Compatible systems of mod $p$ Galois representations: II

Chandrashekhar Khare

1 Introduction

Compatible systems of $n$-dimensional, mod $p$ representations of absolute Galois groups of number fields were considered by Serre in his study of openness of images of adelic Galois representations arising from elliptic curves in [S2]. In [K] the author considered abstract compatible systems of $n$-dimensional, mod $p$ representations of the absolute Galois group of $\mathbb{Q}$ and determined them in the one-dimensional case.

1.1 Definition and main theorem

We begin with the definition of compatible systems of $n$-dimensional, mod $p$ representations of the absolute Galois group of a number field $K$.

Definition 1. Let $K$ and $L$ be number fields and $S,T$ finite sets of places of $K$ and $L$ respectively. An $L$-rational (resp., $L$-integral) strictly compatible system $\{\rho_\wp\}$ of $n$-dimensional mod $\wp$ representations of $G_K := \text{Gal}(\overline{K}/K)$ with defect set $T$ and ramification set $S$, consists of giving for each finite place $\wp$ of $L$ not in $T$ a continuous, semisimple representation $\rho_\wp : G_K \to GL_n(F_\wp)$, for $F_\wp$ the residue field of $O_L$ at $\wp$ of characteristic $p$, that is

- unramified at the places outside $S \cup \{\text{places of } K \text{ above } p\}$
- there is a monic polynomial $f_r(X) \in L[X]$ (resp., $f_r(X) \in O_L[X]$) such that for each place $r$ of $K$ not in $S$ and for all places $\wp$ of $L$ not in $T$, coprime to the residue characteristic of $r$, and such
that \( f_r(X) \) has coefficients that are integral at \( \wp \), the characteristic polynomial of \( \rho_{\wp}(Frob_r) \) is the reduction of \( f_r(X) \) mod \( \wp \), where \( Frob_r \) is the conjugacy class of the Frobenius at \( r \) in the Galois group of the extension of \( K \) that is the fixed field of the kernel of \( \rho_{\wp} \).

- If the prime to \( p \) part of the Artin conductor of \( \rho_{\wp} \) is bounded independently of \( \wp \) we say that the system \( \{\rho_{\wp}\} \) has bounded conductor.

In the course of the paper we will often suppress the sets \( S, T \) from the notation. It might be prudent to impose the condition of integrality of the roots of \( f_r(X) \) outside primes coprime to the residue characteristic of primes in \( T \) and \( r \) in Definition \( \mathbb{I} \), as only then are the \( \rho_{\wp} \)'s determined a priori by the compatible system data \( \{f_r(X)\} \). But as we do not need this in the main result of the paper, Theorem \( \mathbb{I} \) below, we stick to our less stringent requirements.

One would like to prove that a strictly compatible system arises motivically, i.e., “from the mod \( p \) étale cohomology of a variety \( X/K \) as \( p \) varies”. In the case of one-dimensional strictly compatible systems this is interpreted as saying that it arises from a Hecke character. It is only in the one-dimensional case thanks to class field theory that at the moment one has a realistic chance of describing compatible sysytems in any degree of generality. We do this below and prove in Section 4.2 the following theorem that is the main result of the paper (for any unexplained terms see Section 4.1, in particular Definition \( \mathbb{I} \)).

**Theorem 1** A \( L \)-rational strictly compatible system \( \{\rho_{\wp}\} \) of one-dimensional mod \( \wp \) representations of \( \text{Gal}(\overline{K}/K) \) arises from a Hecke character.

**Corollary 1** An \( L \)-rational strictly compatible system of one-dimensional \( \wp \)-adic representations \( \{\rho_{\wp,\infty}\} \) as in I-11 of \([S]\) arises from a Hecke character.

**Proof of corollary:** This follows from the general observation that, given a strictly compatible system of \( \wp \)-adic representations as in I-11 of \([S]\), we can reduce it mod \( \wp \) and semisimplify to get a rational strictly compatible system in the sense of Definition \( \mathbb{I} \) such that \( f_r(X) \) is integral away from the primes that have the same residue characteristic as \( r \). Thus the resulting strictly compatible system of mod \( \wp \) representations determines the system of \( \wp \)-adic representations. From this observation and Theorem \( \mathbb{I} \) the corollary follows.
The corollary is Proposition 1.4 in [Sch]. It is deduced there as a consequence of a deep result of Waldschmidt in transcendental number theory that as a corollary proves the much stronger result that even a single, algebraic, one-dimensional \( \varphi \)-adic representation arises from a Hecke character (see [H]). The corollary in a weaker form, assuming a supplementary purity hypothesis, is also an old theorem of Taniyama (see Theorem 1 of [Tani]) that he proved using methods quite different from the present paper. But both in [Tani] and in the present work the compatibility hypotheses are used to describe 1-dimensional compatible systems using purely algebraic methods, and Theorem \( \mathbb{H} \) can be addressed using only such methods.

We point out a corollary that is immediate from the theorem but that sets us up to formulate two conjectures in the higher dimensional case.

**Corollary 2** An \( L \)-rational strictly compatible system \( \{ \rho_\varphi \} \) of one-dimensional \( \text{mod} \ \varphi \) representations of \( \text{Gal}(\overline{K}/K) \) lifts to a strictly compatible system of \( \varphi \)-adic representations. Further it is of bounded conductor, and satisfies the purity and integrality properties of Conjecture 2 below.

### 1.2 Conjectures and their discussion

Guided by the results above we make the following two conjectures. In the first we propose a reciprocity law for compatible systems of \( \text{mod} \ \varphi \) representations as the following “meta-conjecture”:

**Conjecture 1** Any strictly compatible \( L \)-integral system as in Definition 1 arises motivically.

To be more specific we propose the following purely Galois theoretic conjecture:

**Conjecture 2** Let \( \{ \rho_\varphi \} \) be a strictly compatible system as in Definition 4.

- *(Lifting)* It lifts to (i.e., is the reduction up to semisimplification of) a strictly compatible system of semisimple \( \varphi \)-adic representations.

- *(Bounded conductor)* It is of bounded conductor: more precisely the prime-to-\( p \) part of the Artin conductor of \( \rho_\varphi \) is independent of \( \varphi \) for almost all \( \varphi \).
(Purity) Assume that $\rho_\wp$ is irreducible for almost all $\wp$. Then the roots of $f_r(X)$ for primes $r$ not in $S$ are of absolute value $|\text{Nm}(r)|^t$ with respect to all embeddings of $\overline{\mathbb{Q}}$ in $\mathbb{C}$, and for an integer or half-integer $t$ that is independent of $r$. Here $\text{Nm}$ is the norm map to $\mathbb{Q}$.

(Integrality) $\{\rho_\wp \otimes \text{Nm}_m^m\}$ is integral where $\text{Nm}_\wp$ is the $\wp$-adic cyclotomic character for some integer $m$.

