FINITE DIMENSIONAL NICHOLS ALGEBRAS OVER THE
SUZUKI ALGEBRAS I: SIMPLE YETTER-DRINFELD
MODULES OF $A_{N/2n}^{\mu \lambda}$

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ABSTRACT. The Suzuki algebra $A_{N/2n}^{\mu \lambda}$ was introduced by Suzuki Satoshi in 1998, which is a class of cosemisimple Hopf algebras. It is not categorically Morita-equivalent to a group algebra in general. In this paper, the author gives a complete set of simple Yetter-Drinfeld modules over the Suzuki algebra $A_{N/2n}^{\mu \lambda}$ and investigates the Nichols algebras over those simple Yetter-Drinfeld modules. The involved finite dimensional Nichols algebras of diagonal type are of Cartan type $A_1$, $A_1 \times A_1$, $A_2$, $A_2 \times A_2$, Super type $A_2(q; I_2)$ and the Nichols algebra $ufo(8)$. There are 64, $4m$ and $m^2$-dimensional Nichols algebras of non-diagonal type over $A_{N/2n}^{\mu \lambda}$. The 64-dimensional Nichols algebras are of dihedral rack type $D_4$. The $4m$ and $m^2$-dimensional Nichols algebras $B(V_{abe})$ discovered first by Andruskiewitsch and Giraldi can be realized in the category of Yetter-Drinfeld modules over $A_{N/2n}^{\mu \lambda}$. By using a result of Masuoka, we prove that $\dim B(V_{abe}) = \infty$ under the condition $b^2 = (ae)^{-1}$, $b \in \mathbb{C}_m$ for $m \geq 5$.

1. Introduction

Let $\mathbb{k}$ be an algebraically closed field of characteristic 0. The motivation of the paper is to make some contributions to the following project.

Problem 1.1. How to classify all finite dimensional Hopf algebras over the Suzuki algebra $A_{N/2n}^{\mu \lambda}$?

There are only a few works to deal with the problem. In 2004, Menini and coauthors studied the quantum lines over $A_{4m}$ and $B_{4m}$, which are isomorphic to $A_{1m}^{++}$ and $A_{1m}^{-+}$ respectively $\cite{9}$. In 2019, the author $\cite{27}$ classified
finite dimensional Hopf algebras over the Kac-Paljutkin algebra $A_{12}^{++}$ and Fantino et al. [10] classified finite dimensional Hopf algebras over the dual of dihedral group $\mathbb{k}D_{2m}$ of order $2m$, with $m = 4a \geq 12$, where $\mathbb{k}D_{2m}$ is a 2-cocycle deformation of $A_{12}^{++}$ [21].

Why are we interested in the Suzuki algebras? Firstly, our classification project is different with the classification of pointed Hopf algebras, since the Suzuki algebras are not categorically Morita-equivalent to group algebras in general. The Suzuki algebras are non-trivial semisimple unless $(n, \lambda) = (2, +1)$. Two semisimple Hopf algebras $K$ and $H$ are categorically Morita-equivalent iff ${}_k^k\mathcal{YD}$ and ${}_H^H\mathcal{YD}$ are equivalent as braided tensor categories. In [4], Andruskiewitsch and coauthors constructed new Hopf algebras with semisimple Hopf algebras as coradicals which are categorically Morita-equivalent to group algebras, including group-theoretical Hopf algebras, in particular those from abelian extensions. The Suzuki algebras can be obtained by abelian extensions [29, Page 18], so they are group-theoretical [24, Theorem 1.3]. But they are not categorically Morita-equivalent to group algebras in general, for example $A_{12}^{++}$ [23, Section 5.2].

Secondly, the study of Nichols algebras with non-group type braidings is rare and it is an interesting problem to find new finite-dimensional Nichols algebras over non-trivial semisimple Hopf algebras. In the past decades, the study of Nichols algebras are mainly focus on the Yetter-Drinfeld categories of group algebras. In [5, section 3.7], Andruskiewitsch and Giraldi found two classes of $4m$ and $m^2$-dimensional Nichols algebras (we call those Nichols algebras are of type $V_{abu}$) which generally cannot be realized in the Yetter-Drinfeld categories of group algebras. Together with the results of [28], we will see that those Nichols algebras can be realized in the Yetter-Drinfeld category of $A_{Nn}^{\mu\lambda}$.

Thirdly, it is meaningful to provide many examples of Yetter-Drinfeld modules with non-group type braidings. The classification of finite dimensional Nichols algebras of group type has archived great success. For examples, Nichols algebras of diagonal type with finite dimension were classified completely by Heckenberger [13] based on the theory of reflections [14] and Weyl groupoid [12]; the classification of Nichols algebras of non-simple semisimple Yetter-Drinfeld modules over non-abelian groups were almost finished by Heckenberger and Vendramin [16] [15]. To generalize the method of group type to non-group type, one obstacle is to realize braidings of non-group type in Yetter-Drinfeld categories. In case of group type, Andruskiewitsch and Graña [6] built connections between Nichols algebras...
of group-type and racks. It is convenient to realize braidings of rack type in Yetter-Drinfeld categories of finite groups. For this reason, the theory of reflections was also used to prove that the Nichols algebras of rack type \( D \) is infinite dimensional [3]. Let \((V, c)\) be a rigid braided vector space. It was shown by Schauenburg that \((V, c)\) can be realized on a coquasitriangular Hopf algebra \((H, \sigma)\) as a right \( H \)-comodule and \( c \) arising from \( \sigma \) [26] [30]. The realization is complicated in general. For example, the Suzuki algebras give a realization of \((V_{abe}, c)\) [29]. As for the realization of non-group type braidings is not easy, it is meaningful to provide examples for observation.

Our classification project over the Suzuki algebras is based on the lifting method introduced by Andruskiewitsch and Schneider [7]. The lifting method is a general framework to classify finite dimensional non-semisimple Hopf algebras with a fixed sub-Hopf algebra \( H \) as coradical. One crucial step of the lifting method is to find out all Yetter-Drinfeld modules \( V \) over \( H \) such that the Nichols algebra \( \mathcal{B}(V) \) has finite dimension. In this paper, we deal with the Nichols algebras over simple Yetter-Drinfeld modules of \( A_{Nn}^{\mu \lambda} \) with \( n \) even. And in the sequel [28], we will study the case with \( n \) odd.

To investigate Nichols algebras, first we should know how to construct all Yetter-Drinfeld modules over a finite dimensional Hopf algebra. Majid [20] identified the Yetter-Drinfeld modules with the modules of the Drinfeld double via the category equivalence \( H^\mathcal{YD} \cong H^{\text{cop}} \mathcal{YD}^{H^{\text{cop}}} \cong D(H^{\text{cop}}) \mathcal{M} \). And many mathematicians have contributed to the construction of Yetter-Drinfeld modules, for example [11] [8] [19] [25] [18] [17]. We take Radford’s method. There are exactly \( 8N^2 \) one-dimensional, \( 2N^2(4n^2 - 1) \) two-dimensional and \( 8N^2 \) \( 2n \)-dimensional non-isomorphic Yetter-Drinfeld modules over \( A_{N2n}^{\mu \lambda} \), see the Theorem 3.1.

The involved Nichols algebras in the paper are of diagonal type, rack type, type \( V_{abe} \), and the Nichols algebra \( \mathcal{B} \left( \mathcal{H}_{jk,p}^{\mathcal{S}} \right) \). The finite dimensional Nichols algebras of diagonal type over simple Yetter-Drinfeld modules of \( A_{N2n}^{\mu \lambda} \) are of Cartan type \( A_1, A_1 \times A_1, A_2, A_2 \times A_2 \), Super type \( A_2(q; I_2) \) and the Nichols algebra \( ufo\) of \( 8 \). As a summary, we have the following theorem.

**Theorem 1.2.** Let \( M \) be a simple Yetter-Drinfeld module over \( A_{N2n}^{\mu \lambda} \). If \( \mathcal{B}(M) \) is of diagonal type and \( \dim \mathcal{B}(M) < \infty \), then \( \mathcal{B}(M) \) can be summarized as follows.

1. Cartan type \( A_1 \), see Lemmas 4.1 and 4.3;
2. Cartan type \( A_1 \times A_1 \), see Lemmas 4.5, 4.6, 4.7, 4.10, 4.19, 4.22, 4.23, 4.13 and 4.17;
(3) Cartan type $A_2$, see Lemmas 4.5, 4.6, 4.7, 4.10, 4.19, 4.22, 4.23, 4.13 and 4.17;

(4) Cartan type $A_2 \times A_2$, see Lemmas 4.22 and 4.23;

(5) Super type $A_2(q; \mathbb{I}_2)$, see Lemmas 4.7 and 4.10;

(6) The Nichols algebra $\mathfrak{ufo}(8)$, see Lemmas 4.7 and 4.10.

The Nichols algebra $B\left(\mathcal{R}_{jk,p}\right)$ is of rack type. If $n > 2$, then it is of type $D$, see the Lemma 4.21. If $B\left(\mathcal{R}_{jk,p}\right)$ is finite dimensional and $n = 2$, then it is of dihedral rack type $D_4$ or Cartan type $A_2 \times A_2$, see the Lemma 4.22.

The Nichols algebras $B\left(\mathcal{H}_{jk,p}^s\right)$ and $\mathfrak{B}\left(\mathcal{A}_{jk,p}^s\right)$ are of type $V_{abe}$, see the section 4.2. If $ae = b^2$, then $\mathfrak{B}(V_{abe})$ is of diagonal type. If $ae \neq b^2$, according to [5, section 3.7] and the Corollary 4.15, we have

$$
\dim \mathfrak{B}(V_{abe}) = \begin{cases}
4m, & b = -1, ae \in \mathbb{G}_m, \\
m^2, & ae = 1, b \in \mathbb{G}_m \text{ for } m \geq 2, \\
\infty, & b^2 = (ae)^{-1}, b \in \mathbb{G}_m \text{ for } m \geq 5, \\
\infty, & b \notin \mathbb{G}_m \text{ for } m \geq 2, \\
\text{unknown, otherwise}, &
\end{cases}
$$

where $\mathbb{G}_m$ denotes the set of $m$-th primitive roots of unity.

If $B\left(\mathcal{H}_{jk,p}^s\right)$ is finite dimensional and $n = 1$, then it is of Cartan type $A_1 \times A_1$ or $A_2$. If $B\left(\mathcal{H}_{jk,p}^s\right)$ is finite dimensional, $\lambda = 1$, and $n = 2$, then it is of Cartan type $A_2 \times A_2$. If $n > 2$, then the Nichols algebra $B\left(\mathcal{H}_{jk,p}^s\right)$ is complicated, see section 4.4.2 for the case $n = 3$.

The two unsolved cases in the paper are difficult in general.

**Problem 1.3.** Determine the dimensions of the following Nichols algebras.

1. the unknown case in the formula (1.1);
2. the unknown case for $\mathfrak{B}\left(\mathcal{H}_{jk,p}^s\right)$, see the section 4.4.

The paper is organized as follows. In the section 1, we introduce the motivation and background of the paper and summarize our main results. In the section 2, we make an introduction for the Suzuki algebra and construct all simple representations of $A_{N2n}^{\mu \lambda}$. In the section 3, we construct all simple Yetter-Drinfeld modules over $A_{N2n}^{\mu \lambda}$ by using Radford’s method and put the construction of those Yetter-Drinfeld modules in the appendix. In the section 4, we calculate Nichols algebras over simple Yetter-Drinfeld modules of $A_{N2n}^{\mu \lambda}$ in cases: diagonal type, type $V_{abe}$, the Nichols algebras
\[ \mathfrak{B}(s_{jk}^p) \text{ and } \mathfrak{B}(s_{jk,p}). \] Finite dimensional Nichols algebras of diagonal type or non-diagonal type are obtained, and there are two cases unsolved.

