Input distinguishability of linear dynamic control systems

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1. Introduction

The switched system is an important case of a hybrid system. As a special kind of linear switched systems has been extensively investigated [1–5]. When we consider the observability of switched systems composed by time-invariant subsystems, distinguishability plays a crucial role (see [6]). Among the references about distinguishability of hybrid systems, we would like to refer the readers to the papers [7–9]. Distinguishability of switched systems is concerned with recovering the initial state as well as the switching signal from the output (and input) and has been widely studied, see e.g. [10] for continuous linear control homogenous systems, Lou et al. [4] for continuous linear control inhomogeneous systems and recently, relatively easy equivalent conditions to verify distinguishability are presented in [3]. For discrete switched case, Baglietto et al. [1,2], considered the problem of identifying a discrete-time nonlinear system, within a finite family of possible models, from data sequences of a finite length. The problem is approached by resorting to the notion of output distinguishability.

However, there are applications of switched systems where their temporal nature cannot be represented by the continuous line or a discrete uniform time domain [11,12]. Indeed, a closed-loop system consisting of a continuous-time system and an intermittent controller is one application [13,14]. The consensus problem under intermittent information due to communication obstacles and limitations of sensors is another example.

Time scale theory is very useful since it is an appropriate tool to study continuous and discrete-time systems in a uniform framework [7,15–18]. The objective of this paper is to extend the distinguishability results of a class of continuous-time linear control switched systems in [3,4]. Necessary and sufficient conditions for input distinguishability of linear switched dynamic systems on time scales.

The rest of the paper is organized as follows. Section 2 recalls some preliminaries on time scale theory. The studied class of systems, input distinguishability concept, necessary and sufficient conditions for input distinguishability for linear control dynamic switched systems are obtained in Section 3.

2. Preliminaries

We recall some basics on time scale theory (for more details see [16,17]). A nonempty subset of real line R is called time scales and it is denoted by T.

Let T be a time scale. As usual, for t ∈ T ⊂ R, σ(t) := inf{s ∈ T : t < s}, ρ(t) := sup{s ∈ T : t > s}, μ(t) := σ(t) − t and ν(t) := t − ρ(t) define and denote its forward jump operator, backward jump operator, forward graininess function and backward graininess function.

A point t ∈ T, is called right-scattered (right-dense), left-scattered (left-dense) and isolated (dense), if σ(t) > t(σ(t) = t), ρ(t) < t(ρ(t) = t) and σ(t) > t > ρ(t)(σ(t) = t = ρ(t)), respectively.

A set T^k is derived from a time scale T: if T has a left-scattered maximum M, then T^k = T − {M}, otherwise T^k = T.

Given a time scale interval [a, b]_T := {t ∈ T : a ≤ t ≤ b}, then [a, b]_T^k denotes the interval [a, b]_T if
$a < \rho(b) = b$ and denotes the interval $[a, b)_T$ if $a < \rho(b) < b$. In fact, $[a, b)_T = [a, \rho(b)_T]$. Also, for $a \in T$, we define $[a, \infty)_T = [a, \infty] \cap T$. If $T$ is a bounded time scale, then $T$ can be identified with $[\inf T, \sup T_T]$. 

If $t_0 \in T$ and $\delta > 0$, then we define the following neighbourhoods of $t_0$: $U_T(t_0, \delta) := (t_0 - \delta, t_0 + \delta) \cap T$, $U_T^+(t_0, \delta) := [t_0, t_0 + \delta) \cap T$ and $U_T^-(t_0, \delta) := (t_0 - \delta, t_0]$ \cap T.

Let us consider some examples of time scales (see [17]).

(i) If $h > 0$, $T = h\mathbb{Z} = \{hk : k \in \mathbb{Z} \}$ is a time scale. Then we have $\sigma(t) = t + h$ and $\rho(t) = t - h$ for all $t \in h\mathbb{Z}$. Hence each point $t \in h\mathbb{Z}$ is isolated, $\mu(t) = h$ for all $t \in h\mathbb{Z}$ and $T^\sigma = T$.

(ii) Let $T = F_{1,1} = \cup_{k \in \mathbb{Z}} [2k, 2k + 1]$. Then 

$$\sigma(t) = \begin{cases} 
  t + 1 & \text{if } t \in \cup_{k \in \mathbb{Z}} [2k + 1] \\
  t & \text{if } t \in \cup_{k \in \mathbb{Z}} [2k, 2k + 1] \end{cases}$$

and

$$\rho(t) = \begin{cases} 
  t - 1 & \text{if } t \in \cup_{k \in \mathbb{Z}} [2k] \\
  t & \text{if } t \in \cup_{k \in \mathbb{Z}} [2k, 2k + 1] 
\end{cases}$$

and

$$\mu(t) = \begin{cases} 
  1 & \text{if } t \in \cup_{k \in \mathbb{Z}} [2k + 1] \\
  0 & \text{if } t \in \cup_{k \in \mathbb{Z}} [2k, 2k + 1] \end{cases}.$$ 

For a function $f : T \to \mathbb{R}^n$, we define $f^\Delta(t) \in \mathbb{R}$ (provided it exists) with the property that for every $\varepsilon > 0$, there exists $\delta > 0$ such that

$$||f(\sigma(t)) - f(s) - f^\Delta(t)(\sigma(t) - s)|| \leq \varepsilon ||\sigma(t) - s||$$

for all $s \in U_T(t, \delta)$. We call $f^\Delta(t)$ the delta derivative ($\Delta$-derivative for short) of $f$ at $t_0$. Moreover, we say that $f$ is delta differentiable ($\Delta$-differentiable for short) on $T^\sigma$ provided $f^\Delta(t)$ exists for all $t \in T^\sigma$.

