On exponential Yang-Mills fields and $p$-Yang-Mills fields

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Abstract. We introduce normalized exponential Yang-Mills energy functional $\mathcal{YM}_e^0$, stress-energy tensor $S_{\mathcal{YM}_e^0}$ associated with the normalized exponential Yang-Mills energy functional $\mathcal{YM}_e^0$, $e$-conservation law. We also introduce the notion of the $e$-degree $d_e$ which connects two separate parts in the associated normalize exponential stress-energy tensor $S_{\mathcal{YM}_e^0}$ (cf. (3.10) and (4.15)), derive monotonicity formula for exponential Yang-Mills fields, and prove a vanishing theorem for exponential Yang-Mills fields. These monotonicity formula and vanishing theorem for exponential Yang-Mills fields augment and extend monotonicity formula and vanishing theorem for $F$-Yang-Mills fields in [DW] and [W11, 9.2]. We also discuss an average principle (cf. Proposition 8.1), isoperimetric and Sobolev inequalities, convexity and Jensen’s inequality, $p$-Yang-Mills fields, an extrinsic average variational method in the calculus of variation and $\Phi_{(3)}$-harmonic maps, from varied, coupled, generalized viewpoints and perspectives (cf. Theorems 6.1, 7.1, 9.1, 9.2, 10.1, 10.2, 11.13, 11.14, 11.15).

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1. Introduction

The Yang–Mills functional, brought to mathematics by physics is broadly analogous to functionals such as the length functional in geodesic theory, the area

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functional in minimal surface, or minimal submanifold theory, the energy (resp. $p$-energy) functional in harmonic (resp. $p$-harmonic) map theory, or the mass functional in stationary or minimal current, geometric measure theory (cf., e.g., [F, L, HoW]). A critical point of the Yang-Mills functional with respect to any compactly supported variation in the space of smooth connections $\nabla$ on the adjoint bundle is called a Yang-Mills connection. Its associated curvature field $\nabla\mathcal{F}$ is known as Yang-Mills field and is “harmonic”, i.e., a harmonic 2-form with values in the vector bundle. The Euler-Lagrange equation for the Yang-Mills functional is Yang-Mills equation. Whereas Hodge theory of harmonic forms is motivated in part by Maxwell’s equations of unifying magnetism with electricity in a physics world, and harmonic forms are privileged representatives in a de Rham cohomology class picked out by the Hodge Laplacian, harmonic maps can be viewed as a nonlinear generalization of harmonic 1-form and Yang-Mills field can be viewed as a nonlinear generalization of harmonic 2-form. On the other hand, Yang-Mills equation which can be viewed as a non-abelian generalization of Maxwell’s equations, has had wide-ranging consequences, and influenced developments in other fields such as low-dimensional topology, particularly the topology of smooth 4-manifolds. For example, M. Freedman and R. Kirby first observed the startling fact that there exists an exotic $\mathbb{R}^4$, i.e., a manifold homeomorphic to, but not diffeomorphic to, $\mathbb{R}^4$ (cf. [K, p. 95], [D, FK, Go]). This is in stunning contrast to a phenomenal theorem of J. Milnor in compact high-dimensional topology which shows that there exist exotic seven-spheres $S^7$, i.e., manifolds that are homeomorphic to, but not diffeomorphic to, the standard Euclidean $S^7$ (cf. [M]).

In [DW], we unify the concept of minimal hypersurfaces in Euclidean space $\mathbb{R}^{n+1}$, maximal spacelike hypersurfaces in Minkowski space $\mathbb{R}^{n,1}$, harmonic maps, $p$-harmonic maps, $F$-harmonic maps, Yang-Mills fields, and introduce $F$-Yang-Mills fields, $F$-degree, and generalized Yang-Mills-Born-Infeld fields (with the plus sign or with the minus sign) on manifolds, where

$$F : [0, \infty) \to [0, \infty)$$

is a strictly increasing $C^2$ function with $F(0) = 0$. (1.1)

When

$$F(t) = t, p^{-1}(2t)^{\frac{2}{p}}, \sqrt{1+2t} - 1, \text{and } 1 - \sqrt{1-2t},$$

$F$-Yang-Mills field becomes an ordinary Yang-Mills field, $p$-Yang-Mills field, a generalized Yang-Mills-Born-Infeld field with the plus sign, and a generalized Yang-Mills-Born-Infeld field with the minus sign on a manifold respectively (cf. [BI, BL, BLS, CCW, D, LA, LWW, LY, SisyTa, W12, Ya]). When

$$F(t) = t, e^t, p^{-1}(2t)^{\frac{2}{p}}, \sqrt{1+2t} - 1, \text{and } 1 - \sqrt{1-2t},$$

$F$-harmonic map or the graph of $F$-harmonic map becomes an ordinary harmonic map, exponentially harmonic map, $p$-harmonic map, minimal hypersurface in Euclidean space $\mathbb{R}^{n+1}$, and maximal spacelike hypersurface in Minkowski space $\mathbb{R}^{n,1}$ respectively (cf. [ES, WY, EL, Ar, WWZ]).

We use ideas from physics - stress-energy tensors and conservation laws to simplify and unify various properties in $F$-Yang-Mills fields, $F$-harmonic maps, and more generally differential $k$-forms, $k \geq 0$ with values in vector bundles.

In this paper, we introduce normalized exponential Yang-Mills energy functional $\mathcal{YM}_e^0$, (resp. exponential Yang-Mills energy functional $\mathcal{YM}_e$), stress-energy tensor $S_{e,\mathcal{YM}}$ associated with the normalized exponential Yang-Mills energy functional $\mathcal{YM}_e^0$, (resp. stress-energy tensor $S_{e,\mathcal{YM}}$ associated with the exponential
Yang-Mills energy functional \( \mathcal{Y} \mathcal{M}^0 \). (A critical point of \( \mathcal{Y} \mathcal{M}^0 \), i.e. a normalized exponential Yang-Mills connection, and its associated normalized exponential Yang-Mills field are just the same as the Yang-Mills connection and its associated exponential Yang-Mills field). We also introduce the notion of the \( e \)-degree \( d_e \) which connects two separate parts in the associated normalized exponential stress-energy tensors \( S_{e,0} \) (cf. (4.15)).

These stress-energy tensors arise from calculating the rate of change of various functionals when the metric of the domain or base manifold is changed and are naturally linked to various conservation laws. For example, we prove that every normalized exponential Yang-Mills field or every exponential Yang-Mills field \( R^\mathcal{Y} \) satisfies an \( e \)-conservation law (cf. Theorem 3.11). Every normalized exponential Yang-Mills connection or exponential Yang-Mills connection satisfies the exponential Yang-Mills equation (cf. Corollary 3.7). We then prove monotonicity formulae, via the coarea formula and comparison theorems in Riemannian geometry (cf. [GW, DW, HLRW, W11]). Whereas a “microscopic” approach to some of these monotonicity formulae leads to celebrated blow-up techniques due to E. de-Giorgi ([Gi]) and W.L. Fleming ([F1]), and regularity theory in geometric measure theory (cf. [A, Al, FF, HL, Lu, PS, SU], for example, the regularity results of Allard ([A]) depend on the monotonicity formulae for varifolds. Monotonicity properties are also dealt with by Price and Simon ([PS]), Price ([P]) for Yang-Mills fields, and by Hardt-Lin ([HL]) and Luckhaus ([Lu]) for \( p \)-harmonic maps), a “macroscopic” version of these monotonicity formulae enable us to derive some vanishing theorems under suitable growth conditions on Cartan-Hadamard manifolds or manifolds which possess a pole with appropriate curvature assumptions. In particular, we have Theorem 5.1 - the monotonicity formula for exponential Yang-Mills fields and Theorem 6.1 - the vanishing theorem for exponential Yang-Mills fields.

These monotonicity formula and vanishing theorem for exponential Yang-Mills fields augment and extend vanishing theorems for \( F \)-Yang-Mills fields in [DW] and [W11]. We note that even when

\[
F(t) = e^t \quad \text{or} \quad F(t) = e^t - 1 \quad \text{for} \quad t = \frac{|R^\mathcal{Y}|^2}{2},
\]

\( F \)-Yang-Mills field becomes exponential Yang-Mills field, the following vanishing theorem for \( F \)-Yang-Mills fields are not applicable to exponential Yang-Mills fields. This is due to the fact that for \( F(t) = e^t \), the degree of \( F \),

\[
d_F := \sup_{t \geq 0} \frac{tF'(t)}{F(t)} = \infty,
\]

and the \( F \)-Yang-Mills energy functional growth condition (1.3) is not satisfied for \( \lambda = -\infty \) in (1.4). To overcome this difficulty in getting estimates, we introduce the notion of \( e \)-degree \( d_e \), for a given curvature tensor \( R^\mathcal{Y} \) (cf. (4.15)).
Theorem A (Vanishing theorem for F-Yang-Mills fields ([DW, W11])). Suppose that the radial curvature $K(r)$ of $M$ satisfies one of the seven conditions

(i) \[ -\alpha^2 \leq K(r) \leq -\beta^2 \] with $\alpha > 0$, $\beta > 0$ and $(n - 1)\beta - 4\alpha d_F \geq 0$;

(ii) \[ K(r) = 0 \] with $n - 4d_F > 0$;

(iii) \[ -\frac{A}{(1 + r^2)^{1+\epsilon}} \leq K(r) \leq \frac{B}{(1 + r^2)^{1+\epsilon}} \] with $\epsilon > 0$, $A \geq 0$, $0 < B < 2\epsilon$, and
\[ n - (n - 1)\frac{B}{2\epsilon} - 4\epsilon^2 d_F > 0; \]

(iv) \[ -\frac{A}{r^2} \leq K(r) \leq -\frac{A_1}{r^2} \] with $0 \leq A_1 \leq A$, and
\[ 1 + (n - 1)\frac{1 + \sqrt{1 + 4A_1}}{2} - 2(1 + \sqrt{1 + 4A_1})d_F > 0; \]

(v) \[ -\frac{A(A - 1)}{r^2} \leq K(r) \leq -\frac{A_1(A_1 - 1)}{r^2} \] and $A \geq A_1 \geq 1$, and
\[ 1 + (n - 1)A_1 - 4Ad_F > 0; \]

(vi) \[ \frac{B_1(1 - B_1)}{r^2} \leq K(r) \leq \frac{B(1 - B)}{r^2}, \] with $0 \leq B$, $B_1 \leq 1$, and
\[ 1 + (n - 1)(|B - \frac{1}{2}| + \frac{1}{2}) - 2(1 + \sqrt{1 + 4B_1(1 - B_1)})d_F > 0; \]

(vii) \[ \frac{B_1}{r^2} \leq K(r) \leq \frac{B}{r^2} \] with $0 \leq B_1 \leq B \leq \frac{1}{4}$, and
\[ 1 + (n - 1)\frac{1 + \sqrt{1 - 4B}}{2} - (1 + \sqrt{1 + 4B_1})\|R^\nabla\|^2_\infty > 0. \] (1.2)

If $R^\nabla \in \mathbb{A}^2(Ad(P))$ is an F-Yang-Mills field and satisfies
\[ \int_{B_\rho(x_0)} F\left(\frac{\|R^\nabla\|^2}{2}\right) \, dv = o(\rho^\lambda) \quad \text{as } \rho \to \infty, \] (1.3)

where $\lambda$ is given by

\[
\begin{aligned}
\lambda &\leq \begin{cases} 
 n - 4\frac{A}{d_F} & \text{if } K(r) \text{ obeys (i)} \\
 n - 4d_F & \text{if } K(r) \text{ obeys (ii)} \\
 n - (n - 1)\frac{B}{2\epsilon} - 4\epsilon^2 d_F & \text{if } K(r) \text{ obeys (iii)} \\
 1 + (n - 1)\frac{1 + \sqrt{1 + 4A_1}}{2} - 2(1 + \sqrt{1 + 4A_1})d_F & \text{if } K(r) \text{ obeys (iv)} \\
 1 + (n - 1)A_1 - 4Ad_F & \text{if } K(r) \text{ obeys (v)} \\
 1 + (n - 1)\frac{1 + \sqrt{1 - 4B}}{2} - (1 + \sqrt{1 + 4B_1})d_F & \text{if } K(r) \text{ obeys (vi)} \\
 1 + (n - 1)\frac{1 + \sqrt{1 - 4B}}{2} - (1 + \sqrt{1 + 4B_1})d_F & \text{if } K(r) \text{ obeys (vii)}. 
\end{cases} 
\end{aligned}
\] (1.4)

Then $R^\nabla \equiv 0$ on $M$. In particular, every F-Yang-Mills field $R^\nabla$ with finite F-Yang-Mills energy functional vanishes on $M$.

We also discuss An Average Principle (cf. Proposition 8.1) and Jensen’s inequality from varied, generalized viewpoints and perspectives of exponential Yang-Mills fields, $p$-Yang-Mills fields, and Yang-Mills fields. (Theorems 7.1, 9.1, 9.2, 10.1, 10.2).
In the context of harmonic maps, the stress-energy tensor was introduced and studied in detail by Baird and Eells ([BE]). Following Baird-Eells ([BE], Sealey [Se2] introduced the stress-energy tensor for vector bundle valued $p$-forms and established some vanishing theorems for $L^2$ harmonic $p$-forms (cf. [DLW, Se1, Xi1]). In a more general frame, Dong and Wei use a unified method to study the stress-energy tensors and yields monotonicity inequalities, and vanishing theorems for vector bundle valued $p$-forms ([DW]). The idea and methods can be extended and unified in $\sigma_2$-version of harmonic maps - $\Phi$-Harmonic maps (cf. [HW]). These are the second elementary symmetric function of a pull-back tensor, whereas harmonic maps are the first elementary symmetric function of a pull-back tensor.

