EVEN MAPS, THE COLIN DE VERDIEÎRE NUMBER AND REPRESENTATIONS OF GRAPHS

VOJTĚCH KALUŽA*, MARTIN TANCER†

Received February 7, 2020
Revised March 5, 2021
Online First May 19, 2022

Van der Holst and Pendavingh introduced a graph parameter $\sigma$, which coincides with the more famous Colin de Verdière graph parameter $\mu$ for small values. However, the definition of $\sigma$ is much more geometric/topological directly reflecting embeddability properties of the graph. They proved $\mu(G) \leq \sigma(G) + 2$ and conjectured $\mu(G) \leq \sigma(G)$ for any graph $G$. We confirm this conjecture. As far as we know, this is the first topological upper bound on $\mu(G)$ which is, in general, tight.

Equality between $\mu$ and $\sigma$ does not hold in general as van der Holst and Pendavingh showed that there is a graph $G$ with $\mu(G) \leq 18$ and $\sigma(G) \geq 20$. We show that the gap appears at much smaller values, namely, we exhibit a graph $H$ for which $\mu(H) \leq 7$ and $\sigma(H) \geq 8$. We also prove that, in general, the gap can be large: The incidence graphs $H_q$ of finite projective planes of order $q$ satisfy $\mu(H_q) \in O(q^{3/2})$ and $\sigma(H_q) \geq q^2$.

An earlier version of this work appeared in the proceedings of the 2020 ACM-SIAM Symposium on Discrete Algorithms (SODA20) [14]. The earlier version contains full proofs of the results mentioned in the abstract. In the present version we alter the core definition of our work (a semivalid representation; see Definition 5). In our opinion, the new definition is conceptually simpler and fits better into the existing theory. We use it to introduce a new graph parameter, which we call $\eta$ and which did not appear in the previ-

* V. K. gratefully acknowledges the support of Austrian Science Fund (FWF): P 30902-N35. This work was done mostly while he was employed at the University of Innsbruck. During the early stage of this research, V. K. was partially supported by Charles University project GAUK 926416.

† M. T. is supported by the grant no. 19-04113Y of the Czech Science Foundation (GAČR) and partially supported by Charles University project UNCE/SCI/004.
ous version. It allows us to establish an extended version of the main result showing that $\mu(G) \leq \eta(G) \leq \sigma(G)$ holds for every graph $G$.

1. Introduction

In 1990 Colin de Verdière [2] (see [3] for an English translation) introduced a graph parameter $\mu(G)$. It arises from the study of the multiplicity of the second smallest eigenvalue of certain matrices associated to a graph $G$ (discrete Schrödinger operators); however, it turns out that this parameter is closely related to geometric and topological properties of $G$. In particular, this parameter is minor monotone, and moreover, it satisfies:

(i) $\mu(G) = 0$ if and only if $G$ embeds in $\mathbb{R}^0$;
(ii) $\mu(G) \leq 1$ if and only if $G$ embeds in $\mathbb{R}^1$;
(iii) $\mu(G) \leq 2$ if and only if $G$ is outerplanar;
(iv) $\mu(G) \leq 3$ if and only if $G$ is planar; and
(v) $\mu(G) \leq 4$ if and only if $G$ admits a linkless embedding into $\mathbb{R}^3$.

The characterization up to the value 3 as well as the minor monotonicity of $\mu$ was shown by Colin de Verdière [2,3]. The characterization of graphs with $\mu(G) \leq 4$ was established by Lovász and Schrijver [18]. Beyond this, any description is known only for the classes of graphs with $\mu(G) \geq |V(G)| - k$ for $k = 1, 2, 3$, and partial results are known also for $k = 4, 5$; see [16]. It used to be an open problem whether the graphs with $\mu(G) \leq 5$ coincide with knotless embeddable graphs [4, Sec. 14.5], [33, Sec. 7]. However, a graph $H$ constructed by Foisy [6] satisfies $\mu(H) \leq 5$ whereas it is not knotless embeddable.\footnote{The inequality $\mu(H) \leq 5$ follows from the fact that there is a vertex $v$ of $H$ such that $H - v$ is a linkless embeddable graph, that is, $\mu(H - v) \leq 4$.} We are very thankful to Rose McCarty for sharing this example with us [21].

Due to the aforementioned properties, the study of $\mu$ gained a lot of popularity (e.g., [1,8,10,16,18,11,19,17,13,7,28,20,31]). A precise definition of the parameter $\mu$ is given at the end of Subsection 2.1.

Later, in 2009, van der Holst and Pendavingh [12] introduced another minor monotone parameter $\sigma(G)$, which is directly defined through topological properties of $G$. Roughly speaking, $\sigma(G)$ is the minimal integer $k$ such that every CW-complex $\mathcal{C}$ whose 1-skeleton is $G$ admits a so-called even mapping into $\mathbb{R}^k$. This is a mapping $f$ such that whenever $\vartheta$ and $\tau$ are disjoint cells of $\mathcal{C}$, then $f(\vartheta) \cap f(\tau) = \emptyset$ if $\dim \vartheta + \dim \tau < k$, and $f(\vartheta)$ and $f(\tau)$ cross in an even number of points if $\dim \vartheta + \dim \tau = k$. For a precise definition, we refer to [12].
It turns out that $\sigma(G) \leq k$ if and only if $\mu(G) \leq k$ for $k \in \{0, 1, 2, 3, 4\}$. In addition, van der Holst and Pendavingh [12, Conj. 43] conjectured that this is true also for $k = 5$. However, in general, $\sigma$ and $\mu$ differ. They provide an example of a graph with $\mu(G) \leq 18$, but $\sigma(G) \geq 20$ based on a previous work of Pendavingh [22]. On the other hand, van der Holst and Pendavingh [12, Cor. 41] proved that $\mu(G) \leq \sigma(G) + 2$, while they conjectured that $\mu(G) \leq \sigma(G)$. We confirm this conjecture.

**Theorem 1.** For any graph $G$, $\mu(G) \leq \sigma(G)$.

Our tools that we use for the proof of Theorem 1 also allow us to show that the gap between $\mu$ and $\sigma$ appears at much smaller values.

**Theorem 2.** There is a graph $G$ such that $\mu(G) \leq 7$ and $\sigma(G) \geq 8$.

We remark here that adding a new vertex to a graph $G$ and connecting it to all vertices of $G$ increases both $\mu(G)$ and $\sigma(G)$ by exactly one (unless $G$ is the complement of $K_2$); see [11, Thm. 2.7] and [12, Thm. 28]. Consequently, Theorem 2 immediately implies that for every $k \geq 7$ there is a graph $G_k$ with $\mu(G_k) \leq k$ and $\sigma(G_k) \geq k + 1$.

The key step in the proof of Theorem 2 is to provide a lower bound on $\sigma$; otherwise we follow [22]. We remark that the example of $G$ with $\mu(G) \leq 18$ but $\sigma(G) \geq 20$ coming from [12,22] is the highly regular Tutte’s 12-cage. The important property is that the second largest eigenvalue of the adjacency matrix of Tutte’s 12-cage has very high multiplicity. We use instead the incidence graphs of finite projective planes, which enjoy the same property. Namely, if $H_q$ is the incidence graph of a finite projective plane of order $q$, we will show that $\mu(H_3) \leq 9$, whereas $\sigma(H_3) \geq 11$; see Proposition 25. Then, by further modification of this graph, we obtain the graph from Theorem 2.

As a complementary result, based on properties of finite projective planes, we also show that the gap between $\mu$ and $\sigma$ is asymptotically large.

**Theorem 3.** Let $q \in \mathbb{N}$ be such that a finite projective plane of order $q$ exists\(^2\). Then $\mu(H_q) \in O\left(q^{3/2}\right)$, while $\sigma(H_q) \geq \lambda(H_q) \geq q^2$, where $\lambda$ is the graph parameter of van der Holst, Laurent, and Schrijver [9], which we overview in Section 4.

---

\(^2\) This includes all prime powers $q$ (see, e.g., [30, Sec. 2.3]).
geometric description\textsuperscript{3} in $\mathbb{R}^k$, then $\mu(G) \leq k$. This is tight in general because $\mu(K_n) = \sigma(K_n) = n - 1$, where $K_n$ is the complete graph on $n$ vertices [11,12]. To the best of our knowledge, this is the first tight upper bound on $\mu(G)$ in terms of embeddability properties of $G$ for general value of the parameter.\textsuperscript{4}

On the other hand, we argue that the parameter $\sigma$ deserves as much attention as $\mu$.

First of all, it provides a much more direct geometric generalization of graph planarity than $\mu$; more in the spirit of the Hanani–Tutte type characterization of graph planarity (see, e.g., [27]).

Next, it seems that it might be computationally more tractable to determine the graphs with $\sigma \leq k$ when compared to graphs with $\mu \leq k$. From now on, let $G_{\mu \leq k}$ and $G_{\sigma \leq k}$ denote the class of graphs with $\mu \leq k$ and $\sigma \leq k$, respectively. Of course, once we fix an integer $k$, there is a polynomial time algorithm for recognition of graphs in $G_{\mu \leq k}$ and $G_{\sigma \leq k}$ via the Robertson–Seymour theory [24,25] as there is a finite list of forbidden minors for these classes. The minors are well known if $k \leq 4$; however the catch of this approach is that finding the minors for $k \geq 5$ seems to be out of reach.

Let us focus on the interesting case $k = 5$. We are not aware of any explicit algorithm for determining the graphs in $G_{\mu \leq 5}$ in the literature. The best algorithm we could come up with is a PSPACE algorithm based on the existential theory of the reals. (This algorithm recognizes the graphs in $G_{\mu \leq k}$ for general $k \in \mathbb{N}$. For an interested reader, we describe this algorithm in Appendix A of the arXiv version of this paper [15].) On the other hand, we are able to get an explicit polynomial time certificate for $\sigma(G) > 5$, that is, a certificate for the co-NP membership in $G_{\sigma \leq k}$. (This is discussed in detail only in the introduction and Appendix B of the arXiv version of this paper [15].) Now, if the conjecture $G_{\mu \leq 5} = G_{\sigma \leq 5}$ of van der Holst and Pendavingh is true, then the certificate above also certifies graphs with $\mu > 5$. Theorem 1 gives one inequality.

