EXPONENTIAL CONVERGENCE TO NON-DEGENERATE REEB CHORDS

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1. Introduction

Let $Y^{2n-1}$ be a smooth manifold equipped with a contact form $\alpha$, i.e. a non-vanishing 1-form with the property that $d\alpha$ is non-degenerate on $\ker \alpha$. The hyperplane field $\xi := \ker \alpha$ is called the contact distribution. Note that $(\xi, d\alpha)$ can be thought of as a symplectic vector bundle.

Recall that the Reeb vector field $R$ for $(Y, \alpha)$ is defined by the equations:

$$R \cdot \alpha = 1 \quad \text{and} \quad R \cdot d\alpha = 0.$$ 

A periodic trajectory of $R$ is called a Reeb orbit. We say that a Reeb orbit

$$\gamma : \mathbb{R}/T\mathbb{Z} \to Y$$

is non-degenerate provided the $\mathbb{R}/T\mathbb{Z}$-family of orbits $\{\gamma(- + \tau_0) : \tau_0 \in \mathbb{R}/T\mathbb{Z}\}$ has a neighborhood (in the $C^0$ topology on the space of smooth loops) which contains no other Reeb orbits.

The symplectization of $(Y, \alpha)$ is defined to be $\mathbb{R} \times Y$ equipped with the symplectic form $\omega = d(e^\sigma \text{pr}^*\alpha)$ where $\sigma : \mathbb{R} \times Y \to \mathbb{R}$ and $\text{pr} : \mathbb{R} \times Y \to Y$ are the coordinate projections.

The tangent bundle of the symplectization splits into three factors

$$T(\mathbb{R} \times Y) = \partial_\sigma \mathbb{R} \oplus R\mathbb{R} \oplus \text{pr}^*\xi,$$

where:

(i) $\partial_\sigma$ is the vector field defined by $\partial_\sigma \cdot d\sigma = 1$ and $\partial_\sigma \cdot d\text{pr} = 0$,

(ii) $R$ is the extension of the Reeb field, extended by $R \cdot d\sigma = 0$, $d\text{pr} \cdot R = R \circ \text{pr}$, and

(iii) $\text{pr}^*\xi$ is identified with the locus $\ker d\sigma \cap \ker \text{pr}^*\alpha$. 

5.1. Bootstrapping the $W^{1,2}$ exponential estimates to $C^k$ estimates

5.2. Estimating the $\sigma$ coordinate

5.3. Simplifying the results

References
We say that almost complex structure $J$ on $\mathbb{R} \times Y$ is called *admissible* provided that
(a) $J$ is invariant under translation by $\partial_\sigma$,
(b) $J(\partial_\sigma) = R$,
(c) $J(\text{pr}^*\xi) = \text{pr}^*\xi$, and
(d) $d\alpha(-, J-)$ defines a Riemannian metric on the sub-bundle $\text{pr}^*\xi$ (i.e. $J|\text{pr}^*\xi$ is *compatible* with $d\alpha$).

1.1. **Asymptotic convergence to Reeb orbits.** In [Hof93] and [HWZ96], Hofer and Hofer-Wysocki-Zehnder established a relationship between punctured $J$-holomorphic curves $u$ and Reeb orbits, under the assumption that $J$ is admissible. Namely, (assuming that all the Reeb orbits were non-degenerate) the authors showed that any punctured holomorphic curve with finite energy$^1$ is asymptotic at its punctures to trivial cylinders over Reeb orbits. The reader is refered to [Hof93, Theorem 31] and [HWZ96, Theorem 1.2] for the original statements. This asymptotic behaviour is illustrated in Figure 1.

![Figure 1](image)

**Figure 1.** A punctured holomorphic disk (outlined in black) is asymptotic to a trivial cylinder over Reeb orbit $\gamma$ (shown in blue). There are two possibilities, either the $\sigma$ coordinate converges to $+\infty$ (shown on the left) or the $\sigma$ coordinate converges to $-\infty$.

**Remark 1.** The fact that the complex structure is admissible implies that trivial cylinders over Reeb orbits are complex submanifolds. Indeed, the map $v : \mathbb{R} \times \mathbb{R}/\mathbb{Z}$ defined by

\[(1) \quad \sigma \circ v(s, t) = Ts + \sigma_0 \text{ and } \text{pr} \circ v(s, t) = \gamma(Tt)\]

$^1$See Definition 6 for the definition of Hofer’s energy.
is holomorphic provided $\gamma : \mathbb{R}/T\mathbb{Z} \to Y$ satisfies $\gamma'(t) = R(\gamma(t))$. △

Moreover, [HWZ96, Theorem 1.4] establishes that the asymptotic convergence is exponential in nature. We paraphrase part of their result here. Suppose that $u(s,t)$ is a finite energy holomorphic half-cylinder, defined for $s + it \in [0, \infty) \times \mathbb{R}/\mathbb{Z}$. Then there exists a non-degenerate Reeb orbit $\gamma$, a choice of $\sigma_0$, and numbers $C, \delta > 0$ so that

$$|\sigma \circ u(s,t) - Ts - \sigma_0| < Ce^{-bs} \text{ and } \text{dist}(\text{pr} \circ u(s,t), \gamma(t)) < Ce^{-bs}.$$  

Moreover, similar exponential convergence holds for the higher derivatives of $u$. The authors also prove an asymptotic formula for $u$ in terms of the eigenfunctions of the asymptotic operator (see Section 4.1 for the definition in the Reeb chord case).

**Remark 2.** Let us comment on the strategy used by [HWZ96], as it is similar to how we will argue in this paper. First, they argue using more “geometric” techniques (such as bubbling analysis) to show that $u(s,t)$ must converge to a trivial cylinder $v(s,t)$ in the $C^0$ topology. Then, using a carefully constructed coordinate centered on the trivial cylinder (which $u(s,t)$ must eventually enter) they analyze the holomorphic curve PDE using more “analytic” techniques to conclude the exponential convergence. The previous paragraph was not entirely honest. In [Hof93] and [HWZ96] they do not actually establish $C^0$ convergence in the first “geometric” step. First they show something weaker: they deduce that $\text{pr} \circ u(s,-)$ eventually enters any fixed $S^1$-invariant neighborhood of the orbit $\text{pr} \circ v(s,-)$. Roughly speaking, they conclude that $\text{pr} \circ u(s,-)$ is converging as an unparametrized orbit, but not as a parametrized orbit. However, this weaker convergence is sufficient to ensure that $u(s,t)$ enters the coordinate chart they construct. Only after analyzing the coordinate representation of the map do they conclude that $u(s,t)$ converges to $v(s,t)$ in the $C^0$ topology. △

In [HWZ02], Hofer-Wysocki-Zehnder generalize the earlier work to higher dimensions, and prove exponential estimates for finite length holomorphic cylinders. This work plays a large role in the SFT compactness theory from [BEH+03].

1.2. **Asymptotic convergence to Reeb chords.** Suppose that $\Lambda \subset Y^{2n-1}$ is a Legendrian submanifold, i.e. $T\Lambda \subset \xi|_{\Lambda}$ is a Lagrangian sub-bundle of $\xi$. Note that $\mathbb{R} \times \Lambda \subset \mathbb{R} \times Y$ is a Lagrangian submanifold of the symplectization. This forces $\dim \Lambda = n - 1$. 

A map \( c : [0, T] \to Y \) is called a Reeb chord of \( \Lambda \) provided that \( c(0), c(1) \in \Lambda \) and \( c'(\tau) = R(c(\tau)) \). If we let \( \mathcal{F} \) denote the image of \((0, \infty) \times \Lambda \) under the Reeb flow, then the Reeb chords are in 1-to-1 correspondence with the intersection points of \( \mathcal{F} \) with \( \Lambda \). If the intersection point in \( \mathcal{F} \cap \Lambda \) is a transverse intersection then we say the corresponding Reeb chord is non-degenerate.

In the paper [Abb99], Abbas shows that holomorphic half-strips

\[
u : \mathbb{R} \times [0, 1] \to (\mathbb{R} \times Y, \mathbb{R} \times \Lambda)\]

are asymptotic to Reeb chords of \( \Lambda \). As in [HWZ96], Abbas works in the setting \( \dim(Y) = 3 \) and assumes the chords are non-degenerate. Abbas also shows that the asymptotic convergence is exponential in nature, and provides an asymptotic formula in terms of the eigenfunctions of the asymptotic operator.

1.3. Statement of results. One of the goals of the current paper is to prove exponential convergence results for Reeb chords for higher dimensional contact manifolds (analogous to those in [HWZ02]). Results in this vein are conjectured in the textbook [Abb14, pp. 206].

Our first result concerns \( C^0 \) convergence to Reeb chords and uniform bounds on the higher derivatives. We allow Reeb chords of zero length (these are constants), although they are degenerate.

**Theorem 3.** Let \( u : [0, \infty) \times [0, 1] \to \mathbb{R} \times Y \) be a \( J \)-holomorphic map with finite Hofer energy. We have the following:

(a) The derivative \( du \) satisfies uniform \( C^k \) bounds in the sense that

\[
\sup_{s,t} |\nabla^k du(s, t)| < \infty
\]

for each \( k \).

(b) If \( s_n \to \infty \) is any sequence, there is a subsequence (still denoted \( s_n \)) so that

\[
\lim_{n \to \infty} \text{pr} \circ u(s_n, t) = c(Tt) \text{ or } c(T(1 - t)).
\]

for some Reeb chord \( c : [0, T] \to Y \). The convergence is in the \( C^\infty \) topology on the space of chords of \( \Lambda \).
(c) If $c$ is a non-degenerate Reeb chord and (2) holds then we have
\[
\lim_{s \to \infty} \text{pr} \circ u(s, t) = c(Tt) \text{ or } c(T(1 - t)).
\]

To state the next result, we need to first explain the coordinate systems we use near a non-degenerate Reeb chord $c : [0, T] \to Y$. In Section 3 we pick a codimension 1 embedding $\Phi_0 : B(1)^{2n-2} \to Y$ so that $\Phi_0(0) = c(0)$ and $\text{im}(d\Phi_0(0)) = \xi_{c(0)}$.

Let $\varphi_t$ denote the time $t$ flow by $R$ and $\psi_s$ the time $s$ flow by $\partial_\sigma$. In Section 3.1 we show that any curve $u$ with $\text{pr} \circ u(s, t)$ sufficiently close to $c(Tt)$ can be expressed as
\[
u(s, t) = \psi_{Ts+\sigma(s, t)} \circ \varphi_{Tt+\tau(s, t)} \circ \Phi_0(x(s, t)),
\]
for $\mathbb{R}$-valued functions $\sigma$, $\tau$ and an $\mathbb{R}^{2n-2}$ valued function $x(s, t)$. In Section 3.2 we introduce the $t$-dependent coordinate change $x(s, t) = \mu_t(y(s, t))$, with $\mu_t(0) = 0$.

Note that trivial cylinders over $c$ are described by the equations
\[
\sigma = \text{const}, \quad \tau = 0 \quad \text{and} \quad y = 0.
\]

The following results are stated in terms of the three coordinates $\sigma, \tau, y$. In words, the results say that if $\text{pr} \circ u(s, -)$ is $C^1$ close to $t \mapsto c(Tt)$, then $u(s, t)$ is exponentially close to a trivial cylinder (3).

**Theorem 4.** Let $c$ be a non-degenerate Reeb chord and fix $\theta > 0$. There is a $C^1$ open set $U$ around $t \mapsto c(tT)$ and constants $M_k > 0$, $d > 0$ with the following property: if $u : [-r - 2, r + 2] \to \mathbb{R} \times Y$ is a holomorphic curve and $\text{pr} \circ u(s, -) \in U$ for all $s$, then
\[
\sum_{\ell=0}^{k} \left| \nabla^\ell (\sigma(s, t) - \sigma(0, 0)) \right| + \left| \nabla^\ell y(s, t) \right| + \left| \nabla^\ell \tau(s, t) \right| \leq M_k \theta (e^{-d(r+s)} + e^{-d(r-s)})
\]
for $(s, t) \in [-r, r] \times [0, 1]$.

**Corollary 5.** Let $c$ be a non-degenerate Reeb chord and fix $\theta > 0$. There is a $C^1$ open set $U$ around $t \mapsto c(tT)$ and constants $M_k$ with the following property: if $u : [-2, \infty) \to \mathbb{R} \times Y$ is a holomorphic curve and $\text{pr} \circ u(s, -) \in U$ for all $s$, then
\[
\sum_{\ell=0}^{k} \left| \nabla^\ell (\sigma(s, t) - \sigma(0, 0)) \right| + \left| \nabla^\ell y(s, t) \right| + \left| \nabla^\ell \tau(s, t) \right| \leq M_k \theta e^{-ds}
\]
for $(s, t) \in [0, \infty) \times [0, 1]$. 
2. $C^0$ convergence and uniform $C^k$ bounds.

Consider the data $(Y, \Lambda, \alpha, J)$ of:

(i) a compact manifold $Y^{2n-1}$,
(ii) a contact form $\alpha$,
(iii) a closed Legendrian submanifold $\Lambda$, and
(iv) an admissible complex structure $J$ on the symplectization $\mathbb{R} \times Y$.

We associate to this data the Riemannian metric $g$ defined by

$$g = e^{-\sigma} \omega(-, J-) = d\sigma^2 + pr^*\alpha^2 + pr^*d\alpha(-, J-),$$

where $\omega = d(e^\sigma pr^*\alpha)$. This defines a translation invariant distance function on $\mathbb{R} \times Y$.

Our main goal in this section is to prove that any holomorphic strip $u : \mathbb{R} \times [0, 1] \to (\mathbb{R} \times Y, \mathbb{R} \times \Lambda)$ with finite Hofer energy is asymptotic to a trivial cylinder over a Reeb chord. To be a bit more precise, we will show that the chords $pr \circ u(s, -)$ converge to $c(T-)$ in the $C^\infty$ topology for a Reeb chord $c : [0, T] \to \Lambda$ as $s \to \infty$. We will prove this result under a non-degeneracy assumption on the Reeb chord. Namely, we will assume that the image of $(0, \infty) \times \Lambda$ under the Reeb flow (which defines an immersed $n$ manifold in $Y$) intersects $\Lambda$ transversally.

We will also prove that $du$ satisfies uniform $C^k$ bounds for each $k \geq 0$. The main techniques we will use to establish the $C^0$ convergence of $pr \circ u$ and the $C^k$ bounds on $du$ are (i) a bubbling argument and (ii) elliptic bootstrapping.

To state the precise result we will prove, we need to first give the definition of the Hofer energy:

**Definition 6.** Suppose that $u : \Sigma \to \mathbb{R} \times Y$ is a smooth map. We define

$$\text{Hofer Energy of } u = \sup_{f \in \mathcal{P}} \int_{\Sigma} u^*d(e^{f(\sigma)}pr^*\alpha),$$

where $\mathcal{P}$ is the class of increasing diffeomorphisms $f : \mathbb{R} \to (0, 1)$. 
Another energy quantity which will play a role in our proof is the \( d\alpha \)-energy, defined by

\[
\text{\( d\alpha \)-energy of } u = \int_{\Sigma} u^* \, d\alpha.
\]

It is straightforward to show that the Hofer energy of \( u \) is positive for every non-constant holomorphic curve \( u \). Similarly, the \( d\alpha \)-energy is non-negative for all holomorphic curves. It is also not hard to show that the \( d\alpha \)-energy of a holomorphic curve \( u \) is bounded from above by the Hofer energy of \( u \).

**Theorem 3.** Let \( u : [0, \infty) \times [0, 1] \to \mathbb{R} \times Y \) be a \( J \)-holomorphic map with finite Hofer energy. We have the following:

(a) The derivative \( du \) satisfies uniform \( C^k \) bounds in the sense that

\[
\sup_{s,t} |\nabla^k du(s, t)| < \infty
\]

for each \( k \).

(b) If \( s_n \to \infty \) is any sequence, there is a subsequence (still denoted \( s_n \)) so that

\[
\lim_{n \to \infty} \text{pr} \circ u(s_n, t) = c(Tt) \text{ or } c(T(1 - t))
\]

for some Reeb chord \( c : [0, T] \to Y \). The convergence is in the \( C^\infty \) topology on the space of chords of \( \Lambda \).

(c) If \( c \) is a non-degenerate Reeb chord and (2) holds then we have

\[
\lim_{s \to \infty} \text{pr} \circ u(s, t) = c(Tt) \text{ or } c(T(1 - t)).
\]

**Remark 7.** It seems to be an interesting question as to whether or not the limit

\[
\lim_{s \to +\infty} \text{pr} \circ u(s, -)
\]

exists without the non-degeneracy assumption.

It is illuminating to compare with the case where \((W, L, J)\) is a compact symplectic manifold with Lagrangian \( L \) and compatible almost complex structure \( J \). It can be shown that every finite energy holomorphic curve \( w : [0, \infty) \times [0, 1] \to (W, L) \) satisfies

\[
\lim_{s \to \infty} w(s, -) = p \in L \text{ (see [Can21])}.
\]

In other words, every holomorphic strip has a well-defined asymptotic limit, even though the set of asymptotics is *not* discrete (i.e. the asymptotics are degenerate). Rather, the set of asymptotics forms the manifold \( L \). In some sense, the set of asymptotics is degenerate in a controlled way, analogous to how the critical points
of a Morse-Bott function are degenerate. Perhaps one would conjecture that the asymptotic limit of \( \text{pr} \circ u(s, -) \) exists provided the Reeb chords satisfy a Morse-Bott condition. We will not pursue this question in this paper.

