AN EFFICIENT ABSTRACT METHOD FOR THE STUDY OF AN INITIAL BOUNDARY VALUE PROBLEM ON SINGULAR DOMAIN

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Abstract. The present work is devoted to the study of a boundary value problem for second order linear differential equation set on singular cylindrical domain. This problem can be regarded via a natural change of variables as an elliptic abstract differential equation with variable operators coefficients subject to some anti-periodic conditions. The complete study of this abstract version allows us to establish some interesting regularity results for our problem. The study is performed in the framework of Hölder spaces.

1. Introduction and preliminaries

In [2] and [3], the solvability of some BVP’s set on cusp domain was discussed. The authors opted for the use of the abstract differential equations theory and some regularity results for these problems were successfully established in the framework of Little Hölder spaces. In the same direction, we will show that this approach can be exploited in order to give a complete study of an initial boundary value problem involving the Laplace operator and which is also posed on nonsmooth domain. More precisely, we consider a particular conical domain Ω given by

$$\Pi = [0, T] \times \Omega,$$

where

$$\Omega = \left\{ (x, y) \in \mathbb{R}^2 : \sqrt{x^2 + y^2} \leq \varphi(t) \right\}.$$ 

Here, $\varphi$ is a positive real-valued function of parametrization defined on $[0, 1]$ such that

$$\varphi(0) = \varphi'(0) = 0.$$ 

In $\Pi$, we consider the following problem

$$(1.1) \quad \partial_t^2 u + \Delta u - \lambda u = h, \quad \lambda > 0,$$

corresponding to the following initial conditions

$$(1.2) \quad u|_{t=0} + u|_{D(T, \varphi(T))} = 0, \quad \partial_t u|_{t=0} + \partial_t u|_{D(T, \varphi(T))} = 0,$$

where $D(T, \varphi(T))$ denotes the disc of radius $\varphi(T)$ centred at $(T, 0, 0)$.

We accompany (1.1)-(1.2) with additional conditions

$$(1.3) \quad u|_{\partial\Pi \setminus D(T, \varphi(T))} = 0, \quad u|_{\partial\Pi \setminus \{0\}} = 0.$$ 

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We assume that the right hand term of (1.1) is taken in the anisotropic Hölder space \( C^{2\theta} ([0, T]; C(\Omega)) \), \( 0 < 2\theta < 1 \), defined by
\[
\left\{ \phi \in C([0, T]; C(\Omega)) : \lim_{\varepsilon \to 0^+} \sup_{0 < |t - t'| \leq \varepsilon} \frac{\|\phi(t) - \phi(t')\|_{C(\Omega)}}{|t - t'|^{2\theta}} < \infty \right\}.
\]
Now, we consider the following change of variables
\[
T : \Pi \to Q,
\]
(1.4) \( (t, x, y) \mapsto (t, \xi, \eta) = \left( t, \frac{x}{\varphi(t)}, \frac{y}{\varphi(t)} \right) \),
where \( Q = [0, T] \times D \),
with \( D := D(0, 1) = \{(\xi, \eta) \in \mathbb{R}^2 : \xi^2 + \eta^2 \leq 1\} \).
Define the following change of functions
(1.5) \( u(t, x, y) = v(t, \xi, \eta) \) and \( h(t, x, y) = f(t, \xi, \eta) \).
It follows from the change of functions (1.5) that the new version of problem (1.1) is given by
\[
\begin{align*}
\partial_t^2 v + L(t) v - \lambda v &= f, & \text{in } Q \\
v|_{[0 \times D} + v|_{(T) \times D} &= 0, \\
\partial_t v|_{[0 \times D} + \partial_t v|_{(T) \times D} &= 0, \\
v|_{[0, T] \times \partial D} &= 0,
\end{align*}
\]
where \( L \) is the linear operator with singular coefficients given by
\[
L(t) = \frac{1}{\varphi^2(t)} \Delta + \frac{\varphi'(t)}{\varphi(t)} \{ \xi \partial_\xi + \eta \partial_\eta \}, \quad 0 \leq t \leq T.
\]
The following lemma is needed in order to clarify the impact of the change of variables (1.4) on the functional framework of Hölder Spaces.

