Remarks on GJMS operator of order six

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Abstract

We study analysis aspects of the sixth order GJMS operator $P_6^g$. Under conformal normal coordinates around a point, the expansions of Green’s function of $P_6^g$ with pole at this point are presented. As a starting point of the study of $P_6^g$, we manage to give some existence results of prescribed $Q$-curvature problem on Einstein manifolds. One among them is that for $n \geq 10$, let $(M^n, g)$ be a closed Einstein manifold of positive scalar curvature and $f$ a smooth positive function in $M$. If the Weyl tensor is nonzero at a maximum point of $f$ and $f$ satisfies a vanishing order condition at this maximum point, then there exists a conformal metric $\tilde{g}$ of $g$ such that its $Q$-curvature $Q_6^{\tilde{g}}$ equals $f$.

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1 Introduction

Recently, some remarkable developments have been achieved in the existence theory of positive constant $Q$-curvature problem associated to Paneitz-Branson operator. One key ingredient in such works is that a strong maximum principle for the fourth order Paneitz-Branson operator is discovered under a hypothesis on the positivity of some conformal invariants or $Q$-curvature of the background metric. The readers are referred to [8, 9, 10, 13] and the references therein. This naturally stimulates us to study GJMS operator of order six and its associated $Q$-curvature problem, the analogue to the Yamabe problem and $Q$-curvature problem for Paneitz-Branson operator. Except for the aforementioned cases, due to the lack of a maximum principle for higher order elliptic equations in general, the existence theory of such problems needs to be developed. Until an analogue of Aubin’s result in [2] for the Yamabe problem is verified in Proposition 3.1 below, by adapting some ideas for Paneitz-Branson operator from [4, 3] we establish some existence results of prescribed $Q$-curvature problem on Einstein manifolds, in which case the sixth order GJMS operator is of constant coefficients.

The conformally covariant GJMS operators with principle part $(-\Delta_g)^k$, $k \in \mathbb{N}$ are discovered by Graham-Jenne-Mason-Sparling [6]. In particular, the GJMS operator of order six and the associated $Q$-curvature are given as follows (cf. [11, 17]): on manifolds $(M^n, g)$ of dimension $n \geq 3$ and $n \neq 4$, denote by $\sigma_k(A_g)$ the $k$-th elementary symmetric function of Schouten

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tensor $A_{ij} = \frac{1}{n-2}(R_{ij} - \frac{R_g}{2(n-1)}g_{ij})$. Denote by

$$C_{ijk} = \nabla_k A_{ij} - \nabla_j A_{ik}, \quad B_{ij} = \Delta_g A_{ij} - \nabla^k \nabla_j A_{ik} - A^{kl} W_{kij} = \nabla^k C_{ijk} - A^{kl} W_{kij}$$

the Cotton tensor and Bach tensor, respectively. Let

$$T_2 = (n - 2)\sigma_1(A_g) - 8A_g = -\frac{8}{n-2}\text{Ric}_g + \frac{n^2 - 4n + 12}{2(n-1)(n-2)}R_g g;$$

$$T_4 = -\frac{3n^2 - 12n - 4}{4}\sigma_1(A_g)^2 g + 4(n - 4)|A|^2_g + 8(2n - 2)\delta(A_g, A_g)A_g$$

$$+ (n - 6)\Delta_g \sigma_1(A_g) - 48A^2_g - \frac{1}{n-4}B_g;$$

$$v_6 = -\frac{1}{8}\sigma_3(A_g) - \frac{1}{24(n-4)}B(A, A)_g.$$

Then, the $Q$-curvature $Q^6_g$ is defined by

$$Q^6_g = -3!2^6v_6 - \frac{n + 2}{2}\Delta_g(\sigma_1(A_g)^2) + 4\Delta_g|A|^2 - 8\sigma(A_g)\sigma(A_g) + \Delta^2_2\sigma(A_g)$$

$$- \frac{n - 6}{2}\sigma_1(A_g)\Delta_g \sigma_1(A_g) - 4(n - 6)\sigma(A_g)|A|^2_g + \frac{(n - 6)(n + 6)}{4}\sigma(A_g)^3$$

(1.1)

and the GJMS operator of sixth order $P^6_g$ is given by

$$-P^6_g = \Delta^3_6 + \Delta_g T_2 d + \delta T_2 d \Delta_g + \frac{n - 2}{2n} T_2 \sigma(A_g) \Delta_g + \delta T_2 d - \frac{n - 6}{2}Q^6_g,$$

(1.2)

where $-\delta d = \Delta_g$. The operator $P^6_g$ is conformally covariant in the sense that if $\bar{g} = u^{-\frac{4}{n-2}}g$, $0 < u \in C^\infty(M)$ with $n \geq 3$ and $n \neq 4, 6$,

$$u^{\frac{n-6}{n-2}} P^6_g \varphi = P^6_g (u \varphi)$$

(1.3)

and in dimension 6,

$$P^6_{e^{2u}g} \varphi = e^{-6u} P^6_g \varphi$$

for all $\varphi \in C^\infty(M)$. When $(M, g)$ is Einstein, $P^6_g$ is of constant coefficients, explicitly,

$$Q^6_g = \frac{n^4 - 20n^2 + 64}{32n^2(n-1)^3}R^3_g,$$

$$-P^6_g = \Delta^3_6 + \frac{3n^2 - 6n + 32}{4n(n-1)}R_g \Delta^2_6 + \frac{3n^4 - 12n^3 - 52n^2 + 128n + 192}{16n^2(n-1)^2}R^2_g \Delta_6 - \frac{n - 6}{2}Q^6_g.$$

Obviously, when $n \geq 7$, $Q^6_g$ is a positive constant whenever the scalar curvature $R_g$ is positive. Through a direct computation, the GJMS operator $P^6_g$ has the following factorization:

$$P^6_g = \left(-\Delta_g + \frac{(n - 6)(n + 4)}{4n(n-1)}R_g\right)\left(-\Delta_g + \frac{(n - 4)(n + 2)}{4n(n-1)}R_g\right)\left(-\Delta_g + \frac{n - 2}{4(n-1)}R_g\right).$$

(1.4)

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1The definition of $P^6_g$ differs from the formula (10.15) in [11] by a minus sign.
In general, as shown in [5] and [7], on Einstein manifolds the GJMS operator of order $2k$ for all positive integers $k$ satisfies the above property as

$$P_g^{2k} = \prod_{i=1}^{k} \left( -\Delta_g + \frac{R_g}{4n(n-1)}(n+2i)(n-2i) \right).$$

In particular, choose $M^n = S^n, g = g_{S^n}$, then

$$Q_{S^n}^6 = \frac{n(n^4 - 20n^2 + 64)}{32},$$

$$P_{S^n}^6 = -\Delta^3_{S^n} - \frac{3n^2 + 6n + 32}{4} \Delta^2_{S^n} - \frac{3n^3 - 12n^2 - 52n + 192}{16} \Delta_{S^n} + \frac{n-6}{2} Q_{S^n}^6$$

$$= \left( -\Delta_{S^n} + \frac{(n-6)(n+4)}{4} \right) \left( -\Delta_{S^n} + \frac{(n-4)(n+2)}{4} \right) \left( -\Delta_{S^n} + \frac{n(n-2)}{4} \right).$$

From now on, we set $P_g = P_{S^n}^6$ and $Q_g = Q_{S^n}^6$ in the whole paper unless stated otherwise. Then, for any $\varphi \in H^3(M, g)$, we get

$$\int_M \varphi P_g \varphi d\mu_g$$

$$= \int_M \left( |\nabla_\varphi|^2 - 2T_2(\nabla_\varphi, \nabla_\varphi) - \frac{n-2}{2} \sigma_1(A)(\Delta_g \varphi)^2 - T_4(\nabla_\varphi, \nabla_\varphi) + \frac{n-6}{2} Q_g \varphi^2 \right) d\mu_g.$$

As a starting point of the study on the sixth order GJMS operator, we obtain some existence results of conformal metrics with positive $Q$-curvature candidates on closed Einstein manifolds under some additional natural assumptions.

**Theorem 1.1.** Suppose $(M^n, g)$ is a closed Einstein manifold of dimension $n \geq 10$ and has positive scalar curvature. Let $f$ be a smooth positive function in $M$. Assume the Weyl tensor $W_g$ is nonzero at a maximum point $p$ of $f$ and $f$ satisfies the vanishing order condition at $p$:

$$\begin{align*}
\Delta_g f(p) &= 0, & \text{if } n = 10, \\
\nabla^k f(p) &= 0, & k = 2, 3, 4, & \text{if } n \geq 11.
\end{align*}$$

Then there exists a smooth solution to the $Q$-curvature equation

$$P_g u = fu^{\frac{n+6}{n-6}}, \quad u > 0 \text{ in } M.$$

We remark that the condition (1.5) imposed on the $Q$-curvature candidates $f$ is conformally invariant. The condition that $(M, g)$ is Einstein is only used to seek a *positive* solution. Theorem 1.1 is a special case of a generalized Theorem 3.1.

This paper is organized as follows. In section 2, the expansions of Green’s function for $P_g$ when $n \geq 7$ are presented under conformal normal coordinates around a point. The technique used here is basically inspired by Lee-Parker [12], see also [10]. The complicate computations of the term $P_g(r^{6-n})$ are left to Appendix A, where $r$ is the geodesic distance from this point. In section 3, we prove an analogue (cf. Proposition 3.1) of Aubin’s result for any closed manifold of dimension $n \geq 10$, which is not locally conformal flat. Based on this result, using the Mountain Pass Lemma we state in Theorem 3.1 some results of prescribed $Q$-curvature problem associated to the sixth order GJMS operator on Einstein manifolds. Then our main Theorem 1.1 directly follows from Theorem 3.1.
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2 Expansion of Green’s function of $P_g$

Based on the survey paper by Lee-Parker [12] on the Yamabe problem, the method of deriving expansions of Green’s function of $P_g$ is more or less standard except for careful computations on some lower order terms involved in $P_g$. One may also refer to [10] for the Paneitz-Branson operator case. Green’s functions of conformally covariant operators play an important role in the solvability of the constant curvature problems, for instance, the Yamabe problem (cf. [12] etc.) and the constant $Q$-curvature problem for Paneitz-Branson operator (cf.[3, 4, 8, 10], etc.).

Especially, F. Hang and P. Yang [10] set up a dual variational method of the minimization for the Paneitz-Branson functional to seek a positive maximizer of the dual functional, such a scheme heavily relies on the positivity and expansion of its Green’s function. We expect that the expansion of Green’s function for $P_g^6$ will be useful to some possible future applications.

Throughout the whole paper, we use the following notation: $2^2 = 2^{n-6}$, $\omega_n = \text{vol}(S^n, g_{S^n})$ and when $n > 6$, $c_n = \frac{2n}{n-6}, \omega_n = \text{vol}(S^n, g_{S^n})$.

For $m \in \mathbb{Z}_+$, let

$$ P_m := \{\text{homogeneous polynomials in } \mathbb{R}^n \text{ of degree } m\} $$

and

$$ H_m := \{\text{harmonic polynomials in } \mathbb{R}^n \text{ of degree } m\}. $$

Moreover, $P_m$ has the following decomposition (cf. [16], P. 68-70)

$$ P_m = \bigoplus_{k=0}^{[\frac{m}{2}]} (r^{2k}H_{m-2k}). $$

**Proposition 2.1.** Assume $n > 6$ and $\ker P_g = 0$. Let $G_p(x)$ be the Green’s function of sixth order GJMS operator at the pole $p \in M^n$ with the property that $P_g G_p = c_n \delta_p$ in the sense of distribution. Then, under the conformal normal coordinates around $p$ with conformal metric $g$, $G_p(x)$ has the following expansions:

(a) If $n$ is odd, then

$$ G_p(x) = r^{6-n}(1 + \sum_{k=1}^{n} \psi_k) + A + O(r), $$

where $A$ is a constant and $\psi_k \in P_k$.

(b) If $n$ is even, then

$$ G_p(x) = r^{6-n}(1 + \sum_{k=1}^{n} \psi_k) + r^{6-n}\left(\sum_{k=n-4}^{n} \varphi_k \right) \log r + r^{6-n}\left(\sum_{k=n-4}^{n} \varphi'_k \right) \log^2 r $$
\[ + r^{6-n} \left( \sum_{k=n-2}^{n} \varphi'''_k \right) \log^3 r + \varphi''_n \log^4 r + A + O(r), \]

where \( A \) is a constant and \( \psi_k, \varphi_k, \varphi'_k, \varphi''_k, \varphi'''_k \in P_k \). Moreover, we may restate some of the above results in another way.

(c) If \( n = 7, 8, 9 \) or \( M \) is conformally flat near \( p \), then
\[ G_p(x) = c_n r^{6-n} + A + O(r), \]
where \( A \) is a constant.

