A PROPOSAL FOR STRINGS AT $D > 1$

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ABSTRACT

Using the reduced formulation of large-$N$ Quantum Field Theories we study strings in space-time dimensions higher than one. We present results on possible string susceptibilities, macroscopic loop operators, $1/N^2$-corrections and other general properties of the model.

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1. INTRODUCTION AND CONCLUSIONS

One of the most interesting open problems in String Theory and Quantum Field Theory is the study of the systems with string excitations in dimensions below the critical dimension. In the context of lattice gauge theories the problem can be formulated in terms of the Schwinger-Dyson equations satisfied by the Wilson-loop observables. In the context of large-$N$ field theory, it was noticed in the early eighties that due to the factorization properties of most gauge invariant observables, the loop-equations of the theory were equivalent to those of a theory on a single hypercube with periodic boundary conditions. These models are known as the large-$N$ reduced models [1,2,3].

In this paper we use the techniques of the reduced models to study non-critical strings in dimensions higher than one. We will find an action in terms of a single matrix which exactly reproduces the partition function of a string moving in 2, 3, 4 space-time dimensions. The model does not appear yet to be exactly solvable, and we can at the moment make only general preliminary remarks concerning its critical properties and its possible continuum limit.

The first lattice formulations of string theories in arbitrary dimensions appeared in the mid-eigthies [4,5,6,7], and there was a good deal of activity in solving some two-dimensional models on random triangulated surfaces [10]. A breakthrough took place with the work in ref. [8] which provided a description in the continuum of the coupling of minimal conformal models to two-dimensional gravity in the light-cone gauge. Their subsequent formulation in the conformal gauge, and the generalization of the results to surfaces of arbitrary topology appeared in [9]. With the discovery of the double scaling limit [11] we begun to understand some of the perturbative and non-pertubative properties of non-critical strings in dimensions below or equal to one. Little progress has been made however in going beyond $d = 1$ apart from some evidence that it is possible to construct Liouville Quantum Field Theory in some special dimensions $d = 7, 13, 19$. Some interesting recent results appear in [13].
Some interesting problems involving the presence of external field have also been studied [14,15,16,17]. In particular the work in [17] attempted to generalize the standard one-matrix model to include the presence of local curvatures in the triangulated surfaces described by the large-\(N\) limit. We will make contact with this work later in sections three and four.

The outline of this paper is as follows: In section two we formulate the problem of strings in arbitrary dimensions in general, we present the general methodology of reduced actions, and the explicit form of the one-matrix model action which reproduces exactly the properties of planar string in \(D\)-dimensions. Some details are given concerning the way to carry out perturbative computations in the context of reduced models.

In section three we continue the analysis of our model and show how in the simplest possible approximation one obtains a second order phase transition between a pure gravity phase with string susceptibility \(\gamma_{st} = -1/2\) and a branched polymer phase with \(\gamma_{st} = +1/2\). This is in agreement with the analysis carried out in [17] for the one-matrix model with world-sheet curvature.

In section four we analyze some general effective actions which could be taken as approximations to our theory. We solve the planar limit of these theories exactly by looking for one-cut solutions. We find generically a second order phase transition between a pure gravity-like phase with negative \(\gamma_{st}\) (with the value \(\gamma_{st} = -1/(m + 1)\), as in the Kazakov multicritical points [7], and a phase that resembles branched polymers with positive string susceptibility \(\gamma_{st} = n/(n+m+1)\), with \(n, m\) arbitrary positive integers. In the approximation we envisage, the \(D\)-dimensional embedding of the string is effectively replaced by some kind of polymeric matter whose role is to break the world-sheet into surfaces oscullating at several points. We have the following phases: a phase where gravity becomes critical and matter remains away from criticality, in which case we obtain the pure gravity regime; a phase where matter becomes critical while gravity remains non-critical, leading to a theory of branched polymers; and finally the most interesting phase where both become
critical simultaneously. This phase is described by the positive string susceptibility \( \gamma_{\text{st}} = n/(n + m + 1) \).

To understand better the geometrical properties of this phase, we compute correlators of macroscopic loop operators on the surface in section five. We show that it is possible to define macroscopic loops in the continuum limit only for a subclass of critical points with \( n = 1 \). We find it quite remarkable that only for the \( n = 1 \) critical points is there enough room on the surface to open arbitrary numbers of loops of arbitrary macroscopic length. For higher values of \( n \), the polymerization of the surface is presumably so complete that no room is left to construct such operators. This leads us to believe that only in this phase we may be able to recover some of the basic properties of non-critical strings.

The continuum limit for \( n = 1 \) contains an extra state with respect to the standard pure gravity case, representing the breaking of the surface into two pieces touching at one point. This extra state is probably the shadow of the effect of the tachyon that one expect to find as soon as string theories are formulated in dimensions higher than one. Whether this feature remains when more realistic effective actions are considered remains to be seen, but we find it rather encouraging. We also explore some of the scaling operators appearing in truncations of the reduced model.

To make sure that the scaling limit we take indeed defines a string theory, in section six we solve the Schwinger-Dyson equations for the approximate effective action on genus one. We find that the David-Distler-Kawai dependence of the string susceptibility on the genus of the surface is recovered to this order. This is encouraging evidence that the continuum limit that we define does describe some kind of non-critical strings.

Although we find the results just summarized encouraging, there are several obvious points in our approximations which should be clarified. First of all, in the effective actions studied in section four there is little evidence of the dependence of the string susceptibility on the string embedding dimensions. In particular, in the
reduced model one of the couplings represents the two-dimensional cosmological constant. There is a similar coupling in the effective action for the reduced model which plays a similar role. What needs to be done is to show that the relation between these two couplings is analytic, at least for some range of parameters. The conclusions we draw in sections four and five depend implicitly on this assumption, and this is an important caveat to keep in mind when considering the work presented. The most natural thing to do seems to be to try to settle these questions by a direct renormalization group analysis of the reduced model along similar lines to those suggested in [18], or to construct an approximate Migdal-Kadanoff-type transformation. Work in this direction is in progress. We hope to convince the reader that the study of non-critical strings in dimensions higher than one, using reduced large-$N$ model is worth pursuing.

While this work was being completed we received several preprints having some overlap with our work [19].

2. LARGE-$N$ REDUCED THEORIES AND MATRIX MODELS

In the study of strings propagating in flat $D$-dimensional space-time, we represent the sum over two-dimensional metrics in Polyakov's approach to string theory [20] by a sum over triangulated surfaces. The quantity we would like to evaluate [4,6,7] is

$$Z = \sum_{g} \kappa^2 (g-1) \sum_{T} \frac{e^{-\mu|T|}}{|n(T)|} \int \prod_{i \in T_0} \sigma_i^0 d^D X_i \prod_{\langle ij \rangle \in T_1} G(X_i - X_j) \quad \quad (2.1)$$

Where $g$ is the genus of the triangulation. For a given triangulation $T$, $T_0, T_1, T_2$ are respectively the sets of vertices, edges and faces of $T$, $n(T)$ is the order of the symmetry group of $T$, $|T|$ is the total area of $T$ counting that every triangle has unit area. $\mu$ is the bare two-dimensional cosmological constant, $X_i$ describes the embedding of the triangulation into $D$-dimensional flat space-time and $G(X-Y)$ is the propagator factor for each link. We have included also the local volume factor.
\[ \sigma_i^\alpha \] at site \( i \) to represent the effect of local curvature. Usually we take \( \alpha = D/2 \), but it is more convenient to leave this exponent arbitrary to include the effect of local curvature terms on the world-sheet. For a triangulation, if \( q_i \) is the local coordination number at site \( i \) (the number of triangles sharing this vertex),

\[ \sigma_i = \frac{q_i}{3}, \]

and the volume factor in the measure can be shown to be related in the continuum to an expansion of the form:

\[ \sum_i \ln \sigma_i = c_0 + c_1 \int \sqrt{g} R + c_2 \int \sqrt{g} R^2 \ldots, \]

thus representing some of the effects of world-sheet curvature. The general features of the phase diagram in the \((D, \alpha)\) plane were studied for example in [21]. In the standard Polyakov formulation, the propagator is Gaussian,

\[ G(X) = e^{-X^2/2} \]

By standard large-N analysis the sum (2.1) can be transformed [6,7] into the large-\( N \) expansion of a matrix field theory in \( D \)-dimensions

\[ Z = \lim_{N \to \infty} \log \int D\phi \exp \left( -N \int d^DXT(\frac{1}{2}\phi G^{-1}\phi + \frac{1}{3}g\phi^3) \right) \quad (2.2) \]

It is difficult to proceed very far with Gaussian propagators. If we work in dimensions \( D < 6 \) it should not matter whether we replace the Gaussian propagator by the Feynman propagator. There is numerical evidence that this change does not affect the critical properties of the theory below six dimensions [22]. Notice that in this construction the exponent \( \alpha \) in (2.1) is set to zero. If we want to include the effect of local world-sheet curvature, we can follow [17] and change the kinetic
term in (2.2). The two-dimensional cosmological constant is here represented by 
\( g, g = e^{-\mu} \). To summarize, we want to study the critical properties of the action:

\[
Z = \int D\phi(X) \exp \left( -N \int d^D X Tr \left( \frac{1}{2} \partial_\mu \phi \partial^\mu \phi + \frac{g}{3} \phi^3 \right) \right), \quad (2.3)
\]

or

\[
Z = \int D\phi(X) \exp \left( -N \int d^D X Tr \left( \frac{1}{2} A \partial_\mu \phi A \partial^\mu \phi + \frac{g}{3} \phi^3 \right) \right), \quad (2.4)
\]

where \( A \) is a constant \( N \times N \) matrix and \( \phi \) is an \( N \times N \) matrix field. The reason why the \( A \)-matrix in the kinetic term simulates the effect of local curvature can be seen by writing the propagator in (2.4) explicitly

\[
\langle \phi_{ij}(X) \phi_{kl}(Y) \rangle = A_{jk}^{-1} A_{li}^{-1} G(X - Y).
\]

If we ignore for the time being the propagator factor \( G \), for every closed index loop in a generic \( \phi^3 \) graph made of \( q \) propagators, we obtain a contribution of \( Tr A^{-q} \). Since the \( \phi^3 \) graphs are dual to triangulation, this means that we are associating a curvature factor of \( Tr A^{-q} \) to the vertex dual to the face considered. In this way we can simulate the presence of the \( \sigma^\alpha \) term in the measure in (2.2). We are interested in particular in the computation of the string susceptibility exponent

\[
\chi \sim \frac{\partial^2 F}{\partial g^2} \sim (g - g_c)^{-\gamma s}, \quad (2.5)
\]

where \( F \) is the free energy of the system, and \( g_c \) is the critical value of \( g \) indicating the location of the critical point. Apart from the string susceptibility we would also like to compute the spectrum of scaling operators and the properties of the quantum geometry implied by (2.3)(2.4).