**Lemma 1.1.** Let \( 0 < 2\theta < 1 \). Then
1. \( h \in C^{2\theta} ([0, T]; C(\Omega)) \Rightarrow f \in C^{2\theta} ([0, T]; C(D)) \).
2. \( f \in C^{2\theta} ([0, T]; C(D)) \Rightarrow h \in C^{2\theta} ([0, T]; C(\Omega)) \) with \( C^{2\theta} ([0, T]; C(\Omega)) = \left\{ h \in C^{2\theta} ([0, T]; C(\Omega)) : (\varphi(\cdot))^{2\theta} h \in C^{2\theta} ([0, T]; C(\Omega)) \right\} \).

**Proof.** See Proposition 3.1 in [2]. \( \square \)

Due to the presence of a singular coefficients, we must approximate the cylinder \( \Pi \) by a sequence of regular subdomains. As in [7], we perform the following regular change of variables given by
\[
\Pi_n := [t_n, T] \times \Omega \to Q_n := [t_n, T] \times D(0, 1),
\]
(1.4) \( (t, x, y) \mapsto (t, \xi, \eta) = \left( t, \frac{x}{\varphi(t)}, \frac{y}{\varphi(t)} \right) \).
Here, \( (t_n)_{n \in \mathbb{N}} \) is a decreasing sequence such that \( 0 \leq t_n \leq 1 \) and \( \lim_{n \to +\infty} t_n = 0 \).
Set
\[
\begin{align*}
v_n &= v|_{Q_n}, \\
f_n &= f|_{Q_n}.
\end{align*}
\]
Summing up, we are confronted to the study of following problems

\[
\begin{align*}
\frac{\partial^2_l}{\partial t^2} v_{n} + L(t) v_{n} - \lambda v_{n} &= f_{n}, & & \text{in } Q_{n}, \\
v_{n}|_{\{t_n \times D} + v_{n}|_{\{T \times D} &= 0, \\
\partial_t v_{n}|_{\{t_n \times D} + \partial_t v_{n}|_{\{T \times D} &= 0, \\
v_{n}|_{\{t_n, T \times \partial D} &= 0.
\end{align*}
\]

(1.7)

Here, we just briefly note that

\[
\lim_{n \to +\infty} v_{n} = v \mid_{\lim_{n \to +\infty} Q_{n}} = v|_{Q} = v.
\]

In the next section, we will show that our transformed problems (1.7) can be formulated as a second order abstract differential equation of elliptic type with variable operators coefficients.

2. THE ABSTRACT FORMULATION OF THE PROBLEMS (1.7)

Let us introduce the following vector-valued functions:

\[
\begin{align*}
v_{n} & : [t_n, T] \to E; \ t \mapsto v_{n}(t); \ v_{n}(t)(\xi, \eta) = v_{n}(t, \xi, \eta), \\
f_{n} & : [t_n, T] \to E; \ t \mapsto f_{n}(t); \ f_{n}(t)(\xi, \eta) = f_{n}(t, \xi, \eta),
\end{align*}
\]

with \(E = C(D)\). So, the transformed problem (1.7) can be formulated as follows

\[
\begin{align*}
v''_{n}(t) & + A(t) v_{n}(t) - \lambda v_n(t) = f_n(t), \quad t_n \leq t \leq T, \\
v_{n}(t_n) + v_{n}(T) &= 0, \\
v'_{n}(t_n) + v'_{n}(T) &= 0.
\end{align*}
\]

(2.1)

Here, \((A(t))_{t_n \leq t \leq T}\) is a family of closed linear operators with domains \(D(A(t))\) (which are not dense) defined by

\[
\begin{align*}
D(A(t)) & := \{\phi \in W^{2,p}(D) \cap C_{0}(D), \ p > 2 : L(t) \phi \in C_{0}(D)\}, \ t_n \leq t \leq T, \\
(A(t)) \phi(\xi, \eta) & := (L(t)) \phi(\xi, \eta).
\end{align*}
\]

