SCATTERING INTO CONES
AND FLUX ACROSS SURFACES
IN QUANTUM MECHANICS:
A PATHWISE PROBABILISTIC APPROACH

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Abstract. We show how the scattering-into-cones and flux-across-surfaces theorems in Quantum Mechanics have very intuitive pathwise probabilistic versions based on some results by Carlen about large time behaviour of paths of Nelson’s diffusions. The quantum mechanical results can be then recovered by taking expectations in our pathwise statements.
1. Introduction.

The problem of finding the basic mathematical relationships between theoretical previsions and experimental observable quantities has been, for a long time, an open problem in quantum theory of scattering.

In this direction there exists two relevant theorems. The first one is due to Dollard in 1969 (see [D]) and states that the probability of asymptotically observing the particle in some cone $C \subset \mathbb{R}^3$ with vertex in the scattering center is equal to the probability of finding its asymptotic momentum exactly in the same cone, i.e.

$$\lim_{t \uparrow \infty} \int_C dx \left| \psi_t(x) \right|^2 = \lim_{t \uparrow \infty} \int_{C \cap B^c_R} dx \left| \psi_t(x) \right|^2 = \int_{C} dk \left| \hat{\psi}_{\text{out}}(k) \right|^2,$$  

(1.1)

where $B^c_R$ is the complement of $B_R$, the ball of radius $R$, $\hat{\cdot}$ denotes the Fourier transform and $\psi_{\text{out}} := \Omega^+ \ast \psi_0$ is the outgoing state, $\Omega^+$ being the wave operator. It is well known that the differential cross section for the time-independent scattering theory can be derived from the right hand side of (1.1). Nevertheless the importance of (1.1) is primarily conceptual since the probability of observation which it refers to is a time-asymptotic one.

Instead, in the usual experimental situation, the detector being sufficiently far away from the scattering centre, one actually measures the probability that the particle crosses the active surface of the detector $C \cap S_R$ ($S_R$ denoting the sphere of radius $R$) at some random time. The theorem which takes care of this experimental setting is the so-called flux-across-surfaces theorem. It was conjectured in 1975 by Combes, Newton and Shtokhamer (see [CNS]) under the form of the following relation:

$$\lim_{R \uparrow \infty} \int_{t_0}^{+\infty} dt \int_{C \cap S_R} d\sigma(x) \ J^{\psi_t}(x) \cdot n(x) = \int_{C} dk \left| \hat{\psi}_{\text{out}}(k) \right|^2,$$  

(1.2)

where $J^{\psi_t} := \text{Im} \psi_t^* \nabla \psi_t$ is the quantum probability current density, $n$ denotes the outward unit normal vector along $C \cap S_R$ and $\sigma$ is the surface measure.

No rigorous proof of this conjecture was known until 1996 when Daumer, Dürr, Goldstein and Zanghì (see [DDGZ1]) proved the flux-across-surfaces theorem in the free case. Successively the result has been extended to the interacting case by Amrein and Zuleta (see [AZ]) and by Teufel, Dürr and Münch-Berndl (see [TDM-B]) for short range potentials, and by Amrein and Pearson (see [AP]) for long range potentials. The case with zero-energy resonances or eigenvalues has been treated by Dell’Antonio and Panati in [DPa] and the case with a delta interaction by Panati and Teta in [PT].

In view of our approach, the most interesting proof is the one given in [AP]. From such a paper one can extract the following clarifying scheme:
Integrating with respect to time the equation of continuity for quantum probability density
\[ \frac{\partial}{\partial t} |\psi_t|^2 + \nabla J^{\psi_t} = 0 \] (1.3)
and inserting the result into the relation (1.1) given by Dollard’s theorem one obtains
\[ \int_C dk |\hat{\psi}_{\text{out}}(k)|^2 = \int_{C \cap B_R^c} dx |\psi_{t_0}(x)|^2 - \lim_{t \uparrow \infty} \int_{t_0}^t ds \int_{C \cap B_R^c} dx \nabla J^{\psi_s}(x). \]

Then by taking the limit \( R \uparrow \infty \) and by Gauss-Green divergence theorem one has
\[ \int_C dk |\hat{\psi}_{\text{out}}(k)|^2 = \lim_{R \uparrow \infty} \lim_{t \uparrow \infty} \int_{t_0}^t ds \int_{(C \cap S_R) \cup (\partial C \cap B_R^c)} d\sigma(x) n(x) \cdot J^{\psi_s}(x), \]
and so the flux-across-surfaces theorem is a consequence of the scattering-into-cones theorem plus the condition
\[ \lim_{R \uparrow \infty} \lim_{t \uparrow \infty} \int_{t_0}^t ds \int_{\partial C \cap B_R^c} d\sigma(x) n(x) \cdot J^{\psi_s}(x) = 0 \] (1.4)
i.e. the flux across the lateral boundary of the cone asymptotically vanishes.

In this paper we give a pathwise formulation of scattering-into-cones and flux-across-surfaces theorems following in some way the pathwise analogue of the above analytic argument. This has the advantage of giving a pictorial view of the scattering behaviour. In doing that we exploit the relevant results, obtained by Carlen in 1985, about potential scattering in Stochastic Mechanics (see [C4]).

It is known that Stochastic Mechanics, introduced by Nelson in 1966 (see [N1-3]), allows a pathwise approach to Quantum Mechanics by providing a suitable class of diffusion processes. Indeed to a solution \( \psi_t \) of the Schrödinger equation there is associated a well defined (see Theorem 1) diffusion process \( X_t \) solution of the stochastic differential equation
\[ dX_t = b(t, X_t) dt + dB_t \]
where \( B_t \) is a Brownian motion and the drift vector field \( b_t(x) \equiv b(t, x) \) is given by
\[ b_t = |\psi_t|^{-2} (\nabla |\psi_t|^2 + J^{\psi_t}). \]
Moreover the probability density of the process \( X_t \) is given by \( |\psi_t|^2 \) and it satisfies the continuity (or Fokker-Planck) equation (1.3). In connection with
the problem of potential scattering, Carlen studied the time evolution of the process \( \frac{1}{t} X_t \) proving that (see Theorem 3):

1. the scattering diffusions (i.e. the ones associated to the scattering states of the corresponding Schrödinger equation) are such that the limit

\[
\lim_{t \uparrow \infty} \frac{1}{t} X_t = p_+
\]

exists almost surely;

2. the random variable \( p_+ \) is square integrable and has the same distribution as does the quantum mechanical final momentum.

