BULK-EDGE CORRESPONDENCE AND THE COBORDISM INVARIANCE OF THE INDEX

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Abstract. The bulk-edge correspondence for two-dimensional type A topological insulators and topological superconductors is proved by using the cobordism invariance of the index. The idea of G. M. Graf and M. Porta to use some vector bundle is developed from the viewpoint of $K$-Theory and index theory. Such vector bundle is defined without using the solvability of some second order difference equation, by treating it as a family index of a family of Fredholm operators.

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1. Introduction

It is well known, especially in the field of condensed matter physics, a correspondence between some topological invariant for a gapped Hamiltonian of an infinite system without edge and some topological invariant for a Hamiltonian of a system with edge, which appeared in the theoretical study of the quantum Hall effect. In 1980, K. von Klitzing, G. Dorda and M. Pepper discovered a quantization of the Hall conductance of a two-dimensional electron gas in a strong magnetic field [25]. Such quantization of the Hall conductance was explained by D. J. Thouless, M. Kohmoto, M. P. Nightingale and M. den Nijs from the topological point of view [33]. They showed that the Hall conductance in the infinite system corresponds to a first Chern number (called the TKNN number) of the Bloch bundle on the Brillouin torus, by using the Kubo formula. Y. Hatsugai considered such phenomena on a system with edge, and showed that the Hall conductance of such system corresponds to a winding number (or a spectral flow) counted on a Riemann surface [19]. These correspondences of Hall conductances with integer valued topological invariants explain the quantization and the robustness of the observed Hall conductance.

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The equality between the first Chern number defined for such system without edge (called the bulk index) and the winding number defined for such system with edge (called the edge index) was proved by Hatsugai [18]. This is called the bulk-edge correspondence.

G. M. Graf and M. Porta proved the bulk-edge correspondence in [17] by using another vector bundle. They defined a vector bundle over some torus for a system without edge by using (formal) solutions of the Hamiltonian which decays as it goes to a specific direction. This vector bundle is closely related to the winding number defined for a system with edge, and by using this bundle, they showed the bulk-edge correspondence for quantum spin Hall systems, which was pointed out by C. L. Kane and E. J. Mele [21], and also for quantum Hall systems. They defined such vector bundles by using the solvability of some second-order difference equation by an initial condition, and so they need to assume some invertibility conditions for hopping matrices of the Hamiltonian to ensure such solvability. They also assumed that the band would be non-degenerated by some technical reasons. Note that such vector bundles consist of decaying solutions were also considered in [32]. Graf and Porta gave in [17] another proof of the bulk-edge correspondence for quantum Hall systems based on Levinson’s theorem in scattering theory. This point of view is not treated in this paper.

In this paper, we develop Graf–Porta’s proof of the bulk-edge correspondence to use another vector bundle, from the point of view of $K$-theory and index theory. We give a proof of the bulk-edge correspondence for some two-dimensional type A topological insulators and topological superconductors in the Altland-Zirnbauer classification [2], based on the cobordism invariance of the indices. We see that Graf–Porta’s vector bundle can be understood as a family index (more precisely, kernels) of some family of Fredholm operators, and define such bundles for Hamiltonians not necessarily have such invertibility conditions. In this way, we do not need to use the solvability of some second order difference equation by an initial condition, nor to assume the non-degeneracy of the band.

The proof goes as follows. We first construct two elements of some compactly supported $K$-groups, and see that they map to, by inverse of Thom isomorphisms, the $K$-class of the Bloch bundle, and the $K$-class of the difference of the class of Graf–Porta’s vector bundle and that of a trivial bundle, respectively. We then construct a cobordism between them. By using the cobordism invariance of the index, we obtain an equality between the bulk index and the first Chern number of Graf–Porta’s vector bundle. We next show the equality between Graf–Porta’s index and the edge index by using the excision property of the index.

There are many other works about the bulk-edge correspondence, especially form the point of view of $K$-theory and index theory. Starting from the pioneering work of J. Bellissard, an approach from the point of view of the noncommutative geometry was developed, and the bulk-edge correspondence was proved by Kellendonk–Richter–Schulz-Baldes, by using the six-term exact sequence for $K$-theory of $C^*$-algebras [10, 11, 31, 23]. Avron–Seiler–Simon considered an index theorem for a pair of projections in this context [9], and Elbau–Graf proved the bulk-edge correspondence by generalizing their method [15]. There are also recent works, by Bourne–Carey–Rennie to use the Kasparov theory [13], by Mathai–Thiang to use the T-duality [28], by Y. Kubota to use the coarse Mayer-Vietoris exact sequence
Although we use K-theory and index theory to prove the bulk-edge correspondence, the proof presented here is different from these methods and rather elementary. What we need is just basics of topological K-theory and index theory. The technique in the elementary proof of the Bott periodicity theorem by M. F. Atiyah and R. Bott [3] is used to calculate the family index. Note that the connection between the bulk-edge correspondence and the Bott periodicity theorem was pointed out in [31]. Note also that K-theory is also used in the classification of topological phases [24, 16].

Graf–Porta considered in [17] the bulk-edge correspondence mainly for quantum spin Hall systems. Actually we can also prove the bulk-edge correspondence for such systems almost in the same way as presented in this paper, by using KQ-theory introduced by J. L. Dupont [14]. The proof of the Bott periodicity theorem for such theory was given in [4]. Since the proof goes in the same way, we decided not to contain this case and consider just quantum Hall systems, in order to make this paper appropriate length to introduce our idea.

This paper is organized as follows. In Sect. 2, some basic facts about topological K-theory which will be needed in this paper, is collected. In order to fix notations, the definition of the spectral flow by J. Phillips [29] is included. In Sect. 3, we fix a class of two-dimensional single-particle lattice Hamiltonians which we consider in this paper. In Sect. 4, the bulk index and the edge index for our system is defined and our main theorem is stated. In Sect. 5, we define Graf–Porta’s vector bundle for our Hamiltonians. By using this bundle, we prove our main theorem in Sect. 6, since the cobordism invariance of the index is the key to prove the bulk-edge correspondence, a simple proof of this property by using the Whitney embedding theorem and the Bott periodicity is contained.

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2. Preliminaries

In this section, we collects some basic facts about topological K-theory needed in this paper. We refer the reader to [5, 7, 22, 27] for the details. We also review one definition of the spectral flow by Phillips [29].

2.1. Topological K-Theory. Let X be a compact Hausdorff space. The isomorphism classes of finite rank complex vector bundles over X makes an abelian semigroup by taking direct sum as its binary operation. We define the K-group \( K^0(X) \) for X as the Grothendieck group of this abelian semigroup. We denote \([E]\) for the class of a vector bundle E. \( K^0(X) \) is a commutative ring with unit by taking tensor product as its multiplication. For a finite dimensional complex
vector space $V$, we denote $V$ for the product bundle $X \times V$ over $X$. Let $x$ be a point of $X$. We define the reduced $K$-group for a based space $(X, x)$ by $K^0_\text{red}(X) := \ker(i^*: K^0(X) \to K^0(\{x\}))$, where $i: \{x\} \to X$ is a natural inclusion. Let $A$ be a closed subspace of $X$. We define the $K$-group for a pair $(X, A)$ by $K^0(X, A) := \tilde{K}^0(X/A)$, where we take $A/A$ as a base point of $X/A$. For a locally compact Hausdorff space $Y$, we take its one-point compactification $Y^\ast$. Then $(Y^\ast, +)$ is a based compact Hausdorff space. We define $K^0_\text{cpt}(Y) := \tilde{K}^0(Y)$, which is called the compactly supported $K$-group for $Y$.

