FACES AND BASES: BOOLEAN INTERVALS

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Abstract. We consider redundant analogues of the $f$- and $h$-vectors of simplicial complexes and present bases of $\mathbb{R}^{m+1}$ related to these “long” $f$- and $h$-vectors describing the face systems $\Phi \subseteq 2^{\{1,\ldots,m\}}$; we list the corresponding change of basis matrices. The representations of the long $f$- and $h$-vectors of a face system with respect to various bases are expressed based on partitions of the system into Boolean intervals.

1. Introduction and preliminaries

Let $V$ be a finite set and let $2^V$ denote the simplex $\{F : F \subseteq V\}$. A family $\Delta \subseteq 2^V$ is called an abstract simplicial complex (or a complex) on the vertex set $V$ if, given subsets $A$ and $B$ of $V$, the inclusions $A \subseteq B \in \Delta$ imply $A \in \Delta$, and if $\{v\} \in \Delta$, for any $v \in V$; see, e.g., [4, 5, 6, 7, 8, 14, 15, 17]. If $\Gamma$ is a complex such that $\Gamma \subseteq \Delta$ (that is, $\Gamma$ is a subcomplex of $\Delta$) then the family $\Delta - \Gamma$ is called a relative simplicial complex, see [15, §III.7].

If $\Psi$ is a relative complex then the sets $F \in \Psi$ are called the faces of $\Psi$. The dimension $\dim(F)$ of a face $F$ by definition equals $|F| - 1$; the cardinality $|F|$ is called the size of $F$. Let $\#$ denote the number of sets in a family. If $\#\Psi > 0$ then the size $d(\Psi)$ of $\Psi$ is defined by $d(\Psi) := \max_{F \in \Psi} |F|$, and the dimension $\dim(\Psi)$ of $\Psi$ by definition is $d(\Psi) - 1$.

The row vector $f(\Psi) := (f_0(\Psi), f_1(\Psi), \ldots, f_{\dim(\Psi)}) \in \mathbb{N}^{d(\Psi)}$, where $f_i(\Psi) := \#\{F \in \Psi : |F| = i + 1\}$, is called the $f$-vector of $\Psi$. The row $h$-vector $h(\Psi) := (h_0(\Psi), h_1(\Psi), \ldots, h_{d(\Psi)}) \in \mathbb{Z}^{d(\Psi)+1}$ of $\Psi$ is defined by

$$\sum_{i=0}^{d(\Psi)} h_i(\Psi) \cdot y^{d(\Psi) - i} := \sum_{i=0}^{d(\Psi)} f_{i-1}(\Psi) \cdot (y - 1)^{d(\Psi) - i}. \quad (1.1)$$

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In this note we consider redundant analogues $f(\Phi; |V|) \in \mathbb{N}^{|V|+1}$ and $h(\Phi; |V|) \in \mathbb{Z}^{|V|+1}$ of the $f$- and $h$-vectors that can be used in some situations for describing the combinatorial properties of arbitrary face systems $\Phi \subseteq 2^V$.

For a positive integer $m$, let $[m]$ denote the set $\{1, 2, \ldots, m\}$. We relate to a face system $\Phi \subseteq 2^{[m]}$ the row vectors $f(\Phi; m) := (f_0(\Phi; m), f_1(\Phi; m), \ldots, f_m(\Phi; m)) \in \mathbb{N}^{m+1}$, \hspace{1cm} (1.2) $h(\Phi; m) := (h_0(\Phi; m), h_1(\Phi; m), \ldots, h_m(\Phi; m)) \in \mathbb{Z}^{m+1}$, \hspace{1cm} (1.3)

where $f_i(\Phi; m) := \# \{ F \in \Phi : |F| = i \}$, for $0 \leq i \leq m$, and the vector $h(\Phi; m)$ is defined by

$$\sum_{i=0}^{m} h_i(\Phi; m) \cdot y^{m-i} := \sum_{i=0}^{m} f_i(\Phi; m) \cdot (y - 1)^{m-i}.$$ \hspace{1cm} (1.4)

Note that if $\Psi \subset 2^{[m]}$ is a relative complex then we set $f_0(\Psi; m) := f_{-1}(\Psi) := \# \{ F \in \Psi : |F| = 0 \} \in \{0, 1\}$, $f_i(\Psi; m) := f_{i-1}(\Psi)$, for $1 \leq i \leq d(\Psi)$ and, finally, $f_1(\Psi; m) := 0$, for $d(\Psi) + 1 \leq i \leq m$.

