Asymptotic symmetry for a class of nonlinear fractional reaction-diffusion equations

Sven Jarohs∗ and Tobias Weth†

December 12, 2013

Abstract

We study the nonlinear fractional reaction-diffusion equation
\[ \partial_t u + (-\Delta)^s u = f(t, x, u) \quad \text{in} \quad \Omega \]
\[ u = 0 \quad \text{on} \quad \partial \Omega \]
in a bounded domain \( \Omega \) together with Dirichlet boundary conditions on \( \mathbb{R}^N \setminus \Omega \). We prove asymptotic symmetry of nonnegative globally bounded solutions in the case where the underlying data obeys some symmetry and monotonicity assumptions. More precisely, we assume that \( \Omega \) is symmetric with respect to reflection at a hyperplane, say \( \{ x_1 = 0 \} \), and convex in the \( x_1 \)-direction, and that the nonlinearity \( f \) is even in \( x_1 \) and nonincreasing in \( |x_1| \). Under rather weak additional technical assumptions, we then show that any nonzero element in the \( \omega \)-limit set of nonnegative globally bounded solution is even in \( x_1 \) and strictly decreasing in \( |x_1| \). This result, which is obtained via a series of new estimates for antisymmetric supersolutions of a corresponding family of linear equations, implies a strong maximum type principle which is not available in the non-fractional case \( s = 1 \).

Keywords: Fractional Laplacian · Asymptotic Symmetry · Moving Hyperplanes · Harnack Inequality

Mathematics Subject Classification: 35K58 · 35B40

1 Introduction

We consider the nonlinear fractional diffusion problem
\[
(P) \quad \begin{cases}
\partial_t u + (-\Delta)^s u = f(t, x, u) & \text{in} \ (0, \infty) \times \Omega, \\
u = 0 & \text{on} \ (0, \infty) \times (\mathbb{R}^N \setminus \Omega),
\end{cases}
\]
where \( \Omega \) is a bounded Lipschitz domain in \( \mathbb{R}^N \), \( s \in (0, 1) \) and \( f \) is a nonlinearity defined on \( (0, \infty) \times \Omega \times \mathbb{R} \). Here and in the following, \( \mathcal{B} \subset \mathbb{R} \) is an open interval (further assumptions on \( f \) are to be specified later). Moreover, \((-\Delta)^s\) denotes the fractional Laplacian, which for functions \( u \in H^2(\mathbb{R}^N) \) is defined via Fourier transform:
\[
(-\Delta)^s u(\xi) = |\xi|^{2s} \hat{u}(\xi) \quad \text{for a.e. } \xi \in \mathbb{R}^N.
\] (1)

Fueled by various applications in physics, biology or finance, linear and nonlinear equations of the form given in \( P \) or of similar type have received immensely growing attention recently. In particular, evolution equations involving the fractional Laplacian appear in the quasi-geostrophic equations (see

∗Institut für Mathematik, Goethe-Universität, Frankfurt, Robert-Mayer-Straße 10, D-60054 Frankfurt, jarohs@math.uni-frankfurt.de.
†Institut für Mathematik, Goethe-Universität, Frankfurt, Robert-Mayer-Straße 10, D-60054 Frankfurt, weth@math.uni-frankfurt.de.
Asymptotic symmetry for a class of nonlinear fractional reaction-diffusion equations

e.g. [11, 33] and in the fractional porous medium equation (see [30]), while further applications in the context of stable processes are considered e.g. in [2, 21]. Very recently, the fractional Laplacian has been studied in conformal geometry, see [14, 22]. In order to incorporate the Dirichlet boundary condition on \( \mathbb{R}^N \setminus \Omega \) in \((P)\), the operator \((-\Delta)^s\) has to be replaced by the Friedrichs extension of the restriction of \((-\Delta)^s\) to the space \(C_c^\infty(\Omega) \subset L^2(\Omega)\) of test functions. Here and in the following, we identify \(L^2(\Omega)\) with the space \( \{ u \in L^2(\mathbb{R}^N) : u \equiv 0 \text{ on } \mathbb{R}^N \setminus \Omega \} \). This new operator, which we will also denote by \((-\Delta)^s\) in the following, has the form domain \( \mathcal{H}_0^s(\Omega) = \{ u \in H^s(\mathbb{R}^N) : u \equiv 0 \text{ on } \mathbb{R}^N \setminus \Omega \} \), and it is widely used in analysis and probability theory. In particular, it has recently been considered in the context of semilinear problems, see e.g. [3, 35, 28] and the references therein. In probabilistic terms, the operator coincides with the generator of the 2s-stable process in \( \Omega \) killed upon leaving \( \Omega \). We note that for \( u \in C_c^\infty(\Omega) \) we have the representation

\[
(-\Delta)^s u(x) = c_{N,s} \text{P.V.} \int_{\mathbb{R}^N} \frac{u(x) - u(y)}{|x-y|^{N+2s}} dy = c_{N,s} \text{P.V.} \int_{\mathbb{R}^N} \frac{u(x) - u(y)}{|x-y|^{N+2s}} dy + u(x) \kappa(x)
\]

for \( x \in \Omega \), where \( \text{P.V.} \) stands for the principal value integral and

\[
c_{N,s} = s(1-s)\pi^{-N/2}4^s \frac{\Gamma(\frac{N+2s}{2})}{\Gamma(2-s)} , \quad \kappa(x) := c_{N,s} \int_{\mathbb{R}^N \setminus \Omega} \frac{|x-y|^{-N-2s}}{dy} \text{ for } x \in \Omega. \tag{3}
\]

see e.g. [8] Remark 3.11.

The focus of the present paper is the asymptotic shape of global bounded solutions of \((P)\), i.e., the symmetry (and monotonicity) of elements in the corresponding \( \omega \)-limit sets. For this we will use a weak formulation for solutions of \((P)\). The quadratic form corresponding to \((-\Delta)^s\) on \( \mathcal{H}_0^s(\Omega) \) is given by

\[
\mathcal{E}(u,v) = \left\langle (-\Delta)^s u, (-\Delta)^s v \right\rangle_{L^2(\mathbb{R}^N)} = \frac{c_{N,s}}{2} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{(u(x) - u(y))(v(x) - v(y))}{|x-y|^{N+2s}} dx \, dy. \tag{4}
\]

Since \( \Omega \) is bounded, \( \mathcal{E} \) defines a scalar product on \( \mathcal{H}_0^s(\Omega) \) which is equivalent to the standard scalar product induced from the embedding \( \mathcal{H}_0^s(\Omega) \hookrightarrow H^s(\mathbb{R}^N) \). Consider the space \( C_0(\Omega) := \{ u \in C(\mathbb{R}^N) : u \equiv 0 \text{ on } \mathbb{R}^N \setminus \Omega \} \) endowed with the usual \( L^\infty \)-norm. We say that a function \( u : (0, \infty) \times \mathbb{R}^N \to \mathbb{R} \) is a solution of \((P)\) if \( u \in C((0,\infty), \mathcal{H}_0^s(\Omega) \cap C_0(\Omega)) \cap C^1((0,\infty), L^2(\Omega)) \), \( u(t,x) \in \mathcal{B} \) for every \((t,x) \in (0,\infty) \times \Omega \) and

\[
\mathcal{E}(u(t), \varphi) = \int_{\Omega} (f(t,x,u) - \partial_t u) \varphi \, dx \quad \text{for all } \varphi \in \mathcal{H}_0^s(\Omega), t \in (0,\infty). \tag{5}
\]

For a solution \( u \) of \((P)\), we define the \( \omega \)-limit set (with respect to the norm \( \| \cdot \|_{L^\infty} \)) as

\[
\omega(u) := \{ z \in C_0(\Omega) : \| u(t_k) - z \|_{L^\infty} \to 0 \text{ for some } t_k \to \infty \}
\]

To state our main result, we introduce the following assumptions.

(D1) \( \Omega \) is bounded with a Lipschitz boundary. Moreover, \( \Omega \) is convex and symmetric in \( x_1 \), i.e., for every \( x \in \Omega \) and \( s \in [-1,1] \) we have \((sx_1, sx_2, \ldots, sx_N) \in \Omega\).

(D2) For every \( \lambda > 0 \), the set \( \Omega_\lambda := \{ x \in \Omega : x_1 > \lambda \} \) has at most finitely many connected components.

(F1) \( f : (0,\infty) \times \mathcal{B} \to \mathbb{R} \) is continuous. Moreover, for every bounded subset \( K \subset \mathcal{B} \) there exists \( L = L(K) > 0 \) such that

\[
\sup_{x \in \Omega, t > 0} |f(t,x,u) - f(t,x,v)| \leq L|u - v| \text{ for } u, v \in K.
\]
Asymptotic symmetry for a class of nonlinear fractional reaction-diffusion equations

(F2) \( f \) is symmetric in \( x_1 \) and nonincreasing in \( |x_1| \), i.e., for every \( t \in (0, \infty) \), \( u \in \mathcal{B} \), \( x \in \Omega \) and \( s \in [-1, 1] \) we have \( f(t, sx_1, x_2, \ldots, x_N, u) \geq f(t, x, u) \).

We note that (D2) is a technical assumption which is needed for some but not all of our results. The main result of this paper is the following.

**Theorem 1.1.** Let (D1), (F1), (F2) be satisfied, and let \( u \) be a nonnegative global solution of (P) satisfying the following conditions:

(1) There is \( c_u > 0 \) such that \( \|u(t)\|_{L^\infty} \leq c_u \) for every \( t > 0 \).

(2) The functions \( u(\tau + \cdot, \cdot) \), \( \tau \geq 1 \) are uniformly equicontinuous on \([0, 1] \times \overline{\Omega} \) that is

\[
\lim_{h \to 0} \sup_{\tau \geq 1} \left| \frac{u(t, x) - u(t, \bar{x})}{\|x - \bar{x}\|} \right| = 0.
\]

Suppose in addition that (D2) holds or that \( z \neq 0 \) for every \( z \in \omega(u) \). Then \( u \) is asymptotically symmetric in \( x_1 \), i.e., for all \( z \in \omega(u) \) we have \( z(-x_1, x') = z(x_1, x') \) for all \( (x_1, x') \in \Omega \).

Moreover, for every \( z \in \omega(u) \) we have the following alternative: Either \( z \equiv 0 \) on \( \Omega \), or \( z \) is strictly decreasing in \( |x_1| \) and therefore strictly positive in \( \Omega \).

We immediately deduce the following corollary for equilibria and time-periodic solutions.

**Corollary 1.2.** Let (D1) be satisfied for \( \Omega \).

(i) Let \( f : \overline{\Omega} \times \mathcal{B} \to \mathbb{R} \), \( (x,u) \mapsto f(x,u) \) be such that

(i.1) \( f \) is continuous in \( x \in \overline{\Omega} \) and locally Lipschitz in \( u \) uniformly with respect to \( x \);

(i.2) \( f \) is symmetric in \( x_1 \) and nonincreasing in \( |x_1| \), i.e., for every \( u \in \mathcal{B} \), \( x \in \Omega \) and \( s \in [-1, 1] \) we have \( f(sx_1, x_2, \ldots, x_N, u) \geq f(x, u) \).

Moreover, let \( u \in C_0(\Omega) \cap \mathcal{H}_0^1(\Omega) \) be a nonnegative nontrivial weak solution of the elliptic problem

\[
(-\Delta)u = f(x,u) \quad \text{in} \ \Omega, \quad u = 0 \quad \text{on} \ \mathbb{R}^N \setminus \Omega, \quad (6)
\]

i.e., \( u(x) \in \mathcal{B} \) for a.e. \( x \in \Omega \) and \( \mathcal{E}(u, \varphi) = \int_{\Omega} f(x,u(x))\varphi(x) dx \) for every \( \varphi \in \mathcal{H}_0^1(\Omega) \). Then \( u \) is symmetric in \( x_1 \) and strictly decreasing in \( |x_1| \).

(ii) Suppose that \( f : (0, \infty) \times \Omega \times \mathcal{B} \to \mathbb{R} \) satisfies (F1), (F2) and is periodic in \( t \), i.e. there is \( T > 0 \) such that \( f(t + T, x, u) = f(t, x, u) \) for all \( t, x, u \). Suppose furthermore that \( u \) is a nontrivial nonnegative T-periodic solution of (P), i.e., \( u(t + T, x) = u(t, x) \) for all \( x \in \Omega, t \in (0, \infty) \). Suppose finally that either (D2) holds or that \( u(t, \cdot) \neq 0 \) on \( \Omega \) for all \( t \). Then \( u(t, \cdot) \) is symmetric in \( x_1 \) and strictly decreasing in \( |x_1| \) for all times \( t \in (0, \infty) \).

**Remark 1.3.** (i) The nonnegativity assumption on \( u \) in Theorem 1.1 can be weakened in special cases. More precisely, if the other assumptions of Theorem 1.1 are satisfied, \( u(t_0, \cdot) \) is nonnegative on \( \Omega \) for some \( t_0 > 0 \) and \( f(t, \cdot, 0) \geq 0 \) for all \( t \geq t_0 \), then \( u(t, \cdot) \) is nonnegative for \( t \geq t_0 \) as a consequence of the weak maximum principle in the form discussed in Remark 2.6 below. Thus Theorem 1.1 applies to \( u \) after a time shift.

(ii) Assumption (U2) implies that \( \{u(t, \cdot) : t > 0\} \subset C_0(\overline{\Omega}) \) is relatively compact and therefore \( \omega(u) \) is nonempty. In Proposition 4.1 below we give sufficient conditions for (U2) to hold.
(iii) In the case where, in addition to the assumptions of Theorem 1.1, \( \Omega \subset \mathbb{R}^N \) is a ball centered at zero and \( f \) is radially symmetric, i.e., \( f(t,x,u) = f(t,|x|,u) \), it follows – by the invariance of the equation under rotations – from Theorem 1.1 that every \( x \in \omega(u) \) is radially symmetric as well. In the special case of equilibria, i.e., solutions of (6), this has been proved in 3 under more restrictive assumptions on the nonlinearity.

(iv) We point out that we do not require an a priori positivity assumption on elements in \( \omega(u) \) in Theorem 1.1 and thus we also do not need to assume strict positivity of solutions of (6) in Corollary 1.2. This is a special feature of the nonlinear problems (P) and (6). The strong maximum principle given by Theorem 1.1 for elements \( x \in \omega(u) \) and by Corollary 1.2 for nonnegative solutions of (6) is a consequence of the monotonicity of the nonlinearity, and it is derived as a byproduct of the method proving the symmetry results (see in particular Lemma 3.2 below). This contrasts with the local case \( s = 1 \), where counterexamples show that such a strong maximum principle is false, see [29, Theorem 1.1], [27, Section 5] and the references therein. In this case, an additional positivity assumption as e.g. in [27, Theorem 2.2] is necessary to obtain asymptotic symmetry.

The proof of Theorem 1.1 is based on a parabolic variant of the moving plane method. As far as the main structure of the argument is concerned, we follow the strategy elaborated by Poláčik [27,28] in the context of Dirichlet problems for fully nonlinear parabolic differential equations, but we need new and quite different tools. We recall that the moving plane method has its roots in a classical work of Alexandrov [1] on constant mean curvature surfaces and Serrin [32] on overdetermined boundary value problems, whereas Gidas, Ni and Nirenberg [20] provided the framework to consider Dirichlet problems for nonlinear elliptic differential equations. In the case where the underlying domain is \( \mathbb{R}^N \), the method of moving plane has been applied in integral form in [16,18] to deduce symmetry and classification results for solutions of semilinear elliptic equations involving the fractional Laplacian. Birker, López-Mimbela and Wakolbinger [3] used a variant of the moving plane method, paired with probabilistic methods, to prove radial symmetry of all equilibria of (P) in the case where the underlying domain is the unit ball \( B \) and the nonlinearity \( f \) is nonnegative, independent of \( t \) and \( x \), and nondecreasing in \( u \). Up to the authors’ knowledge, our results are the first symmetry results for parabolic boundary value problems involving the fractional Laplacian and even for the elliptic problem if \( f \) depends on \( x \) or the domain is more general than a ball. We point out that – in comparison with the elliptic case – proving asymptotic symmetry in the parabolic setting with the moving plane approach requires much finer – time dependent – estimates. This is already evident from the seminal work of Poláčik [27,28] for the case of nonlinear differential equations. One key requirement is a special version of a parabolic Harnack inequality related to a linear fractional diffusion equation. Felsinger and Kassmann derived a parabolic Harnack inequality in [19], which requires nonnegativity of the solutions in the entire space. This global nonnegativity assumption is not technical since – already in the elliptic case – the Harnack inequality for the fractional Laplacian is not valid in a purely local form, see e.g. [23, Theorem 2.2] for a counterexample. However, since the moving plane method consists in studying the difference between the reflection of a solution of (P) at a hyperplane and the solution itself, we need to derive a corresponding Harnack inequality for antisymmetric (and therefore sign changing) supersolutions of a class of linear problems in the present paper. Another (closely related) problem in the fractional setting is the lack of local comparison principle to derive estimates via sub- or supersolutions. Here much finer quantitative arguments are needed to control the nonlocal effects and exclude the appearance of intersections in finite time. We will establish such estimates in two steps in Section 2.3 below, passing first to the Caffarelli-Silvestre extension of the solution \( u \), which is defined, for each fixed time, on the half space \( \mathbb{R}^{N+1}_+ \) (see [9]).

It seems worthwhile to note that another type of Dirichlet boundary conditions has also been assigned to the fractional Laplacian in the literature. In [10,12,35], the authors consider the \( s \)-th power of the Dirichlet Laplacian in spectral theoretic sense, which – in the case of a bounded domain \( \Omega \) – is given by

\[
\mathcal{A}_\Omega^s u := \sum_{k=1}^\infty \mu_k^s u_k e_k.
\]

Here \( \mu_k = \mu_k(\Omega) \) are the eigenvalues of the Dirichlet Laplacian on \( \Omega \) in increasing
order (counted with multiplicity), \( e_k, k \in \mathbb{N} \) are the corresponding eigenfunctions and \( u_k := \int_{\Omega} u e_k \, dx \) the corresponding Fourier coefficients of \( u \).

