Phonon-drag magnetoquantum oscillations in graphene

S S Kubakaddi¹, Tutul Biswas² and Tarun Kanti Ghosh³

¹ Department of Physics, K. L. E. Technological University, Hubballi-580 031, Karnataka, India
² Department of Physics, Vivekananda Mahavidyalaya, Burdwan-713 103, West Bengal, India
³ Department of Physics, Indian Institute of Technology-Kanpur, Kanpur-208 016, Uttar Pradesh, India

E-mail: sskubakaddi@gmail.com, tb tutulm53@gmail.com and tk ghosh@iitk.ac.in

Received 3 March 2017, revised 7 May 2017
Accepted for publication 22 May 2017
Published 20 June 2017

Abstract
A theory of low-temperature phonon-drag magnetothermopower $S_{xx}^g$ is presented in graphene in a quantizing magnetic field. $S_{xx}^g$ is found to exhibit quantum oscillations as a function of magnetic field $B$ and electron concentration $n_e$. The amplitude of the oscillations is found to increase (decrease) with increasing $B$ ($n_e$). The behavior of $S_{xx}^g$ is also investigated as a function of temperature. A large value of $S_{xx}^g$ ($\sim$ few hundreds of $\mu V K^{-1}$) is predicted. Numerical values of $S_{xx}^g$ are compared with the measured magnetothermopower $S_{xx}$ and the diffusion component $S_{xx}^d$ from the modified Girvin-Jonson theory.

Keywords: thermoelectric phenomena, electronic transport in Graphene, electron–phonon interaction

(Some figures may appear in colour only in the online journal)

1. Introduction
Graphene, a monolayer of carbon atoms arranged in a honeycomb lattice of hexagons, has a unique band structure. Its electronic states, at the points $K$ and $K'$ of the Brillouin zone, have a linear dispersion relation, described by the Dirac equation. It is an ambipolar material with zero effective mass of the carriers and zero energy gap. Its electrical transport properties have been studied extensively [1, 2], since its discovery [3–5], with host of intriguing phenomena due to its unusual band structure. In a quantizing magnetic field $B$, graphene exhibits the integer quantum Hall effect (QHE) [4, 5], with novel features different from those in conventional two-dimensional electron gas (2DEG), particularly at $n = 0$ Landau level (LL).

Thermopower $S$, an electric field $E$ generated in a sample due to unit temperature gradient i.e. $S = E/(-\nabla T)$, has been another powerful tool for probing carrier transport. Application of the magnetic field $B$, in addition to a temperature gradient $\nabla T$, provides a valuable experimental tool to investigate magnetothermoelectric effects. In a 2DEG, in the $xy$ plane, a temperature gradient $\nabla T$ in $x$-axis and magnetic field $|B|$ in $z$-axis generates electric field $E$, in the sample, with components $E_x = S_{xx}(-\nabla T)$, and $E_y = S_{xy}(-\nabla T)$, where the thermopower $S_{xx}$ and the Nernst–Ettingshausen coefficient $S_{xy}$ are the tensor components of $S$. Quantization effects due to the magnetic field are reflected in thermopower.

In conventional 2DEG of GaAs heterojunctions (HJs) and Si-MOSFETs, magnetothermopower is investigated in detail, experimentally and theoretically, in the quantum Hall regime [6–9]. Measured magnetothermopower tensor components $S_{xx}$ and $S_{xy}$ exhibited oscillations as a function of magnetic field, arising due to the crossing of the Landau level by the Fermi level due to either change in carrier concentration or magnetic field. There are two additive and independent contributions to thermopower $S$. In a $\nabla T$, diffusion component $S^d$ arises due to diffusion of carriers and the phonon-drag component $S^f$ arises due to the non-equilibrium phonons transferring some of their momentum to the electrons via electron–phonon (el–ph) scattering. The oscillatory behavior of diffusion component is explained by the theory of Jonson and Girvin [10] and Oji [11]. It is established that for about $0.1 < T < 10 K$, the $S^f$ contribution dominates $S$ in GaAs HJs[6–8]. $S^f$ is important because it gives directly el–ph coupling and is independent of impurity scattering, unlike mobility.

The study of phonon-drag magnetothermopower $S^f$, in conventional 2DEG, began with pioneering experimental work of
Fletcher et al [12] showing the quantum oscillations as a function of B. Its B and T dependence were explained by developing the theory of $S^I$ [13–15], by modifying the Boltzmann theory of phonon-drag in bulk semiconductors [16, 17], following the II-approach due to Herring [18].

