Multivariable $\rho$-contractions

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Abstract. We suggest a new version of the notion of $\rho$-dilation ($\rho > 0$) of an $N$-tuple $A = (A_1, \ldots, A_N)$ of bounded linear operators on a common Hilbert space. We say that $A$ belongs to the class $C_{\rho,N}$ if $A$ admits a $\rho$-dilation $\tilde{A} = (\tilde{A}_1, \ldots, \tilde{A}_N)$ for which $\zeta \tilde{A} := \zeta_1 \tilde{A}_1 + \cdots + \zeta_N \tilde{A}_N$ is a unitary operator for each $\zeta := (\zeta_1, \ldots, \zeta_N)$ in the unit torus $T^N$. For $N = 1$ this class coincides with the class $C_{\rho,1}$ of B. Sz.-Nagy and C. Foiaș. We generalize the known descriptions of $C_{\rho,1} = C_{\rho}$ to the case of $C_{\rho,N}$, $N > 1$, using so-called Agler kernels. Also, the notion of operator radii $w_{\rho} > 0$, is generalized to the case of $N$-tuples of operators, and to the case of bounded (in a certain strong sense) holomorphic operator-valued functions in the open unit polydisk $D^N$, with preservation of all the most important their properties. Finally, we show that for each $\rho > 1$ and $N > 1$ there exists an $A = (A_1, \ldots, A_N) \in C_{\rho,N}$ which is not simultaneously similar to any $T = (T_1, \ldots, T_N) \in C_{1,N}$, however if $A \in C_{\rho,N}$ admits a uniform unitary $\rho$-dilation then $A$ is simultaneously similar to some $T \in C_{1,N}$.

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1. Introduction

Linear pencils of operators $L_A(z) := A_0 + z_1 A_1 + \cdots + z_N A_N$ on a Hilbert space which take contractive (resp., unitary or $J$-unitary for some signature operator $J = J^* = J^{-1}$) values for all $z = (z_1, \ldots, z_N)$ in the unit torus $T^N := \{ \zeta \in \mathbb{C}^N : |\zeta_k| = 1, k = 1, \ldots, N \}$ serve as one of possible generalizations of a single contractive (resp., unitary, $J$-unitary) operator on a Hilbert space. They appear in constructions of Agler’s unitary colligation and corresponding conservative (unitary) scattering $N$-dimensional discrete-time linear system of Roesser type [1, 2], and also of Fornasini–Marchesini type [3], and dissipative (contractive), conservative...
(unitary) or $J$-conservative ($J$-unitary) scattering $N$-dimensional linear systems of one more form introduced in our paper [17] and studied in [17] [18] [19] [20] [21] [7]. These constructions, in particular, provide the transfer function realization formulae for certain classes of holomorphic functions [1] [8] [17] [19] [21] [7], the solutions to the Nevanlinna–Pick interpolation problem [2] [8], the Toeplitz corona problem [2] [8], and the commutant lifting problem [6] in several variables.

In [18] we developed the dilation theory for multidimensional linear systems, and in particular gave a necessary and sufficient condition for such a system to have a conservative dilation. As a special case, this gave a criterion for the existence of a unitary dilation of a contractive (on $T^N$) linear pencil of operators on a Hilbert space. Linear pencils of operators satisfying this criterion inherit the most important properties of single contraction operators on a Hilbert space (note that, due to [22], not all linear pencils which take contractive operator values on $T^N$ satisfy this criterion).

The purpose of the present paper is to develop the theory of $\rho$-contractions in several variables in the framework of “linear pencils approach”. We introduce the notion of $\rho$-dilation of an $N$-tuple $\mathbf{A} = (A_1,\ldots,A_N)$ of bounded linear operators on a common Hilbert space by means of a simultaneous $\rho$-dilation, in the sense of B. Sz.-Nagy and C. Foiaş [32] [34], of the values of a homogeneous linear pencil of operators $z\mathbf{A} := \sum_{k=1}^{N} z_k A_k$. The class $C_{\rho,N}$ consists of those $N$-tuples of operators $\mathbf{A} = (A_1,\ldots,A_N)$ ($\rho$-contractions) for which there exists a $\rho$-dilation $\tilde{\mathbf{A}} = (\tilde{A}_1,\ldots,\tilde{A}_N)$ such that the operators $\zeta \tilde{\mathbf{A}} = \sum_{k=1}^{N} \zeta_k \tilde{A}_k$ are unitary for all $\zeta = (\zeta_1,\ldots,\zeta_N) \in T^N$. On the one hand, this class generalizes the class $C_{\rho,1} = C_\rho$ of Sz.-Nagy and Foiaş [32] [34] consisting of operators which admit a unitary $\rho$-dilation to the case $N > 1$. On the other hand, this class generalizes the class of $N$-tuples of operators $\mathbf{A}$ for which the associated linear pencil of operators $z\mathbf{A}$ admits a unitary dilation in the sense of [18] (this corresponds to $\rho = 1$) to the case of $N$-tuples of operators $\mathbf{A}$ which have a unitary $\rho$-dilation for $\rho \neq 1$.

The paper is organized as follows. Section 2 gives preliminaries on $\rho$-contractions for the case $N = 1$. Namely, we recall the relevant definitions, the known criteria for an operator to be a $\rho$-contraction, i.e., to belong to the class $C_\rho$ of Sz.-Nagy and Foiaş, the notion of operator radii $w_\rho$, and their properties, and the theorem on similarity of $\rho$-contractions to contractions. In Section 3 we give the definitions of a $\rho$-dilation of an $N$-tuple of operators, and of the class $C_{\rho,N}$ of $\rho$-contractions for the case $N > 1$, and prove a theorem which generalizes the criteria of $\rho$-contractiveness to this case, as well as to the case $0 < \rho \neq 1$. Some properties of classes $C_{\rho,N}$ are discussed. Then it is shown that the notions of a $\rho$-contraction and of the corresponding class $C_{\rho,N}$, as well as the theorem just mentioned, can be extended to holomorphic functions on the open unit polydisk $D^N := \{z \in \mathbb{C}^N : |z_k| < 1, k = 1,\ldots,N\}$ that are bounded in a certain strong sense, though the notion of unitary $\rho$-dilation is not relevant any more in this case. In Section 4 we define operator radii $w_{\rho,N}$ of $N$-tuples of operators, and operator-function radii $w_{\rho,N}^{(\infty)}$ of bounded holomorphic functions on $D^N$, $\rho > 0$. These radii
generalize \( w_\rho \)'s and inherit all the most important properties of them. In Section 3, we prove that for each \( \rho > 1 \) and \( N > 1 \) there exists an \( A = (A_1,\ldots,A_N) \in C_{\rho,N} \) which is not simultaneously similar to any \( T = (T_1,\ldots,T_N) \in C_{1,N} \). Then we introduce the classes \( C_{\rho,N}^u, \rho > 0 \), of \( N \)-variable \( \rho \)-contractions \( A = (A_1,\ldots,A_N) \) which admit a uniform unitary \( \rho \)-dilation. We prove that if \( A \in C_{\rho,N}^u \) for some \( \rho > 1 \) then \( A \) is simultaneously similar to some \( T \in C_{1,N}^u \). Note, that since the class \( C_{\rho,N}^u \) (as well as \( C_{\rho,N} \)) increases as a function of \( \rho \), for any \( \rho \leq 1 \) an \( A \in C_{\rho,N}^u \) (resp., \( A \in C_{\rho,N} \)) belongs to \( C_{1,N}^u \) (resp., \( C_{1,N} \)) itself. We show the relation of our results to ones of G. Popescu \[30\] where a different notion of multivariable \( \rho \)-contractions has been introduced, and the relevant theory has been developed. The classes \( C_{\rho,N}^u, \rho > 0 \), which appear in Section 3 in connection with the similarity problem discussed there, certainly deserve a further investigation.

2. Preliminaries

Let \( L(\mathcal{X},\mathcal{Y}) \) denote the Banach space of bounded linear operators mapping a Hilbert space \( \mathcal{X} \) into a Hilbert space \( \mathcal{Y} \), and \( L(\mathcal{X}) := L(\mathcal{X},\mathcal{X}) \). For \( \rho > 0 \), an operator \( A \in L(\mathcal{X}) \) is said to be a \( \rho \)-dilation of an operator \( A \in L(\mathcal{X}) \) if \( \tilde{A} \supset \mathcal{X} \) and

\[
A^n = \rho P_X \tilde{A}^n |\mathcal{X}^\prime, \quad n \in \mathbb{N},
\]

where \( P_X \) denotes the orthogonal projection onto the subspace \( \mathcal{X} \) in \( \mathcal{X}^\prime \). If, moreover, \( \tilde{A} \) is a unitary operator then \( \tilde{A} \) is called a unitary \( \rho \)-dilation of \( A \). In \[32\] (see also \[31\]) B. Sz.-Nagy and C. Foiaş introduced the classes \( C_{\rho} \), \( \rho > 0 \), consisting of operators which admit a unitary \( \rho \)-dilation. Due to B. Sz.-Nagy \[31\], the class \( C_1 \) is precisely the class of all contractions, i.e., operators \( A \) such that \( \|A\| \leq 1 \). C. A. Berger \[9\] showed that the class \( C_2 \) is precisely the class of all operators \( A \in L(\mathcal{X}) \), for some Hilbert space \( \mathcal{X} \), which have the numerical radius

\[
w(A) = \sup \{ \|Ax, x\| : x \in \mathcal{X}, \|x\| = 1 \}
\]

equal to at most one. Thus, the classes \( C_{\rho}, \rho > 0 \), provide a framework for simultaneous investigation of these two important classes of operators.

Recall that the Herglotz (or Caratheodory) class \( \mathcal{H}(\mathcal{X}) \) (respectively, the Schur class \( \mathcal{S}(\mathcal{X}) \)) consists of holomorphic functions \( f \) on the open unit disk \( \mathbb{D} \) which take values in \( L(\mathcal{X}) \) and satisfy \( \text{Re} f(z) = f(z) + f(z)^* \geq 0 \) in the sense of positive semi-definiteness of an operator (resp., \( \|f(z)\| \leq 1 \)) for all \( z \in \mathbb{D} \). Let us recall some known characterizations of the classes \( C_{\rho} \).

**Theorem 2.1.** Let \( A \in L(\mathcal{X}) \) and \( \rho > 0 \). The following statements are equivalent:

(i): \( A \in C_{\rho} \);

(ii): the function \( k^A_\rho(z,w) := \rho I_X - (\rho - 1)((zwA + (wA)^*) + (\rho - 2)(wA)^*zA \) satisfies \( k^A_\rho(z,z) \geq 0 \) for all \( z \in \text{ clos}(\mathbb{D}) \);

(iii): the function \( \psi^A_\rho(z) := (1 - \frac{z}{\rho})I_X + \frac{z}{\rho}(I_X - zA)^{-1} \) belongs to \( \mathcal{H}(\mathcal{X}) \);

(iv): the function \( \varphi^A_\rho(z) := zA ((\rho - 1)zA - \rho I_X)^{-1} \) belongs to \( \mathcal{S}(\mathcal{X}) \).
Conditions (ii) and (iii) of Theorem 2.1 each characterizing the class \( C_\rho \) appear in [32], while condition (iv) is due to C. Davis [11].

**Corollary 2.2.** Condition (ii) in Theorem 2.1 can be replaced by

\[
\text{(ii'): } k^A(C, C) := \rho I_X \otimes I_{H_C} - (\rho - 1)(A \otimes C + (A \otimes C)^*) + (\rho - 2)(A \otimes C)^*(A \otimes C) \succeq 0
\]

for any contraction \( C \) on a Hilbert space \( H_C \).

**Proof.** Indeed, (ii')\( \Rightarrow \) (ii), hence (ii')\( \Rightarrow \) (i). Conversely, if \( A \in C_\rho \cap L(X) \) then for any contraction \( C \) on \( H_C \) one has \( A \otimes C \in C_\rho \) because, by [31], \( C \) admits a unitary dilation \( \tilde{C} \), and \( A \) admits a unitary \( \rho \)-dilation \( \tilde{A} \), thus \( \tilde{A} \otimes \tilde{C} \) is a unitary \( \rho \)-dilation of \( A \otimes C \):

\[
(A \otimes C)^n = A^n \otimes C^n = (\rho P_X \tilde{A}^n|X) \otimes (P_{H_C} \tilde{C}^n|H_C)
\]

\[
= \rho P_X \otimes H_C (\tilde{A}^n \otimes \tilde{C}^n)|X \otimes H_C
\]

\[
= \rho P_X \otimes H_C (\tilde{A} \otimes \tilde{C})^n|X \otimes H_C, \quad n \in \mathbb{N}.
\]

Therefore, \( k^A(C, C) = k^A \otimes C(1, 1) \succeq 0 \), i.e., (ii') is valid. \( \square \)

**Corollary 2.3.** Condition

\( (v): A \otimes C \in C_\rho \) for any contraction \( C \) on a Hilbert space,

is equivalent to each of conditions (i)–(iv) of Theorem 2.1.

