Lost in translation: topological singularities in group field theory

Razvan Gurau

Perimeter Institute for Theoretical Physics, Waterloo, ON N2 L 2Y5, Canada
E-mail: rgurau@perimeterinstitute.ca

Received 8 June 2010, in final form 29 September 2010
Published 15 November 2010
Online at stacks.iop.org/CQG/27/235023

Abstract
Random matrix models generalize to group field theories (GFTs) whose Feynman graphs are dual to gluings of higher dimensional simplices. It is generally assumed that GFT graphs are always dual to pseudo manifolds. In this paper, we prove that already in dimension three (and in all higher dimensions), this is not true due to subtle differences between simplicial complexes and gluings dual to GFT graphs. We prove however that, fortunately, the recently introduced ‘colored’ GFT models (Gurau R 2009 arXiv:0907.2582 [hep-th]) do not suffer from this problem and only generate graphs dual to pseudo manifolds in any dimension.

PACS numbers: 04.60, Pp, 04.60, −m, 02.40.Re

1. Introduction: group field theory

Group field theories (GFTs) [2, 3] or [4–6] are quantum field theories over group manifolds. They generalize random matrix models and random tensor models [7, 8] (see also [9, 10]). GFTs encode much of the mathematical structure common to several discrete approaches to quantum gravity such as Regge calculus [11], dynamical triangulations [12] or spin foam models [13] (see [14] for further details) providing a possible common framework.

The Feynman graphs of GFT are built from vertices encoding the connectivity dual to an \( n \) simplex, and propagators encoding the connectivity dual to the gluing of \( n \) simplices along the boundary \( (n - 1) \) simplices. A graph is dual to a ‘gluing of simplices’ yielding some \( n \)-dimensional topological space, and GFTs generalize the familiar matrix models [15] to a theory of random higher dimensional topological spaces.

In discrete approaches to quantum gravity [14, 16, 17], a gluing of simplices is interpreted as a spacetime background making GFT a combinatorial, background independent theory, whose perturbative development generates spacetimes. This is further supported as, for the simplest choice of vertex and propagator, the Feynman amplitude of a graph reproduces the
partition function of a BF theory discretized on the gluing of simplices \([2, 18–20]\). Moreover, in \([24]\), the discretized BF path integral is recovered for general GFT graphs (representing spacetimes with boundaries). BF theory becomes Einstein gravity after implementing the Plebanski constraints and it is natural to suppose that a some more involved GFT model will reproduce the partition function of the latter. Working at the level of individual GFT graphs (spin foams), one has a good control over the constraints, and their implementation leads to several alternative propositions ([25, 26] or [27, 28]) of vertex kernels. The semiclassical limit [29, 30] of these models has been analyzed with encouraging results, and related, albeit in a slightly artificial three-dimensional framework [31], to ‘graviton’ propagation. Alternatively, one can try to implement the constraints directly at the level of the action ([32, 33] or [6, 24, 34]) or include matter fields [35–37]. Effective field theories which admit an interpretation as non-commutative matter fields have been derived [38, 39] by developing the GFT action around particular instanton solutions. Recently, GFTs and spin foams have been adapted to the study of loop quantum cosmology [40, 41].

Irrespective of the details, all discrete approaches to quantum gravity build a quantum superposition (a statistical average) of geometry micro states. As a function of the geometry micro states one considers, the fundamental question in all such approaches is ‘to sum or not to sum?’. According to the answer to this question, one distinguishes several possibilities.

Regge calculus and spin foam models sum over metrics at fixed triangulation, dynamical triangulations sum over subclasses of triangulations\(^2\) at fixed topology, while GFTs sum over everything. The weights of different topologies, triangulations and metrics are completely fixed by the Feynman rules. In two dimensions, GFTs reduce to matrix models some of which [42, 43] are ultraviolet complete [44, 45]. This opens up the tantalizing possibility that some particular GFT models\(^3\) could be consistent and complete quantum field theories.

The scenario of GFT as a fundamental quantum field theory recently received renewed attention. Partial power counting theorems [46] and upper bounds for the graph amplitudes [47] have been obtained for the simplest GFT models. For the recently introduced ‘colored GFTs’ [1, 48], bounds [49] as well as estimates (relying on a linearization of the base group) [50] have been obtained. The linearized estimates have been reproduced for non-colored models [51].

However, there is a fundamental aspect of GFTs which has been little addressed so far but has the potential to completely invalidate them: the topology of the gluings dual to GFT graphs. It has been noted from some time [52] that GFTs generate not only manifolds but also pseudo manifolds\(^4\). As it is clear that spacetime is a manifold, this is a rather unpleasant feature of GFTs. However it is not critical: pseudo manifolds are related (one to one) to manifolds with boundary\(^5\), and gravity makes perfect sense on the latter. In a highly conjectural approach [53], the topological defects of pseudo manifolds are interpreted as matter coupled to the gravitational background.

This paper addresses, in its full generality, the problem of the topology of gluings dual to GFT graphs. An in-depth study of this question will reveal a very serious issue which has been largely ignored up to now: in all dimensions, including three, there exist GFT graphs dual to gluings which are not pseudo manifolds but correspond to much more singular topologies.

---

1 In algebraic combinatorics, this leads to new topological invariants [21] and advances on the volume conjecture [22, 23].

2 With metric fixed for a given triangulation.

3 Which should first be proven renormalizable.

4 Manifolds with a specific class on singularities, defined precisely in section 2 below. In three dimensions a pseudo manifold is a manifold with conical singularities.

5 Each connected component of the boundary being the scar left after eliminating the singularity by surgery (two-dimensional closed surfaces in three dimensions).
Figure 1. A graph not dual to a pseudo manifold.

The simplest example of such a graph is presented in figure 1. We will present in detail this example in section 4, but for now it suffices to say that the Euler characteristic of its dual gluing is $-1$. It is a fundamental result (which we recall in section 2 for completeness) that the Euler characteristic of three-dimensional pseudo manifolds is always greater than or equal to zero; thus, this graph cannot correspond to a pseudo manifold.

The pathological singularities we identify in this paper are generic, appearing at arbitrary order in perturbations, and dominate in power counting. The tadpoles and tadfaces discussed in [49] (and proved to yield worse power counting estimates) are a first example of this generic problem. The extended singularities are an important problem for GFTs. To understand why, recall the case of matrix models.

The perturbative development of the identically distributed matrix models is dominated by the planar graphs because they possess the largest number of independent sums (one per face) at fixed order of perturbations. A nontrivial propagator and vertex kernel translate in constraints (and weight) on these independent sums but irrespective of the details, the graph with the largest number of available sums will still dominate the effective regime. Indeed, the Grosse Wulkenhaar model, although a priori very different from the identically distributed matrix models, is dominated by planar graphs [42, 43]. The situation is parallel in GFTs, where (in representation space) divergences arise also from independent sums. Order by order the most divergent graphs of the simplest GFT models, (that is, those with the largest number of independent sums) exhibit extended singularities, and for the same reason as in matrix models will also dominate GFTs with nontrivial vertex and propagator kernels. In retrospect, when compared to these pathologies, the pseudo manifolds seem just a small nuisance one can live with.

As the problems the extended singularities pose do not depend on the details of the propagator and vertex kernels, we will consider in this paper only the simple GFTs associated with the BF theory and refer to them as the ‘usual GFT’ models, as opposed to the ‘colored GFT’ models.

The usual GFTs however miss their target only by an inch. If one assumes that the gluing dual to a GFT graph is a simplicial complex, then all the pathologies disappear and the gluing is a pseudo manifold. In order to improve the effective behavior of the usual models, one must find some way to eliminate the singular topologies. Restricting by some condition at the level of the action the allowed gluings, one can hope that only pseudo manifolds are created. However, finding a good restriction is a subtle question. For instance, requiring that the identification of two $(n - 1)$ simplices respects orientations is largely insufficient: their $(n - 2)$, $(n - 3)$ etc subsimplices are identified in a completely arbitrary way and generate pathologies.

