THE SIMPLE TYPE CONJECTURE FOR MOD 2 SEIBERG–WITTEN INVARIANTS

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Abstract. We prove that, under a simple condition on the cohomology ring, every closed 4-manifold has mod 2 Seiberg–Witten simple type. This result shows that there exists a large class of topological 4-manifolds such that all smooth structures have mod 2 simple type, and yet some have non-vanishing (mod 2) Seiberg–Witten invariants. As corollaries, we obtain adjunction inequalities and show that, under a mild topological condition, every geometrically simply connected closed 4-manifold has the vanishing mod 2 Seiberg–Witten invariant for at least one orientation.

1. Introduction

The Seiberg–Witten invariant [27] of a smooth 4-manifold has played a significant role in the study of 4-manifolds over the past 25 years, and has produced many striking applications to low dimensional topology. Although the invariants have been computed for various 4-manifolds, in general it still seems out of reach to compute. The simple type conjecture, posed in 1990s, states a fundamental constraint on the invariant (e.g. [17, Conjecture 1.6.2]).

Conjecture 1.1 (Simple type conjecture). Every closed, connected, oriented, smooth 4-manifold with $b_2^+ > 1$ has Seiberg–Witten simple type.

Here a closed, connected, oriented, smooth 4-manifold $X$ with $b_2^+ > 1$ is called of Seiberg–Witten simple type if the (integer valued) Seiberg–Witten invariant $SW_X(s)$ (see [19, 17]) of a spin$^c$ structure $s$ on $X$ is zero whenever the virtual dimension $d_X(s)$ of the Seiberg–Witten moduli space for $s$ is non-zero. We note that $d_X(s) = \frac{1}{4}(c_1(s)^2 - 2\chi(X) - 3\sigma(X))$, where $\chi$ and $\sigma$ respectively denote the Euler characteristic and the signature. Due to [28], the conjecture is equivalent to the following: if $SW_X(s) \neq 0$, then $c_1(s)$ is the first Chern class of an almost complex structure on $X$.

In the case where $b_2^+ - b_1 \equiv 0 \pmod{2}$, $SW_X(s) = 0$ by the definition, and hence the conjecture is trivial. In the case where $b_2^+ - b_1 \equiv 1 \pmod{2}$, the conjecture has been proved for many smooth 4-manifolds under smooth restrictions such as the existence of a symplectic structure ([23]). However, the conjecture remains open for general smooth structures on any topological 4-manifold.

In this paper, we discuss the mod 2 version of the conjecture. A closed, connected, oriented, smooth 4-manifold $X$ with $b_2^+ > 1$ will be called of mod 2 Seiberg–Witten simple type if $SW_X(s) \equiv 0 \pmod{2}$ whenever $d_X(s) \neq 0$. Here we prove the mod 2 simple type conjecture under a simple condition on the cohomology ring.

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Theorem 1.2. Let $X$ be a closed, connected, oriented, smooth 4-manifold with $b_2^+ - b_1 > 1$ and $b_2^+ - b_1 \equiv 3 \pmod{4}$, and let $\{\delta_1, \delta_2, \ldots, \delta_k\}$ be a generating set of $H^1(X; \mathbb{Z})$. If each cup product $\delta_i \cup \delta_j$ is either torsion or divisible by 2, then $X$ has mod 2 Seiberg–Witten simple type.

This result gives the first examples of topological 4-manifolds such that all smooth structures are of mod 2 Seiberg–Witten simple type, and yet some have non-vanishing (mod 2) Seiberg–Witten invariants. As easily seen, this theorem provides a large class of such topological 4-manifolds.

Corollary 1.3. Let $X$ be a closed, connected, oriented, smooth 4-manifold with $b_2^+ - b_1 > 1$ and $b_2^+ - b_1 \equiv 3 \pmod{4}$. Suppose that the cohomology ring is isomorphic to that of a connected sum of (possibly more than two) closed oriented 4-manifolds, each summand of which satisfies either $b_1 \leq 1$ or $b_2 = 0$. Then $X$ has mod 2 Seiberg–Witten simple type.