**Proof.** See the proof of Corollary 2.2 \( \square \)

Any operator \( A \in C_\rho \) is power-bounded:

\[
\|A^n\| \leq \rho, \quad n \in \mathbb{N}, \tag{2.2}
\]

moreover, its spectral radius

\[
\nu(A) = \lim_{n \to +\infty} \|A^n\|^{\frac{1}{n}} \tag{2.3}
\]

is at most one. In [32] an example of a power-bounded operator which is not contained in any of the classes \( C_\rho, \rho > 0 \), is given. However, J. A. R. Holbrook [15] showed that any bounded linear operator \( A \) with \( \nu(A) \leq 1 \) can be approximated in the operator norm topology by elements of the classes \( C_\rho \). More precisely, if \( C_\infty \) denotes the class of bounded linear operators with spectral radius at most one, and \( X \) is a Hilbert space, then

\[
C_\infty \cap L(X) = \text{clos} \left\{ \bigcup_{0 < \rho < \infty} C_\rho \cap L(X) \right\}. \tag{2.4}
\]

For a fixed Hilbert space \( X \), the class \( C_\rho \) as a function of \( \rho \) increases [32]:

\[
C_\rho \subset C_\rho' \quad \text{for } \rho < \rho'. \tag{2.5}
\]

Moreover, it was shown by E. Durszt [13] that \( C_\rho \) increases strictly for \( \dim X \geq 2 \):

\[
C_\rho \neq C_\rho' \quad \text{for } \rho \neq \rho'.
\]
Proposition 2.4. For $\mathcal{X} = \mathbb{C}$, the classes $C_\rho$ coincide for all $\rho \geq 1$, and strictly increase for $0 < \rho < 1$:

$$C_\rho \subsetneq C_{\rho'} \text{ for } 0 < \rho < \rho' \leq 1.$$  

Proof. If $a \in \mathbb{C} \cong L(\mathbb{C})$ belongs to $C_\rho$ then $\|a\| = |a| = \nu(a) \leq 1$. Hence $C_\rho \subset C_1$ for any $\rho > 0$. Since (2.5) implies $C_\rho \supseteq C_1$ for $\rho \geq 1$, we get $C_\rho = C_1$ for this case, that proves the first part of this proposition.

For the proof of the second part, we will show that for any $\varepsilon, \rho : 0 < \varepsilon < \rho < 1$, one has

$$a := \frac{\rho}{2 - \rho} \in C_\rho \setminus C_{\rho - \varepsilon}.$$  

If $0 \leq \varepsilon < \rho$ then, by condition (ii) in Theorem 2.1, the inclusion $a \in C_{\rho - \varepsilon}$ is equivalent to

$$\rho - \varepsilon - (\rho - \varepsilon - 1) |a z + \bar{a} \bar{z}| + (\rho - \varepsilon - 2) |a z|^2 \geq 0, \; \; z \in \operatorname{clos}(\mathbb{D}),$$

which for $a = \frac{\rho}{2 - \rho}$ turns into

$$\rho - \varepsilon - 2(\rho - \varepsilon - 1) \frac{\rho r}{2 - \rho} \cos \theta + (\rho - \varepsilon - 2) \left(\frac{\rho r}{2 - \rho}\right)^2 \geq 0, \; \; r \in [0, 1], \; \theta \in [0, 2\pi).$$

Since $\rho - \varepsilon - 1 < 0$, the left-hand side of this inequality, as a function of $\theta$ for a fixed $r$, has a minimum at $\theta = \pi$, so the latter condition turns into

$$\rho - \varepsilon + 2(\rho - \varepsilon - 1) \frac{\rho r}{2 - \rho} + (\rho - \varepsilon - 2) \left(\frac{\rho r}{2 - \rho}\right)^2 \geq 0, \; \; r \in [0, 1].$$

The left-hand side attains its minimum at $r = 1$, thus the latter inequality turns into

$$\rho - \varepsilon + 2(\rho - \varepsilon - 1) \frac{\rho}{2 - \rho} + (\rho - \varepsilon - 2) \left(\frac{\rho}{2 - \rho}\right)^2 = -\frac{4\varepsilon}{(2 - \rho)^2} \geq 0,$$

which is possible if and only if $\varepsilon = 0$. Thus, (2.6) is true. \qed

The properties of the classes $C_\rho$ become more clear due to the following numerical characteristics of operators. J. A. R. Holbrook [15] and J. P. Williams [35], independently, introduced for any $A \in L(\mathcal{X})$ the operator radii

$$w_\rho(A) := \inf \{u > 0 : \frac{1}{u} A \notin C_\rho\}. $$  

(2.7)

Theorem 2.5. $w_\rho(\cdot)$ has the following properties:

(i): $w_\rho(A) < \infty$;

(ii): $w_\rho(A) > 0$ unless $A = 0$, moreover, $w_\rho(A) \geq \frac{1}{\rho} \|A\|$;

(iii): $\forall \mu \in \mathbb{C}, \; w_\rho(\mu A) = |\mu| w_\rho(A);$  

(iv): $w_\rho(A) \leq 1$ if and only if $A \in C_1$;

(v): $w_\rho(\cdot)$ is a norm on $L(\mathcal{X})$ for any $\rho : 0 < \rho \leq 2$, and not a norm on $L(\mathcal{X})$, $\dim \mathcal{X} \geq 2$, for any $\rho > 2$;

(vi): $w_1(A) = \|A\|$ (of course, here $\|\cdot\|$ is the operator norm on $L(\mathcal{X})$ with respect to the Hilbert-space metric on $\mathcal{X}$);
implies (xvii).

Finally, property (xvii) easily follows from (x) and (xvi). Indeed, for $0 < \rho < 1$ one has $w_{\rho}(A) > w_{\infty}(A)$, hence $w_{\rho}(A) > w_{\infty}(A)$, and follows also from (xvi) and (xviii).

Properties (i)–(xii), (xviii), and (xx) were proved by J. A. R. Holbrook [15]. Properties (xiii)–(xvi) were discovered by T. Ando and K. Nishio [4]. Property (xix) was shown by K. Okubo and T. Ando [26], and follows also from (xvi) and (xviii).

Let us note that properties of the classes $C_\rho$ discussed before Theorem 2.5 including Proposition 2.4 can be deduced from properties (iv), (vi)–(x) in Theorem 2.5. Due to property (iv) in Theorem 2.5 operators from the classes $C_\rho$ are called $\rho$-contractions.

Any $A \in C_\rho$ satisfies the following generalized von Neumann inequality [32]:

$$\|p(A)\| \leq \max_{|z| \leq 1} |zp(z) + (1 - \rho)p(0)|.$$  \hspace{1cm} (2.8)
Let \( A \in L(\mathcal{X}), B \in L(\mathcal{Y}) \). Then \( A \) is said to be similar to \( B \) if there exists a bounded invertible operator \( S \in L(\mathcal{X}, \mathcal{Y}) \) such that
\[
A = S^{-1}BS.
\]

(2.9)

B. Sz.-Nagy and C. Foiaş proved in [33] (see also [34]) that any \( A \in C_\rho \) is similar to some \( T \in C_1 \), i.e., any \( \rho \)-contraction is similar to a contraction.

To conclude this section, let us remark that the classes \( C_\rho \) are of continuous interest, e.g., see recent works [12, 10, 5, 24, 27]. In [30] the classes \( C_\rho \) were extended to a multivariable setting; we shall discuss this generalization in Section 5.

3. The classes \( C_{\rho,N} \)

Let \( \rho > 0 \). We will say that an \( N \)-tuple of operators \( \tilde{A} = (\tilde{A}_1, \ldots, \tilde{A}_N) \in L(\tilde{\mathcal{X}})^N \) is a \( \rho \)-dilation of an \( N \)-tuple of operators \( A = (A_1, \ldots, A_N) \in L(\mathcal{X})^N \) if \( \tilde{\mathcal{X}} \supset \mathcal{X} \), and for any \( z = (z_1, \ldots, z_N) \in \mathbb{C}^N \) the operator \( z\tilde{A} = \sum_{k=1}^N z_k \tilde{A}_k \) is a \( \rho \)-dilation, in the sense of [32], of the operator \( zA = \sum_{k=1}^N z_k A_k \), i.e.,
\[
(zA)^n = \rho_P(\rho)(zA)^n |\mathcal{X}|, \quad z \in \mathbb{C}^N, \quad n \in \mathbb{N}.
\]

(3.1)

These relations are equivalent to
\[
A^t = \rho P_X \tilde{A}^t |\mathcal{X}|, \quad t \in \mathbb{Z}_+^N := \{ \tau \in \mathbb{Z}^N : \tau_k \geq 0, k = 1, \ldots, N \},
\]

(3.2)

where \( A^t, t \in \mathbb{Z}_+^N \), are symmetrized multi-powers of \( A \):
\[
A^t := \frac{t!}{|t|!} \sum_{\sigma} A_{[\sigma(1)]} \cdots A_{[\sigma(|t|)]},
\]

and analogously for \( \tilde{A} \). Here for a multi-index \( t = (t_1, \ldots, t_N) \), \( t! := t_1! \cdots t_N! \) and \( |t| := t_1 + \cdots + t_N \); \( \sigma \) runs over the set of all permutations with repetitions in a string of \( |t| \) numbers from the set \( \{1, \ldots, N\} \) such that the \( \kappa \)-th number \( [\kappa] \in \{1, \ldots, N\} \) appears in this string \( t_{[\kappa]} \) times. Say, if \( t = (1, 2, 0, \ldots, 0) \) then
\[
A^t = A_1^2 + A_2 A_1 A_2 + A_2^2 A_1.
\]

In the case of a commutative \( N \)-tuple \( A \) one has \( A^t = A_1^t \cdots A_N^t \), i.e., a usual multi-power.

Note 3.1. Compare (3.1) and (3.2) with (2.1).

In the case \( \rho = 1 \) the notion of \( \rho \)-dilation of an \( N \)-tuple of operators \( A = (A_1, \ldots, A_N) \) coincides with the notion of dilation of \( A \) (or corresponding linear pencil \( zA \)) as defined in [18].

We will call \( \tilde{A} \in L(\tilde{\mathcal{X}})^N \) a unitary \( \rho \)-dilation of \( A \in L(\mathcal{X})^N \) if \( \tilde{A} \) is a \( \rho \)-dilation of \( A \) and for any \( \zeta \in \mathbb{T}^N \) the operator \( \zeta \tilde{A} = \sum_{k=1}^N \zeta_k \tilde{A}_k \) is unitary. The class of operator \( N \)-tuples which admit a unitary \( \rho \)-dilation will be denoted by \( C_{\rho,N} \)
Let $C^N$ denote the family of all $N$-tuples $C = (C_1, \ldots, C_N)$ of commuting strict contractions on a common Hilbert space $H_C$, i.e., $C_k C_j = C_j C_k$ and $\|C_k\| < 1$ for all $k,j \in \{1, \ldots, N\}$. An $L(\mathcal{X})$-valued function
\[
k(z, w) = \sum_{(t,s) \in \mathbb{Z}_+^N \times \mathbb{Z}_+^N} \hat{k}(t,s) \bar{w}^t z^t, \quad (z, w) \in \mathbb{D}^N \times \mathbb{D}^N,
\]
which is holomorphic in $z \in \mathbb{D}^N$ and anti-holomorphic in $w \in \mathbb{D}^N$, will be called an Agler kernel if
\[
k(C, C) := \sum_{(t,s) \in \mathbb{Z}_+^N \times \mathbb{Z}_+^N} \hat{k}(t,s) \otimes C^* C^t \succeq 0, \quad C \in C^N,
\]
where the series converges in the operator norm topology on $L(\mathcal{X} \otimes H_C)$. The Agler–Herglotz class $\mathcal{AH}_N(\mathcal{X})$ (resp., the Agler–Schur class $\mathcal{AS}_N(\mathcal{X})$) is the class of all $L(\mathcal{X})$-valued functions $f$ holomorphic on $\mathbb{D}^N$ for which $k(z, w) = f(z) + f(w)^*$ (resp., $k(z, w) = I_{\mathcal{X}} - f(w)^* f(z)$) is an Agler kernel. Agler kernels, as well as the classes $\mathcal{AH}_N(\mathcal{X})$ and $\mathcal{AS}_N(\mathcal{X})$, were defined and studied by J. Agler in [1].