These facts imply that almost surely the diffusion paths are definitively inside or outside the cone \( C \), a pathwise analogue of (1.4). Then the following pathwise version of Dollard’s theorem immediately follows:

\[
\lim_{t \uparrow \infty} \chi_C(X_t) = \lim_{t \uparrow \infty} \chi_{C \cap S_R}(X_t) = \chi_C(p_+),
\]

\( \chi_D \) denoting the characteristic function of the set \( D \); the usual quantum mechanical version is then obtained by taking expectations (see Theorems 4 and 5).

As regards the flux-across-surfaces theorem the situation is almost equally simple. If \( N_{C \cap S_R} \) were finite, where

\[
N_{C \cap S_R} := N_{C \cap S_R}^+ - N_{C \cap S_R}^-,
\]

\( N_{C \cap S_R}^+ (\gamma) \) (resp. \( N_{C \cap S_R}^- (\gamma) \)) denoting the number of outward (resp. inward) crossing by the path \( t \mapsto \gamma(t) \) of \( C \cap S_R \), then, again by 1 and 2 above, one would obtain the following pathwise version of the flux-across-surfaces theorem:

\[
\lim_{R \uparrow \infty} N_{C \cap S_R} = \chi_C(p_+).
\] (1.5)

Let us remark here that the relevance of \( N_{C \cap S_R} \) for the flux-across-surfaces theorem was already pointed out (in the framework of Bohmian Mechanics) in [DDGZ2]. The problem here is that almost surely the diffusion \( X_t \) intersects \( C \cap S_R \) on a set of times that has no isolated point and is uncountable. Therefore the definition of \( N_{C \cap S_R} \) given above makes no sense in general. However, by a suitable redefinition of \( N_{C \cap S_R} \) as the total mass of an almost surely compactly supported random distribution (see section 3 for the details), (1.5) can be made rigorous (see Theorem 6). After showing (see Theorem 7) how to explicitly compute, by using the continuity equation (1.3), the expectation of \( N_{C \cap S_R} \) in terms of the quantum probability current density
$J^{\psi_t}$, the flux-across-surfaces theorem then follows by taking expectation in (1.5) (see Theorem 9).

In our opinion these results show how the probabilistic approach we use is very fruitful and extremely intuitive from the physical point of view.

As regards the analytical hypotheses we impose, our proofs of the pathwise results need, beside the existence of the asymptotic velocity (see hypotheses h.3, h.4 in definition 2), the following condition on the quantum evolution:

$$\int_{t_0}^{+\infty} \frac{dt}{t} \left\| \left( P - \frac{Q}{t} \right) \psi_t \right\|_{L^2} < +\infty , \quad t_0 > 0 . \quad (1.6)$$

where $P\psi(x) := -i\nabla \psi(x)$ and $Q\psi(x) = x\phi(x)$ denote the usual momentum and position operators of Quantum Mechanics in Schrödinger representation.

Let us remark that the original results by Carlen were obtained by requiring the existence and completeness of wave operators, which is a hypothesis stronger than our h.3 and h.4. It is not clear to us if our weaker hypotheses together with (1.6) in any case imply existence and completeness of wave operators. Therefore it could be interesting to find examples (if any) of cases in which the pathwise scattering-into-cones and flux-across-surfaces theorems hold true notwithstanding there are no wave operators.

In order to obtain then the quantum mechanical results by taking expectations, (1.6) is still sufficient to get Dollard’s theorem, whereas the flux-across-surfaces theorem requires that the property of paths of being definitively always inside or outside the cone $C$ holds not only pathwise but in the mean, i.e, as we already know, (1.4) must be true. This condition is a consequence of

$$\int_{t_0}^{+\infty} dt \left\| \theta(Q)\psi_t \right\|_{H^1} \left\| \left( P - \frac{Q}{t} \right) \psi_t \right\|_{H^1} < +\infty . \quad (1.7)$$

where $\theta \in C^2_b(\mathbb{R}^3; \mathbb{C})$, $\theta = 1$ on a neighbourhood of $\partial C \cap B^C_R$ for some $R > 0$ and $H^s(\mathbb{R}^3)$ denotes the Sobolev space of tempered distributions with a Fourier transform which is integrable with respect to the measure with density $(1 + |x|^2)^s$.

Conditions (1.6) and (1.7) both follows from propagation estimates on $\psi_t$. This is a well-known topic in mathematical physics and a large literature exists on them. Thus by using known results on time-decay of the solutions of the Schrödinger equation it is possible to deduce (1.6) and (1.7) from explicit conditions imposed on the initial state $\psi_0$ and on the potential $V$, which are the natural prescriptions for a physicist. In particular, beside some technical condition on the initial state $\psi_0$, (1.6) holds true with potential functions decaying at infinity like $\|x\|^{-\epsilon}$, $\epsilon > 0$, whereas (1.7) requires potentials decaying
faster that \( \|x\|^{-2/3} \) (see section 5 for more details). These conditions on \( \psi_0 \) and \( V \) also lead to existence and completeness of (modified) wave operators.

Our probabilistic proof remains unchanged in the case of the presence either of a time-dependent potential or of a magnetic field, the only difference, if \( A \) denotes the magnetic potential, being the replacement of \( P \) by \( P - A \) and of \( J^{\psi_t} \) by \( J^{\psi_t} - |\psi_t|^2 A \). We plan to work out the details in a future work.

Finally let us remark that all our results hold true every time we can find a stochastic process \( X_t \) having \( |\psi_t|^2 \) as its density and for which Theorem 3 can be proven. By Theorem 1 we realized such a process as a Nelson Diffusion, but this is not the only possible choice. Another one is given by Bohmian Mechanics (see [DGZ] for a thorough introduction to the subject), where one considers the stochastic process \( \tilde{X}_t \), solution of the ordinary differential equation

\[
\frac{d}{dt} \tilde{X}_t = |\psi_t(\tilde{X}_t)|^{-2} J^{\psi_t}(\tilde{X}_t)
\]

with a random initial condition with density \( |\psi_0|^2 \). Also in this case, under the same hypotheses plus the technical condition \( \psi_{t_0} \in C^\infty(\mathbb{R}^3) \) (the Bohmian analogue of Theorem 1, see [BDGPZ], needs more regularity), Theorem 3 holds true, the proof being essentially the same, and so all our results can be stated in a Bohmian context. We decided to work with Nelson’s stochastic mechanics since it does not necessarily need the Schrödinger equation in its formulation. Indeed it can be derived either from a stochastic analogue of Newton’s law (see [N1], [N2]) or from a stochastic variational principle (see [GM], [N3]).

