AN INVERSE PROBLEM FOR A SEMILINEAR ELLIPTIC EQUATION ON CONFORMALLY TRANSVERSALLY ANISOTROPIC MANIFOLDS

ALI FEIZMOHAMMADI, TONY LIIMATAINEN, AND YI-HSUAN LIN

Abstract. Given a conformally transversally anisotropic manifold \((M, g)\), we consider the semilinear elliptic equation

\[ (-\Delta_g + V)u + qu^2 = 0 \quad \text{on} \ M. \]

We show that an a priori unknown smooth function \(q\) can be uniquely determined from the knowledge of the Dirichlet-to-Neumann map associated to the semilinear elliptic equation. This extends the previously known results of the works [FO20, LLLS21a]. Our proof is based on analyzing higher order linearizations of the semilinear equation with non-vanishing boundary traces and also the study of interactions of two or more products of the so-called Gaussian quasimode solutions to the linearized equation.

Keywords. Inverse problems, boundary determination, semilinear elliptic equation, Riemannian manifold, conformally transversally anisotropic, Gaussian quasimodes, WKB construction.

Contents

1. Introduction 2
1.1. Previous literature 3
1.2. Outline of the key ideas 4
1.3. Organization of the paper 4
2. Preliminaries 5
2.1. Reduction to the case \(c = 1\) 5
2.2. Higher order linearization method with boundary values 5
2.3. Integral identities for the inverse problem 6
3. Complex geometrical optics and Gaussian beam quasimodes 7
4. Solutions for the linearized equations 9
4.1. Solutions for the second order linearization 9
4.2. Solutions for the third linearization 15
5. Proof of Theorem 1.1 18
5.1. Proof of \(q_1^2 = q_2^2\) 18
5.2. Choices of initial vectors for the third linearization 19
5.3. Proof of \(q_1^2 = q_2^2\) (continued) 22
5.4. Proof of \(q_1 = q_2\) and fourth order linearization 24
5.5. Choices of vectors for the fourth order linearization 26
5.6. Proof of \(q_1 = q_2\) (continued) 29
Appendix A. Boundary determination 30
Appendix B. Proof of the Carleman estimate with boundary terms 32
Appendix C. Computations of \(D_{ik}\) 34
References 37
1. Introduction

Let \((M, g)\) be a smooth compact Riemannian manifold of dimension \(n \geq 3\) with a smooth boundary. We assume that \((M, g)\) is conformally transversally anisotropic (CTA), that is to say,
\[
M \Subset I \times M_0,
\]
and the metric \(g\) has a smooth extension to \(\mathbb{R} \times M_0\) so that
\[
g = c(x_1, x') (dx_1 \otimes dx_1 + g_0(x')),
\]
where \((M_0, g_0)\) is a compact \((n - 1)\)-dimensional Riemannian manifold with a smooth boundary \(\partial M_0\) [DSFKSU09]. Let \(q, V\) be real-valued smooth functions on \(M\) and consider the semi-linear elliptic equation:
\[
\begin{cases}
(-\Delta g + V) u + qu^2 = 0, & \text{in } M \\
u u = f & \text{on } \partial M
\end{cases}
\]
We make the standing assumption that 0 is not a Dirichlet eigenvalue for the operator \(-\Delta g + V\). As shown in [LLLS21a, Proposition 2.1], equation (1.3) is well-posed for sufficiently small Dirichlet data \(f\). Precisely, given any \(\alpha \in (0, 1)\), there exists \(C, \delta > 0\) such that for all
\[
f \in U_\delta = \{ h \in C^{2,\alpha}(\partial M) \mid \|h\|_{C^{2,\alpha}(\partial M)} \leq \delta \},
\]
the equation (1.3) has a unique solution \(u\) in the set
\[
\{ w \in C^{2,\alpha}(M) \mid \|w\|_{C^{2,\alpha}(M)} \leq C\delta \}.
\]
Moreover,
\[
\|u\|_{C^{2,\alpha}(M)} \leq C\|f\|_{C^{2,\alpha}(\partial M)}.
\]
We define the associated Dirichlet-to-Neumann map (DN map in short) for (1.3) by
\[
\Lambda_q f = \partial_n u|_{\partial M} \text{ for } f \in U_\delta,
\]
where \(u\) is the unique solution to (1.3) that lies in the set (1.4) and \(\nu\) denotes the unit outward normal vector field on \(\partial M\).

In this paper, we consider the following inverse problem: Given an a priori fixed CTA manifold \((M, g)\) and a smooth zeroth order coefficient \(V\), is it possible to recover an a priori unknown function \(q\) given the knowledge of the map \(\Lambda_q\)? We show that this is indeed possible under the following minor technical assumption on the transversal manifold \((M_0, g_0)\):

(H1) Given any \(p \in M_0\), there exists a non-tangential geodesic \(\gamma\), which has no self-intersections and passes through \(p\).

Precisely, we prove the following uniqueness result.

**Theorem 1.1.** Let \((M, g)\) be a conformally transversally anisotropic manifold of the form (1.1)–(1.2) and suppose that (H1) is satisfied. Let \(V \in C^\infty(M)\) and assume that zero is not a Dirichlet eigenvalue for \(-\Delta g + V\) on \(M\). Let \(q_1, q_2 \in C^\infty(M)\) and assume that for some \(\delta > 0\) sufficiently small and for any \(f \in U_\delta\)
\[
\Lambda_{q_1} f = \Lambda_{q_2} f.
\]
Then
\[
q_1 = q_2 \text{ in } M.
\]

We will provide a discussion of the assumption (H1) as well as the main novelties of Theorem 1.1 in Section 1.2.
1.1. Previous literature. Inverse problems for non-linear partial differential equations is a topic with a vast literature. When the manifold is assumed to be Euclidean, the first result goes back to the work Isakov and Sylvester in [IS94] where the authors considered the equation

\[-\Delta u + F(x, u) = 0,\]

on a Euclidean domain of dimension greater than or equal to three and studied the problem of recovering a class of non-linear functions \(F(x, u)\) that satisfy a homogeneity property as well as certain monotonicity and growth conditions on its partial derivatives. The analogous problem in dimension two was first solved by Isakov and Nachman in [IN95]. For further results in Euclidean geometries, we refer the reader to the works [Sun04, Sun10, LLLS21a, LLLS21b, KU20c] in the context semilinear elliptic equations, to [Sun96, SU97, HS02, MuU20, LLS20, CF20, CFK+21, CF21, CNV19, Sha21] in the context of quasilinear elliptic equations and to [LL22, Lin21] for fractional semilinear elliptic equations. We also mention the early work [Isa93] and the work [KU20a] on similar results on Euclidean geometries for parabolic equations.

Most of the results discussed above are based on the idea of higher order linearization of nonlinear equations. The idea of a first or a second order linearization was initiated by in [Isa93, IS94] and the idea of higher order linearizations was introduced and developed fully by Kurylev, Lassas and Uhlmann [KLU18] in the context of nonlinear hyperbolic equations over Lorentzian geometries. There, the authors showed that in geometric settings, it is possible to solve certain classes of inverse problems for nonlinear hyperbolic equations in a much broader geometric generality compared to analogous inverse problems stated for linear hyperbolic equations. We refer the reader to the works [WZ19, LUW17, LUW18, UZ21b, HUZ20, UZ21a, FLO21, LLPMT21] for more examples of inverse problems for nonlinear hyperbolic equations solved in broad Lorentzian geometries. We also point out the simultaneous recovery results [LLL21, LLLZ21] in inverse problems for semilinear parabolic and hyperbolic equations in the Euclidean space.

Recently, the works [FO20, LLLS21a] introduced a similar higher order linearization approach in the context of semilinear elliptic equations on CTA manifolds. We also refer the reader to the more recent works [KU20b, LLST22] on study of similar inverse problems for nonlinear elliptic equations stated on CTA manifolds. In [FO20, LLLS21a], it was proved that for elliptic semilinear equations of the form

\[-\Delta g u + F(x, u) = 0 \quad \text{on } M,\]  

(1.5)

with non-linear functions \(F(x, z)\) that depend analytically on \(z\), the problem of recovering the differentials \(\partial_k^z F(x, 0)\) with \(k \geq 3\) is equivalent to the question of injectivity of products of four solutions to the linearized equation

\((-\Delta_g + V) u = 0 \quad \text{on } M.\)

This density property was subsequently proved in [FO20, LLLS21a] without imposing any geometric assumptions on the transversal manifold \((M_0, g_0)\), through studying products of four Gaussian quasimode solutions to the linear equation. The underlying theme discovered in the latter works is that one can solve inverse problems for nonlinear elliptic equations in CTA manifolds without imposing additional strong assumptions on the transversal manifold \((M_0, g_0)\). This is in sharp contrast to the study of inverse problems for linear elliptic equations on CTA manifolds [FKSU09, FKLS16] where additional strong assumptions must be imposed on the transversal manifold such as simplicity or existence of a strictly convex function on \((M_0, g_0)\).
In this paper, we have considered an extension of [FO20, LLLS21a] that allows non-linearities $F(x, u)$ in (1.5) that have a quadratic term in $u$. As far as we know, the only previous result that is concerned with recovery of quadratic non-linear functions on CTA manifolds is [FO20, Theorem 2] in the context of three and four dimensional CTA manifolds under additional geometric assumptions on the transversal manifold.

1.2. Outline of the key ideas. One of the key themes in the recent works that study inverse problems for nonlinear equations of the form

$$-\Delta_g u + q u^m = 0, \quad \text{on } M,$$

on CTA manifolds $(M, g)$ with any integer $m \geq 2$ is the reduction from the problem of recovering the unknown coefficient $q$ to the density problem of showing that the products of $m+1$ harmonic functions on $(M, g)$ forms a dense set in $L^\infty(M)$. This reduction is based on an $m$-fold linearization argument for the nonlinear equation.

When $m \geq 3$, the latter density problem involves the product of four harmonic functions. Following the arguments of [LLLS21a] and choosing harmonic functions based on Gaussian quasimode constructions near four intersecting geodesics on the transversal factor $(M_0, g_0)$, the density property can be proved. However, when $m = 2$, one only obtains product of three Gaussian quasimodes and this is not a sufficiently reach set to conclude our desired density claim.

In this paper, we introduce a method to solve the coefficient determination problem concerning the case $m = 2$, by considering further linearizations of the equation up to fourth order, rather than just considering the second order linearization of the equation. In this sense we over-differentiate the nonlinearity. This will allow us to implicitly obtain products of more harmonic functions. We remark that in analyzing the third and fourth order linearization of the equation (1.3), one important step is to try to understand the interactions of two Gaussian quasimode solutions to the linearized equation, namely we need to analyze an equation of the form

$$-\Delta_g w = f u_1 u_2, \quad \text{on } M,$$

where $u_1$ and $u_2$ are two Gaussian quasimodes. To the best of our knowledge, a treatment of the latter equation does not exist in the literature. We show that the latter equation can be solved asymptotically with respect to the semi-classical parameter of the Gaussian quasimodes and in doing so we obtain precise closed form expressions for $w$ modulo a small correction term, see Section 4. This will be partly based on a WKB approximation for $w$ as well as a new Carleman estimate on CTA manifolds with boundary terms. As the correction term has a non-vanishing trace on $\partial M$, we also need to introduce a variant of the higher order linearization method with a family of Dirichlet data that also depend on additional powers of the involved small parameters (see Section 2.2 and also Section 5).

Let us remark in closing that our assumption (H1) simplifies the presentation of the Gaussian quasimode solutions to the linearized equation (2.4). It is well known that Gaussian quasimodes for equation (2.4) can also be constructed in the absence of (H1), see for example [FKLS16]. However, in this paper we impose the mild assumption (H1) to better convey the key ideas discussed above.

1.3. Organization of the paper. The paper is organized as follows. In Section 2, we reduce our the setup of our study to a case where the conformal factor $c$ in (1.2) of the CTA manifold is constant 1. There we also review the higher order linearization method, and derive the linearized equations and associated integral identities we use. We review suitable Gaussian quasimodes for the first linearized equations in Section 3. In Section 4, we find solution formulas for the special solutions of the second and third linearized equations. In Section 5 we prove Theorem 1.1 by
utilizing these solutions. Finally, we prove a boundary determination result, derive Carleman estimates and compute coefficients related to products of solutions in Appendix.

2. Preliminaries

2.1. Reduction to the case $c = 1$. We show that for our purposes we can assume without any loss in generality that $c \equiv 1$. This is standard, see e.g. [DSFKSU09] or [FO20, Section 2.3]. To see this, let us define $\tilde{g} = (dx_1)^2 + g$ so that $g = c\tilde{g}$. Using the transformation law for changes of the Laplace-Beltrami operator under conformal rescalings of the metric, we write
\[c^{-\frac{n-2}{2}}(-\Delta_g u + V u + qu^2) = -\Delta_\tilde{g}v + \tilde{V}v + \tilde{q}v^2,\] (2.1)
where $v = c^{\frac{n-2}{2}} u$, $\tilde{V} = c V - (c^{\frac{n-2}{2}} \Delta_g c^{-\frac{n-2}{2}})$ and $\tilde{q} = c^{-\frac{n-2}{2}}q$. This shows that there exists a one to one correspondence between solutions to (1.3) with $f \in U_\delta$ and solutions to the following equation
\[\begin{cases}
(-\Delta_\tilde{g} + \tilde{V})v + \tilde{q}v^2 = 0, & x \in M \\
v = h, & x \in \partial M
\end{cases}\] (2.2)
provided that $\|c^{-\frac{n-2}{2}}h\|_{C^1(\partial M)} \leq \delta$. Hence, the DN map for (1.3) determines the DN map for (2.2). Thus the problem of unique recovery of $q$ from the DN map for (1.3) is equivalent to that of determining $\tilde{q}$ from the DN map for (2.2). With this observation in mind, for the remainder of this paper and without loss of generality, we assume that $c \equiv 1$ so that $g = dx_1 \otimes dx_1 + g_0$.

2.2. Higher order linearization method with boundary values. In this section, we discuss the higher order linearization method of equation (1.3). Our method is slightly different from the, by now standard, one [LLLS21a, FO20]. The difference is that we include boundary terms, which are not linear in the used small parameters.

Let $\epsilon_i \in \mathbb{R}$ and $f_i, f_{ij}, f_{ijk} \in C^{2,\alpha}(\partial M)$, for some $0 < \alpha < 1$ and for $i, j, k = 1, \ldots, 4$, and $\epsilon = (\epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4)$. In the most general case of this paper, we take boundary values $f$ to be of the form
\[f_\epsilon := \sum_{i=1}^4 \epsilon_i f_i + \sum_{i,j=1}^4 \epsilon_i \epsilon_j f_{ij} + \sum_{i,j,k=1}^4 \epsilon_i \epsilon_j \epsilon_k f_{ijk}\] on $\partial M$. (2.3)
Observe that $f_\epsilon \in U_\delta$ for sufficiently parameters $\epsilon_i$, where $U_\delta$ is defined by
\[U_\delta := \{ f \in C^{2,\alpha}(\partial M) \| f \|_{C^{2,\alpha}(\partial M)} < \delta \},\]
for some sufficiently small number $\delta > 0$. By using the implicit function theorem and the Schauder estimate for linear second order elliptic equations, one can show that the solution $u_f$ to the nonlinear equation (1.3) depends smoothly (in the Fréchet sense) on the parameters $\epsilon_1, \ldots, \epsilon_4$ (see [FO20, LLLS21a, Section 2] for detailed arguments).

The first linearization of the equation (1.3) at the zero boundary value is
\[\begin{cases}
(-\Delta_\partial + V)u^{(i)} = 0 & \text{in } M, \\
u^{(i)} = f_i & \text{on } \partial M,
\end{cases}\] (2.4)
for $i = 1, 2, 3, 4$. Here
\[v^{(i)} := \partial_{\epsilon_i} u_f,\]
where we have denoted $\epsilon = 0$ for the case $\epsilon_1 = \epsilon_2 = \epsilon_3 = \epsilon_4 = 0$. The second linearization
\[ w^{(ij)} := \partial^2_{\epsilon_i \epsilon_j} \bigg|_{\epsilon=0} u_f \]
of $u_f$ satisfies the second linearized equation (2.4)
\[
\begin{cases}
(-\Delta_g + V)w^{(ij)} = -2q v^{(i)} v^{(j)} & \text{in } M, \\
w^{(ij)} = f_{ij} & \text{on } \partial M,
\end{cases}
\]
for different $i, j = 1, 2, 3, 4$, where the functions $v^{(i)} := \partial_{\epsilon_i} |_{\epsilon=0} u_f$ are the unique solutions to the first linearized equation.

We denote
\[ w^{(ijk)} := \partial^3_{\epsilon_i \epsilon_j \epsilon_k} \bigg|_{\epsilon=0} u_f \quad \text{and} \quad w^{(1234)} := \partial^3_{\epsilon_1 \epsilon_2 \epsilon_3 \epsilon_4} \bigg|_{\epsilon=0} u_f, \]
and see that they satisfy
\[
\begin{cases}
(-\Delta_g + V)w^{(ijk)} = -2q (v^{(i)} w^{(jk)} + v^{(j)} w^{(ik)} + v^{(k)} w^{(ij)}) & \text{in } M, \\
w^{(ijk)} = f_{ijk} & \text{on } \partial M,
\end{cases}
\]
for different $i, j, k = 1, 2, 3, 4$, and
\[
\begin{cases}
(-\Delta_g + V)w^{(1234)} = -2q (v^{(1)} w^{(234)} + v^{(2)} w^{(134)}
+ v^{(3)} w^{(124)} + v^{(4)} w^{(123)}
+ w^{(12)} w^{(34)} + w^{(13)} w^{(24)} + w^{(14)} w^{(23)}) & \text{in } M, \\
w^{(1234)} = 0 & \text{on } \partial M.
\end{cases}
\]

We will construct special solutions for the above linearized equations in Section 3.

2.3. Integral identities for the inverse problem. Let us consider two potentials $q_1, q_2 \in C^\infty(M)$. Let $v^{(i)}, w^{(ij)}_\beta, w^{(ijk)}_\beta$ and $w^{(1234)}_\beta$ be the respective solutions of (2.4), (2.5), (2.6) and (2.7), where the index $\beta = 1, 2$ refers to the potentials $q = q_\beta$, and $i, j, k = 1, 2, 3, 4$. We denote by $v^{(5)}$ an additional solution to the linearized equation:
\[
\begin{cases}
(-\Delta_g + V)v^{(5)} = 0 & \text{in } M, \\
v^{(5)} = f_5 & \text{on } \partial M.
\end{cases}
\]

We record the integral identities for the second, third and fourth order linearized equations.