Consider the natural change of function

\[
w_{n}(t) = v_{n}(t + t_n) \text{ and } g_{n}(t) = f_{n}(t + t_n);
\]

then

\[
g_{n} \in C^{2\theta}([0, T]; C(D));
\]

and \(w_{n}\) is the eventual solution of

\[
\begin{align*}
w''_{n}(t) + A(t + t_n)w_{n}(t) - \lambda w_{n}(t) &= g_{n}(t), \quad 0 \leq t \leq T, \\
w_{n}(0) + w_{n}(T) &= 0, \\
w'_{n}(0) + w'_{n}(T) &= 0.
\end{align*}
\]

(2.4)

From [1] p. 60, we know that the family \((A(t + t_n))_{0 \leq t \leq T}\) enjoys the following three properties:

\[
\begin{align*}
(1) \quad & \exists M > 0, \ \forall \varepsilon > 0, \ \forall t \in [0, T] \quad \| (A_{n}(t + t_n) - \varepsilon)^{-1} \|_{L(E)} \leq \frac{M}{\varepsilon + 1}; \\
(2.5) \quad & \exists M > 0, \ \forall \varepsilon > 0, \ \forall t \in [0, T] \quad \| (A_{n}(t + t_n) - \varepsilon)^{-1} \|_{L(E)} \leq \frac{M}{\varepsilon + 1};
\end{align*}
\]
(2) For all \( z \geq 0 \), the application \( t \mapsto (A_n(t + t_n) - \lambda - z)^{-1} \) defined on \([0, T]\) is in \( C^2([0, T]; L(E)) \) and there exist \( C > 0 \) such that:

\[
\forall z \geq 0, \forall t \in [0, T] \quad \left\| \frac{\partial}{\partial t} (A(t + t_n) - \lambda - zI)^{-1} \right\|_{L(E)} \leq \frac{C}{z + 1},
\]

and

\[
\forall z \geq 0, \forall t \in [0, T] \quad \left\| \frac{\partial^2}{\partial t^2} (A(t + t_n) - \lambda - zI)^{-1} \right\|_{L(E)} \leq \frac{C}{z + 1};
\]

(3) Moreover, one has: \( \forall z \geq 0, \forall t, s \in [0, T] \)

\[
\left\| \frac{\partial^2}{\partial t^2} (A(t + t_n) - \lambda - z)^{-1} - \frac{\partial^2}{\partial s^2} (A(s + s_n) - \lambda - z)^{-1} \right\|_{L(E)} \leq \frac{C |t - s|^{2\theta}}{z + 1},
\]

\[
\left\| \frac{\partial^2}{\partial t^2} (A(t + t_n) - \lambda - z)^{-1} - \frac{\partial^2}{\partial s^2} (A(s + s_n) - \lambda - z)^{-1} \right\|_{L(E)} \leq \frac{C |t - s|^{2\theta}}{z + 1}.
\]

Remark 2.1. In the sequel the symbol \( C \) stands for a generic positive constant except when other dependence is stated explicitly. On the other hand, it is important to note that all the constants given above are independent of \( t \) and consequently of \( n \).

3. SOME REGULARITY RESULTS

We are concerned with a study of the problem

\[
\left\{ \begin{array}{ll}
 w''_n(t) + A_n(t)w_n(t) - \lambda w_n(t) = g_n(t), & 0 \leq t \leq T, \\
 w_n(0) + w_n(T) = 0, & \\
 w'_n(0) + w'_n(T) = 0,
\end{array} \right.
\]

(3.1)

where:

- \( g_n \in C^{2\theta}([0, T], E), 2\theta \in ]0, 1[, \)
- \( A_n(t) := A(t + t_n). \)

Our purpose is to establish some results about the existence, uniqueness and maximal regularity of a strict solution \( w_n \) for Problem (3.1) by building explicitly a representation of the solution \( w_n(t) \) and studying its optimal regularity. Recall here that a strict solution is a function \( w_n \) such that

\[
\left\{ \begin{array}{ll}
 w_n \in C^2([0, T], E), & \\
 w_n(t) \in D(A(t)) \text{ for every } t \in [0, T], & \\
 t \mapsto (A_n(t) - \lambda)w_n(t) \in C([0, T], E), &
\end{array} \right.
\]

and satisfying the anti-periodic boundary conditions

\[
 w_n(0) + w_n(T) = 0,
\]

\[
 w'_n(0) + w'_n(T) = 0.
\]

