EXCESS DIMENSIONS OF BRILL–NOETHER SCHEMES OF RANK TWO STABLE BUNDLES

ALI BAJRAVANI

ABSTRACT. We use results of M. Aprodu and E. Sernesi to extend a result by Fulton–Harris–Lazarsfeld in Brill–Noether theory of line bundles to Brill–Noether loci of stable bundles inside the moduli space of rank two stable vector bundles on a smooth projective algebraic curve. As a consequence; if \( B^2_{k,d} \) is of expected dimension, then we prove that its smoothness is equivalent with the smoothness of \( B^2_{k,d+1} \).

1. Introduction

Let \( C \) be a smooth projective algebraic curve of genus \( g \). For integers \( r \) and \( d \) with \( 0 \leq 2r \leq d \), the Brill–Noether scheme of line bundles, \( W^r_d \), parameterizes line bundles of degree \( d \) with the space of global sections of dimension at least \( r + 1 \). Brill–Noether theory (see for example [13]) computes the expected dimensions of these schemes in terms of \( d \), \( r \) and \( g \), and shows that this dimension is attained for a general curve. However, the dimension may be strictly larger when \( C \) is special in the sense of moduli. The basic Martens’ theorem bounds the dimension of \( W^r_d \) in terms of \( d \) and \( r \), on arbitrary smooth projective curves. Variations of this basic result have been obtained in the literature, see [22], [9], [6] and [3]. See also [7, Theorem 3.5] and [7, Theorem 3.7]. Another result concerning dimensions of Brill–Noether schemes of line bundles, which holds on arbitrary curves, is the excess dimension inequality. William Fulton, Joe Harris and Rob Lazarsfeld proved in [12] that the difference of dimensions of \( W^r_d \) and \( W^{r+1}_d \), (resp., of \( W^r_d \) and \( W^{r+1}_d \)), are controllable in terms of \( r \) (resp., in terms of \( r \), \( d \) and \( g \)). A significant consequence of their result is that if \( \dim W^r_d \geq r + 1 \), then \( W^r_{d-1} \) would be non-empty.

Marian Aprodu and Edoardo Sernesi have extended the excess dimension inequalities to the schemes of secant loci of very ample line bundles. See [1]. Michael Kemeny used their result in studying the syzygies of curves of arbitrary gonality, see [16]. As well, in [5], the author uses Aprodu-Sernesi’ excess dimension result to obtain a Mumford type theorem for schemes of secant loci.

Since some bundles appeared in constructing \( W^r_d \)’s turns out to be ample, no non-emptyness prerequisite is needed in Fulton-Harris-Lazarsfeld framework. But, this positivity property fails to be hold in Aprodu and Sernesi’ general setting. So they enter a non-emptiness assumption, which is slightly a weaker hypothesis.
If $X$ is an integral algebraic scheme and

$$
0 \longrightarrow \mathcal{L} \longrightarrow \mathcal{F}_1 \xrightarrow{\pi} \mathcal{F} \longrightarrow 0,
$$

is an arbitrary diagram of bundles on $X$ with $\text{rk}(\mathcal{E}) = m$, $\text{rk}(\mathcal{F}) = n$, $\text{rk}(\mathcal{F}_1) = n + 1$, then Aprodu and Sernesi compare dimensions of $D_k(\sigma)$, $D_{k+1}(\sigma)$ and $D_k(\pi\sigma)$, where for a morphism $\sigma : \mathcal{E} \rightarrow \mathcal{F}$ of vector bundles on $X$, the loci $D_k(\sigma)$ is defined as

$$D_k(\sigma) := \{x \in X \mid \text{rk}(\sigma_x) \leq k\}.$$

Consequently Fulton–Harris–Lazarsfeld’s result is a special case of Aprodu–Sernesi’ result.

