REGULAR FUNCTIONS ON THE $K$-NILPOTENT CONE

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Abstract. Let $G$ be a complex reductive algebraic group with Lie algebra $\mathfrak{g}$ and let $G_\mathbb{R}$ be a real form of $G$ with maximal compact subgroup $K_\mathbb{R}$. Associated to $G_\mathbb{R}$ is a $K \times \mathbb{C}^\times$-invariant subvariety $\mathcal{N}_\theta$ of the (usual) nilpotent cone $\mathcal{N} \subset \mathfrak{g}^\ast$. In this article, we will derive a formula for the ring of regular functions $\mathbb{C}[\mathcal{N}_\theta]$ as a representation of $K \times \mathbb{C}^\times$.

Some motivation comes from Hodge theory. In [SV11], Schmid and Vilonen use ideas from Saito’s theory of mixed Hodge modules to define canonical good filtrations on many Harish-Chandra modules (including all standard and irreducible Harish-Chandra modules). Using these filtrations, they formulate a conjectural description of the unitary dual. If $G_\mathbb{R}$ is split, and $X$ is the spherical principal series representation of infinitesimal character 0, then conjecturally $\text{gr}(X) \simeq \mathbb{C}[\mathcal{N}_\theta]$ as representations of $K \times \mathbb{C}^\times$. So a formula for $\mathbb{C}[\mathcal{N}_\theta]$ is an essential ingredient for computing Hodge filtrations.

1. Introduction

Let $G$ be a complex connected reductive algebraic group and let $G_\mathbb{R}$ be a real form of $G$. Choose a maximal compact subgroup $K_\mathbb{R}$ of $G_\mathbb{R}$ and let $\theta : G \to G$ be the corresponding Cartan involution. Let $K$ be the group of $\theta$-fixed points (i.e. the complexification of $K_\mathbb{R}$). Note that $K$ is a complex reductive algebraic group (it is often disconnected). Write $\mathfrak{g}, \mathfrak{k}$ for the Lie algebras and let $\mathfrak{p} = \mathfrak{g}^{\text{adj}}$. There is a Cartan decomposition

\[(1.0.1) \quad \mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}\]

Using (1.0.1), we can identify $\mathfrak{p}^\ast$ with $(\mathfrak{g}/\mathfrak{k})^\ast$, a subspace of $\mathfrak{g}^\ast$.

Let $G \times \mathbb{C}^\times$ act on $\mathfrak{g}^\ast$ by the usual formula

\[(g, z) \cdot \zeta = z \text{Ad}^\ast(g)\zeta, \quad g \in \hat{G}, \; z \in \mathbb{C}^\times, \; \zeta \in \mathfrak{g}^\ast.\]

Note that $\mathfrak{p}^\ast$ is stable under $K \times \mathbb{C}^\times \subset G \times \mathbb{C}^\times$. Let $\mathcal{N}$ denote the nilpotent cone in $\mathfrak{g}^\ast$. Recall that $G \times \mathbb{C}^\times$ acts on $\mathcal{N}$ (with finitely many orbits). The $K$-nilpotent cone is the $K \times \mathbb{C}^\times$-invariant subvariety

\[\mathcal{N}_\theta = \mathcal{N} \cap \mathfrak{p}^\ast \subset \mathfrak{p}^\ast.\]

This subvariety (and the $K \times \mathbb{C}^\times$-action on it) is closely related to the representation theory of $G_\mathbb{R}$, see [Vog91]. The main result of this paper is an explicit description of the ring of regular functions $\mathbb{C}[\mathcal{N}_\theta]$ as a representation of $K \times \mathbb{C}^\times$ (in the case when $G_\mathbb{R}$ is split modulo center). Some motivation for this problem will be discussed at the end of Section 2.

Since $K$ is (in general) a disconnected group, its irreducible representations cannot be easily parameterized using the theory of highest weights. In [Vog07], Vogan gives a parameterization of a very different flavor (making essential use of the fact that $K$ is a symmetric subgroup of $G$). We will recall some of the details in Section 2.
There are several well-known results regarding the structure of \( \mathbb{C}[\mathcal{N}_\theta] \) as a \( K \)-representation. In [KR71], it is shown that for \( G_\mathbb{R} \) quasi-split, \( \mathbb{C}[\mathcal{N}_\theta] \) is isomorphic as a \( K \)-representation to the induced representation \( \text{Ind}_M^K \) \( \text{triv} \), where \( M \) is central in \( K \) of a maximal abelian subspace \( a \subset p \). Equivalently, \( \mathbb{C}[\mathcal{N}_\theta] \) is isomorphic as a \( K \)-representation to a spherical principal series representation of \( G_\mathbb{R} \). Importantly, these results do not provide any information about the \( \mathbb{C}^* \)-action on \( \mathbb{C}[\mathcal{N}_\theta] \). To understand this structure, we must adopt a different approach. First, we relate the ring of regular functions \( \mathbb{C}[\mathcal{N}_\theta] \) (regarded as a representation of \( K \times \mathbb{C}^* \)) to the ring of regular functions \( \mathbb{C}[\mathcal{N}] \) (regarded as a representation of \( G \times \mathbb{C}^* \)). For \( G_\mathbb{R} \) split modulo center, we prove the following formula in Corollary 6.0.2

\[
\mathbb{C}[\mathcal{N}_\theta]|_{K \times \mathbb{C}^*} = \mathbb{C}[\mathcal{N}]|_{K \times \mathbb{C}^*} \otimes \langle \wedge(\mathfrak{t}) \rangle
\]

Here \( \langle \wedge(\mathfrak{t}) \rangle \) denotes the signed graded exterior algebra associated to \( \mathfrak{t} \) (this formula takes place in the Grothendieck group of admissible representations of \( K \times \mathbb{C}^* \)). The proof of this result is not purely formal—we make essential use of the fact that \( \mathcal{N} \) is Cohen-Macaulay and that \( \mathcal{N}_\theta \subset \mathcal{N} \) is a complete intersection of codimension \( \dim(\mathfrak{t}) \) (for the latter assertion, we use that \( G_\mathbb{R} \) is split modulo center). The structure of \( \mathbb{C}[\mathcal{N}] \) as a \( G \times \mathbb{C}^* \)-representation is well-known (it can be computed using Lusztig’s \( q \)-analog of Kostant’s partition function, see [McG89]). We then ‘restrict’ this description to \( K \times \mathbb{C}^* \) to obtain a formula for \( \mathbb{C}[\mathcal{N}_\theta] \).

The final result is Theorem 7.0.2

Along the way, we introduce a restriction map on equivariant K-theory

\[
K^G(\mathcal{N}) \to K^K(\mathcal{N}_\theta)
\]

(this map is defined and studied in Section 3). In fact, this map arises as the ‘associated graded’ of a map from representations of \( G \) (regarded as a real group) to representations of \( G_\mathbb{R} \) (see Remark 3.0.8 for more details). We will pursue this point in future work.

1.1. Notation. Let \( R \) be an algebraic group. An algebraic \( R \)-representation \( V \) is admissible if every irreducible \( R \)-representation appears in \( V \) with finite multiplicity. Consider the abelian categories

\[
\begin{align*}
\text{Rep}(R) &= \text{algebraic representations of } R \\
\text{Rep}_a(R) &= \text{admissible algebraic representations of } R \\
\text{Rep}_f(R) &= \text{finite-dimensional algebraic representations of } R
\end{align*}
\]

There are obvious embeddings \( \text{Rep}_f(R) \subset \text{Rep}_a(R) \subset \text{Rep}(R) \).

