THE SPECTRAL SYMMETRY OF WEAKLY IRREDUCIBLE NONNEGATIVE TENSORS AND CONNECTED HYPERGRAPHS

YI-ZHENG FAN, TAO HUANG, YAN-HONG BAO, CHEN-LU ZHUAN-SUN, AND YA-PING LI

ABSTRACT. Let \( A \) be a weakly irreducible nonnegative tensor with spectral radius \( \rho(A) \). Let \( D \) (respectively, \( D^{(0)} \)) be the set of normalized diagonal matrices arising from the eigenvectors of \( A \) corresponding to the eigenvalues with modulus \( \rho(A) \) (respectively, the eigenvalue \( \rho(A) \)). It is shown that \( D \) is an abelian group containing \( D^{(0)} \) as a subgroup, which acts transitively on the set \( \{ \frac{2\pi j}{\ell} : j = 0, 1, \ldots, \ell - 1 \} \), where \( |D/D^{(0)}| = \ell \) and \( D^{(0)} \) is the stabilizer of \( A \). The spectral symmetry of \( A \) is characterized by the group \( D/D^{(0)} \), and \( A \) is called spectral \( \ell \)-symmetric. We obtain the structural information of \( A \) by analyzing the property of \( D \), especially for connected hypergraphs we get some results on the edge distribution and coloring. If moreover \( A \) is symmetric, we prove that \( A \) is spectral \( \ell \)-symmetric if and only if it is \( (m, \ell) \)-colorable. We characterize the spectral \( \ell \)-symmetry of a tensor by using its generalized traces, and show that for an arbitrarily given integer \( m \geq 3 \) and each positive integer \( \ell \) with \( \ell | m \), there always exists an \( m \)-uniform hypergraph \( G \) such that \( G \) is spectral \( \ell \)-symmetric.

1. Introduction

A real tensor (also called hypermatrix) \( A = (a_{i_1 i_2 \ldots i_m}) \) of order \( m \) and dimension \( n \) refers to a multidimensional array with entries \( a_{i_1 i_2 \ldots i_m} \in \mathbb{R} \) for all \( i_j \in [n] := \{1, 2, \ldots, n\} \) and \( j \in [m] \). Surely, if \( m = 2 \), then \( A \) is a square matrix of dimension \( n \). A subtensor of \( A \) is a multidimensional array with entries \( a_{i_1 i_2 \ldots i_m} \) such that \( i_j \in S_j \subseteq [n] \) for some \( S_j \)'s and \( j \in [m] \), denoted by \( A[S_1][S_2] \cdots [S_m] \). Let \( \rho(A) \) be the spectral radius of \( A \), and \( \text{Spec}(A) \) be the spectrum of \( A \). The circle centered at the origin of the complex plane with radius \( \rho(A) \) is called the spectral circle of \( A \).

By the famous Perron-Frobenius theorem, for a nonnegative irreducible matrix \( A \) of dimension \( n \), if it has \( k \) eigenvalues with modulus \( \rho(A) \), then those \( k \) eigenvalues are equally distributed on the spectral circle, i.e. they are \( \rho(A)e^{\frac{2\pi j i}{k}} \), \( j = 0, 1, \ldots, k - 1 \). Furthermore, the spectrum of \( A \) keeps invariant under a rotation of angle \( \frac{2\pi}{k} \) of the complex plane, i.e. \( \text{Spec}(A) = e^{\frac{2\pi j i}{k}} \text{Spec}(A) \). Under the above
spectral symmetry, \( A \) has a cyclic structure via a permutation matrix \( P \), i.e.

\[
P^T A P = \begin{bmatrix}
O & A_{12} & O & \cdots & O \\
O & O & A_{23} & \cdots & O \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
O & O & O & \cdots & A_{k-1,1}
A_{k1}
\end{bmatrix},
\]

where the diagonal blocks are all square zero matrices of suitable sizes. Equivalently, we have a partition \( [n] = V_1 \cup \cdots \cup V_k \) such that if \( j \not\equiv i + 1 \pmod{k} \), then

\[
A[V_i|V_j] = 0.
\]

The Perron-Frobenius theorem for nonnegative tensor is established by Chang et.al \[1\], Friedland et.al \[6\] and Yang et.al \[22, 23, 24\]. From those work, the spectral symmetry of an irreducible or weakly irreducible nonnegative tensor \( A \) is investigated. The eigenvalues of \( A \) with modulus \( \rho(A) \) are also equally distributed on the spectral circle. However, the structure of \( A \) receives little attention. Does \( A \) has a similar structure to that in (1.1)? The key difference is that an irreducible nonnegative matrix has a unique (positive) eigenvector corresponding to the spectral radius up to a scalar (called Perron vector), but an irreducible or weakly irreducible nonnegative tensor may have more than one eigenvector corresponding to the spectral radius up to a scalar.

In order to obtain the structural information of weakly irreducible nonnegative tensors, we start from the discussion of spectral symmetry of tensors.

**Definition 1.1.** Let \( A \) be an \( m \)-th order \( n \)-dimensional tensor, and let \( \ell \) be a positive integer. The tensor \( A \) is called spectral \( \ell \)-symmetric if

\[
\text{Spec}(A) = e^{i2\pi\ell}\text{Spec}(A).
\]

Suppose that \( A \) is spectral \( \ell \)-symmetric. The maximum number \( \ell \) such that (1.2) holds is called the cyclic index of \( A \) and denoted by \( c(A) \) \[2\], and \( A \) is called spectral \( c(A) \)-cyclic. Obviously, if \( A \) is spectral \( k \)-cyclic, it is spectral \( k \)-symmetric; and for any positive integer \( \ell \) such that \( \ell | k \), it is also spectral \( \ell \)-symmetric. In particular, if \( A \) is spectral 2-symmetric, then \( \lambda \) is an eigenvalue of \( A \) if and only \(-\lambda \) is an eigenvalue of \( A \); in this case, we say \( A \) has a symmetric spectrum. If \( c(A) = 1 \), then \( A \) is spectral 1-cyclic, and is also said spectral nonsymmetric.

If a nonnegative tensor \( A \) holds one of the following properties: (1) \( A \) is positive \[22\]; (2) \( A \) is primitive \[2\]; (3) \( A \) is irreducible \[23\] or weakly irreducible \[24\] with positive trace, then \( A \) is spectral nonsymmetric. Nikiforov \[15\] characterize a symmetric weakly irreducible nonnegative tensor with a symmetric spectrum by introducing the odd-coloring of a tensor, where an odd-coloring is exactly an \((m, 2)\)-coloring in the following definition for \( m \) being even.

**Definition 1.2.** Let \( m \geq 2 \) and \( \ell \geq 2 \) be integers such that \( \ell | m \). An \( m \)-th order \( n \)-dimensional tensor \( A \) is called \((m, \ell)\)-colorable if there exists a map \( \phi : [n] \to [m] \) such that if \( a_{i_1 \ldots i_m} \neq 0 \), then

\[
\phi(i_1) + \cdots + \phi(i_m) \equiv \frac{m}{\ell} \pmod{m}.
\]

Such \( \phi \) is called an \((m, \ell)\)-coloring of \( A \).
By the results in [23, 24], for a weakly irreducible nonnegative tensor \( A \) of order \( m \), if \( A \) is spectral \( \ell \)-symmetric, then there exists a diagonal matrix \( D \) such that 
\[
A = e^{-i\frac{\pi}{2}} D^{-\frac{1}{2}} (m-1) AD,
\]
where \( D \) is constructed from an eigenvector corresponding to the eigenvalue \( \rho(A)e^{i\frac{\pi}{2}} \). Let
\[
\mathcal{D} = \bigcup_{j=0}^{\ell-1} \mathcal{D}^{(j)},
\]
\[
\mathcal{D}^{(j)} = \{ D : A = e^{-i\frac{\pi}{2}} D^{-\frac{1}{2}} (m-1) AD, d_{11} = 1, j = 0, 1, \ldots, \ell - 1 \}.
\]
In the case of \( A \) being a matrix, for \( j = 0, 1, \ldots, \ell - 1 \), each \( \mathcal{D}^{(j)} \) contains only one element, and \( \mathcal{D} \) is a group of order \( \ell \) under the usual matrix multiplication. However, in the general case of \( A \) being a tensor, each \( \mathcal{D}^{(j)} \) may have more than one element, and contains more rich content. We show that \( \mathcal{D} \) is a finite abelian group containing \( \mathcal{D}^{(0)} \) as a subgroup. Let \( S = \{ e^{i\frac{\pi}{2}} : i = 0, 1, \ldots, \ell - 1 \} \). Then \( \mathcal{D} \) acts on \( S \) as a permutation group, where \( \mathcal{D}^{(0)} \) acts as a stabilizer of \( A \), and the quotient group \( \mathcal{D}/\mathcal{D}^{(0)} \) acts as a rotation over \( S \). The spectral symmetry of \( A \) is characterized by \( \mathcal{D}/\mathcal{D}^{(0)} \).

In this paper we mainly investigate the structure of a weakly irreducible nonnegative tensor \( A \) by the group \( \mathcal{D} \). The paper divides into two parts. First we give some properties of \( \mathcal{D} \) defined in (1.4), and then obtain the structural information of \( A \) similar to (1.1) with application to the edge distribution and coloring of connected hypergraphs. Consequently, we prove that a symmetric weakly irreducible nonnegative tensor is spectral \( \ell \)-symmetric if and only if it is \((m, \ell)\)-colorable, which generalizes the result of Nikiforov [15]. In the second part, we characterize the spectral \( \ell \)-symmetry and the cyclic index of a tensor by its generalized traces, which generalizes the result of Shao et al. [21]. We also prove that for an arbitrarily given integer \( m \geq 3 \) and for each positive integer \( \ell \) such that \( \ell \mid m \), there always exists an \( m \)-uniform hypergraph \( G \) such that its adjacency tensor is spectral \( \ell \)-symmetric.

2. Preliminaries

2.1. Notions. Let \( A \) be a real tensor of order \( m \) and dimension \( n \). The tensor \( A \) is called symmetric if its entries are invariant under any permutation of their indices. Given a vector \( x \in \mathbb{R}^n \), \( Ax^m \in \mathbb{R} \) and \( Ax^{m-1} \in \mathbb{R}^n \), which are defined as follows:
\[
Ax^m = \sum_{i_1, i_2, \ldots, i_m \in [n]} a_{i_1i_2\ldots i_m} x_{i_1}x_{i_2}\cdots x_{i_m},
\]
\[
(Ax^{m-1})_i = \sum_{i_2, \ldots, i_m \in [n]} a_{i_2\ldots i_m} x_{i_2}\cdots x_{i_m}, i \in [n].
\]
Let \( \mathcal{I} = (i_1i_2\ldots i_m) \) be the identity tensor of order \( m \) and dimension \( n \), that is, \( i_1i_2\ldots i_m = 1 \) if and only if \( i_1 = i_2 = \cdots = i_m = [n] \) and \( i_1i_2\ldots i_m = 0 \) otherwise.