**Lemma 2.1** (Integral identities). Let $f_1, \ldots, f_5 \in C^{2,\alpha}(\partial M)$ and $i, j, k \in \{1, 2, 3, 4\}$. The following integral identities hold:

(1) The second order integral identity
\[
\int_{\partial M} \partial^2_{\epsilon_i \epsilon_j} \bigg|_{\epsilon=0} (\Lambda q_i f_k - \Lambda q_k f_i) f_k dS = 2 \int_M (q_1 - q_2) v^{(i)} v^{(j)} v^{(k)} dV.
\]
(2) The third order integral identity
\[
\begin{align*}
\int_{\partial M} \partial^3_{\epsilon_i \epsilon_j \epsilon_k} \bigg|_{\epsilon=0} (\Lambda q_i f_k - \Lambda q_k f_i) f_m dS \\
= 2 \int_M \left\{ q_1 (v^{(i)} w^{(jk)}_1 + v^{(j)} w^{(ik)}_1 + v^{(k)} w^{(ij)}_1) \\
- q_2 (v^{(i)} w^{(jk)}_2 + v^{(j)} w^{(ik)}_2 + v^{(k)} w^{(ij)}_2) \right\} v^{(l)} dV.
\end{align*}
\]
(3) The fourth order integral identity
\[
\int_{\partial M} \frac{\partial^4}{\partial^4 t^2 \varepsilon^2 s^4} |_{\varepsilon=0} (\Lambda q_1 f_\varepsilon - \Lambda q_2 f_\varepsilon) f_5 \, dS
\]
\[
= 2 \int_M \left\{ q_1 \left( v^{(1)} w_1^{(234)} + v^{(2)} w_1^{(134)} + v^{(3)} w_1^{(124)} + v^{(4)} w_1^{(123)} \right)
- q_2 \left( v^{(1)} w_2^{(234)} + v^{(2)} w_2^{(134)} + v^{(3)} w_2^{(124)} + v^{(4)} w_2^{(123)} \right)
\right\} v^{(5)} \, dV,
\]
\[(2.10)\]

Proof. The proof is based on integration by parts. We only prove (2.8) explicitly. The other two integral identities follow similarly.

(1) Let us consider the second linearized equation (2.5) with \( q = q_\beta \) for \( \beta = 1, 2 \). Integrating by parts yields
\[
\int_{\partial M} \frac{\partial^2}{\partial^2 t \varepsilon^2} |_{\varepsilon=0} (\Lambda q_1 f_\varepsilon - \Lambda q_2 f_\varepsilon) f_k \, dS
\]
\[
= \int_{\partial M} \left( \partial_k w_1^{(ij)} - \partial_k w_2^{(ij)} \right) f_k \, dS
\]
\[
= \int_M \left( \Delta w_1^{(ij)} - \Delta w_2^{(ij)} \right) v^{(k)} \, dV + \int_M \nabla \left( w_1^{(ij)} - w_2^{(ij)} \right) \cdot \nabla v^{(k)} \, dV
\]
\[
= \int_M V \left( w_1^{(ij)} - w_2^{(ij)} \right) v^{(k)} \, dV + 2 \int_M (q_1 - q_2) v^{(i)} v^{(j)} v^{(k)} \, dV
\]
\[
- \int_M \left( w_1^{(ij)} - w_2^{(ij)} \right) \Delta v^{(k)} \, dV
\]
\[
= 2 \int_M (q_1 - q_2) v^{(i)} v^{(j)} v^{(k)} \, dV,
\]
where we have utilized \( w_1^{(ij)} = f_{ij} = w_2^{(ij)} \) on \( \partial M \) and \( -\Delta_g + V ) v^{(k)} = 0 \) in \( M \).

(2) We have
\[
\int_{\partial M} \frac{\partial^2}{\partial^2 t \varepsilon^2 s^4} |_{\varepsilon=0} (\Lambda q_1 f_\varepsilon - \Lambda q_2 f_\varepsilon) f_i \, dS = \int_{\partial M} \left( \partial_i w_1^{(ijk)} - \partial_i w_2^{(ijk)} \right) f_j \, dS.
\]
The above integration by parts combined with the equations (2.4) and (2.6) results in the claimed identity. Proof of (3) is obtained similarly. \( \square \)

3. Complex geometrical optics and Gaussian beam quasimodes

Let us introduce the complex geometric optics type solutions for the first order linearized equation. These are solutions to the linearized equation (2.4) that concentrate on planes of the form \( I \times \gamma \), where \( I \) is an interval and \( \gamma \) is an inextendible non-tangential geodesic on \( M_0 \). We call them CGOs in short. We also assume in this paper for simplicity that \( \gamma \) does not have self-intersections.

We recall the Gaussian quasimode construction for the equation (2.4) that originated from [FKLS16, Section 3] in the setting of CTA manifolds. We follow the constructions [FO20, Section 4.1, Proposition 5.1] and [LLLS21a, Section 5 and Appendix] that allow a zeroth order term \( V \) in (2.4) as well as providing decay estimates in higher order Sobolev spaces. We refer to these works for details of the constructions in this section.
Throughout the remainder of this paper and for the sake of brevity of notation, we write \( \Delta \equiv \Delta_g \) and \( \nabla \equiv \nabla_g \). We also denote by \( \nabla' \) the gradient operator in the transversal variables on \( M_0 \). We first consider a unit speed non-tangential geodesic \( \gamma : [l_1, l_2] \to M_0 \) that connects two points on the boundary \( \partial M_0 \). We assume that \( \gamma \) does not have self-intersections for simplicity. We write \((\hat{M}_0, g_0)\) for an artificial smooth extension of \((M_0, g_0)\) into a slightly larger smooth Riemannian manifold and denote by \((t, y)\) the Fermi coordinates in a tubular neighborhood of the geodesic \( \gamma \), where \( t \in [l_1, l_2] \) and \( y \in B_{\delta'}(\mathbb{R}^{n-2}) \) for some \( \delta' > 0 \) sufficiently small. We refer the reader to [FKLS16, Section 3] for the details of the construction of Fermi coordinates.

We define the complex parameter
\[
s = \tau + i\lambda, \quad \tau > 0, \quad \lambda \in \mathbb{R},
\]
where \( i = \sqrt{-1} \), \( \lambda \) is to be viewed as a fixed parameter, and \( \tau > 0 \) is an asymptotic parameter that tends to infinity. Given any \( K > 0 \) and \( N > 0 \), there exists a positive integer \( N' \) depending on \( K, N \) (see (3.5) for the precise choice) and solutions \( v_s \) to the linear equation
\[
(-\Delta + V)v_s = 0 \quad \text{in } M,
\]
of the form
\[
v_s(x_1, t, y) = e^{\pm \tau x_1} \left( r_{\frac{N-2}{2}} e^{i \lambda \psi(t, y)} a_s(x_1, t, y) + r_s(x_1, t, y) \right), \tag{3.1}
\]
where \( \chi \) is a cutoff function supported in a \( \delta' \)-neighborhood of the origin and each term in the right hand side has certain properties that we will describe next. Alternatively, we denote \( v_s, a_s \) and \( r_s \) by \( v, a \) and \( r \) respectively.

The phase function \( \psi(t, y) \) satisfies
\[
\psi(\gamma(t)) = t, \quad \nabla \psi(\gamma(t)) = \dot{\gamma}(t), \quad \Im(D^2\psi(\gamma(t))) \geq 0, \quad \Im(D^2|\gamma(t)|) > 0. \tag{3.2}
\]
More explicitly, in terms of the Fermi coordinates we can write
\[
\psi(t, y) = t + \frac{1}{2} \sum_{j, k=1}^{n-2} H_{jk}(t) y_j y_k + O(|y|^3),
\]
where the complex-valued symmetric matrix \( H(t) = (H_{jk}(t))_{j, k=1}^{n-2} \) is given by the expression
\[
H(t) = \hat{Y}(t)Y^{-1}(t), \quad \text{for any } t \in [l_1, l_2],
\]
and \( Y \) is a non-degenerate matrix that solves the second order linear differential equation
\[
\dot{Y} + DY = 0 \quad \text{for any } t \in [l_1, l_2].
\]
Here, the symmetric matrix \( D \) is given by \( D_{jk} = \frac{1}{2} \partial^2 \chi \partial x \partial x \) for each \( j, k = 1, \ldots, n-2 \). The matrix \( H \) additionally satisfies
\[
\Im(H)(t) > 0 \quad \text{for any } t \in [l_1, l_2],
\]
and
\[
\det(\text{Im}(H(t))) \cdot |\det Y(t)|^2 = 1.
\]

Next we describe the amplitude function in the expansion (3.1). The amplitude \( a_s(x_1, t, y) \) is of the form
\[
a_s(x_1, t, y) = \left( a_0(t, y) + \frac{a_1^s(x_1, t, y)}{s} + \cdots + \frac{a_{N-1}^s(x_1, t, y)}{s^{N-1}} \right) \chi \left( \frac{|y|}{\delta'} \right), \tag{3.3}
\]
where the principal amplitude \( a_0(t, y) \) itself is given by the expression
\[
a_0(t, y) = a_{0,0}(t) + a_{0,1}(t, y) + \cdots + a_{0,N-1}(t, y).
\]
Here, $a_{0,0}(t)$ is an explicit positive function on $\gamma$ given by the expression
\[ a_{0,0}(t) = (\det Y(t))^{-\frac{1}{2}}, \quad (3.4) \]
and the subsequent terms $a_{0,j}(t,y)$ with $j = 1, 2, \ldots, N' - 1$ are homogeneous polynomials of degree $j$ in the $y$-coordinates. These terms arise as solutions to certain transport equations along the geodesic $\gamma$ on $M_0$.

The remaining amplitude terms $a_k^\pm$, for $k = 1, 2, \ldots, N' - 1$, have analogous expressions of the form
\[ a_k^\pm(x_1, t, y) = a_{k,0}^\pm(x_1, t, y) + a_{k,1}^\pm(x_1, t, y) + \cdots + a_{k,N'-1}^\pm(t, y), \]
where $a_{k,j}^\pm$ are homogeneous polynomials of degree $j$ in the $y$-coordinates, for $j = 0, 1, \ldots, N' - 1$. These amplitudes arise as solutions to certain complex transport equations on the plane $y = 0$ on $M$ (see [FO20, Section 4] for more details).

Finally, using [FO20, Proposition 2, Lemma 4] and fixing the order
\[ N' = 2 + 2N + 2K, \quad (3.5) \]
for the Gaussian quasimode construction, it follows that the remainder term in (3.1) satisfies the decay estimate
\[ \|r_s\|_{H^K(M)} \lesssim \tau^{-N}. \quad (3.6) \]

4. Solutions for the linearized equations

We discussed CGO solutions for the first order linearization of the equation $(-\Delta + V)u + gu^2 = 0$ at the zero solution in the previous section. In this section, we construct solutions for the second and third order linearizations of the equation.

4.1. Solutions for the second order linearization. In what follows, we assume that geodesics do not have self-intersections. Let $p_0 \in M_0$ and let $\gamma_1$ be a nontangential geodesic passing through $p_0$ in some direction $v \in S_{p_0}M_0$. Here $S_{p_0}M_0$ stands for unit length vectors of $T_{p_0}M_0$. We will use the following definition.

**Definition 4.1.** We say that two geodesics intersect properly if they intersect and are not reparametrizations of each other.

Assume that $\gamma_2$ is another nontangential geodesic, and that it intersects $\gamma_1$ properly at $p_0$. If $v' \in S_{p_0}M_0$ is the velocity vector of $\gamma_2$ at $p_0$, then $v'$ is linearly independent of $v$ due to the uniqueness of geodesics. Due to a compactness argument (see e.g. [LLLS21a]), the geodesics $\gamma_1$ and $\gamma_2$ can only intersect at a finite number of points.

We consider CGO solutions $v^{(1)} = v_1^{(1)}$ and $v^{(2)} = v_\tau^{(2)}$ to the equation (2.4) corresponding to geodesics $\gamma_1$ and $\gamma_2$, respectively. That is, the CGOs $v^{(k)}$ for $k = 1, 2$ are of the form (3.1):
\[ v^{(k)} = e^{\pm s_k x_1} \left( e^{\frac{2\pi i k}{2N}} e^{i s_k \frac{d_{\tau}(k)}{\tau}} + r^{(k)} \right), \quad (4.1) \]
where $\psi, d_{\tau}(k)$ and $r^{(k)}$ have the properties described in the previous section. We have also denoted
\[ s_k = c_k \tau + i\lambda_k, \quad (4.2) \]
where $c_k, \lambda_k \in \mathbb{R}$, and $\tau > 0$ is a (large) parameter.

In the next lemma, we construct solutions for the second linearized equation. After proving the lemma, in Proposition 4.4, we show that if the DN map is known, the boundary value of the solutions can be fixed.
Lemma 4.2. Let $K, N \in \mathbb{N} \cup \{0\}$. Assume that $v^{(1)}$ and $v^{(2)}$ are CGOs, which correspond to properly intersecting geodesics on $M_0$ and are of the form (4.1). If the restrictions of the amplitudes $v^{(k)}$ to $M_0$ are supported in small enough neighborhoods of the geodesics $\gamma_k$ for $k = 1, 2$, and $N' = N'(K, N)$ is large enough, then the equation

$$(-\Delta_g + V)w = -2q v^{(1)} v^{(2)} \text{ in } M$$

(4.3)

has a smooth solution $w$ up to the boundary $\partial M$ with the following properties: The solution $w$ is of the form

$$w = w_0 + e^{\Psi} R,$$

where

$$w_0 = \frac{2q}{\tau} e^{(\pm s_1 \pm s_2)x_1 + k(s_1 \psi_1 + s_1 \psi_1)} b_\tau,$$

with

$$b_\tau = \frac{1}{\tau^2} b_{-2} + \frac{1}{\tau^3} b_{-3} + \cdots + \frac{1}{\tau^{2N}} b_{-2N},$$

and

$$b_{-2} = \frac{2q}{(|\pm c_1 + c_2|^2 - \left| c_1 \nabla \psi_1 + c_2 \nabla \psi_2 \right|^2)} a_0^{(1)} a_0^{(2)}.$$  

(4.4)

The function $\Psi$ is given by

$$\Psi = (\pm c_1 \pm c_2)x_1 + i c_1 \psi_1 + i c_2 \psi_2.$$  

(4.5)

and $R = R_\tau$ is a remainder term that satisfies

$$\|R_\tau\|_{H^{N}(M)} \lesssim \tau^{-N}.$$  

Proof. We first find an approximate solution for the equation

$$(-\Delta + V)\tilde{w}_0 = -2q V^{(1)} V^{(2)} \text{ in } M,$$

(4.6)

where

$$V^{(k)}_\tau = e^{\pm s_k x_1} e^{i k \psi_k} a^{(k)}_\tau.$$  

After that, we scale $\tilde{w}_0$ and correct it by using either Carleman or elliptic estimates to a solution of (4.3). Here $\psi_k$ and $a^{(k)}$ are constructed with respect to geodesics $\gamma_k$ that intersect properly on the transversal manifold $M_0$.

We shorthand our notation and write

$$e^{\pm s_1 x_1} e^{i \psi_1} e^{\pm s_2 x_1} e^{i \psi_2} := e^{\psi} e^\Lambda,$$

where $\Psi$ is as in (4.5) and

$$\Lambda = i(\pm \lambda_1 \pm \lambda_2)x_1 - \lambda_1 \psi_1 - \lambda_2 \psi_2.$$  

Using the expressions for the amplitude functions (3.3), the equation (4.6) can be written as

$$(-\Delta_g - V)\tilde{w}_0 = e^{\psi} e^{\Lambda} \sum_{k=0}^{2(N'-1)} E_{-k} \frac{E_0}{x^k},$$

where the functions $E_{-k} \in C^\infty(M)$, $k = 0, 1, \ldots, 2(N'-1)$, are supported near the intersection points of the geodesics $\gamma_1$ and $\gamma_2$. We have

$$E_0 = 2q a_0^{(1)} a_0^{(2)}.$$  

Let us consider a WKB ansatz for $\tilde{w}_0$ of the form

$$e^{\psi} \tilde{b}.$$  

A direct calculation shows that

$$(-\Delta - V) \left( e^{\psi} \tilde{b} \right) = e^{\psi} \left( \tau^2 (\nabla \psi, \nabla \tilde{b}) + \tau \left| 2 (\nabla \tilde{b}, \nabla \psi) + \tilde{b} (\Delta \psi) \right| \right) + (\Delta - V) \tilde{b},$$  

(4.7)
where $\langle \eta, \zeta \rangle$ denotes the complexified Riemannian inner product. At the center of normal coordinates $(\eta, \zeta) = \eta \cdot \zeta = \sum_{i=1}^{n} \eta_i \zeta_i$, for any $\eta, \zeta \in \mathbb{C}^n$. Note that $\langle \cdot, \cdot \rangle$ is not a Hermitian inner product of complex vectors. Especially $\langle \eta, \eta \rangle = 0$ does not imply $\eta = 0$. We assume that $\hat{b}_r$ is an amplitude function of the form

$$\hat{b}_r = \frac{1}{\tau^2} b_{-2} + \frac{1}{\tau^3} \hat{b}_{-3} + \cdots + \frac{1}{\tau^N} \hat{b}_{-2N}.$$  

(4.7)

At an intersection point of the geodesics, we have by the properties of Gaussian beams (see (3.2)) that

$$\nabla \Psi = (\pm c_1 \pm c_2) e_1 + i c_1 \nabla' \psi_1 + i c_2 \nabla' \psi_2 = (\pm c_1 \pm c_2) e_1 + i (c_1 \gamma_1 + c_2 \gamma_2),$$

where $\gamma_1$ and $\gamma_2$ are the velocity vectors of $\gamma_1$ and $\gamma_2$ at the intersection point. Here $c_1 = \partial x, \tau$ for $x = (x_1, \ldots, x_n)$. Since the geodesics $\gamma_1$ and $\gamma_2$ intersect properly

$$\langle \nabla \Psi, \nabla \Psi \rangle = (\pm c_1 \pm c_2)^2 - |c_1 \nabla' \psi_1 + c_2 \nabla' \psi_2|^2$$

$$= c_1^2 \pm 2 c_1 c_2 + c_2^2 - c_1^2 - 2 c_1 c_2 (\gamma_1, \gamma_2) - c_2^2$$

$$= -2 c_1 c_2 ((\gamma_1, \gamma_2) = 1) \neq 0$$

at the intersection points of the geodesics. By the above and assuming that $a^{(1)}$ and $a^{(2)}$ are supported in small enough neighborhoods of $\gamma_1$ and $\gamma_2$ we have

$$|\langle \nabla \Psi, \nabla \Psi \rangle| \geq \text{constant} > 0$$  

(4.8)

on the support of each $E_{-k}$ for all $k = 0, 1, \ldots, 2(N' - 1)$.

