Arbitrary high-order structure-preserving methods for the quantum Zakharov system

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Abstract
In this paper, we present a new methodology to develop arbitrary high-order structure-preserving methods for solving the quantum Zakharov system. The key ingredients of our method are as follows: (i) the original Hamiltonian energy is reformulated into a quadratic form by introducing a new quadratic auxiliary variable; (ii) based on the energy variational principle, the original system is then rewritten into a new equivalent system which inherits the mass conservation law and a quadratic energy; and (iii) the resulting system is discretized by symplectic Runge–Kutta method in time combining with the Fourier pseudo-spectral method in space. The proposed method achieves arbitrary high-order accurate in time in a periodic domain and can exactly preserve the discrete mass and original Hamiltonian energy. Moreover, an efficient iterative solver is presented to solve the resulting discrete nonlinear equations. Finally, ample numerical examples are presented to demonstrate the theoretical claims and illustrate the efficiency of our methods.

Keywords Quantum Zakharov system · Symplectic Runge–Kutta method · Structure-preserving method · Quadratic auxiliary variable approach

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1 Introduction

The quantum Zakharov system \([20, 27]\) is widely used to describe the nonlinear interaction between the Langmuir waves and the ion-acoustic waves. In this paper, we consider the following quantum Zakharov system (QZS):

\[
\begin{align*}
    i \partial_t E(x,t) + \Delta E(x,t) - \varepsilon^2 \Delta^2 E(x,t) &= N(x,t) E(x,t), \\
    \partial_{tt} N(x,t) - \Delta N(x,t) + \varepsilon^2 \Delta^2 N(x,t) &= \Delta |E(x,t)|^2, \quad x \in \Omega, \quad 0 < t \leq T,
\end{align*}
\]

(1.1)

where \(i = \sqrt{-1}\) is the complex unit, \(t\) is the time variable, \(x\) is the spatial variable, the complex-valued function \(E := E(x,t)\) denotes the slowly varying envelope of the rapidly oscillatory electric field, the real-valued function \(N := N(x,t)\) represents the low-frequency variation of the density of the ions, \(\Delta\) is the usual Laplace operator, the quantum effect \(\varepsilon > 0\) is the ratio of the ion plasma and the temperature of electrons, and \(E_0(x), N_0(x)\) and \(N_1(x)\) are given initial conditions. In what follows, the QZS (1.1) will be supplemented by periodic boundary conditions, and \(N_1(x)\) satisfies the following compatibility condition \([8, 22]\):

\[
\int_{\Omega} N_1(x) dx = 0.
\]

In the special case \(\varepsilon = 0\), it reduces to the classical Zakharov system (ZS), which have been widely applied to various physical problems, such as plasmas \([25, 56]\), hydrodynamics \([15]\), the theory of molecular chains \([14]\), and so on. When either the electrons temperature is low or the ion-plasma frequency is high, the quantum effect can be characterized by the fourth-order perturbation with a quantum parameter \(\varepsilon\). For more details, please refer to Refs. \([26, 38, 41]\). With the periodic boundary conditions, the QZS (1.1) conserves the mass

\[
\mathcal{M}(t) = \int_{\Omega} |E|^2 dx \equiv \mathcal{M}(0), \quad t \geq 0,
\]

(1.2)

and the energy

\[
\mathcal{H}(t) = \int_{\Omega} \left( |\nabla E|^2 + \frac{1}{2} (|\nabla v|^2 + N^2) + \varepsilon^2 |\Delta E|^2 + \frac{\varepsilon^2}{2} |\nabla N|^2 + N|E|^2 \right) dx \equiv \mathcal{H}(0), \quad t \geq 0.
\]

(1.3)

where \(\Delta v = N_t\).

Extensive mathematical and numerical studies have been carried out for the above QZS (1.1) in the literature. Along the mathematical front, Fang et al. \([18]\) showed that the QZS (1.1) is locally well-posed in \(L^2(\mathbb{R}^d)\) data for dimension up to eight, together with global existence for dimensions up to five, which is different from the classical ZS where the global and local well-posedness of the Cauchy problem is known only for \(1 \leq d \leq 3\) \([17, 21]\). The blow-up in finite time of the solution for high-dimensional...
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classical ZS was investigated in [39]. Along the numerical front, the numerical studies of the classical or generalized ZS are very rich, such as time splitting methods [2, 3, 33, 34], scalar auxiliary variable approach [49], finite difference methods [1, 9, 22, 54], and discontinuous Galerkin method [53]. Recently, there has been growing interest in developing accurate and efficient numerical methods for the QZS (1.1). Baumstark and Schratz [5] developed a new class of asymptotic preserving trigonometric integrators for the QZS (1.1). Meanwhile, it is shown rigorously in mathematics that the scheme converges to the classical ZS in the limit $\varepsilon \to 0$ uniformly in the time discretization parameter. Zhang [58] proposed an explicit mass-conserving time-splitting exponential wave integrator Fourier pseudo-spectral (TS-EWI-FP) method for the QZS (1.1). However, these mentioned schemes cannot conserve the energy (1.3) of the QZS (1.1). It has been shown that in the numerical simulation of the collision of solitons, the solution of mass- and energy-conserving schemes cannot produce nonlinear blow-up [45, 57]. Later on, Zhang and Su [59] proposed and analyzed a linearly-implicit conservative compact finite difference method for the QZS (1.1). Cai et al. [8] proposed a novel of mass- and energy-conserving compact finite difference scheme for the QZS (1.1). However, all of them are only second-order accurate in time. In [13, 24, 31], it is clear to observe that high-order accurate structure-preserving scheme will provide much smaller numerical error and more robust than the second-order accurate one as the large time step is chosen. Thus, it is desirable to propose high-order accurate mass- and energy-preserving methods for solving the QZS (1.1).

As a matter of the fact, in the past few decades, how to devise high-order accurate energy-preserving schemes for conservative systems has attracted plenty of attention. The excellent ones include Hamiltonian Boundary Value Methods (HBVMs) [6, 7], energy-preserving variant of collocation methods [12, 28], high-order averaged vector field (AVF) methods [35, 40, 43], functionally fitted energy-preserving methods [37, 52], energy-preserving continuous stage Runge–Kutta (RK) methods [42, 50], etc. These methods can be easily extended to propose high-order accurate energy-preserving scheme for the QZS (1.1), which however cannot simultaneously conserve the mass [4, 36]. Based on the basic principle of the structure-preserving method where the numerical method should preserve the intrinsic properties of the original problems as much as possible, it is valuable to expect that the high-order mass and energy-preserving discretizations for the QZS (1.1) will produce richer information on the continuous system. Very recently, inspired by the ideas of the invariant energy quadratization (IEQ) approach [55], a new class of high-order accurate energy-preserving methods are proposed in [11, 23], which was generalized more recently by Tapley [51]. Especially, the term “quadratic auxiliary variable (QAV) approach” was coined by Gong et al. in [11]. The key differences between the IEQ approach and the QAV approach are as follows: (i) the auxiliary variable introduced by the QAV approach shall be quadratic and (ii) as a high-order quadratic invariant-preserving method in time is applied to the reformulated system, the resulting method can preserve the original Hamiltonian energy instead of a modified energy [19, 30, 32]. In addition to the Hamiltonian energy, numerical methods based on the QAV
approach and the quadratic invariant-preserving method also conserve the quadratic invariants of the reformulated system, thus the QAV approach combining with high-order quadratic invariant-preserving methods will be an efficient strategy to develop high-order accurate mass- and energy-preserving schemes for the QZS (1.1); however, to our knowledge, there has been no reference considering this issue.

In this paper, motivated by the QAV approach, we first introduce a quadratic auxiliary variable to transform the Hamiltonian energy into a quadratic form, and the original system is then reformulated into a new equivalent system using the energy variational principle. Finally, a fully-discrete method is obtained by using the RK method in time and the Fourier pseudo-spectral method in space for the reformulated system. We show that when the symplectic RK method is selected, the proposed method can conserve the discrete mass and original Hamiltonian energy exactly. In addition, to solve the discrete nonlinear equations of our methods efficiently, a fixed-pointed iteration method is presented.

The rest of the paper is organized as follows. In Sect. 2, we first reformulate the original QZS (1.1) into an equivalent form, and the mass and energy of the reformulated system are then investigated. In Sect. 3, a new class of high-order structure-preserving schemes are proposed based on the symplectic RK method in time and the Fourier pseudo-spectral method in space, respectively. In Sect. 4, an efficient fixed-pointed iteration method for solving the nonlinear equations of the proposed schemes is presented. In Sect. 5, various numerical experiments for the QZS (1.1) are carried out to illustrate the capability and accuracy of the schemes, and show some complex dynamical behaviors. Finally, we draw conclusions in Sect. 6.

2 Model reformulation

Denote \( u = (E, N, v)^T \), the QZS (1.1) can be written into the following energy-conserving system

\[
\partial_t u = S \frac{\delta \mathcal{H}}{\delta \bar{u}},
\]

where \( \bar{u} \) is the complex conjugate of \( u \),

\[
S = \begin{pmatrix}
  i & 0 & 0 \\
  0 & 0 & 1 \\
  0 & -1 & 0
\end{pmatrix}
\]

is the skew-adjoint operator, and the Hamiltonian energy functional

\[
\mathcal{H}[u] = \int_\Omega \left( -|\nabla E|^2 - \frac{1}{2} \left(|\nabla v|^2 + N^2\right) - \epsilon^2 |\Delta E|^2 - \frac{\epsilon^2}{2} |\nabla N|^2 - N |E|^2 \right) dx.
\]

Next, based on the idea of the QAV approach, we introduce a quadratic auxiliary variable, as follows:

\[
q := q(x, t) = |E|^2,
\]
the original energy (2.2) is then transformed into the following quadratic form
\[ E[u, q] = \int_{\Omega} \left( -|\nabla E|^2 - \frac{1}{2} \left( |\nabla v|^2 + N^2 \right) - \varepsilon^2 |\Delta E|^2 - \frac{\varepsilon^2}{2} |\nabla N|^2 - Nq \right) dx. \] (2.3)

According to the energy variational principle, we rewritten the QZS (2.1) into a new equivalent system
\[ E_t = i(\Delta E - \varepsilon^2 \Delta^2 E - NE), \]
\[ N_t = \Delta v, \]
\[ v_t = N - \varepsilon^2 \Delta N + q, \]
\[ q_t = 2\text{Re}(E_t \cdot \bar{E}), \] (2.4)

with the consistent initial conditions
\[ E(x, 0) = E_0(x), \quad N(x, 0) = N_0(x), \]
\[ q(x, 0) = |E_0(x)|^2, \quad N_t(x, 0) = N_1(x), \]
\[ \Delta v(x, 0) = N_1(x), \quad \int_{\Omega} v(x, 0) dx = 0, \quad x \in \Omega, \] (2.5)

where \( \text{Re}(\bullet) \) represents the real part of \( \bullet \), and we impose \( \int_{\Omega} v(x, 0) dx = 0 \) in the equivalent form so that \( v(x, t) \) is uniquely defined in the second equality of (2.4) [8].