To simplify the arguments we concentrate on the study of planar configurations (spherical topologies). From work dating back to the late seventies [23], and due to the factorization properties of the leading large-\( N \) approximation, the planar
limit is dominated by a single constant configuration; Witten’s master field (a master orbit in the case of gauge theories). This idea is explicitly realized in 0-dimensional matrix models [24]. In the lattice gauge theories one can think of the reduced Eguchi-Kawai (EK)[1] or Twisted Eguchi-Kawai (TEK)[2] models as explicit descriptions of the master field. We now briefly describe the main ideas behind the reduced models. Further details can be found in the previous references and in the review article [3]. These models were originally formulated to describe the large-$N$ properties of lattice gauge theories.

Due to the factorization property in the large-$N$ limit, the Schwinger-Dyson equations for Wilson-loops form a closed system of equations in the planar limit. For gauge theories we get an infinite set of polynomial equations for the Wilson-loops [25]. In terms of the standard link variables $U_\mu(x)$, the EK proposal consists of making them independent of $x$, $U_\mu(x) \mapsto U_\mu$, and the action of the reduced model becomes

$$S_{EK} = \frac{1}{\text{Vol.}} S(U_\mu(x) \mapsto U_\mu). \quad (2.6)$$

Some of the properties of this action are:

1). We obtain a theory on a single hypercube with periodic boundary conditions. This is an important simplification of the problem.

2). In the case of gauge theories, the gauge symmetry becomes a global symmetry, $U_\mu \mapsto \Omega U_\mu \Omega^{-1}$.

3). There is an extra $U(1)^D$ symmetry in the case of $U(N)$ lattice gauge theory $U_\mu \mapsto e^{i\theta_\mu} U_\mu$. In the case of $SU(N)$ the symmetry becomes $Z_N^D$.

4). It is possible to show that the planar Schwinger-Dyson equations following from the reduced action coincide with those obtained from the original Wilson theory as long as open loop expectation values vanish. This can be shown to be true at strong coupling, but it does not hold at weak coupling [26]. Without this problem we would have a beautiful implementation of the Master Field idea, because the loop equations for the 1-site EK model are identical with the standard
ones. The problem is connected with the fact that at weak coupling the extra $U(1)^D$ symmetry is broken. Open loops have non-trivial charge with respect to this symmetry. Only if the symmetry remains unbroken are we guaranteed to maintain the loop equations without extraneous terms.

The resolution of this problem motivated the formulation of the TEK model [2]. It is inspired on 'tHooft's use of twisted boundary conditions in gauge theories [27]. For any matrix theory the TEK prescription is exceedingly simple. We reduce according to

$$\phi(x) \mapsto D(x)\phi D(x)^{-1}, \quad (2.7)$$

where $D(x)$ is a projective representation of the $D$-dimensional lattice translation group. We have to choose a set of $D$ $N$-dimensional matrices $\Gamma_\mu$ such that:

$$D(x) = \prod_\mu \Gamma_\mu^{x_\mu}, \quad x = (x_1, x_2, \ldots, x_D), \quad x_i \in \mathbb{Z}, \quad (2.8)$$

since $D(x)$ has an adjoint action on the fields $\phi$, the $\Gamma_\mu$'s are required to commute only up to an element of the center of $SU(N)$,

$$\Gamma_\mu \Gamma_\nu = Z_{\nu \mu} \Gamma_\nu \Gamma_\mu, \quad (2.9)$$

and the integers $n_{\mu \nu}$ are defined mod $N$. The reduced action prescription is now

$$S_{TEK} = \frac{1}{Vol.} S(\phi(x) \mapsto D(x)\phi D(x)^{-1}), \quad (2.10)$$

and similarly for expectation values.

To avoid open loop expectation values and a mismatch between the original and reduced Schwinger-Dyson equations, the matrices $\Gamma_\mu$ must verify some conditions. In particular they should generate an irreducible representation of the group of
lattice translations. For a lattice with $L^D$ sites this requires $N = L^{D/2}$. Hence $N = L$ for $D = 2$ and $N = L^2$ for $D = 4$. Hence, if we want to simulate a two-dimensional lattice with $64 \times 64$ sites it suffices to consider the group $SU(64)$. The choice of twist matrix $n_{\mu\nu}$ depends on the dimensionality. In two-dimensions, the simplest choice is given by

$$n_{\mu\nu} = \epsilon_{\mu\nu}. \quad (2.11)$$

The explicit form of the four and higher dimensional twists can be found in the quoted literature.

We now set aside lattice gauge theory and return to our problem. The twisted reduced version of (2.3) becomes:

$$S = \frac{1}{2} \sum_{\mu} Tr(\Gamma_\mu \phi \Gamma_\mu^{-1} - \phi)^2 + TrV(\phi). \quad (2.12)$$

The equivalence of the planar approximation to (2.3) with (2.12) follows from earlier work on reduced models [2,28]. For simplicity we consider the two-dimensional case. First, since for $k_\mu = \epsilon_{\mu\nu} q_\nu$ and $q_\nu$ defined modulo $N$,

$$A(q) = \Gamma_1^{k_1} \Gamma_2^{k_2}$$

are $N^2$ linearly independent matrices, we can expand the matrix $\phi$ in the $\Gamma$-basis according to

$$\phi = \sum \phi_q A(q).$$

Some useful properties of the $A$-matrices are:

$$A(q)^\dagger = A(-q)e^{2 \pi i N \langle k|k \rangle}, \quad \langle k^i|k^j \rangle = \sum_{\mu<\nu} n_{\mu\nu} k^i_\mu k^j_\nu = k^i_2 k^j_1, \quad (2.13)$$

$$A(q^1, \ldots, q^n) = A(q^1 + \cdots + q^n) \exp \left( \frac{2\pi i}{N} \sum_{i<j} \langle k^i|k^j \rangle \right), \quad (2.14)$$
and
\[ \Gamma_{\mu}A(q)\Gamma_{\mu}^\dagger = e^{2\pi i q_{\mu}/L}A(q), \]
explicitly showing how the \( \Gamma \)'s implement the lattice translation group. Furthermore,
\[ Tr(A(q)^\dagger A(q')) = N\delta_L(q-q'), \]
\[ Tr(A(q^1)\ldots A(q^n)) = N\delta(\sum q^i)\exp\left(\frac{2\pi i}{N}\sum_{i<j}\langle k^i|k^j\rangle\right), \]
where the \( \delta \)-function is defined mod \( L \). Using these properties, the kinetic term of our action becomes
\[ \frac{1}{2}\sum_{\mu} Tr(\Gamma_{\mu}\phi\Gamma_{\mu}^{-1} - \phi)^2 + \frac{1}{2}m^2 Tr\phi^2 \sim \]
\[ N\sum_q \left[ \frac{1}{2}m^2 + \sum_{\mu}(1 - \cos(\frac{2\pi q_{\mu}}{L})) \right]\phi_q\phi_{-q}e^{-2\pi i\langle k|k\rangle/N}, \quad (2.13) \]
coinciding with the standard lattice propagator of a scalar field on a size \( L D \)-dimensional lattice up to the last phase factor in (2.13).

The role of this and similar phase factors appearing in the vertices of the theory is to restore the topological expansion in large-\( N \). We have now obtained an effective field theory with scalar variables \( \phi_q \). The phase factors keep track of the fact that the theory came from a matrix field theory and that the expansion in powers of \( 1/N \) can be represented in terms of a topological expansion where the leading order corresponds to spherical topologies, and higher orders come from surfaces of increasing Euler number.

In particular, the phases of all planar graphs vanish. The easiest way to see that is to note that the phase factor does not change when we contract all the propagators connecting two vertices. The two vertices contribute factors \( Tr[A(q_1)\ldots A(q_n)A(q'_1)\ldots A(q'_{n'})] \).
and \( Tr[A(-q'_1) \ldots A(-q'_{n'})A(q''_r) \ldots A(q''_{n''})] \) while the phase factor coming from the propagators is \( \exp \left( \frac{2\pi i}{N} \sum_{i'=1}^{n'} \langle k'_i|k'_j \rangle \right) \). All together, the phase factor is

\[
\delta(\sum_i q_i + \sum_i q'_i) \delta(-\sum_i q'_i + \sum_i q''_i) \exp \left[ \frac{2\pi i}{N} \left( \sum_{i<j} \langle k_i|k_j \rangle + \sum_{i,j} \langle k'_i|k'_j \rangle \right) \right.
\]

\[
+ \sum_{i<j} \langle k'_i|k'_j \rangle + \sum_i \langle k'_i|k''_j \rangle - \sum_{i,j} \langle k'_i|k''_j \rangle + \sum_{i<j} \langle k''_i|k''_j \rangle + \sum_{i<j} \langle k'_i|k''_j \rangle \right]
\]

\[ \propto \exp \left[ \frac{2\pi i}{N} \left( \sum_{i<j} \langle k_i|k_j \rangle + \sum_{i,j} \langle k'_i|k'_j \rangle + \sum_{i<j} \langle k''_i|k''_j \rangle \right) \right], \]

where we used the \( \delta \)-function constraints. This is precisely the phase factor of a single vertex with \( \{k_i\}, \{k''_j\} \) propagators attached. Provided we are dealing with a planar graph, we can repeat this procedure until we end up with a single vertex with no propagators attached whose phase factor is trivially zero.

In case of non-planar graphs we will not be able to contract the propagators winding around non-trivial homology cycles. For example, it is easy to see that the phase corresponding to the torus diagram consisting of two propagators with momenta \( q_1 \) and \( q_2 \), each following one of the two non-trivial homology cycles and meeting in one vertex, is

\[
\exp \left[ \frac{2\pi i}{N} (\langle k_1|k_2 \rangle - \langle k_2|k_1 \rangle) \right].
\]

The non-vanishing phases of non-planar diagrams result in their \( 1/N^2 \)-suppression. Finally, note that (2.13) shows that the space-time degrees of freedom of the original theory are coded in the Fourier transforms of the “internal” \( SU(N) \) indices.

The construction we have just carried out works only for even dimensions. For odd-dimensions we can leave one of the dimensions unreduced, and apply the reduction prescription to the remaining even number of dimensions. This yields an action

\[
S = \int dt \frac{1}{2} Tr \dot{\phi}(t)^2 + \frac{1}{2} \int dt Tr(\Gamma_\mu \dot{\phi}(t)\Gamma_{\mu}^{-1} - \phi)^2 + \int dt Tr V(\phi(t)).
\]

We can consider a slightly more general model by including a general coupling in
the hopping term:

$$-S(\phi) = \sum_{\mu} \frac{1}{2} R^2 Tr \phi \Gamma_\mu \phi \Gamma_\mu^\dagger - \frac{1}{2} m^2 \phi^2 - Tr V_0(\phi),$$

$$V_0(\phi) = \frac{g_3}{\sqrt{N}} \phi^3 + \frac{g_4}{N} \phi^4 + \ldots, \quad m^2 = 2D R^2 + m_0^2. \quad (2.14)$$

The hopping term is the first term in $S(\phi)$. Adding the effect of world-sheet curvature is easy, we simply replace the hopping term according to

$$Tr(\Gamma_\mu \phi \Gamma_\mu^\dagger - \phi)^2 \rightarrow Tr A(\Gamma_\mu \phi \Gamma_\mu^\dagger - \phi) A(\Gamma_\mu \phi \Gamma_\mu^\dagger - \phi) \quad (2.15)$$

Our proposal to describe strings in dimensions higher than one is to investigate the properties of (2.14) and (2.15). We have effectively reduced the problem to a one-matrix model. The difficulty lies however in the presence of the twist matrices. We begin our analysis in the following section.

