Existence and uniqueness results to positive solutions of integral boundary value problem for fractional $q$-derivatives

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**Abstract**

In this paper, we are interested in the existence and uniqueness of positive solutions for integral boundary value problem with fractional $q$-derivative:

\[
D^\alpha_q u(t) + f(t, u(t), u(t)) + g(t, u(t)) = 0, \quad 0 < t < 1,
\]

\[
u(0) = D_q u(0) = 0, \quad u(1) = \mu \int_0^1 u(s) d_q s,
\]

where $D^\alpha_q$ is the fractional $q$-derivative of Riemann–Liouville type, $0 < q < 1, 2 < \alpha \leq 3$, and $\mu$ is a parameter with $0 < \mu < [\alpha]_q$. By virtue of fixed point theorems for mixed monotone operators, we obtain some results on the existence and uniqueness of positive solutions.

**Keywords**: Positive solution; Mixed monotone operator; Fractional $q$-difference equation; Existence and uniqueness

1 Introduction

The theory that fractional differential equations arise in the fields of science and engineering such as physics, chemistry, mechanics, economics, and biological sciences, etc.; see, for example, [1–6]. The $q$-difference calculus or quantum calculus is an old subject that was put forward by Jackson [7, 8]. The essential definitions and properties of $q$-difference calculus can be found in [9, 10]. Early development for $q$-fractional calculus can be seen in the papers by Al-Salam [11] and Agarwal [12] on the existence theory of fractional $q$-difference. These days the fractional $q$-difference equation have given fire to increasing scholars’ imaginations. Some works considered the existence of positive solutions for nonlinear $q$-fractional boundary value problem [13–32]. For example, Ferreira [13] studied the existence of positive solutions to the fractional $q$-difference equation

\[
\begin{cases}
D^\alpha_q u(t) + f(t, u(t)) = 0, & 0 < t < 1, 1 < \alpha \leq 2, \\
u(0) = u(1) = 0.
\end{cases}
\]
Ferreira [14] also considered the existence of positive solutions to the nonlinear $q$-difference boundary value problem

$$\begin{align*}
\left\{ \begin{array}{l}
D_{q}^\alpha u(t) + f(t, u(t)) = 0, & 0 < t < 1, 1 < \alpha \leq 3, \\
u(0) = D_{q}u(0) = 0, & D_{q}u(1) = \beta \geq 0.
\end{array}\right.
\tag{1.2}
\end{align*}$$

EI-Shahed and AI-Askar [15] studied the existence of a positive solution to the fractional $q$-difference equation

$$\begin{align*}
\left\{ \begin{array}{l}
\overset{c}{D}_{q}^\alpha u(t) + a(t)f(t) = 0, & 0 \leq t \leq 1, 2 < \alpha \leq 3, \\
u(0) = D_{q}^2(0) = 0, & \gamma D_{q}u(1) + \beta D_{q}^2u(1) = 0,
\end{array}\right.
\tag{1.3}
\end{align*}$$

where $\gamma, \beta \leq 0$, and $\overset{c}{D}_{q}^\alpha$ is the fractional $q$-derivative of Caputo type.

Darzi and Agheli [16] studied the existence of a positive solution to the fractional $q$-difference equation

$$\begin{align*}
\left\{ \begin{array}{l}
D_{q}^\alpha u(t) + a(t)f(t) = 0, & 0 \leq t \leq 1, 3 < \alpha \leq 4, \\
u(0) = D_{q}u(0) = D_{q}^2u(0) = 0, & D_{q}^2u(1) = \beta D_{q}^2u(\eta),
\end{array}\right.
\tag{1.4}
\end{align*}$$

where $0 < \eta < 1$ and $1 - \beta \eta^{3-\alpha} > 0$.

The methods used in the papers mentioned are mainly the Krasnoselskii fixed point theorem, the Schauder fixed point theorem, the Leggett–Williams fixed point theorem, and so on. Differently from methods used in the literature mentioned, on the basis of the enlightenment of the works [17, 18, 26], we will use fixed point theorems for mixed monotone operators to demonstrate the existence and uniqueness of positive solutions for integral boundary value problems of the form

$$\begin{align*}
\left\{ \begin{array}{l}
D_{q}^\alpha u(t) + f(t, u(t), u(t)) + g(t, u(t)) = 0, & 0 < t < 1, \\
u(0) = D_{q}u(0) = 0, & u(1) = \mu \int_{0}^{1} u(s) d_{q}s,
\end{array}\right.
\tag{1.5}
\end{align*}$$

where $D_{q}^\alpha$ is the fractional $q$-derivative of Riemann–Liouville type, $0 < q < 1$, $2 < \alpha \leq 3$, $0 < \mu < [\alpha]_{q}$. Our results ensure the existence of a unique positive solution. Moreover, an iterative scheme is constructed for approximating the solution. As far as we know, there are still very few works utilizing the fixed point results for mixed monotone operators to study the existence and uniqueness of a positive solution for fractional $q$-derivative integral boundary value problems.

