Abstract

We define a homotopy algebra associated to classical open-closed strings. We call it an open-closed homotopy algebra (OCHA). It is inspired by Zwiebach’s open-closed string field theory and also is related to the situation of Kontsevich’s deformation quantization. We show that it is actually a homotopy invariant notion; for instance, the minimal model theorem holds. Also, we show that our open-closed homotopy algebra gives us a general scheme for deformation of open string structures ($A_{\infty}$-algebras) by closed strings ($L_{\infty}$-algebras).
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1 Introduction

In this paper we define a strong homotopy algebra inspired by Zwiebach’s classical open-closed string field theory [61] and examine its homotopy algebraic structures. It is known that classical closed string field theory has an $L_\infty$-structure [60, 52, 34] and classical open string field theory has an $A_\infty$-structure [12, 61, 46, 30]. As described by Zwiebach [60, 61] and others, string field theory is presented in terms of decompositions of moduli spaces of the corresponding Riemann surfaces into cells. The associated Riemann surfaces are (respectively) spheres with (closed string) punctures and disks with (open string) punctures on the boundaries. That is, classical closed string field theory is related to the conformal plane $\mathbb{C}$ with punctures and classical open string field theory is related to the upper half plane $H$ with punctures on the boundary from the viewpoint of conformal field theory. The algebraic structure that the classical open-closed string field theory has is similarly interesting since it is related to the upper half plane $H$ with punctures both in the bulk and on the boundary, which also appeared recently in the context of deformation quantization [36, 5].

In operad theory (see [44]), the relevance of the little disk operad to closed string theory is known, where a (little) disk is related to a closed string puncture on a sphere in the Riemann surface picture above. The homology of the little disk operad defines a Gerstenhaber algebra [6, 16], in particular, a suitably compatible graded commutative algebra structure and graded Lie algebra structure. The framed little disk operad is in addition equipped with a
BV-operator which rotates the disk boundary $S^1$. The algebraic structure on the homology is then a BV-algebra [14], where the graded commutative product and the graded Lie bracket are related by the BV-operator. Physically, closed string states associating to each disk boundary $S^1$ are constrained to be the $S^1$-invariant parts, the kernel of the BV-operator. This in turn leads to concentrating on the Lie algebra structure, where two disk boundaries are identified by twist-sewing as Zwiebach did [60]. On the other hand, he worked at the chain level (‘off shell’), discovering an $L_\infty$-structure. This was important since the multi-variable operations of the $L_\infty$-structure provided correlators of closed string field theory. Similarly for open string theory, the little interval operad and associahedra are relevant, the homology corresponding to a graded associative algebra, but the chain level reveals an $A_\infty$-structure giving the higher order correlators of open string field theory.

The corresponding operad for the open-closed string theory is the Swiss-cheese operad [58] that combines the little disk operad with the little interval operad; it was inspired also by Kontsevich’s approach to deformation quantization. The algebraic structure at the homology level has been analyzed thoroughly by Harrelson [24]. In contrast, our work in the open-closed case is at the level of strong homotopy algebra, combining the known but separate $L_\infty$- and $A_\infty$-structures. There are interesting relations (not yet fully explored) between an algebra over the Swiss-cheese operad and the homotopy algebra we define in the present paper. In particular, we leave for later work the inclusion of the appropriate homotopy algebra corresponding to the graded commutative product and the BV-operator. For possible structures to be added to our structure, see [54].

We call our structure an open-closed homotopy algebra (OCHA) (since it captures a lot of the operations in existing open-closed string field theory algebra structure [32]). We show that this description is a homotopy invariant algebraic structure, i.e. that it transfers well under homotopy equivalences or quasi-isomorphisms. Also, we show that an open-closed homotopy algebra gives us a general scheme for deformation of open string structures ($A_\infty$-algebras) by closed strings ($L_\infty$-algebras).

We first present our notion of open-closed homotopy algebras in section 2. An open-closed homotopy algebra consists of a direct sum of graded vector spaces $\mathcal{H} = \mathcal{H}_c \oplus \mathcal{H}_o$. It has an $L_\infty$-structure on $\mathcal{H}_c$ and reduces to an $A_\infty$-algebra if we set $\mathcal{H}_c = 0$. Moreover, the operations that intertwine the two are a generalization of the strong homotopy analog of H. Cartan’s notion of a Lie algebra $\mathfrak{g}$ acting on a differential graded algebra $E$ [4, 9].

We present the basics of these notions of homotopy algebra from three points of view: multi-variable operations, coderivation differentials and tree diagrams. In a more physically oriented paper [32], we give an alternate interpretation in the language of homological vector fields on a supermanifold.

The motivating physics of string interactions suggests that the homotopy algebra should be appropriately cyclic [44, 17]. In section 3, we make it precise in terms of an odd symplectic/cyclic structure which is strictly invariant with respect to the OCHA structure. It would be worth investigating a strongly homotopy invariant analog in the sense of [55, 56], which we do not discuss in this paper.

One of the key theorems in homotopy algebra is the minimal model theorem which was first
proved for $A_\infty$-algebras by Kadeishvili [29]. The minimal model theorem states the existence of minimal models for homotopy algebras analogous to Sullivan’s minimal models [53] for differential graded commutative algebras introduced in the context of rational homotopy theory. For an $A_\infty$- or $L_\infty$-algebra, the minimal model theorem is now combined with various stronger results; those employing the techniques of homological perturbation theory (HPT) (for instance see [19, 27, 20, 21, 22, 23]), what is called the decomposition theorem in [31, 33], Lefèvre’s approach [40], etc. These theorems are very powerful and make clear the homotopy invariant nature of the algebraic properties (for instance [42, 28, 33]). In section 4 we describe these theorems for our open-closed homotopy algebras, pointing out subtleties of the open-closed case in addition to those for $L_\infty$-algebras in comparison to the existing versions for $A_\infty$-algebras.

In section 5, we show that an open-closed homotopy algebra gives a general scheme of deformation of the $A_\infty$-algebra $H_\alpha$ as controlled by $H_c$. A particular example of the deformation point of view applied in an open-closed setting occurs in analyzing Kontsevich’s deformation quantization theorem, which we shall explain explicitly in the sequel to this paper [32]. We discuss this deformation theory also from the viewpoint of generalized Maurer-Cartan equations for an open-closed homotopy algebra and the moduli space of their solution space.

We include an appendix by M. Markl, where $A_\infty$-algebras over $L_\infty$-algebras are interpreted as a colored version of strongly homotopy algebras in the sense in [44].

We have taken care to provide the detailed signs which are crucial in calculations, but which are conceptually unimportant and can be ignored at first reading. The majority of this paper is entirely mathematics, and in the sequel [32] we show how our structures are related to those in Zwiebach [61], deformation quantization by Kontsevich [36], as well as those discussed in [25, 26] where an open-closed homotopy algebra is applied to topological open-closed strings. It should be very interesting to investigate the application to homological mirror symmetry [59, 35, 3, 26].

2 Strong homotopy algebra

An open-closed homotopy algebra, as we propose in this paper, is a strong homotopy algebra (or $\infty$-algebra) which combines two typical strong homotopy algebras, an $A_\infty$-algebra and an $L_\infty$-algebra. Let us begin by recalling those definitions. We restrict our arguments to the case that the characteristic of the field $k$ is zero. We further let $k = \mathbb{C}$ for simplicity.

There are various equivalent way of defining/describing strong homotopy algebras: in terms of multi-variable operations and relations among them, in terms of a coderivation differential of square zero on an associated coalgebra or as a representation of a particular operad of trees. We will treat all three of these in turn. The reader who is familiar with these approaches to the ‘classical’ $A_\infty$-algebras and $L_\infty$-algebras can move ahead to subsection 2.7, being warned that the definitions we give are different from the original ones [49, 39, 44] in the degrees of the multi-linear maps and hence of the relevant signs. Both are in fact equivalent and related by suspension [44], as we explain further below.
2.1 Strong homotopy associative algebras

**Definition 1** (*A_∞*-algebra (strong homotopy associative algebra)\([49]\)) Let \( A \) be a \( \mathbb{Z} \)-graded vector space \( A = \bigoplus_{r \in \mathbb{Z}} A^r \) and suppose that there exists a collection of degree one multi-linear maps
\[
m := \{ m_k : A^\otimes k \to A \}_{k \geq 1}.
\]

\((A, m)\) is called an \( A_\infty \)-algebra when the multi-linear maps \( m_k \) satisfy the following relations
\[
\sum_{k+l=n+1} \sum_i (-1)^{o_1+\cdots+o_i-1} m_k(o_1, \cdots, o_i, o_{i+1}, \cdots, o_{i+l-1}), o_{i+l}, \cdots, o_n) = 0 \tag{2.1}
\]

for \( n \geq 1 \), where \( o_j \) on \(-1\) denotes the degree of \( o_j \).

A \textit{weak} \( A_\infty \)-algebra consists of a collection of degree one multi-linear maps
\[
m := \{ m_k : A^\otimes k \to A \}_{k \geq 0}
\]
satisfying the above relations, but for \( n \geq 0 \) and in particular with \( k, l \geq 0 \).

**Remark 1** The relation (2.1) is different from the original one \([49]\) in the definition of the degrees of the multi-linear maps \( m_k \) and hence of the signs. Both are in fact equivalent and related by \textit{desuspension} \([44]\). In \([49]\), the \( m_k \) are multi-linear maps on \( \downarrow A \) where \((\downarrow A)^{r+1} = A^r\); we denote desuspension by \( \downarrow \). (The algebraic geometry tradition would use \([-1]\). ) Note that, in that notation \([49]\), a differential graded \( (dg) \) algebra is an \( A_\infty \)-algebra with a differential \( m_1 \), a product \( m_2 \), and \( m_3 = m_4 = \cdots = 0 \).

The ‘weak’ version is fairly new, inspired by physics, where \( m_0 : \mathbb{C} \to A \), regarded as an element \( m_0(1) \in A \), is related to what physicists refer to as a ‘background’. The augmented relation then implies that \( m_0(1) \) is a cycle, but \( m_1 m_1 \) need no longer be 0, rather \( m_1 m_1 = \pm m_2(0 \otimes 1) \pm m_2(1 \otimes m_0) \).

**Definition 2** (*\( A_\infty \)-morphism*) For two \( A_\infty \)-algebras \((A, m)\) and \((A', m')\), suppose that there exists a collection of degree zero (degree preserving) multi-linear maps
\[
f_k : A^\otimes k \to A', \quad k \geq 1.
\]
The collection \( \{ f_k \}_{k \geq 1} : (A, m) \to (A', m') \) is called an \( A_\infty \)-\textit{morphism} iff it satisfies the following relations:
\[
\sum_{1 \leq k_1 < k_2 \cdots < k_j = n} m_j'(f_{k_1}(o_1, \cdots, o_{k_1}), f_{k_2-k_1}(o_{k_1+1}, \cdots, o_{k_2}), \cdots, f_{n-k_{j-1}}(o_{k_{j-1}+1}, \cdots, o_n)) = \sum_{k+l=n+1} \sum_i (-1)^{o_1+\cdots+o_i-1} f_k(o_1, \cdots, o_{i-1}, m_l(o_i, \cdots, o_{i+l-1}), o_{i+l}, \cdots, o_n) \tag{2.2}
\]

for \( n \geq 1 \).
If \((A, m)\) and \((A', m')\) are weak \(A_\infty\)-algebras, then a weak \(A_\infty\)-morphism consists of multi-linear maps \(\{f_k\}_{k \geq 0}\), where \(f_0 : C \to A'\), satisfying the above conditions for \(n \geq 0\). In particular, the condition for \(n = 0\) is:

\[
f_1 \circ m_0 = \sum_{k \geq 0} m_k(f_0, \ldots, f_0).
\]

### 2.2 The coalgebra description and the Gerstenhaber bracket

The maps \(m_k\) can be assembled into a single map, also denoted \(m\), from the tensor space \(T^c A = \bigoplus_{k \geq 0} A^\otimes k\) to \(A\) with the convention that \(A^\otimes 0 = C\). The grading implied by having the maps \(m_k\) all of degree one is the usual grading on each \(A^\otimes k\). We can regard \(T^c A\) as the tensor coalgebra by defining

\[
\triangle(o_1 \otimes \cdots \otimes o_n) = \Sigma_{p=0}^{n} (o_1 \otimes \cdots \otimes o_p) \otimes (o_{p+1} \otimes \cdots \otimes o_n).
\]

A map \(f \in \text{Hom}(T^c A, T^c A)\) is a graded coderivation means \(\triangle f = (f \otimes 1 + 1 \otimes f) \triangle\), with the appropriate signs and dual to the definition of a graded derivation of an algebra. Here \(1\) denotes the identity \(1 : A \to A\). We then identify \(\text{Hom}(T^c A, A)\) with \(\text{Coder}(T^c A)\) by lifting a multi-linear map as a coderivation [51]. Analogously to the situation for derivations, the composition graded commutator of coderivations is again a coderivation; this graded commutator corresponds to the Gerstenhaber bracket on \(\text{Hom}(T^c A, A)\) [13, 51]. Notice that this involves a shift in grading since Gerstenhaber uses the traditional Hochschild complex grading. Thus \(\text{Coder}(T^c A)\) is a graded Lie algebra and in fact a dg Lie algebra with respect to the bar construction differential, which corresponds to the Hochschild differential on \(\text{Hom}(T^c A, A)\) in the case of an associative algebra \((A, m)\)[13]. Using the bracket, the differential can be written as \([m, \cdot]\).