Now let \( \tilde{R} = R \times \mathbb{C}^* \). Then \( \text{Rep}(\tilde{R}), \text{Rep}_a(\tilde{R}), \text{and} \text{Rep}_f(\tilde{R}) \) can be defined as above. We will also consider the category

\[
\text{Rep}_{aa}(\tilde{R}) = \text{algebraic representations of } \tilde{R} \text{ which are admissible as representations of both } R \text{ and } \mathbb{C}^*
\]

There are obvious embeddings \( \text{Rep}_f(\tilde{R}) \subset \text{Rep}_{aa}(\tilde{R}) \subset \text{Rep}_a(\tilde{R}) \subset \text{Rep}(\tilde{R}) \). We will denote the Grothendieck groups by \( K(R), K_a(R), K_f(R), K_{aa}(\tilde{R}) \), and so on. Write \( \text{Irr}(R) \) for the set of (equivalence classes of) irreducible representations of \( R \). Then

\[
\text{Irr}(\tilde{R}) = \{ \tau q^n \mid \tau \in \text{Irr}(R), n \in \mathbb{Z} \},
\]
where \( q^n \) denotes the degree-\( n \) character of \( \mathbb{C}^x \) and \( \tau q^n \) is shorthand for the irreducible \( \hat{R} \)-representation \( \tau \otimes q^n \). The group \( K_f(R) \) (resp. \( K_f(\hat{R}) \)) can be identified with (finite) integer combinations of \( \tau \in \text{Irr}(R) \) (resp. \( \tau q^n \in \text{Irr}(\hat{R}) \)). The group \( K_a(R) \) (resp. \( K_a(\hat{R}) \)) can be identified with formal integer combinations of \( \tau \in \text{Irr}(R) \) (resp. \( \tau q^n \in \text{Irr}(\hat{R}) \)). Finally, \( K_{aa}(\hat{R}) \) can be identified with formal integer combinations of \( \tau \) such that for each \( \tau \in \text{Irr}(\hat{R}) \) and for each \( n \in \mathbb{Z} \) only finitely many \( \tau q^n \) appear with nonzero multiplicity.

The tensor product of representations turns \( K_f(R) \) into a commutative ring, \( K_a(R) \) into a \( K_f(R) \)-module, and \( K_{aa}(\hat{R}) \) into a \( K_a(\hat{R}) \)-module.

If \( X \) is a scheme equipped with an algebraic \( R \)-action, we write \( \text{Coh}^R(X) \) for the category of (strongly) \( R \)-equivariant coherent sheaves on \( X \) and \( K^R(X) \) for its Grothendieck group.

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### 2. Irreducible Representations of \( K \)

In this section, we will recall a parameterization of \( \text{Irr}(K) \) due to Vogan ([Vog07]). Suppose \( H \) is a \( \theta \)-stable maximal torus in \( G \). Write \( \Delta = \Delta(G, H) \) for the roots of \( H \) on \( \mathfrak{g} \). Since \( H \) is \( \theta \)-stable, \( \theta \) acts on \( \Delta \) by \( \alpha \mapsto \alpha \circ \theta \). A root \( \alpha \in \Delta \) is imaginary (resp. real, resp. complex) if \( \theta \alpha = \alpha \) (resp. \( \theta \alpha = -\alpha \), resp. \( \theta \alpha \notin \{ \pm \alpha \} \)). If \( \alpha \) is imaginary, then \( \theta \) acts on the root space \( \mathfrak{g}_\theta \) by \( \pm \text{Id} \). We say that \( \alpha \) is compact (resp. noncompact) if \( \theta|_{\mathfrak{h}_\alpha} = \text{Id} \) (resp. \( \theta|_{\mathfrak{h}_\alpha} = -\text{Id} \)). So \( \Delta \) is partitioned

\[
\Delta = \Delta_{iR} \sqcup \Delta_{R} \sqcup \Delta_{C}
\]

into imaginary, real, and complex roots, and \( \Delta_{iR} \) is partitioned

\[
\Delta_{iR} = \Delta_{c} \sqcup \Delta_{n}
\]

into compact and noncompact roots. Note that \( \Delta_{iR} \) is a root system. If we choose a positive system \( \Phi^+ \subset \Delta_{iR} \), we can define the element \( \rho_{iR} = \frac{1}{2} \sum \Phi^+ \in \mathfrak{h}^* \).

**Definition 2.0.1** ([Sec 6, AVTV20]). A continued Langlands parameter for \( (G, K) \) is a triple \( (H, \gamma, \Phi^+) \) where

1. \( H \) is a \( \theta \)-stable maximal torus in \( G \).
2. \( \gamma \) is a formal sum \( \gamma_0 + \rho_{iR} \), where \( \gamma_0 \) is a one-dimensional \( (\mathfrak{h}, \mathfrak{h}^\theta) \)-module (we define \( d\gamma = d\gamma_0 + \rho_{iR} \)).
3. \( \Phi^+ \) is a positive system for \( \Delta_{iR} \).

There is a \( K \)-action on the set of continued Langlands parameters. Two parameters are equivalent if they are conjugate under \( K \). A continued Langlands parameter is standard if

4. For every \( \alpha \in \Phi^+ \), \( \langle d\gamma, \alpha\rangle \geq 0 \).

A standard Langlands parameter is nonzero if

5. For every \( \alpha \in \Phi^+ \) which is simple and compact, \( \langle d\gamma, \alpha\rangle \neq 0 \).
A nonzero Langlands parameter is final if

\[(6) \text{ If } \alpha \in \Delta_\mathbb{R} \text{ and } \langle d\gamma, \alpha^\vee \rangle = 0, \text{ then } \alpha \text{ does not satisfy the Speh-Vogan parity condition } (SV80).\]

Write \(\mathcal{P}_L(G, K)\) for the set of equivalence classes of final Langlands parameters.

To the equivalence class of a continued Langlands parameter \(\Gamma\), one can associate a virtual Harish-Chandra module \(r_I^p\Gamma^q\). If \(\Gamma\) is standard, then \(r_I^p\Gamma^q\) is represented by a distinguished \(\mathfrak{p}\)-module \(I^p\Gamma^q\). If \(\Gamma\) is nonzero, then \(I^p\Gamma^q \neq 0\). If \(\Gamma\) is final, there is a unique irreducible quotient \(I^p\Gamma^q \to J^p\Gamma^q\).

The following is a version of the Langlands classification.

**Theorem 2.0.2** (Thm 6.1, [AVTV20]). The map \(\Gamma \mapsto J(\Gamma)\) defines a bijection between \(\mathcal{P}_L(G, K)\) and isomorphism classes of irreducible \((\mathfrak{g}, K)\)-modules.

For any continued Langlands parameter \(\Gamma\), the infinitesimal character of the virtual \((\mathfrak{g}, K)\)-module \([I(\Gamma)]\) corresponds, under the Harish-Chandra isomorphism, to the \(W\)-orbit of \(d\gamma\). If \(\Gamma\) is final, then \(I(\Gamma)\) is tempered, i.e. \(I(\Gamma) = J(\Gamma)\).

