Thermal Equilibration of Brane-Worlds

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We analyze the thermodynamical properties of brane-worlds, with a focus on the second model of Randall and Sundrum. We point out that during an inflationary phase on the brane, black holes will tend to be thermally nucleated in the bulk. This leads us to ask the question: Can the black hole - brane-world system evolve towards a configuration of thermal equilibrium? To answer this, we generalize the second Randall-Sundrum scenario to allow for non-static bulk regions on each side of the brane-world. Explicitly, we take the bulk to be a \( Vaidya-AdS \) metric, which describes the gravitational collapse of a spherically symmetric null dust fluid in Anti-de Sitter spacetime. Using the background subtraction technique to calculate the Euclidean action, we argue that at late times a sufficiently large black hole will relax to a point of thermal equilibrium with the brane-world environment. These results have interesting implications for early-universe cosmology.

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I. INTRODUCTION: THE THERMODYNAMICS OF INFATING BRANE-WORLDS

Recently \cite{1}, Randall and Sundrum presented a model in which the universe is realized as a \( \mathbb{Z}_2 \)-symmetric positive tension domain wall, or brane, in \( AdS_5 \). They analyzed the linearized equation for a graviton propagating in this spacetime. They proved that there is a solution describing a ‘bound state’, i.e., an integrable wave function corresponding to a massless graviton which is confined to the domain wall. For low energy processes this bound state dominates over the Kaluza-Klein (KK) states, so that Newtonian gravity is recovered as long as the length scale of the AdS space is sufficiently small.

Soon after this model appeared, it was pointed out \cite{2} that the AdS/CFT correspondence gives rise to the Randall-Sundrum model in a certain limit. More precisely, the AdS/CFT correspondence relates the Randall-Sundrum model to an equivalent four dimensional theory consisting of general relativity coupled to a strongly interacting conformal field theory.

Now, the Hawking-Page phase transition \cite{3} manifests itself in the AdS/CFT duality \cite{4}. Explicitly, there is a critical temperature, \( T_c \), past which thermal radiation in AdS is unstable to the formation of a Schwarzschild black hole. (In fact, for \( T>T_c \) there are two values of the black hole mass at which the Hawking radiation can be in equilibrium with the thermal radiation of the background. The lesser of these two masses is a point of unstable equilibrium (it has negative specific heat), whereas the greater mass is a point of stable equilibrium.)

Since the second Randall-Sundrum model may be understood by looking at the AdS/CFT duality in the non-gravity decoupled limit, one would expect that a similar phase transition should occur in that setting. In particular, during an inflationary phase on the brane, the brane-world is a de Sitter hyperboloid embedded in \( AdS_5 \), and it will generate an acceleration horizon in the bulk. This horizon will have a temperature, and so we would expect that inflating brane-worlds would be unstable to the creation of bulk (five-dimensional) black holes. In fact, this process was discussed in a recent paper by Garriga and Sasaki \cite{5}. There, the authors studied the ‘thermal instantons’ which correspond to black holes in AdS, and showed that these instantons describe the thermal nucleation of Schwarzschild-AdS black holes in the presence of a pre-existing \( \mathbb{Z}_2 \) symmetric inflating brane-world.

For us, this is evidence that we should assume that a bulk black hole will be formed during the inflationary epoch of a brane-world universe. It therefore behooves us to answer the question: Can an inflating brane-world and a bulk black hole ever attain thermal equilibrium?

This question is very subtle, because we have to decide what boundary conditions to impose on the bulk fields at the location of the brane-world. Here we assume that the brane-world is a \( \mathbb{Z}_2 \) symmetric domain wall, so that the brane acts like a reflecting mirror for massless bulk modes. In other words, for us the brane-world is like an accelerating mirror: A ‘black box’ with perfectly reflecting, accelerating walls. Certainly, this is the standard boundary condition for bulk modes when the wall is located at the boundary of AdS, and in particular it is the condition assumed by Hawking and Page \cite{6}. Because the wall acceleration will not be constant (as the wall climbs out of the potential well of the black hole), the wall should emit a flux of radiation with some temperature. There
are thus naively three basic cases to consider. First, the black hole temperature ($T_{BH}$) may be greater than the brane-world temperature ($T_{BW}$), in which case the hole has negative specific heat and it will evaporate in a finite time and have no effect on the late time evolution of the system. Alternatively, the system may be fine-tuned so that $T_{BH} = T_{BW}$ exactly; this would describe thermal equilibrium between the two competing temperatures. Finally, it may be the case that $T_{BW} > T_{BH}$\(^1\); clearly, this case is of interest because it would appear that the black hole might be unstable to some runaway process where the mass increases indefinitely. One approach to this problem is to actually explicitly compute the brane-world temperature exactly; however, this task is rather difficult and one is inevitably led to estimates which are hard to control. In this paper, we will instead approach this problem using the more elegant techniques of Euclidean quantum gravity. In order to motivate this analysis, we will begin with a discussion of brane-world evolution in a non-static bulk with collapsing null dust fluid. We work with the signature convention $(-, +, +, +, +, +, +)$.\(^2\)

