ENERGY FUNCTIONALS FOR THE PARABOLIC
MONGE-AMPERE EQUATION

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1. Introduction

Because of its close connection with the Kähler-Ricci flow, the parabolic complex Monge-Ampère equation on complex manifolds has been studied by many authors. See, for instance, [Cao85, CT02, PS06]. On the other hand, theories for complex Monge-Ampère equation on both bounded domains and complex manifolds were developed in [BT76, Yau78, CKNS85, Kol98]. In this paper, we are going to study the parabolic complex Monge-Ampère equation over a bounded domain.

Let \( \Omega \subset \mathbb{C}^n \) be a bounded domain with smooth boundary \( \partial \Omega \). Denote \( Q_T = \Omega \times (0, T) \) with \( T > 0 \), \( B = \Omega \times \{0\} \), \( \Gamma = \partial \Omega \times \{0\} \) and \( \Sigma_T = \partial \Omega \times (0, T) \). Let \( \partial_p Q_T \) be the parabolic boundary of \( Q_T \), i.e. \( \partial_p Q_T = B \cup \Gamma \cup \Sigma_T \).

Consider the following boundary value problem:

\[
\begin{aligned}
\frac{\partial u}{\partial t} - \log \det (u_{,\alpha \bar{\beta}}) &= f(t, z, u) \quad \text{in} \ Q_T, \\
u &= \varphi \quad \text{on} \ \partial_p Q_T.
\end{aligned}
\]

where \( f \in C^\infty(\mathbb{R} \times \bar{\Omega} \times \mathbb{R}) \) and \( \varphi \in C^\infty(\partial_p Q_T) \). We will always assume that

\[
\frac{\partial f}{\partial u} \leq 0.
\]

Then we will prove that

**Theorem 1.** Suppose there exists a spatial plurisubharmonic (psh) function \( \underline{u} \in C^2(Q_T) \) such that

\[
\begin{aligned}
\underline{u}_t - \log \det (u_{,\alpha \bar{\beta}}) &\leq f(t, z, \underline{u}) \quad \text{in} \ Q_T, \\
\underline{u} &\leq \varphi \quad \text{on} \ B \quad \text{and} \quad \underline{u} = \varphi \quad \text{on} \ \Sigma_T \cap \Gamma.
\end{aligned}
\]

Then there exists a spatial psh solution \( u \in C^\infty(Q_T) \) of (1) with \( u \geq \underline{u} \) if following compatibility condition is satisfied: \( \forall z \in \partial \Omega \),

\[
\begin{aligned}
\varphi_t - \log \det (\varphi_{,\alpha \bar{\beta}}) &= f(0, z, \varphi(z)), \\
\varphi_{tt} - (\log \det (\varphi_{,\alpha \bar{\beta}}))_t &= f_t(0, z, \varphi(z)) + f_u(0, z, \varphi(z)) \varphi_t.
\end{aligned}
\]
Motivated by the energy functionals in the study of the Kähler-Ricci flow, we introduce certain energy functionals to the complex Monge-Ampère equation over a bounded domain. Given \( \varphi \in C^\infty(\partial \Omega) \), denote

\[
P(\Omega, \varphi) = \{ u \in C^2(\overline{\Omega}) \mid u \text{ is psh, and } u = \varphi \text{ on } \partial \Omega \},
\]

then define the \( F^0 \) functional by following variation formula:

\[
\delta F^0(u) = \int_\Omega \delta u \det (u_{\alpha \bar{\beta}}).
\]

We shall show that the \( F^0 \) functional is well-defined. Using this \( F^0 \) functional and following the ideas of [PS06], we prove that

**Theorem 2.** Assume that both \( \varphi \) and \( f \) are independent of \( t \), and

\[
f_u \leq 0 \quad \text{and} \quad f_{uu} \leq 0.
\]

Then the solution \( u \) of (1) exists for \( T = +\infty \), and as \( t \) approaches \( +\infty \), \( u(\cdot, t) \) approaches the unique solution of the Dirichlet problem

\[
\begin{cases}
\det (v_{\alpha \bar{\beta}}) = e^{-f(z,v)} & \text{in } Q_T, \\
v = \varphi & \text{on } \partial_p Q_T,
\end{cases}
\]

in \( C^{1,\alpha}(\Omega) \) for any \( 0 < \alpha < 1 \).