**Definition 2.0.3.** A final Langlands parameter \((H, \gamma, \Phi^+\)\) is tempered with real infinitesimal character if \(d\gamma|_{\mathfrak{h}-\mathfrak{d}\theta}\) is imaginary. In this case, \(I(\Gamma)\) is irreducible, i.e. \(I(\Gamma) \neq J(\Gamma)\).

**Corollary 2.0.4.** The map \(\Gamma \mapsto J(\Gamma)\) defines a bijection between \(\mathcal{P}_L^{t, \mathbb{R}}(G, K)\) and isomorphism classes of irreducible tempered \((\mathfrak{g}, K)\)-modules with real infinitesimal character.

Suppose \(\mu\) is an irreducible representation of \(K\). Choose a maximal torus \(T \subset K\) and a positive system \(\Phi^+_T\) for \(\Delta(K, T)\). Let \(\rho_c = \frac{1}{2} \sum \Phi^+_c \in \mathfrak{t}^*\). Let \(\lambda \in \mathfrak{t}^*\) be a highest weight of \(\mu\) (if \(K\) is disconnected, \(\lambda\) need not be unique). The \(K\)-norm of \(\mu\) is defined by the formula

\[|\mu|^2 := \langle \lambda + 2\rho_c, \lambda + 2\rho_c \rangle\]

It is not hard to see that \(|\mu|\) is independent of \(\Phi^+_T\) and \(\lambda\).

We say that \(\mu\) is a lowest \(K\)-type in a a \((\mathfrak{g}, K)\)-module \(X\) if \(\mu\) appears in \(X\) with nonzero multiplicity and \(|\mu|\) is minimal among all irreducible representations of \(K\) with this property.

**Theorem 2.0.5** (Thm 11.9, [Vog07]). The following are true:

(i) If \(\Gamma \in \mathcal{P}_L^{t, \mathbb{R}}(G, K)\), then \(I(\Gamma)\) contains a unique lowest \(K\)-type \(\mu(\Gamma)\).

(ii) The map \(\Gamma \mapsto \mu(\Gamma)\) defines a bijection between \(\mathcal{P}_L^{t, \mathbb{R}}(G, K)\) and \(\text{Irr}(K)\).

(iii) There is a total order on \(\mathcal{P}_L^{t, \mathbb{R}}(G, K)\) such that the \((\text{infinite})\) square matrix \(m(\Gamma, \Gamma')\) defined by the formula

\[|I(\Gamma)| = \sum_{\Gamma' \in \mathcal{P}_L^{t, \mathbb{R}}} m(\Gamma, \Gamma') \mu(\Gamma')\]

is upper triangular with 1’s along the diagonal.
(iv) In particular, this matrix $m(\Gamma, \Gamma')$ is invertible. Write $M(\Gamma, \Gamma')$ for its inverse (which is also upper triangular).

(v) The entries of the matrices $m(\Gamma, \Gamma')$ and $M(\Gamma, \Gamma')$ can be computed by an algorithm.

The algorithm in (v) is described in [Vog07].

Now suppose that $G_R$ is split modulo center. This means that there is a maximal torus $H_s$ in $G$ such that $\Delta(G, H_s) = \Delta_R(G, H_s)$ (and so $\Delta_R(G, H_s) = \emptyset$). Let $\Gamma_0$ be the parameter

$$\Gamma_0 := (H_s, 0, \emptyset) \in \mathcal{P}_L^m(G, K)$$

By a result of Kostant ([Kos69]) there is an identity in $K_a(K)$

$$\mathbb{C}[\mathcal{N}_\theta]|_K = I(\Gamma_0)|_K$$

So by Theorem 2.0.5 we can write $\mathbb{C}[\mathcal{N}_\theta]$ as a formal integer sum of the irreducible $K$-representations $\mu(\Gamma)$ (this idea has been implemented in the atlas software). This is quite useful information, but it does not give us the grading on $\mathbb{C}[\mathcal{N}_\theta]$, which is part of what we’re after.

In [SV11], Schmid and Vilonen define canonical good filtrations on all Harish-Chandra modules of ‘functorial origin’, including all standard and irreducible Harish-Chandra modules. These canonical good filtrations (and their associated gradeds) should be closely related to questions of unitarity.

The associated graded of a Harish-Chandra module with respect to a good filtration can be regarded as a representation of $\tilde{K}$ (in fact, as a class in $K_a(\tilde{K})$). It is conjectured in [SV11] that

$$\mathbb{C}[\mathcal{N}_\theta]|_{\tilde{K}} = \text{gr } I(\Gamma_0)|_{\tilde{K}}$$

It is also suggested that the Hodge filtration on an arbitrary standard module (of an arbitrary group) can be reduced to this case (via cohomological induction and a deformation argument). So computing the class $\mathbb{C}[\mathcal{N}_\theta]|_{\tilde{K}}$ in the case when $G_R$ is split is central to the program of computing Hodge filtrations. In Theorem 7.0.2 we will give a formula for $\mathbb{C}[\mathcal{N}_\theta]|_{\tilde{K}}$ in terms of the classes $I(\Gamma)q^n$.

3. A restriction map $K^G(\mathcal{N}) \to K^\tilde{K}(\mathcal{N}_\theta)$

In this section, we will define a restriction map

$$K^\tilde{G}(\mathcal{N}) \to K^\tilde{K}(\mathcal{N}_\theta)$$

Since $\mathcal{N}$ and $\mathcal{N}_\theta$ are singular, we cannot proceed directly. Instead, we follow the standard approach outlined (for example) in [CG10, Sec 5.3]: we first regard $\mathcal{N}$ (resp. $\mathcal{N}_\theta$) as a subvariety of $\mathfrak{g}^*$ (resp. $\mathfrak{p}^*$) and then apply the restriction map $K^G(\mathfrak{g}^*) \to K^\tilde{K}(\mathfrak{p}^*)$ (defined in the usual way, as an alternating sum of Tor functors).

Our first proposition describes the relationship between $K^\tilde{G}(\mathcal{N})$, $K^\tilde{G}(\mathfrak{g}^*)$, $K_{aa}(\tilde{G})$, and $K_f(\tilde{G})$.

**Proposition 3.0.1.** The following are true:
(i) If $\mathcal{E} \in \text{Coh}^\wedge(G(N))$, then $\Gamma(N, \mathcal{E})|_{\tilde{G}} \in \text{Rep}_{aa}(\tilde{G})$. This defines an exact functor $\text{Coh}^\wedge(G(N)) \to \text{Rep}_{aa}(\tilde{G})$, and hence a group homomorphism

$$\Gamma(\bullet)|_{\tilde{G}} : K^\wedge(G(N)) \to K_{aa}(\tilde{G})$$

(ii) The homomorphism in (i) is injective.

