The $\beta$-function for Yukawa theory at large $N_f$

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ABSTRACT: We compute the $\beta$-function for a massless Yukawa theory in a closed form at the order $O(1/N_f)$ in the spirit of the expansion in a large number of flavours $N_f$. We find an analytic expression with a finite radius of convergence, and the first singularity occurs at the coupling value $K = 3$. 
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1 Introduction

The success of the Standard Model in describing the electroweak scale phenomena notwithstanding the apparent problems with the high-energy behaviour have lead to revival of interest in better understanding the UV properties of general gauge-Yukawa theories, see e.g. Refs [1–3]. In particular, gauge-Yukawa theories with a large number of fermion flavours, $N_f$, provide interesting candidates within the asymptotic-safety framework as opposed to the traditional asymptotic-freedom paradigm [4, 5].

The groundwork for these considerations was laid few decades ago with the computation of the leading large-$N_f$ behaviour of the gauge $\beta$-functions [6–8] for $N_f$ fermion charged under the gauge group; see also Refs [9, 10]. The leading $1/N_f$ contribution to the $\beta$-function is obtained by resumming the gauge self-energy diagrams with ever increasing chain of fermion bubbles constituting a power series in $K = \alpha N_f / \pi$. It was noticed that this series has a finite radius of convergence: in the case of U(1) gauge group $K = 15/2$. Furthermore, the leading $1/N_f$ contribution to the U(1) $\beta$-function has a negative pole at $K = 15/2$, thereby suggesting that this behaviour could cure the Landau-pole behaviour of the SM U(1) coupling, see e.g. Refs [9, 11, 12].

Recently, a further step towards a more complete understanding of these models was achieved by working out the leading $1/N_f$ contribution from the gauge sector to a Yukawa coupling [13]; an extension to semi-simple gauge groups was discussed in Ref. [14]. However, only a single fermion flavour was assumed to couple to the scalar, and the scalar self-energy remained unaffected by the $N_f$ fermion bubbles. Our work is the first step to bridge this remaining gap: we provide the leading $1/N_f$ $\beta$-function for pure Yukawa theory, where $N_f$ flavours of fermions couple to the scalar field via Yukawa interaction. We leave the more detailed study within a general gauge-Yukawa framework for future work. Interestingly, the pure Yukawa model is closely related to the Gross–Neveu–Yukawa model, whose critical exponents have been recently computed up to $1/N_f^2$ [15, 16]; see also the earlier studies on the Gross–Neveu model e.g. Refs [17, 18].

The paper is organized as follows: In Sec. 2 we introduce the framework and notations and in Sec. 3 give the expressions for the renormalization constants. In Sec. 4 we perform the resummations of the bubble chains and give closed form expressions for the renormalization constants. In Sec. 5 we collect the results, and write down the final expression for the $\beta$-function, and in Sec. 6 we conclude. Explicit formulas for the loop integrals are given in Appendix A.

2 The framework and definitions

We consider the massless Yukawa theory for a real scalar field, $\phi$, and a fermionic multiplet, $\psi$, consisting of $N_f$ flavours interacting through the usual Yukawa interaction:

$$\mathcal{L}_{\text{Yuk}} = g \bar{\psi} \gamma \psi \phi.$$  \hfill (2.1)

We define the rescaled coupling,

$$K \equiv \frac{g^2}{4\pi^2} N_f,$$  \hfill (2.2)
(a) Scalar self-energy corrections.

(b) Fermion self-energy correction.

(c) Vertex correction.

Figure 1: Scalar self-energy, fermion self-energy, and vertex corrections due to a chain of fermion bubbles.

which is kept constant in the limit $N_f \to \infty$. The $\beta$-function of the rescaled coupling, $K$, can then be expanded in powers of $1/N_f$ as

$$\beta(K) \equiv \frac{dK}{d\ln \mu} = K^2 \left[ F_0 + \frac{1}{N_f} F_1(K) \right] + \mathcal{O}\left(1/N_f^2\right). \quad (2.3)$$

The purpose of this paper is to compute $F_0$ and $F_1(K)$. The former is entirely fixed at the one-loop level and can be derived just by rescaling the well-known result for the $\beta$-function at that order, while the evaluation of $F_1(K)$ requires the resummation of diagrams in Fig. 1 involving all-order fermion-bubble chains.

