DYNAMICS OF BREATHERS IN THE GARDNER HIERARCHY: UNIVERSALITY OF THE VARIATIONAL CHARACTERIZATION

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ABSTRACT. We present a new variational characterization of breather solutions of any equation of the focusing Gardner hierarchy. This hierarchy is characterized by a nonnegative index \( n \), and \( 2n + 1 \) represents the order of the corresponding PDE member. In this paper, we first show the existence of such breathers, and that they are solutions of the \((2n+1)\)th-order Gardner equation. Then we prove a variational universality property, in the sense that all these breather solutions satisfy the same fourth order stationary elliptic ODE, regardless the order of the hierarchy member. This fact also characterizes them as critical points of the same Lyapunov functional, that we also construct here. As by product of our approach, we find breather solutions of the hierarchy of \((2n+1)\)th-order mKdV equations, as well as a respective characterization of them as solutions of a fourth order stationary elliptic ODE. We also extend part of these results to the periodic setting, presenting new breather solutions for the 5th and 7th mKdV members of the hierarchy. Finally, we prove ill-posedness results for the whole Gardner hierarchy, by using appropriately their breather solutions.

1. INTRODUCTION

In this work we are concerned with the focusing Gardner hierarchy that we define as follows

\[
  u_t = -\frac{\partial}{\partial x} \left( -i \frac{\partial}{\partial x} + 2(\mu + u) \right) \mathcal{L}_n[u_x + (\mu + u)^2],
\]

\[
  u(t, x) \in \mathbb{R}, \quad \mu \in \mathbb{R}^+, \quad n \in \mathbb{N},
\]

which we will call it hereafter as \((2n+1)\)th-order focusing Gardner equations. Here \( \mathcal{L}_n \) are the Lenard operators defined recursively by

\[
  \frac{\partial}{\partial x} \mathcal{L}_{n+1}[v] = \left( \frac{\partial^3}{\partial x^3} + 4i \frac{\partial}{\partial x} + 2v_x \right) \mathcal{L}_n[v], \quad n \in \mathbb{N}.
\]

For instance, starting with \( \mathcal{L}_0[v] = \frac{1}{x} \), we get
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\[ L_1[v] = v, \]
\[ L_2[v] = v_{xx} + 3v^2, \]
\[ L_3[v] = v_{4x} + 10vv_{xx} + 5v_x^2 + 10v^3, \] (1.3)

and
\[ L_4[v] = v_{6x} + 14vv_{4x} + 28v_xv_{3x} + 21v_x^2 + 70v^2v_{xx} + 70vv_x^2 + 35v^4, \]
\[ L_5[v] = v_{8x} + 18v_{6x} + 54v_xv_{5x} + 114v_{2x}v_{4x} + 126v^2u_{4x} + 69v_{3x}^2 \]
\[ + 504v_xv_{3x} + 420v^3v_{xx} + 390vv_x^2 + 630v^2v_x^2 \]
\[ + 462v_x^2v_{2x} + 966v_x^2v_{3x} + 1260v^5v_x^2 + 126v^5. \] (1.4)

This recursion relation (1.1) generates the whole focusing Gardner hierarchy. Note that we will only work with the focusing version (via the complex Miura transformation \( iu_x + (\mu + u)^2 \)) since we are interested in real valued and regular solutions (namely, solitons and breathers), instead of the singular structures which appear in the defocusing Gardner hierarchy, and which, on the other hand, was already studied by Gomes et al \([16]\). Moreover, only positive values for the parameter \( \mu \) will be considered in the present work, since as it was recently discovered by Muñoz and Ponce \([24]\), Gardner breathers do not exist when \( \mu \in \mathbb{R}^- \). See below for more details of breather solutions \([16,10]\).

Using the Lenard operators \( L_n \) \([1.2]-[1.3]-[1.4]\) and a suitable spatial translation to remove linear terms, a first few members of the Gardner hierarchy are (see appendix \([5.1]\) for 9th and 11th order Gardner equations):

the focusing Gardner \((n = 1)\)
\[ u_t + (u_{xx} + 6\mu u^2 + 2u^3)_x = 0, \] (1.5)

the 5th-order Gardner \((n = 2)\)
\[ u_t + u_{5x} + 10u^2u_{3x} + 20\mu uu_{3x} + 10u^2u_{3x} + 120u^3uu_x \]
\[ + 180\mu^2u^2u_x + 120\mu^3u_x + 10u_x^3 + 30u^4u_x + 40\mu u_x u_{xx} + 40uu_x u_{xx} = 0, \] (1.6)

the 7th-order Gardner \((n = 3)\)
\[ u_t + u_{7x} + 14u^2u_{5x} + 28\mu uu_{5x} + 14u^2u_{5x} + 70u^4u_{3x} + 280u^3uu_{3x} \]
\[ + 420\mu^2u^2u_{3x} + 280\mu^3u_{3x} + 70u^4u_{3x} + 840\mu uu_{x} + 2100u^4u_{x} + 2800\mu^3u^3u_x \]
\[ + 420\mu^2u_x^3 + 2100\mu^2u_x^3 + 840\mu uu_x^3 + 840\mu uu_x^5 + 2800\mu u_x^3 + 140u^6u_x \]
\[ + 84\mu uu_x u_{xx} + 84uu_{xx}u_{4x} + 4u_{xx}u_{4x} + 140uu_{xx}u_{3x} + 140uu_{xx}u_{3x} + 126u_x^2u_{3x} + 560\mu^3u_x u_{xx} \]
\[ + 1680\mu^2u_x u_{xx} + 1680\mu uu_x u_{xx} + 182u_x u_{xx}^2 + 560u^3u_x u_{xx} = 0. \] (1.7)

Note, moreover that selecting \( \mu = 0 \) in (1.1), we get the focusing mKdV hierarchy of equations, defined by
\[ u_t = -\frac{\partial}{\partial x} \left( -i\frac{\partial}{\partial x} + 2u \right) L_n[iu_x + u^2], \quad n \in \mathbb{N}. \] (1.8)

This recursion relation (1.8) generates the whole mKdV hierarchy, recovering the one depicted in \([23,21]\). Many previous works have shown explicit solutions of hierarchies. For the mKdV hierarchy, Matsuno \([21]\) proved the existence and
built explicitly the $N$-soliton solution. More recently, Gomes et al [16] dealt with the defocusing mKdV with non vanishing boundary conditions (NVBC) and the associated defocusing Gardner hierarchy, showing multisolitonic structures. Unfortunately, many of the zero boundary value solutions are singular.

With respect to breather solutions, they are defined as localized in space and periodic in time (up to symmetries of the equation) functions. For instance, in the Gardner case, they are shortly defined as follows:

**Definition 1.1 (Gardner breather).** Let $\alpha, \beta \in \mathbb{R}\backslash\{0\}$, $\mu \in \mathbb{R}^+\setminus\{0\}$ such that $\Delta = \alpha^2 + \beta^2 - 4\mu^2 > 0$. A breather solution $B_\mu$ of the classical Gardner equation (1.5) is given by the formula

$$B_\mu \equiv B_{\alpha,\beta,\mu}(t, x; x_1, x_2) := 2\partial_x \left[ \arctan \left( \frac{\beta \sqrt{\alpha^2 + \beta^2}}{\alpha \sqrt{\Delta}} \sin(\alpha y_1) - h_1(t, x) \right) \right],$$

(1.9) where $h_1, h_2$ are precise trigonometric and hyperbolic functions.

See Theorem 1.1 for a detailed and complete definition. Breathers were previously studied for mKdV [6, 7], Gardner [8], sine-Gordon [10] and NLS [9] equations and also in many other nonlinear models (see [18, 19, 13, 14, 15]). In some of these works, breather solutions are indeed characterized as solutions of a precise fourth order ODE (mKdV and Gardner) or a system of ODEs (sine-Gordon), and indeed being defined as local minimizers of suitable Lyapunov functionals built as linear combinations of conserved quantities up to $H^2$ level. In the case of higher order equations, some stability results were proved for breather solutions of the 5th, 7th and 9th mKdV equations in [2] and also for the breather solution of the 5th Gardner equation, once the global in time behavior of its solutions was well understood [5, 3]. However, with respect to Gardner or mKdV hierarchies and as far as we know, no real regular breather solutions were shown or mentioned explicitly in the literature [20, 16].

In this work we present breather solutions for the whole Gardner and mKdV hierarchies (1.1)-(1.8), and we will show that in fact they share the same functional profile for the whole hierarchy, up to corresponding speed parameters, which will depend on the level of the hierarchy considered (see (1.10) for a detailed definition). This is for us a nonexpected and surprising universality property. Even more, in the periodic mKdV setting, we obtain a detailed description of periodic breather solutions of the corresponding 5th and 7th order mKdV equations (see Section 6). We believe in fact that a similar complete description of periodic breather solutions for the whole mKdV and Gardner hierarchies is feasible.

Also the main aim of this work is to show that all breather solutions of Gardner and mKdV hierarchies satisfy a universal fourth order ODE, which is the same for any breather solution of the corresponding equation member of the considered hierarchy. This universality expands the variational characterization of these breather solutions, meaning that for any member of the Gardner and mKdV hierarchies, a breather solution is a critical point of a precise Lyapunov functional defined in the Sobolev space $H^2$.

Therefore, the goal of this work is twofold: firstly we are going to define and to characterize variationally regular breather and N-soliton solutions of the Gardner...
and mKdV hierarchies of equations. Secondly, the most important result will be to show that any higher order Gardner (mKdV) breather solution of the Gardner (mKdV) hierarchy of equations (1.1)-(1.8) satisfies the same fourth-order stationary elliptic ODE and it is a critical point of a Lyapunov functional defined in $H^2$. That means that the breather solution of the corresponding equation member of the Gardner (mKdV) hierarchy holds a fourth order ODE, which is the same for all breather solutions of any higher order Gardner (mKdV) equation of the Gardner (mKdV) hierarchy, and therefore the same universal ODE independently of the level of the Gardner (mKdV) hierarchy considered. Using these higher order breather solutions of the Gardner hierarchy, we will prove the ill-posedness of the Gardner hierarchy, determining the critical Sobolev index depending on the level of the hierarchy. Finally, and for the shake of completeness, we will provide a complete description of periodic breather solutions of 5th and 7th order mKdV equations.