**Definition 2.1.** [17] Let \( A \) be an \( m \)-th order \( n \)-dimensional real tensor. For some \( \lambda \in \mathbb{C} \), if the polynomial system \((AX - A)x^{m-1} = 0\), or equivalently \( Ax^{m-1} = \lambda x^{m-1} \), has a solution \( x \in \mathbb{C}^n \setminus \{0\} \), then \( \lambda \) is called an eigenvalue of \( A \) and \( x \) is an eigenvector of \( A \) associated with \( \lambda \), where \( x^{m-1} := (x_{1}^{m-1}, x_{2}^{m-1}, \ldots, x_{n}^{m-1}) \).

If \( x \) is a real eigenvector of \( A \), surely the corresponding eigenvalue \( \lambda \) is real. In this case, \( \lambda \) is called an \( H \)-eigenvalue of \( A \). The spectral radius of \( A \) is defined as
\[
\rho(A) = \max\{|\lambda| : \lambda \text{ is an eigenvalue of } A\}.
\]
Definition 2.2 ([11]). A tensor \( \mathcal{A} = (a_{i_1i_2...i_m}) \) of order \( m \) and dimension \( n \) is called reducible if there exists a nonempty proper index subset \( I \subset [n] \) such that \( a_{i_1i_2...i_m} = 0 \) for any \( i_1 \in I \) and any \( i_2, ..., i_m \notin I \). If \( \mathcal{A} \) is not reducible, then it is called irreducible.

For a tensor \( \mathcal{A} \) of order \( m \) and dimension \( n \), we associate it with a directed graph \( D(\mathcal{A}) \) on vertex set \([n]\) such that \((i, j)\) is an arc of \( D(\mathcal{A})\) if and only if there exists a nonzero entry \( a_{i_1i_2...i_m} \) such that \( j \in \{i_2, ..., i_m\} \). The tensor \( \mathcal{A} \) is called weakly irreducible if \( D(\mathcal{A}) \) is strongly connected; otherwise it is called weakly reducible [2].

It is known that if \( \mathcal{A} \) is irreducible, then it is weakly irreducible; but the converse is not true.

A hypergraph \( G = (V(G), E(G)) \) consists of a vertex set \( V(G) = \{v_1, v_2, ..., v_n\} \) and an edge set \( E(G) = \{e_1, e_2, ..., e_l\} \) where \( e_j \subseteq V(G) \) for \( j \in [l] \). If \( |e_j| = m \) for each \( j \in [l] \), then \( G \) is called an \( m\)-uniform hypergraph. In particular, the 2-uniform hypergraphs are exactly the classical simple graphs. The degree \( d_G(v) \) or simply \( d(v) \) of a vertex \( v \in V(G) \) is defined as \( d(v) = |\{e_j : v \in e_j \in E(G)\}| \). A walk \( W \) in \( G \) is a sequence of alternate vertices and edges: \( v_0, e_1, v_1, e_2, ..., e_l, v_l \), where \( \{v_i, v_{i+1}\} \subseteq e_{i+1} \) for \( i = 0, 1, ..., l-1 \). The hypergraph \( G \) is connected if every two vertices of \( G \) are connected by a walk, and is called \( k\)-colorable if there exists a map \( \phi : V(G) \to [k] \) such that each edge contains at least two vertices with different colors, or equivalently, the vertices can be partitioned into \( k \) subsets such that each edge intersects at least two subsets. The chromatic number \( \chi(G) \) is the smallest \( k \) such that \( G \) is \( k\)-colorable.

The adjacency tensor \( \mathcal{A}(G) \) of the hypergraph \( G \) is defined as \( \mathcal{A}(G) = (a_{i_1i_2...i_m}) \), an \( m\)-th order \( n\)-dimensional tensor, where

\[
a_{i_1i_2...i_m} = \begin{cases} \frac{1}{(m-1)!}, & \text{if } \{v_{i_1}, v_{i_2}, ..., v_{i_m}\} \in E(G); \\ 0, & \text{otherwise}. \end{cases}
\]

In this paper, the eigenvalues of a hypergraph \( G \) always refer to those of its adjacency tensor. Let \( D(G) \) be an \( m\)-th order \( n\)-dimensional diagonal tensor, where \( d_{i_1...i} = d(v_i) \) for \( i \in [n] \). The tensor \( \mathcal{L}(G) = D(G) - \mathcal{A}(G) \) is called the Laplacian tensor of \( G \), and \( \mathcal{Q}(G) = D(G) + \mathcal{A}(G) \) is called the signless Laplacian tensor of \( G \).

Observe that the adjacency (Laplacian, signless Laplacian) tensor of a hypergraph \( G \) is symmetric, and it is weakly irreducible if and only if the \( G \) is connected [16, 24]. However, even if \( G \) is connected, the tensor \( \mathcal{A}(G) \) (\( \mathcal{L}(G) \), \( \mathcal{Q}(G) \)) is always reducible when \( m \geq 3 \); for taking an arbitrary proper subset \( I \subset [n] \) with cardinality not less than \( n - (m-2) \), we always have \( a_{i_1i_2...i_m} = 0 \) for all \( i_1 \in I \) and all \( i_2, ..., i_m \notin I \) since there must exist repeated indices among \( i_2, ..., i_m \).

Let \( I_X \) be the indicator function of a set \( X \subset [n] \), and let \( \mathcal{A} \) be a tensor of order \( m \) and dimension \( n \). A set \( X \subset [n] \) is called an odd transversal of \( \mathcal{A} \) if \( a_{i_1...i_m} \neq 0 \) implies that

\[
I_X(i_1) + \cdots + I_X(i_m) \equiv 1 \mod 2.
\]

A tensor \( \mathcal{A} \) with an odd transversal is called an odd transversal tensor [15]. Odd transversal tensors were also named weakly odd-bipartite tensors by Chen an Qi [3], and were called odd-bipartite tensors in case of \( m \) being even.

When we say a hypergraph is \( \ell\)-symmetric (or \( \ell\)-cyclic, \( (m, \ell)\)-colorable, odd-colorable, odd transversal), it always referred to its adjacency tensor. An even uniform hypergraph is called odd-bipartite if its vertices can be partitioned into two subsets such that every edge intersects with each subset with an odd number of vertices.
Nikiforov [15] proved that if \( m \) is even, then an \( m \)-th order tensor with an odd transversal is always odd-colorable. Furthermore, if \( m \equiv 2 \mod 4 \), then these two notions are equivalent. However, if \( m \equiv 0 \mod 4 \), they construct two classes of \( m \)-uniform hypergraphs to illustrate that they are odd-colorable but not odd transversal. We also construct a class of non-odd-bipartite generalized power hypergraphs to illustrate the above fact [5].

Finally we introduce some special classes of hypergraphs. An \( m \)-uniform hypergraph \( G \) is called \( p \)-hm bipartite if the vertex set has a bipartition \( V(G) = V_1 \cup V_2 \) such that each edge of \( G \) intersects \( V_1 \) with exactly \( p \) vertices [21]. The notion of \( p \)-hm hypergraphs generalizes the hm-bipartite hypergraphs [3] (i.e. \( 1 \)-hm hypergraphs), and \( m \)-partite hypergraphs [1]. A cored hypergraph is one such that each edge contains one vertex of degree one [9], which is also hm-bipartite. An \( m \)-th power of a simple graph \( G \), denoted by \( G^m \), is obtained from \( G \) by replacing each edge (a 2-set) with a \( k \)-set by adding \( (k - 2) \) additional vertices [9], which is a cored hypergraph and hence hm-bipartite. A generalized power hypergraph \( G^{m,s} \) is constructed from a simple graph \( G \) by Khan and Fan [12] and from a hypergraph \( G \) by Kang et.al [11]. In particular, if \( t|m \), then \( G^{m,T} \) is simply obtained from \( G \) by blowing up each vertex into an \( \frac{m}{t} \)-set.

\[ \text{Definition 2.3 (11).} \] Let \( G = (V,E) \) be a \( t \)-uniform hypergraph. For any integers \( m,s \) such that \( m > t \) and \( 1 \leq s \leq \frac{m}{t} \), the generalized power of \( G \), denoted by \( G^{m,s} \), is defined as the \( m \)-uniform hypergraph with the vertex set \( (\cup_{v \in V} v) \cup (\cup_{e \in E} e) \), and the edge set \( \{u_1 \cup \ldots \cup u_i \cup e : e = \{u_1, \ldots, u_i \} \in E(G)\} \), where \( v \) denotes an \( s \)-set corresponding to \( v \) and \( e \) denotes an \( (m-ts) \)-set corresponding to \( e \), and all those sets are pairwise disjoint.

2.2. Characteristic polynomial of tensors. Let \( \mathcal{A} \) be an \( m \)-th order \( n \)-dimensional real tensor. The determinant of \( \mathcal{A} \), denoted by \( \det \mathcal{A} \), is defined as the resultant of the polynomials \( \mathcal{A}^m-1 \), and the characteristic polynomial \( \varphi_{\mathcal{A}}(\lambda) \) of \( \mathcal{A} \) is defined as \( \det(\lambda \mathcal{I} - \mathcal{A}) \) [2, 17]. It is known that \( \lambda \) is an eigenvalue of \( \mathcal{A} \) if and only if it is a root of \( \varphi_{\mathcal{A}}(\lambda) \). The algebraic multiplicity of \( \lambda \) as an eigenvalue of \( \mathcal{A} \) is defined as the multiplicity of \( \lambda \) as a root of \( \varphi_{\mathcal{A}}(\lambda) \). The spectrum of \( \mathcal{A} \), denoted by \( \text{Spec}(\mathcal{A}) \), is the multi-set of the roots of \( \varphi_{\mathcal{A}}(\lambda) \), including multiplicity. Denote by \( H\text{Spec}(\mathcal{A}) \) the set of distinct \( H \)-eigenvalues of \( \mathcal{A} \).

Morozov and Shakirov [14] give a formula for calculating \( \det(\mathcal{I} - \mathcal{A}) \) using Schur polynomials in the generalized traces of \( \mathcal{A} \). Let \( \mathcal{A} \) be an auxiliary matrix of order \( n \) with distinct variables \( a_{ij} \) as entries. The generalized d-th order trace of \( \mathcal{A} \) is defined by

\[
\text{Tr}_d(\mathcal{A}) = (m-1)^{n-1} \sum_{d_1 + \cdots + d_n = d} \frac{1}{(d_1(m-1))!} \left( \sum_{y \in [n]^{m-1}} t_{iy} \frac{\partial}{\partial a_{iy}} \right)^{d_i} \text{Tr}(\mathcal{A}^{d(m-1)}),
\]

where \( d_1, \ldots, d_n \) are nonnegative integers, \( t_{iy} = t_{i_1 \cdots i_m} \) and \( \frac{\partial}{\partial a_{iy}} = \frac{\partial}{\partial a_{i_1 \cdots i_2}} \cdots \frac{\partial}{\partial a_{i_1 \cdots i_m}} \) if \( y = i_1 \cdots i_m \). By the results in [1, 7, 21]

\[
\varphi_{\mathcal{A}}(\lambda) = \sum_{i=0}^{D} P_i \left( -\frac{\text{Tr}_1(\mathcal{A})}{1}, -\frac{\text{Tr}_2(\mathcal{A})}{2}, \ldots, -\frac{\text{Tr}_i(\mathcal{A})}{i} \right) \lambda^{D-i}, \quad (D = n(m-1)^{n-1}),
\]
where the Schur polynomial
\[
P_d(t_1, \ldots, t_d) = \sum_{m=1}^{d} \sum_{d_1 + \cdots + d_m = d, d_i \in \mathbb{Z}^+} \frac{t_{d_1} \cdots t_{d_m}}{m!}.
\]

It was proved in [7, 21] that
\[
\text{Tr}_d(A) = \sum_{i=1}^{D} \lambda_i^d,
\]
where \(\lambda_1, \ldots, \lambda_D\) are all eigenvalues of \(A\).

