The Boundary Conformal Field Theories of the 2D Ising critical points

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Abstract. We present a new method to identify the Boundary Conformal Field Theories (BCFTs) describing the critical points of the Ising model on the strip. It consists in measuring the low-lying excitation energies spectra of its quantum spin chain for different boundary conditions and then to compare them with those of the different boundary conformal field theories of the \((A_2, A_3)\) minimal model.

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1. Introduction
The critical properties of the two-dimensional (2D) Ising model on the plane are known to be described by the \((A_2, A_3)\) conformal model, which is a unitary minimal conformal field theory of the ADE classification. On two dimensional geometries with boundaries they should be described by conformal field theories (CFTs) defined on the half plane or equivalently on the strip (for review see [1] [3], [4],[5],[6] and [7]). For each boundary conditions, there exists a boundary conformal field theory (BCFT) having a spectrum formed by one or more of the Verma modules of the corresponding CFT on the plane. The main object of this work is a numerical study of the Ising linear quantum chain at criticality to identify the BCFTs of the \((A_2, A_3)\) conformal model.

The paper is organized as follows. In the first section we introduce briefly the subject of boundary conformal field theories and give their predictions (boundary states, partition functions, operators contents and energy spectrum) concerning the \((A_2, A_3)\) conformal model. In section two we introduce the methods used to make the identification of the different BCFTs for the 2D Ising model with the different allowed boundary conditions. The last section is consacred for presenting our numerical data for the quantum linear chain realizations of the Ising singularity of the 2D Ising model with different boundary conditions and to identify them with BCFTs candidates of \((A_2, A_3)\) conformal model.

2. BCFT predictions
A conformal field theory constructed on the complex upper half plane will be strongly constrained by the presence of the real axe. In this case only symmetries preserving this boundary are concerned. So that, the independence of the holomorphic and the anti-holomorphic components of the Virasoro algebra is broken and only the holomorphic symmetries for example are considered. As a consequence, the partition function of a BCFT on the half plane is no more
a sesquilinear combination of the Virasoro characters, as it was the case in the plane geometry. The partition functions $Z_{\alpha|\beta}$ describing the evolution of the system between the boundary states $|\alpha\rangle$ and $|\beta\rangle$ become linear on the characters, as follows ([1], [3]):

$$Z_{\alpha|\beta}(q) = \sum_j n_{j\alpha}^\beta \chi_j(q)$$

(1)

in this equation, the indices $j$ runs over all the characters $\chi_j(q)$ of the Virasoro algebra and the multiplicities $n_{j\alpha}^\beta$ are positive integers giving the operator content of the BCFT. They are obtained from the Verlinde formula [9] :

$$n_{j\alpha}^\beta = \sum_i S_{ji}^\alpha S_{\alpha i}^\beta S_{1i}^\beta$$

(2)

The allowed physical boundary states are some combinations of Ishibashi states [2] of the following form :

$$|\alpha\rangle = \sum_j \frac{S_{nj}}{\sqrt{S_{1j}}} |j\rangle$$

(3)

For the $(A_2, A_3)$ conformal model we have three representations generated by the three primary fields

$$\phi_{(1,1)} = I , \quad \phi_{(1,2)} = \sigma \quad \text{and} \quad \phi_{(2,1)} = \epsilon$$

Their conformal weights are respectively $h_{1,1} = 0 , \quad h_{1,2} = \frac{1}{16} \quad \text{and} \quad h_{2,1} = \frac{1}{2}$. In this case the modular matrix is

$$S = \frac{1}{2} \begin{pmatrix} 1 & 1 & \sqrt{2} \\ 1 & 1 & -\sqrt{2} \\ \sqrt{2} & -\sqrt{2} & 0 \end{pmatrix}$$

Applying the relation (3) we obtain the three allowed boundary states

$$|I\rangle = \frac{1}{\sqrt{2}} |I\rangle + \frac{1}{\sqrt{2}} |\epsilon\rangle + \frac{1}{(2)^{1/4}} |\sigma\rangle$$
$$|\epsilon\rangle = \frac{1}{\sqrt{2}} |I\rangle + \frac{1}{\sqrt{2}} |\epsilon\rangle - \frac{1}{(2)^{1/4}} |\sigma\rangle$$
$$|\sigma\rangle = |I\rangle - |\epsilon\rangle$$

From (1) and (2) we obtain the allowed partition functions

$$Z_{\sigma|\sigma} = \chi_{(1,1)} + \chi_{(2,1)} = \chi_I + \chi_\sigma$$
$$Z_{I|I} = Z_{\epsilon|\epsilon} = \chi_{(1,1)} = \chi_I$$
$$Z_{\sigma|I} = Z_{\sigma|\epsilon} = \chi_{(1,2)} = \chi_\sigma$$
$$Z_{\epsilon|I} = Z_{I|\epsilon} = \chi_{(2,1)} = \chi_\epsilon$$

