THE PRIME IDEALS IN EVERY CLASS CONTAIN ARBITRARY LARGE TRUNCATED CLASSES

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ABSTRACT. We prove that the prime ideals in every class contain arbitrary large truncated classes.

1. Introduction

Green-tao [GT1] proved the following epoch-making theorem.

Theorem 1.1 (Green-Tao’s PAP theorem). The primes contains arbitrary long arithmetic progressions.

We shall prove a generalization of Green-Tao’s PAP theorem to number fields.

Let \( K \) be any number field. We embed it into its Minkowski space

\[ K_\infty = \prod_{\sigma \mid \infty} K_\sigma, \]

where \( K_\sigma \) is the completion of \( K \) at the archimedean place \( \sigma \). The metric on \( K_\infty \) is given by the formula

\[ \| (x_\sigma) \|_{\text{Min}}^2 = \sum_{\sigma \mid \infty} [K_\sigma : \mathbb{R}] \cdot \| x_\sigma \|^2. \]

So the balls

\[ B_r = \{ a \in K \mid \| a \|_{\text{Min}} < r \}, \quad r > 0 \]

form a fundamental system of neighborhoods of 0.

We view an arithmetic progression as a truncated residue class in \( \mathbb{Z} \). The ideal-theoretic generalization of the notion of residue classes in \( \mathbb{Z} \) to number fields is the notion of equivalence classes of ideals. Let \( O_K \) be the ring of integers in \( K \).

Definition 1.2. Let \( m, a, b \) be nonzero ideals of \( O_K \). If there is a nonzero number \( \xi \in 1 + ma^{-1} \) such that

\[ b = (\xi)a, \]

then \( b \) is said to be equivalent to \( a \) modulo \( m \).

Definition 1.3. Let \( m \) and \( b \) be nonzero fractional ideals of \( O_K \) such that \( m \subseteq b \). Let \( a \in b \) and \( r > 0 \). We call

\[ \{ \xi \in b \mid \xi \equiv a (\text{mod} m), \| \xi - a \|_{\text{Min}} < r \} \]
a truncated residue class of $b$ modulo $m$. We call it a truncated principal residue class of $b$ if $m$ is principal.

**Definition 1.4.** Let $m, b$ be nonzero fractional ideals of $O_K$ such that $m \subseteq b$, and let $A$ be a truncated residue class of $b$ modulo $m$. We call

$$\{\xi b^{-1} \mid \xi \in A\}$$

a truncated generalized class.

We shall prove the following generalization of Green-tao’s PAP theorem.

**Theorem 1.5.** The prime ideals in every class contain arbitrary large truncated generalized classes.

The proof of Theorem 1.5 is a generalization of the arguments of Green-Tao in [GT1]. A positive density version of the above theorem can be proved similarly.

## 2. Pseudo-random measures on inverse systems

In this section we establish the relationship between two kinds of measures on inverse systems.

Let $b$ a fixed nonzero fractional ideal of $K$. For the sake of convenience, we take $b$ to be the inverse of a nonzero integral ideal. Let $k$ be a fixed positive integer, and $I$ the set of positive integers which are prime to every nonzero number in $O_K \cap B_{2k}$. Then $\{(b/(N_b))_{N \in I}\}$ is an inverse system of finite groups. For each $j \in O_K \cap B_k$, we write $e_j = (O_K \cap B_k) \setminus \{j\}$. Then $(O_K \cap B_k, \{e_j\}_{j \in O_K \cap B_k})$ is a hyper-graph. To each hyper-edge $e_j$, we associate the inverse system $\{(b/(N_b))^{e_j}\}_{N \in I}$. Thus the system $\{(b/(N_b))^{e_j}\}_{N \in I, j \in O_K \cap B_k}$ maybe regarded as an inverse system on the hyper-graph $(O_K \cap B_k, \{e_j\}_{j \in O_K \cap B_k})$.

For each $j \in O_K \cap B_k$, and for each $N \in I$, let $\vartheta_{N,j}$ be a nonnegative function on $(b/(N_b))^{e_j}$.

**Definition 2.1.** The system $\{\vartheta_{N,j}\}_{N \in I, j \in O_K \cap B_k}$ is called a pseudo-random system of measures on the system $\{(b/(N_b))\}_{N \in I, j \in O_K \cap B_k}$ if the following conditions are satisfied.

1. For all $j \in O_K \cap B_k$, and for all $\Omega_j \subseteq \{0, 1\}^{e_j} \setminus \{0\}$, one has

$$\frac{1}{N^{[e_j]/[K:Q]}} \sum_{x^{(1)} \in (b/(N_b))^{e_j}} \prod_{\omega \in \Omega_j} \vartheta_{N,j}(x^{(\omega)}) = O(1),$$

uniformly for all $x^{(0)} \in (b/(N_b))^{e_j}$.

2. Given any choice $\Omega_j \subseteq \{0, 1\}^{e_j}$ for each $j \in O_K \cap B_k$, one has

$$\frac{1}{N^{2[O_K \cap B_k]/[K:Q]}} \sum_{x^{(0)}, x^{(1)} \in (b/(N_b))^{O_K \cap B_k}} \prod_{j \in O_K \cap B_k} \prod_{\omega \in \Omega_j} \vartheta_{N,j}(x^{(\omega)}) = 1 + o(1),$$

as $N \to \infty$ in $I$. 
Definition 2.2. The system on the inverse system $s$

Definition 2.4. If $\nu \in I$, let $\tilde{\nu}_N$ be a nonnegative function on $b/(Nb)$.

Definition 2.3. The system $\{\tilde{\nu}_N\}$ is said to satisfy the $k$-auto-correlation condition if, given any positive integers $s \leq |O_K \cap B_k|_1$, and for all mutually independent linear forms $\psi_1, \ldots, \psi_s$ in $m$ variables whose coefficients are numbers in $O_K \cap B_2$, we have

$$
\frac{1}{N^{2(|c|)}} \sum_{x \in (b/Nb)^s \cap \Omega_j} \prod_{\omega \in \Omega_j} \tilde{\nu}_{N,j}(x^{(\omega)}) = O(1).
$$

For each positive integer $N \in I$, let $\tilde{\nu}_N$ be a nonnegative function on $b/(Nb)$.