In graphene, the experimental and theoretical investigations of thermoelastic effects in a zero and quantizing magnetic field are being intensively pursued [19]. In monolayer graphene, experimental data of S versus T, in zero magnetic field, in temperature regime 10–300 K show largely linear behavior suggesting that the mechanism for thermopower is diffusive [20]. Thermopower measurements in quantum Hall regime are carried out as a function of the magnetic field for a different gate voltage (i.e. for different carrier concentration) and temperatures [20–23]. Magnetothermopower $S_{xx}$ and Nernst–Ettingshausen coefficient $S_{yx}$, have shown the oscillatory behavior as a function of magnetic field. The behavior of $S_{xx}$ and $S_{yx}$ are in agreement with the generalized Mott relation, extending the theory of Jonson and Girvin [10] to graphene [20]. The peak values of $S_{xx}$ are predicted to be given by $S_{peak}^{xx} = (k_B/e)ln2n$, noting that Jonson–Girvin theory fails for Landau level $n = 0$. Similar observations are made in high mobility samples of graphene [23]. Zero-field, non-linear $T$ dependence is attributed to the screening [24]. Measured strong quantum oscillations as a function of $B$ are understood by the evolution of the density of states at the Fermi level and $S_{xx}$ becoming zero when the Fermi level lies in the localized states, because of absence of diffusion [23]. In all these measurements, it is observed that phonon-drag thermopower component is absent and there is no evidence of phonon-drag magnetoquantum oscillations, even at low temperatures, attributing to the weak el–ph coupling. We believe that, an equally important reason for the absence of phonon-drag component in these samples may be due to their small size ($\sim 300 \text{nm}$). It is about $10^4$ times smaller than the samples of GaAs HJs ($\sim \text{mm}$ in which $S_{xx}$ is large and about $\sim \text{mV K}^{-1}$) [6]. At low temperatures, the smaller dimension of the sample sets the limit for phonon mean free path $\Lambda$, in the boundary scattering regime, as $S^I \sim \Lambda$. We expect the phonon-drag to be significant in large samples (few $\mu$m) for example in the samples of Nika et al [25]. Moreover, to know the significant contribution of phonon-drag contribution, more data of $S$ is required at low temperature, covering the sub-Kelvin region in pure samples.

The theory [26] of zero magnetic field $S^I$ has been developed in monolayer graphene, in the boundary scattering regime, as a function of temperature $T (\lesssim 10 \text{ K})$ and electron concentration $n_e$ for the phonon mean free path $\Lambda \sim 10 \mu\text{m}$ (closer to the samples of Nika et al [25]). At about 10K, $S^I \sim 10 \mu\text{V K}^{-1}$. This value is nearly the same order of magnitude as that of predicted $S^I$ with the peak values few tens of $\mu\text{V K}^{-1}$, by the modified Jonson–Girvin formula. We have to note that, unlike $S^I$, the latter is independent of sample size.

It would be interesting to study the effects of magnetic field quantization on phonon-drag thermopower $S^\alpha_{dc}$. In the present work, we theoretically investigate the phonon-drag magnetothermopower as a function of magnetic field $B$, electron concentration $n_e$, and temperature $T$. We explore the circumstances and possibilities of its significant contribution to the measured magnetothermopower, by tuning the parameters $B$, $n_e$, $T$, and $\Lambda$. For comparison, we also compute the diffusion component $S^\alpha_{dc}$. The qualitative comparison of our calculations is made with the experimental observations.

This paper is organized as follows. In section 2, we provide formalism of phonon-drag thermopower in presence of quantizing magnetic field. In section 3, we present our results and discussion. A summary of our work is provided in section 4.

## 2. Formalism of phonon-drag magnetothermopower

In the following we proceed with the calculations by appropriately modifying the theory of Fromhold et al [15] for the monolayer graphene. We consider an isotropic and homogeneous 2DEG of graphene in the $xy$-plane with the magnetic field $B = (0, 0, B)$ along the $z$-direction. In presence of an electric field $E$ (along $x$-axis) electrons are assumed to be accelerated isothermally ($\nabla T = 0$). In the steady state, the non-equilibrium distribution of the electrons in state $\alpha$ is given by $f_\alpha = f^{0}_\alpha + f^{1}_\alpha$, where $f^{0}_\alpha = [\exp((E_\alpha - \mu_F)/k_BT) + 1]^{-1}$ is the thermal equilibrium distribution function in absence of the electric field for state $\alpha$, $\mu_F$ is the chemical potential and $f^{1}_\alpha$ is the first-order perturbation due to electric field $E$. These non-equilibrium electrons transfer some of their momentum to the 2D phonons through the el–ph coupling. This causes perturbation in the phonon distribution which is given by $N_q = N^0_q + N^1_q$, where $N^0_q = [\exp(h\omega_q/k_BT) - 1]^{-1}$ is the equilibrium distribution of the phonons of energy $h\omega_q$ and the wave vector $q$. Here, $N^1_q$ is the perturbation in the phonon distribution, due to electric field, producing the heat current density $U$.

We confine our attention to the linear transport regime at liquid helium temperature. Then it is necessary to consider only acoustic phonons, with a 2D character, which interact weakly with the 2D electrons. The phonon heat current density, noting that $N^0_q$ does not contribute, is given by

$$U = A_0^{-1} \sum_q N^1_q h\omega_q \nu_q,$$

(1)

where $A_0 = L_x L_y$ is the area of graphene sample and $\nu_q$ is the phonon group velocity.