Because of known properties of Galois representations which arise from geometry one expects that Conjecture 1 implies Conjecture 2 in the integral case. Note that the first part of Conjecture 2 implies integrality of the roots of $f_r(X)$ at primes coprime to the residue characteristic of primes in $T$ and $r$, while the last part of the conjecture implies integrality of the roots of $f_r(X)$ at primes coprime to the residue characteristic of $r$. Haruzo Hida has also pointed out that one might expect that the minimal field of rationality of a strictly compatible system as in Definition 1 is totally real or CM: this is again verified in the one-dimensional case by Theorem 1. Our conjectures may be regarded as an analog for compatible mod $p$ systems of Galois representations of the well-known conjectures of Fontaine and Mazur, cf. [FM]. Proving that a strictly compatible system $\{\rho_\wp\}$ arises motivically is much stronger than proving that an individual $\rho_\wp$ arises motivically: this is in contrast to what happens in the case of compatible systems of $\wp$-adic representations. Nevertheless in the mod $\wp$ case too, there is a link between reciprocity conjectures about the individual representations $\rho_\wp$, being motivic and the compatible system arising motivically, if we grant the properties of the compatible system $\{\rho_\wp\}$ of Conjecture 2. The reader is invited to consult [S1] where such a link is established which allows Serre to deduce the Shimura-Taniyama conjecture as a consequence of the conjectures in [S1].

Theorem 1 proves these conjectures for 1-dimensional strictly compatible systems. On the other hand there are pairs of 1-dimensional mod $p$ and mod $q$ representations that do not arise simultaneously from a Hecke character. In [KK] it is determined when a given pair of one-dimensional mod $p$ and mod $q$ representations arises from a Hecke character.

Here is the plan of the paper: In Section 2 we prove a descent result (Proposition 1 below) related to the lifting property. In Section 3 we generalise a result of [CS] that is needed for the proof of Theorem 1. The proof is carried out in Section 4. In Section 5 we partially generalise the results of Section 4 to abelian semisimple compatible systems. Although we do not
have a complete result in this case, as there is an essential difficulty that we are unable to overcome, the methods used in this section may be of independent interest. In particular we would like to draw attention to an analog of Artin’s conjecture on primitive roots, for the ring that arises from restriction of scalars to $\mathbb{Z}$ of the ring of integers of a number field, that we formulate in this section.

2 Descent results

We prove a descent result in the context of the lifting statement of Conjecture 2. We first define what we mean by weakly compatible systems:

Definition 2  
• Let $K$ and $L$ be number fields and $S, T$ finite sets of places of $K$ and $L$ respectively. A $L$-rational (resp., $L$-integral) weakly compatible system $\{\rho_{\wp, \infty}\}$ of $n$-dimensional $\wp$-adic representations of $\text{Gal}(\overline{K}/K)$ with defect $T$ and ramification set $S$, consists of giving for each finite place $\wp$ of $L$ not in $T$ a continuous semsimple representation 

$$\rho_{\wp, \infty} : G_K \rightarrow GL_n(L_{\wp}),$$

for $L_{\wp}$ the completion of $L$ at $\wp$ whose residue field has characteristic $p$, that is

- unramified at the primes outside $S \cup \{\text{places of } K \text{ above } p\}$
- for a place $r$ of $K$ not in $S$ there is a monic polynomial $f_r(X) \in L[X]$ (resp., $f_r(X) \in \mathcal{O}_L[X]$) such that for almost all places $\wp$ of $L$, the characteristic polynomial of $\rho_{\wp, \infty}(\text{Frob}_r)$ is $f_r(X)$, where $\text{Frob}_r$ is the conjugacy class of the Frobenius at $r$ in the Galois group of the extension of $K$ that is the fixed field of the kernel of $\rho_{\wp, \infty}$.

• A $L$-rational (resp., $L$-integral) weakly compatible system $\{\rho_{\wp}\}$ of $n$-dimensional mod $\wp$ representations of $\text{Gal}(\overline{K}/K)$ with defect $T$ and ramification set $S$, consists of giving for each finite place $\wp$ of $L$ not in $T$ a continuous semsimple representation 

$$\rho_{\wp} : G_K \rightarrow GL_n(F_{\wp}),$$

for $F_{\wp}$ the residue field of $\mathcal{O}_L$ at $\wp$ of characteristic $p$, that is
unramified at the primes outside \( S \cup \{ \text{places of } K \text{ above } p \} \)

for a prime \( r \) of \( K \) not in \( S \) there is a monic polynomial \( f_r(X) \in L[X] \) (resp., \( f_r(X) \in \mathcal{O}_L[X] \)) such that for almost all places \( \varphi \) of \( L \), the characteristic polynomial of \( \rho_{\varphi}(\text{Frob}_r) \) is the reduction of \( f_r(X) \mod \varphi \), where \( \text{Frob}_r \) is the conjugacy class of the Frobenius at \( r \) in the Galois group of the extension of \( K \) that is the fixed field of the kernel of \( \rho_{\varphi} \).

We do not know if there are weakly compatible systems in the mod \( \varphi \) setting that are not strictly compatible for some large but finite defect set. (Note that we might use the equivalent words place and prime below.)

**Proposition 1** Let \( K' \) be a finite Galois extension of \( K \). Consider a \( L \)-integral strictly compatible system \( \{ \rho_{\varphi} \} \) of mod \( \varphi \) representations of \( G_K := \text{Gal}(K/K) \), with the further property that \( \rho_{\varphi}|_{G_{K'}} \) is absolutely irreducible for almost all \( \varphi \). Then if the strictly compatible system \( \{ \rho_{\varphi}|_{G_{K'}} \} \) lifts to a strictly compatible \( L' \)-integral \( \varphi \)-adic system \( \{ \rho_{\varphi}|_{G_{K'},n} \} \) for some number field \( L' \) with finite defect and exceptional sets, then the system \( \{ \rho_{\varphi} \} \) lifts to a weakly compatible \( L' \)-integral \( \varphi \)-adic system \( \{ \rho_{\varphi,K,n} \} \).

**Proof:** We only consider the \( \rho_{\varphi} \)'s for \( \varphi \)'s such that their residue characteristic is prime to the degree \([K':K]\) and for which \( \rho_{\varphi}|_{G_{K'}} \) is irreducible. We assume without loss of generality that \( L' \) contains \( L \) and that we are considering \( L' \)-rational compatible systems and denote the places of \( L' \) by \( \varphi \) etc. Then:

- \( \rho_{\varphi}|_{G_{K'}} \) lifts to \( \rho_{\varphi,K',n} \) the mod \( \varphi^n \) representation that is the reduction of \( \rho_{\varphi,K',\infty} \).

Note that the residue characteristic of \( \varphi \) is prime to the degree \([K':K]\), and \( \rho_{\varphi}|_{G_{K'}} \) extends to the representation \( \rho_{\varphi} \) of \( G_K \). It is easy to see from this, by computing cohomological obstructions (see Proposition 1.1 of [Clo] for a similar argument), that for each \( n \), the representation \( \rho_{\varphi,K',n} \) extends to a representation \( \rho_{\varphi,K,n} \) of \( G_K \), using that \( \rho_{\varphi,K',n} \) satisfies descent data as \( \{ \rho_{\varphi,K',\infty} \} \) is a strictly compatible system and \( \{ \rho_{\varphi,K'} \} \) comes by restriction from \( \{ \rho_{\varphi} \} \). Then we are done by invoking Carayol’s theorem, see [Ca], as in [Clo].