The advantage of this point of view is that the component maps \(m_k\) assemble into a single map \(m\) in \(\text{Coder}(T^c A)\) and relations (2.1) can be summarized by

\[
[m, m] = 0 \quad \text{or, equivalently,} \quad D^2 = 0,
\]

where \(D = [m, \cdot]\). In fact, \(m \in \text{Coder}(T^c A)\) is an \(A_\infty\)-algebra structure on \(A\) iff \([m, m] = 0\) and \(m\) has no constant term, \(m_0 = 0\). If \(m_0 \neq 0\), the structure is a weak \(A_\infty\)-algebra. The \(A_\infty\)-morphism components similarly combine to give a single map of dg coalgebras \(f : T^c A \to T^c A'\), \((f \otimes f) \triangle = \triangle f\). In particular, (2.2) is equivalent to \(f \circ m = m' \circ f\).

### 2.3 The tree description

There are some advantages to indexing the maps \(m_k\) and their compositions by planar rooted trees; e.g. \(m_k\) will correspond to the corolla with \(k\) leaves all attached directly to the root. The composite \(m_k \bullet_i m_l\) then corresponds to grafting the root of \(m_l\) to the \(i\)-th leaf of \(m_k\), reading from left to right (see Figure 1). This is the essence of the planar rooted tree operad [44]. Multi-linear maps compose in just this way, so relations (2.1) can be phrased as saying we have a map from planar rooted trees to multi-linear maps respecting the \(\bullet\) ‘products’, the essence of a map of operads [44]. More precisely, let \(A_\infty(n), n \geq 1\) be a graded vector space spanned by
Figure 1: The grafting $m_k \bullet_i m_l$ of the $l$-corolla $m_l$ to the $i$-th leaf of $k$-corolla $m_k$, where $j = i + l - 1$ and $n = k + l - 1$.

planar rooted trees of $n$ leaves with identity $e \in A_\infty(1)$. For a planar rooted tree $T \in A_\infty(n)$, its grading is introduced by the number of the vertices contained in $T$, which we denote by $v(T)$. A tree $T \in A_\infty(n)$, $n \geq 2$, with $v(T) = 1$ is the corolla $m_n$. Any tree $T$ with $v(T) = 2$ is obtained by the grafting of two corollas as in eq. (2.3). Grafting of any two trees is defined in a similar way, with an appropriate sign rule, and any tree $T$ with $v(T) \geq 2$ can be obtained recursively by grafting a corolla to a tree $T'$ with $v(T') = v(T) - 1$. One can define a differential $d$ of degree one, which acts on each corolla as

$$d(m_n) = - \sum_{k,l \geq 2, k+l=n+1} \sum_{i=1}^k m_k \bullet_i m_l$$

and extends to one on $A_\infty := \oplus_{n \geq 1} A_\infty(n)$ by the following rule:

$$d(T \bullet_i T') = d(T) \bullet_i T' + (-1)^{v(T)} T \bullet_i d(T') .$$

If we introduce the contraction of internal edges, that is, indicate by $T' \to T$ that $T$ is obtained from $T'$ by contracting an internal edge, the differential is equivalently given by

$$d(T) = \sum_{T' \to T} \pm T'$$

with an appropriate sign $\pm$. Thus, one obtains a dg operad $A_\infty$, which is known as the $A_\infty$-operad. An algebra $A$ over $A_\infty$ is obtained by a representation $\phi : A_\infty(k) \to \text{Hom}(A^\otimes k, A)$, though we use the same notation $m_k$ for the $k$-corolla and its image by $\phi$. This map $\phi$ should be defined so that it is compatible with respect to $\bullet_i$ and also the differentials. Here, identifying elements in $\text{Hom}(A^\otimes k, A)$ with those in $\text{Coder}(T^c A)$, the differential on the algebra side is $[m_1, \ ]$.

In particular, for each corolla one gets

$$\phi(d(m_k)) = m_1 \phi(m_k) + \sum_{i=1}^k \phi(m_k) \circ (1^\otimes (i-1) \otimes m_1 \otimes 1^\otimes (k-i)) .$$

It is clear that this equation combined with eq. (2.4) implies the $A_\infty$-relations (2.1).

Note that this grading of trees we introduced here corresponds to that in the suspended notation of $A_\infty$-algebras. In unsuspended notation $\downarrow A$, the grading of a tree $T \in A_\infty(n)$ should
be replaced by \( \text{int}(T) + 2 - n \), where \( \text{int}(T) = v(T) - 1 \) denotes the number of the internal edges. Each tree \( T \in A_\infty(n) \) corresponds to a codimension \( \text{int}(T) \) boundary piece of associahedron \( K_n \) [49], that is, \( \text{int}(T) + 2 - n \) is equal to minus the dimension of the boundary piece.

### 2.4 \( L_\infty \)-algebras and morphisms

Since an ordinary Lie algebra \( g \) is regarded as ungraded, the defining bracket is regarded as skew-symmetric. If we regard \( g \) as all of degree one, then the bracket would be graded symmetric. For dg Lie algebras and \( L_\infty \)-algebras, we need graded symmetry, which refers to symmetry with signs determined by the grading. The basic relation is

\[
\tau : x \otimes y \mapsto (-1)^{|x||y|} y \otimes x .
\] (2.5)

Also we adopt the convention that tensor products of graded functions or operators have the signs built in; e.g. \( (f \otimes g)(x \otimes y) = (-1)^{|g||x|} f(x) \otimes g(y) \). By decomposing permutations as a product of transpositions, there is then defined the sign of a permutation of \( n \) graded elements, e.g. for any \( c_i \in V, 1 \leq i \leq n \) and any \( \sigma \in \mathfrak{S}_n \), the permutation of \( n \) graded elements, is defined by

\[
\sigma(c_1, \cdots, c_n) = (-1)^{\epsilon(\sigma)} (c_{\sigma(1)}, \cdots, c_{\sigma(n)}) .
\] (2.6)

The sign \((-1)^{\epsilon(\sigma)}\) is often referred to as the Koszul sign of the permutation.

**Definition 3 (Graded symmetry)** A graded symmetric multi-linear map of a graded vector space \( V \) to itself is a linear map \( f : V^{\otimes n} \to V \) such that for any \( c_i \in V, 1 \leq i \leq n \), and any \( \sigma \in \mathfrak{S}_n \) (the permutation group of \( n \) elements), the relation

\[
f(c_1, \cdots, c_n) = (-1)^{\epsilon(\sigma)} f(c_{\sigma(1)}, \cdots, c_{\sigma(n)})
\] (2.7)

holds.

Since we will have many formulas with such indices and their permutations, we will use the notation \( I = (i_1, \ldots, i_n) \) and

\[c_I = c_{i_1} \otimes \cdots \otimes c_{i_n} .\]

Then, for any \( \sigma \in \mathfrak{S}_n \), we use \( \sigma(I) \) to denote \((\sigma(i_1), \ldots, \sigma(i_n))\) and hence

\[c_{\sigma(I)} = c_{\sigma(i_1)} \otimes \cdots \otimes c_{\sigma(i_n)} .\]

**Definition 4** By a \((k, l)\)-unshuffle of \( c_1, \ldots, c_n \) with \( n = k + l \) is meant a permutation \( \sigma \) such that for \( i < j \leq k \), we have \( \sigma(i) < \sigma(j) \) and similarly for \( k < i < j \leq k + l \). We denote the subgroup of \((k, l)\)-unshuffles in \( \mathfrak{S}_{k+l} \) by \( \mathfrak{S}_{k,l} \) and by \( \mathfrak{S}_{k+l} = n \), the union of the subgroups \( \mathfrak{S}_{k,l} \) with \( k + l = n \). Similarly, a \((k_1, \cdots, k_i)\)-unshuffle means a permutation \( \sigma \in \mathfrak{S}_n \) with \( n = k_1 + \cdots + k_i \) such that the order is preserved within each block of length \( k_1, \cdots, k_i \). The subgroup of \( \mathfrak{S}_n \) consisting of all such unshuffles we denote by \( \mathfrak{S}_{k_1, \cdots, k_i} \).
Definition 5 (**L*-algebra (strong homotopy Lie algebra)** [39]) Let \( L \) be a graded vector space and suppose that a collection of degree one graded symmetric linear maps \( l := \{ l_k : L^\otimes k \to L \}_{k \geq 1} \) is given. \((L, l)\) is called an \( L^-\text{algebra} \) iff the maps satisfy the following relations:

\[
\sum_{\sigma \in S_k \cup \{1\}} (-1)^{\varepsilon(\sigma)} l_{1+l}(l_k(c_{\sigma(1)}, \cdots, c_{\sigma(k)}, c_{\sigma(k+1)}, \cdots, c_{\sigma(n)})) = 0
\]

for \( n \geq 1 \). Using the multi-index notation \( I \), this can be written

\[
\sum_{\sigma \in S_k \cup \{1\}} (-1)^{\varepsilon(\sigma)} l_{1+l}(l_k \otimes 1 \otimes l)(c_{\sigma(I)}) = 0
\]

for \( n \geq 1 \). A weak \( L^-\text{algebra} \) consists of a collection of degree one graded symmetric linear maps \( l := \{ l_k : L^\otimes k \to L \}_{k \geq 0} \) satisfying the above relations, but for \( n \geq 0 \) and with \( k, l \geq 0 \).

Remark 2 The alternate definition in which the summation is over all permutations, rather than just unshuffles, requires the inclusion of appropriate coefficients involving factorials. Recall that the signs we use correspond to the suspension of the original definition.

Remark 3 A dg Lie algebra is expressed as the desuspension of an \( L^-\text{algebra} \) \((L, l)\) where \( l_1 \) and \( l_2 \) correspond to the differential and the Lie bracket, respectively, and higher multi-linear maps \( l_3, l_4, \cdots \) are absent.

Remark 4 For the ‘weak’ version, remarks analogous to those for weak \( A^-\text{algebras} \) apply.

Definition 6 (**L*-morphism**) For two \( L^-\text{algebras} \) \((L, l)\) and \((L', l')\), suppose that there exists a collection of degree zero (degree preserving) graded symmetric multi-linear maps \( f_k : L^\otimes k \to L' \), \( l \geq 0 \). Here \( f_0 \) is a map from \( \mathbb{C} \) to a degree zero subvector space of \( L \). The collection \( \{ f_k \}_{k \geq 1} : (L, l) \to (L', l') \) is called an \( L^-\text{morphism} \) iff it satisfies the following relations

\[
\sum_{\sigma \in S_k \cup \{1\}} (-1)^{\varepsilon(\sigma)} f_{1+l}(l_k \otimes 1 \otimes l)(c_{\sigma(I)}) = \sum_{\sigma \in S_{k+j} \cup \{1\}} (-1)^{\varepsilon(\sigma)} l'_{1+l}(f_{k_1} \otimes f_{k_2} \otimes \cdots \otimes f_{k_j})(c_{\sigma(I)})
\]

for \( n \geq 1 \).

When \((L, l)\) and \((L', l')\) are weak \( L^-\text{algebras} \), then a weak \( L^-\text{morphism} \) consists of multi-linear maps \( \{ f_k \}_{k \geq 0} \) satisfying the above conditions and in addition \( f_1 c l_0 = \sum_k (1/k!) l_k (f_0, \cdots, f_0) \).