The $\beta$-function can be obtained from

$$\beta = K^2 \frac{\partial G_1(K)}{\partial K}, \quad (2.4)$$

where $G_1$ is defined by

$$\ln Z_K \equiv \ln (Z_S^{-1} Z_F^{-2} Z_V^2) = \sum_{n=1}^{\infty} \frac{G_n(K)}{\epsilon^n}, \quad (2.5)$$

and $Z_S$, $Z_F$, and $Z_V$ are the renormalization constants for the scalar wave function, the fermion wave function, and the 1PI vertex, respectively. The scalar wave function renormalization constant is determined via

$$Z_S = 1 - \text{div}\{Z_S \Pi_0(p^2, K_0, \epsilon)\}, \quad (2.6)$$

where $\Pi_0(p^2, K_0, \epsilon)$ is the scalar self-energy divided by $p^2$, where $p$ is the external momentum. Here and in the following, $\text{div} X$ denotes the poles of $X$ in $\epsilon$. The self-energy can be
written as
\[
\Pi_0(p^2, K_0, \epsilon) = K_0 \Pi^{(1)}(p^2, \epsilon) + \frac{1}{N_f} \sum_{n=2}^{\infty} K_0^n \Pi^{(n)}(p^2, \epsilon),
\tag{2.7}
\]
where \(\Pi^{(1)}\) gives the one-loop result, and \(\Pi^{(n)}\) the \(n\)-loop part containing \(n - 2\) fermion bubbles in the chain, and summing over the topologies given in Fig. 1a. Other contributions are of higher order in \(1/N_f\) and are thus omitted.

For the fermion self-energy and vertex renormalization constants, the lowest non-trivial contributions are already \(O(1/N_f)\), and we, therefore, have
\[
Z_F = 1 - \text{div} \left\{ \Sigma_0(p^2, Z_K K, \epsilon) \right\},
\tag{2.8}
\]
\[
\Sigma_0(p^2, K_0, \epsilon) = \frac{1}{N_f} \sum_{n=1}^{\infty} K_0^n \Sigma^{(n)}(p^2, \epsilon),
\tag{2.9}
\]
where \(\Sigma^{(n)}\) is depicted in Fig. 1b with \(n - 1\) fermion bubbles. Similarly,
\[
V_0(p^2, K_0, \epsilon) = \frac{1}{N_f} \sum_{n=1}^{\infty} K_0^n V^{(n)}(p^2, \epsilon),
\tag{2.11}
\]
where \(V^{(n)}\) again contains \(n - 1\) fermion bubbles and is shown diagrammatically in Fig 1c.

Finally, we briefly comment on the scalar three-point and four-point functions, assuming that they are generated via fermion loops: the former exactly vanishes for massless fermions, while the latter is found to be already \(O(1/N_f)\) at the lowest order. Therefore, they can be neglected for the purpose of our analysis.

### 3 Renormalization constants

In this section our goal is to extract the contributions to the renormalization constants that are \(O(1/N_F)\) and relevant for the computation of the \(\beta\)-function.

Our starting point for \(Z_S\) is Eq. (2.6). Using the expansion of the scalar self-energy, Eq. (2.7), we obtain
\[
Z_S = 1 - \text{div} \left\{ Z_S Z_K K \Pi^{(1)}(p^2, \epsilon) + \frac{1}{N_f} \sum_{n=2}^{\infty} Z_S (Z_K K)^n \Pi^{(n)}(p^2, \epsilon) \right\}.
\tag{3.1}
\]
Recalling that \(Z_K \equiv Z_S^{-1} Z_F^{-2} Z_V^2\) and substituting Eqs (2.8) and (2.10), the first term between brackets can be written as
\[
\text{div} \left\{ Z_S Z_K \Pi^{(1)}(p^2, \epsilon) K \right\}
= K \text{div} \left\{ \Pi^{(1)} \right\} + \frac{1}{N_f} \text{div} \left\{ 2K \text{div} \left\{ \Sigma_0(p^2, Z_K K, \epsilon) - V_0(p^2, Z_K K, \epsilon) \right\} \Pi^{(1)}(p^2, \epsilon) \right\}.
\tag{3.2}
\]
The \( \Pi^{(1)} \) part corresponds to the one-loop diagram and is given by

\[
\Pi^{(1)}(p^2, \epsilon) \equiv \text{div} \left\{ \Pi^{(1)} \right\} + \Pi^{(1)}_F(p^2, \epsilon) = \frac{1}{(4\pi)^{d/2-2}} \frac{G(1,1)}{2} (-p^2)^{d/2-2},
\]

(3.3)

where \( d = 4 - \epsilon \), the loop function, \( G(1,1) \), is defined in Eq. (A.2) in Appendix A.1, and we have introduced the notation \( \Pi^{(1)}_F \) to indicate the finite part of \( \Pi^{(1)} \). Then, \( \text{div} \{ Z_{S} Z_{K} \Pi^{(1)}(p^2, \epsilon) K \} \)

\[
\frac{K}{\epsilon} + \frac{1}{N_f} \text{div} \left\{ 2K \text{div} \left\{ \Sigma_0(p^2, Z_{K} K, \epsilon) - V_0(p^2, Z_{K} K, \epsilon) \right\} \right. \\
\times \left( \text{div} \left\{ \Pi^{(1)} \right\} + \Pi^{(1)}_F(p^2, \epsilon) \right) \right\}
\]

