Majorana qubit rotations in microwave cavities

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Majorana bound states have been proposed as building blocks for qubits on which certain operations can be performed in a topologically protected way using braiding. However, the set of these protected operations is not sufficient to realize universal quantum computing. We show that the electric field in a microwave cavity can induce Rabi oscillations between adjacent Majorana bound states. These oscillations can be used to implement an additional single-qubit gate. Supplemented with one braiding operation, this gate allows to perform arbitrary single-qubit operations.

The search for Majorana bound states (MBSs) [1] in solid-state systems is currently receiving a lot of attention from theorists and experimentalists alike [2–4]. Shortly after the prediction that MBSs can be realized in semiconductor nanowires with strong Rashba spin-orbit coupling [5–7], various experiments have indeed reported signatures of Majorana fermions in such systems, which are currently being scrutinized [8–11]. MBSs appear at the phase boundaries between topologically trivial and non-trivial sections of the nanowire. The latter can be created in the presence of a proximity-induced superconducting gap $\Delta$ and a magnetic field $B$: if $\mu_0$ denotes the (position-dependent) chemical potential, the wire is in the topologically nontrivial (trivial) phase in regions where $\mu_0^2 < B^2 - \Delta^2$ ($\mu_0^2 > B^2 - \Delta^2$) [7].

MBSs are interesting in their own right because they constitute the simplest quasiparticles with non-Abelian exchange statistics. They have also been suggested as resources for topological quantum computing [12] where the fact that a single qubit can be encoded in two or four spatially separated Majorana fermions offers protection from some common sources of decoherence.

Current proposals for topological quantum computing are striving to overcome two obstacles: First, it has been shown that the protection of MBSs from decoherence in realistic systems is less than ideal [13–15]. Indeed, quasiparticle poisoning is probably a ubiquitous source of decoherence in all proposed setups to date. If one assumes that this problem can be overcome, a second, more fundamental difficulty arises: topologically protected operations on Majorana fermions, i.e., braiding, are not sufficient to achieve universal quantum computing [12]. They have to be complemented by other single-qubit and two-qubit gates – as well as read-out operations – which will still have to be performed in a topologically unprotected way. There have been proposals on how to perform such operations, e.g., in $p$-wave superfluids [16, 17] or using conventional superconducting qubits [18].

In this paper, we will focus on the second problem. To find a way to perform operations on a Majorana qubit in a topologically unprotected, but minimally invasive way, we will investigate two MBSs which are coupled via a gapped system of length $L$. In setups involving semiconductor nanowires, such a topologically trivial gapped region can be created using appropriate gating [7, 8]. Virtual cotunneling processes lead to an energy splitting between the two MBSs which is proportional to $e^{-L/\xi}$, where $\xi(\mu)$ is a model-dependent decay length which depends on the band gap $|\mu|$ and increases for $\mu \to 0$. We shall show that by coupling the MBSs to an electric field, it becomes possible to tune $\xi$, thus allowing for a coherent manipulation of the MBSs.

To see the benefits of such an endeavor, let us consider a popular proposal for a Majorana-based qubit. Four Majorana modes $\gamma_1, \gamma_2, \gamma_3, \gamma_4$, which satisfy $\gamma_j^\dagger = \gamma_j$ and $\{\gamma_j, \gamma_k\} = 2\delta_{jk}$, can be used to define two Dirac fermion operators $\psi_L = (\gamma_1 + i\gamma_2)/2$ and $\psi_R = (\gamma_3 + i\gamma_4)/2$. Since the topological protection relies on a conserved fermion parity, the computational basis should contain states with the same parity. Therefore, one can use

$$|\downarrow\rangle = \psi_L^\dagger |0\rangle, \quad |\uparrow\rangle = \psi_R^\dagger |0\rangle$$

as the two logical states of the qubit. Certain topologically protected single-qubit gates can be realized by braiding. For instance, exchanging the positions of the MBSs $\gamma_1$ and $\gamma_2$ ($\gamma_3$ and $\gamma_4$), which can be realized in semiconductor-nanowire based setups using $T$-shaped junctions [19], corresponds to the transformations

$$U_{12} = \exp\left(\frac{i\pi}{4} \sigma_z\right), \quad U_{34} = \exp\left(-\frac{i\pi}{4} \sigma_z\right),$$

respectively, where $\sigma_z$ is a Pauli matrix in the basis $\{|\uparrow\rangle, |\downarrow\rangle\}$. Similarly, exchanging $\gamma_2$ and $\gamma_3$ corresponds to $U_{23} = \exp(i\pi \sigma_x/4)$. However, it is easy to see that these three operations are not sufficient to reach arbitrary points on the Bloch sphere spanned by all normalized linear combinations of $|\uparrow\rangle$ and $|\downarrow\rangle$.