In the linear response regime ($N^1_q \propto E$), the heat current density is given by $U = ME = \Pi T$, where $M$ is the thermo-electric tensor, $\Pi = M/\sigma$ is the Peltier coefficient tensor and $\sigma$ is the electrical conductivity tensor. From the Onsager relation, the thermopower tensor is defined as $S = \Pi/T$. Using the Onsager symmetry relations it has been shown that [15] $TS_{xx} = \rho_\perp M_{xx} - \rho_\parallel M_{xx}$ and $TS_{yx} = \rho_\parallel M_{xx} + \rho_\perp M_{xx}$, where $S_{xx}$ and $S_{yx}$ are, respectively, thermopower and Nernst–Ettingshausen coefficient, $M_{xx}$ and $M_{yx}$ are the components of tensor $M$ and $\rho_\parallel$ and $\rho_\perp$ are the components of electrical resistivity tensor $\rho$. Hence, the calculation of $U$ will facilitate the calculation of $M$ and hence $S_{xx}$ and $S_{yx}$.
The solution for $N_q^1$, in the linear response regime, is found to be

$$N_q^1 = \frac{g}{k_bT} \sum_{a,\alpha'} \gamma_{\alpha,\alpha'}(q) \left[ \frac{f^\dagger_{\alpha}(\partial f_{\alpha}/\partial E_q) - f^\dagger_{\alpha'}(\partial f_{\alpha'}/\partial E_q)}{(\partial f_{\alpha}/\partial E_q) - (\partial f_{\alpha'}/\partial E_q)} \right],$$  

(2)

where $g = g_{\alpha\alpha'}, \gamma(f_q)$ is the spin (valley) degeneracy, and

$$\gamma_{\alpha,\alpha'}(q) = P_{\alpha,\alpha'}^0(q) f_{\alpha}(1 - f_{\alpha'}),$$  

(3)

$$P_{\alpha,\alpha'}^0(q) = \frac{2\pi}{\hbar} \left| C_{\alpha,\alpha'}(q) \right|^2 N_q^0(E_q - E_{\alpha'} - \hbar\omega_q).$$  

(4)

Here, $P_{\alpha,\alpha'}^0(q)$ is the transition probability, in equilibrium, for the electron scattering from state $\alpha$ to state $\alpha'$ by absorbing a phonon and $\left| C_{\alpha,\alpha'}(q) \right|^2$ is the square of the electron-acoustic phonon interaction matrix element. Equation (2) is general to the extent that, it is independent of the electronic structure and the type of el–ph coupling.

In a quantizing magnetic field $B = (0, 0, B)$ with the Landau gauge $A = (0, Bx, 0)$, the eigenfunctions and energy eigenvalues are given in [27]. The energy eigenvalues are $E_{\alpha}^0 = E_{\alpha}^0(k_y, B) = \hbar\omega_q \sqrt{2n}$, where $\alpha \equiv (n, k_y), n = 0, 1, 2, ...$ is the Landau level quantum number, $k_y$ is the electron wave vector in the $y$-direction, $\omega_q = v_F/\hbar l_0$ is the cyclotron frequency, $v_F = 1 \times 10^6 \text{ m s}^{-1}$ is the Fermi velocity of electron in graphene, and $l_0 = \sqrt{\hbar/(eB)}$ is the magnetic length. The el–ph matrix element is given by (see appendix for details)

$$\left| C_{\alpha,\alpha'}(q) \right|^2 = \left| C(q) \right|^2 \left| J_{\alpha,\alpha'}(u) \right|^2 \delta_{k_y+k',q},$$  

(5)

where $\left| C(q) \right|^2$ is the matrix element which describes the el–ph coupling strength and $\left| J_{\alpha,\alpha'}(u) \right|^2/2$ is the matrix element describing the scattering between Landau levels.

In presence of a crossed electric field $E = (E, 0, 0)$ and magnetic field $B = (0, B, B)$, the energy spectrum of graphene can be found exactly [28, 29]. In the linear response regime, where the applied electric field is low enough, one can obtain energy eigenvalue for the magnetic state $\alpha$, approximately, as $E_{\alpha} \approx E_{\alpha}^0 + eE_{\alpha} x_\alpha$ by expanding the exact expression given in [28, 29] up to the first-order in $E/(v_F B)$. Here, $x_\alpha = k_y f_0^2$. This is nothing but the first-order energy correction due to the weak external electric field. Further, assuming that the form of distribution function retains the same with the modified energy, we expand $f_{\alpha} = f_0(E_{\alpha}) = f_0(E_{\alpha}^0) - eE_{\alpha} x_\alpha \delta f(E_{\alpha}^0)/\delta E_{\alpha} \delta(E_{\alpha}^0 + f_0^1) \left(= f_0(E_{\alpha}^0) + f_\alpha^1 \right)$, which gives $\delta f(E_{\alpha}^0)/\delta E_{\alpha} \delta(E_{\alpha}^0 + f_\alpha^1) = -eE_{\alpha} x_\alpha$. Then, equation (2), using the momentum conservation $k'_y = k_y + q_y$, for the chosen Landau gauge gives,

$$N_q^1 = \frac{geE_{\alpha}^2}{k_bT} \sum_{\alpha,\alpha'} \gamma_{\alpha,\alpha'}(q) q_{\alpha},$$  

(6)

