Algebraicity of cycles on smooth manifolds

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Abstract According to the Nash–Tognoli theorem, each compact smooth manifold $M$ is diffeomorphic to a nonsingular real algebraic set, called an algebraic model of $M$. It is interesting to investigate to what extent algebraic and differential topology of compact smooth manifolds can be transferred into the algebraic-geometric setting. Many results, examples and counterexamples depend on the detailed study of the homology classes represented by algebraic subsets of $X$, as $X$ runs through the class of all algebraic models of $M$. The present paper contains several new results concerning such algebraic homology classes. In particular, a complete solution in codimension 2 and strong results in codimensions 3 and 4.

Keywords Real algebraic set · Algebraic homology class · Algebraic model of a smooth manifold

Mathematics Subject Classification (2010) 14P05 · 14P25 · 57R19

1 Introduction and main results

There is a large research program whose goal is to transfer, as far as possible, algebraic and differential topology of compact smooth (of class $C^\infty$) manifolds into the algebraic-geometric setting. The origins of this program go back to 1952 and the celebrated paper of J. Nash on real algebraic manifolds [53] (cf. also [16, Theorem 14.1.8]). Nash’s result and conjectures inspired several mathematicians, but despite their efforts, no significant progress was made for 20 years (cf. [34] for historical
As explained in [63], a breakthrough in 1973 by Tognoli [63], who proved one of Nash’s conjectures (cf. also [16, Theorem 14.1.10]). According to Tognoli’s theorem, every compact smooth manifold $M$ is diffeomorphic to a nonsingular real algebraic set (in $\mathbb{R}^n$ for some $n$), called an algebraic model of $M$. A projective version of this theorem was proved in 1976 by King [35]. Actually, both [63] and [35] contain much stronger results, concerning approximation of smooth manifolds by algebraic sets, as suggested in [53]. Remarkable refinements of [35,63] can be found in the contributions from the 1980s and 1990s of two pairs of researchers, Akbulut–King [1–3,6–8] and Benedetti–Tognoli [13,14]. If some topological objects such as smooth submanifolds, vector bundles, homology or cohomology classes are attached to $M$, it is interesting to investigate whether or not there exists an algebraic model of $M$ on which the corresponding objects admit an algebraic description. Important positive results are known for smooth submanifolds [2,13] and vector bundles [13,14]. Contrary to initial expectations, expressed explicitly in [2,3], the situation is drastically different for homology and cohomology classes, where obstructions appear [12,20,41,42,61].

This on the one hand imposes limitations and on the other hand leads to challenging problems considered below.

Let $X$ be a compact nonsingular real algebraic set. A homology class in $H_d(X; \mathbb{Z}/2)$ is said to be algebraic if it can be represented by a $d$-dimensional algebraic subset of $X$ (cf. [27] and [8,16,22]). The set $H^\text{alg}_d(X; \mathbb{Z}/2)$ of all algebraic homology classes in $H_d(X; \mathbb{Z}/2)$ forms a subgroup. Early papers dealing with algebraic homology classes provided examples of $X$ with $H^\text{alg}_d(X; \mathbb{Z}/2) \neq H_d(X; \mathbb{Z}/2)$ for some $d$, $1 \leq d \leq \dim X - 1$ (cf. [3,14,15,36,54,57]). For technical reasons, it is often preferable to work with cohomology rather than homology. The subgroup $H^k(X; \mathbb{Z}/2)$ of algebraic cohomology classes in $H^k(X; \mathbb{Z}/2)$ is by definition the inverse image of $H^\text{alg}_{n-k}(X; \mathbb{Z}/2)$ under the Poincaré duality isomorphism $H^k(X; \mathbb{Z}/2) \rightarrow H_{n-k}(X; \mathbb{Z}/2)$, where $n = \dim X$. In particular, $H^0_{\text{alg}}(X; \mathbb{Z}/2) = H^0(X; \mathbb{Z}/2)$. The direct sum

$$H^*_\text{alg}(X; \mathbb{Z}/2) = \bigoplus_{k \geq 0} H^k_{\text{alg}}(X; \mathbb{Z}/2)$$

is a subring of the cohomology ring $H^*(X; \mathbb{Z}/2)$, containing the Stiefel–Whitney classes $w_k(X)$ of $X$ for $k \geq 0$ (cf. [27] and, for purely topological proofs, [4,15,56]). Consequently, $H^*_{\text{alg}}(X; \mathbb{Z}/2)$ contains the subring of $H^*(X; \mathbb{Z}/2)$ generated by $H^n(X; \mathbb{Z}/2)$ and $w_k(X)$ for $k \geq 0$. What other, if any, cohomology classes belong to $H^*_\text{alg}(X; \mathbb{Z}/2)$ depends in a very subtle way on the algebraic-geometric properties of $X$ (cf. [21,31,49–51,58,65,66]). The groups $H^\text{alg}_d(\cdot; \mathbb{Z}/2)$ and $H^k_{\text{alg}}(\cdot; \mathbb{Z}/2)$ are closely related via the cycle maps to the Chow groups of quasiprojective schemes over $\mathbb{R}$ and to the equivariant cohomology of the set of complex points of such schemes (cf. [27,28,33,36,48,66]). They play a crucial role in the research program described at the beginning (cf. [1,3–5,8–18,20,23–26,37–48,55,56,61,64] and, for a short survey, [22]).

Numerous results, examples and counterexamples in the papers just cited required information on algebraic homology and cohomology classes on various algebraic
models of a given compact smooth manifold $M$. According to [19], $M$ has an uncountable family of pairwise nonisomorphic algebraic models whenever $\dim M \geq 1$. However, $M$ may not admit any algebraic model $X$ with $H^*_{\text{alg}}(X; \mathbb{Z}/2) = H^*(X; \mathbb{Z}/2)$ (see the remarks preceding Corollary 1.3). In order to avoid awkward repetitions, if $X$ is an algebraic model of $M$ and $\varphi : X \to M$ is a smooth diffeomorphism, the pair $(X, \varphi)$ will also be called an algebraic model of $M$. A subring $A$ of $H^*(M; \mathbb{Z}/2)$ (only subrings containing the identity element 1 are considered) is said to be algebraically realizable if there exists an algebraic model $(X, \varphi)$ of $M$ with $\varphi^*(A) \subseteq H^*_{\text{alg}}(X; \mathbb{Z}/2)$. An important algebraically realizable subring of $H^*(M; \mathbb{Z}/2)$ is identified in [13, Theorem 4, Remark 8]. It is the subring $A(M)$ generated by the Stiefel–Whitney classes of all real vector bundles on $M$ and the cohomology classes corresponding via the Poincaré duality to the homology classes represented by all the Stiefel–Whitney classes of the vector bundles in $B$. For any full subring $A$ contained in $A(M)$, when there are no obstructions to algebraic realizability of $A$. This paper provides partial solutions for a large class of subrings of $A(M)$ (cf. Theorems 1.1, 1.7, 2.10 and Corollaries 1.2, 1.4, 2.5, 1.8, 1.10).

The analogous problem of finding, for a fixed positive integer $r$, a characterization of the subgroups $G$ of $H^r(M; \mathbb{Z}/2)$ for which there exists an algebraic model $(X, \varphi)$ of $M$ with $\varphi^*(G) = H^r_{\text{alg}}(X; \mathbb{Z}/2)$ is more tractable. It is completely solved in [17, Theorems 1.2 and 1.3] and [42, Corollary 1.12] for $r = 1$. The present paper contains a complete solution, under the assumption $\dim M \geq 5$, for $r = 2$ (cf. Corollary 1.3) and several partial results for $r \geq 3$ (cf. Corollaries 1.6, 1.9 and 1.11). A necessary condition for the existence of such model $(X, \varphi)$ is that all cup products $w_{i_1}(M) \cup \cdots \cup w_{i_p}(M)$ be in $G$, where $i_1, \ldots, i_p$ are nonnegative integers with $i_1 + \cdots + i_p = r$.

As the initial step, a suitable class of subrings of $H^*(M; \mathbb{Z}/2)$ will be defined.

If $h : M \to P$ is a smooth map into a compact smooth manifold $P$, then a standard transversality argument implies that $h^*(A(P)) \subseteq A(M)$ (cf. also [27, Proposition 2.15]). A subring $B$ of $H^*(M; \mathbb{Z}/2)$ is said to be full if $B = h^*(H^*(P; \mathbb{Z}/2))$ for some $h : M \to P$ with $A(P) = H^*(P; \mathbb{Z}/2)$. Every full subring is contained in $A(M)$.

For any collection $F$ of real vector bundles on $M$, the subring $F(M)$ generated by the Stiefel–Whitney classes of the vector bundles in $F$ is a full subring of $H^*(M; \mathbb{Z}/2)$. Indeed, the collection $F$ can be assumed to be finite, the set $H^*(M; \mathbb{Z}/2)$ being finite, and hence, the assertion readily follows by making use of smooth classifying maps and Künneth’s theorem (cf. [30,32,59]).

For any subring $B$ and any subset $T$ of $H^*(M; \mathbb{Z}/2)$, let $B[T]$ denote the extension of $B$ by $T$, that is, the subring of $H^*(M; \mathbb{Z}/2)$ generated by $B$ and $T$. A cohomology class in $H^*(M; \mathbb{Z}/2)$ will be called regular if it corresponds via the Poincaré duality...
to a homology class in $H_*(M; \mathbb{Z}/2)$ represented by a compact smooth submanifold of $M$. The subset $T$ will be called regular if each cohomology class in $T$ is regular. A subring of $H^*(M; \mathbb{Z}/2)$ that is the extension of a full subring by a regular subset is said to be admissible. An admissible subring $A$ is said to be $r$-admissible, where $r$ is a nonnegative integer, if it can be written as $A = B[T]$ for some full subring $B$ and some regular subset $T$, with $T$ disjoint from $H^i(M; \mathbb{Z}/2)$ for $0 \leq i \leq r - 1$. Thus, admissible is the same as 0-admissible. By a transversality argument, each admissible subring $A$ can be written as $A = F(M) [T]$, where $F$ is a finite collection of real vector bundles and $T$ is a regular subset. In particular, the definitions of an admissible subring used here and in [45] are equivalent. The largest admissible subring is $A(M)$.

If $\dim M \leq 5$, then each cohomology class in $H^*(M; \mathbb{Z}/2)$ is regular [62, Théorème II.26], and hence, every subring of $H^*(M; \mathbb{Z}/2)$ is admissible.

Relationships between admissible subrings and $H^*_{\text{alg}}(\cdot; \mathbb{Z}/2)$ are investigated below. The main results, whose proofs are postponed until Sect. 2, are Theorems 1.1 and 1.7. Their significance is elaborated upon in a series of corollaries. Some simple topological facts, contained in Proposition 1.12, are also required for the derivation of the corollaries.

As usual, the $i$th Steenrod square operation will be denoted by $\text{Sq}^i$. Only $\text{Sq}^1$ is used in Sect. 1.