A surprisingly simple solution to this problem is provided by the recently introduced ‘colored’ GFT (CGFT) models. This model completely eliminates all the pathologies.

---

6 They are however only the first naive example: extended singularities appear generically in graphs with neither tadpoles nor tadfaces.

7 Or that it becomes one after subdivision.
yielding only pseudo manifolds in any dimension by a unique prescription. Establishing this result is the ‘raison d’être’ of this paper.

We will prove this in all the technical detail in section 5, but the profound reason which makes the colored models work is very intuitive. If we denote a strand (a solid line in figure 2) by the colors of the two halflines to which it belongs, the colored GFT lines will always conserve the labels of the strands. In turn, this will guarantee that all subsimplices (of any dimension) are identified respecting their orientations. In retrospect the colored prescription is very natural: it is the simplest one which ensures this. We view this result as a very strong argument in support of the colored GFT models.

This paper is organized as follows. In section 2 we review some definitions and classical results concerning normal simplicial pseudo manifolds. In section 3 we describe in detail the simplest GFTs and their graphs, and introduce the link graphs in subsection 3.1. We detail at length the pathological wrapping singularities plaguing the usual GFTs in section 4. In section 5 we recall the colored GFT models and prove that they only generate graphs dual to normal simplicial pseudo manifolds. Finally, in section 6, we review the implications of our result. We will only deal with closed GFT graphs, the generalization to open graphs [48] being immediate.

2. Simplicial pseudo manifolds

In this section we review some definitions and properties of normal simplicial pseudo manifolds (following the notations of [54]) relevant to our subsequent analysis of GFTs.

A finite abstract simplicial complex\(^8\) is a finite set \(A\) together with a collection \(\Delta\) of subsets such that if \(X \in \Delta\) and \(Y \subseteq X\), then \(Y \in \Delta\).

An element \(v \in A\) such that \([v] \in \Delta\) is called a vertex of \(\Delta\), and the set of all vertices of \(\Delta\) is denoted as \(V(\Delta)\). An element \(\sigma \in \Delta\) is called a simplex. The proper subsets \(\tau\) of a simplex \(\sigma\) (\(\tau \subset \sigma\), \(\sigma \setminus \tau \neq \emptyset\)) are called faces or subsimplices of \(\sigma\). Note that \(\Delta\) is not a set but a collection (or a multiset), meaning that the same simplex can appear several times in \(\Delta\). A subcomplex \(\Delta'\) of \(\Delta\) is a simplicial complex such that \(\sigma \in \Delta' \Rightarrow \sigma \in \Delta\). With any simplex in a simplicial complex, one canonically associates several simplicial subcomplexes of \(\Delta\).

- The deletion of \(\tau\) is the abstract simplicial subcomplex of \(\Delta\):
  \[
  dl_\Delta(\tau) = \{\sigma \in \Delta| \tau \not\subseteq \sigma\}. \tag{1}
  \]

- The link of \(\tau\) is the abstract simplicial subcomplex of \(\Delta\):
  \[
  lk_\Delta(\tau) = \{\sigma \in \Delta| \sigma \cap \tau = \emptyset \text{ and } \sigma \cup \tau \in \Delta\}. \tag{2}
  \]

- The closed star of \(\tau\) is the abstract simplicial subcomplex of \(\Delta\):
  \[
  star_\Delta(\tau) = \{\sigma \in \Delta| \sigma \cup \tau \in \Delta\}. \tag{3}
  \]

\(^8\) Or simplicial complex, for brevity.
The link and the closed star of a simplex $\tau$ are related by
\[
\text{lk}_\Delta(\tau) = \text{star}_\Delta(\tau) \cap \text{dl}_\Delta(V(\tau)),
\]
(4)
as $\sigma \not\subseteq \{v\}, \forall v \in V(\tau) \Rightarrow \sigma \cap \tau = \emptyset$.

For any vertex $v$ of $\Delta$, and any simplex $\sigma \in \Delta$, either $\{v\} \not\subseteq \sigma$ or $\{v\} \cup \sigma = \sigma \in \Delta$; hence,
\[
\Delta = \text{star}_\Delta(v) \cup \text{dl}_\Delta(v).
\]
(5)

A simplex $\tau$ of a simplicial complex $\Delta$ has a \textit{dimension} $n$ (it is an $n$ simplex) if it has cardinality $n + 1$. For instance, the vertices of $\Delta$ have dimension 0. We denote the number of simplices of dimension $p$ in $\Delta$ by $f^p(\Delta)$ (hence $f^0(\Delta) = |V(\Delta)|$) and its Euler characteristic by
\[
\chi(\Delta) = \sum_{p \geq 0} (-1)^p f^p(\Delta).
\]
(6)

For any vertex $v$, equations (4) and (5) imply that the Euler characteristic of a simplicial complex respects
\[
\chi(\Delta) = \chi(\text{star}_\Delta(v)) + \chi(\text{dl}_\Delta(v)) - \chi(\text{lk}_\Delta(v))
\]
\[
= 1 - \chi(\text{lk}_\Delta(v)) + \chi(\text{dl}_\Delta(v)),
\]
(7)
where in the last line we used $\chi(\text{star}_\Delta(v)) = 1$ (see equation (6)).

An \textit{n-dimensional simplicial pseudo manifold} is a finite abstract simplicial complex with the following properties.

- It is \textit{non-branching}: each $(n - 1)$ simplex is a face of precisely two $n$ simplices.
- It is \textit{strongly connected}: any two $n$ simplices can be joined by a 'strong chain' of $n$ simplices in which each pair of neighboring simplices have a common $(n - 1)$ simplex.
- It is \textit{pure} (it has \textit{dimensional homogeneity}): each simplex is a face of some $n$ simplex.

A pseudo manifold is called \textit{normal} if all its links are pseudo manifolds. This condition can fail (see equation (6)) because the links of a pseudo manifold, while always being pure, non-branching simplicial complexes, are not in general strongly connected. Crucial in what follows is the following property of three-dimensional normal pseudo manifolds.

\textbf{Proposition 1.} \textit{The Euler character of a three-dimensional normal pseudo manifold $\Delta$ respects}
\[
\chi(\Delta) = |V(\Delta)| - \frac{1}{2} \sum_i \chi(\text{lk}_\Delta(v_i)),
\]
(8)
\textit{and $\chi(\text{lk}_\Delta(v_i)) \leq 2$; thus $\chi(\Delta) \geq 0$.}

\textbf{Proof.} Counting subsets of fixed cardinality shows that a pure simplicial complex respects
\[
\sum_i f^2(\text{lk}_\Delta(v_i)) = 4 f^3(\Delta),
\]
\[
\sum_i f^1(\text{lk}_\Delta(v_i)) = 3 f^2(\Delta),
\]
\[
\sum_i f^0(\text{lk}_\Delta(v_i)) = 2 f^1(\Delta).
\]
(9)

In a non-branching simplicial complex in three dimensions, every 3 simplex is bounded by four 2 simplices and every 2 simplex belongs to exactly two 3 simplices; hence
\[
4 f^3(\Delta) = 2 f^2(\Delta);
\]
(10)
therefore, the Euler characteristic of a three-dimensional pure, non-branching simplicial complex respects
\[
\chi(\Delta) = f^0(\Delta) - \frac{1}{2} \sum_i \chi(\text{lk}_\Delta(v_i)). \tag{11}
\]

If, furthermore, \( \Delta \) is a normal pseudo manifold, all the links of its vertices are two-dimensional pseudo manifolds. A link, \( \text{lk}_\Delta(v_i) \), is strongly connected; hence, there exists a ‘strong tree’ of 1 simplices connecting all its 2 simplices. If one deletes the 1 simplices in the strong tree (and glues the 2 simplices into a patch), the 0 simplices are still connected by (at least a tree of) the remaining 1 simplices; thus
\[
f^1(\text{lk}_\Delta(v_i)) \geq [f^2(\text{lk}_\Delta(v_i)) - 1] + [f^0(\text{lk}_\Delta(v_i)) - 1], \tag{12}
\]
which achieves the proof. \( \square \)

3. GFT graphs

In this section, we detail the Feynman graphs of the usual GFT models in \( n \) dimensions and relate them to normal simplicial pseudo manifolds.