A function $f$ is called rd-continuous provided that it is continuous at right-dense points in $T$, and has a finite limit at left-dense points, and the set of rd-continuous functions are denoted by $C_{rd}(T, \mathbb{R}^n)$. The set of functions $C_{rd}^p(T, \mathbb{R}^n)$ includes the functions whose derivative is in $C_{rd}(T, \mathbb{R}^n)$.

It is known [9] that for every $\delta > 0$ there exists at least one partition $P : a = t_0 < t_1 < \cdots < t_n = b$ of $[a, b)_T$ such that for each $i \in \{1, 2, \ldots, n\}$ either $t_i - t_{i-1} \leq \delta$ or $t_i - t_{i-1} > \delta$ and $\rho(t_i) = t_{i-1}$. For given $\delta > 0$ we denote by $P([a, b)_T, \delta)$ the set of all partitions $P : a = t_0 < t_1 < \cdots < t_n = b$ that possess the above property.

Let $f : T \to \mathbb{R}$ be a bounded function on $[a, b)_T$, and let $P : a = t_0 < t_1 < \cdots < t_n = b$ be a partition of $[a, b)_T$. In each interval $[t_{i-1}, t_i)_T$, where $1 \leq i \leq n$, choose an arbitrary point $\xi_i$ and form the sum

$$S = \sum_{i=1}^{n} (t_i - t_{i-1}) f(\xi_i).$$

We call $S$ a Riemann $\Delta$-sum of $f$ corresponding to the partition $P$.

We say that $f$ is Riemann $\Delta$-integrable from $a$ to $b$ (or on $[a, b)_T$) if there exists a number $l$ with the following property: for each $\varepsilon > 0$ there exists $\delta > 0$ such that $|S - l| < \varepsilon$ for every Riemann $\Delta$-sum $S$ of $f$ corresponding to a partition $P \in P([a, b)_T, \delta)$ independent of the way in which we choose $\xi_i \in [t_{i-1}, t_i)_T$, $i = 1, 2, \ldots, n$. It is easily seen that such a number $l$ is unique. The number $l$ is the Riemann $\Delta$-integral of $f$ from $a$ to $b$, and we will denote it by $\int_a^b f(t) \Delta t$.

A bounded function $f : [a, b)_T \to \mathbb{R}$ is Riemann $\Delta$-integrable on $[a, b)_T$ if and only if the set of all right-dense points of $[a, b)_T$ at which $f$ is discontinuous is a set of $\Delta$-measure zero.

Assume that $a, b \in T$, $a < b$ and $f : T \to \mathbb{R}$ is rd-continuous. Then the integral has the following properties.

(i) If $T = \mathbb{R}$, then $\int_a^b f(t) \Delta t = \int_a^b f(t) \, dt$, where the integral on the right-hand side is the Riemann integral.

(ii) If $T$ consists of isolated points, then

$$\int_a^b f(t) \Delta t = \sum_{t \in \{a,b\}_T} \mu(t)f(t).$$

A function $g : T \to \mathbb{R}$ is called a $\Delta$-antiderivative of $f : T \to \mathbb{R}$ if $g^\Delta(t) = f(t)$ for all $t \in T^\sigma$. It is well known that each rd-continuous function has a $\Delta$-antiderivative [17, Theorem 1.74].

Let $f : T \to \mathbb{R}$ be Riemann $\Delta$-integrable function on $[a, b)_T$. If $f$ has a $\Delta$-antiderivative $g : [a, b)_T \to \mathbb{R}$, then $\int_a^b f(t) \Delta t = g(b) - g(a)$. In particular, $\int_a^b f(t) \Delta s = \mu(t)f(t)$ for all $t \in [a, b)_T$ (see [17, Theorem 1.75]).

Let $f : T \to \mathbb{R}$ be a function which is Riemann $\Delta$-integrable from $a$ to $b$. For $t \in [a, b)_T$, let $g(t) = \int_a^t f(t) \Delta t$. Then $g$ is continuous on $[a, b)_T$. Further, let $t_0 \in [a, b)_T$ and let $f$ be arbitrary at $t_0$ if $t_0$ is right-scattered, and let $f$ be continuous at $t_0$ if $t_0$ is right-dense. Then $g$ is $\Delta$-differentiable at $t_0$ and $g^\Delta(t_0) = f(t_0)$ (see [8, Theorem 4.3]).