The plan of the paper is as follows. In Sect. 2, we give not only basic definitions of $q$-fractional integral, but also some properties of certain Green’s functions, which play a fundamental role in the process of proofs. In Sect. 3, in light of some sufficient conditions, we obtained some results on the existence and uniqueness of positive solutions to problem (1.5). At the closing part, two examples are given to demonstrate the serviceability of our main results in Sect. 4.
2 Preliminaries

For convenience of the reader, on one hand, we recall some well-known facts on $q$-calculus and, on the other hand, some notations and lemmas that will be used in the proofs of our theorems.

A nonempty closed convex set $P \subset E$ is a cone if (1) $x \in P, r \geq 0 \Rightarrow rx \in P$ and (2) $x \in P, -x \in P \Rightarrow x = \theta$ ($\theta$ is the zero element of $E$), where $(E, \| \cdot \|)$ is a real Banach space. For all $x, y \in E$, if there exist $\mu, \nu > 0$ such that $\mu x \leq y \leq \nu x$, then we write $x \sim y$. Obviously, $\sim$ is an equivalence relation. Let $P_h = \{x \in E | x \sim h, h > \theta\}$.

Let $q \in (0, 1)$. Then the $q$-number is given by

$$[a]_q = \frac{1 - q^a}{1 - q}, \quad a \in \mathbb{R}.$$  

The $q$-analogue of the power function $(a - b)^{(\alpha)}$ with $n \in \mathbb{N}_0$ is

$$(a - b)^{(0)} = 1, \quad (a - b)^{(\alpha)} = \prod_{k=0}^{n-1} (a - bq^k), \quad n \in \mathbb{N}, a, b \in \mathbb{R}.$$  

More generally, if $\alpha \in \mathbb{R}$, then

$$(a - b)^{(\alpha)} = a^\alpha \prod_{k=0}^{\infty} \frac{a - bq^k}{a - bq^{\alpha+k}}, \quad \alpha \neq 0.$$  

Note that if $b = 0$, then $a^{(\alpha)} = a^\alpha$. The $q$-gamma function is defined by

$$\Gamma_q(x) = \frac{(1 - q)^{(x-1)}}{1 - q^{x-1}}, \quad x \in \mathbb{R},$$  

and satisfies $\Gamma_q(x + 1) = [x]_q \Gamma_q(x)$.

The $q$-derivative of a function $f$ is defined by

$$(D_q f)(x) = \frac{f(qx) - f(x)}{(q - 1)x}, \quad (D_q f)(0) = \lim_{x \to 0} (D_q f)(x),$$  

and $q$-derivatives of higher order by

$$(D^n_q f)(x) = f(x), \quad (D^n_q f)(x) = D_q (D^{n-1}_q f)(x), \quad n \in \mathbb{N}.$$  

The $q$-integral of a function $f$ defined in the interval $[0, b]$ is given by

$$\int_0^b f(s) d_q s = x(1 - q) \sum_{k=0}^{\infty} f(xq^k)q^k, \quad x \in [0, b].$$  

If $a \in [0, b]$ and $f$ is defined in the interval $[0, b]$, then its integral from $a$ to $b$ is defined by

$$\int_a^b f(s) d_q s = \int_0^b f(s) d_q s - \int_0^a f(s) d_q s.$$
Similarly to the derivatives, the operator \( I^0_q \) is given by
\[
(I^0_q f)(x) = f(x), \quad (I^n_q f)(x) = I_q (I^{n-1}_q f)(x), \quad n \in \mathbb{N}.
\]
The fundamental theorem of calculus applies to the operators \( I_q \) and \( D_q \), that is,
\[
(D_q I_q f)(x) = f(x),
\]
and if \( f \) is continuous at \( x = 0 \), then
\[
(I_q D_q f)(x) = f(x) - f(0).
\]
The following formulas will be used later (\( D_q \) denotes the derivative with respect to variable \( t \)):
\[
\begin{align*}
_{t}D_q(t-s)^{(\alpha)} &= [\alpha]_q (t-s)^{\alpha-1}, \\
\left(D_q \int_0^x f(x, t) d_q t \right)(x) &= \int_0^x D_q f(x, t) d_q t + f(qx, x).
\end{align*}
\]
**Definition 2.1** (see [4]) Let \( \alpha \geq 0 \), and let \( f \) be a function defined on \([0, 1]\). The fractional \( q \)-integral of the Riemann–Liouville type is defined by
\[
(I^\alpha_q f)(x) = f(x) \quad \text{and} \quad (I^{\alpha}_q f)(x) = \frac{1}{\Gamma(\alpha)} \int_0^x (x - qt)^{(\alpha-1)} f(t) d_q t, \quad \alpha > 0, x \in [0, 1].
\]
**Definition 2.2** (see [10]) The fractional \( q \)-derivative of the Riemann–Liouville type is defined by
\[
(D^\alpha_q f)(x) = f(x), \quad (D^\alpha_q f)(x) = (D^{\alpha}_q I^{\alpha}_q f)(x), \quad \alpha > 0,
\]
where \( p \) is the smallest integer greater than or equal to \( \alpha \).

**Lemma 2.1** (see [10]) Let \( \alpha, \beta \geq 0 \), and let \( f \) be a function defined on \([0, 1]\). Then the following formulas hold:

(i) \( (I^\alpha_q I^\beta_q f)(x) = (I^{\alpha+\beta}_q f)(x) \),

(ii) \( (D^\alpha_q I^\alpha_q f)(x) = f(x) \).