In order to explain the role of \( A^s_{\Omega} \) in terms of stochastic processes, we recall that the \( 2s \)-stable process is constructed by subordinating Brownian motion with a \( s \)-stable subordinator, see [2, Chapter 1.3]. On the other hand, the process generated by \( A^s_{\Omega} \) is obtained by first killing Brownian motion upon leaving \( \Omega \) and then subordinating this process with a \( s \)-stable subordinator, see e.g. [34]. Hence the order of killing and subordination is reversed in this case. It is easy to see that the corresponding operators coincide only if \( \Omega = \mathbb{R}^N \) (where the Dirichlet boundary conditions are not present). For more information related to these stochastic processes and their generators, we refer the reader to [4], [21] or [2, Chapter 3]. It is more involved in the present setting, and this is the only stage where we had to pass to the Caffarelli-Silvestre extensions of the solutions.

Moreover, we let \( \text{inrad} \subset \mathbb{R}^N \) be a subset and \( \text{diam} \) denote the diameter of \( A \). This notation – taken from [27] – differs slightly from the usual one \( x \in \Omega \), where \( x \in \mathbb{R}^N \) are understood in \( \text{inrad} \), \( x \in \mathbb{R}^N \) are always understood in \( \text{diam} \). For elliptic semilinear problems involving the operator \( A^s_{\Omega} \), symmetry and monotonicity results have been proved recently in special cases in [10][12] by applying the moving plane method to the Caffarelli-Silvestre extensions of the solutions.

The article is organized as follows. In Section 2, we develop the new tools we need to carry out the moving plane method for the fractional parabolic problem (P). We believe that the results of this Section could be of interest for other problems as well. Since, as already noted, the moving plane method consists in studying the difference between the reflection of a solution of (P) at a hyperplane and the solution itself, we are led to study antisymmetric supersolutions of linear problems here. Due to the nonlocality of the fractional Laplacian, it is important to estimate the influence of the negative part of these functions. This is one of the key differences in comparison with local problems involving classical differential operators. The first part of this Section is concerned with a parabolic small volume maximum principle. In Section 2.2 we establish, based on recent results in [19], a parabolic Harnack inequality for antisymmetric supersolutions of a class of linear fractional problems. Section 2.3 is devoted to a generalized subsolution estimate. The idea to control the positive part of the solution by comparing with suitable subsolutions is inspired by [27]. However, as mentioned above, the argument is essentially more involved in the present setting, and this is the only stage where we had to pass to the Caffarelli-Silvestre extension. In Section 2.4 we combine all estimates obtained so far to deduce our main result on antisymmetric supersolutions for a class of linear problems. This result should be seen as an analogue of [27, Theorem 3.7] for the fractional case. The moving plane argument is then carried out in Section 3. Here we follow the main structure of the argument in [27, Chapter 4], but we need to implement some new ideas at key points (see in particular the proof of Lemma 3.2) in the nonlocal setting. In the appendix, we present a sufficient condition for \( (U2) \), and we discuss a specific example to which Theorem 1.1 applies.

### 1.1 Notation

The following notation is used throughout the paper. For \( x \in \mathbb{R}^N \) and \( r > 0 \), \( B_r(x) \) is the open ball centered at \( x \) with radius \( r \) and \( \omega_N \) will denote the volume of the \( N \)-dimensional ball with radius 1. For any subset \( M \subset \mathbb{R}^N \), we denote by \( 1_M : \mathbb{R}^N \to \mathbb{R} \) the indicator function of \( M \) and \( \text{diam}(M) \) the diameter of \( M \). Moreover, we let \( \text{inrad}(M) \) denote the supremum of all \( r > 0 \) such that every connected component of \( M \) contains a ball \( B_r(x_0) \) with \( x_0 \in M \). This notation – taken from [27] – differs slightly from the usual one but is very convenient in our setting. If \( T \subset \mathbb{R}, \Omega \subset \mathbb{R}^N \) are subsets and \( u : T \times \Omega \to \mathbb{R}, (t, x) \mapsto u(t, x) \) is a function, we frequently write \( u(t, \cdot) : \Omega \to \mathbb{R} \) for \( t \in T \). If \( M \subset \mathbb{R}^N \) resp. \( M \subset \mathbb{R}^{N+1} \) is a subset and \( w : M \to \mathbb{R} \) is a function, the inequalities \( w \geq 0 \) and \( w > 0 \) are always understood in pointwise sense. Moreover, \( w^+ = \max\{w, 0\} \) resp. \( w^- = -\min\{w, 0\} \) denote the positive and negative part of \( w \), respectively. If \( M \) is measurable with \( |M| > 0 \) (where \( |\cdot| \) always stands for Lebesgue measure)
and \( w \in L^1(M) \), we put
\[
[w]_{L^1(M)} := \frac{1}{|M|} \int_M w(x) \, dx,
\]
respectively, to denote the mean of \( w \) over \( M \). If \( D, U \subset \mathbb{R}^N \) are subsets, the notation \( D \subset U \) means that \( D \) is compact and contained in the interior of \( U \). Moreover, we set
\[
\operatorname{dist}(D, U) := \inf \{ |x - y| : x \in D, y \in U \},
\]
so this notation does not stand for the usual Hausdorff distance. If \( D = \{x\} \) is a singleton, we simply write \( \operatorname{dist}(x, U) \) in place of \( \operatorname{dist}(\{x\}, U) \). Finally, when we call an interval \( T \subset \mathbb{R} \) a time interval, we assume that it consists of more than one point.

## 2 Antisymmetric supersolutions of a corresponding linear problem

Throughout this section, we consider a fixed open half space \( H \) and the reflection \( Q : \mathbb{R}^N \rightarrow \mathbb{R}^N \) at \( \partial H \). We will call a function \( w : \mathbb{R}^N \rightarrow \mathbb{R} \) antisymmetric if \( w(Q(x)) = -w(x) \) for every \( x \in \mathbb{R}^N \), i.e., \( w \) is antisymmetric with respect to \( Q \). We first fix notions of supersolutions. For an open subset \( U' \subset \mathbb{R}^N \), we introduce the function space
\[
\mathcal{V}^\delta(U') := \{ u \in L^\infty(\mathbb{R}^N) : \int_{U' \times U'} \frac{|u(x) - u(y)|^2}{|x - y|^{N+2\delta}} \, dxdy < \infty \},\tag{7}
\]
edowed with the norm
\[
\|u\|_{\mathcal{V}^\delta(U')} := \|u\|_{L^\infty(\mathbb{R}^N)} + \left( \int_{U' \times U'} \frac{|u(x) - u(y)|^2}{|x - y|^{N+2\delta}} \, dxdy \right)^{\frac{1}{2}}.
\]
We note that if \( U \subset \subset U' \) is a pair of open sets and \( u \in \mathcal{V}^\delta(U') \), \( v \in \mathcal{H}^\delta_0(U) \), then \( \delta'(u, v) \) is well defined by (4).

**Definition 2.1.** Let \( U \subset \mathbb{R}^N \) be a bounded open subset, \( T \) a time interval and \( c, g \in L^\infty(T \times U) \). We call a function \( v : T \times \mathbb{R}^N \rightarrow \mathbb{R} \) a supersolution of
\[
\partial_t v + (-\Delta)^\delta v = c(t,x) v + g(t,x) \tag{8}
\]
on \( T \times U \) if \( v \in C(T, \mathcal{V}^\delta(U')) \cap C^1(T, L^2(U)) \) for some open set \( U' \subset \mathbb{R}^N \) with \( U \subset \subset U' \) and
\[
\delta'(v(t), \varphi) \geq \int_{U} (c(t,x)v(t) + g(t,x) - \partial_t v(t)) \varphi \, dx
\]
for all \( \varphi \in \mathcal{H}^\delta_0(U) \), \( \varphi \geq 0 \) and a.e. \( t \in T \). If, in addition, \( U \subset H \) and \( v \) is antisymmetric, we call \( v \) an antisymmetric supersolution. A supersolution of (8) on \( T \times U \) will be called an entire supersolution if \( v \geq 0 \) on \( T \times (\mathbb{R}^N \setminus U) \). If \( U \subset H \), an antisymmetric supersolution of (8) on \( T \times U \) will be called an entire antisymmetric supersolution if \( v \geq 0 \) on \( T \times (H \setminus U) \).
Remark 2.2. (i) Note that an entire antisymmetric supersolution \( v \) of (8) on \( T \times U \) may take negative values in \( \mathbb{R}^N \setminus H \), so in general it is not an entire supersolution of (8).

(ii) Let \( T, U \) and \( c \) be as in the definition above. We will mostly consider the case \( g \equiv 0 \) in the remainder of the paper, i.e., we consider supersolutions of

\[
\partial_t v + (-\Delta)^s v = c(t, x)v
\]

on \( T \times U \). We briefly explain the connection between (P) and (9). Suppose that (F1) is satisfied and that

\[
\begin{align*}
H \cap \Omega \neq \emptyset, & \quad Q(H \cap \Omega) \subset \Omega \\
f(t, Q(x), u) \geq f(t, x, u) & \quad \text{for every } t \in (0, \infty), x \in U \text{ and } u \in \mathcal{B}.
\end{align*}
\]

Let \( u \) be a nonnegative solution of (P), and let \( v(t, x) = u(t, Q(x)) - u(t, x) \) for \( x \in \mathbb{R}^N, t \geq 0 \). Then \( v \) is an entire antisymmetric supersolution of (9) with \( T = (0, \infty), U = H \cap \Omega \) and

\[
c(t, x) = \begin{cases} 
\frac{f(t, x, u(t, Q(x)) - f(t, x, u(t, x))}{v(x)}, & u(t, Q(x)) \neq u(t, x); \\
0, & u(t, Q(x)) = u(t, x).
\end{cases}
\]

Indeed, by (10) we have \( v \geq 0 \) on \( T \times (H \setminus U) \). Moreover, for \( \varphi \in \mathcal{H}_0^1(U), \varphi \geq 0 \) and \( t \in (0, \infty) \) we have

\[
\begin{align*}
\mathcal{E}(v(t), \varphi) &= \mathcal{E}(u(t) \circ Q - u(t), \varphi) = \mathcal{E}(u(t), \varphi \circ Q - \varphi) \\
&= \int_{\Omega} (f(t, x, u) - \partial_t u)(\varphi \circ Q - \varphi) \, dx \\
&= \int_{U} [f(t, Q(x), u(t, Q(x))) - f(t, x, u(t, x)) - \partial_t (u \circ Q - u)] \varphi \, dx \\
&\geq \int_{U} [c(t, x)v - \partial_t v] \varphi \, dx,
\end{align*}
\]

where (11) was used in the last step.

The following observation will be useful in the sequel.

Lemma 2.3. For any \( \varphi \in \mathcal{H}_0^1(H) \) and every antisymmetric \( v \in H^s(\mathbb{R}^N) \) we have

\[
\begin{align*}
\mathcal{E}(v, \varphi) &= \frac{1}{2} \int_{\mathbb{R}^N} \left( v(x) - v(y) \right) (\varphi(x) - \varphi(y)) J(x, y) \, dx \, dy + 2 \int_H \kappa_H(x) v(x) \varphi(x) \, dx \\
&= \frac{CN_s}{|x-y|^{N+2s}} \int_{\mathbb{R}^N} \frac{CN_s}{|x-Q(y)|^{N+2s}} \, dy = \frac{4^s \Gamma\left(\frac{1}{2} + s\right)}{\sqrt{\pi} \Gamma\left(1 - s\right)} |\text{dist}(x, \partial H)|^{-2s}
\end{align*}
\]

for \( x, y \in H \), where \( c_{N,s} \) is given in (3). Moreover,

\[
J(x, y) \geq \frac{CN_s}{|x-y|^{N+2s}} \frac{[1 - 5^{-N/2 - s}]}{|x-y|^{N+2s}}
\]

for \( x, y \in H \) with \( |x - y| \leq \min\{\text{dist}(x, \partial H), \text{dist}(y, \partial H)\} \).
Proof. It is convenient to write \( \bar{x} \) in place of \( Q(x) \) for \( x \in \mathbb{R}^N \) in the following. For \( \varphi \in \mathcal{H}_0^1(\mathbb{R}) \) and an antisymmetric \( v \in H^1(\mathbb{R}) \) we then have

\[
\mathcal{E}(v, \varphi) = \frac{c_{N,s}}{2} \left( \int_H \int_{\mathbb{R}^N} \frac{(v(x) - v(y))(\varphi(x) - \varphi(y))}{|x-y|^{N+2s}} \, dx \, dy \right.
\]

\[
+ \int_H \int_{\mathbb{R}^N} \ldots \, dx \, dy + \int_H \int_{\mathbb{R}^N} \ldots \, dx \, dy \bigg) 
\]

\[
= \frac{c_{N,s}}{2} \int_H \int_{\mathbb{R}^N} \frac{(v(x) - v(y))(\varphi(x) - \varphi(y))}{|x-y|^{N+2s}} \, dx \, dy
\]

\[
- (v(\bar{x}) - v(y))\varphi(y) - \frac{(v(x) - v(y))\varphi(x)}{|x-y|^{N+2s}} \bigg) \, dx \, dy
\]

\[
= \frac{1}{2} \int_H \int_{\mathbb{R}^N} (v(x) - v(y))(\varphi(x) - \varphi(y))J(x,y) \, dx \, dy 
\]

\[
+ 2c_{N,s} \int_H \int_{\mathbb{R}^N} \frac{v(y)\varphi(y)}{|x-y|^{N+2s}} \, dx \, dy
\]

\[
= \frac{1}{2} \int_H \int_{\mathbb{R}^N} (v(x) - v(y))(\varphi(x) - \varphi(y))J(x,y) \, dx \, dy + 2 \int_H \kappa_H(x)v(x)\varphi(x) \, dx
\]

with \( J \) and \( \kappa_H \) as defined above, as claimed. To see (13), let \( d > 0 \) and \( x, y \in H \) with \( |x-y| \leq d \leq \min\{\text{dist}(x, \partial H), \text{dist}(y, \partial H)\} \). Then \( |x-y|^2 \geq |x-y|^2 + 4d^2 \) and therefore

\[
\frac{|x-y|^2}{|x-y|^2} \leq \frac{|x-y|^2}{|x-y|^2 + 4d^2} \leq \frac{1}{5},
\]

which implies that

\[
\frac{J(x,y)|x-y|^{N+2s}}{c_{N,s}} = \left( 1 - \left( \frac{|x-y|^2}{|x-y|^2} \right)^\frac{N+2s}{s} \right) \geq 1 - 5^{-N/2-s}
\]

as claimed in (13). \( \Box \)

2.1 A small volume maximum principle

The main result of this subsection is the following.

**Proposition 2.4.** For every \( c_\infty, \gamma > 0 \) there exists \( \delta = \delta(N,s,c_\infty, \gamma) > 0 \) such that for any bounded open subset \( U \subset H \) with \( |U| \leq \delta \), any time interval \( T := [t_0, t_1] \), any \( c \in L^\infty(T \times U) \) with \( \|c^+\|_{L^\infty} \leq c_\infty \) and any entire antisymmetric supersolution \( v \) of (9) on \( T \times U \) we have

\[
\|v^-(t,\cdot)\|_{L^\infty(H)} \leq e^{-\gamma(t-t_0)}\|v^-(t_0,\cdot)\|_{L^\infty(H)} \quad \text{for all } t \in T.
\]
Lemma 2.5. For every measurable $A \subset \mathbb{R}^N$ and every $x \in \mathbb{R}^N$ we have
\[
\int_{\mathbb{R}^N \setminus A} \frac{1}{|x - y|^{N+2s}} \, dy \geq K|A|^{-\frac{s}{N}}
\]
with $K = K(N,s) = \frac{N}{2s} \omega_N^{1+2s/N}$.