The usual \( n \)-dimensional GFT model is defined for a scalar field \( \phi : G^n \to \mathbb{R} \), (with \( G \) some Lie group), invariant under simultaneous left multiplication of its arguments
\[
\phi(hg_{a_1}, \ldots, hg_{a_n}) = \phi(g_{a_1}, \ldots, g_{a_n}), \quad \forall h \in G. \tag{13}
\]
The field \( \phi \) is sometimes chosen invariant under arbitrary permutations of its arguments [32], [38] or ‘hermitic’ (invariant under even permutations and changing into its conjugate under odd permutations [52]). The GFT action in \( n \) dimensions is [52]
\[
S = \frac{1}{2} \int [dg] \phi_{a_0\alpha_1\ldots\alpha_n} \phi_{a_0\alpha_1\ldots\alpha_n} + S_{\text{int}},
\]
\[
S_{\text{int}} = \frac{\lambda}{n+1} \int [dg] \phi_{a_0\alpha_{0_1}a_{-1}\ldots\alpha_0} \phi_{a_0\alpha_{0_2}a_{-1}\ldots\alpha_1} \cdots \times \phi_{a_0\alpha_{0_n}a_{-1}\ldots\alpha_{n-1}}, \tag{14}
\]
where \( \phi_{a_0\alpha_1\ldots\alpha_n} \equiv \phi(g_{a_0}, g_{a_1}, \ldots, g_{a_n}) \), and \( g_{a_i} = g_{a_i} \) in \( S_{\text{int}} \). Note that for all fields in \( S_{\text{int}} \) the argument \( a_{\rho} \) is absent, that is, the fields have \( n \) arguments (as they must in \( n \) dimensions). The GFT vertex generated by \( S_{\text{int}} \) is represented in figure 3.

Each field \( \phi \) in \( S_{\text{int}} \) is associated with a half line of the GFT vertex. Every two fields in \( S_{\text{int}} \) share a group element; consequently, every two half lines of the GFT vertex share a strand (depicted as a solid line in figure 3). We label the half lines of the GFT vertex 0, 1 up to \( n \) and each strand by the (unordered) couple of labels of the two half lines which share it (that is, the strand \( ij \) is shared by the half lines \( i \) and \( j \)). Note that the two labels of a strand are always different \( i \neq j \); thus, a halfline \( i \) has \( n \) strands, \( 0i, 1i, \ldots, i-1, i, i+1, \ldots, in \).

The GFT vertex \( (0, \ldots, n) \) is dual to an \( n \) simplex \( \{A_0, \ldots, A_n\} \). The half lines of the vertex represent the \( (n-1) \) simplices bounding the \( n \) simplex, namely the half line \( i \) represents the simplex opposite to the vertex \( A_i \):
\[
\{A_0, \ldots, A_n\} \setminus \{A_i\} \equiv \{A_0, \ldots, \hat{A}_i, \ldots, A_n\}. \tag{15}
\]
The strand \( ij \) represents the \( (n-2) \) simplex shared by the two \( (n-1) \) simplices \( \{A_0, \ldots, \hat{A}_i, \ldots, A_n\} \) and \( \{A_0, \ldots, \hat{A}_j, \ldots, A_n\} \), that is,
\[
\{A_0, \ldots, \hat{A}_i, \ldots, \hat{A}_j, \ldots, A_n\}. \tag{16}
\]

Throughout this paper, we denote by a hat the absence of a symbol in a list.
In the following, the GFT vertex will be called a *stranded vertex*, to emphasize its internal strand structure. Any \( p \) subsimplex of an \( n \) simplex is itself dual to some GFT vertex in the appropriate dimension. The precise relation between the GFT vertices dual to subsimplices and the original GFT vertex is detailed in section 3.1.

The GFT propagator, generated by the quadratic part of action (14), connects two GFT vertices via an arbitrary permutation of the strands. Some possible choices of GFT lines are presented in figure 4. The GFT lines represent the identification of two \((n - 1)\) simplices and each permutation of the strands encodes one of the \( n! \) possible ways to do this. Like the GFT vertices, the GFT lines are stranded and have an internal structure.

With a GFT line \( \ell_{v_Av_B} \) connecting two GFT vertices \( v_A \) and \( v_B \) (and oriented from \( v_A \) to \( v_B \)) dual to the \( n \) simplices

\[
\sigma^n_A = \{A_0, \ldots, A_n\}, \quad \sigma^n_B = \{B_0, \ldots, B_n\},
\]

we associate a function \( \ell_{v_Av_B} : [0, \ldots, n] \rightarrow [0, \ldots, n] \) defined as follows. The line connects the half line \( i \) of \( v_A \) with some half line, say \( k \), of \( v_B \). We set \( \ell_{v_Av_B}(i) = k \). Also, the line connects the strand \( ij \) of \( v_A \) to some strand, say \( kl \) of \( v_B \). We set \( \ell_{v_Av_B}(j) = l \). The function \( \ell_{v_Av_B} \) encodes the identification of the two \((n - 1)\) simplices

\[
\{A_0, \ldots, \hat{A}_i, \ldots, A_n\}, \quad \{B_0, \ldots, \hat{B}_{\ell_{v_Av_B}(i)}, \ldots, B_n\},
\]

and all their faces via \( A_j = B_{\ell_{v_Av_B}(j)} \), \( \forall j \neq i \).

The perturbative development of GFT is indexed by stranded Feynman graphs \( \mathcal{G} \) generalizing the ribbon graphs of matrix models. A GFT graph \( \mathcal{G} \) is dual to some gluing of \( n \) simplices, denoted in the following \( \Delta^\mathcal{G} \).

The gluing \( \Delta^\mathcal{G} \) is a collection of \( n \) simplices (and all their faces) modulo the identifications encoded in the lines. Clearly \( X \subseteq \Delta^\mathcal{G} \) and \( Y \subseteq X \), then \( Y \subseteq \Delta^\mathcal{G} \); thus \( \Delta^\mathcal{G} \) is very close to a simplicial complex. However, in general, \( \Delta^\mathcal{G} \) is not a simplicial complex. Performing the
identifications encoded in the lines, one can end up identifying two \textit{a priori} distinct vertices on the same \( n \) simplex. Consequently, the elements \( X \in \Delta^G \) are not sets, but multisets\(^{10} \).

This is not always a problem. It is possible that, even if \( \Delta^G \) is not a simplicial complex, it is still topologically equivalent to some simplicial complex \( \tilde{\Delta}^G \). What is much less obvious is that sometimes \( \Delta^G \) is not equivalent to \textit{any} simplicial complex. This in turn leads to some very pathological singularities.

3.1. Link graphs

The links defined for simplicial complexes generalize immediately to gluings. The link of a \( p \) simplex is a gluing of \( (n - p - 1) \) simplices; hence, it is dual to a GFT graph in \( (n-p-1) \) dimensions. We call this graph a \textit{link graph}. To construct it, consider the \( p \) simplex \( \sigma^p = \{ A_{i_0}, \ldots, A_{i_p} \} \) in a gluing. The contribution of the \( n \) simplex \( \sigma^n = \{ A_0, \ldots, A_n \} \) to its link consists of the simplex \( \sigma^{n-p-1} = \sigma^n \setminus \sigma^p \) and all its faces. The \( n \) simplex \( \sigma^n \) is dual to the GFT vertex \( (0, \ldots, n) \); therefore, \( \sigma^{n-p} \) is dual to the GFT vertex \( (0, \ldots, \hat{i_0}, \ldots, \hat{i_p}, \ldots, n) \) obtained by deleting all the half lines \( i_0, \ldots, i_p \) together with all their strands in the initial GFT vertex \( (0, \ldots, n) \). We call \( (0, \ldots, \hat{i_0}, \ldots, \hat{i_p}, \ldots, n) \) a descendant vertex of \( (0, \ldots, n) \). The \( (n-p-1) \) link graphs are obtained by connecting the descendant \( (n-p-1) \) vertices of all initial \( n \)-dimensional GFT vertices as dictated by the GFT lines.

