Multiscale model reduction of fluid flow based on the dual porosity model

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Abstract. Construction of multiscale method for double porosity model is considered. Mathematical modelling of such type of models widely used for describe the flow in fractured porous media. Idea of double porosity model based on the interaction mechanism between fractured and porous continua. Each continuum considered as a separate homogeneous medium which the expressed through filtration equation with additional terms corresponding to the exchange flux. The solution method is based on the use of a generalized multiscale finite element method. The main idea is to build local-domain solutions that can be used to effectively solve the coarse grid.

1. Introduction

In this paper the physical filtration process of single-phase slightly compressible fluid in a fractured-porous medium is considered. To describe the filtration process in such a medium we use double porosity model. Double porosity model appear in many fields such as oil and gas producing, engineering, medicine, space craft etc. The double porosity model [1] is a popular approach for describing fluid flow in fractured porous media. This model based on a number of assumptions and it is suitable for describing natural fracturing. It is assumed that the scale of the medium is large enough to neglect the size of the cracks and they are located quite densely so that fracture can be treated similarly to the pores.

In this model a fractured-porous medium usually considered as a system of pore blocks separated from each other by a system of fractures. Also we suppose that the liquid is found both in cracks and in pore blocks. Cracks act as a transport carrier while the pores allow absorb a much larger amount of liquid. In this case, we taking into account next assumption – at each point of the medium there is a simultaneous flow in the fractured medium and in the porous medium. The mass transfer by the way provided with a flow function, which in this particular case based on the pressure difference.

In this work we will consider the applicability of the generalized multiscale method to the double porosity model [2–5]. The formulation of the problem constructed by authors in such a way that the novelty of the problem related to the presence of subdomains with high permeability. This feature can be observed in many applied problems, for example, quasi-hollow spaces in oil reservoirs, inclusions in composite materials, etc. Thus, we apply the idea of projecting features on a small scale to a rough scale to the double porosity model that is already present by itself the result of homogenization. With this approach it is necessary to control the loss of accuracy since errors can be accumulated additively. In connection with this fact should be noted that we are dealing with a system of coupled parabolic
equations which is usually implies the possibility of using splitting schemes. In this case we are limited to particular case and we talking about solving the problem based on constructing a complete matrix in order to avoid the occurrence of an additional error [6,7].

2. Model problem
We consider double porosity model of the single-phase fluid filtration problem in a fractured porous medium. Double porosity model for such problems can be expressed as system of differential parabolic equations:

\[
\begin{align*}
    c_1(x) \frac{\partial u_1}{\partial t} - \text{div}(d_1(x) \text{grad } u_1) + r(x)(u_1 - u_2) &= f_1(x, t), \\
    c_2(x) \frac{\partial u_2}{\partial t} - \text{div}(d_2(x) \text{grad } u_2) + r(x)(u_2 - u_1) &= f_2(x, t).
\end{align*}
\]

Here, the indices 1 and 2 denote the parameters of the fractured and porous media respectively. The coefficient \( r(x)(u_1 - u_2) \) describes the exchange flux between media, \( d_i = \frac{k_i}{\mu} \) where \( k_i \) is permeability, \( \mu \) is fluid viscosity.

Physical considerations for the coefficient of exchange flux can be written as inequality:

\[
r(x) \geq 0, \quad x \in \Omega.
\]

The system of equations is supplemented by the initial conditions:

\[
\begin{align*}
    u_1(x, 0) &= u^0_1(x), \\
    u_2(x, 0) &= u^0_2(x).
\end{align*}
\]

Next geometry and boundary condition are taken for problem, note that inside circles we consider subdomains with greater value of permeability (Figure 1):

\[
\begin{align*}
    u_\alpha(x, t) &= g_1(x, t), \quad x \in \Gamma_1, \\
    u_\alpha(x, t) &= g_2(x, t), \quad x \in \Gamma_3.
\end{align*}
\]

For simplicity this model considered in two-dimensional form. Such limitation provide sufficient quality and visually conveys common behavior of the model. Also for fine-scale problem some basic estimates are derived. For example, differential problem satisfies next a priori estimate that guarantee stability on initial and boundary conditions:
\[ ||u_1||^2_{c_1} + ||u_2||^2_{c_2} \leq ||u_1^0||^2_{c_1} + ||u_2^0||^2_{c_2} + \frac{T}{2} \int_0^T \left( ||f_1||^2_{d_1} + ||f_2||^2_{d_2} \right) dt, \]

where
\[ ||u_1||_{c_1} = \sqrt{\int_\Omega c_1 u_1 \cdot u_1 dx}, \quad ||u_2||_{c_2} = \sqrt{\int_\Omega c_2 u_2 \cdot u_2 dx}, \]
\[ ||f_1||_{d_1} = \sqrt{\int_\Omega d_1 (\text{grad } f_1, \text{grad } f_1) dx}, \quad ||f_2||_{d_2} = \sqrt{\int_\Omega d_2 (\text{grad } f_2, \text{grad } f_2) dx}. \]

