An Elementary Proof of Eigenvalue Preservation for the Co-rotational Beris-Edwards System

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Received: 5 April 2018 / Accepted: 3 October 2018 / Published online: 13 October 2018
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Abstract
We study the corotational Beris-Edwards system modeling nematic liquid crystals and revisit the eigenvalue preservation property discussed in Wu et al. (Arch Rational Mech Anal, 2018. https://doi.org/10.1007/s00205-018-1297-2). We give an alternative but direct proof to the eigenvalue preservation of the initial data for the $Q$-tensor. It is noted that our proof is not only valid in the whole-space case, but in the bounded-domain case as well.

Keywords Corotational Beris-Edwards system · Q-tensor · Eigenvalue preservation

Mathematics Subject Classification 35Q35 · 35Q30

1 Introduction

In this paper, we study the eigenvalue preservation property of solutions for a hydrodynamic system modeling the evolution of nematic liquid crystals. Mathematically speaking, this system is composed of a coupled incompressible Navier–Stokes equations with anisotropic forces and Q-tensor equations of a parabolic type that describes
the evolution of the liquid crystal director field, which is called the Beris-Edwards system (Beris and Edwards 1994). In the Landau-de Gennes theory (Ball 2012; Gennes and Prost 1993), the basic element is a symmetric, traceless tensor \( Q \) that is a tensor-valued function taking values in the five-dimensional \( Q \)-tensor space

\[
S_0^{(3)} \equiv \{ Q \in \mathbb{M}^{3 \times 3}, \ Q' = Q, \ \text{tr}(Q) = 0 \}.
\]

The simplest form of the free energy in the Landau-de Gennes theory takes the following form:

\[
F(Q) \overset{\text{def}}{=} \int_{\Omega} \frac{L}{2} |\nabla Q|^2 + \frac{a}{2} \text{tr}(Q^2) - \frac{b}{3} \text{tr}(Q^3) + \frac{c}{4} \text{tr}^2(Q^2) \, dx,
\]

where \( \Omega \subset \mathbb{R}^3 \) is a smooth and bounded domain. Above in (1.1), we use the one constant approximation of the Oseen-Frank energy, and \( L, a, b, c \) are material-dependent constants that satisfy (Majumdar 2010; Majumdar and Zarnescu 2010)

\[
L > 0, \ b > 0, \ c > 0.
\]

The simplified Beris-Edwards system we study reads

\[
u_t + u \cdot \nabla u - \nu \Delta u + \nabla P = \lambda L \nabla \cdot (Q \Delta Q - \Delta QQ) - \lambda L \nabla \cdot (\nabla Q \otimes \nabla Q),
\]

\[\nabla \cdot u = 0,\]

\[Q_t + u \cdot \nabla Q - \omega Q + Q\omega = \Gamma \left( L \Delta Q - a Q + b \left[ Q^2 - \frac{\text{tr}(Q^2)}{3} I \right] - c Q \text{tr}(Q^2) \right),\]

with the following initial and boundary conditions

\[u(0, x) = u_0(x) \text{ with } \nabla \cdot u_0 = 0, \quad Q(0, x) = Q_0(x) \in S_0^{(3)},\]

\[u(t, x)|_{\partial \Omega} = 0, \quad Q(t, x)|_{\partial \Omega} = Q_0(x)|_{\partial \Omega} = \tilde{Q}(x).\]

Above \( u(t, x) : (0, +\infty) \times \Omega \to \mathbb{R}^3 \) stands for the incompressible fluid velocity field, \( Q(t, x) : (0, +\infty) \times \Omega \to S_0^{(3)} \) represents the order parameter of the liquid crystal molecules and \( \omega = \frac{\nabla u - \nabla^T u}{2} \) denotes the skew-symmetric part of the rate of strain tensor. The positive constants \( \nu, \lambda, \text{ and } \Gamma \) denote the fluid viscosity, the competition between kinetic energy and elastic potential energy, and macroscopic elastic relaxation time for the molecular orientation field, respectively (Wu et al. 2018). This simplified system is at time referred to as the “co-rotational” Beris-Edwards system (Paicu and Zarnescu 2012) in the literature, whose related mathematical study can be found in Abels et al. (2016), Anna (2017), Dai et al. (2016), Guillén-González and Rodríguez-Bellido (2014), Guillén-González and Rodríguez-Bellido (2015) and Paicu
and Zarnescu (2012). On the other hand, the full system is also called the “non co-
rotational” Beris-Edwards system, and we refer interested readers to (Abels et al. 2014;
Cavaterra et al. 2016; Anna and Zarnescu 2016; Paicu and Zarnescu 2011; Zhao and
Wang 2016; Zhao et al. 2017) for its relevant PDE and numeric work.