Our proof Theorem \( \text{3} \) has four steps. The first step will reduce the proof of the \( C^k \) bound for every \( k \) to the case \( k = 0 \). The main technique in this step will be *elliptic bootstrapping*. The next step will be to show that if \( |du| \) is unbounded, then a *non-constant* holomorphic plane or half-plane with boundary on \( \mathbb{R} \times \Lambda \) with finite Hofer energy and *zero* \( d\alpha \)-energy exists. The third step will be to show that there are *no* non-constant planes or half-planes with finite Hofer energy and *zero* \( d\alpha \)-energy. The first three steps together prove the uniform \( C^k \) bounds. Finally, in the fourth step, we will investigate the \( C^0 \) convergence of \( \text{pr} \circ u(s, -) \).

### 2.1. Elliptic bootstrapping and bounding higher derivatives

In this section we will prove the following lemma:

**Lemma 8.** Let \( u_n : D(z_n, \frac{1}{2}) \to (\mathbb{R} \times Y, \mathbb{R} \times \Lambda) \) be a sequence of holomorphic curves whose derivatives are uniformly bounded. Then \( \sup_n \left| \nabla^k du_n(z_n) \right| < \infty \) for each \( k \).

Here \( D(z, r) \) is the domain \( D(z, r) \cap \mathbb{R} \times [0, 1] \), as shown in the figure below for various values of \( z \).

![Diagram](image)

**Figure 2.** The domain \( D(z, \frac{1}{2}) \) is a partial disk. Shown for three points \( z_1, z_2, z_3 \).

**Remark 9.** In the statement of the lemma we use metric \( g \) from \( \{6\} \) to measure sizes. We use the Levi-Civita connection \( \nabla \) associated to \( g \) to take the higher derivatives. Any translation invariant metric will suffice for this lemma.

In our proof of Lemma \( \text{8} \) we will require two analytical results: the *Sobolev embedding theorem* and the *elliptic estimates for the Laplacian*. We state these prerequisites here.

\(^2\text{Note that the Morse-Bott condition for } f \text{ is not just that the critical points form a manifold } L; \text{ it also requires that the Hessian } \nabla df \text{ is non-degenerate when restricted to the normal bundle of } L.\)
Lemma 10 (Sobolev embedding theorem). For every bounded Lipschitz domain $\Omega \subset \mathbb{R}^2$ there exists constants $c_2(\Omega), c_1(\Omega) > 0$ so that

$$\|f\|_{C^0(\Omega)} \leq c_1 \|f\|_{W^{1,4}(\Omega)} \leq c_2 \|f\|_{W^{2,2}(\Omega)}.$$

**Proof.** See [MS12, Theorem B.1.11] for a more general result. \hfill $\square$

Lemma 11 (Elliptic estimates for the Laplacian). For every pair of domains $\Omega_1, \Omega_2 \subset \mathbb{H}$ with $\overline{\Omega_1} \subset \Omega_2$ there exists a constant $c(k, \Omega_1, \Omega_2)$ so that

$$\|u\|_{W^{k+2,2}(\Omega_1)} \leq c(\|\Delta u\|_{W^{k,2}(\Omega_2)} + \|u\|_{W^{k+1,2}(\Omega_2)}),$$

for all smooth functions $u : \Omega_2 \to \mathbb{R}^d$ satisfying the Dirichlet boundary conditions $u(\mathbb{R} \cap \Omega_2) = 0$ or the Neumann boundary conditions $\partial_t u(\mathbb{R} \cap \Omega_2) = 0$.

**Proof.** See [RS01, Lemma C.2] for a short proof. \hfill $\square$

Note that we will only consider $\Omega = D(0, r)$ or $\Omega = D(0, r) \cap \mathbb{H}$, when applying Lemmas 10 and 11.

**Proof (of Lemma 8).** This argument is inspired by the proof of [RS01, Lemma C.3].

In search of a contradiction, let us suppose that $|\nabla^k u_n(z_n)|$ is unbounded. Then, passing to a subsequence, we may assume that $\lim_{n \to \infty} |\nabla^k u_n(z_n)| = \infty$.

Let us redefine our curves by translating in the vertical direction $u_n := T_n \circ u_n$. This does not change the sizes of the derivatives. We pick $T_n$ so that $u_n(z_n)$ converges to some point $p \in \mathbb{R} \times Y$, after potentially taking a further subsequence.

Write $z_n = s_n + it_n$. By passing to a further subsequence, we may suppose that $t_n$ converges to a point $t_\infty \in [0, 1]$. By replacing $u_n(s, t) := u_n(-s, -t)$, we may suppose that $t_\infty \in [0, \frac{1}{2}]$. We consider two cases, either $t_\infty = 0$ or $t_\infty \in (0, \frac{1}{2}]$. We will prove the case $t_\infty = 0$, i.e. when the points $z_n$ are converging to the boundary, and leave the other (simpler) case to the reader.

Consider the function $v_n(s + it) = u_n(s_n + s + it)$. Since $z_n = s_n + it_n$ and $t_n$ converges to 0, eventually $v_n$ is defined on the half-disk $D(0, \frac{1}{2})$. 
Figure 3. The half disk $D(0, \frac{1}{3})$ is eventually contained in $D(it_n, \frac{1}{2})$.

Since $v_n(it_n) = u_n(z_n)$, we conclude that $v_n(it_n)$ and $v_n(0)$ both converge to $p$. Therefore $p$ lies on $\mathbb{R} \times \Lambda$. Choose now a coordinate chart $\varphi : \overline{U} \to \overline{B} \subset \mathbb{R}_{2n}$ centered at $p$ which identifies $(\mathbb{R} \times \Lambda) \cap \overline{U}$ with $(\mathbb{R}^n \times \{0\}) \cap \overline{B}$ and so that the induced complex structure $d\varphi \cdot J \cdot d\varphi^{-1}$ is equal to $J_0$ along $\mathbb{R}^n \times \{0\}$. To see that such a coordinate chart exists one can, e.g., pick the first $n$ coordinates $x_1, \cdots, x_n$ for $\mathbb{R} \times \Lambda$ and then define the remaining coordinates $y_1, \cdots, y_n$ by exponentiating the vector fields $J \partial_{x_i}$ (which are transverse to $\mathbb{R} \times \Lambda$ since $J$ is compatible with $\omega$).

By the assumed $C^1$ bound, we conclude that $v_n$ eventually maps $D(0, \delta)$ into $U$. Thus we may (eventually) define the $\mathbb{R}^{2n}$-valued function $w_n(z) = \varphi \circ v_n(z)$.

Then, abusing notation and letting $J := d\varphi \cdot J \cdot d\varphi^{-1}$, we conclude that $w_n$ satisfies the boundary value problem:

\[
\left\{ \begin{array}{l}
\partial_s w_n + J(w_n) \cdot \partial_t w_n = 0 \\
w_n(s, 0) \in \mathbb{R}^n \times \{0\}
\end{array} \right.
\]

We decompose $w_n(s, t)$ into its real and imaginary parts:

\[w_n(s, t) = \begin{bmatrix} X_n(s, t) \\ Y_n(s, t) \end{bmatrix}.
\]

We easily compute that $Y_n(s, 0) = 0$ and $0 = \partial_s Y_n(s, 0) = -\partial_t X_n(s, 0)$. This means that $X_n$ satisfies the Neumann boundary conditions and $Y_n$ satisfies the Dirichlet boundary conditions. Therefore we conclude from Lemma 11 that, for $k \geq 2$, $w_n$ satisfies the elliptic estimates:

\[
\|w_n\|_{W^{k, 2}(\mathcal{D}(\delta/k))} \leq c_k \left( \|\Delta w_n\|_{W^{k-2, 2}(\mathcal{D}(\delta/(k-1)))} + \|w_n\|_{W^{k-1, 2}(\mathcal{D}(\delta/(k-1)))} \right).
\]

Here we abbreviate $\mathcal{D}(r) := \mathcal{D}(0, r)$. 
In order to use (8), we compute

\[(\partial_s - J(w_n)\partial_t)(\partial_s w_n + J(w_n)\partial_t w_n) = 0 \]

\[\implies \Delta w_n = \partial_t [J(w_n)]\partial_s w_n - \partial_s [J(w_n)]\partial_t w_n.\]

Our strategy will be to use (8) and (9) to bootstrap the initial $C^1$ bound to a $W^{k,2}$ bound on the disk $\mathcal{D}(\delta/k)$, for all $k$. To be more precise, we will prove:

\[(10) \sup_n \|w_n\|_{W^{k,2}(\mathcal{D}(\delta/k))} < \infty\]

by induction on $k$. The base case $k = 1$ holds from the initial $C^1$ bound.

Since $w_n$ is uniformly bounded in $C^1$, we conclude from (9) that $\Delta w_n$ is uniformly bounded in $L^2(\mathcal{D}(\delta))$. Therefore (8) implies that (10) holds with $k = 2$.

It is well-known that

\[(11) \sup_n \|w_n\|_{W^{k,2}(\mathcal{D}(r))} < \infty \implies \sup_n \|J(w_n)\|_{W^{k,2}(\mathcal{D}(r))} < \infty,\]

for all $k \geq 0$, since $J$ is smooth.

It is also easy to see that the following quadratic estimate holds:

\[(12) \|fg\|_{W^{1,2}(\Omega)} \leq \|f\|_{W^{1,2}(\Omega)} \|g\|_{C^0(\Omega)} + \|f\|_{C^0(\Omega)} \|g\|_{W^{1,2}(\Omega)}.\]

In particular, applying (12) to (9) implies that

\[\sup_n \|\Delta w_n\|_{W^{1,2}(\mathcal{D}(\delta/2))} = \sup_n \|\partial_t [J(w_n)]\partial_s w_n - \partial_s [J(w_n)]\partial_t w_n\|_{W^{1,2}(\mathcal{D}(\delta/2))} < \infty,\]

since we know $J(w_n), w_n$ are uniformly bounded in $W^{2,2}(\mathcal{D}(\delta/2))$ and $C^1$.

Then we easily conclude from the elliptic estimates (8) that the desired result (10) holds with $k = 3$. 

**Figure 4.** Nested disks $\mathcal{D}(\delta/k)$. 
To continue the bootstrapping argument, we will require another quadratic estimate; for $W^{k,2}$ with $k \geq 2$ we can use the following estimate:

\begin{equation}
\|fg\|_{W^{k,2}(\Omega)} \leq C_k \|f\|_{W^{k,2}(\Omega)} \|g\|_{W^{k,2}(\Omega)}.
\end{equation}

It is easy establish (13) for $k > 2$ by induction using

\[
\|fg\|_{W^{k,2}(\Omega)} \leq \|fg\|_{W^{k-1,2}(\Omega)} + \|\nabla f \cdot g\|_{W^{k-1,2}(\Omega)} + \|f \cdot \nabla g\|_{W^{k-1,2}(\Omega)}.
\]

The base case when $k = 2$ follows from a similar observation:

\[
\|fg\|_{W^{2,2}(\Omega)} \leq \|fg\|_{W^{1,2}(\Omega)} + \|\nabla f \cdot \nabla g\|_{L^2} + \|\nabla^2 f \cdot g\|_{L^2} + \|f \cdot \nabla g\|_{L^2}.
\]

The first term above can be estimated using our first quadratic estimate (12), together with the Sobolev embedding for $C^0 \subset W^{2,2}$. The last two terms can be estimated using $\|ab\|_{L^2} \leq \|a\|_{L^2} \|b\|_{C^0}$ and the Sobolev embedding theorem. The hard term to estimate is $\|\nabla f \cdot \nabla g\|_{L^2}$. To do so, we will use the Sobolev embedding theorem for $W^{1,4} \subset W^{2,2}$, and the H"{o}lder-type inequality

\[
\|\nabla f \cdot \nabla g\|_{L^2} \leq \|\nabla f\|_{L^4} \|\nabla g\|_{L^4}.
\]

Returning to our bootstrapping argument, we can now conclude from (11) and (13) that

\[
\sup_n \|\Delta w_n\|_{W^{2,2}(D(\delta/2))} = \sup_n \|\partial_t[J(w_n)]\partial_s w_n - \partial_s[J(w_n)]\partial_t w_n\|_{W^{2,2}(D(\delta/2))} < \infty.
\]

Then applying the elliptic estimates (8) proves (10) in the case $k = 4$. The argument repeats, without any further modification, to conclude (10) for all $k$.

We are almost finished with the proof. Recall that we assumed that

\[
\lim_{n \to \infty} \left|\nabla^k d u_n(z_n)\right| = \infty
\]

in search of a contradiction. Since $\varphi \circ u_n(z) = w_n(z - s_n)$, we conclude that $w_n$ is also unbounded in the $C^{k+1}$ norm near $z_n - s_n = it_n$ (since $\varphi$ is a diffeomorphism between compact domains, it distorts the $C^{k+1}$ size by a bounded amount).

Since $t_n$ converges to 0, $it_n$ eventually enters the disk $D(\delta/(k + 3))$. However, the $C^{k+1}(D(\delta/(k + 3)))$ norm is bounded by the $W^{k+3,2}(D(\delta/(k + 3)))$ norm, by the Sobolev embedding theorem. Then (10) with $k$ replaced by $k + 3$ contradicts the fact that the $C^{k+1}$ size of $w_n$ is unbounded. This contradiction completes the proof. \(\square\)
2.2. The bubbling argument. In this section we will prove the following lemma:

**Lemma 12.** Let \( u_n : D(z_n, \frac{1}{2}) \to (\mathbb{R} \times Y, \mathbb{R} \times \Lambda) \) be a sequence of holomorphic curves. Then we have the following alternative: either \( \sup_n |d_{u_n}(z_n)| < \infty \), or there exists a non-constant holomorphic plane \( v_\infty : \mathbb{C} \to \mathbb{R} \times Y \) or half-plane \( v_\infty : \mathbb{H} \to (\mathbb{R} \times Y, \mathbb{R} \times \Lambda) \) with

\[
(\text{Hofer energy of } v_\infty) \leq \limsup_{n \to \infty} (\text{Hofer energy of } u_n)
\]

\[
(\text{do-energy of } v_\infty) \leq \limsup_{n \to \infty} (\text{do-energy of } u_n).
\]

**Remark 13.** The case that will be of interest to us is the following: suppose we have a single holomorphic curve \( u : [0, \infty) \times [0, 1] \to \mathbb{R} \times Y \) with finite Hofer energy. Suppose that the derivative \( u \) is unbounded. Then we will set

\[ u_n = u|_{D(z_n, \frac{1}{2})} \]

for a sequence of points with \( \sup_n |d_{u_n}(z_n)| = \infty \). This forces \( \lim_{n \to \infty} s(z_n) = \infty \), and so

\[ \limsup_{n} (\text{do-energy of } u_n) = 0. \]

Then Lemma 12 will imply the existence of a finite Hofer energy plane or half-plane with zero do-energy. In Section 2.3 we will show that such planes/half-planes cannot exist. This argument shows \( u \) must have a bounded derivative. \( \triangle \)

One technical result needed in the proof of Lemma 12 is known as “Hofer’s lemma,”

**Lemma 14** (Hofer’s Lemma). Let \( d : X \to [0, \infty) \) be a continuous function on a complete metric space; and let \( \epsilon' > 0 \) and \( x' \in X \). One can find \( 0 < \epsilon \leq \epsilon' \) and \( x \in X \) so that

(i) \( \text{dist}(x, x') < 2\epsilon' \),

(ii) \( d(y) \leq 2d(x) \) for all \( y \in D(x, \epsilon) \).

(iii) \( \epsilon d(x) \geq \epsilon' d(x') \),

Hofer’s lemma was introduced in [HV92, Lemma 3.3] (moreover, they show that the lemma gives a characterization of completeness). We will give the proof here for the reader’s convenience.

**Proof** (of Lemma 14). Let \( \epsilon_n = 2^{-n} \epsilon' \), and define a (potentially terminating) sequence \( x_n \) as follows: let \( x_0 = x' \), and choose \( x_{n+1} \in D(x_n, \epsilon_n) \) so that \( d(x_{n+1}) > 2 \epsilon_{n+1} \). Then...
2d(x_n). If no such x_{n+1} exists (i.e. the sequence terminates at x_n), then we conclude that, for all y \in D(x_n, \epsilon_n) we have d(y) \leq 2d(x_n), so (ii) is satisfied with x = x_n, \epsilon = \epsilon_n. By construction, we have
\[
\epsilon_n d(x_n) \geq 2\epsilon_n d(x_{n-1}) = \epsilon_{n-1} d(x_{n-1}) \geq \cdots \geq \epsilon_0 d(x_0) = \epsilon' d(x'),
\]
so (iii) would also be satisfied. Since dist(x_0, x_n) \leq \epsilon_0 + \cdots + \epsilon_n \leq 2\epsilon_0, we conclude (i) also holds.

Thus the proof of the lemma is reduced to proving that the above recursion terminates. In search of a contradiction suppose it does not converge. Then the sequence x_n converges, however, d(x_n) is unbounded since d(x_n) > 2d(x_{n-1}). This is impossible, and so we complete the proof. \(\square\)

**Proof** (of Lemma 12). Let u_n : D(z_n, \frac{1}{2}) \to \mathbb{R} \times Y be a sequence of holomorphic curves. Without loss of generality, let us suppose that the derivative du_n(z_n) is unbounded. By passing to a subsequence, we may suppose that R' := |du_n(z_n)| satisfies \(\lim_{n \to \infty} R'_n = +\infty\).

