BIG QUANTUM COHOMOLOGY OF ORBIFOLD SPHERES

LINO AMORIM, CHEOL-HYUN CHO, HANSOL HONG, AND SIU-CHEONG LAU

ABSTRACT. We construct a Kodaira-Spencer map from the big quantum cohomology of a sphere with three orbifold points to the Jacobian ring of the mirror Landau-Ginzburg potential function. This is constructed via the Lagrangian Floer theory of the Seidel Lagrangian and we show that Kodaira-Spencer map is a ring isomorphism.

CONTENTS

1. Introduction 2
Acknowledgments 5
2. Bulk deformed Floer theory of Seidel Lagrangian in $\mathbb{P}^1_{a,b,c}$ 6
   2.1. $\mathbb{P}^1_{a,b,c}$ and its orbifold quantum cohomology 6
   2.2. Immersed Lagrangian Floer theory 7
   2.3. Orbi-discs and Lagrangian Floer theory for orbifolds 8
   2.4. Bulk deformed Fukaya algebra 10
3. Weakly unobstructedness for bulk-deformed Fukaya algebra 11
4. Bulk-deformed potential function and change of variables 14
   4.1. Gauss–Bonnet theorem and convergence 14
   4.2. Fukaya algebra of $L$ 22
5. Dependence of the potential on chain level representatives of bulk 24
6. The Kodaira–Spencer map 27
   6.1. Definition of $KS_\tau$ and well-definedness 27
   6.2. Ring homomorphism 29
7. $KS_\tau$ is an isomorphism 33
   7.1. Surjectivity 33
   7.2. Jacobian ring of the leading order potential 35
   7.3. Deforming Jac($W_\tau$) 36
7.4. Injectivity 38
8. Calculations 38
   8.1. Euler vector field 38
   8.2. Versality of the potential 39

arXiv:2002.11180v1 [math.SG] 25 Feb 2020
1. Introduction

Orbifold projective lines $\mathbb{P}^1_{a,b,c}$ are two-dimensional spheres with three orbifold singular points as drawn in Figure 1. They provide a simple yet very interesting class of geometries. Despite low dimensionality, their orbifold Gromov-Witten theory is surprisingly rich. Satake-Takahashi [ST11] computed the Gromov-Witten invariants and Frobenius structures for elliptic $\mathbb{P}^1_{a,b,c}$ (where $1/a+1/b+1/c = 1$), which involves many interesting number theoretic power series. Rossi [Ros10] obtained analogous results for spherical $\mathbb{P}^1_{a,b,c}$ (where $1/a+1/b+1/c > 1$). More recently, Ishibashi-Takahashi-Shiraishi [IST19] proved that the Frobenius structure from the Gromov-Witten invariants of hyperbolic $\mathbb{P}^1_{a,b,c}$ (where $1/a+1/b+1/c < 1$) is isomorphic to the one from their associated affine cusp polynomials. In this paper, we provide a geometric approach to study closed-string mirror symmetry for $X = \mathbb{P}^1_{a,b,c}$ in all three cases, with help of Lagrangian Floer theory. Namely, we will construct a Kodaira-Spencer map from orbifold quantum cohomology of $X$ with bulk deformations to the Jacobian ring of the mirror potential function and show that it is an isomorphism.

Lagrangian Floer theory has provided a purely mathematical approach to construct and prove mirror symmetry. A typical example is a compact toric manifold, whose mirror can be nicely constructed from Lagrangian Floer theory. In the Fano case, the second-named author and Yong-Geun Oh [CO06] classified the holomorphic discs bounded by toric fibers and showed that the LG mirror $W$ can be formulated as the count of these discs. Later Fukaya-Oh-Ohta-Ono [FOOO10, FOOO11, FOOO16b] used Lagrangian deformation theory to construct the LG mirrors in general. They also constructed the Kodaira-Spencer map (or closed-open map) which produces close-string mirror symmetry for all compact toric manifolds. This provides a mirror construction from the first principle, which has the advantage that Kontsevich’s homological mirror symmetry conjecture [Kon95] can be canonically derived (See [CHL19] for Fano cases).

For $X = \mathbb{P}^1_{a,b,c}$, the Landau-Ginzburg (LG for short) mirrors $W$ were uniformly constructed in [CHL12] based on Lagrangian Floer theory of a certain immersed Lagrangian $L$, which was first used by Seidel [Sei15]. Moreover, homological mirror symmetry for the elliptic and hyperbolic cases was derived by a family version of a Yoneda functor naturally coming with the construction. In the hyperbolic case, the LG mirror is an infinite series in variables $x, y, z$. [CHKL17] found an inductive algorithm to compute the explicit expressions in all cases. In this article, we consider a bulk-deformed version of such LG mirrors. The deformed potentials have the same leading order terms as the ones in [CHKL17].
In this approach to constructing the mirror, it is crucial to find a large space of solutions to the weak Maurer-Cartan equation. For the immersed Lagrangian $L$ we show (see Proposition 3.1) that any linear combination of the odd-degree immersed points gives a solution of the Maurer-Cartan equation. This extends the result in [CHL12] to the case of bulk deformations by orbi-sectors. The key ingredient is an anti-symplectic involution on $P^{1}_{a,b,c}$, which makes holomorphic polygons appearing in pairs and their contributions to the even-degree immersed sectors cancel.

In order to relate the Gromov-Witten invariants of $P^{1}_{a,b,c}$ with the Jacobian ring of the bulk-deformed mirror potential, we use the method of Kodaira-Spencer map invented by [FOOO16b], which gives a homomorphism from the quantum cohomology of $X$ to the Jacobian ring of the mirror $W_\tau$. The following is the main theorem.

**Theorem 1.1.** Let $X = P^{1}_{a,b,c}$ and $W_\tau$ be its bulk-deformed disc potential by $\tau \in H^*(X, \Lambda_+)$. Let $\text{Jac}(W_\tau)$ be the completed Jacobian ring over the Novikov field $\Lambda$ in a certain choice of coordinates. Denote the big quantum cohomology of $X$ over $\Lambda$ with quantum product $\cdot_\tau$ by $\text{QH}^*_{\text{orb}}(X, \tau)$. The Kodaira-Spencer map $KS_\tau : \text{QH}^*_{\text{orb}}(X, \tau) \to \text{Jac}(W_\tau)$ is a ring isomorphism.

We also show that the map $KS_\tau$ identifies the Euler vector field on the big quantum cohomology (see Theorem 8.1 for details) with the Euler vector field on $\text{Jac}(W_\tau)$, which is the class $[W_\tau]$.

The construction of Kodaira-Spencer map [FOOO16b] crucially depends on the existence of $T^m$-action, hence the definition is still missing in general cases. The above theorem provides the first class of examples of Kodaira-Spencer map beyond toric manifolds.

In fact, there is a crucial difference between our case of $P^{1}_{a,b,c}$ and that of toric manifolds. Namely, we need to enlarge the domain of LG potential to make the above theorem hold true. Maurer-Cartan formalism of Lagrangian Floer theory provides a natural set of coordinates $\tilde{x}, \tilde{y}, \tilde{z} \in \Lambda_0$. Namely, they are the coordinates of the Maurer-Cartan space which are dual to the immersed sectors of $L$. Given the bulk deformed mirror potential $W_\tau(\tilde{x}, \tilde{y}, \tilde{z})$, one can define the Jacobian ring as in Definition 6.1 as the completed power series ring $\Lambda \ll \tilde{x}, \tilde{y}, \tilde{z} \gg$ modulo Jacobian ideal of $W_\tau(\tilde{x}, \tilde{y}, \tilde{z})$. With this Jacobian ring, $KS_\tau$ is not an isomorphism in general hence the above theorem fails. In Section 8.3, we give an explicit counter-example.

In this paper, we will make the change of variables

\[
\begin{align*}
x &= T^3 \tilde{x}, \\
y &= T^3 \tilde{y}, \\
z &= T^3 \tilde{z}.
\end{align*}
\]

and consider $x, y, z \in \Lambda_0$. In terms of old variables, this is equivalent to allowing

\[\text{val}(\tilde{x}), \text{val}(\tilde{y}), \text{val}(\tilde{z}) \geq -3.\]

In terms of non-archimedean norm $e^{-\text{val}}$, $\tilde{x}, \tilde{y}, \tilde{z}$ are functions on a disc $D(1)$ of radius $1 = e^0$, and $x, y, z$ are functions on a disc $D(e^3)$ which contains $D(1)$. In the above counter example, critical points of the potential $W_\tau(\tilde{x}, \tilde{y}, \tilde{z})$ lie on $D(e^3) \setminus D(1)$ as shown in Proposition 8.6. Thus we need the bigger disc $D(e^3)$ to match the number of critical points with the rank of the quantum cohomology ring. See 8.3 for related discussions.
However, this necessary enlargement of domain is the main source of complication almost in every steps of the proof of the main theorem. Namely, Lagrangian Floer theory for bounding cochains of negative valuation does not work in general and we need to take care of convergence issues in each step of the proof.

We give another perspective of the above coordinate change. For readers convenience, we first recall the case of toric manifolds briefly. For a compact toric n-fold, which can be understood as a compactification of $\mathbb{C}^n$, $W$ takes the form

$$z_1 + \ldots + z_n + \sum_{i=1}^{\infty} T^{A_i} Z_i + h.o.t.$$ 

where $A_i > 0$ and $Z_i$ are monomials in $z_1, \ldots, z_n$, and $h.o.t.$ consists of higher-order terms in $T$. Under the Kodaira-Spencer map, the images of the toric divisors $D_1, \ldots, D_n$, which are compactifications of the coordinate hyperplanes of $\mathbb{C}^n$, are sent to $z_1, \ldots, z_n$, which generate (a suitable completion of) $\Lambda[z_1, \ldots, z_n]$ and hence the Jacobian ring. Thus surjectivity of the Kodaira-Spencer map is automatic in this case.

On the other hand for $\mathbb{P}_a^1$, the potential $W$ (with $\tau = 0$) takes the form

$$W(\tilde{x}, \tilde{y}, \tilde{z}) = -T\tilde{x}\tilde{y}\tilde{z} + T^{3a}\tilde{x}^a + T^{3b}\tilde{y}^b + T^{3c}\tilde{z}^c + h.o.t.$$ 

whereas, in new coordinates, the leading terms of the above become

$$W_{\text{lead}} := -T^{-8}xyz + x^a + y^b + z^c.$$ 

The images of the orbifold points $[1/a], [1/b], [1/c]$ are $T^3\tilde{x}, T^3\tilde{y}, T^3\tilde{z}$ respectively, but in new coordinates these orbifold points map to $x, y, z$ which generates $\Lambda(x, y, z)$. This is one of key ingredient in proving surjectivity of the KS map. Therefore the coordinate change is also quite natural in this perspective as well.

Once we establish surjectivity of $\text{KS}_\tau$, we match the dimension of the Jacobian ring of the bulk-deformed potential with that of $\text{QH}^*_\text{orb}(X, \tau)$ to show that $\text{KS}_\tau$ is injective, where the former is given as $a + b + c - 1$. For this, we argue with the deformation invariance of the dimension, as it is relatively easy to analyze the leading order terms. In fact, the rank of the Jacobian ring for $-T^{-8}xyz + x^a + y^b + z^c$ is already quite nontrivial, as one needs to additionally take into account the convergence issue when working over $\Lambda$. For this reason, the computation for leading order terms is somewhat lengthy which we will see in Appendix B. Then we prove that the leading order terms and the actual potential can be interpolated by a flat deformation. This involves a delicate induction step together with some nontrivial algebraic facts.

While the necessity of the coordinate change is now clear, it results in the analytic difficulty that we need to insure convergence throughout the construction under this coordinate change, which a priori is not at all obvious. Even though the construction in Floer theory has automatic $T$-adic convergence for bounding cochains in $\Lambda_+$, this coordinate change has an effect that our bounding cochains lie in $\Lambda_{-3}$. Hence we need a better control in areas to have convergence. First we will show that in the coordinates $x, y, z$, every term of $W_{\tau}$ has non-negative valuation (Lemma 4.4). Then we use an orbifold version of Gauss-Bonnet theorem (Theorem 4.5) to show that $W_{\tau}$ actually converges in $T$-adic topology.
Theorem 1.2 (Theorem 4.8). The bulk-deformed potential $W_\tau$ is a convergent series in new variables $x, y, z$ as in (1.1), that is, it is an element of $\Lambda\langle\langle x, y, z \rangle\rangle$.

In the orbifold setting, the twisted sectors have fractional degrees. For $X = \mathbb{P}^1_{a,b,c}, H^2_{\text{orb}}(X)$ is spanned by the fundamental class $1_X$ and the twisted sectors $[i/a], [j/b], [k/c]$ for $0 < i < a, 0 < j < b, 0 < k < c$. The compatibility of $\text{KS}_\tau : \text{QH}^*_\text{orb}(X, \tau) \to \text{Jac}(W_\tau)$ with ring structures follows from the standard cobordism argument as in [FOOO16b], but it still requires a careful analysis on the associated virtual perturbation scheme in our context. The details will be provided in 6.2.

The main theorem is particularly interesting in the hyperbolic case, which belongs to the class of general-type manifolds whose mirror symmetry is mostly conjectural. Theorem 1.1 together with the result in [CHKL17] provides the first class of manifolds in general-type whose small quantum cohomology has a presentation which can be explicitly computed. Even in the toric case, $W$ is a highly non-trivial series due to obstructed non-constant sphere bubbling with negative Chern number. There is no general algorithm to compute $W$ for toric manifolds of general type. On the other hand, for hyperbolic $\mathbb{P}^1_{a,b,c}$ with no bulk deformation (that is $\tau = 0$), there is an algorithm to compute the series $W_\tau$ by [CHKL17], which in turn gives an explicit presentation of the small quantum cohomology $\text{QH}^*_\text{orb}(X, 0)$. (Note that there is no non-constant smooth sphere in hyperbolic $\mathbb{P}^1_{a,b,c}$ and so there is no obstruction in the disc moduli for computing $W$.)

Finally in the last section, we exhibit several interesting properties of the bulk-deformed potential as well as a few explicit calculations for $\text{KS}_\tau$. Most importantly, we show that the bulk-deformation of the Floer theory of $L$ produces a versal deformation of the mirror potential. More specifically,

Theorem 1.3 (Theorem 8.2). Consider $P \in \Lambda\langle\langle x, y, z \rangle\rangle$ with $\text{val}(P - W_{\text{lead}}) > 0$ where $W_{\text{lead}} = -T^{-8}xyz + x^a + y^b + z^c$. Then there exist $\tau' \in H^0_{\text{orb}}(\mathbb{P}^1_{a,b,c}, \Lambda_0)$ and a coordinate change $(x', y', z')$ such that $P(x', y', z') = W_{\tau'}$.

Note that this is analogous to the versality statement in toric case proven in [FOOO16b, Theorem 2.8.1]. The proof is based on the induction argument on energy, which is similar to the one used to establish surjectivity of $\text{KS}_\tau$.

The organization of the paper is as follows. In Section 2, we review Floer theory of the Lagrangian $L$ in $\mathbb{P}^1_{a,b,c}$ and its bulk-deformation including orbi-sectors. In Section 3, we prove the weakly unobstructedness of $L$ after the bulk-deformation, and in Section 4, we study the resulting bulk-deformed potential and its convergence after coordinate change. In Section 5, we prove that the bulk-deformed potential changes by an explicit coordinate change for different choices of cohomology representatives, and hence its well-definedness follows. Throughout Section 6 and 7, we show that $\text{KS}_\tau$ is a ring homomorphism that is surjective and injective, which proves our main theorem. Finally, we provide some concrete calculations of $\text{KS}_\tau$, and prove the versality theorem in Section 8.

Acknowledgments. The authors express their gratitude to Kenji Fukaya and Yong-Geun Oh for useful discussions on virtual perturbation schemes. C.-H. Cho was supported by the NRF grant funded by the Korea government(MSIT) (No. 2017R1A2B4009488). H. Hong
is supported by the Yonsei University Research Fund of 2019 (2019-22-0008). S.-C. Lau is supported by Simons Collaboration Grant.

2. Bulk deformed Floer theory of Seidel Lagrangian in $\mathbb{P}^1_{a,b,c}$

In this section, we recall orbifold quantum cohomology and immersed Lagrangian Floer theory mainly to set the notations. In short, we will consider orbifold quantum cohomology by Chen-Ruan [CR02] and a de Rham version of immersed Lagrangian Floer theory (defined by Akaho-Joyce [AJ10] and Fukaya [Fuk17]). One can enhance the latter by including orbi-discs following the work of the second author and Poddar [CP14]. This gives bulk deformations by twisted sectors.

2.1. $\mathbb{P}^1_{a,b,c}$ and its orbifold quantum cohomology. Let $\mathbb{P}^1_{a,b,c}$ be an orbifold sphere with three orbifold points with isotropy groups $\mathbb{Z}/a$, $\mathbb{Z}/b$, $\mathbb{Z}/c$, where $a, b, c \geq 2$. We take the Kähler form $\omega$ descended from the universal cover of $\mathbb{P}^1_{a,b,c}$ with constant curvature. For later convenience we scale it such that the total area of $\mathbb{P}^1_{a,b,c}$ is 8. The orbifold Euler characteristic is given by

$$\chi(\mathbb{P}^1_{a,b,c}) = \frac{1}{a} + \frac{1}{b} + \frac{1}{c} - 1.$$ 

Depending on $\chi$ being positive, zero or negative, the universal cover of $\mathbb{P}^1_{a,b,c}$ is $S^2$, $\mathbb{R}^2$ or $\mathbb{H}^2$. We refer to these as the spherical, elliptic and hyperbolic respectively. In all cases, $\mathbb{P}^1_{a,b,c}$ can be constructed as a global quotient of a Riemann surface $\Sigma$ by a finite group. In the spherical case $\Sigma$ is a sphere, in the elliptic case $\Sigma$ is an elliptic curve and in the hyperbolic case $\Sigma$ is a surface of genus $\geq 2$.

Recall that the Chen-Ruan orbifold cohomology of an orbifold $X$, as a vector space, is given by the singular cohomology group of the inertia orbifold. In particular, its degree $d$ part (where $d \in \mathbb{Q}$) is given by

$$H^d_{orb}(X) = \bigoplus_g H^{d-2\iota(g)}_*(X(g))$$

where the sum is over all twisted sectors $g$. The degree-shifting $\iota(g) \in \mathbb{Q}$ is called the age of the twisted sector in literature.

For $H^*_\text{orb}(\mathbb{P}^1_{a,b,c})$, we have cohomology classes $1_X, [pt] \in H^2(\mathbb{P}^1_{a,b,c}, \mathbb{R})$, as well as the twisted sectors

$$(2.1) \quad \frac{1}{a}, \ldots, \frac{a-1}{a}, \frac{1}{b}, \ldots, \frac{b-1}{b}, \frac{1}{c}, \ldots, \frac{c-1}{c}$$

where $\left\lfloor \frac{k}{a} \right\rfloor$ has degree $\frac{2k}{a}$. Let us denote by $H^w(X)$ the span of the twisted sectors.

By local computations, the classical part of Chen-Ruan product of $\left\lfloor \frac{j}{a} \right\rfloor$ and $\left\lfloor \frac{k}{a} \right\rfloor$ is $\left\lfloor \frac{j+k}{a} \right\rfloor$ if $j+k < a$, and is $\frac{1}{a}[pt]$ if $j+k = a$ and zero otherwise. These are the products from constant orbi-spheres.
There are non-trivial contributions from non-constant orbi-spheres as well for the quantum cohomology $\text{QH}^{*}_{\text{orb}}(X, \tau)$. They can be written as follows via the orbifold Poincaré pairing:

$$\langle 1_X, [\text{pt}] \rangle_{PD_X} = 1, \quad \langle \left\lfloor \frac{j}{a} \right\rfloor, \left\lfloor \frac{a - j}{a} \right\rfloor \rangle_{PD_X} = \frac{1}{a}.$$  

Fix $\tau \in H^{*}_{\text{orb}}(X, \Lambda_+)$ and for each $A, B \in H(X, \Lambda_0)$ the bulk deformed quantum product $A \bullet_{\tau} B$ is defined by

$$\langle A \bullet_{\tau} B, C \rangle_{PD_X} = \sum_{l=0}^{\infty} \frac{1}{l!} GW_{l+3}(A, B, C, \tau, \ldots, \tau).$$

where $GW_{l+3}$ is the orbifold Gromov-Witten invariant with $l + 3$ inputs ([CR02]). The above sum converges over $\Lambda$ by our choice of $\tau$.

2.2. **Immersed Lagrangian Floer theory.** Immersed Lagrangian Floer theory was introduced by Akaho-Joyce [AJ10] using singular chains, extending the embedded case of Fukaya, Oh, Ohta Ono [FOOO09]. A different version using Morse function (and pearl complex) was given by Seidel [Sei11] and Sheridan [She15, She11]. In our previous work [CHL12], we used the definition by Seidel to prove homological mirror symmetry. In this paper, we work with de Rham version of immersed Lagrangian Floer theory (by Fukaya [Fuk17]) since we use Kuranishi structures to deal with orbifold quantum cohomology. We refer the readers to the above references for general definitions.

Seidel [Sei11] constructed an immersed circle $\mathbb{S}^1 \hookrightarrow \mathbb{P}^{1}_{a,b,c}$ with three transversal (double) self-intersections (see Figure 1), and we refer it as the Seidel Lagrangian. We assume that the image of $\mathbb{L}$ is invariant under reflection with respect to the equator (which passes through the three orbifold points), which is crucial for weakly unobstructedness in the next section. The image of $\mathbb{L}$ and the equator divide the sphere into eight regions: two triangles and six bigons. We take $\mathbb{L}$ such that each of these regions have area 1. The Lagrangian $\mathbb{L}$ is equipped with a non-trivial spin structure (this is needed for weakly unobstructedness in Section 3). This is given by fixing a point in $\mathbb{L}$ (which is not the immersed point) and any holomorphic disc contribution through this point gets a (-1) sign for each $A_{\infty}$-operation.
One associates to the Seidel Lagrangian its Fukaya algebra $\mathcal{F}(L)$, which is a filtered $A_\infty$-algebra with the underlying $\mathbb{Z}_2$-graded vector space

$$\mathcal{F}^*(L) := \left( \Omega^*(\mathbb{S}^1) \oplus \bigoplus_{X,Y,Z} \Lambda_{\mathbb{C}}^{\oplus 2} \right) \hat{\otimes} \Lambda_0.$$ 

Here $\Omega^*(\mathbb{S}^1)$ is the classical de Rham cochain algebra of $\mathbb{S}^1$ (the domain of $L$) with coefficients in $\mathbb{C}$. Each of the intersection points gives rise to two generators in $\mathcal{F}(L)$, one even and one odd. We denote by $X$, $Y$ and $Z$ the odd ones and by $\bar{X}$, $\bar{Y}$ and $\bar{Z}$ the even generators.

The $A_\infty$-operations are defined using the moduli space of pseudo-holomorphic polygons as in Fukaya [Fuk17], to which we refer readers for details. For the case where inputs and the output are immersed generators ($X,Y,Z$, $\bar{X},\bar{Y},\bar{Z}$), the corresponding $A_\infty$-operation is given by the signed count of rigid immersed polygons in $\mathbb{P}^1_{a,b,c}$ with prescribed (convex) corners. In this case, by automatic regularity of (holomorphic) polygons in Riemann surface (see [Sei08, Part II, Section 13]), they are already transversal, and hence it is legitimate to use them for counting. Also we remark that the interior of a polygon may cover orbifold points of $\mathbb{P}^1_{a,b,c}$. There exists a manifold cover of $\mathbb{P}^1_{a,b,c}$, where the lifts of $L$ are embedded Lagrangians, and one may count polygons in the cover. Orbifold insertions will be considered in bulk deformations, and we explain them in the next subsection.

When some of the inputs or the output are differential forms of $\mathbb{S}^1$ (the domain of $L$), we follow Fukaya to define $A_\infty$-operations using pull-back and push-forward of differential forms over the moduli spaces. In general one needs the technique of a continuous family of multi-sections to define push-forwards.

2.3. Orbi-discs and Lagrangian Floer theory for orbifolds. We recall how to incorporate orbi-discs into the story. In our case, the Seidel Lagrangian stays away from the orbifold points of $\mathbb{P}^1_{a,b,c}$, which can be handled as in the case of toric orbifolds [CP14].

Let us first recall the definition of an orbi-disc, adapted for an immersed Lagrangian boundary condition. Let $T$ be the index set of inertia components of $X$, where $0 \in T$ corresponds to the underlying topological space of $X$. Let $R$ be the index set of the immersed sectors of $L$, where $+ \in R$ corresponds to the underlying immersed Lagrangian.

**Definition 2.1.** Let $\beta \in H_2(X,\mathbb{L})$ be a disc class, $\gamma : \{0,\ldots,k\} \to R$ a specification of immersed sectors of $\mathbb{L}$ and $\nu : \{1,\ldots,l\} \to T$ a specification of twisted sectors of $X$. The moduli space $\mathcal{M}^{\text{main}}_{k+1,l}(\beta;\nu;\gamma)$ consists of elements of the form $(\Sigma,\bar{z}^+,\bar{m},\bar{z},u)$ such that

- $(\Sigma,\bar{z}^+,\bar{m})$ is a (prestable) bordered orbifold Riemann surface with genus zero, where $\bar{z}^+ = (z_1^+,\ldots,z_l^+) \in (\Sigma - \partial\Sigma)^l$ is a sequence of interior orbifold marked points which are not (orbi-)nodes, and $\bar{m} = (m_1,\ldots,m_l) \in \mathbb{N}^l$ specifies the multiplicities of the uniformizing chart at these orbifold points.
- $u : (\Sigma,\partial\Sigma) \to (X,\mathbb{L})$ is a holomorphic map on each component, that is, $u$ is a continuous map which is holomorphic in the interior of $\Sigma$ away from the orbifold points, and around each orbifold point $z_i^+$, $u$ can be locally lifted to be a holomorphic map from the uniformizing chart at $z_i^+$ to a uniformizing chart of $X$ at $f(z_i^+)$. Moreover, $z_i^+$ is mapped to the twisted sector $X_{\nu(i)}$ for $i = 1,\ldots,l$.
- $u$ is good and representable as an orbifold morphism.
• \( \mathcal{Z} = (z_0, \ldots, z_k) \in (\partial \Sigma)^{k+1} \) is a sequence of boundary marked points obeying the cyclic ordering of \( \partial \Sigma \). Moreover, \( z_i \) is mapped to the immersed sector labeled by \( \gamma(i) \) for \( i = 0, \ldots, k \).