We use Aprodu and Sernesi’ general machinery to study the excess-dimension problem for Brill–Noether loci of stable vector bundles of rank 2 and degree $d$ with the space of global sections at least of dimension $k$ inside the moduli space of rank two stable vector bundles, $U(2, d)$. The closure of some suitable loci, denoted by $B_{k,d+1}^j$, consisting of general dual elementary transformations of general bundles in $B_{2,d+1}^1$ is a divisor in $U(2, d+1)$ for $1 \leq d \leq 2(g-1) - 1$. On a suitable dense sub-locus $U \subseteq U(2, d + 1)$, consisting of general dual elementary transformations of general bundles in $U(2, d)$, we obtain a diagram of vector bundles. As Lemma 2.2(i) indicates, there exists a dense subset of $U(r, d + 1)$ consisting of general dual elementary transformations of general bundles in $U(r, d)$. But it is not known if this loci intersects the loci $B_{k,d}^r$ for $r \geq 3$. As for the case $r = 2$, we observe that such a dense subset has non-empty intersection with any non-empty component of $B_{k,d}^2$. See lemma 2.2(ii). So comparing dimensions of degeneracy loci of the diagram gives a right comparison of dimensions of various $B_{k,d}^r$’s. The diagram we use, diagram 2.1, is a “dual” version of Aprodu-Sernesi’ diagram 1.1. We apply theorem 2.4 by which we obtain our main result. See Theorem 2.6. Using theorem 2.6 together with a comparison of dimension of tangent spaces we relate the smoothness of $B_{k,d}^r$ to those of $B_{k,d+1}^r$. See corollary 2.8.

**Notation.** For a sheaf $\mathcal{F}$ on $C$, we abbreviate $H^i(C, \mathcal{F})$ and $h^i(C, \mathcal{F})$ to $H^i(\mathcal{F})$ and $h^i(\mathcal{F})$, respectively. If $\sigma : E \rightarrow F$ is a morphism of locally free sheaves $E$, $F$ of ranks $\alpha$, $\beta$ on an algebraic variety $X$, respectively, then for $p \in X$ the $\alpha \times \beta$ matrix $\sigma|_p$ acts on an $\alpha$-vector $\nu \in E|_p$ by $\nu \cdot \sigma|_p$.

# 2. Excess dimensions of Brill-Noether loci of rank two stable bundles

## 2.1. A dual version of Aprodu and Sernesi’ result

Let $X$ be an integral algebraic variety and consider a diagram of vector bundles

$$
0 \longrightarrow \mathcal{E} \longrightarrow \mathcal{F}_1 \xrightarrow{\pi} \mathcal{F} \longrightarrow 0,
$$

**Theorem 2.1.** Suppose $\mathcal{E}, \mathcal{F}_1$ and $\mathcal{F}$ are vector bundles of ranks $m$, $n+1$ and $n$, respectively, which are set in a diagram as 2.1. Assume moreover that no irreducible component of $D_k(\sigma i)$ is contained in $D_k(\sigma)$.
(1) If $D_k(\sigma)$ is nonempty, then
$$\dim D_k(\sigma) \geq \dim D_{k+1}(\sigma) - (n - k),$$

(2) If $D_k(\sigma)$ is nonempty, then
$$\dim D_k(\sigma) \geq \dim D_k(\sigma) - (m - k),$$
$$\dim D_k(\sigma) \geq \dim D_{k+1}(\sigma) - (m + n - 2k).$$

**Proof.** For a morphism $\gamma : E \to F$ of vector bundles on an integral algebraic scheme $X$, one has $D_k(\gamma) = D_k(\gamma^*)$, where $\gamma^* : F^* \to E^*$ is the dual homomorphism of $\gamma$. The theorem follows, because diagram (2.1) is a dual version of the diagram considered in proposition [11 Th. 3.1]. \qed

2.2. A divisorial component in $B_{r,d+1}^1$. Assume that $g \geq 2$. Let $U(r, d)$ denote the moduli space of stable vector bundles of rank $r$ and degree $d$. Then, $U(r, d)$ admits an étale finite cover on which there exists a Poincaré bundle. Precisely, there exists an étale finite cover $\pi : U \to U(r, d)$ and a bundle $E_d \to U$ such that for each $E \in U$ one has $E_d \times U \cong \pi(E)$. Furthermore, for any $\lambda \in \mathbb{N}$, $U(r, d) \cong U(r, d + \lambda r)$ by an isomorphism $\theta$, where $\theta : U(r, d + \lambda r) \to U(r, d)$ is given by $E \mapsto E \otimes \mathcal{O}(-D)$ with $D := p_1 + \ldots + p_h$, for some fixed $p_i \in C$.