In short, we list our main results as follows:

**Theorem 1.2** (Existence of Gardner hierarchy breathers). For all $n \in \mathbb{N}$ there exists a breather solution of the corresponding $(2n+1)$-th order Gardner equation of the Gardner hierarchy (1.1). More precisely

1. Structure: all these breather solutions have the same functional structure, namely given $\alpha, \beta \in \mathbb{R}\setminus\{0\}$ and $\mu \in \mathbb{R}^+\setminus\{0\}$ such that $\Delta = \alpha^2 + \beta^2 - 4\mu^2 > 0$, and $x_1, x_2 \in \mathbb{R}$ we have

$$B_\mu := 2\partial_x \left[ \arctan \left( \frac{\beta\sqrt{\alpha^2+\beta^2} \sin(\alpha y_1)}{\alpha\sqrt{\Delta}} - \frac{2\mu \beta \sin(\beta y_2)}{\Delta} \right) \right],$$

(1.10)

with $y_1 = x + \delta_{2n+1,\mu} t + x_1$ and $y_2 = x + \gamma_{2n+1,\mu} t + x_2$.

2. Velocities: The velocities $\gamma_{2n+1,\mu}$ and $\delta_{2n+1,\mu}$ are the only parameters depending on the level of the hierarchy considered, in the following explicit form:

$$\gamma_{2n+1,\mu} := -\frac{1}{\beta} \text{Re} \left[ \sum_{p=1}^{n} a_{p,n}(\beta + i\alpha)^{2p+1}\mu^{2(n-p)} \right],$$

$$\delta_{2n+1,\mu} := -\frac{1}{\alpha} \text{Im} \left[ \sum_{p=1}^{n} a_{p,n}(\beta + i\alpha)^{2p+1}\mu^{2(n-p)} \right],$$

where $a_{p,n} \equiv$ coefficient of the term $\mu^{2(n-p)}u_{(2p+1)x}$ in the $(2n+1)$th-order Gardner equation (1.1).

3. Smoothness: Each breather solution of the Gardner hierarchy above presented is smooth in time and space, and belong to the Schwartz class in space.

4. Convergence to mKdV: For all $n \in \mathbb{N}$, these breather solutions of the Gardner hierarchy reduce to breather solutions of the mKdV hierarchy (1.8) as $\mu \to 0$, namely

$$B = B_{\alpha,\beta,n}(t, x; x_1, x_2) := 2\partial_x \left[ \arctan \left( \frac{\beta \sin(\alpha y_1)}{\alpha \cosh(\beta y_2)} \right) \right],$$

(1.11)
with $y_1$ and $y_2$

$$y_1 = x + \delta_{2n+1}t + x_1, \quad y_2 = x + \gamma_{2n+1}t + x_2,$$  \hspace{1cm} (1.12)

and with velocities

$$\delta_{2n+1} := -\frac{1}{\alpha} \text{Im} \left[ (\beta + i\alpha)^{2n+1} \right], \quad \gamma_{2n+1} := -\frac{1}{\beta} \text{Re} \left[ (\beta + i\alpha)^{2n+1} \right].$$

Remark 1.1. As far as we know, this is the first example of breather solutions of the mKdV and Gardner hierarchies.

Our second result is a characterization of each new Gardner breather as solution of a nonlinear fourth order ODE. Note that this ODE does not depend on the order of the hierarchy. Furthermore, any breather solution of the hierarchy is a critical point of the same Lyapunov functional.

**Theorem 1.3** (Universal nonlinear ODE and variational characterization). Any breather solution $B_\mu$ of the $(2n+1)$-th order Gardner equation (1.1) satisfies, for any $n \in \mathbb{N}$, the same fourth order elliptic equation

$$B_{\mu,xx} + 2(\alpha^2 - \beta^2)B_{\mu,xxx} + (\alpha^2 + \beta^2)^2 B_{\mu} + 10B_{\mu}^2B_{\mu,xxx} + 10B_{\mu}B_{\mu,xx}^2 + 6B_{\mu,xxx}^2 + 10B_{\mu,x}^2B_{\mu,x}^2 + 20B_{\mu}B_{\mu,xx}B_{\mu,x} + 40B_{\mu}^2B_{\mu,xx}^3 + 30B_{\mu}^4 = 0.$$  \hspace{1cm} (1.13)

Moreover, for any $n \in \mathbb{N}$, the breather solution $B_\mu$ of the Gardner hierarchy (1.1) is a critical point of a universal Lyapunov functional $\mathcal{H}_\mu$ (1.14), which is written as linear combination of three conserved laws in the following way

$$\mathcal{H}_\mu[u(t)] := F_\mu[u](t) + 2(\beta^2 - \alpha^2)E_\mu[u](t) + (\alpha^2 + \beta^2)^2 M[u](t).$$  \hspace{1cm} (1.14)

Remark 1.2. The conserved functionals in (1.14) can be explicitly found in (4.1), (4.2) and (4.3).

Remark 1.3. This nonlinear ODE was already found in the case of the classical Gardner equation in [8]. See also [6] for a first proof in the mKdV case. The surprise here is that every breather member of the hierarchy is solution of the same nonlinear ODE.

Remark 1.4 (About the stability of the hierarchy Gardner breathers). Note that with this variational characterization for breather solutions of the whole Gardner hierarchy, and applying the same ideas pointed out in [8] (see also [3]), a suitable **stability result** for breather solutions of the Gardner hierarchy (1.1) can be presented, provided a well-posedness theory is available, which is not the case today. Currently, only the stability result for breather solutions of the 5th order Gardner equation has been proved, see [5]. The well-posedness of the remaining members of the hierarchy is an interesting open problem. Compare with Theorem 1.5 below, which shows weak ill-posedness depending on the index $n$ of the member.

As a direct corollary of the main result Theorem 1.2, we get, when $\mu = 0$ in (1.13), the fourth-order elliptic ODE satisfied by all breather solutions of the mKdV hierarchy.
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Corollary 1.4 (Universality for mKdV hierarchy breathers). Any breather solution $B$ of the $(2n + 1)$-th order mKdV equation (1.8) satisfies for any $n \in \mathbb{N}$ the same fourth order elliptic equation

$$\begin{align*}
B_{4x} + 2(\alpha^2 - \beta^2)(B_{xx} + 2B^3) + (\alpha^2 + \beta^2)^2 B \\
+ 10B^2B_{xx} + 10BB_x + 6B^5 &= 0.
\end{align*}$$

Moreover, mKdV breathers of the hierarchy are classical critical point of an associated functional, just as in [1.14], but with $\mu = 0$.

This corollary complements and completes the main result in [2, Th.2.4], which essentially considered the 5th, 7th and 9th mKdV equations only.

Our third result is related to well-posedness issues. Indeed, following the key idea introduced by Kenig, Ponce and Vega [17] on NLS and mKdV, and generalized to the classical Gardner equation in [4] and the 5th order Gardner equation in [3], we are able to study these higher order breather solutions, and show ill-posedness for the whole hierarchy in the following sense:

Theorem 1.5 (Growing ill-posedness for the Gardner hierarchy). If $s < \frac{2n - 1}{4}$, the mapping data-solution $u_0 \to u(t)$, with $u(t)$ a solution of the IVP for the $(2n+1)$st-order Gardner equation (1.1), is not uniformly continuous.

Remark 1.5. For the sake of completeness, we write explicitly the critical threshold index $s_c(n)$ for well-posedness, according to Theorem 1.5 for $n = 1$, $s_c(1) = \frac{1}{4}$ (classical Gardner), for $n = 2$, $s_c(2) = \frac{3}{4}$ (5th order, see [2]), for $n = 3$, $s_c(3) = \frac{5}{4} > 1$ (7th order, above standard energy space $H^1$), for $n = 4$, $s_c(4) = \frac{7}{4} < 2$ (9th order), and $s_c(5) = \frac{9}{4} > 2$ (11th order, above $H^2$ energy space), and so on.

Note that the larger is $n$, the higher is the Sobolev index $s$ for which we have ill-posedness below that regularity. Additionally, for $n > 4$ we have ill-posedness above $H^2$, the natural space for breathers stability. Consequently, we cannot expect variational stability of breathers for members of the hierarchy of order $2n + 1 = 11$ or higher.

Also, a completely similar ill-posedness result can be proved for the mKdV hierarchy. See Corollary 5.1. The Sobolev index inequality $s < \frac{2n - 1}{4}$ is the same.

Finally, our last result considers the mKdV hierarchy in the periodic setting. We have obtained periodic breather solutions of higher order mKdV equations, as follows:

Theorem 1.6 (Existence of higher order periodic breathers). The 5th and 7th order mKdV equations have periodic breather solutions, of the following form:

$$B = B(t, x; \alpha, \beta, k, m, x_1, x_2) := 2\partial_x\left[\arctan\left(\frac{\beta \text{ sn}(\alpha y_1, k)}{\alpha \text{ nd}(\beta y_2, m)}\right)\right],$$

where $\text{sn}(\cdot, k)$ and $\text{nd}(\cdot, m)$ are the standard Jacobi elliptic functions of elliptic modulus $k$ and $m$, and

$$y_1 := x + \delta_{i,m} t + x_1, \quad y_2 := x + \gamma_{i,m} t + x_2, \quad i = 5, 7.$$  (1.17)

Remark 1.6. See (6.5) and (6.6) in Section 6 for an explicit expression for velocities $(\delta_{i,m}, \gamma_{i,m})$ in the 5th order case and $(\delta_{i,m}, \gamma_{i,m})$ in the 7th order respectively.
Remark 1.7. See \cite{11} for a detailed account on the elliptic functions involved in (1.16).

The validity Theorem 1.6 follows directly, after cumbersome computations, or the use of a standard symbolic software. We skip the details for the interested reader.

We believe that these periodic breathers are completely new for the 5th and 7th order setting. In the classical mKdV periodic case, these solutions were found by Kevrekidis et al. \cite{18,19}. Also, in \cite{11}, we presented stability properties of these solutions, which could be applied to these new breathers after some work. The variational structure of these solutions is an interesting open problem.

We believe that the whole mKdV hierarchy has the same functional expression for periodic breather solutions, varying with velocities, but the form of higher order speeds than 5th and 7th order has escaped to us, and we were not able yet to obtain a complete description.

