Semilattices of Rectangular Bands and Groups of Order Two.

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Abstract. We prove that a semigroup $S$ is a semilattice of rectangular bands and groups of order two if and only if it satisfies the identity $x = x^3$ and $xyx \in \{xy^2x, y^2x^2y\}$ ($x, y \in S$).

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1. Introduction

Semigroup theory has a certain symmetric elegance that links it to group theory. For example, a semigroup $S$ is a group if (and only if) for every $x \in S$, $Sx = S = xS$. In addition, a semigroup $S$ is a union of groups if (and only if) for every $x \in S$, $x \in Sx^2 \cap xS^2$ [3]. The powerful result that a semigroup is a union of groups if and only if it is a semilattice of completely simple semigroups is so well known that its beauty can almost be overlooked [3]. In this paper we apply this result to prove that the collection of all semigroups that are semilattices of semigroups that are either rectangular bands or groups of order two is a semigroup inclusion class [4]. Precisely, a semigroup is a semilattice of rectangular bands and groups of order two if and only if it satisfies the identity $x = x^3$ and $xyx \in \{xy^2x, y^2x^2y\}$ ($x, y \in S$).

2. Notation, definitions and preliminary results

Definition. Let $G$ be a group and $\Lambda \times I$ a non-empty set. Let $P: \Lambda \times I \to G$, with $P(\mu, j)$ denoted by $p_{\mu j}$. Let $S = I \times G \times \Lambda$ and define a product on $S$ as follows: $(i, a, \mu)(j, b, \lambda) = (i, ap_{\mu j}, b, \lambda)$. Then $S$ endowed with this product is called a Rees $I \times \Lambda$ matrix semigroup (over the group $G$ with the sandwich matrix $P$).

The well known definition above is repeated here because it will be used extensively throughout this paper. All other terminology and notation can be found in [3]. Other well known results that will be used here follow.

Result 1. [5, Corollary IV.2.8] A semigroup is completely simple if and only if it is isomorphic to a Rees matrix semigroup.

Result 2. [3, Theorem 4.3] A semigroup $S$ is a union of groups if and only if for every $x \in S$, $x \in x^2 S \cap Sx^2$.

Result 3. [3, Theorem 4.6] A semigroup is a union of groups if and only if it is a semilattice $Y$ of completely simple semigroups $S_\alpha = I_\alpha \times G_\alpha \times \Lambda_\alpha$ ($\alpha \in Y$).

Result 4. [2, Proposition 1] A semilattice $Y$ of completely simple semigroups is a semilattice of rectangular groups if and only if the product of two idempotents is an idempotent.

Definition. We will denote the collection of all left zero, right zero and rectangular bands by $L_0$, $R_0$ and $RB$ respectively and that of all groups of order two by $G_2$. If $S$ is a union of groups and $x \in S$ then $I_x$ will denote the identity element of any group to which $x$ belongs. If $G$ is a group then the identity element of $G$ is
denoted by \(1\) or \(1_G\). Also if \(x\) is an element of a semigroup \(S\) then \(x^{-1}\) or \((x^{-1})_G\) denotes the inverse of \(x\) in any subgroup \(G\) of \(S\). A rectangular group is the direct product of a rectangular band and a group.

Note that if an element \(x\) of a semigroup belongs to two groups, \(G\) and \(H\), then 
\[
1 = (x^{-1})_G = x(x^{-1})_G = x = x(x^{-1})_H = 1, \quad \text{and therefore } 1_x \text{ is well-defined.}
\]
Similarly, 
\[
1 = (x^{-1})_G = x(x^{-1})_G = x = x(x^{-1})_H = 1, \quad \text{and so } x^{-1} \text{ is well-defined.}
\]

**Definition.** [4] An inclusion class of semigroups is a collection of all semigroups that satisfy a given set of \(k\) number of inclusions as follows:

1. \(W_1 = \{w_{1,1}, w_{1,2}, \ldots, w_{1,n_1}\} \subseteq \{t_{1,1}, t_{1,2}, \ldots, t_{1,m_1}\} = T_1\);
2. \(W_2 = \{w_{2,1}, w_{2,2}, \ldots, w_{2,n_2}\} \subseteq \{t_{2,1}, t_{2,2}, \ldots, t_{2,m_2}\} = T_2\);
3. \(W_k = \{w_{k,1}, w_{k,2}, \ldots, w_{k,n_k}\} \subseteq \{t_{k,1}, t_{k,2}, \ldots, t_{k,m_k}\} = T_k\), where the \(w\)’s and the \(t\)’s are semigroup words over some alphabet.

