Flag-transitive point-primitive 2-designs with socle A₅

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Abstract. Let D be a non-trivial 2-(v, k, λ) design with a flag-transitive, point-primitive automorphism group G of almost simple type. If Soc(G) = A₅, one of the alternating groups, then there are precisely 3 such non-isomorphic 2-designs.

1. Introduction

A 2-(v, k, λ) design is an incidence system (v, k, λ, b, r) in which a set P of v points is equipped with a family B of b different subsets (blocks) in such a way that any two points determine λ blocks with k points in each block, and each point is contained in r different blocks. A flag in a design is an incident point-block pair. If 2 < k < v − 2 holds, then we speak of a non-trivial 2-(v, k, λ) design, written as simply 2-design [1]. If b = v, then the 2-design is called a symmetric design. If the block set B contains all the k-subset of P, then the 2-design is called a full design.

The five parameters v, k, λ, b, r are not independent, but satisfy the three relations:

vr = bk  \hspace{1cm} (1)
λ(v − 1) = r(k − 1)  \hspace{1cm} (2)
b ≥ v (Fisher’s inequality)  \hspace{1cm} (3)

For an automorphism group G ≤ Aut(D), the design D = (P, B) is called point-primitive if G is primitive on P, and flag-transitive if G is transitive on the set of flags.

In 1988, Zieschang [2] proved that if G is a flag-transitive automorphism group of a 2-design with (r, λ) = 1 and T is a minimal normal subgroup of G , then T is abelian, or simple and C_G(T) = 1. In 2015, Tian and Zhou [3] completely classified all flag-transitive point-primitive 2-(v, k, λ) symmetric designs with sporadic socle. In [4], Alavi S, Bayat M and Daneshkhah A classified symmetric designs admitting flag-transitive and point primitive automorphism groups associated to two dimensional projective special groups. After that, Zhan and Zhou [5] considered the case of the non-symmetric designs with (r, λ) = 1 . In [6], Zhu, Tian and Zhou considered flag-transitive point-primitive 2-(v, k, λ) symmetric designs with λ at most 100 and alternating socle. In [7], Zhou and Wang considered flag-transitive non-symmetric 2-designs with (r, λ) = 1 and alternating socle. Classification work is a very important thing [8, 9]. Hence it is interesting to consider the case when the automorphism group G is one of almost simple type. In this paper, we study the 2-designs with Soc(G) = A₅ for general λ , and obtain the complete classification of flag-transitive point-primitive 2-designs with socle A₅.
Our main result of complete classification is the following.

**Theorem 1.** Let \( D \) be a non-trivial \( 2-(v,k,\lambda) \) design with a flag-transitive, point-primitive automorphism group \( G \) of almost simple type. If \( \text{Soc}(G) = A_5 \), then one of the following holds:

(i) \( D \) is \( 2-(6,3,2) \) design with \( G = A_5 \) and the point stabilizer \( G_x = D_{10} \);

(ii) \( D \) is \( 2-(6,3,4) \) design with \( G = S_3 \) and the point stabilizer \( G_x = Z_5 : Z_4 \);

(iii) \( D \) is \( 2-(10,4,2) \) design with \( G = S_5 \) and the point stabilizer \( G_x = D_{12} \).

2. **Some preliminary results**

In this section we give some preliminary results which are used throughout the paper. The following lemma is well-known.

**Lemma 2.1.** Let \( D = (P, B) \) be a non-trivial \( 2-(v,k,\lambda) \) design and \( G \) be an automorphism group of \( D \). For any point \( x \in P \) and block \( B \in B \), the following three statements are equivalent:

(i) \( G \) acts flag-transitively on \( D \);

(ii) \( G \) acts point-transitively on \( D \) and \( G_x \) acts transitively on \( B(x) \), where \( B(x) \) denotes the set of all blocks which are incident with a point \( x \);

(iii) \( G \) acts block-transitively on \( D \) and \( G_B \) acts transitively on the points of \( B \).

**Lemma 2.2.** Let \( D = (P, B) \) be a non-trivial flag-transitive \( 2-(v,k,\lambda) \) design and \( G \) be an automorphism group of \( D \). Then the following hold:

(i) \( r > \lambda \), \( r^2 > \lambda v \);

(ii) \( r \mid |G_x| \), where \( G_x \) is any point-stabilizer of \( G \).

**Proof.** Since \( D \) is a non-trivial 2-design, \( 2 < k < v - 2 \) holds. By \( \lambda(v - 1) = r(k - 1) \), we get \( r > \lambda \).

Fisher’s inequality implies that \( r \geq k \), then \( r^2 \geq rk > rk + (\lambda - r) = r(k - 1) + \lambda = \lambda v \).