Vectors (1.2) and (1.3) go back to analogous constructions that appear, e.g., in \cite{12, 13}. In some situations, these “long” $f$- and $h$-vectors either can be used as an intermediate description of face systems or they can independently be involved in combinatorial problems and computations, see, e.g., \cite{11}. Since the maps $\Phi \mapsto f(\Phi; m)$ and $\Phi \mapsto h(\Phi; m)$ from the Boolean lattice $\mathcal{D}(m)$ of all face systems (ordered by inclusion) to $\mathbb{Z}^{m+1}$ are \textit{valuations} on $\mathcal{D}(m)$, the long $f$- and $h$-vectors can also be used in the study of decomposition problems; here, a basic construction is a \textit{Boolean interval}, that is, the family $[A, C] := \{ B \in 2^{[m]} : A \subseteq B \subseteq C \}$, for some faces $A \subseteq C \subseteq [m]$.

We consider the vectors $f(\Phi; m)$ and $h(\Phi; m)$ as elements from the real Euclidean space $\mathbb{R}^{m+1}$ of row vectors. We present several bases of $\mathbb{R}^{m+1}$ related to face systems and list the corresponding change of basis matrices.

See, e.g., \cite{11} \S IV.4 on valuations, \cite{14} Chapter 5] on Alexander duality, \cite{21} \S VI.6, \cite{3} Chapter 5], \cite{6} \S II.5], \cite{7} \S\S 1.2, 3.6, 8.6], \cite{8} \S III.11], \cite{12} \S 5.1], \cite{15} \S II.3, II.6, III.6], \cite{16} \S 3.14], \cite{17} \S 8.3] on the Dehn-Sommerville relations, and \cite{9} on matrix analysis.

2. Notation

Throughout this note, $m$ means a positive integer; all vectors are of dimension $(m+1)$, and all matrices are $(m+1) \times (m+1)$ matrices. The components of vectors as well as the rows and columns of matrices are indexed starting with zero. For a vector $w$, $w^\top$ denotes its transpose.
If $\Phi$ is a face system, $\#\Phi > 0$, then its size $d(\Phi)$ is defined by $d(\Phi) := \max_{F \in \Phi} |F|$.

We denote the empty set by $0$, and we use the notation $\emptyset$ to denote the family containing no sets. We have $\#\emptyset = 0$, $\#\{0\} = 1$, and

$$f(\emptyset; m) = h(\emptyset; m) = (0, 0, \ldots, 0),$$

$$f(\{0\}; m) = h(2^m; m) = (1, 0, \ldots, 0).$$

$I(m)$ is the identity matrix.

$U(m)$ is the backward identity matrix whose $(i, j)$th entry is the Kronecker delta $\delta_{i+j,m}$.

$T(m)$ is the forward shift matrix whose $(i, j)$th entry is $\delta_{j-i,1}$.

If $B := (b_0, \ldots, b_m)$ is a basis of $\mathbb{R}^{m+1}$ then, given a vector $w \in \mathbb{R}^{m+1}$, we denote by $[w]_B := (\kappa_0(w, B), \ldots, \kappa_m(w, B)) \in \mathbb{R}^{m+1}$ the $(m+1)$-tuple satisfying the equality $\sum_{i=0}^m \kappa_i(w, B) \cdot b_i = w$.

3. The long $f$- and $h$-vectors

We recall the properties of vectors (1.2) and (1.3) described in [10].

(i) The maps $\Phi \mapsto f(\Phi; m)$ and $\Phi \mapsto h(\Phi; m)$ are valuations $\mathcal{D}(m) \rightarrow \mathbb{Z}^{m+1}$ on the Boolean lattice $\mathcal{D}(m)$ of all face systems (ordered by inclusion) contained in $2^m$.