Now pick 0 < \epsilon'_n < 1/6 so that \(\lim_{n \to \infty} \epsilon'_n = 0\) but \(\lim_{n \to \infty} \epsilon'_n R'_n = +\infty\).

Introduce the function \(d_n(z) = |du_n(z)|\), and apply Hofer’s lemma with \(\epsilon' = \epsilon'_n\) and \(x' = 0\) to conclude \(\epsilon_n \leq \epsilon'_n\) and \(x_n\) so that

(i) \(x_n \in D(z_n, 2\epsilon'_n)\),
(ii) \(|du_n(y)| \leq 2|du_n(x_n)|\) for \(y \in D(x_n, \epsilon_n)\),
(iii) \(\epsilon_n |du_n(x_n)| \geq \epsilon'_n R'_n\).

The reader may complain that \(d_n\) is not defined on a complete metric space, but it is easy to see that every point and ball considered in the recursive proof of Hofer’s lemma will remain entirely in \(D(z_n, 3\epsilon'_n)\). Since we chose \(\epsilon'_n < 1/6\), we see that we can cut off \(d_n\) outside of \(D(z_n, 3\epsilon'_n)\) (and obtain a continuous function defined on all of \(\mathbb{R} \times [0, 1]\)) without affecting our conclusions.

We abbreviate \(R_n := |du_n(x_n)|\). Note that by item (iii) \(R_n\) is still diverging to \(\infty\).

The idea now is to rescale the domains of \(u_n\) by the factor of \(R_n^{-1}\); we introduce
\[
v_n : D(0, R_n \epsilon_n) \cap R_n (\mathbb{R} \times [0, 1] - x_n) \text{ given by } v_n(z) = u_n(x_n + R_n^{-1}z).
\]
The domain of $v_n$ seems a bit awkward, but we can simplify it by writing $x_n = s_n + it_n$ and observing that

$$R_n(\mathbb{R} \times [0, 1] - x_n) = \mathbb{R} \times [-t_n R_n, (1 - t_n) R_n].$$

We can pass to a further subsequence so that $t_n R_n$ and $(1 - t_n) R_n$ converge in $[0, \infty]$. There are then three cases to consider: if either $t_n R_n$ converges to a finite number, then the domains of $v_n$ converge to an upper half-plane. On the other hand, if $(1 - t_n) R_n$ converges to a finite number, then the domains of $v_n$ converge to a lower half-plane.

To be precise, by “converge to a half plane” we mean that there exists a half plane $H$ with the property that any compact set in $H$ is eventually contained in the domain of $v_n$.

If both $t_n R_n$ and $(1 - t_n) R_n$ diverge to $\infty$, the domains of $v_n$ converge to the entire complex plane $\mathbb{C}$.

It is straightforward to conclude that

$$(14) \quad |dv_n(0)| = 1 \text{ and } |dv_n(z)| \leq 2$$

for all $z$ in the domain of $v_n$ (using (ii)). As we did in the proof of Lemma 8, we now replace $v_n := T_n \circ v_n$ where $T_n$ is a sequence of vertical translations. This does not affect (14). We choose $T_n$ so that $v_n(0)$ converges (taking a subsequence if necessary).

The Arzéla-Ascoli theorem implies that $v_n$ converges in $C^0_{\text{loc}}$ to a continuous function $v_\infty : \Omega \to \mathbb{R} \times Y$ with where $\Omega$ is either a half-plane or $\mathbb{C}$. If $\Omega$ is a half-plane, then the aforementioned $C^0_{\text{loc}}$ convergence implies that $v_\infty$ maps the boundary onto $\mathbb{R} \times \Lambda$. 
By our elliptic bootstrapping lemma (Lemma 8), we conclude that the higher derivatives of $v_n$ are uniformly bounded on compact sets. Here we use the fact that if $z$ lives in a compact set of $\Omega$, then eventually $z$ has a neighborhood in the domain of $v_n$ identical to one of the partial disks considered in Lemma 8.

The $C^k_{\text{loc}}$ bounds allow us to upgrade the conclusion of the Arzéla-Ascoli theorem to conclude that (i) the limit map $v_\infty$ is smooth and (ii) $v_n$ actually converges in $C^\infty_{\text{loc}}$ to $v_\infty$. In particular, $v_\infty$ is holomorphic. Moreover, by the $C^1_{\text{loc}}$ convergence, we conclude that $|dv_\infty(0)| = 1$, and hence $v_\infty$ is non-constant.

It remains only to prove the bounds on the energies of $v_\infty$. This follows from a fairly standard argument, which we will briefly explain. If $v_\infty$ has energy greater the $E$, then for $\epsilon > 0$ there is a compact domain $K \subset \Omega$ on which $v_\infty$ has energy greater than $E - \epsilon$. Eventually $v_n$ is defined on $K$, and since $v_n$ converges to $v_\infty$ in $C^1(K)$, we conclude that the energy of $v_n$ on $K$ is eventually greater than $E - 2\epsilon$. Therefore

$$E - 2\epsilon < \lim sup_{n \to \infty} (\text{energy of } v_n).$$

Since $E$ was an arbitrary number less than the energy of $v_\infty$, and $\epsilon > 0$ was also chosen arbitrarily, we conclude that

$$\text{energy of } v_\infty < \lim sup_{n \to \infty} (\text{energy of } v_n).$$

This argument works verbatim replacing “energy” with “$d\alpha$-energy.” This argument also applies if we set

“energy” of $u = \int u^* d(e^f(\sigma)d\alpha).$

Then we can take the supremum over all $f$ as required by the definition of the Hofer energy. This completes the proof of the lemma. $\square$

2.3. Holomorphic curves with zero $d\alpha$-energy. Our main goal in this section is to prove that there are no holomorphic planes or half-planes with finite Hofer energy and zero $d\alpha$-energy. As a first step, we prove the following lemma:

Lemma 15. Let $\Sigma$ be a connected Riemann surface, and let $u : \Sigma \to \mathbb{R} \times Y$ be a holomorphic map with zero $d\alpha$-energy. Then there is a leaf $L \to \mathbb{R} \times Y$ of the Reeb foliation (the foliation spanned by $\partial_\sigma$ and $R$) so that $u$ factors smoothly through $L$. 

Proof. Since the $d\alpha$-energy is the integral of $u^*\text{pr}^*\!d\alpha$, and $\text{pr}^*\!d\alpha$ is a $J$-compatible symplectic form on the contact distribution $\text{pr}^*\xi \subset T(\mathbb{R} \times Y)$, we conclude that 

$$\text{im}(du) \subset \ker \text{pr}^*\!d\alpha = \mathbb{R}\partial_\sigma \oplus \mathbb{R}R.$$ 

Pick a point $z \in \Sigma$ and let $u(z) = p$. Choose coordinates $x_1, x_2, y_1, \ldots, y_{2n-2}$ centered on $p$ so that $\partial_{x_1} = \partial_\sigma$ and $\partial_{x_2} = R$ (this is possible since $\partial_\sigma$ and $R$ commute). On an open set around $z$ we conclude that $d(y_i \circ u) = 0$ for all $i$, since 

$$\text{im}(du) \subset \text{span}\{\partial_{x_1}, \partial_{x_2}\}.$$ 

In particular $u$ factors smoothly through the locus where $y_1 = \cdots = y_{2n-2} = 0$, which is evidently part of some leaf $\mathcal{L}$.

This argument shows that set of points $z \in \Sigma$ which land in $\mathcal{L}$ is an open set. However, since the complement of $\mathcal{L}$ is a union of other leaves, we conclude by the same argument that the set of points which don’t land in $\mathcal{L}$ is also an open set. By the connectedness of $\Sigma$, we conclude that all points $u(\Sigma) \subset \mathcal{L}$. Our argument also shows the factorization of $u$ through the inclusion $\mathcal{L} \rightarrow \mathbb{R} \times Y$ is smooth. This completes the proof. \hfill \Box 

It is easy to classify the leaves of the Reeb foliation: each leaf is of the form $\mathbb{R} \times \gamma$ where $\gamma$ is a Reeb flow line. In particular, if $\mathcal{L}$ is a leaf, then

(15) \hspace{1cm} \Gamma: \sigma + i\tau \in \mathbb{C} \mapsto (\sigma, \gamma(\tau)) \in \mathcal{L}

is either a diffeomorphism or the universal cover (depending on whether $\gamma$ is a closed orbit or not). Also note that the fact that $J$ is assumed to be admissible implies that (15) is holomorphic.

We compute the following formula:

$$\Gamma^*d(e^{f(\sigma)}\text{pr}^*\!\alpha) = f'(\sigma)e^{f(\sigma)}d\sigma \wedge d\tau.$$ 

Combining Lemma 15 with (15) allows us to prove the following:

**Corollary 16.** Let $\Sigma$ be a simply-connected Riemann surface. If $u: \Sigma \rightarrow \mathbb{R} \times Y$ is a holomorphic curve with zero $d\alpha$-energy, then there exists a holomorphic map $w: \Sigma \rightarrow \mathbb{C}$ so that $\Gamma \circ w = u$. If $u$ has finite Hofer energy, then

(16) \hspace{1cm} \sup_{f \in \mathcal{D}} \int_{\Sigma} w^*(f'(\sigma)e^{f(\sigma)}d\sigma \wedge d\tau) < \infty,$
Lemma 17. There are no non-constant holomorphic planes $u : \mathbb{C} \to \mathbb{R} \times Y$ or half-planes $u : \mathbb{H} \to (\mathbb{R} \times Y, \mathbb{R} \times \Lambda)$ with finite Hofer energy and zero $d\alpha$-energy.

Proof. We argue by contradiction. By Corollary 16, we conclude a either a map $w : \mathbb{C} \to \mathbb{C}$ or a map $w : \mathbb{H} \to \mathbb{C}$ so that the energy (16) is finite.

Consider the case when $u : \mathbb{H} \to \mathbb{R} \times Y$. Since $u(\mathbb{R}) \subset \mathbb{R} \times \Lambda$ and the inverse image $\Gamma^{-1}(\mathbb{R} \times \Lambda)$ is a union of horizontal lines $\tau = \text{const}$, we conclude that the induced map $w$ satisfies $w(\mathbb{R}) \subset L$ for some horizontal line $L$. Then $w$ can be doubled by the Schwarz reflection principle to obtain a holomorphic plane $w : \mathbb{C} \to \mathbb{C}$.

The doubling process increases (16) by a factor of 2, since the energy only depends on the horizontal coordinate which is unchanged when we reflect.

Thus it suffices to prove the case $w : \mathbb{C} \to \mathbb{C}$. We observe that the integral of $f'(\sigma)e^{f(\sigma)}d\sigma \wedge dt$ over a region of the form $[a, b] \times \mathbb{R}$ is always infinite. In particular, if $w$ surjects on $[a, b] \times \mathbb{R}$, then the integral (16) would be infinite.

Picard’s little theorem asserts that $w$ must surject onto $\mathbb{C}$ or $\mathbb{C}$ minus a single point. In particular, we can find $a < b$ so that $w$ surjects onto $[a, b] \times \mathbb{R}$. This completes the proof.

If the reader does not like using Picard’s theorem, we can also argue as follows. Recall that under the stereographic projection $p : \mathbb{C} \to \mathbb{CP}^1$, the Fubini-Study form $\omega_{FS}$ is pulled back to

$$p^*\omega_{FS} = \frac{d\sigma \wedge d\tau}{(1 + \sigma^2 + \tau^2)^2}.$$  
(See [MST17, Exercise 4.3.4]). Now observe that

$$\int_{-\infty}^{+\infty} \frac{1}{(1 + \sigma^2)^2} = c < \infty.$$

If we define

$$f'(\sigma) = \frac{1}{c(1 + \sigma^2)^2},$$

\text{The $\sigma$ coordinate is the real coordinate on $\mathbb{C}$, even though it corresponds to the “vertical” coordinate in the symplectization. This conflict between “horizontal” and “vertical” is a bit unfortunate.}
and $f(\sigma) = \int_{-\infty}^{\sigma} f'(\sigma') d\sigma'$, then $f : \mathbb{R} \to (0, 1)$ is an increasing diffeomorphism, and hence is in the family $\mathcal{P}$ of functions from Definition 6. Moreover, it is clear that

$$p^* \omega_{FS} = \frac{d\sigma \wedge d\tau}{(1 + \sigma^2 + \tau^2)^2} \leq c e^{f(\sigma)} f'(\sigma) d\sigma \wedge d\tau.$$  

In particular, we conclude that the composite $p \circ w : \mathbb{C} \to \mathbb{CP}^1$ has finite Fubini-Study area. By the removal of singularities theorem (see [MS12, Theorem 4.1.2]) we conclude that $p \circ w$ extends to a holomorphic map $\mathbb{CP}^1 \to \mathbb{CP}^1$. In particular, $p \circ w$ is surjective, and hence $w : \mathbb{C} \to \mathbb{C}$ is surjective.

Thus $w$ surjects onto $\mathbb{R} \times [a, b]$, contradicting (16) (as we already explained above). This completes the proof.$^4$

In Remark 13, we explained that if a holomorphic curve $u : [0, \infty) \times [0, 1] \to (\mathbb{R} \times Y, \mathbb{R} \times \Lambda)$ with finite Hofer energy had an unbounded first derivative, then the bubbling lemma (Lemma 12) would produce a finite energy plane or half-plane with zero $d\alpha$-energy. This conclusion is obviously incompatible with Lemma 17 above. Thus we have concluded the first part of Theorem 3, namely that for every holomorphic curve $u : [0, \infty) \times [0, 1] \to (\mathbb{R} \times Y, \mathbb{R} \times \Lambda)$ with finite Hofer energy we have

$$\sup_{s, t} |\nabla^k u(s, t)| < \infty,$$

for all $k$.

**Proof.** Remark 13 together with Lemma 17 imply the $C^1$ bound ($k = 0$). The result for $k \geq 1$ follows from Lemma 8.$^\square$

Before we end this section, we wish to prove one more lemma concerning holomorphic curves with zero $d\alpha$-energy.

**Lemma 18.** Let $u : \mathbb{R} \times [0, 1] \to (\mathbb{R} \times Y, \mathbb{R} \times \Lambda)$ be a holomorphic curve with finite Hofer energy and zero $d\alpha$-energy. Then there exists a Reeb chord $c : [0, T] \to (Y, \Lambda)$ and a real number $\sigma_0$ so that either

$\sigma \circ u(s, t) = Ts + \sigma_0$ and $pr \circ u(s, t) = c(Tt)$,
or

\[ \sigma \circ u(s, t) = -Ts + \sigma_0 \text{ and } \text{pr} \circ u(s, t) = c(T(1 - t)). \]

**Proof.** If \( u \) is constant, then we can take \( T = 0 \) and \( c \) to be a constant map. Thus let us suppose that \( u \) is non-constant. Since the strip \( \mathbb{R} \times [0, 1] \) is simply connected, we can apply Corollary 16 to conclude a holomorphic map

\[ w : \mathbb{R} \times [0, 1] \to \mathbb{C} \text{ satisfying } \Gamma \circ w = u. \]

Here \( \Gamma : \mathbb{C} \to \mathbb{R} \times Y \) is defined by \( \sigma \circ \Gamma(\sigma, \tau) = \sigma \) and \( \text{pr} \circ \Gamma(\sigma, \tau) = c(\tau) \) for some Reeb flow line \( c \).

It follows from 18 that \( w \) has bounded derivatives.

Since \( u \) has boundary on \( \mathbb{R} \times \Lambda \), \( w \) must have boundary on \( \Gamma^{-1}(\mathbb{R} \times \Lambda) \). As shown in Figure 6, \( \Gamma^{-1}(\mathbb{R} \times \Lambda) \subset \mathbb{C} \) is a collection of horizontal lines \( \tau = \text{const} \) corresponding to the places where \( \gamma(\tau) \) intersects \( \Lambda \).

Without loss of generality, suppose that \( w(\mathbb{R} \times \{0\}) \) lies in the line \( \tau = 0 \), and \( w(\mathbb{R} \times \{1\}) \) lies in the line \( \tau = T \). If \( T < 0 \), then we replace \( w(s, t) := w(-s, 1 - t) \), so now \( T > 0 \).

---

![Figure 6](image_url)

**Figure 6.** \( \Gamma^{-1}(\mathbb{R} \times \Lambda) \) is a collection of horizontal lines. The holomorphic map \( w : \mathbb{R} \times [0, 1] \to \mathbb{C} \) (shown in orange) has boundary on \( \Gamma^{-1}(\mathbb{R} \times \Lambda) \).

We apply the Schwarz Reflection principle repeatedly to reflect \( w \) across horizontal lines until we have extended \( w \) to a map \( w : \mathbb{C} \to \mathbb{C} \), with the property that \( w \) still has bounded derivatives. To give some details of the construction, the first step is extend \( w \) to a map \( \mathbb{R} \times [-1, 1] \to \mathbb{C} \) by defining \( w(s, -t) = \overline{w}(s, t) \). The subsequent steps are similar, and we leave them to the reader. See Figure 7 for an illustration of the construction.
Note that if $T = 0$, (i.e. both boundaries of $w$ lie on the same line), then the extension $w : \mathbb{C} \to \mathbb{C}$ has a bounded imaginary part (noting that the original map $w$ has a bounded real part because its derivative is bounded). However, it is well-known that there are no non-constant functions with bounded imaginary part. \footnote{proof: apply $z \mapsto e^{iz}$ to obtain a bounded holomorphic function.} Since we assume that $w$ is non-constant, we conclude that $T > 0$.

