 1, x \neq y \]
holds for some constant $C > 0$. By the same reason as in (a) we conclude that $|\nabla P_t f|$ is bounded on $[0, T] \times M$.

(c) Finally, for $f \in C^1_b(M)$ there exist $\{f_n\}_{n \geq 1} \subset C^\infty_b(M)$ such that $f_n \to f$ uniformly as $n \to \infty$ and $\|\nabla f_n\|_\infty \leq \|\nabla f\|_\infty + 1$ for any $n \geq 1$. Therefore, the proof is complete by the same reason as in (a) and (b). \qed

### 3 Proof of Theorem 1.1

The boundedness of $P_t^Q$ under the uniform norm is ensured by Lemma 2.1. Since a bounded continuous function can be uniformly approximated by bounded smooth functions, due to Lemma 2.1 we may and do assume that $Q \in C^\infty_b(\partial M)$. To handle the integral $\int_0^t Q(X_s) dl_s$, we shall also need the upper bound of $L\rho_{\partial M}$.

**Lemma 3.1.** Let $\bar{I} \geq -\sigma$ and (1.1) hold. Then
\[ L\rho_{\partial M} \leq (d - 1)\sigma + \sup_{\partial M}(Z, N) + K\rho_{\partial M}, \quad \rho_{\partial M} < i_{\partial M}. \]
Proof. Let $x \in M$ such that $\rho_{\partial M}(x) < i_{\partial M}$. Then there exist a unique $x_0 \in \partial M$ and the minimal geodesic $x : [0, \rho_{\partial M}(x)] \to M$ linking $\partial M$ and $x$. By (1.1) we have

$$\text{Ric}(\dot{x}_s, \dot{x}_s) \geq \langle \nabla \dot{x}_s Z, \dot{x}_s \rangle - K =: R(s).$$

Let $h$ solve the equation

$$h''(s) + \frac{R(s)}{d-1}h(s) = 0, \quad h(0) = 1, h'(0) = \sigma.$$ 

By the Laplacian comparison theorem (see [6, Theorem 1]),

$$\Delta \rho_{\partial M}(x) \leq \frac{(d-1)h'}{h}(\rho_{\partial M}(x)) = (d-1)\sigma + (d-1) \int_0^{\rho_{\partial M}(x)} \left( \frac{h'}{h} \right)'(s)ds$$

$$= (d-1)\sigma - (d-1) \int_0^{\rho_{\partial M}(x)} \left( \frac{R(s)}{d-1} + \frac{(h')^2}{h^2}(s) \right)ds$$

$$\leq (d-1)\sigma + K \rho_{\partial M}(x) - \int_0^{\rho_{\partial M}(x)} \langle \nabla \dot{x}_s Z, \dot{x}_s \rangle ds.$$

Then the proof is completed by noting that

$$Z \rho_{\partial M}(x) = \langle Z, N \rangle(x_0) + \int_0^{\rho_{\partial M}(x)} \langle \nabla \dot{x}_s Z, \dot{x}_s \rangle ds.$$

3.1 The strong Feller property

As explained after Theorem 1.1, when $M$ is compact the solution to (1.2) is bounded in $x \in M$, so that (1.3) holds. In particular, $P_t^Qf$ is differentiable for $f \in \mathcal{B}_b(M)$ and thus, $P_t^Q$ is strong Feller. When $M$ is noncompact, this argument does not apply due to the lack of boundedness of $u(t, \cdot)$. Below we provide a different proof for the strong Feller property.

a) We first prove the Feller property. Since by Lemma 2.1 for $f \in C_b(M)$ the function $P_t^Qf$ is bounded, it suffices to show that

$$\lim_{y \to x} P_t^Q f(y) = P_t^Q f(x), \quad x \in M.$$ 

For any $y \in M$, let $(X_s, Y_s)$ be the coupling constructed in the proof of Lemma 2.4 via time changes. We shall first prove

$$\lim_{y \to x} \max_{s \in [0, t]} \rho(X_s, Y_s) = 0.$$
Using the notations in the proof of Lemma 2.4 and adopting (2.9) and (2.10), there exists a constant $c(t) > 0$ such that for any $s \in [0, t]$

$$
\rho(X_s, Y_s) = \rho(X'_{\xi_x(s)}', Y'_{\xi_x(s)}) \leq \rho(X'_{\xi_x(s)}', Y'_{\xi_x(s)}) + \rho(X'_{\xi_y(s)}, X'_{\xi_y(s)}) \\
\leq c(t)\rho(x, y) + \sup \{\rho(X'_{s_1}, X'_{s_2}) : s_1, s_2 \in [0, t + c(t)\rho(x, y)], |s_1 - s_2| \leq c(t)\rho(x, y)\}.
$$

By the continuity of the reflecting diffusion process we prove (3.2).