- \( \rho_{\varphi}|_{G_{K'}} \) being irreducible, any extension of \( \rho_{\varphi}|_{G_{K'}} \) to \( G_K \) is unique up to twisting by characters of \( \text{Gal}(K'/K) \), and thus we may assume that for each \( n \), \( \rho_{\varphi,K,n} \) reduces mod \( \varphi \) to \( \rho_{\varphi} \).
Thus for all but finitely many \( \wp \), \( \rho_\wp \) lifts to a \( \wp \)-adic representation \( \rho_{\wp,K,\infty} \) whose restriction to \( G_{K'} \) is \( \rho_{\wp,K',\infty} \). Now we claim that the \( \{ \rho_{\wp,K,\infty} \} \)'s for such \( \wp \)'s form the desired weakly compatible lift of the \( \rho_\wp \)'s. First observe that \( \rho_{\wp,K,\infty} \) is unramified outside primes above \( p \) and a fixed finite set \( S' \) that does not depend on \( p \), that we see using the fact that \( K'/K \) is ramified at only finitely many primes. Then observe that for almost all primes \( r \) of \( K \) there are only finitely many possibilities for the roots of the characteristic polynomial of \( \text{Frob}_r \) in \( \rho_{\wp,K,\infty} \) as \( \wp \) varies, and \( r \) is fixed. This is because \( \{ \rho_{\wp,K',\infty} \} \) is in particular a weakly compatible system. Thus using the strictly compatible system \( \{ \rho_\wp \} \) it follows that the characteristic polynomial of \( \rho_{\wp,K,\infty}(\text{Frob}_r) \) is \( f_r(X) \) for almost all primes \( \wp \) and fixed \( r \).

**Remark:** We unfortunately do not know how to prove the more desirable result that under the conditions above \( \{ \rho_\wp \} \) lifts to a strictly compatible system of \( \wp \)-adic representations, although the weakly compatible \( \wp \)-adic system constructed above should be strictly compatible for a suitably large defect set. This is because the compatible system \( \{ \rho_\wp \} \) serves as rigidifying data in the sense that for almost all primes, more precisely all primes of residue characteristic prime to \( [K':K] \), the extension \( \rho_{\wp,K',\infty} \) of \( \rho_{\wp,K,\infty} \) is determined by the requirement that it reduces to \( \rho_\wp \); this follows from the irreducibility hypothesis on \( \{ \rho_\wp|_{G_{K'}} \} \).

We now prove a more specific descent result in the context of Conjecture \( \mathcal{H} \) for 1-dimensional representations that will be useful in the proof of Theorem \( \mathcal{H} \) (see Definition \( \mathcal{H} \) of Section 4.1 below for unexplained terms).

**Lemma 1** Let \( K' \) be a finite extension of \( K \). Then a \( L \)-rational strictly compatible system \( \{ \rho_\wp \} \) of 1-dimensional mod \( \wp \) representations of \( G_K := \text{Gal}(K/K) \) with defect \( T \) and exceptional set \( S \) and with bounded conductor arises from a Hecke character if and only if the strictly compatible system \( \{ \rho_\wp|_{G_{K'}} \} \) arises from a Hecke character.

**Proof:** Only one direction needs a proof. We may assume without loss of generality that \( K' \) is a Galois extension of \( \mathbb{Q} \). So assume that the strictly compatible system \( \{ \rho_\wp|_{G_{K'}} \} \) arises from a Hecke character \( \chi \) of the idele group of \( K' \) and of infinity type \( (m_\sigma)_{\sigma \in \text{Gal}(K'/\mathbb{Q})} \). Using that the strictly compatible system \( \{ \rho_\wp|_{G_{K'}} \} \) arises by restriction from a strictly compatible system of \( G_K \) we see that \( \chi^\sigma = \chi \) for all \( \sigma \in \text{Gal}(K'/\mathbb{Q}) \). Consider a principal prime ideal \( (a) \) of \( K' \) that lies above a prime of \( \mathbb{Q} \) that splits completely in \( K' \).
Then as \( \chi(\sigma(a)) = \chi(a) \) for all \( \sigma \in \text{Gal}(K'/K) \) we deduce that \( m_\sigma = m_{\sigma'} \) whenever the restrictions of \( \sigma, \sigma' \in \text{Gal}(K'/\mathbb{Q}) \) to \( K \) are equal, and thus for each embedding \( \sigma \) of \( K \) we can put without ambiguity \( m_\sigma = m_{\sigma'} \) for any \( \sigma' \in \text{Gal}(K/\mathbb{Q}) \) that extends \( \sigma \). Now consider the algebraic character \( K^* \to \mathbb{C}^* \) given by \( k \to \Pi_\sigma \sigma(k)^{m_\sigma} \) where \( \sigma \) runs through embeddings of \( K \) in \( \mathbb{C} \). It is easy to see that this character is trivial on a subgroup of finite index of the units \( \mathcal{O}_K^* \). Thus it is trivial by Théorème 1 of [C] on units congruent to \( 1 \) mod \( n \) of \( \mathcal{O}_K^* \) for some ideal \( n \) of \( \mathcal{O}_K \). It follows that there is a Hecke character \( \chi' \) for \( K \) such that \( \rho_\wp \otimes \tilde{\chi'}_{\wp}^{-1} \) where \( \{ \chi'_{\wp} \} \) is the compatible mod \( \wp \)-system that \( \chi' \) gives rise to as described in the next section factors through a fixed finite extension of \( K \). Observe that strictly compatible systems as in Definition 1 that factor through the Galois group \( \text{Gal}(K''/K) \) of a fixed finite extension \( K'' \) of \( K \) arise as reductions mod primes of a continuous representation of \( G_K \) into \( GL_n(\mathcal{O}) \) with finite image: here \( \mathcal{O} \) the ring of integers of a number field. This finishes the proof.

We say that a compatible system \( \{ \rho_\wp \} \) as in Definition 1 arises from an Artin representation if it arises from reducing a continuous representation \( \rho : G_K \to GL_n(\mathcal{O}) \) with finite image, where \( \mathcal{O} \) the ring of integers of a number field \( L \), modulo the primes of \( L \) and semisimplifying. We quote a result in Section 8 of Deligne-Serre (cf. [DS]; we thank C.S. Rajan for this reference) that characterises compatible systems \( \{ \rho_\wp \} \) that arise from Artin representations and refines the observation towards the end of the proof above.

**Proposition 2 (Deligne-Serre)** Let \( \{ \rho_\wp \} \) be a \( L \)-integral compatible system where we allow the defect set \( T \) to be any set whose complement in the set of places of \( L \) is infinite. If \( |\text{im}(\rho_\wp)| \) is bounded independently of \( \wp \), then \( \{ \rho_\wp \} \) arises from an Artin representation.

**Remarks:**

1. One can ask for another subtler characterisation of integral strictly compatible systems \( \{ \rho_\wp \} \) that arise from an Artin representation as those that are unramified outside a fixed finite set of places that is independent of \( \wp \).

2. Compatible systems arising from Artin representations were used to prove that Serre’s conjectures in [S1] imply the modularity of 2-dimensional irreducible odd complex representations of \( G_\mathbb{Q} \) in [K1].
3 A result of Corrales and Schoof

We need the following straightforward generalisation of Theorem 1 of [CS].

**Proposition 3** Let \( a_1, \ldots, a_n, c \) be a finite set of elements of \( K^* \) and \( \ell \) a rational prime. If for almost all primes \( \varphi \) of \( K \) the subgroup generated by the image of \( a_i \)'s in \((O_k/\varphi)^*\) contains \( c^{\varphi} \) for some integer \( t_\varphi \) that depends on \( \varphi \) and is prime to \( \ell \), then \( c = \prod_{i=1}^{n} a_i^{m_i} \) for some integers \( m_i, i = 1, \ldots, n \). If further \( t_\varphi \) can be taken to be 1 for almost all \( \varphi \), then \( c = \prod_{i=1}^{n} a_i^{m_i} \) for some integers \( m_i, i = 1, \ldots, n \).