Corollary 1.4. Every closed, connected, oriented, smooth 4-manifold with $b_2^+ > 1$, $b_2^+ - b_1 \equiv 3 \pmod{4}$ and $b_1 \leq 1$ has mod 2 Seiberg–Witten simple type.

We note that there are many 4-manifolds with non-vanishing mod 2 Seiberg–Witten invariants satisfying the assumption of Corollary 1.3. See, for example, [8, 22, 1, 25, 30, 26]. Also, the normal connected sum formula [18, Corollary 3.3] provides many 4-manifolds with non-vanishing mod 2 Seiberg–Witten invariants for which the mod 2 simple type conjecture is difficult to prove (without our results).

Remark 1.5. The $b_1 = 0$ case of Corollary 1.3 can be alternatively derived from results of Bauer and Furuta [3, Corollary 3.6 and Theorem 3.7]. We note that the proofs of these results are homotopy theoretic and hence very different from ours.

We give simple applications of these results. We first discuss adjunction inequalities. A second cohomology class $K$ of a 4-manifold $X$ will be called a mod 2 Seiberg–Witten basic class if there exists a spin$^c$ structure $s$ on $X$ satisfying $K = c_1(s)$ and $SW_X(s) \not\equiv 0 \pmod{2}$. Here we assume that every immersed sphere intersects itself only at transverse double points. Due to the generalized adjunction formula of Fintushel and Stern [7], Theorem 1.2 implies the following adjunction inequality for immersed spheres.

Theorem 1.6. Let $X$ be a closed, connected, oriented, smooth 4-manifold satisfying the assumption of Theorem 1.2. Suppose that a second homology class $\alpha$ is represented by an immersed sphere having exactly $p_+$ positive double points and $p_-$ negative double points. If $p_+ > 0$ and $\alpha \cdot \alpha < 0$, then any mod 2 Seiberg–Witten basic class $K$ satisfies

$$|\langle K, \alpha \rangle| + \alpha \cdot \alpha \leq 2p_+ - 2.$$

We note that this theorem holds for any 4-manifold satisfying the assumption of Corollary 1.3. For embedded surfaces, Corollary 1.4 implies the following adjunction inequality due to the generalized adjunction formula of Ozsváth and Szabó [20].

Theorem 1.7. Let $X$ be a closed, connected, oriented, smooth 4-manifold with $b_2^+ > 1$, $b_2^+ - b_1 \equiv 3 \pmod{4}$ and $b_1 \leq 1$. Suppose that a second homology class $\alpha$ is represented by a smoothly embedded, closed, oriented surface of genus $g$. If $g > 0$ and $\alpha \cdot \alpha < 0$, then any mod 2 Seiberg–Witten basic class $K$ satisfies

$$|\langle K, \alpha \rangle| + \alpha \cdot \alpha \leq 2g - 2.$$
We next discuss the following conjecture, which states that the choice of an orientation of a 4-manifold imposes a strong constraint on the Seiberg–Witten invariant.

**Conjecture 1.8** (cf. Kotschick [13], see [5]). Every simply connected, closed, oriented, smooth 4-manifold with \( b_2^+ > 1 \) and \( b_2^- > 1 \) has the vanishing Seiberg–Witten invariant for at least one orientation.