In order to make $N_q^1$ linear in $E$, we set all terms in $\gamma_{\alpha,\alpha'}(q)$ independent of $E$. Inserting equation (6) into equation (1) we write $U = ME$ and take the phonon group velocity components $v_{q^0} = (q, q) v_{\alpha} \nu$, and $\nu_{q^0} = (q, q) \nu_y$, $\nu_y$ being the acoustic phonon velocity in graphene. Then, the two components of the thermoelectric tensor $M$ are given by

$$M_{\alpha\alpha} = \frac{ge\omega_{q}}{k_b T} \sum_{q} \gamma_{\alpha,\alpha'}(q) (q \nu_{\alpha} + \nu_{\alpha'}) \hbar\omega_q,$$

(7)

$$M_{\alpha\alpha'} = \frac{ge\omega_{q^0}}{k_b T} \sum_{q} \gamma_{\alpha,\alpha'}(q) \delta(E_{\alpha}^0 - E_{\alpha'} - \hbar\omega_q).$$

(8)

where

$$\gamma_{\alpha,\alpha'}(q) = \sum_{\alpha'} \gamma_{\alpha,\alpha'}(q).$$

(9)

When we carry out the angular integration, it can be seen that $M_{\alpha\alpha} = 0$ because of the $xy$ isotropy. However, this is shown as the limitation of this theory as the experimental results of $M_{\alpha\alpha}$ in conventional 2DEG show its non-zero value [15]. Later calculations in these systems, taking into account of anisotropy of electrons and phonons, remove this limitation [30]. However, in the present work we undertake the evaluation of only $M_{\alpha\alpha}$, as we have considered $xy$ isotropy of the system.

Using equations (3) and (4), equation (9) turns out to be

$$\gamma_{\alpha,\alpha'}(q) = \frac{2\pi}{\hbar} \sum_{\alpha} \left| C_{\alpha,\alpha'}(q) \right|^2 N_q^0(E_{\alpha}^0) \times [1 - f^0(E_{\alpha}^0)] \delta(E_{\alpha}^0 - E_{\alpha'} - \hbar\omega_q).$$

(10)

Summation over $k'_y$ is carried out by replacing it by $k_y + q_y$. Since the integrand is dependent of $k_y$, summation over $k_y$ simply gives $A_0/\sqrt{\pi l_0^2}$.

In the presence of disorder, the energy levels (in zero electric field) $E_{n,k_y}^0$ and $E_{n,k_y+q_y}^0$ are randomized by LL broadening. A simple system average is taken by integrating over $\epsilon = E_{n,k_y}^0$ and $\epsilon' = E_{n,k_y+q_y}^0$, with the weight factor $\rho(\epsilon - E_{n,k_y}^0)\rho(\epsilon' - E_{n,k_y+q_y}^0)$. When $\delta(E_{\alpha}^0)$ is the energy of the $n(n'\text{th})$ LL in absence of disorder and $\rho(x)$ is the LL density of states with convenient line shape function. Now equation (10) becomes

$$\gamma_{\alpha,\alpha'}(q) = \frac{A_0}{2\pi l_0^2} \sum_{n,n'} \left| C_{n,n'}(q) \right|^2 N_q^0 \times \int d\epsilon d\epsilon' f^0(\epsilon) f^0(\epsilon') \delta(\epsilon' - \epsilon - \hbar\omega_q) \rho(\epsilon - E_{n,k_y}^0)\rho(\epsilon' - E_{n,k_y+q_y}^0).$$

(11)

Integration with respect to $\epsilon'$, using the Dirac delta function, gives

$$\gamma(q) = \frac{A_0}{\hbar l_0^2} \sum_{n} \sum_{n'} \left| C(q) \right|^2 \left| J_{n,n}(u) \right|^2 N_q^0 \rho(\epsilon - E_n^0),$$

(12)

where

$$I_n(\hbar\omega_q) = \int d\epsilon f^0(\epsilon) [1 - f^0(\epsilon + \hbar\omega_q)] \rho(\epsilon - E_n^0) \times \rho(\epsilon + \hbar\omega_q - E_n^0).$$

(13)
Since the phonon-drag thermopower is important at low temperature, the energy of acoustic phonons involved is so small that only intra LL scattering is possible. Thus we set $n = n'$ in equation (12). Then, with

$$ |J_{m}(u)|^2 = \frac{e^{-u}}{4}|L_m(u) + L_{n-1}(u)|^2, \quad (14) $$

equation (12) gives

$$ \Gamma(q) = \frac{A_0}{h_0^2}|C(q)|^2 N_0 \sum_n |J_m(u)|^2 I_m(h\omega_q). \quad (15) $$

Intra LL transitions are possible as the energy levels are broadened.

Using equation (15) in equation (8), we get

$$ M_{xx} = \frac{e^2v_B}{4\pi\hbar^2k_BT} \sum_q \tau_q(q^2/q)\hbar\omega_q|C(q)|^2 N_0 \sum_n |J_m(u)|^2 I_m(h\omega_q). \quad (16) $$

The summation over $q$ is converted into integration as

$$ \sum_q \rightarrow \frac{A_0}{(2\pi)^2} \int_0^\infty dq-q \int_0^{2\pi} d\theta. \quad (17) $$

Angular integration coming through $q_y = q \sin \theta$ gives $\pi$. Then

$$ M_{xx} = \frac{eD^2}{8\pi\hbar^2k_BT} \int_0^\infty d(h\omega_q)\tau_q(h\omega_q)^4|C(q)|^2 N_0 \sum_n |J_m(u)|^2 I_m(h\omega_q). \quad (18) $$

