MHD Kelvin-Helmholtz instability in non-hydrostatic equilibrium

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Abstract. The present work deals with the linear stability of a magnetohydrodynamic shear flow so that a stratified inviscid fluid rotating about a vertical axis when a uniform magnetic field is applied in the direction of the streaming or zonal flow.

In geophysical flow, the stability of the flow is determined by taking into account the non-hydrostatic condition depending on Richardson number $R_i$ and the deviation $\delta$ from hydrostatic equilibrium.

According to Stone [1], it is shown that such deviation $\delta$ decreases the growth rates of three kinds of instability which can appear as geostrophic (G), symmetric (S) and Kelvin-Helmholtz (K-H) instabilities.

To be specific, the evolution of the flow is therefore considered in the light of the influence of magnetic field, particularly, on K-H instability. The results of this study are presented by the linear stability of a magnetohydrodynamic, with horizontal free-shear flow of stratified fluid, subject to rotation about the vertical axis and uniform magnetic field in the zonal direction. Results are discussed and compared to previous works as Chandrasekhar [2] and Stone [1].

1. Introduction

Large-scale geophysical flows in planetary atmosphere are often dominated by horizontal basic shear flow in consequence of the influence of rotation, stratification and magnetic field. In many cases the variation of velocity plays an essential role in the dynamics and which may give rise to the instability. First Chandrasekhar [2] has established the equilibrium criteria with the aid of linear theory for the stratified non rotating heterogeneous fluid when different layers are in relative motion. It seems that the magnetic field parallel to the streaming has a stabilized effect. Among the important works related to rotation of geophysical fluid, Stone [1] has given accurate results for non geostrophic instability such as symmetric instability, and short waves similar as Kelvin-Helmholtz instability in non hydrostatic conditions. Another important works of baroclinic instabilities are viewed by Pierrehumbert et al [4] and more recently by Plougonven et al [5] have studied the model of Eady, under the condition of the critical layer.

Our concern is to solve the linear stability problem of an unidirectional shear flow under rotating system. The uniform magnetic field is assumed to be parallel to the streaming as in the geophysical case so as to the uniform magnetic field combined to the rotation of the system are expected to have competitive effects on the stability. We choose a hyperbolic-tangent profile as used by Drazin [3], which represents a general shear flow. As a consequence we search to understand the development of
the K-H instability undergoing under the condition of non hydrostatic equilibrium and with the influence of zonal magnetic field.

2. Formulation of the problem

2.1 General equations

The MHD equations in rotating system for a stratified inviscid incompressible Boussinesq fluid are expressed in time \( t \) and rectangular coordinates \( (x, y, z) \), respectively zonal, meridional and vertical directions.

**Continuity equations**

\[
\frac{\partial \rho}{\partial t} + \nabla \cdot \vec{V} = 0, \tag{1}
\]

\[
\nabla \cdot \vec{V} = 0. \tag{2}
\]

**Momentum equation,**

\[
\frac{\partial \vec{V}}{\partial t} + (\vec{V} \cdot \nabla) \vec{V} - \frac{\mu}{4\pi \rho_0} \left( \nabla \times \vec{H} \right) \times \vec{V} + 2\Omega \times \vec{V} = \frac{1}{\rho_0} \nabla \Pi \tag{3}
\]

**Induction equation,**

\[
\frac{\partial \vec{H}}{\partial t} + (\vec{V} \cdot \nabla) \vec{H} = (\vec{H} \cdot \nabla) \vec{V} + \lambda_m \nabla^2 \vec{H} \tag{4}
\]

**Continuity of the magnetic field,**

\[
\nabla \cdot \vec{H} = 0. \tag{5}
\]

Where \( \vec{V} = (U, V, W) \) denotes the fluid velocity, \( \vec{\Omega} = (\Omega_x, \Omega_y, \Omega_z) \) the angular velocity, \( \vec{H} = (H_x, H_y, H_z) \) the magnetic field, \( \rho \) is the density of the fluid, \( \rho_0 \) a reference density, \( \mu \) the magnetic permeability of the vacuum, \( \lambda_m \) the magnetic diffusion coefficient and \( g \) the gravity.

\[
\frac{\Pi}{\rho_0} = \frac{P}{\rho_0} + \frac{\mu H^2}{8\pi} + \frac{1}{2} \left( \vec{\Omega} \times \vec{r} \right)^2, \tag{6}
\]

represents the generalized pressure, where \( P \) is the static pressure. \( \vec{r} = (x, y, z) \), is the position vector and \( \nabla^2 = \left( \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right) \), the differential operator.

