Why Disks Shine: the Transport of Angular Momentum in Hot, Thin Disks

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ABSTRACT

I review recent work on the radial transport of angular momentum in ionized, Keplerian accretion disks. Proposed mechanisms include hydrodynamic and MHD local instabilities and long range effects mediated by wave transport. The most promising models incorporate the Velikhov-Chandrasekhar instability, caused by an instability of the magnetic field embedded in a differentially rotating disk. This has the important feature that the induced turbulent motions necessarily transport angular momentum outward. By contrast, convective modes may transport angular momentum in either direction. Combining the magnetic field instability with an $\alpha-\Omega$ dynamo driven by internal waves leads to a model in which the dimensionless viscosity scales as $(H/r)^{4/3}$. However, this model has a phenomenology which is quite different from the $\alpha$ disk model. For example, an active disk implies some source of excitation for the internal waves. In binary systems with a mass ratio of order unity the most likely exciting mechanism is a parametric instability due to tidal forces. This implies that in systems where the accretion stream is intermittent, like MV Lyrae or TT Ari, epochs when the mass flow is absent or very small will be epochs in which the disk shrinks and becomes relatively inactive and dark. This model also implies that forced vertical mixing is important, even in convectively stable disks. I discuss various observational tests of this model and the focus of current theoretical work.

Subject headings: accretion, binaries

1. Introduction

One of the outstanding problems in the theory of interacting binary stars, in fact in modern astrophysical theory, is the question of how angular momentum is transported in accretion disks. This cannot be explained by microscopic viscosity, i.e. atoms carrying momentum around. It must be some sort of collective effect. The fundamental physical forces involved are all well understood. Nevertheless, this is a difficult problem for at least three reasons. First, the relevant physics is probably very complicated, defying any straightforward analytic treatment. Second, the relevant computer models are always grossly simplified. No one runs high resolution three dimensional
magnetohydrodynamics simulations with realistic radiative transfer. Third, the observational constraints are hard to apply. Partly because unambiguous theoretical predictions are so hard to come by, and partly because those aspects of an accretion disk that present themselves to an observer are often only tangentially related to the essential physics of these objects. The traditional solution is to invoke a dimensional estimate, i.e.

$$\nu = \alpha c_s H$$  \hspace{1cm} (1)

as suggested by Shakura & Sunyaev (1973). Here $H$ is the disk thickness, $c_s$ is the average sound speed, and $\alpha$ is (repeatedly!) adjusted according to the demands of the observations. Curiosity about the underlying physics, and the apparent plasticity of $\alpha$ drive continued work in this area.

In the work presented here my collaborators and I have employed several simplifying assumptions. First, we assume we are considering geometrically thin disks, i.e. $H \ll r$, where $r$ is the disk radius. With this assumption it becomes possible to use $H/r$ as an ordering parameter. Second, we consider only disks with negligible self-gravity. Both the radial and vertical gravitational forces within the disk are assumed to be dominated by the gravity of the central star, which implies that $\Omega(r) \propto r^{-3/2}$, where $\Omega$ is the orbital frequency as a function of radius, $c_s \sim H\Omega$, and vertical gravity is $-z\Omega^2$. Third, we will only consider disks composed of perfectly conducting gas. This excludes large regions within protostellar disks. Finally, we will neglect all external magnetic fields. Each disk generates its own internal field. Elsewhere we have argued that this internal field dominates the dynamics of the disk except under exceptional circumstances. Each of these conditions need not apply everywhere in a disk. The ideas presented here will apply to those parts of those disks where these conditions are satisfied.

In what follows I will identify the critical processes which are important for angular momentum transport in disks, and briefly discuss why others are not. I will then present the internal wave driven dynamo model for angular momentum transport in accretion disks and discuss some of the implications of this model. Most of the points discussed here are explained at length elsewhere (Vishniac & Diamond 1989, Vishniac, Jin, & Diamond 1990, Vishniac & Diamond 1992, Vishniac & Diamond 1993).

## 2. Perturbation Modes in an Accretion Disk

Our discussion will be based on a consideration of the local modes that might reasonably be found in a magnetized accretion disk. I will assume that the large scale magnetic field is largely azimuthal, due to the strong shearing in the disk. Two families of disk perturbations are closely related to hydrodynamic modes, i.e. sound waves and internal waves (or g-modes and p-modes to stellar structure theorists). The former have the dispersion relation

$$\bar{\omega}^2 \approx c_s^2 k^2,$$  \hspace{1cm} (2)
where \( \bar{\omega} = \omega + m\Omega \) is the frequency measured by an observer moving in a circular orbit with frequency \( \Omega(r) \). The radial propagation of these waves is severely hampered by refraction effects. They tend to bend towards regions of low sound speed, like the disk atmosphere, and steepen into weak shocks which rapidly dissipate. They may contribute to the heating of the disk corona, but are unlikely to play a major role in angular momentum transport.