Proof of Proposition 2.4. For given $c_\infty > 0$ we put $\delta := \left( \frac{c_\infty K}{\gamma} \right)^{\frac{s}{N}}$, where $K$ is given in Lemma 2.5. By assumption and Lemma 2.5 we then have
\[
k_U(x) := \int_{\mathbb{R}^N \setminus U} \frac{c_{N,s}}{|x - y|^{N+2s}} \, dy \geq \gamma + c_\infty \quad \text{for every } x \in \mathbb{R}^N.
\]

Without loss of generality, we may assume that $t_0 = 0$. Let $d := \|v^-(0)||_{L^2(U)}$, and define $u(t,x) := e^{\beta_1 y(t,x)}$ for $t \in [0,t_1], x \in \mathbb{R}^N$. Then $u$ is an antisymmetric supersolution of $u + (\Delta)^s u = \tilde{c}(t,x)u$ on $U$ with $\tilde{c}(t,x) = c(t,x) + \gamma$. We need to show that
\[
u(t,x) \geq -d \quad \text{for } x \in H \text{ and } t \in [0,t_1].
\]

For $0 \leq t \leq t_1$, we consider the function $\varphi(t) = \varphi(t,\cdot) : \mathbb{R}^N \to \mathbb{R}$ defined by $\varphi(t,x) = (u(t,x) + d)^{-1}H(x)$. Since $u(t) \in \mathcal{F}'(U')$ for some open set $U'$ with $U \subset U'$ and $u \geq 0$ in $H \setminus U$, it follows from [17, Lemma 5.1] that $\varphi(t) \in \mathcal{D}'(U)$ for $0 \leq t \leq t_1$. We then have
\[
\delta'(u(t),\varphi(t)) \geq \int_U (\tilde{c}(t,x)u - \Delta u)\varphi \, dx \geq \int_U (c_\infty + \gamma)u(t)\varphi(t) \, dx + \frac{1}{2} \frac{d}{dt} \int_U \varphi(t)^2 \, dx.
\]

We first claim that
\[
\delta'(u(t),\varphi(t)) \leq -\delta'(u^-(t)1_H,\varphi(t)) \quad \text{for } (t,x) \in [0,t_1].
\]

Indeed, for $(t,x) \in [0,t_1] \times \mathbb{R}^N$ we have
\[
(u(t,x) - u(t,y)) \left( \varphi(t,x) - \varphi(t,y) \right) + (u^-(t,x)1_H(x) - u^-(t,y)1_H(y)) \left( \varphi(t,x) - \varphi(t,y) \right)
= - \left[ \varphi(t,x) \left( u(t,y) + u^-(t,y)1_H(y) \right) + \varphi(t,y) \left( u(t,x) + u^-(t,x)1_H(x) \right) \right].
\]

Thus we find, using the symmetry of the kernel and the antisymmetry of $u$,
\[
\delta'(u(t),\varphi(t)) + \delta'(u^-(t)1_H,\varphi(t)) = -c_{N,s} \int \varphi(t,y) \int_{\mathbb{R}^N} \frac{u(t,x) + 1_H(x)u^-(t,x)}{|x - y|^{N+2s}} \, dx \, dy
= -c_{N,s} \int H \varphi(t,y) \int_{\mathbb{R}^N} \left( \frac{u^+(t,x)}{|x - y|^{N+2s}} - \frac{u^-(t,x)}{|Q(x) - y|^{N+2s}} \right) \, dx \, dy
\]
and hence $\delta'(u(t),\varphi(t)) + \delta'(u^-(t)1_H,\varphi(t)) \leq 0$ for $t \in [0,t_1]$, since $|x - y| \leq |Q(x) - y|$ for $x,y \in H$ and $\varphi$ is nonnegative. This shows (18). We now put
\[
A_1(t) := \{ x \in H : u(t,x) \leq -d \} \quad \text{and} \quad A_2(t) := (\mathbb{R}^N \setminus H) \cup \{ y \in H : u(t,y) > -d \}.
\]
We fix \( h(t) \) with \( h(0) = 0 \), and we consider a function \( u \), \( v \) entire supersolution of (9) on \( T \times U \) with \( \| v \|_{L^\infty(U)} \leq c_w \) and any entire supersolution \( \varphi \) of (9) on \( T \times U \) we have
\[
\| v^-(t, \cdot) \|_{L^\infty(U)} \leq e^{-\gamma(t-t_0)} \| v^-(t_0, \cdot) \|_{L^\infty(U)} \quad \text{for all } t \in T.
\] (20)
As a consequence, we may readily derive the following weak maximum principle: If \( \Omega \subset \mathbb{R}^N \) is a bounded open subset, \( T := [t_0, t_1] \) a time interval, \( c \in L^\infty(T \times \Omega) \) and \( \nu \) an entire supersolution of (9) on \( T \times \Omega \) such that \( \nu(t_0, x) \geq 0 \) for a.e. \( x \in \Omega \), then also \( \nu(t, x) \geq 0 \) for all \( t \in T \) and almost every \( x \in \Omega \).

### 2.2 A Harnack inequality for antisymmetric supersolutions

In this part we state a Harnack inequality for antisymmetric supersolutions of (9). We will derive this inequality – via a reformulation of the problem – from a recent result in [19]. We need to introduce some notation. Denote by \( \Delta = \{(x, x) : x \in \mathbb{R}^N\} \) the diagonal in \( \mathbb{R}^N \times \mathbb{R}^N \). We fix \( r_0 \in (0, 1] \) and \( C_1, C_2 > 0 \), and we consider a function \( k : \mathbb{R}^N \times \mathbb{R}^N \setminus \Delta \to [0, \infty) \) satisfying, for every \( x, y \in \mathbb{R}^N \) with \( x \neq y \),

\begin{align*}
k(x, y) &= k(y, x); \\
k(x, y) &\leq C_1 |x - y|^{-N-2s}; \\
k(x, y) &\geq C_2 |x - y|^{-N-2s} \quad \text{if } |x - y| \leq r_0.
\end{align*}
(21)
The quadratic form corresponding to this kernel is given by
\[ \delta_k(u, v) = \frac{1}{2} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} (u(x) - u(y))(v(x) - v(y))k(x,y)\, dx \, dy, \] for \( u, v \in H^s(\mathbb{R}^N) \).

Recall the definition of \( \mathcal{I}^s(U') \) in (7). If \( U \subset \mathbb{R}^N \) is a bounded open subset, \( T \subset \mathbb{R} \) a time interval with nonempty interior and \( g \in L^2(T \times U) \), we say that a function \( v \) is a supersolution of the problem
\[ \partial_t v(t, x) - \mathcal{P}V. \int_{\mathbb{R}^N} (v(t,y) - v(t,x))k(x,y)\, dy = g \] on \( T \times U \) if \( v \in C(T, \mathcal{I}^s(U')) \cap C^1(T, L^2(U)) \) for some open set \( U' \subset \mathbb{R}^N \) with \( U \subset U' \) and
\[ \delta_k(v(t), \varphi) \geq \int_U [g(t,x) - \partial_t v(t,x)]\varphi(x)\, dx \]
for \( \varphi \in \mathcal{D}'(U) \), \( \varphi \geq 0 \) and a.e. \( t \in T \). Next we introduce notation for parabolic cylinders. For \( t_0 \in \mathbb{R} \), \( x_0 \in \mathbb{R}^N \), \( r, \vartheta > 0 \) we put \( Q(r, \vartheta, t_0, x_0) := (t_0, t_0 + 8 \vartheta) \times B_r(x_0) \) and
\[ Q^-(r, \vartheta, t_0, x_0) := (t_0, t_0 + \vartheta) \times B_r(x_0), \quad Q^+(r, \vartheta, t_0, x_0) := (t_0 + 7 \vartheta, t_0 + 8 \vartheta) \times B_r(x_0). \]
In view of the scaling properties of (22), the following is a mere reformulation of a special case of [19, Theorem 1.1], see also [19, Remark after Theorem 1.2]. We point out that the notion of supersolution considered in [19] is weaker than the one considered here.

**Theorem 2.7.** Let \( r_0 \in (0, 1) \) and \( \vartheta, C_1, C_2 > 0 \) be given. Then there are constants \( c_i > 0 \), \( i = 1, 2 \) depending on \( N, s, r_0, \vartheta, C_1, C_2 \) such that for any \( k : \mathbb{R}^N \times \mathbb{R}^N \setminus \Delta \rightarrow [0, \infty) \) satisfying (22), any \( (t_0, x_0) \in \mathbb{R}^{N+1} \), any \( g \in L^2(Q(r_0, \vartheta, t_0, x_0)) \) and any supersolution \( v \) of (22) on \( Q(r_0, \vartheta, t_0, x_0) \) which is nonnegative in \( (t_0, t_0 + 8 \vartheta) \times \mathbb{R}^N \) we have
\[ \inf_{(r, x) \in Q^*(r_0, \vartheta, t_0, x_0)} v \geq c_1[r]\|L^1(Q^-(r_0, \vartheta, t_0, x_0)) - c_2\|g\|L^2(Q(r_0, \vartheta, t_0, x_0))]. \] (23)

By an argument based on building chains of cylinders, we deduce the following Harnack inequality for general pairs of domains. We include the proof here since the argument is not completely standard. A similar argument has been detailed in [27, Appendix], but we need to argue somewhat differently since the triples of parabolic cylinders in Theorem 2.7 have a smaller overlap than the ones considered in [27].

**Corollary 2.8.** Let \( r_0 \in (0, 1) \), \( R, \tau, \varepsilon > 0 \) and \( C_1, C_2 > 0 \) be given. Then there exist positive constants \( c_i = c_i(N, s, r_0, C_1, C_2, R, \tau, \varepsilon) \), \( i = 1, 2 \) with the following property:
Let \( k : \mathbb{R}^N \times \mathbb{R}^N \setminus \Delta \rightarrow [0, \infty) \) satisfy (22), and let \( D \subset U \subset \mathbb{R}^N \) be a pair of bounded domains such that \( \text{dist}(D, \partial U) \geq 2r_0, |D| \geq \varepsilon \) and \( \text{diam}(D) \leq R \). Moreover, let \( g \in L^\infty(T \times U) \) and a supersolution \( v \) of (22) on \( T \times U \) be given such that \( v \) is nonnegative in \( T \times \mathbb{R}^N \), where \( T = [t_0, t_0 + 4\tau] \) for some \( t_0 \in \mathbb{R} \). Then we have
\[ \inf_{(r, x) \in T_+ \times D} v(r, x) \geq c_1\|L^1(T_+ \times D) - c_2\|g\|L^2(T \times \mathbb{R}^N)), \] (24)
where \( T_+ = [t_0 + 3\tau, t_0 + 4\tau] \) and \( T_- = [t_0 + \tau, t_0 + 2\tau] \).

**Proof.** We first note that there exist \( n = n(N, R, r_0) \in \mathbb{N} \) and \( \mu = \mu(N, R, r_0) > 0 \) such that the following holds:
For every subset \( D \subset \mathbb{R}^N \) with \( \text{diam} D \leq R \) there exists a subset \( S_D \subset D \) of \( n + 1 \) points such that \( D \) is
covered by the balls $B_{r_0}(x_j), x \in S_D$, and for every two points $x_s, x^* \in S_D$ there exists a finite sequence $x_j \in S_D, j = 0, \ldots, n$ such that
\[
x_0 = x_s, \quad x_n = x^* \quad \text{and} \quad |B_{r_0}(x_j) \cap B_{r_0}(x_{j+1})| \geq \mu \quad \text{for} \quad j = 0, \ldots, n - 1.
\] (25)

We now fix $D \subset \subset U \subset H$ as in the assertion, and we fix $n, \mu$ and a set $S_D$ with the property above. Next, we put $\vartheta = \frac{r}{4} \min\{\frac{r}{4}, \frac{1}{t^* - \vartheta}\}$, and we claim the following:

For given $t_s \in [t_0 + \vartheta, t_0 + 2\tau]$ and $t^* \in [t_0 + 3\vartheta, t_0 + 4\tau]$ there exists a finite sequence $t_s = s_0 < \ldots < s_m = t^* - \vartheta$ such that
\[
s_j + \frac{15}{2} \vartheta \leq s_{j+1} \leq s_j + 15 \vartheta \quad \text{for} \quad j = 0, \ldots, m - 1
\] (26)
and
\[
\max\{14, n\} \leq m \leq \max\{51, 3(n + 3)\} \quad \text{(27)}
\]
Indeed, let $m \in \mathbb{N}$ and $\sigma \in [0, 7\vartheta]$ be such that $t_s + 7m\vartheta + \sigma = t^* - \vartheta$. The definition of $\vartheta$ and the restrictions on $t_s, t^*$ then force (27), and (26) holds with $s_j := t_s + j\left(\frac{7\vartheta + \sigma}{\vartheta}\right)$ for $j = 0, \ldots, m$. Next, we fix $t_s \in [t_0 + \vartheta, t_0 + 2\tau], x_s \in S_D$ such that
\[
\|v\|_{L^1(Q_{t_s}(\rho, t_s, x_s))} = \max\{\|v\|_{L^1(Q_{t_s}(\rho, t_s, x_s))} : x_s \in S_D, t_0 + \vartheta \leq t \leq t_0 + 2\tau\}.
\]
Since the cylinders
\[
Q_{t_s}(\rho, t_s, x_s) = Q_{t_s}(\rho, t_s, x_s) \quad \text{for} \quad l \in \mathbb{N} \cup \{0\}, l \leq \frac{r}{\vartheta}, x_s \in S_D
\]
cover $[t_0 + \vartheta, t_0 + 2\tau] \times D$, we have
\[
\|v\|_{L^1([t_0 + \vartheta, t_0 + 2\tau] \times D)} = \frac{1}{|D|}\|v\|_{L^1([t_0 + \vartheta, t_0 + 2\tau] \times D)} \leq \frac{(n + 1)(\frac{7}{\vartheta} + 1)}{t^* - \vartheta}||v||_{L^1(Q_{t_s}(\rho, t_s, x_s))}
\]
\[
= \frac{(n + 1)(\frac{7}{\vartheta} + 1)}{t^* - \vartheta} ||v||_{L^1(Q_{t_s}(\rho, t_s, x_s))} \\leq \kappa_1 ||v||_{L^1(Q_{t_s}(\rho, t_s, x_s))} \quad \text{with} \quad \kappa_1 := \frac{2(n + 1)|B_{r_0}(0)|}{\vartheta} \quad \text{(28)}
\]
We now consider $t^* \in [t_0 + 3\vartheta, t_0 + 4\tau], x \in D$ arbitrary. Then we choose $x^* \in S_D$ such that $x \in B_{r_0}(x^*)$, and we choose $s_j, j = 0, \ldots, m$ with the properties (26) and (27). Moreover, we fix a sequence of points $x_j \in S_D, j = 0, \ldots, m$ such that (25) holds with $m$ in place of $n$. This may be done, since $m \geq n$, by repeating some of the points in the chain if necessary. We now define
\[
Q_j := Q_{t_0}(\rho, t_0, s_j, x_j) \quad \text{and} \quad Q_j^+ := Q_{t_0}(\rho, s_j, x_j) \quad \text{for} \quad j = 0, \ldots, m.
\]
We note that, by (25) and (26), we have
\[
|Q_j^+ \cap Q_j^-| \geq \frac{\mu \vartheta}{2} \quad \text{for} \quad j = 0, \ldots, m - 1.
\]
Hence we may estimate, using Theorem 2.7 and the fact that $Q_j \subset T \times U$ for $j = 0, \ldots, m$,
\[
c_1 ||v||_{L^1(Q_j^+ \setminus Q_j^-)} \leq \inf_{Q_j} v + c_2 ||g||_{L^\infty(Q_j \setminus Q_j^-)} \leq ||v||_{L^1(Q_j^+ \setminus Q_j^-)} + c_2 ||g||_{L^\infty(T \times U)} \leq \frac{|Q_j^+ \cap Q_j^-|}{|Q_j^+ \cup Q_j^-|} ||v||_{L^1(Q_j^+ \setminus Q_j^-)} + c_2 ||g||_{L^\infty(T \times U)}
\]
Asymptotic symmetry for a class of nonlinear fractional reaction-diffusion equations

\[
\leq \frac{2|B_n(0)|}{\mu} |v|_{L^1(Q_{s+1}^n)} + c_2 \|g\|_{L^\infty(T \times U)}.
\]

Iterating this estimate \(m\) times and using Theorem 2.7 once more, we obtain

\[
|v|_{L^1(Q_0^m)} \leq \left(\frac{2|B_n(0)|}{c_1 \mu}\right)^m |v|_{L^1(Q_0)} + c_2 \sum_{k=0}^{m-1} \left(\frac{2|B_n(0)|}{c_1 \mu}\right)^k \|g\|_{L^\infty(T \times U)}
\]

\[
\leq \left(\frac{2|B_n(0)|}{\mu}\right)^m c_1^{-(m+1)} \inf_{Q_0} v + c_2 \sum_{k=0}^{m} \left(\frac{2|B_n(0)|}{c_1 \mu}\right)^k \|g\|_{L^\infty(T \times U)}.
\]

Hence, since \((t^*, x^*) \in Q_m^n\), we conclude by (28) that

\[
v(t^*, x^*) \geq \inf_{Q_m^n} v \geq \hat{c}_1 |v|_{L^1(Q_0)} - \hat{c}_2 \|g\|_{L^\infty(T \times U)} \geq \frac{\hat{c}_1}{K_1} |v|_{L^1([t_0 + \tau_0 + 2\tau] \times D)} - \hat{c}_2 \|g\|_{L^\infty(T \times U)}
\]

with

\[
\hat{c}_1 = \left(\frac{2|B_n(0)|}{\mu}\right)^{-m} c_1^{m+1} \quad \text{and} \quad \hat{c}_2 = \frac{c_2}{c_1} \sum_{k=0}^{m} \left(\frac{2|B_n(0)|}{c_1 \mu}\right)^k.
\]

Hence the claim follows with \(\hat{c}_1 = \frac{\hat{c}_1}{K_1}\) and \(\hat{c}_2\) as above. Note that \(\hat{c}_1\) and \(\hat{c}_2\) only depend – via \(n, m, \mu, c_1, c_2\) and \(K_1\) – on the given quantities \(N, s, r_0, R, \tau, C_1\) and \(C_2\). \(\square\)

The main goal of this subsection is to deduce the following Harnack inequality for entire antisymmetric supersolutions of (2).

**Theorem 2.9.** Let \(r_0 \in (0, 1], c_{\infty}, R, \tau, \varepsilon > 0\) be given. Then there exist positive constants \(K_i > 0, i = 1, 2\) depending on \(N, s, r_0, c_{\infty}, R, \tau\) with the following property:

If \(D \subset \subset U \subset H\) is a pair of bounded domains with \(\text{dist}(D, \partial U) \geq 4r_0, \text{diam}(D) \leq R, |D| \geq \varepsilon, and \(v\) is entire antisymmetric supersolution of (2) on \(T \times U\) with \(T = [t_0, t_0 + 4\tau]\) for some \(t_0 \in \mathbb{R}\) and \(c \in L^\infty(T \times U)\) with \(|c|_{L^\infty} \leq c_{\infty}\) such that \(v(t) \in H^s(\mathbb{R}^N)\) for all \(t \in T\), then

\[
\inf_{(t, x) \in T_+ \times D} v(t, x) \geq K_1 |v|_{L^1(T_+ \times D)} - K_2 ||v||_{L^\infty(T \times U)},
\]

where \(T_+ = [t_0 + 3\tau, t_0 + 4\tau]\) and \(T_- = [t_0 + \tau, t_0 + 2\tau]\).

The first step in the derivation of this result is the following lemma.

