Determination of the critical plane and durability estimation for a multiaxial cyclic loading

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Abstract. An analytical procedure is proposed to determine the critical plane orientation according to the Findley criterion for the multiaxial cyclic loading. The cases of in-phase and anti-phase cyclic loading are considered. Calculations of the stress state are carried out for the system of the gas turbine engine compressor disk and blades for flight loading cycles. The formulas obtained are used for estimations of the fatigue durability of this essential element of structure.

1. Introduction
Modern approaches to the construction of multi-axial fatigue fracture criteria often use the notion of the critical plane [1-6]. The first criterion with determination of critical plane for classical low-cyclic (LCF, conditionally \(10^5 < N < 10^6\)) and high-cyclic (HCF, conditionally \(10^6 < N < 10^8\)) fatigue failure was proposed by Findley [7]. According to this criterion, the destruction in the cyclic loading process occurs along so called critical plane where the maximum of the combination reaches a certain critical value. To determine service life of a sample up to its fatigue failure under the uniaxial cyclic loading, there is the Baskin relation [8], which is an analytical representation of the fatigue curve for various coefficients of the asymmetry of the cycle. The generalization of the Baskin relation in the case of the multiaxial stress state for the Findley criterion has the form:

\[
(\Delta \tau_n / 2 + \alpha \sigma_n)_{\text{MAX}} = S_F + A_F N^{\beta_F}, \quad \sigma_n = n \cdot \sigma \cdot n, \quad \tau_n = |\sigma \cdot n - (n \cdot \sigma \cdot n)n|
\]

Here \(\sigma\) is the stress tensor, which determines the stress state in the point of the sample, parameters \(\beta_F < 0, \alpha, S_F, A_F\) are defined experimentally, \(N\) is the number of cycles to failure, \(\Delta \tau_n\) is the range of the tangential stress per cycle, \(\sigma_n\) is the normal stress on critical plane, \(n\) is the unit normal vector to the critical plane.

Calculation of durability by the Findlay criterion requires the determination of the orientation \((n)\) of the critical plane in a given material point. It means to find the plane \((n)\) where during cycle
$t \in [0,T]$ the Findlay’s function $F = \Delta \tau_n + 2\alpha_F \max_{t \in [0,T]} \sigma_n$ is reached its maximum. For a multiaxial stressed state this is not easy task, which, as a rule, is to be solved numerically. Below we propose analytical solution of this problem for in-phase and anti-phase cyclic loading.

2. **Determination of the critical plane for in-phase multiaxial stressed state**

Let us consider a case of triaxial cyclic loading in a coordinate system connected with the principal axes of the stress tensor. We assume that these axes are not changed during the cycle and the principal values of the stress tensor vary according to a harmonic law without a phase shift relative to each other:

\[
\sigma_1(t) = \sigma_{1m} + \sigma_{1a} \sin \omega t, \quad \sigma_2(t) = \sigma_{2m} + \sigma_{2a} \sin \omega t, \quad \sigma_3(t) = \sigma_{3m} + \sigma_{3a} \sin \omega t, \quad \sigma_{1,2,3} \geq 0
\]

where the additional index “$m$” denotes the average values of the stresses per cycle, and the index “$a$” denotes their amplitudes.

The ranges of the principal stresses in the cycle are

\[
\Delta \sigma_1 = 2\sigma_{1a} \geq 0, \quad \Delta \sigma_2 = 2\sigma_{2a} \geq 0, \quad \Delta \sigma_3 = 2\sigma_{3a} \geq 0.
\]

We choose the principal axes so that the maxima of the principal stresses satisfy the inequalities

\[
\Sigma_1 \geq \Sigma_2 \geq \Sigma_3, \quad \Sigma_1 = \sigma_{1m} + \sigma_{1a}, \quad \Sigma_2 = \sigma_{2m} + \sigma_{2a}, \quad \Sigma_3 = \sigma_{3m} + \sigma_{3a}.
\]

We use the following notation:

\[
\Sigma_{12} = \sigma_1 - \sigma_2, \quad \Sigma_{13} = \sigma_1 - \sigma_3, \quad \Sigma_{23} = \sigma_2 - \sigma_3, \quad \Delta \sigma_{12} = \Delta \sigma_1 - \Delta \sigma_2, \quad \Delta \sigma_{13} = \Delta \sigma_1 - \Delta \sigma_3, \quad \Delta \sigma_{23} = \Delta \sigma_2 - \Delta \sigma_3.
\]

In the coordinate system associated with the principal axes of the stress tensor we have the following formulas for the normal and tangential stresses on the plane with normal $n_k$ (summation over the repeated indices $k$ and $l$):

\[
\sigma_n = \sigma_{kl} n_k n_l = \sigma_k n_k^2, \quad \tau_i = \sigma_{ik} n_k - \sigma_k n_i = (\sigma_i - \sigma_n) n_i
\]