A function $f \in C^q_{rd}(T, \mathbb{R}^n)$ is called regressive if $1 + \mu(t)f(t)$ is invertible for all $t \in T^\sigma$. The set of regressive functions is denoted by $R_{rd}(T, \mathbb{R}^n)$ (or shortly denoted by $R$). For simplicity, we denote by $R_T(T, \mathbb{R})$ the set of complex regressive constants and similarly, we define the set of $R_T^+(T, \mathbb{C})$.

For the definition of the exponential function on time scales see [16]. A function $p : T \to \mathbb{R}$ is called to be positively regressive if $1 + \mu(t)p(t) > 0$ for all $t \in T$. If $p : T \to \mathbb{R}$ is a positively regressive function and $t_0 \in T$, then
then (see [16]) the exponential function \(e_p(t, t_0)\) is the unique solution of the initial value problem
\[
y^A = p(t)y, \quad y(t_0) = 1.
\]
In particular, if \(p \in \mathbb{R}\) is such that \(1 + \mu(t)p > 0\) for all \(t \in \mathbb{T}\), we have \(e_p(t, 0) = e^{pt}\) if \(\mathbb{T} = \mathbb{R}\), \(e_p(t, s) = (1 + ph)^t/h\) if \(\mathbb{T} = \mathbb{Z}\) with \(h > 0\).

The definition of generalized monolayer on time scales (see [17, Section 1.6]) \(h_n : \mathbb{T} \times \mathbb{T} \to \mathbb{R}\) is given as
\[
h_n(t, s) = \begin{cases} 1 & \text{if } n = 0, \\ t \int_{t^{-1}} h_n^{-1}(r, s) \Delta r & \text{if } n \in \mathbb{N}, \end{cases}
\]
for \(s, t \in \mathbb{T}\). It follows that
\[
h_n^A(t, s) = h_n(t, s) \quad \text{for all } n \in \mathbb{N},
\]
where \(h_n^A\) denotes \(\Delta\)-derivative of the \(h_n\) with respect to \(t\).

Using induction, it is easy to see that \(h_n(t, s) \geq 0\) holds for all \(n \in \mathbb{N}_0\) and all \(s, t \in \mathbb{T}\) with \(t \geq s\) and \((-1)^n h_n(t, s) \leq 0\) holds for all \(n \in \mathbb{N}\) and all \(s, t \in \mathbb{T}\) with \(t < s\).

Throughout this paper, we assume that \(\sup \mathbb{T} = \infty\). The next definitions and results were given in [19].

Let \(s \in \mathbb{T}\). A function \(f \in C_d(\mathbb{T}, \mathbb{C})\) has exponential order \(\alpha\) on \([s, \infty)_\mathbb{T}\), if \(\alpha \in \mathcal{R}_k^\Delta(\mathbb{T}, \mathbb{R})\) and there exists \(K > 0\) such that \(|f(t)| \leq Ke^\alpha(t, s)\) for all \(t \in [s, \infty)_\mathbb{T}\). For \(\alpha \in \mathcal{R}_k^\Delta(\mathbb{T}, \mathbb{R})\), it is easy to see that \(e_{\alpha}(s, t)\) is of exponential order \(\alpha\) on \([s, \infty)_\mathbb{T}\).

Let \(f(t) = \sum_{k=0}^{\infty} a_k h_k(t, s)\) for \(t \in [s, \infty)_\mathbb{T}\). If \(M, \alpha > 0\) with \(|a_k| \leq Me^\alpha\) for all \(k \in \mathbb{N}_0\), then \(|f(t)| \leq Me^\alpha(t, s)\) for all \(t \in [s, \infty)_\mathbb{T}\), so that \(f\) is of exponential order \(\alpha\).

Now we give the definition of the Laplace transform (see [19, Definition 4.5]):

Let \(f \in C_d(\mathbb{T}, \mathbb{C})\) be a function. Then the Laplace transform \(\mathcal{L}(f)(s)\) about the point \(s \in \mathbb{T}\) of the function \(f\) is defined by
\[
\mathcal{L}(f)(z) := \int_s^\infty f(t)e^{\Delta t}(\sigma(t), s) \Delta t \quad \text{for } z \in D, \tag{1}
\]
where \(D \subset \mathbb{C}\) such that \(z \in \mathcal{R}_e(\mathbb{T}, \mathbb{C})\) for which the improper integral converges.

For \(h > 0\), let
\[
\mathbb{R}_h := \mathbb{C}_h \cap \mathbb{R} = \left\{ \lambda \in \mathbb{R} : \lambda \neq -\frac{1}{h} \right\},
\]
and \(\mathbb{R}_0 = \mathbb{R}\). The minimal graininess function \(\mu_\bullet : \mathbb{T} \to \mathbb{R}_0^+\) by
\[
\mu_\bullet(s) := \inf_{\tau \in [s, \infty)_\mathbb{T}} \mu(\tau) \quad \text{for } s \in \mathbb{T},
\]
and for \(h \geq 0\), define
\[
\mathop{\text{Cp}}\mu(\lambda) := \{ z \in \mathbb{C}_h : Re(z) > \lambda \}.
\]
By using equation (1), \(\mathcal{L}(h_n(t, s))(z) = \frac{z}{z + 1}\), \((z \in \mathop{\text{Cp}}\mu(s)(0))\).

