Origin of primeval seed magnetism in spinning astrophysical bodies

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(Dated: April 27, 2022)

We show that a primeval seed magnetic field arises as a natural consequence of spin-degeneracy breaking of fermions caused by the dragging of inertial frames in the curved spacetime of spinning astrophysical bodies. This seed magnetism would arise even due to electrically neutral fermions such as neutrons. As an example, we show that an ideal neutron star spinning at 500 revolutions per second, having mass 0.83 M⊙ and described by an ensemble of degenerate neutrons would have 0.12 Gauss seed magnetic field at its center arising through the breaking of spin-degeneracy.

PACS numbers: 91.25.Cw, 98.35.Eg

A. Introduction

Magnetic fields are observed in the universe on widely different scales. The field strengths are seen to vary from being as small as 10^{-16} Gauss in the voids of intergalactic medium [1–4] to as large as 10^{15} Gauss in a spinning neutron star [5]. Given a small seed magnetic field, there exists mechanism such as the turbulent dynamo [6, 7] that can rapidly amplify the magnetic field strength in astrophysical bodies. This seed magnetism would arise even due to electrically neutral fermions such as neutrons. As an example, we show that an ideal neutron star spinning at 500 revolutions per second, having mass 0.83 M⊙ and described by an ensemble of degenerate neutrons would have 0.12 Gauss seed magnetic field at its center arising through the breaking of spin-degeneracy.

In this article, we show that a natural answer to the question of seed magnetism arises from two fundamental theories of nature, namely Einstein’s general theory of relativity and Dirac’s theory of fermions when they are put together. This answer does not require any new exotic physics but a proper reconciliation of the Dirac theory together with the general relativity through the methods of quantum field theory in the curved spacetime. In particular, we show that the genesis of seed magnetism is a direct consequence of spin-degeneracy breaking of fermions caused by the curved spacetime of spinning astrophysical bodies, principally due to the dragging of inertial frames. This seed magnetism would arise even due to electrically neutral fermions such as neutrons.

B. Fermions in curved spacetime

In Fock-Weyl formulation, dynamics of a free Dirac fermion $\psi$ in a generally curved spacetime is described by an invariant action

$$S_{\psi} = -\int d^4x\sqrt{-g} \bar{\psi}[i\gamma^\mu e^\mu_a D_a + m] \psi,$$

where Dirac adjoint $\bar{\psi} = \psi^\dagger \gamma^0$ and $m$ is the mass of the fermion. Here $e^\mu_a$ are the tetrad components defined in terms of the global metric $g_{\mu\nu}$ as $g_{\mu\nu} e^a e^b = \eta_{ab}$ where $\eta_{ab} = \text{diag}(-1,1,1,1)$ is the Minkowski metric. Spin-covariant derivative for the fermion field $\psi$ is defined as $D_\mu \psi \equiv \partial_\mu \psi + \Gamma_\mu \psi$ together with the spin connection

$$\Gamma_\mu = -\frac{1}{2} \eta_{ac} e^c (\partial_\mu e^b_a + \Gamma^\nu_{\mu\nu} e^b) [\gamma^a, \gamma^b],$$

where $\Gamma^\nu_{\mu\nu}$ are the Christoffel connections and $\gamma^a$ are the Dirac matrices in Minkowski spacetime, satisfying the Clifford algebra $\{\gamma^a, \gamma^b\} = -2\eta^{ab}$. The minus sign in front of the metric $\eta_{ab}$ here ensures that usual relations $(\gamma^a)^2 = 1$ and $(\gamma^k)^2 = -1$ for $k = 1, 2, 3$ holds true for the chosen metric signature. The Lagrangian density $\mathcal{L}$ corresponding to the action (1) is expressed as $S_{\psi} = \int d^4x\sqrt{-g} \mathcal{L}$.