Because the extension $w : \mathbb{C} \to \mathbb{C}$ has a bounded first derivative, $w$ must be an affine function $w(z) = az + \sigma_0$. Since $w$ sends the lines $t = 0$ to $\tau = 0$ and $t = 1$ to $\tau = T$, we must have $a = T$, and $\sigma_0 \in \mathbb{R}$. Thus we conclude that

$$w(s, t) = Ts + iTt + \sigma_0 \implies \sigma \circ u(s, t) = Ts + \sigma_0 \text{ and } \text{pr} \circ u(s, t) = c(Tt),$$

or

$$\sigma \circ u(s, t) = -Ts + \sigma_0 \text{ and } \text{pr} \circ u(s, t) = c(T(1 - t)),$$

depending on whether or not we replaced $w(s, t)$ by $w(-s, 1 - t)$.

It is clear from this construction that $c : [0, T] \to \Lambda$ is a Reeb chord. This completes the proof. \hfill \Box

\begin{figure}[h]
\centering
\begin{tikzpicture}
\draw[fill=green!20] (0,0) rectangle (1,1) node at (0.5,0.5) {$w(s,t)$};
\draw[fill=orange!20] (0,-1) rectangle (1,0) node at (0.5,0) {$\overline{w}(s,-t)$};
\draw[fill=blue!20] (0,-2) rectangle (1,-1) node at (0.5,-1.5) {$w(s,t+2) - 2iT$};
\end{tikzpicture}
\caption{Extending a function $w(s,t)$ with $w(s,0) \in \{0\} \times \mathbb{R}$ and $w(s,1) \in \{T\} \times \mathbb{R}$ to a holomorphic function $\mathbb{C} \to \mathbb{C}$.}
\end{figure}
2.4. $C^0$ convergence to Reeb chords. In this section we will analyze the convergence of the chords $\text{pr} \circ u(s,-)$. Our goal is to prove the rest of Theorem 3. Let $u : [0, \infty) \times [0,1] \rightarrow (\mathbb{R} \times Y, \mathbb{R} \times \Lambda)$ be a holomorphic map with finite Hofer energy. Pick a sequence $s_n$ tending to $+\infty$. Consider the translated curves $v_n : [-\frac{1}{2}s_n, \infty) \times [0,1] \rightarrow (\mathbb{R} \times Y, \mathbb{R} \times \Lambda)$ given by $v_n(s,t) = u(s + s_n, t)$.

Note that $v_n(0,-) = u(s_n,-)$.

It is clear from (18) that $\sup_{s,t} |\nabla^k d v_n(s,t)| < C_k$ for constants $C_k$ independent of $n$. Moreover, the Hofer energy of $v_n$ is bounded from above.

The $\alpha$-energy of $v_n$ equals the $\alpha$-energy of $u_n$ on the region $[\frac{1}{2}s_n, \infty) \times [0,1]$. As a consequence, the $\alpha$-energy of $v_n$ is tending to zero.

As we have done before, replace $v_n := T_n \circ v_n$, where $T_n$ are vertical translations of $\mathbb{R} \times Y$, and $T_n$ are chosen so that $v_n(0,0)$ converges.

The Arzélà-Ascoli theorem implies that $v_n$ converges in $C^\infty_{\text{loc}}$ to a limiting holomorphic map $v_\infty : \mathbb{R} \times [0,1] \rightarrow \mathbb{R} \times Y$. By the same argument given in the proof of Lemma 12, the $\alpha$-energy of $v_\infty$ is zero, and its Hofer energy is finite.

Therefore Lemma 18 applies, and so we conclude that

$$\text{pr} \circ v_\infty(s,t) = c(Tt) \text{ or } \text{pr} \circ v_\infty(s,t) = c(T(1-t))$$

for some Reeb chord $c$. Note that $c$ may be constant (in which case $T = 0$).

It is clear that $\text{pr} \circ v_n(0,-) = \text{pr} \circ u_n(s_n,-)$. Therefore the $C^\infty_{\text{loc}}$-convergence of $v_n$ to $v_\infty$ implies that

$$\lim_{n \to \infty} \text{pr} \circ u_n(s_n,t) = c(Tt) \text{ or } c(T(1-t)) \text{ in the } C^\infty([0,1],Y) \text{ topology.}$$

Thus we have proved the second assertion of Theorem 3.

To complete the proof of Theorem 3, we need to upgrade the convergence of (19) by removing the dependence on the subsequence $s_n$. For this part we assume that the limit Reeb chord $c$ is non-degenerate. In particular, we assume that $T > 0$.

For concreteness, let us suppose that in (19) $\text{pr} \circ u_n(s_n,t)$ converges to $c(Tt)$; the case $c(T(1-t))$ is similar. We argue by contradiction: if $\lim_{n \to \infty} \text{pr} \circ u_n(s_n,t) \text{ does not converge} \text{ to } c(Tt)$, then we can find another subsequence $s'_n \to \infty$ so that $\lim_{n \to \infty} \text{pr} \circ u_n(s'_n,t)$ converges to a different Reeb chord $c'(T't)$.
By further taking subsequences, we may suppose that $s_n < s'_n < s_{n+1}$. We consider $\text{pr} \circ u_n(s, -)$ for $s \in [s_n, s'_n]$ as a path of chords joining $\text{pr} \circ u_n(s_n, -)$ and $\text{pr} \circ u_n(s'_n, -)$. Since $c(t)$ is assumed to be a non-degenerate Reeb chord, there is a $C^1$ neighborhood $U$ of $t \mapsto c(Tt)$ (in the space of chords of $\Lambda$) so that the only Reeb chord in $\overline{U}$ is $c$. The $C^1$ topology is metrizable, and hence we can find a smaller open set $U'$ so that $c \in U'$, $U' \subset \overline{U}$.

Since $\mathcal{(U')}^c$ is open around $c'$ and $U'$ is open around $c$, eventually $\text{pr} \circ u(s_n, -) \in U'$ and $\text{pr} \circ u(s'_n, -) \in (\mathcal{U'})^c$. Then we conclude $s''_n \in (s_n, s'_n)$ so that $\text{pr} \circ u(s''_n, -)$ lies in $U \setminus U'$.

By the same argument leading to (19), some subsequence of $\text{pr} \circ u(s''_n, -)$ must converge to a map $t \mapsto c''(T''t)$ for some Reeb chord $c''$; moreover $\text{pr} \circ u(s''_n, -)$ converges in $\mathcal{U} \setminus U'$. This contradicts the construction of $\mathcal{U} \setminus U'$.

Therefore $\lim_{s \to \infty} u(s, t)$ converges to $c(Tt)$ in the $C^\infty$ topology. This completes the proof of Theorem 3.

3. Coordinate systems near a Reeb chord

Let $c : [0, T] \to Y$ be a non-degenerate Reeb chord. In this section we will construct a coordinate system centered on $c$. In order to simplify the holomorphic curve equation in our coordinates, we will use the Reeb flow when constructing our coordinates.

We start with a codimension-1 embedding $\Phi_0 : \overline{B(1)^{2n-2}} \to Y$ so that

(i) $\Phi_0(0) = c(0)$,

(ii) $\Phi_0^{-1}(\Lambda) = \mathbb{R}^{n-1} \cap \overline{B(1)^{2n-2}}$, and

(iii) $d\Phi_0(0) : \mathbb{R}^{2n-2} \to \xi_c(0)$ is a linear symplectomorphism, and $d\Phi_0(x)$ is transverse to the Reeb field $R$ for all $x$.

We then extend $\Phi_0$ to a map $\Phi : \mathbb{R} \times \overline{B(1)^{2n-2}} \to Y$ using the Reeb flow $\varphi_\tau$:

$$\Phi(\tau, x) := \varphi_\tau(\Phi_0(x)).$$

To see why, observe that if $c_k(T_k-)$ was a sequence of different Reeb chords converging to $c(T-)$ in $C^1$ then (i) the periods $T_k$ of $c_k$ converge to $T$ and (ii) the starting points $c_k(0)$ converge to $c(0)$. This implies that the points $(T_k, c_k(0)) \in \mathbb{R} \times \Lambda$ lie in the inverse image $\varphi^{-1}(\Lambda)$ for the Reeb flow $\varphi$. However, the transversality assumption implies that $(T, c) \in \varphi^{-1}(\Lambda)$ is an isolated point of $\varphi^{-1}(\Lambda)$. This contradicts the convergence of the sequence $(T_k, c_k(0))$. 
Note that $\Phi(\tau, 0) = c(\tau)$.

The following lemma will simplify some of the future analysis:

**Lemma 19.** The embedding $\Phi_0$ can be chosen so that $\Phi$ satisfies

$$(iv) \Phi^{-1}(\Lambda) \cap ((T - \epsilon, T + \epsilon) \times B(1)) = \{T\} \times \Gamma,$$

where $\Gamma \subset B(1)$ is a closed $(n - 1)$-dimensional submanifold with $TT_0$ transverse to $\mathbb{R}^{n-1}$.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure8.png}
\caption{Illustration of $\Phi^{-1}(\Lambda)$ (shown in red) near $\tau = 0$ and $\tau = T$.}
\end{figure}

**Proof.** Since $\Phi: \mathbb{R} \times B(1) \to Y$ has full rank, $\Phi$ is transverse to $\Lambda$. Therefore $\Phi^{-1}(\Lambda)$ is a closed submanifold. Clearly $(T, 0) \in \Phi^{-1}(\Lambda)$. Moreover, since $\{T\} \times B(1)$ is mapped to $\xi_c(T)$ via $d\Phi$, we conclude that $\Pi := T\Phi^{-1}(\Lambda)(T,0)$ is a $(n-1)$-dimensional plane inside of $\{0\} \times B(1)$.

By the implicit function, near $(T,0)$, $\Phi^{-1}(\Lambda)$ can be written as a graph over $\Pi$:

$$\Phi^{-1}(\Lambda) \cap ((T - \epsilon, T + \epsilon) \times B(\epsilon)) \subset \{(\tau, \pi, \pi^\perp): \tau = f(\pi) \text{ and } \pi^\perp = F(\pi) \text{ for } \pi \in \Pi\}$$

where $(\pi, \pi^\perp)$ are orthogonal linear coordinates on $\mathbb{R}^{2n-2}$. Moreover, $df(0) = 0$, $dF(0) = 0$, $F(0) = 0$ and $f(0) = T$. By the assumed non-degeneracy of $c$, we know that $\Pi$ is transverse to $\mathbb{R}^{n-1} \times \{0\}$.

Shrinking $\epsilon$ if necessary we may assume that

$$\Phi^{-1}(\Lambda) \cap ((-\epsilon, \epsilon) \times B(1)) \subset \{0\} \times \mathbb{R}^{n-1}.$$ 

This can be seen as a mild upgrade to property $[ii]$. This observation will be needed in moment.

We will now correct our initial choice of embedding $\Phi_0$ so that $f \equiv T$. Then property $[iv]$ will be true. The tricky part is to do this correction while preserving the first three properties $[i][iii]$. 

First of all, let us rescale our initial embedding $\Phi_0$ so that $B(\epsilon)$ is replaced by $B(1)$ in (20). Shrinking the embedding further, we may assume that the function $f - T$ from (20) is bounded by $\epsilon/2$.

Define $\Gamma = \{ (\pi, \pi^\perp) : \pi^\perp = F(\pi) \text{ for } \pi \in \Pi \}$. Note that $f$ can be thought of as a function on $\Gamma$ and (20) implies that

$$\Phi^{-1}(\Lambda) \cap ((-\epsilon, \epsilon) \times B(1)) = \{ (f(y), y) : y \in \Gamma \}.$$

Consider the following union of two transverse closed submanifolds:

$$X = (\mathbb{R}^{n-1} \cup \Gamma) \cap B(1);$$

here $X$ is a closed subset of $B(1)$. Let $h : X \to \mathbb{R}$ be the piecewise smooth function given $h|_{\mathbb{R}^{n-1}} = 0$ and $h|_{\Gamma} = f - T$. Since $f(0) = T$, $h$ extends to a smooth function on $B(1)$. Moreover, we can extend $h$ so that it is bounded by $\epsilon/2$. Moreover it is easy to see that $dh(0) = 0$ (since $df(0) = 0$).

Now we modify our initial embedding by defining

$$\tilde{\Phi}_0(x) = \varphi_{h(x)}(\Phi_0(x)).$$

Note that since $h$ vanishes on $\mathbb{R}^{n-1}$, we still have properties (i) and (ii) (we also use (21)). Since $dh(0) = 0$ we still have property (iii).

We compute

$$(\tau, x) \in \tilde{\Phi}^{-1}(\Lambda) \iff (\tau + h(x), x) \in \Phi^{-1}(\Lambda),$$

since $\tilde{\Phi}(\tau, x) = \varphi_{\tau+h(x)}\Phi_0(x) = \Phi(\tau + h(x), x)$. This implies that near $(\tau, x) = (T, 0)$ we have

$$\tilde{\Phi}^{-1}(\Lambda) \subset \{ (\tau + h(x), x) = (f(y), y) \text{ for } y \in \Gamma \}.$$

Simplifying, we conclude that

near $(T, 0)$ we have $\tilde{\Phi}^{-1}(\Lambda) \subset \{ (\tau, x) = (f(x) - h(x), x) = (T, x) \text{ for } x \in \Gamma \}.$

Redefining $\Phi := \tilde{\Phi}$ completes the proof. □

The next lemma tells us that chords nearby $c$ admit coordinate descriptions in terms of $\Phi$. 

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Lemma 20. There exists a $C^0$ open neighborhood $U$ of $t \mapsto c(Tt)$ so that if $a \in U$ then we may write

$$a(t) = \Phi(Tt + \tau(t), x(t)),$$

for smooth functions $x : [0, 1] \to B(1)$ and $\tau : [0, 1] \to \mathbb{R}$. Moreover $\tau$ vanishes at both endpoints and $x(0) \in \mathbb{R}^{n-1}$ and $x(1) \in \Gamma$.

Proof. Let $b(t) = \varphi_{-Tt}(a(t))$. If $a(t)$ is sufficiently close to $c(Tt)$, then $b(t)$ will be sufficiently close to the constant map $c(0)$. In particular, if $U$ is chosen sufficiently small, $b(t)$ will lie in the open neighborhood $\Phi((-\epsilon, \epsilon) \times B(1))$.

Thus we can write $b(t) = \Phi(\tau(t), x(t))$ for smooth $(\tau, x)$. Since $b(0) \in \Lambda$ we must have $\tau(0) = 0$ by (21).

It is clear that

$$a(t) = \Phi(Tt + \tau(t), x(t)).$$

Since $\tau(t) \in (-\epsilon, \epsilon$, $(T + \tau(1), x(1)) \in \Phi^{-1}(\Lambda) \cap (T - \epsilon, T + \epsilon) \times B(1)$. Thus property $(iv)$ implies that $\tau(1) = 0$ as desired. \qed

3.1. Transverse and tangential coordinates for holomorphic curves. Throughout this section, we fix a holomorphic strip $u : [s_0, s_1] \times [0, 1] \to \mathbb{R} \times Y$ with the property that $\text{pr} \circ u(s, t)$ remains close to $c(Tt)$ for a non-degenerate Reeb chord $c$. More precisely, fix an open set $U$ around $t \mapsto c(Tt)$ satisfying Lemma 20, and suppose that $\text{pr} \circ u(s, -)$ remains in $U$ for all $s$.

Identifying $Y$ with $\{0\} \times Y$, and letting $\psi_\sigma : \mathbb{R} \times Y \to \mathbb{R} \times Y$ be the time $\sigma$ flow by $\partial_\sigma$, Lemma 20 guarantees that we can express $u$ in the form:

$$u(s, t) = \psi_{T_t + \sigma(s, t)}(\Phi(Tt + \tau(s, t), x(s, t))).$$

where $\tau(s, 0) = \tau(s, 1) = 0$, and $x(s, 0) \in \mathbb{R}^{n-1}$ and $x(s, 1) \in \Gamma$.

The reason we have separated $Tt$ and $Tt$ is so that trivial cylinders are described by $\sigma =$ const and $\tau = 0$.

To simplify the notation, we let

$$\Phi_{s,t}(x) := \psi_s(\Phi(t, x)) = \psi_s(\varphi_t(\Phi_0(x))).$$

Then the holomorphic curve $u$ considered above can be be written as

$$u(s, t) = \Phi_{T_t + \sigma(s, t), T_t + \tau(s, t)}(x(s, t)).$$

(22)
We will refer to \( x \) as the *transverse coordinate*, and \((\sigma, \tau)\) as the *tangential coordinates*.

We introduce the following notation for the derivatives of \( \Phi_{s,t} \):

(i) \( d\Phi_{s,t}(x) \) is the derivative of \( h \mapsto \Phi_{s,t}(x + h) \) at \( h = 0 \).

(ii) \( \nabla_s \Phi_{s,t}(x) \) is the derivative of \( h \mapsto \Phi_{s,t+h}(x) \) at \( h = 0 \).

(iii) \( \nabla_t \Phi_{s,t}(x) \) is the derivative of \( t \mapsto \Phi_{s,t+h}(x) \) at \( h = 0 \).

