NOTES ON TWO-PARAMETER QUANTUM GROUPS, (I)

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Abstract. A simpler definition for a class of two-parameter quantum groups associated to semisimple Lie algebras is given in terms of Euler form. Their positive parts turn out to be 2-cocycle deformations of each other under some conditions. An operator realization of the positive part is given.

1. Introduction

The notion of quantum groups was introduced by V. Drinfel’d and M. Jimbo, independently, around 1985 in their study of the quantum Yang-Baxter equations. Quantum groups $\mathcal{U}_q(g)$, depending on a single parameter $q$, are certain families of Hopf algebras that are deformations of universal enveloping algebras of symmetrizable Kac-Moody algebras. In the early 90s of the last century, much work had been done on their multiparameter generalizations, which can be obtained by twisting the algebra structure via a 2-cocycle on an indexed free abelian group (see [1]) or by twisting the coalgebra structure in the spirit of Drinfeld (see [2], [3]). Note that a 2-cocycle (or a Drinfeld twist) deformation is an important method to yield new (twisted) bialgebras from old ones.

Motivated by the work on down-up algebras [4], Benkart and Witherspoon, et al [5, 7, 8] investigated the two-parameter quantum groups of the general linear Lie algebra $\mathfrak{gl}_n$ and the special linear Lie algebra $\mathfrak{sl}_n$. Later on, Bergeron, Gao and Hu [9, 10] developed the corresponding theory for two-parameter quantum orthogonal and symplectic groups. Recently, Hu et al continued this project (see [11, 12] for exceptional types $G, E$, [13, 14, 15] for restricted types $B, C, D$, and [16, 17] for untwisted affine types, etc.).

In this note, we give a simpler definition for a class of two-parameter quantum groups $\mathcal{U}_{r,s}(g)$ associated to semisimple Lie algebras in terms of the Euler form (or say, Ringel form). As in [5, 7, 12, 16, 17], these quantum groups also possess the Drinfel’d double structures and the triangular decompositions (see Section 2). As a main point of this note, we show that the positive parts of quantum groups under consideration are 2-cocycle deformations of each other as $Q^+$-graded associative $\mathbb{C}$-algebras if the parameters satisfy certain conditions (see Section 3). This affords an insight into the interrelation between the two-parameter quantum groups we defined and the one-parameter Drinfeld-Jimbo ones. In Section 4, we get an operator realization of the positive part of $\mathcal{U}_{r,s}(g)$ by assigning the canonical generators $e_i$’s with some skew differential operators in the sense of Kashiwara ([18]).

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2. Euler form and definition of two-parameter quantum groups

Let $C = (a_{ij})_{i,j \in I}$ be a Cartan matrix of finite type and $g$ the associated semisimple Lie algebra over $\mathbb{Q}$. Let $\{d_i \mid i \in I\}$ be a set of relatively prime positive integers such that $d_i a_{ij} = d_j a_{ij}$ for $i, j \in I$. Let $Q(r, s)$ be the function field in two variables $r, s$ over the field $\mathbb{Q}$ of rational numbers. Denote $r_i = r^{d_i}, s_i = s^{d_i}$ for $i \in I$.

Let $\langle -, - \rangle$ be the bilinear form, which is called the Euler form (or Ringel form), on the root lattice $Q$ defined by

$$\langle i, j \rangle := \langle \alpha_i, \alpha_j \rangle = \begin{cases} d_i a_{ij} & i < j, \\ d_i & i = j, \\ 0 & i > j. \end{cases}$$

**Definition 2.1.** The two-parameter quantum group $U_{r,s}(g)$ is a unital associative algebra over $\mathbb{Q}(r, s)$ generated by $e_i, f_i, \omega_i^{\pm 1}, \omega_i'^{\pm 1}, i \in I$, subject to the following relations:

1. $\omega_i^{\pm 1} \omega_j^{\pm 1} = \omega_j^{\pm 1} \omega_i^{\pm 1}$, $\omega_i'^{\pm 1} \omega_j'^{\pm 1} = \omega_j'^{\pm 1} \omega_i'^{\pm 1}$,
2. $\omega_i e_j \omega_i^{-1} = r^{(j,i)} s^{-(i,j)} e_j$, $\omega_i' e_j \omega_i'^{-1} = r^{-(i,j)} s^{(j,i)} e_j$,
3. $\omega_i f_j \omega_i^{-1} = r^{-(j,i)} s^{(i,j)} f_j$, $\omega_i' f_j \omega_i'^{-1} = r^{(j,i)} s^{-(j,i)} f_j$,
4. $e_k f_k - f_k e_k = \delta_{i,j} \frac{\omega_i - \omega_i'}{r_i - s_i}$,
5. $\sum_{k=0}^{1-a_{ij}} (-1)^k \binom{1-a_{ij}}{k} c_{i,j}^{(k)} e_i^{-a_{ij} - k} e_j e_i^{k} = 0, \quad (i \neq j),$
6. $\sum_{k=0}^{1-a_{ij}} (-1)^k \binom{1-a_{ij}}{k} c_{i,j}^{(k)} f_i^{k} f_j^{1-a_{ij} - k} = 0, \quad (i \neq j),$

where $c_{i,j}^{(k)} = (r_i s_i^{-1})^{\binom{k}{2}} f^{k(j,i)} s^{-k(i,j)}$, for $i \neq j$, and for a symbol $v$, we set the notations:

$$n_v = \frac{n^n - 1}{v - 1}, \quad (n)_v! = (1)_v (2)_v \cdots (n)_v,$$

$$\binom{n}{k}_v = \frac{(n)_v!}{(k)_v! (n-k)_v!} \quad \text{for } n \geq k \geq 0,$$

and $(0)_v! = 1$.

