ROUND FOLD MAPS ON 3–MANIFOLDS

NAOKI KITAZAWA AND OSAMU SAEKI

Abstract. We show that a closed orientable 3–dimensional manifold admits a round fold map into the plane, i.e. a fold map whose critical value set consists of disjoint simple closed curves isotopic to concentric circles, if and only if it is a graph manifold, generalizing the characterization for simple stable maps into the plane. Furthermore, we also give a characterization of closed orientable graph manifolds that admit directed round fold maps into the plane, i.e. round fold maps such that the number of regular fiber components of a regular value increases toward the central region in the plane.

1. Introduction

Let \( M \) be a smooth closed manifold of dimension \( \geq 2 \). It is known that if a smooth map \( f : M \to \mathbb{R}^2 \) is generic enough, then it has only fold and cusps as its singularities \([5, 6, 12]\). Furthermore, if \( M \) has even Euler characteristic (e.g. if \( \dim M \) is odd), then the cusps can be eliminated by homotopy. In particular, every smooth closed orientable 3–dimensional manifold admits a smooth map into \( \mathbb{R}^2 \) with only fold singularities, i.e. a fold map.

In \([10, 9]\), the second author considered the following smaller class of generic smooth maps. A fold map \( f : M \to \mathbb{R}^2 \) on a smooth closed orientable 3–dimensional manifold \( M \) is a simple stable map if for every \( q \in \mathbb{R}^2 \), each component of \( f^{-1}(q) \) contains at most one singular point and \( f|_{S(f)} \) is an immersion with normal crossings, where \( S(f) \subset M \) denotes the set of singular points of \( f \). Note that if \( f \) is a fold map, then \( S(f) \) is a regular closed submanifold of \( M \). In particular, if \( f|_{S(f)} \) is an embedding, then \( f \) is a simple stable map. In \([10]\), it has been proved that for a smooth closed orientable 3–dimensional manifold \( M \), the following three are equivalent to each other:

1. \( M \) admits a fold map \( f : M \to \mathbb{R}^2 \) such that \( f|_{S(f)} \) is an embedding,
2. \( M \) admits a simple stable map into \( \mathbb{R}^2 \),
3. \( M \) is a graph manifold, i.e. it is a finite union of \( S^1 \)–bundles over compact surfaces attached along their torus boundaries.

Thus, for example, if \( M \) is hyperbolic, then \( M \) never admits such a fold map.

On the other hand, the first author introduced the notion of a round fold map \([3, 4]\): a smooth map \( f : M \to \mathbb{R}^2 \) is a round fold map if it is a fold map and \( f|_{S(f)} \) is an embedding onto the disjoint union of some concentric circles in \( \mathbb{R}^2 \) (for details, see \([2]\)). As has been studied by the first author, round fold maps have various nice properties.

The first main result of this paper is Theorem 2.8, which states that every graph 3–manifold admits a round fold map into \( \mathbb{R}^2 \). This generalizes the characterization result obtained in \([10]\) for simple stable maps mentioned above.
It is not difficult to observe that if \( f : M \to \mathbb{R}^2 \) is a round fold map of a closed orientable 3–dimensional manifold, then the number of components of the fiber over a regular value changes exactly by one when the regular value crosses the critical value set. We can thus put a normal orientation to each component of the critical value set in such a way that the orientation points in the direction that increases the number of components of a regular fiber. Then, a round fold map is said to be directed if all the circles in the critical value set are directed inward. The second main result of this paper (Theorem 2.9) characterizes those graph 3–manifolds which admit directed round fold maps. It will turn out that the class is strictly smaller than that of closed orientable graph 3–manifolds.

The paper is organized as follows. In §4 we prepare several definitions and a lemma concerning round fold maps and graph 3–manifolds, and state our main theorems. We also give an observation on fibered links or open book structures associated with round fold maps and give some examples. In §3 we prove the main theorems. Basically, we will follow the proof given in [10, Theorem 3.1]: however, in some steps we need to modify the strategy for the constructions of round fold maps. In §4, we give some corollaries and show that the class of 3–manifolds that admit directed round fold maps is strictly smaller than that of all graph 3–manifolds, using results obtained in [2, 8].