For an integer $\delta$ with $\delta := d + \lambda r > 2r(g - 1)$, consider the projections

$$U(r, \delta) \times C \xrightarrow{\pi_1} C$$
$$\xrightarrow{\pi_2} U(r, \delta),$$

and set $E_\delta := (\pi_2)_*E_\delta$, $F_\delta := (\pi_2)_*E_\delta \otimes \mathcal{O}_D(D))$. The bundles $E_\delta$ and $F_\delta$ would be of ranks $\delta - r(g - 1)$ and $\lambda r$, respectively. See [14].

The loci of stable vector bundles $E$, of rank $r$ and degree $d$ with the space of global sections of dimension at least $k$ is up to the isomorphism $\theta$ the $(\delta - r(g - 1) - k)$-th degeneracy locus of the evaluating morphism of bundles

$$\gamma_\delta : E_\delta \to F_\delta.$$

Observe that for $E \in U(r, \delta)$ with $\delta > 2r(g - 1)$ one has $h^0(E) = \delta - 2r(g - 1)$ and so the locus $E \in U(r, \delta)$ with $\delta > 2r(g - 1)$ and $h^0(E) = 1$ might be empty. By abuse of notations, we denote by $B_{r,\delta}^1$ the loci $\theta^{-1}(B_{r,\delta}^1)$.

Recall that in an extension of vector bundles $0 \to E \to F \to \mathcal{O}_p \to 0$, the bundle $E$ is called an elementary transformation of $F$ by $p$ and $F$ is called a dual elementary transformation of $E$ by $p$.

The Segre invariant $s_1(E)$ is defined to be

$$s_1(E) := \min\{\deg(E) - 2\deg(L) : L \subset E \text{ a line subbundle}\}.$$}

Then $E$ is stable if and only if $s_1(E) \geq 1$.

**Lemma 2.2.** (i) Assume that $1 \leq d \leq n(g - 1) - 1$ and $p \in C$ is generic. Then, a general extension as $0 \to E \to F \to \mathcal{O}_p \to 0$, with $E$ a general element of $B_{r,d}^1$, $0 \leq i \leq 1$, is stable.

(ii) Suppose $E$ is an arbitrary stable rank two vector bundle and $p \in C$ generic. Then, a general dual elementary transformation of $E$ by $p$ is stable.

(iii) Assume $p \in C$ is generic and $E$ an arbitrary stable rank two vector bundle. Then, a general elementary transformation of $E$ by $p$ is stable.
Prove. (i) Observe that for $0 \leq d \leq r(g-1)$, the loci $B^1_{r,d}$ and $B^0_{r,d} := U(r,d)$ are irreducible by [23, Theorem II.3.1]. So, according to the openness of stability, it is enough to prove that there exists one special extension in each of them with the stated property.

We prove the case $i = 1$. The other case is similar. Set $d = r\alpha + t$ and observe then that $\alpha \leq g-1$. Take line bundles $L_1, \ldots, L_{r-1}$ in the loci $B^1_{1,\alpha} \setminus B^1_{1,\alpha}$ and a line bundle $H \in B^1_{1,\alpha} \setminus B^1_{1,\alpha}$, with the property that no two of these line bundles are isomorphic. For generic points $p_1, \ldots, p_t$ in $C$, a general extension as

\[
0 \to \bigoplus_{i=1}^{r-1} L_i \oplus H \to F \to \bigoplus_{i=1}^t \mathcal{O}_{p_i} \oplus \mathcal{O}_p \to 0,
\]