From the physical point of view, the main feature of nematic liquid crystals is the
locally preferred orientation of the nematic molecule directors. To this end, $Q$-tensors
are introduced, which are considered suitably normalized second-order moments of
the probability distribution function. Specifically, if $\mu_x$ is a probability measure on
the unit sphere $S^2$ representing the orientation of liquid crystal molecules at a point $x$
in space, then a $Q$-tensor denoted by $Q(x)$ is a symmetric and traceless $3 \times 3$
matrix defined by

$$Q(x) = \int_{S^2} \left( p \otimes p - \frac{1}{3} I \right) \, d\mu_x(p).$$

Indeed, it is a crude measure (from the viewpoint of statistical theory) of how the
second-moment tensor associated with a given probability measure deviates from its
isotropic value (Mottram and Newton 2014; Virga 1994). It is noted that (1.7) imposes
a constraint such that (see Mottram and Newton 2014)

$$-\frac{1}{3} \leq \lambda_i(Q) \leq \frac{2}{3}, \quad \forall 1 \leq i \leq 3.$$ 

Hence, it is easy to check that not every symmetric and traceless $3 \times 3$ matrix is a
physical $Q$-tensor but only those whose eigenvalues range in $[-\frac{1}{3}, \frac{2}{3}]$.

Motivated by the physical interpretation of the $Q$-tensors, it seems to be of great
importance to understand how the fluid dynamics would affect the behavior of eigen-
values of the $Q$-tensors as time evolves. Partially motivated by this question, in Wu
et al. (2018), the authors proved that certain eigenvalue constraints of the initial data
$Q_0$ are preserved by the evolution problem (1.3)–(1.6) when the domain is either the
entire Euclidean space or a periodic box. Inspired by the idea in Wu et al. (2018), in
this paper we give an alternative but direct proof whose argument works well both in
the whole-space case and in the bounded-domain case.

Our main result is stated as follows.

**Theorem 1.1** For any $u_0 \in H^1_0(\Omega), \nabla \cdot u_0 = 0$, $Q_0 \in H^2(\Omega; S^{(3)}_0)$ and $\tilde{Q} \in H^1(\partial \Omega)$,
let $(u(t, x), Q(t, x))$ be the unique local strong solution to the evolution problem (1.3)–
(1.6) on $[0, T]$. We assume

$$0 \leq a \leq \frac{b^2}{24c},$$

and the initial data $Q_0$ and the boundary data $\tilde{Q}$ satisfy

$$\lambda_i(Q_0(x)) \in \left[ -\frac{b + \sqrt{b^2 - 24ac}}{12c}, \frac{b + \sqrt{b^2 - 24ac}}{6c} \right], \quad \forall x \in \Omega, \ 1 \leq i \leq 3.$$ 

Then, for any $t \in (0, T]$ and $x \in \Omega$, the eigenvalues of $Q(t, x)$ stay in the same
interval.
Remark 1.1 By Theorem 1.1 in Liu and Wang (2016), the existence and uniqueness of local strong solutions to the evolution problem (1.3)–(1.6) is ensured and satisfies

\[ u \in H^1(0, T; H^1(\Omega)) \cap L^\infty(0, T; H^2(\Omega)), \quad \nabla \cdot u = 0, \]
\[ Q \in H^2(0, T; L^2(\Omega; S_0^{(3)})) \cap H^1(0, T; H^2(\Omega; S_0^{(3)})) \cap L^\infty(0, T; H^3(\Omega; S_0^{(3)})). \]

Remark 1.2 Compared to Wu et al. (2018), one extra assumption in Theorem 1.1 is \( a \geq 0 \) which captures a regime of physical interest but not the deep nematic regime (Anna and Zarnescu 2018). This assumption is only used to get the same lower bound \( -b + \sqrt{b^2 - 24ac} \), but not needed to achieve the upper bound \( b + \sqrt{b^2 - 24ac} \). We also want to point out that this assumption (1.8) is different from its counterpart imposed in Wu et al. (2018) because the bulk part is dealt with in different ways.

The idea of the proof is to proceed by contradiction and to exploit the variational characterization of the eigenvalues in relation to the evolution problem (1.3)–(1.6), which works for the solution \( Q \) with \( C^{1,2} \) regularity. If we were able to show the solutions to the Beris-Edwards system were regular enough, we would be done; however, this seems out of reach at the moment, though an interesting problem on its own. Fortunately, we can bypass this difficulty by using a regularization argument discussed in Wu et al. (2018) that preserves the eigenvalue constraints (the eigenvalues converge pointwise in fact in the whole domain).