Shao, Qi and Hu [21] give a graph theoretic formula for the trace \(\text{Tr}_d(A)\). Denote
\[
F_d = \{((i_1, \alpha_1), \ldots, (i_d, \alpha_d)) : 1 \leq i_1 \leq \cdots \leq i_d \leq n; \alpha_1, \ldots, \alpha_n \in [n]^{m-1}\}.
\]
For each \(F = ((i_1, \alpha_1), \ldots, (i_d, \alpha_d)) \in F_d\), define a directed graph \(D(F)\) with arc multi-set \(E(F) = \bigcup_{j=1}^{d} E_j(F)\), where 
\[
E_j(F) = \{(i_j, v_1), (i_j, v_2), \ldots, (i_j, v_{m-1})\}
\]
if \(\alpha_j = (v_1, \ldots, v_{m-1})\). Here, for each tuple \((i_j, \alpha_j)\), \(i_j\) is called the primary index and the indices in \(\alpha_j\) are called secondary indices.

In the directed graph \(D(F)\), denote by \(b(F)\) the product of the factorials of the multiplicities of all arcs of \(E(F)\), by \(c(F)\) the product of the factorials of the out-degrees of all vertices incident to the arcs of \(E(F)\), and by \(W(F)\) the set of all closed walks with the arc multi-set \(E(F)\). Denote by \(\Pi_F(A) = \prod_{j=1}^{d} t_{i_j, \alpha_j}\). Then
\[
\text{Tr}_d(A) = (m-1)^{n-1} \sum_{F \in F_d} \frac{b(F)}{c(F)} |\Pi_F(A)| W(F)|.
\]

If one summand of (2.3) is nonzero for some \(F = ((i_1, \alpha_1), \ldots, (i_d, \alpha_d)) \in F_d\), then \(\Pi_F(A) = \prod_{j=1}^{d} t_{i_j, \alpha_j}\) is \(m\)-valent, i.e. each index occurs in the monomial \(\Pi_F(A)\) in times of multiple of \(m\). [4, 21]; furthermore, the directed graph \(D(F)\) contains an Eulerian directed circuit, i.e. \(W(F) \neq \emptyset\), or equivalently \(D(F)\) is connected and each vertex of \(D(F)\) has the same in-degree and out-degree. If, in addition, \(A = A(G)\) for some hypergraph \(G\), then, by omitting the order of tuples in \(F\), the above \(F\) can be written as a set
\[
F = \{e_1(i_1), \ldots, e_d(i_d)\},
\]
where \(e_j(i_j)\) denotes an edge \(e_j\) of \(G\) with primary index (vertex) \(i_j\) for \(j \in [d]\). If we write \(\varphi_A(\lambda) = \sum_{i=0}^{D} a_i \lambda^{D-i}\), where \(a_0 = 1\), then
\[
a_i = P_i \left( \frac{\text{Tr}_1(A)}{1}, \frac{\text{Tr}_2(A)}{2}, \ldots, \frac{\text{Tr}_i(A)}{i} \right), i = 1, \ldots, D.
\]

2.3. Perron-Frobenius theorem for nonnegative tensors. Chang et.al [1] generalize the Perron-Frobenius theorem for nonnegative matrices to nonnegative tensors. Yang and Yang [22, 23, 24] get some further results for Perron-Frobenius theorem, especially for the spectral symmetry. Friedland et.al [6] also get some results for weakly irreducible nonnegative tensors. We combine those results in the following theorem, where an eigenvalue is called \(H^+\)-eigenvalue (respectively \(H^{++}\)-eigenvalue) if it is associated with a nonnegative (respectively positive) eigenvector.

**Theorem 2.4** (The Perron-Frobenius Theorem for nonnegative tensors).

(1) (Yang and Yang [22]) If \(A\) is a nonnegative tensor of order \(m\) and dimension \(n\), then \(\rho(A)\) is an \(H^+\)-eigenvalue of \(A\).
(2) (Friedland, Gaubert and Han [6]) If furthermore \( A \) is weakly irreducible, then \( \rho(A) \) is the unique \( H^{++} \)-eigenvalue of \( A \), with the unique positive eigenvector, up to a positive scalar.

(3) (Chang, Pearson and Zhang [1]) If moreover \( A \) is irreducible, then \( \rho(A) \) is the unique \( H^{+} \)-eigenvalue of \( A \), with the unique nonnegative eigenvector, up to a positive scalar.

According to the definition of tensor product in [19], for a tensor \( A \) of order \( m \) and dimension \( n \), and two diagonal matrices \( P, Q \) both of dimension \( n \), the product \( PAQ \) has the same order and dimension as \( A \), whose entries are defined by

\[
(PAQ)_{i_1i_2...i_m} = p_{i_1}a_{i_2...i_m}q_{i_3i_4...i_m}.
\]

If \( P = Q^{-1} \), then \( A \) and \( P^{m-1}AQ \) are called diagonal similar. It is proved that two diagonal similar tensors have the same spectrum [19].

**Theorem 2.5** ([24]). Let \( A \) and \( B \) be two \( m \)-th order \( n \)-dimensional real tensors with \( |B| \leq |A| \). Then

1. \( \rho(B) \leq \rho(A) \).
2. Furthermore, if \( A \) is weakly irreducible and \( \rho(B) = \rho(A) \), where \( \lambda = \rho(A) e^{i\theta} \) is an eigenvalue of \( B \) corresponding to an eigenvector \( y \), then \( y \) contains no zero entries, and \( B = e^{-i\theta} D^{-(m-1)} AD \), where \( D = \text{diag}(\frac{y_1}{|y_1|}, \ldots, \frac{y_n}{|y_n|}) \).

**Theorem 2.6** ([24]). Let \( A \) be an \( m \)-th order \( n \)-dimensional weakly irreducible nonnegative tensor. Suppose \( A \) has \( k \) distinct eigenvalues with modulus \( \rho(A) \) in total. Then these eigenvalues are \( \rho(A)e^{\frac{i2\pi j}{k}} \), \( j = 0, 1, \ldots, k-1 \). Furthermore,

\[
A = e^{-i\frac{2\pi j}{k}} D^{-(m-1)} AD,
\]

and the spectrum of \( A \) keeps invariant under a rotation of angle \( \frac{2\pi}{k} \) (but not a smaller positive angle) of the complex plane.

Suppose \( A \) be as in Theorem 2.6. By Theorem 2.5 and Theorem 2.6, if \( \text{Spec}(A) \) is invariant under a rotation of angle \( \theta \) of the complex plane, then \( \theta = \frac{2\pi j}{k} \) for some positive \( k \) and some \( j \in [k] \). So

\[
\text{Spec}(A) = e^{i\frac{2\pi j}{k}} \text{Spec}(A).
\]

This is the motivation of our Definition 1.1. The number \( k \) in Theorem 2.6 is exactly the cyclic index of \( A \). In addition, if \( A \) is spectral \( \ell \)-symmetric, Then \( \ell \mid c(A) \) by Theorem 2.6. We have a more generalized result as follows.

**Lemma 2.7.** Let \( A \) be an \( m \)-th order \( n \)-dimensional tensor. If \( A \) is spectral \( \ell \)-symmetric, then \( \ell \mid c(A) \).

**Proof.** Let \( c := c(A) \). Assume to the contrary, \( \ell \nmid c \). Then there exists an integer \( h \) such that \( \frac{2\pi h}{c} < \frac{2\pi}{\ell} < \frac{2\pi(h+1)}{c} \), and hence \( \theta := \frac{2\pi}{\ell} - \frac{2\pi h}{c} < \frac{2\pi}{c} \). Write \( \theta = \frac{2\pi q}{p} \), where \( (p, q) = 1 \) and \( p > c \). As \( A \) is both spectral \( \ell \)-symmetric and spectral \( c \)-symmetric, we have

\[
\text{Spec}(A) = e^{i\frac{2\pi h}{c}} \text{Spec}(A) = e^{i\frac{2\pi q}{p}} \text{Spec}(A).
\]

Since \( (p, q) = 1 \), there exist integers \( h_1, h_2 \) such that \( ph_1 + qh_2 = 1 \). So

\[
\text{Spec}(A) = e^{i\frac{2\pi h_1}{p}} \text{Spec}(A) = e^{i\frac{2\pi h_2}{c}} \text{Spec}(A) = e^{i\frac{2\pi}{c}} \text{Spec}(A),
\]

which implies that \( A \) is spectral \( p \)-symmetric, a contradiction as \( p > c = c(A) \).
3. Structure of nonnegative weakly irreducible tensors

In this section we will first analyze the property of the group $\mathcal{D}$ defined in (1.4) by the theory of finite abelian group. Then we obtain some structural information of a weakly irreducible nonnegative tensor, with application to the edge distribution and coloring of connected hypergraphs.

**Lemma 3.1** ([24]). Let $\mathcal{A}$ be an $m$-th order $n$-dimensional weakly irreducible nonnegative tensor. Let $y$ be an eigenvector of $\mathcal{A}$ corresponding to an eigenvalue $\lambda$ with $|\lambda| = \rho(\mathcal{A})$. Then $|y|$ is the unique positive eigenvector corresponding to $\rho(\mathcal{A})$ up to a scalar.

Let $r$ be a positive integer and let $p$ be a prime number. Denote by $r_p$ the maximum power of $p$ that divides $r$. Surely, if $p \nmid r$, then $r_p = 1$.

**Lemma 3.2.** Let $\mathcal{A}$ be an $m$-th order $n$-dimensional weakly irreducible nonnegative tensor, which is spectral $\ell$-symmetric. Suppose $\mathcal{A}$ has $r$ distinct eigenvectors corresponding to $\rho(\mathcal{A})$ in total up to a scalar. Let $\mathcal{D}$ and $\mathcal{D}^{(j)}$ be as defined in (1.4) for $j = 0, 1, \ldots, \ell - 1$. Then the following results hold.

1. $\mathcal{D}$ is a finite abelian group of order $r\ell$ under matrix multiplication, where $\mathcal{D}^{(0)}$ is a subgroup of $\mathcal{D}$ of order $r$, and $\mathcal{D}^{(j)}$ is a coset of $\mathcal{D}^{(0)}$ in $\mathcal{D}$ for $j = 1, \ldots, \ell - 1$.
2. For each prime factor $p$ of $\ell$, there exists a matrix $D \in \mathcal{D} \setminus \mathcal{D}^{(0)}$ such that $D^{p|p} = I$ and hence $D^{p|\ell} = D^p = D^{\ell} = I$.
3. If further $\mathcal{A}$ is symmetric, then $\ell | m$, and $D^m = I$ for any $D \in \mathcal{D}$. In particular, each elementary divisor of $\mathcal{D}$ divides $m$ and each prime factor of $r\ell$ divides $m$.