(3.4)

\[
\frac{K}{\epsilon} + \frac{1}{N_f} \text{div} \left\{ 2K \Pi^{(1)}_F(p^2, \epsilon) \left[ \Sigma_0(p^2, Z_{K} K, \epsilon) - V_0(p^2, Z_{K} K, \epsilon) \right] \right\}
\]

+ \frac{1}{N_f} \times \text{higher poles},

where the higher poles, i.e., higher than \( 1/\epsilon \), arise from the product of two divergent parts and will be omitted because they play no role in what follows. Then, at the lowest order in \( 1/N_f \),

\[
Z_{S} = 1 - \frac{K}{\epsilon} + \mathcal{O}(1/N_f).
\]

(3.5)

Therefore, every time \( Z_{K} K \) appears in the argument of \( \Sigma_0 \) and \( V_0 \), it can be replaced by \( K \left( 1 - \frac{K}{\epsilon} \right)^{-1} \); the additional contributions are higher order in \( 1/N_f \). For Eq. (3.4), we arrive at

\[
\text{div} \left\{ Z_{S} Z_{K} \Pi^{(1)}(p^2, \epsilon) K \right\}
\]

\[
\frac{K}{\epsilon} + \sum_{n=1}^{\infty} K^{n+1} \text{div} \left\{ 2K \Pi^{(1)}_F(p^2, \epsilon) \left( 1 - \frac{K}{\epsilon} \right)^{-n} \left[ \Sigma^{(n)}(p^2, \epsilon) - V^{(n)}(p^2, \epsilon) \right] \right\}.
\]

(3.6)

Similarly, the second term of Eq. (3.1) reads

\[
\frac{1}{N_f} \text{div} \left\{ \sum_{n=2}^{\infty} Z_{S}(Z_{S}^{-1} K)^{n} \Pi^{(n)}(p^2, \epsilon) \right\} = \frac{1}{N_f} \sum_{n=2}^{\infty} K^{n} \text{div} \left\{ \left( 1 - \frac{K}{\epsilon} \right)^{1-n} \Pi^{(n)}(p^2, \epsilon) \right\}.
\]

(3.7)

Altogether, we can write \( Z_{S} \) as

\[
Z_{S} = 1 - \frac{K}{\epsilon} - \frac{1}{N_f} \sum_{n=2}^{\infty} K^{n} \left\{ \left( 1 - \frac{K}{\epsilon} \right)^{1-n} \left[ 2\Pi^{(1)}_F \left[ \Sigma^{(n-1)} - V^{(n-1)} \right] + \Pi^{(n)} \right] \right\},
\]

(3.8)

where the explicit functional dependence on \( (p^2, \epsilon) \) has been omitted to lighten the notation. Using the binomial expansion,

\[
\left( 1 - \frac{K}{\epsilon} \right)^{1-n} = \sum_{i=0}^{\infty} \binom{n + i - 2}{i} \frac{K^i}{\epsilon^i}
\]

(3.9)
and performing a shift in the summation, \( n \to n - i \), we find our final expression for \( Z_S \):

\[
Z_S = 1 - \frac{K}{\epsilon} - \frac{1}{N_f} \sum_{n=2}^{\infty} K^n \text{div} \left\{ \sum_{i=0}^{n-2} \left( \frac{n-2}{i} \right) \frac{1}{\epsilon^i} \left( 2\Pi_F^{(1)} \left( \Sigma^{(n-i-1)} - V^{(n-i-1)} \right) + \Pi^{(n-i)} \right) \right\}.
\]

(3.10)

We notice that Eq. (3.10) differs essentially from its counterpart in the QED [7] because of the contribution from the fermion self-energy and the vertex, which exactly cancel in QED because of the Ward identity.

The expression for \( Z_F \) can be derived from Eq. (2.8) in a similar manner:

\[
Z_F = 1 - \frac{1}{N_f} \sum_{n=1}^{\infty} \text{div} \left\{ (Z_K K)^n \Sigma^{(n)}(p^2, \epsilon) \right\}
\]

\[
= 1 - \frac{1}{N_f} \sum_{n=1}^{\infty} K^n \text{div} \left\{ \left( 1 - \frac{K}{\epsilon} \right)^{-n} \Sigma^{(n)}(p^2, \epsilon) \right\}
\]

\[
= 1 - \frac{1}{N_f} \sum_{n=1}^{\infty} K^n \text{div} \left\{ \sum_{i=0}^{n-1} \left( \frac{n-1}{i} \right) \frac{1}{\epsilon^i} \Sigma^{(n-i)}(p^2, \epsilon) \right\},
\]

(3.11)

where we have again performed the same shift \( n \to n - i \) in the last line. The derivation of \( Z_V \) is completely analogous, and we can readily write the expression for \( Z_V \):

\[
Z_V = 1 - \frac{1}{N_f} \sum_{n=1}^{\infty} K^n \text{div} \left\{ \sum_{i=0}^{n-1} \left( \frac{n-1}{i} \right) \frac{1}{\epsilon^i} V^{(n-i)}(p^2, \epsilon) \right\}.
\]

(3.12)

4 Resummation

In this section we provide closed formulas for Eqs (3.10), (3.11), and (3.12).