Internal waves have the dispersion relation

\[
 k_r^2 + \left( \frac{m}{r} \right)^2 \approx k_z^2 \left( \frac{\Omega^2 - \bar{\omega}^2}{\bar{\omega}^2 - N^2} \right),
\]

where \( N^2 = z\Omega^2 \frac{d}{dz} \ln(P^{1/\gamma}/\rho) \) is the vertical buoyancy frequency. Note that \( N^2 = 0 \) at the disk midplane. In general it will become large far from the midplane. This dispersion relation implies that \( N^2(z) < \bar{\omega}^2 < \Omega^2 \). Since \( \bar{\omega} \) is a function of \( r \) for non-axisymmetric waves this implies that internal waves are confined vertically and radially. The vertical trapping concentrates these waves in regions of low \( N^2 \) (like the midplane). One of the radial boundaries is defined by \( \bar{\omega}^2 = \Omega^2 \), where the wave undergoes reflection. At the other boundary, as \( \bar{\omega} \to 0, k_r \to \infty \). The wave slows down without reflecting and its energy becomes concentrated towards the disk midplane and the radial boundary. In practice this implies nonlinear dissipation of the wave energy, a phenomenon referred to as resonant absorption. The existence of an absorbing boundary would seem to make such waves distinctly unpromising as a means of long range transport, but there is one important exception. A slightly non-axisymmetric internal wave, \( |m| = 1 \), with a small \( \omega \) can have a resonant absorption radius far outside the disk and a reflecting boundary at \( r = 0 \). For example, in a binary system such waves could be excited either by the impact of the accretion stream or by tidal forces with a frequency of \( \Omega_b \), which is necessarily less than \( \Omega \) at the outer edge of the disk.

Recently, Goodman (1993) has pointed out that there is a generic parametric instability driven by tidal forces which will produce exactly this kind of wave. Of course, \( m = 0 \) waves can also move within the disk without encountering reflecting or absorbing radial boundaries. However, since these waves are axisymmetric they do not carry angular momentum and cannot twist up magnetic fields (so as to cause dynamo action). Furthermore, a low frequency wave propagating inward will necessarily become increasingly confined to the disk midplane.

It’s worth noting that in addition to their other virtues, \( m = 1 \) internal waves undergo linear amplification as they propagate inward. This comes about because their total wave energy flux is the sum of two parts, i.e.

\[
 F_r = \langle \delta P v_r \rangle + \Omega \mathcal{L}_r, \tag{4}
\]

where \( \mathcal{L} \) is the angular momentum flux. Both \( F_r \) and \( \mathcal{L}_r \) are conserved to linear order. An inward moving wave with \( m = 1 \) will have \( \mathcal{L}_r \) positive but \( \langle \delta P v_r \rangle \) negative so that both increase together as \( r \to 0 \) and \( \Omega \to \infty \). Consequently, \( \langle v_r^2 \rangle \) also increases and the waves grow to a limit imposed by nonlinear processes. The growth rate is just \( V_{\text{group}}/r \sim (H/r)\Omega \), the rate at which the waves travel a significant radial distance.

Convection is closely related to internal waves, in the sense that they appear in Eq. (3) for
\[ N^2 < 0. \] For \( \Gamma_{\text{conv}} \equiv |N| \) small compared to \( \Omega \) typical convective modes have \( k_r \sim (\Omega/\Gamma_{\text{conv}})k_z \) and \( m \sim rk_z \). The angular momentum transport induced by such cells can be as large as an ‘\( \alpha \)’ of \( (\Gamma_{\text{conv}}/\Omega)^3 \) (assuming \( k_z \sim H^{-1} \)). This will not be important unless \( \alpha \) is otherwise quite small or convection is quite strong. Worse, the sign of this effect is not yet known. Ryu & Goodman (1992) have argued that the dominant modes transport angular momentum inward. Lin, Papaloizou & Kley (1993) have argued that modes with transport effects of either sign must occur and that correctly accounting for boundary conditions will inevitably lead to the outward transport of angular momentum. Diamond, Vishniac & Luo (1993) have demonstrated that although modes with both signs of angular momentum transport do occur, they differ in their systematic properties in ways that suggest that the inward transport of angular momentum may be the net effect. The conservative position is probably to demand a fully nonlinear treatment of the problem.