Substituting for $|C(q)|^2 = D^2\hbar\omega_q/(2\rho_m A_0 v_B^2)$, where $D$ is the acoustic phonon deformation potential coupling constant and $\rho_m$ is the areal mass density of graphene, we obtain

$$ M_{xx} = \frac{eD^2}{8\pi\rho_m^3v_B^4k_BT} \int_0^\infty d(h\omega_q)\tau_q(h\omega_q)^4 N_0 \sum_n |J_m(u)|^2 I_m(h\omega_q). \quad (19) $$

From Onsager relation, with $M_{xx} = 0$, we have $T S_{xx}^g = -\rho_{xx} M_{xx}$. Taking $\rho_{xx} = B/(n_e e)$ [31], where $n_e$ is the electron density and expressing $B$ in terms of $\rho_0$, we get

$$ S_{xx}^g = -\frac{gk_BT^2}{8\pi\rho_m^3v_B^4k_BT} \int_0^\infty d(h\omega_q)\tau_q(h\omega_q)^4 N_0 \sum_n |J_m(u)|^2 I_m(h\omega_q). \quad (20) $$

We note that this equation can also be obtained by following the method of Kubakaddi et al [13] for conventional 2DEG ignoring $\Gamma(q)$ compared to $1/\tau_q$.

3. Results and discussion

Since $S_{xx}^g \sim \Lambda$ and $D^2$, it is essential to choose the reasonable values of these parameters. We numerically evaluate $S_{xx}^g$, for $T \leq 20$ K in the boundary scattering regime for which $\tau_q = \Lambda/\nu_B$, where $\Lambda$ is the phonon mean free path. Normally, $\Lambda$ is taken to be the smaller dimension of the sample. Thermal conductivity calculations are demonstrated with $\Lambda$ chosen in the range of 3–30 $\mu$m and the choice of $\Lambda = 5$ $\mu$m provides reasonable agreement with the measured thermal conductivity [25, 32].

Nika et al [25], to fit the thermal conductivity data, use the effective phonon mean free path $\lambda_{\text{eff}} = \Lambda(1 + p)/(1 - p)$ by modulating the smallest dimension of the sample using a specular parameter $p = 0.9$, which enhances $\Lambda$ by a factor of 20. The value of $0 \leq p \leq 1$ is determined by the roughness of the graphene edges. To present our calculations we choose a reasonable value of $\Lambda = 10$ $\mu$m.

In the literature there is a range of $D = 3–30$ eV. We chose $D = 20$ eV which is closer to the values of $D$, for unscreened el–ph interaction, used to fit the experimental data of some of the transport properties [33–36]. The line shape function $\rho(x)$ of LL is taken to be Lorentzian with the width $\Gamma_x = (C/\sqrt{B})$. Other parameter values used are:

$$ \rho_m = 7.6 \times 10^{-7} \text{Kg m}^{-2} \text{ and } v_f = 2 \times 10^4 \text{ m s}^{-1}. $$

In figure 1, $S_{xx}^g$ is shown as a function of $B$, for $T = 2, 4, 6, 8, 10$ K for $n_e = 1 \times 10^{10}$ m$^{-2}$. We see that $S_{xx}^g$ is oscillatory with the height of the peak increasing with the increasing $B$. The position of the peak occurs when the Fermi energy is in the localized state of LL. Interestingly, our calculations show large peak values of the order of a few hundreds of $\mu$V K$^{-1}$, which is closer to the values observed in GaAs HJs [12, 37].

We would like to point out that the sample size in GaAs HJs (∼few mm) is about two orders of magnitude larger than the size of the graphene sample chosen here. The size of the graphene sample (300 nm) in the experiment of Zuev et al [20] is about 30 times smaller than the value of $\Lambda$ used in the present calculation. Scaling $\Lambda$ down by 30 times, we get the peak values of $S_{xx}^g$, a few tens of $\mu$V K$^{-1}$ which is comparable to the measured values.

Dependence of $S_{xx}^g$ on $n_e$ is shown in figure 2 for three different magnetic fields, namely $B = 2.82, 3.88$, and 6.20 Tesla, at $T = 5$ K taking $\Lambda = 10$ $\mu$m. Again, the behavior is found to be oscillatory. This is similar to the behavior observed (as function of gate voltage) in the experiment of Zuev et al [20]. The peak value of $S_{xx}^g$ decreases with the increasing $n_e$. This is similar to the $n_e$ dependence of zero field $S^g$ and $S'$ [26]. Also, it is found that the peak values are smaller for smaller $B$. The number of oscillations contained in $S_{xx}^g$ is reduced with the increase of magnetic field. This is due to the increase of the separation between LLs with increasing magnetic field. Interestingly, we note that the position of the peak values corresponding to three different $B$ are coinciding at $n_e = (1, 4, 7$ and $10) \times 10^{16}$ m$^{-2}$.