**Proof.** (1) Surely the identity matrix $I \in \mathcal{D}^{(0)}$. For any two matrices $D^{(j_1)} \in \mathcal{D}^{(j_1)}$ and $D^{(j_2)} \in \mathcal{D}^{(j_2)}$, we have

$$\mathcal{A} = e^{-\frac{2\pi ij_1}{\ell}}D^{(j_1)}(m-1)AD^{(j_1)}, \quad \mathcal{A} = e^{-\frac{2\pi ij_2}{\ell}}D^{(j_2)}(m-1)AD^{(j_2)}.$$ 

Then

$$\mathcal{A} = e^{-\frac{2\pi j_1 j_2 \ell}{\ell}}(D^{(j_1)}D^{(j_2)})^{-1}(m-1)A(D^{(j_1)}D^{(j_2)}).$$

So $D^{(j_1)}D^{(j_2)} \in \mathcal{D}^{(j_1+j_2)}$ where the superscript is taken modulo $\ell$. It is seen that $(D^{(j_1)})^{-1} \in \mathcal{D}^{(-j_1)}$. So $\mathcal{D}$ is a group under the usual matrix multiplication.

Following the same routine, one can verify that $\mathcal{D}^{(0)}$ is a subgroup of $\mathcal{D}$. Taking a $D^{(j)} \in \mathcal{D}^{(j)}$, one can show that $\mathcal{D}^{(j)} = \mathcal{D}^{(0)}D^{(j)}$, i.e. $\mathcal{D}^{(j)}$ is a coset of $\mathcal{D}^{(0)}$ by verifying $\bar{D}^{(j)}D^{(j)}^{-1} \in \mathcal{D}^{(0)}$ and $D^{(0)}D^{(j)} \in \mathcal{D}^{(j)}$ for any $\bar{D}^{(j)} \in \mathcal{D}^{(j)}$ and $D^{(0)} \in \mathcal{D}^{(0)}$.

Next we will show $\mathcal{D}^{(0)}$ has $r$ elements, and hence $\mathcal{D}$ has $r\ell$ elements by the above discussion. Let $y^{(01)}, \ldots, y^{(0r)}$ be the $r$ distinct eigenvectors of $\mathcal{A}$ corresponding to $\rho(\mathcal{A})$ up to a scalar, each of which contains no zero entries by Lemma 3.1.

Without loss of generality, assume $y^{(0j)} = 1$ for $j \in [r]$. Define

$$D^{(0j)} = \text{diag} \left( \begin{array}{c} \frac{y^{(0j)}_1}{|y^{(0j)}_1|}, \ldots, \frac{y^{(0j)}_n}{|y^{(0j)}_n|} \end{array} \right), \quad j = 1, \ldots, r.$$  


By Theorem 2.5, $D^{(0)} \in \mathcal{D}^{(0)}$ for $j \in [r]$. By Theorem 2.4(2), we may assume $y^{(01)} > 0$ so that $D^{(01)} = I$. Now suppose that $D \in \mathcal{D}^{(0)}$. From the equalities

$$A(y^{(01)})^{m-1} = \rho(A)(y^{(01)})^{m-1},$$

we get

$$A(Dy^{(01)})^{m-1} = \rho(A)(Dy^{(01)})^{m-1}.$$ 

So $Dy^{(01)}$ is an eigenvector of $A$ corresponding to $\rho(A)$ with $(Dy^{(01)})_{1} = 1$, which implies that $Dy^{(01)} = y^{(01)}$ for some $l \in [r]$ and $D = D^{(0)}$.

(2) Let $p$ be a prime factor of $\ell$. First assume $r_{[p]} = 1$, i.e. $p \nmid r$. By Cauchy Theorem, $\mathcal{D}$ contains an element $D$ of order $p$. Since $p \nmid r$ and $\mathcal{D}^{(0)}$ has order $r$, $D \notin \mathcal{D}^{(0)}$ by Lagrangian Theorem.

Next suppose that $p \mid r$. Let $\mathcal{D}(p), \mathcal{D}^{(0)}(p)$ be the Sylow $p$-subgroups of $\mathcal{D}$ and $\mathcal{D}^{(0)}$ respectively. Observe that $\mathcal{D}^{(0)}(p)$ is a proper subgroup of $\mathcal{D}(p)$, and

$$\mathcal{D}(p) \cong \mathbb{Z}_{p^{e_{1}}} \oplus \cdots \oplus \mathbb{Z}_{p^{e_{r}}}, \quad \mathcal{D}^{(0)}(p) \cong \mathbb{Z}_{p^{f_{1}}} \oplus \cdots \oplus \mathbb{Z}_{p^{f_{\xi}}},$$

where $e_{1} \geq \cdots \geq e_{r} \geq 1$ and $f_{1} \geq \cdots \geq f_{\xi} \geq 1$. Note that $e_{1} \geq f_{1}$. If $e_{1} > f_{1}$, then $\mathcal{D} \setminus \mathcal{D}^{(0)}$ contains an element $D$ of order $p^{f_{1}}$. So $D^{p^{f_{1}}-1} = I$, and hence $(D^{p^{f_{1}}-1})^{p^{f_{1}+1}} = I$. Let $\tilde{D} = D^{p^{f_{1}}-1}$. Then $\tilde{D}$ has order $p^{f_{1}+1}$ that divides $r_{[p]}p$, implying that $\tilde{D} \notin \mathcal{D}^{(0)}$, as desired. Otherwise, $e_{1} = f_{1}$. We consider $\mathbb{Z}_{p^{f_{2}}}$ and $\mathbb{Z}_{p^{f_{2}}}$, and repeat the above process. Finally we have two cases: (i) there exists a $j \in [r]$ such that $e_{j} > f_{j}$, (ii) $\tau \geq \xi + 1$, and for each $j \in [\xi]$, $e_{j} = f_{j}$. If the case (i) occurs, we will obtain a desired matrix like $\tilde{D}$ as in the above. Otherwise, there is an element $D \in \mathcal{D} \setminus \mathcal{D}^{(0)}$ of order $p^{\tau+1}$, implying that $\mathcal{D} \setminus \mathcal{D}^{(0)}$ contains an element of order $p$, as desired.

(3) Suppose $A$ is symmetric. From the equality $A = e^{-12\pi i}D^{-(m-1)}AD$, letting $d_{ii} = e^{i\theta_{i}}$ for $i \in [n]$ where $\theta_{1} = 0$, if $a_{i_{1} \ldots i_{m}} \neq 0$, we have

$$\frac{2\pi j}{\ell} + m\theta_{i_{1}} \equiv \theta_{i_{1}} + \cdots + \theta_{i_{m}} \mod 2\pi.$$

As $A$ is symmetric, replacing $i_{1}$ in left side of (3.2) by $i_{1}$ and summing over all $l = 1, \ldots, m$,

$$\frac{2\pi jm}{\ell} + m \sum_{l=1}^{m} \theta_{i_{l}} \equiv m \sum_{l=1}^{m} \theta_{i_{l}} \mod 2\pi.$$ 

So we have $\frac{2\pi jm}{\ell} \equiv 0 \mod 2\pi$. If taking $j = 1$, we have $\ell \mid m$. Also from (3.2), we have

$$m\theta_{i_{1}} = \ldots = m\theta_{i_{m}} \mod 2\pi.$$ 

As $A$ is weakly irreducible and $\theta_{1} = 0$, we get $m\theta_{l} \equiv 0 \mod 2\pi$. So $D^{\ell} = I$ for any $D \in \mathcal{D}$. The result follows. \(\square\)

**Remark 3.3.** Under the assumption of Theorem 3.2 if taking $\ell = c(A)$, then all the results also hold. In particular, if $A$ is symmetric, then $c(A) = m$, which is proved in [22]. Let $S = \{e^{i2\pi i}A : i = 0, 1, \ldots, \ell - 1\}$. Then $\mathcal{D}$ acts on $S$ as a permutation group, where $\mathcal{D}^{(0)}$ acts as a stabilizer of $A$, and the quotient $\mathcal{D}/\mathcal{D}^{(0)}$ acts as a rotation over $S$. The spectral symmetry of $A$ is characterized by $|\mathcal{D}/\mathcal{D}^{(0)}|$-symmetric.
Denote \( s(A) := |\mathcal{D}(0)| \) and \( s(G) := s(A(G)) \) for a uniform hypergraph \( G \), which are exactly the number of distinct eigenvectors of \( A \) and \( A(G) \) corresponding to their spectral radii up to a scalar, respectively.

**Corollary 3.4.** Let \( A \) be an \( m \)-th order \( n \)-dimensional weakly irreducible nonnegative tensor with \( s(A) = r \), which is spectral \( \ell \)-symmetric. Let \( \lambda_j = \rho(A)e^{\frac{2\pi i}{\ell^r}} \) be an eigenvalue of \( A \) corresponding to an eigenvector \( y^{(j)} \) with \( y_1^{(j)} = 1 \) for \( j = 0, 1, \ldots, \ell - 1 \).

1. If \( j = 0 \), then \( y^{[r]} > 0 \).
2. For each \( j \in [\ell - 1] \), \( (y^{(j)})^{[r\ell]} > 0 \).
3. For each prime factor \( p \) of \( \ell \), there exists some \( j \in [\ell - 1] \) such that \( (y^{(j)})^{[r\mid p]} > 0 \).
4. If further \( A \) is symmetric, then for each \( j = 0, 1, \ldots, \ell - 1 \), \( (y^{(j)})^{[m]} > 0 \).

**Proof.** Similar to (3.1), we can construct a diagonal matrix \( D \) by the eigenvector \( y^{(j)} \):

\[
D = \text{diag} \left( \frac{y_1^{(j)}}{y^{(j)}}, \ldots, \frac{y_n^{(j)}}{y^{(j)}} \right).
\]

By the results in [22, 24], we have \( A = e^{-\frac{2\pi i}{\ell}} D^{-m-1} A D \). If \( j = 0 \), then \( D \in \mathcal{D}(0) \) and \( D^r = I \) by Lemma 3.2(1), yielding the result (1). For each \( j \in [\ell - 1] \), as \( \mathcal{D} \) has order \( r\ell \), we have \( D^{r\ell} = I \), implying the result (2). The results (3) and (4) can be obtained similarly by Lemma 3.2(2-3).