In addition to these hydrodynamic modes, the presence of a large scale azimuthal magnetic field implies the possibility of Alfvén waves and related modes. The most important of these are long wavelength (i.e. \( \lambda > V_A/\Omega \)) instabilities related to radially polarized Alfvén waves. This instability was described in the context of couette flow more than 30 years ago (Velikhov 1959, Chandrasekhar 1961), but its significance for accretion disks has only been recognized recently (Balbus & Hawley 1991). It has usually been described in terms of a vertical magnetic field, but also occurs for an azimuthal field. It appears whenever \( \partial_r \Omega \neq 0 \), which is necessarily true in a thin accretion disk. Moreover, it automatically moves angular momentum so as to minimize the gradient in \( \Omega \) (rather than in \( r^2 \Omega \)). Consequently it is guaranteed to move angular momentum outward in an accretion disk, a necessary part of any successful model. The growth rate is roughly \( mV_A/r \) for small \( m \), but cuts off at a maximum value somewhat less than \( \Omega \) on a scale of \( V_A/\Omega \). On smaller scales it disappears altogether. The instability saturates in a turbulent state with velocities of order \( V_A \) on scales of \( V_A/\Omega \). Smaller \( m \) modes are largely suppressed by the small scale turbulence so that the turbulent diffusion can be approximately modeled by a diffusion coefficient of \( D \sim V_A^2/\Omega \). Similarly, the angular momentum can be modeled by an effective \( \alpha \sim (V_A/c_s)^2 \). Although the driving mechanism for this instability comes from radial motions, shearing stresses and the condition of approximate incompressibility lead to approximately isotropic turbulence. There is now widespread agreement that this modes are critically important for angular momentum transport in accretion disks. However, an important question is how the magnetic field necessary to drive the Velikhov-Chandrasekhhar instability is maintained.

One traditional objection to the presence of significant magnetic fields in accretion disks is that buoyancy will tend to eject any magnetic field on a time scale much shorter than radial infall time. This comes about because magnetic fields supply pressure, but not mass, so that flux tubes tend to be buoyant. In practice the dominant buoyant mode involves vertical ripples which allow the matter tied to the field lines to sink while the magnetic field forms rising bubbles. These bubbles will almost certainly reconnect as they rise, leading to a net loss of flux. This is the Parker instability (Parker 1971, 1979) and while it can be suppressed in some circumstances it is quite difficult to avoid in general. Even if the entropy gradient of the gas leads to a stabilization,
small conductive effects will still allow this process to continue, albeit at a reduced rate. These modes can be thought of as long azimuthal wavelength, vertically polarized Alfvén waves. The critical modes have azimuthal wavelengths comparable to the height of the disk and grow at a rate of $V_A/H$. If this process were effective it would lead to a flux loss rate of approximately $V_A/H$. However, this process is disrupted by the Velikhov-Chandrasekhar instability, which necessarily accompanies the existence of an azimuthal field. This follows from a simple qualitative argument. An instability in an accretion disk with a growth rate $\Gamma$ has a radial wavenumber constrained by the condition that

$$k_r > \frac{\partial \bar{\omega}}{\Gamma} \sim \frac{m\Omega}{\Gamma},$$

(5)

since smaller wavenumbers will get sheared out in less than one e-folding time of the instability. The Parker instability has $m \sim r/H$ so

$$k_r > \frac{\Omega}{V_A}.$$ 

(6)

On such radial scales the Velikhov-Chandrasekhar instability will mix the vertical momentum of rising and falling flux tubes at a rate $\sim \Omega$, i.e. much faster than the growth time of the Parker instability. Of course, the buoyancy of the field lines cannot be entirely suppressed, but the speed with which flux tubes can rise will be reduced by a factor of $\Gamma/\Omega$ so that the magnetic flux loss rate from a disk will be of order $(V_A/c_s)^2\Omega$. This is approximately what one would expect from turbulent diffusion due to the Velikhov-Chandrasekhar instability, the only difference being that the buoyant motions have a preferred direction. In other words, the Parker instability will operate at greatly reduced efficiency if $V_A < c_s$, and the magnetic flux loss rate will be roughly what is expected from turbulent diffusion.