Next, to describe $\int_0^T Q(X_s)\,ds$ we shall apply the Itô formula to a proper reference function of $X_s$. To this end, we first extend $Q$ to a smooth function on $M$. By assumption (A), one may find a function $\tilde{Q} \in C^\infty(M)$ such that $\tilde{Q}|_{\partial M} = Q, N\tilde{Q}|_{\partial M} = 0$ and $|\nabla \tilde{Q}| + |L\tilde{Q}|$ is bounded. This can be realized by using the polar coordinates

$$
\partial M \times [0, r) \ni (\theta, s) \mapsto \exp[sN\theta]
$$

for small enough $r > 0$ such that $\rho_0$ is smooth on $\partial M$. From this one may take $\tilde{Q}(\theta, s) = Q(\theta)h(s)$ on $\partial M$ for some $h \in C^\infty([0, \infty)$ such that $h(0) = 1, h'(0) = 0$ and $h(s) = 0$ for $s \geq r$, and let $\tilde{Q} = 0$ outside $\partial M$. This $\tilde{Q}$ meets our requirements since $L\rho_0$ is bounded on $\partial M$ according to (2.11) and Lemma 3.1.

Let $\Phi \in C^\infty_0([0, \infty))$ be such that $0 \leq \Phi \leq 1, \Phi(s) = 1$ for $s \in [0, 1]$ and $\Phi(s) = 0$ for $s \geq 2$. Let

$$
\psi_n = \frac{1}{n} \int_0^{\rho_{\partial M}/n} \Phi(s)\,ds.
$$

Then $0 \leq \psi_n \leq 2n^{-1}, \psi_n = \rho_{\partial M}$ for $\rho_{\partial M} \leq n^{-1}, \psi_n$ is constant for $\rho_{\partial M} \geq 2n^{-1}$ and $|\nabla \psi_n| \leq 1$. Moreover, $\psi_n \in C^\infty(M)$ for large $n$. Since $\nabla \psi_n = N$ and $N\tilde{Q} = 0$ on $\partial M$, by the Itô formula we have

$$
\langle \tilde{Q}\psi_n \rangle(X_t) = M_n(t) + \int_0^t L(\psi_n \tilde{Q})(X_s)\,ds + \int_0^t Q(X_s)\,dl_s,
$$

where $M_n(t)$ is a martingale with

$$
\langle M_n(t) \rangle = \int_0^t |\nabla (\tilde{Q}\psi_n)|^2(X_s)\,ds.
$$

Note that $L(\tilde{Q}\psi_n)$ is bounded since so is $|\nabla \tilde{Q}| + |L\tilde{Q}| + 1_{\partial M}|L\rho_{\partial M}|$. Similarly, let $l^y_s$ be the local time of $Y_s$ on $\partial M$, we have

$$
\langle \tilde{Q}\psi_n \rangle(Y_t) = M^y_n(t) + \int_0^t L(\psi_n \tilde{Q})(Y_s)\,ds + \int_0^t Q(Y_s)\,dl^y_s
$$

for some martingale $M^y_n(t)$ with
\[(M_n(t)) = \int_0^t |\nabla (\bar{Q} \psi_n)|^2(Y_s) ds.\]

Combining these with (3.2) and using the dominated convergence theorem, we obtain

\[
P_t^Q f(x) = \lim_{y \to x} \mathbb{E} \{ f(Y_t) e^{(\bar{Q} \psi_n)(Y_t) - \int_0^t L(\bar{Q} \psi_n)(Y_s) ds - M_n(t)} \}
\]

and

\[
I_n(y) := \left| P_t^Q f(y) - \mathbb{E} \{ f(Y_t) e^{(\bar{Q} \psi_n)(Y_t) - \int_0^t L(\bar{Q} \psi_n)(Y_s) ds - M_n(t)} \} \right|
\]

for some constant \(c_1 > 0\). Since by the construction of \(\psi_n\), (3.3) and (3.4) we conclude that \(M_n(t)\) and \(M_n(t)\) are exponentially integrable uniformly in \(y\) and

\[
(M_n(t)) + (M_n(t)) \leq c_2 \int_0^t \{ 1_{\partial_{2/n} M(X_s)} + 1_{\partial_{2/n} M(Y_s)} \} ds
\]

holds for some constant \(c_2 > 0\), it follows from (3.2) and (3.6) that

\[
\lim_{n \to \infty} \lim_{y \to x} I_n(y) = 0.
\]