3. PROPERTIES OF THE REDUCED STRING ACTION

Some general features of (2.14) and (2.15) are easy to extract. If the hopping term is very small, we are in the limit where the space-time lattice points are very far apart and the whole surface is mapped into a single point, or various points incoherently. This is the pure gravity regime, $\gamma_{st}$ is expected to be $-1/2$ and the induced metric plays no role. As the hopping term is increased, it becomes more likely for the surface to occupy more and more space-time lattice sites, the induced metric and singular embeddings in principle begin to play important roles, and we can expect a transition to a phase different from pure gravity at some critical value of the hopping coupling $R^2$. We will show presently that in the most na"ive approximation we are driven to a branched polymer phase with $\gamma_{st} = 1/2$, and at the transition point $\gamma_{st} = 1/3$. A more careful analysis will show a much broader spectrum of possibilities. Note that the role played by $R$ is similar to the compactification radius in the study of $c = 1$ strings propagating in a circle of radius $R$ [29].
To deal with (2.14) we decompose the matrix $\phi$ into its eigenvalues and angular variables as usual

$$\phi = U^{-1} \lambda U \quad \lambda = \text{diag}(\lambda_1, \lambda_2, \ldots, \lambda_N),$$

The angular integration produces an effective action

$$e^{\Gamma_{\text{eff}}(\phi)} = \int dU e^{R^2 \sum_\mu Tr \lambda \Gamma_\mu(U) \lambda^{\dagger}}(U)$$

$$\Gamma_\mu(U) = U \Gamma_\mu U^{-1}$$

$$Z = \int d\phi e^{\Gamma_{\text{eff}}[\phi] - V_0(\phi)}$$

(3.1)

The simplest representation of $\Gamma_{\text{eff}}$ is given in terms of the leading order expansion in $R^2$:

$$Z = \int d\phi e^{-N(R^2 \frac{2}{N} (Tr \phi)^2 - V_0(\phi))}.$$  

(3.2)

The critical properties of (3.2) are very similar to those of the curvature model in [17]. What the first term in $\Gamma_{\text{eff}}$ is describing is the breaking of the surface. The Feynman graph representation of a term like $(Tr \phi^n)^2$ is a vertex where two surfaces osculate at one point, and the coordination number of the touching vertices in each of the surfaces is $n$. If we have instead a term of the form $(Tr \phi^n)^p$, it represents $p$-surfaces sharing one point. When there are no products of traces, the Feynman graph expansion for the free energy or the connected correlators involves only the sum over connected orientable surfaces. When products of traces are included we obtain an expansion where we have surfaces of different topologies touching at points determined by the new vertices. The new graphs look like “daisies”. In the original curvature model introduced in [17] the initial partition function is

$$Z = \int d\phi e^{-N(\frac{1}{2} \phi A \phi A + g Tr \phi^4)}$$

(3.3)

The angle integral is again very difficult to carry out in this model, and the simplest
non-trivial approximation is to study the action:

\[ S_{\text{eff}} = N \left( \frac{1}{2} Tr\phi^2 + g Tr\phi^4 + \frac{g'}{N}(Tr\phi^2)^2 \right). \]

For real \( A \), the coupling \( g' = (TrA^2/N - (TrA)^2/N^2) \) is always positive. In this case as we will see later one cannot escape the pure gravity phase. To leading order in \( 1/N \), the graphs have the topology of spheres touching at one point, where any two spheres in the graph can share at most one common point, and no closed loop of spheres can appear. A generic graph looks like a tree of spheres. In the planar approximation it was shown in [17] that we have the following possibilities:

1. \( g' > 0, \gamma_{st} = -1/2 \) and we are in the pure gravity regime.

2. \( g_c < g' < 0 \). Again \( \gamma_{st} = -1/2 \) and we continue in the pure gravity case. The branching is still unimportant, and the leading contributions to the free energy come from graphs with few big spheres and few small branchings.

3. At \( g' = g_c = -9/256, \gamma_{st} = +1/3 \) and the number of branchings become competitive with the number of smooth spheres.

4. For \( g < g_c \) we obtain \( \gamma_{st} = +1/2 \) and the dominating term is the one with a product of two traces, thus the notion of a smooth surface disappears and we end up with a branch polymer phase.

The same analysis can be carried out with our naïve approximation (3.2), and we will only note the differences with respect to the analysis in [17] we have just reviewed. The basic differences are:

1. Our potential is cubic, \( V_0 = \frac{1}{2} \phi^2 + g\phi^3/\sqrt{N} \), the one-cut solution is asymmetrical, and the new vertex is just given by \((Tr\phi)^2\). These are purely technical differences and nothing in the analysis changes substantially.

2. We are always in the \( g' < 0 \) region. In the original model in [17] we can reach this region only after analytically continuing the matrix \( A \mapsto iA \) in order to obtain the polymer phase. However, in this case the interpretation of the matrix \( A \) in terms of local world-sheet curvature loses its meaning.
Our simplified model then contains a pure gravity phase, an intermediate phase with $\gamma_{st} = 1/3$, and a branched polymer phase with $\gamma_{st} = 1/2$. We should like to remark that the exact solution to the pure curvature model (3.3) is known only for Penner’s potential [30], however in this case the full theory has the symmetry $A \mapsto \text{const.} A^{-1}$ making the interpretation of $A$ in terms of world-sheet curvature rather doubtful.

4. PRELIMINARY ANALYSIS OF THE EFFECTIVE ACTION

We have learned that the basic problem is the evaluation of the angular integral

$$e^{\Gamma_{\text{eff}}} = \int dU e^{R^2 \sum_\mu \text{Tr} \lambda U \Gamma_\mu U^{-1} \lambda U \Gamma_\mu^\dagger U^{-1}}.$$  \hspace{1cm} (4.1)

The naïve approximation embodied in (3.2) already gave us a polymer phase and a critical point in-between pure gravity and polymers. To understand the behavior of strings in dimensions higher than one, we need to obtain information about the general critical properties of the effective action in (4.1). Since $R^2$ is a hopping parameter, we can write the hopping term as $L_\mu = \phi \Gamma_\mu \phi \Gamma_\mu^\dagger$ and in analogy with standard lattice analysis, we could write an approximation to the effective action in terms of a sum over connected graphs of angular averages over the hopping terms generating the graph:

$$\Gamma_{\text{eff}} = \sum_{C,\text{Connected}} \langle \text{Tr} \prod_{l \in C} L(l) \rangle_U,$$

where the subscript $U$ indicates that the average is taken only over the angular variables. To every connected graph the average over angles associates a set of products of traces. We can interpret these traces as different ways of fracturing and breaking the loop $C$. They describe how the embedded surface is collapsed into pieces. If we add a local curvature term (the $A$-matrix), we will obtain more general operators. Since we do not have an explicit way of evaluating (4.1) and
since we are interested in the critical behavior of (2.14) and not in the fine details of
lattice dynamics, it is worth exploring the type of universality classes of potentials
with arbitrary numbers of products of traces. There are two qualitative classes of
potentials, depending on whether they contain a finite or infinite number of traces.
So far we have analyzed the case of finite numbers of traces.

Defining
\[ t_i = \frac{1}{N} Tr \phi^{2i}, \]  
(4.2)
we take the effective action to be a general function of an arbitrary, but finite
number of \( t_i \) variables,
\[ \Gamma_{\text{eff}} = V(t_1, t_2, \ldots, t_n) = V\left(\frac{1}{N} Tr \phi^2, \frac{1}{N} Tr \phi^4, \ldots, \frac{1}{N} Tr \phi^{2n}\right). \]
(4.3)
Thus we are faced with the analysis of the critical properties in the planar limit of
the action
\[ S(\phi) = N^2 V(t_i) + N T r V_0(\phi), \]
(4.4)
\[ V_0 = \sum_k g_k Tr \phi^{2k}. \]
The world-sheet cosmological constant is related to the the coupling \( g_2 = e^{-\mu B} \).
We have made the simplifying assumption of restricting our considerations to the
case of even potentials. This is just a question of technical simplicity, and nothing
changes if we take general potentials. The analysis can be carried out for any of
the Kazakov potentials [31].

The solution of the planar approximation of (4.4) uses the Hartree-Fock (HF)
approximation, which becomes exact in the planar limit. We first write down
the saddle-point equations for (4.4), and then we replace the traces (4.2) by their
planar vacuum expectation values \( x_i \equiv \langle \frac{1}{N} Tr \phi^{2i} \rangle \). This reduces the problem to
the original pure gravity case studied in [24]. Finally we fix the variables \( x_i \) self-
consistently. The saddle-point equations and the HF conditions are given by

\[
\frac{1}{2} \sum_p 2p \tilde{g}_p \lambda^{2k-1} = \text{P.V.} \int d\mu \frac{\rho(\mu)}{\lambda - \mu}, \tag{4.5}
\]

\[
\tilde{g}_p = g_p + \frac{\partial V(x)}{\partial x_k},
\]

\[
x_k = \int \rho(\mu) \mu^{2k}. \tag{4.6}
\]

\(\rho(\mu)\) is the density of eigenvalues. We will denote by \(\tilde{V}\) the potential with the shifted couplings (4.5).

Following the pure gravity case, we look for one-cut solutions of (4.5). This means that the loop operator

\[
F(p) = \int d\mu \frac{\rho(\mu)}{p - \mu}
\]

is given by

\[
F(p) = \frac{1}{2} \tilde{V}'(p) - M(p) \sqrt{p^2 - R}. \tag{4.7}
\]

The polynomial \(M(p)\) is determined by requiring that at large values of \(|p|\), \(F(p) \sim 1/p + \mathcal{O}(1/p^3)\). Since \(V\) depends only on a finite number of traces, the potential \(\tilde{V}\) is a polynomial, and there are no ambiguities in the determination of \(M\).