Subsequently, we focus on investigating the structure-preserving properties of the reformulated system (2.4)-(2.5).

**Theorem 2.1** Under the periodic boundary conditions, the system (2.4)-(2.5) possesses the following conservation laws:

- **The mass**
  \[ M(t) \equiv M(0), \quad t \geq 0, \] (2.6)

  where \( M(t) \) is defined by (1.2).

- **The two quadratic invariants**
  \[ q(x, t) - |E(x, t)|^2 = q(x, 0) - |E(x, 0)|^2 \equiv 0, \quad x \in \Omega, \quad t \geq 0, \]
  \[ E(t) \equiv E(0), \quad t \geq 0, \] (2.7)

  where \( E(t) \) is defined by (2.3).

**Proof** We make the inner product of the first equality of (2.4) with \( E \) and then take the real part of the resulting equation to obtain
\[ \frac{d}{dt}(|E|^2, 1) = 2\text{Re} \int_{\Omega} i \left( -|\nabla E|^2 - \varepsilon^2 |\Delta E|^2 - N|E|^2 \right) dx = 0, \]

which implies that the system (2.4) satisfies (2.6).
According to the initial condition \( q(x, 0) = |E(x, 0)|^2 \) and the fourth equality of the system (2.4), we can deduce

\[
\partial_t(q - |E(x, t)|^2) = 0,
\]

which yields (2.7).

Using the periodic boundary conditions and the system (2.4), we then obtain

\[
\frac{d}{dt}E(t) = 2\text{Re}\left(\Delta E - \varepsilon^2 \Delta^2 E - NE_t, E_t\right) + \left(-N + \varepsilon^2 \Delta N - q + v_t, N_t\right)
\]

\[
= 2\text{Re}(-iE_t, E_t)
\]

\[
= 0,
\]

where \((f, g) = \int_\Omega f\bar{g}dx\), for any \(f, g \in L^2(\Omega)\) is denoted as the continuous \(L^2\) inner product. This completes the proof. 

\(\square\)

3 High-order mass- and energy-preserving schemes

In this section, the symplectic RK methods are first used to discretize the system (2.4) in time and a class of semi-discrete RK methods are proposed, which can conserve the mass and original energy of the QZS (1.1). Subsequently, the Fourier pseudo-spectral method is then employed to discretize the spatial variables of the semi-discrete schemes and a class of fully-discrete high-order mass- and energy-preserving schemes are presented.

3.1 Temporal semi-discretization

Let the time step \(\tau = \frac{T}{J}\) and denote \(t_n = n\tau\) for \(0 \leq n \leq J\). Let \(w^n\) and \(w_{ni}\) be the numerical approximations of the function \(w(x, t)\) at \(t_n\) and \(t_n + c_i \tau\), respectively.

Applying an \(s\)-stage RK method to discrete the system (2.4) in time and the following semi-discrete scheme is presented.

**Scheme 3.1** Let \(b_i, a_{ij}(i, j = 1, \cdots, s)\) be real numbers and let \(c_i = \sum_{j=1}^s a_{ij}\). For given \((E^n, N^n, v^n, q^n)\), an \(s\)-stage RK method is given by

\[
\begin{align*}
E_{ni} &= E^n + \tau \sum_{j=1}^s a_{ij}k^1_j, \quad k^1_i = i(\Delta E_{ni} - \varepsilon^2 \Delta^2 E_{ni} - N_{ni} E_{ni}), \\
N_{ni} &= N^n + \tau \sum_{j=1}^s a_{ij}k^2_j, \quad k^2_i = \Delta v_{ni}, \\
v_{ni} &= v^n + \tau \sum_{j=1}^s a_{ij}k^3_j, \quad k^3_i = N_{ni} - \varepsilon^2 \Delta N_{ni} + Q_{ni}, \\
Q_{ni} &= q^n + \tau \sum_{j=1}^s a_{ij}k^4_j, \quad k^4_i = 2\text{Re}(\bar{E}_{ni} \cdot k^1_i).
\end{align*}
\]  

(3.1)
Then, \((E^{n+1}, N^{n+1}, v^{n+1}, q^{n+1})\) is updated by

\[
\begin{align*}
E^{n+1} &= E^n + \tau \sum_{i=1}^s b_i k_1^i, \\
N^{n+1} &= N^n + \tau \sum_{i=1}^s b_i k_2^i, \\
v^{n+1} &= v^n + \tau \sum_{i=1}^s b_i k_3^i, \\
q^{n+1} &= q^n + \tau \sum_{i=1}^s b_i k_4^i.
\end{align*}
\tag{3.2}
\]

**Theorem 3.1** If the coefficients of the RK method in (3.1)-(3.2) satisfy

\[
b_i a_{ij} + b_j a_{ji} = b_i b_j, \quad \forall \ i, j = 1, \ldots, s,
\tag{3.3}
\]

and the periodic boundary conditions are considered, **Scheme 3.1** preserves the following semi-discrete conservation laws

- **The semi-discrete mass**
  
  \[
  \mathcal{M}^{n+1} = \mathcal{M}^n, \quad \mathcal{M}^n = (|E^n|^2, 1), \quad n = 0, 1, 2, \ldots, J.
  \tag{3.4}
  \]

- **The two semi-discrete quadratic invariants**
  
  \[
  q^n - |E^n|^2 = 0, \\
  \mathcal{E}^{n+1} = \mathcal{E}^n, \quad n = 0, 1, 2, \ldots, J,
  \tag{3.5}
  \]
  
  where
  
  \[
  \mathcal{E}^n = (\Delta E^n, E^n) - e^2 (\Delta E^n, \Delta E^n) - \frac{1}{2} (N^n, N^n)
  \]
  
  \[
  + \frac{e^2}{2} (\Delta N^n, N^n) - (N^n, q^n) + \frac{1}{2} (\Delta v^n, v^n).
  \]

**Proof** According to the first equality of (3.2), we have

\[
\mathcal{M}^{n+1} = \mathcal{M}^n + \tau \sum_{i=1}^s b_i (k_1^i, E^n) + \tau \sum_{i=1}^s b_i (E^n, k_1^i) + \frac{\tau^2}{2} \sum_{i,j=1}^s b_i b_j (k_1^i, k_1^j).
\tag{3.7}
\]

Plugging \(E^n = E_{ni} - \tau \sum_{j=1}^s a_{ij} k_j^1\) into the right hand side of (3.7) and using (3.3), we have

\[
\mathcal{M}^{n+1} = \mathcal{M}^n + \tau \sum_{i=1}^s b_i (E_{ni}, k_1^i) + \tau \sum_{i=1}^s b_i (k_1^i, E_{ni}).
\tag{3.8}
\]

Furthermore, we can deduce
\[
\tau \sum_{i=1}^{s} b_i(E_{ni}, k^1_i) + \tau \sum_{i=1}^{s} b_i(k^1_i, E_{ni}) \\
= 2\tau \sum_{i=1}^{s} b_i \text{Re}(k^1_i, \bar{E}_{ni}) \\
= 2\tau \sum_{i=1}^{s} b_i \text{Re}\left(i(\Delta E_{ni} - \varepsilon^2 \Delta^2 E_{ni} - N_{ni} E_{ni}), \bar{E}_{ni}\right) \\
= 2\tau \sum_{i=1}^{s} b_i \text{Re}\left(i\left(-|\nabla E_{ni}|^2 - \varepsilon^2 |\Delta E_{ni}|^2 - N_{ni} |E_{ni}|^2\right), 1\right) \\
= 0.
\]

Then, combining the above equation with (3.8), we obtain the discrete mass (3.4). Based on (3.2), (3.3) and \(E^n = E_{ni} - \tau \sum_{j=1}^{s} a_{ij} k^1_j\), we get

\[
|E^{n+1}|^2 - |E^n|^2 = E^{n+1} \cdot \bar{E}^{n+1} - E^n \cdot \bar{E}^n \\
= \tau \sum_{i=1}^{s} b_i(k^1_i \cdot \bar{E}^n) + \tau \sum_{i=1}^{s} b_i(\bar{k}^1_i \cdot E^n) + \tau^2 \sum_{i,j=1}^{s} b_i b_j(k^1_i \cdot \bar{k}^1_j) \\
= \tau \sum_{i=1}^{s} b_i(k^1_i \cdot \bar{E}_{ni}) + \tau \sum_{i=1}^{s} b_i(\bar{k}^1_i \cdot E_{ni}) \\
+ \tau^2 \sum_{i,j=1}^{s} (-b_i a_{ij} - b_j a_{ji} + b_i b_j)(k^1_i \cdot \bar{k}^1_j) \\
= \tau \sum_{i=1}^{s} b_i(k^1_i \cdot \bar{E}_{ni}) + \tau \sum_{i=1}^{s} b_i(\bar{k}^1_i \cdot E_{ni}).
\]

(3.9)

Noticing that

\[
q^{n+1} - q^n = \tau \sum_{i=1}^{s} b_i k^4_i \\
= 2\tau \sum_{i=1}^{s} b_i \text{Re}(k^1_i \cdot \bar{E}_{ni}) \\
= \tau \sum_{i=1}^{s} b_i(k^1_i \cdot \bar{E}_{ni}) + \tau \sum_{i=1}^{s} b_i(\bar{k}^1_i \cdot E_{ni}).
\]

(3.10)

It follows from (3.9) and (3.10) that

\[
q^{n+1} - |E^{n+1}|^2 = q^n - |E^n|^2.
\]
With the help of the initial condition \( q^0 = |E^0|^2 \), we can obtain (3.5).