For simplicity, we set the eigenvalues of matrix \( Q \)

\[ \lambda_i(t, x) \overset{\text{def}}{=} \lambda_i(Q(t, x)), \quad 1 \leq i \leq 3. \]

Without loss of generality, we assume

\[ \lambda_1(t, x) \geq \lambda_2(t, x) \geq \lambda_3(t, x), \quad \forall (t, x) \in \tilde{\Omega} \times [0, T] \]

As a matter of fact, we may establish the following more general result based on Theorem 1.1.

Corollary 1.1 For any given \( u_0 \in H_0^1(\Omega), \nabla \cdot u_0 = 0, Q_0 \in H^2(\Omega; S_0^{(3)}) \) and \( \tilde{Q} \in H^{5/2}(\partial \Omega) \), the unique local strong solution \((u(t, x), Q(t, x))\) to the evolution problem (1.3)–(1.6) on \([0, T]\) satisfies

\[ \lambda_1(t, x) \leq \max \left[ \frac{b + \sqrt{b^2 - 24ac}}{6c}, \max_{\Omega} \lambda_1(Q_0) \right], \]
\[ \lambda_3(t, x) \geq \min \left[ \frac{-b + \sqrt{b^2 - 24ac}}{12c}, \min_{\Omega} \lambda_3(Q_0) \right], \]

for any \( t \in (0, T) \) and \( x \in \Omega \).

Remark 1.3 Analogously, Theorem 1.1 and Corollary 1.1, also valid in the static case, provided the corresponding solution \( Q \in C(\tilde{\Omega}) \cap C^2(\Omega) \). However, this regularity issue cannot be solved directly by following the approximation argument in the Appendix part and henceforth is beyond the scope of our paper.
The proof of Theorem 1.1 and Corollary 1.1 is given in Sect. 2, while a related technical regularization lemma is presented in Appendix.

2 Proof of Theorem 1.1

Here and after, we let $|Q|$ denote the Frobenius norm of $Q \in \mathcal{S}_0(3)$, that is, $|Q| = \sqrt{tr(Q^TQ)}$ where $tr$ denotes the trace of a matrix. Also, because of the traceless property of $Q$-tensors, for all $(t, x) \in \bar{\Omega} \times [0, T]$ one has

$$\lambda_1(t, x) + \lambda_2(t, x) + \lambda_3(t, x) = 0. \tag{2.1}$$

To begin with, we see from Ryan (1969) and Nomizu (1973) that

Lemma 2.1 For $1 \leq i \leq 3$, $\lambda_i(t, x) \in C(\bar{\Omega} \times [0, T])$

Now we are ready to prove Theorem 1.1.

Proof Due to Lemma 2.2 in Appendix, we may assume $Q(t, x) \in C^{1,2}((0, T) \times \Omega) \cap C([0, T] \times \bar{\Omega})$.

Step 1 Let

$$\lambda_1(t_0, x_0) = \max_{(t, x) \in [0, T] \times \bar{\Omega}} \lambda_1(t, x). \tag{2.2}$$

We shall show that $\lambda_1(t_0, x_0) \leq \frac{b + \sqrt{b^2 - 24ac}}{6c}$. We prove by a contradiction argument. Suppose

$$(t_0, x_0) \in (0, T] \times \Omega, \text{ and } \lambda_1(t_0, x_0) > \frac{b + \sqrt{b^2 - 24ac}}{6c}. \tag{2.3}$$

Let $\tilde{v} \in \mathbb{S}^2$ be the corresponding unit eigenvector, such that $Q(t_0, x_0)\tilde{v} = \lambda_1(t_0, x_0)\tilde{v}$. Meanwhile, we denote

$$f(t, x) = \langle Q(t, x)\tilde{v}, \tilde{v} \rangle_{\mathbb{R}^3};$$

then, it is easy to check from (1.9) and (2.2) that

$$f(t_0, x_0) = \max_{(t, x) \in [0, T] \times \bar{\Omega}} f(t, x) \tag{2.4}$$

Next, we take the matrix inner product of equation (1.5) with $\tilde{v}\tilde{v}^T$ and evaluate the resultant at $(t_0, x_0)$. Note that $u \cdot \nabla f = 0$

$$\omega^{ik} Q^{kj} \tilde{v}^i \tilde{v}^j = \omega^{ik}(Q^{kj} \tilde{v}^j)\tilde{v}^i = \lambda_1 \omega^{ik} \tilde{v}^k \tilde{v}^l = 0,$$