The following results about the Laplace transform were proved in [19].

**Theorem 2.1:** Let \(f \in C_d([s, \infty)_\mathbb{T}, \mathbb{C})\) be of exponential order \(\alpha\). Then the Laplace transform \(\mathcal{L}(f)(s)\) exists on \(\mathop{\text{Cp}}\mu(s)(\alpha)\) and converges absolutely.

**Theorem 2.2:** Let \(f \in C_d([s, \infty)_\mathbb{T}, \mathbb{C})\) be of exponential order \(\alpha_1, \alpha_2\), respectively. Then for any \(c_1, c_2 \in \mathbb{R}\), we have \(\mathcal{L}(c_1 f_1 + c_2 f_2) = c_1 L(f_1) + c_2 L(f_2)\) on \(\mathop{\text{Cp}}\mu(s)(\max(\alpha_1, \alpha_2))\).

Recently, Zada et al. [20], proved the following spectral decomposition theorem on time scales:

Let \(A\) be a regressive matrix of order \(n\). For each \(w \in \mathbb{C}^n\) there exist \(z_j \in \ker(A - \lambda_j)\) \((j = 1, 2, \ldots, k)\) such that
\[
e^{A}(s, 0)w = e^{A}(s, 0)z_1 + e^{A}(s, 0)z_2 + \cdots
+ e^{A}(s, 0)z_k, \quad s \in \mathbb{T}.
\]

Moreover, if \(z_j(s) := e^{A}(s, 0)z_j\) then \(z_j(s) \in \ker(A - \lambda_j)\) \((j = 1, 2, \ldots, k)\), for all \(s \in \mathbb{T}\) and there exist \(\mathbb{C}^n\)-valued polynomials \(t_j(s)\) with \(\deg(t_j) < n_j - 1\) such that
\[
z_j(s) = e^{A}(s, 0)t_j(s), \quad s \in \mathbb{T} \quad \text{and} \quad (j = 1, 2, \ldots, k).
\]

It follows that
\[
e^{A}(s, 0)w = e^{A}(s, 0)t_1(s) + e^{A}(s, 0)t_2(s) + \cdots
+ e^{A}(s, 0)t_k(s), \quad s \in \mathbb{T}, \tag{2}
\]

By using Theorem 2.3 and first translation theorem (see [21, Theorem 3.2]) of the Laplace transform, the form (2) becomes proper rational function.

On the other hand, if we have a proper rational function, then by using partial fractions method and applying Laplace inverse it is easy to get the form (2).

For various properties of the Laplace transform on time scales, we refer to [16,17,19,21].

Before addressing the problem of distinguishability, it helps to recall the solution frame work assumptions with the following time-invariant linear dynamic system
\[
x^A(t) = Ax(t) + Bu(t), \tag{3}
\]
where \(x(\cdot) \in \mathbb{R}^n, u(\cdot) \in \mathbb{R}^m\) are the state and input vectors and \(A \in \mathbb{R}^{n \times n}, B \in \mathbb{R}^{n \times m}\) are constant matrices.

For a regressive matrix \(A\) and rd-continuous input \(u\), the time-invariant dynamic linear system (3) with initial condition \(x(t_0) = x_0\) has a unique solution of the form
\[
x(t) = e^A(t, t_0)x_0 + \int_{t_0}^t e^A(t, \sigma(s))Bu(s)\Delta s.
\]
3. Distinguishability

Consider a switched dynamic system composed by time-invariant subsystems \( i = 1, 2, \ldots, q \):

\[
S_j: \begin{cases}
x^\Delta(t) = A_j x(t) + B_j u(t), \\
y(t) = C_j x(t),
\end{cases}
\]

where \( x(\cdot) \in \mathbb{R}^n, u(\cdot) \in \mathbb{R}^m \) and \( y(\cdot) \in \mathbb{R}^p \). Naturally, \( A_i \in \mathbb{R}^{n \times n}, B_i \in \mathbb{R}^{n \times m} \) and \( C_i \in \mathbb{R}^{p \times n} \).

Without loss of generality, we can assume only two subsystems i.e. \( i = 1, 2 \). Denote

\[
A = \begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix}, \quad B = \begin{pmatrix} B_1 \\ B_2 \end{pmatrix}, \quad \text{and} \quad C = \begin{pmatrix} C_1 \\ -C_2 \end{pmatrix}
\]

and

\[
X_0 = \begin{pmatrix} x_{10} \\ x_{20} \end{pmatrix}, \quad Y(\cdot) = y_1(\cdot) - y_2(\cdot).
\]

Recently, in [4], the authors gave a notion of distinguishability for linear non-autonomous systems and yielded a necessary and sufficient condition for distinguishability of two linear systems.

We refer the reader to [8,9,16,22] for a broad introduction to \( \Delta \)-measure and integration theory.