The techniques used here are essentially based on the Dunford functional calculus and the methods applied in [1], [5]-[6] and [9]. We know that if \( A_n(t) \) is a constant operator satisfying (2.5), the representation of the solution \( w_n \) is given by the formula

\[
w_n(t) = -\frac{1}{2\pi i} \int_{\gamma} \int_{0}^{T} K(t, s) (A_n - \lambda - z)^{-1} g_n(s)dsdz
\]

(3.2)
Proposition 3.3. Suppose that $g_t \in \mathbb{R}$ to obtain a strict solution $w$. There exists a solvent, the previous assumptions hold true in the sector $\Pi_{\delta}$.

Remark 3.2. By using a classical argument of analytic continuation on the resolvent, the previous assumptions hold true in the sector $\Pi_{\delta_0}$, and then, on $\gamma$. Furthermore, we can replace $z$ by $z + \lambda$.

Keeping in mind the constant case (see formula (3.2)), we look for a solution of Problem (3.1) in the following form:

\[
(3.5) \quad w_n(t) = -\frac{1}{2\pi i} \int_\gamma \int_0^T K_{\sqrt{-z}}(t, s) (A_n(t) - \lambda - z)^{-1} g_n^*(s) ds dz,
\]

where $g_n^*$ is an unknown function to be determined in some adequate space in order to obtain a strict solution $w_n$ of Problem (3.1), when $g_n \in C^{2\theta}([0, T], E)$.

Our first result concerning the vector valued function $w_n(t)$ given by (3.5) is

Proposition 3.3. Suppose that $g_n^* \in C^{2\theta}([0, T], E)$, $0 < 2\theta < 1$. Then, for all $t \in [0, T]$:

$w_n \in C^2([0, T], E)$,

and

$w_n(t) \in D(A_n(t)).$

Proof. First, observe that the vector valued function $w_n$ is well defined. In fact, using a direct computation on the kernel (3.3), one has

\[
\left\| \int_0^T K_{\sqrt{-z}}(t, s) (A_n(t) - \lambda - z)^{-1} g_n^*(s) ds \right\| \\
\leq \left( \sup_{t \in [0, T]} \int_0^T |K_{\sqrt{-z}}(t, s)| \left\| (A_n(t) - \lambda - z)^{-1} \right\|_{L(E)} ds \right) \|g_n^*\|_{C([0, T]; E)} \\
\leq \frac{C}{|z|} \|g_n^*\|_{C([0, T]; E)}.
\]
Now, we write \( w_n \) as follows:

\[
    w_n(t) = -\frac{1}{2\pi} \int_0^T \int_\gamma K_{\sqrt{\pi}}(t, s)(A_n(t) - \lambda - z)^{-1} (g_n^*(s) - g_n^*(t)) 
    ds\, dz 
\]

which becomes:

\[
    w_n(t) = -\frac{1}{2\pi} \int_0^T \int_\gamma K_{\sqrt{\pi}}(t, s)(A_n(t) - \lambda - z)^{-1} (g_n^*(s) - g_n^*(t)) \, ds 
    \, dz 
\]

where

\[
    c_{\sqrt{\pi}}(t) = \frac{e^{-\sqrt{\pi}t} + e^{-\sqrt{\pi}(T-t)}}{(1 + e^{-\sqrt{\pi}t})}.
\]

Thanks to (2.5), we have

\[
    \left\| \frac{1}{2\pi} \int_\gamma \int_0^T K_{\sqrt{\pi}}(t, s)(A_n(t) - \lambda - z)^{-1} (g_n^*(s) - g_n^*(t)) \, ds \, dz \right\| 
\]

\[
    \leq \int_\gamma \int_0^T \left| K_{\sqrt{\pi}}(t, s) \right| |t - s|^{10} \| g_n^* \|_{C^{10}([0, T]; E)} \, ds \, dz 
\]

\[
    \leq C \int_\gamma \frac{|dz|}{|z|^{10}} \| g_n^* \|_{C^{10}([0, T]; E)} 
\]

\[
    \leq C \| g_n^* \|_{C^{10}([0, T]; E)}.
\]

