High Temperature Limit of the $N = 2$ IIA Matrix Model

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The high temperature limit of a system of two D-0 branes is investigated. The partition function can be expressed as a power series in $\beta$ (inverse temperature). The leading term in the high temperature expression of the partition function and effective potential is calculated exactly. Physical quantities like the mean square separation can also be exactly determined in the high temperature limit. We comment on $SU(3)$ IIB matrix model and the difficulties to study it.

1. Introduction

The study of string theory at finite temperature has received renewed attention recently. In finite temperature the number density of string states increases exponentially with energy or temperature $\rho_b(m) = m^{-\frac{5}{2}} \exp(4\pi\sqrt{\alpha' m})$. So, partition function diverges for $T > T_H$, where $T_H$ is the Hagedorn temperature. In perturbative string theory it is not clear whether $T_H$ is limiting temperature or phase transition temperature, so non-perturbative formulation like matrix model is needed. In a recent paper one of us [2] attempted to elucidate the nature of the Hagedorn transition [1] using the matrix model and found similarities with the deconfinement transition in gauge theories. This was also investigated in a subsequent paper using the AdS/CFT correspondence [4]. It is clear that recent developments in non-perturbative string theory or M-theory [6–10] have some bearing on our understanding of the high temperature behavior of strings. For all these reasons the study of matrix models at high temperature is worthwhile.

A related model of D-instantons, the IKKT matrix model [5], which is 0+0 dimensional has been solved exactly for $N=2$ [6]. The D-0-brane action that we are interested in, is a quantum mechanical one (i.e. 0+1 dimensional). However after compactifying the Euclideanised time, if one takes the high temperature limit, it reduces to a 0+0 dimensional model. There is thus a hope of solving this model order by order in $\beta$ but to all orders in $g$ using the same techniques as [6].

One can then calculate physical quantities such as the mean square separation of the D-0-branes - a measure of the size of the bound states. This is what is attempted in this paper. We obtain the leading behavior in $\beta$. We can also estimate, the corrections to the leading result. The noteworthy feature being that each term is exact in its dependence on the string coupling constant. For details and references see [6].

2. High Temperature $SU(2)$ matrix Model

2.1. The action

The BFSS Lagrangian is given by 10 dimensional SYM lagrangian reduced in (0 + 1) dimension

\[
L = \frac{1}{2g_s} \text{tr} \left[ \dot{X}^\mu \dot{X}^\mu + 2i \dot{\theta} \dot{\bar{\theta}} - \frac{1}{2l_s^2} [X^\mu, X^\nu]^2 - \frac{2}{l_s^2} \bar{\theta} \gamma_\mu [\theta, X^\mu] - i \bar{\theta} [X^0, X^i] X^i \right] \quad (2.1.1)
\]

where $i = 1, ..., 9; \mu = 0, ..., 9 ; X^\mu$ and $\theta$ are $N \times N$ hermitian matrices. $X^\mu$ is 10 dimensional vector and $\theta$ is 16 component Majorana-Weyl spinor in 10 dimension. We write $X^\mu, \theta$ in terms of the group generators $T^a$

\[
X^\mu = \sum_{a=1}^{N^2-1} T^a X^a, \quad \theta = \sum_{a=1}^{N^2-1} T^a \theta^a \quad (2.1.2)
\]

$X^a, \theta^a$ are real fields. For $N = 2$, $T^a = \frac{1}{2} \sigma^a$, $\sigma^a$ are the hermitian Pauli matrices.

If we Euclideanize ($t \to it, X^0 \to iX^0$) and compactify time on a circle of circumference $\beta$, the action becomes