(d) If \( n = 10 \), then
\[ G_p(x) = c_n r^{-4} + \frac{1}{17280} |W(p)|^2 \log r + O(1). \]

(e) If \( n \geq 11 \), then
\[ G_p(x) = c_n r^{6-n} + \psi_4 r^{6-n} + O(r^{11-n}), \]
where \( \psi_4 \in P_4 \) and
\[ \psi_4(x) = \frac{1}{135(n-2)} \left[ \sum_{k,l} (W_{iklj}(p)x^i x^j)^2 - \frac{r^2}{n+4} \sum_{k,l,s} ((W_{ikls}(p)+W_{ilks}(p))x^i)^2 \right. \]
\[ + \frac{3}{2(n+4)(n+2)} |W(p)|^2 r^4 \]
\[ + \frac{3n-20}{270(n+4)(n-4)(n-8)} r^2 \left[ \sum_{k,l,s} ((W_{ikls}(p)+W_{ilks}(p))x^i)^2 - \frac{3}{n} |W(p)|^2 r^2 \right] \]
\[ - \frac{5n^2 - 66n + 224}{120(n-8)(n-4)} r^2 \sigma_1(A)_{ij}(p)x^i x^j + \frac{|W(p)|^2}{12(n-1)} r^2 \right] \]
\[ + \frac{3n^4 - 16n^3 - 164n^2 + 400n + 2432}{576(n+4)(n+2)n(n-1)} |W(p)|^2 r^4 \right] \}

Before starting to derive the expansion of Green’s function of \( P_g \), we first need to introduce some notation. For \( \alpha \in \mathbb{R} \), set
\[ A_{\alpha} = r^2 \Delta_0 + \alpha r \partial_r + \alpha(\alpha + n - 2), \]
\[ A_{\alpha,g} = r^2 \Delta_g + \alpha r \partial_r + \alpha(\alpha + n - 2), \]
where \( \Delta_0 \) denotes the Euclidean Laplacian, and
\[ B_{\alpha} = \frac{\partial}{\partial \alpha} A_{\alpha} = 2r \partial_r + 2\alpha + n - 2. \]

For \( k \in \mathbb{Z}_+ \), a straightforward computation yields (also see [10, Lemma 2.4])
\[ A_{\alpha}(\varphi \log^k r) = A_{\alpha} \varphi \log^k r + k B_{\alpha} \varphi \log^{k-1} r + k(k - 1) \varphi \log^{k-2} r. \]
From this, for $\alpha, \beta, \gamma \in \mathbb{R}$ we get

$$A_\gamma A_\beta A_\alpha (\varphi \log^k r)$$

$$= A_\gamma A_\beta A_\alpha \varphi \log^k r$$

$$+ k(B_\gamma A_\beta A_\alpha + A_\gamma B_\beta A_\alpha + A_\gamma A_\beta B_\alpha)\varphi \log^{k-1} r$$

$$+ k(k - 1)(A_\beta A_\alpha + B_\gamma A_\beta A_\alpha + A_\gamma B_\beta A_\alpha + A_\gamma A_\beta B_\alpha + A_\gamma A_\beta B_\alpha)\varphi \log^{k-2} r$$

$$+ k(k - 1)(k - 2)(B_\gamma A_\alpha + A_\gamma B_\alpha + B_\gamma A_\alpha + B_\gamma B_\beta A_\alpha + B_\gamma B_\beta A_\alpha + A_\gamma B_\alpha + A_\gamma B_\alpha)\varphi \log^{k-3} r$$

$$+ k(k - 1)(k - 2)(k - 3)(A_\alpha + A_\beta + A_\gamma + B_\gamma B_\beta A_\alpha + B_\gamma B_\beta A_\alpha)\varphi \log^{k-4} r$$

$$+ k(k - 1)(k - 2)(k - 3)(k - 4)(B_\alpha + B_\beta + B_\gamma)\varphi \log^{k-5} r$$

$$+ k(k - 1)(k - 2)(k - 3)(k - 4)(k - 5)\varphi \log^{k-6} r.$$  \hfill (2.1)

A direct computation yields

$$\Delta_0^3(r^\alpha \varphi) = r^{\alpha - 2} A_\alpha \varphi,$$

$$\Delta_0^2(r^\alpha \varphi) = \Delta_0^3(r^{\alpha - 2} A_\alpha \varphi) = r^{\alpha - 4} A_\alpha - 2 A_\alpha \varphi,$$

$$\Delta_0^3(r^\alpha \varphi) = r^{\alpha - 6} A_\alpha - 4 A_\alpha - 2 A_\alpha \varphi.$$  \hfill (2.1)

In particular,

$$\Delta_0^3(r^{6-n} \varphi) = r^{-n} A_{2-n} A_{4-n} A_{6-n} \varphi.$$  \hfill (2.1)

Define

$$M_g := \Delta_g \delta T_2 d + \delta T_2 d \Delta_g + \frac{n - 2}{2} \Delta_g (\sigma_1 (A) \Delta_g) + \delta T_4 d,$$

then rewrite (1.2) as $-P_g = (\Delta_g)^3 + M_g - \frac{n - 6}{2} Q_g.$ Notice that

$$A_{\alpha, g} = A_\alpha + r^2 (\Delta_g - \Delta_0) = A_\alpha + r^2 \partial_i ((g^{ij} - \delta^{ij}) \partial_j),$$

$$-P_g(r^\alpha \varphi) = r^{\alpha - 6} (A_{\alpha - 4} A_{\alpha - 2} A_\alpha \varphi + K_\alpha \varphi),$$

where

$$K_\alpha \varphi = r^2 (\Delta_g - \Delta_0) A_{\alpha - 2} A_\alpha \varphi + A_{\alpha - 4} (r^2 (\Delta_g - \Delta_0)) A_\alpha \varphi + A_{\alpha - 4} A_{\alpha - 2} (r^2 (\Delta_g - \Delta_0)) \varphi$$

$$+ r^{6-\alpha} M_g (r^\alpha \varphi) - \frac{n - 6}{2} r^6 Q_g \varphi.  \hfill (2.2)$$

We first state the expression of $P_g(r^{6-n})$ and leave complicate computations to Appendix A.

**Lemma 2.1.** Under conformal normal coordinates around $p$ with metric $g$, there holds

$$-P_g(r^{6-n})$$

$$= -c_\alpha \delta_p + (n - 6) r^{-n} \left\{ \frac{64(n - 4)}{9} \sum_{k,l} (W_{ikl}(p) x^i x^j)^2 - \frac{r^2}{n + 4} \sum_{k,l,s} ((W_{ikls}(p) + W_{ilksp}(p)) x^j)^2 \right\}$$

$$+ \frac{3}{2(n + 4)(n + 2)} |W(p)|^2 r^4 + \frac{16(3n - 20)}{9(n + 4)} r^2 \sum_{k,l,s} ((W_{ikls}(p) + W_{ilksp}(p)) x^j)^2 - \frac{3}{n} |W(p)|^2 r^2 \right\}$$
\[-4(5n^2 - 66n + 224)r^2 \left[ \sigma_1(A)_{ij}(p)x^ix^j + \frac{|W(p)|^2}{12n(n-1)}r^2 \right] + \frac{3n^4 - 16n^3 - 164n^2 + 400n + 2432}{3(n+4)(n+2)n(n-1)}|W(p)|^2r^4 \right\} + O(r^{5-n}),\]

where $W_{ijkl}$ is the Weyl tensor of metric $g$ and each term in square brackets on the right hand side of the identity is harmonic polynomial.

Consequently, we rewrite the above equation in Lemma 2.1 as

$$P_g(r^{6-n}) = c_n\delta_p + r^{-n}f$$

with $f = O(r^4)$.

Observe that for $i = 0, 1, \cdots, \lfloor \frac{m}{2} \rfloor$,

$$A_\alpha|_{r^{2i}H_{m-2i}} = (\alpha + 2i)(2m - 2i + \alpha + n - 2)$$

and

$$B_\alpha|_{r^{2i}H_{m-2i}} = 2m + 2\alpha + n - 2,$$

then it yields

$$A_{2-n}A_{4-n}A_{6-n}|_{r^{2i}H_{m-2i}} = (6 - n + 2i)(4n + 2i)(2 - n + 2i)(2m + 4 - 2i)(2m + 2 - 2i)(2m - 2i). \quad (2.3)$$

We start to find a formal asymptotic solution like $G_p(x) = r^{6-n}(1 + \sum_{k=1}^{n} \psi_k) + \varphi$ with $\psi_k \in \mathcal{P}_k$. If we can find $\bar{\psi} = \sum_{k=1}^{n} \psi_k$ such that

$$A_{2-n}A_{4-n}A_{6-n}\bar{\psi} + K_{6-n}\bar{\psi} + f = O(r^{n+1}), \quad (2.4)$$

the regularity theory for elliptic equations gives that there exists a solution $\varphi \in C^{6,\alpha}_{loc}$ for any $0 < \alpha < 1$ to

$$P_g(\varphi) = -r^{-n}(A_{2-n}A_{4-n}A_{6-n}\bar{\psi} + K_{6-n}\bar{\psi} + f) \in C^{\alpha}_{loc}.$$

Thus it only remains to seek $\bar{\psi}$ satisfying (2.4) via induction. For any nonnegative integer $k$, it is not hard to see from the definition (2.2) for $K_{6-n}$ that $K_{6-n}\varphi \in \mathcal{P}_{k+2}$ when $\varphi \in \mathcal{P}_k$. We first set $\psi_1 = \psi_2 = \psi_3 = 0$ by (2.4) and define

$$f_3 = f = O(r^4).$$

**Case 1.** $n$ is odd.

If we have found $\psi_1, \cdots, \psi_k$ for $3 \leq k \leq n - 1$ with $\psi_k \in \mathcal{P}_k$ and

$$f_k = A_{2-n}A_{4-n}A_{6-n}\left( \sum_{i=1}^{k} \psi_i \right) + K_{6-n}\left( \sum_{i=1}^{k} \psi_i \right) + f := b_{k+1} + O(r^{k+2}),$$

- 4(5n^2 - 66n + 224)r^2 \left[ \sigma_1(A)_{ij}(p)x^ix^j + \frac{|W(p)|^2}{12n(n-1)}r^2 \right] + \frac{3n^4 - 16n^3 - 164n^2 + 400n + 2432}{3(n+4)(n+2)n(n-1)}|W(p)|^2r^4 \right\} + O(r^{5-n}),\]
then it follows from (2.3) that $A_{2-n}A_{4-n}A_{6-n}$ is invertible on $\mathcal{P}_{k+1}$ for $0 \leq k \leq n - 1$. Thus there exists a unique $\psi_{k+1} \in \mathcal{P}_{k+1}$ such that

$$A_{2-n}A_{4-n}A_{6-n}\psi_{k+1} + b_{k+1} = 0.$$ 

This implies that

$$f_{k+1} = A_{2-n}A_{4-n}A_{6-n} \left( \sum_{i=1}^{k+1} \psi_i \right) + K_{6-n} \left( \sum_{i=1}^{k+1} \psi_i \right) + f$$

$$= f_k + A_{2-n}A_{4-n}A_{6-n}\psi_{k+1} + K_{6-n}\psi_{k+1}$$

$$= O(r^{k+2}).$$

This finishes the induction and assertion (a) follows.

**Case 2.** $n$ is even and not less than 10.

Since $A_{2-n}A_{4-n}A_{6-n}$ is invertible on $\mathcal{P}_k$ for $0 \leq k \leq n - 7$, by the same induction in Case 1, we may find $\psi_1, \cdots, \psi_{n-7}$ such that

$$f_{n-7} = A_{2-n}A_{4-n}A_{6-n} \left( \sum_{k=1}^{n-7} \psi_k \right) + K_{6-n} \left( \sum_{k=1}^{n-7} \psi_k \right) + f = O(r^{n-6})$$

$$= b_{n-6} + O(r^{n-5}).$$

Let $\psi_{n-6}^{(0)} = \alpha_{n-6}^{(0)}(x) + \beta_{n-6}^{(0)}(x) \log r$, where $\alpha_{n-6}^{(0)}(x) \in \mathcal{P}_{n-6} \setminus r^{n-6}\mathcal{H}_0$ and $\beta_{n-6}^{(0)}(x) \in r^{n-6}\mathcal{H}_0$, then it yields from (2.1) that

$$A_{2-n}A_{4-n}A_{6-n}\psi_{n-6}^{(0)}$$

$$= A_{2-n}A_{4-n}A_{6-n}\alpha_{n-6} + (B_{2-n}A_{4-n}A_{6-n} + A_{2-n}B_{4-n}A_{6-n} + A_{2-n}A_{4-n}B_{6-n})\beta_{n-6}^{(0)}.$$ 

Notice that for $0 \leq i \leq \frac{n-8}{2}$, there hold

$$A_{2-n}A_{4-n}A_{6-n} \mid_{r^i\mathcal{H}_{n-2i}} \neq 0$$

by (2.3) and

$$(B_{2-n}A_{4-n}A_{6-n} + A_{2-n}B_{4-n}A_{6-n} + A_{2-n}A_{4-n}B_{6-n}) \mid_{r^{n-6}\mathcal{H}_0} = 8(n-2)(n-4)(n-6) \neq 0.$$ 

Hence there exists a unique $\psi_{n-6}^{(0)} \in \mathcal{P}_{n-6} + \mathcal{P}_{n-6}\log r$ to

$$A_{2-n}A_{4-n}A_{6-n}\psi_{n-6}^{(0)} + b_{n-6} = 0.$$ 

This indicates that

$$f_{n-6} = f_{n-7} + A_{2-n}A_{4-n}A_{6-n}\psi_{n-6}^{(0)} + K_{6-n}\psi_{n-6}^{(0)}$$

$$= O(r^{n-5}) + (K_{6-n}\beta_{n-6}^{(0)}) \log r.$$
\begin{align*}
&:= b_{n-5} + O(r^{n-4}) \log r + O(r^{n-4}). \tag*{9}
\end{align*}

Let \( \psi_{n-5}^{(0)} = \alpha_{n-5}^{(0)} + \beta_{n-5}^{(0)} \log r \), where \( \alpha_{n-5}^{(0)} \in \mathcal{P}_{n-5} \setminus r^{n-6} \mathcal{H}_1 \) and \( \beta_{n-5}^{(0)} \in r^{n-6} \mathcal{H}_1 \), then it yields
\[
A_{2-n} A_{4-n} A_{6-n} \psi_{n-5}^{(0)} = A_{2-n} A_{4-n} A_{6-n} \alpha_{n-5}^{(0)} + (B_{2-n} A_{4-n} A_{6-n} + A_{2-n} B_{4-n} A_{6-n} + A_{2-n} A_{4-n} B_{6-n}) \beta_{n-5}^{(0)}.
\]

By similar arguments, there exists a unique \( \psi_{n-5}^{(0)} \in \mathcal{P}_{n-5} + r^{n-6} \mathcal{H}_1 \log r \) such that
\[
A_{2-n} A_{4-n} A_{6-n} \psi_{n-5}^{(0)} + b_{n-5} = 0.
\]