Concerning the second integral, we have

\[
    \left\| \frac{1}{2\pi} \int_\gamma c_{\sqrt{\pi}}(t) (A_n(t) - \lambda)^{-1} (A_n(t) - \lambda - z)^{-1} \frac{1}{z} g_n^*(t) \, dz \right\| 
    \leq C \| g_n^* \|_{C([0, T]; E)}.
\]

On the other hand, by Cauchy Theorem, we deduce that

\[
    \frac{1}{2\pi} \int_\gamma (A_n(t) - \lambda - z)^{-1} \frac{1}{z} g_n^*(t) \, dz = -(A_n(t) - \lambda)^{-1} g_n^*(t).
\]

Summing up, we deduce that

\[
    \forall t \in [0, T], \ w_n(t) \in D(A_n(t))
\]

and

\[
    (A_n(t) - \lambda)^{-1} w_n(t) 
\]

\[
    = \frac{-1}{2\pi} \int_0^\delta \int_\gamma K_{\sqrt{\pi}}(t, s)(A_n(t) - \lambda)^{-1} (A_n(t) - \lambda - z)^{-1} (g_n^*(s) - g_n^*(t)) \, ds \, dz 
\]

\[
    - \frac{1}{2\pi} \int_\gamma c_{\sqrt{\pi}}(t) (A_n(t) - \lambda)^{-1} (A_n(t) - \lambda - z)^{-1} \frac{1}{z} g_n^*(t) \, dz 
\]

\[
    - g_n^*(t).
\]

\[ \square \]
Proposition 3.4. Suppose that $g_n^* \in C^{2\theta}([0, T], E)$, $0 < 2\theta < 1$. Then, the abstract equation

$$w''_n(t) + A_n(t)w_n(t) - \lambda w_n(t) = g_n^*(t) - R_\lambda(g_n^*)(t)$$

is satisfied, where

$$R_\lambda(g_n^*)(t) = \frac{1}{2\pi i} \int_\gamma \int_0^T \frac{\partial}{\partial t} K_{\sqrt{-\bar{z}}}(t, s) \frac{\partial}{\partial t} (A_n(t) - \lambda - z)^{-1} g_n^*(s)dsdz$$

Proof. Step 1. First, regarding the derivative $w'_n(t)$, we have:

$$w'_n(t) = \frac{1}{2\pi i} \int_\gamma \int_0^t \frac{e^{-\sqrt{-z}(t-s)} + e^{-\sqrt{-z}(T-t+s)}}{2(1 + e^{-T\sqrt{-z}})} \frac{\partial}{\partial t} (A_n(t) - \lambda - z)^{-1} g_n^*(s)dsdz$$

$$- \frac{1}{2\pi i} \int_\gamma \int_t^T \frac{e^{-\sqrt{-z}(T-t)} + e^{-\sqrt{-z}(T-t+s)}}{2(1 + e^{-T\sqrt{-z}})} \frac{\partial}{\partial t} (A_n(t) - \lambda - z)^{-1} g_n^*(s)dsdz$$

$$- \frac{1}{2\pi i} \int_\gamma \int_0^t \frac{e^{-\sqrt{-z}(t-s)} - e^{-\sqrt{-z}(T-t+s)}}{2\sqrt{-z}(1 + e^{-T\sqrt{-z}})} \frac{\partial}{\partial t} (A_n(t) - \lambda - z)^{-1} g_n^*(s)dsdz$$

$$- \frac{1}{2\pi i} \int_\gamma \int_t^T \frac{e^{-\sqrt{-z}(T-t-s)} - e^{-\sqrt{-z}(T-t+s)}}{2\sqrt{-z}(1 + e^{-T\sqrt{-z}})} \frac{\partial}{\partial t} (A_n(t) - \lambda - z)^{-1} g_n^*(s)dsdz.$$
\[ \Pi_{n,\varepsilon}^2(t) = \frac{1}{2\pi} \int_\gamma \frac{e^{-\sqrt{-z}w} + e^{-\sqrt{-z}(T-\varepsilon)}}{2 \left(1 + e^{-T\sqrt{-z}}\right)} (A_n(t) - \lambda - z)^{-1} g_n^*(t - \varepsilon) dz \]
\[ + \frac{1}{2\pi} \int_\gamma \frac{e^{-\sqrt{-z}w} + e^{-\sqrt{-z}(T-\varepsilon)}}{2 \left(1 + e^{-T\sqrt{-z}}\right)} (A_n(t) - \lambda - z)^{-1} g_n^*(t + \varepsilon) dz, \]