Let us set $b_0 = \hat{b}_{-1} = 0$ and define the coefficients $\hat{b}_{-k}$ for $k = 2, \ldots, 2N'$ recursively by the formula

$$\hat{b}_{-k} = \frac{e^{\Lambda} E_{-k+2} - [2(\nabla \hat{b}_{-k+1}, \nabla \Psi) + \hat{b}_{-k+1} \Delta \Psi] - (\Delta - V) \hat{b}_{-k+2}}{\langle \nabla \Psi, \nabla \Psi \rangle}.$$  

(4.9)

Especially,

$$\hat{b}_{-2} = e^{\Lambda} \frac{2 \eta}{\langle \nabla \Psi, \nabla \Psi \rangle} a_{(1)}^{(1)} a_{(2)}^{(2)}.$$  

We also see by a recursive inspection that $b_k$ is supported on the set where (4.8) holds. Thus $\hat{b}_k$ are well-defined. It follows by reindexing sums and using $\hat{b}_{-1} = \hat{b}_0 = 0$, such that

$$(\Delta - V) \left( e^{\tau \Psi} \hat{b}_r \right) = 2 \eta V^{(1)} V^{(2)}$$

$$= e^{\tau \Psi} \sum_{k=2}^{2N'} \left[ e^{-k} \langle \nabla \Psi, \nabla \Psi \rangle \hat{b}_{-k} + \tau^{1-k} [2(\nabla \hat{b}_{-k}, \nabla \Psi) + \hat{b}_{-k} (\Delta \Psi)]ight.$$

$$+ \tau^{-k} (\Delta - V) \hat{b}_{-k} - e^{\Lambda} E_{-k+2} \left. \right]$$

$$= e^{\tau \Psi} \sum_{k=2}^{2N'} \frac{1}{\tau^{k-1}} \left[ \langle \nabla \Psi, \nabla \Psi \rangle \hat{b}_{-k} + [2(\nabla \hat{b}_{-k+1}, \nabla \Psi) + \hat{b}_{-k+1} (\Delta \Psi)]ight.$$

$$+ (\Delta - V) \hat{b}_{-k+2} - e^{\Lambda} E_{-k+2} \left. \right]$$

$$+ e^{\tau \Psi} \left( \tau^{-2N'+1} \left[ 2(\nabla \hat{b}_{-2N'}, \nabla \Psi) + \hat{b}_{-2N'} (\Delta \Psi) \right] + (\Delta - V) \hat{b}_{-2N'+1} \right)$$

$$+ \tau^{-2N'} (\Delta - V) \hat{b}_{-2N'} \right)$$

$$= e^{\tau \Psi} \mathcal{O}_{H^c M} (\tau^{-2N'+1}),$$  

(4.10)
since all the terms of the first sum after the second to last equality are zero by (4.9). Here \( \ell \) can be taken to be any number 0, 1, 2, \ldots, since \( \Psi \) and \( \hat{b}_{-k} \) are smooth.

Next we scale and correct \( e^{r\Psi} \hat{b}_r \) so that it solves (4.3). We write

\[
w = \tau \frac{1}{r^2} e^{r\Psi} \hat{b}_r + \tilde{R}_r.
\]

Note that

\[
qv(1)v(2) = q\tau \frac{1}{r^2} V(1) V(2) + qe^{r\Psi} r,
\]

where \( r \) corresponds to the correction terms \( r^{(1)} \) and \( r^{(2)} \) given by

\[
r = r^{(1)}_r \tau \frac{1}{r^2} e^{i\psi \omega_2} a^{(2)} + r^{(2)}_r \tau \frac{1}{r^2} e^{i\psi \omega_1} a^{(1)} + r^{(3)}_r \tau \frac{1}{r^2}.
\]

Hence, \( \tilde{R}_r \) solves

\[
(\Delta - V) \tilde{R}_r = -\left((\Delta - V) r \frac{1}{r^2} e^{r\Psi} \hat{b}_r - 2\tau \frac{1}{r^2} qV(1) V(2)\right) + 2qe^{r\Psi} r,
\]

whenever \( w \) solves (4.3). Now, if \( N' \) is chosen large enough, i.e.

\[
N' \geq 2 + 2N + 4K,
\]

then we have \( r = O_{H^k(M)}(\tau^{-N}) \) by combining the bounds (3.6) for the correction terms \( r^{(1)}, \beta = 1, 2 \) together with the bounds

\[
\tau \frac{1}{r^2} \|e^{i\psi \omega_2}\|_{H^k(M)} + \tau \frac{1}{r^2} \|e^{i\psi \omega_1}\|_{H^k(M)} \lesssim \tau^K.
\]

(4.12)

For example, see [FO20, Lemma 4] for the estimate (4.12).

Redefining \( N' \) larger, if necessary, the equation (4.11) for \( \tilde{R} \) together with (4.10) implies

\[
(\Delta - V) \tilde{R}_r = e^{r\Psi} O_{H^k(M)}(\tau^{-N}).
\]

By writing

\[
R_r = e^{-r\Psi} \tilde{R}_r \quad \text{and} \quad b_r = e^{-\Lambda} \hat{b}_r,
\]

the claim follows from Carleman estimates [DSFKSU09] if

\[
\pm c_1 \pm c_2 \neq 0.
\]

Alternatively, if

\[
\pm c_1 \pm c_2 = 0,
\]

we may impose zero boundary conditions for \( \tilde{R} \) and use standard elliptic estimates. This completes the proof.

\[\square\]

**Remark 4.3.** We remark that in the case \( \pm c_1 \pm c_2 \neq 0 \), the correction term \( R \) is a smooth function defined on an open manifold \( U \) such that \( M \subseteq U \), which satisfies

\[
\|R\|_{H^k(U)} \lesssim \tau^{-N},
\]

see [DSFKSU09]. In the case \( \pm c_1 \pm c_2 = 0 \), the correction term \( R \) has zero boundary values.

In addition, let us consider the second linearized equation (2.5) for two possibly different potentials \( q_1 \) and \( q_2 \). We show that if \( \Lambda_{q_1} = \Lambda_{q_2} \), then the solutions of Lemma 4.2 corresponding to potentials \( q_1 \) and \( q_2 \) can be taken to have same boundary values.

**Proposition 4.4.** Assume as in Lemma 4.2 and adopt its notation, and assume that \( \Lambda_{q_1} = \Lambda_{q_2} \) additionally. Then the second linearized equations

\[
(-\Delta + V)w^{(\beta)} = -2q_\beta v^{(1)}v^{(2)}, \quad \beta = 1, 2,
\]

have solutions of the form

\[
u^{(\beta)} = u_0^{(\beta)} + e^{r\Psi} R^{(\beta)}.
\]
Here
\[
\begin{align*}
  w^{(β)}_0 &= \tau^{\frac{2qβ}{qβ - qβ}} e^{(±c_1 ± c_2) x_1 + (s_1 \psi_1 + s_2 \psi_2)} b^2(α), \\
  b^{(β)} &= τ^{−2b^{(α)}} + \cdots + τ^{−2N}\beta b^{(β)}_{−2N}, \\
  b_{−2}^{(β)} &= \frac{2qβ}{(±c_1 ± c_2)^2 - |c_1 \nabla^t \psi_1 + c_2 \nabla^t \psi_2|^2} a^{(1)}_{0} (2).
\end{align*}
\]
Moreover \( R^{(β)} = O_{L^2(M)}(τ^{−N}) (β = 1, 2) \) and
\[
w^{(1)}|_{∂M} = w^{(2)}|_{∂M}.
\]

In order to prove Proposition 4.4, we need a boundary determination result:

**Proposition 4.5** (Boundary determination). For \( m ≥ 2, m ∈ \mathbb{N} \) let \((M, g)\) be a compact Riemannian manifold with \( C^∞ \) boundary \( ∂M \) and consider the boundary value problem
\[
\begin{align*}
(-Δ_g + V)u + qu^n &= 0 & \text{in } M, \\
u &= f & \text{on } ∂M.
\end{align*}
\]
Assume that the DN map \( Λ_q \) of the equation (4.14) is known for small boundary values. Then \( Λ_q \) determines the formal Taylor series of \( q \) on the boundary \( ∂M \).

In addition, if \( f ∈ C^{2,α}(∂M) \) is so small that (4.14) has a unique small solution, the DN map determines the formal Taylor series of the solution \( u = uf \) at any point on the boundary.

We also need the following Carleman estimate with boundary terms.

**Lemma 4.6** (Carleman estimate with boundary terms). Let \((M, g)\) be a compact, smooth, transversally anisotropic Riemannian manifold with a smooth boundary. Let \( V ∈ L^∞(M) \). There exists constants \( τ_0 > 0 \) and \( C > 0 \) depending only on \((M, g)\) and \( ∥V∥_{L^∞(M)} \) such that given any \( |τ| > τ_0 \), and any \( v ∈ C^{2}(M) \), there holds
\[
\begin{align*}
C|τ| ∥v∥_{L^2(M)} \leq ∥e^{−τt} (-Δ_g + V)(e^{τt} v)∥_{L^2(M)} + |τ|^\frac{q}{2} ∥v∥_{W^{2,∞}(∂M)} \\
+ |τ|^\frac{q}{2} ∥∂v∥_{W^{1,∞}(∂M)} + |τ|^\frac{q}{2} ∥∂^2v∥_{L^∞(∂M)}.
\end{align*}
\]

We have placed the proofs of the above two results in the the Appendix A and B, respectively. The proof of Proposition 4.5 uses a standard boundary determination result for linearized second order elliptic equations. The proof of Lemma 4.6 is by integration by parts and using standard elliptic estimates. In this paper, the preceding Carleman estimate with the \( L^2(M) \) bound is sufficient in deriving the upper bound for the correction term \( R^{(β)} \) in Proposition 4.4 for \( β = 1, 2 \); however let us also mention that analogous Carleman estimates with boundary terms can be obtained in higher Sobolev spaces \( H^k(M) \), for \( k ∈ \mathbb{N} \).

**Proof of Proposition 4.4.** Let us first consider the case \( ±c_1 ± c_2 ≠ 0 \). By Lemma 4.2 we have a solution of the form
\[
w^{(2)} = w^{(2)}_0 + e^{τΨ} R^{(2)}
\]
for the equation
\[
(-Δ + V)w^{(2)} = -2qβ w^{(1)} v^{(2)}.
\]
In general, controlling the boundary value of \( R^{(2)} \) is hard. As remarked in Remark 4.3, we have that \( R^{(2)} \) is a smooth function defined on an open manifold \( U \) such that \( M ∈ U \), which satisfies
\[
∥R^{(2)}∥_{H^k(U)} ≤ \frac{C}{τ^N}
\]
if $N' = N'(K, N)$ was chosen large enough.

By redefining $K$ as $K + 5/2$ (and thus also redefining also $N'$ larger) and using trace theorem

$$R^{(2)}|_{\partial M} = \mathcal{O}_{H^{K}(\partial M)}(\tau^{-N})$$

and

$$\partial_{\nu} R^{(2)}|_{\partial M} = \mathcal{O}_{H^{K}(\partial M)}(\tau^{-K}), \quad \partial_{\nu}^{2} R^{(2)}|_{\partial M} = \mathcal{O}_{H^{K}(\partial M)}(\tau^{-K}).$$

Let us then consider the equation (4.13) for $q_1$ with boundary value $w^{(2)}|_{\partial M}$.

By the standard elliptic theory, we know that there is a unique solution $w^{(1)}$ to the equation

$$\begin{cases}
(-\Delta + V) w^{(1)} = -2q_{1} v^{(1)} v^{(2)} & \text{in } M, \\
w^{(1)} = w^{(2)}|_{\partial M} & \text{on } \partial M.
\end{cases}$$

We write

$$w^{(1)} = w^{(1)}_{0} + e^{\tau \Psi} R^{(1)},$$

where $w^{(1)}_{0} = \tau a_{w} e^{(\pm a_{1} \pm a_{2}) \cdot x} + \text{higher-order terms}$ is the WKB ansatz given as in Lemma 4.2 such that

$$(\Delta - V) w^{(1)} - 2q_{1} v^{(1)} v^{(2)} = e^{\tau \Psi} F.$$

Here

$$F = \mathcal{O}_{H^{K}(M)}(\tau^{-N}),$$

which can be derived by making the WKB ansatz $w^{(1)}_{0}$ precise enough (i.e. $N'$ large enough). Since $w^{(1)}$ solves $(-\Delta + V) w^{(1)} = -2q_{1} v^{(1)} v^{(2)}$, we have that $R^{(1)}$ solves the conjugated equation

$$e^{-\tau \Psi} (\Delta - V) e^{\tau \Psi} R^{(1)} = \mathcal{O}_{H^{K}(M)}(\tau^{-N}).$$

Unfortunately, we can not directly deduce from standard Carleman estimates that the correction term $\|R^{(1)}\|_{L^{2}(M)}$ is small.

As matter of fact, in order to obtain that $\|R^{(1)}\|_{L^{2}(M)}$ is small, we use the assumption $\Lambda_{q_{1}} = \Lambda_{q_{2}}$, which implies that the DN maps of the second linearized equations (4.4) for $q_{1}$ and $q_{2}$ are the same. By additionally using the boundary determination result (Proposition 4.5), we have that

$$q_{1} = q_{2} \quad \text{on } \partial M$$

up to infinite order. The ansatz $w^{(1)}_{0}$ and $w^{(2)}_{0}$ depend on $(M, g)$ and the potentials $q_{1}$ and $q_{2}$ respectively. The dependence on the potentials is local. That is, the dependence on pointwise values of the potentials and their derivatives, see (4.4).

It follows that

$$(4.20)$$

and also

$$\partial_{\nu} w^{(1)}_{0} = \partial_{\nu} w^{(2)}_{0}. \quad \text{and} \quad \partial_{\nu}^{2} w^{(1)}_{0} = \partial_{\nu}^{2} w^{(2)}_{0}. \quad (4.21)$$

Consequently, by using $w_{1}|_{\partial M} = w_{2}|_{\partial M}$, we have that

$$R^{(1)}|_{\partial M} = e^{-\tau \Psi} |_{\partial M} (w^{(1)} - w^{(2)}_{0})|_{\partial M} = e^{-\tau \Psi} |_{\partial M} (w^{(2)} - w^{(2)}_{0})|_{\partial M} = R^{(2)}|_{\partial M}.$$

By (4.16) we thus have that

$$R^{(1)}|_{\partial M} = \mathcal{O}_{H^{K}(\partial M)}(\tau^{-N}).$$
Furthermore, we have $\partial_\nu w_1|_{\partial M} = \partial_\nu w_2|_{\partial M}$ since $\Lambda_{q_1} = \Lambda_{q_2}$. Consequently, by (4.21) we have
\[
\partial_\nu R^{(1)}|_{\partial M} = \partial_\nu \left( e^{-\tau\Psi}(w^{(1)}_1 - w^{(1)}_0) \right)|_{\partial M} = \partial_\nu \left( e^{-\tau\Psi}(w^{(2)}_0 - w^{(2)}_0) \right)|_{\partial M} = \partial_\nu R^{(2)}|_{\partial M} = O_{H^\infty(\partial M)}(\tau^{-N}).
\]
By the boundary determination result of solutions on the boundary in Proposition 4.5, we have $\partial_\nu^2 w_1|_{\partial M} = \partial_\nu^2 w_2|_{\partial M}$. Thus, combining (4.17) and (4.21) shows $\partial_\nu^2 R^{(1)}|_{\partial M} = O_{H^\infty(\partial M)}(\tau^{-N})$. In conclusion, we have that $R^{(1)}$ solves
\[
\begin{cases}
    e^{-\tau\Psi}(\Delta - V)e^{\tau\Psi}R^{(1)} = O_{H^\infty(M)}(\tau^{-N}) & \text{in } M, \\
    \partial_\nu^2 R^{(1)} = O_{H^\infty(\partial M)}(\tau^{-N}) & \text{on } \partial M, \quad \ell = 0, 1, 2.
\end{cases}
\]
(4.22)
Now, it follows from Lemma 4.6 by taking $K = \frac{q_1 + 1}{2}$ and using the Sobolev embedding $H^K(\partial M) \subset L^\infty(\partial M)$, and finally redefining $N$ as $N - 2$ that
\[
\|R^{(1)}\|_{L^2(M)} = O(\tau^{-N}).
\]
In the remaining case $\pm c_1 \pm c_2 = 0$, the correction terms $R^{(1)}$ and $R^{(2)}$ have zero boundary values by Remark 4.3. Since we also have $w^{(1)}_0|_{\partial M} = w^{(2)}_0|_{\partial M}$ by (4.20), the claim follows also in this case. \hfill \square

4.2. Solutions for the third linearization. In this section, we consider solutions for the third linearizations of $(-\Delta + V)u + qu^2 = 0$ at the zero solution. Recalling that the third linearized equation is of the form
\[
(-\Delta + V)\omega^{(ijk)} = -2q \left( v(i)w^{(jk)} + v(j)w^{(ik)} + v \left( -q_2 \right) w^{(ij)} \right) \text{ in } M,
\]
(4.23)
where $v^{(i)}$ and $w^{(kl)}$, are solutions to (2.5) and (2.6), respectively, for different $i, j, k = 1, 2, 3$. Again, we assume that the solutions $v^{(k)}$ are CGOs of the form (4.1):
\[
v^{(k)} = e^{\pm s_k x_1} \left( e^{\frac{\tau}{\tau + e} \nu \Psi k} a^{(k)} \right),
\]
where $s_k$ corresponds to a nontangential geodesics $\gamma_k$ of $(M_0, g_0)$. Here $s_k = c_k \tau + 1 \lambda_k$ also as before. We assume that $\gamma_1, \gamma_2$ and $\gamma_3$ intersect at the point $p_0$. We also assume that the supports of $v^{(k)}$ restricted to $M_0$ are so small that the mutual support of $v^{(1)}$, $v^{(2)}$ and $v^{(3)}$ does not intersect the points on the geodesics $\gamma_k$ where only two of the geodesics intersect. Lastly, we assume that all the pairs of geodesics $\gamma_i$ and $\gamma_k$, $i \neq k$, intersect properly.

In order to analyze the solution ansatz for the third linearized equation (4.23), we can simply consider the case $i = 1, j = 2$ and $k = 3$. By Lemma 4.2, the equation $(-\Delta + V)u^{(23)} = -qu^{(2)}v^{(3)}$ has a solution of the form
\[
u^{(23)} = u^{(23)} = u^{(23)}_0 + e^{\tau\Psi}(e^{\lambda H} + \rho).
\]
Here $u^{(23)}_0$ is given by the WKB ansatz
\[
u^{(23)}_0 = \frac{\tau^{N_q}}{\lambda^{N_q}} e^{(s_2 + s_3)x_1 + (s_2 s_3)} h^{(23)},
\]
\[
h^{(23)} = \tau^{-2} h^{(23)}_0 + \cdots + \tau^{-2N} h^{(23)}_{-2N},
\]
\[
 h^{(23)}_{-2N} = \frac{2q}{(s_2 \pm c_3)^2 + c_2 \nabla \nu \Psi_2 + c_3 \nabla \nu \Psi_3} a^{(2)}_0 a^{(3)}_0.
\]
We take the solutions $w^{(13)}$ and $w^{(12)}$ to be ones given by similar formulas as $w^{(23)}$. Using these formulas for $w^{(ik)}$ and $v^{(j)}$ we see that (4.23) can be written as
\[
(\Delta_g - V)\omega = \tau^{3(n-2)} e^{\tau\Psi}(e^{\lambda H} + \rho),
\]
where $\omega \equiv \omega^{(123)}$, and
\[
\tilde{\Psi} = (\pm c_1 \pm c_2 \pm c_3)x_1 + i c_1 \psi_1 + i c_2 \psi_2 + i c_3 \psi_3 \\
\tilde{\Lambda} = i (\pm \lambda_1 \pm \lambda_2 \pm \lambda_3)x_1 - \lambda_1 \psi_1 - \lambda_2 \psi_2 - \lambda_3 \psi_3
\]
\[
H = \sum_{k=2}^{3N'-1} \frac{H_{-k}}{\tau^k},
\]
\[
\rho = \mathcal{O}_{H^k(M)}(\tau^{-N}).
\]

The amplitude $H \in C^\infty(M)$ is supported on neighborhoods of the points where all the geodesics $\gamma_1$, $\gamma_2$ and $\gamma_3$ intersect and which do not contain points where only two of the geodesics intersect. The order $3N' - 1$ of the amplitude $H$ is a consequence of the respective orders $2N'$ and $N' - 1$ of the expansions of $w^{(ij)}$ and $a^{(k)}$. We have also assumed $N'$ to be large enough so that the condition for $\rho$ in (4.24) holds. Meanwhile, the factor $\tau^{\frac{n(n-2)}{2}}$ is a result of the product of the respective normalization factors $r^{\pm}$ and $r^{\pm}$ of $w^{(ij)}$ and $v^{(k)}$. The functions $H_{-k}$ depend on $q$ only in terms of the pointwise values $q$ and its derivatives.