I note in passing that the buoyant rise of an axisymmetric flux tube (cf. Sakimoto & Coroniti 1989) will still involve the turbulent entrainment of the neighboring fluid due to the Velikhov-Chandrasekhar instability so that the magnetic flux loss rate is given by the preceding estimate (Vishniac 1993).

Applying similar considerations to the theory of convection in a magnetized disk, we find that a sufficiently powerful magnetic field will disrupt the convective cells. Assuming that the circulation within the cells will create a locally ordered field whose strength is just sufficient to modify the nature of the convective flow, one finds that the angular momentum transport associated with the Velikhov-Chandrasekhar instability of this field is comparable in magnitude to the angular momentum transport associated with purely hydrodynamic convection. The only qualitative difference is that this contribution is guaranteed to be positive. Evidently a complete theory of convection in cataclysmic variable disks must include some consideration of the magnetic field swept up in the convective motions.

3. The Internal-Wave Driven Dynamo
The idea behind the internal-wave driven dynamo model for angular momentum transport is to combine these perturbative modes with a dynamo model in order to produce a complete model of accretion disk dynamics. The chain of causation in this model runs from the tidal forces that excite $m = 1$ internal waves, through the waves themselves and the processes that determine their amplitude as a function of radius, including the generation of subharmonics by nonlinear wave interactions, through a magnetic dynamo mediated by the full spectrum of internal waves, and culminates in the appearance of magnetic field instabilities which saturate the dynamo action and induce positive angular momentum transport. A flow chart for this model looks like this:

EXCITATION OF INTERNAL WAVES - tidal forces cf. Goodman

↓ (+ Linear Amplification)

↓ (+ Nonlinear Wave Interactions)

SATURATED WAVE SPECTRUM

↓ (+ Shearing)

$\alpha - \Omega$ DYNAMO

↓

GROWTH OF $B_\theta$

↓ (+ VC Instability)

SMALL SCALE TURBULENCE

↓

↓

↓

↓

SATURATION TURBULENT TRUNCATION SUPPRESSION

OF $B_\theta$ TRANSPORT OF INTERNAL OF PARKER WAVE INSTABILITY SPECTRUM
I’ve already alluded to most of the steps in this process. The major gap lies in the discussion of the dynamo itself. Typical proposals for disk dynamos are turbulent $\alpha - \Omega$ dynamos. In this kind of scheme an azimuthal field is acted on by some local, turbulent velocity field so that the unperturbed field line is replaced by a spirally flux tube centered on the original field line position. If a vertical stack of such flux tubes then undergo reconnection a vertical gradient in field strength (or the amplitude of the velocity field) results in the production of radial flux. The differential shearing of the disk results in a growth rate for $B_\theta$ of $-(3/2)\Omega B_r$, thereby closing the loop. The growth rate for $B_r$ is $-\partial_z \alpha_{\theta\theta} B_\theta$ (neglecting dissipative terms), where

$$\alpha_{\theta\theta} = \langle v_z^* \frac{1}{r} \partial \int^t v_r(t')dt' - v_r^* \frac{1}{r} \partial \int^t v_z(t')dt' \rangle$$  \hspace{1cm} (7)$$

Note that an isotropic velocity field will give $\alpha_{\theta\theta} = 0$, even if the disk turbulence is quite strong. Moreover, whatever asymmetry is present must be quite strong since the velocity field will also tend to have a dissipative effect.

In the wave-driven dynamo model the turbulence is replaced by an ensemble of internal waves, consisting of the $m = 1$ waves that are generated in the outer parts of the disk, and the higher order waves generated locally through nonlinear interactions. Waves of infinitesimal amplitude will have $\int^t \tilde{v} dt' = 0$, since they can be represented as the sum of perfectly periodic motions. However, if the waves are balanced between nonlinear dissipation and linear amplification then they will not be periodic, but suffer from a slight loss of coherence due to nonlinear effects. Such an effect is roughly analogous to the collisional broadening of spectral lines. In this case the decoherence rate is equal to the linear amplification rate, $(H/r)\Omega$. Consequently the time integral becomes $\int^t \tilde{v} dt' \approx \tilde{v}(\bar{\omega}^2 \tau_{\text{decoherence}})^{-1}$. The necessary asymmetry in the velocity field is supplied by the dominance of ingoing waves. This dominance is guaranteed by the fact that the ingoing waves amplify and the excitation region for the waves is near the outer edge of the disk. In fact, even in the outer parts of the disk there will be such an asymmetry since the growth rate for the tidal instability is strong function of radius (Goodman 1993). One other distinctive feature of this kind of dynamo is that the waves that drive it have long wavelength symmetries. Unlike typical turbulent dynamos the wave-driven dynamo involves motions with vertical and radial wavelengths comparable to the disk thickness, and an azimuthal wavelength of $2\pi r$. The emergence of large scale structure in the magnetic field is less mysterious under such circumstances.