Let $n$ be a non-negative integer. We define $K^{-n}(X, A) := \tilde{K}^0(\sum^n(X/A))$ where $\sum^n(X/A)$ is the $n$-fold reduced suspension. Let $\emptyset$ be the empty set, then we have $K^{-n}(X) = K^{-n}(X, \emptyset)$. We have the following long exact sequence.

$$\cdots \to K^{-1}(X, A) \to K^{-1}(X) \to K^{-1}(A) \to K^0(A) \to K^0(X) \to K^0(A).$$

There is an alternative description of $K$-groups. $K^0_\text{cpt}(Y)$ can be defined as an equivalence classes of the isomorphism classes of triples $(E, F; f)$, where $E$ and $F$ are finite rank complex vector bundles over $Y$ and $f: E \to F$ is a bundle isomorphism invertible outside a compact set. Its equivalence relation is generated by stabilization and homotopy. We denote $[E, F; f]$ for its class.

Let $GL(\infty, \mathbb{C})$ be the inductive limit of a sequence $GL(1, \mathbb{C}) \to GL(2, \mathbb{C}) \to \cdots$, where $GL(n, \mathbb{C}) \to GL(n + 1, \mathbb{C})$ is given by $A \mapsto \text{diag}(A, 1)$. Then we have a natural isomorphism $[X, GL(\infty, \mathbb{C})] \cong K^{-1}(X)$. We denote $[f]$ for the homotopy class of a continuous map $f: X \to GL(\infty, \mathbb{C})$.

Let $H$ be a separable Hilbert space. We denote $\text{Fred}(H)$ for the space of bounded linear Fredholm operators on $H$ with norm topology. Then there is a bijection index: $[X, \text{Fred}(H)] \to K^0(X)$ given by taking the family index, which was shown by Atiyah and K. Jänich independently. We refer the reader to [9] for the construction of this map. Instead of explaining the construction, we note that, if a continuous map $f: X \to \text{Fred}(H)$ consists of a family $f(x)$ of Fredholm operators whose dimension of kernels are constant, then $\text{index}(f) = [\ker(f)] - [\text{coker}(f)]$, where $\ker(f)$ and $\text{coker}(f)$ are vector bundles over $X$ whose fibers at a point $x$ in $X$ are $\ker(f(x))$ and $\text{coker}(f(x))$ respectively.

Let $E$ be an even dimensional spin$^c$ vector bundle on $Y$. Then we have an isomorphism $K^0_\text{cpt}(Y) \to K^0_\text{cpt}(E)$, called the Thom isomorphism, given by taking a cup product with the Thom class associated to the spin$^c$ structure of $E$. If $E = Y \times \mathbb{C}$ and if we consider for $E$ a spin$^c$ structure naturally induced by the complex structure of $E$, the Thom isomorphism $K^0_\text{cpt}(Y) \to K^0_\text{cpt}(Y \times \mathbb{C})$ is given by taking a cup product with the Thom class $[\mathbb{C}]_\mathbb{C}^{[x]} \in K^0_\text{cpt}(\mathbb{C})$.

Let $X$ and $Y$ be manifolds possibly with boundary and let $f: X \to Y$ be a neat embedding. Let $N$ be the normal bundle of this embedding. We assume that $N$ is even dimensional and equipped with a spin$^c$ structure. We take a tubular neighborhood of the embedded manifold, and identify it with $N$. Then we can define a push-forward map $f_1: K^0_\text{cpt}(X) \to K^0_\text{cpt}(Y)$ by the composition of the Thom isomorphism $K^0_\text{cpt}(X) \to K^0_\text{cpt}(N)$ and the map $K^0_\text{cpt}(N) \to K^0_\text{cpt}(Y)$ induced by the homotopy $f_2: N \to Y$.

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1. Note that the notation $\prod V$ can mean product bundles over different base spaces. We use this notation for simplicity. Its base space will be clear from the context.
2. An embedding $f: X \to Y$ is said to be neat if $X \cap \partial Y = \partial X$ and the space $T_x(X)$ is not contained in $T_x(\partial Y)$ for any point $x \in \partial X$ (see [20]).
the collapsing map \( Y^+ \to Y^+/\langle Y^+ - N \rangle \cong N^+ \). The push-forward map \( f_i \) is independent of the choice of a tubular neighborhood.

Let \( X \) be an even-dimensional spin\(^c\) manifold without boundary. We take an embedding \( j \) of \( X \) into an even dimensional Euclidean space \( \mathbb{R}^{2n} \). We fix a spin\(^c\) structure on \( \mathbb{R}^{2n} \). Then the normal bundle \( N \) of this embedding has a naturally induced spin\(^c\) structure, and we have a push-forward map \( j: K^{cpt}_0(X) \to K^{cpt}_0(\mathbb{R}^{2n}) \).

We take an orientation preserving linear isomorphism \( c \in \text{Spin}(\mathbb{C}^n) \), and consider the inverse of Thom isomorphism \( \beta^{-1}: K^{cpt}_0(\mathbb{R}^{2n}) \cong K^{cpt}_0(\mathbb{C}^n) \to K^0(\{pt\}) \cong \mathbb{Z} \). We define a homomorphism \( \text{ind}_X: K^0(X) \to \mathbb{Z} \) by \([E] \mapsto \beta^{-1} \circ j(\langle E \rangle \otimes [L(X)]) \), where \( L(X) \) is the determinant line bundle associated to the spin\(^c\) structure of \( X \).

Remark 2.1. The map \( \text{ind}_X \) coincides with the composite of the Thom isomorphism \( K^0(X) \to K^0(\text{TX}) \) associated to the spin\(^c\) structure of \( X \), and the topological index \( K^{cpt}_0(\text{TX}) \to \mathbb{Z} \) defined by Atiyah and I. M. Singer in \[7, 8\].

2.2. Spectral Flow. Spectral flow is, roughly speaking, the net number of crossing points of eigenvalues of the family of self-adjoint Fredholm operators with zero\(^3\). For a precise definition of the spectral flow, we find it convenient to follow the definition given by Phillips \[29\].

Definition 2.2 (Spectral Flow \[29\]). Let \( \mathcal{F}_{s.a.} \) be the space of self-adjoint Fredholm operators on a fixed separable complex Hilbert space with norm topology. Let \( B: [-\pi, \pi] \to \mathcal{F}_{s.a.} \) be a continuous path. We can choose a partition \( -\pi = t_0 < t_1 < \cdots < t_n = \pi \) and positive numbers \( c_1, c_2, \ldots, c_n \) so that for each \( i = 1, 2, \ldots, n \), the function \( t \mapsto \chi_{[-c_i, c_i]}(B_t) \) is continuous and finite rank on \([t_{i-1}, t_i] \), where \( \chi_{[a, b]} \) denotes the characteristic function of \([a, b] \). We define the spectral flow of \( B \) by

\[
\text{sf}(B) := \sum_{i=1}^{n} (\text{rank}_c(\chi_{[0, c_i]}(B_{t_i})) - \text{rank}_c(\chi_{[0, c_i]}(B_{t_{i-1}}))) \in \mathbb{Z}.
\]

Spectral flow is independent of the choice made and depends only on the homotopy class of the path \( B \) leaving the endpoints fixed.