For any nonnegative integer $k$ and any subring $A$ of $H^*(M; \mathbb{Z}/2)$, let

$$A^k := A \cap H^k(M; \mathbb{Z}/2).$$

**Theorem 1.1** Let $M$ be a compact connected smooth manifold and let $r$ be a positive integer with $2r + 1 \leq \dim M$. For an $r$-admissible subring $A$ of $H^*(M; \mathbb{Z}/2)$ with $A^i = 0$ for $1 \leq i \leq r - 2$, the following conditions are equivalent:

(a) There exists an algebraic model $(X, \varphi)$ of $M$ satisfying

$$\varphi^*(A) \subseteq H^*(X; \mathbb{Z}/2) \text{ and } \varphi^*(A^k) = H^k_{\text{alg}}(X; \mathbb{Z}/2) \text{ for } 0 \leq k \leq r.$$  

(b) $w_k(M)$ is in $A^k$ for $0 \leq k \leq r$.

Of course, the condition $A^i = 0$ for $1 \leq i \leq r - 2$ is vacuous if $r = 1$ or $r = 2$. If $r = 1$, then Theorem 1.1 is a minor improvement upon [17, Theorems 1.2 and 1.3]. The case $r = 2$ is much more interesting.

**Corollary 1.2** Let $M$ be a compact connected smooth manifold of dimension at least 5. For an admissible subring $A$ of $H^*(M; \mathbb{Z}/2)$, the following conditions are equivalent:

(a) There exists an algebraic model $(X, \varphi)$ of $M$ satisfying

$$\varphi^*(A) \subseteq H^*_{\text{alg}}(X; \mathbb{Z}/2) \text{ and } \varphi^*(A^k) = H^k_{\text{alg}}(X; \mathbb{Z}/2) \text{ for } k = 0, 1, 2.$$  

(b) $w_k(M)$ is in $A^k$ for $k = 0, 1, 2$.

**Proof** According to Proposition 1.12(p1), each admissible subring is 2-admissible, and hence, it suffices to apply Theorem 1.1 with $r = 2$. $\square$
Corollary 1.2 was proved in [45] for $M$ with homology group $H_{m-2}(M; \mathbb{Z})$ having no 2-torsion, where $m = \dim M$. This additional assumption removed the main difficulty in the proof.

It is interesting to extract from Corollary 1.2 and previously known results information on the behavior of $H^2_{\text{alg}}(\ast; \mathbb{Z}/2)$. Let $A^r(M) := A(M)^r$. According to [20,61], for any compact smooth manifold $M$, the group $A^2(M)$ can be described as follows: $A^2(M) = W^2(M)$, where

$$W^2(M) := \{v \in H^2(M; \mathbb{Z}/2) \mid v = w_2(\xi) \text{ for some real vector } \xi \text{ on } M \text{ with } w_1(\xi) = 0\}$$

and $w_k(\xi)$ denotes the $k$th Stiefel–Whitney class of $\xi$ for $k \geq 0$. Thus, $W^2(M) = H^2(M; \mathbb{Z}/2)$ if $\dim M \leq 5$. However, for each integer $n \geq 6$, there exists an $n$-dimensional compact connected smooth manifold $N$ with $W^2(N) \neq H^2(N, \mathbb{Z}/2)$ [61]. On the other hand,

$$H^2_{\text{alg}}(X; \mathbb{Z}/2) \subseteq W^2(X)$$

for every compact nonsingular real algebraic set $X$ (cf. [12,18] and, for an elementary topological proof, [23]). In particular, $H^2_{\text{alg}}(Y; \mathbb{Z}/2) \neq H^2(Y; \mathbb{Z}/2)$ for every algebraic model $Y$ of $N$.

Corollary 1.3 Let $M$ be a compact connected smooth manifold of dimension at least 5. For a subgroup $G$ of $H^2(M; \mathbb{Z}/2)$, the following conditions are equivalent:

(a) There exists an algebraic model $(X, \varphi)$ of $M$ satisfying

$$\varphi^*(G) = H^2_{\text{alg}}(X; \mathbb{Z}/2).$$

(b) $w_1(M) \cup w_1(M)$ and $w_2(M)$ are in $G$, and $G \subseteq W^2(M)$.

Proof If (a) holds, then $w_1(X) \cup w_1(X)$ and $w_2(X)$ belong to $\varphi^*(G)$, and $\varphi^*(G) \subseteq W^2(X)$. Hence, (b) follows.

Suppose that (b) holds. For each cohomology class $v$ in $W^2(M)$, let $\xi_v$ be a real vector bundle on $M$ with $w_1(\xi_v) = 0$ and $w_2(\xi_v) = v$. Let $F$ be the collection consisting of the tangent bundle to $M$ and the $\xi_v$ for $v$ in $G$. The subring $A := F(M)$ of $H^*(M, \mathbb{Z}/2)$ is admissible, $A^2 = G$, and $w_i(M)$ is in $A^i$ for $i \geq 0$. Hence, Corollary 1.2 implies that (a) is satisfied. □

Corollary 1.3 was already conjectured in [20], but proved there only for $M$ orientable, that is, $w_1(M) = 0$. In [40], Corollary 1.3 was proved under very restrictive assumptions on the group $H^{m-2}(M; \mathbb{Z}/2)$, where $m = \dim M$. The methods used in [20,40] do not work without these extra hypotheses.

There is also a version of Corollary 1.2 for an arbitrary, not necessarily admissible, subring.

Corollary 1.4 Let $M$ be a compact connected smooth manifold of dimension at least 5. For a subring $A$ of $H^*(M; \mathbb{Z}/2)$, the following conditions are equivalent:
(a) There exists an algebraic model \((X, \varphi)\) of \(M\) satisfying

\[ \varphi^*(A^k) = \alg^k(X; \mathbb{Z}/2) \quad \text{for } k = 0, 1, 2. \]

(b) \(w_k(M)\) is in \(A^k\) for \(k = 0, 1, 2,\) and \(A^2 \subseteq W^2(M)\).

**Proof** It is already explained that (a) implies (b).

Suppose now that (b) holds. Each cohomology class \(u\) in \(H^1(M; \mathbb{Z}/2)\) can be written as \(u = w_1(\gamma_u)\) for some real line bundle \(\gamma_u\) on \(M\). Similarly, each cohomology class \(v\) in \(W^2(M)\) can be written as \(v = w_2(\xi_v)\) for some real vector bundle \(\xi_v\) on \(M\) with \(w_1(\xi_v) = 0\). Let \(F\) be the collection consisting of \(\gamma_u\) for \(u\) in \(A^1\), \(\xi_v\) for \(v\) in \(A^2\) and the tangent bundle to \(M\). The subring \(C := F(M)\) of \(H^*(M; \mathbb{Z}/2)\) is admissible with \(C^k = A^k\) for \(k = 0, 1, 2\). Corollary 1.2 applied to the subring \(C\) implies (a). 

Theorem 1.1 with \(r = 3\) implies the following:

**Corollary 1.5** Let \(M\) be a compact connected orientable smooth manifold of dimension at least 7. For an admissible subring \(A\) of \(H^*(M; \mathbb{Z}/2)\) with \(A^1 = 0\), the following conditions are equivalent:

(a) There exists an algebraic model \((X, \varphi)\) of \(M\) satisfying

\[ \varphi^*(A) \subseteq \alg^*(X; \mathbb{Z}/2) \quad \text{and} \quad \varphi^*(A^k) = \alg^k(X; \mathbb{Z}/2) \quad \text{for } k = 0, 1, 2, 3. \]

(b) \(w_i(M)\) is in \(A^i\) for \(i = 2, 3,\) and \(\Sq^1(A^2) \subseteq A^3\).

**Proof** According to [4, Theorem 6.6], \(\Sq^1(H^2_\alg(-; \mathbb{Z}/2)) \subseteq H^3_\alg(-; \mathbb{Z}/2)\), and therefore, (a) implies (b).

Suppose now that (b) holds. By Proposition 1.12(p2), there exists a 3-admissible subring \(\overline{A}\) of \(H^*(M; \mathbb{Z}/2)\) such that \(A \subseteq \overline{A}\) and \(A^k = \overline{A}^k\) for \(k = 0, 1, 2, 3\). The orientability of \(M\) implies \(w_1(M) = 0\). Hence, (a) follows by applying Theorem 1.1 with \(r = 3\) to the subring \(\overline{A}\).

It would be interesting, but very hard, to extend Corollary 1.3 to subgroups of \(H^*(M; \mathbb{Z}/2)\) with \(r \geq 3\). The following partial result is available.

**Corollary 1.6** Let \(M\) be a compact connected smooth manifold and let \(r \geq 3\) be an integer with \(2r + 1 \leq \dim M\). Assume that \(w_i(M) = 0\) for \(1 \leq i \leq r - 2\). If \(G\) is a subgroup of \(H^*(M; \mathbb{Z}/2)\) generated by some regular cohomology classes and \(w_r(M)\), then there exists an algebraic model \((X, \varphi)\) of \(M\) satisfying \(\varphi^*(G) = \alg^*(X; \mathbb{Z}/2)\).

**Proof** The subring \(A\) of \(H^*(M; \mathbb{Z}/2)\) generated by \(G\) and the cohomology classes \(w_k(M)\) for \(k \geq 0\) is \(r\)-admissible. Moreover, \(A^i = 0\) for \(1 \leq i \leq r - 2\) and \(A^r = G\). It remains to apply Theorem 1.1.

If \(r = 3\) in Corollary 1.6, then the condition \(w_i(M) = 0\) for \(1 \leq i \leq r - 2\) is equivalent to the orientability of \(M\).
**Theorem 1.7** Let $M$ be a compact connected smooth manifold whose homology group $H_{r-1}(M; \mathbb{Z})$ has no 2-torsion for some integer $r \geq 3$ with $2r + 1 \leq \dim M$. Let $A$ be an $r$-admissible subring of $H^*(M; \mathbb{Z}/2)$ with $A^i = 0$ for $1 \leq i \leq r - 4$. Assume that $w_j(M)$ is in $A^j$ for $0 \leq j \leq r$. Then, there exists an algebraic model $(X, \varphi)$ of $M$ satisfying

$$\varphi^*(A) \subseteq H^*_{\text{alg}}(X; \mathbb{Z}/2) \quad \text{and} \quad \varphi^*(A^k) = H^k_{\text{alg}}(X; \mathbb{Z}/2) \quad \text{for } k \in \{0, 1, \ldots, r - 2, r\} \cup \{2\}.$$ 

Moreover, the last equality holds for $0 \leq k \leq r$ if $r \geq 4$, and either the homology group $H_{r-2}(M; \mathbb{Z})$ has no 2-torsion or $A^{r-3} = 0$.

Clearly, the condition $A^i = 0$ for $1 \leq i \leq r - 4$ is vacuous if $r = 3$ or $r = 4$. The case $r = 3$ is of particular interest.

**Corollary 1.8** Let $M$ be a compact connected smooth manifold of dimension at least 7, whose homology group $H_2(M; \mathbb{Z})$ has no 2-torsion. For an admissible subring $A$ of $H^*(M; \mathbb{Z}/2)$, the following conditions are equivalent:

(a) There exists an algebraic model $(X, \varphi)$ of $M$ satisfying

$$\varphi^*(A) \subseteq H^*_{\text{alg}}(X; \mathbb{Z}/2) \quad \text{and} \quad \varphi^*(A^k) = H^k_{\text{alg}}(X; \mathbb{Z}/2) \quad \text{for } k = 0, 1, 2, 3.$$ 

(b) $w_k(M)$ is in $A^k$ for $k = 0, 1, 2, 3$.

