THE SIZE OF BETTI TABLES OF EDGE IDEALS ARISING FROM BIPARTITE GRAPHS

NURSEL EREY AND TAKAYUKI HIBI

Abstract. Let \(\text{pd}(I(G))\) and \(\text{reg}(I(G))\) respectively denote the projective dimension and the regularity of the edge ideal \(I(G)\) of a graph \(G\). For any positive integer \(n\), we determine all pairs \((\text{pd}(I(G)), \text{reg}(I(G)))\) as \(G\) ranges over all connected bipartite graphs on \(n\) vertices.

1. Introduction

Let \(G\) be a finite simple graph with the vertex set \(V(G) = \{x_1, \ldots, x_n\}\). Let \(S = k[x_1, \ldots, x_n]\) be the polynomial ring in \(n\) variables over a field \(k\). The edge ideal of \(G\), denoted by \(I(G)\), is the monomial ideal generated by the monomials \(x_ix_j\) such that \(\{x_i, x_j\}\) is an edge of \(G\). Edge ideals of bipartite graphs were studied in the literature for several purposes. Fernández-Ramos and Gimenez [6] gave a characterization of bipartite graphs whose edge ideal has regularity 3. When \(G\) is an unmixed bipartite graph, Kummini [22] described the regularity of \(I(G)\) in terms of the induced matching number of \(G\) and Kimura [21] gave a combinatorial description of the projective dimension of \(I(G)\) via complete bipartite subgraphs satisfying certain conditions. Van Tuyl [25] provided a formula for the regularity of the edge ideal of a sequentially Cohen-Macaulay bipartite graph in terms of the induced matching number of the graph. Jayanthan et al. [18] computed the regularity of powers of edge ideals for several subclasses of bipartite graphs. Herzog and Hibi [10] classified all bipartite graphs which are Cohen-Macaulay and Van Tuyl and Villarreal [26] classified those which are shellable.

In a recent article, Hà and Hibi [8] considered the following problem: Given a graph \(G\) on \(n\) vertices, what are the possible values of \((\text{pd}(S/I(G)), \text{reg}(S/I(G)))\)? They determined all such pairs when \(\text{pd}(S/I(G))\) attains its minimum possible value \(2\sqrt{n} - 2\) or when \(\text{reg}(S/I(G))\) attains its minimum possible value 1. Hibi et al. [12] determined all tuples consisting of the values of depth, regularity, dimension and the degree of the \(h\)-polynomial of \(S/I(G)\) as \(G\) ranges over all Cameron-Walker graphs on \(n\) vertices. Similar type of problems were studied recently in [5, 13, 14, 15].

2020 Mathematics Subject Classification. 05C69, 05C70, 05E40, 13D02.

Key words and phrases. bipartite graph, Castelnuovo-Mumford regularity, edge ideal, matching number, projective dimension.
In this article, we determine all pairs \((\text{pd}(I(G)), \text{reg}(I(G)))\) as \(G\) ranges over all connected bipartite graphs on \(n\) vertices. To state our main result precisely, for any positive integer \(n\) we denote by \(\text{BPT}(n)\) the set of connected bipartite graphs on the vertices \(\{x_1, \ldots, x_n\}\). We define

\[
\text{BPT}^{\text{reg}}(n) = \{(\text{pd}(S/I(G)), \text{reg}(S/I(G))) : G \in \text{BPT}(n)\}
\]

which is the set of sizes of Betti tables of \(S/I(G)\) as \(G\) ranges over all connected bipartite graphs on \(n\) vertices. Our main result is then the following theorem:

**Theorem 1.1 (Theorem 3.14).** Let \(n \geq 4\) be an integer. Then

\[
\text{BPT}^{\text{reg}}(n) = \{(p, r) \in \mathbb{Z}^2 : 1 \leq r < \left\lceil \frac{n}{2} \right\rceil, \left\lfloor \frac{n}{2} \right\rfloor \leq p \leq n - 2\} \cup \{(n - 1, 1)\} \cup A_n
\]

where \(A_n = \emptyset\) if \(n\) is even and, \(A_n = \{([n/2], [n/2])\}\) if \(n\) is odd.

We make use of the graph parameters (induced) matching number, co-chordal cover number and maximum size of minimal vertex covers to bound the regularity and projective dimension. Along the way, we describe all the pairs \((\text{pd}(I(G)), \text{reg}(I(G)))\) as \(G\) ranges over all trees on \(n\) vertices.

2. Preliminaries

2.1. Graph theory background. Given a finite simple graph \(G\), we denote by \(V(G)\) and \(E(G)\) respectively the vertex set and the edge set of \(G\). We say a vertex \(u\) is neighbor of (or, adjacent to) another vertex \(v\) if \(\{u, v\} \in E(G)\). We denote by \(N(u)\) the set consisting of all neighbors of \(u\) in \(G\). We define \(N[u]\) by \(N[u] = N(u) \cup \{u\}\). We call a vertex isolated if it has no neighbors.