### 2. Potential Scattering in Stochastic Mechanics

At first let us recall that, by Nelson’s Stochastic Mechanics (see [N1-3]), it is possible to associate to a solution \( \psi_t \) of the Schrödinger equation a diffusion process which has \( |\psi_t|^2 \) as its density. More precisely one has the following (in the reference [C1] \( V \) is a Rellich-class potential but the results obtained there can be extended to the more general potentials used here by proceeding as in [DP, Theorem 2.1])

**Theorem 1.** ([C1]) Let \( V = V_1 + V_2 \), with \( V_1 \) bounded from below and \( V_2 \) \((-\Delta)\)-form-bounded with relative bound smaller than one. Let \( H = -\frac{1}{2} \Delta + V \) be defined as a sum of quadratic forms, and let \( \psi_0 \) be a normalized state in its form domain \( \mathcal{Q}(H) = H^1(\mathbb{R}^3) \cap \mathcal{Q}(V_1) \). If \( \psi_t := e^{-itH} \psi_0 \), define then

\[
b(t, x) := b_t(x) , \quad b_t := |\psi_t|^{-2} (\nabla |\psi_t|^2 + J^{\psi_t}) .
\]

Consider the measurable space \((\Omega, \mathcal{F})\), with \( \Omega = C([t_0, +\infty); \mathbb{R}^d) \), \( t_0 \geq 0 \), \( \mathcal{F} \) the Borel \( \sigma \)-algebra, and let \((\Omega, \mathcal{F}, \mathcal{F}_t, X_t)\) be the evaluation stochastic process \( X_t(\gamma) := \gamma(t) \), with \( \mathcal{F}_t = \sigma(X_s, t_0 \leq s \leq t) \) the natural filtration. Then there exists a unique Borel probability measure \( \mathbb{P} \) on \((\Omega, \mathcal{F})\) such that:
- \((\Omega, \mathcal{F}, \mathcal{F}_t, X_t, \mathbb{P})\) is a Markov process;
- the image of \(\mathbb{P}\) under \(X_t\) has density \(|\psi_t|^2\);
- \(B_t := X_t - X_{t_0} - \int_{t_0}^{t} ds \, b(s, X_s)\) is a \((\mathbb{P}, \mathcal{F}_t)\)-Brownian motion, i.e. \(\mathbb{P}\) is a weak solution of the stochastic differential equation \(dX_t = b(t, X_t) \, dt + dB_t\) with initial density \(|\psi_{t_0}|^2\).

From now on we will assume \(t_0 > 0\) and \(d = 3\).

**Definition 2.** With the same notation and hypotheses as in Theorem 1, let us call the couple \((\psi_0, V)\) weakly admissible if

h.1) \(\psi_0\) is in \(\mathcal{H}_c\), the spectral subspace corresponding to the continuous spectrum of \(H\);

h.2) \[
\int_{t_0}^{+\infty} \frac{dt}{t} \left\| \left( P - \frac{Q}{t} \right) \psi_t \right\|_{L^2} < +\infty .
\]

h.3) the asymptotic velocity exists in the following sense:

\[
\forall g \in C^\infty_c(\mathbb{R}^3), \quad \text{w-lim} \,_{t \to \infty} \Pi_c e^{itH} g \left( \frac{Q}{t} \right) e^{-itH} \Pi_c = \Pi_c g(P_+) \Pi_c ,
\]

where w-lim means the limit in the weak operator norm topology, \(\Pi_c\) denotes the projection onto \(\mathcal{H}_c\) and \(P_+\) is a vector of commuting self-adjoint operators;

A weakly admissible couple \((\psi_0, V)\) is then called admissible if moreover

h.4) \(\psi_0\) is in the spectral subspace corresponding to the absolutely continuous spectrum of \(P_+\).

**Remark.** If the (modified) wave operators \(\Omega_{\pm}\) exist and are complete then

\[
P_+ \Pi_c = \Omega_+ P \Omega_+^* \Pi_c
\]

and, given \(\psi_0 \in \mathcal{H}_c\),

\[
\langle \psi_0, \chi_A(P_+)\psi_0 \rangle = \langle \psi_0, \Omega_+ \chi_A(P) \Omega_+^* \psi_0 \rangle = \int_{\mathcal{A}} |\hat{\psi}_{\text{out}}(k)|^2 \, dk .
\]

Thus in this case h.4 holds true for all \(\psi_0 \in \mathcal{H}_c\). Therefore hypotheses h.3 and h.4 can be interpreted as a weaker substitute for existence and completeness of wave operators.
For explicit conditions on $\psi_0$ and $V$ ensuring admissibility the reader is referred to section 5.

The above definitions permit us to extend (with the same proof) Carlen’s results (see [C4], the free case $V = 0$ was already studied in [S]) to the case where the hypothesis of existence of the wave operators is replaced by the weaker h.3:

**Theorem 3.** Let $(\psi_0, V)$ be weakly admissible and let $(\Omega, \mathcal{F}, \mathcal{F}_t, X_t, \mathbb{P})$ be as in Theorem 1. Then

$$\lim_{t \uparrow \infty} \frac{1}{t} X_t = p_+ \quad \mathbb{P}\text{-a.s.},$$

for some random variable $p_+$. Moreover $p_+$ is $\mathbb{P}$-square integrable and it has, under $\mathbb{P}$, the same distribution as does the quantum mechanical final momentum $P_+$, i.e., for every Borel set $A$ one has

$$\mathbb{E}(\chi_A(p_+)) = \langle \psi_0, \chi_A(P_+)\psi_0 \rangle,$$

where $\mathbb{E}$ denotes expectation with respect to $\mathbb{P}$.