\[ \Pi_{n,\varepsilon}^3(t) = -\frac{1}{2\pi} \int_\gamma \frac{e^{-\sqrt{-z}w} - e^{-\sqrt{-z}(T-\varepsilon)}}{2 \left(1 + e^{-T\sqrt{-z}}\right)} \frac{\partial}{\partial t} (A_n(t) - \lambda - z)^{-1} g_n^*(t) ds dz \]
\[ + \frac{1}{2\pi} \int_\gamma \frac{e^{-\sqrt{-z}w} - e^{-\sqrt{-z}(T-\varepsilon)}}{2 \left(1 + e^{-T\sqrt{-z}}\right)} \frac{\partial}{\partial t} (A_n(t) - \lambda - z)^{-1} g_n^*(t + \varepsilon) ds dz. \]

\[ \Pi_{n,\varepsilon}^4(t) = \frac{1}{2\pi} \int_\gamma \left[ \int_0^{t-\varepsilon} e^{-\sqrt{-z}(s-t)} + e^{-\sqrt{-z}(T-t+s)} \frac{\partial}{\partial t} (A_n(t) - \lambda - z)^{-1} g_n^*(s) ds \right] dz \]
\[ + \frac{1}{2\pi} \int_\gamma \left[ \int_{t+\varepsilon}^{t+T} e^{-\sqrt{-z}(s-t)} + e^{-\sqrt{-z}(T-t+s)} \frac{\partial}{\partial t} (A_n(t) - \lambda - z)^{-1} g_n^*(s) ds \right] dz \]
\[ - \frac{1}{2\pi} \int_\gamma \left[ \int_{t+\varepsilon}^{t+T} e^{-\sqrt{-z}(s-t)} + e^{-\sqrt{-z}(T-t+s)} \frac{\partial}{\partial t} (A_n(t) - \lambda - z)^{-1} g_n^*(s) ds \right] dz, \]

\[ \Pi_{n,\varepsilon}^5(t) = -\frac{1}{2\pi} \int_\gamma \left[ \int_0^{t-\varepsilon} e^{-\sqrt{-z}(s-t)} - e^{-\sqrt{-z}(T-t+s)} \frac{\partial^2}{\partial t^2} (A_n(t) - \lambda - z)^{-1} g_n^*(s) ds \right] dz \]
\[ - \frac{1}{2\pi} \int_\gamma \left[ \int_{t+\varepsilon}^{t+T} e^{-\sqrt{-z}(s-t)} - e^{-\sqrt{-z}(T-t+s)} \frac{\partial^2}{\partial t^2} (A_n(t) - \lambda - z)^{-1} g_n^*(s) ds \right] dz. \]

Taking into account all properties \((2.3)\) to \((2.8)\), it is easy to see all these integrals are absolutely convergent and can be treated similarly using the Lebesgue dominated convergence theorem. Then, we obtain the strong convergence

\[ w_{n,\varepsilon}'(t) \to w_n'(t) \quad \text{and} \quad w_{n,\varepsilon}''(t) \to -(A_n(t) - \lambda - z)^{-1} w_n(t) + R_\lambda(g_n^*) + g_n^*(t), \]

as \(\varepsilon \to 0\). Hence

\[ w_{n,\varepsilon}''(t) = -(A_n(t) - \lambda - z)^{-1} w_n(t) + R_\lambda(g_n^*) + g_n^*(t), \]

or

\[ w_n''(t) + (A_n(t) - \lambda - z)^{-1} w_n(t) = g_n^*(t) + R_\lambda(g_n^*(t)). \]

\[ \square \]

The relationship between the vectorial functions \(g_n\) and \(g_n^*\) is given by the following