Combining this with (3.5) we derive (3.1).

b) Let \(f \in \mathcal{B}_b(M)\). By Remark A.1 in the Appendix, the Neumann semigroup is strong Feller. So, \(f_\varepsilon := P_\varepsilon f \in C_b(M)\) for any \(\varepsilon > 0\). Combining this with the Feller property of \(P_t^Q\), it suffices to prove

\[
\lim_{\varepsilon \to 0} \| P_t^Q f - P_t^Q f_\varepsilon \|_\infty = 0.
\]

Since \(Q\) is bounded and \(l_s\) is continuous in \(s\) according to (2.4) and the continuity of \(X_s\), Lemma 2.1 implies that

\[
\lim_{\varepsilon \to 0} \sup_{x \in M} \mathbb{E}^x \left| f(X_t) e^{\int_0^t Q(X_s) ds} (1 - e^{\int_{t-\varepsilon}^t Q(X_s) ds}) \right| = 0.
\]

Next, let \(\{\mathcal{F}_s\}_{s \geq 0}\) be the natural filtration of \(X_s\). By the Markov property we have
\[ P^Q_{t-\varepsilon}f = \mathbb{E}\{(P_{\varepsilon}f)(X_{t-\varepsilon})e^{I^t_{0-\varepsilon}Q(X_s)dl_s}\} \]
\[ = \mathbb{E}\{e^{I^t_{0-\varepsilon}Q(X_s)dl_s}\mathbb{E}(f(X_t)|\mathcal{F}_{t-\varepsilon})\} \]
\[ = \mathbb{E}\{f(X_t)e^{I^t_{0-\varepsilon}Q(X_s)dl_s}\}. \]

Combining this with (3.8) we prove (3.7).

3.2 The generator

We first prove that

\[ \lim_{t \to 0} \|P^Q_f - f\|_\infty = 0, \quad f \in C_0(M). \]

Since a function in \( C_0(M) \) can be uniformly approximated by functions in \( C_0^\infty(M) \) satisfying the Neumann boundary condition, we may assume that \( f \in C_0^\infty(M) \) with \( Nf|_{\partial M} = 0 \). By the Itô formula we have

\[ \|P_t f - f\|_\infty \leq \sup_{M} \int_0^t P_s|Lf|ds \leq t\|Lf\|_\infty \]

which goes to zero as \( t \to 0 \). Noting that

\[ |P^Q_t f - f| \leq |P_t f - f| + |P^Q_t f - P_t f| \leq |P_t f - f| + \|f\|_\infty \sup_{x \in M} \mathbb{E}^x(e^{\|Q\|_\infty l_t} - 1), \]

by Lemma 2.1 for \( \lambda = \|Q\|_\infty \) and letting first \( t \to 0 \) then \( r \to 0 \), we obtain (3.9).

Next, let \( f \in C_0^2(M) \) satisfy the boundary condition \( Nf + Qf = 0 \). We intend to prove that

\[ \lim_{t \to 0} \left\| \frac{P^Q_t f - f}{t} - Lf \right\|_\infty = 0. \]

By the Itô formula and the boundary condition, we have

\[ d\{f(X_t)e^{I^t_{0}Q(X_s)dl_s}\} = dM_t + (Lf(X_t))e^{I^t_{0}Q(X_s)dl_s}dt + \left\{ Nf(X_t) + (Qf)(X_t) \right\}e^{I^t_{0}Q(X_s)dl_s}dl_t \]
\[ = dM_t + (Lf(X_t))e^{I^t_{0}Q(X_s)dl_s}dt \]

for some martingale \( M_t \). This implies
\[ P_t^Q f(x) = f(x) + \int_0^t \mathbb{E}_x \{ (L f(X_s)) e^{\int_0^s Q(X_r) dr} \} ds. \]

So,
\[
\left\| \frac{P_t^Q f - f}{t} - L f \right\|_{\infty} \leq \frac{1}{t} \int_0^t \| P_s^Q (L f) - L f \|_{\infty} ds,
\]
and hence (3.10) follows from (3.9) since \( L f \in C_0(M) \).

