A non-perturbative formulation of $\mathcal{N} = 4$ super Yang-Mills theory
based on the large-$N$ reduction

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Abstract
We study a non-perturbative formulation of $\mathcal{N} = 4$ super Yang-Mills theory (SYM)
on $R \times S^3$ proposed in [arXiv:0807.2352]. This formulation is based on the large-$N$
reduction, and the theory can be described as a particular large-$N$ limit of the plane
wave matrix model (PWMM), which is obtained by dimensionally reducing the original
theory over $S^3$. In this paper, we perform some tests for this proposal. We construct an
operator in the PWMM that corresponds to the Wilson loop in SYM in the continuum
limit and calculate the vacuum expectation value of the operator for the case of the
circular contour. We find that our result indeed agrees with the well-known result first
obtained by Erickson, Semenoff and Zarembo. We also compute the beta function at
the 1-loop level based on this formulation and see that it is indeed vanishing.

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1 Introduction

The AdS/CFT correspondence [1–3] has been intensively studied and fruitfully extended to various directions over a decade. However, a complete proof of this conjecture is still missing even for the most typical example, the correspondence between $\mathcal{N} = 4$ super Yang-Mills theory (SYM) and type IIB superstring theory on $AdS_5 \times S^5$. This is partially because the correspondence is a strong/weak one. Namely, the region on the string theory side in which the supergravity approximation or the classical string approximation is valid is mapped to the strongly coupled region in the planar limit on the gauge theory side. In order to study $\mathcal{N} = 4$ SYM in the strongly coupled region, one needs to have a non-perturbative formulation such as the lattice gauge theory. In fact, there are considerable developments in the lattice theories of $\mathcal{N} = 4$ SYM [4–11]. However, any lattice formulations of $\mathcal{N} = 4$ SYM proposed so far seem to require the fine-tuning of at least three parameters

In [12], a non-perturbative formulation of $\mathcal{N} = 4$ SYM on $R \times S^3$ in the planar ('t Hooft) limit was proposed by using the plane wave matrix model (PWMM) [13] (for earlier discussions, see [14,15]). Note that $\mathcal{N} = 4$ SYM on $R^4$ at a conformal point has the $PSU(2,2|4)$ symmetry with thirty-two supercharges and is equivalent to $\mathcal{N} = 4$ SYM on $R \times S^3$ through the conformal mapping. The PWMM can be obtained by dimensionally reducing $\mathcal{N} = 4$ SYM on $R \times S^3$ over $S^3$ [16,17]. In the formulation, $\mathcal{N} = 4$ SYM on $R \times S^3$ is retrieved by taking an appropriate large-$N$ limit for the theory around a particular vacuum of the PWMM. Thus the formulation is viewed as an extension of the large-$N$ reduction [18], which asserts that the planar limit of a gauge theory is described by its dimensionally reduced model. The formulation provides a matrix regularization of the planar $\mathcal{N} = 4$ SYM on $R \times S^3$, where the matrix size plays a role of the ultraviolet cutoff. Remarkably, the formulation preserves sixteen supercharges and the gauge symmetry. Note that this number of the preserved supercharges is optimal, because any regularization must break the conformal symmetry so that the number is inevitably reduced to less than or equal to sixteen from thirty-two. Since the formulation preserves so many supercharges and provides a massive theory, no fine-tuning should be needed in taking the continuum limit. This is advantageous to the lattice formulations of $\mathcal{N} = 4$ SYM, while the latter may also be used

\footnote{It was shown recently that only a fine-tuning of one parameter is needed at least to one-loop order in a lattice formulation [11]. It is discussed that no fine-tuning is required in the recent proposals [9,10].}
to study the finite $\mathcal{N} = 4$ SYM. Thus the formulation gives a feasible way to simulate the planar $\mathcal{N} = 4$ SYM by computer using the methods in [19,20] and study its strong coupling dynamics.

To check the validity of the formulation, some one-loop perturbative calculations were performed in [12]. It was shown that the tadpole vanishes and that the fermion one-loop self-energy agrees with the one in the continuum theory. In [27,28], the known result of the confinement-deconfinement transition in the planar $\mathcal{N} = 4$ SYM on $R \times S^3$ in the weak coupling limit at finite temperature was reproduced in the formulation. In [29], a two-loop calculation in the high temperature limit was done, and the result was consistent with the continuum theory. In [30,31], based on the earlier work [32,33], the same large-$N$ reduction for $S^3$ was applied to Chern-Simons theory on $S^3$ to obtain a matrix regularization of the theory in the planar limit, and the exact results on Chern-Simons theory on $S^3$ were reproduced. Extension of the formulation to other supersymmetric gauge theories was discussed in [34,35]. In [36,37], large-$N$ reduction on general group manifolds and coset spaces was found. Applying the case of $SU(2) \simeq S^3$ to planar $\mathcal{N} = 4$ SYM on $R \times S^3$ yields another non-perturbative formulation of the theory in terms of the PWMM. Recently, a matrix model regularization of $\mathcal{N} = 4$ SYM on $S^4$ was proposed in [38].

In this paper, we make a further check by calculating two quantities: the vacuum expectation value (VEV) of a half-BPS Wilson loop and the beta function.

Erickson, Semenoff and Zarembo gave a successful example of calculation in the strongly coupled regime of $\mathcal{N} = 4$ SYM [39] (see also [40]). They considered (locally) supersymmetric Wilson loops in $\mathcal{N} = 4$ SYM on $R^4$. It is conjectured that the logarithms of the VEV of such a Wilson loop in $\mathcal{N} = 4$ SYM corresponds to minus the area of the minimal surface of a string world sheet in the $AdS_5 \times S^5$ such that the boundary of the world sheet coincides with the loop [41,42]. In the planar limit, they evaluated the VEV of the circular Wilson loop, which is half-BPS, by summing up all ladder diagrams of all orders in the perturbative expansion. They indeed showed that in the strong coupling region the result agrees with the prediction on the gravity side [43,44]. Thus, they gave a nontrivial check of the AdS/CFT correspondence. Their result was reproduced through the localization method in [45] to

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2Monte Carlo simulation of the correlation functions of the chiral primary operators in $\mathcal{N} = 4$ SYM and comparison of the results with the prediction from the gravity side will be reported in [21]. Preliminary results including the one for the Wilson loop are seen in [22,23]. For related work, see [24,26].
show that it is indeed the exact result. In this paper, we reproduce their result in the above formulation of $\mathcal{N} = 4$ SYM on $R \times S^3$. We first construct an operator in terms of the matrices in such a way that it coincides with the Wilson loop in $\mathcal{N} = 4$ SYM on $R \times S^3$ in the continuum limit. Then, we concentrate on the circular loop, and calculate its VEV by summing up all the planar ladder diagrams. We find that the VEV exactly agrees with the result in [39].

We also evaluate the 1-loop beta function and find that it indeed vanishes, which is consistent with restoration of the superconformal invariance.

This paper is organized as follows. In section 2, we review the non-perturbative formulation of the planar $\mathcal{N} = 4$ SYM on $R \times S^3$ proposed in [12]. In section 3, after reviewing Wilson loops in the AdS/CFT correspondence, we construct operators in the non-perturbative formulation which correspond to the Wilson loops in $\mathcal{N} = 4$ SYM on $R \times S^3$ in the continuum limit. In section 4, we calculate the VEV of the operator in the half-BPS case and show that it agrees with the known exact result. In section 5, we calculate the 1-loop beta function and verify that it indeed vanishes. Section 6 is devoted to summary and discussion. In appendices, some details are gathered.

2 Large-$N$ reduction for $\mathcal{N} = 4$ SYM on $R \times S^3$

In this section, we review the novel large-$N$ reduction for $\mathcal{N} = 4$ SYM on $R \times S^3$ proposed in [12]. In section 2.1, as a warm-up, we review the large-$N$ reduction on $S^1$ developed in [12], by taking the $\phi^3$ theory on $S^1$ as an example. In section 2.2, to explain the mechanism of the large-$N$ reduction on $S^3$, we consider the large $N$ reduction for the $\phi^3$ theory on $S^3$. In section 2.3, we apply the mechanism to $\mathcal{N} = 4$ SYM on $R \times S^3$.

2.1 Large-$N$ reduction on $S^1$

We consider the $\phi^3$ matrix quantum mechanics at finite temperature with the inverse temperature $R$. In other words, it is a one-dimensional matrix field theory on $S^1$ with the radius $R$. The action is

$$S = \frac{1}{g^2} \int_0^{2\pi R} dx \, \text{Tr} \left( \frac{1}{2} \left( \frac{d\phi}{dx} \right)^2 + \frac{\xi^2}{2R^2} \phi^2 + \frac{1}{3R} \phi^3 \right),$$

(2.1)

where $0 \leq x < 2\pi R$, $\phi$ is an $N \times N$ hermitian matrix, and $\xi$ is a dimensionless mass.
To diagonalize the quadratic part in (2.1), we make the Fourier expansion
\[ \phi(x) = \sum_n \phi_n e^{i n R x} \quad (2.2) \]
The action (2.1) is rewritten in terms of the Fourier modes as
\[ S = \frac{V_{S^1}}{g^2} \left[ \frac{1}{2 R^2} \sum_n (n^2 + \xi^2) \phi_n \phi_{-n} + \frac{1}{3 R} \sum_{n_1,n_2,n_3} \delta_{n_1+n_2+n_3,0} \phi_n \phi_{n_2} \phi_{n_3} \right] \quad (2.3) \]
where \( V_{S^1} = 2\pi R \) is the volume of \( S^1 \).

Here, for instance, we consider the following correlation function
\[ \langle \frac{1}{N} \text{Tr}(\phi(x_1)\phi(x_2)) \rangle \quad (2.4) \]
By using the translation invariance of the theory and the Fourier expansion (2.2), we see that the above correlation function is equal to
\[ \frac{1}{2\pi R} \int dy \left\langle \frac{1}{N} \text{Tr}(\phi(x_1-x_2+y)\phi(y)) \right\rangle = \sum_n \left\langle \frac{1}{N} \text{Tr}(\phi_n \phi_{-n}) \right\rangle e^{i n R (x_1-x_2)} \quad (2.5) \]
We calculate a diagram in Fig. 1 which appears in the perturbative expansion of \( \langle \frac{1}{N} \text{Tr}(\phi_n \phi_{-n}) \rangle \):
\[ \left( \frac{g^2 N}{V_{S^1}} \right)^2 R^6 \left( \frac{1}{n^2 + \xi^2} \right)^2 \sum_l \frac{1}{l^2 + \xi^2} \frac{1}{(n+l)^2 + \xi^2} \quad (2.6) \]
This survives in the planar limit where
\[ N \rightarrow \infty \quad \text{with} \quad g^2 N \quad \text{fixed} \quad (2.7) \]
In this limit, all the planar diagrams such as Fig. 1 contribute to (2.4), while other non-planar diagrams such as Fig. 2 do not.

To obtain the reduced model of (2.1), we introduce a constant diagonal matrix with the eigenvalues uniformly distributed:
\[ P = \frac{1}{R} \text{diag} \left( -\nu + \frac{1}{2}, -\nu + \frac{3}{2}, \cdots, \nu - \frac{1}{2} \right) \otimes 1_k \quad (2.8) \]
where \( \nu \) and \( k \) are integers satisfying
\[ N = \nu k \quad (2.9) \]
\[ ^3 \text{The tensor product of } 1_k \text{ is needed for the theory on } S^1 \text{ to extract the planar contribution through the } k \rightarrow \infty \text{ limit [12]. It is not needed for the theory on } R. \]
\( \nu \) turns out to play the role of the UV cutoff. The rule for obtaining the reduced model is

\[
\phi(x) \rightarrow e^{iP_x} \phi e^{-iP_x}, \quad V_{S1} \rightarrow v, \tag{2.10}
\]

where \( \phi \) is an \( N \times N \) hermitian matrix independent of \( x \). \( v \) is determined shortly such that the reduced model reproduces the original theory. The rule (2.10) implies that

\[
\frac{d\phi(x)}{dx} \rightarrow i e^{iP_x} [P, \phi] e^{-iP_x}. \tag{2.11}
\]

Then, applying the rule (2.10) to (2.1) yields the reduced model of (2.1)

\[
S_r = \frac{v}{g^2} \text{Tr} \left( -\frac{1}{2} [P, \phi]^2 + \frac{\xi^2}{2R^2} \phi^2 + \frac{1}{3R} \phi^3 \right). \tag{2.12}
\]

If \( \phi \) is decomposed to a tensor product of a \( \nu \times \nu \) matrix and a \( k \times k \) matrix following (2.8), (2.12) is expressed as

\[
S_r = \frac{v}{g^2} \text{tr} \left( \frac{1}{2R^2} \sum_{s,t} ((s - t)^2 + \xi^2)\phi^{(s,t)}\phi^{(t,s)} + \frac{1}{3R} \sum_{s,t,u} \phi^{(s,t)}\phi^{(t,u)}\phi^{(u,s)} \right), \tag{2.13}
\]

where \( \phi^{(s,t)} \) is a \( k \times k \) matrix, and tr stands for the trace over \( k \times k \) matrices. The range of the indices \( s, t, u \) is

\[
-\nu + \frac{1}{2} \leq s, t, u \leq \nu - \frac{1}{2}, \tag{2.14}
\]

and the indices \( s, t, u \) run integers for odd \( \nu \) and half-integers for even \( \nu \).