By the similar argument as Eq. (3.8), we can obtain that

\[
\begin{align*}
(\Delta E^{n+1}, E^n) - (\Delta E^n, E^n) &= 2\tau \sum_{i=1}^{s} b_i \text{Re}(k^1_i, \Delta E_{ni}), \\
(\Delta E^{n+1}, \Delta E^n) - (\Delta E^n, \Delta E^n) &= 2\tau \sum_{i=1}^{s} b_i \text{Re}(k^1_i, \Delta^2 E_{ni}), \\
(N^{n+1}, N^n) - (N^n, N^n) &= 2\tau \sum_{i=1}^{s} b_i (N_{ni}, k^2_i), \\
(\Delta N^{n+1}, N^n) - (\Delta N^n, N^n) &= 2\tau \sum_{i=1}^{s} b_i \left( \Delta N_{ni}, k^2_i \right), \\
(N^{n+1}, q^n) - (N^n, q^n) &= \tau s \sum_{i=1}^{s} b_i \left[ (k^2_i Q_{ni}) + (N_{ni}, k^4_i) \right], \\
(\Delta v^{n+1}, v^n) - (\Delta v^n, v^n) &= 2\tau \sum_{i=1}^{s} b_i \left( \Delta v_{ni}, k^3_i \right).
\end{align*}
\]

Thus, together with (3.1), we can derive

\[
E^{n+1} - E^n = \tau \sum_{i=1}^{s} b_i \left[ 2\text{Re}(k^1_i, \Delta E_{ni}) - 2\varepsilon^2 \text{Re}(k^1_i, \Delta^2 E_{ni}) - (N_{ni}, k^2_i) \\
+ \varepsilon^2 \left( \Delta N_{ni}, k^2_i \right) - (N_{ni}, k^4_i) - (k^2_i, Q_{ni}) + (\Delta v_{ni}, k^3_i) \right]
\]

\[
= \tau \sum_{i=1}^{s} b_i \left[ 2\text{Re}(k^1_i, \Delta E_{ni} - \varepsilon^2 \Delta^2 E_{ni} - N_{ni} E_{ni}) - (N_{ni} - \varepsilon^2 \Delta N_{ni} + Q_{ni} - k^3_i, k^2_i) \right]
\]

\[
= \tau \sum_{i=1}^{s} b_i \left[ 2\text{Re}(k^1_i, -ik^1_i) - (N_{ni} - \varepsilon^2 \Delta N_{ni} + Q_{ni} - k^3_i, k^2_i) \right]
\]

\[
= 0,
\]

which implies that Scheme 3.1 satisfies (3.6).

\[\square\]

**Theorem 3.2** Under the consistent initial conditions (2.5), the periodic boundary conditions and the condition (3.3), Scheme 3.1 conserves the original Hamiltonian energy at each time step, as follows:

\[
\mathcal{H}^n \equiv \mathcal{H}^0, \quad n = 1, 2, \ldots, J,
\]

where

\[
\mathcal{H}^n = (\Delta E^n, E^n) - \varepsilon^2 (\Delta E^n, \Delta E^n) - \frac{1}{2} (N^n, N^n)
+ \frac{\varepsilon^2}{2} (\Delta N^n, N^n) - (N^n, |E^n|^2) + \frac{1}{2} (\Delta v^n, v^n).
\]
Proof As the consistent initial conditions (2.5), the periodic boundary conditions and the condition (3.3) are considered, we can obtain from Theorem 3.1 that

\[ q^n = |E^n|^2, \quad n = 0, 1, 2, \cdots, J, \] (3.12)

and

\[ E^{n+1} = E^n, \quad n = 0, 1, 2, \cdots, J, \] (3.13)

where

\[ E^n = (\Delta E^n, E^n) - \varepsilon^2 (\Delta E^n, \Delta E^n) - \frac{1}{2} (N^n, N^n) + \frac{\varepsilon^2}{2} (\Delta N^n, N^n) - (N^n, q^n) + \frac{1}{2} (\Delta v^n, v^n). \]

Then, substituting (3.12) into (3.13), we obtain (3.11). We finish the proof. □

Scheme 3.2 As the symplectic RK method is selected, Scheme 3.1 is equivalent to the following s-stage RK (using (3.5) of Theorem 3.1):

\[
\begin{align*}
E_{ni} &= E^n + \tau \sum_{j=1}^{s} a_{ij} k_j^1, \\
k_i^1 &= i(\Delta E_{ni} - \varepsilon^2 \Delta^2 E_{ni} - N_{ni} E_{ni}), \\
N_{ni} &= N^n + \tau \sum_{j=1}^{s} a_{ij} k_j^2, \\
k_i^2 &= \Delta v_{ni}, \\
v_{ni} &= v^n + \tau \sum_{j=1}^{s} a_{ij} k_j^3, \\
k_i^3 &= N_{ni} - \varepsilon^2 \Delta N_{ni} + |E^n|^2 + 2\tau \sum_{j=1}^{s} a_{ij} \Re(\bar{E}_{nj} \cdot k_j^1),
\end{align*}
\]

where \((E^{n+1}, N^{n+1}, v^{n+1})\) is obtained by

\[
\begin{align*}
E^{n+1} &= E^n + \tau \sum_{i=1}^{s} b_i k_i^1, \\
N^{n+1} &= N^n + \tau \sum_{i=1}^{s} b_i k_i^2, \\
v^{n+1} &= v^n + \tau \sum_{i=1}^{s} b_i k_i^3,
\end{align*}
\]

which implies that the QAV approach need to introduce an auxiliary variable, but the auxiliary variable can be eliminated in practical computations. Thus, it cannot increase additional computational costs.

Remark 3.1 We should note that the Gauss method where the RK coefficients \(c_1, c_2, \cdots, c_s\) are chosen as the Gaussian quadrature nodes, i.e., the zeros of the
s-th shifted Legendre polynomial \( \frac{d^s}{dx^s} (x^s(x - 1)^s) \) satisfies the condition (3.3) [29]. In particular, the coefficients of the Gauss methods of order 2, 4 and 6 can be given [29, 44], respectively, (see Table 1).

**Remark 3.2** If the Gauss method of order 2 (see Table 1) is selected, Scheme 3.2 reduces to the following semi-discrete Crank-Nicolson scheme (CNS) [8]

\[
\begin{align*}
    i\delta_t E^n + \Delta E^{n+\frac{1}{2}} - \varepsilon^2 \Delta^2 E^{n+\frac{1}{2}} - N^{n+\frac{1}{2}} E^{n+\frac{1}{2}} &= 0, \\
    \delta_t N^n &= \Delta v^{n+\frac{1}{2}}, \\
    \delta_t v^n - N^{n+\frac{1}{2}} + \varepsilon^2 \Delta N^{n+\frac{1}{2}} - \frac{1}{2}(|E^{n+1}|^2 + |E^n|^2) &= 0, \quad n = 0, 1, 2, \ldots, J,
\end{align*}
\]

where

\[
\delta_t w^n = \frac{w^{n+1} - w^n}{\tau}, \quad w^{n+\frac{1}{2}} = \frac{w^{n+1} + w^n}{2}.
\]

**Remark 3.3** It is well-known that the scalar auxiliary variable (SAV) approach [47, 48] is also an efficient method for developing high-order accurate structure-preserving methods for the conservative systems [13, 31]. However, we should note that it is challenging for introducing a special scalar auxiliary variable to construct high-order accurate methods in time which can preserve the original Hamiltonian energy of the system.

### 3.2 Full discretization

In this subsection, the Fourier pseudo-spectral method is employed to discretize Scheme 3.2 in spatial variables and a class of fully discrete structure-preserving schemes are presented.

**Table 1** The Gauss methods of 2 (s=1, up), 4 (s=2, middle) and 6 (s=3, bottom)

|       | \( \frac{1}{2} \) | \( \frac{1}{2} \) |
|-------|-----------------|-----------------|
| \( \frac{1}{2} \) | \( \frac{1}{4} + \frac{\sqrt{3}}{6} \) | \( \frac{1}{4} \) |
| \( \frac{1}{2} \) | \( \frac{5}{36} + \frac{\sqrt{15}}{24} \) | \( \frac{5}{36} + \frac{\sqrt{15}}{24} \) |
| \( \frac{1}{2} \) | \( \frac{3}{18} \) | \( \frac{3}{18} \) |
Let the domain $\Omega = [a, b] \times [c, d]$ be uniformly partitioned with spatial steps $h_x = \frac{b-a}{N_x}$ and $h_y = \frac{d-c}{N_y}$, where $N_x$ and $N_y$ are two positive even integers. Then, we denote the spatial grid points as

$$\Omega_h = \{(x_j, y_k) | x_j = a + jh_x, \ y_k = c + kh_y, \ 0 \leq j \leq N_x - 1, \ 0 \leq k \leq N_y - 1\}.$$ 

Let $w_{j,k}$ be the numerical approximation of $w(x_j, y_k, t)$ on $\Omega_h$, and denote

$$w := (w_{0,0}, w_{1,0}, \cdots, w_{N_x-1,0}, w_{0,1}, w_{1,1}, \cdots, w_{N_x-1,1}, \cdots, w_{0,N_y-1}, w_{1,N_y-1}, \cdots, w_{N_x-1,N_y-1})^T$$

be the solution vector; we also define discrete inner product and norms as

$$\langle u, w \rangle_h = h_x h_y \sum_{j=0}^{N_x-1} \sum_{k=0}^{N_y-1} u_{j,k} w_{j,k}, \ \|w\|_h = (w, w)^{1/2}_h, \ \|w\|_{h,\infty} = \max_{(x_j, y_k) \in \Omega_h} |w_{j,k}|.$$ 

In addition, we denote $\cdot \cdot \cdot$ as the element-wise product of vectors $u$ and $w$, that is

$$u \cdot w = (u_{0,0} w_{0,0}, \cdots, u_{N_x-1,0} w_{N_x-1,0}, \cdots, u_{0,N_y-1} w_{0,N_y-1}, \cdots, u_{N_x-1,N_y-1} w_{N_x-1,N_y-1})^T.$$ 

For brevity, we denote $w \cdot w$ and $w \cdot \tilde{w}$ as $w^2$ and $|w|^2$, respectively.

