Recent Developments in the Analysis of Galaxy Surveys

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Summary. — These are advanced lecture notes covering recent developments in the methodology used to analyse galaxy surveys. The focus is particularly on direct measurements of the galaxy power spectrum although I also discuss its Fourier transform, the correlation function for comparison. These 2-point statistics, under the assumption that the overdensity field has Gaussian statistics on large-scales, contain the majority of the cosmological signal available from the galaxy distribution. Recent developments in multipole measurements, dealing with systematics, convolving theoretical models with the survey window function, the approximation of covariance matrices, and weighting schemes for measuring evolution with redshift are considered. The focus is on analytic explanation of the issues involved rather than on recent analyses or simulation results. These notes only loosely follow the lectures I gave in Varenna, which were more wide ranging and contained more introductory material. However, I have in the not-too-distant-past created lecture notes for an introductory course on galaxy survey analysis [1], and I did not want to duplicate those notes, but instead write something new for these proceedings.

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1. – Introduction

This review of recent developments in the analysis of galaxy surveys is meant to be useful for someone who already has a basic understanding of the field. For a general introduction see, for example, [1].

2. – The overdensity field

The dimensionless overdensity field is defined as

\[
\delta(s) = \frac{\rho(s) - \langle \rho(s) \rangle}{\langle \rho(s) \rangle},
\]

where \( \rho(s) \) is the observed galaxy density and \( \langle \rho(s) \rangle \) is the expected density.

At early times, and on large-scales at present day, \( \delta(s) \) has a distribution that is close to that of Gaussian, adiabatic fluctuations [2], and in these limits the statistical distribution is completely described by the two-point functions of this field.

[3] (commonly referred to as the FKP paper) presented the first full analysis pipeline for a galaxy survey in Fourier space. The start of this pipeline is to define the function,

\[
F(s) = \frac{w(s)}{I^{1/2}} \left[ n(s) - \langle n(s) \rangle \right],
\]

where \( n(s) \) is the observed number density of galaxies around location \( s \). The expected value \( \langle n(s) \rangle \) is commonly determined by means of a synthetic catalog of random points, Poisson sampled with the same mask and selection function as the survey. In this case,
$(n(s)) \equiv \alpha n_r(s)$, where $\alpha$ normalizes the weighted random catalogue with density $n_r(s)$ to match the weighted galaxy catalogue. The random catalogue is usually set to have $\sim 50$ times as many points as galaxies in order that the shot noise contribution is sub-dominant compared to that of the galaxy catalogue.

In contrast to $\delta(s)$ in Eq. (1), the denominator in the expression for $F(s)$ in Eq. (2) is not a function of $s$. Thus, fluctuations in the expected density of galaxies across the survey are not normalised out, and we have to accept that the power-spectrum of the field $F(s)$ is convolved with a window function. In general, it is not easy to divide by the density when calculating the power spectrum as regions outside of the survey, or where no galaxies are expected because of discreteness effects when using a random catalogue to specify the survey selection function, would cause divide-by-zero problems in the calculation of $F(s)$.

The factor $I$ normalizes this expression such that the observed monopole moment of the power has the correct amplitude in a universe with no window, $I \equiv \int ds \bar{w}^2 \bar{n}^2(s)$.

The match between galaxy and random catalogues creates what is known as the Integral Constraint (IC), forcing the average density within the survey region to be zero, ignoring larger-than-survey fluctuations in density. In Section 11, I discuss how to calculate $\alpha$ in such a way that the effect of this IC can be included in models to be fitted to the data.

The FKP paper showed how the galaxies in the survey should be optimally weighted to allow for variations in density across a survey, balancing sample variance and shot noise. Each galaxy is weighted by

$$w_{FKP}(r) = \frac{1}{1 + \bar{n}(r) \bar{P}(k)},$$

where $\bar{n}(r)$ is the expected density of galaxies, and $\bar{P}(k)$ the expected power spectrum, usually fixed at a fiducial value. This optimal weight depends on a number of assumptions - particularly that the galaxies Poisson sample the density field, and that the galaxies all have the same clustering strength. Revised weighting schemes have been proposed for more realistic models of these effects (e.g. [4, 5]).

3. – Line-of-sight assumptions

Most theoretical models (e.g. [6]) make predictions for an idealised survey in which the line-of-sight (LOS) to every galaxy is assumed to be parallel. However, for a real galaxy survey the LOS to different galaxies are not parallel, and analysing a survey under the global plane-parallel assumption only gives results close to the theory for distant surveys with small angular coverage (e.g. WiggleZ, [7]). For surveys covering a wide angular region (e.g. the Baryon Oscillation Spectroscopic Survey BOSS, [8]), such a global approximation gives a poor match to the standard theory.

In order to make a measurement that can be matched to theory (in which it is assumed that LOS to galaxies are parallel), it is far better to make a local plane-parallel
approximation when analysing a survey: that is, when measuring 2-point clustering we split the survey into pairs of galaxies, and make a local plane-parallel approximation for each pair. This approximation will still break down for pairs of galaxies separated by a wide-angle, but gives results closer to the global plane-parallel clustering amplitude for pairs whose angular separation is small. There is a subtlety in that when analysing data, we can choose whether to define the single line of sight as matching that for one galaxy in a pair, or as the direction to the pair centre, and then furthermore define how the pair centre is calculated. However, this only produces a minor perturbation on the overall effect.