A graph \(H\) is called a subgraph of \(G\) if \(V(H) \subseteq V(G)\) and \(E(H) \subseteq E(G)\). A subgraph \(H\) of \(G\) is called an induced subgraph if for any two vertices \(u, v\) in \(H\), \(\{u, v\} \in E(H)\) if \(\{u, v\} \in E(G)\). If \(U\) is a subset of \(V(G)\), we define the induced subgraph of \(G\) on \(U\) as the subgraph whose vertex set is \(U\) and whose edge set is \(\{\{x, y\} : x, y \in U\text{ and }\{x, y\} \in E(G)\}\). Moreover, for any \(W \subseteq V(G)\), we denote by \(G - W\) the induced subgraph of \(G\) on \(V(G) \setminus W\). To simplify the notation, if \(W = \{x\}\) consists of a single vertex, then we write \(G - x\) for \(G - W\). The complement of \(G\), denoted by \(G^c\), is a graph that has the same vertices as \(G\) such that \(\{x, y\} \in E(G^c)\) if and only if \(\{x, y\} \notin E(G)\).

A graph \(G\) is called connected if for every pair of vertices \(x\) and \(y\), there is a path in \(G\) that starts at \(x\) and ends at \(y\). A maximal connected subgraph of \(G\) is called a connected component of \(G\). We say \(G\) is a forest if \(G\) has no cycle subgraphs. A connected forest is called a tree. It is well-known that every tree on \(n\) vertices has exactly \(n - 1\) edges. An independent set in \(G\) is a subset of vertices which contain no edges of \(G\). A bipartite graph is a graph that contains no odd cycles. The vertex set of a bipartite graph can be partitioned to two independent sets. A bipartite graph \(G\) with vertex bipartition \(V(G) = A \cup B\) is called a complete bipartite graph if every
vertex in $A$ is adjacent to every vertex in $B$. A graph is called chordal if it has no induced cycles of length greater than three. A graph $G$ is called co-chordal if $G^c$ is chordal.

A matching of $G$ is a collection of edges which are pairwise disjoint. The matching number of $G$, denoted by $\text{mat}(G)$, is defined by

$$\text{mat}(G) = \max\{|M| : M \text{ is a matching of } G\}.$$ 

A matching $M = \{e_1, \ldots, e_k\}$ of $G$ is called an induced matching of $G$ if the induced subgraph of $G$ on $\bigcup_{i=1}^k e_i$ has exactly $k$ edges. The induced matching number of $G$, denoted by $\text{indm}(G)$, is the maximum cardinality of an induced matching of $G$. A perfect matching of $G$ is a matching $M$ such that each vertex of $G$ belongs to some edge in $M$. The co-chordal cover number of $G$, denoted by $\text{cochord}(G)$, is the minimum number of co-chordal subgraphs required to cover the edges of $G$, i.e.,

$$\text{cochord}(G) = \min\{r : E(G) = \bigcup_{i=1}^r E(H_i), \text{ each } H_i \text{ is a co-chordal subgraph of } G\}.$$ 

For any positive integer $n$, we denote $\{1, \ldots, n\}$ by $[n]$.

A vertex cover $C$ of a graph $G$ is a subset of vertices such that every edge of $G$ contains a vertex from $C$. A vertex cover is called minimal if no proper subset of it is a vertex cover. The maximum cardinality of a minimal vertex cover of $G$ is denoted by $\tau_{\text{max}}(G)$.

2.2. Algebra background. Let $G$ be a graph with the vertex set $V(G) = \{x_1, \ldots, x_n\}$. Let $\mathbb{k}$ be a field and let $S = \mathbb{k}[x_1, \ldots, x_n]$ be the polynomial ring in $n$ variables over $\mathbb{k}$. The edge ideal of $G$, denoted by $I(G)$, is the monomial ideal defined by

$$I(G) = (x_ix_j : \{x_i, x_j\} \text{ is an edge of } G).$$

Let $M$ be a finitely generated graded $S$-module. Then $M$ has a minimal graded free resolution of the form

$$0 \longrightarrow \bigoplus_{j \in \mathbb{Z}} S(-j)^{b_{i,j}(M)} \longrightarrow \cdots \longrightarrow \bigoplus_{j \in \mathbb{Z}} S(-j)^{b_{i,j}(M)} \longrightarrow M \longrightarrow 0.$$ 

The numbers $b_{i,j}(M)$ are called the graded Betti numbers of $M$. The projective dimension of $M$, denoted by $\text{pd}(M)$, is defined by

$$\text{pd}(M) = \max\{i : b_{i,j}(M) \neq 0 \text{ for some } j\}.$$ 

The (Castelnuovo-Mumford) regularity of $M$, denoted by $\text{reg}(M)$, is defined by

$$\text{reg}(M) = \max\{j - i : b_{i,j}(M) \neq 0\}.$$
Theorem 2.1. [1, Corollary 3.8] If $G$ is a graph with connected components $G_1, \ldots, G_r$, then

$$\text{reg}(S/I(G)) = \sum_{i=1}^r \text{reg}(S/I(G_i)).$$

The following bounds on the regularity and projective dimension of edge ideals are well-known, see for example [2, Lemma 3.1] and [3, Lemma 3.2].