The von Neumann inequality [25] implies that $AS$ implies $\rho, N \in$ Corollary 2.2, is an Agler kernel (by condition (ii') from Corollary 2.2, and added condition (v) from Corollary 2.3.

Proof of Theorem 3.3.

(i) Let $A \in L(\mathcal{X})^N$, $\rho > 0$. The following conditions are equivalent:

(i): $A \in C_{\rho, N}$;

(ii): the function $k^A_{\rho, N}(z, w) := \rho I_{\mathcal{X}} - (\rho - 1) ((z A + (w A)^*) + (\rho - 2) (w A)^* z A)$ is an Agler kernel on $\mathbb{D}^N \times \mathbb{D}^N$;

(iii): the function $\psi^A_{\rho, N}(z) := (1 - \frac{2}{\rho} I_{\mathcal{X}} + \frac{2}{\rho} (I_{\mathcal{X}} - z A)^{-1})$ belongs to $\mathcal{AH}_N(\mathcal{X})$;

(iv): the function $\phi^A_{\rho, N}(z) := z A ((\rho - 1) z A - \rho I_{\mathcal{X}})^{-1}$ belongs to $\mathcal{AS}_N(\mathcal{X})$;

(v): $A \otimes C := \sum_{k=1}^N A_k \otimes C_k \in C_{\rho, 1}$ for all $C \in C^N$.

Remark 3.4. This theorem generalizes Theorem [24] with condition (ii) replaced by condition (ii') from Corollary 2.2 and added condition (v) from Corollary 2.3.

Proof of Theorem 3.3 (i)⇔(iii). The proof of this part combines the idea of B. Sz.-Nagy and C. Foiaş [32] for the proof of the equivalence (i)⇔(iii) in Theorem 2.1 (see Remark 3.3) with Agler’s representation of functions from $\mathcal{AH}_N(\mathcal{X})$ [1]. Let $A = (A_1, \ldots, A_N) \in C_{\rho, N} \cap L(\mathcal{X})^N$, and $\tilde{A} = (\tilde{A}_1, \ldots, \tilde{A}_N) \in L(\tilde{\mathcal{X}})^N$ be a unitary $\rho$-dilation of $A$. By Corollary 4.3 in [13], the linear function $I_{\tilde{\mathcal{X}}} \otimes z \tilde{A}$ belongs to the class $\mathcal{AS}_N(\tilde{\mathcal{X}})$. Since for any $C \in C^N$ one has $(1 + \varepsilon) C \in C^N$ for a sufficiently small $\varepsilon > 0$, the operator $\tilde{A} \otimes C$, as well as $\tilde{A} \otimes (1 + \varepsilon) C$, is contractive. Thus, $\tilde{A} \otimes C$ is a strict contraction, and the series
\[
I_{\tilde{\mathcal{X}}} \hat{\otimes} H_C + 2 \sum_{n=1}^{\infty} (\tilde{A} \otimes C)^n
\]
converges in the $L(\tilde{X} \otimes \mathcal{H}_C)$-norm to
\[(I_{\tilde{X} \otimes \mathcal{H}_C} + \tilde{A} \otimes C)(I_{\tilde{X} \otimes \mathcal{H}_C} - \tilde{A} \otimes C)^{-1}.\]
Moreover,
\[\text{Re}[(I_{\tilde{X} \otimes \mathcal{H}_C} + \tilde{A} \otimes C)(I_{\tilde{X} \otimes \mathcal{H}_C} - \tilde{A} \otimes C)^{-1}] \geq 0.\] (3.4)
Therefore,
\[P_{X \otimes \mathcal{H}_C}(I_{\tilde{X} \otimes \mathcal{H}_C} + \tilde{A} \otimes C)(I_{\tilde{X} \otimes \mathcal{H}_C} - \tilde{A} \otimes C)^{-1}|X \otimes \mathcal{H}_C\]
\[= P_{X \otimes \mathcal{H}_C} \left( I_{\tilde{X} \otimes \mathcal{H}_C} + 2 \sum_{n=1}^{\infty} (\tilde{A} \otimes C)^n \right) |X \otimes \mathcal{H}_C\]
\[= I_{X \otimes \mathcal{H}_C} + 2 \sum_{n=1}^{\infty} \sum_{|t|=n} \frac{n!}{t!} (P_X \otimes I_{\mathcal{H}_C})(\tilde{A}^t \otimes C^t)|X \otimes \mathcal{H}_C\]
\[= I_{X \otimes \mathcal{H}_C} + 2 \sum_{n=1}^{\infty} \sum_{|t|=n} \frac{n!}{t!} \tilde{A}^t \otimes C^t = I_{X \otimes \mathcal{H}_C} + 2 \sum_{n=1}^{\infty} (A \otimes C)^n\]
\[= (1 - \frac{2}{\rho}) I_{X \otimes \mathcal{H}_C} + \frac{2}{\rho} (I_{X \otimes \mathcal{H}_C} - A \otimes C)^{-1} = \psi_{\rho,N}^A(C),\]
and (3.4) implies $\text{Re} \psi_{\rho,N}^A(C) \geq 0$. Since the function $(I_{\tilde{X}} + z\tilde{A})(I_{\tilde{X}} - z\tilde{A})^{-1}$ is well-defined and holomorphic on $\mathbb{D}^N$, so is
\[\psi_{\rho,N}^A(z) = P_X(I_{\tilde{X}} + z\tilde{A})(I_{\tilde{X}} - z\tilde{A})^{-1}|X, \quad z \in \mathbb{D}^N,\] (3.5)
and we obtain $\psi_{\rho,N}^A \in \mathcal{AH}_N(X)$.

Conversely, let $\psi_{\rho,N}^A \in \mathcal{AH}_N(X)$. Since $\psi_{\rho,N}^A(0) = I_X$, according to [1], there exist a Hilbert space $X' \supset X$, its subspaces $X_1, \ldots, X_N$ satisfying $X = \bigoplus_{k=1}^N X_k$, and a unitary operator $U \in L(X)$ such that
\[\psi_{\rho,N}^A(z) = P_X(I_{\tilde{X}} + U(zP))(I_{\tilde{X}} - U(zP))^{-1}|X, \quad z \in \mathbb{D}^N,\] (3.6)
where $zP := \sum_{k=1}^N z_k P_{\tilde{X}_k}$, i.e., we get (3.5) with $\tilde{A}_k = U P_{\tilde{X}_k}$, $k = 1, \ldots, N$. Note that for each $\zeta \in T^N$ the operator $\zeta \tilde{A}$ is unitary. Developing both parts of (3.6) into the series in homogeneous polynomials convergent in the operator norm, we get
\[I_X + 2 \sum_{n=1}^{\infty} (zA)^n = I_X + 2 \sum_{n=1}^{\infty} P_X(z\tilde{A})^n|X, \quad z \in \mathbb{D}^N,\]
that implies the relations
\[(zA)^n = \rho P_X(z\tilde{A})^n|X, \quad n \in \mathbb{N},\]
for all $z \in \mathbb{D}^N$, and hence for all $z \in \mathbb{C}^N$. Thus, $\tilde{A}$ is a unitary $\rho$-dilation of $A$, and $A \in C_{\rho,N}$. The equivalence (i)$\Leftrightarrow$(iii) is proved.

Note that in this proof we have established that each Agler representation (3.6) of $\psi_{\rho,N}^A$ gives rise to a unitary $\rho$-dilation $\tilde{A}$ of $A$, and vice versa. Indeed, we
already showed that \([3.3]\) determines \(\tilde{A}\). Conversely, if \(\tilde{A} \in L(\tilde{X})^N\) is a unitary \(\rho\)-dilation of \(A\), then \([3.3]\) holds. Set \(U := \sum_{k=1}^N \tilde{A}_k \in L(\tilde{X})\) and \(\tilde{A} := \tilde{A}_k\tilde{X}\), \(k = 1, \ldots, N\). Then \(U\) is unitary, \(\tilde{X}\) is a closed subspace in \(\tilde{X}\) for each \(k = 1, \ldots, N\), the subspaces \(\tilde{X}\) are pairwise orthogonal, and \(\tilde{X} = \bigoplus_{k=1}^N \tilde{X}_k\) (see Proposition 2.4 in [17]). Thus, \([3.3]\) turns into \([3.6]\).

\(\nu \Leftrightarrow \iota\). Let \((\nu)\) be true. By Theorem 2.1 applied for \(A \otimes C\) with a \(C \in \mathcal{C}^N\), one has \(\varphi_{A \otimes C} \in \mathcal{S}(\mathcal{X} \otimes \mathcal{H}_C)\). For \(\varepsilon > 0\) small enough, \((1 + \varepsilon)C \in \mathcal{C}^N\), hence \(A \otimes (1 + \varepsilon)C \in \mathcal{C}_\rho\), and \(\varphi_{A \otimes (1 + \varepsilon)C} \in \mathcal{S}(\mathcal{X} \otimes \mathcal{H}_C)\). Thus,

\[
\psi_{\rho, N}(C) = A \otimes C((\rho - 1)A \otimes C - \rho I_{\mathcal{X} \otimes \mathcal{H}_C})^{-1} = \psi_{\rho}^{A \otimes (1 + \varepsilon)C}\left(\frac{1}{1 + \varepsilon}\right)
\]

is a contraction on \(\mathcal{X} \otimes \mathcal{H}_C\). In particular, \(\varphi_{\rho, N}(z)\) is well-defined, holomorphic and contractive on \(\mathbb{D}^N\). Finally, \(\varphi_{\rho, N} \in \mathcal{AS}_N(\mathcal{X})\).

Conversely, if \((\iota)\) is true then for any \(C \in \mathcal{C}^N\):

\[
\varphi_{\rho \otimes C}^\rho(\lambda) = \lambda A \otimes C((\rho - 1)A \otimes C - \rho I_{\mathcal{X} \otimes \mathcal{H}_C})^{-1} = \varphi_{\rho, N}^A(\lambda C)
\]

is well-defined, holomorphic and contractive for \(\lambda \in \mathbb{D}\). Thus, \(\varphi_{\rho \otimes C}^\rho \in \mathcal{S}(\mathcal{X} \otimes \mathcal{H}_C)\), and by Theorem 2.1 \(A \otimes C \in \mathcal{C}_\rho\).

\((\nu) \Leftrightarrow (\iota)\) and \((\nu) \Leftrightarrow (\iota)\) are proved analogously, using the following relations for \(C \in \mathcal{C}^N\), \(\lambda \in \mathbb{D}\):

\[
\psi_{\rho, N}(C) = \psi_{\rho}^{A \otimes (1 + \varepsilon)C}\left(\frac{1}{1 + \varepsilon}\right), \quad \psi_{\rho}^{A \otimes C}(\lambda) = \psi_{\rho, N}(A \lambda C),
\]

\[
k_{\rho, N}(C) = k_{\rho}^{A \otimes (1 + \varepsilon)C}(1, 1), \quad k_{\rho}^{A \otimes C}(\lambda, \lambda) = k_{\rho, N}(A \lambda C, \lambda C).
\]

The proof is complete. \(\square\)

**Remark 3.5.** For the case \(\rho = 1\) each of conditions (ii)–(v) in Theorem 3.3 means that for any \(C \in \mathcal{C}^N\) the operator \(A \otimes C\) is a contraction. In other words,

\[
A \in C_{1,N} \cap L(\mathcal{X})^N \iff L_A \in \mathcal{AS}_N(\mathcal{X}),
\]

that coincides with in [13] Corollary 4.3] (here \(L_A(z) := zA\), \(z \in \mathbb{C}^N\)).