2. The Hopf Algebra \( A_{Nn}^{\mu \lambda} \) and the Representations of \( A_{N2n}^{\mu \lambda} \)

Suzuki introduced a family of cosemisimple Hopf algebras \( A_{Nn}^{\mu \lambda} \) which is parametrized by integers \( N \geq 1, n \geq 2 \) and \( \mu, \lambda = \pm 1 \), and investigated various properties and structures of them [29]. As explained in [31, Example 13.3], the suzuki algebras give the braided vector space \((V_{ahe}, c)\) a realization in Yetter-Drinfeld category. Wakui studied the Suzuki algebra \( A_{Nn}^{\mu \lambda} \) in perspectives of polynomial invariant [33], braided Morita invariant [34] and coribbon structures [32]. The Hopf algebra \( A_{Nn}^{\mu \lambda} \) is generated by \( x_{11}, x_{12}, x_{21}, x_{22} \) subject to the relations:

\[
\begin{align*}
x_{11}^2 &= x_{22}, \quad x_{12}^2 = x_{21}, \quad x_n^n = \lambda x_{12}^n, \quad x_{11}^n = x_{22}^n, \\
x_{11}^{2N} + \mu x_{12}^{2N} &= 1, \quad x_{ij}x_{kl} = 0 \text{ whenever } i + j + k + l \text{ is odd,}
\end{align*}
\]

where we use the following notation for \( m \geq 1, \)

\[
\begin{align*}
\chi_{11}^m &:= x_{11}x_{12}x_{21} \cdots, \\
\chi_{22}^m &:= x_{22}x_{11}x_{21} \cdots, \\
\chi_{12}^m &:= x_{12}x_{21}x_{11} \cdots, \\
\chi_{21}^m &:= x_{21}x_{12}x_{22} \cdots.
\end{align*}
\]

The Hopf algebra structure of \( A_{Nn}^{\mu \lambda} \) is given by

\[
\Delta(x_{ij}^k) = x_{ij}^k \otimes x_{ij}^k + x_{12}^k \otimes x_{21}^k, \quad \epsilon(x_{ij}) = \delta_{ij}, \quad S(x_{ij}) = x_{ji}^{4N-1},
\]

for \( k \geq 1, i, j = 1, 2. \)

Let \( \{i, i+1, i+2, \cdots, i+j\} \) be an index set. Then the basis of \( A_{Nn}^{\mu \lambda} \) can be represented by

\[
\{ x_{11}^{i_1}x_{22}^{i_2}, x_{12}^{i_3}x_{21}^{i_4} \mid s \in \overline{1, 2N}, t \in \overline{0, n-1} \}.
\]

Thus for \( s, t \geq 0 \) with \( s + t \geq 1, \)

\[
\begin{align*}
\Delta(x_{11}^{s}x_{22}^{t}) &= x_{11}^{s}x_{22}^{t} \otimes x_{11}^{s}x_{22}^{t} + x_{12}^{s}x_{21}^{t} \otimes x_{21}^{s}x_{12}^{t}, \\
\Delta(x_{12}^{s}x_{21}^{t}) &= x_{11}^{s}x_{22}^{t} \otimes x_{21}^{s}x_{12}^{t} + x_{12}^{s}x_{21}^{t} \otimes x_{22}^{s}x_{11}^{t}.
\end{align*}
\]

The cosemisimple Hopf algebra \( A_{Nn}^{\mu \lambda} \) is decomposed to the direct sum of simple subcoalgebras such as \( A_{Nn}^{\mu \lambda} = \bigoplus_{g \in G} \mathbb{K}g \oplus \bigoplus_{0 \leq s \leq N-1} \mathbb{K}c_{st} \) [29, Theorem 3.1][33, lemma 5.5], where

\[
G = \left\{ x_{11}^{2s} \pm x_{12}^{2s} \pm x_{11}^{2s+1} x_{22}^{-1} \pm \sqrt{\lambda} x_{12}^{2s+1} x_{21}^{-1} \mid s \in \overline{1, N} \right\},
\]
The set \( \{ k \mid g \in G \} \cup \{ \kappa x_{11}^{2s} + \kappa x_{12}^{2s} + \kappa x_{11}^{2t} + \kappa x_{12}^{2t} \mid s \in 1, N, t \in 1, n-1 \} \) is a full set of non-isomorphic simple left \( A_{Nn}^{\mu \lambda} \)-comodules, where the coactions of the comodules listed above are given by the coproduct \( \Delta \). Denote the comodule \( \kappa x_{11}^{2s} + \kappa x_{12}^{2s} \) by \( \Lambda_{st} \). That is to say the comodule \( \Lambda_{st} = \kappa w_1 + \kappa w_2 \) is defined as

\[
\rho (w_1) = x_{11}^{2s} \otimes w_1 + x_{12}^{2s} \otimes w_2, \quad \rho (w_2) = x_{11}^{2t} \otimes w_2 + x_{12}^{2t} \otimes w_1.
\]

**Proposition 2.1.** Let \( \omega \) be a primitive \( 8nN \)-th root of unity. Set

\[
\tilde{\mu} = \begin{cases} 
1, & \mu = 1 \\
\omega^2, & \mu = -1
\end{cases}, \quad \tilde{\mu} = \begin{cases} 
1, & \mu = 1 \\
\omega^4, & \mu = -1
\end{cases}.
\]

Then a full set of non-isomorphic simple left \( A_{N2n}^{\mu \lambda} \)-modules is given by

1. \( V_{ijk} = \kappa v, i, j \in \mathbb{Z}_2, k \in 0, N-1 \). The action of \( A_{N2n}^{\mu \lambda} \) on \( V_{ijk} \) is given by

\[
x_{12} \mapsto 0, \quad x_{21} \mapsto 0, \quad x_{11} \mapsto (-1)^j \omega^{4nk}, \quad x_{22} \mapsto (-1)^j \omega^{4nk};
\]

2. \( V_{ijk}' = \kappa v, i = 1, j \in \mathbb{Z}_2, k \in 0, N-1 \). The action of \( A_{N2n}^{\mu \lambda} \) on \( V_{ijk}' \) is given by

\[
x_{11} \mapsto 0, \quad x_{22} \mapsto 0, \quad x_{12} \mapsto (-1)^i \omega^{4nk} \tilde{\mu}, \quad x_{21} \mapsto (-1)^i \omega^{4nk} \tilde{\mu};
\]

3. \( V_{jk} = \kappa v_1 \oplus \kappa v_2, k \in 0, N-1, \frac{i}{2} \in 0, n-1 \). The action of \( A_{N2n}^{\mu \lambda} \) on the row vector \((v_1, v_2)\) is given by

\[
x_{11} \mapsto \begin{pmatrix} 0 & \omega^{2(4kn-jN)} \\ \omega^{2(jN)} & 0 \end{pmatrix}, \quad x_{12}, x_{21} \mapsto 0, \quad x_{22} \mapsto \begin{pmatrix} 0 & \omega^{8kn} \\ 1 & 0 \end{pmatrix};
\]

4. \( V_{jk}' = \kappa v_1' \oplus \kappa v_2', k \in 0, N-1, \left\{ \begin{array}{ll} \frac{j}{2} \in 1, n-1, & \lambda = 1, \\ \frac{j+1}{2} \in 1, n, & \lambda = -1. \end{array} \right. \) The action of \( A_{N2n}^{\mu \lambda} \) on the row vector \((v_1', v_2')\) is given by

\[
x_{21} \mapsto \begin{pmatrix} 0 & \tilde{\mu} \omega^{2(4kn-jN)} \\ \omega^{2(jN)} & 0 \end{pmatrix}, \quad x_{11}, x_{22} \mapsto 0, \quad x_{12} \mapsto \begin{pmatrix} 0 & \tilde{\mu} \omega^{8kn} \\ 1 & 0 \end{pmatrix}.
\]

We leave the proof to the reader since it’s easy and tedious.
3. Yetter-Drinfeld modules over $A_{N2n}^{\mu\lambda}$

Similarly according to Radford’s method [25, Proposition 2], any simple left Yetter-Drinfeld module over a Hopf algebra $H$ could be constructed by the submodule of tensor product of a left module $V$ of $H$ and $H$ itself, where the module and comodule structures are given by:

\[
\begin{align*}
(3.1) & \quad h \cdot (\ell \otimes g) = (h(2) \cdot \ell) \otimes h(1) g S(h(3)), \\
(3.2) & \quad \rho(\ell \otimes h) = h(1) \otimes (\ell \otimes h(2)), \forall h, g \in H, \ell \in V.
\end{align*}
\]

Here we use $\otimes$ instead of $\otimes$ to avoid confusion by using too many symbols of the tensor product. We construct all simple left Yetter-Drinfeld modules over $A_{N2n}^{\mu\lambda}$ in this way and put them in the appendix.

Let $V$ be a simple left $A_{N2n}^{\mu\lambda}$ module, we decompose $V \otimes A_{N2n}^{\mu\lambda}$ into small Yetter-Drinfeld modules. The left $A_{N2n}^{\mu\lambda}$-module structure of $V \otimes A_{N2n}^{\mu\lambda}$ is decided by formulas in the Figure 1.

We can decompose $V \otimes A_{N2n}^{\mu\lambda}$ into small Yetter-Drinfeld modules as

\[
V \otimes A_{N2n}^{\mu\lambda} \cong \bigoplus_{s=1}^{N} \left[ M_{s}^{\mu\lambda}_{ij} \oplus N_{s}^{\mu\lambda}_{ij} \oplus \bigoplus_{t=-1}^{n-1} (V \otimes C_{2t+2}) \right],
\]

where

\[
C_{s0} := 2s_{11}^{2} + 2s_{12}^{2},
\]

\[
C_{s2n} := 2s_{11}^{2} + 2s_{12}^{2},
\]

and

\[
\begin{align*}
V_{ij} \otimes C_{2t+2} & \cong \begin{cases} 
C_{st}^{i,j,k,0} \oplus C_{st}^{i,j,k,1}, & t \in \{0, n-2\}, \\
B_{ij}^{st}, & t = -1, j = i + 1, \\
A_{st}^{i,k,0} \oplus A_{st}^{i,k,1}, & t = -1, j = i, \\
C_{st}^{i,j,0}, & t = -1, \text{ see Table 1}, \\
A_{st}^{i,j,0} \oplus A_{st}^{i,j,1}, & t = -1, i = j \text{ (or } j + 1), \lambda = 1 \text{ (or } -1). \end{cases}
\end{align*}
\]

Similarly, we have

\[
V_{jk} \otimes A_{N2n}^{\mu\lambda} \cong \bigoplus_{s=1}^{N} \left[ \bigoplus_{p=0}^{1} \left( J_{p}^{s} \oplus J_{p}^{s} \right) \oplus \bigoplus_{t=-1}^{n-1} (V \otimes C_{2t+2}) \right],
\]

where

\[
V_{jk} \otimes C_{2t+2} \cong \begin{cases} 
\bigoplus_{p=0}^{1} \left( D_{jk,p}^{st} \oplus D_{jk,p}^{st+1} \right), & 0 \leq t \leq n-2, \\
D_{jk,0}^{st} \oplus D_{jk,1}^{st}, & t = n-1, \\
E_{jk,0}^{st} \oplus E_{jk,1}^{st}, & t = -1; \end{cases}
\]

\[
V_{jk} \otimes A_{N2n}^{\mu\lambda} \cong \bigoplus_{s=1}^{N} \left[ \bigoplus_{p=0}^{1} \left( K_{p}^{s} \oplus L_{p}^{s} \right) \oplus \bigoplus_{t=0}^{n-1} (V' \otimes C_{2t+1}) \right],
\]
where $V'_{ij} \otimes C_{s2t+1} \simeq \bigoplus_{p=0}^1 \left( \mathfrak{S}^{st}_{jk,p} \oplus \mathcal{H}^{st}_{jk,p} \right)$; 

$$V'_{ijk} \otimes A^\mu_{N,2n} \simeq \bigoplus_{s=1}^N \left[ \mathcal{L}^s_{ijk,0} \oplus \mathcal{L}^s_{ijk,1} \oplus \bigoplus_{t=0}^{n-1} (V'_{ijk} \otimes C_{s2t+1}) \right],$$

where $V'_{ijk} \otimes C_{s2t+1} \simeq \bigoplus_{p=0}^1 \mathcal{E}^{st}_{ijk,p}$.