Throughout the paper, all manifolds and maps between them are smooth of class \( C^\infty \) unless otherwise specified. For a space \( X \), \( \text{id}_X \) denotes the identity map of \( X \). The symbol “\( \cong \)” denotes a diffeomorphism between smooth manifolds.

2. Round fold maps

Let \( M \) be a closed orientable 3–dimensional manifold and \( f : M \to \mathbb{R}^2 \) a smooth map.

**Definition 2.1.** A point \( p \in M \) is a singular point of \( f \) if the rank of the differential \( df_p : T_p M \to T_{f(p)} \mathbb{R}^2 \) is strictly less than two. We denote by \( S(f) \) the set of all singular points of \( f \). A point \( p \in S(f) \) is a definite fold point (resp. an indefinite fold point, or a cusp point) if \( f \) is represented by the map

\[
(u, x, y) \mapsto (u, x^2 + y^2) \quad \text{(resp. } (u, x^2 - y^2), \text{ or } (u, y^2 + ux - x^3))
\]

around the origin with respect to certain local coordinates around \( p \) and \( f(p) \). We call a point \( p \in S(f) \) a fold point if it is a definite or an indefinite fold point. A smooth map \( f : M \to \mathbb{R}^2 \) is called a fold map if it has only fold points as its singular points. Note that then \( S(f) \) is a closed 1–dimensional submanifold of \( M \) and that \( f|_{S(f)} \) is an immersion.

**Definition 2.2.** Let \( C \) be a finite disjoint union of simple closed curves in \( \mathbb{R}^2 \). We say that \( C \) is concentric if for every pair \( c_0, c_1 \) of distinct components of \( C \), exactly one of them, say \( c_i \), is contained in the bounded region of \( \mathbb{R}^2 \setminus c_{1-i} \) (see Fig. 1). (In this case, we say that \( c_i \) (or \( c_{1-i} \)) is an inner component (resp. an outer component) with respect to \( c_{1-i} \) (resp. \( c_i \)).) In other words, \( C \) is isotopic to a set of concentric circles in \( \mathbb{R}^2 \).

**Definition 2.3.** We say that a smooth map \( f : M \to \mathbb{R}^2 \) of a closed 3–dimensional manifold \( M \) into the plane is a round fold map if it is a fold map and \( f|_{S(f)} \) is an embedding onto a concentric family of simple closed curves. Note that a round fold map is a simple stable map in the sense of [11, 12]. Note also that the outermost circle component of \( f(S(f)) \) consists of the images of definite fold points.

In the following, \( A \) denotes the annulus \( S^1 \times [-1, 1] \), and \( P \) denotes the compact surface obtained from the 2–sphere by removing three open disks: in other words, \( P \) is a pair of pants.
Let $f : M \to \mathbb{R}^2$ be a round fold of a closed orientable 3–dimensional manifold $M$. For a component $c$ of $f(S(f))$, take a small arc $\alpha \cong [-1, 1]$ in $\mathbb{R}^2$ that intersects $f(S(f))$ exactly at one point in $c$ transversely. Then, $f^{-1}(\alpha)$ is a compact surface with boundary $f^{-1}(a) \cup f^{-1}(b)$, which is diffeomorphic to a finite disjoint union of circles, where $a$ and $b$ are the end points of $\alpha$. Furthermore, $f|_{f^{-1}(\alpha)} : f^{-1}(\alpha) \to \alpha$ can be regarded as a Morse function with exactly one critical point. As $M$ is orientable, we see that $f^{-1}(\alpha)$ is diffeomorphic to the union of $D^2$ (or $P$) and a finite number of copies of $A$ (see [11], for example). Therefore, the number of components of $f^{-1}(\alpha)$ differs from that of $f^{-1}(b)$ exactly by one. If $f^{-1}(\alpha)$ has more components than $f^{-1}(b)$, then we normally orient $c$ from $b$ to $a$: otherwise, we orient $c$ from $a$ to $b$. It is easily shown that this normal orientation is independent of the choice of $\alpha$. In this way, each component of $f(S(f))$ is normally oriented. If the normal orientation points inward, then the component is said to be inward-directed; otherwise, outward-directed.