Kotschick [14] proved this conjecture for a large class of complex surfaces (see also [5]). We remark that this conjecture has counterexamples if we remove the simply connected condition (e.g. the 4-torus). To state our result, let us recall that a compact, connected, smooth manifold is called **geometrically simply connected** if it admits a handle decomposition without 1-handles. We note that a geometrically simply connected manifold is simply connected. Also, we say that a 4-manifold \( X \) has the **vanishing mod 2 Seiberg–Witten invariant** if \( \text{SW}_X(s) \equiv 0 \pmod{2} \) for any spin \( c \) structure \( s \) on \( X \). In [29], the third author showed that every geometrically simply connected, closed 4-manifold with \( b_2^+ \not\equiv 1 \) and \( b_2^- \not\equiv 1 \pmod{4} \) admits no symplectic structure for at least one orientation. Improving this result, Corollary 1.4 implies the mod 2 version of Conjecture 1.8 under a mild condition.

**Theorem 1.9.** Every geometrically simply connected, closed, oriented, smooth 4-manifold with \( b_2^+ \not\equiv 1 \) and \( b_2^- \not\equiv 1 \pmod{4} \) has the vanishing mod 2 Seiberg–Witten invariant for at least one orientation.

We note that many simply connected, closed 4-manifolds including a large class of complex surfaces are geometrically simply connected (see [8, 29]). If this theorem does not hold without the condition “geometrically”, then this theorem guarantees the existence of a counterexample to a long-standing open problem whether every simply connected, closed, smooth 4-manifold is geometrically simply connected ([12, Problem 4.18]). For background on this problem, we refer to [29]. In fact, we prove this theorem under a more general condition, which holds for many 4-manifolds including non-simply connected ones. Furthermore, this condition is much easier to verify. See Theorem 2.7.

## 2. Proofs

### 2.1. Mod 2 Seiberg–Witten simple type.

For a closed, connected, oriented 4-manifold \( X \), let \([X]\) denote the fundamental class of \( X \). We first prove the following theorem.

**Theorem 2.1.** Let \( X \) be a closed, connected, oriented, smooth 4-manifold with \( b_2^+ - b_1 > 1 \) and \( b_2^- - b_1 \equiv 3 \pmod{4} \), and let \( \{\delta_1, \delta_2, \ldots, \delta_k\} \) be a generating set of \( H^1(X; \mathbb{Z}) \). Suppose that a spin \( c \) structure \( s \) on \( X \) satisfies the following conditions.

- \( \text{SW}_X(s) \equiv 1 \pmod{2} \).
- \( \langle c_1(s) \cup \delta_i \cup \delta_j, [X] \rangle \equiv 0 \pmod{4} \) for any \( i, j \).

Then \( d_X(s) = 0 \).

**Proof.** Suppose, to the contrary, that \( d_X(s) \neq 0 \). Put \( K = c_1(s) \). Due to the assumption \( \text{SW}_X(s) \neq 0 \), it follows from the definition of \( \text{SW}_X(s) \) that \( d_X(s) = 2n \) for some positive integer \( n \) (e.g. [19] Section 2.3)). By the blow-up formula ([7, 19]), we see that \( X_n := X \# n\mathbb{CP}^2 \) has a mod 2 Seiberg–Witten basic class \( K_n := K + 3E_1 + 3E_2 + \cdots + 3E_n \), where each \( E_i \) denotes the Poincaré dual of the second homology class \( e_i \) of the \( i \)-th \( \mathbb{CP}^2 \) represented by the exceptional sphere.
We thus have a spin$^c$ structure $s_0$ on $X_0$ satisfying $c_1(s_0) = K_{n_0}$, $SW_{X_0}(s_0) \equiv 1 \pmod{2}$, and $d_{X_0}(s_0) = 0$. Hence, we see that $(X_0, s_0)$ is BF admissible in the sense of Ishida and Sasahira [11, Definition 2], due to the assumption on $(X, s)$. One can also check that $(K_3, t)$ is BF admissible, where $(K_3, t)$ denotes the $K_3$ surface equipped with a spin$^c$ structure $t$ with $c_1(t) = 0$. By a result of Ishida and Sasahira [11, Theorem A] (see also [11, Theorem 23 and Proposition 14]) on the Bauer–Furuta invariant [3], we see that $K_n$ is a Bauer–Furuta basic class of $Z := X_n \# K_3$, and thus a monopole class of $Z$ due to [10, Proposition 6].