3. Settings

Let \( \mathbb{T} \) be the unit circle in the complex plane. We denote \( t \) for an element of \( \mathbb{T} \). We fix the counter-clockwise orientation on the circle \( \mathbb{T} \). This oriented circle has a unique spin\(^c\) structure up to isomorphism. Let \( V \) be a finite dimensional Hermitian vector space. We denote \( N \) for the complex dimension of \( V \), and \( \| \cdot \|_V \) for its norm. Let \( \mathbb{L}^2(\mathbb{Z}; V) \) be the space of sequences \( \varphi = \{ \varphi_n \}_{n \in \mathbb{Z}} \) where each \( \varphi_n \) is an element of \( V \), and satisfies \( \sum_{n \in \mathbb{Z}} \| \varphi_n \|_V^2 < +\infty \). Let \( S: \mathbb{L}^2(\mathbb{Z}; V) \to \mathbb{L}^2(\mathbb{Z}; V) \) be the shift operator defined by \( (S\varphi)_n = \varphi_{n-1} \). Let \( A_j: \mathbb{T} \to \text{End}_c(V) \) be continuous maps which satisfies \( \sum_{j \in \mathbb{Z}} \| A_j \|_\infty \times \sum_{n \in \mathbb{Z}} \| A_n \|_V \) < +\infty, where \( \| A_j \|_\infty = \sup_{t \in \mathbb{T}} \| A_j(t) \|_V \). We consider a tight-binding Hamiltonian on the lattice \( \mathbb{Z} \times \mathbb{Z} \) with the periodic potential.

Definition 3.1 (Bulk Hamiltonian). For each \( t \) in \( \mathbb{T} \), we define a linear map

\[
H(t): \mathbb{L}^2(\mathbb{Z}; V) \to \mathbb{L}^2(\mathbb{Z}; V)
\]

\(^3\)The determinant line bundle \( L(X) \) associates to the principal spin\(^c\) bundle of the spin\(^c\) structure by the homomorphism \( \text{Spin}^{c}(2n) = (\text{Spin}(2n) \times \text{U}(1))/\{\pm 1\} \) \( \cong \{\lambda, z \mapsto z^2 \} \in \text{U}(1) \).

\(^4\)There exists different sign conventions in K-theory. If we use another conventions, the expression of \( \text{ind}_X \) can be different, for example. In this paper, we follow the one used in \[7\].

\(^5\)Counted positively for increasing eigenvalue crossings with respect to the parameter.
by \((H(t)\phi)_n = \sum_{j \in \mathbb{Z}} A_j(t) \phi_{n-j}\). We call \(H(t)\) a bulk Hamiltonian. \(H(t)\) is a bounded linear operator and commutes with the shift operator \(S\).

We assume that \(H(t)\) is a self-adjoint operator for any \(t\) in \(\mathbb{T}\). Thus its spectrum \(\sigma(H(t))\) is contained in \(\mathbb{R}\). We assume further that our bulk Hamiltonian has a spectral gap at the Fermi level \(\mu\) for any \(t\) in \(\mathbb{T}\), i.e., \(\mu\) is not contained in \(\sigma(H(t))\). We call this condition as a spectral gap condition. We take a positively oriented simple closed smooth loop \(\gamma\) through \(\mu\) in the complex plane, which satisfies that, for any \(t\) in \(\mathbb{T}\), the set \(\sigma(H(t)) \cap (\infty, \mu)\) is contained inside the loop \(\gamma\), and \(\sigma(H(t)) \cap (\mu, \infty)\) is contained outside \(\gamma\) (see Figure 1). This oriented loop has a unique spin\(^c\) structure up to isomorphism. Let \(S_{0}^1\) be the unit circle in the complex plane, and \(H(\eta, t) \in \text{End}_{\mathbb{C}}(V)\) be a continuous family of Hermitian operators on \(S_{0}^1 \times \mathbb{T}\) given by \(H(\eta, t) = \sum_{j \in \mathbb{Z}} A_j(t)\eta^j\). Since the bulk Hamiltonian commutes with the shift operator \(S\), by the Fourier transform \(L^2(\mathbb{Z}; V) \cong L^2(S_{0}^1; V)\), \(H(t)\) can be expressed as a multiplication operator \(M_{H(\eta, t)}\) on \(L^2(S_{0}^1; V)\) generated by \(H(\eta, t)\). Note that the shift operator \(S\) corresponds to \(M_{\eta}\) by the Fourier transform.\(^6\)

**Lemma 3.2.** For \((\eta, z, t) \in S_{0}^1 \times \gamma \times \mathbb{T}, \ H(\eta, t) - z\) is invertible.

**Remark 3.3.** As we will see later, taking this loop \(\gamma\) is substantial for the definition of the bulk index and the edge index. In this sense, we do not need to choose this loop in order to formulate the bulk-edge correspondence for our system. However, the choice of this loop is needed for the definition of Graf–Porta’s vector bundle.

### 4. Bulk Index, Edge Index and Bulk-Edge Correspondence

We define the bulk index and the edge index, and formulate our main theorem.

**Definition 4.1 (Bloch Bundle).** By the spectral gap condition, Riesz projections give a continuous family of projections on \(V\) parametrized by \(S_{0}^1 \times \mathbb{T}\), given by \(\frac{1}{2\pi i} \int_{\gamma} (1 - H(\eta, t))^{-1} d\lambda\). The images of this family makes a complex subvector bundle \(E_{B}\) of the product bundle \(V = (S_{0}^1 \times \mathbb{T}) \times V\). \(E_{B}\) is called the Bloch bundle.

The Bloch bundle defines an element \([E_{B}]\) of a \(K\)-group \(K^0(S_{0}^1 \times \mathbb{T})\). We fix a counter-clockwise orientation on \(S_{0}^1\). This oriented circle has a unique spin\(^c\) structure up to isomorphism. We consider the product spin\(^c\) structure on \(S_{0}^1 \times \mathbb{T}\). By using this spin\(^c\) structure, we have the map \(\text{ind}_{S_{0}^1 \times \mathbb{T}}: K^0(S_{0}^1 \times \mathbb{T}) \to \mathbb{Z}\).

**Definition 4.2 (Bulk Index).** We define the bulk index of our system by

\[
\mathcal{I}_{\text{Bulk}} := \text{ind}_{S_{0}^1 \times \mathbb{T}}([E_{B}]).
\]

**Remark 4.3.** Note that the determinant line bundle associated to this spin\(^c\) structure on \(S_{0}^1 \times \mathbb{T}\) is trivial. By using the Atiyah-Singer index formula \([8]\), it can be easily checked that our bulk index coincides with the first Chern number of the Bloch bundle.

\(^6\)Note that, Graf and Porta used in [17], the operator \(\tilde{S}\) defined by \((\tilde{S}\phi)_n = \varphi_{n+1}\) instead of our shift operator \(S\). Consequently, our choice of the orientation of \(S_{0}^1\) (in Sect. 3) is different from [17], and the definition of the bulk index is different by sign \(-1\). Although our choice of \(S\) causes such confusing differences with [17], we decided to use this \(S\) for the shift operator (and the orientation on \(S_{0}^1\)), since this choice is consistent with the one used in the field of Toeplitz operators [12], and we use a family index of a family of Toeplitz operators in the proof of Lemma 0.9.
For each $k \in \mathbb{Z}$, let $\mathbb{Z}_{\geq k} := \{k, k + 1, k + 2, \ldots \}$ be the set of all integers greater than or equal to $k$. We regard $l^2(\mathbb{Z}_{\geq k}; V)$ as a closed subspace of $l^2(\mathbb{Z}; V)$ in a natural way. For each $k \in \mathbb{Z}$, we denote $P_{\geq k}$ for the orthogonal projection of $l^2(\mathbb{Z}; V)$ onto $l^2(\mathbb{Z}_{\geq k}; V)$.