By applying the rule (2.10), we obtain the observable corresponding to (2.4)

\[
\left\langle \frac{1}{N} \text{Tr}(e^{iP_{x_1}} \phi e^{-iP_{x_1}} e^{iP_{x_2}} \phi e^{-iP_{x_2}}) \right\rangle_r, \tag{2.15}
\]
where \( \langle \cdots \rangle_r \) stands for the VEV in the reduced model. By using (2.8), this is rewritten as
\[
\frac{1}{\nu} \sum_{s,t} \left\langle \frac{1}{k} \text{Tr}(\phi^{(s,t)}(t,s)) \right\rangle_r e^{i \frac{s-t}{r}(x_1-x_2)}.
\] (2.16)

We calculate again the diagram in Fig. 1 which appears in the perturbative expansion of \( \langle \frac{1}{k} \text{Tr}(\phi^{(s,t)}(t,s)) \rangle_r \):
\[
\sum_u \left( \frac{g^2 N}{\nu v} \right)^2 R^6 \left( \frac{1}{(s-t)^2 + \xi^2} \right)^2 \frac{1}{(t-u)^2 + \xi^2} \frac{1}{(u-s)^2 + \xi^2}.
\] (2.17)

Comparing (2.6) and (2.17), we put
\[
v = \frac{V_{S^1}}{\nu},
\] (2.18)
and take a limit
\[
k \to \infty, \quad \nu \to \infty \text{ with } g^2 N \text{ fixed}.
\] (2.19)

Note that (2.19) implies (2.7) because \( N = \nu k \). We make an identification \( n = s - t, \)
\( l = t - u \) and \(-n - l = u - s\), and find that (2.17) agrees with (2.6) in the limit (2.19). This correspondence holds for all the planar diagrams such as Fig. 1. As for the non-planar diagrams such as Fig. 2, such correspondence does not hold. However, because of \( k \to \infty \),
only the planar diagrams contribute to (2.16). Furthermore, we compare (2.16) with (2.5). While the summation over \( s \) and \( t \) in (2.16) is redundant compared to the summation over \( n \) in (2.5), this redundancy is canceled by a factor \( 1/\nu \) in (2.16). Thus we conclude that (2.15) in the limit (2.19) agrees with (2.4) in the planar limit.

In a similar manner, one can easily show that in the limit (2.19)
\[
\left\langle \frac{1}{N} \text{Tr}(e^{iP_{x_1} \phi} e^{-iP_{x_2} \phi} e^{iP_{x_2} \phi} e^{-iP_{x_2} \phi} \cdots e^{iP_{x_r} \phi} e^{-iP_{x_r} \phi}) \right\rangle_r = \left\langle \frac{1}{N} \text{Tr}(\phi(x_1)\phi(x_2)\cdots\phi(x_r)) \right\rangle_r.
\] (2.20)

One can also show the following equality for the free energies of both theories in the limit (2.19):
\[
\frac{F_r}{N^2 v} = \frac{F}{N^2 V_{S^1}}.
\] (2.21)

This gives an interpretation of \( v \) that it is the volume of space on which the reduced model is defined. Thus, in the limit (2.19), the reduced model (2.12) with (2.18) retrieves the planar limit of the original theory (2.1).
2.2 Large-\(N\) reduction on \(S^3\)

To illustrate the large-\(N\) reduction on \(S^3\), we consider the \(\phi^3\) matrix field theory on \(S^3\):

\[
S = \frac{V_{S^3}}{g^2} \int \frac{d\Omega_3}{2\pi^2} \text{Tr} \left( -\frac{\mu^2}{2} (\mathcal{L}_i \phi)^2 + \frac{\xi^2 \mu^2}{2} \phi^2 + \frac{\mu}{3} \phi^3 \right). \tag{2.22}
\]

Here the radius of the \(S^3\) is \(2/\mu\), and its volume \(V_{S^3}\) is given by

\[
V_{S^3} = \frac{16\pi^2}{\mu^3}. \tag{2.23}
\]

\(d\Omega_3\) represents the volume element of unit \(S^3\). \(\phi(\Omega_3)\) is an \(N \times N\) hermitian matrix scalar field on \(S^3\). \(\mathcal{L}_i\) is the Killing vector on unit \(S^3\). Note that \(S^3\) is identified with the \(SU(2)\) group manifold. From this point of view, \(d\Omega_3\) is the Haar measure which is left and right invariant, and \(\mathcal{L}_i\) is the generator of the left translation on the group manifold. For an explicit form of \(d\Omega_3\) and \(\mathcal{L}_i\), see (A.7) and (A.11), respectively. \(\xi\) is a dimensionless mass as before.

In what follows, the argument goes parallel to the one in the previous subsection. Indeed, \(S^3\) is an \(S^1\)-bundle over \(S^2\), and this \(S^1\) is a counterpart of the \(S^1\) on which the \(\phi^3\) theory is defined in the previous subsection.

To diagonalize the quadratic part in (2.22), we expand \(\phi\) in terms of the spherical harmonics on \(S^3\) defined in (A.12):

\[
\phi(\Omega_3) = \sum_J \sum_{m, \tilde{m} = -J} \phi_{Jm\tilde{m}} Y_{Jm\tilde{m}}(\Omega_3), \tag{2.24}
\]

where \(J\) take non-negative integers and half-integers. The properties of the spherical harmonics on \(S^3\) are summarized in appendix A. \(\tilde{m}\) is viewed as the momentum along the \(S^1\) fiber [12,14,15]. By using them, the action (2.22) is rewritten in terms of the modes as

\[
S = \frac{V_{S^3}}{g^2} \left[ \frac{\mu^2}{2} \sum_{Jm\tilde{m}} (-1)^{m-\tilde{m}} (J(J+1) + \xi^2) \text{Tr}(\phi_{Jm\tilde{m}} \phi_{J-m-\tilde{m}}) \right. \\
\left. + \frac{\mu}{3} \sum_{J_1m_1\tilde{m}_1, J_2m_2\tilde{m}_2, J_3m_3\tilde{m}_3} \mathcal{C}_{J_1m_1\tilde{m}_1, J_2m_2\tilde{m}_2, J_3m_3\tilde{m}_3} \text{Tr}(\phi_{J_1m_1\tilde{m}_1} \phi_{J_2m_2\tilde{m}_2} \phi_{J_3m_3\tilde{m}_3}) \right], \tag{2.25}
\]

where \(\mathcal{C}_{J_1m_1\tilde{m}_1, J_2m_2\tilde{m}_2, J_3m_3\tilde{m}_3}\) is defined in (A.18).
By identifying the $S^3$ with the $SU(2)$ group manifold, we denote $\phi(\Omega_3)$ by $\phi(g)$, where $g \in SU(2)$. We consider the following correlation function which is a counterpart of (2.4):

$$\left\langle \frac{1}{N} \text{Tr}(\phi(g_1)\phi(g_2)) \right\rangle.$$  \hspace{1cm} (2.26)

The invariance of the theory under the right translation implies that (2.26) is equivalent to

$$\int \frac{d\Omega_3}{2\pi^2} \left\langle \frac{1}{N} \text{Tr}(\phi(g_1^{-1})\phi(g_2)) \right\rangle.$$  \hspace{1cm} (2.27)

We substitute (2.24) into the above quantity:

$$\sum_{Jm\tilde{m},J^\prime m^\prime \tilde{m}^\prime} \int \frac{d\Omega_3}{2\pi^2} \left\langle \frac{1}{N} \text{Tr}(\phi_{Jm\tilde{m}}\phi_{J^\prime m^\prime \tilde{m}^\prime}) \right\rangle R_J(g_2 g_1^{-1})_{m^\prime m} Y_{Jm\tilde{m}}(\Omega_3) Y_{J^\prime m^\prime \tilde{m}^\prime}(\Omega_3)$$

$$= \sum_{Jm\tilde{m},J^\prime m^\prime \tilde{m}^\prime} (-1)^{m^\prime + \tilde{m}^\prime} R_J(g_2 g_1^{-1})_{-m^\prime m} \left\langle \frac{1}{N} \text{Tr}(\phi_{Jm\tilde{m}}\phi_{J^\prime m^\prime -\tilde{m}^\prime}) \right\rangle;$$  \hspace{1cm} (2.28)

where $R_J(g)$ is the representation matrix of $g$ in the spin $J$ representation, and we have used (A.12) and (A.16). We have obtained a counterpart of (2.5). We calculate a diagram in Fig.4 which appears in the perturbative expansion of ($\frac{1}{N} \text{Tr}(\phi_{Jm\tilde{m}}\phi_{J^\prime m^\prime -\tilde{m}^\prime})$):

$$\sum_{Jm\tilde{m},J_2 m_2 \tilde{m}_2 J_3 m_3 \tilde{m}_3} (-1)^{m+m^\prime} \frac{1}{\mu^6} \left( \frac{g^2 N}{V_{S^3}} \right)^2 \left( \frac{1}{J(J+1) + \xi^2} \right)^2 \frac{(-1)^{m_2-\tilde{m}_2}}{J_2(J_2+1) + \xi^2} \frac{(-1)^{m_3-\tilde{m}_3}}{J_3(J_3+1) + \xi^2} \times C_{J-m-\tilde{m}} J_2 m_2 \tilde{m}_2 J_3 m_3 \tilde{m}_3 C_{J-m'-\tilde{m}'} J_2 m_2 \tilde{m}_2 J_3 m_3 \tilde{m}_3 J_2 m_2 \tilde{m}_2.$$  \hspace{1cm} (2.29)

This survives in the planar limit (2.7). As before, in this limit, all the planar diagrams such as Fig.4 contribute to (2.26), while other non-planar diagrams such as Fig.4 do not.

Next we construct the reduced model of (2.22). For this it is convenient to introduce an $N$-dimensional reducible representation of $SU(2)$ in which the generators take the form

$$L_i = \bigoplus_{s=-(\nu-1)/2}^{(\nu-1)/2} \left( L_i^{[j]} \otimes 1_{k_s} \right),$$  \hspace{1cm} (2.30)

which represents multiple fuzzy spheres. Here $L_i^{[j]}$ are the spin $j$ representation matrices of the $SU(2)$ generators obeying

$$[L_i^{[j]}, L_j^{[j]}] = i \epsilon_{ijk} L_k^{[j]}.$$  \hspace{1cm} (2.31)
\( \nu \) is the number of different irreducible representations of \( SU(2) \), and the index \( s \) runs integers for odd \( \nu \) and half-integers for even \( \nu \). The parameters \( j_s \) and \( k_s \) in (2.30) satisfy the relation

\[
\left( \frac{\nu - 1}{2} \right) \sum_{s = -(\nu - 1)/2}^{(\nu - 1)/2} (2j_s + 1)k_s = N. \tag{2.32}
\]

In particular, in order to construct a reduced model of (2.22), we choose the following parameters:

\[
2j_s + 1 = n + s, \quad k_s = k. \tag{2.33}
\]

\( n \) is the centered value of the dimension of the representations, so that

\[
N = n\nu k. \tag{2.34}
\]

The rule for obtaining the reduced model, which is a counterpart of (2.10), is

\[
\phi(g) \rightarrow G^{-1}\phi G, \quad V_{S^3} \rightarrow v, \tag{2.35}
\]

where \( G \) is the representation matrix of \( g \) in the reducible representation specified by (2.33), and \( \phi \) is an \( N \times N \) hermitian matrix independent of \( \Omega_3 \). \( v \) is determined shortly such that the reduced model reproduces the original theory. By using (A.9), it is easy to see that

\[
\mathcal{L}_{i}\phi \rightarrow G^{-1}[L_{i}, \phi]G. \tag{2.36}
\]

Then applying the rule to (2.22) yields the reduced model of (2.22)

\[
S_r = \frac{\nu}{g^2} \text{Tr}\left( -\frac{\mu^2}{2}[L_{i}, \phi]^2 + \frac{\xi^2\mu^2}{2}\phi^2 + \frac{\mu}{3}\phi^3 \right). \tag{2.37}
\]

To see how this reduced model retrieves the original theory, we expand the \((s, t)\) block of \( \phi \) we denote by \( \phi^{(s,t)} \) in terms of the fuzzy spherical harmonics defined in (B.3):

\[
\phi^{(s,t)} = \sum_{J = |j_s - j_t|}^{j_s + j_t} \sum_{m = -J}^{J} \phi^{(s,t)}_{Jm} \otimes \hat{Y}_{Jm(j_s,j_t)}. \tag{2.38}
\]

\(^4\)The correspondence between the notation here and that in [12] is \((n)_{\text{here}} = (N_0)_{\text{there}}, \ (\nu)_{\text{here}} = (T + 1)_{\text{there}}, \ (k_s)_{\text{here}} = (N_s)_{\text{there}}, \ (k)_{\text{here}} = (N)_{\text{there}}.\)
where $\phi_{(s,t)}$ is a $k \times k$ matrix. The properties of the fuzzy spherical harmonics are summarized in appendix B. We will see shortly that $j_s - j_t = (s - t)/2$ is identified with $\tilde{m}$ as in the case of the $\phi^3$ theory on $S^1$. Using the modes, (2.37) is expressed as

$$S_r = \frac{\nu^4}{g^2} \left[ \frac{\mu^2}{2} \sum_{s,t} \sum_{J_m} (-1)^{m - (j_s - j_t)} (J(J + 1) + \xi^2) \text{tr}(\phi_{J_m}^{(s,t)} \phi_{J_m}^{(t,s)}) \right] + \frac{\mu}{3} \sum_{s,t,u} \sum_{J_m J_n J_m J_n} \hat{C}_{J_m J_n J_m J_n} \text{tr}(\phi_{J_m}^{(s,t)} \phi_{J_n}^{(t,u)} \phi_{J_m}^{(u,s)}) \right] \right], \quad (2.39)$$

where $\hat{C}_{J_m J_n J_m J_n}$ is defined in (B.8), and $\text{tr}$ stands for the trace over $k \times k$ matrices.