Consider an example of \( n \)-dimensional GFT whose vertex and dual three simplex (tetrahedron) are presented in figure 5.

A vertex (say \( A_0 \)) of the tetrahedron \( \sigma^3 = \{ A_0, A_1, A_2, A_3 \} \) is opposite to a triangle \( \sigma^2 = \sigma^3 \setminus \{ A_0 \} = \{ A_1, A_2, A_3 \} \). This triangle is represented by a half line (the half line 0) in the GFT graph. Two triangles (say \{ \( A_1, A_2, A_3 \) \} and \{ \( A_0, A_2, A_3 \) \}) share and edge on the tetrahedron (the edge \{ \( A_2, A_3 \) \}). This edge is represented by the strand common to the two half lines (the strand 01, common to the half lines 0 and 1).

Consider an example of the GFT graph in figure 6. Its dual gluing, \( \Delta^G \), consists of two tetrahedra \{ \( A_0, A_1, A_2, A_3 \) \} and \{ \( B_0, B_1, B_2, B_3 \) \}, and four identifications (hence four functions) associated with the lines

\[
\begin{align*}
\ell_{1,4}^1 (2) &= 0; & \ell_{1,4}^1 (1) &= 1; & \ell_{1,4}^1 (0) &= 2; & \ell_{1,4}^1 (3) &= 3, \\
\ell_{2,4}^1 (1) &= 1; & \ell_{2,4}^1 (2) &= 0; & \ell_{2,4}^1 (3) &= 3; & \ell_{2,4}^1 (0) &= 2, \\
\ell_{3,4}^1 (0) &= 2; & \ell_{3,4}^1 (1) &= 1; & \ell_{3,4}^1 (2) &= 0; & \ell_{3,4}^1 (3) &= 3, \\
\ell_{4,4}^1 (3) &= 3; & \ell_{4,4}^1 (0) &= 2; & \ell_{4,4}^1 (1) &= 1; & \ell_{4,4}^1 (2) &= 0,
\end{align*}
\]

(19)

\(^{10}\)To add to the confusion, recall that \( \Delta \) itself \textit{is} a multiset. Its elements \( X \in \Delta \), however, must be sets.
where the first column is the half line of $v_A'$ from which each line originates, and the subsequent columns indicate the various identifications of $A$'s with $B$'s. The line $\ell_1^{(1)}$ for example encodes the identifications
\[
\{A_1, A_0, A_3\} = \{B_1, B_2, B_3\} \quad \{A_1, A_0\} = \{B_1, B_2\} \\
\{A_1, A_3\} = \{B_1, B_3\} \quad \{A_0, A_3\} = \{B_2, B_3\} \\
\{A_1\} = \{B_1\} \quad \{A_0\} = \{B_2\} \quad \{A_3\} = \{B_3\}.
\]
(20)

The reader can convince himself that, after performing also the identifications corresponding to $\ell_2^{(2)}, \ell_3^{(3)}$, and $\ell_4^{(4)}$, the gluing writes
\[
\Delta^\varphi = \emptyset, \{A_0\}, \{A_1\}, \{A_2\}, \{A_3\}, \{A_0, A_1\}, \{A_0, A_2\}, \{A_0, A_3\}, \\
\{A_1, A_2\}, \{A_1, A_3\}, \{A_2, A_3\}, \{A_0, A_1, A_2\}, \\
\{A_0, A_1, A_3\}, \{A_0, A_2, A_3\}, \{A_1, A_2, A_3\}, \\
\{A_0, A_1, A_2, A_3\}. \quad \quad \quad \quad (21)
\]

Note that $\Delta^\varphi$ is a multiset (the two 3 simplices have exactly the same vertices) and one can check that this gluing is a simplicial complex. The link of $A_0$, for instance, writes
\[
\text{lk}_{\Delta^\varphi}(A_0) = \emptyset, \{A_1\}, \{A_2\}, \{A_3\}, \{A_1, A_2\}, \{A_1, A_3\}, \{A_2, A_3\}, \\
\{A_1, A_2, A_3\}. \quad \quad \quad \quad (22)
\]

It is in fact easier to access directly the link graph dual to $\text{lk}_{\Delta^\varphi}(A_0)$ starting from $\varphi$. To build the link graph dual to $\text{lk}_{\Delta^\varphi}(A_0)$, we distinguish the labels on the vertices $V_A$ and $V_B$ by a lower index. Take the descendant vertex $1_42_43_4$ obtained by deleting the half line $0_A$ (and all its strands) of $\{0_A, 1_4, 2_4, 3_4\}$. The half lines of the descendant vertex inherit the labels of the corresponding GFT half lines, $1_A, 2_A$ and $3_A$, and a pair of descendant half lines share a strand ($1_A$ and $2_A$ share the strand $1_42_4$, etc).

The half line $1_A$ of the descendant vertex $1_42_43_4$ connects to the half line $1_B$ of the descendant vertex $1_B0_B3_B$ (obtained by deleting $2_B$) of $0_B1_B2_B3_B$. Similarly, the half line $2_A$ connects to $0_B$ and $3_A$ to $3_B$ of the same descendant vertex $1_B0_B3_B$. The two descendant’s vertices thus form a connected graph, dual to the link $\text{lk}_{\Delta^\varphi}(A_0)$.

By construction, every GFT vertex in three dimensions has four descendant vertices in the link graphs; thus the dual gluing of any GFT graph respects
\[
\sum_i f^2(\text{lk}_{\Delta^\varphi}(v_i)) = 4f^3(\Delta). \quad \quad \quad \quad (23)
\]
Also, any GFT line always has three descendants (any two strands of a GFT line yield a descendant line in some link graph); hence, the dual gluing respects
\[ \sum f^1(\text{lk}_\Delta(v_i)) = 3 f^2(\Delta). \] (24)

Moreover, each strand on the GFT vertex has two descendant strands in the link graphs.

The main result of this section is synthesized in the following lemma.

**Lemma 1.** If a gluing $\Delta^G$ dual to an $n$-dimensional connected GFT graph $\mathcal{G}$ is a simplicial complex, then it is a normal pseudo manifold.

Let us comment on this lemma before proving it. In the mathematical literature, there are numerous results concerning pseudo manifolds (notoriously, for example, in three dimensions they only present isolated singularities). Although these results hold for some graphs, they fail in general. For instance, the Betti numbers and boundary operators, relevant for power counting estimates, can be defined only for graphs dual to pseudo manifolds. They make no sense for arbitrary GFT graphs.

**Proof.** A gluing dual to a connected GFT graph is always pure and strongly connected. The GFT lines either connect two different GFT vertices or are tadpole lines (they start and end on the same GFT vertex). Thus, in the dual gluing the $(n - 1)$ simplices either separate two distinct $n$ simplices or belong twice to the same $n$ simplex.

If an $(n - 1)$ simplex belongs twice to the same $n$ simplex, then in its corresponding gluing at least two a priori distinct vertices of the $n$ simplex are identified. Thus, the $n$ simplex is not represented by a set in $\Delta^G$, but by a multiset and $\Delta^G$ is not a simplicial complex. Consequently, if $\Delta^G$ is a simplicial complex, then all its $(n - 1)$ simplices bound exactly two $n$ simplices; therefore, $\Delta^G$ is non-branching thus a simplicial pseudo manifold.