The difference between clustering measurements made using the local plane-parallel approximation (see next section), and the model provided in the global plane-parallel framework, is commonly called the wide-angle effect. This gets worse for wide-angle pairs, and the severity of the problem depends on the distribution of pair separation angles in a survey. Note that the plane-parallel approximation is also commonly called the distant observer approximation.

In order to allow for wide-angle effects, one could imagine trying to model the power spectrum calculated under the local plane-parallel assumption using the full wide-angle theory [9, 10]. However, it is not clear that there would be a gain compared with using the correlation function, for which it would be simple to split pair-counts into bins in separation, pair-centre angle to the LOS, and angular separation, and then model these directly.

4. – Multipole moments

Models of the Alcock-Packynski (AP; [11]) effect and of linear Redshift-Space Distortions (RSD; [6]) show that, to first order, under the global plane-parallel assumption, the cosmological information of interest is contained within the first three even power-law moments of the correlation function or power spectrum with respect to $\mu$, the cosine of the angle that the pair or that the Fourier mode makes with respect to the line-of-sight (LOS). Decomposing into a Legendre polynomial basis instead of power-law moments has the advantage of giving independent moments of the power spectrum in the absence of a window function in the global plane-parallel limit. Remembering that the first 3 even Legendre polynomials are

\begin{align*}
L_0(\mu) &= 1, \\
L_2(\mu) &= \frac{1}{2}(3\mu^2 - 1), \\
L_4(\mu) &= \frac{1}{8}(35\mu^4 - 30\mu^2 + 3),
\end{align*}

we see that the Legendre polynomial moments can be determined by the trivial linear combination of power-law moments. Thus we use both interchangeably in the following depending on which is simplest to adopt.
The Legendre polynomial moments of the unconvolved correlation function and power spectrum in the global plane-parallel approximation are related by

\begin{equation}
\xi_\ell(s) = (2\ell + 1) \int_0^1 d\mu \xi(s) L_\ell(\mu s),
\end{equation}

\begin{equation}
P_\ell(k) = (2\ell + 1) \int_0^1 d\mu P(k) L_\ell(\mu k),
\end{equation}

with inverse formulae

\begin{equation}
\xi(s) = \sum_\ell \xi_\ell(s) L_\ell(\mu s),
\end{equation}

\begin{equation}
P(k) = \sum_\ell P_\ell(k) L_\ell(\mu k).
\end{equation}

The power spectrum and correlation function moments are related by the Hankel transform

\begin{equation}
P_\ell(k) = 4\pi i^\ell \int ds s^2 \xi_\ell(s) j_\ell(\nu k).
\end{equation}

In the following quantities that include the survey window function are denoted by a prime, while unconvolved quantities are not. No distinction is made between measured and model quantities as this should be clear from the context.

5. – Correlation function estimators in the local plane-parallel formalism

The correlation function is most commonly measured using the Landy-Szalay estimator [12]

\begin{equation}
\bar{\xi}(s, \mu) = \frac{DD(s, \mu) - 2DR(s, \mu) + RR(s, \mu)}{RR(s, \mu)},
\end{equation}

where \(\mu\) is defined with respect to the LOS to the pair centre. \(DD(s)\) is the number of galaxy-galaxy pairs within a bin with centre \(s\) normalised to the maximum possible number of galaxy-galaxy pairs, and \(RR(s)\) and \(DR(s)\) are the normalised number of random-random pairs, and galaxy-random pairs respectively.

This estimate is biased by the integral constraint, a consequence of the fact that the total number of galaxy-galaxy pairs is estimated from the sample itself (equivalent to determining \(\alpha\) in Eq. 2 from the galaxies).

\begin{equation}
\langle 1 + \bar{\xi}(s, \mu) \rangle = \frac{1 + \xi(s, \mu)}{1 + \xi_{\Omega}(s)},
\end{equation}

where \(\xi_{\Omega}(s)\) is the mean of the two-point correlation function over the mask [12]. For modern large galaxy surveys this correction is negligibly small unless we are interested
in the clustering on very large scales. See Section 11 to see how this affects the power spectrum measurement.

From this, we can calculate the multipoles by integrating over $\mu$ as in Eq. (7). Note that this integral should be carried out separately from the calculation of $\xi(s, \mu)$ of Eq. (12) because the distribution of pairs in a survey is usually not uniformly distributed in $\mu$ as required by the integral in Eq. (7). An alternative is to define a pseudo-multipole estimator where the kernel is not exactly the Legendre polynomial, which generally complicates the analysis.

Note that Eq. (12) estimates the unconvolved correlation function, and so can be directly compared with models. As we will see later, this is not necessarily true for direct estimators of the Power Spectrum.

6. Power spectrum estimators in the global plane-parallel formalism

The Quadratic Maximum Likelihood (QML) estimator [13] correctly accounts for correlations between modes when optimally measuring the power spectrum from data. In the limit of uncorrelated modes with equal noise per mode in each bin this simplifies to the FKP estimator [3].