**Proof.** The existence of the limit $\lim_{t \uparrow \infty} \frac{1}{t} X_t$ is proven in [C4, lemma 1]. For the convenience of the reader we reproduce here the main steps of such a proof. Defining the stochastic process $\pi_t := \frac{1}{t} X_t$, one has the following stochastic differential equation:

$$d\pi_t = \frac{1}{t} (b(t, X_t) - \pi_t) dt + \frac{1}{t} dB_t.$$

This implies

$$\mathbb{P} \left( \sup_{t > T} \| \pi_t - \pi_T \| > \epsilon \right) \leq \mathbb{P} \left( \int_T^{+\infty} \frac{dt}{t} \| (b(t, X_t) - \pi_t) \| > \epsilon \right) + \mathbb{P} \left( \sup_{t > T} \left\| \int_T^t \frac{1}{t} dB_t \right\| > \epsilon \right).$$

By Doob’s martingale maximal inequality and Chebychev inequality the second term on the right can be estimated by $2\epsilon^{-2} T^{-1}$. As regards the first
term, by the definition of \( b \) one has

\[
\mathbb{P} \left( \int_{T}^{+\infty} \frac{1}{t} \| (b(t, X_t) - \pi_t) \| \, dt > \epsilon \right) \\
\leq \frac{1}{\epsilon} \int_{T}^{+\infty} \frac{1}{t} \mathbb{E} \left( \| (b(t, X_t) - \pi_t) \| \right) \, dt \\
\leq \frac{1}{\epsilon} \int_{T}^{+\infty} \frac{1}{t} \mathbb{E} \left( \| (b(t, X_t) - \pi_t) \|^2 \right)^{1/2} \, dt \\
\leq \frac{\sqrt{2}}{\epsilon} \int_{T}^{+\infty} \left\| \left( -i\nabla - \frac{x}{t} \right) \psi_t \right\|_{L^2} \frac{dt}{t}.
\]

The above estimates and h.2 say that we can find a \( T_n \) large enough that

\[
\mathbb{P} \left( \bigcup_{s, t > T_n} \{ \| \pi_t - \pi_s \| > \frac{1}{n} \} \right) < \frac{1}{2^n}.
\]

Then, by Borel-Cantelli lemma, one has

\[
\mathbb{P} \left( \bigcap_{m=1}^{\infty} \bigcap_{n>m} \bigcup_{s, t > T_n} \{ \| \pi_t - \pi_s \| > \frac{1}{n} \} \right) = 0,
\]

which exactly means that \( \lim_{t \uparrow \infty} \pi_t \) exists \( \mathbb{P} \)-a.s.

By a density argument \( p_{+} \) has the same distribution as does the quantum mechanical final momentum \( P_{+} \) if \( \mathbb{E}(g(p_{+})) = \langle \psi_0, g(P_{+})\psi_0 \rangle \) for all \( g \in C_c^\infty(\mathbb{R}^3) \). By h.3 there follows

\[
\mathbb{E}(g(p_{+})) = \lim_{t \uparrow \infty} \mathbb{E}(g(\pi_t)) = \lim_{t \uparrow \infty} \langle \psi_t, g(Q/t)\psi_t \rangle = \langle \psi_0, g(P_{+})\psi_0 \rangle,
\]

and the proof is done. \( \square \)

**Remark.** The proof of the above theorem shows that

\[
h.2 \quad \Rightarrow \quad \frac{1}{t} X_t \to p_{+} \quad \text{almost surely,}
\]

\[
h.3 \quad \iff \quad \frac{1}{t} X_t \to p_{+} \quad \text{in distribution.}
\]

**Remark.** Under the stronger hypothesis
h.2.1)
\[
\int_{t_0}^{+\infty} \left\| \left( P - \frac{Q}{t} \right) \psi_t \right\|_{L^2}^2 dt < +\infty
\]

it is possible to prove (see [C5]) that the random variable \( p_+ \) generates the tail \( \sigma \)-algebra
\[
\mathcal{T} := \bigcap_{t > t_0} \sigma(X_s, s \geq t) .
\]

This is the probabilistic analogue of the fact that in Quantum Mechanics the only scattering observables are functions of the final momentum \( P_+ \). However we will not need such a nice result here.

Under hypothesis h.2.1, according to [C5], the proof of Theorem 3 becomes simpler:

Let \( \tilde{P} \) be the weak solution of the simple stochastic differential equation
\[
dX_t = \frac{1}{t} X_t dt + d\tilde{B}_t .
\]

Therefore
\[
d\left( \frac{1}{t} X_t \right) = -\frac{1}{t^2} X_t dt + \frac{1}{t} dX_t = \frac{1}{t} \tilde{B}_t
\]

and so
\[
\frac{1}{t} X_t = \frac{1}{t_0} X_{t_0} + \int_{t_0}^{t} \frac{1}{s} d\tilde{B}_s .
\]

Since
\[
\tilde{E} \left( \int_{t_0}^{+\infty} \frac{1}{s} d\tilde{B}_s \right)^2 = \int_{t_0}^{+\infty} \frac{ds}{s^2} < +\infty ,
\]
by Doob’s martingale convergence theorem one gets \( \tilde{P} \)-a.s. convergence of \( \frac{1}{t} X_t \). Thus the proof of Theorem 3 is then concluded by observing that h.2.1 implies
\[
\mathbb{E} \left( \int_{t_0}^{+\infty} dt \| b(t, X_t) - X_t/t \|^2 \right) < +\infty
\]

and so, by [E, prop. 2.11], \( \mathbb{P} \) is absolutely continuous with respect to \( \tilde{P} \).

3. The pathwise scattering-into-cones and flux-across-surfaces theorems.

From now on by an open cone \( C \) we will mean a set of the kind
\[
\{ \lambda x \in \mathbb{R}^3 : x \in \Sigma, \lambda > 0 \} ,
\]
where $\Sigma$ is an open subset of the unit sphere with $\partial\Sigma$ a finite union of $C^1$ manifolds.

In the framework of Stochastic Mechanics, thanks to Theorem 3, the pathwise version of Dollard’s scattering-into-cones theorem (see [D]) is obvious:

**Theorem 4.** Let $(\psi_0,V)$ be admissible and let $(\Omega,\mathcal{F},\mathcal{F}_t,X_t,\mathbb{P})$ be as in Theorem 1. Then for every open cone $C$ and for every ball $B_R$ of radius $R$ one has

$$\lim_{t \uparrow \infty} \chi_{C \cap B_R^c}(X_t) = \lim_{t \uparrow \infty} \chi_C(X_t) = \chi_C(p_+) \quad \mathbb{P}\text{-a.s.} \ .$$

**Proof.** By Theorem 3 and h.4 $p_+ \notin \partial C$, $\mathbb{P}$-a.s.. Thus by (2.1) $X_t$ is $\mathbb{P}$-a.s. definitively either in $C$ or in $C^c$ for every open cone $C$. Moreover, by (2.1) again, being $p_+ \neq 0$ $\mathbb{P}$-a.s. by h.4, we have

$$\lim_{t \uparrow \infty} \|X_t\| = +\infty \quad \mathbb{P}\text{-a.s.} \ . \quad (3.1)$$

Therefore

$$\lim_{t \uparrow \infty} \chi_{C \cap B_R^c}(X_t) = \lim_{t \uparrow \infty} \chi_C(X_t) = \lim_{t \uparrow \infty} \chi_C\left(\frac{1}{t}X_t\right) = \chi_C(p_+) \quad \mathbb{P}\text{-a.s.} \ .$$

\[\square\]

Let us now come to the flux-across-surfaces theorem.