**Lemma 2.2** (see [10]) Let \( \alpha > 0 \), and let \( p \) be a positive integer. Then the following equality holds:
\[
(I^\alpha_q D^{p}_q f)(x) = (D^{p}_q I^\alpha_q f)(x) - \sum_{k=0}^{p-1} \frac{x^{\alpha-p+k}}{\Gamma(\alpha + k - p + 1)} (D^k_q f)(0).
\]

**Lemma 2.3** (see [27]) Let \( 2 < \alpha \leq 3 \) and \( 0 < \mu < [\alpha]_q \). Let \( x \in C[0, 1] \). Then the boundary value problem
\[
\begin{align*}
D^\alpha_q u(t) + x(t) &= 0, \quad 0 < t < 1, \quad (2.1) \\
u(0) = D_q u(0) &= 0, \quad u(1) = \mu \int_0^1 u(s) d_q s, \quad (2.2)
\end{align*}
\]
has a unique solution

\[ u(t) = \int_0^1 G(t, qs)x(s) \, ds, \]

where

\[
G(t, s) = \begin{cases} 
\frac{\mu^{-1}((1-s)\alpha^{-1}(\alpha - (\mu + q\mu^{-1})) + (\alpha - \mu)(t-s)^{\mu^{-1}} - 1)(\alpha - \mu)^{\alpha-1}}{(\alpha - \mu)^{\alpha-1}(\alpha - \mu)^{\alpha-1}}, & 0 \leq s \leq t \leq 1, \\
\alpha^{-1}(1-s)^{\alpha-1}(\alpha - \mu)(t-s)^{\mu-1}, & 0 \leq t \leq s \leq 1.
\end{cases}
\]

Lemma 2.4 (see [27]) The function \( G(t, qs) \) defined by (2.3) has the following properties:

(i) \( G(t, qs) \) is a continuous function and \( G(t, qs) \geq 0; \)

(ii) \( \frac{\mu^{-1}(1-q)}{(\alpha - \mu)^{\alpha-1}} \leq G(t, qs) \leq \frac{M_0(1-q)}{(\alpha - \mu)^{\alpha-1}}, t, s \in [0, 1], \)

where \( M_0 = \max\{[(\alpha - 1)^{\alpha} - \mu] + \mu q^{\alpha}, q^{\alpha-1}[\alpha]_q \}. \)

Definition 2.3 (see [17]) An operator \( A : P \times P \to P \) is said to be a mixed monotone operator if \( A(x, y) \) is increasing in \( x \) and decreasing in \( y \), that is, \( x_i, y_i \in P \) \((i = 1, 2), x_1 \leq x_2, y_1 \geq y_2 \) imply \( A(x_1, y_1) \leq A(x_2, y_2) \). An element \( x \in P \) is called a fixed point of \( A \) if \( A(x, x) = x \).

Definition 2.4 (see [18]) An operator \( A : P \to P \) is said to be subhomogeneous if

\[ A(tx) \geq tA(x) \quad \text{for any } t \in (0, 1), x \in P. \]

Definition 2.5 (see [18]) Let \( h = \gamma \) be a real number with \( 0 \leq \gamma < 1 \). An operator \( A : D \to D \) is said to be \( \gamma \)-concave if it satisfies

\[ A(tx) \geq t^\gamma A(x) \quad \text{for any } t \in (0, 1), x \in D. \]

Lemma 2.5 (see [17]) Let \( h > \theta \) and \( \gamma \in (0, 1) \).

Let \( A : P \times P \to P \) be a mixed monotone operator satisfying

\[ A(tx, t^{-1}y) \geq t^{\gamma}A(x, y) \quad \text{for any } t \in (0, 1), x, y \in P, \]

and let \( B : P \to P \) be an increasing subhomogeneous operator. Assume that

(i) there is \( h_0 \in P_h \) such that \( A(h_0, h_0) \in P_h \) and \( Bh \in P_h; \)

(ii) there exists a constant \( \delta_0 \) such that \( A(x, y) \geq \delta_0 Bx \) for any \( x, y \in P \).

Then:

(1) \( A : P_h \times P_h \to P_h \) and \( B : P_h \to P_h; \)

(2) there exist \( u_0, v_0 \in P_h \) and \( r \in (0, 1) \) such that

\[ rv_0 \leq u_0 < v_0, \quad u_0 \leq A(u_0, v_0) + Bu_0 \leq A(v_0, u_0) + Bv_0 \leq v_0; \]

(3) the operator equation \( A(x, x) + Bx = x \) has a unique solution \( x^* \in P_h; \)

(4) for any initial values \( x_0, y_0 \in P_h, \) constructing successively the sequences

\[ x_n = A(x_{n-1}, y_{n-1}) + Bx_{n-1}, \quad y_n = A(y_{n-1}, x_{n-1}) + By_{n-1}, \quad n = 1, 2, \ldots, \]

we have \( x_n \to x^* \) and \( y_n \to x^* \) as \( n \to \infty. \)
Remark 2.1 When $B = \theta$ in Lemma 2.5, then the corresponding conclusion still holds.