**Remark:** Similar energy functionals have been studied in [Bak83, Ts90, Wan94, TW97, TW98] for the real Monge-Ampère equation and the real Hessian equation with homogeneous boundary condition \( \varphi = 0 \), and the convergence for the solution of the real Hessian equation was also proved in [TW98]. Our construction of the energy functionals and the proof of the convergence also work for these cases, and thus we also obtain an independent proof of these results. Li [Li04] and Blocki [Bl05] studied the Dirichlet problems for the complex \( k \)-Hessian equations over bounded complex domains. Similar energy functional can also be constructed for the parabolic complex \( k \)-Hessian equations and be used for the proof of the convergence.

## 2. A priori \( C^2 \) estimate

By the work of Krylov [Kry83], Evans [Eva82], Caffarelli etc. [CKNS85] and Guan [Gua98], it is well known that in order to prove the existence and smoothness of (1), we only need to establish the a priori \( C^{2,1}(\overline{Q}_T) \) estimate, i.e. for solution \( u \in C^{4,1}(\overline{Q}_T) \) of (1) with

\[
u(\cdot, t) = u \quad \text{on } \Sigma_T \cup \Gamma \quad \text{and} \quad u \geq u \quad \text{in } Q_T,
\]

then

\[
\| u \|_{C^{2,1}(Q_T)} \leq M_2,
\]

where \( M_2 \) only depends on \( Q_T, u, f \) and \( \| u(\cdot, 0) \|_{C^2(\Omega)} \).

\(^{1}C^{m,n}(Q_T)\) means \( m \) times and \( n \) times differentiable in space direction and time direction respectively, same for \( C^{m,n} \)-norm.
Proof of (10). Since $u$ is spatial psh and $u \geq u$, so

$$\underline{u} \leq u \leq \sup_{\Sigma_T} u$$

i.e.

(11)  $||u||_{C^0(\bar{Q}_T)} \leq M_0$.

Step 1. $|u_t| \leq C_1$ in $\bar{Q}_T$.

Let $G = u_t(2M_0 - u)^{-1}$. If $G$ attains its minimum on $\bar{Q}_T$ at the parabolic boundary, then $u_t \geq -C_1$ where $C_1$ depends on $M_0$ and $u_t$ on $\Sigma$. Otherwise, at the point where $G$ attains the minimum,

(12)  $G_t \leq 0$ i.e. $u_{tt} + (2M_0 - u)^{-1}u_t^2 \leq 0$,

$G_\alpha = 0$ i.e. $u_{t\alpha} + (2M_0 - u)^{-1}u_t u_\alpha = 0$,

$G_{\bar{\beta}} = 0$ i.e. $u_{t\bar{\beta}} + (2M_0 - u)^{-1}u_t u_{\bar{\beta}} = 0$,

and the matrix $G_{\alpha\bar{\beta}}$ is non-negative, i.e.

(13)  $u_{t\alpha\bar{\beta}} + (2M_0 - u)^{-1}u_t u_{\alpha\bar{\beta}} \geq 0$.

Hence

(14)  $0 \leq u^{\alpha\bar{\beta}}(u_{t\alpha\bar{\beta}} + (2M_0 - u)^{-1}u_t u_{\alpha\bar{\beta}}) = u^{\alpha\bar{\alpha}} u_{t\alpha\bar{\beta}} + n(2M_0 - u)^{-1}u_t$,

where $(u^{\alpha\bar{\beta}})$ is the inverse matrix for $(u_{\alpha\bar{\beta}})$, i.e.

$$u^{\alpha\bar{\beta}} u_{\gamma\bar{\beta}} = \delta^\alpha_\gamma.$$

Differentiating (11) in $t$, we get

(15)  $u_{tt} - u^{\alpha\bar{\beta}} u_{t\alpha\bar{\beta}} = f_t + f_u u_t$,

so

$$ (2M_0 - u)^{-1}u_t^2 \leq -u_{tt}$$

$$ = -u^{\alpha\bar{\beta}} u_{t\alpha\bar{\beta}} - f_t - f_u u_t$$

$$ \leq n(2M_0 - u)^{-1}u_t - f_u u_t - f_t,$$

hence

$$u_t^2 - (n - (2M_0 - u)f_u)u_t + f_t(2M_0 - u) \leq 0.$$

Therefore at point $p$, we get

(16)  $u_t \geq -C_1$

where $C_1$ depends on $M_0$ and $f$.