For all $j \in O_K \cap B_k$, for all $i \in \varepsilon_j$, and for all $\Omega_j \subseteq \{0,1\}^e$, and for all $M \in \mathbb{N}$, we have

$$
\frac{1}{N^{2(|\varepsilon|)}} \sum_{x^{(0)},x^{(1)} \in (b/Nb)^s} \left( \frac{1}{N^{2(|\varepsilon|)}} \sum_{x^{(0)},x^{(1)} \in b/Nb} \prod_{\omega \in \Omega_j} \tilde{\nu}_{N,j}(x^{(\omega)}) \right)^M = O(1).
$$

For all $j \in O_K \cap B_k$, for all $i \in \varepsilon_j$, and for all $\Omega_j \subseteq \{0,1\}^e$, and for all $M \in \mathbb{N}$, we have

$$
\frac{1}{N^{2(|\varepsilon|)}} \sum_{x^{(0)},x^{(1)} \in (b/Nb)^s} \left( \frac{1}{N^{2(|\varepsilon|)}} \sum_{x^{(0)},x^{(1)} \in b/Nb} \prod_{\omega \in \Omega_j} \tilde{\nu}_{N,j}(x^{(\omega)}) \right)^M = O(1).
$$

For each positive integer $N \in I$, let $\tilde{\nu}_N$ be a nonnegative function on $b/(Nb)$.

Definition 2.3. The system $\{\tilde{\nu}_N\}$ is said to satisfy the $k$-auto-correlation condition if, given any positive integers $s \leq |O_K \cap B_k|_1$, and for all mutually independent linear forms $\psi_1, \ldots, \psi_s$ in $m$ variables whose coefficients are numbers in $O_K \cap B_2$, we have

$$
\frac{1}{N^{2(|\varepsilon|)}} \sum_{x \in (b/Nb)^s \cap \Omega_j} \prod_{\omega \in \Omega_j} \tilde{\nu}_{N,j}(x^{(\omega)}) = O(1), \quad N \to \infty
$$

uniformly for all $b_1, \ldots, b_s \in b/(Nb)$.

Definition 2.4. The system $\{\tilde{\nu}_N\}$ is called a $k$-pseudo-random system of measure on the inverse system $\{b/(Nb)\}$ if it satisfies the $k$-correlation condition and the $k$-auto-correlation condition.

From now on we assume that

$$
\tilde{\nu}_{N,j}(x) := \tilde{\nu}_N \left( \sum_{i \in \varepsilon_j} (i - j)x_i \right).
$$

Theorem 2.5. If $\{\tilde{\nu}_N\}$ is $k$-pseudo-random, then $\{\tilde{\nu}_{N,j}\}$ is pseudo-random.

Proof First, we show that, for all $j \in O_K \cap B_k$, and for all $\Omega_j \subseteq \{0,1\}^e \setminus \{0\}$,

$$
\frac{1}{N^{2(|\varepsilon|)}} \sum_{x^{(1)} \in (b/Nb)^s \cap \Omega_j} \prod_{\omega \in \Omega_j} \tilde{\nu}_{N,j}(x^{(\omega)}) = O(1),
$$

uniformly for all $x^{(0)} \in (b/Nb)^s$. For each $\omega \in \Omega_j$, set

$$
\psi_\omega(x^{(1)}) = \sum_{i \in \varepsilon_j, \omega_i = 1} (i - j)x_i^{(1)},
$$

and

$$
b_\omega = \sum_{i \in \varepsilon_j, \omega_i = 0} (i - j)x_i^{(0)}.$$

Then
\[
\frac{1}{N^{|e_j|}[K:\mathbb{Q}]^2} \sum_{x^{(1)} \in (b/Nb)^{e_j}} \prod_{\omega \in \Omega_j} \tilde{\nu}_{N,j}(x^{(\omega)}) = \frac{1}{N^{|e_j|}[K:\mathbb{Q}]^2} \sum_{x^{(1)} \in (b/Nb)^{e_j}} \prod_{\omega \in \Omega_j} \tilde{\nu}_{N,j}(\psi_{\omega}(x^{(1)}) + b_\omega) = O(1).
\]

Secondly, we show that, given any choice \(\Omega_j \subseteq \{0,1\}^{e_j}\) for each \(j \in O_K \cap B_k\),
\[
\frac{1}{N^2[O_K \cap B_k][K:\mathbb{Q}]^2} \sum_{x^{(0)},x^{(1)} \in (b/Nb)^{O_K \cap B_k}} \prod_{j \in O_K \cap B_k} \prod_{\omega \in \Omega_j} \tilde{\nu}_{N,j}(x^{(\omega)}) = 1 + o(1),
\]
as \(N \to \infty\) in \(I\). For each pair \((j, \omega)\) with \(j \in O_K \cap B_k\) and \(\omega \in \Omega_j\), set
\[
\psi_{(j,\omega)}(x) = \sum_{i \in O_K \cap B_k, \omega_i = \delta} (i - j)x_i^{(\delta)}.
\]
Then
\[
\frac{1}{N^2[O_K \cap B_k][K:\mathbb{Q}]^2} \sum_{x^{(0)},x^{(1)} \in (b/Nb)^{O_K \cap B_k}} \prod_{j \in O_K \cap B_k} \prod_{\omega \in \Omega_j} \tilde{\nu}_{N,j}(\psi_{(j,\omega)}(x)) = 1 + o(1),
\]
as \(N \to \infty\) in \(I\).