C. Axially symmetric stationary spacetime

In order to describe the spacetime around a spinning astrophysical body, we consider its matter distribution to be axially symmetric and stationary. In natural units $c = \hbar = 1$, the metric due to an axially symmetric, stationary matter distribution can be expressed as [13]

$$ds^2 = -H^2 dt^2 + Q^2 dr^2 + r^2 K^2 d\theta^2 + \sin^2 \theta (d\varphi - L dt)^2,$$

where metric functions $H$, $Q$, $K$, and $L$ depend on the coordinates $r$ and $\theta$. Here we require these functions to be such that the metric (3) is asymptotically flat. The function $L$ represents acquired angular velocity of a freely-falling observer from infinity, also known as the frame-dragging angular velocity. These metric functions are governed by Einstein’s equation $G_{\mu\nu} = 8\pi G T_{\mu\nu}$. We represent the matter distribution here by a perfect fluid with the stress-energy tensor

$$T_{\mu\nu} = (\rho + P) u_\mu u_\nu + P g_{\mu\nu},$$

where $u^\mu$ is the 4-velocity of the fluid, $P$ is its pressure and $\rho$ is its energy density.
D. Reduced action

In the Einstein equation, the pressure and the energy density are considered to be variables functions of the coordinates in general. However, in equilibrium statistical mechanics, these thermodynamical quantities are considered to be uniform within a given system. These two seemingly disparate aspects therefore need to be combined consistently by using the notion of local thermodynamical equilibrium. In essence, one has to consider two distinct scales — the astrophysical scale over which Einstein’s equation operates and the scale of microscopic physics over which local thermodynamical equilibrium is achieved. In order to ensure a local thermodynamical equilibrium, here we consider a small region around every spatial point such that the variation of the metric functions within the region can be neglected.

For definiteness, let us consider a small spatial box whose center is located at the coordinates \((r_0, \theta_0)\). By defining a new set of local coordinates as \(X = Q_0 r \sin \theta \cos \varphi, \ Y = Q_0 r \sin \theta \sin \varphi, \) and \(Z = Q_0 r \cos \theta\) where \(\theta = \xi_1 \theta, \ \varphi = \xi_0 \varphi\) with \(\xi_1 = (K_0/Q_0), \) \(\xi_0 = (\xi_1 \sin \theta_0)/\sin(\xi_1 \theta_0),\) the metric (5) can be reduced within the box to be

\[
g_{\mu \nu} = \begin{bmatrix} -N^2 & \omega Y & -\omega X & 0 \\ \omega Y & 1 & 0 & 0 \\ -\omega X & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},
\]

where \(\omega = \xi_0 L_0\) and \(N^2 = H_0^2 - (K_0 \sin \theta_0/\xi_0)^2 r_0^2 \omega^2.\)

Here we have assumed that the metric functions take a fixed set of values \(H_0, Q_0, K_0\) and \(L_0\) within the box such that \(Q_0 \geq K_0 \) i.e. \(\xi_1 \leq 1\). Additionally, we have approximated \(\sin^2 \theta \varphi \): \(\sin^2(\theta + \delta \theta) \varphi \approx \sin^2 \theta \varphi \) for all points within the box. In order to ensure \(N^2 > 0\), we note that for a given set of values for \(H_0, Q_0, K_0\), there should be an upper limit on the values of \((r_0 L_0)\).

For most spinning astrophysical systems the frame-dragging angular velocity is small such that \((r_0 L_0) \ll 1\). So for simplicity here onward we shall drop terms which are \(O(\omega^2).\) The non-vanishing tetrad components corresponding to the metric (5), can be expressed as \(e^X_1 = e^Y_2 = e^Z_3 = 1, e^t_0 = N^{-1}, e^X_0 = -\omega Y N^{-1}, e^Y_0 = \omega X N^{-1}\) whereas non-vanishing components of the inverse tetrad \(e^a_\mu\) can be written as \(e^X_1 = e^Y_2 = e^Z_3 = 1, e^t_0 = N, e^t_1 = \omega Y, e^t_2 = -\omega X.\) Non-vanishing components of the Christoffel connection are given by \(\Gamma^X_{Yt} = \Gamma^Y_{Xt} = -\Gamma^Y_{Xt} = \omega.\) Consequently, only non-vanishing component of the spin-connection \(\Gamma^i_{\mu}\) is

\[
\Gamma_t = -\omega \gamma^1, \gamma^2 = \frac{i \omega}{2} \Sigma_3,
\]

where \(\Sigma_3 = \sigma^3 \otimes I_2\) with \(\sigma^3\) being the third Pauli matrix.