Lemma 2.2. [2, 3] For any vertex $x$ of a graph $G$, the short exact sequence

$$0 \to \frac{S}{I(G):(x)}(-1) \to \frac{S}{I(G)} \to \frac{S}{I(G)+(x)} \to 0$$

gives the following bounds for the regularity and projective dimension:

1. $\text{reg}(S/I(G)) \leq \max\{\text{reg}(S/I(G-x)), \text{reg}(S/I(G-N[x])) + 1\}$,
2. $\text{pd}(S/I(G)) \leq \max\{\text{pd}(S/I(G-x)) + 1, \text{pd}(S/I(G-N[x])) + |N(x)|\}$.

The Stanley-Reisner ideal of a simplicial complex $\Delta$ is the squarefree monomial ideal generated by the monomials corresponding to non-faces of $\Delta$. The following theorem of Hochster [16] provides a formula for the graded Betti numbers of Stanley-Reisner ideals.

Theorem 2.3 (Hochster’s Formula). [16] Let $I_\Delta$ be the Stanley-Reisner ideal of a simplicial complex $\Delta$. If $i \geq 0$ and $u$ is a squarefree monomial, then

$$b_{i,u}(I_\Delta) = \dim_k \hat{H}_{\deg u - i - 2}(\Delta[u]; k)$$

where $\Delta[u] = \{ \sigma \in \Delta : \sigma \subseteq U \}$ and $U$ consists of those vertices that correspond to the variables dividing $u$.

If $G$ is a graph, the independence complex of $G$ is a simplicial complex whose faces are independent sets of $G$. The edge ideal $I(G)$ of $G$ is the Stanley-Reisner ideal of the independence complex of $G$.

By a theorem of Terai [24], $\text{pd}(S/I(G))$ is equal to the regularity of the Alexander dual of $I(G)$. Therefore, the projective dimension problem for edge ideals is equivalent to the regularity problem for so-called cover ideals. The next theorem can be deduced from [23, Corollary 3.3] or [11, Corollary 8.2.14] both of which are stated in the more general setting of monomial ideals but in dual terms.

Theorem 2.4. [11, 23] For any graph $G$, $\text{pd}(S/I(G)) \geq \tau_{\text{max}}(G)$. Moreover, the equality holds when $S/I(G)$ is sequentially Cohen-Macaulay.

Since forests are known to be sequentially Cohen-Macaulay (see [4] or [7]), we have an exact formula for $\text{pd}(S/I(G))$ when $G$ is a forest. This formula was proved independently by several authors in the literature.

Theorem 2.5. [20, 28] If $G$ is a forest, then $\text{pd}(S/I(G)) = \tau_{\text{max}}(G)$. 
The following lower bound was also proved several times in the literature:

**Theorem 2.6.** \[9, 19, 20, 28\] For any graph \(G\), \(\text{reg}(S/I(G)) \geq \text{indm}(G)\).

When \(G\) is a forest, the regularity can be described by \(\text{indm}(G)\).

**Theorem 2.7.** \[9, 20, 28\] If \(G\) is a forest, then \(\text{reg}(S/I(G)) = \text{indm}(G)\).

Hà and Van Tuyl \[9\] gave an upper bound for the regularity of edge ideal of any graph via the matching number of the graph:

**Theorem 2.8.** \[9\] For any graph \(G\), \(\text{reg}(S/I(G)) \leq \text{mat}(G)\).

Woodroofe \[27\] improved the upper bound in the previous theorem by replacing the matching number with the co-chordal cover number:

**Theorem 2.9.** \[27\] For any graph \(G\), \(\text{reg}(S/I(G)) \leq \text{cochord}(G)\).

### 3. Edge Ideals of Bipartite Graphs

In the following lemma, we provide a rough estimate for the possible values of regularity and projective dimension of edge ideals of bipartite graphs.

**Lemma 3.1.** Let \(G\) be bipartite graph on \(n \geq 2\) vertices which has no isolated vertices. Then

1. \([n/2] \leq \text{pd}(S/I(G)) \leq n - 1\),
2. \(1 \leq \text{reg}(S/I(G)) \leq \lfloor n/2 \rfloor\).

**Proof.** Since \(I(G)\) is generated in degree two, it is clear that \(\text{pd}(S/I(G)) \leq n - 1\) and \(\text{reg}(S/I(G)) \geq 1\). Let \(V(G) = A \cup B\) be a bipartition of the vertex set of \(G\). Then either \(A\) or \(B\) has cardinality at least \([n/2]\). Since \(G\) has no isolated vertices, both \(A\) and \(B\) are minimal vertex covers. Hence \(\tau_{\text{max}}(G) \geq [n/2]\) and \(\text{pd}(S/I(G)) \geq [n/2]\) follows from Theorem 2.4. Lastly, \(\text{reg}(S/I(G)) \leq [n/2]\) follows from Theorem 2.8 as \(\text{mat}(G)\) cannot exceed the minimum of the cardinalities of \(A\) and \(B\). \(\square\)

We will determine when the regularity upper bound in Lemma 3.1 can be realized by connected bipartite graphs. It turns out that when \(n\) is an even integer greater than two, no connected graph attains the regularity value in the upper bound. On the other hand, when \(n\) is odd, we will see that the regularity upper bound is sharp and, in such case, the projective dimension is uniquely determined.