The algebra $U_{r,s}(g)$ has a Hopf algebra structure with the comultiplication, the counit and the antipode given by:

$$\Delta(\omega_i^{\pm 1}) = \omega_i^{\pm 1} \otimes \omega_i^{\pm 1}, \quad \Delta(\omega_i'^{\pm 1}) = \omega_i'^{\pm 1} \otimes \omega_i'^{\pm 1},$$

$$\Delta(e_i) = e_i \otimes 1 + \omega_i \otimes e_i, \quad \Delta(f_i) = 1 \otimes f_i + f_i \otimes \omega_i',$$

$$\varepsilon(\omega_i^{\pm 1}) = \varepsilon(\omega_i'^{\pm 1}) = 1, \quad \varepsilon(e_i) = \varepsilon(f_i) = 0,$$

$$S(\omega_i^{\pm 1}) = \omega_i^{\mp 1}, \quad S(\omega_i'^{\pm 1}) = \omega_i'^{\mp 1},$$

$$S(e_i) = -\omega_i^{-1} e_i, \quad S(f_i) = -f_i \omega_i'^{-1}.$$
Remark 2.2. (i) Let $r = q, s = q^{-1}$. Then $U_{q, q^{-1}}$ modulo the Hopf ideal generated by $\omega'_i - \omega^{-1}_i (i \in I)$ is isomorphic to the standard one-parameter quantum group $U_q(\mathfrak{g})$ defined by Drinfel'd and Jimbo (see [19]).

(ii) Let $r = q^2, s = 1$. Then $U_{q^2, 1}^+$ is isomorphic to the (nontwisted) generic Hall algebra introduced by Ringel (20).

(iii) For the type $D$ case, the definition above is distinct from that given in [9].

(iv) Definition 2.1 might be adopted to define the affine cases, but the resulting quantum groups in this fashion (because of no Drinfeld realization to be found in these cases) are different from that given in [16].

Let $U_{r, s}^+$ (resp., $U_{r, s}^-$) be the subalgebra of $U_{r, s} := U_{r, s}(\mathfrak{g})$ generated by the elements $e_i$ (resp., $f_i$) for $i \in I$, and $U^0$ the subalgebra of $U_{r, s}$ generated by $\omega_i^{\pm 1}$, $\omega_i^{\pm 1}$ for $i \in I$. Moreover, let $U_{r, s}^0$ (resp., $U_{r, s}^{\leq 0}$) be the subalgebra of $U_{r, s}$ generated by the elements $e_i$, $\omega_i^{\pm 1}$ for $i \in I$ (resp., $f_i$, $\omega_i^{\pm 1}$ for $i \in I$). For each $\mu \in Q$ (the root lattice of $\mathfrak{g}$), we define elements $\omega_\mu$ and $\omega'_\mu$ by

$$\omega_\mu = \prod_{i \in I} \omega_i^{\mu_i}, \quad \omega'_\mu = \prod_{i \in I} \omega_i'^{\mu_i}, \quad \text{for } \mu = \sum_{i \in I} \mu_i \alpha_i \in Q.$$

For $\beta \in Q^+$ (a fixed positive root lattice), let

$$U_{r, s}^{\pm \beta} = \left\{ x \in U_{r, s}^\pm \mid \omega_\mu x \omega_{-\mu} = r^{(\beta, \mu)} s^{-(\mu, \beta)} x, \omega'_\mu x \omega'_{-\mu} = r^{-(\mu, \beta)} s^{(\beta, \mu)} x, \forall \mu \in Q \right\},$$

then $U_{r, s}^{\pm \beta}$ are $Q^+$-graded.

Proposition 2.3. For any $i \in I$, we have the $\mathbb{Q}$-algebra automorphism $\Phi$ and the $\mathbb{Q}(r, s)$-algebra anti-automorphism (or say, involution) $\Psi$ of $U_{r, s}(\mathfrak{g})$ defined by

$$\Phi(r) = s^{-1}, \quad \Phi(s) = r^{-1}, \quad \Phi(e_i) = f_i, \quad \Phi(f_i) = r_i s_i e_i, \quad \Phi(\omega_i) = \omega_i', \quad \Phi(\omega'_i) = \omega_i.$$

$$\Psi(e_i) = f_i, \quad \Psi(f_i) = e_i, \quad \Psi(\omega_i) = \omega_i, \quad \Psi(\omega'_i) = \omega'_i.$$

Proof. It is straightforward to check the statements. \qed

Proposition 2.4. There exists a unique bilinear skew pairing

$$(,): U_{r, s}^{\leq 0} \times U_{r, s}^{\geq 0} \to \mathbb{Q}(r, s)$$

such that for all $x, x' \in U_{r, s}^{\geq 0}$, $y, y' \in U_{r, s}^{\leq 0}$, $\mu, \nu \in Q$, and $i, j \in I$,

$$(y, x x') = (\Delta(y), x' \otimes x),$$

$$(y y', x) = (y \otimes y', \Delta(x)),$$

$$(f_i, e_j) = \delta_{i, j} \frac{1}{s_i - r_i},$$

$$(\omega'_\mu, \omega_\nu) = r^{(\mu, \nu)} s^{-(\nu, \mu)},$$

$$(\omega'_\mu, e_i) = 0,$$

$$(f_i, \omega_\mu) = 0.$$

Proof. The coalgebra structure of $U_{r, s}(\mathfrak{g})$ defines an algebra structure on $(U_{r, s}^{\geq 0})^*$ by

