Abstract

Chern–Simons type Lagrangians in $d = 3$ dimensions are analyzed from the point of view of their covariance and globality. We use the transgression formula to find out a new fully covariant and global Lagrangian for Chern–Simons gravity: the price for establishing globality is hidden in a bimetric (or biconnection) structure. Such a formulation allows to calculate from a global and simpler viewpoint the energy-momentum complex and the superpotential both for Yang–Mills and gravitational examples.

1 Introduction and Preliminaries

It is well known that Einstein’s gravity is trivial in dimension $d = 2$, since the curvature tensor reduces essentially to a scalar. Also in dimension $d = 3$ Einstein theory of gravitation is somehow trivial, since the Riemann tensor reduces essentially to the Einstein tensor. Because of this a generalization of the standard Hilbert Lagrangian was suggested in $d = 3$, by introducing, in full analogy with gauge theories, additional terms of the Chern–Simons type \[1\]. In a previous paper of ours \[2\] we have thence tackled with the specific problem of conservation laws for Chern–Simons type Lagrangians, both in the Yang–Mills and in the gravitational case. In particular, we have calculated the relevant energy–momentum complex and the superpotential for Chern–Simons gravity in dimension $d = 3$ (see also \[3\] in this context). Another technique to compute superpotentials for Chern–Simons gauge theory which is based on the so-called cascade equation formalism \[4\] has been recently proposed in \[5\].

Let us recall that Chern–Simons Lagrangians for gravity are non–covariant (and non–global in general) due to the presence of cubic terms in the connection and to a non–covariant coupling of curvature and connection, although field equations turn out to be global and covariant. Because of this and for the sake of simplicity, our result of \[2\] were
obtained in a non–covariant framework as well as by assuming explicitly that spacetime had a trivial topology, i.e. assuming it to be globally diffeomorphic to an open subset of \( \mathbb{R}^3 \). The aim of the present note is thence to provide a “covariantised” version of our previous calculations, by relying on the “background connection method”, a covariantisation procedure which has revealed itself to be rather useful in the case of first order gravity \([6, 7, 8]\). In fact, the present paper is based on and should be considered as a direct continuation of \([2]\). In particular, we shall use the methods for computing currents and superpotentials as presented therein.

It is known that natural, i.e. generally covariant, Lagrangians lead to covariant Euler–Lagrange equations of motion. The inverse statement is, in general, not true. For example, as it was mentioned before, this holds for the following non–global (in general) and non–invariant metric Lagrangian in dimension \( d = 3 \):

\[
L_{CSG} = \frac{1}{2} \varepsilon^{\mu \nu \rho} (R^\beta_{\beta \mu \nu} \Gamma^\alpha_{\alpha \rho} - \frac{2}{3} \Gamma^\alpha_{\beta \mu} \Gamma^\beta_{\sigma \nu} \Gamma^\sigma_{\alpha \rho})
\]  

(1)

where \( \Gamma^\alpha_{\beta \mu} \) and \( R^\beta_{\beta \mu \nu} \) are the Christoffel symbols and the Riemann curvature tensor of a metric \( g_{\mu \nu} \) respectively; here \( \alpha, \mu \ldots = 1, 2, 3 \). The Lagrangian (1) leads, when varied with respect to the metric, to the following global and covariant tensorial Euler–Lagrange equations

\[
C^{\alpha \beta} \equiv 2 \varepsilon^{\mu \nu (\alpha} R^\beta_{\mu \nu)} = 0
\]

(2)

where \( C^{\alpha \beta} \) is called the York–Cotton tensor density and semicolon denotes metric covariant derivative. This symmetric and traceless tensor density vanishes if and only if \( R_{\alpha \beta} \) is the Ricci curvature tensor of a locally conformally flat metric \( g \) (see \([9, 10]\)). Here and above \( \varepsilon^{\mu \nu \alpha} \) denotes the relevant skew–symmetric Levi–Civita tensor density. It has been shown in \([11]\) that the Lagrangian (1) is the only obstruction to the equivariant inverse problem in \( d = 3 \). This Lagrangian is the gravitational counterpart of Chern–Simons Lagrangians of gauge theories \([1, 2]\).

A similar situation occurs in fact for the case of Chern–Simons gauge Lagrangian

\[
L_{CS} = \frac{1}{2} \varepsilon^{\mu \nu \rho} \text{tr}(F_{\mu \nu} A_\rho - \frac{2}{3} A_\mu A_\nu A_\rho)
\]

(3)

where \( A_\mu \) is a matrix–valued gauge potential, its curvature 2–form \( F_{\mu \nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu] \) being the gauge field strength and \( \text{tr} \) denoting the trace operation for matrices (in any suitable matrix group). The Lagrangian (3) is not gauge–covariant although the corresponding field equations \( F = 0 \) are. For this reason such type of Lagrangians are sometimes called quasi (or almost) invariant.

