Full thermomechanical coupling in modelling of micropolar thermoelasticity

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Abstract. The present paper is devoted to plane harmonic waves of displacements and microrotations propagating in fully coupled thermoelastic continua. The analysis is carried out in the framework of linear conventional thermoelastic micropolar continuum model. The reduced energy balance equation and the special form of the Helmholtz free energy are discussed. The constitutive constants providing fully coupling of equations of motion and heat conduction are considered. The dispersion equation is derived and analysed in the form bi-cubic and bi-quadratic polynomials product. The equation are analyzed by the computer algebra system Mathematica. Algebraic forms expressed by complex multivalued square and cubic radicals are obtained for wavenumbers of transverse and longitudinal waves. The exact forms of wavenumbers of a plane harmonic coupled thermoelastic waves are computed.

Preliminaries & backgrounds
The concept of micropolar continua take its origin from the classical E. and F. Cosserat’s study [1]. Micropolar continuum theories include not only translational displacements but also microrotations. Contrary to conventional elasticity a micropolar continuum is described by the asymmetric strain and stress tensors known from many previous discussions and surveys [2–7]. Thus the asymmetric elastic theory is characterized by a comparatively large number of constitutive elastic constants need to be determined from the experimental observations. By now there is no fully coupled model of micropolar thermoelastic behaviour taking accounts of torsion–bending processes of heat conduction.

The conventional thermoelastic micropolar continuum may be described in terms of field formalism, for example, from positions of the Green–Naghdi thermoelasticity [8, 9]. They are based on different modifications of the classical Fourier law of heat conduction. The refinements aim at derivations of hyperbolic partial differential equations of coupled thermoelasticity. Those are to simultaneously fulfill the following conditions:

(i) Finiteness of the heat signal propagation velocity;
(ii) The ability of the spatial propagation of the thermoelastic waves without attenuation;
(iii) Existence of distortionless wave forms akin to the classical d’Alembert type waves.
CTEM theory is based on the classical Fourier law of heat conduction with an infinite velocity of propagation of an exponentially decaying heat signal. Hyperbolic thermoelasticity theory is characterised by the energy conservation and the finite propagation velocity of thermal waves known as second sound waves.

In [4, 10–13] a non-linear mathematical model of thermoelastic micropolar continuum has been presented in terms of 4-covariant field theoretical formalism. In-depth study of weak and strong discontinuities in macropolar thermoelastic continua is given in [14–19]. In [20] problems concerning plane harmonic wavenumbers of coupled type-III thermoelastic waves are discussed.

1. Requisites equations of micropolar thermoelasticity

A linear thermoelastic micropolar continuum model (see [21] for details) is used throughout the paper. For such continuum, equations of motion of micropolar media are written in direct tensor representation for the case of the absence of mass forces and mass moments

$$\begin{align*}
\nabla \cdot \sigma &= \rho \ddot{u}, \\
\nabla \cdot m + \epsilon \cdot \sigma &= J \ddot{\phi}.
\end{align*}$$

(1)

where $\nabla$ is the three-dimensional Hamiltonian differential operator (the nabla symbol), $\rho$ is the mass density, $J$ is a scalar dynamic characteristic of continuum (the rotational inertia), superimposed dot denotes partial differentiation with respect to time at fixed spatial coordinates, $u$ is the translational displacement vector and $\phi$ rotation vector, $\epsilon$ is the three-dimensional Levi–Civita symbol (permutation symbol, antisymmetric symbol, or alternating symbol).

Displacements $u$ and microrotations $\phi$ are associated with asymmetric strain tensor $e$ and bending–torsion tensor $\Gamma$ by formulae

$$\begin{align*}
e &= \nabla \otimes u - \epsilon \cdot \phi, \\
\Gamma &= \nabla \otimes \phi.
\end{align*}$$

(2)

The equation (2) are represented in a rectangular co-ordinate system as

$$\begin{align*}
e_{ji} &= \partial_j u_i - \epsilon_{jik} \phi_k, \\
\Gamma_{ji} &= \partial_j \phi_i.
\end{align*}$$

(3)

The reduced energy balance equation can be furnished by

$$-(\dot{\psi} + s \dot{\theta}) + \text{tr}(\sigma \cdot \dot{e}) + \text{tr}(m \cdot \dot{\Gamma}) - h \cdot \nabla \theta = \theta \xi$$

(4)

where $\psi = \psi(e, \Gamma, \theta)$ is the Helmholtz free energy per unit of volume, $s$ denotes the entropy referred to the unit of volume, $h$ is the heat flux vector, $\theta$ is the actual temperature, $\xi$ denotes the internal entropy product.