**Overview of our proofs.** Here we briefly overview the key steps in our main proofs. We start with Theorem 1. On high level, we follow the strategy of

\textsuperscript{3} In fact, Theorem 30 of [12] reveals that an even mapping of a CW-complex $\mathcal{C}$ (in the definition of $\sigma$) can be exchanged with an even mapping of the $\lfloor k/2 \rfloor$-skeleton of $\mathcal{C}$ into $\mathbb{R}^{k-1}$, provided that in addition $\mathcal{C}$ is a so-called closure (which can be assumed in the definition of $\sigma$). This explains the shift of the dimension in the geometric description of the classes with $\mu(G) \leq 3$ or $\mu(G) \leq 4$, equivalently, the classes with $\sigma(G) \leq 3$ or $\sigma(G) \leq 4$.

\textsuperscript{4} For comparison, there is a result of Izmestiev [13] providing a quite different lower bound on $\mu$: If $G$ is the 1-skeleton of convex $d$-polytope, then $\mu(G) \geq d$. However, as Izmestiev points out, this result already follows from the minor monotonicity of $\mu$ and the fact that the 1-skeleton of a $d$-polytope contains $K_{d+1}$ as a minor.
Lovász and Schrijver [18], who showed that if $G$ is a linklessly embeddable graph, then $\mu(G) \leq 4$. However, we need to rework many details as we need many claims in somewhat stronger form. We also work in a more general setting than what is strictly needed for the proof of Theorem 1, which we use in the proof of Theorem 2. First we sketch their strategy (in our words) and then we point out the important differences.

For contradiction, Lovász and Schrijver assume that there is linklessly embeddable $G$ with $\mu(G) \geq 5$. According to the definition of $\mu$ (given in the next section), there is a certain matrix $M \in \mathbb{R}^{V \times V}$ of corank 5 associated to $G = (V, E)$ which witnesses $\mu(G) \geq 5$. Given a vector $x \in \mathbb{R}^V$, we denote by $\text{supp}(x)$ the set $\{v \in V : x_v \neq 0\}$. Correspondingly, we define $\text{supp}_+(x) := \{v \in V : x_v > 0\}$ and $\text{supp}_-(x) := \{v \in V : x_v < 0\}$. Then $\ker(M)$, the kernel of $M$, can be decomposed into equivalence classes of vectors for which $\text{supp}_+$ and $\text{supp}_-$ coincide. Each equivalence class is a (relatively open) cone (see Definition 11). We extend the notation $\text{supp}_ \pm$ to sets lying inside a single such cone. Then, by choosing carefully a fine enough set of unit vectors in each of the cones and taking the convex hull, Lovász and Schrijver obtain a 5-dimensional polytope $P$ such that every relatively open face of $\partial P$ is in one of the cones.

Given a linkless embedding of $G$ (more precisely, a flat embedding), it is possible now to define an embedding $f$ of the 1-skeleton $P^{(1)}$ into $\mathbb{R}^3$ in such a way that for every vertex $u$ of $P$, which is also a vector of $\ker(M)$, $f(u)$ is mapped close to a vertex of $\text{supp}_+(u)$ (this vertex is embedded in $\mathbb{R}^3$ by the given linkless/flat embedding of $G$).

Also, for every edge $e = uw$ of $P$, we have

$$\text{supp}_+(e) \supseteq \text{supp}_+(u) \cup \text{supp}_+(w).$$

If $G[\text{supp}_+(e)]$, the subgraph induced by $\text{supp}_+(e)$, is connected for every such $e$, then Lovász and Schrijver pass $f(e)$ close to some path connecting $f(u)$ and $f(w)$ in $G[\text{supp}_+(e)]$. An existence of such $f$ then reveals that the original embedding of $G$ was not linkless via a Borsuk–Ulam type theorem by Lovász and Schrijver [18], which is the required contradiction.

It, however, still remains to resolve the case when some edges $e$ do not satisfy that $G[\text{supp}_+(e)]$ is connected. Such edges are called broken edges and it is the main technical part of the proof to handle them. Via structural properties of $G$, including the usage of one of the forbidden minors for linkless embeddability (see [23] for the list of minimal such graphs), Lovász and Schrijver show how to route the broken edges without introducing new linkings, which again yields the required contradiction.
Our main technical contribution is that we design a strategy how to route broken edges without any requirements on the structure of $G$—this is a must if we want to get a result for a general value of $\mu$. Namely, we show that if we make several very careful choices in the very beginning when placing the vertices of $P$ as well as if we carefully route the nonbroken edges of $P$, then we are able to make enough space for the broken edges as well. (For this we need to know more about the structure of broken edges and about $P$ than what appears in [18]; see Definition 18 and Proposition 19). An important property for routing the edges is that when $F$ and $F'$ are (so-called) antipodal faces, then the edges of $F$ and the edges of $F'$ are routed close to disjoint subgraphs. (The precise statement is given by Proposition 21, and we actually map $P^{(1)}$ into the graph $G$.)

Now, we could aim to conclude in a similar way as Lovász and Schrijver via a suitable Borsuk–Ulam type theorem, which would require to extend the map to higher skeletons and to perturb it a bit. However, we instead use a lemma of van der Holst and Pendavingh [12] tailored to such a setting, which they used in the proof of the inequality $\mu(G) \leq \sigma(G) + 2$ (see the proof of Proposition 20).

Last but not least, instead of working directly with matrices from the definition of $\mu$, we abstract their properties required in the proof of Theorem 1 into a notion of semivalid representation; see Definition 5. (The main difference is that we replace the so-called Strong Arnold hypothesis by more combinatorial properties.) This abstraction turns out to be very useful in the proof of Theorem 2 because then it is possible to provide lower bounds on $\sigma$ also with aid of matrices not satisfying the Strong Arnold hypothesis, which is essential if we want to separate $\mu$ and $\sigma$.

Recall that by $H_q$ we denote the incidence graph of a finite projective plane of order $q$. We add a short description of how our bound on $\sigma(G)$ is used in the proof of Theorem 2; here we only sketch how to show a slightly weaker result $\sigma(H_3) \geq 11$, discussed below the statement of Theorem 2. We first note that semivalid representations are defined as certain linear subspaces of $\mathbb{R}^V$ and we will introduce a parameter $\eta(G)$ which is the maximal dimension of a semivalid representation. We will also show $\mu(G) \leq \eta(G) \leq \sigma(G)$, where $\mu(G) \leq \eta(G)$ follows easily from the known results on $\mu$ whereas showing the inequality $\eta(G) \leq \sigma(G)$ is the core of the proof of Theorem 1.

Now let us consider a matrix $M_3$ which is a suitable shift of the adjacency matrix of $H_3$. This matrix satisfies $\text{corank}(M_3) = \dim \ker(M_3) = 12$ and $\ker M_3$ is ‘almost’ a semivalid representation of $G$. Namely, by a trick that we learnt from Pendavingh [22] we can find a codimension 1 subspace $L$ of
ker(M) which is a semivalid representation. This shows \( \eta(H_3) \geq 11 \) and the key inequality \( \sigma(G) \geq \eta(G) \) gives the required bound \( \sigma(H_3) \geq 11 \).

The proof of Theorem 3 follows the same high-level strategy as the proof of Theorem 2, except we do not work there with a semivalid representation, but rather with a so-called valid representation, which is a concept used to define the parameter \( \lambda \) (see Subsection 2.2). We use a simple general position argument to show that if \( G \) has a low maximum degree, then a large subspace of \( \ker(M_q) \) has to be a valid representation of \( G \) where \( M_q \) is, in analogy to the previous case, a suitable shift of the adjacency matrix of \( H_q \).

**Organization.** In Section 2 we overview (or introduce) various representations of graphs and establish some of their properties. Then we prove Theorem 1 in Section 3 and Theorems 2 and 3 in Section 4.

### 2. Representations of graphs

#### 2.1. The Colin de Verdière graph parameter

If not stated otherwise, we work with a graph \( G = (V, E) \). We use the usual graph-theoretic notation \( N(v) \) for all vertices adjacent to \( v \in V \) and \( N(S) \) for all vertices in \( V \setminus S \) adjacent to a vertex in \( S \subseteq V \). Moreover, we use \( G[S] \) to denote the subgraph of \( G \) induced by \( S \). For a set \( S \subseteq V \) we denote by \( x_S \) the restriction of the vector \( x \) to the subset \( S \), that is, \( x_S := (x_v)_{v \in S} \).

Let \( M(G) \) be the set of symmetric matrices \( M \in \mathbb{R}^{V \times V} \) satisfying

(i) \( M \) has exactly one negative eigenvalue, of multiplicity one,

(ii) for any \( u \neq v \in V \), \( uv \in E \) implies \( M_{uv} < 0 \) and \( uv \notin E \) implies \( M_{u,v} = 0 \).

Matrices satisfying (ii) are sometimes called discrete Schrödinger operators in the literature.

Note that there is no condition on the diagonal entries of \( M \). Despite this, a part of the Perron–Frobenius theory is still applicable to \( M \in M(G) \), assuming that \( G \) is connected. (In the disconnected case the same reasoning can be applied component-wise.) This is because the matrix \( -M + cI_V \), where \( I_V \) denotes the identity matrix of size \( V \times V \), has nonnegative entries for large enough \( c > 0 \). Since this transformation preserves all eigenspaces, the Perron–Frobenius theorem implies that the smallest eigenvalue of \( M \) has multiplicity one and the corresponding eigenvector is strictly positive (or strictly negative). For instance, as \( M \) has an orthogonal eigenbasis, this implies that for every nonzero vector \( x \in \ker(M) \) both \( \text{supp}_+(x) \) and \( \text{supp}_-(x) \) must be nonempty.
A matrix $M \in \mathcal{M}(G)$ satisfies the so-called Strong Arnold hypothesis (SAH), if 

$$MX = 0 \implies X = 0$$

for every symmetric $X \in \mathbb{R}^{V \times V}$ such that $X_{u,v} = 0$ whenever $u = v$ or $uv \in E$. The Colin de Verdière graph parameter $\mu(G)$ is defined as the maximum of corank($M$) over matrices $M \in \mathcal{M}(G)$ satisfying SAH.

In the following lemma we collect some easy (but important) properties of matrices in $\mathcal{M}(G)$ in the following lemma. The proofs can be found, for instance, in a survey by van der Holst, Lovász, and Schrijver [11, Sec. 2.5].

**Lemma 4.** Let $G = (V,E)$ be a connected graph and $M \in \mathcal{M}(G)$. If $x \in \ker(M)$ is nonzero, then

1. $N(\text{supp}(x)) = N(\text{supp}_-(x)) \cap N(\text{supp}_+(x))$,
2. if $G[\text{supp}_+(x)]$ is disconnected, then there is no edge between $\text{supp}_+(x)$ and $\text{supp}_-(x)$, and moreover, for every connected component $C$ of $G[\text{supp}(x)]$ we have $N(C) = N(\text{supp}(x))$,
3. if $\text{supp}(x)$ is inclusion-minimal among nonzero vectors in $\ker(M)$, then both $G[\text{supp}_+(x)]$ and $G[\text{supp}_-(x)]$ are nonempty and connected.

