On Quantum Field Theories in Operator and Functional Integral Formalisms

Aba Teleki
Department of Theoretical Physics, Faculty of Mathematics, Physics and Computer Sciences, Comenius University, SK-842 48 Bratislava, Slovakia
(Dated: March 27, 2022)

Milan Noga
Department of Physics, Faculty of Natural Sciences, Constantine the Philosopher University, SK-949 74 Nitra, Slovakia

Relations and isomorphisms between quantum field theories in operator and functional integral formalisms are analyzed from the viewpoint of inequivalent representations of commutator or anticommutator rings of field operators. A functional integral in quantum field theory cannot be regarded as a Newton-Lebesgue integral but rather as a formal object to which one associates distinct numerical values for different processes of its integration. By choosing an appropriate method for the integration of a given functional integral, one can select a single representation out of infinitely many inequivalent representations for an operator whose trace is expressed by the corresponding functional integral. These properties are demonstrated with two exactly solvable examples.

PACS numbers: 03.70.+k, 05.30.-d, 11.10.-z, 05.30.Ch, 74.20.Fg, 02.90.+p

I. INTRODUCTION

In quantum field theory based on the functional integral formalism, the partition function \( Z \) and quantum expectation values \( \langle A \rangle \) of physical observables \( A \) are given by the formal functional integrals

\[
Z = \int e^{-S(a^*, a)} \mathcal{D}(a^*, a) \tag{1.1a}
\]

and

\[
\langle A \rangle = \frac{1}{Z} \int A(a^*, a)e^{-S(a^*, a)} \mathcal{D}(a^*, a), \tag{1.1b}
\]

respectively, where \( S(a^*, a) \) is an action functional of fields \( a^* \) and \( a \), \( A(a^*, a) \) is a function of \( a^* \) and \( a \), and \( \mathcal{D}(a^*, a) \) denotes a formal measure on a space of fields \( a^* \) and \( a \). All problems of quantum field theory are thus reduced to problems of finding a correct definition and a computation method of the functional integrals \( Z \). However, the results of the integration of \( Z \) are not unique and depend on the chosen method for carrying out the computation of these functional integrals. This is why in their monograph the mathematicians Kobzarev and Manin have expressed the following statement about \( Z \) and \( \langle A \rangle \): “From a mathematician’s viewpoint almost every such computation is in fact a half-baked and ad hoc definition, but a readiness to work heuristically with such a priori undefined expressions as \( Z \) is a must in this domain.” The most standard method for the integrations of \( Z \) is their reduction to Gaussian integrals, whose theory is the only developed chapter of infinite-dimensional integration, and then to use an appropriate perturbation expansion. In this method one divides the action functional \( S(a^*, a) \) into a sum of two terms

\[
S(a^*, a) = S_0(a^*, a) + S_I(a^*, a), \tag{1.2}
\]

where \( S_0 \) has a bilinear form in the fields \( a^* \) and \( a \). Thus the corresponding partition function

\[
Z_0 = \int e^{-S_0(a^*, a)} \mathcal{D}(a^*, a) \tag{1.3}
\]

is a Gaussian integral and can be exactly and explicitly evaluated. The total partition function \( Z \) can be expressed in the form

\[
Z = Z_0 \langle \exp(-S_I) \rangle_0, \tag{1.4}
\]

where

\[
\langle \exp(-S_I) \rangle_0 = \sum_{\nu=0}^{\infty} \frac{(-1)^\nu}{\nu!} \langle S_I^\nu \rangle_0 \tag{1.5}
\]

is the infinite perturbation series involving quantum expectation values of \( \langle S_I^\nu \rangle_0 \) evaluated with respect to the chosen action \( S_0 \). The separation of the action functional \( S(a^*, a) \) as the sum of terms \( S_0 \) and \( S_I \) is, however, not unique. There are infinitely many ways to select \( S_0 \). To each selected \( S_0 \) one obtains a different result for the perturbation series \( \langle S_I^\nu \rangle_0 \). Thus the results of functional integrations of \( Z \) as expressed by \( \langle \exp(-S_I) \rangle_0 \) seem to be indeed as “half-baked and ad hoc definitions” from the mathematician’s viewpoint.

If quantum field theory based on the functional integrals \( Z \) is equivalent to the same quantum field theory formulated in the operator formalism, then the relations

\[
\int e^{-S(a^*, a)} \mathcal{D}(a^*, a) = \text{Tr} e^{-\beta H(a^*, a)}, \tag{1.6a}
\]

are satisfied.

*Electronic address: ateleki@ukf.sk
†Electronic address: Milan.Noga@fmph.uniba.sk
and
\[ \int A(a^+, a) e^{-S(a^+, a)} \mathcal{D}(a^*, a) = \text{Tr} \left\{ A(a^+, a) e^{-\beta H(a^+, a)} \right\} \] (1.6b)

must be satisfied, where \( H(a^+, a) \) is the Hamiltonian operator corresponding to the action \( S(a^+, a) \), the operator \( A(a^+, a) \) corresponds to its normal symbol \( A(a^*, a) \), and \( a^+ \) and \( a \) are field operators, and \( \beta = 1/T \) is the inverse temperature.

From the last relations it follows that the functional integrals \( \mathcal{Z} \) cannot be defined in such a way as to give a unique result after a process of their integrations because the right-hand sides of the relations (1.6) are distinct for each inequivalent representation of the commutator or anticommutator ring of field operators \( a^+ \) and \( a \) entering the Hamiltonian \( H(a^+, a) \) and the operator \( A(a^+, a) \). The reasons are given by the following arguments.