Define

\[
G(t) = pF(p)|_{p^2=t} = 1 + x_1(R)t + x_2(r)t^2 + \cdots.
\]

The condition \(x_0 = 1\) is the string equation. Since \(G(t)\) is analytic around \(t = 0\),

\[
\oint_C t^l G(t)(1-tR)^{-1/2} = 0, \tag{4.8}
\]
for \( l \geq 0 \) and \( C \) a circle around \( t = 0 \) with radius less than \( 1/R \). Using (4.7) and

\[
\frac{1}{\sqrt{1 - Rt}} = \sum_{k \geq 0} \binom{2k}{k} \left( \frac{Rt}{4} \right)^k,
\]

the equation (4.8) results in \( M(p) = \sum_{m \geq 0} M_mp^m \) with

\[
M_m = \sum_{k \geq 0} \binom{2k}{k} \left( \frac{R}{4} \right)^k (m + k + 1)\bar{g}_{m+k+1}.
\]

The density of eigenvalues is given by

\[
\rho(\lambda) = \frac{1}{2\pi} M(\lambda) \sqrt{R - \lambda^2}, \quad \lambda \in [-\sqrt{R}, \sqrt{R}].
\]

In the following we will also need expressions for \( x_p(R) = \langle \frac{1}{N} Tr\phi^{2p} \rangle \). To derive those, we find it convenient to distinguish between two sources of \( R \)-dependence: the explicit \( R \)-dependence that would be here even for pure gravity, i.e. for \( \bar{g}_k = g_k \), and the implicit \( R \)-dependence of \( x_p(R) \) (and therefore of \( \bar{g}_k \)), generated by the HF feedback. Consequently, we will use two kinds of derivatives w.r.t. \( R \): the usual derivative \( \partial/\partial R \equiv \partial_R \), taking into account both sources of \( R \)-dependence, and \( D_R \), derivative w.r.t. \( R \) carried out while ignoring the implicit HF dependence of \( x_k \). In particular,

\[
D_RG(t) = \frac{1}{\sqrt{1 - Rt}} \left[ \frac{M(1/t)}{2} - D_R M(1/t) \left( \frac{1}{t} - R \right) \right].
\]

Since \( \sqrt{1 - Rt}D_RG(t) \) is a series in positive powers of \( t \), the only term from the square brackets on the r.h.s. of (4.12) that contributes is \( M_0/2 + RD_RM_0 \). Thus,

\[
D_Rx_p = \int t^{-p-1} \frac{M_0/2 + RD_RM_0}{\sqrt{1 - Rt}}
\]

\[
= \left( \frac{M_0}{2} + RD_RM_0 \right) \binom{2p}{p} \left( \frac{R}{4} \right)^p,
\]

\[
= \left( \frac{2p}{p} \right) \left( \frac{R}{4} \right)^p D_Rx_0.
\]
Finally, using the explicit expression for $M_0$ from (4.10) we arrive at

$$x_p(R) = \sum_{k \geq 1} \frac{k^2}{p + k} \binom{2p}{p} \binom{2k}{k} \left( \frac{R}{4} \right)^{k+p} \tilde{g}_k.$$ (4.14)

The string equation is given by (4.14) with $p = 0$. We are interested in the analysis of the critical points of (4.4), the computation of string susceptibilities, and in the computation of correlation functions of macroscopic loop operators. If we denote by $\beta$ the coupling $g_2$, the string susceptibility is determined from

$$\chi = \frac{dx_2}{d\beta} = \frac{\partial x_2}{\partial \beta} + \frac{dR \partial x_2}{d\beta \partial R}.$$ (4.15)

This is because the coupling $\beta$ multiplies the operator $t_2$ in the action. Therefore $\partial \ln Z/\partial \beta$ is proportional to the expectation value $x_2$, and the second derivative of the logarithm of the partition function is the string susceptibility.

The dependence of $R$ on $\beta$ is read off from the string equation $x_0 = 1$. In particular,

$$0 = \frac{dx_0}{d\beta} = \frac{\partial x_0}{\partial \beta} + \frac{dR \partial x_0}{d\beta \partial R}$$

allows us to read the equation determining the critical points,

$$\frac{d\beta}{dR} = -\frac{\partial R x_0}{\partial \beta x_0}.$$ (4.16)

Since the denominator in (4.16) is proportional to $R^2$, the critical points are determined by the zeroes of the numerator. We can similarly write $\chi$ according to

$$\chi = \partial_\beta x_2 - \partial_\beta x_0 \frac{\partial R x_2}{\partial R x_0}.$$ (4.17)

The singularities in the behavior of $\chi$ come from the ratio of derivatives in (4.17). For the standard Kazakov critical points [31] (4.17) does not blow up at
the critical point $R_c$, and $\chi - \chi_c \sim R - R_c$. Since at the $m$-th critical point $\beta - \beta_c \sim (R - R_c)^m$ this yields $\gamma_{st} = -1/m$. As we will show presently, the critical points we describe have $\gamma_{st} > 0$ and $\chi$ will be singular at the critical point. All the phase transitions are second order.

To derive the criticality conditions, use (4.14) to write the complete derivative of $x_p$ w.r.t. $R$ as

$$\partial_R x_p = D_R x_p + \sum_{k \geq 1} \frac{k^2}{k + p} \binom{2k}{k} \binom{2p}{p} \left( \frac{R}{4} \right)^{k+p} \partial_k g_k$$

$$= D_R x_p + \sum_{k \geq 1} \frac{k^2}{k + p} \binom{2p}{p} \left( \frac{R}{4} \right)^p \binom{2k}{k} \left( \frac{R}{4} \right)^k \sum_q \partial_k V \partial_R x_q.$$  \hspace{1cm} (4.18)

Defining the matrix

$$U''_{pq} = \sum_{k \geq 1} \frac{k^2}{k + p} \binom{2p}{p} \left( \frac{R}{4} \right)^p \binom{2k}{k} \left( \frac{R}{4} \right)^k \partial_k V,$$

we can compute $\partial_R x_p$ in terms of $D_R x_0$:

$$\partial_R x_p = [(1 - U'')^{-1} A]_p D_R x_0,$$  \hspace{1cm} (4.20)

where $A$ is a vector whose $p$-th component equals $\binom{2p}{p} (\frac{R}{4})^p$. When $p = 0$, we obtain the criticality condition implied by (4.16):

$$\partial_R x_0 = [(1 - U'')^{-1} A]_0 D_R x_0 = 0.$$  \hspace{1cm} (4.21)

Notice that the criticality condition splits into two terms. The first one is related to the function including the couplings with multiple traces,

$$[(1 - U'')^{-1} A]_0 = 0,$$  \hspace{1cm} (4.22)

and the second is equivalent to the criticality condition for the Kazakov critical
points:

$$D_R x_0 = 0. \quad (4.23)$$

The term in (4.17) which may lead to singularities in $\chi$ and to positive $\gamma_{st}$ is given by

$$\frac{\partial Rx_2}{\partial Rx_0} = \left[(1 - U'')^{-1}A\right]_2 \left[(1 - U'')^{-1}A\right]_0. \quad (4.24)$$

In (4.21) there are three possibilities: 1). The polymer couplings in $U$ become critical but the gravity part does not. The zeroes of $d\beta/dR$ come only from the polymer contribution. 2). The gravity contribution is the only one generating zeroes of $d\beta/dR$. 3). Both terms become critical. In the first case we have a theory of polymers, and little is remembered of the coupling to gravity. In the second case the polymer degrees of freedom are completely frozen and we reproduce the string susceptibility and exponents of the Kazakov critical points. The third and more interesting case is when both the “polymer” matter and gravity become critical simultaneously. This is the case where we can find novel behavior. Notice also that unless we tune the numerator in (4.24) we will generically obtain $\gamma_{st} > 0$ in cases 1) and 3).

If $R_c, \beta_c$ are the critical values of $R$ and $\beta$, and if near the critical point

$$[(1 - U'')^{-1}A]_0 \sim (R - R_c)^n \quad D_Rx_0 \sim (R - R_c)^m, \quad (4.25)$$

then

$$\beta - \beta_c \sim (R - R_c)^{n+m+1}. \quad (4.26)$$

If we do not tune the numerator of (4.24) to partially cancel the zeroes in the denominator, we obtain

$$\chi \sim \frac{1}{(R - R_c)^n}. \quad (4.25)$$
The previous two equation give a string susceptibility for the \((n, m)\)-critical point:

\[
\gamma_{st} = \frac{n}{n + m + 1}.
\]

(4.27)

By tuning the numerator in (4.24) we can change the numerator in (4.27) to any positive integer smaller than \(n\)

\[
\gamma_{st} = \frac{p}{n + m + 1}.
\]

We will show below that only in the case \(n = 1\) the model maintains some resemblance with the properties one would expect of a non-critical string. We can define operators creating macroscopic loop only in that case. For \(n > 1\) the polymerization of the surface is so strong that there is no room left to open macroscopic loops.

To illustrate the preceding discussion, we present an explicit example. The simplest case to study is the one where the function \(V\) depends on a single trace \(V = V(t_1)\). In this case the matrix \(U''\) becomes:

\[
U''_{pq} = \delta_{ql}U''_{pl} = \delta_{ql} \frac{l^2}{l + p} \binom{2p}{p} \left( \frac{R}{4} \right)^p \binom{2l}{l} \left( \frac{R}{4} \right)^l V'', \quad p = 1, 2, \ldots.
\]

The matrix \(U''\) contains a single non-vanishing column in position \(l\). The inverse of \(1 - U''\) is:

\[
(1 - U'')^{-1} = 1 + \frac{U''}{1 - \frac{l}{2} \left( \frac{2l}{l} \left( \frac{R}{4} \right)^l \right)^2 V''},
\]

therefore,

\[
[(1 - U'')^{-1} A]_p =
\]

\[
\binom{2p}{p} \left( \frac{R}{4} \right)^p + \frac{1}{1 - \frac{l}{2} \left( \frac{2l}{l} \left( \frac{R}{4} \right)^l \right)^2 V''} p + \frac{l^2}{1} \left( \frac{2l}{l} \left( \frac{R}{4} \right)^l \right)^2 \binom{2p}{p} \left( \frac{R}{4} \right)^p V'',
\]

22
\[
[(1 - U'')^{-1} A]_0 = \frac{1 + \frac{l}{2} \left(\frac{2l}{l} \left(\frac{R}{4}\right)^l\right)^2 V''}{1 - \frac{l}{2} \left(\frac{2l}{l} \left(\frac{R}{4}\right)^l\right)^2 V''},
\]

The criticality condition (4.22) is now
\[
1 + \frac{l}{2} \left(\frac{2l}{l} \left(\frac{R}{4}\right)^l\right)^2 V'' \propto (R - R_c)^n.
\]

We may simplify the expressions for the moments by eliminating \(V'\) from the string equation \(x_0 = 1\) to get
\[
x_k = \frac{1}{l + k} \left(\frac{2k}{k} \left(\frac{R}{4}\right)^l\right)^k \left[l + \sum_p \frac{p k (p - l)}{p + k} \left(\frac{2p}{p}\right) \left(\frac{R}{4}\right)^p g_p\right].
\]

