Sum rules for spin-1/2 quantum gases in states with well-defined spins: II. Spin-dependent two-body interactions.

Vladimir A. Yurovsky
School of Chemistry, Tel Aviv University, 6997801 Tel Aviv, Israel
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Sums of matrix elements of spin-dependent two-body momentum-independent interactions and sums of their products are calculated analytically in the basis of many-body states with given total spin — the states built from spin and spatial wavefunctions belonging to multidimensional irreducible representations of the symmetric group, unless the total spin has the maximal allowed value. As in the first part of the series [V. A. Yurovsky, Phys. Rev. A 91, 053601 (2015)], the sum dependence on the many-body states is given by universal factors, which are independent of the Hamiltonians of non-interacting particles. The sum rules are applied to perturbative analysis of energy spectra and to calculation of two-body spin-dependent local correlations.

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INTRODUCTION

The present paper continues analyses [1] (see this work for a more comprehensive introduction and definitions) of the sum rules for many-body systems of indistinguishable spin-1/2 particles. The particles can be composite, e.g., atoms or molecules, and the spin can be either a real angular momentum of the particle or a formal spin, whose projections are attributed to the particle’s internal states (e.g. hyperfine states of atoms). In the latter case, the spin 1/2 means that only two internal states are present in the system. This formal spin is not related to the real, physical spin of the particles, which can be either bosons or fermions.

The many-body wavefunctions are represented here, as well as in [1], as a sum of products of the collective spin and spatial functions. These functions depend on spin projections and coordinates, respectively, of all particles and belong to multidimensional, non-Abelian representations of the symmetric group (see [2–5]), unless the total spin has the maximal allowed value. For spin-1/2 particles, the representation is unambiguously determined by the total many-body spin. This approach is applicable to the Hamiltonians which are separable to spin-independent and coordinate-independent parts. It differs from the conventional approach (see [6] and [1]), where each particle is characterized by its spin projection and coordinate, and the total wavefunction is symmetrized for bosons or antisymmetrized for fermions over permutations of all particles. The total many-body spin is undefined in this case. As follows from the Heitler’s results [7] (see also [1]), an exact wavefunction for particles with spin-independent interactions can not be obtained in the approach with defined individual spin projections. Many-body states with defined total spin [8–15], including the collective spin and spatial wavefunctions [16–19], were applied to spinor quantum gases (see [20–30]). Such states were also proposed for implementation of permutation quantum computers [31]. Other kings of entangled states with non-trivial symmetry have been analyzed for quantum-degenerate gases of spin-1 [32] and spin-2 [33] bosons. A more comprehensive review is presented in Ref. [1].

The well-known mean-field approach (see [6, 24]), where the interactions between particles are replaced by the self-consistent field, is generally applied to states with defined individual spin projections and undefined total spins. It can be applied to certain states with defined total spins if the interactions are spin-independent. For bosons, the many-body mean-field wavefunction contains a single spatial orbital, determined by the Gross-Pitaevskii equation (see [24]). For fermions, the Hartree-Fock wavefunction (see [6]) contains double-occupied spatial orbitals. Such wavefunctions have the total spin \( S = N/2 \) or \( S = 0 \) for \( N \) spin-1/2 bosons or fermions, respectively, and describe quantum-degenerate gases with multiple occupations of the spatial orbitals. The present approach describes states with arbitrary total spins and is suitable for non-degenerate gases.

For spin-independent two-body interactions between particles, sums of matrix elements directly follow from the Heitler’s results [7], while the sum of the matrix element squared moduli was calculated in [1]. Matrix elements of spin-dependent interactions depend on the total spin projections of the coupled many-body states. For spin-dependent one-body interactions with external fields, this dependence was factorized using the Wigner-Eckart theorem and sums of the matrix elements and their squared moduli were calculated [1].

The present work is devoted to spin-dependent two-body interactions between particles. As well as in Ref. [1], the matrix elements are calculated in the basis of non-interacting particles with single occupation of spatial modes. The interactions are expressed in terms of irreducible spherical tensors and their matrix elements are related to ones for the maximal allowed projections of the total spins using the Wigner-Eckart theorem in Sec. I. Sums of these matrix elements and their squared moduli are calculated in Sec. II for zero-range spin-dependent interactions. The sum rules are applied to perturbative analysis of energy spectra in Sec. III. Average two-body spin-dependent correlations are analyzed in this section.
too. These correlations allow to distinguish between the many-body states with defined total spins and individual spin projections. Appendices contains calculation of sums, used in Sec. II.

References to equations in the previous paper [1] are started here from I. The present paper, as well as [1], uses the following notation for the universal factors. In the ratios of the 3j-Wigner symbols $X^{(s,s',q)}_{S_{z},k}$, giving the dependence on the total spin projections, $S$ and $S'$ are the maximal and minimal total spins of the coupled states, $q$ is the rank of the spherical tensor, and $S_2$ and $S_z + k$ are the spin projections for $S$ and $S'$, respectively. The factors $Y_{i}^{(S,m)}[A]$ and $Y_{i}^{(s,m)}[A,B]$, respectively, appear in the sum rules for the matrix elements of $A$ and products of the matrix elements of $A$ and $B$ for the maximal allowed spin projections. Here $A$ and $B$ are the maximal total spin of the coupled states and $m$ is the number of changed spatial quantum numbers (it is omitted if the factor is independent of $m$). If the sum rule contains several factors $Y$, they are specified by the subscript $i$.

I. THE SPIN-PROJECTION DEPENDENCE

Let $| \uparrow \rangle$ and $| \downarrow \rangle$ denote two spin states of the particles. Permutation-invariant momentum-independent two-body interactions between particles with arbitrary spin-dependence can be decomposed into irreducible spherical tensors of ranks 0, 1, and 2 using general relations [34] between irreducible spherical and Cartesian tensors. In the present case the Cartesian tensors are proportional to the products $\mathbf{s}_a(j)\mathbf{s}_{a'}(j')$ of the spin components of two particles, where $a$ and $a'$ can be either $x$, $y$, or $z$. The $z$-component of the $j$th particle spin is

$$\mathbf{s}_z(j) = \frac{1}{2}(| \uparrow \rangle\langle \uparrow | - | \downarrow \rangle\langle \downarrow |)$$

while $\mathbf{s}_x(j) = \frac{1}{2}(| \uparrow \rangle\langle \downarrow | + | \downarrow \rangle\langle \uparrow |)$ and $\mathbf{s}_y(j) = -i\frac{1}{2}(| \uparrow \rangle\langle \downarrow | - | \downarrow \rangle\langle \uparrow |)$ are expressed in terms of the spin raising and lowering operators

$$\mathbf{s}_+(j) = | \uparrow \rangle\langle \downarrow |, \quad \mathbf{s}_-(j) = | \downarrow \rangle\langle \uparrow |$$

There are two zero-rank tensors (spherical scalars),

$$\mathbf{\hat{V}} = \sum_{j \not= j'} V^+(r_j - r_{j'})$$

and

$$\mathbf{\hat{V}}^{(0)}_{0} = -\frac{1}{\sqrt{3}}(\mathbf{\hat{V}}_{zz} + \mathbf{\hat{V}}^{+}_{+-})$$

where

$$\mathbf{\hat{V}}_{zz} = \sum_{j \not= j'} V^+(r_j - r_{j'})\mathbf{s}_z(j)\mathbf{s}_z(j')$$

$$\mathbf{\hat{V}}^{+}_{+-} = \sum_{j \not= j'} V^+(r_j - r_{j'})\mathbf{s}_+(j)\mathbf{s}_-(j').$$

Here

$$V^\pm(r) = \frac{1}{2}(V(r) \pm V(-r))$$

are the even and odd parts of the two-body potential function $V(r)$.

The three components of the rank 1 spherical tensor $\mathbf{\hat{V}}(1)$ can be expressed as

$$\mathbf{\hat{V}}_{0}^{(1)} = \frac{1}{\sqrt{2}} \sum_{j \not= j'} V^-(r_j - r_{j'})\mathbf{s}_+(j)\mathbf{s}_-(j')$$

$$\mathbf{\hat{V}}^{\pm}_{\pm1} = -\sum_{j \not= j'} V^-(r_j - r_{j'})\mathbf{s}_z(j)\mathbf{s}_\pm(j').$$

Finally,

$$\mathbf{\hat{V}}_{0}^{(2)} = \frac{1}{\sqrt{6}}(2\mathbf{\hat{V}}_{zz} - \mathbf{\hat{V}}^{+}_{+-})$$

$$\mathbf{\hat{V}}^{(2)}_{\pm1} = \mp \sum_{j \not= j'} V^+(r_j - r_{j'})\mathbf{s}_z(j)\mathbf{s}_\pm(j')$$

$$\mathbf{\hat{V}}_{0}^{(2)} = \frac{1}{2} \sum_{j \not= j'} V^+(r_j - r_{j'})\mathbf{s}_z(j)\mathbf{s}_z(j')$$

are the five components of the rank 2 spherical tensor $V^{(2)}$. The even or odd rank tensors depend, respectively, on the even or odd components $V^\pm(r)$ of the potential. Therefore the rank 1 tensor vanish in the case of even two-body interaction, while the scalars and rank 2 tensor vanish for the odd interaction.

The interactions conserving the $z$-projection of the total many-body spin are expressed in terms of the scalars and zero-components of the tensors,

$$\mathbf{\hat{V}}^{\parallel} = \sum_{j \not= j'} V(r_j - r_{j'})| \uparrow \rangle\langle \uparrow |(j')\langle \uparrow |(j)| \uparrow \rangle\langle \uparrow |(j')\langle \uparrow |(j)| \uparrow \rangle\langle \uparrow |(j')\langle \uparrow |(j)| \uparrow \rangle\langle \uparrow |(j')\langle \uparrow |(j)| \uparrow \rangle\langle \uparrow |(j')\langle \uparrow |(j)| \uparrow \rangle\langle \uparrow |(j')\langle \uparrow |(j)| \uparrow \rangle\langle \uparrow |(j')\langle \uparrow |(j)| \uparrow \rangle\langle \uparrow |(j')\langle \uparrow |(j)| \uparrow \rangle\langle \uparrow |(j')\langle \uparrow |(j)| \uparrow \rangle\langle \uparrow |(j')\langle \uparrow |(j)| \uparrow \rangle\langle \uparrow |(j')\langle \uparrow |(j)| \uparrow \rangle\langle \uparrow |(j')\langle \uparrow |(j)| \uparrow \rangle\langle \uparrow |(j')\langle \uparrow |(j)| \uparrow \rangle\langle \uparrow |(j')\langle \uparrow |(j)| \uparrow \rangle\langle \uparrow |(j')\langle \uparrow |(j)| \uparrow \rangle\langle \uparrow |(j')\langle \uparrow |(j)| \uparrow \rangle\langle \uparrow |(j')\langle \uparrow |(j)| \uparrow \rangle\langle \uparrow |(j')\langle \uparrow |(j)| \upar
are zero-components of spherical vectors (see [34]). Equations (4a) and (4b) lead to the relation
\[
\hat{V}_0^+ = \frac{1}{2} \left( \hat{V}_\| - \hat{V}_\perp \right).
\]

The interactions changing spin of one of the colliding particles,
\[
\hat{V}_{-\uparrow} \equiv \sum_{j \neq j'} V(r_j - r_{j'}) \downarrow (j) \uparrow (j') \| (j'\downarrow (j) \uparrow (j')]
= \hat{V}_{-\uparrow}^{(2)} + \hat{V}_{-\uparrow}^{(1)} + \frac{1}{\sqrt{2}} \hat{V}_{-\downarrow}^{(1)},
\]
\[
\hat{V}_{-\downarrow} \equiv \sum_{j \neq j'} V(r_j - r_{j'}) \downarrow (j) \downarrow (j') \| (j') \uparrow (j)\]
\[
= -\hat{V}_{-\downarrow}^{(2)} - \hat{V}_{-\downarrow}^{(1)} + \frac{1}{\sqrt{2}} \hat{V}_{-\downarrow},
\]
involve also
\[
\hat{V}_{\pm 1} = \mp \frac{1}{\sqrt{2}} \sum_{j \neq j'} V(r_j - r_{j'}) \hat{s}_\pm (j),
\]
which form a spherical vector together with \(\hat{V}_0 = \hat{V}_\uparrow + \hat{V}_\downarrow\) (see [34]).

The interaction
\[
\hat{V}_{-\downarrow} \equiv \sum_{j \neq j'} V(r_j - r_{j'}) \downarrow (j) \downarrow (j') \| (j') \uparrow (j)\]
\[
= 2\hat{V}_{-\downarrow}^{(2)}
\]
changes spins of both colliding particles. Other spin-changing interactions are obtained by the Hermitian conjugation of Eqs. (6) and (7), taking into account that
\[
(\hat{V}_{-\downarrow}^{(2)})^\dagger = \hat{V}_{-\downarrow}^{(2)}, (\hat{V}_{-\downarrow}^{(1)})^\dagger = -\hat{V}_{-\downarrow}^{(2)}
\]
\[
(\hat{V}_{-\uparrow}^{(1)})^\dagger = \hat{V}_{-\uparrow}^{(1)}, \hat{V}_{-\uparrow}^{(1)} = -\hat{V}_{-\uparrow}^{(1)}.
\]

Consider matrix elements between wavefunctions \(\Psi_{nS_n}^{(S)}\) with the defined projection \(S_n\) of the total spin \(S\). The explicit form of \(\Psi_{nS_n}^{(S)}\), given by Eq. (1.16), is not used here. The multi-index \(n\) labels different functions with the same \(S\) and \(S_n\). According to the Wigner-Eckart theorem (see [35]), the matrix elements of the spherical scalars are diagonal in spins and independent of the spin projection,
\[
\langle \Psi_{n'S_n'}^{(S')} | \hat{V}_k | \Psi_{nS_n}^{(S)} \rangle = \delta_{S,S'} \delta_{S_n,S_n'} \langle \Psi_{nS_n}^{(S)} | \hat{V}_k | \Psi_{nS_n}^{(S)} \rangle
\]
\[
\langle \Psi_{n'S_n'}^{(S')} | \hat{V}_{0}^{(0)} | \Psi_{nS_n}^{(S)} \rangle = \delta_{S,S'} \delta_{S_n,S_n'} \langle \Psi_{nS_n}^{(S')} | \hat{V}_{0}^{(0)} | \Psi_{nS_n}^{(S)} \rangle.
\]

The matrix elements of the spherical vectors and the rank 1 tensor follow the same relations (1.23) as the one-body interactions,
\[
\langle \Psi_{nS_n}^{(S)} | \hat{A}_{k} | \Psi_{nS_n}^{(S')} \rangle = \delta_{S,S} \delta_{S_n,S_n} \langle \Psi_{nS_n}^{(S')} | \hat{A}_{k} | \Psi_{nS_n}^{(S)} \rangle,
\]
where the factors
\[
X_{S_{1/2},k}^{(S,S')}(q) = ( -1 )^{S'-S} \left( \begin{array}{ccc} S & S' & q \\ S_z & -S_z - k & k \end{array} \right)
\]
\[
\times \left( \begin{array}{ccc} S & S' & q \\ S & S' & S - S' \end{array} \right)^{-1}
\]
are expressed in terms of the 3j-Wigner symbols. Here \(\hat{A}_0\) can be either \(\hat{V}_0^+\), \(\hat{V}_0^\perp\), or \(\hat{V}_0^{(1)}\), and \(\hat{A}_\pm\) can be \(\hat{V}_{\pm 1}\), or \(\hat{V}_{\pm 1}^{(1)}\). According to the properties of the 3j-Wigner symbols, the matrix elements (9) vanish if \(|S - S'| > 1\) (in agreement to the selection rules [19]). The factors \(X_{S_{1/2},k}^{(S,S',1)}\) for \(S' \leq S\) are presented in Table I in Ref. [1]. Hermitian conjugate of Eq. (9), together with Eq. (8) and relations \(\hat{V}_{0}^{(2)} = \hat{V}_{0}^{(2)}\), \(\hat{V}_0 = \hat{V}_0\), and \(\hat{V}_{0}^{(1)} = \hat{V}_{0}^{(1)}\), give us the matrix elements for \(S' = S + 1\).