When \( L' \) is (a suspension of) a strict dg Lie algebra, the formula simplifies greatly since, on the right hand side, we have \( j = 1 \) or \( 2 \) only ([38] Definition 5.2).
2.5 The symmetric coalgebra description

The graded symmetric coalgebra on a graded vector space $V$ is naturally the subcoalgebra $S^c V \subset T^c V$ consisting of the graded symmetric elements in each $V^\otimes n$. By not regarding $S^c V$ as a quotient of $T^c V$, certain complicated factorial coefficients do not appear in our formulas. Also, in rational homotopy theory, $S^c V$ is often denoted $\Lambda^c V$, due to a historical accident. To avoid possible confusion, we will use neither, but instead $C(V)$, as in [47].

Again, the sum of the maps $l = \bigoplus_k l_k$ provides a coderivation differential $l$ (with $[l, l] = 0$) on the full tensor coalgebra. Because of the graded symmetry of the $l_k$, the structure can be identified with a coderivation differential on the graded symmetric coalgebra $C(L)$, see [38]. That is, $l \in \text{Coder}^1(C(L))$ is an $L_\infty$-algebra structure on $L$ iff $[l, l] = 0$ and $l$ has no constant term: $l_0 = 0$. If $l_0 \neq 0$, the structure is a weak $L_\infty$-algebra.

Also, if we assemble the $L_\infty$-morphism components $\{f_k\}$ to a single $f : T^c L \to L'$ and lift it to a coalgebra map $f : C(L) \to C(L')$, then (2.10) is equivalent to $f \circ l = l' \circ f$.

2.6 The tree description

The tree operad description of $L_\infty$-algebras uses non-planar rooted trees with leaves numbered $1, 2, \ldots$ arbitrarily [44]. Namely, a non-planar rooted tree can be expressed as a planar rooted tree but with arbitrary ordered labels for the leaves. In particular, corollas obtained by permuting the labels are identified (Figure 2). Let $L_\infty(n), n \geq 1$ be a graded vector space generated by

\[
\begin{array}{ccc}
1 & 2 & 3 \\
\vdots & \vdots \end{array}
\quad \quad \quad
\begin{array}{c}
k \\
\vdots \sigma(1) \sigma(2) \sigma(3) \cdots \sigma(k) \end{array}
\]

Figure 2: Nonplanar $k$-corolla corresponding to $l_k$. Since edges are non-planar, it is symmetric with respect to the permutation of the edges.

those non-planar rooted trees of $n$ leaves. For a tree $T \in L_\infty(n)$, a permutation $\sigma \in \mathfrak{S}_n$ of the labels for leaves generates a different tree in general, but sometimes the same one because of the symmetry of the corollas above. The grafting, $\circ$, to the $i$-th leaf is defined as in the planar case in subsection 2.3, and any non-planar rooted tree is obtained by grafting corollas $\{l_k\}_{k \geq 2}$ recursively, as in the planar case, together with the permutations of the labels for the leaves. A degree one differential $d : L_\infty(n) \to L_\infty(n)$ is given in a similar way; for $T' \to T$ indicating that $T$ is obtained from $T'$ by the contraction of an internal edge,

\[
d(T) = \sum_{T' \to T} \pm T'.
\]
In particular, for each corolla one gets

\[
\begin{pmatrix}
1 & 2 & 3 & \cdots & n
\end{pmatrix}
\]

\[
\begin{pmatrix}
\sigma(1) & \cdots & \sigma(k) & \cdots & \sigma(n)
\end{pmatrix}
\]

and \(d(T \circ_i T') = d(T) \circ_i T' + (-1)^{v(T)} T \circ_i d(T')\) again holds. Thus, \(L_\infty := \oplus_{n \geq 1} L_\infty(n)\) forms a dg operad, called the \(L_\infty\)-operad.

An algebra \(L\) over \(L_\infty\) obtained by a map \(\phi : L_\infty(k) \to \text{Hom}(L^\otimes k, L)\) then forms an \(L_\infty\)-algebra \((L, l)\). If we use double desuspended notation \(\downarrow\downarrow L\) (physics notation \([39, 60]\); see \([34]\)), the degree of a multi-linear map \(l_k\) turns into \(1 - 2(k - 1) = 3 - 2k\). The grading of a tree \(T \in L_\infty(n)\) should be replaced by \(\text{int}(T) + (3 - 2k)\), where the dimension of the corresponding boundary piece of the compactified moduli space of a sphere with \((k + 1)\) marked points is \(-(\text{int}(T) + (3 - 2k)) - 1\) (see \([34]\)).

### 2.7 Open-closed homotopy algebra (OCHA)

For our open-closed homotopy algebra, we consider a graded vector space \(\mathcal{H} = \mathcal{H}_c \oplus \mathcal{H}_o\) in which \(\mathcal{H}_c\) will be an \(L_\infty\)-algebra and \(\mathcal{H}_o\), an \(A_\infty\)-algebra. An open-closed homotopy algebra includes various sub-structures, or reduces to various simpler structures as particular cases. An important such structure is the action of \(\mathcal{H}_c\) as an \(L_\infty\)-algebra on \(\mathcal{H}_o\) as a dg vector space. This is the appropriate strong homotopy version of the action of an ordinary Lie algebra \(L\) on a vector space \(M\), also described as \(M\) being a module over \(L\) or a representation of \(L\). Thus we can also speak of \(\mathcal{H}_o\) as a strong homotopy module over \(\mathcal{H}_c\) or as a strong homotopy representation of \(\mathcal{H}_c\) (cf. \([50]\)). Moreover, we will need the strong homotopy version of an algebra \(A\) over a Lie algebra \(L\), that is, an action of \(L\) by derivations of \(A\), so that the map \(L \to \text{End}(A)\) takes values in the Lie sub-algebra \(\text{Der}A\). We first arrange these known structures, and then define our open-closed homotopy algebra as an extension of them.

Lada and Markl ([38] Definition 5.1) provide the definition of an \(L_\infty\)-module at the desuspended level where it is easier to see the structure as satisfying the relations for a Lie module only up to homotopy. At the suspended level with which we are working in this paper, adjusting degrees and signs, the definition looks as follows:

**Definition 7 (sh-L-module)** Let \(L = (L, l_1)\), be an \(L_\infty\)-algebra, and let \(M\) be a differential graded vector space with differential denoted by \(k_1\). Then a left \(L_\infty\)-module structure on \(M\) is a collection \(\{k_n | 1 < n < \infty\}\) of graded linear maps of degree one

\[
k_n : L^\otimes (n-1) \otimes M \longrightarrow M ,
\]
which are graded symmetric in $L$ and such that
\[
\sum_{\sigma \in S_{p+q=n}} (-1)^{\epsilon(\sigma)} k_{1+q+1}(l_p(\xi_{\sigma(1)}, \cdots, \xi_{\sigma(p)}), \xi_{\sigma(p+1)}, \cdots, \xi_{\sigma(n)}, \xi_{n+1})
\]
\[
+ \sum_{\sigma \in S_{p+q=n}} (-1)^{\epsilon(\sigma)} (-1)^{\sigma(1)+\cdots+\sigma(p)} k_{p+1}(\xi_{\sigma(1)}, \cdots, \xi_{\sigma(p)}, k_{q+1}(\xi_{\sigma(p+1)}, \cdots, \xi_{\sigma(n)}, \xi_{n+1})) = 0,
\]
(2.13)

where $\xi_1, \cdots, \xi_n \in L$ and $\xi_{n+1} \in M$. 

Several comments are in order. For clarity, let $d_L = l_1$ and $d_M = k_1$, then the first few relations are:

- $d_M k_2 = -k_2(d_L \otimes 1 + 1 \otimes d_M)$, that is, $k_2$ is a chain map,

- $d_M k_3 + k_3(d_L \otimes 1 \otimes 1 + 1 \otimes d_L \otimes 1 + 1 \otimes 1 \otimes d_M) = -k_2(l_2 \otimes 1 + (1 \otimes k_2) \circ (\tau \otimes 1) + 1 \otimes k_2)$,

where $\tau$ is the interchange operator (2.5); that is, $k_2$ is a Lie action up to homotopy.

Of course, the fundamental example of such a structure occurs in the situation in which $M = L$ and each $k_i = l_i$, i.e., $L$ is an $L_\infty$-module over itself.

According to Lada and Markl [38], we have the usual relationship between homomorphisms and module structures. Let $\text{End}(M)$ denote the differential graded Lie algebra of linear maps from $M$ to $M$ with bracket given by the composition graded commutator and differential induced by the differential $k_1$ on $M$. After shifting the grading, their theorem reads as follows:

**Theorem 1 (Theorem 5.4 [38])** Suppose that $L$ is an $L_\infty$-algebra and that $M = (M, k_1)$ is a differential graded vector space. Then there exists a natural one-to-one correspondence between sh-L-module structures on $M$ and $L_\infty$-maps $L \rightarrow \uparrow \text{End}(M)$.

**Remark 5** The following is phrased with the traditional grading. Fortunately, $\text{End}(M)$ and $\text{End}(\uparrow M)$ are isomorphic as graded Lie algebras, as are the Lie subalgebras $\text{Der}A$ and $\text{Der}(\uparrow A)$.

In his groundbreaking “Notions d’algèbre différentielle; ...” [4], Henri Cartan formalized several dg algebra notions related to his study of the deRham cohomology of principal fibre bundles, in particular, that of a Lie group $G$ acting in (‘dans’) a differential graded algebra $E$. The action uses only the Lie algebra $g$ of $G$. Cartan’s action includes both graded derivations, the Lie derivative $\theta(X)$ and the inner derivative $i(X)$ for $X \in g$. We need only the analog of the $\theta(X)$, (which we denote $\rho(X)$ since by $\theta$ we denote the image by $\rho$ of an element $X \in g$) for the following definition.

**Definition 8** For a dg Lie algebra $g$, a dg associative algebra $A$ is a $g$-algebra if $g$ acts by derivations of $A$, i.e. there is given a representation of $g$,

$$\rho : g \rightarrow \text{Der}A$$

which is a Lie map and a chain map.
Remark 6 The concept was later reintroduced by Flato, Gerstenhaber and Voronov [9] under the name *Leibniz pair*, cf. also [2].

We have seen that an $L_\infty$-module structure is defined in terms of relations on the maps $L^{\otimes p} \otimes A \to A$. A $g$-algebra (or Leibniz pair) extends this in the sense that it includes a relation (2.14) on $g \otimes A^{\otimes 2} \to A$ where $g = \downarrow L$. For an extension to $L \otimes A^{\otimes q} \to A$, a relevant notion is that of a homotopy derivation; that is, given $\theta_1 : A \to A$ and $m_2 : A \otimes A \to A$, we ask for a homotopy $\theta_2 : A \otimes A \to A$ between $\theta_1 m_2$ and $m_2(\theta_1 \otimes 1) + m_2(1 \otimes \theta_1)$. Further higher homotopies follow the usual pattern.

**Definition 9 (Strong homotopy derivation)** A *strong homotopy derivation* of degree one of an $A_\infty$-algebra $(A,m)$ consists of a collection of multi-linear maps of degree one

$$\theta := \{\theta_q : A^{\otimes q} \to A\}_{q \geq 1}$$

satisfying the following relations

$$0 = \sum_{r+s=q+1} \sum_{i=0}^{r-1} (-1)^{\beta(s,i)} \theta_r(a_1, \ldots, a_i, m_s(a_{i+1}, \ldots, a_{i+s}), \ldots, o_q)$$

$$(2.17) + (-1)^{\beta(s,i)} m_r(a_1, \ldots, a_i, \theta_s(a_{i+1}, \ldots, a_{i+s}), \ldots, o_q).$$

Here the sign $\beta(s, i) = o_1 + \cdots + o_i$ results from moving $m_s$, respectively $\theta_s$, past $(o_1, \ldots, o_i)$.

Remark 7 The formulas are equivalent (as suggested by Markl [43]) to seeing $\theta$ as a coderivation $\theta$ of $T^eA$ with no constant term and such that

$$[m, \theta] = 0.$$

If we extend $\theta_q : A^{\otimes q} \to A$ to a map $\rho_q : L \otimes A^{\otimes q} \to A$ by $\theta_q := \rho_1(\uparrow X)$, the appropriate defining equation is then replaced by

$$\rho(d_q(\uparrow X)) = [m, \rho(\uparrow X)],$$

$$(2.18)$$

where $\rho(\uparrow X)$ is the lift of $\sum_q \rho_q(\uparrow X)$ to a coderivation. This can be read as the condition for $\rho$ to be a chain map regarding $[m, ]$ as a differential on $\uparrow \text{Coder}(T^eA)$.