In order to perform arbitrary single-qubit rotations, one needs to supplement this set of gates by topologically unprotected operations. If the MBSs $\gamma_2$ and $\gamma_3$ are subject to a coupling Hamiltonian

$$V = \frac{i\varepsilon}{2} \gamma_2 \gamma_3$$

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$$V = \frac{i\varepsilon}{2} \gamma_2 \gamma_3$$
Our starting point is the minimal system which contains the two MBSs $\gamma_2$ and $\gamma_3$, which are coupled indirectly by tunneling to a common, gapped central region (CR). The system is depicted schematically in Fig. 1. It can be described by the Hamiltonian

$$H_0 = \sum_k \epsilon(k) d_k^\dagger d_k - \frac{t_c}{2} \gamma_2 [d(0) - d^\dagger(0)] - \frac{it_c}{2} [d(L) + d^\dagger(L)] \gamma_3.$$  

(6)

We assume that the generic single-particle spectrum of the CR is $\epsilon(k) = k^2/(2m) - \mu$, where $\mu < 0$. Here, $k$ is the wave number which labels the eigenmodes $d_k$, and $m > 0$ is an effective mass determined by the band curvature. The CR has a length $L$, and tunneling (with amplitude $t_c$) from and to the MBSs occurs at $x = 0$ and $x = L$. In order to impose the proper boundary conditions, we use the conventional “unfolding” transformation to map the CR onto a chiral system with length $2L$ containing only right-movers. One can then impose periodic boundary conditions on this doubled system, which leads to the momentum quantization $k = \pi n/L$, where $n \geq 0$. The fermion operators in real space are given by $d(x) = (1/\sqrt{2L}) \sum_k e^{ikx} d_k$.

The Hamiltonian $H_0$ is quadratic and can easily be solved exactly. One finds that tunneling leads to a nonzero overlap between $\gamma_2$ and $\gamma_3$. For small $t_c$, the resulting level splitting due to virtual cotunneling processes is

$$\varepsilon = \frac{t_c^2}{2|\mu|\xi} e^{-L/\xi},$$  

(7)

with the decay length $\xi = (2m|\mu|)^{-1/2}$. At energies small compared to $|\mu|$, the effective coupling between the MBSs can be described by the Hamiltonian (6). Therefore, the operation $U_z(t)$ could in principle be realized by tuning the chemical potential $\mu$ time-dependently. However, it is unlikely that this can be done on short timescales and accurately enough to realize qubit rotations reliably. Moreover, tuning $|\mu|$ to small values is undesirable. Once real tunneling processes become possible ($\mu > 0$), the Majorana qubit will quickly decohere.

A possible way to realize a coupling between MBSs in a more controllable fashion is to use microwave photons to activate the tunnel process. Indeed, the induced gaps in recent experiments are in the microwave regime [5]. Moreover, hybrid structures involving semiconductor nanostructures and microwave cavities have recently been realized experimentally [21]. It may at first be surprising that Majorana fermions, which are by definition uncharged, should be susceptible to electromagnetic radiation. However, today’s solid-state versions are superpositions of particles and holes, and are chargeless only on average. The existence of a coupling of MBSs to a vector potential $A(x)$ can be derived explicitly by using the minimal-coupling substitution $p \rightarrow p - eA(x)$ in the Hamiltonian for semiconductor-nanowire based proposals [6] [7]. A schematic picture of a semiconductor nanowire
Let us first assume that the cavity field is coupled to a thermal environment \[23\]. The time-dependence of \( a(t) \) is then governed by the Langevin equation, \( \dot{a} = (-\Omega - \kappa/2) a + \sqrt{\kappa} a_{in} \), where \( \kappa \) is the cavity decay rate and \( a_{in}(t) \) is the fluctuating thermal mode, satisfying \( \langle a_{in}(\omega_a) a_{in}^\dagger(\omega') \rangle = 2\pi n_{th}(\omega) \delta(\omega - \omega') \), where \( n_{th}(\omega) \) is the Bose distribution. Approximating \( n_{th}(\omega) \approx n_{th}(\Omega) = n_{ph} \), one finds for \( n_{ph} \gg 1 \),
\[
A(\omega) = -i \sum_{\eta = \pm} \frac{n_{ph}\kappa}{(\kappa/2)^2 + (\omega - \eta \Omega)^2}.
\]

The same result formally applies if we assume that the cavity is coherently driven at its resonance frequency \( \Omega \), but in that case, \( \kappa \) should be interpreted as the linewidth of the input field and \( n_{ph} \) is proportional to the number of photons in the cavity. For \( \kappa = 0 \), \( A(\omega) \) consists of two delta peaks at \( \omega = \pm \Omega \). A nonzero linewidth turns the latter into Lorentzians of width \( \kappa \).