In figure 3(a), we have shown $S_{xx}^g$, as a function of $n_e$ for $\Lambda = 300$ nm (as taken in [20]) at temperatures $T = 4.2, 10$, and 20 K for a magnetic field $B = 8.8$ Tesla. For comparison, we have calculated diffusion component $S_{xx}^d$ as a function of $n_e$, for the same $T$ and $B$, using modified Girvin–Jonsion theory (see [21]) and it is shown in figure 3(b). Note that $S_{xx}^d$ is also found to decrease with increasing $n_e$. According to the Girvin–Jonsion theory, the peak values due to diffusion component are given by $(k_B T e n B / 2 n^2) / S_{xx}^d$ is found to be much greater than $S_{xx}^g$. For example, for $n_e = 1.5 \times 10^{12}$ cm$^{-2}$, at 4.2 K (10 K)
$S_{xx}^d$ is nearly ten (three) times greater than $S_{xx}^g$. The total $S_{xx} = S_{xx}^d + S_{xx}^g$ is shown as a function of $n_e$ in figure 3(c) and it is increasing with $T$.

We would like to point out that in graphene, the peak values of diffusion thermopower $S_{xx}^d$ are quantized as $(k_B/e) \ln 2/n$ which differs from the peak value quantization $(k_B/e) \ln 2/(n + 1/2)$ corresponding to the conventional 2DEG. This difference is attributed to the existence of a non-trivial Berry phase $\pi$ in graphene [21]. Unlike diffusion thermopower, it is difficult to establish such peak value quantization for $S_{xx}^g$. However, a careful observation of figures 3(a) and (b) dictates to us that $S_{xx}^g$ follow $S_{xx}^d$ with respect to the locations of peaks. Moreover, the locations of thermopower peaks in 2DEG and graphene are expected to be different due to the different Landau level structures as was found in the case of conductivity oscillations [27].

In figure 4, we have shown $S_{xx}^g$ as function of $T$ for $B = 2.82$, 3.88 and 6.20 Tesla (corresponding to three peak values in figure 1). $S_{xx}^g$ increases with increasing $T$, more rapidly at lower $T$. At higher $T$, the increase is slower and shows almost independent behavior for about $T > 10$ K. This behavior is similar to the observations in conventional 2DEG [12, 14]. The faster increase of $S_{xx}^g$ with $T$, at low $T$, may be attributed to the increasing number of phonons linearly with $T$. For a given magnetic field, maximum momentum transfer takes place when $\hbar v q \approx \Gamma$ setting limit on $q$. As $T$ increases further, the allowed $q$ is limited by the width of LL ($\Gamma \sim B^{1/2}$) and fewer phonons will exchange momentum. In zero magnetic field, such behavior is generally interpreted [6, 8] in terms of the dominant phonon wave vector $q_D$ and Fermi wave vector $2k_F$. At a given $T$, $S_{xx}^g$ is larger for larger $B$. This is consistent with the findings in conventional 2DEG [12, 14].
Inset of figure 4, expressing \( \gamma \sim \gamma_{STxx} \), shows the behavior of exponent \( \gamma \) as a function of \( T \) for different \( B \). It is found to decrease and tending to zero with increasing \( T \). Moreover, \( \gamma \) is found to be larger for larger \( B \). We observe that for \( T \) closer to 1 K, \( \gamma \) is greater than 2 which is signature of phonon-drag thermopower. When \( S_{xx}^{d} \) is calculated as a function of \( T \) (not shown in the figure) for different \( n_e \) we expect it to increase with increasing \( T \), but to be smaller for larger \( n_e \). These curves are expected to show nearly independent behavior again at higher \( T \), more so for smaller \( n_e \). Exponent \( \gamma \), in this case \( n_e \) dependent, is again expected to decrease with increasing \( T \).

An important point we notice is that \( S_{xx}^{d} \) can be tailored to be as large as few mV K\(^{-1} \) by reducing \( n_e \) and working at larger \( B \). We suggest that the enhanced phonon-drag contribution \( S_{xx}^{d} \) can be achieved by polishing the edge of the sample. It is characterized by a specular parameter \( p \) with its value \( 0 < p < 1 \). The perfect reflecting edge gives \( p = 1 \) and the very rough edge corresponds to \( p = 0 \) (diffusive scattering).

Figure 3. Plots of thermopower versus carrier density \( n_e \) at different temperature for \( B = 8.8 \) Tesla and \( \Lambda = 300 \) nm as considered in [20]. Here, (a) \( S_{xx}^{g} \), (b) \( S_{xx}^{d} \), (c) \( S_{xx} = S_{xx}^{g} + S_{xx}^{d} \).

Figure 4. Temperature dependence of phonon-drag thermopower at electron density \( n_e = 10^{16} \) m\(^{-2} \). Inset shows variation of the exponent \( \gamma = d \ln S_{xx}^{d} / d \ln T \) with temperature.
Besides, the larger samples can be grown on piezoelectric substrates. Woszczyńska et al. [38] have shown that the graphene samples as large as 150 x 30 μm² on GaAs substrate can be prepared.

We would like to point out that the screening of el–ph coupling in magnetic field is ignored, although justification is for zero magnetic field case. In conventional 2DEG screening is found to reduce the phonon drag thermopower significantly both in a zero and quantizing magnetic field [6–9]. However, screening of el–ph interaction in graphene in magnetic field is yet to be established. Low temperature experimental measurements, for $S_{xx}$, may throw some light on significance of screening. One can extract the experimental phonon drag $S_{xx}$ from the experimentally measured $S_{xx}$ by subtracting diffusion component using generalized Mott formula [20].