In the operator approach one assumes to have a complete set of annihilation and creation operators \( a_{k,\sigma} \) and \( a_{k,\sigma}^+ \) of particles in quantum states denoted by quantum numbers \((k, \sigma)\), as, for example, by the momentum \( k \) and the spin \( \sigma \). In any quantum field theory the number of the pairs of the operators \((a_{k,\sigma}, a_{k,\sigma}^+)\) is infinite. These operators satisfy the canonical commutation or anticommutation relations
\[ \{a_{k,\sigma}, a_{k',\sigma'}^+\} = \delta_{kk'} \delta_{\sigma\sigma'}, \] (1.7a)
\[ \{a_{k,\sigma}, a_{k',\sigma'}\} = \{a_{k,\sigma}^+, a_{k',\sigma'}^+\} = 0. \] (1.7b)
The operators act on state vectors \( \Psi \) which span a Hilbert space \( \mathcal{H} \). In order to achieve a unique specification of the commutator or anticommutator ring of the operators \((a_{k,\sigma})\) one requires in addition to \( \mathcal{H} \) the existence of a vacuum state \( \Phi_0 \) for which
\[ a_{k,\sigma} \Phi_0 = 0 \] (1.8)
for all \((k, \sigma)\).

In this case, the Hilbert space \( \mathcal{H} \) is the space for a representation of the commutator or anticommutator ring of the operators \( a_{k,\sigma} \) with the auxiliary condition \( \mathcal{H} \). Since the operators \( a_{k,\sigma} \) and \( a_{k,\sigma}^+ \) form a complete set of operators, a Hamiltonian \( H \) of a given system can be expressed as a given function of \( a_{k,\sigma} \) and \( a_{k,\sigma}^+ \), i.e.
\[ H = H(a^+, a). \] (1.9)
The partition function \( \mathcal{Z} \) of the system is expressed as the trace of the density matrix
\[ \rho = e^{-\beta H}, \] (1.10)
i.e.,
\[ \mathcal{Z} = \text{Tr} e^{-\beta H}, \] (1.11a)
and the statistical average values corresponding to physical observables associated with the operators \( A(a^+, a) \) are given by
\[ \langle A \rangle = \frac{1}{\mathcal{Z}} \text{Tr} \left\{ A(a^+, a) e^{-\beta H} \right\}. \] (1.11b)

As long as the number of the operators \( a_{k,\sigma} \) entering the commutator or anticommutator ring \( \mathcal{H} \) is finite, there is only one inequivalent representation of the relations (1.7) and (1.8). However, in quantum field theories describing systems with an infinite number of degrees of freedom, the algebraic structure \( \mathcal{H} \) has infinitely many inequivalent representations \( [\mathcal{H}] \). Intuitively speaking, one can say that there exists infinitely many different matrix realizations of the operators \( a_{k,\sigma} \) and \( a_{k,\sigma}^+ \) satisfying the same algebraic structure \( \mathcal{H} \). The situation reminds us distantly of a Lie algebra of a non-compact group which has infinitely many unitary irreducible representations realized by sets of infinite-dimensional matrices. Thus, to each inequivalent representation of the commutator or anticommutator ring \( \mathcal{H} \) one has to associate the corresponding representation of the Hamiltonian \( \mathcal{H} \) and the density matrix \( \rho \).

Intuitively speaking, one can say that the matrix form of the Hamiltonian \( \mathcal{H} \) and the density matrix \( \rho \) are distinct for each inequivalent representation of \( \mathcal{H} \). This implies that the partition function \( \mathcal{Z} \) and the average values \( \langle A \rangle \) as given by \( \mathcal{Z} \) can give rise to various results depending on the chosen inequivalent representations of the ring \( \mathcal{H} \). This non-uniqueness of the value of the partition function \( \mathcal{Z} \) and of the average values \( \langle A \rangle \) associated with the given Hamiltonian present in the operator formalism \( \mathcal{H} \) should be preserved in quantum field theory based on the functional integrals \( \mathcal{Z} \) if these two approaches are equivalent. They should be equivalent because the integrand \( e^{-S(a^*, a)} \) entering the functional integral \( \mathcal{Z} \) is in fact the kernel of the density matrix \( \rho = e^{-\beta H} \). For these reasons, the functional integral \( \mathcal{Z} \) or \( \mathcal{Z} \) can be as well defined as is permitted by a freedom present in the operator approach to quantum field theory. This freedom is associated with the existence of the inequivalent representations of the commutator or anticommutator ring \( \mathcal{H} \) of field operators.

For a detailed understanding of the fact that the computation of the functional integrals \( \mathcal{Z} \) can give rise to various different results, we will analyze the process of their integrations from three different aspects. Firstly, we analyze a certain class of inequivalent representations of the anticommutator ring \( \mathcal{H} \) and show that to each inequivalent representation one has to associate a distinct action functional \( S(a^*, a) \). Thus, even the explicit form of the action functional \( S(a^*, a) \) corresponding to a given Hamiltonian \( H \) is not unique, but depends on the chosen inequivalent representation of the anticommutator ring of field operators. Secondly, we study perturbation series in both the operator and functional integral formalisms of quantum field theories in order to show that
by selecting an unperturbed part of a Hamiltonian and the corresponding unperturbed action functional, one in fact selects one inequivalent representation of the commutator or anticommutator ring \( \{a_{k,\sigma}, a_{k',\sigma'}^\dagger\} \) of field operators. Thirdly, we demonstrate explicitly the properties of the functional integrals mentioned above on a toy model of quantum field theory. We construct a simple action functional \( S(a^*, a) \) which permits the exact evaluation of the functional integral \( \mathcal{Z}_1 \), but nonetheless it leads to infinitely many different results corresponding to infinitely many perturbation series.