Therefore, the Dirac action (1) within the box reduces to

\[
S_\psi = -\int d^4 x \bar{\psi} \gamma^0 \partial_t \psi + N(\gamma^k \partial_k + m) - \omega \gamma^0 \bar{\psi} \psi,
\]

where \(\bar{J}_Z = (\bar{L}_Z + \frac{1}{2} \Sigma_3)\) with \(\bar{L}_Z = -i(\bar{X} \partial_Y - Y \partial_X).\) We note that \(\bar{J}_Z\) can be naturally interpreted as the total angular momentum operator associated with the dragging of inertial frames where \(L_Z\) is the orbital angular momentum operator and \(\sigma^3\) is the third Pauli matrix which is the spin operator along \(Z\)-direction. One may arrive at the reduced action (7) also by considering the transformation of the spinor field \(\psi(x) \rightarrow e^{i \Omega Z \omega t} \psi(x)\) under a rotation around \(Z\) axis with angular velocity \(\omega,\) starting from a non-rotating configuration.

E. Partition function

We consider an ensemble of fermions in the box which is in a local thermodynamical equilibrium. Here we define the scale of temperature \(T\) in the frame of an asymptotic observer where \(N^{-1} \rightarrow 1\) and \(\omega \rightarrow 0.\) This choice allows us to treat the reduced action (7) as an effective action written in the Minkowski spacetime. It leads to a simpler computation as well as it helps to avoid the issues related to Wick rotation in a general curved spacetime [16]. The effective action nevertheless includes the effects of curved spacetime through the fixed parameters \(N\) and \(\omega\) in the given box. In order to compute the partition function, here we follow similar methods as used by the authors for computing equation of states in the curved spacetime of spherical stars [14, 15]. Using coherent states of the Grassmann fields [17–19], the partition function corresponding to the action (7) can be expressed as \(\mathcal{Z}_0 = \int D\bar{\psi} D\psi e^{-S_\psi}\)

where \(S_\psi = \int_0^\beta d\tau \int d^3 x (\mathcal{L}^E - \mu \bar{\psi} \gamma^0 \psi).\) Here \(\mu\) is the chemical potential of the fermion and \(\beta = 1/k_B T\) with \(k_B\) being the Boltzmann constant. The Euclidean Lagrangian density is obtained through a Wick rotation as \(\mathcal{L}^E = -\mathcal{L}(t \rightarrow -it).\)

In a thermal equilibrium, the fermion field \(\psi\) is subjected to the anti-periodic boundary condition \(\psi(\tau, x) = -\psi(\tau + \beta, x)\). Consequently, in Fourier domain the Dirac field \(\psi\) can be written as

\[
\psi(\tau, x) = \frac{1}{\sqrt{V}} \sum_{l, k} e^{-i(\omega_l \tau + k \cdot x)} \bar{\psi}(l, k),
\]

where Matsubara frequencies are \(\omega_l = (2l + 1)\pi \beta^{-1}\) with \(l\) being an integer and \(V = \int d^3 x \sqrt{-g}\) is the volume of the box. Using the reduced action (7) it is convenient to express the partition function as

\[
\mathcal{Z}_\psi = \ln \mathcal{Z}_0 + \ln \mathcal{Z}_l
\]

where \(\mathcal{Z}_0 = \int D\bar{\psi} D\psi e^{-S_\psi}\) and

\[
S_\psi^\beta = \sum_{l, k} \bar{\psi} \psi \beta \left[ \gamma^0 \frac{p + m}{2} \right] \psi,
\]
with \( \tilde{m} = m \mathcal{N} \), \( \tilde{p} = \gamma^0 (i \omega_l - \mu - \frac{2}{3} \Sigma_3) + \gamma^k (k_3 \mathcal{N}) \).