Our strategy is to break $V \otimes A^\mu_{N,2n}$ into small sub-Yetter-Drinfeld modules which can’t break any more, and single out a complete set of simple Yetter-Drinfeld modules over $A^\mu_{N,2n}$ from those submodules. According to the above decompositions, we see the dimension distribution of those

$$x_{pq} \cdot (v \otimes x_{i1}^s x_{22}^t) = \begin{cases} (-1)^j \omega^{4nk} v \otimes x_{11}^s, & pq = 11, t = 0, \\ (-1)^j \omega^{4nk} v \otimes x_{11}^{s+1} x_{22}^t, & pq = 11, t \text{ even}, t > 0, \\ (-1)^j \omega^{4nk} v \otimes x_{11}^{s-1} x_{22}^t, & pq = 11, t \text{ odd}, \\ (-1)^j \omega^{4nk} v \otimes x_{11}^{s+1} x_{22}^t, & pq = 22, s \text{ odd}, t \text{ odd}, \\ (-1)^j \omega^{4nk} v \otimes x_{11}^{s-3} x_{22}^t, & pq = 22, s \text{ odd}, t \text{ even}, \\ (-1)^j \omega^{4nk} v \otimes x_{11}^s, & pq = 22, s \text{ even}, t = 0, \\ (-1)^j \omega^{4nk} v \otimes x_{11}^s x_{22}, & pq = 22, s \text{ even}, t = 1, \\ (-1)^j \omega^{4nk} v \otimes x_{11}^{s+1} x_{22}^t, & pq = 22, s \text{ even}, 0 < t \text{ even}, \\ (-1)^j \omega^{4nk} v \otimes x_{11}^{s+3} x_{22}^t, & pq = 22, s \text{ even}, 1 < t \text{ odd}, \\ 0, & \text{otherwise}, \\ \end{cases}$$

$$x_{pq} \cdot (v \otimes x_{12}^s x_{21}^j) = \begin{cases} (-1)^j \omega^{4nk} v \otimes x_{12}^s, & pq = 11, t = 0, \\ (-1)^j \omega^{4nk} v \otimes x_{12}^{s+1} x_{21}^t, & pq = 11, t \text{ even}, t > 0, \\ (-1)^j \omega^{4nk} v \otimes x_{12}^{s-1} x_{21}^t, & pq = 11, t \text{ odd}, \\ (-1)^j \omega^{4nk} v \otimes x_{12}^{s+1} x_{21}^t, & pq = 22, s \text{ odd}, t \text{ odd}, \\ (-1)^j \omega^{4nk} v \otimes x_{12}^{s-3} x_{21}^t, & pq = 22, s \text{ odd}, t \text{ even}, \\ (-1)^j \omega^{4nk} v \otimes x_{12}^s, & pq = 22, s \text{ even}, t = 0, \\ (-1)^j \omega^{4nk} v \otimes x_{12}^s x_{21}, & pq = 22, s \text{ even}, t = 1, \\ (-1)^j \omega^{4nk} v \otimes x_{12}^{s+1} x_{21}^t, & pq = 22, s \text{ even}, 0 < t \text{ even}, \\ (-1)^j \omega^{4nk} v \otimes x_{12}^{s+3} x_{21}^t, & pq = 22, s \text{ even}, 1 < t \text{ odd}, \\ 0, & \text{otherwise}. \\ \end{cases}$$
small Yetter-Drinfeld modules is 1, 2 and 2n. From the appendix and table 1, it is not difficult to see that there are $8N^2$ pairwise non-isomorphic Yetter-Drinfeld modules of dimension one and $2N^2(4n^2 - 1)$ pairwise non-isomorphic Yetter-Drinfeld modules of dimension two, see the Theorem 3.1. There are seven classes of Yetter-Drinfeld modules of dimension $2n$ in total and they have the following relations.

1. $\mathcal{J}^s_{pjk} \simeq \mathcal{J}^s_{p'jk}$ in case of $j \equiv j' \mod 4$.
2. If $n$ is even, then $\mathcal{M}^s_{ijk} \simeq \mathcal{J}^s_{pjk}$ in case of $i = p, j' | 4$.
3. If $n$ is odd, then $\mathcal{M}^s_{ijk} \simeq \mathcal{J}^s_{pjk}$ in case of $i = p$ and
   \[ j' \equiv \begin{cases} 
   0 \mod 4, & \text{if } i + j \text{ is even}, \\
   2 \mod 4, & \text{if } i + j \text{ is odd}.
   \end{cases} \]
4. If $n$ is even, then $\mathcal{N}^s_{ijk} \simeq \mathcal{J}^s_{pjk}$ in case of $j = p$ and
   \[ j' \equiv \begin{cases} 
   0 \mod 4, & \text{if } \lambda = 1, \\
   2 \mod 4, & \text{if } \lambda = -1.
   \end{cases} \]
5. If $n$ is odd, then $\mathcal{N}^s_{ijk} \simeq \mathcal{J}^s_{pjk}$ in case of $j = p$ and
   \[ j' \equiv \begin{cases} 
   0 \mod 4, & \text{if } \lambda = 1, i + j \text{ is even}, \\
   2 \mod 4, & \text{if } \lambda = 1, i + j \text{ is odd}, \\
   2 \mod 4, & \text{if } \lambda = -1, i + j \text{ is even}, \\
   0 \mod 4, & \text{if } \lambda = -1, i + j \text{ is odd}.
   \end{cases} \]
6. $\mathcal{J}^s_{pjk} \simeq \mathcal{J}^s_{p'jk}$ in case of $j \equiv \begin{cases} 
   j' \mod 4, & \text{if } \lambda = 1, \\
   j' + 2 \mod 4, & \text{if } \lambda = -1.
   \end{cases} \]
7. $\mathcal{H}^s_{ijk,p} \simeq \mathcal{H}^s_{jk,p}$ in case of $j \equiv j' \mod 4$.
8. $\mathcal{H}^s_{ijk,p} \simeq \mathcal{L}^s_{jk,p}$ in case of $j + j' \equiv 0 \mod 4$.
9. $\mathcal{D}^s_{ijk,p} \simeq \mathcal{D}^s_{jk,p}$ in case of
   \[ j' \equiv \begin{cases} 
   0 \mod 4, & \text{if } i + j \text{ is even and } n \text{ is odd}, \\
   2 \mod 4, & \text{if } i + j \text{ is odd and } n \text{ is odd}, \\
   0 \mod 4, & \text{if } n \text{ is even}.
   \end{cases} \]

Since $8N^2 \cdot 1^2 + 2N^2(4n^2 - 1) \cdot 2^2 + 8N^2 \cdot (2n)^2 = (8Nn)^2$, all the simple Yetter-Drinfeld modules are given by the following theorem.

**Theorem 3.1.** A full set of non-isomorphic simple Yetter-Drinfeld modules over $A^\mu_{N,2n}$ is given by the following list.

1. There are $8N^2$ non-isomorphic Yetter-Drinfeld modules of dimension one:
   a. $\mathcal{S}^s_{ijk,p}$, $i, p \in \mathbb{Z}_2$, $s \in \overline{1,N}$, $k \in \overline{0,N-1}$;
   b. $\mathcal{S}^s_{ii+1,k,p}$, $i, p \in \mathbb{Z}_2$, $s \in \overline{1,N}$, $k \in \overline{0,N-1}$.
(2) There are $2N^2(4n^2 - 1)$ non-isomorphic Yetter-Drinfeld modules of dimension two:

- $(a) \mathcal{B}_{01k}^s, s \in \overline{1, N}, k \in \overline{0, N-1};$
- $(b) \mathcal{C}_{ijk,p}^r, i j = 00 or 01, k \in \overline{0, N-1}, p \in \mathbb{Z}_2, s \in \overline{1, N}, t \in \overline{0, n-2};$
- $(c) \mathcal{C}_{ijk,p}^s, i = 0, j = \begin{cases} \frac{i + 1}{i}, & \text{if } \lambda = 1, \\ \frac{i}{i}, & \text{if } \lambda = -1, \end{cases} k \in \overline{0, N-1}, s \in \overline{1, N},$
  \hspace{1cm} p = 0, t = n - 1;
- $(d) \mathcal{D}_{ijk,p}^s, i j = \frac{1}{2} \in \overline{1, n-1}, k \in \overline{0, N-1}, p \in \mathbb{Z}_2, s \in \overline{1, N}, t \in \overline{0, n-1};$
- $(e) \mathcal{E}_{ijk,p}^s, i j = \frac{1}{2} \in \overline{1, n-1}, k \in \overline{0, N-1}, p \in \mathbb{Z}_2, s \in \overline{1, N}, t \in \overline{0, n-1};$
- $(f) \mathcal{F}_{ijk,p}^s, \begin{cases} \frac{i + 1}{i} \in \overline{1, n-1}, & \text{if } \lambda = 1, \\ \frac{i}{i} \in \overline{1, n}, & \text{if } \lambda = -1, \end{cases} k \in \overline{0, N-1}, p \in \mathbb{Z}_2, s \in \overline{1, N}, t \in \overline{0, n-1};$
- $(g) \mathcal{G}_{ijk,p}^s, \begin{cases} \frac{i + 1}{i} \in \overline{1, n-1}, & \text{if } \lambda = 1, \\ \frac{i}{i} \in \overline{1, n}, & \text{if } \lambda = -1, \end{cases} k \in \overline{0, N-1}, p \in \mathbb{Z}_2, s \in \overline{1, N}, t \in \overline{0, n-1};$
- $(h) \mathcal{H}_{ijk,p}^s, i j = 00 or 01, k \in \overline{0, N-1}, p \in \mathbb{Z}_2, s \in \overline{1, N}, t \in \overline{0, n-1}.$

(3) There are $8N^2$ non-isomorphic Yetter-Drinfeld modules of dimension $2n$:

- $(a) \mathcal{J}_{ijk}^s, j = 2 or 4, k \in \overline{0, N-1}, p \in \mathbb{Z}_2, s \in \overline{1, N};$
- $(b) \mathcal{K}_{ijk}^s, j = \begin{cases} 1 or 3, & \text{if } \lambda = -1, \\ 2 or 4, & \text{if } \lambda = 1, \end{cases} k \in \overline{0, N-1}, p \in \mathbb{Z}_2, s \in \overline{1, N}.$

**Remark 3.2.** As for the description of those Yetter-Drinfeld modules, please see the Appendix. To simplify the notations, we allow $j$ takes values $2$ and $4$ for $\mathcal{J}_{ijk}^s$ and the similar settings for $\mathcal{K}_{ijk}^s$.

4. Nichols algebras over simple Yetter-Drinfeld modules

In this section, we investigate Nichols algebras over simple Yetter-Drinfeld modules of $A_{N_20}^{\lambda}$. So the Yetter-Drinfeld modules discussed in the section are those listed in Theorem 3.1. For the knowledge about Nichols algebras, please refer to [14] [1].