### 3. Fine-scale approximation

A uniform grid in time with step \( \tau \) are used for an approximate solution of the problem of nonstationary filtration process:
\[ \bar{\phi}_r = \phi_t \cup \{T\} = \{t^n = n \tau, \quad n = 0, 1, ..., N, \quad \tau N = T\}, \]
and denote by \( y^n = y(t^n), \quad t^n = n \tau \). For space approximation finite element method is used. For this research finite element method consists standard Lagrange finite elements of order 1. This method flexible enough for take into account geometry with subdomains inside. In domain \( \Omega \) is carried out a Delauney triangulation. On this computational grid a finite-dimensional space of finite element \( V_\tau \in H^2(\Omega) \) are defined. For pretend to better accuracy we will consider only implicit scheme in time and didn’t construct any splitting schemes.

Further, we can formulate variational form: find \((y_1, y_2) \in (V_1, V_2)\) such that
\[ c_1(x) \int_\Omega \frac{y_1^{k+1} - y_1^k}{\tau} v_1 dx + \int_\Omega d_1(x) \text{grad } y_1^{k+1} \text{grad } v_1 dx + \int_\Omega r(x) (y_1^{k+1} - y_2^{k+1}) v_1 dx = 0, \]
\[ c_2(x) \int_\Omega \frac{y_2^{k+1} - y_2^k}{\tau} v_2 dx + \int_\Omega d_2(x) \text{grad } y_2^{k+1} \text{grad } v_2 dx + \int_\Omega r(x) (y_2^{k+1} - y_1^{k+1}) v_2 dx = 0, \]
where \( \forall v_1 \in V_1 \) and \( \forall v_2 \in V_2 \).

Same analogue estimate connected with discrete problem can be derived:
\[ ||y_1^H||^2_{c_1} + ||y_2^H||^2_{c_2} \leq ||y_1^{H,0}||^2_{c_1} + ||y_2^{H,0}||^2_{c_2} + \frac{T}{2} \int_0^T (||f_1^H||^2_{K_1} + ||f_2^H||^2_{K_2}) dt. \]

### 4. Generalized multiscale finite element method

In this section, we describe construction local reduction of a model on the snapshot space by solving some local spectral problems using GMsFEM [6, 7].

Let \( \mathcal{T}_H \) is a coarse grid in domain \( \Omega \), such that \( \mathcal{T}_H = \cup_{i=1}^{N_c} K_i \), where \( K_i \) is coarse cell. The \( \mathcal{T}_h \) is fine grid with \( H \geq h \geq 0 \), where \( h \) is size of fine grid. Each coarse-grid block \( K_i \) consists of connected union of fine-grid blocks. Further, we construct some domain for this denote by \( N_c \) the coarse nodes number, by \( \{x_i\}_{i=1}^{N_c} \) the vertices of the coarse mesh and define the neighborhood of the node \( x_i \):
\[ \omega_i = \cup \{K_i \in \mathcal{T}_H; \quad x_i \in \overline{K_i}\}. \]

Next steps need to be implemented for GMsFEM realization:

1. Coarse grid generation \( \mathcal{T}_H \);
2. Offline space construction;
   - Construction of snapshot space that will be used to compute an offline space,
- Construction of a small dimensional offline space by performing dimensional reduction in the space of local snapshots.

3. Solution of a coarse-grid problem for any force term and boundary condition.

In the first step of GMsFEM (Offline stage), we need construct the “snapshots” space, a large dimensional snapshots space of local solutions.

In the “snapshots” space, we consider the following:

\[-\text{div}(k_\alpha \nabla \psi_{\alpha,i}) = 0, \quad x \in \omega_i,\]
\[\psi_{\alpha,j} = \delta_j(x), \quad x \in \partial \omega_i,\]

where \(\alpha = 1, 2, \delta_j(x)\)are some set of function defined on \(\partial \omega_i\), here \(j = 1, J_\omega\). The \(J_\omega\) is number of fine grid edges on \(\omega\).