(iii) If $\mathcal{E} \in \text{Coh}^\wedge(G(\mathfrak{g}^*))$, then $\Gamma(\mathfrak{g}^*, \mathcal{E})|_{\tilde{G}} \in \text{Rep}_{a}(\tilde{G})$. This defines an exact functor $\text{Coh}^\wedge(G(\mathfrak{g}^*)) \to \text{Rep}_{a}(\tilde{G})$, and hence a group homomorphism

$$\Gamma(\bullet)|_{\tilde{G}} : K^\wedge(G(\mathfrak{g}^*)) \to K_{a}(\tilde{G})$$

(iv) Restriction along $\{0\} \subset \mathfrak{g}^*$ induces an exact functor $\text{Coh}^\wedge(G(\mathfrak{g}^*)) \to \text{Coh}^\wedge(\{0\}) \simeq \text{Rep}_{f}(\tilde{G})$, which in turn induces a group isomorphism

$$|_{\{0\}} : K^\wedge(G(\mathfrak{g}^*)) \overset{\sim}{\longrightarrow} K_{f}(\tilde{G})$$

(v) The direct image along the closed embedding $j : N \hookrightarrow \mathfrak{g}^*$ induces an exact functor $\text{Coh}^\wedge(G(N)) \to \text{Coh}^\wedge(G(\mathfrak{g}^*))$, and therefore a group homomorphism

$$j_* : K^\wedge(G(N)) \to K^\wedge(G(\mathfrak{g}^*))$$

(vi) If $V \in \text{Rep}_{f}(\tilde{G})$, then $V \otimes \mathbb{C}[\mathfrak{g}^*] \in \text{Rep}_{a}(\tilde{G})$. This defines an exact functor $\text{Rep}_{f}(\tilde{G}) \to \text{Rep}_{a}(\tilde{G})$, and hence a group homomorphism

$$\phi_{\mathfrak{g}^*} : K_{f}(\tilde{G}) \to K_{a}(\tilde{G})$$

(vii) The following diagram commutes:

$$\begin{array}{ccc}
K_{a}(\tilde{G}) & \leftarrow & K_{f}(\tilde{G}) \\
\uparrow & & \uparrow \\
K_{aa}(\tilde{G}) & \leftarrow & K_{a}(\tilde{G}) \\
\uparrow \downarrow & & \uparrow \downarrow \\
\Gamma(\bullet)|_{\tilde{G}} & \leftarrow & \Gamma(\bullet)|_{\tilde{G}} \\
\downarrow & & \downarrow \\
K^\wedge(G(N)) & \longrightarrow & K^\wedge(G(\mathfrak{g}^*)) \\
\end{array}$$

(viii) The homomorphism in (v) is injective.

Proof.

(i) This is [AV19, (5.4f)] (it is an immediate consequence of the following facts: $\tilde{G}$ is reductive, $N$ is affine, and $\tilde{G}$ acts on $N$ with finitely many orbits).
(ii) This is [AV19, Cor 7.4].

(iii) Since \( g^* \) is affine, the functor \( \Gamma(\bullet)|_{\hat{G}} : \text{Coh}^{\hat{G}}(g^*) \to \text{Rep}(\hat{G}) \) is exact. It suffices to show that its image is contained in \( \text{Rep}_{a}(\hat{G}) \). Let \( E \in \text{Coh}^{\hat{G}}(g^*) \). Since \( g^* \) is smooth, there is a \( \hat{G} \)-equivariant vector bundle \( V \) on \( g^* \) and a surjection \( V \to E \) in \( \text{Coh}^{\hat{G}}(g^*) \).

Since \( \Gamma(\bullet)|_{\hat{G}} \) is exact, we get a surjection \( \Gamma(g^*, V)|_{\hat{G}} \to \Gamma(g^*, E)|_{\hat{G}} \) in \( \text{Rep}(\hat{G}) \). So it suffices to show that \( \Gamma(g^*, V)|_{\hat{G}} \in \text{Rep}_{a}(\hat{G}) \). Let \( V = V|_{(0)} \). Then \( V \in \text{Rep}_f(\hat{G}) \) and

\[
V \simeq V \otimes \mathcal{O}_{g^*}
\]

in \( \text{Coh}^{\hat{G}}(g^*) \). So \( \Gamma(g^*, V)|_{\hat{G}} = V \otimes \mathbb{C}[g^*]|_{\hat{G}} \). Note that each graded component of \( \mathbb{C}[g^*] \), and hence of \( V \otimes \mathbb{C}[g^*] \), is finite-dimensional. So \( V \otimes \mathbb{C}[g^*]|_{\hat{G}} \in \text{Rep}_{a}(\hat{G}) \), as required.

(iv) This is a well-known fact from equivariant \( K \)-theory, see [Tho87, Thm 4.1].

(v) This follows from the fact that \( j \) is closed, and hence affine.

(vi) See the proof of (iii).

(vii) It suffices to show that the top triangle is commutative (the commutativity of the bottom triangle is obvious). If \( V \simeq V \otimes \mathcal{O}_{g^*} \in \text{Coh}^{\hat{G}}(g^*) \) is a vector bundle, then

\[
\Gamma(g^*, V)|_{\hat{G}} \simeq V \otimes \mathbb{C}[g^*]|_{\hat{G}} \simeq V|_{(0)} \otimes \mathbb{C}[g^*]|_{\hat{G}}
\]

But since \( g^* \) is smooth, \( K^{\hat{G}}(g^*) \) is spanned by vector bundles. So the upper triangle is commutative.

(viii) By (vii), the map

\[
K^{\hat{G}}(\mathcal{N}) \to K_{aa}(\hat{G}) \subset K_a(\hat{G})
\]

coincides with the composition

\[
K^{\hat{G}}(\mathcal{N}) \to K^{\hat{G}}(g^*) \to K_a(\hat{G}).
\]

By (ii), \((3.0.1)\) is injective. Hence, \( j^* : K^{\hat{G}}(\mathcal{N}) \to K^{\hat{G}}(g^*) \) must be injective as well.

\[\square\]

**Remark 3.0.2.** We note that (ii) of Proposition [3.0.1] (and hence (viii), which is a consequence) is a very deep assertion—the proof of (ii) in [AV19] makes essential use of the Langlands classification.

**Remark 3.0.3.** It is worth considering what happens if we forget about the \( \mathbb{C}^* \)-actions. Arguing exactly as in Proposition [3.0.1], we get a commutative diagram
We can compute these restriction maps in terms of a graded Koszul identity in $K$. However, $K(G) = 0$ (indeed, every algebraic $G$-representation $V$ satisfies $V \otimes V^\infty \cong V^\infty$, and therefore has image 0 in $K(G)$). So we cannot deduce that $j_\ast : K^G(N) \to K^G(g^*)$ is injective (and in fact, it is not: the skyscraper sheaf at $\{0\}$ with trivial $G$-action lies in the kernel of the map $j_\ast : K^G(N) \to K^G(g^*)$. So, the $\mathbb{C}^* \cong \mathbb{C}$-actions are essential for the proposition above.

If we replace $\tilde{\mathcal{G}}$ with $\tilde{K}$, $g^*$ with $p^*$, and so on, we can prove a result which is completely analogous to Proposition 3.0.1 (the only change in the proof is that in (iii) we use [AV19, Cor 10.9] instead of [AV19, Cor 7.4]).