**Notation.** We write \(\{W_1 \subseteq T_1; W_2 \subseteq T_2; \ldots; W_k \subseteq T_k\}\) to denote the inclusion class determined by the set \(\{W_i \subseteq T_i\}\) of inclusions. If \(S\) is a semilattice and \(\{\alpha, \beta\} \subseteq S\) then we write \(\alpha \leq \beta\) if \(\alpha = \alpha \beta\) and \(\alpha < \beta\) if \(\alpha = \alpha \beta\) and \(\alpha \neq \beta\).

**Definition.** A semigroup \(S\) is a semilattice if \(S \in \{x = x^2; \ xy = yx\}\). A semilattice \(S\) is a chain if \(\{\alpha, \beta\} \subseteq S\) and \(\alpha \neq \beta\) implies \(\alpha \beta \in \{\alpha, \beta\}\).

**Definition.** A semigroup \(S\) is a semilattice \(Y\) of semigroups \(S_\alpha (\alpha \in Y)\) if \(S\) is a disjoint union of the \(S_\alpha (\alpha \in Y)\) and for every \(\{\alpha, \beta\} \subseteq Y, S_\alpha S_\beta \subseteq S_{\alpha \beta}\).

### 3. Some inclusion classes of semilattices of rectangular bands and groups of order two.

**Theorem 1.** The following statements are equivalent:

1. \(S \in \{xyx \in \{x, y\}\}\) and

2. \(S\) is a chain \(Y\) of semigroups \(S_\alpha (\alpha \in Y)\) where

   (1) \(S_\alpha \in RB \cup G_2 (\alpha \in Y)\),

   (2) for any \(\alpha < \beta\) \((\alpha, \beta \in Y)\) and any \(x \in S_\alpha\), \(y \in S_\beta\), \(xy = yx = x\) and

   (3) for any \(\alpha < \beta\) \((\alpha, \beta \in Y)\) with \(S_\alpha \in G_2, S_\alpha = \{1\}\).
Proof: \((1 \Rightarrow 2)\) Let \(S \subseteq \{xyx \in \{x, y\}\}\). Then clearly, \(S \subseteq \{x = x^3\}\) and so by Result 2, \(S\) is a union of groups and, therefore, by Result 3, \(S\) is a semilattice \(Y\) of completely simple semigroups \(S_{\alpha} (\alpha \in Y)\).

Let \(x \in S_{\alpha}\) and \(y \in S_{\beta} (\alpha, \beta \in Y)\). Then either (a) \(xyx = x\) and \(yxy = y\) or (b) \(xyx = x\) and \(yxy = x\) or (c) \(xyx = y\) and \(yxy = y\) or (d) \(xyx = y\) and \(yxy = x\). In cases (a) and (d), \(\alpha = \alpha \beta = \beta\). So if \(\alpha \neq \beta\) then either case (b) or (c) holds. Therefore, \(\alpha \neq \beta\) implies that either \(\alpha \beta = \alpha\) or \(\alpha \beta = \beta\). Hence, \(Y\) is a chain.

We now show that \(S_{\alpha} \subseteq RB \cup G_2 (\alpha \in Y)\). Each \(S_{\alpha} = I_{\alpha} \times G_{\alpha} \times \Lambda_{\alpha} (\alpha \in Y)\). Assume that either \(|I_{\alpha}| \neq 1\) or \(|\Lambda_{\alpha}| \neq 1\). Then let \(x = (i, g, \lambda)\) and \(y = (j, h, \mu)\), where either \(i \neq j\) or \(\lambda \neq \mu\) and where \(g\) and \(h\) are arbitrary elements of \(G_{\alpha}\). Now (b), (c) and (d) all imply that \(i = j\) and \(\lambda = \mu\). Thus, (a) holds.

Now \(x = x^3\) and \(x^2 = 1_x\). So \((i, g, \lambda)^2 = (i, gp_{\alpha}g, \lambda) = (i, (p_{\alpha})^{-1}, \lambda)\). Hence, \(gp_{\alpha}g = (p_{\alpha})^{-1}\). Let \(g'\) be an arbitrary element of \(G_{\alpha}\). Since \(g\) was arbitrary we can let \(g = g'(p_{\alpha})^{-1}\). Then \((p_{\alpha})^{-1} = g'(p_{\alpha})^{-1} p_{\alpha} g = (p_{\alpha})^{-1}\)
and hence \((g')^2\) is the identity element of \(G_{\alpha}\) and therefore \(G_{\alpha} \subseteq G_2\). Hence, \(G_{\alpha}\) is abelian and \(p_{\alpha} g = (p_{\alpha})^{-1}\).