By (4.4), the leading order coefficient of $H$ satisfies
\[
H_{-2} = 4q^2 a_0^{(1)} a_0^{(2)} a_0^{(3)}
\]
\[
\times \left( \frac{1}{(\pm c_1 \pm c_2)^2 - |c_1 \nabla' \psi_1 + c_2 \nabla' \psi_2|^2} - \frac{1}{(\pm c_1 \pm c_3)^2 - |c_1 \nabla' \psi_1 + c_3 \nabla' \psi_3|^2} + \frac{1}{(\pm c_2 \pm c_3)^2 - |c_2 \nabla' \psi_2 + c_3 \nabla' \psi_3|^2} \right).
\]

If we additionally assume that
\[
|\langle \nabla \tilde{\Psi}, \nabla \tilde{\Psi} \rangle| \geq \text{constant} > 0
\]
on the support of $H$, it makes sense to try an ansatz
\[
\tau^{\frac{n(n-2)}{2}} e^{r \tilde{\Psi}} e^{i \tilde{\Lambda}} B
\]
for a solution $\omega$ of (4.23), where
\[
B = \sum_{k=4}^{3N'+1} \frac{B_{-k}}{\tau^k}
\]

Here $B_{-k}$, $k = 4, 3, \ldots, 2(N' + 2)$ are given by the recursive formula
\[
B_{-k} = \frac{e^{i \tilde{\Lambda}} H_{-k+2} - [2(\nabla B_{-k+1}, \nabla \Psi) + B_{-k+1} \Delta \Psi] - (\Delta - V) B_{-k+2}}{\langle \nabla \tilde{\Psi}, \nabla \tilde{\Psi} \rangle}
\]

and setting $B_{-2} = B_{-3} = 0$. Especially
\[
B_{-4} = \frac{H_{-2}}{\langle \nabla \tilde{\Psi}, \nabla \tilde{\Psi} \rangle},
\]

where $H_{-2}$ is given in (4.25). The support of $B$ is the mutual support of $v^{(k)}$.

We obtain the following result. We omit the proof as it is a direct adaptation of the proof of Lemma 4.2.

**Lemma 4.7.** Let $K, N \in \mathbb{N} \cup \{0\}$. Assume that $v^{(1)}, v^{(2)}, v^{(3)}$ are CGOs of the form (4.1) corresponding to geodesics $\gamma_1, \gamma_2, \gamma_3$ on $M_0$, respectively, such that the
pairs of geodesics $\gamma_k$ and $\gamma_i$ intersect properly for $i, k = 1, 2, 3$ and $i \neq k$. Assume additionally that $\Psi$ given by (4.24) satisfies

$$(\nabla \tilde{\Psi}, \nabla \tilde{\Psi}) \neq 0$$

at the points where all the geodesics $\gamma_1, \gamma_2$ and $\gamma_3$ intersect. If the restrictions of the amplitudes $a^{(k)}$ of $v^{(k)}$ to $M_0$ are supported in small enough neighborhoods of the geodesics $\gamma_k$, and $N' = N'(K, N)$ is large enough, then the equation

$$(-\Delta_g + V)\omega = -2q \left( v^{(1)} w^{(23)} + v^{(2)} w^{(13)} + v^{(3)} w^{(12)} \right) \text{ in } M,$$  \hspace{1cm} (4.29)

where $w^{(1k)}$ is given as in Lemma 4.2 has a smooth solution $\omega$ up to the boundary $\partial M$ with the following properties: The solution $w$ is of the form

$$\omega = \omega_0 + e^{r\tilde{\Psi}} \tilde{R},$$

where the function $\omega_0$ is of the form

$$\omega_0 = \tau^{\frac{2(n-2)}{8}} e^{r\tilde{\Psi}} e^\lambda B,$$

where $\tilde{\lambda}$ and $B = B_\tau$ are given by (4.24) and (4.26) respectively. Especially $B_{-4}$ is given by (4.28). The amplitude $B$ depends on $q$ only in terms of the pointwise values $q$ and its derivatives. The remainder term $\tilde{R} = \tilde{R}_\tau$ satisfies

$$\|\tilde{R}_\tau\|_{H^N(M)} \lesssim \tau^{-N}.$$  

As stated, the amplitude $B$ depends on $q$ only in terms of the pointwise values $q$ and its derivatives. Thus, by assuming that we know the DN map of $(-\Delta + V)u + qu^2 = 0$, we may determine the value of $\omega_0$ on the boundary by boundary determination result (Proposition 4.5). Consequently, by using the Carleman estimate with boundary terms (Lemma 4.6), we have the following analogous result of Proposition 4.4. Note that

$$(\nabla \tilde{\Psi}, \nabla \tilde{\Psi}) = (\pm c_1 \pm c_2 \pm c_3)^2 - |c_1 \nabla' \psi_1 + c_2 \nabla' \psi_2 + c_3 \nabla' \psi_3|^2.$$  

**Proposition 4.8.** Assume as in Lemma 4.7 and adopt its notation. Assume additionally that $\Lambda_{\gamma_1} = \Lambda_{\gamma_2}$. If the restrictions of the amplitudes $a^{(k)}$ of $v^{(k)}$ to $M_0$ are supported in small enough neighborhoods of the geodesics $\gamma_k$, and $N' = N'(K, N)$ is large enough, then the third linearized equations

$$(-\Delta_g + V)\omega^{(\beta)} = -2q_\beta \left( v^{(1)} w^{(23)}_{\beta} + v^{(2)} w^{(13)}_{\beta} + v^{(3)} w^{(12)}_{\beta} \right) \text{ in } M,$$  \hspace{1cm} (4.30)

where $w^{(1k)}_{\beta}, \text{ for } \beta = 1, 2$ and different $i, k = 1, 2, 3$, are given as in Proposition 4.4. Moreover, the solution $\omega^{(\beta)}$ is of the form

$$\omega^{(\beta)} = \omega^{(\beta)}_0 + e^{r\tilde{\Psi}} \tilde{R}^{(\beta)},$$

where

$$\omega^{(\beta)}_0 = \tau^{\frac{2(n-2)}{8}} e^{r\tilde{\Psi}} e^\lambda B^{(\beta)},$$

$$B^{(\beta)} = \tau^{-4} B_{-4}^{(\beta)} + \cdots + \tau^{-3N'+1} B_{-3N'+1}^{(\beta)}.$$  

Here $\tilde{\Lambda}$ and $\tilde{\Psi}$ are given by (4.24). Especially, the quantity $B_{-4}$ in (4.28) can be written as

$$B_{-4} = 4q_3^3 a_0 a_0 a_0 \left( \frac{1}{(\pm c_1 \pm c_2 \pm c_3)^2 - |c_1 \nabla^r \psi_1 + c_2 \nabla^r \psi_2 + c_3 \nabla^r \psi_3|^2} \right) \times \left( \frac{1}{(\pm c_1 \pm c_2)^2 - |c_1 \nabla^r \psi_1 + c_2 \nabla^r \psi_2|^2} \right) \times \left( \frac{1}{(\pm c_1 \pm c_3)^2 - |c_1 \nabla^r \psi_1 + c_3 \nabla^r \psi_3|^2} \right) \times \left( \frac{1}{(\pm c_2 \pm c_3)^2 - |c_2 \nabla^r \psi_2 + c_3 \nabla^r \psi_3|^2} \right),$$

and $\tilde{R}^{(\beta)} = O_{L^2(M)}(\tau^{-N})$, for $\beta = 1, 2$. Moreover

$$\omega^{(1)} \mid_{\partial M} = \omega^{(2)} \mid_{\partial M}.$$

We skip the proof of Proposition 4.8 as it can be obtained from the proof of Proposition 4.4 by replacing $w$ by $\omega$ and $\Psi$ by $\tilde{\Psi}$ etc. The function $F$ in (4.19) in the proof also needs to be replaced by a function of the class $O_{L^2(M)}(\tau^{-N})$ since $R^{(2)}$ in Proposition 4.4 is $O_{L^2(M)}(\tau^{-N})$. We remark that by deriving Carleman estimates similar to those in Lemma 4.6 for higher Sobolev spaces, we could in fact have that $\tilde{R}$ is of the size $\tau^{-N}$ also in higher Sobolev spaces $H^K(M)$ by taking $N'$ large enough.

5. Proof of Theorem 1.1

In this section we prove Theorem 1.1. We will see that it is possible to deduce

$$q_1^2 = q_2^2$$

in $M$ from third order linearizations and the DN map of the equation $(-\Delta + V)u + qu^2 = 0$. Our method for the third linearized equation however does not imply $q_1 = q_2$ in general. To show that

$$q_1 = q_2$$

we in fact need to consider fourth order linearized equations. To give a proof of Theorem 1.1, we could consider the fourth order linearization from the begininging. However, we first consider third order linearizations and prove $q_1^2 = q_2^2$ to better explain the main ideas of the proof.

5.1. Proof of $q_1^2 = q_2^2$. Let $p_0 \in M_0$, and let $\gamma_1$ be a non-tangential geodesic that has no self-intersections. We consider the equation

$$\begin{cases} (-\Delta + V)u_a + q_a u_a^3 = 0 & \text{in } M, \\ u_a = f & \text{on } \partial M, \end{cases}$$

for $a = 1, 2$, where $f = f e \in C^{2,\alpha}(\partial M)$ is of the form

$$f_e := \sum_{i=1}^4 \epsilon_i f_i + \sum_{i,j=1}^4 \epsilon_i \epsilon_j f_{ij} \quad \text{on } \partial M.$$  

Let us recall the linearizations (5.1) from Section 2.2. The first linearization reads

$$\begin{cases} (-\Delta + V)v_\beta^{(1)} = 0 & \text{in } M, \\ v_\beta^{(1)} = f_e & \text{on } \partial M, \end{cases}$$

and

$$\omega^{(1)} \mid_{\partial M} = \omega^{(2)} \mid_{\partial M}.$$
where $v^{(i)}_\beta = \partial_{e_i}|_{e=0} u_\beta$ for $\beta = 1, 2$, and $i = 1, 2, 3, 4$. By the uniqueness of solutions to (5.3), we obtain

$v^{(i)} := v^{(i)}_1 = v^{(i)}_2$ in $M$,

for $i = 1, 2, 3, 4$. The second linearization of (5.1) satisfies

$$\begin{cases} (-\Delta + V)w^{(j)}_\beta = -2q_\beta v^{(j)} v^{(j)} & \text{in } M, \\ w^{(j)}_\beta = f_{jj} & \text{on } \partial M, \end{cases}$$

(5.4)

where

$$w^{(j)}_\beta = \partial^2_{e_i e_j} u_\beta |_{e=0},$$

for $\beta = 1, 2$ and different $i, j \in \{1, 2, 3\}$. Lastly, the third linearization of (5.1) satisfies

$$\begin{cases} (-\Delta + V)w^{(jk)}_\beta = -2q_\beta \left( v^{(i)} w^{(jk)}_\beta + v^{(j)} w^{(ik)}_\beta + v^{(k)} w^{(ij)}_\beta \right) & \text{in } M, \\ w^{(jk)}_\beta = 0 & \text{on } \partial M, \end{cases}$$

(5.5)

where

$$w^{(jk)}_\beta = \partial^3_{e_i e_j e_k} u_\beta |_{e=0}.$$  

Since $\Lambda_{q_1} = \Lambda_{q_2}$

$$0 = \partial^3_{e_i e_j e_k} |_{e=0} (\Lambda_{q_1} - \Lambda_{q_2}) (f_r)$$

(5.6)

Thus, by Lemma 2.1 we have

$$0 = \int_M \left\{ q_1 \left( v^{(i)} w^{(jk)}_1 + v^{(j)} w^{(ik)}_1 + v^{(k)} w^{(ij)}_1 \right) - q_2 \left( v^{(i)} w^{(jk)}_2 + v^{(j)} w^{(ik)}_2 + v^{(k)} w^{(ij)}_2 \right) \right\} v^{(i)} dV,$$

(5.7)

where $v^{(i)}$ and $w^{(jk)}_\beta$ are the solutions of (5.3) and (5.4), respectively, for different $i, j, k = 1, 2, 3, 4$ and $\beta = 1, 2$.

We choose $v^{(i)}$ to be CGOs corresponding geodesics on $(M_0, g_0)$, which intersect properly pairwise. We show that the integrand on the right hand side of (5.7) restricted to a neighborhood of $p_0$ in $M_0$ is close to a multiple of the delta function. We let $v^{(i)}$ correspond to the geodesic $\gamma_1$ and choose the other 3 geodesics next.

5.2. Choices of initial vectors for the third linearization. Let $\delta \in (0, 1)$, and we denote the initial data of $\gamma_1$ by $\xi_1 \in S_{p_0} M_0$. By perturbing $\xi_1$, we find $\xi_2 \in S_{p_0} M_0$ such that the associated geodesic $\gamma_2$ is also non-tangential, has no self-intersections, and that

$$|\xi_1| = |\xi_2| = 1$$

and

$$\langle \xi_1, \xi_2 \rangle = 1 - \delta.$$

Let us define

$$\xi_3 = -\frac{1}{1 + \delta} (\xi_1 + \delta \xi_2) \in S_{p_0} M_0 \quad \text{and} \quad \xi_4 = -\frac{1}{1 + \delta} (\delta \xi_1 + \xi_2) \in S_{p_0} M_0.$$

A direct computation shows

$$\sum_{i=1}^4 \xi_i = \xi_1 + \xi_2 - \frac{1}{1 + \delta} \xi_1 - \frac{\delta}{1 + \delta} \xi_2 = \frac{\delta}{1 + \delta} \xi_1 - \frac{1}{1 + \delta} \xi_2 = 0.$$  

(5.8)

We redefine $\delta$ smaller, if necessary, so that the geodesics $\gamma_3$ and $\gamma_4$ corresponding to $\xi_3$ and $\xi_4$ are also nontangential.

Note that $\xi_1$ is not proportional to $\xi_2$ as $\xi_1$ and $\xi_2$ are linearly independent. Similarly, for $k = 3, 4$, the vector $\xi_k$ is neither proportional to $\xi_1$ nor to $\xi_2$. Lastly,
\( \xi_1 \) is not proportional to \( \xi_4 \). Indeed, if \( A \in \mathbb{R} \) is such that \( \xi_3 = A \xi_4 \), we have that \( 1 = \delta A \) and \( \delta = A \), implying that \( \delta = \pm 1 \). However, \( \delta \in (0, 1) \). This means that all the pairs of the geodesics corresponding to initial data \( \xi_k/|\xi_k| \) intersect properly.

Note also that since \( |\xi_1| = |\xi_2| = 1 \), we have

\[
|\xi_3|^2 = \frac{1}{(1 + \delta)^2} \left( |\xi_1|^2 + \delta^2 |\xi_2|^2 + 2\delta \langle \xi_1, \xi_2 \rangle \right) = \frac{1}{(1 + \delta)^2} \left( |\xi_2|^2 + \delta^2 |\xi_1|^2 + 2\delta \langle \xi_2, \xi_1 \rangle \right) = |\xi_4|^2.
\]

That is

\[
|\xi_3|^2 = |\xi_4|^2 = \frac{1}{(1 + \delta)^2} \left( 1 + \delta^2 + 2\delta(1 - \delta) \right) = \frac{1}{(1 + \delta)^2} \left( 1 + 2\delta - \delta^2 \right) \tag{5.9}
\]

Let us then define vectors \( \vec{\xi}_k \in TM, k = 1, 2, 3, 4 \), by

\[
\vec{\xi}_1 = |\xi_1|e_1 + ik_1, \quad \vec{\xi}_2 = -|\xi_2|e_1 + ik_2 \quad \vec{\xi}_3 = |\xi_3|e_1 + ik_3, \quad \vec{\xi}_4 = -|\xi_4|e_1 + ik_4. \tag{5.10}
\]

Then

\[
\sum_{k=1}^{4} \vec{\xi}_k = 0. \tag{5.11}
\]

Note also that

\[
\langle \vec{\xi}_k, \vec{\xi}_k \rangle = 0, \quad k = 1, \ldots, 4. \tag{5.12}
\]

Related to these vectors \( \vec{\xi}_k \), we will consider in the proof of Theorem 1.1 CGOs, which can be written of the form

\[
u_0^{(k)}(\xi_k) = e^{\Re(\vec{\xi}_k)x_1} \left( \frac{\delta}{\kappa + \delta} e^{i|\xi_k|\psi_a s + r_s} \right) .
\]

Here the phase functions \( \psi_k \) are constructed with respect to the geodesics \( \gamma_k \) with initial data \( \gamma_k(0) = p_0 \) and \( \gamma_k(0) = \frac{\xi_k}{|\xi_k|} \). We note that

\[
\nabla \left( \Re(\vec{\xi}_k)x_1 + i|\xi_k|\psi_k \right) \bigg|_{\gamma_k(0)} = \vec{\xi}_k. \tag{5.13}
\]

Consequently, the ansatz \( u_0^{(k)} \) in Lemma 4.2 for the solutions of \((-\Delta + V)u^{(k)} = -2\mu^{(k)}u^{(k)} \) have amplitudes with a factor that divides by

\[
\langle \nabla \left( \Re(\vec{\xi}_i + \vec{\xi}_k)x_1 + i|\xi_i|\psi_i \right) , \nabla \left( \Re(\vec{\xi}_k)x_1 + i|\xi_k|\psi_k \right) \rangle ,
\]

for different \( i, k = 1, 2, 3 \). At an intersection point of the geodesics \( \gamma_1 \) and \( \gamma_2 \) the above equals

\[
\langle \vec{\xi}_i + \vec{\xi}_k, \vec{\xi}_i + \vec{\xi}_k \rangle = 2 \langle \nabla \left( \Re(\vec{\xi}_i)x_1 + i|\xi_i|\psi_i \right) , \nabla \left( \Re(\vec{\xi}_k)x_1 + i|\xi_k|\psi_k \right) \rangle = 2 \langle \vec{\xi}_i, \vec{\xi}_k \rangle.
\]