Definition 4.4 (Edge Hamiltonian). For each $t$ in $T$, we consider an operator $H^\#(t)$ given by the compression of $H(t)$ onto $l^2(\mathbb{Z}_{\geq 0}; V)$, that is,

$$H^\#(t) := P_{\geq 0}H(t)P_{\geq 0} : l^2(\mathbb{Z}_{\geq 0}; V) \to l^2(\mathbb{Z}_{\geq 0}; V).$$

We call $H^\#(t)$ an edge Hamiltonian.

$H^\#(t)$ is the Toeplitz (or the discrete Wiener-Hopf) operator with continuous symbol $H(q, t)$. Since $H(q, t)$ is Hermitian, $\{H^\#(t) - \mu\}_{t \in T}$ is a norm-continuous family of self-adjoint Fredholm operators on the Hilbert space $l^2(\mathbb{Z}_{\geq 0}; V)$.

Definition 4.5 (Edge Index). We define the edge index of our system as the spectral flow of the family $\{H^\#(t) - \mu\}_{t \in T}$.

$$\mathcal{I}_{\text{Edge}} := \text{sf}(\{H^\#(t) - \mu\}_{t \in T}).$$

We now revisit the definition of the spectral flow explained in Sect. 2.2. In order to count $\text{sf}(\{H^\#(t) - \mu\}_{t \in T})$, we need to choose $t_i$ and $c_i$. These data will be used to prove our main theorem, and we need to choose such data in the following specific way. We regard $\{H^\#(t) - \mu\}_{t \in T}$ as a family parametrized by $[-\pi, \pi]$, and consider...
the following path in $\mathbb{C} \times [-\pi, \pi]$ (or a loop in $\mathbb{C} \times \mathbb{T}$),

$$l := \bigcup_{i=0}^{n-1} I(\tilde{c}_i, \tilde{c}_{i+1}) \times \{t_i\} \cup \{\tilde{c}_{i+1}\} \times [t_i, t_{i+1}] \bigcup I(\tilde{c}_n, \tilde{c}_0) \times \{t_n\}$$

where $c_0 := 0$, and we denote $\tilde{c}_i := c_i + \mu$, for simplicity. We denote $I(a, b)$ for the closed interval in $\mathbb{R}$ between $a$ and $b$. The loop $l$ may have intersections with spectrums of edge Hamiltonians. If we choose $t_i$ and $c_i$ as in Sect. 2.2, crossing points appear only at intervals of the form $I(\tilde{c}_i, \tilde{c}_{i+1}) \times \{t_i\}$. We take $t_i$ and $c_i$ so that crossing points appear only at open intervals of the form $I^o(\tilde{c}_i, \tilde{c}_{i+1}) \times \{t_i\}$ ($i = 1, \ldots, n - 1$), where $I^o(a, b)$ denotes for the open interval in $\mathbb{R}$ between $a$ and $b$, or at half-open intervals $[\tilde{c}_0, \tilde{c}_1) \times \{t_0\}$ and $[\tilde{c}_n, \tilde{c}_0) \times \{t_n\}$

We see that, when we consider $\text{sf}((H^H(t) - \mu)_{t \in \mathbb{T}})$, we do not need to consider about crossing points at $\{\tilde{c}_0\} \times \{t_0\}$ and $\{\tilde{c}_0\} \times \{t_n\}$. This is because crossing points at these points, if exist, do not contribute to the spectral flow (actually, they cancel out, in our case). Thus, we assume that there are no crossing points at these points, for simplicity. Then, $\text{sf}((H^H(t) - \mu)_{t \in \mathbb{T}})$ is the net number of crossing points $\{u_1, \ldots, u_m\}$ counted with multiplicity, where crossing points in $I(\tilde{c}_i, \tilde{c}_{i+1}) \times \{t_i\}$ with $c_i > c_{i+1}$ are counted with positive sign, and that of $c_i < c_{i+1}$ are counted with negative sign. We fix an orientation on $l$ which is compatible with the natural one on the interval $[\tilde{c}_{i+1}] \times [t_i, t_{i+1}]$ (see Figure 2).

Remark 4.6. At each crossing point, there exist edge states which mean eigenvectors of our edge Hamiltonian of eigenvalue $\mu$. In this sense, the edge index is defined by counting edge states.

Our main theorem, the bulk-edge correspondence for two-dimensional type A topological insulators and topological superconductors, is the following:

**Theorem 4.7.** The bulk index coincides with the edge index. That is,

$$I_{\text{Bulk}} = I_{\text{Edge}}.$$

To give a proof of this theorem is the purpose of the rest of this paper.

5. Graf–Porta’s Vector Bundle

In this section, we define a vector bundle over $\gamma \times \mathbb{T}$, which is a generalization of the one considered by Graf and Porta in [17].

**Lemma 5.1.** There exists a positive integer $K$ such that for any integer $k \geq K$ and $(z, t)$ in $\gamma \times \mathbb{T}$, the following map is surjective

$$P_{\geq k}(H(t) - z)P_{\geq 0}: \ell^2(\mathbb{Z}_{\geq 0}; V) \to \ell^2(\mathbb{Z}_{\geq k}; V).$$

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7 The existence of $t_i$ and $c_i$ which satisfies these conditions follows by the argument in [29].
8 Compared with the original bulk-edge correspondence by Hatugai [18], our bulk-edge correspondence states that the minus of the Hall conductance of an infinite system coincides with the minus of the Hall conductance of a system with edge. Therefore, from the point of view of the original bulk-edge correspondence, we should call the minus of $I_{\text{Bulk}}$ (resp. $I_{\text{Edge}}$) as the bulk (resp. the edge) index, as in [17]. However, we decided not to do so, in order to avoid extra minus signs.
Proof: For a non-negative integer $k$, we define a linear map $H_k(t): l^2(\mathbb{Z}; V) \to l^2(\mathbb{Z}; V)$ by $(H_k(t)\varphi)_n = \sum_{j=-k}^k A_j(t)\varphi_{n-j}$. Let $s_k := H_k(\eta, t) - z$, then $H_k(t) - z = M_{s_k}$. We first show that there exists some integer $K'$ such that for any $k \geq K'$ and for any $(z, t) \in \gamma \times \mathbb{T}$, the operator $P_{\geq k}(H_k(t) - z)P_{\geq 0}: l^2(\mathbb{Z}_{\geq 0}; V) \to l^2(\mathbb{Z}_{\geq k}; V)$ is surjective.

Note that $||(H(\eta, t) - z) - (H_k(\eta, t) - z)|| \leq \sum_{|j| > k} ||A_j||_{\infty} \to 0$ as $k \to \infty$.