The full set of equations (1) to (4) must satisfy boundary layers that it should be indicated latter.

2.2 Hypothesis

Consider a basic state of an unbounded incompressible inviscid geophysical fluid, with an initial shear flow \( \vec{V} = (\vec{U}(z), 0, 0) \) in the x direction and a variable density \( \vec{\rho} \) (figure 1). The whole system is subject to a constant angular velocity around z axis, \( \vec{\Omega} = (0, 0, \Omega) \) and a uniform applied magnetic field \( \vec{H}_0 = (H_0, 0, 0) \).
The velocity satisfy to a hyperbolic-tangent representation of the form $\vec{U}(z) = U_0 \tanh \left( \frac{z}{d} \right)$, where $U_0$ and $d$ are respectively the characteristic velocity and depth of the shear layer. The stratified ambient buoyancy is in thermal wind balance, we have, $\frac{\rho(y,z)}{\rho_0} = -\beta z - 2 \frac{\Omega D \vec{U}}{g} y$, where $D$ denotes the derivative symbol $\frac{d}{dz}$.

The continuity of the magnetic field is implicitly included in the induction equations which assume a very small magnetic diffusion coefficient ($\lambda_m \ll 1$) so that the influence of the Joule effect is neglected.

When the Kelvin-Helmholz instability appears, the whole flow consists in a superposition of the basic flow and the associated perturbations of velocity, pressure, density and magnetic field.

2.3 Basic flow

From the general equations (1) to (4) and by considering the above hypothesis, the initial basic state is given by the mean velocity field $\vec{U} = U_0 \tanh \left( \frac{z}{d} \right)$, and the mean total pressure

$$\frac{\Pi}{\rho_0} = \left( \beta g \frac{z^2}{2} - 2 \Omega \vec{U} y + \frac{\mu H_0^2}{8\pi} + \frac{\Omega^2}{2} (x^2 + y^2) \right) + C_0.$$

Where, $C_0$ is a constant of integration.

2.4 Equations of the perturbation

On the basis of previous hypothesis and neglecting the quadratic terms of perturbation the system of equations (1)-(4) is reduced to,
Continuity equations
\[ \frac{\partial \rho^*}{\partial t} + U \frac{\partial \rho^*}{\partial x} + w \frac{\partial \rho^*}{\partial z} + v \frac{\partial \rho^*}{\partial y} = 0 \] (6)
\[ \frac{\partial u^*}{\partial t} + U \frac{\partial u^*}{\partial x} + \frac{\partial \rho^*}{\partial t} = 0 \] (7)

Momentum equations
\[ \frac{\partial \rho^*}{\partial t} + U \frac{\partial \rho^*}{\partial x} + w \frac{\partial U}{\partial z} - 2\Omega \dot{v} = -\frac{1}{\rho_0} \frac{\partial p^*}{\partial x} \] (8)
\[ \frac{\partial v^*}{\partial t} + U \frac{\partial v^*}{\partial x} - \frac{\mu H_0}{4\pi \rho_0} \left( \frac{\partial h_x}{\partial x} - \frac{\partial h_x}{\partial y} \right) + 2\Omega \dot{u} = -\frac{1}{\rho_0} \frac{\partial p^*}{\partial y} \] (9)
\[ \frac{\partial v^*}{\partial t} + U \frac{\partial v^*}{\partial x} - \mu H_0 \left( \frac{\partial h_x}{\partial x} - \frac{\partial h_x}{\partial y} \right) = -\frac{1}{\rho_0} \frac{\partial p^*}{\partial y} - \frac{g}{\rho_0} \dot{\rho}^* \] (10)

Where \( \rho^* \) and \( p^* \) denote respectively, the perturbations of both density and pressure.