Finally, \( G_{ij}(t) \) is the unperturbed Green’s function of the CR. Defining \( G_d(x,x',t) = -i \langle T d(x,t) d^\dagger(x',0) \rangle_0 \), one finds
\[
G_{22}(t) = G_{33}(t) = G_d(0,0,t) - G_d(0,0,-t),
\]
\[
G_{32}(t) = -G_{32}(-t) = -i G_d(0,t) - i G_d(L,t).
\]

The Fourier transform of the Green’s function \( G_d(x,x',t) \) has the standard form, \( G_d(x,x',\omega) = \frac{1}{2\pi} \sum_k \theta(k) e^{i k (x-x')/\omega - \epsilon(k) + i \delta^k} \), where we took into account that \( \text{sgn}(\epsilon(k)) = 1 \) due to the gapped spectrum. Equations \[10\]-\[13\] can be used to obtain the MBS Green’s function,
\[
D_{22}(t) = \int \frac{d\omega}{2\pi} \frac{e^{-i\omega t} \left[ \frac{n}{2} - \Sigma_{22}(\omega) \right]}{\left[ \frac{n}{2} - \Sigma_{22}(\omega) \right]^2 + \Sigma_{23}(\omega) \Sigma_{23}(\omega)}. 
\]

The residue theorem states that this Fourier transform at positive \( t \) should be determined by the poles of the integrand in the complex lower half plane. For \( g_c = 0 \), there is a single pole at \( \omega = 0 \). For small \( g_c \), the pole is split and shifted by an amount proportional to \( g_c^2 \). Since \( \Sigma_{ij}(\omega) \) are regular functions near \( \omega = 0 \), we can replace \( \Sigma_{ij}(\omega) \) by \( \Sigma_{ij}(0) \). These functions are given by \( \Sigma_{22}(0) = 0 \) and
\[
\Sigma_{23}(0) = -i n_{ph} g_c^2 mL \sum_{\eta = \pm} \frac{1}{\sqrt{\phi_0 \sin(\sqrt{\phi_0})}}, 
\]
where we defined \( \phi_\pm = (\pm \Omega + \mu + i\kappa/2)/\epsilon_L \) and \( \epsilon_L = (2mL^2)^{-1} \) is proportional to the energy of the lowest excited state in the CR. Eventually, one finds,
\[
D_{22}(t > 0) = -e^{-2|\Gamma_R| t} e^{-2i \text{sgn}(\Gamma_R) \Omega_R t},
\]
\[
D_{23}(t > 0) = \text{sgn}(\Gamma_R) e^{-2|\Gamma_R| t} e^{-2i \text{sgn}(\Gamma_R) \Omega_R t},
\]
where \( \Gamma_R = \text{Re}[\Sigma_{23}(0)] \) and \( \Omega_R = \text{Im}[\Sigma_{23}(0)] \) denote the real and imaginary parts of the self-energy, respectively.
and only virtual tunneling processes are allowed. In this regime, the photon energy is insufficient to overcome the band gap, \( E \). First, let us focus on the regime \( \gamma < |\mu| \), where \( E_\gamma \) and \( \gamma_\gamma \) for \( \gamma > |\mu| \), see Eq. (15), and \( \gamma > |\mu| \), see Eq. (16), respectively. Solid green and red lines correspond to the solutions for the limits \( \Omega < |\mu| \), see Eq. (13), and \( \Omega > |\mu| \), see Eq. (14). Lower panel: The ratio between Rabi frequency and damping, \( \Omega_R/\Gamma_R \), determines the fidelity of qubit rotations.