**Theorem 3.2.** Let \(G\) be a connected graph on \(n \geq 4\) vertices where \(n\) is even. If the matching number of \(G\) is \(n/2\), then \(\text{cochord}(G) < n/2\).

**Proof.** Let \(\{e_1, \ldots, e_{n/2}\}\) be a matching of \(G\). Then it is a perfect matching. Since \(G\) is connected, we may assume that there is an edge \(e\) such that \(e \cap e_1 \neq \emptyset\) and \(e \cap e_2 \neq \emptyset\). Let \(H_1\) denote the induced subgraph of \(G\) on \(e_1 \cup e_2\). Furthermore, for each \(2 \leq i \leq n/2 - 1\) let \(H_i\) be the subgraph of \(G\) which consists of those edges \(f \in E(G)\) with \(f \cap e_{i+1} \neq \emptyset\). Then each \(H_i\) is co-chordal and \(E(G) = E(H_1) \cup \cdots \cup E(H_{n/2-1})\). Hence \(\text{cochord}(G) \leq n/2 - 1\). \(\square\)
Corollary 3.3. If \( G \) is a connected graph on \( n \geq 4 \) vertices where \( n \) is even, then \( \text{reg}(S/I(G)) < n/2 \).

Proof. By Theorem 2.8 we have \( \text{reg}(S/I(G)) \leq \text{mat}(G) \leq n/2 \). Assume for a contradiction \( \text{reg}(S/I(G)) = n/2 \). Then \( \text{mat}(G) = n/2 \). From Theorem 2.9 and Theorem 3.2 it follows that \( \text{reg}(S/I(G)) \leq \text{cochord}(G) < n/2 \), contradiction. \( \Box \)

We can actually classify all graphs \( G \) on even number of vertices for which \( \text{reg}(S/I(G)) \) is equal to half the number of vertices:

Corollary 3.4. Let \( G \) be a graph on \( n \) vertices where \( n \) is even. Then \( \text{reg}(S/I(G)) = n/2 \) if and only if \( G \) consists of \( n/2 \) disjoint edges.

Proof. Let us assume that \( n \geq 4 \) as the statement is clear otherwise. If \( G \) consists of \( n/2 \) disjoint edges, then \( \text{reg}(S/I(G)) = n/2 \) follows from Theorem 2.1. To show the converse, let \( \text{reg}(S/I(G)) = n/2 \). Then by Theorem 2.8 the matching number of \( G \) is \( n/2 \) and \( G \) has a perfect matching. By Corollary 3.3 the graph \( G \) is disconnected. Then every connected component of \( G \) has a perfect matching. Let \( G_1, \ldots, G_r \) with \( r \geq 2 \) be the connected components of \( G \). Let \( |V(G_i)| = 2k_i \) for each \( i \in [r] \). Assume for a contradiction one of the connected components has at least two edges. We may assume that \( k_1 \geq 2 \). Then by Corollary 3.3 we have \( \text{reg}(S/I(G_1)) < k_1 \). Moreover, \( \text{reg}(S/I(G_i)) \leq \text{mat}(G_i) = k_i \) for each \( i = 2, \ldots, r \). Using Theorem 2.1 we get

\[
\text{reg}(S/I(G)) = \sum_{i=1}^{r} \text{reg}(S/I(G_i)) < \sum_{i=1}^{r} k_i = n/2
\]

which is a contradiction. \( \Box \)

We can now investigate the regularity upper bound in Lemma 3.1 when \( n \) is an odd integer. The following theorem classifies all bipartite graphs \( G \) on \( n \) vertices with \( \text{reg}(S/I(G)) = (n - 1)/2 \).

Theorem 3.5. Let \( G \) be a bipartite graph on \( n \) vertices where \( n \) is an odd number. Then \( \text{reg}(S/I(G)) = \lfloor n/2 \rfloor \) if and only if \( \text{indm}(G) = \lfloor n/2 \rfloor \).

Proof. Let \( n = 2k + 1 \). If \( \text{indm}(G) = k \), then \( G \) is a forest and \( \text{reg}(S/I(G)) = k \) by Theorem 2.7. Now, suppose that \( \text{reg}(S/I(G)) = k \). Since \( \text{mat}(G) \geq \text{reg}(S/I(G)) \) by Theorem 2.8 there exists a matching \( M = \{e_1, \ldots, e_k\} \) of \( G \). Let \( x \) be the vertex of \( G \) which does not belong to any edge in \( M \). We may assume that \( k \geq 2 \) because \( \text{indm}(G) = 1 \) is clear when \( k = 1 \). We consider two cases:

Case 1: Suppose that \( x \) is an isolated vertex of \( G \). Then

\[
k = \text{reg}(S/I(G)) = \text{reg}(S/I(G - x))
\]

and by Corollary 3.4 it follows that \( M \) is an induced matching of \( G - x \). Hence \( M \) is an induced matching of \( G \).
that the matching number of $G$ follows from Theorem 2.8. Thus we must have $\text{reg}(S/I_G)$ since $\text{mat}(G)$. Hence $N$.