**Proof** It suffices to prove that (b) implies (a). According to Proposition 1.12(p3), the subring $A$ is 3-admissible, and hence, it suffices to apply Theorem 1.7 with $r = 3$. \hfill \Box

A much weaker version of Corollary 1.8 was proved in [45] for a spin manifold $M$ whose homology group $H_i(M; \mathbb{Z})$ has no 2-torsion for $i = 1, 2$. By definition, $M$ is a spin manifold if $w_1(M) = 0$ and $w_2(M) = 0$, which automatically implies $w_3(M) = 0$ (cf. [52, p. 94]).

**Corollary 1.9** Let $M$ be a compact connected smooth manifold of dimension at least 7, whose homology group $H_2(M; \mathbb{Z})$ has no 2-torsion. For an admissible subring $A$ of $H^*(M; \mathbb{Z}/2)$, the following conditions are equivalent:

(a) There exists an algebraic model $(X, \varphi)$ of $M$ satisfying

$$\varphi^*(A^3) = H^3_{\text{alg}}(X; \mathbb{Z}/2).$$ 

(b) $w_1(M) \cup w_1(M) \cup w_1(M), w_1(M) \cup w_2(M)$ and $w_3(M)$ are in $A^3$.

**Proof** It is already known that (a) implies (b).

Suppose now that (b) holds. According to Proposition 1.12(p3), the subgroup $A^3$ of $H^3(M; \mathbb{Z}/2)$ is generated by regular cohomology classes. Hence, the subring $C$ of $H^*(M; \mathbb{Z}/2)$ generated by $A^3$ and $w_i(M)$ for $i \geq 0$ is admissible and $C^3 = A^3$. Condition (a) follows by applying Corollary 1.8 to the subring $C$. \hfill \Box

Theorem 1.7 with $r = 4$ takes the following form:
Corollary 1.10 Let $M$ be a compact connected smooth manifold of dimension at least 9, whose homology group $H_3(M; \mathbb{Z})$ has no 2-torsion. For a 4-admissible subring $A$ of $H^*(M; \mathbb{Z}/2)$, the following conditions are equivalent:

(a) There exists an algebraic model $(X, \varphi)$ of $M$ satisfying

$$\varphi^*(A) \subseteq H^k_{\text{alg}}(X; \mathbb{Z}/2) \quad \text{and} \quad \varphi^*(A^j) = H^k_{\text{alg}}(X; \mathbb{Z}/2) \text{ for } k = 0, 1, 2, 4.$$

(b) $w_j(M)$ is in $A^j$ for $j = 0, 1, 2, 3, 4$.

Moreover, $k = 3$ can be added in condition (a) if either the homology group $H_2(M; \mathbb{Z})$ has no 2-torsion or $A^1 = 0$.

Proof Since the ring $A$ is 4-admissible, it readily follows that $\text{Sq}^1(A^2) \subseteq A^3$. By Wu’s formula [52, p. 94], $\text{Sq}^1(w_2(M)) = w_1(M) \cup w_2(M) + w_3(M)$. Consequently, if $w_j(M)$ is in $A^j$ for $j = 1, 2$, then $w_3(M)$ is in $A^3$. If (a) holds, then $w_j(M)$ is in $A^j$ for $j = 0, 1, 2, 4$, and hence, (b) is satisfied. According to Theorem 1.7 with $r = 4$, condition (b) implies (a). \hfill \Box

It is an open problem whether or not Corollary 1.10 remains true if the homology group $H_i(M; \mathbb{Z})$ has no 2-torsion for $i = 2, 3$ and the subring $A$ is admissible, but not necessarily 4-admissible. No result similar to Corollary 1.10 is available in the literature.

Under an additional assumption on $M$, Corollary 1.6 can be strengthened as follows.

Corollary 1.11 Let $M$ be a compact connected smooth manifold whose homology group $H_{r-1}(M; \mathbb{Z})$ has no 2-torsion for some integer $r \geq 3$ with $2r + 1 \leq \dim M$. Let $G$ be a subgroup of $H^*(M; \mathbb{Z}/2)$ generated by some regular cohomology classes and all cup products $w_{i_1}(M) \cup \cdots \cup w_{i_p}(M)$, where $i_1, \ldots, i_p$ are nonnegative integers with $i_1 + \cdots + i_p = r$. If $w_i(M) = 0$ for $1 \leq i \leq r - 4$, then there exists an algebraic model $(X, \varphi)$ of $M$ satisfying $\varphi^*(G) = H_{\text{alg}}(X; \mathbb{Z}/2)$.

Proof The subring $A$ of $H^*(M; \mathbb{Z}/2)$ generated by $G$ and $w_j(M)$ for $j \geq 0$ is $r$-admissible, $A^i = 0$ for $1 \leq i \leq r - 4$, and $A^r = G$. Hence, it suffices to apply Theorem 1.7. \hfill \Box

In Corollary 1.11, the condition $w_i(M) = 0$ for $1 \leq i \leq r - 4$ is vacuous if $r = 3$ or $r = 4$, while it is equivalent to the orientability of $M$ if $r = 5$. It follows from Proposition 1.12(p3) that Corollary 1.9 is equivalent to Corollary 1.11 with $r = 3$.

The properties of admissible rings used in the proofs of the corollaries above are contained in the following:

Proposition 1.12 Let $M$ be a compact connected smooth manifold. Any admissible subring $M$ of $H^*(M; \mathbb{Z}/2)$ has the following properties:

(p1) $A$ is 2-admissible.

(p2) If $\text{Sq}^1(A^2) \subseteq A^3$, then there exists a 3-admissible subring $\overline{A}$ of $H^*(M; \mathbb{Z}/2)$ satisfying $A \subseteq \overline{A}$ and $A^i = \overline{A}^i$ for $i = 0, 1, 2, 3$.

(p3) If the homology group $H_2(M; \mathbb{Z})$ has no 2-torsion, then $A$ is 3-admissible and the subgroup $A^3$ of $H^3(M; \mathbb{Z}/2)$ is generated by regular cohomology classes.
Proof By Künneth’s theorem, each subring of $H^*(M; \mathbb{Z}/2)$ that is generated by two full subrings is also full.

The admissible subring $A$ can be written as $A = B[T]$, where $B$ is a full subring and $T$ is a regular subset of $H^*(M; \mathbb{Z}/2)$. Let $T^i := T \cap H^i(M; \mathbb{Z}/2)$ for $i \geq 0$. One has $A^0 = B^0 = H^0(M; \mathbb{Z}/2)$, the manifold $M$ being connected, and hence, it can be assumed that $T^0 = \emptyset$.

For each cohomology class $u$ in $H^1(M; \mathbb{Z}/2)$, let $\gamma_u$ be a real line bundle on $M$ with $w_1(\gamma_u) = u$. Let $F_1 := \{\gamma_u | u \in T^1\}$. The subring $B(F_1)$ of $H^*(M; \mathbb{Z}/2)$ generated by $B$ and $F_1(M)$ is full. Property $(p_1)$ follows since $A = B(F_1)[T \setminus T^1]$.

According to Wu’s formula [52, p. 94], for each real vector bundle $\xi$ on $M$,

$$\text{Sq}^1(w_2(\xi)) = w_1(\xi) \cup w_2(\xi) + w_3(\xi).$$  

For each cohomology class $v$ in $W^2(M)$, let $\xi_v$ be a real vector bundle on $M$ with $w_1(\xi_v) = 0$ and $w_2(\xi_v) = v$. The admissibility of $A$ implies that $A^2$ is contained in $A^2(M) = W^2(M)$. In particular, the set $F_2 := \{\xi_v | v \in T^2\}$ is well defined. The subring $B(F_1, F_2)$ of $H^*(M; \mathbb{Z}/2)$ generated by $B$ and $(F_1 \cup F_2)(M)$ is full, and the subring $\overline{A} := B(F_1, F_2)[T \setminus (T^1 \cup T^2)]$ is 3-admissible. Moreover, $A \subseteq \overline{A}$ and $A^i = \overline{A}^i$ for $i = 0, 1, 2$. If $\text{Sq}^1(A^2) \subseteq A^3$, then $(*)$ with $\xi = \xi_v$ implies that $w_3(\xi_v) = \text{Sq}^1(v)$ is in $A^3$ for $v$ in $T^2$. Consequently, $A^3 = \overline{A^3}$. Property $(p_2)$ is proved.

Suppose now that the homology group $H_2(M; \mathbb{Z})$ has no 2-torsion. According to the universal coefficient theorem, the cohomology group $H^3(M; \mathbb{Z})$ has no 2-torsion and the reduction modulo 2 homomorphism $\rho : H^2(M; \mathbb{Z}) \to H^2(M; \mathbb{Z}/2)$ is surjective. For each cohomology class $z$ in $H^2(M; \mathbb{Z})$, let $\lambda_z$ be a complex line bundle on $M$ whose first Chern class is $z$. Regarding $\lambda_z$ as a rank 2 real vector bundle, one gets $w_1(\lambda_z) = 0$ and $w_2(\lambda_z) = \rho(z)$. Consequently, $W^2(M) = H^2(M; \mathbb{Z}/2)$, and it can be assumed that for each $v$ in $H^2(M; \mathbb{Z}/2)$, the vector bundle $\xi_v$ above is of rank 2. In particular, $w_j(\xi_v) = 0$ for $j \geq 3$. It follows that then $A$ is equal to the subring $\overline{A}$ constructed above, and hence, $A$ is 3-admissible. It remains to prove that $A^3$ is generated by regular cohomology classes. Each cohomology class in $H^1(M, \mathbb{Z}/2)$ is regular. Similarly, each cohomology class $v$ in $H^2(M, \mathbb{Z}/2)$ is regular since it is Poncaré dual to the homology class represented by the zero locus of an arbitrary smooth section of $\xi_v$ that is transverse to the zero section (cf. [27, Proposition 2.15]). The homomorphism $\text{Sq}^1 : H^2(M; \mathbb{Z}/2) \to H^3(M; \mathbb{Z}/2)$ is zero, the homomorphism $\rho$ being surjective [52, p. 182], and hence, $(*)$ gives $w_3(\xi) = w_1(\xi) \cup w_2(\xi)$. The proof is complete since cup product of regular cohomology classes is a regular class. □

Convention Henceforth, smooth submanifolds are assumed to be closed subsets of the ambient manifold.

2 Proofs and further results

The language of real algebraic geometry, as in [16], is used throughout this section. The term real algebraic variety designates a locally ringed space isomorphic to an algebraic subset of $\mathbb{R}^n$, for some $n$, endowed with the Zariski topology and the sheaf
of real-valued regular functions (such objects are called affine real algebraic varieties in [16]). The Grassmannian $\mathbb{G}_{n,r}(\mathbb{R})$ of $r$-dimensional vector subspaces of $\mathbb{R}^n$ is a real algebraic variety in this sense [16, Theorem 3.4.4]. Morphisms between real algebraic varieties are called regular maps. Every real algebraic variety carries also the Euclidean topology, which is induced by the usual matric on $\mathbb{R}$. Unless explicitly stated otherwise, all topological notions relating to real algebraic varieties refer to the Euclidean topology.

A topological real vector bundle on a real algebraic variety $X$ is said to admit an algebraic structure if it is isomorphic to an algebraic subbundle of the trivial vector bundle on $X$ with total space $X \times \mathbb{R}^p$ for some $p$.