Recalling that \( \psi \) and \( \varphi \) are the (commuting) flows of \( \partial \sigma \) and \( R \), respectively, we easily conclude that

\[
\nabla_s \Phi_{s,t}(x) = \partial_\sigma(\Phi_{s,t}(x)) \quad \text{and} \quad \nabla_t \Phi_{s,t}(x) = R(\Phi_{s,t}(x)).
\]

We compute

\[
\frac{\partial u}{\partial s} = d\Phi \circ \frac{\partial x}{\partial s} + T \partial_\sigma + \frac{\partial \sigma}{\partial s} \partial_\sigma + \frac{\partial \tau}{\partial s} R,
\]

\[
\frac{\partial u}{\partial t} = d\Phi \circ \frac{\partial x}{\partial t} + TR + \frac{\partial \sigma}{\partial t} \partial_\sigma + \frac{\partial \tau}{\partial t} R.
\]

where we abbreviate \( d\Phi = d\Phi_{T_s+\sigma(s,t),T_t+\tau(s,t)}(x(s,t)) \). Therefore the Cauchy-Riemann equation can be written as

\[
(23) \quad \frac{\partial u}{\partial s} + J(u) \frac{\partial u}{\partial t} = d\Phi \circ \frac{\partial x}{\partial s} + J(\Phi(x)) d\Phi \circ \frac{\partial x}{\partial t} + (\frac{\partial \sigma}{\partial s} - \frac{\partial \tau}{\partial t}) \partial_\sigma + (\frac{\partial \tau}{\partial s} + \frac{\partial \sigma}{\partial t}) R.
\]

We wish to split this equation into three equations according to the splitting of the tangent bundle \( T(\mathbb{R} \times Y) = \partial_\sigma \mathbb{R} \oplus R \mathbb{R} \oplus \text{im}(d\Phi) \). Note that this splitting is *not* the usual splitting, since \( \text{im}(d\Phi) \neq \xi \) in general.

Indeed, \( \text{im}(d\Phi) \) may not be \( J \)-invariant. However, we can write

\[
J(\Phi_{s,t}(x)) = \begin{bmatrix}
0 & -\partial_\sigma \otimes \alpha & \partial_\sigma \otimes (B(t,x) \circ d\Phi^{-1}) \\
R \otimes d_\sigma & 0 & 0 \\
0 & 0 & d\Phi \circ \tilde{J}(t,x) \circ d\Phi^{-1}
\end{bmatrix},
\]

with respect to this splitting. Here \( \tilde{J} \) is an almost complex structure on \( \mathbb{R}^{2n-2} \), and \( B : \mathbb{R}^{2n-2} \to \mathbb{R} \) is a linear functional. Moreover, \( \tilde{J} \) and \( B \) depend only on the \( t \) and \( x \) coordinates, and not on the \( s \) coordinate. Moreover, since \( \text{im}(d\Phi_{s,t}(0)) = \xi \), we conclude that \( B(t,0) = 0 \).

Therefore \( (23) \) reduces to three equations:

\[
(24) \quad \frac{\partial x}{\partial s} + \tilde{J}(Tt + \tau(s,t),x) \frac{\partial x}{\partial t} = 0,
\]
\( \frac{\partial \sigma}{\partial s} - \frac{\partial \tau}{\partial t} = -B(Tt + \tau(s, t), x) \frac{\partial x}{\partial t}, \) and \( \frac{\partial \sigma}{\partial t} + \frac{\partial \tau}{\partial s} = 0. \)

3.2. Fine-tuning the transverse coordinate. In this section we will describe a coordinate change for the transverse coordinate which will simplify (24) and the boundary conditions for \( x(s, t) \).

Before we begin, we wish to remark that \( \tilde{J}(Tt, 0) \) is compatible with the standard symplectic structure on \( \mathbb{R}^{2n-2} \), because of property \( \text{[iii]} \) for \( \Phi_0 \). As a consequence, we can find a family of linear symplectomorphisms \( d \mu_t(0) \) with the property that

\[
d \mu_t(0)^{-1} J(Tt, 0) d \mu_t(0) = J_0,
\]

where \( J_0 \) is the standard complex structure on \( \mathbb{R}^{2n-2} \). This is because every unitary vector bundle over an interval (i.e. a bundle with a fiberwise symplectic structure and compatible complex structure) is trivializable. Moreover, we may suppose that \( d \mu_0(0) \mathbb{R}^{n-1} = \mathbb{R}^{n-1} \) and \( d \mu_1(0) \mathbb{R}^{n-1} = T \Gamma_1. \)

There exists a family of diffeomorphisms \( \mu_t : \mathcal{B}(1) \to \mathcal{B}(1) \) whose derivative equals \( d \mu_t(0) \). Moreover, we may suppose that \( \mu_0(\mathbb{R}^{n-1}) = \mathbb{R}^{n-1} \) and \( \mu_1(\mathbb{R}^{n-1}) = \Gamma_1. \)

Introduce the complex structure

\[
J_t(\tau, y) = d \mu_t(y)^{-1} \tilde{J}(Tt + \tau, \mu_t(y)) d \mu_t(y).
\]

If we set \( x(s, t) = \mu_t(y(s, t)) \), then (24) and the boundary conditions for \( x(s, t) \) simplify to

\[
\begin{cases}
\frac{\partial y}{\partial s} + J_t(\tau, y) \frac{\partial y}{\partial t} + J_t(\tau, y) d \mu_t(y)^{-1} \nabla \mu_t(y) = 0, \\
y(s, 0), y(s, 1) \text{ both lie in } \mathbb{R}^{n-1}.
\end{cases}
\]

Here \( d \mu_t(y) \) is the derivative of \( h \mapsto \mu_t(y + h) \) at \( h = 0 \), and \( \nabla \mu_t(y) \) is the derivative of \( h \mapsto \mu_t+h(y) \).

Our choice of \( \mu_t \) implies \( J_t(0, 0) = J_0 \) is the standard complex structure, however we can actually make \( J_t(0, y) \) standard for all \( y \in \mathbb{R}^{n-1} \times \{0\} \) as the following argument shows.

Let \( y = (q_1, \ldots, q_{n-1}, p_1, \ldots, p_{n-1}) \) denote the coordinates of \( \mathbb{R}^{2n-2} \). Consider the vector fields \( P_i(q) = J_t(0, (q, 0)) \partial_{q_i} \). Since \( J_t(0, 0) \) is standard, \( P_i(0) = \partial_{p_i} \).
Now define a change of coordinates by

\[ \nu_t(q, p) = (q, \sum_{i=1}^{n-1} p_i P_i(q)). \]

The previous paragraph shows that the \( \nu_t \) equals the identity on

\[ (\mathbb{R}^{n-1} \times \{0\}) \cup (\{0\} \times \mathbb{R}^{n-1}). \]

In particular, \( \text{d} \nu_t(0) = \text{id} \).

We redefine \( y := \nu_t(y) \), and define the complex structure \( J_t(\tau, y) \) as we did above.

Our construction of \( \nu_t \) has the property that \( J_t(0, y) \) is standard for \( y \in \mathbb{R}^{n-1} \times \{0\} \), and not just for \( y = 0 \). We summarize what we have done:

**Lemma 21.** There exists a family of open embeddings \( \mu_t : \overline{B}(1) \to B(1) \) so that \( \mu_t(0) = 0 \) and if we set \( x(s, t) = \mu_t(y(s, t)) \) then (24) implies

\[
\begin{aligned}
\frac{\partial y}{\partial s} + J_t(\tau, y) \frac{\partial y}{\partial t} + J_t(\tau, y) \text{d}\mu_t(y)^{-1} \nabla \mu_t(y) &= 0, \\
y(s, 0), y(s, 1) \text{ both lie in } \mathbb{R}^{n-1}.
\end{aligned}
\]

for the complex structure

\[ J_t(\tau, y) := \text{d}\mu_t(y)^{-1} \tilde{J}(T\tau + \tau, \mu_t(y)) \text{d}\mu_t(y). \]

Moreover \( \mu_t \) can be chosen so that the following properties hold:

(i) \( J_t(\tau, y) \) is a complex structure and \( J_t(0, y) = J_0 \) if \( y \in \mathbb{R}^{n-1} \times \{0\} \),

(ii) \( \text{d}\mu_t(0) \) is a linear symplectomorphism.

(iii) \( \mu_0^{-1}(\mathbb{R}^{n-1}) = \mathbb{R}^{n-1}, \mu_1^{-1}(\Gamma_1) = \mathbb{R}^{n-1}. \)

(iv) \( S(t) := J_0 \text{d}\mu_t(0)^{-1} \nabla \text{d}\mu_t(0) \) is symmetric,

where \( \nabla \text{d}\mu_t(0) \) is the derivative of \( \text{d}\mu_{t+h}(0) \) at \( h = 0 \).

**Proof.** The equation (26) is a simple consequence of the definition \( x(s, t) = \mu_t(y(s, t)) \) and \( J_t(\tau, y) \). We have already shown that \( \mu_t \) can be chosen so that the first three items hold.

We will now prove the fourth item. It is a direct consequence of the second item, as follows: let \( g_0 \) be the standard Riemannian metric on \( \mathbb{R}^{n-1} \), so that \( g_0(J_0 - , -) = \omega_0 \).
is the standard symplectic structure. Then for fixed vectors $\xi, \eta$ we compute:

$$g(S(t)\xi, \eta) = \omega_0(d\mu_t(0)^{-1}\nabla d\mu_t(0)\xi, \eta) = \omega_0(\nabla d\mu_t(0)\xi, d\mu_t(0)\eta)$$

$$= -\omega_0(d\mu_t(0)\xi, \nabla d\mu_t(0)\eta) = -\omega_0(\xi, d\mu_t(0)^{-1}\nabla d\mu_t(0)\eta)$$

$$= \omega_0(\xi, J_0^2 d\mu_t(0)^{-1}\nabla d\mu_t(0)\eta) = g_0(\xi, S(t)\eta).$$

In the last equality we have used $\omega_0(-, J_0^-) = g_0$. This completes the proof. □

4. Exponential estimates via differential inequalities

In this section our strategy will be to establish differential inequalities for various
integral quantities, and then deduce exponential estimates from said differential
inequalities.

Our strategy has a few steps: first we will analyze the transverse coordinate $y$ (defined
in Section 3.2). We will prove (Lemma 23) that, if $pr \circ u(s, t)$ is sufficiently close to
c($Tt$) that

$$\gamma(s) := \frac{1}{2} \int_0^1 |y(s, t)|^2 \, dt \text{ satisfies } \gamma''(s) - \delta^2 \gamma(s) \geq 0.$$ 

As we will explain in Section 4.4, this differential inequality will imply that $\gamma(s)$ satisfies certain exponential estimates. The main analytic input we will need in Lemma 23 is the non-degeneracy of the asymptotic operator (defined in Section 4.1).

In Section 4.3 we will analyze the tangential coordinates.

4.1. The asymptotic operator associated to a Reeb chord. Our strategy for analyzing the holomorphic curve equation (26) will be to write it as a perturbation of a linear equation. We begin by expanding one of the non-linear terms:

$$\nabla \mu_t(y) = d\nabla \mu_t(0) \cdot y - E(t, y) \cdot y^2 = \nabla d\mu_t(0) \cdot y - E(t, y) \cdot y^2.$$ 

Here $d\nabla \mu_t(0)$ is the derivative of $h \mapsto \nabla \mu_t(h)$, while (recall) $\nabla d\mu_t(0)$ is the derivative of $h \mapsto d\mu_{t+h}(0)$. The final equality above follows from the equality of mixed partial derivatives for $\mathbb{R}^{2n-2}$ valued functions. Here $E(t, y)(-, -)$ is a smooth family of bilinear maps $(\mathbb{R}^{2n-2}) \otimes (\mathbb{R}^{2n-2}) \to \mathbb{R}^{2n-2}$, and $E(t, y) \cdot y^2$ is obtained by inserting $y$ into both entries of $R_t(y)$. A precise formula can be obtained for $E(t, y)$ from the fundamental theorem of calculus. We will only need the fact that $(t, y) \mapsto E(t, y)$ satisfies $C^k$ bounds which is obvious since it is defined on a compact set $[0, 1] \times \overline{B}(1).$
We can use this expansion to rewrite (26) as
\[ \frac{\partial y}{\partial s} + J_t(\tau, y)\frac{\partial y}{\partial t} + J_t(\tau, y)\text{d}\mu(y)^{-1}\nabla\text{d}\mu(0) \cdot y = J_t(\tau, y)\text{d}\mu(y)^{-1}E(t, y) \cdot y^2. \]

The left hand side of the above equation is still non-linear. We will simplify the equation further by defining the following three remainder terms
\[ R_1 = -(J_t(\tau, y) - J_0)\frac{\partial y}{\partial t} \]
\[ R_2 = -(J_t(\tau, y)\text{d}\mu(y)^{-1} - J_0\text{d}\mu(0)^{-1})\nabla\text{d}\mu(0) y \]
\[ R_3 = J_t(\tau, y)\text{d}\mu(y)^{-1}E(t, y) \cdot y^2. \]

Then equation (26) simplifies to
\[ \frac{\partial y}{\partial s} + J_0\frac{\partial y}{\partial t} + J_0\text{d}\mu(0)^{-1}\nabla\text{d}\mu(0) \cdot y = R_1 + R_2 + R_3. \]

We now introduce the asymptotic operator:
\[ A(y) = -J_0\frac{\partial y}{\partial t} - J_0\text{d}\mu(0)^{-1}\nabla\text{d}\mu(0) \cdot y = -J_0\frac{\partial y}{\partial t} - S(t) \cdot y, \]
so that the above equation can be written as
\[ \frac{\partial y}{\partial s} - A(y) = R_1 + R_2 + R_3. \]

We think of \( A \) as bounded linear operator
\[ A : W^{1,2}([0, 1], \mathbb{R}^{2n-2}, \mathbb{R}^{n-1}) \to L^2([0, 1], \mathbb{R}^{2n-2}). \]

Here the domain of \( A \) consists of \( W^{1,2} \) sections (which are easily seen to be continuous) which take values in \( \mathbb{R}^{n-1} \) on both endpoints.

**Lemma 22.** The asymptotic operator \( A \) defined in (28) is a self-adjoint isomorphism
\[ W^{1,2}([0, 1], \mathbb{R}^{2n-2}, \mathbb{R}^{n-1}) \to L^2([0, 1], \mathbb{R}^{2n-2}). \]

By self-adjoint we mean that \( \langle Av, w \rangle = \langle v, Aw \rangle \) for all \( v, w \in W^{1,2}([0, 1], \mathbb{R}^{2n-2}, \mathbb{R}^{n-1}) \).

This lemma will play a key role in our estimates establishing the exponential convergence of \( \text{pr} \circ u(s, t) \) to \( c(Tt) \).

**Proof.** A similar argument can be found in the proof of [RS01, Lemma 2.3]. Since \( C^\infty([0, 1], \mathbb{R}^{2n-2}, \mathbb{R}^{n-1}) \) is dense in \( W^{1,2}([0, 1], \mathbb{R}^{2n-2}, \mathbb{R}^{n-1}) \) it suffices to prove the self-adjointness for smooth functions \( u, v \). This follows from a straightforward
integration-by-parts computation, using the fact that the matrix $S(t)$ is symmetric, and $J_0$ is anti-symmetric. We leave this computation to the reader. Note that it is crucial that both $u, v$ take boundary values in the Lagrangian subspace $\mathbb{R}^{n-1}$, otherwise the integration by parts will fail.

To show that $A$ is an isomorphism, it suffices to prove that it is a bijection, since continuous bijections between Banach spaces are isomorphisms. First we will prove that $A$ is injective. We observe that

$$A(\xi) = 0 \implies \partial_t \xi(t) + d\mu_t(0)^{-1} \nabla d\mu_t(0) \cdot \xi(t) = 0$$

$$\implies d\mu_t(0) \partial_t \xi(t) + \nabla d\mu_t(0) \cdot \xi(t) = 0$$

$$\implies \frac{\partial}{\partial t}(d\mu_t(0) \cdot \xi(t)) = 0.$$ 

This implies that $d\mu_t(0) \cdot \xi(t) = c$ is a constant vector. Note that the pointwise aspects of this computation are justified since elements in the kernel of $A$ are easily seen to be smooth by a linear elliptic bootstrapping argument.

Recall from the construction of $\mu$ (Lemma 21) that $d\mu_0(0)$ takes $\mathbb{R}^{n-1}$ to $\mathbb{R}^{n-1}$, while $d\mu_1$ takes $\mathbb{R}^{n-1}$ onto $T\Gamma$. In particular, $c$ lies in $\mathbb{R}^{n-1} \cap T\Gamma$. However, Lemma 19 implies $T\Gamma$ and $\mathbb{R}^{n-1}$ are transverse. Therefore $c = 0$, and hence $\xi(t) = 0$, as desired.