4.1. Nichols algebras of diagonal type. Let $V = \bigoplus_{i \in I} \mathbb{k}v_i$ be a vector space with a braiding $c(v_i \otimes v_j) = q_{ij}v_j \otimes v_i$, $q_{ij} \in \mathbb{k}^\times$, then the Nichols algebra $\mathfrak{B}(V)$ is of diagonal type. Our results in this section heavily rely on
| YD-mod       | dim | parameters                                                                 | mod               | comod                          |
|--------------|-----|-----------------------------------------------------------------------------|-------------------|--------------------------------|
| $\mathcal{B}^s_{ijk}$ | 2   | $i, j \in \mathbb{Z}_2, j = i + 1, s \in 1, N, k \in 0, N - 1$            | $V_{ijk} \oplus V_{jik}$ | $\mathbb{k}g^+_s \oplus \mathbb{k}g^-_s$ |
| $\mathcal{C}^s_{ijk,p}$ | 2   | $i, j, p \in \mathbb{Z}_2, s \in 1, N, t \in 0, n - 2, k \in 0, N - 1$   | $V_{ijk} \oplus V_{i+1,j+1,k}$ | $\Lambda_{s,2r+2}$ |
| $\mathcal{C}^s_{ijk,p}$ | 2   | $j = \begin{cases} i + 1, & \lambda = 1, \\ i, & \lambda = -1, \end{cases}, s \in 1, N, t = n - 1, k \in 0, N - 1$ | $V_{ijk} \oplus V_{i+1,j+1,k}$ | $\mathbb{k}h^+_s \oplus \mathbb{k}h^-_s$ |
| $\mathcal{D}^s_{jk,p}$ | 2   | $s \in 1, N, t \in 0, n - 1, j = \begin{cases} \frac{i}{2}, s \in 1, n - 1, k \in 0, N - 1, \end{cases}, p \in \mathbb{Z}_2, t \neq n - 1, k \in 0, N - 1$ | $V_{jk}$ | $\Lambda_{s,2r+2}$ |
| $\mathcal{E}^s_{jk,p}$ | 2   | $s \in 1, N, t \in 0, n - 1, j = \begin{cases} \frac{i}{2}, s \in 1, n - 1, k \in 0, N - 1, \end{cases}, p \in \mathbb{Z}_2, t \neq 0, k \in 0, N - 1$ | $V_{jk}$ | $\Lambda_{s,2r}$ |
| $\mathcal{G}^s_{jk,p}$ | 2   | $\begin{cases} \frac{i}{2}, s \in 1, n - 1, & \lambda = 1, \\ \frac{i+1}{2}, s \in 1, n - 1, & \lambda = -1, \end{cases}, j = \begin{cases} \frac{i}{2}, s \in 1, n - 1, k \in 0, N - 1, \end{cases}, p \in \mathbb{Z}_2$ | $V'_{jk}$ | $\Lambda_{s,2r+1}$ |
| $\mathcal{H}^s_{jk,p}$ | 2   | $\begin{cases} \frac{i}{2}, s \in 1, n - 1, & \lambda = 1, \\ \frac{i+1}{2}, s \in 1, n - 1, & \lambda = -1, \end{cases}, j = \begin{cases} \frac{i}{2}, s \in 1, n - 1, k \in 0, N - 1, \end{cases}, p \in \mathbb{Z}_2$ | $V'_{jk}$ | $\Lambda_{s,2r+1}$ |
| $\mathcal{D}^s_{ijk,p}$ | 2   | $\lambda = 1, i, j, p \in \mathbb{Z}_2, s \in 1, N, t \in 0, n - 1, k \in 0, N - 1$ | $V'_{ijk} \oplus V'_{i+1,j+1,k}$ | $\Lambda_{s,2r+1}$ |

Table 1. Two dimensional simple Yetter-Drinfeld modules over $A_{N,2n}^{\mu,\lambda}$. Here $g^s = x_{11}^{2s} \pm x_{12}^{2s}, h^s = x_{11}^{2s}x_{11}^{2n} \pm \sqrt{\lambda}x_{12}^{2s}x_{12}^{2n}$. 


Lemma 4.1. \( \dim \mathcal{B} (\mathcal{A}^s_{iik,p}) = \left\{ \begin{array}{ll} \infty, & N | ks, \\ \frac{N}{(N, ks)}, & N \nmid k s. \end{array} \right. \)

Proof. \( c(w \otimes w) = \left[ (-1)^{i} \omega^{8kN} \right]^{2s} w \otimes w = \omega^{8nk} W \otimes W. \)

Corollary 4.2. (1) If \( N = 1 \), then \( \dim \mathcal{B} (\mathcal{A}^s_{iik,p}) = \infty. \)

(2) If \( N \) is a prime, then \( \dim \mathcal{B} (\mathcal{A}^s_{iik,p}) = \left\{ \begin{array}{ll} \infty, & k = 0 \text{ or } s = N, \\ N, & \text{otherwise.} \end{array} \right. \)

Lemma 4.3. \( \dim \mathcal{B} (\mathcal{A}^s_{ijk,p}) = \left\{ \begin{array}{ll} \infty, & \lambda = 1, N | k(s + n), \\ \frac{N}{(N, k(s + n))}, & \lambda = 1, N \nmid k(s + n), \\ \infty, & \lambda = -1, 2N | [Nn + 2k(s + n)], \\ \frac{2N}{(2N, Nn + 2k(s + n))}, & \lambda = -1, 2N \nmid [Nn + 2k(s + n)]. \end{array} \right. \)

Proof. \( c(w \otimes w) = (-1)^{(i+j)n} \omega^{8kn(s+n)} W \otimes W. \)

Corollary 4.4. If \( N = 1 \), then \( \dim \mathcal{B} (\mathcal{A}^s_{ijk,p}) = \left\{ \begin{array}{ll} 2, & \lambda = -1, n \text{ odd,} \\ \infty, & \text{otherwise.} \end{array} \right. \)

Lemma 4.5. \( \dim \mathcal{B} (\mathcal{B}^s_{ijk}) = \left\{ \begin{array}{ll} 4, & N | 2ks, N \nmid ks, \text{(Cartan type } A_1 \times A_1), \\ 27, & N | 3ks, N \nmid ks, \text{(Cartan type } A_2), \\ \infty, & \text{otherwise.} \end{array} \right. \)

Proof. \( c(w_\alpha \otimes w_\beta) = \omega^{8nk} W_\beta \otimes W_\alpha \) for \( \alpha, \beta \in \mathbb{T}_2. \)

Lemma 4.6. Denote \( d = 2(k(s + t + 1) + N(i + j)(t + 1)), \)

\( \dim \mathcal{B} (\mathcal{C}^s_{ijk,p}) = \left\{ \begin{array}{ll} 4, & N | d, 2N \nmid d, \text{(Cartan type } A_1 \times A_1), \\ 27, & 2N | 3d, 2N \nmid d, \text{(Cartan type } A_2), \\ \infty, & \text{otherwise.} \end{array} \right. \)

Proof. \( c(w_\alpha \otimes w_\beta) = qw_\beta \otimes w_\alpha \) for \( \alpha, \beta \in \mathbb{T}_2, q = (-1)^{(i+j)(t+1)} \omega^{8nk(s+t+1)}. \)

\( \dim \mathcal{B} (\mathcal{C}^s_{ijk,p}) < \infty \iff \mathcal{B} (\mathcal{C}^s_{ijk,p}) \text{ is of Cartan type } A_1 \times A_1 \text{ or } A_2. \)

Lemma 4.7. Denote \( \alpha = 8nk(s + t + 1) - 2jN(t + 1), \beta = 8nk(s + t + 1) + 2jN(t + 1). \)

\( \dim \mathcal{B} (\mathcal{D}^s_{jk,p}) < \infty \) if and only if one of the following conditions holds.
(1) \(8nN \not| \alpha, 8nN \mid 2\beta\), Cartan type \(A_1 \times A_1\);
(2) \(8nN \not| \alpha, 8nN \mid (\alpha + 2\beta)\), Cartan type \(A_2\);
(3) \(\alpha \equiv 4nN \mod 8nN, 2\beta \not\equiv 0 \text{ and } 4nN \mod 8nN\), Super type \(A_2(q; \overline{1}_2)\);
(4) \(\alpha - 4\beta \equiv 12\beta \equiv 4nN \mod 8nN, 8\beta \not\equiv 0 \mod 8nN\). The Nichols algebras \(u\varphi(8)\), see [1, Page 561].

Proof. The braiding is given by
\[
\begin{align*}
    c(w_1 \otimes w_1) &= \omega^\alpha w_1 \otimes w_1, & c(w_1 \otimes w_2) &= \omega^\beta w_2 \otimes w_1, \\
    c(w_2 \otimes w_1) &= \omega^\beta w_1 \otimes w_2, & c(w_2 \otimes w_2) &= \omega^\alpha w_2 \otimes w_2.
\end{align*}
\]
□

Corollary 4.8. If \(\mathcal{B}\left(\mathcal{D}_{jk,p}^{st}\right)\) is isomorphic to the Nichols algebras \(u\varphi(8)\), \(5 \not| N \) and \(17 \not| n\), then \(8 \mid N, 4 \mid n\).

Remark 4.9. When \(n = 8, N = 48, j = 2 \) and \(k \leq 11\), then \(\mathcal{B}\left(\mathcal{D}_{jk,p}^{st}\right)\) is isomorphic to the Nichols algebras \(u\varphi(8)\) in case that \((k, s, t)\) is in the set
\[
\left\{ (1, 18, 0), (1, 22, 2), (1, 26, 4), (1, 30, 6), (1, 34, 0), (1, 38, 2), (1, 42, 4),
    (1, 46, 6), (5, 2, 2), (5, 6, 0), (5, 10, 6), (5, 14, 4), (5, 22, 0), (5, 30, 4),
    (5, 34, 2), (5, 42, 6), (7, 4, 0), (7, 12, 6), (7, 20, 4), (7, 28, 2), (7, 28, 6),
    (7, 36, 0), (7, 36, 4), (7, 44, 2), (11, 8, 2), (11, 8, 4), (11, 24, 0), (11, 24, 4),
    (11, 24, 6), (11, 40, 0), (11, 40, 2), (11, 40, 6) \right\}
\]

Proof. Since \(\alpha - 4\beta \equiv 12\beta \equiv 4nN \mod 8nN\),
\[
\Rightarrow 8nN \not| 15 \times 8nk(s + t + 1) + 17 \times 2jN(t + 1) \equiv 0 \mod 8nN,
\Rightarrow 8nN \not| 12 \times 8nk(s + t + 1) + 12 \times 2jN(t + 1) \equiv 4nN \mod 8nN,
\Rightarrow 8nN \not| 8n | 17 \times 2jN(t + 1), \ 4N \not| 15 \times 8nk(s + t + 1),
\Rightarrow 8nN \not| 17 \times 2jN(t + 1) = 8nr_1, \ 15 \times 8nk(s + t + 1) = 4N r_2,
\Rightarrow \left\{ \begin{align*}
    12 \times 8nk(s + t + 1) + 12 \times \frac{8nr_1}{17} & \equiv 4nN \mod 8nN, \\
    12 \times \frac{4N r_2}{15} + 24jN(t + 1) & \equiv 4nN \mod 8nN,
\end{align*} \right.
\Rightarrow 32n \not| 4nN, \ 16N \not| 4nN,
\Rightarrow 8 \mid N, \ 4 \mid n.
\]
□

Lemma 4.10. Denote \(\alpha = 8nk(s + t) + 2jNt, \beta = 8nk(s + t) - 2jNt\). \(\dim \mathcal{B}\left(\mathcal{D}_{jk,p}^{st}\right) < \infty\) if and only if one of the following conditions holds.
(1) \(8nN \not| \alpha, 8nN \mid 2\beta\), Cartan type \(A_1 \times A_1\);
(2) \(8nN \not| \alpha, 8nN \mid (\alpha + 2\beta)\), Cartan type \(A_2\);
(3) $\alpha \equiv 4nN \mod 8nN$, $2\beta \not\equiv 0$ and $4nN \mod 8nN$, Super type $A_2(q; \mathbb{I}_2)$;
(4) $\alpha - 4\beta \equiv 12\beta \equiv 4nN \mod 8nN$, $8\beta \not\equiv 0 \mod 8nN$. The Nichols algebras $\mathfrak{U}(\mathfrak{so}(8))$.

Proof. The braiding is given by
\[
c(w_1 \otimes w_1) = \omega^c w_1 \otimes w_1, \quad c(w_1 \otimes w_2) = \omega^\beta w_2 \otimes w_1, \\
c(w_2 \otimes w_1) = \omega^\beta w_1 \otimes w_2, \quad c(w_2 \otimes w_2) = \omega^c w_2 \otimes w_2.
\]

\[\square\]

Corollary 4.11. If $\mathfrak{B} \left( E_{jk,p}^{st} \right)$ is isomorphic to the Nichols algebras $\mathfrak{U}(\mathfrak{so}(8))$, $5 \nmid N$ and $17 \nmid n$, then $8 \mid N$, $4 \mid n$.