On the other hand, \( Z_L \) can be expressed as a perturbative series \( \ln Z_L = \ln(1 + \sum_{i=1}^{\infty} \frac{\omega_l^i}{i} (-S_L^i)) \) where \( S_L = \int_0^\beta d\tau \int d^4x \tilde{\psi} [\gamma^0 \tilde{L}_z] \psi \). It can be shown that \( \ln Z_L \sim \mathcal{O}(\omega^2) \) and hence it can be ignored. By employing the Dirac representation of \( \gamma^a \) matrices and the results of Gaussian integral over Grassmann fields, one can evaluate the partition function as \( \ln Z_0 = \ln Z_+ + \ln Z_- \) where

\[
\ln Z_\pm = \sum_k \left[ \ln \left( 1 + e^{-\beta (\varepsilon - \mu \pm)} \right) + \ln \left( 1 + e^{-\beta (\varepsilon + \mu \pm)} \right) \right] ,
\]

with \( \varepsilon^2 = \mathcal{N}(k^2 + m^2) \) and \( \mu \pm = \mu \pm (\omega/2) \). To arrive at the expression (10), formally divergent terms including the zero-point energy are dropped. In the equation (10), the first and the second terms correspond to the \textit{particle} and \textit{anti-particle} sectors respectively. Here we consider the spinning astrophysical body to be made of only \textit{particles} for which the partition function becomes

\[
\ln Z_\psi = \sum_k \left[ \ln \left( 1 + e^{-\beta (\varepsilon - \mu \pm)} \right) + \ln \left( 1 + e^{-\beta (\varepsilon + \mu \pm)} \right) \right] .
\]

The presence of Pauli matrices \( \sigma^3 \) in \( \tilde{p} \) leads the partition function (11) to split up in two parts with different energy levels corresponding to the spin-up and the spin-down fermions respectively which in turns breaks the spin-degeneracy of fermions. In the absence of dragging of inertial frames \( i.e. \) if one takes \( \omega \to 0 \) limit, then the partition function (11) reduces to its usual form with the spin-degeneracy factor of 2. However, for any non-vanishing \( \omega \), there exists a gap between the energy levels of the spin-up and the spin-down fermions having same \( \varepsilon \). The energy gap nevertheless is very small. For example, if the frame-dragging angular velocity \( \omega \) is one revolution per second then for neutrons the ratio \( (\mu_+ - \mu_-)/\mu = (\omega/\mu) < (\omega/m) \sim 10^{-24} \).

\[ \text{F. Primeval magnetic field} \]

The number density of fermions that follows from the partition function (11) as \( n = (\beta V)^{-1} (\partial \ln Z_\psi/\partial \mu) \), can be expressed as

\[
n = n_+ + n_- , \quad \text{with } n_\pm = \frac{1}{V} \sum_k \frac{1}{e^{\beta (\varepsilon - \mu \pm)} + 1} .
\]

We note the number densities for the spin-up and the spin-down fermions are different. Consequently, it gives rise to a net magnetic moment \( \mathfrak{M} = \mu_\mathcal{D} (n_+ - n_-) \) where \( \mu_\mathcal{D} \) is the \textit{magnitude} of the magnetic moment of a spin-up Dirac fermion. The corresponding magnetic moment then can be expressed as

\[
\mathfrak{M} = \mu_\mathcal{D} \frac{\beta \omega}{V} \sum_k \frac{e^{\beta (\varepsilon - \mu)}}{e^{(e^{\beta (\varepsilon - \mu)} + 1)^2} + \mathcal{O}(\omega^2)} .
\]