According to the postulate of the thermodynamics of irreversible processes the equation and inequality is valid

$$\xi = -h \cdot \nabla \theta \geq 0.$$  

(5)

The inequality (5) is satisfied by using the Fourier’s law of heat conduction, which states the proportionality of the heat flux vector $h$ and negative spatial temperature gradient $\theta$

$$h = -\Lambda \nabla \theta,$$

(6)
where Λs is the thermal conductivity coefficient (thermal diffusion coefficient), Λs > 0.

The following equation may be derived by transforming (4)

\[ \dot{\psi} = \sigma_{ji} \dot{e}_{ji} + \mu_{ji} \dot{\Gamma}_{ji} - \theta \dot{s}. \]  

(7)

Since the free energy is the function of the independent variables Γji, eji, θ then

\[ \dot{\psi} = \frac{\partial \psi}{\partial e_{ji}} \dot{e}_{ji} + \frac{\partial \psi}{\partial \Gamma_{ji}} \dot{\Gamma}_{ji} + \frac{\partial \psi}{\partial \theta} \dot{\theta}. \]  

(8)

Thus following relations are valid

\[ \sigma_{ji} = \frac{\partial \psi}{\partial e_{ji}}, \quad \mu_{ji} = \frac{\partial \psi}{\partial \Gamma_{ji}}, \quad s = - \frac{\partial \psi}{\partial \theta}. \]  

(9)

Let us expand the free energy ψ into the Taylor series in the vicinity of the natural state \( e = 0, \theta = 0 \), disregarding the terms of higher order than the second one. The following form of the expansion is obtained for isotropic, homogeneous and centrosymmetric bodies

\[ \psi = \frac{\mu + \eta}{2} e_{ji} e_{ji} + \frac{\mu - \eta}{2} e_{ji} e_{ij} + \frac{\lambda}{2} e_{kk} e_{nn} + \frac{\gamma + \varepsilon}{2} \Gamma_{ji} \Gamma_{ji} + \frac{\gamma - \varepsilon}{2} \Gamma_{ji} \Gamma_{ij} + \frac{\beta}{2} \Gamma_{nn} - \alpha e_{kk} \theta - \zeta \Gamma_{kk} \theta - \frac{\Lambda_s}{2} \theta^2. \]  

(10)

Asymmetric tensors of the force stresses \( \sigma \) and the moment stresses \( m \) are determined according to the constitutive law of GNI/-thermoelasticity

\[ \sigma = (\mu + \eta) e + (\mu - \eta) e^T + (\lambda \text{tr} e - \alpha \theta) \mathbf{I}, \]

\[ m = (\gamma + \varepsilon) \Gamma + (\gamma - \varepsilon) \Gamma^T + (\beta \text{tr} \Gamma - \zeta \theta) \mathbf{I}, \]

\[ s = \alpha \text{tr} e + \zeta \text{tr} \Gamma + \Lambda_s \theta. \]  

(11)

wherein \( \theta \) denotes the temperature increment over the referential temperature; \( \lambda, \mu, \eta, \gamma, \beta, \varepsilon \) are isothermal constitutive constants of type-I micropolar thermoelastic continuum; \( \alpha, \zeta \) are constitutive constants providing coupling of equations of motion and heat conduction. Constants \( \alpha, \zeta \) depend not only on the mechanical properties of the continuum, but also depend on the thermal properties.

Dynamic equations (1) should be supplemented by heat conduction equation. After substituting the stress tensors \( \sigma \) and \( m \) from the formulas (11) in the equations of motion (1) and taking account of (2) the system of coupled partial differential equations of motion and heat conduction for a linear isotropic type-I micropolar thermoelastic continuum in the absence of mass forces, moments, and heat sources can be written as [?, 21]:

\[
\begin{align*}
\left\{ \begin{array}{l}
(\lambda + \mu - \eta) \nabla \nabla \cdot u + (\mu + \eta) \nabla \cdot \nabla u + 2\eta \nabla \times \phi - \alpha \nabla \theta - \rho \ddot{u} = 0, \\
(\beta + \gamma - \varepsilon) \nabla \nabla \cdot \phi + (\gamma + \varepsilon) \nabla \cdot \nabla \phi - 4\eta \phi + 2\eta \nabla \times u - \zeta \nabla \theta - \zeta \phi = 0,
\end{array} \right.
\end{align*}
\]

\[
\begin{align*}
\nabla^2 \theta - \frac{\kappa}{\Lambda_s} \dot{\theta} - \frac{\alpha}{\Lambda_s} \nabla \cdot \dot{u} - \frac{\zeta}{\Lambda_s} \nabla \cdot \phi = 0.
\end{align*}
\]

(12)

Hereafter \( \kappa \) is the heat capacity (per unit volume) at constant (zero) strains.