Let $i : p^* \hookrightarrow g^*$ be the inclusion. The restriction functor $i^* : \text{Coh}^\tilde{\mathcal{G}}(g^*) \to \text{Coh}^\tilde{\mathcal{K}}(p^*)$ is not exact in general (if $E \in \text{Coh}^\tilde{\mathcal{G}}(g^*)$, then $i^*E$ corresponds to the $\mathbb{C}[p^*]$-module $\mathbb{C}[p^*] \otimes_{\mathbb{C}[g^*]} E$. The functor $\mathbb{C}[p^*] \otimes_{\mathbb{C}[g^*]} \bullet$ is only right exact). Write $L_n i^*$ for its higher derived functors (if $E \in \text{Coh}^\tilde{\mathcal{G}}(g^*)$, then $L_n i^*E$ corresponds to the $\mathbb{C}[p^*]$-module $\text{Tor}^\mathbb{C}[g^*]_n(\mathbb{C}[p^*], E)$). Since $g^*$ is smooth, $L_n i^*E = 0$ for $n$ very large (see e.g. [CG10, Prop 5.1.28]). So we can define a homomorphism

$$i^* : K^\tilde{\mathcal{G}}(g^*) \to K^\tilde{\mathcal{K}}(p^*), \quad i^*[E] = \sum_{n=0}^{\infty} (-1)^n [L_n i^*E].$$

If $[E]$ is supported in $N$, then $i^*[E]$ is supported in $N_\theta$. So $i^* : K^\tilde{\mathcal{G}}(g^*) \to K^\tilde{\mathcal{K}}(p^*)$ restricts to a (unique) homomorphism $i^* : K^\tilde{\mathcal{G}}(N) \to K^\tilde{\mathcal{K}}(N_\theta)$

$$K^\tilde{\mathcal{G}}(N) \xrightarrow{j^*} K^\tilde{\mathcal{G}}(g^*) \xrightarrow{i^*} K^\tilde{\mathcal{K}}(N_\theta) \xrightarrow{j^*} K^\tilde{\mathcal{K}}(p^*)$$

(3.0.2)

We can compute these restriction maps in terms of $\tilde{K}$-representations. The key ingredient is a graded Koszul identity in $K_a(\tilde{K})$. Consider the class

$$[\land(t)] := \sum_{n=0}^{\infty} (-1)^n [\land^n(t)] \in K_f(\tilde{K})$$

Here, as usual, we put $t$ in degree 1

**Lemma 3.0.4.** There is an identity in $K_a(\tilde{K})$

$$[\mathbb{C}[t^*]|_{\tilde{K}} \otimes [\land(t)] = \text{triv}$$
Proof. Consider the Koszul resolution of the trivial \( \mathbb{C}[\mathfrak{t}^*] \)-module

\[
0 \rightarrow \mathbb{C}[\mathfrak{t}^*] \otimes \wedge^{\dim(t)}(\mathfrak{t}) \rightarrow \cdots \rightarrow \mathbb{C}[\mathfrak{t}^*] \otimes \wedge^1(\mathfrak{t}) \rightarrow \mathbb{C}[\mathfrak{t}^*] \otimes \wedge^0(\mathfrak{t}) \rightarrow \mathbb{C} \rightarrow 0
\]

We can regard each term as a representation of \( K \), and it is easy to check that the differentials are \( K \)-equivariant. Restricting to \( K \) we get an exact sequence in \( \text{Rep}_{a}K \)

\[
0 \rightarrow \mathbb{C}[\mathfrak{t}^*]|_{K} \otimes \wedge^{\dim(t)}(K) \rightarrow \cdots \rightarrow \mathbb{C}[\mathfrak{t}^*]|_{K} \otimes \wedge^1(K) \rightarrow \mathbb{C}[\mathfrak{t}^*]|_{K} \otimes \wedge^0(K) \rightarrow \text{triv} \rightarrow 0
\]

Now the required identity follows from the Euler-Poincare principle

\[
\mathbb{C}[\mathfrak{t}^*]|_{K} \otimes [\wedge(t)] = \sum_{n} (-1)^{n} \mathbb{C}[\mathfrak{t}^*]|_{K} \otimes [\wedge^n(t)] = \text{triv}
\]

If \( V \in \text{Rep}_{a}(\tilde{G}) \), then \( V|_{K} \in \text{Rep}_{a}(\tilde{K}) \). This defines an exact functor \( \text{Rep}_{a}(\tilde{G}) \rightarrow \text{Rep}_{a}(\tilde{K}) \), and hence a group homomorphism

\[
|_{K} : K_{a}(\tilde{G}) \rightarrow K_{a}(\tilde{K})
\]

Tensoring with the class \( [\wedge(t)] \in K_{f}(\tilde{K}) \), we obtain a further homomorphism

\[
r : K_{a}(\tilde{G}) \rightarrow K_{a}(\tilde{K}), \quad r[V] = [V]|_{K} \otimes [\wedge(t)]
\]

Lemma 3.0.5. The following diagram is commutative

\[
\begin{array}{ccc}
K_{a}(\tilde{G}) & \xleftarrow{\phi_{\mathfrak{g}^*}} & K_{f}(\tilde{G}) \\
|_{r} & & \downarrow|_{r} \\
K_{a}(\tilde{K}) & \xleftarrow{\phi_{\mathfrak{p}^*}} & K_{f}(\tilde{K})
\end{array}
\]

Proof. Let \( V \in K_{f}(\tilde{G}) \). Then by Lemma 3.0.4 we have

\[
\phi_{\mathfrak{g}^*}(V)|_{K} \otimes [\wedge(t)] = (V \otimes \mathbb{C}[\mathfrak{g}^*])|_{K} \otimes [\wedge(t)]
\]

\[
= V|_{K} \otimes \mathbb{C}[\mathfrak{p}^*]|_{K} \otimes \mathbb{C}[\mathfrak{t}^*]|_{K} \otimes [\wedge(t)]
\]

\[
= V|_{K} \otimes \mathbb{C}[\mathfrak{p}^*]|_{K}
\]

\[
= \phi_{\mathfrak{p}^*}(V|_{K})
\]

as desired. \( \square \)

Lemma 3.0.6. The following diagram is commutative

\[
\begin{array}{ccc}
K^\tilde{G}(\mathfrak{g}^*) & \xrightarrow{\phi_{\mathfrak{g}^*}} & K_{f}(\tilde{G}) \\
|_{r^*} & & \downarrow|_{r} \\
K^\tilde{K}(\mathfrak{p}^*) & \xrightarrow{\phi_{\mathfrak{p}^*}} & K_{f}(\tilde{K})
\end{array}
\]
Proof. If $V \simeq V \otimes O_{g^*} \in \text{Coh} \tilde{G}(g^*)$ is a vector bundle, then $L_n i^* V = 0$ for $n > 0$ ($V$ corresponds to a flat, and hence projective, $\mathbb{C}[g^*]$-module, so all higher Tor groups vanish). Consequently

\[
(i^*[V])|_{\{0\}} = [i^*V]|_{\{0\}} \\
= (V|_{\tilde{K}} \otimes O_{g^*})|_{\{0\}} \\
= V|_{\tilde{K}} \\
= ([V]|_{\{0\}})|_{\tilde{K}}
\]

So the diagram commutes on vector bundles. But since $g^*$ is smooth, $K\tilde{G}(g^*)$ is spanned by vector bundles. This completes the proof. $\square$

The next result gives a method for computing $i^*: K\tilde{G}(\mathcal{N}) \to K\tilde{K}(\mathcal{N}_\theta)$ on the level of $\tilde{K}$-representations.