We would like to define the function

$$N_{C \cap S_R}(\gamma) := N^+_{C \cap S_R}(\gamma) - N^-_{C \cap S_R}(\gamma) ,$$

where $N^+_{C \cap S_R}(\gamma)$ (resp. $N^-_{C \cap S_R}(\gamma)$) denotes the number of outward (resp. inward) crossing by $[t_0, +\infty) \ni t \mapsto \gamma(t)$ of $C \cap S_R$, the intersection of the cone $C$ with $S_R$, the sphere of radius $R$. The problem is that the above definition makes no sense since $\mathbb{P}$-a.s. the set $\{t : X_t \in C \cap S_R\}$ has no isolated point and is uncountable. Therefore we are forced to proceed in an alternative way:

Let us observe that if $\# \{t : \gamma(t) \in C \cap S_R\} < +\infty$ then $N_{C \cap S_R}(\gamma)$ is the total mass of the random distribution

$$\sum_{t \in \{s : \gamma(s) \in C \cap S_R\}} c(t) \delta_t ,$$
where \( c(t) = +1 \) if \( t \) corresponds to an outward crossing and \( c(t) = -1 \) if \( t \) corresponds to an inward crossing. Since \( t \mapsto \gamma(t) \) is definitively either in \( C \) or in \( \overline{C} \) by h.4 and (2.1), if \( R \) is sufficiently large (then we will consider the limit \( R \uparrow \infty \)) one has

\[
\sum_{t \in \{ s : \gamma(s) \in C \cap S_R \}} c(t) \delta_t = \sum_{t \in \{ s : \gamma(s) \in (C \cap S_R) \cup (\partial C \cap B_R^c) \}} c(t) \delta_t
\]

\[
= \frac{d}{dt} \chi_{C \cap \overline{B_R^c}}(\gamma(t)),
\]

where the derivative has to be intended in distributional sense. The advantage of this rewriting is that for every path \( \gamma \) the distribution \( \frac{d}{dt} \chi_{C \cap \overline{B_R^c}}(\gamma(t)) \) is well defined.

**Definition.** Given an open domain \( D \), we define the random distribution

\[
\mu_D : \Omega \to \mathcal{D}'(\mathbb{R})
\]

by

\[
\mu_D(\gamma) := \frac{d}{dt} \chi_{D}(\tilde{\gamma}(t)), \quad \tilde{\gamma}(t) := \begin{cases} 
\gamma(t), & \text{for } t \geq t_0 \\
\gamma(t_0), & \text{for } t < t_0,
\end{cases}
\]

i.e. for every test function \( \phi \in \mathcal{D}(\mathbb{R}) \equiv C_c^\infty(\mathbb{R}) \)

\[
\langle \mu_D(\gamma), \phi \rangle := -\chi_{D}(\gamma(t_0)) \phi(t_0) - \int_{t_0}^{+\infty} dt \chi_{D}(\gamma(t)) \dot{\phi}(t).
\]

Note that \( \text{supp}[\mu_D(\gamma)] = \gamma^{-1}(\partial D) \). In the case \( \mu_D(\gamma) \in \mathcal{E}'(\mathbb{R}) \), i.e. it has compact support, we define as usual its mass by

\[
M_D(\gamma) := \langle \mu_D(\gamma), \phi_\gamma \rangle,
\]

where \( \phi_\gamma \) is a test function such that \( \phi_\gamma = 1 \) on a neighbourhood of \( \text{supp}[\mu_D(\gamma)] \).

By the previous definition and by Theorem 3 we have then the following pathwise version of the flux-across-surfaces theorem:

**Theorem 5.** Let \( (\psi_0, V) \) be admissible and let \( (\Omega, \mathcal{F}, \mathcal{F}_t, X_t, \mathbb{P}) \) be as in Theorem 1. Then

\[
\mu_{C \cap \overline{B_R^c}} \in \mathcal{E}'(\mathbb{R}) \quad \mathbb{P}\text{-a.s.}
\]

and, defining \( N_{C \cap S_R} := M_{C \cap \overline{B_R^c}} \), one has

\[
\lim_{R \uparrow \infty} N_{C \cap S_R} = \chi_C(p+) \quad \mathbb{P}\text{-a.s.}
\]
Proof. Let
\[
\tau_R(\gamma) := \sup \{ t \in \mathbb{R} : X_t(\gamma) \in \partial(C \cap \bar{B}_R^c) \} .
\]
By (3.1) and since \(X_t\) is \(\mathbb{P}\)-a.s. definitively either in \(C\) or in \(\bar{C}^c\) by h.4 and Theorem 3, one has \(\tau_R < +\infty, \mathbb{P}\)-a.s. Thus \(\mu_{C\cap\bar{B}_R^c} \in \mathcal{E}'(\mathbb{R}), \mathbb{P}\)-a.s., being \(\text{supp}[\mu_{C\cap\bar{B}_R^c}] \subseteq [t_0, \tau_R(\gamma)]\).

Let \(\phi_\gamma \in \mathcal{D}(\mathbb{R})\) such that \(\phi_\gamma = 1\) on a neighbourhood of \([t_0, \tau_R(\gamma)]\). By the definition of \(\mu_{C\cap\bar{B}_R^c}\) one has
\[
\langle \mu_{C\cap\bar{B}_R^c}(\gamma), \phi_\gamma \rangle = -\chi_{C\cap\bar{B}_R^c}(\gamma(t_0)) - \chi_C(p_+(\gamma)) \int_{\tau_R(\gamma)}^{+\infty} dt \dot{\phi}_\gamma(t) \\
= -\chi_{C\cap\bar{B}_R^c}(\gamma(t_0)) + \chi_C(p_+(\gamma)),
\]
and the thesis then immediately follows by taking the limit \(R \uparrow \infty\). \(\square\)

4. The scattering-into-cones and flux-across-surfaces theorems in Quantum Mechanics.

By taking expectations in Theorem 4 and by dominated convergence theorem one immediately obtains Dollard’s theorem:

**Theorem 6.** For every open cone \(C\), every ball \(B_R\) of radius \(R\), and for every admissible couple \((\psi_0, V)\), one has
\[
\lim_{t \uparrow \infty} \int_{C \cap \bar{B}_R^c} dx |\psi_t(x)|^2 = \lim_{t \uparrow \infty} \int_C dx |\psi_t(x)|^2 = \langle \psi_0, \chi_C(P_+) \psi_0 \rangle .
\]

In order to prove the flux-across-surfaces theorem we need now to compute the expectation of \(\mu_{C\cap\bar{B}_R^c}\). To this end we state the following

**Theorem 7.** Let \(\psi_t\) and \(\mathbb{P}\) be as in Theorem 1, with \(\psi_0 \in H^2(\mathbb{R}^3)\) and \(V\) a \((-\Delta)\)-operator-bounded potential, with relative bound smaller than one. For every open domain \(D\), with \(\partial D\) a finite union of \(C^1\) manifolds, and for every test function \(\phi\) one has
\[
\mathbb{E}(\langle \mu_D, \phi \rangle) = -\int_{t_0}^{+\infty} dt \phi(t) \int_{\partial D} d\sigma(x) J^{\psi_t}(x) \cdot n(x),
\]
where \(n\) denotes the outward unit normal vector along \(\partial D\) and \(\sigma\) is the surface measure.
Proof. Since $|\psi_t|^2$ is the density of $X_t$ under $P$,

$$E(\mu_D, \phi) = -\phi(t_0) \int_D dx |\psi_{t_0}(x)|^2 - \int_{t_0}^{+\infty} dt \dot{\phi}(t) \int_D dx |\psi(t,x)|^2 .$$

Since $\psi_t$ solves the Schrödinger equation, one has (see e.g. [C1]), for all $f \in C^1_b(\mathbb{R}^3)$ and for a.e. $t$, the continuity equation

$$\frac{d}{dt} \int_{\mathbb{R}^3} dx |\psi_t(x)|^2 f(x) = \int_{\mathbb{R}^3} dx J_{\psi_t}(x) \cdot \nabla f(x) .$$

Since $\psi_t \in H^2(\mathbb{R}^3)$ by our hypotheses on $\psi_0$ and $V$, $\nabla J_{\psi_t}$ is an integrable function. Therefore one has, integrating by parts, for all $f \in C^1_b(\mathbb{R}^3)$

$$\int_{t_0}^{+\infty} dt \dot{\phi}(t) \int_{\mathbb{R}^3} dx |\psi_t(x)|^2 f(x) = -\phi(t_0) \int_{\mathbb{R}^3} dx |\psi_{t_0}(x)|^2 f(x)$$

$$+ \int_{t_0}^{+\infty} dt \phi(t) \int_{\mathbb{R}^3} dx \nabla J_{\psi_t}(x) f(x) .$$

Taking now a uniformly bounded sequence $\{f_n\}_1^\infty \subset C^1_b(\mathbb{R}^3)$, pointwise converging to $\chi_D$, by the dominated convergence theorem one obtains

$$\int_{t_0}^{+\infty} dt \dot{\phi}(t) \int_D dx |\psi_t(x)|^2$$

$$= -\phi(t_0) \int_D dx |\psi_{t_0}(x)|^2 + \int_{t_0}^{+\infty} dt \phi(t) \int_D dx \nabla J_{\psi_t}(x) .$$

Since $\psi_t \in H^2(\mathbb{R}^3)$, one has $\nabla \psi_t \in H^1(\mathbb{R}^3)$, so that both $\psi_t$ and $\nabla \psi_t$ have traces in $L^2(\partial D)$ (see e.g. [B, chap. 5]). Thus $J_{\psi_t}$ has a trace in $L^1(\partial D)$ by

$$\|J_{\psi_t}\|_{L^1(\partial D)} \leq \|\psi_t\|_{L^2(\partial D)} \|\nabla \psi_t\|_{L^2(\partial D)} \leq C \|\psi_t\|_{H^1(D)} \|\nabla \psi_t\|_{H^1(D)} ,$$

and the proof is then concluded by the Gauss-Green theorem. \hfill \Box

Definition 8. The admissible couple $(\psi_0, V)$ is said to be strongly admissible if

h.5) $\psi_0 \in H^2(\mathbb{R}^3)$, $V$ is a $(-\Delta)$-operator-bounded potential and

$$\int_{t_0}^\infty dt \|\theta(Q)\psi_t\|_{H^1} \left\| \left( P - \frac{Q}{t} \right) \psi_t \right\|_{H^1} < +\infty .$$
where \( \theta \in C^2_b(\mathbb{R}^3; \mathbb{C}) \) such that \( \theta = 1 \) on a neighbourhood of \( \partial C \cap B_R^c \) for some \( R > 0 \).

In the next section we will give explicit conditions on \( \psi_0 \) and \( V \) ensuring strong admissibility.

By combining Theorems 5 and 7 the flux-across-surfaces theorem now follows:

**Theorem 9.** For every open cone \( C \) and for every strongly admissible couple \((\psi_0, V)\) one has

\[
\lim_{R \uparrow \infty} \lim_{T \uparrow \infty} \int_{t_0}^T dt \int_{C \cap S_R} d\sigma(x) J^{\psi_t}(x) \cdot n(x) = \langle \psi_0, \chi_C(P_+) \psi_0 \rangle.
\]

**Proof.** By pointwise approximating, on the compact interval \([t_0, \tau_R(\gamma)]\),
\( t \mapsto \gamma(t) \) with a sequence of polynomials paths, the wildly oscillating function \( t \mapsto \chi_{C \cap B_R^c}(\gamma(t)) \) can be pointwise approximated with a sequence \( \{\chi_n\}^\infty_{n=1} \) of characteristic functions of finite unions \( \bigcup_{k=0}^{m(n)} [s_k^{(n)}, t_k^{(n)}] \) of disjoint intervals.