Now (a) holds and so \(xyx = x\). So \((i, g, \lambda) = (i, gp_{\alpha}h \mu g, \lambda')\) and therefore \(g = gp_{\alpha} h \mu g\) and \(1_s = p_{\alpha} h \mu g\). But \(G_{\alpha}\) is abelian and so \(g = gp_{\alpha} h \mu g = g^2 p_{\alpha} h \mu h = h\).

Therefore \(G_{\alpha} = \{1\}\). This implies \(S_{\alpha} \subseteq RB\).

Now if \(I_{\alpha} = \{i\}\) and \(\Lambda_{\alpha} = \{\lambda\}\) then the mapping \(g \mapsto (i, g(p_{\alpha})^{-1}, \lambda)\) is an isomorphism between \(G_{\alpha}\) and \(S_{\alpha}\).

As shown two paragraphs above, \(G_{\alpha} \subseteq G_2\) and so \(S_{\alpha} \subseteq G_2\). We have therefore shown that \(S_{\alpha} \subseteq RB \cup G_2 (\alpha \in Y)\) and so \((1)\) is valid.

Now assume that \(\alpha < \beta (\alpha, \beta \in Y)\) and let \(x \in S_{\alpha}\) and \(y \in S_{\beta}\). Then (b) holds and so \(xyx = x = yxy\).

Then \(1_s = x^{-1}x = x^{-1}xyx = 1_s 1_s 1_s\) and \(1_s = xx^{-1} = yxyx^{-1} = 1_s 1_s\).

If \(S_{\alpha} \subseteq G_2\) then \(S_{\alpha}\) is commutative and so

\[
\begin{align*}
xy = (xyx) y = [x (xy)] y = x^2 y^2 = 1_s 1_s 1_s = (yx)^2 = y (xyx) = yx.
\end{align*}
\]

However, if \(z \in S_{\alpha}\) then \((xy) z = 1_s z = z = x (yz) = xl_{z} = x\). Therefore \(S_{\alpha} = \{x\}\) and so \(1_s = x = xy = yx\). (We have shown that \((3)\) is valid.)

If \(S_{\alpha} \subseteq RB\) then \(1_s = x = x^2 = (xyx)(yxy) = (yx)^2 = x\). Also, from (b), \(yx = (yx)^3 = (yx)(yxy) = x^2 = x = xy\).

Hence, \(x = xy = yx\). We have shown that \((2)\) is valid and this completes the proof of \((1 \Rightarrow 2)\).

\((2 \Rightarrow 1)\) Assume that \(S\) is a chain \(Y\) of semigroups \(S_{\alpha} (\alpha \in Y)\) where \((1), (2)\) and \((3)\) in Theorem 1 are valid.

We wish to show that \(xyx \in \{x, y\}\) for any \(\{x, y\} \subseteq S\). Since \(S_{\alpha} \in RB \cup G_2 (\alpha \in Y)\), we can assume that \(\alpha \neq \beta\).
Case 1. Suppose that \( \alpha < \beta \) and \( S_\alpha \in G_2 \). Then by (3), \( S_\alpha = \{ x \} \) and so \( xyx \in S_\alpha = \{ x \} \).

Case 2. Suppose that \( \alpha < \beta \) and \( S_\alpha \in RB \). Then \( xyx = (xy)x = (xy)x^2 = x(\alpha x)x = x^2 = x \).

Case 3. If \( \beta < \alpha \) and \( S_\beta \in G_2 \) then, by (3), \( S_\beta = \{ y \} \). Therefore, \( xyx \in S_\beta = \{ y \} \).

Case 4. As in the proof of case 2, \( \beta < \alpha \) and \( S_\beta \in RB \) implies \( yxy = y \). Then
\[
\begin{align*}
xyx &= x(\alpha xy)x = x(\alpha xy)^2 = xyx = (xy)y = y^2 = y.
\end{align*}
\]

We have therefore shown that \( xyx \in \{ x, y \} \), completing the proof of \( (2 \Rightarrow 1) \). This completes the proof of Theorem 1.

Note that we have shown that any \( S \in \{ xyx \in \{ x, y \} \} \) is a chain \( Y \) of rectangular bands, except possibly if \( Y \) has a maximal element \( \alpha \) and \( S_\alpha \in G_2 \) with \( |S_\alpha| > 1 \). The question arises as to whether the collection of chains of semigroups that are either rectangular bands or groups of order 2 is an inclusion class. In Theorem 5 below we prove this question in the affirmative when the word “chain” is replaced by “semilattice”.