Remark 4.12. The proof is similar to the Corollary 4.8. If $n = 8$, $N = 48$, $j = 2$ and $k \leq 11$, then $\mathfrak{B} \left( E_{jk,p}^{st} \right)$ is isomorphic to the Nichols algebras $\mathfrak{U}(\mathfrak{so}(8))$ in case that $(k, s, t)$ is in the set
\[
\left\{ (1, 4, 3), (1, 4, 7), (1, 12, 1), (1, 12, 5), (1, 20, 1), (1, 36, 7), \\
(1, 44, 5), (5, 8, 3), (5, 8, 5), (5, 24, 1), (5, 24, 5), (5, 24, 7), (5, 40, 1), \\
(5, 40, 3), (5, 40, 7), (7, 2, 5), (7, 6, 7), (7, 10, 1), (7, 14, 3), (7, 18, 5), \\
(7, 22, 7), (7, 42, 1), (7, 46, 3), (11, 2, 3), (11, 6, 1), (11, 10, 7), (11, 14, 5), \\
(11, 22, 1), (11, 30, 5), (11, 34, 3), (11, 42, 7) \right\}
\]

4.2. The Nichols algebras of type $V_{abe}$. Let $V_{abe} = \mathbb{K}v_1 \oplus \mathbb{K}v_2$ be a vector space with a braiding given by
\[
c(v_1 \otimes v_1) = av_2 \otimes v_2, \quad c(v_1 \otimes v_2) = bv_2 \otimes v_1, \\
c(v_2 \otimes v_1) = bv_2 \otimes v_1, \quad c(v_2 \otimes v_2) = ev_2 \otimes v_1,
\]
then we call that the braided vector space $(V_{abe}, c)$ is of type $V_{abe}$. The braided vector space $V_{abe}$ is isomorphic to $V_{abe}^{\otimes 1}$ via $v_1 \mapsto \sqrt{c}v_1, v_2 \mapsto v_2$.

If $ae = b^2$, then $V_{abe}$ is of diagonal type and
\[
\dim \mathfrak{B}(V_{abe}) = \begin{cases} 4, & b = -1, \ (\text{Cartan type } A_1 \times A_1), \\
27, & b^3 = 1 \neq b, \ (\text{Cartan type } A_2), \\
\infty, & \text{otherwise}. \end{cases}
\]

If $b^2 \neq ae$, $\mathfrak{B}(V_{abe})$ is obviously not of rack type, please refer to the formula (1.1) for more details.

Lemma 4.13. $\mathfrak{B} \left( E_{jk,p}^{st} \right)$ is of type $V_{abe}$, where
\[
ae = \mu^2 x^{2t+1} \omega^{4kn(4s+4t+2)+jN(-2-4t)} \mu^x t^{s+\frac{1}{2}} \omega^{4nk(2s+2t+1)+jN(2t+1)}.
\]
Remark 4.14. Notice that \( \frac{ae}{p^t} = \omega^{-4jN(2t+1)} \), we have \( b = \omega^{2jN(2t+1)} \) for some suitable \( p \in \mathbb{Z}_2 \) under the case \( ae = 1 \). Furthermore, if we take \( t = 0 \) and \( j = \begin{cases} 2, & \lambda = 1, \\ 1, & \lambda = -1, \end{cases} \) then \( b \in \begin{cases} \mathbb{G}_{2n}, & \lambda = 1, \\ \mathbb{G}_{4n}, & \lambda = -1. \end{cases} \) So \( \dim \mathcal{B}(\mathcal{G}_{jk,p}^{st}) = \begin{cases} (2n)^2, & \lambda = 1, \\ (4n)^2, & \lambda = -1, \end{cases} \) for suitable choice of \((n, N, j, s, t, k, p)\).

Similarly, if \( b = -1 \), then \( \dim \mathcal{B}(\mathcal{G}_{jk,p}^{st}) = \begin{cases} 4n, & \lambda = 1, \\ 8n, & \lambda = -1, \end{cases} \) for suitable choice of \((n, N, j, s, t, k, p)\).

Corollary 4.15. Suppose \( ae \neq b^2 = (ae)^{-1} \) and \( b \in \mathcal{G}_n \) for \( n \geq 5 \), then \( \dim \mathcal{B}(V_{abe}) = \infty \).

Proof. \( \mathcal{B}(\mathcal{G}_{jk,p}^{st}) \) is of type \( V_{abe} \), and \( b = (-1)^p \omega^{j(2t+1)} \), \( ae = \omega^{-2j(2t+1)} \) in case \( \mu = \lambda = 1 = N \). According to [2], Nichols algebras associated with two dimensional Yetter-Drinfeld modules, over the dihedral group \( D_{4n} \) of order \( 4n \), are either 4-dimension or infinite dimension. And \( A_{4n}^{\pm} \) is isomorphic to a 2-cocycle deformation of \( \mathcal{B}^{D_{4n}} \) [21], so \( \dim \mathcal{B}(\mathcal{G}_{jk,p}^{st}) = \infty \) under the provided conditions. If we take \( j = 8 \), then \( b = (-1)^p \omega^{8(2t+1)} \) covers all \( n \)-th primitive roots of unity for \( n \geq 5 \).

Remark 4.16. The corollary provided a correct proof for \( \dim \mathcal{B}(W_1^a) = \infty = \dim \mathcal{B}(W_2^a) \), see [27, Page 278], where the braiding of \( \mathcal{B}(W_1^a) \) should be corrected as

\[
\begin{align*}
  c\left( w_1^{(1)} \otimes w_1^{(1)} \right) &= -\theta w_2^{(1)} \otimes w_2^{(1)}, \\
  c\left( w_1^{(1)} \otimes w_2^{(1)} \right) &= -\theta w_1^{(1)} \otimes w_2^{(1)}, \\
  c\left( w_2^{(1)} \otimes w_1^{(1)} \right) &= -\theta w_2^{(1)} \otimes w_1^{(1)}, \\
  c\left( w_2^{(1)} \otimes w_2^{(1)} \right) &= \theta w_1^{(1)} \otimes w_1^{(1)}.
\end{align*}
\]

The parameters \( \theta = \pm \frac{\sqrt{2}(\sqrt{7}-1)}{2} \) are 8-th primitive roots of unity.

Lemma 4.17. \( \mathcal{B}(\mathcal{H}_{jk,p}^{st}) \) is of type \( V_{abe} \) with

\[
ae = \tilde{\mu}^{2s+2t+1} \omega^{4kn(4r+4s+2)+jN(4r+2)}, \quad b = (-1)^p \tilde{\mu}^{s+r+\frac{1}{2}} \omega^{4kn(2t+2s+1)+jN(-1-2r)}.
\]

Remark 4.18. From observation, we have \( \frac{ae}{p^t} = \omega^{4jN(2t+1)} \).

(1) Suppose \( ae = 1 \), then \( \dim \mathcal{B}(\mathcal{H}_{jk,p}^{st}) = \begin{cases} (4n)^2, & \lambda = -1, \\ (2n)^2, & \lambda = 1, \end{cases} \) under suitable choice of \((n, N, j, s, t, k, p)\).
(2) Suppose $b = -1$, then $\dim \mathcal{B}(\mathcal{H}_{ijk,p}^s) = \begin{cases} 8n, & \lambda = -1, \\ 4n, & \lambda = 1, \end{cases}$ under suitable choice of $(n, N, j, s, t, k, p)$.

**Lemma 4.19.** Denote $q = (-1)^{p+(i+j)(t+1)+j}z^{2s+2t+1}w^{4nk(2s+2t+1)}$, then

$$\dim \mathcal{B}(\mathcal{P}_{ijk,p}^{st}) = \begin{cases} 4, & q = -1, \text{(Cartan type } A_1 \times A_1), \\ 27, & q^3 = 1 \neq q, \text{(Cartan type } A_2), \\ \infty, & \text{otherwise}. \end{cases}$$

**Proof.** $\mathcal{P}_{ijk,p}^{st}$ is of type $\mathcal{V}_{qqq}$, so $\mathcal{B}(\mathcal{P}_{ijk,p}^{st})$ is finite dimensional iff $q = -1$ or $q^3 = 1 \neq q$. \hfill $\Box$

### 4.3. The Nichols algebras over $\mathcal{J}_{pjk}^s$

**Lemma 4.20.** Let $(V, c)$ be a braided vector space such that $c(x \otimes y) \in k f_x(y) \otimes x$, where the map $f_x : V \to V$ is bijective for any $x \in V$ under a fixed basis. Then $(V, \triangleright)$ is a rack with $x \triangleright y = f_x(y)$ under the fixed basis.

**Proof.** For any $x, y, z$ in a fixed basis of $V$,

$$c_1c_2c_1(x \otimes y \otimes z) \in k[(x \triangleright y) \triangleright (x \triangleright z)] \otimes (x \triangleright y) \otimes x,$$

$$c_2c_1c_2(x \otimes y \otimes z) \in k[x \triangleright (y \triangleright z)] \otimes (x \triangleright y) \otimes x.$$  
So $x \triangleright (y \triangleright z) = (x \triangleright y) \triangleright (x \triangleright z)$. \hfill $\Box$

**Lemma 4.21.** If $n > 2$, then $\dim \mathcal{B}(\mathcal{J}_{pjk}^s) = \infty$.

**Proof.** In case $n > 2$, let $X = \{w_r, m_r \mid r \in \overline{1,n}\}$, then $(X, \triangleright)$ is a rack as defined in Lemma 4.20. It’s easy to see that $\{w_r \mid r \in \overline{1,n}\}$ and $\{m_r \mid r \in \overline{1,n}\}$ are two subracks of $X$. $X$ is of type $D$, since

$$w_1 \triangleright (m_1 \triangleright (w_1 \triangleright m_1)) = \begin{cases} m_4, & \text{if } n > 3, \\ m_3, & \text{if } n = 3. \end{cases}$$

According to [3, Theorem 3.6], $\dim \mathcal{B}(\mathcal{J}_{pjk}^s) = \infty$. \hfill $\Box$

**Lemma 4.22.** Let $q = (-1)^{p}w^{4kn(2s+1)}$.

1. If $n = 1$, then

$$\dim \mathcal{B}(\mathcal{J}_{pjk}^s) < \infty \iff \begin{cases} \lambda q^2 = 1 \neq q, \text{(Cartan type } A_1 \times A_1), \\ \lambda q^3 = 1 \neq q, \text{(Cartan type } A_2). \end{cases}$$
(2) If $n = 2$, then
\[
\dim \mathfrak{B}(\mathcal{J}^s_{pjk}) = \begin{cases} 
64, & q = -1, j = 2 \text{ and } \lambda = 1, \text{(dihedral rack type } \mathbb{D}_4), \\
64, & q = -1, j = 4 \text{ and } \lambda = 1, \text{(Cartan type } A_2 \times A_2), \\
\infty, & \text{otherwise.}
\end{cases}
\]

**Proof.** If $n = 1$, then the braiding of $\mathcal{J}^s_{pjk}$ is given by
\[
c(w_1 \otimes w_1) = qw_1 \otimes w_1, \quad c(w_1 \otimes m_1) = \lambda q^2 \omega^{2n/N(2s+1)} m_1 \otimes w_1,
\]
\[
c(m_1 \otimes w_1) = q\omega^{2n/N} w_1 \otimes m_1, \quad c(m_1 \otimes m_1) = qm_1 \otimes m_1.
\]
So $\mathfrak{B}(\mathcal{J}^s_{pjk})$ is of diagonal type and its Dynkin diagram is
\[
\begin{tikzpicture}
    \node[anchor=east] at (0,0) {$\circ$};
    \node[anchor=west] at (1,0) {$\circ$};
    \node at (0.5,0) {$\lambda q^2$};
\end{tikzpicture}
\]
if $\lambda q^2 \neq 1$.