Applying the rule (2.38), we obtain the observable in the reduced model corresponding to (2.26):

$$\left\langle \frac{1}{N} \text{Tr}(G_1^{-1} \phi G_1 G_2^{-1} \phi G_2) \right\rangle_r, \quad (2.40)$$

where $\langle \cdots \rangle_r$ stands for the VEV in the reduced model. By using (2.38), (B.4) and (B.6), we calculate (2.40):

$$\frac{1}{\nu} \sum_{s,t} \sum_{J_m J'_m} \left\langle \frac{1}{k} \text{tr}(\phi_{J_m}^{(s,t)} \phi_{J'_m}^{(t,s)}) \right\rangle_r R_J(g_2 g_1)_{m'm} \frac{1}{n} \text{tr}(\hat{Y}_{J_m'}(j_{s,t}) \hat{Y}_{J_m}(j_{s,t}))$$

$$= \frac{1}{\nu} \sum_{s,t} \sum_{J_m J'_m} (-1)^{m' - (j_{s,t})} R_J(g_2 g_1)_{m'm} \left\langle \frac{1}{k} \text{tr}(\phi_{J_m}^{(s,t)} \phi_{J_m'}^{(t,s)}) \right\rangle_r. \quad (2.41)$$

We calculate again the diagram in Fig. 4 which appears in the perturbative expansion of $\left\langle \frac{1}{k} \text{tr}(\phi_{J_m}^{(s,t)} \phi_{J_m'}^{(t,s)}) \right\rangle_r$:

$$\sum_u \sum_{J_{m' m} J_{m' m} J_{m' m}} \left( -1 \right)^{m + m'} \frac{g^2 N}{\mu^6 \left( v^2 \nu \right)^2} \left( \frac{1}{J(J + 1) + \xi^2} \right)^2 \frac{(-1)^{m_2 - (j_s - j_t)}}{J_2 \left( J_2 + 1 \right) + \xi^2} \frac{(-1)^{m_3 - (j_s - j_t)}}{J_3 \left( J_3 + 1 \right) + \xi^2}$$

$$\times \hat{C}_{J_{m' m} J_{m' m}} \hat{C}_{J_{m' m} J_{m' m}} \hat{C}_{J_{m' m} J_{m' m}} \hat{C}_{J_{m' m} J_{m' m}}. \quad (2.42)$$

Comparison of (2.29) with (2.42) leads us to put

$$v = \frac{V_{S^3}}{n^2 \nu}, \quad (2.43)$$

and take a limit

$$k \to \infty, \ \nu \to \infty, \ n/\nu \to \infty \ \text{with} \ g^2 N \ \text{fixed}. \quad (2.44)$$
Note again that (2.44) implies (2.7). Then, in the limit (2.44), we can make an identification
\[ \tilde{m} = j_s - j_t, \quad \tilde{m}_2 = j_s - j_u, \quad \tilde{m}_3 = j_u - j_t \] as anticipated and show by using (B.9) that
\[ \hat{C}_{J−m(jt,js)}J_{2m2(js,ju)}J_{3m3(ju,jt)} \to \hat{C}_{J−m−\tilde{m}}J_{2m2\tilde{m}_2}J_{3m3\tilde{m}_3} \]
\[ \hat{C}_{J−m'(jt,js)}J_{3−m3(js,ju)}J_{2m2(js,ju)} \to \hat{C}_{J−m'm−\tilde{m}}J_{3−m3−\tilde{m}_3}J_{2m2−\tilde{m}_2}. \] (2.45)

We therefore find that (2.42) agrees with (2.29). As before, this correspondence holds for all the planar diagrams such as Fig.4, while it does not hold for the non-planar diagrams such as Fig.5. However, because of \( k \to \infty \), only the planar diagrams contribute to (2.41). Note also that the UV/IR mixing on fuzzy spheres does not exist in the planar contribution, so non-commutativity does not remain in the \( n \to \infty \) limit. Furthermore, while the summation over \( s \) and \( t \) in (2.41) is redundant compared to the summation over \( \tilde{m} \) in (2.28), this redundancy is canceled by a factor \( 1/\nu \) in (2.41). Thus we conclude that (2.40) in the limit (2.44) agrees with (2.26) in the planar limit.

As in the case of the \( \phi^3 \) theory on \( S^1 \), one can easily show that counterparts of (2.20) and (2.21) hold in the limit (2.44):
\[ \left\langle \frac{1}{N} \text{Tr}(G_1^{-1}\phi G_1 G_2^{-1}\phi G_2 \cdots G_r^{-1}\phi G_r) \right\rangle_r = \left\langle \frac{1}{N} \text{Tr}(\phi(g_1)\phi(g_2) \cdots \phi(g_r)) \right\rangle_r. \] (2.46)

and
\[ \frac{F_r}{N^2v} = \frac{F}{N^2V_{S^3}}, \] (2.47)

Thus, in the limit (2.44), the reduced model (2.37) with (2.43) retrieves the planar limit of the original theory (2.22).\cite{footnote1}

What we have done is understood as follows. \( S^3 \) is viewed as an \( S^1 \)-bundle over \( S^2 \). The momenta along the \( S^1 \) is given by \( \tilde{m} \) in (2.24). By making the Kaluza-Klein (KK) reduction along the \( S^1 \), one obtains the KK modes on \( S^2 \) with the KK momenta given by \( \tilde{m} \). The KK mode with the KK momentum \( \tilde{m} \) behaves as in a situation that the monopole with the monopole charge \( \tilde{m} \) is located at the origin of the \( S^2 \). Reflecting the angular momentum possessed by the monopole, the angular momentum \( J \) of the KK mode on \( S^2 \) is restricted to \( |\tilde{m}| \leq J \). The identification \( \tilde{m} = j_s - j_t = (s - t)/2 \) matches the range of \( J \) in

\footnote{Another large-N reduction on \( S^3 \) was developed in \cite{footnote2}, where the planar limit of the original theory is retrieved by setting \( n = (\nu + 1)/2 \) and \( k_s = (2j_s + 1)k \) in (2.30) instead of (2.33) and taking the limit in which \( \nu \to \infty \) and \( k \to \infty \) with \( v = V_{S^3}k/N \).}
in the $n \to \infty$ limit. Actually, the monopole spherical harmonics for the monopole charge $q$ is regularized by the fuzzy spherical harmonics which is an $(l + q) \times l$ rectangular matrix \[12\]. The relation (2.45) reflects this fact. We obtain the base space $S^2$ through the continuum limit of the fuzzy sphere and the $S^1$ fiber through the mechanism of the large-$N$ reduction on $S^1$ explained in the previous subsection. $n$ plays the role of the UV cutoff on $S^2$, while $\nu$ plays the role of the UV cutoff on $S^1$.

2.3 $\mathcal{N} = 4$ SYM on $R \times S^3$ from PWMM

We first see that $\mathcal{N} = 4$ SYM on $R^4$ at a conformal point, where the VEVs of all the scalar fields vanish, is equivalent to $\mathcal{N} = 4$ SYM on $R \times S^3$ through the conformal map. In this subsection, for simplicity, we suppress the terms including fermion fields. Then, the action of $\mathcal{N} = 4$ $U(N)$ SYM on $R^4$ takes the form

$$S_{YM} = \frac{1}{g^2} \int d^4 x \text{Tr} \left( F^2_{\mu\nu} + \frac{1}{2} (D_{\mu}\phi_m)^2 - \frac{1}{4} [\phi_m, \phi_n]^2 \right),$$

(2.48)

where $D_{\mu} = \partial_{\mu} + i [A_{\mu}, \cdot]$. There are six scalars $\phi_m$, and for later convenience we make $m$ and $n$ run from 4 to 9. This theory possesses the $PSU(2, 2|4)$ superconformal symmetry with 32 supercharges. Applying the Weyl transformation defined by

$$A_{\mu} \to A_{\mu},$$

$$\phi_m \to e^{-\frac{\rho(x)}{2}} \phi_m,$$

$$\delta_{\mu\nu} \to g_{\mu\nu} = e^{\rho(x)} \delta_{\mu\nu}.$$

(2.49)

to (2.48) yields $\mathcal{N} = 4$ SYM on a curved space endowed with a metric $g_{\mu\nu}$:

$$S_{YM} = \frac{1}{g^2} \int d^4 x \sqrt{g} \text{Tr} \left( \frac{1}{4} g^{\mu\lambda} g^{\nu\rho} F_{\mu\nu} F_{\lambda\rho} + \frac{1}{2} g^{\mu\nu} D_\mu \phi_m D_\nu \phi_m + \frac{1}{12} R \phi_m^2 - \frac{1}{4} [\phi_m, \phi_n]^2 \right),$$

(2.50)

where $R$ is the Ricci scalar constructed from $g_{\mu\nu}$. $R^4$ is transformed to $R \times S^3$ by the Weyl transformation. In fact, if one starts with the metric of $R^4$ in the polar coordinate and rewrites it as

$$ds^2_{R^4} = dr^2 + r^2 d\Omega_3^2 = e^{\mu r} \left( d\tau^2 + \left( \frac{2}{\mu} \right)^2 d\Omega_3^2 \right),$$

(2.51)
one obtains the metric of $R \times S^3$ up to the conformal factor $e^{\mu \tau}$. Namely, $\rho = -\mu \tau$ in this case. Here we have changed the coordinate from $r$ to $\tau \equiv \frac{2}{\mu} \log \left( \frac{r}{\mu} \right)$ where $2/\mu$ is the radius of the resultant $S^3$ as we have adopted so far. Thus $N = 4$ SYM on $R^4$ at the conformal point is equivalent to $N = 4$ SYM on $R \times S^3$.

Next, using the equations in appendix A, we rewrite the action (2.50) for $R \times S^3$ in terms of the Killing vector. We expand the gauge field on $S^3$ as

$$A = X_i e^i,$$  \hspace{1cm} (2.52)

where $e^i (i = 1, 2, 3)$ are defined in (A.3), and put $\phi_m = X_m$. By using (A.5), we rewrite (2.50) in terms of $X_i$ and $X_m$:

$$S = V_{S^3} \frac{g^2}{2\pi^2} \int d\tau \frac{d\Omega_3}{2\pi^2} \text{Tr} \left( \frac{1}{2} (D_\tau X_i - i\mu L_i A_\tau)^2 + \frac{1}{2} \left( \mu X_i + i\epsilon_{ijk} \left( \mu L_j X_k + \frac{1}{2} [X_j, X_k] \right) \right) \right)^2 + \frac{1}{2} (D_\tau X_m)^2 - \frac{1}{2} \left( \mu L_i X_m + [X_i, X_m] \right) \right)^2 + \frac{\mu^2}{8} X_m^2 - \frac{1}{4} [X_m, X_n]^2 \right), \hspace{1cm} (2.53)$$

where $D_\tau = \partial_\tau + i[A_\tau, ]$. Applying the rule (2.35) to the above action, we obtain a reduced model of $N = 4$ SYM on $R \times S^3$:

$$S_{YM,r} = \frac{V}{g^2} \int d\tau \text{Tr} \left( \frac{1}{2} (D_\tau X_i - i\mu [L_i, A_\tau])^2 + \frac{1}{2} \left( \mu X_i + i\epsilon_{ijk} \left( \mu [L_j, X_k] + \frac{1}{2} [X_j, X_k] \right) \right) \right)^2 + \frac{1}{2} (D_\tau X_m)^2 - \frac{1}{2} \left( \mu [L_i, X_m] + [X_i, X_m] \right) \right)^2 + \frac{\mu^2}{8} X_m^2 - \frac{1}{4} [X_m, X_n]^2 \right). \hspace{1cm} (2.54)$$

Note that the matrices still depend on $\tau$.

Here we can absorb $\mu L_i$ into $X_i$:

$$\mu L_i + X_i \rightarrow X_i. \hspace{1cm} (2.55)$$

In other words, we can regard $\mu L_i$ as a background of $X_i$. Note that one can not perform such an absorption in the case of the $\phi^3$ theory in the previous subsection. The resulting action is nothing but the plane wave matrix model [13]:

$$S_{PW} = \frac{1}{g_{PW}^2} \int d\tau \text{Tr} \left( \frac{1}{2} (D_\tau X_M)^2 - \frac{1}{4} [X_M, X_N]^2 + \frac{\mu^2}{2} X_i^2 + \frac{\mu^2}{8} X_m^2 + i\mu \epsilon_{ijk} X_i X_j X_k \right), \hspace{1cm} (2.56)$$
where $M$ runs from 1 to 9 and we put

$$\frac{g^2}{\nu} = g^2_{\text{PW}}. \quad (2.57)$$

This action is obtained by dimensionally reducing $\mathcal{N} = 4$ SYM on $R \times S^3$ over $S^3$ [16,17].

The PWMM possesses the $SU(2|4)$ supersymmetry, which includes 16 supercharges. The PWMM possesses many discrete vacua given by $X_i = \mu L_i$, where $L_i$ is given in (2.30), and they are degenerate. There are no classical moduli, and these vacua are classically stable. Furthermore, all of these vacua preserve the $SU(2|4)$ supersymmetry, and the theories around these vacua are massive. Hence they are also quantum mechanically stable at least at perturbative level [46]. We pick up a particular one specified by (2.33) from these vacua and take the limit (2.44) to obtain the reduced model of $\mathcal{N} = 4$ SYM on $R \times S^3$. In this limit, this vacuum is stable even at non-perturbative level, because tunneling to other vacua by an instanton effect is suppressed by the $k \to \infty$ limit. The global gauge invariance (depending only on $\tau$) of the PWMM is translated to the local gauge invariance of $\mathcal{N} = 4$ SYM. The $SU(2|4)$ symmetry is expected to enhance to the $PSU(2,2|4)$ symmetry in the continuum limit. Thus the reduced model (2.54) retrieves the planar limit of $\mathcal{N} = 4$ SYM on $R \times S^3$ as the reduced model (2.37) does the planar limit of (2.22).

Note that obtaining a reduced model of $\mathcal{N} = 4$ SYM on $R^4$ is quite non-trivial. For instance, dimensionally reducing it to $R$ yields the D0-brane effective theory or the Matrix theory [47], which is obtained by putting $\mu = 0$ in (2.56). This theory possesses infinitely many continuous vacua, which are characterized by $[X_M, X_N] = 0$. All of these vacua preserve sixteen supercharges as well. However, there are one-dimensional massless fields around these vacua at least perturbatively, so whether each vacuum is stable at quantum level heavily depends on dynamics. Thus one may not naively expect that a theory around a vacuum where $X_i$ are simultaneously diagonal and their eigenvalues distribute uniformly in $R^3$ retrieves the planar limit of $\mathcal{N} = 4$ SYM on $R^4$.

