Centrally Essential Semirings

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Received December 24, 2021; revised January 26, 2022; accepted February 10, 2022

Abstract—By a semiring, we mean a structure that differs from an associative ring, possibly, by the irreversibility of the additive operation. A semiring $S$ is said to be centrally essential if $S$ either is commutative or any non-central element of $S$ becomes a nonzero central element after multiplying by a central element. We study centrally essential semirings and their properties. We give some examples of non-commutative centrally essential semirings and describe properties of additively cancellative centrally essential semirings. It is proved that an additively cancellative reduced semiring is commutative if and only if its ring of differences is a centrally essential ring. In addition, in an additively cancellative centrally essential semiring $S$, any complemented idempotent is central. Additively cancellative semisubtractive centrally essential semiring without non-zero nilpotent elements is commutative. We give an example of a noncommutative additively cancellative reduced centrally essential group semiring without zero-divisors. A left (resp., right) multiplicatively cancellative centrally essential semiring is commutative.

DOI: 10.1134/S199508022206021X

Keywords and phrases: centrally essential semiring; additively cancellative semiring; reduced semiring.

1. INTRODUCTION

A semiring is a system consisting of a set $R$ together with two binary operations on $R$ called addition and multiplication such that

1. $(R, +)$ is a commutative monoid with identity element $0$;
2. $(R, \cdot)$ is a monoid with identity element $1$;
3. Multiplication distributes over addition from either side;
4. $0 \cdot r = 0 = r \cdot 0$ for all $r \in R$.

We will add one more axiom to eliminate the trivial case

5. $1 \neq 0$ (otherwise $r = r \cdot 1 = r \cdot 0 = 0$ for all $r \in R$, i.e. $R = \{0\}$).

For a semiring $S$, the center of $S$ is the set $C(S) = \{s \in S: ss' = s's$ for all $s' \in S\}$. This set is not empty, since it contains 0 and 1; we also have that $C(S)$ is a subsemiring in $S$. A semiring is said to be centrally essential if for every non-zero element $x$, there exist non-zero central elements $y, z$ with $xy = z$.

Centrally essential rings with non-zero 1 are studied, for example, in [9–13]. Every centrally essential semiprime ring with $1 \neq 0$ is commutative; see [9, Proposition 3.3]. If $F$ is the field of order 2 and $Q_8$ is the quaternion group of order 8, then the group algebra $FQ_8$ is a finite non-commutative centrally essential ring; see [9]. In addition, the Grassman algebra over a three-dimensional vector space over the field of order 3 also is a finite non-commutative centrally essential ring; see [10]. In [13], a centrally essential ring $R$ is constructed such that the factor ring $R/J(R)$ with respect to the Jacobson radical

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$J(R)$ is not a PI ring (in particular, the ring $R/J(R)$ is not commutative). Matrix centrally essential algebras are studied in [7]. Abelian groups with centrally essential endomorphism rings are considered in [6] and [8].

**Example 1.** We consider a semigroup $(M, \cdot)$ with multiplication table

\[
\begin{array}{ccc}
\cdot & a & b & c \\
1 & 1 & a & b \\
a & a & a & a \\
b & b & b & b \\
c & c & c & c \\
\end{array}
\]

For a quick test of associativity, it is convenient to use the Light's associativity test; see [2, p. 7]. Let $S = \text{Sub}(M)$ be the set of all subsets of the semigroup $M$. For any $A, B \in S$, operations $A + B = A \cup B$ and $AB = \{ab : a \in A, b \in B\}$ are defined; then $S$ is a semiring with zero $\emptyset$ and the identity element $1 = \{1_M\}$; see [3, Example 1.10]. We have $|S| = 2^4 = 16$. We note that $S$ does not have zero sums, i.e., the relation $A + B = \emptyset$ implies the relation $A = B = \emptyset$. In addition, $S$ is additively idempotent and multiplicatively idempotent. The center $C(S)$ is of the form $C(S) = \{\emptyset, \{1\}, \{c\}, \{1, c\}\}$. If $\emptyset \neq A \in S$, then $A \cdot \{c\} = \{c\} \in C(S)$. Consequently, $S$ is a non-commutative centrally essential semiring.