1.1. Organization of this paper. This paper is organized as follows: in Section 2 we introduce the solitons for the Gardner hierarchy. In Section 3 we prove existence of breathers, Theorem 1.2. Section 4 deals with the proof of the variational characterizations of Gardner breathers, Theorem 1.3. Section 5 is devoted to the proof of Ill-posedness of the Gardner and mKdV hierarchies, Theorem 1.5. Finally, Section 6 provides further information on Theorem 1.6.

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2. Preliminaries

2.1. Formulae for 1-solitons. As a consequence of the complete integrability of (1.1), and following Matsuno \cite{21} (3.11)-\cite{22} and the Inverse Scattering Method, it is possible to see that the Gardner hierarchy (1.1) has explicit 1-soliton solutions. For the sake of completeness, since we have not found a formal statement of this result in the literature, we include it here:

Definition 2.1. The higher order 1-soliton solution \( Q_c = Q_{c,n} \) of any equation member of the Gardner hierarchy (1.1), i.e. of any \((2n + 1)\)-th order Gardner equation is given by

\[
\begin{align*}
Q_\mu(t,x) & := Q_{\mu,c}(x-v_{\mu,c}t), \\
Q_{\mu,c}(z) & := \frac{c^2}{2\mu + \sqrt{4\mu^2 + c^2} \cosh(cz)}, \quad c > 0,
\end{align*}
\]

(2.1)

with

\[
v_{\mu,c} := \sum_{p=1}^{n} a_{p,n} c^{2p} \mu^{2(n-p)}, \quad \forall n \in \mathbb{N},
\]

and \( a_{p,n} \) as in the Definition 1.10 above.
Moreover, it is easy to see that any 1-soliton solution $Q_{\mu,c}$ of any equation member of the Gardner hierarchy (1.1) satisfies the same nonlinear stationary elliptic equation:

\begin{equation}
Q''_\mu - cQ'_\mu + 6\mu Q^2_\mu + 2Q^3_\mu = 0, \quad Q_\mu > 0, \quad Q_\mu \in H^1(\mathbb{R}).
\end{equation}

**Proof.** By substituting directly $Q_\mu$ in (2.2) for all $n \in \mathbb{N}$. \hfill \Box

Finally note that selecting the parameter $\mu = 0$ in (2.1) we reduce to the mKdV limit, and we get from the 1-soliton solution of the Gardner hierarchy (2.1), the 1-soliton solution of the mKdV hierarchy, as it was depicted by Matu no [21].

**Definition 2.3** (1-soliton solution of the mKdV hierarchy). The 1-soliton solution of the $(2n+1)$th-order mKdV equation (1.8) is given by

$$
Q_\mu(t,x) := Q_{\mu,c}(x-v_n t),
$$

where

$$
Q_{\mu,c}(z) := \text{sech}(cz), \quad c > 0,
$$

with

$$
v_n := c^{2n}, \quad \forall n \in \mathbb{N}.
$$

In addition to these higher order 1-soliton solution (2.1), we are also able to obtain breather solutions of the Gardner hierarchy (1.1).

### 3. Higher order breathers. Proof of Theorem 1.2

Note that item (4) in Theorem 1.2 follows directly from proving items (1), (2) and (3). On the other hand, item (3) is a direct consequence of item (1). Given the speeds in item (2), we are only left to prove item (1) in Theorem 1.2.

Our first step will be to present the following identity valid for any solution of the Gardner hierarchy (1.1), and that will be a key tool in the proof of item (1) in Theorem 1.2.

**Lemma 3.1.** Let $u(t,x) = i\partial_x \log(F_\mu + iG_\mu) = 2\partial_x \arctan(G_\mu/F_\mu)$ be any solution of the Gardner hierarchy (1.1) for any $t \in \mathbb{R}$. Then $u$ satisfies

$$
u^2 - \frac{\partial^2}{\partial x^2} \log(F^2_\mu + G^2_\mu) + 2\mu u = 0.
$$

**Proof.** We select a solution $u$ of (1.1) in the form

$$
u(t,x) := \phi_x, \quad \phi(t,x) := i \log \left( \frac{G(t,x)}{\text{F}(t,x)} \right),
$$

where

$$
\text{F} := F_\mu + iG_\mu, \quad \text{G} = F_\mu - iG_\mu = \text{F}^*.
$$

First note that here $F_\mu$ and $G_\mu$ are not necessarily the same functions introduced in (1.10) but generic ones for this ansatz. Note moreover, that the left part of the
Gardner hierarchy (1.1) can be rewritten as a finite sum of terms like
\[ \prod_{l=0}^{L} (u_l x^l), \]
with \( u_l = \frac{\partial^l u}{\partial x^l} \) and \( d_l, \ l = 0, \ldots, L \) are nonnegative integers. Then, substituting the above expression in (1.1), and using Hirota’s \( D \)-operators \( D_t, D_x \), we arrive to the following conditions on \( G \) and \( F \):

\[ D_x^2(GF) - 2i\mu D_x(GF) = 0, \quad (3.4) \]

and

\[ D_t(GF) + \bar{a}_{n,\mu} D_x^{2n+1}(GF) + \sum_{j=0}^{n-1} \bar{b}_{n,\mu} D_t D_x^{2(n-j)}(GF) = 0, \]

where \( \bar{a}_{n,\mu}, \bar{b}_{n,\mu} \) are coefficients depending on \( \mu \) and associated to each member of (1.1). Then, dividing by \( GF \) the first equation (3.4), and taking into account the following identity

\[ D_x^2(GF) = \partial_x^2 \log(GF) + \left( \partial_x \log \left( \frac{G}{F} \right) \right)^2, \quad (3.5) \]

we obtain:

\[ \frac{D_x^2(GF)}{GF} - 2i\mu \frac{D_x(GF)}{GF} = \partial_x^2 \log(GF) + \left( \partial_x \log \left( \frac{G}{F} \right) \right)^2 - 2i\mu \partial_x \log \left( \frac{G}{F} \right) \]
\[ = \partial_x^2 \log(GF) + \left( \frac{u}{k} \right)^2 - 2\mu u = 0. \]

Hence,

\[ u^2 = \frac{\partial^2}{\partial x^2} \log(G \cdot F) - 2\mu u. \]

We will also need the following Lemma

**Lemma 3.2.** Let \( B_\mu(t, x) = i\partial_x \log \left( \frac{F_\mu - iG_\mu}{F_\mu + iG_\mu} \right) = 2\partial_x \arctan \left( \frac{G}{F_\mu} \right) \). Assume that \( B_\mu \) satisfies (3.1) and has velocities \( \gamma_{2n+1,\mu} \) and \( \delta_{2n+1,\mu} \):

\[ \gamma_{2n+1,\mu} := -\frac{1}{\beta} \text{Re} \left[ \sum_{p=1}^{n} a_{p,n} (\beta + i\alpha)^{2p+1} \mu^{2(n-p)} \right], \]
\[ \delta_{2n+1,\mu} := -\frac{1}{\alpha} \text{Im} \left[ \sum_{p=1}^{n} a_{p,n} (\beta + i\alpha)^{2p+1} \mu^{2(n-p)} \right]. \]

Then \( B_\mu \) is a solution of the Gardner hierarchy (1.1) for any \( t \in \mathbb{R} \).

**Proof.** By hypothesis \( B_\mu \) satisfies (3.1), namely

\[ B_\mu^2 = \frac{\partial^2}{\partial x^2} \log(G \cdot F) - 2\mu B_\mu. \]

Since, the Gardner hierarchy (1.1) is equivalent to system (3.4)-(3.5), it is enough to see that actually \( B_\mu \) holds both. First of all, since \( B_\mu(t, x) := i\partial_x \log \left( \frac{G(t,x)}{F_\mu(t,x)} \right) \), with \( F := F_\mu + iG_\mu, \ G = F_\mu - iG_\mu = F^* \), and resorting to (3.5), we rewrite the above identity as:

\[ \text{e.g. } D_t f \cdot g = (\partial_{t'} - \partial_t) f(t')g(t)|_{t'=t} = f_t g - fg_t. \]
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\[ B_\mu^2 = (i \partial_x \log \left( \frac{G(t,x)}{F(t,x)} \right))^2 = \frac{\partial^2}{\partial x^2} \log(G \cdot F) - \frac{D_x^2(GF)}{GF} \]

\[ \quad \quad = \frac{\partial^2}{\partial x^2} \log(G \cdot F) - 2\mu(i \partial_x \log(G(t,x) \cdot F(t,x))), \]

and therefore

\[ \quad \quad \frac{D_x^2(GF)}{GF} - 2\mu(i \partial_x \log(G(t,x) \cdot F(t,x))) = 0, \]

and multiplying by \( GF \), we get

\[ D_x^2(GF) - 2\mu D_x(GF) = 0, \]

which is the first equation (3.4) that one obtains after application of the Hirota’s operators to the Gardner hierarchy (1.1). In order to describe the time evolution, we should check that (3.5) holds. Instead, we are going to compute directly the velocities, driving the temporal part of (1.1) and which is equivalent to (3.5). Namely, using a matching method with ansatz \( B_\mu (1.10) \) and free parameters \( \gamma_{2n+1, \mu} \) and \( \delta_{2n+1, \mu} \), we proceed substituting directly \( B_\mu (1.10) \) into the 5th and 7th order Gardner equations (1.6)-(1.7), and upon lengthy algebraic manipulations we find that \( B_\mu \) is indeed a breather solution of the 5th and 7th order Gardner equations (1.6)-(1.7) respectively, provided that

\[ \delta_{5, \mu} := -\alpha^4 + 10\alpha^2 \beta^2 - 5\beta^4 - 10\mu^2(3\beta^2 - \alpha^2) \]

\[ = -\frac{1}{\alpha} \Im \left[ \sum_{p=1}^{2} a_{p,2}(\beta + i\alpha)^{2p+1}\mu^{2(2-p)} \right], \]

\[ \gamma_{5, \mu} := -\beta^4 + 10\alpha^2 \beta^2 - 5\alpha^4 - 10\mu^2(\beta^2 - 3\alpha^2) \]

\[ = -\frac{1}{\beta} \Re \left[ \sum_{p=1}^{2} a_{p,2}(\beta + i\alpha)^{2p+1}\mu^{2(2-p)} \right], \]