### 3.3 The symmetry and \( C_0 \) property

Let \( Z = \nabla V \). Since by Lemma 2.1
\[(P_t^Q f)^2 \leq (P_t f^2) e^{2t\|Q\|_\infty} \leq c(t) P_t f^2\]
holds for some constant \( c(t) > 0 \), the boundedness of \( P_t^Q \) in \( L^2(\mu) \) follows from the fact that \( \mu \) is \( P_t \)-invariant. Moreover, since
\[
|P_t^Q f - f|^2 \leq 2|P_t^Q f - P_t f|^2 + 2|P_t f - f|^2 \\
\leq 2|P_t f - f|^2 + 2(P_t f^2) \sup_{x \in M} \mathbb{E}_x (e^{\|Q\|_\infty t} - 1)^2,
\]
by Lemma 2.1 and the strong continuity of \( P_t \) in \( L^2(\mu) \), we conclude that \( P_t^Q \) is strongly continuous in \( L^2(\mu) \) as well. So, it remains to prove that for any \( f, g \in C_0(M) \)

\[
\mu(g P_t^Q f) = \mu(f P_t^Q g).
\]

We shall prove (3.11) by using symmetric Schrödinger semigroups to approximate \( P_t^Q \).

Let \( \tilde{Q} \) and \( \psi_n \) be constructed above. We have

\[
\mathbb{E}_x^z [f(X_t) e^{\int_0^t Q(X_s) ds}] = \mathbb{E}_x^z [f(X_t) e^{-\int_0^t L(\tilde{Q} \psi_n)(X_s) ds}] + \varepsilon_n,
\]
where
\[
\varepsilon_n := \mathbb{E}_x^z [f(X_t) e^{-\int_0^t Q(X_s) ds} (1 - e^{-M_n(t) - (Q \psi_n)(X_t)})],
\]
which goes to zero uniformly in \( x \) as \( n \to \infty \) according to Lemma 2.1 and the properties of \( \tilde{Q} \) and \( \psi_n \). Let \( P_t^{(n)} \) be the Schrödinger semigroup generated by

\[
L_n := L - L(\tilde{Q} \psi_n).
\]

Since \( L(\tilde{Q} \psi_n) \) is bounded, by the Feynman-Kac formula

\[
P_t^{(n)} f(x) = \mathbb{E}_x^z [f(X_t) e^{-\int_0^t L(\tilde{Q} \psi_n)(X_s) ds}] \]
and $P^{(n)}_t$ is symmetric in $L^2(\mu)$. So,

$$\int_M g(x)E^x[f(X_t)e^{-\int_0^t L(\tilde{Q}_\psi)(X_s)ds}]\mu(dx) = \int_M g(x)P^{(n)}_t f(x)\mu(dx)$$

$$= \int_M f(x)P^{(n)}_t g(x)\mu(dx) = \int_M f(x)E^x[g(X_t)e^{-\int_0^t L(\tilde{Q}_\psi)(X_s)ds}]\mu(dx)$$

$$= \int_M f(x)E^x[g(X_t)e^{\int_0^t L(\tilde{Q}_\psi)(X_s)ds}]\mu(dx).$$

(3.13)

Obviously, $M_n(t) - (\tilde{Q}_\psi)(X_t) \to 0$ a.s. as $n \to \infty$ and is exponentially integrable uniformly in $x$. So, by Lemma 2.1, (3.12), (3.13) and using the dominated convergence theorem we arrive at

$$\mu(gP^Q_t f) = \lim_{n \to \infty} \left\{ \int_M f(x)E^x[g(X_t)e^{\int_0^t Q(X_s)ds + M_n(t) - (\tilde{Q}_\psi)(X_t)]} \mu(dx) + \varepsilon_n \right\} = \mu(fP^Q_t g).$$