By Lemma 2.1(ii), \( G_x \) acts transitively on the set of all blocks which are incident with a point \( x \), thus \( r \mid |G_x| \) holds.

3. **Proof of Theorem 1**

We prove Theorem 1 in the following three subsections.

3.1. **Feasible parameters of 2-designs**

In this subsection we describe briefly our approach to search for feasible parameters of \( 2-(v,k,\lambda) \) designs.

Suppose that there is a non-trivial 2-design \( D \) admitting a flag-transitive, point-primitive almost simple automorphism group \( G \) with socle \( A_5 \). Then \( G = A_5 \) or \( G = S_5 \). It is well-known that \( G \) is point-primitive on \( P \) if and only if the stabilizer \( G_x \) is a maximal subgroup of \( G \), where \( x \in P \) [10].

Thus \( v = |G : G_x| \). The list of the maximal subgroups of \( G \) is given in the ATLAS [11].

We compute all possible parameters \((v,k,\lambda,b,r)\) satisfying the following conditions:

(i) \( G = A_5 \) (or \( S_5 \)) and \( G_x \) is one of its maximal subgroups [11];

(ii) \( v \in \{5,6,10\} \) when \( G = A_5 \) (or \( S_5 \));

(iii) \( 2 < k < v - 2 \), \( 1 \leq \lambda < r \);

(iv) \( r \mid |G_x| \), and \( r^2 > \lambda v \) (Lemma 2.2);

(v) \( vr = bk \), \( \lambda(v - 1) = r(k - 1) \);
(vi) \( v \leq b \leq \frac{v!}{k!(v-k)!} \), and \( b \mid |G| \) ([Lemma 2.1]).

By using the computer algebra system GAP [12], we get 7 five-tuples of parameters \((v, k, \lambda, b, r)\) listed in Table 1. Note that in the 5-th column, “Lengths” means the orbit lengths of subgroup \(H\) with index \(b\), and the notation \(s'\) means that the degree \(s\) appears with multiplicity \(t\). In the following, “Case i” denotes the \(i\)-th line of Table 1.

\[
\begin{array}{|c|c|c|c|c|}
\hline
\text{CASE} & G & Gx & (v, k, \lambda, b, r) & \text{Lengths} & \text{Reference} \\
\hline
1 & A_5 & D_{10} & (6, 3, 2, 10, 5) & 3^5 & D_1 \\
\hline
2 & & & (6, 3, 4, 20, 10) & 3^2 & \text{3.2.1} \\
\hline
3 & S_3 & & (6, 3, 2, 10, 5) & 6 & \text{3.2.1} \\
\hline
4 & S_3 & Z_5:Z_4 & (6, 3, 4, 20, 10) & --- & D_3 \\
\hline
5 & & & (10, 4, 2, 15, 6) & 2, 4 \^ 2 & D_4 \\
\hline
6 & D_{12} & & (10, 4, 30, 12) & --- & 3.2.2 \\
\hline
\end{array}
\]

### 3.2. Rule out 4 potential cases

Now, we will rule out 4 potential cases listed in Table 1.

#### 3.2.1. Rule out Cases 2-4

By Lemma 2.1, the block-stabilizer \(G_B\) acts transitively on the points of \(B\). There exists at least one orbit \(O\) of \(G_B\) with size \(k\) and \(|O^G| = b\). But Case 2, 3 and 4 do not satisfy this fact.

For Case 2, the group \(G = A_5\) contains only one conjugacy class of subgroups with index 20, denoted by \(H\). The generator of subgroup \(H\) is \((1, 5, 3)(2, 6, 4)\). The orbits of \(H\) acting on \(\Omega = \{1, 2, ..., 6\}\) are \(O_1 = \{1, 4, 5\}\) and \(O_2 = \{2, 3, 6\}\). But \(|O^G_i| = 10\) \((i = 1, 2)\), both of them are not equal to \(b = 20\).

For Case 3, the generators of subgroup \(H\) with index 15 are \((1, 8)(2, 6)(4, 10)(7, 9)\) and \((1, 8)(3, 5)(4, 9)(7, 10)\). The orbits of \(H\) acting on \(\Omega = \{1, 2, ..., 10\}\) are \(O_1 = \{1, 8\}\), \(O_2 = \{2, 6\}\), \(O_3 = \{3, 5\}\) and \(O_4 = \{4, 7, 9, 10\}\). Unfortunately, \(|O^G_i| = 5 \neq 15\).

For Case 4, the subgroup \(H\) of \(G = S_3\), the conjugacy class of subgroups with index 10, is transitive on \(\Omega = \{1, 2, ..., 6\}\). So, there exists no one orbit \(O\) with size \(k = 3\).