II. INEQUIVALENT REPRESENTATIONS

For the sake of simplicity, we start by considering a complete set of annihilation and creation field operators \( a_{k,\sigma} \) and \( a_{k,\sigma}^\dagger \) of the fermion type. Let the index \( k \) run over integer numbers over the interval \( k \in [-N/2, N/2] \) and \( \sigma \) denote spin \( \frac{1}{2} \) projection of a fermion, i.e.,

\[ \sigma = \uparrow, \downarrow = +, - \]

In order to have a quantum field theory, we take the limit \( N \to \infty \). The field operators \( a_{k,\sigma} \) and \( a_{k,\sigma}^\dagger \) satisfy the anticommutator ring

\[
\{a_{k,\sigma}, a_{k',\sigma'}^\dagger\} = \delta_{kk'}\delta_{\sigma\sigma'} \quad (2.1a)
\]

\[
\{a_{k,\sigma}, a_{k',\sigma'}\} = \{a_{k,\sigma}^\dagger, a_{k',\sigma'}^\dagger\} = 0 \quad (2.1b)
\]

with the subsidiary condition

\[
a_{k,\sigma} \Phi_0 = 0 \quad (2.2)
\]

on the vacuum state \( \Phi_0 \) for all \( (k, \sigma) \). The representation space for the anticommutator ring \( \{2.1\} \) with the subsidiary condition \( \{2.2\} \) can be chosen to be the Hilbert space \( \mathcal{H} \) spanned by the basis vectors \( \Psi_{\{n_{k,\sigma}\}} \) defined by the formula

\[
\Psi_{\{n_{k,\sigma}\}} = \lim_{N \to \infty} \prod_{k,\sigma} (a_{k,\sigma}^\dagger)^{n_{k,\sigma}} \Phi_0. \quad (2.3)
\]

where \( n_{k,\sigma} = 0, 1 \) are the occupation numbers of fermions in states \((k, \sigma)\), and \( \{ n_{k,\sigma} \} \) denotes an array with an infinite number of items 0 and 1. Each such infinite array \( \{ n_{k,\sigma} \} \) specifies one of the basis vectors of the Hilbert space \( \mathcal{H} \).

The Hamiltonian \( H \) governing a physical system in a quantum field theory is a function of the operators \( a_{k,\sigma} \) and \( a_{k,\sigma}^\dagger \). Let its normal form be denoted by

\[
H = H(a^+, a). \quad (2.4)
\]

To the normal form of \( H(a^+, a) \) one assigns the normal symbol

\[
H = H(a^*(\tau), a(\tau)) \quad (2.5)
\]

of the operator \( \{2.4\} \) in which every operator \( a_{k,\sigma} \) and \( a_{k,\sigma}^\dagger \) is replaced by the Grassmann generators \( a_{k,\sigma}(\tau) \) and \( a_{k,\sigma}^\dagger(\tau) \), respectively \( \{1, 2, 3, 4, 5\} \). These generators are thus enumerated also by the continuous parameter \( \tau \) in addition to the quantum numbers \((k, \sigma)\). The action functional \( S = S(a^*, a) \) is then defined by the integral

\[
S(a^*, a) = \int_0^\beta d\tau \lim_{N \to \infty} \left\{ \sum_{k,\sigma} a_{k,\sigma}^*(\tau) a_{k,\sigma}(\tau) + H(a^*(\tau), a(\tau)) \right\}, \quad (2.6)
\]

where \( a_{k,\sigma}(\tau) \) denotes the “time” derivative of \( a_{k,\sigma}(\tau) \).

The partition function \( \mathcal{Z} \) of the system is expressed as

\[
\mathcal{Z} = \text{Tr} e^{-\beta H(a^+, a)} \quad (2.7)
\]

in the operator formalism, or as the integral

\[
\mathcal{Z} = \int e^{-S(a^*, a)} \mathcal{D}(a^*, a) \quad (2.8)
\]

in the functional integral formalism of the quantum field theory. From the relations \( \{2.7\} \) and \( \{2.8\} \), one evidently sees that \( -\beta S(a^*, a) \) is in fact the kernel of the density matrix \( -\beta H(a^+, a) \) operator, and therefore the expressions \( \{2.7\} \) and \( \{2.8\} \) should give the same result if these two approaches to the quantum field theory are equivalent.

Next, we study a class of inequivalent representations of the anticommutator ring \( \{2.1\} \). We start from the operators \( a_{k,\sigma} \) and \( a_{k,\sigma}^\dagger \) obeying \( \{2.1\} \) and introduce the “unitary” transformations

\[
c_{k,\sigma} = e^{iQ} a_{k,\sigma} e^{-iQ}, \quad c_{k,\sigma}^\dagger = e^{iQ} a_{k,\sigma}^\dagger e^{-iQ}, \quad (2.9)
\]

where \( Q \) is the Hermitian operator

\[
Q = \lim_{N \to \infty} \sum_k \alpha_k T_k, \quad (2.10a)
\]

\[
T_k = i(a_{k,\sigma}^\dagger a_{-k,\sigma}^\dagger - a_{k,\sigma} a_{-k,\sigma}), \quad (2.10b)
\]

and \( \alpha_k \) are arbitrary real parameters. The anticommutation relations for the transformed operators \( c_{k,\sigma} \) and \( c_{k,\sigma}^\dagger \) are, of course, the same as those given by \( \{2.1\} \). The operator \( e^{iQ} \) can be expressed as the infinite product

\[
e^{iQ} = \lim_{N \to \infty} \prod_{k=-N/2}^{N/2} \left[ 1 + i T_k \sin \alpha_k - T_k^2 (1 - \cos \alpha_k) \right], \quad (2.11)
\]

where

\[
T_k^2 = 2a_{k,\sigma}^\dagger a_{-k,\sigma}^\dagger a_{-k,\sigma} a_{k,\sigma} - a_{k,\sigma}^\dagger a_{-k,\sigma} a_{-k,\sigma}^\dagger a_{k,\sigma} + 1. \quad (2.12)
\]
The transformations (2.9), if elaborated with (2.11), are similar to the well-known Bogoliubov-Valatin transformations (3)

\[ \begin{align*}
    c_{k,+} &= u_k a_{k,+} + v_k a_{k,-}^\dagger, \\
    c_{k,-} &= u_k a_{k,-} - v_k a_{k,+}^\dagger, \\
    c_{k,+}^\dagger &= u_k a_{k,+}^\dagger + v_k a_{k,-}, \\
    c_{k,-}^\dagger &= u_k a_{k,-}^\dagger - v_k a_{k,+},
\end{align*} \tag{2.13a-d}

where

\[ u_k = \cos \alpha_k \quad \text{and} \quad v_k = \sin \alpha_k. \tag{2.13e} \]

In the limit \( N \to \infty \), the operator \( Q \) given by (2.10) is not a proper operator, but transforms every vector \( \Psi \) of the Hilbert space \( \mathcal{H} \) into \( \Psi' = e^{iQ} \Psi \) of the Hilbert space \( \mathcal{H}' \) with unexpected properties.