Case 2: Suppose that $\{x, y\}$ is an edge of $G$ where $e_k = \{y, z\}$. By Lemma 2.2
\[ \text{reg}(S/I(G)) \leq \max\{\text{reg}(S/I(G - y)), \text{reg}(S/I(G - N[y])) + 1\}. \]
Hence either $\text{reg}(S/I(G - y)) \geq k$ or $\text{reg}(S/I(G - N[y])) \geq k - 1$. Observe that $G - y$ is a bipartite graph such that one side of the bipartition has $k - 1$ vertices. This implies that the matching number of $G - y$ is at most $k - 1$. Therefore, $\text{reg}(S/I(G - y)) < k$ follows from Theorem 2.8. Thus we must have $\text{reg}(S/I(G - N[y])) \geq k - 1$. Similarly, since $\text{mat}(G - N[y]) \leq k - 1$, it follows from Theorem 2.8 that
\[ \text{reg}(S/I(G - N[y])) = k - 1 = \text{mat}(G - N[y]). \]
Hence $N(y) = \{x, z\}$. Corollary 3.4 implies that $\{e_1, \ldots, e_{k-1}\}$ is an induced matching of $G - N[y]$ and thus it is induced matching of $G$. If $y$ is the only neighbor of $x$, then $\{e_1, \ldots, e_{k-1}, \{x, y\}\}$ is an induced matching of $G$ and nothing is left to show. Suppose that $x$ has at least two neighbors, say $y$ and $u$. Without loss of generality, we may assume that $e_{k-1} = \{u, v\}$. By Lemma 2.2
\[ \text{reg}(S/I(G)) \leq \max\{\text{reg}(S/I(G - y)), \text{reg}(S/I(G - N[x])) + 1\}. \]
Hence either $\text{reg}(S/I(G - x)) \geq k$ or $\text{reg}(S/I(G - N[x])) \geq k - 1$. Observe that $G - N[x]$ is a bipartite graph such that one side of the bipartition has at most $k - 2$ vertices. This implies that the matching number of $G - N[x]$ is at most $k - 2$. Therefore, $\text{reg}(S/I(G - N[x])) < k - 1$ by Theorem 2.8. Thus we must have $\text{reg}(S/I(G - x)) \geq k$. In fact, $\text{reg}(S/I(G - x)) = k$ because the matching number of $G - x$ is equal to $k$. Then Corollary 3.4 implies that $G - x$ consists of $k$ disjoint edges. Then $M$ is an induced matching of $G - x$. Thus $M$ is an induced matching of $G$.

Remark 3.6. In Theorem 3.5 the bipartite assumption cannot be dropped. Indeed, if $G$ is a cycle graph of length 5, then $\text{reg}(S/I(G)) = 2$ but $\text{indm}(G) = 1$.

Corollary 3.7. Let $G$ be a connected bipartite graph on $n = 2k + 1$ vertices such that $k \geq 1$. Suppose that $\text{reg}(S/I(G)) = k$. Then $\text{pd}(S/I(G)) = k + 1$.

Proof. By Theorem 3.5 the induced matching number of $G$ is $k$. Let $M = \{e_1, \ldots, e_k\}$ be an induced matching of $G$. Let $x$ be the vertex of $G$ that does not belong to any edge in $M$. Then since $G$ is connected, for every $i \in [k]$, the vertex $x$ is adjacent to exactly one endpoint of $e_i$. Hence $G$ is a tree with $\tau(G) = k + 1$ and the proof is complete because of Theorem 2.5.

The next result describes all connected bipartite graphs for which the projective dimension upper bound in Lemma 3.1 can be realized.

Proposition 3.8. Let $G$ be a connected bipartite graph on $n \geq 2$ vertices. Then $\text{pd}(S/I(G)) = n - 1$ if and only if $G$ is a complete bipartite graph. Moreover, in such case, $\text{reg}(S/I(G)) = 1$. 

□
For some a and the edge set n

Moreover, vertex set G

One can easily see that G in two steps (Theorem 3.9 and Theorem 3.13) as our construction depends on whether Theorem 3.9.

Observe that G

Proof. If G is a complete bipartite graph, then pd(S/I(G)) = n−1 and reg(S/I(G)) = 1 was proved in [17]. Now, suppose that pd(S/I(G)) = n−1. Since G is connected, I(G) ≠ (0). Then n−2 = pd(S/I(G))−1 = pd(I(G)). Then there exists a squarefree monomial u such that bn_{2,a}(I(G)) ≠ 0. Since I(G) is generated in degree two, the degree of u must be n. Theorem 2.3 implies dim_k H_0(Δ;k) ≠ 0 where Δ is the independence complex of G. Then Δ is disconnected. Let V(G) = A ∪ B be a bipartition of the vertex set of G. Since G has no isolated vertices, both A and B are facets of Δ. Assume for a contradiction G is not complete bipartite. Then there exists a ∈ A and b ∈ B such that {a, b} /∈ E(G). We now show that Δ is connected.