$$(\gamma_1 \gamma_2)(x) := (\gamma_1 \otimes \gamma_2)(\Delta(x)), \quad \forall \gamma_1, \gamma_2 \in (U_{r, s}^{\geq 0})^*, \quad x \in U_{r, s}^{\geq 0}. $$
The identity element is given by $e$. We define the linear functionals $\gamma_\mu, \xi_i \in (U_{r,s}^{\geq 0})^*$ for any $i \in I, \mu \in Q$ by
\[
\gamma_\mu(x\omega_\nu) = r^{(\mu,\nu)} s^{-(\nu,\mu)} \epsilon(x), \quad (\forall \ x \in U_{r,s}^+, \ \nu \in Q),
\xi_i(x\omega_\nu) = 0, \quad (\forall \ x \in U_{r,s}^{+\beta}, \ \beta \in Q^+ - \{\alpha_i\}, \ \nu \in Q),
\xi_i(e_i\omega_\nu) = \frac{1}{s_i - r_i}, \quad (\forall \ \nu \in Q).
\]
Define a linear map
\[
\phi : U_{r,s}^{\leq 0} \rightarrow (U_{r,s}^{\geq 0})^*
\]
by $\phi(\omega_\mu') = \gamma_\mu, \ \phi(f_i) = \xi_i$ and extending it algebraically. It is straightforward to check that $\phi$ is well-defined. Now we can define the pairing
\[
(\cdot, \cdot) : U_{r,s}^{\leq 0} \times U_{r,s}^{\geq 0} \rightarrow \mathbb{Q}(r,s)
\]
by $(x, y) := \phi(x)(y)$, for any $x \in U_{r,s}^{\leq 0}, \ y \in U_{r,s}^{\geq 0}$. The condition
\[
(yy', x) = (y \otimes y', \Delta(x)), \quad (y, xx') = (\Delta(y), x' \otimes x),
\]
for $x \in U_{r,s}^{\geq 0}, \ y, y' \in U_{r,s}^{\leq 0}$ can be proved by induction. The remaining conditions are obvious. Moreover, it is clear that the bilinear form $(\cdot, \cdot)$ is uniquely determined. \hfill \square

Based on Proposition 2.4, similarly to the proof of Theorem 2.5 in [9], we have

**Corollary 2.5.** $U_{r,s}(g)$ can be realized as a Drinfel’d double of Hopf subalgebras $U_{r,s}^{\geq 0}$ and $U_{r,s}^{\geq 0}$ with respect to the pairing $(\cdot, \cdot)$, that is,
\[
U_{r,s}(g) \cong D(U_{r,s}^{\geq 0}, U_{r,s}^{\leq 0}).
\]

As a consequence of the Drinfel’d double structure, with the same argument of Corollary 2.6 in [9], we have

**Corollary 2.6.** $U_{r,s}(g)$ has the standard triangular decomposition
\[
U_{r,s}(g) \cong U_{r,s}^- \otimes U_{r,s}^0 \otimes U_{r,s}^+, \quad \text{where } U_{r,s}^0 = \bigoplus_{\mu, \nu \in \mathbb{Q}} \mathbb{Q}(r,s) \omega'_\mu \omega_\mu \text{ and } U_{r,s}^{\pm, \beta} = \bigoplus_{\beta \in \mathbb{Q}^+} U_{r,s}^{\pm, \beta}.
\]

### 3. Cocycle deformations of $U_{r,s}^+$

**Lemma 3.1.** Let $A = \bigoplus_{g \in G} A_g$ be a $G$-graded associative algebra over a field $k$, where $G$ is an abelian group. Let $\psi : G \times G \rightarrow k^*$ be a 2-cocycle of the group $G$. We introduce a new multiplication $*$ on $A$ as follows: For any $x \in A_g, \ y \in A_h$, where $g, h \in G$, we define
\[
x * y = \psi(g, h) \ xy.
\]
Denote this new algebra by $A^\psi$. Then $A^\psi$ is a $G$-graded associative algebra, owing to $\psi$ being a 2-cocycle. The algebra $A^\psi$ is called a cocycle deformation of the algebra $A$ by $\psi$.

In this section, let us take the parameters $r, s, r', s' \in \mathbb{C}^*$ and consider both algebras $U_{r,s}^+$ and $U_{r',s'}^+$ to be defined over the field $\mathbb{C}$ of complex numbers. Note that both algebras are $Q$-graded. In view of Lemma 3.1, the argument of the following main result is interesting.
Theorem 3.2. $U^+_{r,s}$ and $U^+_{r',s'}$ are 2-cocycle deformations of each other if $rs^{-1} = r's'^{-1}$ or $rs^{-1} = r'^{-1}s'$.

**Proof.** (I) Assume that $rs^{-1} = r's'^{-1}$. In this case, we define a new product $\ast$ on $U^+_{r,s}$ as follows

$$x \ast y = \psi(\mu, \nu) x y = (r^{-1}r')(\mu, \nu) x y,$$

for any $x \in U^+_{r,s}, y \in U^+_{r',s'}$, where $\psi : Q \times Q \rightarrow \mathbb{C}^*$ such that $\psi(\mu, \nu) = (r^{-1}r')(\mu, \nu)$.

Note that $\psi$ is a bicharacter on $Q \times Q$, which is obviously a 2-cocycle of the abelian group $Q$. This fact ensures that the new $\ast$-product is associative.