The analogue of the criticality condition (4.23) is therefore
\[
RD_R x_0 = l + \sum_p p (p - l) \left(\frac{2p}{p}\right) \left(\frac{R}{4}\right)^p g_p \propto (R - R_c)^m.
\]

Note that we still need the string equation for \(l \neq 2\), since \(\beta = g_2\) appears on the right hand side of these expressions. Hence, the HF feedback is explicitly solvable only in the case \(l = 2\). On the other hand, direct differentiation of (4.30) gives
\[
\frac{\partial x_k}{\partial R} = \frac{k}{l + k} A_k D_R x_0.
\]

This means that the total derivative \(dx_k/dR\) is proportional to the criticality condition (4.31), thus restricting the critical points accessible with the simple potentials \(V(t_l)\) to \((n, m = 0)\) or \((n = 1, m)\). The criticality condition (4.29) for \(n = 2\) implies
\[
\left.\frac{dW}{dR}\right|_{R=R_c} = l^2 \left(\frac{2l}{l} \left(\frac{R_c}{4}\right)^l\right)^2 V''(x_l(R_c)) + \frac{l}{2} \left(\frac{2l}{l} \left(\frac{R_c}{4}\right)^l\right)^2 V'''(x_l(R_c)) \left[\frac{dx_l}{dR}\right]_{R=R_c},
\]

\[
= -2l + \frac{l}{4} \left(\frac{2l}{l} \left(\frac{R_c}{4}\right)^l\right)^3 V'''(x_l(R_c)) D_R x_0(R_c) = 0,
\]

where \(W \equiv \frac{l}{2} \left(\frac{2l}{l} \left(\frac{R}{4}\right)^l\right)^2 V''(x_l(R))\). Obviously, the last equation is inconsistent with \(D_R x_0(R_c) = 0\), the condition for \(m > 0\). One can show, following the same
line of reasoning that we used here, that a more general class of potentials with
\( V = V(\sum \alpha_r t_r) \) can accommodate \( n > 1, m > 0 \) points, with a judicious choice of
\( \alpha_r \)'s. Since the physically interesting case will anyhow turn out to be \( n = 1 \), we
will relegate the discussion of this more general class of potentials to appendix A.

Finally, let us display a few explicit potentials resulting in a critical behaviour
of the new kind that we are studying. Specializing to \( V = V(t_2) \), the criticality
conditions (4.22) and (4.23) become

\[
1 + 36 \left( \frac{R}{4} \right)^4 V''(x_2) \propto x^n,
\]

\[
2 + \sum_{p \geq 1} p(p-2)g_p \left( \frac{2p}{p} \right) \left( \frac{R}{4} \right)^p \propto x^m,
\]

where \( R = R_c(1 + x) \) and \( x \to 0 \) is the critical point. Another useful identity is

\[
x_2 = 3 \left( \frac{R}{4} \right)^2 + 3 \sum_{p \geq 1} \frac{p(p-2)}{p+2} g_p \left( \frac{2p}{p} \right) \left( \frac{R}{4} \right)^{p+2}.
\]

It will be sufficient to look at the potential

\[ U = g_1 t_1 + \beta t_2 + h t_2^2. \]

We choose \( R_c = 4 \). To obtain \( n = 1, m = 0 \) critical point all we need now is
\( h = -1/72 \). The string equation \( x_0 = 1 \) fixes \( \beta = 1/18 + x^2/6 + \mathcal{O}(x^3) \) which
implies \( \gamma = 1/2 \). The eigenvalue density (4.11) is (neglecting \( \mathcal{O}(x^2) \) in \( M(\lambda) \)),

\[
\rho(\lambda) = \frac{1}{\pi} \left( \frac{1}{2} - \frac{x}{3} - \frac{x}{6} \lambda^2 \right) \sqrt{4(1+x) - \lambda^2},
\]

which for \( x \to 0 \) reduces precisely to Wigner’s semicircle law. For \( n = m = 1 \) (4.33)
gives \( h = -1/72 \) and \( g_1 = 1 \), whereas \( x_0 = 1 \) gives \( \beta = -1/18 - 2x^3/9 + \mathcal{O}(x^4) \)
implying \( \gamma = 1/3 \). The critical eigenvalue density is exactly equal to the pure gravity one (\( \gamma = -1/2 \))

\[
\rho(\lambda) = \frac{1}{6\pi} (4 - \lambda^2)^{3/2}.
\]

Note that the critical eigenvalue distribution reflects only the Kazakov part of the criticality conditions, while it is insensitive to (4.22), the condition responsible for divergence of \( \chi \) and for positive string susceptibility.

An argument similar to the one presented at the end of [17] shows that the critical point for the transition between the polymer and the pure gravity phases is generically second order. To see this, we look at the partition function for "daisies" at a fixed number of Feynmann vertices (or elementary plaquettes in the dual version). We can extract this quantity from the asymptotics

\[
Z \sim \sum_n n^{\gamma_a-3} \left( \frac{\beta}{\beta_c} \right)^n
\]

So that, in the thermodinamic limit \( n \to \infty \) we have the free energy

\[
f = \log|\beta_c|
\]

where \( \beta_c(g, V) \) is the convergence radius of the planar expansion as a function of the couplings in the matrix-model Lagrangian. \( g \) denotes collectively the linear couplings except \( g_2 = \beta \), while \( V \) corresponds to non-linear terms like \((Tr\phi^2)^2\). We are interested in singularities of \( \beta_c \) as a function of \( V \), since these couplings act as the "temperature" for the gas of splitting points. A first order phase transition would imply coexistence of the smooth and polymer phases, while a second order transition represents global condensation of daisies at the critical point. We calculate the function \( \beta_c(g, V) \) as

\[
\beta_c(g, V) = \beta(R_c(g, V)), g, V)
\]

where \( \beta(R, g, V) \) is the solution to the self-consistency conditions (equations for the moments \( x_k \)) plus the string equation. It depends on the planar solution
and it is a regular function of the couplings \( g, V \). \( R_c(g, V) \) is the position of the critical point as a function of the couplings, and we can find it from the criticality conditions (4.21):

\[
\frac{\partial \beta}{\partial R_c}(R_c, g, V) = 0 \tag{4.36}
\]

Tuning \( g \) and \( V \) to some submanifold of appropriate codimension we can arrange zeroes or any order in (4.36). Further change of \( g, V \) within these universal submanifolds amounts simply to move the location of the critical point \( R_c \). In our case (4.36) has a factorized structure as in (4.21), and we have two branches of critical points: \( R^I_c \) and \( R^{II}_c \), determining two functions in (4.35); \( \beta^I_c \) and \( \beta^{II}_c \). The smaller of these values determines the radius of the convergence disk for the planar perturbation series around \( \beta = 0 \). Hence we are interested in the behaviour of the function

\[
\beta_c(g, V) = \min\{|\beta^I_c|, |\beta^{II}_c|\}
\]

near the coalescing region \( R^I_c \sim R^{II}_c \sim R^*_c \), reached by tuning \( V \).

\( \beta_c(g, V) \) is continuous across the transition, and the first derivative is

\[
\frac{d\beta_c}{dV}(g, V) = \frac{\partial \beta}{\partial V}(R_c, g, V) + \frac{\partial \beta}{\partial R_c}(R_c(g, V), g, V) \frac{\partial R_c}{\partial V}(g, V)
\]

The first term on the right hand side is continuous. In the second term, \( \frac{\partial R_c}{\partial V} \) may have some isolated singularities, but we can assume smooth behaviour in open sets. On the other hand \( \frac{\partial \beta}{\partial R_c}(R_c) \) vanishes identically for all critical points, not only for the coalescing one. So the second term is zero along the critical submanifold and \( \frac{d\beta_c}{dV} \) is continuous at \( V^* \).

The second derivative contains unprotected terms like \( \frac{\partial \beta}{\partial \partial V}(R_c(g, V), g, V) \). These are generically discontinuous since \( R^I_c \) depends on the first derivative of the potential, while \( R^{II}_c \) depends on the second derivative. Thus we are led to at least second order phase transitions and a continuum limit is possible as expected.
We have presented the general analysis of effective actions in the planar limit containing arbitrary, but finite numbers of traces (4.4). We were able to show that beyond some critical coupling, $\gamma_{st}$ becomes positive and takes values of the form $\gamma_{st} = n/(n + m + 1)$ for any positive integers $n, m$. We should continue exploring the properties of these critical points and their scaling operators. Obviously the more challenging case of studying arbitrary functions $V$ depending on an infinite number of traces is crucial before we can draw any conclusions concerning the properties of our model. This together with the study of general properties of (4.1) should shed further light into the properties of non-critical strings in interesting dimensions $D = 2, 3, 4$. It is very important to obtain in what way the effective actions studied in the previous sections depend on dimensionality.

5. MACROSCOPIC LOOPS

To explore the new critical points in some detail, we compute the correlation functions of macroscopic loop operators. The simplest loops to compute are the ones analogous to those appearing in pure gravity (see for instance [32]). They are obtained by taking the limit $k \to \infty$ of $Tr \phi^{2k}$. In terms of the original reduced model these are loop on the surface whose position is averaged over the target space. One could also define loops with definite positions in the target space. The expectation values of the latter would certainly give crucial information on the reduced formulation of string theory, but we have not yet calculated them. For the simpler, pure gravity loops of length $l$ we take the definition

$$\langle w(l) \rangle = \lim_{k \to \infty} \sqrt{\frac{\pi}{l}} \langle Tr \phi^{2k} \rangle = N \lim_{k \to \infty} \sqrt{\frac{\pi}{l}} x_k. \quad (5.1)$$

As a matter of convenience we choose the critical values for $\beta$ and $R$ as $\beta_c = -1, R_c = 1$. In the Kazakov critical points, $\gamma_{st} = -1/m$ and the scaling limit is taken to be $Na^{2-\gamma_{st}} = \kappa^{-1}$. It is a simple but non-trivial fact that the correct way to define a loop of length $l$ in this case is to scale $k$ according to $ka^{2\gamma_{st}} = l$ as
$k \to \infty$ (see for example [32] and references therein). In our case we will restrict for simplicity our computations to the case of potentials $V = V(t_l)$.