$$Q^{ik} \omega^{kj} \tilde{v}^i \tilde{v}^j = (Q^{ik} \tilde{v}^i) \omega^{kj} \tilde{v}^j = \lambda_1 \omega^{kj} \tilde{v}^k \tilde{v}^l = 0;$$
hence, we get
\[
\partial_t f = \Delta f - a\lambda_1 - c \text{tr}(Q^2)\lambda_1 + b \left( \frac{\lambda_2^2}{3} - \frac{\text{tr}(Q^2)}{3} \right)
\]
\[
= \Delta f - \lambda_1 \left[ a + c(\lambda_1^2 + \lambda_2^2 + \lambda_3^2) \right] + b \left( \lambda_1^2 - \frac{\lambda_1^2 + \lambda_2^2 + \lambda_3^2}{3} \right)
\]
at \((t_0, x_0)\).
\tag{2.5}
\]

Using Cauchy–Schwarz inequality and \(Q \in S_0^{(3)}\), we get
\[
\lambda_1^2 + \lambda_2^2 + \lambda_3^2 \geq \lambda_1^2 + \frac{(\lambda_2 + \lambda_3)^2}{2} = \frac{3}{2} \lambda_1^2,
\]
which combined with (2.5) at \((t_0, x_0)\) gives
\[
\partial_t f \mid_{(t_0, x_0)} \leq \Delta f - a\lambda_1 - \frac{3c}{2} \lambda_1^3 + \frac{b}{3} \lambda_1^2 \mid_{(t_0, x_0)} \leq - \frac{3c}{2} \lambda_1 \left( \lambda_1^2 - \frac{b}{3c} \lambda_1 + \frac{2a}{3c} \right) \mid_{(t_0, x_0)} < 0.
\]

Above in the last inequality, we used (2.3). However, (2.4) indicates that
\[
\partial_t f \mid_{(t_0, x_0)} \geq 0,
\]
which is a contradiction. Therefore, we conclude that
\[
\lambda_1(t, x) \leq \frac{b + \sqrt{b^2 - 24ac}}{6c}, \quad \forall (t, x) \in (0, T] \times \Omega.
\tag{2.6}
\]

**Step 2** Let
\[
\lambda_3(\tilde{t}, \tilde{x}) = \min_{(t, x) \in [0, T] \times \tilde{\Omega}} \lambda_3(t, x).
\tag{2.7}
\]

We shall again show that \(\lambda_3(\tilde{t}, \tilde{x}) \geq \frac{-b - \sqrt{b^2 - 24ac}}{12c}\), by contradiction. Suppose
\[
(\tilde{t}, \tilde{x}) \in (0, T] \times \Omega, \quad \text{and} \quad \lambda_3(\tilde{t}, \tilde{x}) < \frac{-b - \sqrt{b^2 - 24ac}}{12c}.
\tag{2.8}
\]

Let \(\tilde{w} \in S^2\) be the corresponding unit eigenvector, such that \(Q(\tilde{t}, \tilde{x})\tilde{w} = \lambda_3(\tilde{t}, \tilde{x})\tilde{w}\). Meanwhile, we denote
\[
g(t, x) = \langle Q(t, x)\tilde{w}, \tilde{w} \rangle_{\mathbb{R}^3};
\]
then, we see from (1.9) and (2.7) that
\[
g(\tilde{t}, \tilde{x}) = \min_{(t, x) \in [0, T] \times \tilde{\Omega}} g(t, x)
\tag{2.9}
\]
After taking the matrix inner product of equation (1.5) with \( \vec{w} \vec{w}' \) and evaluating at \((\tilde{t}, \tilde{x})\), it gives

\[
\mathbf{u} \cdot \nabla g = 0,
\]

\[
\omega^{ik} Q^{kj} \vec{w}^i \vec{w}^j = \omega^{ik}(Q^{kj} \vec{w}^j) \vec{w}^i = \lambda_1 \omega^{ik} \vec{w}^k \vec{w}^j = 0,
\]

\[
Q^{ik} \omega^{kj} \vec{w}^i \vec{w}^j = (Q^{ik} \vec{w}^i) \omega^{kj} \vec{w}^j = \lambda_1 Q^{ik} \vec{w}^k \vec{w}^j = 0.
\]