A semiring $S$ is said to be reduced if $x = y$ for all $x, y \in S$ with $x^2 + y^2 = xy + yx$; see [1]. If $S$ is a ring, this is equivalent to the property that $S$ has no non-zero nilpotent elements. A semiring $S$ is said to be additively cancellative if the relation $x + z = y + z$ is equivalent to the relation $x = y$ for all $x, y, z \in S$. A ring $D(S)$ is called the ring of differences of the semiring $S$ if $S$ is a subsemiring in $D(S)$ and every element $a \in D(S)$ is the difference $a = x - y$ of some elements $x, y \in S$. The class of additively cancellative semirings contains all rings. The ring of differences is unique up to isomorphism over $S$; see [5, Chapter II] for details. An element $a$ of the semiring $R$ is called a left zero-divisor if $ab = 0$ for some $0 \neq b \in S$. Similar to [12, Lemma 2.2], it can be proved that one-sided zero-divisors are two-sided zero-divisors in a centrally essential semiring. Other semiring-theoretical notions and designations can be found in [3, 5].

In this work, we study properties of additively cancellative centrally essential semirings. The main result of this paper is the following theorem.

**Theorem 1.** There exists a non-commutative additively cancellative reduced centrally essential semiring without zero-divisors. An additively cancellative reduced semiring $S$ is commutative if and only if the ring of differences of $S$ is a centrally essential ring.

Note that the semiring $S$ in Example 1 is not additively cancellative, because $S$ is additively idempotent.

**2. ADDITIVELY CANCELLATIVE CENTRALLY ESSENTIAL SEMIRINGS**

A semiring $S$ is said to be semiprime if $S$ does not have nilpotent ideals. A semiring $S$ is said to be semisubtractive if for all $a, b \in S$ with $a \neq b$, there exists an element $x \in S$ such that $a + x = b$ or $b + x = a$.

**Proposition 1.** Let $S$ be an additively cancellative semisubtractive centrally essential semiring with center $C$. The following conditions are equivalent.

- $S$ is a semiprime semiring;
- $C$ is a semiprime semiring;
- $S$ does not have non-zero nilpotent elements;
- $S$ is a commutative semiring without non-zero nilpotent elements.

**Proof.** It is well known that a semiring $S$ can be embedded in the ring of differences $D(S)$ if and only if $S$ is additively cancellative. In addition, the relation $D(S) = -S \cup S$ holds if and only if $S$ is a semisubtractive semiring; see [5, Chapter II, Remark 5.12]. Then the assertion follows from [11, Proposition 2.8].

**Remark 1.** It follows from Example 1 that the assertion of Proposition 1 is not true without the assumptions of additive cancellativity and semisubtractivity. In Example 4 a non-commutative centrally essential semiring without zero-divisors is constructed; this semiring is additively cancellative but is not semisubtractive.

It is known that every idempotent of a centrally essential ring is central; see [9, Lemma 2.3]. For semirings, a similar result is not true: in Example 1, the semiring $S$ is non-commutative multiplicatively idempotent centrally essential semiring. For a semiring $S$, an idempotent $e$ of $S$ is said to be *complemented* if there exists an idempotent $f \in S$ with $e + f = 1$.

**Proposition 2.** In an additively cancellative centrally essential semiring $S$, any complemented idempotent is central.

**Proof.** Let $e^2 = e$ and $e + f = 1$ for some $f \in S$. Since $S$ is an additively cancellative semiring, it follows from $e = e + fe$ that $fe = 0$. Similarly, we have $ef = 0$. Let $x \in S$. Then $x = ex + fx$ and $xe = exe + fxe$.