(ii) Let $\Psi \subseteq 2^m$ be a relative complex.

$$h_l(\Psi) = \sum_{k=0}^l \left( m - d(\Psi) - 1 + l - k \right) \frac{(m - d(\Psi) - 1 + l - k)}{l - k} h_k(\Psi; m), \quad 0 \leq l \leq d(\Psi); \quad (3.1)$$

$$h_l(\Psi; m) = (-1)^l \sum_{k=0}^l (-1)^k \left( m - d(\Psi) - 1 + l - k \right) \frac{(m - d(\Psi) - 1 + l - k)}{l - k} h_k(\Psi), \quad 0 \leq l \leq m. \quad (3.2)$$

(iii) Let $\Phi \subseteq 2^m$.

(a)

$$h_l(\Phi; m) = (-1)^l \sum_{k=0}^l (-1)^k \left( m - k \right) \frac{(m - k)}{l - k} h_l(\Phi; m), \quad 0 \leq l \leq m; \quad (3.3)$$

$$f_l(\Phi; m) = \sum_{k=0}^l \left( m - k \right) \frac{(m - k)}{l - k} h_l(\Phi; m), \quad 0 \leq l \leq m. \quad (3.4)$$
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(b)  
\[ h_0(\Phi; m) = f_0(\Phi; m), \]  
\[ h_1(\Phi; m) = f_1(\Phi; m) - mf_0(\Phi; m), \]  
\[ h_m(\Phi; m) = (-1)^m \sum_{k=0}^{m} (-1)^k f_k(\Phi; m), \]  
\[ h(\Phi; m) \cdot \nu(m)^\top = f_m(\Phi; m). \]  

(c)  
\[ h(\Phi; m) \cdot \tau(m)^\top = f(\Phi; m) \cdot \nu(m)^\top = \#\Phi. \]  

(d) Consider the face system
\[ \Phi^* := \{ [m] - F : F \in 2^{|m|}, F \not\in \Phi \} \]  
“dual” to the system \( \Phi \).

\[ h_l(\Phi; m) + (-1)^l \sum_{k=l}^{m} \binom{k}{l} h_{k}(\Phi^*; m) = \delta_{l,0}, \quad 0 \leq l \leq m; \]  
\[ h_m(\Phi; m) = (-1)^{m+1} h_m(\Phi^*; m). \]  

If \( \Delta \) is a complex on the vertex set \([m]\) then the complex \( \Delta^* \) is called its Alexander dual. If \( \#\Delta > 0 \) and \( \#\Delta^* > 0 \) then
\[ h_l(\Delta; m) = 0, \quad 1 \leq l \leq m - d(\Delta^*) - 1, \]  
\[ h_{m-d(\Delta^*)}(\Delta; m) = -f_d(\Delta^*)(\Delta^*; m). \]  

4. BASES, AND CHANGE OF BASIS MATRICES

We relate to the simplex \( 2^{[m]} \) three pairs of bases of the space \( \mathbb{R}^{m+1} \).
Let \( \{F_0, \ldots, F_m\} \subset 2^{[m]} \) be a face system such that \( |F_k| = k \), for \( 0 \leq k \leq m \); here, \( F_0 := \emptyset \) and \( F_m := [m] \).

The first pair consists of the bases \( \{ f(\{F_0\}; m), f(\{F_1\}; m), \ldots, f(\{F_m\}; m) \} \) and \( \{ h(\{F_0\}; m), h(\{F_1\}; m), \ldots, h(\{F_m\}; m) \} \).

The bases \( \{ f([F_0, F_0]; m), f([F_0, F_1]; m), \ldots, f([F_0, F_m]; m) \} \) and \( \{ h([F_0, F_0]; m), h([F_0, F_1]; m), \ldots, h([F_0, F_m]; m) \} \) compose the second pair.