This allow us reduce the snapshot space via some spectral procedure to offline space. Offline space is constructed using the following local spectral problems in the snapshots space:

\[\overline{A}_{\alpha,\omega} \overline{\psi}_k^\alpha = \lambda_k \overline{S}_{\alpha,\omega} \overline{\psi}_k^\alpha,\]

where \(\overline{A}_{\alpha,\omega} = R_{\alpha,\omega} A_{\alpha,\omega} R_{\alpha,\omega}^T, \overline{S}_{\alpha,\omega} = R_{\alpha,\omega} S_{\alpha,\omega} R_{\alpha,\omega}^T\) and

\[A_{\alpha,\omega} = [a^\alpha_{mn}] = \int_{\omega_i} (k^\alpha_m \nabla \psi_m, \nabla \psi_n) \, dx,\]
\[S_{\alpha,\omega} = [s^\alpha_{mn}] = \int_{\omega_i} (k^\alpha_m \psi_m, \psi_n) \, dx.\]

To generate the offline space we choose the largest \(M^\alpha_{i1}\) eigenvalues and find the corresponding eigenvectors in the space by multiplication, \(\Psi_k^\alpha = R_{\alpha,\omega} \overline{\psi}_k^\alpha\) for \(k = 1, 2, \ldots, M^\alpha_{i1}\).

After this procedure we can obtain conforming basis functions in the space:

\[V_{\text{off}}^\alpha = \text{span}\{\Psi_k^\alpha, 1 \leq k \leq M^\alpha_{i1}, \quad 1 \leq i \leq N\}, \quad \alpha = 1, 2.\]

Further, we define the projection matrix

\[R = \begin{pmatrix} R_1 & 0 \\ 0 & R_2 \end{pmatrix},\]

where \(R_\alpha^T = [\Psi_1^\alpha, \ldots, \Psi_{M^\alpha_{i1}}^\alpha], \alpha = 1, 2.\)

Using constructed multiscale space, we solve coarse-scale system

\[M \left( \frac{\partial u_c}{\partial t} \right) + A_c u_c = F_c,\]

where \(A_c = R_{\alpha} A_{\alpha} R_{\alpha}^T\) and \(F_c = R_{\alpha} F_f\).

After solving coarse-scale solution, we can calculate solution on the fine grid \(u_{ms} = R_{\alpha}^T u_c\).

5. Numerical results

In this section numerical results of simulation are presented. The computational domain consists of several layers with different properties: for layer 1 (fractures \(\alpha = 1\)) we apply \(d_1 = 1.0, c_1 = 1.0\) and for layer 2 (porous matrix \(\alpha = 2\)) \(d_2 = 0.01, c_2 = 0.1\), for circles we apply \(k = 1000.0, c = 0.01\) (Figure 2) and interflow parameter \(r = 0.001\). Also we set source on left site is equal to 1 (applying Dirichlet boundary condition). Numerical parameters of model problem: \(t_{\text{max}} = 0.1\) with time step \(\tau = 0.001\) and enough detailed mesh for fine mesh problem (15338 vertices and 30674
elements). Note that GMsFEM method allow to perform calculations on very rude meshes (5x5, 10x10).

![Computational domain](image)

**Figure 2:** Computational domain.

In Table 1, we present relative errors for GMsFEM that demonstrate convergence of method on coarse grid refinement and increasing number of basis functions. Table demonstrate the results for different number of multiscale basis functions separately for each continuum.

| \(M\) | DOF | \(L_2(c_1)\) | \(H_1(c_1)\) | \(L_2(c_2)\) | \(H_2(c_2)\) |
|---|---|---|---|---|---|
| Coarse grid 10 x 10 |
| 1 | 242 | 15,19 | 92,50 | 36,46 | 78,97 |
| 2 | 484 | 5,68 | 62,09 | 13,78 | 52,43 |
| 4 | 968 | 0,21 | 6,07 | 0,9 | 7,81 |
| 8 | 1936 | 0,06 | 2,76 | 0,17 | 2,80 |
| 16 | 3872 | 0,02 | 1,16 | 0,08 | 1,62 |
| Coarse grid 5 x 5 |
| 1 | 72 | 19,77 | 53,34 | 116,35 | 70,63 |
| 2 | 144 | 11,16 | 69,87 | 74,15 | 84,54 |
| 4 | 288 | 0,84 | 11,02 | 7,69 | 22,04 |
| 8 | 576 | 0,24 | 5,13 | 1,05 | 7,98 |
| 16 | 1152 | 0,11 | 2,97 | 0,41 | 4,43 |

**Table 1:** Numerical results of relative errors (%) at the final simulation time.

On Figure 3 we present pressure distribution in background medium.
We observe that the error on 10x10 coarse grid is better than 5x5 coarse grid. When we increase number of multiscale basis function the error decreases.

6. Summary
Double porosity model for single-phase fluid filtration process is considered. Fine mesh and GMsFEM realizations are numerically investigated. Based on numerical results we can conclude following:

- A priori estimates guarantee some stability results for fine mesh solution;
- Numerical investigation of GMsFEM method demonstrate that multiscale solution converge to fine mesh solution;
- Fractures has better properties and closer to fine mesh solution.

Using finite element method allows extending these results on complex geometries and 3D dimension case. Moreover, GMsFEM final efficiency consist from constructing basis functions and solving the reduced problem on the coarse grid. This approach much more effective if calculations can be performed using already constructed basis functions.

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