Consider the interpolation space

$$S'' = \text{span}\{X_j(x)Y_k(y) | 0 \leq j \leq N_x - 1, \ 0 \leq k \leq N_y - 1\},$$

where

$$X_i(x) = \frac{1}{N_x} \sum_{m=-N_x/2}^{N_x/2} \frac{1}{a_m} e^{im\mu_x (x-x_i)}, \ Y_j(y) = \frac{1}{N_y} \sum_{m=-N_y/2}^{N_y/2} \frac{1}{b_m} e^{im\mu_y (y-y_j)},$$

$$\mu_x = \frac{2\pi}{b-a}, \ a_m = \begin{cases} 1, & |m| < \frac{N_x}{2}, \\ 2, & |m| = \frac{N_x}{2} \end{cases}, \ \mu_y = \frac{2\pi}{d-c}, \ b_m = \begin{cases} 1, & |m| < \frac{N_y}{2}, \\ 2, & |m| = \frac{N_y}{2} \end{cases}.$$ 

According to [10, 46], we define the interpolation operator $I_N : C(\Omega) \to S''$

$$I_N w(x, y, t) = \sum_{j=0}^{N_x-1} \sum_{k=0}^{N_y-1} w(x_j, y_k, t) X_j(x) Y_k(y).$$

Computing partial derivatives in $x$ and $y$, respectively at the collocation points $w(x_j, y_k)$ yields

$$\frac{\partial^2}{\partial x^2} I_N w(x_j, y_k, t) = \sum_{l=0}^{N_x-1} w(x_l, y_k, t) \frac{d^2}{dx^2} X_l(x_j) = \sum_{l=0}^{N_x-1} (D_x^2)_{j,l} w(x_l, y_k, t).$$

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\[
\frac{\partial^2}{\partial y^2} I_N w(x_j, y_k, t) = \sum_{l=0}^{N_y-1} w(x_j, y_l, t) \frac{d^2}{d y^2} Y_l(y_k) = \sum_{l=0}^{N_y-1} w(x_j, y_l, t)(D^2_y)_{k,l},
\]

where \([10]\)

\[
(D^2_x)_{j,k} = \begin{cases} 
\frac{1}{2} \mu_x^2 (-1)^{j+k+1} \csc^2 \left( \frac{\mu_x (y_j - y_k)}{2} \right), & j \neq k, \\
- \mu_x^2 \frac{N_x^2 + 2}{12}, & j = k,
\end{cases}
\]

\[
(D^2_y)_{j,k} = \begin{cases} 
\frac{1}{2} \mu_y^2 (-1)^{j+k+1} \csc^2 \left( \frac{\mu_y (y_j - y_k)}{2} \right), & j \neq k, \\
- \mu_y^2 \frac{N_y^2 + 2}{12}, & j = k.
\end{cases}
\]

**Lemma 3.1** For the matrix \(D^2_\theta (\theta = x \text{ or } y)\), there exists the following relation \([46]\)

\[
D^2_\theta = \mathcal{F}_{N_\theta}^H \Lambda_\theta \mathcal{F}_{N_\theta},
\]

where \(\mathcal{F}_{N_\theta}\) denotes the discrete Fourier transform (DFT) matrix, and satisfies \(\mathcal{F}_{N_\theta}^H\) is the conjugate transpose matrix of \(\mathcal{F}_{N_\theta}\),

\[
\Lambda_\theta = - \mu_\theta^2 \text{diag}[0^2, 1^2, \cdots, (\frac{N_\theta}{2})^2, (-\frac{N_\theta}{2} + 1)^2, \cdots, (-2)^2, (-1)^2].
\]

Then, applying the Fourier pseudo-spectral method to **Scheme 3.2**, we obtain the following fully discrete scheme.

**Scheme 3.3** Let \(b_i, a_{ij}(i, j = 1, \cdots, s)\) be real numbers and let \(c_i = \sum_{j=1}^{s} a_{ij}\). For given \((E^n, N^n, v^n)\), an \(s\)-stage RK Fourier pseudo-spectral method is given by

\[
\begin{align*}
E_{ni} & = E^n + \tau \sum_{j=1}^{s} a_{ij} k^1_j, \\
k^1_i & = i(\Delta_h E_{ni} - \varepsilon^2 \Delta^2_t E_{ni} - N_{ni} \cdot E_{ni}), \\
N_{ni} & = N^n + \tau \sum_{j=1}^{s} a_{ij} k^2_j, \\
k^2_i & = \Delta_h v_{ni}, \\
v_{ni} & = v^n + \tau \sum_{j=1}^{s} a_{ij} k^3_j, \\
k^3_i & = N_{ni} - \varepsilon^2 \Delta_h N_{ni} + |E^n|^2 + 2 \tau \sum_{j=1}^{s} a_{ij} \text{Re}(\tilde{E}_{nj} \cdot k^1_j),
\end{align*}
\]

where \(\Delta_h = I_y \otimes D^2_x + D^2_y \otimes I_x\). Then \((E^{n+1}, N^{n+1}, v^{n+1})\) is updated by

\[
E^{n+1} = E^n + \tau \sum_{i=1}^{s} b_i k^1_i,
\]

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\[ N^{n+1} = N^n + \tau \sum_{i=1}^{s} b_i k_i^2, \]
\[ v^{n+1} = v^n + \tau \sum_{i=1}^{s} b_i k_i^3, \]

with the initial conditions
\[ E^0 = E_0(x), \ N^0 = N_0(x), \ \Delta_h v^0 = N_1(x). \]

We note that based on the condition \( \int_{\Omega} v(x, 0) \, dx = 0 \), we can obtain the zeroth Fourier coefficient on \( v^0 \) is zero, so that the Poisson equation \( \Delta_h v^0 = N_1(x) \) with periodic boundary conditions only has a unique solution.

For Scheme 3.3, we can obtain the following theorem which can be carried out similarly as Theorems 3.1 and 3.2, respectively.

**Theorem 3.3** If the RK coefficients of Scheme 3.3 satisfy the condition (3.3), then Scheme 3.3 satisfies the following discrete conservative properties

- **The discrete mass**
  \[ M^{n+1}_h = M^n_h, \ M^n_h = \langle E^n_h, E^n_h \rangle_h, \ n = 0, 1, 2, \ldots, J. \]

- **The discrete quadratic invariant**
  \[ q^n - |E^n|^2 = 0, \ n = 0, 1, 2, \ldots, J. \]

- **The discrete Hamiltonian energy**
  \[ H^{n+1}_h = H^n_h, \ n = 0, 1, 2, \ldots, J, \]
  where
  \[ H^n_h = \langle \Delta_h E^n_h, E^n_h \rangle_h - \epsilon^2 \langle \Delta_h E^n_h, \Delta_h E^n_h \rangle_h \]
  \[ - \frac{1}{2} \langle N^n_h, N^n_h \rangle_h + \frac{\epsilon^2}{2} \langle \Delta_h N^n_h, N^n_h \rangle_h - \langle N^n_h, |E^n|^2 \rangle_h + \frac{1}{2} \langle \Delta_h v^n_h, v^n_h \rangle_h. \]

### 4 Implementation of numerical schemes

In this section, we will present an efficient fixed-point iteration method to solve the resulting nonlinear equations of Scheme 3.3, which is mainly based on the matrix diagonalization method (see Ref. [46] and references therein) and the discrete Fourier transform algorithm. For simplicity, we only consider the one dimensional QZS (1.1). The generalizations to two dimensional case is straightforward for tensor product grids.
and the results remain valid with modifications. In addition, we only take \( s = 2 \) as an example, in which **Scheme 3.3** can be rewritten into

\[
\begin{align*}
  k_1^1 &= i(\Delta_h - \varepsilon^2 \Delta_n^2)E_{n1} - iN_{n1} \cdot E_{n1}, \quad k_2^1 &= i(\Delta_h - \varepsilon^2 \Delta_n^2)E_{n2} - iN_{n2} \cdot E_{n2}, \\
  k_1^2 &= \Delta_h v_{n1}, \quad k_2^2 &= \Delta_h v_{n2}, \\
  k_1^3 &= (1 - \varepsilon^2 \Delta_h)N_{n1} + Q_1, \quad k_2^3 &= (1 - \varepsilon^2 \Delta_h)N_{n2} + Q_2, \\
  E_{n1} &= E^n + \tau a_{11} k_1^1 + \tau a_{12} k_1^2, \quad E_{n2} = E^n + \tau a_{21} k_1^1 + \tau a_{22} k_2^2, \\
  N_{n1} &= N^n + \tau a_{11} k_1^3 + \tau a_{12} k_2^3, \quad N_{n2} = N^n + \tau a_{21} k_1^3 + \tau a_{22} k_2^3, \\
  v_{n1} &= v^n + \tau a_{11} k_1^3 + \tau a_{12} k_2^3, \quad v_{n2} = v^n + \tau a_{21} k_1^3 + \tau a_{22} k_2^3,
\end{align*}
\]

where

\[
\begin{align*}
  Q_1 &= |E^n|^2 + \tau a_{11} k_1^4 + \tau a_{12} k_2^4, \quad Q_2 = |E^n|^2 + \tau a_{21} k_1^4 + \tau a_{22} k_2^4, \\
  k_1^4 &= 2\text{Re}(\overline{E}_{n1} \cdot k_1^1), \quad k_2^4 = 2\text{Re}(\overline{E}_{n2} \cdot k_2^1).
\end{align*}
\]

Then, \( E^{n+1} \), \( N^{n+1} \) and \( v^{n+1} \) are obtained by

\[
\begin{align*}
  E^{n+1} &= E^n + \tau b_1 k_1^1 + \tau b_2 k_2^1, \quad N^{n+1} = N^n + \tau b_1 k_1^2 + \tau b_2 k_2^2, \\
  v^{n+1} &= v^n + \tau b_1 k_1^3 + \tau b_2 k_2^3.
\end{align*}
\]

According to (4.1) and (4.4), we have

\[
\begin{align*}
  \begin{bmatrix}
  1 - \tau i(\Delta_h - \varepsilon^2 \Delta_n^2)a_{11} & -\tau i(\Delta_h - \varepsilon^2 \Delta_n^2)a_{12} \\
  -\tau i(\Delta_h - \varepsilon^2 \Delta_n^2)a_{21} & 1 - \tau i(\Delta_h - \varepsilon^2 \Delta_n^2)a_{22}
  \end{bmatrix}
  \begin{bmatrix}
  k_1^1 \\
  k_2^1
  \end{bmatrix}
  &= \begin{bmatrix}
  i(\Delta_h - \varepsilon^2 \Delta_n^2)E^n - iN_{n1} \cdot E_{n1} \\
  i(\Delta_h - \varepsilon^2 \Delta_n^2)E^n - iN_{n2} \cdot E_{n2}
  \end{bmatrix}.
\end{align*}
\]