3.4 The Dirichlet form

Again let $Z = \nabla V$ for $V \in C^2(M)$. Let $Q \leq 0$ such that $\mathcal{E} \geq 0$. Since by (1.3) and the integration by parts formula, we have

$$\mathcal{E}(f, g) = -\int_M fLg d\mu, \quad f, g \in \mathcal{D}_0,$$

the form $\mathcal{E}, \mathcal{D}_0$ is closable. Moreover, as in §3.2 for $f \in \mathcal{D}_0$ one has $P^Q_t f \to Lf$ in $L^2(\mu)$ as $t \to 0$, it remains to show that $(\mathcal{E}, \mathcal{D}_0)$ is a pre-Dirichlet form in $L^2(\mu)$. Firstly, to understand that $(\mathcal{E}, \mathcal{D}_0)$ is well defined in $L^2(\mu)$, i.e. $\mathcal{E}(f, g)$ is independent of $\mu$-versions of $f$ and $g$, for a bounded continuous extension $\hat{Q}$ of $Q$ on $M$, we rewrite

$$\mu_\theta(Qf g) = \lim_{r \to 0} \frac{\mu(\hat{Q}f g \mathbb{1}_{\partial_r M})}{r}.$$

Since $Q \leq 0$ implies the nonnegativity and the normal contraction property of $\mathcal{E}$, it remains to show that $\mathcal{D}_0$ is dense in $L^2(\mu)$.

It is well known that the class of functions in $C^\infty_0(M)$ satisfying the Neumann boundary condition is dense in $L^2(\mu)$, it suffices to prove that for any $f \in C^\infty_0(M)$ with $Nf|_{\partial M} = 0$, there exists a sequence $\{f_n\} \subset \mathcal{D}_0$ such that $\mu(|f_n - f|^2) \to 0$ as $n \to \infty$. To this end, for any $\varepsilon > 0$, let $h_\varepsilon \in C^\infty_0([0, \infty))$ such that $h_\varepsilon(0) = 0, h'_\varepsilon(0) = 1, h_\varepsilon(s) = 0$ for $s \geq r_0$ and $\|h_\varepsilon\| \leq \varepsilon/\|Q\|_\infty$. Here $r_0 > 0$ is such that $\rho_{\partial M}$ is smooth on $\partial_r M$. Then $\psi_\varepsilon(\theta, s) := 1 + Q(\theta)h_\varepsilon(s)$ defined under the polar coordinates

$$\partial M \times [0, r_0) \ni (\theta, s) \mapsto \partial_r M$$
is smooth and can be naturally extended smoothly on \( M \) by letting \( \psi_{\varepsilon} = 1 \) on \( M \setminus \partial r_0 M \). Obviously, we have \( \psi_{\varepsilon}|_{\partial M} = 1, N\psi_{\varepsilon}|_{\partial M} = Q \) and \( |\psi_{\varepsilon} - 1| \leq \varepsilon \). Thus, \( \psi_{\varepsilon} f \in \mathcal{D}_0 \) and \( \psi_{\varepsilon} f \to f \) in \( L^2(\mu) \) as \( \varepsilon \to 0 \).

4 Proof of Theorem 1.2

By Lemma 2.1, it remains to verify (1.12). Let \( f \in C^1_b(M) \) and \( t > 0 \). We have

\[
\frac{d}{ds} P_s \{(P_{t-s}f)^2 \log P_{t-s}f^2\} = P_s \frac{|\nabla P_{t-s}f|^2}{P_{t-s}f^2}, \quad s \in [0, t].
\]

By (1.11) and the Schwartz inequality we have

\[
\frac{|\nabla P_{t-s}f|^2}{P_{t-s}f^2}(y) \leq 4e^{2K(t-s)} \mathbb{E} \{ |\nabla f|^2 (X_{t-s}) e^{\sigma l_{t-s}} \} = 4e^{2K(t-s)} g_s(y), \quad s \in [0, t], y \in M.
\]

Combining this with (4.1) we obtain

\[
P_t (f^2 \log f^2) \leq (P_t f^2) \log P_t f^2 + 4 \int_0^t e^{2K(t-s)} P_s g_s ds.
\]

Since \( \mu \) is an invariant measure of \( P_t \), taking integral for both sides with respect to \( \mu \) we arrive at

\[
\mu(f^2 \log f^2) \leq \mu((P_t f^2) \log P_t f^2) + 4 \int_0^t e^{2K(t-s)} \mu(g_s) ds.
\]