In what follows, it suffices to prove the relations below:

$$\sum_{k=0}^{1-a_{ij}} (-1)^k \binom{1-a_{ij}}{k} c_{ij}^{(k)} e_i^{(1-a_{ij}-k)} \ast e_j \ast e_i^k = 0, \quad (i \neq j);$$

$$\sum_{k=0}^{1-a_{ij}} (-1)^k \binom{1-a_{ij}}{k} c_{ij}^{(k)} f_i^{k} \ast f_j \ast f_i^{(1-a_{ij}-k)} = 0, \quad (i \neq j),$$

where

$$c_{ij}^{(k)} = (r_is_i^{-1})^{\frac{k(k-1)}{2}} r_{(j,i)} s_{-k(i,j)}, \quad (i \neq j).$$

By the definition of $\ast$-product, we have

$$e_i^{(m-k)} \ast e_j \ast e_i^k = (s^{-1}s')^{\frac{m(m-1)}{2} d_i + m-k} c_{i,j} e_i^{m-k} e_j e_i^k.$$

**Case (1):** $i < j$, i.e., $\langle j, i \rangle = 0$: when $m = 1 - a_{ij}$, we have

$$\sum_{k=0}^{m} (-1)^k \binom{m}{k} c_{ij}^{(k)} e_i^{(m-k)} \ast e_j \ast e_i^k$$

$$= \sum_{k=0}^{m} (-1)^k \binom{m}{k} c_{ij}^{(k)} (s^{-1}s')^{\frac{m(m-1)}{2} d_i + m-k} c_{i,j} e_i^{m-k} e_j e_i^k$$

$$= (s_i^{-1}s_j')^{\frac{m(m-1)}{2} d_i} \sum_{k=0}^{m} (-1)^k \binom{m}{k} (r_is_i^{-1})^{\frac{k(k-1)}{2} d_i} c_{i,j} e_i^{m-k} e_j e_i^k$$

$$= (s_i^{-1}s_j')^{\frac{m(m-1)}{2} d_i} \sum_{k=0}^{m} (-1)^k \binom{m}{k} (r_is_i^{-1})^{\frac{k(k-1)}{2} d_i} c_{i,j} e_i^{m-k} e_j e_i^k$$

$$= (s_i^{-1}s_j')^{\frac{m(m-1)}{2} d_i} \sum_{k=0}^{m} (-1)^k \binom{m}{k} c_{i,j} e_i^{m-k} e_j e_i^k$$

$$= 0. \quad ((r', s')-Serre relations in $U^+_{r',s'}$)
Case (2): \( i > j \), i.e., \( \langle i, j \rangle = 0 \): when \( m = 1 - a_{ij} \), we have
\[
\sum_{k=0}^{m} (-1)^{k} \binom{m}{k} r_{i} s_{i}^{-1} c_{ij}^{(k)} e_{i}^{* (m-k)} * e_{j} * e_{i}^{k}
\]
\[
= \sum_{k=0}^{m} (-1)^{k} \binom{m}{k} c_{ij}^{(k)} \left( s_{i}^{-1} s_{i}' \right)^{m(m-1)/2} (i,i) + (m-k)(i,j) + k(j,i) e_{i}^{m-k} e_{j} e_{i}^{k}
\]
\[
= \sum_{k=0}^{m} (-1)^{k} \binom{m}{k} \left( r_{i} s_{i}^{-1} \right)^{k(k-1)/2} r_{i}^{k} a_{ij} \left( s_{i}^{-1} s_{i}' \right) e_{i}^{m-k} e_{j} e_{i}^{k}
\]
\[
= \left( s_{i}^{-1} s_{i}' \right)^{m(m-1)/2} \sum_{k=0}^{m} (-1)^{k} \binom{m}{k} \left( r_{i} s_{i}^{-1} \right)^{k(k-1)/2} \left( r_{i}^{k} a_{ij} e_{i}^{m-k} e_{j} e_{i}^{k}
\]
\[
= \left( s_{i}^{-1} s_{i}' \right)^{m(m-1)/2} \sum_{k=0}^{m} (-1)^{k} \binom{m}{k} \left( r_{i} s_{i}^{-1} \right)^{k(k-1)/2} \left( r_{i}^{k} a_{ij} e_{i}^{m-k} e_{j} e_{i}^{k}
\]
\[
= \left( s_{i}^{-1} s_{i}' \right)^{m(m-1)/2} \sum_{k=0}^{m} (-1)^{k} \binom{m}{k} r_{i}^{k} e_{i}^{m-k} e_{j} e_{i}^{k} = 0.
\]

Hence, \( U_{r,s}^{+} \) and \( U_{r,s}'^{+} \) are 2-cocycle deformations of each other.

(II) Assume that \( r s^{-1} = r'^{-1} s' \). In this case, we can define another new product \(*\) on \( U_{r,s}^{+} \). \( \psi : Q \times Q \rightarrow \mathbb{C}^{*} \) such that \( \psi(\mu, \nu) = (r' s^{-1})^{\langle \mu, \nu \rangle} \). Thus we have the following