The matrix elements of the components of the rank 2 spherical tensor \(\hat{V}_k^{(2)}\) can be expressed in terms of the matrix elements for the maximal allowed spin projections \((S' \leq S)\) in the same way,
\[
\langle \Psi_{nS_n}^{(S)} | \hat{V}_k^{(2)} | \Psi_{nS_n}^{(S')} \rangle = \delta_{S,S} \delta_{S_n,S_n} \langle \Psi_{nS_n}^{(S')} | \hat{V}_k^{(2)} | \Psi_{nS_n}^{(S)} \rangle.
\]

According to the properties of the 3j-Wigner symbols, the matrix elements (10) vanish if \(|S - S'| > 2\) (in agreement to the selection rules [19]). The non-vanishing factors \(X_{S_{1/2},k}^{(S,S',2)}\), calculated with the 3j-Wigner symbols [6, 35], are presented in Table I. The symmetry properties of the 3j-Wigner symbols [6, 35] lead to the relation
\[
X_{S_{1/2},k}^{(S,S',q)} = ( -1 )^{S'-S'+q} X_{S_{1/2},-k}^{(S,S',q)}.
\]
providing the factors $X_{S,k}^{(S',S';2)}$ for $k < 0$. The matrix elements for $S + 1 \leq S' \leq S + 2$ are given by Hermitian conjugation of Eq. (10), taking into account Eq. (8) and the relation $(\hat{V}_0^{(2)})^\dagger = -\hat{V}_0^{(2)}$.

Thus, each permutation-invariant two-body interaction between particles is expressed in terms of irreducible spherical tensors. Their matrix elements for arbitrary total spin projections are related to ones for the maximal allowed spin projections using Wigner-Eckart theorem. The next section deals with the later matrix elements.

II. SUM RULES

A. Matrix elements for zero-range interactions

The sums of the matrix elements and sums of their squared moduli will be evaluated here for zero-range spin-dependent two-body interactions with the even potential function

$$V(r) = \delta(r),$$

where $r$ is a $D$-dimensional vector. This function is generally used for description of interactions of cold atoms in free space, when $D = 3$, and under tight pancake- or cigar-shape confinement, when $D = 2$ or $D = 1$, respectively. In the three- and two-dimensional cases the $\delta$-function have to be properly renormalized.

For the even potential function, we have $V^+(r) = V(r)$, $V^-(r) = 0$, and, therefore, $\hat{V}_k^{(1)} = \hat{V}_0^{-} = 0$, $\hat{V}_+ = \hat{V}_-^+$. Besides, the identity $| \uparrow (j)\rangle \langle \uparrow (j)| + | \downarrow (j)\rangle \langle \downarrow (j)| = 1$ leads to the relation

$$\hat{V}_{\uparrow\downarrow} = \frac{1}{2} (\hat{V} - \hat{V}_{\uparrow\downarrow}).$$

The zero range of interaction allows to relate the matrix elements of $\hat{V}_{\uparrow\downarrow}$ and $\hat{V}_{\uparrow\downarrow}$, defined by (4c) and (4d), respectively, in the following way. Wavefunctions of indistinguishable particles $\Psi$ obey to the quantum exclusion principle $P_{jj'}\Psi = \pm \Psi$, where the sign $+$ or $-$ is taken for bosons or fermions, respectively. Therefore

$$\langle \Psi' | \hat{V}_{\uparrow\downarrow} | \Psi \rangle = \pm \sum_{j \neq j'} \langle \Psi' | \uparrow (j)\rangle \langle \downarrow (j') | \delta(r_j - r_{j'}) \times \langle \downarrow (j') | \langle \uparrow (j) | P_{jj'} | \Psi \rangle.$$ 

The permutation operator $P_{jj'}$ permutes both spins and the coordinates $r_j$ and $r_{j'}$. However, the $\delta$-function sets $r_j = r_{j'}$ and the coordinate permutation has no effect. Acting to the left, permutation $P_{jj'}$ of spins is restricted by the bra $| \downarrow (j') \rangle \langle \uparrow (j) |$ of the interaction operator, therefore

$$\langle \Psi' | \hat{V}_{\uparrow\downarrow} | \Psi \rangle = \pm \sum_{j \neq j'} \langle \Psi' | \uparrow (j)\rangle \langle \downarrow (j') | \delta(r_j - r_{j'}) \times \langle \downarrow (j') | \langle \uparrow (j) | \Psi \rangle = \pm \langle \Psi' | \hat{V}_{\uparrow\downarrow} | \Psi \rangle.$$ 

Then (4c) and (4d) for the even potential function, together with (12), lead to

$$\langle \Psi' | \hat{V}_0^{(0)} | \Psi \rangle = -\frac{1}{4 \sqrt{3}} \langle \Psi' | \hat{V} | \Psi \rangle,$$

$$\langle \Psi' | \hat{V}_0^{(2)} | \Psi \rangle = \sqrt{\frac{3}{2}} \langle \Psi' | \frac{1}{2} (\hat{V}_{\uparrow\downarrow} + \hat{V}_{\downarrow\uparrow}) + \frac{1}{3} \hat{V} | \Psi \rangle,$$

(13)

$$\langle \Psi' | \hat{V}_0^{(1)} | \Psi \rangle = \frac{1}{2} \langle \Psi' | \hat{V}_{\uparrow\downarrow} - \hat{V}_{\downarrow\uparrow} | \Psi \rangle.$$

for bosons. Thus, matrix elements of scalars and zero-components of the vector and tensors are expressed in term of $\hat{V}$, $\hat{V}_{\uparrow\downarrow}$, and $\hat{V}_{\downarrow\uparrow}$. The non-zero components of the vector and tensor are obtained from (6) and (7)

$$\hat{V}_{\uparrow\downarrow} = \frac{1}{2} (\hat{V}_{\uparrow\downarrow} - \hat{V}_{\downarrow\uparrow}),$$

$$\hat{V}_{\uparrow\downarrow} = \frac{1}{\sqrt{2}} (\hat{V}_{\uparrow\downarrow} + \hat{V}_{\downarrow\uparrow}),$$

(14)

and their Hermitian conjugates.

For fermions, matrix elements of $\hat{V}_{\uparrow\downarrow}$, and $\hat{V}_{\downarrow\uparrow}$ are equal to zero, in agreement with the Pauli principle — two particles with the same spin cannot have equal coordinates. This leads to the zero matrix elements of the spherical vector and tensors. The matrix elements of the spherical scalars are related as

$$\langle \Psi' | \hat{V}_0^{(0)} | \Psi \rangle = \frac{\sqrt{3}}{4} \langle \Psi' | \hat{V} | \Psi \rangle.$$ 

(15)

The sums of the matrix elements of $\hat{V}$ and sums of their squared moduli are derived in Ref. [1]. Other relevant interactions are analyzed below.

B. Matrix elements for non-interacting bosons

Let us evaluate matrix elements of $\hat{V}_{\uparrow\downarrow}$, $\hat{V}_{\downarrow\uparrow}$, $\hat{V}_{\uparrow\downarrow}$, and $\hat{V}_{\downarrow\uparrow}$ between wavefunctions

$$\tilde{\psi}_r^{(S)}(S)_{\uparrow\downarrow} = f_{S}^{1/2} \sum_t \tilde{\psi}_{tr(n)}^{(S)}(S)_t,$$

[see Eq. (I.17)] of non-interacting bosons with defined projection $S_z$ of the total spin $S$, where $f_S$ is the dimension of the respective irreducible representation of the symmetric group. The representations and functions within the representations, respectively, are labeled by the standard Young tableau $r$ and $t$ of the shape $\lambda = \left[\lambda_1, \lambda_2, \ldots, \lambda_n\right]$ (see [4, 5]). For bosons, the spatial functions of $N$ non-interacting particles (1.11) are represented in the form

$$\tilde{\psi}_{tr(n)}^{(S)} = \left(\frac{f_S}{N!}\right)^{1/2} \sum_{p} D_{r(n)}^{N}(P) \prod_{j=1}^{N} \varphi_{n,p_j}(r_j),$$

(16)

[1]
where \( \mathcal{P} \) are permutations of \( N \) symbols, \( \varphi_n(r) \) are the spatial orbitals, and the relation for the Young orthogonal matrices

\[
D_{rl}^{[\lambda]}(\mathcal{P}^{-1}) = D_{rl}^{[\lambda]}(\mathcal{P})
\]  

[see (I.7)] is used (see Ref. [1] for other notation). Each matrix element, e.g., the one of \( \hat{V} \), can be decomposed into the spatial and spin parts,

\[
\langle \Phi_{\lambda_1} | \Phi_{\lambda_2} \rangle = \sum_{\lambda} D_{\lambda_1 \lambda_2}^{([\lambda])}(\mathcal{P}) D_{\lambda_1 \lambda_2}^{([\lambda])}(\mathcal{P}^{-1})
\]

Using Eq. (17), the spatial matrix elements can be expressed as

\[
\langle \Phi_{\lambda_1}^{(S')} | V | \Phi_{\lambda_2}^{(S')} \rangle = \frac{\sqrt{fsfs'}}{N!} \sum_{\mathcal{R}, \mathcal{Q}} D_{\lambda_1 \lambda_2}^{[\lambda]}(\mathcal{Q}) D_{\lambda_1 \lambda_2}^{[\lambda]}(\mathcal{R}) \times \int d^D r_i d^D r_i' \phi_{\lambda_i}^{(S')} (r_i) \phi_{\lambda_i}^{(S')} (r_i') V(r_i - r_i') \phi_{\lambda_i}^{(S')} (r_i) \phi_{\lambda_i}^{(S')} (r_i') \prod_{i' \neq i'' \neq \lambda_i} \delta_{\lambda_i^{(S')}, \lambda_i^{(S')}}.
\]

The Kronecker \( \delta \)-symbols appear here due to the orthogonality of the spatial orbitals \( \varphi_n \) and the absence of equal quantum numbers in each of the sets \( \{n\} \) and \( \{n'\} \). Due to the \( \delta \)-symbols, all but two spatial quantum numbers remain unchanged. Supposing that the unchanged \( n_{i'} \) are in the same positions in the sets \( \{n\} \) and \( \{n'\} \), one can see that the Kronecker symbols allow only \( \mathcal{Q} = \mathcal{R} \) or \( \mathcal{Q} = \mathcal{R} \mathcal{P}_{i'i'} \). Therefore

\[
\langle \Phi_{\lambda_1}^{(S')} | V | \Phi_{\lambda_2}^{(S')} \rangle = \frac{\sqrt{fsfs'}}{N!} \sum_{\mathcal{R}} D_{\lambda_1 \lambda_2}^{[\lambda]}(\mathcal{R}) \left[ D_{\lambda_1 \lambda_2}^{[\lambda]}(\mathcal{R}) + D_{\lambda_1 \lambda_2}^{[\lambda]}(\mathcal{R} \mathcal{P}_{i'i'}) \right] \langle n_{i'} n_{i'}' | V | n_{i'} n_{i'}' \rangle \prod_{i' \neq j' \neq \omega_i} \delta_{\lambda_i^{(S')}, \lambda_i^{(S')}}
\]

where for the zero-range potential (11) the matrix elements

\[
\langle n_{1} n_{2} | V | n_{1} n_{2} \rangle = \langle n_{2} n_{1} | V | n_{1} n_{2} \rangle = \int d^D r \phi_{n_{i}}^{*} (r) \phi_{n_{j}}^{*} (r) \phi_{n_{i}} (r) \phi_{n_{j}} (r)
\]

are invariant over permutations of \( n_{1} \) and \( n_{2} \), as well as of \( n_{1} \) and \( n_{2} \).