In the case of toric orbifolds, such an orbi-disc with boundary on a Lagrangian torus fiber was studied and classified. The orbi-disc potential for a toric Calabi-Yau orbifold or a compact semi-Fano toric orbifold was computed using the mirror map in [CCLT16, CCLT14].

To state the dimension formula for the moduli spaces, we use two related notions of the Maslov index. The first one is the desingularized Maslov index \( \mu^{de} \) (following [CR02]). Given an orbi-disc with a Lagrangian boundary condition, the pull-back bundle data is an orbi-bundle together with Lagrangian boundary data. This bundle cannot be trivialized due to the non-trivial orbifold structure. On the other hand there is an associated smooth bundle, called the desingularized bundle, which has the same set of local holomorphic sections. The latter property enables us to compute the virtual dimension. The other one is the Chern-Weil Maslov index \( \mu_{CW} \). It was shown in [CS16] that

\[
\mu^{de} = \mu_{CW} - 2 \sum_i \iota(\nu(i))
\]

where \( \iota(\nu(i)) \) is the degree shifting number associated to the twisted sector labeled by \( \nu(i) \).

Let us also explain how to handle \( J \)-holomorphic polygons with transversally intersecting Lagrangian boundary conditions. Given two Lagrangian subspaces \( L_i, L_{i+1} = JL_i \), there exist a positive path of Lagrangian subspaces from \( L_i \) to \( L_{i+1} \) given by \( e^{\pi J t/2} L_0 \) for \( t \in [0, 1] \). Given a holomorphic polygon, we get a loop of Lagrangians along the boundary by adding these positive paths at each corner. The resulting Maslov index of the Lagrangian loop is called the topological Maslov index. If there is in addition an orbifold point in the interior, we can first desingularize it as above, and add positive paths to define the topological Maslov index, which is also denoted by \( \mu^{de} \).

**Remark 2.2.** For the definition of the Chern-Weil index, we choose a unitary connection, which asymptotically sends \( L_i \) to \( JL_i \) at the puncture along the positive path. Then the relation (2.2) also holds for polygons. We remark that there is an error in [CS16] Proposition 5.6. Namely, the formula (23) in [CS16] holds true for connections which are trivial near the puncture, but it does not hold for general connections. Rather we have (2.2) with asymptotic conditions given by positive paths.

It is well-known that the Fredholm index of the \( \overline{\partial} \) operator on discs with smooth Lagrangian boundary condition equals \( n + \mu^{de} \). For transversely intersecting Lagrangians, we can glue orientation operators of positive paths (this has index \( n \)) at the punctures to obtain a formula

\[
\text{Ind}(\overline{\partial}) + (k + 1)n = n + \mu^{de}
\]

(If we had used negative paths instead of positive paths to define the topological Maslov index, the term \( (k + 1)n \) will disappear. In this sense, it would be more convenient to use negative paths. We follow the usual convention to use positive paths.)

Hence the dimension of the moduli space of \( J \)-holomorphic polygons are given by (adding the effects of \( l \) interior and \( k + 1 \) boundary marked points and equivalences)

\[
\text{Ind}(\overline{\partial}) + 2l + (k + 1) - 3 = n + \mu^{de} - (k + 1)n + 2l + k - 2
\]
In our case of \( n = 1 \) the moduli space \( \mathcal{M}_{l,k+1}(\beta; \nu; \gamma) \) has virtual dimension
\[
\mu^{de} + 2l - 2 = \mu_{CW} + 2 \sum_j (1 - \iota(j)) - 2
\]
For \( l = 0 \), it is simply given by \( \mu^{de} - 2 \).

2.4. Bulk deformed Fukaya algebra. The Fukaya algebra with bulk deformations by twisted sectors can be defined as follows. First, we see how to adapt the definition of \( q \) operator to our orbifold setting. Let \( T_1, \ldots, T_m \) denote the twisted sectors in (2.1). For each multi-index \( I = (i_1, \ldots, i_l) \) we define the specification of twisted sectors \( \nu_I \) as
\[
\nu_I(z_j^+) = T_{i_j}^+.
\]
We denote the corresponding moduli space \( \mathcal{M}^{main}_{k+1,l}(\beta, \nu_I, \gamma) \) as \( \mathcal{M}^{main}_{k+1,l}(\beta, T_I, \gamma) \). This space has a \( \mathbb{Z}/2 \)-equivariant Kuranishi structure, and we can take a continuous family of multi-sections which are \( \mathbb{Z}/2 \)-equivariant and transversal to the zero-section, compatible with other multi-sections given at the boundaries as in [FOOO16b], [Fuk17]. There is an evaluation map
\[
ev^I : \mathcal{M}^{main}_{k+1,l}(\beta, T_I, \gamma) \to \prod_{i=1}^k L(\gamma(i))
\]
as well as \( \ev_0^I \) where \( L(\gamma(i)) \) is the corresponding immersed sector for \( i \neq + \), and \( L(+) = \mathbb{L} \). We can define
\[
q_{l,k,\beta}(T_I; h_1 \otimes \cdots \otimes h_k) = (\ev_0^I)_*(\ev^I)^*(h_1 \times \cdots \times h_k)
\]
\[
q_{l,k}^\rho(T_I; h_1 \otimes \cdots \otimes h_k) = \sum_{\beta} T^{\beta \cap \omega/2\pi} \rho(\partial \beta) q_{l,k,\beta}(T_I; h_1 \otimes \cdots \otimes h_k).
\]
The way to handle the unitary line bundle \( \rho \) (on \( \mathbb{L} \)) is very standard, and we will omit the superscript \( \rho \) from now on.

Given a cohomology class \( [\tau] \in H^*_c(X; \Lambda_0) \) we pick a representative \( \tau = \tau^01_X + \tau^2p + \tau_{tw} \), where \( p \) is a \( \mathbb{Z}_2 \)-invariant cycle (away from \( \mathbb{L} \) and the orbi-points) representing \( [pt] \in H^2(X, \Lambda_0) \) and \( \tau_{tw} = \sum_k \tau_k T_k \). We define
\[
m^\tau_k(h_1, \cdots, h_k) = \sum_{\beta} \exp(\tau^2 p \cap \beta) \sum_{l=0}^\infty \frac{T_{l}^{\beta \cap \omega}}{l!} q_{l,k,\beta}(\tau^l_{tw}; h_1, \cdots, h_k)
\]
for \(k > 0\), and \(m^r_0 = \tau_0 \cdot 1_L + \sum_\beta \exp(\tau^2 p \cap \beta) \sum_{l=0}^\infty \frac{\tau^{\beta \cap \omega} q_{l,0,\beta}(\tau^l_1)}{l!} q_{l,k,\beta}(T_1; h_1, \cdots, h_k).\)

**Remark 2.3.** Here we are slightly abusing notation. The expression \(q_{l,k,\beta}(\tau^l_1; h_1, \cdots, h_k)\) really stands for

\[
\sum_{I = (i_1, \cdots, i_l)} \tau_{i_1} \cdots \tau_{i_l} q_{l,k,\beta}(T_I; h_1, \cdots, h_k).
\]

The above formulas define, for each \(\tau\), a unital filtered \(A_\infty\)-algebra which we denote by \(F(L, \tau)\). Like usual, given a Maurer-Cartan element \(b \in F(L, \tau)\) we denote the deformed \(A_\infty\) operations by \(m^{\tau,b}\).

Recall that the Seidel Lagrangian together with the equator divides \(X\) into eight regions with equal area (say 1). For our convenience, we may take \(p\) to be \(\lambda\) times the sum of eight points, one in each region (for \(\lambda \in \mathbb{Q}\)). We have \(\tau^2 p \cap \beta = \lambda \cdot (\omega \cap \beta)\) since the area is given by the number of regions. Then \(\exp(\tau^2 p \cap \beta) = t^{\omega \cap \beta}\) where \(t := e^\lambda\) and \(\omega \cap \beta \in \mathbb{Z}_{\geq 0}\). We will see in Proposition 5.2 that the choice of a representative of \(p\) does not affect our calculation significantly.

### 3. Weakly unobstructedness for bulk-deformed Fukaya algebra

In this section, we show weakly unobstructedness of the Seidel Lagrangian in bulk-deformed Floer theory. The main geometric idea behind this result is the anti-symplectic involution of the orbifold sphere \(\mathbb{P}^1_{a,b,c}\). Let \(\iota\) be the anti-symplectic involution on the orbifold sphere. The Seidel Lagrangian \(i : S^1 \to \mathbb{P}^1_{a,b,c}\) is chosen so that the immersion \(i\) is equivariant (with the involution on the domain \(S^1\) by \(\pi\)-rotation). Note that \(\iota\) preserves the orientation and spin structure of the Seidel Lagrangian. The bulk inputs that we will consider are orbifold cohomology representatives of \(\mathbb{P}^1_{a,b,c}\). The twisted sectors and the fundamental cycle are invariant under the involution. A representative of the point class will be chosen to be invariant under \(\iota\).

Recall that the Seidel Lagrangian is shown to be weakly unobstructed in [CHL12]. We extend it to the case of bulk deformations.

**Proposition 3.1.** Fix \(\tau \in H^*_\text{orb}(X, \Lambda_+),\) let \(\bar{x}, \bar{y}, \bar{z} \in \Lambda_+\) and define \(b = \bar{x}X + \bar{y}Y + \bar{z}Z \in F^*(L)\). Any such \(b\) is a weak Maurer-Cartan element (that is, a weak bounding cochain). In other words we have

\[
m^r_0\tau,b = \sum_{k \geq 0} m^r_k(b, \cdots, b) = \mathcal{P}(\tau, b)1_L,
\]

where \(1_L\) is the unit in \(F^*(L)\) and \(\mathcal{P}(\tau, b)\) is some element in \(\Lambda\).

**Remark 3.2.** Here \(\bar{x}, \bar{y}, \bar{z}\) are regarded as a scalar. In later sections they will be regarded as variables. In particular, we will investigate convergence problems.

**Proof.** The main idea of proof is similar to that of [CHL12], namely, any non-unit output of \(m^r_k(b, \cdots, b)\) vanishes due to cancellation from the \(\mathbb{Z}/2\)-involution. We will see that the argument of [CHL12] still works with bulk deformations, and explain why we can also work with de Rham model instead of Morse model of [Sei11], [CHL12].
From $\mathbb{Z}/2$-grading of Lagrangian Floer theory, $m_0^b$ is given by a linear combination of even-degree immersed generators and the unit $1_L$. The weak Maurer-Cartan equation is satisfied if all outputs except $1_L$ vanish. The anti-symplectic involution was used to show that any immersed output of $m_k(b, b, \cdots, b)$ cancels out with the opposite polygon. In addition, $L$ should be equipped with a non-trivial spin structure which brings the exact cancellation of signs (It is not weakly unobstructed with the trivial spin structure).

First we will show that sign cancellation still works for the bulk deformed theory, so that $m_0^\tau b$ does not involve even-degree immersed generators. In the de Rham model, this means that $m_0^\tau b$ is a zero-form, that is a function on $S^1$ (the normalization of $L$). We will show that it is simply a constant function on $S^1$, which proves the weakly unobstructedness.

Since moduli spaces of sphere bubbles attached to the interior of stable discs carry complex orientations, the cancellation for the case without (orbi)-sphere bubbles will imply the general cases. We will first consider the combinatorial sign rule of Seidel [Sei11] and later argue that we may use them for our computation.

Let us consider a orbi-polygon $P$ that produces an immersed output of $m_k^\tau(b, \cdots, b)$ (in particular, such a $P$ should have $k + 1$ edges). By applying the reflection about the equator of $P_{a,b,c}$ to $P$, we get another polygon $P^{op}$. The $A_\infty$-operations for $P$ and $P^{op}$ give the same output (in $\mathbb{Z}/2$-graded theory) but if $P$ contributes to $m_k^\tau(X_1, \cdots, X_k)$ then $P^{op}$ contributes to $m_k^\tau(X_k, \cdots, X_1)$. We claim that these two contributions have the opposite signs to each other.

Without loss of generality, let us assume that the boundary orientation of $P$ is coherent with that of $L$. Then the boundary operation of $P^{op}$ is opposite (for each edges of $P^{op}$) to that of $L$ since the reflection preserves the orientation of $L$ whereas it reverses boundary orientations of holomorphic polygons. From the sign rule of [Sei11], There is a sign difference of $(-1)^k$ between $P$ and $P^{op}$. Another source of sign difference is how many times $P$ and $P^{op}$ pass through the point that represents the nontrivial spin structure. Let $s_1$ and $s_2$ denote these numbers, respectively. We now show that $s_1 - s_2$, or equivalently, $s_1 + s_2$ has the same parity as $k + 1$. We first claim that $\partial P \cup \partial P^{op}$ evenly covers $L$. To see this, let us divide $L$ into 6 minimal arcs, which are edges joining one corner with another without passing through other corners. We will denote these arcs by $\overrightarrow{XY}_\pm, \overrightarrow{YZ}_\pm, \overrightarrow{ZX}_\pm$ as in the left of Figure 3.

![Figure 3](image-url)
Suppose \( p \) is a point on the boundary of \( P \) that lies in \( \tilde{XY}_\pm \), then its reflection image, say \( p' \), is located on \( \tilde{XY}_\mp \). When \( p \) travels along \( \partial P \), the pair \((p, p')\) first covers both of \( \tilde{XY}_\pm \). And then the pair starts covering both of \( \tilde{YZ}_\pm \) afterward, regardless of having corners at \( Y \) or not (see the right of Figure 3. Since \( p \) starts at and comes back to the same point when we go along \( \partial P \) once, we see that \((p, p')\) covers \( L \) evenly.

Having this, let \( \partial P \cup \partial P^{op} = s[L] \), which implies \( s_1 + s_2 = s \). If \((k+1)\) edges of \( \partial P \) consists of \( a_1, a_2, \ldots, a_{k+1} \) minimal arcs, we have

\[
6s = 2(a_1 + a_2 + \cdots + a_{k+1}) \Rightarrow 3s = a_1 + a_2 + \cdots + a_{k+1}
\]

since \( \partial P \cup \partial P^{op} \) has \( 6s \) minimal arcs. On the other hand, it is easy to see that each edge of \( \partial P \) (and \( \partial P^{op} \)) consists of an odd number of minimal arcs i.e., \( a_i \) are all odd, and hence we conclude that the parity of \( s \) is the same as \( k+1 \), which completes the proof of the claim.

Now, let us argue that the combinatorial sign of Seidel is compatible with the de Rham model we use in this paper. In [Sei08, Part II section 13], it is shown that the sign of an \( A_\infty \)-operation defined using Floer theory (orientation operators) and that defined by combinatorial convention can be identified. Seidel showed that in this surface case, the sign in Floer theory is local, and hence depends only on absolute indices of intersection points. On the other hand, there is a combinatorial way of giving sign in this case. Seidel constructs \( \gamma(k) \) which is a linear isomorphism for each Floer group \( CF^k \) which makes these two signs compatible. We use the existence of this isomorphism to show cancellations. Note that the combinatorial sign only depends on the parity of the intersection points.

On the other hand, one can also show that the Floer sign also depends only on the parity of the absolute indices of corners. Note that a choice of path of Lagrangian subspaces from \( T_pL_1 \) to \( T_pL_2 \) for \( p \in L_1 \cap L_2 \) defines an orientation operator, which can be used to define its absolute index as well as associated orientation space (determinant of the orientation operator). Absolute indices from different choice of paths may differ by even integer, and one can fix a canonical isomorphism between two different choices using gluing of discs with Lagrangian loop of the difference of paths. It can be shown that this gluing provides a canonical way to relate orientation spaces corresponding to different paths, which gives rise to the same sign for associated polygons. In this way, one can observe that the Floer theoretic sign only depends on the parity of the absolute indices at the intersection points based on the above isomorphism of orientation spaces.

Thus \( m_k^p(b, \cdots, b) \) does not involve even-degree immersed generators and hence is a zero-form. It remains to show that the non-immersed output is a multiple of the unit (namely a constant function on \( S^1 \)).

**Lemma 3.3.** The expression \( m_k^p(b, \cdots, b) \) is a constant (as a function on \( x, y, z \)) \( k \geq 0 \).

**Proof.** We proceed by induction on \( k \). Note that \( m_0 = 0 \) in the elliptic and hyperbolic cases, and \( m_0 \) can be given by the contribution of two hemispheres in the spherical cases [CHKL17, Section 12]. In any cases, it is always a constant multiple of a unit.

Let us assume the statement for \( k = i \) and prove the case of \( i + 1 \). By degree reason, \( m_{i+1}(b, \cdots, b) \) is even, and hence it is either a smooth function, or an immersed sector. By the above reflection argument we know that the output in immersed sectors cancels out.
Hence, it is enough to show that the output is a constant function. Observe that \( m_1 \) on a function \( f \) is given by \( df \) by construction. Hence, in order to show that the output is constant, it is enough to prove that \( m_1(m_{i+1}(b, \cdots, b)) = 0 \). This follows from the \( A_\infty \)-identity and the induction hypothesis (and the property of a unit).

□

This proves the weakly unobstructedness of the Seidel Lagrangian \((\mathbb{L}, b)\).

□

It follows from Lemma 3.3 that each \( b \) determines a deformation of \( \mathcal{F}(\mathbb{L}) \) with central curvature \( m_{\tau,b}^0 \). This means that \( m_{\tau,b}^1 \) is a differential and its cohomology \( HF^*(\mathbb{L}, \tau, b) \) is an algebra with product \( m_{\tau,b}^2 \). We will describe this algebra in Section 4.2.

4. Bulk-deformed potential function and change of variables

The previous section asserts that the Lagrangian Floer potential function \( P(b) \) is a formal power series in \( \tilde{x}, \tilde{y}, \) and \( \tilde{z} \) with coefficients in the Novikov ring \( \Lambda_0 \), where \( b = \tilde{x}X + \tilde{y}Y + \tilde{z}Z \). As explained in the introduction, it is essential for the purpose of studying the Kodaira-Spencer map that we work with the following change of variables.

\[
\begin{cases}
  x = T^3 \tilde{x}, \\
  y = T^3 \tilde{y}, \\
  z = T^3 \tilde{z}.
\end{cases}
\]

with \( x, y, z \in \Lambda_0 \). From now on we denote

\[ W_\tau(x, y, z) = P(\tau, b) \]

and call this the potential function. Notice that \( b \) on the right hand side is now given by \( b = T^{-3}xX + T^{-5}yY + T^{-3}zZ \).

This coordinate change will be essential in our study of Kodaira-Spencer map, and at the same time it is the main source of complication.

After the coordinate change the term of minimal valuation in the potential \( W_\tau \) is \( T^3 \tilde{x} \tilde{y} \tilde{z} = T^{-8}xyz \). This negative energy term should be handled in a delicate way as we will see in our proof of the Kodaira-Spencer map being an isomorphism. For this reason, we will need a better control on the energy of the terms appearing in the related Floer operations and algebraic manipulations.

We first examine the potential function and its convergence in new variables. When there is no bulk deformation (i.e. \( \tau = 0 \)), [CHKL17] gives closed formulas for \( W \) in the spherical and elliptic cases - in these cases, \( W \) is simply a polynomial on \( x, y, z \). In the hyperbolic case (again when \( \tau = 0 \)), an algorithm that computes \( W \) is given in [CHKL17].

4.1. Gauss–Bonnet theorem and convergence. Recall that \( b = \tilde{x}X + \tilde{y}Y + \tilde{z}Z = T^{-3}(xX + yY + zZ) \) where \( \tilde{x}, \tilde{y}, \tilde{z} \) are the dual variables to the immersed generators \( X, Y, Z \). Gromov compactness ensures the boundary deformed \( A_\infty \) algebra is convergent when \( \text{val} \tilde{x}, \text{val} \tilde{y}, \text{val} \tilde{z} > 0 \). We will show that it is still convergent for \( \text{val} \tilde{x}, \text{val} \tilde{y}, \text{val} \tilde{z} \geq -3 \) (that is \( \text{val} x, \text{val} y, \text{val} z \geq 0 \)).
Definition 4.1. A **convergent power series** in $x, y, z$ is a series of the form

$$
\sum_{i,j,k \in \mathbb{Z}_{\geq 0}} c_{i,j,k} x^i y^j z^k,
$$

with $c_{i,j,k} \in \Lambda$ and $\lim_{i+j+k \to \infty} \nu(c_{i,j,k}) = +\infty$, where $\nu$ is the usual valuation in $\Lambda$. We denote by $\Lambda\langle\langle x,y,z \rangle\rangle$ the ring of convergent power series.

Recall that the valuation $\nu$ in $\Lambda$ determines a non-archimedean metric by the formula $|\xi| = e^{-\nu(\xi)}$. The condition above then states that the coefficients of the series converge to zero in this norm. Therefore the ring just defined is a special case of the Tate algebra, see [BGR84].

Note that any element $P \in \Lambda\langle\langle x,y,z \rangle\rangle$ is indeed convergent (in the unit disc), in the sense that it determines a map $P : \Lambda^3_0 \rightarrow \Lambda$.

In the remainder of this section we will show that $W_\tau(x,y,z)$ is a convergent power series for each $\tau$ with $\text{val}(\tau) > 0$, and hence, is an element in $\Lambda\langle\langle x,y,z \rangle\rangle$. We begin by establishing a complete classification of the non-positive energy terms of $W_\tau(x,y,z)$. First of all, we have $T^{-8}xyz$ from the minimal triangle. In addition, slices of discs for $x^a, y^b, z^c$-terms give rise to $x^i, y^j, z^l$ with $1 \leq i \leq a - 1, 1 \leq j \leq b - 1, 1 \leq l \leq c - 1$ with energy zero coefficients (which are all 1). These discs have exactly one interior orbi-insertion, and can be viewed as the first-order contribution of orbi-sectors in $\text{QH}^*_\text{orb}(X, \tau)$ to the potential. We give a precise description on such discs by the lifting argument below, which is valid for general orbi-discs although their liftings are the maps defined on higher genus (bordered) Riemann surfaces in most of cases.

Lemma 4.2. Suppose $D$ is an orbifold disc with interior orbifold marked points $p_1, \cdots, p_k$ where $p_i$ is $\mathbb{Z}/k_i$ cone point. Consider a map $\pi : U \rightarrow D$ between Riemann surfaces with boundary (mapping boundaries to boundaries) and suppose that $p_i$ is a branch point of $\pi$ of multiplicity $k_i$ for each $i$. For any orbifold holomorphic disc $u : (D, \partial D) \rightarrow (X, L)$, the composition $u \circ \pi : (U, \partial U) \rightarrow (X, L)$ is a holomorphic disc.

Proof. Note that $u \circ \pi$ is a holomorphic disc away from $\pi^{-1}(p_i)$ by definition. Near each $\pi^{-1}(p_i)$, $u \circ \pi$ is nothing but the lift to a uniformizing cover, hence it is holomorphic. □

Applying the lemma to orbi-discs with a single orbi-insertion, we obtain the following.

Corollary 4.3. A holomorphic orbi-disc $u$ with one orbifold marked point has a holomorphic lift $\tilde{u} : U \rightarrow X$, where $U$ is a disc.

Therefore we see that all such orbi-discs are slices of the discs that contribute to non-bulk-deformed potential. In particular, the slice of the discs for $x^a, y^b, z^c$ will be called the **basic orbi-discs** from now on.

The following lemma gives a complete description of the low energy orbi-discs contributing to the potential. It is a kind of energy quantization at the corners of the discs. A similar result in dimension greater or equal than two appears in [PW19, Lemma 4.2].

Lemma 4.4. Except for the single term corresponding to the minimal triangle, every term of the bulk-deformed orbifold potential in $x, y, z$ variable (4.1) has non-negative $T$-exponent.
Moreover, it has $T$-exponent being 0 exactly for basic orbi-discs, and the $T$-exponents are positive for the rest (except the minimal triangle).

Proof. Let $u : (D, \partial D) \to (\mathbb{P}_a^1, \mathbb{L})$ be a non-constant holomorphic orbi-disc which is not the minimal triangle. We may assume that the counter-clockwise orientation of $\partial D^2$ agrees with the orientation of $\mathbb{L}$ under $u$. (The other case can be handled similarly). $\mathbb{P}_a^1$ is decomposed into 8 pieces by $\mathbb{L}$ and the equator. We denote by $M_u, M_l$ the upper and lower middle triangle piece and denote by $A_u, B_u, C_u$ (resp. $A_l, B_l, C_l$) the triangles with one of their corners at $a, b, c$ orbifold points respectively and lies in the upper (resp. lower) hemisphere. We may decompose the domain of the orbi-disc $D$ according to the above decomposition under the map $u$. Suppose $u$ has an immersed corner mapping to $X, Y$ or $Z$ contributing to the monomial of the potential. By our choice of orientation, the map $u$ covers the piece $M_u$. (One can check that we cannot turn corners at $M_l$ in this case). We consider the part of $u$ which maps to $M_u$ as in Figure 4. We argue that in the neighborhood of each such corner, we have additional regions (in $D$) of area 2 which are distinct for each corner. Note that the piece $M_u$ should be attached to exactly one of $A_u, B_u, C_u$, say the piece $A_u$, since it involves a corner of the disc. In this case, this $A_u$ cannot be attached to any other preimages of $M_u$ or $M_l$ in $D$. Now, $A_u$ is attached to two of $A_l$ pieces $A_1^l, A_2^l$. Let us further cut these $A_l$ pieces into halves.

Hence each corner of $D$ at least covers $M_u, A_u, \frac{1}{2}A_1^l, \frac{1}{2}A_2^l$, which gives the area 3 (or $T^3$). Given two different corners, the above local pieces do not overlap since $A_u$ can be attached to the only one corner piece $M_u$. This proves the first part of the proposition. Suppose that after the coordinate change, it has no $T$-component. This means that the holomorphic orbi-disc consists of these $T^3$ pieces only. It is elementary to see that the only orbifold holomorphic discs that we can make in this way are the basic ones. This proves the lemma.