is stable by the Theorem of Mercat. See [20, p. 76]. A general extension $F$ as in [23] is mapped by $\mu$ to a general element of $\text{Ext}^1 \left( \bigoplus_{i=1}^t \mathcal{O}_{p_i}, \left( \bigoplus_{i=1}^{r-1} L_i \right) \oplus H \right)$, where

\[
\mu : \text{Ext}^1 \left( \bigoplus_{i=1}^t \mathcal{O}_{p_i} \oplus \mathcal{O}_p, \left( \bigoplus_{i=1}^{r-1} L_i \right) \oplus H \right) \to \text{Ext}^1 \left( \bigoplus_{i=1}^t \mathcal{O}_{p_i}, \left( \bigoplus_{i=1}^{r-1} L_i \right) \oplus H \right)
\]

is the pull back morphism of the inclusion of sheaves $\bigoplus_{i=1}^t \mathcal{O}_{p_i} \to \left( \bigoplus_{i=1}^t \mathcal{O}_{p_i} \right) \oplus \mathcal{O}_p$. Observe that $\mu$ is the second connecting homomorphism in the long exact cohomology sequence associated to the exact sequence

\[
0 \to \bigoplus_{i=1}^t \mathcal{O}_{p_i} \to \left( \bigoplus_{i=1}^t \mathcal{O}_{p_i} \right) \oplus \mathcal{O}_p \to \mathcal{O}_p \to 0.
\]

So $\text{Coker}(\mu) \subseteq \text{Ext}^2 \left( \bigoplus_{i=1}^t \mathcal{O}_{p_i}, \left( \bigoplus_{i=1}^{r-1} L_i \right) \oplus H \right)$ and it has to vanish. Consequently $\nu$ would be surjective. Therefore, $\dim \text{Im} \mu = \dim \text{Ext}^1 \left( \bigoplus_{i=1}^t \mathcal{O}_{p_i}, \left( \bigoplus_{i=1}^{r-1} L_i \right) \oplus H \right)$.

Since, again by Mercat’s theorem, a general element of $\text{Ext}^1 \left( \bigoplus_{i=1}^t \mathcal{O}_{p_i}, \left( \bigoplus_{i=1}^{r-1} L_i \right) \oplus H \right)$ is stable, we obtain an extension as $0 \to E \to F \to \mathcal{O}_p \to 0$, with $E$ and $F$ stable.

The genericity of $p_i, \ p$ with $1 \leq i \leq t$ and the genericity of $E$ together with lemma [21, 2.5] implies that $h^0(E) = h^0 \left( \left( \bigoplus_{i=1}^{r-1} L_i \right) \oplus H \right) = h^0(F)$. So $h^0(E) = h^0(F) = 1$, as required.

(ii) Either the Segre invariant $s_1(E) \geq 2$ or $s_1(E) = 1$. If $s_1(E) \geq 2$, then for any $L \subset E$ we have

\[
(\deg(E) + 1) - (2 \deg(L) - 2) \geq 1;
\]

that is,

\[
\deg(F) - 2 \deg(L(p)) = s_1(F) \geq 1,
\]

for any elementary transformation $0 \to E \to F \to \mathcal{O}_p \to 0$. Thus $F$ is stable.

If $s_1(E) = 1$, then by [17] Prop. 4.2] the number of maximal line subbundles of $E$ is finite. Let $0 \to E \to F \to \mathcal{O}_p \to 0$ be an elementary transformation with $\text{Ker}(E_\mu \to F_\mu) \neq L_\mu$ for any maximal subbundle $L$. So if $M \subset E$ is a line subbundle with $M_\mu = \Lambda$, then $\deg(E) - 2 \deg(M) \geq 3$. As above, $(\deg(E) + 1) - (2 \deg(M) - 2) \geq 1$; that is,

\[
\deg(F) - 2 \deg(M(p)) = s_1(F) \geq 1.
\]