The normalized Low-Lying excitation spectra of these four CFT candidates are given in table(1)
Table 1. excitation spectra of the BCFTs candidates in the ($A_2, A_3$) minimal model

| CFT of $h_{1,1}$ verma module | excitation energy | degeneracy |
|-------------------------------|------------------|------------|
|                               | 1 2 3 4 5 6 7 8 9 | 1 1 2 2 3 3 5 5 |

| CFT of $h_{1,2}$ verma module | excitation energy | degeneracy |
|-------------------------------|------------------|------------|
|                               | 1 2 3 4 5 6 7 8 | 1 1 2 2 3 4 5 6 |

| CFT of $h_{2,1}$ verma module | excitation energy | degeneracy |
|-------------------------------|------------------|------------|
|                               | 1 2 3 4 5 6 7 8 | 1 1 1 2 3 4 5 |

| CFT of $(h_{1,1} \oplus h_{2,1})$ verma module | excitation energy | degeneracy |
|------------------------------------------------|------------------|------------|
|                                               | 1 3 4 5 6 7 8 9 | 1 1 1 1 2 2 |

3. The Methods

The Hamiltonian limit of the statistical 2D Ising model provides a quantum spin chain model having the same critical behavior. We will focus on the Ising singularity of such quantum spin chain model including special boundary magnetic fields at the first and the last sites. That is, different boundary conditions are imposed on the quantum spin chain. We will determine its low lying excitation spectrum for different chain lengths and at special values of the spin-spin coupling constant which we will call "pseudo-critical" values. The phenomenological renormalization group (PRG) will fix these special values that define the Ising singularity (see [8] and the references therein).

For a given boundary condition, the numerical measurements of the spectrum for different chain lengths lead to series of values. When fitted to leading scaling behavior, these values will correspond to the energy levels at criticality.

In the last step, this will be compared with the BCFT predictions given in table(1).

The general form of the hamiltonian describing the quantum Ising chain with magnetic fields at the first and the last sites is given by

$$H = -\sum_{n=1}^{N-1} t\sigma_x(n)\sigma_x(n+1) - \sum_{n=1}^{N} \sigma_x(n) - h_1\sigma_z(1) - h_2\sigma_z(N)$$  \hspace{1cm} (4)

In equation (4), $N$ represents the number of sites in the chain, $\sigma_x(n)$ and $\sigma_z(n)$ are the 2x2 Pauli spin matrices at site "n", $h_1$ and $h_2$ are respectively the external magnetic fields applied at the first and the last sites and "t" is the ferromagnetic spin-spin coupling. We distinguish four cases:

(i) The free-free boundary conditions. In this case, one sets $h_1 = h_2 = 0$.
(ii) The fixed-parallel boundary conditions. In this case one sets $h_1 = h_2 = h_0$.
(iii) The fixed anti-parallel boundary conditions. In this cases one sets, $h_1 = -h_2 = h_0$
(iv) The free-fixed boundary conditions. In this cases one sets $h_1 = 0$ and $h_2 = h_0$

At the Ising singularity, the special coupling constants solve the PRG equation [13]

$$[N - 1]m(t(N), N - 1) - [N]m(t(N), N) = 0$$  \hspace{1cm} (5)
where $m(t, N)$ is the energy gap $[E_1(t, N) - E_0(t, N)]$. The energies $E_0(t, N)$ and $E_1(t, N)$ are the energies of the respective ground state "0" and first excited state "1" for the Ising quantum spin chain of length $N$. As $N \to \infty$, the special values of $t$ solving the PRG equation, converge to the Ising singularity. The scaling behavior of physical quantities at solutions of the PRG equation provide the scaling behavior of the same physical quantities near the corresponding critical point.

4. Numerical Check of the BCFT Predictions
For all the boundary conditions, we have measured the energy spectrum of the Ising quantum chains of lengths $N = 6$ to $N = 12$. The study of these spectra for different values of $N$ permits to us to obtain the series describing the energy levels. Then we have extrapolated them for $N \to \infty$ to obtain the energy spectra at criticality.