#### 3.2.2. Rule out Case 7

For Case 7, the group \(G = S_3\) contains three conjugacy classes of the subgroups with index \(b = 30\), denoted by \(H\), \(J\) and \(K\), respectively.

Firstly, we list the generators of subgroup \(H\):

\[
g_1 = (1, 9)(2, 7)(3, 10)(6, 8); \quad g_2 = (2, 7)(3, 6)(4, 5)(8, 10).
\]

There are 4 orbits with lengths 2, 2 and 4, respectively:

\[
O_1 = \{1, 9\}; \quad O_2 = \{2, 7\}; \quad O_3 = \{4, 5\}; \quad O_4 = \{3, 6, 8, 10\}.
\]

In fact, \(|O^G_i| = 5\), thus there exists one orbit \(O_5\) of \(H\) with size \(k = 4\), but \(|O^G_i| \neq b\).

Secondly, we list the generators of subgroup \(J\):

\[
g_1 = (1, 4, 2, 3)(5, 10)(6, 7, 9, 8); \quad g_2 = (1, 2)(3, 4)(6, 9)(7, 8).
\]

There are 3 orbits with lengths 2, 4 and 4, respectively:

\[
O_1 = \{5, 10\}; \quad O_2 = \{1, 2, 3, 4\}; \quad O_3 = \{6, 7, 8, 9\}.
\]

Here \(|O_2| = |O_3| = 4\), but \(|O^G_1| = 5\), \(|O^G_2| = 15\), both of them are not equal to \(b = 30\).
Thirdly, we list the generators of subgroup $K$:

\[ g_1 = (1, 2)(6, 8)(7, 9); \quad g_2 = (1, 2)(3, 4)(6, 9)(7, 8). \]

There are 5 orbits with lengths 1, 1, 2, 2 and 4, respectively:

\[ O_1 = \{5\}; \quad O_2 = \{10\}; \quad O_3 = \{1, 2\}; \quad O_4 = \{3, 4\}; \quad O_5 = \{6, 7, 8, 9\}. \]

Here $|O^G_i| = 15 \neq b$.

3.3. Three 2-designs.

We list the generators of $G = A_5$ acting on $\Omega = \{1, 2, \ldots, 6\}$:

\[ g_1 = (1, 6, 2)(3, 4, 5); \quad g_2 = (3, 4, 5)(6, 2, 1). \]

The generators of $G = S_5$ acting on $\Omega = \{1, 2, \ldots, 6\}$ are:

\[ g_1 = (1, 6, 2)(3, 4, 5); \quad g_2 = (3, 4, 5, 6). \]

The generators of $G = S_5$ acting on $\Omega = \{1, 2, \ldots, 10\}$ are:

\[ g_1 = (1, 5, 7)(2, 9, 4)(3, 8, 10); \quad g_2 = (1, 8)(2, 5, 6, 3)(4, 9, 7, 10). \]

3.3.1. Design $D_1$ coming from Case 1. For Case 1, the group $G = A_5$ contains only one conjugacy class of the subgroups with index 10, denoted by $H$. We list the generators of $H$:

\[ g_1 = (1, 6)(2, 3); \quad g_2 = (1, 4, 6)(2, 3, 5). \]

There are 2 orbits with lengths 3 and 3, respectively:

\[ O_1 = \{1, 4, 6\}; \quad O_2 = \{2, 3, 5\}. \]

It follows that $|O^H_i| = 10 \ (i = 1, 2)$ and each 2-subset \{j, k\} is incident with 2 blocks in $O^G_i$ (for any $1 \leq j < k \leq 6$). Thus there are really two non-trial flag-transitive point-primitive 2-designs, denoted by $D_{11}$ and $D_{12}$. The basic block of $D_H$ is $O_i = \{1, 2, \ldots, 6\}$.

By the Magma-command `IsIsomorphic`, we get that $D_{11}$ and $D_{12}$ are isomorphic. So up to isomorphism we take $D_{11}$ and $D_{12}$ as the same design and denoted it by $D_1$.

3.3.2. Design $D_2$ coming from Case 5. For Case 5, the group $G = S_5$ contains three conjugacy classes of the subgroups with index $b = 20$, denoted by $H$, $J$, and $K$, respectively.

(i) The generators of subgroup $H$ are:

\[ g_1 = (1, 4)(2, 3)(5, 6); \quad g_2 = (1, 6, 3)(2, 5, 4). \]

Obviously, the subgroup $H$ is transitive on $\Omega = \{1, 2, \ldots, 6\}$. Thus there exists no one orbit $O$ with size $k = 3$.

(ii) The generators of subgroup $J$ are:

\[ g_1 = (1, 4)(2, 3)(5, 6); \quad g_2 = (1, 2, 6)(3, 5, 4). \]

In fact, the subgroup $J$ is also transitive on $\Omega = \{1, 2, \ldots, 6\}$. So there exists no one orbit $O$ with size $k = 3$, too.