Magnetic field equations,
\[ \frac{\partial h_x}{\partial t} + U \frac{\partial h_x}{\partial x} = H_0 \frac{\partial u^*}{\partial x} + \tilde{h} \frac{\partial U}{\partial z} \] (11)
\[ \frac{\partial h_y}{\partial t} + U \frac{\partial h_y}{\partial x} = H_0 \frac{\partial v^*}{\partial x} \] (12)
\[ \frac{\partial h_z}{\partial t} + U \frac{\partial h_z}{\partial x} = H_0 \frac{\partial w^*}{\partial x} \] (13)

The associated boundary conditions are
\[ \tilde{w}(z) = 0 \text{ as } z \rightarrow \pm \infty \] (14)

3. Stability analysis
Analyzing the disturbance into normal modes as Chandrasekhar [2], we seek perturbation solutions whose dependence on \( x, y \) and \( t \) is of the form,
\[ (\tilde{u}, \tilde{v}, \tilde{w}, \tilde{p}, \tilde{h}, \tilde{\rho}) = (u(z), v(z), w(z), p(z), h(z), \rho(z)) \text{Exp}[ikx + i\lambda y + i\sigma t] \], where \( k, \lambda \) and \( i\sigma \) are respectively the wave numbers in \( x \) and \( y \) directions and the growth rate.
Substituting these expressions into equations (6) to (14), we obtain the following set of equations,
\[ i(\sigma + kU)\rho^* = -D\rho^* + \frac{2\rho_0 \Omega \lambda D\bar{U}}{g} v \] (15)
\[ i(ku + \lambda v) = -Dw \] (16)
\[ \rho_0 (\rho + kU)u + \rho_0 (D\bar{U})w - 2\rho_0 \Omega v = -ikp \] (17)
\[ i\rho_0 (\sigma + k \overline{U}) \dot{v} - \frac{\mu H_0}{4\pi} (ikh_y - i\lambda h_x) + 2\rho_0 \Omega u = -i\lambda p \]  \hspace{1cm} (18)

\[ i\rho_0 (\sigma + k \overline{U}) w - \frac{\mu H_0}{4\pi} (ikh_z - D h_x) = -D p - g \rho' \]  \hspace{1cm} (19)

\[ h_x = \frac{k H_0}{\sigma + k \overline{U}} \left( \mu - i \frac{D \overline{U} w}{\sigma + k \overline{U}} \right) \]  \hspace{1cm} (20)

\[ h_y = \frac{k H_0}{\sigma + k \overline{U}} v \]  \hspace{1cm} (21)

\[ h_z = \frac{k H_0}{\sigma + k \overline{U}} w \]  \hspace{1cm} (22)

By eliminating \( u, v \) and \( p \) from the system (15-22), we are leading to a unique differential equation expressed only in function of \( w(z) \),

\[
\begin{align*}
\left[ (c + \overline{U}) - \frac{A_i^2}{(c + \overline{U})} \right] D^2 w + \left[ (c + \overline{U}) \left( \frac{D \rho}{\rho_0} \right) - \frac{4i\Omega \lambda (D \overline{U})}{(c + \overline{U})^2} + \frac{2 A_i^2 (D \overline{U})}{(c + \overline{U})} \right] D w + \left[ -K^2 (c + \overline{U}) - D^2 \overline{U} - \left( D \overline{U} \right) \left( \frac{D \rho}{\rho_0} \right) + \frac{4i\Omega \lambda (D \overline{U})^2}{K^2} + A_i^2 D^2 \overline{U} \right] w = 0
\end{align*}
\]

\[ \text{where } K^2 = k^2 + \lambda^2, \quad c = \frac{\sigma}{k} \text{ and } A_i = \sqrt{\frac{\mu H_0^2}{4\pi \rho_0}}. \]