All matrix elements are related by the Wigner-Eckart theorem to ones for the maximal allowed spin projection, \( S_2 = S' \), \( S = S \). The spinor matrix elements include projections of the spin wavefunctions (1.14), derived in [36],

\[
\Xi_{iS}^{(S)} = C_{SS} \sum_{\mathcal{P}} D_{t[0]}^{[\lambda]}(\mathcal{P}) \prod_{j=1}^{N/2+S} \prod_{j'=1}^{N} \left| \uparrow (P_{i}) \right|, \prod_{j'=1}^{N} \left| \downarrow (P_{j}) \right|
\]

with the normalization factor

\[
C_{SS} = \frac{1}{(N/2+S)!} \frac{(2S+1)(S+S_2)!}{(N/2+S+1)(2S)(S-S_2)!}
\]

The projection

\[
\langle \uparrow (i) | \uparrow (i') | \Xi_{iS}^{(S)} \rangle = C_{SS} \sum_{\mathcal{Q}} \sum_{l \neq l'} D_{t[0]}^{[\lambda]}(\mathcal{Q} \mathcal{P}_{i} \mathcal{P}_{l \lambda_2} \mathcal{P}_{l' \lambda_1}) \prod_{l \neq j \neq l'} \delta_{\lambda_1 \lambda_2} \delta_{\lambda_1 \lambda_2} \delta_{\lambda_1 \lambda_2} \prod_{j=1}^{\lambda_1+1} \prod_{j=1}^{\lambda_2+1} \left| \uparrow (Q_{j}) \right|, \prod_{j'=1}^{N} \left| \downarrow (Q_{j}) \right|
\]

(recall, \( \lambda_{1,2} = N/2 \pm S \)) can be transformed, using substitution \( \mathcal{P} = \mathcal{Q} \mathcal{P}_{l_{1} \lambda_{1}} \mathcal{P}_{l' \lambda_{1}} \), to the form

\[
\langle \uparrow (i) | \uparrow (i') | \Xi_{iS}^{(S)} \rangle = C_{SS} \sum_{\mathcal{Q}} \sum_{l \neq l'} D_{t[0]}^{[\lambda]}(\mathcal{Q} \mathcal{P}_{l_{1} \lambda_{1}} \mathcal{P}_{l' \lambda_{1}}) \prod_{l \neq j \neq l'} \delta_{\lambda_1 \lambda_2} \delta_{\lambda_1 \lambda_2} \delta_{\lambda_1 \lambda_2} \prod_{j=1}^{\lambda_1+1} \prod_{j=1}^{\lambda_2+1} \left| \uparrow (Q_{j}) \right|, \prod_{j'=1}^{N} \left| \downarrow (Q_{j}) \right|
\]
The projections involved into matrix elements for other interactions are evaluated in the same way,

\[
\langle \uparrow (i) | \uparrow (i') | \Xi_{S}^{(S)} \rangle = \lambda_2 (\lambda_2 - 1) C_{SS} \sum_{Q} D_{t[0]}^{[1]}(Q) \delta_{i, Q (\lambda_1 + 1)} \delta_{i', Q (\lambda_1 + 1)} \prod_{j=1}^{\lambda_1-2} \uparrow (Q_j) \prod_{j=\lambda_1+1}^{N} \downarrow (Q_j).
\]

The matrix elements of the operators except the last one are calculated in the same way,

\[
\langle \uparrow (i') | \Xi_{S}^{(S)} \rangle = \lambda_2 (\lambda_2 - 1) C_{SS} \sum_{Q} D_{t[0]}^{[1]}(Q) \delta_{i, Q (\lambda_1 + 1)} \delta_{i', Q (\lambda_1 + 1)} \prod_{j=1}^{\lambda_1-2} \uparrow (Q_j) \prod_{j=\lambda_1+1}^{N} \downarrow (Q_j).
\]

In the spin matrix elements of \( \hat{V}_{\uparrow} \),

\[
\langle \Xi_{S}^{(S)} | \uparrow (i) \rangle \uparrow (i') \rangle = \lambda_1 (\lambda_1 - 1) C_{SS}^{2} \sum_{Q} D_{t[0]}^{[1]}(Q) \delta_{i, Q (\lambda_1 - 1)} \delta_{i', Q (\lambda_1 - 1)} \sum_{Q} D_{t[0]}^{[1]}(Q) \delta_{i, Q (\lambda_1 + 1)} \delta_{i', Q (\lambda_1 + 1)} \prod_{j=1}^{\lambda_1-2} \uparrow (Q_j) \prod_{j=\lambda_1+1}^{N} \downarrow (Q_j).
\]

Let us substitute this equation and Eq. (21) into Eq. (19), perform the summation over \( t \) and \( t' \), using the relation

\[
\sum_{t} D_{t[0]}^{[1]}(P) D_{t'[0]}^{[1]}(Q) = D_{t[0]}^{[1]}(P Q),
\]

[see Eq. (1.6)] and substitute \( P = Q^{-1} R^{-1}, j = R i, \) and \( j' = R i'. \) Then the Kronecker symbols lead to \( \lambda_1 - 1 \) and \( \lambda_2 \) ones. As the permutations \( P' \) and \( P'' \) do not permute symbols between rows in the Young tableau \([0]\), the equality \( D_{t[0]}^{[1]}(P P') = D_{t[0]}^{[1]}(Q) \) [see Eq. (1.8)] can be applied, leading to

\[
\langle \Xi_{S}^{(S)} | \uparrow (i) \rangle \uparrow (i') \rangle = \lambda_1 (\lambda_1 - 1) C_{SS}^{2} \sum_{Q} D_{t[0]}^{[1]}(Q) D_{t'[0]}^{[1]}(Q) \delta_{i, Q (\lambda_1 - 1)} \delta_{i', Q (\lambda_1 - 1)} \prod_{j=1}^{\lambda_1-2} \uparrow (Q_j) \prod_{j=\lambda_1+1}^{N} \downarrow (Q_j).
\]

Matrix elements of other operators are calculated in the same way,

\[
\langle \Xi_{S}^{(S)} | \uparrow (i') \rangle = \lambda_1 (\lambda_1 - 1) C_{SS}^{2} \sum_{Q} D_{t[0]}^{[1]}(Q) \delta_{i, Q (\lambda_1 - 1)} \delta_{i', Q (\lambda_1 - 1)} \prod_{j=1}^{\lambda_1-2} \uparrow (Q_j) \prod_{j=\lambda_1+1}^{N} \downarrow (Q_j).
\]

C. Sums of the matrix elements and their squares and products

For the matrix elements which are diagonal in the total spin and \( r \), one can calculate their sums and write out
them in the form

\[
\sum_r \langle \tilde{\Psi}^{(S)}_{r'} | V_a | \tilde{\Psi}^{(S)}_r \rangle S = Y^{(S)}[V_a] \frac{2J_S}{N(N-1)}
\]

\[
\times \sum_{j\neq j'} \langle n_j' n_j'' | V | n_j n_j' \rangle \prod_{j \neq j'' \neq j} \delta_{n_j', n_j''}, \quad (28a)
\]

where \( \tilde{V}_a \) is any two-body interaction, which does not change the spin projection. In each term of the sum over \( j \) and \( j' \), only two spatial quantum numbers can be changed. For the operators \( \tilde{V}_\parallel \) and \( \tilde{V}_\perp \) the factors

\[
Y^{(S)}[\tilde{V}_\parallel] = 2\lambda_1(\lambda_1 - 1), \quad Y^{(S)}[\tilde{V}_\perp] = 2\lambda_2(\lambda_2 - 1),
\]

calculated using the Eqs. (25), (27), (22), (26), (23), and (18), are proportional to the numbers \( \lambda_1(\lambda_1 - 1) \) and \( \lambda_2(\lambda_2 - 1) \) of particle pairs with spins \( \uparrow \) and \( \downarrow \), respectively. For the spherical tensor components, the factors

\[
\sum_{r', r''} \sum_{n'} V^{[\lambda][\lambda]}_{r', r''}(l_1, l_1') V^{[\lambda][\lambda]}_{r' n'}(l_2, l_2') = \sum_{j_1 \neq j_2 \neq j_2'} \Sigma^{(S, S)}_{j_1 j_2 j_2'}(l_1, l_1', l_2, l_2')
\]

\[
\times \langle n_j' n_j'' | V | n_j n_j' \rangle \prod_{j_1 \neq j_2 \neq j_2'} \delta_{n_j', n_j''} \langle n_j' n_j'' | V | n_j n_j' \rangle^* \prod_{j_1 \neq j_2 \neq j_2'} \delta_{n_j', n_j''}, \quad (29)
\]

where

\[
\Sigma^{(S, S)}_{j_1 j_2 j_2'}(l_1, l_1', l_2, l_2') = \sum_{r', r''} \sum_{n'} \sum_{\lambda} D^{[\lambda]}_{[0 \lambda]}(P) D^{[\lambda]}_{[0 \lambda]}(Q) \delta_{l_1, r_1} \delta_{l_1', r_1'} \sum_{\lambda} D^{[\lambda]}_{[0 \lambda]}(Q) \delta_{l_2, r_2} \delta_{l_2', r_2'}, \quad (30)
\]

The sums of squared moduli of the matrix elements of the spin-independent interactions (I.45a) can be expressed for bosons and the zero-range potentials (11) in the form (28a) too with

\[
Y^{(S)}[\tilde{V}] = \frac{3}{2} N(N-2) + 2S(S+1). \quad (28c)
\]

The sums of squared moduli of the matrix elements (25) and (27) and their products are proportional to the sums of products of the functions (26), which can be expressed as

\[
\sum_{r', r''} V^{[\lambda][\lambda]}_{r', r''}(l_1, l_1') V^{[\lambda][\lambda]}_{r' n'}(l_2, l_2') Y^{(S)}[\tilde{V}] = \sum_{j_1 \neq j_2 \neq j_2'} \Sigma^{(S, S)}_{j_1 j_2 j_2'}(l_1, l_1', l_2, l_2')
\]

\[
\times \langle n_j' n_j'' | V | n_j n_j' \rangle \prod_{j_1 \neq j_2 \neq j_2'} \delta_{n_j', n_j''} \langle n_j' n_j'' | V | n_j n_j' \rangle^* \prod_{j_1 \neq j_2 \neq j_2'} \delta_{n_j', n_j''}, \quad (31)
\]

where

\[
\Sigma^{(S)}_{j_1 j_2 j_2'}(l, l') = \sum_{r', r''} \sum_{\lambda} D^{[\lambda]}_{[0 \lambda]}(P) D^{[\lambda]}_{[0 \lambda]}(Q) \delta_{l_1, r_1} \delta_{l_1', r_1'} \sum_{\lambda} D^{[\lambda]}_{[0 \lambda]}(Q) \delta_{l_2, r_2} \delta_{l_2', r_2'}, \quad (32)
\]

is calculated in Appendix B. Equation (31) contains \( \Sigma^{(S)}_{j_1 j_2 j_2'}(l_1, l_1', l_2, l_2') \) for \( \tilde{V}_\parallel \) and \( \tilde{V}_\perp \), respectively. The sums of squared moduli of the matrix elements and their products are expressed in different forms if the set of spatial quantum numbers is

changed \((\{n\} \neq \{n'\})\) or conserved \(\{n\} = \{n'\}\).

D. Changing set of spatial quantum numbers

If the sets of spatial quantum numbers \(\{n\}\) and \(\{n'\}\) are different by two elements, the product of Kronecker symbols in (29) and (31) does not vanish only if either \(j_1 = j'_2, j'_1 = j_2\) or \(j_1 = j'_2, j'_1 = j_2\). Since \(\Sigma_{jj'jj'j}^{(S)_j} (l_1, l'_1, l_2, l'_2)\) is independent of particular values of \(j\) and \(j'\) (see Appendix A), the sum (29) attains the form

\[
\sum_{r, r'} V_{r, r'}^{[n]} \langle r, n | l_1, l_1' \rangle V_{r, r'}^{[n']} \langle r, n' | l_2, l_2' \rangle = 2 \Sigma_{2}^{(S)} (l_1, l_1', l_2, l_2') \sum_{j < j'} \left(\langle n'_j n'_j' | V | n_j n_j'\rangle\right)^2 \prod_{j \neq j' \neq j} \delta_{n'_j, n_j, n_j'}
\]

with \(\Sigma_{2}^{(S)} (l_1, l_1', l_2, l_2') = \Sigma_{jj'jj'}^{(S)} (l_1, l_1', l_2, l_2') + \Sigma_{jjjj'}^{(S)} (l_1, l_1', l_2, l_2'). \quad (33)\)

Then for any two-body spin-dependent interactions \(\hat{V}_a\) and \(\hat{V}_b\), the sums of squared moduli of the matrix elements and their products can be written out in the form

\[
\sum_{r, r'} \langle \tilde{\Psi}_{r}^{(S')} | \hat{V}_a | \tilde{\Psi}_{r}^{(S)} \rangle \langle \tilde{\Psi}_{r'}^{(S')} | \hat{V}_b | \tilde{\Psi}_{r'}^{(S)} \rangle^{*} = Y(S, 2) [\hat{V}_a, \hat{V}_b] \frac{2 f_S}{N(N - 1)} \sum_{j < j'} \left(\langle n'_j n'_j' | V | n_j n_j'\rangle\right)^2 \prod_{j \neq j' \neq j} \delta_{n'_j, n_j, n_j'} \quad (34a)
\]

with \(S' \leq S\). Each term in the sum above changes two of the spatial quantum numbers, conserving other ones. Since \(S\) and \(S'\) are equal to the spin projections, \(S'\) is unambiguously determined by the operators \(\hat{V}_a\) and \(\hat{V}_b\), such that \(S' = S\) for \(\hat{V}_a\) and \(\hat{V}_b\), and \(S' = S + k\) for \(\hat{V}_k(\hat{V}_k')\) and \(\hat{V}_k\). The factors \(Y(S, 2) [\hat{V}_a, \hat{V}_b]\) are expressed in terms of the sums \(\Sigma_{2}^{(S, S)} (l_1, l_1', l_2, l_2')\). For example, the factor \(Y(S, 2) [\hat{V}_a, \hat{V}_b]\) takes the form

\[
Y(S, 2) [\hat{V}_a, \hat{V}_b] = 4 \left[\lambda_2 | \lambda_1 (\lambda_1 - 1) C_{SS}^2 S \right] \frac{N(N - 1)}{f_S} \Sigma_{2}^{(S)} (l_1 - 1, \lambda_1 - 1, \lambda_1 - 1, \lambda_1) \quad (34g)
\]

Similarly, the sum (31) leads to the factor

\[
Y(S, 2) [\hat{V}_a, \hat{V}_b] = 4 \lambda_1 | \lambda_2 (\lambda_1 - 1) C_{SS}^2 S \frac{N(N - 1)}{f_S} \left[(N - 2)! + \Sigma_{jjjj'}^{(S)} (\lambda_1 - 1, \lambda_1)\right]. \quad (34k)
\]

The factors in the sums of squared moduli and products of matrix operators of other operators are expressed in a similar form. Explicit expressions for the factors are obtained using the normalization factors (22) and sums \(\Sigma_{jjjj'}^{(S, S)}\) and \(\Sigma_{jjjj'}^{(S)}\), calculated in Appendices A and B, respectively. For example,

\[
Y(S, 2) [\hat{V}_a, \hat{V}_b] = \frac{2}{2S + 3} \left[\frac{N^2(2S + 1)}{S + 1} + 2(N(4S^3 + 8S^2 + S - 1) + 2S(2S^3 + 5S^2 - 5))\right].
\]

Equations (13) and (14) lead to the following factors in the sums of squared moduli of the matrix elements of spherical tensor components and their products

\[
Y(S, 2) [\hat{V}_0(\hat{V}_0), \hat{V}_0] = \frac{S(2S - 1)}{6(2S + 3)} \left(3 \frac{(N + 2)^2}{S + 1} - 4S\right) \quad (34b)
\]

\[
Y(S, 2) [\hat{V}_0, \hat{V}_0] = 3 \frac{(N + 2)S(2S - 1)}{S + 1} \quad (34c)
\]

\[
Y(S, 2) [\hat{V}_0, \hat{V}_0] = \sqrt{\frac{2}{3}} \frac{(N + 2)S(2S - 1)}{S + 1} \quad (34d)
\]

\[
Y(S, 2) [\hat{V}_0, \hat{V}_0] = 4 \sqrt{\frac{2}{3}} S(2S - 1) \quad (34e)
\]

\[
Y(S, 2) [\hat{V}_0, \hat{V}_0] = 8S(N - 1) \quad (34f)
\]

The sum of squared moduli of the matrix elements of the spin-independent interactions (I.47a) can be expressed in the case of bosons with the zero-range potentials (11) in the form (34a) too with

\[
Y(S, 2) [\hat{V}, \hat{V}] = 6N(N - 2) + 8S(N + 1). \quad (34k)
\]

The case of a single changed quantum number will be considered elsewhere.