## 2 Transgression Formula and Covariant Chern–Simons Lagrangians

Let \( G \) be any Lie group and let us denote by \( \mathfrak{g} \) the corresponding Lie algebra. For simplicity we shall think of \( G \) as a matrix group and \( \mathfrak{g} \) as a matrix algebra with the commutator \([, ]\) as a Lie bracket. Consider a principal \( G \)–bundle \( P \) over a manifold \( M \) (which, for the moment is arbitrary) with a principal connection \( \omega \) on \( P \). Its curvature 2–form is defined

\[
\]
by Ω = dω + ω ∧ ω and fulfills the Bianchi identities \( DΩ \equiv dΩ + [ω, Ω] = 0 \). Recall (see e.g. [12, 13]) that ω is a \( g \)-valued and \( G \)-equivariant 1-form that lives on the total space \( P \) and which is not defined on the base manifold \( M \). Choosing a (local) section \( e : M \to P \) of \( P \) we get via pull-back a (local) matrix-valued 1-form \( A^{(e)} \equiv e^*ω \) which lives on an open domain \( U \subseteq M \). This is the familiar gauge potential (or Yang-Mills gauge field).

In local coordinates \( \{x^μ\} \) on \( M \) it reads as \( A^{(e)} = A^{(e)}_μ dx^μ \). A change of the local section \( e \mapsto e' = eu \), with \( u \in C^∞(U, G) \), implies a non-tensorial transformation law for the corresponding (local) gauge potentials \( A^{(e)} \mapsto A^{(e')} = u^{-1}A^{(e)} + u^{-1}du; \) \( u \) is also called a gauge transformation. The (local) Yang–Mills field strength 2–form \( F^{(e)} \equiv e^*Ω \), however, undergoes a tensorial transformation rule \( F^{(e)} \mapsto F^{(e')} = u^{-1}F^{(e)}u \).

Because of this Ω is called a tensorial 2–form (see e.g. example 5.2, p. 76 in [12], for the correspondence between tensorial forms on \( P \) and vector–valued forms on the base \( M \)). In local coordinates we shall write \( F^{(e)} = \frac{1}{2}F^{(e)}_{μν}dx^μ ∧ dx^ν \). On the contrary, \( ω \) is a non-tensorial (but vertical) 1–form. By an abuse of notation from now on we shall drop all upper indication to the section \( e \).

For any two principal connection 1–forms \( ω \) and \( \bar{ω} \) on \( P \), Chern and Simons [9, 10] have established the famous transgression formula

\[
\text{tr}(Ω ∧ Ω) − \text{tr}(\bar{Ω} ∧ \bar{Ω}) = d [Q_T(ω, \bar{ω})]
\]

expressing the difference between two tensorial 4–forms. Here

\[
Q_T(ω, \bar{ω}) \equiv \text{tr}(2Ω ∧ α − dα ∧ α − 2ω ∧ α ∧ α + \frac{2}{3}α ∧ α ∧ α ∧ α)
\]

denotes the so-called transgression 3–form (see e.g. [13] p. 348), with \( α = ω − \bar{ω} \) and \( \bar{Ω} = d\bar{ω} + \bar{ω} ∧ \bar{ω} \). Notice that \( \text{tr}(Ω ∧ Ω) \) is a tensorial scalar–valued 4–form on \( P \).

Therefore, it uniquely determines the corresponding 4–form on the base manifold \( M \), since \( e^*(\text{tr}(Ω ∧ Ω)) = \frac{1}{4} \text{tr}(F_{μν}F_{ρσ})dx^μ ∧ dx^ν ∧ dx^ρ ∧ dx^σ \) does not depend on \( e \). The transgression form is an interesting and intriguing object by its own. It can be easily re–expressed as

\[
Q_T(ω, \bar{ω}) = \text{tr}(2\bar{Ω} ∧ α + \bar{D}α ∧ α + \frac{2}{3}α ∧ α ∧ α ∧ α)
\]

where \( \bar{D}α = dα + [\bar{ω}, α] \) denotes the covariant derivative of \( α \) with respect to the connection \( \bar{ω} \). Since \( α \) is tensorial, being the difference of two connections, the form \( Q_T(ω, \bar{ω}) \) is also a tensorial scalar–valued 3–form on \( P \) which uniquely determines the corresponding 3–form on the base manifold \( M \).

The formula expresses a well known fact: although the Chern 4–form \( \text{tr}(Ω ∧ Ω) \) itself depends on the connection its cohomology class \( [\text{tr}(Ω ∧ Ω)] \in H^4(M, \mathbb{R}) \) in the de Rahm cohomology of \( M \) is connection–independent since the difference \( \text{tr}(Ω ∧ Ω) − \text{tr}(\bar{Ω} ∧ \bar{Ω}) \) is exact. In more physical terms we can also say that the Chern form, when considered as a Lagrangian, whenever \( \dim M = 4 \), is variationally trivial since its variation

\[
δ \text{ tr}(Ω ∧ Ω) = 2d \text{ tr}(δω ∧ Ω)
\]

is a total divergence.