Since $H(\eta, t) - z$ is invertible by Lemma \ref{lem:invertibility}, there exists a positive integer $K'$ such that for any $k \geq K'$ and $(\eta, z, t)$ in $S^1_{\eta} \times \gamma \times \mathbb{T}$, $H_k(\eta, t) - z$ is invertible. Note that $SP_{\geq k} = P_{\geq k+1}S$, and we have $P_{\geq 0} = (S^*)^kP_{\geq k}$. Since $s_k\eta^{-k}$ does not contain the term of $\eta^m$ ($m > 0$), $(s_k\eta^{-k})^{-1}$ does not contain the term of $\eta^m$ ($m < 0$), and so $M(s_k\eta^{-k})^{-1}$ maps $l^2(\mathbb{Z}_{\geq k}; V)$ to $l^2(\mathbb{Z}_{\geq k}; V)$. Thus for $\varphi \in l^2(\mathbb{Z}_{\geq k}; V)$,

$$(P_{\geq k}M_{s_k}P_{\geq 0})(P_{\geq 0}M_{s_k^{-1}}P_{\geq k})(\varphi) = P_{\geq k}M_{s_k}(S^*)^kP_{\geq k}M_{s_k^{-1}}(\varphi) = P_{\geq k}M_{s_k}\eta^{-k}P_{\geq k}M(\eta^{-k})^{-1}(\varphi) = P_{\geq k}M_{s_k}\eta^{-k}M(s_k\eta^{-k})^{-1}(\varphi) = \varphi.$$ 

Thus $P_{\geq k}M_{s_k}P_{\geq 0}$ has a right inverse, and it is surjective.9

As is shown above, the composite of $P_{\geq 0}M_{s_k^{-1}}P_{\geq k}$ and $P_{\geq k}M_{s_k}P_{\geq 0}$ is the identity map. Thus $P_{\geq 0}M_{s_k^{-1}}P_{\geq k}$ is bounded below, and so closed range. We also have

$$\text{Im}(P_{\geq 0}M_{s_k^{-1}}P_{\geq k}) \cap \text{Ker}(P_{\geq k}M_{s_k}P_{\geq 0}) = \{0\},$$

and

$$\text{Im}(P_{\geq 0}M_{s_k^{-1}}P_{\geq k}) \oplus \text{Ker}(P_{\geq k}M_{s_k}P_{\geq 0}) = l^2(\mathbb{Z}_{\geq k}; V).$$

Let $X_k := \text{Im}(P_{\geq 0}M_{s_k^{-1}}P_{\geq k})$, and denote $P_{X_k}$ for the orthogonal projection onto $X_k$. Then by the open mapping theorem, $P_{\geq k}M_{s_k}P_{X_k}: X_k \to l^2(\mathbb{Z}_{\geq k}; V)$ is invertible. On the other hand, we have $P_{\geq 0}M_{s_k^{-1}}P_{\geq k} = P_{X_k}M_{s_k^{-1}}P_{\geq k}$, and

$$1 = (P_{\geq k}M_{s_k}P_{\geq 0})(P_{\geq 0}M_{s_k^{-1}}P_{\geq k}) = (P_{\geq k}M_{s_k}P_{X_k})(P_{X_k}M_{s_k^{-1}}P_{\geq k}).$$

Therefore, the inverse of $P_{\geq k}M_{s_k}P_{X_k}$ is $P_{X_k}M_{s_k^{-1}}P_{\geq k}$.

Since $s_k$ converges to $H(\eta, t) - z$ as $k \to \infty$ uniformly with respect to $(\eta, z, t)$ in $S^1_{\eta} \times \gamma \times \mathbb{T}$, the operator $M_{s_k}$ converges to $H(t) - z$ uniformly, and so $M_{s_k^{-1}}$ converges to $(H(t) - z)^{-1}$ uniformly with respect to $(z, t)$. Thus there exist some $L > 0$ and an integer $K''$ such that for any $k \geq K''$ and $(z, t) \in \gamma \times \mathbb{T}$, we have $||M_{s_k^{-1}}|| < L$. For $k \geq K''$, we have $||(P_{\geq k}M_{s_k}P_{X_k})^{-1}||^{-1} > ||M_{s_k^{-1}}||^{-1} > L^{-1} > 0$. On the other hand, we have

$$||P_{\geq k}(H(t) - z)P_{X_k} - P_{\geq k}M_{s_k}P_{X_k}|| \leq ||H(t) - H_k(t)|| \leq \sum_{|j| > k} ||A_j||_{\infty} \to 0.$$

Thus there exists an positive integer $K$ such that for any $k \geq K$ and $(z, t)$ in $\gamma \times \mathbb{T}$, we have

$$||P_{\geq k}(H(t) - z)P_{X_k} - P_{\geq k}M_{s_k}P_{X_k}|| \leq ||(P_{\geq k}M_{s_k}P_{X_k})^{-1}||^{-1},$$

\footnote{9Such argument was used by Atiyah and Bott to compute the family index of a family of Toeplitz operators \cite{Atiyah1983, Bott1983}.}
which means that $P_{\geq k}(H(t) - z)P_{\geq k}$ is invertible. Thus $P_{\geq k}(H(t) - z)P_{\geq 0}$ is surjective for such $k \geq K$ and $(z, t)$ in $\gamma \times T$.

We choose such an integer $k \geq K$, and denote

$$(E_{\text{GP}})_{z,t} := \text{Ker}(P_{\geq k}(H(t) - z)P_{\geq 0}).$$

Remark 5.2. For $n \geq k$, we have

$$(\langle P_{\geq k}(H(t) - z)P_{\geq 0} \rangle_n = (\langle H(t) - z \rangle(\phi))_n.$$ 

Thus $(E_{\text{GP}})_{z,t}$ consists of sequences $\phi = \{\phi_n\}_{n \geq 0}$ in $I^2(\mathbb{Z}_{\geq 0}; V)$ which satisfies $(H(t)(\phi))_n = z\phi_n$ for $n \geq k$.

We define an operator $H'(z, t): I^2(\mathbb{Z}_{\geq 0}; V) \to I^2(\mathbb{Z}_{\geq 0}; V)$ to be the composite of $P_{\geq k}(H(t) - z)P_{\geq 0}$ and the inclusion $I^2(\mathbb{Z}_{\geq 0}; V) \hookrightarrow I^2(\mathbb{Z}_{\geq 0}; V)$.

Lemma 5.3. For any $(z, t)$ in $\gamma \times T$, $H'(z, t)$ is a Fredholm operator whose Fredholm index is zero. Moreover, Ker $H'(z, t)$ and $(E_{\text{GP}})_{z,t}$ are the same, and Coker $H'(z, t)$ is naturally isomorphic to $V^\oplus k$. Thus the dimension of $(E_{\text{GP}})_{z,t}$ is $kN$ which is constant with respect to the parameter.

Proof. For $(z, t) \in \gamma \times T$, $(H'(t) - z) - H'(z, t)$ is a finite rank operator, and $H'(t) - z$ is a Fredholm operator. Thus, $H'(z, t)$ also is a Fredholm operator, and their Fredholm indices are the same. Since $H'(t) - z$ can be connected to a self-adjoint Fredholm operator $H'(t) - \mu$ by a continuous path of Fredholm operators, the homotopy invariance of the Fredholm index says that the Fredholm index of $H'(t) - z$ is zero. Thus the Fredholm index of $H'(z, t)$ is zero, and so the dimensions of its kernel and cokernel are the same. On the other hand, Ker $H'(z, t)$ coincides with $(E_{\text{GP}})_{z,t}$ by definition. By Lemma 5.1, Coker $H'(z, t)$ is naturally isomorphic to the closed subspace of $I^2(\mathbb{Z}_{\geq 0}; V)$ which consists of the sequences with $\phi_n = 0$ for $n \geq k$, which is naturally isomorphic to $V^\oplus k$. \hfill $\Box$

Definition 5.4. By Lemma 5.3, we have a vector bundle $E_{\text{GP}}$ over $\gamma \times T$ whose fiber at a point $(z, t)$ is given by $(E_{\text{GP}})_{z,t}$. The rank of this vector bundle is $kN$. We call this vector bundle as Graf–Porta’s vector bundle.