Now if $A$ is an $L_\infty$-module over $L$, the analog of Cartan’s second condition would be for $\rho : L \to \uparrow \text{Coder}(T^eA)$ to be an $L_\infty$-map. We already have the homotopies $k_3 : L \otimes L \otimes A \to A$ and $\rho_2 : L \otimes A \otimes A \to A$. The next stage of a *strong* homotopy version looks at the various compositions giving rise to maps $L^{\otimes p} \otimes A^{\otimes q} \to A$ with $p + q = 4$ and so forth.
Definition 10 (A∞-algebra over an L∞-algebra) Let L be an L∞-algebra and A an A∞-algebra which as a dg vector space is an sh-L module. That A is an A∞-algebra over L means that the module structure map \( \rho : L \to \text{End}(A) \), regarded as in \( \text{Coder}(T^c A) \), extends to an \( L_\infty \) map \( L \to \text{Coder}(T^c A) \), where \( \text{Coder}(T^c A) \) is the suspension of \( \text{Coder}(T^c A) \) as the dg Lie algebra stated in subsection 2.2.

Theorem 2 That A is an A∞-algebra over the L∞-algebra L is equivalent to having a family of maps \( n = \{ n_{p,q} : L \otimes^p A \otimes^q \to A \} \) for \( p \geq 0 \) but \( q > 0 \) satisfying the compatibility conditions:

\[
0 = \sum_{\sigma \in \mathcal{S}_{p+r=n}} (-1)^{\epsilon(\sigma)} n_{1+r,m} (l_p(c_{\sigma(1)}, \cdots, c_{\sigma(p)})), c_{\sigma(p+1)}, \cdots, c_{\sigma(n)}, o_1, \cdots, o_m)
+ \sum_{\sigma \in \mathcal{S}_{p+r=n}} (-1)^{\mu_{p,i}(\sigma)} n_{p,i+1+j} (c_{\sigma(1)}, \cdots, c_{\sigma(p)}, o_1, \cdots, o_i, n_{r,s}(c_{\sigma(p+1)}, \cdots, c_{\sigma(n)}), o_{i+1}, \cdots, o_{i+s+1}, \cdots, o_m).
\]

(2.19)

For an A∞-algebra A over an L∞-algebra L, the substructure \( (A, \{ n_{0,k} \}_{k \geq 1}) \) forms an A∞-algebra and the substructure \( (L \oplus A, \{ n_{p,1} \}_{p \geq 0}) \) makes A an L∞-module over \( (L, 1) \). More precisely, \( n_{0,k} = m_k, n_{p,1} = k_{p+1} \); the \( n_{1,q \geq 0} \) map \( L \to \text{Coder}(T^c A) \) and the rest extend that to an \( L_\infty \)-map. This is just the higher homotopy structure a mathematician would construct by the usual procedures of strong homotopy algebra (see the Appendix by M. Markl).

Here the sign exponent \( \mu_{p,i}(\sigma) \) is given explicitly by

\[
\mu_{p,i}(\sigma) = \epsilon(\sigma) + (c_{\sigma(1)} + \cdots + c_{\sigma(p)}) + (o_1 + \cdots + o_i) + (o_1 + \cdots + o_i)(c_{\sigma(p+1)} + \cdots + c_{\sigma(n)}),
\]

(2.20)
corresponding to the signs effected by the interchanges. The sign can be seen easily in the coalgebra and tree expressions. We can also write the defining equation (2.19) in the following shorthand expression,

\[
0 = \sum_{\sigma \in \mathcal{S}_{p+r=n}} (-1)^{\epsilon(\sigma)} n_{1+r,m} \left( (l_p \otimes 1^c_{\sigma} \otimes 1^o_{\sigma}) (c_{\sigma(1)}; o_1, \cdots, o_m) \right)
+ \sum_{\sigma \in \mathcal{S}_{p+r=n}} (-1)^{\epsilon(\sigma)} n_{p,i+1+j} \left( (1^c_{\sigma} \otimes 1^o_{\sigma}) \otimes n_{r,s} \otimes 1^o_{\sigma} (c_{\sigma(1)}; o_1, \cdots, o_m) \right),
\]

where the complicated sign is absorbed into this expression. Note that the rule for the action of tensor products of graded multi-linear maps on \( (\mathcal{H}_c)^{\otimes n} \otimes (\mathcal{H}_o)^{\otimes m} \) is determined in a canonical way; for instance for \( (f \otimes \cdots) (c_1, \cdots, c_n; o_1, \cdots, o_m) \) with the first multi-linear map \( f : (\mathcal{H}_c)^{\otimes k} \otimes (\mathcal{H}_o)^{\otimes l} \to \mathcal{H} \), we may bring \( (c_1, \cdots, c_k; o_1, \cdots, o_l) \) to the first \( f \) with the associated sign, do the same thing for the next multi-linear map in \( \cdots \) and repeat this in order.

String field theory suggests that an open-closed homotopy algebra includes the addition of the maps \( n_{p,0} : L^{\otimes p} \to A \) and in particular \( n_{1,0} : L \to A \) corresponding to the opening of a closed string to an open one.
Definition 11 (Open-Closed Homotopy Algebra (OCHA)) An open-closed homotopy algebra (OCHA)\(^1\) \((\mathcal{H} = \mathcal{H}_c \oplus \mathcal{H}_o, l, n)\) consists of an \(L_\infty\)-algebra \((\mathcal{H}_c, l)\) and a family of maps \(n = \{n_{p,q} : \mathcal{H}_c^{\otimes p} \otimes \mathcal{H}_o^{\otimes q} \to \mathcal{H}_o\}\) for \(p,q \geq 0\) with the exception of \((p,q) = (0,0)\) satisfying the compatibility conditions (2.19):

\[
0 = \sum_{\sigma \in \mathcal{S}_{p+r} \circ \mathcal{S}_{n}} (-1)^{\epsilon(\sigma)} n_{1+r,m}(l_p(c_{\sigma(1)}, \cdots, c_{\sigma(p)}), c_{\sigma(p+1)}, \cdots, c_{\sigma(n)}; o_1, \cdots, o_m)
\]

\[
+ \sum_{\sigma \in \mathcal{S}_{p+r} \circ \mathcal{S}_{n} \atop i+s+j=m} (-1)^{\mu(\sigma)} n_{p,i+1+j}(c_{\sigma(1)}, \cdots, c_{\sigma(p)}; o_1, \cdots, o_i, n_{r,s}(c_{\sigma(p+1)}, \cdots, c_{\sigma(n)}; o_{i+1}, \cdots, o_{i+s+1}, \cdots, o_m)
\]

(2.21)

for the full range \(n,m \geq 0, (n,m) \neq (0,0)\).

A weak OCHA consists of a weak \(L_\infty\)-algebra \((\mathcal{H}_c, l)\) with a family of maps \(n = \{n_{p,q} : \mathcal{H}_c^{\otimes p} \otimes \mathcal{H}_o^{\otimes q} \to \mathcal{H}_o\}\) now for \(p,q \geq 0\) satisfying the analog of the above relation.

For an OCHA \((\mathcal{H}, l, n)\), the multi-linear maps \(\{n_{p,q}\}_{p \geq 1, q \geq 0}\) still correspond to an adjoint \(L_\infty\)-map \(\mathcal{H}_c \to \text{Coder}(T^e \mathcal{H}_o)\), as in the case of an \(A_\infty\)-algebra over an \(L_\infty\)-algebra. This has a particular importance in terms of deformation theory, cf. subsection 5.1, where the addition of maps \(n_{0,0}\) leads in turn to the deformation of the \(A_\infty\)-structure \(m\) to a weak \(A_\infty\)-structure.

Definition 12 (Open-closed homotopy algebra (OCHA) morphism) For two weak OCHAs \((\mathcal{H}, l, n)\) and \((\mathcal{H}', l', n')\), consider a collection \(f\) of degree zero (degree preserving) multi-linear maps

\[
f_k : (\mathcal{H}_c)^{\otimes k} \to \mathcal{H}_c', \quad \text{for } k \geq 0,
\]

\[
f_{l,t} : (\mathcal{H}_c)^{\otimes l} \otimes (\mathcal{H}_o)^{\otimes t} \to \mathcal{H}_o', \quad \text{for } k, l \geq 0,
\]

where \(f_k\) and \(f_{l,t}\) are graded symmetric with respect to \((\mathcal{H}_c)^{\otimes k}\). We call \(f : (\mathcal{H}, l, n) \to (\mathcal{H}', l', n')\) a weak OCHA-morphism when \(\{f_k\}_{k \geq 0} : (\mathcal{H}_c, l) \to (\mathcal{H}_c', l')\) a weak \(L_\infty\)-morphism and \(\{f_{l,t}\}_{l,t \geq 0}\) further satisfies the following relations:

\[
\sum_{\sigma \in \mathcal{S}_{p+r} \circ \mathcal{S}_{n}} (-1)^{\epsilon(\sigma)} f_{1+r,m}(l_p(1_c^{\otimes} \otimes 1_o^{\otimes m})(c_{\sigma(I)}; o_1, \cdots, o_m))
\]

\[
+ \sum_{\sigma \in \mathcal{S}_{p+r} \circ \mathcal{S}_{n} \atop i+s+j=m} (-1)^{\epsilon(\sigma)} f_{p,i+1+j}(1_c^{\otimes p} \otimes 1_o^{\otimes i} \otimes n_{r,s}(1_o^{\otimes j})(c_{\sigma(I)}; o_1, \cdots, o_m))
\]

\[
= \sum_{\sigma \in \mathcal{S}_{(r_1+\cdots+r_i)+(p_1+\cdots+p_j)=n} \atop (q_1+\cdots+q_j)=m} \frac{(-1)^{\epsilon(\sigma)}}{i!} n_{i,j}(f_{r_1} \otimes \cdots \otimes f_{r_i} \otimes f_{p_1,q_1} \otimes \cdots \otimes f_{p_j,q_j})(c_{\sigma(I)}; o_1, \cdots, o_m).
\]

(2.22)

The right hand side is written explicitly as

\[
n_{i,j}'(f_{r_1} \otimes \cdots \otimes f_{r_i} \otimes f_{p_1,q_1} \otimes \cdots \otimes f_{p_j,q_j})(c_{\sigma(I)}; o_1, \cdots, o_m)
\]

\[
= (-1)^{\sum_{1 \leq i < j \leq n} \epsilon(\sigma_i \sigma_j)} n_{i,j}'(f_{r_1}(c_{\sigma(1)}, \cdots, c_{\sigma(r_1)}), \cdots, f_{r_i}(c_{\sigma(r_i-1+1)}, \cdots, c_{\sigma(r_i)});
\]

\[
f_{p_1,q_1}(c_{\sigma(r_1+1)}, \cdots, c_{\sigma(p_1)}; o_1, \cdots, o_{q_1}), \cdots, f_{p_j,q_j}(c_{\sigma(p_j+1)}, \cdots, c_{\sigma(p_j)}; o_{q_j+1}, \cdots, o_{q_j})).
\]

\(^1\)The authors worked with the acronym for several weeks before realizing it is Japanese for 'tea'.

where \( \bar{r}_k := r_1 + \cdots + r_k, \bar{p}_k := \bar{r}_i + p_1 + \cdots + p_k, \bar{q}_k := q_1 + \cdots + q_k \) and \( \tau_{\bar{p}', \bar{q}}(\sigma) \) is given by

\[
\tau_{\bar{p}', \bar{q}}(\sigma) = \sum_{k=1}^{j-1} (c_{\sigma(\bar{p}_k+1)} + \cdots + c_{\sigma(\bar{p}_{k+1})})(o_1 + \cdots + o_{\bar{q}_k}).
\]

In particular, if \( (H, \mathfrak{l}, n) \) and \( (H', \mathfrak{l}', n') \) are OCHAs and if \( f_0 = f_{0,0} = 0 \), we call it an OCHA-morphism.

**Definition 13 (OCHA-quasi-isomorphism)** Given two OCHAs \( (H, \mathfrak{l}, n) \), \( (H', \mathfrak{l}', n') \) and an OCHA-morphism \( f : (H, \mathfrak{l}, n) \to (H', \mathfrak{l}', n') \), \( f \) is called an OCHA-quasi-isomorphism if \( f_1 + f_{0,1} : H \to H' \) induces an isomorphism between the cohomology spaces of the complexes \( (H, d := l_1 + n_{0,1}) \) and \( (H', d') \). In particular, if \( f_1 + f_{0,1} \) is an isomorphism, we call \( f \) an OCHA-isomorphism.