This implies that
\[
f_{n-5} = f_{n-6} + A_{2-n} A_{4-n} A_{6-n} \psi_{n-5}^{(0)} + K_{6-n} \psi_{n-5}^{(0)} = O(r^{n-4}) \log r + O(r^{n-4}) := b_{n-4}^{(1)} \log r + O(r^{n-4}) + O(r^{n-3}) \log r.
\]

Choose \( \psi_{n-4}^{(1)} = \alpha_{n-4}^{(1)} \log r + \beta_{n-4}^{(1)} \log^2 r \in \mathcal{P}_{n-4} \log r + (r^{n-6} \mathcal{H}_2 + r^{n-4} \mathcal{H}_0) \log^2 r \), then (2.1) gives
\[
A_{2-n} A_{4-n} A_{6-n} \psi_{n-4}^{(1)} = \left[ A_{2-n} A_{4-n} A_{6-n} \alpha_{n-4}^{(1)} + 2(B_{6-n} A_{4-n} A_{2-n} + A_{6-n} B_{4-n} A_{2-n} + A_{6-n} A_{4-n} B_{2-n}) \beta_{n-4}^{(1)} \right] \log r
\]
\[
+ A_{2-n} A_{4-n} A_{6-n} \beta_{n-4}^{(1)} \log^2 r + O(r^{n-4}).
\]

Since
\[
(B_{6-n} A_{4-n} A_{2-n} + A_{6-n} B_{4-n} A_{2-n} + A_{6-n} A_{4-n} B_{2-n})|_{r^{n-6} \mathcal{H}_2} = 8(n + 2)n(n - 2) \neq 0;
\]
\[
(B_{6-n} A_{4-n} A_{2-n} + A_{6-n} B_{4-n} A_{2-n} + A_{6-n} A_{4-n} B_{2-n})|_{r^{n-4} \mathcal{H}_0} = -4n(n - 2)(n - 4) \neq 0
\]
and \( A_{2-n} A_{4-n} A_{6-n} |_{r^{2i} \mathcal{H}_{n-4-2i}} \neq 0 \) for \( 0 \leq i \leq \frac{n-8}{2} \), then there exists a unique \( \psi_{n-4}^{(1)} \) such that
\[
A_{2-n} A_{4-n} A_{6-n} \alpha_{n-4}^{(1)} + 2(B_{6-n} A_{4-n} A_{2-n} + A_{6-n} B_{4-n} A_{2-n} + A_{6-n} A_{4-n} B_{2-n}) \beta_{n-4}^{(1)} + b_{n-4}^{(1)} = 0
\]
and
\[
f_{n-4}^{(1)} = f_{n-5} + A_{2-n} A_{4-n} A_{6-n} \psi_{n-4}^{(1)} + K_{6-n} \psi_{n-4}^{(1)} = O(r^{n-4}) + O(r^{n-3}) \log r + O(r^{n-2}) \log^2 r
\]
\[
:= b_{n-4}^{(0)} + O(r^{n-3}) \log r + O(r^{n-3}) + O(r^{n-2}) \log^2 r.
\]

Choose \( \psi_{n-4}^{(0)} \in \mathcal{P}_{n-4} + (r^{n-6} \mathcal{H}_2 + r^{n-4} \mathcal{H}_0) \log r \) to remove the term \( b_{n-4}^{(0)} \) and set
\[
f_{n-4}^{(0)} = f_{n-4}^{(1)} + A_{2-n} A_{4-n} A_{6-n} \psi_{n-4}^{(0)} + K_{6-n} \psi_{n-4}^{(0)} = O(r^{n-3}) \log r + O(r^{n-3}) + O(r^{n-2}) \log^2 r.
\]
By similar arguments and (2.1), we get

\[
\begin{align*}
\psi_{n-3}^{(1)} & \in \mathcal{P}_{n-3} \log r + (r^{n-6} \mathcal{H}_3 + r^{n-4} \mathcal{H}_1) \log^2 r; \\
\psi_{n-3}^{(0)} & \in \mathcal{P}_{n-3} + (r^{n-6} \mathcal{H}_3 + r^{n-4} \mathcal{H}_1) \log r; \\
\psi_{n-2}^{(i)} & \in \mathcal{P}_{n-2} \log^i r + (r^{n-6} \mathcal{H}_4 + r^{n-4} \mathcal{H}_2 + r^{n-2} \mathcal{H}_0) \log^{i+1} r, \quad \text{for } i = 0, 1, 2; \\
\psi_{n-1}^{(i)} & \in \mathcal{P}_{n-1} \log^i r + (r^{n-6} \mathcal{H}_5 + r^{n-4} \mathcal{H}_3 + r^{n-2} \mathcal{H}_1) \log^{i+1} r, \quad \text{for } i = 0, 1, 2; \\
\psi_n^{(i)} & \in \mathcal{P}_n \log^i r + (r^{n-6} \mathcal{H}_6 + r^{n-4} \mathcal{H}_4 + r^{n-2} \mathcal{H}_2) \log^{i+1} r, \quad \text{for } i = 0, 1, 2, 3.
\end{align*}
\]

Now we set

\[
\psi_{n-6}^{(i)} = \psi_{n-6}^{(0)}, \psi_{n-5}^{(i)} = \psi_{n-5}^{(0)}, \psi_{n-4}^{(i)} = \psi_{n-4}^{(0)}, \psi_{n-3}^{(i)} = \psi_{n-3}^{(0)} + \psi_{n-3}^{(1)}
\]

and

\[
\begin{align*}
\psi_n^{(i)} &= \sum_{i=0}^{2} \psi_n^{(i)}, \quad \psi_{n-1}^{(i)} = \sum_{i=0}^{2} \psi_{n-1}^{(i)}, \quad \psi_n = \sum_{i=0}^{3} \psi_n^{(i)}.
\end{align*}
\]

Eventually, we obtain

\[
\begin{align*}
f_n &= A_{2-n} A_{4-n} A_{6-n} \left( \sum_{k=1}^{n} \psi_k \right) + K_{6-n} \left( \sum_{k=1}^{n} \psi_k \right) + f \\
&= O\left(r^{n+1}\right)\left(\log^3 r + \log^2 r + \log r + 1\right) + O\left(r^{n+2}\right)\log^4 r.
\end{align*}
\]

Hence, \(r^{-n}f_n \in C^\alpha\) for any \(0 < \alpha < 1\). This finishes the induction and we obtain assertion (b) as desired.

**Case 3.** \(n = 8\).

Notice that

\[
P_g\left(G_p(x) - c_n r^{-2}\right) = O\left(r^{-4}\right) \in L^p,
\]

for some \(\frac{8}{5} < p < 2\), then it follows from regularity theory of elliptic equations that \(G_p(x) - c_n r^{-2} \in C^{6-\frac{8}{p}}_{\text{loc}}\). From this, we have

\[
G_p(x) = c_n r^{-2} + A + O(r).
\]

**Case 4.** \(M\) is locally conformal flat.

One may choose \(g\) flat near \(p\) and \(P_g = -\Delta_0^3\). Hence, one has \(P_g\left(G(x) - c_n r^{6-n}\right) = 0\) and then \(G_p(x) - c_n r^{6-n}\) is smooth near \(p\).

Therefore, the assertion (c) follows from cases 1, 3, 4. In some special cases, the leading term \(\psi_4\) can be computed with the help of Lemma 2.1. The proof of Proposition 2.1 is complete.
3  \( n \geq 10 \) and not locally conformally flat

Similar to the Yamabe constant, for \( n \geq 3 \) and \( n \neq 4, 6 \), we define

\[
Y^+_6(M, g) = \inf_{0 < u \in H^3(M, g)} \frac{\int_M u P_g u d\mu_g}{\left( \int_M u^{\frac{2n}{n-6}} d\mu_g \right)^{\frac{n-6}{n}}}
\]

It follows from (1.3) that \( Y^+_6(M, g) \) is a conformal invariant. However, due to the lack of a maximum principle for higher order elliptic equations in general, we first study another conformally invariant quantity

\[
Y_6(M, g) = \inf_{u \in H^3(M, g) \setminus \{0\}} \frac{\int_M u P_g u d\mu_g}{\left( \int_M |u|^{\frac{2n}{n-6}} d\mu_g \right)^{\frac{n-6}{n}}}
\]

In particular, we have \( Y_6(S^n) = Y^+_6(S^n) = \frac{n-6}{2} Q_{S^n, \omega_{S^n}} \). For \( w \in C^\infty_c(\mathbb{R}^n) \), let

\[
\|w\|_{D^{3,2}} := \sum_{|\beta|=3} \|D^\beta w\|_{L^2(\mathbb{R}^n)} \approx \|\nabla \Delta w\|_{L^2(\mathbb{R}^n)}
\]

and let \( D^{3,2}(\mathbb{R}^n) \) denote the completion of \( C^\infty_c(\mathbb{R}^n) \) under this norm. The equivalence of the above last two norms can be easily deduced by the formula (3.4) below. We first recall an optimal Euclidean Sobolev inequality (cf. [15, p.154-165], [14]).

**Lemma 3.1.** For \( n \geq 7 \), the following sharp Sobolev embedding inequality holds

\[
Y_6(S^n) \left( \int_{\mathbb{R}^n} |w|^{\frac{2n}{n-6}} dy \right)^{\frac{n-6}{n}} \leq \int_{\mathbb{R}^n} |\nabla \Delta w|^2 dy \text{ for all } w \in D^{3,2}(\mathbb{R}^n).
\]

The equality holds if and only if \( w(y) = \left( \frac{2}{2+|y|^2} \right)^{\frac{n-6}{2}} \) up to any nonzero constant multiple, as well as all translations and dilations.

**Proposition 3.1.** On a closed Riemannian manifold \((M^n, g)\) of dimension \( n \geq 10 \), if there exists \( p \in M^n \) such that the Weyl tensor \( W_g(p) \neq 0 \), then \( Y_6(M^n) < Y_6(S^n) \).

**Proof.** Recall the definition of \( P_g \):

\[
-P_g = \Delta_9^3 + \Delta_9 \delta T_2 d + \delta T_2 d \Delta_9 + \frac{n-2}{2} \Delta_9 (\sigma_1(A) \Delta_9) + \delta T_4 d - \frac{n-6}{2} Q_g,
\]

then for all \( \varphi \in H^3(M, g) \),

\[
\int_M \varphi P_g \varphi d\mu_g = \int_M |\nabla \Delta \varphi|^2 d\mu_g - 2 \int_M T_2 (\nabla \varphi, \nabla \Delta \varphi) d\mu_g - \frac{n-2}{2} \int_M \sigma_1(A)(\Delta \varphi)^2 d\mu_g
\]

\[
- \int_M T_4 (\nabla \varphi, \nabla \varphi) d\mu_g + \frac{n-6}{2} \int_M Q_g \varphi^2 d\mu_g.
\]

Fix \( \rho > 0 \) small and choose test functions

\[
\varphi(x) = \eta_0(x) u_\epsilon(x), \quad u_\epsilon(x) = \left( \frac{2\epsilon}{\epsilon^2 + |x|^2} \right)^{\frac{n-6}{2}}, \quad \epsilon > 0,
\]
where \( r = |x| = d_g(x, p) \) and

\[
\eta_\rho \in C^\infty_c, \ 0 \leq \eta_\rho \leq 1, \ \eta_\rho \equiv 1 \ \text{in} \ B_\rho \ \text{and} \ \eta_\rho \equiv 0 \ \text{in} \ B^c_{2\rho}.
\]

It is known from Lee-Park [12] that up to a conformal factor, under conformal normal coordinates around \( p \) with metric \( g \), for all \( N \geq 5 \) there hold

\[
\sigma_1(A_g)(p) = 0, \ \sigma_1(A_g)_t(p) = 0, \ \Delta_g \sigma_1(A_g)(p) = -\frac{|W(p)|_g^2}{12(n-1)}
\]

and

\[
\sqrt{\det g} = 1 + O(r^N).
\]

Our purpose is to estimate

\[
\int_M \varphi P_g \varphi d\mu_g \ \text{and} \ \int_M \varphi^{2n} d\mu_g.
\]

A direct computation shows

\[
u'_\epsilon = -(n-6)\epsilon \frac{r}{\epsilon^2 + r^2}, \quad u''_\epsilon = -(n-6)\epsilon \frac{\epsilon^2 - (n-5)r^2}{(\epsilon^2 + r^2)^2}
\]

and

\[
\Delta_0 u_\epsilon = -(n-6)\frac{u_\epsilon}{(\epsilon^2 + r^2)^2}(n\epsilon^2 + 4r^2),
\]

\[
(\Delta_0 u_\epsilon)' = (n-6)(n-4)\frac{u_\epsilon r}{(\epsilon^2 + r^2)^3}[(n+2)\epsilon^2 + 4r^2].
\]

We start with \( \int_M |\nabla \Delta \varphi|^2_g d\mu_g \) and divide its integral into two parts \( \int_M = \int_{B_\rho} + \int_{M \setminus B_\rho} \). Compute with

\[
\int_{B_\rho} |\nabla \Delta \varphi|^2_g d\mu_g = \int_{B_\rho} g^{ij}(\Delta \varphi)_i(\Delta \varphi)_j d\mu_g
\]

\[
= \int_{B_\rho} (\delta^{ij} + O(r^2))((\Delta_0 \varphi + O(r^{-1}))(\Delta_0 \varphi + O(r^{-1}))(\varphi)_i(\varphi)_j(1 + O(r^N)) dx
\]

\[
= \int_{B_\rho} |(\nabla \Delta) \varphi|^2 d\mu_g + \int_{B_\rho} (\Delta_0 \varphi)'(O(r^{-2}) \varphi' + O(r^{-1}) \varphi'') dx
\]

and

\[
\int_{\mathbb{R}^n \setminus B_\rho} |(\nabla \Delta) \varphi|^2 dx = (n-6)^2(n-4)^2 \int_{\mathbb{R}^n \setminus B_\rho} \frac{u_\epsilon^2 r^2}{(\epsilon^2 + r^2)^6}[(n+2)\epsilon^2 + 4r^2]^2 dx
\]

\[
\leq C \int_{\rho/\epsilon}^\infty \sigma^{5-n} d\sigma = O(\epsilon^{n-6}).
\]