For \( \Delta \)-measurable set \( E \subset T \), a \( \Delta \)-measurable vector-valued function \( f : E \rightarrow \mathbb{R}^n \) belongs to \( L^1_\Delta(\mathbb{R}^n; \mathbb{R}^n) \) provided that

\[
\int_E ||f(\cdot)||(\cdot) \Delta s < \infty.
\]

**Definition 3.1:** (see [4]) Let \( J = [0, T]_T \) and \( J^0 = [0, T]_T \). We say that \( S_1 \) and \( S_2 \) are said to be distinguishable on \( J \) if for any non-zero

\[
(x_{10}, x_{20}, u(\cdot)) \in \mathbb{R}^n \times \mathbb{R}^n \times L^1_\Delta(J^0; \mathbb{R}^m),
\]

the corresponding outputs \( y_1(\cdot) \) and \( y_2(\cdot) \) cannot be identical to each other on \( J \).

To study the distinguishability of two subsystems, some auxiliary concepts of distinguishability have been stated here:

**Definition 3.2:** (see [4]) Let \( \mathcal{U} \subseteq L^1_\Delta(J^0; \mathbb{R}^m) \) be a function space. We say that \( S_1 \) and \( S_2 \) are \( \mathcal{U} \) input distinguishable on \( J \) if for any non-zero

\[
(x_{10}, x_{20}, u(\cdot)) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathcal{U},
\]

the outputs \( y_1(\cdot) \) and \( y_2(\cdot) \) cannot be identical to each other on \( J \).

Especially, when \( \mathcal{U} \) is the set of generalized polynomial function class, the set of analytic function class and the set of smooth function class \( C^\infty(J; \mathbb{R}^m) \), then the corresponding distinguishability is called “generalized polynomial input distinguishability”, “analytic input distinguishability” and “smooth input distinguishability”, respectively.

The distinguishability of \( S_1 \) and \( S_2 \) on \( J \) is equivalent to that for the following system:

\[
S: \begin{cases}
x^\Delta = AX(t) + Bu(t), \\
y(0) = x_0, \\
Y(t) = CX(t),
\end{cases}
\]

\((x_0, u(\cdot)) \neq 0 \) implies that \( Y(\cdot) \neq 0 \) on \( J \).

Thus the problem relates to the notion of zero dynamics.

The closed form solution with initial condition of the system (8) is as follows

\[
X(t) = e_{A}(t, 0)X_0 + \int_0^t e_{A}(t, \sigma(s))Bu(s) \Delta s
\]

for all \( t \in [0, \infty)_T \).

Let us define the following infinite order matrices

\[
\mathcal{M} := \begin{pmatrix} C & 0 & 0 & 0 & \cdots \\ CA & CB & \cdots & 0 & 0 \\ CA^2 & CAB & \cdots & CB & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots 
\end{pmatrix}
\]

and

\[
\mathcal{M}_N := \begin{pmatrix} C & 0 & 0 & 0 \\ CA & CB & \cdots & 0 \\ CA^2 & CAB & \cdots & CB \\ \vdots & \vdots & \vdots & \vdots 
\end{pmatrix}
\]

Our first result characterized generalized polynomial input distinguishability:

**Theorem 3.3:** The generalized polynomial input distinguishability of \( S_1 \) and \( S_2 \) is independent of \( T > 0 \) which is equivalent to that of every sub-matrix composed of the left finite column vector of \( \mathcal{M} \) has full column rank.

Moreover, the necessary and sufficient conditions for the \( N \)-th generalized polynomial input distinguishability of \( S_1 \) and \( S_2 \) is that \( \mathcal{M}_N \) has full column rank. While the necessary and sufficient conditions for the generalized polynomial input distinguishability of \( S_1 \) and \( S_2 \) is that for any \( N \geq 1 \), \( \mathcal{M}_N \) has full column rank.

**Proof:** For the given input \( u(\cdot) \in \mathbb{R}^m \), the corresponding output of the system (8) is as follows

\[
Y(t) = C e_{A}(t, 0)X_0 + \int_0^t C e_{A}(t, \sigma(s))Bu(s) \Delta s
\]

for all \( t \in J \).
Let \( u(\cdot) \) be an \( N \)-th \( \mathbb{R}^m \)-valued generalized polynomial on \( J \):

\[
u(t) = \sum_{j=0}^{N} \alpha_j h_j(t,0),
\]

where \( \alpha_j \in \mathbb{R}^m \).