Let us denote by \( \varphi_{(n_k)} \) any basis vector of \( \mathcal{H} \) given by the formula

\[ \varphi_{(n_k)} = \lim_{N \to \infty} \frac{1}{N/2} \prod_{k=-N/2}^{N/2} (a_{k,+}^\dagger a_{k,-}^\dagger)^{n_k} \phi_0, \tag{2.14} \]

where \( n_k = n_{k,+} = n_{k,-} = 0,1 \). All the basis vectors \( \varphi_{(n_k)} \) form a subspace of \( \mathcal{H} \). The transformation \( e^{iQ} \) transforms every basis vector \( \varphi_{(n_k)} \) into one \( \varphi'_{(n_k)} = e^{iQ} \varphi_{(n_k)} \) of \( \mathcal{H}' \), given by the formula

\[ \varphi'_{(n_k)} = \lim_{N \to \infty} \frac{1}{N/2} \prod_{k=-N/2}^{N/2} \left\{ \delta_{n_k,1} - (a_{k,+}^\dagger a_{k,-}^\dagger)^n \right\} \sin \alpha_k + (a_{k,+}^\dagger a_{k,-}^\dagger)^n \cos \alpha_k \phi_0. \tag{2.15} \]

The result is that the scalar product \( (\Psi, e^{iQ} \varphi_{(n_k)}) \) of every basis vector \( \Psi \) given by (2.8) of \( \mathcal{H} \) is either identically equal to zero or equal to the infinite product

\[ (\Psi_{(n'_k)}', e^{iQ} \varphi_{(n_k)}) = \lim_{N \to \infty} \prod_{k,k'} (D_k \sin \alpha_k + D_c \cos \alpha_k), \tag{2.16a} \]

where

\[ \begin{align*}
    D_k &= \delta_{1,n_k} \delta_{0,n'_k} - \delta_{0,n_k} \delta_{1,n'_k}, \\
    D_c &= \delta_{n_k,n'_k} \delta_{k,k'},
\end{align*} \tag{2.16b-c} \]

which also diverges to zero in the limit \( N \to \infty \) for any suitable parameters \( \alpha_k \). Thus the Hilbert space \( \mathcal{H}' \) contains a subspace of the state vectors \( \varphi'_{(n_k)} \) which are orthogonal to every vector \( \Psi \) of \( \mathcal{H} \). To make the conclusion as in Haag’s paper (7), \( c_{k,+} \) and \( c_{k,-} \) given by (2.9) are operators satisfying the same canonical anticommutator ring as (2.11), i.e.,

\[ \begin{align*}
    \{ c_{k,\sigma}, c_{k',\sigma'}^\dagger \} &= \delta_{kk} \delta_{\sigma\sigma'}, \tag{2.17a} \\
    \{ c_{k,\sigma}, c_{k',\sigma'} \} &= \{ c_{k,\sigma}, c_{k',\sigma'}^\dagger \} = 0, \tag{2.17b}
\end{align*} \]

but there is no proper unitary transformation connecting these two operator systems. In other words, they belong to inequivalent representations of the same anticommutator ring (2.11) or (2.17) of the field operators. Each inequivalent representation of (2.17) is specified by the chosen infinite set of parameters \( \alpha_k \) entering the transformations (2.9)-(2.13). Thus, the number of the inequivalent representations of (2.17) is infinite.

For each inequivalent representation of (2.17), we can construct the transformed Hamiltonian

\[ \hat{H}(c^+, c) = e^{iQ} \hat{H}(a^+, a) e^{-iQ} \tag{2.18} \]

in its normal form by employing the anticommutation relations (2.14). The corresponding partition function

\[ \mathcal{Z} = \text{Tr} e^{-\beta \hat{H}(c^+, c)} \tag{2.19} \]

can be different from that given by (2.16) because the operators \( \hat{H}(a^+, a) \) and \( \hat{H}(c^+, c) \) act in different Hilbert spaces \( \mathcal{H} \) and \( \mathcal{H}' \), respectively.

In the functional formalism of quantum field theory, we construct the normal symbol \( \hat{H}(c^+(\tau), c(\tau)) \) of the operator \( \hat{H}(c^+, c) \) and the action functional

\[ \tilde{S}(c^+, c) = \int_0^\beta d\tau \left\{ \lim_{N \to \infty} \sum_{k,\sigma} c_{k,\sigma}^* \dot{c}_{k,\sigma}(\tau) \right. \]

\[ \left. + \hat{H}(c^+(\tau), c(\tau)) \right\} \tag{2.20} \]

for each inequivalent representation of the anticommutator ring (2.17) of the field operators \( c_{k,\sigma} \) and \( c_{k,\sigma}^\dagger \). The corresponding partition function \( \mathcal{Z} \) in the functional integral form

\[ \mathcal{Z} = \int e^{-\tilde{S}(c^+, c)} d\mathcal{G}(c^+, c) \tag{2.21} \]

gives the distinct result for each inequivalent representation. One can be easily convinced by studies of concrete Hamiltonians with interactions between fields that the functional integral (2.21) cannot be transformed into that of (2.8) by the transformations (2.13),

\[ \begin{align*}
    c_{k,+}(\tau) &= u_k a_{k,+}(\tau) + v_k a_{k,-}^+(\tau), \tag{2.22a} \\
    c_{k,-}(\tau) &= u_k a_{k,-}(\tau) - v_k a_{k,+}^+(\tau), \tag{2.22b} \\
    c_{k,+}^\dagger(\tau) &= u_k a_{k,+}^\dagger(\tau) + v_k a_{k,-}(\tau), \tag{2.22c} \\
    c_{k,-}^\dagger(\tau) &= u_k a_{k,-}^\dagger(\tau) - v_k a_{k,+}(\tau). \tag{2.22d}
\end{align*} \]

of the integration variables.