### 2.8 The coalgebra description

Consider an OCHA \( H = H_c \oplus H_o \). Recall that the separate \( L_\infty \)- and \( A_\infty \)-structures are described by coderivation differentials on, respectively, \( C(H_c) \) and \( T^c(H_o) \). The defining multi-linear maps for \( H \) are to be extended to coderivations of \( C(H_c) \otimes T^c(H_o) \). The coproduct on \( T^c(H_c) \otimes T^c(H_o) \) is the standard tensor product coproduct defined by

\[
\Delta((c_1 \otimes \cdots \otimes c_m) \otimes (o_1 \otimes \cdots \otimes o_n)) = \sum_{p=0}^{m} \sum_{q=0}^{n} (-1)^{\eta(p,q)} (c_1 \otimes \cdots \otimes c_p \otimes o_1 \otimes \cdots \otimes o_q) \otimes (c_{p+1} \otimes \cdots \otimes c_m \otimes o_{q+1} \otimes \cdots \otimes o_n),
\]

(2.23)

where \( \eta(p, q) = (c_{p+1} + \cdots + c_m)(o_1 + \cdots + o_q) \).

The relevant subcoalgebra is \( C(H_c) \otimes T^c(H_o) \). Now we define the total coderivation \( \mathfrak{l} + \mathfrak{n} \) by lifting

\[
\sum_{k \geq 1} (l_k + m_k) + \sum_{p \geq 1, q \geq 0} n_{p,q}, \tag{2.24}
\]

with \( m_k = n_{0,k} \). Thus we have an OCHA iff \( \mathfrak{l} + \mathfrak{n} \) is a codifferential:

\[
(\mathfrak{l} + \mathfrak{n})^2 = 0. \tag{2.25}
\]

If this is true with the addition of \( l_0 \) and \( m_0 \), we have a weak OCHA.

Also, given two OCHAs \( (H, \mathfrak{l}, n) \) and \( (H', \mathfrak{l}', n') \), an OCHA-morphism \( f : (H, \mathfrak{l}, n) \to (H', \mathfrak{l}', n') \) can be lifted to the coalgebra homomorphism \( f : C(H_c) \otimes T^c(H_o) \to C(H'_c) \otimes T^c(H'_o) \) and the condition for an OCHA-morphism is written as \( f \circ (\mathfrak{l} + \mathfrak{n}) = (\mathfrak{l}' + \mathfrak{n}') \circ f \).

### 2.9 The tree description

We associated the \( k \)-corolla of planar rooted trees to the multi-linear map \( m_k \) of an \( A_\infty \)-algebra, and the \( k \)-corolla of non-planar rooted trees to the graded symmetric multi-linear map \( l_k \) of an
For an OCHA \((\mathcal{H}, l, n)\), the corolla corresponding to \(n_{k,l}\) should be expressed as the following mixed corolla,

\[
\begin{array}{c}
n_{k,l} \\ \rightarrow \\
\end{array}
\begin{array}{c}
1 \cdots k \\
\vdots \\
1 \cdots l \\
\vdots \\
\end{array},
\]

which is partially symmetric (non-planar), that is, only symmetric with respect to the \(k\) leaves. Let us consider such corollas for \(2k + l + 1 \geq 3\) together with non-planar corollas \(\{l_{k}\}_{k \geq 2}\). Since we have two kinds of edges, we have two kinds of grafting; grafting of edges associated to \(\mathcal{H}_{c}\) (closed string edges) and those for \(\mathcal{H}_{o}\) (open string edges). We denote them by \(\circ\) and \(\bullet\), respectively. For these corollas, we have three types of the composite; in addition to the composite \(l_{1+k} \circ_{i} l_{i} \) in \(L_{\infty}\), there is a composite \(n_{k,m} \circ_{i} l_{p}\) described by

\[
\begin{array}{c}
1 \cdots k \\
\vdots \\
1 \cdots m \\
\vdots \\
1 \cdots p \\
\vdots \\
\end{array}
\begin{array}{c}
\circ_{i} \\
\bullet_{i} \\
\end{array}
\]

where in the right hand side the labels are given by \([i, \cdots, i+p-1][1, \cdots, i-1, i+p, \cdots, p+k-1](1, \cdots, m)\), and the composite \(n_{p,q} \bullet_{i} n_{r,s}\)

\[
\begin{array}{c}
1 \cdots p \\
\vdots \\
1 \cdots q \\
\vdots \\
1 \cdots r \\
\vdots \\
1 \cdots s \\
\vdots \\
\end{array}
\begin{array}{c}
\circ_{i} \\
\bullet_{i} \\
\end{array}
\]

with labels \([1, \cdots, p](1, \cdots, i-1)[p+1, \cdots, p+r](i, \cdots, i+s-1)(i+s, \cdots, q+s-1)\). To these resulting trees, grafting of a corolla \(l_{k}\) or \(n_{k,l}\) can be defined in a natural way, and we can repeat this procedure. Let us consider tree graphs obtained in this way, that is, by grafting the corollas \(l_{k}\) and \(n_{k,l}\) recursively, together with the action of permutations of the labels for closed string leaves. Each of them has a closed string root edge or an open string root edge. The tree graphs with closed string root edge, with the addition of the identity \(e_{c} \in L_{\infty}(1)\), generate \(L_{\infty}\) as stated in subsection 2.6. On the other hand, the tree graphs with open string root edge are new; the graded vector space generated by them with \(k\) closed string leaves and \(l\) open string leaves we denote by \(N_{\infty}(k; l)\). In particular, we formally add the identity \(e_{o}\) generating \(N_{\infty}(0; 1)\), and \(N_{\infty}(1; 0)\) is generated by a corolla \(n_{1,0}\). For \(N_{\infty} := \oplus_{k,l} N_{\infty}(k; l)\), the tree operad relevant here is then \(OC_{\infty} := L_{\infty} \oplus N_{\infty}\). For each tree \(T \in OC_{\infty}\), its grading is given by the number of the vertices \(v(T)\).
For trees in $\mathcal{OC}_\infty$, let $T' \to T$ indicate that $T$ is obtained from $T'$ by contracting a closed or an open internal edge. A degree one differential $d: \mathcal{OC}_\infty \to \mathcal{OC}_\infty$ is given by

$$d(T) = \sum_{T' \to T} \pm T',$$

so that the following compatibility holds:

$$d(T \circ_i T') = d(T) \circ_i T' + (-1)^{v(T)} T \circ_i d(T'), \quad d(T \bullet_i T'') = d(T) \bullet_i T'' + (-1)^{v(T)} T \bullet_i d(T'').$$

Thus, $\mathcal{OC}_\infty$ forms a dg operad. In particular, $d(l_k)$ is given by eq. (2.12), and $d(n_{n,m})$ is as follows:

$$- \sum_{\sigma \in \mathcal{O}_{p+r=n}} \begin{bmatrix} \cdots \\ \vdots \\ \cdots \end{bmatrix} \begin{bmatrix} \cdots \\ \vdots \\ \cdots \end{bmatrix} \begin{bmatrix} \cdots \\ \vdots \\ \cdots \end{bmatrix}, \quad \sigma \in \mathcal{O}_{p+r=n}$$

where the labels for the first and the second terms are $[\sigma(1), \cdots, \sigma(p)] [\sigma(p+1), \cdots, \sigma(n)] (1, \cdots, m)$ and $[\sigma(1), \cdots, \sigma(p)(1, \cdots, i) [\sigma(p+1), \cdots, \sigma(n)] (i+1, \cdots, i+s) (i+s+1, \cdots, m)$, respectively.

An algebra $\mathcal{H} := \mathcal{H}_c \oplus \mathcal{H}_o$ over $\mathcal{OC}_\infty$ is obtained by a representation

$$\phi : \mathcal{L}_\infty(k) \to \text{Hom}(\mathcal{H}_c^{\otimes k}, \mathcal{H}_c), \quad \phi : \mathcal{N}_\infty(k;l) \to \text{Hom}(\mathcal{H}_c^{\otimes k} \otimes (\mathcal{H}_o)^{\otimes l}, \mathcal{H}_o)$$

which is compatible with respect to the grafting $\circ_i, \bullet_i$ and the differential $d$. Here, regarding elements in both $\text{Hom}(\mathcal{H}_c^{\otimes k}, \mathcal{H}_c)$ and $\text{Hom}(\mathcal{H}_c)^{\otimes k} \otimes (\mathcal{H}_o)^{\otimes l}, \mathcal{H}_o)$ as those in $\text{Coder}(C(\mathcal{H}_c) \otimes T^c(\mathcal{H}_o))$, the differential in the algebra side is given by $[l_1 + n_{0,1}, ]$. By combining it with eq. (2.29) one can recover the condition of an OCHA (2.19).

If we adjust the notation for grading as $\downarrow \downarrow \mathcal{H}_c$ and $\downarrow \mathcal{H}_o$, the degree of the multi-linear map $l_k$ is $3 - 2k$ as stated previously and the degree of $n_{k,l}$ turns out to be $1 + (1 - l) - 2k = 2 - (2k + l)$. The grading of a tree $T \in \mathcal{N}_\infty(k;l)$ is then replaced by $\text{int}(T) + (2 - 2k - l)$, which is equal to minus the dimension of the corresponding boundary piece of the compactified moduli space of a disk with $k$ points interior and $l$ points on the boundary (see [61]).

3 Cyclic structures

Now we consider an additional structure, cyclicity, on open-closed homotopy algebras. Algebras with invariant inner products ($\langle ab, c \rangle = \langle a, bc \rangle$ or $\langle [a, b], c \rangle = \langle a, [b, c] \rangle$) are very important in mathematical physics; the analogous definition for strong homotopy algebras is straightforward (cf. [44], sections II.5.1 and II.5.2). The string theory motivation for this additional structure is that punctures on the boundary of the disk inherit a cyclic order from the orientation of the disk and the operations are to respect this cyclic structure, just as the $L_\infty$-structure reflects the symmetry of the punctures in the interior of the disk or on the sphere.
In our context, cyclicity is defined in terms of constant symplectic inner products. (The terminology is that used for symplectic structures on supermanifolds [1]; see also [31] and references therein. These inner products are also essential to the description of the Lagrangians appearing in string field theory.)

**Definition 14 (Constant symplectic structure)** Bilinear maps, $\omega_c : \mathcal{H}_c \otimes \mathcal{H}_c \to \mathbb{C}$ and $\omega_o : \mathcal{H}_o \otimes \mathcal{H}_o \to \mathbb{C}$, are called **constant symplectic structures** when they have fixed integer degrees $|\omega_c|, |\omega_o| \in \mathbb{Z}$ and are non-degenerate and skew-symmetric. Here 'skew-symmetric' indicates that $\omega^s := \omega_c \oplus \omega_o$.

Suppose that an OCHA $(\mathcal{H}, l, n)$ is equipped with constant symplectic structures $\omega_c : \mathcal{H}_c \otimes \mathcal{H}_c \to \mathbb{C}$ and $\omega_o : \mathcal{H}_o \otimes \mathcal{H}_o \to \mathbb{C}$ as in Definition 14. For $\{l_k\}_{k \geq 1}$ and $\{n_{p,q}\}_{p+q \geq 1}$, let us define two kinds of multi-linear maps by

$$V_{k+1} = \omega_c(l_k \otimes 1_c) : (\mathcal{H}_c)^{\otimes(k+1)} \to \mathbb{C} \quad \text{and} \quad V_{p,q+1} = \omega_o(n_{p,q} \otimes 1_o) : (\mathcal{H}_c)^{\otimes p} \otimes (\mathcal{H}_o)^{\otimes(q+1)} \to \mathbb{C}$$

or more explicitly

$$V_{k+1}(c_1, \cdots, c_{k+1}) = \omega_c(l_k(c_1, \cdots, c_k), c_{k+1})$$

and

$$V_{p,q+1}(c_1, \cdots, c_p; o_1, \cdots, o_{q+1}) = \omega_o(n_{p,q}(c_1, \cdots, c_p; o_1, \cdots, o_q), o_{q+1}) \, .$$

The degree of $V_{k+1}$ and $V_{p,q+1}$ are $|\omega_c| + 1$ and $|\omega_o| + 1$.