E. Conserving set of spatial quantum numbers

If the set of spatial quantum numbers is unchanged, \(\{n\} = \{n'\}\), the Kronecker symbols in sums (29) and (31) are equal to one for any \(j_1, j_2, j'_1,\) and \(j'_2\). Then these sums contain
where the matrix elements of zero-range interactions (11) are symmetric over permutations of \( n_j \) and \( n_j' \),

\[
V_{jj'} = \langle n_j n_j'|V|n_j n_j'\rangle = \langle n_j n_j'|V|n_j n_j\rangle = \langle n_j n_j'|V|n_j n_j\rangle = \int d^3r \varphi_{n_j}(r) \varphi_{n_j}(r)^2
\]

and \( \Sigma_{jj,jj'} \) can be either \( \Sigma^{(S', S)}_{jj,jj'}(l_1, l_1', l_2, l_2') \) or \( \Sigma^{(S)}_{jj,jj'}(l, l') \) [see Eqs. (30) and (32)] with arbitrary superscripts and arguments. These functions depend on relations between their subscripts, rather than the subscript specific values (see Appendices A and B). Then \( \Sigma_4 = \Sigma_{jj,jj'} \) for \( j_1 \neq j_2 \neq j_1' \) and \( j_1 \neq j_2' \neq j_1' \) or \( j_1 \neq j_2 \neq j_1' \), \( \Sigma_3 = \Sigma_{jj,jj''} + \Sigma_{jj''j} + \Sigma_{jj'''} \) for \( j_1 \neq j \neq j'' \neq j' \), and \( \Sigma_2 = \Sigma_{jj,jj''} + \Sigma_{jj''j} \) for \( j \neq j' \). The sum (35) can be further transformed to

\[
\sum_{j_1 \neq j_1', j_2 \neq j_2'} \Sigma_{jj,jj'} V_{jj,jj'} = N(N-1)(N-2)(N-3)\Sigma_4 + (N-2)\Sigma_3 + \Sigma_2|V|^2 + N(N-1)^2(\Sigma_4 - 4\Sigma_4)|\Delta_1 V|^2 + N(N-1)(\Sigma_2 - \Sigma_3 + 2\Sigma_4)|\Delta_2 V|^2.
\]

Here

\[
\langle V \rangle = \frac{2}{N(N-1)} \sum_{j < j'} V_{jj'}
\]

is the average value of the matrix elements (36) and

\[
\langle \Delta_1 V \rangle^2 = \frac{1}{N} \sum_{j=1}^{N} \left( \frac{1}{N-1} \sum_{j' \neq j} V_{jj'} - \langle V \rangle \right)^2
\]

\[
\langle \Delta_2 V \rangle^2 = \frac{2}{N(N-1)} \sum_{j < j'} (V_{jj'} - \langle V \rangle)^2
\]

measure their average deviations [in consistency with (1.48)]. Then the sums of squared moduli of the matrix elements and their products can be written out in the form

\[
\sum_{r, r'} (\tilde{\Psi}^{(S')}_{r'\{n\}S'}|\tilde{\psi}^{(S)}_{r\{n\}S}\rangle|\tilde{\psi}^{(S')}_{r\{n\}S}\rangle)^* = f_{SS'} Y^{(S)}_0[\tilde{\psi}_a, \tilde{\psi}_b]|\tilde{\psi}^{(S)}_a\rangle Y^{(S)}_0|\tilde{\psi}^{(S)}_b\rangle
\]

with \( S' \leq S \). The factors \( Y^{(S, 0)}_0 \) here are calculated using results of Appendices A, B, and C. The first factor can be represented for all considered \( \tilde{\psi}_a \) and \( \tilde{\psi}_b \) as

\[
Y^{(S, 0)}_0[\tilde{\psi}_a, \tilde{\psi}_b] = \delta_{SS'} Y^{(S)}[\tilde{\psi}_a] Y^{(S)}[\tilde{\psi}_b]
\]

[see Eq. (28)]. Since \( S' - S \) is unambiguously determined by \( \tilde{\psi}_a \) and \( \tilde{\psi}_b \) \( (S' = S \) for \( \tilde{\psi} \) and \( S' = S + k \) for \( \tilde{\psi}^{(2)}_k \) and \( \tilde{\psi}_k \) \), \( Y^{(S, 0)}_0[\tilde{\psi}_a, \tilde{\psi}_b] = 0 \) for \( k \neq 0 \) components of the spherical vectors and tensors, when \( S \neq S' \). Other factors are expressed as

\[
Y^{(S, 0)}_1[\tilde{\psi}^{(2)}_0, \tilde{\psi}^{(2)}_0] = -(N-1)(N-2S)(N+2S+2)S(2S-1) \left( \frac{3}{S+1} - 8S \right)
\]

\[
Y^{(S, 0)}_2[\tilde{\psi}^{(2)}_0, \tilde{\psi}^{(2)}_0] = \frac{(N-2S)(N+2S+2)S(2S-1)}{6(N-2)(N-3)(2S+3)} \left( \frac{3}{S+1} - 8S \right)
\]

\[
Y^{(S, 0)}_1[\tilde{\psi}_0, \tilde{\psi}_0] = \frac{(N-1)(N-2S)(N+2S+2)S}{S+1}
\]
dependent interaction between the particles.

The sum rules, combined with the

\[
Y_1^{(S,0)}[\hat{V}_0^{(2)}, \hat{V}_0] = \frac{\sqrt{2}}{3} \frac{(N-1)(N-2S)(N+2S+2)\sqrt{S(N-1)}}{(N-2)(S+1)}
\]

(39f)

\[
Y_1^{(S,0)}[\hat{V}_0^{(2)}, \hat{V}] = 2\sqrt{\frac{2}{3}} \frac{(N-1)(N-2S)(N+2S+2)\sqrt{S(N-1)}}{(N-2)(N-3)}
\]

(39g)

\[
Y_2^{(S,0)}[\hat{V}_0^{(2)}, \hat{V}] = -\sqrt{\frac{2}{3}} \frac{(N-2S)(N+2S+2)\sqrt{S(N-1)}}{(N-2)(N-3)}
\]

(39h)

\[
Y_1^{(S,0)}[\hat{V}_0, \hat{V}] = 2\frac{(N-1)(N-2S)(N+2S+2)S}{N-2}
\]

(39i)

\[
Y_1^{(S,0)}[\hat{V}_{-1}^{(2)}, \hat{V}_{-1}^{(2)}] = (N-1)(N-2S+2)(S-1)\frac{2(N+1)S^2 - N(N+3)}{(N-2)(N-3)(S+1)}
\]

(39j)

\[
Y_2^{(S,0)}[\hat{V}_{-1}^{(2)}, \hat{V}_{-1}^{(2)}] = \frac{(N-1)(N-2S)(N+2S)(N-2S+2)(S-1)}{2(N-2)(N-3)(S+1)}
\]

(39k)

\[
Y_1^{(S,0)}[\hat{V}_{-1}, \hat{V}_{-1}] = N(N-1)(N-2S+2)
\]

(39l)

\[
Y_1^{(S,0)}[\hat{V}^{(2)}, \hat{V}_{-1}] = \sqrt{\frac{2}{3}} N(N-1)(N-2S+2)(S-1)
\]

(39m)

\[
Y_1^{(S,0)}[\hat{V}_{-2}^{(2)}, \hat{V}_{-2}^{(2)}] = -\frac{(N-1)^2(N-2S+2)(N-2S+4)}{(N-2)(N-3)}
\]

(39n)

\[
Y_2^{(S,0)}[\hat{V}_{-2}^{(2)}, \hat{V}_{-2}^{(2)}] = \frac{(N-1)(N-2S+2)(N-2S+4)}{2(N-3)}
\]

(39o)

The vanishing factors are omitted above, namely \(Y_2^{(S,0)}[\hat{V}_a, \hat{V}_b] = 0\) when \(\hat{V}_a\) or \(\hat{V}_b\) is \(\hat{V}_k\). The sum of squared moduli of the matrix elements of the spin-independent interactions (1.47b) can be expressed for bosons and the zero-range potentials (11) in the form (39a) too with

\[
Y_1^{(S,0)}[\hat{V}, \hat{V}] = -\frac{(N-1)(N-2S)(N+2S+2)}{(N-3)} \left(3 - 4S \frac{S+1}{N-2}\right)
\]

(39p)

\[
Y_2^{(S,0)}[\hat{V}, \hat{V}] = (N-2S)(N+2S+2) \frac{3N(N-4) - 4S(S+1) + 12}{2(N-2)(N-3)}
\]

(39q)

They are equal to the factors (1.47c).

---

### III. Multiplet energies and correlations for weakly-interacting gases

#### A. Average multiplet energies and energy widths

Consider a general spin-dependent two-body interaction, conserving the particle spins,

\[
\hat{V}_{\text{tot}} = \frac{1}{2} g_{\|} \hat{V}_{\|} + \frac{1}{2} g_{\perp} \hat{V}_{\perp} + g_{\perp} \hat{V}_{\perp},
\]

(40)

where the potentials are defined by Eqs. (4a), (4b), and (4c) with the zero-range potential function (11). Here the interaction strengths \(g_{\sigma\sigma'}\) are proportional to the s-wave elastic scattering lengths \(a_{\sigma\sigma'}\) for corresponding pairs of spin states. For example, in three-dimensional geometry \(g_{\sigma\sigma'} = 4\pi\hbar^2 a_{\sigma\sigma'}/m\), where \(m\) is the particle’s mass. The factors \(\frac{1}{2}\) in Eq. (40) appears due to double-counting of the interacting pairs in \(\hat{V}_{\|}\) (4a) and \(\hat{V}_{\perp}\) (4b).
In the case of weak interaction, the average multiplet energy can be evaluated in the zero-order of the degenerate perturbation theory, in the same way as in the case of spin-independent interactions [see derivation of Eq. (1.52)]

\[ \bar{E}_{SS_2} = \frac{1}{J} \sum_r \langle \tilde{\Psi}_r^{(S)} \mid \tilde{V}_{\text{tot}} \mid \tilde{\Psi}_r^{(S)} \rangle \]  

(in the case of spin-dependent interactions the energies depend on the total spin projection \(S_z\)).

For bosons with zero-range interactions, matrix elements of the potentials are related to ones of the irreducible spherical tensor components by Eqs. (4a), (4b), (4c), and (13),

\[
\langle \Psi' \mid \tilde{V}_{||} \mid \Psi \rangle = \langle \Psi' \mid \sqrt{\frac{2}{3}} \sqrt{\hat{S}_0^2} + \hat{V}_0 + \frac{1}{3} \hat{V} \mid \Psi \rangle \\
\langle \Psi' \mid \tilde{V}_{\perp} \mid \Psi \rangle = \langle \Psi' \mid \sqrt{\frac{2}{3}} \sqrt{\hat{S}_0^2} - \hat{V}_0 + \frac{1}{3} \hat{V} \mid \Psi \rangle \\
\langle \Psi' \mid \tilde{V}_{+} \mid \Psi \rangle = \langle \Psi' \mid - \sqrt{\frac{2}{3}} \sqrt{\hat{S}_0^2} + \frac{1}{6} \hat{V} \mid \Psi \rangle.
\]

These equations allow to expand the interaction \(\tilde{V}_{\text{tot}}\) in terms of irreducible spherical tensors, which matrix elements can be related to the ones for the maximal allowed spin projections by Eqs. (9) and (10). Then the sum rules (28) lead to

\[ \bar{E}_{SS_2} = \frac{1}{2} \left( g Y^{(S)} [\hat{V}] + g_- X^{(S,S,1)}_{S,0} Y^{(S)} [\hat{V}_0] \right) \]

where \(g = (g_{||} + g_{\parallel} + g_{\perp})/3, g_+ = g_{||} + g_{\parallel} - 2g_{\perp}, g_- = g_{||} - g_{\parallel}, \) and the average matrix element \(\langle \tilde{V} \rangle\) is defined by Eq. (37). Substituting the coefficients \(X\) and \(Y\) from Table 1 in [1], Table I, and Eq. (28) one gets

\[ \bar{E}_{SS_2} = g \left[ \frac{3}{4} N (N - 2) + S (S + 1) - \frac{1}{3} S (S + 1) \frac{g_+}{g} \right. \]

\[ + S_z (N - 1) \frac{g_-}{g} + S_z \frac{g_+}{g} \]  

Here the first two terms in the square brackets provide the average multiplet energy (1.52) for spin-independent interactions. Spin-dependence of the interactions leads the third term, which is independent of the total spin projection \(S_z\), as well as to the linear and quadratic in \(S_z\) shifts (the fourth and fifth terms, respectively). The corrections are proportional to the ratios \(g_\pm/g\), which are determined by the scattering length. For example, 

\[ g_+/g \approx -0.001 \quad \text{and} \quad g_-/g \approx -0.049 \quad \text{for} \quad ^{87}\text{Rb}. \]

In this case, the two states, generally used in experiments, \(|\uparrow\rangle = |F = 2, m_f = -1\rangle \) and \(|\downarrow\rangle = |F = 1, m_f = 1\rangle\), have the scattering lengths [37] \(a_\uparrow \approx 95.5 a_B\), \(a_\downarrow \approx 100.4 a_B\), and \(a_{1\uparrow} \approx 98.0 a_B\), where \(a_B\) is the Bohr radius. One-body spin-dependent interactions with external fields lead only to linear shifts [see Eq. (1.54)].