Let us also note that using [13] Corollary 4.3 one can deduce \((\nu)\) from \((\iota)\) directly. Indeed, if \(\tilde{A} \in L(\tilde{X})^N\) is a unitary \(\rho\)-dilation of \(A \in L(\mathcal{X})^N\) then for any \(C \in \mathcal{C}^N\) by [13] Corollary 4.3 the operator \(\tilde{A} \otimes C\) is a contraction. Therefore, due to \([31]\) \(\tilde{A} \otimes C \in L(\tilde{X} \otimes \mathcal{H}_C)\) has a unitary dilation \(U \in L(\mathcal{K})\), \(\mathcal{K} \supset \tilde{X} \otimes \mathcal{H}_C\). Then for any \(n \in \mathbb{N}\):

\[
(A \otimes C)^n = \rho P_{\mathcal{X} \otimes \mathcal{H}_C}(\tilde{A} \otimes C)^n|\mathcal{X} \otimes \mathcal{H}_C
\]

\[
= \rho P_{\mathcal{X} \otimes \mathcal{H}_C}(P_{\tilde{X} \otimes \mathcal{H}_C}U^n|\tilde{X} \otimes \mathcal{H}_C)|\mathcal{X} \otimes \mathcal{H}_C
\]

\[
= \rho P_{\mathcal{X} \otimes \mathcal{H}_C}U^n|\mathcal{X} \otimes \mathcal{H}_C,
\]

i.e., \(U\) is a unitary \(\rho\)-dilation of the operator \(A \otimes C\). Thus, \(A \otimes C \in \mathcal{C}_\rho\).
Let us define the numerical radius of an $N$-tuple of operators $A \in L(\mathcal{X})^N$ as

$$w^{(N)}(A) := \sup_{C \in \mathcal{C}^N} w(A \otimes C). \quad (3.7)$$

For $N = 1$, $w^{(1)}(A) = w(A)$. Indeed,

$$w^{(1)}(A) = \sup_{||C|| < 1} w(A \otimes C) \geq \sup_{0 < \varepsilon < 1} w(A \otimes (1 - \varepsilon)I_{\mathcal{H}_C}) = \sup_{0 < \varepsilon < 1} (1 - \varepsilon)w(A) = w(A);$$

$$w^{(1)}(A) = \sup_{||C|| < 1} w(A \otimes C) \leq \sup_{||C|| < 1} w(A)||C|| = w(A).$$

Here we used the properties $w(A \otimes I_{\mathcal{H}}) = w(A)$ and $w(A \otimes B) \leq w(A)||B||$ valid for any $A \in L(\mathcal{X})$, $B \in L(\mathcal{H})$ (see, e.g., [13]).

**Proposition 3.6.** $A \in C_{2,N} \iff w^{(N)}(A) \leq 1$.

**Proof.** By Theorem 3.3, $A \in C_{2,N}$ if and only if $A \otimes C \in C_2 = C_{2,1}$ for any $C \in \mathcal{C}^N$. This, in turn, means that $w(A \otimes C) \leq 1$ for any $C \in \mathcal{C}^N$ (by Berger’s result mentioned in Section 2), i.e., $w^{(N)}(A) \leq 1$.

**Theorem 3.7.** If $A \in C_{\rho,N} \cap L(\mathcal{X})^N$ for a $\rho > 0$, then $L_A \in \rho \mathcal{A}S_N(\mathcal{X})$. For any $\rho > 0$ such that $\rho \neq 1$, there exists an $A \in L(\mathcal{X})^N$ such that $L_A \in \rho \mathcal{A}S_N(\mathcal{X})$ and $A \notin C_{\rho,N}$.

**Proof.** Let $A \in C_{\rho,N} \cap L(\mathcal{X})^N$ for some $\rho > 0$, and $C \in \mathcal{C}^N$. Then $A$ has a unitary $\rho$-dilation $\tilde{A} \in L(\bar{\mathcal{X}})^N$, and

$$||A \otimes C|| = \left\| \sum_{k=1}^N A_k \otimes C_k \right\| = \left\| \rho(P_{\mathcal{X}} \otimes I_{\mathcal{H}_C}) \left( \sum_{k=1}^N A_k \otimes C_k \right) \right\| \mathcal{X} \otimes \mathcal{H}_C \leq \rho \left\| \sum_{k=1}^N \tilde{A}_k \otimes C_k \right\| = \rho \left\| \tilde{A} \otimes C \right\| \leq \rho$$

(here we used again Corollary 4.3 in [18]). Thus, $L_A \in \rho \mathcal{A}S_N(\mathcal{X})$.

Now, let $0 < \rho \neq 1$, and $A \in L(\mathcal{X})^N$ be such that $\frac{1}{\rho} L_A(\zeta) = \frac{1}{\rho} \zeta A$ is a unitary operator for each $\zeta \in \mathbb{T}_N$. Then, again by Corollary 4.3 in [18], $L_A \in \rho \mathcal{A}S_N(\mathcal{X})$. Suppose there exists a unitary $\rho$-dilation $\tilde{A} \in L(\bar{\mathcal{X}})^N$ of $A$. Then for any $\zeta \in \mathbb{T}_N$, $L_A(\zeta) = \zeta A = \rho P_{\mathcal{X}}(\zeta \tilde{A})\mathcal{X}$. Hence, for any $\zeta \in \mathbb{T}_N$ and $x \in \mathcal{X}$,

$$\|\zeta \tilde{A} x\| = \|x\| = \frac{1}{\rho} \|\zeta Ax\| = \|P_{\mathcal{X}}(\zeta \tilde{A})x\|,$$

that is possible only if $\zeta \tilde{A} x \in \mathcal{X}$ for all $\zeta \in \mathbb{T}_N$ and $x \in \mathcal{X}$. Therefore, for $n > 1$,

$$\rho^n \|x\| = \|((\zeta A)^n x\| = \|\rho P_{\mathcal{X}}(\zeta \tilde{A})^n x\| = \rho \|((\zeta \tilde{A})^n x\| = \rho \|x\|,$$

that is impossible for $x \neq 0$. Thus, $A \notin C_{\rho,N}$.

**Note 3.8.** Compare Theorem 3.7 with Remark 3.5.
The same argument as in the proof of the first part of Theorem 3.7 shows that, for $A \in C_{\rho,N}$,
\[
\| (A \otimes C)^n \| \leq \rho, \quad n \in \mathbb{N}, \quad C \in \mathcal{C}^N.
\]  
(3.8)

**Note 3.9.** Compare (3.8) with (2.2).

This uniform (in $C \in \mathcal{C}^N$) power-boundedness of an $N$-tuple of operators $A$ is, in our setting, a generalization of power-boundedness of a single operator. Let us define the *spectral radius of an $N$-tuple of operators $A \in L(\mathcal{X})^N$* as
\[
\nu^{(N)}(A) := \lim_{n \to +\infty} \left( \sup_{C \in \mathcal{C}^N} \| (A \otimes C)^n \| \right)^{\frac{1}{n}}.
\]  
(3.9)

**Note 3.10.** Compare (3.9) with (2.3).

In other words, $\nu^{(N)}(A) = \nu^{(N,\infty)}(L_A)$, where $\nu^{(N,\infty)}(f)$ is the *spectral radius of an element $f$ of the Banach algebra $\mathcal{H}_N^{\infty}(\mathcal{X})$ consisting of holomorphic $L(\mathcal{X})$-valued functions $f$ on $D_N$* which satisfy
\[
\| f \|_{\infty,N} := \sup_{C \in \mathcal{C}^N} \| f(C) \| < \infty
\]
(this algebra was introduced in [1]). Here $f(C)$ is defined in the same manner as $k(C,C)$ in (3.3), i.e., for
\[
\begin{align*}
  f(z) &= \sum_{t \in \mathbb{Z}_N^+} \hat{f}_t z^t, \quad z \in D_N, \\
  f(C) &= \sum_{t \in \mathbb{Z}_N^+} \hat{f}_t \otimes C^t, \quad C \in \mathcal{C}^N,
\end{align*}
\]
where the latter series converges in the $L(\mathcal{X} \otimes \mathcal{H}_C)$-norm. For $N = 1$, $\nu^{(1)}(A) = \nu(A)$. Indeed,
\[
\begin{align*}
  \nu^{(1)}(A) &= \lim_{n \to +\infty} \left( \sup_{\| C \| < 1} \|(A \otimes C)^n \| \right)^{\frac{1}{n}} = \lim_{n \to +\infty} \left( \sup_{\| C \| < 1} \| A^n \otimes C^n \| \right)^{\frac{1}{n}} \\
  &= \lim_{n \to +\infty} \left( \| A^n \| \sup_{\| C \| < 1} \| C^n \| \right)^{\frac{1}{n}} = \lim_{n \to +\infty} \| A^n \|^{\frac{1}{n}} = \nu(A).
\end{align*}
\]

**Remark 3.11.** For any $A \in C_{\rho,N}$, by virtue of (3.8), $\nu^{(N)}(A) \leq 1$.

**Theorem 3.12.** *For a fixed Hilbert space $\mathcal{X}$ and any $N \geq 1$ the class $C_{\rho,N}$ increases as a function of $\rho$:

$C_{\rho,N} \subset C_{\rho',N}$ for $\rho < \rho'$.

Moreover, for $\dim \mathcal{X} \geq 2$, $C_{\rho,N}$ increases strictly:

$C_{\rho,N} \not\subset C_{\rho',N}$ for $\rho \neq \rho'$.

For $\dim \mathcal{X} = 1$ the classes $C_{\rho,N}$ coincide for all $\rho \geq 1$, and strictly increase for $0 < \rho < 1$. 
Proof. For $N = 1$ this theorem is true (see Section 2). For $N > 1$ it follows from the equivalence (i)$\iff$(v) in Theorem 3.3.

\textbf{Theorem 3.13.} For any $A \in C_{\rho,N}$, $C \in C^{N}$, and a polynomial $p$ of one variable, \[\|p(A \otimes C)\| \leq \max_{|z| \leq 1} |zp(z) + (1 - \rho)p(0)|.\]

Proof. This result follows from the generalized von Neumann inequality (2.8) and the equivalence (i)$\iff$(v) in Theorem 3.3.

Let us remark that results of this section on N-tuples of operators from the classes $C_{\rho,N}$ can be extended to elements of $H_{\infty}^{N}(\mathcal{X})$, though the notion of unitary $\rho$-dilation no longer makes sense for this case. Define $C_{\rho,N}^{(\infty)}$ as a class of functions $f \in H_{\infty}^{N}(\mathcal{X})$ such that $f(C) \in C_{\rho} = C_{\rho,1}$ for any $C \in C^{N}$. Then, in particular, Theorem 3.3 implies that $A \in C_{\rho,N}$ if and only if $L_{A} \in C_{\rho,N}^{(\infty)}$. The following analogue of Theorem 3.3 is easily obtained.

\textbf{Theorem 3.14.} Let $f \in H_{\infty}^{N}(\mathcal{X})$ and $\rho > 0$. The following conditions are equivalent:

(i): $f \in C_{\rho,N}^{(\infty)}$;

(ii): the function $k^{f}_{\rho,N}(z,w) := \rho I_{X} - (\rho - 1)(f(z) + (f(w)^{\ast}) + (\rho - 2)f(w)^{\ast}f(z)$ is an Agler kernel on $\mathbb{D}^{N} \times \mathbb{D}^{N}$;

(iii): the function $\psi^{f}_{\rho,N}(z) := (1 - \frac{1}{\rho})I_{X} + \frac{2}{\rho}(I_{X} - f(z))^{-1}$ belongs to $\mathcal{A} \mathcal{H}_{N}(\mathcal{X})$;

(iv): the function $\varphi^{f}_{\rho,N}(z) := f(z)((\rho - 1)f(z) - \rho I_{X})^{-1}$ belongs to $\mathcal{A} \mathcal{S}_{N}(\mathcal{X})$.

Clearly, $H_{\infty}^{N}(\mathcal{X}) \cap C_{1,N}^{(\infty)} = \mathcal{A} \mathcal{S}_{N}(\mathcal{X})$. Set \[w^{(N,\infty)}(f) := \sup_{C \in C^{N}} w(f(C)). \tag{3.10}\]

\textbf{Note 3.15.} Compare (3.10) with (3.7).