Similarly, we estimate

\[
\int_{M \setminus B_\rho} |\nabla \Delta \varphi|^2_g d\mu_g = O(\epsilon^{n-6}).
\]
Thus, we obtain
\[ \int_M |\nabla \Delta \varphi|^2 d\mu_g = \int_{\mathbb{R}^n} |\nabla \Delta_0 u|_2^2 dx + O(\epsilon^{n-6}). \]

Secondly, we compute
\[
\int_{B_{\rho}} \sigma_1(A)(\Delta \varphi)^2 d\mu_g = \int_{B_{\rho}} \left( \frac{1}{2} \sigma_1(A)_{ij}(p)x^ix^j + O(r^3)(\Delta_0 \varphi + O(r^{N-1}) \varphi')^2 (1 + O(r^N)) \right) dx
\]
\[
= \int_{B_{\rho}} \frac{1}{2n} \Delta \sigma_1(A)(p) |x|^2 (\Delta_0 \varphi)^2 dx + \int_{B_{\rho}} O(r^3) \frac{u^2}{(\epsilon^2 + r^2)^4} (n\epsilon^2 + 4r^2)^2 dx
\]
\[
= - \frac{(n - 6)^2 |W(p)|^2}{24n(n - 1)} \omega_{n-1} \int_0^\rho \frac{(n\epsilon^2 + 4r^2)^2}{(\epsilon^2 + r^2)^4} u^2_\rho r^{n+1} dr + \int_{B_{\rho}} O(r^3) u^2
\]
\]
\]
\]
\]
\]
\]
\]

and for some large enough \( N \)
\[
\int_{B_{2\rho}\setminus B_{\rho}} \sigma_1(A)(\Delta \varphi)^2 d\mu_g \leq C \int_{B_{2\rho}\setminus B_{\rho}} [\Delta_0 \varphi + O(r^{N-1}) \varphi']^2 (1 + O(r^N)) dx
\]
\[
\leq C \int_{B_{2\rho}\setminus B_{\rho}} [(\Delta_0 \varphi)^2 + O(r^{2(N-1)}) |\varphi'|^2] dx
\]
\[
\leq C \int_{B_{2\rho}\setminus B_{\rho}} \left( u^2_\rho \Delta_0 \eta_\rho + 2 \nabla u_\epsilon \cdot \nabla \eta_\rho + \eta_\rho \Delta_0 u_\epsilon \right)^2 dx + O(\epsilon^{n-6})
\]
\[
\leq C \int_{B_{2\rho}\setminus B_{\rho}} \frac{2\rho (n\epsilon^2 + 4r^2)^2}{(\epsilon^2 + r^2)^4} u^2_\rho r^{n+1} dr + O(\epsilon^{n-6})
\]
\[
\leq C \epsilon^2 \int_{\rho/\epsilon}^{2\rho/\epsilon} \frac{(n + 4\sigma^2)^2}{1 + \sigma^2} \sigma^{n-1} d\sigma + O(\epsilon^{n-6})
\]
\[
\leq C \epsilon^2 \frac{(\rho/\epsilon)^{8-n}}{1} + O(\epsilon^{n-6}) = O(\epsilon^{n-6}).
\]

Observe that
\[
\int_{B_{\rho}} \frac{r^3 u^2_\rho}{(\epsilon^2 + r^2)^2} dx = \begin{cases} O(\epsilon^4) & \text{if } n = 10, \\ O(\epsilon^5 |\log \epsilon|) & \text{if } n = 11, \\ O(\epsilon^5) & \text{if } n \geq 12. \end{cases}
\]

Hence,
\[
- \frac{n - 2}{2} \int_M \sigma_1(A)(\Delta \varphi)^2 d\mu_g
\]
\[
= \frac{(n - 6)^2(n - 2)|W(p)|^2}{48n(n - 1)} \omega_{n-1} \int_0^\rho \frac{(n\epsilon^2 + 4r^2)^2}{(\epsilon^2 + r^2)^4} u^2_\rho r^{n+1} dr + \begin{cases} O(\epsilon^4) & \text{if } n = 10, \\ O(\epsilon^5 |\log \epsilon|) & \text{if } n = 11, \\ O(\epsilon^5) & \text{if } n \geq 12. \end{cases}
\]

Thirdly, we compute \( \int_M T_2(\nabla \varphi, \nabla \Delta \varphi) d\mu_g \).
\[
\int_{B_{\rho}} T_2(\nabla \varphi, \nabla \Delta \varphi) d\mu_g = \int_{B_{\rho}} \left[ (n - 2) \sigma_1(A) \langle \nabla \varphi, \nabla \Delta \varphi \rangle - 8 A_{ij} \varphi_{,i} \varphi_{,j} (\Delta \varphi) \right] d\mu_g.
\]
Observe that $u_{\epsilon,i} = \frac{x^i}{r} u_{\epsilon}'$ and $(\Delta_0 u_{\epsilon})_i = \frac{x^i}{r} (\Delta_0 u_{\epsilon})'$. Then we get

\[
\begin{aligned}
(n - 2) \int_{B_\rho} \sigma_1(A)(\nabla \varphi \cdot \nabla \Delta \varphi) \, d\mu_g \\
= (n - 2) \int_{B_\rho} \left( \frac{1}{2} \sigma_1(A)_{ij}(p) x^i x^j + O(r^3) \right) g^{kl} \varphi_{,k}(\Delta \varphi)_{,l} \, d\mu_g \\
= (n - 2) \int_{B_\rho} \left( \frac{1}{2} \sigma_1(A)_{ij}(p) x^i x^j + O(r^3) \right) (\delta^{kl} + O(r^2)) \varphi_{,k}(\Delta_0 \varphi + O(r^{N-1}) \varphi')_{,l} \, d\mu_g \\
= \frac{n - 2}{2} \int_{B_\rho} \frac{1}{n} \Delta \sigma_1(A)(p) |x|^2 \varphi_{,i}(\Delta_0 \varphi)_i \, dx + \int_{B_\rho} O(r^3) |\varphi'| |(\Delta_0 \varphi)'| \, dx \\
= -\frac{(n - 2)|W(p)|^2}{24n(n - 1)} \int_{B_\rho} \left\{ -(n - 6)^2(n - 4) \frac{u_{\epsilon}^2 r^4}{(\epsilon^2 + r^2)^4} \left[ (n + 2) \epsilon^2 + 4r^2 \right] \right\} \, dx \\
+ \int_{B_\rho} \frac{O(r^3) u_{\epsilon}^2}{(\epsilon^2 + r^2)^2} \, dx \\
= \frac{(n - 2)(n - 4)(n - 6)^2}{24n(n - 1)} |W(p)|^2 \int_{B_\rho} \frac{r^4}{(\epsilon^2 + r^2)^4} u_{\epsilon}^2 \left[ (n + 2) \epsilon^2 + 4r^2 \right] \, dx \\
+ \int_{B_\rho} \frac{O(r^3) u_{\epsilon}^2}{(\epsilon^2 + r^2)^2} \, dx,
\end{aligned}
\]

and

\[
-8 \int_{B_\rho} A_{ij} \varphi_{,i}(\Delta \varphi)_{,j} \, d\mu_g \\
= -8 \int_{B_\rho} \left( A_{ij,k}(p) x^k + \frac{1}{2} A_{ij,kl}(p) x^k x^l + O(r^3) \right) \varphi_{,i}(\Delta_0 \varphi + O(r^{N-1}) \varphi')_{,j} \, d\mu_g \\
= -4 \int_{B_\rho} A_{ij,kl}(p) x^k x^l x^j \left[ -(n - 4)(n - 6)^2 \frac{u_{\epsilon}^2}{(\epsilon^2 + r^2)^4} \left[ (n + 2) \epsilon^2 + 4r^2 \right] \right] \, dx \\
+ \int_{B_\rho} O(r^3) |\varphi'| |(\Delta_0 \varphi)'| \, dx \\
= 4(n - 4)(n - 6)^2 \int_{B_\rho} \left[ -\frac{1}{9n - 2} \sum_{k,l} (W_{ki,j}(p) x^i x^j)^2 - \frac{\sigma_1(A)_{ij}(p) x^i x^j r^2}{n - 2} \right] \times \frac{u_{\epsilon}^2}{(\epsilon^2 + r^2)^4} \left[ (n + 2) \epsilon^2 + 4r^2 \right] \, dx \\
+ \int_{B_\rho} \frac{O(r^3) u_{\epsilon}^2}{(\epsilon^2 + r^2)^2} \, dx \\
= -\frac{8(n - 4)(n - 6)^2}{9(n - 2)} \int_{B_\rho} \left[ \sum_{k,l} (W_{ki,j}(p) W_{kl,j}(p) x^i x^j x^k x^l) \frac{u_{\epsilon}^2}{(\epsilon^2 + r^2)^4} \left[ (n + 2) \epsilon^2 + 4r^2 \right] \right] \, dx \\
- \frac{4(n - 4)(n - 6)^2}{n(n - 2)} \int_{B_\rho} \frac{\Delta \sigma_1(A)(p) r^4}{(\epsilon^2 + r^2)^4} u_{\epsilon}^2 \left[ (n + 2) \epsilon^2 + 4r^2 \right] \, dx \\
+ \int_{B_\rho} \frac{O(r^3) u_{\epsilon}^2}{(\epsilon^2 + r^2)^2} \, dx \\
= -\frac{(n - 4)(n - 6)^2}{(n - 1)n(n + 2)} \omega_{n-1} |W(p)|^2 \int_0^\rho \frac{r^{n+3} u_{\epsilon}^2}{(\epsilon^2 + r^2)^4} \left[ (n + 2) \epsilon^2 + 4r^2 \right] \, dr \\
+ \int_{B_\rho} \frac{O(r^3) u_{\epsilon}^2}{(\epsilon^2 + r^2)^2} \, dx,
Then we have

\[
\sum_{k,l} W_{iklj}(p) W_{sklt}(p) \int_{B_{\rho}} x^i x^j x^s x^t \frac{u^2}{(\epsilon^2 + r^2)^4} [(n + 2)\epsilon^2 + 4r^2] dx
\]

\[
= \sum_{k,l} W_{iklj}(p) W_{sklt}(p) \int_{S_{n-1}} \xi^i \xi^j \xi^s \xi^t d\mu_{S_{n-1}} \int_0^\rho r^{n+3} \frac{u^2}{(\epsilon^2 + r^2)^4} [(n + 2)\epsilon^2 + 4r^2] dr
\]

\[
= -\frac{\omega_{n-1}}{n(n+2)} \sum_{k,l} W_{iklj}(p) W_{sklt}(p) \int_{S_{n-1}} \delta_{ij} \delta_{st} + \delta_{is} \delta_{jt} + \delta_{it} \delta_{js} \int_0^\rho r^{n+3} \frac{u^2}{(\epsilon^2 + r^2)^4} [(n + 2)\epsilon^2 + 4r^2] dr
\]

\[
= \frac{\omega_{n-1}}{n(n+2)} \left[ |W(p)|^2 + W_{iklj}(p) W_{jkl}(p) \right] \int_0^\rho r^{n+3} \frac{u^2}{(\epsilon^2 + r^2)^4} [(n + 2)\epsilon^2 + 4r^2] dr
\]

\[
= \frac{3}{2} \frac{\omega_{n-1}}{n(n+2)} |W(p)|^2 \int_0^\rho r^{n+3} \frac{u^2}{(\epsilon^2 + r^2)^4} [(n + 2)\epsilon^2 + 4r^2] dr.
\]

Then we have

\[
-2 \int_{B_{\rho}} T_2(\nabla \varphi, \nabla \Delta \varphi) d\mu_g = -\frac{\omega_{n-1}}{n(n+2)} |W(p)|^2 \omega_{n-1} \int_0^\rho r^{n+3} \frac{u^2}{(\epsilon^2 + r^2)^4} [(n + 2)\epsilon^2 + 4r^2] dr
\]

\[
+ \int_{B_{\rho}} O(r^3) u^2 \frac{1}{(\epsilon^2 + r^2)^2} dx.
\]

By a similar argument, one has

\[
\left| \int_{B_{2\rho}\setminus B_{\rho}} T_2(\nabla \varphi, \nabla \Delta \varphi) \right| \leq C \int_{B_{2\rho}\setminus B_{\rho}} |\nabla \varphi| |\nabla \Delta \varphi| d\mu_g
\]

\[
\leq C \int_{B_{2\rho}\setminus B_{\rho}} |u'| |(\Delta u)| dx + O(\epsilon^{n-6}) = O(\epsilon^{n-6}).
\]

Fourthly, we compute $\int_M T_4(\nabla \varphi, \nabla \varphi) d\mu_g$.

\[
(n - 6) \int_{B_{\rho}} \Delta \sigma_1(A) |\nabla \varphi|^2 d\mu_g
\]

\[
= (n - 6) \int_{B_{\rho}} (\Delta \sigma_1(A)(p) + O(r)) (|\varphi'|^2 + O(r^2) |\varphi|^2) dx
\]

\[
= -(n - 6)^3 |W(p)|^2 \int_{B_{\rho}} \frac{u^2 r^2}{(\epsilon^2 + r^2)^2} dx + \int_{B_{\rho}} O(r^3) u^2 \frac{1}{(\epsilon^2 + r^2)^2} dx.
\]