Here we used (5.13) and (5.12). Motivated by this, we define

\[
\mathbf{C}_{ik} := 2 \langle \vec{\xi}_i, \vec{\xi}_k \rangle.
\]

The coefficient \( \mathbf{C}_{ik} \) can be collectively written as

\[
\mathbf{C}_{ik} = 2|\xi_i| |\xi_k| \left( -1 \right)^{i+k} \frac{\langle \xi_i, \xi_k \rangle}{|\xi_i||\xi_k|} .
\]

We calculate expansions for \( \mathbf{C}_{ik} \) for small \( \delta > 0 \) parameter. A direct computation shows that

\[
\langle \xi_1, \xi_1 \rangle = \left( 1, \frac{1}{1 + \delta} (\xi_1 + \delta\xi_2) \right) = -\frac{1}{1 + \delta} (|\xi_1|^2 + \delta(\xi_1, \xi_2)) = -\frac{1}{1 + \delta} (1 + \delta - \delta^2)
\]

and

\[
\langle \xi_2, \xi_3 \rangle = -\frac{1}{1 + \delta} (\xi_1, \xi_2) + \delta |\xi_2|^2 = -\frac{1}{1 + \delta}.
\]
where we used $\langle \xi_1, \xi_2 \rangle = 1 - \delta$. We also have

$$\frac{\langle \xi_1, \xi_3 \rangle}{|\xi_1||\xi_3|} = -\frac{1}{1 + \delta} \left(1 + \delta - \delta^2\right)^{1/2} = 1 + \mathcal{O}(\delta)$$

and

$$\frac{\langle \xi_2, \xi_3 \rangle}{|\xi_2||\xi_3|} = -\frac{1}{1 + \delta} \left(1 + \delta - \delta^2\right)^{1/2} = -1 + \mathcal{O}(\delta).$$

Here we have utilized the Taylor expansions

$$(1 + r)^{1/2} = 1 + r/2 + \mathcal{O}(r^2) \quad \text{and} \quad (1 + r)^{-1} = 1 - r + \mathcal{O}(r^2),$$

which hold for small $|r|$. Combining the above formulas yields

$$C_{12} = 2|\xi_1||\xi_2| \left(1 - \frac{\langle \xi_1, \xi_2 \rangle}{|\xi_1||\xi_2|}\right) = -4 + \mathcal{O}(\delta),$$

$$C_{13} = 2|\xi_1||\xi_3| \left(1 - \frac{\langle \xi_1, \xi_3 \rangle}{|\xi_1||\xi_3|}\right) = 2 \left(1 + 2\delta - \delta^2\right)^{1/2} \left(1 - \left(1 + \mathcal{O}(\delta)\right)\right)$$

$$= 4 + \mathcal{O}(\delta),$$

$$C_{23} = 2|\xi_2||\xi_3| \left(1 - \frac{\langle \xi_2, \xi_3 \rangle}{|\xi_2||\xi_3|}\right) = 2 \left(1 + 2\delta - \delta^2\right)^{1/2} \left(-1 - \left(1 + \mathcal{O}(\delta)\right)\right)$$

$$= \mathcal{O}(\delta).$$

(5.14)

We also remark here that $C_{ik} \neq 0$, $i \neq k$, for $\delta > 0$ since

$$|C_{ik}| = 2|\xi_i||\xi_k| \left((-1)^{i+k} - \frac{\langle \xi_i, \xi_k \rangle}{|\xi_i||\xi_k|}\right)$$

(5.15)

and

$$\frac{\langle \xi_i, \xi_k \rangle}{|\xi_i||\xi_k|} \in (-1, 1),$$

because the pairs of vectors $\xi_i$ and $\xi_k$ are linearly independent. Finally, we note that

$$\left| \frac{1}{C_{12}} + \frac{1}{C_{13}} + \frac{1}{C_{13}} \right| = \left| \frac{1}{-4 + \mathcal{O}(\delta)} + \frac{1}{4 + \mathcal{O}(\delta)} + \frac{1}{\mathcal{O}(\delta)} \right| \to \infty,$$

when $\delta \to 0$. Thus

$$\frac{1}{C_{12}} + \frac{1}{C_{13}} + \frac{1}{C_{13}} \neq 0,$$

(5.16)

for all small enough $\delta > 0$.

**Remark 5.1.** Let us define a Lorentz metric $\eta$ for $M$ by the formula

$$\eta(c_1e_1 + V_1, c_2e_1 + V_2) := \langle c_1e_1 + V_1, c_2e_1 + V_2 \rangle,$$

where $c_1, c_2 \in \mathbb{R}$ and $V_1, V_2 \in TM_0$. We required that the vectors $\xi_1, \ldots, \xi_4$ in (5.10) are lightlike vectors with respect to $\eta$ and sum up to 0. The former requirement is because the corresponding phase functions need to satisfy the complex eikonal equation. The latter requirement is discussed in the next section.

A fact is that three $\eta$-lightlike vectors can only sum up to 0, if the parts in $TM_0$ of two of them are linearly dependent. This would correspond to geodesics that do not intersect properly. Due to this geometric fact, we overdifferentiate in this paper the nonlinearity $q^2$ to obtain integral identities that consider more than three CGOs.
5.3. Proof of $q_1^2 = q_2^2$ (continued). Let us then return to proving $q_1^2 = q_2^2$. Let $\xi_k$, $k = 1, 2, 3, 4$, be as in \((5.10)\). We set
\[ c_k = |\xi_k| \]
and
\[ s_1 = c_1 \tau + i \lambda \quad \text{and} \quad s_\ell = c_\ell \tau, \quad \text{for } \ell = 2, 3, 4. \]
Then the corresponding CGOs are of the form
\[
\begin{align*}
\psi^{(1)} &= e^{i(|\xi_1| \tau + i \lambda)x_1} \left( r^{\alpha^2} e^{i(|\xi_1| \tau + i \lambda)\psi^4 a_1 + r_1} \right), \\
\psi^{(2)} &= e^{-|\xi_2| \tau x_1} \left( r^{\alpha^2} e^{i|\xi_2| \tau \psi^2 a_2 + r_2} \right), \\
\psi^{(3)} &= e^{i|\xi_3| \tau x_1} \left( r^{\alpha^2} e^{i|\xi_3| \tau \psi^3 a_3 + r_3} \right), \\
\psi^{(4)} &= e^{-|\xi_4| \tau x_1} \left( r^{\alpha^2} e^{i|\xi_4| \tau \psi^4 a_4 + r_4} \right).
\end{align*}
\]
We may assume that $\psi^{(k)}$, $k = 1, \ldots, 4$ are supported in small enough neighborhoods of the corresponding geodesics $\gamma_k$ so that the mutual support of $\psi^{(k)}$ belongs to neighborhoods of the points where all the geodesics $\gamma_k$ intersect and where any pair of the geodesics intersect only once. Let us denote the points where all the geodesics $\gamma_k$ intersect by $p_0, p_1, \ldots, p_Q$.

Let $i \neq j \in \{1, 2, 3, 4\}$, $i \neq j$ and $\beta = 1, 2$. By assumption, the DN maps of the equation \((1.3)\) for the potentials $q_1$ and $q_2$ satisfy $\Lambda_{q_1} = \Lambda_{q_2}$. By Proposition 4.4 there are boundary values $f_{ij}$, which are the same for both $q_1$ and $q_2$, such that the solutions of the second linearized equations \((5.4)\) are of the form
\[
w^{(ij)}_{\beta} = w^{(ij)}_0 + e^{\tau \psi^{(ij)} R^{(ij)}_{\beta}},
\]
where the ingredients are as follows:
\[
\begin{align*}
\psi^{(ij)} &= ((-1)^{i+j} c_i + (-1)^{i+j} c_j) x_1 + i(c_i \psi_i + c_j \psi_j), \\
w^{(ij)}_0 &= r^{\alpha^2} e^{i(-1)^{i+j} x_1 + i(s_i \psi_i + s_j \psi_j) b^{(ij)}_\beta}, \\
b^{(ij)}_\beta &= r^{-2} b^{(ij)}_{-2, \beta} + \cdots + r^{-2N'} b^{(ij)}_{-2N', \beta}, \\
b^{(ij)}_{-2, \beta} &= \frac{2q_\beta}{\xi_j} \left( (-1)^{i+j} c_i + (-1)^{i+j} c_j \right)^2 - |c_i \nabla \psi_i + c_j \nabla \psi_j|^2 a^{(i)}_0 a^{(j)}_0 .
\end{align*}
\]
By \((5.13)\), at points of the form $(x_1, p_0) \in M$ we have
\[
b^{(ij)}_{-2, \beta} = \frac{2q_\beta}{\xi_j} a^{(i)}_0 a^{(j)}_0 .
\]
Here $a^{(i)}_0$ and $a^{(j)}_0$ are independent of the variable $x_1 \in \mathbb{R}$.
To ease the following calculations, let us denote
\[
\psi_{1234} = \sum_{k=1}^4 ((-1)^{1+k} c_k x_1 + i c_k \psi_k) = i \sum_{k=1}^4 c_k \psi_k
\]
and
\[
\Lambda_{1234} = \lambda_1 (ix_1 - \psi_1) .
\]
We see that
\[
\psi^{(12)} + \psi^{(34)} = \psi^{(13)} + \psi^{(24)} = \psi^{(14)} + \psi^{(23)} = i \sum_{k=1}^4 c_k \psi_k = \Psi_{1234}.
\]

\[(5.18)\]
Since $\Psi_{1234}$ is purely imaginary at the intersection points $p_k$, $b = 0, \ldots, Q$, the exponentially large linear factors will cancel in terms of the form $v^{(i)} w^{(jk)} v^{(l)}$ appearing the integral identity for the third linearization (5.5). We also have at the intersection point $p_0$ of the geodesics that

$$
\nabla (\Psi^{(12)} + \Psi^{(34)})\bigg|_{p_0} = \nabla (\Psi^{(13)} + \Psi^{(24)})\bigg|_{p_0} = \nabla (\Psi^{(14)} + \Psi^{(23)})\bigg|_{p_0} = 0.
$$

This implies that $p_0$ is a critical point of the phase functions of functions of the form $v^{(i)} w^{(jk)} v^{(l)}$. The critical point is also nondegenerate by (3.2) in Section 3 and thus we will be able to apply stationary phase in the asymptotic parameter $\tau$.

Let us first consider the case $p_0 = 0$ is the only point where all the geodesics $\gamma_1, \ldots, \gamma_4$ intersect. With the above preparations and using $A_q = A_{2q}$, the integral identity (5.7) of the third order linearization reads

$$
0 = \int_M \left[ q_1 \left( v^{(i)} w_1^{(ik)} + v^{(k)} w_1^{(il)} + v^{(l)} w_2^{(ik)} \right) - q_2 \left( v^{(i)} w_2^{(ik)} + v^{(k)} w_2^{(il)} + v^{(l)} w_2^{(ik)} \right) \right] v^{(m)} dV.
$$

By (5.18) the exponentially large factors of the integrand cancel. Recall that the dimension of $M_0$ is $n - 1$.

We multiply the integral identity (5.20) by $\tau^{1/2}$ and $\tau^2$. This achieves the correct normalization $\tau^{\text{dim}(M_0)/2}$ for stationary phase. By (5.19), at the intersection point $p_0$ of the geodesics $\gamma_k$ for $x_1 \in I \subset \mathbb{R}$ holds

$$
\nabla \Psi_{1234}(x_1, p_0) = 0.
$$

In normal coordinates $(y^1, \ldots, y^{n-1})$ centered at the point $p_0$ in $M_0$

$$
\text{Re} \Psi_{1234}(y) = \sum_{j,k=1}^{n-1} A_{jk} y^j y^k + \mathcal{O}(|y|^3),
$$

for some negative definite matrix $A$ by the properties (3.2) of the phase functions. Note also that

$$
\tau^{\frac{n-1}{2}} \int_{\mathbb{R}^{n-1}} e^{-|y|^2} dy = \mathcal{O}(1) \quad \text{and} \quad \tau^{\frac{n-1}{2}} \int_{\mathbb{R}^{n-1}} |y| e^{-|y|^2} dy = \mathcal{O}(\tau^{-\frac{1}{2}}).
$$

Thus, stationary phase shows that the limit $\tau \to \infty$ of (5.20) equals

$$
c_A \left( a_0^{(1)} a_0^{(2)} a_0^{(3)} a_0^{(4)} \right)_{p_0} \left( C^{-1}_{12} + C^{-1}_{13} + C^{-1}_{23} \right) \times \int_{\mathbb{R}} e^{A_{1234}(x_1, p_0)} (q_1^2(x_1, p_0) - q_2^2(x_1, p_0)) dx_1,
$$

where

$$
c_A = \int_{\mathbb{R}^{n-1}} e^{x^A} dx \neq 0.
$$

We refer to [LLLS21a, Proof of Theorem 5.1, Step 4] for more details on this stationary phase argument. Here we have also used that $a_0^{(k)}$, $k = 1, \ldots, 4$, depend
only on the transversal variables. We also continued \( q_1 \) and \( q_2 \) by zero from \( I \) to \( \mathbb{R} \) in the \( x_1 \) variable.

The geodesics \( \gamma_k \) were parametrized so that \( \gamma_k(0) = p_0 \). Thus \( \psi_k(p_0) = 0 \) and we have

\[
e^{i\lambda x_1} = e^{i\lambda x_1}.
\]

Since \( C_{12}^{-1} + C_{13}^{-1} + C_{23}^{-1} \neq 0 \) and \( a_0^{(k)}\gamma_k \neq 0 \) by (5.16) and (3.4) respectively, combining the above shows that

\[
\int_{\mathbb{R}} e^{i\lambda x_1} (q_1^2(x_1,p_0) - q_2^2(x_1,p_0)) \, dx_1 = 0.
\]

Inverting the Fourier transformation in the \( x_1 \) variable shows that \( q_1^2(x_1,p_0) = q_2^2(x_1,p_0) \). Since \( p_0 \) was arbitrary, this completes the proof in the case \( p_0 \) was the only point where all the geodesics intersect.

Consider then the remaining case where are several points \( p_b, b = 0, \ldots, Q \), where all all the geodesics \( \gamma_1, \gamma_2, \gamma_3, \gamma_4 \) intersect. Note also that outside (disjoint) neighborhoods \( U_b \) of \( p_b \) the function \( e^{i\Psi_{1234}} \) is exponentially small. Thus, for different \( i, j, k, l, \) by normalizing and taking limit \( \tau \to \infty \) of (5.20) we obtain

\[
\sum_{b=0}^{Q} \int_{U_b} q_1^2(x_1,p_b) - q_2^2(x_1,p_b) \, dx_1 = 0.
\]

Here we have denoted

\[
C_{ik}(p_b) = \left\langle \nabla \left((-1)^{1+i}c_ix_1 + i\xi_i\psi_i\right), \nabla \left((-1)^{1+k}c_kx_1 + i\xi_k\psi_k\right) \right\rangle_{p_b} \neq 0.
\]

Note that \( C_{ik}(p_b) \neq 0 \), since \( \gamma_i \) and \( \gamma_k \), \( i \neq k \), intersect properly, cf. (5.15). Therefore, by applying stationary phase to (5.21) it follows that

\[
\sum_{b=0}^{Q} \hat{h}_b(\lambda)e^{i\lambda} = 0, \quad \lambda \in \mathbb{R}.
\]

Here \( c_b \) are the distinct geodesic parameter times of \( \gamma_i \) where \( \gamma_i(c_b) = p_b \) and

\[
\hat{h}_b(\lambda) := \mathcal{F}(\gamma_i) \left( a_0^{(1)} \cdots a_0^{(4)} (C_{12}^{-1} + C_{13}^{-1} + C_{23}^{-1}) \right)_{p_b} (q_1^2(x_1,p_b) - q_2^2(x_1,p_b))
\]

where \( \mathcal{F}(\gamma_i) \) is the Fourier transform in \( x_1 \) variable. By [LLS21a, Lemma 6.2]

\[
h_0 = \cdots = h_Q = 0.
\]

Especially \( q_1^2(x_1,p_b) = q_2^2(x_1,p_b) \), which concludes the proof of \( q_1^2 = q_2^2 \) also in the case where there are several points where the geodesics \( \gamma_k \) all intersect.

5.4. Proof of \( q_1 = q_2 \) and fourth order linearization. We proved \( q_1^2 = q_2^2 \) using third order linearizations of the equation \( (-\Delta + V)u + qu^2 = 0 \). Here we consider fourth order linearizations of the equation and use it to complete the proof of Theorem 1.1. Most of the steps here will be similar to those we used to prove \( q_1^2 = q_2^2 \). However, the steps are somewhat more complicated.
Let \( p_0 \in M_0 \), and let \( \gamma_1 \) be a non-tangential geodesic, which has no self-intersections. Let \( \xi_1 \in \mathcal{S}_{p_0}M_0 \) by the initial data if \( \gamma_1 \). Let us consider the equation
\[
\left\{ \begin{aligned}
(\Delta + \Lambda) u_{\beta} + q_{\beta} u_\beta^2 &= 0 &\text{in } M, \\
u_{\beta} &= f \\
\end{aligned} \right.
\]
\( (5.22) \)

this time with boundary values \( f \in C^\infty(\partial M) \) of the form \( (2.3) \), for \( \beta = 1, 2 \). The first and second linearized equations are the same as before and read
\[
(\Delta + \Lambda) v^{(i)} = 0,
\]
\[
(\Delta + \Lambda) w^{(ij)} = -2q_{\beta} v^{(i)} v^{(j)}.
\]

Here \( v^{(i)} \) and \( w^{(ij)} \), \( i \neq j \in \{1, \ldots, 4\} \), \( \beta = 1, 2 \), have boundary values \( f_i \) and \( f_{ij} \) respectively. The solutions \( v^{(i)} \) are the same for both potentials \( q_1 \) and \( q_2 \). The third order linearizations \( w^{(ijk)} \) now have (possibly) non-zero boundary values and satisfy
\[
\left\{ \begin{aligned}
(\Delta + \Lambda) w^{(ijk)} &= -2q_{\beta} \left( v^{(i)} w^{(jk)}_{\beta} + v^{(j)} w^{(ik)}_{\beta} + v^{(k)} w^{(ij)}_{\beta} \right) &\text{in } M, \\
w^{(ijk)} &= f_{ijk} &\text{on } \partial M,
\end{aligned} \right.
\]
where \( w^{(ijk)}_{\beta} = \partial_{\epsilon, \epsilon, \epsilon} u_{\beta} \), for \( \beta = 1, 2 \) and different \( i, j, k \in \{1, \ldots, 4\} \). The boundary values \( f_{ijk} \) are the same for both of the equations \( (5.5) \), which correspond to the potentials \( q_1 \) and \( q_2 \).