More recently, Feng-Han-Li-Wei use stress-energy tensors to unify properties in $\Phi_{S,p}$-harmonic maps (cf. [FHLW]), Feng-Han-Wei extend and unify results in $\Phi_{S,p}$-harmonic maps (cf. [FHW]), and Feng-Han-Jiang-Wei further extend and unify results in $\Phi_{S,p}$-harmonic maps (cf. [FHW]). Whereas we can view harmonic maps as $\Phi_{(1)}$-harmonic maps (involving $\sigma_1$) and $\Phi$-harmonic maps as $\Phi_{(2)}$-harmonic maps (involving $\sigma_2$), $\Phi_{(3)}$-harmonic maps involve $\sigma_3$, the third elementary symmetric function of the pullback tensor. In fact, an extrinsic average variational method in the calculus of variation can be carried over to more general settings by which we introduce a notion of $\Phi_{(3)}$-harmonic map and find a large class of manifolds, $\Phi_{(3)}$-superstrongly unstable ($\Phi_{(3)}$-SU) manifolds, introduce notions of a stable $\Phi_{(3)}$-harmonic map, and $\Phi_{(3)}$-strongly unstable ($\Phi_{(3)}$-SSU) manifolds (cf. Theorems 11.8, 11.9, 11.10, and 11.11).

By an extrinsic average variational method in the calculus variations proposed in [W3], we find multiple large classes of manifolds with geometric and topological properties in the setting of varied, coupled, generalized type of harmonic maps, and summarize some of the results in Table 1. For some details, related ideas, techniques, we refer to [CW3], [W1]-[W12], [WLW].

Table 1. An Extrtrinsic Average Variational Method

| Mappings | Functionals | New manifolds found | Geometry | Topology |
|----------|-------------|---------------------|----------|----------|
| $\Phi_{(1)}$-harmonic map | $E_{\Phi_{(1)}}$ | $\Phi_{(1)}$-SSU manifolds or $\Phi_{(1)}$-SU | $\pi_1 = \pi_2 = 0$ |
| $p$-harmonic map | $E_p$ | $p$-SSU manifolds or $p$-SU | $\pi_1 = \pi_2 = 0$ |
| $\Phi_{(2)}$-harmonic map | $E_{\Phi_{(2)}}$ | $\Phi_{(2)}$-SSU manifolds or $\Phi_{(2)}$-SU | $\pi_1 = \pi_2 = 0$ |
| $\Phi_{(3)}$-harmonic map | $E_{\Phi_{(3)}}$ | $\Phi_{(3)}$-SSU manifolds or $\Phi_{(3)}$-SU | $\pi_1 = \pi_2 = 0$ |

2. Fundamentals in vector bundles and principal $G$-bundle

This section is devoted to a brief discussion of the fundamental notions in vector bundles and principal $G$-bundle.

**Definition 2.1.** A (differentiable) vector bundle of rank $n$ consists of a total space $E$, a base $M$, and a projection $\pi : E \to M$, where $E$ and $M$ are differentiable manifolds, $\pi$ is differentiable, each fiber $E_x := \pi^{-1}(x)$ for $x \in M$, carries the structure of an $n$-dimensional (real) vector space, with the following local triviality:
For each \( x \in M \), there exist a neighborhood \( U \) and a diffeomorphism
\[
\varphi : \pi^{-1}(U) \to U \times \mathbb{R}^n
\]
such that for every \( y \in U \)
\[
\varphi_y := \varphi_{|E_y} : E_y \to \{y\} \times \mathbb{R}^n
\]
is a vector space isomorphism. Such a pair \((\varphi, U)\) is called a bundle chart.

Note that local trivializations \( \varphi_\alpha, \varphi_\beta \) with \( U_\alpha \cap U_\beta \neq \emptyset \) determines transition maps
\[
\varphi_{\beta \alpha} : U_\alpha \cap U_\beta \to \text{Gl}(n, \mathbb{R})
\]
by
\[
\varphi_\beta \circ \varphi_\alpha^{-1}(x, v) = (x, \varphi_{\beta \alpha}(x)v) \quad \text{for} \quad x \in M, v \in \mathbb{R}^n,
\]
where \( \text{Gl}(n, \mathbb{R}) \) is the general linear group of bijective linear self maps of \( \mathbb{R}^n \).

As direct consequences, the transition maps satisfy:
\[
\varphi_{\alpha \alpha}(x) = \text{id}_{\mathbb{R}^n} \quad \text{for} \quad x \in U_\alpha;
\]
\[
\varphi_{\alpha \beta}(x)\varphi_{\beta \alpha}(x) = \text{id}_{\mathbb{R}^n} \quad \text{for} \quad x \in U_\alpha \cap U_\beta;
\]
\[
\varphi_{\alpha \gamma}(x)\varphi_{\gamma \beta}(x)\varphi_{\beta \alpha}(x) = \text{id}_{\mathbb{R}^n} \quad \text{for} \quad x \in U_\alpha \cap U_\beta \cap U_\gamma.
\]
(cf. [3]) A vector bundle can be reconstructed from its transition maps
\[
E = \coprod_\alpha U_\alpha \times \mathbb{R}^n / \sim,
\]
where \( \coprod \) denotes disjoint union, and the equivalence relation \( \sim \) is defined by
\[
(x, v) \sim (y, w) :\iff x = y \text{ and } w = \varphi_{\beta \alpha}(x)v \quad (x \in U_\alpha, y \in U_\beta, v, w \in \mathbb{R}^n).
\]  
(2.1)

**Definition 2.2.** Let \( G \) be a subgroup of \( \text{Gl}(n, \mathbb{R}) \), for example the orthogonal group \( O(n) \) or special orthogonal group \( \text{SO}(n) \). By a vector bundle has the structure group \( G \), we mean there exists an atlas of bundle charts for which all transition maps have their values in \( G \).

**Definition 2.3.** Let \( G \) be a Lie group. A principal \( G \)-bundle consists of a base \( M \), the total space \( P \) of the bundle, and a differentiable projection \( \pi : P \to M \), where \( P \) and \( M \) are differentiable manifolds, with an action of \( G \) on \( P \) satisfying
(i) \( G \) acts freely on \( P \) from the right: \((q, p) \in P \times G \) is mapped to \( qp \in P \), and \( qp \neq q \) for \( q \neq e \).

The \( G \) action then defines an equivalence relation on \( P : p \sim q :\iff \exists g \in G \) such that \( p = gq \).

(ii) \( M \) is the quotient of \( P \) by this equivalence relation, and \( \pi : P \to M \) maps \( q \in M \) to its equivalence class. By (i), each fiber \( \pi^{-1}(x) \) can then be identified with \( G \).

(iii) \( P \) is locally trivial in the following sense:
For each \( x \in M \), there exists a neighborhood \( U \) of \( x \) and a diffeomorphism
\[
\varphi : \pi^{-1}(U) \to U \times G
\]
of the form \( \varphi(p) = (\pi(p), \psi(g)) \) which is \( G \)-equivariant, i.e. \( \varphi(pg) = (\pi(p), \psi(p)g) \) for all \( g \in G \).

**Example 2.4.** We have the following results.
(i) The projection \( S^n \to P^n(\mathbb{R}) \) of the \( n \)-sphere to the real projective space is a principal bundle with group \( G = O(1) = \mathbb{Z}_2 \).

(ii) The Hopf map \( S^{2n+1} \to P^n(\mathbb{C}) \) of the \( 2n+1 \)-sphere to the complex projective space is a principal bundle with group \( G = U(1) = S^1 \).

(iii) The Hopf map \( S^{4n+1} \to P^n(\mathbb{Q}) \) of the \( 4n+1 \)-sphere to the quaternionic projective space is a principal bundle with group \( G = Sp(1) = S^3 \).

(iv) Hopf fibrations: \( S^1 \to S^1, S^3 \to S^2, S^7 \to S^4, \) and \( S^{15} \to S^8 \).

For \( k = 1, 2, 4, 8 \), the Hopf construction is defined by
\[
(z, w) \mapsto u(z, w) = (|z|^2 - |w|^2, 2z \cdot w) : \mathbb{R}^k \times \mathbb{R}^k \to \mathbb{R}^{k+1}.
\]

In fact, Hopf fibrations are \( p \)-harmonic maps and \( p \)-harmonic morphisms for every \( p > 1 \) (c.f., e.g., [W8, CW2]).

We recall a \( C^2 \) map \( u : M \to N \) is said to be a \( p \)-harmonic morphism if for any \( p \)-harmonic function \( f \) defined on an open set \( V \) of \( N \), the composition \( f \circ u \) is \( p \)-harmonic on \( u^{-1}(V) \).

\textbf{Example 2.5.} If \( E \to M \) is a vector bundle with fiber \( V \), the bundle of bases of \( E, B(E) \to M \) is a principle bundle with group \( \text{Gl}(V) \).

\textbf{2.1. Reversibility of principal and vector bundles.} (\( \Rightarrow \)) Given a principal \( G \)-bundle \( P \to M \) and a vector space \( V \) on which \( G \) acts from the left, we construct the associated vector bundle \( E \to M \) with fiber \( V \) as follows:

We have a free action of \( G \) on \( P \times V \) from the right:
\[
P \times V \times G \to P \times V \quad \quad (p, v) \cdot g = (p \cdot g, g^{-1}v).
\]

If we divide out this \( G \)-action, i.e. identify \( (p, v) \) and \( (p, v) \cdot g \), the fibers of \( (P \times V)/G \to P/G \) becomes vector spaces isomorphic to \( V \), and
\[
E := P \times_G V := (P \times V)/G \to M
\]
is a vector bundle with fiber \( G \times_G V := (G \times V)/G = V \) and structure group \( G \).

The transition functions for \( P \) also give transition functions for \( E \) via the left action of \( G \) on \( V \).

(\( \Leftarrow \)) Conversely, given a vector bundle \( E \) with structure group \( G \), we construct a principal \( G \)-bundle as
\[
\prod_\alpha U_\alpha \times G \sim
\]
with
\[
(x_\alpha, g_\alpha) \sim (x_\beta, g_\beta) \quad \iff \quad x_\alpha = x_\beta \in U_\alpha \cap U_\beta \quad \text{and} \quad g_\beta = \varphi_{\beta \alpha}(x)g_\alpha
\]
where \( \{U_\alpha\} \) is a local trivialization of \( E \) with transition functions \( \varphi_{\beta \alpha} \) as in (2.1).

\textbf{Example 2.6.} We have the following assertions.

(i) The canonical line bundles (real, complex and quaternionic) over the projective spaces \( P^n(\mathbb{R}) \), \( P^n(\mathbb{C}) \) and of the \( P^n(\mathbb{Q}) \) are the \textit{associated bundles} of the principal bundles in Example 2.4 (i) – (iii) via the canonical actions of \( O(1), U(1) \) and \( Sp(1) \) on \( \mathbb{R}, \mathbb{C} \) and \( \mathbb{Q} \) respectively.
(ii) Let $E \to M$ be a bundle with fiber $F$ and structure group $G$ and $f : N \to M$ be a map between manifolds $N$ and $M$. Then the pull-back of $E \to M$ is a bundle $f^{-1}E \to M$ with fiber $F$, structure group $G$, and bundle charts $(\varphi \circ f, f^{-1}(U))$, where $\varphi(U)$ are bundle charts of $E$. The pull-back $f^{-1}E \to M$ is called the pull-back bundle.

3. Normalized exponential Yang-Mills functionals and $\epsilon$-conservation laws

Our basic set-up is the following: We consider a Riemannian manifold $M$, and a principal bundle $P$ with compact structure Lie group $G$ over $M$. Let $Ad(P)$ be the adjoint bundle

$$Ad(P) = P \times_{Ad} G,$$

where $G$ is the Lie algebra of $G$. Every connection $\rho$ on $P$ induces a connection $\nabla$ on $Ad(P)$. A connection $\nabla$ on the vector bundle $Ad(P)$ is a rule that equips us to take derivatives of smooth cross sections of $Ad(P)$. We also have the Riemannian connection $\nabla^M$ on the tangent bundle $TM$, and the induced connection on the tensor product $\Lambda^2 T^*M \otimes Ad(P)$, where $\Lambda^2 T^*M$ is the second exterior power of the cotangent bundle $T^*M$. An $Ad_G$ invariant inner product on $G$ induces a fiber metric on $Ad(P)$ and makes $Ad(P)$ and $\Lambda^2 T^*M \otimes Ad(P)$ into Riemannian vector bundles. Denote by $\Gamma(\Lambda^2 T^*M \otimes Ad(P))$ the (infinite-dimensional) vector space of smooth sections of $\Lambda^2 T^*M \otimes Ad(P)$.