As pointed out the nonzero constitutive constants \( \alpha, \zeta \) provide coupling of micropolar thermoelasticity equations. It is usually assumed that \( \zeta = 0 \) (see for example [21]). This constitutive constant is kept in all further considerations.

The scalar equation in the system (12) is called a generalized heat conduction equation conjugate to the equations of motion (the first and the second equations in (12)).
2. Plane harmonic waves propagation

Plane harmonic coupled micropolar thermoelastic wave manifests a fairly simple analytical structure:

\[
\begin{align*}
\mathbf{u} &= A e^{i(k \cdot \mathbf{r} - \omega t)}, \\
\phi &= S e^{i(k \cdot \mathbf{r} - \omega t)}, \\
\theta &= B e^{i(k \cdot \mathbf{r} - \omega t)},
\end{align*}
\]

where \( \mathbf{k} \) is a wave vector, \( \omega \) is a cyclic frequency; \( A, S \) are the polarization wave vectors; \( B \) is an amplitude of the deviation of temperature from the referential temperature. \( A, S, B \) are the values of weak discontinuities of displacements, microrotations and temperature, respectively.

Wavenumber \( k \) (the modulus of the wave vector \( k \) or the waves propagation constant) can be real as well as complex variable.

Substituting the expression (13) into the differential equation (12), and given that \( \nabla = i k \), \( \partial / \partial t = -i \omega \), the following coupled system of equations of wave vector \( k \), the cyclic frequency \( \omega \), the polarization vectors of the plane wave \( A, S \) and the amplitude \( B \) can be obtained:

\[
\begin{align*}
\begin{cases}
(\rho \omega^2 - (\mu + \eta)k^2)A - (\lambda + \mu - \eta)k(\mathbf{k} \cdot \mathbf{A}) + 2\eta k \times \mathbf{S} - \alpha i k B = 0, \\
(3\omega^2 - 4\eta - (\gamma + \varepsilon)k^2)S - (\beta + \gamma - \varepsilon)k(k \cdot \mathbf{S}) + 2\eta k \times \mathbf{A} - \varsigma i k B = 0, \\
\left(\frac{k}{\Lambda_s}i\omega - k^2\right)B - \frac{\alpha}{\Lambda_s}k \cdot \mathbf{A} - \frac{\varsigma}{\Lambda_s}k \cdot \mathbf{S} = 0.
\end{cases}
\end{align*}
\]

The dispersion equation for the system (14) furnishes

\[
\left(\frac{\rho \omega^2}{\rho} - (\mu + \eta)k^2 \right)\left(\frac{\rho \omega^2}{\rho} - \omega^2 - \Omega^2\right) - \frac{\Omega^2k^2\eta}{\rho} \times \left(\frac{\rho \omega^2}{\rho} - (\lambda + 2\mu)k^2 \right) \left(\frac{\rho \omega^2}{\rho} - (\beta + 2\gamma)k^2 \right) - \frac{i\omega k^2}{T_s^2} - \frac{\varsigma^2k^2\omega}{\Lambda_s^2} (\rho \omega^2 - (\lambda + 2\mu)k^2) = 0,
\]

where

\( \Omega^2 = \frac{4\eta}{\beta} \).