with \( a_{1,2} = 10, a_{2,2} = 1, \) and

\[ \delta_{7, \mu} := \alpha^6 - 21\alpha^4 \beta^2 + 35\alpha^2 \beta^4 - 7\beta^6 + 14\mu^2(-\alpha^4 + 10\alpha^2 \beta^2 - 5\beta^4) \]

\[ - 70\mu^4(3\beta^2 - \alpha^2) = -\frac{1}{\alpha} \Im \left[ \sum_{p=1}^{3} a_{p,3}(\beta + i\alpha)^{2p+1}\mu^{2(3-p)} \right], \]

\[ \gamma_{7, \mu} := -\beta^6 + 21\alpha^2 \beta^4 - 35\alpha^4 \beta^2 + 7\alpha^6 + 14\mu^2(-\beta^4 + 10\alpha^2 \beta^2 - 5\alpha^4) \]

\[ - 70\mu^4(\beta^2 - 3\alpha^2) = -\frac{1}{\beta} \Re \left[ \sum_{p=1}^{3} a_{p,3}(\beta + i\alpha)^{2p+1}\mu^{2(3-p)} \right], \]

with \( a_{1,3} = 70, a_{2,3} = 14, a_{3,3} = 1. \) By induction for any order \( 2n + 1, n \in \mathbb{N} \), we get that
\[\gamma_{2n+1,\mu} := -\frac{1}{\beta} \Re \left[ \sum_{p=1}^{n} a_{p,n}(\beta + i\alpha)^{2p+1} \mu^{2(n-p)} \right],\]
\[\delta_{2n+1,\mu} := -\frac{1}{\alpha} \Im \left[ \sum_{p=1}^{n} a_{p,n}(\beta + i\alpha)^{2p+1} \mu^{2(n-p)} \right],\]

where \(a_{p,n}\) is the coefficient of the term \(\mu^{2(n-p)}u_{(2p+1)x}\) in the \((2n+1)\)th-order Gardner equation.

\[\square\]

**Proof of Theorem 1.2** (1) Structure: Let \(\alpha, \beta \in \mathbb{R} \setminus \{0\}\) and \(\mu \in \mathbb{R}^{+} \setminus \{0\}\) such that \(\Delta = \alpha^2 + \beta^2 - 4\mu^2 > 0\), and \(x_1, x_2 \in \mathbb{R}\). We check now that \(B_{\mu}\) in (1.10) satisfies, at \(t = 0\), the identity (3.1) valid for solutions of the Gardner hierarchy (1.1). For ease of notation let us use the following expression for \(B_{\mu}\) (1.10):

\[B_{\mu} = 2\partial_x \arctan \left[ \frac{a_1 \sin(y_1) - a_2 e^{y_2}}{\cosh(y_2) - a_3(\alpha \cos(y_1) - \beta \sin(y_1))} \right] = \frac{H(t,x)}{N(t,x)},\] (3.10)

with

\[H(t,x) = H = 2 - a_3(\alpha^2 a_1 + a_2 (\beta^2 - \alpha^2) e^{y_2} \sin(y_1) - 2\alpha a_2 e^{y_2} \cos(y_1)) + \beta \sinh(y_2)(a_2 e^{y_2} - a_1 \sin(y_1)) + \cosh(y_2)(\alpha a_1 \cos(y_1) - a_2 e^{y_2}),\]
\[N(t,x) = N = F_{\mu}^2 + G_{\mu}^2 = (a_2 e^{y_2} - a_1 \sin(y_1))^2 + \cosh(y_2)^2 a_3 \sin(y_1) - a_3 \alpha \cos(y_1))^2,\] (3.11)

where

\[a_1 = \frac{\beta \sqrt{\alpha^2 + \beta^2}}{\alpha \sqrt{\Delta}}, \quad a_2 = \frac{2\beta \mu}{\Delta}, \quad a_3 = \frac{2\beta \mu}{\alpha \sqrt{\Delta} \sqrt{\alpha^2 + \beta^2}}\] (3.12)

Then, having in mind notation \(N, N_x, N_{xx}, N_{3x}, N_{4x}\) and \(H, H_x, H_{xx}, H_{3x}, H_{4x}\) of Appendix A 3.1 simplifies as

\[u^2 - \frac{\partial^2}{\partial x^2} \log(F_{\mu}^2 + G_{\mu}^2) + 2\mu u = \frac{1}{N^2} \left( H^2 + N_x^2 - N_{xx}N + 2\mu HN \right).\] (3.13)

Finally, substituting explicitly \(H's\) and \(N's\) terms, we verify, using the symbolic software *Mathematica*, simple trigonometric and hyperbolic identities and some rearrangements, that

\[H^2 + N_x^2 - N_{xx}N + 2\mu HN = 0,\] (3.14)

and we conclude.

Now, in order to get a complete dynamical description of \(B_{\mu}\) (1.10) at any \(t\), we have to obtain the velocities \(\gamma_{2n+1,\mu}\) and \(\delta_{2n+1,\mu}\).

(2) Velocities: Now, we determine the velocities \(\gamma_{2n+1,\mu}\) and \(\delta_{2n+1,\mu}\). Using a matching method with ansatz \(B_{\mu}\) as in Lemma (3.5), we conclude that
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\[
\gamma_{2n+1,\mu} := -\frac{1}{\beta} \text{Re} \left[ \sum_{p=1}^{n} a_{p,n}(\beta + i\alpha)^{2p+1}\mu^{2(n-p)} \right],
\]

\[
\delta_{2n+1,\mu} := -\frac{1}{\alpha} \text{Im} \left[ \sum_{p=1}^{n} a_{p,n}(\beta + i\alpha)^{2p+1}\mu^{2(n-p)} \right],
\]

where \(a_{p,n}\) is the coefficient of the term \(\mu^{2(n-p)}u(2p+1)x\) in the \((2n+1)\)th-order Gardner equation.

**Remark 3.1.** See Appendix B.1 for a few additional examples of these higher order mKdV and Gardner breather solutions in cases 9th, 11th and 13th order.

### 4. Universality in the variational characterization. Proof of Theorem 1.3

The Gardner hierarchy (1.1), as being a completely integrable scheme of equations, has infinitely many conserved quantities. Some standard conservation laws at the \(H^1\)-level are the mass

\[
M[u](t) := \frac{1}{2} \int_{\mathbb{R}} u^2(t, x)dx = M[u](0),
\]

the energy

\[
E_{\mu}[u](t) := \int_{\mathbb{R}} \left( \frac{1}{2}u_x^2 - 2\mu u^3 - \frac{1}{2} u^4 \right) (t, x)dx = E_{\mu}[u](0),
\]

and the higher order energy, defined respectively in \(H^2(\mathbb{R})\),

\[
F_{\mu}[u](t) := \int_{\mathbb{R}} \left( \frac{1}{2} u_{xx}^2 - 10\mu uu_x^2 + 10\mu^2 u^4 - 5\mu^2 u_x^2 + 6\mu u^5 + u^6 \right) (t, x)dx = F_{\mu}[u](0).
\]

Now, considering \(M[u]\), \(E_{\mu}[u]\) and \(F_{\mu}[u]\), we define the following Lyapunov functional, as in (1.14):

\[
\mathcal{H}_{\mu}[u(t)] := F_{\mu}[u](t) + 2(\beta^2 - \alpha^2)E_{\mu}[u](t) + (\alpha^2 + \beta^2)^2 M[u](t).
\]

Therefore, \(\mathcal{H}_{\mu}[u]\) is a real-valued conserved quantity, well-defined for \(H^2\)-solutions of the Gardner hierarchy (1.1), provided they exist.
4.1. **Proof of Theorem 1.3.** We first prove (1.13): we recast the l.h.s. of (1.13) as follows:

\[ G_\mu[B_\mu] := \partial^2_x (B_{\mu,xx} + 6\mu B^2 + 2B^3) + 2(\alpha^2 - \beta^2) (B_{\mu,xx} + 6\mu B^2 + 2B^3) + 8\mu B_\mu B_{\mu,xx} - 2B\mu B^2_{\mu,xx} + 4B^2_\mu B_{\mu,xx} - 2\mu B^2_\mu + (\alpha^2 + \beta^2)^2 \mu B_\mu + 4\mu^2 B^3_\mu + 30\mu B^4_\mu + 6B^5_\mu = \partial^2_x (B_{\mu,xx} + 6\mu B^2 + 2B^3) + 2(\mu + B_\mu)(B^2_{\mu,xx} + 4\mu B^3 + B^4_\mu) + (\alpha^2 + \beta^2)^2 B_\mu + (4B^2_\mu + 8\mu B_\mu + 2(\alpha^2 - \beta^2)(B_{\mu,xx} + 6\mu B^2 + 2B^3) - 2(\mu + B_\mu)(B^2_{\mu,xx} + 4\mu B^3 + B^4_\mu) + (\alpha^2 + \beta^2)^2 B_\mu. \]

(4.5)

Now we compute explicitly the last line in (4.5). For simplicity, we use the same notation (3.10) and \( A \) as in the proof of Theorem 1.2 in 3.

\[ \frac{B_\mu(t, x)}{H(t, x)} = \frac{H(t, x)}{N(t, x)}, \]

(4.6)

with \( H, \ N \) as in (3.11) and \( a_i, \ i = 1, 2, 3 \) already defined in (3.12). We first compute the term

\[ B_{xx} + 6\mu B^2 + 2B^3 = \frac{1}{N^3} \left( 2H^3 + 6H^2 \mu N + H(2N^2_x - N_{xx}N) + N(H_{xx}N - 2H_x N_x) \right), \]

(4.7)

and we get

\[ \partial^2_x (B_{xx} + 6\mu B^2 + 2B^3) = \frac{R_1}{N^5}, \]

(4.8)

with

\[ R_1 := \left( 6H^3(4N^2_x - N_{xx}N) - 6H^2 N(6H_x N_x - N(H_{xx} - 2\mu N_{xx}) - 6\mu N^3_x) + H(2N^2(6H^2_x + 24H_x N_x + 4N_x N_{3x} + 3N^2_x) + N^3(12H_{xx} N - 4N^3_x) + 24N^4_x - 36N^3_x N_{xx} N) + N(2N^2(6H^2_x - 2H_x N_{3x} - 3H_{xx} N_{xx} - 2N_x N_x) + 12N_x N(2H_x N_{xx} + H_{xx} N_x) - 24H_x N^3_x + H_{xx} N^3 \right). \]