For $k \geq 0$ set

$$A^k(Ad(P)) = \Gamma(\Lambda^k T^*M \otimes Ad(P))$$

be the space of smooth $k$-forms on $M$ with values in the vector bundle $Ad(P)$. Although $\rho$ is not a section of $A^1(Ad(P))$, via its induced connection $\nabla$, the associated curvature tensor $R^\nabla$, given by

$$R^\nabla_{X,Y} = [\nabla_X, \nabla_Y] - \nabla_{[X,Y]},$$

is in $A^2(Ad(P))$. Let $C$ be the space of smooth connections $\nabla$ on $Ad(P)$, and $dv$ be the volume element of $M$. Recall the Yang-Mills functional is the mapping $\mathcal{YM} : C \to \mathbb{R}^+$ given by

$$\mathcal{YM}(\nabla) = \int_M \frac{1}{2} ||R^\nabla||^2 dv,$$

(3.2)

the $p$-Yang-Mills functional, for $p \geq 2$ (resp. the $F$-Yang-Mills functional) is the mapping $\mathcal{YM}_p : C \to \mathbb{R}^+$ given by

$$\mathcal{YM}_p(\nabla) = \int_M \frac{1}{p} ||R^\nabla||^p dv$$

$$\quad (resp. \quad \mathcal{YM}_F(\nabla) = \int_M F \left( \frac{1}{2} ||R^\nabla||^2 \right) dv),$$

(3.3)

where the norm is defined in terms of the Riemannian metric on $M$ and a fixed $Ad_G$-invariant inner product on the Lie algebra $G$ of $G$. That is, at each point $x \in M$, its norm

$$||R^\nabla||_x^2 = \sum_{i<j} ||R^\nabla_{e_i,e_j}||^2_x$$

(3.4)
where \( \{ e_1, \ldots, e_n \} \) is an orthonormal basis of \( T_x(M) \) and the norm of \( R_{e_i,e_j}^\nabla \) is the standard one on \( \text{Hom}(Ad(P),Ad(P)) \)-namely,
\[
(S,U) \equiv \text{trace } (S^T \circ U).
\]

A connection \( \nabla \) on the adjoint bundle \( Ad(P) \) is said to be a Yang-Mills connection (resp. \( p \)-Yang-Mills connection, \( p \geq 2 \), \( F \)-Yang-Mills connection) and its associated curvature tensor \( R^\nabla \) is said to be a Yang-Mills field (resp. \( p \)-Yang-Mills field, \( p \geq 2 \), \( F \)-Yang-Mills field), if \( \nabla \) is a critical point of \( \mathcal{YM} \) (resp. \( \mathcal{YM}_p \), \( \mathcal{YM}_F \)) with respect to any compactly supported variation in the space of smooth connections on \( Ad(P) \). We now introduce

**Definition 3.1.** The normalized exponential Yang-Mills energy functional is the mapping \( \mathcal{YM}_e^0 : C \rightarrow \mathbb{R}^+ \) given by
\[
\mathcal{YM}_e^0(\nabla) = \int_M \left( \exp\left( \frac{1}{2} ||R^\nabla||^2 \right) - 1 \right) dv, \tag{3.5}
\]
the exponential Yang-Mills energy functional is the mapping \( \mathcal{YM}_e : C \rightarrow \mathbb{R}^+ \) given by
\[
\mathcal{YM}_e(\nabla) = \int_M \exp\left( \frac{1}{2} ||R^\nabla||^2 \right) dv, \tag{3.6}
\]
on \( M \), the uniform norm \( ||R^\nabla||_\infty \) is given by
\[
||R^\nabla||_\infty^2 = \sup_{x \in M} ||R^\nabla||_x^2. \tag{3.7}
\]

The normalized exponential Yang-Mills energy functional \( \mathcal{YM}_e^0 \) has the following simple and useful advantage.

**Proposition 3.2.**
\[
\mathcal{YM}_e^0(\nabla) \geq 0 \quad \text{and} \quad \mathcal{YM}_e^0(\nabla) = 0 \iff R^\nabla \equiv 0. \tag{3.8}
\]
This is an analog of \( p \)-Yang-Mills functional, for \( p \geq 2 \),
\[
\mathcal{YM}_p(\nabla) \geq 0 \quad \text{and} \quad \mathcal{YM}_p(\nabla) = 0 \iff R^\nabla \equiv 0. \tag{3.9}
\]

**Definition 3.3.** The stress-energy tensor \( S_{e,\mathcal{YM}_e^0} \) associated with the normalized exponential Yang-Mills energy functional \( \mathcal{YM}_e^0 \) and the stress-energy tensor \( S_{e,\mathcal{YM}_e} \) associated with the exponential Yang-Mills energy functional \( \mathcal{YM}_e \) are defined respectively as follows:
\[
S_{e,\mathcal{YM}_e^0}(X,Y) = \left( \exp\left( \frac{||R^\nabla||^2}{2} \right) - 1 \right) g(X,Y) - \exp\left( \frac{||R^\nabla||^2}{2} \right) \langle i_X R^\nabla, i_Y R^\nabla \rangle, \tag{3.10}
\]
\[
S_{e,\mathcal{YM}}(X,Y) = \exp\left( \frac{||R^\nabla||^2}{2} \right) \left( g(X,Y) - \langle i_X R^\nabla, i_Y R^\nabla \rangle \right) \tag{3.11}
\]
where \( \langle \, , \, \rangle \) is the induced inner product on \( A^1(Ad(P)) \), and \( i_X R^\nabla \) is the interior multiplication by the vector field \( X \) given by
\[
(i_X R^\nabla)(Y_1) = R^\nabla(X,Y_1), \tag{3.12}
\]
for any vector fields \( Y_1 \) on \( M \).
We calculate the rate of change of the normalized exponential Yang-Mills energy functional $\mathcal{YM}^0_{e,g}$ and exponential Yang-Mills energy functional $\mathcal{YM}_{e,g}$ when the metric $g$ on the domain or base manifold is changed. To this end, we consider a compactly supported smooth one-parameter variation of the metric $g$, i.e. a smooth family of metrics $g_s$ such that $g_0 = g$. Set $\delta g = \frac{dg}{ds}|_{s=0}$. Then $\delta g$ is a smooth 2-covariant symmetric tensor field on $M$ with compact support. These give birth to their associated stress-energy tensors.

**Lemma 3.4.** With the same notations as above, we have

\[
\frac{d}{ds}\mathcal{YM}^0_{e,g_s}(\nabla)|_{s=0} = \frac{1}{2} \int_M \langle S_{e,\mathcal{YM}^0}, \delta g \rangle dv_g 
\]

(3.13)

\[
\frac{d}{ds}\mathcal{YM}_{e,g_s}(\nabla)|_{s=0} = \frac{1}{2} \int_M \langle S_{e,\mathcal{YM}}, \delta g \rangle dv_g 
\]

(3.14)

where $S_{e,\mathcal{YM}^0}$ and $S_{e,\mathcal{YM}}$ are as in (3.8) and (3.9) respectively.

**Proof.** From ([Ba]), we obtain

\[
\frac{d}{ds} |R^\nabla|^2 |_{s=0} = \sum_{i,j} \langle i_{e_i}R^\nabla_i, i_{e_j}R^\nabla_j \rangle \delta g(e_i, e_j) 
\]

(3.15)

and

\[
\frac{d}{ds} dv_g |_{s=0} = \frac{1}{2} \langle g, \delta g \rangle dv_g .
\]

(3.16)

Then by the chain rule, (3.15), (3.16), and (3.10), we have

\[
\frac{d}{ds} \mathcal{YM}^0_{e,g_s}(\nabla)|_{s=0} = \int_M \frac{d}{ds} \left( \exp\left( \frac{|R^\nabla|^2}{2} \right) - 1 \right) dv_g 
\]

\[
= \int_M \left( \exp\left( \frac{|R^\nabla|^2}{2} \right) \frac{d}{ds} \left( \frac{|R^\nabla|^2}{2} \right) \right) |_{s=0} dv_g 
\]

\[
+ \int_M \left( \exp\left( \frac{|R^\nabla|^2}{2} \right) - 1 \right) \frac{d}{ds} dv_g |_{s=0} 
\]

\[
= \frac{1}{2} \int_M \left( \exp\left( \frac{|R^\nabla|^2}{2} \right) - 1 \right) \langle g, \delta g \rangle 
\]

\[
- \exp\left( \frac{|R^\nabla|^2}{2} \right) \sum_{i,j} \langle i_{e_i}R^\nabla_i, i_{e_j}R^\nabla_j \rangle \delta g(e_i, e_j) \right) dv_g 
\]

(3.17)

Similarly, we can calculate $\frac{d}{ds} \mathcal{YM}_{e,g_s}(\nabla)|_{s=0}$ and obtain the desired (3.14). □

The exterior differential operator $d^\nabla : A^1(Ad(P)) \to A^2(Ad(P))$ relative to the connection $\nabla$ is given by

\[
(d^\nabla \sigma)(X_1, X_2) = (\nabla_{X_1} \sigma)(X_2) - (\nabla_{X_2} \sigma)(X_1) .
\]

(3.18)

Relative to the Riemannian structures of $Ad(P)$ and $TM$, the codifferential operator $\delta^\nabla : A^2(Ad(P)) \to A^1(Ad(P))$ is characterized as the adjoint of $d$ via the formula

\[
\int_M \langle d^\nabla \sigma, \rho \rangle dv_g = \int_M \langle \sigma, \delta^\nabla \rho \rangle dv_g ,
\]

(3.19)
Thus, exponential Yang-Mills connection and its associated curvature tensor $\nabla$ on $\text{YM}$ functional to be an exponential Yang-Mills field via $\nabla$ is the volume element associated with the metric $g$ on $TM$. Then

$$
(\delta \nabla \rho)(X_1) = - \sum_i (\nabla_{e_i} \rho)(e_i, X_1).
$$

**Definition 3.5.** A connection $\nabla$ on the adjoint bundle $Ad(P)$ is said to be an exponential Yang-Mills connection and its associated curvature tensor $R_{\nabla}$ is said to be an exponential Yang-Mills field, if $\nabla$ is a critical point of $\mathcal{Y}M_e$ with respect to any compactly supported variation in the space of connections on $Ad(P)$.

**Lemma 3.6 (The first variation formula for normalized exponential Yang-Mills functional $\mathcal{Y}M^0$ or $\mathcal{Y}M_e$).** Let $A \in A^1(Ad(P))$ and $\nabla^t = \nabla + tA$ be a family of connections on $Ad(P)$. Then

$$
\frac{d}{dt} \mathcal{Y}M^0_e(\nabla^t) \big|_{t=0} = \frac{d}{dt} \mathcal{Y}M^0_e(\nabla^t) \big|_{t=0} = \int_M (\delta \nabla (\exp(\frac{1}{2}||R_{\nabla}^t||^2)R_{\nabla}^t), A) \, dv. \tag{3.21}
$$

Furthermore, The Euler-Lagrange equation for $\mathcal{Y}M^0$ or $\mathcal{Y}M_e$ is

$$
\exp\left(\frac{1}{2}||R^t||^2\right)\delta \nabla R^t - i_{\text{grad}}(\exp(\frac{1}{2}||R^t||^2)) R^t = 0, \tag{3.22}
$$

or

$$
\delta \nabla (\exp(\frac{1}{2}||R^t||^2)R^t) = 0. \tag{3.23}
$$

**Proof.** By assumption, the curvature of $\nabla^t$ is given by

$$
R_{\nabla^t} = R^t + t(d^\nabla A) + t^2[A, A], \tag{3.24}
$$

where $[A, A] \in A^2(Ad(P))$ is given by $[A, A]_{X, Y} = [A_X, A_Y]$. Indeed, for any local vector fields $X, Y$ on $M$, with $[X, Y] = 0$, we have via (3.18)

$$
R_{X, Y} = (\nabla_X + tA_X)(\nabla_Y + tA_Y) - (\nabla_Y + tA_Y)(\nabla_X + tA_X)
= R_{X, Y}^t + t[\nabla_X, A_Y] - t[\nabla_Y, A_X] + t^2[A_X, A_Y]
= R_{X, Y}^t + t\nabla_X (A_Y) - t\nabla_Y (A_X) + t^2[A, A]_{X, Y}
= R_{X, Y}^t + t(d^\nabla A)_{X, Y} + t^2[A, A]_{X, Y}. \tag{3.25}
$$

Thus,

$$
\exp \left(\frac{1}{2}||R^t||^2\right) = \exp \left(\frac{1}{2}||R^t||^2 + t(R^t, d^\nabla A) + \varepsilon(t^2)\right), \tag{3.26}
$$

where $\varepsilon(t^2) = o(t^2)$ as $t \to 0$. Therefore,

$$
\mathcal{Y}M_e(\nabla^t) = \int_M \exp \left(\frac{1}{2}||R^t||^2 + t(R^t, d^\nabla A) + \varepsilon(t^2)\right) dv \tag{3.27}
$$

and via (3.19), we have

$$
\frac{d}{dt} \mathcal{Y}M^0_e(\nabla^t) \big|_{t=0} = \frac{d}{dt} \mathcal{Y}M_e(\nabla^t) \big|_{t=0}
= \int_M \exp \left(\frac{1}{2}||R^t||^2\right)(R^t, d^\nabla A) \, dv \tag{3.28}
= \int_M (\delta \nabla (\exp(\frac{1}{2}||R^t||^2)R^t), A) \, dv.
$$
This derives the Euler-Lagrange equation for $\mathcal{YM}_p$ or $\mathcal{YM}_c$ by (3.20) as follows

$$0 = \delta^\nabla \left( \exp \left( \frac{1}{2} \| R^\nabla \|^2 \right) R^\nabla \right)$$

$$= -\sum_{i=1}^m (\nabla_{e_i} \exp \left( \frac{1}{2} \| R^\nabla \|^2 \right) R^\nabla)(e_i, \cdot) \quad (3.29)$$

$$= \exp \left( \frac{1}{2} \| R^\nabla \|^2 \right) \delta^\nabla R^\nabla - i_{\text{grad} \left( \exp \left( \frac{1}{2} \| R^\nabla \|^2 \right) \right) R^\nabla}.$$

\[\square\]

**Corollary 3.7.** Every normalized exponential Yang-Mills connection or every exponential Yang-Mills connection $\nabla$ satisfies (3.29).

Dong and Wei derive

**Theorem B ([DW])** (i) The Euler-Lagrangian equation for $F$-Yang-Mills functional $\mathcal{YM}_F$ is

$$F'(\frac{1}{2} \| R^\nabla \|^2) \delta^\nabla R^\nabla - i_{\text{grad} \left( F'(\frac{1}{2} \| R^\nabla \|^2) \right) R^\nabla} = 0 \quad (3.30)$$

or

$$\delta^\nabla \left( F'(\frac{1}{2} \| R^\nabla \|^2) R^\nabla \right) = 0.$$

(ii) The Euler-Lagrangian equation for $p$-Yang-Mills functional $\mathcal{YM}_p, p \geq 2$ is

$$\delta^\nabla (\| R^\nabla \|^{p-2} R^\nabla) = 0 \quad (3.31)$$

or

$$\| R^\nabla \|^{p-2} \delta^\nabla R^\nabla - i_{\text{grad} (\| R^\nabla \|^{p-2}) R^\nabla} = 0.$$

(3.30) is also due to C. Gherghe ([G]).