First, we assume that $fxe = 0$, i.e., $xe = exe$. Since $x = xe + xf$, we have $ex = exe + ef$. If $ef \neq 0$, then there exist $c, d \in C(S)$ with $(ef)c = d \neq 0$. Since $d = (ef)c$ is a central element and $e$ is an idempotent, $ed = d = de$. Then

$$0 \neq d = ed = de = (ef)c = (exc)f = 0;$$

this is a contradiction. Consequently, $ef = 0$ and $ex = xe = exe$.

Now let $fxe \neq 0$. Then $0 \neq (fxe)c = d$ for some non-zero elements $c, d \in C(S)$. In this case,

$$0 \neq d = de = ed = ef(xec) = 0;$$

this is a contradiction.

**Corollary 1.** If $S$ is an additively cancellative semiring, then the semiring $M_n(S)$ of all matrices and the semiring $T_n(S)$ of all upper triangular matrices over $S$ are not centrally essential for $n \geq 2$.

**Proof.** For the identity matrices of the above semirings, we have $E = E_{11} + \ldots + E_{nn}$, where $E_{11}, \ldots, E_{nn}$ are matrix units. It follows from [3, Example 4.19] that $M_n(S)$ is an additively cancellative semiring. The idempotents $E_{11}, \ldots, E_{nn}$ are non-central complemented idempotents. Consequently, the semirings $M_n(S)$ and $T_n(S)$ are not centrally essential.

As it was mentioned above, additively cancellative semirings $S$ coincide with semirings $S$ which can be embedded in the rings of differences $D(S)$ whose elements are of the form $x - y$, where $x, y \in S$.

**Example 2.** We consider the semiring $S$ generated by the matrices

$$\begin{pmatrix} \alpha & a & b \\ 0 & \alpha & c \\ 0 & 0 & \alpha \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \alpha \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \alpha \end{pmatrix}, \begin{pmatrix} \alpha & 0 & 0 \\ 0 & \alpha & 0 \\ 0 & 0 & \alpha \end{pmatrix}.$$
where $\alpha, a, b, c \in \mathbb{Z}^+$. Let $A = (a_{ij})$ and $B = (b_{ij})$, where $a_{12} = b_{23} = a, b_{12} = a_{23} = c, a \neq c$, and the remaining components are equal to each other. Then $AB \neq BA$, i.e., $S$ is a non-commutative semiring. It is directly verified that the center $C(S)$ consists of matrices of the form

$$\begin{pmatrix}
\alpha & 0 & b \\
0 & \alpha & 0 \\
0 & 0 & \alpha
\end{pmatrix},$$

where $\alpha, b \in \mathbb{Z}^+ \cup \{0\}$. Since $0 \neq AD \in C(S)$, where $0 \neq A \in S \setminus C(S), 0 \neq D \in C(S)$ with $\alpha = 0$, we have that $S$ is a non-commutative centrally essential semiring. However, the ring of differences $D(S) = M_3(\mathbb{Z})$ is not a centrally essential ring, since the ring has non-central idempotents. In addition, any centrally essential subalgebra of triangular $3 \times 3$ matrix algebra is commutative; this is proved in [7, Proposition 3.1].

We give an example of a centrally essential ring $R$ which is the ring of differences for two proper semirings $S_1$ and $S_2$ of $R$ such that $S_1$ is not a centrally essential semiring and $S_2$ is a centrally essential semiring.

Example 3. Let $R$ be the ring consisting of matrices of the form

$$\begin{pmatrix}
\alpha & a & b & c & d & e & f \\
0 & \alpha & 0 & b & 0 & 0 & d \\
0 & 0 & \alpha & 0 & 0 & e & 0 \\
0 & 0 & 0 & \alpha & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \alpha & 0 & a \\
0 & 0 & 0 & 0 & 0 & \alpha & b \\
0 & 0 & 0 & 0 & 0 & 0 & \alpha
\end{pmatrix},$$