(4.9)

Moreover, we have that

\[ (4B^2 + 8\mu B + 2(\alpha^2 - \beta^2))(B_{xx} + 6\mu B^2 + 2B^3) = \frac{R_2}{N^5}, \]

(4.10)

with

\[ R_2 := 2(2H^2 + 4\mu N + N^2(\alpha^2 - \beta^2))(2H^3 + 6H^2 \mu N + H(2N^2_x - N_{xx}N) + (H_{xx} N - 2H_x N_x)), \]

(4.11)

and

\[ -2(B + \mu)(B^2_x + 4\mu B^3 + B^4) + (\alpha^2 + \beta^2)^2 B = \frac{R_3}{N^5}, \]

(4.12)
with
\[ R_3 := H N^4 \left( \alpha^2 + \beta^2 \right)^2 - 2 \left( H + \mu N \right) \left( H^4 + 4 \mu H^3 N + (H_N - H x_N)^2 \right). \]
(4.13)
Hence, we get the following simplification of (1.13):
\[ G_\mu[B_\mu] = \frac{R_1 + R_2 + R_3}{N^5}, \]
(4.14)
with \( R_1, R_2, R_3 \) in (4.9), (4.11) and (4.13) respectively. In fact, we verify, using
the symbolic software *Mathematica*, that after substituting \( H \)'s and \( N \)'s terms
explicitly (as they were shown in the Appendix A in (1.14), lengthy rearrangements
and basic trigonometric and hyperbolic identities, we get
\[ R_1 + R_2 + R_3 = 0, \]
(4.15)
and we conclude the proof of (1.13).

Now, we prove that \( B_\mu \) is actually a critical point of \( H_\mu \). We will show that
\[ H_\mu[B_\mu + z] - H_\mu[B_\mu] = \frac{1}{2} Q[z] + N[z], \]
(4.16)
with \( Q \) being the quadratic form defined in (4.17), and \( N[z] \) satisfying \( |N[z]| \leq K \|z\|^3_{H^2(\mathbb{R})} \). We evaluate and expand the Lyapunov functional \( H_\mu \) in terms of a
perturbation of the breather \( B_\mu \), with \( z \in H^2(\mathbb{R}) \). A direct computation with
integration by parts yields
\[
F_\mu[B_\mu + z] = F_\mu[B_\mu] + \int \left[ B_{\mu,xx} + 10 \mu B_{\mu}^2 + 20 \mu B_{\mu,xx} + 10 B_{\mu} B_{\mu,xx} + 10 B_{\mu}^2 B_{\mu,xx} + 40 \mu B_{\mu}^3 \right. \\
\left. + 30 \mu B_{\mu}^4 + 6 B_{\mu}^5 \right] z + \frac{1}{2} \int \left( z_{xx} + \left( 20 \mu B_{\mu} + 10 B_{\mu}^2 \right) z_{xx} - 20 \left( \mu B_{\mu,xx} + B_{\mu} B_{\mu,xx} \right) z_{xx} \right. \\
\left. + \left( -10 B_{\mu,xx} + 120 \mu B_{\mu}^2 + 120 \mu B_{\mu}^3 \right) z_{xx} + 20 B_{\mu}^3 z_{xx} \right) z + \int \left( -10 \mu z_{xx}^2 - 10 B_{\mu} z_{xx} - 10 B_{\mu,xx} z_{xx} + 40 \mu z_{xx}^3 + 60 B_{\mu} z_{xx}^3 + 20 B_{\mu}^3 z_{xx}^3 \right).
\]
Similarly,
\[
E_\mu[B_\mu + z] = E_\mu[B_\mu] - \int (B_{\mu,xx} + 6 \mu B_{\mu}^2 + 2 B_{\mu}^3) z - \frac{1}{2} \int \left( z_{xx} + (12 \mu B_{\mu} + 6 B_{\mu}^2) z \right) - \int \left( 2 \mu z^3 + B_{\mu} z^3 + \frac{1}{2} z^4 \right),
\]
and
\[
M[B_\mu + z] = M[B_\mu] + \int B_{\mu} z + \frac{1}{2} \int z^2.
\]
Collecting all, we get that
\[ H_\mu[B_\mu + z] = H_\mu[B_\mu] + \int G_\mu[B_\mu] z + \frac{1}{2} Q_\mu[z] + N_\mu[z], \]
where the quadratic form
\[ Q_\mu[z] := \int z L_\mu[z], \]
(4.17)
associated to the linearized operator $L_\mu$ given by
\[
L_\mu := \partial_x^4 + (2\mu B_\mu + 10B_\mu^2 - 2(\beta^2 - \alpha^2))\partial_x^2 - 20(\mu B_{\mu,x} + B_\mu B_{\mu,x})\partial_x
\]
\[
+ (-10B_\mu^2 + 120\mu B_\mu^2 + 120\mu B_\mu^3 + 30B_\mu^4 - 2(\beta^2 - \alpha^2)(12\mu B_\mu + 6B_\mu^2)
\]
\[
+ (\alpha^2 + \beta^2)^2).
\]

Gathering all higher order terms (with respect to $z$) in $N_\mu[z]$ we get
\[
N_\mu[z] := \int \left(-10\mu z z_x^2 - 10B_\mu z z_x - 10B_{\mu,x} z x z_x^2 + 40\mu^2 B_\mu z^3 + 60\mu B_\mu^2 z^3
\]
\[
+ 20B_\mu^3 z^3\right) - 2(\beta^2 - \alpha^2) \int \left(2\mu z^3 + 2B_\mu z^3 + \frac{1}{2} z^4\right).
\]

Part (i) above guarantees $\int_R G_\mu(B_\mu) z = 0$, and then we have $H_\mu'[B_\mu] = 0$. Moreover, from direct estimates, one has $N_\mu[z] = O(\|z\|_{L^2(R)}^3)$, and we conclude.

Proof. of Corollary 1.4: it follows directly from the above proof for the Gardner case, when we select $\mu = 0$. 

5. ILL-POSEDNESS OF THE GARDNER AND mKdV HIERARCHIES. PROOF OF THEOREM 1.5

Now, we prove the ill-posedness result presented in Theorem 1.5 for the whole Gardner hierarchy (1.5), having in mind the explicit breather solution (1.10). Note firstly that from (1.10), the explicit breather solution can be approximated in the limit $\alpha \gg \beta$, namely let $\mu$ fixed and suppose that $\frac{\alpha}{\beta} \ll 1$. From (3.10), we see that the breather solution (1.10) reduces to the function
\[
B_{\alpha,\beta,\mu,n}(t, x) \approx \sqrt{2} \text{Re}[e^{i(\alpha(x + \delta_{2n+1} t))} Q_\beta(x + \gamma_{2n+1} t)],
\]

or simply
\[
B_{\alpha,\beta,\mu,n}(t, x) \approx \sqrt{2} \text{Re}[e^{i(\alpha(x + \delta_{2n+1} t))} Q_\beta(x + \gamma_{2n+1} t)],
\]

where $Q$ denotes the solution of the nonlinear ODE
\[
Q'' - Q + Q^3 = 0,
\]

with
\[
Q(\xi) = \sqrt{2} \text{sech}(\xi)
\]

and
\[
Q_\beta(\xi) = \beta Q(\beta \xi).
\]

Proof. We consider the IVP for the $(2n+1)$th-order Gardner equation with initial data given by the breather solution (1.10).
\[
\begin{cases}
  u_t = -\frac{\partial}{\partial x} \left( -\frac{\partial}{\partial x} + 2(\mu + u) \right) L_n[iu_x + (\mu + u)^2], \\
  u(0, x) = B_{\alpha,\beta,\mu,n}(0, x).
\end{cases}
\]
With \( \mu \) fixed, we take the parameter \( \alpha \) large enough, such that \( \frac{2}{\alpha} \ll 1 \). Then, from (5.2), the initial data reads
\[
B_{\alpha, \beta, \mu, n}(0, x) \approx \sqrt{2} \text{Re}[e^{i\alpha x} Q_{\beta}(x)],
\]
with \( Q_{\beta} \) defined in (5.5). We take
\[
\beta = \alpha^{-2s} \quad \text{and} \quad \alpha_1, \alpha_2 \sim \alpha. \tag{5.7}
\]

Observe that \( \hat{Q}_{\beta}(\cdot) \) concentrates in the ball \( B_{\beta}(0) = \{ \xi \in \mathbb{R}; |\xi| < \beta \} \). First, we calculate the \( H^s \)-norm of two different initial data for the \((2n+1)\)th-order Gardner equation in the regime with \( \alpha \) large enough, such that \( \frac{2}{\alpha} \ll 1 \):

\[
\| B_{\alpha_1, \beta, \mu, n}(0) \|_{H^s}^2 \approx \| (1 + |\xi|^2)^{s/2} \hat{Q}_{\beta}(\xi - \alpha_1) \|_{L^2}^2 \approx C \alpha^{2s} \beta = C, \quad j = 1, 2, \tag{5.8}
\]
where \( C \) denotes a constant.