### 2.2. Semivalid representations of graphs

Motivated by the parameter $\mu$, van der Holst, Laurent, and Schrijver [9] introduced the invariant $\lambda(G)$ defined as follows. We say that a linear subspace $X \subseteq \mathbb{R}^V$ is a valid representation of the graph $G$, if for every nonzero $x \in X$ the graph $G[\text{supp}_+(x)]$ is nonempty and connected. Then $\lambda(G)$ is defined as the maximum of dim$(X)$ over all valid representations $X$ of $G$.

Among other properties, van der Holst, Laurent, and Schrijver [9] proved that $\lambda$ is minor monotone and characterized the classes of graphs with $\lambda(G) \leq 1, 2, 3$. From this characterization it follows that the parameters $\lambda$ and $\mu$ differ already for those small values. In general, $\lambda$ can be both greater or smaller than $\mu$ (see [9, 22]).

Somewhat analogously to the notion of a valid representation, we introduce the following definition:

---

5 A global convention of [11, Sec. 2.5] is that the matrices $M$ considered there satisfy SAH. However, SAH is not used in the proof of the properties asserted in Lemma 4.
6 This part is originally due to van der Holst [8].
Definition 5 (Semivalid representation). Given a connected graph $G = (V,E)$ we call a linear subspace $L \subseteq \mathbb{R}^V$ a semivalid representation\(^7\) of $G$ if, for every nonzero $x \in L$,

(i) both $\text{supp}_+(x)$ and $\text{supp}_-(x)$ are nonempty,
(ii) the graph $G[\text{supp}_+(x)]$ is either connected, or $G[\text{supp}_+(x)]$ has two connected components and $G[\text{supp}_-(x)]$ is connected,
(iii) if $x$ has inclusion-minimal support in $L$, then both $G[\text{supp}_+(x)]$ and $G[\text{supp}_-(x)]$ are nonempty and connected,
(iv) if $G[\text{supp}_+(x)]$ is disconnected, then there is no edge between $\text{supp}_+(x)$ and $\text{supp}_-(x)$, and moreover, for every connected component $C$ of $G[\text{supp}(x)]$ we have $N(C) = N(\text{supp}(x))$.

We will use semivalid representations of $G$ as a substitute for $\ker(M)$ in case we want to work with $M$ not necessarily satisfying SAH; this will be especially useful for the proof of Theorem 2. This is enabled by the following lemma taken from Pendavingh [22], which together with Lemma 4 implies that the kernel of $M \in \mathcal{M}(G)$ satisfying SAH defines a semivalid representation of $G$:

Lemma 6 ([22, Lem. 3]). Let $G$ be a connected graph and $M \in \mathcal{M}(G)$. Let $x \in \ker(M)$ and set

$$D := \{ y \in \ker(M) : \text{supp}(y) \subseteq \text{supp}(x) \} \,.$$

If $G[\text{supp}(x)]$ is disconnected, it has exactly $\dim(D) + 1$ connected components. If, in addition, $M$ satisfies SAH, then $\dim(D) \leq 2$.

Similarly to the graph parameter $\lambda$ introduced by van der Holst, Laurent, and Schrijver [9], we define an analogous parameter $\eta(G)$:

Definition 7. Let $G$ be a graph. If $G$ is connected, we define

$$\eta(G) := \max \{ \dim(L) : L \text{ is a semivalid representation of } G \} \,.$$

For convenience, we also extend the definition to disconnected graphs $G$. If $G$ has at least one edge, then we define

$$\eta(G) := \max_C \eta(G[C]) \,,$$

where the maximum is taken over connected components $C$ of $G$. If $G$ is disconnected and does not have any edge, then we set $\eta(G) := 1$.

---

\(^7\) In the first version of the present work [14], we were using a notion of an extended representation with a very similar definition: it had the same properties as in the current definition, but in addition it was assumed to lie in $\ker(M)$ of some $M \in \mathcal{M}(G)$. We found this extra assumption somewhat unpleasant, thus we spent an extra effort on removing it from this key definition. But this does not mean that the proofs of the main results are more complicated—only a few details are slightly different.
Lemmas 4 and 6 show that $\mu(G) \leq \eta(G)$ for every connected graph $G$. The definition of $\eta$ for disconnected graphs is chosen in a way that agrees precisely with the behavior of $\mu$: In [11, Thm. 2.5] it is shown that $\mu(G)$ is equal to the maximum of $\mu$ over the connected components of $G$ if $G$ has at least one edge. Moreover, it is easy to see that $\mu$ of the empty graph on $n \geq 2$ vertices is 1 (or see, e.g., [11, Sec. 1.2]).

Comparing the definitions of valid and semivalid representations, it is clear that every valid representation is also a semivalid representation. Since for disconnected graphs $\lambda$ exhibits exactly the same type of behavior as $\mu$ and $\eta$ with respect to the connected components, which is easy to see directly from the definition of $\lambda$, we get that $\eta(G)$ is always an upper bound on $\lambda(G)$ for any graph $G$. Summarizing, we get the following:

**Observation 8.** For every graph $G$ it holds that $\max \{\mu(G), \lambda(G)\} \leq \eta(G).$

### 2.3. Topological preliminaries

**Polyhedra.** A set $\tau' \subset \mathbb{R}^k$ is a closed (convex) polyhedron if it is an intersection of finitely many closed half-spaces. A closed face of a polyhedron $\tau'$ is a subset $\eta' \subseteq \tau'$ such that there exists a hyperplane $h$ satisfying that $\eta' = h \cap \tau'$ and $\tau'$ belongs to one of the closed half-spaces determined by $h$.

A relatively open polyhedron is the relative interior $\tau$ of a closed polyhedron $\tau'$ (the relative interior is taken with respect to the affine hull of $\tau'$).

**Important convention.** In the sequel, when we say polyhedron, we mean relatively open polyhedron. This is nonstandard, but it will be very convenient for our considerations. We apply this convention also to polytopes; that is, a polytope is a bounded relatively open polyhedron. Given a polyhedron $\tau$, by $\overline{\tau}$ we denote the closure of $\tau$, that is, the corresponding closed polyhedron. We also say that a (relatively open) polyhedron $\eta$ is a face of $\tau$ if $\overline{\eta}$ is a closed face of $\overline{\tau}$.

**Polyhedral complexes.** A polyhedral complex is a collection $\mathcal{C}$ of polyhedra satisfying:

1. If $\tau \in \mathcal{C}$ and $\eta$ is a face of $\tau$, then $\eta \in \mathcal{C}$.
2. If $\theta, \tau \in \mathcal{C}$, then $\overline{\theta \cap \tau}$ is a closed face of $\overline{\theta}$ as well as a closed face of $\overline{\tau}$.

The body of a polyhedral complex $\mathcal{C}$ is defined as $|\mathcal{C}| := \bigcup \mathcal{C}$. Due to our convention that we consider relatively open polyhedra, $|\mathcal{C}|$ is a disjoint union of polyhedra contained in $\mathcal{C}$. A polyhedral complex $\mathcal{D}$ is a subdivision
of a polyhedral complex $C$ if $|C| = |D|$ and for every $\eta \in D$, there is $\tau$ in $C$ containing $\eta$.

Given a polyhedron $\tau$, by $\partial \tau$ we denote the boundary of $\tau$. With a slight abuse of notation, depending on the context, this may be understood both as a polyhedral complex formed by the proper faces of $\tau$ as well as the topological boundary of $\tau$, that is, the body of the former one.

The $k$-skeleton of a polyhedral complex $C$ is the subcomplex $C^{(k)}$ consisting of all faces of $C$ of dimension at most $k$.

In our considerations, we will need two special classes of polyhedra: fans and simplicial complexes.

**Fans.** A cone is a polyhedron $\alpha \subseteq \mathbb{R}^k$ such that $rx \in \alpha$ whenever $x \in \alpha$ and $r \in (0, \infty)$. A polyhedral complex $F$ is a fan if each polyhedron in $F$ is a cone, and moreover, if $F$ contains a nonempty polyhedron, then $F$ contains the origin as a polyhedron. A fan is complete if $|F| = \mathbb{R}^k$.

Now, let $P \subseteq \mathbb{R}^k$ be a polytope such that the origin is in the interior of $P$. Then $P$ defines a complete fan $F(P)$ formed by the cones over the proper faces of $P$ (plus the empty set). Again, we consider the faces of $P$ relatively open. With a slight abuse of terminology, we say that $P$ subdivides a fan $F'$ if $F(P)$ subdivides $F'$; see Figure 1.

**Simplicial complexes and their barycentric subdivisions.** A polyhedral complex is a simplicial complex if each polyhedron in the complex is a simplex. (Consistently with our convention above, by a simplex we mean a relatively open simplex.) Now let $K$ be a simplicial complex. For every nonempty simplex $\tau \in K$ let $b_\tau$ be the barycenter of $\tau$. For two faces $\eta$ and $\tau$ of $K$, let $\eta \prec \tau$ denote that $\eta$ is a proper face of $\tau$. The barycentric subdivision of $K$, denoted $\text{sd}K$, is a simplicial complex obtained so that for every chain $\Gamma = \theta_1 \prec \theta_2 \prec \cdots \prec \theta_m$ of nonempty faces of $K$ we add a simplex, denoted $\Delta(\Gamma)$, with vertices $b_{\theta_1}, \ldots, b_{\theta_m}$ into $\text{sd}K$. It is well known that $\text{sd}K$ subdivides $K$. In particular, $\Delta(\Gamma) \subset \theta_m$.

**Observation 9.** Let $K$ be a simplicial complex and $\Delta$ be a simplex of the barycentric subdivision $\text{sd}K$. Let $\Delta_1$ and $\Delta_2$ be two faces of $\Delta$ and $\eta_1 \supseteq \Delta_1$ and $\eta_2 \supseteq \Delta_2$ be two faces of $K$. Then either $\eta_1$ is a face of $\eta_2$ or $\eta_2$ is a face of $\eta_1$.