Analogously, we can obtain from (4.2)-(4.3) and (4.5)-(4.6) that

\[
\begin{align*}
  \begin{bmatrix}
  1 - \tau^2(a_{11}^2 + a_{12} a_{21})(\Delta_h - \varepsilon^2 \Delta_n^2) & -\tau^2(a_{11}a_{12} + a_{12} a_{22})(\Delta_h - \varepsilon^2 \Delta_n^2) \\
  -\tau^2(a_{21} a_{11} + a_{22} a_{21})(\Delta_h - \varepsilon^2 \Delta_n^2) & 1 - \tau^2(a_{21}a_{12} + a_{22} a_{22})(\Delta_h - \varepsilon^2 \Delta_n^2)
  \end{bmatrix}
  \begin{bmatrix}
  k_1^2 \\
  k_2^2
  \end{bmatrix}
  &= \begin{bmatrix}
  \tau(a_{11} + a_{12})(\Delta_h - \varepsilon^2 \Delta_n^2)N^n + \Delta_h v^n + \tau a_{11} \Delta_h Q_1 + \tau a_{12} \Delta_h Q_2 \\
  \tau(a_{21} + a_{22})(\Delta_h - \varepsilon^2 \Delta_n^2)N^n + \Delta_h v^n + \tau a_{21} \Delta_h Q_1 + \tau a_{22} \Delta_h Q_2
  \end{bmatrix}.
\end{align*}
\]

It is clear to see that if \( k_i^1 \) and \( k_i^2 \) (\( i = 1, 2 \)) are obtained after solving the nonlinear equations (4.11) and (4.12), respectively, and \( E_{ni} \) (\( i = 1, 2 \)) and \( N_{ni} \) (\( i = 1, 2 \)) are updated by (4.4) and (4.5), respectively. Subsequently, \( k_i^3 \) (\( i = 1, 2 \)) and \( k_i^4 \) (\( i = 1, 2 \)) are updated by (4.3) and (4.8), respectively. Thus, for **Scheme 3.3**, we only need to solve the nonlinear equations (4.11) and (4.12), respectively. For the nonlinear algebraic equations (4.11) and (4.12), we apply the following fixed-point iteration method.
\[
\begin{bmatrix}
1 - \tau i(\Delta_h - \varepsilon^2 \Delta_h^2)a_{11} & -\tau i(\Delta_h - \varepsilon^2 \Delta_h^2)a_{12} \\
-\tau i(\Delta_h - \varepsilon^2 \Delta_h^2)a_{21} & 1 - \tau i(\Delta_h - \varepsilon^2 \Delta_h^2)a_{22}
\end{bmatrix}
\begin{bmatrix}
(k_1)^{l+1} \\
(k_2)^{l+1}
\end{bmatrix}
= \begin{bmatrix}
i(\Delta_h - \varepsilon^2 \Delta_h^2)E^n - iN_{n1}^l \cdot E_{n1}^l \\
i(\Delta_h - \varepsilon^2 \Delta_h^2)E^n - iN_{n2}^l \cdot E_{n2}^l
\end{bmatrix},
\]
and
\[
\begin{bmatrix}
1 - \tau^2(a_{11}^2 + a_{12}a_{21})(\Delta_h - \varepsilon^2 \Delta_h^2) & -\tau^2(a_{11}a_{12} + a_{12}a_{22})(\Delta_h - \varepsilon^2 \Delta_h^2) \\
-\tau^2(a_{21}a_{11} + a_{22}a_{21})(\Delta_h - \varepsilon^2 \Delta_h^2) & 1 - \tau^2(a_{21}a_{12} + a_{22}^2)(\Delta_h - \varepsilon^2 \Delta_h^2)
\end{bmatrix}
\begin{bmatrix}
(k_1)^{l+1} \\
(k_2)^{l+1}
\end{bmatrix}
= \begin{bmatrix}
\tau(a_{11} + a_{12})(\Delta_h - \varepsilon^2 \Delta_h^2)N^n + \Delta_h v^n + \tau a_{11} \Delta_h Q_1^l + \tau a_{12} \Delta_h Q_2^l \\
\tau(a_{21} + a_{22})(\Delta_h - \varepsilon^2 \Delta_h^2)N^n + \Delta_h v^n + \tau a_{21} \Delta_h Q_1^l + \tau a_{22} \Delta_h Q_2^l
\end{bmatrix},
\]
where
\[
E_{n1}^l = E^n + \tau a_{11}(k_1^l) + \tau a_{12}(k_2^l), \quad E_{n2}^l = E^n + \tau a_{21}(k_1^l) + \tau a_{22}(k_2^l),
\]
\[
N_{n1}^l = N^n + \tau a_{11}(k_1^l) + \tau a_{12}(k_2^l), \quad N_{n2}^l = N^n + \tau a_{21}(k_1^l) + \tau a_{22}(k_2^l),
\]
\[
Q_1^l = |E^n|^2 + 2\tau a_{11} \text{Re}(\bar{E}_{n1}^l \cdot (k_1^l)^l) + 2\tau a_{12} \text{Re}(\bar{E}_{n2}^l \cdot (k_1^l)^l),
\]
\[
Q_2^l = |E^n|^2 + 2\tau a_{21} \text{Re}(\bar{E}_{n1}^l \cdot (k_1^l)^l) + 2\tau a_{22} \text{Re}(\bar{E}_{n2}^l \cdot (k_1^l)^l).
\]
Then, it follows from Lemma 3.1 that
\[
\begin{bmatrix}
1 - \tau i(\Lambda_x - \varepsilon^2 \Lambda_x^2)a_{11} & -\tau i(\Lambda_x - \varepsilon^2 \Lambda_x^2)a_{12} \\
-\tau i(\Lambda_x - \varepsilon^2 \Lambda_x^2)a_{21} & 1 - \tau i(\Lambda_x - \varepsilon^2 \Lambda_x^2)a_{22}
\end{bmatrix}
\begin{bmatrix}
(k_1)^{l+1} \\
(k_2)^{l+1}
\end{bmatrix}
= \begin{bmatrix}
i(\Lambda_x - \varepsilon^2 \Lambda_x^2)\tilde{E}^n - iN_{n1}^l \cdot \tilde{E}_{n1}^l \\
i(\Lambda_x - \varepsilon^2 \Lambda_x^2)\tilde{E}^n - iN_{n2}^l \cdot \tilde{E}_{n2}^l
\end{bmatrix},
\tag{4.13}
\]
and
\[
\begin{bmatrix}
1 - \tau^2(a_{11}^2 + a_{12}a_{21})(\Lambda_x - \varepsilon^2 \Lambda_x^2) & -\tau^2(a_{11}a_{12} + a_{12}a_{22})(\Lambda_x - \varepsilon^2 \Lambda_x^2) \\
-\tau^2(a_{21}a_{11} + a_{22}a_{21})(\Lambda_x - \varepsilon^2 \Lambda_x^2) & 1 - \tau^2(a_{21}a_{12} + a_{22}^2)(\Lambda_x - \varepsilon^2 \Lambda_x^2)
\end{bmatrix}
\begin{bmatrix}
(k_1)^{l+1} \\
(k_2)^{l+1}
\end{bmatrix}
= \begin{bmatrix}
\tau(a_{11} + a_{12})(\Lambda_x - \varepsilon^2 \Lambda_x^2)\tilde{N}^n + \Lambda_x \tilde{v}^n + \tau a_{11} \Lambda_x \tilde{Q}_1^l + \tau a_{12} \Lambda_x \tilde{Q}_2^l \\
\tau(a_{21} + a_{22})(\Lambda_x - \varepsilon^2 \Lambda_x^2)\tilde{N}^n + \Lambda_x \tilde{v}^n + \tau a_{21} \Lambda_x \tilde{Q}_1^l + \tau a_{22} \Lambda_x \tilde{Q}_2^l
\end{bmatrix},
\tag{4.14}
\]
where \(\tilde{\bullet} = \mathcal{F}_{\Lambda_x} \bullet \) and \(\mathcal{F}_{\Lambda_x} \) is the discrete Fourier transform matrix. Then we rewrite (4.13) and (4.14) into component-wise form, as follows:
\[
\begin{bmatrix}
1 - \tau i(\Delta_j - \varepsilon^2 \Delta_j^2)a_{11} & -\tau i(\Delta_j - \varepsilon^2 \Delta_j^2)a_{12} \\
-\tau i(\Delta_j - \varepsilon^2 \Delta_j^2)a_{21} & 1 - \tau i(\Delta_j - \varepsilon^2 \Delta_j^2)a_{22}
\end{bmatrix}
\begin{bmatrix}
((k_1)^{l+1})_j \\
((k_2)^{l+1})_j
\end{bmatrix}
\]
\[
\begin{align*}
&= \left[ i(\Delta_j - \varepsilon^2 \Delta^2_j)(\overline{E}^n)_{j} - (iN'_{n1} \cdot E^l_{n1})_{j} \right] \\
&\quad + \left[ i(\Delta_j - \varepsilon^2 \Delta^2_j)(\overline{E}^n)_{j} - (iN'_{n2} \cdot E^l_{n2})_{j} \right],
\end{align*}
\]

and

\[
\begin{bmatrix}
1 - \tau^2(a_{11}^2 + a_{12}a_{21})(\Delta_j - \varepsilon^2 \Delta^2_j) - \tau^2(a_{11}a_{12} + a_{12}a_{22})(\Delta_j - \varepsilon^2 \Delta^2_j) \\
-\tau^2(a_{21}a_{11} + a_{22}a_{21})(\Delta_j - \varepsilon^2 \Delta^2_j) 1 - \tau^2(a_{21}a_{12} + a_{22}^2)(\Delta_j - \varepsilon^2 \Delta^2_j)
\end{bmatrix}
\begin{bmatrix}
((k_{1}^{(l+1)})_{j} \\
((k_{2}^{(l+1)})_{j}
\end{bmatrix}
\]

\[
\begin{bmatrix}
\tau(a_{11} + a_{12})(\Delta_j - \varepsilon^2 \Delta^2_j)(\widetilde{N}^n)_{j} + \Delta_j(\overline{Q}_{1}^{l})_{j} + \tau a_{11} \Delta_j(\overline{Q}_{1}^{l})_{j} + \tau a_{12} \Delta_j(\overline{Q}_{2}^{l})_{j} \\
\tau(a_{21} + a_{22})(\Delta_j - \varepsilon^2 \Delta^2_j)(\widetilde{N}^n)_{j} + \Delta_j(\overline{Q}_{1}^{l})_{j} + \tau a_{21} \Delta_j(\overline{Q}_{1}^{l})_{j} + \tau a_{22} \Delta_j(\overline{Q}_{2}^{l})_{j}
\end{bmatrix},
\]

where \( \Delta_j = (\Delta_x)_j \) and \( j = 0, 1, 2, \ldots, N_{x} - 1 \).