The link graphs of a GFT graph $\mathcal{G}$ are also GFT graphs (of lower dimensions). If $\mathcal{G}$ has no tadpole lines, none of its links can have tadpole lines (the lines of link graphs are descendants of lines of $\mathcal{G}$). The same reasoning as before holds for all the link graphs. Thus, all the links of $\Delta^G$ are also pseudo manifolds. Therefore, $\Delta^G$ is a normal pseudo manifold. \(\square\)

As a last remark, note that we used the fact that if a GFT graph is a simplicial complex, then it has no tadpole line. If, however, a graph has no tadpole lines, its dual gluing might still not be simplicial complex: two vertices on an $n$ simplex could be identified after a longer sequence of gluings of lines (see section 4 for examples).

### 4. Wrapping singularities in GFT graphs

We will detail the singularities of GFT graphs in three dimensions. We will present several examples of three-dimensional GFT graphs whose dual gluings do not respect proposition 1, namely
\[ \chi(\Delta) \neq |V(\Delta)| - \frac{1}{2} \sum_i \chi(\text{lk}_\Delta(v_i)). \] (25)

As the Euler characteristic is a topological invariant, these gluings are not homeomorphic to pseudo manifolds. We will prove that whenever a GFT graph presents a certain type of singularity (we baptize wrapping singularity) it will not respect proposition 1. We will show that these singularities are generic (they appear at arbitrary high order in perturbations). These problems reappear in all higher dimensions, as similar singularities in the link graphs prevent any higher dimensional gluing from being a normal pseudo manifold.
Consider the GFT graph $\mathcal{G}_1^1$ represented in figure 7. The lines applications are

$$
\ell_{vA}^{(1)}(0) = 1, \quad \ell_{vA}^{(1)}(1) = 0, \quad \ell_{vA}^{(1)}(2) = 3, \quad \ell_{vA}^{(1)}(3) = 2,
\ell_{vA}^{(2)}(2) = 3, \quad \ell_{vA}^{(2)}(0) = 1, \quad \ell_{vA}^{(2)}(1) = 0, \quad \ell_{vA}^{(2)}(3) = 2,
$$

(26)

where, again, the first column presents the half lines identified by the lines $\ell_{vA}^{(1)}$ and $\ell_{vA}^{(2)}$.

Denoting $A_0 = A_1 = \alpha, A_2 = A_3 = \beta$, the dual gluing writes

$$
\Delta^{\mathcal{G}_1^1} = \{ [\alpha, \alpha, \beta, \beta], [\alpha, \alpha, \beta], [\alpha, \beta, \beta], [\alpha, \beta], [\beta], [\alpha], [\beta], \emptyset \}.
$$

(27)

Note that the 3 simplex of this gluing is not a set; hence $\Delta^{\mathcal{G}_1^1}$ is not a simplicial complex. The Euler characteristic of $\Delta^{\mathcal{G}_1^1}$ is $\chi(\Delta^{\mathcal{G}_1^1}) = -1 < 0$ which breaks proposition 1. Therefore, $\Delta^{\mathcal{G}_1^1}$ is a first example of a gluing not homeomorphic to a pseudo manifold.

Proposition 1 fails again for the graph $\mathcal{G}_2^1$ of figure 8, which is in fact related by symmetry to $\mathcal{G}_1^1$. The Feynman amplitude of these graphs is

$$
A_{\mathcal{G}_1^1} = A_{\mathcal{G}_2^1} = [\delta^A(e)]^2,
$$

(28)

where $\delta^A$ is a suitable cutoffed delta function on the group $G$ and $e$ is the identity element of $G$ (see [46, 47] for details on the computation of Feynman amplitudes in GFT).

The graph $\mathcal{G}_3^1$ in figure 9 has a planar link (with Euler characteristic 2) and a non-orientable one (with Euler characteristic 1); hence $V - \frac{1}{2} \sum_i \chi(lk v_i) = 1/2$ which is not even an integer. Its amplitude is

$$
A_{\mathcal{G}_3^1} = \delta^A(e).
$$

(29)

At first order, one also has a GFT graph dual to a gluing homeomorphic to a pseudo manifold (in fact homeomorphic to the manifold $S^3$), presented in figure 10.

The reader should not be distracted by the twists of the lines in the link graphs: they can be undone by flipping either of the end vertices. We prefer to represent the twists explicitly so that the reader can easily identify the descendant vertices in the link graphs. In detail the line applications are

$$
\ell_{vA}^{(1)}(0) = 1, \quad \ell_{vA}^{(1)}(3) = 3, \quad \ell_{vA}^{(1)}(2) = 2, \quad \ell_{vA}^{(1)}(1) = 0,
\ell_{vA}^{(2)}(2) = 3, \quad \ell_{vA}^{(2)}(1) = 1, \quad \ell_{vA}^{(2)}(0) = 0, \quad \ell_{vA}^{(2)}(3) = 2.
$$

(30)
The gluing dual to this graph is written as (denoting $A_0 = A_1 = \alpha$ and $A_2 = A_3 = \beta$)

$$\Delta G^1 = \{[\alpha, \alpha, \beta, \beta], [\alpha, \alpha, \beta], [\alpha, \beta, \beta], [\alpha, \alpha], [\alpha, \beta], [\beta, \beta], [\alpha, \emptyset], \emptyset\},$$

(31)
to be compared with equation (27). This amplitude of $G^4$ is

$$A_{G^4} = \delta^A(e).$$

(32)

The analysis of these first four examples of graphs leads to the flowing conclusions.

- At first order, graphs not dual to pseudo manifolds are larger in power counting than graphs dual to pseudo manifolds. At arbitrary order, a graph obtained by star subdivisions (‘one–four moves’) of $G^1$ will consistently have one extra power of $\delta_A(e)$ with respect to the similar graph obtained from $G^2$.
- Restricting the permutations of strands allowed on the three-dimensional GFT lines does not solve the problem: there exist singular graphs generated by even as well as odd permutations of the strands. Although (as we will see in the following) this idea is part of the solution, by itself it is insufficient.
As all the examples we presented so far exhibit tadpole lines, the reader might still hope that the singularities are just an artifact of these tadpoles. This is not true; the example of figure 11 presents a graph with no tadpole lines, whose links have Euler characteristics 1, 2 and 2; hence \( V - \frac{1}{2} \sum \chi(\text{lk} v_i) = \frac{1}{2} \) again.

Proposition 1 fails for arbitrary GFT graphs because the detailed balance crucial for its proof,

- each GFT vertex has four descendant vertices in the link graphs;
- each GFT line has three descendant lines in the link graphs;
- each GFT face has two descendant faces in the link graphs,

\textit{does not hold} in general. The attentive reader will recall the correct balance encoded in equations (23) and (24),

- each GFT vertex has four descendant vertices in the link graphs;
- each GFT line has three descendant lines in the link graphs;
- each \textit{strand} on a GFT vertex has two descendant \textit{strands} in the link graphs.

The faces are \textit{closed strands}, and in all the singular cases we presented that the two descendants of some strand on a GFT vertex belong to \textit{only one} face in the link graph. For instance, in figure 7 we denoted by \( F_1 \) and \( F_2 \) two GFT faces, and \( f_1 \) and \( f_2 \) their unique descendant faces in the link graphs. The faces of the link graphs \((f_1 \text{ and } f_2)\) wrap twice around the GFT faces \( (F_1 \text{ and } F_2) \), hence the name ‘wrapping singularities’. The reader can check that this phenomenon is present in all the examples we presented.

Whenever such singularities are present, \( \sum f^0(\text{lk}_{x^G}(v_i)) < 2 f^1(\Delta^G) \) and \( \Delta^G \) does not respect proposition 1; hence, it is \textit{not} homeomorphic to a pseudo manifold. The wrapping singularities are generic in GFT: a graph having subgraphs like the ones in figures 12 and 13 will have a wrapping singularity.

In figure 14 we give an example of a four-dimensional graph having a wrapping singularity. The face \( F_{01} \) has only two three-dimensional descendants (instead of three), denoted both by \( f_{01} \), and one of them wraps twice around \( F_{01} \).
The situation looks bleak for GFTs. The graphs with wrapping singularities are large in power counting and generic. Singular graphs dominate the ‘low-energy’ effective behavior of GFTs and render them unsatisfactory.