We summarize the prescription of the large-$N$ reduction for $\mathcal{N} = 4$ SYM on $R \times S^3$. We expand the PWMM (2.56) around the vacuum $X_i = \mu L_i$ with (2.30) and (2.33), and take the limit

$$k \to \infty, \quad \nu \to \infty, \quad n/\nu \to \infty \quad \text{with} \quad \frac{g^2_{\text{PW}} k}{n} = g^2 N \frac{V_{S^3}}{S^3} \text{ fixed}. \quad (2.58)$$
Then, the planar limit of $\mathcal{N} = 4$ SYM on $R \times S^3$ with the 't Hooft coupling $g^2 N$ is retrieved. The correspondence is explicitly given in (2.46) and (2.47). Thus the planar limit of $\mathcal{N} = 4$ SYM on $R \times S^3$ is regularized by the PWMM. This regularization preserves the $SU(2|4)$ symmetry including sixteen supercharges and the gauge symmetry. This number of the preserved supercharges is optimal. In the next section, we will see the correspondence between Wilson loops.

3 Wilson loops

In this section, we examine the correspondence between Wilson loops in $\mathcal{N} = 4$ SYM and its reduced model we reviewed in the previous section. In section 3.1, we review the Wilson loops considered in the gauge theory side in the context of the AdS/CFT correspondence. In section 3.2, we construct the corresponding Wilson loop in the reduced model.

3.1 Wilson loop in the AdS/CFT correspondence

The Wilson loop considered in the AdS/CFT correspondence takes the form

$$W(C) = \frac{1}{N} \text{Tr} \mathcal{P} \exp \left( i \int_0^1 ds \{ \dot{x}^\mu(s) A_\mu(x(s)) + i |\dot{x}(s)| \Theta^m(s) \phi_m(x(s)) \} \right),$$

(3.1)

where the function $x^\mu(s) : [0,1] \to C$ specifies the contour $C$, and $\Theta^m(s)$ satisfies $\delta_{mn} \Theta^m \Theta^n = 1$. The Wilson loop is invariant under the Weyl transformation (2.49), namely takes the same form both in $R^4$ and $R \times S^3$. The contour $C$ on $R^4$ is mapped to the corresponding contour $C'$ on $R \times S^3$ given by the coordinate transformation used in (2.51), and the following equality holds:

$$\langle W(C) \rangle_{R^4} = \langle W(C') \rangle_{R \times S^3}. \tag{3.2}$$

We next consider the supersymmetric property of the Wilson loop on $R \times S^3$ for later convenience. Its infinitesimal variation under the supersymmetry transformation is proportional to $(\dot{x}^\mu \Gamma_\mu + i |\dot{x}| \Theta^m \Gamma_m) \epsilon(x)$ where $\mu = \tau, \theta, \phi, \psi$ and $\epsilon(x)$ is a ten-dimensional Killing spinor on $R \times S^3$ satisfying the Killing spinor equation \cite{49,50},

$$\nabla_a \epsilon(x) = \pm \frac{\mu}{4} \Gamma_a \Gamma^1 \Gamma^2 \Gamma^3 \epsilon(x). \tag{3.3}$$
Here \( a = \tau, 1, 2, 3 \) is the local Lorentz index. This equation is solved by

\[
\epsilon_1(x) = e^{\frac{i}{2} \Gamma^{123}\tau} \eta_1, \quad \epsilon_2(x) = e^{-\frac{i}{2} \Gamma^{123}\tau} e^{-\frac{i}{2} \Gamma^{31}\theta} e^{-\frac{i}{2} \Gamma^{12}\psi} \eta_2,
\]

where \( \epsilon_1 \) and \( \epsilon_2 \) are the solutions to the upper and the lower signs of (3.3), respectively, and \( \eta_1 \) and \( \eta_2 \) are constant spinors. If the Wilson loop operator is invariant under some of the supersymmetries, there exist some non-zero components of \( \eta_{1,2} \) such that the following two equations hold,

\[
(\dot{x}^\mu \Gamma_\mu + i |\dot{x}| \Theta \Gamma^m \eta_1 = 0,
(\dot{x}^\mu \Gamma_\mu + i |\dot{x}| \Theta \Gamma^m \eta_2 = 0.
\]

Introducing local projection operators as

\[
P_{1\pm} = \frac{1}{2} e^{-\frac{i}{2} \Gamma^{123}\tau} (1 \pm \frac{i \dot{x}^\mu}{|\dot{x}|} \Theta^m \Gamma_{\mu m}) e^{\frac{i}{2} \Gamma^{123}\tau},
P_{2\pm} = \frac{1}{2} e^{\frac{i}{2} \Gamma^{12}\psi} e^{\frac{i}{2} \Gamma^{31}\theta} e^{\frac{i}{2} \Gamma^{12}\tau} (1 \pm \frac{i \dot{x}^\mu}{|\dot{x}|} \Theta^m \Gamma_{\mu m}) e^{-\frac{i}{2} \Gamma^{123}\tau} e^{-\frac{i}{2} \Gamma^{12}\theta} e^{-\frac{i}{2} \Gamma^{12}\psi},
\]

one can rewrite (3.5) simply as

\[
P_{1\pm} \eta_1 = 0, \quad P_{2\pm} \eta_2 = 0.
\]

The number of independent non-zero components of \( \eta_1 \) and \( \eta_2 \) which satisfy (3.7) (or equivalently (3.5)) is just the number of supersymmetries preserved by the insertion of the operator.

As an example, let us consider the case where \( \dot{x}^\mu(s) \) and \( \Theta(s) \) are constant in \( s \) and the contour does not extend to the \( \tau \) direction. In this case, the projection operator \( P_1 \) is a constant operator on the contour, so that the half components of \( \eta_1 \) which are projected by \( P_{1\pm} \) into 0 can be non-zero constants. Therefore, the operator preserves at least 8 supersymmetries (1/4 BPS). We can also consider a special case of this example in which the path is given by a great circle on \( S^3 \) at a fixed value of \( \tau \),

\[
C : (\tau(s), \theta(s), \varphi(s), \psi(s)) = (\tau_0, 0, 0, 4\pi s),
\]

where \( \tau_0 \) is a constant. In addition to the above mentioned 8 supersymmetries, another 8 supercharges are preserved in this case because \( P_2 \) is also a constant operator on the contour (3.8) as well as \( P_1 \). Hence, the Wilson loop on the great circle (3.8) is a half-BPS operator.
The contour (3.8) is mapped to the circular loop, whose center is located at the origin on $R^4$, by the Weyl transformation (2.49). For the circular contour with the unit radius, $x^\mu(s)$ can be parametrized as

$$\{x^\mu(s)\} = (\cos(2\pi s), \sin(2\pi s), 0, 0).$$  \hspace{1cm} (3.9)

Because of (3.2), the VEV of the Wilson loop with the contour (3.8) on $R \times S^3$ coincides with the VEV of the Wilson loop with the contour (3.9) on $R^4$.

The VEV of the BPS circular Wilson loop on $R^4$ can be computed including all orders in the perturbative expansion in the planar limit \cite{39,40}. Let us review the result of this computation. The VEV of (3.1) for the path (3.9) turns out to be given by the following VEV in the Gaussian matrix model,

$$\langle W(\text{circle}) \rangle = \left\langle \frac{1}{k} \text{Tr} \exp(M) \right\rangle \equiv \frac{1}{Z} \int DM \frac{1}{k} \text{Tr} \exp(M) \exp \left(-\frac{k}{\lambda} \text{Tr} M^2\right) = \sqrt{\frac{2}{\lambda}} I_1(\sqrt{2\lambda}),$$  \hspace{1cm} (3.10)

where $\lambda$ is the 't Hooft coupling and $I_1$ is a Bessel function. The strong coupling expansion of this result reproduces the correct coupling behavior $e^{\sqrt{\lambda}}$ observed in the gravity side.

### 3.2 Wilson loop in the reduced model

By using the relation (2.46), we can construct an operator in the reduced model (2.54) corresponding to the Wilson loop (3.1):

$$\hat{W}(C) = \frac{1}{N} \text{Tr} \left[ \mathcal{P} \exp \left\{ i \int_0^1 ds \left( \dot{x}^\tau(s) G^{-1}(s) A_\tau(\tau(s)) G(s) 
+ \dot{x}^\bar{\mu}(s) e^\bar{i}_\bar{\mu}(x^\bar{\mu}(s)) G^{-1}(s) X_\bar{i}(\tau(s)) G(s) 
+ i|\dot{x}(s)||\Theta^m(s) G^{-1}(s) X_m(\tau(s)) G(s) \right) \right\} \right],$$  \hspace{1cm} (3.11)

where $\bar{\mu} = \theta, \varphi, \psi$ and $e^i$ is the right invariant 1-form on $S^3$ defined in \cite{A3}. $G(s)$ is given by

$$G(s) = P \exp \left[ i \int_0^s ds' \dot{x}^\bar{\mu}(s') e^i_\bar{\mu}(x^\bar{\mu}(s')) \mu L_i \right],$$  \hspace{1cm} (3.12)

\footnote{Our convention of the 't Hooft coupling is slightly different from that in \cite{39}. The relation is given by $\lambda_{\text{ours}} = \lambda_{\text{ESZ}}/2$.}
where $L_i$ is given by (2.30) and (2.33). Note that $G(1) = 1$. By referring to an argument in [48], we can show that (3.11) is equal to [32]

\[
\hat{W}(C) = \frac{1}{N} \text{Tr} \left[ \mathcal{P} \exp \left\{ i \int_0^1 ds \left( \dot{x}^\tau(s) A_\tau(\tau(s)) + \dot{x}^\mu(s) e^i_{\mu}(x^\mu(s))(\mu L_i + X_i(\tau(s))) \right) + i|\dot{x}(s)|\Theta^m(s)X_m(\tau(s)) \right\} \right].
\]

(3.13)

In fact, (3.11) can be viewed as $1/N$ times the trace of the time evolution kernel for a time-dependent Hamiltonian $H(s) = -\{ \dot{x}^\mu(s)e^i_{\mu}(x^\mu(s))\mu L_i + \dot{x}^\tau(s)A_\tau(\tau(s)) + \dot{x}^\mu(s)e^i_{\mu}(x^\mu(s))X_i(\tau(s)) \} + i|\dot{x}(s)|\Theta^m(s)X_m(\tau(s))$. (3.14)

If one regards the first term as a free part and the others as an interaction part and switches the picture to the interaction one, one obtains (3.11). Note that applying (2.55) to (3.13) results in the dimensional reduction of (3.1) from $R \times S^3$ to $R$. In other words, we expand $X_i$ around $\mu L_i$ in the dimensionally reduced Wilson loop. Such a correspondence for the Wilson loops is seen commonly in the large-$N$ reduction.

From (2.46), we expect

\[
\langle \hat{W}(C) \rangle_r = \langle W(C) \rangle.
\]

(3.15)

In the next section, we verify this relation in the case of the half-BPS Wilson loop with the contour $C$ given in (3.8).

4 VEV of the half-BPS Wilson loop from PWMM

In this section, we calculate VEV of the operator (3.13) with the contour (3.8).

For the calculation of the circular BPS Wilson loop in SYM on $R^4$, it is convenient to take the Feynman gauge where the computation is simplified such that the non-ladder diagrams cancel out [39]. One can expect that this simplification also occurs in the PWMM if one takes a particular gauge which corresponds to the Feynman gauge in the continuum limit. This leads us to take this particular gauge in the PWMM for the following computation.

The corresponding gauge fixing term on $R \times S^3$ can be obtained by applying the Weyl transformation. Although the original action is invariant under the Weyl transformation
the gauge fixing term is not invariant in general. In fact, if one takes the Feynman gauge on $R^d$, the gauge fixing term is transformed as

$$
\frac{1}{2} \sqrt{g_{R^4}} \left( g_{\mu R^4} A_{R^4} \right)^2 = \frac{1}{2} \sqrt{g_{R^4}} \left\{ \partial_\mu \left( \sqrt{g_{R^4}} g_{\mu R^4} A_{R^4} \right) \right\}^2 = \frac{e^{-2\mu R}}{2 \sqrt{g_{R^4}}} \left\{ \partial_\mu \left( e^{\mu R} \sqrt{g_{R^4 S^3}} g_{\mu R^4 S^3} A_{R^4 S^3} + \mu A_{R^4 S^3} \right) \right\}^2
$$

(4.1)

Hence, the corresponding gauge fixing term on $R \times S^3$ should be given by the last line of (4.1).

We then apply the reduction rule (2.35) to the above gauge fixing term plus appropriate ghost terms to obtain those in the PWMM,

$$S_{gf + gh} = \frac{1}{g_{PW}} \int d\tau \text{Tr} \left\{ \frac{1}{2} (\partial_\tau A_\tau + i\mu [L_i, X_i] + \mu A_\tau)^2 - i\tilde{c} (\partial_\tau D_\tau + \mu D_\tau) c + \mu c [L_i, i\mu [L_i, c] + i[c, X_i]] \right\},$$

(4.2)

where $c$ and $\tilde{c}$ are the ghost fields.