Consequently, we obtain

\[
\partial_t g = \Delta g - a \lambda_3 - c \text{tr}(Q^2) \lambda_3 + b \left[ \frac{\lambda_3^2}{3} - \frac{\text{tr}(Q^2)}{3} \right]
\]

\[
= \Delta g - \lambda_3 \left[ a + c(\lambda_1^2 + \lambda_2^2 + \lambda_3^2) \right] + b \left( \frac{\lambda_2^2 + \lambda_3^2 + \lambda_3^2}{3} \right) \text{ at } (\tilde{t}, \tilde{x}). \quad (2.10)
\]

We claim

\[
\partial_t g |_{(\tilde{t}, \tilde{x})} \geq - \frac{3c}{2} \lambda_3 \left( \lambda_3^2 + \frac{b}{6c} \lambda_3 + \frac{a}{6c} \right) |_{(\tilde{t}, \tilde{x})} . \quad (2.11)
\]

Here, we focus on the proof of the theorem and the proof of the claim will be postponed to the end of the section. Combining (2.8) and the claim (2.11), we get

\[
\partial_t g |_{(\tilde{t}, \tilde{x})} > 0.
\]

On the other hand, however, based on (2.9) one can deduce that

\[
\partial_t g |_{(\tilde{t}, \tilde{x})} \leq 0,
\]

which is again a contradiction. Thus,

\[
\lambda_3(t, x) \geq - \frac{b + \sqrt{b^2 - 24ac}}{12c} , \quad \forall (t, x) \in (0, T] \times \Omega. \quad (2.12)
\]

The proof is complete by combining (2.6) and (2.12).

\[\Box\]

**Proof** We divide the proof of (2.11) into three cases. By (2.1) and (2.7), we know that \(|\lambda_2| \leq |\lambda_3|\).

**Case 1** \(|\lambda_2|_{(\tilde{t}, \tilde{x})} \geq 0\)

First of all, note that

\[
\frac{3}{2} \lambda_3^2 = \frac{(\lambda_1 + \lambda_2)^2}{2} + \lambda_3^2 \leq \lambda_1^2 + \lambda_2^2 + \lambda_3^2 \leq (\lambda_1 + \lambda_2)^2 + \lambda_3^2 = 2\lambda_3^2,
\]

which together with (2.10) and the fact that \(g\) attains minimum at \((\tilde{t}, \tilde{x})\) yields
\begin{align*}
\partial_t g|_{(\tilde{t}, \tilde{x})} & \geq \Delta g - a\lambda_3 - \frac{3c}{2} \lambda_3^2 + \frac{b}{3} \lambda_3^3 |_{(\tilde{t}, \tilde{x})} \\
& \geq -a\lambda_3 - \frac{3c}{2} \lambda_3^2 + \frac{b}{3} \lambda_3^3 |_{(\tilde{t}, \tilde{x})} \\
& \geq -\frac{3c}{2} \lambda_3 \left( \frac{\lambda_3^2 + b}{6c} \lambda_3 + \frac{a}{6c} \right) |_{(\tilde{t}, \tilde{x})}.
\end{align*}

**Case 2** \( \frac{(\sqrt{5} - 1)}{2} \lambda_3 |_{(\tilde{t}, \tilde{x})} \leq \lambda_2 |_{(\tilde{t}, \tilde{x})} < 0 \)

In this case, again thanks to (2.1) \(|\lambda_1| > |\lambda_3|\) at \((\tilde{t}, \tilde{x})\), we have

\begin{align*}
2\lambda_3^2 < \lambda_1^2 + \lambda_2^2 + \lambda_3^2 &= (\lambda_2 + \lambda_3)^2 + \lambda_2^2 + \lambda_3^2 = 2(\lambda_2^2 + \lambda_3^2 + \lambda_2 \lambda_3) \\
&= 2(\lambda_2^2 + \lambda_2 \lambda_3 - \lambda_3^2) + 4\lambda_3^2 \leq 4\lambda_3^2.
\end{align*}

Hence,

\begin{align*}
\partial_t g|_{(\tilde{t}, \tilde{x})} & \geq \Delta g - a\lambda_3 - 2c\lambda_3^3 - \frac{b}{3} \lambda_3^2 |_{(\tilde{t}, \tilde{x})} = -2c\lambda_3 \left( \frac{\lambda_3^2 + b}{6c} \lambda_3 + \frac{a}{2c} \right) |_{(\tilde{t}, \tilde{x})} \\
& \geq -\frac{3c}{2} \lambda_3 \left( \frac{\lambda_3^2 + b}{6c} \lambda_3 + \frac{a}{6c} \right) |_{(\tilde{t}, \tilde{x})}.
\end{align*}