4.1 The vertex

By explicit computation, the \( n \)-loop contribution to \( V_0 \) is

\[
V^{(n)}(p^2, \epsilon) = \frac{(-1)^n}{4} \left( \frac{1}{(4\pi)^{d/2-2}} \right)^n \left( \frac{G(1, 1)}{2} \right)^{n-1} (-p^2)^{n(d/2-2)}
\times G(1, 1 - (n - 1)(d/2 - 2)),
\]

(4.1)

where \( G(n_1, n_2) \) is defined in Eq. (A.2). We notice that, as in Ref. [7], Eq. (4.1) allows for the following expansion:

\[
V^{(n)}(p^2, \epsilon) = (-1)^n \frac{1}{n!} \frac{v(p^2, \epsilon, n)}{\epsilon^n} v(p^2, \epsilon, n)
\]

(4.2)

where

\[
v(p^2, \epsilon, n) = \sum_{j=0}^{\infty} v_j(p^2, \epsilon)(n\epsilon)^j,
\]

(4.3)
and \( v_j(p^2, \epsilon) \) are regular in the limit \( \epsilon \to 0 \) for all \( j \). In particular, \( v_0(\epsilon) \) is independent of \( p^2 \) and is explicitly given by

\[
v_0(\epsilon) = \frac{2\Gamma(2-\epsilon)}{\Gamma \left( 1 - \frac{d}{2} \right) \Gamma \left( 2 - \frac{d}{2} \right) \Gamma \left( \frac{d}{2} \right)} \epsilon.
\] (4.4)

Substituting Eqs (4.1) and (4.2) in Eq. (3.12), we find:

\[
Z_V = 1 - \frac{1}{N_f} \sum_{n=1}^{\infty} (-K)^n \text{div} \left\{ \sum_{j=0}^{n-1} \frac{1}{\epsilon^{n-j}} \sum_{i=0}^{n-1} \binom{n-1}{i} (-1)^i (n-i)^{j-1} v_j(p^2, \epsilon) \right\}.
\] (4.5)

Then, by using the result of Ref. [7],

\[
\sum_{i=0}^{n-1} \binom{n-1}{i} (-1)^i (n-i)^{j-1} = -\delta_{j0} \frac{(-1)^n}{n}, \quad j = 0, \ldots, n - 1,
\] (4.6)

Eq. (4.5) gets simplified to

\[
Z_V = 1 + \frac{1}{2N_f} \sum_{n=1}^{\infty} K^n \frac{v_0(\epsilon)}{n}.
\] (4.7)

Expanding \( v_0(\epsilon) \) as

\[
v_0(\epsilon) = \sum_{j=0}^{\infty} v_0^{(j)} \epsilon^j
\] (4.8)

and keeping only the \( 1/\epsilon \) pole of Eq. (4.7), we find the closed formula for \( Z_V \):

\[
Z_V = 1 + \frac{1}{2\epsilon N_f} \sum_{n=1}^{\infty} K^n \frac{v_0^{(n-1)}}{n} = 1 + \frac{1}{2\epsilon N_f} \int_0^K v_0(t) dt.
\] (4.9)

4.2 The fermion self-energy

The \( n \)-loop contribution to \( \Sigma_0 \) is found to be

\[
\Sigma^{(n)}(p^2, \epsilon) = -\frac{(-1)^n}{8} \left( \frac{1}{(4\pi)^{d/2-2}} \right)^n \left( \frac{G(1, 1)}{2} \right)^{n-1} (-p^2)^{n(d/2-2)} \times \left[ G(1, 1 - (n-1)(d/2-2)) - G(1, -(n-1)(d/2-2)) \right].
\] (4.10)