Also, if $L$ is a maximal subbundle of $E$, then by the choice of $\Lambda$, also $L$ is a subbundle of $F$, and

\[
\deg(F) - (2 \deg(L)) \geq 2.
\]

Thus $s_1(F) \geq 1$. Hence a general elementary transformation $0 \to E \to F \to \mathcal{O}_p \to 0$ is a stable vector bundle.
(iii) This is immediate by dualizing (ii). \qed

Motivated by Lemma 2.2 for a generic point \( p \in C \) and \( i = 0, 1 \), we set

\[
(2.6) \quad B^1_{r,d+1,p} := \{ F \in B^1_{r,d+1} \mid F \text{ is a general extension } 0 \to E \to F \to \mathcal{O}_p \to 0, \ E \in B^1_{r,d} \},
\]

where we have considered \( B^1_{r,d} \) the same as its preimage via étale map \( U \to U(r,d) \).

Furthermore, we set

\[
(2.7) \quad B^0_{r,\delta+1,p} := \theta^{-1}(B^0_{r,d+1,p}) = B^1_{r,\delta+1,p} := \theta^{-1}(B^1_{r,d+1,p}).
\]

Observe that the superscript numbers 0 and 1 in \( B^0_{r,\delta+1,p} \) and \( B^1_{r,\delta+1,p} \), respectively, don’t refer to the number of linearly independent global sections of the bundles therein.

Taking Lemma 2.2 into account one obtains \( B^0_{r,\delta+1,p} = U(r,\delta + 1) \). As for \( B^1_{r,\delta+1,p} \) we have the following

**Theorem 2.3.** Suppose \( 1 \leq d \leq n(g - 1) - 1 \). Then, the loci \( B^1_{r,\delta+1,p} \) is an irreducible closed subscheme of \( B^1_{r,\delta+1} \) of codimension 1.

**Proof.** Since \( B^1_{r,\delta+1,p} = \theta^{-1}(B^1_{r,d+1,p}) \) by our convention in 2.6 and 2.7. Lemma 2.2 implies that \( B^1_{r,\delta+1,p} \) is a non-empty subset of \( B^1_{r,\delta+1} \), so \( B^1_{r,\delta+1,p} \) would be non-empty as well. Restrict the Poincare bundle \( E_\delta \) to \( B^1_{r,\delta} \times C \) and denote its restriction by the same letter. Set

\[
(2.8) \quad V_\delta := R^1(\pi_1)_* \left( \mathcal{E}xt^1((\pi_2)_*\mathcal{O}_p, E_\delta) \right),
\]

where \( \mathcal{E}xt^i(F,G) \) denotes the \( i \)-th “ext” sheaves of \( F \) and \( G \). The sheaf \( V_\delta \) is a vector bundle on \( B^1_{r,\delta} \) of rank \( r \), by [2] pp. 166-167]; and the projective bundle

\[
(2.9) \quad \alpha_\delta : \mathbb{P}(V_\delta) \to B^1_{r,\delta}
\]

is the family of \( \text{Ext}^1(\mathcal{O}_p, E)'s \) as \( E \) varies in an open subset of \( B^1_{r,\delta} \). Then, one has

\[
B^1_{r,\delta+1,p} = \text{Im}(\nu_\delta),
\]

where the morphism \( \nu_\delta : \mathbb{P}(V_\delta) \to B^1_{r,\delta+1} \) assigns the stable bundle \( F \) to each extension \( 0 \to E \to F \to \mathcal{O}_p \to 0 \). Now, in order to prove that \( B^1_{r,\delta+1,p} \) is of codimension 1 in \( B^1_{r,\delta+1} \), observe that the fibers of both morphisms \( \alpha_\delta \) and \( \nu_\delta \) are \( r \)-dimensional. Indeed, for an extension \( e : 0 \to E \to F \to \mathcal{O}_p \to 0 \) and for \( E \in B^1_{r,\delta} \), we have

\[
\dim \nu_\delta^{-1}(e) = \dim \text{Ext}^1(\mathcal{O}_p, F^*) = r, \quad \dim \alpha_\delta^{-1}(E) = \dim \text{Ext}^1(\mathcal{O}_p, E) = r.
\]

Therefore

\[
\dim B^1_{r,\delta+1,p} = \dim B^1_{r,\delta} = \dim B^1_{r,d} = \dim B^1_{r,d+1} - 1 = \dim B^1_{r,\delta+1} - 1,
\]

by [23, Theorem II.3.1], as required.