We add (4.2) to the original action of the PWMM expanded around the background (2.30) with the parameters given by (2.33) and construct the Feynman rule of this theory. We make a mode expansion of the matrices as in (C.9) and read off the propagators for each momentum mode as

$$\langle x_{jm\rho}(p) | x_{jm'\rho'}(p') \rangle_{kl} = \begin{cases} \frac{g_{PW}^2}{n} (-1)^{m-(j-s-j')} \frac{\delta_{J,l}}{\delta_{J',l'}} \frac{\delta_{m,l}}{\delta_{m',l'}} \frac{\delta_{\rho,l}}{\delta_{\rho',l'}} \delta_{ts} \delta_{tt'} \delta_{ij} \delta_{j'k} 2\pi \delta(p + p') \frac{1}{p^2 + \omega^2_{j}} & (\rho \neq 0) \\ \frac{g_{PW}^2}{n} (-1)^{m-(j-s-j)} \frac{\delta_{J,l}}{\delta_{J',l'}} \frac{\delta_{m,l}}{\delta_{m',l'}} \frac{\delta_{\rho,l}}{\delta_{\rho',l'}} \delta_{ts} \delta_{tt'} \delta_{ij} \delta_{j'k} 2\pi \delta(p + p') \frac{p^2 + \mu^2 J(J + 1)}{(p^2 + \mu^2 J(J + 1))} & (\rho = \rho' = 0) \end{cases},$$

(4.3)

where $\omega^\phi_j, \omega^\psi_j$ and $\omega^\phi_{\bar{j}}$ are defined in (C.11). We can also read off the interaction vertices in this mode expansion. In particular the vertices used in this section take the same form as in (C.13) although a different gauge is taken in appendix C.
In general, the tadpole is expected to vanish as in the continuum theory. Let us check it at the 1-loop level in the present gauge. The only possibly nonzero contribution is the truncated 1-point function for $x_{j_{mp}}^{(s,t)}(p)_{ij}$, where $i,j$ run from 1 to $k$ and $p$ is the momentum along the $\tau$ direction. The 1-point function takes the form,

$$2\pi \delta(p)\delta_{st}\delta_{ij}\delta_{\rho-1}\delta_{J0}\delta_{m0}\delta_{ij} \Upsilon^{(s)}.$$

There are six diagrams for the 1-loop correction to (4.4) as shown in Fig. 3. Note that all these diagrams are planar ones. The diagrams $(T - a)$ and $(T - b)$ completely cancel each other. The diagram $(T - f)$ vanishes under the integration of $p$ because it is an odd function of $p$. Below we list the value of $\Upsilon^{(s)}$ for each of the remaining diagrams,

$$(T - c) = 4 \frac{g^2_{PW}}{\sqrt{n}} \sum_{t,R} k_t (-1)^{R+j_s+j_t} \sqrt{R(R+1)(2R+1)} \left\{ 1_{j_t j_s j_s} R R R \right\},$$

$$(T - d) = 12 \frac{g^2_{PW}}{\sqrt{n}} \sum_{t,R} k_t (-1)^{R+j_s+j_t} \sqrt{R(R+1)(2R+1)} \left\{ 1_{j_t j_s j_s} R R \right\},$$

$$(T - e) = -16 \frac{g^2_{PW}}{\sqrt{n}} \sum_{t,R} k_t (-1)^{R+j_s+j_t} \sqrt{R(R+1)(2R+1)} \left\{ 1_{j_t j_s j_s} R R \right\},$$

where the last factor in the above equations is the 6-j symbol. We therefore find that the 1-loop contribution to the tadpole in the present gauge is indeed vanishing without taking the continuum limit.

Let us calculate the VEV of the Wilson loop (3.11) with the contour given by a great circle (3.8). Thanks to the $SO(6)_R$ rotational invariance, we can set $\Theta^m = \delta^{m4}$ without loss.
of generality. Then, the operator (3.11) for the great circle can be written in a relatively simple form by substituting (A.4) for the circular contour,

\[ \hat{W} = \frac{1}{\nu nk} \text{Tr} \left( e^{4\pi i(L_3 + H)} \right), \tag{4.8} \]

where we have defined a dimensionless complex matrix \( H \) as

\[ H = \frac{1}{\mu} (X_3 + iX_4). \tag{4.9} \]

\( H \) is expanded in terms of the fuzzy spherical harmonics as

\[ H(s,t) = \frac{1}{\mu} \sum_{R,m} \left( iC_{Rm0}^{R+1m} x_{Rm1}^{(s,t)} + C_{Rm0}^{Rm} x_{Rm0}^{(s,t)} - iC_{Rm0}^{R-1m} x_{Rm-1}^{(s,t)} + i(\phi_{AB,Rm}^{(s,t)} + \phi_{23,Rm}^{(s,t)}) \right) \otimes \hat{Y}_{Rm}^{(s,t)}, \tag{4.10} \]

where we have used (B.10), (C.2) and (C.9). Here the second term \( x_{Rm0}^{(s,t)} \) and the third term \( x_{Rm-1}^{(s,t)} \) in the right-hand side of the first line are summed over \( R \geq \frac{1}{2} \) and \( R \geq 1 \), respectively, while the first term \( x_{Rm1}^{(s,t)} \) and the last term \( \phi_{AB,Rm}^{(s,t)} \) are summed over \( R \geq 0 \).

Using (4.3) and the explicit expression of the Clebsch-Gordan coefficients

\[ C_{Rm0}^{R+1m} = \left[ \frac{(R + 1 + m)(R + 1 - m)}{(2R + 1)(R + 1)} \right]^{1/2}, \]
\[ C_{Rm0}^{Rm} = \left[ \frac{m}{R(R + 1)} \right]^{1/2}, \]
\[ C_{Rm0}^{R-1m} = \left[ \frac{(R + m)(R - m)}{R(2R + 1)} \right]^{1/2}, \tag{4.11} \]

we can calculate the propagator for \( H \). In particular, the equal-time propagator is simply given by

\[ \langle H_{Rm}^{(s,t)}(\tau)_{ij} H_{R'm'}^{(s',t')}^{(s,t)}(\tau)_{kl} \rangle = -\delta_{R0} \delta_{R'0} \delta_{m0} \delta_{m'0} \delta_{s,t} \delta_{s',t'} \delta_{\alpha,i} \delta_{\beta,j} \frac{g^2}{2\mu^3 n}. \tag{4.12} \]

Namely, only the zero mode contributes to the propagator of \( H \) in the present gauge. We expand the VEV of the Wilson loop as

\[ \langle \hat{W} \rangle = \frac{1}{\nu nk} \sum_{q=0}^{\infty} \frac{1}{q!} \left( \frac{d^q}{da^q} \text{Tr}(e^{4\pi i(L_3 + aH)}) \right)_{a=0} \]
\[ = \frac{1}{\nu nk} \sum_{q=0}^{\infty} \int_0^1 d\alpha_1 \int_0^1 d\alpha_2 \cdots \int_0^1 d\alpha_{q+1} \delta(1 - \alpha_1 - \alpha_2 - \cdots - \alpha_{q+1}) \]

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\[ \langle \hat{W} \rangle = \frac{1}{\nu k} \sum_{q=0}^{\infty} \sum_{\{s_i\}} \int_0^1 \cdots \int_0^1 d\alpha_{q+1} \delta(1 - \alpha_1 - \alpha_2 - \cdots - \alpha_{q+1}) \times e^{4\pi i \alpha_1 r_1} e^{4\pi i \alpha_2 r_2} \cdots e^{4\pi i \alpha_{q+1} r_{q+1}} (4\pi i)^q (-1)^{r_1 + r_2 + r_3 + \cdots + r_q - j_q} \times C_{R_1, m_1}^{R_2, m_2} \cdots C_{R_q, m_q}^{R_1, m_1} \text{Tr} \langle H^{(s_1, s_2)}_{R_1, m_1} H^{(s_2, s_3)}_{R_2, m_2} \cdots H^{(s_{q}, s_1)}_{R_q, m_q} \rangle, \]  

(4.14)

where \( C^{R_m}_{j_{1r_1}j_{2r_2}}^{R'_{m'}} \) is the Clebsch-Gordan coefficient.

In the following, we assume that only the planar ladder diagrams (i.e. planar diagrams without vertices, see Fig. 4) contribute to the VEV as in the case of the continuum theory. This assumption is reasonable since we are working in the gauge corresponding to the Feynman gauge on \( R^4 \). In the Feynman gauge, the ladder approximation gives an exact result in the case of the circular Wilson loop in SYM on \( R^4 \). Hence, also for the PWMM, we believe that the non-ladder diagrams do not contribute to the VEV in the continuum limit in the present gauge.

Within this assumption, only the modes with \( R = m = 0 \) contribute to \( \langle \hat{W} \rangle \) because the tree-level propagator (4.12) is vanishing unless \( R = m = 0 \). Furthermore, only the terms with \( q \) even contribute. Thus, we obtain,

\[ \langle \hat{W} \rangle = \frac{1}{\nu k} \sum_{q=0}^{\infty} \sum_s \frac{(4\pi i)^2q}{(2q)!} \left( \frac{n}{2j_s + 1} \right)^{q-1} \text{Tr} \langle (H^{(s,s)}_{00})^{2q} \rangle, \]  

(4.15)

\[ \times (4\pi i)q \text{Tr} \langle e^{4\pi i \alpha_1 L_3} H e^{4\pi i \alpha_2 L_3} H \cdots e^{4\pi i \alpha_q L_3} H e^{4\pi i \alpha_{q+1} L_3} \rangle, \]  

(4.13)
where we have used $C^0_{jrjr} = \frac{1}{\sqrt{2j+1}}$. Taking the limit (2.58), we obtain

\[ \langle \hat{W} \rangle \to \frac{1}{k} \sum_{q=0}^{\infty} \frac{1}{(2q)!} \langle \text{Tr}(4\pi i H_{00}^{(0,0)})^{2q} \rangle \]

\[ = \frac{1}{Z} \int \mathcal{D}M \frac{1}{k} \text{Tr} \exp(M) \exp \left( -\frac{k}{\lambda} \text{Tr} M^2 \right). \tag{4.16} \]

On the right hand side of the first line of (4.16), we have put $(s, s)$ to $(0, 0)$ because the propagator does not depend on $s$. In the second line, we have put $\lambda = \frac{V_{esg^2}{W}^k}{n}$ and identified $4\pi i H_{00}^{(0,0)}$ with a $k \times k$ hermitian matrix $M$. Although $H_{00}^{(0,0)}$ is a complex matrix and time-dependent, this identification is possible within the ladder approximation where only the equal-time propagator in the free theory (4.12) is needed for the computation of the Wilson loop. Since the equal-time propagator of $4\pi i H_{00}^{(0,0)}$ takes the same value as that of $M$ if we take the weight of the integral of $M$ as in (4.16), the computational rules are identical between the two expressions. The result (4.16) agrees with (3.10).

Here, we again emphasize that in a general gauge one needs to add also the planar non-ladder diagrams (See Fig. 5) to obtain the correct result (3.10) in the continuum theory. In fact, we checked that in a different gauge which will be introduced in the next section the sum over only the planar ladder diagrams does not coincide with (3.10). This is consistent with the fact that the value of each Feynman diagram depends on a gauge choice and the non-ladder diagrams cancel out only in the Feynman gauge on $R^4$.

5 Beta function

In this section, we compute the beta function at the 1-loop level in the PWMM expanded around the background specified by (2.33) and show that it is indeed vanishing. In this section, we take the gauge shown in appendix C for convenience which is different from what we used in the previous section. We consider only the planar diagrams in the following.

We first compute the wave function renormalization of the $SO(6)$ scalar field. The self-energy is given by the truncated two-point function $\langle \phi_{ABJm}(p)\phi_{CDJ'm'}(p') \rangle$ which takes the form,

\[ 2\pi \delta(p - p') \delta_{sr} \delta_{s'r'} \frac{1}{2} \epsilon_{ABC} \delta_{ih} \delta_{jk} \delta_{JJP'} \delta_{m-m'} \Xi^{(s,t)} (p). \tag{5.1} \]

Here, the limit $\nu \to \infty$ is not used explicitly. However this limit should be needed for the non-ladder diagrams (See Fig. 5) to cancel out.
There are six diagrams contributing to the 1-loop correction to the self-energy as shown in Fig. 6. For example, one can compute $(S - 2)$ in Fig. 6 as follows. In terms of the Feynman rules shown in appendix C, the contribution of $(S - 2)$ to $\Xi^{(\kappa_1)}_j(p)$ in (5.1) can be written as,

$$-2k \sum_{j,m_1,k_1,j_2m_2k_2} \int \frac{dq}{2\pi i} = \frac{\kappa_1}{\omega_{j_1}^2} \frac{i(p - q) + \kappa_2}{(\omega_{j_2}^2)^2} \hat{F}_{j_1 \rightarrow \mu} \hat{F}_{j_2 \rightarrow \nu} \cdot \hat{F}_{j_1 \rightarrow \mu} \hat{F}_{j_2 \rightarrow \nu},$$

(5.2)

where the ranges of the variables in $\sum_{j,m_1,k_1}$ are given by $\kappa_i = \pm 1$, $m_i = -J_i, -J_i + 1, \ldots, J_i$ $(i = 1, 2)$, $J_1 - \kappa_1 = |j_s - j_u|, |j_s - j_u| + 1, \ldots, j_s + j_u$ and $J_2 - \kappa_2 = |j_t - j_u|, |j_t - j_u| + 1, \ldots, j_t + j_u$. By performing the integration, substituting the explicit form of $\hat{F}$ shown in (B.17) and finally taking the summation over $m_1, m_2, k_1$ and $k_2$, one can obtain,

$$32\mu k(-1)^{m-(j_s-j_t)} \sum_{R_1=|j_s-j_u|}^{j_s+j_u} \sum_{R_2=|j_t-j_u|}^{j_t+j_u} (2J + 1)(2R_1 + 1)(2R_2 + 1) \times \left[ \frac{(R_1 + 1)(R_2 + 1)(R_1 + R_2 + \frac{3}{2})}{p^2 + \mu^2 (R_1 + R_2 + \frac{3}{2})^2} \right] \left[ \frac{R_1 + \frac{1}{2}}{R_2 + \frac{1}{2}} \right] \left[ \frac{R_1}{R_2} \right] \left[ \frac{R_1 + \frac{1}{2}}{R_2 + \frac{1}{2}} \right] \left[ \frac{R_1}{R_2} \right] \left[ \frac{R_1 + \frac{1}{2}}{R_2 + \frac{1}{2}} \right] \left[ \frac{R_1}{R_2} \right] \left[ \frac{R_1 + \frac{1}{2}}{R_2 + \frac{1}{2}} \right] \left[ \frac{R_1}{R_2} \right] \left[ \frac{R_1 + \frac{1}{2}}{R_2 + \frac{1}{2}} \right] \left[ \frac{R_1}{R_2} \right] \left[ \frac{R_1 + \frac{1}{2}}{R_2 + \frac{1}{2}} \right] \left[ \frac{R_1}{R_2} \right],$$