**Theorem 2.** The following statements are equivalent:

1. \( S \in \{ xyx \in \{ y, xy \} \} \) and

2. \( S \) is a semilattice \( Y \) of semigroups \( S_\alpha (\alpha \in Y) \) where
   
   (1) \( (S_\alpha)^2 \in R_0 \cup G_2 (\alpha \in Y) \),
   
   (2) \( \alpha < \beta \) and \( (S_\alpha)^2 \in G_2 \) implies \( (S_\alpha)^2 = \{ 1 \} \),
   
   (3) \( S_\alpha - (S_\alpha)^2 \neq \emptyset \) implies \( (S_\alpha)^2 = \{ 1 \} \),
   
   (4) for each \( \alpha \in Y \) and \( x \in S_\alpha \) there is a mapping \( \theta : S \to S \) such that for any \( \beta \in Y \),
   
   \[ \theta : S_\beta \to (S_\alpha)^2 \]
   
   is a homomorphism, satisfying
   
   (4.1) if \( (S_\alpha)^2 \in G_2 \) then for any \( g \in (S_\alpha)^2 \) and \( \{ x, y \} \subseteq S_\alpha \), \( \theta_g = \theta_x = \theta_y = \theta_x \) on \( S_\alpha \),
   
   (4.2) if \( \{ \alpha, \beta, \gamma \} \subseteq Y \) and \( (S_\alpha y)^2 \in R_0 \) then \( \theta_{(\alpha, \beta, \gamma)} = \theta_y \theta_x \) on \( S_\gamma \) and
   
   (4.3) for every \( x \in S_\alpha \) and \( y \in S_\beta \), \( xy = (\theta_x y)(\theta_y x) \).

**Proof:** \( (1 \Rightarrow 2) \) Assume that \( S \in \{ xyx \in \{ y, xy \} \} \). First we will prove that \( S \in \{ xy = (xy)^3 \} \). Let \( \{ x, y \} \subseteq S \). Then either (a) \( xyx = y \) and \( yxy = x \) or (b) \( xyx = y \) and \( yxy = xy \) or (c) \( xyx = xy \) and \( yxy = x \) or (d) \( xyx = xy \) and \( yxy = xy \). Note that since \( S \in \{ xyx \in \{ y, xy \} \} \), \( (xy)^3 \in \{ xy, (xy)^2 \} \). So in each case we can assume that \( (xy)^3 = (xy)^2 \).

Case (a): \( xy = (xy)(xyx) = (xy)^3 = [(xy)(xyx)]^3 = (xy)^9 = (xy)^6 = (xy)^4 = (xy)^3 \).
Case (b): \( xy = y(xyx)(yxy) = (xy)^3 \).

Case (c): \((xy)^2 = (xyx)y = yxy = x = (yxy) = x^2 \). But then \( x = (xy)^2 \) implies \( x = (xy)^2 x \) and so \( xy = \left[(xy)^2 x\right]y = (xy)^3 \).

Case (d): \( xy = yxy = (yx)y = (xyx)y = (xy)^2 = (xy)^3 \).

By Results 2 and 3, \( S^2 \) is a semilattice \( Y \) of completely simple semigroups \( \left(S^2\right)_\alpha (\alpha \in Y) \). We now show that the product of two idempotents of \( S \) is an idempotent. Let \( \{e, f\} \subseteq E_S \). Then \( efe \in \{f, fe\} \). If \( efe = f \) then \( ef = e(efe) = efe = f \in E_S \). If \( efe = fe \) then \( (efe)^2 = (efe) f = fef \in \{e, ef\} \). So we can assume that \( (ef)^2 = e \).

Hence \( ef = (ef)^2 f = (ef)^2 \) and this completes the proof that the product of two idempotents is idempotent.

Now by Result 4, \( S^2 \) is a semilattice \( Y \) of rectangular groups \( \left(S^2\right)_\alpha (\alpha \in Y) \). We want to show now that each \( \left(S^2\right)_\alpha \) is either a right-zero semigroup or a group of order 2.

Let \( \{x, y\} \subseteq \left(S^2\right)_\alpha \) with \( x = (i, g, \lambda) \) and \( y = (j, h, \delta) \). Then \( xxy \{y, yx\} \) and so \( i = j \). So \( \left(S^2\right)_\alpha = G \times R \).

We have shown that an arbitrary rectangular group component of \( S^2 \) is a right group \( G \times R \).

