Correction: Non-Self-Adjoint Toeplitz Matrices Whose Principal Submatrices Have Real Spectrum

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Abstract
We announce an error in the proof of Theorem 8 of Constr. Approx. 48(2) (2019) 191–226.

Correction to:
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In [3, Eq. (7)], the false equality
\[ \langle u, T_n(a)u \rangle = \langle fu, \bar{a}fu \rangle \gamma \]
has been used in the course of the proof of Theorem 8. If the terms are interpreted using the notation from [3], the right-hand side equals the integral
\[ \frac{1}{2\pi i} \int_{-\pi}^{\pi} \frac{a(\gamma(t))fu(\gamma(t))\bar{fu}(\gamma^*(t))\dot{\gamma}(t)}{\gamma(t)} \, dt, \]
while the left-hand side coincides with \( \langle u, T_n^*(a)u \rangle \) and can be expressed as the integral
\[ \frac{1}{2\pi i} \int_{-\pi}^{\pi} \frac{a(\gamma^*(t))fu(\gamma(t))\bar{fu}(\gamma^*(t))\dot{\gamma}(t)}{\gamma(t)} \, dt. \]

The two integrals do not coincide in general. Unfortunately, this error is an essential problem for the idea of the proof of Theorem 8, that was based on a contour integral.

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representation of the quadratic form of a Toeplitz matrix $T_n(a)$, where the integration path is chosen as the Jordan curve $\gamma$ on which the symbol $a$ is real-valued.

Recall that [3, Thm. 1], which is the main result of the paper, claims that the following 3 statements are equivalent:

(i) $\Lambda(b) \subset \mathbb{R}$,
(ii) $b^{-1}(\mathbb{R})$ contains a Jordan curve,
(iii) $\text{spec}(T_n(b)) \subset \mathbb{R}$ for all $n \in \mathbb{N}$,

where $b$ is a Laurent polynomial, $T_n(b)$ the $n \times n$ Toeplitz matrix given by the symbol $b$, and $\Lambda(b)$ is the set of limit points of eigenvalues of $T_n(b)$, as $n \to \infty$; see [3] for details.

The logic of the proof of Theorem 1 was to establish implications (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) $\Rightarrow$ (i). The currently unproven Theorem 8 was used to prove (ii) $\Rightarrow$ (iii). As we are not able to find an alternative proof at the moment, the implication (ii) $\Rightarrow$ (iii) remains unproven.

However, we strongly believe that the currently unproven implication, or at least its weaker form (ii) $\Rightarrow$ (i), is true. The implication (ii) $\Rightarrow$ (iii) is supported by numerous numerical experiments that we have performed. Professors Yi Zhang and Hao-Yan Chen, to whom we are sincerely grateful for noticing the latter error in the proof, expressed a similar opinion.

Recall, by [3, Def. 2], that the Laurent polynomial $b$ is said to belong to the class $\mathcal{R}$, i.e. $b \in \mathcal{R}$, if and only if $\Lambda(b) \subset \mathbb{R}$. We list several places in [3], where arguments based on the claim of [3, Thm. 8] have been used:

(a) Remark 10.

(b) Sec. 4.1: Example 1. The symbol $b(z) = z^{-1} + az$, where $a \in \mathbb{C}\{0\}$, belongs to the class $\mathcal{R}$, if and only if $a > 0$.

(c) Sec. 4.2: Example 2. The symbol $b(z) = z^{-1} + az + \beta z^2$, where $a \in \mathbb{C}$ and $\beta \in \mathbb{C}\{0\}$, belongs to the class $\mathcal{R}$, if and only if $\beta \in \mathbb{R}\{0\}$ and $a^3 \geq 27\beta^2$.

(d) Sec. 4.3: Example 3. The symbol $b(z) = z^{-r}(1 + az)^{r+s}$, where $r, s \in \mathbb{N}$ and $a \in \mathbb{R}\{0\}$, belongs to the class $\mathcal{R}$.

(e) The second paragraph of Sec. 4.4.

The claim of [3, Rem. 10] was drawn as a direct consequence of Theorem 8 and remains open, too. The other points (b)–(e) comprise concrete examples of symbols, which belong to $\mathcal{R}$ for specific restrictions of the parameters. As the main argument for these claims, we found in [3] an explicit parametrization of the Jordan curve $\gamma$, for which $b \circ \gamma$ is real-valued in each of the cases. We will prove at least partly the claims without using Theorem 8.

The equivalence in (b) can be verified directly since the eigenvalues of $T_n(b)$ can be computed fully explicitly

$$\lambda_k = 2(-1)^n \sqrt{a} \cos \frac{\pi k}{n+1}, \quad k = 1, 2, \ldots, n,$$

with the standard branch of the square root. This example also exhibits (iii), if $a > 0$.

The point (c) is left conjectural as inessential for the paper in its generality. Its relevant particular case, $b \in \mathcal{R}$ for $\alpha = 3a^2$ and $\beta = a^3$, with $a \in \mathbb{R}\{0\}$, will be checked as a a special case of the point (d) in the end of this corrigendum. Lastly, a
description of a possible construction of further examples of not necessarily banded Toeplitz matrices with real spectra from the point (e) is heavily based on the falsely proven Theorem 8 and also remains open at this point.

As a final correction, we briefly indicate the proof of the implication

\[ b(z) = \frac{1}{z^r} (1 + az)^{r+s} \text{ with } r, s \in \mathbb{N} \text{ and } a \in \mathbb{R} \setminus \{0\} \Rightarrow b \in \mathcal{R}. \]

without using the fact that \( b \) is real on a Jordan curve. This is the claim (d). In fact, one can prove the stronger claim that the eigenvalues of \( T_n(b) \) are all real, for all \( n \in \mathbb{N} \), i.e. (iii). The argument relies on the theory of oscillatory matrices [2] and has been used for the special case \( r = s = 1 \) in the proof of [1, Thm. 2.8]. Without loss of generality, we may assume \( a = 1 \) since two Toeplitz matrices \( T_n(b) \) and \( T_n(b_a) \), where \( b_a(z) := b(az) \), are similar, if \( a \neq 0 \). Thus, suppose \( b(z) = z^{-r} (1 + z)^{r+s} \). Notice the bidiagonal matrix \( T_n(1 + z) \) is totally non-negative, see [2, Def. 4, p. 74]. Since \( T_n(b) \) is a submatrix of \( T_n^{r+s} (1 + z) \), it is also totally non-negative; see [2, p. 74]. Further, without going into details, let us mention the determinant of \( T_n(b) \) can be explicitly computed in terms of binomial coefficients as follows:

\[
\det T_n(b) = \prod_{j=0}^{r-1} \frac{(n + s + j)}{s} \binom{s + j}{j} = \prod_{j=0}^{r-1} \frac{(n + s + j)!}{(n + j)!(s + j)!}.
\]

The determinant is obviously non-vanishing and therefore \( T_n(b) \) is non-singular. By [2, Thm. 10, p. 100], \( T_n(b) \) is oscillatory and hence eigenvalues of \( T_n(b) \) are all positive (and simple), see [2, Thm. 6, p. 87].

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