We will start by calculating $dx_k/dg_j$. From (4.14) we have

$$\frac{\partial x_l}{\partial g_j} = \frac{j^2 (2l) \binom{2j}{l} (\frac{R}{4})^j}{1 - W},$$

$$\frac{\partial x_k}{\partial g_j} = \frac{j^2 (2k) \binom{2j}{k} (\frac{R}{4})^j}{j + k} \frac{1 - \frac{(k-l)(j-l)}{(k+l)(j+l)} W}{1 - W}, \tag{5.2}$$

where we have introduced the notation $W \equiv \frac{1}{2} \left( \frac{2l}{l} \left( \frac{R}{4} \right)^l \right)^2 V''(x_l)$. From (4.28) we have

$$\frac{\partial R x_k}{\partial R x_0} = \left( \frac{2k}{k} \left( \frac{R}{4} \right)^k \frac{1 - \frac{k-l}{k+l} W}{1 + W} \right). \tag{5.3}$$

Putting it all together,

$$\frac{dx_k}{dg_j} = \frac{\partial x_k}{\partial g_j} - \frac{\partial x_0}{\partial g_j} \frac{\partial R x_k}{\partial R x_0},$$

$$= -\frac{jk}{j + k} \left( \frac{2k}{k} \left( \frac{R}{4} \right)^j \right)^{k+j} + \frac{jk}{(j + l)(k + l)} \left( \frac{2k}{k} \left( \frac{R}{4} \right)^j \right)^{k+j} \frac{2lW}{1 + W}. \tag{5.4}$$

For $V'' = 0$, up to irrelevant numerical factors this answer coincides with the one-loop expectation value in the Kazakov critical points.

To go to the continuum limit we have to choose the scaling variables. Recall that in the double scaling limit [11] the free energy takes the form

$$F = \sum_{g \geq 0} N^{2-2g} (\mu - \mu_c)^{(2-\gamma_s)(1-g)} F_g$$

where $\mu = -\log \beta$, and $g$ is the genus of the triangulations summed over. The renormalized cosmological constant is introduced according to $\mu - \mu_c = a^2 t$ where $a$ is the microscopic lattice spacing. Then the effective string coupling constant is defined by $Na^{2-\gamma_s} = \kappa^{-1}$. Hence the scaling variable associated with the string
susceptibility is \( \chi = a^{-2\gamma s} u \). At the \((n, m)\)-critical point \( \chi \) behaves as \( \chi \sim (1 - \mathcal{R})^{-n} \). Thus the scaling variables are

\[
\chi \sim \frac{1}{(1 - \mathcal{R})^n} = a^{-2\gamma s} u, \quad 1 - \mathcal{R} = a^{2\gamma s/n} u^{-1/n},
\]

\[
\beta - \beta_c = \beta + 1 = a^2 t. \quad (5.5)
\]

For an \((n, m)\)-critical point we have

\[
1 - \mathcal{R} = a^{\frac{2}{n+m+1}} u^{-\frac{1}{n}}, \quad \chi = u a^{-\frac{2n}{n+m+1}}, \quad N a^{2 - \frac{n}{n+m+1}} = \kappa^{-1}. \quad (5.6)
\]

An insertion of a loop operator is a string interaction which brings in an extra power of \( \kappa \), the string coupling constant. Therefore, in general we expect

\[
\langle w(\ell_1) \ldots w(\ell_M) \rangle_c = \kappa^{M-2} \langle \hat{w}(\ell_1) \ldots \hat{w}(\ell_M) \rangle_c, \quad (5.7)
\]

and in particular

\[
\langle w(\ell) \rangle = \frac{1}{\kappa} \langle \hat{w}(\ell) \rangle. \quad (5.8)
\]

This will allow us to arrive at the proper definition of the length stick, \textit{i.e.} to fix \( y \) in \( \ell = k a^y \). Setting \( j = 2 \) in (5.4) \((g_2 \equiv \beta, 1 + W \propto (R_c - R)^n)\), we easily obtain, up to a non-universal positive multiplicative constant

\[
\frac{d}{dt} \langle Tr \phi^{2k} \rangle = N a^2 \left( \frac{a^{y/2}}{\sqrt{\pi}} \left( 1 - a^{\frac{2}{n+m+1}} u^{-1/n} \right)^{\ell/a^y} u a^{-\frac{2n}{n+m+1}} \right). \quad (5.9)
\]

Identifying the r.h.s. with \( \frac{1}{\kappa} \frac{d}{dt} \langle \hat{w}(\ell) \rangle \) leads to

\[
\ell \equiv k a^{\frac{2n}{n+m+1}}, \quad \frac{d}{dt} \langle w(\ell) \rangle = -\frac{u}{\kappa \ell} e^{-\ell/u} \quad \text{for } n = 1, \quad (5.10)
\]

\[
\langle w(\ell) \rangle = 0 \quad \text{for } n > 1.
\]

In the phases with \( n > 1 \) the polymer couplings dominate completely the critical
behavior, in spite of the fact that gravity becomes also critical, in such a way that there is not enough room to open macroscopic loops.

In the case of $n = 1$ we can also calculate two-loop and multiloop correlation functions. Let $\ell_1 = ka^{2n+m+1}$, $\ell_2 = ja^{2n+m+1}$. For $n = 1, 1 + W = 2la^{2m+1}u^{-1}$. For $k, j \to \infty$ and $a \to 0$, (5.4) gives the continuum two-loop operator

$$\langle w(\ell_1)w(\ell_2) \rangle \equiv \lim_{k,j \to \infty} \frac{\pi}{\ell_1 \ell_2} \left( \left< Tr\phi^{2k}Tr\phi^{2j} \right> \right)_{c},$$

$$= \frac{e^{-\ell_2/u}}{\ell_1 + \ell_2} + \frac{u}{\ell_1 \ell_2} e^{-(\ell_1 + \ell_2)/u}. \tag{5.11}$$

The lesson we can draw from (5.11) is that in the simple case of $n = 1$ the effect of the polymer couplings is to contribute an extra state (the last term in (5.11) ) which resembles very much the contribution one would expect of a tachyon. This term represents the breaking of the cylinder interpolating between the two loop of lengths $l_1, l_2$ into two disks osculating at one point. We can represent the last term in (5.11) as

$$\langle w(l_1)P \rangle \frac{1}{\langle PP \rangle} \langle Pw(l_2) \rangle$$

where $P$ is the puncture operator.

With (5.10) and (5.11) it is easy to obtain the multi-loop correlators as well. We can write $\langle w(\ell_1)w(\ell_2) \rangle$ in two ways:

$$\langle w(\ell_1)w(\ell_2) \rangle = \lim_{k_1 \to \infty} \left(-\frac{1}{N}\right) \frac{1}{\kappa \ell_2} \int \left( 1 + \frac{\ell_2}{u} \right) \frac{du}{dg_{k_1}} e^{-\ell_2/u} dt', \tag{5.12}$$

and

$$\langle w(\ell_1)w(\ell_2) \rangle = -\frac{1}{\ell_1 \ell_2} \int \left( 1 + \frac{\ell_1}{u} \right) \left( 1 + \frac{\ell_2}{u} \right) \frac{du}{dt'} e^{-(\ell_1 + \ell_2)/u} dt'. \tag{5.13}$$

* Strictly speaking, we should have put this discussion in the framework of potentials with $V(\sum \alpha_r r)$, the simplest ones with $n > 1$ critical points. As shown in the appendix A, the much more complicated formulas in that case lead again to (5.9) and to the same conclusion that $\langle w(\ell) \rangle = 0$ for $n > 1$. 
Comparing (5.12) and (5.13) we have
\[
\lim_{k \to \infty} \sqrt{\frac{\pi}{\ell}} \left( -\frac{1}{N} \right) \frac{du}{d\ell} = -\frac{\kappa}{\ell} \left( 1 + \frac{\ell}{u} \right) \frac{du}{dt} e^{-\ell/u}.
\]
Define the operator of insertion of a macroscopic loop \( D(\ell) \) as
\[
D(\ell) \equiv -\frac{\kappa}{\ell} \sum_{p \geq 0} \frac{d^p}{dt^p} \left\{ \left( 1 + \frac{\ell}{u} \right) e^{-\ell/u} \frac{du}{dt} \right\} \frac{\partial}{\partial u^{(p)}}.
\]
Where \( u^{(p)} = \frac{d^p u}{dt^p} \). As it should be, \( D(\ell) \) is proportional to \( \kappa \), and we can write the \( M \)-loop amplitude as
\[
\langle w(\ell_1) \ldots w(\ell_M) \rangle = \left( \prod_{i=2}^{M} D(\ell_i) \right) \langle w(\ell_1) \rangle.
\]
The pattern is reminiscent of the residue of the tachyon pole in the Belavin-Knizhnik theorem [33]. This interpretation, though tempting, has to be taken with several grains of salt because in the types of effective actions we are studying it is hard to see any dependence on dimensionality. Nevertheless we take this result as encouraging evidence that our approximation captures some of the expected properties of non-critical strings for \( D > 1 \).

The form of the two-loop operator (5.11) does not depend on the form of the potential \( V(t_1, \ldots, t_n) \) as long as we consider \( (1, m) \)-critical points. Hence only the quadratic part of \( V \) determines the critical properties of macroscopic loops.

To conclude this section we mention that we can study, at least in part, the spectrum of the scaling operators by perturbing the criticality conditions. The string equation is then modified according to
\[
1 + \beta = (1 - R)^{n+m+1} + \sum \tau_k^B (1 - R)^{k+1}
\]
The subleading contributions depend on both the polymer and Kazakov perturbations of the critical conditions (4.21). The bare couplings of the scaling operators
are the $\tau_k^B$ parameters. Using (5.6) the string equation becomes

$$t = u^{-\frac{n+m+1}{n}} + \sum_k \tau_k u^{-\frac{k+1}{n}}, \quad (5.14)$$

and the $\tau_k$ are the renormalized couplings. In the original model these couplings would hardly exhaust the scaling operators, although they may represent a significant subset associated to the simplest loop operators. If we call $\sigma_k$ the scaling operator associated to the coupling $\tau_k$, its planar correlators at the $(n, m)$-critical point follow from (5.14)

$$\frac{du}{d\tau_k} = -u^{-\frac{k+1}{n}} \frac{du}{dt}$$

and

$$\langle \sigma_k PP \rangle = \frac{n}{n+m+1} t^{-\frac{k+m-2n}{n+m+1}}.$$

6. $1/N^2$-CORRECTIONS, DISCRETE AND CONTINUUM LOOP EQUATIONS

In this section we will use Schwinger-Dyson equations to evaluate $1/N^2$-corrections to our planar results. The $1/N^2$-corrections will prove to be consistent with string perturbation expansion and with double scaling. In particular, we will calculate the string susceptibility on the torus to be

$$\gamma_{\text{torus}} = +2.$$

It is amusing to note that this is in agreement with DDK formula $\gamma_h = 2 + (1-h)\gamma_0$ for string susceptibility at genus $h$, where $\gamma_0$ (a complex number for $c > 1$) drops out at $h = 1$.