**Proof:** Without loss of generality we can and will assume \( \sqrt{-1} \in K \). The proof is basically the same as that of Theorem 1 of [CS]: we will briefly sketch the proof using the same breakdown into steps as in [CS].

- **Step 1:** Let \( q \) be a power of \( \ell \). Consider \( K(\zeta_q, 1^{1/q}, \ldots, a_n^{1/q}) \) and \( K(\zeta_q, c^{1/q}) \). Observe that for all but finitely many exceptions a prime \( \varphi \) of \( K \) that lies above a prime of \( \mathbb{Q} \) that splits completely in \( K/\mathbb{Q} \) splits in \( K(\zeta_q, c^{1/q}) \) (resp., \( K(\zeta_q, a_1^{1/q}, \ldots, a_n^{1/q}) \)) if and only if \( p \) (the characteristic of the residue field at \( \varphi \)) is 1 mod \( q \), and \( c \), or equivalently \( c^{\varphi} \) as \( t_\varphi \) is prime to \( \ell \), is a \( q \)th power in \( \mathbb{F}_p^* \) (resp., \( a_1, \ldots, a_n \) are \( q \)th powers in \( \mathbb{F}_p^* \)). Thus by assumptions of the proposition almost all split primes of \( K \) that split in \( K(\zeta_q, c^{1/q}) \) also split in \( K(\zeta_q, c^{\varphi/q}) = K(\zeta_q, c^{1/q}) \). Thus by the Frobenius density theorem we conclude that \( K(\zeta_q, c^{1/q}) \subset K(\zeta_q, a_1^{1/q}, \ldots, a_n^{1/q}) \).

- **Step 2:** Using Kummer theory and duality for finite abelian groups we deduce from Step 1 that the subgroup generated by the images of \( a_i \)'s in \( K(\zeta_q)^*/K(\zeta_q)^{q} \) contains the image of \( c \). From Lemma 2.1 (ii) of [CR] (see also remark on page 39 of [C]), we have that the natural map \( K^*/K^{q} \rightarrow K(\zeta_q)^*/K(\zeta_q)^{q} \) is injective, and thus we deduce that the subgroup generated by the images of \( a_i \)'s in \( K^*/K^{q} \) contains the image of \( c \).

- **Step 3:** Consider \( A := \mathcal{O}_{K,T}^*/\langle a_1, \ldots, a_n \rangle \) where \( \mathcal{O}_{K,T}^* \) are the \( T' \)-units with \( T' \) consisting of all places dividing any of the \( a_i \)'s or \( c \), the infinite places and the places where the hypotheses of the statement of the proposition are not satisfied. We know from Step 2 that \( c \) is in \( A^q \) for all \( q \)'s that are powers of \( \ell \). Using the fact that \( \mathcal{O}_{K,T}^* \) is finitely
generated (Dirichlet unit theorem), we conclude that the image of \( c \) in \( A \) is torsion of order prime to \( \ell \) and thus we are done.

The last line of the proposition follows from the fact that assuming the \( t_v \)'s to be 1 we can work with all prime powers \( q \).

4 Reciprocity for one-dimensional compatible systems

4.1 Generalities about Hecke characters

We recall the definition of Hecke characters and the association of compatible systems of \( p \)-adic representations to them (see Chapter II of [S], or [W] and [Tani]). There are equivalent ways of looking at Hecke characters that we will need below and we briefly recall these. We index as usual the real (resp., complex) places of \( K \) by embeddings \( \sigma \) (resp., pairs of elements \( \{ \sigma, c\sigma \} \) where \( c \) is complex conjugation) of \( K \) in \( \mathbb{C} \). Let \( I \) be the group of ideles of \( K \) and \((K_\infty^\times)^0\) be the connected component of the identity of the product of the completions of \( K \) at the archimedean places.

Definition 3 A Hecke character \( \chi \) is a continuous homomorphism \( \chi : I/K^* \to \mathbb{C}^* \) such that

\[
\chi|_{(K_\infty^\times)^0}(x) = \prod_{\sigma \text{ real}} x_{\sigma}^{n_{\sigma}} \prod_{\sigma \text{ complex}} x_{\sigma}^{n_{\sigma}} \overline{x_{\sigma}^{n_{c\sigma}}}
\]

for integers \( n_{\sigma}, n_{c\sigma} \) and with \( x_{\sigma} \) the components of \( x \). We say that the tuple of integers \( (n_{\sigma})_\sigma \) is the infinity type of \( \chi \), and say that \( \chi \) is unramified at a finite place \( v \) if the units \( U_v \) at \( v \) are in the kernel of \( \chi \). The conductor of \( \chi \) is the largest ideal \( n \) such that elements of the finite ideles \( I^{(\infty)} \) congruent to 1 mod \( n \) are in the kernel of \( \chi \).

From \( \chi \) we get a continuous homomorphism \( \chi_0 : I/(K_\infty^\times)^0 \to \mathbb{C}^* \) defined by \( \chi_0(x) = \chi(x)\prod_{\sigma \text{ real}} x_{\sigma}^{-n_{\sigma}} \prod_{\sigma \text{ complex}} x_{\sigma}^{-n_{\sigma}} \overline{x_{\sigma}^{-n_{c\sigma}}} \) whose kernel is open and takes values in a sufficiently large subfield \( L \) of \( \mathbb{C} \) which is a finite extension of \( \mathbb{Q} \).

For any finite place \( \wp \) of \( L \) above a prime \( p \) of \( \mathbb{Q} \) the morphism \( \psi : K^* \to L_\wp^* \), \( \psi(x) = \Pi_{\sigma} \sigma(x)^{n_{\sigma}} \), extends to a continuous morphism \( \psi_p : (K \otimes \mathbb{Q}_p)^* \to L_\wp^* \), and we define \( \chi_\wp : I/K^*(K_\infty^\times)^0 \to L_\wp^* \) by \( \chi_\wp = \chi_0(\psi_p \alpha_p) \) where \( \alpha_p \) is the projection of \( I \) to the components at places above \( p \). The kernel of \( \chi_\wp \)
is open in $I^{(p)}$ the ideles concentrated at places away from those dividing $p$. Using the isomorphism of class field $G_K^{ab} \simeq I/K^*(K^{\times})^0$, we see that $\{\chi_\wp\}$ forms a compatible system of 1-dimensional $\wp$-adic representations of $G_K$ in a natural way. Since $G_K$ is compact, $\chi_\wp$ takes values in the units $O_\wp^*$ of $L_\wp^*$ and thus can be reduced mod $\wp$.

**Definition 4** We say that a strictly compatible system of (one-dimensional) mod $\wp$ representations $(\rho_\wp)$ as in Definition 4 arises from a Hecke character $\chi$ if $\rho_\wp = \tilde{\chi}_\wp$ where $\tilde{\chi}_\wp$ is the reduction of $\chi_\wp$ mod $\wp$ at all primes $\wp$ not in $T$.

