RECTIFIABLE CURVES IN PROXIMALLY SMOOTH SETS

GRIGORY IVANOV AND MARIANA LOPUSHANSKI

ABSTRACT. We provide an algorithm of constructing a rectifiable curve between two sufficiently close points of a proximally smooth set in a uniformly convex and uniformly smooth Banach space. Our algorithm returns a reasonably short curve between two sufficiently close points of a proximally smooth set, is iterative and uses a certain modification of the metric projection. We estimate the length of a constructed curve and its deviation from the segment with the same endpoints. These estimates coincide up to a constant factor with those for the geodesics in a proximally smooth set in a Hilbert space.

1. Introduction

Weakly convex sets have been studied in non-smooth analysis for several decades. Several established mathematicians proposed their own definition of a weakly convex set, among them are Federer [Fed59], Efimov and Stechkin [ES58], Vial [Via83], Rockafellar (for akin classes of functions) [Roc81]. In this paper we stick to the most convenient in our opinion definition of a weakly convex set due to Clarke, Stern and Wolenski [CSW95]. A closed set in a Banach space is called proximally smooth with constant $R$ if distance to it from a point of the space is continuously differentiable in the open $R$-neighborhood of this set excluding the set itself (this and other definitions used in the introduction are formally given below in Section 2). As for other definitions, each of them characterizes weakly convex sets as the set with a certain property, e.g. differentiability of the distance function, supporting by balls, hypomonotonicity of the normal cone, etc. It turns out that many of these definitions are equivalent in Hilbert space, which allows using weakly convex sets in different applications. See, for example, [Bal17] and [Iva06b]. However, everything is a bit trickier in Banach spaces. Different classes of weakly convex sets in Banach spaces were studied in [BTZ06], [BTZ11], [Ali12]. As was shown in [BI09], some of the definitions are still equivalent in uniformly smooth and uniformly convex Banach spaces, but some not. The first author [Iva17] showed that the hypomonotonicity of the normal cone of a closed set fails to be equivalent to the proximally smoothness in any Banach space that is not isomorphic to a Hilbert space (in a Hilbert space these two properties are equivalent, see [Iva06a, Theorem 1.9.1] and [PRT00, Corollary 2.2]).

Another equivalent to the proximal smoothness property of a closed set in a Hilbert space was given in [Iva06a, Theorem 1.14.2]. We formulate it as follows.

**Proposition 1.1.** Let $A$ be a closed set in a Hilbert space and $R > 0$. The following conditions are equivalent:

1. The set $A$ is proximally smooth with constant $R$.
2. For any two different points $x_0, x_1 \in A$ with $|x_0 - x_1| < 2R$, there exists a curve $\gamma$ in $A$ with endpoints $x_0$ and $x_1$, whose length is at most

$$2R \arcsin \left( \frac{|x_0 - x_1|}{2R} \right).$$

This result plays a crucial role in proofs of many other important results related to the properties and applications of proximally smooth sets. For example, it implies the existence and uniqueness of the shortest path connecting two sufficiently close points of a proximally smooth set in a
Hilbert space, and that a “locally” proximally smooth set in a Hilbert space is proximally smooth.

However, even in a sufficiently smooth and convex Banach space, the existence of rectifiable curve between to sufficiently close points of a proximally smooth set has been unknown yet. One might argue that the definition of a proximally smooth set implies locally connectedness of a proximally smooth set since the metric projection of a sufficiently short segment with endpoints in a proximally smooth set onto the set itself has to be continuous (to be more precise, this argument works in a uniformly smooth and uniformly smooth spaces for segments strictly shorter than $2R$). It is not clear whether the curve constructed in such a way is rectifiable since the metric projection onto a proximally smooth set is Hölder continuous [Iva15, Theorem 3.2], but not Lipschitz continuous in a Banach space not isomorphic to a Hilbert space. Even if one can show that such a curve is rectifiable, the same rather unfortunate property of the metric projection implies that this curve has quite nasty behavior.

In this paper we provide an algorithm of constructing a rectifiable curve between two sufficiently close points of a proximally smooth set in a uniformly convex and uniformly smooth Banach space. Our Algorithm 1 returns a reasonably short curve between two sufficiently close points of a proximally smooth sets, is iterative and uses a certain modification of the metric projection. We collect two important properties of the curve constructed with the use of our algorithm in the two following Theorems.

**Theorem 1.** Let $X$ be a uniformly convex and uniformly smooth Banach space whose modulus of smoothness is of power type $s$. Then there are positive constants $\beta_L$ and $L$ satisfying the following property. Let $A \subset X$ be a proximally smooth set with positive constant $R$, and let $x_0, x_1 \in A$ with $\frac{|x_0 - x_1|}{R} < \beta_L$. Then Algorithm 1 returns the curve $\gamma$ in $A$ with endpoints $x_0$ and $x_1$ such that inequality

$$\text{length}(\gamma) \leq \|x_0 - x_1\| \left(1 + L \left(\frac{\|x_0 - x_1\|}{R}\right)^{s(s-1)}\right)$$

holds.

**Theorem 2.** Let $X$ be a uniformly convex and uniformly smooth Banach space whose modulus of smoothness is of power type $s$. Then there are positive constants $\beta_I$ and $L_I$ satisfying the following property. Let $A \subset X$ be a proximally smooth set with positive constant $R$, and let $x_0, x_1 \in A$ with $\frac{|x_0 - x_1|}{R} < \beta_I$. Then Algorithm 1 returns the curve $\gamma$ in $A$ with endpoints $x_0$ and $x_1$ such that inclusion

$$\gamma \subset \text{conv} \left\{x_0, B_r \left(\frac{x_0 + x_1}{2}\right), x_1\right\}$$

holds, where $B_r \left(\frac{x_0 + x_1}{2}\right)$ is the closed ball centered at $\frac{x_0 + x_1}{2}$ of radius

$$r = L_I \|x_0 - x_1\| \left(\frac{\|x_0 - x_1\|}{R}\right)^{s-1}.$$

We note that we will estimate constants $\beta_L, L, \beta_I, L_I$ using constants related to the smoothness of Banach space $X$. Moreover, we do not use the uniform convexity of $X$ directly; the reason for using this condition on a space is being able to use different definitions of a weakly convex set, which are equivalent to the proximally smoothness in a uniformly convex and uniformly smooth Banach space. This condition might be relaxed, for example, in a finite dimensional space, but it will add complications and little to the ideas. Also, since the complement of the interior of the unit ball is proximally smooth with constant one and by basic properties of the modulus of smoothness, the bound on $r$ in Theorem 2 is asymptotically tight. The bound on the length of the constructed curve in Theorem 1 coincides up to a constant factor with the bound on the shortest path between two points on a fixed distance in a proximally smooth set.
in uniformly smooth space whose modulus of smoothness is of power type 2 (for example, in a Hilbert space and in $L_p$ spaces with $p \geq 2$).

Since the metric projection onto a proximally smooth set in a uniformly convex and uniformly smooth Banach space is Hölder continuous [Iva15, Theorem 3.2], we get the following result as an immediate consequence of Theorem 1.