For ranges of the tangential stress components and their modulus, we have:

\[
\Delta \tau_i = (\Delta \sigma_i - \Delta \sigma_n) n_i, \quad \Delta \tau = \sqrt{\sum_{i=1}^{3} (\Delta \sigma_i - \Delta \sigma_n)^2 n_i^2}, \quad \Delta \sigma_n = \Delta \sigma_k n_k^2
\]

Finally

\[
\Delta \tau = \sqrt{\Delta \sigma_i^2 n_i^2 - (\Delta \sigma_k n_k^2)^2}
\]

This formula can be transformed to the form:

\[
\Delta \tau = \sqrt{(\Delta \sigma_{12})^2 n_2^2 + (\Delta \sigma_{13})^2 n_3^2 + (\Delta \sigma_{23})^2 n_2^2 n_3^2}
\]

In the notation adopted, we obtain the compact formula

\[
\Delta \tau_n = \sqrt{(\Delta \sigma_{12})^2 n_2^2 + (\Delta \sigma_{13})^2 n_3^2 + (\Delta \sigma_{23})^2 n_2^2 n_3^2}
\]

Excluding $n_i^2$ we obtain the formula used to calculate the maximum of the function $F$:
\[
\Delta \tau = \sqrt{\Delta \sigma_{12}^2 n_2^2 + \Delta \sigma_{13}^2 n_3^2 - (\Delta \sigma_{12} n_2^2 + \Delta \sigma_{13} n_3^2)^2}
\]

Taking these formulas into account, we define the orientation of the critical plane by the components of the normal:

\[
x_1 = n_1^2 \geq 0, \quad x_2 = n_2^2 \geq 0, \quad x_3 = n_3^2 \geq 0, \quad x_1 + x_2 + x_3 = 1.
\]

It can be shown that the problem of determining the critical plane for a triaxial stressed state reduces to determining the maximum of the function

\[
F(x_2, x_3) = \frac{\Delta \sigma_{12}^2 x_2 + \Delta \sigma_{13}^2 x_3 - (\Delta \sigma_{12} x_2 + \Delta \sigma_{13} x_3)^2}{2} + 2\alpha \sigma_1 - 2\alpha \left( \sum_{i=12} x_2 + \sum_{i=13} x_3 \right)
\]

under constraints \(0 \leq x_2 + x_3 \leq 1, \ x_2 \geq 0, \ x_3 \geq 0\).

Below we present the results of solving this problem for all possible values of the maxima and variations of the principal stresses.

Firstly, we define the extremum of a function inside the zone of the constraints \(0 \leq x_2 + x_3 \leq 1, \ x_2 \geq 0, \ x_3 \geq 0\):

\[
\frac{\partial F(x_2, x_3)}{\partial x_2} = 0, \quad \frac{\partial F(x_2, x_3)}{\partial x_3} = 0
\]

Depending on the magnitude of the principal stresses, we have the following cases.

2.1. Case 1

If \(\sum_{i=12}/\Delta \sigma_{12} \neq \sum_{i=13}/\Delta \sigma_{13}, \ \Delta \sigma_{12} \neq 0, \ \Delta \sigma_{13} \neq 0\) then

I-a. For the case \(\Delta \sigma_{12} \neq \Delta \sigma_{13}\) conditions of extremum for \(F(x_2, x_3)\):

\[
x_2 = \Delta_2/(\Delta \sigma_{12} \Delta \sigma_{13}), \quad x_3 = \Delta_3/(\Delta \sigma_{13} \Delta \sigma_{23})
\]

I-b. For the case \(\Delta \sigma_{12} = \Delta \sigma_{13} \neq 0\) conditions of extremum for \(F(x_2, x_3)\):

\[
x_2 + x_3 = \Delta_4/\Delta \sigma_{12} \Delta
\]

where

\[
\beta_{12} = 4\alpha \sum_{i=12}/\Delta \sigma_{12} \ \beta_{13} = 4\alpha \sum_{i=13}/\Delta \sigma_{13}, \ \Delta = 2(\beta_{13} - \beta_{12}) \neq 0
\]

\[
\Delta_4 = \Delta \sigma_{12} \beta_{13} - \Delta \sigma_{13} \beta_{12}, \ \Delta_r = 2(\Delta \sigma_{13} - \Delta \sigma_{12})
\]

\[
\Delta_2 = \Delta \sigma_{13} \Delta_5/\Delta - (\Delta_5^2 + \Delta_7^2)/\Delta^2, \ \Delta_3 = (\Delta_5^2 + \Delta_7^2)/\Delta^2 - \Delta \sigma_{12} \Delta_5/\Delta
\]

2.2. Case 2

If \(\sum_{i=12}/\Delta \sigma_{12} = \sum_{i=13}/\Delta \sigma_{13}\) then \(\Delta \sigma_{12} = \Delta \sigma_{13} \neq 0, \ \sum_{i=12} = \sum_{i=13}\).