Now let $\alpha$ be a second homology class of the $K_3$ surface represented by a smoothly embedded, closed, oriented surface of genus $g > 1$ satisfying $\alpha \cdot \alpha = 2g - 2$. As easily seen, there are many examples of such $\alpha$ (e.g. [9, Theorem 1.1]). We note that the class $\alpha - e_1$ of the 4-manifold $Z$ is represented by a closed surface of genus $g$ with non-negative self-intersection number. Applying the adjunction inequality of Kronheimer [16, p. 53] to $Z$, we obtain the inequality

$$|\langle K_n, \alpha - e_1 \rangle| + (\alpha - e_1) \cdot (\alpha - e_1) \leq 2g - 2,$$

which shows $2g \leq 2g - 2$. Since this is a contradiction, we obtain $d_X(s) = 0$. \qed

Remark 2.2. (1) The role of the $K_3$ surface in this proof can be replaced by any closed, connected, oriented, smooth 4-manifold $Y$ with $b_2^+ \equiv 3 \pmod{4}$ and $b_1 = 0$ satisfying the following conditions: (i) $Y$ has a mod 2 Seiberg–Witten basic class; (ii) $Y$ has a smoothly embedded, closed surface of genus $g > 1$ satisfying $\alpha \cdot \alpha = 2g - 2$. This can be easily checked by using the adjunction inequality (21). We remark that there are many examples of such $Y$.

(2) We used a connected sum formula of the Bauer–Furuta invariant to obtain a restriction on the smooth structure of a connected summand. A similar idea was used by the third author [29] to impose constraints on geometrically simply connected 4-manifolds and, more generally, on 4-manifolds admitting a non-torsion second homology class represented by a 2-handle neighborhood. We remark that the $b_1 = 0$ condition of [29, Theorem 2.4] can be relaxed to conditions similar to Theorem 1.2 and hence Corollaries 1.3 and 1.4 of this paper without changing the proof, except that the connected sum formula of [11] is used instead of the formula of [11].

Proof of Theorem 1.3. We note that $(\delta_i \cup \delta_j) \cup (\delta_i \cup \delta_j) = 0$ for any $i, j$ (see also the proof of Lemma 2.3). It is thus easy to see that every spin$^c$ structure $s$ on $X$ satisfies $\langle c_1(s) \cup \delta_i \cup \delta_j, [X] \rangle \equiv 0 \pmod{4}$ for any $i, j$, since $c_1(s)$ is characteristic, and any $\delta_i \cup \delta_j$ is either torsion or divisible by 2. Hence Theorem 1.2 follows from Theorem 2.1. \qed

We here observe the lemma below to prove Corollaries 1.3 and 1.4.

Lemma 2.3. Let $X$ be a closed, connected, oriented, smooth 4-manifold with $b_1 \leq 1$. Then for any classes $\gamma, \delta$ of $H^1(X; \mathbb{Z})$, the cup product $\gamma \cup \delta$ is zero.

Proof. By the universal coefficient theorem, we see that $H^1(X; \mathbb{Z})$ has no torsion. Due to the assumption $b_1(X) \leq 1$, it suffices to prove $\gamma \cup \gamma = 0$ for any class $\gamma$ of $H^1(X; \mathbb{Z})$. We note that the Poincaré dual $PD(\gamma)$ is represented by a closed oriented codimension one submanifold of $X$ having a trivial normal bundle. This implies that $PD(\gamma \cup \gamma)$ is represented by the empty set, showing $\gamma \cup \gamma = 0$. \qed
On the other hand, dξ and Seiberg–Witten invariants, and was later extended to the case of arbitrary relation to Witten’s conjecture \([27, 6]\) on the relationship between the Donaldson conjecture.