**Corollary 3.8.** Let $\| R^\nabla \| = \text{constant}$. Then the following are equivalent:

(i) A curvature tensor $R^\nabla$ is a normalized exponential Yang-Mills field.

(ii) A curvature tensor $R^\nabla$ is a Yang-Mills field.

(iii) A curvature tensor $R^\nabla$ is a $p$ - Yang-Mills field, $p \geq 2$.

(iv) A curvature tensor $R^\nabla$ is an exponential Yang-Mills field.

(v) A curvature tensor $R^\nabla$ is an $F$-Yang-Mills field.

**Proof.** This follows at once from (3.29)-(3.31). \[\square\]

**Lemma 3.9.** Let $S_{e,\mathcal{YM}_0}$ and $S_{e,\mathcal{YM}}$ be the stress-energy tensors defined by (3.9) and (3.10) respectively, then for any vector field $X$ on $M$, we have

$$(\text{div} S_{e,\mathcal{YM}_0})(X) = (\text{div} S_{e,\mathcal{YM}})(X)$$

$$= \exp \left( \frac{1}{2} \| R^\nabla \|^2 \right) \langle \delta^\nabla R^\nabla, i_X R^\nabla \rangle + \exp \left( \frac{1}{2} \| R^\nabla \|^2 \right) \langle i_{\text{grad} \left( \exp \left( \frac{1}{2} \| R^\nabla \|^2 \right) \right) R^\nabla, i_X R^\nabla \rangle$$

$$- \langle i_{\text{grad} \left( \exp \left( \frac{1}{2} \| R^\nabla \|^2 \right) \right) R^\nabla, i_X R^\nabla \rangle,$$

(3.33)

where $\text{grad} (\bullet)$ is the gradient vector field of $\bullet$. 

DEFINITION 3.10. A curvature tensor \( R^V \in A^2(Ad(P)) \) is said to satisfy an \( e \)-conservation law if \( S_{e,YM^0} \) is divergence free, i.e.,
\[
\text{div} S_{e,YM^0} = \text{div} S_{e,YM} = 0. \tag{3.34}
\]

THEOREM 3.11. Every normalized exponential Yang-Mills field or every exponential Yang-Mills field \( R^V \) satisfies an \( e \)-conservation law.

PROOF. It is known that \( R^V \) satisfies the Bianchi identity
\[
d^V R^V = 0. \tag{3.35}
\]
Therefore, by Corollary 3.7, Lemma 3.9 and (3.35), we immediately derive the desired (3.34). \( \square \)

4. Comparison theorems in Riemannian geometry

In this section, we will discuss comparison theorems with applications on Cartan-Hadamard manifolds or more generally on complete manifolds with a pole. We recall a Cartan-Hadamard manifold is a complete simply-connected Riemannian manifold of nonpositive sectional curvature. A pole is a point \( x_0 \in M \) such that the exponential map from the tangent space to \( M \) at \( x_0 \) into \( M \) is a diffeomorphism. By the radial curvature \( K \) of a manifold with a pole, we mean the restriction of the sectional curvature function to all the planes which contain the unit vector \( \partial(x) \) in \( T_xM \) tangent to the unique geodesic joining \( x_0 \) to \( x \) and pointing away from \( x_0 \).

Let the tensor \( g - dr \otimes dr = 0 \) on the radial direction \( \partial \), and is just the metric tensor \( g \) on the orthogonal complement \( \partial \perp \).

THEOREM 4.1. (Hessian comparison theorem [GW, DW, HLRW, W11]) Let \((M, g)\) be a complete Riemannian manifold with a pole \( x_0 \). Denote by \( K(r) \) the radial curvature of \( M \). Then
\[
-\alpha^2 \leq K(r) \leq -\beta^2 \quad \text{with} \quad \alpha > 0, \beta > 0 \quad \text{(i [GW])}
\]
\[
\Rightarrow \beta \coth(\beta r)(g - dr \otimes dr) \leq \text{Hess}(r) \leq \alpha \coth(\alpha r)(g - dr \otimes dr) \quad \text{(4.1)}
\]
\[
K(r) = 0 \quad \Rightarrow \frac{1}{r}(g - dr \otimes dr) = \text{Hess}(r) \quad \text{(ii [GW])}
\]
\[
-\frac{A}{(1 + r^2)^{1+\epsilon}} \leq K(r) \leq \frac{B}{(1 + r^2)^{1+\epsilon}} \quad \text{with} \quad \epsilon > 0, A \geq 0, \text{ and } 0 \leq B < 2\epsilon \quad \text{(iii [GW], [DW], Lemma 4.1.(iii))}
\]
\[
\Rightarrow \frac{1 - B}{r}(g - dr \otimes dr) \leq \text{Hess}(r) \leq \frac{e^\frac{B}{r}}{r}(g - dr \otimes dr) \quad \text{(4.2)}
\]
\[
-\frac{A}{r^2} \leq K(r) \leq -\frac{A_1}{r^2} \quad \text{with} \quad 0 \leq A_1 \leq A \quad \text{(iv [HLRW], [W11, Theorem A])}
\]
\[
\Rightarrow \frac{1 + \sqrt{1 + 4A_1}}{2r} (g - dr \otimes dr) \leq \text{Hess}(r) \leq \frac{1 + \sqrt{1 + 4A_1}}{2r} (g - dr \otimes dr) \quad \text{(4.4)}
\]
\[
-\frac{A(A - 1)}{r^2} \leq K(r) \leq -\frac{A_1(A_1 - 1)}{r^2} \quad \text{with} \quad A \geq A_1 \geq 1 \quad \text{(v [W11, Corollary 3.1])}
\]
\[
\Rightarrow \frac{A_1}{r} (g - dr \otimes dr) \leq \text{Hessr} \leq \frac{A}{r} (g - dr \otimes dr) \quad \text{(4.5)}
\]
\[
\frac{B_1(1 - B_1)}{r^2} \leq K(r) \leq \frac{B(1 - B)}{r^2}, \text{ with } 0 \leq B, B_1 \leq 1
\]
\[
\Rightarrow \quad \frac{B - \frac{1}{r}}{r} + \frac{1}{r} \left( g - dr \otimes dr \right) \leq Hess \leq 1 + \frac{\sqrt{1 + 4B_1(1 - B_1)}}{2r} \left( g - dr \otimes dr \right); \tag{4.6}
\]
\[
\frac{B_1}{r^2} \leq K(r) \leq \frac{B}{r^2} \quad \text{with } 0 \leq B_1 \leq B \leq \frac{1}{4} \quad \text{ (vii) [W11, Theorem 3.5]}
\]
\[
\Rightarrow \quad 1 + \frac{\sqrt{1 - 4B}}{2r} \left( g - dr \otimes dr \right) \leq Hess \leq 1 + \frac{\sqrt{1 + 4B_1}}{2r} \left( g - dr \otimes dr \right); \tag{4.7}
\]
\[
-Ar^{2q} \leq K(r) \leq -Br^{2q}\quad \text{with } A \geq B > 0, q > 0 \quad \text{(viii) [GW]}
\]
\[
\Rightarrow \quad B_0r^q \left( g - dr \otimes dr \right) \leq Hess(r) \leq (\sqrt{A} \coth \sqrt{A})r^q \left( g - dr \otimes dr \right), \text{ for } r \geq 1, \tag{4.8}
\]
where
\[
B_0 = \min \left\{ 1, -\frac{q+1}{2} + \left( B + \left( \frac{q+1}{2} \right)^2 \right)^\frac{1}{2} \right\}. \tag{4.9}
\]

**Proof.** (i), (ii) and (viii) are treated in section 2 of [GW], (iii) is proved in [DW], (iv) is derived in [HLRW, W11], (v) - (vii) are proved in [W11]. \( \square \)

We note there are many applications of this Theorem (cf., e.g., [WW]), (iv) extends the asymptotic comparison theorem in ([GW], [PRS], p.39), and (vii) generalizes ([EF], Lemma 1.2 (b)).

Let \( \mathfrak{h} \) denote the bundle isomorphism that identifies the vector field \( X \) with the differential one-form \( X^\flat \), and let \( \nabla \) be the Riemannian connection of \( M \). Then the covariant derivative \( \nabla X^\flat \) of \( X^\flat \) is a \((0,2)\)-type tensor, given by
\[
\nabla X^\flat(Y,Z) = \nabla_Y X^\flat Z = \langle \nabla_Y X, Z \rangle, \quad \forall X,Y \in \Gamma(M). \tag{4.10}
\]
If \( X \) is conservative, then
\[
X = \nabla f, \quad X^\flat = df \quad \text{and} \quad \nabla X^\flat = \text{Hess}(f). \tag{4.11}
\]
for some scalar potential \( f \) (cf. [CW], p. 1527). A direct computation yields (cf., e.g., [DW])
\[
\text{div}(i_X S_{e,YM^o}) = \langle S_{e,YM^o}, \nabla X^\flat \rangle + \langle \text{div} S_{e,YM^o}(X) \rangle, \quad \forall X \in \Gamma(M). \tag{4.12}
\]
By Theorem 3.11, every normalized exponential Yang-Mills field \( R^\nabla \) satisfies an \( e \)-conservation law. It follows from the divergence theorem that for every bounded domain \( D \) in \( M \) with \( C^1 \) boundary \( \partial D \),
\[
\int_{\partial D} S_{e,YM^o}(X,\nu) ds_\nu = \int_{\partial D} \langle S_{e,YM^o}, \nabla X^\flat \rangle dv_\nu, \tag{4.13}
\]
where \( \nu \) is unit outward normal vector field along \( \partial D \) with \((n-1)\)-dimensional volume element \( ds_\nu \). When we choose scalar potential \( f(x) = \frac{1}{2}r^2(x) \), (4.11) becomes
\[
X = r\nabla r, \quad X^\flat = rdr \quad \text{and} \quad \nabla X^\flat = \text{Hess}(\frac{1}{2}r^2) = dr \otimes dr + r\text{Hess}(r). \tag{4.14}
\]
The conservative vector field \( X \) and \( e \)-conservation law will illuminate that the curvature of the base manifold \( M \) via Hessian Comparison Theorems 4.1 influences the behavior of the stress energy tensor \( S_{e,YM^o} \) and the behavior of the underlying
criticality - curvature field $R^\nabla \in A^2(Ad(P))$ with the help from the following concept (4.15) and estimate (4.20).

Analogous to $F$-degree, we introduce

**Definition 4.2.** For a given curvature field $R^\nabla$, the e-degree $d_e$ is the quantity, given by

$$
    d_e = \sup_{x \in M} \exp \left( \frac{||R^\nabla||^2(x)}{2} \right) - 1.
$$

(4.15)

The $e$-degree $d_e$ will play a role in connecting two separated parts of the normalized stress-energy tensor $S_{e,YM}$. Since $\frac{e^x}{e^x - 1}$ is a decreasing function, with $1 \leq \frac{e^x}{e^x - 1} \leq \infty$, we have

**Proposition 4.3.** Suppose

$$
    \frac{||R^\nabla||^2}{2}(x) \leq c \quad \forall \quad x \in M,
$$

(4.16)

where $c > 0$ is a constant. Then

$$
    d_e \geq \frac{e^c}{e^c - 1}.
$$

(4.17)

**Lemma 4.4.** Let $M$ be a complete $n$-manifold with a pole $x_0$. Assume that there exist two positive functions $h_1(r)$ and $h_2(r)$ such that

$$
    h_1(r)(g - dr \otimes dr) \leq Hess(r) \leq h_2(r)(g - dr \otimes dr)
$$

(4.18)

on $M \setminus \{x_0\}$. If $h_2(r)$ satisfies

$$
    rh_2(r) \geq 1,
$$

(4.19)

and $||R^\nabla|| > 0$ on $M$, then

$$
    \langle S_{e,YM}, \nabla X^\nabla \rangle \geq (1 + (n - 1)r h_1(r) - 2d_e||R^\nabla||^2_{\infty}r h_2(r)) \left( \exp \left( \frac{||R^\nabla||^2}{2} \right) - 1 \right),
$$

(4.20)

where $X = r \nabla r$.