The magnetic field arising due to the magnetic moment (13), can be obtained by computing the magnetic \textit{susceptibility} of the fermionic matter. The susceptibility represents the response of spin degrees of freedom in orienting themselves along the direction of an external magnetic field \( B \) and is defined as \( \chi = (\partial \mathfrak{M} / \partial B)_{B=0} \). In order to compute the magnetic susceptibility, here we consider a test magnetic field \( B \) along the \( Z \)-direction. The coupling between an electrically \textit{neutral} fermion field \( \psi \) and the electromagnetic field \( A_\mu \) is described by Pauli-Dirac interaction term \( S_I = \int d^4x \sqrt{-g} \bar{\psi} \frac{i}{2} \mu_\mathcal{D} \omega \bar{\sigma}^{\mu \nu} F_{\mu \nu} \psi \) where \( \sigma^{\mu \nu} = \frac{i}{2} \epsilon^{\mu \nu \alpha \beta} \gamma_{\alpha} \gamma_{\beta} \) and \( F_{\mu \nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \). With a gauge choice \( A_\mu = (0, 0, B_1 X, 0) \), the interaction term reduces within the box as

\[
S_I = \int d^4x \sqrt{-g} \bar{\psi} \left[ \mu_\mathcal{D} \omega N \Sigma_3 \right] .
\]

For the \textit{particle} sector, the contribution from (14) effectively alters \( \omega \to \bar{\omega} = \omega + 2 \mu_\mathcal{D} N B \) in the action (7) whereas for the \textit{anti-particle} sector it changes \( \omega \to \tilde{\omega} = \omega - 2 \mu_\mathcal{D} N B \). In the context of zero-temperature field theory, it would directly imply that the effect of dragging of inertial frames on the particle sector can be traded off by an external magnetic \( \tilde{B} = \omega/(2 \mu_\mathcal{D} N) \). This aspect holds true even for thermal field theory with non-zero temperature in the leading order in \( \omega \). The magnetic susceptibility now can be computed as \( \chi = 2 \mu_\mathcal{D} \mathcal{N} (\partial \mathfrak{M} / \partial \omega) \) leading to

\[
\chi = 2 \mu_\mathcal{D} \frac{2 \beta \mathcal{N}}{V} \sum_k \frac{e^{\beta (\varepsilon - \mu)}}{(e^{\beta (\varepsilon - \mu)} + 1)^2} + \mathcal{O}(\omega) .
\]

Therefore, the resultant magnetic field in the small box is given by

\[
B = \frac{\mathfrak{M}}{\chi} = \frac{1}{2 \mu_\mathcal{D} \mathcal{N}} \frac{\mathcal{N}_0 L_0}{H_0} + \mathcal{O}(L_0^2) .
\]

The equation (16) establishes that a primeval magnetic field arises spontaneously due to the spin-degeneracy breaking of fermions in the curved spacetime of a spinning astrophysical body. In particular, a non-vanishing frame-dragging angular velocity \( L_0 \) leads to a genesis of the magnetic field \textit{without} requiring any prior magnetic field. This genesis of magnetism works even for electrically neutral fermions such as neutrons. The resultant primeval magnetic field thus can act like a \textit{seed} magnetic field which can be used for subsequent amplification by other astrophysical processes.

\[ \text{G. Slowly rotating neutron star} \]

For quantitative predictions, we now consider a \textit{slowly} rotating ideal neutron star whose degenerate core consists of an ensemble of non-interacting neutrons. This choice also highlights the fact that the seed magnetism arises even due to electrically neutral fermions. In general, a
rotating star has the shape of an oblate sphere. However, for a slowly rotating star its mass and radius can be decomposed into a ‘spherical’ part and a set of non-spherical perturbative corrections which are \(O(\omega^2)\) [13]. So for spherical part we may expand \(\xi\) as \(\xi = 1 + \Delta(\phi)\) and consider only the leading term. Consequently, the spacetime metric of a slowly rotating star can be obtained from the axially symmetric, stationary metric (3) with the following choices of the metric functions

\[
H = e^{\Phi(r)}, \quad Q = e^\nu(r), \quad K = 1, \quad L = \omega(r).
\]

(17)