5. The colored GFT graphs and pseudo manifolds

In this section we prove that the colored GFT (CGFT) model \[1\] completely solves the problem of non-pseudo manifold graphs in a single stroke, in arbitrary dimension. By coloring our quantum field, we introduce a combinatorial constraint in all its graphs and completely eliminate the wrapping singularities. Moreover, once the combinatorial constraints are properly understood, the proof that all CGFT graphs are dual to normal pseudo manifolds is practically tautological. For this reason the CGFT model is, in our opinion, the appropriate GFT model one should always consider when treating GFTs as quantum field theories.

In \( n \) dimensions, the colored GFT model is defined by \( n + 1 \) pairs of fermionic (or complex bosonic) fields \( \psi^p, \bar{\psi}^p : \mathbb{G}^n \rightarrow \mathbb{G} \) or \( \mathbb{C} \), invariant under left group multiplication of the argument, and with no symmetry properties. The action of the colored GFT writes

\[
S = \frac{1}{2} \int [dg] \sum_{p=0}^{n} \psi^p_{a_0, a_1, \ldots, a_{n-1}} \bar{\psi}^p_{a_0, a_1, \ldots, a_{n-1}} + S_{\text{int}} + \bar{S}_{\text{int}}.
\]
\[ S_{\text{int}} = \lambda \int [dg] \psi_0^{\alpha_0} \cdots \psi_p^{\alpha_p} \] 
\[ \times \psi_{n-1}^{\alpha_{n-1}} \cdots \psi_0^{\alpha_0} \] 
(33)

and \( S_{\text{int}} \) has the same form as \( S_{\text{int}} \) with \( \psi \) replaced by \( \bar{\psi} \). The index \( p \) on each field is a color index and we denote the set of all colors \( C_{n+1} = \{0, \ldots, n\} \).

The interaction part of the colored GFT model has two terms and generates two vertices: the positive vertex, involving only \( \psi \)'s, represented in figure 3 (where the labels \( 0, \ldots, n \) become now colors), and the negative vertex, involving only \( \bar{\psi} \), with colors turning anticlockwise around it. The propagator of the model has \( n \) parallel strands and always connects two half lines of the same color, one on a positive and one on a negative vertex. We orient all lines from positive to negative vertices.

The strand structure of the vertex and propagator is rigid; thus, a CGFT graph admits a simplified representation as a colored graph. The colored graph is obtained by collapsing all the strands of the lines in ‘thin’ lines, and all the strands of the vertices in point vertices. Conversely, given a colored graph with thin lines and point vertices, one can reconstruct the stranded graph associated with it. Figure 15 depicts a CGFT graph either as a stranded graph (on the left) or as a colored graph (on the right).

A CGFT graph \( \mathcal{G} \) comes equipped with a natural family of subgraphs, called the \( p \)-bubbles. A \( p \)-bubble is a connected subgraph of \( \mathcal{G} \) made only of lines of colors in \( C^p \) for some subset \( C^p \subset C_{n+1} \) of cardinality \( |C^p| = p \). We denote a \( p \)-bubble with colors \( C^p \) and vertices \( \mathcal{V} \) by \( B_{C^p}^{\mathcal{V}} \).

Clearly the 0-bubbles of a graph are its vertices and the 1-bubbles are its lines. For \( p \geq 2 \), the \( p \)-bubbles admit two graphical representations, either as colored graphs or as stranded graphs. In the stranded graph representation, one only draws the strands common to the lines of colors \( C^p \). The colored and stranded representations of the 3-bubbles of the graph \( \mathcal{G} \) in figure 15 are depicted in figure 16.

The \( p \)-bubbles are themselves colored GFT graphs in \((p - 1)\) dimensions. Comparing figure 16 with figure 6, we note that for this graph the 3-bubbles correspond to the link graphs. This is in fact a general result for \( p \geq 2 \).

**Theorem 1.** For \( p \geq 2 \), the \( p \)-bubbles of a CGFT graph are the link graphs of the \((n - p)\) simplices in the gluing \( \Delta^{n-1} \).

**Proof.** Consider two vertices \( v_A \) (positive) and \( v_B \) (negative) connected by a line of color \( i \) (see figure 17) in an \( n \)-dimensional GFT graph \( \mathcal{G} \).

This drawing essentially proves the result. As the line has only parallel strands and connects opposite vertices, the strand \( ji \), common to the half lines \( j \) and \( i \) on \( v_A \) necessarily connects with the strand \( ij \) on \( v_B \). This holds for all lines; therefore, the labels \( ij \) are conserved all along the strand. This is the fundamental difference between the usual GFT graphs and the CGFT graphs and render the latter much better behaved.
Figure 16. The 3-bubbles of $G$ represented as stranded or as colored graphs.

Figure 17. A colored GFT line.

The vertex $v_A$ is dual to some simplex $\sigma^n = \{A_0, \ldots, A_n\}$. Consider one of its $(n-p)$ subsimplices

$$\sigma^{n-p} = \{A_0, \ldots, A_n\} \setminus \{A_{i_1}, \ldots, A_{i_p}\}. \tag{34}$$

Following subsection 3.1, the contribution of $v_A$ to the link graph of $\sigma^p$ is the $p$-dimensional GFT vertex (descendant of $v_A$) with labels $(i_1, \ldots, i_p)$. But, as the colors of strands are conserved, this vertex will always connect only with the link vertex $(i_1, \ldots, i_p)$ descending from $v_B$.

The link graphs are exactly the connected $p$-dimensional GFT graphs formed by lines and strands with colors $(i_1, \ldots, i_p)$, hence the $p$ bubbles of $G$.

We are now in the position to state and prove the core result of this paper.

**Theorem 2.** Any connected $n$-dimensional CGFT graph is dual to a normal simplicial pseudo manifold.

**Proof.** By lemma 1 it is enough to prove that the gluing $\Delta^G$ dual to any CGFT graph is a simplicial complex. This is trivial once the appropriate notations are introduced.

The 0 simplices (vertices) of the gluing $\Delta^G$ are dual to the $n$-bubbles of the CGFT graph, $V(\Delta^G) = \{E_{n-p}^{\text{strands}}\}$.

The 1 simplices of $\Delta^G$ are dual to the $(n-1)$ bubbles of $G$. Consider one of the $(n-1)$ bubbles of $G$, say $E_{n-1}^{\text{strands}}(p,q)$. This bubble is a subgraph of $G$; hence, there exists a unique
subgraph obtained by adding all the lines of color $p$ for incident on the vertices $V$ and then closing the entire connected component with colors $C^{n+1} \setminus \{q\}$, that is,

$$\forall E^{(n+1)}_{V}(p,q), \exists ! E^{(n+1)}_{V}(q), \quad V \subset \bar{V},$$

(35)

where we denoted by $\bar{V}$ the (unique) maximal set of vertices connected by lines of colors $C^{n+1} \setminus \{q\}$ and containing $V$. The same holds for the lines of color $p$. Pick any vertex $v_A \in V$, dual to an $n$ simplex $\{A_0, \ldots, A_n\}$. The 1 simplex dual to $E^{(n+1)}_{V}(p,q)$ is $\{A_p, A_q\}$, and the 0 simplex dual to $E^{(n+1)}_{V}(q)$ is $\{A_q\}$. Therefore, the 1 simplex dual to $E^{(n+1)}_{V}(p,q)$ writes

$$\{E^{(n+1)}_{V}(q), E^{(n+1)}_{V}(p)\}, \quad V \subset \bar{V} \cap \bar{V}.$$  

Similarly, a $p$ simplex $\sigma^p$ is dual to an $(n-p)$ bubble $E^{(n+1)}_{V}(\{i_0, \ldots, i_p\})$, and for each $(n-p)$ bubble and each color $i_p$, there exists a unique subgraph with $n$ colors obtained by adding the lines of colors, all colors except $i_q$ incident on the vertices $V$ and completing the connected component with lines of all colors except $i_q$ (whose set of vertices we denote $\bar{V}^\alpha$)

$$E^{(n+1)}_{V}(i_q), \quad V \subset \bar{V}^\alpha.$$  

(37)