\(^1\)This analysis is similar to that of Yi [7], who studied black holes which are uniformly accelerated by cosmic strings.

\(^2\)In [9] the authors take $\psi(v, r) = 0$, so that the Einstein equations simplify considerably. We also make this assumption, and so we may follow their analysis and conclude that the source for this metric is a ‘Type II fluid’ [11]. Following [9], it is straightforward to see that if we want to describe gravitational collapse (or the time reverse) in $AdS_5$ then the appropriate mass function, in general, is

$$M(v, r) = \frac{\Lambda}{12} r^4 + m(v) - \frac{q^2(v)}{r^2},$$

(2)

where $\Lambda = -(6/\ell^2)$ is the bulk cosmological constant, and $m(v)$ is an arbitrary function of $v$ which will be determined by the energy density of the radiation in the bulk. $q(v)$ corresponds to the charge of the bulk, if any.

Now that we have a precise idea of what the non-static bulk metric looks like, we turn to the question of how a domain wall moves in such a bulk.

II. GRAVITATIONAL COLLAPSE IN ANTI-DE SITTER SPACE: THE VAIDYA METRIC

In 1951, Vaidya [8] wrote down a metric that represents an imploding (or exploding) null dust fluid with spherical symmetry in asymptotically flat space. Recently [10], this metric has been generalized to describe gravitational collapse in spacetimes with a non-vanishing cosmological constant. Here, we are interested in the metric which describes gravitational collapse in asymptotically Anti-de Sitter (AdS). This metric is written using ‘Eddington-Finkelstein’ coordinates, so that it takes the explicit form

$$ds^2 = -e^{2\psi(v, r)} f(v, r) dv^2 + 2\epsilon e^{\psi(v, r)} dv dr + r^2 d\Omega_3^2,$$

(1)

where

$$f(v, r) = k - \frac{2M(v, r)}{r^2},$$

and

$$d\Omega_3^2 = d\chi^2 + R_k^2(\chi) \left( d\theta^2 + \sin^2 \theta d\phi^2 \right),$$

with $R_{-1}(\chi) = \sinh(\chi)$, $R_0(\chi) = \chi$ and $R_{+1}(\chi) = \sin(\chi)$ is the metric on hyperbolic space, flat space and the round three-sphere respectively. The function $M(v, r)$, called the mass function, is a measure of the total gravitational energy within a radius $r$. The sign $\epsilon = \pm 1$ indicates whether the null coordinate $v$ is advanced or retarded. If $\epsilon = +1$ then $v$ is advanced time, in which case rays of constant $v$ are ingoing. Likewise, $\epsilon = -1$ means that rays of constant $v$ are outgoing. Since we are interested in collapsing radiation, we will assume $\epsilon = +1$.

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III. THE ISRAEL EQUATIONS OF MOTION: THE COSMOLOGY OF BRANE-WORLDS

The equations of motion for a domain wall, when the effects of gravity are included, are given by the Israel junction conditions. These conditions relate the discontinuity in the extrinsic curvature ($K_{AB}$) at the wall to the energy momentum ($t_{AB}$) of fields which live on the wall:

$$[K_{AB} - Kh_{AB}]^{\pm} = \kappa^2_D t_{AB},$$

(3)

(see [13] for a derivation of this equation). The gravitational coupling constant, $\kappa^2_D$, in arbitrary dimension $D$, is given by [14]

$$\kappa^2_D = \frac{2(D - 2)\pi^{D/2}(D - 1)}{(D - 3)(D/2 - 3/2)!} G_D,$$

(4)

where $G_D$ is the $D$-dimensional Newton constant. Here, for instance, $\kappa^2_5 = 3\pi^2 G_5$. Given the form of [3], it is obvious that we need to calculate the extrinsic curvature for timelike hypersurfaces which move in the Vaidya-AdS background.

