S1. ADDITIONAL INFORMATION ON THE SETUP AND EXPERIMENTAL RESULTS

S1.1. Details on the experimental setup

The specimen is a strip cut out from a gray acrylonitrile butadiene styrene (ABS) baseplate (LEGO®, Item 10701) and features 12×5 telescopic resonators arranged in a square lattice architecture. The strip is 5.7 cm wide and the distance between the clamped ends is approximately 46.3 cm. Each resonator is characterized by a rod/pillar (LEGO®, Elem. ID 395726) and a conical brick (LEGO®, Elem. ID 4518029) in prismatic contact. The conical brick can be slid up and down the rod to tune the pillar’s natural frequency; the rod-brick contact is strong enough for the brick to hold its position throughout the tests. Note that the pillars are attached to the baseplate through frictional contact as well (anchoring the base of the rod to one of the protuberances, studs, of the baseplate). This allows for an agile reconfiguration of the brick arrangement. The standing wave excitation signals are transmitted to the structure through an electromechanical shaker (Brüel & Kjær Type 4810) and a stinger. The out of plane velocity time histories of points on the back-side of the plate belonging to a pre-determined grid are recorded via a 3D Scanning Laser Doppler Vibrometer (3D-SLDV, Polytec PSV-400-3D). The acquisition is performed in the frequency domain (the Fast Fourier Transform is performed automatically within the PSV acquisition system). To eliminate non-repeatable noisy features from the response, measurements are repeated 10 times and averaged at each measurement location. As far as the vibrometer channel is concerned, we select a 5 V range and DC coupling. We acquire in the 0–5 kHz frequency range, and we concentrate on the 0–600 Hz band when postprocessing the data. The sampling frequency is $f_s = 12.8$ kHz and the number of FFT lines is 3200, resulting in a frequency resolution of 1.5625 Hz. The selected velocity decoder is the digital VD-08-10 mm/s/V, that allows acquisitions up to 20 kHz. The excitation is a pseudorandom waveform with maximum amplitude of 500 mV. The excitation signal is amplified using a Brüel & Kjær Type 2718 Power Amplifier, with gain set to 30 dB.

The grid of measurement points is shown in Fig. S1. The legend shows which sets of points are used to calculate

\[ \begin{align*}
\bullet & \text{: used to calculate } v_{x,\text{ave}}^\text{in} \\
\circ & \text{: used to calculate } v_{x,\text{ave}}^\text{out} \\
\ast & \text{: used to calculate } v_{z,\text{ave}}^\text{bd} \\
\ast & \text{: other meas. points}
\end{align*} \]

FIG. S1. Measurement grid (left, where the groups of points are used to calculate the quantities reported in the legend). Picture of the experimental setup (right).
certain quantities of interest. Note that the transmissibility is measured as $v_{z,ave}^{out}/v_{z,ave}^{in}$, while $v_{z,ave}^{bd}$ is used to reconstruct the dispersion relation of each configuration. In particular, the reconstruction operation is performed by taking the frequency-space data for all points highlighted in yellow, and performing a 1D discrete Fourier transform. This yields frequency-wavenumber spectral maps. The dispersion branches are extracted by tracing the maxima of the spectral function at each frequency.

Our specimens are extremely flexible and are made of a polymeric material. Moreover, since the pillars are attached to the baseplate via frictional contacts, it is reasonable to assume that, above a certain amplitude of excitation, these contacts can lead to nonlinearities in the response. To rule out nonlinearities from our explanations, we compare the responses of the same architecture featuring only M1 resonators to excitations at different amplitudes. The transmissibility plots for three loading amplitudes (0.1 V, 0.5 V—the amplitude used for all experiments throughout this article—and 1.0 V) are superimposed in Fig. S2. We can see that the only difference between the 0.1 V case and the 0.5 V case is represented by the morphology of the bandgap—with the 0.1 V case being characterized by a jagged “bottom”. With respect to the other two cases, the high amplitude (1.0 V) one is characterized by a different response both before and after the bandgap. We believe this amplitude-dependent behavior to be indeed due to a mix of material and geometric nonlinearities that are triggered when the load is larger than a certain threshold. However, we exclude that nonlinearities affect our observations on bandgap widening, due to the fact that the amplitude of excitation does not significantly affect the extent of the bandgap, whose sharp onset and sloping end are unchanged.