Conditions of extremum \(F(x_2, x_3): \ x_2 + x_3 = \left[1 - \beta_{12}/\sqrt{4 + \beta_{12}^2}\right]/2\)

Values \((x_2, x_3)\) should satisfy inequalities: \(x_2 > 0, \ x_1 > 0, \ x_2 + x_3 < 1\)

and conditions of maximum \(F(x_2, x_3): \Delta \sigma_{12}^2 + 4\Delta \sigma_{13} \Delta \sigma_{23} x_3 > 0, \ \Delta \sigma_{13}^2 - 4\Delta \sigma_{12} \Delta \sigma_{23} x_2 > 0\)

The inequalities follow from the condition

\[
\frac{\partial^2 F}{\partial x_2^2} (dx_2)^2 + 2 \frac{\partial^2 F}{\partial x_2 \partial x_3} dx_2 dx_3 + \frac{\partial^2 F}{\partial x_3^2} (dx_3)^2 < 0
\]

If such values of principal stresses do not exist, then it needs to seek maximum \(F(x_2, x_3)\) on boundaries: \(x_2 = 0, \ or x_3 = 0, \ or x_2 + x_3 = 1\).
2.3. Case 3
In this case maximum \( F(x_2, x_3) \) is reached for values:

\[
\begin{align*}
x_2 &= 0, \quad x_3 = \left(1 - \frac{\beta_{13}}{\sqrt{4 + \beta_{13}^2}}\right)/2 \\
x_2 &= 0, \quad x_3 = \left(1 - \frac{\beta_{12}}{\sqrt{4 + \beta_{12}^2}}\right)/2 \\
x_2 + x_3 &= 1, \quad x_2 = \left(1 + \frac{\beta_{23}}{\sqrt{4 + \beta_{23}^2}}\right)/2, \quad x_3 = \left(1 - \frac{\beta_{23}}{\sqrt{4 + \beta_{23}^2}}\right)/2
\end{align*}
\]

From these three pairs \((x_2, x_3)\) we choose one that provides maximum value of function \( F(x_2, x_3) \).

So, for the case of multiaxial stressed state the critical plane is found analytically. Knowing components of normal to the critical plane it is possible to calculate Findley’s function and corresponding number of cycles to fracture \( N \) [9].

3. Determination of the critical plane for anti-phase multiaxial stressed state
In the coordinate system connected with the principal axes of the stress tensor, an anti-phase harmonic cyclic loading can always be represented in the form:

\[
\sigma_1(t) = \sigma_{1m} + \sigma_{1a} \sin \omega t, \quad \sigma_2(t) = \sigma_{2m} + \sigma_{2a} \sin \omega t, \\
\sigma_3(t) = \sigma_{3m} + \sigma_{3a} \sin (\omega t + \pi) = \sigma_{3m} - \sigma_{3a} \sin \omega t, \quad \sigma_{1,2,3} \geq 0
\]

In this case

\[
\Delta \sigma_1 = 2\sigma_{1a} \geq 0, \quad \Delta \sigma_2 = 2\sigma_{2a} \geq 0, \quad \Delta \sigma_3 = -2\sigma_{3a} \leq 0
\]

and

\[
\Delta \tau_n = \sqrt{(\Delta \sigma_{12})^2 n_1^2 n_2^2 + (\Delta \sigma_{13})^2 n_1^2 n_3^2 + (\Delta \sigma_{23})^2 n_2^2 n_3^2}.
\]

The normal stress on the plane with the normal \( n \) depends on time:

\[
\sigma_n(t) = (\sigma_{1m}n_1^2 + \sigma_{2m}n_2^2 + \sigma_{3m}n_3^2) + (\sigma_{1a}n_1^2 + \sigma_{2a}n_2^2 - \sigma_{3a}n_3^2) \sin \omega t
\]

Therefore to determine \( \max \sigma_n \) we consider two cases.

3.1. Case 1
If \( \sigma_{1a}n_1^2 + \sigma_{2a}n_2^2 - \sigma_{3a}n_3^2 \geq 0 \), then \( \max \sigma_n = \Sigma_1 n_1^2 + \Sigma_2 n_2^2 + \Sigma_3 n_3^2 \) for \( \sin \omega t = 1 \),

where \( \Sigma_1 = \sigma_{1m} + \sigma_{1a}, \quad \Sigma_2 = \sigma_{2m} + \sigma_{2a}, \quad \Sigma_3 = \sigma_{3m} - \sigma_{3a} \).