As for irreducibility of \( B^1_{r,\delta+1,p} \), it suffices to prove the irreducibility of \( B^1_{r,d+p} \). If \( U \subset B^1_{r,d} \) is the set of \( E \in B^1_{r,d} \) in which their general dual elementary transformations are stable, then \( U \) is irreducible, as well. Since the fibers of \( \alpha_\delta \) are irreducible, so \( B^1_{r,\delta+1,p} = \nu_\delta(\alpha_\delta^{-1}(U)) \) would be irreducible, as desired. \qed

**Remark 2.4.** The content of Lemma 2.2 may already exist in the literature, but we were unable to locate a suitable reference. A similar situation is studied in [5] Lemma 1, Lemma 2].
2.3. Excess dimension inequality. In this subsection, we apply the results of subsection 2.1 to Brill-Noether loci of stable rank two vector bundles.

For a general $p \in C$, consider $p$ as an effective divisor of degree 1 on $C$ and set $L := O(\pi^* p)$. Then $L := (\pi_2)_*(L)$ is a line bundle on the sub-locus $U := B^0_{2,\delta+1,p}$. As parts (ii) and (iii) of Lemma 2.5 show, the dense sub-locus $U$ has non-empty intersection with any irreducible component $X \subset B^k_{2,\delta+1}$.

Let

$$0 \to [(\nu_3)_*(\alpha_3)^*(E_3)] \overset{\alpha}{\to} F \to L \to 0,$$

be a general extension of $L$ by $(\nu_3)_*(\alpha_3)^*(E_3)$ and observe that $F$ is not necessarily a Poincaré bundle neither on $U(2, \delta + 1)$ nor on $U$. But it can be used to define a dense sub-locus of B-N loci on $U$, on which we can compare the dimensions of $B^k_{2,\delta}$ and $B^k_{2,\delta+1}$.

Now we consider a diagram of vector bundles on $U$ as

$$(2.11) \quad 0 \longrightarrow [(\nu_3)_*(\alpha_3)^*(E_3)]|_U \longrightarrow (F)|_U \longrightarrow L \longrightarrow 0,$$

where $\gamma_{\delta+1} := (\gamma_{\delta+1}|_U) \circ \alpha$.

Lemma 2.5. For $k$ in the allowable range, we have

$$(2.12) \quad \dim D_{\delta-2(g-1)-k}(\gamma_{\delta+1}) = \dim B^k_{2,d+1}.$$

Proof. At a point $e : 0 \to E \longrightarrow F \to O_p \to 0 \in U$ the stalks of $[(\nu_3)_*(\alpha_3)^*(E_3)]|_U$, $(F)|_U$ and $[F \otimes (\pi_2)_*\pi_1^*(O_D(D))]|_U$ are $H^0(E)$, $H^0(F)$ and $H^0(F_D)$, respectively. Also $\ker(\gamma_{\delta+1})_e = H^0(E(-D))$. So $\dim H^{-1}(D_{\delta-2(g-1)-k}(\gamma_{\delta+1})) = \dim B^k_{2,d}$. This gives the first equality and the proof for another equality is the same.

Now we apply Theorem 2.1 to this situation, by which we obtain:

Theorem 2.6. If $B^k_{2,d}$ is nonempty, then

(a) $\dim B^k_{2,d} \geq \dim B^k_{2,d+1} - k$,

(b) $\dim B^k_{2,d} \geq \dim B^{k-1}_{2,d} - (2g - 1) - d + k$,

(c) $\dim B^k_{2,d+1} \geq \dim B^{k-1}_{2,d} - (2g - 1) - d + 2k$.