**Proof.** The face $\Delta$ corresponds to a chain $\Gamma = \theta_1 \prec \cdots \prec \theta_m$ of faces of $K$. Then $\Delta_1$ corresponds to a subchain $\Gamma_1$ of $\Gamma$ with maximal face $\theta_i$ (for some $i$). Then $\theta_i$ is the (unique) face of $K$ containing $\Delta_1 = \Delta(\Gamma_1)$. Therefore $\eta_1 = \theta_i$. Similarly, $\eta_2 = \theta_j$ for some $j$, from which the conclusion follows. □
Before we state the next lemma, we introduce two more well-known notions. Let $K$ be a simplicial complex and $|L|$ be the body of some subcomplex $L$ of $K$. We define the simplicial neighborhood of $|L|$ in $K$ as

$$\mathcal{N}(|L|, K) := \{ \eta \in K : \eta \subset \tau \text{ for some } \tau \text{ with } \tau \cap |L| \neq \emptyset \}.$$ 

If $L$ consists of a single vertex $a$, then the simplicial neighborhood is known as (closed) star of $a$ in $K$, denoted by $\text{st}(a; K)$.

**Lemma 10.** Let $K$ be a simplicial complex and let $L_1, L_2$ be two subcomplexes of $K$ with $|L_1| \cap |L_2| = \emptyset$. Let $a$ be a vertex of the second barycentric subdivision $\text{sd}^2 K$. Then the closed star $\text{st}(a; \text{sd}^2 K)$ cannot intersect both $|L_1|$ and $|L_2|$.

**Proof.** The closed star $\text{st}(a; \text{sd}^2 K)$ intersects $|L_i|$ only if $a$ belongs to $\mathcal{N}(|L_i|, \text{sd}^2 K) = \mathcal{N}(|\text{sd}^2 L_i|, \text{sd}^2 K)$. The lemma follows from the fact that $\mathcal{N}(|L_1|, \text{sd}^2 K)$ and $\mathcal{N}(|L_2|, \text{sd}^2 K)$ are disjoint. (This is a simple exercise on properties of simplicial/derived/regular neighborhoods using the tools from [26]. An explicit reference for this claim we are aware of is Corollary 4.5 in [32]—embedding in a manifold assumed in [32] plays no role in the proof.)
Stellar subdivisions of polytopes. Let \( P \subseteq \mathbb{R}^k \) be a polytope such that the origin belongs to the interior of \( P \) and let \( F \) be a face of \( P \). Let \( a \) be a point beyond all facets (i.e. maximal faces) \( F' \) of \( P \) such that \( F \subseteq \overline{F'} \) (that is, \( a \) and the origin are on different sides of the hyperplane defining \( F' \)) whereas \( a \) is beneath all other facets (\( a \) and the origin are on the same side of the defining hyperplane). Then the polytope \( P' \) obtained as the convex hull of the set of vertices of \( P \) and \( a \) is called a geometric stellar subdivision of \( P \) [5]. For any \( F \), we can pick \( a \) as above lying inside the cone of \( \mathcal{F}(P) \) containing \( F \). Let \( p: \partial P' \to \partial P \) be the projection towards the origin. Then the complex \( p(\partial P') := \{ p(F'): F' \text{ is a proper face of } P' \} \) is a subdivision of the boundary of \( P \).\(^8\) Consequently, \( \mathcal{F}(P') \) subdivides \( \mathcal{F}(P) \).

If we perform stellar subdivisions gradually on all proper faces of a polytope \( P \) ordered by nonincreasing dimension, we obtain a simplicial polytope. In fact, we get a polytope isomorphic to a barycentric subdivision of \( P \); however, we will use this stronger conclusion only when \( P \) is already simplicial. That is, in this case we obtain a polytope \( P' \) such that the projection \( p: \partial P' \to \partial P \) is a simplicial isomorphism between \( \partial P' \) and \( \text{sd} \partial P \) provided in each step, when performing individual stellar subdivisions over face \( F \), the newly added point \( a \) is on the ray from the origin containing the barycenter of \( F \). For more details on stellar and barycentric subdivisions of polytopes, we refer to [5].

2.4. Fan of a semivalid representation

In line with [18], given a semivalid representation \( L \) of \( G \) we now aim to build a fan \( \mathcal{P} = \mathcal{P}(L) \) (complete in \( L \)) formed by convex polyhedral cones in a way that corresponds to splitting \( L \) by hyperplanes passing through the origin and perpendicular to the standard basis vectors of \( \mathbb{R}^V \).

Definition 11 (Fan \( \mathcal{P}(L) \)). Let \( L \) be a semivalid representation of \( G \) and let us define an equivalence relation \( \sim \) on \( \mathbb{R}^V \) by

\[
  x \sim y \iff \text{supp}_+(x) = \text{supp}_+(y) \text{ and } \text{supp}_-(x) = \text{supp}_-(y).
\]

Each equivalence class \([x]_{\sim}\) is a convex cone in \( \mathbb{R}^V \) (relatively open), and we define \( \mathcal{E} \) to be the fan formed by these cones.

\(^8\) Considering \( \partial P \) as a polytopal complex, \( p(\partial P') \) is exactly the stellar subdivision of \( \partial P \) as defined in [5] on the level of polytopal complexes; see also Exercise 3.0 in [34]. However, we do not need the exact formula explicitly. It is sufficient for us that \( p(\partial P') \) is a subdivision.
Then we define \( P = \mathcal{P}(L) \) as the fan obtained by intersecting \( \mathcal{E} \) with \( L \). In other words, the cones of \( P \) are the equivalence classes of \( \sim \) restricted to \( L \).

If the semivalid representation \( L \) is irrelevant or understood from the context, we omit it from the notation and simply write \( P \). We refer to a \( k \)-dimensional cone as a \( k \)-cone.

We extend the notation of support to cones in \( P \), i.e., if \( \alpha \in P \), then \( \text{supp}_\pm(\alpha) := \text{supp}_\pm(x) \) for some \( x \in \alpha \). Also, if \( A \subseteq \alpha \) for some \( \alpha \) in \( P \), then \( \text{supp}_\pm(A) := \text{supp}_\pm(\alpha) \).

In the remainder of this subsection, we always assume that \( L \subseteq \mathbb{R}^V \) is a semivalid representation of a given graph \( G = (V,E) \) and \( P := \mathcal{P}(L) \) is the fan corresponding to \( L \). We continue with several observations on properties of \( P \).

**Observation 12.** Let \( \alpha, \beta \) be two cones of \( P \). Then

(i) \( \alpha \subseteq \partial \beta \) if and only if \( \text{supp}_+(\alpha) \subseteq \text{supp}_+(\beta) \) and \( \text{supp}_-(\alpha) \subseteq \text{supp}_-(\beta) \) and at least one of the inclusions is strict,

(ii) \( \alpha \) is a 1-cone if and only if the vectors of \( \alpha \) have inclusion-minimal support among nonzero vectors in \( L \).

**Proof.** The equivalence in (i) follows immediately from the facts that \( \partial \beta \subseteq \overline{\beta} \) and \( \overline{\beta} \) contains all \( y \in L \) with \( \text{supp}_+(y) \subseteq \text{supp}_+(\beta) \) and \( \text{supp}_-(y) \subseteq \text{supp}_-(\beta) \). At least one of the inclusions is strict if and only if \( \alpha \neq \beta \).

To prove (ii), first assume that \( \alpha \) is a 1-cone. Then every vector in \( \alpha \) has to have inclusion-minimal support among nonzero vectors in \( L \) by (i). On the other hand, if \( \alpha \) contains vectors \( x \) with inclusion-minimal supports, then \( \dim(\alpha) = 1 \), otherwise there were two linearly independent vectors \( x, y \in \alpha \) and \( x - \varepsilon y \in L \) would have strictly smaller support than \( x, y \) for an appropriate choice of \( \varepsilon > 0 \). 

**Definition 13 ([18]).** If \( G[\text{supp}_+(x)] \) is disconnected for a nonzero \( x \in \mathbb{R}^V \), we call \( x \) a broken vector. The cones of \( P \) consisting of broken vectors are called broken cones.

For the rest of the subsection we additionally assume that \( G \) is connected.

**Lemma 14.** Let \( \beta \) be a broken cone of \( P \) and \( \alpha \) be a cone of \( P \) with \( \alpha \subseteq \partial \beta \). Then

(i) \( \text{supp}_-(\alpha) = \text{supp}_-(\beta) \),

(ii) \( G[\text{supp}_+(\alpha)] \) is equal to a single connected component of \( G[\text{supp}_+(\beta)] \), and
(iii) \( \alpha \) is a 1-cone.

**Proof.** For \( W \subseteq V = V(G) \) by a (connected) component of \( W \) we always mean the vertex set of a connected component of \( G[W] \). Observation 12(i) says that \( \text{supp}_+(\alpha) \subseteq \text{supp}_+(\beta) \), \( \text{supp}_-(\alpha) \subseteq \text{supp}_-(\beta) \), and at least one of the inclusions is strict. This implies that \( G[\text{supp}(\alpha)] \) is a strict subgraph of \( G[\text{supp}(\beta)] \). Throughout the proof, \( a \) is a vector in \( \alpha \) and \( b \) in \( \beta \).

Assume \( C \) is a component of \( \text{supp}(b) \) which has nonempty intersection with both \( \text{supp}(a) \) and \( V \setminus \text{supp}(a) \). Fix \( K > \) large enough. If \( C \) is a part of \( \text{supp}_+(b) \), then \( b - Ka \) is also broken by Definition 5(i) applied to \( a \) and Definition 5(i) applied to \( b \). But the connectedness of \( C \) then contradicts Definition 5(iv) for \( b - Ka \). If \( C = \text{supp}_-(\beta) \), then exactly the same argument applies to \( b - Ka \) or to \( Ka - b \). In any case, we get that no such \( C \) exists. This proves (i) and (ii), as we know from Definition 5 that \( \text{supp}(b) \) has exactly three components; one of them coincides with \( \text{supp}_-(b) \) and two of them form \( \text{supp}_+(b) \).

Finally, all this implies that \( \alpha \) is a 1-cone via Observation 12(ii) as (i) and (ii) (applied to \( \alpha' \)) show that there is no \( \alpha' \) with its support strictly included in \( \alpha \).

The preceding lemma has no analog in [18], because no information on the boundary of broken cones is needed there. But it easily implies the following generalization of part (8) from the proof of [18, Thm. 3].

**Corollary 15 (generalized [18, Claim (8)]).** Let \( \beta \) be a broken cone of \( \mathcal{P}(L) \). Then

(i) \( \dim(\beta) = 2 \) and

(ii) \( \partial \beta \) consists of two 1-cones, which correspond to vectors \( x \in L \) for which \( \text{supp}_-(x) = \text{supp}_-(\beta) \) and \( \text{supp}_+(x) \) is identical with one of the connected components induced by \( \text{supp}_+(\beta) \).