Case (1'): \( i < j \), i.e., \( \langle j, i \rangle = 0 \): when \( m = 1 - a_{ij} \), we have
\[
\sum_{k=0}^{m} (-1)^{k} \binom{m}{k} r_{i} s_{i}^{-1} c_{ij}^{(k)} e_{i}^{* (m-k)} * e_{j} * e_{i}^{k}
\]
\[
= \sum_{k=0}^{m} (-1)^{k} \binom{m}{k} c_{ij}^{(k)} \left( r' s^{-1} \right)^{m(m-1)/2} (i,i) + (m-k)(i,j) + k(j,i) e_{i}^{m-k} e_{j} e_{i}^{k}
\]
\[
= \sum_{k=0}^{m} (-1)^{k} \binom{m}{k} \left( r_{i} s_{i}^{-1} \right)^{k(k-1)/2} r_{i}^{k} a_{ij} \left( r'^{-1} s^{-1} \right)^{m(m-1)/2} d_{i} + (m-k) d_{i} a_{ij} e_{i}^{m-k} e_{j} e_{i}^{k}
\]
\[
= \left( r_{i} s_{i}^{-1} \right)^{m(m-1)/2} + m a_{ij} \sum_{k=0}^{m} (-1)^{k} \binom{m}{k} \left( r_{i} s_{i}^{-1} \right)^{k(k-1)/2} \left( r_{i}^{k} a_{ij} e_{i}^{m-k} e_{j} e_{i}^{k}
\]
\[
= \left( r_{i} s_{i}^{-1} \right)^{m(m-1)/2} + m a_{ij} \sum_{k=0}^{m} (-1)^{k} \binom{m}{k} \left( r_{i} s_{i}^{-1} \right)^{k(k-1)/2} \left( r_{i}^{k} a_{ij} e_{i}^{m-k} e_{j} e_{i}^{k}
\]
\[
= \left( r_{i} s_{i}^{-1} \right)^{m(m-1)/2} + m a_{ij} \sum_{k=0}^{m} (-1)^{k} \binom{m}{k} \left( r_{i} s_{i}^{-1} \right)^{k(k-1)/2} \left( r_{i}^{k} a_{ij} e_{i}^{m-k} e_{j} e_{i}^{k}
\]
\[
= \left( r_{i} s_{i}^{-1} \right)^{m(m-1)/2} + m a_{ij} \sum_{k=0}^{m} (-1)^{k} \binom{m}{k} \left( r_{i} s_{i}^{-1} \right)^{k(k-1)/2} \left( r_{i}^{k} a_{ij} e_{i}^{m-k} e_{j} e_{i}^{k}
\]
\[
= \left( r_{i} s_{i}^{-1} \right)^{m(m-1)/2} + m a_{ij} \sum_{k=0}^{m} (-1)^{k} \binom{m}{k} \left( r_{i} s_{i}^{-1} \right)^{k(k-1)/2} \left( r_{i}^{k} a_{ij} e_{i}^{m-k} e_{j} e_{i}^{k}
\]
\[
= \left( r_{i} s_{i}^{-1} \right)^{m(m-1)/2} + m a_{ij} \sum_{k=0}^{m} (-1)^{k} \binom{m}{k} \left( r_{i} s_{i}^{-1} \right)^{k(k-1)/2} \left( r_{i}^{k} a_{ij} e_{i}^{m-k} e_{j} e_{i}^{k}
\]
\[
= \left( r_{i} s_{i}^{-1} \right)^{m(m-1)/2} + m a_{ij} \sum_{k=0}^{m} (-1)^{k} \binom{m}{k} \left( r_{i} s_{i}^{-1} \right)^{k(k-1)/2} \left( r_{i}^{k} a_{ij} e_{i}^{m-k} e_{j} e_{i}^{k}
\]
\[
= 0.
Proof. Using the triangular decomposition of $\partial R$ in $(i)$, we have
\[ \sum_{k=0}^{m} (-1)^k \binom{m}{k} c_{ij}^{(k)} e_i^{(m-k)} * e_j * e_i^k \]
\[ = \sum_{k=0}^{m} (-1)^k \binom{m}{k} r_i s_i^{-1} e_i^{(m-1)} e_j e_i^k \]
\[ = \sum_{k=0}^{m} (-1)^k \binom{m}{k} (r_i s_i^{-1}) \in e_i^{(m-1)} (i,i) + (m-k)(i,j) + k(j,i) \]
\[ = (r_i s_i^{-1}) \in e_i^{(m-1)} (i,i) + (m-k)(i,j) + k(j,i) \]
\[ = (r_i s_i^{-1}) \in e_i^{(m-1)} (i,i) + (m-k)(i,j) + k(j,i) \]
\[ = (r_i s_i^{-1}) \in e_i^{(m-1)} (i,i) + (m-k)(i,j) + k(j,i) \]
\[ \sum_{k=0}^{m} (-1)^k \binom{m}{k} r_i s_i^{-1} e_i^{(m-1)} (i,i) + (m-k)(i,j) + k(j,i) \]
\[ = (r_i s_i^{-1}) \in e_i^{(m-1)} (i,i) + (m-k)(i,j) + k(j,i) \]
\[ = (r_i s_i^{-1}) \in e_i^{(m-1)} (i,i) + (m-k)(i,j) + k(j,i) \]
\[ = 0. \]

Hence, $U^+_{r,s}$ and $U^+_{r,s,\lambda}$ are 2-cocycle deformations of each other. □

Corollary 3.3. (i) $U^+_{r,s}$ and $U^+_{r,s,\lambda}$ (the so-called associated object of the former in [5]) are 2-cocycle deformations of each other. Moreover, $U^+_{r,s} = U^+_{r,s,\lambda}$ if and only if $rs = 1$.

(ii) In particular, if $rs^{-1} = q^2$, $U^+_{r,s}$, $U^+_{q^2,1}$ and $U^+_{q,q^{-1}}$ are 2-cocycle deformations of each other. □

4. Realization and Kashiwara’s skew differential operators

The following result arises from Kashiwara’s work [15] (in one-parameter case).