Finally we show that, for all \(j \in O_K \cap B_k\), for all \(i \in e_j\), for all \(\Omega_j \subseteq \{0,1\}^{e_j}\), and for all \(M \in \mathbb{N}\),
\[
\frac{1}{N^{2(|e_j|-1)[K:\mathbb{Q}]^2}} \sum_{x^{(0)},x^{(1)} \in (b/Nb)^{e_j} \setminus \{i\}} \left( \frac{1}{N^2[K:\mathbb{Q}]} \sum_{x^{(0)},x^{(1)} \in (b/Nb)} \prod_{\omega \in \Omega_j} \tilde{\nu}_{N,j}(x^{(\omega)}) \right)^M = O(1).
\]
By Cauchy-Schwartz it suffices to show that, for \(a = 0,1\),
\[
\frac{1}{N^{2(|e_j|-1)[K:\mathbb{Q}]^2}} \sum_{x^{(0)},x^{(1)} \in (b/Nb)^{e_j} \setminus \{i\}} \left( \frac{1}{N^2[K:\mathbb{Q}]} \sum_{x^{(0)} \in (b/Nb)} \prod_{\omega \in \Omega_j, \omega_i = a} \tilde{\nu}_{N,j}(x^{(\omega)}) \right)^2M = O(1).
\]
For each \(\omega \in \Omega_j\) with \(\omega_i = a\), set
\[
\psi_{\omega}(x) = \sum_{l \in e_j \setminus \{i\}} (l - j)x_l^{(\omega_l)}.
\]
Then
\[
\frac{1}{N^{2(|e_j|-1)[K:\mathbb{Q}]^2}} \sum_{x^{(0)},x^{(1)} \in (b/Nb)^{e_j} \setminus \{i\}} \left( \frac{1}{N^2[K:\mathbb{Q}]} \sum_{x^{(a)} \in (b/Nb) \setminus \{x_i^{(a)}\}} \prod_{\omega \in \Omega_j, \omega_i = a} \tilde{\nu}_{N,j}(x^{(\omega)}) \right)^2M \leq \frac{1}{N^{2(|e_j|-1)[K:\mathbb{Q}]^2}} \sum_{x^{(0)},x^{(1)} \in (b/Nb)^{e_j} \setminus \{i\}} \sum_{\omega' \in \Omega_j, \omega_i = \omega'_i = a} \tilde{\nu}_{N,j}^{2M}(\psi_{\omega}(x) - \psi_{\omega'}(x)) \leq \frac{1}{N^{[K:\mathbb{Q}]}} \sum_{x \in b/(Nb)} x^{2M}(x) = O(1).
\]
In this section we establish the relationship between measures on inverse systems and measures on nonzero fractional ideals.

Let $A$ be a positive constant. For $N \in I$, let $\nu_N \ll \log^4 N$ be a nonnegative function on $b$.

**Definition 3.1.** The system $\{\nu_N\}$ is said to satisfy the $k$-cross-correlation condition if, given any parallelootope $I$ in $K_\infty$, given any positive integers $s \leq |O_K \cap B_k|2^{2|O_K \cap B_k|}$, given any $N \log^{-2sA} N \leq \lambda < N$, and given any mutually independent linear forms $\psi_1, \cdots, \psi_s$ in $m$ variables whose coefficients are numbers in $O_K \cap B_{2k}$, we have

$$\frac{1}{|b \cap (\lambda I)|} \sum_{x_i \in b \cap (\lambda I)} \prod_{j=1}^s \nu_N(\psi_j(x) + b_j) = 1 + o(1), \quad N \to \infty$$

uniformly for all numbers $b_1, \cdots, b_s \in b$.

**Definition 3.2.** The system $\{\nu_N\}$ is said to satisfy the $k$-auto-correlation condition if given any positive integers $s \leq |O_K \cap B_k|2^{2|O_K \cap B_k|}$, there exists a system $\{\tau_N\}$ of nonnegative functions on $b$ such that, given any parallelootope $I$ in $K_\infty$,

$$\frac{1}{|(NI) \cap b|} \sum_{x \in (NI) \cap b} \tau_N^M(x) = O_M(1), \quad \forall M \in \mathbb{N}$$

and

$$\frac{1}{|(NI) \cap b|} \sum_{x \in (NI) \cap b} \prod_{i=1}^s \nu_N(x + y_i) \leq \sum_{1 \leq i < j \leq s} \tau_N(y_i - y_j).$$

**Definition 3.3.** The system $\{\nu_N\}$ is $k$-pseudo-random if it satisfies the $k$-cross-correlation condition and the $k$-auto-correlation condition.

Let $\eta_1, \cdots, \eta_n$ be a $\mathbb{Z}$-basis of $b$, and set

$$G = \sum_{j=1}^n (-1/2, 1/2] \eta_j \subseteq K_\infty.$$

Let $\varepsilon > 0$ be a sufficiently small constant depending only on $k$ and $b$. From on on we assume that

$$\tilde{\nu}_N(x) = \begin{cases} \nu_N(\hat{x}), & x = \hat{x} + N\mathbb{b}, \hat{x} \in \varepsilon NG, \\ 1, & \text{otherwise}. \end{cases}$$

We now prove the following.

**Theorem 3.4.** If the system $\{\nu_N\}_{N \in I}$ is $k$-pseudo-random, then the system $\{\tilde{\nu}_N\}_{N \in I}$ is also $k$-pseudo-random.

**Proof** First we show that, given any positive integers $s \leq |O_K \cap B_k|2^{2|O_K \cap B_k|}$, $m \leq 2|O_K \cap B_k|$, and given any mutually independent linear forms $\psi_1, \cdots, \psi_s$ in $m$
variables whose coefficients are numbers in $O_K \cap B_{2\kappa}$,

$$
\frac{1}{N^{m|K:Q|}} \sum_{s_1, \ldots, s_m} \prod_{j=1}^s \nu_N(\psi_j(x) + b_j) = 1 + o(1), \quad N \to \infty
$$

uniformly for all $b_1, \ldots, b_s \in b/(Nb)$. It suffices to show that for any $S' \subset \{1, \ldots, s\}$,

$$
\frac{1}{N^{m|K:Q|}} \sum_{s_1, \ldots, s_m} \prod_{j \in S'} (\nu_N(\psi_j(x) + b_j) - 1) = o(1), \quad N \to \infty
$$

uniformly for all $b_1, \ldots, b_s \in b$. Regard $\psi$ as an $\mathbb{R}$-linear map from $K_\infty^m$ to $K_\infty^s$. There is a positive constant $c$ such that for any $x \in K_\infty^s$, the number of translations of $G$ by vectors in $x + b^s$ needed to cover $\psi(G^m)$ is $\leq c$. Hence the number of translations of $NG^s$ by vectors in $-b + Nb^s$ needed to cover $\psi(NG^m)$ is $O(1)$. Therefore it suffices to show that, for any $\beta \in b^S$,

$$
\frac{1}{N^{mn}} \sum_{x \in (NG\cap b)^m} \sum_{s \in S'} (\nu_N(\psi_j(x) + b_j) - 1) = o(1).
$$