The QML estimator is given by

$$P(k_i) = \sum_j N^{-1}_{ij} p_j,$$

where the power is a convolution of the inverse of a normalisation matrix $N_{ij}$ and a weighted two-point function

$$p_j \equiv \sum_{\alpha, \beta} F^*(k_\alpha) E_{\alpha \beta}(k_j) F(k_\beta).$$

The weight is given by the estimator matrix

$$E(k_j) = -\frac{\partial C^{-1}}{\partial P(k_j)},$$

which describes how the inverse of the density field covariance matrix $C$ changes with respect to the prior of the power spectrum of the respective bin. If the QML normalisation is proportional to the Fisher information,

$$N_{ij} = \text{tr} \left\{ C^{-1} \frac{\partial C}{\partial P(k_i)} C^{-1} \frac{\partial C}{\partial P(k_j)} \right\},$$

the QML estimator is the optimal maximum likelihood estimator of the variance of a field that obeys a multivariate Gaussian distribution [13]. I.e. assuming a Gaussian density field, the QML estimator therefore provides an estimate of the power spectrum with minimal errors.
Under the assumption that all modes are independent, the QML estimator reduces to the FKP estimator

\[ P(k_i) = \frac{1}{N_i} \sum_{k, \in \text{bin}_i} |F(k_s)|^2, \]

which simply averages the power in the Fourier modes. The FKP-style estimator is commonly applied even when the assumptions required for optimality are not valid. This concept of averaging rather than performing the optimised combination of on and off-diagonal modes is also used in the estimator in the local plane-parallel formalism given in the next section.

7. – Power spectrum estimators in the local plane-parallel formalism

Following the ethos behind the FKP power spectrum estimator, in the local plane-parallel approximation, we can define as the statistic that we want to reduce the data to as the order-\(n\) power-law moments of the window-convolved power spectrum [14]

\[ P'_n(k) = \frac{1}{4\pi} \int d\Omega_k \int d^3s_1 \int d^3s_2 (\hat{k} \cdot \hat{s}_1)^n F(s_1) F(s_2) e^{i\hat{k} \cdot (s_1 - s_2)} - P_s, \]

where we have assumed that the LOS of the pair of galaxies lies along direction \(s_1\), \(d\Omega_k\) is the solid angle element in \(k\)-space, and we denote the window convolved power spectrum \(P'_n\). \(P_s\) is the shot noise term. The local plane-parallel approximation is used to both define a single LOS to each pair of galaxies, and to define that LOS as the direction to one of the galaxies.

By writing the LOS in terms of only \(s_1\), we can split the integrals in Eq. (19), such that

\[ P'_n(k) = \frac{1}{4\pi} \int d\Omega_k A'_n(k) [A'_0(k)]^* - P_s, \]

where

\[ A'_n(k) = \int d^3s F(s)(\hat{k} \cdot \hat{s})^n e^{i\hat{k} \cdot s}. \]

[15] and [16] showed that these can be solved using Fast Fourier Transforms (FFTs) on a Cartesian grid after substituting the trivial decomposition

\[ \hat{k} \cdot \hat{s} = \frac{k_x s_x + k_y s_y + k_z s_z}{k_s}, \]

into Eq. (21). Recently, [17] proposed a Legendre polynomial decomposition of \(\hat{k} \cdot \hat{s}\) which allowed fewer FFTs to be used than in the simple approach above. Many FFT libraries are available, with FFTW being a commonly used example (http://www.fftw.org/). Thus this statistic can be quickly measured for a given catalogue.
When using FFTs to Fourier transform $F(s)$ (Eq. 2) we must sample $F(s)$ on a grid. Small-scale modes, unresolved due to the finite grid can alias large-scale modes, leading to the wrong power spectrum measurement. One way to avoid this is to use direct summation rather than FFT for the Fourier transform. However this is slow for large galaxy surveys, particularly when applied to a large random catalogue.

The effect of aliasing can be reduced through the choice of grid assignment scheme to interpolate $F(s)$ onto the regular grid, and by use of interlacing. [18] provided an excellent review of these issues, and is the source for the ideas presented in this section.

Considering calculating $F(s)$ at the grid points $s_j$, we see that we can assign galaxies to $n(s)$ as given by

$$n(s_j) = \frac{1}{H^3} \sum_{i=1}^{N_{gal}} W^{(p)}(\Delta s_x/H) W^{(p)}(\Delta s_y/H) W^{(p)}(\Delta s_z/H),$$

where $\Delta s = s_j - s_i = (\Delta s_x, \Delta s_y, \Delta s_z)$, and $H$ is the grid size. If a random catalogue is used to define the survey mask, then the assignment of these points to the grid follows the same procedure.

It is common to consider piecewise polynomial functions for the 1-dimensional functions $W^{(p)}$, which simply correspond to convolving a top-hat function with itself $(p - 1)$ times:

**Nearest Grid Point (NGP)**

$$W^{(1)}(t) = \begin{cases} 1 & \text{for } |t| < \frac{1}{2} \\ 0 & \text{otherwise} \end{cases}$$

**Cloud-In-Cell (CIC)**

$$W^{(2)}(t) = \begin{cases} 1 - |t| & \text{for } |t| < 1 \\ 0 & \text{otherwise} \end{cases}$$

**Triangular Shaped Cloud (TSC)**

$$W^{(3)}(t) = \begin{cases} \frac{4}{3} - t^2 & \text{for } |t| < \frac{1}{2} \\ \frac{1}{2} \left( \frac{3}{2} - |t| \right)^2 & \text{for } \frac{1}{2} \leq |t| < \frac{3}{2} \\ 0 & \text{otherwise} \end{cases}$$

**Piecewise Cubic Spline (PCS)**

$$W^{(4)}(t) = \begin{cases} \frac{1}{6} \left( 4 - 6 t^2 + 3 |t|^3 \right) & \text{for } 0 \leq |t| < 1 \\ \frac{1}{6} \left( 2 - |t| \right)^3 & \text{for } 1 \leq |t| < 2 \\ 0 & \text{otherwise} \end{cases}$$
The interpolation function acts as a convolution in configuration-space, and hence is a multiplicative factor in Fourier space and can be removed by dividing $F(k)$ by the Fourier transform of the window, $W^{(p)}(k_x) W^{(p)}(k_y) W^{(p)}(k_z)$, where

$$W^{(p)}(k) = \left[ \frac{\sin(k H / 2)}{(k H / 2)} \right]^p.$$  

While this corrects for the magnitude of the convolution its effects live-on in the amplitude of the aliasing effect.