Next, we will use a version of the Gauss–Bonnet theorem to compute the valuations of the monomials appearing in $W_\tau$.

Recall that the Seidel Lagrangian $\mathbb{L}$ is taken to be symmetric about the equator, and it subdivides each of the upper and lower hemispheres into four triangles with equal area $A$ (which is set to be 1), which are $M_u, A_u, B_u, C_u$ and $M_l, A_l, B_l, C_l$ respectively in the proof of Lemma 4.4. Let $K$ be the constant curvature of the orbi-sphere. The equator is taken to be a union of three geodesics connecting the three orb-points, and the reflection about the equator is an isometry. Denote by $k$ the geodesic curvature of $\mathbb{L}$.
We arrange \( L \) in such a way that the exterior angles of the minimal triangles at \( X, Y, Z \) are \( 2\pi (1/a - \epsilon), 2\pi (1/b - \epsilon), 2\pi (1/c - \epsilon) \) respectively, where \( \epsilon \) is taken such that the angles are in \((0, \pi)\). \( \epsilon \) can be set to zero in case \( a, b, c \geq 3 \).

The Gauss-Bonnet formula for an orbi-polygon is given as follows. It can be easily proved by making a cut for each orbi-point (along a simple path connecting the orbi-point to a boundary point), and applying the ordinary Gauss-Bonnet formula to the new polygon (where the cutting path appears as part of the boundary).

**Theorem 4.5** (Gauss-Bonnet formula for an orbi-polygon). For an (embedded) orbi-polygon \( P \subset X \) with exterior angles \( \angle_i \), boundary edges \( \gamma_j \), and ages of interior orbi-points being \( \iota_k \),

\[
\int_P KdA + \sum_i \angle_i + \sum_j \int_{\gamma_j} kds + 2\pi \sum_k (1 - \iota_k) = 2\pi.
\]

Closely related to this, the Maslov-index formula for an orbi-polygon class \((\beta, \alpha)\) in terms of curvature (where \( \alpha \) is the collection of immersed generators that the corners hit) is given as follows (see Remark 2.2, [CS16] or [Pac19]). (Here \( \angle X \) denotes the exterior angles).

\[
\mu_{CW}(\beta, \alpha) = \frac{1}{\pi} \left( \int_{\beta} KdA + \int_{\partial \beta} kds + \sum_{X \in \alpha} \angle X \right).
\]

In our case the Maslov index is given as follows.

**Proposition 4.6.** For an orbi-disc bounded by \( \mathbb{L} \) with corners being only \( X, Y \) or \( Z \) (thus contributing to the potential), denote its area by \( mA \) the numbers of \( X, Y, Z \) corners by \( n_1, n_2, n_3 \) respectively. Its Maslov index \( \mu_{CW} \) equals to

\[
2 \left( \frac{n_1}{a} + \frac{n_2}{b} + \frac{n_3}{c} + (m - 3(n_1 + n_2 + n_3)) \cdot \frac{\chi}{8} \right).
\]

In particular the Chern number of an orbi-sphere equals to \( m\chi/8 \).

Recall that \( \chi = -1 + \frac{1}{a} + \frac{1}{b} + \frac{1}{c} \) is the (orbifold) Euler characteristic of \( \mathbb{P}^1_{a,b,c} \).

**Proof.** First of all we find the geodesic curvatures of the edges of the minimal triangles. By Gauss-Bonnet formula applied to the upper hemisphere (which is a triangle bounded by three geodesics segments forming the equator), we have

\[
4AK + \left( \pi - \frac{\pi}{a} \right) + \left( \pi - \frac{\pi}{b} \right) + \left( \pi - \frac{\pi}{c} \right) = 2\pi
\]

(where \( A = 1 \) is the area of the minimal triangle.) Thus

\[
\chi = \left( \frac{1}{a} + \frac{1}{b} + \frac{1}{c} \right) - 1 = \frac{8AK}{2\pi}.
\]

Let \( 2\pi k_{12} \) be the total geodesic curvature along the edge connecting the \( X \) and \( Y \) corners of the minimal triangle contained in the upper hemisphere. Here, the orientation of the Seidel Lagrangian is fixed such that the orientations of the edges agree with the induced ones from the minimal triangle contained in the upper hemisphere. \( k_{23} \) and \( k_{31} \) are similarly defined.
We apply the Gauss-Bonnet formula to the triangle $C_u$ which is the triangle in the upper hemisphere having corners at the point $X, Y$ and the orbi-point $[1/c]$. This gives

$$
\left(\pi - \frac{\pi}{a} + \pi \epsilon\right) + \left(\pi - \frac{\pi}{b} + \pi \epsilon\right) + \left(\pi - \frac{\pi}{c}\right) - 2\pi k_{12} + KA = 2\pi.
$$

Combining with (4.3), we have $k_{12} = \kappa + \epsilon$ where $\kappa = -3\chi/8$. Applying the same argument for the other two triangles contained in the upper hemisphere, we obtain

$$
k_{12} = k_{23} = k_{31} = -\frac{3\chi}{8} + \epsilon = -\frac{3KA}{2\pi} + \epsilon = \kappa + \epsilon.
$$

Then the total geodesic curvatures of the edges of the minimal triangle in the lower hemisphere (in the fixed orientation of the Seidel Lagrangian) are ($2\pi$ times)

$$
k_{12}' = k_{23}' = k_{31}' = -\kappa - \epsilon.
$$

Consider a holomorphic orbi-disc bounded by $L$ with the numbers of $X, Y, Z$ corners being $n_1, n_2, n_3$ respectively. The area is a multiple $mA$ of the area of the minimal triangle for $m \in \mathbb{Z}_{>0}$. Thus

$$
\frac{1}{\pi} \int_{\beta} KdA = \frac{mKA}{\pi} = \frac{m\chi}{4}.
$$

The edges of the orbi-disc are unions of the edge segments of the two minimal triangles in the upper and lower hemispheres. By the property of holomorphic orbi-disc, each side between two corners must consist of an odd number of edge segments. The total geodesic curvatures of the edge segments cancel with each other, except for one edge segment for each side. Such a segment in each side lie in the same hemisphere, and in the boundary orientation of the holomorphic disc its geodesic curvature is $2\pi k_{12} = 2\pi(\kappa + \epsilon)$. Then

$$
\frac{1}{\pi} \int_{\partial\beta} kds = 2(n_1 + n_2 + n_3)k_{12} = \frac{-3\chi \cdot (n_1 + n_2 + n_3)}{4} + 2(n_1 + n_2 + n_3)\epsilon.
$$

Since the exterior angles of $X, Y, Z$ are $2\pi(1/a - \epsilon), 2\pi(1/b - \epsilon), 2\pi(1/c - \epsilon)$ respectively, we have

$$
\frac{1}{\pi} \sum_{X \in \alpha} \angle X = 2 \left(\frac{n_1}{a} + \frac{n_2}{b} + \frac{n_3}{c}\right) - 2(n_1 + n_2 + n_3)\epsilon.
$$

Hence the error term $2(n_1 + n_2 + n_3)\epsilon$ cancels in the sum. Combining the above equations, result follows. \qed

**Corollary 4.7.** Consider a stable orbi-disc bounded by $L$ which contributes to the disc potential. Suppose it has area $mA$, the interior orbi-insertions have ages $\iota_j$, and the numbers of $X, Y, Z$ corners are $n_1, n_2, n_3$ respectively. Then it satisfies

$$
(m - 3(n_1 + n_2 + n_3)) \cdot \frac{-\chi}{8} = \frac{n_1}{a} + \frac{n_2}{b} + \frac{n_3}{c} + \sum_j (1 - \iota_j) - 1.
$$

**Proof.** Recall that an orbi-disc contributes to the potential if it is rigid, or equivalently its Maslov index satisfies

$$
\mu_{CW} + 2 \cdot \sum_j (1 - \iota_j) - 2 = 0.
$$
A stable orbi-disc consist of disc and sphere components. Since Maslov index is additive, Proposition 4.6 applied to each component gives the result. (It can also be seen by taking an orbi-smooth disc representative of the class.) □

From the corollary, the area of the orbi-disc is given in terms of the numbers of corners and ages by

(4.5) \[ mA = \left( 3(n_1 + n_2 + n_3) + 8 \frac{\frac{n_1}{a} + \frac{n_2}{b} + \frac{n_3}{c} + \sum_j (1 - \iota_j) - 1}{1 - \left( \frac{1}{a} + \frac{1}{b} + \frac{1}{c} \right)} \right) A \]

when \( \chi \neq 0 \), which matches the one given in [CHKL17] when there is no orbi-insertions. If \( \chi = 0 \) (i.e. elliptic case), we have \( \frac{n_1}{a} + \frac{n_2}{b} + \frac{n_3}{c} + \sum_j (1 - \iota_j) = 1 \).

Suppose that \( T^a \tau_i x^{a_1} y^{a_2} z^{a_3} \in \Lambda[[\tau, x, y, z]] \) is one of monomials contained in \( W_\tau \). Our discussion so far tells us that the exponent \( a \) satisfies \( a = m - 3(n_1 + n_2 + n_3) \geq 0 \), where \( m \) is given by (4.5) and \( \tilde{k} \) records the ages \( \iota_j \) in the formula. Notice that the coordinate change 4.1 is responsible for the term \(-3(n_1 + n_2 + n_3)\) in \( a \). By Lemma 4.4, \( m - 3(n_1 + n_2 + n_3) \geq 0 \) for all holomorphic orbi-discs other than the minimal triangle.

**Theorem 4.8.** The bulk-deformed potential \( W_\tau \) is a convergent series, that is, it is an element of \( \Lambda\langle \langle x, y, z \rangle \rangle \).

**Proof.** By Gromov compactness, it suffices to show that the area \( mA \) is bounded above once the exponent \( m - 3(n_1 + n_2 + n_3) \) of \( T \) is bounded. For elliptic case, (4.4) gives

\[
\frac{n_1}{a} + \frac{n_2}{b} + \frac{n_3}{c} + \sum_j (1 - \iota_j) = 1.
\]

There are only finitely many possibilities of \( n_i \) and orbi-insertions satisfying this. (And hence \( W_\tau \) is a polynomial in \( x, y, z, \tau \).) In particular \( n_i \) are bounded. Once \( m - 3(n_1 + n_2 + n_3) \) is bounded, \( m \) is bounded.

For spherical case, \( m - 3(n_1 + n_2 + n_3) \geq 0 \) (other than the minimal triangle) implies that

\[
\frac{n_1}{a} + \frac{n_2}{b} + \frac{n_3}{c} + \sum_j (1 - \iota_j) \leq 1.
\]

There are only finitely many possibilities of \( n_i \) and orbi-insertions satisfying this. Thus there are only finitely many possibilities of \( m - 3(n_1 + n_2 + n_3) \), and hence \( m \). \( W_\tau \) just consists of finitely many terms.

For hyperbolic case, if \( m - 3(n_1 + n_2 + n_3) \) is bounded above, then \( \frac{n_1}{a} + \frac{n_2}{b} + \frac{n_3}{c} + \sum_j (1 - \iota_j) \) is also bounded above. This gives finitely many possibilities of \( n_i \) and \( \iota_j \). Hence \( n_1 + n_2 + n_3 \) is bounded above, and so is \( m \). □

**Remark 4.9.** Recall that \( \tau = \tau^0 1_X + \tau^2 p + \tau_w \), and \( \exp(\tau^2 p \cap \beta) = t^{\omega \cap \beta} \) (see the paragraph before Section 3). The above shows that \( W(\tau_0, t, \tau_1, \ldots, \tau_m, x, y, z) \) is an element of \( \Lambda\langle \langle \tau_0, t, \tau_1, \ldots, \tau_m, x, y, z \rangle \rangle \).

Similarly, we can show that every \( m^{\tau,h}(\alpha_1, \ldots, \alpha_k) \) is convergent power series and so we have the following proposition.
Proposition 4.10. The Fukaya algebra $\mathcal{F}(\mathbb{L}, \tau, b)$ is convergent over $\Lambda\langle x, y, z \rangle$.

Proof. Consider an orbi-polygon $P$ with $k_1, k_2, k_3$ numbers of $X, Y, Z$ corners (which have odd degree), and $k^-_1, k^-_2, k^-_3$ numbers of $\bar{X}, \bar{Y}, \bar{Z}$ corners (which have even degree) respectively. Without loss of generality, we assume that in a neighborhood of one of the odd corners, $P$ is contained in the upper hemisphere. (If there is no odd corner, then we assume that in a neighborhood of one of the even corners, $P$ is contained in the lower hemisphere.)

Note that for a side of $P$ between odd and even adjacent corners, the number of minimal edge segments is even. Similarly for an odd-odd side or even-even side, the number of minimal edge segments is odd. Thus two corners adjacent to an odd-odd or even-even side remain in the same hemisphere; an odd-even side connects a corner in the upper hemisphere to a corner in the lower hemisphere. It implies all corners of $P$ are contained in the upper hemisphere.

For an odd-even side, the geodesic curvature of the edge segments cancel among each other; for an odd-odd edge (resp. even-even edge), the geodesic curvature of all but one edge segment cancel, and that edge segment lies in the upper (resp. lower) hemisphere. It follows that the total geodesic curvature of an odd-odd edge, odd-even edge, and even-even edge is $2\pi$ times $k_{12} = \kappa + \epsilon, 0, -k_{12} = -\kappa - \epsilon$ respectively.

We claim that the error term, namely the term which is a multiple of $\epsilon$, is zero in the Maslov index. Recall that the exterior angles of the odd vertices $X, Y, Z$ are $2\pi (1/a - \epsilon), 2\pi (1/b - \epsilon), 2\pi (1/c - \epsilon)$ respectively. As in the proof of Proposition 4.6, each odd vertex contributes $-2\epsilon$ to the error term in the Maslov index. For the even vertices $\bar{X}, \bar{Y}, \bar{Z}$, the exterior angles are $\pi - 2\pi (1/a - \epsilon), \pi - 2\pi (1/b - \epsilon), \pi - 2\pi (1/c - \epsilon)$ respectively. Each even vertex contributes $2\epsilon$ to the error term in the Maslov index.

Let $l_{oo}, l_{oe}, l_{ee}$ be the numbers of odd-odd, odd-even, even-even sides respectively. Then the numbers of odd and even vertices equal $(l_{oe} + 2l_{oo})/2$ and $(l_{oe} + 2l_{ee})/2$ respectively. The total error contribution from the angles of the vertices is

$$-2\epsilon (l_{oe} + 2l_{oo})/2 + 2\epsilon (l_{oe} + 2l_{ee})/2 = -2\epsilon (l_{oo} - l_{ee}).$$

The geodesic curvature of an odd-odd (even-even resp.) edge contributes $2\epsilon (-2\epsilon$ resp.) to the error; the geodesic curvature of an odd-even edge has no error term. Thus the total error contribution from the geodesic curvatures of the sides is

$$2\epsilon (l_{oo} - l_{ee}).$$

We see that the above two error contributions cancel among each other and hence there is no $\epsilon$-term in the Maslov index.

Thus we can throw away the $\epsilon$ terms. The total geodesic curvature (mod $\epsilon$) of the sides equal to $2\pi \kappa (l_{oo} - l_{ee})$, and $l_{oo} - l_{ee}$ equals to the number of odd vertices minus the number of even vertices, that is $k_1 + k_2 + k_3 - k^-_1 - k^-_2 - k^-_3$. The Maslov index formula in Proposition 4.6 generalizes to this situation:

$$\mu_{CW}(P) = 2 \left( \frac{k_1 - k^-_1}{a} + \frac{k_2 - k^-_2}{b} + \frac{k_3 - k^-_3}{c} + \frac{k^-_1 + k^-_2 + k^-_3}{2} \right) + (m - 3(k_1 + k_2 + k_3 - k^-_1 - k^-_2 - k^-_3)) \cdot \frac{\chi}{8}. $$


Now consider a stable orbi-disc contributing to a term of \( m^\tau b(\alpha_1, \ldots, \alpha_k) \) with the monomial \( T^{m-3(n_1+n_2+n_3)}x^{n_1}y^{n_2}z^{n_3} \). The corresponding dimension formula is

\[
\mu_{CW} + \delta + 2 \cdot \sum_j (1 - \iota_j) = 2
\]

where \( \delta = 0 \) if the output is a zero-form or an immersed generator, \( \delta = 1 \) if the output is a one-form. Combining, we have

\[
\left( m - 3 \left( n_1 + n_2 + n_3 + \sum_{i=0}^k s_i \right) \right) \cdot \frac{-\chi}{8} = \frac{n_1}{a} + \frac{n_2}{b} + \frac{n_3}{c} + \sum_{i=0}^k \theta_i + \frac{\delta}{2} - 1 + \sum_j (1 - \iota_j)
\]

where \( s_i = 1, -1, 0 \) depending on the input \( \alpha_i \) for \( i = 1, \ldots, k \) being odd generators, even generators, or point classes respectively, or the output (for \( i = 0 \)) being even generators, odd generators, or \( 1L, pt \) respectively; \( \theta_i = 1/a, 1/b, 1/c \) if \( \alpha_i = X, Y, Z, i > 0 \) or \( \alpha_i = \bar{X}, \bar{Y}, \bar{Z}, i = 0 \); \( \theta_i = 1/2 - 1/a, 1/2 - 1/b, 1/2 - 1/c \) if \( \alpha_i = X, Y, Z, i = 0 \) or \( \alpha_i = \bar{X}, \bar{Y}, \bar{Z}, i > 0 \), \( \theta_i = 0 \) in all other cases.

Then the argument goes in a similar way as the proof of Theorem 4.8. In elliptic case \(-\chi = 0\), so the LHS = 0. \( n_i \) are bounded. The valuation \( m - 3(n_1 + n_2 + n_3) \) is bounded implies \( m \) is bounded. Similarly, in spherical case \(-\chi < 0\), \( m \) is automatically bounded above: otherwise \( m\chi/8 \) is too negative and the above equality has no solution.

Finally, suppose the valuation \( m - 3(n_1 + n_2 + n_3) \) is bounded above, in hyperbolic case \(-\chi > 0\). Then the LHS is bounded. Hence there are just finitely many possibilities of \( n_1, n_2, n_3 \) and \( \iota_j \). Then there is only finitely many possibilities of \( m \), and hence the area is bounded above. It follows from Gromov compactness that there are just finitely many terms satisfying the area bound. \( \square \)

So far, we have discussed the properties of the potential function \( W_\tau \). It can be shown as in [FOOO16b] that if we choose a different representative (\( \mathbb{Z}/2 \)-symmetric points) of point class for \( \tau \), we may get a different potential function, but they are equivalent in the Jacobian ring.

For technical reasons, we choose the representative of the point class away from middle minimal triangle bounded by Seidel Lagrangian for the rest of the paper. Then the result of this section shows that

**Corollary 4.11.** \( W_\tau \) is a convergent series. Moreover, if \( \nu(\tau) > 0 \), we have

\[
W_\tau = -T^{-8}xyz + x^a + y^b + z^c + W_{high},
\]

where \( \text{val}(W_{high}) \geq \lambda > 0 \) for some \( \lambda \) and the representative for \( [pt] \) is chosen not to intersect the minimal triangles.

Note that if we choose a representative of point class in the middle triangle, then we may get bulk deformed contribution of the middle minimal triangle, which have negative valuation. Nevertheless, the coefficient \( \tau_2 \) of \( [pt] \) has a nonnegative valuation, and hence does not affect the valuation of the coefficient of \( xyz \). More specifically, the potential still admits an expression

\[
(4.6) \quad W_\tau = -\xi xyz + x^a + y^b + z^c + W_{high},
\]
and we still have $\text{val}(\xi) = -8$ in this case.

4.2. Fukaya algebra of $\mathbb{L}$. We will carry the argument on the canonical model of $\mathcal{F}(\mathbb{L})$. That is, we use the homological perturbation lemma to transfer the $A_\infty$-algebra structure to $H^*(\mathbb{L})$. In addition, instead of evaluating the $A_\infty$ operations at a specific value of $b$, we will consider the canonical model of $\mathcal{F}(\mathbb{L})$ with values in the ring $\Lambda\langle\langle x, y, z \rangle\rangle$. This will be useful in section 5.

We denote by $H^*(\mathbb{L}, \Lambda\langle\langle x, y, z \rangle\rangle)$ the canonical model of $\mathcal{F}(\mathbb{L})$ with the coefficients in the ring $\Lambda\langle\langle x, y, z \rangle\rangle$ and denote the $A_\infty$ operations by $m_{k,\text{can}}^\tau$. $H^*(\mathbb{L}, \Lambda\langle\langle x, y, z \rangle\rangle)$ should be thought of as a family of $A_\infty$-algebras over $\Lambda$, and over each $(x, y, z)$ in $\Lambda_0^3$ and $\tau$ with $\text{val}(\tau) > 0$ sits a well-defined $A_\infty$-algebra modeled on $H^*(\mathbb{L}, \Lambda)$ (by Proposition 4.10) whose $A_\infty$-operations are deformed by $b = xX + yY + zZ$ and $\tau$.

**Lemma 4.12.** Let $p$ be the odd degree generator of $H^*(\mathbb{S}^1)$. We have the following identities:

$$m_{2,\text{can}}^\tau(X, Y) = \tilde{Z}T + T^3\xi_1 + T^{-2}d_11_L,$$

$$m_{2,\text{can}}^\tau(Y, Z) = \tilde{X}T + T^3\xi_2 + T^{-2}d_21_L,$$

$$m_{2,\text{can}}^\tau(Z, X) = \tilde{Y}T + T^3\xi_3 + T^{-2}d_31_L,$$

$$m_{2,\text{can}}^\tau(X, \tilde{X}) = (1 + c_1)p + \eta_1,$$

$$m_{2,\text{can}}^\tau(Y, \bar{Y}) = (1 + c_2)p + \eta_2,$$

$$m_{2,\text{can}}^\tau(Z, \bar{Z}) = (1 + c_3)p + \eta_3,$$

where each $\xi_i$ is a linear combination of $X, Y, Z$ with $\text{val}(\xi_i) \geq 0$, $\text{val}(d_i) \geq 0$; each $\eta_i$ is a linear combination of $X, Y, Z$ and $c_i$ is an element of $\Lambda_+$. 

**Proof.** The first term in $m_{2,\text{can}}^\tau(X, Y)$ is due to the minimal triangle with $X, Y, Z$-corners in counter-clockwise order. Also, the last term is essentially $\partial^2/\partial x\partial y$ applying to the bulk deformed potential evaluated at $(T^3x, T^3y, z)$. This is because $X = \tilde{x}X|_{\tilde{x}=1}$ should be interpreted as $xX|_{x=T^3} = T^3X$ after change of variables (in particular, when computing in $x, y, z$-variables) and similar for $Y$. Therefore $T^{-2}$ again comes from the minimal triangle, and that is the smallest valuation among the terms in the coefficient of $1_L$ by Corollary 4.11. Finally, $T^3\xi_1$ comes from the polygons apart from the minimal triangle, where one of their corners are used as outputs. Since the output (one of $\tilde{X}, \bar{Y}, \bar{Z}$) in this case has valuation zero unlike variables $x, y, z$, we have additional $T^3$ (which we would lose if the corresponding corner was not an output). The valuations in $m_{2,\text{can}}^\tau(Y, Z)$ and $m_{2,\text{can}}^\tau(Z, X)$ can be estimated in a similar way.

For $m_{2,\text{can}}^\tau(X, \tilde{X})$, constant triangle contributes to $p$ which gives the first term. In general, $m_{2,\text{can}}^\tau(X, \tilde{X})$ should be of odd degree. In fact, there does not exist a polygon whose corners are $X, Y, Z$’s except one of $\tilde{X}, \bar{Y}, \bar{Z}$ corner due to orientation of Lagrangian. Thus any non-trivial polygon contributing to $m_{2,\text{can}}^\tau(X, \tilde{X})$ should have an output in $X, Y, Z$. 

$\square$
Lemma 4.13. Let $R(X,Y,Z)$ be the subring of $H^*(\mathbb{L},\Lambda\langle\langle x,y,z\rangle\rangle)$ generated by $X,Y,Z$. There exists $r,s,t$ in the closure of $R(X,Y,Z)$ and $q_1,q_2,q_3 \in \Lambda\langle\langle x,y,z\rangle\rangle$ such that
\[
\begin{align*}
\bar{X} &= r + q_1 1_L, \\
\bar{Y} &= s + q_2 1_L, \\
\bar{Z} &= t + q_3 1_L.
\end{align*}
\]

Proof. The proof of the three statements is identical, we prove the last one. First we prove by induction, that for each integer $k \geq 1$ there are $t_k \in R(X,Y,Z)$ and $c_k \in \Lambda\langle\langle x,y,z\rangle\rangle$ such that $E_k = \bar{Z} - t_k - c_k 1_L$ is a linear combination of $\bar{X}, \bar{Y}, \bar{Z}$ and
\[
\text{val}(\bar{Z} - t_k - c_k 1_L) \geq 2k, \quad \text{val}(t_k - t_{k-1}) \geq 2k - 3, \quad \text{val}(c_k - c_{k-1}) \geq 2k - 3.
\]

The first equation in Lemma 4.12, gives the case $k = 1$ with $t_1 = T^{-1} m_{2,\text{can}}^{r,b}(X,Y)$ and $c_1 = T^{-3} d_1$. Assuming the statement for $k$, we have
\[
E_k = T^{2k} (\alpha \bar{X} + \beta \bar{Y} + \gamma \bar{Z}),
\]
where $\alpha, \beta, \gamma \geq 0$. Now using the formulas for $\bar{X}, \bar{Y}, \bar{Z}$ given by the first three equations in Lemma 4.12 we can write
\[
E_k = T^{2k} (T^{-3} R + d 1_L + T^2 J),
\]
where $R \in R(X,Y,Z)$, $J$ is a linear combination of $\bar{X}, \bar{Y}, \bar{Z}$ and $\text{val}(R) \geq 0$, $\text{val}(d) \geq -3$, $\text{val}(J) \geq 0$. Therefore we have by induction
\[
\bar{Z} = t_k + T^{2k-3} R + (c_k + d T^{2k}) 1_L + T^{2k+2} J.
\]
Hence we can take $t_{k+1} = t_k + T^{2k-3} R$ and $c_{k+1} = c_k + d T^{2k}$, satisfying the conditions required.