**Definition 15 (Cyclic open-closed homotopy algebra (COCHA))** An OCHA $(\mathcal{H}, l, n)$ is a **cyclic open-closed homotopy algebra (COCHA)** when $V_{k+1}$ is graded symmetric with respect to any permutation of $(\mathcal{H}_c)^{\otimes(k+1)}$ and $V_{p,q+1}$ has cyclic symmetry with respect to cyclic permutations of $(\mathcal{H}_o)^{\otimes(q+1)}$, that is, if

$$V_{k+1}(c_1, \cdots, c_{k+1}) = (-1)^{\varepsilon(\sigma)}V_{k+1}(c_{\sigma(1)}, \cdots, c_{\sigma(k+1)}) \, , \quad \sigma \in \mathfrak{S}_{k+1}$$

and

$$V_{p,q+1}(c_1, \cdots, c_p; o_1, \cdots, o_{q+1}) = (-1)^{o_1(o_2 + \cdots + o_{q+1})}V_{p,q+1}(c_1, \cdots, c_p; o_2, \cdots, o_{q+1}, o_1) \, .$$

The graded commutativity of $V_{p,q+1}$ with respect to permutations of $(\mathcal{H}_c)^{\otimes p}$, that is,

$$V_{p,q+1}(c_1, \cdots, c_p; o_1, \cdots, o_{q+1}) = (-1)^{\varepsilon(\sigma)}V_{p,q+1}(c_{\sigma(1)}, \cdots, c_{\sigma(p)}; o_1, \cdots, o_{q+1}) \, , \quad \sigma \in \mathfrak{S}_p$$

automatically holds by the definition of $n$. 
Since we have non-degenerate inner products $\omega_c$ and $\omega_o$, we can identify $\mathcal{H}$ with its linear dual, then reverse the process and define further maps

$$r_{p-1,q+1} : (\mathcal{H}_c)^{\otimes(p-1)} \otimes (\mathcal{H}_o)^{\otimes(q+1)} \rightarrow \mathcal{H}_c$$

with relations amongst themselves and with the operations already defined, which can easily be deduced from their definition. In particular, for $n_{1,0} : \mathcal{H}_c \rightarrow \mathcal{H}_o$ we have $r_{0,1} : \mathcal{H}_o \rightarrow \mathcal{H}_c$. Namely, for the cyclic case the fundamental object is the multi-linear map $V_{p,q}^{n_{p,q}}$ and $r_{p-1,q+1}$ are equivalent under the relation above. However, we get a codifferential (2.25) since we took $n_{p,q}$ instead of $r_{p-1,q+1}$ for defining an OCHA. Physically, for the multi-linear map $V_{p,q}$, choosing $\mathcal{H}_o$ as a root edge instead of $\mathcal{H}_c$ as in eq.(2.26) is related to a standard compactification of the corresponding Riemann surface (a disk with $p$ points interior and $(q + 1)$ points on the boundary).

4 Minimal model theorem and decomposition theorem

Homotopy algebras are designed to have homotopy invariant properties. A key and useful theorem in homotopy algebras is then the minimal model theorem. For $A_{\infty}$-algebras, it was proved by Kadeishvili [29]. For the construction of minimal models of $A_{\infty}$-structures, in particular on the homology of a differential graded algebra, homological perturbation theory (HPT) is developed by [19, 27, 20, 21, 22, 23], for instance, and the form of a minimal model is also given explicitly and more recently in [45, 37].

There are various results referred to as minimal model theorems: the weakest form asserts the existence of a quasi-isomorphism as $A_{\infty}$-algebras $H(A) \rightarrow A$ for an $A_{\infty}$-structure on $H(A)$, by noticing that all the relevant obstructions vanish because the homology of $A$ and $H(A)$ agree. A stronger result constructs an $A_{\infty}$-structure on $H(A)$ and the quasi-isomorphism, then a decomposition theorem is proved from which the inverse quasi-isomorphism follows [31, 40, 33, 29]. Alternatively, the full strength of homological perturbation theory gives the maps in both directions and the homotopy for the composition $A \rightarrow H(A) \rightarrow A$ all together.

The corresponding theorems for $L_{\infty}$-algebras are more recent: [48] for the two step procedure, [28] for the full HPT treatment. The latter points out that, although $L_{\infty}$-algebras can be constructed by symmetrization of $A_{\infty}$-algebras, the corresponding constructions of the maps and homotopy are more subtle.

It is not surprising that the minimal models and decompositions exist also for our OCHAs. These theorems imply that, for an OCHA $(\mathcal{H}, l, n)$, the higher multi-linear structures $l_k$, $k \geq 2$ and $n_{p,q}$, $(p, q) \neq (0, 1)$ have been transformed to those on $H(\mathcal{H})$, where $H(\mathcal{H})$ is the cohomology of the complex $(\mathcal{H}, d = l_1 + n_{0,1})$. Even though some of those higher structures may have been zero on the original OCHA $\mathcal{H}$, those on $H(\mathcal{H})$ need not be.

We present these statements more precisely below, leaving detailed proofs to the industrious reader.

**Definition 16 (Minimal open-closed homotopy algebra)** An OCHA $(\mathcal{H} = \mathcal{H}_c \oplus \mathcal{H}_o, l, n)$ is called **minimal** if $l_1 = 0$ on $\mathcal{H}_c$ and $n_{0,1} = 0$ on $\mathcal{H}_o$. 
Definition 17 (Linear contractible open-closed homotopy algebra) A linear contractible OCHA \((\mathcal{H}, l, n)\) is a complex \((\mathcal{H}, d = l_1 + n_{0,1})\) which has trivial cohomology, that is, an OCHA \((\mathcal{H} = \mathcal{H}_c \oplus \mathcal{H}_o, l, n)\) such that \(l_i = 0\) for \(i \geq 2\), \(n_{p,q} = 0\) except for \((p, q) = (0, 1)\), and the complexes \((\mathcal{H}_c, l_1), (\mathcal{H}_o, n_{0,1})\) having trivial cohomologies.

Theorem 3 (Decomposition theorem for open-closed homotopy algebras) Any OCHA is isomorphic to the direct sum of a minimal OCHA and a linear contractible OCHA.

A weak version of the minimal model theorem follows from the decomposition theorem above:

Theorem 4 (Minimal model theorem for open-closed homotopy algebras) For a given OCHA \((\mathcal{H}, l, n)\), there exists a minimal OCHA \((H(\mathcal{H}), l', n')\) and an OCHA-quasi-isomorphism \(\tilde{f} : (H(\mathcal{H}), l', n') \to (\mathcal{H}, l, n)\). In particular, the minimal model can be taken so that \(l_2' = H(l_2)\), \(n_{0,2}' = H(n_{0,2})\) and \(n_{1,0}' = H(n_{1,0})\).

To obtain a homotopy equivalence from an initial quasi-isomorphism \(\tilde{f}\) above, one way is to employ the decomposition theorem (Theorem 3). Alternatively, it can be obtained directly by the methods of HPT (see Theorem 5 below). In either approach, for a given OCHA \((\mathcal{H}, l, n)\), one first considers a Hodge decomposition of the complex \((\mathcal{H}, d = l_1 + n_{0,1})\). Namely, decompose \(\mathcal{H}\) into a direct sum isomorphic to \(\mathcal{H} = H(\mathcal{H}) \oplus \mathcal{C}, \mathcal{C} := Y \oplus dY\) with a contracting homotopy \(h : dY \to Y\) of degree minus one. Together with the inclusion \(\iota\) and the projection \(\pi\), let us express these data as

\[
( H(\mathcal{H}) \xrightarrow{\iota} \mathcal{H} \xrightarrow{\pi} \mathcal{H}_c, h ) .
\]

Then the decomposition theorem (Theorem 3) states that there exists an OCHA-isomorphism \(f_{\text{isom}} : (H(\mathcal{H}), l', n') \oplus (\mathcal{C}, d) \to (\mathcal{H}, l, n)\), where \((\mathcal{C}, d)\) is the linear contractible OCHA. The OCHA-isomorphism is obtained by first decomposing the \(L_\infty\)-algebra \((\mathcal{H}_c, l)\) into the direct sum of a minimal part \(H(\mathcal{H}_c)\) and a linear contractible part \(\mathcal{C}_c\), and then decomposing the OCHA \((\mathcal{C}_c, d_c) \oplus (H(\mathcal{H}_c) \oplus \mathcal{H}_o, l', n)\) in a similar way as in the \(A_\infty\) case. Because we have the OCHA-isomorphism, we may consider a homotopy equivalence between \((H(\mathcal{H}), l', n')\) and \((H(\mathcal{H}), l', n') \oplus (\mathcal{C}, d)\). In fact, the maps \(\iota\) and \(\pi\) naturally extend to OCHA-quasi-isomorphisms between them, and the corresponding homotopy is obtained as in the sense in Theorem 5 below (see [31] for the \(A_\infty\) case) or as a path between them with some appropriate compatibility ([33] for the \(A_\infty\) case).

Alternatively, one can refine the standard HPT machinery to function in the category of OCHAs and their morphisms or apply the known results to the \(L_\infty\)-algebra \(\mathcal{H}_c\) and then extend to \(\mathcal{H}\), regarding \(\mathcal{H}_o\) as an analog of an sh-algebra over \(\mathcal{H}_c\). The extra detail of the HPT form of the minimal model theorem is then:

Theorem 5 (HPT minimal model theorem for open-closed homotopy algebras) Given an OCHA \((\mathcal{H}, l, n)\) and a Hodge decomposition with a contraction

\[
( H(\mathcal{H}) \xrightarrow{\iota} \mathcal{H} \xrightarrow{\pi} \mathcal{H}_c, h ) ,
\]
the linear maps \( \pi \) and \( \iota \) can be extended to coalgebra maps and perturbed so that there exists a corresponding contraction of coalgebras

\[
( \ C(H(H_c)) \otimes T^c(H(H_o)) \xrightarrow{\bar{\iota}} C(H_c) \otimes T^c(H_o) , \bar{h} \ ), \tag{4.1}
\]

where \( \bar{h} \) is a degree minus one linear homotopy on \( C(H_c) \otimes T^c(H_o) \), not necessarily a coalgebra map.

In the same way as in the case of \( A_\infty \)-algebras, the minimal model theorem together with these additional theorems implies various corollaries. For instance,

**Corollary 1 (Uniqueness of minimal open-closed homotopy algebras)** For an OCHA \((\mathcal{H},l,n)\), its minimal OCHA \(H(\mathcal{H})\) is unique up to an isomorphism on \(H(\mathcal{H})\).

**Corollary 2 (Existence of an inverse quasi-isomorphism)** For two OCHAs \((\mathcal{H},l,n)\) and \((\mathcal{H}',l',n')\), suppose there exists an OCHA quasi-isomorphism \(f : (\mathcal{H},l,n) \rightarrow (\mathcal{H}',l',n')\). Then, there exists an inverse OCHA quasi-isomorphism \(f^{-1} : (\mathcal{H}',l',n') \rightarrow (\mathcal{H},l,n)\).

In particular, Corollary 2 guarantees that quasi-isomorphisms do in fact define a (homotopy) equivalence relation and in addition give bijective maps between the moduli spaces of the solution space of the corresponding Maurer-Cartan equations for quasi-isomorphic sh-algebras (see Theorem 9).

The same facts should hold also for cyclic OCHAs.

## 5 Deformations and moduli spaces of \( A_\infty \)-structures

### 5.1 Deformations and Maurer-Cartan equations

Consider an OCHA \((\mathcal{H} = \mathcal{H}_c \oplus \mathcal{H}_o, l, n)\). We will show how the combined structure implies the \( L_\infty \)-algebra \((\mathcal{H}_c,l)\) controls some deformations of the \( A_\infty \)-algebra \((\mathcal{H}_o,\{m_k\}_{k \geq 1})\). We will further investigate the deformations of this control as \(\mathcal{H}\) is deformed.

We first review some of the basics of deformation theory from a homotopy algebra point of view. The philosophy of deformation theory which we follow (due originally, we believe, to Grothendieck \(^2\) cf. [47, 18, 7]) regards any deformation theory as ‘controlled’ by a dg Lie algebra \(g\) (unique up to homotopy type as an \(L_\infty\)-algebra).

For the deformation theory of an (ungraded) associative algebra \(A\), the standard controlling dg Lie algebra is \(\text{Coder}(T^c A)\) with the graded commutator as the graded Lie bracket [51]. Under the identification (including a shift in grading) of \(\text{Coder}(T^c A)\) with \(\text{Hom}(T^c A, A)\) (which is the Hochschild cochain complex), this bracket is identified with the Gerstenhaber bracket and the differential with the Hochschild differential, which can be written as \([m,]\) \([13]\).

The generalization to a differential graded associative algebra is straightforward; the differential is now: \([d_A + m_2, \ ]\). For an \(A_\infty\)-algebra, the differential similarly generalizes to \([m,\ ]\).

\(^2\)See [8] for an extensive annotated bibliography of deformation theory.
Deformations of $A$ correspond to certain elements of $\text{Coder}(T^c A)$, namely those that are solutions of an integrability equation, now known more commonly as a Maurer-Cartan equation.