**Lemma 3.5.** Let \( A \) be an \( m \)-th order \( n \)-dimensional weakly irreducible nonnegative tensor, which is spectral \( \ell \)-symmetric. If there exists a diagonal matrix \( D \in \mathcal{D}(j) \) such that \( D \neq I \) and \( D^\sigma = I \), then there exists a partition of \([n] = V_1 \cup \cdots \cup V_s \) for some integer \( s \geq 2 \) and a map \( \phi : [n] \to [\sigma] \) satisfying \( \phi|_{V_i} = l_i \) for \( i \in [s] \), where \( l_1(=\sigma), l_2, \ldots, l_s \) are distinct integers, such that if \( a_{i_1} \cdots i_m \neq 0 \), then

\[
j/\ell + m\phi(i_1)/\sigma = \phi(i_1) + \cdots + \phi(i_m)/\sigma \mod \mathbb{Z};
\]

furthermore, if \( A \) is also symmetric, then \( \sigma \mid ml_i \) for each \( i \in [s] \), and

\[
j/\ell = \phi(i_1) + \cdots + \phi(i_m)/\sigma \mod \mathbb{Z}.
\]

**Proof.** By the definition of \( \mathcal{D}(j) \), as \( D^\sigma = I \), without loss of generality, we write

\[
D = I_{n_1} + e^{\frac{2\pi i}{\sigma}l_2} I_{n_2} + \cdots + e^{\frac{2\pi i}{\sigma}l_s} I_{n_s},
\]

where \( s \geq 2 \) as \( D \neq I \), \( I_t \) denotes an identity matrix of dimension \( t \), and \( l_1(=\sigma), l_2, \ldots, l_s \) are distinct integers not greater than \( \sigma \). So we have a partition of \([n] = V_1 \cup \cdots \cup V_s \) such that \( V_i \) consists of the indices indexed by \( I_{n_i} \) for \( i \in [s] \), and a map \( \phi : [n] \to [\sigma] \) satisfying \( \phi|_{V_i} = l_i \) for \( i \in [s] \). As \( A = e^{-\frac{2\pi i}{\ell}} D^{-m-1} A D \), if \( a_{i_1} \cdots i_m \neq 0 \) for \( i_1 \in V_{i_1}, \ldots, i_m \in V_{i_m} \), then

\[
j/\ell + ml_{i_1}/\sigma = l_{i_1} + \cdots + l_{i_m}/\sigma \mod \mathbb{Z},
\]

yielding (3.3).
If $\mathcal{A}$ is symmetric, replacing $l_{i_j}$ in the left side of (3.5) by $l_{i_j}$ for $j \in [m]$, then (3.5) also holds. So for $j \in [m]$,
$$\frac{ml_{i_j}}{\sigma} \equiv \frac{ml_{i_j}}{\sigma} \mod \mathbb{Z},$$
As $\mathcal{A}$ is weakly irreducible and $l_1 = \sigma$, we have $\frac{ml_{i_j}}{\sigma} \equiv 0 \mod \mathbb{Z}$, which implies that $\sigma \mid ml_j$ for each $j \in [s]$. So, from (3.5) we get
$$\frac{j}{\ell} \equiv \frac{l_{i_1} + \cdots + l_{i_m}}{\sigma} \mod \mathbb{Z},$$
yielding (3.4). \qed

Now we will arrive at a result on hypergraph coloring by using spectral symmetry. By Lemma 3.2(1), for any $D \in \mathcal{D}$, $D^{r\ell} = I$ as $\mathcal{D}$ has order $r\ell$. But, for the coloring problem, we need as few colors as possible. So we will use the matrix $D$ in (2) or (3) of Lemma 3.2.

**Corollary 3.6.** Let $G$ be a connected $m$-uniform hypergraph with $s(G) = r$, which is spectral $\ell$-symmetric ($\ell \geq 2$). Then the following results hold.

1. $G$ is $r_{[p]}p$-colorable for each prime number $p \mid \ell$.
2. $G$ is $(m, \ell)$-colorable and $m$-colorable.
3. $\chi(G) \leq \min\{r_{[p]}p, m\}$, where the minimum is taken over all prime numbers $p$ with $p \mid \ell$.

**Proof.** By Lemma 3.2(2), there exists a diagonal matrix $D \in \mathcal{D}^{(j)}$ for some $j \in [\ell-1]$ such that $D^{r_{[p]}p} = I$. So by Lemma 3.5 and (3.3), there exists a map $\phi : [n] \to [r_{[p]}p]$ such that if $e = \{i_1, \ldots, i_m\} \in E(G)$ (or equivalently $a_{i_1}, \ldots, a_{i_m} \neq 0$), then
$$\frac{j}{\ell} \equiv \frac{\phi(i_1) + \cdots + \phi(i_m)}{r_{[p]}p} \mod \mathbb{Z}.$$

If the vertices contained in $e$ receive the same color from $\phi$, as $(r_{[p]}p) \mid m\phi(i_1)$ by Lemma 3.5, we have
$$\frac{j}{\ell} \equiv \frac{m\phi(i_1)}{r_{[p]}p} \equiv 0 \mod \mathbb{Z},$$
which yields that $\frac{j}{\ell}$ is an integer, a contradiction. So $G$ is $r_{[p]}p$-colorable.

Similarly, as $D^m = I$ for any $D \in \mathcal{D}$ by Lemma 3.2(3), by (3.4) in Lemma 3.3 we have
$$\frac{1}{\ell} \equiv \frac{\phi(i_1) + \cdots + \phi(i_m)}{m} \mod \mathbb{Z},$$
yielding $G$ has an $(m, \ell)$-coloring, and hence $G$ is $m$-colorable. The last result is obtained immediately from the above discussion. \qed

Finally we will investigate the structure of weakly irreducible nonnegative tensor, i.e. the zero entries distribution. When applying to the hypergraphs, we will get the information of edge distribution.

**Corollary 3.7.** Let $\mathcal{A}$ be an $m$-th order $n$-dimensional weakly irreducible nonnegative tensor with $s(\mathcal{A}) = r \geq 2$. Then there exists a partition of $[n] = V_1 \cup \cdots \cup V_s$ for some integer $s \geq 2$ and a map $\phi : [n] \to [r]$ satisfying $\phi|_{V_i} = l_i$ for $i \in [s]$, where $l_1 = r, l_2, \ldots, l_s$ are distinct integers, such that if $ml_{i_j} \not\equiv (l_{i_1} + \cdots + l_{i_m}) \mod r$, then
$$\mathcal{A}[V_{i_1}|V_{i_2}|\cdots|V_{i_m}] = 0.$$
Furthermore, if $\mathcal{A}$ is also symmetric, then if $r \nmid (l_{i_1} + \cdots + l_{i_m})$, then
\begin{equation}
\mathcal{A}[V_{i_1}|V_{i_2}|\cdots|V_{i_m}] = 0;
\end{equation}
or there exists a partition of $[n] = U_1 \cup \cdots \cup U_t$ for some integer $t \geq 2$ and a map
$\psi : [n] \rightarrow [T]$ satisfying $\psi|_{U_i} = q_i$ for $i \in [t]$, where $q_1(= m), q_2, \ldots, q_t$ are
distinct integers, such that if $m \nmid (q_{t_1} + \cdots + q_{t_m})$, then
\begin{equation}
\mathcal{A}[U_{i_1}|U_{i_2}|\cdots|U_{i_m}] = 0.
\end{equation}

\textbf{Proof.} By Lemma 3.2(1), as $r \geq 2$, choose $D \in \mathcal{D}^{(0)}$ such that $D \neq I$ and \( D^r = I \).
By (3.3) of Lemma 3.5, taking $j = 0$ and $\sigma = r$, then there exists a partition of
$[n] = V_1 \cup \cdots \cup V_s$ and a map $\phi : [n] \rightarrow [r]$ satisfying $\phi|_{V_i} = l_i$ for $i \in [s]$, where
$l_1(= r), l_2, \ldots, l_s$ are distinct integers, such that if $a_{i_1\ldots i_m} \neq 0$ for $i_1 \in V_{i_1}, \ldots, i_m \in V_{i_m},$
\[ m \ell_i = \frac{l_{i_1} + \cdots + l_{i_m}}{r} \equiv \ell_i \pmod{Z}, \]
implies (3.6). If $\mathcal{A}$ is also symmetric, by (3.4) of Lemma 3.5, we get $r \mid (l_{i_1} + \cdots + l_{i_m})$ if
$a_{i_1\ldots i_m} \neq 0$ for $i_1 \in V_{i_1}, \ldots, i_m \in V_{i_m}$, yielding (3.7). As $D^m = I$ for
all $D \in \mathcal{D}$, we have a $D \in \mathcal{D}^{(0)}$ such that $D \neq I$ and $D^m = I$. The remaining
discussion is similar by using (3.4) and taking $j = 0$ and $\sigma = m$. \hfill \Box

\textbf{Corollary 3.8.} Let $\mathcal{A}$ be an $m$-th order $n$-dimensional weakly irreducible nonnegative
tensor with $s(\mathcal{A}) = r$, which is spectral $\ell$-symmetric ($\ell \geq 2$). Then there exists
a partition of $[n] = V_1 \cup \cdots \cup V_s$ for some integer $s \geq 2$ and a map $\phi : [n] \rightarrow [r\ell]$ satisfying
$\phi|_{V_i} = l_i$ for $i \in [s]$, where $l_1(= r\ell), l_2, \ldots, l_s$ are distinct integers, such that if $r + m\ell_i \neq (l_{i_1} + \cdots + l_{i_m}) \mod r\ell$, then
\begin{equation}
\mathcal{A}[V_{i_1}|V_{i_2}|\cdots|V_{i_m}] = 0.
\end{equation}
Furthermore, if $\mathcal{A}$ is also symmetric, if $r \nmid (l_{i_1} + \cdots + l_{i_m}) \mod r\ell$, then
\begin{equation}
\mathcal{A}[V_{i_1}|V_{i_2}|\cdots|V_{i_m}] = 0,
\end{equation}
or there exists a partition of $[n] = U_1 \cup \cdots \cup U_t$ for some integer $t \geq 2$ and a coloring
$\psi : [n] \rightarrow [T]$ satisfying $\psi|_{U_i} = q_i$ for $i \in [t]$, where $q_1(= m), q_2, \ldots, q_t$ are
distinct integers, such that if $m \nmid (q_{t_1} + \cdots + q_{t_m}) \mod m$, then
\begin{equation}
\mathcal{A}[U_{i_1}|U_{i_2}|\cdots|U_{i_m}] = 0.
\end{equation}

\textbf{Proof.} By Lemma 3.2(2), there exists a diagonal $D \in \mathcal{D}^{(1)}$ with $D^{r\ell} = I$. Taking
$j = 1$ and $\sigma = r\ell$ in Lemma 3.5 we get the first result from (3.3) and the second result from (3.4).
Also, for any $D \in \mathcal{D}$, $D^m = I$. So, taking $j = 1$ and $\sigma = m$ in Lemma 3.5 we get the third result from (3.4). \hfill \Box

\textbf{Remark 3.9.} Let $\mathcal{A}$ be an $m$-th order $n$-dimensional weakly irreducible nonnegative
tensor. If $m = 2$, then $\mathcal{A}$ is an irreducible nonnegative matrix $\mathcal{A}$, $s(\mathcal{A}) = 1$ and
c(\mathcal{A}) \geq 1. If moreover c(\mathcal{A}) \geq 2, then $\mathcal{A}$ has a structure as in (3.3).
Now for the case of $m \geq 3$, it will happen that $s(\mathcal{A}) \geq 2$ or c(\mathcal{A}) \geq 2. If $s(\mathcal{A}) \geq 2$,
then $\mathcal{A}$ has a structure as in (3.4), (3.7) or (3.8). If c(\mathcal{A}) \geq 2, then $\mathcal{A}$ has a structure as in (3.9), (3.10) or (3.11).
This is the difference between low-dimensional tensors (matrices) and high-dimensional tensors.