(1)

over the ring $\mathbb{Z}$ of integers. In [7, Example 4.1], it is proved that $R$ is a non-commutative centrally essential ring. Let $S_1$ be the semiring generated by matrices of the form (1) over $\mathbb{Z}^+$ and scalar matrices with $\alpha \in \mathbb{Z}^+ \cup \{0\}$ and zeros on the remaining positions. Since $C(S_1)$ consists of scalar matrices, $S_1$ is not a centrally essential semiring. We note that $S_1$ is a semiring without zero-divisors. At the same time, the semiring $S_2$ of matrices of the form (1) over the semiring $\mathbb{Z}^+ \cup \{0\}$ is a centrally essential semiring.

Lemma 1. [5, Chapter II, Theorem 5.13] In a semiring $S$, any central element is contained in the center $C(D(S))$ of its ring of differences.

Proposition 3. Let $S$ be a centrally essential semiring without zero-divisors. If the ring $D(S)$ does not contain zero-divisors, the semiring $S$ is commutative.

Proof. Let $0 \neq a = x - y \in D(S)$. By assumption, $0 \neq xc = d$ and $0 \neq yf = g$ for some $c, d, f, g \in C(S)$. Then

$$a(cf) = (x - y)cf = (xc)f - (yf)c = df - gc.$$

It follows from Lemma 1 that $c, d, f, g \in C(D(S))$ and $ae' \in C(D(S))$, where $e' = cf$. In addition, $ae' \neq 0$, since $D(S)$ does not contain zero-divisors. Then $D(S)$ is a centrally essential ring, which must be commutative; see [9, Proposition 3.3].

LOBACHEVSKII JOURNAL OF MATHEMATICS Vol. 43 No. 3 2022
3. THE PROOF OF THEOREM 1

We recall that the upper central series of a group $G$ is the chain of subgroups

$$\{e\} = C_0(G) \subseteq C_1(G) \subseteq \ldots,$$

where $C_i(G)/C_{i-1}(G)$ is the center of the group $G/C_{i-1}(G)$, $i \geq 1$. For a group $G$, the nilpotence class of $G$ is the least positive integer $n$ with $C_n(G) = G$ provided such an integer $n$ exists.

**Proposition 4.** (cf. [9, Proposition 2.6]) Let $G$ be a finite group of nilpotence class $n \leq 2$ and let $S$ be a commutative semiring without zero-divisors and zero sums. Then $SG$ is a centrally essential group semiring.

**Proof.** If $n = 1$, then the group $G$ is Abelian and $SG$ is a centrally essential group semiring; see [9, Lemma 2.2].

Let $n = 2$. Similar to the case group rings (e.g., see [14, Part 2]), the center $C(SG)$ is a free $S$-semimodule with basis

$$\left\{ \sum_K : K \text{ are the conjugacy classes in the group } G \right\}.$$

It is sufficient to verify that $SG \sum C(G) \subseteq C(SG)$, where $C(G)$ is the center of the group $G$. If $g, h \in G$, then $(gh)^{-1}hg \sum C(G) = \sum C(G)$, since $h^{-1}g^{-1}hg \in G' \subseteq C(G)$.

We give an example of a noncommutative additively cancellative reduced centrally essential semiring without zero-divisors.

**Example 4.** Let $Q_8$ be the quaternion group, i.e., the group with two generators $a, b$ and defining relations $a^4 = 1, a^2 = b^2$ and $aba^{-1} = b^{-1}$; e.g., see [4, Section 4.4]. We have

$$Q_8 = \{e, a, a^2, b, ab, a^3, a^2b, a^3b\},$$

the conjugacy classes of $Q_8$ are

$$K_e = \{e\}, K_{a^2} = \{a^2\}, K_a = \{a, a^3\}, K_b = \{b, a^2b\}, K_{ab} = \{ab, a^3b\},$$

and the center $C(Q_8)$ is $\{e, a^2\}$. We consider the group semiring $SQ_8$, where $S = \mathbb{Q}^+ \cup \{0\}$. Since $Q_8$ is a group of nilpotence class 2, it follows from Proposition 4 that $SQ_8$ is a centrally essential group semiring. To illustrate the above, we have