First, observe that A, {a, b}, B is a chain from A to B. Let σ and τ be two facets of Δ. Since both A and B are facets, we may assume that σ ∩ A ≠ ∅ and τ ∩ A ≠ ∅. Then σ, A, τ is a chain from σ to τ which shows that Δ is connected, a contradiction. □

Now that we have determined when the upper bounds in Lemma 3.1 can be attained by connected bipartite graphs, our next goal is to show that for any integers a, b, τ is a chain from

σ, A, τ is a chain from

G

reg(G−N[x]) = max{1, r−a}. Therefore, by Theorem 2.7 we get

reg(S/I(G)) ≤ max{r, max{1, r−a} + 1} ≤ r
as desired. To evaluate the projective dimension, first observe that by Theorem 2.4
\[ \text{pd}(S/I(G)) \geq \tau_{\text{max}}(G) \geq a + r + t. \]

On the other hand, by Lemma 2.2 we have
\[ \text{pd}(S/I(G)) \leq \max\{ \text{pd}(S/I(G - N[y])), |N(y)|, \text{pd}(S/I(G - y)) + 1 \}. \]

Moreover, \( G - y \) is a forest with \( \tau_{\text{max}}(G - y) = r + 1 \) and \( G - N[y] \) is a forest with \( \tau_{\text{max}}(G - N[y]) = a \). Since \( y \) has exactly \( r + t \) neighbors and \( a \geq 3 \), it follows from Theorem 2.5 that
\[ \text{pd}(S/I(G)) \leq \max\{ a + r + t, r + 2 \} \leq a + r + t \]
which completes the proof. \( \square \)

Remark 3.10. Let \( C \) be a minimal vertex cover of \( G \). Then by the minimality, for every \( v \in C \), there exists an edge \( e \) in \( G \) such that \( e \cap C = \{ v \} \).

Remark 3.11. If \( H \) is a forest on \( n \) vertices, then it has at most \( n - 1 \) edges.

Lemma 3.12. Let \( G \) be a tree on \( n \) vertices with \( \text{indm}(G) = r \) and \( \tau_{\text{max}}(G) = p \). Then \( p \leq n - r - 1 \).

Proof. Assume for a contradiction there exists a minimal vertex cover \( C \) of cardinality at least \( n - r + 1 \). Let \( M = \{ e_1, \ldots, e_r \} \) be an induced matching such that \( \cup_{i=1}^r e_i \subseteq C \) and \( e_j \not\subseteq C \) for each \( j = a + 1, \ldots, r \). Since \( C \) contains at least \( n - r + 1 \) vertices it follows that \( a \geq 1 \). Let \( U = V(G) \setminus \cup_{i=1}^r e_i \). For each \( i \in [a] \), let \( e_i = \{ x_i, y_i \} \). By Remark 3.10 for each \( i \in [a] \) there exists \( u_i, v_i \in U \) such that \( \{ x_i, u_i \}, \{ y_i, v_i \} \in E(G) \) and \( u_i, v_i \not\in C \).

We claim that \( |\cup_{i=1}^a \{ u_i, v_i \}| \geq a + 1 \). Let \( H \) be the induced subgraph of \( G \) on the vertices \( (\cup_{i=1}^a e_i) \cup (\cup_{i=1}^a \{ u_i, v_i \}) \). Since \( H \) is a forest, \( 3a \leq E(H) \leq V(H) - 1 \) by Remark 3.11. This proves the claim as \( H \) has at least \( 3a + 1 \) vertices.

Now, we conclude that \( U \cap C \) has at most \( n - 2r - a - 1 \) elements. On the other hand, \( (V(G) \setminus U) \cap C \) has exactly \( r + a \) elements. Thus \( C \) has at most \( n - r - 1 \) elements, a contradiction. \( \square \)

Let \( \text{TREE}(n) \) denote the set of all trees on the vertices \( \{x_1, \ldots, x_n\} \). We define
\[ \text{TREE}_{\text{pd}}(n) = \{ (\text{pd}(S/I(G)), \text{reg}(S/I(G))) : G \in \text{TREE}(n) \} \]
which consists of all sizes of Betti tables of \( S/I(G) \) as \( G \) ranges over all trees on \( n \) vertices.