**Definition 2.4.** Let $f : M \to \mathbb{R}^2$ be a round fold map. We say that $f$ is **directed** if all the components of $f(S(f))$ are inward-directed. It is easy to see that a round fold map $f$ is directed if and only if the number of components of a regular fiber over a point in the innermost component of $\mathbb{R}^2 \setminus f(S(f))$ coincides with the number of components of $S(f)$.

Let $f : M \to \mathbb{R}^2$ be a round fold map of a closed connected oriented 3–dimensional manifold. In the following, for $r > 0$, $C_r$ denotes the circle of radius $r$ centered at the origin in $\mathbb{R}^2$. We may assume that

$$f(S(f)) = \bigcup_{i=1}^{t} C_i$$

for some $t \geq 1$ by composing a diffeomorphism of $\mathbb{R}^2$ isotopic to the identity if necessary. Set $L = f^{-1}(0)$, which is an oriented link in $M$ if it is not empty. Let $D$ be the closed disk centered at the origin with radius $1/2$. Then, $f^{-1}(D)$ is diffeomorphic to $L \times D$, which can be identified with a tubular neighborhood $N(L)$ of $L$ in $M$. Furthermore, the composition $\varphi = \pi \circ f : M \setminus \text{Int} \ N(L) \to S^1$ is a submersion, where $\pi : \mathbb{R}^2 \setminus \text{Int} \ D \to S^1$ is the standard radial projection and $\varphi|_{\partial N(L)} : \partial N(L) = L \times \partial D \to S^1$ corresponds to the projection to the second factor.
followed by a scalar multiplication. Hence, \( \varphi \) is a smooth fiber bundle and \( L \) is a fibered link. (In other words, \( M \) admits an open book structure with binding \( L \).) The fiber (or the page) is identified with \( F = f^{-1}(J) \), where

\[
J = \left[ \frac{1}{2}, t + 1 \right] \times \{0\} \subset \mathbb{R}^2,
\]

and it is a compact oriented surface. Note that \( g = f|_F : F \to J \) is a Morse function with exactly \( t \) critical points and that a monodromy diffeomorphism of the fibration over \( S^1 \) can be chosen so that it preserves \( g \).

Note that all these arguments work even when \( L = \emptyset \). In this case, \( F \) is a closed orientable surface and \( M \) is the total space of an \( F \)-bundle over \( S^1 \).

Conversely, if we have a compact orientable surface \( F \), a Morse function \( g : F \to \left[ \frac{1}{2}, t + 1 \right] \) such that \( g(\partial F) = \frac{1}{2} \) and that \( g \) has no critical point near the boundary, and a diffeomorphism \( h : F \to F \) which is the identity on the boundary and which satisfies \( g \circ h = g \), then we can construct a round fold map \( f : M \to \mathbb{R}^2 \) in such a way that \( M \) is the union of \( \partial F \times D^2 \) and the total space of the \( F \)-bundle over \( S^1 \) with geometric monodromy \( h \).

**Example 2.5.** Let \( F \) be a compact connected orientable surface with \( \partial F \neq \emptyset \). Let us consider the identity diffeomorphism as the geometric monodromy in the above construction. Then, we see that the source 3–manifold \( M \) of the round fold map is diffeomorphic to \((\partial F \times D^2) \cup (F \times S^1) \cong \partial (F \times D^2)\). By using a handle decomposition argument, we see easily that \( F \times D^2 \) is diffeomorphic to \( D^4 \) or a boundary connected sum of a finite number of copies of \( S^1 \times D^2 \). Therefore, \( M \) is diffeomorphic either to \( S^3 \) or to the connected sum of a finite number of copies of \( S^1 \times S^2 \).

For example, if we start with the Morse function \( g_1 : F_1 \to [1/2, 4] \) as depicted in Fig. 2 (left), then the singular point set \( S(f_1) \) of the resulting round fold map \( f_1 : M_1 \to \mathbb{R}^2 \) has three components and their images coincide with \( C_1 \), \( C_2 \) and \( C_3 \). The first one is outward directed, while the other two are inward directed. Therefore, the fold map \( f_1 \) is not directed. In this example, \( M_1 \) is diffeomorphic to \((S^1 \times S^2)\#(S^1 \times S^2)\).