The fourth order linearization
\[
w^{(1234)}_{\beta} = \partial_{\epsilon, \epsilon, \epsilon, \epsilon} u_{\beta} \]
is the solution of
\[
\left\{ \begin{aligned}
(\Delta + \Lambda) w^{(1234)}_{\beta} &= -2q_{\beta} \left( v^{(1)} w^{(234)}_{\beta} + v^{(2)} w^{(134)}_{\beta} + v^{(3)} w^{(123)}_{\beta} + v^{(4)} w^{(123)}_{\beta} \right) &\text{in } M, \\
w^{(1234)}_{\beta} &= 0 &\text{on } \partial M.
\end{aligned} \right.
\]
\( (5.23) \)

Using \( \Lambda_{q_1} = \Lambda_{q_2} \) we have by Lemma 2.1 the integral identity
\[
0 = \int_M \left\{ q_1 \left( v^{(1)} w^{(234)}_{1} + v^{(2)} w^{(134)}_{1} + v^{(3)} w^{(123)}_{1} + v^{(4)} w^{(123)}_{1} + w^{(12)}_{1} (234) + w^{(13)}_{1} (24) + w^{(14)}_{1} (23) \right) + q_2 \left( v^{(1)} w^{(234)}_{2} + v^{(2)} w^{(134)}_{2} + v^{(3)} w^{(123)}_{2} + v^{(4)} w^{(123)}_{2} + w^{(12)}_{2} (234) + w^{(13)}_{2} (24) + w^{(14)}_{2} (23) \right) \right\} v^{(5)} dV.
\]
\( (5.24) \)

We will use five CGOs as the solutions \( v^{(k)} \), \( k = 1, \ldots, 5 \). As before, these have the form
\[
v^{(k)} = e^{\pm s_k \tau} \left( \tau^{\frac{n-2}{2}} e^{i\lambda_k \sigma} \xi^{(k)} \right),
\]
where \( s_k = c_k \tau + i\lambda_k \). However, the geodesics of \( (M_0, g_0) \) corresponding to the phase functions \( \psi_k \) will be different from what we used earlier. We choose the
geodesics so that each pair of different ones of them intersect properly. This is as before. However, we additionally require the geodesics to be so that
\[(\pm c_i \pm c_j \pm c_k)^2 - |c_i \gamma_i + c_j \gamma_j + c_k \gamma_k|^2 \neq 0,
\]
when all the geodesics \(\gamma_k\) intersect. This is the condition \(\langle \nabla \tilde{\Psi}, \nabla \tilde{\Psi} \rangle \neq 0\) of Lemma 4.7 and Proposition 4.8.

With suitable choices of other geodesics \(\gamma_2, \gamma_3, \gamma_4, \gamma_5\), and coefficients \(c_k, k = 1, \ldots, 5\), we show that the integrand on the right hand side of \((5.24)\) restricted to a neighborhood of \(p_0\) in \(M_0\) is close to a multiple of the delta function at \(p_0\).

5.5. **Choices of vectors for the fourth order linearization.** The fourth order linearization \(w^{(1234)}\) of \((-\Delta_g + V) + qu^2 = 0\) satisfies
\[
(-\Delta_g + V)w^{(1234)} = -2q \left( v^{(1)}w^{(234)} + v^{(2)}w^{(134)} + v^{(3)}w^{(124)} + v^{(4)}w^{(123)} \right)
\]
\[
+ w^{(12)}w^{(34)} + w^{(13)}w^{(24)} + w^{(14)}w^{(23)} \right) \quad \text{in } M.
\]
\(\text{(5.25)}\)

Our aim is to show that the solution \(w^{(1234)}\) behaves like \(v^{(1)}v^{(2)}v^{(3)}v^{(4)}\) up to a multiplication by an amplitude function for \(\tau\) sufficiently large.

- **Failed choices of vectors.** Let us first discuss why the earlier vectors \(\vec{\xi}_i, i = 1, 2, 3, 4\), do not work here. If we use the earlier vectors \((5.10)\) and the corresponding CGOs we will find that for example \(w^{(123)}\) in \((5.25)\) solves
\[
(-\Delta_g + V)w^{(123)} = -2q \left( v^{(1)}w^{(23)} + v^{(2)}w^{(13)} + v^{(3)}w^{(12)} \right)
\]
\[
= e^{\tau \sum_{j=1,2,3}\xi_j (\xi_j + i\psi_j)} \tilde{a},
\]
\(\text{(5.26)}\)

where \(\tilde{a}\) is some amplitude function whose precise form is not important for this discussion. At the intersection points of the geodesics corresponding to \(\xi_a\)
\[
\nabla \left( \sum_{j=1}^3 |\xi_j|((-1)^{j+1}x_1 + i\psi_j) \right) = \sum_{j=1}^3 \xi_j = -\vec{\xi}_4
\]
by \((5.11)\). Now, if we try a WKB ansatz of the form \(e^{\tau \sum_{a=1,2,3,4}|\xi_a|((-1)^{a+1}x_1 + i\psi_a)} \tilde{b}\) to solve \((5.26)\), where \(\tilde{b}\) is an amplitude function, we end up dividing by \(\langle \vec{\xi}_4, \vec{\xi}_4 \rangle\), which is 0. Consequently, the ansatz does not work and we need to use vectors that are different than \(\vec{\xi}_a\).

- **Successful choices of vectors.** We choose new vectors to define the CGOs \(v^{(k)}\), such that we can apply these CGOs to achieve our target. Denote the vectors by
\[
\vec{\xi}_k, \quad k = 1, 2, 3, 4, 5.
\]
Let \(\delta > 0\) and let \(\xi_j, j = 1, 2, 3, 4, \) be as in Section 5.2. Especially \(\langle \xi_1, \xi_2 \rangle = 1 - \delta\) and \(|\xi_1| = |\xi_2| = 1\). Note that the integral identity \((5.24)\) implicitly concerns 5 possibly different \(v^{(k)}\). We choose the vectors \(\xi_k \in T_{p_0} M_0\) as follows
\[
\xi_1 = \xi_1, \quad \xi_2 = \xi_2,
\]
\[
\xi_3 = \left(1 + \sqrt{\frac{2}{2 - \delta}}\right) \xi_3, \quad \xi_4 = \left(1 + \sqrt{\frac{2}{2 - \delta}}\right) \xi_4,
\]
and
\[
\xi_5 = \sqrt{\frac{2}{2 - \delta}} (\xi_1 + \xi_2).
\]
Note that $|\zeta_5| = 2$. We define $\zeta_k$ by

\[
\zeta_1 = |\zeta_1|e_1 + i\zeta_1, \quad \zeta_2 = |\zeta_2|e_1 + i\zeta_2, \\
\zeta_3 = |\zeta_3|e_1 + i\zeta_3, \quad \zeta_4 = -|\zeta_4|e_1 + i\zeta_4, \\
\text{and} \quad \zeta_5 = -|\zeta_5|e_1 + i\zeta_5.
\]

We also define $c_k = |\zeta_k|$. In particular, we have $c_1 = c_2 = 1$ and $c_5 = 2$. Then

\[
\sum_{j=1}^{5} \zeta_j = \xi_1 + \xi_2 - \left(1 + \sqrt{\frac{2}{2 - \delta}}\right) \left(\frac{1}{1 + \delta} \xi_1 + \frac{\delta}{1 + \delta} \xi_2 + \frac{\delta}{1 + \delta} \xi_1 + \frac{1}{1 + \delta} \xi_2\right) \\
+ \sqrt{\frac{2}{2 - \delta}} (\xi_1 + \xi_2) = 0. \quad (5.27)
\]

and

\[
\text{Re} \left(\sum_{j=1}^{5} \zeta_j^\prime\right) = 0. \quad (5.28)
\]

Consequently, the sum of the vectors $\zeta_\alpha$ vanishes:

\[
\sum_{j=1}^{5} \zeta_j = 0. \quad (5.29)
\]

The condition (5.29) will imply that the non-stationary phase at $p_0$ and exponentially growing factors of in the integrand of the integral identity (5.24) will cancel out. We showed in Section 5.2 that the vectors $\xi_1, \ldots, \xi_4$ are pairwise linearly independent. Consequently $\zeta_1, \ldots, \zeta_4$ are pairwise linearly independent. We also see that $\zeta_5$ is not proportional to any of the other vectors $\zeta_1, \ldots, \zeta_4$. It follows that the geodesics of $(M_0, g_0)$ corresponding to $\zeta_1, \ldots, \zeta_5$ intersect properly. Since the vectors $\zeta_2, \ldots, \zeta_5$ are up to scalings small perturbations of $\zeta_1$, the corresponding geodesics are nontangential and they do not have self-interactions.

We then consider how solutions to (5.25), which correspond to the CGOs determined by the vectors $\zeta_j$, $j = 1, 2, 3, 4$, look like. Let us first note that at the intersection points of the corresponding geodesics

\[
(\pm c_i \pm c_j \pm c_k)^2 - |c_i \nabla\psi_i + c_j \nabla\psi_j + c_k \nabla\psi_k|^2 = 0 \quad (5.30)
\]

for all indices $i, j, k \in \{2, 3, 4\}$ which are all different. Indeed, by (5.29) we note that

\[
(\pm c_i \pm c_j \pm c_k)^2 - |c_i \nabla\psi_i + c_j \nabla\psi_j + c_k \nabla\psi_k|^2 = (\zeta_l + \zeta_m, \zeta_l + \zeta_m),
\]

where $l, m \{1, 2, 3, 4, 5\}$ are the unique two different indices, which do not belong to the set $\{j, k, l\}$. Then, we have

\[
(\zeta_l + \zeta_m, \zeta_l + \zeta_m) = 2(\zeta_l, \zeta_m) = 2(\pm c_l c_m - (\zeta_l, \zeta_m)) \neq 0,
\]

since $|\zeta_l, \zeta_m| < |\zeta_l||\zeta_m| = c_l c_m$. Here we have the strict inequality since $\zeta_l$ and $\zeta_m$ are linearly independent.

By (5.30), we may apply Lemma 4.7. Thus, by having the restrictions of the supports of $v^{(k)}$ to $M_0$ in small enough neighborhoods of the corresponding geodesics, the solution $w^{(123)}$ to third linearization

\[
(-\Delta + V)w^{(123)} = -2q \left(v^{(1)}w^{(23)} + v^{(2)}w^{(13)} + v^{(3)}w^{(12)}\right)
\]
up to a correction term is given by a WKB ansatz with amplitude of the form (4.26). The leading order coefficient of the ansatz is

\[
B_{-4}^{(123)} = 4q^3 a_0^{(1)} a_0^{(2)} a_0^{(3)} \frac{1}{(\pm c_1 \pm c_2 \pm c_3)^2 - |c_1 \nabla \psi_1 + c_2 \nabla \psi_2 + c_3 \nabla \psi_3|^2} 
\times \left( \frac{1}{(\pm c_1 \pm c_2)^2 - |c_1 \nabla \psi_1 + c_2 \nabla \psi_2|^2} + \frac{1}{(\pm c_1 \pm c_3)^2 - |c_1 \nabla \psi_1 + c_3 \nabla \psi_3|^2} + \frac{1}{(\pm c_2 \pm c_3)^2 - |c_2 \nabla \psi_2 + c_3 \nabla \psi_3|^2} \right).
\]

Note the power 2 of the potential \(q\) in \(B_{-4}^{(123)}\). Let us define for \(i, j, k \in \{1, \ldots, 4\}\) all different

\[
D_{ij} = \langle \zeta_i + \zeta_j, \zeta_i + \zeta_j \rangle,
\]

and

\[
D_{ijk} = \langle \zeta_i + \zeta_j + \zeta_k, \zeta_i + \zeta_j + \zeta_k \rangle.
\]

(5.31)

At an intersection point \(p_0\) of the geodesics, we thus have

\[
B_{-4}^{(123)}(p_0) = 4q^3 a_0^{(1)} a_0^{(2)} a_0^{(3)} \frac{1}{D_{123}} \left( \frac{1}{D_{12} + D_{13} + D_{23}} \right).
\]

We have similar formulas for the leading order coefficients of the ansatzes for \(w^{(124)}\), \(w^{(134)}\) and \(w^{(142)}\). Furthermore, by (5.29) we have

\[
D_{123} = \langle \zeta_1 + \zeta_2 + \zeta_3, \zeta_1 + \zeta_2 + \zeta_3 \rangle = \langle \zeta_4 + \zeta_5, \zeta_4 + \zeta_5 \rangle = D_{45},
\]

\[
D_{234} = \langle \zeta_4 + \zeta_5, \zeta_4 + \zeta_5 \rangle = D_{15},
\]

\[
D_{134} = D_{25},
\]

\[
D_{124} = D_{35}.
\]

(5.32)

Therefore, by using (5.32), the solution \(w^{(124)}\) to the fourth order linearization will be (up to a correction term) of the form

\[
\left[ \frac{1}{D_{15}} \left( \frac{1}{D_{23} + D_{24} + D_{34}} \right) + \frac{1}{D_{25}} \left( \frac{1}{D_{13} + D_{14} + D_{24}} \right) + \frac{1}{D_{45}} \left( \frac{1}{D_{12} + D_{13} + D_{25}} \right) \right] + \frac{1}{D_{12}} \left( \frac{1}{D_{34} + D_{13} + D_{24}} \right) + \frac{1}{D_{14}} \left( \frac{1}{D_{13} + D_{24} + D_{23}} \right) \times e^{\tau \sum_{i=1,2,3,4} (\text{Re}(\zeta_i) x_1 + i |\zeta_i| \psi_1)} \tilde{A}.
\]

(5.33)

Here \(\tilde{A}\) is an amplitude function which has (up to a multiplication by a power of \(\tau\)) the leading order coefficient

\[
8q^3 a_0^{(1)} a_0^{(2)} a_0^{(3)} a_0^{(4)}.
\]

Note the power 3 of the potential \(q\) here.

Similar to Section 5.2, where we showed that the factor (5.16) of the third order linearization is non-zero, we may show that the coefficient in the brackets of (5.33), call it \(E_d\) is not zero. We have:
Lemma 5.2. The quantity

\[ E_0 = \frac{1}{D_{15}} \left( \frac{1}{D_{23} + D_{24} + D_{34}} \right) + \frac{1}{D_{25}} \left( \frac{1}{D_{13} + D_{14} + D_{34}} \right) + \frac{1}{D_{35}} \left( \frac{1}{D_{12} + D_{14} + D_{24}} \right) + \frac{1}{D_{45}} \left( \frac{1}{D_{12} + D_{13} + D_{23}} \right) \]

(5.34)

for all sufficiently small \( \delta > 0 \).

The proof of the lemma is elementary, but involves rather long calculations. We have placed the proof in Appendix C.

5.6. Proof of \( q_1 = q_2 \) (continued). Let us then return to proving \( q_1 = q_2 \). Let \( \zeta_k, k = 1, \ldots, 5 \), be as in Section 5.5 above. We have

\[ c_k = |\zeta_k| \]

and we set

\[ s_1 = c_1 \tau + i \lambda, \quad s_k = c_k, \quad k = 2, 3, 4, 5. \]

The CGOs corresponding to vectors \( \zeta_k \) are of the form

\[ \begin{align*}
    v^{(1)} &= e^{(\zeta_{\nu_{\lambda}} + i \lambda \tau)x_1} \left( \tau, \frac{a^2}{\tau^{\frac{n}{2}}}, \right) a_1 + r_1, \\
    v^{(2)} &= e^{(\zeta_{\nu_{\lambda}} + i \lambda \tau)x_1} \left( \tau, \frac{a^2}{\tau^{\frac{n}{2}}}, \right) a_2 + r_2, \\
    v^{(3)} &= e^{(\zeta_{\nu_{\lambda}} + i \lambda \tau)x_1} \left( \tau, \frac{a^2}{\tau^{\frac{n}{2}}}, \right) a_3 + r_3, \\
    v^{(4)} &= e^{-|\zeta_{\nu_{\lambda}} + i \lambda \tau|x_1} \left( \tau, \frac{a^2}{\tau^{\frac{n}{2}}}, \right) a_4 + r_4, \\
    v^{(5)} &= e^{-|\zeta_{\nu_{\lambda}} + i \lambda \tau|x_1} \left( \tau, \frac{a^2}{\tau^{\frac{n}{2}}}, \right) a_5 + r_5.
\end{align*} \]

(5.35)

Since \( \Lambda_{q_1}(f_{e}) = \Lambda_{q_2}(f_{e}) \), by Propositions 4.4 and 4.8 there are boundary values \( f_{ik} \) and \( f_{ijk} \), \( i, j, k = 1, 2, 3, 4 \), such that the solutions of the second linearized equations (5.4) and third linearized equations

\[ \begin{cases}
    (-\Delta + V) \omega^{(ijk)}_\beta = -2q_\beta \left( v^{(i)} w^{(jk)}_{\beta} + v^{(j)} w^{(ik)}_{\beta} + v^{(k)} w^{(ij)}_{\beta} \right) & \text{in } M, \\
    \omega^{(ijk)}_\beta = f_{ijk} & \text{on } \partial M,
\end{cases} \]

(5.36)

for \( \beta = 1, 2 \), and \( i, j, k \) all different, which are of the form

\[ \begin{align*}
    w^{(ij)}_\beta &= w^{(ij)}_{0,\beta} + e^r \Psi^{(ij)}_{\beta}, \\
    \omega^{(ijk)}_\beta &= \omega^{(ijk)}_{0,\beta} + e^r \Psi^{(ijk)}_{\beta}.
\end{align*} \]

For given \( K, N \in \mathbb{N} \cup \{0\} \), the correction terms \( R^{(ij)}_{\beta} \) and \( \bar{R}^{(ijk)}_{\beta} \) can be assumed to be \( \mathcal{O}_{L^2(M)}(\tau^{-N}) \) by taking the amplitude expansions of the CGOs \( v^{(k)} \), \( k = 1, 2, 3, 4 \) to be precise enough (i.e. \( N' \) large enough). We refer to Propositions (4.4) and (4.8) for the specifics of \( w^{(ij)}_\beta \) and \( w^{(ijk)}_\beta \).