Similarly, by considering the function $u_t(2M_0 + u)^{-1}$ we can show that

(17)  $u_t \leq C_1$.

Step 2. $|\nabla u| \leq M_1$
Extend $u|_{\Sigma}$ to a spatial harmonic function $h$, then
\[ u \leq u \leq h \text{ in } Q_T \quad \text{and} \quad u = u = h \text{ on } \Sigma_T. \]
So
\[ |\nabla u|_{\Sigma_T} \leq M_1. \]

Let $L$ be the linear differential operator defined by
\[ Lv = \frac{\partial v}{\partial t} - u^{\alpha\beta}v_{\alpha\beta} - fuv. \]
Then
\[ L(\nabla u + e^{\lambda|z|^2}) = L(\nabla u) + Le^{\lambda|z|^2} \]
\[ \leq \nabla f - e^{\lambda|z|^2}(\lambda \sum u^{\alpha\bar{\alpha}} - fu). \]
Noticed that and both $u$ and $\dot{u}$ are bounded and
\[ \det(u_{\alpha\bar{\beta}}) = e^{\dot{u} - f}, \]
so
\[ 0 < c_0 \leq \det(u_{\alpha\bar{\beta}}) \leq c_1, \]
where $c_0$ and $c_1$ depends on $M_0$ and $f$. Therefore
\[ \sum u^{\alpha\bar{\alpha}} \geq nc_1^{-1/n}. \]
Hence after taking $\lambda$ large enough, we can get
\[ L(\nabla u + e^{\lambda|z|^2}) \leq 0, \]
thus
\[ |\nabla u| \leq \sup_{\partial_p Q_T} |\nabla u| + C_2 \leq M_1. \]

**Step 3.** $|\nabla^2 u| \leq M_2$ on $\Sigma$.

At point $(p, t) \in \Sigma$, we choose coordinates $z_1, \cdots, z_n$ for $\Omega$, such that at $z_1 = \cdots = z_n = 0$ at $p$ and the positive $x_n$ axis is the interior normal direction of $\partial \Omega$ at $p$. We set $s_1 = y_1, s_2 = x_1, \cdots, s_{2n-1} = y_n, s_{2n} = x_n$ and $s' = (s_1, \cdots, s_{2n-1})$. We also assume that near $p$, $\partial \Omega$ is represented as a graph
\[ x_n = \rho(s') = \frac{1}{2} \sum_{j,k<2n} B_{jk} s_j s_k + O(|s'|^3). \]
Since $(u - \bar{u})(s', \rho(s'), t) = 0$, we have for $j, k < 2n$,
\[ (u - \bar{u})_{s_js_k}(p, t) = -(u - \bar{u})_{x_n}(p, t)B_{jk}, \]
hence
\[ |u_{s_js_k}(p, t)| \leq C_3, \]
where $C_3$ depends on $\partial \Omega, u$ and $M_1$. 
We will follow the construction of barrier function by Guan [Gua98] to estimate $|u_{x_n s_j}|$. For $\delta > 0$, denote $Q_\delta(p,t) = (\Omega \cap B_\delta(p)) \times (0,t)$.

**Lemma 3.** Define the function

$$ (28) \quad d(z) = \text{dist}(z, \partial \Omega) $$

and

$$ (29) \quad v = (u - \underline{u}) + a(h - \underline{u}) - N d^2. $$

Then for $N$ sufficiently large and $a, \delta$ sufficiently small,

$$ (30) \quad L v \geq \epsilon (1 + \sum |u^{\alpha \bar{\beta}}|) \quad \text{in} \quad Q_\delta(p,t) $$

$$ v \geq 0 \quad \text{on} \quad \partial (B_\delta(p) \cap \Omega) \times (0,t) $$

$$ v(z,0) \geq c_3 |z| \quad \text{for} \quad z \in B_\delta(p) \cap \Omega $$

where $\epsilon$ depends on the uniform lower bound of the eigenvalues of $\{u^{\alpha \bar{\beta}}\}$.