**Proof.** Because \( \mathcal{P}(L) \) is obtained as the intersection of \( \mathcal{E} \) with \( L \) in Definition 11, we get that every \( k \)-cone in \( \mathcal{P}(L) \) has nonempty boundary, which must then be \((k-1)\)-dimensional. Lemma 14(iii) implies that \( \partial \beta \) consists only of 1-cones, which implies (i). Lemma 14(ii) together with the fact that every 2-cone has exactly two 1-cones on its boundary implies (ii).

**Notation.** For \( x \in L \) we write \( S(x) := N(\text{supp}(x)) \) and \( R(x) := V \setminus (\text{supp}(x) \cup S(x)) \). Let \( \beta \in \mathcal{P} \). We write \( S(\beta) := S(x) \) and \( R(\beta) := R(x) \) for any \( x \in \beta \). The notation is motivated by the fact that \( S(x) \) is a ‘separator’
if $x$ is a broken vector and $R(x)$ is the set of vertices of $G$ ‘remote’ from $\text{supp}(x)$; see Figure 2.

All broken vectors $x$ in the proof of [18, Thm. 3] automatically satisfy $R(x) = \emptyset$. Since we work in a more general setting, we cannot assert that; however, the observation below suggests that this will not form a serious obstacle.

**Observation 16.** Let $\beta \in \mathcal{P}$ be broken. Then for every $x \in L$ such that $x_{S(\beta)} = 0$ we have $x_{R(\beta)} = 0$.

**Proof.** Let $y \in \beta$ and assume, for contradiction, that there is $x \in L$ such that $x_{S(\beta)} = 0$ and $x_{R(\beta)} \neq 0$. Since there is no edge between $R(\beta)$ and $\text{supp}(y)$ in $G$, the set $\text{supp}_+(y + \varepsilon x)$ is still disconnected and $\text{supp}(y + \varepsilon x)$ induces at least four connected components, for all $\varepsilon > 0$ small enough. This is incompatible with Definition 5(ii).

The following observation, which is crucial for our subsequent considerations, is new and does not have any analog in [18].

**Observation 17.** Let $\alpha$ be a 1-cone of $\mathcal{P}$. Then there is at most one broken cone $\beta \in \mathcal{P}$ such that $\alpha \subseteq \overline{\beta}$.

**Proof.** Let $\beta, \gamma$ be two broken cones such that $\alpha \subseteq \overline{\beta \cap \gamma}$. By Corollary 15(ii) we get that $\text{supp}_-(\beta) = \text{supp}_-(\alpha) = \text{supp}_-(\gamma)$. Definition 5(iv) then implies that $S(\gamma) = S(\beta)$. Applying Observation 16 finishes the argument.
2.5. Polytopal representation

In analogy with the approach of Lovász and Schrijver [18], we utilize semi-valid representations $L$ of a given connected graph $G$ to build convex polytopes of dimension $\dim(L)$. By a $k$-face (or a $k$-cell) we mean a face (or a cell) of dimension $k$. We refer to a $d$-dimensional polytope as a $d$-polytope.

**Definition 18 (Polytopal representation).** Let $L$ be a semivalid representation of $G$, and $\mathcal{P} = \mathcal{P}(L)$ be the complete fan corresponding to $L$. We say that a polytope $P \subset L$ containing the origin in its interior (relative in $L$) is *polytopal representation* of $G$ if it satisfies the following conditions.

(i) The vertex set of $P$ is centrally symmetric.

(ii) $P$ subdivides $\mathcal{P}$. This in particular means, that for every face $F$ of $P$, there is a unique cone of $\mathcal{P}$ which contains $F$. We denote this cone by $\gamma(F)$.

(iii) $P$ is simplicial, that is, all faces of $P$ are simplices.

(iv) Let $E, F$ be faces of $\partial P$ which are faces of a common face of $\partial \mathcal{P}$. Then either $\gamma(E)$ is a face of $\gamma(F)$ or $\gamma(F)$ is a face of $\gamma(E)$. (This includes the option $\gamma(E) = \gamma(F)$.)

(v) Let us define a *broken edge* as an edge of $P$ lying in a broken cone of $\mathcal{P}$. Then we require: For every $a \in P^{(0)}$ all broken edges of $P$ in $\text{st}(a; P)$ belong to the same broken cone.

We, of course, need to know that a polytopal representation exists. Lovász and Schrijver [18] build a polytope $P$ satisfying (i)–(iii) and a weaker version of (iv) as a convex hull of a sufficiently dense set of unit vectors taken from every cone, without going into details about how to choose this set. As we add extra properties, we want to be more careful and check that all of them can be satisfied.

**Proposition 19.** Given a semivalid representation $L$, a corresponding polytopal representation $P$ always exists.

**Proof.** We start with considering the crosspolytope $C \subseteq \mathbb{R}^V$ whose vertices are the standard basis vectors $e_v \in \mathbb{R}^V$ and their negatives $-e_v$ for $v \in V(G)$. Then the fan of the crosspolytope $\mathcal{F}(C)$ is exactly the fan $\mathcal{E}$ defined in Definition 11. Next we consider the auxiliary polytope $Q := C \cap L$ and we get $\mathcal{P} = \mathcal{F}(Q)$. In particular, $Q$ subdivides $\mathcal{P}$.

Subsequently, we apply a series of geometric stellar subdivisions on $Q$ as described in Subsection 2.3. First we get a simplicial polytope $Q'$ which subdivides $\mathcal{P}$. Then we take $P$ as the second barycentric subdivision of $Q'$,
Figure 3. A picture illustrating property (iv). In this picture, $L$ is 3-dimensional. Left: The faces $E$ and $F$ are a vertex and an edge in a common (small) triangle of $\partial P$. The larger (black) subdivided triangle containing both $E$ and $F$ is a result of applying a barycentric subdivision to (some triangle of) $Q''$. The green outer 'almost' triangle depicts the intersection of $\partial P$ and the cone $\gamma(F)$. Right: The picture shows $E''$ and $F''$ obtained as faces of $Q''$ containing $p''(E)$ and $p''(F)$. In this specific case $p''(E)$ coincides with $E$ and $E''$, thus only $p''(F)$ is depicted.

again by a series of stellar subdivisions. We perform all stellar subdivisions in a centrally symmetric fashion so that we obtain centrally symmetric $P$.

It remains to verify the properties from Definition 18. The properties (i), (ii), and (iii) follow immediately from the construction.

We will show that (iv) follows from Observation 9. Let $Q''$ be the polytope obtained from $Q'$ after the first barycentric subdivision and let $p'': \partial P \rightarrow \partial Q''$ be the projection towards the origin, as in Subsection 2.3. Then $p''(\partial P)$ is a barycentric subdivision of $\partial Q''$. Now, let $E''$ be the face of $Q''$ containing $p''(E)$ and let $F''$ be the face of $Q''$ containing $p''(F)$; see Figure 3. Note that $E'' \subseteq \gamma(E)$. Indeed, $Q''$ subdivides $P$, therefore $E''$ is contained in some cone of $\mathcal{P}$, and $\gamma(E)$ is the only option. Similarly, $F'' \subseteq \gamma(F)$. By Observation 9, $E''$ is a face of $F''$ or vice versa (the observation is applied with $\eta_1 = E''$, $\eta_2 = F''$, $\Delta_1 = E$, and $\Delta_2 = F$). Therefore $\gamma(E)$ is a face of $\gamma(F)$ or vice versa.

Finally, we derive (v) from Lemma 10. This time, we consider the projection $p': P \rightarrow Q'$. Then $p'(\partial P)$ is the second barycentric subdivision of $\partial Q'$. For contradiction, assume that the edges of $\text{st}(a;P)$ belong to two broken cones $\beta_1$ and $\beta_2$. Equivalently, the edges of $\text{st}(p'(a);p'(\partial P)) = \text{st}(p'(a);\text{sd}^2(\partial Q'))$ belong to $\beta_1$ and $\beta_2$. Let $\mathcal{L}_1$ and $\mathcal{L}_2$ be subcomplexes of $\partial Q'$ triangulating $\beta_1$ and $\beta_2$, respectively. Observation 17 implies that $|\mathcal{L}_1| \cap |\mathcal{L}_2| = \emptyset$. Then, by Lemma 10, $\text{st}(p'(a);\text{sd}^2(\partial Q'))$ cannot intersect both $|\mathcal{L}_1|$ and $|\mathcal{L}_2|$, a contradiction.
3. On the relation $\mu(G) \leq \sigma(G)$

The aim of this section is to prove Theorem 1. In fact, we prove that $\eta(G) \leq \sigma(G)$ for every graph $G$. This immediately implies Theorem 1 thanks to Observation 8.

To make our exposition easier to follow, in the present section we refer to vertices and edges of a graph as nodes and arcs, respectively, and reserve the terms vertices and edges for the 0- and 1-faces of polytopes.

Proposition 20. Let $G$ be a connected graph and $L$ be a semivalid representation of $G$. Then $\dim(L) \leq \sigma(G)$.

The key step for the proof of Proposition 20 is to deduce Proposition 21 below. Given a polytope $Q$, two faces $F$ and $F'$ are antipodal if there exist two distinct parallel hyperplanes (relatively in the affine hull of $Q$) $h$ and $h'$ such that $F \subset h$, $F' \subset h'$ and $Q$ is ‘between’ $h$ and $h'$, that is, it belongs to one of the closed halfspaces bounded by $h$ as well as one of the closed halfspaces bounded by $h'$. If $Q$ is centrally symmetric, then $F$ and $F'$ are antipodal if and only if $F$ and $-F'$ belong to the closure of some proper face of $Q$.

Given two polyhedral complexes $C$ and $D$, a map $f : |C| \to |D|$ is cellular if $f(C^{(k)}) \subseteq D^{(k)}$ for every $k \geq 0$. If $C$ and $D$ are graphs, which is the only case we are interested in, then this condition means that every vertex of $C$ is mapped to a vertex of $D$.

Proposition 21. Let $G$ be a connected graph and $P$ a polytopal representation of $G$ (arising from the fan $P = P(L)$, where $L$ is a semivalid representation of $G$). Then, there is a cellular map $f : P^{(1)} \to G$ such that for every pair of antipodal faces $F$ and $F'$, the smallest subgraphs of $G$ containing $f(F^{(1)})$ and $f(F'^{(1)})$, respectively, have no common nodes.

Using the tools of van der Holst and Pendavingh [12], Proposition 21 implies Proposition 20 in a fairly straightforward manner. As this proof is short, we present it before the proof of Proposition 21. Here, we essentially only repeat the proof of [12, Thm. 40].