\* Since the equivalence of the original action (2.3) and its reduced version (2.12) has been demonstrated only in the planar approximation, the $1/N^2$-corrections for (4.4) are one step further removed from (2.3). Still, in order to be able to take the new critical points of our effective action (4.4) more seriously, we want to see if their continuum limit is consistent with a string theory.
We will study matrix integrals of the form

\[ Z = \int D\phi \exp \left\{ -N^2 \left[ \sum_{n=1}^{s} g_n \left( \frac{1}{N} Tr\phi^{2n} \right) + V(t) \right] \right\}, \]

\[ V(t) = \sum_{m=2}^{t} h_m \left( \frac{1}{N} Tr\phi^4 \right)^m. \]

The structure of the calculation follows [34]. In a standard fashion,

\[
0 = \int D\phi \frac{d}{d\phi_{ab}} \left\{ (e^{\ell\phi})_{ab} \exp \left\{ -N^2 \left[ \sum_{n=1}^{s} g_n \left( \frac{1}{N} Tr\phi^{2n} \right) + V(t) \right] \right\} \right\}
\]

leads to the loop equation

\[
\left\langle \frac{\ell}{N} Tr e^{\phi} \frac{1}{N} Tr e^{(\ell-\ell')\phi} \right\rangle = \sum_{n=1}^{s} g_n 2n \left\langle \frac{1}{N} Tr \left( \phi^{2n-1} e^{\ell\phi} \right) \right\rangle \\
+ \sum_{m=2}^{t} h_m 4m \left\langle \left( \frac{1}{N} Tr\phi^4 \right)^{m-1} \frac{1}{N} Tr \left( \phi^3 e^{\ell\phi} \right) \right\rangle. \tag{6.1}
\]

Let us define connected correlation functions \( \Phi_{\ell_1...\ell_M} \) as

\[
\Phi_{\ell_1...\ell_M} \equiv N^{M-2} \left\langle Tr e^{\ell_1\phi} ... Tr e^{\ell_M\phi} \right\rangle_c,
\]

where the explicit factor of \( N \) makes all \( \Phi_{\ell_1...\ell_M} \) of order one. For \( N \) large, (6.1) leads to

\[
\sum_{n=1}^{s} 2n\tilde{g}_n \left( \frac{d}{d\ell} \right)^{2n-1} \left\langle \frac{1}{N} Tr\phi^{\ell}\phi \right\rangle = \int_{0}^{\ell} d\ell' \left\langle \frac{1}{N} Tr\phi^{\ell}\phi \right\rangle \left\langle \frac{1}{N} Tr\phi^{(\ell-\ell')\phi} \right\rangle \\
+ \frac{1}{N^2} \left\langle \int_{0}^{\ell} d\ell' \left\langle Tr e^{\phi} Tr e^{(\ell-\ell')\phi} \right\rangle_c - 4V''(x) \phi^4 \right\rangle_c \\
- 2V'''(x) \phi^4 \left\langle \frac{1}{N} Tr \left( \phi^3 e^{\ell\phi} \right) \right\rangle_c + O(1/N^4), \tag{6.2}
\]

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where
\[ \tilde{g}_n = g_n + V'(x_2)\delta_{n,2}, \]
\[ x_i = \left\langle \frac{1}{N}T_{\phi}^{2i} \right\rangle, \]
\[ \Phi_{n,m} = \langle Tr\phi^n Tr\phi^m \rangle_c. \]

After the Laplace transform, (6.2) becomes
\[ \tilde{V}'(p)G(p) = Q(p) + G^2(p) + \frac{1}{N^2} \left[ \Phi(p,p) - 4V''(x_2)(p^2\Phi_4(p) - \Phi_{2,4}) \right. \]
\[ - 2V''(x_2)\Phi_{4,4}(p^3G(p) - p^2 - x_1) \left. \right] + O(1/N^4). \]

(6.3)

We have introduced the notations
\[ \tilde{V}'(p) = \sum_{n=1}^{s} 2n\tilde{g}_n p^{2n-1}, \]
\[ Q(p) = \sum_{n=1}^{s} 2n\tilde{g}_n \sum_{i=0}^{n-1} p^{2n-2-2i}x_i, \]
\[ G(p) = \int_0^\infty d\ell e^{-\ell p} \left\langle \frac{1}{N}T_{r}\ell_{\phi} \right\rangle = \left\langle \frac{1}{N}T_{r} \frac{1}{p - \phi} \right\rangle, \]
\[ \Phi(p,p') = \int_0^\infty d\ell \int_0^\infty d\ell' e^{-\ell p} e^{-\ell' p'} \Phi_{\ell,\ell'}, \]
\[ \Phi_4(p) = \left\langle Tr\phi^4Tr \frac{1}{p - \phi} \right\rangle_c = \sum_{n=1}^{\infty} \Phi_{4,2n} p^{-2n-1}. \]

The solution to (6.3) is well-known in the planar limit:
\[ G^{(0)}(p) = \frac{1}{2} \tilde{V}'(p) - M(p)\sqrt{p^2 - R}, \]
\[ = \sum_{n=1}^{s} n\tilde{g}_n p^{2n-1} - \sum_{n=1}^{s} p^{2n-2} \sum_{k=0}^{s-n} \binom{2k}{k} \left( \frac{R}{4} \right)^k (k + n)\tilde{g}_{k+n} \sqrt{p^2 - R}, \]

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where $R$ is fixed by the string equation

$$x_0 = 1 = \sum_{n=1}^{s} n \tilde{g}_n \frac{(2n)}{n} \left( \frac{R}{4} \right)^n.$$  

To include $1/N^2$-corrections, write

$$\tilde{V}' = \tilde{V}'(0) + \delta V' = \tilde{V}'(0) + 4V''(x_2)\delta x_2 p^3,$$

$$G = G(0) + \delta G,$$

$$Q = Q(0) + \delta Q,$$

where $\delta(\cdot)$ is $O(1/N^2)$. From (6.3) we obtain the equation for $1/N^2$-corrections

$$4V''(x_2)\delta x_2 p^3 G(0)(p) + 2M(0)(p)\sqrt{p^2 - R} \delta G(p) = \delta Q(p) +$$

$$\frac{1}{N^2} \left[ \Phi(p, p) - 4V''(x_2)(p^3\Phi_4(p) - \Phi_{2,4}) - 2V'''(x_2)\Phi_{4,4}(p^3G(p) - p^2 - x_1) \right]^{(0)},$$

where

$$\delta Q(p) = \sum_{n=2}^{s} 2n \tilde{g}_n \sum_{i=1}^{n-1} p^{2n-2i-2} \delta x_i + 4V''(x_2)\delta x_2 (p^2 + x_1).$$

An important piece of information that we can extract from (6.4) is the dependence of $\delta x_2$ on $R - R_c$. This will give $1/N^2$-corrections to specific heat. To simplify (6.4) we use the fact that, for $s \geq 2$, $M(0)(p)$ vanishes for $p$ equal to some $p_0$. Therefore, at $p = p_0$ (6.4) reduces to

$$4V''(x_2)\delta x_2 p_0^3 G(0)(p_0) = \delta Q(p_0) + \frac{1}{N^2} \left[ \Phi(p_0, p_0) - 4V''(x_2)(p_0^3\Phi_4(p_0) - \Phi_{2,4}) $$

$$- 2V'''(x_2)\Phi_{4,4}(p_0^3G(p_0) - p_0^2 - x_1) \right]^{(0)}.$$

We need the planar expressions for $\Phi(p, p)$ and $\Phi_4(p)$. They are calculated in
The results are

$$\Phi_4(p) = \frac{1}{1 + W} \left[ \frac{8p^4 - 4Rp^2 - R^2}{4\sqrt{p^2 - R}} - 2p^3 \right],$$

$$\Phi(p, p) = \frac{R}{4(p^2 - R)^2} - \frac{4W}{9R^4(1 + W)} \left[ 128p^6 + \frac{16p^3(R^2 + 4Rp^2 - 8p^4)}{\sqrt{p^2 - R}} + \frac{R^3(8p^2 + R)}{p^2 - R} \right].$$

(6.6)

We are ready to look at some explicit examples. As discussed earlier, $V(x_2)$ potentials can describe critical points with either $m = 0$ or $n = 1$. The simplest non-trivial case are potentials with $s = 2$ which allow us to look at $m = 0$, $n \geq 1$ and $m = n = 1$ points.

For $s = 2$ (6.5) gives

$$(p_0^3G(p_0) - p_0^2 - x_1) \left( 4V''(x_2)\delta x_2 + \frac{2}{N^2} V'''(x_2)\Phi_{4,4} \right) =$$

$$= 4\tilde{g}_2 \delta x_1 + \frac{1}{N^2} \left[ \Phi(p_0, p_0) - 4V''(x_2)(p_0^3\Phi_4(p_0) - \Phi_{2,4}) \right].$$

Another relation between the unknowns $\delta x_1$ and $\delta x_2$ comes from the coefficient of $1/p^2$ in (6.4):

$$\delta x_1 = -\frac{2}{\tilde{g}_1} (\tilde{g}_2 + V''(x_2)x_2)\delta x_2 - \frac{1}{N^2} \frac{\Phi_{4,4}}{\tilde{g}_1} (2V''(x_2) + V'''(x_2)x_2).$$

Finally,

$$\delta x_2 = \frac{1}{N^2 D},$$

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where

\[
N \equiv \frac{R}{4(p_0^2 - R)^2} - \frac{9R^4}{16(1 + W)} \left[ -\frac{3R\tilde{g}_2V''(x_2)}{\tilde{g}_1(p_0^2 - R)} + V'''(x_2) \left( \frac{\tilde{G}(p_0)}{2} + \frac{\tilde{g}_2}{\tilde{g}_1} x_2 \right) \right],
\]

\[
D \equiv 4V''(x_2)\tilde{G}(p_0) + \frac{8\tilde{g}_2}{\tilde{g}_1}(\tilde{g}_2 + V''(x_2)x_2),
\]

\[
\tilde{G}(p_0) \equiv p_0^3G(p_0) - p_0^2 - x_1 = \tilde{g}_1p_0^4 + 2\tilde{g}_2p_0^6 - p_0^2 - x_1,
\]

\[
p_0^2 = -\frac{R}{2} - \frac{\tilde{g}_1}{2\tilde{g}_2}.
\]

Specializing further to \(m = 0, n \geq 1\) potentials with \(\gamma_0 = n/(n + 1)\), we set \(R = 4(1 + x), x \ll 1\) and \(\tilde{g}_1 = 1/2\). From \(x_0 = 1\) we get \(\tilde{g}_2 = -x/(12(1 + x)^2)\) and \(p_0^2 = 3/x + \mathcal{O}(1)\). Note that \(\tilde{g}_2\) is \(\mathcal{O}(x)\) and \(p_0^2\) diverges as we approach the critical point, which is as it should be since \(m = 0\) potentials have Gaussian linear part of the potential (\(i.e. \tilde{g}_n = 0, n > 1\)).