On the other hand, if we start with the Morse function \( g_2 : F_2 \to [1/2, 4] \) as depicted in Fig. 2 (right), then we get a round fold map \( f_2 : M_2 \to \mathbb{R}^2 \) with the same singular values: however, this round fold map is directed. We can also show that \( M_2 \) is again diffeomorphic to \((S^1 \times S^2)\#(S^1 \times S^2)\).

**Figure 2.** Morse functions on surfaces with Euler characteristic \(-1\)

**Definition 2.6.** Let \( M \) be a closed orientable 3–dimensional manifold. It is called a **graph manifold** if it is diffeomorphic to a union of \( S^1 \)-bundles over compact surfaces attached along their torus boundaries.
Lemma 2.7. Every closed orientable graph 3–manifold is diffeomorphic to a union of finite numbers of copies of \( P \times S^1 \) and solid tori attached along their torus boundaries.

Proof. It is known that every such 3–manifold is diffeomorphic to a union of a finite number of \( S^1 \)–bundles over compact connected orientable surfaces of genus zero (for example, see [10, Lemma 3.3]). For such a base surface \( B \), if the number of boundary components is greater than or equal to 4, then we can decompose \( B \) into a union of a finite number of copies of \( P \) attached along their circle boundaries. If the number of boundary components is equal to two, then \( B \) is diffeomorphic to the union of \( P \) and a disk. If the surface \( B \) has no boundary, then we can decompose it into two disks. As orientable \( S^1 \)–bundles over \( P \) or a disk are always trivial, the result follows. \( \square \)

As a consequence, a graph manifold can be represented by a (multi-)graph, where each vertex corresponds to \( P \times S^1 \) or a solid torus and each edge corresponds to the gluing along a pair of boundary components. Note that each gluing corresponds to an element of the (orientation preserving) mapping class group of the torus, identified with \( SL(2,\mathbb{Z}) \).

Our main results of this paper are as follows.

Theorem 2.8. Let \( M \) be a closed orientable 3–dimensional manifold. Then, it admits a round fold map into \( \mathbb{R}^2 \) if and only if it is a graph manifold.

In particular, every closed orientable graph 3–manifold admits a fibered link. Compare this with [7].

Theorem 2.9. Let \( M \) be a closed connected orientable graph 3–manifold. Then, it admits a directed round fold map into \( \mathbb{R}^2 \) if and only if it can be decomposed into a union of finite numbers of copies of \( P \times S^1 \) and a solid torus such that the corresponding graph is a tree.

Note that Theorem 2.8 generalizes the characterization for simple stable maps obtained in [10].

3. Proofs

In this section, we prove Theorems 2.8 and 2.9.

Proof of Theorem 2.8. As noted above, a round fold map is a simple stable map. Therefore, if a closed orientable 3–dimensional manifold admits such a map, then it is necessarily a graph manifold by [10].

Now, suppose \( M \) is a graph manifold. We will basically follow the proof of [10, Theorem 3.1] in order to construct a round fold map \( f : M \to \mathbb{R}^2 \), except for the first step, in which a non-singular map is constructed for each \( S^1 \)–bundle piece in [10] while we construct a fold map for each piece, as explained below.

By virtue of Lemma 2.7, we have disjointly embedded tori \( T_1, T_2, \ldots, T_r \) in \( M \) such that each of the components \( X_1, X_2, \ldots, X_s \) of \( M \setminus \cup_{i=1}^{r} \text{Int} N(T_i) \) is diffeomorphic either to \( P \times S^1 \) or to \( D^2 \times S^1 \), where \( N(T_i) \) denotes a small tubular neighborhood of \( T_i \) in \( M \), \( 1 \leq i \leq r \). By inserting pieces diffeomorphic to \( A \times S^1 \cong T^2 \times [-1,1] \) if necessary, we may assume that the decomposition is of a plumbing type (for details, see [10, Lemma 3.4]). Now, each \( X_j \) is diffeomorphic either to \( P \times S^1 \), \( D^2 \times S^1 \), or \( A \times S^1 \).