**Proof.** See the proof of Lemma 2.1 in [Gua98]. $\square$

For $j < 2n$, consider the operator

$$ T_j = \frac{\partial}{\partial s_j} + \rho_{s_j} \frac{\partial}{\partial x_n}. $$

Then

$$ T_j(u - \underline{u}) = 0 \quad \text{on} \quad (\partial \Omega \cap B_\delta(p)) \times (0,t) $$

$$ |T_j(u - \underline{u})| \leq M_1 \quad \text{on} \quad (\Omega \cap \partial B_\delta(p)) \times (0,t) $$

$$ |T_j(u - \underline{u})(z,0)| \leq C_4 |z| \quad \text{for} \quad z \in B_\delta(p) $$

So by Lemma 3, we may choose $C_5$ independent of $u$, and $A >> B >> 1$ so that

$$ L(A v + B |z|^2 - C_5 (u_{y_n} - \underline{u}_{y_n})^2 \pm T_j(u - \underline{u})) \geq 0 \quad \text{in} \quad Q_\delta(p,t), $$

$$ A v + B |z|^2 - C_5 (u_{y_n} - \underline{u}_{y_n})^2 \pm T_j(u - \underline{u}) \geq 0 \quad \text{on} \quad \partial_p Q_\delta(p,t). $$

Hence by the comparison principle,

$$ A v + B |z|^2 - C_5 (u_{y_n} - \underline{u}_{y_n})^2 \pm T_j(u - \underline{u}) \geq 0 \quad \text{in} \quad Q_\delta(p,t), $$

and at $(p,t)$

$$ |u_{x_n y_j}| \leq M_2. $$

To estimate $|u_{x_n x_n}|$, we will follow the simplification in [Tru95]. For $(p,t) \in \Sigma$, define

$$ \lambda(p,t) = \min \{ u_{\xi \bar{\xi}} \mid \text{complex vector } \xi \in T_p \partial \Omega, \text{ and } |\xi| = 1 \} $$

**Claim** $\lambda(p,t) \geq c_4 > 0$ where $c_4$ is independent of $u$.

Let us assume that $\lambda(p,t)$ attains the minimum at $(z_0, t_0)$ with $\xi \in T_{z_0} \partial \Omega$. We may assume that

$$ \lambda(z_0, t_0) < \frac{1}{2} u_{\xi \bar{\xi}}(z_0, t_0). $$
Take a unitary frame $e_1, \cdots, e_n$ around $z_0$, such that $e_1(z_0) = \xi$, and $\Re e_n = \gamma$ is the interior normal of $\partial \Omega$ along $\partial \Omega$. Let $r$ be the function which defines $\Omega$, then

$$(u - \underline{u})_{11}(z, t) = -r_{11}(z)(u - \underline{u})_\gamma(z, t) \quad z \in \partial \Omega$$

Since $u_{11}(z_0, t_0) < \underline{u}_{11}(z_0, t_0)/2$, so

$$-r_{11}(z_0)(u - \underline{u})_\gamma(z_0, t_0) \leq \frac{1}{2}\underline{u}_{11}(z_0, t_0).$$

Hence

$$r_{11}(z_0)(u - \underline{u})_\gamma(z_0, t_0) \geq \frac{1}{2}\underline{u}_{11}(z_0, t_0) \geq c_5 > 0.$$ 

Since both $\nabla u$ and $\nabla \underline{u}$ are bounded, we get

$$r_{11}(z_0) \geq c_6 > 0,$$

and for $\delta$ sufficiently small (depends on $r_{11}$) and $z \in B_\delta(z_0) \cap \Omega$,

$$r_{11}(z) \geq \frac{c_6}{2}.$$ 

So by $u_{11}(z, t) \geq u_{11}(z_0, t_0)$, we get

$$\underline{u}_{11}(z, t) - r_{11}(z)(u - \underline{u})_\gamma(z, t) \geq \underline{u}_{11}(z_0, t_0) - r_{11}(z_0)(u - \underline{u})_\gamma(z_0, t_0).$$