Second, we measure the distance between these initial data
\[
\| B_{\alpha_1, \beta, \mu, n}(0) - B_{\alpha_2, \beta, \mu, n}(0) \|_{H^s}^2 \approx \| (1 + |\xi|^2)^{s/2} (\hat{Q}_{\beta}(\xi - \alpha_1) - \hat{Q}_{\beta}(\xi - \alpha_2)) \|_{L^2}^2 \leq C \alpha^{2s} \| \hat{Q}_{\beta}(\xi - \alpha_1) - \hat{Q}_{\beta}(\xi - \alpha_2) \|_{L^2}^2
\]
\[
\leq C \alpha^{2s} \int_{-\infty}^{+\infty} \left| \int_{\xi - \alpha_2}^{\xi - \alpha_1} \frac{d}{d\xi} \hat{Q}_{\beta}(\rho) d\rho \right|^2 d\xi
\]

Consequently,
\[
\| B_{\alpha_1, \beta, \mu, n}(0) - B_{\alpha_2, \beta, \mu, n}(0) \|_{H^s}^2 \leq C \alpha^{2s} \frac{2|\alpha_1 - \alpha_2|}{\beta^2} \int_{-\infty}^{\infty} \int_{\xi - \alpha_2}^{\xi - \alpha_1} |\hat{Q}_{\beta}(\rho)|^2 d\rho d\xi \leq C \alpha^{2s} \frac{2|\alpha_1 - \alpha_2|^2}{\beta^2} \int_{-\infty}^{+\infty} |\hat{Q}_{\beta}(\rho)|^2 d\rho
\]
\[
\leq C \alpha^{2s} (\alpha_1 - \alpha_2)^2 \beta = C \alpha^{2s} (\alpha_1 - \alpha_2)^2 \alpha^{2s} = C (\alpha^{2s} (\alpha_1 - \alpha_2))^2.
\tag{5.10}
\]

Next, we consider the corresponding solutions \( B_{\alpha_1, \beta, \mu, n}(t) \) and \( B_{\alpha_2, \beta, \mu, n}(t) \) at the time \( t = T \). We can see that
\[
\| B_{\alpha_1, \beta, \mu, n}(T) - B_{\alpha_2, \beta, \mu, n}(T) \|_{L^2}^2 \approx \alpha^{2n} \| B_{\alpha_1, \beta, \mu, n}(T) - B_{\alpha_2, \beta, \mu, n}(T) \|_{L^2}^2. \tag{5.12}
\]

From (5.7), if \( \alpha \) is large enough,
\[
B_{\alpha_1, \beta, \mu, n}(T, x) \approx \sqrt{2} \text{Re}[e^{i(\alpha_1(x + \beta_2n+1T))} \beta Q(\beta(x + \gamma_2n+1T))], \quad j = 1, 2. \tag{5.13}
\]

Moreover, from (5.6), note that
\[
\gamma_{2n+1} = (-1)^n (2n + 1) \alpha^{2n}
\]
\[
+ \sum_{j=0}^{n-1} a_{j,n} P_{(2j)}(\alpha, \beta) \mu^{2(n-j)} + \sum_{j=0}^{n-1} a_{n,n} b_j \beta^{2(n-j)} \alpha^{2j}, \tag{5.14}
\]
where \( a_{j,n} \) and \( b_j \) are convenient real constants and \( P_{(l)}(\alpha, \beta) \) is a polynomial whose its degree is \( l \in \mathbb{N} \). If \( \frac{2}{\alpha} \ll 1 \), we see that
\[
\gamma_{2n+1} \sim (-1)^{n+1} (2n + 1) \alpha^{2n}. \tag{5.15}
\]
and

\[ \alpha_1^{2n} - \alpha_2^{2n} = \left( \sum_{j=0}^{2n-1} \alpha_1^{2n-1-j} \cdot \alpha_2^j \right) (\alpha_1 - \alpha_2) \sim (\alpha_1 - \alpha_2) \alpha^{2n-1}. \]  

(5.16)

The information above shows that \( B_{(\alpha_1, \beta, \mu, n)}(T), \ j = 1, 2 \) concentrates in the ball \( B_{\beta^{-1}}((-1)^n(2n+1)\alpha^{2n}T), \ j = 1, 2 \).

So, we basically have disjoint supports if

\[ \alpha^{2n-1}(\alpha_1 - \alpha_2)T \gg \beta^{-1} = \alpha^{2s}. \]  

(5.17)

Under this condition, we have that

\[ \|B_{(\alpha_1, \beta, \mu, n)}(T) - B_{(\alpha_2, \beta, \mu, n)}(T)\|_L^2 \approx \|B_{(\alpha_1, \beta, \mu, n)}(T)\|_L^2 + \|B_{(\alpha_2, \beta, \mu, n)}(T)\|_L^2 \approx \beta \]  

and

\[ \|B_{(\alpha_1, \beta, \mu, n)}(T) - B_{(\alpha_2, \beta, \mu, n)}(T)\|_{H^s} \geq C\alpha^{2s} \beta = C. \]  

(5.19)

If we select

\[ \alpha_1 = \alpha + \frac{\delta}{2\alpha^{2s}}, \ \alpha_2 = \alpha - \frac{\delta}{2\alpha^{2s}}, \ \alpha_1 - \alpha_2 = \frac{\delta}{\alpha^{2s}}, \]  

(5.20)

we have that

\[ (\alpha^{2s}(\alpha_1 - \alpha_2))^2 = \delta^2 \]  

(5.21)

and, from (5.17),

\[ \alpha^{2n-1} \frac{\delta}{\alpha^{2s}} T \gg \alpha^{2s}. \]  

(5.22)

Finally, from (5.22),

\[ T \gg \frac{\alpha^{4s-2n+1}}{\delta}. \]  

(5.23)

Since \( s < \frac{2n-1}{4} \), given \( \delta, T > 0 \), we can choose \( \alpha \) so large that (5.23) is still valid, and then (5.19) does not satisfy uniform continuity. The proof is complete. \( \square \)

**Corollary 5.1** (Ill-Posedness of the \((2n+1)\)th-order mKdV equation). If \( s < \frac{2n-1}{4} \), the mapping data-solution \( u_0 \to u(t) \), with \( u(t) \) a solution of the IVP for the \((2n+1)\)th-order mKdV equation (1.8) is not uniformly continuous.

**Proof.** Selecting \( \mu = 0 \) in the above theorem, we get the result. \( \square \)

6. Remarks on periodic breathers for the 5th and 7th mKdV equations

In this section we provide further details on the introduction of periodic in space breathers for the mKdV hierarchy, namely, Theorem 1.6.
6.1. 5th and 7th order mKdV. We consider now, from (1.6) and (1.7) when \( \mu = 0 \), the periodic case of the 5th-order mKdV:
\[
  u_t + (u^4_x + f_5(u))_x = 0,
\]
and the 7th-order mKdV
\[
  u_t + (u^6_x + f_7(u))_x = 0,
\]
where
\[
  f_5(u) := 10u^2u_x^2 + 10u^2u_{xx} + 6u^5,
\]
and
\[
  f_7(u) := 14u^2u_x^4u_{3x} + 56uu_x^2u_{3x} + 42uu_x^2u_{xx} + 70u^2u_{xx} + 70u^4u_{xx} + 140u^3u_{xx} + 20u^7.
\]

6.2. Standard and new mKdV periodic breathers. A family of periodic breathers (named KKSH breathers) for the classical mKdV equation was found by Kevrekidis et al by using elliptic functions and a matching of free parameters (see [18, 19, 11] for further reading). For the higher order mKdV equations (6.1) and (6.2), an equivalent expression of periodic breathers is available, by following a similar matching of parameters. Namely, we consider here the 5th and 7th mKdV equations (6.1) and (6.2) where now
\[
  u : \mathbb{R} \times T_x \mapsto \mathbb{R},
\]
is periodic in space, and \( T_x = \mathbb{T} = \mathbb{R}/L\mathbb{Z} = (0, L) \) denotes a torus with period \( L \), to be fixed later. Higher order periodic cases can also be described but for the sake of simplicity, we will keep our discussion with these 5th and 7th orders.

We refer the reader to [1, 12] for a more detailed account on the Jacobi elliptic functions \( \text{sn} \) and \( \text{nd} \) presented below. The proof of Theorem 1.6 is essentially contained in the following

**Proposition 6.1** (Periodic breathers of 5th and 7th mKdV equations). Given \( \alpha, \beta > 0 \), \( x_1, x_2 \in \mathbb{R} \) and \( k, m \in [0, 1] \), the following is satisfied.

1. Periodic breather solutions of the 5th and 7th mKdV equations (6.1) and (6.2), are given by the explicit formula (see [18] and [11] for checking and comparison reasons)
\[
  B = B(t, x; \alpha, \beta, k, m, x_1, x_2)
  := \partial_x B := 2\partial_x \left[ \arctan \left( \frac{\beta \text{sn}(\alpha y_1, k)}{\alpha \text{nd}(\beta y_2, m)} \right) \right],
\]
with \( \text{sn}(\cdot, k) \) and \( \text{nd}(\cdot, m) \) the standard Jacobi elliptic functions of elliptic modulus \( k \) and \( m \), respectively, but now
\[
  y_1 := x + \delta_{i,m}t + x_1, \quad y_2 := x + \gamma_{i,m}t + x_2, \quad i = 5, 7.
\]

2. The velocities \( (\delta_{5,m}, \gamma_{5,m}) \) in the 5th order case and \( (\delta_{7,m}, \gamma_{7,m}) \) in the 7th order case are given by, respectively:
\[
  \delta_{5,m} := -\alpha^4(k^2 - 26k + 1)
  + 10\alpha^2\beta^2(1 + k)(2 - m) - 5\beta^4(m^2 - 16m + 16),
\]
\[
  \gamma_{5,m} := -\beta^4(m^2 + 24m - 24)
  + 10\alpha^2\beta^2(1 + k)(2 - m) - 5\alpha^4(k^2 + 14k + 1),
\]
\[
  \delta_{7,m} := -\alpha^4(3k^2 - 28k + 1)
  + 10\alpha^2\beta^2(1 + k)(2 - m) - 5\beta^4(m^2 - 72m + 64),
\]
\[
  \gamma_{7,m} := -\beta^4(m^2 + 48m - 48)
  + 10\alpha^2\beta^2(1 + k)(2 - m) - 5\alpha^4(k^2 + 28k + 1),
\]
and

\[ \delta_{7,m} := \alpha^6(k^3 + 135k^2 + 135k + 1) + 21\alpha^4\beta^2(-2 + k^2(m - 2) + m + 2k(7m - 6)) + 7\alpha^2\beta^4(1 + k)(5m^2 - 24m + 24) + 7\beta^6(m^3 - 2m^2 + 48m - 48), \]

\[ \gamma_{7,m} := -\beta^6(-m^3 - 254m^2 - 2256m + 2512) + 7\alpha^2\beta^4(5(k^2 + 1)(m - 2) + k(70m + 292)) + 7\beta^6(k^3 + 135k^2 + 135k + 1). \]  

(6.6)

Additionally, in order to be a periodic solution of 5th and 7th-mKdV equations (and also for the classical mKdV), the parameters \( m, k, \alpha \) and \( \beta \) must satisfy the following commensurability conditions on the spatial periods

\[ \frac{\beta^4}{\alpha^4} = \frac{k}{1 - m}, \quad K(k) = \frac{\alpha}{2\beta}K(m), \]  

(6.7)

where \( K \) is the complete elliptic integral of the first kind, defined as

\[ K(r) := \int_0^{\pi/2} (1 - r \sin^2(s))^{-1/2} \, ds \]

\[ = \int_0^1 ((1 - t^2)(1 - rt^2))^{-1/2} \, dt, \]  

(6.8)

and which satisfies

\[ K(0) = \frac{\pi}{2} \quad \text{and} \quad \lim_{k \to 1} K(k) = \infty. \]

(4) The spatial period of the breather is given by

\[ \frac{4}{\alpha}K(k) = \frac{2}{\beta}K(m). \]  

(6.9)

Remark 6.1. Note that conditions (6.7) formally imply that the periodic breather \( B \) [6.3] has only four independent parameters (e.g. \( \beta, k \) and translations \( x_1, x_2 \)). Additionally, if we assume that the ratio \( \beta/\alpha \) stays bounded, we have that \( k \) approaches 0 as \( m \) is close to 1. Using this information, the standard non periodic 5th and 7th-mKdV breathers can be formally recovered as the limit of very large spatial period \( L \to +\infty \), obtained e.g. if \( k \to 0 \).