Now we will prove that $A$ is surjective. Fix a smooth $\eta$, and we attempt to solve $A(\xi) = \eta$ for a smooth $\xi$:

$$J_0 \frac{\partial \xi}{\partial t} + S(t)\xi = -\eta$$

$$\iff \frac{\partial \xi}{\partial t} - J_0 S(t)\xi = J_0 \eta(t).$$

(30)

$$\iff \frac{\partial}{\partial t}(\exp(\Sigma(t))\xi(t)) = \exp(\Sigma(t))J_0 \eta(t),$$

$$\iff \xi(t) = \exp(-\Sigma(t))\xi(0) + \exp(-\Sigma(t)) \int_0^t \exp(\Sigma(t'))J_0 \eta(t')dt'.$$

where $\Sigma'(t) = -J_0 S(t)$ and $\Sigma(0) = 0$. This shows that we can solve $A(\xi) = \eta$ for many different choices of $\xi$, namely there is an $\mathbb{R}^{2n-2}$ dimensional family of solutions corresponding to the choice of $\xi(0)$. We claim that (exactly) one of these solutions will satisfy $\xi(0), \xi(1) \in \mathbb{R}^{n-1}$. To see why, consider the affine map

$$F : \xi(0) \in \mathbb{R}^{n-1} \mapsto \exp(-\Sigma(1))\xi(0) + \exp(-\Sigma(1)) \int_0^1 \exp(\Sigma(t'))J_0 \eta(t')dt' \in \mathbb{R}^{2n-2}.$$
This map parametrizes a \(n - 1\) dimensional affine subspace of \(\mathbb{R}^{2n-2}\). Note that the associated linear subspace \(\exp(-\Sigma(1))\mathbb{R}^{n-1}\) is transverse to \(\mathbb{R}^{n-1}\) (otherwise we could find a vector \(v \in \mathbb{R}^{n-1}\) so \(\exp(-\Sigma(1))v \in \mathbb{R}^{n-1}\), and the above computation with \(\eta = 0\) would imply \(\xi(t) = \exp(-\Sigma(t))v\) lies in the kernel of \(A\)).

Therefore \(F(\mathbb{R}^{n-1})\) intersects \(\mathbb{R}^{n-1}\) in a unique point \(F(\xi(0)) = \xi(1)\). Thus (30) with this special \(\xi(0)\) shows that \(A\) is surjective onto the smooth elements \(\eta\).

To show that \(A\) is surjective in general, it suffices to prove that the image of \(A\) is closed. This follows from the estimate

\[
\|\xi\|_{W^{1,2}} \leq C(\|A(\xi)\|_{L^2} + \|\xi\|_{L^2})
\]

and the fact that \(L^2 \to W^{1,2}\) is a compact inclusion. This completes the proof of the lemma.

As a corollary to 22 we conclude the following estimates

\[
\|\xi\| + \|\partial_s \xi\| \leq C \|A(\xi)\|,
\]

where \(\|\xi\|\) indicates the \(L^2\) norm \((\int |\xi(s,t)|^2 \, dt)^{1/2}\).

4.2. Establishing a differential inequality for the transverse coordinate. The following lemma is the key technical estimate used in our proof of the exponential convergence of \(pr \circ u(s,t)\) to \(c(Tt)\).

**Lemma 23.** Let \(\gamma(s) = \frac{1}{2} \|y\|^2\). There exists a constant \(\epsilon > 0\) with the following property: if the \(C^1\) sizes of \(y, \tau\) are less than \(\epsilon\), then \(\gamma\) satisfies the differential inequality:

\[
\gamma''(s) - \delta^2 \gamma(s) \geq \frac{1}{3}(\|\partial_s y\|^2 + \|A(y)\|^2),
\]

where \(\delta > 0\) is any constant so that

\[
\delta^2 \|\xi\|^2 \leq \frac{2}{3} \|A(\xi)\|^2
\]

for \(\xi \in W^{1,2}([0,1], \mathbb{R}^{2n-2}, \mathbb{R}^{n-1})\).

**Proof.** We will use equation (29) and the estimates (31).

A straightforward computation establishes that

\[
\gamma''(s) = \|\partial_s y\|^2 + \langle y, \partial_s \partial_s y \rangle.
\]
We replace $\partial_s y = Ay + \sum R_i$ using (29), so that
\[
\langle y, \partial_s \partial_s y \rangle = \langle y, \partial_s y + \sum \partial_s R_i \rangle = \langle y, \partial_s y \rangle + \sum \langle y, \partial_s R_i \rangle,
\]
where we have used the fact that $A$ is $s$-independent. Since $y$ and $\partial_s y$ both take boundary values in $\mathbb{R}^{n-1}$, we can use the self-adjointness of $A$ to conclude
\[
\langle y, \partial_s y \rangle = \langle Ay, \partial_s y \rangle = \|Ay\|^2 + \sum \langle Ay, \partial_s R_i \rangle.
\]
Using (33) with $\xi = y(s, -)$ we obtain
\[
\gamma'' - \delta^2 \gamma \geq 1 + \frac{1}{3} (\|\partial_s y\|^2 + \|A(y)\|^2) + \frac{2}{3} (\|\partial_s y\|^2 + \frac{1}{3} \|A(y)\|^2 + \sum \langle Ay, R_i \rangle + \langle y, \partial_s R_i \rangle).
\]
Thus, in order to prove the lemma, it is sufficient to prove that
\[
\sum \langle Ay, R_i \rangle + \langle y, \partial_s R_i \rangle \leq \frac{1}{3} (\|A(y)\|^2 + \|\partial_s y\|^2),
\]
provided the $C^1$ sizes of $y$ and $\tau$ are less than $\epsilon$. We will now proceed to do this. For the reader’s convenience, we reprint the remainder terms (27) here.

\[
\begin{align*}
R_1 &= -(J_t(\tau, y) - J_0) \frac{\partial y}{\partial t} \\
R_2 &= -(J_t(\tau, y) d\mu(y)^{-1} - J_0 d\mu(0)^{-1}) \nabla d\mu_t(0) y \\
R_3 &= J_t(\tau, y) d\mu(y)^{-1} E(t, y) \cdot y^2.
\end{align*}
\]

First we observe that $|R_3| \leq |\partial_s R_3| \leq C \epsilon |y|$ for some constant $C$ (depending on the $C^1$ sizes of $(\tau, y) \mapsto J_t(\tau, y) d\mu(y)^{-1} E(t, y)$. We also assume that $\epsilon$ is less than 1).

In particular,
\[
|\langle Ay, R_3 \rangle + \langle y, \partial_s R_3 \rangle| \leq C \epsilon (\|Ay\| \|y\| + \|y\|^2) \leq \frac{1}{9} \|Ay\|^2
\]
provided $\epsilon$ is sufficiently small (using (31) in the last step).

Next we estimate the $R_2$ terms. Since the first factor of $R_2$ vanishes when $y = \tau = 0$, the first factor is bounded by the $C^1$ size of $y$ and $\tau$. Hence we can estimate
\[
|R_2| \leq C \epsilon |y|.
\]

When we take the $\partial_s$ derivative, we obtain something of the form
\[
\partial_s R_2 = (a_1 \partial_s \tau + a_2 \partial_s y) \nabla d\mu_t(0)y - (J_t(\tau, y) d\mu(y)^{-1} - J_0 d\mu(0)^{-1}) \nabla d\mu_t(0) \partial_s y,
\]
for smooth functions $a_1(t, \tau, y)$ and $a_2(t, \tau, y)$. In particular,

$$|\partial_s R_2| \leq C \epsilon |y| + C \epsilon |\partial_s y| .$$

Thus

$$|\langle A(y), R_2 \rangle + \langle y, \partial_s R_2 \rangle| \leq C \epsilon (\|A(y)\| \|y\| + \|y\|^2 + \|y\| \|\partial_s y\|) \leq \frac{1}{9}(\|A(y)\|^2 + \|\partial_s y\|^2),$$

assuming $\epsilon$ is sufficiently small. We use (31) and the trick $2ab \leq a^2 + b^2$.

Finally we deal with the $R_1$ terms. Arguing as we did for the $R_2$ term, we have

$$R_1 \leq C \epsilon \|\partial_t y\| \|\partial_s y\| + C \epsilon \|y\| (\|\partial_t y\| + \|\partial_s y\|).$$

Using (31) together with the $2ab \leq a^2 + b^2$ trick implies that for $\epsilon$ sufficiently small we have

$$|\langle y, \partial_s R_1 \rangle| \leq \frac{1}{18}(\|\partial_s y\|^2 + \|A(y)\|^2).$$
Combining all of our estimates proves that
\[
\left| \sum_i \langle Ay, R_i \rangle + \langle y, \partial_s R_i \rangle \right| \leq \left( \frac{1}{9} + \frac{1}{9} + \frac{1}{18} \right) \left( \| \partial_s y \|^2 + \| A(y) \|^2 \right),
\]
which gives (34), and this completes the proof. □

4.3. Establishing a differential inequality for the tangential coordinate. In this section we will analyze the tangential coordinates \( \sigma \) and \( \tau \). These coordinates satisfied the equations (25), which we reprint here:
\[
\frac{\partial \sigma}{\partial s} - \frac{\partial \tau}{\partial t} = -B(Tt + \tau(s, t), x) \frac{\partial x}{\partial t}, \quad \text{and} \quad \frac{\partial \sigma}{\partial t} + \frac{\partial \tau}{\partial s} = 0.
\]
Recall that \((t, x) \mapsto B(t, x)\) is a smooth function with \( B(t, 0) = 0 \).

**Lemma 24.** Introduce the quantity:
\[
\Gamma(s) = \frac{1}{2} \int_0^1 |\tau(s, t)|^2 \, dt.
\]
There is \( c > 0 \) (depending only on the constant from the Poincaré lemma for \([0, 1]\)) and \( \epsilon > 0 \) with the following property: if the \( C^0 \) size of \( x \) is less than \( \epsilon \), then
\[
\Gamma''(s) - c^2 \Gamma(s) \geq \| \partial_s \tau \|^2 + \frac{1}{3} \| \partial_t \tau \|^2 - \frac{1}{3} \| A(y) \|^2,
\]
where \( x = \mu_t(y) \) is as in Section 3.2. Here \( \| - \| \) denotes the \( L^2 \) norm over \([0, 1]\).

**Remark 25.** Note that we will use (32) to obtain estimates on \( \| A(y) \|^2 \). These estimates will be combined with (35) to obtain estimates on \( \tau \). This is why we have opted to place \( A(y) \) in the statement of the lemma. △

**Remark 26.** Once we prove that \( \tau \) and \( x \) satisfy exponential estimates, it will be fairly straightforward to use the equations (25) to obtain similar estimates on \( \sigma \). For this reason, we will focus only on \( \tau \) in this section. △

**Proof.** We compute
\[
\Gamma''(s) = \| \partial_s \tau \|^2 + \langle \tau, \partial_s \partial_s \tau \rangle = \| \partial_s \tau \|^2 - \langle \tau, \partial_t \partial_s \sigma \rangle \\
= \| \partial_s \tau \|^2 + \langle \partial_t \tau, \partial_s \sigma \rangle = \| \partial_s \tau \|^2 + \| \partial_t \tau \|^2 - \langle \partial_t \tau, B \partial_t x \rangle.
\]
We have used the fact that \( \tau \) vanishes on both endpoints in order to do the integration by parts.
From the definition of \( y \), we have
\[
\partial_t x = d\mu_t(y) \cdot \partial_t y + \nabla \mu_t(y).
\]
Since \( \mu_t(0) \equiv 0 \) we have \( \nabla \mu(0) = 0 \) and hence \( \| \nabla \mu_t(y) \| \leq C \| y \| \) for some constant \( C \) depending only on the choice of \( \mu_t \). It then follows from (31) that
\[
\| \partial_t x \| \leq C' \| A(y) \|.
\]
Therefore our computation of \( \Gamma''(s) \) implies
\[
\Gamma''(s) \geq \| \partial_s \tau \|^2 + \| \partial_t \tau \|^2 - C' \max_{s,t} |B| \| \partial_t \tau \| \| A(y) \|.
\]
The size of \( B \) is controlled by the \( C^0 \) size of \( x \), and hence we may pick \( \epsilon \) small enough so that \( C' \max_{s,t} |B| \leq 2/3 \). Then
\[
C' \max_{s,t} |B| \| \partial_t \tau \| \| A(y) \| \leq \frac{1}{3} \| \partial_t \tau \|^2 + \frac{1}{3} \| A(y) \|^2.
\]
It follows that
\[
\Gamma''(s) \geq \| \partial_s \tau \|^2 + \frac{2}{3} \| \partial_t \tau \|^2 - \frac{1}{3} \| A(y) \|^2.
\]
Finally, the Poincaré inequality (for functions on \([0,1]\) which vanish on both end-points) implies there is a constant \( d \) so that
\[
\frac{c^2}{2} \| \tau \|^2 \leq \frac{1}{3} \| \partial_t \tau \|^2,
\]
so that
\[
\Gamma''(s) - c^2 \Gamma(s) \geq \| \partial_s \tau \|^2 + \frac{1}{3} \| \partial_t \tau \|^2 - \frac{1}{3} \| A(y) \|^2,
\]
as desired. \( \square \)

4.4. The relationship between \( \gamma'' - \delta^2 \gamma \) and exponential estimates. There are many results in the theory of holomorphic curves which involve quantities decaying exponentially. Many of these results begin by establishing an estimate involving the combination \( \gamma'' - \delta^2 \gamma \), and usually \( \gamma \) is the \( L^2 \) size (squared) of some quantity. We give four examples from the literature, arranged chronologically:

(i) In [Flo89, Lemma 5.2], Floer establishes an estimate of the form

\[
\gamma''(s) - \delta^2 \gamma(s) \geq 0
\]
for a certain quantity $\gamma : \mathbb{R} \to [0, \infty)$. Since $\gamma$ is non-negative and defined on all $\mathbb{R}$, $\gamma$ must be identically zero. This is part of the argument Floer uses to show that the Floer homology of $L$ with $L_f$ (the graph of $df$ in $T^*L$) can be computed in terms of the Morse complex of $f$.

(ii) In [Sal97, Lemma 2.11], Salamon considers the quantity
\[
\gamma(s) := \frac{1}{2} \int_{0}^{1} |\xi(s, t)|^2 \, dt
\]
where $\xi(s, t)$ solves an equation of the form $\partial_s \xi + J_0 \partial_t \xi + S(t) \xi = 0$. He then shows that $\gamma$ satisfies (36) for $s$ sufficiently large, using the assumption that $\xi \mapsto J_0 \partial_t \xi + S(t) \xi$ is an isomorphism. This can be thought of as a linear analogue to our Lemma 23. Salamon then shows that $\gamma(s)$ decays exponentially with rate $\delta$, for $s$ sufficiently large.

(iii) In [RS01, Lemma 3.1], Robbin-Salamon consider a function $\gamma : [0, \infty) \to \mathbb{R}^+$ satisfying inequality of the form
\[
\gamma''(s) - \delta^2 \gamma(s) \geq -c_0 e^{-\epsilon s}.
\]
The authors use to show that $\gamma(s)$ decays exponentially with rate $\delta$, provided $\epsilon > \delta$.

(iv) In [HWZ02, Lemma 3.6], Hofer-Wysocki-Zehnder consider the $L^2$ size
\[
\gamma(s) = \int_{\mathbb{R}/\mathbb{Z}} |z(s, t)|^2 \, dt,
\]
where $z$ is a $\mathbb{R}^{2n-2}$-valued coordinate near a Reeb orbit measuring the directions transverse to $\partial_\sigma$ and $R$. They show that $\gamma : [-r, r] \to \mathbb{R}^+$ satisfies the estimate (36). This is quite similar to our Lemma 23. The authors then use (36) to show that $\gamma(s) \leq Ae^{-\delta(r+s)} + Be^{-\delta(r-s)}$ for appropriately chosen constants $A, B$.

We will prove the following lemma:

**Lemma 27.** Let $\gamma : [-r, r] \to \mathbb{R}^+$, $\alpha : [-r, r] \to \mathbb{R}^+$ and $\kappa : [-r, r] \to \mathbb{R}^+$ be smooth functions satisfying
\[
\gamma'' - \delta^2 \gamma \geq \alpha - \kappa,
\]
for some constant $\delta > 0$. Suppose that $\kappa \leq K_1 e^{-D(r+s)} + K_2 e^{-D(r-s)}$ for $K_1, K_2 \geq 0$ and $D > \delta$. Then
\[
\gamma \leq C_1 e^{-\delta(r+s)} + C_2 e^{-\delta(r-s)} \quad \text{and} \quad \int_{s-0.5}^{s+0.5} \alpha(s) \, ds \leq A_1 e^{-\delta(r+s)} + A_2 e^{-\delta(r-s)},
\]
where the inequality involving $\alpha$ holds only for $s \in [-r+1, r-1]$, while the inequality involving $\gamma$ holds for all $s \in [-r, r]$. The constants are given by

$$C_1 = \gamma(-r) + K_1 + K_2 e^{-2Dr} \over D^2 - \delta^2$$

$$C_2 = \gamma'(r) - \delta \gamma(r) + K_2 \over 2\delta \over D^2 - \delta^2$$

$$A_i = e^\delta (40 + 2\delta^2) C_i + 2e^D K_i$$

![Figure 9. Plots of $C_1 e^{-\delta(r+s)} + C_2 e^{-\delta(r-s)}$ for various values of $C_1$ and $C_2$. The values $C_1, C_2$ are the values taken by the function at the left and right endpoints, respectively.](image)

**Proof.** We begin by introducing the function

$$\beta := \gamma - Be^{-\delta(r-s)} + K_1 e^{-D(r+s)} + K_2 e^{-D(r-s)},$$

for a constant $B$ to be determined at a later stage. It is straightforward to observe that

$$\beta'' - \delta^2 \beta = \gamma'' - \delta^2 \gamma + [K_1 e^{-D(r+s)} + K_2 e^{-D(r-s)}] \geq \alpha \geq 0.$$

The trick now is to observe that

$$\frac{d}{ds} (e^{-\delta s} (\beta' + \delta \beta)) \geq 0 \implies e^{-\delta s} (\beta' + \delta \beta) \text{ is increasing.}$$

Now we will pick the constant $B > 0$ so that $\beta' + \delta \beta$ is non-positive at the right endpoint $s = r$. We compute

$$\beta'(r) + \delta \beta(r) = \gamma'(r) + \delta \gamma(r) - 2\delta B + (D + \delta) K_2 - (D - \delta) K_1 e^{-2Dr} \over D^2 - \delta^2.$$

Therefore

$$\beta'(r) + \delta \beta(r) \leq \gamma'(r) + \delta \gamma(r) + K_2 \over D - \delta - 2\delta B.$$
This leads us to make the choice

\[ B = \frac{\gamma'(r) + \delta \gamma(r)}{2\delta} + \frac{K_2}{2\delta(D - \delta)}. \]

With this choice of \( B \) we can conclude from (38) that \( \beta'(s) + \delta \beta(s) \leq 0 \) holds for all \( s \).