Take a component \( X_j \), \( 1 \leq j \leq s \). Suppose it is diffeomorphic to \( D^2 \times S^1 \). Let \( \delta : D^2 \to [-1,1] \) be the Morse function defined by \( \delta(x,y) = -x^2 - y^2 \), where \( D^2 \) is
identified with the unit 2–disk in \( \mathbb{R}^2 \) (see Fig. 3 left). Then, define \( f|_{X_j} \) to be the composition

\[
\eta_j \circ (\delta \times \text{id}_{S^1}) \circ \varphi_j : X_j \xrightarrow{\varphi_j} D^2 \times S^1 \xrightarrow{\delta \times \text{id}_{S^1}} [-1, 1] \times S^1 \xrightarrow{\eta_j} \mathbb{R}^2,
\]

where \( \varphi_j \) is a diffeomorphism and \( \eta_j \) is an embedding whose image is a small tubular neighborhood of the circle of radius \( j \) centered at the origin. We also arrange \( \eta_j \) in such a way that \( \eta_j((\pm 1) \times S^1) \) coincides with the circle of radius \( j \pm (1/3) \).

Suppose \( X_j \) is diffeomorphic to \( P \times S^1 \). We define \( f|_{X_j} \) by the composition

\[
\eta_j \circ (\iota \times \text{id}_{S^1}) \circ \varphi_j : X_j \xrightarrow{\varphi_j} P \times S^1 \xrightarrow{\iota \times \text{id}_{S^1}} [-1, 1] \times S^1 \xrightarrow{\eta_j} \mathbb{R}^2,
\]

where \( \varphi_j \) is a diffeomorphism, \( \iota : P \to [-1, 1] \) is the standard Morse function with exactly one saddle point as depicted in Fig. 3 (right), and \( \eta_j \) is an embedding as described in the previous paragraph.

Now, suppose \( X_j \) is diffeomorphic to \( A \times S^1 \). In this case, we define \( f|_{X_j} \) by the composition

\[
\eta_j \circ (\rho \times \text{id}_{S^1}) \circ \varphi_j : X_j \xrightarrow{\varphi_j} A \times S^1 \xrightarrow{\rho \times \text{id}_{S^1}} [-1, 1] \times S^1 \xrightarrow{\eta_j} \mathbb{R}^2,
\]

where \( \varphi_j \) is a diffeomorphism, \( \rho : A \cong S^1 \times [-1, 1] \to [-1, 1] \) is the projection to the second factor, and \( \eta_j \) is an embedding as described above.

Now, the map \( f|_{\cup_{j=1}^6 X_j} \) has only fold singular points, and its restriction to the singular point set is an embedding onto a concentric family of circles in \( \mathbb{R}^2 \). Then, we can extend the map to get a round fold map \( f : M \to \mathbb{R}^2 \) by the same procedure as described in [10] Proof of Theorem 3.1. Note that in the proof there, a simple stable map into \( S^2 \) is first constructed: however, in our case, we can directly construct a map into \( \mathbb{R}^2 \) as the singular value set of \( f|_{\cup_{j=1}^6 X_j} \) is a concentric family of circles in \( \mathbb{R}^2 \). Sometimes, we need to use a disk region in the target: in such a case, we can choose the region that does not contain the point \( \infty \in S^2 = \mathbb{R}^2 \cup \{\infty\} \). Note also that we can arrange \( f \) in such a way that \( f|_{S(f)} \) is an embedding by slightly perturbing \( f \) near \( S(f) \) if necessary. \( \square \)

Let us go on to the proof of the second theorem.

Let \( f : M \to \mathbb{R}^2 \) be a round fold map of a closed orientable 3–dimensional manifold. By post-composing an appropriate diffeomorphism of \( \mathbb{R}^2 \), we may assume that \( f(S(f)) \) consists of the circles centered at the origin with radii \( 1, 2, \ldots, t \). Recall that, for \( r > 0 \), \( C_r \) denotes the circle of radius \( r \) centered at the origin in \( \mathbb{R}^2 \). We also put, for \( 0 < a < b \),

\[
C_{[a,b]} = \{(x, y) \in \mathbb{R}^2 | a \leq \sqrt{x^2 + y^2} \leq b\}.
\]
We can observe that \( f^{-1}(C_{k-1}(1/2)) \) is a finite disjoint union of tori for each \( k = 1, 2, \ldots, t \), since \( M \) is orientable. Let \( K \) be the closure of a component of