For a fractional ideal $n$ of $O_K$, which we identify with a sequence of integers $(m_v)$ for $v$ running through finite places of $K$ and $m_v = 0$ for almost all $v$, define the subgroup $U_n$ of the ideles $I$ of $K$ to be the product $U_{v,n}$ where $U_{v,n}$ is the connected component of $K_v^*$ if $v$ is an infinite place, and the units $U_v$ congruent to 1 modulo the $m_v$ th power of the maximal ideal if $v$ is a finite place. Thus $K^* \cap U_n$ are the totally positive units $E_n$ of $O_K$ congruent to 1 mod $n$. Let $I_n$ be the quotient $I/U_n$. We then have an exact sequence $1 \to K^*/E_n \to I_n \to C_n \to 1$ (see loc. cit.) with $C_n$ finite and defined by means of this exact sequence. We can consider the projective system of the $C_{n^r}$’s as $r$ varies and define $C_{n^\infty}$ to be the projective limit. The character $\chi_\wp$ above maybe regarded naturally as a character of $C_{n^\infty}$ where $n$ is the conductor of $\chi$.

Let $\chi$ be a Hecke character. Since the kernel of the associated homomorphism $\chi_0$ is open, there is a fractional ideal $a$, a number field $L$ and a character $I_a \to L^*$ induced by $\chi$. Viewing $K^*$ as an algebraic torus, the pull back to $K^*$ is algebraic and its kernel contains $E$. Conversely, a character $I_a \to L^*$ whose pull-back to $K^*$ is algebraic automatically has $E$ in its kernel and gives rise to a Hecke character. Just reverse the procedure of going from $\chi$ to $\chi_0$ above.

It will be convenient below to switch between ideal-theoretic and idele-theoretic viewpoints. The strict ray class group $\text{Cl}_n$ sits inside the exact sequence $1 \to P_n \to \text{Id}_n \to \text{Cl}_n \to 1$ where $\text{Id}_n$ is the group of fractional ideals prime to $n$ and $P_n$ is the subgroup of principal fractional ideals generated by $\gamma \in K$ with $\gamma$ totally positive and congruent to 1 mod $n$. Note that we have
natural maps between exact sequences

\[
\begin{array}{cccccc}
1 & \longrightarrow & P_n & \longrightarrow & \text{Id}_n & \longrightarrow & \text{Cl}_n & \longrightarrow & 1 \\
& & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
1 & \longrightarrow & K^*/E_n & \longrightarrow & I_n & \longrightarrow & C_n & \longrightarrow & 1
\end{array}
\]  

(1)

that induces an isomorphism from \(\text{Cl}_n\) to \(C_n\).

4.2 Proof of Theorem 1

Let the compatible system \(\{\rho_\wp\}\) as in Theorem 1 have defect set \(T\) and ramification set \(S\).

By Lemma 1 we may assume that \(K\) is a Galois extension of \(\mathbb{Q}\) and is totally complex. Let \(m_\wp\) be the ideal of \(\mathcal{O}_K\) that is the prime to \(p\) part of the Artin conductor of \(\rho_\wp\) for places \(\wp\) of \(L\) not in \(T\). We assume without loss of generality that \(L\) contains \(K\).

\text{A priori} the fixed field of the kernel of \(\rho_\wp\) is contained in the strict ray class field of \(K\) of conductor \(m_\wp\). But as the image of \(\rho_\wp\) has order prime to \(p\), the fixed field of the kernel of \(\rho_\wp\) is in fact contained in the ray class field of conductor \(m_\wp\). By class field theory we regard the data of the strictly compatible system as the giving of homomorphisms \(\rho_\wp: \text{Cl}_{m_\wp} \rightarrow \mathbb{F}_\wp^*\) where \(\text{Cl}_{m_\wp}\) is the strict ray class group of \(K\) of conductor \(m_\wp\), and now using the Artin map we see that the strict compatibility of the \(\rho_\wp\)’s is expressed in terms of the images of prime ideals in \(\text{Cl}_{m_\wp}\) under \(\rho_\wp\). We will for the most part only consider the restriction of \(\rho_\wp: \text{Cl}_{m_\wp} \rightarrow \mathbb{F}_\wp^*\) to the subgroup \(P_{m_\wp}/P_{m_\wp}p\) generated by the principal ideals \(P_{m_\wp}p\) prime to \(m_\wp\). We can also inflate this restriction and regard it as a homomorphism of \((\mathcal{O}_K/m_\wp p\mathcal{O}_K)^*\) which factors through the quotient by the image of the global units \(E\).

Consider any integer \(r\) of \(K\) that generates a prime ideal that lies above a prime ideal of \(\mathbb{Q}\) which splits completely in \(K\) and is unramified in \(L\). Abusing notation we will write \(\rho_\wp(r)\) for \(\rho_\wp((r))\) especially when we are thinking of the homomorphism of \((\mathcal{O}_K/m_\wp p\mathcal{O}_K)^*\) induced by \(\rho_\wp\). We see by the axioms of strictly compatible systems that there is an element \(f_r \in L^*\) such that \(\rho_\wp(r)\) is the reduction mod \(\wp\) of \(f_r\) that depends on \(r\) but is independent of \(\wp\), and is \(-f_r(0)\) using the notation of Definition 1. These considerations apply for all \(r\) prime to \(S\) and for all primes \(\wp\) not in \(S \cup T\) and such that \(p\), the residue characteristic of \(\wp\), does not lie below \((r)\), and is prime to \(f_r\).
Choose a prime $\ell$ of $\mathbb{Q}$ that is coprime to the orders of the multiplicative groups of the residue fields of primes in $S$ and is prime to the residue characteristics of the prime in $S$. We claim that for almost all primes $\varphi$ of $L$ (in particular we exclude the places in $T$ and the places which are ramified in $L$) the subgroup generated by the $\{\sigma(r)\}$'s in $(\mathcal{O}_L/\varphi\mathcal{O}_L)^*$ contains the subgroup generated by $f_r^t\varphi$ (for some integer $t\varphi$ prime to $\ell$) in $(\mathcal{O}_L/\varphi\mathcal{O}_L)^*$, and where $\sigma$ runs through the distinct embeddings of $K$ in $L$. The claim follows from considering the homomorphism of $(\mathcal{O}_K/m_p\mathcal{O}_K)^* (\simeq (\mathcal{O}_K/m_p\mathcal{O}_K)^* \times (\mathcal{O}_K/p\mathcal{O}_K)^*)$ induced by $\rho_\varphi$. We are using here implicitly the fact that the map $\mathcal{O}_K/\varphi' \to \mathcal{O}_L/\varphi$ is injective for any prime $\varphi'$ of $K$ below $\varphi$. To prove the claim note that as $\rho_\varphi(r)$ is the reduction of $f_r$ mod $\varphi$ and $m_p$ is only divisible by primes in $S$, the order of $f_r$ in $\mathbb{F}_p^*$, up to products of powers of primes which divide the orders of the multiplicative groups of the residue fields of primes in $S$ and the residue characteristics of the primes in $S$, divides the order of $r \in (\mathcal{O}_K/p\mathcal{O}_K)^*$. The latter, by transport of structure, is easily seen to be the l.c.m. of the orders of the $\{\sigma(r)\}_{\sigma \in \text{Gal}(K/Q)}$'s in $\mathbb{F}_p^*$.