Finally we simply take the limits $t = \lim_k t_k$ and $c = \lim_k c_k$. \hfill $\square$

We are now ready to prove the main proposition in this subsection.

Proposition 4.14. The image of $m_{1,\text{can}}^{r,b}$ is contained in the Jacobian ideal. That is, we have the following
\[
\text{Im} \left( m_{1,\text{can}}^{r,b} \right) \subset \langle \partial_x W_\tau, \partial_y W_\tau, \partial_z W_\tau \rangle \cdot H^*(\mathbb{L},\Lambda\langle\langle x,y,z\rangle\rangle).
\]

Proof. In order to prove this proposition one first differentiates the Maurer-Cartan equation
\[
\frac{\partial P}{\partial \tau}(\tau, b) \cdot 1_L = \sum_{k_1,k_2 \geq 0} m_{2+k_2+1}^r(b,\ldots,b,X,\ldots,b) = m_{1,\text{can}}^{r,b}(X).
\]

We have analogous identities for $Y$ and $Z$. Taking the change of variables into account we have
\[
m_{1,\text{can}}^{r,b}(X) = T_3 \frac{\partial W_\tau}{\partial x}(b) 1_L, \quad m_{1,\text{can}}^{r,b}(Y) = T_3 \frac{\partial W_\tau}{\partial y}(b) 1_L \quad \text{and} \quad m_{1,\text{can}}^{r,b}(Z) = T_3 \frac{\partial W_\tau}{\partial z}(b) 1_L.
\]

By the previous lemma,
\[
m_{1,\text{can}}^{r,b}(\bar{X}) = m_{1,\text{can}}^{r,b}(r) + q_1 m_{1,\text{can}}^{r,b}(1_L) = m_{1,\text{can}}^{r,b}(r),
\]
Now, given the Leibniz rule for $m_{\tau,b}^{r,1,can}$ and $m_{\tau,b}^{r,2,can}$, we have that $m_{\tau,b}^{r,1,can}(R(X,Y,Z))$ is contained in the Jacobian ideal. Recall from [BGR84, Section 5.2.7], that the Jacobian ideal, like any ideal in the Tate algebra is closed. Therefore $m_{\tau,b}^{r,1,can}$ of the closure of $R(X,Y,Z)$ is also in the Jacobian ideal. Hence $m_{\tau,b}^{r,1,can}(\bar{X})$ is in the Jacobian ideal. The same is true for $\bar{Y}$ and $\bar{Z}$.

Finally, from the fourth equation in Lemma 4.12 we have

$$m_{1,can}^{r,b}(p) = (1 + c_1)^{-1}\left(m_{1,can}^{r,b}(m_{2,can}^{r,b}(X,\bar{X}))) + m_{1,can}^{r,b}(\eta_1)\right),$$

since $1 + c_1$ is invertible. Again, it follows from the Leibniz rule that the first term on the right is in the Jacobian ideal. Recall, by construction $\eta_1$ is a linear combination of $X,Y,Z$ and therefore we conclude that $m_{\tau,b}^{r,1,can}(\eta_1)$ is in the Jacobian ideal, which completes the proof.

□

**Proposition 4.15.** The cohomology $HF^*(\mathbb{L}, \tau, b)$ is nonzero if and only if $(x,y,z)$ (corresponding to $b$) is a critical point of $W_\tau$.

In this case, $HF^*(\mathbb{L}, \tau, b)$ is isomorphic to

$$H^*(\mathbb{L}) := \left(H^*(S^1) \oplus \bigoplus_{x,y,z} \Lambda_0^{\otimes 2}\right) \otimes \Lambda_0,$$

as a vector space.

**Proof.** From the previous proposition we have

$$m_{1,can}^{r,b}(X) = T^3 \frac{\partial W_\tau}{\partial x}(b)1_L, \quad m_{1,can}^{r,b}(Y) = T^3 \frac{\partial W_\tau}{\partial y}(b)1_L \quad \text{and} \quad m_{1,can}^{r,b}(Z) = T^3 \frac{\partial W_\tau}{\partial z}(b)1_L.$$

Since $1_L$ is the identity in $HF^*(\mathbb{L}, \tau, b)$, this immediately implies that $HF^*(\mathbb{L}, \tau, b)$ is zero if $(x,y,z)$ is not a critical point of $W_\tau$.

For the converse, note that Proposition 4.14 implies that $\text{Im}(m_{1,can}^{r,b}) = 0$, when $(x,y,x)$ is a critical point. This implies that $HF^*(\mathbb{L}, \tau, b)$ is isomorphic to $H^*(\mathbb{L})$. □

Moreover, one can show that when $b$ is a critical point, $HF^*(\mathbb{L}, \tau, b)$ is isomorphic, as a ring, to the Clifford algebra associated to the Hessian of $W_\tau$ at the point $b$. But we will not make use of this fact.

## 5. Dependence of the potential on chain level representatives of bulk

In this section, we prove that if we change the $\mathbb{Z}/2$-representative $\tau^2 p$ for the $H^2(\mathbb{P}^1_{a,b,c}; \Lambda_0)$ component of the bulk deformation, the associated potentials are related by a coordinate change. We start with the definition of this notion.

**Definition 5.1.** A coordinate change is a map $\varphi : \Lambda(\langle x', y', z' \rangle) \rightarrow \Lambda(\langle x, y, z \rangle)$ of the form

$$x' \rightarrow c_1 x + u_1, \quad y' \rightarrow c_2 y + u_2, \quad z' \rightarrow c_3 z + u_3,$$

where $c_i \in \mathbb{C}^*$ and $u_i \in \Lambda(\langle x, y, z \rangle)$ satisfy $\text{val}(u_i) > 0$, for $i = 1, 2, 3$. 


Proposition 5.2. Let $\tau_2$ and $\tau'_2$ be two $\mathbb{Z}/2$-invariant representatives of the same class in $H^2(\mathbb{P}^1_{a,b,c}, \Lambda_0)$. Their associated potentials $W_{\tau_2}$ and $W_{\tau'_2}$ are related by a coordinate change $x' = \exp(k_X) x$, $y' = \exp(k_Y) y$, $z' = \exp(k_Z) z$, with $k_X, k_Y, k_Z \in \Lambda_0$, i.e., $W_{\tau_2}(x', y', z') = W_{\tau'_2}(x, y, z)$. The coefficients $k_X, k_Y, k_Z$ are given in Lemma 5.3.

The main ingredient in the proof of this result is the following topological lemma.

Lemma 5.3. Suppose $\tau_2$ and $\tau'_2$ are two reflection-invariant, cohomologous cycles, and let $Q = \tau_2 - \tau'_2$. Given an (orbi-)polygon $\beta$ contributing to the potential $W_\tau$, denote by $\beta(X), \beta(Y)$ and $\beta(Z)$ the numbers of $X$, $Y$ and $Z$ corners respectively. Then there exist $k_X, k_Y$ and $k_Z$ such that

$$Q \cap \beta = k_X \beta(X) + k_Y \beta(Y) + k_Z \beta(Z)$$

for any (orbi-)polygon $\beta$ contributing to the potential $W_\tau$.

Proof. Since $Q = \tau_2 - \tau'_2$ is cohomologous to zero we can choose a 1-(co)cycle $R$ such that $\partial R = Q$. Here, we are abusing notations for cycles and cocycles via Poincaré duality (see Figure 5 (a)). Moreover we can choose $R$ so that it is reflection invariant and avoids $X, Y, Z$. Then we have $Q \cap \beta = -R \cap \partial \beta$ for any (orbi-)polygon class $\beta$ for the potential.

Let $\widehat{XY}_+$ denote the minimal segment of $\mathbb{L}$ between $X$ and $Y$ lying on the upper hemisphere and $\widehat{XY}_-$ denote its reflection image, see Figure 5 (a). We analogously define $\widehat{YZ}_+$ and $\widehat{ZX}_+$. Define $i, j, k$ by $i := R \cap \widehat{XY}_+$, $j := R \cap \widehat{YZ}_+$ and $k := R \cap \widehat{ZX}_+$. Since $R$ is reflection invariant $R \cap \widehat{XY}_- = -R \cap \widehat{XY}_+$, therefore $R \cap \partial \beta$ equals

$$\pm \left( i \left( \beta \left( \widehat{XY}_+ \right) - \beta \left( \widehat{XY}_- \right) \right) + j \left( \beta \left( \widehat{YZ}_+ \right) - \beta \left( \widehat{YZ}_- \right) \right) + k \left( \beta \left( \widehat{ZX}_+ \right) - \beta \left( \widehat{ZX}_- \right) \right) \right)$$

where $\beta \left( \widehat{XY}_\pm \right)$ is the number of $\widehat{XY}_\pm$-segments in $\partial \beta$ (and analogously for $\widehat{YZ}_+$ and $\widehat{ZX}_+$). The plus or minus depends on $\beta$, having its boundary orientation match with that of $\mathbb{L}$ or its opposite. The former was called a positive polygon in [CHKL17], and the latter a negative polygon for this reason.

Let us assume that $\beta$ is positive, we claim that

$$\beta \left( \widehat{XY}_+ \right) - \beta \left( \widehat{XY}_- \right) = \beta(X) + \beta(Y) - \beta(Z).$$

We consider the loop $p_\beta$ in $\pi_1(\mathbb{L})$ obtained by attaching three consecutive minimal segments for each corner of $\partial \beta$. Namely, $p_\beta$ near a corner of $\beta$ is parameterized in such a way that we walk past the corner without turning and coming back to the same vertex, rather than jumping to another branch at the corner. See Figure 5 (b).

Since $p_\beta$ is an integer-multiple of $[\mathbb{L}]$, we have

$$p_\beta \left( \widehat{XY}_+ \right) - p_\beta \left( \widehat{XY}_- \right) = 0.$$

Thus $\beta \left( \widehat{XY}_+ \right) - \beta \left( \widehat{XY}_- \right)$ can be computed from the extra minimal segments that are attached to $\beta$ to obtain $p_\beta$. For instance, the extra segments attached to $X$-corner of $\beta$
Figure 5

Figure 5 consists of $\widehat{XY}_-, \widehat{YZ}_+$ and $\widehat{ZX}_-$, and hence removing these from $p_\beta$ increases (5.3) by 1. Likewise, for each $Y$-corner (resp. $Z$-corner) of $\beta$, removing extra segments from $p_\beta$ increases (resp. decreases) (5.3) by 1. Therefore we conclude the claim.

When $\beta$ is negative, formula (5.2) still holds but now with a minus sign on the right-hand side. There are analogous formulas for the arcs $YZ$ and $ZX$. Combining all of these with (5.1) we obtain

$$R \cap \partial \beta = (i - j + k)\beta(X) + (i + j - k)\beta(Y) + (-i + j + k)\beta(Z),$$

which gives the desired result. \hfill \Box

Proof of Proposition 5.2. Let us expand our bulk-deformed potential as

$$W_\tau^2(x, y, z) = \sum_{\beta} \exp(\tau_2 \cap \beta) c_{\beta, \tau} x^{\beta(X)} y^{\beta(Y)} z^{\beta(Z)} T^{\omega(\beta)}$$

where the sum is taken over the set of all (orbi-)polygon classes. Here, the potential also depends on other bulk parameters as the coefficient $c_{\beta, \tau}$ shows, but we wrote $W_\tau^2$ to highlight its dependence on $[\text{pt}]$ which is what we want to analyze. $W_\tau^2$ can be written analogously.
Using Lemma 5.3, one can compute \( W_{\tau_2}(x', y', z') \) as follows
\[
W_{\tau_2}(x', y', z') = \sum_\beta \exp(\tau_2 \cap \beta) c_{\beta, \tau_2} x^{\beta(X)} y^{\beta(Y)} z^{\beta(Z)} T^{\omega(\beta)}
\]
\[
= \sum_\beta \exp(\tau_2 \cap \beta) c_{\beta, \tau_2} x^{\beta(X)} y^{\beta(Y)} z^{\beta(Z)} T^{\omega(\beta)}
\]
\[
= \sum_\beta \exp(\tau_2 \cap \beta + R \cap \partial \beta) c_{\beta, \tau_2} x^{\beta(X)} y^{\beta(Y)} z^{\beta(Z)} T^{\omega(\beta)}
\]
\[
= \sum_\beta \exp(\tau_2 \cap \beta - Q \cap \beta) c_{\beta, \tau_2} x^{\beta(X)} y^{\beta(Y)} z^{\beta(Z)} T^{\omega(\beta)}
\]
\[
= \sum_\beta \exp(\tau'_2 \cap \beta) c_{\beta, \tau_2} x^{\beta(X)} y^{\beta(Y)} z^{\beta(Z)} T^{\omega(\beta)}
\]
which is the desired result. \( \square \)

6. The Kodaira–Spencer map

In this section we will define the Kodaira-Spencer map in our setting. This is a map \( KS_\tau : \text{QH}^*_\text{orb}(X, \tau) \to \text{Jac}(W_\tau) \) from the quantum cohomology of \( \mathbb{P}_1^{a,b,c} \) to the Jacobian ring of \( W_\tau \), constructed geometrically using \( J \)-holomorphic discs. The only previously known construction of \( KS_\tau \) map is the case of toric manifolds by Fukaya-Oh-Ohta-Ono [FOOO16b]. We will follow the line of their construction. Their construction heavily uses the \( T^n \)-action on the moduli space of holomorphic discs. In our construction, \( \mathbb{Z}/2 \)-action will play an analogous role. We show that \( KS_\tau \) map is well-defined (independent of the choice of cohomology representative) and \( KS \) is a ring homomorphism.

6.1. Definition of \( KS_\tau \) and well-definedness. We start by defining the Jacobian ring of \( W_\tau \). Recall that we use the convergent power series ring \( \Lambda \langle \langle x, y, z \rangle \rangle \) defined in Definition 4.1.

**Definition 6.1.** Consider \( P \in \Lambda \langle \langle x, y, z \rangle \rangle \). We define the Jacobian ring of \( P \) as the ring
\[
\text{Jac}(P) = \frac{\Lambda \langle \langle x, y, z \rangle \rangle}{< \partial_x P, \partial_y P, \partial_z P >}.
\]

We would like to point out that there is no need to take closure of the ideal, since in \( \Lambda \langle \langle x, y, z \rangle \rangle \) (as a Tate algebra) all ideals are closed, see [BGR84, Section 5.2.7].

**Remark 6.2.** We saw in Section 5 that \( W_\tau \) is well defined up to a change of variables. Since a change of variables induces a ring isomorphism on the corresponding Jacobian rings, we see that the Jacobian ring \( \text{Jac}(W_\tau) \) is well defined up to isomorphism.

Let \( w_0, \cdots, w_B \) be coordinates of \( \tau \) with respect to the basis \( \{ f_i \}_{i=0}^B \) (i.e. \( \tau = \sum_i w_i f_i \)). The potential function \( W_\tau \) can be regarded as a function \( W_\tau(w_1, \cdots, w_B, x, y, z) \) with \( w_i \in \Lambda_+ \). Regarding \( \tau \) as an element of \( H^*_\text{orb}(X, \Lambda) \), we identify the tangent space \( T_\tau H^*_\text{orb}(X, \Lambda) \) at \( \tau \) with \( \text{QH}^*_\text{orb}(X, \tau) \).
Definition 6.3. We define the Kodaira-Spencer map 
\( \text{KS}_\tau : \text{QH}^*_\text{orb}(X, \tau) \rightarrow \text{Jac}(W_\tau) \)
by the formula
\[
\text{KS}_\tau \left( \frac{\partial}{\partial w_i} \right) = \frac{\partial W_\tau}{\partial w_i}
\]

There is an ambiguity of the choice of representatives \( f_i \) in \( \text{QH}^*_\text{orb}(X, \tau) \). In our case, the twisted sectors as well as the fundamental cycle have canonical representatives. Hence we only need to consider the choice of \( \tau_2 \).

Lemma 6.4. The map \( \text{KS}_\tau \) is well-defined. In other words, if \( \tau_2 = \partial R \) then \( \text{KS}_\tau \left( \frac{\partial}{\partial \tau_2} \right) = 0 \) in \( \text{Jac}(W_\tau) \).

Proof. As in the proof of Proposition 5.2 we write
\[
W_\tau(x, y, z) = \sum_\beta \exp(\tau_2 \cap \beta) c_{\beta, \tau_2} x^{\beta(X)} y^{\beta(Y)} z^{\beta(Z)} T^{\omega(\beta)}.
\]
Then, by definition we have
\[
\text{KS}_\tau \left( \frac{\partial}{\partial \tau_2} \right) = \sum_\beta (\tau_2 \cap \beta) \exp(\tau_2 \cap \beta) c_{\beta, \tau_2} x^{\beta(X)} y^{\beta(Y)} z^{\beta(Z)} T^{\omega(\beta)}.
\]
By assumption, \( \tau_2 = \partial R \), then by Lemma 5.3, there are \( k_X, k_Y, k_Z \) such that \( Q \cap \beta = k_X \beta(X) + k_Y \beta(Y) + k_Z \beta(Z) \). Therefore
\[
\text{KS}_\tau \left( \frac{\partial}{\partial \tau_2} \right) = \sum_\beta (k_X \beta(X) + k_Y \beta(Y) + k_Z \beta(Z)) \exp(\tau_2 \cap \beta) c_{\beta, \tau_2} x^{\beta(X)} y^{\beta(Y)} z^{\beta(Z)} T^{\omega(\beta)}
\]
\[
= k_X x \frac{\partial W_\tau}{\partial x} + k_Y y \frac{\partial W_\tau}{\partial y} + k_Z z \frac{\partial W_\tau}{\partial z}.
\]
Therefore \( \text{KS}_\tau \left( \frac{\partial}{\partial \tau_2} \right) = 0 \) in \( \text{Jac}(W_\tau) \). \( \Box \)

Here is an alternative description of \( \text{KS}_i \), for \( i \) such that \( f_i \) is a twisted sector. The derivative \( \frac{\partial}{\partial w_i} \) has the effect of removing \( w_i \) in one of the \( \tau = \sum_i w_i f_i \) insertions on the disc. Therefore we have the following expression
\[
\text{KS}_\tau(f_i) = \sum_\beta, k \exp(\tau^2 \cap \beta) \sum_{l=0}^{\infty} \frac{T^{\beta \cap \omega}}{l!} q_{l+1, k, \beta}(f_i, \tau^l_{\text{tw}}, b, \ldots, b).
\]
We would like to have an analogous description for the cases of the fundamental and point classes in \( \mathbb{P}^1_{a,b,c} \). More concretely, let \( Q \) be a \( \mathbb{Z}/2 \)-invariant cycle in \( \mathbb{P}^1_{a,b,c} \), and define the moduli spaces
\[
\mathcal{M}^{\text{main}}_{k+1, l+1}(\beta, Q, \tau_{\text{tw}}, \gamma) \times \mathbb{P}^1_{a,b,c}.
\]
Using these spaces and their evaluation maps, analogously to (2.3) we can define maps \( q_{l+1, k, \beta}(Q, \tau^l_{\text{tw}}, -) \). Then we have the following statement.
Proposition 6.5. Let $Q$ be the cycle representing the fundamental cycle or the point class in $\mathbb{P}^1_{a,b,c}$. Then

$$KS_\tau(Q) = \sum_{\beta,k} \exp(\tau^2 p \cap \beta) \sum_{l=0}^{\infty} \frac{T^{\tau_l \omega}}{l!} q_{l+1,k,\beta}(Q, \tau_l, b, \ldots, b),$$

in $\text{Jac}(W_\tau)$.

This proposition essentially asserts that the $q$ maps are unital and satisfy a version of the divisor axiom in Gromov-Witten theory. Both properties are related to the compatibility of the Kuranishi structures (and perturbations) on the moduli spaces of discs with forgetting interior marked points. It turns out that ensuring this compatibility for all moduli spaces seems a rather complicated problem. We will avoid tackling that problem by taking homotopy between the usual Kuranishi structures on $M_{k+1,l+1}^{\text{main}}(\beta, Q, \tau_l, \gamma)$ and one constructed specifically to ensure this compatibility. Therefore the equality in the statement holds only in the Jacobian ring, but not necessarily at chain-level. We will postpone this proof to Appendix A.

6.2. Ring homomorphism. In this subsection we will prove the following

Theorem 6.6. The map $KS_\tau : QH^*_\text{orb}(X, \tau) \longrightarrow \text{Jac}(W_\tau)$ is a ring homomorphism.

This map is rather surprising in that it identifies complicated quantum multiplication with a standard multiplication of polynomials in Jacobian ring. The geometric idea behind this map is rather well-known. Namely, the closed-open maps in topological conformal field theory are ring homomorphisms from the closed theory to Hochschild cohomology of the open theory. They are explored in Seidel [Sei12], Biran-Cornea [BC13], Fukaya-Oh-Ohta-Ono [FOOO09]. A benefit of this construction is that Hochschild cohomology of the Fukaya category is a very heavy object to handle, whereas the construction of $KS_\tau$ map is rather direct and simple.

Proof. As before, we will follow the line of proof of [FOOO16b] Theorem 2.6.1 and we will use their notation freely to shorten our exposition. The proof is based on a cobordism argument. Consider two cohomology representatives $A, B$ in $QH^*_\text{orb}(X, \tau)$. Consider the forgetful map for the moduli space introduced in Section 2.4

$$\text{forget} : M_{k+1,l+2}^{\text{main}}(\beta, A \otimes B \otimes \tau_l, \gamma) \rightarrow M_{1,2}^{\text{main}}$$

which forgets maps and shrinks resulting unstable components if any, followed by the forgetful map $M_{k+1,l+2}^{\text{main}} \rightarrow M_{1,2}^{\text{main}}$ forgetting the boundary marked points, except the first one and forgetting the interior marked points except the first two. In Lemma 2.6.3 [FOOO16b], $M_{1,2}^{\text{main}}$ is shown to be topologically a disc with some stratification, so that the above $\text{forget}$ is a continuous and stratified smooth submersion.

The idea of proof is to consider a line segment in $M_{1,2}^{\text{main}}$ which connects two point strata of $D^2$. $\Sigma_0$ is a stratum where two interior marked points lie on a sphere bubble, and $\Sigma_{12}$ is a component where there are two disc bubbles each of which contains one of the interior marked points. We will see that integration over $\text{forget}^{-1}(\Sigma_0)$ and $\text{forget}^{-1}(\Sigma_{12})$ correspond to $KS_\tau(A \bullet, B), KS_\tau(A)KS_\tau(B)$ respectively and the pre-image of line segment will define the desired cobordism relation between them. There is a technical issue in that the map forget
is only a stratified submersion. We will explain below how to handle this issue following [FOOO16b].

We first have to construct Kuranishi structures and continuous family of multi-sections on (neighborhoods of) the spaces $\text{forget}^{-1}(\Sigma_0)$ and $\text{forget}^{-1}(\Sigma_{12})$. To describe the neighborhood near $\Sigma_0$, we consider the following moduli space. For $\alpha \in H_2(X, \mathbb{Z})$, let $\mathcal{M}_l(\alpha)$ the moduli space of stable maps from genus zero closed Riemann surface with $l$-marked points and of homology class $\alpha$.

\[ \mathcal{M}_{l+3}(\alpha, A \otimes B \otimes \tau_{lw}^{l1}) = \mathcal{M}_{l+3}(\alpha)_{(ev_1, \ldots, ev_{l+2})} \times X_{l+2}^{l1} (A \otimes B \otimes \tau_{lw}^{l1}) \]

Then $ev_{l+3}$ defines an evaluation map from the above moduli space to $X$. Define the moduli space $\mathcal{M}_{k+1,l_1,l_2}(\alpha, \beta; A, B, K)$ to be the fiber product

\[ (\mathcal{M}_{l+3}(\alpha, A \otimes B \otimes \tau_{lw}^{l1}) \times \mathcal{M}_{k+1,l_2+1}^{\text{main}}(\beta, \tau_{lw}^{l2})) \times (IX \times IX) K \]

for a chain $K$ in $IX \times IX$.

Let us consider the case that $K = \Delta'$ defined as

\[ \Delta' = \{(x, g), (x, g^{-1})\} \subset IX \times IX \]

for the inertia orbifold $IX$. The following is an analogue of Lemma 2.6.9 [FOOO16b], to which we refer readers for the proof.

**Lemma 6.7.** There exist a surjective map

\[ \text{Glue} : \bigcup_{\alpha \geq \beta = \beta} \bigcup_{l_1+l_2 = l} \mathcal{M}_{k+1,l_1,l_2}(\alpha, A, B, \Delta') \to \text{forget}^{-1}(\Sigma_0) \]

which defines an isomorphism outside codimension 2 strata as a space with Kuranishi structure.

The $\text{Glue}$ map gives a way to describe an element of $\text{forget}^{-1}(\Sigma_0)$ as a fiber product of sphere and disc moduli space. In this way, it corresponds to first taking the quantum multiplication and then taking the Kodaira-Spencer map. In the case that there are several sphere bubbles attached to a disc component (codimension higher than 2), there may be several ways of such description. Namely, $\text{Glue}$ map image may overlap in codimension two strata.

The more important issue is the compatibility of Kuranishi perturbations to be chosen, where there are differences between toric and our cases. Recall that the proof of [FOOO09] uses $T^n$-action on moduli space of holomorphic discs in an essential way. Because finite group symmetry is much easier to handle than $T^n$-symmetry, many of the arguments simplify in our $\mathbb{Z}/2$-symmetry case.

First note that finite group symmetry can be easily accommodated by Kuranishi structures. Hence, one may consider $\mathbb{Z}/2$-equivariant Kuranishi structures on moduli space of $J$-holomorphic discs or spheres.