\[
S = i \int_0^\beta Ldt \quad (2.1.3)
\]
Considering the field boundary conditions
\[ X^\mu(0) = X^\mu(\beta), \quad \theta(0) = -\theta(\beta) \]
we can expand the fields \( X^\mu, \theta \) in modes as
\[ X^\mu_a(t) = \sum_{n=-\infty}^{\infty} X^\mu_{a,n} e^{\frac{2\pi i n t}{\beta}}, \quad \theta_a(t) = \sum_{r=-\infty}^{\infty} \theta_{ar} e^{\frac{2\pi i r t}{\beta}} \]
n, (m, l, p) are integers and \( r, s \) are half-integers.
So, the action reduces to
\[ S = S_{b,\text{free}} + S_b + S_f \quad (2.1.4) \]
where, \( S_{b,\text{free}}, S_b \) and \( S_f \) are the free bosonic, bosonic and the fermionic terms
\[ S_{b,\text{free}} = \frac{i}{4g} \left\{ -\sum_{n=-\infty}^{\infty} \frac{4\pi^2 n^2}{\beta^2} X^i_{a,n} X^i_{a,-n} \right\} \quad (2.1.5) \]
\[ S_b = \frac{i}{8g} \left\{ \beta \sum_{n,m,l,p=-\infty}^{\infty} X^\mu_{b,n} X^\nu_{b,m} X^\mu_{a,l} X^\nu_{a,p} - \beta \sum_{n,m,l,p=-\infty}^{\infty} X^\mu_{a,n} X^\nu_{b,m} X^\mu_{a,l} X^\nu_{b,p} \right. \]
\[ + n\pi \epsilon^{abc} \sum_{n+l+p=0}^{\infty} X^0_{a,l} X^i_{b,m} X^i_{c,n} \left. \right\} \quad (2.1.6) \]
\[ S_f = \frac{i}{4g} \left\{ -2i\beta \sum_{r,s,l=-\infty}^{\infty} \theta_{ar} \gamma_0 \gamma_\mu \theta_{bs} X^\mu_{r,s,j} \right. \]
\[ + \sum_{r} 4\pi r \theta_{ar} \epsilon^{abc} \left. \right\} \quad (2.1.7) \]
\[ (X^\mu_{a,n})_{n \neq 0} \text{ and } X^\mu_{a,0} \text{ are of the order } \left( \frac{\sqrt{\gamma}}{n} \right) \]
and (1). In \( \beta \rightarrow 0 \) limit, the zero-modes of \( X^\mu \) will contribute in the leading order to \( Z \). So, we can re-write
\[ S_b = \frac{i\beta}{8g} \left\{ X^2_{a,0} X^2_{b,0} - (X_{a,0} X_{b,0})^2 \right\} \quad (2.1.8) \]
\[ S_f = \frac{i}{2g} \left\{ -i\beta \sum_{r} \theta_{ar} \gamma_0 \gamma_\mu \theta_{bs} X^\mu_{r,0} \right. \]
\[ + \sum_{r} 2\pi r \theta_{ar} \theta_{bs} \epsilon^{abc} \left. \right\} \quad (2.1.9) \]
The partition function is \( Z = \int e^{-iS} \)

2.2. Pfaffian
Following [5], we rotate \( X^\mu_{c,0} \) by a Lorentz transformation so that only \( X^0_{c,0}, X^1_{c,0} \) and \( X^2_{c,0} \) are nonzero. We take the representation of the Gamma matrices, in which
\[ \gamma_0 = i\sigma_2 \otimes 1_8, \quad \gamma_1 = \sigma_3 \otimes 1_8, \quad \gamma_2 = -\sigma_1 \otimes 1_8 \quad (2.2.1) \]
Integrating the fermionic fields we get the pfaffian,
\[ Z_f = g^{-24} \left( O(1) + O(\beta^2 \gamma^2) + O(\beta^4 \gamma^4) + \ldots \right) \]
the above expression has \( SO(3) \) symmetry in spinor indices, and \( SO(2,1) \) symmetry in the vector indices. The \( O(1) \) term gives the free fermionic contribution. Note that it is temperature independent as the Hamiltonian is identically zero for free fermions in \( 0 + 1 \) dimensions.

2.3. Free Bosonic sector
As the higher modes interacting terms do not contribute to the leading order, we can decouple the free bosonic and interacting terms. Integrating the bosonic sector and regularizing , we get
\[ Z_{\text{free}} = (\beta g)^{-27} \quad (2.3.1) \]