**Case 3** \( \lambda_3 |_{(\tilde{t}, \tilde{x})} \leq \lambda_2 |_{(\tilde{t}, \tilde{x})} < \frac{(\sqrt{5} - 1)}{2} \lambda_3 |_{(\tilde{t}, \tilde{x})} < 0 \)

Note that

\begin{align*}
4\lambda_3^2 < 2(\lambda_2^2 + \lambda_2 \lambda_3 - \lambda_3^2) + 4\lambda_3^2 &= 2(\lambda_2^2 + \lambda_3^2 + \lambda_2 \lambda_3) \\
&= (\lambda_2 + \lambda_3)^2 + \lambda_2^2 + \lambda_3^2 = \lambda_1^2 + \lambda_2^2 + \lambda_3^2 \quad \text{ at } (\tilde{t}, \tilde{x}),
\end{align*}

which gives

\begin{align*}
- \lambda_3 [a + c(\lambda_1^2 + \lambda_2^2 + \lambda_3^2)] + b \left( \lambda_3^2 - \frac{\lambda_1^2 + \lambda_2^2 + \lambda_3^2}{3} \right) \\
= -2c\lambda_3 \left[ (\lambda_2^2 + \lambda_3^2 + \lambda_2 \lambda_3) + b \left( \frac{2\lambda_2^2 + 2\lambda_2 \lambda_3 - \lambda_3^2}{\lambda_3} \right) + \frac{a}{2c} \right] \\
\overset{\text{def}}{=} -2c\lambda_3 H(\lambda_2, \lambda_3) \quad \text{ at } (\tilde{t}, \tilde{x}).
\end{align*}

We proceed to show that

\[ H(\lambda_2, \lambda_3) \geq \lambda_3^2 + \frac{b}{6c} \lambda_3 + \frac{a}{6c} \quad \text{ at } (\tilde{t}, \tilde{x}), \quad (2.13) \]

which is equivalent to

\[ \lambda_2^2 + \lambda_2 \lambda_3 + \frac{b}{3c} \frac{(\lambda_2^2 + \lambda_2 \lambda_3 - \lambda_3^2)}{\lambda_3} + \frac{a}{3c} \geq 0 \quad \text{ at } (\tilde{t}, \tilde{x}). \quad (2.14) \]
Let us denote \( \mu = \lambda_2^2 + \lambda_2 \lambda_3 - \lambda_3^2 \big|_{(\tilde{t}, \tilde{x})} \), then \( 0 < \mu \leq \lambda_3^2 \) due to the assumption in Case 3 and (2.14) is reduced to

\[
(1 + \frac{b}{3c} \lambda_3) \mu + \lambda_3^2 + \frac{a}{3c} \geq 0 \quad \text{at } (\tilde{t}, \tilde{x}).
\]  

(2.15)

By (2.8), we have

\[
-3 \leq 1 - \frac{4b}{b + \sqrt{b^2 - 24ac}} \leq \left(1 + \frac{b}{3c} \lambda_3\right)_{(\tilde{t}, \tilde{x})} \leq 1
\]  

(2.16)

If \( 0 \leq \left(1 + \frac{b}{3c} \lambda_3\right)_{(\tilde{t}, \tilde{x})} \leq 1 \), then (2.15) is automatically true. Otherwise, since \( \mu \) is a monotone decreasing, nonnegative function of \( \lambda_2 \) on the given interval, we have

\[
(1 + \frac{b}{3c} \lambda_3) \mu + \lambda_3^2 + \frac{a}{3c} \geq \left(1 + \frac{b}{3c} \lambda_3\right)_{(\tilde{t}, \tilde{x})} > 0 \quad \text{at } (\tilde{t}, \tilde{x}),
\]

where we used (2.8) in the last inequality above. In all, (2.15) is valid, and so is (2.13). As a consequence,

\[
\partial_t g_{(\tilde{t}, \tilde{x})} \geq -2c\lambda_3 \left(\lambda_3^2 + \frac{b}{6c} \lambda_3 + \frac{a}{6c}\right)_{(\tilde{t}, \tilde{x})} \geq -\frac{3c}{2} \lambda_3 \left(\lambda_3^2 + \frac{b}{6c} \lambda_3 + \frac{a}{6c}\right)_{(\tilde{t}, \tilde{x})}.
\]

The proof of claim 2.11 is complete.

After this, Corollary 1.1 can be easily established.