Therefore, we may consider the following Kuranishi structure on $\text{forget}^{-1}(\Sigma_0)$. We choose a component-wise $\mathbb{Z}/2$-equivariant Kuranishi structure and continuous family of multi-sections (CF perturbations) on the moduli spaces $\mathcal{M}_{k+1,l_1+2}^{\text{main}}(\beta, A, B, \tau_{lw}^{l1})$, following [Fuk10]. Here component-wise means that the Kuranishi structure is compatible with the fiber product description of each of the strata of disc-sphere stratification.
Lemma 6.8. There exist component-wise \( \mathbb{Z}/2 \)-equivariant Kuranishi structures and CF perturbations on the moduli spaces \( \mathcal{M}_{k+1,l+2}^{\text{main}}(\beta, A, B, \tau_{l_1}^1) \). This Kuranishi structure and perturbations induce Kuranishi structures and perturbations on \( \text{forget}^{-1}(\Sigma_0) \). These Kuranishi structures and perturbations coincide with the ones induced by the Glue map (6.2) that coincide with each other on the overlapped part.

Remark 6.9. Recall that \( T^n \)-equivariant analogue of this lemma near \( \Sigma_{12} \) is given at Lemma 2.6.23 for toric cases. But this does not hold near \( \Sigma_0 \) for toric cases because the moduli space of \( J \)-holomorphic spheres cannot be made \( T^n \)-equivariant. Therefore, the construction for \( \Sigma_0 \) is much more involved than that of \( \Sigma_{12} \) in toric cases, but since we can impose \( \mathbb{Z}/2 \)-symmetry even for sphere moduli spaces, we can treat both cases in the same way.

Proof. We can choose \( \mathbb{Z}/2 \)-equivariant CF perturbation, following [Fuk10], on each sphere or disc component. On the overlapped part, two Kuranishi structures can be shown to be isomorphic (using the associativity of fiber products of Kuranishi structures). Hence perturbations can be chosen to inductively component-wise to have this compatibility because evaluation map \( ev_0 \) can be made submersive (see Lemma 3.1 [Fuk10]). \( \square \)

The multi-section in the neighborhood of \( \text{forget}^{-1}(\Sigma_{12}) \) can be constructed in a similar way. There exist a surjective map

\[ \text{Glue} : \bigcup_{(\beta)_{(0)}+\beta_{(1)}+\beta_{(2)}=\beta} \left((\mathcal{M}_{k_1+1,l_1+1}(\beta_{(1)}, A \otimes \tau_{l_1}^1) \times \mathcal{M}_{k_2+1,l_2+1}(\beta_{(2)}, B \otimes \tau_{l_2}^1)) \right) \]

\[ \times (ev_0, ev_0) \times (ev_1, ev_j) \mathcal{M}_{k_3+3,l_3}(\beta_{(0)}, \tau_{l_3}^1) \rightarrow \text{forget}^{-1}(\Sigma_{12}) \]

The relationship of Kuranishi structures under the \( \text{Glue} \) map is the same as that of Lemma 2.6.22 [FOOO16b] (we consider \( \mathbb{Z}/2 \)-equivariance instead of \( T^n \)-equivariance), and we can choose \( \mathbb{Z}/2 \)-equivariant CF perturbations as in Lemma 6.8 in this case also.

Now that we have Kuranishi structures and CF perturbations, we can define maps using these moduli spaces. For \( \Sigma \in \mathcal{M}_{1,2}^{\text{main}} \), consider \( \text{forget}^{-1}(\Sigma) \). Following (2.3), we use the evaluation map \( ev : \text{forget}^{-1}(\Sigma) \rightarrow \prod_{i=1}^k L(\alpha(i)) \) to define

\[ \tilde{Z}_{b,\tau}^\Sigma = \sum_{k,\beta,l} T_{b;\beta} \frac{\Gamma}{l!} (ev_0)_*(ev_1^*b \wedge \cdots \wedge ev_l^*b). \]

Lemma 6.10. We have

\[ \tilde{Z}_{b,\tau}^\Sigma = z_{b,\tau}^\Sigma \mathbf{1}_L, \]

for some \( z_{b,\tau}^\Sigma \in \Lambda_0 \)

Proof. The proof is the same as that of Proposition 3.1. Namely, the output given by immersed sectors vanishes by the reflection argument. Hence, the output is in \( \Omega^0(L) \), i.e. a function on \( L \). One can consider the boundary configuration of \( \text{forget}^{-1}(\Sigma) \) to conclude that the output is \( m_{\tau}^*b \)-closed. But \( m_{\tau}^*b \) on a function on \( L \) is given by a total derivative. Hence it is a constant function. \( \square \)

Now, pick \( \Sigma_{3,i} \) (resp. \( \Sigma_{4,i} \)) very close to \( \Sigma_{12} \) (resp. \( \Sigma_0 \)) in \( \mathcal{M}_{1,2}^{\text{main}} \) which converges to \( \Sigma_{12} \) (resp. \( \Sigma_0 \)) as \( i \rightarrow \infty \). We have the following analogue of Lemma 2.6.27 of [FOOO16b].
Proposition 6.11.

\[
\lim_{i \to \infty} (ev_0)_* (\text{forget}^{-1}(\Sigma_{3,i})) = (ev_0)_* (\text{forget}^{-1}(\Sigma_{12}))
\]

\[
\lim_{i \to \infty} (ev_0)_* (\text{forget}^{-1}(\Sigma_{4,i})) = (ev_0)_* (\text{forget}^{-1}(\Sigma_0))
\]

Proof. The proof in our case is easier than that of [FOOO16b], because of Lemma 6.8. Namely, in our case, Glue map is compatible with $\mathbb{Z}/2$-equivariant Kuranishi structures for both $\Sigma_0, \Sigma_{12}$. Therefore, we can just apply Lemma 4.6.5 [FOOO16b] which claims the $C^1$-convergence of perturbations as $i \to \infty$. We remark that this type of convergence was extensively studied in [FOOO16a].

We will now show that $Z_{\Sigma_{3,i}}^{b,\tau}$ and $Z_{\Sigma_{4,i}}^{b,\tau}$ are equal in the Jacobian ring. For this purpose we introduce an additional moduli space: choose a smooth curve $\psi$ on the open stratum of $\mathcal{M}_{1,2}$ connecting $\Sigma_{3,i}$ to $\Sigma_{4,i}$ and define $\mathcal{N}_{k+1,l+2}(\beta) = \text{forget}^{-1}(\psi) \subset \mathcal{M}_{k+1,l+2}^{\text{main}}(\beta, A \otimes B \otimes \tau_{\text{tw}}^0, \alpha)$. Since \text{forget} is a weakly smooth submersion when restricted to the open stratum, the Kuranishi structure defined in Lemma 6.8 induces a Kuranishi structure on $\mathcal{N}_{k+1,l+2}(\beta)$. Using the evaluation map $ev : \mathcal{N}_{k+1,l+2}(\beta) \to \prod_{i=1}^k L(\alpha(i))$, as before, we define

\[
\tilde{Y}_{b,\tau} = \sum_{k,\beta,l} T_{\omega_{\gamma_{\beta}}} \frac{1}{l!} (ev_0)_* (ev_i^* b \wedge \cdots \wedge ev_k^* b).
\]

By the homological perturbation lemma there is an $A_{\infty}$-quasi-isomorphism from $\mathcal{F}(L)$ to its canonical model $H_{\text{can}}^* (L)$. Denote by $\Pi^{b,\tau}$ the arity one (or linear) component of this quasi-isomorphism, by definition we have $\Pi^{b,\tau} \circ m^{\tau,b}_1 = (m^{\tau,b}_1)_{\text{can}} \circ \Pi^{b,\tau}$ and $\Pi^{b,\tau}(1_L) = 1_L$. We define

\[
Y_{b,\tau} := \Pi^{b,\tau}(\tilde{Y}_{b,\tau}).
\]

The following lemma can be proved in the same way as Proposition 4.10 for the maps of $m^{\tau,b}_k$.

Lemma 6.12. $\tilde{Y}_{b,\tau}$ (resp. $Y_{b,\tau}$) is convergent series, more precisely it is as an element of $\mathcal{F}(L, \Lambda \langle \langle x, y, z \rangle \rangle)$ (resp. $H_{\text{can}}^* (L, \Lambda \langle \langle x, y, z \rangle \rangle)$).

Proposition 6.13. We have the following relation in $H_{\text{can}}^* (L)$:

\[
(m^{\tau,b}_1)_{\text{can}} (Y_{b,\tau}) = Z_{\Sigma_{3,i}}^{b,\tau} - Z_{\Sigma_{4,i}}^{b,\tau}.
\]

Therefore $Z_{b,\tau}^{\Sigma_{3,i}} = Z_{b,\tau}^{\Sigma_{4,i}}$ in the Jacobian ring $\text{Jac}(W_{\tau})$.

Proof. First note that the second statement follows from the first together with the fact, proved in Proposition 4.14 that $\text{Im} (m^{\tau,b}_1)_{\text{can}}$ is contained in the Jacobian ideal. Second note that the first statement is equivalent to $m^{\tau,b}_1 (\tilde{Y}_{b,\tau}) = Z_{\Sigma_{3,i}}^{b,\tau} - Z_{\Sigma_{4,i}}^{b,\tau}$, by definition of $\Pi^{b,\tau}$.

In order to prove this relation we describe the boundary of $\mathcal{N}_{k+1,l+2}(\beta)$. As a space with Kuranishi structure the boundary of $\mathcal{N}_{k+1,l+2}(\beta)$ is the union of $\text{forget}^{-1}(\Sigma_{3,i})$, $\text{forget}^{-1}(\Sigma_{4,i})$ and the fiber products

\[
N_{k+1,l+2}(\beta_1)_{ev_0} \times_{ev_i} \mathcal{M}_{k_2+l_2}(\beta_2, \tau_{\text{tw}}^l) \quad \text{and} \quad \mathcal{M}_{k_1+l_1}(\beta_1, \tau_{\text{tw}}^l)_{ev_0} \times_{ev_i} N_{k_1+l_1}(\beta_1),
\]

where $\beta_1 + \beta_2 = \beta$, $k_1 + k_2 = k$, $l_1 + l_2 = l$ and $i \in \{1, \ldots, k+1\}$.
Now, using Stokes theorem [FOOO11, Lemma 12.13] and summing over all $\beta, k, l$ (like in Lemma 2.6.36 [FOOO16b]), we see that the first product in (6.3) gives $m_{\tau, b}^{1} (\tilde{Y}_{b, \tau})$. The second product in (6.3) contributes as zero since $m_{b, \tau}^{0}$ is a multiple of the unit and the perturbation in $N_{k, l}(\beta)$ is compatible with forgetting boundary marked points. Finally, by definition, $\text{forget}^{-1}(\Sigma_{3, i})$ and $\text{forget}^{-1}(\Sigma_{4, i})$ give $\tilde{Z}_{b, \tau}^{\Sigma_{3, i}}$ and $\tilde{Z}_{b, \tau}^{\Sigma_{4, i}}$ respectively. Now the desired relation follows from the Stokes theorem. □

Now we need to relate cohomological intersection product and geometric intersection. Let $\{f_{i}\}_{i=1}^{m}$ be basis of $H^{*}_{\text{orb}}(X)$, $g_{ij} = \langle f_{i}, f_{j} \rangle_{PD}$ and $(g^{ij})$ be its inverse matrix. On this basis, we write $A \bullet_{\tau} B = \sum_{i} c_{i} f_{i}$. Let $R$ be a chain in $IX \times IX$ such that

$$\partial R = \Delta' - \sum_{ij} g^{ij} f_{i} \times f_{j},$$

and consider the moduli space $M_{k+1, l_{1}, l_{2}}(\alpha, \beta; A, B, R)$ defined above. Using the boundary evaluation maps on these moduli spaces we define

$$\tilde{\Xi}(A, B, K, b) = \sum_{\alpha, \beta, l_{1}, l_{2}, k} T_{\omega^\alpha, \omega^\beta} (l_{1} + l_{2})! (ev_{0})_{*}(ev_{1}^{*} b \wedge \cdots \wedge ev_{k}^{*} b).$$

The following lemma can be proved exactly as Lemma 2.6.36 in [FOOO16b].

Lemma 6.14.

$$\sum_{i} c_{i} \text{KS}_{\tau}(f_{i}) 1_{L} - Z_{b, \tau}^{\Sigma_{0}} = m_{1, b}^{\tau} (\tilde{\Xi}(A, B, R, b))$$

Please note that here we are using the description for $\text{KS}_{\tau}$ provided by Proposition 6.5.

Proposition 6.15. $\text{KS}(A \bullet_{\tau} B)$ equals $Z_{b, \tau}^{\Sigma_{0}}$ modulo the Jacobian ideal.

Proof. As before we can show that $\tilde{\Xi}(A, B, R, b)$ is convergent. Then we apply $\Pi_{b, \tau}$ to the equation in the previous lemma to conclude that $\text{KS}_{\tau}(A \bullet_{\tau} B) 1_{L}$ and $Z_{b, \tau}^{\Sigma_{0}} 1_{L}$ differ by an element in the image of $(m_{1, b}^{\tau})_{\text{can}}$. The result now follows from Proposition 4.14. □

Proposition 6.16 (c.f. Lemma 2.6.29 [FOOO16b]). We have

$$\text{KS}_{\tau}(A) \cdot \text{KS}_{\tau}(B) = Z_{b, \tau}^{\Sigma_{12}}.$$

This proposition is completely analogous to Lemma 2.6.29 [FOOO16b]. Now combining Propositions 6.16, 6.15, 6.13 and 6.11 we obtain the proof of Theorem 6.6.

7. $\text{KS}_{\tau}$ IS AN ISOMORPHISM

7.1. Surjectivity. In this subsection we show that $\text{KS}_{\tau}$ is surjective. We start with computations on lower energy contributions.
Lemma 7.1. There is \( \lambda > 0 \) (depending on \( \tau \)) such that:

\[
\begin{align*}
KS_\tau \left( \left\lfloor \frac{1}{a} \right\rfloor \cdot \tau \right) &= x^i \pmod{T^\lambda}, \\
KS_\tau \left( \left\lfloor \frac{1}{b} \right\rfloor \cdot \tau \right) &= y^j \pmod{T^\lambda}, \\
KS_\tau \left( \left\lfloor \frac{1}{c} \right\rfloor \cdot \tau \right) &= z^k \pmod{T^\lambda},
\end{align*}
\]

(7.1)

\[KS_\tau(8pt) = -T^{-8xyz + 3ax^a + 3bx^b + 3cz^c} \pmod{T^\lambda},\]

where \( 1 \leq i < a, 1 \leq j < b \) and \( 1 \leq k < c \).

Proof. The first order term follows from direct computation. For example, \( \frac{1}{a} \)-slice of the disc contributing to \( x^a \) in \( W_\tau \) produces \( x \) in the first equation (see Corollary 4.3 and the preceding discussion for the precise description for these orbi-discs). Thus it suffices to show that all the higher order terms in the above equations have strictly positive powers in \( T \). This directly follows from Lemma 4.4. \( \square \)

Lemma 7.1 together with the fact that \( KS_\tau \) is a ring map, is enough to establish surjectivity.

**Proposition 7.2.** The map \( KS_\tau \) is surjective.

**Proof.** Since any element in \( \text{Jac}(W_\tau) \) can be written as \( T^{-\epsilon}R \) such that \( \epsilon > 0 \) and \( R \) only has positive powers in \( T \), it is enough to prove that any \( R \in \text{Jac}(W_\tau) \) with \( \Lambda_0 \)-coefficients is in the image of \( KS_\tau \). Let \( \lambda \) be the minimum of powers of \( T \) appearing in the higher order terms in (7.1). We claim that for any such \( R \) there exists \( \rho \) with

\[R - KS_\tau(\rho) = T^\lambda U\]

where \( U \) is also an element in \( \text{Jac}(W_\tau) \) with \( \Lambda_0 \)-coefficients only. To see this, write \( R \) as

\[R = \sum_{l=1}^{N} a_l T^{\lambda_l} x^i y^j z^k + T^\lambda \tilde{U}\]

where \( \tilde{U} \) has \( T \) with positive powers (either of summands could be zero even for nonzero \( R \)). We take \( \rho \) to be as follows

\[\rho = \sum_{l=1}^{N} a_l T^{\lambda_l} \left( \frac{1}{a} \right)^{\cdot \cdot \cdot i} \left( \frac{1}{b} \right)^{\cdot \cdot \cdot j} \left( \frac{1}{c} \right)^{\cdot \cdot \cdot k} \cdot \tau\]

Using the fact that \( KS_\tau \) is a ring homomorphism, Lemma 7.1 implies that the valuation of \( R - KS_\tau(\rho) \) is no less than \( \lambda \).

We next use this inductively to prove the surjectivity. For \( R \in \text{Jac}(W_\tau) \) only with \( \Lambda_0 \)-coefficients, there exists \( \rho_1 \) such that

\[R - KS_\tau(\rho_1) = T^\lambda R_1\]

Applying the same to \( R_1 \), we get \( \rho_2 \) such that

\[R - KS_\tau(\rho_1 + T^\lambda \rho_2) = T^\lambda (R_1 - KS_\tau(\rho_2)) = T^{2\lambda} R_3\]
Inductively, one sees that \( \sum_i T^{(i-1)\lambda} \rho_i \) maps to \( R \) under \( KS_\tau \).

### 7.2. Jacobian ring of the leading order potential.

From Corollary 4.11, we can write

\[
T^8 W_\tau = \mathcal{W}_{\text{lead}} + W_+,
\]

where

\[
\mathcal{W}_{\text{lead}} = -xyz + T^8(x^a + y^b + z^c)
\]

and \( W_+ = T^8 \mathcal{W}_{\text{high}} \). In particular, we have \( \text{val}(W_+) = \lambda_0 > 8 \) for some \( \lambda_0 \). The coefficient of \( xyz \) in \( \mathcal{W}_{\text{lead}} \) depends on the choice of a representative of \([\text{pt}]\), but we will only consider the case of (7.2) in this section to make our exposition simpler. In general, one can have

\[
\mathcal{W}_{\text{lead}} = -\tilde{\xi}xyz + T^8(x^a + y^b + z^c)
\]

for some \( \tilde{\xi} \) with \( \text{val}(\tilde{\xi}) = 0 \) (see (4.6) where \( \tilde{\xi} = T^8 \xi \)), but the argument below will still apply for any \( \tilde{\xi} \) without much change, since what essentially matters is its valuation.

We set the following notation

\[
g_1 = \partial_x \mathcal{W}_{\text{lead}}, \ g_2 = \partial_y \mathcal{W}_{\text{lead}}, \ g_3 = \partial_z \mathcal{W}_{\text{lead}}.
\]

Moreover let \( \gamma_1, \ldots, \gamma_N \) denote the following set of elements in \( \Lambda\langle\langle x, y, z \rangle\rangle \):

\[
1, x, x^2, \ldots, x^{a-1}, y, \ldots, y^{b-1}, z, \ldots, z^{c-1}, xyz.
\]

**Theorem 7.3.** Let \( \mathcal{A}_0 \) be the Jacobian ring of \( \mathcal{W}_{\text{lead}} \), that is \( \mathcal{A}_0 := \Lambda\langle\langle x, y, z \rangle\rangle / \langle g_1, g_2, g_3 \rangle \).

We have the following:

1. \( \{\gamma_1, \ldots, \gamma_N\} \) forms a linear basis of \( \mathcal{A}_0 \) over \( \Lambda \).
2. Any \( \rho \in \Lambda\langle\langle x, y, z \rangle\rangle \) with \( \text{val}(\rho) \geq 0 \) can be written as

\[
\rho = \sum_{i=1}^N c_i \gamma_i + \sum_{j=1}^3 t_j g_j
\]

where \( c_i \in \Lambda \) with \( \text{val}(c_i) \geq -8 \) and \( t_j \in \Lambda\langle\langle x, y, z \rangle\rangle \) with \( \text{val}(t_j) \geq -8 \).

Note that the ideal \( \langle g_1, g_2, g_3 \rangle \) is closed since \( \Lambda\langle\langle x, y, z \rangle\rangle \) is a Tate algebra [BGR84, Section 5.2.7], as are other ideals appearing in earlier sections.

It requires a clever usage of relations in the Jacobian ring to see that the condition (2) of Theorem 7.3 holds true, and the argument varies for different types of \((a, b, c)\). We will provide a detailed proof in Appendix B. It implies that the monomials in (7.3) form a generating set for the \( \Lambda \)-vector space \( \mathcal{A}_0 \). Thus, in order to complete the proof of Theorem 7.3, it only remains to show that they are linearly independent.

**Proposition 7.4.** The rank of \( \mathcal{A}_0 \) is \( a + b + c - 1 \).

**Proof.** We need to show that

\[
1, x, x^2, \ldots, x^{a-1}, y, \ldots, y^{b-1}, z, \ldots, z^{c-1}, xyz.
\]

\footnote{An analogous statement over \( \mathbb{C} \) is well-known, but here, we additionally need a careful estimate on the valuation to prove this over \( \Lambda \).}
are linearly independent in $A_0 = \Lambda\langle\langle x, y, z \rangle\rangle/\langle\langle g_1, g_2, g_3 \rangle\rangle$. As in (7.3), we write them as $\gamma_1, \ldots, \gamma_N$. Suppose we have the following equation in $\Lambda\langle\langle x, y, z \rangle\rangle$:

\begin{equation}
\sum_{i=1}^{N} c_i \gamma_i = f_1 g_1 + f_2 g_2 + f_3 g_3 = 0
\end{equation}

where $c_i \in \Lambda$, $f_j \in \Lambda\langle\langle x, y, z \rangle\rangle$. It is enough to show that $c_1 = \cdots = c_N = 0$. From the expression of $g_1, g_2, g_3$ it is easy to see that $f_1 g_i$ cannot have terms like

$$1, x, \ldots, x^{a-2}, y, \ldots, y^{b-2}, z, \ldots, z^{c-2}.$$ 

Thus we find that the coefficients on these monomials should vanish, and the equation (7.5) can be written as

\begin{equation}
c_x x^{a-1} + c_y y^{b-1} + c_z z^{c-1} + c_N x y z + f_1 g_1 + f_2 g_2 + f_3 g_3 = 0.
\end{equation}

If $c_x \in \Lambda$ is non-zero, then $f_1$ must have a nontrivial constant term $f_1^0 \in \Lambda$ in order to cancel $c_x x^{a-1}$ making use of $f_1^0 g_1 = f_1^0 (y z + T^8 x^{a-1})$. However, the monomial $f_1^0 y z$ cannot appear in other expressions of (7.6). Thus $f_1^0 = 0$, and hence $c_x = 0$. In the same way $c_y = c_z = 0$, and the equation (7.6) can be written as

\begin{equation}
c_N x y z + f_1 g_1 + f_2 g_2 + f_3 g_3 = 0.
\end{equation}

If $c_N \neq 0$, then one of $f_1, f_2, f_3$ should have a term of monomial $x, y, z$ respectively. Suppose $f_1$ has a monomial $f_1^1 x$. Then, $f_1^1 x^{a}$ cannot appear in other expressions of (7.7) and thus $f_1^1 = 0$. Similarly $f_2$ and $f_3$ cannot have monomials in $y$ and $z$ respectively, which implies $c_N = 0$. Therefore all the coefficients $c_1, \ldots, c_N$ must vanish, as desired. $\square$

This completes the proof of Theorem 7.3.

7.3. Deforming $\text{Jac}(W_\tau)$. In this subsection we will construct a flat family of rings interpolating between $\text{Jac}(W_{\text{lead}})$ and $\text{Jac}(W_\tau)$.

Recall $T^8 W_\tau = W_{\text{lead}} + W_\tau$, for some $W_\tau$ with $\text{val}(W_\tau) = \lambda_0 > 8$. We define

$$W(s) = W_{\text{lead}} + s W_\tau \in \Lambda\langle\langle s, x, y, z \rangle\rangle,$$

and denote

$$f_1 = \partial_x W(s), \ f_2 = \partial_y W(s), \ f_3 = \partial_z W(s).$$

Note that $f_i = g_i + sh_i$, for some $h_i \in \Lambda\langle\langle x, y, z \rangle\rangle$ with $\text{val}(h_i) \geq \lambda_0 > 8$.

**Proposition 7.5.** Let $A = \Lambda\langle\langle s, x, y, z \rangle\rangle/\langle f_1, f_2, f_3 \rangle$. Then $A$ is a finitely generated $\Lambda\langle\langle s \rangle\rangle$-module.

**Proof.** We will show that (7.3) forms a generating set of $A$. It is enough to show that any convergent series in $x, y, z$ with non-negative valuation is in the $\Lambda\langle\langle s \rangle\rangle$-span of (7.3). Let $\rho$ be such series with $\text{val}(\rho) \geq 0$, by Theorem 7.3 we have

$$\rho = \sum_{i=1}^{N} c_i \gamma_i + \sum_{j=1}^{n} t_j g_j,$$

with $c_i \in \Lambda$, $t_j \in \Lambda\langle\langle x, y, z \rangle\rangle$ and $\text{val}(c_i), \text{val}(t_j) \geq -8$. Rearranging we get
\[
\rho = \sum_{i=1}^{N} c_i \gamma_i + \sum_{j=1}^{n} t_j(f_j - sh_j)
\]

\[
= \sum_{i=1}^{N} c_i \gamma_i + \sum_{j=1}^{n} t_j f_j - T^{\lambda_0 - 8}s \sum_{j=1}^{n} t'_j h'_j
\]

with \(\text{val}(t'_j h'_j) \geq 0\). By setting \(\rho_1 = \sum_{j=1}^{n} t'_j h'_j\), we can repeat the argument to prove that

\[
\rho_1 = \sum_{i=1}^{N} c_i \gamma_i + \sum_{j=1}^{n} t_j f_j - T^{\lambda_0 - 8}s \rho_2,
\]

for some \(\rho_2 \in \Lambda\langle s, x, y, z \rangle\) with non-negative valuation. Combining the two we obtain

\[
\rho = \sum_{i=1}^{N} (c_i - T^{\lambda_0 - 8}s c_i) \gamma_i + \sum_{j=1}^{n} (t_j - T^{\lambda_0 - 8}s t_j f_j + T^{2(\lambda_0 - 8)}s \rho_2.
\]

Note that this process increases the valuation of the error term (each time by \(\lambda_0 - 8 > 0\)), and hence using induction and taking the limit we obtain

\[
\rho = \sum_{i=1}^{N} \tilde{c}_i \gamma_i + \sum_{j=1}^{n} \tilde{t}_j f_j
\]

with \(\tilde{c}_i \in \Lambda\langle s \rangle\) and \(\tilde{t}_j \in \Lambda\langle s, x, y, z \rangle\), which implies the result. \(\square\)

**Proposition 7.6.** \(A\) is a flat \(\Lambda\langle s \rangle\)-module.

**Proof.** Flatness of \(A\) is equivalent to flatness of the localizations \(\Lambda\langle s \rangle_n \to A_m\) for all maximal ideals \(m\) of \(A\), with \(n = m \cap A\), (see [Mat80, Section 3.1]). Note that, since \(A\) is a PID, \(n = \langle s - s_0 \rangle\) for some \(s_0 \in \Lambda_0\).