For any smooth manifolds $N$ and $P$, the space of smooth maps $C^\infty(N, P)$ is endowed with the $C^\infty$ topology [30]. The source manifold will always be assumed to be compact, and hence, the weak $C^\infty$ topology coincides with the strong one. The unoriented bordism group of $P$ is denoted by $\Omega_\ast(P)$. If $W$ is a nonsingular real algebraic variety, then a bordism class in $\Omega_\ast(W)$ is said to be algebraic, provided that it can be represented by a regular map from a compact nonsingular real algebraic variety into $W$. The set $\Omega_\ast^{\text{alg}}(W)$ of all algebraic bordism classes in $\Omega_\ast(W)$ forms a subgroup.

The main approximation theorem of real algebraic geometry, in the form most suitable for this paper, will be recalled first. It is just a reformulation of very similar results proved in [1,8,13,14,64].

**Theorem 2.1** (cf. [42, Theorem 4.4]) Let $M$ be a compact smooth submanifold of $\mathbb{R}^n$ and let $W$ be a nonsingular real algebraic variety. Let $f : M \to W$ be a smooth map whose bordism class in $\Omega_\ast(W)$ is algebraic. Suppose that $M$ contains a (possibly empty) subset $Z$ which is the union of finitely many nonsingular algebraic subsets of $\mathbb{R}^n$, $f|_Z : Z \to W$ is a regular map, and the restriction to $Z$ of the tangent bundle of $M$ admits an algebraic structure. If $2 \dim M + 1 \leq n$, then there exists a smooth embedding $e : M \to \mathbb{R}^n$, a nonsingular algebraic subset $X$ of $\mathbb{R}^n$, and a regular map $g : X \to W$ such that $X = e(M)$, $Z \subseteq X$, $e|_Z : Z \to \mathbb{R}^n$ is the inclusion map, $g|_Z = f|_Z$, and $g \circ \bar{e}$ is homotopic to $f$, where $\bar{e} : M \to X$ is the smooth diffeomorphism defined by $\bar{e}(x) = e(x)$ for all $x$ in $M$. Furthermore, given a neighborhood $U$ of the inclusion map $M \hookrightarrow \mathbb{R}^n$ in the space $C^\infty(M, \mathbb{R}^n)$ and a neighborhood $V$ of $f$ in $C^\infty(M, W)$, the objects $e$, $X$, $g$ can be chosen in such a way that $e$ is in $U$ and $g \circ \bar{e}$ is in $V$.

In favorable situations, the bordism condition in Theorem 2.1 is automatically satisfied.

**Proposition 2.2** Let $V$ and $W$ be compact nonsingular real algebraic varieties. Then:

(i) $\Omega_\ast^{\text{alg}}(V) = \Omega_\ast(V)$ if and only if $H_\ast^{\text{alg}}(V; \mathbb{Z}/2) = H_\ast(V; \mathbb{Z}/2)$.

(ii) The equality $H_\ast^{\text{alg}}(V \times W; \mathbb{Z}/2) = H_\ast(V \times W; \mathbb{Z})$ holds, provided that $H_\ast^{\text{alg}}(V; \mathbb{Z}/2) = H_\ast(V; \mathbb{Z}/2)$ and $H_\ast^{\text{alg}}(W; \mathbb{Z}/2) = H_\ast(W; \mathbb{Z}/2)$.

Moreover, $H_\ast^{\text{alg}}(\mathbb{G}_{n,r}(\mathbb{R}); \mathbb{Z}/2) = H_\ast(\mathbb{G}_{n,r}(\mathbb{R}); \mathbb{Z}/2)$.

**Proof** Condition (i) is a consequence of deep results from topology (cf. [8, Lemma 2.7.1]). Condition (ii) follows from Künneth’s theorem. The last assertion is a standard fact (cf. [16, Proposition 11.3.3]).
The result that will be recalled next is used in constructions of nonalgebraic cohomology classes. For any compact nonsingular real algebraic variety $X$, let $\text{Alg}^k(X)$ denote the set of all elements $u$ in $H^k(X; \mathbb{Z}/2)$ for which there exist a compact nonsingular irreducible real algebraic variety $T$, two points $t_0$ and $t_1$ in $T$ and the cohomology class $z$ in $H^k_{\text{alg}}(X \times T; \mathbb{Z}/2)$ such that

$$u = i^n_t(z) - i^n_{t_0}(z),$$

where $i_t: X \to X \times T$ is defined by $i_t(x) = (x, t)$ for $t \in T$ and $x \in X$. An equivalent description of $\text{Alg}^k(X)$, which immediately implies that $\text{Alg}^k(X)$ is a subgroup of $H^k_{\text{alg}}(X; \mathbb{Z}/2)$, is given in [38,40]. The groups $H^k_{\text{alg}}(-; \mathbb{Z}/2)$ and $\text{Alg}^k(-)$ have the expected functorial property. If $f: X \to Y$ is a regular map between compact nonsingular real algebraic varieties, then the induced homomorphism $f^*: H^*(Y; \mathbb{Z}/2) \to H^*(X; \mathbb{Z}/2)$ satisfies

$$f^*(H^k_{\text{alg}}(Y; \mathbb{Z}/2)) \subseteq H^k_{\text{alg}}(X; \mathbb{Z}/2) \quad \text{and} \quad f^*(\text{Alg}^k(Y)) \subseteq \text{Alg}^k(X)$$

(cf. [27, Section 5] or [4,15] for the former inclusion and [40] for the latter).

**Example 2.3** Let $\Sigma$ be an irreducible nonsingular real algebraic variety with precisely two connected components $\Sigma_0$ and $\Sigma_1$, each diffeomorphic to the unit circle. For example, one can take

$$\Sigma = \{(x_1, x_2) \in \mathbb{R}^2 \mid x_1^4 - 4x_1^2 + x_2^2 + 1 = 0\}.$$ 

Let $z$ be the cohomology class in $H^1(\Sigma \times \Sigma; \mathbb{Z}/2)$ that is Poincaré dual to the homology class in $H_1(\Sigma \times \Sigma; \mathbb{Z}/2)$ represented by the diagonal of $\Sigma \times \Sigma$. For any point $t$ in $\Sigma$, let $i_t: \Sigma \to \Sigma \times \Sigma$ be defined by $i_t(x) = (x, t)$ for all $x$ in $\Sigma$. The cohomology class $i^*_t(z)$ in $H^1(\Sigma; \mathbb{Z}/2)$ is Poincaré dual to the homology class in $H_1(\Sigma; \mathbb{Z}/2)$ represented by the point $t$. Let $t_j$ be a point in $\Sigma_j$ for $j = 0, 1$. The cohomology class $u := i^*_t(z) - i^*_{t_0}(z)$ is in $\text{Alg}^1(\Sigma)$. If $\sigma: \Sigma_0 \leftarrow \Sigma$ is the inclusion map, then $\sigma^*(u)$ generates $H^1(\Sigma_0; \mathbb{Z}/2) \cong \mathbb{Z}/2$ and hence

$$H^1(\Sigma_0; \mathbb{Z}/2) = \sigma^*(H^1(\Sigma; \mathbb{Z}/2)) = \sigma^*(\text{Alg}^1(\Sigma)).$$

Consequently, the functoriality of $\text{Alg}^1(-)$ implies that

$$r^*(H^1(\Sigma; \mathbb{Z}/2)) \subseteq \text{Alg}^1(Y)$$

for every nonsingular real algebraic variety $Y$ and every regular map $r: Y \rightarrow \Sigma$ with $r(Y) \subseteq \Sigma_0$.

As usual, the Kronecker index (scalar product) of cohomology and homology classes will be denoted by $\langle -, - \rangle$. For any $m$-dimensional compact smooth manifold $M$, let $[M]$ denote its fundamental class in $H_m(M; \mathbb{Z}/2)$. 
Proof. Let $U$ follow since the bordism classes of $f$, and let $V$ in the group $\mathcal{N}_s(V)$ is 0. Then, there exists a smooth embedding $\varepsilon: L \to \mathbb{R}^n$, a nonsingular algebraic subset $Y$ of $\mathbb{R}^n$, and a regular map $g: Y \to V$ such that $Y = \varepsilon(L)$, $\varepsilon$ is in $U$, $g \circ \bar{\varepsilon}$ is in $V$, and

$$H^k_{\text{alg}}(Y; \mathbb{Z}/2) \subseteq \{w \in H^k(Y; \mathbb{Z}/2) \mid \langle w, \bar{\varepsilon}_*(\langle K \rangle_L) \rangle = 0\},$$

where $\bar{\varepsilon}: L \to Y$ is the smooth diffeomorphism determined by $\varepsilon$.

Lemma 2.5. Let $L$ be a $(k+1)$-dimensional compact smooth submanifold of $\mathbb{R}^n$ and let $K$ be a $k$-dimensional smooth submanifold of $L$ such that there is a smooth diffeomorphism $\theta: K \times S^1 \to L$ satisfying $\theta(K \times \{z_0\}) = K$ for some point $z_0$ in $S^1$. Let $f: L \to V$ be a smooth map into a nonsingular real algebraic variety $V$. Let $U$ be a neighborhood of the inclusion map $L \hookrightarrow \mathbb{R}^n$ in the space $C^\infty(L, \mathbb{R}^n)$ and let $V$ be a neighborhood of $f$ in $C^\infty(L, V)$. Assume that $2k + 3 \leq n$, the map $f \circ \theta: K \times S^1 \to V$ has a continuous extension $K \times D^1 \to V$, and the bordism class of the map $f|_K: K \to V$ in the group $\mathcal{N}_s(V)$ is 0. Then, there exists a smooth embedding $\varepsilon: L \to \mathbb{R}^n$, a nonsingular algebraic subset $Y$ of $\mathbb{R}^n$, and a regular map $g: Y \to V$ such that $Y = \varepsilon(L)$, $\varepsilon$ is in $U$, $g \circ \bar{\varepsilon}$ is in $V$, and

$$H^k_{\text{alg}}(Y; \mathbb{Z}/2) \subseteq \{w \in H^k(Y; \mathbb{Z}/2) \mid \langle w, \bar{\varepsilon}_*(\langle K \rangle_L) \rangle = 0\},$$

where $\bar{\varepsilon}: L \to Y$ is the smooth diffeomorphism determined by $\varepsilon$.

Proof. Let $\Sigma$ be as in Example 2.3, and let $h_0: S^1 \to \Sigma$ be a smooth embedding onto $\Sigma_0$. If $f_0: K \to V$ is defined by $f_0(x) = f(\theta(x, z_0))$ for all $x$ in $K$, then the bordism class of $f_0 \times h_0: K \times S^1 \to V \times \Sigma$ in the group $\mathcal{N}_s(V \times \Sigma)$ is 0. Indeed, this assertion follows since the bordism classes of $f_0: K \to V$ and $f|_K: K \to V$ in $\mathcal{N}_s(V)$ are equal, and the latter class is 0 by assumption.