\textbf{Corollary 3.10.} Let $\mathcal{A}$ be an $m$-th order $n$-dimensional symmetric weakly irreducible
nonnegative tensor, which is spectral $\ell$-symmetric ($\ell \geq 2$). Then $\mathcal{A}$ is $(m, \ell)$-colorable.
Proof. By Corollary 3.8 and (3.11), we have a map \( \phi : [n] \to [m] \) such that (1.3) holds, implying that \( A \) has an \((m, \ell)\)-coloring. \( \square \)

Lemma 3.11. If an \( m \)-th order \( n \)-dimensional tensor \( A \) is \((m, \ell)\)-colorable, then it is spectral \( \ell \)-symmetric.

Proof. Suppose that \( A \) has an \((m, \ell)\)-coloring \( \phi : [n] \to [m] \). Let
\[
D = \text{diag}(e^{i2\pi/2}, \ldots, e^{i2\pi/2})
\]
It is easy to verify that \( A = e^{-i2\pi}D^{(m-1)}AD \) by (1.3). So \( A \) is spectral \( \ell \)-symmetric.

The following two theorems follow by Corollary 3.10 and Lemma 3.11 immediately, which generalize the Nikiforov’s results on spectral 2-symmetry.

Theorem 3.12. Let \( A \) be a symmetric weakly irreducible nonnegative tensor of order \( m \). Then \( A \) is spectral \( \ell \)-symmetric if and only if \( A \) is \((m, \ell)\)-colorable.

Theorem 3.13. Let \( G \) be a connected \( m \)-uniform hypergraph. Then \( G \) is spectral \( \ell \)-symmetric if and only if \( G \) is \((m, \ell)\)-colorable.

Example 3.14. Let \( A = (a_{ijk}) \) be a tensor of order 3 and dimension 6 (cf. [15]), where \( i, j, k \in [6] \), such that \( a_{123} = a_{234} = a_{345} = a_{456} = a_{561} = a_{612} = 1 \), and all other entries are zero. The eigen-equations of \( A \) are
\[
\lambda x_1^2 = x_2 x_3, \lambda x_2^2 = x_3 x_4, \lambda x_3^2 = x_4 x_5, \lambda x_4^2 = x_5 x_6, \lambda x_5^2 = x_6 x_1, \lambda x_6^2 = x_1 x_2.
\]
If \( \lambda \neq 0 \), then \( \lambda^6 = 1 \), implying \( A \) is spectral 6-cyclic (i.e. \( c(A) = 6 \)). If taking \( x_1 = 1 \) and \( x_2 \) as a parameter, then by the first four equations of (3.12) we get
\[
x_1 = 1, x_2 = x_2, x_3 = \lambda x_2^{-1}, x_4 = x_2^3, x_5 = \lambda^3 x_2^{-5}, x_6 = \lambda^4 x_2^{11}.
\]
From the 5th or 6th equation, we get \( x_2^3 = \lambda^3 \). So, if letting \( \lambda = 1, \tau = e^{i2\pi/21} \) in (3.13), then get 21 different eigenvectors \( x^{(ij)} (j \in [21]) \) corresponding \( \rho(A) \) listed in Table 1 implying \( s(A) = 21 \).

Let \( \omega = e^{i2\pi/6} \) and \( \xi = e^{i2\pi/21} \). Then by a similar discussion, for each \( i \in [5] \), we get 21 different eigenvectors \( x^{(ij)} (j \in [21]) \) corresponding to the eigenvalue \( \omega^i \) listed in Table 3.1.

\begin{tabular}{|c|l|}
\hline
Eigenvalues & Eigenvectors \\
\hline
1 & \( x^{(0j)} = (1, \tau^j, (\tau^j)^{-1}, (\tau^j)^3, (\tau^j)^{-5}, (\tau^j)^{11}) \) \\
\hline
\( \omega \) & \( x^{(1j)} = (1, \tau^j, \omega (\tau^j)^{-1}, (\tau^j)^3, \omega^3 (\tau^j)^{-5}, \omega^3 (\tau^j)^{11}) \) \\
\hline
\( \omega^2 \) & \( x^{(2j)} = (1, \tau^j, \omega^2 (\tau^j)^{-1}, (\tau^j)^3, \omega^2 (\tau^j)^{-5}, \omega^2 (\tau^j)^{11}) \) \\
\hline
\( \omega^3 \) & \( x^{(3j)} = (1, \tau^j, \omega^3 (\tau^j)^{-1}, (\tau^j)^3, \omega^3 (\tau^j)^{-5}, (\tau^j)^{11}) \) \\
\hline
\( \omega^4 \) & \( x^{(4j)} = (1, \tau^j, \omega^4 (\tau^j)^{-1}, (\tau^j)^3, \omega^4 (\tau^j)^{-5}, \omega^4 (\tau^j)^{11}) \) \\
\hline
\( \omega^5 \) & \( x^{(5j)} = (1, \tau^j, \omega^5 (\tau^j)^{-1}, (\tau^j)^3, \omega^5 (\tau^j)^{-5}, \omega^2 (\tau^j)^{11}) \) \\
\hline
\end{tabular}

For each \( i = 0, 1, \ldots, 5 \) and each \( j \in [21] \), we associated the eigenvector \( x^{(ij)} \) with a diagonal matrix \( D^{(ij)} = \text{diag}(x^{(ij)}) = \text{diag}(x_1^{(ij)}, \ldots, x_6^{(ij)}) \), and form a set
Eigenvectors \( x \) corresponding to an eigenvalue with modulus \( x \) with the all-one vector as an eigenvector. By Lemma 3.1, for any eigenvector \( A \rightarrow G \) ample 3.14. Let Example 3.15. So each elementary divisor of \( D \) are \( (j,k) \) for \( D = \{3,2,1\} \). Such \( D \rightarrow I \) is contained in \( D \rightarrow (3), i.e. \)

Writing

\[ D^{(01)} = \text{diag} \left( 1 = e^{i \frac{2\pi}{21}}, e^{i \frac{4\pi}{21}}, e^{i \frac{6\pi}{21}}, e^{i \frac{8\pi}{21}}, e^{i \frac{10\pi}{21}}, e^{i \frac{12\pi}{21}}, e^{i \frac{14\pi}{21}} \right). \]

Then we have a partition \( [n] = V_1 \cup \cdots \cup V_6 \), where \( V_i = \{i\} \) for \( i \in [6] \); and a map \( \phi_1 : [n] \to [21] \), such that

\[ \phi_1(1) = 21, \phi_1(2) = 1, \phi_1(3) = 20, \phi_1(4) = 3, \phi_1(5) = 16, \phi_1(6) = 11. \]

One can verify that \( 3.6 \) of Corollary 3.7 holds. For example, as the solutions of

\[ D \mid D \}

and \( 12 \equiv 0 \) for \( j, k \in [6] \).

As \( 2 \mid 6 \) and \( 2 \nmid 21 \), by Lemma 3.2(2), there exists a \( D \in D \setminus D^{(0)} \) such that \( D^2 = I \). Such \( D \) is contained in \( D^{(3)} \), i.e.

\[ D = \text{diag} \{ x^{(3,10)} \} = \text{diag} \{ 1, -1, 1, -1, 1, -1 \}. \]

**Example 3.15.** Next we consider a symmetrization form of the tensor \( A \) in Example 3.14 Let \( G \) be a 3-uniform hypergraph with vertex set \( [6] \) and edge set

\[ \{1,2,3\}, \{2,3,4\}, \{3,4,5\}, \{4,5,6\}, \{5,6,1\}, \{6,1,2\} \].

Let \( A = A(G) \) be the adjacency tensor of \( G \). Since \( G \) is 3-regular, \( \rho(G) = 3 \), with the all-one vector as an eigenvector. By Lemma 3.1 for any eigenvector \( x \) corresponding to an eigenvalue with modulus \( \rho(G) \), \( |x| = x^{(01)} \) if normalizing \( x_1 = 1 \); and by Corollary 3.4, \( x^{[3]} = x^{(01)} \). Let \( \omega = e^{i \frac{2\pi}{3}} \). By the eigenvector equations of \( A(G) \), the eigenvalues with modulus \( \rho(G) \) are \( \lambda_k = 3\omega^k \) \((k = 0, 1, 2)\), which are corresponding to the eigenvectors \( x^{(k)} \) \((j \in [3]) \) listed as in Table 2 that is, \( s(G) = 3 \) and \( c(G) = 3 \).

**Table 2.** Eigenvalue and eigenvectors of the tensor in Example 3.15

| Eigenvalues | Eigenvectors |
|-------------|--------------|
| 3 | \( x^{(0)} = (1, \omega^j, \omega^{-j}, 1, \omega^j, \omega^{-j}) \) |
| \( 3\omega \) | \( x^{(1)} = (1, \omega^j, \omega^{-j}, 1, \omega^j, \omega^{-j}) \) |
| \( 3\omega^2 \) | \( x^{(2)} = (1, \omega^j, \omega^{-j}, 1, \omega^j, \omega^{-j}) \) |

For each \( i = 0, 1, 2 \) and each \( j \in [3] \), define \( D^{(ij)} = \text{diag}(x^{(ij)}) \), and form a set \( \mathcal{D} = \{ D^{(11)}, D^{(12)}, D^{(13)} \} \). By Lemma 3.2 \( \mathcal{D} = \bigcup_{k=0}^{2} \mathcal{D}^{(0)} \) is a group of order 9, and \( \mathcal{D}^{(0)} \) is a subgroup of order 3. It is easy to verify that each set \( \mathcal{D}^{(i)} \) \((i \in [2]) \) is a coset of \( \mathcal{D}^{(0)} \). Also, we find that

\[ \mathcal{D} \cong \mathbb{Z}_3 \oplus \mathbb{Z}_3, \mathcal{D}^{(0)} \cong \mathbb{Z}_3. \]

So each elementary divisor of \( \mathcal{D} \) and \( \mathcal{D}^{(0)} \) divides \( m \) (here \( m = 3 \)).
Theorem 4.1. A generalized result. By using the generalized traces; see [21, Theorem 3.1]. Following their idea, we get

Proof. (1) \[ \sum_{D}^{\lambda} \equiv \sum_{D}^{\lambda} \phi_1(V_i) + \phi_1(V_j) + \phi_1(V_k) \mod 3, \]

has solutions \( (i, j, k) \) with \( \{i, j, k\} = \{1, 2, 3\} \). So, we have a partition \( V(G) = V_1 \cup V_2 \cup V_3 \) such that each edge of \( G \) intersects those three parts. Equivalently, if one of \( i, j, k \) occurs more than one time, then \( A(G)[V_i, V_j, V_k] = 0 \), which implies that \( G \) contains no edges \( \{1, 2, 4\}, \{1, 2, 5\}, \{1, 3, 4\}, \{1, 3, 6\}, \{1, 4, 5\}, \{1, 4, 6\}, \{2, 3, 5\}, \{2, 3, 6\}, \{2, 4, 5\}, \{2, 5, 6\}, \{3, 4, 6\}, \{3, 5, 6\} \).