The $p$ simplex $\sigma^p$ writes

$$\{E^{(n+1)}_{\bar{V}^\alpha}(j_0), \ldots, E^{(n+1)}_{\bar{V}^\alpha}(j_p)\} \quad V \subset \bar{V}^\beta \cap \ldots \cap \bar{V}^\alpha.$$  

(38)

The proof is now tautological. For any subset $M^k \subset \sigma^p$ of cardinality $(k+1) < (p+1)$,

$$\{E^{(n+1)}_{\bar{V}^\beta}(j_0), \ldots, E^{(n+1)}_{\bar{V}^\beta}(j_k)\} \quad V \subset \bar{V}^\alpha \cap \ldots \cap \bar{V}^\alpha,$$

(39)

with $\{j_0, \ldots, j_k\} \subset \{0, \ldots, p\}$ there exists a $k$-bubble obtained by adding all lines of colors $\{0, \ldots, p\} \setminus \{j_0, \ldots, j_k\}$ to $V$ and completing the graph thus obtained to a bubble of colors $C^{n+1} \setminus \{i_0, \ldots, i_p\} \setminus \{i_{j_0}, \ldots, i_{j_k}\}$. Consequently, $M^k$ is the simplex dual to this bubble, $M^k \in \Delta^\beta$, and the gluing is a simplicial complex. \qed

In retrospect one sees that all the link graphs are orientable, as they are always made of colored lines joining the vertices of opposite orientation. The colored GFT model is the simplest one which guarantees that all subsimplices, of arbitrary dimension, are always identified coherently in all gluings corresponding to the CGFT lines.

6. Conclusion

We started out work by an in-depth study of singularities in the usual GFT models. We concluded that highly pathological GFT graphs whose dual gluings are not homeomorphic to pseudo manifolds dominate in power counting and proliferate in the perturbative development of the usual GFTs. Of course, as long as one analyzes the particular examples of ‘nice’ graphs, one is oblivious to this problem. However, the moment one tries to treat the usual GFTs as fully fledged quantum field theories and take into account all the graphs, the pathological ones dominate. In order to save the GFTs as quantum field theories and obtain a reasonable effective behavior, one must deal one way or another with this problem.

The solution we present in this paper is to use the colored GFT models. The extra structure encoded in the coloring eliminates the wrapping singularities for all graph and in all dimensions in a very natural way.

A large amount of work still remains to be done before establishing the colored GFTs as quantum field theories; most importantly, one would like to find some scaling regime in which their effective behavior is dominated by manifold configurations. On the other
hand, the language of the colored GFTs could be used as a mathematical tool to further the understanding of topology. The encoding of the link graphs into the bubbles provides a bridge between topology and combinatorics opening up the possibility of obtaining, using the latter, new results concerning the former.

Acknowledgments

The author would like to thank an anonymous referee whose numerous comments have led to a definite improvement of the manuscript.

Research at the Perimeter Institute was supported by the Government of Canada through Industry Canada and by the Province of Ontario through the Ministry of Research and Innovation.

Appendix. Remarks on simplicial complexes

Remark 1. The collection $\text{star}_\Delta(v) \setminus \text{lk}_\Delta(v)$ is

\[ \text{star}_\Delta(v) \setminus \text{lk}_\Delta(v) = \{ \{v\} \cup \tau \mid \tau \in \text{lk}_\Delta(v) \}. \]  

(A.1)

Proof. ‘⊃’: Let any $\tau \in \text{lk}_\Delta(v)$. Then $\{v\} \cup \tau \in \Delta$ and $\{v\} \cup \tau \in \text{star}_\Delta(v)$. But, as $\{v\} \in \{v\} \cup \tau$, then $\{v\} \cup \tau / \in \text{lk}_\Delta(v)$; thus $\{v\} \cup \tau \in \text{star}_\Delta(v) \setminus \text{lk}_\Delta(v)$.

‘⊂’: Let $\sigma \in \text{star}_\Delta(v) \setminus \text{lk}_\Delta(v)$. Then $\sigma \cup \{v\} \in \Delta$ and $\sigma \in \sigma$; thus $\sigma \cup \{v\} = \sigma \in \Delta$. Denote $\tau = \sigma \setminus \{v\}$. As $\sigma \in \Delta$, and $\Delta$ is a simplicial complex, $\tau \in \Delta$. Therefore, $\tau \in \Delta$, $\tau \cap \{v\} = \emptyset$ and $\tau \cup \{v\} = \sigma \in \Delta$; hence $\tau \in \text{lk}_\Delta(v)$. □

Remark 2. Let $\Delta$ be a simplicial complex with simplices of maximal dimension $n$. For any vertex $v$ of $\Delta$

\[ \chi(\text{star}_\Delta(v)) = 1. \]  

(A.2)

Proof. The $\text{star}_\Delta(v)$ admits the disjoint decomposition

\[ \text{star}_\Delta(v) = (\text{star}_\Delta(v) \setminus \text{lk}_\Delta(v)) \cup \text{lk}_\Delta(v). \]  

(A.3)

Under this decomposition, all simplices either belong to $\text{star}_\Delta(v) \setminus \text{lk}_\Delta(v)$ or to $\text{lk}_\Delta(v)$.

Let any $p \geq 0$ simplex in the star, $\sigma^p$. Either $\sigma^p \in \text{lk}_\Delta(v)$ or $\sigma^p = \{v\} \cup \sigma^{p-1}$, $\sigma^{p-1} \in \text{lk}_\Delta(v)$. Note that by definition the link has exactly one $-1$ simplex, namely $\emptyset$, and zero $n$ simplices.

Denote the number of $p$ simplices in the star $N_p$, and the number of $p$ simplices in the link $n_p$. Then

\[ f^P(\text{star}_\Delta(v)) = f^P(\text{lk}_\Delta(v)) + f^{P-1}(\text{lk}_\Delta(v)) = f^{P-1}(\text{lk}_\Delta(v)); \]  

(A.4)

hence,

\[ \chi(\text{star}_\Delta(v)) = \sum_{p=0}^{n} (-)^p f^P(\text{star}_\Delta(v)) \]  

(A.5)

\[ = \sum_{p=0}^{P} (-)^p (f^P(\text{lk}_\Delta(v)) + f^{P-1}(\text{lk}_\Delta(v))) = f^{-1}(\text{lk}_\Delta(v)) = 1. \]  

□

Remark 3. The links of a simplicial pseudo manifold $\Delta$ are pure and non-branching.
Proof. Consider a simplex \( \tau^p \) of dimension \( p \) in a simplicial pseudo manifold and denote \( \sigma_{(1)}^n, \ldots, \sigma_{(N)}^n \) all the \( n \)-dimensional simplices to which \( \tau^p \) is a face. Denote also \( \sigma_{(k)}^{n-p-1} = \sigma_{(k)}^n \setminus \tau^p \).

Step 1: \( \text{lk}_A(\tau^p) \) is an \((n-p-1)\)-dimensional pure simplicial complex.

Any \( \sigma \in \text{lk}_A(\tau^p) \), one has \( \sigma \cup \tau \in \Delta \); hence, there exist some \( n \)-dimensional simplex \( \rho^n \supseteq \sigma \cup \tau \). But \( \tau \subseteq \rho^n \); thus \( \rho^n = \sigma_{(k)}^n \) for some \( k \), and \( \sigma \subseteq \sigma_{(k)}^n \setminus \tau^p = \sigma_{(k)}^{n-p-1} \).

Step 2: \( \text{lk}_A(\tau^p) \) is non-branching.