Now, since \(\Lambda\langle s \rangle_n\) is a regular local ring of dimension 1, \(A_m\) is flat over it if and only if \(s - s_0\) is not a zero-divisor in \(A_m\), by Lemma 10.127.2 in [Aut20].

By the previous proposition \(rk_{\Lambda}(A/n) < \infty\), which implies that \(\dim(A/n) = 0\). Therefore

\[
\dim_{\Lambda\langle s, x, y, z \rangle_{m'}}(f_1, f_2, f_3, s - s_0) = 0,
\]

where \(m'\) is the ideal of \(\Lambda\langle s, x, y, z \rangle\) corresponding to \(m\). Which implies that \(f_1, f_2, f_3, s - s_0\) is a system of parameters of \(\Lambda\langle s, x, y, z \rangle_{m'}\). Since this is a regular local ring, Theorem 31 in [Mat80], shows that \(f_1, f_2, f_3, s - s_0\) is a regular sequence in \(\Lambda\langle s, x, y, z \rangle_{m'}\). This immediately implies that \(s - s_0\) is not a zero-divisor in \(A_m\), which gives the desired result. \(\square\)

**Corollary 7.7.** \(A\) is a free, finite dimensional \(\Lambda\langle s \rangle\)-module.

**Proof.** This is an immediate consequence of the two previous propositions, since \(\Lambda\langle s \rangle\) is a PID. \(\square\)
Remark 7.8. It follows from our argument that (7.3) forms a basis of $A$, since this is the case for $s = 0$. In addition to this, it follows from the proof of Proposition 7.5, that any $\rho \in A$ with non-negative valuation, can be written as

$$\rho = \sum_{i=1}^{N} c_i \gamma_i + \sum_{j=1}^{3} t_j f_j$$

with $c_i \in \Lambda$, $\text{val}(c_i) \geq -8$ and $t_j \in \Lambda(\langle s, x, y, z \rangle)$, $\text{val}(t_j) \geq -8$.

7.4. Injectivity. Now we are ready to prove injectivity of the Kodaira-Spencer map.

Proposition 7.9. The Kodaira-Spencer map $KS_{\tau} : QH^*_{orb}(X, \tau) \to \text{Jac}(W_{\tau})$ is injective, and hence a ring isomorphism.

Proof. We have already established that $KS_{\tau}$ is a surjective ring homomorphism, so we need only to compare the ranks of the quantum cohomology and the Jacobian ring, to prove injectivity. It follows from the definition that $H^*_{orb}(\mathbb{P}^1_{a,b,c})$ has rank $a + b + c - 1$.

As a consequence of Corollary 7.7, we have

$$\dim_{\Lambda} \text{Jac}(W_{\text{lead}}) = \dim_{\Lambda} A/\langle s \rangle = \dim_{\Lambda} A/\langle s - 1 \rangle = \dim_{\Lambda} \text{Jac}(W_{\tau}).$$

We know, by Proposition 7.4, that $\text{Jac}(W_{\text{lead}})$ has rank $a + b + c - 1$, therefore $\text{Jac}(W_{\tau})$ also has rank $a + b + c - 1$, which implies the result. \qed

Remark 7.10. In fact we have shown that (7.3) forms a basis of $\text{Jac}(W_{\tau})$. Moreover, as explained in Remark 7.8, it follows that any $\rho \in \text{Jac}(W_{\tau})$ with non-negative valuation, can be written as

$$\rho = \sum_{i=1}^{N} c_i \gamma_i + t_1 \partial_x W_{\tau} + t_2 \partial_y W_{\tau} + t_3 \partial_z W_{\tau}.$$ 

with $c_i \in \Lambda$, $\text{val}(c_i) \geq -8$ and $t_j \in \Lambda(\langle x, y, z \rangle)$, $\text{val}(t_j) \geq 0$.

8. Calculations

8.1. Euler vector field. Let $\chi$ be the Euler characteristic of $\mathbb{P}^1_{a,b,c}$. We have $c_1(\mathbb{P}^1_{a,b,c}) = \chi[\text{pt}]$.

Theorem 8.1. Under the Kodaira-Spencer map $KS_{\tau} : QH^*_{orb}(X, \tau) \to \text{Jac}(W_{\tau})$,

$$KS_{\tau} \left( \chi[\text{pt}] + \sum_{k} (1 - \iota_k) \tau_k T_k \right) = [W_{\tau}]$$

where $T_k$ form a basis of twisted sectors and $\tau_k$ are the corresponding coordinates.

Proof. Recall that $KS_{\tau}(\text{pt})$ is defined by counting discs with one interior point passing through pt and one boundary output point to the unit. Since the total area of $\mathbb{P}^1_{a,b,c}$ equals to $8A$, the image of pt equals to

$$\frac{1}{8A} T \cdot \frac{\partial W_{\tau}(\tilde{x}, \tilde{y}, \tilde{z})}{\partial T}$$

written in terms of the geometric variables $(\tilde{x}, \tilde{y}, \tilde{z})$ corresponding to the immersed generators.
Let $T^m A \left( \prod_j \frac{\partial}{\partial T_j} \right) \bar{x}^{n_1} y^{n_2} z^{n_3}$ be a term in $W_\tau(\bar{x}, \bar{y}, \bar{z})$. Using Proposition 4.6,

$$
\left( \chi \cdot \frac{1}{8A} \cdot T \frac{\partial}{\partial T} + \sum_{j} (1 - \tau_j j \frac{\partial}{\partial \tau_j}) \right) \cdot T^m A \left( \prod_j \frac{\partial}{\partial T_j} \right) \bar{x}^{n_1} y^{n_2} z^{n_3}
$$

$$
= \left( \frac{m \chi}{8} + \sum_{j} \tau_j j (1 - \tau_j) \right) T^m A \left( \prod_j \frac{\partial}{\partial T_j} \right) \bar{x}^{n_1} y^{n_2} z^{n_3}
$$

$$
= \left( 1 - \frac{n_1}{a} - \frac{n_2}{b} - \frac{n_3}{c} + \frac{3 \chi}{8} \cdot (n_1 + n_2 + n_3) \right) T^m A \left( \prod_j \frac{\partial}{\partial T_j} \right) \bar{x}^{n_1} y^{n_2} z^{n_3}
$$

$$
= \left( \left( \frac{\chi}{8} - \frac{1}{a} \right) \frac{\partial}{\partial x} + \left( \frac{\chi}{8} - \frac{1}{b} \right) \frac{\partial}{\partial y} + \left( \frac{\chi}{8} - \frac{1}{c} \right) \frac{\partial}{\partial z} \right) \cdot T^m A \left( \prod_j \frac{\partial}{\partial T_j} \right) \bar{x}^{n_1} y^{n_2} z^{n_3}
$$

$$
+ T^m A \left( \prod_j \frac{\partial}{\partial T_j} \right) \bar{x}^{n_1} y^{n_2} z^{n_3}
$$

Hence

$$
\left( \chi \cdot \frac{1}{8A} \cdot T \frac{\partial}{\partial T} + \sum_{j} (1 - \tau_j j \frac{\partial}{\partial \tau_j}) \right) \cdot W_\tau(\bar{x}, \bar{y}, \bar{z})
$$

$$
= W_\tau(\bar{x}, \bar{y}, \bar{z}) + \left( \left( \frac{\chi}{8} - \frac{1}{a} \right) \frac{\partial}{\partial x} + \left( \frac{\chi}{8} - \frac{1}{b} \right) \frac{\partial}{\partial y} + \left( \frac{\chi}{8} - \frac{1}{c} \right) \frac{\partial}{\partial z} \right) \cdot W_\tau(\bar{x}, \bar{y}, \bar{z}).
$$

Changing back to the variables $x = T^3 \bar{x}$, $y = T^3 \bar{y}$, $z = T^3 \bar{z}$, the left hand side is

$$
\text{KS}_\tau \left( \chi [p] + \sum_k (1 - \tau_k) \tau_k T_k \right).
$$

The right hand side equals to $W_\tau(x, y, z) + \left( \left( \frac{\chi}{8} - \frac{1}{a} \right) \frac{\partial}{\partial x} + \left( \frac{\chi}{8} - \frac{1}{b} \right) \frac{\partial}{\partial y} + \left( \frac{\chi}{8} - \frac{1}{c} \right) \frac{\partial}{\partial z} \right) W_\tau(x, y, z)$ which is in the same class of $W_\tau(x, y, z)$ in the Jacobian ideal.

8.2. Versality of the potential. The goal of this section is to prove the following statement, which describes the power series (up to a coordinate change) that can appear as the bulk deformed potential $W_\tau$.

**Theorem 8.2.** Consider $P \in \Lambda \langle x, y, z \rangle$ with $\text{val}(P - W_{\text{lead}}) > 0$. Then there exist $\tau' \in H^*_{\text{orb}}(\mathbb{P}^1_{a,b,c}, \Lambda_0)$ and a coordinate change $(x', y', z')$ such that

$$
P(x', y', z') = W_\tau'.
$$

In order to prove this proposition we first need three lemmas.

**Lemma 8.3** (Refined surjectivity). For any $P \in \Lambda \langle x, y, z \rangle$ with $\text{val}(P) \geq 0$, there is $\rho \in \text{QH}^*_{\text{orb}}(X, \tau)$ with $\text{val}(\rho) \geq 0$ such that

$$
\text{KS}_\tau(\rho) = P + t_1 \partial_x W_\tau + t_2 \partial_y W_\tau + t_3 \partial_z W_\tau,
$$

for $t_1, t_2, t_3 \in \Lambda \langle x, y, z \rangle$ with $\text{val}(t_1, t_2, t_3) \geq 0$. 
Lemma 8.4. Let \( P \in \Lambda\langle x, y, z \rangle \) with \( \text{val}(P - W_{\text{lead}}) \geq 0 \) and define
\[
G(s, x, y, z, \tau) := W_z(x, y, z) + s(P(x, y, z) - W_{\text{lead}}) \in \Lambda\langle s, x, y, z, \tau \rangle.
\]
There exist \( c_i \in \Lambda(\langle s, \tau \rangle) \), and \( t_1, t_2, t_3 \in \Lambda(\langle s, x, y, z, \tau \rangle) \) with \( \text{val}(c_i), \text{val}(t_j) > 0 \) such that
\[
\frac{\partial G}{\partial s} = \sum_i c_i \frac{\partial G}{\partial \tau_i} + t_1 \frac{\partial G}{\partial x} + t_2 \frac{\partial G}{\partial y} + t_3 \frac{\partial G}{\partial z}.
\]

Proof. We will prove the following more general statement: given \( Q \in \Lambda(\langle x, y, z, \tau, s \rangle) \) with \( \text{val}(Q) \geq 0 \) there are \( c_i \) and \( t_j \) as in the statement with valuation greater or equal than zero such that
\[
Q = \sum_i c_i \frac{\partial G}{\partial \tau_i} + t_1 \frac{\partial G}{\partial x} + t_2 \frac{\partial G}{\partial y} + t_3 \frac{\partial G}{\partial z}.
\]
Which easily implies the lemma. It is enough to consider the case of \( Q \in \Lambda(\langle x, y, z, \tau \rangle) \). We proceed in two steps:

Step 1: At \( s = 0 \), \( \frac{\partial G}{\partial \tau_i} = KS_\tau(c_i) \) and \( \partial_x G = \partial_x W_\tau, \partial_y G = \partial_y W_\tau, \partial_z G = \partial_z W_\tau \). Applying Lemma 8.3 to \( Q \) we obtain \( t_j \) and \( \rho = \sum c_i e_i \) satisfying Equation (8.2).

Step 2: (Similar to Proposition 7.5) By Step 1 , there are \( c_i^{(0)} \) and \( t_j^{(0)} \) satisfying Equation (8.2). By definition, \( \partial_x G = \partial_x W_\tau + sT^\alpha f_1 \) for some \( \alpha > 0 \) and \( f_1 \) with valuation greater or equal than zero. Similarly for \( y \) and \( z \). Hence
\[
Q = \sum c_i^{(0)} \frac{\partial G}{\partial \tau_i} + t_1^{(0)} \frac{\partial G}{\partial x} + t_2^{(0)} \frac{\partial G}{\partial y} + t_3^{(0)} \frac{\partial G}{\partial z} - sT^\alpha \sum t_j^{(0)} f_j.
\]

Next, we apply Step 1 to \( Q^{(1)} := \sum t_j^{(0)} f_j \) gives \( c_i^{(1)} \) and \( t_j^{(1)} \) which allows us to rewrite \( Q \) as
\[
\sum_i (c_i^{(0)} - sT^\alpha c_i^{(1)}) \frac{\partial G}{\partial \tau_i} + (t_1^{(0)} - sT^\alpha t_1^{(1)}) \frac{\partial G}{\partial x} + (t_2^{(0)} - sT^\alpha t_2^{(1)}) \frac{\partial G}{\partial y} + (t_3^{(0)} - sT^\alpha t_3^{(1)}) \frac{\partial G}{\partial z} + s^2 T^\alpha Q^{(2)}
\]
By induction, we construct \( c_i^{(n)} \) and \( t_j^{(n)} \) and define \( c_i := \sum c_i^{(n)} s^n T^\alpha \) and \( t_j := \sum t_j^{(n)} s^n T^\alpha \). From the construction, it is clear these satisfy Equation (8.2). \( \square \)

The next lemma is a general result about the existence of coordinate changes by integrating a vector field in our non-archimedean setting. It should be well known to experts. We include a proof for completeness. We will use the short-hand notation \( \Lambda(\langle x, \tau \rangle) := \Lambda(\langle x_1, \ldots, x_n, \tau_1, \ldots, \tau_m \rangle) \).

Lemma 8.5. Consider \( A_j \in \Lambda(\langle s, x, \tau \rangle) \) and \( B_i \in \Lambda(\langle s, \tau \rangle) \) with valuations \( \geq \epsilon > 0 \) and let \( X \) be the vector field
\[
X := \sum_j A_j \frac{\partial}{\partial x_j} + \sum_i B_i \frac{\partial}{\partial \tau_i}.
\]
Then there exists a coordinate change \( \Phi(s, x, \tau) = (s, \psi(s, x, \tau), \varphi(s, \tau)) \), with \( \Phi(0, x, \tau) = (0, x, \tau) \) and
\[
\frac{d\Phi}{ds}(s, x, \tau) = X(\Phi(s, x, \tau)).
\]

Proof. Simplifying the notation, we have to show that there is \( (\psi_s(x, \tau), \varphi_s(\tau)) \) such that
\[
\frac{d}{ds}((\psi_s(x, \tau), \varphi_s(\tau))) = (A(s, \psi_s(x, \tau), \varphi_s(\tau)), B(s, \varphi_s(\tau))) and (\psi_0(x, \tau), \varphi_0(\tau)) = (x, \tau). This
is equivalent to
\[(8.3) \quad (\psi_s(x, \tau), \varphi_s(\tau)) - (x, \tau) = \left( \int_0^s A(u, \psi_u(x, \tau), \varphi_u(\tau)) \, du, \int_0^s B(u, \varphi_u(\tau)) \, du \right). \]

We define a sequence $\Phi^k := (\psi^k, \varphi^k)$ inductively as $(\psi^0, \varphi^0) = (x, \tau)$ and
\[(\psi_{s+1}^k(x, \tau), \varphi_{s+1}^k(\tau)) = \left( \int_0^s A(u, \psi_u^k(x, \tau), \varphi_u^k(\tau)) \, du, \int_0^s B(u, \varphi_u^k(\tau)) \, du \right). \]

By assumption there is $\epsilon > 0$ such that $\text{val}(A), \text{val}(B) \geq \epsilon$. We claim that $\text{val}(\Phi^k - \Phi^{k-1}) \geq k\epsilon$. We prove it by induction on $k$. First note that
\[\text{val}(\Phi^k - \Phi^{k-1}) = \text{val}(\int_0^s F(u, \Phi^k) - F(u, \Phi^{k-1}) \, du),\]
where $F = (A, B)$. Then by induction $\text{val}(\Phi^{k-1}) = F^{k-2} + T^{(k-1)}\rho$ for some $\rho$ of non-negative valuation. Hence $F(u, \Phi^{k-1}) - F(u, \Phi^{k-2}) = T^k\rho^{k-1}$, for some $\rho$ of non-negative valuation, which conclude the induction step.

This argument shows that $(\psi, \varphi) := (\psi^0, \varphi^0) + \sum_{k \geq 1}(\psi^k, \varphi^k) - (\psi^{k-1}, \varphi^{k-1})$ converges. By construction, it is a coordinate change. Obviously $(\psi, \varphi) = \lim_k (\psi^k, \varphi^k)$ and therefore it solves (8.3). \[\square\]

Proof of Theorem 8.2. Given $P$, define $G$ as in Lemma 8.4 and let $t_j, c_i$ be the series provided by that lemma. Let $X$ be the vector field
\[X := \sum_j t_j \frac{\partial}{\partial x_j} + \sum_i c_i \frac{\partial}{\partial \tau_i},\]
and let $\Phi(s, x, \tau)$ be the coordinate change provided by Lemma 8.5. By construction $X \cdot G = \frac{\partial G}{\partial s}$, which implies $\frac{d}{ds}(G(\Phi(s, x, y, z, \tau))) = 0$. Hence $G(0, x, y, z, \tau) = G(1, x, y, z, \tau)$. Using the notation $\Phi(s, x, y, z, \tau) = (s, \psi_s(x, y, z, \tau), \varphi_s(\tau))$, this is equivalent to
\[G(0, \psi^{-1}_1(x, y, z, \tau), \varphi^{-1}_1(\tau)) = G(1, x, \tau).\]

Evaluating at $\tau = 0$ we obtain $W_{\varphi^{-1}_1(0)}(\psi^{-1}(x, y, z, 0)) = W_{\text{lead}}(x, y, z) + P(x, y, z) - W_{\text{lead}}(x, y, z)$. Denoting $\varphi^{-1}(0) = \tau'$ and $(x', y', z') = \psi(x, y, z, 0)$ we get the desired equality $W_{\tau'}(x, y, z) = P(x', y', z')$. \[\square\]

8.3. Valuation of critical points. Recall that instead of working with geometric variables for the immersed sectors $(\tilde{x}, \tilde{y}, \tilde{z})$, we switched to new variables $(x, y, z)$ which were defined by $x = T^3\tilde{x}, y = T^3\tilde{y}, z = T^3\tilde{z}$. The Jacobian ring for $W_\tau(x, y, z)$ and $W_\tau(\tilde{x}, \tilde{y}, \tilde{z})$ are a priori different. We show that indeed, there is an example that the Kodaira-Spencer map is not an isomorphism if we consider the map to the Jacobian ring of $W_\tau(\tilde{x}, \tilde{y}, \tilde{z})$.

This can happen because of the following. Before bulk deformation, at critical points of $W_\tau(\tilde{x}, \tilde{y}, \tilde{z})$, we have $\text{val}(\tilde{x}), \text{val}(\tilde{y}), \text{val}(\tilde{z}) \geq 0$. But after bulk deformations by twisted sectors $\tau = \tau_{tw}$ with small valuations, critical points of $W_\tau(\tilde{x}, \tilde{y}, \tilde{z})$ may begin to have negative valuations, hence moving away from the disc of Novikov convergence $A^3_{\tau, (\tilde{x}, \tilde{y}, \tilde{z})}$. We illustrate
such a phenomenon in the example \((a, b, c) = (2, 2, r)\). In fact, in this example all the critical points will escape.

**Proposition 8.6.** Consider the case \(X = \mathbb{P}^{1}_{2,2,r}\). There exists \(\tau \in H^*_{\text{orb}}(X, \Lambda_+)\) such that any critical point of the bulk-deformed potential \(W_\tau(x, y, z)\) has at least one coordinate whose valuation is smaller than 3.

Proposition 8.6 implies that every critical point of \(W_\tau(\tilde{x}, \tilde{y}, \tilde{z})\) have at least one coordinate with a negative valuation, and hence the conventional boundary deformation of \(L\) would not capture these points.

**Proof.** Let us take \(\tau = T^\lambda \left\lfloor \frac{1}{2} \right\rfloor_a + T^\lambda \left\lfloor \frac{1}{2} \right\rfloor_b\) supported on the two orbi-points with \(\mathbb{Z}/2\)-singularity. We further assume that \(\lambda < \min \{3, \frac{3r-5}{2}\}\). The associated bulk-deformation only adds two terms \(T^\lambda x + T^\lambda y\) to the non-bulk-deformed potential. To see this, recall that if the term \(x^{n_1}y^{n_2}z^{n_3}\) appears in the potential, the contributing polygon satisfies the inequality

\[
\frac{n_1}{2} + \frac{n_2}{2} + \frac{n_3}{r} + \sum_j (1 - \iota_j) \leq 1.
\]

by the area formula together with Lemma 4.4. In our case \(\iota_j = \frac{1}{2}\), and hence the polygon can have at most one orbi-insertion. Any such polygon lifts to a holomorphic disc in the universal cover of \(\mathbb{P}^{1}_{2,2,r}\) by Corollary 4.3, and the lift must be invariant under the group action corresponding to either \(\left\lfloor \frac{1}{2} \right\rfloor_a\) or \(\left\lfloor \frac{1}{2} \right\rfloor_b\). It is easy to see that the 2-gons responsible for \(x^2\) and \(y^2\) are only such (see diagrams in [CHKL17, Section 12]), and their halves are orbi-discs producing \(T^\lambda x + T^\lambda y\) in \(W_\tau\).

More concretely, the resulting bulk-deformed potential is given as

\[
W_\tau(x, y, z) = -T^{-8}xyz + x^2 + y^2 + z^r + f(z) + T^\lambda x + T^\lambda y
\]

where \(f(z)\) is a polynomial in \(z\) of the form

\[
f(z) = c_1 T^{16} z^{r-2} + c_2 T^{32} z^{r-4} + \cdots = \sum_{k=1}^{|\frac{r}{2}|} c_k T^{16k} z^{r-2k}
\]

for some combinatorially defined integers \(c_k\). The precise value of \(c_k\), which can be found in [CHKL17, Theorem 12.2], is not important to us, but we will use the fact that \(\text{val}(c_k) = 0\) in the argument below.

Given the formula, critical points of \(W_\tau\) satisfy

\[
\begin{align*}
-T^{-8}yz + 2x + T^\lambda &= 0 \\
-T^{-8}xz + 2y + T^\lambda &= 0 \\
-T^{-8}xy + rz^{r-1} + f'(z) &= 0.
\end{align*}
\]

Subtracting the second equation from the first gives

\[(x - y)(T^{-8}z + 2) = 0,
\]

and hence, either \(z = -2T^8\) or \(x = y\).
(i) If $z = -2T^8$, then $2(x + y) = -T^\lambda$ and $T^{-8}xy = CT^{r-1}$ for some constant $C$ with $val(C) = 0$. Thus $x$ and $y$ are solutions of the quadratic equation (in $s$)

$$s^2 + \frac{1}{2}T^\lambda s + CT^{8+(r-1)} = 0,$$

which are

$$-\frac{1}{4}T^\lambda \pm \frac{1}{4}T^\lambda \sqrt{1 - 2CT^{8+(r-1)-2\lambda}}.$$

In particular, one of $x$ or $y$ always has valuation $\lambda$, which is smaller than $3$.

(ii) Consider the case of $x = y$. The second equation in (8.4) reads

$$x(-T^{-8}z + 2) = -T^\lambda.$$

If $val(z) \geq 8$, then $val(x) = \lambda < 3$, we are done.

Now suppose $val(z) < 8$, which implies $val(x) + val(z) = \lambda + 8$. Therefore

$$val(x) = \lambda + 8 - val(z)$$

and the valuation of monomials in $f'(z)$ can be estimated as

$$val(T^{16k}z^{-2k-1}) = 16k + (r - 2k - 1)val(z)$$

$$= (r - 1)val(z) + 2k(8 - val(z)) > (r - 1)val(z).$$

Therefore the right hand side of (8.6) has valuation $(r - 1)val(z)$, and we obtain

$$-8 + 2val(x) = (r - 1)val(z)$$

Combining (8.5) and (8.7), we conclude that $val(z) = \frac{2\lambda + 8}{r+1}$, which is again smaller than $3$.

Therefore, at least one coordinate of any critical point $(x, y, z)$ of $W_\tau$ has valuation smaller than $3$, and this proves the claim. \qed

In $(x, y, z)$ coordinates, these critical points still have $val(x), val(y), val(z) \geq 0$ for $val(\tau) \geq 0$, so it does not violate the isomorphism $KS_\tau : QH^*_\orb(X, \tau) \cong Jac(W_\tau(x, y, z))$.

**Remark 8.7.** Although geometric variables $\tilde{x}, \tilde{y}, \tilde{z}$ with negative valuations are not legitimate in Floer theory of $\mathbb{L}$, we speculate that such a deformation can be replaced by another Lagrangian using the gluing procedure explained in [CHL18].

**8.4. Explicit computation of $KS_\tau$ for $\mathbb{P}^1_{3,3,3}$ without bulk-parameters.** In this section, we give an explicit computation of the Kodaira-Spencer map for $\mathbb{P}^1_{3,3,3}$ without bulk-insertions. Namely, we set $\tau = 0$ throughout the section. For notational simplicity, let us write $X$ for $\mathbb{P}^1_{3,3,3}$ from now on.