For both the free-free and fixed-parallel boundary conditions the solutions $t_c(N)$ of the PRG equation are presented respectively in tables (2) and (3). They converge toward the critical value $t_c(\infty) = 1$ when $(N \to \infty)$.

| Table 2. The critical values of $t$ solving the PRG equation for free-free B.Cs |
|---|
| $N = \begin{array}{ccccccccc}
4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 \\
t_c & 1.09269 & 1.05685 & 1.03846 & 1.02775 & 1.02097 & 1.01640 & 1.01318 & 1.01083 & 1.00905 \\
\end{array}$ |

| Table 3. The critical values of $t$ solving the PRG equation for Fixed Parallel B.Cs |
|---|
| $N = \begin{array}{ccccccccc}
5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 \\
t_c & 1.03284 & 1.04577 & 1.04302 & 1.03743 & 1.03199 & 1.02739 & 1.02353 & 1.02036 \\
\end{array}$ |

The measured low lying excitation energies spectra corresponding to the first case are normalized, organized in series and presented with the extrapolated values (obtained using the BST algorithm [14]) in table (4). For the second case see table (7) for the spectrum at criticality. For the fixed anti-parallel boundary conditions we observed a decreasing and an increasing values of the pseudo-critical coupling constants (see table (5)) which prove that in this case we didn’t reach the asymptotic regime with lengths below $N = 12$. The extrapolated values of the low lying excitation energies corresponding to this case are given in table.

In the special case of free-fixed boundary condition, the PRG equation doesn’t admit any exact solution for finite $N$. But we remark that for the special value of $t = 1$, the PRG equation admits a minimum which converges to 0 when $N$ goes to infinity. This means that the value $t = 1$ is the critical value for infinite chain. We give in table(6) the corresponding values of the PRG equation for finite size chain at $t = 1$ and in table (7) we present the corresponding extrapolated low lying excitation energies spectra.

In conclusion the analysis of the numerical measurement of the energy excitation spectra for the Ising linear chains with different boundary conditions confirm that these models at criticality are in the same universality class as those of the boundary conformal field theories of the $(A_2, A_3)$ minimal model. Note also that in this case all the singularities have been considered.
Table 4. Measured low lying excitation energies of the Ising quantum spin chain with free-free B.Cs

| N=6   | N=7   | N=8   | N=9    | N=10   | N=11   | N=12   | N=∞   |
|-------|-------|-------|--------|--------|--------|--------|-------|
| E_{1} | 1.00000 | 1.00000 | 1.00000 | 1.00000 | 1.00000 | 1.00000 | 1.00000 |
| 2     | 3.22593 | 3.19176 | 3.16683 | 3.14777 | 3.13270 | 3.12047 | 3.11033 |
| 3     | 4.22593 | 4.19176 | 4.16683 | 4.14777 | 4.13270 | 4.12047 | 4.11033 |
| 4     | 5.20053 | 5.19231 | 5.18240 | 5.17229 | 5.16261 | 5.15357 | 5.14526 |
| 5     | 6.20053 | 6.19231 | 6.18240 | 6.17229 | 6.16261 | 6.15357 | 6.14526 |
| 6     | 6.86405 | 6.95879 | 7.01551 | 7.05058 | 7.07267 | 7.08664 | 7.09540 |
| 7     | 7.86405 | 7.95879 | 8.01551 | 8.05058 | 8.07267 | 8.08664 | 8.09540 |
| 8     | 8.42647 | 8.38407 | 8.34923 | 8.32007 | 8.29531 | 8.27405 | 8.25560 |
| 9     | 8.12516 | 8.41845 | 8.60753 | 8.73470 | 8.82314 | 8.88630 | 8.93388 |
| 10/   | 9.42647 | 9.38407 | 9.34923 | 9.32007 | 9.29531 | 9.27405 | 9.25560 |

Table 5. The critical values of t solving the PRG equation for Fixed Anti-parallel B.Cs

| N=5  | 6     | 7     | 8     | 9     | 10    | 11    | 12    |
|------|-------|-------|-------|-------|-------|-------|-------|
| t_{c} | 1.17704 | 1.03605 | 0.99940 | 0.98770 | 0.98421 | 0.98442 | 0.98552 |

Table 6. The value of the PRG equation at t=1 for Free-fixed B.Cs

| N=5  | 6     | 7     | 8     | 9     | 10    | 11    | 12    |
|------|-------|-------|-------|-------|-------|-------|-------|
| PRG  | 0.05930 | 0.03284 | 0.02012 | 0.01325 | 0.00921 | 0.00667 | 0.00384 |

Table 7. The low lying Energy Excitation at Criticality for the different B.Cs

| Fixed Parallel B.Cs | 2.0000 | 3.0043 | 3.9983 | 4.0362 | 5.0098 | 5.0618 | 6.0325 | 6.0272 | 6.0068 |
|---------------------|--------|--------|--------|--------|--------|--------|--------|--------|--------|
| Free-Fixed B.Cs     | 1.0000 | 2.0006 | 3.0087 | 3.0007 | 4.0428 | 4.0088 | 5.0847 | 5.0409 | 5.0052 |
| Fixed AntiParallel B.Cs | 1.0000 | 2.0254 | 3.0155 | 3.9984 | 4.0857 | 5.0265 | 5.0843 |        |        |
| Free-Free B.Cs      | 1.0000 | 3.0018 | 4.0019 | 5.0033 | 6.0031 | 7.0048 | 8.0048 | 8.0052 | 9.0040 | 9.0052 |

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