Proof of Proposition 20. The main tool for this proof is Lemma 37 from [12]. This lemma says that, under the additional assumption that $P$ does not contain parallel faces (that is, faces with disjoint affine hulls such that $F - F$ and $F' - F'$ contain a common nonzero vector), the existence of $f$ from Proposition 21 implies $\sigma(G) \geq \dim P$. (Note that $\dim P = \dim L$.) Our $P$ contains parallel faces. However, as van der Holst and Pendavingh point out,
\( \mathbf{P} \) can be perturbed by a projective transformation to a polytope without antipodal parallel faces preserving the combinatorial structure of the polytope. Similarly as van der Holst and Pendavingh do, we refer to the proof of [18, Thm. 1] for details.

**Notation.** Given \( G, L, \mathcal{P} \) and \( \mathbf{P} \) as in the statement of Proposition 21, we extend the notation \( R(\gamma) \) and \( S(\gamma) \) from cones to faces of \( \mathbf{P} \). Let \( \mathbf{F} \) be a face of \( \mathbf{P} \), which lies in a unique cone \( \gamma(\mathbf{F}) \in \mathcal{P} \) by Definition 18. We define \( S(\mathbf{F}) := S(\gamma(\mathbf{F})) \) and \( R(\mathbf{F}) := R(\gamma(\mathbf{F})) \). Note also that \( \text{supp}(\mathbf{F}) = \text{supp}(\gamma(\mathbf{F})) \) and \( \text{supp}_\pm(\mathbf{F}) = \text{supp}_\pm(\gamma(\mathbf{F})) \) according to our convention above Observation 12.

**Proof of Proposition 21.** During the construction, for each face \( \mathbf{F} \) of \( \mathbf{P} \) we will introduce a set \( W(\mathbf{F}) \), which will be a subset of nodes of \( G \) such that \( f(\mathbf{F}^{(1)}) \subseteq G[W(\mathbf{F})] \). The key property of the construction will be that \( W(\mathbf{F}) \) and \( W(\mathbf{F}') \) are disjoint if \( \mathbf{F} \) and \( \mathbf{F}' \) are antipodal faces of \( \mathbf{P} \). We first define \( f \) and \( W \) on the vertices of \( \mathbf{P} \) and then on the edges of \( \mathbf{P} \). Finally, we extend the definition of \( W \) to higher-dimensional faces and verify the required disjointness condition.

Throughout the proof, we repeatedly use the fact that every broken cone is 2-dimensional according to Corollary 15(i). In particular, faces of \( \mathbf{P} \) lying in a broken cone are either broken edges, or ‘inner’ vertices in a broken 2-cone.

Before we start the construction, for every broken cone \( \beta \) we fix a node \( v(\beta) \in S(\beta) \). We also use the notation \( v(b) := v(\beta) \), where \( b \) is an arbitrary broken edge lying in \( \beta \), that is, \( \gamma(b) = \beta \).

**Dimension 0.** Given \( u \in \mathbf{P}^{(0)} \), Definition 18(v) applied to \( a = -u \) implies that either there is no broken edge antipodal to \( u \), or there is a unique 2-cone \( \beta = \beta(u) \in \mathcal{P} \) such that all broken edges antipodal to \( u \) lie in \( \beta \). In the former case, we let \( f(u) \) be an arbitrary node of \( \text{supp}_+(u) \). In the latter case, we want to avoid \( R(\beta) \) and \( v(\beta) \); thus, we need to check that we can do so.

**Claim 21.1.** If \( \beta = \beta(u) \) exists, then there is a node in \( \text{supp}_+(u) \setminus R(\beta) \) different from \( v(\beta) \).

**Proof.** We distinguish two cases according to whether \( \gamma(-u) \subseteq \beta \) or not.

If \( \gamma(-u) \subseteq \beta \), we get

\[
\text{supp}_+(u) = \text{supp}_-(u) \subseteq \text{supp}_-(\beta)
\]
whereas \( v(\beta) \) does not belong to \( \text{supp}(\beta) \). Therefore the claim follows from the facts that \( \text{supp}_+(u) \) is nonempty by Definition 5(i) and \( R(\beta) \cap \text{supp}(\beta) = \emptyset \).

Now we assume that \( \gamma(-u) \not\subseteq \overline{\beta} \). Let \( b \) be an arbitrary broken edge antipodal to \( u \). We know that \( \beta = \gamma(b) \). We also know that there is a proper face \( F \) of \( P \) such that \( b \) and \(-u\) belong to \( F \). Definition 18(iv) implies that \( \beta \) is a face of \( \gamma(-u) \) or vice versa. Since \( \gamma(-u) \not\subseteq \overline{\beta} \), we obtain that \( \gamma(-u) \) is at least 3-dimensional cone satisfying \( \beta \subseteq \overline{\gamma(-u)} \).

Now we get \( \text{supp}_+(u) = \text{supp}_-(u) \supseteq \text{supp}_-(\beta) \). We also again use that \( v(\beta) \) does not belong to \( \text{supp}(\beta) \). Therefore, the claim follows from the fact that \( \text{supp}_-(\beta) \) is nonempty and \( R(\beta) \cap \text{supp}(\beta) = \emptyset \).

Therefore, if \( \beta = \beta(u) \) exists, by Claim 21.1, we may set \( f(u) \) to be an arbitrary node of \( \text{supp}_+(u) \setminus R(\beta) \) different from \( v(\beta) \).

We also set, somewhat trivially, \( W(u) := \{ f(u) \} \).

**Dimension 1.** Let \( e = uw \) be an edge of \( P \). We want to define \( f \) on \( e \) as well as \( W(e) \). We proceed so that for every edge \( e = uw \) of \( P \) we first suitably define \( W(e) \) in such a way that \( f(u) \) and \( f(w) \) are nodes in the same connected component of \( G[W(e)] \). Then we set \( f(e) \) to be an arbitrary path connecting \( f(u) \) and \( f(w) \) inside \( G[W(e)] \).

If \( e = b \) is a broken edge, then we set \( W(b) := \text{supp}_+(b) \cup \{ v(b) \} \). Then \( f(u) \) and \( f(w) \) are nodes in \( W(b) \) as \( \text{supp}_+(u), \text{supp}_+(w) \subseteq \text{supp}_+(b) \). Also, \( G[W(b)] \) is connected as \( v(b) \) is adjacent to every component of \( G[\text{supp}(b)] \) by Definition 5(iv).

Now, let us assume that \( e \) is not broken. For the connectedness of \( G[W(e)] \) it would suffice to set \( W(e) = \text{supp}_+(e) \). We know that \( G[\text{supp}_+(e)] \) is connected as \( e \) is not broken, and also, \( f(u) \) and \( f(w) \) are nodes of \( G[W(e)] \) by the same argument as above. However, in some cases we want \( W(e) \) to be smaller; namely, if there is a broken edge \( b \) antipodal to \( e \), we want to avoid \( v(b) \). Note that the cone \( \beta := \gamma(b) \) is independent of the choice of \( b \), if \( b \) exists, by Definition 18(v) applied to an arbitrary vertex of \(-e\) in place of \( a \). Then \( v(b) = v(\beta), R(b) = R(\beta) \) and \( S(b) = S(\beta) \) are independent of \( b \) as well. So, we set \( W(e) := \text{supp}_+(e) \) if there is no broken edge antipodal to \( e \), but we set \( W(e) := \text{supp}_+(e) \setminus \{ v(b) \} \) if there is a broken edge \( b \) antipodal to \( e \).

We want to check that \( f(u) \) and \( f(w) \) belong to the same connected component of \( G[W(e)] \). This we already did in the former case, thus it remains to consider the latter case, when \( b \) exists. We observe that since \( e \) is antipodal to \( b \), the vertices \( u \) and \( w \) are antipodal to \( b \) as well. Therefore,
both $f(u)$ and $f(w)$ are distinct from $v(b) = v(\beta)$. In other words, $f(u)$ and $f(w)$ indeed lie in $W(e)$. It remains to show that they belong to the same connected component of $G[W(e)]$.

**Claim 21.2.** Either $b = -e$, or $\gamma(e)$ is at least 3-dimensional, and $-\beta \subseteq \overline{\gamma(e)}$.

**Proof.** Assume that $b \neq -e$. Because $b$ and $e$ are antipodal, there is a face $D$ of $\partial P$ containing $-b$ and $e$. Therefore $\gamma(-b) = -\beta$ is a face of $\gamma(e)$ or vice versa according to Definition 18(iv). Since $-\beta$ is a 2-cone and $\gamma(e)$ is at least 2-dimensional, $-\beta$ must be a face of $\gamma(e)$. It remains to observe that $-\beta \neq \gamma(e)$. For contradiction assume $-\beta = \gamma(e)$. Consider the defining hyperplane for $D$; it contains $-b$ and $e$. Therefore it contains $-\beta$ because $-\beta$ is in the affine hull of $b \cup -e$ if $b \neq -e$ and $-\beta = \gamma(-b) = \gamma(e)$. Consequently, it contains the origin, which is a contradiction. □

We remark that if the former case $b = -e$ occurs, then $v(b) \notin supp_+(e)$ as $v(b) \notin supp(b) = supp(e)$; we already resolved this situation. Thus it remains to consider the case that $\gamma(e)$ is at least 3-dimensional and $-\beta \subseteq \overline{\gamma(e)}$. In addition, we can assume that $v(b) \in supp_+(e)$ (again, the opposite case was already resolved).

Now note that $f(u) \in supp_+(u) \setminus R(\beta)$ and $f(w) \in supp_+(w) \setminus R(\beta)$ due to the definition of $f(u)$ and $f(w)$. This gives $f(u), f(w) \in supp_+(e) \setminus R(\beta)$.

From $-\beta \subseteq \overline{\gamma(e)}$ we also get

$$\text{supp}_+(\beta) = \text{supp}_-(-\beta) \subseteq \text{supp}_-(\gamma(e)) = \text{supp}_-(e).$$

Therefore $f(u), f(w) \notin supp_+(\beta)$, because they belong to $supp_+(e)$. Altogether, both $f(u), f(w) \in supp_-(\beta) \cup S(\beta)$ as they also do not belong to $R(\beta)$. Moreover, each of $f(u)$ and $f(w)$ either belongs to $supp_-(\beta)$ or has a neighbor in $supp_-(\beta)$, since each vertex of $S(\beta)$ is connected to every component of $G[\text{supp}(\beta)]$. We also know that $G[\text{supp}_-(\beta)]$ is connected by Definition 5(ii) as $\beta$ is broken, that $\text{supp}_-(\beta) = \text{supp}_+(\beta) \subseteq \text{supp}_+(\gamma(e)) = \text{supp}_+(e)$, and that $v(\beta) \notin \text{supp}_-(\beta)$. Altogether, $f(u)$ and $f(w)$ can indeed be connected inside $G[\text{supp}_+(e) \setminus \{v(b)\}]$. (See Figure 4 as an example.)