The root-mean-square multiplet width can be evaluated in the same way as in the case of spin-independent interactions [see derivation of Eq. (1.53)]

\[ \langle \Delta E_{SS_2} \rangle^2 \]

Expanding the interaction \(\tilde{V}_{\text{tot}}\) in terms of irreducible spherical tensors, expressing their matrix elements in terms of the ones for the maximal allowed spin projections, and applying the sum rules (39) we get

\[ \langle \Delta E_{SS_2} \rangle^2 \]

Here the matrix element deviations \(\langle \Delta_i V \rangle^2\) are defined by Eq. (38) and the terms proportional to \(\langle V \rangle^2\) are canceled due to relation (39b). The first term in the square brackets gives the width for the spin-independent interactions (1.53). Corrections due to spin-dependence of the interactions are proportional to the small parameters \(g_\pm/g\). Leading terms in the coefficients before each power of \(S_z\) can be obtained using explicit expressions for \(X\) from Table 1 in [1]
and Table I,

\[
(\Delta E_{SS_z})^2 = \frac{g^2}{4} \sum_{i=1}^{2} [Y_i^{(S,0)}(V, \hat{V}) - 2\sqrt{\frac{2}{3}} S + 1] Y_i^{(S,0)}(V_0, \hat{V}) \frac{g}{g} + \frac{2}{S} Y_i^{(S,0)}(V_0, \hat{V}) \frac{g}{g} S_z + \frac{2\sqrt{6}}{S(2S-1)} Y_i^{(S,0)}(V_0, \hat{V}) \frac{g}{g} S_z + \frac{6}{S^2(2S-1)} Y_i^{(S,0)}(V_0, \hat{V}) \frac{g^2}{g} S_z^2 + \frac{6}{S(2S-1)^2} Y_i^{(S,0)}(V_0, \hat{V}) \frac{g^2}{g} S_z^2 \langle \Delta V \rangle^2. \tag{43}
\]

The consideration above was devoted to the case of bosons. For fermions, matrix elements of \( \hat{V}_{\uparrow \uparrow} \) and \( \hat{V}_{\downarrow \downarrow} \) vanish, according to the Pauli principle (see Sec. II A). Then, due to Eqs. (4c) and (15) matrix elements of \( \hat{V}_{\text{tot}} \) are equal to ones of \( g_{\uparrow \uparrow} \hat{V} \). Therefore, the multiplet average energies and energy widths are independent of \( S_z \) and can be calculated using Eqs. (1.52) and (1.53) for spin-independent interactions.

### B. Average correlations

The probabilities of finding two particles with given (either equal or different) spins in the same point, the two-body local spin-dependent correlations are expectation values of operators

\[
\hat{\rho}_{\sigma_1, \sigma_2} = \delta(\mathbf{r}_1 - \mathbf{r}_2) |\sigma_1(1)\rangle \langle \sigma_1(1)| |\sigma_2(2)\rangle \langle \sigma_2(2)|. \tag{44}
\]

The spin projection \( \sigma_j \) can be either \( \uparrow \) or \( \downarrow \). Due to permutation symmetry of the total wavefunctions, the expectation values of \( \hat{\rho}_{\sigma_1, \sigma_2} \) are proportional to matrix elements of the spin-dependent potentials \( \hat{V}_{\sigma_1, \sigma_2} \) with the potential function (11). The multiplet-averaged correlations can be evaluated in the same way as the average multiplet energy (41). However, Eq. (41) already contains all necessary information, as, according to the Hellmann-Feynman theorem [38, 39], the correlations are proportional to derivatives of the average energy over respective coupling constants

\[
\hat{\rho}_{\sigma_1, \sigma_2}^{(S, S_z)} = \frac{1}{S} \sum_{r} \langle \Psi_{r(n)}^{(S)} | \hat{\rho}_{\sigma_1, \sigma_2} | \Psi_{r(n)}^{(S)} \rangle = 1 + \delta_{\sigma_1, \sigma_2} \frac{\partial}{N(N - 1)} \langle \langle \delta SS_z \rangle \rangle.
\]

Then the dependence of the multiplet-averaged correlations

\[
\hat{\rho}_{\uparrow \uparrow}^{(S, S_z)} = \left( \frac{1}{3} Y^{(S)}(V) + X^{(S, S, 1)} Y^{(S)}(V_0) \right) + \frac{1}{N(N - 1)} \langle \rho_2(0) \rangle
\]

\[
\hat{\rho}_{\downarrow \downarrow}^{(S, S_z)} = \left( \frac{1}{3} Y^{(S)}(V) - X^{(S, S, 1)} Y^{(S)}(V_0) \right) + \frac{1}{N(N - 1)} \langle \rho_2(0) \rangle
\]

\[
\hat{\rho}_{\uparrow \downarrow}^{(S, S_z)} = \left( \frac{1}{6} Y^{(S)}(V) - \frac{2}{3} X^{(S, S, 2)} Y^{(S)}(V_0) \right) \times \frac{1}{N(N - 1)} \langle \rho_2(0) \rangle
\]

on the total many-body spin \( S \) and its projection \( S_z \) factorized to the universal factors, which are independent of the spatial Hamiltonian and occupied spatial orbitals. These factors are expressed in terms of the coefficients \( X \) and \( Y \). The dependence on the spatial state is given by the average two-body density [19], \( \langle \rho_2(0) \rangle \), which is independent of \( S \) and \( S_z \) and equal to the average matrix element \( \langle V \rangle \) (37) for the potential function (11), \( \langle \rho_2(0) \rangle = \langle V \rangle \). Similar factorization was proved [19] for spin-independent local correlations of particles with arbitrary spins. Substitution the coefficients \( X \) and \( Y \) from Table I in [1] Table I, and Eq. (28) leads to explicit expressions for the universal factors in terms of \( S \) and \( S_z \),

\[
\hat{\rho}_{\uparrow \uparrow}^{(S, S_z)} = \frac{\langle \rho_2(0) \rangle}{N(N - 1)} \left( \frac{1}{2} N(N - 2) + 2(N - 1)S_z + 2S_z^2 \right)
\]

\[
\hat{\rho}_{\downarrow \downarrow}^{(S, S_z)} = \frac{\langle \rho_2(0) \rangle}{N(N - 1)} \left( \frac{1}{2} N(N - 2) - 2(N - 1)S_z + 2S_z^2 \right)
\]

\[
\hat{\rho}_{\uparrow \downarrow}^{(S, S_z)} = \frac{\langle \rho_2(0) \rangle}{N(N - 1)} \left( \frac{1}{4} N(N - 2) + S(S + 1) - 2S_z^2 \right)
\]

The local spin-independent correlations [19] are multiplet-averaged expectation values of \( \delta(\mathbf{r}_1 - \mathbf{r}_2) \) and can be calculated with the characters from Table 2 in [1] for both bosons and fermions (the signs + and −,
respectively, below)
\[
\bar{\rho}_2^{[N]}(0) = \left(1 + \frac{N^2}{2} \right) \langle \rho_2(0) \rangle
= \left(1 + \frac{4S^2 + N^2 + 4S - 4N}{2N(N-1)} \right) \langle \rho_2(0) \rangle. \tag{46}
\]
They are related to spin-dependent correlations,
\[
\bar{\rho}_2^{[N]}(0) = \rho_{\parallel}^{(S,N)} + \rho_{\perp}^{(S,N)} + 2\rho_{1 \perp}^{(S,N)},
\]
as can be proved in the same way as (12), and are independent of \(S_z\).

In an alternative description, each particle is characterized by its spin projection and coordinate, and the total wavefunction is symmetrized for bosons or antisymmetrized for fermions over permutations of all particles [see Eq. (1.19)],
\[
\bar{\Psi}_{\{n\}\{\sigma\}} = (N!)^{-1/2} \sum_{\mathcal{P}} \text{sgn}(\mathcal{P}) \prod_{j=1}^{N} \varphi_{\sigma_n \sigma_j}(r_j) |\sigma_{\sigma_j}(j)\rangle,
\]
where the factor \(\text{sgn}(\mathcal{P})\) is the permutation parity for fermions and \(\text{sgn}(\mathcal{P}) \equiv 1\) for bosons. Given total spin projection \(S_z\), the set \(\{\sigma\}\) contains \(N_f = N/2 + S_z\) spins \(\uparrow\) and \(N_i = N/2 - S_z\) spins \(\downarrow\). For these wavefunctions the correlations are calculated as expectation values of the operators (44) and averaged over all distinct choices of \(N_f\) particles with spin up, leading to
\[
\bar{\rho}_{\parallel}^{(N_f,N_i)} = \frac{1 + \text{sgn}(\mathcal{P})_{12}}{N(N-1)} \rho_{\parallel}(N_f - 1,1) \langle \rho_2(0) \rangle,
\]
\[
\bar{\rho}_{\perp}^{(N_f,N_i)} = \frac{1 + \text{sgn}(\mathcal{P})_{12}}{N(N-1)} \rho_{\perp}(N_i - 1,1) \langle \rho_2(0) \rangle,
\]
\[
\bar{\rho}_{1 \perp}^{(N_f,N_i)} = \langle \rho_2(0) \rangle \frac{N_f}{N(N-1)} \rho_{1 \perp}.\tag{47}
\]
The transposition parity \(\text{sgn}(\mathcal{P})_{12} = 1\) leads to the factor \(2\) in \(\bar{\rho}_{\parallel}^{(N_f,N_i)}\) and \(\bar{\rho}_{\perp}^{(N_f,N_i)}\) for bosons. For fermions, \(\text{sgn}(\mathcal{P})_{12} = -1\) and correlations of particles with the same spins vanish, according to the Pauli exclusion principle.

Using relations between \(S_z\) and \(N_f,\) one can see that the average correlations of bosons with the same spins are the same as for wavefunctions with defined collective spins and individual spin projections, \(\bar{\rho}_{\parallel}^{(S,S_z)} = \rho_{\parallel}^{(N_f,N_i)}, \rho_{\perp}^{(S,S_z)} = \rho_{\perp}^{(N_f,N_i)}\). However, average correlations of particles with opposite spins are different,
\[
\bar{\rho}_{1 \perp}^{(S,S_z)} = \bar{\rho}_{1 \perp}^{(N_f,N_i)} + \frac{\langle \rho_2(0) \rangle}{N(N-1)} \left( S^2 - S_z^2 + S - \frac{N}{2} \right).
\]
The same is valid for average local spin-independent correlations, calculated as a sum of spin-dependent correlations. For the defined individual spin projections we have for both bosons and fermions (the signs + and −, respectively, below)
\[
\bar{\rho}_{\parallel}^{(N_f,N_i)} + \bar{\rho}_{\perp}^{(N_f,N_i)} + 2\bar{\rho}_{1 \perp}^{(N_f,N_i)}
= \left(1 + \frac{N^2 + 4S^2 - 4N}{2N(N-1)} \right) \langle \rho_2(0) \rangle.
\]
which depends on the total spin projection, unlike \(\rho_{\parallel}^{[N]}(0)\) [see Eq. (46)]. For fermions, the average correlations of particles with opposite spins, being equal to a half spin-independent ones, are different for the two kinds of states too. Thus the spin-independent correlations, as well as the correlations of particles with opposite spins, allow to determine the kind of the many-body state.

### CONCLUSIONS

Matrix elements of spin-dependent two-body interactions [Eqs. (4), (6), and (7)] in the basis with collective spin and spatial wavefunctions (16) can be calculated with group-theoretical methods. These matrix elements agree to the selection rules [19]. The interactions can be decomposed into irreducible spherical tensors, whose explicit dependencies on the total spin projection [Eqs. (9) and (10)] are obtained using the Wigner-Eckart theorem. Analytic expressions are derived for sums of these matrix elements (28) and their squared moduli (34) and (39)] over wavefunctions with the fixed total spin, its projection, and the set of spatial orbitals. Dependence on the many-body states in these sums is given by the 3j Wigner symbols and the universal factors \(Y^{(S)}, Y^{(S,2)}, Y_0^{(S,0)}, Y_1^{(S,0)}, Y_2^{(S,0)}\). These factors are independent of details of one-body Hamiltonians and are expressed in terms of the total spin and number of particles. The sum rules can be applied to the evaluation of changes of the spin-multiplet average energies (42) and energy widths (43) due to weak spin-dependent interactions. Multiplet-averaged two-body spin-dependent correlations (45), calculated with the sum rules, are factorized to universal factors, which are independent of the spatial orbitals, and the average density, which is independent of many-body spins. The difference between these correlations and ones for the many-body states with defined individual spin projections allows identification of the many-body state kind. Other possible applications of the sum rules include estimates of the spin-multiplet depletion rates due to spin-dependent two-body perturbations.