**Proof.** Choose an orthonormal frame $\{e_i, \frac{\partial}{\partial r}\}_{i=1,...,n-1}$ around $x \in M \setminus \{x_0\}$. Take $X = r \nabla r$. Then

$$
    \nabla_{\frac{\partial}{\partial r}} X = \frac{\partial}{\partial r},
$$

(4.21)

$$
    \nabla_{e_i} X = r \nabla_{e_i} \frac{\partial}{\partial r} = r Hess(r)(e_i, e_j) e_j.
$$

(4.22)

Using (3.10), (4.14), (or (4.21), (4.22)), we have

$$
    \langle S_{e,YM}, \nabla X^\nabla \rangle = \left( \exp \left( \frac{||R^\nabla||^2}{2} \right) - 1 \right) (1 + \sum_{i=1}^{n-1} r Hess(r)(e_i, e_i))
$$

$$
    - \sum_{i,j=1}^{n-1} \exp \left( \frac{||R^\nabla||^2}{2} \right) \langle \iota_{e_i} R^\nabla, \iota_{e_j} R^\nabla \rangle r Hess(r)(e_i, e_j)
$$

$$
    - \exp \left( \frac{||R^\nabla||^2}{2} \right) \langle \iota_{\iota R^\nabla}, \iota_{\iota R^\nabla} \rangle .
$$

(4.23)
By (4.18) and (4.15), (4.23) implies that

\[ \langle S_e, YM^0, \nabla X^b \rangle \geq \left( \exp \left( \frac{\|R^\nabla\|^2}{2} \right) - 1 \right) \left( 1 + (n - 1)rh_1(r) \right) \]

\[ - \left( \exp \left( \frac{\|R^\nabla\|^2}{2} \right) - 1 \right) \sum_{i=1}^{n-1} \langle i_{e_i} R^\nabla, i_{e_i} R^\nabla \rangle rh_2(r) \frac{\exp \left( \frac{\|R^\nabla\|^2}{2} \right)}{\left( \exp \left( \frac{\|R^\nabla\|^2}{2} \right) - 1 \right)} \]

\[ - \left( \exp \left( \frac{\|R^\nabla\|^2}{2} \right) - 1 \right) \langle i_{\partial/\partial r} R^\nabla, i_{\partial/\partial r} R^\nabla \rangle \frac{\exp \left( \frac{\|R^\nabla\|^2}{2} \right)}{\left( \exp \left( \frac{\|R^\nabla\|^2}{2} \right) - 1 \right)} \]

\[ \geq \left( \exp \left( \frac{\|R^\nabla\|^2}{2} \right) - 1 \right) \left( 1 + (n - 1)rh_1(r) - 2\|R^\nabla\|^2 rh_2(r) \frac{\exp \left( \frac{\|R^\nabla\|^2}{2} \right)}{\left( \exp \left( \frac{\|R^\nabla\|^2}{2} \right) - 1 \right)} \right) \]

\[ + \left( \exp \left( \frac{\|R^\nabla\|^2}{2} \right) - 1 \right) (rh_2(r) - 1) \langle i_{\partial/\partial r} R^\nabla, i_{\partial/\partial r} R^\nabla \rangle \frac{\exp \left( \frac{\|R^\nabla\|^2}{2} \right)}{\left( \exp \left( \frac{\|R^\nabla\|^2}{2} \right) - 1 \right)} \]

\[ \geq \left( 1 + (n - 1)rh_1(r) - 2\|R^\nabla\|^2 rh_2(r) \right) \left( \exp \left( \frac{\|R^\nabla\|^2}{2} \right) - 1 \right). \]  

The last two steps follow from (4.19) and the fact that

\[ \sum_{i=1}^{n-1} \langle i_{e_i} R^\nabla, i_{e_i} R^\nabla \rangle + \langle i_{\partial/\partial r} R^\nabla, i_{\partial/\partial r} R^\nabla \rangle \]

\[ = \sum_{1 \leq j_1 \leq n} \sum_{i=1}^{n} \langle R^\nabla(e_i, e_{j_1}), R^\nabla(e_i, e_{j_1}) \rangle \]

\[ = 2\|R^\nabla\|^2, \]

where \( e_n = \frac{\partial}{\partial r} \). Now the Lemma follows immediately from (4.24) and (3.7). \( \square \)

### 5. Monotonicity formulae

In this section, we will establish monotonicity formulae on complete manifolds with a pole.

**Theorem 5.1 (Monotonicity formulae).** Let \((M, g)\) be an \(n\)-dimensional complete Riemannian manifold with a pole \(x_0\), \(\text{Ad}(P)\) be the adjoint bundle and the curvature tensor \(R^\nabla \in \mathcal{A}^2(\text{Ad}(P))\) be an exponential Yang-Mills field. Assume that the radial curvature \(K(r)\) of \(M\) and the curvature tensor \(R^\nabla\) satisfy one of the following seven conditions:
(i) \(-\alpha^2 \leq K(r) \leq -\beta^2\) with \(\alpha > 0, \beta > 0\) and \((n-1)\beta - 2d_c\alpha \| R^V \|_\infty^2 \geq 0;\)

(ii) \(K(r) = 0\) with \(-n - 2d_c\| R^V \|_\infty^2 > 0;\)

(iii) \(-\frac{A}{(1 + r^2)^{1+\epsilon}} \leq K(r) \leq \frac{B}{(1 + r^2)^{1+\epsilon}}\) with \(\epsilon > 0\), \(A \geq 0\), \(0 < B < 2\epsilon\), and
\[
\frac{n + (n - 1) B}{2\epsilon} - 2d_c e^{\frac{\beta}{2}} \| R^V \|_\infty^2 > 0;
\]

(iv) \(-\frac{A}{r^2} \leq K(r) \leq -\frac{A_1}{r^2}\) with \(0 \leq A_1 \leq A\), and
\[
1 + (n - 1) \frac{1 + \sqrt{1 + 4A_1}}{2} - d_c(1 + \sqrt{1 + 4A}) \| R^V \|_\infty^2 > 0;
\]

(v) \(-\frac{A(A - 1)}{r^2} \leq K(r) \leq -\frac{A_1(A_1 - 1)}{r^2}\) and \(A \geq A_1 \geq 1\), and
\[
1 + (n - 1) A_1 - 2d_c A \| R^V \|_\infty^2 > 0;
\]

(vi) \(\frac{B_1(1 - B_1)}{r^2} \leq K(r) \leq \frac{B(1 - B)}{r^2}\) with \(0 \leq B, B_1 \leq 1\), and
\[
1 + (n - 1)(|B - \frac{1}{2}| + \frac{1}{2}) - d_c(1 + \sqrt{1 + 4B_1(1 - B_1)}) \| R^V \|_\infty^2 > 0;
\]

(vii) \(\frac{B_1}{r^2} \leq K(r) \leq \frac{B}{r^2}\) with \(0 \leq B_1 \leq B \leq \frac{1}{4}\), and
\[
1 + (n - 1) \frac{1 + \sqrt{1 - 4B}}{2} - d_c(1 + \sqrt{1 + 4B_1}) \| R^V \|_\infty^2 > 0.
\]

Then
\[
\frac{1}{\rho_1} \int_{B_{\rho_1}(x_0)} \left( \exp\left( \frac{\| R^V \|_2^2}{2} \right) - 1 \right) dv \leq \frac{1}{\rho_2} \int_{B_{\rho_2}(x_0)} \left( \exp\left( \frac{\| R^V \|_2^2}{2} \right) - 1 \right) dv,
\]

for any \(0 < \rho_1 \leq \rho_2\), where
\[
\lambda \leq \begin{cases} 
- n - 2d_c \frac{\alpha}{B} \| R^V \|_\infty^2 & \text{if } K(r) \text{ obeys (i)}, \\
- n - 2d_c \| R^V \|_\infty^2 & \text{if } K(r) \text{ obeys (ii)}, \\
1 + (n - 1) \frac{B_1}{2} - 2d_c e^{\frac{\beta}{2}} \| R^V \|_\infty^2 & \text{if } K(r) \text{ obeys (iii)}, \\
1 + (n - 1) A_1 - 2d_c A \| R^V \|_\infty^2 & \text{if } K(r) \text{ obeys (iv)}, \\
1 + (n - 1) \frac{B_1(1 - B_1)}{2} - d_c(1 + \sqrt{1 + 4B_1(1 - B_1)}) \| R^V \|_\infty^2 & \text{if } K(r) \text{ obeys (v)}, \\
1 + (n - 1) \frac{1 + \sqrt{1 - 4B}}{2} - d_c(1 + \sqrt{1 + 4B_1}) \| R^V \|_\infty^2 & \text{if } K(r) \text{ obeys (vi)}, \\
1 + (n - 1) \frac{1 + \sqrt{1 - 4B_1}}{2} - d_c(1 + \sqrt{1 + 4B_1}) \| R^V \|_\infty^2 & \text{if } K(r) \text{ obeys (vii)}.
\end{cases}
\]

**Proof.** Take a smooth vector field \(X = r\nabla r\) on \(M\). If \(K(r)\) satisfies (i), then by Theorem 4.1 and the increasing function \(\alpha \coth(\alpha r) \to 1\) as \(r \to 0\), (4.19) holds. Now Lemma 4.1 is applicable and by (4.20), we have on \(B_\rho(x_0)\), for
every $\rho > 0$,
\[
\langle S_{e, Y, M^0} \nabla X^\flat \rangle \\
\geq (1 + (n-1)\beta r \coth(\beta r) - 2d_e\alpha \coth(\alpha \beta r) \|R^\nabla\|^2_\infty) \left( \exp\left(\frac{\|R^\nabla\|^2}{2}\right) - 1 \right) \\
= (1 + \beta r \coth(\beta r)(n-1) - 2d_e \cdot \frac{\alpha \coth(\alpha \beta r)}{\beta r \coth(\beta r)} \|R^\nabla\|^2_\infty) \left( \exp\left(\frac{\|R^\nabla\|^2}{2}\right) - 1 \right) \\
> (1 + 1 \cdot (n-1 - 2 \cdot d_e \cdot \frac{\alpha}{\beta} \cdot 1) \|R^\nabla\|^2_\infty) \left( \exp\left(\frac{\|R^\nabla\|^2}{2}\right) - 1 \right) \\
\geq \lambda \left( \exp\left(\frac{\|R^\nabla\|^2}{2}\right) - 1 \right),
\]
provided that
\[n - 1 - 2 \cdot d_e \cdot \frac{\alpha}{\beta} \|R^\nabla\|^2_\infty \geq 0,
\]
since
\[
\beta r \coth(\beta r) > 1 \text{ for } r > 0 \text{, and } \frac{\coth(\alpha \beta r)}{\coth(\beta r)} < 1 \text{, for } 0 < \beta < \alpha,
\]
and coth is a decreasing function. Similarly, from Theorem 4.1 and Lemma 4.4, the above inequality holds for the cases (ii) - (vii) on $B_p(x_0)$. Thus, by the continuity of $\langle S_{e, Y, M^0} \nabla X^\flat \rangle$ and $\exp(\|R^\nabla\|^2_\infty)$, and (3.10), we have for every $\rho > 0$,
\[
\langle S_{e, Y, M^0} \nabla X^\flat \rangle \geq \lambda \left( \exp\left(\frac{\|R^\nabla\|^2}{2}\right) - 1 \right) \text{ in } B_{\rho}(x_0) \\
\rho \left( \exp\left(\frac{\|R^\nabla\|^2}{2}\right) - 1 \right) \geq S_{e, Y, M^0}(X, \frac{\partial}{\partial r}) \text{ on } \partial B_{\rho}(x_0).
\]
It follows from (4.13) and (5.5) that
\[
\rho \int_{\partial B_{\rho}(x_0)} \left( \exp\left(\frac{\|R^\nabla\|^2}{2}\right) - 1 \right) ds \geq \lambda \int_{B_{\rho}(x_0)} \left( \exp\left(\frac{\|R^\nabla\|^2}{2}\right) - 1 \right) dv.
\]
Hence, we get from (5.6) the following
\[
\int_{\partial B_{\rho}(x_0)} \left( \exp\left(\frac{\|R^\nabla\|^2}{2}\right) - 1 \right) ds \geq \frac{\lambda}{\rho} \int_{B_{\rho}(x_0)} \left( \exp\left(\frac{\|R^\nabla\|^2}{2}\right) - 1 \right) dv.
\]
The coarea formula implies that
\[
\frac{d}{d\rho} \int_{B_{\rho}(x_0)} \left( \exp\left(\frac{\|R^\nabla\|^2}{2}\right) - 1 \right) dv = \int_{\partial B_{\rho}(x_0)} \left( \exp\left(\frac{\|R^\nabla\|^2}{2}\right) - 1 \right) ds.
\]
Thus we have
\[
\frac{d}{d\rho} \int_{B_{\rho}(x_0)} \exp\left(\frac{\|R^\nabla\|^2}{2}\right) - 1 dv \geq \frac{\lambda}{\rho}
\]
for a.e. $\rho > 0$. By integration (5.9) over $[\rho_1, \rho_2]$, we have
\[
\ln \int_{B_{\rho_2}(x_0)} \left( \exp\left(\frac{\|R^\nabla\|^2}{2}\right) - 1 \right) dv - \ln \int_{B_{\rho_1}(x_0)} \left( \exp\left(\frac{\|R^\nabla\|^2}{2}\right) - 1 \right) dv \geq \ln \rho_2^\lambda - \ln \rho_1^\lambda.
\]
This proves (5.2). □
COROLLARY 5.2. Suppose that \( M \) has constant sectional curvature \(-\alpha^2 \leq 0\) and
\[
\begin{align*}
&\left\{ n - 1 - 2d_\epsilon \| R^\nabla \|_\infty^2 \geq 0 \quad \text{if } \alpha \neq 0; \\
&n - 2d_\epsilon \| R^\nabla \|_\infty^2 > 0 \quad \text{if } \alpha = 0.
\end{align*}
\]
Let \( R^\nabla \in \mathcal{A}(\text{Ad}(P)) \) be an exponential Yang-Mills field. Then
\[
\frac{1}{\rho_1^{n - 2d_\epsilon \| R^\nabla \|_\infty^2}} \int_{B_{\rho_1}(x_0)} \left( \exp\left( \frac{\| R^\nabla \|_\infty^2}{2} \right) - 1 \right) \, dv \\
\leq \frac{1}{\rho_1^{n - 2d_\epsilon \| R^\nabla \|_\infty^2}} \int_{B_{\rho_2}(x_0)} \left( \exp\left( \frac{\| R^\nabla \|_\infty^2}{2} \right) - 1 \right) \, dv,
\]
for any \( x_0 \in M \) and \( 0 < \rho_1 \leq \rho_2 \).