Note that the family index of a family of Fredholm operators $\{H'(z, t)\}_{(z, t) \in \gamma \times T}$ is an element of $K$-group $K^0(\gamma \times T)$, and is given by $[E_{\text{GP}}] - [V^\oplus k]$. Thus Graf–Porta’s vector bundle appears as a family index. As in the definition of the bulk index, we consider the product spin\(^c\) structure on $\gamma \times T$. By using this spin\(^c\) structure, we have the map $\text{ind}_{\gamma \times T}: K^0(\gamma \times T) \to \mathbb{Z}$.

Definition 5.5 (Graf–Porta [17]). We define Graf–Porta’s index by

$$I_{\text{GP}} := \text{ind}_{\gamma \times T}(E_{\text{GP}}) - [V^\oplus k]).$$

Remark 5.6. The conclusion of Lemma 5.3 also holds for $(z, t) \in \mathbb{C} \times T$ which satisfies $z \not\in \sigma(H(t))$. Thus we can define a vector bundle in the same way on the space $\{(z, t) \in \mathbb{C} \times T \mid z \not\in \sigma(H(t))\}$. We denote this vector bundle as $E_{\text{GP}}$. Then we have $E_{\text{GP}} = E_{\text{GP}}|_{\gamma \times T}$. Such extension is used in Sect. 6.2.

Remark 5.7. The determinant line bundle associated to the spin\(^c\) structure on $\gamma \times T$ is trivial. Thus, as in Remark 4.3, we can show that $\text{ind}_{\gamma \times T}(V^\oplus k)) = 0$, and $I_{\text{GP}}$ coincides with the first Chern number of the bundle $E_{\text{GP}}$. 


Remark 5.8. If, for some \( k \), we have \( A_j = 0 \) for \( j < -k \) and \( k < j \), and if \( A_{-k} \) and \( A_k \) are invertible, then the difference equation \( H(t) \phi = z \phi \) is solvable by an initial condition. In this case, each fiber \((E_{GP})_{z,t}\) can be identified with the space of formal solutions of the equation \( H(t) \psi = z \psi \) which decays as \( n \to +\infty \). Graf and Porta considered these spaces to define \( E_{GP} \) in [17].

6. Proof of the Bulk-Edge Correspondence

In this section, we prove Theorem 4.7 by first showing that the bulk index coincides with Graf–Porta’s index (Sect. 6.1) and next showing that Graf–Porta’s index coincides with the edge index (Sect. 6.2).

6.1. Bulk index and Graf–Porta’s index. We show the bulk index coincides with Graf–Porta’s index. The key ingredient is the cobordism invariance of the index.

**Proposition 6.1.** \( I_{\text{Bulk}} = I_{GP} \).

Let \( D^2_\eta \) be the closed unit disk in the complex plane whose boundary is \( S^1_\eta \), and let \( D^2_\gamma \) be the closed domain of the complex plane surrounded by \( \gamma \). We fix, on \( D^2_\eta \) and \( D^2_\gamma \), spin\(^c\) structures naturally induced by the spin\(^c\) structure of the complex structure of the complex plane. Let \( X := (D^2_\eta \times \gamma) \times T \) and \( Y := \partial X = S^1_\eta \times (D^2_\gamma \setminus \gamma) \times T \) and \( Y := \partial X = S^1_\eta \times (D^2_\gamma \setminus \gamma) \times T \). We consider the product spin\(^c\) structure on \( X \) and the boundary spin\(^c\) structure on \( Y \). We take an extension \( f = f(\eta, z, t) : X \to \text{End}_Z(V) \) of \( H(\eta, t) - z \), which is defined on \( S^1_\eta \times \gamma \times T \). We first see that each element maps, through Thom isomorphisms and push-forward maps, to the bulk index (Lemma 6.2) and Graf–Porta’s index (Lemma 6.3). We then prove Proposition 6.1 by using the cobordism invariance of the index (Lemma 6.4).

We denote \( F \) for the composite of the inverse of the Thom isomorphism \( \beta^{-1}_z : K^0_{\text{cpt}}(S^1_\eta \times (D^2_\gamma \setminus \gamma) \times T) \to K^0(S^1_\gamma \times T) \) and \( \text{ind}_{r \times T} : K^0(S^1_\gamma \times T) \to \mathbb{Z} \). The relation between \( \alpha_z \) and \( I_{\text{Bulk}} \) can be stated as follows, which follows easily from Atiyah–Bott’s proof of the Bott periodicity theorem [3].

**Lemma 6.2.** \( \beta^{-1}_z(\alpha_z) = [E_B] \). Thus we have \( F(\alpha_z) = I_{\text{Bulk}} \).

We denote \( F' \) for the composite of the inverse of the Thom isomorphism \( \beta^{-1}_\eta : K^0_{\text{cpt}}((D^2_\eta \setminus S^1_\eta) \times \gamma \times T) \to K^0(\gamma \times T) \) and \( \text{ind}_{r \times T} : K^0(\gamma \times T) \to \mathbb{Z} \). The relation between \( \alpha_\eta \) and \( I_{GP} \) can be stated as follows.

**Lemma 6.3.** \( \beta^{-1}_\eta(\alpha_\eta) = -[E_{GP}] + [V^\oplus k] \). Thus we have \( F'(\alpha_\eta) = -I_{GP} \).

**Proof.** We have the following commutative diagram (see Sect. 7 of [0]),

\[
\begin{array}{ccc}
K^{-1}(S^1_\eta \times \gamma \times T) & \xrightarrow{\partial} & K^0(D^2_\eta \times \gamma \times T, S^1_\eta \times \gamma \times T) \\
\tau \downarrow & & \downarrow -\beta^{-1}_\eta \\
[\gamma \times T, \text{Fred}(I^2(\mathbb{Z}_\eta))] & \xrightarrow{\text{index}} & K^0(\gamma \times T)
\end{array}
\]
where $\partial$ is a boundary map of the long exact sequence for the pair $(S^1_\eta \times \mathbb{D}^2 \times T, S^1_\eta \times \gamma \times T)$, and a map $T$ from $K^{-1}(S^1_\eta \times \gamma \times T)$ to $[\gamma \times T, \text{Fred}(l^2(\mathbb{Z}_{\geq 0}))]$ is given by taking a family of Toeplitz operators. Note that $-\beta_\eta$ is given by taking a cup product with the element $[\mathbb{C}^2] \in K^0_c(\mathbb{D}^2 \setminus S^1_\eta)$. By Lemma 5.2, we have an element $[H(\eta, t) - z] \in K^{-1}(S^1_\eta \times \gamma \times T)$. This element maps to $\alpha_{\eta} \in K^0_c((\mathbb{D}^2_\eta \setminus S^1_\eta) \times \gamma \times T)$ by the boundary map $\partial$. On the other hand, we have $T([H(\eta, t) - z]) = \{H^\#(t) - z\}_{\gamma \times T}$. Now since $H^\#(z, t)$ is a finite rank perturbation of $H^\#(t) - z$, we have $\{H^\#(z, t)\}_{\gamma \times T}$ as an element of $[\gamma \times T, \text{Fred}(H)]$, and by Lemma 3.3 we have index$(\{H^\#(z, t)\}_{\gamma \times T}) = [E_{GP}] - [Y^{\oplus k}]$. By the commutativity of the above diagram, we have $-\beta_\eta^{-1}(\alpha_{\eta}) = [E_{GP}] - [Y^{\oplus k}]$. □

**Proof of Proposition 6.1.** By Lemma 5.2, the endomorphism $f$ defines the following element $\alpha$ of a compactly supported $K$-group, 

$$\alpha := [\mathcal{V}, \mathcal{V}; f(\eta, z, t)] \in K^0_c(X).$$

$\alpha$ gives a cobordism between $\alpha_\eta$ and $\alpha_{\eta}$, that is, let $i: Y \hookrightarrow X$ be the inclusion, then we have $i^*(\alpha) = (\alpha_\eta, \alpha_{\eta})$. By Lemma 5.2 and 5.3 we have $(F \oplus F')(\alpha_\eta, \alpha_{\eta}) = \mathcal{I}_{\text{Bulk}} - \mathcal{I}_{\text{GP}}$. Thus the following Lemma is suffic to prove Proposition 6.1.