\textbf{Remark 3.16.} Proposition 3.14 extends directly to the class $C_{2,N}^{(\infty)}$, with $f \in H_{\infty}^{N}(\mathcal{X})$ in the place of $A \in L(\mathcal{X})^{N}$, and $w^{(N,\infty)}(f)$ in the place of $w^{(N)}(A)$. Remark 3.11 extends directly to $f \in H_{\infty}^{N}(\mathcal{X})$ in the place of $A \in L(\mathcal{X})^{N}$, and $\nu^{(N,\infty)}(f)$ in the place of $\nu^{(N)}(A)$. Also, Theorems 3.12 and 3.13 extend to the classes $C_{\rho,N}^{(\infty)}$.

\section{Multivariable operator and operator-function radii}

In this section we extend the notion of operator radii $w_{\rho}$, $0 < \rho \leq \infty$, to the multivariable case, i.e., to $N$-tuples of bounded linear operators and to elements of the Banach algebra $H_{\infty}^{N}(\mathcal{X})$. Let $0 < \rho < \infty$ and $f \in H_{\infty}^{N}(\mathcal{X})$. Set \[w^{(\infty)}_{\rho,N}(f) := \inf\{u > 0 : \frac{1}{u} f \in C_{\rho,N}^{(\infty)}\},\] and for $A \in L(\mathcal{X})^{N}$, define \[w_{\rho,N}(A) := w^{(\infty)}_{\rho,N}(L_{A}).\]
Due to our remark preceding to Theorem 3.14,
\[ w_{\rho,N}(A) = \inf\{u > 0 : \frac{1}{u} A \in C_{\rho,N}\}. \tag{4.1} \]

**Note 4.1.** Compare (4.1) with (2.7).

Clearly, for \( N = 1 \) and \( A \in L(\mathcal{X}) \), \( w_{\rho,1}(A) = w_{\rho}(A) \).

**Lemma 4.2.** For \( f \in H_{N}^\infty(\mathcal{X}) \), \( A \in L(\mathcal{X})^N \),
\[ w_{\rho,N}^{(\infty)}(f) = \sup_{C \in \mathcal{C}^N} w_{\rho}(f(C)), \tag{4.2} \]
\[ w_{\rho,N}(A) = \sup_{C \in \mathcal{C}^N} w_{\rho}(A \otimes C). \tag{4.3} \]

**Proof.** Let \( f \in H_{N}^\infty(\mathcal{X}) \). Then for \( u > 0 \), \( \frac{1}{u} f \in C_{\rho,N}^{(\infty)} \) if and only if for any \( C \in \mathcal{C}^N \) one has \( \frac{1}{u} f(C) \in C_{\rho} \). Therefore,
\[ w_{\rho,N}^{(\infty)}(f) = \inf\{u > 0 : \frac{1}{u} f \in C_{\rho,N}^{(\infty)}\} = \inf\{u > 0 : \forall C \in \mathcal{C}^N, \frac{1}{u} f(C) \in C_{\rho}\} = \sup_{C \in \mathcal{C}^N} \inf\{u > 0 : \frac{1}{u} f(C) \in C_{\rho}\} = \sup_{C \in \mathcal{C}^N} w_{\rho}(f(C)), \]
i.e., (4.2) is true. Now, (4.3) follows from (4.2) and the definition of \( w_{\rho,N}(A) \). \( \square \)

**Theorem 4.3.** 1. All properties (i)–(xx) listed in Theorem 2.5 are satisfied for \( w_{\rho,N}^{(\infty)}(\cdot) \) in the place of \( w_{\rho}(\cdot) \); \( f, g \in H_{N}^\infty(\mathcal{X}) \) in the place of \( A, B \in L(\mathcal{X}) \); \( w^{1(\infty)}(\cdot) \) in the place of \( w(\cdot) \); and \( \nu^{(\infty)}(\cdot) \) in the place of \( \nu(\cdot) \).

2. Properties (i)–(xvii) listed in Theorem 2.5 are satisfied for \( w_{\rho,N}(\cdot) \) in the place of \( w_{\rho}(\cdot) \); \( A \in L(\mathcal{X})^N \) in the place of \( A \in L(\mathcal{X}) \); \( w^{1}(\cdot) \) in the place of \( w(\cdot) \); and \( \nu^{1}(\cdot) \) in the place of \( \nu(\cdot) \).

**Proof.** 1. Let \( f \in H_{N}^\infty(\mathcal{X}) \). Then \( \|f\|_{\infty,N} = \sup_{C \in \mathcal{C}^N} \|f(C)\| < \infty \). By properties (vi) and (x) in Theorem 2.5 and Lemma 1.2 if \( 0 < \rho \leq 1 \) then
\[ w_{\rho,N}^{(\infty)}(f) = \sup_{C \in \mathcal{C}^N} w_{\rho}(f(C)) \leq \left(\frac{2}{\rho} - 1\right) \sup_{C \in \mathcal{C}^N} w_{1}(f(C)) = \left(\frac{2}{\rho} - 1\right) \sup_{C \in \mathcal{C}^N} \|f(C)\| < \infty, \]
and if \( \rho > 1 \) then
\[ w_{\rho,N}^{(\infty)}(f) = \sup_{C \in \mathcal{C}^N} w_{\rho}(f(C)) \leq \sup_{C \in \mathcal{C}^N} w_{1}(f(C)) = \sup_{C \in \mathcal{C}^N} \|f(C)\| < \infty. \]

Thus, property (i) is fulfilled.

Properties (ii)–(vii), (ix)–(xi), (xiii)–(xv), and (xvii)–(xx) easily follow from the properties in Theorem 2.5 with the same numbers, and Lemma 1.2.

The proof of property (viii) is an adaptation of the proof of Theorem 5.1 in [13] to our case. First of all, let us remark that property (iv) implies that if
$u > \nu_{\rho,N}^{(\infty)}(f)$ then $\frac{1}{u} f \in C_{\rho,N}^{(\infty)}$, and for any $C \in \mathcal{C}^N$ one has $\frac{1}{u} f(C) \in C_{\rho}$. In particular,
\[
\sup_{C \in \mathcal{C}^N} \left\| \left( \frac{f(C)}{u} \right)^n \right\| \leq \rho, \quad n \in \mathbb{N}.
\]
Therefore, $\nu_{\rho,N}^{(N,\infty)} \left( \frac{1}{u} \right) \leq 1$, i.e., $\nu_{\rho,N}^{(N,\infty)}(f) \leq u$. Thus, for any $\rho > 0$, $\nu_{\rho,N}^{(N,\infty)}(f) \leq u_{\rho,N}^{(\infty)}(f)$, moreover,
\[
\nu_{\rho,N}^{(N,\infty)}(f) \leq \lim_{\rho \to +\infty} u_{\rho,N}^{(\infty)}(f)
\]
(note, that due to property (x), $u_{\rho,N}^{(\infty)}(f)$ is a non-increasing and bounded from below function of $\rho$, hence it has a limit as $\rho \to +\infty$).

For the proof of the opposite inequality, let us first show that if $\nu_{\rho,N}^{(N,\infty)}(g) < 1$ for some $g \in H^\infty_{\mathcal{C}}(\mathcal{X}^{\infty})$ then beginning with some $\rho_0 > 0$ (i.e., for all $\rho \geq \rho_0$) one has $g \in C_{\rho,N}^{(\infty)}$. Indeed, in this case there exists an $s > 1$ such that $\nu_{\rho,N}^{(N,\infty)}(sg) < 1$. Then there exists a $B > 0$ such that
\[
s^n \sup_{C \in \mathcal{C}^N} \| g(C) \|^n \leq B, \quad n \in \mathbb{N}.
\]
Hence, for any $C \in \mathcal{C}^N$,
\[
\operatorname{Re} \psi_{\rho,N}^g(C) = \left( 1 - \frac{2}{\rho} \right) I_{X \otimes \mathcal{H}C} + \frac{2}{\rho} \operatorname{Re}(I_{X \otimes \mathcal{H}C} - g(C))^{-1}
\]
\[
= I_{X \otimes \mathcal{H}C} + \frac{2}{\rho} \operatorname{Re} \sum_{n=1}^{\infty} g(C)^n \geq \left( 1 - \frac{2}{\rho} \sum_{n=1}^{\infty} \| g(C) \|^n \right) I_{X \otimes \mathcal{H}C}
\]
\[
\geq \left( 1 - \frac{2}{\rho} \sum_{n=1}^{\infty} \frac{B}{s^n} \right) I_{X \otimes \mathcal{H}C} = \left( 1 - \frac{2B}{\rho(s-1)} \right) I_{X \otimes \mathcal{H}C} \geq 0
\]
as soon as $\rho \geq \frac{2B}{s-1}$. Thus, by Theorem 8.11 $g \in C_{\rho,N}^{(\infty)}$ for any $\rho \geq \frac{2B}{s-1}$.

Now, if $\nu_{\rho,N}^{(N,\infty)}(f) = 0$ then for any $k \in \mathbb{N}$, $\nu_{\rho,N}^{(N,\infty)}(kf) = 0$. Hence, for $\rho \geq \rho_0$ we have $kf \in C_{\rho,N}^{(\infty)}$, and by property (iv), $u_{\rho,N}^{(\infty)}(kf) \leq 1$. Thus,
\[
\lim_{\rho \to +\infty} u_{\rho,N}^{(\infty)}(f) \leq \frac{1}{k}
\]
for any $k \in \mathbb{N}$, and
\[
\lim_{\rho \to +\infty} u_{\rho,N}^{(\infty)}(f) = 0 = \nu_{\rho,N}^{(N,\infty)}(f),
\]
as required.

If $\nu_{\rho,N}^{(N,\infty)}(f) > 0$ then for any $\varepsilon > 0$,
\[
\nu_{\rho,N}^{(N,\infty)} \left( \frac{f}{(1 + \varepsilon) \nu_{\rho,N}^{(N,\infty)}(f)} \right) = \frac{1}{1 + \varepsilon} < 1.
\]
Then for $\rho \geq \rho_0$,
\[
u_{\rho,N}^{(\infty)} \left( \frac{f}{(1 + \varepsilon) \nu_{\rho,N}^{(N,\infty)}(f)} \right) \leq 1,
\]
hence \( w_{\rho,N}^{(\infty)}(f) \leq (1 + \varepsilon)\nu(\rho,\infty)(f) \). Passing to the limit as \( \rho \to +\infty \), and then as \( \varepsilon \downarrow 0 \), we get
\[
\lim_{\rho \to +\infty} w_{\rho,N}^{(\infty)}(f) \leq \nu(\rho,\infty)(f),
\]
as required. Thus, property (viii) is proved.

For the proof of property (xii), it is enough to suppose, by virtue of positive homogeneity of \( w_{\rho,N}^{(\infty)}(\cdot) \) and \( \nu(\rho,\infty)(\cdot) \), that for \( f \in H_N^\infty(\mathcal{X}) \) one has \( w_{\rho,N}^{(\infty)}(f) = 1 \), \( \nu(\rho,\infty)(f) < 1 \), and prove that for any \( \rho > \rho_0 \), \( w_{\rho,N}^{(\infty)}(f) < 1 \). By Theorem \( 2.5 \) and property (iv) in the present theorem,
\[
\frac{\rho_0}{2} w_{\rho_0,N}^{(\infty)}(z) = \left( \frac{\rho_0}{2} - 1 \right) I_{\mathcal{X}} + (I_{\mathcal{X}} - f(z))^{-1} \in \mathcal{A}\mathcal{H}_{\mathcal{N}}(\mathcal{X}),
\]
i.e., for any \( C \in \mathcal{C}^N \),
\[
\text{Re} \left[ \frac{\rho_0}{2} w_{\rho_0,N}^{(\infty)}(C) \right] = \text{Re} \left[ \left( \frac{\rho_0}{2} - 1 \right) I_{\mathcal{X} \otimes \mathcal{H}_C} + (I_{\mathcal{X} \otimes \mathcal{H}_C} - f(C))^{-1} \right] \geq 0,
\]
and for any \( \rho > \rho_0 \),
\[
\text{Re} \left[ \frac{\rho}{2} w_{\rho,N}^{(\infty)}(C) \right] = \text{Re} \left[ \left( \frac{\rho}{2} - 1 \right) I_{\mathcal{X} \otimes \mathcal{H}_C} + (I_{\mathcal{X} \otimes \mathcal{H}_C} - f(C))^{-1} \right] \geq \frac{\rho - \rho_0}{2} I_{\mathcal{X} \otimes \mathcal{H}_C}.
\]
Since the resolvent \( R_f(\lambda) := (\lambda I_{\mathcal{X}} - f)^{-1} \) is continuous in the \( H_N^\infty(\mathcal{X}) \)-norm on the resolvent set of \( f \), and \( \nu(\rho,\infty)(f) < 1 \), for \( \varepsilon > 0 \) small enough, one has \( \nu(\rho,\infty)((1 + \varepsilon)f) < 1 \), and
\[
\text{Re} \left[ \frac{\rho}{2} w_{\rho,N}^{(1+\varepsilon)f}(C) \right] = \text{Re} \left[ \left( \frac{\rho}{2} - 1 \right) I_{\mathcal{X} \otimes \mathcal{H}_C} + (I_{\mathcal{X} \otimes \mathcal{H}_C} - (1 + \varepsilon)f(C))^{-1} \right] \geq 0
\]
for any \( C \in \mathcal{C}^N \), i.e., \( \frac{\rho}{2} w_{\rho,N}^{(1+\varepsilon)f} \in \mathcal{A}\mathcal{H}_{\mathcal{N}}(\mathcal{X}) \), and \( \psi^{(1+\varepsilon)f} \in \mathcal{A}\mathcal{H}_{\mathcal{N}}(\mathcal{X}) \). Hence, by Theorem \( 2.5 \) \((1 + \varepsilon)f \in C_{\rho,N}^{\infty} \) which means, by property (iv), that \( w_{\rho,N}^{(\infty)}((1 + \varepsilon)f) \leq 1 \). Thus, \( w_{\rho,N}^{(\infty)}(f) \leq \frac{1}{1+\varepsilon} < 1 \), as required.