Proposition 4.1. For $P \in U^+_{r,s}$, there exist unique $L, R \in U^+_{r,s}$ satisfying the following equation
\[ [P, f_i] = \frac{\omega_i L - \omega'_i R}{r_i - s_i}, \]
where we define $\partial_i(P) = L$ and $\partial'_i(P) = R$.

Proof. Assume that
\[ \frac{\omega_i L_1 - \omega'_i R_1}{r_i - s_i} = \frac{\omega_i L_2 - \omega'_i R_2}{r_i - s_i}, \]
then we have
\[ \omega_i (L_1 - L_2) - \omega'_i (R_1 - R_2) = 0. \]
Using the triangular decomposition of $U_{r,s}(g)$ in Corollary 2.6, we get $L_1 = L_2$, $R_1 = R_2$. This means that the uniqueness of $L$ and $R$ is clear.

To show the existence of $L$ and $R$, we consider its graded decomposition $U^+_{r,s} = \bigoplus_{\nu \in Q} U^+_{r,s} \nu$. We proceed to prove this by induction on the height of weights,
Then for \( e_j P \in \mathcal{U}_{r,s}^{+}\) with \( \text{ht}(e_j P) = \nu + 1 \), we have

\[
[e_j P, f_i] = e_j \left[ P, f_i \right] + \left[ e_j, f_i \right] P
\]

Then the following lemma can be proved inductively by using Proposition 4.1.

**Theorem 4.3.** For any \( \nu \in \mathbb{Z}_{\geq 0} \), we choose \( L = R = \delta_{i,j} \). Suppose that for \( P \in \mathcal{U}_{r,s}^{+} \), there exist \( L, R \) satisfying

\[
[P, f_1] = \frac{\omega_i L - \omega_i^j R}{r_1 - s_1}.
\]

Then for \( e_j P \in \mathcal{U}_{r,s}^{+\nu+1} \) with \( \text{ht}(e_j P) = \nu + 1 \), we have

\[
[e_j P, f_i] = e_j \left[ P, f_i \right] + \left[ e_j, f_i \right] P
\]

by the induction hypothesis. In particular, we get

\[
\partial_i(e_j P) = r_{-ij} s_{ij} e_j \partial_i(P) + \delta_{i,j} P,
\]

\[
\partial_i'(e_j P) = r_{ij} s_{-ij} e_j \partial_i'(P) + \delta_{i,j} P.
\]

This completes the proof. \( \square \)

From (*), we easily get

\[
\partial_i(e^m) = (m)_{r_s^{-1}} e^{m-1}, \quad \partial_i'(e^m) = (m)_{r_s^{-1}} e^{m-1}.
\]

For \( i \in I \), we introduce the operator \( E_i : \mathcal{U}_{r,s}^{+} \to \mathcal{U}_{r,s}^{+} \) defined by

\[
E_i x = e_i x, \quad \text{for any } x \in \mathcal{U}_{r,s}^{+}.
\]

Then the following lemma can be proved inductively by using Proposition 4.1.

**Lemma 4.2.** For \( m \in \mathbb{Z}_{+} \), \( i, j \in I \), the following commutation relations hold

\[
\partial_i \partial_i' = r_{ij} s_{-ij} \partial_i' \partial_i,
\]

\[
\partial_i^m E_j = r_{m(j,i)} s_{m(j,i)} E_j \partial_i^m + \delta_{i,j} (m)_{r_s^{-1}} \partial_i^{m-1},
\]

\[
\partial_i^m E_j = r_{m(i,j)} s_{m(i,j)} E_j \partial_i^m + \delta_{i,j} (m)_{r_s^{-1}} \partial_i^{m-1}.
\]

**Theorem 4.3.** For any \( i \neq j \in I \), we have

\[
\sum_{k=0}^{1-a_{ij}} (-1)^k \binom{1-a_{ij}}{k} c_{ij}^{(k)} \partial_i^k \partial_j \partial_i^{1-a_{ij}-k} = 0,
\]

\[
\sum_{k=0}^{1-a_{ij}} (-1)^k \binom{1-a_{ij}}{k} c_{ij}^{(k)} \partial_i^k \partial_j \partial_i^{1-a_{ij}-k} = 0,
\]

which give rise to two (operators) realizations of \( \mathcal{U}_{r,s}^{+} \) via assigning the generators \( e_i \)'s of \( \mathcal{U}_{r,s}^{+} \) to the Kashiwara’s skew differential operators \( \partial_i \)'s or \( \partial_i' \)'s respectively.

**Proof.** (i) For any \( u \in \mathcal{U}_{r,s}^{+\nu} \), we will prove the formula

\[
\sum_{k=0}^{1-a_{ij}} (-1)^k \binom{1-a_{ij}}{k} c_{ij}^{(k)} \partial_i^{1-a_{ij}-k} \partial_j \partial_i^k u = 0,
\]

by induction on \( \text{ht}(u) = \mu \). For any \( v \in \mathcal{U}_{r,s}^{+\nu} \) with \( \text{ht}(v) = \nu < \text{ht}(u) = \mu \), we assume that (***) holds. Write \( u = e_{\ell} v = E_{\ell} v \), for some \( \ell \in I \) and some \( v \in \mathcal{U}_{r,s}^{+\nu} \). Now put \( m = 1 - a_{ij} \).
First, we note that

\[ \partial_i^{m-k} \partial_j \partial_i^k E_\ell \]