The growth rate of the dynamo is roughly $(\alpha_{\theta\theta} \Omega/H)^{1/2}$. If we are far enough from the disk edge that the tidal instability can be neglected then $\tau_{\text{decoherence}}^{-1} \sim (H/r)\Omega$. Since the internal wave dispersion relation gives a group velocity which is a strong function of $\bar{\omega}$ it is reasonable to invoke an incoherent nonlinear damping rate, i.e.

$$\tau_{\text{decoherence}}^{-1} \sim \langle v^2 \rangle \frac{1}{\bar{\omega}H^2}.$$  \hspace{1cm} (8)$$

Consequently, $\langle v^2 \rangle \sim (H/r)c_s^2$. Folding these together with the definition of $\alpha_{\theta\theta}$ we find that in the inner parts of the disk the dynamo growth rate due to the $m = 1$ waves will be approximately...
\[(H/r)^{3/2}\Omega\]. However, note that nonlinear interactions between these waves will generate modes with lower \(\bar{\omega}\) and higher \(m\). Consequently, the contribution to \(\alpha_{\theta\theta}\) from subharmonics may very well dominate provided that the asymmetry in the wave distribution is preserved during the nonlinear cascade. Preliminary work by Huang (1992) indicates that it is preserved and an estimate of its effects (Vishniac and Diamond, 1992) suggests that the total dynamo growth rate may be as high as \(\sim (H/r)^{4/3}\Omega\). This implies an equivalent ‘\(\alpha\)’ of \((H/r)^{4/3}\). Towards the outer parts of the disk the angular momentum transport may be higher, due to the direct excitation of internal waves from tidal instabilities.

Ultimately the contribution from subharmonics is limited by the disruptive effect of the small scale turbulence due to the Velikhov-Chandrasekhar instability. As one goes down the weakly turbulent cascade towards small \(\bar{\omega}\) the wave-wave interactions are eventually overwhelmed by the dissipative effects of this shearing instability. This effect provides the stabilizing feedback for the whole system. A stronger magnetic field will act to disrupt a larger part of the internal wave distribution, lowering \(\alpha_{\theta\theta}\) and the dynamo growth rate. Meanwhile, this same turbulence will increase the magnetic flux loss rate from the disk. A weakened magnetic field will have the opposite effect, resulting in a growth of the mean magnetic field strength. At equilibrium the local magnetic field will have a disordered component roughly as strong as the long range component. The average Alfvén speed will be \(\sim \alpha^{1/2}c_s \sim (H/r)^{2/3}c_s\). The typical turbulent eddy size will be \(\alpha^{1/2}H \sim (H/r)^{2/3}H\). It follows that in thin disks the dissipation of orbital energy will occur on scales much smaller than the disk thickness.

4. Observations and Possible Tests

In its present form the internal wave-driven dynamo model can only yield a set of scaling laws. In spite of this there are several points of contact with the observations, and as the model becomes more quantitative we can expect decisive observational tests of the model. Here I will note only a few of the more interesting possible tests.

First, in this model thicker disks imply larger dimensionless viscosities. This should be directly applicable to stationary disks (where the truth of this assertion is hard to judge) and could be incorporated into models of dwarf novae outbursts. Such models typically suggest a larger ‘\(\alpha\)’ during outbursts (e.g. Mineshige & Osaki 1983, 1985, Smak 1984, Meyer 1984, Cannizzo, Wheeler & Polidan 1986), but the nonlocal nature of \(\alpha\) in this model implies that a detailed implementation of this model is necessary before any conclusions can be drawn.

Second, nonthermal heating in a disk atmosphere can come from the dissipation of weak shocks generated within the disk, the eruption of magnetized bubbles of plasma, and the appearance of turbulent eddys whose size (relative to \(H\)) is not small. All of these effects occur in this model. The first involves the diversion of some fraction of the energy in the weakly turbulent wave cascade
into compressive modes. The total flux from this will scale as $\langle \nu^2 \rangle \tau^{-1}$ or $(H/r)^2$. This implies that the fraction of the local heating budget released in this way is $\sim (H/r)^{2/3}$. The flux of magnetic energy will be proportional to $V_A^2(V_A/c_s)^2$, which is a fraction $\sim (H/r)^{1/3}$ of the total energy. The last contribution is dependent on the local structure of the disk, but will certainly increase with $V_A/c_s$. It follows that coronal heating will be a function of the disk geometry, with fatter disks showing stronger coronae. If the fractional local coronal heating rate can be measured, then it should increase with $r$ (since $H/r$ typically does).