Lemma 2.6 (see [18]) Let $h > \theta$ and $\gamma \in (0,1)$.
Let $A : P \times P \to P$ be a mixed monotone operator satisfying
\[
A(tx, t^{-1}y) \geq tA(x, y), \quad \text{for any } t \in (0, 1), x, y \in P, \tag{2.7}
\]
and let $B : P \to P$ be an increasing $\gamma$-concave operator. Assume that
(i) there is $h_0 \in P_h$ such that $A(h_0, h_0) \in P_h$ and $Bh_0 \in P_h$;
(ii) there exists a constant $\delta_0$ such that $A(x, y) \leq \delta_0 Bx$ for any $x, y \in P$.

Then:
(1) $A : P_h \times P_h \to P_h$ and $B : P_h \to P_h$;
(2) there exist $u_0, v_0 \in P_h$ and $r \in (0, 1)$ such that
\[
rv_0 \leq u_0 < v_0, \quad u_0 \leq A(u_0, v_0) + Bu_0 \leq A(v_0, u_0) + Bv_0 \leq v_0;
\]
(3) the operator equation $A(x, x) + Bx = x$ has a unique solution $x^* \in P_h$;
(4) for any initial values $x_0, y_0 \in P_h$, constructing successively the sequences
\[
x_n = A(x_{n-1}, y_{n-1}) + Bx_{n-1}, \quad y_n = A(y_{n-1}, x_{n-1}) + By_{n-1}, \quad n = 1, 2, \ldots,
\]
we have $x_n \to x^*$ and $y_n \to x^*$ as $n \to \infty$.

Remark 2.2 When $A = \theta$ in Lemma 2.6, then the corresponding conclusion still holds.

3 Main results

In this section, we give and prove our main results by applying Lemmas 2.5 and 2.6. We consider the Banach space $X = C[0, 1]$ endowed with standard norm $\|x\| = \sup |x(t)| : t \in [0, 1]|$. Clearly, this space can be equipped with a partial order given by
\[
x, y \in C[0, 1], \quad x \leq y \iff x(t) \leq y(t) \quad \text{for } t \in [0, 1].
\]
We define the cone $P = \{x \in X : x(t) \geq 0, t \in [0, 1]\}$. Notice that $P$ is a normal cone in $C[0, 1]$ and the normality constant is 1.

Theorem 3.1 Suppose that
\begin{itemize}
  \item (F1) a function $f(t, x, y) : [0, 1] \times [0, +\infty) \times [0, +\infty) \to [0, +\infty)$ is continuous, increasing with respect to the second variable, and decreasing with respect to the third variable;
  \item (F2) a function $g(t, x) : [0, 1] \times [0, +\infty) \to [0, +\infty)$ is continuous and increasing with respect to the second variable;
  \item (F3) there exists a constant $\lambda \in (0,1)$ such that $f(t, \lambda x, \lambda^{-1} y) \geq \lambda^\gamma f(t, x, y)$ for any $t \in [0, 1], x, y \in [0, +\infty)$, and $g(t, \lambda x) \geq \lambda g(t, x)$ for $\lambda \in (0, 1), t \in [0, 1], u \in [0, +\infty)$, and $g(t, 0) \equiv 0$;
  \item (F4) there exists a constant $\delta_0 > 0$ such that $f(t, x, y) \geq \delta_0 g(t, x)$, $t \in [0, 1], x, y \geq 0$.
\end{itemize}

Then:
T is a mixed monotone operator. In fact, for $u_1, u_2, v_1, v_2 \in P$ with $u_1 \geq u_2$ and $v_1 \leq v_2$, it is easy to see that $u_1(t) \geq u_2(t)$, $v_1(t) \leq v_2(t)$, $t \in [0, 1]$, and by Lemma 2.4 and (F1),

$$T_1(u_1, v_1)(t) = \int_0^1 G(t, qs)f(s, u_1(s), v_1(s)) \, dq \, s \geq \int_0^1 G(t, qs)f(s, u_2(s), v_2(s)) \, dq \, s = T_1(u_2, v_2)(t).$$  

(3.3)

For any $\lambda \in (0, 1)$ and $u, v \in P$, by (F3) we have

$$T_1(\lambda u, \lambda^{-1} v)(t) = \int_0^1 G(t, qs)f(s, \lambda u(s), \lambda^{-1} v(s)) \, dq \, s \geq \lambda^{-1} \int_0^1 G(t, qs)f(s, u(s), v(s)) \, dq \, s \geq \lambda^{-1} T_1(u, v)(t).$$  

(3.4)

So, the operator $T_1$ satisfies (2.6).
For any \( u_1(t) \geq u_2(t), \ t \in [0,1] \), from \( G(t,qs) \geq 0 \) and \((F_2)\) we know that

\[
T_2u_1(t) = \int_0^1 G(t,qs)g(s,u_1(s)) \, dq = \int_0^1 G(t,qs)g(s,u_2(s)) \, dq = T_2u_2(t).
\]