The first part of property (xvi) in this theorem follows from property (xvi) in Theorem \( 2.5 \) and Lemma \( 4.2 \). For the proof of the second part of (xvi), we use properties (xv) and (xvi) from Theorem \( 2.5 \) property (xv) in the present theorem, and Lemma \( 4.2 \)
\[
\lim_{\rho \downarrow 0} \frac{\rho}{2} w_{\rho,N}^{(\infty)}(f) = \sup_{0 < \rho < 1} \left\{ \frac{\rho}{2} w_{\rho,N}^{(\infty)}(f) \right\} = \sup_{0 < \rho < 1} \sup_{C \in \mathcal{C}^N} \left\{ \frac{\rho}{2} w_{\rho}(f(C)) \right\}
= \sup_{C \in \mathcal{C}^N} \sup_{0 < \rho < 1} \left\{ \frac{\rho}{2} w_{\rho}(f(C)) \right\} = \sup_{C \in \mathcal{C}^N} \left\{ \lim_{\rho \downarrow 0} \frac{\rho}{2} w_{\rho}(f(C)) \right\}
= \sup_{C \in \mathcal{C}^N} w_{2}(f(C)) = w_{2,N}^{(\infty)}(f).
\]
The proof of property (xvi), as well as part 1 of this theorem, is complete.

Part 2 follows from part 1. \( \square \)

Denote by \( C_{\rho,N}^{\infty} \) (resp., \( C_{\rho,N}^{\infty} \)) the class of \( \mathcal{C}^N \)-bounded holomorphic operator valued functions on \( \mathbb{D}^N \) (resp., \( N \)-tuples of bounded linear operators on a common Hilbert space) with spectral radius at most one.
Theorem 4.4. Let \( X \) be a Hilbert space. Then
\[
C^{(\infty)}_{\infty,N} \cap H^{\infty}_N(X) = \text{clos} \left\{ \bigcup_{0<\rho<\infty} (C^{(\infty)}_{\rho,N} \cap H^{\infty}_N(X)) \right\};
\]
\[
C_{\infty,N} \cap L(X)^N = \text{clos} \left\{ \bigcup_{0<\rho<\infty} (C_{\rho,N} \cap L(X)^N) \right\}.
\]

Note 4.5. Compare (4.4) and (4.5) with (2.4).

Proof of Theorem 4.4. The inclusion “\( \supset \)” in (4.4) and (4.5) follows from Remarks 3.11 and 3.16, and the fact that the set of \( C^N \)-bounded holomorphic operator-valued functions on \( \mathbb{D}^N \) (resp., \( N \)-tuples of bounded operators) with spectral radius at most one is closed in \( H^{\infty}_N(X) \) (resp., \( L(X)^N \)).

To show the inclusion “\( \subset \)” in (4.4), observe that for \( f \in C^{(\infty)}_{\infty,N} \cap H^{\infty}_N(X) \) and \( 0 < r < 1 \), \( \nu^{(N,\infty)}(rf) \leq r < 1 \). By property (viii) from Theorem 4.3, for \( \rho_0 > 0 \) big enough, \( w^{(\infty)}_{\rho_0,N}(rf) < 1 \), and by property (iv) from the same theorem,
\[
rf \in C^{(\infty)}_{\rho_0,N} \cap H^{\infty}_N(X) \subset \text{clos} \left\{ \bigcup_{0<\rho<\infty} (C_{\rho,N} \cap H^{\infty}_N(X)) \right\}.
\]
Passing to the limit as \( r \uparrow 1 \), we get
\[
f \in \text{clos} \left\{ \bigcup_{0<\rho<\infty} (C_{\rho,N} \cap H^{\infty}_N(X)) \right\},
\]
and the inclusion “\( \subset \)” in (4.4) follows. Analogously for the inclusion “\( \subset \)” in (4.5).

In view of property (iv) in Theorem 4.3 let us call the elements of the class \( C_{\rho,N} \) (\( N \)-variable) \( \rho \)-contractions.

5. On similarity of \( \rho \)-contractions to 1-contractions in several variables

An \( N \)-tuple of operators \( A = (A_1, \ldots, A_N) \in L(X)^N \) is said to be simultaneously similar to an \( N \)-tuple of operators \( B = (B_1, \ldots, B_N) \in L(Y)^N \) if there exists a boundedly invertible operator \( S \in L(X,Y) \) such that
\[
A_k = S^{-1}B_kS, \quad k = 1, \ldots, N,
\]
or equivalently,
\[
zA = S^{-1}(zB)S, \quad z \in \mathbb{C}^N.
\]

Note 5.1. Compare (5.1) and (5.2) with (2.9).

Theorem 5.2. For any \( \rho > 1 \) and \( N > 1 \), there exists an \( A = (A_1, \ldots, A_N) \in C_{\rho,N} \) which is not simultaneously similar to any \( T = (T_1, \ldots, T_N) \in C_{1,N} \).
Proof. Let \( N = 2 \), and for any \( \varepsilon \geq 0 \) set \( A^{(\varepsilon)} = (A^{(\varepsilon)}_1, A^{(\varepsilon)}_2) \in L(\mathbb{C}^3)^2 \), where
\[
A^{(\varepsilon)}_1 := \begin{bmatrix} 0 & \frac{1+\varepsilon}{\sqrt{2}} & 0 \\ 0 & 0 & 0 \\ -\frac{1+\varepsilon}{\sqrt{2}} & 0 & 0 \end{bmatrix}, \quad A^{(\varepsilon)}_2 := \begin{bmatrix} 0 & 0 & \frac{1+\varepsilon}{\sqrt{2}} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.
\]
Then for any \( \varepsilon \geq 0 \) and \( z \in \mathbb{C}^2 \),
\[
(zA^{(\varepsilon)})^2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & (1+\varepsilon)^2 z_2 & 0 \\ 0 & -\left(\frac{1+\varepsilon}{2}\right)^2 z_1 z_2 & (1+\varepsilon)^2 z_2 \end{bmatrix},
\]
\[
(zA^{(\varepsilon)})^3 = (zA^{(\varepsilon)})^4 = \ldots = 0,
\]
i.e., \( zA^{(\varepsilon)} \) is a nilpotent operator of degree 3. Hence, for any \( \rho > 1 \) and \( z \in \mathbb{D}^2 \),
\[
\|\varphi^{(0)}_{\rho, 2}(z)\| = \|zA^{(0)}((\rho - 1)zA^{(0)} - \rho I)^{-1}\| = \left\| \frac{zA^{(0)}}{\rho} \left( I - \frac{\rho - 1}{\rho} zA^{(0)} \right)^{-1} \right\|
\leq \frac{1}{\rho} \left\| \frac{zA^{(0)}}{\rho} \right\|^2 + \frac{\rho - 1}{\rho^2} \|zA^{(0)}\|^2
\leq \frac{1}{\rho} \left\| \begin{bmatrix} 0 & \frac{z_1}{\sqrt{2}} \\ \frac{z_2}{\sqrt{2}} & 0 & 0 \\ -\frac{z_1}{\sqrt{2}} & 0 & 0 \end{bmatrix} \right\|^2 + \frac{\rho - 1}{\rho^2} \left\| \begin{bmatrix} 0 \\ \frac{z_2}{\sqrt{2}} \\ -\frac{z_1}{\sqrt{2}} \end{bmatrix} \right\|^2
\leq \frac{1}{\rho} + \frac{\rho - 1}{\rho^2} = \frac{2\rho - 1}{\rho^2} < 1.
\]
Then, due to the von Neumann inequality in two variables \( \mathbb{H} \), one has
\[
\|\varphi^{(0)}_{\rho, 2}(C)\| \leq \frac{2\rho - 1}{\rho^2} < 1, \quad C \in \mathbb{C}^2,
\]
i.e., \( \varphi^{(0)}_{\rho, 2} \in AS_2(\mathbb{C}^3) \). Analogously, for \( \varepsilon > 0 \) small enough (the choice of \( \varepsilon \) depends on \( \rho \)), one has
\[
\sup_{C \in \mathbb{C}^2} \|\varphi^{(\varepsilon)}_{\rho, 2}(C)\| = \sup_{z \in \mathbb{D}^2} \|\varphi^{(\varepsilon)}_{\rho, 2}(z)\| \leq \frac{1+\varepsilon}{\rho} + \left(\frac{1+\varepsilon}{\rho}\right)^2 < 1,
\]
i.e., \( \varphi^{(\varepsilon)}_{\rho, 2} \in AS_2(\mathbb{C}^3) \), and by Theorem \( \mathbb{H} \) \( A^{(\varepsilon)} \in C_{\rho, 2} \).
Let us show now that for any \( \varepsilon > 0 \) the pair \( \mathbf{A}^{(\varepsilon)} = (A_1^{(\varepsilon)}, A_2^{(\varepsilon)}) \) is not simultaneous similar to any pair \( \mathbf{T} = (T_1, T_2) \in C_{1,2} \). Observe that

\[
(A_1^{(\varepsilon)} + A_2^{(\varepsilon)})(A_1^{(\varepsilon)} - A_2^{(\varepsilon)}) = \begin{bmatrix}
0 & \frac{1+\varepsilon}{\sqrt{2}} & \frac{1+\varepsilon}{\sqrt{2}} \\
\frac{1+\varepsilon}{\sqrt{2}} & 0 & 0 \\
\frac{1+\varepsilon}{\sqrt{2}} & 0 & 0
\end{bmatrix}
\begin{bmatrix}
0 & \frac{1+\varepsilon}{\sqrt{2}} & \frac{-1+\varepsilon}{\sqrt{2}} \\
\frac{1+\varepsilon}{\sqrt{2}} & 0 & 0 \\
\frac{-1+\varepsilon}{\sqrt{2}} & 0 & 0
\end{bmatrix}
= \begin{bmatrix}
-(1+\varepsilon)^2 & 0 & 0 \\
0 & \frac{(1+\varepsilon)^2}{2} & -\frac{(1+\varepsilon)^2}{2} \\
0 & -\frac{(1+\varepsilon)^2}{2} & \frac{(1+\varepsilon)^2}{2}
\end{bmatrix}.
\]

Then

\[
\lim_{n \to +\infty} \|(A_1^{(\varepsilon)} + A_2^{(\varepsilon)})(A_1^{(\varepsilon)} - A_2^{(\varepsilon)})^n\| = \infty. \tag{5.3}
\]

On the other hand, if \( \mathbf{A}^{(\varepsilon)} = (A_1^{(\varepsilon)}, A_2^{(\varepsilon)}) \) is simultaneously similar to some \( \mathbf{T} = (T_1, T_2) \in C_{1,2} \) then for any \( n \in \mathbb{N} \) one would have

\[
\|(A_1^{(\varepsilon)} + A_2^{(\varepsilon)})(A_1^{(\varepsilon)} - A_2^{(\varepsilon)})^n\| = \|S^{-1}(T_1 + T_2)(T_1 - T_2)^nS\| \leq \|S\|\|S^{-1}\| < \infty,
\]

since \( \|T_1 \pm T_2\| \leq 1 \). We get a contradiction with (5.3).