Then we can integrate \( \beta'(s) + \delta \beta(s) \leq 0 \) to conclude

\[ e^{\delta s} \beta(s) \leq e^{-\delta r} \beta(-r) \text{ for } s \in [-r, r]. \]

Thus \( \beta(s) \leq e^{-\delta(r+s)} \beta(-r) \), and hence

\[ \gamma(s) \leq \beta(-r)e^{-\delta(r+s)} + Be^{-\delta(r-s)}. \]

We estimate \( \beta(-r) \) as follows

\[ \beta(-r) \leq \gamma(-r) + \frac{K_1 + K_2e^{-2Dr}}{D^2 - \delta^2} =: C_1. \]

We set \( C_2 = B \). This completes the first part of the proof.

To estimate the integral \( \int_{s-0.5}^{s+0.5} \alpha(s) \, ds \), we will use a convolution trick. We introduce a smooth symmetric bump function \( \rho \) which is supported in \((-1, 1)\) and which equals 1 on \([-0.5, 0.5]\), as shown in Figure 10.

\[ \text{Figure 10. The bump function } \rho. \]  

We have \( \|\rho\|_{L^1} \leq 2 \). Moreover, this can be achieved with \( \|\rho''\|_{L^2} \leq 4\pi^2 + \epsilon \leq 40 \), because one can take a smooth \( \epsilon \)-approximation of the \( C^2 \) function \( 0.5 - 0.5 \cos(2\pi x) \) which interpolates from 0 to 1 over an interval of size 0.5 (we need two copies of this function).

We convolve both sides of (37) with \( \rho \), yielding

\[ (\rho'') * \gamma - \delta^2 (\rho * \gamma) \geq \rho * \alpha - \rho * \kappa. \]
This holds only on the restricted domain $s \in [-r+1, r-1]$. For functions $f$ supported in $[-1, 1]$ it is simple to estimate

$$|f \ast \gamma|(s) \leq \max_{s' \in [s-1, s+1]} |\gamma(s)| \|f\|_{L^1} \leq e^\delta \|f\|_{L^1} (C_1 e^{-\delta(s+r)} + C_2 e^{-\delta(r-s)}).$$

A similar conclusion holds with $\gamma$ replaced by $\kappa$. We conclude

$$\rho \ast \alpha(s) \leq e^\delta (\|\rho''\|_{L^1} + \delta^2 \|\rho\|_{L^1})(C_1 e^{-\delta(s+r)} + C_2 e^{-\delta(r-s)})$$

$$+ e^D \|\rho\|_{L^1} (K_1 e^{-D(s+r)} + K_2 e^{-D(r-s)}).$$

Using the fact that $D > \delta$, and $\|\rho\|_{L^1} \leq 2$ and $\|\rho''\|_{L^1} \leq 40$, we obtain

$$\rho \ast \alpha(s) \leq \left[ e^\delta(40 + 2\delta^2)C_1 + 2e^D K_1 \right] e^{-\delta(s+r)} + \left[ e^\delta(40 + 2\delta^2)C_2 + 2e^D K_2 \right] e^{-\delta(r-s)}.$$

Setting $A_i := e^\delta(40 + 2\delta^2)C_i + 2e^D K_i$, we conclude

$$\int_{s-0.5}^{s+0.5} \alpha(s) \, ds \leq \rho \ast \alpha(s) \leq A_1 e^{-\delta(s+r)} + A_2 e^{-\delta(r-s)}.$$ 

This completes the proof. □

**Remark 28.** A simpler proof in the case when $\kappa = \alpha = 0$ (with different constants $C_1, C_2$) is possible using a slightly different argument: observe that $\beta = \gamma - c \cosh(\delta s)$ satisfies $\beta'' - \delta^2 \beta \geq 0$, and hence cannot attain a positive interior maximum. Thus if $c$ is chosen so that $\beta$ is non-positive at both endpoints, then $\beta$ must be everywhere non-positive, hence $\gamma \leq c \cosh(\delta s)$, as desired. This is the argument used in [HWZ02, Lemma 3.6]. △

### 4.5. Exponential estimates on the $W^{1,2}$ norm.

Throughout this section let

$$u : [-r-2, r+2] \to \mathbb{R} \times Y$$

be a holomorphic curve so that $pr \circ u(s, t)$ remains sufficiently close to the non-degenerate Reeb chord $c(Tt)$ so that the coordinate system $(\sigma, \tau, x)$ from Section 3.1 and the modification $x = \mu_t(y)$ from Section 3.2 apply.

In this section we will apply Lemma 27 to the $y$ and $\tau$ coordinates. Let $\delta$ and $c$ be constants compatible with Lemma 23 and Lemma 24 respectively. Without loss of generality assume that $c < \delta$.

We have the following result:
Lemma 29. Fix a constant $\theta > 0$. There exists a $C^1$ open set $U$ around $t \mapsto c(Tt)$ with the following property: if $\text{pr} \circ u(s, -) \in U$ for all $s \in [-r - 2, r + 2]$ then the following estimates hold for $s \in [-r, r]$:

\[
\begin{align*}
\int_{s-0.5}^{s+0.5} \int_0^1 |y|^2 + |\partial_s y|^2 + |\partial_t y|^2 \, ds \, dt &\leq \theta (e^{-\delta(s+r)} + e^{-\delta(r-s)}) \\
\int_{s-0.5}^{s+0.5} \int_0^1 |\tau|^2 + |\partial_s \tau|^2 + |\partial_t \tau|^2 \, ds \, dt &\leq \theta (e^{-c(s+r)} + e^{-c(r-s)}),
\end{align*}
\]

where $\delta > c > 0$ are the constants from Lemmas 23 and 24.

Proof. The technique of proof will be to apply Lemma 27 to the differential inequalities (32) (see Lemma 23) and (35) (see Lemma 24). We reprint (32) and (35) here for the reader’s convenience:

\[
\begin{align*}
\gamma''(s) - \delta^2 \gamma(s) &\geq \frac{1}{3} (||\partial_s y||^2 + ||A(y)||^2) \\
\Gamma''(s) - c^2 \Gamma(s) &\geq ||\partial_s \tau||^2 + \frac{1}{3} ||\partial_t \tau||^2 - \frac{1}{3} ||A(y)||^2.
\end{align*}
\]

We begin the proof by picking $U$ small enough so that the $\epsilon$ bounds on the $C^1$ sizes of $y$ and $\tau$ necessary for Lemma 23 and Lemma 24 hold. For the first part of (39) we only need $\text{pr} \circ u(s, -)$ to be in $U$ for $s \in [-r - 1, r + 1]$. We apply Lemma 27 to (32), concluding that for $s \in [-r, r]$

\[
\int_{s-0.5}^{s+0.5} ||\partial_s y||^2 + ||A(y)||^2 \leq 3A_1 e^{-\delta} [e^{-\delta(s+r)} + e^{-\delta(r-s)}],
\]

where

\[
3A_1 e^{-\delta} = 3(40 + 2\delta^2) \gamma(-r - 1) \quad \text{and} \quad 3A_2 e^{-\delta} = 3(40 + 2\delta^2) \frac{\gamma'(r + 1) - \delta \gamma(r + 1)}{2\delta}.
\]

We can now apply the $L^2$ estimates for the asymptotic operator (31) to conclude

\[
\int_{s-0.5}^{s+0.5} ||y||^2 + ||\partial_s y||^2 + ||\partial_t y||^2 \leq 6C^2 A_1 e^{-\delta} [e^{-\delta(s+r)} + e^{-\delta(r-s)}].
\]

Since $\gamma(r + 1), \gamma(-r - 1)$ and $\gamma'(r + 1)$ are all controlled by the $C^1$ size of $y$, we can pick the $C^1$ neighborhood $U$ sufficiently small to ensure that $6C^2 A_1 e^{-\delta} \leq \theta$. This proves the first part of (39).
For the next part of (39) we will need to do a bit more work, since Lemma 27 does not apply directly to (35). However, using a convolution will bring us into a setting where Lemma 27 does apply.

We convolve (35) with the bump function \( \rho \) from Figure 10:

\[
(\rho * \Gamma)'(s) - c^2(\rho * \Gamma)(s) \geq \rho * (\|\partial_s \tau\|^2 + \frac{1}{3} \|\partial_t \tau\|^2) - \frac{1}{3} \rho * \|A(y)\|^2.
\]

This holds for \( s \in [-r - 1, r + 1] \). Then we apply Lemma 27 to this with \( \gamma = \rho * \Gamma \),

\[
\alpha = \rho * (\|\partial_s \tau\|^2 + \frac{1}{3} \|\partial_t \tau\|^2),
\]

and \( \kappa = (1/3)\rho * \|A(y)\|^2 \) to conclude that

\[
\int_{s-0.5}^{s+0.5} \rho * (\|\partial_s \tau\|^2 + \frac{1}{3} \|\partial_t \tau\|^2)ds \leq A_1' e^{-c(r+s)} + A_2' e^{-c(r-s)},
\]

for \( s \in [-r, r] \). Here the constants \( A_1' \) and \( A_2' \) are controlled by the \( C^1 \) sizes of \( \rho * \Gamma \) at \( \pm (r + 1) \) and constants \( K_1, K_2 \) satisfying

\[
\frac{1}{3} \rho * \|A(y)\|^2 \leq K_1 e^{-\delta(r+s)} + K_2 e^{-\delta(r-s)},
\]

for \( s \in [-r - 1, r + 1] \). That such constants \( K_1, K_2 \) exist follows from the proof of Lemma 27 and the estimate (32). Moreover, the proof of Lemma 27 shows that \( K_1 \) and \( K_2 \) are controlled by the \( C^1 \) sizes of \( \gamma(r + 2), \gamma(-r - 2) \) and \( \gamma'(r + 2) \).

Thus, by picking \( U \) sufficiently small, we can make \( A_1' \) and \( A_2' \) as small as desired.

It is straightforward to show that there is some constant \( C_\rho \) depending only on \( \rho \) so that

\[
\int_{s-0.5}^{s+0.5} \|\partial_s \tau\|^2 + \frac{1}{3} \|\partial_t \tau\|^2 ds \leq C_\rho \int_{s-0.5}^{s+0.5} \rho * (\|\partial_s \tau\|^2 + \frac{1}{3} \|\partial_t \tau\|^2)ds.
\]

Next, the Poincaré inequality applied to \( \tau \) guarantees a constant \( c_{pc} \) so that

\[
\int_{s-0.5}^{s+0.5} \int_0^1 |\tau|^2 + |\partial_s \tau|^2 + |\partial_t \tau|^2 dsdt \leq c_{pc} \int_{s-0.5}^{s+0.5} \|\partial_s \tau\|^2 + \frac{1}{3} \|\partial_t \tau\|^2 ds.
\]

Combining everything, we have

\[
\int_{s-0.5}^{s+0.5} \int_0^1 |\tau|^2 + |\partial_s \tau|^2 + |\partial_t \tau|^2 dsdt \leq c_{pc} C_\rho A_1' e^{-c(r+s)} + c_{pc} C_\rho A_2' e^{-c(r-s)}.
\]

Picking \( U \) small enough that \( c_{pc} C_\rho A_1' < \theta \) completes the proof. \( \Box \)
5. $C^k$ exponential estimates.

The goal in this section is to establish $C^k$ exponential estimates on $y$, $\tau$, and $\sigma$. First we will establish estimates for $y$, $\tau$ coordinates (Lemma 32), and in the second section we will deduce similar estimates for the $\sigma$ coordinate.

5.1. Bootstrapping the $W^{1,2}$ exponential estimates to $C^k$ estimates. In this section we will upgrade the $W^{1,2}$ exponential estimates from Lemma 29 to $C^k$ estimates. We begin by reprinting the equations that the tranverse coordinate $y$ and tangential coordinates $(\sigma, \tau)$ satisfy:

\begin{equation}
\frac{\partial y}{\partial s} + J_t(\tau, y) \frac{\partial y}{\partial t} + J_t(\tau, y) \text{d}\mu_t(y)^{-1} \nabla \mu_t(y) = 0.
\end{equation}

\begin{equation}
\frac{\partial \sigma}{\partial s} - \frac{\partial \tau}{\partial t} = -B(t + \tau(s, t), x) \frac{\partial x}{\partial t}, \text{ and } \frac{\partial \sigma}{\partial t} + \frac{\partial \tau}{\partial s} = 0.
\end{equation}

The boundary conditions for $y$ and $\tau$ were:

$$y(s, 0), y(s, 1) \in \mathbb{R}^{n-1} \text{ (Lagrangian) and } \tau(s, 0) = \tau(s, 1) = 0 \text{ (Dirichlet)}$$

To obtain an equation involving only $\tau$, we compute

\begin{equation}
\Delta \tau = \frac{\partial}{\partial t} \left[ \beta_t(\tau, y) \frac{\partial x}{\partial t} \right],
\end{equation}

where $\beta(\tau, y) = -B(Tt + \tau, \mu_t(x))$. We re-express the equation for $y$ as follows:

\begin{equation}
\frac{\partial y}{\partial s} - J_0 \frac{\partial y}{\partial t} = \lambda_t(\tau, y) \frac{\partial y}{\partial t} + \kappa_t(\tau, y),
\end{equation}

where $\lambda_t(\tau, y) = (J_0 - J_t(\tau, y))$ and $\kappa_t(\tau, y) = J_t(\tau, y) \text{d}\mu_t(y)^{-1} \nabla \mu_t(y)$.

We will use the following linear elliptic estimates for $\Delta = \partial_s + J_0 \partial_t$ and $\Delta = \partial_s^2 + \partial_t^2$.

**Lemma 30.** Consider a sequence $0.5 = \rho_0 > \rho_1 > \rho_2 > \cdots > 0.1$. Abbreviate the domains $\Omega_k = [-\rho_k, \rho_k] \times [0, 1]$. There exist constants $E_k$ so that for $k \geq 1$

\begin{equation}
\|y\|_{W^{k,2}(\Omega_k)} \leq E_k(\|\nabla y\|_{W^{k-1,2}(\Omega_{k-1})} + \|y\|_{W^{k-1,2}(\Omega_{k-1})})
\end{equation}

for every smooth function $y : \Omega_0 \to \mathbb{R}^{2n-2}$ with $y(s, 0), y(s, 1) \in \mathbb{R}^{n-1}$. Similarly there are constants $c_k$ so that for $k \geq 2$

\begin{equation}
\|\tau\|_{W^{k,2}(\Omega_k)} \leq E_k(\|\Delta \tau\|_{W^{k-2,2}(\Omega_{k-1})} + \|\tau\|_{W^{k-1,2}(\Omega_{k-1})})
\end{equation}
for every smooth function $\tau : \Omega_0 \to \mathbb{R}$ with $\tau(s, 0) = \tau(s, 1) = 0$.

**Proof.** First we prove the elliptic estimate for $\overline{\partial}$, following [RS01, Lemma C.1]. If $y$ has compact support with $\mathbb{R}^{n-1}$ boundary conditions, we compute

$$\int_{\Omega_{k-1}} |\overline{\partial} y|^2 = \int_{\Omega_{k-1}} |\partial_s y|^2 + |\partial_t y|^2 \, ds dt.$$ 

Now let $\beta_k$ be a bump function which is 1 on $\Omega_k$ and supported in $\Omega_{k-1}$, and compute

$$\|y\|_{W^{1,2}(\Omega_k)} \leq \|\beta y\|_{W^{1,2}(\Omega_{k-1})} \leq \|\beta y\|_{L^2(\Omega_{k-1})} + \|d(\beta y)\|_{L^2(\Omega_{k-1})}$$

$$\leq \|\beta y\|_{L^2(\Omega_{k-1})} + \|\overline{\partial} \beta y\|_{L^2(\Omega_{k-1})} + \|\beta \overline{\partial} y\|_{L^2(\Omega_{k-1})}$$

$$\leq c_k \left( \|\overline{\partial} y\|_{L^2(\Omega_{k-1})} + \|y\|_{L^2(\Omega_{k-1})} \right).$$

A similar estimate holds with $y$ replaced by $\nabla^\ell y$ (since $\overline{\partial}$ commutes with derivatives).

Summing over $\ell = 0, \ldots, k-1$ we conclude

$$\|y\|_{W^{k,2}(\Omega_k)} \leq E_k \left( \|\overline{\partial} y\|_{W^{k-1,2}(\Omega_{k-1})} + \|y\|_{W^{k-1,2}(\Omega_{k-1})} \right),$$

as desired.