A method to partially correct for aliasing based on the interlacing of two grids is discussed in the classic text of [19]. The key idea is to perform an additional, configuration-space interpolation onto a grid shifted by $H/2$ in all spatial directions, and then take the average of the two

$$F(k) = \frac{1}{2} [F_1(k) + F_2(k)],$$  

where $F_1(k)$ and $F_2(k)$ represent the individual transforms. This removes the leading order aliasing terms.

9. – Linking Fourier and Fourier-Bessel bases

The link between a Fourier-space decomposition and a decomposition into a basis consisting of spherical harmonics and spherical Bessel functions (hereafter known as a Fourier-Bessel basis) is given by the Rayleigh expansion of a plane wave. In terms of Legendre polynomials this is written:

$$e^{i \mathbf{k} \cdot \mathbf{s}} = \sum_{\ell} i^\ell (2\ell + 1) j_\ell(k s) \mathcal{L}_\ell(\mathbf{k} \cdot \mathbf{s}),$$

and in terms of Spherical Harmonics:

$$e^{i \mathbf{k} \cdot \mathbf{s}} = 4\pi \sum_{\ell m} i^\ell j_\ell(k s) Y_{\ell m}(\mathbf{k}) Y_{\ell m}^*(\mathbf{s}).$$

Using this, we can see how Eq. (21) can be solved using a Fourier-Bessel basis. As explained by [20], consider differentiating Eq. (31) with respect to $ks$ $n$ times to give:

$$i^n (\mathbf{k} \cdot \mathbf{s})^n e^{i ks} = 4\pi \sum_{\ell m} i^\ell j_\ell^{(n)}(k s) Y_{\ell m}(\mathbf{k}) Y_{\ell m}^*(\mathbf{s}),$$

where $j_\ell^{(n)}(k s)$ is the $n$’th derivative of the spherical Bessel function with respect to $ks$.

[20] used the trick of taking the derivative of the plane wave expansion to directly link power spectrum multipoles to a Fourier-Bessel decomposition. This was applied to the linear model RSD to write $\delta_{\ell m}(k)$ in terms of expansions in spherical Bessel functions.
and their derivatives. The work of [20] was applied by [21, 22] to investigate the impact of wide-angle effects from Fourier-based multipole measurements.

To see how this trick can allow us to measure the statistic in Eq. (21), using a Fourier-Bessel rather than Fourier basis, we substitute the plane-wave expansion into Eq. (21) to give

\begin{equation}
A'_n(k) = 4\pi \int d^3s F(s) \sum_{\ell m} j_{\ell}^{(n)}(ks) Y_{\ell m}(\hat{k}) Y_{\ell m}^*(\hat{s}) .
\end{equation}

If we also expand $F(s)$ in a basis of Spherical Harmonics and the $n$'th derivative of spherical Bessel functions

\begin{equation}
F_{\ell m}^{(n)}(k) = \int d^3s F(s) Y_{\ell m}^*(\hat{s}) j_{\ell}^{(n)}(ks) ,
\end{equation}

then we can write $A_n(k)$ in terms of $F_{\ell m}^{(n)}(k)$ as

\begin{equation}
A'_n(k) = 4\pi \sum_{\ell m} j_{\ell}^{(n)} F_{\ell m}(k) Y_{\ell m}(\hat{k}) .
\end{equation}

To intuitively see how the Fourier and Fourier-Bessel solutions for $A_n(k)$ are related, note that derivatives of spherical Bessel functions can be rewritten as the difference between standard spherical Bessel functions. For example:

\begin{equation}
j_{\ell}^{(1)}(ks) = -j_{\ell+1}(ks) + \frac{\ell}{ks} j_{\ell}(ks) .
\end{equation}

Subsequent derivatives can be related to standard spherical Bessel functions by recursive application of this formula. As $\ell/(ks)$ is equal to $k_{\perp}/k$ defined locally at position $s$ for modes of wavenumber $\ell$ in a Fourier-Bessel function decomposition, we can see that taking the derivatives in Eq. (32) is directly related to the local multipole expansion as considered in Eq. (22). I.e. Instead of using the derivatives of the spherical Bessel functions to get Eq. (35), it would have instead been possible to use the local definition of $k \cdot s$ in the integral to get the same result.

Substituting Eq. (35) for $n$ and $n = 0$ into Eq. (20), and using the orthogonality relations for spherical harmonics removes the angular integral and gives the simple result that

\begin{equation}
P_n'(k) = 4\pi \sum_{\ell m} F_{\ell m}^{(n)}(k) F_{\ell m}^{(0)}(k) Y_{\ell m}(\hat{k}) .
\end{equation}

Thus we see that we can use either Fourier or Fourier-Bessel bases to measure $P_n'(k)$ from a galaxy redshift survey under the local plane-parallel approximation. The use of a Fourier-Bessel basis does not alleviate wide-angle effects: these are built in to the power spectrum multipole definition (Eq. 19). Indeed, wide-angle effects are fundamentally
model dependent, so we cannot remove them completely using a model-independent or fiducial-model based estimator.