From now on we shall assume that the base manifold \( M \) is a 3–manifold. In this case \( Q_T(ω, \bar{ω}) \) is also closed since, of course, any 4–form on a 3–manifold vanishes identically.
Because of this it determines a cohomology class \([Q_T(\omega, \bar{\omega})] \in H^3(M, \mathbb{R})\) which, in general, does not need to be trivial since \(Q_T(\omega, \bar{\omega})\) needs not to be exact.

In particular, by replacing \(\bar{\omega} = 0\) into (5) one immediately recognizes the well known Chern–Simons 3–form:

\[
P_T(\omega) = Q_T(\omega, 0) = \text{tr}(d\omega \wedge \omega + \frac{2}{3} \omega \wedge \omega \wedge \omega) \equiv \text{tr}(\Omega \wedge \omega - \frac{1}{3} \omega \wedge \omega \wedge \omega) \tag{8}
\]

The Chern–Simons form (8) is also a closed, scalar–valued, but non–tensorial 3–form, which lives on the principal bundle \(P\) and not on the base manifold. Therefore it determines a cohomology class \([P_T(\omega)] \in H^3(P, \mathbb{R})\) in the de Rahm cohomology of \(P\), which, in general, may depend on the connection. To see this one can use a type of arguments similar to these presented in [9] (Lemma 3.10). For this purpose we calculate with a bit of algebra the following:

\[
Q_T(\omega, \bar{\omega}) = P_T(\omega) - P_T(\bar{\omega}) - d \text{tr}(\omega \wedge \bar{\omega}) \tag{9}
\]

Now it is clear that the element \([Q_T(\omega, \bar{\omega})] \in H^3(M, \mathbb{R})\) measures, in a certain sense, the difference between the cohomology classes \([P_T(\omega)]\) and \([P_T(\bar{\omega})]\). These classes are called secondary characteristic classes for a manifold with connection.

The local Lagrangians (3) can be obtained from (8) by pull–back along local sections \(e\) of \(P\). If any global section exists, i.e. if \(P\) is a trivializable bundle \(^1\) one can use it to construct a global Lagrangian. In this case the corresponding action integral

\[
A_M(\omega) = \frac{1}{8\pi \kappa} \int e P_T(\omega) \equiv \frac{1}{8\pi \kappa} \int_M e^*(P_T(\omega)) \equiv \frac{1}{8\pi \kappa} \int_M L_{CS}(A) \tag{10}
\]

is multivalued since its value depends on the section chosen \([14, 15, 16]\). In fact, Chern and Simons found that this dependence is up to a homology class of the section \(e\), therefore it must have a non–dynamical character. After introducing an appropriate normalization constant \(\kappa\) it turns out that the actions corresponding to homologically non–equivalent sections differ by integer values (the so–called winding number). Alternatively, one can say that the action depends on the connection and takes its values in the quotient \(\mathbb{R}/\mathbb{Z}\).

In other words it produces a (secondary) characteristic number for a 3–manifold with connection (see [14] for an exhaustive discussion).

To resume, fixing any (global) section, the Chern-Simons form (8) pulls down to \(M\) and gives the Lagrangian (3). This non–invariant Lagrangian produces, however, invariant and geometrically simple equations of motion. Indeed, the variation of (8) gives rise to the following expression (see [2])

\[
\delta P_T = 2 \text{tr}(\delta \omega \wedge \Omega) + d \text{tr}(\delta \omega \wedge \omega) \tag{11}
\]

which of course yields \(\Omega = 0\) as equation of motion. Since \(\delta \omega\) is tensorial, the Euler–Lagrange part is tensorial too and one realizes that the whole non–invariance has passed into the boundary term \(\text{tr}(\delta \omega \wedge \omega)\). This implies that the corresponding canonical Nöther currents and superpotentials are not tensorial (compare formulae (26), (27), (30) and (31))

\(^{1}\)If the group \(G\) is simply connected then any principal \(G\)–bundle over a 3–manifold is trivializable [14].
in [2]). It means that they are gauge (i.e., section) dependent or in other words they live on the total space of the bundle \( P \).

Our main idea in the present note is to use

\[
L_T(A, \bar{A}) = e^\ast (Q_T(\omega, \bar{\omega}))
\]

as a Lagrangian 3–form on \( M \). We stress again that \( L_T \) is a global and covariant object which lives on the base manifold \( M \). This fact is independent of the topologies of \( P \), \( G \) and \( M \). However, the price one has to pay for this is the bi–connection character of the Lagrangian (12). We shall analyze two cases: (i) both connections are dynamical; (ii) only \( \omega \) is dynamical while \( \bar{\omega} \) is a fixed background (non–dynamical) connection.