Using (A.5), we get

\[
-\frac{16}{n-4} \int_{B_{\rho}} B_{ij} \varphi_i \varphi_j d\mu_g = -\frac{16}{n-4} \int_{B_{\rho}} (n-6)^2 u^2 \frac{B_{ij} x^i x^j}{(\epsilon^2 + r^2)^2} dx
\]

\[
= -\frac{16(n-6)^2}{n-4} \int_{B_{\rho}} \left[ -\frac{2}{9} \frac{1}{n-2} \sum_{k,l,s} [(W_{iklt}(p) + W_{ikls}(p)) x^i]^2
\right]
\]
Also we have

\[ \frac{1}{12(n-2)(n-1)}|W(p)|^2 r^2 - \frac{7n-8}{n-2} \sigma_1(A, ij(p)x^i x^j + O(r^3)] \frac{u_x^2}{(\epsilon^2 + r^2)^2} dx \]

\[ = - \frac{16(n-6)^2}{n-4} \left[- \frac{2}{3w(n-2)} + \frac{1}{12(n-2)(n-1)} + \frac{7n-8}{12(n-2)(n-1)n} \right]|W(p)|^2 \int_{B_\rho} \frac{r^2 u_x^2}{(\epsilon^2 + r^2)^2} dx \]

\[ + \int_{B_\rho} O(r^3) u_x^2 \frac{dx}{(\epsilon^2 + r^2)^2} \]

\[ = \int_{B_\rho} O(r^3) u_x^2 \frac{dx}{(\epsilon^2 + r^2)^2} , \]

where the second identity follows from

\[ \sum_{i,k,l,s} (W_{ikls}(p) + W_{ilks}(p))^2 = 2|W(p)|^2 + 2 \sum_{i,k,l,s} W_{ikls}(p) W_{ilks}(p) = 3|W(p)|^2 , \]

in view of

\[ 0 = W_{ikls}(W_{ilks} + W_{iksl} + W_{iisl}) \]

\[ = W_{ikls} W_{ilks} + W_{ikls} W_{iksl} + W_{ilks} W_{iisl} \]

\[ = 2W_{ikls} W_{ilks} - |W|^2 \text{ at } p. \]

Also we have

\[ \int_{B_\rho} T_4(\nabla \varphi, \nabla \varphi) d\mu_g \leq C \int_{B_\rho} |\nabla \varphi|^2 d\mu_g = O(n^{-6}). \]

Hence, collecting the above terms together with (3.1), we obtain

\[ = - \int_{M} T_4(\nabla \varphi, \nabla \varphi) d\mu_g \]

\[ = - (n-6) \int_{B_\rho} \Delta \sigma_1(A) |\nabla \varphi|^2 d\mu_g + \frac{16}{n-4} \int_{B_\rho} B_{ijkl} \varphi_{ij} \varphi_{kl} d\mu_g + O(n^{-6}) \]

\[ = (n-6)^3 \frac{|W(p)|^2}{12(n-1)} \int_{B_\rho} \frac{u_x^2 r^2}{(\epsilon^2 + r^2)^2} dx + \begin{cases} 
O(\epsilon^4) & \text{if } n = 10, \\
O(\epsilon^5 |\log \epsilon|) & \text{if } n = 11, \\
O(\epsilon^5) & \text{if } n \geq 12. 
\end{cases} \]

Finally, we compute \( \frac{n-6}{2} \int_{M} Q_g \varphi^2 d\mu_g \). By the definition (1.1) of \( Q_g \), integration by parts gives

\[ \frac{n-6}{2} \int_{M} Q_g \varphi^2 d\mu_g \]

\[ = \frac{n-6}{2} \int_{M} \Delta^2 \sigma_1(A) \varphi^2 d\mu_g + \int_{B_\rho} \frac{O(r^3) u_x^2}{(\epsilon^2 + r^2)^2} dx + O(n^{-6}) \]

\[ = \frac{n-6}{2} \int_{M} \Delta \sigma_1(A) \Delta \varphi^2 d\mu_g + \int_{B_\rho} \frac{O(r^3) u_x^2}{(\epsilon^2 + r^2)^2} dx + O(n^{-6}) \]

\[ = - \frac{(n-6)^2 |W(p)|^2}{12(n-1)} \int_0^\rho \frac{u_x^2 r^{n-1}}{(\epsilon^2 + r^2)^2} [(n-10)r^2 - nr^2] dr + \begin{cases} 
O(\epsilon^4) & \text{if } n = 10, \\
O(\epsilon^5 |\log \epsilon|) & \text{if } n = 11, \\
O(\epsilon^5) & \text{if } n \geq 12, 
\end{cases} \]
by (3.1), where the last identity follows from
\[
\frac{n - 6}{2} \int_{B_\rho} \Delta \sigma_1(A) \Delta \varphi^2 d\mu_g = \frac{n - 6}{2} \int_{B_\rho} (\Delta \sigma_1(A)(p) + O(r))(\Delta_0 \varphi^2 + O(r^{N-1})(\varphi^2)'dx \\
= \frac{n - 6}{2} \Delta \sigma_1(A)(p) \int_{B_\rho} 2(\varphi \Delta_0 \varphi + |\nabla \varphi|^2)dx + \int_{B_\rho} \frac{O(r)u_\epsilon^2}{\epsilon^2 + r^2} dx \\
= - \frac{(n - 6)^2|W(p)|^2}{12(n - 1)} \int_0^\rho u_\epsilon^2 r^{n-1}[(n - 10)r^2 - n\epsilon^2]dr + \int_{B_\rho} \frac{O(r)u_\epsilon^2}{\epsilon^2 + r^2} dx
\]
and the first identity follows from
\[
\left| \int_{B_\rho \setminus B_\rho} Q_g \varphi^2 d\mu_g \right| \leq C \int_{B_\rho \setminus B_\rho} u_\epsilon^2 dx = O(\epsilon^{n-6}).
\]
Therefore collecting all the above terms together, we obtain
\[
\int_M \varphi P_g \varphi d\mu_g = \int_{\mathbb{R}^n} |\nabla \Delta_0 u_\epsilon|^2 dx + A_{n,\rho,\epsilon}|W(p)|^2\omega_{n-1} + O(\epsilon^{\min\{n-6,5\}},
\]
where \( A_{n,\rho,\epsilon} \) is a constant given by
\[
(n - 6)^2 \left\{ \frac{n - 2}{12(n - 1)} \int_0^\rho \frac{(n\epsilon^2 + 4\rho^2)^2}{(\epsilon^2 + r^2)^4} u_\epsilon^2 r^{n+1}dr + \frac{n - 6}{12(n - 1)} \int_0^\rho \frac{u_\epsilon^2 r^{n+1}}{(\epsilon^2 + r^2)^2} dr \\
- \frac{1}{12(n - 1)} \int_0^\rho \frac{u_\epsilon^2 r^{n+1}}{(\epsilon^2 + r^2)^2} [(n - 10)r^2 - n\epsilon^2]dr \\
- \frac{(n^2 - 28)(n - 4)}{12n(n - 1)(n + 2)} \int_0^\rho r^{n+3} \frac{u_\epsilon^2}{(\epsilon^2 + r^2)^4} [(n + 2)\epsilon^2 + 4\rho^2]dr \right\} \\
= 2^{n-6} \frac{(n - 6)^2}{12(n - 1)} \epsilon^4 \left\{ \frac{n - 2}{4n} \int_0^{\rho/\epsilon} \frac{(n + 4\epsilon^2)^2}{(1 + \epsilon^2)^4} (1 + \epsilon^2)^{-(n-5)}\sigma^{n+1}d\sigma \\
+ (n - 6) \int_0^{\rho/\epsilon} \frac{1}{(1 + \epsilon^2)^2} (1 + \epsilon^2)^{-(n-6)}\sigma^{n+1}d\sigma \\
- \int_0^{\rho/\epsilon} \frac{1}{(1 + \epsilon^2)^2} (1 + \epsilon^2)^{-(n-6)}\sigma^{n-1}[(n - 10)\epsilon^2 - n]d\sigma \\
- \frac{(n^2 - 28)(n - 4)}{n(n + 2)} \int_0^{\rho/\epsilon} \frac{\sigma^{n+3}}{(1 + \epsilon^2)^4} (1 + \epsilon^2)^{-(n-6)}[(n + 2) + 4\epsilon^2]d\sigma \right\}
\]
where \( r = \epsilon \sigma \). When \( n = 10 \), we claim that the leading term of the constant in the brace on the right hand side of the above identity:
\[
\frac{1}{5} \int_0^{\rho/\epsilon} \frac{(4\epsilon^2 + 10)^2}{(1 + \epsilon^2)^4} (1 + \epsilon^2)^{-4}\sigma^{10}d\sigma + \int_0^{\rho/\epsilon} \frac{1}{(1 + \epsilon^2)^2} (1 + \epsilon^2)^{-4}(4\epsilon^2 + 10)\sigma^9 d\sigma \\
- \frac{18}{5} \int_0^{\rho/\epsilon} \frac{1}{(1 + \epsilon^2)^4} (1 + \epsilon^2)^{-4}(4\epsilon^2 + 12)\sigma^{13}d\sigma
\]
With the help of Beta function:

\[ \frac{1}{5} \int_0^\infty \left\{ \sigma^2[(4\sigma^2 + 10)^2 - 18\sigma^2(4\sigma^2 + 12)] + 5(4\sigma^2 + 10)(1 + \sigma^2)^2 \right\} (1 + \sigma^2)^{-8} \sigma^9 d\sigma \]

\[ = \frac{1}{5} \int_0^\infty (-36\sigma^6 - 46\sigma^4 + 220\sigma^2 + 50)(1 + \sigma^2)^{-8} \sigma^9 d\sigma, \]

whose leading term is also a negative constant multiple of \( |\log \epsilon| \). For \( n \geq 11 \), let \( t = \sigma^2 \), the limit of the coefficient of \( |W(p)|^2 \omega_{n-1} \) as \( \epsilon \to 0 \) is

\[ 2^{n-7} \frac{(n-6)^2}{12(n-1)} \epsilon^4 \left\{ \frac{n-2}{4n} \int_0^\infty \frac{(n+4t)^2}{(1+t)^{n-2}} t^{\frac{n}{2}} dt \right. \]

\[ + (n-6) \int_0^\infty \frac{1}{(1+t)^{n-4}} t^{\frac{n}{2}} dt \left. - \int_0^\infty \frac{(n-10)t - n}{(1+t)^{n-4}} t^{\frac{n}{2}} dt \right. \]

\[ - \frac{(n^2 - 28)(n-4)}{n(n+2)} \int_0^\infty (n+2) + 4t \frac{n}{(1+t)^{n-2}} t^{\frac{n}{2}} dt \]

\[ = \frac{(n^2 - 28)(n-4)}{n(n+2)} \int_0^\infty 4(1+t)^2 + (n-6)(1+t) - (n-2) \frac{n}{(1+t)^{n-2}} t^{\frac{n}{2}} dt \]

\[ = - \frac{4(n^2 - 28)(n-4)}{n(n+2)} B(\frac{n}{2} + 1, \frac{n}{2} - 5) - \frac{(n^2 - 28)(n-4)(n-6)}{n(n+2)} B(\frac{n}{2} + 1, \frac{n}{2} - 4) \]

\[ + \frac{(n^2 - 28)(n-4)(n-2)}{n(n+2)} B(\frac{n}{2} + 1, \frac{n}{2} - 3). \]

Hence, the above limit of the coefficient of \( |W(p)|^2 \omega_{n-1} \) is rewritten as

\[ 2^{n-7} \frac{(n-6)^2}{12(n-1)} \epsilon^4 \left\{ nB(\frac{n}{2}, \frac{n}{2} - 4) \right. \]
\[
+ B\left(\frac{n}{2} + 1, \frac{n}{2} - 3\right) \left[\frac{n - 2}{4n} (n - 4)^2 + \frac{(n^2 - 28)(n - 4)(n - 2)}{n(n + 2)}\right] \\
+ B\left(\frac{n}{2} + 1, \frac{n}{2} - 4\right) \left[\frac{2(n - 2)(n - 4)}{n} - \frac{(n^2 - 28)(n - 4)(n - 6)}{n(n + 2)}\right] \\
+ B\left(\frac{n}{2} + 1, \frac{n}{2} - 5\right) \left[\frac{4(n - 2)}{n} - n + 10 + n - 6 - \frac{4(n^2 - 28)(n - 4)}{n(n + 2)}\right] \}
\]

\[
= 2^{n-7} \frac{(n - 6)^2}{12(n - 1)} B\left(\frac{n}{2} + 1, \frac{n}{2} - 5\right) \epsilon^4 \left\{ (n - 10) \\
+ \frac{(n - 2)(\frac{n}{2} - 4)(\frac{n}{2} - 5)}{4n(n + 2)(n - 3)}(5n^2 - 2n - 120) + \frac{\frac{n}{2} - 5}{n(n + 2)}(-n^3 + 8n^2 + 28n - 176) \\
+ \frac{4}{n(n + 2)}(-n^3 + 6n^2 + 30n - 116) \right\}, \tag{3.2}
\]

where we have used some elementary identities

\[
B\left(\frac{n}{2} + 1, \frac{n}{2} - 3\right) = \frac{\Gamma\left(\frac{n}{2} + 1\right) \Gamma\left(\frac{n}{2} - 3\right)}{\Gamma(n - 2)} = \frac{(\frac{n}{2} - 4)(\frac{n}{2} - 5)}{(n - 3)(n - 4)} B\left(\frac{n}{2} + 1, \frac{n}{2} - 5\right),
\]

\[
B\left(\frac{n}{2} + 1, \frac{n}{2} - 4\right) = \frac{n - 10}{n} B\left(\frac{n}{2} + 1, \frac{n}{2} - 5\right),
\]

\[
B\left(\frac{n}{2}, \frac{n}{2} - 4\right) = \frac{\Gamma\left(\frac{n}{2}\right) \Gamma\left(\frac{n}{2} - 4\right)}{\Gamma(n - 4)} = \frac{n - 10}{n} B\left(\frac{n}{2} + 1, \frac{n}{2} - 5\right).
\]

The constant in the last brace of (3.2) when \( n \geq 11 \) is

\[
n - 10 + \frac{1}{16n(n + 2)(n - 3)} \left\{ (n - 2)(n - 8)(n - 10)(5n^2 - 2n - 120) \\
+ 8(n - 3)[(n - 10)(-n^3 + 8n^2 + 28n - 176) + 8(-n^3 + 6n^2 + 30n - 116)] \right\}
\]

\[
= n - 10 + \frac{1}{16n(n + 2)(n - 3)} \left[ -3n^5 + 2n^4 + 228n^3 - 264n^2 - 1760n - 768 \right] \\
= -3n^5 + 18n^4 + 52n^3 - 200n^2 - 800n - 768 < 0.
\]