**Proposition 3.5.** Suppose that \(g_n^* \in L^\infty([0,T]; E)\). Then, there exists \(\lambda^* > 0\) such that for all \(\lambda \geq \lambda^*\), the equation

\[ g_n(t) = g_n^*(t) - R_\lambda(g_n^*) \]

admits a unique solution

\[ g_n^* \in L^\infty([0,T]; E). \]
\textbf{Proof.} Recall that
\begin{align*}
R_\lambda(g_n^*(t)) &= + \frac{1}{\pi} \int \int_{\gamma} K_{\sqrt{-z}}(t, s) \frac{\partial}{\partial t} (A_n(t) - \lambda - z)^{-1} g_n^*(s) ds dz \\
&= - \frac{1}{2\pi} \int \int_{\gamma} K_{\sqrt{-z}}(t, s) \frac{\partial^2}{\partial t^2} (A_n(t) - \lambda - z)^{-1} g_n^*(s) ds dz.
\end{align*}

Then, thanks to (2.6)-(2.8), we see that
\begin{align*}
\left\| \frac{1}{\pi} \int \int_{\gamma} K_{\sqrt{-z}}(t, s) \frac{\partial}{\partial t} (A_n(t) - \lambda - z)^{-1} g_n^*(s) ds dz \right\| \\
&\leq C \int \frac{1}{|z|^{1/2}} \frac{1}{|z + \lambda|^{1/2}} |dz| \left\| g_n^* \right\|_{C(E)} \leq (C/\lambda^2) \left\| g_n^* \right\|_{L^\infty([0, T]; E)}.
\end{align*}

This implies
\begin{equation}
\tag{3.6}
\| R_\lambda \|_{L(L^\infty([0, T]; E))} = C/\lambda.
\end{equation}

Now, to establish the result it suffices to choose \( \lambda^* > 0 \) such that for \( \lambda \geq \lambda^* \)
\begin{equation*}
\| R_\lambda \|_{L(L^\infty([0, T]; E))} < 1.
\end{equation*}

\qed

The following proposition is concerned with the regularity of the operator \( R_\lambda \) needed in order to study the optimal regularity of the solution we are looking for:

\textbf{Proposition 3.6.} Let \( g_n \in C^{2\theta}([0, T]; E) \). Then, there exists \( \lambda^* > 0 \) such that for all \( \lambda \geq \lambda^* \),
\begin{equation*}
R_\lambda(g_n^*) \in C^{2\theta}([0, T]; E).
\end{equation*}

\textbf{Proof.} It suffices to adapt the same techniques delivered in the proof of Proposition 4.3 in [6]. \qed

This justifies the following result:

\textbf{Theorem 3.7.} Let \( g_n \in C^{2\theta}([0, T]; E) \). Then, there exist \( \lambda^* > 0 \) such that for all \( \lambda \geq \lambda^* \), the function \( w_n \) given in the representation (3.7) is the unique strict solution of Problem (3.7) satisfying
\begin{equation}
\tag{3.7}
\max_t \left\| w_n(t) \right\|_E \leq C.
\end{equation}

As a consequence, we have

\textbf{Corollary 3.8.} Let \( g_n^* \in L^\infty([0, T]; E) \). Then there exists \( \lambda^* > 0 \) and \( C > 0 \) such that for all \( \lambda \geq \lambda^* \), the strict solution \( w_n \) given by (3.7) fulfills the estimate
\begin{equation*}
\tag{3.7}
\max_t \left\| w_n(t) \right\|_E \leq C.
\end{equation*}
Sketch of the proof. The calculus are very cumbersome, we just give the main line of the demonstration. We have
\[ w_n''(s) + (A_n(s) - \lambda)^{-1}w_n(s) = g_n(s), \]
then, we may deduce that
\[ w_n(t) = -\frac{1}{2i\pi} \int_{\gamma} \int_{0}^{T} K_{-z}(t, s) (A_n(t) - \lambda - z)^{-1}g_n(s)dsdz \]
\[ = -\frac{1}{2i\pi} \int_{\gamma} \int_{0}^{T} K_{-z}(t, s) (A_n(t) - \lambda - z)^{-1}w_n''(s)dsdz \]
\[ - \frac{1}{2i\pi} \int_{\gamma} \int_{0}^{T} K_{-z}(t, s) (A_n(t) - \lambda - z)^{-1}(A_n(s) - \lambda)^{-1}w_n(s)dsdz. \]