In terms of physically more relevant (but local) quantities \( \omega = A_\mu dx^\mu \), \( \Omega = \frac{1}{2} F_{\mu\nu} dx^\mu \wedge dx^\nu \) and \( \alpha = B_\mu dx^\mu \equiv (A_\mu - \bar{A}_\mu) dx^\mu \), according to (6) and (9) one has

\[
L_T(A, \bar{A}) = \varepsilon^{\mu\nu\rho} \text{tr}(F_{\mu\nu} \delta A_\rho + \bar{F}_{\mu\nu} \delta \bar{A}_\rho - \partial_\mu [\varepsilon^{\mu\nu\rho} \text{tr}(A_\nu \bar{A}_\rho)]) - \partial_\mu [\varepsilon^{\mu\nu\rho} \text{tr}(A_\nu \bar{A}_\rho)]
\]

where \( dx^\mu \wedge dx^\nu \wedge dx^\rho = \varepsilon^{\mu\nu\rho} dx^1 \wedge dx^2 \wedge dx^3 \) has been used and \( \bar{D}_\mu \equiv \partial_\mu + [\bar{A}_\mu, \cdot] \) denotes the directional covariant derivative with respect to the connection \( \bar{\omega} \). Now the Lagrangian \( L_T \) is represented by a scalar density of weight one rather then a 3–form (see [2]).

The variation of (12) is easily calculated from (9) and (11); we get:

\[
\delta Q_T = 2 \text{tr}(\delta \omega \wedge \Omega) - 2 \text{tr}(\delta \bar{\omega} \wedge \bar{\Omega}) + d \text{tr}((\delta \omega + \delta \bar{\omega}) \wedge \alpha)
\]

Accordingly, (14) reads now as

\[
\delta L_T(A, \bar{A}) = \varepsilon^{\mu\nu\rho} \text{tr}(F_{\mu\nu} \delta A_\rho + \bar{F}_{\mu\nu} \delta \bar{A}_\rho - \partial_\mu [\varepsilon^{\mu\nu\rho} \text{tr}(A_\nu \bar{A}_\rho)])
\]

An infinitesimal pure gauge transformation is given by means of a matrix–valued function (0–form) \( \chi \). One has

\[
\delta \chi A_\mu = D_\mu \chi, \quad \delta \chi \bar{A}_\mu = \bar{D}_\mu \chi \quad \text{and} \quad \delta \chi L_T = 0
\]

i.e. \( L_T \) is a gauge scalar. With this in mind we are able to calculate the canonical Nöther current associated with a gauge symmetry as

\[
J_T^\mu(\chi) = \varepsilon^{\mu\nu\rho} \text{tr}(B_\nu(D_\rho \chi + \bar{D}_\rho \chi))
\]

(see [2]). This quantity is weakly conserved. Due to the second Nöther theorem, it decomposes into the so called reduced current (which vanishes on shell) and the superpotential \([2, 4, 5, 6, 17, 18]\). The superpotential is known to represent that part of a current which is identically conserved, does not vanish on shell and which is enough for the computation of conserved quantities (like charges, masses and so on). In this case one gets explicitly (see [2])

\[
J_T^\mu(\chi) = 2 \partial_\nu [\varepsilon^{\mu\nu\rho} \text{tr}(B_\rho \chi)] + \varepsilon^{\mu\nu\rho} \text{tr}((\chi D_\rho B_\nu + \bar{D}_\rho B_\nu))
\]

\[
= \partial_\rho U_T^{\mu\rho} + \varepsilon^{\mu\nu\rho} \text{tr}(\chi F_{\nu\rho} - \chi \bar{F}_{\nu\rho})
\]
where the superpotential \( U^\mu_\rho = -U^\rho_\mu \) takes the very simple form

\[
U^\mu_\rho (\chi) = 2\varepsilon^{\mu\nu\rho} \text{tr}(B_{\nu}\chi)
\]  

The above decomposition can be easily justified by using the identity \((D + \bar{D})\alpha = 2(\Omega - \bar{\Omega})\) and by the following formula

\[
\text{tr}(D\chi \wedge \alpha) = \text{tr}(D(\chi\alpha)) - \text{tr}(\chi D\alpha) = d\text{tr}(\chi\alpha) - \text{tr}(\chi D\alpha)
\]  

which holds true since the trace vanishes on commutators.

A similar analysis can be performed for the diffeomorphism invariance of \( L_T \). Any vectorfield \( \xi = \xi^\mu \partial_\mu \) on \( M \) is just an infinitesimal diffeomorphism. Under diffeomorphisms the gauge potentials \( A_\mu \) and \( \bar{A}_\mu \) behave (at least locally; see the discussion below) as 1-forms and \( L_T \) as a scalar density of weight one. An infinitesimal diffeomorphism transformation acts on any (natural) geometric object over \( M \) by means of the Lie derivative \( L_\xi \). In particular

\[
\delta_\xi A_\mu \equiv L_\xi A_\mu = \xi^\alpha \partial_\alpha A_\mu + A_\alpha \partial_\mu \xi^\alpha, \quad \delta_\xi L_T \equiv L_\xi L_T = \partial_\alpha (\xi^\alpha L_T)
\]  

and similarly for \( \delta_\xi \bar{A}_\mu \equiv L_\xi \bar{A}_\mu \). This leads to the following expression for the Nöther current

\[
J^\mu_T (\xi) = \xi^\mu L_T + \varepsilon^{\mu\nu\rho} \text{tr}[B_{\nu}\partial_\alpha (A_\rho + \bar{A}_\rho)] \xi^\alpha + \varepsilon^{\mu\nu\rho} \text{tr}[B_{\nu}(A_\alpha + \bar{A}_\alpha)] \partial_\rho \xi^\alpha
\]  

and

\[
U^\mu_\rho (\xi) = \varepsilon^{\mu\nu\rho} \text{tr}[B_{\nu}(A_\xi + \bar{A}_\xi)]
\]  

for the corresponding superpotential (compare with formulae (31) and (32) in [2]). Here for a simplicity we introduced the shortcut \( A_\xi \equiv A_\alpha \xi^\alpha \).