Therefore one obtains

\[
|\langle \mu_{C \cap B_R^c}(\gamma), \phi \rangle| \leq \chi_{C \cap B_R^c}(\gamma(t_0)) |\phi(t_0)| + \lim_{n \uparrow \infty} \sum_{k=0}^{m(n)} \int_{s_k^{(n)}}^{t_k^{(n)}} dt \phi(t) \left| \chi_{C \cap B_R^c}(\gamma(t)) |\phi(t)| + \lim_{n \uparrow \infty} \sum_{k=0}^{m(n)} \phi(t_k^{(n)}) - \phi(s_k^{(n)}) \right|
\]

\[
\leq \chi_{C \cap B_R^c}(\gamma(t_0)) |\phi(t_0)| + \text{var}(\phi).
\]

Now let us note that in Theorem 5 we can alternatively define \( N_{C \cap S_R} \) by

\[
N_{C \cap S_R}(\gamma) := \lim_{n \uparrow \infty} \lim_{m \uparrow \infty} \langle \mu_{C \cap B_R^c}(\gamma), \phi_{n,m} \rangle
\]

where \( \{\phi_{n,m}\}_{n,m \geq 1} \) is a double sequence of test functions such that \( \phi_{n,m} = 1 \) on \([t_0, n] \), \( \phi_{n,m} \to \chi_{[t_0,n]} \) pointwise. Then if we choose such test functions \( \phi_{n,m} \) in such a way that their variation is bounded uniformly in \( n \) and \( m \), by the dominated convergence theorem and by Theorem 5 one has

\[
\langle \psi_0, \chi_C(P_+) \psi_0 \rangle = \mathbb{E}(\chi_C(P_+))
\]

\[
= \lim_{R \uparrow \infty} \lim_{n \uparrow \infty} \sum_{m \uparrow \infty} \mathbb{E}(\langle \mu_{C \cap B_R^c}, \phi_{n,m} \rangle)
\]

\[
= \lim_{R \uparrow \infty} \lim_{n \uparrow \infty} \int_{t_0}^n dt \int_{(C \cap S_R) \cup (\partial C \cap B_R^c)} d\sigma(x) J^{\psi_1}(x) \cdot n(x).
\]
The proof is then concluded by proving that

$$\lim_{R \uparrow \infty} \lim_{n \uparrow \infty} \int_{t_0}^n dt \int_{\partial C \cap B_R^c} d\sigma(x) J_{\psi_t}(x) \cdot n(x) = 0. \quad (4.1)$$

Since $n \cdot x = 0$ on $\partial C$ and $\|J_{\psi_t}\| \leq \|\psi_t^* \nabla \psi_t\| = \|\psi_t P \psi_t\|$, one has

$$\left| \int_{t_1}^n dt \int_{\partial C \cap B_R^c} d\sigma(x) J_{\psi_t}(x) \cdot n(x) \right| \leq \int_{t_1}^n dt \int_{\partial C \cap B_R^c} d\sigma(x) \|\psi_t(x) P \psi_t(x)\| \leq \int_{t_1}^n dt \int_{\partial C \cap B_R^c} d\sigma(x) \left\| \psi_t(x) \left( P - \frac{Q}{t} \right) \psi_t(x) \right\|.$$

Thus, since $\psi_t P \psi_t \in L^1(\partial C)$, by the monotone convergence theorem, (4.1) follows from

$$\int_{t_0}^\infty dt \int_{\partial C \cap B_R^c} d\sigma(x) \left\| \psi_t(x) \left( P - \frac{Q}{t} \right) \psi_t(x) \right\| < +\infty \quad (4.2)$$

for some $R > 0$. By trace estimates on functions in $H^1(\mathbb{R}^3)$ of the kind

$$\| \cdot \|_{L^2(\partial C)} \leq c \| \cdot \|_{H^1(\mathbb{R}^3)},$$

(see e.g. [B, chap. 5]) one has

$$\int_{\partial C \cap B_R^c} d\sigma(x) \left\| \psi_t(x) \left( P - \frac{Q}{t} \right) \psi_t(x) \right\| \leq c \|\theta(Q)\psi_t\|_{H^1} \left\| \left( P - \frac{Q}{t} \right) \psi_t \right\|_{H^1}.$$

so that (4.2) is a consequence of h.5.

$$\square$$

5. On the admissibility conditions.

When $V = 0$, $\psi_0 \in H^2(\mathbb{R}^3)$ and $|Q|\psi_0 \in L^2(\mathbb{R}^3)$, by the explicit expression for $e^{it\Delta} \psi_0$ one has

$$\left\| \left( P - \frac{1}{t} Q \right) \psi_t \right\|_{L^2} = \frac{1}{t} \|Q\psi_0\|_{L^2}.$$
and therefore the free case satisfies the admissibility hypothesis h.2-h.4 (with $P_+ = P$).

Now we come to the interacting case. The condition h.3 gives no trouble: it follows from fairly general hypotheses on the potential function $V$. Indeed by [DG, thm. 4.4.1],

$$V(-\Delta + 1)^{-1} \text{ is compact} \quad (5.1)$$

and

$$\int_1^{+\infty} dR \left\| (-\Delta + 1)^{-1} \nabla V \chi_{[1, +\infty)} \left( \frac{\|x\|}{R} \right) (-\Delta + 1)^{-1} \right\|_{L^2, L^2} < +\infty, \quad (5.2)$$

imply (a stronger version of) h.3. By [HS, thm. 14.9], if for all $\epsilon > 0$ we can decompose $V = V_1 + V_2$ with $V_1 \in L^2(\mathbb{R}^3)$ and $V_2 \in L^\infty(\mathbb{R}^3)$, with $\|V_2\|_\infty < \epsilon$, then (5.1) holds true. If, outside some ball, $V$ is differentiable with its first derivatives decaying at infinity faster than $\|x\|^{-1}$ then condition (5.2) follows.

As regards the condition h.2, by the proof of [C4, lemma 4] one has

$$\left\| \left( P - \frac{Q}{t} \right) \psi_t \right\|_{L^2} \leq \frac{1}{t} \|(P - Q)\psi_1\|_{L^2} + \frac{1}{t} \int_1^t ds \|\psi_s \nabla V\|_{L^2}. \quad (5.3)$$

By (5.1) $V$ is infinitesimally $(-\Delta)$-operator-bounded (see e.g. [HS, thm. 14.2]) and so (see [C1, thm. 2.1(iv)]) $\|(i\nabla - x)\psi_t\|_{L^2} < +\infty$ for all $t$ if $\psi_0 \in H^2(\mathbb{R}^3)$, $|Q|\psi_0 \in L^2(\mathbb{R}^3)$. Therefore h.2 follows from

$$\|\psi_t \nabla V\|_{L^2} \leq c (1 + |t|)^{-\sigma}, \quad \sigma > 1. \quad (5.3)$$

We introduce the notations $\langle x \rangle$ for the function $(1 + \|x\|^2)^{1/2}$ and $\langle Q \rangle$ for the corresponding multiplication operator.