We conclude from the claim and Proposition 3 that $f_r^{t(r)} = \Pi_\sigma \sigma(r)^{m_{r,\sigma}}$ for integers $t(r), m_{r,\sigma}$ that a priori depend on $r$, where $\sigma$ runs through the distinct embeddings of $K$ in $L$. Now as $r$ lies above a prime that is unramified in $L$, the $t(r)$'s divide all the $m_{r,\sigma}$'s. From this we conclude that we can write $f_r = \zeta_r \Pi_\sigma \sigma(r)^{m_{r,\sigma}}$ for integers $m_{r,\sigma}$, where $\sigma$ runs through the distinct embeddings of $K$ in $L$, and for some root of unity $\zeta_r$ that because of our assumption that $L$ contains $K$ that is a Galois extension of $\mathbb{Q}$, and as $L$ contains only finitely many roots of unity, has order bounded independently of $r$, and thus divides a fixed integer say $n$.

Now consider two integers $r$ and $r'$ as above prime to $S$, that generate prime ideals that lie above distinct primes ideals of $\mathbb{Q}$ that split completely in $K$ and are unramified in $L$. Then we see by as before that by considering the homomorphisms of $(\mathcal{O}_K/m_p\mathcal{O}_K)^* \to \mathbb{F}_p^*$ induced by $\{\rho_\varphi\}$, and denoted by abuse of notation by the same symbol, that $\rho_\varphi(rr')$ is the reduction mod $\varphi$ of $f_r f_{r'}$. Thus again as before we see that $(f_r f_{r'}) = \Pi_\sigma \sigma((rr')^{m_{r,\sigma}}$ for some integers $m_{r,\sigma}$. Thus we have an equality of ideals

$$\Pi_\sigma \sigma((r))^{m_{r,\sigma}} \Pi_\sigma \sigma((r')^{m_{r',\sigma}} = \Pi_\sigma \sigma((rr')^{m_{r,\sigma} m_{r',\sigma}}$$

where $\sigma$ runs through the distinct embeddings of $K$ in $L$. As $(r), (r')$ are split prime ideals in $K$ lying above distinct primes of $\mathbb{Q}$, we conclude that in fact
the $m_{r,\sigma} = m_{r',\sigma}$. For conjugates $\sigma(r)$ of $r$ apply the argument above now to $\sigma(r)r'$ and thus we conclude that the $m_{r,\sigma}$’s are independent of $r$ and depend only on $\sigma$. We denote the common value by $m_\sigma$. Note that at this point we have proved $f_r$ is integral at all places of $L$ not lying above the prime of $Q$ below $r$.

Recalling that the $\rho_\varphi$’s induce homomorphisms $(\mathcal{O}_K/m_\varphi p\mathcal{O}_K)^* \rightarrow \mathbf{F}_p^*$ which factor through the quotient by the image of the global units $E$, we see that the algebraic character $\chi : K^* \rightarrow K^*$, $\chi(x) = \Pi \sigma(x)^{m_\sigma}$ contains the units $E$ of $\mathcal{O}_K$ in its kernel. Then using the facts recalled in Section 4.1, and especially equation (1), the fact that follows from the Cebotarev density theorem that the images of the split principal prime ideals coprime to $m_\varphi$ generate $P^{m_\varphi}/P_{m_\varphi}$, and the axioms of strictly compatible systems, we see that for some Hecke character $\chi'$ of infinity type $(nm_\sigma)$, $\rho_\varphi^n \otimes \tilde{\chi}_\varphi^{-1}$ factors through the Galois group of a fixed finite extension of $K$: in fact this fixed finite extension can be taken to be the Hilbert class field of $K$. Consequently by Lemma 1 the strictly compatible system $\{\rho_\varphi^n\}$ arises from a Hecke character of infinity type $(nm_\sigma)$.

Hence by inspection as $\{\chi_\varphi^n\}$ has trivial prime to $p$ conductor, we deduce that the compatible system $\{\rho_\varphi\}$ is such that the exponents of the primes dividing the prime to $p$ part of the conductor of $\rho_\varphi$ are bounded independently of $\varphi$ and thus as the exceptional set $S$ is finite the strictly compatible system $\{\rho_\varphi\}$ has bounded conductor in the sense of Definition 1.

Thus we can take the $m_\varphi$’s to be independent of $\varphi$, and denote the common ideal by $m$. Now essentially we have to repeat the argument above. Consider principal prime ideals $(r)$ with $r$ congruent to 1 mod $m$ that lie above primes that split completely in $K$, and repeat the argument above to get that this time $f_r$, which we know a priori is in $K$ using the Frobenius density theorem, is $= \Pi \sigma(r)^{m_\sigma}$ for the same $m_\sigma$’s as above, using the last sentence of Proposition 3. Now observe that the algebraic character $K^* \rightarrow K^*$ that sends $x$ to $\Pi \sigma(x)^{m_\sigma}$ is trivial on units congruent to 1 mod $m$. Then using the facts recalled in Section 4.1, and that such $r$’s, prime to $p$, project surjectively to $(\mathcal{O}_K/p\mathcal{O}_K)^*$ for almost all $p$, we deduce by the axioms of strictly compatible systems, that there is a Hecke character $\chi$ of infinity type $(m_\sigma)$, such that $\rho_\varphi \otimes \tilde{\chi}_\varphi^{-1}$ factors through the Galois group of a fixed finite extension of $K$. The proof of the theorem is now complete by appealing to Lemma 1.
4.3 Some remarks

1. The proof of Theorem [K] follows the general lines of the method of [K] that dealt with the case $K = \mathbb{Q}$. The presentation of the proof in [K] is inaccurate when $S$ is non-empty, as the second line of the proof is unjustified (we thank N. Fakhruddin for pointing this out): nevertheless the proof of loc. cit. can be modified without much difficulty to work in the general case considered there of compatible systems of bounded conductor. As compared to [K], the substantive improvements made in this paper as far as the results about one-dimensional systems are concerned, are that we no longer assume $K = \mathbb{Q}$, we no longer have a bounded conductor hypothesis, and the the proof is simplified to the extent that we no longer appeal to results towards Artin’s conjecture on primitive roots, albeit we will need to use results towards Artin’s conjecture in the next section when studying abelian semisimple compatible systems.

2. In [S2] a similar theorem was proved assuming that “inertial weights” (see Section 1.7 of [S3]) of the $\rho_\wp$’s were bounded independently of $\wp$. In our proof the fact that inertial weights are bounded is proved to be a consequence of the defining properties of a compatible system.

3. The proof works even if we allow the set $T$ in Definition [II] to be a set of places of density 0.

4. The proof also gives that one-dimensional weakly compatible mod $\wp$ systems are (weakly) equivalent in a natural sense to strictly compatible systems that arise from a uniquely determined Hecke character.

5. Although our conjectures seem inaccessible at the moment in the higher dimensional situation, it will be of interest to prove some more accessible “independence of $p$” results for the images of $\rho_\wp(G_K)$ analogous to the case of compatible $\wp$-adic systems studied in [LP]. For instance one might expect that for 2-dimensional strictly compatible systems either for a set of primes $\wp$ of density 1, $\text{im}(\rho_\wp)$ has an abelian subgroup of bounded index, or for a set of primes $\wp$ of density 1, $\text{im}(\rho_\wp)$ contains $SL_2(\mathbb{F}_p)$. This will be a necessary step in studying the adelic images of compatible systems of $p$-adic representations in the abstract case studied in [LP].
5 Reciprocity for abelian semisimple compatible systems

As a first step in addressing the conjectures of the introduction for dimensions greater than 1, it is of interest to generalise Theorem 1 to describe abelian semisimple compatible mod $\wp$ systems of $G_K$ of arbitrary dimensions. Such systems which are integral can be easily classified by using results in the 1-dimensional case. We do not have satisfactory answers for rational abelian compatible systems. The problem in generalising the proof of Theorem 1 given above is that we cannot use the argument given there to conclude that the $m_{i,\sigma}$’s are independent of $r$. By using results towards Artin’s conjecture the case when $K = \mathbb{Q}$ or more generally when $K$ is an abelian extension of $\mathbb{Q}$ can be treated. Below we state what can be proved and only briefly indicate the arguments highlighting the novel features which arise in the case of higher dimensional abelian systems.