After using integration by parts, we deduce that
\[ -\frac{1}{2i\pi} \int_{\gamma} \int_{0}^{T} K_{-z}(t, s) (A_n(t) - \lambda - z)^{-1}g_n(s)dsdz \]
\[ = w_n(t) + \frac{1}{i\pi} \int_{\gamma} \int_{0}^{T} \frac{\partial}{\partial t} K_{-z}(t, s) \frac{\partial}{\partial s}(A_n(s) - \lambda - z)^{-1}w_n(s)dsdz \]
\[ - \frac{1}{2i\pi} \int_{\gamma} \int_{0}^{T} K_{-z}(t, s) \frac{\partial^2}{\partial s^2}(A_n(s) - \lambda - z)^{-1}w_n(s)dsdz, \]
so that
\[ -\frac{1}{2i\pi} \int_{\gamma} \int_{0}^{T} K_{-z}(t, s) (A_n(t) - \lambda - z)^{-1}g_n(s)dsdz = (1 + R_\lambda(w_n))(t). \]

At this level, it is easy to see that the result is a direct consequence of the estimate (3.6). \qed

4. COMING BACK TO THE SINGULAR CYLINDRICAL DOMAIN

Coming back to the problem (2.1) one obtains, for \( t \geq t_n \),
\[ v_n(t) = w_n(t - t_n). \]

Thanks to Proposition 3.7, a classical argument allows us to extract a convergent subsequence
\[ v_{nj} := v(t_{nj}), \]
where
\[ \lim_{n \to +\infty} t_{nj} = 0. \]

Then, after a passage to the limit, we deduce the following important result

**Theorem 4.1.** Let \( g \in C^{2\theta}([0, T]; E) \). Then, there exists \( \lambda^* > 0 \) such that for \( \lambda \geq \lambda^* \), the problem
\[ v''(t) + A(t)v(t) - \lambda v(t) = f(t), \quad t \geq 0, \]
\[ v(0) + v(T) = 0, \]
\[ v'(0) + v'(T) = 0 \]

admits a unique strict solution satisfying
\[ v(.), \ (A(.)-\lambda)^{-1}v(.) \in C^{2\theta}([0, T]; E). \]
Applying all the preceding abstract results and Lemma 1.1 our main results concerning the transformed problem (1.6) are formulated as follows:

**Theorem 4.2.** Let \( f \in C^{2\theta}(Q) \), \( 0 < 2\theta < 1 \). Then, there exists \( \lambda^* > 0 \) such that for \( \lambda \geq \lambda^* \), Problem (1.6) has a unique strict solution \( v \in C^2(Q) \). Moreover, \( v \) satisfies the maximal regularity

\[
\begin{align*}
\partial_t^2 v &\in C^{2\theta}(Q), \\
\frac{1}{\varphi^2(t)} \Delta v + \frac{\varphi'(t)}{\varphi^2(t)} \{\xi \partial_x + \eta \partial_y\} v - \lambda v &\in C^{2\theta}(Q).
\end{align*}
\]

**Theorem 4.3.** Let \( h \in C^{2\theta}([0, T]; C(\Omega)) \), \( 0 < 2\theta < 1 \). Then, there exists \( \lambda^* > 0 \) such that for \( \lambda \geq \lambda^* \), Problem (1.7) has a unique strict solution \( v \in C^2(\Pi) \). Moreover, \( v \) satisfies the maximal regularity

\[
\begin{align*}
\partial_t^2 v &\in C^{2\theta}([0, T]; C(\Omega)), \\
\Delta v - \lambda v &\in C^{2\theta}([0, T]; C(\Omega)),
\end{align*}
\]

where

\[
C^{2\theta}([0, T]; C(\Omega)) = \left\{ h \in C^{2\theta}([0, T]; C(\Omega)) : (\varphi(\cdot))^{2\theta} h \in C^{2\theta}([0, T]; C(\Omega)) \right\}.
\]

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