### Appendix A: Calculation of the sums (30)

The sums (30) and (32) contain the Young orthogonal matrix elements \(D^{[N]}_{[0][0]}(R)\), which have been calculated by Goddard [40] in the following way. Each permutation \(\mathcal{R}\) can be represented as
\[
\mathcal{R} = \prod_{k=1}^{n_{ex}} \mathcal{P}_{i_k}^{(p')} \mathcal{P}' \mathcal{P}'',\tag{A1}
\]
where \(\mathcal{P}'\) are permutations of symbols in the first row of the Young tableau \([0]\) (\(\lambda_1\) first symbols), \(\mathcal{P}''\) are permutations of symbols in the second row (\(\lambda_2\) last symbols), and \(\mathcal{P}_{i_k}^{(p')}\) transpose symbols between the rows as \(i_k < \lambda_1\).
and $i_k'' > \lambda_1$. Then [40] the matrix element is inversely proportional to the binomial coefficient,

$$D_{00}^{[\lambda]}(R) = (-1)^{n_{ex}} \lambda_1^{n_{ex}} (\lambda_1 - n_{ex})!.$$  \hspace{1cm} (A2)

Using relations (23) and (18) and substitution $R = QP^{-1}$, the sum (30) can be represented in the following form

$$\sum_{j_1,j_2,j_2'} (l_1, l_1', l_2, l_2') = \sum_{R} D_{00}^{[\lambda]}(R) D_{00}^{[\lambda]}(R) \sum_{P} \delta_{l_1, P_{j_1}} \delta_{l_1', P_{j_1'}} \delta_{l_2, P_{j_2}} \delta_{l_2', P_{j_2'}} \\
$$

where $\lambda = [N/2 + S, N/2 - S]$ and $\lambda' = [N/2 + S', N/2 - S']$. Only the sums with $j_1 \neq j_1'$ and $j_2 \neq j_2'$ are used here. This implies $l_1 \neq l_1'$ and $l_2 \neq l_2'$. The sum remains unchanged on simultaneous permutation of arguments and corresponding subscripts. $\Sigma_{j_1,j_2,j_2'} (l_1, l_1', l_2, l_2') = \Sigma_{j_1,j_1,j_2} (l_1, l_1', l_2, l_2') = \Sigma_{j_1,j_1,j_2} (l_1, l_1', l_2, l_2')$. The identity $\Sigma_{j_1,j_2,j_2'} (l_1, l_1', l_2, l_2') = \Sigma_{j_1,j_1,j_2} (l_1, l_1', l_2, l_2')$ can be proved by the substitution $R = \Sigma_{j_1,j_1,j_2}$. If $j_1 = j_2$ and $j_1' = j_2'$, we have

$$\sum_{P} \delta_{l_1, P_{j_1}} \delta_{l_1', P_{j_1'}} \delta_{l_2, P_{j_2}} \delta_{l_2', P_{j_2'}} = \delta_{l_2, \Sigma_{j_1,j_2,j_2'}} \delta_{l_2', \Sigma_{j_1,j_2,j_2'}} = \delta_{l_2, \Sigma_{j_1,j_2,j_2'}} \delta_{l_2', \Sigma_{j_1,j_2,j_2'}} = (N - 3)! \delta_{l_2, \Sigma_{j_1,j_2,j_2'}} \delta_{l_2', \Sigma_{j_1,j_2,j_2'}}$$

Then

$$\Sigma_{j_1,j_1,j_2} (l_1, l_1', l_2, l_2') = \sum_{R} D_{00}^{[\lambda]}(R) D_{00}^{[\lambda]}(R) \sum_{P} \delta_{l_1, P_{j_1}} \delta_{l_1', P_{j_1'}} \delta_{l_2, P_{j_2}} \delta_{l_2', P_{j_2'}} = \frac{1}{(N - 1)(N - 2)} \Sigma_{j_1,j_1,j_2} (l_1, l_1', l_2, l_2') - \frac{1}{N - 2} \Sigma_{j_2,j_2,j_2} (l_1, l_1', l_2, l_2')$$

(A4)

is calculated in Appendix C. The sum

$$\Sigma_{j_3,j_3,j_3} (l_1, l_1', l_2, l_2') = \Sigma_{j_3,j_3,j_3} (l_1, l_1', l_2, l_2') + \Sigma_{j_3,j_3,j_3} (l_1, l_1', l_2, l_2') + \Sigma_{j_3,j_3,j_3} (l_1, l_1', l_2, l_2') + \Sigma_{j_3,j_3,j_3} (l_1, l_1', l_2, l_2')$$

in Eq. (35) can be expressed as

$$\Sigma_{j_3,j_3,j_3} (l_1, l_1', l_2, l_2') = \frac{1}{(N - 1)(N - 2)} \Sigma_{j_3,j_3,j_3} (l_1, l_1', l_2, l_2') - \frac{2}{N - 2} \Sigma_{j_2,j_2,j_2} (l_1, l_1', l_2, l_2'),$$

(A5)

where

$$\Sigma_{j_1,j_1,j_2} (l_1, l_1', l_2, l_2') = \Sigma_{j_1,j_1,j_2} (l_1, l_2) + \Sigma_{j_1,j_1,j_2} (l_1, l_2) + \Sigma_{j_1,j_1,j_2} (l_1, l_2) + \Sigma_{j_1,j_1,j_2} (l_1, l_2)$$

(A6)

and $\Sigma_{j_2,j_2,j_2} (l_1, l_1', l_2, l_2')$ is defined by Eq. (33).
Finally, if \( j_1 \neq j_2 \neq j_1' \) and \( j_1 \neq j_2' \neq j_2 \), the sum is expressed using the following identity,

\[
\sum_{p} \delta_{i_1,p_1} \delta_{i_2,p_2} \delta_{i_3,p_3} \delta_{i_4,p_4} = \sum_{l_1 \neq l_1'} \delta_{l_2,Rl l_3,Rl l_4,Rl} \sum_{l_1 \neq l_1'} \delta_{l_2',Rl l_3',Rl l_4',Rl}

= (N-4)! \sum_{l_1, l_1'} \delta_{l_2,Rl l_3,Rl l_4,Rl} \delta_{l_2',Rl l_3',Rl l_4',Rl} (1-\delta_{l_1 l_1}) (1-\delta_{l_1' l_1'}). \]

Then \( \Sigma_{j_1,j_2,j_2'}^{(S',S)}(l_1, l_1', l_2, l_2') \equiv \Sigma_{j_1,j_2,j_2'}^{(S',S)}(l_1, l_1', l_2, l_2') \) [see Eq. (35)] is independent of \( j_1, j_2, j_1', \) and \( j_2' \) and

\[
\Sigma_{j_1,j_2,j_2'}^{(S',S)}(l_1, l_1', l_2, l_2') = \sum_{R} D_{[0][0]}^{[\lambda]}(R) D_{[0][0]}^{[\lambda]}(R) (N-4)! (1-\delta_{l_2,Rl l_1} - \delta_{l_2,Rl l_1'} - \delta_{l_2',Rl l_1} - \delta_{l_2',Rl l_1'} + \delta_{l_2,Rl l_1 l_2} + \delta_{l_2,Rl l_1 l_2'} + \delta_{l_2',Rl l_1 l_2} + \delta_{l_2',Rl l_1 l_2'})

= \frac{N!(N-4)!}{fS} \sum \delta_{\lambda \lambda'} \frac{1}{(N-1)(N-2)(N-3)} \sum \delta_{(S',S)}(l_1, l_1', l_2, l_2') + \frac{1}{(N-2)(N-3)} \sum \delta_{(S',S)}(l_1, l_1', l_2, l_2'). \quad (A7)

[see Eqs. (I.5), (33) and (A6)].

Therefore, each sum (30) is expressed in terms of \( (A3) \) and \( (A4) \).

Let us at first calculate the sums (A3) for \( S = S' \). If \( l_1 = l_2 = \lambda_1 = 1, l_1' = l_2' = \lambda_1 = 1, \) the Kronecker symbols in Eq. (A3) select the permutations of the form \( (A1) \) with \( P' \) which do not affect \( l_1 \) and \( l_1' \) and \( l_1 \neq l_1' \neq l_1' \). Therefore there are \( (\lambda_1 - 2)! \) permutations \( P' \), \( \lambda_2 \) ! permutations \( P'' \), and number of distinct choices of the sets of \( i_k \) and \( i_k' \) are given by the binomial coefficients \( \binom{\lambda_1 - 2}{n_{ex}} \) and \( \binom{\lambda_2}{n_{ex}} \), respectively. Then Eq. (A3) can be transformed as follows,

\[
\Sigma_{j_1,j_2}^{(S,S)}(\lambda_1 - 1, \lambda_1 - 1, \lambda_1 - 1, \lambda_1) = (N-2)! \sum_{n_{ex}} \binom{\lambda_2}{n_{ex}} \binom{\lambda_2-2}{n_{ex}} \binom{\lambda_1}{n_{ex}}^{-2}

= \frac{N!(N-2)!}{fS \lambda_1(\lambda_1-1)} \left[ 1 - \frac{\lambda_2}{(\lambda_1-1)(\lambda_1-\lambda_2+3)} + \frac{4\lambda_2}{\lambda_1(\lambda_1-1)(\lambda_1-\lambda_2+2)(\lambda_1-\lambda_2+3)} \right].

If \( l_1 = l_2 = \lambda_1 - 1, l_1' = l_2 = \lambda_1, \) the permutations \( P = P_{l_1 l_2} \prod_{k=1}^{k_m} P_{i_k i_k'} P' P'' \) (A8)

satisfy the Kronecker symbols if \( P' \) do not affect \( l_1 \) and \( l_1' \) and \( l_1 \neq l_1' \neq l_1' \). Since \( P_{l_1 l_2} \) is a transposition of symbols in the first row of the Young tableau \([0] \), \( n_{ex} = k_m \) and

\[
\Sigma_{j_1,j_2}^{(S,S)}(\lambda_1 - 1, \lambda_1, \lambda_1 - 1, \lambda_1) = \Sigma_{j_1,j_2}^{(S,S)}(\lambda_1 - 1, \lambda_1, \lambda_1 - 1, \lambda_1)

If \( l_1 = l_2 = \lambda_1 + 1, l_1' = l_2 = \lambda_1 + 2, \) the proper permutations are given by (A1) with \( P'' \) which do not affect \( l_1 \) and \( l_1' \) and \( l_1 \neq l_1' \neq l_1' \), and with no additional restrictions to \( P' \) and \( i_k' \). There are \( \lambda_1 \) ! permutations \( P' \), \( \lambda_2 - 2)! \) permutations \( P'' \), and number of distinct choices of the sets of \( i_k \) and \( i_k' \) are given by the binomial coefficients \( \binom{\lambda_1}{n_{ex}} \) and \( \binom{\lambda_2-2}{n_{ex}} \), respectively. Then

\[
\Sigma_{j_1,j_2}^{(S,S)}(\lambda_1 + 1, \lambda_1 + 1, \lambda_1 + 1, \lambda_1 + 1) = (N-2)! \sum_{n_{ex}} \lambda_1(\lambda_2-2)! \binom{\lambda_1}{n_{ex}} \binom{\lambda_2-2}{n_{ex}} \binom{\lambda_1}{n_{ex}}^{-2}

= \frac{N!(N-2)!}{fS \lambda_2(\lambda_2-1)} \left[ 1 - \frac{2}{\lambda_1-\lambda_2+3} \right].

If \( l_1 = l_2' = \lambda_1 + 1, l_1' = l_2 = \lambda_1 + 2, \) the Kronecker symbols are satisfied by the permutations (A8) with the same restrictions to \( P'' \) and \( i_k' \) as in the previous case. Since \( P_{l_1 l_2} \) is a transposition of symbols in the second row of the Young tableau \([0] \), \( n_{ex} = k_m \) and

\[
\Sigma_{j_1,j_2}^{(S,S)}(\lambda_1 + 1, \lambda_1 + 1, \lambda_1 + 2, \lambda_1 + 2, \lambda_1 + 1) = \Sigma_{j_1,j_2}^{(S,S)}(\lambda_1 + 1, \lambda_1 + 2, \lambda_1 + 1, \lambda_1 + 2).\]
If \( l_1 = \lambda_1 - 1, l'_1 = \lambda_1, l_2 = \lambda_1 + 1, \) and \( l'_2 = \lambda_1 + 2, \) the permutations

\[
\mathcal{R} = \mathcal{P}_{i_1} \mathcal{P}_{i'_1} \mathcal{P}_{i'_2} \prod_{k=1}^{k_m} \mathcal{P}_{i''_k} \mathcal{P}' \mathcal{P}''
\]  

(A9)

satisfy the Kronecker symbol if \( \mathcal{P}' \) do not affect \( l_1 \) and \( l'_1 \), and \( l_2 \neq l'_2 \). If \( l_2 \neq i''_k \neq l'_2 \) for any \( k, \mathcal{P}_{i_1} \) and \( \mathcal{P}_{i'_1} \) are additional transpositions between the rows of the Young tableau \([0]\), and \( n_{ex} = k_m + 2 \). Otherwise, if one of \( i''_k \) is equal to \( l_2 \), \( \mathcal{P}_{i_1} \mathcal{P}_{i'_1} \mathcal{P}_{i'_2} \mathcal{P}_{i''_k} \mathcal{P}_{i'_1} \), \( n_{ex} = k_m + 1 \). By the same reason \( n_{ex} = k_m + 1 \) if one of \( i''_k \) is equal to \( l'_2 \). If two of \( i''_k \) are equal to \( l_2 \) and \( l'_2, \) \( n_{ex} = k_m \). Then

\[
\Sigma^{(S,S)}_{jj'jj'}(\lambda_1-1, \lambda_1+1, \lambda_1+2) = (N-2)! \sum_{\lambda_m=0}^{\lambda_2} \left( \lambda_1 - 2 \right)_k! \lambda_2! \left( \begin{array}{c} \lambda_1 - 2 \\ \lambda_m \end{array} \right)_k \lambda_1 \left( \begin{array}{c} \lambda_1 - 2 \\ k_m \end{array} \right) \left( \begin{array}{c} \lambda_1 \\ \lambda_m \end{array} \right)_k + 2 \left( \begin{array}{c} \lambda_1 - 2 \\ k_m - 1 \end{array} \right) \left( \begin{array}{c} \lambda_1 \\ k_m + 1 \end{array} \right)_k = 2N!(N-2)! f_S \lambda_1 (\lambda_1 - 1)(\lambda_1 - \lambda_2 + 2)(\lambda_1 - \lambda_2 + 3).
\]

This derivation is valid for the case of \( l_1 = \lambda_1 - 1, l'_1 = \lambda_1, l_2 = \lambda_1 + 2, \) and \( l'_2 = \lambda_1 + 1 \) as well, giving

\[
\Sigma^{(S,S)}_{jj'jj'}(\lambda_1-1, \lambda_1+2, \lambda_1+1) = \Sigma^{(S,S)}_{jj'jj'}(\lambda_1-1, \lambda_1+1, \lambda_1+2).
\]

Consider now the case of \( S' = S - 1 \). If \( l_1 = l_2 = \lambda_1 - 1, l'_1 = l'_2 = \lambda_1, \) the Kronecker symbols in Eq. (A3) select the permutations of the form (A1) with \( \mathcal{P}' \) which do not affect \( l_1 \) and \( l'_1 \) and \( l_1 \neq i''_k \neq l'_1 \). Then

\[
\Sigma^{(S-1,S)}_{jj'jj'}(\lambda_1-1, \lambda_1-1, \lambda_1-1) = (N-2)! \sum_{\lambda_m=0}^{\lambda_2} \left( \lambda_1 - 2 \right)_k! \lambda_2! \left( \begin{array}{c} \lambda_1 - 2 \\ \lambda_m \end{array} \right)_k \lambda_1 \left( \begin{array}{c} \lambda_1 - 2 \\ k_m \end{array} \right) \left( \begin{array}{c} \lambda_1 \\ \lambda_m \end{array} \right)_k \left( \begin{array}{c} \lambda_1 - 1 \\ n_{ex} \end{array} \right)_k = (N-2)! \left( \begin{array}{c} \lambda_1 - 1 \\ n_{ex} \end{array} \right)_k f_S \lambda_1 (\lambda_1 - 1)^2 (\lambda_1 - \lambda_2 + 2).
\]