If $F: K \times D^1 \to V$ is a continuous extension of $f \circ \theta: K \times S^1 \to V$, then the map $H: K \times S^1 \times [0, 1] \to V$,

$$H(x, z, t) = F(x, (1-t)z + tz_0)$$

for $(x, z, t)$ in $K \times S^1 \times [0, 1]$,

is a homotopy from $f \circ \theta$ to $f_0 \circ \pi$, where $\pi: K \times S^1 \to K$ is the canonical projection. Hence, if $\rho: K \times S^1 \to S^1$ is the canonical projection and $h := h_0 \circ \rho \circ \theta^{-1}$, the map

$$(f, h) \circ \theta = (f \circ \theta, h \circ \theta): K \times S^1 \to V \times \Sigma$$

is homotopic to

$$(f_0 \circ \pi, h_0 \circ \rho) = f_0 \times h_0: K \times S^1 \to V \times \Sigma.$$
By Theorem 2.1 (with \( M = L, Z = \emptyset, \) and \( W = V \times \Sigma \)), there exist a smooth embedding \( \varepsilon: L \to \mathbb{R}^n \), a nonsingular algebraic subset \( Y \) of \( \mathbb{R}^n \), and a regular map \( (h, r): Y \to V \times \Sigma \) such that \( Y = \varepsilon(L) \), \( \varepsilon \) is in \( \mathcal{U} \), and \( (g, r) \circ \varepsilon \) is close to \( (f, h) \) in \( C^\infty(L, V \times \Sigma) \), where \( \varepsilon: L \to Y \) is the smooth diffeomorphism determined by \( \varepsilon \). In particular, \( g \circ \varepsilon \) is in \( \mathcal{V} \), and \( r \) is homotopic to \( h \circ \varepsilon^{-1} \). The proof can be completed as follows. Let \( v \) be the cohomology class in \( H^1(\Sigma; \mathbb{Z}/2) \) that is Poincaré dual to the homology class represented by the point \( y_0 := h_0(z_0) \). Since \( y_0 \) is a regular value of \( h \circ \varepsilon^{-1} \) and \( \varepsilon(K) = (h \circ \varepsilon^{-1})^{-1}(y_0) \), it follows that the cohomology class \( (h \circ \varepsilon^{-1})^*(v) \) is Poincaré dual to the homology class \( [\varepsilon(K)]_Y = \varepsilon_*([K]_L) \) (cf. [27, Proposition 2.15]). Consequently, \( r^*(v) \) is Poincaré dual to \( \varepsilon_*([K]_L) \), the maps \( h \circ \varepsilon^{-1} \) and \( r \) being homotopic. Thus, \( r^*(v) \cap [Y] = \varepsilon_*([K]_L) \) and hence for every cohomology class \( w \) in \( H^1(Y; \mathbb{Z}/2) \),

\[
\langle w, \varepsilon_*([K]_L) \rangle = \langle w, r^*(v) \cap [Y] \rangle = \langle w \cup r^*(v), [Y] \rangle.
\]

Since \( r \) is a regular map and \( r(Y) \subseteq \Sigma_0 \), Example 2.3 implies that \( r^*(v) \) is in \( \text{Alg}^1(Y) \). Hence, according to Theorem 2.4,

\[
H^k_{\text{alg}}(Y; \mathbb{Z}/2) \subseteq \{ w \in H^k(Y; \mathbb{Z}/2) \mid \langle w, \varepsilon_*([K]_L) \rangle = 0 \},
\]

as required. \( \square \)

The ability to verify the bordism hypothesis in Lemma 2.5 is essential for applications. This often requires the following deep result from differential topology.

**Theorem 2.6** (cf. [29, (17.3)]) Let \( f: K \to P \) be a smooth map between compact smooth manifolds. The bordism class of \( f \) in the group \( \mathfrak{N}_q(P) \) is 0 if and only if for every nonnegative integer \( q \) and every cohomology class \( u \) in \( H^q(P; \mathbb{Z}/2) \),

\[
\langle w_{i_1}(K) \cup \cdots \cup w_{i_p}(K) \cup f^*(u), [K] \rangle = 0
\]

for all nonnegative integers \( i_1, \ldots, i_p \) with \( i_1 + \cdots + i_p = k - q \), where \( k = \dim K \).

Henceforth, the following notion will play a crucial role.

**Definition 2.7** Given a compact smooth manifold \( M \) and a subring \( A \) of \( H^* (M; \mathbb{Z}/2) \), a smooth submanifold \( K \) of \( M \) is said to be adapted to \( A \) if for every nonnegative integer \( q \) and every cohomology class \( u \) in \( A^q \),

\[
\langle w_{i_1}(K) \cup \cdots \cup w_{i_p}(K) \cup k^*(u), [K] \rangle = 0
\]

for all nonnegative integers \( i_1, \ldots, i_p \) with \( i_1 + \cdots + i_p = k - q \), where \( k = \dim K \) and \( k: K \hookrightarrow M \) is the inclusion map.

For any smooth manifold \( N \), let \( \tau_N \) denote its tangent bundle.
Lemma 2.8 Let $M$ be a compact smooth manifold and let $K$ be a connected smooth submanifold of $M$ of positive dimension $k$, with $2k + 1 \leq \dim M$. If $K$ is adapted to a subring $A$ of $H^*(M; \mathbb{Z}/2)$ containing the Stiefel–Whitney classes $w_i(M)$ for $0 \leq i \leq k$, then the normal bundle of $K$ in $M$ splits off a trivial vector bundle of rank 2.

Proof If $2k + 2 \leq \dim M$, then the assertion is true without any additional assumptions on $K$.

Suppose now that $2k + 1 = \dim M$. It suffices to prove that the normal bundle $v$ of $K$ in $M$ has two continuous sections that are linearly independent at each point of $K$.

Since $\text{rank } v = k + 1$ and $\dim K = k$, the only obstruction to the existence of such sections is an element $W_k(v)$ in the cohomology group $H^k(X; \Gamma)$, where $\Gamma$ is a local system of coefficients with fiber $\mathbb{Z}$ or $\mathbb{Z}/2$ (cf. [52, p. 140] and [60, pp. 190, 191]).

If $k$ is even, then $\Gamma$ is isomorphic to the constant local system $\mathbb{Z}/2$, and $W_k(v)$ can be identified with $w_k(v)$ (cf. [52, p. 143]).

If $k$ is odd, then the local system $\Gamma$ has fiber $\mathbb{Z}$. The group $H^k(K; \Gamma)$ is isomorphic either to $\mathbb{Z}$ or $\mathbb{Z}/2$. Indeed, the Poincaré duality gives an isomorphism between $H^k(K; \Gamma)$ and the 0th homology group of $K$ with a suitable local system of coefficients with fiber $\mathbb{Z}$. The 0th homology group of $K$ with an arbitrary local system of coefficients with fiber $\mathbb{Z}$ is isomorphic either to $\mathbb{Z}$ or $\mathbb{Z}/2$. If the group $H^k(K; \Gamma)$ is infinite cyclic, then $W_k(v) = 0$ since $W_k(v)$ is an element of order 2 (cf. [60, Theorem 38.11]). If $H^k(K; \Gamma)$ is isomorphic to $\mathbb{Z}/2$, then the reduction modulo 2 homomorphism $\rho : H^k(K; \Gamma) \to H^k(K; \mathbb{Z}/2)$ is an isomorphism. According to [52, Theorem 12.1], $\rho(W_k(v)) = w_k(v)$.

In conclusion, $W_k(v) = 0$ if $w_k(v) = 0$. It remains to prove the equality $w_k(v) = 0$. If $\kappa : K \hookrightarrow M$ is the inclusion map, then the vector bundles $\tau_K \oplus v$ and $\kappa^*(\tau_M)$ are isomorphic, and hence, one gets $w(K) \cup w(v) = \kappa^*(w(M))$ for the total Stiefel–Whitney classes. The last equality implies that $w_k(v)$ belongs to the subring of $H^*(K; \mathbb{Z}/2)$ generated by $w_i(K)$ and $\kappa^*(w_i(M))$ for $0 \leq i \leq k$. Consequently, $\langle w_k(v), [K] \rangle = 0$ since $K$ is adapted to $A$ and $w_i(M)$ is in $A^i$ for $0 \leq i \leq k$. Thus, $w_k(v) = 0$, the manifold $K$ being connected.

The next lemma is included for the sake of completeness. If $M$ is a compact smooth manifold and $N$ is a smooth submanifold of $M$ of codimension $k$, let $[N]^M$ denote the cohomology class in $H^k(M; \mathbb{Z}/2)$ that is Poincaré dual to the homology class $[N]_M$ represented by $N$. That is, $[N]^M \cap [M] = [N]_M$, where $\cap$ denotes the cap product.

Lemma 2.9 Let $M$ be a compact smooth manifold of dimension $m$. Let $K_1, \ldots, K_p$ be pairwise disjoint connected smooth submanifolds of $M$ of dimension $k$, where $1 \leq k \leq m - 1$. Let $N$ be a smooth submanifold of $M$ of codimension $k$. If

$$\langle [N]^M, [K_l]_M \rangle = 0 \text{ for } 1 \leq l \leq p,$$

then there exists a smooth submanifold $N'$ of $M$ of codimension $k$ such that $[N']^M = [N]^M$ and $K_l \cap N' = \emptyset$ for $1 \leq l \leq p$.

Proof Arguing by induction on the number of submanifolds $K_l$ suppose that $j$ is an integer satisfying $0 \leq j \leq p - 1$, and $N_j$ is a smooth submanifold of $M$ with
Let $M$ be a compact connected smooth manifold and let $r$ be a positive integer with $2r + 1 \leq \dim M$. Let $A$ be an $r$-admissible subring of $H^*(M; \mathbb{Z}/2)$ and let $\Delta$ be the set of all integers $k$ such that $1 \leq k \leq r$ and the group $A_k$ is generated by homology classes of the form $[K]_M$, where $K$ is a $k$-dimensional connected smooth submanifold of $M$ adapted to $A$. If $w_i(M)$ is in $A^i$ for $0 \leq i \leq r$, then there exists an algebraic model $(X, \varphi)$ of $M$ satisfying

$$\varphi^*(A) \subseteq H^*_\text{alg}(X; \mathbb{Z}/2) \quad \text{and} \quad \varphi^*(A^k) = H^k_\text{alg}(X; \mathbb{Z}/2) \quad \text{for } k \text{ in } \{0\} \cup \Delta.$$  

**Proof** The subring $A$ can be written as $A = B[T]$, where $B$ is a full subring and $T$ is a regular subset of $H^*(M; \mathbb{Z}/2)$, with $T$ disjoint from $H^c(M; \mathbb{Z}/2)$ for $0 \leq c \leq r - 1$. By definition,

$$B = h^*(H^*(W; \mathbb{Z}/2)), \quad (1)$$  

where $h : M \to W$ is a smooth map into a compact smooth manifold $W$ with $A(W) = H^*(W; \mathbb{Z}/2)$. In view of the last equality, the whole ring $H^*(W; \mathbb{Z}/2)$ is algebraically realizable (cf. Sect. 1), and hence, it can be assumed that $W$ is a nonsingular real algebraic variety satisfying

$$H^*_\text{alg}(W; \mathbb{Z}/2) = H^*(W; \mathbb{Z}/2). \quad (2)$$  

Let $m := \dim M$.