Solving above two linear systems for all \( j = 0, 1, 2, \ldots, N_{x} - 1 \), we obtain \((k_{1}^{(l+1)})_{j}\) and \((k_{2}^{(l+1)})_{j}\), then \((k_{1}^{(l)})_{j}\) and \((k_{2}^{(l)})_{j}\) are updated by \( \mathcal{F}_{N_{x}}^{H}(k_{1}^{(l)})_{j}\) and \( \mathcal{F}_{N_{y}}^{H}(k_{2}^{(l)})_{j}\) \( (i = 1, 2) \), respectively. In our computations, we take the iterative initial value \((k_{1}^{(0)})_{j} = E_{n}^{l}\) and \((k_{2}^{(0)})_{j} = N_{n}^{l}, (i = 1, 2) \), respectively. The iteration terminates when the number of maximum iterative step \( M = 30 \) is reached or the infinity norm of the error between two adjacent iterative steps is less than \( 10^{-14} \), that is

\[
\max \left\{ \max_{1 \leq i \leq 2} \left\{ \| (k_{i}^{(l)})^{(l+1)} - (k_{i}^{(l)})^{(l)} \|_{h, \infty} \right\}, \max_{1 \leq i \leq 2} \left\{ \| (k_{i}^{(l)})^{(l+1)} - (k_{i}^{(l)})^{(l)} \|_{h, \infty} \right\} \right\} < 10^{-14}.
\]

After \( k_{1}^{(l)} \) and \( k_{2}^{(l)} \), \( i = 1, 2 \), are obtained, we then have:

- Step 1: \( E_{n}^{l+1} \) and \( N_{n}^{l+1} \) are obtained from (4.9), \( E_{n_i} \) and \( N_{n_i} \) \( (i = 1, 2) \) are obtained from (4.4) and (4.5);
- Step 2: We first obtain \( k_{1}^{(l)} \) from (4.8) and \( Q_{i} \) \( (i = 1, 2) \) is then calculated by (4.7);
- Step 3: \( k_{3}^{(l)} \) and \( k_{3}^{(l)} \) are then obtained from (4.3) and \( v_{n}^{l+1} \) is updated by (4.10).

**Remark 4.1** Let \( \overline{\mathcal{F}} = \mathcal{F}_{N_{x}} \cdot \mathcal{F}_{N_{y}}^{H} \) where \( \cdot \) represents the matrix of \( N_{x} \times N_{y} \). By the similar argument as the 1D case, the efficient implementation of the proposed scheme for the 2D case can be expressed into the following component-wise form, as \( s = 2 \) is chosen:

\[
\begin{bmatrix}
1 - \tau i(\Delta_{j,k} - \varepsilon^2 \Delta^2_{j,k})a_{11} - \tau i(\Delta_{j,k} - \varepsilon^2 \Delta^2_{j,k})a_{12} \\
-\tau i(\Delta_{j,k} - \varepsilon^2 \Delta^2_{j,k})a_{21} 1 - \tau i(\Delta_{j,k} - \varepsilon^2 \Delta^2_{j,k})a_{22}
\end{bmatrix}
\begin{bmatrix}
((k_{1}^{(l+1)})_{j,k} \\
((k_{2}^{(l+1)})_{j,k}
\end{bmatrix}
\]

\[
\begin{bmatrix}
i(\Delta_{j,k} - \varepsilon^2 \Delta^2_{j,k})(\overline{E}^{n})_{j,k} - (iN'_{n1} \cdot E^l_{n1})_{j,k} \\
i(\Delta_{j,k} - \varepsilon^2 \Delta^2_{j,k})(\overline{E}^{n})_{j,k} - (iN'_{n2} \cdot E^l_{n2})_{j,k}
\end{bmatrix},
\]
and

\[
\begin{bmatrix}
1 - \tau^2(a_{11}^2 + a_{12}a_{21})(\Delta_{j,k} - \varepsilon^2\Delta_{j,k}^2) - \tau^2(a_{11}a_{12} + a_{12}a_{22})(\Delta_{j,k} - \varepsilon^2\Delta_{j,k}^2) \\
-\tau^2(a_{21}a_{11} + a_{22}a_{21})(\Delta_{j,k} - \varepsilon^2\Delta_{j,k}^2) \\
1 - \tau^2(a_{21}a_{12} + a_{22}^2)(\Delta_{j,k} - \varepsilon^2\Delta_{j,k}^2)
\end{bmatrix}
\begin{bmatrix}
\tilde{k}_{1,l}^{(l+1)} \\
\tilde{k}_{2,l}^{(l+1)}
\end{bmatrix}_{j,k}
\]

where \( \Delta_{j,k} = (\Delta_x)_j + (\Delta_y)_k, j = 0, 1, 2, \ldots, N_x - 1, k = 0, 1, 2, \ldots, N_y - 1 \). After we obtain \((k_i^1)^{l+1}\) and \((k_i^2)^{l+1}\), then \((k_i^1)^{l+1}\) and \((k_i^2)^{l+1}\) are updated by \(\mathcal{F}_{N_x}^H(k_i^1)^{l+1}\mathcal{F}_{N_y}\) and \(\mathcal{F}_{N_x}^H(k_i^2)^{l+1}\mathcal{F}_{N_y}\) (i = 1, 2), respectively. We should note that the Fast Fourier Transform (FFT) algorithm is employed to the above process.

5 Numerical results

The purpose of this section is to test validity and efficiency of the newly proposed schemes for solving the QZS (1.1) with the periodic boundary conditions. In particular, when the s-stage Gauss method (see Remark 3.1) is employed for Scheme 3.3, we denote the resulting structure-preserving RK (SPRK) method as SPRK-s. For brevity, in the rest of this paper, Gauss methods of order 4 and 6 (see Table 1) are only used for demonstration purposes. Also, the results are compared with the existing TS-EWI-FP method [58] and CNS [8], where the Fourier pseudo-spectral method takes the place of the compact finite difference method in space. Numerical examples including accuracy tests, convergence rates of the QZS (1.1) to the classical ZS in the semi-classical limit, soliton collisions and pattern dynamics of the QZS (1.1) in 1D. Let \(E_{h,\tau}^n\) and \(N_{h,\tau}^n\) be the numerical solutions of \(E(\cdot, t_n)\) and \(N(\cdot, t_n)\) obtained by SPRK-2 and SPRK-3 with the spatial step \(h\) and the time step \(\tau\), respectively. To quantify the numerical methods, we define the error functions as, respectively

\[
e_\varepsilon(t_n) = \|E(\cdot, t_n) - E_{h,\tau}^n\|_{h,\infty}, \quad n_\varepsilon(t_n) = \|N(\cdot, t_n) - N_{h,\tau}^n\|_{h,\infty}.
\]

Meanwhile, we also define the relative residual functions on the mass and energy as, respectively

\[
RM(t_n) = \left| \frac{\mathcal{M}_{h}^n - \mathcal{M}_{h}^0}{\mathcal{M}_{h}^0} \right|, \quad RH(t_n) = \left| \frac{\mathcal{H}_{h}^n - \mathcal{H}_{h}^0}{\mathcal{H}_{h}^0} \right|.
\]

5.1 One-dimensional case

As we choose \(\varepsilon = 0\), the one-dimensional QZS (1.1) reduces to the classical ZS (cf. [56]), which admits a solitary-wave solution given by [22]

\[
E(x, t) = i\sqrt{2B^2(1 - V^2)} \text{sech}(B(x - x_0 - Vt)) e^{i(V(x - x_0)/2 - V^2/4 - B^2)t},
\]
Table 2 Spatial errors provided by SPRK-2 and SPRK-3 at $T = 1$ for the classical ZS with the initial conditions (5.1)

| Scheme | $\varepsilon = 0$ | $h_0 = 1$ | $h_0/2$ | $h_0/2^2$ | $h_0/2^3$ |
|--------|------------------|-----------|----------|-----------|-----------|
| SPRK-2 | $e_0(1)$         | $4.58e-2$ | $8.90e-5$ | $1.38e-9$ | $3.25e-15$ |
|        | $n_0(1)$         | $9.37e-2$ | $7.15e-4$ | $1.77e-8$ | $5.67e-13$ |
| SPRK-3 | $e_0(1)$         | $4.58e-2$ | $8.90e-5$ | $1.38e-9$ | $3.91e-15$ |
|        | $n_0(1)$         | $9.37e-2$ | $7.15e-4$ | $1.77e-8$ | $7.57e-13$ |

$N(x, t) = -2B^2 sech^2 \left( B (x - x_0 - Vt) \right), \ x \in \Omega,$

where $B$ is a constant, $x_0$ and $V$ represent the initial displacement and the propagation velocity of the soliton, respectively. In this test, the computations are done on the interval $\Omega = [-128, 128)$, and we choose the parameters $B = 1$, $V = \frac{1}{2}$, $x_0 = 0$, as well as the following initial conditions

$E_0(x) = i \sqrt{1.5} \ sech(x) e^{ix/4}$, $N_0(x) = -2 sech^2(x)$, $N_1(x) = -2 sech^2(x) \ tanh(x)$, $x \in \Omega$.

Table 2 reports the spatial errors of SPRK-2 and SPRK-3 at $T = 1$ with a very small time step $\tau = 1/2^{12}$ such that the discretization error in time is negligible. We observe that the spatial errors converge exponentially. Then, to test the temporal discretization errors of the numerical schemes, we fix the Fourier node 2048 so that spatial errors play no role here. In Table 3, we list the temporal errors of SPRK-2 and SPRK-3 for the classical ZS at $T = 1$ with different time steps, respectively, which shows that SPRK-2 and SPRK-3 are fourth and sixth order accurate in time, respectively. Then, in Figs. 1 and 2, we show the relative residuals on the mass and energy calculated by SPRK-2 and SPRK-3 on the time interval $[0, 200]$, respectively, where we choose $h_0 = 1$ and $\tau_0 = 1/4$.