Let $Q = \log^{2A} N$. We analyze the contributions to the left-hand side from the translates of $(\frac{1}{N} G)^m$ by vectors in $(NG \cap b)^m$. The translations whose images under $\psi$ do not intersect with $-b + N\beta + \varepsilon NG^S$ apparently make no contribution. The total contributions from translations whose images under $\psi$ are contained in $-b + N\beta + \varepsilon NG^S$ is equal to

$$
\frac{1}{N^{mn}} \sum_{\psi(x_0 + (\frac{1}{N} G)^m) \subseteq -b + N\beta + \varepsilon NG^S} \sum_{x \in x_0 + (\frac{1}{N} G)^m} \prod_{s \in S'} (\nu_N(\psi_j(x) + b_j - N\beta) - 1),
$$

which is $o(1)$ by the pseudo-randomness of $\{\nu_N\}$. It remains to consider the contribution from translations whose images under $\psi$ intersect with the boundary of $-b + N\beta + \varepsilon NG^S$. The total number of such translations is bounded by $O(Q^{mn-1})$. As each such a translation contributes at most $Q^{-mn} \log^s A N$. The total contribution given by such translations is bounded by $O(\frac{\log^{2A} N}{Q})$.

Secondly we show that, given any positive integers $s \leq |O_K \cap B_{\kappa}|2^{O_K \cap B_{\kappa}}$,

$$
\frac{1}{N^{|K:Q|}} \sum_{x \in b/(Nb)} \prod_{i=1}^s \nu_N(x + y_i) \ll \sum_{1 \leq i < j \leq s} \tilde{\tau}(y_i - y_j),
$$

where

$$
\tilde{\tau}(x) = \tau(x), \quad x \in NG.
$$

Set

$$
g_N(x) = \begin{cases} 
\nu_N(\hat{x}), & x = \hat{x} + Nb, \hat{x} \in \varepsilon NG, \\
0, & \text{otherwise}.
\end{cases}
$$

Then

$$
\frac{1}{N^{|K:Q|}} \sum_{x \in b/(Nb)} \prod_{i=1}^s \nu_N(x + y_i)
$$
We may assume that

\[ \text{For each } \]

In this section we prove the relative Szemerédi theorem for number fields.

Theorem 4.2. The relative Szemerédi theorem for number fields follows from a theorem of Tao in [Tao].

The following version of the relative Szemerédi theorem for number fields follows from a theorem of Tao in [Tao].

Theorem 4.2. If the system \( \hat{\nu}_N \) is pseudo-random, and \( \hat{\nu}_N \) has positive upper density relative to \( \hat{\nu}_N \), then there is a subset \( \hat{A}_N \) and a truncated residue class of \( b \) of size \( |O_K \cap B_k| \) such that

\[ A(\mod N b) \subseteq \hat{A}_N. \]
Theorem 4.3. If the system \( \tilde{\nu}_N \) is \( k \)-pseudo-random, and \( \tilde{A}_N \) has positive upper density relative to \( \tilde{\nu}_N \), then there is a subset \( \tilde{A}_N \) and a truncated residue class of \( b \) of size \( |O_K \cap B_k| \) such that

\[
A \pmod{Nb} \subseteq \tilde{A}_N.
\]

Definition 4.4. For \( N \in I \), let \( A_N \) be a subset of \( b \cap B_N \). The upper density of \( \{A_N\} \) relative to \( \{\nu_r\} \) is defined to be

\[
\limsup_{N \to \infty} \frac{\sum_{x \in A_N} \nu_N(g)}{\sum_{x \in b \cap B_N} \nu_N(x)}.
\]

We now prove the following.

Theorem 4.5. If \( \{\nu_N\} \) is \( k \)-pseudo-random, and \( \{A_N \cap B_{\varepsilon N}\} \) has positive upper density relative to \( \{\nu_N\} \), then there is a subset \( A_N \) that contains a truncated principal residue class of \( b \) of size \( |O_K \cap B_k| \).

Proof We have

\[
\frac{\sum_{x \in A_N \cap B_{\varepsilon N}} \tilde{\nu}_N(x)}{\sum_{x \in b \cap (Nb)} \nu_N(x)} = \frac{1}{N[K:Q]} \sum_{x \in A_N \cap B_{\varepsilon N}} \tilde{\nu}_N(x) + o(1) = \frac{1}{N[K:Q]} \sum_{x \in A_N \cap B_{\varepsilon N}} \nu_N(x) + o(1) = \sum_{x \in A_N \cap B_{\varepsilon N}} \nu_N(x) + o(1).
\]

So \( \{A_N \cap B_{\varepsilon N}(\pmod{Nb})\} \) has positive upper density relative to \( \{\tilde{\nu}_N\} \). By Theorem 4.3, there is a subset \( A_N \cap B_{\varepsilon N}(\pmod{Nb}) \), a truncated residue class of \( b \) of size \( |O_K \cap B_k| \) such that

\[
A \pmod{Nb} \subseteq A_N \cap B_{\varepsilon N}(\pmod{Nb}).
\]

As \( A \) is bounded, and \( \varepsilon \) is sufficiently small, we conclude that

\[
A \subseteq A_N \cap B_{\varepsilon N}.
\]

The theorem follows. \( \blacksquare \)

5. The cross-correlation of the truncated von Mangoldt function

In this section we shall establish the cross-correlation of the truncated von Mangoldt function.

The truncated von Mangoldt function for the rational number field was introduced by Heath-Brown [HB]. The truncated von Mangoldt function for the Gaussian number field was introduced by Tao [Tao]. The cross-correlation of the truncated von Mangoldt function for the rational number field were studied by Goldston-Yıldırım in [GY1, GY2, GY3], and by Green-Tao in [GT1, GT2].

Let \( \varphi : \mathbb{R} \to \mathbb{R}^+ \) be a smooth bump function supported on \([-1, 1]\) which equals 1 at 0, and let \( R > 1 \) be a parameter. We now define the truncated von Mangoldt function for the number field \( K \).
The prime ideals contain arbitrary large truncated classes

Definition 5.1. We define the truncated von Mangoldt function \( \Lambda_{K,R} \) of \( K \) by the formula

\[
\Lambda_{K,R}(n) := \sum_{d \mid n} \mu_K(d) \varphi\left( \frac{\log N(d)}{\log R} \right),
\]

where \( \mu_K \) is the Möbius function of \( K \) defined by the formula

\[
\mu_K(n) = \begin{cases} 
(-1)^k, & n \text{ is a product of } k \text{ distinct prime ideals,} \\
0, & \text{otherwise.}
\end{cases}
\]

Note that \( \Lambda_{K,R}(n) = 1 \) if \( n \) is a prime ideal with norm \( \geq R \).