PROOF. In Theorem 5.1, if we take \( \alpha = \beta \neq 0 \) for the case (i) or \( \alpha = 0 \) for the case (ii), this corollary follows immediately. \( \square \)

PROPOSITION 5.3. Let \( (M, g) \) be an \( n \)-dimensional complete Riemannian manifold whose radial curvature satisfies
\[
(\text{viii}) - Ar^{2q} \leq K(r) \leq -B r^{2q} \quad \text{with } A \geq B > 0 \quad \text{and} \quad q > 0.
\]
Let \( R^\nabla \) be an exponential Yang-Mills field, and
\[
\delta := (n - 1)B_0 - 2d_\epsilon \| R^\nabla \|_\infty^2 \sqrt{A} \coth \sqrt{A} \geq 0,
\]
where \( B_0 \) is as in (4.9). Suppose that (5.18) holds. Then
\[
\frac{1}{\rho_1^{1 + \delta}} \int_{B_{\rho_1}(x_0) - B_{1}(x_0)} \left( \exp\left( \frac{\| R^\nabla \|_\infty^2}{2} \right) - 1 \right) \, dv \\
\leq \frac{1}{\rho_2^{1 + \delta}} \int_{B_{\rho_2}(x_0) - B_{1}(x_0)} \left( \exp\left( \frac{\| R^\nabla \|_\infty^2}{2} \right) - 1 \right) \, dv,
\]
for any \( 1 \leq \rho_1 \leq \rho_2 \).

PROOF. Take \( X = r \nabla r \). Applying Theorem 4.1, (4.19), and (4.20), we have
\[
\langle S_{e,\mathcal{Y},M^0}, \nabla X^\flat \rangle \geq \left( \exp\left( \frac{\| R^\nabla \|_\infty^2}{2} \right) - 1 \right)(1 + \delta r^{q+1})
\]
and
\[
S_{e,\mathcal{Y},M^0}(X, \frac{\partial}{\partial r}) = \exp\left( \frac{\| R^\nabla \|_\infty^2}{2} \right) \left( 1 - \langle \frac{i}{\partial r} R^\nabla, \frac{i}{\partial r} R^\nabla \rangle \right) - 1 \quad \text{on } \partial B_1(x_0)
\]
\[
S_{e,\mathcal{Y},M^0}(X, \frac{\partial}{\partial r}) = \rho \exp\left( \frac{\| R^\nabla \|_\infty^2}{2} \right) \left( 1 - \langle \frac{i}{\partial r} R^\nabla, \frac{i}{\partial r} R^\nabla \rangle \right) - \rho \quad \text{on } \partial B_\rho(x_0).
\]

It follows from (4.13) that
\[
\rho \int_{\partial B_\rho(x_0)} \exp\left( \frac{\| R^\nabla \|_\infty^2}{2} \right) \left( 1 - \langle \frac{i}{\partial r} R^\nabla, \frac{i}{\partial r} R^\nabla \rangle \right) - 1 \, ds \\
- \int_{\partial B_1(x_0)} \exp\left( \frac{\| R^\nabla \|_\infty^2}{2} \right) \left( 1 - \langle \frac{i}{\partial r} R^\nabla, \frac{i}{\partial r} R^\nabla \rangle \right) - 1 \, ds
\geq \int_{B_\rho(x_0) - B_1(x_0)} (1 + \delta r^{q+1}) \left( \exp\left( \frac{\| R^\nabla \|_\infty^2}{2} \right) - 1 \right).
Whence, if
\[
\int_{\partial B_1(x_0)} \exp\left(\frac{||R^\nabla||^2}{2}\right) \left(1 - \langle i\frac{\partial}{\partial r} R^\nabla, i\frac{\partial}{\partial r} R^\nabla \rangle \right) - 1 \, ds \geq 0, \tag{5.18}
\]
then
\[
\rho \int_{\partial B_1(x_0)} \left(1 - \langle i\frac{\partial}{\partial r} R^\nabla, i\frac{\partial}{\partial r} R^\nabla \rangle \right) - 1 \, ds \geq (1 + \delta) \int_{B_\rho(x_0) - B_1(x_0)} \left(1 - \langle i\frac{\partial}{\partial r} R^\nabla, i\frac{\partial}{\partial r} R^\nabla \rangle \right) - 1 \, dv, \tag{5.19}
\]
for any \(\rho > 1\). Coarea formula then implies
\[
d \int_{B_\rho(x_0) - B_1(x_0)} \left(1 - \langle i\frac{\partial}{\partial r} R^\nabla, i\frac{\partial}{\partial r} R^\nabla \rangle \right) - 1 \, dv \geq \frac{1 + \delta}{\rho} d\rho \tag{5.20}
\]
for a.e. \(\rho \geq 1\). Integrating (5.20) over \([\rho_1, \rho_2]\), we get
\[
\ln \left(\int_{B_{\rho_2}(x_0) - B_1(x_0)} \left(1 - \langle i\frac{\partial}{\partial r} R^\nabla, i\frac{\partial}{\partial r} R^\nabla \rangle \right) - 1 \, dv \right) - \ln \left(\int_{B_{\rho_1}(x_0) - B_1(x_0)} \left(1 - \langle i\frac{\partial}{\partial r} R^\nabla, i\frac{\partial}{\partial r} R^\nabla \rangle \right) - 1 \, dv \right) \geq (1 + \delta) \ln \rho_2 - (1 + \delta) \ln \rho_1. \tag{5.21}
\]
Hence we prove the proposition. \(\square\)

**Corollary 5.4.** Let \(K(r)\) and \(\delta\) be as in Proposition 5.3, satisfying (5.12) and (5.13) respectively, \(R^\nabla\) be an exponential Yang-Mills field. Suppose
\[
\exp\left(\frac{||R^\nabla||^2}{2}\right) \left(1 - \langle i\frac{\partial}{\partial r} R^\nabla, i\frac{\partial}{\partial r} R^\nabla \rangle \right) - 1 \geq 0 \tag{5.22}
\]
on \(\partial B_1\). Then (5.14) holds.

**Proof.** The assumption (5.22) implies that (5.18) holds, and the assertion follows from Proposition 5.3. \(\square\)

### 6. Vanishing theorems for exponential Yang-Mills fields

**Theorem 6.1 (Vanishing Theorem).** Suppose that the radial curvature \(K(r)\) of \(M\) satisfies one of the seven growth conditions in (5.1) (i)-(vii), Theorem 5.1. Let \(R^\nabla\) be an exponential Yang-Mills field satisfying the \(\mathcal{YM}_0\)-energy functional growth condition
\[
\int_{B_\rho(x_0)} \left(1 - \langle i\frac{\partial}{\partial r} R^\nabla, i\frac{\partial}{\partial r} R^\nabla \rangle \right) - 1 \, dv = O(\rho^\lambda) \quad \text{as} \quad \rho \to \infty, \tag{6.1}
\]
where \(\lambda\) is given by (5.3). Then \(\exp\left(\frac{||R^\nabla||^2}{2}\right) \equiv 1\), and hence \(R^\nabla \equiv 0\). In particular, every exponential Yang-Mills field \(R^\nabla\) with finite normalized exponential Yang-Mills \(\mathcal{YM}_0\)-energy functional vanishes on \(M\).

**Proof.** This follows at once from Theorem 5.1. \(\square\)

**Proposition 6.2.** Let \((M, g)\) be an \(n\)-dimensional complete Riemannian manifold whose radial curvature satisfies (5.12) (viii), Proposition 5.1. Let \(\delta\) be as in
(5.13) in which $B_0$ is as in (4.9). Suppose (5.18) holds. Then every exponential Yang-Mills field $R^\nabla$ with the growth condition
\[
\int_{B_\rho(x_0)-B_1(x_0)} \left( \exp\left(\frac{\|R^\nabla\|^2}{2}\right) - 1 \right) dv = o(\rho^{1+\delta}) \quad \text{as } \rho \to \infty \tag{6.2}
\]
vansishes on $M - B_1(x_0)$. In particular, if $R^\nabla$ has finite normalized exponential Yang-Mills energy on $M - B_1(x_0)$, then $R^\nabla \equiv 0$ on $M - B_1(x_0)$.

**Proof.** This follows at once from Proposition 5.3. \hfill \Box

7. Vanishing theorems from exponential Yang-Mills fields to $F$-Yang-Mills fields

**Theorem 7.1.** Suppose that the radial curvature $K(r)$ of $M$ satisfies one of the seven growth conditions in (1.1) (i)-(vii), Theorem A, in which $d_F = 1$. Let $R^\nabla$ be an exponential Yang-Mills field with $\|R^\nabla\|$ constant and
\[
\text{Volume}(B_\rho(x_0)) = o(\rho^\lambda) \quad \text{as } \rho \to \infty, \tag{7.1}
\]
where $\lambda$ is given by (1.4), in which $d_F = 1$. Then $R^\nabla \equiv 0$. In particular, every exponential Yang-Mills field $R^\nabla$ with constant $\|R^\nabla\|$ over manifold which has finite volume, $\text{Volume}(M) < \infty$ vanishes.

**Proof.** By Corollary 3.8, this exponential Yang-Mills field $R^\nabla$ is a Yang-Mills field which is a special case of $F$-Yang-Mills field, where $F$ is the identity map. Thus the $F$-degree of the identity map $d_F = 1$. Now we apply $F$-Yang-Mills Vanishing Theorem A in which $F(t) = t, d_F = 1$, the $F$-Yang-Mills functional $YM_F$ growth condition, (1.3) is transformed to the volume of the base manifold growth condition, (7.1), and the conclusion $R^\nabla \equiv 0$ follows. \hfill \Box

8. An average principle, isoparametric and sobelov inequalities

In this section, we state, interpret, and apply an average principle in a simple discrete version, then extend it to a dual (or continuous) version:

**Proposition 8.1 (An average principle of concavity (resp. convexity, linearity)).** Let $f$ be a concave function (resp. convex function, linear function). Then
\[
\begin{align*}
&f(\text{average}) \geq \text{average}(f), \\
&\quad \text{resp. } f(\text{average}) \leq \text{average}(f), \\
&f(\text{average}) = \text{average}(f). \tag{8.1}
\end{align*}
\]

Applying (8.1), where a convex function $f = \exp$ and “average” is taken over two positive numbers with respect to the sum, yields one of the simplest inequalities that has far-reaching impacts
\[
\sqrt{a \cdot b} = \exp\left(\frac{A + B}{2}\right) \quad \text{(Average Principle)} \leq \frac{\exp A + \exp B}{2} = \frac{a + b}{2}. \tag{8.2}
\]
That is,
Example 8.2 (G.M. ≤ A.M.). The Geometric Mean is no greater than the Arithmetic Mean:
\[ \sqrt{a \cdot b} \leq \frac{a + b}{2}, \quad \text{for} \quad a, b > 0 \]
with “=” holds if and only if \( a = b \).

Indeed, Let \( a = \exp(A) \) and \( b = \exp(B) \). Then applying An Average Principle of Convexity (8.1), where \( f = \exp \) yields

**A geometric interpretation of this inequality:**
Among all rectangles on the Euclidean plane with a given perimeter \( \mathcal{L} \), the square has the largest area \( \mathcal{A} \).

By duality, this means parallelly
Among all rectangles on the Euclidean plane with a given area \( \mathcal{A} \), the square has the least perimeter \( \mathcal{L} \).

Indeed,
\[ 16 \mathcal{A} = 16 \cdot a \cdot b \leq (2a + 2b)^2 = \mathcal{L}^2 \]
Equality holds if and only if the rectangles are squares, i.e., \( a = b \).

A dual approach from discreteness to continuity yields

**A sharp isoperimetric inequality for plane curves:** Among all simple closed smooth curves on the Euclidean plane with a given length \( L \), the circle encloses the largest area \( A \).

\[ 4\pi A \leq L^2 \]
Equality holds if and only if the curve encloses a disk.

This is equivalent to

**The Sobolev inequality on \( \mathbb{R}^2 \) with optimal constant:** If \( u \in W^{1,1}(\mathbb{R}^2) \), then
\[ 4\pi \int_{\mathbb{R}^2} |u|^2 \, dx \leq \left( \int_{\mathbb{R}^2} |\nabla u| \, dx \right)^2. \]

Similarly, applying (8.1), where \( f = \exp \) and “average” is averaging the sum of \( n \) positive numbers, \( n \geq 2 \), yields

\[ \sqrt[n]{a_1 \cdot a_n} = \exp(n^{-1} \sum_{j=1}^{n} A_j) \leq n^{-1} \sum_{j=1}^{n} \exp A_j = n^{-1} \sum_{j=1}^{n} a_j. \]

That is,

**Example 8.3 (The geometric mean of the numbers is no greater than the arithmetic mean of \( n \) positive numbers).**
\[ \sqrt[n]{a_1 \cdot a_n} \leq \frac{a_1 + \cdots + a_n}{n}, \quad \text{for} \quad a_1, \cdots, a_n > 0, \]
with “=” holds if and only if \( a_1 = \cdots = a_n \).

For a dual version, let a concave function \( f = \log \), An Average Principle, Proposition 8.1 yields
Example 8.4. Let \( g \) be a nonnegative measurable function on \([0, 1]\). Then
\[
\log \int_0^1 g(t) \, dt \geq \int_0^1 \log (g(t)) \, dt
\] (8.9)
whenever the right side is defined.

Isoperimetric and Sobolev inequalities can be generalized to higher dimensional Euclidean spaces. As in dimension two, the \( n \)-dimensional sharp isoperimetric inequality is equivalent (for sufficiently smooth domains) to:

**The Sobolev inequality on \( \mathbb{R}^n \) with optimal constant**

If \( u \in W^{1,1}(\mathbb{R}^n) \) and \( \omega_n \) is the volume of the unit ball in \( \mathbb{R}^n \), then
\[
\left( \int_{\mathbb{R}^n} |u|^n \, dx \right)^{\frac{1}{n}} \leq \frac{1}{n} \frac{1}{\sqrt[n]{\omega_n}} \int_{\mathbb{R}^n} |\nabla u| \, dx.
\] (8.10)

Isoperimetric and Sobolev inequalities are extended to Riemannian manifolds \( M \) with sharp constants and applications to optimal sphere theorems (cf., e.g., Wei-Zhu [WZ]).