Table 3 Temporal errors provided by SPRK-2 and SPRK-3 at $T = 1$ for the classical ZS with the initial conditions (5.1)

| Scheme | $\varepsilon = 0$ | $\tau_0/1/4$ | $\tau_0/2$ | $\tau_0/2^2$ | $\tau_0/2^3$ | $\tau_0/2^4$ |
|--------|------------------|-------------|----------|-----------|-----------|-----------|
| SPRK-2 | $e_0(1)$         | $2.57e-05$  | $1.34e-06$ | $8.03e-08$ | $4.98e-09$ | $3.11e-10$ |
|        | rate             | -           | $4.27$    | $4.06$    | $4.01$    | $4.00$    |
|        | $n_0(1)$         | $4.04e-05$  | $2.50e-06$ | $1.56e-07$ | $9.77e-09$ | $6.10e-10$ |
|        | rate             | -           | $4.02$    | $4.00$    | $4.01$    | $4.00$    |
| SPRK-3 | $e_0(1)$         | $8.54e-07$  | $1.07e-08$ | $1.13e-10$ | $1.58e-12$ | $2.41e-14$ |
|        | rate             | -           | $6.32$    | $6.55$    | $6.17$    | $6.03$    |
|        | $n_0(1)$         | $2.10e-07$  | $3.43e-09$ | $5.47e-11$ | $8.62e-13$ | $1.49e-14$ |
|        | rate             | -           | $5.94$    | $5.97$    | $5.99$    | $5.85$    |
Fig. 1 The relative residuals on the mass (left) and energy (right) of SPRK-2 for the classical ZS with the initial conditions (5.1), the Fourier node 1024 and the time step \( \tau = 1/20 \), respectively. It can be observed that SPRK-2 and SPRK-3 can exactly preserve the mass and original energy of the classical ZS.

To demonstrate the advantages of the proposed schemes, we fix the Fourier node 4096 and then investigate the numerical error accumulations and robustness of the proposed schemes in long numerical simulation as the large time step \( \tau = 1/20 \) is chosen. The results are summarized in Figs. 3 and 4, respectively. In Fig. 3, we can observe that compared with the TS-EWI-FP method [58], the errors provided by the SPRK-2 and SPRK-3 are not only much smaller at the same time steps, but also have a good numerical performance in long time simulations. In Fig. 4, it is clear to see that the computational results provided by the TS-EWI-FP method is unstable and our schemes are more robust.

Fig. 2 The relative residuals on the mass (left) and energy (right) of SPRK-3 for the classical ZS with the initial conditions (5.1), the Fourier node 1024 and the time step \( \tau = 1/20 \), respectively.
Fig. 3 Time evolution of the numerical errors for the classical ZS with the initial conditions (5.1), the Fourier node 4096 and the time step $\tau = 1/20$, respectively.

5.2 Two-dimensional case

In this subsection, we consider the two-dimensional QZS (1.1) with initial conditions

$$E_0(x, y) = \cos^2 \frac{\pi x}{8} \cos^2 \frac{\pi y}{8}, \quad N_0(x, y) = 0, \quad N_1(x, y) = 0, \quad x \in \Omega,$$

(5.2)
Fig. 4 The numerical solutions at \( T = 100 \) for the classical ZS with the initial conditions (5.1), the Fourier node 4096 and the time step \( \tau = 1/20 \), respectively

where \( \Omega = [-8, 8) \times [-8, 8) \) is the computational domain with the periodic boundary conditions.

Due to the exact solution is not known, we take the numerical solution obtained by the proposed SPRK-3 with the Fourier node 256 \( \times 256 \) and the time step \( \tau = 10^{-3} \) as the “reference solution.” Tables 4 and 5 list the temporal errors of SPRK-2 and SPRK-3 at \( T = 1 \) under different time steps \( \tau \) and \( \epsilon \), where the Fourier node is chosen as 256 \( \times 256 \). It can be clearly observed that SPRK-2 and SPRK-3 are really fourth order accurate and sixth order accurate in time, respectively. Subsequently, the plots for \( |E| \)
Table 4  Temporal errors provided by SPRK-2 at $T = 1$ for the QZS (1.1) with (5.2) and different parameters $\varepsilon$

| $\varepsilon$ (1) | $\tau_0 = 1/4$ | $\tau_0/2$ | $\tau_0/2^2$ | $\tau_0/2^3$ | $\tau_0/2^4$ |
|-------------------|----------------|------------|--------------|--------------|--------------|
| $\varepsilon = 1/2^2$ | 7.05e−5 | 4.99e−6 | 3.32e−7 | 2.11e−8 | 1.32e−9 |
| rate              | 3.82        | 3.91      | 3.98        | 4.00        |              |
| $\varepsilon = 1/2^3$ | 6.03e−5 | 4.50e−6 | 2.93e−7 | 1.85e−8 | 1.16e−9 |
| rate              | 3.75        | 3.94      | 3.99        | 4.00        |              |
| $\varepsilon = 1/2^4$ | 4.67e−5 | 3.45e−6 | 2.24e−7 | 1.41e−8 | 8.83e−10 |
| rate              | 3.76        | 3.95      | 3.99        | 4.00        |              |
| $\varepsilon = 1/2^5$ | 4.30e−5 | 3.16e−6 | 2.04e−7 | 1.29e−8 | 8.07e−10 |
| rate              | 3.77        | 3.95      | 3.99        | 4.00        |              |
| $\varepsilon = 1/2^6$ | 2.30e−5 | 1.51e−6 | 9.59e−8 | 6.00e−9 | 3.75e−10 |
| rate              | 3.92        | 3.98      | 4.00        | 4.00        |              |
| $\varepsilon = 1/2^7$ | 9.79e−6 | 6.20e−7 | 3.89e−8 | 2.43e−9 | 1.52e−10 |
| rate              | 3.98        | 4.00      | 4.00        | 4.00        |              |
| $\varepsilon = 1/2^8$ | 9.33e−6 | 6.00e−07| 3.79e−8 | 2.37e−9 | 1.49e−10 |
| rate              | 3.96        | 3.99      | 4.00        | 4.00        |              |
| $\varepsilon = 1/2^9$ | 1.01e−5 | 6.57e−7 | 4.15e−8 | 2.60e−9 | 1.63e−10 |
| rate              | 3.95        | 3.98      | 4.00        | 4.00        |              |
| $\varepsilon = 1/2^{10}$ | 1.03e−5 | 6.71e−7 | 4.24e−8 | 2.66e−9 | 1.66e−10 |
| rate              | 3.95        | 3.98      | 4.00        | 4.00        |              |
| $\varepsilon = 1/2^{11}$ | 1.04e−5 | 6.74e−7 | 4.27e−8 | 2.68e−9 | 1.67e−10 |
| rate              | 3.95        | 3.98      | 4.00        | 4.00        |              |

and $N$ at $T = 10$ are shown in Fig. 5, which implies the numerical solutions obtained by the proposed method are stable in a long time-integration. In Figs. 6 and 7, we display the relative residuals on the mass and energy in time interval $[0, 200]$ obtained by SPRK-2 and SPRK-3 with the parameter $\varepsilon = 1/2^7$, the Fourier node $256 \times 256$ and the time step $\tau = 1/20$, which implies that the proposed schemes can exactly preserve the mass and original energy of the QZS (1.1).

To verify the efficiency of the proposed schemes, we also investigated the maximum norm errors and the CPU time using SPRK-2, SPRK-3 and CNS [8] at $T = 1$ with the parameter $\varepsilon = 1/2^3$ and the Fourier node $256 \times 256$, respectively. The results are summarized in Table 6. As illustrated in the Table, we can draw that for a given global error, the proposed schemes spend much less CPU time than CNS, which implies that
Table 5 Temporal errors provided by SPRK-3 at $T = 1$ for the QZS (1.1) with (5.2) and different parameters $\varepsilon$

| $\varepsilon$ | $\tau_0 = 1/4$ | $\tau_0/2$ | $\tau_0/2^2$ | $\tau_0/2^3$ |
|--------------|----------------|-------------|--------------|--------------|
| $\varepsilon = \frac{1}{2^4}$ | 2.95e-6 | 6.41e-8 | 1.10e-9 | 1.76e-11 |
| rate | - | 5.53 | 5.87 | 5.97 |
| $\varepsilon = \frac{1}{2^5}$ | 1.63e-6 | 2.95e-8 | 4.78e-10 | 7.54e-12 |
| rate | - | 5.79 | 5.95 | 5.99 |
| $\varepsilon = \frac{1}{2^6}$ | 1.19e-6 | 2.13e-8 | 3.44e-10 | 5.42e-12 |
| rate | - | 5.80 | 5.95 | 5.99 |
| $\varepsilon = \frac{1}{2^7}$ | 1.08e-6 | 1.91e-8 | 3.08e-10 | 4.86e-12 |
| rate | - | 5.82 | 5.95 | 5.99 |
| $\varepsilon = \frac{1}{2^8}$ | 1.05e-6 | 1.86e-8 | 3.00e-10 | 4.72e-12 |
| rate | - | 5.82 | 5.96 | 5.99 |
| $\varepsilon = \frac{1}{2^9}$ | 1.04e-6 | 1.85e-8 | 2.98e-10 | 4.69e-12 |
| rate | - | 5.82 | 5.96 | 5.99 |
| $\varepsilon = \frac{1}{2^{10}}$ | 2.70e-7 | 5.10e-9 | 8.41e-11 | 1.33e-12 |
| rate | - | 5.73 | 5.92 | 5.98 |
| $\varepsilon = \frac{1}{2^{11}}$ | 6.89e-8 | 1.18e-9 | 1.89e-11 | 3.01e-13 |
| rate | - | 5.86 | 5.96 | 5.99 |
| $\varepsilon = \frac{1}{2^{12}}$ | 1.24e-7 | 2.08e-9 | 3.31e-11 | 5.33e-13 |
| rate | - | 5.90 | 5.97 | 5.98 |
| $\varepsilon = \frac{1}{2^{13}}$ | 1.56e-7 | 2.60e-9 | 4.14e-11 | 6.91e-13 |
| rate | - | 5.90 | 5.97 | 5.99 |
| $\varepsilon = \frac{1}{2^{14}}$ | 1.63e-7 | 2.72e-9 | 4.32e-11 | 8.00e-13 |
| rate | - | 5.90 | 5.98 | 5.98 |
| $\varepsilon = \frac{1}{2^{15}}$ | 1.65e-7 | 2.75e-9 | 4.37e-11 | 7.29e-13 |
| rate | - | 5.90 | 5.98 | 5.98 |

our schemes are much more efficient. We note that the “reference solution” is obtained using the proposed SPRK-3 with the parameter $\varepsilon = 1/2^3$, the Fourier node $256 \times 256$ and the time step $\tau = 10^{-3}$, respectively.