Note that, since we have already shown above that \( xy = (xy)^3 \), \( G \in G_2 \) and so \( G \) is commutative and satisfies \( g = g^{-1} \), \( g^2 = 1 \) and \( ghg = h(g, h \in G) \).

Now let \( \{(g, r)(g, s)\} \subseteq G \times R \), with \( r \neq s \). Then \( (g, r) = (g, r)(g, s)(g, r) \in \{(g, r), (g, s), (g^2, r)\} \). Therefore \( g = g^2 = 1 \). So either \( |R| = 1 \) or \( |G| = 1 \). Hence \( G \times R \) is either a right-zero semigroup or a group of order 2.

For \( \alpha \in Y \) we define \( S_\alpha = \left\{x \in S : x^2 \in \left(S^2\right)_\alpha \right\}\). Let \( \{x, y\} \subseteq S \) with \( x^2 \in \left(S^2\right)_\alpha \) and \( y^2 \in \left(S^2\right)_\beta \). We show that \( xy \in \left(S^2\right)_{\alpha \beta} \subseteq \left(S^2\right)_{\alpha \beta} \). First note that \( S \) is a null extension of a union of groups. From the proof of Theorem 5 [1], \( xy \in xy^2 S \cap S x^2 y \). It is then straightforward to show that

\[(1) \ xy = \left(xy_1 \gamma^1\right)\left(xy_1 \gamma^2\right)^{-1} xy \left(1_{\gamma}xy\right)\left(1_{\gamma}xy\right)^{-1} \quad [\ast], \] with \( 1_{\gamma} \in \left(S^2\right)_\alpha \) and \( 1_{\gamma} \in \left(S^2\right)_\beta \).

Assume that \( xy \in \left(S^2\right)_{\gamma} \). Then, since \( S^2 \) is a semilattice of the semigroups \( \left(S^2\right)_{\alpha} (\alpha \in Y) \), it follows from (1) that \( \delta = \delta \alpha \beta \).

Note that \( x^3 \in \left(S^2\right)_{\alpha} \) and \( y^3 \in \left(S^2\right)_{\beta} \). Therefore \( x^3 y^3 = x^2 (xy) y^2 \), which implies that \( \alpha \beta = \alpha \delta \beta = \delta \).

Therefore \( S \) is a semilattice of the semigroups \( S_\alpha (\alpha \in Y) \). It is straightforward to show that \( \left(S_\alpha \right)^2 = \left(S^2\right)_\alpha \). Thus, we have proved part (1) of Theorem 2.
Suppose now that $\alpha < \beta$, $x \in \left( S^2 \right)_{\alpha} \in G_2$ and $y \in \left( S^2 \right)_{\beta}$. Then $xyx \in \{ y, yx \}$ and so $xyx = yx$. But $yx \in \left( S^2 \right)_{\alpha \beta} = \left( S^2 \right)_{\alpha} \in G_2$ and so $x = 1$. Therefore $\left( S^2 \right)_{\alpha} = \{ 1 \}$. This proves part (2) of Theorem 2.

Suppose that $y \in S_\alpha - \left( S_\alpha \right)^2$. Let $x \in \left( S_\alpha \right)^2$. Then by hypothesis $xyx = yx$, and so $x = 1$. This proves part (3) of Theorem 2.

We proceed with the proof of part (4). For any $x \in S_\alpha$ we define $\theta_x : S \to S$ as $y \mapsto xyx \left( y \in S_\beta \right)$. Note that $\theta_x \cdot S_\beta : S \to \left( S_{\alpha \beta} \right)^2$, because $xyx = x \left( yx \right)^3 = \left[ x \left( yx \right)^2 \right] \left( yx \right) \in \left( S^2 \right)_{\alpha \beta} = \left( S_{\alpha \beta} \right)^2$. Let $\gamma \subseteq S_\beta$. If $\left( S_{\alpha \beta} \right)^2 \subseteq G_2$. If $\alpha \neq \beta$ then either $\alpha \beta < \beta$ or $\alpha \beta < \beta$ and so, by (2), $\left( S_{\alpha \beta} \right)^2 = \{ 1 \}$. This implies that $\left( \theta_x \left( yz \right), \theta_x y, \theta_x z \right) \subseteq \left( S_{\alpha \beta} \right)^2 = \{ 1 \}$ and so $\theta_x \cdot S_\beta$ is a homomorphism.