2.4. Leading and Non-leading Interaction Terms.
The terms \( X^0_{a,l} X^1_{b,m} X^2_{c,n} \) and \( X^0_{a,n} X^1_{b,m} X^2_{c,n} \) in the original action are not Lorentz invariant. However, these terms do not contribute to the partition function in leading order . The action with only the zero modes has Lorentz invariance. Hence, as long as we are interested in leading order contribution only, we can work with and assume Lorentz invariance and consider the parametrisation
\[ X_{1,0} = (x_1, \vec{r}_1), \quad X_{2,0} = (x_2, \vec{r}_2), \quad X_{3,0} = (l, 0) \quad (2.4.1) \]
At this stage, we can find the temperature dependence of the partition function and the mean square separation of two D-0 branes from a simple scaling argument. This scaling argument in fact applies to $SU(N)$ also. To see this we note that the leading order $i.e.$ the zero mode bosonic contribution of the partition function comes from the $[X^\mu ,X^\nu]^2$ term in the Lagrangian (2.1.1). And this term is Lorentz invariant and hence just as in $SU(2)$ case can use Lorentz invariance while calculating the leading order contribution to the partition function from this term. Therefore in $SU(N)$ case also we can use a parametrisation similar to (2.4.3)

$$X_{i,0} = (x_i, \vec{r}_i), \quad 1 \leq i \leq N^2 - 2; \quad X_{N^2-1,0} = (l,0)$$

Under the above parametrisation $\frac{1}{2}[X^\mu ,X^\nu]^2$ will be homogeneous in $l, r_i, x_i$, $0 < i \leq N^2 - 2$ and of order 4. So, in general for any $N$, we need to scale these variables by $\beta^{-\frac{1}{4}}g^{\frac{1}{2}}$ to scale out the $\beta$ from the exponent. And the temperature dependence of $\langle l^2 \rangle$ will be $\beta^{-\frac{1}{2}}g^{\frac{1}{2}}$ in the leading order. Under the above scaling the measure in $Z_0$ will pick up a $\beta^{-\frac{1}{4}}g^{\frac{1}{2}}$ factor for $SU(2)$, which comes from $(3 \times 10) X_{a,n}^\mu$. In general for $SU(N)$ in $D$ dimension there will be $D(N^2 - 1) X_{a,n}^\mu$ in the measure. So, the partition function $Z_0$ has temperature dependence $\frac{\beta^{D(N^2 - 1) - d(N^2 - 1)}}{4} g^{(D-1)(N^2-1)}$. And $Z_{free}$ will be proportional to $(\beta g)^{(D-1)(N^2-1)}$. Now we evaluate the partition function for this action.

$$Z_0 = \frac{2^{3415} 5^{15} \pi^5}{\beta^{\frac{1}{2}}} \int_{\infty}^{-\infty} dl I_0(l)$$

where

$$I_0(l) = 24576l^{-7} - 2752512l^{-11} + ....$$

In large $l$ limit and

$$I_0(l) = \sqrt{8\pi}l^7 - \frac{256}{5}l^9 + \left(\sqrt{\frac{\pi}{8}} - \frac{256}{3}\right)l^{11}$$

In small $l$ limit.

Hence, $Z$ converges for both large $l$ and small $l$ and $I_0(l)$ is non-singular and independent of $\beta$. So, $Z_0$ has a temperature dependence of $T^{\frac{D}{2}}$.

Combining the free bosonic, fermionic and the bosonic parts, we can write the $\beta$ dependence as

$$Z = \beta^{\frac{D}{2}} \left(O(1) + O(\beta^{\frac{1}{2}}) + O(\beta^{\frac{3}{2}}) + ... \right)$$

2.5. Effective Potential & Mean-square Separation of the D-0 branes.

For high temperature we have evaluated the partition function both for large and small $l$. Up to leading order the effective potential between two D-0 branes is proportional to $-\log l$ and $\log l$ for small and large $l$ (2.4.3, 2.4.4). We can see that the potential increases at both $l$ ends, though we can not clearly see the nature of the potential in the intermediate region but we can conclude that the potential is a confining potential and binds the D-0 branes.

We can identify $l$ as one of the spatial components and hence as the separation between two D-0 branes. We calculate the mean square separation of two D-0 branes.

$$\langle l^2 \rangle = 6.385 \left(\frac{\beta}{g}\right)^{-\frac{1}{2}}$$

If we assume high temperature expression has a finite radius of convergence, we can conclude that the mean square separation is finite for finite temperature. This implies that there is a confining potential that binds the D-0 branes. As argued earlier the scaling argument that gives the $\beta$ and $g$ dependence in (2.5.1) is valid for all $N$. So we can conclude that $\langle l^2 \rangle \approx \sqrt{2\pi} \frac{1}{\sqrt{\beta}}$ for all $N$.