**Lemma 6.4.** $(F \oplus F')(\alpha_\eta, \alpha_{\eta}) = 0$.

**Proof.** By the Whitney embedding theorem, there exists a neat embedding $E: X \rightarrow \mathbb{R}^M \times [0, \infty)$ for sufficiently large even integer $M$, such that the boundary $Y$ of $X$ maps to $\mathbb{R}^M \times \{0\}^1\). We fix a spin$^c$ structure on $\mathbb{R}^M \times [0, \infty)$. Then a normal bundle of this embedding has a naturally induced spin$^c$ structure. Let $i': \mathbb{R}^M \rightarrow \mathbb{R}^M \times [0, \infty)$ be an inclusion given by $i'(x) = (x, 0)$. We have the following commutative diagram.

$$
\begin{array}{c}
K^0_c(\mathbb{R}^M \times \{0, \infty\}) & \xrightarrow{i^*} & K^0_c(\mathbb{R}^M) \\
\downarrow{E_i} & \downarrow{(E|Y)} & \\
K^0_c(X) & \xrightarrow{i^*} & K^0_c(Y) \\
\end{array}
$$

where $\beta$ is the Thom isomorphism. We have $i^*(\alpha) = (\alpha_\eta, \alpha_{\eta})$ and $K^0_c(\mathbb{R}^M \times \{0, \infty\}) \cong K^0_c(\mathbb{R}^M + 1) = \{0\}$, where $\mathbb{R}^M + 1$ is the $(M + 1)$-dimensional closed ball which is contractible. By the commutativity of this diagram we have $(F \oplus F')(\alpha_\eta, \alpha_{\eta}) = 0$. □

6.2. Graf–Porta’s index and the edge index. We next show Graf–Porta’s index coincides with the edge index. In order to prove this, we use a localization argument for a $K$-class. We localize the support of a $K$-class near the crossing points of the spectrums of edge Hamiltonians and the loop $l$ which we take in Sect. 11. We then use the excision property of the index.

**Proposition 6.5.** $\mathcal{I}_{GP} = \mathcal{I}_{\text{Edge}}$.

---

10For the proof of the Whitney embedding theorem for an embedding of a non-compact smooth manifold with boundary into a half-space, see [1][20]. for example. It is also easy to construct such embedding explicitly for our space $X$. 

---
We identify $V^\oplus k$ with the image of the projection $1 - P_{\geq k}$, that is, the closed subspace of $l^2(\mathbb{Z}_{\geq 0}; V)$ consists of sequences $\varphi = \{\varphi_n\}$ which satisfies $\varphi_n = 0$ for $n \geq k$. For $(z, t)$ in $\gamma \times \mathbb{T}$, we consider the following map,

$$g_{z, t} := (H^\#(t) - z)|_{(E_{GP})_{z, t}}: (E_{GP})_{z, t} \rightarrow V^\oplus k.$$ 

By the definition of $(E_{GP})_{z, t}$, the space $(H^\#(t) - z)((E_{GP})_{z, t})$ is contained in $V^\oplus k$. Therefore we have a bundle homomorphism $g: \tilde{E}_{GP}|_{\mathbb{T}} \rightarrow V^\oplus k$.

**Lemma 6.6.** For $(z, t) \in \gamma \times \mathbb{T}$, we have, $\text{Ker} g_{z, t} = \text{Ker}(H^\#(t) - z)$.

**Proof.** $\varphi$ is an element of $\text{Ker} g_{z, t}$ if and only if $\varphi$ is an element of $l^2(\mathbb{Z}_{\geq 0}; V)$ which satisfies $(H(t)\varphi)_n = z\varphi_n$ for $n \geq k$ (see Remark 5.2), and $(H(t)\varphi)_n = z\varphi_n$ for $0 \leq n \leq k - 1$ (since $\varphi$ is an element of $\text{Ker} g_{z, t}$). This is equivalent to saying that $\varphi$ is an element of $\text{Ker}(H^\#(t) - z)$. \hfill \Box

**Remark 6.7.** If there is no edge states, then $g$ is a bundle isomorphism, and we have a trivialization of the bundle $E_{GP}$.

We now deform the torus $\gamma \times \mathbb{T}$, so that, instead of the loop $\{\mu\} \times \mathbb{T}$, the deformed torus intersects $\mathbb{R} \times \mathbb{T} \subset \mathbb{C} \times \mathbb{T}$ on the loop $\hat{t}$ which we take in Sect. 4 and that, at each crossing point, a neighborhood of the point in the deformed torus is contained in $\mathbb{C} \times \{t\}$ for some $t$ in $\mathbb{T}$ (see Figure 2). We denote this deformed torus by $\mathbb{T}$. Since $\hat{t}$ is contained in the resolvent set of bulk Hamiltonians, such $\mathbb{T}$ is contained in the set $\{(z, t) \in \mathbb{C} \times \mathbb{T} \mid z \notin \sigma(H(t))\}$. Thus we can consider the restriction of $\tilde{E}_{GP}$ onto $\mathbb{T}$ (see Remark 5.2), $\mathbb{T}$ can be deformed continuously to $\gamma \times \mathbb{T}$ (see Figure 2, and consider the “crushing the ledge” map). By using this deformation, $\mathbb{T}$ has a spin$^c$ structure naturally induced by that of $\gamma \times \mathbb{T}$. By using this spin$^c$ structure, we have the map $\text{ind}_{\mathbb{T}}: K^0(\mathbb{T}) \rightarrow \mathbb{Z}$.

**Lemma 6.8.** $\mathcal{I}_{GP} = \text{ind}_{\mathbb{T}}([\tilde{E}_{GP}|_{\mathbb{T}}, V^\oplus k; g])$.

**Proof.** By the naturality of the index, the integer $\text{ind}_{\gamma \times \mathbb{T}}([E_{GP}] - [V^\oplus k])$ coincides with $\text{ind}_{\mathbb{T}}([\tilde{E}_{GP}|_{\mathbb{T}}] - [V^\oplus k])$. Thus, we have

$$\mathcal{I}_{GP} = \text{ind}_{\mathbb{T}}([\tilde{E}_{GP}|_{\mathbb{T}}] - [V^\oplus k]) = \text{ind}_{\mathbb{T}}([\tilde{E}_{GP}|_{\mathbb{T}}, V^\oplus k; 0]) = \text{ind}_{\mathbb{T}}([\tilde{E}_{GP}|_{\mathbb{T}}, V^\oplus k; g]).$$ \hfill \Box