In particular, if $B^k_{2,d}$ is non-empty, then the scheme $B^k_{2,d+1}$ is of expected dimension if and only if $B^k_{2,d}$ is of expected dimension.

Proof. (a) Taking the inequality 2.6 of theorem 2.1 into account, it suffices to prove that no irreducible component of $B^k_{2,d}$ is contained in $B^{k+1}_{2,d}$, which is immediate by [10] Remark 2.3] or by [10] Prop. 1.6.

Part (b) is a direct consequence of part (a) together with the Riemann-Roch theorem and part (c) is a combination of (a) and (b).

Example 1: There are cases in which the inequalities in theorem 2.6 get to be sharp. On general curves of genus $g \geq 1$ and for $d$ with $3 \leq d \leq 2(g - 1)$, M. Teixidor i Bigas computes the dimension of $B^2_{2,d}$ to be $2d - 3$ and $B^3_{2,d}$ to be either empty or of dimension $3d - 2g - 6$. See [24] and [25] Theorem 2]. So, the equality holds for such $B^2_{2,d}$, $B^3_{2,d+1}$, $B^3_{2,d}$ and $B^3_{2,d+1}$, in the allowable range.
Also inequality can hold in theorem 2.4 strictly. Let \( C \) be a general curve of genus 6. Then, the scheme \( B_{2,10}^4 \) is non-empty, because the loci

\[
B_{2,K}^4 := \{ E \in B_{2,10}^4 \mid \text{det}(E) = K \},
\]

is non-empty. Furthermore, \( B_{2,10}^4 \) would have superabundant components of dimension at least 6, provided that \( B_{2,K}^4 \) is strictly contained in \( B_{2,10}^4 \). Assuming this to hold, we would have \( \dim B_{2,10}^4 \geq 6 \). Once again, using \[24\] and \[25\] Theorem 2, we obtain \( \dim B_{2,9}^3 = 9 \). So \[24\)(c) holds strictly. Furthermore \( \dim B_{2,11}^4 = \dim B_{2,9}^3 = 9 \), by residuation. So \[24\](b) holds strictly.

It remains only to prove that not any vector bundle \( E \in B_{2,10}^4 \) belongs to \( B_{2,K}^4 \), to which it suffices to represent a bundle \( E \in B_{2,10}^4 \) with \( \text{det}(E) \neq K \). In order to see this, take line bundles \( L_1, L_2 \in W_2^1 \setminus W_4^2 \) with \( L_1 + L_2 \neq K \) and consider the extensions as

\[
0 \rightarrow L_1 \rightarrow E \rightarrow L_2 \rightarrow 0.
\]

Such an extension, belonging to \( B_{2,10}^4 \setminus B_{2,K}^4 \), exists according to [17] Proposition 3.1 and [18] Lemma 2.18.

**Example 2:** This example shows that Lemma 2.2(ii) fails if the genericity of the dual elementary transformation was removed. Suppose \( E \) is a rank two bundle with \( \text{deg}(E) = 1 \) and has a line subbundle of degree 0. Take a generic point \( p \in C \) and let \( F \) be the dual elementary transformation of \( E \) by \( p \) satisfying \( \text{Ker}(E_p \rightarrow F_p) = L_{|p}. \) Then for \( p_1, \cdots, p_t \in C \) with \( p_i \neq p \) we have a diagram as

\[
0 \rightarrow E(p_1 + \cdots + p_t) \rightarrow F(p_1 + \cdots + p_t) \rightarrow \mathcal{O}_p \rightarrow 0,
\]

\[
0 \rightarrow L(p_1 + \cdots + p_t) \rightarrow L(p_1 + \cdots + p_t + p) \rightarrow \mathcal{O}_p \rightarrow 0.
\]

Then, \( \deg L(p_1 + \cdots + p_t + p) \geq \mu(F(p_1 + \cdots + p_t)) \), so \( F \) is strictly semi-stable.