Theorem 3.13. Let \( G \) be a tree on \( n \geq 4 \) vertices. Then
\[ \text{TREE}_{\text{pd}}(n) = \{ (p, r) \in \mathbb{Z}^2 : 1 \leq r < n/2, \lfloor n/2 \rfloor \leq p \leq n - r \}. \]
Proof. By Lemma 3.1, Corollary 3.3, and Lemma 3.12 we have
\[ \text{TREE}_{\text{pd}}^\text{reg}(n) \subseteq \{(p, r) \in \mathbb{Z}^2 : 1 \leq r < n/2, \left\lfloor n/2 \right\rfloor \leq p \leq n - r \}. \]
To show the equality, let \( 1 \leq r < n/2 \) and \( \left\lfloor n/2 \right\rfloor \leq p \leq n - r \) be fixed. By Theorem 2.5 and Theorem 2.7 it suffices to find a tree \( G \) on \( n \) vertices with \( \text{indm}(G) = r \) and \( \tau_{\text{max}}(G) = p \). Since \( n > 2r \) we may assume that \( n = 2r + a \) for some \( a \geq 1 \). Then since \( p \leq n - r \) we obtain \( p - a \leq r \). So, we may also assume that \( r = p - a + b \) for some \( b \geq 0 \). Since \( p \geq \left\lfloor n/2 \right\rfloor \) we get \( p - r \geq 1 \). Hence \( a - b \geq 1 \). Let \( t = a - b - 1 \).
Let \( G \) be the graph on the vertex set
\[ V(G) = \{u_1, \ldots, u_r, v_1, \ldots, v_r\} \cup \{x, y_1, \ldots, y_b\} \cup \{z_1, \ldots, z_t\} \]
and the edge set
\[ E(G) = \{\{u_i, v_i\} : i \in [r]\} \cup \{\{x, v_i\} : i \in [r]\} \cup \{\{x, y_i\} : i \in [b]\} \cup \{\{v_r, z_i\} : i \in [t]\}. \]
It is clear that \( G \) is a tree on \( n \) vertices. It is not hard to see that \( \{\{u_i, v_i\} : i \in [r]\} \) is an induced matching of maximum cardinality. Moreover, \( \{u_1, \ldots, u_r, x, z_1, \ldots, z_t\} \) is a minimal vertex cover of cardinality \( p \). We will now show that \( \tau_{\text{max}}(G) = p \). Let \( C \) be a minimal vertex cover of \( G \). We consider the following cases.

Case 1: Suppose that \( v_r \notin C \). Then \( \{u_r, z_1, \ldots, z_t, x\} \subseteq C \). Moreover, \( y_i \notin C \) for every \( i \in [b] \). This implies that for each \( i \in [r] \), either \( u_i \in C \) or \( v_i \in C \), but not both. Hence \( |C| = p \).

Case 2: Suppose that \( v_r \in C \). Then \( u_r \notin C \) and \( z_i \notin C \) for each \( i \in [t] \). We consider two cases:

Case 2.1: Suppose that \( x \in C \). Then \( y_i \notin C \) for each \( i \in [b] \). This implies that for each \( i \in [r] \), either \( u_i \in C \) or \( v_i \in C \), but not both. Hence \( |C| = r + 1 \leq p \) as desired.

Case 2.2: Suppose that \( x \notin C \). Then \( C = \{v_1, \ldots, v_r, y_1, \ldots, y_b\} \). Thus we get
\[ |C| = r + b = r + (r - p + a) = 2r - p + a = 2r - p + (n - 2r) = n - p \leq p \]
where the last inequality follows from the assumption that \( \left\lfloor n/2 \right\rfloor \leq p \).

For any positive integer \( n \) let \( \text{BPT}(n) \) denote the set of connected bipartite graphs on the vertices \( \{x_1, \ldots, x_n\} \). We define
\[ \text{BPT}_{\text{pd}}^\text{reg}(n) = \{(\text{pd}(S/I(G)), \text{reg}(S/I(G))) : G \in \text{BPT}(n)\}. \]
Finally, we arrived at our main result:

**Theorem 3.14.** Let \( n \geq 4 \) be an integer. Then
\[ \text{BPT}_{\text{pd}}^\text{reg}(n) = \{(p, r) \in \mathbb{Z}^2 : 1 \leq r < \left\lfloor \frac{n}{2} \right\rfloor, \left\lfloor \frac{n}{2} \right\rfloor \leq p \leq n - 2\} \cup \{(n - 1, 1)\} \cup A_n \]
where \( A_n = \emptyset \) if \( n \) is even and, \( A_n = \{(\left\lfloor n/2 \right\rfloor, \left\lfloor n/2 \right\rfloor)\} \) if \( n \) is odd.
Proof. Keeping Lemma 3.1 in mind, first observe that the set $A_n$ is determined by Corollary 3.3 and Corollary 3.7. By Proposition 3.8, $(n - 1, 1)$ is the only pair in $\text{BPT}^{\text{reg}}(n)$ of the form $(n - 1, r)$. The rest of the proof follows from Theorem 3.9 and Theorem 3.13. □