On the other hand, we have

\[
\int_M \varphi^{2n_{-\delta}} d\mu_g = \int_{B_\rho} u_{e^{-\delta}} d\mu_g + \int_{B_{2\rho}\setminus B_\rho} \varphi^{2n_{-\delta}} d\mu_g = \int_{\mathbb{R}^n} u_{e^{-\delta}} dx + O(\epsilon^n).
\]

Therefore, putting these facts together, we conclude by Lemma 3.1 that

\[
\frac{\int_M \varphi_{P_g} \varphi d\mu_g}{\left(\int_M \varphi^{2n_{-\delta}} d\mu_g\right)^{\frac{n-\alpha}{n}}} = Y_\alpha(S^n) + A_{n,\rho,\epsilon} |W(p)|^2 \omega_{n-1} + \left\{ \begin{array}{ll}
O(\epsilon^4) & \text{if } n = 10, \\
O(\epsilon^5 |\log \epsilon|) & \text{if } n = 11, \\
O(\epsilon^5) & \text{if } n \geq 12,
\end{array} \right.
\]
\[
Y_6(S^n) - C_n |W(p)|^2 e^4 |\log \epsilon| + O(\epsilon^4) \quad \text{if } n = 10,
\]
\[
Y_6(S^n) - C_n |W(p)|^2 e^4 + o(\epsilon^4) \quad \text{if } n \geq 11,
\]
for some positive constant \(C_n > 0\). Consequently, choosing \(\epsilon\) sufficiently small, we obtain \(Y_6(M^n) < Y_6(S^n)\). This finishes the proof. \(\square\)

Given a smooth positive function \(f\) in \(M^n\), we define a “free” energy functional by
\[
E_f[u] = \frac{1}{2} \int_M u P_g u \, d\mu_g - \frac{1}{2^p} \int_M |u|^2 \, d\mu_g.
\]
Let \(u_i\) or \(\nabla_i u\) denote the covariant derivatives of \(u\) with respect to metric \(g\) and \(R^i_{jk}\) be Riemannian curvature tensor of metric \(g\). Notice that
\[
\nabla_j \nabla_i \nabla^i u = \nabla_i \nabla_j \nabla^i u + R^k_{ij} \nabla_k u = \nabla_i \nabla^i \nabla_j u - R^k_j \nabla_k u,
\]
there holds
\[
\int_M |\nabla \Delta u|^2 \, d\mu_g = \int_M |\Delta \nabla_j u - R^k_j \nabla_k u|^2 \, d\mu_g. \tag{3.3}
\]
Under \(g\)-normal coordinates around a point, one gets
\[
\frac{1}{2} \Delta_g |\nabla^2 u|^2_g = |\nabla^3 u|^2_g + \langle \nabla \Delta u, \nabla \nabla^2 u \rangle_g + u_{ij} (R^l_{ij} u_{jk} + R^j_{il} u_{ik} + R^l_{ij} u_{jk} u + R^l_{ijk} u_{ik}).
\]
Integrating the above identity over \(M\) to show
\[
\int_M |\Delta \nabla u|^2 \, d\mu_g = \int_M |\nabla^3 u|^2 \, d\mu_g + \int_M O(|Rm||\nabla^2 u|^2_g + |\nabla Rm||\nabla u|^2_g) |\nabla^2 u|^2 \, d\mu_g. \tag{3.4}
\]
From (3.3) and (3.4), it yields that the following two norms are equivalent:
\[
\|u\|_{H^3} := \left( \int_M (|\nabla \Delta u|^2 \, d\mu_g + |\nabla^2 u|^2_g + |\nabla u|^2_g + u^2) \, d\mu_g \right)^{\frac{1}{2}}
\]
\[
\approx \left( \int_M (|\nabla^3 u|^2 \, d\mu_g + |\nabla^2 u|^2_g + |\nabla u|^2_g + u^2) \, d\mu_g \right)^{\frac{1}{2}}, \quad u \in H^3(M, g).
\]
Let \(\| \cdot \|_p\) denote the norm of \(L^p(M, g)\) for \(1 \leq p \leq \infty\).

A sequence \(\{u_k\}\) in \(H^3(M, g)\) is called a Palais-Smale (P-S) \(\beta\) sequence for \(E_f\) if \(E_f[u_k] \to \beta \in \mathbb{R}\) and \(DE_f[u_k] \to 0\) as \(k \to \infty\). The energy \(E_f\) satisfies (P-S) \(\beta\) condition if any Palais-Smale sequence of \(E_f\) has a strongly convergent subsequence. We call that \(P_g\) is coercive if there exists a constant \(\mu(g) > 0\) such that
\[
\int_M \psi P_g \psi d\mu_g \geq \mu(g) \int_M \psi^2 \, d\mu_g, \quad \text{for all } \psi \in H^3(M, g).
\]

\textbf{Remark 3.1.} If \((M, g)\) is Einstein and of positive constant scalar curvature, from the factorization (1.4) of \(P_g\), the coercivity of \(P_g\) is automatically satisfied.
As an application, we adapt some arguments in Esposito-Robert [4] to show some existence results of prescribed $Q$-curvature equation, whose solution may change signs due to the lack of maximum principles (in general).

**Theorem 3.1.** Let $(M^n, g)$ be a smooth closed manifold of dimension $n \geq 10$ and $f$ be a smooth positive function in $M^n$. Suppose the Weyl tensor $W_g$ is nonzero at a maximum point of $f$ and $f$ satisfies the vanishing order condition (1.5) at this maximum point. Assume $P_g$ is coercive, then there exists a nontrivial $C^{6,\mu}$ ($0 < \mu < 1$) solution to

$$P_g u = f|u|^{2^* - 2} u \text{ in } M. \quad (3.5)$$

In addition, assume $(M, g)$ is Einstein and of positive scalar curvature, then there exists a smooth solution to the $Q$-curvature equation

$$P_g u = f u^{\frac{n+6}{n-6}}, u > 0 \text{ in } M. \quad (3.6)$$

**Proof.** By the assumptions, there exists $p \in M$ such that $f(p) = \max_{x \in M^n} f(x)$, $W_g(p) \neq 0$ and the vanishing order condition (1.5) of $f$ is true at $p$. Let

$$\gamma_\epsilon(t) = t \frac{\varphi}{\|f^\frac{1}{2^*} \varphi\|_{2^*}}$$

where $\varphi = \eta_\rho u_\epsilon$ is the test function chosen in Proposition 3.1. By choosing $t_0$ large enough, we get $E_f[\gamma_\epsilon(t_0)] < 0$. Let

$$\Gamma = \left\{ \gamma(t) \in C([0, t_0], H^3(M, g)) ; \gamma(0) = 0, \gamma(t_0) = t_0 \frac{\varphi}{\|f^\frac{1}{2^*} \varphi\|_{2^*}} \right\}.$$

From the coercivity of $P_g$ and Sobolev embedding theorem, we have

$$E_f\left[ \frac{\varphi}{\|f^\frac{1}{2^*} \varphi\|_{2^*}} \right] = \frac{1}{2} \int_M \varphi P_g \varphi d\mu_g - \frac{1}{2^*} \geq C - \frac{1}{2^*}.$$

It only suffices to estimate the term:

$$\int_M f \varphi^\frac{2n}{n-6} d\mu_g = \int_{B_\rho} \left[ f(p) + \sum_{k=2}^4 \frac{1}{k!} \partial_{t_1 \ldots t_k} f(p) x^{i_1} \cdots x^{i_k} + O(|x|^5) \right] u_\epsilon^{2^*} dx + O(\epsilon^n)$$

$$= f(p) \int_{\mathbb{R}^n} u_\epsilon^{\frac{2n}{n-6}} dx + \left\{ \begin{array}{ll} O(\epsilon^4) & \text{if } n = 10, \\ o(\epsilon^4) & \text{if } n \geq 11, \end{array} \right.$$
we have

Together with the coercivity of

Applying Brezis-Lieb lemma to

is a critical value of

Proposition 1]) that

where \( t^* = \left( \frac{f_{\phi} \varphi_{\phi} \varphi_{\phi}}{\|f \varphi\|_{L^2}} \right)^{\frac{1}{2}} \). Then it follows from Mountain Pass Lemma (cf. [1] or [4, Proposition 1]) that

\[
\beta = \inf_{\gamma \in \Gamma} \sup_{t \geq 0} E_f[\gamma(t)] \leq \sup_{t \geq 0} E_f[\gamma(t)] < \frac{3}{n} Y_6(S^n)^\frac{2}{p} (\max f) \frac{6-n}{6},
\]

is a critical value of \( E_f \) and there exists a (P-S)\(_\beta\) sequence \( \{u_k\} \) of \( E_f \) in \( H^3(M, g) \).

Next we claim that \( E_f \) satisfies (P-S)\(_\beta\) condition. For the above (P-S)\(_\beta\) sequence \( \{u_k\} \) satisfying \( E_f[u_k] \to \beta \) and \( DE_f[u_k] \to 0 \) as \( k \to \infty \), there holds

\[
2\beta + o(\|u_k\|_{H^3}) = 2E_f[u_k] - \langle DE_f[u_k], u_k \rangle = \frac{6}{n} \int_M f |u_k|^2 d\mu_g.
\]

Together with the coercivity of \( P_g \), one has

\[
\mu(g) \|u_k\|_{H^3} \leq 2E_f[u_k] + \frac{2}{2^2} \int_M f |u_k|^2 d\mu_g \leq C + o(\|u_k\|_{H^3}).
\]

From this, we get \( \{u_k\} \) is bounded in \( H^3(M, g) \). Then up to a subsequence, as \( k \to \infty \), \( u_k \rightharpoonup u \) in \( H^3(M, g) \) and \( u_k \to u \) in \( L^p(M, g) \) for \( 1 \leq p < 2^* \). It is easy to verify that \( u \) is a weak solution to (3.5), that is, for all \( \psi \in H^3(M, g) \),

\[
\int_M \psi P_g ud\mu_g = \int_M f |u|^{2^* - 2} u\psi d\mu_g.
\]

Choosing \( \psi = u \), one has

\[
\int_M u P_g ud\mu_g = \int_M f |u|^2 d\mu_g,
\]

whence

\[
E_f[u] = \frac{3}{n} \int_M f |u|^2 d\mu_g \geq 0.
\]

Applying Brezis-Lieb lemma to

\[
\int_M |\nabla \Delta u_k|^2 g d\mu_g = \int_M |\nabla \Delta u|^2 g d\mu_g + \int_M |\nabla \Delta (u - u_k)|^2 g d\mu_g + o(1),
\]

\[
\int_M f |u_k|^2 d\mu_g = \int_M f |u|^2 d\mu_g + \int_M f |u - u_k|^2 d\mu_g + o(1),
\]

we have

\[
E_f[u_k] - E_f[u] = \frac{1}{2} \int_M |\nabla \Delta (u - u_k)|^2 g - \frac{1}{2^2} \int_M f |u - u_k|^2 d\mu_g + o(1)
\]

\[
= E_f[u - u_k] + o(1).
\]
Since $DE_f[u_k] \to 0$ in $(H^3(M, g))'$, we have

$$o(1) = \langle u_k - u, DE_f[u_k] \rangle = \langle u_k - u, DE_f[u_k] - DE_f[u] \rangle = \int_M |\nabla \Delta (u - u_k)|^2_g d\mu_g - \int_M f |u - u_k|^2_g d\mu_g + o(1).$$

Thus, we obtain

$$\frac{3}{n} \int_M |\nabla \Delta (u - u_k)|^2_g d\mu_g + o(1) = E_f[u_k] - E_f[u] + o(1) \leq E_f[u_k] + o(1) \to \beta,$$

as $k \to \infty$, which yields

$$\int_M |\nabla \Delta (u - u_k)|^2_g d\mu_g \leq \frac{n}{3} \beta + o(1). \quad (3.7)$$

Mimicking a cut-and-paste argument as in [3], we obtain that given $\epsilon > 0$, there exists a constant $B_\epsilon > 0$ such that

$$\left( \int_M |\psi|^2 f d\mu_g \right)^{\frac{2}{2^*}} \leq (1 + \epsilon) Y_6(S^n)^{-1} \int_M (|\nabla \Delta \psi|^2_g + |\nabla^2 \psi|^2_g + |\nabla \psi|^2_g) d\mu_g + B_\epsilon \int_M \psi^2 d\mu_g,$$

for all $\psi \in H^3(M, g)$. Choosing $\psi = u_k - u$ and $k$ sufficiently large to get

$$\left( \int_M |u - u_k|^2 f d\mu_g \right)^{\frac{2}{2^*}} \leq (1 + \epsilon) Y_6(S^n)^{-1} \int_M |\nabla \Delta (u - u_k)|^2_g d\mu_g + o(1).$$

Hence we have

$$o(1) \geq \int_M |\nabla \Delta (u - u_k)|^2_g d\mu_g - \int_M f |u - u_k|^2_g d\mu_g \geq \int_M |\nabla \Delta (u - u_k)|^2_g d\mu_g \left[ 1 - (\max_M f)(1 + \epsilon) \right] Y_6(S^n)^{-\frac{2}{2^*}} \left( \int_M |\nabla \Delta (u - u_k)|^2_g d\mu_g \right)^{-\frac{6}{6 - n}}.$$

From (3.7) and $\beta < \frac{3}{n} Y_6(S^n)^{\frac{2}{2^*}} (\max_M f)^{-\frac{6-n}{6}}$, choosing $\epsilon$ sufficiently small, we get

$$o(1) \geq C \int_M |\nabla \Delta (u - u_k)|^2_g d\mu_g.$$

Combining the above inequality and the coercivity of $P_g$ to show that $u_k \to u$ in $H^3(M, g)$. Using the regularity result in Lemma 3.2 below, we know that $u \in C^{6,\mu}(M)$ for any $0 < \mu < 1$.