5.1 Integral compatible systems

We begin by indicating how a dévissage argument reduces the understanding of $L$-integral abelian compatible systems to understanding the 1-dimensional case.

**Theorem 2** Let $\{\rho_\wp\}$ be a $n$-dimensional compatible abelian $L$-integral semisimple of $G_K$ of bounded conductor. Then it arises from a direct sum of Hecke characters.

**Sketch of proof:** For split principal prime ideals $(r)$ of $K$ with $r$ congruent to 1 mod $m$ (m is divisible by the prime to $p$ part of the conductor of $\rho_\wp$) we get as before that the roots of $f_\wp(X)$ are of the form $\Pi_\sigma \sigma(r)^{m_{i,\sigma}}$, $i = 1, \ldots, n$, with the exponents non-negative by the integrality hypothesis.

Looking at the compatible system of representations of $G_K$ given by $\{\det(\rho_\wp)\}$ we get a compatible system of one-dimensional representations of $G_K$ of bounded conductor that by Theorem 1 arises from a Hecke character. From this it is easy to conclude that $\sum_i \sum_\sigma m_{i,\sigma}$ is bounded independently of $r$ and using the non-negativity of the exponents $m_{i,\sigma}$ we get that the “infinity types” $m_{i,\sigma}$’s are bounded independently of $r$. Thus there are only finitely many possibilities for the $m_{i,\sigma}$’s as $i, r, \sigma$ vary and let $N$ be the sum of all these finitely many possibilities. Let $(\alpha)$ be a split prime ideal of $K$ and
choose a prime \( p \) large enough such that whenever \( \prod_{\sigma} \sigma(\alpha)^{m_\sigma} - 1 \) is not coprime to \( p \) and \( |m_\sigma| \leq N \) then all the \( m_\sigma \)'s are 0. Consider integral elements \( \beta \) in \( K \) such that \( \beta \) is congruent to \( \alpha \) mod \( p \), and congruent to 1 mod \( m \), and \( \beta \) generates a split prime ideal. We claim that the roots of \( f_\alpha(X) \) and \( f_\beta(X) \) have the same “infinity type”. This is because the roots of these polynomials, which are of the form \( \prod_{\sigma} \sigma(\alpha)^{m_i,\alpha,\sigma} \) and \( \prod_{\sigma} \sigma(\beta)^{m_i,\beta,\sigma} \) with the latter congruent to \( \prod_{\sigma} \sigma(\alpha)^{m_i,\beta,\sigma} \) mod \( p \) by choice of \( \beta \), are congruent mod \( \wp \) under some ordering, for \( \wp \) a prime above \( p \). From this and the fact that \( p \) was chosen so that, whenever \( \prod_{\sigma} \sigma(\alpha)^{m_\sigma} - 1 \) is not coprime to \( p \) and \( |m_\sigma| \leq N \), then all the \( m_\sigma \)'s are 0, the claim follows. Such elements \( \beta \) surject onto \((O_K/p'O_K)^*\) for almost all primes \( p' \) of \( \mathbb{Z} \). From this we conclude that \( \{\rho_\wp\} \) arises from a direct sum of Hecke characters of infinity types that can be read off from \( f_\alpha(X) \).

**Remark:** The proof extends to the case when we do not assume bounded conductor hypothesis on remarking that ramification indices of primes in \( \mathbb{Q}_M \), the union of all extensions of \( \mathbb{Q} \) of a fixed degree say \( M \), are finite and the number of roots of unity in \( \mathbb{Q}_M \) is finite.

### 5.2 \( K = \mathbb{Q} \)

**Theorem 3** Let \( \{\rho_\wp\} \) be a compatible abelian \( \mathbb{L} \)-rational semisimple system of \( G_K \) with finite defect and exceptional sets and of bounded conductor. Then it arises from a direct sum of Hecke characters.

**Sketch of proof:** Using known results towards Artin’s conjecture (cf., [M] and [HB]) we find a prime \( q \) that is 1 mod \( m \) (with \( m \) as in the preceding paragraph) a primitive root mod \( p \) for infinitely many primes \( p \). By using Proposition 3 as before we conclude that the roots of the polynomial \( f_q(X) \), that is part of the defining data of the compatible system \( \{\rho_\wp\} \), are \( \{q^{m_1}, \ldots, q^{m_n}\} \) for some integers \( m_i \). Then we see by a pigeonhole argument, using the infinitely many primes \( p \) for which \( q \) is a primitive root, that there are Dirichlet characters \( \varepsilon_i \) of conductor dividing \( m \) so that there are infinitely many primes \( \wp \) such that \( \rho_\wp \) is the direct sum \( \bigoplus_{i=1}^n \varepsilon_i,\wp \chi_p^{m_i} \) with \( \chi_p \) the \( p \)-adic cyclotomic character. This proves the theorem.

**Remark:** Using that the closure of the subgroup generated by \( q \) in \( \prod_{\ell \in S} \mathbb{Z}_\ell^* \) has finite index, it is easy to remove the bounded conductor hypothesis.
## 5.3 A version of Artin’s conjecture and abelian extensions \( K \)

The case of rational abelian compatible systems of \( G_K \) with \( K \) a general number field is one we are unable to resolve satisfactorily. In this section we assume that \( K \) is abelian over \( \mathbb{Q} \) and indicate an approach.

We begin by formulating an Artin-type conjecture on primitive roots that may be of independent interest: we formulate a very weak version that suffices for the purposes here.

**Conjecture 1** Let \( K \) be a finite Galois extension of \( \mathbb{Q} \) and \( m \) any ideal of \( \mathcal{O}_K \). Then there is a totally positive integer \( a \in \mathcal{O}_K \), \( \equiv 1 \mod m \), which generates a split prime ideal of \( \mathcal{O}_K \), and infinitely many primes \( p \) of \( \mathbb{Q} \) such that the conjugates of \( a \), \( \{ \sigma(a) \}_{\sigma \in \text{Gal}(K/\mathbb{Q})} \), generate a subgroup of \( (\mathcal{O}_K/p\mathcal{O}_K)^*/\mathcal{O}_K^* \) whose index is bounded independently of \( p \).

**Theorem 4** Assume Conjecture 1 and \( K \) is an abelian extension of \( \mathbb{Q} \). A \( K \)-rational compatible system \( \{ \rho_\wp \} \) of abelian semisimple mod \( \wp \) representations of \( \text{Gal}(\overline{K}/K) \) with bounded conductor arises from a direct sum of Hecke characters.