If \( l_1 = l'_2 = \lambda_1 - 1, l'_1 = l_2 = \lambda_1, \) the permutations (A8) satisfy the Kronecker symbols if \( \mathcal{P}' \) do not affect \( l_1 \) and \( l'_1 \) and \( l_1 \neq i''_k \neq l'_1 \). Since \( \mathcal{P}_{i_1} \mathcal{P}_{i'_1} \mathcal{P}_{i'_2} \mathcal{P}_{i''_k} \mathcal{P}_{i'_1} \mathcal{P}' \mathcal{P}'' \) is an additional transposition between the rows of the Young tableau \([0]\) of the shape \( \lambda' = [\lambda_1 - 1, \lambda_2 + 1], n_{ex} = k_m \) and \( n'_{ex} = k_m + 1 \). As a result,

\[
\Sigma^{(S-1,S)}_{jj'jj'}(\lambda_1-1, \lambda_1-1, \lambda_1-1) = -(N-2)! \sum_{\lambda_m=0}^{\lambda_2} \left( \lambda_1 - 2 \right)_k! \lambda_2! \left( \begin{array}{c} \lambda_1 - 2 \\ \lambda_m \end{array} \right)_k \lambda_1 \left( \begin{array}{c} \lambda_1 - 2 \\ k_m \end{array} \right) \left( \begin{array}{c} \lambda_1 \\ \lambda_m \end{array} \right)_k \left( \begin{array}{c} \lambda_1 - 1 \\ k_m + 1 \end{array} \right)_k = - \frac{N!(N-2)! (\lambda_1 + 2)}{f_S \lambda_1 (\lambda_1 - 1)^2 (\lambda_1 - \lambda_2 + 2)}.
\]

If \( l_1 = l_2 = \lambda_1, l'_1 = l'_2 = \lambda_1 + 1, \) the Kronecker symbols in Eq. (A3) are satisfied by the permutations of the form (A1) with \( \mathcal{P}' \) which do not affect \( l_1, \mathcal{P}' \) which do not affect \( l'_1, i''_k \neq l'_1, \) and \( i''_k \neq l'_1. \) Then

\[
\Sigma^{(S-1,S)}_{jj'jj'}(\lambda_1, \lambda_1 + 1, \lambda_1 + 1, \lambda_1 + 1) = (N-2)! \sum_{n_{ex}=0}^{\lambda_2} (\lambda_1 - 1)!(\lambda_2 - 1)! \lambda_1 \left( \begin{array}{c} \lambda_1 - 1 \\ n_{ex} \end{array} \right)_k \lambda_2 \left( \begin{array}{c} \lambda_2 - 1 \\ n_{ex} \end{array} \right)_k \lambda_1 \left( \begin{array}{c} \lambda_1 - 1 \\ n_{ex} \end{array} \right)_k \left( \begin{array}{c} \lambda_1 - 1 \\ n_{ex} \end{array} \right)_k \left( \begin{array}{c} \lambda_1 - 1 \\ n_{ex} \end{array} \right)_k = \frac{N!(N-2)! (\lambda_1 + 2)}{f_S \lambda_1 \lambda_2 (\lambda_1 - \lambda_2 + 2)}.
\]

If \( l_1 = l'_2 = \lambda_1, l'_1 = l_2 = \lambda_1 + 1, \) the permutations (A8) satisfy the Kronecker symbols if \( \mathcal{P}' \) do not affect \( l_1, \mathcal{P}'' \) do not affect \( l'_1, i''_k \neq l'_1, \) and \( i''_k \neq l'_1. \) Since \( \mathcal{P}_{i_1} \mathcal{P}_{i'_1} \mathcal{P}_{i'_2} \mathcal{P}_{i''_k} \mathcal{P}_{i'_1} \mathcal{P}' \mathcal{P}'' \) is an additional transposition between the rows of the Young tableau \([0]\) of the shape \( \lambda, n_{ex} = k_m + 1 \) and \( n'_{ex} = k_m. \) As a result,

\[
\Sigma^{(S-1,S)}_{jj'jj'}(\lambda_1, \lambda_1 + 1, \lambda_1 + 1, \lambda_1 + 1) = -(N-2)! \sum_{k_m=0}^{\lambda_2} (\lambda_1 - 1)!(\lambda_2 - 1)! \lambda_1 \left( \begin{array}{c} \lambda_1 - 1 \\ k_m \end{array} \right)_k \lambda_2 \left( \begin{array}{c} \lambda_2 - 1 \\ k_m \end{array} \right)_k \lambda_1 \left( \begin{array}{c} \lambda_1 - 1 \\ k_m + 1 \end{array} \right)_k \left( \begin{array}{c} \lambda_1 - 1 \\ k_m \end{array} \right)_k = - \frac{N!(N-2)!}{f_S \lambda_1 \lambda_2 (\lambda_1 - \lambda_2 + 2)}.
\]
If $l_1 = \lambda_1 - 1$, $l'_1 = l'_2 = \lambda_1$, $l'_2$ is an additional transposition between the rows of the Young tableaux $[0]$ of the shapes $\lambda$ and $\lambda'$ and $n_{ex} = n'_{ex} = k_m + 1$, unless $i''_k = \lambda_1 + 1$, when $n_{ex} = n'_{ex} = k_m$. Then

$$
\Sigma_{jj'jj'}^{(S-1,S)}(\lambda_1 - 1, \lambda_1, \lambda_1 + 1, \lambda_1) = (N - 2)! \sum_{k_m = 0}^{\lambda_2} (\lambda_1 - 2)! \lambda_2! \left( \frac{\lambda_2 - 1}{k_m} \right) \left( \frac{\lambda_2 - 1}{k_m + 1} \right)^{-1} \left( \frac{\lambda_1 - 1}{k_m + 1} \right)^{-1} + \left( \frac{\lambda_2 - 1}{k_m - 1} \right) \left( \frac{\lambda_1 - 1}{k_m} \right)^{-1} \left( \frac{\lambda_1 - 1}{k_m + 1} \right)^{-1} = \frac{N!(N - 2)!}{f_\lambda N \lambda(\lambda_1 - 1)}.
$$

If $l_1 = \lambda_1 - 1$, $l'_1 = l_2 = \lambda_1$, the Kronecker symbols are satisfied by the permutations (A9) with the same restrictions to $P'$ and $i'_k$ as in the previous case. Since $P_{\lambda_1 \lambda_1 - 1}P_{\lambda_1 \lambda_1 + 1} = P_{\lambda_1 \lambda_1 + 1}P_{\lambda_1 - 1 \lambda_1 + 1}$, $n_{ex} = n'_{ex} = k_m + 1$, unless $i''_k = \lambda_1 + 1$, when $n_{ex} = n'_{ex} = k_m$, as in the previous case, and

$$
\Sigma_{jj'jj'}^{(S-1,S)}(\lambda_1 - 1, \lambda_1, \lambda_1 + 1, \lambda_1) = \Sigma_{jj'jj'}^{(S-1,S)}(\lambda_1, \lambda_1 - 1, \lambda_1, \lambda_1 + 1).
$$

The last relevant case is $S' = S - 2$. If $l_1 = l_2 = \lambda_1 - 1$, $l'_1 = l'_2 = \lambda_1$, the Kronecker symbols in Eq. (A3) select the permutations of the form (A1) with $P'$ which do not affect $l_1$ and $l'_1$ and $l_1 \neq i'_k \neq l'_2$. Then

$$
\Sigma_{jj'jj'}^{(S-2,S)}(\lambda_1 - 1, \lambda_1, \lambda_1 - 1, \lambda_1) = (N - 2)! \sum_{n_{ex} = 0}^{\lambda_2} (\lambda_1 - 2)! \lambda_2! \left( \frac{\lambda_2 - 1}{n_{ex}} \right) \left( \frac{\lambda_1 - 1}{n_{ex}} \right)^{-1} \left( \frac{\lambda_2 - 1}{n_{ex}} \right)^{-1} = \frac{N!(N - 2)!}{f_\lambda N \lambda(\lambda_1 - 1)}.
$$

If $l_1 = l'_2 = \lambda_1 - 1$, $l'_1 = l_2 = \lambda_1$, the Kronecker symbols are satisfied by the permutations (A8) with the same restrictions to $P'$ and $i'_k$ as in the previous case. Now $P_{l_1l_2}$ is a transposition within the same row of the Young tableaux $[0]$ of the shapes $\lambda$ or $\lambda' = [\lambda_1 - 2, \lambda_2 + 2]$. Therefore

$$
\Sigma_{jj'jj'}^{(S-2,S)}(\lambda_1 - 1, \lambda_1, \lambda_1 - 1, \lambda_1 - 1) = \Sigma_{jj'jj'}^{(S-2,S)}(\lambda_1, \lambda_1 - 1, \lambda_1, \lambda_1 - 1) \text{ in Eq. (35).}
$$

If $j_1 = j_2$, but $j_1 \neq j'_2 \neq j'_1$,

$$
\Sigma_{jj'jj'}^{(S)}(l, l') = \sum_{l \neq l' \neq l''} D_{[0][0]}^{[\lambda]}(P(l'')) \sum_{x, y, z} \delta_{l, x} \delta_{l', y} \delta_{l'', z} = (N - 3)! \sum_{l \neq l' \neq l''} D_{[0][0]}^{[\lambda]}(P(l'')) = -1/\lambda_1 \text{[see Eq. (A2)]. Then}
$$

Similarly,

$$
\Sigma_{jj'jj'}^{(S)}(l, l') = \Sigma_{jj'jj'}^{(S)}(l, l') = \Sigma_{jj'jj'}^{(S)}(l, l') = \Sigma_{jj'jj'}^{(S)}(l, l').
$$

If $l = \lambda_1 - 1, l' = \lambda_1$, for $l'' \leq \lambda_1 - 2, l$ and $l''$ are in the same row of the Young tableau $[0]$ of the shape $\lambda$, and $D_{[0][0]}^{[\lambda]}(P(l'')) = 1$. For $l'' > \lambda_1$, $l$ and $l''$ are in different rows of this Young tableau, and $D_{[0][0]}^{[\lambda]}(P(l'')) = -1/\lambda_1$ [see Eq. (A2)]. Then

$$
\Sigma_{jj'jj'}^{(S)}(\lambda_1 - 1, \lambda_1) = (N - 3)! \left( \frac{\lambda_1 - 2 - \lambda_2}{\lambda_1} \right).
$$

Similarly,

$$
\Sigma_{jj'jj'}^{(S)}(\lambda_1 + 1, \lambda_1 + 2) = (N - 3)! \left( \frac{\lambda_2 - 2}{\lambda_1} \right)
$$

in Eq. (35).

If $j_1 \neq j_2 \neq j'_2 \neq j'_1$, we have

$$
\Sigma_{jj'jj'}^{(S)}(\lambda_1 - 1, \lambda_1) = \sum_{l \neq l' \neq l''} D_{[0][0]}^{[\lambda]}(P(l'')) \sum_{x, y, z} \delta_{l, x} \delta_{l', y} \delta_{l'', z} = (N - 3)! \sum_{l \neq l' \neq l''} D_{[0][0]}^{[\lambda]}(P(l'')) = -1/\lambda_1 \text{[see Eq. (A2)]. Then}
$$

Similarly,

$$
\Sigma_{jj'jj'}^{(S)}(\lambda_1 - 1, \lambda_1) = 4(N - 3)! \left( \frac{\lambda_1 - 2 - \lambda_2}{\lambda_1} \right)
$$

in Eq. (35).

$$
\Sigma_{jj'jj'}^{(S)}(\lambda_1 + 1, \lambda_1 + 2) = (N - 3)! \left( \frac{\lambda_2 - 2}{\lambda_1} \right)
$$

in Eq. (35).

Appendix B: Calculation of the sums (32)

The sum (32) is expressed using the relations (23) and (18) as

$$
\Sigma_{jj'jj'}^{(S)}(l, l') = \sum_{P} D_{[0][0]}^{[\lambda]}(P) \Sigma_{jj'jj'}^{(S)}(P(l'')) \delta_{l, x_1} \delta_{l', x_2} \delta_{l'', x_3}.
$$

Only the sums with $j_1 \neq j'_1$ and $j_2 \neq j'_2$ are used here. This implies $l \neq l'$. If $j_1 = j_2$ and $j_1' = j_2'$, or $j_1 = j_2$ and $j_1' = j_2'$, there are $(N - 2)!$ permutations $P$ which satisfy the Kronecker symbols, and the identity (24) leads to

$$
\Sigma_{jj'jj'}^{(S)}(l, l') = \Sigma_{jj'jj'}^{(S)}(l, l') = (N - 2)! D_{[0][0]}^{[\lambda]}(P(l'')).
$$

Therefore

$$
\Sigma_{jj'jj'}^{(S)}(\lambda_1 - 1, \lambda_1) = \Sigma_{jj'jj'}^{(S)}(\lambda_1 + 1, \lambda_1 + 2) = (N - 2)!,
$$

since $l$ and $l'$ are in the same row of the Young tableau $[0]$ of the shape $\lambda$ [see Eq. (1.8)]. Then

$$
\Sigma_{2}^{(S)}(\lambda_1 - 1, \lambda_1) = \Sigma_{2}^{(S)}(\lambda_1 + 1, \lambda_1 + 2) = 2(N - 2)! \text{ (B1)}
$$

in Eq. (35).
\[
\Sigma_{j_1j_2j_3j_4}(l, l') = \sum_{l \neq l' \neq l' \neq l'' \neq l} \frac{D^{X}_{[0][0]}(P)}{\rho} \sum_{i j_1j_2j_3j_4} \delta_{i, i'} \delta_{l', l''} \delta_{j', j''} \delta_{p_j, p_{j'} \delta_{i', i''}} = (N-4)! \sum_{l \neq l' \neq l'' \neq l' \neq l''} \sum_{l \neq l' \neq l'' \neq l} \frac{D^{X}_{[0][0]}(P)}{\rho}.
\]