In addition, assume $(M, g)$ is Einstein and has positive constant scalar curvature, we define the modified energy in $H^3(M, g)$ by

$$E^+_f[u] = \frac{1}{2} \int_M u P_g u d\mu_g - \frac{1}{2^*} \int_M f u^2_g d\mu_g,$$

where $u_+ = \max\{u, 0\}$. Using the above similar arguments associated with Mountain Pass Lemma and mimicking what we did in Lemma 3.2 below for $E^+_f$, we get that there exists a nontrivial $C^6$-solution $u$ to

$$P_g u = f u^\frac{n+6}{6} \text{ in } M. \quad (3.8)$$
Since $P_g$ is coercive by Remark 3.1, testing equation (3.8) with $u_\varepsilon = \min \{u, 0\}$ we conclude that $u \geq 0$ in $M$. Together with $R_g$ being a positive constant and the factorization (1.4) of GJMS operator:

\[
\left( -\Delta_g + \frac{(n-6)(n+4)}{4n(n-1)} R_g \right) \left( -\Delta_g + \frac{(n-4)(n+2)}{4n(n-1)} R_g \right) u \geq 0
\]

and $u \not\equiv 0$ in $M$, we employ the maximum principle twice and strong maximum principle once for elliptic equations of second order to show that $u$ is a positive solution to the equation (3.6). From this and Schauder estimates for elliptic equations, we conclude that $u \in C^\infty(M)$. This completes the proof. 

We are now concerned with the regularity of mountain pass critical points for $E$.

**Lemma 3.2.** Let $(M, g)$ be a smooth closed Riemannian manifold of dimension $n \geq 7$. Assume $u \in H^3(M, g)$ is a weak solution of equation (3.5). Then $u \in C^{6,\mu}(M)$ for any $0 < \mu < 1$.

**Proof.** Rewrite $P_g = (-\Delta_g)^3 - M_g + \frac{n-6}{2}Q_g$ by (1.2). Let $u \in H^3(M, g)$ be a weak solution of equation (3.5) and rewrite this equation as

\[
(-\Delta_g + 1)^3 u = M_g u + 3\Delta_g^2 u - 3\Delta_g u + (1 - \frac{n-6}{2}Q_g) u + f|u|^{2^*-2} u \\
=: b + f|u|^{2^*-2} u, \tag{3.9}
\]

where $b \in H^{-1}(M, g)$. By the Sobolev embedding theorem we have $u \in L^{2^*}(M, g)$ and $|u|^{2^*-2} \in L^\frac{n}{2}(M, g)$. Given $\varepsilon > 0$, there exist a $K_\varepsilon > 0$ and a decomposition of $f|u|^{2^*-2} = h_\varepsilon + \eta_\varepsilon$ with $\|h_\varepsilon\|_\frac{n}{2} \leq \varepsilon$, $\|\eta_\varepsilon\|_\infty \leq K_\varepsilon$. Inspired by the arguments in [4, Proposition 3], for $s > 1$ we define an operator

\[
H_\varepsilon : v \in L^s(M, g) \to (-\Delta_g + 1)^{-3}(h_\varepsilon v) \in L^s(M, g).
\]

Indeed, from Sobolev embedding theorem, the standard $W^{2,p}$-regularity theory of the elliptic operator $-\Delta_g + 1$ and Hölder’s inequality, we have

\[
\|H_\varepsilon v\|_s \leq C\|(-\Delta_g + 1)^{-3}(h_\varepsilon v)\|_{W^{2,p}(M, g)} \leq C\|h_\varepsilon v\|_{\frac{n}{n-6}}^{\frac{2}{2^*-2}} \leq C\|h_\varepsilon\|_\frac{n}{2}\|v\|_s \leq C\varepsilon\|v\|_s,
\]

where the constant $C$ is independent of $u$. Choose $\varepsilon > 0$ small enough, then the norm of $H_\varepsilon$ on space $L^s(M, g)$ satisfies

\[
\|H_\varepsilon\|_{L^s\to L^s} \leq C\varepsilon \leq \frac{1}{2}.
\]

With the help of the operator $H_\varepsilon$, we rewrite equation (3.9) as

\[
(Id - H_\varepsilon)u = (-\Delta_g + 1)^{-3}(b + \eta_\varepsilon u),
\]

then it is easy to show $Id - H_\varepsilon : L^s \to L^s$ is bounded and invertible. We intend to prove $u \in H^6(M, g)$. To see this, notice that $(-\Delta_g + 1)^{-3}(b + \eta_\varepsilon u) \in H^5(M, g)$ since $b + \eta_\varepsilon u \in H^{-1}(M, g)$. In the following, we first show $u \in H^4(M, g)$. Applying the Sobolev embedding
theorem and the $L^s$ boundedness of the operator $(Id - H_{\delta})^{-1}$ to show that if $n \leq 10$, $u \in L^p(M, g)$ for all $p > 1$, and if $n > 10$, $u \in L^{\frac{2n}{n+6}}(M, g)$. In the latter case we have $|u|^{2^* - 2}u \in L^{\frac{2n}{(n+6)(n-10)}}(M, g)$.

From equation (3.9), we get

$$(-\Delta_g + 1)^2 u = (-\Delta_g + 1)^{-1}b + (-\Delta_g + 1)^{-1}(f|u|^{2^* - 2}u).$$

From $(-\Delta_g + 1)^{-1}(|u|^{2^* - 2}u) \in W^{2, \frac{2n(n-6)}{(n+6)(n-10)}}(M, g) \hookrightarrow L^2(M, g)$ and $(-\Delta_g + 1)^{-1}b \in L^2(M, g)$, we have $u \in H^4(M, g)$ in both cases. Repeat the above step with $u \in H^2(M, g)$ and $b \in L^2(M, g)$ in this situation. Notice that $(-\Delta_g + 1)^{-3}(b + \eta u) \in H^{6}(M, g)$, similar arguments in the above step show that if $n \leq 12$, $u \in L^p(M, g)$ for all $p > 1$ and if $n > 12$, $u \in L^{\frac{2n}{n+12}}(M, g)$. In the latter case, we get $|u|^{2^* - 2}u \in L^2(M, g)$ due to $\frac{2n}{(n+6)(n-12)} > 2$. Hence we obtain $u \in H^6(M, g)$.

Finally we start with the classical bootstrap. We now construct a non-decreasing sequence $s_k \in \mathbb{R} \cup \{+\infty\}$ such that $u \in W^{6, s_k}(M, g)$ for all $k \in \mathbb{N}$. Set $s_0 = 2$, and find $k \geq 0$ such that $u \in W^{6, s_k}(M, g)$. Next we will define $s_{k+1}$ by induction. The Sobolev embedding theorem yields

$$b \in L^{\frac{n s_k}{n-2s_k}}(M, g),$$

with the convention that $\frac{n s_k}{n-2s_k} = +\infty$ if $s_k \geq \frac{n}{2}$, and

$$|u|^{2^* - 2}u \in L^{\frac{n s_k}{(n-6s_k)(n+6)}}(M, g),$$

with the convention that $\frac{n s_k}{n-6s_k} = +\infty$ if $s_k \geq \frac{n}{6}$. In view of equation (3.9), we have $u \in W^{6, s_{k+1}}(M, g)$ with $s_{k+1} = \min\{\frac{n s_k}{n-2s_k}, \frac{n s_k}{(n-6s_k)(n+6)}\}$. If $s_k \in \mathbb{R}$ for all $k \in \mathbb{N}$, it must hold that $s_k \to +\infty$. Then we have $u \in W^{6, p}(M, g)$ for all $1 \leq p < +\infty$. If $s_k = +\infty$ for all $k \geq k_0 + 1$, then $s_{k_0} \geq \frac{n}{6}$, whence $b \in L^{\frac{n}{n+6}}(M, g)$ and $|u|^{2^* - 2}u \in L^q(M, g)$ for all $1 \leq q < +\infty$. The equation (3.9) leads to $u \in W^{6, \frac{n}{6}}(M)$. Repeating the argument twice, we obtain $u \in W^{6, p}(M, g)$ for all $1 \leq p < +\infty$. From this and the Sobolev embedding theorem, we have $u \in C^{5, \nu}(M)$ for all $0 < \nu < 1$. By the regularity theory for the classical solution of the elliptic operator $-\Delta_g + 1$, we get $u \in C^{6, \mu}(M)$ for some $0 < \mu < 1$. This completes the proof. 

\[\Box\]

### A Appendix: Proof of Lemma 2.1

As in Proposition 3.1, one may employ all computations under conformal normal coordinates of metric $g$ around a point in $M$. From Lee-Park [12] that up to a conformal factor, under $g$-conformal normal coordinates around this point, for all $N \geq 5$ there hold

$$\sigma_1(A_g) = 0, \quad \sigma_1(A_g)_i = 0, \quad \Delta_g \sigma_1(A_g) = -\frac{|W|^2}{12(n-1)},$$

at this point and

$$\sqrt{\det g} = 1 + O(r^N) \text{ near this point.}$$

To simplify the notation, we will omit the subscript $g$. Notice that

$$-P_g(r^{6-n}) = \left[\Delta^3 + \Delta \delta T_2 d + \delta T_2 d \Delta + \frac{n-2}{2} \Delta (\sigma_1(A) \Delta) + \delta T_4 d - \frac{n-6}{2} Q_g\right](r^{6-n})$$
Next, we begin to estimate all terms $I_1$-$I_6$.

For $I_1$, let $u = u(r)$ be a radial function, we have

$$\Delta u(r) = \Delta_0 u(r) + O(r^{N-1}u'),$$
$$\Delta^2 u(r) = \Delta_0 (\Delta_0 u(r) + O(r^{N-1})u'') + O(r^{N-1})(\Delta_0 u(r) + O(r^{N-1})u'')'$$
$$= \Delta_0^2 u(r) + O(r^{N-1})u''' + O(r^{N-2})u'' + O(r^{N-3})u';$$
$$\Delta^3 u(r) = \Delta_0^3 u(r) + O(r^{N-1})u(5) + O(r^{N-2})u(4) + O(r^{N-3})u'''$$
$$+ O(r^{N-4})u'' + O(r^{N-5})u'.$$

Hence we obtain

$$I_1 = \Delta^3 (r^{6-n}) = -c_n \delta_p + O(r^{N-n}).$$

To estimate $I_2$. Notice that

$$I_2 = \Delta \delta T_2 d(r^{6-n})$$
$$= - \delta [(T_2)_{ij} (r^{6-n})]_i$$
$$= - \delta [(T_2)_{ij} (r^{6-n})]_j + (T_2)_{ij} (r^{6-n})]_j.$$  \hspace{1cm} (A.1)

Using

$$(r^{6-n})_j = (6-n)r^{4-n}x^j,$$
$$(r^{6-n})_{ji} = (4-n)(6-n)r^{2-n}x^i x^j + (6-n)r^{4-n} \delta_{ij} + O(r^{6-n}),$$

one has

$$(T_2)_{ij} (r^{6-n})_j = (n-10) \sigma_1 (A)_{ij} (6-n)r^{4-n} x^i$$
$$= (n-10)(6-n) \sigma_1 (A)_{ij} x^i r^{4-n}$$

and

$$(T_2)_{ij} (r^{6-n})_{ji}$$
$$= [ (n-2) \sigma_1 (A) g_{ij} - 8 A_{ij} ] (6-n) [(4-n)r^{2-n} x^i x^j + r^{4-n} \delta_{ij} + O(r^{6-n})]$$
$$= (6-n) [ 4(n-4) \sigma_1 (A) r^{4-n} - 8(4-n) A_{ij} x^i x^j r^{2-n} ] + O(r^{7-n}).$$

Hence, we obtain

$$I_2$$
$$= - (6-n) \Delta [(n-10) \sigma_1 (A)_{ij} x^i r^{4-n} + 4(n-4) \sigma_1 (A) r^{4-n} - 8(4-n) A_{ij} x^i x^j r^{2-n}]$$
$$= (n-6) ((n-10)[4(n-4) \sigma_1 (A)_{ij} x^i r^{2-n} + 2(4-n) \sigma_1 (A)_{ij} x^i x^j r^{2-n}$$
$$+ \sigma_1 (A)_{ijk} x^i r^{2-n} + 2 \Delta \sigma_1 (A) r^{4-n}] + O(r^{5-n})$$
Recall that 

\[ \text{Observe that } \]

then it yields 

\[ = n - 6 \left\{ - 4(n - 4)(3n - 26)\sigma_1(A)_{ij}(p)x^i x^j r^{-2n} - \frac{n - 6}{2(n - 1)}|W(p)|^2 r^{4-n} \right. \]

\[ - 2(n - 10)(n - 4)\sigma_1(A)_{ij}(p)x^i x^j r^{-2n} - 4(n - 6)(n - 4)\sigma_1(A)_{ij}(p)x^i x^j r^{-2n} \]

\[ - 16(n - 4)(n - 2)A_{ij,kl}(p)x^i x^j x^k x^l r^{-2n} + 8(n - 4)\Delta A_{ij} x^i x^j r^{-2n} \}

\[ = (n - 6) \left\{ - 2(n - 4)(9n - 74)\sigma_1(A)_{ij}(p)x^i x^j r^{-2n} - \frac{n - 6}{2(n - 1)}|W(p)|^2 r^{4-n} \right. \]

\[ - 16(n - 4)(n - 2)A_{ij,kl}(p)x^i x^j x^k x^l r^{-2n} + 8(n - 4)\Delta A_{ij} x^i x^j r^{-2n} \}

To estimate

\[ I_3 = \delta T_2 d\Delta(r^{6-n}) \]

\[ = - [(T_2)_{ij}((\Delta r^{6-n})_{j})_i \]

\[ = - (T_2)_{ij,i}(\Delta r^{6-n})_{j} - (T_2)_{ij}(\Delta r^{6-n})_{ji} \]