So \( T_2 \) is increasing. Further, for any \( \lambda \in (0,1) \) and \( u \in P \), from hypothesis \((F_3)\) we get

\[
T_2(\lambda u)(t) = \int_0^1 G(t,qs)g(s,\lambda u(s)) \, dq \geq \lambda \int_0^1 G(t,qs)g(s,u(s)) \, dq = \lambda T_2u(t), \tag{3.5}
\]

that is, the operator \( T_2 \) is subhomogeneous. By \((F_1)\) and Lemma 2.4, for any \( t \in [0,1] \), we have

\[
T_1(h,h)(t) = \int_0^1 G(t,qs)f(s,h(s),h(s)) \, dq \\
= \int_0^1 G(t,qs)f(s,s^{q-1},s^{q-1}) \, dq \\
\leq \frac{M_0}{\Gamma_q(\alpha)([\alpha]_q - \mu)} h(t) \int_0^1 f(s,0,0) \, dq \tag{3.6}
\]

and

\[
T_1(h,h)(t) = \int_0^1 G(t,qs)f(s,h(s),h(s)) \, dq \\
= \int_0^1 G(t,qs)f(s,s^{q-1},s^{q-1}) \, dq \\
\geq \frac{\mu q^q}{\Gamma_q(\alpha)([\alpha]_q - \mu)} h(t) \int_0^1 s(1-qs)^{(\alpha-1)}f(s,0,1) \, dq. \tag{3.7}
\]

From \((F_2)\) and \((F_3)\) we have the inequality

\[
f(s,1,0) \geq f(s,0,1) \geq \delta_0 g(s,0) \geq 0.
\]

Since \( g(t,0) \neq 0 \), we also obtain

\[
\int_0^1 f(s,1,0) \, dq \geq \int_0^1 f(s,0,1) \, dq \geq \delta_0 \int_0^1 g(s,0) \, dq > 0. \tag{3.8}
\]

Let

\[
M_1 = \frac{M_0}{\Gamma_q(\alpha)([\alpha]_q - \mu)} \int_0^1 f(s,1,0) \, dq, \\
M_2 = \frac{\mu q^q}{\Gamma_q(\alpha)([\alpha]_q - \mu)} \int_0^1 s(1-qs)^{(\alpha-1)}f(s,0,1) \, dq, \\
M_3 = \frac{\mu q^q}{\Gamma_q(\alpha)([\alpha]_q - \mu)} \int_0^1 s(1-qs)^{(\alpha-1)}g(s,0) \, dq, \\
M_4 = \frac{M_0}{\Gamma_q(\alpha)([\alpha]_q - \mu)} \int_0^1 g(s,1) \, dq.
\]
Thus we have $M_2 h(t) \leq T_1(h, h) \leq M_1 h(t), M_2 h(t) \leq T_2 h \leq M_1 h(t), t \in [0, 1].$ So, $T_1(h, h) \in P_h.$ From $g(t, 0) \equiv 0$ it is easy to see that $T_2 h \in P_h.$ So, there is $h(t) = t^{a-1} \in P_h$ such that $T_1(h, h) \in P_h$ and $T_2 h \in P_h.$

Next, we prove that the operators $T_1$ and $T_2$ satisfy condition (ii) of Lemma 2.5. In fact, for $u, v \in P$ and any $t \in [0, 1],$ by $(F_3)$ we have

$$T_1(u, v)(t) = \int_0^1 G(t, qs) f(s, u(s), v(s)) \, ds \geq \delta_0 \int_0^1 G(t, qs) g(s, u(s)) \, ds = \delta_0(T_2 u)(t).$$  \hspace{1cm} (3.9)

Then we have $T_1(u, v) \geq \delta_0 T_2 u$ for $u, v \in P.$ By Lemma 2.5 we can deduce: there exist $u_0, v_0 \in P_h$ and $r \in (0, 1)$ such that $rv_0 \leq u_0 \leq v_0,$ $u_0 \leq T_1(u_0, v_0) + T_2 u_0 \leq T_1(v_0, u_0) + T_2 v_0 \leq v_0;$ the operator equation $T_1(u, u) + T_2 u = u$ has a unique solution $u^* \in P_h;$ and for any initial values $x_0, y_0 \in P_h,$ constructing successively the sequences

$$x_n = T_1(x_{n-1}, y_{n-1}) + T_2 x_{n-1}, \quad y_n = T_1(y_{n-1}, x_{n-1}) + T_2 y_{n-1}, \quad n = 1, 2, \ldots,$$

we get $x_n \to u^*$ and $y_n \to u^*$ as $n \to \infty.$ We have the following two inequalities:

$$u_0(t) \leq \int_0^1 G(t, qs) [f(s, u_0(s), v_0(s)) + g(s, u_0(s))] \, ds, \quad t \in [0, 1],$$

$$v_0(t) \geq \int_0^1 G(t, qs) [f(s, v_0(s), u_0(s)) + g(s, v_0(s))] \, ds, \quad t \in [0, 1].$$

Thus problem (1.5) has a unique positive solution $u^* \in P_h;$ for any $u_0, v_0 \in P_h,$ constructing successively the sequences

$$x_{n+1}(t) = \int_0^1 G(t, qs) [f(s, x_n(s), y_n(s)) + g(s, x_n(s))] \, ds, \quad n = 0, 1, 2, \ldots,$$

$$y_{n+1}(t) = \int_0^1 G(t, qs) [f(s, y_n(s), x_n(s)) + g(s, y_n(s))] \, ds, \quad n = 0, 1, 2, \ldots,$$

we have $\|x_n - u^*\| \to 0$ and $\|y_n - u^*\| \to 0$ as $n \to \infty.$ \hfill \Box