**Lemma 2.10.** Let \(\beta > 0\) be given, and put \(H_\beta := \{x \in H : \text{dist}(x, \partial H) > \beta\}\). Then there exists a continuous kernel function \(k : \mathbb{R}^N \times \mathbb{R}^N \setminus \triangle \to [0, \infty)\) – depending on \(\beta\) – with the following properties:

(i) \(k(x, y) = k(y, x)\) for all \(x, y \in \mathbb{R}^N, x \neq y\);

(ii) \(0 \leq k(x, y) \leq c_{N, s} |x - y|^{-N - 2s}\), for all \(x, y \in \mathbb{R}^N, x \neq y\);

(iii) \(k(x, y) \geq (1 - 5^{-N/2 - s})c_{N, s} |x - y|^{-N - 2s}\) for \(x, y \in \mathbb{R}^N\) with \(0 < |x - y| \leq \frac{\beta}{2}\);

(iv) For any antisymmetric \(v \in H^s(\mathbb{R}^N)\) and any \(\varphi \in \mathcal{H}_0^s(H_\beta)\) we have

\[
\mathcal{E}(v, \varphi) = \delta_k (\tilde{v}, \varphi) + 2 \int_{H_\beta} k_\beta(x) \tilde{v}(x) \varphi(x) \, dx
\]

with \(k_\beta(x)\) as given in Lemma 2.3 and \(\tilde{v} = v_{1_H} \in \mathcal{H}_0^s(H)\).
Lemma 2.3 gives

\[ \text{Asymptotic symmetry for a class of nonlinear fractional reaction-diffusion equations} \]

Similarly, if \( x \in H \setminus H_\beta \) we have \( s(x, y) = 0 \) and thus \( J(x, y) = g(x, y) = k(x, y) \), while for \( x \in \mathbb{R}^N \setminus H \) we have \( k(x, y) = 0 \). Hence we may rewrite the second integral of the RHS of (34) as

\[
\int_{H_\beta \setminus H_\delta} (\varphi(x) - \varphi(y)) J(x, y) \, dx \, dy = \int_{\mathbb{R}^N \setminus H_\delta} (\varphi(x) - \varphi(y)) J(x, y) \, dx \, dy
\]

\[ \text{Similarly, if } y \in H_\beta, \text{ then for } x \in H \setminus H_\beta \text{ we have \( s(x, y) = 0 \) and thus \( J(x, y) = g(x, y) = k(x, y) \), while for } x \in \mathbb{R}^N \setminus H \text{ we have } k(x, y) = 0 \text{. Hence we may rewrite the second integral of the RHS of (34) as} \]

\[
\int_{H_\beta \setminus H_\delta} (\varphi(x) - \varphi(y)) J(x, y) \, dx \, dy
\]
Asymptotic symmetry for a class of nonlinear fractional reaction-diffusion equations

\[ = \int_{\mathbb{R}^n \setminus H_\beta} (\bar{v}(x) - \bar{v}(y))(\varphi(x) - \varphi(y))k(x, y) \, dxdy \]

\[ = \int_{\mathbb{R}^n} (\bar{v}(x) - \bar{v}(y))(\varphi(x) - \varphi(y))k(x, y) \, dxdy, \]

where the last equality follows again since \( \varphi = 0 \) on \( \mathbb{R}^n \setminus H_\beta \). Combining these identities, we get

\[
\int_{\mathbb{R}^n} (\bar{v}(x) - \bar{v}(y))(\varphi(x) - \varphi(y))J(x, y) \, dxdy
\]

\[ = \int_{\mathbb{R}^n} (\bar{v}(x) - \bar{v}(y))(\varphi(x) - \varphi(y))k(x, y) \, dxdy, \]

and together with (33) it follows that

\[
\mathcal{E}(v, \bar{\varphi}) = \frac{1}{2} \int_{\mathbb{R}^n} (\bar{v}(x) - \bar{v}(y))(\varphi(x) - \varphi(y))k(x, y) \, dxdy + 2 \int_{H_\beta} \kappa_H(x)\bar{v}(x)\varphi(x) \, dx,
\]

as claimed in (30).

We may now complete the

**Proof of Theorem 2.9**

Put \( \beta = 2\sigma_0 \), \( U_0 = \{x \in U : \text{dist}(x, D) < \beta\} \subset U \), and let \( k \) be the function given by Lemma 2.10 for this choice of \( \beta \). Let \( v \) be an antisymmetric supersolution of (9) on \( T \times U \), and consider

\[
\bar{v}(t, x) = \begin{cases} 
  v(t, x), & (t, x) \in T \times H \\
  0, & (t, x) \notin T \times H.
\end{cases}
\]

Since \( U_0 \subset H_\beta \), Lemma 2.10(iv) implies that

\[
\mathcal{E}_k(\bar{v}(t), \varphi) \geq \int_{U_0} \left( [c(t, x) - 2\kappa_H(x)]\bar{v}(t) - \partial_t \bar{v}(t) \right) \varphi \, dx \quad \text{for } \varphi \in \mathcal{M}_0^0(U_0), \varphi \geq 0, t \in T,
\]

where \( 0 \leq \kappa_H(x) \leq 4^{d-1+\delta} \sqrt{\frac{2\gamma(t, x)\beta - 2r}{\gamma(t, x)}} \beta^{-2r} \) for \( x \in H_\beta \) by Lemma 2.3. Let \( d := \frac{2^{d-1+\delta}}{\sqrt{2\gamma(t, x)}} \beta^{-2r} + c_\infty \) and \( \sigma := \|v\|_{L^p(T\times U)} \), and define \( w(t, x) := e^{d(t-\gamma)}[\bar{v}(t, x) + \sigma] \) for \( t \in T, x \in \mathbb{R}^n \). Setting \( w(t) = w(t, \cdot) \) as usual, we observe that \( w(t) \geq 0 \) on \( \mathbb{R}^n \) for all \( t \in T \). Moreover, for any \( t \in T \) and any nonnegative \( \varphi \in \mathcal{M}_0^0(U_0) \) we have

\[
\mathcal{E}_k(w(t), \varphi) = e^{d(t-\gamma)}\mathcal{E}_k(\bar{v}(t), \varphi)
\]

\[ \geq \int_{U_0} \left( [d + c(t, x) - 2\kappa_H(x)]w(t, x) - \partial_t w(t, x) - e^{d(t-\gamma)} \sigma[c(t, x) - 2\kappa_H(x)] \right) \varphi \, dx
\]

\[ \geq \int_{U_0} \left( e^{d(t-\gamma)} \sigma[2\kappa_H(x) - c(t, x)] - \partial_t w(t, x) \right) \varphi \, dx.
\]

Hence \( w \) is a nonnegative supersolution of (22) on \( T \times U_0 \) with

\[ g(t, x) = e^{d(t-\gamma)} \sigma[2\kappa_H(x) - c(t, x)]. \]
Applying Corollary 2.8 with $U_0$ in place of $U$ (noting that $\text{dist}(D, \partial U_0) = \beta = 2r_0$) and using the properties of $k$ given by Lemma 2.10 we find $c_1 = c_1(N, s, r_0, R, \varepsilon, \tau) > 0$ such that

$$\inf_{T \times D} w(t, x) \geq c_1 \|w\|_{L^1(T \times D)} - c_2 \|g\|_{L^\infty(T \times U_0)}$$

We note furthermore that $\|w\|_{L^1(T \times D)} \geq [v + \sigma]_{L^1(T \times D)} \geq [v^+]_{L^1(T \times D)}$ and

$$\inf_{T \times D} w \leq e^{4td} \left( \inf_{T \times D} v + \sigma \right),$$

so that

$$\inf_{T \times D} v \geq c_1 e^{-4td} [v^+]_{L^1(T \times D)} - e^{-4td} c_2 \|g\|_{L^\infty(T \times U_0)} - \sigma$$

Noting furthermore that $\|g\|_{L^\infty(T \times U_0)} \leq e^{4td} \sigma d$, we conclude that

$$\inf_{T \times D} v \geq c_1 e^{-4td} [v^+]_{L^1(T \times D)} - (c_2 d + 1)\sigma.$$ 

Hence the assertion follows with $K_1 = c_1 e^{-4td}$ and $K_2 = c_2 d + 1$. Note that both constants only depend on $N, s, r_0, c_\infty, \varepsilon, R$ and $\tau$. \qed

### 2.3 A lower bound based on a subsolution estimate

The aim of this subsection is to prove the following result.

**Proposition 2.11.** Let $\rho > 0$, and let $\Psi$ denote the unique positive eigenfunction of the problem

$$\begin{cases}
-\Delta \Psi &= \lambda_1 \Psi \quad \text{in } B_\rho(0),
\Psi &= 0 \quad \text{on } \partial B_\rho(0),
\end{cases} \quad (35)$$

**corresponding to the first eigenvalue** $\lambda_1 > 0$ with $\|\Psi\|_{L^\infty(B_\rho(0))} = 1$. Moreover, let $c_\infty > 0$. Then there exist $\gamma = \gamma(N, s, \rho, c_\infty) > 0$ and $q = q(N, s, \rho, c_\infty) > 0$ with the following property. If $T := [t_0, t_1] \subset \mathbb{R}$, $x_0 \in H$ with $\text{dist}(x_0, \partial H) \geq 2\rho$, $\sigma_0 > 0$, $\sigma_1 \geq q\sigma_0$ and an antisymmetric supersolution $v$ of (9) on $T \times B_\rho(x_0)$ with $\|v\|_{L^\infty(T \times B_\rho(x_0))} \leq c_\infty$ are given such that

(i) $v(t) \in H^1(\mathbb{R}^N)$ for $t \in T$.

(ii) $v$ is nonnegative in $T \times B_{2\rho}(x_0)$.

(iii) $\|v(t)\|_{L^\infty(H \times B_{2\rho}(x_0))} \leq \sigma_0 e^{-q(t-t_0)}$ for $t \in T$.

(iv) $v(t_0, x) \geq \sigma_1 \Psi(x - x_0)$ for $x \in B_\rho(x_0)$,

then

$$v(t, x) \geq \sigma_1 e^{-q(t-t_0)} \Psi(x - x_0) \quad \text{for } (t, x) \in T \times B_{\rho}(x_0). \quad (36)$$

To show this estimate, we consider the Caffarelli-Silvestre extension of a function $v \in H^1(\mathbb{R}^N)$ which was introduced in (9). For this we consider the usual half space $\mathbb{R}^{N+1}_+ := \{(x, y) \in \mathbb{R}^N \times \mathbb{R} : y > 0\}$. For a domain $U_+ \subset \mathbb{R}^{N+1}_+$, the weighted Sobolev space $H^1(U_+, y^{1-2s})$ is given as the set of all functions $w \in H^1_{\text{loc}}(U_+)$ such that

$$\int_{U_+} y^{1-2s} \left( |w|^2 + |\nabla w|^2 \right) d(x, y) < \infty.$$
Asymptotic symmetry for a class of nonlinear fractional reaction-diffusion equations

In the following, we only consider the case \( U_+ = U \times (0, \infty) \) for some domain \( U \subset \mathbb{R}^N \). Then we have a well-defined continuous trace map \( \text{tr} : H^1(U_+; y^{1-2s}) \to H^s(U), \) see e.g. [8]. We also recall the following integration by parts formula. If \( h \in H^1(U_+; y^{1-2s}) \cap C(\overline{U} \times (0, \infty)) \) and \( \tilde{w} \in H^1(U_+; y^{1-2s}) \cap C^1(\overline{U} \times (0, \infty)) \) are such that \( h \equiv 0 \) on \( \partial U \times (0, \infty) \) and the limit \( m(x) := \lim_{y \to 0} y^{1-2s} \partial_y \tilde{w}(x,y) \) exists in uniform sense for \( x \in U \), then

\[
\int_{U_+} y^{1-2s} \nabla \tilde{w} \nabla h d(x,y) = - \int_U m \text{tr}(h) dx - \int_{U_+} [\text{div}(y^{1-2s} \tilde{w})] h d(x,y). \tag{37}
\]

Formally introducing the operator \( L_s := \text{div}(y^{1-2s} \nabla) \), we say that a function \( w \in H^1(U_+; y^{1-2s}) \) is (weakly) \( L_s \)-harmonic on \( \mathbb{R}^{N+1}_+ \) if

\[
\int_{\mathbb{R}^{N+1}_+} y^{1-2s} \nabla w \nabla \varphi d(x,y) = 0 \quad \text{for all } \varphi \in H^1(\mathbb{R}^{N+1}_+, y^{1-2s}) \text{ with } \text{tr}(\varphi) = 0.
\]

Standard elliptic regularity then shows that \( w \in C^{\alpha}(\mathbb{R}^{N+1}_+) \) and that \( \text{div}(y^{1-2s} \nabla w) = 0 \) in \( \mathbb{R}^{N+1}_+ \) in pointwise sense. We finally recall that every function \( v \in H^s(\mathbb{R}^N) \) has a \( L_s \)-harmonic extension \( w \in H^1(\mathbb{R}^{N+1}_+, y^{1-2s}) \) given by

\[
w(x,y) = \int_{\mathbb{R}^N} v(z) G(x-z,y) dz \quad \text{for } x \in \mathbb{R}^N, y > 0, \tag{38}
\]

where \( G(x,y) := p_{N,s} y^{2s} \left( |x|^2 + y^2 \right)^{-\frac{N+2s}{2}} \) for \( x \in \mathbb{R}^N, y > 0 \), where \( p_{N,s} \) is a normalization constant, see e.g. [9]. We need the following lemmas.

**Lemma 2.12.** Let \( \rho > 0 \). Then there exists constants \( \tilde{c}_1 = \tilde{c}_1(N,s,\rho) \) and \( \tilde{c}_2 = \tilde{c}_2(N,s,\rho) \) such that the following holds:

If \( x_0 \in H \) satisfies \( \text{dist}(x_0,H) \geq 2\rho \) and \( v \in H^s(\mathbb{R}^N) \) is a continuous antisymmetric function such that \( v \geq 0 \) on \( B_{2\rho}(x_0) \), then

\[
\frac{w(x,y)}{y^{2s}} \geq \tilde{c}_1 \left[ \int_{B_{\rho}(x_0)} (v(z))^\frac{1}{2} dz \right]^2 \tilde{c}_2 \| v^+ \|_{L^\infty(H; B_{2\rho}(x_0))} \quad \text{for } (x,y) \in B_{\rho}(x_0) \times (0,1]. \tag{39}
\]

where \( w \) is the \( L_s \)-harmonic extension of \( v \).

**Proof.** Since \( v \) is antisymmetric, we have, by a simple change of variable,

\[
w(x,y) = \int_{B_r} [G(x-z,y) - G(x-Q(z),y)] v(z) dz \quad \text{for } x \in H. \tag{40}
\]

For \( x,z \in H \) and \( y > 0 \) we have

\[
G(x-z,y) \geq G(x-z,y) - G(x-Q(z),y) = G(x-z,y) \left( 1 - \left( \frac{|x-z|^2 + y^2}{|x-Q(z)|^2 + y^2} \right)^{\frac{N+2s}{N}} \right). \tag{41}
\]

Moreover, for \( x,z \in B_{\rho}(x_0) \) we have \( |x-z|^2 \leq 4\rho^2 \) and \( |x-Q(z)|^2 \geq |x-z|^2 + 4\rho^2 \), so that

\[
G(x-z,y) - G(x-Q(z),y) \geq c_1 G(x-z,y) \quad \text{for } y \in (0,1] \tag{42}
\]
with 
\[ c_1 = 1 - \left(1 + \frac{4\rho^2}{1 + 4\rho^2}\right)^{-\frac{N+2\rho}{2}} = 1 - \left(1 + \frac{4\rho^2}{1 + 8\rho^2}\right)^{\frac{N+2\rho}{2}}. \]

Combining (40), (41) and (42) and using that \( v \geq 0 \) on \( B_{2\rho}(x_0) \), we obtain the estimate
\[
\frac{w(x,y)}{p_{N,s}^{2\rho}} \geq c_1 \int_{B_{\rho}(y)} v(z)(|x-z|^2 + y^2)^{-\frac{N+2\rho}{2}} \, dz - \| v^- \|_{L^\infty(H',B_{2\rho}(x_0))} \int_{H',B_{2\rho}(x_0)} (|x-z|^2 + y^2)^{-\frac{N+2\rho}{2}} \, dz 
\]
\[
\geq c_1 \left( \int_{B_{\rho}(y)} v^+(z) \, dz \right)^2 - c_2 \| v^- \|_{L^\infty(H',B_{2\rho}(x_0))} \int_{B_{\rho}(y)} (|z|^2 + 1)^{\frac{N+2\rho}{2}} \, dz 
\]
for \( x, z \in B_{\rho} \) and \( y \in (0,1] \). Hence the claim follows with \( c_1 = \frac{c_1 p_{N,s}^{2\rho}}{\omega_N} (p^2 + 1)^{-\frac{N+2\rho}{2}} \) and \( c_2 = c_2 p_{N,s} \).

**Lemma 2.13.** Let \( U \subset H \) be a bounded Lipschitz domain, \( T := (t_0,T_0) \), \( t_0 < T_0 \), \( c \in L^\infty(T \times U) \), and let \( v \) be a supersolution of (20) on \( T \times U \) such that \( v(t) \in H^1(\mathbb{R}^N) \) for all \( t \in T \). Moreover, let \( w(t) \in H^1(\mathbb{R}^{N+1},y^{1-s}) \) be the \( L_s \)-harmonic extension of \( v(t) \) given by (33) for each fixed time \( t \in T \). Then for every nonnegative \( \Phi \in H^1(\mathbb{R}^{N+1},y^{1-s}) \) with \( \varphi := \text{tr}(\Phi) \in H^0_0(U) \) and every \( t \in T \) we have
\[
\int_{\mathbb{R}^{N+1}} y^{1-2s} \nabla w \nabla \Phi d(x,y) \geq d_s \int_U (c(t,x)v - \partial_\nu v) \varphi \, dx, \tag{44}
\]
where \( d_s = 2^{1-2s} \Gamma(1-s)/\Gamma(s) \).