We can therefore assume that $\alpha = \beta = \alpha \beta$ and so $\left( S_{\alpha \beta} \right)^2 = \left( S_\alpha \right)^2 = \left( S_\beta \right)^2 \subseteq G_2$. Since, by hypothesis, $x^3 \in \{ x, x^2 \}$, $x^2 = x^4 = 1$. Then, $\left( \theta_x y \right) \left( \theta_x z \right) = xyx^2zx = xyxx = \theta_x \left( yz \right)$ and so $\theta_x \cdot S_\beta$ is a homomorphism in any case. This proves (4).

Let $\left( S_\alpha \right)^2 \in G_2$, $g \in \left( S_\alpha \right)^2$, $1 = 1_g$ and $\{ x, y, z \} \subseteq S_\alpha$. Then $\theta_g \cdot x = gxg = g1x1g = g^21x1 = 1xl = \theta_x \cdot x$ and $\theta_g \cdot \theta_x = x \theta_g \cdot x = y \theta_g \cdot x = y \theta_x \cdot x = y^2 = z^2 \in 1l = \theta_z \cdot z$. Also, $\theta_g \cdot z = xz = 1z = 1z = z^2 = x^2z^2 = 1z = \theta_z \cdot z$.

Hence, $\theta_g = \theta_{\theta_g \cdot x} = \theta_x \theta_z = \theta_x$ and this proves (4.1).

Let $\{ \alpha, \beta, \gamma \} \subseteq Y$ and $\left( S_{\alpha \beta \gamma} \right)^2 \subseteq R_0$. Suppose $x \in S_\alpha$, $y \in S_\beta$ and $z \in S_\gamma$. Then

$\theta_{\left( \theta_x y \right) \left( \theta_x z \right)} = \theta_x \left( \left( \theta_y z \right) \left( \theta_x x \right) \right) = \theta_x \left( \left( \theta_z \left( \theta_y z \right) \left( \theta_x x \right) \right) \right) = \theta_x \left( \left( \theta_x \left( \theta_z \left( \theta_y z \right) \left( \theta_x x \right) \right) \right) \right) \left( \theta_x \left( \theta_x \left( \theta_z \left( \theta_y z \right) \left( \theta_x x \right) \right) \right) \right)$. However, since by (4) $\theta_x$ and $\theta_z$ are homomorphisms on $\left( S_{\alpha \beta} \right)^2$ and $\left( S_\beta \right)^2$ respectively, this equation becomes:

Finally, (4.3) follows from the fact that $xy = \left( xy \right)^3$. This completes the proof of $(1 \Rightarrow 2)$.

$(2 \Rightarrow 1)$ Assume that the hypotheses of the “only if” part of Theorem 2 are valid. We first show that the product defined is associative. Let $x \in S_\alpha$, $y \in S_\beta$, $z \in S_\gamma$. Using (4.3) we need to show that:

$\left[ \theta_{\left( \theta_x y \right) \left( \theta_x z \right)} \right] \left( \theta_x \left( \left( \theta_y z \right) \left( \theta_x x \right) \right) \right) = \left( \theta_x \left( \left( \theta_z \left( \theta_y z \right) \left( \theta_x x \right) \right) \right) \right) \left( \theta_x \left( \theta_x \left( \theta_z \left( \theta_y z \right) \left( \theta_x x \right) \right) \right) \right)$. However, since by (4) $\theta_x$ and $\theta_z$ are homomorphisms on $\left( S_{\alpha \beta} \right)^2$ and $\left( S_\beta \right)^2$ respectively, this equation becomes:

$\left( 2 \right) \left( \theta_{\left( \theta_x y \right) \left( \theta_x z \right)} \right) \left( \theta_x \left( \theta_y z \right) \left( \theta_x x \right) \right) = \left( \theta_x \left( \theta_z \left( \theta_y z \right) \right) \left( \theta_x x \right) \right) \left( \theta_x \left( \theta_x \left( \theta_z \left( \theta_y z \right) \right) \right) \right)$,
with – by (4) again -- each of the 6 terms an element of \((S_{a\beta y})^2\). If \((S_{a\beta y})^2 \in R_0\) then, since by hypothesis 
\((4.2), \theta_{(\theta,z)(\theta,y)} x = \theta_z \theta_y x,\) equation (2) is valid in this case.

So we can assume that \((S_{a\beta y})^2 \in G_2\). We can therefore assume that \(\alpha = \beta = \gamma\), or else there
exists \(\sigma \in \{\alpha, \beta, \gamma\}\) such that \(\alpha \beta \gamma < \sigma\), which implies \((S_{a\beta y})^2 = \{1\}\). [This would imply that equation (2) is
valid.] But if \(\alpha = \beta = \gamma\) then, by (4.1), each side of equation (2) equals, \(\theta_{(\theta,z)(\theta,y)}(\theta_x)\), so (2) is valid.