Similarly to Eq. (4.1), Eq. (4.10) can be expanded as

\[
\Sigma^{(n)}(p^2, \epsilon) = -(-1)^n \frac{1}{n \epsilon^n} \sigma(p^2, \epsilon, n),
\] (4.11)

where

\[
\sigma(n, \epsilon, p^2) = \sum_{j=0}^{\infty} \sigma_j(p^2, \epsilon)(n\epsilon)^j,
\] (4.12)

and \( \sigma_j(p^2, \epsilon) \) are regular for \( \epsilon \to 0 \). Again, \( \sigma_0(\epsilon) \) is independent of \( p^2 \), and it is given by

\[
\sigma_0(\epsilon) = -\frac{2^{5-\epsilon} \Gamma \left( \frac{3}{2} - \frac{\epsilon}{2} \right)}{\sqrt{\pi}(4-\epsilon) \Gamma \left( -\frac{\epsilon}{2} \right)} \frac{\sin \left( \frac{\pi \epsilon}{2} \right)}{\pi \epsilon}.
\] (4.13)
Using the same procedure as in the previous section, we find that only $\sigma_0(\epsilon)$ contributes to $Z_F$. Keeping only the $1/\epsilon$ pole, the closed formula for $Z_F$ is

$$Z_F = 1 - \frac{1}{4\epsilon N_f} \int_0^K \sigma_0(t) dt. \quad (4.14)$$

### 4.3 The scalar self-energy

The evaluation of the bubble diagrams in Fig. 1a is quite cumbersome and is discussed in Appendix A.2. Here, we notice that the expression for $\Pi^{(n)}(p^2, \epsilon)$, $n \geq 2$, allows for the following expansion:

$$\Pi^{(n)} = -\frac{3}{2} \left( -\frac{1}{n-1} \right)^n \pi(p^2, \epsilon, n), \quad (4.15)$$

where

$$\pi(p^2, \epsilon, n) = \sum_{j=0}^{\infty} \pi_j(p^2, \epsilon)(ne)^j, \quad (4.16)$$

and $\pi_j(p^2, \epsilon)$ are regular for $\epsilon \to 0$. Similarly to the previous cases, $\pi_0(\epsilon)$ is independent of $p^2$.

In view of Eq. (3.10), we define

$$2\Pi^{(1)}(p^2, \epsilon) \left( \Sigma^{(n-1)}(p^2, \epsilon) - V^{(n-1)}(p^2, \epsilon) \right) + \Pi^{(n)}(p^2, \epsilon) \equiv \left( -\frac{1}{n-1} \right)^n \xi(p^2, \epsilon, n), \quad (4.17)$$

where

$$\xi(p^2, \epsilon, n) \equiv n\epsilon \Pi^{(1)} \left( \frac{\sigma(p^2, \epsilon, n-1)}{2} + v(p^2, \epsilon, n-1) \right) - \frac{3}{2} \pi(p^2, \epsilon, n), \quad (4.18)$$

and

$$\xi(p^2, \epsilon, n) = \sum_{j=0}^{\infty} \xi_j(p^2, \epsilon)(ne)^j, \quad (4.19)$$

with $\xi_j(\epsilon, p^2)$ regular for $\epsilon \to 0$ for all $j$. In particular, $\xi_0(\epsilon)$ is independent of $p^2$ and is explicitly given by

$$\xi_0(\epsilon) = \Gamma(2-\epsilon) \left( \frac{2\epsilon}{\Gamma(2-\frac{\epsilon}{2}) - \frac{6 - 4\epsilon + \epsilon^2}{\Gamma(1 - \frac{\epsilon}{2}) \Gamma(3 - \frac{\epsilon}{2})}} \right) \sin \left( \frac{\pi \epsilon}{2} \right). \quad (4.20)$$

Then, using the above definitions, Eq. (3.10) can be written as

$$Z_S = 1 - \frac{K}{\epsilon} - \frac{1}{N_f} \sum_{n=2}^{\infty} K^n \text{div} \left\{ \sum_{i=0}^{n-2} \left( \frac{n-2}{i} \right) \frac{(-1)^{n-i}}{\epsilon^{i}(n-i)(n-i-1)} \xi(p^2, \epsilon, n-i) \right\}$$