**Corollary 1.1.** Let $X$ be a uniformly convex and uniformly smooth Banach space whose moduli of smoothness and convexity are of power type. Let $A \subset X$ be a proximally smooth set with positive constant $R$, and let $x_0, x_1 \in A$ with $\frac{\|x_0 - x_1\|}{R} < 2$. Then there is a rectifiable curve $\gamma$ in $A$ with endpoints $x_0$ and $x_1$.

The rest of the paper is organized as follows. In the next Section 2 we give the standard terminology related to weakly convex sets and to the geometry of the unit ball of a Banach space. In Section 3 we study the distance function to a proximally smooth set restricted to a segment with endpoints in this set. In Section 4 we describe our Algorithm 1 for the construction of the rectifiable curve with endpoints in a proximally smooth set, and also summarize the assumptions needed to show the correctness of the algorithm. Then in Section 5 we estimate the length of curve returned by Algorithm 1 and prove Theorem 1. In Section 6 we prove Theorem 2. Finally, in the last section Section 7 we prove several purely technical results used in the proofs.

2. Terminology and basic properties

2.1. Properties of the unit ball. Let $X$ be a real Banach space, and $X^*$ be its conjugate space. We use $\langle p, x \rangle$ to denote the value of a functional $p \in X^*$ at a vector $x \in X$. For $r > 0$ and $c \in X$ we denote by $B_r(c)$ the closed ball with center $c$ and radius $r$.

We will use the notation $[xy]$ for the segment with endpoints $x$ and $y$. Define

$$\delta_X(\varepsilon) = \inf \left\{ 1 - \frac{\|x + y\|}{2} : x, y \in B_1(0), \|x - y\| \geq \varepsilon \right\}$$

and

$$\varrho_X(\tau) = \sup \left\{ \frac{\|x + y\| + \|x - y\|}{2} - 1 : \|x\| = 1, \|y\| = \tau \right\}.$$ 

Functions $\delta_X(\cdot) : [0, 2] \to [0, 1]$ and $\varrho_X(\cdot) : \mathbb{R}^+ \to \mathbb{R}^+$ are referred to as the moduli of convexity and smoothness of $X$, respectively. The modulus of convexity is of power type $s > 0$ if for some constant $C_{cv}$ inequality

$$\delta_X(\varepsilon) \geq C_{cv} \tau^s$$

holds for any $\tau \in [0, 2]$.

A Banach space $X$ is called uniformly convex if $\delta_X(\varepsilon) > 0$ for all $\varepsilon > 0$, and uniformly smooth if $\frac{\varrho_X(\tau)}{\tau} \to 0$ as $\tau \to 0$. We refer the reader to the book [Die75] as a comprehensive survey on these moduli and their geometric properties.

In what follows, we consider only uniformly smooth Banach spaces. In such spaces, for any non-zero vector $x$, there is a unique unit functional $p$ attaining its norm on $x$. Let $x$ be a non-zero vector of $X$ and $p$ be a unit functional attaining its norm on $x$, we use $H_x$ to denote the hyperplane $\{y \in X : \langle y, p \rangle = 0\}$. We will say that $y$ is quasi-orthogonal to vector $x \in X \setminus \{0\}$ and write $y^\perp x$ if $y \in H_x$.

Note that the following conditions are equivalent:

- $y$ is quasi-orthogonal to $x$;
- for any $\lambda \in \mathbb{R}$ vector $x + \lambda y$ lies in the supporting hyperplane to the ball $B_{\|x\|}(0)$ at $x$;
- for any $\lambda \in \mathbb{R}$ inequality $\|x + \lambda y\| \geq \|x\|$ holds;
- $x$ is orthogonal to $y$ in the sense of Birkhoff–James (see [Die75, Chapter 2] and [AMW12]).
2.2. Modulus of smoothness and related functions. The modulus of smoothness of a Banach space is a strictly increasing convex function satisfying the following inequality of Day–Nordlander type (see [Die75, Chapter 3])

\[ \sqrt{1 + \tau^2} - 1 \leq \varphi_X(\tau) \leq \tau \quad \text{for all } \tau \in [0, +\infty). \]

The modulus of smoothness is of power type \( s \) if for some constant \( C_{sm} \),

\[ \varphi_X(\tau) \leq C_{sm} \tau^s \quad \text{for all } \tau \in [0, +\infty). \]

It follows that the modulus of smoothness of a uniformly smooth Banach space might be of power type \( s \) only for some \( s \) in \( (1, 2] \).

In our computations we will use two functions related to the modulus of smoothness of a Banach space.

Define function \( \omega_X : [0, +\infty) \to [0, +\infty) \) by

\[ \omega_X(\tau) = \frac{\varphi_X(\tau)}{\tau}. \]

Since the modulus of smoothness of a uniformly smooth Banach space is a strictly increasing convex function, we conclude that \( \omega_X(\cdot) \) is a strictly increasing function. Thus, the inverse function \( \omega_X^{-1}(\cdot) \) is also strictly increasing.

The second function \( \zeta_X^+ : [0, +\infty) \to [0, +\infty) \) is defined by

\[ \zeta_X^+(\varepsilon) = \sup \{ \|x + \varepsilon y\| : \|x\| = \|y\| = 1, y \parallel x \}. \]

Thus, \( \zeta_X^+(\cdot) - 1 \) bounds the deviation of a point in a supporting hyperplane from the unit ball. This modulus of a Banach space was studied in [IM17], where it was shown that it is equivalent to the modulus of smoothness near zero.

**Proposition 2.1.** Let \( X \) be an arbitrary Banach space. Then

\[ \varphi_X\left( \frac{\varepsilon}{2(1 + \varepsilon)} \right) \leq \zeta_X^+(\varepsilon) - 1 \leq \varphi_X(2\varepsilon), \quad \varepsilon \in \left[ 0, \frac{1}{2} \right]. \]

It is not hard to see that \( \zeta_X^+ \) is strictly increasing, and hence, its inverse function \( (\zeta_X^+)^{-1} \) is well-defined and is strictly increasing.

2.3. Weakly convex sets. The distance from a point \( x \in X \) to a set \( A \subset X \) is defined as

\[ \text{dist}(x, A) = \inf_{a \in A} \|x - a\|. \]

The metric projection of a point \( x \) onto a set \( A \) is defined as any element of the set

\[ P_A(x) = \{ a \in A : \|a - x\| = \text{dist}(x, A) \}. \]

We call the set \( \{ x \in X : 0 < \text{dist}(x, A) < R \} \) the open \( R \)-neighborhood of a set \( A \).

**Definition 2.1.** A set \( A \subset X \) is called proximally smooth with constant \( R \) if it is closed and the distance function \( x \mapsto \text{dist}(x, A) \) is continuously differentiable on the open \( R \)-neighborhood of \( A \).

The geometric properties of proximally smooth sets are hidden in the definition. To clarify these geometrical properties, which are very useful in this paper, we introduce two equivalent (in certain spaces) to the proximal smoothness properties.

**Proposition 2.2** ([BI09]). Let \( X \) be a uniformly convex and uniformly smooth Banach space, let \( A \subset X \) be a closed set, and let \( R > 0 \). The following assertions are equivalent:

1. \( A \) is proximally smooth with constant \( R \).
2. the projection map \( x \mapsto P_A(x) \) is single valued and continuous on the open \( R \)-neighborhood of \( A \).
(3) for any \( u \) in the open \( R \)-neighborhood of \( A \) and any \( x \in \text{PA}(u) \) inequality
\[
\text{dist}\left(x + \frac{R}{\|u - x\|}(u - x), A\right) \geq R
\]
holds.