The third pair consists of the bases \( \{ f([F_m, F_m]; m), f([F_{m-1}, F_m]; m), \ldots, f([F_0, F_m]; m) \} \) and \( \{ h([F_m, F_m]; m), h([F_{m-1}, F_m]; m), \ldots, h([F_0, F_m]; m) \} \) :

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4. BASES, AND CHANGE OF BASIS MATRICES

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1) We use the notation $\mathcal{S}_m$ to denote the standard basis $(\sigma(i; m) : 0 \leq i \leq m)$ of $\mathbb{R}^{m+1}$, where
\[ \sigma(i; m) := (1, 0, \ldots, 0) \cdot T(m)^i. \tag{4.1} \]
We define a basis $\mathcal{H}_m^\bullet := (\vartheta^\bullet(i; m) : 0 \leq i \leq m)$ of $\mathbb{R}^{m+1}$, where
\[ \vartheta^\bullet(i; m) := (\vartheta_0^\bullet(i; m), \vartheta_1^\bullet(i; m), \ldots, \vartheta_m^\bullet(i; m)) \in \mathbb{Z}^{m+1}, \tag{4.2} \]
by
\[ \vartheta_j^\bullet(i; m) := (-1)^{i-j}(m-i)_{j-i}, \quad 0 \leq j \leq m. \tag{4.3} \]

2) Bases $\mathcal{F}_m^\bullet := (\varphi^\bullet(i; m) : 0 \leq i \leq m)$ and $\mathcal{H}_m^\bullet := (\vartheta^\bullet(i; m) : 0 \leq i \leq m)$ of $\mathbb{R}^{m+1}$ are defined in the following way:
\[ \varphi^\bullet(i; m) := (\varphi_0^\bullet(i; m), \varphi_1^\bullet(i; m), \ldots, \varphi_m^\bullet(i; m)) \in \mathbb{N}^{m+1}, \tag{4.4} \]
where
\[ \varphi_j^\bullet(i; m) := \binom{i}{j}, \quad 0 \leq j \leq m, \tag{4.5} \]
and
\[ \vartheta^\bullet(i; m) := (\vartheta_0^\bullet(i; m), \vartheta_1^\bullet(i; m), \ldots, \vartheta_m^\bullet(i; m)) \in \mathbb{Z}^{m+1}, \tag{4.6} \]
where
\[ \vartheta_j^\bullet(i; m) := (-1)^{j-1}(m-i)_{j-i}, \quad 0 \leq j \leq m. \tag{4.7} \]
The notations $\varphi(i; m)$ and $\vartheta(i; m)$ were used in [10] instead of $\varphi^\bullet(i; m)$ and $\vartheta^\bullet(i; m)$, respectively.

3) The third pair consists of bases $\mathcal{F}_m^\blacksquare := (\varphi^\blacksquare(i; m) : 0 \leq i \leq m)$ and $\mathcal{H}_m^\blacksquare := (\vartheta^\blacksquare(i; m) : 0 \leq i \leq m)$ of $\mathbb{R}^{m+1}$ defined as follows:
\[ \varphi^\blacksquare(i; m) := (\varphi_0^\blacksquare(i; m), \varphi_1^\blacksquare(i; m), \ldots, \varphi_m^\blacksquare(i; m)) \in \mathbb{N}^{m+1}, \tag{4.8} \]
where
\[ \varphi_j^\blacksquare(i; m) := \binom{i}{m-j}, \quad 0 \leq j \leq m, \tag{4.9} \]
and
\[ \vartheta^\blacksquare(i; m) := (\vartheta_0^\blacksquare(i; m), \vartheta_1^\blacksquare(i; m), \ldots, \vartheta_m^\blacksquare(i; m)) \in \mathbb{Z}^{m+1}, \tag{4.10} \]
where
\[ \vartheta_j^\blacksquare(i; m) := \delta_{m-i,j}, \quad 0 \leq j \leq m. \tag{4.11} \]
Note that $\mathcal{H}_m^\blacksquare$ is up to rearrangement the standard basis $\mathcal{S}_m$. 
Let $S(m)$ be the change of basis matrix from $S_\bullet_m$ to $S_\bullet_m$:
\[
S(m) := \begin{pmatrix}
\vartheta_\bullet(0; m) \\
\vdots \\
\vartheta_\bullet(m; m)
\end{pmatrix};
\] (4.12)
the $(i, j)$th entry of the inverse matrix $S(m)^{-1}$ is $\binom{m-i}{j-i}$.