**Corollary 3.0.7.** The following diagram is commutative

\[
\begin{array}{cccc}
K_\alpha(\tilde{G}) & \Phi_{g^*} & K_f(\tilde{G}) \\
\downarrow r & \downarrow & \downarrow \\
K\tilde{G}(\mathcal{N}) & \Phi_{p^*} & K\tilde{K}(\mathcal{N}_\theta) \\
\downarrow i^* & \downarrow & \downarrow j^* \\
K_\alpha(\tilde{K}) & \Phi_{p^*} & K\tilde{K}(p^*) \\
\downarrow r & \downarrow & \downarrow \\
K\tilde{G}(g^*) & K\tilde{K}(p^*) & K\tilde{K}(\mathcal{N}_\theta) \\
\end{array}
\]

In particular, for every $[\mathcal{E}] \in K\tilde{G}(\mathcal{N})$, the restriction $i^*[\mathcal{E}]$ is uniquely determined by the following identity in $K_\alpha(\tilde{K})$

\[(3.0.4) \quad \Gamma(i^*[\mathcal{E}])|_{\tilde{K}} = \Gamma([\mathcal{E}])|_{\tilde{K}} \otimes [\wedge(\mathfrak{p})]
\]

Proof. The front face is commutative by the definition of $i^*: K\tilde{G}(\mathcal{N}) \to K\tilde{K}(\mathcal{N}_\theta)$, see [3.0.2]. The right face is commutative by Lemma [3.0.6]. The back face is commutative by Lemma [3.0.5]. The top face is commutative by Proposition [3.0.1]. The bottom face is commutative by its analog for $\tilde{K}$. The commutativity of the left face follows as a formal consequence of the commutativity of the others. $\square$

**Remark 3.0.8.** There are surjective ‘forgetful’ maps

\[K\tilde{G}(\mathcal{N}) \to K^G(\mathcal{N}), \quad K\tilde{K}(\mathcal{N}_\theta) \to K^K(\mathcal{N}_\theta)\]

It is not hard to show that the restriction map $i^*: K\tilde{G}(\mathcal{N}) \to K\tilde{K}(\mathcal{N}_\theta)$ descends to a (necessarily unique) homomorphism

\[i^*: K^G(\mathcal{N}) \to K^K(\mathcal{N}_\theta)\]
Since we will not use this fact, we will not prove it here. Write $K \mathcal{M}(G_\mathbb{R})$ for the Grothendieck group of finite-length admissible $G_\mathbb{R}$-representations. Similarly, write $K \mathcal{M}(G)$ (here $G$ is regarded as a real reductive group by restriction of scalars). There are ‘associated graded’ maps

$$\text{gr} : K \mathcal{M}(G) \to K^G(\mathcal{N}), \quad \text{gr} : K \mathcal{M}(G_\mathbb{R}) \to K^K(\mathcal{N}_0)$$

(see [Vog91] for definitions). An intriguing question, which we will not pursue in this paper, is whether there is a natural homomorphism $K \mathcal{M}(G) \to K \mathcal{M}(G_\mathbb{R})$ such that the following diagram commutes

$$
\begin{array}{ccc}
K \mathcal{M}(G) & \xrightarrow{?} & K \mathcal{M}(G_\mathbb{R}) \\
\downarrow \text{gr} & & \downarrow \text{gr} \\
K^G(\mathcal{N}) & \xrightarrow{\iota^*} & K^K(\mathcal{N}_0)
\end{array}
$$

4. Regular functions on $\mathcal{N}$

In this section, we will recall a (well-known) formula for $C_r^s$ as a representation of $G$. Choose a maximal torus $H \subset G$ and a Borel subgroup $B \subset G$ containing $H$. Let $\Lambda \subset \mathfrak{h}^*$ denote the weight lattice, $\Phi^+ \subset \Lambda$ the positive roots, and $\Lambda^+ \subset \Lambda$ the dominant weights. Then

$$\text{Irr}(G) = \{\tau_\lambda \mid \lambda \in \Lambda^+\},$$

where $\tau_\lambda$ is the irreducible representation of $G$ with highest weight $\lambda$. Recall Kostant’s partition function

$$\mathcal{P} : \Lambda \to \mathbb{Z}, \quad \mathcal{P}(\lambda) = \#\{m : \Phi^+ \to \mathbb{Z} \mid \lambda = \sum_{\alpha \in \Phi^+} m(\alpha)\alpha\}$$

Define

$$\mathcal{M} : \Lambda^+ \times \Lambda \to \mathbb{Z}, \quad \mathcal{M}(\lambda, \mu) = \sum_{w \in W} (-1)^{\ell(w)}\mathcal{P}(w(\lambda + \rho) - (\mu + \rho))$$

where $\ell : W \to \mathbb{Z}_{\geq 0}$ is the length function and $\rho = \frac{1}{2} \sum \Phi^+$. For each $\lambda \in \Lambda^+$, there is an identity in $K_a(H)$

$$\tau_\lambda|_H = \sum_{\mu \in \Lambda} \mathcal{M}(\lambda, \mu)e^\mu$$

(4.0.1)

This is a version of the Weyl character formula.

Lusztig has introduced $q$-analogs of both $\mathcal{P}$ and $\mathcal{M}$ ([Lus83]). The $q$-analog of $\mathcal{P}$ is defined by the formula

$$\mathcal{P}_q : \Lambda \to \mathbb{Z}[q], \quad \mathcal{P}_q(\lambda) = \sum_{n=0}^\infty \mathcal{P}_q^n(\lambda)q^n$$

where

$$\mathcal{P}_q^n(\lambda) = \#\{m : \Phi^+ \to \mathbb{Z} \mid \lambda = \sum_{\alpha \in \Phi^+} m(\alpha)\alpha \text{ and } n = \sum_{\alpha \in \Phi^+} m(\alpha)\}$$
The $q$-analog of $\mathcal{M}$ is
\[ \mathcal{M}_q : \Lambda^+ \times \Lambda \rightarrow \mathbb{Z}[q], \quad \mathcal{M}_q(\lambda, \mu) = \sum_{w \in W} (-1)^{c(w)} \mathcal{P}_q(w(\lambda + \rho) - (\mu + \rho)) \]

The following can be extracted from [McG89].

**Proposition 4.0.1.** There is an identity in $K_a(\tilde{G})$
\[ C[N]|_{\tilde{G}} = \sum_{\lambda \in \Lambda^+} \tau_\lambda \mathcal{M}_q(\lambda, 0) \]

**Sketch of proof.** Consider the Springer resolution $\eta : T^*(G/B) \rightarrow N$. Since $T^*(G/B)$ is symplectic, there is an identification (of $\tilde{G}$-equivariant sheaves)
\[ \mathcal{O}_{T^*(G/B)} \simeq \omega_{T^*(G/B)} \]
where $\omega_{T^*(G/B)}$ is the canonical sheaf on $T^*(G/B)$. So by the theorem of Grauert and Riemenschneider (GR70)
\[ R^i \eta_* \mathcal{O}_{T^*(G/B)} = 0, \quad \forall i > 0 \]
On the other hand, $R^0 \eta_* \mathcal{O}_{T^*(G/B)} \simeq \mathcal{O}_N$. Using the Leray spectral sequence (and the fact that $N$ is affine), we get an identity in $K_a(\tilde{G})$
\[ C[N]|_{\tilde{G}} = \sum_{i} (-1)^i H^i(T^*(G/B), \mathcal{O}_{T^*(G/B)}) \tag{4.0.2} \]

If $p : T^*(G/B) \rightarrow G/B$ is the projection, then $p_* \mathcal{O}_{T^*(G/B)}$ is identified (as a $\tilde{G}$-equivariant sheaf) with (the sheaf of local sections of) the $\tilde{G}$-equivariant vector bundle $G \times_B S(\mathfrak{g}/\mathfrak{b})$. Since $p$ is affine, the direct image functor $p_*$ preserves cohomology, i.e.
\[ H^i(T^*(G/B), \mathcal{O}_{T^*(G/B)}) \simeq H^i(G/B, p_* \mathcal{O}_{T^*(G/B)}), \quad \forall i \geq 0 \tag{4.0.3} \]
Combining (4.0.2) and (4.0.3), we get a further identity in $K_a(\tilde{G})$
\[ C[N]|_{\tilde{G}} = \sum_{i} (-1)^i H^i(G/B, G \times_B S(\mathfrak{g}/\mathfrak{b})) \]
The right hand side can be computed using Borel-Weil-Bott. The result follows. \hfill \Box

5. **Some commutative algebra**

Using the results of Section 3, we get a well-defined class $i^*[\mathcal{O}_N] \in K^\wedge(\mathcal{N}_\partial)$, which can be regarded as the restriction of $\mathcal{O}_N$ to $p^*$. However, it is not at all clear that $i^*[\mathcal{O}_N] = [\mathcal{O}_N]$. For groups split modulo center, we will see that this equality always holds, but the proof will require some commutative algebra.