Roughly speaking, the last property here implies that the set can be supported by a ball of fixed radius \( R \) at a point of its boundary.

2.4. Auxiliary geometric constructions.

**Definition 2.2.** Let \( A \) be proximally smooth with constant \( R \), let \( x_0 \) and \( x_1 \) be two distinct points of \( A \) with \( \frac{\|x_0 - x_1\|}{R} < 2 \), we say that an arbitrary point of the set
\[
P_A([x_0, x_1]) \cap \left(H_{x_0 - x_1} + \frac{x_0 + x_1}{2}\right)
\]
is a slice-projection of the midpoint \( \frac{x_0 + x_1}{2} \) of the segment \([x_0x_1]\) onto \( A \).

**Proposition 2.2** and the separation lemma imply that the slice-projection is non-empty in a uniformly smooth and uniformly convex Banach space.

In this section we bound the distance between the midpoint of a segment with endpoints in \( \text{PA} \) with \( \|x_0 - x_1\| < 2 \) and \( \lambda \in [0, 1] \). Then the following bound on the distance from point \( x_\lambda = (1 - \lambda)x_0 + \lambda x_1 \) to set \( A \) holds
\[
\text{dist}(x_\lambda, A) \leq 8R\lambda(1 - \lambda) g_X\left(\frac{\|x_0 - x_1\|}{R}\right).
\]

**Lemma 3.1.** Let \( X \) be a uniformly convex and uniformly smooth Banach space, let \( R > 0 \) and \( A \subset X \) be a proximally smooth with constant \( R \) set. Let \( x_0, x_1 \in A \) with \( \frac{\|x_0 - x_1\|}{R} < 2 \) and \( \lambda \in [0, 1] \). Then the following bound on the distance from point \( x_\lambda = (1 - \lambda)x_0 + \lambda x_1 \) to set \( A \) holds
\[
\text{dist}(x_\lambda, A) \leq 8R\lambda(1 - \lambda) g_X\left(\frac{\|x_0 - x_1\|}{R}\right).
\]

**Proof.** Proposition 2.2 implies that \( P_A(x_\lambda) \) is nonempty for all \( \lambda \in [0, 1] \). Fix an arbitrary \( \lambda \in (0, 1) \) and consider \( y \in P_A(x_\lambda) \). Using assertion 3 of Proposition 2.2, we get that
\[
\text{dist}\left(y + R\frac{x_\lambda - y}{\|x_\lambda - y\|}, A\right) \geq R.
\]
Hence \( \|y + R\frac{x_\lambda - y}{\|x_\lambda - y\|} - x_0\| \geq R \), and \( \left\|\frac{x_\lambda - y}{\|x_\lambda - y\|} - \frac{y - x_0}{R}\right\| \geq 1 \). By the definition of the modulus of smoothness, we get that
\[
2 g_X\left(\frac{\|y - x_0\|}{R}\right) \geq \left\|\frac{x_\lambda - y}{\|x_\lambda - y\|} + \frac{y - x_0}{R}\right\| + \left\|\frac{x_\lambda - y}{\|x_\lambda - y\|} - \frac{y - x_0}{R}\right\| - 2 \geq \left\|\frac{x_\lambda - y}{\|x_\lambda - y\|} + \frac{y - x_0}{R}\right\| - 1.
\]

Let \( p \) be the unit functional attaining its norm on vector \( x_\lambda - y \). Then
\[
\left\| \frac{x_\lambda - y}{\|x_\lambda - y\|} + \frac{y - x_0}{R} \right\| - 1 \geq \left\langle p, \frac{x_\lambda - y}{\|x_\lambda - y\|} + \frac{y - x_0}{R} \right\rangle - 1 = \frac{1}{R} \langle p, y - x_0 \rangle.
\]

Therefore, we obtain that

\[\langle p, x_0 - y \rangle \leq 2R \varrho_X \left( \frac{\|x_0 - y\|}{R} \right).\]

Since \( y \in P_A(x_\lambda) \) and \( x_0, x_1 \in A \),
\[
\|y - x_\lambda\| = \text{dist}(x_\lambda, A) \leq \min \{ \|x_\lambda - x_0\|, \|x_\lambda - x_1\| \} \leq \min \{ \|x_\lambda - x_0\|, \|x_\lambda - x_1\| \} = \min \{ \lambda, (1 - \lambda) \} \|x_0 - x_1\|.
\]

Therefore,
\[
\|y - x_0\| \leq \|y - x_\lambda\| + \|x_\lambda - x_0\| \leq \min \{ 2\lambda, 1 \} \|x_0 - x_1\|.
\]

This, the monotonicity of the modulus of smoothness and inequality (8) yield inequality

\[\langle p, x_0 - y \rangle \leq \Phi(\lambda),\]

where

\[\Phi(\lambda) = 2R \varrho_X \left( \frac{\min \{ 2\lambda, 1 \} \|x_0 - x_1\|}{R} \right).\]

Similarly,

\[\langle p, x_1 - y \rangle \leq \Phi(1 - \lambda).
\]

Multiplying inequalities (9) and (11) by \((1 - \lambda)\) and \(\lambda\) respectively, and then summing them, one has
\[
\langle p, x_\lambda - y \rangle \leq (1 - \lambda)\Phi(\lambda) + \lambda\Phi(1 - \lambda).
\]

This and the inequality \(\text{dist}(x_\lambda, A) \leq \|x_\lambda - y\| = \langle p, x_\lambda - y \rangle\) imply that

\[\text{dist}(x_\lambda, A) \leq (1 - \lambda)\Phi(\lambda) + \lambda\Phi(1 - \lambda).
\]

Using once more the convexity of function \(\varrho_X(\cdot)\), and the identities \(\varrho_X(0) = 0\) and (10), we get that
\[
\Phi(\lambda) \leq 2R \min \{ 2\lambda, 1 \} \varrho_X \left( \frac{\|x_0 - x_1\|}{R} \right) \leq 4\lambda R \varrho_X \left( \frac{\|x_0 - x_1\|}{R} \right).
\]

Thus, inequality (12) implies that
\[
\text{dist}(x_\lambda, A) \leq 8\lambda R(1 - \lambda) \varrho_X \left( \frac{\|x_0 - x_1\|}{R} \right).
\]

\[
\Box
\]

**Proposition 2.2** and the separation lemma imply the following.

**Lemma 3.2.** Let \(X\) be a uniformly convex and uniformly smooth Banach space, let \(A \subset X\) be a proximally smooth with constant \(R\) set. Then for any \(x_0, x_1 \in A\) with \(\frac{\|x_0 - x_1\|}{R} < \omega_X^{-1}(1/8)\), there exists a slice-projection of the midpoint of \([x_0 x_1]\) onto \(A\).