Hence if we let

$$\Psi(z, t) = \frac{1}{r_{11}(z)}(r_{11}(z_0)(u - \underline{u})_\gamma(z_0, t_0) + \underline{u}_{11}(z, t) - \underline{u}_{11}(z_0, t_0))$$

then

$$(u - \underline{u})_\gamma(z, t) \leq \Psi(z, t) \quad \text{on } (\partial \Omega \cap B_\delta(z_0)) \times (0, T)$$

$$(u - \underline{u})_\gamma(z_0, t_0) = \Psi(z_0, t_0).$$

Now take the coordinate system $z_1, \cdots, z_n$ as before. Then

$$(u - \underline{u})_{x_n}(z, t) \leq \frac{1}{\gamma_n(z)}\Psi(z, t) \quad \text{on } (\partial \Omega \cap B_\delta(z_0)) \times (0, T)$$

(34)

$$(u - \underline{u})_{x_n}(z_0, t_0) = \frac{1}{\gamma_n(z_0)}\Psi(z_0, t_0).$$

where $\gamma_n$ depends on $\partial \Omega$. After taking $C_6$ independent of $u$ and $A >> B >> 1$, we get

$$L(Av + B|z|^2 - C_6(u_{y_n} - \underline{u}_{y_n})^2 + \frac{\Psi(z, t)}{\gamma_n(z)} - T_j(u - \underline{u})) \geq 0 \quad \text{in } Q_\delta(p, t),$$

$$Av + B|z|^2 - C_6(u_{y_n} - \underline{u}_{y_n})^2 + \frac{\Psi(z, t)}{\gamma_n(z)} - T_j(u - \underline{u}) \geq 0 \quad \text{on } \partial \Omega Q_\delta(p, t).$$

So

$$Av + B|z|^2 - C_6(u_{y_n} - \underline{u}_{y_n})^2 + \frac{\Psi(z, t)}{\gamma_n(z)} - T_j(u - \underline{u}) \geq 0 \quad \text{in } Q_\delta(p, t),$$

and

$$|u_{x_n, x_n}(z_0, t_0)| \leq C_7.$$
Therefore at \((z_0, t_0)\), \(u_{\overline{\alpha}\beta}\) is uniformly bounded, hence
\[
u_{11}(z_0, t_0) \geq c_4
\]
with \(c_4\) independent of \(u\). Finally, from the equation
\[
det u_{\alpha\overline{\beta}} = e^{u-f}
\]
we get
\[
|u_{x_n x_n}| \leq M_2.
\]

**Step 4.** \(|\nabla^2 u| \leq M_2\) in \(Q\).

By the concavity of \(\log \det\), we have
\[
L(\nabla^2 u + e^{\lambda |z|^2}) \leq O(1) - e^{\lambda |z|^2}(\lambda \sum u^{\alpha\overline{\alpha}} - f_u)
\]
So for \(\lambda\) large enough,
\[
L(\nabla^2 u + e^{\lambda |z|^2}) \leq 0,
\]
and
\[
(35) \quad \sup |\nabla^2 u| \leq \sup_{\partial_p Q_T} |\nabla^2 u| + C_8
\]
with \(C_8\) depends on \(M_0, \Omega\) and \(f\).

\[\square\]

**3. The Functionals \(I, J\) and \(F^0\)**

Let us recall the definition of \(\mathcal{P}(\Omega, \varphi)\) in (5),
\[
\mathcal{P}(\Omega, \varphi) = \{ u \in C^2(\overline{\Omega} \mid u \text{ is psh, and } u = \varphi \text{ on } \partial \Omega \}.
\]
Fixing \(v \in \mathcal{P}\), for \(u \in \mathcal{P}\), define
\[
(36) \quad I_v(u) = -\int_\Omega (u - v)(\sqrt{-1} \partial \overline{\partial} u)^n.
\]

**Proposition 4.** There is a unique and well defined functional \(J_v\) on \(\mathcal{P}(\Omega, \varphi)\), such that
\[
(37) \quad \delta J_v(u) = -\int_\Omega \delta u((\sqrt{-1} \partial \overline{\partial} u)^n - (\sqrt{-1} \partial \overline{\partial} v)^n),
\]
and \(J_v(v) = 0\).