Let \( \zeta_K(z) \) be the zeta function of \( K \), \( \phi_K(W) := |O_K/(W)^\times| \),

\[
\hat{\varphi}(x) = \int_{-\infty}^{\infty} e^{itx} d_t,
\]

and

\[
c_\varphi := \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \frac{(1+iy)(1+iy')}{(2+iy+iy')} \hat{\varphi}(y)\hat{\varphi}(y') dy dy'.
\]

From now on, for each \( N \in I \), let

\[
\nu_N(x) = \frac{\phi_K(W) \log R \cdot \text{Res}_{z=1} \zeta_K(z)}{c_\varphi W^{|K:2|}} \Lambda_{K,R}^2((W x + \alpha)b^{-1}).
\]

Here

\[
\log R = \frac{\log N}{8|O_K \cap B_k|2|O_K \cap B_k|},
\]

\( W \) is the product of prime numbers \( \leq w := \log \log N \), and \( \alpha \) a number prime to \( W \).

We now prove the following.

Theorem 5.2. The system \( \{\nu_N\} \) satisfies the k-cross-correlation condition.

Proof Given any parallelootope \( I \) in \( K_\infty \), given any positive integers \( s \leq |O_K \cap B_k|2|O_K \cap B_k|, m \leq 2|O_K \cap B_k| \), given any \( N \log^{-2sA} N < \lambda < N \), and given any mutually independent linear forms \( \psi_1, \cdots, \psi_s \) in \( m \) variables whose coefficients are numbers in \( O_K \cap B_{2k} \), we show that

\[
\frac{1}{|b \cap (\lambda I)|^s} \sum_{x_i \in b \cap (\lambda I)} \prod_{j=1}^{s} \nu_N(\psi_j(x) + b_j) = 1 + o(1), \; N \to \infty
\]

uniformly for all numbers \( b_1, \cdots, b_s \in b \).

We define

\[
\mathcal{G} = \sum_{d, d'} \omega(d_1 \cap d'_1 \leq \cdots \leq d_s) \prod_{i=1}^{s} \mu_K(d_i) \mu_K(d'_i) \varphi\left( \frac{\log N d_i}{\log R} \right) \varphi\left( \frac{\log N d'_i}{\log R} \right),
\]

where \( d \) and \( d' \) run over \( s \)-tuples of ideals of \( O_K \), and

\[
\omega(d_1 \leq \cdots \leq d_s) = \left\{|x \in (b/(\cap_{i=1}^{s} d_i))^m : d_i ((W \psi_i(x) + b_i')b^{-1}, \forall i = 1, \cdots, s\right\}
\]

with \( b'_i = W b_i + \alpha \).
Let \( \{ \gamma_j \} \) \( (j = 1, \cdots, [K : \mathbb{Q}]) \) be a \( \mathbb{Z} \)-basis of \( b \) such that \( \{ \lambda_j \gamma_j \} \) is a \( \mathbb{Z} \)-basis of \( b \cdot \cap_{i=1}^s d_i \), where each \( \lambda_i \) is a positive integer. Set

\[
I_0 = \{ x \in K_\infty : x_i \in \sum_{j=1}^{[K : \mathbb{Q}]} (0, 1] \lambda_j \gamma_j \}.
\]

Then

\[
\omega((d_i)_{1 \leq i \leq s}) = \frac{|\{ x \in (I_0 \cap b)^m : d_i | (W \psi_i(x) + b'_i)b^{-1}, \forall i = 1, \cdots, s \}|}{(N \cap_{i=1}^s d_i)^m}.
\]

The number of translates of \( I_0^m \) by vectors in \( (b \cdot \cap_{i=1}^s d_i)^m \) which intersect the boundary of \( \lambda I^m \) is bounded by \( O(\lambda^{m[K : \mathbb{Q}]-1}/(\prod_{j=1}^{[K : \mathbb{Q}]})^m) \). So the number of translates of \( I_0^m \) by vectors in \( (b \cdot \cap_{i=1}^s d_i)^m \) which lie in the interior of \( \lambda I^m \) is

\[
\frac{\text{Vol}(I)^m}{\text{Vol}(I_0)^m}\lambda^{m[K : \mathbb{Q}]} + O(\lambda^{m[K : \mathbb{Q}]-1}N(\cap_{i=1}^s d_i)/(\prod_{j=1}^{[K : \mathbb{Q}]})^m).
\]

It follows that

\[
\frac{|\{ x \in (\lambda I \cap b)^m : d_i | (W \psi_i(x) + b'_i)b^{-1}, \forall i = 1, \cdots, s \}|}{\lambda^{m[K : \mathbb{Q}]}\text{Vol}(I)^m/(\sqrt{|d_K|N(b)^{K[\mathbb{Q}]} \text{Vol}(I)^m}) = \omega((d_1)_{1 \leq i \leq s}) + O\left(\frac{N(\cap_{i=1}^s d_i)}{\lambda}\right),
\]

where \( d_K \) is the discriminant of \( K \). From that estimate one can infer

\[
\left(\frac{|d_K|N(b)}{\lambda^{m[K : \mathbb{Q}]}}\right)^m \sum_{x \in (\lambda I / \cap_{i=1}^s d_i)^m} \prod_{i=1}^s \Lambda'^2 R((W \psi_i(x) + b'_i)b^{-1}) = \mathcal{G} + O\left(\frac{R^{4s}}{\lambda}\right).
\]

Therefore we are is reduced to the following.

\[
\mathcal{G} = (1 + o(1))\frac{c_p W^{[K : \mathbb{Q}]}}{\phi_K(W) \log R \cdot \text{Res}_{z=1} \zeta_K(z)^s}.
\]

We define

\[
F(t, t') = \sum_{\delta, \delta'} \omega((\delta_j \cap \delta'_j)_{1 \leq j \leq s}) \prod_{j=1}^s \frac{\mu_K(\delta_j) \mu_K(\delta'_j)}{N(\delta_j)^{1/\log R} N(\delta'_j)^{1/\log R}} \cdot t, t' \in \mathbb{R}^s,
\]

where \( \delta \) and \( \delta' \) run over \( s \)-tuples of ideals of \( O_K \).