$$= 1 - \frac{K}{\epsilon} - \frac{1}{N_f} \sum_{n=2}^{\infty} (-K)^n \text{div} \left\{ \sum_{j=0}^{n-1} \frac{1}{\epsilon^{n-j}} \xi_j(p^2, \epsilon) \sum_{i=0}^{n-2} \left( \frac{n-2}{i} \right) (-1)^{i} \frac{(n-i)^{j-1}}{n-i-1} \right\}. \quad (4.21)$$
Moreover, we find that
\[ \sum_{i=0}^{n-2} \binom{n-2}{i} (-1)^i \frac{(n-i)^{j-1}}{(n-i-1)} = \begin{cases} \frac{(-1)^n}{n-1} & j = 0 \\ \frac{(-1)^n}{n} & j = 1, \ldots, n-1 \end{cases} \tag{4.22} \]
and therefore the expression for \( Z_S \) can be significantly simplified:
\[
Z_S = 1 - \frac{K}{\epsilon} - \frac{1}{N_f} \sum_{n=2}^\infty K^n \text{div} \left\{ \frac{1}{\epsilon^n} \left( \frac{\xi_0(\epsilon)}{n} + \frac{1}{n-1} \sum_{j=1}^{n-1} \xi_j(p^2, \epsilon) \epsilon^j \right) \right\} = 1 - \frac{K}{\epsilon} - \frac{1}{N_f} \sum_{n=2}^\infty K^n \text{div} \left\{ \frac{1}{\epsilon^n} \left( \frac{\xi_0(\epsilon)}{n} + \frac{1}{n-1} \sum_{j=1}^\infty \xi_j(p^2, \epsilon) \epsilon^j \right) \right\}, \tag{4.23} \]
where in the second line we extended the sum over \( j \) up to \( \infty \) without affecting the result, since all the terms for \( j > n - 1 \) are finite. The function \( \xi(p^2, \epsilon, 1) \), corresponding to \( \xi(p^2, \epsilon, 0) \equiv \infty \sum_{j=0}^\infty \xi_j(p^2, \epsilon) \epsilon^j \), can be evaluated by taking in Eq. (4.18) the limit \( n \to 1 \), although the latter is formally defined for \( n \geq 2 \). We find the following expression:
\[
\xi(p^2, \epsilon, 1) = -\frac{\Gamma(4-\epsilon)}{\Gamma(2-\frac{\epsilon}{2}) \Gamma(3-\frac{\epsilon}{2})} \frac{\sin \left( \frac{\pi \epsilon}{2} \right)}{\pi \epsilon} \equiv \xi(\epsilon, 1). \tag{4.25} \]

Few comments are in order: Eq. (4.25) ensures that \( Z_S \) is independent of the external momentum \( p^2 \), as it should. This result comes from an exact cancellation among the different contributions of the scalar self-energy, the fermion self-energy, and the vertex in Eq. (4.18). In particular, we find that
\[
\pi(p^2, \epsilon, 1) = \frac{2}{3} \left( \frac{\sigma(p^2, \epsilon, 0)}{2} + v(p^2, \epsilon, 0) \right) \left[ 1 + \epsilon \Pi_F^{(1)}(p^2, \epsilon) \right] = \frac{2}{3} \left( \frac{\sigma_0(\epsilon)}{2} + v_0(\epsilon) \right) \left[ 1 + \epsilon \Pi_F^{(1)}(p^2, \epsilon) \right], \tag{4.26} \]
and therefore
\[
\xi(\epsilon, 1) = -\frac{\sigma_0(\epsilon)}{2} - v_0(\epsilon), \tag{4.27} \]
which is equivalent to Eq. (4.25). Interestingly, Eq. (4.26) only holds for \( n = 1 \). All in all, the \( p^2 \) independence of Eq. (4.25) provides a non-trivial check for our computation.

We are now ready to resum the series in Eq. (4.23). By expanding \( \xi_0(\epsilon) \) as
\[
\xi_0(\epsilon) = \sum_{j=0}^\infty \xi_0^{(j)} \epsilon^j, \tag{4.28} \]
the $\frac{1}{n}$ term in Eq. (4.23) is given by
\[
\sum_{n=2}^{\infty} \frac{K^n}{\epsilon^n} \frac{\xi_0(\epsilon)}{n} = \frac{1}{\epsilon} \sum_{n=2}^{\infty} \frac{K^n \zeta_0^{(n-1)}}{n} + \text{higher poles}
\]
\[
= \frac{1}{\epsilon} \left( \sum_{n=0}^{\infty} \frac{K^{n+1}}{n+1} \frac{\zeta_0^{(n)}}{n} - K \zeta_0^{(0)} \right) + \text{higher poles}
\]
\[
= \frac{1}{\epsilon} \int_0^K [\xi_0(t) - \xi_0(0)]\,dt + \text{higher poles.}
\]

As for the $\frac{1}{n-1}$ term, since $\xi(0,1) = \xi_0(0)$, we can write
\[
\xi(\epsilon, 1) - \xi_0(\epsilon) = \sum_{j=1}^{\infty} \tilde{\xi}^{(j)} e^j,
\]
and therefore
\[
\sum_{n=2}^{\infty} \frac{K^n}{\epsilon^n} \frac{\xi(\epsilon, 1) - \xi_0(\epsilon)}{n-1} = \sum_{n=1}^{\infty} \frac{K^{n+1}}{\epsilon^{n+1}} \frac{\xi(\epsilon, 1) - \xi_0(\epsilon)}{n}
\]
\[
= \frac{1}{\epsilon} \sum_{n=1}^{\infty} \frac{K^{n+1}}{n} \tilde{\xi}^{(n)} + \text{higher poles}
\]
\[
= \frac{K}{\epsilon} \int_0^1 \frac{dx}{x} \sum_{n=1}^{\infty} \tilde{\xi}^{(n)}(xK)^n + \text{higher poles}
\]
\[
= \frac{K}{\epsilon} \int_0^K \frac{\xi(t, 1) - \xi_0(t)}{t}\,dt + \text{higher poles},
\]
where we used $\frac{1}{n} = \int_0^1 x^{n-1}\,dx$. Finally, the closed formula for $Z_S$ reads
\[
Z_S = 1 - \frac{K}{\epsilon} - \frac{1}{\epsilon N_f} \int_0^K \left( \xi_0(t) - \xi_0(0) + \frac{\xi(t, 1) - \xi_0(t)}{t} K + \frac{\sigma_0(t)}{2} + v_0(t) \right)\,dt.
\]