We conclude this section by stating that a given Hamiltonian \( H \) of a system leads to distinct results for the partition function \( \mathcal{Z} \), and for statistical average values \( \langle A \rangle \) depending on the chosen inequivalent representation of the anticommutator ring of field operators in both the operator and the functional integral approach to quantum field theory. Our analysis of inequivalent representations of the anticommutator ring of field operators (2.17) presented above can be generalized in a straightforward way to any set of quantum numbers \( (k, \sigma) \) and to many other classes of inequivalent representations.
III. PERTURBATION SERIES

We begin with an elucidation of how one tacitly selects a single inequivalent representation of the commutator or anticommutator ring of field operators in a practical application of quantum field theory. In quantum field theories with interactions between fields, there is not known even one physical example with an exact solution. In all practical calculations, one divides the Hamiltonian $H$ of a system into the sum

$$H = H_0(a^+, a) + H_I(a^+, a), \quad (3.1)$$

where $H_0$ is called the unperturbed Hamiltonian, and the remaining term $H_I$ is called the perturbative part. The unperturbed Hamiltonian $H_0$ is chosen in such a way in order to be exactly diagonalized, and by this fact its effects are treated exactly. Its eigenstates $\Psi_\mu$, where $\mu$ denotes an array with an infinite series of items, form a complete basis of a Hilbert space $\mathcal{H}$. This is the representation space for a single representation of the commutator or anticommutator ring of field operators $a_{k,\sigma}$ and $a_{k,\sigma}^+$ entering the Hamiltonian $H$. The partition function

$$Z_0 = \text{Tr} e^{-\beta H_0(a^+, a)} \quad (3.2)$$

can be exactly evaluated and is typical for the chosen representation. The total partition function $Z$ is expressed as the perturbation series

$$Z = \text{Tr} e^{-\beta H} = Z_0 \left\langle \text{exp} \left\{ - \int_0^\beta d\tau V(\tau) \right\} \right\rangle_0 = Z_0 \sum_{\nu=0}^\infty \frac{(-1)\nu}{\nu!} \left\langle T \left\{ \int_0^\beta d\tau V(\tau) \right\}^\nu \right\rangle_0 \quad (3.3)$$

where the symbol $T$ stands for the time- or temperature-ordered product and

$$V(\tau) = e^{\tau H_0} H_I e^{-\tau H_0}. \quad (3.4)$$

The individual terms of the perturbation series are in one-to-one correspondence with the perturbation series and in the functional integral method. Namely, $\exp\{-S_0(a^+, a)\}$ in (3.3) is the kernel of the unperturbed density matrix $\rho_0 = \exp\{-\beta H_0(a^+, a)\}$ operator. The statistical average values $\left\langle T \left\{ \int_0^\beta d\tau V(\tau) \right\}^\nu \right\rangle_0$ evaluated in the chosen single inequivalent representation correspond to the functional integrals $\left\langle S_I^\nu(a^+, a) \right\rangle_0$ in (3.3). Thus, the perturbation series and in both the functional integral and the operator formalism of quantum field theory should give the same results in the chosen inequivalent representation of the commutator or anticommutator ring of field operators.

However, the splitting of the total Hamiltonian $H(a^+, a)$ as given by (3.1) is not unique. One can equally well divide the same Hamiltonian as

$$H = \bar{H}_0(a^+, a) + \bar{H}_I(a^+, a), \quad (3.5)$$

where the unperturbed Hamiltonian $\bar{H}_0$ is not related to $H_0$ by any proper unitary transformation. In this case, the eigenstates $\Psi'_\nu$ of $\bar{H}_0$ form again a complete basis of a new Hilbert space $\mathcal{H}'$ for another inequivalent representation of the commutator or anticommutator ring of field operators. Thus, in this another inequivalent representation one gets the partition functions

$$\bar{Z}_0 = \text{Tr} e^{-\beta \bar{H}_0(a^+, a)} \quad (3.6)$$

and

$$\bar{Z} = \bar{Z}_0 \sum_{\nu=0}^\infty \frac{(-1)\nu}{\nu!} \left\langle T \left\{ \int_0^\beta d\tau \bar{V}(\tau) \right\}^\nu \right\rangle_0 \quad (3.7)$$

which are distinct from those given by (3.2) and (3.3) because the operators $H_0(a^+, a)$ and $\bar{H}_0(a^+, a)$ act in different Hilbert spaces, $\mathcal{H}$ and $\mathcal{H}'$, respectively.

Two different separations and of the same Hamiltonian $H$ correspond to two different divisions of the action functional $S(a^+, a)$ as given by the formulas

$$S(a^+, a) = S_0(a^+, a) + S_I(a^+, a) \quad (3.8a)$$
$$S(a^+, a) = \bar{S}_0(a^+, a) + \bar{S}_I(a^+, a) \quad (3.8b)$$

The corresponding partition functions

$$Z'_0 = \int \mathcal{D}(a^+, a) e^{-\bar{S}_0(a^+, a)}, \quad (3.9a)$$
$$\bar{Z}_0 = \int \mathcal{D}(a^+, a) e^{-\bar{S}_0(a^+, a)} \quad (3.9b)$$

are, of course, different because the functional integrals contain different integrands which cannot be transformed one into another by any substitution of the integration variables. In the same way, the different results for the corresponding perturbation series

$$Z = \int \mathcal{D}(a^+, a) e^{-S(a^+, a)}$$
$$\bar{Z} = \int \mathcal{D}(a^+, a) e^{-\bar{S}(a^+, a)}$$

can be understood from the viewpoint of inequivalent representations of the commutator or anticommutator ring of field operators. The last two formulæ may seem to represent a paradox. Namely, the result of the integration of the same functional integral depends on the process of its integration by means of perturbation series. For this reason, the mathematicians Kozzarev and Manin regard every such computations of functional integrals done by physicists as “half-baked and ad hoc definitions.”
However, the different results \([4.10a]\) and \([4.10b]\) of the same functional integral are not due to its \textit{a priori} undefined expression, but are due to the existence of infinitely many inequivalent representations of the commutator or anticommutator ring of field operators entering the quantum field theory.

In the next section we demonstrate the properties \([4.10]\) with a toy action functional \(S(a^*, a)\) which permits exact summation of perturbation series and gives rise to infinitely many different results.

