Scattering approach to classical quasi-1D transport

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General dynamical transport of classical particles in disordered quasi-1D samples is viewed in the framework of scattering approach. Simple equation for the transfer-matrix is obtained within this unified picture. In the case of diffusive transport the solution of this equation exactly coincides with the solution of diffusion equation.

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The problem of classical particles transport in disordered quasi-1D system has a long history. In fact, many would claim that the problem itself is history. I want to show in this paper that this is not the case, and that the well known problem can be seen in a new light.

We consider the quasi-1D system. Classical particle is impinging from the left and leaves the sample after the time $t$ from the input face with the differential probability $R(t)$ and from the output face with the differential probability $T(t)$. We want to calculate these probability functions. The problem can be reduced to finding, for any given frequency, two Fourier transforms – $T_\omega$, which we shall call transmission coefficient, and $R_\omega$, which we shall call reflection coefficient (both of course are frequency dependent).

We are going to show that scattering approach which is extensively used in quantum transport, can serve as simple and giving physical insight tool to describe the classical problem also. In the framework of this approach the transport of particles through the sample can be represented as scattering from one of the two input channels into either one of the two output channels (see Fig.1).

$$\hat{j}(\omega(x+L)) = \hat{M}_\omega(L) \cdot \hat{j}(\omega(x)),$$  \hspace{1cm} (1)

where

$$\hat{j}(\omega(x+L)) = \hat{M}_\omega(L) \cdot \hat{j}(\omega(x)),$$  \hspace{1cm} (2)

The transfer-matrix give full description of the transport; transmission and reflection coefficients are simply connected with its matrix elements

$$\hat{M}_\omega = \begin{pmatrix} \frac{1}{T_\omega} & -\frac{R_\omega}{T_\omega} \\ \frac{R_\omega}{T_\omega} & T_\omega - \frac{R_\omega^2}{T_\omega} \end{pmatrix}.$$  \hspace{1cm} (3)

To obtain the equation for the transfer-matrix let us consider $L$ not just as a fixed length of the sample but as a free parameter. Now let us imagine $L$ being divided into two arbitrary parts $L_1$ and $L_2$.

$$L = L_1 + L_2,$$  \hspace{1cm} (4)

The convenience of using the transfer-matrix is due to the fact that the transfer-matrix of the whole sample is the product of the transfer-matrices of its parts

$$\hat{M}_\omega(L_1 + L_2) = \hat{M}_\omega(L_1) \cdot \hat{M}_\omega(L_2).$$  \hspace{1cm} (5)

This equation is quite general. It is valid provided only the sample is homogeneous. (When the system is not in the multiple scattering regime Eq.(5) implies also the condition of the ”diffusive illumination”). The Eq.(5) looks like a symbolic one. But we’ll show that it can be really solved, and e.g. for a diffusive transport its solution exactly coincides with that of the diffusion equation with the appropriate boundary conditions.

Let us first consider the case of zero frequency. In this case, taking into account the conservation law

$$R(L) + T(L) = 1,$$  \hspace{1cm} (6)

we can rewrite Eq.(5) in the form

$$\frac{R}{T}(L_1 + L_2) = \frac{R}{T}(L_1) + \frac{R}{T}(L_2).$$  \hspace{1cm} (7)

The solution is obvious

$$T(L) = \frac{1}{1 + \alpha L},$$  \hspace{1cm} (8)
where $\ell^*$ is some constant, which is determined by the equation

$$R(dL) = 1 - T(dL) = \alpha dL.$$  

(9)

Now let us return to the general case of finite frequency. In this case it is convenient to write Eq. (5) in differential form

$$d\hat{M}_\omega(L) = \hat{K}_\omega \cdot \hat{M}_\omega(L),$$  

(10)

where

$$\hat{K}_\omega = \frac{d\hat{M}_\omega}{dL} \bigg|_{L=0}.$$  

(11)

This equation should be supplemented by the boundary condition

$$\hat{M}_\omega(0) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$  

(12)

Eq. (10) expresses transfer-matrix through two constants, defined by the equations

$$T_\omega(dL) = 1 - \alpha_\omega dL, \quad R_\omega(dL) = \beta_\omega dL.$$  

(13)

Thus

$$\hat{K}_\omega = \begin{pmatrix} \alpha_\omega & -\beta_\omega \\ \beta_\omega & -\alpha_\omega \end{pmatrix}.$$  

(14)

Inserting $\hat{K}_\omega$ into Eq. (10) and using the boundary conditions (12) we get

$$\hat{M}_\omega(L) = \frac{\beta_\omega}{C_\omega} \begin{pmatrix} \sinh [C_\omega (L + \ell_\omega)] & -\sinh (C_\omega L) \\ \sinh (C_\omega L) & -\sinh [C_\omega (L - \ell_\omega)] \end{pmatrix},$$  

(15)

where

$$C_\omega = \sqrt{\alpha_\omega^2 - \beta_\omega^2},$$  

(16)

and $\ell_\omega$ is found from the equation

$$\beta_\omega \sinh (C_\omega \ell_\omega) = C_\omega.$$  

(17)

Eq. (15) is the main result of the paper. We see that the Fourier components of the transmission and reflection coefficients have universal $L$-dependence, whatever the character of the transport and the detailed microscopics of the system is. The $\omega$-dependence of the Fourier components is less universal, so to say something meaningful about it we should specify the character of the transport.

For illustration consider the diffusive transport. Let us start from the static case. To calculate $\alpha$ we should consider the transmission through the part of the sample $dL$ much less than the mean free path $\ell$ at the kinetic level of description, using, say Boltzmann equation. But even without further reasoning it is obvious that

$$R(dL) = 1 - T(dL) = \frac{kdL}{\ell},$$  

(18)

where $k$ is some numerical coefficient of the order of one, exact value of which depends upon the microscopics of the system (similarly to the fact that the exact form of boundary conditions to the diffusion equation depends upon microscopics of the system). Hence we see that apart from numerical coefficient of the order of one $\alpha$ is just the inverse mean free path.

Let us consider now the case of finite frequency. In this case we can connect $\alpha_\omega$ and $\beta_\omega$ (more exactly their difference) with the diffusion coefficient $D$. Eq. (1) in the differential form can be written as

$$\frac{\partial \hat{j}_\omega}{\partial x} = \hat{K}_\omega(L) \cdot \hat{j}_\omega.$$  

(19)

For the physical current $j$

$$j = j^{(1)} - j^{(2)}$$  

(20)

we obtain

$$\frac{\partial^2 j_\omega}{\partial x^2} = (\alpha_\omega^2 - \beta_\omega^2) j_\omega.$$  

(21)

Comparing this equation with the diffusion equation we immediately obtain

$$C_\omega = \alpha_\omega^2 - \beta_\omega^2 = i\omega D.$$  

(22)

On the other hand in diffusive approximation we restrict ourselves by the leading order approximation with respect to small parameter

$$\sqrt{\frac{i\omega D}{\ell}} \ll 1.$$  

(23)

It means that for the diffusive regime we should put into Eq. (15)

$$\beta_\omega = \ell^{-1} = \beta_0 = \alpha.$$  

(24)

After inserting these constants we obtain for the transmission and reflection coefficient results which exactly coincide with those following from the diffusion equation [1].

[1] P. M. Morse and H. Feshbach, *Methods of Theoretical Physics* (McGraw - Hill, New York, 1953).