Examples of \( \mathbf{T} \)-tuples of operators from \( C_{p,N}, p > 1 \), which are not simultaneously similar to any \( \mathbf{T} \in C_{1,N} \) for the case \( N > 2 \) can be obtained from the examples of pairs \( \mathbf{A} = (A_1^{(\varepsilon)}, A_2^{(\varepsilon)}) \) above, for sufficiently small \( \varepsilon > 0 \), by setting zeros for the rest of operators in these \( N \)-tuples: \( \mathbf{\tilde{A}}^{(\varepsilon)} := (A_1^{(\varepsilon)}, A_2^{(\varepsilon)}, 0, \ldots, 0) \). \( \square \)

Let \( \mathbf{A} = (A_1, \ldots, A_N) \in L(\mathcal{X})^N \). Then \( \mathbf{\tilde{A}} = (\tilde{A}_1, \ldots, \tilde{A}_N) \in L(\tilde{\mathcal{X}})^N \) is called a uniform \( \rho \)-dilation of \( \mathbf{A} \) if \( \tilde{\mathcal{X}} \supset \mathcal{X} \) and

\[
\forall n \in \mathbb{N}, \forall i_1, \ldots, i_n \in \{1, \ldots, N\}, \quad A_{i_1} \cdots A_{i_n} = \rho P_X \tilde{A}_{i_1} \cdots \tilde{A}_{i_n}|\mathcal{X}, \tag{5.4}
\]

or equivalently,

\[
\forall n \in \mathbb{N}, \forall z^{(1)}, \ldots, z^{(n)} \in \mathbb{C}^N, \quad z^{(1)} \mathbf{A} \cdots z^{(n)} \mathbf{A} = \rho P_X z^{(1)} \tilde{\mathbf{A}} \cdots z^{(n)} \tilde{\mathbf{A}}|\mathcal{X}. \tag{5.5}
\]

Note 5.3. Compare (5.4) and (5.5) with (5.1) and (5.2).

Clearly, a uniform \( \rho \)-dilation is a \( \rho \)-dilation. If \( \tilde{\mathbf{A}} \in L(\tilde{\mathcal{X}})^N \) is a uniform \( \rho \)-dilation of \( \mathbf{A} \in L(\mathcal{X}) \), and for any \( \zeta \in \mathbb{T}^N \), \( \zeta \tilde{\mathbf{A}} \) is a unitary operator, then \( \tilde{\mathbf{A}} \) is called a uniform unitary \( \rho \)-dilation of \( \mathbf{A} \). Denote by \( C_{p,N}^u \) the class of \( N \)-tuples of operators \( \mathbf{A} = (A_1, \ldots, A_N) \) on a common Hilbert space which admit a uniform unitary \( \rho \)-dilation. Clearly, \( C_{p,N}^u \subset C_{\rho,N} \).

**Theorem 5.4.** Any \( \mathbf{A} = (A_1, \ldots, A_N) \in C_{p,N}^u \) is simultaneously similar to some \( \mathbf{T} = (T_1, \ldots, T_N) \in C_{1,N}^u \).

**Proof.** Let \( \mathbf{A} = (A_1, \ldots, A_N) \in C_{p,N}^u \cap L(\mathcal{X})^N \), and \( \mathbf{U} = (U_1, \ldots, U_N) \in L(\tilde{\mathcal{X}})^N \) be a uniform unitary \( \rho \)-dilation of \( \mathbf{A} \). Let \( \mathcal{A} \subset L(\tilde{\mathcal{X}}) \) be the minimal \( C^* \)-algebra which contains the operators \( I_{\tilde{\mathcal{X}}}, U_1, \ldots, U_N \), and \( \mathcal{B} \subset L(\tilde{\mathcal{X}}) \) be the minimal algebra over
\[ \mathbb{C} \] which contains the operators \( U_1, \ldots, U_N \). Clearly, \( \mathcal{B} \subset \mathcal{A} \). Let \( \varphi : \mathcal{B} \to L(\mathcal{A}) \) be a homomorphism defined on the generators as
\[ \varphi : U_k \mapsto A_k, \quad k = 1, \ldots, N. \]
The algebra \( \mathcal{B} \) consists of operators of the form
\[ p(U) = \sum_{1 \leq k \leq m, \ i_1, \ldots, i_k \in \{1, \ldots, N\}} \alpha_{i_1, \ldots, i_k} U_{i_1} \cdots U_{i_k}, \]
where \( \alpha_{i_1, \ldots, i_k} \in \mathbb{C} \) for all \( i_1, \ldots, i_k \in \{1, \ldots, N\} \). Then
\[ \varphi(p(U)) = \varphi(\sum \alpha_{i_1, \ldots, i_k} U_{i_1} \cdots U_{i_k}) = \sum \alpha_{i_1, \ldots, i_k} \alpha_{i_1} \cdots \alpha_{i_k} \]
\[ = \varphi(U) = \rho P \varphi(U)|\mathcal{A}. \]
Therefore, if \( p(U) = 0 \) then \( \varphi(p(U)) = 0 \), and \( \varphi \) is correctly defined. The homomorphism \( \varphi \) is completely bounded, i.e.,
\[ \| \varphi \|_{cb} := \sup_{n \in \mathbb{N}} \| \text{id}_n \otimes \varphi \| < \infty, \]
where \( \text{id}_n \) is the identical map of the matrix algebra \( M_n(\mathbb{C}) \) onto itself. Moreover, \( \| \varphi \|_{cb} \leq \rho \). Indeed, for any \( n \in \mathbb{N} \) and a polynomial \( n \times n \) matrix of \( N \) non-commuting variables,
\[ P(X) = [p_{ij}(X)]_{i,j=1}^n = \left[ \sum_{1 \leq k \leq m, \ i_1, \ldots, i_k \in \{1, \ldots, N\}} \alpha_{i_1, \ldots, i_k}^{(ij)} X_{i_1} \cdots X_{i_k} \right]_{i,j=1}^n, \]
\[ \| (\text{id}_n \otimes \varphi)(P(U)) \| = \| (\text{id}_n \otimes \varphi)([p_{ij}(U)]_{i,j=1}^n) \| = \| \varphi([p_{ij}(U)]_{i,j=1}^n) \| \]
\[ = \| [p_{ij}(\mathcal{A})]_{i,j=1}^n \| = \| \rho P \varphi([p_{ij}(U)]_{i,j=1}^n) \| \]
\[ = \rho \| ([I_{C^n} \otimes \mathcal{A}][p_{ij}(U)]_{i,j=1}^n) \| \]
\[ \leq \rho \| ([p_{ij}(U)]_{i,j=1}^n) \| = \rho \| P(U) \|. \]
Then, by Theorem 3.1 in [28], there exist a Hilbert space \( \mathcal{N} \), a completely contractive homomorphism \( \gamma : \mathcal{B} \to L(\mathcal{N}) \) (i.e., such that \( \| \gamma \|_{cb} \leq 1 \)), and a boundedly invertible operator \( \gamma = L(\mathcal{N}) \) such that
\[ \varphi(b) = S^{-1} \gamma(b) S, \quad b \in \mathcal{B}. \]
Moreover, as was shown in the proof of Theorem 3.1 in [28], \( \gamma \) can be chosen in the form
\[ \gamma(b) = P_{\mathcal{N}} \pi(b)|\mathcal{N}, \quad b \in \mathcal{B}, \]
where \( \pi : \mathcal{A} \to L(\mathcal{K}) \) is a \( * \)-homomorphism, for some Hilbert space \( \mathcal{K} \supset \mathcal{N} \). In addition, it follows from Theorem 2.7 and the proof of Theorem 2.8 in [28] that one can choose \( \mathcal{K} = \mathcal{K}_1 \oplus \mathcal{K}_1 \), for some Hilbert space \( \mathcal{K}_1 \), and
\[ \pi(a) = \pi_1(a) \oplus 0, \quad a \in \mathcal{A}, \]
where \( \pi_1 : \mathcal{A} \to L(K_1) \) is a unital \( * \)-homomorphism. Set
\[ T_k := \gamma(U_k) \in L(\mathcal{N}), \quad k = 1, \ldots, N. \]
Then

\[ A_k = \varphi(U_k) = S^{-1}T_kS, \quad k = 1, \ldots, N. \]

It remains to show that \( \mathbf{T} = (T_1, \ldots, T_N) \in C^n_{1,N} \). Set

\[ W_k := \pi(U_k) \in L(K), \quad k = 1, \ldots, N. \]

Since for any \( n \in \mathbb{N} \) and \( i_1, \ldots, i_n \in \{1, \ldots, N\} \) one has

\[ T_{i_1} \cdots T_{i_n} = \gamma(U_{i_1} \cdots U_{i_n}) = P_N \pi(U_{i_1} \cdots U_{i_n})|N = P_N W_{i_1} \cdots W_{i_n}|N, \]

\( \mathbf{W} = (W_1, \ldots, W_N) \) is a uniform 1-dilation of \( \mathbf{T} = (T_1, \ldots, T_N) \), however, still not unitary. Actually,

\[ W_k = \pi(U_k) = \pi_1(U_k) \oplus 0 \quad (=: W_k^{(1)} \oplus 0), \quad k = 1, \ldots, N. \]

Since \( \pi_1 \) is a unital \(*\)-homomorphism, and for any \( \zeta \in \mathbb{T}^N \),

\[ (\zeta \mathbf{U})^* \zeta \mathbf{U} = I_{\mathbb{K}} = \zeta \mathbf{U}(\zeta \mathbf{U})^*, \]

one has, for any \( \zeta \in \mathbb{T}^N \),

\[ (\zeta \mathbf{W}^{(1)})^* \zeta \mathbf{W}^{(1)} = I_{\mathbb{K}^1} = \zeta \mathbf{W}^{(1)}(\zeta \mathbf{W}^{(1)})^*, \]

where \( \mathbf{W}^{(1)} = (W_1^{(1)}, \ldots, W_N^{(1)}) \). Set

\[ \tilde{W}_k := W_k^{(1)} \oplus \delta_{1k} V \in L(\mathbb{K} \oplus \mathbb{R}), \quad k = 1, \ldots, N, \]

where \( \delta_{ij} \) is the Kronecker symbol, and \( V \) is a unitary dilation of the zero operator on \( \mathbb{K}^1 \), e.g., the two-sided shift on the space \( \mathbb{R} := l^2(\mathbb{K}^1) = \bigoplus_{-\infty}^{+\infty} \mathbb{K}^1 \) (here we identify the space \( \mathbb{K}^1 \) with the subspace \( \cdots \oplus \{0\} \oplus \{0\} \oplus \mathbb{K}^1 \oplus \{0\} \oplus 0 \oplus \cdots \) in \( \mathbb{R} \)). Then, for any \( \zeta \in \mathbb{T}^N \),

\[ (\zeta \tilde{W})^* \zeta \tilde{W} = I_{\mathbb{K}^1 \oplus \mathbb{R}} = \zeta \tilde{W}(\zeta \tilde{W})^*, \]

and \( \tilde{\mathbf{W}} = (\tilde{W}_1, \ldots, \tilde{W}_N) \in L(\mathbb{K} \oplus \mathbb{R}) \) is a uniform unitary 1-dilation of \( \mathbf{W} = (W_1, \ldots, W_N) \), and therefore, of \( \mathbf{T} = (T_1, \ldots, T_N) \). Thus, \( \mathbf{T} \in C^n_{1,N} \), as required.

\[ \square \]

Theorem 5.4 is similar to the result of G. Popescu [30] on simultaneous similarity of \( \rho \)-contractions to 1-contractions in several variables, however his notion of multivariable \( \rho \)-contractions is different. Let us clarify the relation between these two results. Denote by \( C^n_{\rho,N} \) (we use here this notation instead of just \( C^n_{\rho} \), as in [30]) the \textit{Popescu class} of all \( N \)-tuples \( \mathbf{A} = (A_1, \ldots, A_N) \) of bounded linear operators on a common Hilbert space, say \( \mathcal{X} \), which have a \textit{uniform isometric} \( \rho \)-dilation, i.e., such an \( N \)-tuple of operators \( \mathbf{V} = (V_1, \ldots, V_N) \in L(\tilde{\mathcal{X}})^N, \quad \tilde{\mathcal{X}} \supset \mathcal{X} \), for which

\begin{enumerate}
\item \( V_k^* V_k = I_{\tilde{\mathcal{X}}}, \quad k = 1, \ldots, N; \)
\item \( V_k^* V_j = 0, \quad k \neq j; \)
\item \( \forall n \in \mathbb{N}, \forall i_1, \ldots, i_n, \quad A_{i_1} \cdots A_{i_n} = \rho P_{\mathcal{X}} V_{i_1} \cdots V_{i_n}|\mathcal{X}. \)
\end{enumerate}

Condition (2) in this definition can be replaced by
(2'): \[ \sum_{k=1}^{N} V_k V_k^* \preceq I_{\tilde{X}}, \]
since (1) & (2) \iff (1) & (2'). According to [29], the class \( C_{1,N}^p \) coincides with the class of \( N \)-tuples of operators \( \mathbf{A} = (A_1, \ldots, A_N) \) (\( \in L(\mathcal{X})^N \), for some Hilbert space \( \mathcal{X} \)) such that

\[ \sum_{k=1}^{N} A_k A_k^* \preceq I_{\mathcal{X}}. \]

By Theorem 4.5 in [30], any \( \mathbf{A} = (A_1, \ldots, A_N) \in C_{\rho,N}^\mu, \rho > 0, \) is simultaneously similar to some \( \mathbf{T} = (T_1, \ldots, T_N) \in C_{1,N}^\rho. \) This is a generalization of the theorem of Sz.-Nagy and Foias [33] to several variables. Theorem 5.4 of the present paper is a different generalization of the same result, since our classes \( C_{\rho,N}^u \) are different from Popescu’s classes \( C_{\rho,N}^p \) for \( N > 1. \) More precisely, the following is true.