Therefore, we have

\[
Y(t) = Ce_A(t,0)X_0 + \int_0^t Ce_A(t,\sigma(s))\beta \left( \sum_{j=0}^{N} \alpha_j h_j(s,0) \right) \Delta s.
\]

It follows that \( Y(\cdot) \) is analytic. Therefore \( Y \equiv 0 \) holds if and only if all \( \Delta \)-derivatives of \( Y(\cdot) \) at \( t = 0 \) equal to zero:

\[
y^{(\Delta)}(0) = 0, \quad \text{for all} \quad j = 0, 1, 2, \ldots \quad (13)
\]

Then the equation (13), implies the following system of equations:

\[
\begin{align*}
CX_0 &= 0 \\
CAx_0 + CBx_0 &= 0 \\
CA^2x_0 + CABx_0 + CBx_1 &= 0 \\
&\vdots \\
CA^{N+1}x_0 + CA^NBx_0 + \cdots + CBx_N &= 0
\end{align*}
\]

(14)

We get that (14) is equivalent to the following infinite dimensional equation:

\[
\mathcal{M}_N[X_0; \alpha_0; \alpha_1; \ldots] = 0.
\]

Therefore, the \( N \)th generalized polynomial input distinguishability of \( S_1 \) and \( S_2 \) is equivalent to that (15) admits only trivial solution. That is to say \( \mathcal{M}_N \) has full column rank.

Hence it is generalized polynomial input distinguishable and also independent of \( T \).

The following result is an immediate consequence of the above theorem and proof is similar to [4 Corollary 3.2].

**Corollary 3.4:** If \( S_1 \) and \( S_2 \) are generalized polynomial input distinguishable, then \( 2n \geq m \).

Generalizing the idea of Theorem 3.3, we can obtain the following result:

**Theorem 3.5:** The analytic input distinguishability of \( S_1 \) and \( S_2 \) on \( J \) are equivalent to the following infinite dimensional equation

\[
\mathcal{M}[X_0; \alpha_0; \alpha_1; \ldots] = 0, \quad (16)
\]

having only trivial solution such that the corresponding series

\[
u(t) = \sum_{j=0}^{\infty} \frac{\alpha_j}{j!} h_j(t,0)
\]

converges in an open interval including \( J \).

To prove our next results, we need to recall some concepts from [4].

It is well known that a special kind of \( F \)-type matrix

\[
\mathcal{G} = \begin{pmatrix}
G_1 & 0 & 0 & \cdots \\
G_2 & G_1 & 0 & \cdots \\
G_3 & G_2 & G_1 & \cdots \\
\vdots & \vdots & \vdots & \ddots
\end{pmatrix}
\]

is called \( p \times m \) \( G \)-type matrix if

\[
\{ G_i \}_{i=1}^{\infty} \in Q_{p,m} = \{ |Q_i| \}_{i=1}^{\infty} |Q_i| \in \mathbb{R}^{p \times m}, ||Q_i|| \leq M^t \text{ for some } M > 0 \}.
\]

It is easy to see that the product of two \( p \times m \) \( G \)-type matrices is still a \( p \times m \) \( G \)-type matrix.

In [4], Lou et al. introduced three types of invertible transformations on a \( p \times m \) \( G \)-type matrix \( \mathcal{G} \):

**Type I:** Left-multiply \( \mathcal{G} \) by an invertible \( p \times p \) \( G \)-type matrix \( P \).

**Type II:** If for some \( I = 1, 2, \ldots, p \) and \( J \geq 0 \), all \( (jp+l) \)th row vectors of \( \mathcal{G} \) \( (j = 0, 1, 2, \ldots, J) \) are zero, but the \( ((J+1)p+l) \)th row vector is not zero, then replace \( (jp+l) \)th row by \( ((J+1)p+l) \)th row \( (j = 0, 1, 2, \ldots, J) \).

**Type III:** If for some \( I = 1, 2, \ldots, p \), all \( (jp+l) \)th row vectors \( (j = 0, 1, 2, \ldots) \) are zero, then delete these rows.

Let us recall Lemma 4.6 of [4]:

**Lemma 3.6:** Let

\[
\mathcal{G} = \begin{pmatrix}
G_1 & 0 & 0 & \cdots \\
G_2 & G_1 & 0 & \cdots \\
G_3 & G_2 & G_1 & \cdots \\
\vdots & \vdots & \vdots & \ddots
\end{pmatrix} \neq 0
\]

be a \( p \times m \) \( G \)-type matrix. Then there exist an \( s \leq \min(p,m) \), an invertible \( m \times m \) matrix \( Q \) and an invertible transform \( P \), which is composed by finite transformations
of types I-III, such that
\[
P \begin{pmatrix} G_1 & Q & 0 & 0 & \ldots \\ G_2 & Q & G_1 & 0 & \ldots \\ G_3 & Q & G_2 & Q & G_1 & \ldots \\ \vdots & \vdots & \vdots & \vdots & \ddots & \end{pmatrix} = \begin{pmatrix} l_\delta & 0 & 0 & \ldots \\ 0 & l_\delta & 0 & \ldots \\ 0 & 0 & l_\delta & \ldots \\ \vdots & \vdots & \vdots & \ddots & \end{pmatrix},
\]
when \( s = m \) or
\[
P \begin{pmatrix} G_1 & Q & 0 & 0 & \ldots \\ G_2 & Q & G_1 & 0 & \ldots \\ G_3 & Q & G_2 & Q & G_1 & \ldots \\ \vdots & \vdots & \vdots & \vdots & \ddots & \end{pmatrix} = \begin{pmatrix} l_\delta & 0 & 0 & 0 & \ldots \\ 0 & \tilde{G}_2 & l_\delta & 0 & \ldots \\ 0 & \tilde{G}_3 & 0 & \tilde{G}_2 & l_\delta & \ldots \\ \vdots & \vdots & \vdots & \vdots & \ddots & \end{pmatrix},
\]
when \( s < m \), where \( \tilde{G}_j \) are \( s \times (m-s) \) matrices \( \{ \tilde{G}_j \} \in \mathcal{Q}_{s,(m-s)} \).