From the eigenvector \( x^{(11)} = (e^{i\frac{2\pi}{3}}, e^{i\frac{4\pi}{3}}, e^{i\frac{2\pi}{3}}, e^{i\frac{4\pi}{3}}, e^{i\frac{2\pi}{3}}, e^{i\frac{4\pi}{3}}) \), we have a map \( \phi_2 : [6] \to [3] \) such that \( \phi_2|_{\{1, 3, 4, 6\}} = 3 \) and \( \phi_2|_{\{2, 5\}} = 1 \) holding \( \{3, 11\} \). Let \( U_1 = \{1, 3, 4, 6\} \) and \( U_2 = \{2, 5\} \). Then every edge of \( G \) takes two vertices from \( U_1 \) and one from \( U_2 \). So we get the information of edge distribution of \( G \), in particular we find almost all non-edges of \( G \) in this example.

4. CYCLIC INDEX OF TENSORS AND HYPERGRAPHS

In this section, we will discuss how to characterize the spectral symmetry or the cyclic index of weakly irreducible nonnegative tensors or connected hypergraphs. Shao et. al give a characterization of spectral \( m \)-symmetric \( m \)-uniform hypergraphs by using the generalized traces; see [21, Theorem 3.1]. Following their idea, we get a generalized result.

Theorem 4.1. Let \( \mathcal{A} \) be a tensor of order \( m \) and dimension \( n \), and \( \varphi_{\mathcal{A}}(\lambda) = \sum_{i=0}^{D} a_i \lambda^{D-i} (D = n(m - 1)^{n-1}) \) be the characteristic polynomial of \( \mathcal{A} \). Then the following conditions are equivalent.

1. \( \mathcal{A} \) is spectral \( \ell \)-symmetric.
2. If \( \ell \nmid d \), then \( a_d = 0 \), i.e. \( \varphi_{\mathcal{A}}(\lambda) = \lambda^t f(\lambda^\ell) \) for some nonnegative integer \( t \) and some polynomial \( f \).
3. If \( \ell \nmid d \), then \( \text{Tr}_d(\mathcal{A}) = 0 \).

Proof. (1) \( \implies \) (2). Let \( \epsilon = e^{i\frac{2\pi}{\ell}} \). Then (1) implies that \( \psi_{\mathcal{A}}(\epsilon \lambda) = \epsilon^D \varphi_{\mathcal{A}}(\lambda) \). So

\[
\sum_{d=0}^{D} a_d \epsilon^{D-d} \lambda^{D-d} = \epsilon^D \sum_{d=0}^{D} a_d \lambda^{D-d}.
\]

Then we have \( a_d (\epsilon^d - 1) = 0 \). So, if \( \ell \nmid d \), then \( \epsilon^d - 1 \neq 0 \), and hence \( a_d = 0 \).

It is easily seen that (2) \( \implies \) (1).

(2) \( \implies \) (3). From (2) we have for some integer \( s \),

\[
\varphi_{\mathcal{A}}(\lambda) = \lambda^t (\lambda^\ell - c_1^\ell) \cdots (\lambda^\ell - c_s^\ell).
\]
Let $P$ be a circulant permutation matrix of dimension $\ell$, that is, $p_{ij} = 1$ if and only if $j \equiv i + 1 \pmod{\ell}$. Then $\psi_{cP}(\lambda) = \lambda^\ell - c^\ell$. If letting $B = c_1 P + \cdots + c_s P$, then $\varphi_A(\lambda) = \lambda^\ell \psi_B(\lambda)$, and by (2.2)
\[ \text{Tr}_d(A) = \text{Tr}(B^d) = \text{Tr}((c_1 P)^d) + \cdots + \text{Tr}((c_s P)^d). \]
It is known that if $\ell \nmid d$, then $\text{Tr}((c_i P)^d) = 0$ for $i \in [s]$, and hence $\text{Tr}_d(A) = 0$.

(3) $\implies$ (2). From (2.3) and (2.1), if $a_d \neq 0$, then there exist some positive integers $d_1, \ldots, d_t$ with $d_1 + \cdots + d_t = d$ such that
\[ \text{(4.2)} \quad \text{Tr}_{d_i}(A) \cdots \text{Tr}_{d_t}(A) \neq 0. \]
By the condition (3), we know that $\ell \mid d_i$ for $i \in [t]$, yielding $\ell \mid d$.

**Corollary 4.2.** Let $A$ be a tensor of order $m$ and dimension $n$, and $\varphi_A(\lambda) = \sum_{i=0}^D a_i \lambda^{D-i}$ ($D = n(m-1)^{n-1}$) be the characteristic polynomial of $A$. If $A$ is spectral $\ell$-symmetric, then
\[ \ell \mid \gcd\{d : a_d \neq 0\}, \ell \mid \gcd\{d : \text{Tr}_d(A) \neq 0\}. \]
Furthermore,
\[ \text{(4.3)} \quad c(A) = \gcd\{d : a_d \neq 0\} = \gcd\{d : \text{Tr}_d(A) \neq 0\}. \]

**Proof.** The first result is obtained by the equivalence of (1) and (2) in Theorem 4.1. So, $c(A) \mid \gcd\{d : a_d \neq 0\}$. Let $g := \gcd\{d : a_d \neq 0\}$ and $\epsilon = e^{2\pi i / g}$. Then for all $d$ with $a_d \neq 0$, $g \mid d$ and $a_d(e^{2\pi i d/g} - 1) = 0$. So (4.1) holds and hence $\psi_A(\epsilon^\ell) = \epsilon^D \varphi_A(\lambda)$, which implies that $A$ is spectral $g$-symmetric. By the definition of $c(A)$, $c(A) = g = \gcd\{d : a_d \neq 0\}$.

Now let $\tilde{g} := \gcd\{d : \text{Tr}_d(A) \neq 0\}$. From (4.2), if $a_d \neq 0$, then there exist some positive integer $d_1, \ldots, d_t$ with $d_1 + \cdots + d_t = d$ such that $\text{Tr}_{d_i}(A) \cdots \text{Tr}_{d_t}(A) \neq 0$. Then, $\tilde{g} \mid d_i$ for $i \in [t]$, and hence $\tilde{g} \mid d$, which implies that $\tilde{g} \mid g$. On the other hand, by what we have proved, $g \mid \tilde{g}$, yielding (4.3). \qed

In the remaining part of this paper, we will discuss the spectral symmetry of connected hypergraphs. Cooper and Dutle [4] raised a problem on characterizing the $m$-uniform hypergraphs whose spectra are invariant under multiplication by the $m$-th roots of unity (i.e., the spectral $m$-symmetric $m$-uniform hypergraphs by our definition). Pearson and Zhang posed a more specific problem on characterizing all connected uniform hypergraphs with symmetric spectrum (i.e., the spectral $2$-symmetric $m$-uniform connected hypergraphs). Nikiforov [15] obtains a complete solution to the latter problem, which is exactly the result of Theorem 3.13 for $\ell = 2$.

Shao et al. [20] give a characterization on the symmetry of $H$-spectra of hypergraphs, that is, an $m$-uniform hypergraph $G$ has a symmetric $H$-spectrum if and only if $m$ is even and $G$ is odd-bipartite (or odd transversal). They posed a problem that whether $\text{Spec}(L(G)) = \text{Spec}(Q(G))$ can imply that $H\text{Spec}(L(G)) = H\text{Spec}(Q(G))$, which is equivalently to ask whether $\text{Spec}(A(G)) = -\text{Spec}(A(G))$ can imply that $H\text{Spec}(A(G)) = -H\text{Spec}(A(G))$. Zhou et al. also pose a similar conjecture whether $-\rho(A(G))$ being an eigenvalue of $G$ can imply that $m$ is even and $G$ is odd-bipartite. In fact, $-\rho(A(G))$ is an eigenvalue of $G$ if and only if $G$ has a symmetric spectrum. By Nikiforov’s result [15], this is equivalent to ask whether an odd-colorable hypergraph is odd transversal. They construct two classes of hypergraphs to give a negative answer. We also give a negative answer to the
above problem in [5] by constructing a class of non-odd-bipartite generalized power hypergraphs.

In general, for an $m$-uniform hypergraph $G$, as $\mathcal{A}(G)$ is symmetric, if $G$ is spectral $\ell$-symmetric, then $\ell \mid m$ by Lemma 5.2. If $G$ is $m$-partite [4], or hm-bipartite [8], or $p$-hm bipartite with $(p,m) = 1$ [21], then $G$ is spectral $m$-symmetric. The following result is proved in Lemma 3.2(3), which is re-proved by using the generalized traces as follows.

**Corollary 4.3.** Let $G$ be an $m$-uniform hypergraph on $n$ vertices. If $G$ is spectral $\ell$-symmetric, then $\ell \mid m$.

**Proof.** By Theorem 3.15 in [4], the codegree $m$ coefficient of $\psi_G(\lambda)$ is

$$a_m = -m^{m-2}(m-1)^{n-m}|E(G)| \neq 0.$$ 

So, by Corollary 4.2, $\ell \mid m$. \quad \square

A simplex in an $m$-uniform hypergraph is a set of $m+1$ vertices where every set of $m$ vertices forms an edge.

**Corollary 4.4.** Let $G$ be an $m$-uniform hypergraph on $n$ vertices. If $G$ contains a simplex, then $G$ is spectral 1-cyclic or nonsymmetric.

**Proof.** By Theorem 3.17 in [4], the codegree $m+1$ coefficient of $\psi_G(\lambda)$ is

$$a_{m+1} = -C(m-1)^{n-m}s,$$

where $s$ is the number of simplices in $G$, and $C$ is a positive integer depending only on $m$. By Corollary 4.2 $c(\mathcal{A}(G)) = g.c.d\{m, m+1, \ldots\} = 1$. \quad \square

The following result in the case of $(p,m) = 1$ is given by Shao et al. [21].

**Theorem 4.5.** Let $G$ be an $m$-uniform $p$-hm bipartite hypergraph. Then $G$ is spectral $\frac{m}{(p,m)}$-symmetric.

**Proof.** By Lemma 4.4, it suffices to prove that if $\text{Tr}_d(G) \neq 0$, then $\frac{m}{(p,m)} \mid d$. Suppose that $\text{Tr}_d(G) \neq 0$. By (2.3) and (2.4), there exists $F = \{e_1(i_1), \ldots, e_d(i_d)\} \in \mathbf{F}_d(G)$. Let $H$ be the sub-hypergraph of $G$ induced by those $d$ edges $e_1, \ldots, e_d$. Then the degree $d_H(v)$ of each vertex $v$ of $H$ is $m$-valent. Let $[n] = V_1 \cup V_2$ be a partition of the vertex set $V(G)$ such that each edge intersects $V_i$ with exactly $p$ vertices. Now

$$\sum_{v \in V_i} d_H(v) = dp.$$ 

As each vertex of $H$ has an $m$-valent degree, $m \mid dp$, which implies that $\frac{m}{(p,m)} \mid d$. \quad \square

Finally, we will discuss the spectral symmetry of generalized power hypergraphs, and show that for an arbitrarily given positive integer $m$ and any positive integer $\ell$ with $\ell|m$, there exists an $m$-uniform hypergraph $G$ with $c(G) = \ell$.