**Corollary 3.1** Suppose that $f$ satisfies the conditions of Theorem 3.1 and $g \equiv 0,$ $f(t, 0, 1) \equiv 0.$ Then:

(i) there exist $u_0, v_0 \in P_h$ and $r \in (0, 1)$ such that $rv_0 \leq u_0 < v_0,$ and

$$u_0(t) \leq \int_0^1 G(t, qs) [f(s, u_0(s), v_0(s))] \, ds, \quad t \in [0, 1],$$

$$v_0(t) \geq \int_0^1 G(t, qs) [f(s, v_0(s), u_0(s))] \, ds, \quad t \in [0, 1],$$

where $G(t, qs)$ is defined by (2.3), and $h(t) = t^{a-1}, t \in [0, 1]$;
Similarly to the proof of Theorem 3.1, 

**Theorem 3.2** Suppose that 

\[
(\begin{array}{ll}
F_5 & \text{there exists a constant } \gamma \in (0, 1) \text{ such that } g(t, \lambda u) \geq \lambda^\gamma g(t, u) \text{ for any } t \in [0, 1], \\
\lambda & \in (0, 1), \ u \in [0, +\infty), \ \text{and } f(t, \lambda u, \lambda^{-1} v) \geq \lambda f(t, u, v) \text{ for } \lambda \in (0, 1), \ t \in [0, 1], \ u, v \in [0, +\infty); \\
F_6 & f(t, 0, 1) \neq 0 \text{ for } t \in [0, 1], \ \text{and there exists a constant } \delta_0 > 0 \text{ such that } f(t, u, v) \leq \delta_0 g(t, u), \ t \in [0, 1], \ u, v \geq 0.
\end{array})
\]

Then: 

1. there exist \( u_0, v_0 \in P_h \) and \( r \in (0, 1) \) such that \( rv_0 \leq u_0 \leq v_0 \) and 

\[
u_0 \leq \int_0^1 G(t,qs) \left[f(s, u_0(s), v_0(s)) + g(s, u_0(s))\right] ds, \quad t \in [0, 1],
\]

\[
v_0 \geq \int_0^1 G(t,qs) \left[f(s, v_0(s), u_0(s)) + g(s, v_0(s))\right] ds, \quad t \in [0, 1],
\]

where \( G(t,qs) \) is defined by (2.3), and \( h(t) = t^{r-1}, \ t \in [0, 1]; \)

2. the boundary value problem (1.5) has a unique positive solution \( u^* \) in \( P_h \); and for any \( x_0, y_0 \in P_h, \) the sequences 

\[
x_{n+1} = \int_0^1 G(t,qs) \left[f(s, x_n(s), y_n(s)) + g(s, x_n(s))\right] ds, \quad n = 0, 1, 2, \ldots,
\]

\[
y_{n+1} = \int_0^1 G(t,qs) \left[f(s, y_n(s), x_n(s)) + g(s, y_n(s))\right] ds, \quad n = 0, 1, 2, \ldots,
\]

satisfy \( \|x_n - u^*\| \to 0 \) and \( \|y_n - u^*\| \to 0 \) as \( n \to \infty. \)

**Proof** Similarly to the proof of Theorem 3.1, \( T_1 \) and \( T_2 \) are given in (3.2). From \( (F_1) \) and \( (F_2) \) we know that \( T_1 : P \times P \to P \) is a mixed monotone operator and \( T_2 : P \to P \) is increasing. By \( (F_5) \) we obtain 

\[
T_1(\lambda u, \lambda^{-1} v) \geq \lambda T_1(u, v), \quad T_2(\lambda u) \geq \lambda^\gamma T_2 u, \quad \text{for } \lambda \in (0, 1), u, v \in P.
\]
According to $(F_2)$ and $(F_6)$, we have
\[ f(s,0,1) \leq \delta_0 g(s,0), \quad f(s,0,1) \leq f(s,1,0), \quad s \in [0,1]. \]

From $f(t,0,1) \neq 0$ we get
\[ 0 < \int_0^1 f(s,0,1) \, dq_s \leq \int_0^1 f(s,1,0) \, dq_s, \]
\[ 0 < \frac{1}{\delta_0} \int_0^1 f(s,0,1) \, dq_s \leq \int_0^1 g(s,0) \, dq_s \leq \int_0^1 g(s,1) \, dq_s, \]

and the following inequalities hold:
\[ 0 < \frac{\mu q^\alpha}{\Gamma_q(\alpha)(\alpha - \mu)} \int_0^1 s(1-qs)^{(\alpha-1)} f(s,0,1) \, dq_s \]
\[ \leq \frac{M_0}{\Gamma_q(\alpha)(\alpha - \mu)} \int_0^1 f(s,0,1) \, dq_s, \quad (3.11) \]
\[ 0 < \frac{\mu q^\alpha}{\Gamma_q(\alpha)(\alpha - \mu)} \int_0^1 s(1-qs)^{(\alpha-1)} g(s,0) \, dq_s \]
\[ \leq \frac{M_0}{\Gamma_q(\alpha)(\alpha - \mu)} \int_0^1 g(s,1) \, dq_s. \quad (3.12) \]