\[
M \setminus \bigcup_{k=1}^{t} f^{-1}(C_{k-1}(1/2))
\]

such that \( f(K) \subset C_{[k-(1/2), k+(1/2)]} \). Let \( p_K : K \to S^1 \) be the composition of \( f|_K : K \to C_{[k-(1/2), k+(1/2)]} \) and the radial projection \( C_{[k-(1/2), k+(1/2)]} \to S^1 \). We see easily that \( p_K \) is a submersion and hence is a locally trivial fibration. The fiber is a disjoint union of copies of \( D^2 \), \( A \) and \( P \). Since \( f|_{S(f)} \) is an embedding and \( K \) is connected, the fiber is diffeomorphic to \( D^2 \), \( P \), or a finite disjoint union of copies of \( A \). If the fiber is diffeomorphic to \( D^2 \), then \( K \) is diffeomorphic to \( D^2 \times S^1 \), since \( K \) is an orientable 3–dimensional manifold. If the fiber is diffeomorphic to \( P \), then \( K \) is diffeomorphic either to \( P \times S^1 \) or a non-trivial \( P \)–bundle over \( S^1 \) (see the proof of [10, Lemma 2.4]).

Suppose that \( K \) is a non-trivial \( P \)–bundle over \( S^1 \) and that \( C_k \subset f(S(f)) \) is inward-directed. If \( k = 1 \), then this leads to a contradiction, since \( f \) is a trivial bundle over the innermost region of \( \mathbb{R}^2 \setminus f(S(f)) \). If \( k > 1 \), then a component of \( f^{-1}(C_{[k-(3/2), k-(1/2)]}) \) adjacent to \( K \) is either a non-trivial \( P \)–bundle over \( S^1 \), or a non-trivial \( \langle P \cup P \rangle \)–bundle over \( S^1 \), where \( A \cup A \) is the disjoint union of two copies of \( A \) and the monodromy for the latter bundle interchanges the two components of \( A \cup A \). In the former case, \( C_{k-1} \subset f(S(f)) \) is outward-directed. In the latter case, we can repeat the argument toward inner components to find an outward-directed component.

If \( C_k \subset f(S(f)) \) is outward-directed, then we can also find an inward-directed component outside of \( C_k \), since the outermost component corresponds to \( D^2 \times S^1 \).

Thus we have proved the following.

**Lemma 3.1.** Let \( f : M \to \mathbb{R}^2 \) be a round fold map of a closed orientable 3–dimensional manifold such that \( f(S(f)) = \bigcup_{k=1}^{t} C_k \). If \( f \) is directed, then the closure of a component of

\[
M \setminus \bigcup_{k=1}^{t} f^{-1}(C_{k-1}(1/2))
\]

is never diffeomorphic to the non-trivial \( P \)–bundle over \( S^1 \).

**Proof of Theorem 2.9.** First, suppose that there exists a directed round fold map \( f : M \to \mathbb{R}^2 \). By post-composing an appropriate diffeomorphism of \( \mathbb{R}^2 \), we may assume that the components of \( f(S(f)) \) are circles centered at the origin with radii 1, 2, \ldots, \( t \). Then the disjoint union of tori \( \bigcup_{k=1}^{t} f^{-1}(C_{k-1}(1/2)) \) decomposes \( M \) into a union of copies of \( P \times S^1 \), \( A \times S^1 \), and \( D^2 \times S^1 \) attached along their torus boundaries. Note that by Lemma 3.1 a non-trivial \( P \)–bundle over \( S^1 \) does not appear, since \( f \) is directed. Furthermore, we can ignore the components diffeomorphic to \( A \times S^1 \cong T^2 \times [-1, 1] \) for obtaining a decomposition of \( M \).

As \( f \) is directed and \( M \) is connected, we see that the components diffeomorphic to \( D^2 \times S^1 \) are the outermost component \( (C_{t-(1/2), t+(1/2)} \) together with the components of the innermost part \( f^{-1}(C_{[0,1/2]} \): no other components are diffeomorphic to \( D^2 \times S^1 \). Then, we see easily that the corresponding graph describing this decomposition of \( M \) into copies of \( D^2 \times S^1 \) and \( P \times S^1 \) is a tree, as the number of components of regular fibers strictly increases toward the central region.