**Higher dimensions.** It remains to define $W(F)$ for faces $F$ of $P$ of higher dimensions. We inductively set $W(F) := \bigcup_H W(H)$, where the union is over all proper subfaces $H$ of $F$. As the definition is given inductively, this is equivalent with setting $W(F)$ to $\bigcup_e W(e)$ where the union is over the edges $e$ in $F$. Then we easily get $f(F^{(1)}) \subseteq G[W(F)]$ for any face $F$ of $P$, as required.
It remains to prove that $W(F)$ and $W(F')$ are disjoint for any pair $F$ and $F'$ of antipodal faces of $P$.

For contradiction, let us assume that $W(F) \cap W(F') \neq \emptyset$. Due to the definition of $W(F)$ and $W(F')$, there are faces $e$ in $F$ and $e'$ in $F'$ of dimension at most 1 such that $W(e) \cap W(e') \neq \emptyset$. (We use the edge notation $e$ and $e'$, which corresponds to the ‘typical case’; however, one of $e, e'$ may be a vertex, if $F$ or $F'$ is 0-dimensional.) We remark that $e$ and $e'$ are antipodal as $F$ and $F'$ are antipodal. Therefore, there is a proper face $D$ containing $e$ and $-e'$.

If neither $e$ nor $e'$ is a broken edge, then $W(e) \subseteq \text{supp}_+(e) \subseteq \text{supp}_+(D)$, and $W(e') \subseteq \text{supp}_+(e') \subseteq \text{supp}_-(D)$, which is a contradiction.

Therefore, we can assume that $e$ or $e'$ is a broken edge; say $e'$ is broken. Then $e$ cannot be broken. (Indeed, if $e$ were broken, it would have to be an edge. Therefore, by Definition 18(iv) and Corollary 15(i), $\gamma(e) = \gamma(-e')$, but $\gamma(e')$ and $-\gamma(e')$ cannot be both broken due to Definition 5(ii).) We get $W(e') \subseteq \text{supp}_+(e') \cup \{v(e')\} \subseteq \text{supp}_-(D) \cup \{v(e')\}$. On the other hand, $W(e) \subseteq \text{supp}_+(e) \setminus \{v(e')\} \subseteq \text{supp}_+(D) \setminus \{v(e')\}$. Therefore $W(e)$ and $W(e')$ are disjoint in this case as well.

Proof of Theorem 1. By Observation 8, $\max\{\lambda(G), \mu(G)\} \leq \eta(G)$ for every graph $G$. To prove the inequality $\eta(G) \leq \sigma(G)$, we can assume that $G$ is connected as for disconnected graphs both parameters $\eta$ and $\sigma$ are realized as the maximum of the respective parameter over the components of $G$,\(^9\) if

\(^9\) For the parameter $\sigma$ it follows from its definition.
it contains at least one edge (and the claim follows from the characterization of classes of graphs with $\sigma(G) \leq 0.1$ for graphs without edges; see the introduction). Applying Proposition 20 to any semivalid representation $L$ of $G$ such that $\eta(G) = \dim(L)$, we get that $\eta(G) \leq \sigma(G)$. 

4. On the relations between $\mu, \lambda$ and $\sigma$

In this section, we further investigate the distinction between $\mu, \lambda$ and $\sigma$. Van der Holst and Pendavingh [12, Thm. 40] proved that $\lambda(G) \leq \sigma(G)$ for every graph $G$. Moreover, Pendavingh [22] provided an example of a graph $G$ such that $\mu(G) \leq 18$ and $\lambda(G) \geq 20$. This is the example that we mentioned in the introduction, which shows that the parameters $\sigma$ and $\mu$ are different in general.

Motivated by [22, Lem. 4] establishing lower bound on $\lambda(G - e)$ for $e \in E(G)$ with special properties, we present a similar lemma for the parameter $\eta$.

Lemma 22. Let $G = (V, E)$ be a connected graph and let $M \in \mathcal{M}(G)$. Suppose $F \subseteq E$ is such that

$\bigcup F \cap \text{supp}(x) \neq \emptyset$

for every broken $x \in \ker(M)$ inducing more than three connected components in $G[\text{supp}(x)]$. Then $\text{corank}(M) - |F| \leq \eta(G)$. If, moreover, $G - F$ is connected, then $\text{corank}(M) - |F| \leq \eta(G - F)$.

Proof. Let $L := \{y \in \ker(M) : y_u + y_v = 0 \forall uv \in F\}$. Clearly, $\dim(L) \geq \text{corank}(M) - |F|$. We show that $L$ is a semivalid representation of $G$, and for the ‘moreover’ part, that it is also a semivalid representation of $G - F$.

To verify that $L$ is a semivalid representation of $G$, it is immediate that condition (i) of Definition 5 is satisfied since it holds for every nonzero vector in $\ker(M)$ (e.g., see Lemma 4(i)). Next we check condition (ii) of Definition 5. Assume it is not satisfied. Take a broken $y \in L$, which induces more than three connected components in $G[\text{supp}(y)]$. By the assumption on $F$, there is $uv \in F$ such that $\{u, v\} \cap \text{supp}(y) \neq \emptyset$. This means that $y_u = -y_v \neq 0$. However, this is impossible by Lemma 4(ii).

Now we turn to condition (iii) of Definition 5. Again, we assume that the condition is not satisfied. Take $y \in L$ which has inclusion-minimal support among nonzero vectors in $L$, but at least one of the graphs $G[\text{supp}_{\pm}(y)]$ is not connected. By the definition of $L$ and Lemma 4(ii), $\bigcup F \subseteq V \setminus \text{supp}(y)$. 

However, this means that $D := \{ x \in \ker(M) : \text{supp}(x) \subseteq \text{supp}(y) \}$ is a subspace of $L$. On the other hand, Lemma 6 says that $\dim(D) + 1$ is equal to the number of connected components of $G[\text{supp}(y)]$. This means that $\dim(D) \geq 2$, which implies that there is $x \in D$ with strictly smaller support than $y$; a contradiction.

Lemma 4(ii) proves that the condition (iv) is satisfied as well; thus, $L$ is a semivalid representation of $G$.

To verify that $L$ is a semivalid representation of $G - F$, we first observe that if we take a nonzero $y \in L$, none of the edges $uv \in F$ can have both endpoints in $\text{supp}_+(y)$ or $\text{supp}_-(y)$, since $y_u + y_v = 0$. Therefore, removing $F$ from $G$ cannot disconnect any of the connected components of $G[\text{supp}_+(y)]$. Consequently, $L$ satisfies both requirements (ii) and (iii) of Definition 5 for $G - F$. Moreover, none of the edges of $uv \in F$ can have one endpoint in $\text{supp}(y)$ and the other in $V \setminus \text{supp}(y)$; again, because $y_u + y_v = 0$. Thus, removing $F$ cannot change $N(\text{supp}_+(y))$ nor $N(C)$ for any of the connected components $C$ induced by $\text{supp}(y)$. Therefore, $L$ also satisfies the requirement (iv) of Definition 5; we conclude that $L$ is a semivalid representation of $G - F$.

The next lemma is an easy consequence of Lemma 4(ii). It generalizes [22, Lem. 5].

**Lemma 23.** Let $G = (V, E)$ be a connected graph with maximum degree at most $d$ and let $M \in \mathcal{M}(G)$. Let $x \in \ker(M)$ be a broken vector. Then

1. $G[\text{supp}(x)]$ has at most $d$ connected components,
2. if $G[\text{supp}(x)]$ has exactly $d$ connected components, then $G[V \setminus \text{supp}(x)]$ has no edges and $V \setminus \text{supp}(x) = N(\text{supp}(x))$.

**Proof.** Since $G$ is connected, Lemma 4(ii) implies that $N(\text{supp}(x))$ is nonempty, and moreover, that every vertex in $N(\text{supp}(x))$ is connected to each component of $G[\text{supp}(x)]$; thus, the number of such components cannot be greater than the maximum degree in $G$. This proves the first part.

For the second part, the above argument shows that $G[N(\text{supp}(x))]$ does not contain any edge. Consider a vertex $v \in V \setminus \text{supp}(x) \setminus N(\text{supp}(x))$. Since $G$ is connected and $v \notin N(\text{supp}(x))$, there must be a path from $v$ to $N(\text{supp}(x))$. However, this is not possible, since all vertices in $N(\text{supp}(x))$ have their neighbours only in $\text{supp}(x)$.

We restate here the following theorem of Pendavingh [22, Thm. 5], which is very useful for proving upper bounds on $\mu(G)$.

**Theorem 24 ([22, Thm. 5]).** Let $G = (V, E)$ be a connected graph. Then either $G = K_{3,3}$, or $|E| \geq (\mu(G) + 1)^2$.  

Finite projective planes. Let $H_q$ denote the incidence graph of a finite projective plane of order $q$. It is a $(q+1)$-regular bipartite graph with parts of size $q^2+q+1$. Using Theorem 24, this implies that

$$\mu(H_q) \leq \left\lceil \frac{-1 + \sqrt{1 + 8|E(H_q)|}}{2} \right\rceil = \left\lceil \frac{-1 + \sqrt{1 + 8(q^2+q+1)(q+1)}}{2} \right\rceil.$$ 

Let $A_q$ be the adjacency matrix of $H_q$. It is known that the spectrum of $A_q$ is

$$\left((q+1)^{(1)}, \sqrt{q(q^2+q)}, -\sqrt{q(q^2+q)}, -(q+1)^{(1)}\right);$$

for a reference, see, e.g. [29, Sec. 3.8.1, eq. (3.38)] (for that reference, note that a finite projective plane of order $q$ is a symmetric BIBD with parameters $p, b = q^2+q+1, k, r = q+1, \ell = 1$). We further define $M_q := \sqrt{q}I - A_q$. Clearly, $M_q \in M(H_q)$ and corank$(M_q) = q^2+q$.

Proposition 25. $\mu(H_3) \leq 9$ and $\sigma(H_3) \geq 11$.