The following consequence of Lemma 3.6 is as follows. The proof is similar to the one in [4, Theorem 4.5] and therefore omitted.

**Theorem 3.7:** Let \( p < m \), then \( S_1 \) and \( S_2 \) are not analytic input distinguishable.

The following Lemma is corollary of Lemma 3.6:

**Lemma 3.8:** Let \( l, p, m \geq 1 \). Suppose that
\[
\{G_i\}_{i=1}^{\infty} \in \mathcal{Q}_{p,m}, \{D_i\}_{i=1}^{\infty} \in \mathcal{Q}_{p,l}.
\]
Then if the infinite order linear equation
\[
\begin{pmatrix} D_1 & G_1 & 0 & 0 & \ldots \\ D_2 & G_2 & G_1 & 0 & \ldots \\ D_3 & G_3 & G_2 & G_1 & \ldots \\ \vdots & \vdots & \vdots & \vdots & \ddots & \end{pmatrix} \begin{pmatrix} x_0 \\ \vdots \\ x_1 \\ \vdots \\ x_m \\ \vdots \\ x_{m+k} \\ \vdots \\ x_{(m+1)t} \\ \vdots \\ x_{(m+k)t} \\ \vdots \\ x_{(m+1)(k+1)t} \\ \vdots \\ x_{(m+k)(k+1)t} \end{pmatrix} = 0
\]
adopts non-trivial solutions, it must admit some non-trivial solution such that
\[
\sum_{j=1}^{\infty} \frac{z_j}{j!} h_j(t, 0)
\]
converges in \((-\infty, \infty)\).

Under the light of Lemma 3.8, we have the following interesting and important result.

**Theorem 3.9:** The analytic input distinguishability of \( S_1 \) and \( S_2 \) on \( J \) is equivalent to that (16) admits only trivial solution. Consequently, it is independent of \( T \).

We will now show that Theorem 3.9 implies that the smooth input distinguishability and analytic input distinguishability are equivalent.

**Theorem 3.10:** The analytic input distinguishability of \( S_1 \) and \( S_2 \) is equivalent to the smooth input distinguishability of \( S_1 \) and \( S_2 \).

Theorem 3.9 gives us conditions not only necessary but also sufficient to the smooth input distinguishability of the linear systems. However, the conditions are not easy enough to verify. Let us go further for an equivalent condition which is relatively more easy to verify.

From Theorem 3.7, if \( S_1 \) and \( S_2 \) are not analytic input distinguishable, then there exists a pair \((X_0, u(\cdot))\) such that
\[
(X_0, u(\cdot)) \neq 0
\]
and
\[
Y(t) = 0
\]
and
\[
u(t) = \sum_{j=0}^{\infty} \frac{a_j}{j!} h_j(t, 0) \quad \text{for } t \in [0, \infty),
\]
with
\[
\left| \frac{a_j}{j!} \right| \leq Me^\alpha t, \quad \text{for all } j = 0, 1, \ldots
\]
for some \( M, \alpha > 0 \). It follows that
\[
|u(t)| \leq Me^\alpha t, \quad \text{forall } t \in [0, \infty).
\]

and Laplace transform \( \mathcal{L}(u(\cdot))(s) \) can be defined for any \( s > M \).

Denote
\[
\Phi(s) = \mathcal{L}(Ce^A(\cdot))(s), \quad \Psi(s) = \Phi(s)B,
\]
and consider the Laplace transform of \( Y(t) \) in (19), we obtain the following results. These results are a straightforward generalization from the linear ODE-case [3, Lemma 3.1 and Lemma 3.2].

**Lemma 3.11:** If \( S_1 \) and \( S_2 \) are not distinguishable, then we can find a pair \((\tilde{X}_0, \tilde{u}(\cdot))\) satisfying (19) with
\[
\tilde{u}(\cdot) = e_{\lambda_1}(t, 0)P_1(t) + e_{\lambda_2}(t, 0)P_2(t) + \cdots + e_{\lambda_k}(t, 0)P_k(t),
\]
where \( \lambda_i \in \mathbb{C} \) and \( P_i(\cdot) \) are vector-valued polynomials \( (i = 1, 2, \ldots k) \).

**Lemma 3.12:** If \( S_1 \) and \( S_2 \) are not distinguishable, then we can find a pair \((\tilde{X}_0, \tilde{u}(\cdot))\) satisfying (19) with
\[
\tilde{u}(\cdot) = e_{\lambda}(t, 0)\xi,
\]
where \( \lambda \in \mathbb{C} \) and \( \xi \in \mathbb{C}^m \).