**Proof of Corollary 1.1** Without loss of generality, we may assume

\[
Q(t, x) \in C^{1,2}((0, T) \times \Omega) \cap C([0, T] \times \tilde{\Omega}),
\]

and let

\[
\lambda_1(t_0, x_0) = \max_{(t, x) \in [0, T] \times \tilde{\Omega}} \lambda_1(t, x).
\]  

(2.17)

If

\[
\lambda_1(t_0, x_0) > \max \left[ \frac{b + \sqrt{b^2 - 24ac}}{6c}, \max_{\Omega} \lambda_1(Q_0) \right],
\]

then \( t_0 \in (0, T] \) and \( x_0 \in \Omega \). As a consequence, it follows from the same argument as in the proof of Theorem 1.1 that

\[
0 \leq -\frac{3c}{2} \lambda_1 \left(\lambda_1^2 - \frac{b}{3c} \lambda_1 + \frac{2a}{3c}\right)_{(t_0, x_0)} < 0.
\]
due to the assumption that \( \lambda_1(t_0, x_0) > \frac{b+\sqrt{b^2-24ac}}{6c} \), which is a contradiction. The corresponding lower bound for \( \lambda_3(t, x) \) can be proved in a similar manner. \( \Box \)

**Remark 2.1** With minor modifications, one may check easily that the above arguments are also valid for the whole-space case that is shown in Wu et al. (2018).

**Acknowledgements** We thank the anonymous referees for their careful reading and useful suggestions to improve our paper, especially the observation that leads to Corollary 1.1, which can be considered an improved result of Theorem 1.1. The work of A. Contreras was partially supported by a grant from the Simons Foundation # 426318. And the work of Zhang is supported by the start-up fund from Department of Mathematics, Rutgers University. We want to thank our friends Xavier Lamy, Yuning Liu and Arghir Zarnescu for their kind discussions. In particular, Xu would like to express his gratitude to Arghir for his consistent support and academic communications over a series of topics on the mathematical \( Q \)-tensor theory during the past six years. Without his idea proposed in Wu et al. (2018), this paper would not come out.

**Appendix**

In this appendix section, using the same idea as (Wu et al. 2018), we prove

Lemma 2.2 Let \( u \in H^1((0, T); H^1(\Omega)) \cap L^\infty(0, T; H^2(\Omega)), u_\delta \in C^\infty([0, T] \times \tilde{\Omega}), \nabla \cdot u = \nabla \cdot u_\delta = u \mid_{\partial \Omega} = u_\delta \mid_{\partial \Omega} = 0 \) be such that \( u_\delta \to u \) as \( \delta \to 0 \) strongly in \( H^1((0, T); H^1(\Omega)) \cap L^\infty(0, T; H^2(\Omega)) \). Let \( Q_\delta \) be the unique classical solution in \( C^{1,2}((0, T) \times \Omega) \cap C([0, T] \times \tilde{\Omega}) \) of the system

\[
\begin{align*}
Q_\delta t + u \cdot \nabla Q_\delta - \omega_\delta Q_\delta + Q_\delta \omega_\delta &= \Gamma \left( L \Delta Q_\delta - a Q_\delta + b \left[ (Q_\delta)^2 - \frac{\text{tr}(Q_\delta)^2}{3} I \right] - c Q_\delta \text{tr}(Q_\delta)^2 \right), \\
Q_\delta(t, x) \mid_{\partial \Omega} &= \tilde{Q}(x),
\end{align*}
\]

where \( \omega_\delta = \frac{\nabla u_\delta - \nabla^T u_\delta}{2} \). Assume that

\[
\tilde{m} \leq \lambda_i(Q_\delta(t, x)) \leq \tilde{M}, \quad \forall \ 1 \leq i \leq 3, \quad (t, x) \in [0, T] \times \Omega.
\]

Then, \( Q^{(\delta)}(t, x) \to Q(t, x) \) as \( \delta \to 0, \forall t \geq 0, x \in \Omega \), where \( Q \) is the unique solution in \( H^1((0, T); H^2(\Omega)) \cap L^\infty(0, T; H^3(\Omega)) \) of

\[
\begin{align*}
Q_t + u \cdot \nabla Q - \omega Q + Q \omega &= \Gamma \left( L \Delta Q - a Q + b \left[ Q^2 - \frac{\text{tr}(Q)^2}{3} I \right] - c Q \text{tr}(Q^2) \right), \\
Q(t, x) \mid_{\partial \Omega} &= \tilde{Q}(x).
\end{align*}
\]

Furthermore, we have an eigenvalue constraint on \( Q \) such that

\[ \text{Springer} \]
\[ \dot{m} \leq \lambda_i (Q(t, x)) \leq \bar{M}, \quad \forall 1 \leq i \leq 3, \quad (t, x) \in (0, T) \times \Omega \]  

(2.23)

provided the initial data \( Q_0 \) and boundary data \( \bar{Q} \) have the same constraint.