For any subring $A$ of $H^*(M; \mathbb{Z}/2)$, let

$$A_k := \{\alpha \in H_k(M; \mathbb{Z}/2) \mid \langle u, \alpha \rangle = 0 \quad \text{for all } u \in A^k\}.$$


$\tau_M$ is isomorphic to the pullback $g^*\gamma$ of the universal vector bundle $\gamma$ on $G$. Hence, the subring $g^*(H^*(H; \mathbb{Z}/2))$ of $H^*(M; \mathbb{Z}/2)$ is generated by $w_i(M)$ for $i \geq 0$. The smooth map $(g, h): M \to G \times W$ plays a crucial role. Set

$$D := (g, h)^*(H^*(G \times W; \mathbb{Z}/2))$$

and $C := D[T]$. (3)

Since $w_i(M)$ is in $A^i$ for $0 \leq i \leq r$, by (1) and Künneth’s theorem, the subring $C$ of $H^*(M; \mathbb{Z}/2)$ satisfies

$$A \subseteq C \quad \text{and} \quad A^i = C^i, \quad A_i = C_i \quad \text{for} \ 0 \leq i \leq r. \quad (4)$$

By (2) and Proposition 2.2,

$$\mathcal{M}^\text{alg}_*(W) = \mathcal{M}_*(W) \quad \text{and} \quad \mathcal{M}^\text{alg}_*(G \times W) = \mathcal{M}_*(G \times W). \quad (5)$$

In view of (4), if $p$ is a sufficiently large integer, then for each integer $k$ in $\Delta$, there exist $k$-dimensional connected smooth submanifolds $K_{k1}, \ldots, K_{kp}$ of $M$ such that

- each $K_{kl}$ is adapted to $C$,
- $[K_{k1}]_M, \ldots, [K_{kp}]_M$ generate $C_k = A_k$. (6)

By (6) and Lemma 2.8,

the normal bundle of each $K_{kl}$ in $M$

splits off a trivial vector bundle of rank 2. (8)

If $\kappa_{kl}: K_{kl} \hookrightarrow M$ is the inclusion map, the restriction map $(g, h)|_{K_{kl}}: K_{kl} \to G \times W$ can be written as $(g, h)|_{K_{kl}} = (g, h) \circ \kappa_{kl}$, and hence

$$(g, h)|_{K_{kl}}^*(H^*(G \times W; \mathbb{Z}/2)) = \kappa_{kl}^*((g, h)^*(H^*(G \times W; \mathbb{Z}/2))) \subseteq \kappa_{kl}^*(C),$$

where the inclusion follows from (3). Consequently, by (6) and Theorem 2.6,

the bordism class of $(g, h)|_{K_{kl}}: K_{kl} \to G \times W$ in $\mathcal{M}_*(G \times W)$ is 0. (9)

Let $N_1, \ldots, N_q$ be smooth submanifolds of $M$ such that

$$T = \{[N_1]^M, \ldots, [N_q]^M\} \quad \text{and} \quad \text{codim}_M N_j \geq r \quad \text{for} \ 1 \leq j \leq q, \quad (10)$$

and let

$$N := N_1 \cup \cdots \cup N_q.$$.
The collection of smooth submanifolds of $M$ consisting of all $K_{kl}$ and all $N_j$ can be assumed to be in general position. In particular, the $K_{kl}$ are pairwise disjoint since $2r + 1 \leq m$. Similarly, each $K_{kl}$ with $1 \leq k \leq r - 1$ is disjoint from $N$ since $\text{codim}_M N_j \geq r$ for $1 \leq j \leq q$. Moreover, according to Lemma 2.9, the $N_j$ can be chosen in such a way that $K_{kl} \cap N = \emptyset$ for $k \in \Delta$ and $1 \leq l \leq p$.

One can assume that $M$ is smoothly embedded in $\mathbb{R}^n$ for some $n \geq 2m + 1$. Since (5) holds, according to [13, Theorem 4, Remark 8], it can be assumed that

$$M \text{ is a nonsingular algebraic subset of } \mathbb{R}^n,$$

(N1, ..., Nq are nonsingular algebraic subsets of $M$,

$$(g, h): M \to G \times W \text{ is a regular map}.$$  

Let $U_{kl}$ be a tubular neighborhood of $K_{kl}$ in $M$. The $U_{kl}$ can be chosen to be pairwise disjoint and disjoint from $N$. In view of (8), one can find a smooth embedding $\eta_{kl}: K_{kl} \times S^1 \to U_{kl}$ such that if $L_{kl} := \eta_{kl}(K_{kl} \times S^1)$ and $\theta_{kl}: K_{kl} \times S^1 \to L_{kl}$ is the smooth diffeomorphism determined by $\eta_{kl}$, then $K_{kl} = \theta_{kl}(K_{kl} \times \{z_0\})$ for some point $z_0$ in $S^1$. The smooth map $((g, h)|_{L_{kl}}) \circ \theta_{kl}: K_{kl} \times S^1 \to G \times W$ is a restriction of the smooth map $(g, h) \circ \eta_{kl}: K_{kl} \times S^1 \to G \times W$. Hence, by (9) and Lemma 2.5 (with $K = K_{kl}$, $L = L_{kl}$ and $f = (g, h)|_{L_{kl}}$), there exist a smooth embedding $\varepsilon_{kl}: L_{kl} \to \mathbb{R}^n$, a nonsingular algebraic subset $Y_{kl}$ of $\mathbb{R}^n$, and a regular map $(g_{kl}, h_{kl}): Y_{kl} \to G \times W$ such that $Y_{kl} = \varepsilon_{kl}(L_{kl})$, $\varepsilon_{kl}$ is close to the inclusion map $L_{kl} \hookrightarrow \mathbb{R}^n$ in the space $C^\infty(L_{kl}, \mathbb{R}^n)$, $(g_{kl}, h_{kl}) \circ \varepsilon_{kl}$ is close to $(g, h)|_{L_{kl}}$ in $C^\infty(L_{kl}, G \times W)$, and

$$H^k_{\text{alg}}(Y_{kl}; \mathbb{Z}/2) \subseteq \{w \in H^k(Y_{kl}; \mathbb{Z}/2)|\langle w, \varepsilon_{kl*}([K_{kl}]_{L_{kl}})\rangle = 0\},$$

where $\varepsilon_{kl}: L_{kl} \to Y_{kl}$ is the smooth diffeomorphism determined by $\varepsilon_{kl}$. If each $(g_{kl}, h_{kl}) \circ \varepsilon_{kl}$ is sufficiently close to $(g, h)|_{L_{kl}}$, then one can find a smooth map $(g', h'): M \to G \times W$ that is homotopic to $(g, h): M \to G \times W$ and satisfies

$$(g', h')|_N = (g, h)|_N \quad \text{and} \quad (g', h')|_{L_{kl}} = (g_{kl}, h_{kl}) \circ \varepsilon_{kl} \text{ for } k \in \Delta \quad \text{and} \quad 1 \leq l \leq p.$$  

If each $\varepsilon_{kl}$ is sufficiently close to the inclusion map $L_{kl} \hookrightarrow \mathbb{R}^n$, then

the $Y_{kl}$ are pairwise disjoint and disjoint from $N$,  

and hence, there exists a smooth embedding $\varepsilon: M \to \mathbb{R}^n$ for which $\varepsilon|_{L_{kl}} = \varepsilon_{kl}$ and $\varepsilon|_N$ is the inclusion map $N \hookrightarrow \mathbb{R}^n$. Let $\overline{M} := \varepsilon(M)$ and let $\overline{\varepsilon}: M \to \overline{M}$ be the smooth diffeomorphism determined by $\varepsilon$. The smooth map $((\overline{g}, \overline{h}) := (g', h') \circ \overline{\varepsilon}^{-1}: \overline{M} \to G \times W$ satisfies $(\overline{g}, \overline{h})|_{Y_{kl}} = (g_{kl}, h_{kl})$ and $(\overline{g}, \overline{h})|_N = (g', h')|_N$. Moreover, the algebraic subset

$$Z := N \cup \bigcup_{k,l} Y_{kl}$$
of \( \mathbb{R}^n \) is contained in \( \overline{M} \), and by (12), (13), (15) and (16),

\[
(g, h)|_Z : Z \to G \times W \text{ is a regular map.} \tag{17}
\]

Since \( g : M \to G \) is a classifying map for \( \tau_M \) and \( g' : M \to G \) is homotopic to \( g \), it follows that \( \overline{g} = g \circ \overline{\epsilon}^{-1} : \overline{M} \to G \) is a classifying map for \( \tau_{\overline{M}} \). Consequently, the regular map \( \overline{g}|_Z : Z \to G \) is a classifying map for \( \tau_{\overline{M}}|_Z \) and hence

\[
\tau_{\overline{M}}|_Z \text{ admits an algebraic structure} \tag{18}
\]

(cf. [16, Theorem 12.1.7]). In view of (5), (17) and (18), Theorem 2.1 can be applied to \( \overline{h} : \overline{M} \to W \) and \( Z \subseteq \overline{M} \). Therefore, there exist a smooth embedding \( e : \overline{M} \to \mathbb{R}^n \), a nonsingular algebraic subset \( X \) of \( \mathbb{R}^n \) and a regular map \( \lambda : X \to W \) such that \( X = e(\overline{M}) \), \( Z \subseteq X \), \( e(x) = x \) for all \( x \) in \( Z \), and \( \lambda \circ \overline{e} \) is homotopic to \( \overline{h} \), where \( \overline{e} : \overline{M} \to X \) is the smooth diffeomorphism determined by \( e \). The map \( \varphi := \overline{e}^{-1} \circ \overline{e}^{-1} : X \to M \) is a smooth diffeomorphism, and hence, \( (X, \varphi) \) is an algebraic model of \( M \).

By construction, \( \lambda \) is homotopic to \( \overline{h} \circ \overline{e}^{-1} = h' \circ \varphi \) and \( h' \) is homotopic to \( h \). Consequently, \( \lambda \) is homotopic to \( h \circ \varphi \), and hence,

\[
\lambda^*(H^*(W; \mathbb{Z}/2)) = \varphi^*(h^*(H^*(W; \mathbb{Z}/2))) = \varphi^*(B),
\]

where the last equality follows from (1). This implies that

\[
\varphi^*(B) \subseteq H^*_{alg}(X; \mathbb{Z}/2)
\]

since (2) holds and \( \lambda : X \to W \) is a regular map. The diffeomorphism \( \varphi : X \to M \) satisfies \( \varphi(x) = x \) for all \( x \) in \( N \), which gives \( \varphi^*([N_j]^M) = [N_j]^X \) for \( 1 \leq j \leq q \). Thus, \( \varphi^*([N_j]^M) \) belongs to \( H^*_alg(X; \mathbb{Z}/2) \), each \( N_j \) being an algebraic subset of \( X \). By (10),

\[
\varphi^*(T) \subseteq H^*_{alg}(X; \mathbb{Z}/2).
\]

The last two inclusions imply that

\[
\varphi^*(A) \subseteq H^*_{alg}(X; \mathbb{Z}/2).
\]

Since \( M \) is connected, one has \( A^0 = H^0(M; \mathbb{Z}/2) \) and

\[
\varphi^*(A^0) = H^0(X; \mathbb{Z}/2) = H^0_{alg}(X; \mathbb{Z}/2).
\]

It remains to prove that if \( u \) is a cohomology class in \( H^k(M; \mathbb{Z}/2) \setminus A^k \) for some \( k \in \Delta \), then \( \varphi^*(u) \) is not in \( H^k_{alg}(X; \mathbb{Z}/2) \). Let \( \delta_{kl} : Y_{kl} \hookrightarrow X \) be the inclusion map. The composite map

\[
\varphi \circ \delta_{kl} \circ \overline{\epsilon}_{kl} = \overline{\epsilon}^{-1} \circ \overline{e}^{-1} \circ \delta_{kl} \circ \overline{\epsilon}_{kl} : L_{kl} \to M
\]
is the inclusion map \( L_{kl} \hookrightarrow M \), and hence,

\[
\langle \delta_{kl}^*(\varphi^*(u)), \overline{\varepsilon}_{kl}*([K_{kl}]_{L_{kl}}) \rangle = \langle u, (\varphi \circ \delta_{kl} \circ \overline{\varepsilon}_{kl})*([K_{kl}]_{L_{kl}}) \rangle = \langle u, [K_{kl}]_M \rangle.
\]

Since \( u \) is not in \( A^k \), condition (7) implies the existence of \( l \) with \( \langle u, [K_{kl}]_M \rangle \neq 0 \). For this \( l \), according to (14), \( \delta_{kl}^*(\varphi^*(u)) \) is not in \( H^k_{\text{alg}}(Y_{kl}; \mathbb{Z}/2) \). Consequently, \( \varphi^*(u) \) is not in \( H^k_{\text{alg}}(X; \mathbb{Z}/2) \), the map \( \delta_{kl} \) being regular. The proof is complete. \( \square \)

Recall that a compact smooth manifold is said to be a boundary if it is diffeomorphic to the boundary of a compact smooth manifold with boundary.