**Lemma 5.0.1.** Let $R$ be a Noetherian ring and let $M$ be a finitely-generated $R$-module. Suppose $x_1, \ldots, x_m \in R$ is an $R$-regular sequence which is also $M$-regular. Consider the ideal $I = (x_1, \ldots, x_m) \subset R$. Then
\[ \text{Tor}^R_n(R/I, M) = 0, \quad n > 0 \]
Proof. Since $x_1, ..., x_m$ is $R$-regular, the Koszul complex $K(x_1, ..., x_m; R)$ is a resolution of $R/I$. So $H_n(K(x_1, ..., x_m; R) \otimes_R M)$ is the homology of $K(x_1, ..., x_m; M) := K(x_1, ..., x_m; R) \otimes_R M$

$$H_n(K(x_1, ..., x_m; M)) \simeq \text{Tor}_n^R(R/I, M), \quad \forall n$$

But since $x_1, ..., x_m$ is an $M$-regular sequence, the complex $K(x_1, ..., x_m; M)$ is acyclic, see [Mat87, Thm 16.5(i)]. So $\text{Tor}_n^R(R/I, M) = 0$ for $n > 0$. \qquad \Box

Lemma 5.0.2 (Thm 17.4, [Mat87]). Suppose $A$ is Cohen-Macaulay, and let $I = (x_1, ..., x_n) \subset A$ be an ideal. If

$$\dim(A/I) = \dim(A) - n$$

then $x_1, ..., x_n$ is an $A$-regular sequence.

Proposition 5.0.3. Suppose $X$ is a smooth Noetherian scheme and let $Y$, $Z$ be closed subschemes of $X$. Write $i : Y \hookrightarrow X$ and $j : Z \hookrightarrow X$ for the inclusions and form the Cartesian diagram of schemes

$$
\begin{array}{ccc}
Z & \xrightarrow{j} & X \\
\uparrow & & \uparrow \\
Z \cap Y & \xleftarrow{i} & Y
\end{array}
$$

Assume

(i) $Y$ is smooth.

(ii) $Z$ is Cohen-Macaulay.

(iii) $\dim(Z \cap Y) = \dim(Z) + \dim(Y) - \dim(X)$.

Then

$$L_n i^*(j_* \mathcal{O}_Z) = 0, \quad n > 0$$

Proof. The statement is local in $X$, so we can assume all schemes are affine. Let $X = \text{Spec}(R)$, $Y = \text{Spec}(R/I)$ and $Z = \text{Spec}(A)$, so that $Z \cap Y = \text{Spec}(A/I)$. Since $Y$ is smooth, we can find an $R$-regular sequence $x_1, ..., x_m \in R$ such that $I = (x_1, ..., x_m)$, where $m = \dim(X) - \dim(Y)$. Now by Lemma 5.0.2, $(x_1, ..., x_m)$ is an $A$-regular sequence. So by Lemma 5.0.1 (applied to the $R$-module $M = A$)

$$\text{Tor}_n^R(R/I, A) = 0, \quad n > 0$$

This is equivalent to (5.0.1). \qquad \Box
6. Regular functions on $\mathcal{N}_\theta$

First, we will assume that $G_\mathbb{R}$ is split. This means that $G$ contains a maximal torus $H_s \subset G$ on which $\theta$ acts by inversion. Our first lemma shows that the codimension of $\mathcal{N}_\theta \subset \mathcal{N}$ equals the codimension of $\mathfrak{p}^* \subset \mathfrak{g}^*$.

**Lemma 6.0.1.** $\dim(\mathcal{N}_\theta) = \dim(\mathcal{N}) + \dim(\mathfrak{p}) - \dim(\mathfrak{g})$

**Proof.** By [KR71, Prop 9]

(6.0.1) $\dim(\mathcal{N}_\theta) = \dim(\mathfrak{p}) - \dim(\mathfrak{h}_s)$

By the Iwasawa decomposition

(6.0.2) $\dim(\mathfrak{g}) = \dim(\mathfrak{t}) + \dim(\mathfrak{h}_s) + \dim(\mathfrak{n})$

Finally

(6.0.3) $\dim(\mathcal{N}) = 2 \dim(\mathfrak{n}) = \dim(\mathfrak{g}) - \dim(\mathfrak{h}_s)$

Combining (6.0.1), (6.0.2), and (6.0.3) proves the lemma. $\square$

**Corollary 6.0.2.** There is an equality in $K^R(\mathcal{N}_\theta)$

(6.0.4) $[\mathcal{O}_{\mathcal{N}_\theta}] = i^*[\mathcal{O}_{\mathcal{N}}]$ and hence an equality in $K_{aa}(\mathcal{O})$

(6.0.5) $\mathbb{C}[\mathcal{N}_\theta]|_\mathbb{R} = \mathbb{C}[\mathcal{N}]|_\mathbb{R} \otimes [\wedge(\mathfrak{t})]$

**Proof.** For (6.0.4), we will apply Proposition 5.0.3. So let $X = \mathfrak{g}^*$, $Z = \mathcal{N}$, and $Y = \mathfrak{p}^*$. Clearly, $X$ and $Y$ are smooth. By [Kos63, Thm 0.1], $\mathcal{N}$ is a complete intersection, and therefore Cohen-Macaulay. Condition (iii) of Proposition 5.0.3 is the content of Lemma 6.0.1. So by Proposition 5.0.3 we have

$$L_n i^* (j_* \mathcal{O}_\mathcal{N}) = 0, \quad n > 0$$

and therefore

$$i^*[\mathcal{O}_{\mathcal{N}}] = \sum_n (-1)^n [L_n i^* j_* \mathcal{O}_\mathcal{N}] = [i^* j_* \mathcal{O}_\mathcal{N}] = [\mathcal{O}_{\mathcal{N}_\theta}]$$

For the final equality, we use the well-known fact that scheme-theoretic intersection $\mathcal{N} \cap \mathfrak{p}^*$ is reduced, see [KR71, Theorem 14]. This proves (6.0.4). Now (6.0.5) follows from (6.0.4) and Corollary 3.0.7. $\square$