**Theorem 5.5.** For any \( N > 1 \) and \( \rho > 0, \) \( C_{\rho,N}^u \subsetneq C_{\rho,N}^p. \)

**Proof.** Let \( \mathbf{A} \in C_{\rho,N}^u \cap L(\mathcal{X})^N, \) \( N > 1, \) and \( \mathbf{U} \in L(\tilde{\mathcal{X}})^N \) be a uniform unitary \( \rho \)-dilation of \( \mathbf{A}. \) Since for any \( \zeta \in T^N \) the operator \( \zeta \mathbf{U} \) is unitary, it follows that

\[ \zeta \mathbf{U}(\zeta \mathbf{U})^* = I_{\tilde{X}}, \quad \zeta \in T^N, \]

which implies

\[ \sum_{k=1}^{N} U_k U_k^* = I_{\tilde{X}}. \]

Thus, by [29], \( \mathbf{U} \in C_{1,N}^{\rho} \). Let \( \mathbf{V} \in L(\tilde{\mathcal{X}})^N \) be a uniform isometric 1-dilation of \( \mathbf{U} \) in the sense of Popescu. Then for any \( n \in \mathbb{N} \) and \( i_1, \ldots, i_n \in \{1, \ldots, N\}, \)

\[ A_{i_1} \cdots A_{i_n} = \rho P_{\mathcal{X}} U_{i_1} \cdots U_{i_n} |\mathcal{X}| = \rho P_{\mathcal{X}} (P_{\tilde{\mathcal{X}}} V_{i_1} \cdots V_{i_n} |\tilde{\mathcal{X}}|) |\mathcal{X}| \]

\[ = \rho P_{\mathcal{X}} V_{i_1} \cdots V_{i_n} |\mathcal{X}|, \]

i.e., \( \mathbf{V} \) is a uniform isometric \( \rho \)-dilation of \( \mathbf{A} \) in the sense of Popescu. Thus, \( \mathbf{A} \in C_{\rho,N}^p. \) This proves the inclusion \( C_{\rho,N}^u \subset C_{\rho,N}^p. \)

Let us prove that this inclusion is proper for any \( N > 1 \) and \( \rho > 0. \) Firstly, consider the case \( N = 2. \) Let \( \mathbf{B} \in L(\mathcal{X}_0) \) be any operator of the class \( C_\rho \) with \( \|\mathbf{B}\| = \rho. \) For example,

\[ \mathbf{B} := \begin{bmatrix} 0 & \rho \\ 0 & 0 \end{bmatrix} \in L(\mathbb{C}^2) \]

satisfies \( \mathbf{B}^2 = 0 \) and \( \|\mathbf{B}\| = \rho, \) therefore by properties (iii) and (xi) in Theorem 2.5, \( w_\rho(B) = 1, \) and by property (iv) in the same theorem, \( \mathbf{B} \in C_\rho. \) Set \( \mathcal{X} := \mathcal{X}_0 \oplus \mathcal{X}_0, \)

\[ A_1 := \begin{bmatrix} B & 0 \\ 0 & 0 \end{bmatrix} \in L(\mathcal{X}), \quad A_2 := \begin{bmatrix} 0 & 0 \\ B & 0 \end{bmatrix} \in L(\mathcal{X}). \] (5.6)
Let $U \in L(\tilde{X}_0)$ be a unitary $\rho$-dilation of $B$. Set $	ilde{X} := \tilde{X}_0 \oplus \tilde{X}_0 \oplus \ldots$, and identify $X = X_0 \oplus X_0$ with the subspace $X_0 \oplus X_0 \oplus \{0\} \oplus \{0\} \oplus \ldots$ in $\tilde{X}$. Set

$V_1 := \begin{bmatrix} U & 0 \\ \tilde{U} & \ddots & 0 \\ 0 & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \ddots & \ddots \end{bmatrix} \in L(\tilde{X}), \quad V_2 := \begin{bmatrix} 0 & U \\ \tilde{U} & \ddots & 0 \\ 0 & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \ddots & \ddots \end{bmatrix} \in L(\tilde{X}),$

i.e., the operators $V_1$ and $V_2$ are introduced here as infinite block-diagonal matrices with equal operator blocks $\begin{bmatrix} U & 0 \\ 0 & \ddots \end{bmatrix} \in L(\tilde{X}_0, \tilde{X}_0 \oplus \tilde{X}_0)$ (resp., $\begin{bmatrix} 0 & U \\ \ddots & \ddots \end{bmatrix} \in L(\tilde{X}_0, \tilde{X}_0 \oplus \tilde{X}_0)$) on the main diagonal. We will show that the pair $V = (V_1, V_2)$ is a uniform isometric $\rho$-dilation of the pair $A = (A_1, A_2)$ in the sense of Popescu. First of all, observe that

$V_1^* V_1 = I_{\tilde{X}} = V_2^* V_2, \quad V_1^* V_2 = V_2^* V_1 = 0.$

Next, the following relations hold:

$\forall k \in \mathbb{N}, A_1^k = \begin{bmatrix} B^k & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} \rho P_{X_0} U^k |X_0 & 0 \\ 0 & 0 \end{bmatrix} = \rho P_X V_1^k |X;$

$\forall k, n \in \mathbb{N}, \forall i_1, \ldots, i_n \in \{1, 2\}, A_1^k A_2 A_{i_1} \cdots A_{i_n} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = \rho P_X V_1^k V_2 V_{i_1} \cdots V_{i_n} |X$

(since $A_1^k A_2 = 0, P_{X_0 \oplus \tilde{X}_0 \oplus \{0\} \oplus \{0\} \oplus \ldots} V_1^k V_2 = 0);$

$A_2 = \begin{bmatrix} 0 & 0 \\ 0 & B \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = \rho P_X V_2 |X;$

$\forall k, n \in \mathbb{N}, \forall i_1, \ldots, i_n \in \{1, 2\}, A_2^{k+1} A_{i_1} \cdots A_{i_n} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = \rho P_X V_2^{k+1} V_{i_1} \cdots V_{i_n} |X$

(since $A_2^2 = 0, P_{X_0 \oplus \tilde{X}_0 \oplus \{0\} \oplus \{0\} \oplus \ldots} V_2^2 = 0);$

$\forall k \in \mathbb{N}, A_2 A_1^k = \begin{bmatrix} 0 & 0 \\ P_{X_0} U^{k+1} & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = \rho P_X V_2 V_1^k |X;$

$\forall k \in \mathbb{N}, A_2 A_1^k A_2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = \rho P_X V_2 V_1^k V_2 |X;$

$\forall k, n \in \mathbb{N}, \forall i_1, \ldots, i_n \in \{1, 2\}, A_2 A_1^k A_2 A_{i_1} \cdots A_{i_n} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = \rho P_X V_2 V_1^k V_2 V_{i_1} \cdots V_{i_n} |X$

(since $A_1^k A_2 = 0, P_{X_0 \oplus \tilde{X}_0 \oplus \{0\} \oplus \{0\} \oplus \ldots} V_1^k V_2 = 0).$ Finally, we get

$\forall n \in \mathbb{N}, \forall i_1, \ldots, i_n \in \{1, 2\}, A_{i_1} \cdots A_{i_n} = \rho P_X V_{i_1} \cdots V_{i_n} |X.$
Thus, $V$ is a uniform isometric $\rho$-dilation of $A$ in the sense of Popescu. However, for any $\zeta \in \mathbb{T}^N$,

$$\|\zeta A\| = \left\| \frac{\zeta_1 B}{\zeta_2 B} \right\| = \sqrt{2}\|B\| = \sqrt{2}\rho > \rho.$$ 

Therefore, $\zeta A \notin C_\rho$ for all $\zeta \in \mathbb{T}^N$. We obtain $A \in C_{\rho,2}^\rho \setminus C_{\rho,2}^\nu$ (moreover, $A \notin C_{\rho,2}$).

For the case $N > 2$ (and any $\rho > 0$) an analogous example of $A \in C_{\rho,N}^\rho \setminus C_{\rho,N}^\nu$ is easily obtained from the previous one, by setting zeros for the rest of operators in the $N$-tuple, i.e., $A := (A_1, A_2, 0, \ldots, 0)$, where $A_1$ and $A_2$ are defined in (5.3). In this case the construction of a uniform isometric $\rho$-dilation of $A$ in the sense of Popescu should be slightly changed (we leave this to a reader as an easy exercise).

**Remark 5.6.** The pair $A^{(\varepsilon)} = (A_1^{(\varepsilon)}, A_2^{(\varepsilon)})$ constructed in Theorem 5.2 doesn’t belong to the class $C_{\rho,2}^{\nu}$ for any $\varepsilon > 0$ and $\rho > 1$. Indeed, we have shown in Theorem 5.2 that $A^{(\varepsilon)}$ is not simultaneously similar to any $T = (T_1, T_2) \in C_{1,2}$, not speaking of $T \in C_{1,2}^\nu$. Thus, by Theorem 5.3 $A^{(\varepsilon)} \notin C_{\rho,2}^{\nu}$. This can be shown also by the following estimate: if $A^{(\varepsilon)} \in C_{\rho,2}^{\nu}$ for some $\varepsilon > 0$ and $\rho > 1$, then there exists a uniform unitary $\rho$-dilation $U^{(\varepsilon)} = (U_1^{(\varepsilon)}, U_2^{(\varepsilon)})$ of $A^{(\varepsilon)} = (A_1^{(\varepsilon)}, A_2^{(\varepsilon)})$, and for any $n \in \mathbb{N}$,

$$\|((A_1^{(\varepsilon)} + A_2^{(\varepsilon)})(A_1^{(\varepsilon)} - A_2^{(\varepsilon)}))^n\| = \|\rho P_X ([U_1^{(\varepsilon)} + U_2^{(\varepsilon)}](U_1^{(\varepsilon)} - U_2^{(\varepsilon)}))^n X\| \leq \rho \|[U_1^{(\varepsilon)} + U_2^{(\varepsilon)}](U_1^{(\varepsilon)} - U_2^{(\varepsilon)}))^n\| = \rho < \infty.$$ 

This contradicts to (5.3). Thus, for each $\rho > 1$ we obtain for $\varepsilon > 0$ small enough, $A^{(\varepsilon)} = (A_1^{(\varepsilon)}, A_2^{(\varepsilon)}) \in C_{\rho,2}^{\nu} \setminus C_{\rho,2}^{\rho}$, as well as $A := (A_1, A_2, 0, \ldots, 0) \in C_{\rho,N}^\rho \setminus C_{\rho,N}^\nu$.

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