\[ \text{Remark 2.4.} \]

\[ \text{Corollary 1.4.} \quad \square \]

2.2. Adjunction inequalities.

\[ \text{Proof of Theorem 1.6.} \quad \text{Suppose, to the contrary, that } |\langle K, \alpha \rangle| + \alpha \cdot \alpha > 2p_+ - 2 \text{ for some } K = c_1(s) \text{ and } \alpha. \text{ Then the generalized adjunction formula of Fintushel and Stern } [7, \text{ Theorem 1.3}] \text{ shows that } K' = K + 2\epsilon PD(\alpha) \text{ is a mod 2 Seiberg–Witten basic class, where } \epsilon = \pm 1 \text{ is the sign of } \langle K, \alpha \rangle. \text{ It is straightforward to see that } K' = c_1(s') \text{ satisfies } d_X(s') = d_X(s) + |\langle K, \alpha \rangle| + \alpha \cdot \alpha > 0. \]

Since X is of mod 2 Seiberg–Witten simple type due to Theorem 1.2, this is a contradiction. \( \square \)

Ozsváth and Szabó [20] introduced the Seiberg–Witten invariant of the form

\[ SW_{X,s} : \mathcal{A}(X) \rightarrow \mathbb{Z} \]

for a spin\(^c\) structure s, where \( \mathcal{A}(X) = \bigwedge H_1(X; \mathbb{Z}) \otimes \mathbb{Z}[U] \), \( H_1(X; \mathbb{Z}) \) has grading 1 and \( U \) is a degree 2 generator (cf. [24]). This function and the integer valued invariant have the relation

\[ SW_{X,s}(U^{d_X(s)/2}) = SW_X(s) \]

when \( d_X(s) \) is non-negative and even. Let \( \Sigma \) be a smoothly embedded, closed, oriented surface of genus \( g \) representing \( \alpha \). For such a surface \( \Sigma \), they defined the class \( \xi(\Sigma) \in \mathcal{A}(X) \) by

\[ \xi(\Sigma) = \prod_{i=1}^{g} (U - A_i \cdot B_i), \]

where \( \{A_i, B_i\}_{i=1}^{g} \) are the images in \( H_1(X; \mathbb{Z}) \) of a standard symplectic basis for \( H_1(\Sigma; \mathbb{Z}) \).

\[ \text{Proof of Theorem 1.7.} \quad \text{Suppose } |\langle K, \alpha \rangle| + \alpha \cdot \alpha > 2g - 2. \text{ Then, by [20, Theorem 1.3], the relation } SW_{X,s+\epsilon\alpha}(\xi(\Sigma)U^{m}) = SW_{X,s}(1) \]

holds, where \( \epsilon = \pm 1 \) is the sign of \( \langle K, \alpha \rangle \) and \( 2m = |\langle K, \alpha \rangle| + \alpha \cdot \alpha - 2g. \) Since \( \xi(\Sigma) = U^{g} \) when \( b_1(X) \leq 1 \), we obtain

\[ SW_X(s + \epsilon\alpha) = SW_{X,s+\epsilon\alpha}(U^{m+g}) = SW_{X,s}(1) = SW_X(s) \equiv 1. \]

On the other hand, \( d_X(s + \epsilon\alpha) = d_X(s) + |\langle K, \alpha \rangle| + \alpha \cdot \alpha > 0. \) This contradicts Corollary 1.4. \( \square \)

\[ \text{Remark 2.4.} \quad \text{Conjecture 1.1 was originally posed for 4-manifolds with } b_1 = 0 \text{ in relation to Witten’s conjecture } [27, 6] \text{ on the relationship between the Donaldson and Seiberg–Witten invariants, and was later extended to the case of arbitrary } b_1 \text{ in the literature. Ozsváth and Szabó [20] gave a stronger version of the simple type condition. They call } X \text{ of simple type when the function } SW_{X,s} \text{ is identically zero if } d_X(s) \neq 0. \text{ Taubes [24, Proof of Proposition 2.2] proved that every closed symplectic 4-manifold } X \text{ with } b_2^+(X) > 1 \text{ is of simple type in this strong sense (see also [20, Remark 3.3]). However, in contrast to the case of (ordinary) simple type,} \]

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there are many 4-manifolds which are not of strong simple type. For instance, it follows from the surgery formula of Ozsváth and Szabó [21, Proposition 2.2] that a connected sum $X \# (S^1 \times S^3)$ is such an example when $X$ has a non-vanishing integer Seiberg–Witten invariant.