For the dimensionless values, we choose the following scales, \( U_0 \) for the velocity, \( d \) and \( \frac{1}{d} \) for the vertical height, and the wave numbers. Retaining \( \beta = \frac{D \rho}{\rho_0} \) only for the buoyancy in the case of the marginal stability \( (c = 0) \) and considering \( k \parallel \lambda \), the equation (23) becomes,
\[
\left\{ \frac{U^2 - A}{U^2 - A} \right\} D^2 w + \left[ \frac{2A}{U} + \frac{\delta^2}{k^2} \left( \frac{U^2 + A}{U^2 - A} \right) \right] Dw + \left[ k^2 + \frac{D\overline{U}^2}{U} \left( A - \overline{U}^2 \right) + R_i \right] - \frac{2A(D\overline{U})^2}{U^2} \right\} w = 0
\]

(24)

Where \( R_i = \frac{g\beta_\perp}{U^2_0} \) denotes the Richardson number, \( A^* = \sqrt{A} = \frac{H}{U^2_0} \sqrt{\frac{\mu}{4\pi \rho_0}} \) the Alfvén number and \( \delta = \frac{2\Omega d}{U_0} \) is the deviation from hydrostatic equilibrium.

\[ R_o = \frac{U_0}{2\Omega L} \] and \( \Gamma = \frac{d}{L} \) are respectively the Rossby number and the aspect ratio.

By replacing the variable \( z \) by the dependent variable \( \overline{U} \) in (24) we have,

\[
D^* w + \left[ \frac{2(A - \overline{U}^4) + (1 + \overline{U}^2) \delta^2}{U\overline{U}^2 - 1} \left( A - \overline{U}^2 \right)^2 + \frac{\delta^2 \overline{U}^2}{k^2} \right] Dw + \left[ -\frac{U^2(2 + k^2 + 2\overline{U}^2)}{U^2(2 - 1)} \right] w = 0
\]

(25)

, where \( D^* = \frac{d}{d\overline{U}} \) is a simple differential operator.

The new boundary conditions become,

\[ w = 0 \text{ as } \overline{U} = \pm 1. \]

(26)

4. Results and discussion

4.1 The influence of magnetic field on K-H instability

4.1.1 Method of resolution. The differential equation (25) is solved by analytical method using the Ritz-Rayleigh approach, taking into account the associated boundary conditions (26). Collatz [6] has described this variational procedure. The solution is approached by a finite series of \( w(\overline{U}) \) so that

\[ w = \left( 1 - \overline{U}^2 \right)^{\alpha_1} \overline{U}^{\alpha_2} \sum_{i=1}^{N} b_i \overline{U}^{2(i-1)}, \]

where the unknown coefficients \( b_i \) are determined by performing their functional \( J(w) \), which can be described as the energy involved in the phenomenon in the case of non-rotational shear flow (\( \delta = 0 \)). By definition we have
$$J(w) = \int \left[ \left( 1 - U^2 \right) \left( U^2 - A \right) \left( \frac{dw}{dU} \right)^2 + \left( -2U^6 + \left( 2 - k^2 \right) \frac{1}{U^4} + \left( d + k^2 \right) R \right) \frac{1}{U^2} \right] dU$$

where the function $J(w)$ is depending on unknown constants $b_i$.

Therefore by imposing the minimizing conditions on functional $J$ so as to $\left( \frac{\partial J}{\partial b_i} = 0 \right)$, we can solve the homogeneous system in $b_i$ coefficients. According to the optimization method, we determine $\alpha_1$ and $\alpha_2$, and we reach at the convergent solution for the order $N=3$. The Richardson number $R_i$ is then determined as an eigenvalue problem by setting the determinant of the matrix coefficients equal to zero.

4.1.2 Stability criterion. Better results could be obtained, of course, by using more terms in the expansion, but the result for $N=3$ is actually accurate for the first eigenvalue. For example, as shown in the figure 2, in the absence of magnetic field ($A = 0$), $R_i = 0.25$ and $k = 0.71$. It can be seen that the agreement between our results and those obtained by Drazin [3], constitutes a good approximation.