The Green’s functions thus display damped Rabi oscillations of the quantum state of the MBSs. The frequency \( \Omega_R \) and damping rate \( \Gamma_R \) are determined by the relation between the energy scales \( \epsilon_L, \Omega, \gamma, \) and \( \mu \). For \( \Gamma_R = 0 \), the dynamics of the MBSs \( \gamma_2 \) and \( \gamma_3 \) according to Eq. (16),

\[
D_{22}(t > 0, \Gamma_R = 0) = -i \cos(2\Omega_R t), \\
D_{23}(t > 0, \Gamma_R = 0) = -i \sin(2\Omega_R t),
\]

coincides with the prediction of the effective Hamiltonian [3] for \( \epsilon = 2\Omega_R \). Therefore, Eq. (3) can be regarded as the effective low-energy Hamiltonian which governs the time-evolution of the MBSs \( \gamma_2 \) and \( \gamma_3 \) if \( |\Gamma_R| \ll |\Omega_R| \). For weak damping, the single-qubit rotations [4] can thus be performed by driving the system for a finite time with a microwave frequency.

Since the topological protection of the MBSs relies on a large length of the CR, we shall assume \( \epsilon_L \ll |\mu|, \Omega \). First, let us focus on the regime \( \kappa \ll \Omega < |\mu| \), where the photon energy is insufficient to overcome the band gap, and only virtual tunneling processes are allowed. In this case,

\[
\begin{align*}
|\Omega_R| &= \frac{n_{ph} g^2}{\sqrt{|\mu| (|\Omega| + |\mu|) \kappa}} \exp \left[ -\sqrt{1 - |\Omega|/|\mu|} \frac{L}{\kappa} \right], \\
\frac{\Omega_R}{\Gamma_R} &= \frac{4 \sqrt{|\Omega| + |\mu|} \epsilon_L}{\kappa}
\end{align*}
\]

These functions are plotted in Fig. 3. The Rabi frequency is, as expected, exponentially suppressed in the length of the CR. However, as the photon frequency \( \Omega \) approaches the critical value \( |\mu| \), the prefactor \( \sqrt{1 - |\Omega|/|\mu|} < 1 \) leads to a significant increase of \( \Omega_R \). The damping rate \( \Gamma_R \) is determined by the photon linewidth \( \kappa \). According to Eq. (17), a bit flip operation \( \sigma_x \) can be achieved by rotating the qubit state for a time \( t_* = \pi/(4\Omega_R) \). In the presence of damping, the fidelity of such an operation can be estimated as

\[
\mathcal{F} = e^{-2|\Gamma_R|t_*} = \exp \left[ -(\pi/2)|\Gamma_R/\Omega_R| \right].
\]

Next, we discuss the case \( \Omega > |\mu| \). In this regime, the photon field can excite real electrons from the MBSs into the CR. One still finds \( \Sigma_{22}(0) = 0 \), but the self-energy \( \Sigma_{23}(0) \) now depends sensitively on the level structure of the CR. For \( |\mu| \ll \kappa \ll \Omega \), the sine function in the denominator of Eq. (15) causes Lorentzian resonances whenever the photon frequency matches an eigenenergy of the CR. For \( \Omega \approx n^2 \pi^2 \epsilon_L - |\mu| \),

\[
\begin{align*}
|\Omega_R| &= \frac{n_{ph} g^2}{L} \frac{|\Omega + \mu - n^2 \pi^2 \epsilon_L|}{(\Omega + \mu - n^2 \pi^2 \epsilon_L)^2 + (\kappa/2)^2}, \\
\frac{\Omega_R}{\Gamma_R} &= \frac{|\Omega + \mu - n^2 \pi^2 \epsilon_L|}{\kappa/2}
\end{align*}
\]

The resonances are shown in Fig. 3. Finally, let us remark that similar Lorentzian resonances in \( \Gamma_R \) and \( \Omega_R \) as a function of \( \Omega \) also arise in the limit of a very short CR, where one can replace the CR with a single fermionic level at energy \(-\mu > 0 \).

Conclusion. We have shown that photon-assisted tunneling can have a strong impact on the coupling between two adjacent Majorana bound states (MBSs) which are separated by a gapped region. Absorption and emission of photons cause Rabi oscillations of the MBSs with a frequency that depends sensitively on the difference between the photon frequency and the gap width. Damping of the Rabi oscillations is caused by a nonzero photon linewidth. For subgap photon frequencies \( \Omega < |\mu| \), the Rabi frequency is a monotonic function of \( \Omega \) and increases exponentially for \( \Omega \rightarrow |\mu| \). Such Rabi oscillations can be useful when applied to Majorana-based qubits because, complemented by braiding, they allow for the implementation of arbitrary single-qubit rotations.

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