Let \( M \) be a compact connected smooth manifold and let \( K \) be a smooth submanifold of \( M \). If \( K \) is adapted to a subring of \( H^*(M; \mathbb{Z}/2) \), then the Stiefel–Whitney numbers of \( K \) are all 0, and hence, \( K \) is a boundary [62]. Conversely, if \( K \) is a boundary, then its Stiefel–Whitney numbers are all 0, and hence, \( K \) is adapted to the subring \( H^0(M; \mathbb{Z}/2) \) of \( H^*(M; \mathbb{Z}/2) \). The last observation can be generalized. This is done in the following two lemmas, in which notation \( M \) and \( K \) is preserved, and \( k := \dim K \) is assumed to be positive. Moreover, \( A \) denotes a subring of \( H^*(M; \mathbb{Z}/2) \).

**Lemma 2.11** Assume that the submanifold \( K \) is a boundary and the cohomology class \([K]_M\) belongs to \( A_k \). Then, \( K \) is adapted to \( A \) if one of the following two conditions is satisfied:

1. \((c_1)\) \( 1 \leq k \leq 2 \).
2. \((c_2)\) \( k \geq 3 \), \( \text{Sq}^1(A^{k-1}) \subseteq A^k \), and \( A^i = 0 \) for \( 1 \leq i \leq k - 2 \).

**Proof** By Wu’s theorem [52, Theorem 11.14], the first Wu class of \( K \) is equal to \( w_1(K) \). In particular,

\[
\text{Sq}^1(a) = w_1(K) \cup a \quad \text{for all } a \text{ in } H^{k-1}(K; \mathbb{Z}/2). \tag{1}
\]

Let \( \kappa : K \hookrightarrow M \) be the inclusion map. Since \([K]_M\) is in \( A_k \), for every cohomology class \( z \) in \( A^k \),

\[
\langle \kappa^*(z), [K] \rangle = \langle z, \kappa_*([K]) \rangle = \langle z, [K]_M \rangle = 0. \tag{2}
\]

According to (1), for every cohomology class \( v \) in \( A^{k-1} \),

\[
w_1(K) \cup \kappa^*(v) = \text{Sq}^1(\kappa^*(v)) = \kappa^*(\text{Sq}^1(v)).
\]

Therefore, the inclusion \( \text{Sq}^1(A^{k-1}) \subseteq A^k \) (which is automatically satisfied if \( 1 \leq k \leq 2 \)) and (2) give

\[
\langle w_1(K) \cup \kappa^*(v), [K] \rangle = 0. \tag{3}
\]

Since \( K \) is a boundary, the Stiefel–Whitney numbers of \( K \) are all 0. Consequently, in view of (2) and (3), the submanifold \( K \) is adapted to \( A \), provided that either \((c_1)\) or \((c_2)\) is satisfied. \( \square \)
Lemma 2.12 Assume that the submanifold $K$ is a boundary and the homology class \([K]_M\) belongs to $A_k$. Moreover, assume that $K$ is orientable. Then, $K$ is adapted to $A$ if one of the following two conditions is satisfied:

\begin{align*}
(c_1) & \quad 1 \leq k \leq 4, \\
(c_2) & \quad k \geq 5, \quad \text{Sq}^2(A^{k-2}) \subseteq A^k, \quad \text{Sq}^2(\text{Sq}^1(A^{k-3})) \subseteq A^k, \quad \text{and} \quad A^i = 0 \quad \text{for} \quad 1 \leq i \leq k - 4.
\end{align*}

Proof In view of Lemma 2.11, it can be assumed that $k \geq 3$. Let $v_j(K)$ denote the $j$th Wu class of $K$. The orientability of $K$ implies that $w_1(K) = 0$, and hence by Wu’s theorem [52, Theorem 11.14], $v_1(K) = 0$ and $v_2(K) = w_2(K)$. In particular,

\begin{align*}
\text{Sq}^1(a) &= v_1(K) \cup a = 0 \quad \text{for all} \quad a \in H^{k-1}(K; \mathbb{Z}/2), \quad (2) \\
\text{Sq}^2(b) &= v_2(K) \cup b = w_2(K) \cup b \quad \text{for all} \quad b \in H^{k-2}(K; \mathbb{Z}/2). \quad (3)
\end{align*}

Let $\kappa : K \hookrightarrow M$ be the inclusion map. Since $[K]_M$ is in $A_k$, for every cohomology class $z$ in $A^k$,

$$\langle \kappa^*(z), [K] \rangle = \langle z, \kappa_*([K]) \rangle = \langle z, [K]_M \rangle = 0. \quad (4)$$

According to (3), for every cohomology class $v$ in $H^{k-2}(M; \mathbb{Z}/2)$,

$$w_2(K) \cup \kappa^*(v) = \text{Sq}^2(\kappa^*(v)) = \kappa^*(\text{Sq}^2(v)).$$

If $\text{Sq}^2(v)$ is in $A^k$ (which is automatically satisfied if $3 \leq k \leq 4$), then (4) gives

$$\langle w_2(K) \cup \kappa^*(v), [K] \rangle = 0. \quad (5)$$

According to (2), for every cohomology class $u$ in $H^{k-3}(M; \mathbb{Z}/2)$,

$$\text{Sq}^1(w_2(K) \cup \kappa^*(u)) = 0.$$

On the other hand,

$$\text{Sq}^1(w_2(K) \cup \kappa^*(u)) = \text{Sq}^1(w_2(K)) \cup \kappa^*(u) + w_2(K) \cup \kappa^*(\text{Sq}^1(u)).$$

Consequently,

$$\text{Sq}^1(w_2(K)) \cup \kappa^*(u) = w_2(K) \cup \kappa^*(\text{Sq}^1(u)).$$

By Wu’s formula [52, p. 94], $\text{Sq}^1(w_2(K)) = w_1(K) \cup w_2(K) + w_3(K)$, which in view of (1) gives $\text{Sq}^1(w_2(K)) = w_3(K)$. Thus,

$$w_3(K) \cup \kappa^*(u) = \text{Sq}^1(w_2(K)) \cup \kappa^*(u) = w_2(K) \cup \kappa^*(\text{Sq}^1(u)).$$
If $\text{Sq}^2(\text{Sq}^1(u))$ is in $A^k$ (which is automatically satisfied if $3 \leq k \leq 4$), then (5) gives

$$\langle w_3(K) \cup \kappa^*(u), [K] \rangle = 0. \quad (6)$$

Since $K$ is a boundary, the Stiefel–Whitney numbers of $K$ are all 0. Consequently, in view of (1), (4), (5) and (6), the submanifold $K$ is adapted to $A$, provided that either $(c_1)$ or $(c_2)$ is satisfied.

The assumption in Lemmas 2.11 and 2.12 that $K$ be a boundary is not a serious limitation for applications, as demonstrated below.

The following is a simple modification of a deep result of Thom [62, Théorème II.26].

**Lemma 2.13** (cf. [40, Lemma 4.7]) Let $M$ be a compact connected smooth manifold and let $k$ be a positive integer satisfying $2k \leq \dim M$. Then, each homology class in $H_k(M \mathbb{Z}/2)$ is of the form $[K]_M$, where $K$ is a $k$-dimensional connected smooth submanifold of $M$. Moreover, $K$ can be chosen in such a way that it is a boundary.

Under some additional assumptions, $K$ can be assumed to be orientable.

**Lemma 2.14** Let $M$ be a compact connected smooth manifold and let $k$ be a positive integer satisfying $2k + 1 \leq \dim M$. Assume that the homology group $H_{k-1}(M \mathbb{Z}/2)$ has no 2-torsion. Then, each homology class in $H_k(M \mathbb{Z}/2)$ is of the form $[K]_M$, where $K$ is a $k$-dimensional connected orientable smooth submanifold of $M$. Moreover, $K$ can be chosen in such a way that it is a boundary.

**Proof** By the universal coefficient theorem, the reduction modulo 2 homomorphism $\rho : H_k(M \mathbb{Z}) \to H_k(M \mathbb{Z}/2)$ is surjective. Hence, each homology class $\alpha$ in $H_k(M \mathbb{Z}/2)$ is of the form $\alpha = \rho(\beta)$ for some homology class $\beta$ in $H_k(M \mathbb{Z})$. According to [29, Corollary 15.3], one can find a $k$-dimensional oriented compact smooth manifold $N$, a smooth map $f : N \to M$ and an integer $r$ such that $f_*([\mu_N]) = (2r + 1)\beta$, where $\mu_N$ is the fundamental class of $N$ in $H_k(N \mathbb{Z})$. Since $2k + 1 \leq m := \dim M$, the map $f$ can be assumed to be a smooth embedding (cf. [30, Theorem 2.13]). Hence, $P := f(N)$ is an orientable smooth submanifold of $M$ with $[P]_M = \alpha$. By joining the connected components of $P$ with $k$-dimensional tubes in $M$, one obtains an orientable connected smooth submanifold $L$ of $M$ satisfying $[L]_M = \alpha$. Let $U$ be an open subset of $M \setminus L$, diffeomorphic to $\mathbb{R}^m$. Let $L'$ be a smooth submanifold of $U$, diffeomorphic to $L$. By joining $L$ and $L'$ with a $k$-dimensional tube in $M$, one gets an orientable connected smooth submanifold $K$ of $M$ satisfying $[K]_M = \alpha$. By construction, $K$ is a boundary. □

One more observation is required for the proofs of the main results.

**Lemma 2.15** Let $M$ be a compact smooth manifold and let $A$ be an $r$-admissible subring of $H^*(M \mathbb{Z}/2)$, where $r$ is a positive integer. Then, $\text{Sq}^i(A) \subseteq A^{i+j}$ for all nonnegative integers $i$ and $j$ with $j \leq r - 1$.