(iii) The generators of subgroup $K$ are:

\[ g_1 = (1, 2)(5, 6); \quad g_2 = (1, 2, 4)(3, 5, 6). \]

There are 2 orbits with lengths 3 and 3, respectively:

\[ O_1 = \{1, 2, 4\}; \quad O_2 = \{3, 5, 6\}. \]
By calculating, $O^G = O^G$ and $|O^G| = 20$ $(i = 1, 2)$. It is not hard to check that each 2-subset $\{j, k\}$ is incident with 4 blocks in $O^G$ (for any $1 \leq j < k \leq 6$). Since the blocks are the same, there is the same non-trial flag-transitive point-primitive 2-designs, denoted by $D_2$, and the basic block is $O_1$ or $O_2$. The design $D_2$ contains all the 3-subset of $P$, so it is a full design. In fact, $D_2$ is the well-known design of points and (unordered) triples of points of $PG(1, 5)$.

The blocks of $D_2$ are:

$B_1 = \{1, 2, 3\}$; $B_2 = \{1, 2, 4\}$; $B_3 = \{1, 2, 5\}$; $B_4 = \{1, 2, 6\}$; $B_5 = \{1, 3, 4\}$; $B_6 = \{1, 3, 5\}$; $B_7 = \{1, 3, 6\}$; $B_8 = \{1, 4, 5\}$; $B_9 = \{1, 4, 6\}$; $B_{10} = \{1, 5, 6\}$; $B_{11} = \{2, 3, 4\}$; $B_{12} = \{2, 3, 5\}$; $B_{13} = \{2, 3, 6\}$; $B_{14} = \{2, 4, 5\}$; $B_{15} = \{2, 4, 6\}$; $B_{16} = \{2, 5, 6\}$; $B_{17} = \{3, 4, 5\}$; $B_{18} = \{3, 4, 6\}$; $B_{19} = \{3, 5, 6\}$; $B_{20} = \{4, 5, 6\}$.

3.3.3. Designs $D_3$ coming from Case 6. For Case 6, the group $G = S_4$ contains only one conjugacy class of the subgroups with index 15, denoted by $H$. We list the generators of $H$:

$g_1 = (1, 3)(5, 8)(7, 10)$; $g_2 = (1, 3)(2, 4)(5, 10)(7, 8)$; $g_3 = (1, 2, 3, 4)(6, 9)(7, 8)$.

There are 3 orbits with lengths 2, 4 and 4, respectively.

$O_1 = \{6, 9\}$; $O_2 = \{1, 2, 3, 4\}$; $O_3 = \{5, 7, 8, 10\}$.

By calculating, $|O^G| = 5 \neq b$, $|O^G| = 15$.

It follows that each 2-subset $\{j, k\}$ is incident with 2 blocks in $O^G$ (for any $1 \leq j < k \leq 10$). Thus there is one non-trial flag-transitive point-primitive 2-designs, denoted by $D_3$. The basic block is $O_3$. We can give the geometric description of this design. The points are the (unordered) pairs of elements of a 5-set $S$ and the blocks are pairs $(z_0, \{z_1, z_2, \{z_3, z_4\}\})$, where $z_0$ is an element of $S$ and $\{\{z_1, z_2\}, \{z_3, z_4\}\}$ is a partition of $S - \{z_0\} = \{z_1, z_2, z_3, z_4\}$ in two pairs. A “point” $(x, y)$ is declared to be incident with a “block” $(z_0, \{\{z_1, z_2\}, \{z_3, z_4\}\})$ precisely when $z_0 \neq x, y$ and $\{x, y\} \neq \{z_1, z_2\}, \{z_3, z_4\}$.

We list the blocks of $D_3$:

$B_1 = \{1, 2, 6, 8\}$; $B_2 = \{1, 2, 7, 9\}$; $B_3 = \{1, 3, 5, 8\}$; $B_4 = \{1, 3, 7, 10\}$; $B_5 = \{1, 4, 5, 9\}$; $B_6 = \{1, 4, 6, 10\}$; $B_7 = \{2, 3, 5, 6\}$; $B_8 = \{2, 3, 9, 10\}$; $B_9 = \{2, 4, 5, 7\}$; $B_{10} = \{2, 4, 8, 10\}$; $B_{11} = \{3, 4, 6, 7\}$; $B_{12} = \{3, 4, 8, 9\}$; $B_{13} = \{5, 6, 9, 10\}$; $B_{14} = \{5, 7, 8, 10\}$; $B_{15} = \{6, 7, 8, 9\}$.

This completes the proof of Theorem 1.

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