4. Summary

In summary, we have studied phonon-drag thermopower $S_{xx}$ in graphene subjected to a transverse magnetic field. Based on the method described in [15], a modified theory is developed to calculate $S_{xx}$ quantitatively. Dependence of $S_{xx}$ on magnetic field, electron density, and temperature have been studied. With both magnetic field and density, $S_{xx}$ exhibits oscillatory behavior. Interestingly, we have found an enhanced phonon-drag thermopower with magnitude of the order of few hundreds $\rho V K^{-1}$. This value is closer to that obtained in the case of conventional 2DEG at a GaAs based semiconductor hetero interface. We attribute this enhanced phonon-drag effect is a consequence of taking the high value of phonon–mean free path, namely, $\Lambda = 10 \mu m$. We, thus, suggest that phonon-drag effect may have significant contribution in larger samples of graphene. We have also shown the density dependence of $S_{xx}$ for parameter values which were taken in [20]. The diffusion thermopower has also been calculated for the sake of comparison using modified Girvin–Johnson theory. Moreover, the temperature dependence of $S_{xx}$ is also studied and the exponent of this dependence has been extracted.

Acknowledgments

SSK would like to thank M Tsaousidou and TKG would like to thank A Kundu for useful discussions.

Appendix. Matrix elements of electron–phonon coupling in a magnetic field

For a graphene monolayer, lying in xy plane, with a perpendicular magnetic field $B = (0, B_0, 0)$, the eigen functions, for Landau gauge $A = (0, B_0 x, 0)$, are given by [27]

$$\psi_n(x) = \frac{e^{i k x} \chi_n(x)}{\sqrt{L_y}}$$

(A.1)

with

$$\chi_{n,k}(x) = \frac{1}{\sqrt{2}} \begin{pmatrix} -i \phi_{n-1}(x) \\ \phi_n(x) \end{pmatrix}.$$  

(A.2)

Here, $\alpha \equiv (n, k)$, $n = 0, 1, 2, 3\ldots$ is the Landau level index, $k$ is the y-component of electron wave vector, $\phi_n(x) = \sqrt{1/(2^n n! \sqrt{\pi} h_0)} e^{-x^2 + i x^2/2 h_0} [x(x + x_0)]$ with $x_0 = \hbar k_y/\sqrt{2 m e^2}$, is the harmonic oscillator wave function.

We assume that at low temperature, for the graphene on the substrate, electrons interact with only in-plane acoustic phonons via deformation potential coupling. In suspended graphene, there will be flexural modes, whose contribution is neglected for the graphene on substrate [39]. The deformation potential coupling is assumed to be only due to longitudinal acoustic phonons.

The most general form of electron–phonon interaction Hamiltonian is

$$H_{ep}(r) = \sum_{q^\prime}[V_q e^{i \mathbf{q} \cdot \mathbf{r}} a_{q^\prime} + V_{q}^\dagger e^{-i \mathbf{q} \cdot \mathbf{r}} a_{q^\prime}^\dagger],$$

where $a_{q^\prime}$ ($a_{q^\prime}^\dagger$) is the phonon annihilation (creation) operator and $V_q$ is the matrix element of a particular phonon mode ($\mathbf{q}, s$). For longitudinal acoustic phonon mode corresponding to deformation potential, the form of $V_q$ is given by

$$V_q = D [\hbar \omega_q/(2 \pi \rho_s 
\hbar^2)]^{1/2},$$

where $A_0$ is the area of graphene sample, $D$ is the deformation potential coupling constant, and $\rho_s$ is the areal mass density of graphene. The electron-acoustic phonon matrix element, for the scattering between the states $\alpha \equiv (n, k)$ and $\alpha' \equiv (n', k')$, is given by

$$C_{\alpha, \alpha'}(q) = \int_0^{L_x} \psi_{\alpha}^\ast (r) V_q e^{i \mathbf{q} \cdot \mathbf{r}} \psi_{\alpha'}(r) d^2 r.$$  

(A.3)

Substituting for $\psi_{\alpha}(r)$ and $V_q$, we get equation (5) in which the integrals are given by

$$C_{k', k, \pm q_y} = \frac{V_q}{L_y} \int_0^{L_y} e^{-\mathbf{q}_y \cdot \mathbf{r}} d y = C(q) \delta_{k', k} \delta_{q_y, \pm q_y},$$

(A.4)

and

$$J_{n, u}(u) = \int_0^{\infty} e^{i \mathbf{q} \cdot \mathbf{r}} \chi_{n, k}(x) dx.$$  

(A.5)

At low temperature, the acoustic phonon energy is small and causes only intra-Landau level transitions ($n = n'$). Inter-Landau level transitions are expected at higher temperatures and in the studies such as magnetophonon resonance in which optical phonons are involved [40]. The matrix element corresponding to intra-Landau level transitions is found to be

$$C_{\alpha, \alpha'}(q) = C(q) J_{n, u}(u) \delta_{k', k} \delta_{q_y, \pm q_y},$$