**Lemma 3.3.** Let \(X\) be a uniformly convex and uniformly smooth Banach space, let \(A \subset X\) be a proximally smooth with constant \(R\) set. Let \(x_0, x_1 \in A\) with \(\frac{\|x_0 - x_1\|}{R} < \omega_X^{-1}(1/8)\), the following inclusion holds
\[
P_A([x_0 x_1]) \subset \text{conv} \left\{ x_0, B_{r'} \left( \frac{x_0 + x_1}{2} \right) \cap \left( H_{x_1 - x_0} + \frac{x_0 + x_1}{2} \right), x_1 \right\},
\]
where \(r' = R'(\|x_0 - x_1\|, R)\) (see (6)).
First, we will show that \( x_1 \notin B_r(x_\lambda) \). That is, we need to verify the following inequality:
\[
8(1 - \lambda) \frac{R}{\|x_0 - x_1\|} \varphi_X \left( \frac{\|x_0 - x_1\|}{R} \right) < 1.
\]

Since \( 1 - \lambda \in (0, 1) \), this inequality holds whenever \( \omega_X \left( \frac{\|x_0 - x_1\|}{R} \right) \leq 1/8 \). Thus, \( x_1 \notin B_r(x_\lambda) \).

Denote the intersection point of ray \( x_\lambda x_1 \) with the boundary of the ball \( B_r(x_\lambda) \) by \( v \) and let \( \ell \) be one of the two lines passing through \( x_1 \) supporting \( B_r(x_\lambda) \). The tangent point of \( \ell \) and \( B_r(x_\lambda) \) is denoted by \( w \). Note that \( y \) is the directional vector of the line supporting \( B_r(x_\lambda) \) at \( v \). Therefore, the lines \( \ell \) and \( x_{1/2} + \text{Lin}\{y\} \) are not parallel and their intersection point, denoted by \( z \), lies in the same half-plane with the point \( w \). By similarity, it suffices to set \( r' \) equal to any upper bound on \( \|x_{1/2} - z\| \) that does not depend on \( \lambda \) and \( y \).

Let us estimate \( \|x_{1/2} - z\| \). Denote the intersection point of the ray \( x_1 x_\lambda \) and the line \( w + \text{Lin}\{y\} \) by \( v' \). By similarity, we get
\[
\|x_{1/2} - z\| = \frac{1}{2} \frac{\|x_0 - x_1\|}{\|v' - x_1\|} \|w - v'\|.
\]

Since \( y \perp (x_1 - x_0) \), we have that \( v' \in B_r(x_\lambda) \). Hence, we get
\[
\|w - v'\| \leq 2r \quad \text{and} \quad \|v' - x_1\| \geq \|x_\lambda - x_1\| - r = \lambda \|x_1 - x_0\| - r.
\]

Combining these inequalities with inequality (13), we get
\[
\|x_{1/2} - z\| = \frac{r}{\lambda - \frac{r}{\|x_0 - x_1\|}} = \frac{8R(1 - \lambda) \varphi_X \left( \frac{\|x_0 - x_1\|}{R} \right)}{1 - 8(1 - \lambda) \omega_X \left( \frac{\|x_0 - x_1\|}{R} \right)} < \frac{8R \varphi_X \left( \frac{\|x_0 - x_1\|}{R} \right)}{1 - 8 \omega_X \left( \frac{\|x_0 - x_1\|}{R} \right)} = r'.
\]

This completes the proof.

As an immediate corollary, we get.

**Corollary 3.1.** Let \( X \) be a uniformly convex and uniformly smooth Banach space, let \( A \subset X \) be a proximally smooth with constant \( R \) set. Let \( x_0, x_1 \in A \) with \( \frac{\|x_0 - x_1\|}{R} < \omega_X^{-1}(1/8) \), fix \( \lambda \in [0, 1] \) and set \( x_\lambda = \lambda x_0 + (1 - \lambda) x_1 \). Then there is a point \( z_\lambda \in P_A([x_0, x_1]) \) such that
\[
\|z_\lambda - x_\lambda\| \leq 4 \lambda (1 - \lambda) R' \left( \|x_0 - x_1\|, R \right)
\]
and \( (z_\lambda - x_\lambda)^\perp (x_1 - x_0) \). Moreover, the distance between \( \frac{x_0 + x_1}{2} \) and any point of the slice projection of the midpoint \( \frac{x_0 + x_1}{2} \) of the segment \( [x_0, x_1] \) onto \( A \) is at most \( R' \left( \|x_0 - x_1\|, R \right) \).

**4. Construction of a curve**

**4.1. Assumptions on the distance between the endpoints.** Our algorithm of curve construction between two distinct points \( x_0 \) and \( x_1 \) of a proximally smooth set works when points are sufficiently close. Moreover, we need different bounds to prove the convergence of the algorithm and, for example, to prove the inclusion in Theorem 2. We have decided to collect all the assumptions on the distance between the two starting points.

By definition put
\[
\mu = \zeta_X^+ \left( \frac{2R' \left( \|x_0 - x_1\|, R \right)}{\|x_0 - x_1\|} \right)
\]
and recall the definition of \( R' \tau, R \) (see (6)).

Assumptions on \( \frac{\|x_0 - x_1\|}{R} \):
then these questions for a separate statement and prove it later in Section 7.

4.2. Algorithm for the construction of a curve.

Claim 4.1. Set

\[ \beta_L = \omega_X^{-1} \left( \frac{(\zeta^+_X)^{-1}(2)}{8(2 + (\zeta^+_X)^{-1}(2))} \right). \]

Then \( \beta_L \leq \omega_X^{-1} \left( \frac{1}{8} \right) < 2 \), and for any positive constants \( \tau \) and \( R \) satisfying \( \frac{\tau}{R} < \beta_L \), inequality \( \zeta^+_X \left( \frac{2\tau}{R} \right) < 2 \) holds.

4.2. Algorithm for the construction of a curve.

**Algorithm 1:** Construction of a curve in a proximally smooth set

**Data:** A proximally smooth with constant \( R \) set \( A \subset X \), two distinct points \( x_0, x_1 \) in \( A \) with \( \frac{\|x_0-x_1\|}{R} < \beta_L \), where \( \beta_L \) is given by (15).

**Result:** A rectifiable curve \( f([0,1]) \), where \( f: [0,1] \to A \) is a continuous function with \( f(0) = x_0 \) and \( f(1) = x_1 \).

Set \( S_0 = \{0,1\} \) and \( S_i = \{ \frac{j}{2^i} \mid j \in [2^i]\} \} \cup \{0\} \) for \( i \in \mathbb{N} \).

1. Define \( f \) at points of \( S_0 \) as follows: \( f(0) = x_0 \) and \( f(1) = x_1 \).
2. For every \( i \in \mathbb{N} \), we extend the domain of \( f \) to the set \( S_i \setminus S_{i-1} \) as follows:
   - set the value of \( f \) at \( \frac{j}{2^i} \) to be a slice-projection of the midpoint of the segment \( [f \left( \frac{j-1}{2^i} \right), f \left( \frac{j}{2^i} \right)] \) for all \( j \in [2^{i-1}] \) on \( A \).
3. Continuously extend \( f \) on \([0,1]\).

4.3. Problems needed to be justified. To show the correctness of Algorithm 1, one needs to check:

1. For every \( i \in \mathbb{N} \) and \( j \in [2^{i-1}] \), there exists a slice-projection of the midpoint of a segment \( [f \left( \frac{j-1}{2^i} \right), f \left( \frac{j}{2^i} \right)] \) onto the set \( A \).
2. \( f \) can be continuously extended from the rational numbers of \([0,1]\) to the whole segment.
3. Curve \( f([0,1]) \) is rectifiable.