2.3. A vanishing theorem for mod 2 Seiberg–Witten invariants. We recall a definition and a lemma given in [29].

Definition 2.5. Let $\alpha$ be a second homology class of a smooth 4-manifold $X$. We say that $\alpha$ is represented by a 2-handle neighborhood, if $X$ has a codimension zero submanifold $W$ satisfying the following conditions.

- The submanifold $W$ is diffeomorphic to a 4-manifold obtained from the 4-ball by attaching a single 2-handle. (This submanifold will be called a 2-handle neighborhood.)
- $\alpha$ is the image of a generator of $H_2(W; \mathbb{Z}) \cong \mathbb{Z}$ by the inclusion induced homomorphism $H_2(W; \mathbb{Z}) \to H_2(X; \mathbb{Z})$.

Lemma 2.6 ([29, Lemma 3.1]). Every second homology class of a geometrically simply connected, compact, smooth 4-manifold is represented by a 2-handle neighborhood.

For an oriented 4-manifold $X$, let $\overline{X}$ denote the 4-manifold $X$ equipped with the reverse orientation. We show the following vanishing theorem for mod 2 Seiberg–Witten invariants.

Theorem 2.7. Let $X$ be a closed, connected, oriented, smooth 4-manifold satisfying $b_2^+ - b_1 > 1$, $b_2^- - b_1 > 1$, $b_2^+ - b_1 \not\equiv 1 \pmod{4}$, and let $\{\delta_1, \delta_2, \ldots, \delta_k\}$ be a generating set of $H^1(X; \mathbb{Z})$. Suppose that each cup product $\delta_i \cup \delta_j$ is either torsion or divisible by 2. If $X$ admits a non-torsion second homology class represented by a 2-handle neighborhood, then at least one of $X$ and $\overline{X}$ has the vanishing mod 2 Seiberg–Witten invariant.

We note that the existence of a non-torsion second homology class represented by a 2-handle neighborhood is much easier to verify than the geometrically simply connected condition, since it is often not necessary to decompose an entire 4-manifold into a handlebody. Indeed, many closed 4-manifolds including non-simply connected ones admit such second homology classes. See [29, Section 3] for more background on such 4-manifolds.

The proof of this theorem relies on the following result.

Theorem 2.8 ([29]). Let $X$ be a 4-manifold satisfying the assumption of Theorem 2.7. Then at least one of the following properties holds.

- Every spin$^c$ structure $s$ with $d_X(s) = 0$ satisfies $SW_X(s) \equiv 0 \pmod{2}$.
- Every spin$^c$ structure $s$ with $d_X(s) = 0$ satisfies $SW_{\overline{X}}(s) \equiv 0 \pmod{2}$.

This theorem is implicit in the proof of [29, Theorem 2.4], which states that any 4-manifold with $b_1 = 0$ satisfying the assumption of Theorem 2.7 admits no symplectic structure for at least one orientation. The proof for the $b_1 = 0$ case of Theorem 2.8 is identical with the proof of [29, Theorem 2.4], and the proof for the general case also is identical, except that the connected sum formula of [11] is used instead of the formula of [4].

Proof of Theorem 2.7. This is straightforward from Theorems 2.8 and 1.2. □
Proof of Theorem 1.9. This is straightforward from Theorem 2.7 and Lemma 2.6.

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