Recall that \( T_2 = (n - 2)\sigma_1(A)g - 8A \), then

\[ (T_2)_{ij,i} = (n - 10)\sigma_1(A)_{j} \]

Observe that

\[ \Delta r^{6-n} = 4(6 - n)r^{4-n} + O(r^{N+4-n}), \]

\[ (\Delta r^{6-n})_{j} = 4(6 - n)(4 - n)x^j r^{-2n} + O(r^{N+3-n}) \]

and

\[ (\Delta r^{6-n})_{ji} = 4(6 - n)(4 - n)[(2 - n)x^i x^j r^{-n} + r^{2-n}\delta_{ij}] + O(r^{4-n}) \]

then it yields

\[ (T_2)_{ij}(\Delta r^{6-n})_{ji} \]

\[ = 4(n - 6)(n - 4)\left[(n - 2)\sigma_1(A)_{ij} - 8A_{ij}\right][2 - n)x^i x^j r^{-n} + r^{2-n}\delta_{ij} + O(r^{4-n})] \]

\[ = 4(n - 6)(n - 4)[(n - 2)^2\sigma_1(A)r^{2-n} + n(n - 2)\sigma_1(A)r^{2-n} \]

\[ + 8(n - 2)r^n A_{ij} x^i x^j - 8\sigma_1(A)r^{2-n}] + O(r^{5-n}) \]

\[ = 4(n - 6)(n - 4)\left[2(n - 6)\sigma_1(A)r^{2-n} + 8(n - 2)r^n A_{ij} x^i x^j \right] + O(r^{5-n}) \]

\[ = 4(n - 6)(n - 4)[(n - 6)\sigma_1(A)_{ij}(p)x^i x^j r^{-2n} + 4(n - 2)r^n(A_{ij,kl}(p)x^i x^j x^k x^l)] + O(r^{5-n}) \].
Hence, we obtain
\[
I_3 = -4(n - 6)(n - 4) \left[ (n - 6)\sigma_1(A)_{ij}(p)x^ix^jr^{2-n} + 4(n - 2)r^{-n}(A_{ij,kl}(p)x^ix^jx^kx^l) \right] \\
- 4(n - 6)(n - 4)(n - 10)r^{2-n}\sigma_1(A)_{,i}x^i + O(r^{5-n}) \\
= -8(n - 8)(n - 6)(n - 4)\sigma_1(A)_{ij}(p)x^ix^jr^{2-n} \\
- 16(n - 6)(n - 4)(n - 2)r^{-n}(A_{ij,kl}(p)x^ix^jx^kx^l) + O(r^{5-n}).
\]

We now compute
\[
I_4 = \frac{n - 2}{2} \Delta(\sigma_1(A)\Delta(r^{6-n})) \\
= 2(n - 2)(6 - n)\Delta(\sigma_1(A)r^{4-n}) + O(r^{N+4-n}) \\
= 2(n - 2)(6 - n)r^{2-n}[\Delta\sigma_1(A)r^2 + 2(4 - n)\sigma_1(A)_{,i}x^i + 2(4 - n)\sigma_1(A)] + O(r^{N+2-n}) \\
= 2(n - 2)(n - 6)r^{2-n}\left[ \frac{1}{12(n - 1)}|W(p)|^2 r^2 + 3(n - 4)\sigma_1(A)_{ij}(p)x^ix^j \right] + O(r^{5-n}).
\]

For $I_5$, from (A.1) we have
\[
I_5 = \delta T_4 d(r^{6-n}) \\
= -((T_4)_{ij}r^{6-n})_{,i} \\
= -(T_4)_{ij,i}(r^{6-n})_{,i} - (T_4)_{ij}(r^{6-n})_{,ji} \\
= (n - 6)[r^{4-n}(T_4)_{ij}x^j - (n - 4)r^{2-n}(T_4)_{ij}x^j + r^{4-n}\text{tr}(T_4)] \\
:= (n - 6)[I_1^{(5)} + I_2^{(5)} + I_3^{(5)}].
\]

Also from [12], there holds
\[
\text{Sym}\left(R_{kl,ij} + \frac{2}{9}R_{nklm}R_{nijm}\right)(p) = 0 \text{ and } R_{ij}(p) = 0,
\]
then
\[
R_{kl,ij}(p)x^ix^jx^kx^l = -\frac{2}{9}W_{nklm}(p)W_{nijm}(p)x^ix^jx^kx^l.
\]

Thus we have
\[
A_{kl,ij}(p)x^ix^jx^kx^l = -\frac{2}{9n - 2}\sum_{k,l}(W_{iklj}(p)x^ix^j)^2 - \frac{\sigma_1(A)_{ij}(p)x^ix^j r^2}{n - 2}. \quad (A.2)
\]

To estimate $I_3^{(5)}$. From the definition of $T_4$, one gets
\[
\text{tr}(T_4) = -\frac{3n^3 - 12n^2 - 36n + 64}{4}\sigma_1(A)^2 + 4(n^2 - 4n - 12)|A|^2 + n(n - 6)\Delta\sigma_1(A) \\
= -\frac{n(n - 6)}{12(n - 1)}|W(p)|^2 + O(r).
\]

Thus one obtains
\[
I_3^{(5)} = -\frac{n(n - 6)}{12(n - 1)}|W(p)|^2 r^{4-n} + O(r^{5-n}).
\]
For the term $I_1^{(5)}$, it is easy to see
\[
I_1^{(5)} = r^{4-n}(T_4)_{ij,i}x^j = O(r^{5-n}).
\]

It remains to estimate the term $I_2^{(5)}$, one has
\[
(T_4)_{ij,i}x^j = (n - 6)\Delta_1(A)r^2 \sum_{k,l} B_{i,j}(x^k x^l)\Delta_1(A) + O(r^4).
\] (A.3)

Notice that
\[
B_{i,j}(x^k x^l) = [(A_{i,j,k} - A_{i,k,j})_k - A_{k,l}W_{kij}]x^i x^j
\]
and
\[
\Delta_1(A_{i,j})(x^i x^j) = (A_{i,j,k}x^k x^l + A_{i,j}(x^k \delta_{jk} + x^j \delta_{ik}))_k
\]
\[
= (\Delta_1 A_{i,j})(x^i x^j) + 2A_{i,j}(x^k \delta_{jk} + x^j \delta_{ik}) + 2\sigma_1(A) + O(r^3)
\]
\[
= (\Delta_1 A_{i,j})(x^i x^j) + 4\sigma_1(A), x^i + 2\sigma_1(A) + O(r^3).
\]

By (A.2), one gets
\[
(\Delta A_{i,j})(x^i x^j) = (\Delta_1 A_{i,j})(x^i x^j) - 4\sigma_1(A), x^i + 2\sigma_1(A) + O(r^3)
\]
\[
= \Delta \left[ \frac{1}{2} A_{i,j,k}(p)x^i x^j x^k + O(r^5) \right] - 4\sigma_1(A), x^i + O(r^3)
\]
\[
= \Delta \left[ -\frac{1}{9 n - 2} \sum_{k,l} W_{ilkl}(p)x^i x^j x^l x^k \right] - 5\sigma_1(A), x^i + O(r^3)
\]
\[
= -\frac{2}{9 n - 2} \sum_{k,l} \left[ (W_{ikls}(p) + W_{ilks}(p))x^i x^j x^k x^l \right] + \frac{1}{12(n-2)(n-1)}|W(p)|^2 r^2
\]
\[
- 6\frac{n-1}{n-2} \sigma_1(A), x^i + O(r^3),
\] (A.4)

where the last identity follows from the following two estimates:
\[
\Delta_1 (\sigma_1(A), x^i x^j r^2) = \Delta_1 (\sigma_1(A), x^i x^j)|r^2 + 2\nabla_r (\sigma_1(A), x^i x^j)\nabla_r r^2
\]
\[
+ (\sigma_1(A), x^i x^j)\Delta r^2
\]
\[
= 2\Delta_1 (A)(p)r^2 + 8\sigma_1(A), x^i x^j + 2n\sigma_1(A), x^i x^j + O(r^3)
\]
\[
= -\frac{1}{6(n-1)}|W(p)|^2 r^2 + 2(n+4)\sigma_1(A), x^i x^j + O(r^3)
\]
and
\[
\Delta_1 \sum_{k,l} (W_{iklj}(p)x^i x^j)^2 = 2 \sum_{k,l} \left[ W_{iklj}(p)(x^i \delta_{js} + x^j \delta_{is}) \right]^2
\]
\[
= 2 \sum_{k,l} \left[ (W_{ikls}(p) + W_{ilks}(p))x^i \right]^2.
\]
which follows from

\[
\Delta \left[ \sum_{k,l} (W_{ijkl}(p)x^ix^j)^2 \right] = 2 \sum_{k,l} [(W_{ijkl}(p)x^ix^j)\Delta(W_{sklt}(p)x^sx^t) + |\nabla(W_{ijkl}(p)x^ix^j)|^2]
\]

and \(\Delta(W_{sklt}(p)x^sx^t) = (W_{sklt}(p)(x^s\delta_{it} + x^t\delta_{is}))_{,i} = 2W_{sklt}(p)\delta_{st} = 0\). Using \(A_{ik,jk} = A_{ik,kj} + R_{ikjk}^l A_{lk} + R_{kijk}^l A_{il} = \sigma_1(A)_{ij} + R_{ijjk}^l A_{lk} + R_{ijlk}^l A_{il},\) one has

\[
A_{ik,jk}x^ix^j = \sigma_1(A)_{ij}x^ix^j + R_{ijlk}^l A_{il}x^ix^j + R_{ijlk}^l A_{il}x^ix^j
\]

\[
= (\sigma_1(A)_{ij}(p) + O(r)x^ix^j + (W_{ijjk}(p) + O(r))(A_{ik,m}(p)x^m + O(r^2))x^ix^j + O(r^4)
\]

\[
= \sigma_1(A)_{ij}(p)x^ix^j + O(r^3).
\]

Thus, one obtains

\[
B_{ij}x^ix^j = -\frac{2}{9(n-2)} \sum_{k,l,s} [(W_{ikls}(p) + W_{ilks}(p))x^i] + \frac{1}{12(n-2)(n-1)} |W(p)|^2 r^2
\]

\[
- \frac{7n-8}{n-2} \sigma_1(A)_{ij}(p)x^ix^j + O(r^3).
\]

(A.5)

Inserting the above equations into (A.3), one gets

\[
(T_4)_{ij}x^ix^j
\]

\[
= -\frac{n-6}{12(n-1)} |W(p)|^2 r^2 + \frac{32}{9(n-4)(n-2)} \sum_{k,l,s} [(W_{ikls}(p) + W_{ilks}(p))x^i]^2
\]

\[
- \frac{4}{3(n-4)(n-2)(n-1)} |W(p)|^2 r^2 + \frac{16(7n-8)}{n-4(n-2)} \sigma_1(A)_{ij}(p)x^ix^j + O(r^3),
\]

whence

\[
I_{2}^{(5)} = r^{2-n} \left[ \frac{(n-6)(n-4)}{12(n-1)} |W(p)|^2 r^2 - \frac{32}{9(n-2)} \sum_{k,l,s} [(W_{ikls}(p) + W_{ilks}(p))x^i]^2
\]

\[
+ \frac{4}{3(n-2)(n-1)} |W(p)|^2 r^2 - \frac{16(7n-8)}{n-2} \sigma_1(A)_{ij}(p)x^ix^j \right] + O(r^{5-n}).
\]

Combining all the terms together, one has

\[
I_5 = \left[ -\frac{n^2 - 8n + 8}{3(n-1)(n-2)} |W(p)|^2 r^{4-n} - \frac{32}{9(n-2)} \sum_{k,l,s} [(W_{ikls}(p) + W_{ilks}(p))x^i]^2 r^{2-n}
\]

\[
- \frac{16(7n-8)}{n-2} \sigma_1(A)_{ij}(p)x^ix^j r^{2-n} \right] (n-6) + O(r^{5-n}).
\]

Finally, from the definition of \(Q_g\) in (1.1), it is not hard to show that

\[
I_6 = -\frac{n-6}{2} Q_g r^{6-n} = O(r^{6-n}).
\]
Therefore, collecting all the terms $I_1-I_6$ together with (A.2) and (A.4), we conclude that

\[-P_g(r^{6-n})
= -c_n \delta_p + (n - 6) \left[ -\frac{16}{9} \sum_{k,l,s} ((W_{ikls}(p) + W_{ilks}(p))x^i)^2 r^{2-n} - \frac{2(n - 8)}{3(n - 1)} |W(p)|^2 r^{4-n} \right.
+ \frac{64(n-4)}{9} \sum_{k,l} (W_{iklj}(p)x^i x^j)^2 r^{-n} - 4(5n^2 - 66n + 224) \sigma_1(A)_{ij}(p)x^i x^j r^{2-n} \right] + O(r^{5-n})
\left. - c_n \delta_p + (n - 6) r^{-n} \left\{ \frac{64(n-4)}{9} \left[ \sum_{k,l} (W_{iklj}(p)x^i x^j)^2 - \frac{x^2}{n+4} \sum_{k,l,s} ((W_{ikls}(p) + W_{ilks}(p))x^i)^2 \right]
+ \frac{3}{2(n+4)(n+2)} |W(p)|^2 r^4 \right\} + \frac{16(3n-20)}{9(n+4)} r^2 \sum_{k,l,s} ((W_{ikls}(p) + W_{ilks}(p))x^i)^2 - \frac{3}{n} |W(p)|^2 r^2
- 4(5n^2 - 66n + 224)r^2 \left[ \sigma_1(A)_{ij}(p)x^i x^j + \frac{|W(p)|^2}{12n(n-1)} r^2 \right]
+ \frac{3n^4 - 16n^3 - 164n^2 + 400n + 2432}{3(n+4)(n+2)n(n-1)} |W(p)|^2 r^4 \right\} + O(r^{5-n}) \right]
\]

where each term in square brackets on the right hand side of the last identity is harmonic polynomial. This finishes the proof of Lemma 2.1.

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