**Proof.** Notice that \(\mathcal{P}\) is connected, so we can connect \(v\) to \(u \in \mathcal{P}\) by a path \(u_t, 0 \leq t \leq 1\) such that \(u_0 = v\) and \(u_1 = u\). Define
\[
(38) \quad J_v(u) = -\int_0^1 \int_\Omega \frac{\partial u_t}{\partial t}((\sqrt{-1} \partial \overline{\partial} u_t)^n - (\sqrt{-1} \partial \overline{\partial} v)^n) dt.
\]
We need to show that the integral in (38) is independent of the choice of path \(u_t\). Let \(\delta u_t = w_t\) be a variation of the path. Then
\[
w_1 = w_0 = 0 \quad \text{and} \quad w_t = 0 \quad \text{on} \ \partial \Omega,
\]
and
\[
\delta \int_0^1 \int_\Omega \dot{u} \left((\sqrt{-1}\partial\bar{\partial}u)^n - (\sqrt{-1}\partial\bar{\partial}v)^n\right) dt
\]
\[
= \int_0^1 \int_\Omega \left(\dot{w}((\sqrt{-1}\partial\bar{\partial}u)^n - (\sqrt{-1}\partial\bar{\partial}v)^n) + \dot{u} n\sqrt{-1}\partial\bar{\partial}w(\sqrt{-1}\partial\bar{\partial}u)^{n-1}\right) dt,
\]

Since \( w_0 = w_1 = 0 \), an integration by part with respect to \( t \) gives
\[
\int_0^1 \int_\Omega \dot{w}((\sqrt{-1}\partial\bar{\partial}u)^n - (\sqrt{-1}\partial\bar{\partial}v)^n) dt
\]
\[
= -\int_0^1 \int_\Omega \frac{d}{dt}(\sqrt{-1}\partial\bar{\partial}u)^n dt = -\int_0^1 \int_\Omega \sqrt{-1}n\partial\bar{\partial}u(\sqrt{-1}\partial\bar{\partial}u)^{n-1} dt.
\]

Notice that both \( w \) and \( \dot{u} \) vanish on \( \partial\Omega \), so an integration by part with respect to \( z \) gives
\[
\int_\Omega \sqrt{-1}nw\partial\bar{\partial}\dot{u}(\sqrt{-1}\partial\bar{\partial}u)^{n-1} dt = -\int_\Omega \sqrt{-1}n\partial\bar{\partial}w(\sqrt{-1}\partial\bar{\partial}u)^{n-1} dt
\]
\[
= \int_\Omega \sqrt{-1}n\partial\bar{\partial}w(\sqrt{-1}\partial\bar{\partial}u)^{n-1} dt.
\]

So
\[
(39) \quad \delta \int_0^1 \int_\Omega \dot{u} \left((\sqrt{-1}\partial\bar{\partial}u)^n - (\sqrt{-1}\partial\bar{\partial}v)^n\right) dt = 0,
\]
and the functional \( J \) is well defined. \( \square \)

Using the \( J \) functional, we can define the \( F^0 \) functional as
\[
(40) \quad F^0_v(u) = J_v(u) - \int_\Omega u(\sqrt{-1}\partial\bar{\partial}v)^n.
\]

Then by Proposition 4, we have
\[
(41) \quad \delta F^0_v(u) = -\int_\Omega \delta u(\sqrt{-1}\partial\bar{\partial}u)^n.
\]

**Proposition 5.** The basic properties of \( I, J \) and \( F^0 \) are following:

1. For any \( u \in \mathcal{P}(\Omega, \varphi) \), \( I_v(u) \geq J_v(u) \geq 0 \).
2. \( F^0 \) is convex on \( \mathcal{P}(\Omega, \varphi) \), i.e. \( \forall u_0, u_1 \in \mathcal{P} \),
\[
(42) \quad F^0 \left(\frac{u_0 + u_1}{2}\right) \leq \frac{F^0(u_0) + F^0(u_1)}{2}.
\]
3. \( F^0 \) satisfies the cocycle condition, i.e. \( \forall u_1, u_2, u_3 \in \mathcal{P}(\Omega, \varphi) \),
\[
(43) \quad F^0_{u_1}(u_2) + F^0_{u_2}(u_3) = F^0_{u_1}(u_3).
\]
To prove (42), let $u$ and $(44)$

\[ \text{Compare (44) and (45), it is easy to see that} \]

\[ \int_{0}^{1} \int_{\Omega} \sqrt{-1} n w \partial \bar{w} (\sqrt{-1} \partial \bar{u}_t)^{n-1} dt \geq 0, \]