In the case $\langle Q \rangle^s \nabla V \in L^\infty(\mathbb{R}^3)$ and $\langle Q \rangle^s \psi_0 \in L^2(\mathbb{R}^3)$ for some $s$, (5.3) then follows from

$$\|\langle Q \rangle^{-s} e^{-itH} \langle Q \rangle^{-s}\|_{L^2, L^2} \leq c (1 + |t|)^{-\sigma}. \quad (5.4)$$

Such a kind of estimates were obtained in many paper about propagation estimates for solution of Schrödinger equations (see e.g. [ACS], [CP], [JK],...
For example, by [CP, thm.1], one obtains that h.2 holds true under the following hypotheses:

\[ \psi_0 \in H^2(\mathbb{R}^3), \quad \langle Q \rangle^s \psi_0 \in L^2(\mathbb{R}^3), \quad \phi(H)\psi_0 = \psi_0, \quad (5.5) \]

\[ V = V_S + V_L, \quad V_S \in C^1(\mathbb{R}^3), \quad V_L \in C^{k+3}(\mathbb{R}^3), \quad (5.6) \]

\[ \|D^\alpha V_S(x)\| \leq c \langle x \rangle^{-2k-|\alpha|-\epsilon}, \quad |\alpha| \leq 1, \quad (5.7) \]

\[ \|D^\alpha V_L(x)\| \leq c \langle x \rangle^{-|\alpha|-\epsilon}, \quad |\alpha| \leq k + 1, \quad (5.8) \]

where \( \phi \in C^\infty(0, +\infty) \) is equal to zero on a (arbitrarily small) neighbourhood of zero and

\[ \epsilon > 0, \quad k \geq 3, \quad 1 < s \leq k, \quad s \left(1 - \frac{1}{k}\right) > 1 \quad (5.9) \]

\( (s(1 - 1/k) > 3/2 \) gives h.2.1). Note that under these conditions (5.1) and (5.2) hold true. Moreover, by [HS, thm. 16.1] there are no strictly positive eigenvalues and, by [DG, thm. 4.7.1], one has also existence and completeness of the (modified) wave operators, so that, for every Borel set \( A \),

\[ \langle \psi_0, \chi_A(P_+)\psi_0 \rangle = \int_A dk |\hat{\psi}_{\text{out}}(k)|^2. \]

Thus h.4 holds true and in conclusion

\[ (5.5)-(5.9) \implies \text{admissibility}. \]

Remark. By [JK, thm. 10.3] the low energy cutoff hypothesis can be removed when 0 is neither an eigenvalue nor a resonance, \( \epsilon > 3 \) and \( s > 5/2 \). If 0 is not an eigenvalue but is a resonance then \( \psi_0 \) has to be orthogonal to the function corresponding to the resonance, otherwise in (5.4) one has \( \sigma = 1/2 \) (see [JK, thm. 10.5]).

As regards strong admissibility, i.e. hypothesis h.5, the main point in the paper [AP] by Amrein and Pearson was just to find the conditions on \( \psi_0 \) and \( V \) leading to such an hypothesis. By using again (5.4) and commutator estimates, by [AP, lemmata 5-8] one obtains

\[ (5.5)-(5.9) \text{ with } s > 5/3 \text{ and } \epsilon > 2/3 \implies \text{strong admissibility}. \]
In [AP, section 6] it is then shown how to avoid regularity hypotheses on the short range component of $V$. However in this situation the hypotheses on the initial state $\psi_0$ become less transparent. Indeed there one requires $W_+^* \psi_0 = \varphi(H_1)W_+^* \psi_0$, $\langle Q \rangle^s W_+^* \psi_0 \in L^2(\mathbb{R}^3)$, $s > 2$, $\varphi \in C_c(0, +\infty)$, $H_1 := -\Delta + V_1$, $V_1$ the smooth part of $V$, $W_+$ the relative wave operator $W_+ := \lim_{t \to \infty} e^{-itH} e^{itH_1}$.

We conclude the section by listing the conditions on the couple $(\psi_0, V)$ used in other papers (beside the already quoted [AP]) in order to obtain the flux-across-surfaces theorem ($\mathcal{S}(\mathbb{R}^3)$ denoting the space of functions of rapid decrease):

1. In [DDGZ1] it is assumed that $V = 0$ and $\psi_0 \in \mathcal{S}(\mathbb{R}^3)$.
2. In [AZ] it is assumed that $\langle Q \rangle^s \psi_{\text{out}} \in L^2(\mathbb{R}^3)$, $s > 5/2$, $\psi_{\text{out}} = \varphi(-\Delta) \psi_{\text{out}}$, $\varphi \in C_c^\infty(0, +\infty)$, $V$ either has local singularities and decays faster than $\|x\|^{-2}$ at infinity or is in $C^4(\mathbb{R}^3)$ and decays faster than $\|x\|^{-1}$ (in this case $\hat{\psi}_{\text{out}}$ has to be in $C_c^4(\mathbb{R}^3 \setminus \{0\})$). By [JN], when $V$ is smooth, the condition on the outgoing state $\psi_{\text{out}}$ is implied by a similar one (with $s > 7/2$) on $\psi$.
3. In [TDM-B] it is assumed that $\psi_{\text{out}} \in \mathcal{S}(\mathbb{R}^3)$, that $V \in L^2(\mathbb{R}^3)$ is locally H"older continuous except at a finite number of points and is decaying faster than $\|x\|^{-4}$, and that $0$ is neither an eigenvalue nor a resonance. No energy cutoff condition on $\psi_0$ is required.
4. In [DPa] the results in [TDM-B] are extended to the case in which $0$ is either a zero-energy eigenvalue or resonance. There it is assumed that $\psi_0 \in \mathcal{S}(\mathbb{R}^3)$, $V \in L^2(\mathbb{R}^3)$ is locally H"older continuous except at a finite number of points and is decaying faster than $\|x\|^{-n}$ for all $n \in \mathbb{N}$, $\hat{\psi}_{\text{out}} \in C^5(\mathbb{R}^3 \setminus \{0\})$ and $\|D^\alpha \hat{\psi}_{\text{out}}(k)\| \leq c \langle k \rangle^{-3-|\alpha|}-\epsilon$, $|\alpha| \leq 5$, $\epsilon > 0$, $\|k\| \geq K_\alpha > 0$.
5. In [PT] and [DPa] the flux-across-surfaces theorem is proven in the case in which $\psi_0 \in \mathcal{S}(\mathbb{R}^3)$ and $H$ is the self-adjoint operator describing the Laplacean with a delta point interaction.

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