As above, we begin by writing the metric in Eddington-Finkelstein coordinates:
\[ ds^2 = dv [-f(v,r)dv + 2dr] + r^2d\Omega_3^2, \] \hfill (5)

where we have assumed that the null coordinate \( v \) represents Eddington advanced time. Since we are interested in the cosmological aspects of brane-world evolution, we want the metric induced on the brane-world to assume a manifestly FRW form:

\[ ds^2|_{brane-world} = -d\tau^2 + a^2(\tau)d\Omega^2, \] \hfill (6)

where the coordinates \( x^\mu = (\tau, \chi, \theta, \phi) \) are the coordinates intrinsic to the brane-world. This means that we constrain the timelike hypersurface describing the evolution so that the brane can only shrink or contract as it moves in time, i.e., the position of the brane at a given time should be completely specified by its radial position: \( r = r(v) = a(\tau) \). (In what follows we will let \( a = a(\tau) \) denote the position of the brane, in order to avoid possible confusion between the coordinate \( r \) and the radial position of the brane-world.) In this way, the problem is basically reduced to a 1+1 dimensional system. Furthermore, we need only focus on the contributions to the Riemann tensor which are induced by the \((r, v)\)-sector of the metric. Using the fact that \( \tau \) is the time experienced by observers who move with the brane-world:

\[ d\tau = \left( f - 2\frac{da(\tau)}{dv} \right)^{1/2} dv, \] \hfill (7)

we may express the problem in the \((\tau, a)\) - sector. In particular, if we let \( \dot{a} \) denote differentiation relative to comoving time, i.e., \( \dot{a} = da/d\tau \), then one can find the extrinsic curvature through the following relation

\[ K_{\mu\nu} = -\epsilon^A_{\mu} \epsilon^B_{\nu} \nabla_A n_B, \]

with

\[ \epsilon^A_{\mu} = \left( \frac{1}{f} \right) \left( \dot{a} + \sqrt{\dot{a}^2 + f} \right) \delta^A_{\mu} + \dot{a} \delta^A_{\tau}, \]

as the tetrad at the wall and

\[ n_A = -\dot{a} \delta^v_A + \frac{1}{f} \left( \dot{a} + \sqrt{\dot{a}^2 + f} \right) \delta^A_{\tau}, \]

as the unit normal vector to the hypersurface \( a(\tau) \). \( \nabla_A \) is the covariant derivative associated with the metric \( g_\text{brane-world} \). The nonvanishing components of extrinsic curvature are then

\[ K_{\tau}^\tau = -\frac{1}{2} \frac{2\dot{a} + f'}{\sqrt{f + \dot{a}^2}} + \frac{1}{2} \frac{f}{f'} \left( \dot{a} + \sqrt{f + \dot{a}^2} \right)^2, \]

\[ = -\frac{d}{da} \left( \sqrt{f + \dot{a}^2} \right) + \frac{1}{2} \frac{f}{f'} \left( \dot{a} + \sqrt{f + \dot{a}^2} \right)^2, \] \hfill (8)

\[ K_{\chi}^\chi = K_{\theta}^\theta = K_{\phi}^\phi = -\sqrt{H^2(\tau) + f(\tau, a)}/a^2, \] \hfill (9)

with \( H \equiv (\dot{a}/a) \).

Given these expressions, we can examine how the non-static bulk is affecting cosmology. That is, we assume that the stress-energy tensor describing the fields which propagate in the brane-world is given as

\[ t^B_\mu = \text{diag}(\rho - \rho_\lambda, p - \rho_\lambda, p - \rho_\lambda, p - \rho_\lambda, 0) \]

We emphasize that \( \rho \) and \( p \) are the energy density and pressure of the ordinary matter, respectively, whereas \( \rho_\lambda \) is the contribution from the tension of the brane which is simply a Nambu-Goto term. From the Israel equations we may then derive the Friedmann equation on the brane:

\[ H^2(\tau) = \frac{\Lambda_{eff}}{3} - \frac{k}{a^2} + \left( \frac{8\pi G_4}{3} \right) \rho \]

\[ + \left( \frac{\pi G_5}{a^2} \right)^2 \rho^2 + \frac{2m(\tau,a)}{a^4} - \frac{\dot{a}^2}{a^6}, \] \hfill (10)

Henceforth, we shall set \( q = 0 \) and \( k \) the (spatial) curvature of the brane to unity and \( G_5 \) to \( \sqrt{1/4 G_4/3 \pi \rho_\lambda} \). \( \Lambda_{eff} \) is the 4-dimensional cosmological constant on the brane, which is given in terms of the brane tension \( \rho_\lambda \) and the bulk cosmological constant \( \Lambda \): \( \Lambda_{eff} \equiv \Lambda_4 = (\frac{1}{2} + 4\pi G_4 \rho_\lambda) \).