The phase functions \( \Psi^{(ij)} \) and \( \bar{\Psi}^{(ijk)} \) satisfy at the point \( p_0 \) where all the geodesics \( \gamma_1, \ldots, \gamma_5 \) intersect

\[ \begin{align*}
    \Psi^{(ij)}(p_0) &= \zeta_i + \zeta_j, \\
    \bar{\Psi}^{(ijk)}(p_0) &= \zeta_i + \zeta_j + \zeta_k.
\end{align*} \]

(5.37)
The leading order coefficients of the amplitudes of $w^{(ik)}_\beta$ and $\omega^{(kl)}_\beta$ are
\begin{align}
b^{(ik)}_{2,\beta}(p_0) &= \frac{2q_\beta}{D_k} a_0^{(i)} a_0^{(k)}, \\
P^{(kl)}_{4,\beta}(p_0) &= 4q_\beta^2 a_0^{(i)} a_0^{(k)} a_0^{(l)} \frac{1}{D_{ikl}} \left( \frac{1}{D_{ik} + D_{il} + D_{kl}} \right). \tag{5.38}
\end{align}

Let us denote $\Psi_{12345}$ as the sum of all the phase functions of $\psi^{(1)}, \ldots, \psi^{(5)}$ in (5.35), where $\tau$ is a parameter. More precisely, $\Psi_{12345}$ is given as
\begin{equation}
\Psi_{12345} = \left( |\zeta_1| + |\zeta_2| + |\zeta_3| - |\zeta_4| - |\zeta_5| \right) x_1 + i \left( |\zeta_1| \psi_1 + |\zeta_2| \psi_2 + |\zeta_3| \psi_3 + |\zeta_4| \psi_4 + |\zeta_5| \psi_5 \right).
\end{equation}

Let us also set
\[ \Lambda_{1234} = \lambda_1 (ix_1 - \psi_1). \]
At the point $p_0$ where all the geodesics $\gamma_1, \ldots, \gamma_5$ intersect
\[ \nabla \Psi_{12345}(x_1, p_0) = 0 \] (5.39)
for $x \in I \subset \mathbb{R}$ by (5.27), and
\[ \text{Re}(\Psi_{12345})(x_1, p_0) = 0 \] (5.40)
by (5.28). The condition (5.40) implies that $\Psi_{12345}$ is not exponentially growing in $\tau$. Moreover, by (5.39) we have that $p_0$ is a critical point of $\Psi_{12345}$. By the properties of $\psi_1$, the point $p_0$ is also nondegenerate, see (3.2).

We multiply the right hand side of the integral identity of the fourth order linearization (5.24) by $\tau^4 \tau^{1/2}$ and take the limit $\tau \to \infty$. In the case $p_0$ is the only point where all the geodesics $\gamma_1, \ldots, \gamma_5$ intersect, by stationary phase the limit tends to
\[ 0 = c_A E_5 \left( a_0^{(1)} a_0^{(2)} a_0^{(3)} a_0^{(4)} a_0^{(5)} \right) \int_{\mathbb{R}} e^{i\lambda x_1} (q_1^3(x_1, p_0) - q_2^3(x_1, p_0)) dx_1, \]
where $E_5$ is the coefficient of $w^{(1234)}$ in Lemma 5.2 and $c_A \neq 0$ is given by a similar formula as $c_A$ in Section 5.3. Here we also used that $a_0^{(1)}, \ldots, a_0^{(5)}$ are independent of $x_1$. By Lemma 5.2, the coefficient $E_5 \neq 0$ for all small enough $\delta > 0$. Inverting, the Fourier transformation in the variable $x_1$ shows that $q_1^3(x_1, p_0) = q_2^3(x_1, p_0)$. Thus
\[ q_1(x_1, p_0) = q_2(x_1, p_0) \]
for $x_1 \in \mathbb{R}$. If there were several points where $\gamma_1, \ldots, \gamma_5$ intersect, we argue similarly as in Section 5.3 by using [LLLS21, Lemma 6.2]. Since $p_0$ was arbitrary, this completes the proof.

**Appendix A. Boundary determination**

We prove that the DN map of the semilinear elliptic equation
\[ (-\Delta g + V)u + qu^m = 0 \text{ in } M, \quad u = f \text{ on } \partial M \]
on a compact smooth Riemannian manifold with boundary determines the formal Taylor series (the jet) of the coefficient $q$ (in the boundary normal coordinates) on the boundary. Here, $m \geq 2$ is an integer, and $V$ and $q$ are smooth functions on $M$. We assume also that zero is not a Dirichlet eigenvalue for the operator $-\Delta_g + V$ on $M$.

We expect this result to be well-known to experts on the field, but could not find a reference on it, so we offer detailed characterization and its proof.
**Proposition A.1** (Boundary determination). For \( m \geq 2 \), \( m \in \mathbb{N} \), let \((M, g)\) be a compact Riemannian manifold with \( C^\infty \) boundary \( \partial M \) and consider the boundary value problem

\[
\begin{aligned}
(-\Delta_g + V) u + qu^m &= 0 \quad \text{in } M, \\
u &= f \quad \text{on } \partial M.
\end{aligned}
\]  

(A.1)

Assume that the DN map \( \Lambda_q \) of the equation (A.1) is known for small boundary values. Then \( \Lambda_q \) determines the formal Taylor series of \( q \) on the boundary \( \partial M \).

In addition, if \( f \in C^{2,\alpha}(\partial M) \) is so small that (A.1) has a unique small solution, the DN map determines the formal Taylor series of the solution \( u = u_f \) at any point on the boundary.

**Proof.** Determination of Taylor expansion of \( q \):

Let \( f \in C^{2,\alpha}(\partial M) \). We consider boundary values \( f_0 \in C^{2,\alpha}(\partial M) \) and \( f_t = f_0 + tf \) and assume that \( \|f_0\|_{C^{2,\alpha}(\partial M)} \) and \( |t| \) are sufficiently small so that the DN maps of \( f_0 \) and \( f_t \) are both well-defined. We denote by \( u_0 \) and \( u_t \), the unique solutions of (A.1) with boundary data \( f_0 \) and \( f_t \) on \( \partial M \), respectively. By linearizing the equation (A.1) at \( t = 0 \), we obtain

\[
\begin{aligned}
(-\Delta_g + V) v + mqu^{m-1}_0 &= 0 \quad \text{in } M, \\
v &= f \quad \text{on } \partial M,
\end{aligned}
\]  

(A.2)

where \( v = \lim_{t \to 0} \frac{u_t - u_0}{t} \) and \( u_0 \) solves

\[
\begin{aligned}
(-\Delta_g + V)u_0 + qu^m_0 &= 0 \quad \text{in } M, \\
u_0 &= f_0 \quad \text{on } \partial M.
\end{aligned}
\]  

(A.3)

Moreover, \( v \) is the solution of

\[
\begin{aligned}
-\Delta_g v + \tilde{q} v &= 0 \quad \text{in } M, \\
v &= f \quad \text{on } \partial M.
\end{aligned}
\]  

(A.4)

where

\[
\tilde{q} := V + mqu^{m-1}_0 \quad \text{in } M.
\]

Since we know the DN map of the boundary value problem (A.1), we know the DN map of the linearized problem (A.2). This is proven in [LLLS21a, Proposition 2.1], where it is shown that the DN map is \( C^\infty \) in the Frechét sense. It follows by [FKSU09, Theorem 8.4.] that we know the formal Taylor series of \( \tilde{q} \) on \( \partial M \). In particular, by choosing

\[
u_0 = f_0 = \varepsilon_0 > 0 \quad \text{on } \partial M,
\]

for some sufficiently small constant \( \varepsilon_0 > 0 \), and noting that

\[
q = \frac{\tilde{q} - V}{m\varepsilon_0^{m-1}} \quad \text{on } \partial M,
\]

it follows that we know the of \( q \) on the boundary \( \partial M \).

Next we determine first order derivatives of \( q \) on the boundary. Given a point \( x_0 \in \partial M \), let \( x = (x_1, \ldots, x_n) \in \partial M \) be boundary normal coordinates near \( x = x_0 \) in \( M \). Differentiating (A.4) yields

\[
\begin{aligned}
\partial_{x_n} \tilde{q} &= m\partial_{x_n}(u_0^{m-1})q + m(\partial_{x_n}q)u_0^{m-1} + \partial_{x_n}V \\
&= m(m-1)u_0^{m-2}(\partial_{x_n}u_0)q + m(\partial_{x_n}q)u_0^{m-1} + \partial_{x_n}V.
\end{aligned}
\]  

(A.5)

Since we have already determined the Taylor series of \( \tilde{q} \) on the boundary and

\[
\partial_{x_n}u_0 = \Lambda_q(f_0),
\]
we may determine $\partial_{x_n} q$ by solving it from (A.5). Since we also know the derivatives of $q$ in tangential directions $x_k$, where $k = 1, \ldots, n - 1$, we have determined all first order derivatives of $q$ on the boundary.

To determine higher order derivatives of $q$ on the boundary, we follow an argument similar to [LLS16, Lemma 3.4]. On a neighborhood of $x_0$ in $M$ we may write

$$Q u_0 := (-\Delta_q + V) u_0 + q u_0^m = -\partial_{x_n}^2 u_0 + P u_0,$$

where $P$ is a non-linear partial differential operator containing derivatives in $x'$ up to order 2 and in $x_n$ up to order 1. The coefficients of $P$ depend on pointwise values of $q$. By expressing

$$\partial_{x_n}^2 = P - Q$$

we obtain

$$\partial_{x_n}^2 u_0 = P u_0 - Q u_0 = P u_0. \quad (A.6)$$

Since we already know the quantities

$$u_0, \partial_{x'} u_0, \partial_{x_n}^2 u_0, \partial_{x'} x_n u_0, \partial_{x_n} q, \partial_{x'} q \text{ and } \partial_{x_n} q,$$

it follows from (A.6) that the second derivative $\partial_{x_n}^2 u_0$ can be also determined. By using this and differentiating (A.5), we may determine second order derivatives of $q$ on the boundary. The higher order derivatives of $q$ on the boundary can be determined by differentiating (A.6) and using (A.5) in succession, and by using induction.

**Determination of Taylor expansions of solutions:** Let then $f \in C^{2,\alpha}(\partial M)$ be small enough so that (A.1) has a unique small solution $u = u_f$. Since we have determined the formal Taylor series of $q$ on the boundary, the formal Taylor series of $u$ on the boundary is determined by differentiating (A.6) with $u$ in place of $u_0$.

**Appendix B. Proof of the Carleman estimate with boundary terms**

In this section, we proceed to prove Lemma 4.6. Let $(M, g)$ be a compact, smooth, transversally anisotropic Riemannian manifold with a smooth boundary and let $V \in L^\infty(M)$. There exists constants $\tau_0 > 0$ and $C > 0$ depending only on $(M, g)$ and $\|V\|_{L^\infty(M)}$ such that given any $|\tau| > \tau_0$ and any $v \in C^2(M)$, the following Carleman estimate holds

$$\|e^{-\tau|\cdot|}(-\Delta_g + V)(e^{\tau|\cdot|}v)\|_{L^2(M)} + |\tau|^{\frac{3}{2}} v\|_{W^{2,\infty}(\partial M)} + |\tau|^{\frac{3}{2}} \|\partial v\|_{W^{1,\infty}(\partial M)}$$

$$+ |\tau|^{\frac{3}{2}} \|\partial_x^2 v\|_{L^\infty(M)} \geq C |\tau| \|v\|_{L^2(M)}. \quad (B.1)$$

**Proof of Lemma 4.6.** We may assume without loss of generality that $v$ is real-valued and also that $\tau > 0$. The proof for the case $\tau < 0$ follows analogously. Throughout this proof, we use the notation $C$ to stand for a generic positive constant that is independent of the parameter $\tau$. We also write $\hat{v}$ to stand for a $C^2$-extension of the function $v$ into a slightly larger manifold $\hat{M} \subseteq \mathbb{R} \times M_0$ with smooth boundary, such that $v \in C^2(\hat{M})$ and that holds

$$\|\hat{v}\|_{W^{2,\infty}(\hat{M}\setminus M)} \leq C (\|\partial^2 v\|_{L^\infty(M)} + \|\partial v\|_{W^{1,\infty}(M)} + \|v\|_{W^{2,\infty}(M)}), \quad (B.2)$$

for some constant $C > 0$, only depending on $(\hat{M}, g)$. We remark that this extension of $v$ can always be achieved using a Taylor series approximation together with the boundary normal coordinates near $\partial M$. We define

$$P_\tau v = e^{-\tau t} \Delta_g (e^{\tau t} v), \quad (B.3)$$

and note that

$$P_\tau v = \partial_t^2 v + \Delta_{\tau t} v + 2\tau \partial_t v + \tau^2 v.$$
Thus, using (B.2) we deduce that

\[ \int_M P_t v \partial_t v \, dV_g = 2\tau \int_M |\partial_t v|^2 \, dV_g + \int_M \partial_t^2 v \partial_t v \, dV_g + \int_M \Delta_{g_t} v \partial_t v \, dV_g + \int_M \tau^2 v \partial_t v \, dV_g. \]

We claim that

\[ \left| \int_M P_t v \partial_t v \, dV_g \right| + C\tau^2 \|v\|_{W^2,\infty(\partial M)}^2 + C\tau^2 \|\partial_t v\|_{W^1,\infty(\partial M)}^2 \]

\[ + C\tau^2 \|\partial_t^2 v\|_{L^\infty(\partial M)}^2 \geq 2\tau \|\partial_t v\|_{L^2(M)}^2. \quad (B.4) \]

To show (B.4) we begin by writing

\[ \int_M P_t v \partial_t v \, dV_g = 2\tau \int_M |\partial_t v|^2 \, dV_g + \int_M \Delta_{g_t} v \partial_t v \, dV_g + \int_M \partial_t^2 v \partial_t v \, dV_g + \int_M \tau^2 v \partial_t v \, dV_g. \]

Note that \( M \in \mathbb{R} \times M_0 \) and \( dV_g = dt \, dV_{g_0} \). We can use integration by parts to bound each of the terms I–III as follows. For I, we first note that

\[ \int_M \partial_t^2 v \partial_t v \, dV_g = \frac{1}{2} \int_M \partial_t (|\partial_t v|^2) \, dV_g = 0. \]

Together with the estimate (B.2), we obtain

\[ |I| = \left| \int_{\hat{M} \setminus M} \partial_t^2 v \partial_t v \, dV_g \right| \leq C \left( \|\partial_t^2 v\|_{L^\infty(\partial M)}^2 + \|\partial_t v\|_{W^1,\infty(\partial M)}^2 + \|v\|_{W^2,\infty(\partial M)}^2 \right). \]

For II, since [\partial_t, \Delta_g] = 0 on (\hat{M}, g), we may apply integration by parts again to deduce that

\[ \int_{\hat{M}} \Delta_{g_t} v \partial_t v \, dV_g = 0. \]

Thus, using (B.2), we can show analogously to term I that

\[ |II| \leq C \left( \|\partial_t^2 v\|_{L^\infty(\partial M)}^2 + \|\partial_t v\|_{W^1,\infty(\partial M)}^2 + \|v\|_{W^2,\infty(\partial M)}^2 \right). \]

Finally for the term III we first note that

\[ \tau^2 \int_M \partial_t v \partial_t v \, dV_g = \frac{\tau^2}{2} \int_M \partial_t (\partial_t^2 v) \, dV_g = 0. \]

Thus, using (B.2), we have

\[ |III| \leq C\tau^2 \left( \|\partial_t^2 v\|_{L^\infty(\partial M)}^2 + \|\partial_t v\|_{W^1,\infty(\partial M)}^2 + \|v\|_{W^2,\infty(\partial M)}^2 \right). \]

Combining the previous three bounds yields the claimed estimate (B.4). Using (B.4) and applying the Cauchy-Schwarz inequality

\[ \left| \int_M P_t v \partial_t v \, dV_g \right| \leq \frac{1}{4\tau} \|P_t v\|_{L^2(M)}^2 + \tau \|\partial_t v\|_{L^2(M)}^2, \]

we deduce that

\[ \|P_t v\|_{L^2(M)}^2 + C\tau^3 \|v\|_{W^2,\infty(\partial M)}^2 + C\tau^3 \|\partial_t v\|_{W^1,\infty(\partial M)}^2 \]

\[ + C\tau^2 \|\partial_t^2 v\|_{L^\infty(\partial M)}^2 \geq \tau^2 \|\partial_t v\|_{L^2(M)}^2. \quad (B.5) \]

We recall that by the standard Poincaré inequality on \( \hat{M} \), there exists \( C > 0 \) such that

\[ \|\partial_t w\|_{L^2(M)} \geq C\|w\|_{L^2(M)} \quad \forall w \in H^1_0(\hat{M}). \]

Using the latter bound together with a bound analogous to (B.2) for extending \( C^1(M) \) functions into \( C^1_c(\hat{M}) \), we can derive the following Poincaré type inequality

\[ \|\partial_t v\|_{L^2(M)} \geq C_1 \|v\|_{L^2(M)} - C_2 \|v\|_{W^1,\infty(\partial M)} - C_3 \|\partial_t v\|_{L^\infty(\partial M)}, \quad (B.6) \]
for all $v \in C^1(M)$, where the positive constants $C_1$, $C_2$ and $C_3$ only depend on $(M, g)$.