Conversely, suppose that the graph describing the decomposition of \( M \) into copies of \( P \times S^1 \) and \( D^2 \times S^1 \) is a tree. By inserting pieces diffeomorphic to \( A \times S^1 \) if necessary, we may assume that the decomposition is of a plumbing type. Then, the graph \( \Gamma \) describing this new decomposition is also a tree. Note that then \( \Gamma \) has at least one vertex of degree one. Let \( s \) denote the number of vertices of \( \Gamma \). We label the vertices by \( \{1, 2, \ldots, s\} \) in such a way that
(1) the labeling gives a one-to-one correspondence between the set of vertices and the set \( \{1, 2, \ldots, s\} \),

(2) a vertex of degree one has the label \( s \),

(3) for each \( j \in \{1, 2, \ldots, s\} \), the vertices of labels \( \geq j \) together with the edges connecting them constitute a connected subgraph of \( \Gamma \).

This is possible, since \( \Gamma \) is a tree with only vertices of degrees one, two or three.

Then, we follow the procedure as in the proof of Theorem 2.8 for constructing a round fold map on \( M \), except for the components corresponding to vertices of degree one whose label is different from \( s \). Note that in the process described in the proof of [10, Theorem 3.1], we do not need to use \( h_1 : S^1 \times S^1 \times [-1, 1] \rightarrow \mathbb{R} \) in our situation. Furthermore, when we use \( h_2 \), we make sure that the corresponding image is contained in \( C_{[0, s]} \). Finally, for the components corresponding to vertices of degree one with label \( < s \), we just consider the projection \( D^2 \times S^1 \rightarrow D^2 \), where the target \( D^2 \) should be enlarged depending on the label. This matches with the construction for the adjacent component.

Now, it is not difficult to see that the resulting map \( f : M \rightarrow \mathbb{R}^2 \) is a directed round fold map. This completes the proof.

We do not know how to generalize Theorems 2.8 and 2.9 for non-orientable 3–manifolds. We also do not know how to classify the right-left equivalence classes of (directed) round fold maps on a given 3–manifold (see a certain classification result for simple stable maps given in [10]).

4. Corollaries and examples

In this section, we give some corollaries of our main theorems. We also show that the class of 3–manifolds that admit directed round fold maps is strictly smaller than that of all graph 3–manifolds.

**Corollary 4.1.** Let \( M \) be a closed connected orientable graph 3–manifold. If \( H_1(M; \mathbb{Q}) = 0 \), then it admits a directed round fold map into \( \mathbb{R}^2 \).

**Proof.** Let \( G \) be the graph corresponding to a decomposition of \( M \) into \( P \times S^1 \) and \( D^2 \times S^1 \) as described in Lemma 2.7. Then, we can naturally construct a continuous map \( \gamma : M \rightarrow G \) in such a way that for each piece, the complement of a small collar neighborhood of the boundary is mapped to the corresponding vertex. Then, we can show that \( \gamma \) induces a surjection \( \gamma_* : \pi_1(M) \rightarrow \pi_1(G) \). Since \( H_1(M; \mathbb{Q}) = 0 \), we see that \( G \) is a tree. Then, the result follows from Theorem 2.9.

Since every closed orientable Seifert 3–manifold over the 2–sphere admits a decomposition into a union of a finite number of copies of \( P \times S^1 \) and a solid torus such that the corresponding graph is a tree, we have the following.

**Corollary 4.2.** Every closed orientable Seifert 3–manifold over \( S^2 \) admits a directed round fold map into \( \mathbb{R}^2 \).

By virtue of the realization result due to [1], as a corollary, we see that every linking form can be realized as that of a 3–manifold admitting a directed round fold map into \( \mathbb{R}^2 \). Thus, the linking form cannot detect the non-existence of a directed round fold map.

On the other hand, as to the cohomology ring, we have the following.

**Corollary 4.3.** If a closed orientable 3–manifold \( M \) admits a directed round fold map into \( \mathbb{R}^2 \), then for every pair \( \xi, \eta \in H^1(M; \mathbb{Q}) \), their cup product \( \xi \smile \eta \) vanishes in \( H^2(M; \mathbb{Q}) \).
The above corollary follows from [2, Theorem 5.2].