It is easy to see that, for all \( B > 0 \),

\[
e^x \varphi(x) = \int_{-\sqrt{\log R}}^{\sqrt{\log R}} \hat{\varphi}(t)e^{-ixt} dt + O((\log R)^{-B}).
\]

It follows that for all \( B > 0 \),

\[
\mathcal{G} = \int_{[-\sqrt{\log R}, \sqrt{\log R}]^s} \int_{[-\sqrt{\log R}, \sqrt{\log R}]^s} F(t, t') \hat{\varphi}(t) \hat{\varphi}(t') dt dt' + O((\log R)^{-B} \cdot \sum_{\delta, \delta'} \omega((\delta_j \cap \delta'_j)_{1 \leq j \leq s}) \prod_{i=1}^s \frac{|\mu_K(\delta_j) \mu_K(\delta'_j)|}{N(\delta_j)^{1/\log R} N(\delta'_j)^{1/\log R}}).
\]

Hence we are reduced to prove the following.

\[
\sum_{\delta, \delta'} \omega((\delta_j \cap \delta'_j)_{1 \leq j \leq s}) \prod_{j=1}^s \frac{|\mu_K(\delta_j) \mu_K(\delta'_j)|}{N(\delta_j)^{1/\log R} N(\delta'_j)^{1/\log R}} \ll \log^{O(1)} R,
\]
and, for \( t, t' \in [-\sqrt{\log R}, \sqrt{\log R}] \),

\[
F(t, t') = (1 + o(1))\left( \frac{W^{[K:Q]}}{\phi_K(W) \log R \cdot \text{Res}_{z=1} \zeta_K(z)} \right)^s \prod_{j=1}^s \frac{(1 + it_j)(1 + it'_j)}{(2 + it_j + it'_j)}.
\]

We prove the equality first. Applying the Chinese remainder theorem, one can show that

\[
\omega((d_j)_{1 \leq j \leq s}) = \prod_{\wp} \omega((d_j, \wp)_{1 \leq j \leq s}),
\]

where \( \wp \) runs over nonzero prime ideals of \( \mathcal{O}_K \). One can also show that

\[
\omega(((d_j, \wp))_{1 \leq j \leq s}) = \begin{cases} 1, & \prod_{j=1}^s (d_j, \wp) = (1), \\ 0, & \prod_{j=1}^s (d_j, \wp) \neq (1), \wp | W. \end{cases}
\]

And, if \( \wp \nmid W \) and \( W \) is sufficiently large, then one can show that

\[
\omega(((d_j, \wp))_{1 \leq j \leq s}) = \begin{cases} 1/N\wp, & \prod_{j=1}^s (d_j, \wp) = \wp \leq 1/N\wp^2, \\ \wp^2 & \prod_{j=1}^s (d_j, \wp). \end{cases}
\]

It follows that

\[
F(t, t') = \prod_{\wp} \sum_{d_j, d'_j | \wp, j=1,\ldots,s} \omega((d_j \cap d'_j)_{1 \leq j \leq s}) \prod_{j=1}^s \frac{\mu_K(d_j)\mu_K(d'_j)}{N_{d_j}^{1+it_j} N_{d'_j}^{1+it'_j}}
\]

\[
= \prod_{\wp | W} (1 + \sum_{j=1}^s -N\wp^{-1-\frac{1+it_j}{\log R}} - N\wp^{-1-\frac{1+it'_j}{\log R}} + N\wp^{-\frac{2+it_j+it'_j}{\log R}} + O_s(\frac{1}{N\wp^2}))
\]

\[
= \prod_{\wp | W} (1 + O_s(\frac{1}{p^e})) \prod_{j=1}^s \prod_{\wp | W} \left( 1 - N\wp^{-1-\frac{1+it_j}{\log R}} \right) \frac{\zeta_K(1 + \frac{2+it_j+it'_j}{\log R})}{\zeta_K(1 + \frac{1+it_j}{\log R})}\frac{\zeta_K(1 + \frac{1+it'_j}{\log R})}{\zeta_K(1 + \frac{1+it_j+it'_j}{\log R})} \prod_{\wp | W} \left( 1 - N\wp^{-1-\frac{2+it_j+it'_j}{\log R}} \right)
\]

From the estimate

\[
\zeta_K(z) = \frac{\text{Res}_{z=1} \zeta_K(z)}{z-1} + O(1), \quad z \to 1,
\]

and the estimate

\[
e^z = 1 + O(z), \quad z \to 0,
\]

we infer that

\[
F(t, t') = (1 + O(\frac{1}{\log R})) \cdot \prod_{\wp | W} (1 + O(\frac{\log N\wp}{N\wp \log^{1/2} R})).
\]

\[
(\frac{W^{[K:Q]}}{\phi_K(W) \log R \cdot \text{Res}_{z=1} \zeta_K(z)})^s \prod_{j=1}^s \frac{(1 + it_j)(1 + it'_j)}{(2 + it_j + it'_j)}.
\]

Applying the estimate

\[
\prod_{\wp | W} (1 + \frac{\log N\wp}{N\wp}) = O(e^{\log^2 w}),
\]
we arrive at
\[ F(t, t') = (1 + o(1)) \left( \frac{W^{[K:Q]}}{\phi_K(W) \log R \cdot \text{Res}_{z=1} \zeta_K(z)} \right)^s \prod_{j=1}^s \frac{(1 + it_j)(1 + it_j')}{(2 + it_j + it_j')} \]
as required.

We now turn to prove the estimate
\[
\sum_{d, d'} \omega((d_j \cap d'_j)_{1 \leq j \leq s}) \prod_{j=1}^s \frac{|\mu_K(d_j)\mu_K(d'_j)|}{N(d_j)^{1/\log R}N(d'_j)^{1/\log R}} \ll \log^{O(1)} R.
\]
We have
\[
\begin{align*}
&\sum_{d, d'} \omega((d_j \cap d'_j)_{1 \leq j \leq s}) \prod_{j=1}^s \frac{|\mu_K(d_j)\mu_K(d'_j)|}{N(d_j)^{1/\log R}N(d'_j)^{1/\log R}} \\
&= \prod_{W} \sum_{d_j, d'_j : y_j = 1, \ldots, s} \omega((d_j \cap d'_j)_{1 \leq j \leq s}) \prod_{j=1}^s \frac{1}{N(d_j)^{1/\log R}N(d'_j)^{1/\log R}} \\
&= \prod_{W} (1 + N^{-1} \log R)^{O(1)} = \zeta(1 + \frac{1}{\log R})^{O(1)} \ll \log^{O(1)} R.
\end{align*}
\]
This completes the proof of the theorem.