We use the following notations for generators of $QH^*_\orb(X) := QH^*_\orb(X, 0)$. We set $1_X$ to be the unit class, and denote twisted sectors by $\Delta_{1/3}^i$ and $\Delta_{2/3}^i$ for $i = 1, 2, 3$ where $i$ indicates orbifold points. We introduce these new notations to avoid the potential confusion due to the coincidence $a = b = c = 3$ in this case. In the previous terminology, they all happen to
be \( \left\lfloor \frac{i}{3} \right\rfloor \). \( \Delta_i^{1/3} \) has degree shifting number \( \iota(\Delta_i^{1/3}) = 1/3 \), and \( \Delta_i^{2/3} \) has degree shifting number \( \iota(\Delta_i^{2/3}) = 2/3 \).

For the point class, we take 8-generic points \( pt_1, \cdots, pt_8 \) on \( \mathbb{P}^1_{3,3,3} \) which are not orbifold singular points, as shown in Figure 6. Then we define the point class pt to be the average of these 8 points, i.e. \( pt := \frac{1}{8} \sum pt_i \). This is to make the calculation of \( KS([pt]) \) easier. Notice that such a choice makes the number of \([pt]\)-insertions proportional to the symplectic area of the contributing polygons. Then \( 1_X, [pt], \Delta_i^{1/3}, \Delta_i^{2/3} \) for \( i = 1, 2, 3 \) form a basis of \( \text{QH}^*(X) \).

On the other hand, in [CHL17], a Morse model was adopted for \( CF(\mathbb{L}, \mathbb{L}) \) together with the combinatorial sign rule in [Sei11], which produces the explicit formula for the potential \( W \) given as

\[
W = \tilde{\phi}(T)(\tilde{x}^3 + \tilde{y}^3 + \tilde{z}^3) - \tilde{\psi}(T)\tilde{x}\tilde{y}\tilde{z}
\]

where \( T \) is the (exponentiated) area of the minimal triangle as before and

\[
\tilde{\phi}(T) = \sum_{k=0}^{\infty} (-1)^k(2k + 1)T^{(6k+3)^2}
\]

\[
\tilde{\psi}(T) = T + \sum_{k=1}^{\infty} (-1)^k \left( (6k + 1)T^{(6k+1)^2} - (6k - 1)T^{(6k-1)^2} \right)
\]

See [CHL17] for detailed computation\(^2\). We then make the change of coordinate (4.1) to obtain

\[
W(x, y, z) = \phi(T)(x^3 + y^3 + z^3) - \psi(T)xyz
\]

with

\[
\phi(T) = \sum_{k=0}^{\infty} (-1)^k(2k + 1)T^{36k^2+36k}
\]

\[
\psi(T) = T^{-8} \left( 1 + \sum_{k=1}^{\infty} (-1)^k \left( (6k + 1)T^{36k^2+12k} - (6k - 1)T^{36k^2-12k} \right) \right).
\]

\(^2\)To be more precise, (8.8) is obtained by changing \( \tilde{x} \) and \( \tilde{z} \) to \( -\tilde{x} \) and \( -\tilde{z} \) in the formula therein. \( \tilde{x}, \tilde{y}, \tilde{z} \) did not appear to be symmetric in [CHL17], due to some asymmetric choice of the input data for the combinatorial sign rule.
Recall that the map $\chi_S : QH^*_{orb}(X) \to \text{Jac}(W)$ is defined by sending a cycle $C$ to the polynomial class in $\text{Jac}(W)$ represented by
\begin{equation}
\sum_{\beta} \sum_{k \geq 0} q^\beta n_{1,k}(\beta; C; b, \ldots, b)
\end{equation}
where $b = xX + yY + zZ (= T^3 \bar{x}X + T^3 \bar{y}Y + T^3 \bar{z}Z)$. (8.10) is a series in $x, y, z$ in general, but we will see from explicit calculations that it is just a polynomial (over the Novikov field) for $C = 1_X, \text{pt}, \Delta_1^{1/3}, \Delta_2^{2/3}, i = 1, 2, 3$. By dimension counting (2.3) restricted to this case, $n_{1,k}(\beta; \Delta_1^{1/3}; b, \ldots, b)$ (resp. $n_{1,k}(\beta; \Delta_2^{2/3}; b, \ldots, b)$) is non-zero only when $\mu_{CW} = 2/3$ (resp. $\mu_{CW} = 4/3$).

On the other hand, $X$ has a manifold cover $\tilde{X}$, (which is the unique elliptic curve $E$ that admits $\mathbb{Z}/3$-action), and the preimage $\tilde{L}$ of $L$ in $\tilde{X}$ is a $\mathbb{Z}/3$-orbit of a circle in $\tilde{L}$, which are represented as dotted lines along three different directions in Figure 6. Recall that any orbi-disc in our interest should lift to $\tilde{X}$ by Corollary 4.3. We will need the following classification of orbi-discs for explicit calculations:

**Proposition 8.8.** Let $u$ be a holomorphic orbi-disc in $X$ bounded by $L$ of Maslov index $2/3$ with one interior orbi-marked point passing through the twisted sector $\Delta_1^{1/3}$ (or $\Delta_2^{1/3}, \Delta_3^{1/3}$), and suppose that $u$ only passes through the immersed sectors $X, Y, Z$ (but not $X, Y, Z$). Then, it can be lifted to a holomorphic disc in $\tilde{X}$ bounded by $L$ of Maslov index two whose boundary passes through the (preimage of) immersed sector $X$ (or $Y, Z$ respectively) three times.

If $u$ is a holomorphic orbi-disc in $X$ bounded by $L$ of Maslov index $4/3$ with one interior orbi-marked point passing through the twisted sector $\Delta_1^{2/3}$ (or $\Delta_2^{2/3}, \Delta_3^{2/3}$), it can be lifted to a holomorphic disc in $\tilde{X}$ bounded by $\tilde{L}$ of Maslov index four whose boundary either passes through $X$ (or $Y, Z$ respectively) six times or passes through both $(Y, Z)$ for three times (or $(X, Z), (X,Y)$ respectively).

**Proof.** Let $u$ be a holomorphic orbi-disc in $X$ bounded by $L$ of Maslov index $2/3$ with one interior orbi-marked point passing through the twisted sector $\Delta_1^{1/3}$. The orbi-marked point in the domain of $u$ must have isotropy group $\mathbb{Z}/3$ because of the injectivity between local groups. Thus $u$ can be lifted to a $\mathbb{Z}/3$-equivariant holomorphic map $\tilde{u} : (\Delta, \partial \Delta) \to (\tilde{X}, \tilde{L})$ with the domain of $\tilde{u}$ covers the domain of $u$ by the map $\zeta = \bar{\zeta}^3$. Since $\tilde{u}$ has Maslov index $2/3$, $\tilde{u}$ has Maslov index two. Moreover $\tilde{u}$ only passes through the immersed sectors $X, Y, Z$. Each of these immersed sectors contribute $2/3$ to the Maslov index. Thus $\tilde{u}$ can only pass through three of them. By the $\mathbb{Z}/3$-equivariance these three immersed sectors are the same. This forces the immersed sectors that $\tilde{u}$ pass through are all $X$.

The proof for the second statement is similar. Consider the uniformizing disc $\tilde{u}$ of the orbi-disc of Maslov index $4/3$. $\tilde{u}$ can only pass through six immersed sectors by the constraint on Maslov index. Unless $\tilde{u}$ is constant, $\mathbb{Z}/3$ acts non-trivially on $\tilde{u}$, and hence the six corners of $\tilde{u}$ can only pass through at most two distinct immersed sectors. Thus the immersed sectors $\tilde{u}$ passes through are either all $X$ (or all $Y$ or all $Z$) or three copies of $Y$ and $Z$ (or three copies of $X$ and $Y$, or three copies of $Z$ and $X$). (See Figure 8 for the shape of these orbi-discs.)

To see this, we first fix the twisted sector $\Delta_1^{2/3}$, and choose one of its pre-images in $\mathbb{C}$, say $p_\alpha$. Pick any point $\tilde{Y} \in \mathbb{C}$ corresponding to the immersed sector $Y$. Then, if $\tilde{u}$ pass through
By calculating contribution from each orbi-disc in the above classification, we can explicitly calculate the map $KS: QH_{orb}^*(X) \to \text{Jac}(W)$ as follows.

**Proposition 8.9.** The map $KS$ from the orbifold quantum cohomology of $X$ to the Jacobian ring of $W$ defined in (8.10) is given by

\[
1_X \mapsto 1, \quad [\text{pt}] \mapsto \frac{1}{8}T \frac{\partial}{\partial T}W.
\]

where

\[
P(T) = \sum_{k=0}^{\infty} (-1)^k (2k+1)T^{12k^2+12k},
\]

\[
Q(T) = \sum_{k=0}^{\infty} (2k+1)T^{24k^2+24k} + \sum_{k=1}^{\infty} k-1 \sum_{i=0}^{k-1} (-1)^{3k-i}(6k-2i+2)T^{36k^2+36k-12i^2-12i},
\]

\[
R(T) = \sum_{k=1}^{\infty} \sum_{i=0}^{k-1} (-1)^{3k-i}T^{-8}((6k-2i)T^{36k^2+12k-12i^2-12i} - (6k-2i-2)T^{36k^2-12k-12i^2-12i}).
\]

The proof will be given in Appendix C.

**Remark 8.10.** Satake-Takahashi [ST11] gave an explicit description of the genus zero Gromov-Witten potential of $\mathbb{P}^1_{3,3,3}$, which, in particular, determines the structure constants for the product structure on $QH_{orb}^*(\mathbb{P}^1_{3,3,3})$. For instance, one of the interesting quantum products is given by

\[
\Delta_1^{1/3} \cdot 0 \Delta_1^{1/3} = f_1(q)\Delta_1^{1/3},
\]

where $f_1(q)$ given in [ST11] is an expression involving Dedekind eta function. This gives a complicated looking identity on $\text{Jac}(W)$-side through our explicit map, which is a priori highly nontrivial.

**Appendix A. Proof of Proposition 6.5**

Proposition 6.5 is equivalent to the following equalities in the Jacobian ring from the Definition 6.3 of Kodaira-Spencer map and the potential.

(A.1) \[
\sum_{\beta,k,l} \text{exp}(\tau^2 p \cap \beta) \frac{T^{3\beta \cap \omega}}{l!} q_{l+1,k,\beta}(1_X, \tau^l_{\text{inf}}, b^k) = 1_L
\]
The proof of both statement is similar, we will start with the second one. The main technical issue is that the Kuranishi structure which is used to define bulk deformation as well as Kodaira-Spencer map may not be compatible with the operation of forgetting interior marked point. To overcome this problem, we will construct Kuranishi structures and CF perturbations on the spaces $\mathcal{M}_{k+1,l+1}^{para}(\beta, Q, \tau_{tw}, \gamma) := \mathcal{M}_{k+1,l+1}(\beta, Q, \tau_{tw}, \gamma) \times [0, 1]$. First note these have the following boundary decomposition

$$
\partial \mathcal{M}_{k+1,l+1}^{para}(\beta, Q, \tau_{tw}, \gamma) \times [0, 1] = \mathcal{M}_{k+1,l+1}(\beta, Q, \tau_{tw}, \gamma) \times \{0\} \bigcup \mathcal{M}_{k+1,l+1}(\beta, Q, \tau_{tw}, \gamma) \times \{1\}
$$

We proceed as follows. Given the Kuranishi neighborhood $(V, E, \phi)$ we take $V_u \cong V \times B$ where $B$ is a ball in $\mathbb{C}$ and $\varphi_{uw} : V_u \to V_v$ is $h_{uw}$-equivariant continuous map, strata-wise smooth; an isomorphism $E_u \cong \varphi_{uw}^* E_v \oplus N$ where $N$ is a rank two bundle; the $\varphi_{uw}^* E_v$ component of $s_u$ equals $\varphi_{uw}^* s_v$; $\varphi \circ \psi_v = \psi_u \circ \varphi_{uv}$ on $s_u^{-1}(0)/\Gamma_u$.

Lemma A.1. There are Kuranishi structures on the moduli spaces $\mathcal{M}_{k+1,l+1}^{para}(\beta, Q, \tau_{tw}, \gamma)$ which respect the boundary decomposition (A.3) and have the compatibilities described above.

Proof. With the exception of the compatibility with the forgetful map $\pi$, the construction of such Kuranishi structure is standard by now, see [Fuk10] for example. To ensure compatibility with $\pi$ we proceed as follows. Given the Kuranishi neighborhood $(V_v, E_v, \Gamma_v, s_v, \psi_v)$ we take $V_u \cong V_v \times B$ where $B$ parameterizes the position of the additional marked point $z_u^+$. Then the map $\varphi_{uw}$ is locally modeled on a forgetful map $\Pi : \mathcal{M}_{k+1,l+1} \to \mathcal{M}_{k+1,l}$, see the proof of Proposition 4.2 in [Amo17] for an analogous argument. Then taking the obstruction bundle
$\varphi^*_w E_v$ would give a Kuranishi neighborhood in $\mathcal{M}_{k+1,l+1}(\beta, \tau_{tw}^l, \gamma)$, that is, without incidence relation with $Q$. We include this restriction by identifying a neighborhood of $ev_{z^1_i}(u)$ (which includes no other component of $Q$) in $\mathbb{P}^1_{a,b,c}$ with a ball in $\mathbb{R}^2 \cong N$. Then the $N$ component of the obstruction map $s_u(x)$ is $ev_{z^1_i}(x) - ev_{z^1_i}(u)$. It is not hard to see this satisfies all the properties.

**Remark A.2.** As explained in Appendix A.1.4 in [FOOO09], when constructing Kuranishi structures on moduli spaces of discs one has to take a special smooth structure near nodal discs. Due to this choice, forgetful maps are continuous but smooth only when we restrict to a stratum of the stratification according to combinatorial type of the underlying disc. This is the reason $\varphi_w$ is only strata-wise smooth.

Now consider a continuous family of multisections $(U_\alpha, W_\alpha, S_\alpha)_{\alpha \in I}$ on $\mathcal{M}^{para}_{k+1,l+1}(\beta, Q, \tau_{tw}^l, \gamma)$. We say it is compatible with $\pi$ if its restriction to $\mathcal{M}_{k+1,l+1}(\beta, Q, \tau_{tw}^l, \gamma) \times \{1\}$ is the pull-back of a continuous family of multisections $(V_\beta, W_\beta, S_\beta)_{\beta \in I}$ on $\mathcal{M}_{k+1,l}(\beta, \tau_{tw}^l, \gamma)$. By pull-back we mean there are maps of Kuranishi neighborhoods from $U_\alpha$ to $V_\beta$, $W_\alpha = W_\beta$, $\theta_\alpha = \theta_\beta$ and $\varphi_{\alpha\beta}$ induces a covering map $S_{\alpha,i,j}^{-1}(0) \to S_{\beta,i,j}^{-1}(0)$.

**Lemma A.3.** There are continuous families of multisections $(U_\alpha, W_\alpha, S_\alpha)_{\alpha \in I}$ on the moduli spaces $\mathcal{M}^{para}_{k+1,l+1}(\beta, Q, \tau_{tw}^l, \gamma)$ which, given the decomposition of the boundary A.3, the restriction of the multisections to the boundary agrees with the fiber product of multisections on the right-hand side of A.3. Moreover they are compatible with $\pi$ and the evaluation at the 0-th boundary marked point maps $(ev_0)_\alpha|_{S^{-1}_\alpha(0)}$ are submersions.

**Proof.** Once again the proof follows the strategy in [Fuk10] and [Amo17, Proposition 4.4]. We take the continuous family of multisections on $\mathcal{M}^{para}_{k+1,l}(\beta, \tau_{tw}^l, \gamma)$ and define $W_\alpha = W_\beta$, $\theta_\alpha = \theta_\beta$. On the $\varphi^*_{\alpha\beta} E_\beta$ component we take the map $S_\alpha = S_\beta \circ \varphi_{\alpha\beta}$. On the $N$ component we take a generic perturbation of $Q$ so that it becomes transversal to the image of $ev_{z^1_i}$. With this choice, $\varphi_{\alpha\beta}$ induces a natural covering map $S_{\alpha,i,j}^{-1}(0) \to S_{\beta,i,j}^{-1}(0)$. Please note if consider the union over all $\alpha$ over a fixed $\beta$ the resulting covering map has exactly $Q \cap \beta$ sheets. The remainder of the proof follows the usual induction argument on energy, see [Fuk10].

Equipped with these CF of perturbations on $\mathcal{M}^{para}_{k+1,l+1}(\beta, Q, \tau_{tw}^l, \gamma)$ we can use the evaluation maps at the boundary marked points to define new operations on the Fukaya algebra and set

$$F_{b,r} = \sum_{k,\beta,l} \frac{T^\omega \cap \beta}{l!} (ev_0)_\ast (ev_{z^1_i}^* b \wedge \cdots \wedge ev_{z^1_i}^* b).$$

Now we apply the Stokes theorem [FOOO11, Lemma 12.13] to $\mathcal{M}^{para}_{k+1,l+1}(\beta, Q, \tau_{tw}^l, \gamma)$. Please note that the terms coming from the second line in (A.3) contribute with zero since $m_{1}^{r,b}$ is unital. Also the terms in the first line of (A.3) give respectively the left and right-hand side of Equation A.2, by the previous proof. Therefore we conclude that $m_{1}^{r,b}(F_{b,r})$ is exactly the difference between the two sides of (A.2). Combining this with Proposition 4.14 proves the desired statement.

The proof of (A.1) is entirely analogous. The main difference is that in this case when describing compatibility with $\pi$ there is no extra component $N$ is the obstruction bundle.
Therefore the induced map \( S_{\alpha,i,j}^{-1}(0) \rightarrow S_{\beta,i,j}^{-1}(0) \) is a submersion of positive codimension and the corresponding operation gives zero (see [Amo17, Proposition 3.7]). There is one exception, when \( \beta = 0 \) and \( k = l = 0 \) in which case it is easy to see that we obtain \( 1_L \). This proves that the two sides of A.1) differ by a \( m_i^{r,b} \) coboundary which proves the result.

**Appendix B. Proof of Theorem 7.3 (2)**

We give a proof of Theorem 7.3 (2), here. First observe that we have the following basic relations in \( \mathcal{A}_0 \) from the ideal \( \langle g_1, g_2, g_3 \rangle \):

\[
aT^8 x^{a-1} = yz, \quad bT^8 y^{b-1} = zx, \quad cT^8 z^{c-1} = xy.
\]

We will refer to these as the Jacobi relations.

**Definition B.1.** Given a monomial in \( \Lambda(x,y,z) \), the operation replacing \( yz, zx, xy \) in the monomial by \( T^8 x^{a-1}, T^8 y^{b-1}, T^8 z^{c-1} \) will be referred to as type I replacement, and replacing \( x^{a-1}, y^{b-1}, z^{c-1} \) by \( T^{-8} yz, T^{-8} zx, T^{-8} xy \) will be referred to as type II replacement. Each of individual replacements (as well as their corresponding relations in \( \langle g_1, g_2, g_3 \rangle \)) will be called by \( I_x, I_y, I_z, II_x, II_y, II_z \), respectively.

Hence, if we perform type I replacement \( a \)-times and type II replacement \( b \)-times, then the original exponent of \( T \) is increased by \( 8a - 8b \). We will use the following properties of \( \langle g_1, g_2, g_3 \rangle \), frequently.

**Lemma B.2.** If an expression \( x^p y^q z^r - T^m x^{p+i} y^{q+j} z^{r+k} \) lies in the ideal \( \langle g_1, g_2, g_3 \rangle \) with \( p, q, r, i, j, k \geq 0 \) and \( m > 0 \), then so does \( x^p y^q z^r \) itself.

**Proof.** We have \( x^p y^q z^r - T^m x^{p+i} y^{q+j} z^{r+k} = x^p y^q z^r (1 - T^m x^i y^j z^k) \) and \( (1 - T^m x^i y^j z^k) \) is invertible in \( \Lambda(x,y,z) \). Hence the lemma follows. \( \square \)

**Lemma B.3.** For \( p, q, r, i, j, k \geq 0 \), if an expression \( x^p y^q z^r \) transforms to another expression \( x^i y^j z^k \) by performing type I or II replacements then their difference lies in the ideal \( \langle g_1, g_2, g_3 \rangle \). i.e.

\[
x^p y^q z^r - x^i y^j z^k = \sum_{j=1}^{3} t_j g_j
\]

for some \( t_j \) with \( \text{val}(t_j) \geq s \), where \( s \) is the minimum valuation of the intermediate expressions (including both ends of the operation sequence).

**Proof.** It directly follows from the fact that both of replacements are trivial modulo relations in the ideal \( \langle g_1, g_2, g_3 \rangle \). \( \square \)

Let us now begin the proof of Theorem 7.3 (2). We divide the proof into a few different cases (Lemma B.4, B.5, B.6 and B.7 below) depending on the type of \( (a, b, c) \).

**Lemma B.4.** If \( a, b, c \geq 3 \), then Theorem 7.3 (2) holds.
Proof. Consider a monomial \( x^{i'} y^{j'} z^{k'} \) (with \( i', j', k' \geq 0 \)) which does not appear in the basis. By symmetry, we may assume that \( i' \geq j' \geq k' \). First we consider the case that \( k' \neq 0 \). Then \( i' \geq 2 \) as otherwise \( x^{i'} y^{j'} z^{k'} \) would be \( xyz \in \{ \gamma_1, \ldots, \gamma_N \} \). Thus we can write \( i' = i + 2, j' = j + 1, k' = k \) (with \( i, j, k \geq 0 \)). By using Jacobian relation \( I_z, I_y, I_x \) successively, we have

\[
x^{2+i} y^{1+j} z^k = c x^{i} y^{j} z^k (xT^8 z^{c-1}) = c x^{i} y^{j} z^k + c - 2T^8 (zx) \\
= (bc) x^{i} y^{j} z^k + c - 2T^{16} y^{b-1} = (bc) x^{i} y^{j} z^k + c - 3T^{16} yz \\
= (abc) x^{i+a-1} y^{j+b-2} z^{k+c-3} T^{24}
\]

Hence the difference

\[
x^{2+i} y^{1+j} z^k - (abc)x^{i+a-1} y^{j+b-2} z^{k+c-3} T^{24} = x^{2+i} y^{1+j} z^k (1 - (abc)x^{a-3} y^{b-3} z^{c-3} T^{24})
\]

lies in the ideal. Lemma B.3 tells us that the term in the right hand side lies in \( \langle g_1, g_2, g_3 \rangle \). Therefore \( x^{i'} y^{j'} z^{k'} = x^{2+i} y^{1+j} z^k \) with \( k' \neq 0 \) belongs to \( \langle g_1, g_2, g_3 \rangle \) by Lemma B.2.

Now, let us consider the case for \( k' = 0 \). If \( i' \geq 2, j' \geq 1 \), then we can apply exactly the same argument as above to prove that \( x^{i'} y^{j'} \) lies in the ideal. If \( i' = 1, j' = 1 \), then we have \( x^{i'} y^{j'} = xy \equiv cT^8 z^{c-1} \subset cT^{8} \{ \gamma_1, \ldots, \gamma_N \} \) and thus the claim holds.

We are left with the case when \( j' = k' = 0 \) and \( i' \geq a \). If \( i' = a \), then \( x^a = \frac{1}{a} T^{-8} xyz \) and hence the claim still holds. If \( i' = a + i \) with \( i \geq 1 \), then

\[
x^{i'} = x^{a+i} = (1/a) T^{-8} xyz \cdot x^i = (1/a) T^{-8} x^{i+1} yz
\]

We have already shown that \( x^{i+1} yz \) is an element in the ideal which can be written as \( \sum t_j g_j \) with \( val(t_j) \geq 0 \). Therefore

\[
x^{i'} = (1/a) T^{-8} x^{i+1} \left( \frac{aT^8 x^{a-1}}{y z} - yz \right) + T^{-8} \sum_{j=1}^{3} t_j g_j = \sum_{j=1}^{3} t'_j g_j
\]

with \( val(t'_j) \geq -8 \). \( \square \)

Lemma B.5. If \( (a, b, c) = (2, 2, c) \) \( (c \geq 2) \), then Theorem 7.3 (2) holds.

Proof. For simplicity, we represent the monomial \( x^{i} y^{j} z^k \) by its exponent vector \( (i, j, k) \) in what follows. Our argument splits into a few different cases depending on which entries of the vector vanish. Below, \( i, j, k \) are all assumed to be positive integers.

\( (i, 0, k) \): By first applying \( I_y \) and later \( I_z \) (or \( I_x \)), we can make it into \( (i, 0, k-2) \): (or \( (i-2, 0, k) \)) Repeating the procedure, we can reduce it to one of the basis element (by type I replacements only).

\( (0, j, k) \): This follows from the previous case by symmetry of \( (2, 2, n) \). Again, we only need type I replacements.

\( (0, j, 0) \): We may assume \( j \geq 3 \). We first apply \( I_y \) to get \( (1, j - 1, 1) \), followed by \( I_z \) to get \( (0, j - 2, c) \). Since we have applied each of type I and II exactly once, the exponent of \( T \) remains zero, and we can now apply the previous case of \( (0, j, k) \). The same argument can be used for \( (i, 0, 0) \).

\( (0, 0, k) \): We may assume \( k \geq c + 1 \). We can proceed as

\[
(0, 0, k) \sim^{I_x} (1, 1, k - c + 1) \sim^{I_y} (0, 2, k - c)
\]

to go back to one of the previous cases.
We may assume $i \geq 2$. We have $(i, j, 0) \sim_1 (i - 1, j - 1, c - 1) \sim_3^2 (i - 2, j, c - 2)$. We can then apply $I_2$ as many times as needed to get $(*, 0, *)$ or $(0, *, *)$ and we go back to one of the previous cases.

$(i, j, k)$: We use induction on $i + j + k$ and $(i, j, k) \sim_1 (i + 1, j - 1, k - 1)$. We can apply either induction hypothesis to $(i + 1, j - 1, k - 1)$ if all entries are non-zero or one of the above steps otherwise.

Lemma B.6. If $(a, b, c) = (2, 3, c)$, then Theorem 7.3 (2) holds.