Proof. The bound on $\mu(H_q)$ above gives $\mu(H_3) \leq 9$. Furthermore, corank$(M_3) = 12$. Now choose any edge $e$ of $H_3$. Since $H_3$ is 4-regular, $e \cap \text{supp}(x) \neq \emptyset$ for every broken $x \in \ker(M_3)$ inducing more than three connected components in $H_3[\text{supp}(x)]$ by Lemma 23. Thus, by Lemma 22 and Proposition 20 we see that $\sigma(H_3) \geq \sigma(H_3 - e) \geq \eta(H_3 - e) \geq 11$.

The separation between $\mu$ and $\sigma$ can be pushed even further by removing a small part from $H_3$ to obtain a graph $G$ with $\mu(G) \leq 7$ and $\sigma(G) \geq 8$, as was announced in Theorem 2 in the introduction.

Proof of Theorem 2. We choose three vertices $v_1, v_2, v_3$ of $H_3$ corresponding to three points of the finite projective plane of order 3 not lying on a single line. Let $G' := H_3 - \{v_1, v_2, v_3\}$. We observe that $G'$ contains three vertices of degree two, since every two points of a projective plane lie on a single line. Next, we choose an edge $e \in E(G')$ adjacent to a vertex of degree three in $G'$ and set $G := G' - e$.

Observe that $G$ contains four vertices of degree two; for each of these four vertices we choose one of the two edges incident to it and put it into a set $F$. We write $G/F$ for the graph resulting from a contraction of the edges of $F$ in $G$. Since a subdivision of edges preserves $\mu(H)$ for graphs $H$ with $\mu(H) \geq 3$ by [11, Thm. 2.12], we get that $\mu(G) = \mu(G/F)$. The graph $G/F$ has $4 \times 13 - 12 - 1 - 4 = 35$ edges. This means that $\mu(G) = \mu(G/F) \leq 7$ by Theorem 24. On the other hand, a removal of a vertex can decrease $\sigma$ by at most 1 [12, Thm. 28]. As $\sigma(H_3 - e) \geq 11$ (this was substantiated in the proof of Proposition 25 above), we deduce that $\sigma(G) \geq 8$. 

\[\square\]
The proof of the following proposition is a direct generalization of the proof of [22, Thm. 1]. To recall the definition of $\lambda$, we refer the reader to the beginning of Subsection 2.2.

**Proposition 26.** Let $G = (V,E)$ be a connected graph of maximum degree at most $d$ and $M \in \mathcal{M}(G)$. Then $\lambda(G) \geq \text{corank}(M) - d + 1$.

**Proof.** Let $x \in \ker(M)$ be a broken vector. The subspace

$$D(x) := \{y \in \ker(M) : \text{supp}(y) \subseteq \text{supp}(x)\}$$

has dimension at most $d - 1$ by Lemma 6 and Lemma 23(i). Let $B \subseteq \ker(M)$ be a set consisting of all broken vectors $x$ with inclusion-maximal support among broken vectors in $\ker(M)$. This implies that for every broken $y \in \ker(M)$ there is $x \in B$ such that $y \in D(x)$. Therefore, every broken vector in $\ker(M)$ is contained in a linear subspace of $\ker(M)$ of dimension at most $d - 1$.

Since the number of different subsets $\text{supp}(x) \subseteq V$ is finite, the number of distinct subspaces $D(x)$ for $x \in B$ is finite as well. Therefore, there is a linear subspace $L \subseteq \ker(M)$ of dimension at least $\text{corank}(M) - d + 1$ such that for every $x \in B$ it holds that $L \cap D(x) = \{0\}$. Consequently, $L$ is a valid representation of $G$, which finishes the proof.

Applying this proposition to the finite projective planes we immediately obtain an asymptotic separation of order $\mu(H_q) \in O(q^{3/2})$ and $\sigma(H_q) \geq \lambda(H_q) \geq q^2$, which was stated in Theorem 3 in the introduction.

**Proof of Theorem 3.** Since $\text{corank}(M_q) = q^2 + q$ and the degree of every vertex in $H_q$ is $q + 1$, Proposition 26 implies that $\lambda(H_q) \geq q^2$. The fact that $\lambda(G) \leq \sigma(G)$ for every graph $G$ was proven by van der Holst and Pendavingh [12, Thm. 40], as was already mentioned before.

The upper bound on $\mu(H_q)$ follows directly from Theorem 24.

**Acknowledgment.** We would like to thank Radek Hušek and Robert Šámal for general discussions on the parameter $\mu$. We would also like to thank Arnaud de Mesmay for pointing us to the paper [12] and Rose McCarty for pointing us to [6].
References

[1] R. Bacher and Y. Colin de Verdière: Multiplicités des valeurs propres et transformations étoile-triangle des graphes, *Bulletin de la Société Mathématique de France* **123** (1995), 517–533.

[2] Y. Colin de Verdière: Sur un nouvel invariant des graphes et un critère de planarité, *Journal of Combinatorial Theory, Series B* **50** (1990), 11–21.

[3] Y. Colin de Verdière: On a new graph invariant and a criterion for planarity, in: N. Robertson and P.D. Seymour, editors, *Graph Structure Theory*, volume 147 of *Contemporary Mathematics*, pages 137–147. American Mathematical Society, 1991.

[4] V. Dujmović and S. Whitesides: Three-dimensional drawings, in: R. Tamassia, editor, *Handbook of Graph Drawing and Visualization*, Discrete Mathematics and Its Applications. CRC Press, 2013.

[5] G. Ewald and G. C. Shephard: Stellar subdivisions of boundary complexes of convex polytopes, *Math. Ann.** 210 (1974), 7–16.

[6] J. Foisy: A newly recognized intrinsically knotted graph, *J. Graph Theory* **43** (2003), 199–209.

[7] F. Goldberg: Optimizing Colin de Verdière matrices of $K_{4,4}$, *Linear Algebra and its Applications* **438** (2013), 4090–4101.

[8] H. van der Holst: A Short Proof of the Planarity Characterization of Colin de Verdière, *Journal of Combinatorial Theory, Series B* **65** (1995), 269–272.

[9] H. van der Holst, M. Laurent and A. Schrijver: On a Minor-Monotone Graph Invariant, *Journal of Combinatorial Theory, Series B* **65** (1995), 291–304.

[10] H. van der Holst, L. Lovász and A. Schrijver: On the invariance of Colin de Verdière’s graph parameter under clique sums, *Linear Algebra and its Applications* **226** (1995), 509–517.

[11] H. van der Holst, L. Lovász and A. Schrijver: The Colin de Verdière graph parameter, 29–85, Bolyai Society Mathematical Studies. János Bolyai Mathematical Society, Hungary, 1999.

[12] H. van der Holst and R. Pendavingh: On a graph property generalizing planarity and flatness, *Combinatorica* **29** (2009), 337–361.

[13] I. Izmestiev: The Colin de Verdière number and graphs of polytopes, *Israel Journal of Mathematics* **178** (2010), 427–444.

[14] V. Kaluža and M. Tancer: Even maps, the Colin de Verdière number and representations of graphs, in: *Proceedings of the 2020 ACM-SIAM Symposium on Discrete Algorithms*, 2642–2657, 2020.

[15] V. Kaluža and M. Tancer: Even maps, the Colin de Verdière number and representations of graphs, arXiv:1907.05055, 2019.

[16] A. Kotlov, L. Lovász and S. Vempala: The Colin de Verdière number and sphere representations of a graph, *Combinatorica* **17** (1997), 483–521.

[17] L. Lovász: Steinitz representations of polyhedra and the Colin de Verdière number, *Journal of Combinatorial Theory, Series B* **82** (2001), 223–236.

[18] L. Lovász and A. Schrijver: A Borsuk theorem for antipodal links and a spectral characterization of linklessly embeddable graphs, *Proceedings of the American Mathematical Society* **126** (1998), 1275–1285.

[19] L. Lovász and A. Schrijver: On the null space of a Colin de Verdière matrix, *Annales de l’Institut Fourier* **49** (1999), 1017–1026.
[20] R. McCarty: The extremal function and Colin de Verdière graph parameter, *Electronic Journal of Combinatorics*, 25(2):P2.32, 2018.
[21] R. M. McCarty: Personal communication, 2019.
[22] R. Pendavingh: On the Relation Between Two Minor-Monotone Graph Parameters, *Combinatorica* 18 (1998), 281–292.
[23] N. Robertson, P. Seymour and R. Thomas: Sachs’ linkless embedding conjecture, *Journal of Combinatorial Theory, Series B* 64 (1995), 185–227.
[24] N. Robertson and P. D. Seymour: Graph minors. XIII. The disjoint paths problem, *J. Combin. Theory Ser. B* 63 (1995), 65–110.
[25] N. Robertson and P. D. Seymour: Graph minors. XX. Wagner’s conjecture, *J. Combin. Theory Ser. B* 92 (2004), 325–357.
[26] C. P. Rourke and B. J. Sanderson: *Introduction to piecewise-linear topology*, Springer Study Edition. Springer-Verlag, Berlin-New York, 1982. Reprint.
[27] M. Schaefer: Hanani–Tutte and related results, in: *Geometry—intuitive, discrete, and convex*, volume 24 of *Bolyai Soc. Math. Stud.*, 259–299. János Bolyai Math. Soc., Budapest, 2013.
[28] A. Schrijver and B. Sevenster: The strong arnold property for 4-connected flat graphs, *Linear Algebra and its Applications* 522 (2017), 153–160.
[29] Z. Stanić: *Regular Graphs: A Spectral Approach*, De Gruyter Series in Discrete Mathematics and Applications, De Gruyter, 2017.
[30] D. R. Stinson: *Combinatorial Designs: Construction and Analysis*, Springer-Verlag New York, 2004.
[31] M. Tait: The Colin de Verdière parameter, excluded minors, and the spectral radius, *Journal of Combinatorial Theory, Series A* 166 (2019), 42–58.
[32] M. Tancer and D. Tonkonog: Nerves of good covers are algorithmically unrecognizable, *SIAM J. Comput.* 42 (2013), 1697–1719.
[33] R. Thomas: Recent excluded minor theorems for graphs, in: *In surveys in combinatorics*, 1999, Ed. by J. D. Lamb and D. A. Preece. London Mathematical Society Lecture Note Series. Cambridge University Press, 1999, 201–222.
[34] G. M. Ziegler: *Lectures on polytopes*, volume 152 of *Graduate Texts in Mathematics*, Springer-Verlag, New York, 1995.

Vojtěch Kaluža
*Institute of Science and Technology*
*Austria*
vojtech.kaluza@ist.ac.at

Martin Tancer
*Department of Applied Mathematics*
*Charles University*
*Prague, Czech Republic*
MySurname@kam.mff.cuni.cz