6. The auto-correlation of the truncated von Mangolt function

In this section we shall establish the auto-correlation of the truncated von Mangolt function.

The auto-correlation of the truncated von Mangolt function for the rational number field was studied by Goldston-Yıldırım in [GY1, GY2, GY3], and by Green-Tao in [GT1, GT2].

We now prove the following.

**Theorem 6.1.** The system \( \{\nu_N\} \) satisfies the \( k \)-auto-correlation condition.

The above theorem follows from the following lemma.

**Lemma 6.2.** Let \( I \) be any parallelootope in \( K_\infty \). Then
\[
\frac{1}{\|(NI) \cap b\|} \sum_{x \in (NI) \cap b} \prod_{i=1}^s \nu_N(x + y_i) \ll \prod_{1 \leq i < j \leq s} (1 + O_N(\frac{1}{N^\theta}))
\]
uniformly for all \( s \)-tuples \( y \in b^s \) with distinct coordinates.
Proof We define

$$\mathcal{G}_2 = \sum_{0, s} \omega_2((d_0 \cap d'_1 \cdots d'_s) \cap \lambda I) \prod_{i=1}^{s} \mu_{K}(d_i) \mu_{K}(d'_i) \phi(\frac{\log N \delta_i}{\log R}) \phi(\frac{\log N \delta'_i}{\log R}),$$

where $d$ and $d'$ run over $s$-tuples of ideals of $O_K$, and

$$\omega_2((d_i)_{1 \leq i \leq s}) = \frac{|\{x \in b/(\cap_{i=1}^{s} d_i) : d_i[(Wx + h_i)b^{-1}, \forall i = 1, \cdots, s]\}|}{(N \cap_{i=1}^{s} d_i)},$$

where $h_i = W b(y) + W y_i + \alpha$.

Let $\{\gamma_j\} (j = 1, \cdots, [K : \mathbb{Q}])$ be a $\mathbb{Z}$-basis of $b$ such that $\{\lambda_i \gamma_j\}$ is a $\mathbb{Z}$-basis of $b \cap \cap_{i=1}^{s} d_i$, where each $\lambda_i$ is a positive integer. Set

$$I_0 = \{x \in K : x_i = \sum_{j=1}^{[K : \mathbb{Q}]} (0, 1) \lambda_j \gamma_j\}.$$

Then

$$\omega_2((d_i)_{1 \leq i \leq s}) = \frac{|\{x \in I_0 \cap b : d_i[(Wx + h_i)b^{-1}, \forall i = 1, \cdots, s]\}|}{(N \cap_{i=1}^{s} d_i)}.$$

The number of translates of $I_0$ by vectors in $b \cap \cap_{i=1}^{s} d_i$ which intersect the boundary of $\lambda I$ is bounded by $O(\lambda [K : \mathbb{Q}]^{-1})$. So the number of translates of $I_0$ by vectors in $b \cap \cap_{i=1}^{s} d_i$ which lie in the interior of $\lambda I$ is

$$\frac{\text{Vol}(I)}{\text{Vol}(I_0)} \lambda^{[K : \mathbb{Q}]} + O(\lambda [K : \mathbb{Q}]^{-1}/N(\cap_{i=1}^{s} d_i)^{[K : \mathbb{Q}]^{-1}}).$$

It follows that

$$\frac{N \alpha \sqrt{d_K} |\{x \in \lambda I \cap b : d_i[(Wx + h_i)b^{-1}, \forall i = 1, \cdots, s]\}|}{\lambda^{[K : \mathbb{Q}]} \text{Vol}(I)} = \frac{\text{Vol}(I)}{\text{Vol}(I_0)} \lambda^{[K : \mathbb{Q}]} + O\left(\frac{N}{\lambda} \frac{\cap_{i=1}^{s} d_i}{\Delta}\right).$$

From that estimate one can infer

$$\left(\frac{W^{[K : \mathbb{Q}]} \alpha}{\phi_K(W) \log R}\right)^{s} \frac{1}{|\cap_{i=1}^{s} d_i|} \prod_{x \in (N \cap b)} \mu_N(x + y_i) = \mathcal{G}_2 + O\left(\frac{R^{is}}{\lambda}\right).$$

So we are reduced to proving that

$$\mathcal{G}_2 \ll \left(\frac{W^{[K : \mathbb{Q}]} \alpha}{\phi_K(W) \log R}\right) \prod_{\nu|\Delta} (1 + O\left(\frac{1}{N^\nu}\right)), $$

whenever

$$\Delta := \prod_{i \neq j} (y_i - y_j) \neq 0.$$

We define

$$F_2(t, t') = \sum_{d, d'} \omega_2((d_0 \cap d'_1 \cdots d'_s) \cap \lambda I) \prod_{j=1}^{s} \frac{\mu_{K}(d_j) \mu_{K}(d'_j)}{N(d_j)^{1 + \frac{1}{\log R}} N(d'_j)^{1 + \frac{1}{\log R}}}, \; t, t' \in \mathbb{R}^s,$$

where $d$ and $d'$ run over $s$-tuples of ideals of $O_K$.
For all $B > 0$, we have

$$
\mathfrak{S}_2 = \int_{-\sqrt{\log R}, \sqrt{\log R}} \cdots \int_{-\sqrt{\log R}, \sqrt{\log R}} F_2(t, t') \psi(t) \psi(t') dt dt'
$$

$$
+ O_B((\log R)^{-B}) \cdot \sum_{d, d'} \omega_2((d_i \cap d'_j)_{1 \leq i \leq s}) \prod_{j=1}^s \frac{\mu_K(d_j) \mu_K(d'_j)}{N(d_j)^{1/\log R} N(d'_j)^{1/\log R}}
$$

Hence we are reduced to prove the following.