**Proof.** In case \( \Phi \) is the \( L_s \)-harmonic extension of \( \varphi \), we have
\[
\int_{\mathbb{R}^{N+1}} y^{1-2s} \nabla w \nabla \Phi d(x,y) = d_s \delta(w,\varphi)
\]
Asymptotic symmetry for a class of nonlinear fractional reaction-diffusion equations

with $d_t$ as stated (see e.g. [9] or [8] Remark 3.11) and therefore (44) is true. On the other hand, since $w$ is $L_\gamma$-harmonic,

$$\int_{\mathbb{R}^{N+1}} y^{1-2s} \nabla w \nabla \Theta d(x,y) = 0 \quad \text{for every } \Theta \in H^1(\mathbb{R}^N, y^{1-2s}) \text{ with } \Theta(0) = 0.$$ 

Hence the assertion follows. □

**Lemma 2.14.** Let $\rho > 0$, $c_\omega > 0$, and let $\Psi$ be defined as in Proposition 2.17 w.r.t. $\rho$. Then there exists $\gamma = \gamma(N,s,\rho,c_\omega) > 0$ such that the following holds:

If $T = [t_0,t_1] \subset \mathbb{R}$, $x_0 \in H$ with $\text{dist}(x_0, \partial H) \geq 2\rho$ and $c \in L^\infty([t_0,t_1] \times B_\rho(x_0))$ with $\|c\|_{L^\infty} \leq c_\omega$ are given and $v$ is an antisymmetric supersolution of (9) on $[t_0,t_1] \times B_\rho(x_0)$ such that $v(t) \in H^s(\mathbb{R}^N)$ for $t \in [t_0,t_1]$,

$$v(t_0,x) \geq \sigma \Psi(x-x_0) \quad \text{for } x \in B_\rho(x_0) \text{ with some constant } \sigma > 0,$$

and the $L_\gamma$-harmonic extension $w(t)$ of $v(t)$ is nonnegative on $B_\rho(x_0) \times [0,1]$ for all $t \in [t_0,t_1]$,

then

$$v(t,x) \geq \sigma e^{-\gamma(t-t_0)} \Psi(x-x_0) \quad \text{for } (t,x) \in T \times B_\rho(x_0).$$

**Proof.** Without loss of generality, we may assume that $x_0 = 0$, $t_0 = 0$ and $\sigma = 1$, and we put $B_\rho = B_\rho(0)$. Let $\lambda_1 > 0$ be defined by (35), and let $f : [0,\infty) \to \mathbb{R}$ denote the solution of the initial value problem

$$f'' + \frac{1-2s}{y} f' - \lambda_1 f = 0,$$

$$f(0) = 1,$$

$$\lim_{y \to \infty} f(y) = 0,$$

which is uniquely given by

$$f(y) = \kappa_1 y^{2s} \int_0^\infty \frac{\cos(\tau)}{\left(\tau^2 + \lambda_1 y^2\right)^{\frac{1-2s}{2}}} d\tau, \text{ for } y \geq 0 \text{ with } \kappa_1 = \lambda_1^s d_s,$$

with $d_s$ as in Lemma 2.13. We note that $f$ is a scalar multiple of a rescaled Macdonald function (or modified Bessel function of the second kind), see e.g. [30]. We also note that $f$ is strictly decreasing on $[0,\infty)$. Moreover, the limit

$$\kappa_2 := \lim_{y \to 0} \frac{y^{1-2s} f'(y)}{1 - f(1)} \leq 0$$

exists and only depends on $s$ and $\rho$ (via $\lambda_1$). We now put $\gamma = c_\omega - \kappa_2 + 1$, and we let

$$\tilde{w} : [0,t_1] \times \mathbb{R}^N \to \mathbb{R} \text{ be defined by } \tilde{w}(t,x,y) = \begin{cases} e^{-\gamma t} \Psi(x) \frac{f(y) - f(1)}{1 - f(1)}, & x \in B_\rho; \\ 0, & x \notin B_\rho. \end{cases}$$

Putting $\tilde{w}(t) = \tilde{w}(t,\cdot,\cdot)$ as usual, we then have

$$L_\gamma \tilde{w}(t) = \frac{1}{1 - f(1)} e^{-\gamma t} \left( y^{1-2s} \Delta_y \tilde{w} + (1 - 2s)y^{-2s} \partial_y \tilde{w} + y^{1-2s} \partial_{yy} \tilde{w} \right).$$

(47)
Hence the consequence of (48), we conclude that

$$v_{\text{consequence}}(0, x, 0) = \Psi(x) \leq v(0, x)$$

for $$x \in B_\rho$$. Moreover, we have

$$\lim_{y \to 0} v^{1/2 - \varepsilon} \partial_t \hat{w}(t, x, 0) = \kappa_2 e^{-\gamma} \Psi(x) = \kappa_2 \hat{w}(t, x, 0) \quad \text{for } x \in B_\rho, t \in [0, t_1],$$

and

$$\hat{w}(t) \equiv 0 \leq w(t) \quad \text{on } \partial B_\rho \times (0, 1) \cup B_\rho \times \{1\}$$

by assumption and by construction of $$\hat{w}$$. In the following, we consider

$$h(t) \in H^1 \times (\mathbb{R}_+^N, y^{1/2}), \quad h(t, x) = \begin{cases} (w - \hat{w})(t, x, y), & (x, y) \in B_\rho \times [0, 1], \\ 0, & \text{elsewhere}. \end{cases}$$

Moreover, we will write $$g$$ resp. $$\hat{g}$$ for the traces of $$h$$ and $$\hat{w}$$, respectively. Then, as a consequence of (47), (48), (49) and (50), we have

$$0 \geq -d_s^{-1} \int_{\mathbb{R}_+^N} y^{1/2} |\nabla h|^2 d(x, y)$$

$$= d_s^{-1} \int_{\mathbb{R}_+^N} y^{1/2} \nabla w \nabla d(x, y) - d_s^{-1} \int_{\mathbb{R}_+^N} y^{1/2} \nabla \hat{w} \nabla d(x, y)$$

$$\geq \int_{B_\rho} \left[ (c(t, x) v - \partial_t \hat{g}) g + \kappa_2 \hat{g} \right] d(x, y)$$

$$= \int_{B_\rho} \left[ (c(t, x) (v - \hat{v}) - \partial_t (v - \hat{v})) |g| + [\kappa_2 + c(t, x) + \gamma] \hat{g} \right] d(x, y)$$

$$\geq -c_\infty \int_{B_\rho} g^2(t, x, d) + \frac{1}{2} \frac{d}{dt} \int_{B_\rho} g^2(t, x, d)$$

for $$t \in [0, t_1]$$.

Hence $$\int_{B_\rho} g^2(t, x) d(x, y) \leq 2c_\infty \int_{B_\rho} g^2(t, x) d(x, y)$$ for $$t \in [0, t_1]$$. Since furthermore $$\int_{B_\rho} g^2(0, x) d(x, y) = 0$$ as a consequence of (48), we conclude that $$\int_{B_\rho} g^2(t, x) d(x, y) = 0$$ and therefore $$g(t) \equiv 0$$ on $$B_\rho$$ for all $$t \in T$$.

Hence $$v(t, x) \geq e^{-\eta} \Psi(x)$$ for $$(x, t) \in T \times B_\rho(0)$$, as claimed.

We may now complete the proof.

**Proof of Proposition 2.11** For given $$\rho, c_\infty > 0$$, let $$\tilde{c}_i, i = 1, 2$$ be given by Lemma 2.12 and let $$\gamma$$ be given by Lemma 2.14. Moreover, let

$$q = \frac{2\tilde{c}_2}{\tilde{c}_1} \left[ \int_{B_\rho(0)} \Psi(z) \frac{1}{2} \right]^{-2}$$

Next, let $$T := [t_0, t_1] \subset R, \sigma_0 > 0$$ and $$\sigma_1 \geq q \sigma_0$$, and let $$v$$ be an antisymmetric supersolution of (49) on $$T \times B_\rho(x_0)$$ satisfying assumptions (i)- (iii). Suppose by contradiction that

$$v(t, x) = \sigma_2 e^{-\eta(t_t - \sigma)} \Psi(x - x_0) \quad \text{for some } \sigma_2 \in (\frac{\sigma_1}{2}, \sigma_1), t_0 \in T \text{ and } x_0 \in B_\rho(x_0).$$

(50)
implies that every connected component of $D$ is bounded open sets with $\partial D\not\subseteq U$ and an antisymmetric supersolution $v$ of $\mp$ on $[t_0,\infty)\times U$. Next, we consider $\mp\in\mathbb{R}$ sufficiently small such that $\left(\frac{K_1}{\mu} - K_2\right) > q$ and $\left(K_1|B_{2\rho}(0)| - K_2\right)|\Psi|_{L^1(B_{2\rho}(0))} - K_2 > 0$.}

where $\Psi$ is given in Proposition 2.11 depending on $\rho$. Next, we consider $D \subset\subset U \subset H$ and an antisymmetric supersolution $v$ of $\mp$ on $[t_0,\infty)\times U$ with the properties stated in the theorem, which implies in
and particular that $\varepsilon \leq |D_x| \leq (2R)^N$ for every connected component $D_x$ of $D$. We put $\sigma_0 = \|v^-\|_{L^\infty(\Omega)}$ and $T_0 := \sup \{ t \geq t_0 + 8 \tau : v > 0 \text{ in } [t_0, t] \times \overline{D} \}$, so that $t_0 + 8 \tau \leq T_0 \leq \infty$ by assumption. Applying Proposition 2.4 we get

$$
\|v^-(t)\|_{L^\infty(U)} = \|v^-(t)\|_{L^\infty(\Omega \setminus \overline{D})} \leq \sigma_0 e^{-y(t-\theta)} \quad \text{for all } t \in [t_0, T_0). \tag{53}
$$

To prove (i), we suppose by contradiction that $T_0 < \infty$. Then there exists a connected component $D_x$ of $D$ and $x_0 \in \partial D_x$ such that

$$
v > 0 \text{ in } [t_0, T_0) \times \overline{D}_x \quad \text{and} \quad v(T_0, x_0) = 0. \tag{54}
$$

Let $U$, be the connected component of $U$ with $D_x \subset U_x$. Since $v \geq 0$ on $[t_0, t_0 + 8 \tau] \times \partial D_x$, we have, by Theorem 2.9 (51) and Proposition 2.4,

$$
\inf_{[t_0 + 4 \tau, t_0 + 4 \tau] \times \partial D_x} v \geq K_1 |v^+|_{L^1([t_0 + 4 \tau, t_0 + 4 \tau] \times D_x)} - K_2 \|v^-\|_{L^\infty([t_0 + 4 \tau, t_0 + 4 \tau] \times U)}
\geq K_1 |v|_{L^1([t_0 + 4 \tau, t_0 + 4 \tau] \times U)} - K_2 \|v^-\|_{L^\infty(U)} - K_2 \|v^-\|_{L^\infty(\Omega \setminus \overline{D})} = \left( \frac{K_1}{\mu} - K_2 \right) \sigma_0 =: \sigma_1. \tag{55}
$$

We fix $x_0 \in D_x$ such that $B_{2R}(x_0) \subset D_x$, which is possible by assumption. Since $\sigma_1 \geq q \sigma_0$ by (55), the estimates (55) and (53) allow us to apply Proposition 2.14 with $t_0 + 4 \tau$ in place of $t_0$, which yields

$$
v(t, x) \geq \sigma_1 e^{-\gamma(t-\theta)} \Psi(x - x_0) \quad \text{for every } x \in B_{2R}(x_0), t \in [t_0 + 4 \tau, T_0]. \tag{56}
$$

With the help of Theorem 2.9 (53) and (56), we find that

$$
v(T_0, x_0) \geq K_1 |v|_{L^1([t_0 - 2 \tau, t_0 - 2 \tau] \times D_x)} - K_2 \|v^-\|_{L^\infty([t_0 - 4 \tau, t_0 - 4 \tau] \times U)}
\geq K_1 \sigma_1 e^{-\gamma(T_0 - 4 \tau - t_0)} \left| B_{2R}(0) \right|_{D_x} - K_2 \|v^-\|_{L^\infty(\Omega \setminus \overline{D})} - K_2 \|v^-\|_{L^\infty(\Omega \setminus \overline{D})} - K_2 \|v^-\|_{L^\infty(\Omega \setminus \overline{D})}
\geq \sigma_0 e^{-\gamma(T_0 - 4 \tau - t_0)} \left[ \frac{K_1 |B_{2R}(0)|}{(2R)^N} \left( \frac{K_1}{\mu} - K_2 \right) \frac{\Psi(0)}{\left| B_{2R}(0) \right|} \right] > 0,
$$

by our choice of $\mu$ in (52), contradicting (54). We conclude that $T_0 = \infty$. In particular, (i) holds, and (ii) follows since, by (55),

$$
\|v^-(t)\|_{L^\infty(U)} = \|v^-(t)\|_{L^\infty(\Omega \setminus \overline{D})} \leq \sigma_0 e^{-y(t-\theta)} \quad \text{for all } t \in [t_0, \infty). \tag{57}
$$

\[\square\]

3 Proof of the main symmetry result

In this section we complete the proof of Theorem 1.1. With the tools developed in Section 2, we may follow the main lines of the moving plane method as developed by Poláčik in [27], but some steps in the argument – in particular the proofs of Lemma 3.2 and Proposition 3.5 below – differ significantly from [27]. This is due to the fact that, contrary to [27], we do not a priori assume the existence of an element $\varphi \in \omega(u)$ with $\varphi > 0$. For $\lambda \in \mathbb{R}$, we use the notations

$$
\Omega_\lambda = \{ x \in \Omega : x_1 > \lambda \}, \quad H_\lambda := \{ x \in \mathbb{R}^N : x_1 > \lambda \}, \quad T_\lambda = \partial H_\lambda \quad \text{and} \quad \Gamma_\lambda = T_\lambda \cap \Omega.
$$
Moreover, we let \( Q_{\lambda} : \mathbb{R}^N \to \mathbb{R}^N \) denote the reflection at \( T_{\lambda} \) given by \( Q_{\lambda}(x) = (2\lambda - x_1, x_2, \ldots, x_N) \). For a function \( z : \mathbb{R}^N \to \mathbb{R} \), we put
\[
\zeta^\lambda = z \circ Q_{\lambda} : \mathbb{R}^N \to \mathbb{R}
\]
and
\[
V_{\lambda} z : \mathbb{R}^N \to \mathbb{R}, \quad V_{\lambda} z(x) = \zeta^\lambda(x) - z(x).
\]
We now assume that the hypotheses \((D1)\) and \((F1),(F2)\) are satisfied, and we let \( u \) be a nonnegative global solution of \((P)\) satisfying \((U1)\) and \((U2)\). We set
\[
I := \max \{ z_1 : (x_1, x') \in \Omega \quad \text{for some } x' \in \mathbb{R}^{N-1} \},
\]
and we fix \( \lambda \in [0, l) \) for the moment. As discussed in Remark 2.2, the function \( \nu := V_{\lambda} u \) is an entire antisymmetric supersolution of the problem
\[
\partial_t \nu + (-\Delta)^{s} \nu = c_{\lambda}(t,x) \nu \tag{58}
\]
in \((0, \infty) \times \Omega_{\lambda}\) with
\[
c_{\lambda}(x,t) = \begin{cases} 
\frac{f(t,x,u^\lambda(x)) - f(t,x,u(x))}{u^\lambda(x) - u(x)}, & u^\lambda(t,x) \neq u(t,x); \\
0, & u^\lambda(t,x) = u(t,x).
\end{cases}
\]
Here the term entire antisymmetric supersolution refers to the notion defined in the beginning of Section 2 with respect to the half space \( H = H_{\lambda} \). Indeed, for \( \lambda \in [0, l) \) and this choice of \( H \), \((10)\) and \((11)\) are satisfied as a consequence of assumptions \((D1)\) and \((F2)\). Moreover, as a consequence of \((F1)\) and \((U1)\), there exists \( c_\infty > 0 \) such that
\[
\|c_\lambda\|_{L^\infty((0,\infty) \times \Omega_{\lambda})} \leq c_\infty \quad \text{for every } \lambda \in [0, l).
\]
In the following, we fix \( c_\infty \) with this property. We also note that \([V_{\lambda} u](t) \in H^s(\mathbb{R}^N)\) for all \( t \in (0, \infty) \). For \( \lambda \in [0, l) \), we now consider the following statement:
\[
(S_\lambda) \quad \|V_{\lambda} u(t)\|_{L^\infty(\Omega_{\lambda})} \to 0 \quad \text{as } t \to \infty.
\]
Our aim is to show, via the method of moving planes, that \((S_\lambda)\) holds for every \( \lambda \in [0, l) \). We need the following lemmas.

**Lemma 3.1.** There is \( \delta > 0 \) such that for each \( \lambda \in [0, l) \) the following statement holds. If \( K \) is a closed subset of \( \Omega_{\lambda} \) with \( |\Omega_{\lambda} \setminus K| < \delta \) and there is \( t_0 \geq 0 \) such that \( V_{\lambda} u(t) \geq 0 \) on \( K \) for all \( t \geq t_0 \), then
\[
\|V_{\lambda} u(t)\|_{L^\infty(\Omega_{\lambda})} \leq e^{-\delta(t-t_0)}\|V_{\lambda} u(t_0)\|_{L^\infty(\Omega_{\lambda})},
\]
for all \( t \geq t_0 \). In particular \((S_\lambda)\) holds if \( \lambda < l \) is sufficiently close to \( l \).

**Proof.** This follows immediately by applying Proposition 2.4 to \( \gamma = 1 \), \( c_\infty > 0 \) as fixed above, \( H = H_{\lambda} \) and \( U = \Omega_{\lambda} \setminus K \). Note that the number \( \delta > 0 \) given by Proposition 2.4 in this case does not depend on \( \lambda \) and \( K \). The second statement of the lemma follows since \( |\Omega_{\lambda}| < \delta \) if \( \lambda \) is close to \( l \). \( \square \)

**Lemma 3.2.** Suppose \( \lambda_0 \in [0, l) \) is such that \((S_\lambda)\) holds for all \( \lambda \in (\lambda_0, l) \). Then we have:

(i) \((S_{\lambda_0})\) holds.

(ii) For each \( z \in \partial(\nu) \) we have either \( V_{\lambda_0} z > 0 \) on \( \Omega_{\lambda} \) or \( V_{\lambda_0} z \equiv 0 \) on \( \mathbb{R}^N \).
(iii) If $\lambda_0 > 0$, then for each $z \in \omega(u)$ we have either $V_{\lambda_0} z > 0$ on $\Omega_{\lambda_0}$ or $z \equiv 0$ on $\mathbb{R}^N$.

**Proof.** (i) Since the set $\{ u(t) : t \geq 0 \}$ is relatively compact in $C(\bar{\Omega})$, the statement $(S_2)$ is equivalent to $V_{\lambda} z \geq 0$ on $\Gamma_\lambda$ for all $z \in \omega(u)$. Hence $(S_{2_{\lambda_0}})$ holds by assumption and continuity of all $z \in \omega(u)$.