Now we need to prove that for \(x \in S_a\) and \(y \in S_\beta, xy \in \{y, yx\}\). If \((S_{a\beta})^2 \in R_0\) then, since \(\{xy, yxy\} \subseteq (S_{a\beta})^2\),
\(xy = (xy)^2 = (xyyx)(yx) = yx\). We can assume, therefore, that \((S_{a\beta})^2 \in G_2\).

If \(\alpha \neq \beta\) then either \(\alpha \beta < \alpha\) or \(\alpha \beta < \beta\). In either case \(\{xy, yx\} \subseteq (S_{a\beta})^2 = \{1\}\) and so \(xy = yx\). We can assume
therefore that \(\alpha = \beta\). We can also assume that \(y \in (S_a)^2\), or else by (3), \(\{xy, yx\} \subseteq (S_a)^2 = \{1\}\). Note that
since \(xy = (\theta_x y)(\theta_y x) \in (S_a)^2 \in G_2\), \((S_a)^2\) is an abelian group. Therefore, \(xy = (\theta_x y)(\theta_y x)(\theta_x y) = yx\).
Also, \(x^2 = (\theta_x x)^2 = 1\). Hence, \(xy = yx^2 = y1 = y\). This completes the proof that \(S \in \{xy \in \{y, yx\}\}\).

So the proof of Theorem 2 is complete.

**Corollary 3.** The following statements are equivalent:

1. \(S\) is a semilattice \(Y\) of semigroups \(S_{\alpha}(\alpha \in Y)\) where

\[
(1.1) S_{\alpha} \in R_0 \cup G_2 (\alpha \in Y),
\]

\[
(1.2) \{\alpha, \beta\} \subseteq Y, \alpha < \beta \text{ and } S_{\alpha} \in G_2 \text{ implies } S_{\alpha} = \{1\}
\]

\[
(1.3) \text{there is a collection of mappings } \{\theta_x : S \to S / x \in S\} \text{ satisfying the following properties:}
\]

(a) for \(\{\alpha, \beta\} \subseteq Y\) each \(\theta_x / s_\beta : S_\beta \to S_{a\beta}\) is a homomorphism,

(b) \(\{\alpha, \beta, \gamma\} \subseteq Y, x \in S_a, y \in S_\beta\) and \(S_{a\beta y} \in R_0\) imply \(\theta_{(\theta,y)(\theta,x)} = \theta_{\gamma} \theta_x\) on \(S_y\) and

(c) if \(S_{\alpha} \in G_2\) and \(\{x, y\} \subseteq S_{\alpha}\) then \(\theta_{\gamma} = \theta_x \theta_y = \theta_x\) on \(S_{\alpha}\) and

2. \(S \in \{xy \in \{y, yx\}; x = x^3\}\).
Theorem 4. The following statements are equivalent:

1. $S$ is a semilattice $Y$ of semigroups $S_\alpha (\alpha \in Y)$ with $S_\alpha \in R_0 \cup G_2$ and

2. $S \subseteq \{xyx \in \{yx, y^2x^2y\}; x = x^3\}$.

Proof: ($1 \Rightarrow 2$) For any $x \in R_0 \cup G_2$, $x = x^3$ and therefore $S \subseteq \{x = x^3\}$. Suppose that $x \in S_\alpha$ and $y \in S_\beta (\alpha, \beta \in Y)$.

If $S_{\alpha\beta} \in R_0$ then $xyx = (xyx)^2 = (xyx^2)yx = yx$. Suppose that $S_{\alpha\beta} \in G_2$. Then,

\[
xy = (xy)^3 = x(yxy)xy = x(xy)yx = x^2y^2xy = x^2(y^21)xy = x^2xyy^1 = x^3yy^1 = y^21x^3y = y^2x^31 = y^21y^2x^3 = y^2x^3y = yx.
\]

Also, $1 = (1y)^2 = 1y1y = 1yy = y^21y = y^21$. Then,

\[
xyx = xxy = 1x^2y = (y^1)^1x^2y = y^2x^2y.
\]

We have proved therefore that $S \subseteq \{xyx \in \{yx, y^2x^2y\}\}$. Hence, $S \subseteq \{x = x^3\} \cap \{xyx \in \{yx, y^2x^2y\}\}$.

($2 \Rightarrow 1$) If $\{e, f\} \subseteq E_3$ then, since $efe \in \{fe, fef\}$, $ef = (ef)^3 = ef(efe)f \in \{effe, fef\} = \{(ef)^2\}$.