(A.6)

where

$$J_{n, u}(u) = \frac{1}{2} e^{i \mathbf{q}_y \cdot \mathbf{r}} \int [L_{n-1}(u) + L_{n}(u)]$$

(A.7)

with $u = q^2 l_0^2/2$. The equation for $J_{n, u}(u)^2$ given in equation (14) is similar to the one obtained in [41, 42].
References

[1] Castro Neto A H, Guinea F, Peres N M R, Novoselov K S and Geim A K 2009 Rev. Mod. Phys. 81 109
[2] Das Sarma S, Adam S, Hwang E H and Rossi E 2011 Rev. Mod. Phys. 83 30
[3] Novoselov K S, Geim A K, Morozov S V, Jiang D, Zhang Y, Dubonos S V, Grigorieva I V and Firsov A A 2004 Science 306 666
[4] Novoselov K S, Geim A K, Morozov S V, Jiang D, Katnelson M I, Grigorieva I V, Dubonos S V and Firsov A A 2005 Nature 438 197
[5] Zhang Y, Tan Y W, Stormer H L and Kim P 2005 Nature 438 201
[6] Gallagher B L and Butcher P N 1992 Handbook on Semiconductors vol 1, ed P T Landsberg (Amsterdam: Elsevier) p 817
[7] Fletcher R 1999 Semicond. Sci. Technol. 14 R1
[8] Fletcher R, Zaremba E and Zeitler U 2003 Electron-Phonon Interactions in Low-Dimensional Structures ed L Challis (Oxford: Oxford Science Publications) p 149
[9] Tsaousidou M 2010 The Oxford Handbook of Nanoscience and Technology vol II, ed A V Narlikar and Y Y Yu (Oxford: Oxford University Press) p 477
[10] Jonson M and Girvin S M 1984 Phys. Rev. B 29 1939
[11] Oji H 1984 J. Phys. C: Solid State Phys. 17 3059
[12] Fletcher R, Maan J C, Ploog K and Weimann G 1986 Phys. Rev. B 33 7122
[13] Kubakaddi S S, Butcher P N and Mulimani B G 1989 Phys. Rev. B 40 1377
[14] Lyo S K 1989 Phys. Rev. B 40 6458
[15] Fromhold T M, Butcher P N, Qin G, Mulimani B G, Oxley J P and Gallagher B L 1993 Phys. Rev. B 48 5326
[16] Jay-Gerin J P 1975 Phys. Rev. B 12 1418
[17] Puri S M 1965 Phys. Rev. 139 A995
[18] Herring C 1954 Phys. Rev. 96 1163
[19] Sankey S S and Kubakaddi S S and Mulimani B G 2016 Graphene Science Handbook: Electrical and Optical Properties vol 18, ed M Aliokhazraei et al (New York: CRC Press) p 273
[20] Zuev Y M, Chang W and Kim P 2009 Phys. Rev. Lett. 102 096807
[21] Checkelsky J G and Ong N P 2009 Phys. Rev. B 80 081413
[22] Wei P, Bao W, Yu P, Lau C N and Shi J 2009 Phys. Rev. Lett. 102 166808
[23] Wu X, Yu Y, Ruan M, Madwoman N K, Berger C and de Heer W A 2011 Appl. Phys. Lett. 99 133102
[24] Hwang E H, Rossi E and Das Sarma S 2009 Phys. Rev. B 80 235415
[25] Nika D L, Pekhtelov E P, Askervel A S and Balandin A A 2009 Phys. Rev. B 79 155413
[26] Kubakaddi S 2009 Phys. Rev. B 79 075417
[27] Matulis A and Peeters F M 2007 Phys. Rev. B 75 125429
[28] Lukose V, Shankar R and Baskaran G 2007 Phys. Rev. Lett. 98 116802
[29] Peres N M R and Castro E V 2007 J. Phys.: Condens. Matter 19 406231
[30] Butcher P N and Tsaousidou M 1998 Phys. Rev. Lett. 80 1718
[31] Tiwari R P and Stroud D 2009 Phys. Rev. B 79 165408
[32] Ghosh S, Nika D L, Pekhtelov E P and Balandin A A 2009 New J. Phys. 11 095012
[33] Baker A M R et al 2013 Phys. Rev. B 87 045414
[34] Huang J et al 2015 J. Phys.: Condens. Matter 27 164202
[35] Bastzitzer R and MacDonald A H 2009 Phys. Rev. B 80 085109
[36] DaSilva A M, Zou K, Jain J K and Zhu J 2010 Phys. Rev. Lett. 104 236601
[37] Tieke B, Fletcher R, Zeitler U, Henini M and Maan J C 1998 Phys. Rev. B 58 2017
[38] Wozyczyna M, Friedemann M, Götz M, Pesel E, Pierz K, Weimann T and Ahlers F J 2012 Appl. Phys. Lett. 100 164106
[39] Mariani E and von Oppen F 2010 Phys. Rev. B 82 195403
[40] Goerbig M O, Fuchs J N, Kechedzhi K and Falko V I 2007 Phys. Rev. Lett. 99 087402
[41] Nomura K and MacDonald A H 2006 Phys. Rev. Lett. 96 256602
[42] Kandemir B S and Mogulkoc A 2015 Phys. Lett. A 379 2120