By using the above results, it implies that the necessary and sufficient conditions for 0-th generalized
polynomial input distinguishability and the \( N \)-th generalized polynomial input distinguishable are equivalent. Hereafter, we can see that for any \( N \geq 0 \), the matrix
\[
\begin{pmatrix}
C & 0 & 0 & \cdots \\
CA & CB & 0 & \cdots \\
C(A)^2 & CAB & 0 & \cdots \\
\vdots & \vdots & \vdots & \ddots \\
C(A)^N & C^{N+1} B & \cdots & \cdots & CB
\end{pmatrix}
\]
has full column rank if and only if
\[
\begin{pmatrix}
C & 0 \\
CA & CB \\
C(A)^2 & CAB \\
\vdots & \vdots \\
C(A)^N & C^{N+1} B
\end{pmatrix}
\]
has full column rank.

From the classical Cayley–Hamilton theorem (22) is equivalent to that
\[
\begin{pmatrix}
C & 0 \\
CA & CB \\
C(A)^2 & CAB \\
\vdots & \vdots \\
C(A)^n & CA^n B
\end{pmatrix}
\]
has a full column rank.

For \( \lambda \in \mathbb{C} \), consider
\[
\bar{S}_1 : \quad \begin{align*}
\dot{X}(t) &= (A - \lambda I)\bar{X}(t) + B\bar{u}(t), \\
\bar{y}(t) &= C\bar{X}(t).
\end{align*}
\]
Similar to previous notations (see e.g. (6) and (7)), it follows that
\[
\begin{align*}
\dot{X}(t) &= (A - \lambda I)\bar{X}(t) + B\bar{u}(t) \\
\bar{X}(0) &= \bar{X}_0 \\
\bar{y}(t) &= C\bar{X}(t).
\end{align*}
\]
(23)

We claim for any \( \bar{X}_0 \in \mathbb{C}^{2n} \) and \( \zeta \in \mathbb{C}^m \), \( (\bar{X}_0, \zeta) \neq 0 \), the solution of (23) corresponding to \( \bar{X}_0 \) and \( \bar{u}(t) \equiv \zeta \) satisfies \( \bar{y}(t) \neq 0 \) on \([0, \infty)\). Equivalently \( \bar{S}_1 \) and \( \bar{S}_2 \) are \( 0 \)th polynomial input distinguishable.

If it is not the case, then we have \( (\bar{X}_0, \zeta) \neq 0 \) such that the corresponding \( \bar{y}(t) \equiv 0 \). Let
\[
X(t) = e_{\bar{X}_0}(t, 0)\bar{X}(t), \quad Y(t) = e_{\bar{X}_0}(t, 0)\bar{y}(t),
\]
Then \((X(\cdot), Y(\cdot))\) solves (8) with
\[
X_0 = \bar{X}_0 \quad \text{and} \quad u(t) = e_{\bar{X}_0}(t, 0)\bar{u}(t).
\]
Since \( Y(t) = e_{\bar{X}_0}(t, 0)\bar{y}(t) = 0 \), by considering the real part or imaginary part of \( X_0, u(\cdot), X(\cdot) \) and \( Y(\cdot) \), it follows that \( S_1 \) and \( S_2 \) are not distinguishable. This is a contradiction.

Therefore, Theorem 3.3, implies that
\[
\mathcal{M}_\lambda = \begin{pmatrix}
C & 0 \\
C(A - \lambda I) & CB \\
\vdots & \vdots \\
C((A - \lambda I)^{2n} & C(A - \lambda I)^{2n-1} B
\end{pmatrix}
\]
has full column rank.

Summarizing the above arguments, we obtain our main result.

\textbf{Theorem 3.13:} Subsystems \( S_1 \) and \( S_2 \) are analytic input distinguishable if and only if for any \( \lambda \in \mathbb{C} \), the matrix \( \mathcal{M}_\lambda \) has a full column rank.

\textbf{Example 3.14:} Consider \( S_1, S_2 \) with their matrices being
\[
A_1 = 1, \quad B_1 = 3, \quad C_1 = -1, \\
A_2 = 3, \quad B_2 = -2, \quad C_2 = 1.
\]

It is easy to see that
\[
\tilde{\mathcal{M}} := \begin{pmatrix}
-1 & -1 & 0 & 0 & \cdots \\
-1 & -3 & -1 & 0 & \cdots \\
-1 & -3 & -3 + 2.3 & -1 & \cdots \\
\cdots & \cdots & \cdots & \cdots & \cdots
\end{pmatrix}
\]
Then \( S_1 \) and \( S_2 \) are generalized polynomial input distinguishable. However,
\[
\mathcal{M}_\lambda = \begin{pmatrix}
-1 & -1 & 0 \\
(\lambda - 1) & \lambda - 3 & -1 \\
-1 & -1 & \lambda + 3
\end{pmatrix}
\]
does not satisfy the full column rank condition for \( \lambda = 7 \). Therefore, \( S_1 \) and \( S_2 \) are not analytic input distinguishable. It implies that generalized polynomial input distinguishability is weaker than the analytic input distinguishability.

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