**Theorem 8.5 (A sharp isoperimetric inequality [Du, WZ])**. For every domain \( \Omega \) (in \( M \)), there exists a constant \( C(M) \) depending on \( M \) such that
\[
P^n \geq n^n \omega_n V^{n-1}(1 - C(M) V^{\frac{n}{n-1}}),
\] (8.11)
where \( P = \text{vol}(\partial \Omega), V = \text{vol}(\Omega) \), and \( \omega_n \) is the volume of the unit ball in \( \mathbb{R}^n \).

Furthermore, on simply connected Riemannian manifolds of dimension \( n \) with Ricci curvature bounded from below by \( n - 1 \), the best \( C(M) \) one can take in the above inequality (8.10) is greater than or equal to
\[
C_0 = \frac{n(n-1)}{2(n+2)\omega_n^\frac{2}{n}}.
\] (8.12)

It is then by a standard technique, via coarea formula and Cavalieri’s principle, that (8.11) is equivalent to the following:

**Theorem 8.6 (A sharp Sobolev inequality [WZ])**. There exists a constant \( A = A(M) \) such that \( \forall \varphi \in W^{1,1}(M) \),
\[
\left( \int_M |\varphi|^{n+1} \, dv \right)^{\frac{1}{n+1}} \leq K(n,1) \left( \int_M |\nabla \varphi| \, dv \right) + A(M) \left( \int_M |\varphi|^{n+1} \, dv \right)^{\frac{n+1}{n+1}},
\] (8.13)
where
\[
K(n,1) = \lim_{p \to +1} K(n,p) = \frac{1}{n\omega_n^\frac{n}{n-1}}.
\]

This isoperimetric inequality (8.11) certainly has its roots in global analysis and partial differential equations (see, e.g., [AuL]). Furthermore, the optimal constants in (8.11) will have some geometric and even topological applications. An immediate example is that sharp estimate on \( C(M) \) recaptures

**Theorem 8.7 (Bernstein isoperimetric inequality [Ber])**. On the 2-sphere \( S^2 \),
\[
L^2 \geq 4\pi A(1 - \frac{1}{4\pi} A) \quad " = \quad \text{if and only if the domain in question is a disk.}
\] (8.14)
Remark 8.8. For a generalization of isoperimetric inequality to $n$-dimensional integer multiplicity rectifiable current in $\mathbb{R}^{n+k}$, which follows from the deformation theorem in geometric measure theory, we refer to Federer and Fleming ([FF]).

9. Convexity and Jensen’s inequalities

We note by Proposition 8.1, every convex function $f$ enjoys an Average Principle of Convexity and Jensen’s inequality in an average sense. From the duality between discreteness and continuity, we consider Jensen’s inequality involving normalized exponential Yang-Mills energy functional $\mathcal{Y} \mathcal{M}_e^0$.

Let $M$ be a compact manifold and $E$ be a vector bundle over $M$. Denote $L^p_1(E)$ the Sobolev space of connections of $E$ which are $p$-integrable and so are their first derivatives. Set

$$\mathcal{W}(E) = \bigcap_{p \geq 1} L^p_1(E) \cap \{ \nabla : \mathcal{Y} \mathcal{M}_e^0(\nabla) < \infty \}. \quad (9.1)$$

**Theorem 9.1.** (Jensen’s inequality involving normalized exponential Yang-Mills energy functional $\mathcal{Y} \mathcal{M}_e^0$) Let $\nabla$ be a connection in $\mathcal{W}(E)$. Then (applying (8.1) yields)

$$\exp\left( \frac{1}{\text{Volume}(M)} \int_M ||R^\nabla||^2 \, dv \right) - 1 \leq \frac{1}{\text{Volume}(M)} \int_M \left( \exp\left( \frac{||R^\nabla||^2}{2} \right) - 1 \right) \, dv. \quad (9.2)$$

That is, \[
\exp\left( \frac{1}{\text{Volume}(M)} \mathcal{Y} \mathcal{M}(\nabla) \right) - 1 \leq \frac{1}{\text{Volume}(M)} \mathcal{Y} \mathcal{M}_e^0(\nabla). \quad (9.3)
\]

Equality is valid if and only if $||R^\nabla||$ is constant almost everywhere.

**Proof.** This is a form of Jensen’s inequality for the convex function $e^t - 1$ (c.f. [Mo, p.21]). \[ \square \]

**Theorem 9.2.** Let $\nabla$ be a minimizer in $\mathcal{W}(E)$ of the Yang-Mills functional $\mathcal{YM}$, and the norm $||R^\nabla||$ be constant almost everywhere. Then the same connection $\nabla$ is a minimizer of the normalized exponential Yang-Mills functional $\mathcal{Y} \mathcal{M}_e^0$, and for any minimizer $\tilde{\nabla}$ of the normalized exponential Yang-Mills functional $\mathcal{Y} \mathcal{M}_e^0$ in $\mathcal{W}(E)$, the norm $||R^{\tilde{\nabla}}||$ is almost everywhere constant.

**Proof.** By the definition of minimizer $\nabla$, the monotone of $t \mapsto e^t - 1$, and Jensen’s inequality (9.3), we have for each $\nabla$ in $\mathcal{W}(E)$,

$$\exp\left( \frac{1}{\text{Volume}(M)} \mathcal{Y} \mathcal{M}(\nabla) \right) - 1 \leq \exp\left( \frac{1}{\text{Volume}(M)} \mathcal{Y} \mathcal{M}(\tilde{\nabla}) \right) - 1 \leq \frac{1}{\text{Volume}(M)} \mathcal{Y} \mathcal{M}_e^0(\tilde{\nabla}). \quad (9.4)$$

so that

$$\exp\left( \frac{1}{\text{Volume}(M)} \mathcal{Y} \mathcal{M}(\nabla) \right) - 1 \leq \inf_{\tilde{\nabla} \in \mathcal{W}(E)} \frac{1}{\text{Volume}(M)} \mathcal{Y} \mathcal{M}_e^0(\tilde{\nabla}). \quad (9.5)$$

On the other hand, since $||R^\nabla||$ = constant a.e.,

$$\frac{1}{\text{Volume}(M)} \mathcal{Y} \mathcal{M}_e^0(\nabla) = \exp\left( \frac{||R^\nabla||^2}{2} \right) - 1 = \exp\left( \frac{1}{\text{Volume}(M)} \mathcal{Y} \mathcal{M}(\nabla) \right) - 1 \quad (9.6)$$
Now we assume that $\tilde{\nabla}$ is any minimizer of the normalized exponential Yang-Mills functional $YM_0^\epsilon$ in $W(E)$. Then

$$\frac{1}{\text{Volume}(M)}YM_0^\epsilon(\tilde{\nabla}) \leq \frac{1}{\text{Volume}(M)}YM_0^\epsilon(\nabla) \quad (9.7)$$

and combining $(9.7)$, $(9.6)$ and $(9.4)$, allows us to improve all inequalities in $(9.4)$ to equalities, so that we are ready to apply Theorem 9.1 and conclude that $||R^\nabla||$ is constant almost everywhere.

\[ \square \]

### 10. $p$-Yang-Mills fields

Similarly, we set

$$W^p(E) = L^p_1(E) \cap L^2_1(E), p \geq 2$$

and obtain via $(8.1)$

**Theorem 10.1** (Jensen’s inequality involving $p$-Yang-Mills energy functional $YM_p$, $p \geq 2$). Let $\nabla$ be a connection in $W^p(E)$. Then

$$\frac{1}{p} \left( \frac{2}{\text{Volume}(M)} \int_M \frac{||R^\nabla||^2}{2} dv \right)^{\frac{p}{2}} \leq \frac{1}{\text{Volume}(M)} \int_M \frac{||R^\nabla||^p}{p} dv. \quad (10.1)$$

That is, $$\frac{1}{p} \left( \frac{2}{\text{Volume}(M)} YM(\nabla) \right)^{\frac{p}{2}} \leq \frac{1}{\text{Volume}(M)} YM_p(\nabla). \quad (10.2)$$

Equality is valid if and only if $||R^\nabla||$ is constant almost everywhere.

**Proof.** This is a form of Jensen’s inequality for the convex function $t \mapsto \frac{1}{p}(2t)^{\frac{p}{2}}, p \geq 2$ (c.f. [Mo, p.21]).

**Theorem 10.2.** Let $\nabla$ be a minimizer in $W^p(E)$ of the Yang-Mills functional $YM$, and the norm $||R^\nabla||$ be constant almost everywhere. Then the same connection $\nabla$ is a minimizer of the $p$-Yang-Mills functional $YM_p$, and for any minimizer $\tilde{\nabla}$ of the $p$-Yang-Mills functional $YM_p$ in $W^p(E)$, the norm $||R^{\tilde{\nabla}}||$ is almost everywhere constant.

**Proof.** By the definition of minimizer $\nabla$, and Jensen’s inequality (10.2), we have for each $\tilde{\nabla}$ in $W^p(E)$,

$$\frac{1}{p} \left( \frac{2}{\text{Volume}(M)} YM(\nabla) \right)^{\frac{p}{2}} \leq \frac{1}{p} \left( \frac{2}{\text{Volume}(M)} YM(\tilde{\nabla}) \right)^{\frac{p}{2}} \leq \frac{1}{\text{Volume}(M)} YM_p(\tilde{\nabla}). \quad (10.3)$$

so that

$$\frac{1}{p} \left( \frac{2}{\text{Volume}(M)} YM(\nabla) \right)^{\frac{p}{2}} \leq \frac{1}{\text{Volume}(M)} YM_p(\nabla). \quad (10.4)$$

On the other hand, since $||R^\nabla|| = \text{constant a.e.}$,

$$\frac{1}{\text{Volume}(M)} YM_p(\nabla) = \frac{||R^\nabla||^p}{p} = \frac{1}{p} \left( \frac{2}{\text{Volume}(M)} YM(\nabla) \right)^{\frac{p}{2}} \quad (10.5)$$
so that $\nabla$ is also a minimizer of the $p$-Yang-Mills functional $\mathcal{YM}_p$.

Now we assume $\tilde{\nabla}$ is any minimizer of the $p$-Yang-Mills functional $\mathcal{YM}_p$ in $W^p(E)$. Then
\[
\frac{1}{\text{Volume}(M)} \mathcal{YM}_p(\tilde{\nabla}) \leq \frac{1}{\text{Volume}(M)} \mathcal{YM}_p(\nabla)
\]
and combining (10.6), (10.5) and (10.3) allows us to improve all inequalities in (10.3) to equalities, so that we are ready to apply Theorem 9.1 and conclude that $||R^{\nabla}||$ is constant almost everywhere.

\[\square\]

**Remark 10.3.** J. Eells and L. Lemaire first derive Jensen’s inequality and establish its optimality in the setting of exponentially harmonic maps ([EL]). F. Matsuura and H. Urakawa show
\[
\exp \left( \frac{\mathcal{YM}(\nabla)}{\text{Volume}(M)} \right) \leq \frac{\mathcal{YM}_e(\nabla)}{\text{Volume}(M)} \quad \text{for any } \nabla \in W(E),
\]
and the validity of equality ([MU]).

### 11. An extrinsic average variation method and $\Phi_{(3)}$-harmonic maps

We propose an extrinsic, average variational method as an approach to confront and resolve problems in global, nonlinear analysis and geometry (cf. [W1, W3]). In contrast to an average method in PDE that we applied in [CW3] to obtain sharp growth estimates for warping functions in multiply warped product manifolds, we employ an *extrinsic average variational method* in the calculus of variations ([W3]), find a large class of manifolds of positive Ricci curvature that enjoy rich properties, and introduce the notions of *superstrongly unstable* (SSU) manifolds and $p$-superstrongly unstable ($p$-SSU) manifolds ([W5, W2, W4, WY]).

**Definition 11.1.** A Riemannian manifold $M$ with its Riemannian metric $\langle \cdot, \cdot \rangle_M$ is said to be **superstrongly unstable** (SSU), if there exists an isometric immersion of $M$ in $(\mathbb{R}^q, \langle \cdot, \cdot \rangle_{\mathbb{R}^q})$ with its second fundamental form $B$, such that for every unit tangent vector $v$ to $M$ at every point $x \in M$, the following symmetric linear operator $Q^M_x$ is negative definite.
\[
\langle Q^M_x(v), v \rangle_M = \sum_{i=1}^m \left( 2 \langle B(v, e_i), B(v, e_i) \rangle_{\mathbb{R}^q} - \langle B(v, v), B(e_i, e_i) \rangle_{\mathbb{R}^q} \right)
\]
and $M$ is said to be $p$-superstrongly unstable ($p$-SSU) for $p \geq 2$ if the following functional is negative valued.
\[
F_{p,x}(v) = (p - 2) \langle B(v, v), B(v, v) \rangle_{\mathbb{R}^q} + \langle Q^M_x(v), v \rangle_M,
\]
where $\{e_1, \ldots, e_m\}$ is a local orthonormal frame on $M$.

We prove, in particular that every compact SSU manifold must be strongly unstable (SU), i.e., (a) A compact SSU manifold cannot be the target of any non-constant stable harmonic maps from any manifold, (b) The homotopic class of any map from any manifold into a compact SSU manifold contains elements of arbitrarily small energy $E$, (c) A compact SSU manifold cannot be the domain of any nonconstant stable harmonic map into any manifold, and (d) The homotopic class of any map from a compact SSU manifold into any manifold contains elements of arbitrarily small energy $E$ (cf. [HoW2, Theorem 2.2, p.321]).
11.1. Harmonic maps and $p$-harmonic maps, from a viewpoint of the first elementary symmetric function $\sigma_1$.

We recall at any fixed point $x_0 \in M$, a symmetric 2-covariant tensor field $\alpha$ on $(M, g)$ in general, or the pullback metric $u^*$ in particular, has the eigenvalues $\lambda$ relative to the metric $g$ of $M$; i.e., the $m$ real roots of the equation
\[
\det(g_{ij} - \lambda \alpha_{ij}) = 0 \quad \text{where} \quad g_{ij} = g(e_i, e_j), \quad \alpha_{ij} = \alpha(e_i, e_j),
\]
and $\{e_1, \ldots, e_m\}$ is a basis for $T_{x_0}(M)$ (cf., e.g., [HW]).