3. Attempt to study $SU(3)$ IIB matrix model

As a step towards understanding the large $N$ matrix model, after studying $SU(2)$ matrix model, we try to study the $SU(3)$ matrix model. But, $SU(3)$ has 8 generators, so the $SO(10)$ rotation will leave at least 8 of the $X^\mu$ non-vanishing resulting non-zero coefficient for 8 $\gamma$ matrices in the expression $\gamma_\mu X^\mu$, unlike the $SU(2)$ case, where we had 3 generators and 3 non-zero coefficients. In $SU(3)$ case, we have to work with $16 \otimes 16$ $\gamma$ matrices, unlike $SU(2)$, where we choose the suitable form as in (2.2.1). Pfaffian for $SU(3)$ is determinant of a $128 \otimes 128$ matrix. Moreover, we
have at least 24 parameters to integrate to calculate $Z$. All these makes SU(3) difficult to study. But we can $X^a \tau^a = X^a \tau^a + X^a \tau^a + X^a \tau^a$ and $\theta_\tau \tau^a = \theta_\tau \tau^a + \theta_\tau \tau^a + \theta_\tau \tau^a$ and look the perturbative solution in the limit $X^a \tau^a \rightarrow 0$, which will give the SU(3) partition function as SU(2) partition function plus modifications. We are working on that presently and will report soon.

4. Conclusion

In this paper we have attempted to study a system of two $d0$ branes at distance $l$ kept in high temperature. The leading nontrivial term of the partition function has been calculated exactly. The non-leading terms can also be systematically calculated although we haven’t attempted to work them out in this paper. From a scaling argument we have also determined the $\beta$ and $g$ dependence of the leading term for any $N$. This complements the work of 3, where the one loop partition function was calculated with the entire $\beta$ dependence. We find that $\langle l^2 \rangle \propto \sqrt{\frac{g}{\beta}}$ (eqn. 2.5.1) (true for any $N$), the finiteness of which shows that there must be a potential between D-0 branes that binds them. In 4 also a logarithmic and attractive potential were found. The present calculation being exact in $g$ is valid at all distances. Thus unlike in 3, the (finite temperature) logarithmic potential found here is attractive at long distances and repulsive at short distances so it has a minimum at non-zero separation. In 3 it was found that at high temperatures, the configuration with all the D-0 branes clustered at the origin i.e. with the zero separation, had lower free energy than the one where they were spread out. However, that was a large $N$ calculation and also restricted to one loop. It is therefore possible that more exact calculation will resolve this issue.

REFERENCES

1. R. Hagedorn, Nuovo Cimento Suppl. 3 (1965) 147.
2. B. Sathiapalan, Mod. Phys. Lett. A 13 (1998) 2085-2094, [hep-th/9805120]
3. S. Kalvan Rama and B. Sathiapalan, [hep-th/9810069]
4. J. Ambjorn, Y. M. Makeenko and G. W. Semenoff, [hep-th/9810170]
5. T. Suyama and A. Tsuchiya, Prog. Theo. Phys. 99 (1998) 321-325, [hep-th/9711073]
6. S. Bal, B. Sathiapalan, Mod. Phys. Lett. A14(1999) 2753, [hep-th/9902087].
7. T. Banks, W. Fischler, S. Shenker and L. Susskind, Phys. Rev. D 55 (1997) 5112 - 5128, [hep-th/9610043].
8. N. Ishibashi, H. Kawai, Y. Kitazawa and A. Tsuchiya, Nucl Phys B498 (1997) 467-491, [hep-th/9612117].
9. M. Fukuma, H. Kawai, Y. Kitazawa and A. Tsuchiya, Nucl. Phys. B 510 (1998) 158 - 174, [hep-th/9705128].
10. H. Aoki, S. Iso, H. Kawai, Y. Kitazawa and T. Tada, Prog. Theor. Phys. 99 (1998) 713 - 746, [hep-th/9802084].
11. M. R. Douglas, D. Kabat, P. Pouliot and S. H. Shenker, Nucl. Phys. B485 (1997) 85, [hep-th/9608024].