5 The $\beta$-function

Using the results of the previous section together with Eq. (2.5), we can finally proceed to evaluating the $\beta$-function. First, we find that
\[
G_1(K) = K + \frac{1}{N_f} \int_0^K \left( \xi_0(t) - \xi_0(0) + \frac{\xi(t, 1) - \xi_0(t)}{t} K + \frac{\sigma_0(t)}{2} + v_0(t) \right)\,dt.
\]

Now, it is straightforward to compute the $\beta$-function:
\[
\beta(K) = K^2 + \frac{K^2}{N_f} \left\{ \xi_0(K) - \xi_0(0) + \xi(K, 1) - \xi_0(K) + \frac{\sigma_0(K)}{2} + v_0(K) \right\}
\]
\[
+ \int_0^K \frac{\xi(t, 1) - \xi_0(t)}{t}\,dt.
\]
Recalling Eq. (4.27) and using $\xi_0(0) = -\frac{3}{2}$, Eq. (5.2) can be further simplified to

$$\frac{\beta(K)}{K^2} = 1 + \frac{1}{N_f} \left\{ \frac{3}{2} + \int_0^K \frac{\xi(t, 1) - \xi_0(t)}{t} \, dt \right\}. \quad (5.3)$$

Finally, by comparison with Eq. (2.3), we see that $F_0 = 1$ and

$$F_1(K) = \frac{3}{2} + \int_0^K \frac{\xi(t, 1) - \xi_0(t)}{t} \, dt = \frac{3}{2} + \int_0^K I_1(t) \, dt, \quad (5.4)$$

where

$$I_1(t) = \frac{(3 - 2t + \frac{t^2}{2}) \Gamma(3 - t) - \Gamma(4 - t) - 2\Gamma(2 - t) \Gamma \left( \frac{3 - t}{2} \right) t \times \frac{\sin \left( \frac{\pi t}{2} \right)}{\pi t}}{\Gamma \left( \frac{2 - \frac{t}{2} \Gamma \left( \frac{3 - \frac{t}{2} \right)}{2} \right) t}. \quad (5.5)$$

We plot the integrand, $I_1(t)$, in Fig. 2. We have checked that our $\beta$-function agrees at the leading order in $N_f$ up to four-loop level by comparing with the result of Ref. [19].

Finally, let us comment on the pole structure: the integrand, $I_1(t)$, has the first pole occurring at $t = 3$, which results in a logarithmic singularity for $F_1(K)$ around $K = 3$. Due to the sign of $I_1(t)$, we see that $F_1(K)$ approaches large negative values for $K \to 3^-$. This suggests the existence of a UV fixed point at $K_{UV} \lesssim 3$ such that $F_1(K_{UV}) = -N_f$. Moreover, it can be shown that there is an infrared fixed point $K_{IR}$, symmetric to $K_{UV}$ with respect to $K = 3$; see e.g. Ref. [9]. The same qualitative analysis applies to the singularity at $K = 5$.

6 Conclusions

We have computed the leading $1/N_f$ contribution for the $\beta$-function in Yukawa theory with $N_f$ fermion flavours coupling to a real scalar. We obtained a closed form expression for the $\beta$-function up to order $O(1/N_f)$. This expression has a finite radius of convergence, and the first singularity occurs at $K = 3$.

The present result adds an interesting ingredient to models with a large number of fermions, and makes a contribution to better understand the UV behaviour of gauge-Yukawa theories. Curiously, the location of the first singularity of the $\beta$-function is the
same as for the non-abelian gauge theory at the order $1/N_f$ [8, 9], which might suggest an interplay of the gauge and Yukawa contributions. This study is, however, left for future work.

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A Loop integrals

We here provide some explicit formulas. We follow closely the notations of Ref. [20].