For any $i \in \mathbb{N}$, $i \leq m$, we have
\[
\vartheta_\bullet(i; m) = \sigma(i; m) \cdot S(m),
\] (4.13)
\[
\vartheta^\bullet(i; m) = \varphi^\bullet(i; m) \cdot S(m),
\] (4.14)
\[
\vartheta^\triangledown(i; m) = \varphi^\triangledown(i; m) \cdot S(m).
\] (4.15)

For any face system $\Phi \subseteq 2^\mathbb{N}$, we have
\[
f(\Phi; m) = h(\Phi; m) \cdot S(m) = \sum_{l=0}^{m} f_l(\Phi; m) \cdot \vartheta_\bullet(l; m),
\] (4.16)
\[
h(\Phi; m) = f(\Phi; m) \cdot S(m)^{-1}.
\] (4.17)

The change of basis matrices corresponding to the bases defined above are collected in Table 1.

5. **Representations of the long $f$- and $h$-vectors with respect to some bases**

If $\Phi \subseteq 2^\mathbb{N}$ then we by (4.16) have
\[
f(\Phi; m) = [h(\Phi; m)]_{S_\bullet_m},
\] (5.1)
and several observations follow:
\[
[h(\Phi; m)]_{S_\bullet_m} = [f(\Phi; m)]_{S_\bullet_m};
\] (5.2)
\[
[h(\Phi; m)]_{S_\bullet_m} = [f(\Phi; m)]_{S_\bullet_m}
= h(\Phi; m) \cdot U(m);
\] (5.3)
\[
[f(\Phi; m)]_{S_\bullet_m} = f(\Phi; m) \cdot U(m)
= [h(\Phi; m)]_{S_\bullet_m} \cdot U(m).
\] (5.4)

6. **Partitions of face systems into Boolean intervals, and the long $f$- and $h$-vectors**

If
\[
\Phi = [A_1, B_1] \cup \cdots \cup [A_\theta, B_\theta]
\] (6.1)
is a partition of a face system $\Phi \subseteq 2^\mathbb{N}$, $\#\Phi > 0$, into Boolean intervals $[A_k, B_k]$, $1 \leq k \leq \theta$, then we call the collection $\mathcal{P}$ of positive integers
The profile of partition (6.1). If \( \theta = \#\Phi \) then \( p_{0l} = f_l(\Phi; m) \) whenever \( f_l(\Phi; m) > 0 \). Table 2 collects the representations of the vectors \( f(\Phi; m) \) and \( h(\Phi; m) \) with respect to various bases.

7. Appendix: Dehn-Sommerville type relations

The \( h \)-vector of a complex \( \Delta \) satisfies the Dehn-Sommerville relations if it holds

\[
h_l(\Delta) = h_{d(\Delta)-l}(\Delta) , \quad 0 \leq l \leq d(\Delta)
\]

or, equivalently (see, e.g., [12, p. 171]),

\[
h_l(\Delta; m) = (-1)^{m-d(\Delta)} h_{m-l}(\Delta; m) , \quad 0 \leq l \leq m .
\]

We say, for brevity, that a face system \( \Phi \subset 2^{[m]} \) is a DS-system if the Dehn-Sommerville type relations

\[
h_l(\Phi; m) = (-1)^{m-d(\Phi)} h_{m-l}(\Phi; m) , \quad 0 \leq l \leq m
\]

hold. The systems \( \emptyset \) and \( \{\hat{0}\} \) are DS-systems.

If \( \#\Phi > 0 \), then define the integer

\[
\eta(\Phi) := \left\{ \begin{array}{ll}
|\bigcup_{F \in \Phi} F|, & \text{if } |\bigcup_{F \in \Phi} F| \equiv d(\Phi) \pmod{2}, \\
|\bigcup_{F \in \Phi} F| + 1, & \text{if } |\bigcup_{F \in \Phi} F| \not\equiv d(\Phi) \pmod{2}.
\end{array} \right.
\]

(7.4)

Note that, given a complex \( \Delta \) with \( v \) vertices, \( v > 0 \), we have

\[
\eta(\Delta) = \left\{ \begin{array}{ll}
v, & \text{if } v \equiv d(\Delta) \pmod{2}, \\
v + 1, & \text{if } v \not\equiv d(\Delta) \pmod{2}.
\end{array} \right.
\]

(7.5)

Equality (7.3) and definition (7.4) lead to the following observation: A face system \( \Phi \) with \( \#\Phi > 0 \) is a DS-system if and only if for any \( n \in \mathbb{P} \) such that

\[
\eta(\Phi) \leq n , \quad n \equiv d(\Phi) \pmod{2} ,
\]

it holds

\[
h_l(\Phi; n) = h_{n-l}(\Phi; n) , \quad 0 \leq l \leq n ,
\]

or, equivalently,

\[
h(\Phi; n) = h(\Phi; n) \cdot U(n) ,
\]

that is, \( h(\Phi; n) \) is a left eigenvector of the \( (n+1) \times (n+1) \) backward identity matrix corresponding to the eigenvalue 1.