If $n = 2$, then the braiding of $\mathcal{J}^s_{pjk}$ is given by
\[
c(w_1 \otimes w_1) = qw_1 \otimes w_1, \quad c(w_1 \otimes w_2) = q\beta w_2 \otimes w_1,
\]
\[
c(w_2 \otimes w_1) = q\beta w_1 \otimes w_2, \quad c(w_2 \otimes w_2) = qw_1 \otimes w_2,
\]
\[
c(w_1 \otimes m_1) = \alpha m_2 \otimes w_1, \quad c(w_1 \otimes m_2) = q^2 \alpha^{-1} m_1 \otimes w_1,
\]
\[
c(w_2 \otimes m_1) = \lambda \alpha \beta m_2 \otimes w_2, \quad c(w_2 \otimes m_2) = \lambda q^2 \alpha^{-1} \beta m_1 \otimes w_2,
\]
\[
c(m_1 \otimes m_1) = qm_1 \otimes m_1, \quad c(m_1 \otimes m_2) = q\lambda \beta m_2 \otimes m_1,
\]
\[
c(m_2 \otimes m_1) = q\lambda \beta m_1 \otimes m_2, \quad c(m_2 \otimes m_2) = qm_2 \otimes m_2,
\]
\[
c(m_1 \otimes w_1) = \alpha w_2 \otimes m_1, \quad c(m_1 \otimes w_2) = q^2 \alpha^{-1} w_1 \otimes m_1,
\]
\[
c(m_2 \otimes w_1) = \alpha \beta w_2 \otimes m_2, \quad c(m_2 \otimes w_2) = q^2 \alpha^{-1} \beta w_1 \otimes m_2,
\]
where $\alpha = \omega^{8kn}$, $\beta = \omega^{2n/N} = \pm 1$. Both $k w_1 \otimes k w_2$ and $k m_1 \otimes k m_2$ are braided subspaces of diagonal type. If $q^2 \neq 1$, then their Dynkin diagrams are given by
\[
\begin{tikzpicture}
    \node[anchor=east] at (0,0) {$\circ$};
    \node[anchor=west] at (1,0) {$\circ$};
    \node at (0.5,0) {$q^2$};
\end{tikzpicture}
\]
So $\dim \mathfrak{B}(\mathcal{J}^s_{pjk}) < \infty$ iff $q = -1$ or $q^3 = 1 \neq q$.

It’s easy to see $\mathfrak{B}(\mathcal{J}^s_{pjk}) \cong \mathfrak{B}(\mathbb{D}_4, c_q)$, where $\mathbb{D}_4$ is the dihedral rack and $c_q$ is some 2-cocycle over $\mathbb{D}_4$. According to [16], if $\dim \mathfrak{B}(\mathcal{J}^s_{pjk}) < \infty$, then $\dim \mathfrak{B}(\mathcal{J}^s_{pjk}) = 64$. This only could be happened in case $\lambda = 1$ and $q = -1$. The relations of the 64-dimensional Nichols algebras are given by
\[
w_1 w_2 + \beta w_2 w_1 = 0, \quad w_1^2 - w_2^2 = 0, \quad m_1 m_2 + \beta m_2 m_1 = 0, \quad m_1^2 = m_2^2 = 0,
\]
\[
w_1 m_1 - \alpha m_2 w_1 + \alpha^2 \beta w_2 m_2 - \alpha m_1 w_2 = 0,
\]
\[
w_1 m_2 - \alpha^{-1} m_1 w_1 + w_2 m_1 - \alpha \beta m_2 w_2 = 0,
\]
\[ w_1m_1w_1m_1 + \beta m_1w_1m_1 = 0, \]
\[ w_1m_1w_2m_1 + w_2m_1w_1m_1 + m_1w_1m_1w_2 + m_1w_2m_1w_1 = 0. \]

In particular, if \( j = 4 \), then the Nichols algebra is of Cartan type \( A_2 \times A_2 \), which was appeared first in [22, Example 6.5]. Denote \( u_1 = w_1 + \alpha w_2 \), \( u_2 = m_1 - \alpha m_2 \), \( u_3 = w_1 - \alpha w_2 \), \( u_4 = m_1 + \alpha m_2 \), we can see this from the relations: \( u_i^2 = 0 \) for \( i \in \{1, 2, 3, 4 \} \) and
\[
(u_1u_2)^2 + (u_2u_1)^2 = 0, \quad u_1u_3 + u_3u_1 = 0, \quad u_1u_4 - u_4u_1 = 0,
\]
\[
(u_3u_4)^2 + (u_4u_3)^2 = 0, \quad u_3u_2 + u_2u_3 = 0, \quad u_2u_4 + u_4u_2 = 0.
\]

\[ \Box \]

4.4. The Nichols algebras over \( \mathcal{H}_{jk,p}^s \). Let
\[
b + 2a - 2 = 2nr + d, \quad r \in \mathbb{N}, \quad 0 \leq d \leq 2n - 1,
\]
\[
2n + 1 - b + 2a - 2 = 2ne + f, \quad e \in \mathbb{N}, \quad 0 \leq f \leq 2n - 1,
\]
then the braiding of \( \mathcal{H}_{jk,p}^s \) is given by
\[
c(w_a \otimes w_b) =
\begin{cases}
(-1)^p \left( \mu\omega^{8nk} \right)^s w_b \otimes w_1, & a = 1, \\
(-1)^p \lambda^r \mu^{s+n+r-2} \omega^{2n(r-2)(4nk+jN)+8nk} w_{2n} \otimes w_{2n-a+2}, & a > 1, d = 0, 2 | (a + b), \\
(-1)^p \lambda^r \mu^{s+n+r-1} \omega^{2n(r-1)(4nk+jN)+8nk} w_d \otimes w_{2n-a+2}, & a > 1, d > 0, 2 | (a + b), \\
(-1)^p \lambda^r \omega^{s-ne-2+2a} \omega^{-2jNne} w_1 \otimes w_{2n-a+2}, & a > 1, f = 0, 2 \not| (a + b), \\
(-1)^p \lambda^r \omega^{s-ne-2+2a} \omega^{-2jNne} w_{2n+1-f} \otimes w_{2n-a+2}, & a > 1, f > 0, 2 \not| (a + b). 
\end{cases}
\]

4.4.1. The Nichols algebras over \( \mathcal{H}_{jk,p}^s \) for \( n = 1, 2 \).

Lemma 4.23. Let \( q = (-1)^p \mu^{s} \omega^{8kn}. \)

(1) If \( n = 1 \), then
\[
\dim \mathcal{B} \left( \mathcal{H}_{jk,p}^s \right) < \infty \iff \begin{cases} \lambda q^2 = 1 \neq q, \quad \text{(Cartan type } A_1 \times A_1), \\ \lambda q^3 = 1 \neq q, \quad \text{(Cartan type } A_2). \end{cases}
\]

(2) If \( n = 2 \), then \( \dim \mathcal{B} \left( \mathcal{H}_{jk,p}^s \right) < \infty \implies q = -1 \) or \( q^3 = 1 \neq q \). In particular, if \( \lambda = 1 \), then \( \dim \mathcal{B} \left( \mathcal{H}_{jk,p}^s \right) = \begin{cases} 64, & q = -1, \\ \infty, & \text{otherwise}. \end{cases} \)

Proof. If \( n = 1 \), then the braiding of \( \mathcal{B} \left( \mathcal{H}_{jk,p}^s \right) \) is given by
\[
c(w_1 \otimes w_1) = qw_1 \otimes w_1, \quad c(w_1 \otimes w_2) = qw_2 \otimes w_1,
\]
\(c(w_2 \otimes w_1) = q\omega^{-4jN}w_1 \otimes w_2, \quad c(w_2 \otimes w_2) = qw_2 \otimes w_2.\)

\(\mathfrak{B}(\mathcal{H}_{jk,p}^s)\) is of diagonal type and its Dynkin diagram is

\[\begin{array}{c}
q \\
\lambda q^2 \\
q
\end{array}\]

in case \(\lambda q^2 \neq 1.\)

Denote \(a = \bar{\mu}^2 \omega^{16kn}, \quad b = \omega^{4jN}.\) If \(n = 2,\) then the braiding of \(\mathfrak{B}(\mathcal{H}_{jk,p}^s)\) is given by

\[
\begin{align*}
&c(w_1 \otimes w_1) = qw_1 \otimes w_1, \quad c(w_1 \otimes w_2) = qw_2 \otimes w_1, \\
&c(w_1 \otimes w_3) = qw_3 \otimes w_1, \quad c(w_1 \otimes w_4) = qw_4 \otimes w_1, \\
&c(w_2 \otimes w_1) = qa^{-1}b^2w_3 \otimes w_4, \quad c(w_2 \otimes w_2) = q\lambda a^{-1}b^{-1}w_4 \otimes w_4, \\
&c(w_2 \otimes w_3) = q\lambda b^{-1}w_1 \otimes w_4, \quad c(w_2 \otimes w_4) = qw_2 \otimes w_4, \\
&c(w_3 \otimes w_1) = qw_1 \otimes w_3, \quad c(w_3 \otimes w_2) = qb^2w_2 \otimes w_3, \\
&c(w_3 \otimes w_3) = qw_3 \otimes w_3, \quad c(w_3 \otimes w_4) = qb^2w_4 \otimes w_3, \\
&c(w_4 \otimes w_1) = qbw_3 \otimes w_2, \quad c(w_4 \otimes w_2) = qw_4 \otimes w_2, \\
&c(w_4 \otimes w_3) = qab^2w_1 \otimes w_2, \quad c(w_4 \otimes w_4) = q\lambda abw_2 \otimes w_2.
\]

\(W_1 = k_{w_1} \oplus k_{w_3}\) and \(W_2 = k_{w_2} \oplus k_{w_4}\) are braided vector spaces. \(\mathfrak{B}(W_1)\) is of diagonal type and its Dynkin diagram is

\[\begin{array}{c}
q \\
q^2 \\
q
\end{array}\]

if \(q^2 \neq 1.\)

\(\mathfrak{B}(W_1)\) (or \(\mathfrak{B}(W_2)\)) is finite dimensional iff \(q = -1\) or \(q^3 = 1 \neq q.\)

In case \(\lambda = 1,\) then \(b^2 = 1.\) Denote \(u_i = w_2 + (-1)^{i+1}\sqrt{\frac{1}{ab}}w_4\) for \(i = 1, 2,\) and \(u_j = w_1 + (-1)^{j+1}\sqrt{\frac{1}{ab}}w_3\) for \(j = 3, 4.\) Then the Nichols algebra is diagonal type and its Dynkin diagram is given by the Figure 2 if \(q^2 \neq \pm 1.\) According to Heckenberger’s classification result [13], we have \(\text{dim} \mathfrak{B}(\mathcal{H}_{jk,p}^s) = \infty\) in case \(\lambda = 1\) and \(q \neq -1.\) In case \(q = -1\) and \(\lambda = 1,\) it’s easy to see that \(\mathfrak{B}(\mathcal{H}_{jk,p}^s)\) is of Cartan type \(A_2 \times A_2\) and \(\text{dim} \mathfrak{B}(\mathcal{H}_{jk,p}^s) = 64.\) \(\square\)

**Remark 4.24.** The relations of the 64-dimensional Nichols algebra of Cartan type \(A_2 \times A_2\) is given by

\[
\begin{align*}
w_1^2 = 0, \quad w_3^2 = 0, \quad w_1w_3 + w_3w_1 = 0, \\
w_2w_4 = 0, \quad w_4w_2 = 0, \quad w_2^2 + \frac{b}{a}w_4^2 = 0, \\
w_1w_2 + w_2w_1 + a^{-1}w_3w_4 + a^{-1}w_4w_3 = 0,
\end{align*}
\]
\[w_1 w_4 + w_4 w_1 + bw_3 w_2 + bw_2 w_3 = 0,\]
\[w_1 w_2 w_3 w_4 = aw_2 w_1 w_2 w_1,\]
\[w_2 w_3 w_2 w_1 = bw_1 w_2 w_1 w_4 + \frac{1}{a} w_3 w_2 w_3 w_4 - w_2 w_1 w_2 w_3,\]

where \(a = \bar{\omega}^{16kn}\), \(b = \omega^{4jN} = \pm 1\).