According to Lemma 3.2 to show the existence of a slice-projection at each step, it suffices to show that the length of segment \( [f \left( \frac{j-1}{2^i} \right), f \left( \frac{j}{2^i} \right)] \) is less than \( R\omega_X^{-1}(1/8) \). We will justify these questions for \( x_0, x_1 \) and \( R \) satisfying assumption 2.
5. Bound on length

Theorem 1 is an immediate consequence of the following theorem.

**Theorem 3.** Under the condition of Theorem 1, additionally let the modulus of smoothness of $X$ satisfy inequality (5) and
\[
\frac{\|x_0 - x_1\|}{R} < \beta_L = \omega_X^{-1}\left(\frac{(\zeta_X^+)^{-1}(2)}{8(2 + (\zeta_X^+)^{-1}(2))}\right).
\]
Then Algorithm 1 returns curve $\gamma$ satisfying the following inequality:
\[
\text{length } (\gamma) \leq \|x_0 - x_1\| \exp\left[\left(\frac{16}{5}\right)^{s} C_{sm}^{s+1} \frac{1 - \left(\frac{\mu}{2}\right)^{s(s-1)}}{2R} \left(\|x_0 - x_1\|^{s(s-1)}\right)\right],
\]
where $\mu$ is given by (14).

**Proof.** We denote by $\gamma_i$ the polygonal curve with consecutive vertices of $\{f(t) \mid t \in S_i\}$ and by $\Delta_i$ the largest length of a segment of $\gamma_i$, $i \in \mathbb{N} \cup \{0\}$.

We start with an upper bound on $\Delta_i$. By construction, we have that
\[
\Delta_i \leq \frac{\Delta_{i-1}^+}{2} \left(\frac{2R'(\Delta_{i-1}, R)}{\Delta_{i-1}}\right).
\]
To have a meaningful bound, one needs to guarantee that the argument of $\zeta_X^+(\cdot)$ is less than one. Using Claim 4.1, we see that $\Delta_i < \Delta_{i-1}$ starting with $i = 1$. Hence, we have

**Lemma 5.1.**
\[
\Delta_i \leq \frac{\mu}{2} \Delta_{i-1} \leq \left(\frac{\mu}{2}\right)^i \Delta_0 \to 0 \quad \text{as } i \to \infty.
\]
Denote $\psi(\tau) = \zeta_X^+\left(\frac{2R'(\tau, R)}{\tau}\right) - 1$. Using (17) in (16), we obtain
\[
\Delta_i \leq \frac{\Delta_{i-1}}{2} \left(1 + \psi\left(\left(\frac{\mu}{2}\right)^i \Delta_0\right)\right) \leq \frac{\Delta_{i-1}}{2} \exp\left[\psi\left(\left(\frac{\mu}{2}\right)^i \Delta_0\right)\right].
\]
Therefore, we have
\[
\Delta_i \leq \frac{\Delta_0}{2^i} \exp\left[\sum_0^i \psi\left(\left(\frac{\mu}{2}\right)^i \Delta_0\right)\right].
\]
Finally,
\[
\text{length } (\gamma_i) \leq 2^i \Delta_i \leq \Delta_0 \exp\left[\sum_0^i \psi\left(\left(\frac{\mu}{2}\right)^i \Delta_0\right)\right] \leq \Delta_0 \exp\left[\sum_0^\infty \psi\left(\left(\frac{\mu}{2}\right)^i \Delta_0\right)\right].
\]
We need to bound the series in the rightmost part in (18). It is a purely technical task that involves only routine computations. We formulate the following inequality and prove it later in Section 7.
Proof. We will greedy estimate \( s \) such a constant exists. Let \( \gamma_i \) the polygonal curve with consecutive vertices of \( \{ f(t) \mid t \in S_i \} \) and by \( \Delta_i \) the largest length of a segment of \( \gamma_i \), \( i \in \mathbb{N} \cup \{0\} \). Denote an intersection point of the hyperplane \( H_{x_0-x_1} + f(t) \) and the line \( x_0x_1 \) by \( g_2(t) = ||x_0 - g(t)|| \) and \( g_1(t) = ||f(t) - g(t)|| \).

We choose \( \beta_i \) in such a way that inequality \( \frac{|x_0 - x_1|}{R} < \beta_i \) implies assumption 3. By Lemma 4.1 such a constant exists.

To bound \( g_1 \) from above and \( g_2 \) from below, we need the following purely technical result, which we prove in the next section.

**Claim 5.1.** Let \( 0 < \Delta_0 < R \cdot \beta_L \). Then

\[
\exp \left[ \sum_0^\infty \psi \left( \left( \frac{\mu}{2} \right)^i \Delta_0 \right) \right] \leq \exp \left[ \left( \frac{16}{5} \right)^s C_{sm+1} \frac{1}{1 - \left( \frac{\mu}{2} \right)^s} (\Delta_0)^{s(s-1)} \right].
\]

Thus, passing to the limit in (18) as \( i \) tends to infinity and using Claim 5.1, we get that the lengths of \( \gamma_i \) are uniformly bounded. Therefore, function \( f \) constructed above is uniformly continuous on the rational numbers of interval \([0, 1]\). By routine, it can be extended to the continuous function on the whole interval with values in \( A \). Thus, \( f \) defines a continuous curve in \( A \). Using (18) again, one sees that the first variation of \( \gamma = f([0, 1]) \) is bounded. Thus, \( \gamma \) is rectifiable. \( \square \)

6. Inclusion

In this Section we prove Theorem 2. The proof consists of several steps. Firstly, we show that the curve returned by Algorithm 1 is in a cylinder of a certain radius around line \( x_0x_1 \). Then we show that the second part of the curve, that is \( f([1/2, 1]) \), belongs to a certain convex cone with apex at \( x_0 \). Finally, we prove that all parts of the curve of the form \( f([1/2^k, 1/2^{k-1}]) \) are in a certain cone with apex at \( x_0 \).

Under the conditions of Theorem 2, additionally let the modulus of smoothness of \( X \) satisfy inequality (5) and let \( \frac{|x_0 - x_1|}{R} \) satisfy assumption 3, then inequality

\[
\sum_{j=0}^k \frac{R'(\Delta_j, R)}{2^{k-j}} < \frac{24C_{sm}}{1 - \frac{\mu^s}{2^{k-1}}} \cdot \left( \frac{\Delta_0}{R} \right)^{s-1} \cdot \frac{\Delta_0}{2^k}
\]

holds.

**Lemma 6.1.** Under the condition of Theorem 2, additionally let the modulus of smoothness of \( X \) satisfy inequality (5) and \( \frac{|x_0 - x_1|}{R} \) satisfy assumption 3. For any \( t \in [0, 1] \), inequality

\[
g_1(t) \leq \frac{48C_{sm}}{1 - \frac{\mu^s}{2^{k-1}}} \cdot \left( \frac{\Delta_0}{R} \right)^{s-1} \Delta_0
\]

holds.