$$
\sum_{d, d'} \omega_2((d_i \cap d'_j)_{1 \leq i \leq s}) \prod_{j=1}^s \frac{\mu_K(d_j) \mu_K(d'_j)}{N(d_j)^{1/\log R} N(d'_j)^{1/\log R}} \ll \log^{O_s(1)} R,
$$

and, for $t, t' \in [-\sqrt{\log R}, \sqrt{\log R}]^s$,

$$
F_2(t, t') \ll \left( \frac{W^{[K:Q]}}{\phi_K(W) \log R} \right)^s \prod_{\nu \mid \Delta, \nu \mid W} \left( 1 + O_s \left( \frac{1}{N^{W}} \right) \right) \prod_{j=1}^s \frac{1 + |t_j| (1 + |t'_j|)}{2 + |t_j| + |t'_j|}.
$$

We prove the second inequality but omit the proof of first one. Applying the Chinese remainder theorem, one can show that

$$
\omega_2((d_i)_{1 \leq i \leq s}) = \prod_{\nu} \omega_2((d_i, \nu)_{1 \leq i \leq s}).
$$

One can also show that

$$
\omega_2((d_i, \nu)_{1 \leq i \leq s}) = \begin{cases} 1, & \prod_{i=1}^s (d_i, \nu) = (1), \\
0, & \prod_{i=1}^s (d_i, \nu) \neq (1), \nu \mid W.
\end{cases}
$$

And, if $\nu \nmid W$ and $w$ is sufficiently large, then one can show that

$$
\omega_2((d_i, \nu)_{1 \leq i \leq s}) \begin{cases} = 1/N^{w}, & \prod_{i=1}^s (d_i, \nu) = \nu \\
= 0, & \nu^2 \mid \prod_{i=1}^s (d_i, \nu), \nu \nmid \Delta, \\
\leq 1/N^{w}, & \nu^2 \mid \prod_{i=1}^s (d_i, \nu), \nu \mid \Delta.
\end{cases}
$$

It follows that

$$
F_2(t, t') = \prod_{\nu} \sum_{d_i, \nu \mid W, \nu \mid = 1, \cdots, s} \omega_2((d_i \cap d'_j)_{1 \leq i \leq s}) \prod_{j=1}^s \frac{\mu_K(d_j) \mu_K(d'_j)}{N(d_j)^{1/\log R} N(d'_j)^{1/\log R}}
$$

$$
= \prod_{\nu \mid \Delta, \nu \mid W} \left( 1 + \sum_{j=1}^s -N^{\nu^{-1} - \frac{1+i t_j}{e\log R}} - N^{\nu^{-1} - \frac{1+i t'_j}{e\log R}} + N^{\nu^{-1} - \frac{2+i t_j + i t'_j}{e\log R}} \right) \prod_{\nu \mid W \setminus \nu \mid} \left( 1 + O_s \left( \frac{1}{N^{W}} \right) \right)
$$

$$
\ll \left( \frac{W^{[K:Q]}}{\phi_K(W) \log R} \right)^s \prod_{\nu \mid \Delta, \nu \mid W} \left( 1 + O_s \left( \frac{1}{N^{W}} \right) \right) \prod_{j=1}^s \frac{\zeta_K(1 + 1 + \frac{1 + i t_j + i t'_j}{e\log R})}{\zeta_K(1 + 1 + \frac{1 + i t_j}{e\log R})} \zeta_K(1 + 1 + \frac{1 + i t'_j}{e\log R})
$$

This completes the proof of the lemma.
7. Proof of the main theorem

In this section we prove Theorem 1.5.

For each $N \in I$, and for each $\alpha \in b$ with $(\alpha, Wb) = b$, set

$$A_{N,\alpha} = \{ x \in b \cap B_N \mid (Wx + \alpha)b \text{ is prime } \}.$$  

By Theorem 4.5, Theorem 1.5 follows from the following theorem.

**Theorem 7.1.** For each $N \in I$, there is a number $\alpha_N \in (WG) \cap b$ with $(\alpha_N, Wb) = b$ such that the system $|\{ A_{N,\alpha} \cap B_{\varepsilon N} \}$ has positive upper density relative to $\{ \nu_N \}$.

**Proof** Let $S_{K,\infty}$ the set of infinite places of $K$. One can prove that there is a positive constant $c_K$ such that every principal fractional ideal of $K$ has a generator $\xi$ satisfying

$$|\sigma(\xi)| \leq c_K(N(\xi))^{1/[K:Q]} \quad \forall \sigma \in S_{K,\infty}.$$  

It follows that, for each $N \in I$, and for any prime ideal $\wp \in [b^{-1}]$ satisfying $(\wp, W) = 1$ and $N\wp \leq c_K^{-1}N^{-1} \cdot (NW\varepsilon/2)^{[K:Q]}$, there is a number $\alpha \in b \cap (WG)$ with $(\alpha, Wb) = b$, and a number $x \in A_{N,\alpha} \cap B_{\varepsilon N}$ such that $\wp = (Wx + \alpha)b^{-1}$. So

$$\sum_{\substack{(\alpha, Wb) = b \\ \alpha \in b \cap (WG)}} \sum_{x \in A_{N,\alpha} \cap B_{\varepsilon N}} \Lambda^2_{K,R}((Wx + \alpha)b^{-1}) \geq \sum_{\wp \in [b^{-1}], (\wp, W) = 1 \atop c/2 < N\wp < (NW)^{[K:Q]} / c} \Lambda^2_{K,R}(\wp) \gg (NW)^{[K:Q]} / \log N,$$

where $c = c_K^{-1}N^{-1} \cdot (\varepsilon/2)^{[K:Q]}$. The theorem now follows by the pigeonhole principle.  

**References**

[GY1] D. Goldston and C.Y. Yildirim, *Higher correlations of divisor sums related to primes, I: Triple correlations*, Integers 3 (2003) A5, 66pp.
[GY2] D. Goldston and C.Y. Yildirim, *Higher correlations of divisor sums related to primes, III: k-correlations*, preprint (available at AIM preprints)
[GY3] D. Goldston and C.Y. Yildirim, *Small gaps between primes, I*, preprint.
[GT1] B. Green, T. Tao, *The primes contain arbitrarily long arithmetic progressions*, Ann. Math. 167 (2008), 481-547.
[GT2] B. Green, T. Tao, *Linear equations in primes*, Ann. Math. 171 (2010), 1753-1850.
[HB] D. R. Heath-Brown, *The ternary Goldbach problem*, Rev. Mat. Iberoamericana, 1 (1985), 45-59.
[Tao] T. Tao, *The Gaussian primes contain arbitrary shaped constellations*, J. d. Analyse Mathematique 99 (2006), 109-176.

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