![Graph showing transmissibility plots for three loading amplitudes](image)

FIG. S2. Response of a uniform M1 architecture to three different loading amplitudes.
S1.2. Response of uniform architectures

In this section, we report additional results on the response of uniform architectures. The comparison between the experimentally-reconstructed dispersion curves of a plate with no pillars and of a plate with $12 \times 5$ identical M1-type pillars is shown in Fig. S3(a,b). Fig. S3(c) shows the frequency range of interest, where only one mode exists.

FIG. S3. (a), (b) Experimentally-reconstructed dispersion relation for a plate with no pillars, and for a plate with $12 \times 5$ identical M1-type pillars, respectively. (c) Low-frequency detail of (b), highlighting the hybridization bandgap of interest. Mode shapes for the case with M1 pillars, measured along the centerline of the plate strip and recorded at frequencies before, within and after the hybridization bandgap.

Fig. S3(a,b), on the other hand, show a much wider frequency range, where multiple modes are present. While we don’t have an explanation for all the modes in this range, we can see that the influence of the bricks is also significant at higher frequencies.

In Fig. S4, we report the low-frequency reconstructed band diagrams of all uniform configurations. We can see that the bandgap consistently shifts towards higher frequencies as we lower the conical brick along the pillar. Here, we define the bandgap as that frequency range where $k_x = 0$. While identifying the bandgaps is trivial for the T, M1, M2, M3, M4 configurations, things are not so clear for M5. This configuration seems to feature a wide gap split by a horizontal mode, that could be due to the mechanics of the pillar when the conical brick is located near its base. Note that this phenomenon is not captured numerically, when the brick is approximated as a point mass. For the reader’s convenience, all bandgap ranges extracted from these experimental results are tabulated in Table S1.

| Configuration | T   | M1  | M2  | M3  | M4  | M5  |
|---------------|-----|-----|-----|-----|-----|-----|
| $f_{\text{onset}}$ (Hz) | 228 | 242 | 259 | 284 | 298 | 327 |
| $f_{\text{end}}$ (Hz) | 289 | 311 | 339 | 355 | 366 | 423* |

TABLE S1. Experimental bandgap ranges. *: This bandgap is split by a mode of unknown origin at around 380 Hz.
FIG. S4. From left to right: reconstructed band diagrams of configurations featuring no pillars, pillars of the T, M1, M2, M3, M4, M5 type. The dots on the band diagrams are the maxima of the spectral function at each frequency.
S1.3. More responses of heterogeneous architectures

In this section, we report additional results related to the attenuation capabilities of architectures featuring graded and disordered spatial arrangements of heterogeneous resonators. In the article, we reported on various architectures displaying 10 resonators of each of the following types: T, M1, M2, M3, M4, M5. All the results we obtained with these sets of resonators are shown in Fig. S5. In Fig. S6, we show that similar considerations can be made for architectures comprising 15 M1, 15 M2, 15 M3 and 15 M4 resonators. We can see that bandgaps for the graded architectures, shown in Figs. S6a-b, span two of the reference bandgaps (yellow and green) and present a similar morphology (sharp onset and sloping end). On the other hand, the bandgaps of configurations featuring disordered brick arrangements are wider than their graded counterparts, spanning three or four individual bandgaps as shown in Figs. S6c-e, and also present different morphological characteristics (they have a “jagged” profile, while also being less deep).

In Fig. S7, we show the response of architectures comprising 20 M1, 20 M2 and 20 M3 resonators. Similar considerations as in the previous case apply, and we still observe a slight widening due to randomization.