(5.3)

where $R_i$ are defined as $R_i = J_i - \kappa_i$ for $\kappa_i = \pm 1$. Then, if one sets $J = 0$ and $s = t = 0$, it is easy to compute the divergent part of (5.3). In this case, the explicit formula for the $6j$-symbol and $9j$-symbol can be found in [56]. Substituting those explicit forms, one finds that the divergent part is given as

$$\frac{k}{\mu} \sum_{u=-(\nu-1)/2}^{(\nu-1)/2} \sum_{R=|u|/2}^{n-1+u/2} \left\{ 8 - \frac{2}{R^2} \left( \frac{p^2}{\mu^2} + \frac{1}{4} \right) \right\},$$

(5.4)

Similarly, the divergent parts in the other diagrams in Fig. 6 are evaluated as

$$(S - 1) = \frac{k}{\mu} \sum_{u=-(\nu-1)/2}^{(\nu-1)/2} \sum_{R=|u|/2}^{n-1+u/2} (-5),$$

$$(S - 3) = \frac{k}{\mu} \sum_{u=-(\nu-1)/2}^{(\nu-1)/2} \sum_{R=|u|/2}^{n-1+u/2} \left( -3 + \frac{3}{8R^2} \right),$$

$$(S - 4) = \frac{k}{\mu} \sum_{u=-(\nu-1)/2}^{(\nu-1)/2} \sum_{R=|u|/2}^{n-1+u/2} \left\{ \frac{1}{2} - \frac{1}{8R^2} \left( \frac{p^2}{\mu^2} + \frac{1}{4} \right) \right\},$$

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Here, it is important that the quadratic divergence cancels when one takes the sum over all the diagrams in Fig. 6. Note also that there is no divergence even if one takes the large-$n$ limit. This limit corresponds to the commutative limit of fuzzy sphere in which the theory on $R \times S^2$ is realized. The fact that there is no divergence due to $n$ is consistent with the super renormalizability of (2+1) dimensional gauge theory.

From (5.4) and (5.5), we find that the wave function renormalization for the $SO(6)$ scalars is given by

$$Z_\phi = 1 - \frac{4g^2_{ PW}}{\mu^2 n} \log \nu. \quad (5.6)$$

Here, we have introduced the following expression for the logarithmic divergence,

$$\sum_{u=-(n-1)/2}^{(n-1)/2} \sum_{R=|u/2|}^{\infty} \frac{1}{R^2} \equiv 4 \log \nu. \quad (5.7)$$
The wave function renormalization of the fermions can be computed in the same way \[12\].

The result is given by

\[ Z_{\psi} = 1 - \frac{16g_{PW}^2 k}{\mu^3 n} \log \nu. \] (5.8)

Next, we compute the renormalization of the coupling which can be read off from the truncated three point function of fermions and a scalar, \( \langle \phi_{ABJ_1m_1}(p_1)\psi_{CJ_2m_2\kappa_2}(p_2)\psi_{DJ_3m_3\kappa_3}(p_3) \rangle \).

We take the simplest choice of the external momenta: \( J_i = 0, p_i = 0, s_i = t_i = 0 \) \((i = 1, 2, 3)\), \( m_1 = 0 \) and \( \kappa_1 = \kappa_2 = 1 \). In Fig. 7 all the diagrams contributing to the 1-loop corrections are listed. By following the same calculation that we described for the self-energy, we find the following values for the divergent part of each diagram,

\[
\begin{align*}
(Y - 1) &= \frac{k}{\mu^3} \epsilon_{ABCD}(-1)^{m+\frac{1}{2}} \sum_{u=-(\nu-1)/2}^{(\nu-1)/2} \sum_{R=|u|/2}^{n-1+u/2} \frac{2}{R^2}, \\
(Y - 2) &= \frac{k}{\mu^3} \epsilon_{ABCD}(-1)^{m+\frac{1}{2}} \sum_{u=-(\nu-1)/2}^{(\nu-1)/2} \sum_{R=|u|/2}^{n-1+u/2} \frac{3}{2R^2}, \\
(Y - 3) &= \frac{k}{\mu^3} \epsilon_{ABCD}(-1)^{m+\frac{1}{2}} \sum_{u=-(\nu-1)/2}^{(\nu-1)/2} \sum_{R=|u|/2}^{n-1+u/2} \frac{1}{2R^2}, \\
(Y - 4) &= \frac{k}{\mu^3} \epsilon_{ABCD}(-1)^{m+\frac{1}{2}} \sum_{u=-(\nu-1)/2}^{(\nu-1)/2} \sum_{R=|u|/2}^{n-1+u/2} \frac{3}{4R^2}, \\
(Y - 5) &= \frac{k}{\mu^3} \epsilon_{ABCD}(-1)^{m+\frac{1}{2}} \sum_{u=-(\nu-1)/2}^{(\nu-1)/2} \sum_{R=|u|/2}^{n-1+u/2} \left( -\frac{1}{4R^2} \right). 
\end{align*}
\] (5.9)

Comparing the sum of the above diagrams with the tree level result, one obtains the renormalization for the coupling at the 1-loop level,

\[ Z_{g_{PW}} = 1 - \frac{18g_{PW}^2 k}{\mu^3 n} \log \nu. \] (5.10)

This satisfies \( Z_{g_{PW}} = Z_{\psi}Z_{\phi}^{\frac{1}{2}} \) and therefore the beta function is vanishing at the 1-loop level.

6 Conclusion and discussion

In this paper, we study a non-perturbative formulation of \( \mathcal{N} = 4 \) SYM on \( R \times S^3 \) proposed in \[12\]. After reviewing the formulation, we introduce Wilson loop operators in the formulation
which correspond to Wilson loops in $\mathcal{N} = 4$ SYM studied in the context of the AdS/CFT correspondence. We calculate the VEV of the half-BPS Wilson loop by summing up all the planar ladder diagrams as done in the continuum theory in [39, 40] and reproduce the known result. We also calculate the one-loop beta function and verify that it vanishes in the continuum limit, which is consistent with restoration of the superconformal invariance. Our results serves as a check of the formulation.

We should check that the planar non-ladder diagrams for the half-BPS Wilson loop are indeed canceled out at lower orders in the perturbative expansion, as done in [39], while the calculation in our case seems hard work. It is also desirable to derive the result using a method such as localization as done in the continuum theory in [45]. It is interesting to do the calculation of this paper in another non-perturbative formulation of $\mathcal{N} = 4$ SYM on $R \times S^3$ proposed in [36]. It is quite important to calculate the VEV of non-BPS Wilson loops such as a rectangular Wilson loop on $R^4$ in the strong coupling regime in the present formulation and reproduce the prediction from the gravity side [42]. This kind of calculation on the gauge theory side not done yet would serve as a highly nontrivial test of the AdS/CFT correspondence.

We hope that our calculation in this paper will trigger development in the study of the
Note

Preliminary result on the beta function in this paper was announced by G. I. and A. T. at the International Conference on Progress of String Theory and Quantum Fields Theory, Osaka City University, 7-10 December 2007 [57, 58].

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A \(S^3\) and the SU(2) group manifold

In this appendix, we summarize some useful facts about \(S^3\) and the SU(2) group manifold (see also [12,15]). We regard \(S^3\) as the SU(2) group manifold. We parameterize an element of SU(2) in terms of the Euler angles as

\[
g = e^{-i\varphi J_3}e^{-i\theta J_2}e^{-i\psi J_3},
\]

where \(J_i\) are the generators of the SU(2) algebra, and \(0 \leq \theta \leq \pi\), \(0 \leq \varphi < 2\pi\), \(0 \leq \psi < 4\pi\). The periodicity with respect to these angle variables is expressed as

\[
(\theta, \varphi, \psi) \sim (\theta, \varphi + 2\pi, \psi + 2\pi) \sim (\theta, \varphi, \psi + 4\pi).
\]

The isometry of \(S^3\) is \(SO(4) = SU(2) \times SU(2)\), and these two SU(2)’s act on \(g\) from left and right, respectively. We construct the right-invariant 1-forms,

\[
dgg^{-1} = -i\mu e^J_i,
\]
where the radius of $S^3$ is $2/\mu$. They are explicitly given by
\begin{align*}
e^1 &= \frac{1}{\mu}(-\sin \varphi d\theta + \sin \theta \cos \varphi d\psi), \\
e^2 &= \frac{1}{\mu}(\cos \varphi d\theta + \sin \theta \sin \varphi d\psi), \\
e^3 &= \frac{1}{\mu}(d\varphi + \cos \theta d\psi),
\end{align*}
(A.4)
and satisfy the Maurer-Cartan equation
\begin{equation}
de^i - \frac{\mu}{2} \epsilon_{ijk} e^j \wedge e^k = 0.
\end{equation}
(A.5)
The metric is constructed from $e^i$ as
\begin{equation}
ds^2 = e^i e^i = \frac{1}{\mu^2} \left(d\theta^2 + \sin^2 \theta d\varphi^2 + (d\psi + \cos \theta d\varphi)^2\right).
\end{equation}
(A.6)
The Haar measure is defined through above metric as
\begin{equation}
d\Omega_3 = \frac{1}{8} \sin \theta d\theta d\phi d\psi,
\end{equation}
(A.7)
which is left and right invariant. The Killing vectors dual to $e^i$ are given by
\begin{equation}
\mathcal{L}_i = -\frac{i}{\mu} e^\mu \partial_\mu,
\end{equation}
(A.8)
where $\bar{\mu} = \theta, \varphi, \psi$ and $e^\mu_i$ are inverse of $e^i_{\bar{\mu}}$. It follows from (A.3) and (A.8) that
\begin{equation}
\mathcal{L}_i g = -J_i g,
\end{equation}
(A.9)
which indicates that $\mathcal{L}_i$ are the generators of the left translation and satisfy
\begin{equation}
[\mathcal{L}_i, \mathcal{L}_j] = i \epsilon_{ijk} \mathcal{L}_k.
\end{equation}
(A.10)
The explicit form of the Killing vectors are
\begin{align*}
\mathcal{L}_1 &= -i \left(-\sin \varphi \partial_\theta - \cot \theta \cos \varphi \partial_\varphi + \frac{\cos \varphi}{\sin \theta} \partial_\psi\right), \\
\mathcal{L}_2 &= -i \left(\cos \varphi \partial_\theta - \cot \theta \sin \varphi \partial_\varphi + \frac{\sin \varphi}{\sin \theta} \partial_\psi\right), \\
\mathcal{L}_3 &= -i \partial_\varphi.
\end{align*}
(A.11)
The spherical harmonics on $S^3$ is defined through Wigner’s D-function (see also [14,50]):
\begin{equation}
Y_{Jm\bar{m}}(\Omega_3) = (-1)^{J-m} \sqrt{2J+1} D_J(-m, \bar{m})
\end{equation}
where $g$ is given in (A.1). The complex conjugate is given by
\begin{align*}
Y^*_{J m \bar{m}}(\Omega_3) &= (-1)^{J-m} \sqrt{2J + 1} \langle J - m | g | J \bar{m} \rangle \\
&= (-1)^{J-m} \sqrt{2J + 1} e^{i m \phi - im \psi} \langle J - m | e^{-i \theta J_z} | J \bar{m} \rangle,
\end{align*}
(A.12)

where $m$ is given in (A.1). The complex conjugate is given by
\begin{align*}
Y^*_{J m \bar{m}}(\Omega_3) &= (-1)^{J-m} \sqrt{2J + 1} e^{-i m \phi + im \psi} \langle J - m | e^{i \theta J_z} | J \bar{m} \rangle \\
&= (-1)^{J-m} \sqrt{2J + 1} e^{-i m \phi + im \psi} (-1)^{m-\bar{m}} \langle Jm | e^{i \theta J_z} | J - \bar{m} \rangle \\
&= (-1)^{m-\bar{m}} Y_{J-m-\bar{m}}(\Omega_3).
\end{align*}
(A.13)

It follows from (A.9) that
\begin{align*}
\mathcal{L}^2_{J} Y_{J m \bar{m}}(\Omega_3) &= J(J + 1) Y_{J m \bar{m}}(\Omega_3), \\
\mathcal{L}_\pm Y_{J m \bar{m}}(\Omega_3) &= \sqrt{(J \pm m)(J \pm m + 1)} Y_{J m \pm 1 \bar{m}}, \\
\mathcal{L}_3 Y_{J m \bar{m}} &= m Y_{J m \bar{m}}.
\end{align*}
(A.14)

By using the orthogonality relation
\begin{equation}
\int dg \langle J \bar{m} | g^{-1} | J m \rangle \langle J' m' | g | J' \bar{m}' \rangle = \frac{1}{2J + 1} \delta_{J,J'} \delta_{m,m'} \delta_{\bar{m},\bar{m}'},
\end{equation}
(A.15)

where $dg = d\Omega_3/(2\pi^2)$ is the Haar measure, one can show the following equalities:
\begin{equation}
\int \frac{d\Omega_3}{2\pi^2} Y^*_{J m_1 \bar{m}_1}(\Omega_3) Y_{J m_2 \bar{m}_2}(\Omega_3) = \delta_{J_1,J_2} \delta_{m_1,m_2} \delta_{\bar{m}_1,\bar{m}_2}
\end{equation}
(A.16)

and
\begin{align*}
\mathcal{C}_{J m_1 \bar{m}_1 J m_2 \bar{m}_2 J m_3 \bar{m}_3} &= \int \frac{d\Omega_3}{2\pi^2} Y^*_{J m_1 \bar{m}_1}(\Omega_3) Y_{J m_2 \bar{m}_2}(\Omega_3) Y_{J m_3 \bar{m}_3}(\Omega_3) \\
&= \frac{(2J_2 + 1)(2J_3 + 1)}{2J_1 + 1} \mathcal{C}_{J m_2 \bar{m}_2 J m_3 \bar{m}_3} \mathcal{C}_{J m_1 \bar{m}_1},
\end{align*}
(A.17)

\begin{align*}
\mathcal{C}_{J m_1 \bar{m}_1 J m_2 \bar{m}_2 J m_3 \bar{m}_3} &= \int \frac{d\Omega_3}{2\pi^2} Y_{J m_1 \bar{m}_1}(\Omega_3) Y_{J m_2 \bar{m}_2}(\Omega_3) Y_{J m_3 \bar{m}_3}(\Omega_3) \\
&= (-1)^{m_1-\bar{m}_1} \mathcal{C}_{J m_2 \bar{m}_2 J m_3 \bar{m}_3},
\end{align*}
(A.18)

where $\mathcal{C}_{J m_1 \bar{m}_1 J m_2 \bar{m}_2 J m_3 \bar{m}_3}$ is the Clebsch-Gordan coefficient.