(ii) Step one: We first claim that on each connected component $U$ of $\Omega_{\lambda_0}$ we either have $V_{\lambda_0} z > 0$ on $U$ or $V_{\lambda_0} z \equiv 0$ on $U$. To prove this, we fix $z \in \omega(u)$ and a connected component $U$ of $\Omega_{\lambda_0}$ such that $V_{\lambda_0} z \neq 0$ on $\Omega_{\lambda_0}$. Since $V_{\lambda_0} z \geq 0$, there exists $x_0 \in U$ and $\rho > 0$ such that $B := B_{\rho}(x_0) \subset \subset \Omega_{\lambda_0}$ and $V_{\lambda_0} z > 0$ on $\bar{B}$. Since $z \in \omega(u)$, there exists a sequence of numbers $t_n > 0$ such that $t_n \to \infty$ and $u(t_n) \to z$ in $C(\bar{\Omega})$, hence also $V_{\lambda_0} u(t_n) \to V_{\lambda_0} z$ in $C(\bar{\Omega}_{\lambda_0})$ as $n \to \infty$. Consequently, there exists $\sigma > 0$ and $n_0 \in \mathbb{N}$ such that

$$V_{\lambda_0} u(t_n, x) > 2\sigma \quad \text{for } x \in \bar{B}, n > n_0.$$  

By the equicontinuity property $(U2)$, there exists $\tau > 0$ such that

$$V_{\lambda_0} u(t, x) > \sigma \quad \text{for } x \in \bar{B}, t \in [t_n - 4\tau, t_n], n > n_0. \quad (60)$$

Now fix a subdomain $D \subset U$. Applying Proposition $[2, 9]$ with $U = \Omega_{\lambda_0}$, $t_0 = t_n - 4\tau$ and using $(60)$, we get

$$\inf_{x \in D} V_{\lambda_0} u(t_n, x) \geq K_1 \| (V_{\lambda_0} u)^+ \|_{L^1([t_n - 4\tau, t_n - 3\tau]; D)} - K_2 \sup_{x \in \partial D} \| (V_{\lambda_0} u)^- \|_{L^\infty(U)}$$

$$\geq K_1 \sigma |B| - K_2 \| v^- \|_{L^\infty(U)}$$

for $n > n_0$

with suitable constants $K_1, K_2 > 0$ independent of $n$. Since $(S_{2_{\lambda_0}})$ holds, we conclude that

$$\inf_{x \in D} V_{\lambda_0} z = \lim_{n \to \infty} \inf_{x \in D} V_{\lambda_0} u(t_n, x) \geq K_1 \sigma |B| > 0.$$  

Since $D \subset U$ was chosen arbitrarily, we conclude that $V_{\lambda_0} z > 0$ in $U$. This shows the claim.

Step two: Let $z \in \omega(u)$ be such that

$$U_z := \{ x \in \Omega_{\lambda_0} : [V_{\lambda_0} z](x) = 0 \}$$

is nonempty. To finish the proof of (ii), we need to show that $V_{\lambda_0} z \equiv 0$ on $\mathbb{R}^N$. We suppose by contradiction that this is false; then there exists a compact set $\mathcal{K} \subset H_{\lambda_0} \setminus U_z$ of positive measure such that

$$\inf_{\mathcal{K}} V_{\lambda_0} z > 0.$$  

(61)

By Step one above, $U_z$ is an open set. Hence we may fix a nonnegative function $\varphi \in C_c^\infty(U_z)$, $\varphi \neq 0$, and we set $D := \text{supp } \varphi$. Moreover, we fix $\rho > 0$ with $\text{dist}(D, \partial U_z) > 2\rho$, and we note that there exists $M > 0$ such that

$$\left| \int_{D_{\rho}(x)} \frac{\varphi(x) - \varphi(y)}{|x-y|^{N+2\kappa}} \, dy \right| < M \quad \text{for all } x \in \mathbb{R}^N,$$

(62)

see e.g. [17, Lemma 3.5]). In the following, we put $v = V_{\lambda_0} u$ and $H = H_{\lambda_0}$. Moreover, we consider $J$ and $\kappa$ as defined in Lemma $[2, 3]$ for this choice of $H$. By Lemma $[2, 3]$ we have

$$\mathcal{E}(v(t), \varphi) = \frac{1}{2} \int_H \int_H (v(t,x) - v(t,y))(\varphi(x) - \varphi(y))J(x,y) \, dxdy$$

$$+ 2 \int_H v(t,x)\kappa_H(x)\varphi(x) \, dx,$$  

(63)
where
\[
\int_{H} v(t,x)\kappa(x)\varphi(x)\,dx \leq \kappa_{s}\|\varphi\|_{L^{1}(\Omega)}\|v(t)\|_{L^{\infty}(\Omega_{x})}
\] with \(\kappa_{s} := \frac{4^{s}\Gamma(\frac{1}{s} + 1)}{\sqrt{\pi\Gamma(1 - s)}}(2\rho)^{-2s}\).

To estimate the double integral on the right hand side of (63), we put
\[
\mathcal{H}_{1} := \{(x,y) \in H \times H : |x - y| \leq \delta\}, \quad \mathcal{H}_{2} := H \times H \setminus \mathcal{H}_{1}
\]
and \(D_{\rho} := \{x \in \mathbb{R}^{N} : \text{dist}(x,D) \leq \rho\}\). Then
\[
\int_{\mathcal{H}_{1}} (v(t,x) - v(t,y))(\varphi(x) - \varphi(y))J(x,y)\,dxdy
\]
\[
= \int_{|x - y| \leq \delta,\ x,y \in D_{\rho}} (v(t,x) - v(t,y))(\varphi(x) - \varphi(y))J(x,y)\,dxdy
\]
\[
= c_{N,s} \left(2 \int_{D_{\rho}} v(t,x) \int_{B_{\rho}(x)} \frac{\varphi(x) - \varphi(y)}{|x - y|^{N + 2s}}\,dydx - \int_{|x - y| \leq \delta,\ x,y \in D_{\rho}} (v(t,x) - v(t,y))(\varphi(x) - \varphi(y))\frac{\rho}{|x - Q(x_{0})|^{N + 2s}}\,dxdy\right)
\]
\[
\leq 2c_{N,s}M||D_{\rho}\||\|v(t)\|_{L^{\infty}(\Omega_{x})} + \frac{4|D_{\rho}|^{2}}{(2\rho)^{N + 2s}}\|\varphi\|_{L^{\infty}(\Omega_{x})}\|v(t)\|_{L^{\infty}(\Omega_{x})},
\]
where we used the fact that \(|x - Q_{x_{0}}(y)| \geq 2\rho\) for every \(x, y \in D_{\rho}\). To estimate the integral over \(\mathcal{H}_{2}\), we first note that
\[
\sup_{x \in H} \int_{H \setminus B_{\rho}(x)} J(x,y)\,dxdy \leq c_{N,s} \int_{\mathbb{R}^{N} \setminus B_{\rho}(0)} |y|^{-N - 2s}\,dy = \frac{N\omega_{N}C_{1,s}}{(2\rho)^{2s}}d =: J_{N,s}
\]
Hence
\[
\int_{\mathcal{H}_{2}} (v(t,x) - v(t,y))(\varphi(x) - \varphi(y))J(x,y)\,dxdy
\]
\[
= 2 \int_{D} \varphi(x) \int_{H \setminus B_{\rho}(x)} (v(t,x) - v(t,y))J(x,y)\,dydx
\]
\[
= 2 \int_{D} \varphi(x) \left\{ \int_{H \setminus B_{\rho}(x)} J(x,y)\,dydx - \int_{H \setminus B_{\rho}(x) \setminus \mathcal{H}_{1}} v(t,y)J(x,y)\,dydx \right\}
\]
\[
\leq 2J_{N,s}||\varphi||_{L^{1}(\Omega_{x})}\left(\|v(t)\|_{L^{\infty}(\Omega_{x})} + \|v^{-}(t)\|_{L^{\infty}(H)}\right) - dm(t)
\]
where in the last step we have set
\[
m(t) := \inf_{y \in \mathcal{X}} v(y,t) \quad \text{and} \quad d := \int_{\mathcal{X}} \varphi(x) \int_{\mathcal{X}} J(x,y)\,dydx > 0.
\]
We now consider the function \( t \mapsto h(t) = \int_{\Omega} v(t,x)\varphi(x)\,dx \) for \( t > 0 \). Combining the estimates above and using (68), we get

\[
\begin{align*}
h'(t) &= \int_{\Omega} \partial_t v(t,x)\varphi(x)\,dx \\
&\geq c_0 \|v(t)\|_{L^1(\Omega)} - \varepsilon'(v(t),\varphi) \\
&\geq c_1 \|v(t)\|_{L^1(\Omega)} - C_2 \|v^{-\ast}(t)\|_{L^p(\Omega)} + m(t) d
\end{align*}
\]

with \( C_1 := \|\varphi\|_{L^1(\Omega)} [2\kappa_c + c_NM |D\varphi| + J_{N,s}] + \frac{2|D\varphi|^2}{|D\varphi|^2 + |\varphi|^2} \|\varphi\|_{L^p(\Omega)} \) and \( C_2 := J_{N,s} \|\varphi\|_{L^1(\Omega)} \). We now consider a sequence \( (t_k)_k \subset (0,\infty) \) such that \( t_k \to \infty \) and \( u(t_k) \to z \) in \( L^p(\Omega) \) as \( k \to \infty \), which yields in particular that \( h(t_k) \to 0 \) as \( k \to \infty \). Using (63) and the equicontinuity property (U2), we find \( \delta > 0 \) and \( k_0 \in \mathbb{N} \) such that

\[
m_* := \inf \{m(t) : t \in [t_k - \delta,t_k + \delta], k \geq k_0 \} > 0.
\]

Moreover, making \( \delta > 0 \) smaller and \( k_0 \in \mathbb{N} \) larger if necessary, we may assume that

\[
\|v(t)\|_{L^p(\Omega)} \leq \|v(t) - v(t_k)\|_{L^p(\Omega)} + \|v(t_k)\|_{L^p(\Omega)} \leq \frac{m_* d}{4C_1}
\]

for \( t \in [t_k - \delta,t_k + \delta] \) and \( k \geq k_0 \). Furthermore, using that \( \|v^{-\ast}(t)\|_{L^p(\Omega)} \to 0 \) as \( t \to \infty \) as a consequence of (S$_{\#}$), we may again make \( k_0 \in \mathbb{N} \) larger such that

\[
\|v^{-\ast}(t)\|_{L^p(\Omega)} \leq \frac{m_* d}{4C_2} \quad \text{for } t \in [t_k - \delta,t_k + \delta], k \geq k_0.
\]

Combining (64), (65) and (66), we thus obtain

\[
h'(t) \geq \frac{m_* d}{2} \quad \text{for } t \in [t_k - \delta,t_k + \delta], k \geq k_0.
\]

This implies that

\[
\limsup_{k \to \infty} h(t_k - \delta) \leq \lim_{k \to \infty} \left( h(t_k) - \frac{\delta m_* d}{2} \right) = -\frac{\delta m_* d}{2},
\]

contradicting the fact that \( \|v^{-\ast}(t)\|_{L^p(\Omega)} \to 0 \) as \( t \to \infty \) and thus \( \liminf_{t \to \infty} h(t) \geq 0 \). The proof of (ii) is finished.

(iii) Suppose that \( \lambda_0 > 0 \), and let \( z \in \omega(u) \) such that \( V_{\lambda_0}z \equiv 0 \) on \( \mathbb{R}^N \). In view of (ii), we need to show that \( z \equiv 0 \) on \( \mathbb{R}^N \). For this we consider the reflected functions

\[
\begin{align*}
\tilde{u} : (0,\infty) \times \mathbb{R}^N &\to \mathbb{R}, & \tilde{u}(t,x) &= u(t,Q_\lambda(x)) \\
\tilde{z} : \mathbb{R}^N &\to \mathbb{R}, & \tilde{z}(x) &= z(Q_\lambda(x)).
\end{align*}
\]

Since \( \Omega \) and the nonlinearity \( f \) are symmetric in the \( x_1 \)-variable, \( \tilde{u} \) is also a solution of (P) satisfying the same hypotheses as \( u \). Moreover, \( \tilde{z} \in \omega(u) \). Putting \( \tilde{\lambda} := l - 2\lambda_0 \in (-l,l) \), it follows from \( V_{\tilde{\lambda}}z \equiv 0 \) on \( \mathbb{R}^N \) that \( \tilde{z} \equiv 0 \) on \( \Omega_{\tilde{\lambda}} \), and therefore

\[
V_{\tilde{\lambda}}z \equiv 0 \quad \text{in } \Omega_{\tilde{\lambda}} \quad \text{for every } \tilde{\lambda} \in \left(\frac{\lambda_0 + l}{2},l\right).
\]

For \( \lambda \in \left(\frac{\lambda_0 + l}{2},l\right) \) sufficiently close to \( l \), it also follows from Lemma 3.1 that (S$_{\lambda}$) holds for \( \tilde{u} \) in place of \( u \), so that (63) and (ii) imply that

\[
V_{\lambda}z \equiv 0 \text{ on } \mathbb{R}^N \text{ for } \lambda < l \text{ sufficiently close to } l.
\]

From this we easily conclude that \( \tilde{z} \equiv 0 \) and therefore \( z \equiv 0 \) on \( \mathbb{R}^N \), as claimed.
Lemma 3.3. Suppose \( \lambda_0 \in (0, l) \) is such that \((S, \lambda)\) holds for all \( \lambda \in (\lambda_0, l) \). Suppose furthermore that one of the following conditions hold:

(i) \( z \not\equiv 0 \) on \( \Omega \) for all \( z \in \omega(u) \).

(ii) \( \Omega \) fulfills (D2) and \( V_{\lambda_0}z > 0 \) on \( \Omega_{\lambda_0} \) for some \( z \in \omega(u) \).

Then there exists \( \varepsilon > 0 \) such that \((S, \lambda)\) holds for each \( \lambda \in (\lambda_0 - \varepsilon, \lambda_0] \).

For the proof of this lemma, the following observation is useful.

Lemma 3.4. Let \( M \subset C_0(\Omega) \) be a bounded and equicontinuous subset, and let

\[
I_{\lambda}(M) := \inf_{u \in M, x \in \Omega_{\lambda}} V_{\lambda}u(x) \quad \text{for} \quad \lambda \in (0, l).
\]

Then the map \( \lambda \mapsto I_{\lambda}(M) \) is left continuous, i.e., for \( \lambda_0 \in (0, l) \) we have \( I_{\lambda}(M) \to I_{\lambda_0}(M) \) as \( \lambda \to \lambda_0, \lambda < \lambda_0 \).

Proof. We first note that

\[
I_{\lambda}(M) \leq 0 \quad \text{for all} \quad \lambda \in (0, l), \tag{70}
\]

since \( \overline{\Omega_{\lambda}} \cap T_{\lambda} \neq \emptyset \) by assumption (D1). Since \( \Omega_{\lambda_0} \subset \Omega_{\lambda} \) for \( \lambda < \lambda_0 \) and \( V_{\lambda}z \to V_{\lambda_0}z \) uniformly on \( \Omega_{\lambda_0} \) for every \( z \in M \), we have \( \limsup I_{\lambda}(M) \leq I_{\lambda_0}(M) \). Now suppose by contradiction that there exists sequences of numbers \( \lambda_n \in (0, \lambda_0) \), of functions \( u_n \in M \) and of points \( x^n \in \Omega_{\lambda_n} \) such that

\[
\lambda_n \to \lambda \quad \text{and} \quad V_{\lambda_n}u_n(x^n) \to c < I_{\lambda_0}(M) \quad \text{for} \quad n \to \infty.
\]

By compactness and equicontinuity, we may assume that there exists \( x \in \overline{\Omega} \) with \( x_1 \geq \lambda_0 \) and \( \bar{u} \in \overline{M} \subset C_0(\Omega) \) such that

\[
x^n \to x \quad \text{and} \quad \|u_n - \bar{u}\|_{L^\infty(\Omega)} \to 0 \quad \text{as} \quad n \to \infty,
\]

where \( \overline{M} \) denotes the closure of \( M \) in \( C_0(\Omega) \) with respect to \( \| \cdot \|_{L^\infty} \). Consequently,

\[
Q_{\lambda_n}(x^n) = (2\lambda_n - x_1^n, x_2^n, \ldots, x_N^n) \to (2\lambda_0 - x_1, x_2, \ldots, x_N) = Q_{\lambda_0}(\bar{x})
\]

and therefore

\[
u_n(x^n) \to \bar{u}(\bar{x}) \quad \text{and} \quad u_n(Q_{\lambda_0}(x^n)) \to \bar{u}(Q_{\lambda_0}(\bar{x})) \quad \text{as} \quad n \to \infty.
\]

Hence

\[
V_{\lambda_0}\bar{u}(\bar{x}) = \lim_{n \to \infty} V_{\lambda_0}u_n(x_n) = c < I_{\lambda_0}(M) \tag{71}
\]

We now distinguish two cases. If \( \bar{x} \in \bar{\Omega} \), then \( \bar{x} \in \overline{\Omega_{\lambda_0}} \), and we conclude that

\[
V_{\lambda_0}\bar{u}(\bar{x}) \geq \inf_{u \in \overline{M}, x \in \Omega_{\lambda_0}} V_{\lambda_0}u(x) = \inf_{u \in \overline{M}, x \in \Omega_{\lambda_0}} V_{\lambda_0}u(x) = I_{\lambda_0}(M).
\]

If, on the other hand, \( \bar{x} \in \partial \Omega \setminus \overline{\Omega_{\lambda_0}} \), then \( \bar{x}_1 = \lambda_0 \) and therefore

\[
V_{\lambda_0}\bar{u}(\bar{x}) = 0 \geq I_{\lambda_0}(M)
\]

by (70). Since in both cases we arrived at a statement contradicting (71), the proof is finished. \( \square \)
Lemma 3.2: This implies that \( V_{\lambda_0}z > 0 \) in \( \Omega_{\lambda_0} \) for all \( z \in \omega(u) \). Let \( \delta > 0 \) be such that the conclusion of Lemma 3.1 holds, and let \( K \subset \Omega_{\lambda_0} \) be a compact subset and \( \epsilon_1, \lambda_0 \) be chosen such that
\[
\|\Omega_4 \setminus K\| < \delta \quad \text{for} \quad \lambda \in (\lambda_0 - \epsilon_1, \lambda_0]. \tag{72}
\]
Since \( V_{\lambda_0}z > 0 \) in \( \Omega_{\lambda_0} \) for all \( z \in \omega(u) \) and \( \omega(u) \) is a compact subset of \( C(\overline{\Omega}) \), we may choose \( \epsilon \in (0, \epsilon_1) \) such that
\[
\inf_{z \in \omega(u), x \in K} V_{\lambda}z(x) > 0 \quad \text{for all} \quad \lambda \in (\lambda_0 - \epsilon, \lambda_0]. \tag{73}
\]
Let \( \lambda \in (\lambda_0 - \epsilon, \lambda_0], \) then (74) implies that there exists \( t_0 = t_0(\lambda) \) such that
\[
V_{\lambda}u(t,x) \geq 0 \quad \text{for} \quad x \in K, \ t \geq t_0.
\]
Hence \( \| (V_{\lambda}u)(t) \|_{L^\infty(\mu_k)} \to 0 \) as \( t \to \infty \) by Lemma 3.1. Thus (52) holds for \( \lambda \in (\lambda_0 - \epsilon, \lambda_0], \) as claimed.