IV. TWO EXPLICITLY SOLVABLE EXAMPLES

For the purpose of demonstrating the conclusions of the two previous sections, we consider the action functional

\[
S_N(a^*, a) = \sum_{\sigma = \pm} \sum_h \varepsilon a^*_{k, \sigma} a_{k, \sigma} - \frac{1}{\gamma^2} \left( \gamma^2 \Delta^* - \gamma^2 \Delta - \sum_k a^*_{k,+} a_{k,+} \right) \left( \gamma^2 \Delta - \gamma^2 \Delta - \sum_k a_{k,-} a_{k,-} \right)
\]

\[
= \gamma^2 \Delta^* \Delta + \sum_{\sigma} \sum_k \varepsilon a^*_{k, \sigma} a_{k, \sigma} - \sum_k \left( \Delta a^*_{k,+} a_{k,-} + \Delta^* a_{k,-} a_{k,+} \right) - \frac{1}{\gamma^2} \left( \gamma^2 \Delta^* - \sum_k a^*_{k,+} a_{k,-} \right) \left( \gamma^2 \Delta - \sum_k a_{k,-} a_{k,+} \right),
\]

(4.2)

where

\[
\gamma = \sqrt{\frac{N}{g}}
\]

and \(\Delta, \Delta^*\) are arbitrary complex numbers. We separate the action \(S_N(a^*, a)\) into a sum of two terms,

\[
S_N(a^*, a) = S_{0,N}(a^*, a) + S_{I,N}(a^*, a),
\]

(4.4)

where the unperturbed action \(S_{0,N}\) is chosen to be

\[
S_{0,N} = \gamma^2 \Delta \Delta + \sum_k \varepsilon \left( a^*_{k,+} a_{k,+} + a^*_{k,-} a_{k,-} \right) - \Delta a^*_{k,+} a_{k,-} - \Delta^* a_{k,-} a_{k,+},
\]

(4.5a)

and

\[
S_{I,N} = \left[ \frac{1}{\gamma} \left( \gamma^2 \Delta^* - \sum_k a^*_{k,+} a_{k,-} \right) \right] \times \left[ \gamma^2 \Delta - \sum_k a_{k,-} a_{k,+} \right].
\]

(4.5b)

is the perturbative term which will be treated by the perturbation series. Thus we have infinitely many unperturbed actions \([4.5a]\) enumerated by arbitrary complex numbers \(\Delta\) and \(\Delta^*\).

The corresponding unperturbed partition function \(Z_{0,N}\) can be exactly and explicitly calculated with the result

\[
Z_{0,N} = E^{2N} \exp \{ -\gamma^2 \Delta \Delta^* \}, \quad E = (\varepsilon^2 + \Delta \Delta^*)^{1/2}.
\]

(4.6a)

(4.6b)

The total partition function \(Z_N\) is given by the infinite perturbation series

\[
Z_N = Z_{0,N} \sum_{\nu=0}^{\infty} \frac{(-1)^{\nu}}{\nu!} \langle S_{I,N}(a^*, a) \rangle_0.
\]

(4.7)

For the purpose of making an explicit calculation of \(Z_N\) it is convenient to introduce the generating action functional \(Z_{0,N}(a^*, a; b^*, b)\) defined by

\[
Z_{0,N}(a^*, a; b^*, b) = S_{0,N} + \frac{b^*}{\gamma} \left( \gamma^2 \Delta - \sum_k a^*_{k,-} a_{k,+} \right) + \frac{b}{\gamma} \left( \gamma^2 \Delta^* - \sum_k a_{k,+} a^*_{k,-} \right),
\]

(4.8)

where \(b^*\) and \(b\) are two complex variables. With \(Z_{0,N}(a^*, a; b^* b)\) we define the generating partition func-

---

**Figure Caption:**

The figure shows a page from a scientific document discussing quantum field theory and perturbation series. The text is formatted in a standard academic style, with equations and mathematical expressions presented clearly. The page contains several paragraphs explaining theorems, derivations, and examples, along with notation and symbols typical of theoretical physics. The content is logically structured, allowing for a thorough understanding of the concepts discussed. The page is a valuable resource for anyone studying quantum field theory or perturbation theory in physics.
tation \( \mathcal{Z}_{0,N}(b^*, b) \) by the functional integral

\[
\mathcal{Z}_{0,N}(b^*, b) = \int \mathcal{D}(a^*, a) \exp \left\{ -\mathcal{S}_{0,N}(a^*, a; b, b^*) \right\},
\]

which is a Gaussian integral and can be explicitly calculated with the result

\[
\mathcal{Z}_{0,N}(b^*, b) = \left\{ e^2 + \left( \Delta + \frac{1}{\gamma} b \right) \left( \Delta^* + \frac{1}{\gamma} b^* \right) \right\}^N \times \exp\left\{ -\gamma (\Delta^* b + b^* \Delta) \right\}. \tag{4.10}
\]

By introducing the ratio

\[
W_N(b^*, b) = \frac{\mathcal{Z}_{0,N}(b^*, b)}{\mathcal{Z}_{0,N}} \left\{ 1 + \frac{1}{N} \frac{g}{E^2} (\gamma \Delta^* b + \gamma b^* b + b^* b) \right\}^N \times \exp\left\{ -\gamma (\Delta^* b + b^* \Delta) \right\}, \tag{4.11}
\]

we express the total partition function \( \mathcal{Z}_N \) in the form

\[
\mathcal{Z}_N = \lim_{b, b^* \to 0} \mathcal{Z}_{0,N} \exp \left\{ \frac{g^2 \sigma^2}{\partial b \partial b^*} \right\} W_N(b^*, b). \tag{4.12}
\]

Now we are prepared to take the limit \( N \to \infty \) in order to get the results \( \mathcal{Z}_{0,N} \), \( \mathcal{Z}_{0,N}(b^*, b) \) for the functional integrals with infinitely many integration variables. In this limit, the asymptotic formula for the ratio \( \mathcal{Z}_{0,N}(b^*, b) \) has the form

\[
W_N(b^*, b) = \exp\left\{ \gamma \left( \frac{g}{E^2} - 1 \right) (\Delta^* b + \Delta b) \right\} + O(N^{-1/2}). \tag{4.13}
\]