Subsequently, by taking the initial conditions (5.2), the Fourier node $256 \times 256$ and the time step $\tau = 1/100$, we apply SPRK-2 and SPRK-3 to study the convergence rates of the QZS (1.1) to its limiting model, i.e., the classical ZS ($\varepsilon = 0$), respectively. Figure 8 shows the maximum norm errors calculated by using SPRK-2 and SPRK-3 between the QZS (1.1) and the corresponding classical ZS at $T = 10$ with different parameters $\varepsilon = 2^{-(2j+1)}$, $j = 2, 3, 4, 5, 6, 7$. As illustrated in the figures, the solution of the QZS (1.1) with the initial conditions (5.2) converges to the classical ZS quadratically with respect to $\varepsilon$, which is consistent with the existing theoretical result presented in [16].
Fig. 5  The numerical solutions at $t = 10$ for the QZS (1.1) with the initial conditions (5.2), the parameter $\varepsilon = \frac{1}{23}$, the Fourier node $256 \times 256$ and the time step $\tau = 1/20$, respectively.

Fig. 6  The relative residuals on the mass (left) and energy (right) of SPRK-2 for the QZS (1.1) with the initial conditions (5.2), the parameter $\varepsilon = \frac{1}{27}$, the Fourier node $256 \times 256$ and the time step $\tau = 1/20$, respectively.
**Fig. 7** The relative residuals on the mass (left) and energy (right) of SPRK-3 for the QZS (1.1) with the initial conditions (5.2), the parameter $\varepsilon = \frac{1}{27}$, the Fourier node $256 \times 256$ and the time step $\tau = 1/20$, respectively.

**Table 6** Numerical errors and computational CPU times using three numerical schemes solving the QZS (1.1) with the initial conditions (5.2) at $T = 1$, the parameter $\varepsilon = 1/23$ and the Fourier node $256 \times 256$, respectively.

| Scheme     | $\tau$       | $e_{\varepsilon}(1)$ | $n_{\varepsilon}(1)$ | CPU time (s) |
|------------|---------------|-----------------------|-----------------------|--------------|
| CNS [8]    | $1.0 \times 10^{-5}$ | 7.18e−10              | 1.22e−09              | 224.19       |
| SPRK-2     | $1.0 \times 10^{-2}$ | 1.94e−10              | 2.55e−11              | 5.27         |
| SPRK-3     | $5.0 \times 10^{-2}$ | 1.26e−10              | 4.99e−12              | 2.68         |

**Fig. 8** Convergence rates of SPRK-2 (left) and SPRK-3 (right) between the QZS (1.1) and the classical ZS at $T = 10$, respectively, where we choose the initial conditions (5.2), the Fourier node $256 \times 256$ and the time step $\tau = 1/100$, respectively.
5.3 Dynamic simulations

In this subsection, the dynamic simulations of one-dimensional case are investigated using the proposed schemes. For the first example, we choose the initial conditions as
Fig. 10  Inelastic collision between two solitons for the QZS (1.1) with the initial conditions (5.3) for Case II
Fig. 11 Inelastic collision between two solitons for the QZS (1.1) with the initial conditions (5.3) for Case III.
Fig. 12  The relative residuals on the mass of SPRK-2 for the QZS (1.1) with the initial conditions (5.3) for Case I, II and III (from left to right), respectively where we choose the parameter $\epsilon = \frac{1}{22}$, the Fourier node 4000 and the time step $\tau = 1/20$.

$$E_0(x) = i \sum_{j=1}^{2} \sqrt{2 \left(1 - V_j^2\right) \text{sech}(x - x_j)} e^{iV_j(x-x_j)/2},$$

$$N_0(x) = -2 \sum_{j=1}^{2} \text{sech}^2(x - x_j),$$

$$N_1(x) = -4 \sum_{j=1}^{2} V_j \text{sech}^2(x - x_j) \tanh(x - x_j), \quad x \in \Omega,$$

where $x_1$ and $x_2$ are initial locations of the two solitary waves, and $V_1$ and $V_2$ are the propagation velocity and the sign means moving to the right or left. For brevity, all computations are performed by using SPRK-2 with the Fourier node 4000 and the

Fig. 13  The relative residuals on the energy of SPRK-2 for the QZS (1.1) with the initial conditions (5.3) for Case I, II and III (from left to right), respectively where we choose the parameter $\epsilon = \frac{1}{22}$, the Fourier node 4000 and the time step $\tau = 1/20$, respectively.
time step $\tau = 1/20$, where the computational domain is chosen as $\Omega = [-200, 200)$. Moreover, we take the following parameters as:

- Case I: $x_1 = -x_2 = -30$, $V_1 = -V_2 = \frac{1}{2}$;
- Case II: $x_1 = -x_2 = -30$, $V_1 = \frac{3}{4}$, $V_2 = -\frac{1}{2}$;
- Case III: $x_1 = -x_2 = -5$, $V_1 = \frac{3}{4}$, $V_2 = -\frac{1}{2}$.

The surface plots of the interaction of two solitary waves for QZS (1.1) and the classical ZS ($\varepsilon = 0$) for Case I, II, and III are demonstrated in Figs. 9, 10, and 11, respectively, which implies that: (i) all the collisions between two solitons are not elastic; (ii) when two initially well-separated solitons with opposite propagation velocities (cf. Case I in Fig. 9) or different propagation speeds (cf. Case II in Fig. 10), they collide and fuse into a new soliton with the strengthened amplitude and the narrower width; (iii) the amplitude-weakened solitons with propagation speeds changed and some small radiation are generated during the collision; and (iv) when the initial locations are not initially well-separated (cf. Case III in Fig. 11), the dynamics is considerably more complicated, it seems that a periodic perturbation on the position of some localized

![Fig. 14](image_url) The contours of $|\sqrt{E(x,t)}|$ for pattern dynamics of the QZS (1.1) provided by SPRK-2 with the initial conditions (5.4)
pulse. In addition, from Figs. 9, 10, and 11, we also observe that the soliton-soliton collisions of the QZS (1.1) are more unstable than the corresponding classical ZS after collision, and the quantum effect makes the chaos much more obvious. The larger the quantum effect is, the more obvious the spatiotemporal chaos is. In particular, for the strong quantum regime $\varepsilon = 1$, there are small outgoing waves emitting before colliding, and the chaos is much more obvious than the classical one. In Figs. 12 and 13, we display the relative residuals on mass and energy provided by SPRK-2, which shows that the proposed method can exactly conserve the discrete mass and original energy.

Then, for the second example, the initial conditions are chosen as (cf. [41])

$$E_0(x) = E_0(1 + \beta \cos(kx)), \quad N_0(x) = -\sqrt{2}E_0k\beta \cos(kx), \quad N_1(x) = 0, \quad x \in \Omega, \quad (5.4)$$

Fig. 15 The relative residuals on mass of the QZS (1.1) provided by SPRK-2 with the initial conditions (5.4), the different parameters $\varepsilon$, the Fourier node 2000 and the time step $\tau = 1/20$, respectively.
where \( E_0 = \left( \frac{k}{\sqrt{2}} \right)(1 + \varepsilon^2 k^2) \) is the amplitude of the pump Langmuir wave, \( 0 < k < \sqrt{2} E_0 \) and \( \beta \) represents a relatively small constant to emphasize that the perturbation is small.

We take the computational domain \( \Omega = [-100, 100) \) and choose the parameters \( k = 0.7 \) and \( \beta = 0.001 \). For brevity, all computations are performed by using SPRK-2 over the time interval \([0, 200]\) with the Fourier node 2000 and the time step \( \tau = 1/20 \), respectively. Figure 14 shows that many solitary patterns can be generated and excited through the modulational instability of unstable harmonic modes. It can be clearly observed that the QZS (1.1) is more unstable than the corresponding classical ZS. In particular, for the strong quantum regime \( \varepsilon = 1 \), numerical simulation also indicates that the motion of the trains leads to more collision among the neighboring coherent solitary patterns, and fuse into the newer pattern with strengthened amplitude. This space-time evolution reveals that the spatiotemporal chaos is more obvious than the classical one. The numerical results are in good agreement with the results given in [41]. Figures 15 and 16 display the relative residuals on the mass and energy provided

\[
\begin{align*}
\text{Fig. 16} & \quad \text{The relative residuals on energy of the QZS (1.1) provided by SPRK-2 with the initial conditions} \\
& \quad \text{(5.4), the different parameters} \ \varepsilon, \ \text{the Fourier node 2000 and the time step} \ \tau = 1/20, \ \text{respectively.}
\end{align*}
\]
by SPRK-2 with the different parameters \( \varepsilon \), which implies that SPRK-2 can exactly preserve the mass and energy.

6 Conclusions

In this paper, we develop a novel class of high-order accurate structure-preserving schemes for solving the QZS (1.1), which is based on the idea of the QAV approach, the symplectic RK method together with the standard Fourier pseudo-spectral method in space. We show that the proposed schemes can exactly preserve the discrete mass and original Hamiltonian energy. In addition, an efficient fixed-point iteration is presented to solve the resulting nonlinear equations of the proposed schemes. Numerical experiments for the QZS (1.1) are carried out to illustrate the capability and accuracy of the new schemes. We also use our new schemes to numerically simulate the soliton-soliton interaction and the pattern dynamics of the QZS (1.1) in 1D. Moreover, it is numerically shown that the numerical solution of the QZS (1.1) converges to the classical Zakharov system quadratically in the semi-classical limit. As far as we know, there are some works (e.g., see Refs. [8, 59]) on optimal error estimates of second-order mass- and energy-conserving schemes for the QZS (1.1), but the error estimate of high-order ones is still not available. Thus, how to establish optimal error estimates for the proposed schemes will be an interesting topic for future studies.

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Author contribution Contributions Gengen Zhang: formal analysis, investigation, software, validation, writing—original draft. Chaolong Jiang: conceptualisation, supervision, methodology, writing—review and editing.

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Data Availability The data that support the findings of this study are available from the corresponding author upon reasonable request.

Declarations

Conflict of interest The authors declare that they have no conflict of interest.

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