A harmonic map $u : (M, g) \to (N, h)$ can be viewed as a critical point of the energy functional, given by the integral of a half of first elementary symmetric function $\sigma_1$, of eigenvalues relative to the metric $g$, or the trace of the pullback metric tensor $u^*h$, with respect to $g$, where $\{e_1, \ldots, e_m\}$ is an local orthonormal frame field on $M$. That is,
\[
E(u) = \int_M \frac{1}{2} \sum_{i=1}^{m} h(du(e_i), du(e_i)) \, dv = \int_M \frac{1}{2} (\sigma_1(u^*)) \, dv. \tag{11.3}
\]

A $p$-harmonic map can be viewed as a critical point of the $p$-energy functional $E_p(u)$, given by the integral of $\frac{1}{p}$ times $\sigma_1$ or the trace of the pullback metric tensor to the power $p^2$, i.e.,
\[
E(u) = \int_M \frac{1}{p} \left( \sum_{i=1}^{m} h(du(e_i), du(e_i)) \right)^{\frac{p}{2}} \, dv = \int_M \frac{1}{p} (\sigma_1(u^*))^{\frac{p}{2}} \, dv. \tag{11.4}
\]

For the study of the stability of harmonic maps (resp. $p$-harmonic maps), Howard and Wei ([HoW2]) (resp. Wei and Yau ([WY])) introduce the following notions:

**Definition 11.2.** A Riemannian manifold $M$ is said to be strongly unstable (SU) (resp. $p$-strongly unstable ($p$-SU)) if $M$ is neither the domain nor the target of any nonconstant smooth stable harmonic map, (resp. stable $p$-harmonic map), and the homotopic class of maps from or into $M$ contains a map of arbitrarily small energy $E$ (resp. $p$-energy $E_p$).

This definition leads to

**Theorem 11.3.** Every compact superstrongly unstable ($SSU$)-manifold (resp. $p$-superstrongly unstable ($p$-$SSU$)) manifold is strongly unstable (SU). (resp. $p$-strongly unstable ($p$-$SU$)).

And, we make the following classification.

**Theorem 11.4 ([O, HoW]).** Let $M$ be a compact irreducible symmetric space. The following statements are equivalent:

1. $M$ is $SSU$.
2. $M$ is $SU$.
3. $M$ is $U$; i.e. Id$_M$ is an unstable harmonic map.
4. $M$ is one of the following:
(i) the simply connected simple Lie groups \((A_l)_{l \geq 1}, \quad B_2 = C_2\) and \((C_l)_{l \geq 3};\)
(ii) \(SU(2n)/Sp(n), \quad n \geq 3;\)
(iii) Spheres \(S^k, \quad k > 2;\)
(iv) Quaternionic Grassmannians \(Sp(m+n)/Sp(m) \times Sp(n), m \geq n \geq 1;\)
(v) \(E_6/F_4;\)
(vi) Cayley Plane \(F_4/\text{Spin}(9).\)

\[ (11.5) \]

**Theorem 11.5 (Topological Vanishing Theorem).** Suppose that \(M\) is a compact \(SSU\) (resp. \(p-SSU\)) manifold. Then \(M\) is \(SU\) and
\[
\pi_1(M) = \pi_2(M) = 0
\]
(resp. \(\pi_1(M) = \cdots = \pi[p] = 0\)).

Furthermore, the following three statements are equivalent:
(a) \(\pi_1(M) = \pi_2(M) = 0\).
(b) the infimum of the energy \(E\) is 0 among maps homotopic to the identity on \(M\).
(c) the infimum of the energy \(E\) is 0 among maps homotopic to a map from \(M\).

That is,
\[
\pi_1(M) = \pi_2(M) = 0 \iff \inf\{E(u') : u' \text{ is homotopic to } \text{Id on } M\},
\]
\[
\iff \inf\{E(u') : u' \text{ is homotopic to } u : M \to \bullet\}. \tag{11.8}
\]

(Cf. [W3, the diagram on p.58].)

**11.2. \(\Phi\)-harmonic maps, from a viewpoint of the second elementary symmetric function \(\sigma_2\) ([HW]).**

We introduce the notion of \(\Phi\)-harmonic map which is the second symmetric function \(\sigma_2\) of the pullback metric tensor \(u^*h\), an analogue of \(\sigma_1\) in the above subsection 11.1.

In [HW], Han and Wei show that the extrinsic average variational method in the calculus of variations employed in the study of harmonic maps, \(p\)-harmonic maps, \(F\)-harmonic maps and Yang-Mills fields can be extended to the study of \(\Phi\)-harmonic maps. In fact, we find a large class of manifolds with rich properties, \(\Phi\)-superstrongly unstable \((\Phi\text{-SSU})\) manifolds, establish their links to \(p\)-SSU manifolds and topology, and apply the theory of \(p\)-harmonic maps, minimal varieties and Yang-Mills fields to study such manifolds. With the same notations as above, we introduce the following notions:

**Definition 11.6.** A Riemannian manifold \((M^m, g)\) with a Riemannian metric \(g\) is said to be \(\Phi\)-superstrongly unstable \((\Phi\text{-SSU})\) if there exists an isometric immersion \(\mathbb{R}^q\) such that, for all unit tangent vectors \(v\) to at every point \(x \in M^m\), the following functional is always negative:
\[
F_{\Phi_2}(v) = \sum_{i=1}^m \left(4(B(v, e_i), B(v, e_i)) - (B(v, v), B(e_i, e_i))\right), \tag{11.9}
\]
where $B$ is the second fundamental form of $M^m$ in $\mathbb{R}^q$, and \{$e_1, \cdots, e_m$\} is a local orthonormal frame on $M$ near $x$.

**Definition 11.7.** A Riemannian manifold $M$ is $\Phi$-strongly unstable ($\Phi$-SU) if it is neither the domain nor the target of any nonconstant smooth $\Phi$-stable stationary map, and the homotopic class of maps from or into $M$ contains a map of arbitrarily small energy.

**Theorem 11.8.** Every compact $\Phi$-superstrongly unstable ($\Phi$-SSU) manifold is $\Phi$-strongly unstable ($\Phi$-SU).

11.3. $\Phi_S$-harmonic maps, from a viewpoint of an extended second symmetric function $\sigma_2$ ([FHLW]).

We introduce the notion of $\Phi_S$-harmonic maps, which is a $\sigma_2$ version of the stress energy tensor $S$.

In [FHLW], Feng, Han, Li, and Wei show that the extrinsic average variational method in the calculus of variations employed in the study of $\sigma_1$ and $\sigma_2$ versions of the pullback metric $u^* h$ on $M$ can be extended to the study of a $\sigma_2$ version of the stress energy tensor $S$. In fact, we find a large class of manifolds, $\Phi_S$-superstrongly unstable ($\Phi_S$-SSU) manifolds, introduce the notions of a stable $\Phi_S$-harmonic map, $\Phi_S$-strongly unstable ($\Phi_S$-SU) manifolds, and prove

**Theorem 11.9.** Every compact $\Phi_S$-superstrongly unstable ($\Phi_S$-SSU) manifold is $\Phi_S$-strongly unstable ($\Phi_S$-SU).

11.4. $\Phi_{S,p}$-harmonic maps, from a viewpoint of a combined extended second symmetric function $\sigma_2$ ([FHW]).

We introduce the notion of $\Phi_{S,p}$-harmonic maps, which is a combined generalized $\sigma_2$ version of the stress energy tensor $S$, and a $\sigma_1$ version of the pullback $u^*$.

In [FHLW], Feng, Han, Li, and Wei show that the extrinsic average variational method in the calculus of variations employed in the study of $\sigma_1$ and $\sigma_2$ versions of the pullback metric $u^* h$ on $M$ and stress-energy tensor can be extended to the study of a combined extended second symmetric function $\sigma_2$ version. In fact, we find a large class of manifolds, $\Phi_{S,p}$-superstrongly unstable ($\Phi_{S,p}$-SSU) manifolds, introduce the notions of a stable $\Phi_{S,p}$-harmonic map, $\Phi_{S,p}$-strongly unstable ($\Phi_{S,p}$-SU) manifolds, and prove

**Theorem 11.10.** Every compact $\Phi_{S,p}$-superstrongly unstable ($\Phi_{S,p}$-SSU) manifold is $\Phi_{S,p}$-strongly unstable ($\Phi_{S,p}$-SU).

11.5. $\Phi_{(3)}$-harmonic maps, from a viewpoint of the third elementary symmetric function $\sigma_3$ ([FHJW]).

We introduce the notion of of $\Phi_{(3)}$-harmonic maps, which is a $\sigma_3$ version of the pullback $u^*$.

In fact, Feng, Han, Jiang, and Wei show that the extrinsic average variational method in the calculus of variations employed in the study of $\sigma_1$ and $\sigma_2$ versions of the pullback metric $u^* h$ on $M$ can be extended to the study of the third symmetric function $\sigma_3$ version. Whereas we can view harmonic maps as $\Phi_{(1)}$-harmonic maps (involving $\sigma_1$) and $\Phi$-harmonic maps as $\Phi_{(2)}$-harmonic maps (involving $\sigma_2$), we introduce the notion of a $\Phi_{(3)}$-harmonic map and find a large class of manifolds, $\Phi_{(3)}$-superstrongly unstable ($\Phi_{(3)}$-SSU) manifolds, introduce the notions of a stable $\Phi_{(3)}$-harmonic map, $\Phi_{(3)}$-strongly unstable ($\Phi_{(3)}$-SU) manifolds, and prove
**Theorem 11.11 ([FHJW]).** Every compact $\Phi_3$-superstrongly unstable ($\Phi_3$-SSU) manifold is $\Phi_3$-strongly unstable ($\Phi_3$-SU).

**Definition 11.12 ([FHJW]).** A Riemannian manifold $M^m$ is said to be $\Phi_3$-superstrongly unstable ($\Phi_3$-SSU) if there exists an isometric immersion of $M^m$ in $\mathbb{R}^q$ with its second fundamental form $B$ such that for all unit tangent vectors $v$ to $M^m$ at every point $x \in M^m$, the following functional is negative valued.

$$F_{\Phi_3}(v) = \sum_{i=1}^{m} \left( 6\langle B(v, e_i), B(v, e_i) \rangle_{\mathbb{R}^q} - \langle B(v, v), B(e_i, e_i) \rangle_{\mathbb{R}^q} \right) = 0$$

where $\{e_1, \ldots, e_m\}$ is a local orthonormal frame field on $M^m$ near $x$.

**Theorem 11.13.** Every $\Phi_3$-SSU manifold $M$ is $p$-SSU for any $2 \leq p \leq 6$.

**Proof.** By Definition 11.12, $\Phi_3$-SSU manifold enjoys

$$F_{\Phi_3}(v) = \sum_{i=1}^{m} \left( 6\langle B(v, e_i), B(v, e_i) \rangle_{\mathbb{R}^q} - \langle B(v, v), B(e_i, e_i) \rangle_{\mathbb{R}^q} \right) < 0$$

for all unit tangent vector $v \in T_x(M)$. It follows from (11.2) and (11.10) that

$$F_{p,x}(v) = (p-2)\langle B(v, v), B(v) \rangle_{\mathbb{R}^q} + \langle Q^M_x(v), v \rangle_M$$

$$\leq (p-2) \sum_{i=1}^{n} \left( 2\langle B(v, e_i), B(v, e_i) \rangle_{\mathbb{R}^q} \right)$$

$$+ \sum_{i=1}^{n} \left( 2\langle B(v, e_i), B(v, e_i) \rangle_{\mathbb{R}^q} - \langle B(v, v), B(e_i, e_i) \rangle_{\mathbb{R}^q} \right)$$

$$\leq \sum_{i=1}^{n} \left( p\langle B(v, e_i), B(v, e_i) \rangle - \langle B(v, v), B(e_i, e_i) \rangle \right)$$

$$\leq \sum_{i=1}^{n} \left( 6\langle B(v, e_i), B(v, e_i) \rangle - \langle B(v, v), B(e_i, e_i) \rangle \right) < 0,$$

for $2 \leq p \leq 6$. So by Definition 11.1, $M$ is $p$-SSU for any $2 \leq p \leq 6$.

**Theorem 11.14.** Every compact $\Phi_3$-SSU manifold $M$ is 6-connected, i.e.,

$$\pi_1(M) = \cdots = \pi_6(M) = 0.$$  

**Proof.** Since every compact $p$-SSU is $[p]$-connected (cf. [W5, Theorem 3.10, p. 645]), and $p = 6$ by the previous Theorem, the result follows.

**Theorem 11.15** (Sphere Theorem). Every compact $\Phi_3$-SSU manifold $M$ of dimension $m \leq 13$ is homeomorphic to an $m$-sphere.

**Proof.** In view of Theorem 11.13, $M$ is 6-connected. By the Hurewicz isomorphism theorem, the 6-connectedness of $M$ implies homology groups $H_1(M) = \cdots = H_6(M) = 0$. It follows from Poincare Duality Theorem and the Hurewicz Isomorphism Theorem ([SP]) again, $H_{m-6}(M) = \cdots = H_{m-1}(M) = 0$, $H_m(M) \neq 0$ and $M$ is $(m-1)$-connected. Hence $N$ is a homotopy $m$-sphere. Since $M$ is $\Phi_2$-SSU, $m \geq 7$. Consequently, a homotopy $m$-sphere $M$ for $m \geq 5$ is homeomorphic to an $m$-sphere by a Theorem of Smale ([Sm]).
We summarize some of new manifolds found and these results obtained by an extrinsic average method in Table 1 in Section 1.

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