**Definition 18 (The classical Maurer-Cartan equation)** In a dg Lie algebra $(\mathfrak{g}, d, [ , ])$, the classical Maurer-Cartan equation is

$$d\theta + \frac{1}{2} [\theta, \theta] = 0$$

(5.1)

for $\theta \in \mathfrak{g}^1 = (\downarrow L)^1$.

For an $A_\infty$-algebra $(A, m)$ and $\theta \in \text{Coder}^1(T^c A)$, a deformed $A_\infty$-structure is given by $m + \theta$ iff

$$(m + \theta)^2 = 0.$$  

Teasing this apart, since we start with $m^2 = 0$, we have equivalently

$$D\theta + 1/2[\theta, \theta] = 0$$

(5.2)

hence the Maurer-Cartan name. (Here $D$ is the natural differential on $\text{Coder}(T^c A) \subset \text{End}(T^c A)$, i.e. $D\theta = [m, \theta]$. ) Notice that we call this the Maurer-Cartan equation for the dg Lie algebra $(\text{Coder}(T^c A), D, [ , ])$ but not for $(A, m)$.

For $L_\infty$-algebras, the analogous remarks hold, substituting the Chevalley-Eilenberg complex for that of Hochschild, i.e. using $\text{Coder} C(L) \simeq \text{Hom}(C(L), L)$.

In any case, formal deformation theory controlled by a dg Lie algebra $(\mathfrak{g}, d, [ , ])$ proceeds as follows. Consider a formal solution $\theta$ of the Maurer-Cartan equation (5.1) in $\theta \in \mathfrak{g}^1 \otimes \hbar \mathbb{C}[[\hbar]]$, where $\hbar$ is a formal parameter. We express it as $\theta = \theta^{(1)} \hbar + \theta^{(2)} \hbar^2 + \cdots$, where $\theta^{(i)} \in \mathfrak{g}^1$. The Maurer-Cartan equation holds separately in different powers of $\hbar$, so we have

$$(\hbar)^1: \quad d\theta^{(1)} = 0$$

(5.3)

$$(\hbar)^2: \quad d\theta^{(2)} + \frac{1}{2} [\theta^{(1)}, \theta^{(1)}] = 0$$

(5.4)

$$(\hbar)^3: \quad d\theta^{(3)} + [\theta^{(1)}, \theta^{(2)}] = 0$$

(5.5)

\[ \cdots \quad \cdots \quad \cdots \]

The first order solution $\theta^{(1)}$ is defined by the first equation (5.3), that is, $\theta^{(1)}$ is a cocycle. This is also known as an infinitesimal deformation. We may proceed to second order if there is some $\theta^{(2)}$ satisfying the second equation (5.4). Similarly, we can ask for $\theta^{(3)}$ satisfying the third equation (5.5), etc. .

Since deformation theory is controlled by a dg Lie algebra up to homotopy (see Theorem 8), the Maurer-Cartan equation should be extended to that for an $L_\infty$-algebra. We present the definition in the suspended ($L = \uparrow \mathfrak{g}$) notation.

In addition to the convergence problem which would occur in the dg Lie algebra case, for an $L_\infty$-algebra on $L$ the Maurer-Cartan equation itself does not make sense in general since it consists of an infinite sum (see below). One way to avoid these problems is again to consider...
formal deformation theory; one usually considers a homotopy algebra on a graded vector space \( V \) over \( h\mathbb{C}[[h]] \), or more generally a finite dimensional nilpotent commutative associative algebra. In particular, for an Artin algebra \( A \) and its maximal ideal \( m_A \), the standard way is to consider \( V \otimes m_A \), where the degree of \( A \) is set to be zero. The multi-linear operations on \( V \) are extended to those on \( V \otimes m_A \) trivially. From now on, we shall assume but not mention explicitly that any homotopy algebra \( V \) we consider has been tensored with \( m_A \) for some fixed \( m_A \) and denote the result also by \( V \).

**Definition 19 (The strong homotopy Maurer-Cartan equation)** In an \( L_\infty \)-algebra \((L, l)\), the (generalized) Maurer-Cartan equation is

\[
\sum_{k \geq 1} \frac{1}{k!} l_k(\bar{c}, \cdots, \bar{c}) = 0
\]

for \( \bar{c} \in L^0 \).

Note that the degree of \( \bar{c} \) is zero since \( g^1 = L^0 \). We denote the set of solutions of the Maurer-Cartan equation as \( MC(L, l) \). In the same sense as in the dg Lie algebra case, a cocycle \( \bar{c} \in L^0 \), \( l_1 \bar{c} = 0 \) play the role of a first order solution.

Recall that for an OCHA \((\mathcal{H}, l, n)\), the adjoints of the maps \( n_{p,q} \) constitute an \( L_\infty \)-map \( \rho : \mathcal{H}_c \rightarrow \mathrm{Coder}(T^c\mathcal{H}_o) \). (5.6)

Since it is known that an \( L_\infty \)-morphism preserves the solutions of the Maurer-Cartan equations, we obtain the following:

**Theorem 6** If \( \bar{c} \in \mathcal{H}_c \) is a Maurer-Cartan element, then \( \rho(\bar{c}) \in \mathrm{Coder}(T^c\mathcal{H}_o) \) gives a deformation of \( \mathcal{H}_o \) as a weak \( A_\infty \)-algebra.

In particular, a first order solution for \( \bar{c} \in \mathcal{H}_c \) is preserved to be a first order solution in \( \mathrm{Coder}(T^c\mathcal{H}_o) \) by an \( L_\infty \)-morphism. The corresponding situation is the chain map (2.18) considered previously:

\[
\rho(dg(\uparrow X)) = [m, \rho(\uparrow X)] ,
\]

where \( \uparrow X \in \mathcal{H}_c \) and \( \rho(\uparrow X) \in \mathrm{Coder}(T^c\mathcal{H}_o) \). If \( \uparrow X \) is a first order solution, the chain map gives us \( [m, \rho(\uparrow X)] = 0 \). However, notice that this \( \rho(\uparrow X) \), a first order solution in \( \mathrm{Coder}(T^c\mathcal{H}_o) \), in general includes a constant term \( C \rightarrow \mathcal{H}_o \) coming from \( n_{1,0} \). Namely, the first order deformation of \( m \) turns out to be a 'weak' homotopy derivation, a natural extension of a strong homotopy derivation in Definition 9 by including a map \( \theta_0 : C \rightarrow A \).

Rather than treat this result in isolation, we look at more general deformations of \( \mathcal{H} \) as an OCHA. In order to do it, let us first explain another aspect of the Maurer-Cartan equation for a dg Lie algebra or an \( L_\infty \)-algebra more generally.

**Lemma 1** For an \( L_\infty \)-algebra \((L, l)\) and a graded vector space \( L' \), consider a coalgebra isomorphism \( f : C(L') \rightarrow C(L) \), that is, a collection of degree zero graded symmetric maps \( \{f_0, f_1, \cdots \} \).
such that $f_1 : L' \to L$ is an isomorphism. Then, the inverse of $f$ exists, and a unique weak $L_\infty$-structure $l'$ is induced by $l' = (f)^{-1} \circ l \circ f$ so that $f : (L', l') \to (L, l)$ is a weak $L_\infty$-isomorphism.

It is clear by definition that $l'$ is a degree one coderivation and $(l')^2 = 0$.

Moreover, if we take $\{f_0 = \bar{c} \in L^0, f_1 = 1, f_2 = \cdots = 0\}$ for $f$, the explicit form of $l'$ is given as follows (see Getzler [15] and Schuhmacher [48]):

$$l'_l(c_1, \cdots, c_l) := \sum_{n \geq 0} \frac{1}{n!} l'_{n+l}(\bar{c}^\otimes n, c_1, \cdots, c_l), \quad l \geq 0.$$  \hspace{1cm} (5.7)

Here recall that $f_0 : C \to \mathcal{H}_c$ so we identify $f_0$ with its image $\bar{c}$. Notice that $l'_0 = \sum_{k \geq 1} \frac{1}{k!} l_k(\bar{c}^\otimes k)$. Thus, $l'$ gives a (strict) $L_\infty$-structure iff $\bar{c} \in MC(L, l)$.

In this argument, we can also begin with a weak $L_\infty$-algebra $(L, l)$ together with a straightforward modification of the Maurer-Cartan equation for a weak $L_\infty$-algebra.

The same fact holds true also for (weak) $A_\infty$-algebras, as explained in subsection 2.4 in [31] (the explicit form of the deformed $A_\infty$-structures can be found in [10, 11]).

Let us consider the same story for an OCHA.

**Lemma 2** For an OCHA $(\mathcal{H}, l, n)$ and a graded vector space $\mathcal{H}'$, consider a coalgebra map $\bar{f} : C(\mathcal{H}') \otimes T^c(\mathcal{H}') \to C(\mathcal{H}_c) \otimes T^c(\mathcal{H}_c)$ such that $f_1 + f_{0,1} : \mathcal{H}' \to \mathcal{H}$ is an isomorphism. Then, a unique weak OCHA-structure $l' + n'$ is induced by $l' + n' = (f)^{-1} \circ (1 + n) \circ \bar{f}$ so that $\bar{f} : (\mathcal{H}', l', n') \to (\mathcal{H}, l, n)$ is a weak OCHA-isomorphism.

Again, the fact that $\bar{f}$ is a coalgebra map and $l + n$ is a degree one coderivation implies that $l' + n'$ is in fact a degree one coderivation, and $(l' + n')^2 = 0$ follows from $(1 + n)^2 = 0$. The reason the structure is *weak* is the presence of the operations $l'_0$ and $n'_{0,0}$.

In particular, when we take a weak OCHA-isomorphism $\bar{f}$ given by $f_0 = \bar{c} \in \mathcal{H}_c^0, \quad f_{0,0} = \bar{d} \in \mathcal{H}_o^0, \quad f_1 = 1_c, \quad f_{0,1} = 1_o$

and other higher multi-linear maps set to be zero, the deformed weak OCHA structure is given by $l'$ in eq.(5.7) and

$$n'_{p,q}(c_1, \cdots, c_p; o_1, \cdots, o_q) := \sum_{n, m_0, \cdots, m_k \geq 0} \frac{1}{n!} n_{n+p, m_0+\cdots+m_k+q}(\bar{c}^\otimes n, c_1, \cdots, c_p; \bar{o}^\otimes m_0, o_1, \bar{o}^\otimes m_1, \cdots, \bar{o}^\otimes m_{q-1}, o_q, \bar{o}^\otimes m_q)$$  \hspace{1cm} (5.8)

for $p \geq 0$ and $q \geq 0$.

Now, we can spell out generalized Maurer-Cartan equations for OCHAs.

**Definition 20 (Maurer-Cartan equations for $(\mathcal{H}, l, n)$)** For an OCHA $(\mathcal{H}, l, n)$ and degree zero elements $\bar{c} \in \mathcal{H}_c$ and $\bar{o} \in \mathcal{H}_o$, we define

$$I_*(\bar{c}) := \sum_k \frac{1}{k!} l_k(\bar{c}, \cdots, \bar{c}), \quad n_*(\bar{c}; \bar{o}) := \sum_{k,l} \frac{1}{k!} n_{k,l}(\bar{c}, \cdots, \bar{c}; \bar{o}, \cdots, \bar{o}).$$  \hspace{1cm} (5.9)
We call the following pair of equations
\[ 0 = l_*(\bar{c}) \quad , \quad 0 = n_*(\bar{c}; \bar{o}) \]
the \textit{Maurer-Cartan equations} for the OCHA \((\mathcal{H}, l, n)\).

The solution space of the Maurer-Cartan equations is denoted by
\[ \mathcal{MC}(\mathcal{H}, l, n) = \{ (\bar{c}, \bar{o}) \in (\mathcal{H}^0_c, \mathcal{H}^0_o) \mid l_*(\bar{c}) = 0, \quad n_*(\bar{c}; \bar{o}) = 0 \} . \]

The Maurer-Cartan equations (5.10) are nothing but the condition that \( l'_0 = 0 \) and \( n'_0 = 0 \), since \( l'_0 = l_*(\bar{c}) \) and \( n'_0 = n_*(\bar{c}; \bar{o}) \). In particular, the first equation is just the Maurer-Cartan equation for the \( L_\infty \)-algebra \((H, c)\).

Theorem 7 (Maurer-Cartan elements as deformations) \((\bar{c}, \bar{o}) \in \mathcal{MC}(\mathcal{H}, l, n)\) gives a deformation of \((H_0, \mathcal{M})\) as a (strict) \( A_\infty \)-algebra.