By Lemma 6.6, the support of the bundle homomorphism $g: \tilde{E}_{GP}|_{\mathbb{T}} \rightarrow V^\oplus k$ is contained in the spectrum of edge Hamiltonians, and coincides with the set of crossing points $\{u_1, \ldots, u_m\}$. For each crossing point $u_a$, we take an open disk neighborhood $U_a$ of $u_a$ in $\mathbb{T}$ which is also contained in $\mathbb{C} \times \{t\}$ for some $t$ in $\mathbb{T}$ (remember our choice of $t_i$ and $c_i$ in Sect. 4). We take $U_a$ small enough so that $U_a$ does not intersect one another. Then we have following elements of compactly supported $K$-groups,

$$[\tilde{E}_{GP}|_{U_a}, V^\oplus k; g] \in K^0_{\text{cpt}}(U_a), \quad a = 1, \ldots, m.$$ 

For each $a$, we consider a spin$^c$ structure on $U_a$ which is the restriction of the spin$^c$ structure of $\mathbb{T}$. By using this spin$^c$ structure, we have a homomorphism $\text{ind}_{U_a, \mathbb{T}}: K^0_{\text{cpt}}(U_a) \rightarrow \mathbb{Z}$. We later consider another spin$^c$ structure on $U_a$, so we
write subscript \((U_a, \tilde{T})\) in order to indicate which spin\(^c\) structure is used. By the excision property of the index, we have,

\[
\text{ind}_{\tilde{T}} ([\tilde{E}_{GP}|_{U_a}, V^{\oplus k}; g]) = \sum_{a=1}^{m} \text{ind}_{U_a, \tilde{T}} ([E_{GP}|_{U_a}, V^{\oplus k}; g]).
\]

Figure 2. Eigenvalues which cross the Fermi level \(\mu\), and the deformed torus \(\tilde{T}\).

In order to prove Proposition 6.5, it is enough to show the following Lemma.

**Lemma 6.9.** For each crossing point \(u_a\), the following holds.

(I) If \(u_a\) is contained in the interval \(I(c_i, c_{i+1}) \times \{t_i\}\) where \(c_i > c_{i+1}\) and if its multiplicity is \(r_a\), then \(\text{ind}_{U_a, \tilde{T}} ([E_{GP}|_{U_a}, V^{\oplus k}; g]) = +r_a\).

(II) If \(u_a\) is contained in the interval \(I(c_i, c_{i+1}) \times \{t_i\}\) where \(c_i < c_{i+1}\) and if its multiplicity is \(r_a\), then \(\text{ind}_{U_a, \tilde{T}} ([E_{GP}|_{U_a}, V^{\oplus k}; g]) = -r_a\).

In order to prove Lemma 6.9, we need some Lemmas. We now consider on \(U_a\) a complex structure induced by the inclusion \(U_a \subset C \times \{t\} \cong C\).

**Lemma 6.10.** \(E_{GP}|_{U_a} \rightarrow U_a\) is a holomorphic vector bundle.

**Proof.** Let us consider the following operator,

\[
D(z, t) := \begin{pmatrix} 0 & H^1(z, t)^* \\ H^0(z, t) & 0 \end{pmatrix}.
\]

\(D(z, t)\) is a bounded linear self-adjoint operator on \(l^2(\mathbb{Z}_{\geq 0}; V) \oplus l^2(\mathbb{Z}_{\geq 0}; V)\). We have \(\text{Ker } D(z, t) = \text{Ker } H^0(z, t) \oplus \text{Ker } H^1(z, t)^* \cong (E_{GP})_{z, t} \oplus V^{\oplus k}\). By Lemma 5.3, the rank of \(\text{Ker } D(z, t)\) is constant on \(U_a\). By considering the spectral decomposition of the self-adjoint operator \(D(z, t)\), it is easy to see that we can take a positively oriented smooth simple closed curve \(C\) in \(C\), which does not intersects with \(\sigma(D(z, t))\) and contains just the zero eigenvalue inside \(C\) for any \((z, t) \in U_a\). Then Riesz projections give a holomorphic family of projections \(p(z, t) := \frac{1}{2\pi i} \int_C (\lambda I - D(z, t))^{-1} d\lambda\).
is also holomorphic. We take a
have two induced spin
orientations are the same, and in the case (II), they are the oppo
In the case where the assumption of (I) holds, these two
Proof of Lemma 6.9. Since \( \tilde{E}_{GP}|_{U_a} \) is a holomorphic vector bundle, \( g: \tilde{E}_{GP}|_{U_a} \rightarrow \mathbb{C}^{\oplus k} \) is also holomorphic. We take a holomorphic trivialization of \( \det \tilde{E}_{GP}|_{U_a} \), then \( \det g \) can be regarded as a holomorphic map on \( U_a \) which is zero at \( u_a \) whose order is \( r_a \). By considering the Taylor series, \( \det g \) can be expressed as \( z^{r_a}h(z) \) where \( h(z) \) is a nowhere vanishing function on \( U_a \). Thus we have,

\[
[\tilde{E}_{GP}|_{U_a}, \mathbb{C}^{\oplus k}; g] = [\det \tilde{E}_{GP}|_{U_a}, \mathbb{C}; \det g] = [\mathbb{C}, \mathbb{C}; z^{r_a}h(z)] = [\mathbb{C}, \mathbb{C}; z^{r_a}].
\]

We remind the reader that \( U_a \) is considered as a subspace in two different ways. One is as a subspace of \( \mathbb{C} \cong \mathbb{C} \times \{t\} \), and the other is as a subspace of \( \mathbb{T} \). Thus we have two induced spin\(^c\) structures (and orientations) on \( U_a \), which can be different.

Proof of Lemma 6.11. Let \( z \) be the complex coordinate of \( U_a \), where \( U_a \) is considered as a subspace of \( \mathbb{C} \). Then \( [\tilde{E}_{GP}|_{U_a}, \mathbb{C}^{\oplus k}; g] \), the element of the K-group \( K^0_{cpt}(U_a) \), can be expressed as, \( [\tilde{E}_{GP}|_{U_a}, \mathbb{C}^{\oplus k}; g] = [\mathbb{C}, \mathbb{C}; z^{r_a}] \).

Proof of Proposition 6.5. In the case where the assumption of (I) holds, these two orientations are the same, and in the case (II), they are the opposite. Thus, by Lemma 6.11 we have

\[
\text{ind}_{U_a, \mathbb{T}}([\tilde{E}_{GP}|_{U_a}, \mathbb{C}^{\oplus k}; g]) = \begin{cases} 
+ r_a, & \text{if (I) holds.} \\
- r_a, & \text{if (II) holds.}
\end{cases}
\]

Proof of Proposition 6.9. By Lemma 6.5 and the definition of the spectral flow given at Sect. 2.2, we have \( \sum_{\tau=1}^{m} \text{ind}_{U_a, \mathbb{T}}([\tilde{E}_{GP}|_{U_a}, \mathbb{C}^{\oplus k}; g]) = \text{sf}(H^\#(t) - \mu) \). Thus by Lemma 6.9 and the equation (1), we have

\[
\mathcal{I}_{GP} = \text{ind}_{\mathbb{T}}([\tilde{E}_{GP}|_{\mathbb{T}}, \mathbb{C}^{\oplus k}; g]) = \sum_{a=1}^{m} \text{ind}_{U_a, \mathbb{T}}([\tilde{E}_{GP}|_{U_a}, \mathbb{C}^{\oplus k}; g])
\]

\[
= \text{sf}(H^\#(t) - \mu) = \mathcal{I}_{\text{Edge}}.
\]

This completes the proof of Theorem 4.7.

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