4.4.2. The Nichols algebras over \(\mathcal{H}_{jk,p}^s\) for \(n = 3\). Denote \(q = (-1)^p \bar{\mu}^s \omega^{8kn}\), \(\alpha = \lambda \bar{\omega}^{8kn}\), \(\beta = \omega^{6jN}\), then the braiding of \(\mathcal{H}_{jk,p}^s\) is given by

\[
c(w_1 \otimes w_1) = qw_1 \otimes w_1, \quad c(w_1 \otimes w_2) = qw_2 \otimes w_1, \quad c(w_1 \otimes w_3) = qw_3 \otimes w_1, \quad c(w_1 \otimes w_4) = qw_4 \otimes w_1, \quad c(w_1 \otimes w_5) = qw_5 \otimes w_1, \quad c(w_1 \otimes w_6) = qw_6 \otimes w_1, \quad c(w_2 \otimes w_1) = \frac{q}{\beta^2 \alpha^4} w_5 \otimes w_6, \quad c(w_2 \otimes w_2) = \frac{q}{\beta \alpha^3} w_4 \otimes w_6, \quad c(w_2 \otimes w_3) = \frac{q}{\beta} w_1 \otimes w_6, \quad c(w_2 \otimes w_4) = \frac{q}{\beta \alpha^3} w_6 \otimes w_6, \quad c(w_2 \otimes w_5) = \frac{q}{\beta \alpha} w_3 \otimes w_6, \quad c(w_2 \otimes w_6) = qw_2 \otimes w_6, \quad c(w_3 \otimes w_1) = \frac{q}{\beta \alpha^3} w_5 \otimes w_5, \quad c(w_3 \otimes w_2) = \frac{q}{\beta^2 \alpha^2} w_4 \otimes w_5, \quad c(w_3 \otimes w_3) = qw_1 \otimes w_5, \quad c(w_3 \otimes w_4) = \frac{q}{\beta^2 \alpha^2} w_6 \otimes w_5, \quad c(w_3 \otimes w_5) = qw_3 \otimes w_5, \quad c(w_3 \otimes w_6) = \frac{q \alpha}{\beta} w_2 \otimes w_5,\]

\]
\[ c(w_4 \otimes w_1) = \frac{q}{\beta^2} w_1 \otimes w_4, \quad c(w_4 \otimes w_2) = q w_2 \otimes w_4, \]
\[ c(w_4 \otimes w_3) = \frac{q}{\beta^2} w_3 \otimes w_4, \quad c(w_4 \otimes w_4) = q w_4 \otimes w_4, \]
\[ c(w_4 \otimes w_5) = \frac{q}{\beta^2} w_5 \otimes w_4, \quad c(w_4 \otimes w_6) = q w_6 \otimes w_4, \]
\[ c(w_5 \otimes w_1) = q w_3 \otimes w_3, \quad c(w_5 \otimes w_2) = \frac{q}{\beta^3 \alpha} w_6 \otimes w_3, \]
\[ c(w_5 \otimes w_3) = q w_5 \otimes w_3, \quad c(w_5 \otimes w_4) = \frac{q \alpha^2}{\beta^2} w_2 \otimes w_3, \]
\[ c(w_5 \otimes w_5) = \beta q \alpha^3 w_1 \otimes w_3, \quad c(w_5 \otimes w_6) = \frac{q \alpha^2}{\beta^2} w_4 \otimes w_3, \]
\[ c(w_6 \otimes w_1) = \frac{q \alpha}{\beta^2} w_3 \otimes w_2, \quad c(w_6 \otimes w_2) = q w_6 \otimes w_2, \]
\[ c(w_6 \otimes w_3) = \frac{q \alpha}{\beta^2} w_5 \otimes w_2, \quad c(w_6 \otimes w_4) = \beta q \alpha^3 w_2 \otimes w_2, \]
\[ c(w_6 \otimes w_5) = \frac{q \alpha^4}{\beta^2} w_1 \otimes w_2, \quad c(w_6 \otimes w_6) = \beta q \alpha^3 w_4 \otimes w_2. \]

If \( q = -1, \beta^2 = 1 \), then the Nichols algebra has the following relations:

\[
\begin{align*}
w_1 w_2 + w_2 w_1 + \frac{1}{\alpha^4} w_5 w_6 + \frac{1}{\alpha^4} w_6 w_5 + \frac{1}{\alpha^2} w_3 w_4 + \frac{1}{\alpha^2} w_4 w_3 &= 0, \\
w_1 w_6 + w_6 w_1 + \alpha \beta w_2 w_3 + \alpha \beta w_3 w_2 + \frac{\beta}{\alpha} w_4 w_5 + \frac{\beta}{\alpha} w_5 w_4 &= 0, \\
w_6 w_2 = w_5 w_3 &= w_4 = w_3 w_5 = w_2 w_6 = w_1^2 = 0, \\
w_1 w_4 + w_4 w_1 &= 0, \quad w_2 w_5 + \frac{1}{\alpha \beta} w_3 w_6 = 0, \quad w_5 w_2 + \frac{1}{\alpha \beta} w_6 w_3 = 0, \\
w_1 w_5 + w_5 w_1 + w_3^2 &= 0, \quad w_1 w_3 + w_3 w_1 + \frac{1}{\alpha \beta} w_5 w_5 = 0, \\
w_2^2 + \frac{1}{\alpha \beta} w_4 w_6 + \frac{1}{\alpha \beta} w_6 w_4 &= 0, \quad w_2 w_4 + w_4 w_2 + \frac{1}{\alpha \beta} w_6^2 = 0.
\end{align*}
\]

5. Appendix

Here is a list of Yetter-Drinfeld modules over \( A_{N/2n}^{\mu} \) with structures decided by the formulae 3.1 and 3.2.
5.1. One and two dimensional Yetter-Drinfeld modules over $\mathcal{A}_{N2n}^\mu$

(1) $\mathcal{A}_{i,k}^{s,p} = \mathbb{K}w$, where $w = v \triangleright [x_{11}^{2s} + (-1)^p x_{12}^{2s}]$, $s \in 1, N$, $p \in \mathbb{Z}_2$, $\mathcal{V}_{i,k} = \mathbb{K}v$.

(2) $\mathcal{A}_{i,j,k}^{s,p} = \mathbb{K}w$, where $w = v \triangleright [x_{11}^{s+1} x_{22}^{2n+1} + (-1)^p \sqrt{\lambda} x_{12}^{s+1} x_{21}^{2n-1}]$, $s \in 1, N$, $p \in \mathbb{Z}_2$, $\mathcal{V}_{i,j,k} = \mathbb{K}v$, \begin{align*} i &= j, & \lambda &= 1, \\
&i &= j + 1, & \lambda &= -1, \text{ and } i, j \in \mathbb{Z}_2.
\end{align*}

(3) $\mathcal{B}_{i,j,k}^{w} = \mathbb{K}w_1 \oplus \mathbb{K}w_2$, where $s \in 1, N$, $i, j \in \mathbb{Z}_2$, $i = j + 1$, $\mathcal{V}_{i,j,k} = \mathbb{K}v$,
\begin{align*}
w_1 &= v \triangleright [x_{11}^{2s} + x_{12}^{2s}], \\
w_2 &= v \triangleright [x_{11}^{2s} - x_{12}^{2s}].
\end{align*}

(4) $\mathcal{C}_{i,j,k}^{st} = \mathbb{K}w_1 \oplus \mathbb{K}w_2$, where $s \in 1, N$, $t \in 0, n - 1$, $\mathcal{V}_{i,j,k} = \mathbb{K}v$, $i, j, p \in \mathbb{Z}_2$,
\begin{align*}
w_1 &= v \triangleright [x_{11}^{2s+1} x_{22}^{2t+1} + (-1)^p \sqrt{(-1)^t+j} x_{12}^{2s+1} x_{21}^{2t+1}], \\
w_2 &= v \triangleright [x_{11}^{2s} x_{22}^{2t+2} + \frac{(-1)^p}{\sqrt{(-1)^t+j}} x_{12}^{2s} x_{21}^{2t+2}].
\end{align*}

(5) $\mathcal{D}_{i,j,k}^{st} = \mathbb{K}w_1 \oplus \mathbb{K}w_2$, where $\mathcal{V}_{i,j,k} = \mathbb{K}v_1 \oplus \mathbb{K}v_2$, $s \in 1, N$, $t \in 0, n - 1$, $p \in \mathbb{Z}_2$,
\begin{align*}
w_1 &= v_1 \otimes x_{11}^{2s+1} x_{22}^{2t+1} + (-1)^p \omega^{jN-4kn} v_2 \otimes x_{12}^{2s+1} x_{21}^{2t+1}, \\
w_2 &= v_2 \otimes x_{11}^{2s} x_{22}^{2t+2} + (-1)^p \omega^{4kn-jN} v_1 \otimes x_{12}^{2s} x_{21}^{2t+2}.
\end{align*}

(6) $\mathcal{E}_{i,j,k}^{st} = \mathbb{K}w_1 \oplus \mathbb{K}w_2$, where $\mathcal{V}_{i,j,k} = \mathbb{K}v_1 \oplus \mathbb{K}v_2$, $s \in 1, N$, $t \in 0, n - 1$, $p \in \mathbb{Z}_2$,
\begin{align*}
w_1 &= v_1 \otimes x_{11}^{2s+1} x_{22}^{2t} + (-1)^p \omega^{jN-4kn} v_2 \otimes x_{12}^{2s} x_{21}^{2t}, \\
w_2 &= v_2 \otimes x_{11}^{2s} x_{22}^{2t} + (-1)^p \omega^{4kn-jN} v_1 \otimes x_{12}^{2s} x_{21}^{2t}.
\end{align*}

(7) $\mathcal{F}_{i,j,k}^{st} = \mathbb{K}w_1 \oplus \mathbb{K}w_2$, where $\mathcal{V}_{i,j,k} = \mathbb{K}v_1 \oplus \mathbb{K}v_2$, $s \in 1, N$, $p \in \mathbb{Z}_2$,
\begin{align*}
w_1 &= v_1 \otimes x_{11}^{2s} x_{22}^{2t} + (-1)^p \omega^{jN-4kn} v_2 \otimes x_{12}^{2s}, \\
w_2 &= v_2 \otimes x_{11}^{2s} x_{22}^{2t} + (-1)^p \omega^{4kn-jN} v_1 \otimes x_{12}^{2s}.
\end{align*}

(8) $\mathcal{G}_{i,j,k}^{st} = \mathbb{K}w_1 \oplus \mathbb{K}w_2$, where $\mathcal{V}_{i,j,k}' = \mathbb{K}v_1' \oplus \mathbb{K}v_2'$, $s \in 1, N$, $t \in 0, n - 1$, $p \in \mathbb{Z}_2$,
\begin{align*}
w_1 &= v_1' \otimes x_{11}^{2s+1} x_{22}^{2t} + \frac{(-1)^p \omega^{jN-4kn}}{\sqrt{\mu}} v_2' \otimes x_{12}^{2s+1} x_{21}^{2t}, \\
w_2 &= v_2' \otimes x_{11}^{2s} x_{22}^{2t+1} + (-1)^p \sqrt{\mu} \omega^{4kn-jN} v_1' \otimes x_{12}^{2s} x_{21}^{2t+1}.
\end{align*}
(9) \( H_{jk,p} = \mathbb{k}w_1 \oplus \mathbb{k}w_2 \), where \( V_{jk} = \mathbb{k}v_1 \oplus \mathbb{k}v_2 \), \( s \in 1, N \), \( t \in 0, n - 1 \), \( p \in \mathbb{Z}_2 \),

\[
\begin{align*}
w_1 &= v_1 \otimes x_1^{2s} x_2^{2r+1} + \frac{(-1)^p \omega^{jN-4kn}}{\sqrt{\mu}} v_2 \otimes x_1^{2s} x_2^{2r+1}, \\
w_2 &= v_2 \otimes x_1^{2s+1} x_2^{2r} + (-1)^p \sqrt{\mu} \omega^{4kn-jN} v_1 \otimes x_1^{2s+1} x_2^{2r}.
\end{align*}
\]

(10) \( \mathcal{P}_{ij,k,p} = \mathbb{k}w_1 \oplus \mathbb{k}w_2 \), where \( \lambda = 1 \), \( V_{ij,k} = \mathbb{k}v \), \( s \in 1, N \), \( t \in 0, n - 1 \), \( p \in \mathbb{Z}_2 \),

\[
\begin{align*}
w_1 &= v \otimes \left[ x_1^{2s+1} x_2^{2r} + \frac{(-1)^p}{\sqrt{(-1)^{j+1}}} x_1^{2s} x_2^{2r} \right], \\
w_2 &= v \otimes \left[ x_1^{2s+1} x_2^{2r} + \sqrt{(-1)^{j+1}(-1)^p} x_1^{2s} x_2^{2r} \right].
\end{align*}
\]

5.2. 2n-dimensional Yetter-Drinfeld modules over \( A_{N}^{\mu,N} \).