Let \( \sigma_{n-p-2} \) be an \((n-p-2)\) simplex in \( \text{lk}_A(\tau^p) \). It is a face of \( r \) simplices of dimension \((n-p-1)\)\( \ldots, \sigma_{(N)}^{n-p-1} \). Then \( \tau^p \cup \sigma_{n-p-2} \) is an \((n-1)\)-dimensional simplex of \( \Delta \) which is a face of the \( r \) simplices \( \tau^p \cup \sigma_{(k)}^{n-p-1} \) of dimension \( n \). As \( \Delta \) is non-branching, \( r = 2 \); hence \( \text{lk}_A(\tau^p) \) is non-branching.

However, the links of simplicial pseudo manifolds are not necessarily strongly connected. For example, in the simplicial pseudo manifold

\[
\{(v, a_1, a_2), (v, a_1, a_3), (v, a_2, a_3), (v, b_1, b_2), (v, b_1, b_3), (v, b_2, b_3) \}
\]

\[
\{b_1, b_2, m_{12}, \{a_1, a_2, m_{12}\}, \{b_1, a_1, m_{12}\}, \{b_2, a_2, m_{12}\} \}
\]

\[
\{b_1, b_3, m_{13}, \{a_1, a_3, m_{13}\}, \{b_1, a_1, m_{13}\}, \{b_3, a_3, m_{13}\} \}
\]

\[
\{b_2, b_3, m_{23}, \{a_2, a_3, m_{23}\}, \{b_2, a_2, m_{23}\}, \{b_3, a_3, m_{23}\} \}. \tag{A.6}
\]

the link of \( v \) is not even connected, much less strongly connected.

References

[1] Gurau R 2009 arXiv:0907.2582 [hep-th]
[2] Boulatov D V 1992 Mod. Phys. Lett. A 7 1629 (arXiv:hep-th/9202074)
[3] Freidel L 2005 Int. J. Theor. Phys. 44 1769 (arXiv:hep-th/0505016)
[4] Oriti D 2006 arXiv:gr-qc/0607032
[5] Oriti D 2007 Quantum Gravity ed B Fauser, J Tolksdorf and E Zeidler (Basel: Birkhaeuser) (arXiv: gr-qc/0512103)
[6] Oriti D 2009 arXiv:0912.2441 [hep-th]
[7] Gross M 1992 Nucl. Phys. Proc. Suppl. 25A 144
[8] Ambjorn J, Durhuus B and Jonsson T 1991 Mod. Phys. Lett. A 6 1133
[9] Sasakura N 1991 Mod. Phys. Lett. A 6 2613
[10] Sasakura N 2010 arXiv:1005.3088 [hep-th]
[11] Williams R in [14]
[12] Ambjorn J, Jurkiewicz J and Loll R 2005 Phys. Rev. D 72 064014 (arXiv:hep-th/0505154)
[13] Oriti D 2001 Rept. Prog. Phys. 64 1489 (arXiv:gr-qc/0106091)
[14] Oriti D (ed.) 2009 Approaches to Quantum Gravity: Toward a New Understanding of Space, Time and Matter (Cambridge: Cambridge University Press)
[15] David F 1985 Nucl. Phys. B 257 543
[16] Hammer H W 2007 arXiv:0704.2895 [hep-th]
[17] Loll R 2003 Lect. Notes Phys. 631 137 (arXiv:hep-th/0212340)
[18] Barrett J W and Boalch-Vauclair 2009 Class. Quantum Grav. 26 157801 (arXiv:0803.3319 [gr-qc])
[19] Freidel L and Loupape D 2004 Class. Quantum Grav. 21 5685 (arXiv:hep-th/0401076)
[20] Kawamoto N, Nielsen H B and Sato N 1999 Nucl. Phys. B 555 629 (arXiv:hep-th/9902165)
[21] Turaev V G and Viro O Y 1992 Topology 31 865
[22] Abdesselam A 2002/04 Sém. Lothar. Combin. 49 Art. B49c, 45 pp (electronic) (arXiv:math-ph/0212121)
[23] Abdesselam A 2009 arXiv:0904.1734v2[math.GT]
[24] Baratia A and Oriti D 2010 arXiv:1002.4723 [hep-th]
[25] Engle J, Pereira R and Rovelli C 2007 Phys. Rev. Lett. 99 161301 (arXiv:0705.2388 [gr-qc])
[26] Engle J, Pereira R and Rovelli C 2008 Nucl. Phys. B 798 251 (arXiv:0708.1236 [gr-qc])
[27] Livine E R and Speziale S 2007 Phys. Rev. D 76 084028 (arXiv:0705.0674 [gr-qc])
[28] Freidel L and Krisnov K 2008 Class. Quantum Grav. 25 125018 (arXiv:0708.1595 [gr-qc])
[29] Conrady F and Freidel L 2008 Phys. Rev. D 78 104023 (arXiv:0809.2280 [gr-qc])
[30] Barrett J W, Dowdall R J, Fairbairn W J, Hellmann F and Pereira R 2010 Class. Quantum Grav. 27 165009 (arXiv:0907.2440 [gr-qc])
[31] Bonzom V, Livine E R, Smerlak M and Speziale S 2008 Nucl. Phys. B 804 507 (arXiv:0802.3983 [gr-qc])
[32] Pietri R De, Freidel L, Krasnov K and Rovelli C 2000 Nucl. Phys. B 574 785 (arXiv:hep-th/9907154)
[33] Perez A and Rovelli C 2001 Nucl. Phys. B 599 255 (arXiv:gr-qc/0006107)
[34] Ori D and Tlas T 2010 Class. Quantum Grav. 27 135018 (arXiv:0912.1546 [gr-qc])
[35] Freidel L, Ori D and Ryan J 2005 arXiv:gr-qc/0506067
[36] Ori D and Ryan J 2006 Class. Quantum Grav. 23 6543 (arXiv:gr-qc/0602010)
[37] Dowdall R J 2009 arXiv:0911.2391 [gr-qc]
[38] Fairbairn W J and Livine E R 2007 Class. Quantum Grav. 24 5277 (arXiv:gr-qc/0702125)
[39] Di Mare A and Ori D 2010 arXiv:1001.2702 [gr-qc]
[40] Ashtekar A, Campiglia M and Henderson A 2009 Phys. Lett. B 681 347 (arXiv:0909.4221 [gr-qc])
[41] Ashtekar A, Campiglia M and Henderson A 2010 Class. Quantum Grav. 27 135020 (arXiv:1001.5147 [gr-qc])
[42] Grosse H and Wulkenhaar R 2005 Commun. Math. Phys. 256 305 (arXiv:hep-th/0401128)
[43] Gurau R, Magnen J, Rivasseau V and Vignes-Tourneret F 2006 Commun. Math. Phys. 267 515 (arXiv:hep-th/0512271)
[44] Disertori M, Gurau R, Magnen J and Rivasseau V 2007 Phys. Lett. B 649 95 (arXiv:hep-th/0612251)
[45] Geloun J B, Gurau R and Rivasseau V 2009 Phys. Lett. B 671 284 (arXiv:0805.4362 [hep-th])
[46] Freidel L, Gurau R and Ori D 2009 Phys. Rev. D 80 044007 (arXiv:0905.3772 [hep-th])
[47] Magnen J, Noui K, Rivasseau V and Smerlak M 2009 Class. Quantum Grav. 26 185012 (arXiv:0906.5477 [hep-th])
[48] Gurau R 2009 arXiv:0911.1945 [hep-th]
[49] Geloun J B, Magnen J and Rivasseau V 2009 arXiv:0911.1719 [hep-th]
[50] Geloun J B, Krajewski T, Magnen J and Rivasseau V 2010 arXiv:1002.3592 [hep-th]
[51] Bonzom V and Smerlak M 2010 arXiv:1004.5196 [gr-qc]
[52] De Pietri R and Petronio C 2000 J. Math. Phys. 41 6671 (arXiv:gr-qc/0004045)
[53] Alexander S, Crane L and Sheppeard M D 2003 arXiv:gr-qc/0306079
[54] Kozlov D 2008 Combinatorial Algebraic Topology (Berlin: Springer)