Given the same end point for the measurements, the Fourier approach is preferred as it allows the use of FFTs to perform the transforms required, saving computational resources. Using a Fourier-Bessel basis may help when calculating the cross-power spectrum of a galaxy redshift survey with an angular survey.

10. – Window convolution of models

Given that we measure the power spectrum convolved with the window function, we need a fast mechanism to convolve models to be compared to $P'_n(k)$. [23] showed that we can perform the required 3D convolution quickly using the Hankel transform relation between the window-convolved multipole moments in configuration and Fourier space.

$$P'_n(k) = 4\pi^2 \int ds \, s^2 \xi'_{2n}(s) j_n(sk).$$

Crucially, this equation holds for both the unconvolved and convolved power spectrum and correlation function pairs in both the global plane-parallel [23] and local plane-parallel [24] limits. The Hankel transform can be quickly solved using a 1D FFT, although care has to be taken given the oscillatory nature of the integrand, as described in [25].

By using this transform we can calculate the convolved model power spectra using multiplications in real space. The Legendre moments of the convolved correlation function are defined

$$\xi'_{2n}(s) = \frac{2\ell + 1}{4\pi} \int d\Omega_s \, \xi(s) W^2(s) L_\ell(s \cdot \hat{x}_1),$$

where $\xi(s)$ and $W^2(s)$ are the anisotropic correlation function and window function. $L_\ell$ is the Legendre polynomial of order $\ell$, here written as a function of the LOS to one galaxy in each pair $x_1$, matching the assumptions of Eq. (19).

We define the moments of the window function as

$$W^2_p(s) = \frac{2p + 1}{4\pi} \int d\Omega_s \int dx_1 W(x_1) W(x_1 + s) L_p(\mu_s),$$

which can be calculated by using the random catalogue to perform the integrations with a Monte-Carlo based technique. Substituting this into Eq. (39), and also expanding the unconvolved correlation function in Legendre moments means that we can rewrite this equation as

$$\xi'_{2n}(s) = (2\ell + 1) \sum_L \xi_L(s) \sum_p \frac{1}{2p + 1} W^2_p(s) a_{L,\ell}^p,$$

where $a_{L,\ell}^p$ are the solutions to the Equation $L_\ell(\mu)L_L(\mu) = \sum_p a_{p,\ell}^L L_p(\mu)$, and can be obtained by substituting in the polynomials and equating powers of $\mu$. The window
convolution spreads the linear information to $\xi'_\ell(s)$ with $\ell > 4$ and the $\xi'_\ell(s)$ modes with $\ell \leq 4$ depend on higher order moments. However, it is common to only fit to the first three even multipoles, ignoring the potential information at higher orders, and furthermore only apply the window convolution to the linear model, which is reasonable as the window effect diminishes to smaller scales. Expanding Eq. (41) gives that the relevant expansion components are

$$
\xi'_0(s) = \xi_0 W_0^2 + \frac{1}{5} \xi_2 W_2^2 + \frac{1}{9} \xi_4 W_4^2,
$$

$$
\xi'_2(s) = \xi_0 W_2^2 + \xi_2 \left[ W_0^2 + \frac{2}{7} W_2^2 + \frac{2}{7} W_4^2 \right] + \xi_4 \left[ \frac{2}{7} W_2^2 + \frac{100}{693} W_4^2 + \frac{25}{143} W_6^2 \right],
$$

$$
\xi'_4(s) = \xi_0 W_4^2 + \xi_2 \left[ \frac{18}{35} W_2^2 + \frac{20}{77} W_4^2 + \frac{45}{143} W_6^2 \right] + \xi_4 \left[ \frac{102}{431} W_2^2 + \frac{20}{77} W_4^2 + \frac{490}{2431} W_6^2 \right],
$$

keeping $\xi_\ell$ terms with $\ell \leq 4$, and including all of the relevant window multipole moments. This matches the premise that linear theory is complete to $\ell = 4$, but the window function has no such constraint. Given $\xi'_\ell(s)$, the model power spectrum multipoles can be calculated using Eq. (38).

11. – Power spectrum Integral Constraint

This formalism for the window also makes it easy to see how the integral constraint can be included in models [24]. For the power spectrum, the integral constraint is relevant because, to formulate $F(s)$ as in Eq. (2), we have matched $\langle n(s) \rangle$ to the actual observed density of galaxies. We can assume that the variations in the expected distribution of $\langle n(s) \rangle$ as a function of $s$ are known, but that the normalisation is incorrect, so

$$
\langle n(s) \rangle_{\text{assumed}} = (1 + C) \langle n(s) \rangle_{\text{true}},
$$

where $C$ is a constant. The multiplicative nature of $C$ with $\langle n(s) \rangle$ means that the constant is inside the window function convolution and is equivalent to an additive contribution to $\xi_0$. We ignore the possibility that a mistake has also been made in the calculation of $I$.