To establish the elliptic estimate for $\Delta \tau$, we insert an intermediate domain $\Omega_k \subset \Omega' \subset \Omega_{k-1}$ and compute

$$\|i\tau\|_{W^{k,2}(\Omega_k)} \leq c' \left( \|\overline{\partial} i\tau\|_{W^{k-1,2}(\Omega')} + \|i\tau\|_{W^{k-1,2}(\Omega')} \right).$$

Observe that $\partial = \partial_s - J_0 \partial_t$ is conjugate to $\overline{\partial}$, and hence satisfies the same elliptic estimates. Since $\overline{\partial}(i\tau)$ is real along the boundary, we can apply these estimates to $\overline{\partial}(i\tau)$. Thus

$$\|\overline{\partial} \tau\|_{W^{k-1,2}(\Omega')} \leq c'' \left( \|\overline{\partial} \overline{\partial} \tau\|_{W^{k-2,2}(\Omega_{k-1})} + \|\overline{\partial} \tau\|_{W^{k-2,2}(\Omega_{k-1})} \right).$$

It is easy to see that $\partial \overline{\partial} = \Delta$, and hence the combination of our two estimates yields the desired result (43).

□

In order to apply the linear elliptic estimates to equations (40) and (41), we introduce the following lemma allowing us to estimate the $W^{k,2}$ norms of $\beta_t(\tau, y)$, $\lambda_t(\tau, y)$ and $\kappa_t(\tau, y)$.

**Lemma 31.** Let $F_t : \mathbb{R}^N \to \mathbb{R}^M$, $t \in [0, 1]$, be a smooth family of functions which vanish at the origin. Let $\Omega = [s_0, s_1] \times [0, 1]$. Fix a constant $c_k \geq 0$. 


There is a constant $C_k$ depending only on $F_t$ and $c_k$ so that for any smooth function $X : \Omega \rightarrow \mathbb{R}^N$ satisfying the pointwise bound

$$\sum_{\ell=0}^k |\nabla^\ell X| \leq c_k$$

we have

$$\|F_t(X(s,t))\|_{W^{k,2}(\Omega)} \leq C_k \|X\|_{W^{k,2}(\Omega)}.$$  

Here we use the shorthand $\nabla X = (\partial_s X, \partial_t X)$. The constant $C_k$ depends only on the $C_k+1$ size of $(t,x) \mapsto F_t(x)$ and the constant $c_k$.

**Proof.** We begin by computing

$$F_t(X(s,t)) = \int_0^1 dF_t(rX(s,t))dr \cdot X(s,t).$$

It is well-known that $\|fg\|_{W^{k,2}} \leq \|f\|_{C^k} \|g\|_{W^{k,2}}$, and hence

$$\|F_t(X(s,t))\|_{W^{k,2}} \leq \int_0^1 \|dF_t(rX(s,t))\|_{C^k} dr \cdot \|X\|_{W^{k,2}}.$$  

It is easy to prove (by induction) that

$$\|dF_t(rX(s,t))\|_{C^k} \leq G(r c_k),$$

where $G$ is a degree $k$ polynomial function whose coefficients involve the derivatives of $F_t$. The integral of $G(r c_k)$ over $[0,1]$ defines the constant $C_k$. This completes the proof. \(\square\)

Here is the main result of this section

**Lemma 32.** Let $c$ be a non-degenerate Reeb chord, and pick a number $\theta > 0$. There exists a $C^1$ open set $U$ of $t \mapsto c(Tt)$ with the following property. If $c_i \geq 0$ is any sequence of positive numbers, and $u : [-r-2, r+2] \times [0,1] \rightarrow \mathbb{R} \times Y$ is a $J$-holomorphic curve satisfying

(a) $\text{pr} \circ u(s,-) \in U$ for all $s \in [-r-2, r+2]$, and

(b) $\sum_{\ell=1}^k |\nabla^\ell y(s,t)| + |\nabla^\ell \tau(s,t)| \leq c_k$, for $(s,t) \in [-r-0.5, r+0.5] \times [0,1]$  

then

$$\sum_{\ell=1}^k |\nabla^\ell y(s,t)| + |\nabla^\ell \tau(s,t)| \leq D_k \theta (e^{-d(r+s)} + e^{-d(r-s)}),$$

where $D_k$ is a constant depending only on $k$. This completes the proof.
for $s \in [-r, r]$ and $t \in [0,1]$, where $d = \frac{1}{2}c < \frac{1}{2}\delta$. Here the constant $D_{k-2}$ depends only on:

(i) the constant appearing in the Sobolev embedding theorem $C^{k-2} \subset W^{k,2}$ for the domain $[-0.1, 0.1] \times [0,1]$.

(ii) the constants $c_1, \ldots, c_k$,

(iii) the constants $E_1, \ldots, E_k$ appearing in elliptic estimates from Lemma 30,

(iv) the $C^k$ sizes of functions $\beta_t, \lambda_t, \kappa_t$ appearing in the equations (40) and (41), and the coordinate transformation $\mu_t$ used to define $x = \mu_t(y)$ in Section 3.2.

**Proof.** First, pick $U$ so that Lemma 29 holds, and let $\Omega_k$ be the domains from Lemma 30. We claim that there are constants $P_k$, depending only on (ii), (iii), and (iv), so that

$$\|\tau\|_{W^{k,p}(\Omega_k)} + \|y\|_{W^{k,p}(\Omega_k)} \leq P_k \theta (e^{-d(r+s)} + e^{-d(r-s)}).$$

The case $k = 1$ was already established in 29 and we can take $P_1 = 2$.

Consider the equation (40) for $\tau$

$$\Delta \tau = \frac{\partial}{\partial t} \left[ \beta_t(\tau, y) \frac{\partial x}{\partial t} \right] \implies \|\Delta \tau\|_{W^{k-1,2}(\Omega_k)} \leq \left\| \beta_t(\tau, y) \frac{\partial x}{\partial t} \right\|_{W^{k,2}(\Omega_k)}$$

$$\leq Q_k (\|\tau\|_{W^{k,2}(\Omega_k)} + \|y\|_{W^{k,2}(\Omega_k)}),$$

where we have bounded the $C^k$ size of $\partial x / \partial t$ using $c_k$ and the $C^k$ size of $\mu_t$. We then applied Lemma 31 to estimate $\|\beta_t(\tau, y)\| \leq C_k (\|\tau\| + \|y\|)$ (here $C_k$ depends on the $C^{k+1}$ size of $\beta$).

Then we conclude that from the elliptic estimates

$$\|\tau\|_{W^{k+1,2}(\Omega_{k+1})} \leq E_{k+1}((Q_k + 1) \|\tau\|_{W^{k,2}(\Omega_k)} + Q_k \|y\|_{W^{k,2}(\Omega_k)}).$$

Consider next the equation (41) for $y$. We compute

$$\|\partial y\|_{W^{k,2}(\Omega_k)} = \|\lambda_t(\tau, y)\|_{W^{k,2}(\Omega_k)} \|\partial y\|_{C^k} + \|\kappa_t(\tau, y)\|_{W^{k,2}(\Omega_k)}.$$

We have $\|\partial y\|_{C^k} \leq c_k$. Using Lemma 31 we estimate

$$\|\partial y\|_{C^k} \|\lambda_t(\tau, y)\|_{W^{k,2}(\Omega_k)} + \|\kappa_t(\tau, y)\|_{W^{k,2}(\Omega_k)} \leq (c_k C_k + C_k) (\|y\|_{W^{k,2}(\Omega_k)} + \|\tau\|_{W^{k,2}(\Omega_k)}).$$
where \( R_k = c_k C_k + C_k \) depends on \( c_k \) and the \( C^{k+1} \) sizes of \( \tau \) and \( \kappa \). Then by the elliptic estimates for \( \partial \) we have
\[
\| y \|_{W^{k+1,2}(\Omega_{k+1})} \leq E_{k+1}(R_k + 1) \| y \|_{W^{k,2}(\Omega_k)} + E_{k+1} R_k \| \tau \|_{W^{k,2}(\Omega_k)}
\]
Combining the our \( W^{k+1,2} \) estimates together we have
\[
\| \tau \|_{W^{k+1,2}(\Omega_{k+1})} + \| y \|_{W^{k+1,2}(\Omega_{k+1})} \leq p_{k+1}(\| \tau \|_{W^{k,2}(\Omega_k)} + \| y \|_{W^{k,2}(\Omega_k)}).
\]
Thus, setting \( P_{k+1} = p_{k+1} \cdots p_2 P_1 \), we have
\[
\| \tau \|_{W^{k+1,2}(\Omega_{k+1})} + \| y \|_{W^{k+1,2}(\Omega_{k+1})} \leq P_{k+1} \theta(e^{-d(r+s)} + e^{-d(r-s)}).
\]
It is straightforward to verify that \( P_k \) only depends on (ii), (iii), and (iv). Finally, the Sobolev embedding theorem gives the desired result (45). This completes the proof. \( \square \)

5.2. Estimating the \( \sigma \) coordinate. We restate the holomorphic curve equation relating \( \sigma \) and \( \tau \):
\[
\frac{\partial \sigma}{\partial s} - \frac{\partial \tau}{\partial t} = \beta(\tau, y) \frac{\partial x}{\partial t},
\]
using the same notation \( \beta(\tau, y) = -B(Tt + \tau, \mu_t(x)) \) from last section. As a straightforward consequence of these equations, we deduce the following lemma

Lemma 33. Assume the setup, hypotheses, and conclusion of Lemma 32. Then there are constants \( S_k \) so that
\[
\sum_{\ell=0}^{k} \| \nabla^\ell (\sigma - \sigma(0,0)) \|_{W^{k+1,2}(\Omega_{k+1})} \leq S_k D_k \theta(e^{-d(r+s)} + e^{-d(r-s)}).
\]
Here \( S_k \) depends only on the \( C^{k+1} \) sizes of \( y \) and \( \tau \) (i.e. the numbers \( c_{k+1} \)), the \( C^{k+1} \) size of \( \beta \), \( D_{k+1} \), and the number \( d \).

Proof. The equations relating \( \sigma \) and \( \tau \) easily establish
\[
\sum_{\ell=1}^{k} \| \nabla^\ell (\sigma - \sigma(0,0)) \| = \sum_{\ell=1}^{k} \| \nabla^\ell \sigma \| \leq S_k' D_k \theta(e^{-d(r+s)} + e^{-d(r-s)}).
\]
The only part which remains is to estimate
\[ |\sigma - \sigma(0, 0)| \leq S_0 D_0 \theta (e^{-d(r+s)} + e^{-d(r-s)}). \]
This follows from the exponential bounds on \( \partial_s \sigma \) and \( \partial_t \sigma \). Indeed, even finding \( \sigma_0 \) so that \( |\sigma - \sigma_0| \) remains bounded is not obvious without the exponential bounds.

We compute for \( s_0, t_0 \in [0, r] \times [0, 1] \)
\[ |\sigma(s_0, t_0) - \sigma(0, 0)| \leq \int_0^{s_0} |\partial_s \sigma(s, t_0)| \, ds + \int_0^{t_0} |\partial_t \sigma(0, t)| \, dt \]
The second integral is bounded by \( S_1' D_1 \theta (2e^{-dr}) \leq 2S_1' D_1 \theta (e^{-d(r+s_0)} + e^{-d(r-s_0)}) \). We estimate the first integral:
\[
\int_0^{s_0} |\partial_s \sigma(s, t_0)| \, ds 
\leq S_1' D_1 \theta \int_0^{s_0} e^{-d(r+s)} + e^{-d(r-s)} \, ds 
= \frac{S_1' D_1 \theta}{d} (e^{-dr} - e^{-d(r+s_0)} + e^{-d(r-s_0)} - e^{-dr}) 
= \leq \frac{S_1' D_1 \theta}{d} (e^{-d(r+s_0)} + e^{-d(r-s_0)}) .
\]
A similar estimate holds if \( -r \leq s_0 \leq 0 \).
\[ |\sigma(s_0, t_0) - \sigma(0, 0)| \leq (S_1' D_1 (1 + d^{-1}) \theta) (e^{-d(r+s_0)} + e^{-d(r-s_0)}). \]
Then we set
\[ S_0 = \frac{S_1' D_1 (1 + d^{-1})}{D_0} , \]
and let \( S_k = S_0 \frac{D_0}{D_k} + S_k' \), we obtain the desired result \( (4) \). \( \Box \)

5.3. **Simplifying the results.** The results of the previous two sections can be combined and simplified, leading to the following theorem.

**Theorem 4.** Let \( c \) be a non-degenerate Reeb chord and fix \( \theta > 0 \). There is a \( C^1 \) open set \( U \) around \( t \mapsto c(tT) \) and constants \( M_k > 0, d > 0 \) with the following property: if \( u : [-r - 2, r + 2] \rightarrow \mathbb{R} \times Y \) is a holomorphic curve and \( \text{pr} \circ u(s, -) \in U \) for all \( s \), then
\[
(4) \quad \sum_{\ell=0}^{k} \left| \nabla^\ell (\sigma(s, t) - \sigma(0, 0)) \right| + \left| \nabla^\ell y(s, t) \right| + \left| \nabla^\ell \tau(s, t) \right| \leq M_k \theta (e^{-d(r+s)} + e^{-d(r-s)}) \]
for \( (s, t) \in [-r, r] \times [0, 1] \).
Remark 34. In the statement \( d = \frac{1}{2} c < \frac{1}{2} \delta \) where \( c, \delta \) are from Lemmas 23 and 24.

Note that \( M_k \) does not depend on \( u \) or \( r \), and we have removed the assumption of the \( C^k \) bound of \( y, \tau \) by \( c_k \). \( \triangle \)

Proof. First of all, pick \( U \) so that \( pr \circ u(s, -) \in U \) implies \( y, \tau \) are uniformly bounded by 1 in the \( C^1 \) norm. By the equation relating \( \sigma \) and \( \tau \), we conclude that \( \sigma \) is also bounded by some constant in the \( C^1 \) norm. We also pick \( U \) so that Lemmas 23, 24, and 32 hold.

Now consider the set \( \mathcal{U} \) of all holomorphic curves \( u : [-r - 2, r + 2] \to \mathbb{R} \times Y \) which satisfy \( pr \circ u(s, -) \in U \). Note that there is some constant \( K \) so that if \( u \in \mathcal{U} \) then \( |du(s, t)| \leq K \) for all \((s, t)\). Here \( K \) depends on the derivatives of the coordinate transformation relating \( u \) with \( \sigma, \tau, y \).

We claim that there are constants \( c_k \) so that all curves \( u \in \mathcal{U} \) satisfy
\[
\sum_{\ell=1}^{k} \left| \nabla^{\ell} y(s, t) \right| + \left| \nabla^{\ell} \tau(s, t) \right| \leq c_k,
\]
for \((s, t) \in [-r - 1, r + 1] \times [0, 1] \).

To see why, suppose not. Then we could find a sequence \( u_n \in \mathcal{U} \) and points \( z_n \) in \([-r_n - 1, r_n + 1] \times [0, 1] \) so that the \( k \)th derivative of \( u_n \) at \( z_n \) diverges. However, this contradicts our earlier bootstrapping lemma (Lemma 8).

Then we can apply Lemmas 32 and 33 with this choice of \( c_k \) to conclude constants \( D_k \) and \( S_k \) so that
\[
\sum_{\ell=0}^{k} \left| \nabla^{\ell}(\sigma - \sigma(0, 0)) \right| + \left| \nabla^{\ell} y \right| + \left| \nabla^{\ell} \tau \right| \leq (S_k D_k + D_k) \theta(e^{-d(r+s)} + e^{-d(r-s)}),
\]
in the region \([-r, r] \times [0, 1] \).

For fixed \( k \), the constants \( D_k, S_k \) depend only on (i) finitely many of the constants \( c_1, c_2, \ldots \), (ii) finitely many of the derivatives of the auxiliary functions \( \mu_t, \kappa_t, \lambda_t, \beta_t \) appearing in the equations (41) and (40), (iii) finitely many of constants appearing the elliptic estimates for \( \Delta \) and \( \tilde{\Delta} \) for the regions \( \Omega_0, \Omega_1, \ldots \), (iv) the constant \( d \), and (v) finitely many of the constants appearing in the Sobolev embedding theorem. In particular, since \( c_1, c_2, \ldots \) only depend on \( U \) (and not on \( u \)), we conclude that \( M_k = (S_k D_k + D_k) \) is independent of \( u \). This completes the proof. \( \square \)
Corollary 5. Let $c$ be a non-degenerate Reeb chord and fix $\theta > 0$. There is a $C^1$ open set $U$ around $t \mapsto c(tT)$ and constants $M_k$ with the following property: if $u : [-2, \infty) \to \mathbb{R} \times Y$ is a holomorphic curve and $\text{pr} \circ u(s, -) \in U$ for all $s$, then

$$\sum_{\ell=0}^{k} \left| \nabla^\ell (\sigma(s, t) - \sigma(0, 0)) \right| + \left| \nabla^\ell y(s, t) \right| + \left| \nabla^\ell \tau(s, t) \right| \leq M_k \theta e^{-ds}$$

for $(s, t) \in [0, \infty) \times [0, 1]$.

Proof. The constants $M_k$ from Theorem 4 do not depend on $r$. Identify the domain $[-r - 2, r + 2]$ with $[-2, 2r + 2]$, and then take the limit as $r \to \infty$. \hfill \Box

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