Hence we can easily check that $T_1(h,h) \in P$, $T_2h \in P$, $t \in [0,1]$, and, by using $(F_6)$, we have
\[ T_1(u,v)(t) = \int_0^1 G(t,s) f(s,u(s),v(s)) \, dq_s \]
\[ \leq \delta_0 \int_0^1 G(t,s) g(s,u(s)) \, dq_s = \delta_0 T_2u(t). \quad (3.13) \]

Then we have $T_1(u,v) \leq \delta_0 T_2u$ for $u,v \in P$. Thus, from Lemma 2.6 we get that there exist $u_0,v_0 \in P_h$ and $r \in (0,1)$ such that $r v_0 \leq u_0 \leq v_0$, $u_0 \leq T_1(u_0,v_0) + T_2u_0 \leq T_1(v_0,u_0) + T_2v_0 \leq v_0$; the operator equation $T_1(u,u) + T_2u = u$ has a unique solution $u^* \in P_h$; and for any initial values $x_0,y_0 \in P_h$, the sequences
\[ x_n = T_1(x_{n-1},y_{n-1}) + T_2x_{n-1}, \quad y_n = T_1(y_{n-1},x_{n-1}) + T_2y_{n-1}, \quad n = 1,2,\ldots, \]

satisfy $x_n \rightarrow u^*$ and $y_n \rightarrow u^*$ as $n \rightarrow \infty$. That is,
\[ u_0(t) \leq \int_0^1 G(t,qs) \left[ f(s,u_0(s),v_0(s)) + g(s,u_0(s)) \right] \, dq_s, \quad t \in [0,1], \]
\[ v_0(t) \geq \int_0^1 G(t,qs) \left[ f(s,v_0(s),u_0(s)) + g(s,v_0(s)) \right] \, dq_s, \quad t \in [0,1], \]
The boundary value problem (1.5) has a unique positive solution $u^* \in P_h$; for $u_0, v_0 \in P_h$, the sequences

\[
x_{n+1}(t) = \int_0^1 G(t, qs) \left[ f(s, x_n(s), y_n(s)) + g(s, x_n(s)) \right] ds, \quad n = 0, 1, 2, \ldots,
\]

\[
y_{n+1}(t) = \int_0^1 G(t, qs) \left[ f(s, y_n(s), x_n(s)) + g(s, y_n(s)) \right] ds, \quad n = 0, 1, 2, \ldots,
\]

satisfy $\|x_n - u^*\| \to 0$ and $\|y_n - u^*\| \to 0$ as $n \to \infty$. □

**Corollary 3.2** Suppose that $g$ satisfies the conditions of Theorem 3.2, $f \equiv 0$, and $g(t, 0) \not\equiv 0$ for $t \in [0, 1]$. Then:

(i) there exist $u_0, v_0 \in P_h$ and $r \in (0, 1)$ such that $rv_0 \leq u_0 < v_0$ and

\[
u_0(t) \geq \int_0^1 G(t, qs) \left[ g(s, v_0(s)) \right] ds, \quad t \in [0, 1],
\]

where $G(t, qs)$ is defined by (2.3), and $h(t) = t^{\alpha-1}$, $t \in [0, 1]$;

(ii) the BVP

\[
\begin{aligned}
D^\alpha u(t) + g(t, u(t)) &= 0, \quad 0 < t < 1, 2 < \alpha \leq 3, \\
u(0) &= D^\alpha u(0) = 0, \quad u(1) = \mu \int_0^1 u(s) ds,
\end{aligned}
\]

(3.14)

has a unique positive solution $u^*$ in $P_h$; and for any $x_0, y_0 \in P_h$, the sequences

\[
x_{n+1} = \int_0^1 G(t, qs) g(s, x_n(s)) ds, \quad n = 0, 1, 2, \ldots,
\]

\[
y_{n+1} = \int_0^1 G(t, qs) g(s, y_n(s)) ds, \quad n = 0, 1, 2, \ldots,
\]

satisfy $\|x_n - u^*\| \to 0$ and $\|y_n - u^*\| \to 0$ as $n \to \infty$.

### 4 Example

Now, we give two examples to illustrate our results.

**Example 4.1** Consider the following boundary value problem:

\[
\begin{aligned}
-D^{5/2}_t u(t) &= u(t)^{1/2} + [u(t) + 1]^{-1/2} + \frac{u(t)}{1 + u(t)} t^3 + t^2 + 4, \quad 0 < t < 1, \\
u(0) &= D^{5/2}_t u(0) = 0, \quad u(1) = \mu \int_0^1 u(s) ds.
\end{aligned}
\]

(4.1)

In this example, we let

\[
f(t, u, v) = u^{1/2} + [v + 1]^{-1/2} + t^2 + 2, \quad g(t, u) = \frac{u}{1 + u} t^3 + 2,
\]

$\gamma = \frac{1}{2}$, $\mu = \frac{1}{2}$. 