The size of the correction depends on how the randoms have been matched to the galaxies. If the total number of weighted pairs has been matched, then we have forced that $P'_0(0) = 0$. In this case, the model to be compared to the data is

$$
P'_{\ell, \text{ic-corrected}}(k) = P'_\ell(k) - \frac{P'_0(k = 0)}{W'_0(k = 0)} W'_\ell(k),
$$
where

\[ W^2(k) = 4\pi \int ds \, s^2 W^2(s) j_0 (sk). \]

Considering this from a different standpoint, by defining \( \alpha \) by matching the total number of weighted pairs in the estimator, we have a simple expression for the integral constraint to be included in the model to match the measurement. Thus this is the preferred method for calculating \( \alpha \).

### 12. Covariance matrix under Gaussian assumption

In order to make statistical inferences from the measured power, we need to model the distribution from which it is drawn. It is common to assume that the power spectrum multipoles are drawn from a multi-variate Gaussian population, in which case the Likelihood for the power spectrum is

\[
\mathcal{L}(x|p, \Psi_{\text{true}}) = \frac{|\Psi_{\text{true}}|}{\sqrt{2\pi}} \exp \left[ -\frac{1}{2} \chi^2(x, p, \Psi_{\text{true}}) \right],
\]

where

\[
\chi^2(x, p, \Psi_{\text{true}}) \equiv [x^d - x(p)]\Psi_{\text{true}}[x^d - x(p)].
\]

In the example considered in these notes, the data \( x^d \), and model for the data \( x(p) \), would be the power spectra (or correlation function), with the parameter \( p \) being the cosmological parameters of interest. \( \Psi_{\text{true}} \) is the true inverse covariance matrix.

For power spectrum measurements the covariance is

\[
\text{Cov} [P_n'(k_i), P_{n'}'(k_j)] = \langle P_n'(k_i) P_{n'}'(k_j) \rangle - \langle P_n'(k_i) \rangle \langle P_{n'}'(k_j) \rangle,
\]

where \( P_n'(k_i) \) is the window-convolved power-law moment of the power spectrum, calculated in the local plane-parallel approximation, and binned into \( k \)-bin \( i \). Each of the window convolution (including weighting), local plane-parallel geometry, power-law moment and binning effects will complicate the covariance from the simple form without these, which is given by

\[
\text{Cov} [P(k), P(k')] = \frac{2}{V} \left[ P(k) + \frac{1}{\bar{n}} \right]^2 \delta^D(k - k').
\]

Here, \( \delta^D \) is the Dirac delta function and the shot noise term assumes that the galaxies Poisson sample the underlying matter field. \( V \) is the volume of the survey. This expression follows from Wick’s theorem for a Gaussian random field with zero mean

\[
\langle \delta_1 \delta_2 \delta_3 \delta_4 \rangle = \langle \delta_1 \delta_2 \rangle \langle \delta_3 \delta_4 \rangle + \langle \delta_1 \delta_3 \rangle \langle \delta_2 \delta_4 \rangle + \langle \delta_1 \delta_4 \rangle \langle \delta_2 \delta_3 \rangle.
\]
and the standard expression for the variance of a Poisson sampling.

For the local plane-parallel power spectrum estimator of Eq. 19, applying Wick’s theorem to \( A_n'(k) \) leads to

\[
(53) \quad \text{Cov} \left[ P_n'(k_i), P_{n'}'(k_j) \right] = \frac{1}{(4\pi)^2} \int d\Omega_k \int d\Omega_{k'} \langle A_n'(k) A_{n'}'(k')^* \rangle \langle A_0'(k) A_{0'}'(k')^* \rangle + \langle A_n'(k) A_{0'}'(k')^* \rangle \langle A_0'(k) A_{n'}'(k')^* \rangle.
\]

No way of writing this equation in a form that allows its calculation using FFTs and Hankel transforms has yet been found without also applying simplifying assumptions. A common assumption to make is that the power spectrum is constant over the extent of the window function, so that the convolution breaks-down \cite{3, 26, 27}. \cite{27} showed that this results in a form for the covariance that can be solved using only FFTs.

Because of these complications it is common to estimate the covariance matrix in a brute-force way, from a large set of mock catalogues that match the survey geometry and analysis method

\[
(54) \quad \text{Cov} \left[ P_n'(k_i), P_{n'}'(k_j) \right] = \frac{1}{N_m - 1} \sum_{m=1}^{N_m} \left[ P_{n,m}'(k_i) - \overline{P}_n'(k_i) \right] \left[ P_{n',m}'(k_j) - \overline{P}_{n'}'(k_j) \right],
\]

where the sum is over \( N_m \) mock catalogues, and \( \overline{P}_n' \) is the mean power spectrum over those mocks. This method automatically includes all of the linear effects discussed above, and can also include non-linear effects to the extent that they are included in the method used to create the mocks, and the Gaussian assumption holds. Given fast methods for creating the mocks and complications due to non-linear shot-noise, galaxy bias and Redshift Space Distortions (RSD) there are many advantages to such an approach.

The disadvantages include that this covariance matrix lacks fluctuations caused by \( k \)-modes larger than the simulation box - so-called supersample covariance \cite{28}. This can be included by adjusting the mocks to force each to have slightly different cosmological parameters \cite{26}.

An additional problem is that the estimate of the covariance matrix has errors that systematically distort the likelihood. Formally, Eq. (54) gives an estimate drawn from a Wishart distribution. We can allow for the bias induced using a perturbative analysis \cite{29, 30}, which has to be adjusted if parameter errors are estimated from the likelihood surface \cite{31}. Alternatively, as demonstrated by \cite{32}, one should perform a joint likelihood analysis of both the power spectrum and covariance matrix. With the assumption of a Jeffreys prior allowing us to use Bayes theorem to determine the distribution of the true covariance matrix given our estimate from mocks, we need to adjust the Likelihood of Eq. (48), to

\[
(55) \quad L(x^d | x(p), \Psi) \propto \left[ 1 + \frac{[x^d - x(p)] \Psi [x^d - x(p)]}{N_m - 1} \right]^{-\frac{N_m}{2}}.
\]
This offers a neater and statistically more rigorous method for correcting for the approximate covariance matrix of Eq. (54) compared with the perturbative solution.