Notice that the expressions (22) and (23) are not gauge–covariant since they do contain gauge non–covariant terms such as \( A_\xi \) and \( \bar{A}_\xi \) as well as terms involving the partial derivatives. A similar situation is also known in Yang–Mills theory, since the formal Lie derivative (21) does not fill the matrix degrees of freedom. Strictly speaking, the group \( \text{Diff}(M) \) of all diffeomorphisms of \( M \) is not valid as a global invariance group for the theory. The most general symmetry group is the group \( \text{Aut}_G(P) \) which consists of all \( G \)-invariant bundle authomorphisms \( \Phi : P \to P \), i.e. the so–called principal \( G \)-authomorphisms of \( P \). The group \( \text{Gauge}(P) \) of all pure gauge transformations is in a natural way a subgroup of \( \text{Aut}_G(P) \), while \( \text{Diff}(M) \) is not. One has instead a surjective group homomorphism from \( \text{Aut}_G(P) \) onto \( \text{Diff}(M) \). The kernel of this homomorphism is of course \( \text{Gauge}(P) \). It is clear that an infinitesimal authomorphism \( \phi \) of the principal bundle \( P \) is generated by the corresponding \( G \)-invariant projectable vectorfield on \( P \) and it can be represented (at least locally) as a pair \( \phi = (\xi, \chi) \) with a vectorfield \( \xi \) uniquely defined [19] (see also discussion in [3]). Fixing some background principal connection \( \omega_o \equiv a_\mu dx^\mu \) on \( P \) and choosing \( \chi = -a_\xi \) we may use the formula

\[
\delta_{(\xi,\chi)} A_\mu \equiv L_\xi A_\mu - D_\mu a_\xi = \xi^\alpha F_{\mu\alpha} + D_\mu (A_\xi - a_\xi)
\]  

in order to lift the vectorfield \( \xi \) on \( M \) into the corresponding \( G \)-invariant projectable vectorfield on \( P \). Such a lifting is not canonical, being background dependent, but it is
global. Moreover, $\omega_0$ is flat if and only if the corresponding lift is a Lie algebra map. This remark generalizes a so called improved diffeomorphism technique presented in [20, 21, 22].

We can conclude this part by saying that the diffeomorphism invariance of a Yang–Mills type Lagrangian is encoded into the invariance with respect to the lifted diffeomorphisms, i.e. the corresponding principal authomorphisms of $P$. In our case, the total superpotential related to the diffeomorphism invariance of the Lagrangian (6) has the form

$$U_T^{\mu\rho}(\xi, a) = \varepsilon^{\mu\nu\rho} \text{tr}[B_\nu(A_\xi + \bar{A}_\xi - 2a_\xi)]$$

which is covariant but depends on the background. Finally, choosing $a = \bar{A}$ we find

$$U_T^{\mu\rho}(\xi, \bar{A}) = \varepsilon^{\mu\nu\rho} \text{tr}(B_\nu B_\xi)$$

which is fully covariant and background independent, provided $\bar{A}$ is a dynamical connection. Notice, that for a diagonal solution $A = \bar{A}$ all expressions for the superpotential automatically vanish, while the limit $\bar{A} \to 0$ reproduces the results previously given in [2].

Alternatively, let us now assume that the connection $\bar{\omega}$ is a fixed background (non–dynamical) connection. Thus $\delta \bar{\omega} = 0$ in (14). The theory in this case has only one dynamical field but the class of symmetries is more restrictive: the gauge transformations have to keep the background unchanged, i.e. $\delta_\chi \bar{A}_\mu \equiv \bar{D}_\mu \chi = 0$. This implies $\bar{D}_\mu \chi = [B_\mu, \chi]$ and

$$J_T^\mu(\chi) = 2\varepsilon^{\mu\nu\rho} \text{tr}(B_\nu B_\rho \chi) \equiv \varepsilon^{\mu\nu\rho} \text{tr}([B_\nu, B_\rho] \chi)$$

This last expression vanishes identically in the case of an Abelian gauge group.

Recently, the so called mixed Chern-Simons term based on two independent $U(1)$–gauge fields, one of electromagnetic origin and the other statistical, has been successfully applied in 2–dimensional superconductivity (see [23] and references quoted therein)².

3 Bi–metric Chern–Simons gravity

A particularly interesting situation appears when $P$ is the bundle of linear frames $LM$, so that the group $G$ is the general linear group $GL(3, \mathbb{R})$. Linear connections on $M$ are principal connections in $LM$.