It is not difficult to find that $f(t,x,y) : [0,1] \times [0, +\infty) \times [0, +\infty) \rightarrow [0, +\infty)$ is continuous, increasing with respect to the second variable, and decreasing with respect to the third variable and that $g(t,x) : [0,1] \times [0, +\infty) \rightarrow [0, +\infty)$ is continuous with $g(t,0) = 2 > 0$ and increasing with respect to the second variable. We also have

$$g(t,\lambda u) = \frac{\lambda u}{1 + \lambda u} t^3 + 2 \geq \frac{\lambda u}{1 + u} t^3 + 2 = \lambda g(t,u), \quad \lambda \in (0,1),$$

$$f(t,\lambda u,\lambda^{-1} v) = \lambda \frac{1}{2} u^\frac{1}{2} + \lambda \frac{1}{2} [v + \lambda]^{-\frac{1}{2}} + t^3 \geq \lambda \frac{1}{2} \left\{ u^{\frac{1}{2}} + [v + 1]^{-\frac{1}{2}} + t^3 \right\}$$

$$= \lambda f(t,u,v).$$

Further, if we take $\delta_0 \in (0, \frac{2}{3}]$, then we easily get

$$f(t,u,v) = u^{\frac{1}{3}} + [v + 1]^{-\frac{1}{3}} + t^3 \geq \frac{2}{3} \cdot 3$$

$$\geq \delta_0 \left[ \frac{u}{1 + u} t^3 + 2 \right] = \delta_0 g(t,u).$$

So $f$ and $g$ satisfy the conditions of Theorem 3.1. Thus by Theorem 3.1 the boundary value problem (4.1) has a unique positive solution in $P_h$, where $h(t) = t^{\alpha - 1} = t^{\frac{3}{2}}$, $t \in [0,1]$.

**Example 4.2** Consider the following boundary value problem:

$$\begin{cases}
-D^{\frac{3}{2}} \frac{1}{2} u(t) = \left( \frac{u(t)}{1 + |u(t)|} \right)^{\frac{1}{2}} + \left[ u(t) + 1 \right]^{-\frac{1}{2}} + t^3 + u(t)^{\frac{1}{2}} + t^2 + 1, & 0 < t < 1, \\
u(0) = D^\frac{1}{2} u(0) = 0, & u(1) = \mu \int_0^1 u(s) d^{\frac{1}{2}} s.
\end{cases} \quad (4.2)$$

We let

$$f(t,u,v) = \left( \frac{u}{1 + u} \right)^{\frac{1}{2}} + [v + 1]^{-\frac{1}{2}} + t^3, \quad g(t,u) = u^{\frac{1}{3}} + t^3 + 1, \quad \gamma = \frac{1}{3}, \quad \mu = \frac{1}{2}.$$

It is not difficult to find that $f(t,x,y) : [0,1] \times [0, +\infty) \times [0, +\infty) \rightarrow [0, +\infty)$ is continuous, increasing with respect to the second variable, and decreasing with respect to the third variable and that $g(t,x) : [0,1] \times [0, +\infty) \rightarrow [0, +\infty)$ and increasing with respect to the second variable. We also have

$$g(t,\lambda u) = \lambda \frac{1}{2} u^\frac{1}{2} + t^3 + 1 \geq \lambda \frac{1}{2} \left\{ u^{\frac{1}{2}} + t^3 + 1 \right\} \geq \lambda \frac{1}{2} g(t,u), \quad \lambda \in (0,1),$$

$$f(t,\lambda u,\lambda^{-1} v) = \left( \frac{\lambda u}{1 + \lambda u} \right)^{\frac{1}{2}} + \left[ \lambda^{-1} v + 1 \right]^{-\frac{1}{2}} + t^3$$

$$\geq \lambda \frac{1}{2} \left\{ \left( \frac{u}{1 + u} \right)^{\frac{1}{2}} + [v + \lambda]^{-\frac{1}{2}} + t^3 \right\}$$

$$\geq \lambda \left\{ \left( \frac{u}{1 + u} \right)^{\frac{1}{2}} + [v + 1]^{-\frac{1}{2}} + t^3 \right\}$$

$$= \lambda f(t,u,v).$$
If we take \( \delta_0 = 1 > 0 \), then we have

\[
f(t, u, v) = \left( \frac{u}{1 + u} \right)^{\frac{1}{4}} + [v + 1]^{-\frac{1}{4}} + t^3 \leq u^{\frac{1}{4}} + t^2 + 1 \leq u^{\frac{1}{3}} + t^2 + 1 = \delta_0 g(u, t).
\]

So \( f \) and \( g \) satisfy the conditions of Theorem 3.2. Thus by Theorem 3.2 the boundary value problem (4.2) has a unique positive solution in \( P_h \), where \( h(t) = t^{\alpha - 1} = t^{\frac{3}{2}} \), \( t \in [0, 1] \).

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Competing interests

The authors declare that they have no competing interests.

Authors’ contributions

FG carried out the molecular genetic studies, participated in the sequence alignment, and drafted the manuscript. SK conceived the study and participated in its design and coordination. FC helped to draft the manuscript. All authors read and approved the final manuscript.

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