We renumber the values \( \Sigma_k \), \( k = 1,2,3 \) so that they satisfy the inequalities \( \Sigma_1 \geq \Sigma_2 \geq \Sigma_3 \). After that the same analysis and calculation of the orientation of the critical plane are carried out for the renumbered values \( \Sigma_k \), \( \Delta \sigma_k \), \( k = 1,2,3 \) as for in-phase cycle loading.

The resulting values of the normal components are verified to satisfy condition \( \sigma_{1a}n_1^2 + \sigma_{2a}n_2^2 - \sigma_{3a}n_3^2 \geq 0 \). If this condition is not satisfied we consider the following alternative case.

3.2. Case 2
If \( \sigma_{1a}n_1^2 + \sigma_{2a}n_2^2 - \sigma_{3a}n_3^2 \leq 0 \), then \( \max \sigma_n = \Sigma_1 n_1^2 + \Sigma_2 n_2^2 + \Sigma_3 n_3^2 \) for \( \sin \omega t = -1 \),

where \( \Sigma_1 = \sigma_{1m} - \sigma_{1a}, \quad \Sigma_2 = \sigma_{2m} - \sigma_{2a}, \quad \Sigma_3 = \sigma_{3m} + \sigma_{3a} \).
In this case we also renumber the values $\Sigma_k$, $k = 1,2,3$ so that they satisfy the inequalities $\Sigma_1 \geq \Sigma_2 \geq \Sigma_3$. After that the critical plane is defined by renumbered values $\Sigma_k$, $\Delta \sigma_k$, $k = 1,2,3$ in the same manner as for in-phase cycle loading. The resulting values of the normal components are verified to satisfy condition $\sigma_{1\alpha} n_1^k + \sigma_{2\alpha} n_2^k - \sigma_{3\alpha} n_3^k \leq 0$.

If both conditions are satisfied the orientation of the chosen unit normal should provide the maximal value of the Findley function on the plane.

4. Durability estimates for gas turbine engine compressor disc in flight cycles

As an example let us consider fatigue durability calculation for typical compressor disk during flight cycles (low cycle fatigue mode). It is assumed that during loading cycle, the maximum loads are achieved at the cruising speed of the aircraft and the corresponding rotational speeds of the compressor disk. A multiaxial Findley’s criterion for LCF regime is used to calculate the safe operation time of the disk (number of flight cycles $N$ to fracture).

Using to the procedure proposed in [10] the parameters of the fatigue criterion were determined from the results of two uniaxial fatigue tests with the coefficients of asymmetry of the cycle $R = -1$, $R = 0$. For this purpose calculation of the stress-strain state of the disc with blades under action of centrifugal forces was performed.
The following input data for loading were used: angular rotation velocity $\omega = 314 \, \text{rad/s}$ (that corresponds to 3000 RPM), velocity head at infinity $\rho v^2 / 2 = 26000 \, \text{N/m}^2$ (that corresponds to cruising speed 200 m/s and air density 1.3 kg/m$^3$).

Elastic material properties were taken as: $E = 116 \, \text{MPa}$, $\nu = 0.32$, $\rho = 4370 \, \text{kg/m}^3$ for disk (titanium alloy), $E = 69 \, \text{MPa}$, $\nu = 0.33$, $\rho = 2700 \, \text{kg/m}^3$ for blades (aluminum alloy) and $E = 207 \, \text{MPa}$, $\nu = 0.27$, $\rho = 7860 \, \text{kg/m}^3$ for fixing pins (steel).

Firstly, stress-strain state (SSS) was calculated for the whole disk and 32 blades. Secondly, SSS was recalculated using refined grid for typical sector of the disk with blade that is shown in Figure 1-a.

Calculations showed that safe operation life reaches its minimum near the contact zone between the disk and the blades (in a dovetail-type connection). Figure 1-b shows the zone of concentration of the maximum tensile stresses in the left (rounded) corner of the groove of contact connection. So the lowest durability was obtained for left corner of the groove.

The left corner of the groove is shown in Figure 2-a using fat continuous lines. Figure 2-b shows the calculated values $N$ (the number of flight cycles to failure) for the chosen criterion for multiaxial fatigue failure. In Figure 2-b the dimensionless coordinate of the curvature of the left corner of the connection groove is plotted along the horizontal axis, and the dimensionless coordinate along the groove depth is plotted along the vertical axis. The smallest value was $\sim 30000 \, \text{cycles}$. If we take that the average flight cycle time is equal to 2 hours then in real time the durability of the safe exploitation for disk-blade structural element will be 60 000 hours.

5. Conclusions
The critical plane orientation in the Findley’s criterion is found analytically for the cases of in-phase and anti-phase cyclic multiaxial loading. This solution is used to estimate the durability of safe operation life for gas turbine engine compressor disk under cyclic flight loading.

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