In this case $\omega$ is a $\mathfrak{gl}(3, \mathbb{R})$–valued 1–form on the bundle $LM$ representing a linear connection on $M$ and $\Omega$ is its Riemann curvature 2–form. We can use a coordinate section (gauge) $\{\partial_\mu\}$ to write down $\omega = \Gamma_\mu dx^\mu$ and $\Omega = \frac{1}{2} R_{\mu\nu} dx^\mu \wedge dx^\nu$, where $\Gamma_\mu \equiv \Gamma_\mu^{\alpha}$ and $R_{\mu\nu} \equiv R_{\beta\mu\nu}$, are the standard local expressions for the connection coefficients and its Riemann curvature tensor represented now as $3 \times 3$ matrices. Alternatively, we can also use a local (but not necessarily coordinate) section $\{E_i = E_i^\mu \partial_\mu\}$, the so–called dreibein. In this case, the matrix indices $\Gamma_\mu \equiv \Gamma_i^j \partial_\mu$ and $R_{\mu\nu} \equiv R_i^{j\mu\nu}$, so called “world indices”, are inherited from the dreibein $\{E_i\}$.

The Chern–Simons 3–form (8) lives then on the bundle of linear frames $LM$ and the (local) Lagrangian (1) can be obtained from (8) by pull–back along a coordinate section $\{\partial_\mu\}$ of $LM$. Having chosen a coordinate atlas on the base manifold, with any coordinate

\[\text{This comment is due to Ashoke Das.}\]
neighborhood one can associate such a local Lagrangian. On the intersection of two neighborhoods both Lagrangians differ by a total derivative. This defines a 0–cochain of local Lagrangians in the sense of Čech cohomology. Conservation laws for this type of non–global Lagrangians will be investigated in detail in [24]. If the manifold \( M \) is parallelizable (i.e. \( LM \) is a trivial bundle, what is always the case for a compact, oriented 3–manifold), one can also use a global (but probably no longer coordinate) dreibein to obtain a global but not invariant Lagrangian.

Assuming that the linear connection \( \Gamma \) is the Levi–Civita connection of some metric \( g \) on \( M \):
\[
\Gamma^\alpha_{\beta\mu} = \frac{1}{2} g^{\alpha\sigma} (\partial_\beta g_{\mu\sigma} + \partial_\mu g_{\sigma\beta} - \partial_\sigma g_{\beta\mu})
\] (28)
i.e. considering \( g \) instead of \( \Gamma \) as the dynamical variable, we thus obtain Chern–Simons gravity theory. The corresponding action \([10]\) is metric dependent and it produces the secondary invariant of a Riemannian manifold \((M, g)\) (see [9, 10]).

The transgression form \([6]\) gives then a new global and bimetric Lagrangian density for a Chern–Simons gravity. The Lagrangian \([133]\) takes now the form
\[
L_{TG}(g, \bar{g}) = \varepsilon^{\mu\nu\rho} \text{tr}(\bar{R}_{\mu\nu}^\alpha N_\rho + (\bar{\nabla}_\mu N_\nu) N_\rho + \frac{2}{3} N_\mu N_\nu N_\rho)
\]
\[
\equiv \frac{1}{2} \varepsilon^{\mu\nu\rho} \text{tr}(R_{\mu\nu} \Gamma_\rho - \bar{R}_{\mu\nu} \bar{\Gamma}_\rho - \bar{\Gamma}_\mu \bar{\Gamma}_\nu \Gamma_\rho + \frac{2}{3} \bar{\Gamma}_\mu \bar{\Gamma}_\nu \bar{\Gamma}_\rho - \partial_\mu [\varepsilon^{\mu\nu\rho} \text{tr}(\bar{\Gamma}_{\nu \rho})] \] (29)

where \( N_\mu \equiv \Gamma_\mu - \bar{\Gamma}_\mu \) and \( \bar{\nabla}_\mu \equiv \partial_\mu + [\bar{\Gamma}_\mu, \cdot] \) denotes the Levi–Civita covariant derivative with respect to the metric \( \bar{g} \). Again, since the difference of two connections is a tensorial 1–form \( N^i_\mu \text{d}x^\mu \) one plays exclusively with tensorial objects. Therefore, there is no need to distinguish between the world and the local indices. Accordingly, the Lagrangian density \([29]\) is a global and dreibein independent 3–form on \( M \). It is even fully covariant (i.e. natural) if one considers both metrics \((g, \bar{g})\) as dynamical fields. Now, it is well justified to use the local expression
\[
L_{TG}(g, \bar{g}) = \varepsilon^{\mu\nu\rho} (\bar{R}_{\beta\mu\nu}^\alpha N_\rho + (\bar{\nabla}_\mu N_\nu^\alpha) N_\rho + \frac{2}{3} N_\mu^\alpha N_\nu^\beta N_\rho^\sigma)
\]
\[
\equiv \frac{1}{2} \varepsilon^{\mu\nu\rho} (R_{\beta\mu\nu} \Gamma_\rho^\alpha - \bar{R}_{\beta\mu\nu} \bar{\Gamma}_\rho^\alpha - \bar{\Gamma}_\mu \bar{\Gamma}_\nu \Gamma_\rho^\alpha + \frac{2}{3} \bar{\Gamma}_\mu \bar{\Gamma}_\nu \bar{\Gamma}_\rho^\alpha - \partial_\mu [\varepsilon^{\mu\nu\rho} \Gamma_{\nu \rho}^\alpha \bar{\Gamma}_\alpha^\beta] \] (30)