Via the bounds (B.5)–(B.6), we deduce that

$$
\|(P_T - V)v\|_{L^2(M)}^2 + C\tau^3 \|v\|_{W^{2, \infty}(\partial M)}^2 + C\tau^2 \|\partial_{\nu}v\|_{L^2(\partial M)}^2 + C\tau^3 \|\partial_{\nu}^2v\|_{L^2(\partial M)}^2 \geq \tau^2 \|v\|_{L^2(M)}^2.
$$

(B.7)

This proves the assertion.

\[\square\]

**Appendix C. Computations of $D_{ik}$**

In the end of this paper, we compute the values $D_{ik}$, for different sub-indices $i, k \in \{1, 2, 3, 4, 5\}$. Recalling that

$$
\zeta_1 = \xi_1, \quad \zeta_2 = \xi_2,
$$

$$
\zeta_3 = \left(1 + \sqrt{\frac{2}{2 - \delta}}\right) \xi_3, \quad \zeta_4 = \left(1 + \sqrt{\frac{2}{2 - \delta}}\right) \xi_4,
$$

$$
\zeta_5 = \sqrt{\frac{2}{2 - \delta}} (\xi_1 + \xi_2),
$$

and

$$
\overline{\zeta}_1 = |\xi_1| e_1 + i \xi_1, \quad \overline{\zeta}_2 = |\xi_2| e_1 + i \xi_2,
$$

$$
\overline{\zeta}_3 = |\xi_3| e_1 + i \xi_3, \quad \overline{\zeta}_4 = -|\xi_4| e_1 + i \xi_4,
$$

$$
\overline{\zeta}_5 = -|\xi_5| e_1 + i \xi_5,
$$

where

$$
|\xi_1| = |\xi_2| = 1, \quad \langle \xi_1, \xi_2 \rangle = 1 - \delta,
$$

$$
\xi_3 = \frac{1}{1 + \delta} (\xi_1 + \delta \xi_2), \quad \xi_4 = \frac{1}{1 + \delta} (\delta \xi_1 + \xi_2).
$$

Via straightforward computations, we have

$$
\langle \xi_1, \xi_2 \rangle = 1 - \delta, \quad \langle \xi_1, \xi_3 \rangle = \frac{1 + \delta - \delta^2}{1 + \delta}, \quad \langle \xi_1, \xi_4 \rangle = \frac{1}{1 + \delta},
$$

$$
\langle \xi_2, \xi_3 \rangle = -\frac{1}{1 + \delta}, \quad \langle \xi_2, \xi_4 \rangle = \frac{1 + \delta - \delta^2}{1 + \delta}, \quad \text{and} \quad \langle \xi_3, \xi_4 \rangle = \frac{1 + \delta + \delta^2 - \delta^3}{(1 + \delta)^2}.
$$

By

$$
D_{ij} = \langle \xi_i + \xi_j, \xi_i + \xi_j \rangle,
$$

for different $i, k \in \{1, 2, 3, 4, 5\}$, direct computations yield that

$$
D_{12} = (|\xi_1| e_1 + i \xi_1 + |\xi_2| e_1 + i \xi_2) \cdot (|\xi_1| e_1 + i \xi_1 + |\xi_2| e_1 + i \xi_2) = 2 (|\xi_1||\xi_2| - \langle \xi_1, \xi_2 \rangle) = 2 (|\xi_1||\xi_2| - \langle \xi_1, \xi_2 \rangle) = 2 \delta,
$$

(C.1)

$$
D_{13} = (|\xi_1| e_1 + i \xi_1 + |\xi_3| e_1 + i \xi_3) \cdot (|\xi_1| e_1 + i \xi_1 + |\xi_3| e_1 + i \xi_3) = 2 (|\xi_1||\xi_3| - \langle \xi_1, \xi_3 \rangle) = 2 \left(1 + \sqrt{\frac{2}{2 - \delta}}\right) (|\xi_1||\xi_3| - \langle \xi_1, \xi_3 \rangle) = 2 \left(1 + \sqrt{\frac{2}{2 - \delta}}\right) \left(2 + 2\delta + \mathcal{O}(\delta^2)\right),
$$

(C.2)
\[ D_{14} = (|\zeta_4|e_1 + i\zeta_4 - |\zeta_4|e_1 + i\zeta_4) \cdot (|\zeta_4|e_1 + i\zeta_4 - |\zeta_4|e_1 + i\zeta_4) \\
= -2 \left( 1 + \sqrt{\frac{2}{2 - \delta}} \right) (|\zeta_1||\zeta_4| + \langle \xi_1, \xi_4 \rangle) \\
= -2 \left( 1 + \sqrt{\frac{2}{2 - \delta}} \right) \delta + O(\delta^2), \quad (C.3) \]

\[ D_{15} = (|\zeta_4|e_1 + i\zeta_4 - |\zeta_5|e_1 + i\zeta_5) \cdot (|\zeta_4|e_1 + i\zeta_4 - |\zeta_5|e_1 + i\zeta_5) \\
= -2 (|\zeta_1||\zeta_5| + \langle \xi_1, \xi_5 \rangle) \\
= -2 \sqrt{\frac{2}{2 - \delta}} (|\zeta_1||\xi_1 + \xi_2| + \langle \xi_1, \xi_1 + \xi_2 \rangle) \\
= -8 + \frac{\delta}{2} + O(\delta^2), \quad (C.4) \]

\[ D_{23} = (|\zeta_2|e_1 + i\zeta_2 + |\zeta_3|e_1 + i\zeta_3) \cdot (|\zeta_2|e_1 + i\zeta_2 + |\zeta_3|e_1 + i\zeta_3) \\
= 2 \left( 1 + \sqrt{\frac{2}{2 - \delta}} \right) (|\zeta_2||\zeta_3| - \langle \xi_2, \xi_3 \rangle) \\
= 2 \left( 1 + \sqrt{\frac{2}{2 - \delta}} \right) \left( 2 + \delta + O(\delta^2) \right), \quad (C.5) \]

In order to compute \( D_{24} \) more carefully, let us recall the Taylor expansion of \( \sqrt{1 + \delta} = 1 + \frac{\delta}{2} - \frac{\delta^2}{8} + O(\delta^3) \), then we have

\[ D_{24} = (|\zeta_2|e_1 + i\zeta_2 - |\zeta_4|e_1 + i\zeta_4) \cdot (|\zeta_2|e_1 + i\zeta_2 - |\zeta_4|e_1 + i\zeta_4) \\
= -2 (|\zeta_2||\zeta_4| + \langle \xi_2, \xi_4 \rangle) \\
= -2 \left( 1 + \sqrt{\frac{2}{2 - \delta}} \right) (|\zeta_2||\zeta_4| - \langle \xi_2, \xi_4 \rangle) \\
= -2 \left( 1 + \sqrt{\frac{2}{2 - \delta}} \right) \left( \sqrt{1 + 2\delta - \delta^2} - (1 + \delta - \delta^2) \right) \\
= -2 \left( 1 + \sqrt{\frac{2}{2 - \delta}} \right) \left( 1 + \delta - \delta^2 + O(\delta^3) - (1 + \delta - \delta^2) \right) \\
= -2 \left( 1 + \sqrt{\frac{2}{2 - \delta}} \right) O(\delta^3) / (1 + \delta), \quad (C.6) \]

\[ D_{25} = (|\zeta_2|e_1 + i\zeta_2 - |\zeta_5|e_1 + i\zeta_5) \cdot (|\zeta_2|e_1 + i\zeta_2 - |\zeta_5|e_1 + i\zeta_5) \\
= -2 (|\zeta_2||\zeta_5| + \langle \xi_2, \xi_5 \rangle) \\
= -2 \sqrt{\frac{2}{2 - \delta}} \left( \sqrt{4 - 2\delta + 2 - \delta} \right) \\
= -8 + \delta + O(\delta^2), \quad (C.7) \]
\[ D_{34} = (|\zeta_3|e_1 + i\zeta_3 - |\zeta_4|e_1 + i\zeta_4) \cdot (|\zeta_5|e_1 + i\zeta_5 - |\zeta_4|e_1 + i\zeta_4) \\
= -2 (|\zeta_3||\zeta_4| + \langle \zeta_3, \zeta_4 \rangle) \\
= -2 \left( 1 + \sqrt{\frac{2}{2-\delta}} \right) \left( \frac{2}{2-\delta} \right)^2 (2 + 3\delta - \delta^3), \quad \text{(C.8)} \]

\[ D_{35} = (|\zeta_3|e_1 + i\zeta_3 - |\zeta_5|e_1 + i\zeta_5) \cdot (|\zeta_3|e_1 + i\zeta_3 - |\zeta_5|e_1 + i\zeta_5) \\
= -2 (|\zeta_3||\zeta_5| + \langle \zeta_3, \zeta_5 \rangle) \\
= -2 \left( 1 + \sqrt{\frac{2}{2-\delta}} \right) \left( \frac{2}{2-\delta} \right)^2 \left( \sqrt{4 - 2\delta(1 + \mathcal{O}(\delta^2) - 2 - \delta + \delta^2) \right) \\
= -2 \left( 1 + \sqrt{\frac{2}{2-\delta}} \right) \left( \frac{2}{2-\delta} \right)^2 \left( \delta + \mathcal{O}(\delta^3) \right) \quad \text{(C.9)} \]

and similarly,

\[ D_{45} = (-|\zeta_4|e_1 + i\zeta_4 - |\zeta_5|e_1 + i\zeta_5) \cdot (-|\zeta_4|e_1 + i\zeta_4 - |\zeta_5|e_1 + i\zeta_5) \\
= 2 (|\zeta_4||\zeta_5| - \langle \zeta_4, \zeta_5 \rangle) \\
= 2 \left( 1 + \sqrt{\frac{2}{2-\delta}} \right) \left( \frac{2}{2-\delta} \right)^2 \left( |\zeta_4||\zeta_1 + \zeta_2| - \langle \zeta_4, \zeta_1 + \zeta_2 \rangle \right) \\
= 2 \left( 1 + \sqrt{\frac{2}{2-\delta}} \right) \left( \frac{2}{2-\delta} \right)^2 \left( \sqrt{4 - 2\delta(1 + \mathcal{O}(\delta^2)) + 2 + \delta - \delta^2} \right) \\
= 2 \left( 1 + \sqrt{\frac{2}{2-\delta}} \right) \left( \frac{2}{2-\delta} \right)^2 \left( 4 + \frac{5}{2}\delta + \mathcal{O}(\delta^2) \right). \quad \text{(C.10)} \]

**Proof of Lemma 5.2.** With (C.1)–(C.10) at hand, let us split the analysis into two cases.

(1) By using (C.5), (C.6) and (C.8), we have that \( \frac{1}{D_{12} + D_{14} + D_{24}} \) is bounded as \( \delta \to 0 \). Similarly, (C.2), (C.3) and (C.8) imply that \( \frac{1}{D_{15} + D_{14} + D_{24}} \) is also bounded as \( \delta \to 0 \). Similarly, \( \frac{1}{D_{12} + D_{13} + D_{23}} \) is bounded as \( \delta \to 0 \). On the other hand, by (C.1), (C.3) and (C.6), we observe that \( \frac{1}{D_{12} + D_{13} + D_{23}} = \mathcal{O}(\delta^{-1}) \). Meanwhile, \( D_{15}^{-1}, D_{25}^{-1} \) and \( D_{35}^{-1} \) are bounded as \( \delta \to 0 \), but \( D_{35}^{-1} = \mathcal{O}(\delta^{-1}) \).

(2) Similarly, \( \frac{1}{D_{12} D_{14}} = \mathcal{O}(\delta^{-1}), \frac{1}{D_{15} D_{25}} = \mathcal{O}(\delta^{-3}) \) and \( \frac{1}{D_{14} D_{23}} = \mathcal{O}(\delta^{-1}) \).
Therefore, combining the above, we conclude that

$$E_\delta = \frac{1}{D_{15}} \left( \frac{1}{D_{23} + D_{24} + D_{34}} \right) + \frac{1}{D_{25}} \left( \frac{1}{D_{13} + D_{14} + D_{34}} \right) + \frac{1}{D_{35}} \left( \frac{1}{D_{12} + D_{14} + D_{24}} \right) + \frac{1}{D_{12} D_{34}} + \frac{1}{D_{13} D_{24}} + \frac{1}{D_{14} D_{23}}$$

$$\geq \frac{C_0}{\delta^3} - \frac{C_1}{\delta^2} - C_2 > 0,$$

for all sufficiently small $\delta > 0$, where $C_0$, $C_1$ and $C_2$ are some positive constants independent of $\delta$. Hence, the coefficient $E_\delta = \mathcal{O}(\delta^{-3}) \neq 0$ for all sufficiently small $\delta > 0$. □

Acknowledgment. A.F gratefully acknowledges support of the Fields institute for research in mathematical sciences. T.L. was supported by the Academy of Finland (Centre of Excellence in Inverse Modeling and Imaging, grant numbers 284715 and 309963). The work of Y.-H. Lin is partially supported by the Ministry of Science and Technology Taiwan, under the Columbus Program: MOST-110-2636-M-009-007.

REFERENCES

[CF20] Cătălin I. Cărstea and Ali Feizmohammadi. A density property for tensor products of gradients of harmonic functions and applications. arXiv preprint arXiv:2009.11217, 2020.

[CF21] Cătălin I. Cărstea and Ali Feizmohammadi. An inverse boundary value problem for certain anisotropic quasilinear elliptic equations. J. Differential Equations, 284:318–349, 2021.

[CFK+21] Cătălin I. Cărstea, Ali Feizmohammadi, Yavar Kian, Katya Krupchyk, and Gunther Uhlmann. The Calderón inverse problem for isotropic quasilinear conductivities. Adv. Math., 391:Paper No. 107956, 31, 2021.

[CNV19] Cătălin I. Cărstea, Gen Nakamura, and Manmohan Vashisth. Reconstruction for the coefficients of a quasilinear elliptic partial differential equation. Appl. Math. Lett., 98:121–127, 2019.

[DSFKSU09] David Dos Santos Ferreira, Carlos Kenig, Mikko Salo, and Gunther Uhlmann. Limiting carleman weights and anisotropic inverse problems. Inventiones mathematicae, 178(1):119–171, Oct 2009.

[FKLS16] David Dos Santos Ferreira, Yaroslav Kurylev, Matti Lassas, and Mikko Salo. The Calderón problem in transversally anisotropic geometries. J. Eur. Math. Soc. (JEMS), 18:2579–2626, 2016.

[FSU09] David Dos Santos Ferreira, Carlos Kenig, Mikko Salo, and Gunther Uhlmann. Limiting carleman weights and anisotropic inverse problems. Inventiones mathematicae, 178(1):119–171, 2009.

[FLO21] Ali Feizmohammadi, Matti Lassas, and Lauri Oksanen. Inverse problems for nonlinear hyperbolic equations with disjoint sources and receivers. Forum Math. Pi, 9:Paper No. e10, 2021.

[FO20] Ali Feizmohammadi and Lauri Oksanen. An inverse problem for a semi-linear elliptic equation in Riemannian geometries. Journal of Differential Equations, 269(6):4683–4719, 2020.

[HS02] David Hervas and Ziqi Sun. An inverse boundary value problem for quasilinear elliptic equations. Comm. Partial Differential Equations, 27(11-12):2449–2450, 2002.

[HUZ20] Peter Hisz, Gunther Uhlmann, and Jian Zhai. An inverse boundary value problem for a semilinear wave equation on lorentzian manifolds. arXiv preprint arXiv:2005.10447, 2020.

[IN95] Victor Isakov and A Nachman. Global uniqueness for a two-dimensional elliptic inverse problem. Trans. of AMS, 347:3375–3391, 1995.
[IS94] Victor Isakov and John Sylvester. Global uniqueness for a semilinear elliptic inverse problem. *Communications on Pure and Applied Mathematics*, 47(10):1403–1410, 1994.

[Isa93] Victor Isakov. On uniqueness in inverse problems for semilinear parabolic equations. *Archive for Rational Mechanics and Analysis*, 124(1):1–12, 1993.

[KLU18] Yaroslav Kurylev, Matti Lassas, and Gunther Uhlmann. Inverse problems for Lorentzian manifolds and non-linear hyperbolic equations. *Inventiones mathematicae*, 212(3):781–857, 2018.

[KU20a] Yavar Kian and Gunther Uhlmann. Recovery of nonlinear terms for reaction-diffusion equations from boundary measurements. *arXiv preprint arXiv:2011.06689*, 2020.

[KU20b] Katya Krupchyk and Gunther Uhlmann. Inverse problems for nonlinear magnetic Schrödinger equations on conformally transversally anisotropic manifolds. *arXiv e-prints arXiv:2009.05089*, 2020.

[KU20c] Katya Krupchyk and Gunther Uhlmann. A remark on partial data inverse problems for semilinear elliptic equations. *Proc. Amer. Math. Soc.*, 148(2):681–685, 2020.

[Lin21] Yi-Hsuan Lin. Monotonicity-based inversion of fractional semilinear elliptic equations. *Calculus of Variations and Partial Differential Equations*, accepted for publication, 2021.

[LL22] Ru-Yu Lai and Yi-Hsuan Lin. Inverse problems for fractional semilinear elliptic equations. *Nonlinear Analysis*, 216:112699, 2022.

[LLL21] Yi-Hsuan Lin, Hongyu Liu, and Xu Liu. Determining a nonlinear hyperbolic system with unknown sources and nonlinearity. *arXiv preprint arXiv:2107.10919*, 2021.

[LLL21a] Matti Lassas, Tony Liimatainen, Yi-Hsuan Lin, and Mikko Salo. Inverse problems for elliptic equations with power type nonlinearities. *Journal de mathématiques pures et appliquées*, 145:44–82, 2021.

[LLL21b] Matti Lassas, Tony Liimatainen, Yi-Hsuan Lin, and Mikko Salo. Partial data inverse problems and simultaneous recovery of boundary and coefficients for semilinear elliptic equations. *Revista Matemática Iberoamericana*, 37:1553–1580, 2021.

[LLL22] Yi-Hsuan Lin, Hongyu Liu, Xu Liu, and Shen Zhang. Simultaneous recoveries for semilinear parabolic systems. *arXiv preprint arXiv:2111.05242*, 2021.

[LLPMT21] Matti Lassas, Tony Liimatainen, Leyter Potenciano-Machado, and Teemu Tyni. Stability estimates for inverse problems for semi-linear wave equations on lorentzian manifolds. *arXiv e-prints 1612.07939*, 2016.

[LLS16] Matti Lassas, Tony Liimatainen, and Mikko Salo. The Calderón problem for the conformal Laplacian. *arXiv e-prints 1612.07939*, 2016.

[LLS20] Matti Lassas, Tony Liimatainen, and Mikko Salo. The Poisson embedding approach to the Calderón problem. *Math. Ann.*, 377(1-2):19–67, 2020.

[LLST22] Tony Liimatainen, Yi-Hsuan Lin, Mikko Salo, and Teemu Tyni. Inverse problems for elliptic equations with fractional power type nonlinearities. *J. Differential Equations*, 306:189–219, 2022.

[LUW17] Matti Lassas, Gunther Uhlmann, and Yiran Wang. Determination of vacuum spacetimes from the Einstein-Maxwell equations. *arXiv preprint arXiv:1703.10704*, 2017.

[LUW18] Matti Lassas, Gunther Uhlmann, and Yiran Wang. Inverse problems for semilinear wave equations on Lorentzian manifolds. *Communications in Mathematical Physics*, 360:555–609, 2018.

[MuU20] Claudio Muñoz and Gunther Uhlmann. The Calderón problem for quasilinear elliptic equations. *Ann. Inst. H. Poincaré Anal. Non Linéaire*, 37(5):1143–1166, 2020.

[Sha21] Ravi Shankar. Recovering a quasilinear conductivity from boundary measurements. *Inverse Problems*, 37(1):Paper No. 015014, 24, 2021.

[SU97] Ziqi Sun and Gunther Uhlmann. Inverse problems in quasilinear anisotropic media. *American Journal of Mathematics*, 119(4):771–797, 1997.

[Sun06] Ziqi Sun. On a quasilinear inverse boundary value problem. *Math. Z.*, 221(2):293–305, 1996.

[Sun04] Ziqi Sun. Inverse boundary value problems for a class of semilinear elliptic equations. *Advances in Applied Mathematics*, 32(4):791–800, 2004.

[Sun10] Ziqi Sun. An inverse boundary-value problem for semilinear elliptic equations. *Electronic Journal of Differential Equations (EJDE)[electronic only]*, 37:1–5, 2010.

[UZ21a] Gunther Uhlmann and Jian Zhai. Inverse problems for nonlinear hyperbolic equations. *Discrete Contin. Dyn. Syst.*, 41(1):455–469, 2021.

[UZ21b] Gunther Uhlmann and Jian Zhai. An inverse boundary value problem for a nonlinear elastic wave equation. *J. Math. Pures Appl. (9)*, 153:114–136, 2021.
Yiran Wang and Ting Zhou. Inverse problems for quadratic derivative nonlinear wave equations. *Communications in Partial Differential Equations*, 44(11):1140–1158, 2019.

The Fields Institute for Research in Mathematical Sciences, Toronto, Canada

*Current address:* 222 College St, Toronto, ON M5T 3J1

*Email address:* afeizmob@fields.utoronto.ca

Department of Mathematics and Statistics, University of Helsinki, Helsinki, Finland

*Email address:* tony.liimatainen@helsinki.fi

Department of Applied Mathematics, National Yang Ming Chiao Tung University, Hsinchu, Taiwan

*Email address:* yihsuanlin3@gmail.com