**Remark 2.2** Theorem 1.1 in Liu and Wang (2016) is only one step away from the existence of local classical solutions to the evolution problem (1.3)–(1.6), which nevertheless cannot be improved using the method therein. However, it remains to be an interesting question to study.

**Proof** Without loss of generality, we set \( \Gamma = L = 1 \). To begin with, we show a priori \( L^\infty \) bound on \( Q \). Since \( c > 0 \), there exists \( \eta_0 > 0 \), such that

\[ -a |M|^2 + b \text{tr}(M)^3 - c |M|^4 \leq 0, \quad \forall |M| \geq \eta_0, \ M \in S_0^3 \]

Let \( \eta \overset{\text{def}}{=} \max\{\|Q_0\|_{L^\infty(\Omega)}, \|\bar{Q}\|_{L^\infty(\Omega)}, \eta_0\} \). Multiplying (2.21) with \( Q(|Q|^2 - \eta)_+ \), then integrating over \( \Omega \) and using integration by parts, we obtain

\[
\frac{1}{2} \frac{d}{dt} \int_{\Omega} (|Q|^2 - \eta)_+^2 = -\int_{\Omega} |\nabla Q|^2 (|Q|^2 - \eta)_+ \, dx - \int_{\Omega} |\nabla (|Q|^2 - \eta)_+|^2 \, dx \\
+ \int_{\Omega} (-a |Q|^2 + b \text{tr}(Q^3) - c |Q|^4) (|Q|^2 - \eta)_+ \leq 0
\]

Thus,

\[ \|Q(t, \cdot)\|_{L^\infty(\Omega)} \leq \eta, \quad \forall 0 \leq t \leq T. \]  

(2.24)

In the same way, we conclude

\[ \|Q^\delta(t, \cdot)\|_{L^\infty(\mathbb{R}^3)} \leq \eta, \quad \forall 0 \leq t \leq T, \ \forall \delta > 0. \]  

(2.25)

Let \( R^\delta \overset{\text{def}}{=} Q^\delta - Q \in S_0^3 \). Then it is easy to see

\[
\partial_t R^\delta + u_\delta \nabla R^\delta - \omega_\delta R^\delta + R^\delta \omega_\delta \\
= \Delta R^\delta - a R^\delta + (u - u_\delta) \nabla Q - (\omega - \omega_\delta) Q + Q(\omega - \omega_\delta) \\
+ b \left[ R^\delta Q^\delta + Q R^\delta - \frac{\text{tr}(R^\delta Q^\delta + Q R^\delta)}{3} \right] \\
- c \left[ |Q^\delta|^2 R^\delta + \text{tr}(R^\delta Q^\delta + Q R^\delta) Q \right],
\]

with initial and boundary datum \( R^\delta_0 = R^\delta|_{\partial \Omega} \equiv 0 \). Multiplying the above equation with \( R^\delta \), integrating over \( \Omega \), by (2.24) and (2.25), we get

\[
\frac{d}{dt} \int_{\Omega} |R^\delta|^2 \, dx \\
\leq C \int_{\Omega} |R^\delta|^2 \, dx + C \int_{\Omega} [(u - u_\delta) \nabla Q - (\omega - \omega_\delta) Q + Q(\omega - \omega_\delta)] R^\delta \, dx
\]
\[ \leq C\|R^\delta\|_{L^2}^2 + C(\|\nabla Q\|_{L^2}\|u - u_\delta\|_{L^2} + \|Q\|_{L^2}\|\omega(s) - \omega_\delta(s)\|_{L^2}) \]
\[ \leq C\|R^\delta\|_{L^2}^2 + C(\|u - u_\delta\|_{L^2} + \|\omega(s) - \omega_\delta(s)\|_{L^2}). \]

(2.26)

Hence, Gronwall’s inequality gives
\[ \|R^\delta(t, \cdot)\|_{L^2}^2 \leq Ce^{Ct}\int_0^t \|u - u_\delta\|_{H^1}(s) \, ds \to 0, \quad \text{as} \; \delta \to 0 \]
due to the assumption that \( u_\delta \to u \) strongly in \( L^\infty(0, T; H^2(\Omega)) \). Therefore, combined with the fact that \( Q^\delta, Q \in C([0, T] \times \overline{\Omega}) \), we get
\[ Q^\delta(t, x) \to Q(t, x), \quad \forall \, (t, x) \in [0, T] \times \Omega \]  
(2.27)

Moreover, (2.23) follows from (2.27) and Lemma 2.1

\[ \square \]

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