![Figure 2](image)

**Figure 2.** Marginal stability curve for the standard case ($c = 0$), in the absence of magnetic field ($A = 0$).

The Richardson number $R_i$ was evaluated as a function of the dimensionless zonal wave number $k$. In figure 3, there are represented curves of marginal state ($c = 0$) for different values of Alfvén number $A$. For all parameter examined, it was found that within the interval of $A$, $0 \leq A \leq 0.5$ there exists $R_c$ and $k$ indicating instability.

An interesting feature is the peak of the curves, that means $R_{ic}$ has a maximum value for $k = k_c$. As mentioned by Drazin [3] and Chandrasekhar [2], the maximum value of $R_{ic}$ determines the wave number $k_c$ of the developing instability.

It is worth noting that when $R_i$ tends to zero, for $k = k_c = 0.37$ we can evaluate $A = A_{max} \approx 0.6$. The analysis reveals that for a minimum value of $k_c \left( A_{max} \right)$ close to $\frac{1}{2} k_c$ corresponding to $A=0$, the motion exhibit a limit or some threshold value. Beyond the onset of the instability of Kelvin-Helmholtz the latter becomes non-sensitive whatever the influence of increasing magnetic effect. As
the result of $A = A_{\text{max}} \approx 0.5$, the magnetic energy is close to half of the actual inertial energy and therefore the action of magnetic field is physically limited.

![Figure 3](image)

**Figure 3.** Evolution of the marginal curves labeled for different values of the Alfven number $A$.

4.2 The influence of rotation on K-H instability

4.2.1 Equation of stability (baroclinic problem). Substituting in the equation (23) the new scale $\frac{U_0}{2\Omega}$ the zonal wave number $k$ instead of $\frac{1}{d}$, we obtain the dimensionless zonal wave number $k^*$. If we consider a small scale $k^* \ll 1$, as Stone [1], we get a dimensionless equation,

$$
\left[ U - \frac{A}{U} \right] D^2 w + \frac{1}{U} [2A] D^2 w + \left[ - \varepsilon^2 U - D U^2 + \frac{(A \varepsilon^2 + R_i)}{U} + \frac{ADU^2}{U^2} - \frac{2ADU^2}{U} \right] = 0
$$

(28)

4.2.2 Marginal stability criteria. We notice that the term $\varepsilon = k^* \delta$ in the equation (28) is similar to $k$ in the equation (24). The same results for the non-rotational system with respect to $k$ are valid for the rotational case (K-H Baroclinic instability) with respect to $\varepsilon = k^* \delta$.

The figure 4 summarizes the results of the behavior of the Richardson number $R_i$ versus the parameter $\delta$ (non-hydrostatic conditions), for different values of $A$ at given $k^* = 10$. For this flow system satisfying a hyperbolic-tangent shear flow with inflexion point, the results are in agreement with those predicted by Stone [1] for a linear shear layer, the non-hydrostatic effect is significant for Kelvin-Helmholtz instability, for such a small $\delta$. It is important to notice that the curves ($A = 0$) decrease more rapidly.
5. Concluding remarks
Based on the linear theory, we have extended the Stone’s model accounting for a uniform magnetic field applied to velocity direction. Under the hypothesis of non hydrostatic equilibrium of the flow and using the hyperbolic profile as Drazin, we derive the linear stability equation for the vertical perturbation featuring K-H instability.

By means of Ritz-Rayleigh method we are leading to an eigenvalue problem in order to find out the Richardson number $R_i$ expressed in function of control parameters of the flow $\delta$ and $A$, and the characteristic values of the instability as the zonal wave number $k^*$ and the growth rate $\sigma$.

Therefore for a given wave number $k^*$, we have established a diagramm of K-H instability for the marginal state ($\sigma = 0$), in such way that the Richardson number $R_i$ is function of the modified Rossby number $\delta$ for different values of Alfven number $A$.

As a consequence, a significant property is that the stability diagram reveals that a maximum value of the Richardson number $R_{i_{\text{max}}}$ decrease as $A$ increase and therefore we deduce that the contribution of magnetic field may able to reduce the mass transfer process, for example, in dusty or polluted atmosphere. In general, we suggest applying this model to specific problems as astrophysical situations, ranging from the interaction of the solar wind with the magnetospheric boundary, to the dynamics of accretion disks and young stellar objects.

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