and

\[ J_v(u) = \int_{0}^{1} \int_{\Omega} w((\sqrt{-1} \partial \bar{u}_t)^{n} - (\sqrt{-1} \partial \bar{v})^{n}) dt \]

\[ = \int_{0}^{1} \int_{\Omega} w(\int_{0}^{t} \frac{ds}{ds} (\sqrt{-1} \partial \bar{u}_s)^{n} ds) dt \]

\[ = \int_{0}^{1} \int_{\Omega} \int_{0}^{t} \sqrt{-1} n w \partial \bar{w} (\sqrt{-1} \partial \bar{u}_s)^{n-1} ds dt \]

\[ = \int_{0}^{1} \int_{\Omega} (1 - s) \sqrt{-1} n \partial \bar{w} \bar{w} \wedge (\sqrt{-1} \partial \bar{u}_s)^{n-1} ds \geq 0. \]

Compare (44) and (45), it is easy to see that

\[ I_v(u) \geq J_v(u) \geq 0. \]

To prove (42), let $u_t = (1 - t)u_0 + tu_1$, then

\[ F^0(u_1/2) - F^0(u_0) = - \int_{0}^{\frac{1}{2}} \int_{\Omega} (u_1 - u_0) (\sqrt{-1} \partial \bar{u}_t)^{n} dt, \]

\[ F^0(u_1) - F^0(u_1/2) = - \int_{\frac{1}{2}}^{1} \int_{\Omega} (u_1 - u_0) (\sqrt{-1} \partial \bar{u}_t)^{n} dt. \]

Since

\[ \int_{0}^{\frac{1}{2}} \int_{\Omega} (u_1 - u_0) (\sqrt{-1} \partial \bar{u}_t)^{n} dt - \int_{\frac{1}{2}}^{1} \int_{\Omega} (u_1 - u_0) (\sqrt{-1} \partial \bar{u}_t)^{n} dt. \]

\[ = \int_{0}^{\frac{1}{2}} \int_{\Omega} (u_1 - u_0) ((\sqrt{-1} \partial \bar{u}_t)^{n} - (\sqrt{-1} \partial \bar{u}_{t+1/2})^{n}) dt \]

\[ = 2 \int_{0}^{\frac{1}{2}} \int_{\Omega} (u_{t+1/2} - u_t) ((\sqrt{-1} \partial \bar{u}_t)^{n} - (\sqrt{-1} \partial \bar{u}_{t+1/2})^{n}) dt \geq 0. \]

So

\[ F^0(u_1) - F^0(u_1/2) \geq F^0(u_{1/2}) - F^0(u_0). \]

The cocycle condition is a simple consequence of the variation formula.
4. The Convergence

In this section, let us assume that both \( f \) and \( \varphi \) are independent of \( t \). For \( u \in \mathcal{P}(\Omega, \varphi) \), define

\[
F(u) = F_0(u) + \int_{\Omega} G(z, u) dV,
\]

where \( dV \) is the volume element in \( \mathbb{C}^n \), and \( G(z, s) \) is the function given by

\[
G(z, s) = \int_{s}^{0} e^{-f(z, t)} dt.
\]

Then the variation of \( F \) is

\[
\delta F(u) = -\int_{\Omega} \delta u (\det(u_{\alpha\bar{\beta}}) - e^{-f(z, u)}) dV.
\]

**Proof of Theorem 2.** We will follow Phong and Sturm’s proof of the convergence of the Kähler-Ricci flow in [PS06]. For any \( t > 0 \), the function \( u(\cdot, t) \) is in \( \mathcal{P}(\Omega, \varphi) \). So by (47)

\[
\frac{d}{dt} F(u) = -\int_{\Omega} \dot{u} (\det(u_{\alpha\bar{\beta}}) - e^{-f(z, u)})
\]

Thus \( F(u(\cdot, t)) \) is monotonic decreasing as \( t \) approaches \( +\infty \). On the other hand, \( u(\cdot, t) \) is uniformly bounded in \( C^2(\Omega) \) by (10), so both \( F_0(u(\cdot, t)) \) and \( f(z, u(\cdot, t)) \) are uniformly bounded, hence \( F(u) \) is bounded. Therefore

\[
\int_{0}^{\infty} \int_{\Omega} (\log \det(u_{\alpha\bar{\beta}}) + f(z, u)) (\det(u_{\alpha\bar{\beta}}) - e^{-f(z, u)}) dt < \infty.
\]