**Proof.** We will greedy estimate \( g_1(t) \), \( t \in (0, 1) \) from above. We proceed by induction on \( k \) and will prove the following

\[
g_1 \left( \frac{2j - 1}{2^k} \right) \leq 2 \sum_{i=0}^k R'(\Delta_i, R) \quad \forall k \in \mathbb{N}, \quad j \in [2^{k-1}].
\]

Lemma 3.3 yields the case \( k = 1 \). Suppose inequality (19) holds for \( k - 1 \). Let us prove it for \( k \). Fix \( j \in [2^{k-1}] \) and denote

\[
a = \frac{f(\frac{2j-2}{2^k}) + f(\frac{2j}{2^k})}{2} \quad \text{and} \quad b = x_0x_1 \cap (H_{x_0-x_1} + a).
\]
Then, by the triangle inequality, we get
\[ g_1 \left( \frac{2j - 1}{2^k} \right) = \| g \left( \frac{2j - 1}{2^k} \right) - f \left( \frac{2j - 1}{2^k} \right) \| \leq \| g \left( \frac{2j - 1}{2^k} \right) - b \| + \| b - a \| + \| a - f \left( \frac{2j - 1}{2^k} \right) \|. \]
Again, by the triangle inequality,
\[ \| b - a \| \leq \frac{g_1 \left( \frac{2j - 2}{2^k} \right) + g_1 \left( \frac{2j}{2^k} \right)}{2}, \]
and by construction, we have
\[ \| \frac{2j - 1}{2^k} \| \leq \frac{g_1 \left( \frac{2j - 2}{2^k} \right) + g_1 \left( \frac{2j}{2^k} \right)}{2}. \]

Thus, by the triangle inequality, we get
\[ \| g \left( \frac{2j - 1}{2^k} \right) - b \| \leq \| f \left( \frac{2j - 1}{2^k} \right) - a \| \leq R'(\Delta_k, R). \]

Thus,
\[ g_1 \left( \frac{2j - 1}{2^k} \right) \leq 2 \| f \left( \frac{2j - 1}{2^k} \right) - a \| + \frac{g_1 \left( \frac{2j - 2}{2^k} \right) + g_1 \left( \frac{2j}{2^k} \right)}{2} \leq 2R'(\Delta_k, R) + 2 \sum_{i=0}^{k-1} R'(\Delta_i, R). \]

Inequality (19) is proven.
Thus, by Claim 6.1 and continuity, we have that
\[ g_1(t) \leq \frac{48C_{sm}}{1 - \frac{\mu}{2^{t-1}}} \cdot \left( \frac{\Delta_0}{R} \right)^{s-1} \Delta_0 \]
for any \( t \in [0, 1] \).

\[ \square \]

Lemma 6.1 says that the curve returned by the algorithm lies in a certain cylinder around line \( x_0x_1 \). To understand curve’s behavior near endpoints, we need a more subtle argument.

**Lemma 6.2.** Under the condition of Theorem 2, additionally let the modulus of smoothness of \( X \) satisfy inequality (5) and \( \frac{\| x_0 - x_1 \|}{R} \) satisfy assumption 3. For any \( t \in \left[ \frac{1}{2}, 1 \right] \), inequality \( g_2(t) \geq \frac{\Delta_0}{2} \) holds.

**Proof.** We will greedy estimate \( g_2(t), t \in \left( \frac{1}{2}, 1 \right) \) from below. We proceed by induction on \( k \) and will prove the following
\[ g_2 \left( \frac{2j - 1}{2^k} \right) \geq \frac{\Delta_0}{2} - \sum_{i=0}^{k} R'(\Delta_i, R) \quad \forall \ k \in \mathbb{N}, \ j \in [2^{k-1}], \ j > 2^{k-2}. \]

The construction of the curve and definition of \( g_2(\cdot) \) yield that
\[ g_2 \left( \frac{1}{2} \right) = \frac{\Delta_0}{2} \geq \frac{\Delta_0}{2} - R'(\Delta_0, R). \]

Thus, we have the induction basis. Suppose now that (20) holds for some \( k - 1 \). Let us now prove that it holds for \( k \).

Fix \( j \in [2^{k-1}], \ j > 2^{k-2} \) and denote
\[ a = \frac{f \left( \frac{2j - 2}{2^k} \right) + f \left( \frac{2j}{2^k} \right)}{2} \quad \text{and} \quad b = x_0x_1 \cap (H_{x_0 - x_1} + a). \]

Note that \( b \) and \( g(t) \) lie on the line \( x_0x_1 \).

By the triangle inequality, we obtain that
\[ g_2 \left( \frac{2j - 1}{2^k} \right) = \| x_0 - g \left( \frac{2j - 1}{2^k} \right) \| \geq \| x_0 - b \| - \| g \left( \frac{2j - 1}{2^k} \right) - b \|. \]
Thus, \( 2a = f\left(\frac{1}{2}\right) + f(1) = f\left(\frac{1}{2}\right) + x_1 \). The dashed lines denotes hyperplanes parallel to \( H_{x_0-x_1} \).

By the definition of \( b \), we have
\[
\|b - x_0\| = \frac{g_2\left(\frac{2j-2}{2^k}\right) + g_2\left(\frac{2j}{2^k}\right)}{2}.
\]

As \( f(t) - g(t) \) is quasi-perpendicular to \( x_0 - x_1 \), we have that
\[
\left\| g\left(\frac{2j-1}{2^k}\right) - b \right\| \leq \left\| f\left(\frac{2j-1}{2^k}\right) - a \right\|.
\]

Thus,
\[
g_2\left(\frac{2j-1}{2^k}\right) \geq \frac{g_2\left(\frac{2j-2}{2^k}\right) + g_2\left(\frac{2j}{2^k}\right)}{2} - \left\| f\left(\frac{2j-1}{2^k}\right) - a \right\| \geq \frac{\Delta_0}{2} - R'(\Delta_k, R) - \sum_{i=0}^{k-1} R'(\Delta_i, R).
\]

Inequality (20) is proven.

Claim 6.1, the assumptions on \( \Delta_0 \), and the continuity of \( g(\cdot) \) imply that \( g_2(t) \geq \frac{\Delta_0}{2} \) for all \( t \in \left[\frac{1}{2}, 1\right] \).

**Corollary 6.1.** Under the condition of Theorem 2, additionally let the modulus of smoothness of \( X \) satisfy inequality (5) and \( \frac{||x_0-x_1||}{R} \) satisfy assumption 3. For any \( t \in \left[\frac{1}{2}, 1\right] \), the set \( \{f(t) : t \in \left[\frac{1}{2}, 1\right]\} \) is a subset of
\[
\text{cone}(x_0, B_{L_1 \Delta_0}(x_1) \cap (H_{x_0-x_1} + x_1)),
\]
where
\[
L_1 = \frac{400C_{sm}}{1 - \mu^s} t \left(\frac{\Delta_0}{R}\right)^{s-1}.
\]

Define
\[
r_i = \frac{1}{1 - \mu^{s-1}} \cdot \frac{400C_{sm}}{1 - \mu^{s-1}} \left(\frac{\Delta_i}{R}\right)^{s-1} \Delta_i \quad \text{and} \quad G_i = \text{cone}\left(x_0, B_{r_i}\left(f\left(\frac{1}{2^{i+1}}\right)\right)\right), \quad i \in \mathbb{N}.
\]

**Lemma 6.3.** Under the condition of Theorem 2, additionally let the modulus of smoothness of \( X \) satisfy inequality (5) and \( \frac{||x_0-x_1||}{R} \) satisfy assumption 3. Then \( f([0, 1]) \subseteq G_1 \).