**Proof** It suffices to observe that for each full subring $B$ of $H^*(M \mathbb{Z}/2)$, one has $\text{Sq}^j(B) \subseteq B^{i+j}$ for all nonnegative integers $i$ and $j$. □
Proof of Theorem 1.1 It is already known that (a) implies (b). If \( k \) is an integer satisfying \( 1 \leq k \leq r \), then according to Lemma 2.13, each homology class in \( A_k \) is of the form \([K]_M\), where \( K \) is a \( k \)-dimensional connected smooth submanifold of \( M \) that is a boundary. By Lemmas 2.11 and 2.15, \( K \) is adapted to \( A \). Hence, (b) implies (a) in view of Theorem 2.10. □

Proof of Theorem 1.7 If \( k \) is an integer satisfying \( 1 \leq k \leq r \), then according to Lemma 2.13, each homology class in \( A_k \) is of the form \([K]_M\), where \( K \) is a \( k \)-dimensional connected smooth submanifold of \( M \) that is a boundary. Moreover, according to Lemma 2.14, \( K \) can be assumed to be an orientable manifold if the homology group \( H_{k-1}(M; \mathbb{Z}) \) has no 2-torsion. By Lemmas 2.11 and 2.15, \( K \) is adapted to \( A \) if \( k \) is in \( \{1, \ldots, r-2\} \cup \{2\} \). By Lemmas 2.12 and 2.15, \( K \) is adapted to \( A \) if \( r \geq 4 \) and either \( A^{r-3} = 0 \) or the homology group \( H_{r-2}(M; \mathbb{Z}) \) has no 2-torsion. The proof is complete in view of Theorem 2.10. □

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References

1. Akbulut, S., King, H.: The topology of real algebraic sets with isolated singularities. Ann. Math. 113, 425–446 (1981)
2. Akbulut, S., King, H.: A relative Nash theorem. Trans. Am. Math. Soc. 267, 465–481 (1981)
3. Akbulut, S., King, H.: The topology of real algebraic sets. Enseign. Math. 29, 221–261 (1983)
4. Akbulut, S., King, H.: Submanifolds and homology of nonsingular real algebraic varieties. Am. J. Math. 107, 45–83 (1985)
5. Akbulut, S., King, H.: A resolution theorem for homology cycles of real algebraic varieties. Invent. Math. 79, 589–601 (1985)
6. Akbulut, S., King, H.: On approximating submanifolds by algebraic sets and a solution to the Nash conjecture. Invent. Math. 107, 87–98 (1992)
7. Akbulut, S., King, H.: Algebraicity of immersions in \( \mathbb{R}^n \). Topology 31, 701–712 (1992)
8. Akbulut, S., King, H.: Topology of Real Algebraic Sets. Math. Sci. Res. Inst. Publ. 25, Springer (1992)
9. Akbulut, S., King, H.: Transcendental submanifolds of \( \mathbb{R}^n \). Comment. Math. Helv. 68, 308–318 (1993)
10. Akbulut, S., King, H.: Transcendental submanifolds of \( \mathbb{R}P^n \). Comment. Math. Helv. 80, 427–432 (2005)
11. Benedetti, R.: On a resolution theorem for homology classes of a real algebraic variety. Boll. Un. Mat. Ital. A(6) 4, 459–466 (1985)
12. Benedetti, R., Dedò, M.: Counterexamples to representing homology classes by real algebraic subvarieties up to homeomorphism. Compos. Math. 53, 143–151 (1984)
13. Benedetti, R., Tognoli, A.: Approximation theorems in real algebraic geometry. Boll. Un. Mat. Ital. Suppl. 2, 209–228 (1980)
14. Benedetti, R., Tognoli, A.: On real algebraic vector bundles. Bull. Sci. Math.(2) 104, 89–112 (1980)
15. Benedetti, R., Tognoli, A.: Remarks and counterexamples in the theory of real vector bundles and cycles. In: Géométrie algébrique réelle et formes quadratiques, Lecture Notes in Math. 959, Springer, 198–211 (1982)
16. Bochnak, J., Coste, M., Roy, M.-F.: Real Algebraic Geometry, Ergeb. Math. Grenzgeb. 36. Springer, Berlin (1998)
17. Bochnak, J., Kucharz, W.: Algebraic models of smooth manifolds. Invent. Math. 97, 585–611 (1989)
18. Bochnak, J., Kucharz, W.: \( K \)-theory of real algebraic surfaces and threefolds. Math. Proc. Camb. Philos. Soc. 106, 471–480 (1989)
19. Bochnak, J., Kucharz, W.: Nonisomorphic algebraic models of a smooth manifold. Math. Ann. 290, 1–2 (1991)
20. Bochnak, J., Kucharz, W.: Algebraic cycles and approximation theorems in real algebraic geometry. Trans. Am. Math. Soc. 337, 463–472 (1993)
21. Bochnak, J., Kucharz, W.: Real algebraic hypersurfaces in complex projective varieties. Math. Ann. 301, 381–397 (1995)
22. Bochnak, J., Kucharz, W.: On homology classes represented by real algebraic varieties. In: Singularities Symposium—Lojasiewicz 70, Banach Center Publ. 44, Inst. Math., Polish Acad. Sci. 21–35 (1998)
23. Bochnak, J., Kucharz, W.: A topological proof of the Grothendieck formula in real algebraic geometry. Enseign. Math. 48, 237–258 (2002)
24. Bochnak, J., Kucharz, W.: On approximation of smooth manifolds by nonsingular real algebraic subvarieties. Ann. Sci. École Norm. Sup. (4) 36, 685–690 (2003)
25. Bochnak, J., Kucharz, W.: Real algebraic morphisms represent few homotopy classes. Math. Ann. 337, 909–921 (2007)
26. Bochnak, J., Kucharz, W., Shiota, M.: On algebraicity of global real analytic sets and functions. Invent. Math. 70, 115–156 (1982/1983)
27. Borel, A., Haefliger, A.: La classe d'homologie fondamentale d'un espace analytique. Bull. Soc. Math. France 89, 461–513 (1961)
28. Bröcker, L.: Reelle Divisoren. Arch. Math. (Basel) 35, 140–143 (1980)
29. Conner, P.E.: Differentiable Periodic Maps, 2nd edn, Lecture Notes in Math. 738. Springer (1979)
30. Hirsch, M.: Differential Topology, Graduate Texts in Math. vol. 33. Springer, New York, Heidelberg, Berlin (1976)
31. Huisman, J.: Real abelian varieties with complex multiplication, Ph.D. Thesis, Vrije Universiteit Amsterdam (1992)
32. Husemoller, D.: Fibre Bundles. Springer, Berlin, New York (1975)
33. Ischebeck, F., Schülting, H.-W.: Rational and homology equivalence for real cycles. Invent. Math. 94, 307–316 (1988)
34. Ivanov, N.: Approximation of smooth manifolds by real algebraic sets. Russ. Math. Surv. 37, 1–59 (1982)
35. King, H.: Approximating submanifolds of real projective space by varieties. Topology 15, 81–84 (1976)
36. Krasnov, V.A.: On the equivariant Grothendieck cohomology of a real algebraic variety and its application. Izv. Ross. Akad. Nauk Ser. Mat. 58, 36–52 (1994) (Russian); English transl. Russ. Acad. Sci. Izv. Math. 44, 461–477 (1995)
37. Kucharz, W.: On homology of real algebraic sets. Invent. Math. 82, 19–26 (1985)
38. Kucharz, W.: Algebraic equivalence and homology classes of real algebraic cycles. Math. Nachr. 160, 135–140 (1996)
39. Kucharz, W.: Algebraic morphisms into rational real algebraic surfaces. J. Algebraic Geom. 8, 569–579 (1999)
40. Kucharz, W.: Algebraic cycles and algebraic models of smooth manifolds. J. Algebraic Geom. 11, 101–127 (2002)
41. Kucharz, W.: Nash cohomology of smooth manifolds. Ann. Polon. Math. 87, 193–205 (2005)
42. Kucharz, W.: Homology classes of real algebraic sets. Ann. Inst. Fourier (Grenoble) 58, 989–1022 (2008)
43. Kucharz, W.: Transcendental submanifolds of projective space. Comment. Math. Helv. 84, 127–133 (2009)
44. Kucharz, W.: Cycles on Nash algebraic models of smooth manifolds. Proc. Am. Math. Soc. 137, 1899–1906 (2009)
45. Kucharz, W.: Cycles on algebraic models of smooth manifolds. J. Eur. Math. Soc. (JEMS) 11, 393–405 (2009)
46. Kucharz, W.: Rational maps in real algebraic geometry. Adv. Geom. 9, 517–539 (2009)
47. Kucharz, W., Kurdyka, K.: Algebraicity of global real analytic hypersurfaces. Geom. Dedic. 119, 141–149 (2006)
48. Kucharz, W., van Hamel, J.: Transcendental manifolds in real projective space and Stiefel–Whitney classes. Ann. Sc. Norm. Super. Pisa Cl. Sci. 5(8), 267–277 (2009)
49. Mangolte, F.: Cycles algébriques sur les surfaces K3 réelles. Math. Z. 225, 559–576 (1995)
50. Mangolte, F.: Cycles algébriques et topologie des surfaces bielliptiques réelles. Comment. Math. Helv. 78, 385–393 (2003)
51. Mangolte, F., van Hamel, J.: Algebraic cycles and topology of real Enriques surfaces. Compos. Math. 110, 215–237 (1998)
52. Milnor, J., Stasheff, J.: Characteristic Classes. Ann. Math. Stud. 76. Princeton University Press, Princeton, NJ (1974)
53. Nash, J.: Real algebraic manifolds. Ann. Math. 56, 405–421 (1952)
54. Risler, J.-J.: Sur l’homologie des surfaces algébriques réelles. In: Géométrie algébrique réelle et formes quadratiques, Lecture Notes in Math. 959, 381–385. Springer (1982)
55. Shiota, M.: Equivalence of differentiable functions, rational functions and polynomials. Ann. Inst. Fourier (Grenoble) 32, 167–204 (1982)
56. Shiota, M.: Real algebraic realization of characteristic classes. Publ. Res. Inst. Math. Sci. 18, 995–1008 (1982)
57. Silhol, R.: A bound on the order of $H^{n-1}_{DR}(X, \mathbb{Z}/2)$ on a real algebraic variety. In: Géométrie algébrique réelle et formes quadratiques, Lecture Notes in Math. 959, 443–450. Springer (1982)
58. Silhol, R.: Real Algebraic Surfaces, Lecture Notes in Math. 1392. Springer (1989)
59. Spainer, E.: Algebraic Topology. Springer, New York (1966)
60. Steenrod, N.: The Topology of Fibre Bundles. Princeton University Press, Princeton, NJ (1951)
61. Teichner, P.: 6-dimensional manifolds without totally algebraic homology. Proc. Am. Math. Soc. 123, 2909–2914 (1995)
62. Thom, R.: Quelques propriétés globales de variétés différentiables. Comment. Math. Helv. 28, 17–86 (1954)
63. Tognoli, A.: Su una congettura di Nash. Ann. Sc. Norm. Sup. Pisa Sci. Fis. Mat. 3(27), 167–185 (1973)
64. Tognoli, A.: Algebraic approximation of manifolds and spaces. In: Séminaire Bourbaki, 32e année, 1979/1980, no. 548, Lecture Notes in Math. vol. 842, 73–94. Springer (1981)
65. van Hamel, J.: Real algebraic cycles on complex varieties. Math. Z. 225, 177–198 (1997)
66. van Hamel, J.: Algebraic Cycles and Topology of Real Algebraic Varieties, CWI Tract, vol. 129. Stichting Mathematisch Centrum, Amsterdam (2000)