\[ = \partial_i^{m-k} \partial_j \left\{ r^{-k(\ell,i)} s^{k(i,\ell)} E_\ell \partial_i^k + \delta_{i,\ell} (k) r^{-1}_{s_i} \partial_i^{k-1} \right\} \]

\[ = \partial_i^{m-k} \left\{ r^{-k(\ell,i)} s^{k(i,\ell)} \left( r^{-k(\ell,j)} s^{j(\ell,i)} E_\ell \partial_j + \delta_{\ell,j} \right) \partial_i^k + \delta_{i,\ell} (k) r^{-1}_{s_i} \partial_j \partial_i^{k-1} \right\} \]

\[ = r^{-k(\ell,i)} s^{k(i,\ell)} r^{-k(\ell,j)} s^{j(\ell,i)} \times \]

\[ \left\{ r^{(m-k)(\ell,i)} s^{(m-k)(i,\ell)} E_\ell \partial_i^{m-k} + \delta_{i,\ell} (m-k) r^{-1}_{s_i} \partial_i^{m-k-1} \right\} \partial_j \partial_i^k \]

\[ + \delta_{\ell,j} r^{-k(\ell,i)} s^{k(i,\ell)} \partial_i^{m-k} \partial_i^k + \delta_{i,\ell} (k) r^{-1}_{s_i} \partial_i^{m-k} \partial_j \partial_i^{k-1} \]

\[ = r^{-m(\ell,i)} s^{m(\ell,i)} r^{-k(\ell,j)} s^{j(\ell,i)} \partial_i^{m-k} \partial_j \partial_i^k \]

\[ + \delta_{\ell,i} r^{-k(\ell,i)} s^{k(i,\ell)} r^{-k(i,j)} s^{k(i,j)} \partial_i^{m-k} \partial_j \partial_i^k \]

Consequently, we obtain

\[ \sum_{k=0}^{m} (-1)^k \binom{m}{k} r_{s_i}^{c_{i,j}} \partial_i^k \partial_j \partial_i^{m-k} E_\ell \]

\[ = r^{-m(\ell,i)} s^{m(\ell,i)} r^{-k(\ell,j)} s^{j(\ell,i)} E_\ell \sum_{k=0}^{m} (-1)^k \binom{m}{k} r_{s_i}^{c_{i,j}} \partial_i^{m-k} \partial_j \partial_i^k \]

\[ + \delta_{\ell,i} r^{-k(i,j)} s^{j(i,j)} \sum_{k=0}^{m} (-1)^k \binom{m}{k} r_{s_i}^{c_{i,j}} \partial_i^{m-k} \partial_j \partial_i^k \]

\[ + \delta_{\ell,j} \sum_{k=0}^{m} (-1)^k \binom{m}{k} r_{s_i}^{c_{i,j}} \partial_i^{m-k} \partial_j \partial_i^k \]

\[ + \delta_{\ell,j} \sum_{k=0}^{m} (-1)^k \binom{m}{k} r_{s_i}^{c_{i,j}} r^{-k(j,i)} s^{k(i,j)} \partial_i^{m-k} \partial_j \partial_i^k \]

\[ = S_1 + S_2 + S_3 + S_4 = S_1, \]

where \( S_2 = -S_3, S_4 = 0 \) (by Lemma 4.4 below), and

\[ S_1 = r^{-m(\ell,i)} s^{m(\ell,i)} r^{-k(\ell,j)} s^{j(\ell,i)} E_\ell \sum_{k=0}^{m} (-1)^k \binom{m}{k} r_{s_i}^{c_{i,j}} \partial_i^{m-k} \partial_j \partial_i^k, \]

\[ S_2 = \delta_{\ell,i} \sum_{k=0}^{m-1} (-1)^k \binom{m}{k} r_{s_i}^{c_{i,j}} \partial_i^{m-k} \partial_j \partial_i^k, \]

\[ S_3 = \delta_{\ell,j} \sum_{k=1}^{m} (-1)^k \binom{m}{k} r_{s_i}^{c_{i,j}} \partial_i^{m-k} \partial_j \partial_i^k, \]

\[ S_4 = \delta_{\ell,j} \partial_i^{m} \sum_{k=0}^{m-1} (-1)^k \binom{m}{k} r_{s_i}^{c_{i,j}} r^{-k(j,i)} s^{k(i,j)} \partial_i^{m-k} \partial_j \partial_i^k, \]

Now according to the inductive hypothesis, we get \( S_1 v = 0 \). So we proved the equality (**) This means that \( U^+_r \) can be realized by identifying the generators
$e_i$'s with the skew differential operators $\partial_i$'s, that is, the algebra generated by the $\partial_i$'s is a homomorphic image of $U_{r,s}^+$. 

(ii) Similarly, we can prove another identity (ii), which shows that $U_{r,s}^+$ can be realized by the skew differential operators $\partial_i$'s.