Proof. This is the most elaborate case. We claim that a given type of monomial is either equivalent to a basis element or to zero element modulo $\langle g_1, g_2, g_3 \rangle$ by applying type I and II replacements. Since we also need to control the valuation of $t_j$ in (7.4), the type II replacement should be applied carefully. It will be always coupled with the type I to compensate the energy. Here, we only consider $c \geq 3$ since the case with $c = 2$ has already been covered by Lemma B.5. $i, j, k$ are all assumed to be positive integers, below.

$(0, j, k)$: We further divide the case into two.

(i) $j \leq 2$: The lowest possibly non-basis element is $(01, 2)$, and since $(0, 1, 2) \sim_1^2 (1, 0, 1)$, it is equivalent to a basis element. Now for $k \geq 3$, observe that $(0, 1, k) \sim_1^2 (1, 0, k - 1) \sim_3^2 (0, 2, k - 2)$. Thus it suffices to consider $(0, 2, k)$ for $k \geq 1$, for which we have

$$(0, 2, k) \sim_1^2 (1, 1, k - 1) \sim_1^2 (2, 0, k - 2) \sim_3^2 (1, 2, k - 3) \sim_1^2 (0, 1, k + c - 4) \sim_1^2 (1, 0, k + c - 5) \sim_3^2 (0, 2, k + c - 6).$$

(For $k = 1, 2$ case, we stop at 2nd and 3rd equality.) If $c \geq 6$, by applying Lemma B.2 to the first and the last term, we obtain the claim. If $c = 3, 4$ or 5, $(0, 2, k) \sim (0, 2, k - 3), (0, 2, k) \sim (0, 2, k - 2)$ or $(0, 2, k) \sim (0, 2, k - 1)$. In any case, we can reduce it to either of $(0, 2, 0), (0, 2, 1), (0, 2, 2)$ which was covered in the first step. Note that we only uses type I in this case.

(ii) $j \geq 3$: If $j = 3$, then $(0, 3, k) \sim_1^2 (1, 1, k + 1) \sim_3^2 (0, 2, k)$ and we are done by (i). Consider $j \geq 4$. The same argument as above shows that $(0, j, k) \sim (0, j, k + c - 6)$ and for $c \geq 6$, this shows the vanishing of the monomial modulo the relations by Lemma B.2. Thus it is enough to consider the case that $3 \leq c \leq 5$. For $c = 3$, we run an induction on $j$ to get $(0, j, 0), (0, j, 1), (0, j, 2)$ as in (i). Finally,

$$(0, j, 0) \sim_3^2 (1, j - 2, 1) \sim_1^2 (0, j - 3, c),$$
$$(0, j, 1) \sim_3^2 (1, j - 2, 2) \sim_1^2 (0, j - 3, c + 1),$$
$$(0, j, 2) \sim_3^2 (1, j - 2, 3) \sim_1^2 (0, j - 3, c + 2),$$

and inductively, we go back to the case $j \leq 3$. The other case $c = 4, 5$ is similar. Note that we sometimes used type II exactly once, but immediately followed by type I in this case.

$(0, j, 0)$: It can be done as in the last paragraph. $(0, j, 0) \sim_3^2 (0, j - 3, c)$ uses type II, but the latter can be reduced without further energy loss. So this proves the claim.

$(0, 0, k)$: We can transform it to the first case since

$$(0, 0, k) \sim_3^2 (1, 1, k - c + 1) \sim_1^2 (0, 3, k - c).$$
(i, 0, k): If \( i \geq 3 \),
\[
(i, 0, k) \sim^{I_y} (i - 1, 2, k - 1) \sim^{I_z} (i - 3, 0, a - 1 + 2c - 2),
\]
so we can run induction on \( i \) to make \( i = 1 \) or \( i = 2 \). The case with \( i = 1, i = 2 \) can be handled easily as follows.
\[
(1, 0, k) \sim^{I_y} (0, 2, k - 1), \quad (2, 0, k) \sim^{I_y} (1, 2, k - 1) \sim^{I_z} (0, 1, c + k - 2).
\]

(i, 0, 0): The claim follows from
\[
(i, 0, 0) \sim^{I_z} (i - 1, 1, 1) \sim^{I_z} (i - 2, 0, c)
\]
where the last term was covered in the previous step.

(i, j, 0): Observe that for \( c \geq 6 \)
\[
(i, j, 0) \sim^{I_z} (i - 1, j - 1, c - 1) \sim^{I_z} (i, j - 2, c - 2) \sim^{I_y} (i - 1, j, c - 3)
\sim^{I_z} (i, j - 1, c - 4) \sim^{I_z} (i + 1, j - 2, c - 5) \sim^{I_y} (i, j, c - 6).
\]
Thus, for \( c \geq 6 \), we have the vanishing of the monomial modulo \( \langle g_1, g_2, g_3 \rangle \) by comparing two ends. If \( c = 3 \), then \((i, j, 0) \sim^{I_z} (i - 1, j, c - 3) = (i - 1, j, 0)\)  Thus we run induction on \( i \). For \( c = 4, 5 \), we similarly run induction on \( j \).

(i, j, k): We run induction on \( i + j + k \) for \((i, j, k)\). Since \((i, j, k) \sim^{I_z} (i + 1, j - 1, k - 1)\), we can make either \( j \) or \( k \) vanish by applying this operation repeatedly.

\(\square\)

**Lemma B.7.** If \((a, b, c) = (2, b, c)\) with \(b, c \geq 4\), then Theorem 7.3 (2) holds.

**Proof.** We again divide the argument by the type of the exponent of a monomial. Like before, \(i, j, k\) below are positive integers.

(0, j, k): Note that
\[
(0, j, k) \sim^{I_z} (1, j - 1, k - 1) \sim^{I_z} (0, j + b - 2, k - 2)
\sim^{I_z} (1, j + b - 3, k - 3) \sim^{I_z} (0, j + b - 4, k + c - 4),
\]
and since \(b, c \geq 4\), this shows that the monomial \((0, j, k)\) is trivial modulo \(\langle g_1, g_2, g_3 \rangle\) if \(i \geq 1, j \geq 3\). The remaining case can be handled by
\[
(0, 2, 1) \sim^{I_z} (1, 1, 0), \quad (0, j, 1) \sim^{I_z} (1, j - 1, 0) \sim^{I_z} (0, j - 2, c - 1),
\quad (0, j, 2) \sim^{I_z} (1, j - 1, 1) \sim^{I_z} (0, j - 2, c).
\]

(0, j, 0): If \(j \leq b\), then it is a basis element, so we only consider \(j \geq b + 1\). In this case, we have \((0, j, 0) \sim^{I_y} (1, j - b + 1, 1) \sim^{I_x} (0, j - b, c)\).

(0, 0, k): We only need to consider \(k \geq c + 1\) for which \((0, 0, k) \sim^{I_z} (1, 1, k - c + 1) \sim^{I_y} (0, b, k - c)\).

(i, 0, k): Let us use induction on \(i\). Observe that \((i, 0, k) \sim^{I_y} (i - 1, b - 1, k - 1)\). We repeatedly apply \(I_z\) (which adds \((-1, -1, c)\)) until either the first or the second entry become 0, depending on the relative sizes of \(i\) and \(b\). In the former, we have \((0, *, *)\) which was already covered. In the latter case, we obtain \((i - b, 0, k - 1 + (b - 1)(c - 1))\), and hence can apply the induction.

(i, 0, 0): We only need to consider \(i \geq 3\), in which case \((i, 0, 0) \sim^{I_z} (i - 1, 1, 1) \sim^{I_z} (i - 2, 0, c)\).
(i, j, 0): We proceed as
\[ (i, j, 0) \sim I_z (i - 1, j - 1, c - 1) \sim I_x (i, j - 2, c - 2) \]
\[ \sim I_y (i - 1, j + b - 3, c - 3) \sim I_x (i, j + b - 4, c - 4). \]
Thus we can say that \((i, j, 0)\) is equivalent to 0 if \(i \geq 1, j \geq 2\). Also note that \((2, 1, 0) \sim I_z (1, 0, c - 1)\).

(i, j, k): It can be handled by using induction on \((i + j + k)\) based on the relation \((i, j, k) \sim I_x (i + 1, j - 1, k - 1)\).

\[ \Box \]

**Appendix C. Proof of Proposition 8.9**

Proposition 8.9 can be shown by directly counting the contributing orbi-discs, which is tedious, but elementary. Below is a reformulation of Proposition 8.9 in \(\tilde{x}, \tilde{y}, \tilde{z}\)-variables which are more accessible in actual disc counting. In addition, we choose

(C.1) \[ b = -\tilde{x}X + \tilde{y}Y - \tilde{z}Z \]

in order to make the signs in the formula more symmetric. It is not difficult to check that Proposition C.1 is equivalent to the original statement in Proposition 8.9.

**Proposition C.1.** The map \(KS\) from the orbifold quantum cohomology of \(X\) to the Jacobian ring of \(W(\tilde{x}, \tilde{y}, \tilde{z})\) defined in (8.10) is given by

\[ 1_X \mapsto 1; \]
\[ [\text{pt}] \mapsto \frac{1}{8A} T \frac{\partial}{\partial T} W(\tilde{x}, \tilde{y}, \tilde{z}); \]
\[ \Delta_1^{1/3} \mapsto \tilde{x} \sum_{k=0}^{\infty} (-1)^k (2k + 1) \phi_k(T); \]
\[ \Delta_2^{1/3} \mapsto \tilde{y} \sum_{k=0}^{\infty} (-1)^k (2k + 1) \phi_k(T); \]
\[ \Delta_3^{1/3} \mapsto \tilde{z} \sum_{k=0}^{\infty} (-1)^k (2k + 1) \phi_k(T); \]
\[
\Delta_1^{2/3} \mapsto \bar{z}^2 \sum_{k=0}^{\infty} (2k + 1) \phi_k(T^2) + \bar{z}^2 \sum_{i=0}^{k-1} (-1)^{3k-i} (6k - 2i + 2) \frac{\phi_k(T^3)}{\phi_i(T)} \\
+ \bar{y} \sum_{i=0}^{k-1} \left( (-1)^{3k-i} (6k - 2i) \frac{\psi_k^+(T)}{\phi_i(T)} + (-1)^{3k-i-1} (6k - 2i - 2) \frac{\psi_k^-(T)}{\phi_i(T)} \right);
\]
\[
\Delta_2^{2/3} \mapsto \bar{y}^2 \sum_{k=0}^{\infty} (2k + 1) \phi_k(T^2) + \bar{y}^2 \sum_{i=0}^{k-1} (-1)^{3k-i} (6k - 2i + 2) \frac{\phi_k(T^3)}{\phi_i(T)} \\
+ \bar{x} \sum_{i=0}^{k-1} \left( (-1)^{3k-i} (6k - 2i) \frac{\psi_k^+(T)}{\phi_i(T)} + (-1)^{3k-i-1} (6k - 2i - 2) \frac{\psi_k^-(T)}{\phi_i(T)} \right);
\]
\[
\Delta_3^{2/3} \mapsto z^2 \sum_{k=0}^{\infty} (2k + 1) \phi_k(T^2) + z^2 \sum_{i=0}^{k-1} (-1)^{3k-i} (6k - 2i + 2) \frac{\phi_k(T^3)}{\phi_i(T)} \\
+ xz \sum_{i=0}^{k-1} \left( (-1)^{3k-i} (6k - 2i) \frac{\psi_k^+(T)}{\phi_i(T)} + (-1)^{3k-i-1} (6k - 2i - 2) \frac{\psi_k^-(T)}{\phi_i(T)} \right)
\]

where \( \phi_k(T) = T^{12k^2+12k+3} \), \( \psi_k^+(T) = T^{(6k+1)^2} \), \( \psi_k^-(T) = T^{(6k-1)^2} \).

**Proof.** \( KS(1_X) = 1 \) by the unital property of \( KS \), and \( KS([pt]) \) was already computed in the proof of Theorem 8.1. It only remains to compute the image of \( \Delta_i^{1/3} \) for \( i = 1, 2 \) (other cases can be calculated in a similar way). In the computation below, we will use the Morse model with the combinatorial sign rule following [Sei11]. For this reason we choose a perfect Morse function on \( \mathbb{L} \) with the minimum \( e \) which serves as the unit class in \( CF(\mathbb{L}, \mathbb{L}) \). In addition, we choose a generic point which is close to \( e \) that represents a nontrivial structure put on \( \mathbb{L} \). Readers may consult [CHL17, 3.4] for more details on the disc counting in this setting.

(1) \( KS(\Delta_1^{1/3}) \): From our earlier lifting argument, the holomorphic triangles counted for the potential \( W \) can be regarded as uniformizing covers of \([1/3]\) orb-disks which contribute to \( KS(\Delta_1^{1/3}) \), as shown in Figure 7. Comparing with (8.9), there are sequences \( \Delta_{x,k} \) and \( \Delta_{x,k}^{op} \) of such orb-disks with sizes \( \phi_k(T) \), \( k = 0, 1, 2, \ldots \). Here, we set \( \Delta_{x,k} \) to be a positive triangle, and \( \Delta_{x,k}^{op} \) a negative one.

We also need to count the number of times in which the discs meet the minimum \( e \). By direct counting, we see that for \( \Delta_{x,k} \) and \( \Delta_{x,k}^{op} \) of size \( \phi_k(T) \), there are \( k + 1 \) and \( k \) many \( e \)'s on their boundaries, respectively. Taking signs into account \( (s(\Delta_{x,k}) = (-1)^{k+1}, s(\Delta_{x,k}^{op}) = (-1)^{k}(-1)^{k+1}) \), the element \( \Delta_1^{1/3} \) of \( QH^*_\text{orb}(X) \) maps to
\[
\tilde{x} \sum_{k=0}^{\infty} (-1)^{k} (2k + 1) \phi_k(T) = \phi(T) \tilde{x}
\]
as desired.\(^3\)

\(^3\) Notice that \((-1)^{k+1} \) in \( s(\Delta_{x,k}) = s(\Delta_{x,k}^{op}) = (-1)^{k+1} \) turns into \((-1)^{k} \) due to (C.1).
(2) $\text{KS} (\Delta_{1}^{2/3})$: From Proposition 8.8, there are two types of such orbi-discs, corresponding to either $\tilde{x}^2$ or $\tilde{y} \tilde{z}$. The images of liftings of orbi-discs can be triangles or immersed hexagons as depicted in Figure 8. We first consider the case when the images are triangles, which occurs only for $\tilde{x}^2$-type orbi-discs. Similarly to (1), we have two sequences $\Delta_{x^2,k}$ and $\Delta_{x^2,k}^{op}$ for such discs. Namely, we can take two third of such triangles to get desired orbi-discs, and their count is given by

$$\tilde{x}^2 \sum_{k=0}^{\infty} (2k + 1) \phi_k (T^2).$$

Here, the two triangles $\Delta_{x^2,k}$ and $\Delta_{x^2,k}^{op}$ have the common size $\phi_k (T^2)$. Also, they have $2k + 2$ and $2k$ many $e$’s on their boundaries respectively, but because of the rotation symmetry (which gives an automorphism on the moduli) we should count them as $k + 1$ and $k$. Signs of contribution are given by

$$s(\Delta_{x^2,k}) = (-1)^{2k+2} = 1, \quad s(\Delta_{x^2,k}^{op}) = (-1)^{|X|+|X|}(-1)^{2k} = (-1)^{2k+2} = 1.$$
We next consider orbi-discs whose liftings become immersed hexagons. Again, there are two types of such orbi-discs corresponding to either \( \tilde{x}^2 \) or \( \tilde{y} \tilde{z} \).

(i) \( \tilde{x}^2 \): In this case, we count the orbi-discs \( \Delta_{x^3,k} \setminus \Delta_{x,i} \) \( (i = 0, \ldots, k - 1) \) of size \( \phi_k(T^3) / \phi_1(T) \), which has \( 3k + 3 - (i + 1) = 3k - i + 2 \) many \( e \)'s on its boundary, and \( s(\Delta_{x^3,k} \setminus \Delta_{x,i}) = (-1)^{3k-i+2} \). Its reflection image \( (\Delta_{x^3,k} \setminus \Delta_{x,i})^{op} \) has \( 3k - i \) many \( e \)'s on the boundary, and \( s ((\Delta_{x^3,k} \setminus \Delta_{x,i})^{op}) = (-1)^{|X|+|\Delta|}(-1)^{3k-i} = (-1)^{3k-i} \). In total, they produce

\[
\tilde{x}^2 \sum_{i=0}^{k-1} (-1)^{3k-i+2}(3k - i + 2) + (-1)^{3k-i}(3k - i) \frac{\phi_k(T^3)}{\phi_1(T)}.
\]

(ii) \( \tilde{y} \tilde{z} \): Denote the two positive triangles contributing to the \( k \)-th terms in 8.9 by \( \Delta_{xyz,k,\pm} \). The only contribution to \( \tilde{y} \tilde{z} \) is from the count of \( \Delta_{xyz,k,\pm} \setminus \Delta_{x,i} \) \( (i = 0, 1, \ldots, k - 1) \) and their reflection image, both of which have with size \( \psi_{k}^{\pm}(T) \).

\[
\Delta_{xyz,k,+} \setminus \Delta_{x,i} \text{ has } 3k + 1 - (i + 1) = 3k - i \text{ many } e \text{'s along the boundary, and}
\]

\[
s(\Delta_{xyz,k,+} \setminus \Delta_{x,i}) = (-1)^{3k-i}. \text{ For } (\Delta_{xyz,k,+} \setminus \Delta_{x,i})^{op}, \text{ we have } 3k - i \text{ many } e \text{'s, and}
\]

\[
s ((\Delta_{xyz,k,+} \setminus \Delta_{x,i})^{op}) = (-1)^{|Y|+|Z|}(-1)^{3k-i} = (-1)^{3k-i}. \text{ So, these two discs give}
\]

\[
\tilde{y} \tilde{z} \left( (-(1)^{3k-i}(3k - i) + (-1)^{3k-i}(3k - i) \right) \frac{\psi_{k}^{+}(T)}{\phi_1(T)}.
\]

Similarly, \( \Delta_{xyz,k,-} \setminus \Delta_{x,i} \) and its reflection image contribute

\[
\tilde{y} \tilde{z} \left( (-(1)^{3k-i-1}(3k - i - 1) + (-1)^{3k-i-1}(3k - 1 - i) \right) \frac{\psi_{k}^{+}(T)}{\phi_1(T)}.
\]

\[\square\]

References

[AJ10] Manabu Akaho and Dominic Joyce, Immersed Lagrangian Floer theory, J. Differential Geom. 86 (2010), no. 3, 381–500.

[Amo17] Lino Amorim, The Kinneth theorem for the Fukaya algebra of a product of Lagrangians, Internat. J. Math. 28 (2017), no. 4, 1750026, 38. MR 3639044

[Aut20] The Stacks Project Authors, Stacks project, https://stacks.math.columbia.edu/ (2020).

[BC13] Paul Biran and Octav Cornea, Lagrangian cobordism, I, J. Amer. Math. Soc. 26 (2013), no. 2, 295–340.

[BGR84] Siegfried Bosch, Ulrich Güntzer, and Reinhold Remmert, Non-Archimedean analysis, Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 261, Springer-Verlag, Berlin, 1984, A systematic approach to rigid analytic geometry. MR 746961

[CCLT14] Kwokwai Chan, Cheol-Hyun Cho, Siu-Cheong Lau, and Hsian-Hua Tseng, Lagrangian Floer superpotentials and crepant resolutions for toric orbifolds, Comm. Math. Phys. 328 (2014), no. 1, 83–130.

[CCLT16] ______, Gross fibrations, SYZ mirror symmetry, and open Gromov-Witten invariants for toric Calabi-Yau orbifolds, J. Differential Geom. 103 (2016), no. 2, 207–288.

[CHKL17] Cheol-Hyun Cho, Hansol Hong, Sang-Hyun Kim, and Siu-Cheong Lau, Lagrangian floer potential of orbifold spheres, Advances in Mathematics (2017), no. 306, 344–426.
[CHL12] Cheol-Hyun Cho, Hansol Hong, and Sangwook Lee, Examples of matrix factorizations from SYZ, SIGMA Symmetry Integrability Geom. Methods Appl. 8 (2012), Paper 053, 24.

[CHL17] Cheol-Hyun Cho, Hansol Hong, and Siu-Cheong Lau, Localized mirror functor for Lagrangian immersions, and homological mirror symmetry for $\mathbb{P}^1_{\alpha,\beta,\gamma}$, J. Differential Geom. 106 (2017), no. 1, 45–126. MR 3640007

[CHL18] Cheol-Hyun Cho, Hansol Hong, and Siu-Cheong Lau, Gluing localized mirror functors, arXiv preprint (2018), arXiv: 1810.02045.

[CHL19] Cheol-Hyun Cho, Hansol Hong, and Siu-Cheong Lau, Localized mirror functor constructed from a lagrangian torus, Journal of Geometry and Physics 136 (2019), 284–320.

[CO06] Cheol-Hyun Cho and Yong-Geun Oh, Floer cohomology and disc instantons of Lagrangian torus fibers in Fano toric manifolds, Asian J. Math. 10 (2006), no. 4, 773–814.

[CP14] Cheol-Hyun Cho and Mainak Poddar, Holomorphic orbi-discs and lagrangian Floer cohomology of symplectic toric orbifolds, J. Differential Geom. 98 (2014), no. 1, 21–116.

[CR02] Weimin Chen and Yongbin Ruan, Orbifold Gromov-Witten theory, Orbifolds in mathematics and physics (Madison, WI, 2001), Contemp. Math., vol. 310, Amer. Math. Soc., Providence, RI, 2002, pp. 25–85.

[CS16] Cheol-Hyun Cho and Hyung-Seok Shin, Chern-Weil Maslov index and its orbifold analogue, Asian Journal of Math. 20 (2016), no. 1, 1–20.

[FOOO09] Kenji Fukaya, Yong-Geun Oh, Hiroshi Ohta, and Kaoru Ono, Lagrangian intersection Floer theory: anomaly and obstruction. Parts I and II, AMS/IP Studies in Advanced Mathematics, vol. 46, American Mathematical Society, Providence, RI, 2009.

[FOOO10] , Lagrangian Floer theory on compact toric manifolds. I, Duke Math. J. 151 (2010), no. 1, 23–174.

[FOOO11] , Lagrangian Floer theory on compact toric manifolds II: bulk deformations, Selecta Math. (N.S.) 17 (2011), no. 3, 609–711. MR 2827178

[FOOO16a] , Exponential decay estimates and smoothness of the moduli space of pseudoholomorphic curves, preprint (2016).

[FOOO16b] , Lagrangian Floer theory and mirror symmetry on compact toric manifolds, Astérisque (2016), no. 376, vi+340. MR 3460884

[Fuk10] Kenji Fukaya, Cyclic symmetry and adic convergence in Lagrangian Floer theory, Kyoto J. Math. 50 (2010), no. 3, 521–590. MR 2723862

[Fuk17] Kenji Fukaya, Unobstructed immersed lagrangian correspondence and filtered a infinity functor, arXiv preprint (2017), arXiv:1706.02131.

[IST19] Yoshihisa Ishibashi, Yuuki Shiraishi, and Atsushi Takahashi, Primitive forms for affine cusp polynomials, Tohoku Mathematical Journal 71 (2019), no. 3, 437–464.

[Kon95] Maxim Kontsevich, Homological algebra of mirror symmetry, Proceedings of the International Congress of Mathematicians, Vol. 1, 2 (Zürich, 1994) (Basel), Birkhäuser, 1995, pp. 120–139.

[Mat80] Hideyuki Matsumura, Commutative algebra, second ed., Mathematics Lecture Note Series, vol. 56, Benjamin/Cummings Publishing Co., Inc., Reading, Mass., 1980. MR 575344

[Pac19] Tommaso Pacini, Maslov, chern–weil and mean curvature, Journal of Geometry and Physics 135 (2019), 129 – 134.

[PW19] Joseph Palmer and Chris Woodward, Invariance of immersed floer cohomology under lagrangian surgery, arXiv:1903.01943 (2019).

[Ros10] Paolo Rossi, Gromov-Witten theory of orbicurves, the space of tri-polynomials and symplectic field theory of Seifert fibrations, Math. Ann. 348 (2010), no. 2, 265–287.

[Sei08] Paul Seidel, Fukaya categories and Picard-Lefschetz theory, Zurich Lectures in Advanced Mathematics, European Mathematical Society (EMS), Zürich, 2008. MR 2441780

[Sei11] Paul Seidel, Homological mirror symmetry for the genus two curve, J. Algebraic Geom. 20 (2011), no. 4, 727–769.

[Sei12] Paul Seidel, Fukaya $A_\infty$-structures associated to Lefschetz fibrations. I, J. Symplectic Geom. 10 (2012), no. 3, 325–388.

[Sei15] , Homological mirror symmetry for the quartic surface, Mem. Amer. Math. Soc. 236 (2015), no. 1116, vi+129. MR 3364859
N. Sheridan, On the homological mirror symmetry conjecture for pairs of pants, J. Differential Geom. 89 (2011), no. 2, 271–367.

N. Sheridan, Homological mirror symmetry for Calabi-Yau hypersurfaces in projective space, Invent. Math. 199 (2015), no. 1, 1–186.

Ikuo Satake and Atsushi Takahashi, Gromov-Witten invariants for mirror orbifolds of simple elliptic singularities, Ann. Inst. Fourier (Grenoble) 61 (2011), no. 7, 2885–2907. MR 3112511

Department of Mathematics, Kansas State University, 138 Cardwell Hall, 1228 N. 17th Street, Manhattan, KS 66506, USA

E-mail address: lamorim@ksu.edu

Department of Mathematical Sciences, Research Institute of Mathematics, Seoul National University, San 56-1, Shinrimdong, Gwanakgu, Seoul 47907, Korea

E-mail address: chocheol@snu.ac.kr

Department of Mathematics, Yonsei University, 50 Yonsei-Ro, Seodaemun-Gu, Seoul 03722, Korea

E-mail address: hansolhong@yonsei.ac.kr

Department of Mathematics and Statistics, Boston University, 111 Cummington Mall, Boston, MA 02215, USA

E-mail address: lau@math.bu.edu