Thus, for example, for every closed orientable surface $\Sigma$ of genus $\geq 1$, the 3–manifold $\Sigma \times S^1$ never admits a directed round fold map into $\mathbb{R}^2$, although it is a graph manifold.

Let $M$ be a closed connected orientable 3-dimensional manifold. If $M$ is a graph manifold, then it can be represented by a plumbing graph whose vertices and edges have certain weights [8]. Such a graph is not unique as the decompositions of $M$ into $S^1$–bundles over surfaces are not unique. However, we have the notion of a normal form, and then we have the existence and uniqueness of such a normal form for a given closed connected orientable graph manifold [8, Theorem 4.1].

The following lemma can be proved by following the proof of [8, Theorem 4.1].

**Lemma 4.4.** Let $M$ be a closed connected orientable graph 3–manifold. If it can be decomposed into a union of finite numbers of copies of $P \times S^1$ and a solid torus in such a way that the corresponding graph is a tree, then its normal form plumbing graph is a finite disjoint union of trees.

As a corollary, we have the following.

**Corollary 4.5.** Let $M$ be a closed connected orientable graph 3–manifold whose normal form plumbing graph contains a loop. Then, $M$ admits a round fold map into $\mathbb{R}^2$ but does not admit a directed round fold map into $\mathbb{R}^2$.

For example, some torus bundles over $S^1$ as described in [8, Theorem 6.1] satisfy the assumption of the above corollary. (More precisely, those torus bundles over $S^1$ whose monodromy matrix has trace $\geq 3$ or $\leq -3$, give such examples.)

**Acknowledgment**

The authors would like to thank Professor Yuya Koda for stimulating discussions which motivated the theme of this paper. This work was supported by JSPS KAKENHI Grant Number JP17H06128.

**References**

[1] J. Bryden and F. Deloup, *A linking form conjecture for 3–manifolds*, in “Advances in topological quantum field theory”, pp. 253–265, NATO Sci. Ser. II Math. Phys. Chem., 179, Kluwer Acad. Publ., Dordrecht, 2004.
[2] M.I. Doig and P.D. Horn, *On the intersection ring of graph manifolds*, Trans. Amer. Math. Soc. 369 (2017), 1185–1203.
[3] N. Kitazawa, *Fold maps with singular value sets of concentric spheres*, Hokkaido Math. J. 43 (2014), 327–359.
[4] N. Kitazawa, *Constructions of round fold maps on smooth bundles*, Tokyo J. Math. 37 (2014), 385–403.
[5] H.I. Levine, *Elimination of cusps*, Topology 3 (1965), suppl. 2, 263–296.
[6] H. Levine, *Classifying immersions into $\mathbb{R}^4$ over stable maps of 3–manifolds into $\mathbb{R}^2$*, Lecture Notes in Math., Vol. 1157, Springer–Verlag, Berlin, 1985.
[7] R. Myers, *Open book decompositions of 3–manifolds*, Proc. Amer. Math. Soc. 72 (1978), 397–402.
[8] W.D. Neumann, *A calculus for plumbing applied to the topology of complex surface singularities and degenerating complex curves*, Trans. Amer. Math. Soc. 268 (1981), 299–344.
[9] O. Saeki, *Simple stable maps of 3–manifolds into surfaces II*, J. Fac. Sci. Univ. Tokyo 40 (1993), 73–124.
[10] O. Saeki, *Simple stable maps of 3–manifolds into surfaces*, Topology 35 (1996), 671–698.
[11] O. Saeki, *Topology of singular fibers of differentiable maps*, Lecture Notes in Math., Vol. 1854, Springer–Verlag, 2004.
[12] H. Whitney, *On singularities of mappings of euclidean spaces. I. Mappings of the plane into the plane*, Ann. of Math. (2) 62 (1955), 374–410.
Institute of Mathematics for Industry, Kyushu University, Motooka 744, Nishi-ku, Fukuoka 819-0395, Japan

Email address: n-kitazawa@imi.kyushu-u.ac.jp
Email address: saeki@imi.kyushu-u.ac.jp