\begin{lemma}
\begin{enumerate}
\item $S_2 = -S_3$.
\item $S_4 = 0$.
\end{enumerate}
\end{lemma}

\begin{proof}
(ii) follows from the identity below:
\[
\sum_{k=0}^{m} (-1)^k \binom{m}{k} c_{ij}^{(k)} r^{k(i,j)} s^{k(i,j)} = \sum_{k=0}^{m} (-1)^k \binom{m}{k} r_{r,s_i}^{-1} (r_i s_i^{-1})^{k(k-1)} = 0.
\]
As for (i), we notice that $(n)_{q-1} = q^{1-n}(n)_q$, $1 - m = a_{ij}$, and
\[
c_{ij}^{(k-1)} = c_{ij}^{(k)} r^{-(j,i)} s^{(i,j)} (r_i s_i^{-1})^{1-k},
\]
\[
r^{-(j,i)-(i,j)} s^{(i,j)+(j,i)} = (r_i s_i^{-1} a_{ij}) = (r_i s_i^{-1})^{m-1},
\]
\[
\binom{m}{k-1} \binom{m-k+1}{r_i s_i^{-1}} s_i = \binom{m}{k-1} \binom{m-k+1}{r_i s_i^{-1}} (r_i s_i^{-1})^{k-m}
\]
\[
= \binom{m}{k} \frac{(k)(r_i s_i^{-1})^{k-m}}{r_i s_i^{-1} (r_i s_i^{-1})^{2k-m-1}},
\]
so that
\[
c_{ij}^{(k-1)} r^{-(j,i)} s^{(j,i)} (r_i s_i^{-1})^{k-1} = c_{ij}^{(k)} (r_i s_i^{-1})^{m-1} + (1-k).
\]
Consequently, we obtain
\[
S_2 = \delta_{\ell,i} \sum_{k=0}^{m-1} (-1)^k \binom{m}{k} c_{ij}^{(k)} r^{-(j,i)} s^{(j,i)} (r_i s_i^{-1})^{k} \partial_i^{m-k-1} \partial_j \partial_i^{k}
\]
\[
= \delta_{\ell,i} \sum_{k=1}^{m} (-1)^k \binom{m}{k-1} c_{ij}^{(k)} r^{-(j,i)} s^{(j,i)} (r_i s_i^{-1})^{k-1} \partial_i^{m-k} \partial_j \partial_i^{k-1}
\]
\[
= -\delta_{\ell,i} \sum_{k=1}^{m} (-1)^k \binom{m}{k} c_{ij}^{(k)} \partial_i^{m-k} \partial_j \partial_i^{k-1}
\]
\[
= -S_3.
\]
This completes the proof.
\end{proof}

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References

[1] Artin M, Schelter W, and Tate J. Quantum deformations of $GL(n)$, Comm. Pure Appl. Math. 44 (1991), 879–895
[2] Reshetikhin N. Multiparameter quantum groups and twisted quasitriangular Hopf algebras, Lett. Math. Phys. 20, (1990), pp. 331–335
[3] Hu N, Wang X. Quantizations of the generalized-Witt algebra and of Jacobson-Witt algebra in the modular case, arXiv:Math.QA/0602261, J. Algebra, 312 (2007), 902–929
[4] Benkart G and Witherspoon S. A Hopf structure for down-up algebras, Math. Z. 238 (3), (2001), 523–553
[5] Benkart G and Witherspoon S. Two-parameter quantum groups and Drinfel’d doubles, Algebr. Represent. Theory, 7 (2004), 261–286
[6] Benkart G and Witherspoon S. Representations of two-parameter quantum groups and Schur-Weyl duality, Hopf algebras, pp. 65–92, Lecture Notes in Pure and Appl. Math., 237, Dekker, New York, 2004
[7] Benkart G and Witherspoon S. Restricted two-parameter quantum groups, Fields Institute Communications, “Representations of Finite Dimensional Algebras and Related Topics in Lie Theory and Geometry”, vol. 40, Amer. Math. Soc., Providence, RI, 2004, pp. 293–318
[8] Benkart G, Kang S-J, Lee K H. On the center of two-parameter quantum groups, Proc. Roy. Soc. Edingburg Sect. A, 136 (3), (2006), 445–472
[9] Bergeron N, Gao Y, Hu N. Drinfel’d doubles and Lusztig’s symmetries of two-parameter quantum groups, arXiv:math.RT/0505614 J. of Algebra, 301 (2006), 378–405
[10] Bergeron N, Gao Y, Hu N. Representations of two-parameter quantum orthogonal groups and symplectic groups, arXiv:math.QA/0510121 AMS/IP Studies in Advanced Mathematics, 39 (2007), 1–21
[11] Hu N, Shi Q. The two-parameter quantum group of exceptional type $G_2$ and Lusztig’s symmetries, arXiv:math.QA/0601444 Pacific J. Math. Vol. 230 (2), (2007), 327–346
[12] Bai X, Hu N. Two-parameter quantum groups of exceptional type E-series and convex PBW type basis, arXiv:math.QA/0605179 Algebra Colloquium (in press)
[13] Hu N, Wang X. Two-parameter Lusztig’s small quantum groups of type $B$ and their ribbon elements, Preprint 2006, (42 pages)
[14] Chen R, Hu N, Wang, X. Two-parameter Lusztig’s small quantum groups of type C and their ribbon elements, Preprint 2007, (43 pages)
[15] Bai X, Hu N. Two-parameter Lusztig’s small quantum groups of type $D$ and their ribbon elements, Preprint 2006, (36 pages)
[16] Hu N, Rosso M, Zhang H. Two-parameter affine quantum group $U_{r,s}(\hat{sl}_n)$, Drinfeld realization and quantum affine Lyndon basis, Comm. Math. Phys. (in press)
[17] Hu N, Zhang H. Vertex representations of two-parameter quantum affine algebras $U_{r,s}(\hat{g})$: the simply laced cases, Preprint 2006, (40 pages)
[18] Kashiwara M. On crystal bases of the q-analogue of universal enveloping algebras, Duke Math. J. 63 (1991), 465–516
[19] Jantzen J C. Lectures on Quantum Groups, Graduate Studies in Mathematics 6, Amer. Math. Soc. Providence, 1996
[20] Ringel C. Hall algebras and quantum groups, Invent. Math. 101 (1990), 583–591

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