The last result is inserted into \( \mathcal{Z}_{0,N} \), and by interchanging the order of the limits \( N \to \infty \) and \( b \to 0, b^* \to 0 \), we get the asymptotic formula

\[
Z_N = E^{2N} \exp \left\{ \frac{N}{g} \Delta \Delta^* \left[ \left( \frac{g}{E^2} \right)^2 \left( 1 - \frac{g}{E^2} \right) \right] \times \left( 1 - \frac{g}{E^2} + 2 \frac{g}{E^2} \Delta \Delta^* \right)^{-1} \right\} + O(1). \tag{4.14}
\]

With the restrictive condition for the parameter \( \Delta \Delta^* \) in the form

\[
\left( 1 - \frac{g}{E^2} \right) \left( 1 - \frac{g}{E^2} + 2 \frac{g}{E^2} \Delta \Delta^* \right) > 0.
\]

At the end we define the "density of the grand canonical potential" \( \Omega \) in the "thermodynamic limit" as

\[
\Omega = -\lim_{N \to \infty} \frac{1}{N} \ln \mathcal{Z}_N. \tag{4.15}
\]

By using the last formula, we obtain the final result

\[
\Omega = \frac{-\Delta \Delta^*}{g} \left\{ \left( 1 - \frac{g}{E^2} \right)^2 \left( 1 - \frac{g}{E^2} + 2 \frac{g}{E^2} \Delta \Delta^* \right)^{-1} \right\} - 1 \right\} - 2 \ln E, \tag{4.16}
\]

which explicitly shows that the functional integral \( \mathcal{Z}_{0,N} \) with the given action functional \( \mathcal{S}_{0,N} \) gives infinitely many grand canonical potentials \( \Omega \) enumerated by arbitrary complex numbers \( \Delta \) and \( \Delta^* \). This simple exactly solvable example demonstrates explicitly that functional integrals in quantum field theories cannot be regarded as Newton-Lebesgue integrals. Different results corresponding to distinct processes of their integrations of the same functional integral should not be regarded as ad hoc definitions for \( a \ priori \) undefined expressions as \( \mathcal{Z}_{0,N} \). The distinct results associated with the same functional integral correspond to the existence of inequivalent representations of the commutator or anticommutator ring of field operators in the operator approach to quantum field theory.

For demonstration purposes presented in this section, we have selected the toy action functional \( \mathcal{S}_{0,N} \) which reminds us of an oversimplified BCS model of superconductivity \( \mathcal{S}_{0,N} \). The method presented in this chapter can be straightforwardly generalized and applied, e.g., to the realistic BCS model of superconductivity with the action functional

\[
S(a^*, a) = \int \frac{d\tau}{\beta} \left\{ \sum_{\mathbf{k}, \sigma} \left[ a^*_{\mathbf{k}, \sigma}(\tau) \hat{a}_{\mathbf{k}, \sigma}(\tau) + \xi_\mathbf{k} a_{\mathbf{k}, \sigma}^*(\tau) a_{\mathbf{k}, \sigma}(\tau) \right] \right.
\]

\[
- \frac{g}{V} \sum_{\mathbf{k}, \mathbf{k'}} \left[ a^*_{\mathbf{k}, \sigma}(\tau) a^*_{-\mathbf{k}, -\sigma}(\tau) a_{-\mathbf{k'}, \sigma}(\tau) a_{\mathbf{k'}, \sigma}(\tau) \theta(\hbar \omega_D - |\xi_\mathbf{k}|) \right], \tag{4.17}
\]

where \( \mathbf{k} \) is the wave vector, \( \sigma \) is the spin \( \frac{1}{2} \) projection of an electron,

\[
\xi_\mathbf{k} = \frac{\hbar^2 \mathbf{k}^2}{2m} - \mu
\]

is the kinetic energy of an electron counted from the chemical potential \( \mu \), \( \omega_D \) is the Debye frequency, \( g \) is
the squared electron-phonon coupling constant, and $V$ is the volume of the system of electrons. By the same steps as presented by the relations [14.14 - 14.16], however, with more involved calculations, one can derive the exact density for the grand canonical potential

$$\Omega(T, \mu) = -\lim_{V\to\infty} \frac{1}{V}\ln Z$$

in the form

$$\Omega(T, \mu, \Delta^*) = \left. \frac{\Delta \Delta^*}{g} \right|_{(1-D)^2(1-D-2\Delta \Delta^* C)^{-1}} \left[ 1 - \frac{2}{(2\pi)^3} \int d^3k \left\{ \ln \left[ \frac{2\cosh \beta E_k}{2} \right] - \beta \xi_k \right\} \right] ,$$

(4.19)

where

$$E_k = \sqrt{\xi_k^2 + \Delta \Delta^* \theta(h\omega_D - |\xi_k|)} + \xi_k \theta(|\xi_k| - h\omega_D)$$

(4.20)

is the energy spectrum of elementary excitations

$$D = \frac{g}{(2\pi)^3} \int d^3k \frac{\theta(h\omega_D - |\xi_k|) \tanh \frac{\beta E_k}{2}}{2}$$

(4.21)

and

$$C = \frac{\partial D}{\partial (\Delta \Delta^*)}$$

with the restrictive condition on the parameter $\Delta \Delta^*$ in the form

$$(1-D)(1-D-2\Delta \Delta^* C) > 0.$$  

The result [14.19] shows again that the functional integral [14.16] with the given action [14.17] gives infinitely many densities of the grand-canonical potential $\Omega(T, \mu)$ enumerated by arbitrary complex values $\Delta$ and $\Delta^*$ which are called the gap functions.