Using the same values of the Young matrix elements as in the previous case, one gets

\[
\Sigma_{4}^{(s)}(\lambda_1 - 1, \lambda_1) = \Sigma_{4}^{(s)}(\lambda_1 - 1, \lambda_1) = (N-4)! \left[ (\lambda_1 - 2)(\lambda_1 - 3) + \lambda_2(\lambda_2 - 1) - 2(\lambda_2 - 2) \frac{\lambda_2}{\lambda_1} \right]
\]

(B3)

Appendix C: Calculation of the sums (A4)

The sum (A4), expressed as

\[
\Sigma_{j_1j_2j_3j_4}^{(s', s)}(l, l') = (N-1)! \sum_{l \neq l'} D_{[0][0]}^{X}(R) D_{[0][0]}^{X} \delta_{l', \lambda_1},
\]

(C1)

is denoted for consistency with the sum \(\Sigma_{j_1j_2j_3j_4}^{(s', s)}\) [see Eq. (I.1.1)], which is its special case for \(l = l' = \lambda_1\). Since \(\frac{D_{[0][0]}^{X}(R)}{\rho} = \frac{D_{[0][0]}^{X}(R)}{\rho} = \delta_{l', l} \rho_{l', \lambda_1} \), the substitution \(\rho = \rho_{l', \lambda_1} \) transposes arguments \(l\) and \(l'\). Therefore

\[
\Sigma_{j_1j_2j_3j_4}^{(s', s)}(l, l') = \Sigma_{j_1j_2j_3j_4}^{(s', s)}(l', l).
\]

(C2)

Consider at first the case \(S = S'\). If \(l = l' \leq \lambda_1\), the Kronecker symbol in Eq. (C1) selects the permutations of the form (A1) with \(P'\) which do not affect \(l\) and \(l'\). Therefore there are \((\lambda_1 - 1)!\) permutations \(P'\), \(\lambda_2!\) permutations \(P''\), and number of distinct choices of the sets of \(i'_{k}\) and \(i''_{k}\) are given by the binomial coefficients \(\left(\frac{\lambda_1}{n_{ex}}\right)^{-1}\) and \(\left(\frac{\lambda_2}{n_{ex}}\right)^{-1}\), respectively. Then this sum, being independent of the particular values of \(l\), is equal

\[
\Sigma_{j_1j_2j_3j_4}^{(s', s)}(l, l') = (N-1)! \sum_{n_{ex}=0}^{\lambda_2} \lambda_1(\lambda_2 - 1)! \left(\frac{\lambda_1}{n_{ex}}\right) \left(\frac{\lambda_2 - 1}{n_{ex}}\right)^{-2} = (N-1)!((\lambda_2 - 1))^{2} \sum_{n_{ex}=0}^{\lambda_2-1} \frac{(\lambda_1 - n_{ex})!}{(\lambda_2 - 1 - n_{ex})!} \frac{1}{f_{S} \lambda_2} \left[ 1 - \frac{1}{\lambda_1 - \lambda_2 + 2} \right].
\]

(C4)

If \(l \neq l'\), but \(l > \lambda_1\) and \(l' > \lambda_1\), the Kronecker symbol in Eq. (C1) selects permutations (C3) with the same restrictions to \(P'\) and \(P''\) as in the case of \(l = l' > \lambda_1\). Since both \(l\) and \(l'\) are in the second row of the Young tableau \([0]\), \(\Sigma_{j_1j_2j_3j_4}^{(s', s)}(l, l') = \Sigma_{j_1j_2j_3j_4}^{(s', s)}(l, l)\) in this case.

If \(l = l' > \lambda_1\), the Kronecker symbol in Eq. (C1) selects the permutations (A1) with \(P''\) which do not affect \(l\) and \(l'\). There are \(\lambda_1!\) permutations \(P'\), \(\lambda_2!\) permutations \(P''\), and number of distinct choices of the sets of \(i'_{k}\) and \(i''_{k}\) are given by the binomial coefficients \(\left(\frac{\lambda_1}{n_{ex}}\right)^{-1}\) and \(\left(\frac{\lambda_2}{n_{ex}}\right)^{-1}\), respectively. Then Eq. (A4) can be represented as,

\[
\Sigma_{j_1j_2j_3j_4}^{(s', s)}(l, l') = (N-1)! \sum_{n_{ex}=0}^{\lambda_2} (\lambda_1 - 1)! \lambda_2! \left(\frac{\lambda_1}{k_{m}}\right) \left(\frac{\lambda_2 - 1}{k_{m} + 1}\right)^{-2} \left(\frac{\lambda_2 - 1}{k_{m}}\right)^{-2} = \frac{N!(N-1)!}{f_{S} \lambda_1} \left(\frac{\lambda_1 - \lambda_2 + 2}{\lambda_1 - \lambda_2 + 2}\right).
\]
The next case is $S' = S - 1$. If $l = l' = \lambda_1$, the sum was calculated in Appendix in Ref. [1],
\[
\Sigma_{ij}^{(S-1,S)}(\lambda_1, \lambda_1) = \sum_{\rho} D_{ij_0}^{[\lambda]}(\rho) D_{\lambda_1\rho}^{[\lambda']}(\rho) = \frac{N!(N-1)!}{f_S \lambda_1}.
\]

If $l = l' = \lambda_1 - 1$, the Kronecker symbol in Eq. (C1) selects permutations
\[
R = \prod_{k=1}^{k_m} P'_{\lambda_k} P_{\lambda_k'} P_{\lambda_k''} P''_{\lambda_k'},
\]
if $P'$ are permutations of the $\lambda_1 - 2$ first symbols, $P''$ are permutations of the $\lambda_2$ last symbols, $i_0 \neq \lambda_1 - 1, i_k' \leq \lambda_1 - 2$, and $i_k'' > \lambda_1$. The numbers of transpositions $n_{ex}$ and $n'_{ex}$, between rows of the Young tableaux $[0]$ of the shapes $\lambda$ and $\lambda'$, respectively, depend on $i_0$. If $i_0 = \lambda_1$, $n_{ex}(i_0) = n'_{ex}(i_0) = k_m$. If $i_0 \leq \lambda_1 - 2$, $n_{ex}(i_0) = k_m$, $n'_{ex}(i_0) = k_m + 1$, unless $i_0 = i_k'$ for any $k$. In the last case, $n_{ex}(i_0) = k_m$ and, since $P_{\lambda_k} P_{\lambda_k'} P_{\lambda_k''} = P_{\lambda_k} P_{\lambda_k'} P_{\lambda_k''}$, $n'_{ex}(i_0) = k_m$. Similarly, if $i_0 > \lambda_1$, $n_{ex}(i_0) = k_m$, $n_{ex}(i_0) = k_m + 1$, unless $i_0 = i_k''$ when $n_{ex}(i_0) = k_m$. Thus for $2k_m + 1$ values of $i_0$ we have $n_{ex}(i_0) = n'_{ex}(i_0) = k_m$, for $\lambda_2 - k_m$ values $n_{ex}(i_0) = k_m$, $n'_{ex}(i_0) = k_m + 1$, and for $\lambda_2 - k_m$ values $n_{ex}(i_0) = k_m + 1, n'_{ex}(i_0) = k_m$. Then the sum (A4) is expressed as
\[
\Sigma_{ij}^{(S-1,S)}(\lambda_1 - 1, \lambda_1 - 1) = (N - 1)! \sum_{k_m=0}^{k_m} (\lambda_2 - 2)!) \lambda_2! \left[ \lambda_1 \choose k_m \right] \left( \lambda_1 \choose k_m \right) \left( 2k_m + 1 \choose k_m \right) (\lambda_1 - 1)\left( \lambda_1 - 1 \right) - (\lambda_1 - 2 - k_m) (\lambda_1 - 1)\left( \lambda_1 - 1 \right) - (\lambda_2 - k_m) (\lambda_1 - 2) (\lambda_1 - 1)\left( \lambda_1 - 1 \right)\left( \lambda_1 - 1 \right) = \frac{N!(N-1)!}{f_S \lambda_1 (\lambda_1 - 1)^2}.
\]

If $l = \lambda_1, l' = \lambda_1 - 1$, the Kronecker symbol in Eq. (C1) selects permutations (C3) if $P'$ are permutations of the $\lambda_1 - 1$ first symbols, $\bar{P}''$ are permutations of the $\lambda_2$ last symbols, $i_k' \leq \lambda_1 - 1$, and $i_k'' > \lambda_1$. Now $n_{ex} = k_m$, $n_{ex} = k_m + 1$ unless $i_k' = \lambda_1 - 1$ for any $k$, when $n_{ex} = n'_{ex} = k_m$. Then
\[
\Sigma_{ij}^{(S-1,S)}(\lambda_1, \lambda_1 - 1) = (N - 1)! \sum_{k_m=0}^{k_m} (\lambda_2 - 2)!) \lambda_2! \left[ \lambda_1 \choose k_m \right] \left( \lambda_1 \choose k_m \right) \left( 2k_m + 1 \choose k_m \right) (\lambda_1 - 1)\left( \lambda_1 - 1 \right) - (\lambda_1 - 2 - k_m) (\lambda_1 - 1)\left( \lambda_1 - 1 \right) - (\lambda_2 - k_m) (\lambda_1 - 2) (\lambda_1 - 1)\left( \lambda_1 - 1 \right)\left( \lambda_1 - 1 \right) = -\frac{N!(N-1)!}{f_S \lambda_1 (\lambda_1 - 1)^2}.
\]

A general relation can be derived for $l \geq \lambda_1 + 1$ and arbitrary $l'$, when the Kronecker symbol in Eq. (C1) is satisfied by
\[
R = \prod_{\nu} P_{\nu} N P_{\nu N}
\]
with arbitrary $\rho \in \mathcal{S}_{N-1}$, such that $\rho N = N$. Then $D_{[\lambda]}^{[\lambda]}(\rho) = D_{[\lambda]}^{[\lambda]}(\rho)$, since both $l$ and $N$ are in the second row of the Young tableau $[0]$ of the shape $\lambda$ [see Eq. (1.8)]. If $S' < S$ and $\lambda_1 < \lambda_2$, $D_{[\lambda]}^{[\lambda]}(\rho) = D_{[\lambda]}^{[\lambda]}(\rho)$ for the same reason. Using Eq. (23), the sum can be expressed as
\[
\Sigma_{ij}^{(S',S)}(l, l') = (N - 1)! \sum_{\rho, \rho'} D_{[\lambda]}^{[\lambda]}(\rho) D_{[\lambda]}^{[\lambda]}(\rho) \prod_{\rho \in \mathcal{S}_{N-1}} D_{[\lambda]}^{[\lambda]}(\rho) D_{[\lambda]}^{[\lambda]}(\rho).
\]
As $\rho$ are elements of the subgroup $\mathcal{S}_{N-1}$ of permutations of $N - 1$ first symbols, a reduction to subgroup (see [4]) can be used, $D_{[\lambda]}^{[\lambda]}(\rho) = D_{[\lambda]}^{[\lambda]}(\rho)$, where the Young tableaux $\bar{\tau}$ and $\bar{l}$, corresponding to the same Young diagram, $\lambda$, are obtained by the removal of the symbol $N$ from the tableaux $\tau$ and $t$, respectively. $(D_{[\lambda]}^{[\lambda]}(\rho) = 0$ if $\bar{\tau}$ and $\bar{t}$ correspond to different Young diagrams due to different placement of the symbol $N$ in $\tau$ and $t$.) The summation over $\rho$ can be then performed using the orthogonality relation (see [4, 5])
\[
(D_{[\lambda]}^{[\lambda]}(\rho) D_{[\lambda]}^{[\lambda]}(\rho)) = \frac{N!}{f_{\lambda_1-\lambda_2} (N-1)!} \delta_{\lambda_1-\lambda_2} \delta_{\bar{\lambda}_1-\bar{\lambda}_2},
\]
where $f_S(N-1)$ is the representation dimension for $N - 1$ particles. The symbol $N$ is placed in the end of the second row in the Young tableau $[0]$. Therefore, $\lambda = [\lambda_1, \lambda_2 - 1], \lambda' = [\lambda_1', \lambda_2' - 1]$, and $\bar{\lambda} = \bar{\lambda}'$ only if $\lambda_1 = \lambda_1'$, or $S = S'$. As a result,
\[
\Sigma_{ij}^{(S',S)}(l, l') \propto \delta_{\lambda\lambda'} = 0
\]
whenever $l \geq \lambda_1 + 1$ and $S' < S$. Due to Eq. (C2), the sum vanish whenever $l' \geq \lambda_1 + 1$ for arbitrary $l$ too.
This general relation provides the sums appearing in the present calculations

\[
\Sigma_{jj}^{(S-1,S)}(\lambda_1-1,\lambda_1+1) = \Sigma_{jj}^{(S-1,S)}(\lambda_1,\lambda_1+1) = \Sigma_{jj}^{(S-1,S)}(\lambda_1+1,\lambda_1+1) = 0.
\]

Another general relation restricts difference between \( S \) and \( S' \). Equations (23), (C6) , and the reduction to subgroup lead to

\[
D_{[0]}^{[\lambda]}(\mathcal{P}_1[\mathcal{N}])D_{[0]}^{[\lambda]}(\mathcal{P}_1[\mathcal{N}]) D_{[t]}^{[\lambda]}(\mathcal{P}_1[\mathcal{N}]) = \sum_{t',t} D_{[t]}^{[\lambda]}(\mathcal{P}_1[\mathcal{N}]) D_{[t']}^{[\lambda]}(\mathcal{P}_1[\mathcal{N}]) D_{[t']}^{[\lambda]}(\mathcal{P}_1[\mathcal{N}]).
\]

Then the sum (C1) contains

\[
\sum_{\mathcal{P}_1[\mathcal{N}]} D_{[t]}^{[\lambda]}(\mathcal{P}_1[\mathcal{N}]) D_{[t]}^{[\lambda]}(\mathcal{P}_1[\mathcal{N}]) \propto \delta_{\lambda\lambda'}.
\]

[see Eq. (C7)]. Now the symbol \( N \) can be placed in the end of either row of the Young tableaux. Then both \( \lambda = [\lambda_1, \lambda_2-1] \) and \( \lambda = [\lambda_1-1, \lambda_2] \) are allowed, as well as two similar \( \lambda' \). Therefore, \( |\lambda_1 - \lambda'_1| \leq 1 \), or \( |S - S'| \leq 1 \). This provides the sums appearing in the present calculations

\[
\Sigma_{jj}^{(S-2,S)}(\lambda_1-1,\lambda_1-1) = \Sigma_{jj}^{(S-2,S)}(\lambda_1-1,\lambda_1) = 0.
\]

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