The equations (16, 17) then lead to a seed magnetic field \(B = \omega e^{-\phi}/(2\mu_D)\). We consider a stellar fluid which is uniformly rotating with angular velocity \(\frac{d\omega}{dr} = \Omega\) with respect to an observer at infinity. Then the 4-velocity of the stellar fluid is \(u^\mu = (e^{-\phi}, 0, 0, \Omega e^{-\phi})\) and non-vanishing components of its co-vector \(u_\mu\) are \(u_t = -e^\phi\) and \(u_r = r^2 \sin^2 \theta (\Omega - \omega) e^{-\phi}\). Here slow rotation means \(\Omega R \ll 1\) with \(R\) being radius of the ‘spherical’ part. The metric function \(\omega\) is governed by \(t-\phi\) component of Einstein’s equation given by [13]

\[
\frac{1}{r^2} \frac{d}{dr} \left( r^2 j \frac{d\omega}{dr} \right) + \frac{4}{r} \frac{d\omega}{dr} (\omega - \Omega) = 0,
\]

(18)

where \(j = e^{-(\nu+r+\phi)}\). The metric function \(\nu\) satisfies

\[
e^{-2\nu} = 1 - 2GM/r \quad \text{with} \quad dM = 4\pi r^2 \rho dr
\]

where \(\rho\) is the density of the fluid at the surface. The equations for \(\Phi\) and the pressure \(P\) are given by

\[
d\Phi = \frac{GM + 4\pi r^3 P}{r(r - 2GM)} \quad \text{and} \quad \frac{dP}{dr} = -(\rho + P) \frac{d\Phi}{dr}.
\]

(19)

The interior metric solutions must match with the exterior vacuum solutions \(e^{2\Phi} = 1 - 2GM/r\) and \(\omega = 2GJ/r^3\) at the star surface. It leads to the following boundary conditions \(e^{2\Phi(R)} = (1 - 2GM/R)\) and \(\omega'(R) = -3\omega(R)/R\) where \(\omega' = (d\omega/dr)\), mass \(M = M(R)\) and angular momentum is \(J\). The requirement of regularity at the center of the star additionally demands \(\omega'(0) = 0\).

By using the curved spacetime of a slowly rotating star, the equation of state for an ensemble of degenerate neutrons has been computed in an accompanying article [20]. Additionally, a numerical method for solving corresponding Einstein’s equation together with the constraints is also described there. By using the said method, the interior solutions for \(\omega\) and \(\Phi\) are plotted in the FIG. 1 for an ideal neutron star with \(\Omega = 500\) revolutions per second. The resultant primeval seed magnetic field is plotted in the FIG. 2 and it can be seen that the seed field could be as high as 0.12 Gauss at the center.

The frame-dragging angular velocity \(\omega\) is non-vanishing even in the exterior of the star. Therefore the equation (16) implies that a fermionic cloud even far away from the star would have a non-vanishing seed magnetic field falling-off as \(\sim (R/r)^3/\sqrt{1 - 2GM/r}\). For example, at a distance of million km from the given neutron star, the seed magnetic field strength would be \(\sim 10^{-17}\) Gauss. Here we may emphasize that the seed magnetism would arise for all spinning astrophysical bodies, not necessarily just for stars. However, for quantitative predictions, one needs to solve the Einstein equation along with its constraints to find the required metric functions for the spinning bodies.

**H. Discussions**

In summary, we have shown that a primeval seed magnetic field naturally arises due to the breaking of spin-degeneracy of fermions, caused by the dragging of inertial
frames in the curved spacetime of spinning astrophysical bodies. This seed magnetism would arise even due to electrically neutral fermions such as neutrons. Further, it shows that the abundance of magnetic field is deeply connected with the abundance of spinning bodies in the universe. We may note from the actions (7, 14) that in the particle sector, the frame-dragging effect changes the energy of a spin-up or a spin-down fermion in the same way as an external magnetic field. However, in the anti-particle sector the frame-dragging effect changes the energy of a spin-up or a spin-down fermion in an opposite manner compared to an external magnetic field.

Acknowledgments

SM thanks IISER Kolkata for support through a doctoral fellowship. GMH acknowledges support from the grant no. MTR/2021/000209 of SERB, Government of India.

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