Case one: We assume that (D2) holds, and that \( V_{\lambda_0}z > 0 \) on \( \Omega_{\lambda_0} \) for some \( z \in \omega(u) \). By (D2), the set \( \Omega_{\lambda_0} \) has only finitely many connected components, and hence \( \rho := \text{inrad}(\Omega_{\lambda_0})/4 > 0. \)
Let \( \gamma = \gamma(N,s,p,c_w) > 0, \ q = q(N,s,p,c_w) > 0 \) as in Proposition 2.11 and let \( \delta > 0 \) be such that the conclusions of Proposition 2.4 hold with \( \gamma + 1 \) in place of \( \gamma. \)
Choose \( D \subset \subset \Omega_{\lambda_0} \) such that \( D \) intersects each connected component of \( \Omega_{\lambda_0} \) and
\[
|\Omega_{\lambda_0} \setminus D| < \frac{\delta}{2}, \quad \text{inrad}(D) > 2 \rho. \tag{74}
\]
Fix \( z \in \omega(u) \) such that \( V_{\lambda_0}z > 0 \) in \( \Omega_{\lambda_0}, \) and let \( t_n \to \infty \) be a sequence with \( h(t_n) \to z. \) Using the equicontinuity property (U2) we can find \( r_1 > 0, \ \tau \in (0, \frac{\rho}{2}) \) and \( n_0 \) such that
\[
V_{\lambda_0}u(t,x) > 2r_1, \quad \text{for all} \quad x \in \overline{D}, \ t \in [t_n - 8 \tau, t_n], \ n > n_0. \tag{75}
\]
Let \( r_0 := \frac{1}{2} \text{dist}(\overline{D}, \partial \Omega_{\lambda_0}), \ R = \text{diam}(D) \) and choose \( \mu \) as in Theorem 2.15 for these parameter values. We first fix \( \epsilon_1 > 0 \) such that
\[
|\Omega_4 \setminus \Omega_{\lambda_0}| < \frac{\delta}{2}, \quad \text{for} \quad \lambda \in [\lambda_0 - \epsilon_1, \lambda_0]. \tag{76}
\]
From the equicontinuity assumption (U2) we may deduce that
\[
\sup_{n \in \mathbb{N}} \sup_{[t_n - 8 \tau, t_n] \times D} |V_{\lambda}u - V_{\lambda_0}u| \to 0 \quad \text{as} \quad \lambda \to \lambda_0. \tag{77}
\]
This and (75) imply the existence of \( \epsilon_2 \in (0, \epsilon_1) \) such that
\[
V_{\lambda}u(t) > r_1, \quad \text{for all} \quad x \in \overline{D}, \ t \in [t_n - 8 \tau, t_n], \ n > n_0, \ \lambda \in [\lambda_0 - \epsilon_2, \lambda_0]. \tag{78}
\]
By (S_{\lambda_0}), we can find \( n_1 > n_0 \) such that for all \( n > n_1 \) we have
\[
\| (V_{\lambda_0}u)^{-}(t_n - 8 \tau) \|_{L^\infty(\Omega_{\lambda_0})} \leq \frac{\mu r_1}{2}.
\]
Using the equicontinuity of the functions \( x \mapsto u(t_n - 8 \tau, x), \ n \in \mathbb{N} \) and Lemma 3.2 we may choose \( \epsilon \in (0, \epsilon_2) \) such that
\[
\| (V_{\lambda}u)^{-}(t_n - 8 \tau) \|_{L^\infty(\Omega_{\lambda})} \leq \mu r_1 \quad \text{for} \quad \lambda \in [\lambda_0 - \epsilon, \lambda_0]. \tag{79}
\]
We now fix \( n \geq n_1 \) and \( \lambda \in [\lambda_0 - \epsilon, \lambda_0], \) and we claim that the assumptions of Theorem 2.15 are satisfied with \( t_0 = t_n - 8 \tau, \ U = \Omega_{\lambda}, \ D \) as above and \( v = V_{\lambda}u. \) Indeed, \( \text{dist}(\overline{D}, \partial U) \geq \text{dist}(\overline{D}, \partial \Omega_{\lambda_0}) \geq 4r_0 \) and
Asymptotic symmetry for a class of nonlinear fractional reaction-diffusion equations

\[ |Ω \setminus D| < \delta \] by (74) and (76). Moreover, \( \text{inrad}(D) > 2\rho \) and \( \text{diam} D \leq R \) by our choice of \( D \) and the definition of \( R \). Moreover, by (78), \( V_λ u \) is nonnegative on \([t_n - 8\tau, t_n]\times D\), and by (78) and (79) we have

\[ \|uλ - (t_n - 8\tau)\|_{L^∞(Ω)} \leq \mu \|V_λ u\|_{L^1([t_n - 7\tau, t_n - 6\tau] \times D,)}, \]

for each connected component \( D_λ \) of \( D \). An application of Theorem 2.15(ii) with these parameters therefore yields that \( (S) \) holds for all \( λ \in [λ_0 - ε, λ_0] \). The proof is finished.

The following Proposition evidently completes the Proof of Theorem 2.17.

\[ \text{Proposition 3.5. Suppose that (D2) holds or that } z \neq 0 \text{ on } Ω \text{ for all } z \in Ω(u). \text{ Then we have:} \]

(i) \( V₀ z \equiv 0 \) on \( \mathbb{R}^N \) for every \( z \in Ω(u) \).

(ii) For every \( z \in Ω(u) \), we either have the following alternative. Either \( z \equiv 0 \) on \( Ω \), or \( z \) is strictly decreasing in \( |x_1| \) and therefore strictly positive in \( Ω \).

\[ \text{Proof. (i) We define } \lambda₀ := \inf\{ μ > 0 : (S) \text{ holds for all } λ > μ \}, \]

and we first claim that \( λ₀ = 0 \). By Lemma 3.1 we have \( λ₀ < l \). If \( z \equiv 0 \) on \( Ω \) for all \( z \in Ω(u) \), then Lemma 3.3 immediately implies that \( λ₀ = 0 \). If (D2) holds and we assume – on the contrary – \( λ₀ > 0 \), then Lemma 3.2(iii) and Lemma 3.3(ii) readily imply that \( z \equiv 0 \) on \( \mathbb{R}^N \) for every \( z \in Ω(u) \), which then also yields \( λ₀ = 0 \). Hence we conclude in both cases that \( λ₀ = 0 \), and therefore \( (S)_0 \) is true by Lemma 3.3(i). This implies that \( V₀ z \geq 0 \) on \( Ω₀ \) for every \( z \in Ω(u) \). Since the analogous statement can also be shown for the reflected solution \( \tilde{u} \) defined in (67), we also have that \( V₀ z \leq 0 \) on \( Ω₀ \) for every \( z \in Ω(u) \). Hence for every \( z \in Ω(u) \) we have \( V₀ z \equiv 0 \) on \( Ω₀ \) and thus also on \( \mathbb{R}^N \), since \( z \equiv 0 \) on \( \mathbb{R}^N \setminus Ω \).

(iii) Let \( z \in Ω(u) \) be given such that \( z \) is strictly decreasing in \( |x_1| \). Then there exists \( λ > 0 \) such that \( V_λ z \) is not strictly positive in \( Ω_λ \). By Lemma 3.2(ii), applied to \( λ \) in place of \( λ₀ \), we then have that \( V_λ z \equiv 0 \) on \( \mathbb{R}^N \). By (ii), \( z \) therefore has two different parallel symmetry hyperplanes. This implies that \( z \equiv 0 \), since \( z \) vanishes outside a bounded subset of \( \mathbb{R}^N \). □

4 Appendix

As announced in the introduction, we derive – based on recent results in [19] and [31] – a sufficient criterion for condition (U2). For a similar result in the context of local parabolic boundary value problems, see [27, Prop. 2.7].

\[ \text{Proposition 4.1. Let } Ω \subset \mathbb{R}^N \text{ be a bounded domain, and suppose that the nonlinearity } f \text{ satisfies (F1). Suppose furthermore that } 0 \in Ω, \text{ and that } f(\cdot, 0) \text{ is bounded on } (0, \infty) \times Ω. \text{ Then for any solution } u \text{ of (P) satisfying (U1)} \text{ we have:} \]

(i) For any domain \( G \subset Ω \) there exist \( α > 0 \) such that

\[ \sup_{\substack{t > 1, \ x, \ x̅, \ x̅̅ \in \ G, \ x̅̅ ≠ x \ x̅ \}} \frac{|u(t, x) - u(\bar{t}, \bar{x})|}{|x - \bar{x}| + |t - \bar{t}|^{1/2}} < \infty. \]  

(ii) If, in addition, \( Ω \) fulfills the exterior sphere condition and, for some \( t₀ > 0, C₁ > 0 \),

\[ |u(t₀, x)| ≤ C₁ \text{dist}(x, \partial Ω)^α \quad \text{for all } x \in Ω, \]

(81)
then
\[
\sup_{t \geq 0, x \in \Omega} \frac{|u(t,x)|}{\text{dist}(x, \partial \Omega)^t} < \infty
\] (82)

In particular, (U2) holds in this case.

In the special case \( f \equiv 0 \), the interior regularity estimate (80) is an immediate consequence of [19, Theorem 1.2], but we could not find any reference where the case \( f \neq 0 \) is considered. Before giving the proof of this proposition, we discuss an example.

**Remark 4.2.** Let \( \Omega \subset \mathbb{R}^N \) be a bounded domain satisfying the exterior sphere condition. We consider an Allen-Cahn-type nonlinearity
\[
f : [0, \infty) \times \Omega \times \mathbb{R} \to \mathbb{R}, \quad f(t,x,u) = a(t)u - b(t)u^3 = u[a(t) - b(t)u^2]
\] (83)
Here \( a, b : [0, \infty) \to \mathbb{R} \) are continuous functions with \( a(t) \leq b(t) \) for \( t \geq 0 \). Then \( f \) satisfies \((F1)\) with \( \mathcal{B} = \mathbb{R} \), and it trivially satisfies \((F2)\) if \( \Omega \) satisfies \((D1)\). Moreover, the constant 1 is a supersolution of problem \((P)\), whereas 0 is a solution. Hence, if \( \varphi \in C_0(\Omega) \cap \mathcal{H}_0^2(\Omega) \) is such that \( 0 \leq \varphi(x) \leq 1 \) for all \( x \in \Omega \), then \((U2)\) is also satisfied by Proposition [4.1ii]. We remark that the solution \( u \) can be found as the unique mild solution of \((84)\), i.e., the unique solution of the nonlinear integral equation
\[
u \in C([0, \infty), C_0(\Omega)), \quad u(t) = S_A(t)\varphi + \int_0^t S_A(t-\tau)F(\tau, u(\tau)) \, d\tau \quad \text{for} \ t \in [0, \infty).
\] (85)
Here \( S_A \) denotes the semigroup generated by the \( m \)-dissipative operator
\[
A : \text{dom}(A) \subset C_0(\Omega) \to C_0(\Omega), \quad Au := -(-\Delta)^s u
\]
where \( \text{dom}(A) \) is the space of all functions \( u \in \mathcal{H}_0^s(\Omega) \cap C_0(\Omega) \) such that \( (-\Delta)^s u \), defined in distributional sense, is contained in \( C_0(\Omega) \). Moreover, \( F : [0, \infty) \times C_0(\Omega) \to C_0(\Omega) \) is the substitution operator given by \( [F(t,w)](x) = f(t,x,w(x)) \) for \( t \in [0, \infty), x \in \Omega \). The \( m \)-dissipativity of the operator \( A \) in \( C_0(\Omega) \) is essentially a consequence of the following recent regularity result given in [11] Proposition 1.1: If \( \Omega \subset \mathbb{R}^N \) is a bounded domain satisfying the exterior sphere condition and \( w \in L^m(\Omega) \), then the unique weak solution \( u \in \mathcal{H}_0^s(\Omega) \) of the equation \( -\Delta u = w \) belongs to \( C_0(\Omega) \). Another important fact needed for the local existence and uniqueness of solutions of \((P)\) is the local uniform (in time) Lipschitz continuity of \((63)\), one may essentially argue as in [13] for the semilinear heat equation, noting the following additional useful property of the substitution operator \( F \): if \( M \subset C_0(\Omega) \cap \mathcal{H}_0^s(\Omega) \) is bounded with respect to \( \| \cdot \|_\infty \), then \( F(M) \subset \mathcal{H}_0^s(\Omega) \), and there exists \( L = L(M) > 0 \) such that
\[
\|F(t,u), F(t,u)\| \leq L\|u, u\| \quad \text{for all} \ u \in M, \ t > 0.
\] (86)
This property can be checked immediately by using \((F2)\) and the definition of the quadratic form \( E \),

Note that \((83)\) is just a particular example of a nonlinearity which admits an ordered pair of a bounded subsolution \( \varphi_* \), and a bounded supersolution \( \varphi^* \) which satisfies \((F1)\) with \( \mathcal{B} = \mathbb{R} \). In such a setting, an initial condition \( \varphi \in C_0(\Omega) \cap \mathcal{H}_0^s(\Omega) \) always gives rise to a global bounded solution of \((P)\).
The remainder of this appendix is devoted to the proof of Proposition 4.1. The assertion (80) on interior regularity will be deduced from the Harnack inequality of Felsinger and Kaufmann [19]. More precisely, we will use the following rescaled variant of a special case of [19 Corollary 5.2].

**Proposition 4.3.** Let
\[ D_0 := (-2^{2r+1}, -2^{2r+1} + 1) \times B_1(0) \quad \text{and} \quad D_\beta := (-1, 0) \times B_1(0). \]

There exists \( \varepsilon_0, \delta > 0 \) such that for every nonnegative supersolution
\[ w : (-2^{2r+1}, 0) \times \mathbb{R}^N \to \mathbb{R} \]
of the equation
\[ \partial_t w + (-\Delta)^s = -\varepsilon_0 \quad \text{in} \ (-2^{2r+1}, 0) \times B_0(0) \]
in the sense of Definition 2.7 with the property that
\[ |D_\beta \cap \{ w \geq 1 \}| \geq \frac{1}{2} |D_\beta| \quad (87) \]
we have \( w \geq \delta \) a.e. on \( D_\beta \).