The explanations are as follows. First of all, for a weak OCHA \((\mathcal{H}' = \mathcal{H}'_c \oplus \mathcal{H}'_o, l', n')\) given in eq.(5.7) and eq.(5.8), let us consider its restriction to \( \mathcal{H}'_c = 0 \), that is, consider the defining equation for a (weak) OCHA \((2.19)\) and set \( c_1 = \cdots = c_n = 0 \). Then, only the equations for \( n = 0 \) survive, which are given by
\[ 0 = n'_{1, m}(l'_0; o_1, \cdots, o_m) + \sum_{i+s+j=m} (-1)^{\beta(s,i)}n'_{0,i+1+j}(\emptyset; o_1, \cdots, o_i, n'_{0,s}(\emptyset; o_{i+1}, \cdots, o_{i+s}), o_{i+s+1}, \cdots, o_m) . \]

Here \( l' = 0 \) iff \( \bar{c} \in \mathcal{MC}(\mathcal{H}_c, l) \), then the first term in the right hand side drops out and the second term turns out to be the defining equation for a weak \( A_\infty \)-algebra. This is just the situation of Theorem 6, where \( \rho(\bar{c}) \) is given explicitly by
\[ \downarrow \rho(\bar{c}) = \sum_{p \geq 1, q \geq 0} \frac{1}{p!} n_{p,q}(\bar{c}^\otimes p; \cdots , ) \in \text{Hom}(T^c\mathcal{H}_o, \mathcal{H}_o) . \]

Note that, \((\bar{c}, 0) \in (\mathcal{H}^0_c, \mathcal{H}^0_o)\) need not belong to \( \mathcal{MC}(\mathcal{H}, l, n) \) even if \( \bar{c} \in \mathcal{MC}(\mathcal{H}, l) \) because of the existence of \( n_{k,0}(\bar{c}, \cdots , \bar{c}) \) terms in the second equation in eq.(5.9). Alternatively, for \( \bar{c} \in \mathcal{MC}(\mathcal{H}_c, l) \), if we can find an element \( \bar{o} \) such that \((\bar{c}, \bar{o}) \in \mathcal{MC}(\mathcal{H}, l, n)\), \( n'_{0,0} \) also vanishes and one gets a deformed (strict) \( A_\infty \)-algebra. Thus we obtain Theorem 7 above.

5.2 Gauge equivalence and moduli spaces

Continuing with the general philosophy of deformation theory, we regard two deformations as equivalent if they are related by \textit{gauge equivalence}, that is, if they differ by the action of the group obtained by exponentiating the action of \( g^0 \) of the controlling dg Lie algebra \( g \).

For the case of \( L_\infty \)-algebras instead of dg Lie algebras, it is more subtle to show that the gauge equivalence given in a similar way in fact defines an equivalence relation, that is,
the composition of gauge transformations is a gauge transformation. In order to avoid such conceptually irrelevant subtlety, we give a definition of gauge equivalence in a more formal way in terms of piecewise smooth paths, though these definitions should be equivalent under some appropriate assumptions (see [11]).

**Definition 21 (Gauge equivalence)** Given an $L_\infty$-algebra $(\mathcal{H}_c, l)$, two elements $\bar{c}_0 \in \mathcal{MC}(\mathcal{H}_c, l)$ and $\bar{c}_1 \in \mathcal{MC}(\mathcal{H}_c, l)$ are called *gauge equivalent* iff there exists a piecewise smooth path $\bar{c}_t \in \mathcal{MC}(\mathcal{H}_c, l)$, $t \in [0, 1]$ such that

$$
\frac{d}{dt} \bar{c}_t = \sum_{k \geq 0} \frac{1}{k!} l_{1+k}(\alpha(t), \bar{c}_t \otimes \bar{c}_t) \quad (5.11)
$$

for a degree minus one element $\alpha(t) \in \mathcal{H}_c^{-1}$.

By this definition, it is clear that the gauge equivalence actually defines an equivalence relation. One can also express this gauge transformation in terms of a path ordered integral as $c_1 = c_1(\{l_k\}, c_0, \alpha(t)) = c_0 + \cdots$ [31, 33].

**Definition 22 (Moduli space)** For an $L_\infty$-algebra $(\mathcal{H}_c, l)$ and the solution space of its Maurer-Cartan equation $\mathcal{MC}(\mathcal{H}_c, l)$, the corresponding moduli space $\mathcal{M}(\mathcal{H}_c, l)$ is defined as

$$
\mathcal{M}(\mathcal{H}_c, l) := \mathcal{MC}(\mathcal{H}_c, l)/\sim ,
$$

where $\sim$ is the gauge equivalence in Definition 21.

The moduli space for an $A_\infty$-algebra $(\mathcal{H}_o, m)$ is also defined in a similar way and denoted by $\mathcal{M}(\mathcal{H}_o, m) := \mathcal{MC}(\mathcal{H}_o, m)/\sim$.

The following classical fact is known (for instance see [36, 10, 11, 31, 33]; some of these include the case of $A_\infty$-algebras, for which a similar fact holds).

**Theorem 8** For two $L_\infty$-algebras $(\mathcal{H}_c, l)$ and $(\mathcal{H}_c', l')$, suppose there exists an $L_\infty$-morphism $f : (\mathcal{H}_c, l) \to (\mathcal{H}_c', l')$. Then there exists a well-defined map

$$
f_\sim : \mathcal{M}(\mathcal{H}_c, l) \to \mathcal{M}(\mathcal{H}_c', l')
$$

and in particular $f_\sim$ gives an isomorphism if $f$ is an $L_\infty$-quasi-isomorphism.

Then, as a corollary of Theorem 6 we have the following:

**Corollary 3 (A_\infty-structure parameterized by the moduli space of L_\infty-structures)** For an $L_\infty$-algebra $(\mathcal{H}_c, l)$ and an $A_\infty$-algebra $(\mathcal{H}_o, m)$, suppose there exists an OCHA $(\mathcal{H} = \mathcal{H}_c \oplus \mathcal{H}_o, l, n)$ such that $(\mathcal{H}_o, \{n_{0,k}\}) = (\mathcal{H}_o, m)$. Also, let $(\mathcal{H}_c', l')$ be an $L_\infty$-algebra obtained by the suspension of the dg Lie algebra Coder$(T^c A)$ with $D = [m, ~]$ and Lie bracket $[~, ~]$. The OCHA $(\mathcal{H}, l, n)$ then gives a map from $\mathcal{M}(\mathcal{H}_c, l)$ to $\mathcal{M}(\mathcal{H}_c', l')$ and it is in particular an isomorphism if the $L_\infty$-morphism $(\mathcal{H}_c, l) \to (\mathcal{H}_c', l')$ is an $L_\infty$-quasi-isomorphism.
In a similar way as in the $A_\infty$ or $L_\infty$ case, we can define the moduli space of the solution space of the Maurer-Cartan equations for an OCHA.

**Definition 23 (Open-closed gauge equivalence)** Given an OCHA $(\mathcal{H}, l, n)$, we call two elements $(\bar{c}_0, \bar{o}_0) \in \mathcal{MC}(\mathcal{H}, l, n)$ and $(\bar{c}_1, \bar{o}_1) \in \mathcal{MC}(\mathcal{H}, l, n)$ gauge equivalent iff there exists a piecewise smooth path $(\bar{c}_t, \bar{o}_t) \in \mathcal{MC}(\mathcal{H}, l, n)$, $t \in [0, 1]$ such that $\bar{c}_t$ satisfies differential equation (5.11) and $\bar{o}_t$ satisfies

\[
\frac{d}{dt} \bar{o}_t = \sum_{p, q \geq 0} \frac{1}{p!} n_{1+p+q} (\alpha(t), \bar{c}_t^{\otimes p}; \bar{o}_t^{\otimes q}) + \sum_{p, q, q' \geq 0} \frac{1}{p!} n_{p+1+q+q'} (\bar{c}_t^{\otimes p}; \bar{o}_t^{\otimes q}, \beta(t), \bar{o}_t^{\otimes q'})
\]

for degree minus one elements $(\alpha(t), \beta(t)) \in (\mathcal{H}_c^{-1}, \mathcal{H}_o^{-1})$.

By definition, when $(\bar{c}_0, \bar{o}_0)$ and $(\bar{c}_1, \bar{o}_1)$ are gauge equivalent in the sense of an OCHA, $\bar{c}_0$ and $\bar{c}_1$ are gauge equivalent in the sense of the $L_\infty$-algebra.

**Definition 24 (Moduli space for an OCHA)** For an OCHA $(\mathcal{H}, l, n)$ and the solution space of its Maurer-Cartan equations $\mathcal{MC}(\mathcal{H}, l, n)$, the moduli space for the OCHA $(\mathcal{H}, l, n)$ is defined by

\[
\mathcal{M}(\mathcal{H}, l, n) := \mathcal{MC}(\mathcal{H}, l, n)/\sim,
\]

where $\sim$ is the gauge equivalence in Definition 23.

Then, due to the theorems in section 4 and in particular Corollary 2, the following theorem is obtained in a similar way as in the $A_\infty$ and $L_\infty$-cases.

**Theorem 9** Suppose we have an OCHA homomorphism $f : (\mathcal{H}, l, n) \to (\mathcal{H}', l', n')$ between two OCHAs. Then, $f$ induces a well-defined map between two moduli spaces $f_\sim : \mathcal{M}(\mathcal{H}, l, n) \to \mathcal{M}(\mathcal{H}', l', n')$. Furthermore, if $f$ is an OCHA quasi-isomorphism, it induces an isomorphism between the two moduli spaces.

Thus, the moduli space $\mathcal{M}(\mathcal{H}, l, n)$ is also a homotopy invariant notion and in particular the equivalence class of deformations given by Theorem 7 is described by $\mathcal{M}(\mathcal{H}, l, n)$.

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**Appendix (by M. Markl): Operadic interpretation of $A_\infty$-algebras over $L_\infty$-algebras**

This part of the paper assumes some knowledge of the language of operads and related notions, see the book [44], namely Section II.3.7 of this book. We explain here how $A_\infty$-algebras over $L_\infty$-algebras can be interpreted using a ‘colored’ version of the standard theory of strong homotopy
algebras in the form formulated in [44, Proposition II.3.88]. Let us briefly recall some necessary background material.

Assume that $\mathcal{P}$ is a quadratic Koszul operad governing the algebraic structure we have in mind (such as associative algebra, Lie algebra, etc.) and let $\mathcal{P}^!$ denote the quadratic dual of $\mathcal{P}$. Proposition II.3.88 of [44] then says that a strongly homotopy $\mathcal{P}$-algebra on a graded vector space $V$ is the same as a degree +1 differential on the cofree nilpotent $\mathcal{P}^!$-coalgebra $T_{\mathcal{P}^!}(\downarrow V)$ on the desuspension of $V$. Using a colored version of this proposition, we show that $A_\infty$-algebras over $L_\infty$-algebras are in fact strongly homotopy Leibniz pairs.

Let $\rho : \mathfrak{g} \to \text{Der} A$ be a Leibniz pair as in Definition 8. These Leibniz pairs are algebras over a two-colored operad $\text{Leib}$, with the white color denoting inputs/output in $\mathfrak{g}$ and the black color inputs/output in $A$. The operad $\text{Leib}$ is a quadratic $\{\circ, \bullet\}$-colored operad generated by one antisymmetric binary operation $l$ of type $(\circ, \circ) \to \circ$ for the Lie multiplication in $\mathfrak{g}$, one binary operation $m$ of type $(\bullet, \bullet) \to \bullet$ for the associative multiplication in $A$, and one binary operation $\rho$ of type $(\circ, \bullet) \to \bullet$ for the action of $\mathfrak{g}$ on $A$. The relations defining $\text{Leib}$ as a quadratic colored operad can be easily read off from eq.(2.14) and eq.(2.15).

We may safely leave as an exercise to verify that the quadratic dual $\text{Leib}^!$ of the operad $\text{Leib}$ describes objects $(\mathcal{C}, \mathfrak{A})$ consisting of a commutative associative algebra $\mathcal{C}$, an associative algebra $\mathfrak{A}$ and an action of $\mathcal{C}$ on $\mathfrak{A}$ that satisfies

$$X(ab) = X(a)b = aX(b), \quad \text{for } X \in \mathcal{C} \text{ and } a, b \in \mathfrak{A},$$

and

$$(X\Psi)(a) = X(\Psi(a)) = \Psi(X(a)), \quad \text{for } X, \Psi \in \mathcal{C} \text{ and } a \in \mathfrak{A}.$$
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