Thus, we find that a time-dependent mass in the bulk gives rise to a time-dependent term that scales like radiation. In other words, the collapse of radiation (to form a black hole) in the bulk gives rise to a component of ‘Hot Dark Matter’ on the brane. While the form of (10) is what one might naively expect given the adS/CFT duality [3], our calculation actually shows explicitly how a time-dependent mass term in the bulk will affect the brane-world.

**IV. THERMODYNAMICS OF THE SYSTEM**

We have seen that the thermal nucleation of a black hole in the bulk gives rise to a time-dependent radiation term on the brane. To understand how this term will affect brane-world cosmology, we would like to solve for the back-reaction generated by 1-loop effects in the bulk. As this is a rather difficult problem, we will simply employ the technology of Euclidean quantum gravity.

We want to argue that there is no ‘runaway’ process, whereby a black hole can keep absorbing more energy than it emits. Furthermore, we are only interested in the situation where the brane-world remains outside of the horizon; from the analysis of [13] it is easy to see that this is true only when \( r^2 > 4m^2 \). In other words, the brane-world has to be sufficiently large relative to the black hole to accomodate the black hole.

The Euclidean action is given as
\[ I = -\frac{1}{k^2} \int_M d^5x \sqrt{\gamma} (R - 2\Lambda) - \frac{1}{8\pi} \int_{\partial M} d^4x \sqrt{\gamma} K \quad (11) \]

where \( M \) is the bulk region of the spacetime, with boundary \( \partial M \) (i.e., the boundary is where the brane-world is located). In the usual situation (where the brane is actually the boundary of AdS) the boundary term does not contribute; however, in this situation the boundary is at a finite distance and does contribute. In order to show that the system equilibrates, we first calculate the action of the Euclidean section of the brane-world with a black hole in the bulk (denoted \( I_{BH} \)), then we calculate the action for the Euclidean section of the brane-world with no black hole (denoted \( I_{BW} \)), and we form the difference:

\[ \Delta I = I_{BH} - I_{BW} \]

Clearly, a phase transition occurs when \( \Delta I \) changes sign (i.e., when the system with a black hole has less action), and likewise the system is thermally stable when the specific heat, \( H \), is positive:

\[ H = -\beta \partial_\beta I \]

where \( \beta = T^{-1} \) is the inverse temperature, or period of the Euclidean section of the relevant solution. The only subtle point in performing this calculation lies in matching the two solutions in the asymptotic region, near the brane-world. Once this is done, it is straightforward to see that the modification to the ordinary Hawking-Page analysis is negligible, as long as the brane-world remains far from the horizon (in particular, as long as \( r^4 > 4m^2 \)). More precisely, the two boundary contributions differ by \( \approx \sqrt{\kappa m} \). Furthermore, even when this correction is non-negligible, it does not affect the phase structure found by Hawking and Page.

From the point of view of the AdS/CFT duality, this makes sense. Truncating AdS\(_5\) with a de Sitter brane introduces a UV cutoff in the CFT defined on the brane, but in our approximation it does not affect the boundary conditions on the bulk radiation (i.e., we assumed the same boundary conditions as in the original Hawking-Page paper [2]). As long as sufficiently large bulk black holes are allowed, we already knew that these holes can be in thermal equilibrium with thermal radiation. The main shortcoming of this analysis is that we assumed reflecting boundary conditions for all bulk modes at the brane-world. Presumably, there may exist fields on the brane which can be excited by bulk gravitons (for example). This would mean that the boundary conditions at the brane would have to allow for some absorption. We will consider such boundary conditions in an upcoming paper [1].

To summarize: We have shown that the Friedmann equations of the second Randall-Sundrum model will generically contain a hot dark radiation term which has an intensity that is variable but will settle down to a fixed value during inflation. Once inflation ends, the mass function will presumably change in some way; it remains to be shown that the system will still equilibrate once the equation of state on the brane is no longer pure vacuum. It is amusing to note that this varying mass function is reminiscent of the ‘C-field’ introduced by Hoyle and Narlikar in their steady state model and later on in the quasi-steady state model of the universe (see e.g. [13] and references therein). Both our dynamical mass function and their C-field can pump energy into the universe, so that total energy on the brane is not conserved. We will have more to say about the cosmology of this model in an upcoming paper [16].

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