In Fig. S8, we show the response of architectures comprising 30 M1 and 30 M2 resonators. As we decrease the degree of heterogeneity among resonator characteristics, we can see that the results for graded and disordered architectures do not differ much. Even though it is challenging to comment on the bandgap width, we are able to see that the bandgaps in Figs. S8a,b have a more regular morphology than those in Figs. S8c-e.
FIG. S6. Influence of the spatial arrangement of heterogeneously tuned resonators on wave attenuation. In all cases, 15/60 resonators are programmed to the brick configurations M1, M2, M3, and M4. (a), (b) Transmissibilities for graded and (c), (d), (e) spatially randomized configurations, marked by thick black lines.
FIG. S7. Influence of the spatial arrangement of heterogeneously tuned resonators on wave attenuation. In all cases, 20/60 resonators are programmed to the brick configurations M1, M2 and M3. (a), (b) Transmissibilities for graded and (c), (d), (e) spatially randomized configurations, marked by thick black lines.
FIG. S8. Influence of the spatial arrangement of heterogeneously tuned resonators on wave attenuation. In all cases, 30/60 resonators are programmed to the brick configurations M1 and M2. (a), (b) Transmissibilities for graded and (c), (d), (e) spatially randomized configurations, marked by thick black lines.
S2. ADDITIONAL INFORMATION ON THE MODEL AND NUMERICAL RESULTS

S2.1. Material properties for the numerical model

Table S2 summarizes the material properties used for numerical simulation of the metamaterial system. The Young’s modulus and density are denoted by $E$ and $\rho$, the conical mass of each pillar with $m_0$, Poisson’s ratio with $\nu$ and the value of structural damping with $\eta$. The value of Poisson’s ratio was chosen based on known data for ABS, the material from which the plate and pillars are made. The value of structural damping was chosen such that the simulated and measured transfer functions are of the same order of magnitude. Other parameters in Table S2 are based on measurements.

| $E_{\text{plate}}$ | $\rho_{\text{plate}}$ | $E_{\text{pillar}}$ | $\rho_{\text{pillar}}$ | $m_0$   | $\nu$  | $\eta$  |
|-------------------|----------------------|---------------------|---------------------|--------|--------|--------|
| 5.5 GPa           | $8.16 \times 10^{-4}$ g/mm$^3$ | 27.2 GPa           | $11.0 \times 10^{-4}$ g/mm$^3$ | 0.219 g | 0.35   | 0.015 Ns$^2$/m |
S2.2. Computation of dispersion relations

To compute the diversion curves of our metamaterial system, we define as our unit cell a portion of the plate that contains one column of 5 pillars. The units cell has a length of 11.25 mm along the $x$ axis and a width of 56.25 mm along the $y$ axis. The top and bottom edges of the unit cell are free, and Bloch boundary conditions are applied to the right and left edges. This unit cell is appropriate for capturing the dispersion properties of the uniform configurations due to the locally-resonant nature of the bandgap.

Fig. S9a shows the dispersion diagram of the tall configuration (T) containing all the modes up to 1 kHz (the first 14 modes of the unit cell). An inspection of the modes up to 600 Hz (not reported for brevity) reveals that the majority of mode shapes exhibit considerable twist/torsional motion where the centreline (along the $x$ axis) remains relatively motionless. While the modes with torsional motion do exist, they are not excited in our metamaterial system because of the two fixed boundary conditions as well as the mid-plane excitation. Out experimental reconstruction of the dispersion relation in Figs. 2b and S4 confirms this claim. Keeping only the mode shapes with non-negligible out-of-plane motion along the centreline, we are left with two branches in the dispersion diagram (modes 1 and 11). These branches are highlighted in thick red curves in Fig. S9a, and agree with measurements of Fig. S4. Fig. S9b shows the computed dispersion diagrams for the 6 uniform configurations.