13. – 1-point Systematics

The measurement of the power spectrum from a galaxy survey is likely to be contaminated by systematic effects that alter the observed galaxy density such that the fluctuations are not driven only by astrophysical processes. Due to the way most galaxy redshift surveys to date have been constructed by spectroscopic follow-up observation of targets selected from imaging data, it is natural to think that there is a split in contaminants between angular and radial directions. However, this is not necessarily the case: for example, removing faint targets in a patch of the sky would tend to remove high-redshift galaxies in an apparent magnitude-limited sample. Consequently, we do not make any such separation here.

Using the observed target distribution, coupled with maps of the distribution of causes of potential problems - for example, imaging depth maps, or maps of bright star locations - one can look for fluctuations in target density. We expect the cosmological fluctuations to be independent of the systematics, and so any statistically significant correlation is indicative of a problem in the sample. [33, 34] undertook a careful analysis of potential systematics in BOSS, and developed a set of multiplicative weights that correct the galaxy density for these fluctuations, in effect creating a different window for the galaxies compared to the random catalogue, so that Eq. (2) is changed to

$$F(s) = \frac{w(s)}{1/2}[n_{\text{sys}}(s) - \langle n(s) \rangle],$$

where $w_{\text{sys}}$ are the systematic weights. These weights increase the noise in our estimator. This would be reduced by weighting instead the randoms to match the galaxies [35]. To see this, consider the toy example of a Poisson distribution with mean and variance $N$, weighted by a set of weights with variance $\sigma_w^2$. The variance of the weighted sample is $N(1 + \sigma_w^2)$, and so is always greater than the unweighted field.

Weighting the randoms rather than the galaxies also shows that this multiplicative weighting is equivalent to an additive contaminant [36]

$$F(s) = \frac{w(s)}{1/2}[n(s) - \langle n(s) \rangle] + f_{\text{sys}}(s).$$

The benefit of writing the effect like this is that we can equate the weighting applied to correct for systematics to the mode deprojection technique. Mode deprojection works by setting the covariance of modes to be removed to be infinite in the covariance matrix of the unbinned power

$$\text{Cov} [P'(k), P'(k')] \to \text{Cov} [P'(k), P'(k')] + \lim_{\sigma \to \infty} \sigma f_{\text{sys}}(k)f_{\text{sys}}(k').$$
[37] showed that this is mathematically equivalent to weighting the randoms, modulo making the correct normalisation when calculating the power spectrum. The standard procedure can easily be modified to include the required renormalisation when measuring the power - adjusting the effective number of modes as required.

It is possible to extend these ideas to remove multiple contaminants, and to remove sets of contaminants that span a space where systematic errors are suspected. However, as discussed in [36], the problem in general lies not in removing the contaminants, but knowing which modes are affected: the removal of contaminants only works for known unknowns, and fails for unknown unknowns.

14. – 2-point Systematics

We now consider a common problem induced by the mechanics of fiber-fed multi-object spectrographs. Most have a physical limit on how close the ends of fibers can be placed in the focal plane of the telescope such that they cannot observe close targets in a single pass of the instrument on the sky. Thus there is a geometrical difference between the samples selected for observation and the parent sample from which it was selected. Furthermore, in a sample of galaxies, close pairs tend to have a higher bias as they are located in higher mass haloes compared with isolated galaxies. Thus the lack of close pairs of galaxies due to fibre collisions changes the clustering between observed and parent samples even at large separations due to the change in mean bias. This would not be a problem if this difference occurred uniformly across the sky as we would then be simply selecting a lower bias galaxy population compared with the full target sample. However, this lack of pairs is often avoided in regions of overlapping observations, leading to an anisotropic mean bias in the observed sample. In addition, obviously, the small-scale clustering is strongly affected as we lose small-separation pairs, such that there are no angular pairs with separation smaller than the instrumental cut-off in 1-pass regions.

The problem described above is inherently of higher order than the issues raised in Section 13. In that section, we considered issues that were equivalent to changing the window through which the survey was observed. In contrast, close-pair effects are 2-point in origin as they depend on the overdensity at two positions: we cannot observe one galaxies if there is another nearby. To correct for these, we cannot easily use the techniques described in Section 13. Surveys such as DESI [38, 39] have more complicated, but related problems due to experimental limitations on how the fibres can be placed in the focal plane of the telescope.

[40] proposed the Pairwise Inverse Probability (PIP) method to correct for the kind of 2-point systematic arising from hardware limitations. For this we need a complete parent sample from which the observed subsample is selected. The PIP method estimates the probability that a pair of objects can be observed by counting how many times it is observed in a set of possible surveys selected for observation from the parent sample. This set can be created by translating or rotating the survey, or by rerunning the algorithm used to select the observed subsample with different randomly chosen priorities for different objects. The key thing is that all surveys in the set are equally likely, and the
number of pairs of objects in the parent sample that are never observed in any realisation of the survey is negligible.