Hereafter we use following dimensionless notation:

wavenumbers of the transverse waves

\[\kappa_{\perp}^{\mu\nu} = \frac{\omega}{c_{\perp}^{\mu\nu}}, \quad \kappa_{\perp}^{\mu} = \frac{\omega}{c_{\perp}^{\mu}};\]

ratios of the transverse waves velocities

\[\tilde{\kappa}_{\perp}^{2} = \frac{c_{\perp}^{2}}{c_{\mu\nu}^{2}}, \quad \tilde{\kappa}_{\perp}^{2} = \frac{c_{\perp}^{2}}{c_{\perp}^{2}};\]

wavenumbers of the longitudinal waves

\[k_{\parallel}^{\mu\nu} = \frac{\omega}{c_{\parallel}^{\mu\nu}}, \quad k_{\parallel} = \frac{\omega}{c_{\parallel}};\]
ratios of the longitudinal waves velocities
\[ \tilde{k}_\parallel = \frac{k^{\mu\mu}_\parallel}{c^{\mu\mu}_\parallel} = \frac{c_\parallel}{c^{\mu\mu}_\parallel}; \]

constants elucidating the coupling of the mechanical and thermal features
\[ s^2_1 = \frac{\alpha^2}{\omega \Lambda_\gamma \rho}, \quad s^2_1 = \frac{s^2_2}{\omega \Lambda_\gamma \beta}; \]
specific dimensionless parameters
\[ h^2_1 = \left( \frac{\Omega}{\omega} \right)^2, \quad h^2_2 = h^2_1 k^{-2}_\perp - 1 - k^{-2}_\perp. \]
\[ \tilde{k}_1 = k_\parallel (1 - h^2_1), \quad Q^2_1 = \frac{\omega}{(2k_\parallel^2)}, \]
\[ Q^2_2 = Q^2_1 + k^2_\parallel s^2_1 + s^2_1, \quad Q^3_3 = Q^2_1 (\tilde{k}_1^2 + 1) + k^2_\parallel (s^2_1 + s^2_1 (1 - h^2_1)). \]

We can solve the dispersion equation separately as the bi-quadratic and bi-cubic equations and obtain the wavenumbers for the transverse waves
\[ \sqrt{2} \frac{k_{1,2,3,4}}{k^\mu_\perp} = \pm \sqrt{-h^2_2 \pm \sqrt{h^4_2 - 4(1 - h^2_1) k^2_\perp}}; \]
and for the longitudinal waves
\[ \frac{k_{1,2,3,4}}{k_\parallel} = \pm \sqrt{-\frac{1}{2} \left( a + b \right) \pm i \frac{\sqrt{3}}{2} \left( a - b \right) + \frac{1 + \tilde{k}_1^2 + iQ^2_2}{3}. \]
\[ \frac{k_{5,6}}{k_\parallel} = \pm \sqrt{a + b + 1 + \frac{\tilde{k}_1^2 + iQ^2_2}{3}.} \]

Here we use the following notations
\[ a = \sqrt{-\frac{q}{2} + \sqrt{\frac{q^2}{4} + \frac{p^2}{27}}}, \quad b = \sqrt{-\frac{q}{2} - \sqrt{\frac{q^2}{4} + \frac{p^2}{27}}}, \]
\[ \text{Rep} = \tilde{k}_1^2 - \frac{1}{32} \left( 1 + \tilde{k}_1^2 \right)^2 - Q^4_1, \]
\[ \text{Imp} = Q^2_3 - \frac{2}{3} Q_3^2 \left( 1 + \tilde{k}_1^2 \right), \]
\[ \text{Req} = \left( -\frac{2}{27} \tilde{k}_1^2 + \frac{1}{9} \tilde{k}_1^2 + \frac{1}{9} + \frac{2}{9} Q^2_1 \right) \tilde{k}_1^2 - \frac{2}{27} - \frac{1}{3} Q_3^2 Q_3^2 + \frac{2}{27} Q^4_2, \]
\[ \text{Imq} = \left( \frac{1}{3} Q^2_3 - Q^2_1 + \frac{1}{9} Q^2_3 - \frac{2}{9} \tilde{k}_1^2 Q^2_3 \right) \tilde{k}_1^2 - \frac{2}{9} Q_3^2 + \frac{1}{3} Q^3_3 + \frac{2}{27} Q^6_2. \]

3. Conclusions
(i) Thermomechanical coupling modelling has been developed within the frameworks of the conventional micropolar thermoelasticity taking account of full thermomechanical coupling.
(ii) The reduced energy balance equation have been discussed.
(iii) Constitutive equations and conservation laws have been derived from balance equations and the thermodynamics principles.

(iv) A plane harmonic coupled thermoelastic displacements and microrotations waves have been discussed.

(v) The dispersion equation have been derived and analysed in the form bi-cubic and bi-quadratic polynomials product.

(vi) Wavenumbers have been obtained and discriminated for a transverse and longitudinal thermoelastic waves.

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