**Proof.** By inequality (17) and assumption (3), we have that \( \Delta_i \leq \Delta_{i-1} \). Hence, \( R'(\Delta_i, R) \leq R'(\Delta_{i-1}, R) \). Thus, Corollary 6.1 implies that \( f([1/2^i, 1/2^{i-1}]) \subseteq G_i \) for every \( i \in \mathbb{N} \). Thus, to prove the lemma, it suffices to show that \( G_i \subseteq G_1 \) for all \( i \in \mathbb{N} \). By construction, the curve \( f([0, 1/2^i]) \) coincides with the curve returned by Algorithm 1 applied to points \( x_0 \) and \( f(1/2^i) \).

Let us prove that \( G_i \subseteq G_{i-1} \). By assumption (3) and by inequality (17),
\[
r_i \leq \frac{1}{2} \cdot \frac{\mu^s}{2^{s-1} r_{i-1}}.
\]

**Figure 2.** Illustration for the proof of Lemma 6.1. Here, \( k = 2 \) and \( j = 2 \).
Denote the midpoint of a segment \([x_0, f(1/2^{i-1})]\) by \(a_i\). By the triangle inequality,
\[
G_{i+1} \subset \text{cone}(x_0, B_{r_{i+1}}f(1/2^i) - a_i(a_i)) \text{ for all } i \in \mathbb{N}.
\]

By similarity,
\[
\text{cone}(x_0, B_{r_{i+1}}f(1/2^i) - a_i(a_i)) = \text{cone}(x_0, B_{2r_{i+1}}f(1/2^i) - a_i(f(1/2^{i-1}))) \text{ for all } i \in \mathbb{N}.
\]

Applying Corollary 3.1 with \(x_0 = x_0\) and \(x_1 = f(1/2^i)\), we get that
\[
\|f(1/2^i) - a_i\| \leq R'(\|f(1/2^{i-1}) - x_0\|, R) \leq R'(\Delta_i, R).
\]

Since \(\Delta_i \leq \Delta_0 \leq \beta_I\) and by Claim 6.1,
\[
\|f(1/2^i) - a_i\| \leq \frac{24C_{sm}}{1 - \frac{\mu}{2^{s-1}}} \left(\frac{\Delta_i}{R}\right)^{s-1} \Delta_i \leq \left(1 - \frac{\mu^3}{2^{s-1}}\right) r_{i-1}.
\]

This and inequality (21) imply that
\[
2r_i + 2\|f(1/2^i) - a_i\| \leq r_{i-1}.
\]

Consequently, by the triangle inequality, one has
\[
G_{i+1} \subset \text{cone}(x_0, B_{r_{i+1}}f(1/2^i) - a_i(a_i)) \subset G_i \text{ for all } i \in \mathbb{N}.
\]

Hence, we conclude that \(G_i \subset G_1\) for all \(i \in \mathbb{N}\), completing the proof of the lemma. \(\square\)

By symmetry and by Lemma 6.3, we get the following result which implies Theorem 2.

**Theorem 4.** Under the condition of Theorem 2, additionally let the modulus of smoothness of \(X\) satisfy inequality (5) and \(\beta_I\) satisfy inequality
\[
\frac{\nu^s}{2^{s-1}} < 1, \quad \text{where} \quad \nu = \zeta^\chi \left(\frac{2R'(\beta_I, R)}{\beta_I}\right).
\]

Then the curve \(\gamma\) returned by Algorithm 1 satisfies inclusion
\[
\gamma \subset \text{conv}\left\{x_0, B_r\left(\frac{x_0 + x_1}{2}\right), x_1\right\},
\]

where
\[
r = \frac{1}{1 - \frac{\mu^s}{2^{s-1}}} \cdot \frac{400C_{sm}}{1 - \frac{\mu^s}{2^{s-1}}} \|x_0 - x_1\| \left(\frac{\|x_0 - x_1\|}{R}\right)^{s-1}.
\]

7. PROOFS OF TECHNICAL RESULTS

**Proof of Claim 4.1.** Denote \(\zeta^{-1} = \left(\zeta^\chi\right)^{-1}(2)\). The definition of \(\zeta^\chi_+(-)\) implies that
\[1 + \tau \geq \zeta^\chi_+(\tau) \geq 1.\]

Since \(\omega^{-1}_X(\cdot)\) is an increasing function, one has \(\omega^{-1}_X(3/40) < \omega^{-1}_X(1/8)\). Thus, to show that \(\beta_L \leq \omega^{-1}_X(1/8)\), it suffices to show that \(\frac{\zeta^{-1}}{2 + \zeta^{-1}} \leq \frac{3}{8}\). By (22) and by monotonicity, we obtain that
\[
\frac{\zeta^{-1}}{2 + \zeta^{-1}} \leq \left.\frac{\tau + 1}{2 + \tau + 1}\right|_{\tau = \frac{3}{8}} = \frac{3}{5}.
\]

Since \(R'(\tau, R) = \tau \frac{8\omega_X\left(\frac{\tau}{R}\right)}{1 - 8\omega_X\left(\frac{\tau}{R}\right)}\) (see (6)), we have
\[
\zeta^\chi_+ \left(\frac{2R'(\tau, R)}{\tau}\right) = \zeta^\chi_+ \left(\frac{16\omega_X\left(\frac{\tau}{R}\right)}{1 - 8\omega_X\left(\frac{\tau}{R}\right)}\right).
\]
Since $\frac{7}{R} < \beta_L$ and functions $\zeta^+_X(\cdot)$ and $\omega_X(\cdot)$ are increasing, we obtain that
\[
\zeta^+_X\left(\frac{16\omega_X\left(\frac{T}{R}\right)}{1 - 8\omega_X\left(\frac{T}{R}\right)}\right) < \zeta^+_X\left(\frac{16\omega_X(\beta_L)}{1 - 8\omega_X(\beta_L)}\right) = \zeta^+_X\left(\frac{16\zeta^{-1}_X(\tau)}{8(2+\zeta^{-1}_X(\tau))}\right) = 2
\]
completing the proof of Claim 4.1.

Let us prove inequality $\omega^{-1}_X(1/8) < 2$. By monotonicity of $\omega^{-1}_X(\cdot)$ and by the definition of $\omega(\cdot)$, we have the following chain
\[
\omega^{-1}_X(1/8) < 2 \iff 1/8 < \omega_X(2) \iff 1/4 < \varrho_X(2).
\]
The last inequality follows from (4).