for the corresponding global 3–form on \( M \). Variation of \([30]\) with respect to the connections \((\Gamma, \bar{\Gamma})\) yields (compare with \([133]\)):
\[
\delta L_{TG} = \varepsilon^{\mu\nu\rho} (R_{\beta\mu\nu}^\alpha \delta \Gamma_\rho^\alpha - \bar{R}_{\beta\mu\nu}^\alpha \delta \bar{\Gamma}_\rho^\alpha - \partial_\mu [\varepsilon^{\mu\nu\rho} N_\nu^\alpha (\delta \bar{\Gamma}_\rho^\beta + \delta \bar{\Gamma}_\alpha^\beta)]
\] (31)

In fact, the Lagrangian \([30]\) and its variation \([31]\) can be alternatively analyzed from a first-order (á la Palatini) point of view, i.e. having just two linear connections \((\Gamma, \bar{\Gamma})\) as dynamical variables \footnote{Of course, the bi-metric and bi-connection approaches are not equivalent since they lead to non-equivalent equations of motion.}. As a symmetry transformation consider then a 1–parameter
group of diffeomorphisms generated by the vectorfield $\xi = \xi^\alpha \partial_\alpha$. The Lie derivative of an arbitrary (non-symmetric) linear connection $\Gamma$ reads (see e.g. [25, 26])

$$\delta_\xi \Gamma_\rho \equiv L_\xi \Gamma^\beta_{\alpha \rho} = \xi^\alpha R^\beta_{\alpha \sigma \rho} + \nabla_\rho \nabla_\alpha^* \xi^\beta$$  \hspace{1cm} (32)

where $\nabla^*_\alpha \xi^\beta = \partial_\alpha \xi^\beta + \Gamma^\beta_{\sigma \alpha} \xi^\sigma$ (remember that $\nabla_\alpha \xi^\beta = \partial_\alpha \xi^\beta + \Gamma^\beta_{\sigma \alpha} \xi^\sigma$). It defines a canonical natural lift from any vectorfield on $\mathcal{M}$ to the corresponding invariant projectable vectorfield on an appropriate bundle of geometric objects over $\mathcal{M}$. In other words, the difference between this case and the general one discussed in the previous section is that the Lie transport provides now a canonical (i.e. background independent) embedding of $\text{Diff}(\mathcal{M})$ into the group of principal authomorphisms of $\mathcal{L}\mathcal{M}$ with a gauge part represented by $\chi = \nabla^* \xi$. Applying formula (23) to the present case one might be tempted to write

$$U^\mu\rho_T(\xi) = \varepsilon^{\mu \nu \rho \sigma} N^\alpha_{\beta \gamma}(\nabla_\gamma + \nabla_\gamma^*) \xi^\beta$$  \hspace{1cm} (33)

for the corresponding superpotential. This is wrong since variation (32) is a second order differential operator in $\xi$ (see [2, 6, 7]).

It is now convenient to assume that both connections are symmetric (i.e. torsion free) linear connections on $\mathcal{M}$. Thus following the same steps as for computations of formula (60) in [2] but this time in a covariant manner, i.e. having replaced $\Gamma^\alpha_{\beta \mu}$ by $N^\alpha_{\beta \mu}$ and the partial derivatives $\partial_\mu$ by the covariant ones $\nabla_\mu$ or $\nabla_\mu$ respectively one gets

$$U^\mu\rho_T(\xi) = \frac{1}{6} \varepsilon^{\mu \nu \rho \sigma}(\nabla_\sigma + \nabla_\sigma)(3N^\alpha_{\nu \sigma} - \delta^\sigma_{\nu} N^\beta_{\alpha \sigma})\xi^\alpha - \frac{1}{3} \varepsilon^{\mu \nu \rho}(3N^\sigma_{\nu \rho} - \delta^\sigma_{\rho} N^\beta_{\alpha \rho})(\nabla_\sigma + \nabla_\sigma) \xi^\alpha$$  \hspace{1cm} (34)