**Corollary 6.0.2** can be easily extended to the case when $G_\mathbb{R}$ is split modulo center.

**Example 6.0.3.** Let $G = \text{SL}_2(\mathbb{C})$ and let $\theta(g) = (g^{-1})^t$ (this is the involution corresponding to split real form $\text{SL}_2(\mathbb{R})$). Then $K = \text{SO}_2(\mathbb{C})$. Write

$$\text{Irr}(G) = \{ \tau_m \mid m = 0, 1, 2, \ldots \}, \quad \text{Irr}(K) = \{ \chi_n \mid n \in \mathbb{Z} \}$$
(here \(\tau_m\) is the irreducible with highest weight \(m\) and \(\chi_n\) is the degree-\(n\) character of \(\text{SO}_2(\mathbb{C})\). We have the following branching rules

\[
\tau_m|_K = \chi_{-2m} + \chi_{-2m+2} + \ldots + \chi_{2m}.
\]

From Proposition 4.0.1 we deduce

\[
\mathbb{C}[\mathcal{N}]|_G = \sum_{m=0}^{\infty} \tau_m q^m
\]

Also

\[
[\wedge(\mathfrak{t})] = \chi_0 - \chi_0 q.
\]

So by Corollary 6.0.2

\[
\mathbb{C}[\mathcal{N}_\theta]|_K = \mathbb{C}[\mathcal{N}]|_K \otimes [\wedge(\mathfrak{t})]
\]

\[
= \left( \sum_{m=0}^{\infty} (\chi_{-2m} + \ldots + \chi_{2m}) q^m \right) \otimes (\chi_0 - \chi_0 q)
\]

\[
= \sum_{m=0}^{\infty} ((\chi_{-2m} + \ldots + \chi_{2m}) - (\chi_{-2m+2} + \ldots + \chi_{2m-2})) q^m
\]

\[
= \sum_{m=0}^{\infty} (\chi_{2m} + \chi_{-2m}) q^m
\]

7. Branching to \(K\)

In this section, we will use Corollary 6.0.2 to compute \(\mathbb{C}[\mathcal{N}_\theta]|_K\) as a formal integer combination of classes of the form \(I(H)q^n\).

Suppose \((H, \gamma, \Phi^+)\) is a continued Langlands parameter (see Definition 2.0.1) and let \(\chi \in K_f(K)\). It is easy to compute the tensor product \([I(H, \gamma, \Phi^+)]|_K \otimes \chi\) as a representation of \(K\).

Lemma 7.0.1 (Lem 12.13, [Vog07]). Choose a finite multiset \(S_H(\chi)\) in \(X^*(H)\) such that

\[
\chi|_{H^\theta} = \sum_{\mu \in S_H(\chi)} \mu|_{H^\theta}
\]

Then there is an identity in \(K_\alpha(K)\)

\[
[I(H, \gamma, \Phi^+)]|_K \otimes \chi = \sum_{\mu \in S_H(\chi)} [I(H, \gamma + \mu, \Phi^+)]
\]

We will use Lemma 7.0.1 (together with Zuckerman’s character formula for the trivial representation) to compute \(\mathbb{C}[\mathcal{N}_\theta]|_K\) in terms of the classes \(I(H)q^n\). For an arbitrary class \(\chi \in K_f(K)\), it may be difficult to find a multiset \(S_H(\chi)\) as in Lemma 7.0.1. Fortunately, in our setting, \(\chi\) is not arbitrary. For the problem at hand, we will need to compute \(S_H(\chi)\) in the following two cases:
(1) \( \chi \) is the restriction to \( K \) of an irreducible representation \( \tau_\lambda \) of \( G \).

(2) \( \chi \) is the class \( \wedge^n(\mathfrak{t}) \).

First, suppose \( \chi = \tau_\lambda|_K \). By the Weyl character formula (4.0.1), we have

\[
\tau_\lambda|_H = \sum_{\mu \in X^*(H)} \mathcal{M}(\lambda, \mu)e^{\mu}
\]

So we can take

\[
S_H(\tau_\lambda) = \{\mathcal{M}(\lambda, \mu)e^{\mu} \mid \mu \in X^*(H)\}
\]

(the coefficients \( \mathcal{M}(\lambda, \mu) \) above denote the multiset multiplicities. We will use similar notation below).

Next, suppose \( \chi = \mathfrak{k} \). Choose a subset \( \Delta'_C \subset \Delta_C \) such that for each \( \alpha \in \Delta_C \), exactly one of \( \{\alpha, \theta\alpha\} \) appears in \( \Delta'_C \). Then there is a decomposition of \( \mathfrak{k} \) into weight spaces for \( H^\theta \)

\[
\mathfrak{k} \simeq \bigoplus_{\alpha \in \Delta_C} g_\alpha \oplus \bigoplus_{\alpha \in \Delta'_C} (1 + d\theta)g_\alpha \oplus \bigoplus_{\alpha \in \Delta_C^+} (1 + d\theta)g_\alpha
\]

So we can take

\[
S_H(\mathfrak{k}) = \{e^\alpha \mid \alpha \in \Delta_c\} \cup \{e^\alpha \mid \alpha \in \Delta'_C\} \cup \{\Delta^+_C|e^0\}
\]

More generally

\[
S_H(\wedge^n(\mathfrak{k})) = \{\sum R \mid R \subseteq S_H(\mathfrak{k}), |R| = n\}
\]

Zuckerman’s character formula for the trivial representation (see [Vog81, Thm 9.4.16]) can be interpreted as an identity in \( K_a(\tilde{K}) \)

\[
\text{triv} = \sum_{H} \sum_{\Delta^+} (-1)^{\ell(\Delta^+)}[I(H, \rho_{i\mathbb{R}}, \Delta^+_i)]|_K
\]

The outer sum runs over \( K \)-conjugacy classes of \( \theta \)-stable maximal tori \( H \subset G \) and the inner sum over \( W(K, H^\theta) \)-conjugacy classes of positive systems \( \Delta^+ \subset \Delta(G, H) \). The integer \( \ell(\Delta^+) \) is the codimension of the \( K \)-orbit on the flag variety containing the Borel subalgebra \( b \subset g \) corresponding to the positive system \( \Delta^+ \). Now assume that \( G_\mathbb{R} \) is split modulo center. Using Corollary 6.0.2 we obtain an identity in \( K_a(\tilde{K}) \)

\[
\mathbb{C}[\mathcal{N}_\theta]|_{\tilde{K}} = \left( \sum_{H} \sum_{\Delta^+} (-1)^{\ell(\Delta^+)}[I(H, \rho_{i\mathbb{R}}, \Delta^+_i)]|_K \right) \otimes \mathbb{C}[\mathcal{N}]|_{\tilde{K}} \otimes [\wedge(\mathfrak{t})]
\]

Using Proposition 4.0.1 and Lemma 7.0.1 we can rewrite the right hand side in terms of classes of the form \([I(\Gamma)]q^n\)

**Theorem 7.0.2.** Assume \( G_\mathbb{R} \) is split modulo center. Then there is an identity in \( K_a(\tilde{K}) \)

\[
\mathbb{C}[\mathcal{N}_\theta]|_{\tilde{K}} = \sum_{H} \sum_{\Delta^+} \sum_{\lambda \in \Lambda^+} \sum_{\mu \in \Lambda} \sum_{R \subseteq S_H(\mathfrak{k})} (-1)^{\ell(\Delta^+)+|R|}\mathcal{M}(\lambda, \mu)\mathcal{M}_q(\lambda, 0)[I(H, \rho_{i\mathbb{R}} + \mu + |R|, \Delta^+_i)]|_K |R|
\]

Note that the terms on the right are not final. We can rewrite the sum in terms of final parameters using the Hecht-Schmid identities (this sort of thing is easy to do in atlas).
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