Proof of Claim 5.1. By Proposition 2.1 and the definition of $R'(\tau, R)$ (see (6)), we get
\[
\psi(\tau) = \zeta^+_X\left(\frac{2R'(\tau, R)}{\tau}\right) - 1 \leq \varrho_X(4R'(\tau, R)\tau) = \varrho_X\left(\frac{32\omega_X\left(\frac{T}{R}\right)}{1 - 8\omega_X\left(\frac{T}{R}\right)}\right).
\]
Claim 4.1 and inequality $\varrho_X(\tau) \leq C_{sm}\tau^s$ imply that
\[
\varrho_X\left(\frac{32\omega_X\left(\frac{\tau}{R}\right)}{1 - 8\omega_X\left(\frac{\tau}{R}\right)}\right) \leq \varrho_X\left(\frac{32\omega_X\left(\frac{T}{R}\right)}{1 - 8\omega_X\left(\frac{T}{R}\right)}\right) \leq \varrho_X\left(\frac{32C_{sm}\omega_X\left(\frac{T}{R}\right)}{1 - 8\omega_X\left(\frac{T}{R}\right)}\right) \leq \left(\frac{16}{5}\right)^s C_{sm}^s \left(\frac{\tau}{R}\right)^{s(s-1)}.
\]
Thus,
\[
\sum_0^\infty \psi\left(\left(\frac{\mu}{2}\right)^i \Delta_0\right) \leq \left(\frac{16}{5}\right)^s C_{sm}^s \Delta_0^{(s-1)} \sum_0^\infty \left(\frac{\mu}{2}\right)^{i(s(s-1))} = \left(\frac{16}{5}\right)^s C_{sm}^s \Delta_0^{(s-1)} \frac{1}{1 - \left(\frac{\mu}{2}\right)^{s(s-1)}}.
\]

Proof of Claim 6.1. Denote
\[
S_k = \frac{1}{8R} \sum_{j=0}^k \frac{R'(\Delta_j, R)}{2^{k-j}} = \frac{1}{2} \sum_{j=0}^k \frac{\varrho_X\left(\frac{\Delta_j}{R}\right)}{2^{k-j} \left(1 - 8\omega_X\left(\frac{\Delta_j}{R}\right)\right)}.
\]
Taking into account that $1 - 8\omega_X\left(\frac{\Delta_j}{R}\right) \geq \frac{2}{R}$, we obtain that
\[
S_k \leq \frac{5}{2} \sum_{j=0}^k \frac{1}{2^{k-j}} \varrho_X\left(\frac{\Delta_j}{R}\right).
\]
Hence, considering that $\Delta_j \leq \left(\frac{\mu}{2}\right)^j \Delta_0$ and $\varrho_X(\tau) \leq C_{sm}\tau^s$, we get
\[
S_k \leq \frac{5}{2} \sum_{j=0}^k \frac{C_{sm}}{2^{k-j}} \left(\left(\frac{\mu}{2}\right)^j \frac{\Delta_0}{R}\right)^s = \frac{5}{2} \cdot \frac{C_{sm}}{2^k} \left(\frac{\Delta_0}{R}\right)^s \sum_{j=0}^k \frac{\mu^j s}{2^{j s-j}} < \frac{5}{2} \cdot \frac{C_{sm}}{2^k} \left(\frac{\Delta_0}{R}\right)^s \sum_{j=0}^\infty \left(\frac{\mu^s}{2^{s-1}}\right)^j.
\]
Assumption (3) $\left(\frac{\mu^s}{2^{s-1}} < 1\right)$ yields that
\[
\sum_{j=0}^\infty \left(\frac{\mu^s}{2^{s-1}}\right)^j = \frac{1}{1 - \frac{\mu^s}{2^{s-1}}} < \infty.
\]
Finally, we obtain that
\[
S_k < \frac{3C_{sm}}{1 - \frac{\mu^s}{2^{s-1}}} \cdot \left(\frac{\Delta_0}{R}\right)^s \cdot \frac{1}{2^k},
\]
completing the proof of Claim 6.1.
REFERENCES

[Ali12] A. R. Alimov. Monotone path-connectedness of r-weakly convex sets in spaces with linear embedding. *Mathematical Notes*, 3(2):21–30, 2012.

[AMW12] J. Alonso, H. Martini, and S. Wu. On Birkhoff orthogonality and isosceles orthogonality in normed linear spaces. *Aequationes Math.*, 83(1-2):153–189, 2012.

[Bal17] M. V. Balashov. About the gradient projection algorithm for a strongly convex function and a proximally smooth set. *J. Convex Analysis*, 24(2):493–500, 2017.

[BI09] M. V. Balashov and G. E. Ivanov. Weakly convex and proximally smooth sets in Banach spaces. *Izv. RAN. Ser. Mat.*, 73(3):23–66, 2009.

[BTZ06] F. Bernard, L. Thibault, and N. Zlateva. Characterizations of prox-regular sets in uniformly convex banach spaces. *J. Convex Anal.*, 13:525–559, 2006.

[BTZ11] F. Bernard, L. Thibault, and N. Zlateva. Prox-regular sets and epigraphs in uniformly convex Banach spaces: Various regularities and other properties. *Trans. Amer. Math. Soc.*, 363:2211–2247, 2011.

[CSW95] F. H. Clarke, R. J. Stern, and P. R. Wolenski. Proximal Smoothness and Lower–c^2 Property. *J. Convex Anal.*, 2(1):117–144, 1995.

[Die75] J. Diestel. *Geometry of Banach Spaces - Selected Topics*, volume 485. Springer-Verlag Berlin Heidelberg, 1975.

[ES58] Nikolai Vladimirovich Efimov and Sergei Borisovich Stechkin. Some properties of chebyshev sets. In *Doklady Akademii Nauk*, volume 118, pages 17–19. Russian Academy of Sciences, 1958.

[Fed59] Herbert Federer. Curvature measures. *Transactions of the American Mathematical Society*, 93(3):418–491, 1959.

[IM17] G. M. Ivanov and Horst Martini. New moduli for Banach spaces. *Annals of Functional Analysis*, 8(3):350–365, 2017.

[Iva06a] G. E. Ivanov. *Weakly Convex Sets and Functions. Theory and Applications. (in Russian).* Moscow, 2006.

[Iva06b] G. E. Ivanov. Weakly convex sets and their properties. *Mathematical Notes*, 79:55–78, 2006.

[Iva15] G. E. Ivanov. Sharp estimates for the moduli of continuity of metric projections onto weakly convex sets. *Izvestiya: Mathematics*, 79(4):668, 2015.

[Iva17] G. M. Ivanov. Hypomonotonicity of the normal cone and proximal smoothness. *Journal Of Convex Analysis*, 24(4):27. 1313–1339, 2017.

[PRT00] R. Poliquin, R. Rockafellar, and L. Thibault. Local differentiability of distance functions. *Trans. Amer. Math. Soc.*, 352(11):5231–5249, 2000.

[Roc81] R. T. Rockafellar. Favorable classes of Lipschitz continuous functions in subgradient optimization. 1981.

[Via83] J.-P. Vial. Strong and weak convexity of sets and functions. *Math. Ops. Res.*, 8(2):231–259, 1983.

GRIGORY IVANOV: INSTITUTE OF SCIENCE AND TECHNOLOGY AUSTRIA (IST AUSTRIA), KLEUSTENEUBURG, 3400, AUSTRIA; LABORATORY OF COMBINATORIAL AND GEOMETRICAL STRUCTURES, MOSCOW INSTITUTE OF PHYSICS AND TECHNOLOGY, MOSCOW, 141701, RUSSIA

Email address: GRIMIVANOV@GMAIL.COM

STEKLOV MATHEMATICAL INSTITUTE OF THE RUSSIAN ACADEMY OF SCIENCES, MOSCOW, RUSSIA

Email address: MASHA.ALEXANDRA@GMAIL.COM