**Corollary 4.4.** Let \( r_0 \in (0, 1] \), \( c_u > 0 \) and \( f_\infty > 0 \). Then there exist constants \( \alpha \in (0, 1) \) and \( C_2 > 0 \) depending on \( N, s, f_\infty, c_u, r_0 \) with the following property:

If \( T := (t_0 - r_0^2, t_0) \) for some \( t_0 \in \mathbb{R}, x_0 \in \mathbb{R}^N \), \( f \in L^\infty(T \times B_{r_0}(x_0)) \) with \( \| f \|_{L^\infty(T \times B_{r_0}(x_0))} \leq f_\infty \) are given and
\[ u \in C(T, H^s(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N) \cap C(\overline{B_{r_0}(x_0)})) \cap C^1(T, L^2(B_{r_0}(x_0))) \]
with \( \| u \|_{L^\infty(T \times \mathbb{R}^N)} \leq c_u \) is a solution of
\[ \partial_t u + (-\Delta)^s u = f(t, x) \quad \text{in} \ T \times B_{r_0}(x_0) \]
in the sense that
\[ \mathcal{E}(u(t), \varphi) = \int_{B_{r_0}(x_0)} [f(t, x) - \partial_t u(t, x)] \varphi(x) \, dx \]
for every \( \varphi \in \mathcal{H}_0^s(B_{r_0}(x_0)) \) and a.e. \( t \in T \), then we have
\[ \text{osc}_u \leq C_2 r^\alpha \quad \text{for} \ r \in (0, r_0], \text{ where } Q(r) := (t_0 - r^2, t_0) \times B_r(x_0). \quad (88) \]

***Proof.*** Without loss, we may assume that \( t_0 = 0 \) and \( x_0 = 0 \). Moreover, we may assume by normalization that \( c_u = \frac{1}{2} \). In this case we will prove (88) with \( C_2 = 1 \) for some suitable \( \alpha \in (0, 1) \). Suppose by contradiction that the statement is false. Then there exist, for every \( k \in \mathbb{N} \), functions \( f_k \in L^\infty(T \times B_{r_0}(0)) \) with \( \| f_k \|_{L^\infty(T \times B_{r_0}(0))} \leq f_\infty \) and \( u_k \in C(T, H^s(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N) \cap C(\overline{B_{r_0}(0)}) \cap C^1(T, L^2(B_{r_0}(0)))) \) with
\[ \| u_k \|_{L^\infty(T \times \mathbb{R}^N)} \leq \frac{1}{4} \]
solving
\[ \partial_t u_k + (-\Delta)^s u_k = f_k(t, x) \quad \text{in} \ T \times B_{r_0}(0) \]
as well as \( \alpha_k \in (0, 1) \) and \( r_k \in (0, r_0] \) such that \( \alpha_k \to 0 \) as \( k \to \infty \) and
\[ \text{osc}_{Q(r_k)} u_k \geq r_k^{\alpha_k} \quad \text{for every} \ k \in \mathbb{N}. \]
Passing to a subsequence, we also have
\[
\osc_{T \times \mathbb{R}^N} u_k \leq 2\|u_k\|_{L^\infty(T \times \mathbb{R}^N)} \leq \frac{1}{2} \leq r_{0_k}^q \quad \text{for every } k \in \mathbb{N}.
\]
By making \( r_k \in (0, r_0) \) larger if necessary, we may therefore assume that
\[
\osc_{Q(r_k)} u_k = r_{0_k}^q \quad \text{for every } k \in \mathbb{N}
\]
and
\[
\osc_{Q(r)} u_k \leq r_{0_k}^q \quad \text{for } r \in [r_k, r_0] \text{ and } k \in \mathbb{N}.
\]
Since also \( \osc_{Q(r)} u_k \leq \frac{1}{2} \) for every \( k \in \mathbb{N} \), we conclude that \( r_k \to 0 \) as \( k \to \infty \). We now define \( T_k := (-\left(\frac{r_k}{r_0}\right)^{2s}, 0) \) and
\[
v_k : T_k \times \mathbb{R}^N \to \mathbb{R}, \quad v_k(t, x) = 2r_{0_k}^{-\alpha}u_k(r_{0_k}^{2s}t, r_kx)
\]
for \( k \in \mathbb{N} \). Then we have
\[
\tilde{\partial} v_k + (-\Delta)^s v_k = \tilde{f}_k(t, x) \quad \text{in } D_k := T_k \times B_{\frac{r_0}{r_k}}(0)
\]
with
\[
\tilde{f}_k(t, x) = 2r_{0_k}^{-\alpha}f_k(r_{0_k}^{2s}t, r_kx).
\]
Without loss, we may assume that \( \frac{r_0}{r_k} \geq \max\{2^{1+\frac{1}{s}} \cdot 5\} \) for every \( k \in \mathbb{N} \), so that \( (-2^{2s+1}, 0) \times B_5(0) \subset D_k \) for every \( k \in \mathbb{N} \). Moreover, we have
\[
\osc_{Q(1)} v_k = 2,
\]
\[
\osc_{Q(r)} v_k \leq 2r_{0_k}^q \quad \text{for } r \in [1, \frac{r_0}{r_k}], k \in \mathbb{N} \quad (89)
\]
and
\[
\osc_{Q(1)} v_k \leq 2 \left(\frac{r_0}{r_k}\right)^{-\alpha} \quad \text{for } k \in \mathbb{N}. \quad (90)
\]
By adding a constant to \( v_k \) if necessary, we may assume that
\[
\sup_{Q(1)} v_k = 1 \quad \text{and} \quad \inf_{Q(1)} v_k = -1. \quad (91)
\]
After passing to a subsequence, we may also assume that, replacing \( v_k \) by \(-v_k\) and \( \tilde{f}_k \) by \(-\tilde{f}_k\) if necessary, \(|D_{\emptyset} \cap \{v_k \geq 0\}| \geq \frac{1}{2}|D_{\emptyset}|\).

Here and in the following, \( D_{\emptyset} \) and \( D_{\emptyset} \) are defined as in Proposition\[4,3\] Note that by \( (89), (90) \) and \( (91) \) we have
\[
v_k(t, x) \geq \min\{-1, 1 - 2|\alpha|_q\} \quad \text{for } x \in \mathbb{R}^N, \ t \in (-2^{2s+1}, 0).
\]
We now consider
\[
w_k : T_k \times \mathbb{R}^N \to \mathbb{R}, \quad w_k(t, x) := v_k(t, x) + 2 \cdot 5^{\alpha_q} - 1.
\]
Then
\[
w_k(t, x) \geq \min\{0, 2(5^{\alpha_q} - |\alpha q|)\} \quad \text{for } x \in \mathbb{R}^N, \ t \in (-2^{2s+1}, 0).
\]
In particular, we have \( w_k \geq 0 \) in \((-2^{2r+1}, 0) \times B_2(0)\), and for \( x \in B_2(0) \) we have
\[
|(-\Delta)^r w_k(t,x)| \leq 2 \int_{\mathbb{R}^N \setminus B_1(0)} |y|^q \nu dy \leq \int_{\mathbb{R}^N \setminus B_1(0)} (|y| - 4)^q \nu dy,
\]
where the latter integral tends to zero as \( k \to \infty \) by Lebesgue’s theorem. Hence
\[
\lim_{k \to \infty} \|(-\Delta)^r w_k\|_{L^\infty((-2^{2r+1}, 0) \times B_2(0))} = 0.
\] (92)

We now note that the function \( w_k^+ \) is a nonnegative solution of
\[
\partial_t w_k^+ + (-\Delta)^r w_k^+ = g_k \quad \text{in } (-2^{2r+1}, 0) \times B_2(0)
\]
with \( g_k := \tilde{f}_k + (-\Delta)^r w_k^- \), whereas \( \|g_k\|_{L^\infty((-2^{2r+1}, 0) \times B_2(0))} \to 0 \) as \( k \to \infty \) as a consequence of (92) and the fact that
\[
\|\tilde{f}_k\|_{L^\infty((-2^{2r+1}, 0) \times B_2(0))} \leq 2^{-2r-6} f_\infty.
\]
Consequently, there exists \( k_0 \in \mathbb{N} \) such that \( \|g_k\|_{L^\infty((-2^{2r+1}, 0) \times B_2(0))} \leq \varepsilon_0 \), where \( \varepsilon_0 \) is given by Lemma 4.3. On the other hand, since \( D_0 = Q(1) \), we infer from (91) that
\[
\inf_{D_0} w_k^+ = \inf_{D_0} w_k = 2 \cdot 5^{k_0} - 2 \to 0 \quad \text{as } k \to \infty.
\]
This contradicts Proposition 4.3 applied to \( w = w_k^- \). The proof is thus finished. \( \blacksquare \)

**Proof of Proposition 4.7 (completed).** (i) We note that \( u \) satisfies
\[
\partial_t u(t,x) + (-\Delta)^r u(t,x) = \tilde{f}(t,x)
\]
with \( \tilde{f}(t,x) = f(t,x, u(t,x)) \), and by assumption \( u \) and \( f \) are bounded on \((0, \infty) \times \mathbb{R}^N\). Hence, for given \( G \subset \subset \Omega \), we may choose \( r_0 > 0 \) such that \( r_0 < \min(\text{dist}(G, \partial \Omega), 1) \), and we may apply Corollary 4.4 to every point \( x_0 \in G, r_0 \geq 1 \). From this (90) easily follows.

(ii) We use barrier functions as constructed in the elliptic setting in [31]. Put \( B_r := B_r(0) \) for \( r > 0 \), and recall the definition of the space \( \mathcal{Y}^s(U') \) in (7). By [31] Lemma 2.6 there exists a function \( \varphi \in \mathcal{Y}^s(\mathbb{R}^N) \) satisfying
\[
\begin{cases}
(-\Delta)^s \varphi \geq 1 & \text{in } B_4 \setminus B_1, \\
\varphi \equiv 0 & \text{in } B_1, \\
0 \leq \varphi(x) \leq c_0(|x| - 1)^s & \text{for } x \in B_4 \setminus B_1;
\end{cases}
\] (93)
as well as
\[
\varphi(x) \geq d_0 \text{dist}(x, \partial B_1)^s \quad \text{for } x \in B_4 \setminus B_1
\] (94)
with some constants \( c_0, d_0 > 0 \). In fact, it is not stated explicitly in [31] that \( \varphi \in \mathcal{Y}^s(\mathbb{R}^N) \) and that (94) holds, but this follows from the construction in [31] Appendix. Now since \( \Omega \) satisfies the exterior sphere condition, there exists \( \rho > 0 \) such that every point in \( \partial \Omega \) can be touched from outside by a ball of radius \( \rho \). Fixing such a ball \( B_\rho(y) \) for some \( y \in \mathbb{R}^N \setminus \Omega \), we may define the function
\[
\psi \in H_{\text{loc}}^s(\mathbb{R}^N), \quad \psi(x) = \lambda \varphi(\frac{x-y}{\rho}).
\]
Here, using (81), (92), (94) and the assumption that \( u \) satisfies (U1), we may choose \( \lambda > 0 \) sufficiently large so that
\[
\begin{cases}
(-\Delta)^s \psi \geq \sup_{t \geq t_0, x \in \Omega} f(t, x, u(t,x)) & \text{in } B_{4\rho}(y) \setminus B_\rho(y), \\
\psi \geq \sup_{t \geq t_0, x \in \Omega} u(t,x) & \text{in } \mathbb{R}^N \setminus B_{4\rho}(y) \\
\psi(x) \geq u(t_0, x) & \text{for } x \in \Omega \cap B_{4\rho}(y).
\end{cases}
\] (95)
Asymptotic symmetry for a class of nonlinear fractional reaction-diffusion equations

Let \( w(t, x) = \psi(x) - u(t, x) \). By the properties (95), \( w \) is an entire supersolution of \( \partial_t w + (-\Delta)^s w = 0 \) in \([t_0, \infty) \times (\Omega \cap B_{4\rho}(y))\) in the sense of Definition 2.1, and \( w(t_0) \) is nonnegative on \( \mathbb{R}^N \). Hence, by the weak maximum principle as stated in Remark 2.6, \( w(t, x) \geq 0 \) for \( x \in \Omega, t \geq t_0 \) and therefore

\[
u(t, x) \leq \psi(x) \leq \frac{\lambda c_0}{\rho^s}(|x - y| - \rho)^s \quad \text{for} \ x \in \Omega \cap B_{4\rho}(y), \ t \geq t_0.
\]

Since the parameter \( \lambda \) in the definition of \( \phi \) can be chosen uniformly with respect to the \( \rho \)-balls touching \( \Omega \) from outside, we find – using also the boundedness of \( u \) on \([t_0, \infty) \times \Omega \) – a constant \( C' > 0 \) such that

\[
u(t, x) \leq C' \text{dist}(x, \partial \Omega)^s \quad \text{for} \ x \in \Omega, \ t \geq t_0. \tag{96}
\]

Repeating the same argument with \(-u\) in place of \( u \), we find a constant \( C'' > 0 \) such that

\[
u(t, x) \geq -C'' \text{dist}(x, \partial \Omega)^s \quad \text{for} \ x \in \Omega, \ t \geq t_0. \tag{97}
\]

Combining (96) and (97), we obtain (82), as claimed. Now (U2) follows easily by combining (80) and (82).

\section*{Acknowledgments}

The authors would like to thank Mouhamed Moustapha Fall and Peter Poláčik for helpful discussions.

\section*{References}

[1] (MR0143162) A. D. Alexandrov, \textit{A characteristic property of the spheres}, Annali di Matematica Pura ed Applicata. Series IV, 58 (1962), 303–315.

[2] (MR2512800) D. Applebaum, \textit{Lévy Processes and Stochastic Calculus}, Cambridge University Press, Cambridge, 2009.

[3] (MR2114412) M. Birkner, J. A. López-Mimbela, and A. Wakolbinger, \textit{Comparison results and steady states for the Fujita equation with fractional Laplacian}, Annales de L’Institut Henri Poincaré 22 (2005), 83–97.

[4] (MR2365478) K. Bogdan, T. Kulczycki, and M. Kwaśnicki, \textit{Estimates and structure of \( \alpha \)-harmonic functions}, Probability Theory and Related Fields 140 (2008), 345–381.

[5] (MR3023003) C. Brändle, E. Colorado and A. de Pablo, \textit{A concave-convex elliptic problem involving the fractional Laplacian}, Proceedings of the Royal Society of Edinburgh. Section A. Mathematics 143 (2013), 39–71.

[6] (MR2257732) E. Chasseigne, M. Chaves and J. D. Rossi, \textit{Asymptotic behavior for nonlocal diffusion equations}, Journal de Mathématiques Purés et Appliquées. 86 (2006) 271–291.

[7] (MR2784330) L. Caffarelli, C. H. Chan and A. Vasseur, \textit{Regularity theory for parabolic nonlinear integral operators}, Journal of the American Mathematical Society 3 (2011), 849–869.

[8] X. Cabré and Y. Sire, \textit{Nonlinear equations for fractional Laplacians I: Regularity, maximum principles, and Hamiltonian estimates}, available online at \url{http://arxiv.org/abs/1012.0867}.

[9] (MR2354493) L. Caffarelli and L. Silvestre, \textit{An Extension Problem Related to the Fractional Laplacian}, Communications in Partial Differential Equations 32 (2007), 1245–1260.
[10] (MR2646117) X. Cabr´e and J. Tan, Positive solutions of nonlinear problems involving the square root of the Laplacian, Advances in Mathematics 224 (2010), 2052–2093.

[11] (MR2680400) L. Caffarelli and A. Vasseur, Drift diffusion equations with fractional diffusion and the quasi-geostrophic equation, Annals of Mathematics. Second Series 171 (2010), 1903–1930.

[12] (MR2825595) A. Capella, J. D´avila, L. Dupaigne, and Y. Sire, Regularity of radial extremal solutions for some non local semilinear equations, Communications in Mathematical Physics 8 (2011), 1353–1384.

[13] (MR1691574) T. Cazenave and A. Haraux, An Introduction to Semilinear Evolution Equations, Oxford Science Publications, Oxford, 1998.

[14] (MR2737789) S.-Y. A. Chang and M. del Mar Gonz´alez, Fractional Laplacian in conformal geometry, Advances in Mathematics 226 (2011), 1410–1432.

[15] H. A. Chang Lara and G. D´avila, Regularity for solutions of non local parabolic equations, available online at [http://arxiv.org/abs/1109.3247](http://arxiv.org/abs/1109.3247).

[16] (MR2200258) W. Chen, C. Li and B. Ou: Classification of solutions for an integral equation, Communications on Pure and Applied Mathematics 59 (2006), 330–343.

[17] (MR2944369) E. di Nezza, G. Palatucci, and E. Valdinoci, Hitchhiker’s Guide to the Fractional Sobolev Spaces, Bulletin des Sciences Mathématiques 136 (2012), 521–573.

[18] (MR3002595) P. Felmer, A. Quaas and J. Tan, Positive solutions of nonlinear Schr¨odinger equation with the fractional Laplacian, Proceedings of the Royal Society of Edinburgh. Section A. Mathematics 142.2 (2012), 1237–1262.

[19] M. Felsinger and M. Kassmann, Local regularity for parabolic nonlocal operators, preprint, available online at [http://arxiv.org/abs/1203.2126](http://arxiv.org/abs/1203.2126).

[20] (MR544879) B. Gidas, W. N. Ni and L. Nirenberg, Symmetry and related properties via the maximum principle, Communications in Mathematical Physics 68.3 (1979), 209-243.

[21] (MR2158336) N. Jacob, Pseudo Differential Operators and Markov Processes, Vol. I, II, III, Imperial College Press, London, 2005.

[22] T. Jin and J. Xiong, A fractional Yamabe flow and some applications, available online at [http://arxiv.org/abs/1110.5664](http://arxiv.org/abs/1110.5664).

[23] (MR2817382) M. Kassmann, A new formulation of Harnack’s inequality for nonlocal operators, Comptes Rendus Mathématique. Académie des Sciences. Paris 1.349 (2011), 637–640.

[24] (MR1465184) G. M. Lieberman, Second Order Parabolic Differential Equations, World Scientific Publishing, Singapore, 2005.

[25] (MR3002745) R. Servadei and E. Valdinoci, Variational methods for non-local operators of elliptic type. Discrete and Continuous Dynamical Systems 33.5 (2013), 2105–2137.

[26] (MR2182305) P. Poláˇcik, Symmetry properties of positive solutions of parabolic equations on $\mathbb{R}^N$: I. Asymptotic symmetry for the Cauchy problem, Communications in Partial Differential Equations 30 (2005), 1567–1593.

[27] (MR2259340) P. Poláˇcik, Estimates of Solutions and Asymptotic Symmetry for Parabolic Equations on Bounded Domains, Archive for Rational Mechanics and Analysis 183 (2007), 59–91.
Asymptotic symmetry for a class of nonlinear fractional reaction-diffusion equations

[28] (MR2532926) P. Poláčik, Symmetry Properties of Positive Solutions of Parabolic Equations: A Survey, World Scientific 2009 (2009), 170–208.

[29] P. Poláčik and S. Terracini, Nonnegative solutions with a nontrivial nodal set for elliptic equations on smooth symmetric domains, to appear in Proceedings of the American Mathematical Society.

[30] (MR2737788) A. de Pablo, F. Quirós, A. Rodríguez and J. L. Vázquez, A fractional porous medium equation, Advances in Mathematics 226 (2011), 1378–1409.

[31] X. Ros-Oton and J. Serra, The Dirichlet Problem for the fractional Laplacian: Regularity up to the boundary, preprint, available online at http://arxiv.org/abs/1207.5985.

[32] (MR0333220) J. Serrin, A symmetry problem in potential theory, Archive for Rational Mechanics and Analysis 43 (1971), 304-318.

[33] (MR2001105) M. E. Schonbek and T. P. Schonbek, Asymptotic Behavior to Dissipative Quasi-Geostrophic Flows, SIAM Journal on Mathematical Analysis 35 (2003), 357–375.

[34] (MR1974415) R. Song and Z. Vondraček, Potential theory of subordinate killed Brownian motion in a domain, Probability Theory and Related Fields 123 (2003), 578–592.

[35] (MR2819627) J. Tan, The Brezis-Nirenberg type problem involving the square root of the Laplacian, Calculus of Variations and Partial Differential Equations 42 (2011), 21-41.

[36] (MR1349110) G. N. Watson, A Treatise on the Theory of Bessel Functions, Cambridge University Press, Cambridge, 1922.