We come to the following conclusion:
Let $\Phi$ be a DS-system with $\#\Phi > 0$, and let $n$ be a positive integer satisfying conditions (7.6). Let $l \in \mathbb{N}$, $l \leq n$.

(i) 
\begin{align*}
\kappa_l (h(\Phi; n), \mathcal{S}_n^\uparrow) &= \kappa_l (f(\Phi; n), \mathcal{S}_n^\uparrow) \\
&= (-1)^{n-l} f_l (\Phi; n) \quad (7.9) \\
\kappa_l (h(\Phi; n), \mathcal{S}_n^\downarrow) &= \kappa_l (f(\Phi; n), \mathcal{S}_n^\downarrow) \\
&= h_l (\Phi; n) = h_{n-l}(\Phi; n) \quad (7.10) \\
\kappa_l (h(\Phi; n), \mathcal{S}_n^\uparrow) &= \kappa_l (h(\Phi; n), \mathcal{S}_n^\downarrow) \quad (7.11)
\end{align*}

(ii) If $P$ is the profile of a partition of $\Phi$ into Boolean intervals then the following equalities hold:
\begin{align*}
\sum_{i,j} p_{ij} \cdot (-1)^{i+j} \binom{j}{l-i} &= (-1)^n \sum_{i,j} p_{ij} \cdot \binom{i}{l-j} \quad (7.12) \\
\sum_{i,j} p_{ij} \cdot (-1)^j \binom{n-i-j}{l-i} &= (-1)^n \sum_{i,j} p_{ij} \cdot (-1)^j \binom{n-i-j}{l-j} \quad (7.13) \\
\sum_s \binom{s}{l} \sum_{i,j} p_{ij} \cdot (-1)^j \binom{n-i-j}{s-j} &= (-1)^n \sum_s \binom{n-s}{l} \sum_{i,j} p_{ij} \cdot (-1)^j \binom{n-i-j}{s-j} \quad (7.14)
\end{align*}
Table 1. Change of basis matrices