**B  Fuzzy spherical harmonics**

In this appendix, we summarize some useful properties of the fuzzy spherical harmonics (See [12][14], and also [46][51][55]).
Given \((2j + 1) \times (2j' + 1)\) rectangular complex matrices, we can generally express them as

\[ M = \sum_{r, r'} M_{rr'} |jr\rangle \langle j'r'|, \quad (B.1) \]

where \(|jr\rangle\) are the basis of the spin \(j\) representation of \(SU(2)\) algebra. \(\{|jr\rangle \langle j'r'| : r = -j, \cdots, j; r' = -j', \cdots, j'\}\) form a basis of \((2j + 1) \times (2j' + 1)\) matrices. Let us consider linear maps, which map the set of \((2j + 1) \times (2j' + 1)\) complex matrices into itself, defined by

\[ L_i \circ |jr\rangle \langle j'r'| \equiv L_i^{[j]} |jr\rangle \langle j'r'| - |jr\rangle \langle j'r'| L_i^{[j']}, \quad (B.2) \]

where \(L_i^{[j]}\) are the spin \(j\) representation matrices of the \(SU(2)\) generators. \(L_i\circ\) satisfy the \(SU(2)\) algebra: \([L_i\circ, L_j\circ] = i\epsilon_{ijk} L_k\circ\).

By changing the basis, we can obtain the more appropriate basis for the action of \(L_i\circ\), which is called the fuzzy spherical harmonics:

\[ \hat{Y}_{Jm(jj')} = \sqrt{n} \sum_{r, r'} (-1)^{-j + r'} C_{jr-j'r'}^{Jm} |jr\rangle \langle j'r'|, \quad (B.3) \]

where \(C_{jr-j'r'}^{Jm}\) is the Clebsch-Gordan coefficient and \(n\) is a positive constant, which is taken to be an integer as an ultraviolet cutoff in section 2. For a fixed \(J\), the fuzzy spherical harmonics form the basis of the spin \(J\) irreducible representation of \(SU(2)\) under \(L_i\circ\),

\[ (L_i\circ)^2 \hat{Y}_{Jm(jj')} = J(J + 1) \hat{Y}_{Jm(jj')}, \]

\[ L_\pm \circ \hat{Y}_{Jm(jj')} = \sqrt{(J \mp m)(J \pm m + 1)} \hat{Y}_{Jm\pm 1(jj')}, \]

\[ L_3 \circ \hat{Y}_{Jm(jj')} = m \hat{Y}_{Jm(jj')} \quad (B.4) \]

The hermitian conjugates of the fuzzy spherical harmonics are given by

\[ \left( \hat{Y}_{Jm(jj')} \right)^\dagger = (-1)^{m-j} \hat{Y}_{J-m(j'-j)} \quad (B.5) \]

The orthonormality condition is

\[ \frac{1}{n} \text{tr} \left\{ \left( \hat{Y}_{Jm(jj')} \right)^\dagger \hat{Y}_{J'm'(jj')} \right\} = \delta_{J,J'} \delta_{mm'}, \quad (B.6) \]
where “tr” stands for the trace over \((2j' + 1) \times (2j' + 1)\) matrices. The trace of three fuzzy spherical harmonics is evaluated as

\[
\hat{C}_{Jm1(j')} Jm2(j''), Jm3(j''') \equiv \frac{1}{n} \text{tr}\left\{ \left( \hat{Y}_{Jm1(j''')} \right)^\dagger \hat{Y}_{Jm2(j'')} \hat{Y}_{Jm3(j''')} \right\} = (-1)^{J_1 + j' + j'''} \sqrt{n(2J_2 + 1)(2J_3 + 1)} C_{Jm1}\{ J_1, J_2, J_3 \}
\]

(B.7)

\[
\hat{C}_{Jm1(j''), Jm2(j'''), Jm3(j''''')} \equiv \frac{1}{n} \text{tr}\left\{ \hat{Y}_{Jm1(j'')} \hat{Y}_{Jm2(j'''')} \hat{Y}_{Jm3(j''''')} \right\} = (-1)^{m_1 - (j'' - j')} \hat{C}_{Jm2(j'''), Jm3(j'''''}
\]

(B.8)

where the last factor of (B.7) is the 6-\(j\) symbol. The 6-\(j\) symbol is related asymptotically to the Clebsch-Gordan coefficient: for \(R \gg 1\),

\[
\left\{ \begin{array}{ccc} a & b & c \\ d + R & e + R & f + R \end{array} \right\} \approx \frac{(-1)^{-a+b+2c-d+e+2f+2R}}{\sqrt{2R(2c+1)}} C_{a-e+f, b-f+d}^{c-d-e}.
\]

(B.9)

We also introduce the vector fuzzy spherical harmonics \(\hat{Y}_{Jm(j')i}^\rho\) and the spinor fuzzy spherical harmonics \(\hat{Y}_{Jm(j')\alpha}^\kappa\), where \(\rho\) takes \(-1, 0, 1\) and \(\kappa\) takes \(-1, 1\). They are defined in terms of the scalar spherical harmonics as

\[
\hat{Y}_{Jm(j')i}^\rho = i^\rho \sum_{n,p} V_{in} \bar{C}_{Qm}^{Qn} \hat{Y}_{Qp(j')i};
\]

\[
\hat{Y}_{Jm(j')\alpha}^\kappa = \sum_p U_{\bar{p}p} \bar{C}_{Qm}^{Qn} \hat{Y}_{Qp(j')\alpha};
\]

(B.10)

where \(Q = J + \delta_\rho 1\), \(\bar{Q} = J + \delta_{\rho - 1}\) and \(U = J + \frac{1}{2} \delta_{\kappa 1}\), \(\bar{U} = J + \frac{1}{2} \delta_{\kappa - 1}\). \(V\) is an unitary matrix given by

\[
V = \frac{1}{\sqrt{2}} \begin{pmatrix} -1 & 0 & 1 \\ -i & 0 & -i \\ 0 & \sqrt{2} & 0 \end{pmatrix}.
\]

(B.11)

The vector fuzzy spherical harmonics and the spinor fuzzy spherical harmonics satisfy

\[
L_i \circ \hat{Y}_{Jm(j')i}^\rho = \sqrt{J(J + 1)} \delta_\rho 1 \hat{Y}_{Jm(j')i};
\]

\[
i \epsilon_{ijk} L_j \circ \hat{Y}_{Jm(j')k}^\rho + \hat{Y}_{Jm(j')i}^\rho = \rho(J + 1) \hat{Y}_{Jm(j')i}^\rho;
\]

\[
\left( (\sigma_i)_{\alpha\beta} L_i \circ + \frac{3}{4} \delta_{\alpha\beta} \right) \hat{Y}_{Jm(j')\beta}^\kappa = \kappa \left( J + \frac{3}{4} \right) \hat{Y}_{Jm(j')\alpha}^\kappa.
\]

(B.12)
Their hermitian conjugates are given by
\[
\left( \hat{Y}^\rho_{Jm(jj')i} \right)^\dagger = (-1)^{m-(j-j')+1} \hat{Y}^\rho_{J-m(jj')i},
\]
\[
\left( \hat{Y}^\kappa_{Jm(jj')\alpha} \right)^\dagger = (-1)^{m-(j-j')\kappa+1} \hat{Y}^\kappa_{J-m(jj')-\alpha},
\]  
(B.13)

and the orthonormality conditions are
\[
\frac{1}{n} \text{tr} \left\{ \left( \hat{Y}^\rho_{Jm(jj')i} \right)^\dagger \hat{Y}^{\rho'}_{Jm'(jj')i} \right\} = \delta_{J,J'} \delta_{m,m'} \delta_{\rho,\rho'},
\]
\[
\frac{1}{n} \text{tr} \left\{ \left( \hat{Y}^\kappa_{Jm(jj')\alpha} \right)^\dagger \hat{Y}^{\kappa'}_{Jm'(jj')\alpha} \right\} = \delta_{J,J'} \delta_{m,m'} \delta_{\kappa,\kappa'},
\]  
(B.14)

The trace of three fuzzy spherical harmonics, including the vector harmonics and/or the spinor harmonics, are evaluated as
\[
\hat{D}^{jm(j')}_{Jm_1(j'j'')\rho_1 Jm_2(j'')\rho_2} = \frac{1}{n} \text{tr} \left\{ \left( \hat{Y}^{j_1}_{Jm(j')} \right)^\dagger \hat{Y}^{j_2}_{Jm_1(j'')} \hat{Y}^{j_3}_{Jm_2(j'')} \right\}
\]
\[
= \sqrt{3n(2J+1)(2J_1+1)(2J_2+1)}(2J_3+1)(2J_2+2J_3+1)
\times (-1)^{\frac{\rho_1^2+\rho_2^2+\rho_3^2+1}{2}} \hat{Q}_1 \hat{Q}_2 \hat{Q}_3 \left\{ Q_1 \hat{Q}_1 \hat{Q}_2 \hat{Q}_3 \right\}
\]
\[
\hat{J}^{j_1}_{Jm_1(j'')} \hat{J}^{j_2}_{Jm_2(j'')} \hat{J}^{j_3}_{Jm_3(j'')}
\]  
(B.15)

\[
\hat{G}^{j_1}_{Jm_1(j'')} \hat{G}^{j_2}_{Jm_2(j'')} \hat{G}^{j_3}_{Jm_3(j'')}
\]
\[
= \sqrt{2n(2J+1)(2J+2J_2+1)(2J_3+2J_2+1)}
\times (-1)^{\hat{U}_1+\hat{U}_2+\hat{U}_3} \left\{ U_1 \hat{U}_1 \hat{U}_2 \hat{U}_3 \right\}
\]
\[
\hat{J}^{j_1}_{Jm(j')} \hat{J}^{j_2}_{Jm(j')} \hat{J}^{j_3}_{Jm(j')}
\]  
(B.17)

where \( \{ \cdots \} \) with 9 slots is the 9-\( j \) symbol.
C Harmonic expansion of PWMM

In this appendix, we make a harmonic expansion of the PWMM around the vacuum with (2.30) and (2.33) for the perturbative analysis in section 4 and 5.

The complete form of the action of the PWMM including fermionic part is given by

\[
S_{\text{PW}} = \frac{1}{g_{\text{PW}}^2} \int d\tau \text{Tr} \left( \frac{1}{2} (D_\tau X_i)^2 + \frac{1}{2} \left( X_i + \frac{i}{2} \epsilon_{ijk} [X_j, X_k] \right)^2 + \frac{1}{2} D_\tau \Phi_{AB} D_\tau \Phi^{AB} + \frac{\mu^2}{8} \Phi_{AB} \Phi^{AB} ight. \\
- \left. \frac{1}{2} [X_i, \Phi_{AB}] [X_i, \Phi^{AB}] - \frac{1}{4} \{ \Phi_{AB}, \Phi_{CD} \} [\Phi^{AB}, \Phi^{CD}] + \psi^\dagger_A D_\tau \psi^A + \frac{3\mu}{4} \psi^\dagger_A \psi^A \\
+ \psi^\dagger_A \sigma^i [X_i, \psi^A] + \psi^\dagger_A \sigma^2 [\Phi^{AB}, (\psi^A)^T] - (\psi^A)^T \sigma^2 [\Phi_{AB}, \psi^B]. \right) \tag{C.1}
\]

where \( A, B \) are indices of 4 of \( SU(4) \). \( \psi^A \) and \( \psi^\dagger_A \) are two-component spinors and \( \Phi_{AB} \) are \( SO(6) \) scalars \( X_m \) in (2.56) rewritten in terms of \( SU(4) \) notation in the following way

\[
\Phi_{i4} = \frac{1}{2} (X_{i+3} + iX_{i+6}) \quad (i = 1, 2, 3), \\
\Phi_{AB} = -\Phi_{BA}, \quad \Phi^{AB} = -\Phi^{BA} = \Phi^\dagger_{AB}, \quad \Phi^{AB} = \frac{1}{2} \epsilon^{ABCD} \Phi_{CD}. \tag{C.2}
\]

For this expression, the harmonic expansion can be easily performed.