Observed that by the Mean Value Theorem, for \( x, y \in \mathbb{R} \),

\[(x + y)(e^x - e^y) = (x + y)^2 e^\eta \geq e^{\min(x, -y)} (x - y)^2,
\]

where \( \eta \) is between \( x \) and \( -y \). Thus

\[(\log \det(u_{\alpha\bar{\beta}}) + f) (\det(u_{\alpha\bar{\beta}}) - e^{-f}) \geq C_9 (\log \det(u_{\alpha\bar{\beta}}) + f)^2 = C_9 |\dot{u}|^2
\]

where \( C_9 \) is independent of \( t \). Hence

\[
\int_{0}^{\infty} \|\dot{u}\|^2_{L^2(\Omega)} dt \leq \infty.
\]

Let

\[
Y(t) = \int_{\Omega} |\dot{u}(\cdot, t)|^2 \det(u_{\alpha\bar{\beta}}) dV,
\]

then

\[
\dot{Y} = \int_{\Omega} (2\dddot{u} + \dot{u}^2 u_{\alpha\bar{\beta}} \dddot{u}_{\alpha\bar{\beta}}) \det(u_{\alpha\bar{\beta}}) dV.
\]

Differentiate (11) in \( t \),

\[
\dddot{u} - u_{\alpha\beta} \dddot{u}_{\alpha\bar{\beta}} = f_u u,
\]
so

\[
\dot{Y} = \int_{\Omega} \left( 2\dot{u}_{\alpha\beta} u^{\alpha\beta} + \ddot{u}^2 \left( 2f_u + \ddot{u} - f_u \dot{u} \right) \right) \det(u_{\alpha\beta}) \, dV \\
= \int_{\Omega} \left( \dot{u}^2 \left( 2f_u + \ddot{u} - f_u \dot{u} \right) - 2\dot{u}_{\alpha} \dot{u}_{\beta} u^{\alpha\beta} \right) \det(u_{\alpha\beta}) \, dV
\]

From (51), we get

\[
\ddot{u} - u^{\alpha\beta} \ddot{u}_{\alpha\beta} - f_u \ddot{u} \leq f_{uu} \ddot{u}^2
\]

Since \( f_u \leq 0 \) and \( f_{uu} \leq 0 \), so \( \ddot{u} \) is bounded from above by the maximum principle. Therefore

\[
\dot{Y} \leq C_{10} \int_{\Omega} \dot{u}^2 \det(u_{\alpha\beta}) \, dV = C_{10}Y,
\]

and

\[
Y(t) \leq Y(s) e^{C_{10}(t-s)} \quad \text{for } t > s,
\]

where \( C_{10} \) is independent of \( t \). By [49], (52) and the uniform boundedness of \( \det(u_{\alpha\beta}) \), we get

\[
\lim_{t \to \infty} \| u(\cdot, t) \|_{L^2(\Omega)} = 0.
\]

Since \( \Omega \) is bounded, the \( L^2 \) norm controls the \( L^1 \) norm, hence

\[
\lim_{t \to \infty} \| u(\cdot, t) \|_{L^1(\Omega)} = 0.
\]

Notice that by the Mean Value Theorem,

\[
|e^x - 1| < e^{|x|}|x|
\]

so

\[
\int_{\Omega} |e^{\ddot{u}} - 1| \, dV \leq e^{\sup |\ddot{u}|} \int_{\Omega} |\ddot{u}| \, dV
\]

Hence \( e^{\ddot{u}} \) converges to 1 in \( L^1(\Omega) \) as \( t \) approaches \( +\infty \). Now \( u(\cdot, t) \) is bounded in \( C^2(\overline{\Omega}) \), so \( u(\cdot, t) \) converges to a unique function \( \ddot{u} \), at least sequentially in \( C^1(\overline{\Omega}) \), hence \( f(z, u) \to f(z, \ddot{u}) \) and

\[
\det(\ddot{u}_{\alpha\beta}) = \lim_{t \to \infty} \det(u(\cdot, t)_{\alpha\beta}) = \lim_{t \to \infty} e^{\ddot{u} - f(z, u)} = e^{-f(z, \ddot{u})},
\]

i.e. \( \ddot{u} \) solves (8).

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