By weighting each pair by the inverse of this probability when counting pairs in order to estimate the correlation function, we recover pair counts with the same expected value as those of the full parent sample. As an example, consider the situation where close pairs are only observed in parts of the survey. In order to ensure that there are no zero-probability pairs in the parent catalogue, we need to move the survey when determining the set of samples, so that any close-pair in the parent has a chance of falling into an overlap region in some surveys in the set. Because they are only observed in selected regions, close-pairs will be given a lower probability than wide-separation pairs, leading the counts to be upweighted in the sums for any realisation, correcting for this effect.

One issue with the technique is the time it takes to perform the calculation. For a galaxy survey with $10^7$ galaxies, there are $10^{14}$ pairs, and if we create a set of $10^3$ possible survey realisations, the PIP calculation is of order $10^{17}$. The computational burden can be minimised by calculating the weights on-the-fly while pair counting to estimate the correlation function, based on storing the selection of galaxies in each survey in a bit-wise way (it’s a yes/no decision on whether each object in the parent makes it into a particular survey). The weight can then be quickly calculated using a bitwise sum [40].

[41] showed how the angular clustering measurement in the parent can be incorporated into the method to improve signal, while [42] and [43] showed that the method works for the Dark Energy Spectroscopic Survey mocks and the VIMOS Public Extragalactic Redshift Survey (VIPERS), respectively.

A similar method to debias power spectrum measurements using only FFTs and Hankel transforms has yet to be developed, and it would also be useful to have a method to correct the overdensity field as used in reconstruction [44] for such effects.

15. – Binning in redshift and redshift-dependent weighting

Future surveys such as DESI [38, 39] and Euclid [45] will cover a wide range in redshift, such that there will be significant evolution in the populations of galaxies observed. Thus we either need to allow for this evolution when analysing the data, or divide the survey in redshift prior to analysis. Dividing galaxies based on their redshifts into shells will tend to miss pairs of galaxies where galaxies are in different bins. It would also be possible to split instead by radial pair-centre rather than galaxy position, which also mitigates for the effect of the window on RSD measurements [46].

An alternative is to perform multiple analyses of the full sample using sets of weights optimised to measure the evolving quantities of interest. I.e. binning can be seen as using a set of top-hat weights in redshift, and this is not necessarily the optimal choice. Using Fisher matrix based techniques, one can find sets of weights optimised for BAO [47], RSD [48], and primordial non-Gaussianity [49] measurements.

These ideas were recently applied to the quasar sample in the extended-Baryon Oscillation Spectroscopic Survey (eBOSS), recovering BAO and RSD based measurements on evolving parameters using multiple analyses with different sets of weights [50, 51, 52, 53].
16. – Reconstruction

While the bulk motion of material in the Universe drives structure growth, it also acts to smooth the primordial overdensity field, leading to the degradation of the BAO feature on small scales. The basic idea behind reconstruction is to move the late-time over-densities back to their initial positions, sharpening the BAO peak [44]. In terms of information, the bulk motion moves the small-scale 2-point information into higher order terms, and reconstruction recovers this information [54].

Reconstruction requires us to know the displacement field linking Eulerian and Lagrangian positions. This displacement field can be approximated using the standard Zeldovich displacements,

\[ \nabla \cdot \Psi + \frac{f}{b} \nabla (\Psi \cdot \hat{r}) \hat{r} = -\frac{\delta_g}{b}. \]

where \( \Psi \) is the displacement field, \( f = d \ln D(a)/d \ln a \), \( D(a) \) is the linear growth rate, and \( \sigma_8 \) normalises the amplitude of the linear power spectrum.

Solving Eq. (59) is complicated by the LOS-dependent RSD term, and that the RSD field has a non-zero curl component. Two approaches have been proposed: solving this equation on a grid spanning the survey using finite difference techniques [55], and a FFT based technique that iteratively solves for the LOS-dependence [56]. Both of these techniques have been successfully applied to the analysis of data (e.g. [57]). Eq. 59 is solved after smoothing the observed field in order to focus on the large-scale bulk motions. This changes the shape of the recovered power spectrum, requiring changes in BAO fitting routines [58].

It is easy to imaging that we can do better than solving Eq. 59 as a way to recover the bulk motions. For example there is extra information available - such as that the initial distribution of over-densities was homogeneous. The development of algorithms to reconstruct the initial density-field from an evolved field has a long history (e.g. [59]), stretching back even before the improvement of BAO observations was considered. Many methods have been proposed. Recent highlights include: Iterative reconstruction [60, 61], which removes the need to specify a smoothing scale. More complicated schemes have been proposed based on limiting the information used, allowing perturbation-theory based solutions [62, 63, 64]. The extension of such methods to biased tracers [65] and including RSD [66], have also been recently considered. Clearly, for the next generation of experiment, reconstruction will be improved compared to the algorithms used for BOSS, and we will have many methods to choose from.

17. – Conclusions

This update on the lecture notes I provided 5 years ago at a previous graduate school in Varenna [1] clearly shows that the best analysis method applied to measure clustering in galaxy surveys continues to change, with better techniques being developed alongside improvements in the experiments themselves. The techniques being used in analyses
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now are very different and more robust than those used 5 years ago, and are resulting in more accurate measurements. Given the excellent data becoming available in the next few years from DESI and Euclid, it is clear that there is a strong driver for techniques to continue to be developed. I look forward to writing the next set of notes in 5 years time, if I’m invited back to lecture again at a Varenna school.

* * *

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