Since by Theorem 1.1 \( P_{t-s}^Q \) for \( Q = 2\sigma \) is symmetric in \( L^2(\mu) \), we have

\[
\mu(g_s) = \int_M |\nabla f|^2(x) \mathbb{P}^x e^{2\sigma l_{t-s}} \mu(dx) \leq \mu(|\nabla f|^2) \eta_{2\sigma}(t-s),
\]

it follows that

\[
\mu(f^2 \log f^2) \leq \mu((P_t f^2) \log P_t f^2) + 4\mu(|\nabla f|^2) \int_0^t e^{2K_s \eta_{2\sigma}}(s) ds.
\]

This is an extension of (1.8) to the nonconvex case.

On the other hand, we intend to establish an analogous to (1.9) for the present situation. For any \( x, y \in M \), let \( x : [0, 1] \to M \) be the minimal curve linking \( x \) and \( y \) with constant speed. We have \( |\dot{x}_s| = \rho(x, y) \). Let \( h \in C^1([0, t]) \) be such that \( h_0 = 1, h_t = 0 \). Then by (1.11) which follows from Proposition 2.2 we have
\[ P_t \log f^2(x) - \log P_t f^2(y) = \int_0^t \frac{d}{ds} P_s(\log P_{t-s} f^2)(x_{h_{t-s}})ds \]

\begin{align*}
&\leq \int_0^t \left\{ |h_{t-s}| \rho(x,y) \frac{|\nabla P_{t-s} f^2|}{P_{t-s} f^2} (X_s)e^{K(t-s)+\sigma_{t-s}} - \frac{|\nabla P_{t-s} f^2|^2}{(P_{t-s} f^2)^2} (X_s) \right\} ds \\
&\leq \frac{\rho(x,y)^2}{4} \int_0^t h_s^2 e^{2K_s \eta_{2\sigma}(s)} ds =: c(t) \rho(x,y)^2.
\end{align*}

(4.3)

Now, let \( \mu(f^2) = 1 \) and \( \pi \in \mathcal{C}(f^2 \mu, \mu) \) be the optimal coupling for \( W_2(f^2 \mu, \mu) \). It follows from the symmetry of \( P_t \) and (4.3) that

\[ \mu((P_t f^2) \log f^2) = \mu(f^2 P_t \log f^2) = \int_{M \times M} P_t(\log f^2)(x) \pi(dx,dy) \]

\[ \leq \int_{M \times M} \log P_{2t} f^2(y) + c(t) \rho(x,y)^2 \pi(dx,dy) \]

\[ = \mu(\log P_{2t} f^2) + c(t) W_2(f^2 \mu, \mu)^2 \leq c(t) W_2(f^2 \mu, \mu)^2, \]

where in the last step we have used the Jensen inequality that

\[ \mu(\log P_{2t} f^2) \leq \log \mu(P_{2t} f^2) = 0. \]

Combining this with (4.2) we obtain

\[ \mu(f^2 \log f^2) \leq 4\mu(|\nabla f|^2) \int_0^t e^{2K_s \eta_{2\sigma}(s)} ds + \frac{W_2(f^2 \mu, \mu)^2}{4} \int_0^t h_s^2 e^{2K_s \eta_{2\sigma}(s)} ds. \]

Then the proof is completed by taking

\[ h_s = \frac{\int_s^t e^{-2\kappa u \eta_{2\sigma}(u)} \frac{du}{s}}{\int_0^t e^{-2\kappa u \eta_{2\sigma}(u)} \frac{du}{s}}, \quad s \in [0,t]. \]

5 Appendix: the Bismut formula

By using a formula for the gradient of \( P_t \) derived in [5], one obtains the following Bismut type formula (5.2) as in [10], which in particular implies the strong Feller property of \( P_t \) as explained in Remark A.1 below.
Because of the exponential integrability of \( l_t \) ensured by Lemma 2.1, it is easy to see that the argument in [5] for compact \( M \) works also for the present case under assumption (A) and condition (1.1). To state the formula for the gradient of \( P_t \) obtained in [5], let us first introduce the SDE for the horizontal lift of the reflecting \( L \)-diffusion process.