The gap functions $\Delta$ and $\Delta^*$ determine the parameters $\alpha_k$ in the transformations [14.14 - 14.16] by the relation

$$\sin^2 \alpha_k = \frac{1}{2} \left( 1 - \frac{\xi_k}{\sqrt{\xi_k^2 + \Delta \Delta^*}} \right) \theta(h\omega_D - |\xi_k|).$$

(4.22)

The given infinite set of the parameters $\alpha_k$ specifies the corresponding inequivalent representation of the anticommutator ring [14.17] of the field operators $a_{k,\sigma}$ and $a_{k,\sigma}^\dagger$. Thus, the values of the functional integral [14.16] with the action functional [14.17] leading to the density [14.19] enumerated by given gap functions $\Delta$ and $\Delta^*$ correspond to distinct inequivalent representations of the anticommutator ring [14.17] of the field operators $a_{k,\sigma}^\dagger$ and $a_{k,\sigma}$.

The second law of thermodynamics, however, requires the density $\Omega(T, \mu)$ at given values of the thermodynamical variables $T$ and $\mu$ to be minimal with respect to any free parameters on which $\Omega$ is dependent. Therefore, the second law of thermodynamics restricts the class of admissible inequivalent representations by the condition

$$\left( \frac{\partial \Omega}{\partial (\Delta \Delta^*)} \right)_{T,\mu} = -\frac{\Delta \Delta^*}{g} \left( 1-D^2 \right)^{-1} \frac{\partial C}{\partial (\Delta \Delta^*)} \times \left( 3C + 2\Delta \Delta^* \frac{\partial C}{\partial (\Delta \Delta^*)} \right) = 0$$

(4.23)

The last condition admits only two solutions

$$\Delta \Delta^* = 0$$

(4.24a)

and $1-D = 0$, i.e.

$$1 = \frac{g}{(2\pi)^3} \int d^3k \frac{\beta E_k}{2} \tanh \frac{\beta h\omega - |\xi_k|}{2}$$

(4.24b)

because the expression

$$3C + 2\Delta \Delta^* \frac{\partial C}{\partial (\Delta \Delta^*)}$$

is always negative.

The last relation is the well-known gap equation of the BCS theory of superconductivity [11]. Its solution gives the gap functions $\Delta$ and $\Delta^*$ as certain functions of the temperature $T$ and the chemical potential $\mu$ at $T < T_c$, where $T_c$ is the critical temperature. Thus, the superconducting state of the electron system described by the action functional [14.17] is associated at each value of $T$ with the corresponding inequivalent representation of the anticommutator ring of the field operators $a_{k,\sigma}$ and $a_{k,\sigma}^\dagger$ specified by the set of the parameters $\alpha_k$ given by [14.22].

The result of the functional integral [14.17] with the action functional [14.17] as given by [14.22] is not the only one. One can, in fact, also find for it another class of inequivalent representations of the anticommutator ring of electron field operators as discussed in [12].

V. CONCLUSIONS

In quantum field theories, each operator $A(a^+, a)$, as a function of the field operators $a_{k,\sigma}^+$ and $a_{k,\sigma}$ satisfying the commutator or anticommutator ring of the field operators, is an abstract object. It can be represented in infinitely many inequivalent representations of the commutator or anticommutator ring of the field operators. Its trace $\text{Tr} A(a^+, a)$, is a number which is distinct for each inequivalent representation.

In the functional integral formalism of quantum field theories, one associates with each operator $A(a^+, a)$ its kernel $\hat{A}(a^+, a)$. The trace of the operator $A(a^+, a)$ is expressed by the functional integral [11]

$$\text{Tr} A(a^+, a) = \int \hat{A}(a^+, a) e^{-a^+ a} \mathcal{D}(a^+, a),$$

(5.1)
which should be regarded as an abstract object. All problems of quantum field theories based on the functional integral formalism can be thought of as problems of finding correct definitions and computational methods for the functional integrals of the type (5.1). By the selection of a method for the evaluation of the functional integral (5.1), one selects tacitly an inequivalent representation of the commutator or anticommutator ring of field operators. From this viewpoint, functional integrals in quantum field theories cannot be regarded as Newton-Lebesgue integrals giving unique values as one expects in ordinary integral calculus. Distinct values corresponding to the same functional integrals in quantum field theories reflect one of the fundamental properties of such theories, namely, the existence of infinitely many inequivalent representations for the same operator. From this viewpoint, the unexpected properties of functional integrals in quantum field theories should not be associated as with their a priori undefined expressions, but with the fundamental structure of quantum field theories.

Acknowledgements

The author (M.N.) is very grateful to Prof. C. Cronström for many stimulating discussions on the relations between functional integrals and inequivalent representations in quantum field theories.

[1] F.A.Berezin, The Method of Second Quantization, vol. 24 of Pure and Applied Physics (Academic Press, New York, 1966).
[2] L.D.Faddeev and A.A.Slavnov, Gauge Fields: Introduction to a Quantum Theory (Benjamin-Cummings, Reading Mass., 1980).
[3] V.N.Popov, Functional Integrals in Quantum Field Theory and Statistical Physics (Reidel, Dordrecht, 1983).
[4] A.N.Vasiliev, Functional Methods in Quantum Field Theory and Statistical Physics (Gordon and Breach, Amsterdam, 1988).
[5] M.Chaichian and A.Demichev, Path Integrals in Physics, vol. II. Quantum Field Theory Statistical Physics and other Modern Applications (Inst. of Physics Publishing, Bristol, 2001).
[6] I. Y. Kobzarev and Y. I. Manin, Elementary Particles: Mathematics, Physics and Philosophy, Fundamental Theories of Physics (Kluwer Academic Publishers, Dordrecht-Boston-London, 1989), p.196.
[7] R Haag, Dan. Math. Fys. Medd. 29, 1 (1955).
[8] N.N.Bogoliubov, Sov. Phys. JETP 7, 41 (1958).
[9] J.G.Valatin, Nuovo Cimento 7, 843 (1958).
[10] J.Bardeen, L.N.Cooper, and J.R.Schrieffer, Phys.Rev. 108, 1175 (1957).
[11] A.L.Fetter and J.D.Walecka, Quantum Theory of Many-Particle Systems (McGraw-Hill Inc., New York, 1971), p. 446.
[12] M.Matejka and M.Noga, (to be published).