Coming back to the purely metric formalism we wish to perform the variation of (30) with respect to the metrics $(g, \tilde{g})$. For this reason one has to replace $\delta \Gamma_\rho$ in the first term of (31) by means of the “Palatini formula”

$$\delta \Gamma^\alpha_{\beta \rho} = \frac{1}{2} g^{\alpha \sigma}(\nabla_\beta \delta g_\sigma - \nabla_\rho \delta g_\sigma - \nabla_\sigma \delta g_\rho)$$  \hspace{1cm} (35)

and the same for $\delta \Gamma_\rho$. Accordingly, after some computation (see also [2]) the bimetric first variational formula reads now as

$$\delta L_{TG} = \mathcal{C}^\alpha_{\rho \sigma} \delta \tilde{g}_{\alpha \rho} - \mathcal{C}^\alpha_{\rho \sigma} \delta g_{\alpha \rho} + \partial_\mu [\varepsilon^{\mu \nu \rho}(2R^\alpha_{\nu \rho} \delta g_\alpha - 2\tilde{R}^\alpha_{\nu \rho} \delta \tilde{g}_{\alpha \rho} - N^\alpha_{\beta \nu}(\delta \Gamma^\beta_{\alpha \rho} + \delta \tilde{\Gamma}^\beta_{\alpha \rho})]$$  \hspace{1cm} (36)

where the York–Cotton tensor density $\mathcal{C}^\alpha_{\rho}$ (resp. $\tilde{\mathcal{C}}^\alpha_{\rho}$) is given by (2). The Euler–Lagrange field equations are $\mathcal{C}^\alpha_{\rho} = \tilde{\mathcal{C}}^\alpha_{\rho} = 0$. We recall that the York–Cotton tensor density is symmetric, traceless, divergence–free and it vanishes if and only if the corresponding metric is conformally flat.

Again, as a symmetry transformation let us consider a flow of diffeomorphisms generated by the vectorfield $\xi = \xi^\alpha \partial_\alpha$. In this case the Lie derivative operators

$$\delta_\xi g \equiv L_\xi g_{\alpha \rho} = \nabla_\alpha \xi_\rho + \nabla_\rho \xi_\alpha$$  \hspace{1cm} (37)

and

$$\delta_\xi \Gamma_\rho \equiv L_\xi \Gamma^\beta_{\alpha \rho} = \xi^\alpha R^\beta_{\alpha \sigma \rho} + \nabla_\rho \nabla_\alpha \xi^\beta$$  \hspace{1cm} (38)
represent the infinitesimal variations.

Consequently, the formulae (36), (37) and (38) allow us to calculate the canonical energy–momentum complex and superpotential in both covariant (bi–metric) and background connection (δ\overline{\Gamma} ≡ 0) formalisms. To this end we make use of the computations already performed in [2]. Only terms under the divergence in (36) will contribute into the superpotential. We see that the first two terms correspond to the formula (56) in [2]. Therefore, combining with (34) we arrive to the following expression:

\[ U^{\mu\rho}_{\Gamma}(\xi) = \varepsilon^{\mu\nu\rho}[(3R_{\nu\alpha} - Rg_{\nu\alpha} - 3\bar{R}_{\nu\alpha} + \bar{R}\bar{g}_{\nu\alpha})\xi^\alpha + \frac{1}{6}\varepsilon^{\mu\nu\rho}[(\nabla_\sigma + \bar{\nabla}_\sigma)(3N^\sigma_{\nu\alpha} - \delta^\sigma_\nu N^\beta_{\alpha\beta})(\nabla_\sigma + \bar{\nabla}_\sigma)\xi^\alpha - \frac{1}{3}\varepsilon^{\mu\nu\rho}(3N^\sigma_{\nu\alpha} - \delta^\sigma_\nu N^\beta_{\alpha\beta})(\nabla_\sigma + \bar{\nabla}_\sigma)\xi^\alpha] \] (39)

As a concrete example one can consider a solution (g, \bar{g}) consisting of a flat metric \bar{g}_{\mu\nu} = \eta_{\mu\nu} while g_{\mu\nu} = \exp(2\phi) \eta_{\mu\nu} being conformal to \eta with a conformal factor \phi. Having chosen \xi^\alpha = \eta^{\alpha\beta}\phi,_{\beta} one calculates

\[ U = dF + Fd\phi \] (40)

as a 1–form, where \( F = -\eta^{\alpha\beta}\phi,_{\alpha}\phi,_{\beta}. In particular, for \phi = r \equiv \sqrt{\eta_{\alpha\beta}x^\alpha x^\beta} we obtain

\[ U^{\mu\rho} = -\varepsilon^{\mu\nu\rho} \frac{x_\nu}{r} \] (41)

4 Conclusions

We have considered the Chern–Simons type models in three dimensions. Exploiting the Chern–Simons transgression 3-form enables us to find a new global Lagrangian density which unlike, the local Chern-Simons Lagrangian is generally covariant. However, in this approach the covariant Lagrangian has bi-connection character and the corresponding theory is getting lost some of its topological properties. Particulary, the action functional becomes insensitive for topology of underlaying 3-manifolds. The formalism has been used for calculation of conserved Nöther currents and their identically conserved parts – superpotentials. Two special cases are of particular interest: the case of two connection being dynamical and the case when one of the connections is given as a fixed background while the only the other one is dynamical. Finally, the Chern-Simons gravity has been treated in a similar way. In this sense the present paper generalizes the results of our previous paper [2] obtained for non-covariant Chern–Simons Lagrangians (see also [3]). Recently, this covariant formalism has been successfully applied to explicit numerical calculations of conserved quantities for BTZ black hole solutions in AdS3 Chern–Simons gravity of the Witten type [27].

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