Let \( O(M) \) be the bundle of orthonormal frames over \( M \) and let \( \pi : O(M) \to M \) be the natural projection. Then \( X_t \) and its horizontal lift \( u_t \) on \( O(M) \) solve the following equations:

\[
\begin{align*}
\mathrm{d}u_t &= H_{u_t} \circ \mathrm{d}X_t, \\
\mathrm{d}X_t &= \sqrt{2} u_t \circ \mathrm{d}B_t + Z(X_t) \mathrm{d}t + N(X_t) \mathrm{d}l_t,
\end{align*}
\]

where \( B_t \) is the \( d \)-dimensional Brownian motion and \( H_u : T_{\pi u}M \to T_uO(M) \) is the horizontal lift at \( u \in O(M) \). Next, let \( \mathbb{M}_t \) be the \( \mathbb{R}^d \otimes \mathbb{R}^d \)-valued process solving the equation

\[
\mathrm{d}\mathbb{M}_t = -\mathbb{M}_t R_{u_t} \mathrm{d}t, \quad \mathbb{M}_0 = I,
\]

where

\[
R_{u}(a,b) = \text{Ric}(ua,ub) - \langle \nabla ua Z, ub \rangle, \quad u \in O(M), a, b \in \mathbb{R}^d.
\]

Then for any \( f \in C_0^\infty(M) \) we have (cf. the proof of [5, Theorem 5.1])

\[
(5.1) \quad u_0^{-1} \nabla P_t f = \mathbb{E}\{\mathbb{M}_t u_t^{-1} \nabla f(X_t)\}.
\]

Since by Lemma 2.1 and Proposition 2.2 \( |\nabla P.f| \) is bounded on \([0,T] \times M\) for any \( T > 0 \), this follows since according to [5, Theorem 3.7] the process \( \{\mathbb{M}_s F(u_s, t - s)\}_{s \in [0,t]} \) is a martingale for \( F(u, s) = u^{-1} \nabla P_s f(\pi u) \). Due to (5.1) we have the following result on the Bismut type formula.

**Proposition 5.1.** Assume (A) and (1.1). Then for any \( f \in C_0^\infty(M) \) and any increasing function \( h \in C^1([0,t]) \) such that \( h(0) = 0, h(t) = 1 \),

\[
(5.2) \quad \nabla P_t f(x) = \frac{u_0}{\sqrt{2}} \mathbb{E}\left[ f(X_t) \int_0^t h'(s) \langle \mathbb{M}_s, \mathrm{d}B_s \rangle \right]
\]

holds for \( x \in M \) and \( X_t, u_t \) start from \( x, u_0 \in O_x(M) \) respectively.

**Proof.** By Itô’s formula we have

\[
\mathrm{d}P_{t-s}f(X_s) = \sqrt{2} \langle \nabla P_{t-s} f(X_s), u_s \mathrm{d}B_s \rangle.
\]

Then
\[ f(X_t) = P_t f + \sqrt{2} \int_0^t \langle \nabla P_{t-s} f(X_s), u_s dB_s \rangle. \]

Combining this with (5.1), for any \( a \in \mathbb{R}^d \),

\[
\frac{1}{\sqrt{2}} \mathbb{E} \left\{ f(X_t) \int_0^t h'(s) \langle M_s a, dB_s \rangle \right\} = \mathbb{E} \int_0^t \langle \nabla P_{t-s} f(X_s), M_s a \rangle h'(s) ds
= \int_0^t \langle u_0 a, \nabla P_t f \rangle h'(s) ds = \langle u_0 a, \nabla P_t f \rangle.
\]

This completes the proof since \( a \in \mathbb{R}^d \) is arbitrary.

\[ \square \]

**Remark A.1.** By (1.1) and letting \( I \geq -\sigma \), we have

\[ \| M_s \| \leq e^{K_s + \sigma I_s}, \quad s \geq 0. \]

So, by Lemma 2.1 and (5.2), for any \( t > 0 \) there exists a constant \( C(t) > 0 \) such that

\[ \| \nabla P_t f \|_{\infty} \leq \| f \|_{\infty} C(t), \quad t > 0, f \in C_0^\infty(M). \]

This implies

\[ |P_t f(x) - P_t f(y)| \leq C(t) \| f \|_{\infty} \rho(x, y), \quad x, y \in M, f \in C_0^\infty(M). \]

By the monotone class theorem, this inequality holds indeed for all \( f \in \mathcal{B}_b(M) \) and thus, \( P_t \) is strong Feller.

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