Remark 6.2. Note that these breathers can be written using only two parametric variables, say \( \beta \) and \( k \), and have a characteristic period \( L = \tilde{L}(\beta,k) \), with \( L \to +\infty \) as \( k \to 0 \). Moreover, compare the periodic higher order velocities \( (\delta_{i,m}, \gamma_{i,m}) \), \( i = 5, 7 \), with the equivalent periodic ones in the simpler classical mKdV case \[ \text{[11 Def.1.1]} \]:

\[ \delta := \alpha^2(1 + k) + 3\beta^2(m - 2), \quad \gamma := 3\alpha^2(1 + k) + \beta^2(m - 2), \]  

(6.10)

and with velocities \( (\delta_i, \gamma_i), i = 5, 7 \), \[ \text{[6.3, 6.8]} \] when \( \mu = 0 \) in the non periodic case:

\[ \delta_5 := -\alpha^4 + 10\alpha^2\beta^2 - 5\beta^4, \quad \gamma_5 := -\beta^4 + 10\alpha^2\beta^2 - 5\alpha^4, \]  

(6.11)
and

\[
\delta_7 := \alpha^6 - 21\alpha^4\beta^2 + 35\alpha^2\beta^4 - 7\beta^6, \\
\gamma_7 := -\beta^6 + 21\alpha^2\beta^4 - 35\alpha^4\beta^2 + 7\alpha^6. 
\]  \hspace{1cm} (6.12)

**APPENDIX A. Notation in proof of Theorem 1.3**

We will use the following notation for the sake of simplicity:

\[
N_x := 2\alpha \left( a_1^2 - \alpha^2 a_3^2 + a_3^2 \beta^2 \right) \sin (y_1) \cos (y_1) - 2\alpha a_1 a_2 e^{y_2} \cos (y_1) \\
- 2a_1 a_2 \beta e^{y_2} \sin (y_1) + 2a_2^2 \beta e^{y_2} + 2\alpha^2 a_3^2 \beta \left( \sin (y_1) \right)^2 \\
- 2\alpha^2 a_3^2 \beta \left( \cos (y_1) \right)^2 + 2\alpha^2 a_3 \sin (y_1) \cosh (y_2) - 2\alpha a_3 \beta \cos (y_1) \sinh (y_2) \\
+ 2\alpha a_3 \beta \cos (y_1) \cosh (y_2) + 2a_3 \beta^2 \sin (y_1) \sinh (y_2) + 2\beta \sinh (y_2) \cosh (y_2), 
\]  \hspace{1cm} (A.1)

\[
N_{xx} := 8\alpha^3 a_3^2 \beta \sin (y_1) \cos (y_1) - 4\alpha a_1 a_2 \beta e^{y_2} \cos (y_1) + 4a_2^2 \beta^2 e^{y_2} \\
+ 2\alpha^2 \left( a_1^2 - \alpha^2 a_3^2 + a_3^2 \beta^2 \right) \left( \cos (y_1) \right)^2 \\
- 2\alpha^2 \left( a_1^2 - \alpha^2 a_3^2 + a_3^2 \beta^2 \right) \left( \sin (y_1) \right)^2 \\
- 2a_3 \beta \left( \alpha^2 - \beta^2 \right) \sin (y_1) \cosh (y_2) + 4\alpha a_3^2 \beta \cos (y_1) \sinh (y_2) \\
+ 4\alpha^2 a_3 \beta \sin (y_1) \sinh (y_2) + 2\beta^2 \left( \cosh (y_2) \right)^2 + 2\beta^2 \left( \sinh (y_2) \right)^2 \\
+ 2a_1 a_2 \left( \alpha^2 - \beta^2 \right) e^{y_2} \sin (y_1) + 2\alpha a_3 \left( \alpha^2 - \beta^2 \right) \cos (y_1) \cosh (y_2) 
\]  \hspace{1cm} (A.2)

\[
N_{3x} := -8\alpha^3 \left( a_1^2 - \alpha^2 a_3^2 + a_3^2 \beta^2 \right) \sin (y_1) \cos (y_1) + 8a_2^2 \beta^2 e^{y_2} \\
+ 2a_1 a_2 \alpha \left( \alpha^2 - 3\beta^2 \right) e^{y_2} \cos (y_1) + 8\alpha^4 a_3^2 \beta \left( \cos (y_1) \right)^2 \\
- 8\alpha^4 a_3^2 \beta \left( \sin (y_1) \right)^2 \\
+ 8\beta^3 \sinh (y_2) \cosh (y_2) - 2\alpha^2 a_3 \left( \alpha^2 - 3\beta^2 \right) \sin (y_1) \cosh (y_2) \\
- 2\beta^2 a_3 \left( 3\alpha^2 - \beta^2 \right) \sin (y_1) \sinh (y_2) + 2a_1 a_2 \beta \left( 3\alpha^2 - \beta^2 \right) \cos (y_1) \\
- 2\alpha a_3 \beta \left( \alpha^2 - 3\beta^2 \right) \cos (y_1) \cosh (y_2) + 2a_3 \alpha \beta \left( 3\alpha^2 - \beta^2 \right) \cos (y_1) \sinh (y_2) 
\]  \hspace{1cm} (A.3)

\[
N_{4x} := -32\alpha^5 a_3^2 \beta \cos (y_1) \sin (y_1) + 8\alpha a_1 a_2 \beta \left( \alpha^2 - \beta^2 \right) e^{y_2} \cos (y_1) \\
- 2a_1 a_2 \left( \alpha^4 + \beta^4 - 6\alpha^2 \beta^2 \right) e^{y_2} \sin (y_1) + 16a_2^2 \beta^4 e^{y_2} + 8\beta^4 \left( \sinh (y_2) \right)^2 \\
+ 8\alpha^4 \left( a_1^2 - \alpha^2 a_3^2 + a_3^2 \beta^2 \right) \left( \sin (y_1) \right)^2 \\
+ 8\beta^2 \left( \cosh (y_2) \right)^2 \\
- 2\alpha a_3 \left( \alpha^4 + \beta^4 - 6\alpha^2 \beta^2 \right) \cos (y_1) \cosh (y_2) \\
- 8\alpha a_3 \beta^2 \left( \alpha^2 - \beta^2 \right) \cos (y_2) - 8\alpha^4 \left( a_1^2 - \alpha^2 a_3^2 + a_3^2 \beta^2 \right) \left( \cos (y_1) \right)^2 \\
+ 2a_3 \beta \left( \alpha^4 + \beta^4 - 6\alpha^2 \beta^2 \right) \sin (y_1) \cosh (y_2) - 8\alpha^2 a_3 \left( \alpha^2 - \beta^2 \right) \sin (y_1) \beta \sinh (y_2) 
\]  \hspace{1cm} (A.4)

and

\[
H_x := 2 \left( \alpha^2 + \beta^2 \right) a_2 a_3 \alpha e^{y_2} \cos (y_1) - a_1 \cosh (y_2) \sin (y_1) - a_2 a_3 \beta e^{y_2} \sin (y_1)), 
\]  \hspace{1cm} (A.5)

\[
H_{xx} := -2 \left( \alpha^2 + \beta^2 \right) \alpha a_1 \cosh (y_2) \cos (y_1) + a_1 \beta \sinh (y_2) \sin (y_1) \\
+ a_2 a_3 \left( \alpha^2 + \beta^2 \right) e^{y_2} \sin (y_1)), 
\]  \hspace{1cm} (A.6)
Higher order members of the mKdV and Gardner hierarchies \((1.8)-(1.1)\). We start
\[
H_{3x} := 2a_1 (\alpha^4 - \beta^4) \cosh(y_2) \sin(y_1) - 4a_1 \alpha \beta (\alpha^2 + \beta^2) \sinh(y_2) \cos(y_1)
- 2a_2a_3 \beta (\alpha^2 + \beta^2)^2 e^{y_2} \sin(y_1) - 2a_2a_3 \alpha (\alpha^2 + \beta^2)^2 e^{y_2} \cos(y_1),
\]
(A.7)
\[
H_{4x} := 2a_1 (\alpha^4 - 2\alpha^2\beta^2 - 3\beta^4) \cosh(y_2) \cos(y_1)
- 4a_2a_3 \alpha \beta (\alpha^2 + \beta^2)^2 e^{y_2} \cos(y_1)
+ 2a_1 \beta (3\alpha^4 + 2\alpha^2\beta^2 - \beta^4) \sinh(y_2) \sin(y_1)
+ 2a_2a_3 (\alpha^6 + \alpha^4\beta^2 - 2\alpha^2\beta^4 - \beta^6) e^{y_2} \sin(y_1).
\]
(A.8)