A.1 The vertex and the fermion self-energy

As shown in Eqs (4.1) and (4.10), the 1PI vertex and the fermion self-energy involve only the function $G(n_1, n_2)$, independently of the number of bubbles. This corresponds to the one-loop integral

$$
\int \frac{d^d k}{(2\pi)^d} \frac{1}{D_1^{n_1} D_2^{n_2}} = \frac{1}{(4\pi)^{d/2}} \left(-p^2\right)^{d/2-n_1-n_2} (-1)^{n_1+n_2} G(n_1, n_2),
$$

(A.1)

where $D_1 = (k + p)^2$ and $D_2 = k^2$. Explicitly,

$$
G(n_1, n_2) = \frac{\Gamma(-d/2 + n_1 + n_2) \Gamma(d/2 - n_1) \Gamma(d/2 - n_2)}{\Gamma(n_1) \Gamma(n_2) \Gamma(d - n_1 - n_2)}.
$$

(A.2)

A.2 The scalar self-energy

Unlike the 1PI vertex and the fermion self-energy, the $n$-loop contribution to the scalar self-energy, $\Pi_0$, indicated by $\Pi^{(n)}$, cannot be written in terms of $G(n_1, n_2)$ functions only. In fact, $\Pi^{(n)}$ is given by ($n \geq 2$):

$$
p^2 \Pi^{(n)}(p^2, \epsilon) = - (4\pi)^2 (-1)^n \left(\frac{1}{(4\pi)^{d/2-2}} \frac{G(1, 1)}{2}\right)^n (-1)^\alpha \int \frac{d^d k_1}{(2\pi)^d} \int \frac{d^d k_2}{(2\pi)^d} \left\{ \begin{array}{c} - \frac{2}{2p^2} + \frac{2}{k_1^2(p + k_1)^2 k_2^2((1 - k_2)^2 - \alpha)} - \frac{2}{k_1^2(p + k_1)^2 k_2^2((1 - k_2)^2 - \alpha) - \alpha} \\
\end{array} \right\},
$$

(A.3)

where $\alpha = (n - 2)(d/2 - 2) = -(n - 2)\epsilon/2$. Eq. (A.3) requires two-loop integrals which can be performed according to the formula in Ref. [20]:

$$
\int \frac{d^d k_1}{(2\pi)^d} \int \frac{d^d k_2}{(2\pi)^d} \frac{1}{D_1^{n_1} D_2^{n_2} D_3^{n_3} D_4^{n_4} D_5^{n_5}} = (-1)^{1+\sum n_i} \frac{\pi^d(-p^2)^{d-\sum n_i}}{(2\pi)^{2d}} G(n_1, n_2, n_3, n_4, n_5),
$$

(A.4)
where $D_1 = (k_1 + p)^2$, $D_2 = (k_2 + p)^2$, $D_3 = k_1^2$, $D_4 = k_2^2$, $D_5 = (k_1 - k_2)^2$. The functions $G(n_1, n_2, n_3, n_4, n_5)$ are symmetric with respect to the following index exchanges: $(1 \leftrightarrow 2, 3 \leftrightarrow 4)$ and $(1 \leftrightarrow 3, 2 \leftrightarrow 4)$. Moreover, they reduce to a product of $G(n_1, n_2)$ if at least one of the entries is zero:

$$G(n_1, n_2, n_3, n_4, 0) = G(n_1, n_3)G(n_2, n_4),$$ (A.5)

$$G(0, n_2, n_3, n_4, n_5) = G(n_3, n_5)G(n_2, n_3 + n_4 + n_5 - d/2).$$ (A.6)

It turns out that the first four integrals in Eq. (A.3) can always be written in terms of $G(n_1, n_2)$ making use of Eqs (A.5) and (A.6).

However, the last integral in Eq. (A.3) involves $G(1, 1, 1, 1, (n - 2)\epsilon/2)$ and, for $n > 2$, its expression can be obtained in terms of hypergeometric functions $\text{$_3F_2$}$ by means of the Gegenbauer technique [21]:

$$G(1, 1, 1, 1, (n - 2)\epsilon/2) = \frac{2\Gamma(1 - \epsilon/2)\Gamma(-1 + n\epsilon/2)\Gamma(1 - (n - 1)\epsilon/2)}{(-2 + (n - 1)\epsilon)\Gamma(2 - \epsilon)\Gamma((n - 2)\epsilon/2)\Gamma(2 - (n + 1)\epsilon/2)}$$

$$\times \left\{ (2 - (n - 1)\epsilon)(2\Gamma(1 - \epsilon/2)^2\Gamma(-\epsilon)$$

$$-(2 - (n - 2)\epsilon)\Gamma(1 - \epsilon)\Gamma(-\epsilon/2) \text{$_3F_2$}(1, -\epsilon/2, 1 - (n - 1)\epsilon/2, 2 - (n + 1)\epsilon/2, 2 - (n - 1)\epsilon/2, 1) \right\}. \tag{A.7}$$

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