REFERENCES

[1] A. Banerjee, S.K. Beyarslan, H.T. Hà, Regularity of edge ideals and their powers, Advances in algebra, 17–52, Springer Proc. Math. Stat., 277, Springer, Cham, 2019.
[2] H. Dao, C. Huneke, J. Schweig, Bounds on the regularity and projective dimension of ideals associated to graphs, J. Algebraic Combin. 38 (2013), no. 1, 37–55.
[3] H. Dao, J. Schweig, Projective dimension, graph domination parameters, and independence complex homology, J. Combin. Theory Ser. A 120 (2013), no. 2, 453–469.
[4] S. Faridi, Simplicial trees are sequentially Cohen-Macaulay, J. Pure Appl. Algebra 190 (2004), no. 1-3, 121–136.
[5] G. Favacchio, G. Keiper, A. Van Tuyl, Regularity and $h$-polynomials of toric ideals of graphs, Proc. Amer. Math. Soc. 148 (2020), no. 11, 4665–4677.
[6] O. Fernández-Ramos, P. Gimenez, Regularity 3 in edge ideals associated to bipartite graphs, J. Algebraic Combin. 39 (2014), no. 4, 919–937.
[7] C.A. Francisco, A. Van Tuyl, Sequentially Cohen-Macaulay edge ideals, Proc. Amer. Math. Soc. 135 (2007), no. 8, 2327–2337.
[8] H.T. Hà, T. Hibi, MAX MIN Vertex Cover and the Size of Betti Tables, Ann. Comb. (2021). https://doi.org/10.1007/s00026-020-00521-4
[9] H.T. Hà, A. Van Tuyl, Monomial ideals, edge ideals of hypergraphs, and their graded Betti numbers, J. Algebraic Combin. 27 (2008), 215–245.
[10] J. Herzog, T. Hibi, Distributive lattices, bipartite graphs and Alexander duality, J. Algebraic Combin. 22 (2005), no. 3, 289–302.
[11] J. Herzog, T. Hibi, Monomial Ideals, Springer-Verlag, London, 2011.
[12] T. Hibi, H. Kanno, K. Kimura, K. Matsuda, A. Van Tuyl, Homological invariants of Cameron–Walker graphs, to appear in Tran. Amer. Math. Soc.
[13] T. Hibi, H. Kanno, K. Matsuda, Induced matching numbers of finite graphs and edge ideals, J. Algebra 532 (2019), 311–322.
[14] T. Hibi, K. Kimura, K. Matsuda, A. Van Tuyl, The regularity and $h$-polynomial of Cameron-Walker graphs, arXiv:2003.07416 [math.CO]
[15] T. Hibi, K. Matsuda, A. Van Tuyl, Regularity and $h$-polynomials of edge ideals, Electron. J. Combin. 26 (2019), no. 1, Paper No. 1.22, 11 pp.
[16] M. Hochster, Cohen-Macaulay rings, combinatorics, and simplicial complexes, In: Ring theory, II (Proc. Second Conf., Univ. Oklahoma, Norman, Okla., 1975). Lecture Notes in Pure and Appl. Math. Vol. 26: 171–223. Dekker, New York, 1977.
[17] S. Jacques, Betti numbers of graph ideals, PhD Thesis, Universite of Sheffield, 2004.
[18] A.V. Jayanthan, N. Narayanan, S. Selvaraja, Regularity of powers of bipartite graphs, J. Algebraic Combin. 47, 17–38 (2018).
[19] M. Kutzman, Characteristic-independence of Betti numbers of graph ideals, J. Combin. Theory Ser. A 113 (2006), 435–454.
[20] K. Kimura, Non-vanishingness of Betti numbers of edge ideals, (English summary) Harmony of Gröbner bases and the modern industrial society, 153–168, World Sci. Publ., Hackensack, NJ, 2012.
[21] K. Kimura, *Nonvanishing of Betti numbers of edge ideals and complete bipartite subgraphs*, Comm. Algebra 44 (2016), no. 2, 710–730.
[22] M. Kummini, *Regularity, depth and arithmetic rank of bipartite edge ideals*, J. Algebraic Combin. 30 (2009), 429–445.
[23] S. Morey, R. Villarreal, *Edge ideals: algebraic and combinatorial properties*, Progress in commutative algebra 1, 85–126, de Gruyter, Berlin, 2012.
[24] N. Terai, *Alexander duality theorem and Stanley-Reisner rings*, Free resolutions of coordinate rings of projective varieties and related topics (Japanese) (Kyoto, 1998), Sūrikaisekikenkyūsho Kōkyūroku, No. 1078 (1999), 174–184.
[25] A. Van Tuyl, *Sequentially Cohen–Macaulay bipartite graphs: vertex decomposability and regularity*, Arch. Math. 93 (2009), 451–459.
[26] A. Van Tuyl, R.H. Villarreal, *Shellable graphs and sequentially Cohen-Macaulay bipartite graphs*, J. Combin. Theory Ser. A, 2008; 115 (5): 799–814.
[27] R. Woodroofe, *Matchings, coverings, and Castelnuovo-Mumford regularity*, J. Commut. Algebra 6 (2014), no. 2, 287–304.
[28] X. Zheng, *Resolutions of facet ideals*, Comm. Algebra 32 (2004), 2301–2324.

NURSEL EREY, Gebze Technical University, Department of Mathematics, 41400 Kocaeli, Turkey
Email address: nurselerey@gtu.edu.tr

TAKAYUKI HIBI, Department of Pure and Applied Mathematics, Graduate School of Information Science and Technology, Osaka University, Suita, Osaka 565–0871, Japan
Email address: hibi@math.sci.osaka-u.ac.jp