By replacing \( X_i \rightarrow \mu L_i + X_i \) in (C.1) and adding the gauge fixing and the Fadeev-Popov terms

\[
S_{gf+gh} = \frac{1}{g_{\text{PW}}^2} \int d\tau \text{Tr} \left\{ \frac{1}{2} \left( \partial_\tau A_r + i\mu [L_i, X_i] \right)^2 - i\bar{c} \partial_\tau c \mu \bar{c} [L_i, i\mu [L_i, c] + i [c, X_i]] \right\} \tag{C.3}
\]

then we obtain

\[
S_{PW+gf+gh} = S_{\text{gauge}}^{\text{PW,free}} + S_{\text{gauge}}^{\text{PW,int}} + S_{\text{matter}}^{\text{PW,free}} + S_{\text{matter}}^{\text{PW,int}}, \tag{C.4}
\]

where

\[
S_{\text{gauge}}^{\text{PW,free}} = \frac{1}{g_{\text{PW}}^2} \int d\tau \text{Tr} \left( \frac{1}{2} (\partial_\tau X_i)^2 - \frac{\mu^2}{2} [L_i, A_r]^2 + \frac{1}{2} (\partial_\tau A_r)^2 \\
+ \frac{\mu^2}{2} (X_i + i\epsilon_{ijk} [L_j, X_k])^2 - \frac{\mu^2}{2} [L_i, X_i]^2 + i\bar{c} \partial_\tau c - i\mu^2 \bar{c} [L_i, [L_i, c]] \right), \tag{C.5}
\]

\[
S_{\text{gauge}}^{\text{PW,int}} = \frac{1}{g_{\text{PW}}^2} \int d\tau \text{Tr} \left( i(\partial_\tau X_i) [A_r, X_i] + \mu [A_r, X_i] [L_i, A_r] + \frac{1}{2} [A_r, X_i]^2 \\
+ i\mu \epsilon_{ijk} (X_i + i\epsilon_{ilm} [L_l, X_m]) X_j X_k - \frac{1}{2} \epsilon_{ijk} \epsilon_{ilm} X_j X_k X_l X_m \right)
\]
We perform a mode expansion for each \((s, t)\) block for each field in terms of the fuzzy spherical harmonics defined in appendix A:

\[
\begin{align*}
S_{\text{PW,free}}^{\text{matter}} &= \frac{1}{g_{\text{PW}}^2} \int d\tau \text{Tr} \left( \frac{1}{2} \Phi_{AB} \partial_{\tau} \Phi_{AB} + \frac{\mu^2}{2} \Phi_{AB} \Phi_{AB} - \frac{\mu^2}{2} [L_i, \Phi_{AB}][L_i, \Phi_{AB}] \\
&\quad + \psi_A^\dagger \partial_{\tau} \psi_A + \mu \psi_A^\dagger (\frac{3}{4} \psi_A + \sigma [L_i, \psi_A]) \right), \\
S_{\text{PW,int}}^{\text{matter}} &= \frac{1}{g_{\text{PW}}^2} \int d\tau \text{Tr} \left( i(\partial_{\tau} \Phi_{AB}) [A_r, \Phi_{AB}^A] - \frac{1}{2} [A_r, \Phi_{AB}] [A_r, \Phi_{AB}^A] - \mu [L_i, \Phi_{AB}] [X_i, \Phi_{AB}^A] \\
&\quad - \frac{1}{2} [X_i, \Phi_{AB}] [X_i, \Phi_{AB}^A] - \frac{1}{4} [\Phi_{AB}, \Phi_{CD}] [\Phi_{AB}^A, \Phi_{CD}^A] + i \psi_A^\dagger [A_r, \psi_A] \\
&\quad + \psi_A^\dagger \sigma [X_i, \psi_A^A] + \psi_A^\dagger \sigma^2 [\Phi_{AB}^A, (\psi_A^A)^T] - (\psi_A^A)^T \sigma^2 [\Phi_{AB}^A, \psi_B^A] \right).
\end{align*}
\]

We perform a mode expansion for each \((s, t)\) block for each field in terms of the fuzzy spherical harmonics defined in appendix A:
of (C.4) is expressed in terms of the modes as:

\[
\sum_{J=|j_s-j_t|-1}^{j_s+j_t-1} \sum_{m=-J}^{J} x_{jm}^{(s,t)} \otimes \hat{Y}_{jm}^{-1}(j_s,j_t) \tag{C.9}
\]

Note that the modes in the right-hand sides are \((2j_s + 1) \times (2j_t + 1)\) matrices. The free part of (C.4) is expressed in terms of the modes as:

\[
S_{PW, free}^{\text{gauge}} + S_{PW, free}^{\text{matter}} = \frac{n}{g_{PW}^2} \int d\tau \text{ tr} \left( \frac{1}{2} (-1)^{m-(j_s-j_t)+1} x_{jm}^{(s,t)} \right) J_{m \rho}^{(t,s)} - \partial_\tau^2 + \mu^2 J (J+1) \right) x^{(t,s)}_{jm} - \partial_\tau^2 + \mu^2 J (J+1) \right) B_{jm}^{(t,s)} + i (-1)^{m-(j_s-j_t)} \epsilon_{jm}^{(s,t)} \right) \delta_\tau^2 - \mu^2 J (J+1) \right) C_{jm}^{(t,s)} + \frac{1}{4} (-1)^{m-(j_s-j_t)} \epsilon^{ABCD} \phi_{AB,jm}^{(s,t)} \left( - \partial_\tau^2 + \omega_{j_s}^2 \right) \phi_{CD,jm}^{(s,t)} + \psi_{A,jm}^{(s,t)} \left( \partial_\tau + \kappa \omega_{j_t}^2 \right) \psi_{A,jm}^{(s,t)} \right), \tag{C.10}
\]

where

\[
\omega_j^0 \equiv \mu (J+1), \quad \omega_j^\psi \equiv \mu \left( J + \frac{3}{4} \right), \quad \omega_j^\phi \equiv \mu (J + \frac{1}{2}). \tag{C.11}
\]

From (C.10), the propagators can be read off as

\[
\langle x_{jm}^{(s,t)}(p) x_{jm'}^{(s',t')}(p') \rangle_{kl} = \begin{cases} \frac{g_{PW}^2}{n} (-1)^{m-(j_s-j_t)+1} \delta_{J,J'} \delta_{m-m'} \delta_{p-p'} \delta_{\omega_{j_s}^2} \delta_{\omega_{j_t}^2} 2 \delta(p+p') \frac{1}{p^2 + \mu^2 J (J+1)} & (\rho \neq 0) \\ \frac{g_{PW}^2}{n} (-1)^{m-(j_s-j_t)+1} \delta_{J,J'} \delta_{m-m'} \delta_{p-p'} \delta_{\omega_{j_s}^2} \delta_{\omega_{j_t}^2} 2 \delta(p+p') \frac{1}{p^2 + \mu^2 J (J+1)} & (\rho = \rho' = 0) \end{cases},
\]

\[
\langle B_{jm}^{(s,t)}(p) B_{jm'}^{(s',t')}(p') \rangle_{kl} = \frac{g_{PW}^2}{n} (-1)^{m-(j_s-j_t)+1} \delta_{J,J'} \delta_{m-m'} \delta_{p-p'} \delta_{\omega_{j_s}^2} \delta_{\omega_{j_t}^2} 2 \delta(p+p') \frac{1}{p^2 + \mu^2 J (J+1)},
\]

\[
\langle c_{jm}^{(s,t)}(p) c_{jm'}^{(s',t')}(p') \rangle_{kl} = \frac{g_{PW}^2}{n} (-1)^{m-(j_s-j_t)+1} \delta_{J,J'} \delta_{m-m'} \delta_{p-p'} \delta_{\omega_{j_s}^2} \delta_{\omega_{j_t}^2} 2 \delta(p+p') \frac{1}{p^2 + \mu^2 J (J+1)}.
\]

The gauge interaction terms in (C.4) are rewritten as

\[
S_{PW, int}^{\text{gauge}}
\]

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\[
\begin{align*}
\text{The matter interaction terms in (C.4) are rewritten as} \\
S_{\text{matter, int}}^{\text{PW}} &= \frac{n}{g_{\text{PW}}} \int d\tau \text{ tr} \left[ i \hat{D}_{J_{m_1}(j_{s_1}) J_{m_2}(j_{s_2}) J_{m_3}(j_{s_3})} B^{(s,t)}_{J_{m_1}} (\partial \tau \chi_{J_{m_1}} \chi_{J_{m_2}} \chi_{J_{m_3}} - \chi_{J_{m_1}} \partial \tau \chi_{J_{m_2}} \chi_{J_{m_3}}) \\ &- \mu \sqrt{J_2(J_2+1)} \hat{D}_{J_{m_1}(j_{s_1}) J_{m_2}(j_{s_2}) J_{m_3}(j_{s_3})} - \sqrt{J_1(J_1+1)} \hat{D}_{J_{m_2}(j_{s_2}) J_{m_3}(j_{s_3})} \right] \\
&= \frac{n}{g_{\text{PW}}} \int d\tau \text{ tr} \left[ i \hat{D}_{J_{m_1}(j_{s_1}) J_{m_2}(j_{s_2}) J_{m_3}(j_{s_3})} B^{(s,t)}_{J_{m_1}} (\partial \tau \chi_{J_{m_1}} \chi_{J_{m_2}} \chi_{J_{m_3}} - \chi_{J_{m_1}} \partial \tau \chi_{J_{m_2}} \chi_{J_{m_3}}) \\ &- \mu \sqrt{J_2(J_2+1)} \hat{D}_{J_{m_1}(j_{s_1}) J_{m_2}(j_{s_2}) J_{m_3}(j_{s_3})} - \sqrt{J_1(J_1+1)} \hat{D}_{J_{m_2}(j_{s_2}) J_{m_3}(j_{s_3})} \right].
\end{align*}
\]
We first show that for $p_1, \ldots, p_k > 0$

$$
\int_0^1 d\tau_1 \cdots \int_0^1 d\tau_k \delta(1 - \tau_1 - \cdots - \tau_k) \tau_1^{p_1-1} \cdots \tau_k^{p_k-1} = \frac{\Gamma(p_1) \cdots \Gamma(p_k)}{\Gamma(p_1 + \cdots + p_k)}. \tag{D.1}
$$

The left-hand side of (D.1) equals

$$
\int_0^1 d\tau_1 \int_0^{1-\tau_1} d\tau_2 \cdots \int_0^{1-\tau_1 - \cdots - \tau_{k-2}} d\tau_{k-1} \tau_1^{p_1-1} \cdots \tau_{k-1}^{p_{k-1}-1} (1 - \tau_1 - \cdots - \tau_{k-1})^{p_{k-1}-1}. \tag{D.2}
$$

By putting $\tau_{k-1} = (1 - \tau_1 - \cdots - \tau_{k-2}) \sigma_{k-1}$, we calculate this as

$$
\beta(p_{k-1}, p_k) \int_0^1 d\tau_1 \int_0^{1-\tau_1} d\tau_2 \cdots \int_0^{1-\tau_1 - \cdots - \tau_{k-3}} d\tau_{k-2} \tau_1^{p_1-1} \cdots \tau_{k-2}^{p_{k-2}-1} (1 - \tau_1 - \cdots - \tau_{k-2})^{p_{k-1}+p_{k-2}-1}. \tag{D.3}
$$

By repeating this procedure, we see that the left-hand side of (D.1) equals

$$
\beta(p_{k-1}, p_k) \beta(p_{k-2}, p_{k-1} + p_k) \cdots \beta(p_1, p_2 + \cdots + p_k). \tag{D.4}
$$

By using

$$
\beta(a, b) = \frac{\Gamma(a) \Gamma(b)}{\Gamma(a + b)} \tag{D.5}
$$

we see that this is indeed equal to the right-hand side of (D.1).

Here we would like to show that

$$
\frac{1}{m!} \frac{d^m}{dg^m} e^{A+gB} \bigg|_{g=0}
$$

$$
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$$
\[
\int_0^1 d\alpha_1 \cdots \int_0^1 d\alpha_{m+1} \delta(1 - \alpha_1 - \cdots - \alpha_{m+1}) e^{\alpha_1 A} B e^{\alpha_2 A} B \cdots B e^{\alpha_m A} B e^{\alpha_{m+1} A}
\]
\[
= \int_0^1 d\alpha_1 \int_0^{1-\alpha_1} d\alpha_2 \cdots \int_0^{1-\alpha_1-\cdots-\alpha_{m-1}} d\alpha_m e^{\alpha_1 A} B e^{\alpha_2 A} B \cdots B e^{\alpha_m A} B e^{(1-\alpha_1-\cdots-\alpha_m) A}. \tag{D.6}
\]

By expanding the exponential and using (D.1), we calculate the second line of (D.6) as

\[
\sum_{l_1, \ldots, l_{m+1}=0}^{\infty} \frac{1}{l_1! l_{m+1}!} \int_0^1 d\alpha_1 \cdots \int_0^1 d\alpha_{m+1} \delta(1 - \alpha_1 - \cdots - \alpha_{m+1}) \alpha_1^{l_1} \cdots \alpha_{m+1}^{l_{m+1}}
\]
\[
\times A^{l_1} B A^{l_2} B \cdots B A^{l_m} B A^{l_{m+1}}
\]
\[
= \sum_{l_1, \ldots, l_{m+1}=0}^{\infty} \frac{1}{l_1! l_{m+1}!} \frac{\Gamma(l_1 + 1) \cdots \Gamma(l_{m+1} + 1)}{\Gamma(l_1 + \cdots + l_{m+1} + m + 1)} A^{l_1} B A^{l_2} B \cdots B A^{l_m} B A^{l_{m+1}}
\]
\[
= \sum_{l_1, \ldots, l_{m+1}=0}^{\infty} \frac{1}{(l_1 + \cdots + l_{m+1} + m)!} A^{l_1} B A^{l_2} B \cdots B A^{l_m} B A^{l_{m+1}}
\]
\[
= \sum_{n=m}^{\infty} \frac{1}{n!} \sum_{l_1+\cdots+l_{m+1}=n-m, l_1, \ldots, l_{m+1} \geq 0} A^{l_1} B A^{l_2} B \cdots B A^{l_m} B A^{l_{m+1}}. \tag{D.7}
\]

This agrees with the left-hand side of (D.6).

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