**2.4. A byproduct:** As a consequence, we relate the smoothness of rank two Brill-Noether schemes with different degrees. In order to do this, we first make a comparison among the dimension of the tangent spaces of these schemes. See also [11 Th. 3.1], [14] and [15].

**Lemma 2.7.** Let \( p \in C \) be a general point, \( E \in B_{r,d}^k \setminus B_{r,d}^{k+1} \) and \( F \) be a general stable dual elementary transformation of \( E \) by \( p \). Then, for \( 1 \leq d \leq r(g-1) \) and \( 1 \leq k \leq d + 1 \), we have

\[
\dim T_{E}(B_{r,d}^k) \geq \dim T_{F}(B_{r,d+1}^k) - k.
\]

**Proof.** Recall that

\[
T_{E}(B_{r,d}^k) = [\text{Im } \{ \mu_E : H^0(E) \otimes H^0(K \otimes E^*) \rightarrow H^0(K \otimes E \otimes E^*) \}]^\perp,
\]

where for a sub-vector space \( W \subseteq V \), by \( W^\perp \) we mean the annihilator vector space of \( W \) inside the dual of \( V \) and \( \mu_E \) denotes the Petri map of \( E \).

Notice that, under the hypothesis, the bundle \( E \) is special. So lemma [21] 2.5 can be applied to obtain \( H^0(E) = H^0(F) \), therefore \( F \in B_{r,d+1}^k \setminus B_{r,d+1}^{k+1} \). There exists now a diagram as

\[
\begin{array}{ccc}
H^0(E) & \otimes H^0(K \otimes E^*) & \xrightarrow{\mu_E} H^0(K \otimes E \otimes E^*) \\
\uparrow & & \downarrow \alpha \\
H^0(F) & \otimes H^0(K \otimes F^*) & \xrightarrow{\mu_F} H^0(K \otimes F \otimes F^*) \\
& \xrightarrow{\beta} H^0(K \otimes F \otimes E^*)
\end{array}
\]
where the morphisms $i, \alpha$ and $\beta$ are all injective. Therefore, we will have $\dim \ker(\mu_E) \geq \dim \ker(\mu_F)$. The Riemann-Roch theorem implies that $h^0(K \otimes F^*) = h^0(K \otimes E^*) - 1$. Therefore,

$$kh^0(K \otimes E^*) - \dim \ker(\mu_E) \leq kh^0(K \otimes F^*) - \dim \ker(\mu_F) + k,$$

implying $\dim \Im (\mu_E) \leq \dim \Im (\mu_F) + k$, as required.

**Corollary 2.8.** Let $E \in B_{2,d}^k$ and $F$ be a general dual elementary transformation of $E$ by a general point $p \in C$. Assume moreover that $B_{2,d}^k$ is of expected dimension. Then, for $1 \leq d \leq 2(g - 1)$ and $1 \leq k \leq \frac{d}{2} + 1$ the scheme $B_{2,d}^k$ is smooth at $E$ if and only if $B_{2,d+1}^k$ is smooth at $F$.

**Proof.** Since $B_{2,d}^k$ is of expected dimension so is the scheme $B_{2,d+1}^k$, and vice versa; by theorem 2.7. The smoothness of $B_{2,d}^k$ at $E \in B_{2,d}^k$ implies that $\dim B_{2,d}^k = \dim T_E B_{2,d}^k$. This, by Lemma 2.7, together with the fact that $B_{2,d+1}^k$ is of expected dimension, gives rise to the assertion.

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Department of Mathematics, Faculty of Basic Sciences, Azarbaijan Shahid Madani University, Tabriz, I. R. Iran., P. O. Box: 53751-71379.
Email address: bajravani@azaruniv.ac.ir

School of Mathematics, Institute for Research in Fundamental Sciences (IPM), P.O. Box: 19395-5746. Tehran, Iran.