(1) Let \( V_{jk} = \mathbb{k}v_1 \oplus \mathbb{k}v_2 \), \( s \in 1, N \), \( p \in \mathbb{Z}_2 \). Denote

\[
\begin{align*}
w_r &= \begin{cases} 
  v_1 + (-1)^p \omega^{2jN-2kn} v_2 \otimes x_1^{2s+1}, & r = 1, \\
  x_1^{r-1} \cdot w_1, & r \text{ even}, \\
  x_1^{r-1} \cdot w_1, & r \text{ odd},
\end{cases} \\
m_r &= \begin{cases} 
  v_1 + (-1)^p \omega^{2jN-2kn} v_2 \otimes x_1^{2s} x_2, & r = 1, \\
  x_1^{r-1} \cdot m_1, & r \text{ even}, \\
  x_1^{r-1} \cdot m_1, & r \text{ odd},
\end{cases}
\end{align*}
\]

then \( \mathcal{J}_{jk} = \bigoplus_{r=1}^{n} (\mathbb{k}w_r \oplus \mathbb{k}m_r) \) is a 2n-dimensional Yetter-Drinfeld module over \( A_{N}^{\mu,N} \) with the module structure given by

\[
\begin{align*}
x_{11} \cdot w_r &= \begin{cases} 
  (-1)^p \omega^{4kn} w_1, & r = 1, \\
  w_{r+1}, & r \text{ even}, 1 < r < n, \\
  \omega^{8kn} w_{r-1}, & r \text{ odd}, 1 < r \leq n, \\
  (-1)^p \omega^{2n(2k+jN)} w_n, & r = n \text{ even},
\end{cases} \\
x_{22} \cdot w_r &= \begin{cases} 
  \omega^{8kn} w_{r-1}, & r \text{ even}, 1 < r \leq n, \\
  w_{r+1}, & r \text{ odd}, 1 \leq r < n, \\
  (-1)^p \omega^{2n(2k+jN)} w_n, & r = n \text{ odd},
\end{cases} \\
x_{pq} \cdot w_r &= 0, \quad pq = 12 \text{ or } 21, 1 \leq r \leq n, \\
x_{11} \cdot m_r &= \begin{cases} 
  \omega^{8kn} m_{r-1}, & r \text{ even}, 1 < r \leq n, \\
  m_{r+1}, & r \text{ odd}, 1 \leq r < n, \\
  \lambda(-1)^p \omega^{2n(2k+jN)} m_n, & r = n \text{ odd},
\end{cases}
\end{align*}
\]
\[ x_{pq} \cdot m_r = 0, \quad pq = 12 \text{ or } 21, 1 \leq r \leq n, \]
\[ x_{22} \cdot m_r = \begin{cases} (-1)^p \omega^{4kn}m_1, & r = 1, \\ m_{r+1}, & r \text{ even, } 1 < r < n, \\ \omega^{8kn}m_{r-1}, & r \text{ odd, } 1 < r \leq n, \\ \lambda(-1)^p \omega^{2n(2k+jN)}m_n, & r = n \text{ even,} \end{cases} \]

and the comodule structure given by
\[
\rho(w_r) = \begin{cases} x_{11}^{2(s-r+1)}x_{22}^{-1} \otimes w_r + x_{12}^{2(s-r+1)}x_{21}^{-1} \otimes m_r, & r \text{ even}, \\ x_{11}^{2(s-r+1)}x_{12}^{-1} \otimes w_r + x_{12}^{2(s-r+1)}x_{21}^{-1} \otimes m_r, & r \text{ odd}, \end{cases}
\]
\[
\rho(m_r) = \begin{cases} x_{11}^{2(s-r+1)}x_{12}^{-1} \otimes m_r + x_{12}^{2(s-r+1)}x_{21}^{-1} \otimes w_r, & r \text{ even}, \\ x_{11}^{2(s-r+1)}x_{22}^{-1} \otimes m_r + x_{12}^{2(s-r+1)}x_{21}^{-1} \otimes w_r, & r \text{ odd}. \end{cases}
\]

(2) Let \( V'_{jk} = \mathbb{k}v'_1 \oplus \mathbb{k}v'_2, s \in \overline{1,N}, p \in \mathbb{Z}_2, \) and denote
\[ w_r = \begin{cases} v'_1 \otimes \left[ x_{11}^{2s} + (-1)^p x_{12}^{2s} \right], & r = 1, \\ x_{12}^{2s-1} \cdot w_1, & r \text{ even and } 2 \leq r \leq 2n, \\ x_{21}^{2s-1} \cdot w_1, & r \text{ odd and } 1 \leq r \leq 2n, \end{cases} \]
then \( \mathcal{Y}^{s}_{jk,p} = \bigoplus_{r=1}^{2n} w_r \) is a 2\( n \)-dimensional Yetter-Drinfeld module over \( A_{N2n}^\mu \) with the module structure given by
\[
x_{12} \cdot w_r = \begin{cases} w_{r+1}, & r \text{ odd and } 1 \leq r < 2n, \\ \bar{\mu} \omega^{8kn}w_{r-1}, & r \text{ even and } 2 \leq r \leq 2n, \\ \lambda \bar{\mu}^{-1-n} \omega^{8kn-2n(4kn+jN)}w_{2n}, & r = 1, \end{cases}
\]
\[
x_{21} \cdot w_r = \begin{cases} \lambda \bar{\mu} \omega^{8kn}w_{r-1}, & r \text{ odd and } 3 \leq r < 2n, \\ w_{r+1}, & r \text{ even and } 2 \leq r \leq 2n, \\ \bar{\lambda} \bar{\mu}^{p} \omega^{2n(4kn+jN)}w_1, & r = 2n, \end{cases}
\]
\[ x_{pq} \cdot w_r = 0, \quad pq = 11 \text{ or } 22, 1 \leq r \leq 2n, \]

and the comodule structure given by
\[
\rho(w_r) = \begin{cases} x_{11}^{2(s-r+1)}x_{12}^{2r-2} \otimes w_r + \Omega_1 x_{12}^{2(s-r+1)}x_{21}^{2r-2} \otimes w_{2n-r+2}, & 2 \leq r \text{ even}, \\ x_{11}^{2(s-r+1)}x_{22}^{2r-2} \otimes w_r + \Omega_1 x_{12}^{2(s-r+1)}x_{21}^{2r-2} \otimes w_{2n-r+2}, & 3 \leq r \text{ odd}, \\ x_{11}^{2s} + (-1)^p x_{12}^{2s} \otimes w_1, & r = 1, \end{cases}
\]
where \( \Omega_1 = (-1)^p \lambda \bar{\mu}^{-1-n} \omega^{8kn(r-1-n)-2jNn} \).
(3) $\mathcal{M}_{ijk} = \bigoplus_{r=1}^{n} (\mathbb{k}w_r \oplus \mathbb{k}m_r)$, where $s \in \overline{1,N}$, $r \in \overline{1,n}$ and $V_{ijk} = \mathbb{k}v,$

$$w_r = \begin{cases} \nu \otimes x_{12}^{2s+1}, & r = 1, \\ x_{22}^{-r} \cdot w_1, & r \text{ even}, \\ x_{11}^{-r} \cdot w_1, & r \text{ odd}, \end{cases}$$

$$m_r = \begin{cases} \nu \otimes x_{12}^{2s}, & r = 1, \\ x_{11}^{-r} \cdot m_1, & r \text{ even}, \\ x_{22}^{-r} \cdot m_1, & r \text{ odd}. \end{cases}$$

(4) $\mathcal{N}_{ijk} = \bigoplus_{r=1}^{n} (\mathbb{k}w_r \oplus \mathbb{k}m_r)$, where $s \in \overline{1,N}$, $V_{ijk} = \mathbb{k}v,$

$$w_r = \begin{cases} \nu \otimes x_{12}^{2s+1}, & r = 1, \\ x_{22}^{-r} \cdot w_1, & r \text{ even}, \\ x_{11}^{-r} \cdot w_1, & r \text{ odd}, \end{cases}$$

$$m_r = \begin{cases} \nu \otimes x_{12}^{2s}, & r = 1, \\ x_{11}^{-r} \cdot m_1, & r \text{ even}, \\ x_{22}^{-r} \cdot m_1, & r \text{ odd}. \end{cases}$$

(5) $\mathcal{J}_{pjk}^{s} = \bigoplus_{r=1}^{n} (\mathbb{k}w_r \oplus \mathbb{k}m_r)$, where $V_{jk} = \mathbb{k}v_1 \oplus \mathbb{k}v_2$, $s \in \overline{1,N}$, $p \in \mathbb{Z}_2$,

$$w_r = \begin{cases} [v_1 + (-1)^p \omega^{-4kn} v_2] \otimes x_{12}^{2s+1}, & r = 1, \\ x_{22}^{-r} \cdot w_1, & r \text{ even}, \\ x_{11}^{-r} \cdot w_1, & r \text{ odd}, \end{cases}$$

$$m_r = \begin{cases} [v_1 + (-1)^p \omega^{-4kn} v_2] \otimes x_{12}^{2s}, & r = 1, \\ x_{11}^{-r} \cdot m_1, & r \text{ even}, \\ x_{22}^{-r} \cdot m_1, & r \text{ odd}. \end{cases}$$

(6) $\mathcal{L}_{jk,p}^{s} = \bigoplus_{r=1}^{2n} \mathbb{k}w_r$, where $V_{jk}^{r} = \mathbb{k}v_1' \oplus \mathbb{k}v_2'$, $s \in \overline{1,N}$, $p \in \mathbb{Z}_2$,

$$w_r = \begin{cases} v_2' \otimes \left[ x_{11}^{2s} + (-1)^p x_{12}^{2s} \right], & r = 1, \\ x_{12}^{-r} \cdot w_1, & r \text{ even and } 2 \leq r \leq 2n, \\ x_{21}^{-r} \cdot w_1, & r \text{ odd and } 1 \leq r \leq 2n. \end{cases}$$

(7) $\mathcal{Q}_{ijk,p}^{s} = \bigoplus_{r=1}^{2n} w_r$, where $V_{ijk}^{r} = \mathbb{k}v$, $s \in \overline{1,N}$, $i,j,p \in \mathbb{Z}_2$, $k \in \overline{0,N-1}$,

$$w_r = \begin{cases} v \otimes \left[ x_{11}^{2s} + (-1)^p x_{12}^{2s} \right], & r = 1, \\ x_{12}^{-r} \cdot w_1, & r \text{ even and } 2 \leq r \leq 2n, \\ x_{21}^{-r} \cdot w_1, & r \text{ odd and } 1 \leq r \leq 2n. \end{cases}$$

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