Third, in some systems (like MV Lyrae or TT Ari) the mass flow is intermittent. When such a system is in an extreme low state the outer edge of the disk should gradually shrink, reducing the role of the tidal excitation of internal waves and thereby diminishing the value of $\alpha$ in the remaining disk. The dynamics of such disks should therefore reflect the persistence of a relatively inactive disk, even after long periods of very low mass flux.

Fourth, instabilities in radiation dominated disks should be profoundly affected by the non-local nature of angular momentum transport (cf. Vishniac 1993). This may be difficult to check using AGN, where the observations are somewhat ambiguous, but disks around galactic black holes may provide useful observational clues.

5. Discussion

This is an active field, and the details presented here are likely to change in the near future. However, it seems likely that the basic points listed below will remain a part of any viable model of angular momentum transport in accretion disks.

First, the Velikhov-Chandrasekhar instability is the only kind of turbulent stirring guaranteed to produce outward transport of angular momentum in non-selfgravitating disks. Turbulence due to other instabilities (e.g. convection) may transport angular momentum in either direction. The basic difficulty here is that pure mixing processes will tend to equalize conserved quantities per particle, like angular momentum. In an accretion disk this will drive angular momentum inward. This means that angular momentum transport in magnetized disks is almost certain to involve the Velikhov-Chandrasekhar instability. Transport in neutral, highly resistive, disks is apt to be much less efficient.

Second, convection is important only when it is strong, but the sign of $\alpha_{\text{conv}}$ is unknown. In a magnetized disk it is apt to be more positive than in a neutral disk and its effects will not be confined to the convectively unstable layer.

Third, magnetic flux escapes from a disk at a rate $\sim (V_A/c_s)^2 \Omega \sim \alpha_{VC} \Omega$. This is substantially slower than previous estimates due to the turbulent mixing of the gas caused by the Velikhov-Chandrasekhar instability. Balancing this loss rate with a dynamo implies...
Fourth, these points imply that a local dynamo gives an $\alpha$ which is a universal constant. There is, as yet, no theoretical reason to believe that such a dynamo emerges from the Velikhov-Chandrasekhar instability (or any other local process), but it is difficult to rule out this possibility. Phenomenological arguments tend to lead to a rejection of this possibility, but the role of convection in real objects could invalidate the models used to date.

Fifth, the internal wave-driven dynamo gives an $'\alpha'$ which is non-local and scales as $\left(\frac{H}{r}\right)^{4/3}$ in a stationary disk. The dynamics of such a model are not yet known.

Sixth, excitation of the internal waves in a binary system will be mostly due to a tidal instability. The efficiency of this process will be a fairly strong function of radius. The presence of tidal resonances will reinforce this process. Systems with moderate mass ratios and intermittent mass flows may shrink and become cold during extended interruptions of the mass flow.

Seventh, the Velikhov-Chandrasekhar instability will produce strong vertical mixing, with an effective vertical diffusion coefficient of $\alpha Hc_s$. To see this one can estimate the heat flux due to vertical mixing as

$$F_{\text{mixing}} \sim P \frac{D_z}{L_S},$$

where $L_S$ is the vertical entropy length scale and I have assumed that the fluid is optically thick. Since $D_z \sim \alpha Hc_s$ and the radiative flux is

$$F_{\text{radiative}} \sim \dot{M} \Omega^2 \sim \alpha P c_s,$$

It follows that

$$F_{\text{mixing}} \sim F_{\text{radiative}} \frac{H}{L_S}.$$  \hfill (11)

In other words, in an optically thick disk with a significant vertical entropy gradient the turbulent mixing will lead to a heat flow which is comparable to the radiative flux, but can have either sign. This will tend to stabilize disk models, since the vertical structure will no longer be as sensitive to the details of the opacity, and reduce the magnitude of the vertical entropy gradients in such models. A simple approximation of this effect could, and should, be incorporated into current models of vertical structure.

The work presented here is the result of an ongoing collaboration involving several other researchers, principally P.H. Diamond. L. Jin, M. Huang, S. Luo, and W. Zhang are also responsible for some of the results summarized here. This work was supported by NASA through research grant NAGW-2418.

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