It then follows from Results 1, 2 and 3 that $S$ is a semilattice $Y$ of rectangular groups $S_\alpha (\alpha \in Y)$. Let $S_\alpha = L_\alpha \times G_\alpha \times R_\alpha$. The fact that $S \subseteq \{xyx \in \{yx, y^2x^2y\}\}$ implies $|L_\alpha| = 1$ and so $S_\alpha = G_\alpha \times R_\alpha$. Since $S \subseteq \{x = x^3\}$, $G_\alpha \subseteq \{x = x^3\}$ and $G_\alpha \subseteq G_2$. Let $x = (g, r), y = (h, s) \in S_\alpha$. Then

\[
xyx = (ghg, r) = (h, s) \in \{(hg, r), (h^2g, h, s)\} = \{(hg, r), (h, s)\}. \]

So $|R_\alpha| > 1$ implies $|G_\alpha| = 1$. Thus, either $|R_\alpha| = 1$ or $|G_\alpha| = 1$. So $S_\alpha \subseteq R_0 \cup G_2$, which is what we needed to prove. This completes the proof of Theorem 4.

Theorem 5: The following statements are equivalent:

1. $S$ is a semilattice $Y$ of semigroups $S_\alpha (\alpha \in Y)$ with $S_\alpha \in RB \cup G_2 (\alpha \in Y)$ and

2. $S \subseteq \{xyx \in \{xy^2x, y^2x^2y\}; x = x^3\}$.

Proof: ($1 \Rightarrow 2$) Clearly, $x = x^3$ for any $x \in S$. Let $\{\alpha, \beta\} \subseteq Y$ with $x \in S_\alpha$ and $y \in S_\beta$.

If $S_{\alpha\beta} \in RB$ then $xyx = (xyx)^3 = xy(x^2yx^2)yxy = xyxyx = xyx^2x$.

So we can assume that $S_{\alpha\beta} \in G_2$. Then

\[
(xy)^2 = 1 = xyxy = x(1y)yx = x1y^2 = x^2y(x1) = y^2x1 = y^21x1 = x^21y^2 = x^2(y^21) = (x^2y^2)x = x^3y^2 = y^2x^3.
\]

Therefore, $1 = x^2y^2 = y^2x^2$.

Then,

\[
xyx = (xyx)^3 = x(y^2x^2y)xy = x(xy^2x^2y)xy = x(xy)(xy^2x^2y)xy = x(xy)(xy^2x^2y)xy = x^3y^2(x^3y^2) = y1xy^2 = y^2x^4 = x^3y^2x^4 = y^2x^3y^2x^2 = y^2x^2y = y^2x^2y = y^2x^2y = y^2x^2y = y1 = 1y1 = 1y = (y^2x^2)y.
\]

Hence, $S \subseteq \{xyx \in \{xy^2x, y^2x^2y\}; x = x^3\}$.

($2 \Rightarrow 1$) Since $S \subseteq \{x = x^3\}$, by Results 2 and 3, $S$ is a semilattice of completely simple semigroups.
We now show that the product $ef$ of the idempotents $e$ and $f$ is idempotent. We have

$$e^2f^2e = e(ef)^2e = (ef)^2ef = (ef)^3 = e.$$ 

If $e^2f^2e = e$ then $ef = (ef)^3 = ef$. So we can assume that $e^2f^2e = e$. Then,

$$ef = f = (ef)^3 = ef.$$ 

Therefore, $ef = (ef)^3 = e(ef)f = ef$. So by Result 3, $S$ is a semilattice $Y$ of rectangular groups $S_\alpha = G_\alpha \times E_\alpha (\alpha \in Y)$, where $E_\alpha \in RB(\alpha \in Y)$.

Fix $\alpha \in Y$ and let $\{x, y\} \subseteq S_\alpha$ with $x = (1, (i, \lambda))$ and $y = (h, (j, \sigma))$ where $h$ is an arbitrary element of $G_\alpha$ and $(i, \lambda)$ and $(j, \sigma)$ are arbitrary elements of $E_\alpha$. Then,

$$xy = (h, (i, \lambda)) \in \{h^2, (i, \lambda), (h, (j, \sigma))\}.$$ 

So either $E_\alpha$ has only one element or $h = h^2$ for every $h \in G_\alpha$. Hence $S_\alpha$ is isomorphic to $G_\alpha$ or to $E_\alpha$. In the former case, since $S \in \{x = x^3\}$, $G_\alpha \in G_2$.

This completes the proof of Theorem 5.

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