Appendix B. Additional higher order mKdV and Gardner equations

For the sake of completeness and forthcoming work by elsewhere, we list below higher order members of the mKdV and Gardner hierarchies \([1.8]-[1.1]\). We start with the 9th order mKdV equation which is written as follows
\[
u_t + \partial_x \left( u_{8x} + 18u^2u_{6x} + 108uu_xu_{5x} + 228uu_{2x}u_{4x} + 210u_x^2u_{4x} + 126uu^4u_{4x}
+ 138u(u_{3x})^2 + 756uu_xu_{3x} + 1008u^3u_{3x} + 182u_x^4 + 756u^3u_{2x}^2
+ 6108u_x^2u_{2x} + 420u_x^6 + 798uu_x^4 + 126uu^5(u_x)^2 + 70u^9 \right) = 0.
\]
(B.1)
The 11th order mKdV equation is written as follows
\[
u_t + \partial_x \left( u_{10x} + 22u^2u_{8x} + 198u^4u_{6x} + 924u^6u_{4x} + 506uu_x^2 + 3036u^3u_{3x}
+ 2310u^8u_{xx} + 8316u^5u_{xx} + 9372u^2u_{3x}^2 + 9240u^7u_x^2 + 26796u^3u_x^4
+ 176uu_xu_{7x} + 484uu_{ux}u_{6x} + 462u_x^2u_{6x} + 836uu_{3x}u_{5x} + 2376u^3u_xu_{5x}
+ 5016u^3u_xu_{4x} + 2706u_x^2u_{4x} + 11220u^2u_x^2u_{4x} + 3498u_xu_{3x}^2 + 11088u^5u_xu_{3x}
+ 54516u^4u_xu_{3x}^2 + 44748uu_x^2u_{3x}^2 + 13398u_x^4u_{3x} + 2376uu_xu_{xx}u_{5x}
+ 2112uu_x^3u_{3x} + 3696uu_xu_{3x}u_{4x} + 39336u^2u_xu_{xx}u_{3x} + 252u_{11} \right) = 0.
\]
(B.2)
We finally present the 13th order mKdV equation
\[
u_t + \partial_x \left( u_{12x} + 1846u_{5x}^2 + 191620u^4u_{xx}^3 + 924u_{13} + 30888u^5u_xu_{5x}
+ 1733160u^8u_{xx}^2u_x^2 + 823680u^3u_{4x}^4u_{4x} + 1398540u^2u_{4x}^4u_{xx}
+ 648648u^6u_{xx}^2u_{xx} + 96096u^7u_{xx}^3u_{xx} + 3172u_{4x}u_{6x} + 1976u_{3x}u_{7x}
+ 884uu_xu_{8x} + 460uu_xu_{9x} + 32432uu_{3x}u_{5x}^2 + 78936uu_{3x}^2u_{5x}
+ 511368uu_{xx}u_{4x} + 919776uu_{xx}u_{xx}^2u_{3x} + 566280u_{xx}^3u_{3x}u_{xx}
+ 17160u_{xx}u_{xx}u_{5x} + 5720u_{xx}u_{xx}u_{7x} + 12012u_{xx}u_{xx}u_{6x} + 33176u_{xx}u_{xx}u_{3x}u_{5x}
+ 21736u^3u_{xx}u_{5x} + 1258u^3u_{xx}u_{6x} + 4576u^3u_{xx}u_{7x} + 29172u^2u_{xx}u_{6x}
+ 231660u^2u_{xx}u_{4x} + 65208u^2u_{xx}u_{4x} + 78078u^4u_{4x} + 403260u^2u_{xx}^3
+ 8866u_{xx}^2u_x + 20306uu_{xx}^2u_{xx} + 858u_{xx}^2u_x + 26598uu_{xx}^2u_{xx}
+ 4684u^5u_x + 60060u^9u_x + 157300uu_x^6 + 13156u^3u_x^2 + 286u^4u_{xx}
\right) = 0.
\]
and the 11th order Gardner equation

\[ u_{transformation} + 39468u^5u_{4x} + 1716u^6u_{6x} + 12012u^{10}u_{xx} + 72072u^7u_{xx}^2 + 6006u^8u_{4x} + 108966u^4u_{10x} + 156156u^2u^2u_{xx}u_{4x} + 197340u^2u_{xx}u_{3x}^2 + 219648u^2u_{xx}u_{3x}u_{4x} + 144144u^2u_{xx}u_{5x} + 806520u^4u_{xx}u_{3x}u_{4x} \]  

(B.4)

Now, associated with these additional higher order mKdV equations, we provide the corresponding breather solutions, as in (1.11):

**Definition B.1** (9th-11th-mKdV breathers). Let \( \alpha, \beta > 0 \) and \( x_1, x_2 \in \mathbb{R} \). The real-valued breather solution associated to the 9th-11th-mKdV equations (B.1) - (B.3) are given explicitly by the formula

\[
B \equiv B_{\alpha, \beta}(t, x; x_1, x_2) := 2\partial_x \left[ \arctan \left( \frac{\beta \sin(\alpha y_1)}{\alpha \cosh(\beta y_2)} \right) \right],
\]

(B.5)

with \( y_1 \) and \( y_2 \)

\[
y_1 = x + \delta_i t + x_1, \quad y_2 = x + \gamma_i t + x_2,
\]

(B.6)

and with velocities \( (\delta_i, \gamma_i), \ i = 9, 11 \)

\[
\delta_9 := -\alpha^8 + 36\alpha^6\beta^2 - 126\alpha^4\beta^4 + 84\alpha^2\beta^6 - 9\beta^8,
\]

\[
\gamma_9 := -\beta^8 + 36\alpha^2\beta^6 - 126\alpha^4\beta^4 + 84\alpha^6\beta^2 - 9\alpha^8,
\]

(B.7)

and

\[
\delta_{11} = \alpha^{10} - 55\alpha^8\beta^2 + 330\alpha^6\beta^4 - 462\alpha^4\beta^6 + 165\alpha^2\beta^8 - 11\beta^{10},
\]

\[
\gamma_{11} = 11\alpha^{10} - 165\alpha^8\beta^2 + 462\alpha^6\beta^4 - 330\alpha^4\beta^6 + 55\alpha^2\beta^8 - \beta^{10}.
\]

(B.8)

Finally, and directly from definition of the Gardner hierarchy (1.1) with \( n = 4 \) or alternatively, from the 9th and 11th order mKdV equations (B.1) - (B.2) with the transformation \( u \to \mu + u \), we present the 9th order Gardner equation:

\[
u_t + \partial_x \left( u_{8x} + 18(\mu + u)^2u_{6x} + 108(\mu + u)u_{5x}u_{5x} + 228(\mu + u)u_{2x}u_{4x} + 210u_2u_4u_{4x} + 126(\mu + u)^4u_{4x} + 138(\mu + u)u_{2x}^2 + 756u_{5x}u_{2x}u_{3x} + 1008u_3u_{3x} + 182u_{3}^2 + 756(\mu + u)^3u_{2x}^2 + 3108(\mu + u)^2u_{2x}^2u_{2x} + 420(\mu + u)^6u_{2x} + 798(\mu + u)u_{2x}^3 + 1260(\mu + u)^5u_{2x}^2 + 70(\mu + u)^9 \right) = 0,
\]

(B.9)

and the 11th order Gardner equation
The corresponding breather solutions of the 9th and 11th order Gardner equations \((B.9)-(B.10)\) are the following

**Definition B.2 (9th-11th-Gardner breathers).** Let \(\alpha, \beta, \mu\) as in definition \((1.10)\) and \(x_1, x_2 \in \mathbb{R}\). Then the real-valued breather solution associated to the 9th and 11th Gardner equations \((B.9)-(B.10)\) is given explicitly by the same formula as in \((1.10)\) but respectively with velocities \((\delta_9, \gamma_9)\) and \((\delta_{11}, \gamma_{11})\):

\[
\begin{align*}
\delta_9 & := \alpha(\alpha^8 - 18\alpha^6(2\beta^2 + \mu^2) + 126\alpha^4(\beta^4 + 3\beta^2\mu^2 + \mu^4)) \\
& \quad - 42\alpha^2(2\beta^6 + 15\beta^4\mu^2 + 30\beta^2\mu^4 + 10\mu^6) + 9(\beta^8 + 14\beta^6\mu^2 + 70\beta^4\mu^4) \\
& \quad + 140\beta^2\mu^6 + 70\mu^8) \\
\gamma_9 & := -\beta(9\alpha^8 - 42\alpha^6(2\beta^2 + 3\mu^2) + 126\alpha^4(\beta^4 + 5\beta^2\mu^2 + 5\mu^4)) \\
& \quad - 18\alpha^2(2\beta^6 + 21\beta^4\mu^2 + 70\beta^2\mu^4 + 70\mu^6) + \beta^8 + 18\beta^6\mu^2 + 126\beta^4\mu^4 \\
& \quad + 420\beta^2\mu^6 + 630\mu^8),
\end{align*}
\]

and

\[
\begin{align*}
\delta_{11} & := \alpha(\alpha^{10} - 11\alpha^8(5\beta^2 + 2\mu^2) + 66\alpha^6(5\beta^4 + 12\beta^2\mu^2 + 3\mu^4)) \\
& \quad - 462\alpha^4(\beta^6 + 15\beta^4\mu^2 + 9\beta^2\mu^4 + 2\mu^6) + 330\alpha^2(5\beta^8 + 56\beta^6\mu^2 + 210\beta^4\mu^4 \\
& \quad + 280\beta^2\mu^6 + 70\mu^8) - 11(\beta^{10} + 18\beta^8\mu^2 + 126\beta^6\mu^4 + 420\beta^4\mu^6 + 630\beta^2\mu^8 \\
& \quad + 252\mu^{10})) \\
\gamma_{11} & := -\beta(-11\alpha^{10} + 33\alpha^8(5\beta^2 + 6\mu^2) - 462\alpha^6(\beta^4 + 4\beta^2\mu^2 + 3\mu^4) \\
& \quad + 66\alpha^4(5\beta^6 + 24\beta^4\mu^2 + 105\beta^2\mu^4 + 70\mu^6) - 11\alpha^2(5\beta^8 + 72\beta^6\mu^2 + 378\beta^4\mu^4 \\
& \quad + 840\beta^2\mu^6 + 630\mu^8) + \beta^{10} + 22\beta^8\mu^2 + 198\beta^6\mu^4 + 924\beta^4\mu^6 + 2310\beta^2\mu^8 \\
& \quad + 2772\mu^{10}).
\end{align*}
\]
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