**Sketch of proof:** Assuming the conjecture, we choose a split principal, prime ideal \( (a) \) that is congruent to 1 mod \( m \) where \( m \) is an ideal invariant under \( \text{Gal}(K/\mathbb{Q}) \) and which is divisible by the prime to \( \wp \) part of the conductor of \( \rho_\wp \) for almost all \( \wp \), such that such that for infinitely many primes \( p \) of \( \mathbb{Q} \), the elements \( \{ \sigma(a) \}_{\sigma \in \text{Gal}(K/\mathbb{Q})} \) generate a subgroup of \( (\mathcal{O}_K/p\mathcal{O}_K)^*/\mathcal{O}_K^* \) whose index is bounded independently of \( p \). Arguing as before we know that the roots of \( f_a(X) \) are of the form \( \prod_{\sigma \in \text{Gal}(K/\mathbb{Q})} \sigma(a)^{m_{i,a,\sigma}} \). Further we know that for each \( i \), the algebraic character of \( K^* \) of infinity type \( (m_{i,a,\sigma}) \) kills a subgroup of finite index of the units \( \mathcal{O}_K^* \).

After having noticed this, the main point of interest of the proof of this theorem is the following proposition which we state for general Galois extensions \( K \) of \( \mathbb{Q} \). The point of it is to “algebraise” an automorphism \( \sigma \) of \( K \) by interpreting it as the Frobenius map mod infinitely many primes using Cebotarev density theorem.

**Proposition 4** Let \( \tau \) be in the centre of \( \text{Gal}(K/\mathbb{Q}) \) with \( K \) a finite Galois extension of \( \mathbb{Q} \). For an element \( a \) of \( \mathcal{O}_K \) as above, for the positive density of primes \( p \) of \( \mathbb{Q} \) such that \( \text{Frob}_p \in \text{Gal}(K/\mathbb{Q}) \) is \( \tau \) for (all) \( \wp \) above \( p \) we...
have that $\rho_\varphi(\text{Frob}_\tau(a)) = \tau(\rho_\varphi(\text{Frob}_a))$. Thus in particular the characteristic polynomial of $\rho_\varphi(\tau(a))$ is $\tau(f_a(X))$, for any $\tau$ in the centre of $\text{Gal}(K/Q)$ and for all $\varphi$ not in $S \cup T$.

**Proof:** For the primes $p$ in the statement, $\tau$ naturally acts on the domain and range of $\rho_p$. The proposition follows from

- $\tau$ maps the image of $a$ in $\text{Cl}_{mp}$ to its $p$th power as $a$ is congruent to 1 mod $m$, and $\tau$ induces the Frobenius map on the residue fields of all primes of $K$ above $p$.
- $\tau$ induces the $p$th power map on $F_p$.

After this the proof of the theorem follows well-rehearsed lines. We use the property of $a$ that it satisfies the conjecture above and the proposition to conclude that the compatible system arises from a Hecke character whose infinity type can be read off from the roots of $f_a(X)$. Namely we first “twist” $\{\rho_\varphi\}$ by the compatible system that arises from the direct sum of the Hecke characters of infinity types that are determined by the roots of $f_a(X)$. Here to make sense of “twisting” we are using that $\{\rho_\varphi\}$ is abelian and we “order” the direct sum of Hecke characters when twisting to match with the ordering of the roots of $f_a(X)$. This twisted system has the property that the characteristic polynomials attached to $\sigma(a)$ are powers of $X - 1$. This uses the fact that for abelian $K$ the algebraic characters of $K^*$ are $\text{Gal}(K/Q)$-equivariant. Now as $a$ was chosen to satisfy the conjecture above we are done by the usual argument that now we have a abelian semisimple system $\{\rho'_\varphi\}$ such that $\rho'_\varphi$ factors through the Galois group of a fixed extension of $K$ for infinitely many $\varphi$, namely those $\varphi$ for which $\langle \sigma(a) \rangle$ generates a subgroup of bounded index of $(\mathcal{O}_K/p\mathcal{O}_K)^*/\mathcal{O}_K^*$.

**Remark:** In this case we do not know how to remove the bounded conductor hypothesis.

6 Acknowledgements

I thank Najmuddin Fakhruddin for his interest, and for the many interesting conversations we’ve had about compatible systems. I thank the referee for useful suggestions to improve the paper.
7 References

[C] Chevalley, C., *Deux théorèmes d’arithmétique*, J. Math. Soc. Japan (1951), 36–44.

[Ca] Carayol, H., *Formes modulaires et représentations galoisiennes avec valeurs dans un anneau local complet*, in *p-adic monodromy and the Birch and Swinnerton-Dyer conjecture*, Contemp. Math. 165 (1994), 213–237.

[Clo] Clozel, L., *Sur la théorie de Wiles et le changement de base nonabélien*, IMRN (1995), no. 9, 437–444.

[CS] Corrales, C., and Schoof, R., *The support problem and its elliptic analogue*, J. of Number Theory 64 (1997), 276–290.

[DS] Deligne, P., Serre, J-P., *Formes modulaires de poids 1*, Annales de l’Ecole Normale Superieure 7 (1974), 507–530.

[FM] Fontaine, J-M., Mazur, B., *Geometric Galois representations*, Elliptic curves, modular forms, and Fermat’s last theorem (Hong Kong, 1993), 41–78, Ser. Number Theory, I, Internat. Press, Cambridge, MA, 1995.

[H] Henniart, G., *Représentations ℓ-adiques abéliennes*, in Séminaire de Théorie des Nombres, Progress in Math. 22 (1982), 107–126, Birkhauser.

[HB] Heath-Brown, D. R., *Artin’s conjecture for primitive roots*, Quart. J. Math. Oxford 37 (1986), no. 145, 27–38.

[K] Khare, C., *Compatible systems of mod $p$ Galois representations*, C. R. Acad. Sci., Paris (1996), t. 323, Série I, 117–120.

[K1] Khare, C., *Remarks on mod $p$ forms of weight one*, International Mathematical Research Notices, vol. 3 (1997), 127–133. (Corrigendum: IMRN 1999, no. 18, pg. 1029.)

[KK] Khare, C., Kiming, I., *Mod $pq$ Galois representations and Serre’s conjecture*, to appear in Journal of Number Theory, preprint available at [http://www.math.utah.edu/~shekhar/papers.html](http://www.math.utah.edu/~shekhar/papers.html)

[LP] Larsen, M., Pink, R., *On ℓ-independence of algebraic monodromy groups in compatible systems of representations*, Invent. Math. 107 (1992), 603–636.

[M] Murty, Ram, *Artin’s conjecture for primitive roots*, Math. Intelligencer,
10 (1988) 59-67.

[S] Serre, J-P., *Abelian ℓ-adic representations and elliptic curves*, Addison-Wesley, 1989.

[S1] Serre, J-P., *Sur les représentations modulaires de degré 2 de Gal(\overline{Q}/Q)*, Duke Math. J. 54 (1987), 179–230.

[S2] Serre, J-P., *Résumés des cours de 1970-71, Oeuvres*, Vol. II, no. 93.

[S3] Serre, J-P., *Propriétés galoisiennes des points d’ordre finies courbes elliptiques*, Invent. Math. 15 (1972), 259–331.

[Sch] Schappacher, N., *Periods of Hecke characters*, SLNM 1301.

[Tani] Taniyama, Y., *L-functions of number fields and zeta functions of abelian varieties*, J. Math. Soc. Japan 9 (1957), 330–366.

[W] Weil, A., *On a certain type of characters of the idele-class group of an algebraic number field*, Collected Papers, volume 2, 255–261.

School of Mathematics, TIFR, Homi Bhabha Road, Mumbai 400 005, INDIA. e-mail: shekhar@math.tifr.res.in
Dept of Math, University of Utah, 155 S 1400 E, Salt Lake City, UT 84112, USA. e-mail: shekhar@math.utah.edu