![Dispersion Diagrams](image)

FIG. S9. Computed dispersion diagrams of the metamaterial system. (a) The first 14 modes of the tall configuration. The branches highlighted in red thick curves correspond to modes that have a non-negligible out-of-plane motion along the centreline. The inset magnifies the long-wavelength portion of the dispersion diagram. (b) Dispersion diagrams of the 6 uniform configurations (cf. Fig. S4). The length of the unit cell is $a = 11.25$ mm.
S2.3. Detailed comparison of bandgaps

We compare the bandgaps of different configurations: 6 uniform, 2 graded and random. Fig. S10a shows the transmissibility curves for the 6 homogeneous pillar populations (T, M1, M2, M3, M4, M5) introduced in Fig. 1. We have defined the bandgap as the frequency range where less than 1% percent of wave energy is transmitted through the plate. This corresponds to the portion of transmissibility curves that lies below the value of 0.1 (see Fig. S10a). Comparison of Fig. S10a to their measured counterparts in Fig. 2d shows that the numerical model closely reproduces the qualitative features of the transmissibility curves. The same is observed when comparing the bandgaps of the graded arrangements in Fig. S10b (simulated) and Figs.3a,3b (measured).

Fig. S10c shows how the bandgap evolves according to the spatial arrangement of the pillars. As the conical mass is slid down the pillars in uniform configurations (from T to M5), the bandgap shifts to higher frequencies and becomes narrower. The shift to higher frequencies occurs because the effective inertia of the sliding mass becomes smaller. The narrowing effect occurs because the relative amplitude of oscillations of the pillar decreases, resulting in weaker coupling between the pillars. The bandgaps of the graded and random arrangements have similar widths, but all three are much wider than bandgaps of the uniform arrangements. It is important to note that the bandgap of the random arrangement is the widest in the ensemble-average sense, meaning that individual realizations could have narrower bandgaps.

Our parametric study of bandgaps revealed that stiffening the plate (Fig. 4c) and increasing the spacing between the resonance frequencies of pillars (Fig. 4d) widens the bandgaps for graded and random configurations. Fig. S11 compares the transmissibilities of graded and random configuration for those parameter values that resulted in the widest bandgaps. As reported in Fig. 4, the random configuration has a wider bandgap than either of the graded configurations. When we consider the standard deviation of the transmissibilities within the ensemble of random configurations, we note that individual realizations may have bandgaps with a similar width to those of graded configurations (see Fig. S11b).
FIG. S11. Comparison of the transmissibilities for those parameters where the bandgap of the random configurations is the widest. The gray area corresponds to the standard deviation of the transmissibility within the ensemble. Panel (a) corresponds to Fig. 4c and panel (b) to Fig. 4d. In both cases, the random configuration has the widest bandgap on average.

Fig. S12 shows the influence of the spacing between pillars (along the $x$ axis) on the transmissibility of the M1 configuration – similar results are obtained for other uniform configurations. Increasing the spacing $d$ makes the coupling between pillars weaker, which results in a narrower bandgap. A similar effect is obtained by softening the plate, as reported in Fig. 4 for random and graded configurations.

FIG. S12. Influence of the spacing between pillars (along $x$) on the transmissibility of the M1 configuration. The experimental setup has a spacing of $d = 11.25$ mm.
S2.4. Spatial profile of the response

We have focused mainly on the influence of the spatial arrangement of pillars on the transmissibility of the metamaterial. In this section, we consider their influence on the spatial profile of the response, as quantified by the velocity amplitudes at the center line of the plate along the x axis. Fig. 2c shows the measured spatial profile of the plate for the uniform arrangement M1 – similar spatial profiles are obtained for other uniform arrangements. Here, we compare the spatial profiles for the graded and random arrangements.

Fig. S13 shows the spatial profiles at three frequencies near the bandgaps – see Fig. S10b for the corresponding transmissibility curves. As expected, the spatial profile of the response is highly dependent on the frequency, specifically within and after the region populated by the pillars. Within the shared bandgap frequencies (Figs. S13a and b), the three configurations have a somewhat similar spatial profile in the region containing the pillars, and their relative response amplitudes on the receiver side agrees well with their transmissibilities in Fig. S10b. At 407 Hz (Fig. S13c), the graded configurations are already well outside their bandgap. Accordingly, its velocity response is markedly lower than the response of the graded configurations within and after the locally resonant region.

**FIG. S13.** Comparison of the spatial profiles of the response for graded and random arrangements. The gray area shows the standard deviation of the black curve. The start and end of the region containing the pillars correspond to $x/L = 0.36$ and $x/L = 0.63$, respectively, where $L = 463$ mm is the total length of the plate.