**Lemma 4.6.** Let $G$ be a simple bipartite graph, and let $m \geq 4$ be an even integer. Then $G^{m/2}$ is spectral $m$-cyclic.

**Proof.** By Corollaries 4.2 and 4.3 it suffices to prove that if $\text{Tr}_d(G^{m/2}) \neq 0$, then $m \mid d$. Let $V(G) = V_1 \cup V_2$ be a bipartition of $V(G)$ such that each edge of $G$ intersects both $V_1$ and $V_2$, which naturally corresponds a bipartition $V(G^{m/2}) = V_1 \cup V_2$, where $V_i$ is obtained from $V_i$ by blowing each vertex $v$ into an $\frac{m}{2}$-set $\mathbf{v}$ for
As each vertex in $i \in [2]$. So we may assume that $V_i \subset V_i$ for $i \in [2]$. Suppose that $\text{Tr}_d(G^{m,s}) \neq 0$. By (23) and (24), we have $d$ edges $F = \{e_1(i_1), \ldots, e_d(i_d)\} \in F_d(G^{m,s})$. Let $H$ be the sub-hypergraph of $G^{m,s}$ induced by the edges in $F$. Then

$$\sum_{v \in V(H)} d_H(v) = d.$$

As each vertex in $H$ is $m$-valent, we have $m \mid d$. □

Lemma 4.7. Let $G$ be a $t$-uniform hypergraph and let $G^{m,s}$ be the generalized power of $G$, where $t \mid m$. Then $c(G^{m,s}) = \frac{m}{t} \cdot l$, where $l$ is a positive integer such that $l \mid c(G)$.

Proof. Let $s := \frac{m}{t}$. Note that each edge $e = \{i_{l_1}, \ldots, i_{l_t}\}$ of $G$ is 1-1 corresponding to the edge $\bar{e} = S_{i_{l_1}} \cup \cdots \cup S_{i_{l_t}}$ of $G^{m,s}$, where $S_{i_j} = \{i_{j_1}, \ldots, i_{j_t}\}$ for $j \in [t]$. Hence we give a labeling of the vertices of $G^{m,s}$. So, if we choose $i_{l_1}, \ldots, i_{l_t}$ from each edge $\bar{e}$ of $G^{m,s}$ with the above expression and form a set $U$, then every edge of $G^{m,s}$ intersects $U$ with exactly $t$ vertices, which implies that $G^{m,s}$ is a $t$-hm bipartite hypergraph. By Theorem 4.5, $G^{m,s}$ is spectral $(m,s)$-symmetric or spectral $s$-symmetric. By Corollary 4.2 if $\text{Tr}_d(G^{m,s}) \neq 0$, then $s \mid d$. So, it is enough to consider the trace $\text{Tr}_{d_s}(G^{m,s})$ for a general positive integer $d$.

Suppose $\text{Tr}_d(G) \neq 0$. By (23) and (24), each nonzero (positive) term of $\text{Tr}_d(G)$ is associated with $d$ edges such that $F = \{e_1(i_1), \ldots, e_d(i_d)\} \in F_d(G)$, and $D(F)$ is a Eulerian directed graph, the latter of which also implies that each index occurring in $F$ will be a primary index and a secondary index as well. Correspondingly, we have $ds$ edges of $G^{m,s}$ such that

$$\bar{F} = \{\bar{e}_1(i_{11}), \ldots, \bar{e}_1(i_{1s}), \ldots, \bar{e}_d(i_{d1}), \ldots, \bar{e}_d(i_{ds})\} \in F_{ds}(G^{m,s}).$$

Here each edge $\bar{e}_j$ occurs $s$ times but with different primary index for $j \in [d]$.

Now we consider the directed graph $D(\bar{F})$. For each fixed $j \in [s]$, let $D_j(\bar{F})$ be the subgraph of $D(\bar{F})$ induced by all possible vertices $i_{pj}$ if $i_p$ occurs in $\bar{F}$, or equivalently, $D_j(\bar{F})$ uses the edges arising from $F_j = \{e_1(i_{1j}), \ldots, e_d(i_{dj})\}$ and the vertices of form $\bar{i}_{pj}$ contained in those edges. The vertex sets of $D_1(\bar{F}), \ldots, D_s(\bar{F})$ form a partition of $D(\bar{F})$, i.e.

$$V(D(\bar{F})) = V(D_1(\bar{F})) \cup \cdots \cup V(D_s(\bar{F})).$$

By the construction of $D(F), D(\bar{F})$ and $G^{m,s}$, for each $j \in [s], D_j(\bar{F})$ is isomorphic to $D(F)$, which implies that $D_j(\bar{F})$ contains an Eulerian directed circuit.

For any fixed $j \in [s]$, we assert that in the directed graph $D(\bar{F})$, each vertex of $D_j(\bar{F})$ has the same number of in-neighbors and out-neighbors outside $D_j(\bar{F})$. We will prove the assertion by considering each primary index $i_{pj}$ occurring in $\bar{F}$ for $p \in [d]$, including multiplicity, which means if considering $i_{pj}$ as a vertex, the (out- or in-)degree of $i_{pj}$ is the sum of (out- or in-)degrees of the index $i_p$ for each appearance of $i_{pj}$ (thinking of all $i_{pj}$ being distinct). Observe that $i_{pj}$ has $t(s - 1)$ out-neighbors outside $D_j(\bar{F})$ arising from the edge $\bar{e}_p(i_{pj})$, where for each $l \in [s] \setminus \{j\}, i_{pj}$ has $t$ out-neighbors in $D_j(\bar{F})$. As $D_j(\bar{F})$ contains a Eulerian directed circuit and $i_{pj}$ has exactly $t - 1$ out-neighbors in $D_j(\bar{F})$ arising from the edge $\bar{e}_p(i_{pj})$, $i_{pj}$ also has exactly $t - 1$ in-neighbors in $D_j(\bar{F})$, say $i_{q_1}, \ldots, i_{q_{t-1}}$, corresponding to $t - 1$ edges $\bar{e}_{q_1}(i_{q_1}), \ldots, \bar{e}_{q_{t-1}}(i_{q_{t-1}})$, each of which contains $i_{pj}$ as a secondary
index. So, by the construction of $\hat{F}$, for each $l \in [t-1]$, we have $s-1$ edges $e_{q_l}(i_{q_l})$’s for $j \in [s]\{i\}$, each of which also contains $i_{q_l}$ as a secondary index. In addition, for each $j \in [s]\{i\}$, the edge $e_{p_j}(i_{p_j})$’s contains $i_{p_j}$ as a secondary index. So, $i_{p_j}$ has $(t-1)(s-1) + (s-1) = t(s-1)$ in-neighbors outside $D_j(\hat{F})$.

By the above discussion, we know $D(\hat{F})$ is weakly connected (considering its underlying undirected graph), and each vertex of $D(\hat{F})$ has the same in-degree and out-degree, which implies that $D(\hat{F})$ is Eulerian, and hence $Tr_{ds}(G^{m,s}) \neq 0$ by (2.3). By Corollary 4.2

\begin{equation}
(4.4) \quad c(G^{m,s}) = g.c.d.\{ds : Tr_{ds}(G^{m,s}) \neq 0\} = s \cdot g.c.d.\{d : Tr_{ds}(G^{m,s}) \neq 0\}.
\end{equation}

Let $l = g.c.d.\{d : Tr_{ds}(G^{m,s}) \neq 0\}$. By what we have proved, if $Tr_d(G) \neq 0$, then $Tr_{ds}(G^{m,s}) \neq 0$, implying that $l | c(G)$.

If $c(G) = 1$, then $c(G^{m,s}) = s \cdot c(G)$ by Lemma 4.7. We pose the following conjecture.

**Conjecture 4.8.**

\begin{equation}
(4.5) \quad c(G^{m,s}) = s \cdot c(G).
\end{equation}

**Corollary 4.9.** Let $G$ be a simple non-bipartite graph, and let $m \geq 4$ be an even integer. Then $G^{m,\overline{2}}$ is spectral $m/2$-cyclic.

**Proof.** As $G$ is non-bipartite, $c(G) = 1$. By Lemma 4.7, $c(G^{m,\overline{2}}) = \frac{m}{2} \cdot l$, where $l | c(G)$. So $l = 1$, and hence $c(G^{m,\overline{2}}) = \frac{m}{2}$.

**Corollary 4.10.** Let $m \geq 2$ be an integer. Then for any positive integer $\ell$ with $\ell | m$, there exists an $m$-uniform hypergraph $G$ such that $G$ is spectral $\ell$-cyclic.

**Proof.** Let $m = \ell \cdot t$. If $\ell = 1$, any $m$-uniform hypergraph with a simplex is as desired by Corollary 4.4. If $\ell = m$, then the hypergraph $G^{m,\overline{2}}$ by taking $G$ be any bipartite graph is spectral $m$-cyclic by Corollary 4.6. Otherwise, $2 \leq \ell \leq \frac{m}{2}$. Let $G$ be a $t$-uniform hypergraph (maybe simple graph) containing a simplex. Then $G$ is spectral $1$-cyclic by Corollary 4.4. By Theorem 1.7, $G^{m,\ell}$ is spectral $\ell$-cyclic.

5. Conclusions

It is known that the Earth has two kinds of movement: one is the rotation around its own axis, the other is the travel around the Sun. Let $\mathcal{A}$ be a weakly irreducible nonnegative tensor of order $m$ and dimension $n$. Governed by the group $\mathfrak{D}$ defined in (1.3) (taking $\ell = c(\mathcal{A})$), the “movement” of $\mathcal{A}$ has a similar behavior: one is self-rotating determined by $\mathfrak{D}^{(0)}$ with “period” $s(\mathcal{A})$, the other is traveling through the orbit $S = \{c(\mathcal{A}) : i = 0, 1, \ldots, c(\mathcal{A}) - 1\}$ determined by $\mathfrak{D} \setminus \mathfrak{D}^{(0)}$ with “period” $c(\mathcal{A})$. So, we have two important parameters $s(\mathcal{A})$ and $c(\mathcal{A})$, from which we know the structural information of $\mathcal{A}$. If $m = 2$ or $\mathcal{A}$ is an irreducible nonnegative matrix, then $s(\mathcal{A}) = 1$. But for the case of $m \geq 3$, it will happen that $s(\mathcal{A}) \geq 2$; see Examples 3.13 and 3.15. So it will be interesting to investigate $s(\mathcal{A})$ and get more structural information of $\mathcal{A}$.
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School of Mathematical Sciences, Anhui University, Hefei 230601, P. R. China
E-mail address: fanyz@ahu.edu.cn

School of Mathematical Sciences, Anhui University, Hefei 230601, P. R. China
E-mail address: huangtao@ahu.edu.cn

School of Mathematical Sciences, Anhui University, Hefei 230601, P. R. China
E-mail address: baoyh@ahu.edu.cn

School of Mathematical Sciences, Anhui University, Hefei 230601, P. R. China
E-mail address: zhuansunc10163.com

School of Mathematical Sciences, Anhui University, Hefei 230601, P. R. China
E-mail address: 18856961415@163.com