| Change of basis matrix | (i, j)th entry | Notation | Case m = 3 |
|------------------------|----------------|----------|-----------|
| from $\mathcal{S}_m$ to $\mathcal{F}^\downarrow_m$ | $(i)$ | $\begin{pmatrix} \varphi^*(0; m) \\ \vdots \\ \varphi^*(m; m) \end{pmatrix}$ | $\begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 \\ 1 & 3 & 3 & 1 \end{pmatrix}$ |
| from $\mathcal{H}_m^*$ to $\mathcal{H}_m^*$ | $(-1)^{i+j} \binom{i}{j}$ | $\begin{pmatrix} \varphi^*(0; m) \\ \vdots \\ \varphi^*(m; m) \end{pmatrix}$ | $\begin{pmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ -1 & 2 & 1 & 0 \\ -1 & 3 & -3 & 1 \end{pmatrix}$ |
| from $\mathcal{F}^\uparrow_m$ to $\mathcal{S}_m$ | $\delta_{m-i,j}$ | $\begin{pmatrix} \varphi^*(0; m) \\ \vdots \\ \varphi^*(m; m) \end{pmatrix}$ | $\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$ |
| Change of basis matrix | \((i, j)\)th entry | Notation | Case \(m = 3\) |
|-------------------------|-------------------|----------|----------------|
| \(S_m \) to \(S_m^*\)   | \((-1)^{j-i} \binom{m-i}{j-i}\) | \(\phi^{*}_{0; m}^{(m-i)}\), or \(\phi^{*}_{m; m}^{(m-i)}\) \(\cdot S(m)\) | \(\begin{pmatrix} 1 & -3 & 3 & -1 \\ 0 & 1 & -2 & -1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{pmatrix}\) |
| \(S_m^* \) to \(S_m\)     | \(\binom{m-i}{j-i}\) | \(\phi^{*}_{(0; m)^{-1}}\), or \(\phi^{*}_{(m; m)}\) \(\cdot S(m)^{-1}\) | \(\begin{pmatrix} 1 & 3 & 3 & 1 \\ 0 & 1 & 2 & 1 \\ 0 & 1 & 4 & 4 \\ 0 & 0 & 1 & 1 \end{pmatrix}\) |
| \(S_m^* \) to \(H_m^*\)   | \((-1)^{j-2m-i} \binom{m-i}{j}\) | \(\phi^{*}_{0; m}^{(m-i)}\), or \(\phi^{*}_{(m; m)}\) \(\cdot S(m)^{-1}\) | \(\begin{pmatrix} 1 & 3 & -3 & 1 \\ 0 & 1 & 2 & -1 \\ 0 & 1 & -4 & 4 \\ 0 & 0 & -1 & 2 \end{pmatrix}\) |
| \(S_m \) to \(F_m^\updownarrow\) | \((-1)^{j-2m-i} \binom{m-i}{j}\) | \(\phi^{*}_{0; m}^{(m-i)}\), or \(\phi^{*}_{(m; m)}\) \(\cdot S(m)^{-1}\) | \(\begin{pmatrix} 1 & 3 & 3 & 1 \\ 0 & 1 & 2 & 1 \\ 0 & 1 & 4 & 4 \\ 0 & 0 & 1 & 1 \end{pmatrix}\) |
| \(F_m^\updownarrow \) to \(H_m^\updownarrow\) | \((-1)^{j-2m-i} \binom{m-i}{j}\) | \(\phi^{*}_{0; m}^{(m-i)}\), or \(\phi^{*}_{(m; m)}\) \(\cdot S(m)^{-1}\) | \(\begin{pmatrix} 1 & 3 & -3 & 1 \\ 0 & 1 & 2 & -1 \\ 0 & 1 & -4 & 4 \\ 0 & 0 & -1 & 2 \end{pmatrix}\) |
| \(H_m^\updownarrow \) to \(F_m^\updownarrow\) | \((-1)^{j-2m-i} \binom{m-i}{j}\) | \(\phi^{*}_{0; m}^{(m-i)}\), or \(\phi^{*}_{(m; m)}\) \(\cdot S(m)^{-1}\) | \(\begin{pmatrix} 1 & 3 & -3 & 1 \\ 0 & 1 & 2 & -1 \\ 0 & 1 & -4 & 4 \\ 0 & 0 & -1 & 2 \end{pmatrix}\) |
Table 2. Representations (based on the profile of a partition of $\Phi \subseteq 2^{|m|}$, $\#\Phi > 0$, into Boolean intervals) of $f(\Phi; m)$ and $h(\Phi; m)$ with respect to various bases

| $i$th component | Expression |
|-----------------|------------|
| $f_i(\Phi; m)$  | $\sum_{i,j} p_{ij} \cdot (i^{-1})$ |
| $\kappa_i(f(\Phi; m), S^*_m)$ | $\sum_{i,j} \left( \binom{m-i}{i} \cdot \sum_{i,j} p_{ij} \cdot (i^{-1}) \right)$ |
| $\kappa_i(f(\Phi; m), S^*_m)$ | $(-1)^i \sum_{i,j} p_{ij} \cdot (i^{-1}) \binom{m-i}{i}$ |
| $h_i(\Phi; m)$  | $\sum_{i,j} p_{ij} \cdot (i^{-1})$ |
| $\kappa_i(h(\Phi; m), S^*_m)$ | $\sum_{i,j} \left( \binom{m-i}{i} \cdot \sum_{i,j} p_{ij} \cdot (i^{-1}) \right)$ |
| $\kappa_i(h(\Phi; m), S^*_m)$ | $(-1)^i \sum_{i,j} p_{ij} \cdot (i^{-1}) \binom{m-i}{i}$ |
| $\kappa_i(h(\Phi; m), S^*_m)$ | $(-1)^i \sum_{i,j} \left( \binom{m-i}{i} \cdot \sum_{i,j} p_{ij} \cdot (i^{-1}) \right)$ |

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