$$a \sum_{C(Q_8)} = \sum_{K_a}, \quad b \sum_{C(Q_8)} = \sum_{K_b}, \quad ab \sum_{C(Q_8)} = \sum_{K_{ab}},$$

$$a^3 \sum_{C(Q_8)} = \sum_{K_a}, \quad a^2b \sum_{C(Q_8)} = \sum_{K_b}, \quad a^3b \sum_{C(Q_8)} = \sum_{K_{ab}}.$$

The group ring of differences $\mathbb{Q}Q_8$ is a reduced ring; see [15, Theorem 3.5]. Then $SQ_8$ is a reduced semiring. Indeed, if $x^2 + y^2 = xy + yx$ and $x \neq y$, then $x^2 + y^2 - xy - yx = (x - y)^2 = 0$ in the ring $\mathbb{Q}Q_8$; this is not true. Thus, $SQ_8$ is a non-commutative reduced centrally essential semiring without zero-divisors. We note that $\mathbb{Q}Q_8$ is not a centrally essential ring, since centrally essential reduced rings are commutative.

3.1. Completion of the Proof of Theorem 1

It follows from Example 4 that there exists a non-commutative additively cancellative reduced centrally essential semiring without zero-divisors.

If a semiring $S$ is commutative, then $D(S)$ is a commutative ring, i.e., $D(S)$ is centrally essential. Conversely, let $D(S)$ be a centrally essential ring. Since $S$ is a reduced semiring, $D(S)$ is a reduced ring. Indeed, let $0 \neq a = x - y \in D(S)$. If $a^2 = 0$, then $x^2 + y^2 = xy + yx$. Therefore, $x = y, a = 0$, and we have a contradiction. Then the ring $D(S)$ is commutative, since $D(S)$ is a reduced centrally essential ring. Consequently, $S$ is a commutative semiring.
4. REMARKS AND OPEN QUESTIONS

An element $x$ of the semiring $S$ is said to be left (resp., right) multiplicatively cancellative if $y = z$ for all $y, z \in S$ with $xy = xz$ (resp., $yx = zx$). A semiring $S$ is said to be left (resp., right) multiplicatively cancellative if every $x \in S \setminus \{0\}$ is left (resp., right) multiplicatively cancellative. A left and right multiplicatively cancellative semiring is said to be multiplicatively cancellative; e.g., see [5, Chapter I].

**Remark 2.** A left (resp., right) multiplicatively cancellative centrally essential semiring $S$ is commutative. Indeed, let $a$ and $b$ be two non-zero elements of the semiring $S$. Since $S$ is a centrally essential semiring, there exists $c \in C(S)$ with $0 \neq ac \in C(S)$. A left multiplicatively cancellative semiring does not contain left zero-divisors; see [5, Chapter I, Theorem 4.4]. Therefore, $acb \neq 0$. Then

$$(ac)b = c(ab) = (ca)b = b(ca) = c(ba),$$

whence we have $ab = ba$. A similar argument is true for right multiplicatively cancellative semirings.

A semiring with division, which is not a ring, is called a *division semiring*. A commutative division semiring is called a *semifield*. It follows from Remark 2 that any centrally essential division semiring is a semifield. Indeed, it follows from [5, Chapter I, Theorem 5.5] that a division semiring with at least two elements is multiplicatively cancellative.

**Question 1.** Are there any non-commutative semisubtractive centrally essential semirings without non-zero nilpotent elements (see Proposition 1)?

**Question 2.** For groups of nilpotence class $n > 2$, are there any non-commutative centrally essential group semirings without zero-divisors?

**FUNDING**

The work of O. V. Lyubimtsev is done under the state assignment no. 0729-2020-0055. A.A. Tuganbaev is supported by Russian Scientific Foundation, project 16-11-10013P.

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