Setting finally \(h_2 = -1/72, h_m = 0, m > 2\) we have fully specified the \(m = 0, n = 1\) potential. We have \(N = -2x/3 + \mathcal{O}(x^2)\) and \(D = -4x^3/9 + \mathcal{O}(x^4)\). Furthermore, \(\beta = 1/18 + x^2/6 + \mathcal{O}(x^3)\). Therefore,

\[
\delta x_2 \propto \frac{1}{N^2 x^2}, \quad \frac{d\delta x_2}{d\beta} \propto \frac{1}{N^2 x^4} \propto \frac{1}{N^2 (\beta - \beta_c)^2}
\]

and

\[
\gamma_1 = +2.
\]

We have derived the first two terms in the genus expansion of the specific heat. In terms of renormalized continuum variables we have

\[
\frac{d^2}{dt^2} \log Z = \kappa^{-2} \left[ \frac{1}{t^{\gamma_0}} + \kappa^2 \frac{1}{t^2} + \cdots \right]. \quad (6.7)
\]

The \(1/N^2\)-correction is consistent with string perturbation expansion and with the double scaling limit.
One can repeat this calculation for \( m = 0, n > 1 \) and one is always led to (6.7). Similarly, for \( n = m = 1 \) potential that we already discussed we have \( \tilde{g}_1 = 1, h_2 = -1/72, h_m = 0, m > 2, \tilde{g}_2 = -\frac{1+2x}{12(1+x)^2}, \beta = -1/18 - 2x^3/9 + \mathcal{O}(x^4), \frac{d\delta x}{d\beta} \propto \frac{1}{N^2} \frac{1}{x^2} \) and we again have \( \gamma_1 = 2 \) and (6.7).

To reach \( n = 1, m = 2 \) we need potentials with \( s = 3 \). Here \( M(p) \) will have two zeroes, \((p_0^\pm)^2\), that will both coalesce with \( R \) at \( x = 0 \). If we take \( \tilde{g}_1 = 3/2, \tilde{g}_3 = 1/60, h_2 = -1/72, h_m = 0, m > 2 \) we obtain \( \beta = -7/30 + x^4/4 + \mathcal{O}(x^5) \) and

\[
(p_0^+_x)^2 = 4 + (-1 + \sqrt{-5})x - 2\sqrt{-5}x^2 + \mathcal{O}(x^3),
(p_0^-_x)^2 = 4 + (-1 - \sqrt{-5})x + 2\sqrt{-5}x^2 + \mathcal{O}(x^3).
\]

Now (6.4) will result in two equations with two unknowns, \( \delta x_1 \) and \( \delta x_2 \). Their solution again leads to \( \gamma_1 = 2 \) and (6.7).

It is interesting that there is a string perturbation expansion (6.7) even for \( n > 1 \). At those critical points, as we have seen through vanishing of the macroscopic loop operators, polymerization is so strong that it is impossible to open big holes in the surface. Since the only continuum objects used in the derivation of (6.7) are microscopic ones like \( d/dt \sim P \), we were able to avoid pathologies due to vanishing continuum macroscopic loop operators.

Finally, let us look at the continuum limit of the loop equation (6.3) itself. Comparing the solution of the continuum limit of the loop equations with the continuum limit of the solution of the discrete loop equations is a useful check of our calculations.

To begin with, use \( \ell = ka^\frac{2n}{n+m+1}, p = p_c + a^\frac{2n}{n+m+1}z \), to determine the scaling for the universal, continuum loop correlation functions. On the one hand, we have

\[
\left< \text{Re}^{\ell_1} \phi \ldots \text{Re}^{\ell_M} \phi \right> = \kappa^{M-2} \int dz_1 \ldots dz_M \left< \hat{w}(z_1) \ldots \hat{w}(z_M) \right>.
\]
On the other hand,

\[
\left\langle \text{Tre}^\ell_1 \phi \ldots \text{Tre}^\ell_M \phi \right\rangle_c = \frac{1}{N^M - 2} \Phi_{\ell_1 \ldots \ell_M} = \\
= \frac{1}{N^M - 2} \int dp_1 \ldots dp_M \langle w(p_1) \ldots w(p_M) \rangle_c.
\]

Consistency demands

\[
\langle w(p_1) \ldots w(p_M) \rangle_c = a^{\frac{2(2m+n+2) - M(2n+2m+2)}{m+n+1}} \langle \hat{w}(z_1) \ldots \hat{w}(z_M) \rangle_c,
\]

with \( n \geq 1, \ m \geq 0 \) and \( M \geq 0 \). In particular,

\[
G(p)|_{\text{universal part}} = a^{\frac{2m-n+2}{m+n+1}} \langle \hat{w}(z) \rangle.
\]

The negative sign of \( n \) in the exponent is the first sign of problems for \( n > 1 \).

We start again with \( n = m = 1 \) point. Set

\[
\beta = \beta_c + a^2 t,
\]

\[
R = R_c (1 + a^{2/3} t^{1/3}),
\]

\[
\tilde{g}_2 = \beta + 2h_2 x_2,
\]

\[
p = p_c + a^{2/3} z,
\]

\[
\tilde{g}_1 = 1 + ca^2 t.
\]

Demanding that universal (non-analytic in \( t \)) pieces in \( x_1 \) and \( x_2 \) scale as \( \langle P \rangle \) fixes \( R_c = 4, \ \beta_c + 2h_2 = -1/12 \). The criticality condition for \( n = 1 \) gives \( h_2 = -1/72 \), \( \beta_c = -1/18 \). Finally, since we want to blow up the critical region between the branch points, we fix \( p_c^2 = R_c = 4 \).

Plugging all of this into solution of discrete planar loop equation we obtain

\[
\langle \hat{w}(z) \rangle_{\text{planar}} = -\frac{2}{3} (t^{1/3} + 2z) \sqrt{z - t^{1/3}}.
\]

On the other hand, scaling the loop equation itself, we arrive at

\[
9(\langle \hat{w}(z) \rangle^2 + \kappa^2 \langle \hat{w}(z) \hat{w}(z) \rangle_c) = (24 + 6c)t - 12 \langle P \rangle z + 16z^3.
\]

With \( c = -14/3 \), \( \langle P \rangle_{\text{planar}} = t^{2/3} \) we recover precisely (6.9). We have shown that
operations of taking the continuum limit and solving the loop equations commute.

Now we turn to \( n = 2, \ m = 0 \). According to (6.8) \( G(p)|_{\text{univ.}} = \langle \hat{w}(z) \rangle \) and universal loop equation should appear at \( \mathcal{O}(1) \) in scaled discrete equation. Indeed, for \( p_c^2 = R_c = 4 \) we obtain

\[
0 = (\hat{w}(z))^2 + \kappa^2 \langle \hat{w}(z) \hat{w}(z) \rangle,
\]

\( \langle \hat{w}(z) \rangle \mid_{\text{planar}}. \) (6.10)

For \( n > 2, \ m = 0 \) negative powers of \( a \) in (6.8) give immediately \( \langle \hat{w}(z) \rangle = 0 \).

7. Appendix A

In this appendix we examine with some detail potentials of the form \( V(\sum \alpha_r t_r) = \sum h_m(\alpha \cdot t)^m \). These potentials already contain complicated couplings between different traces, but they are still explicitly soluble and serve as multicritical potentials for \( m, n > 1 \) critical points.

The general matrix in (4.19) simplifies in this case to

\[
U'' = U \otimes \alpha
\]

where \( \alpha, U \) denote the vectors of components \( \alpha_k \) and

\[
U_k = V'' A_k \sum_p \frac{p^2}{p + k} A_p \alpha_p
\]

(recall \( A_k = \binom{2p}{p} \left( \frac{R}{4} \right)^p \)). This structure makes the inversion of \( 1 - U'' \) trivial. Given that

\[
\partial_R x = D_R x + (U \cdot \alpha) \partial_R x = D_R x_0 A + U(\alpha \cdot \partial_R x)
\]

(7.1)

we may solve \( \partial_R x_k \) as

\[
\partial_R x_k = D_R x_0 \frac{(1 - U \cdot \alpha) A_k + U_k \alpha \cdot A}{1 - U \cdot \alpha}
\]
In particular
\[ \partial_R x_0 = \frac{D_R x_0}{1 - U \cdot \alpha} (1 - U \cdot \alpha + U_0 \alpha \cdot A) \]
and the polymer criticality condition reads:
\[ 1 - U \cdot \alpha + U_0 \alpha \cdot A = 1 + V'' \sum_{p,k} \frac{p^k}{p + k} A_p A_k \alpha_p \alpha_p \sim (R - R_c)^n \quad (7.2) \]

Analogous considerations to those in (7.1) lead to
\[ \partial_\beta x_k = \frac{24}{k + 2} A_k \left( \frac{R}{4} \right)^2 + U_k \frac{\alpha \cdot D_\beta x}{1 - U \cdot \alpha} \]

With these ingredients at hand we can compute \( \frac{dx_k}{d\beta} \), whose scaling for large \( k \) determines the one-loop correlator,
\[ \frac{dx_k}{d\beta} \sim -A_k \frac{D_\beta x_0 (1 - U \cdot \alpha) + U_0 \alpha \cdot D_\beta x}{1 + W} \]
for \( 1 + W \equiv 1 - U \cdot \alpha + U_0 \alpha \cdot A \). Assuming an n-th order critical point for polymers in (7.2) we find the large \( k \) asymptotics
\[ \frac{dx_k}{d\beta} \sim \left( \frac{2k}{k} \right) \left( \frac{R}{4} \right)^k (R - R_c)^{-n} \]
the same scaling as for \( V(x_1) \) potentials, and we conclude \( \frac{dw(\ell)}{dt} = 0 \) here as well, for \( n > 1 \).

Now we come to the definition of \( n > 1, m \geq 1 \) critical points. First we should simplify the expressions for the moments \( x_k \) by solving \( V' \) from the string equation:
\[ V' (\alpha \cdot x) = \frac{1 - \sum p A_p g_p}{\sum p A_p \alpha_p} \]
and we get the functions
\[ x_k = X_k (R, g_p, \alpha_p) = \sum_p \frac{p^2}{p + k} A_p A_k g_p + \frac{1 - \sum p A_p g_p}{\sum p A_p \alpha_p} \sum_p \frac{p^2}{p + k} A_p A_k \alpha_p \quad (7.3) \]
Notice that these functions do depend on \( g_2 = \beta \), so that we still need the string equation to relate \( \beta \) and \( R \).
The Kazakov-like critical points come from
\[
RD_{Rx_0} = \sum_{p,q} p^2qA_pA_q(g_p\alpha_q - \alpha_p g_q) + \sum_p p^2A_p\alpha_p \sim (1 - R)^m
\] (7.4)
In this expression we assume \( g_2 = \beta(R) \) solved from the string equation.

Upon direct differentiation of (7.3) we get
\[
\frac{\partial X_k}{\partial R} = D_{Rx_0} \frac{\sum_p \frac{p}{p+k}A_pA_k\alpha_p}{\sum pA_p\alpha_p}
\]
still proportional to \( D_{Rx_0} \), as in (4.32), but through a non-trivial factor. If \( R_c \) is the critical point in the polymer phase then obviously
\[
\frac{dX_k}{dR} = \frac{\partial X_k}{\partial R}(R_c) + \frac{\partial X_k}{\partial \beta} \frac{\partial \beta}{\partial R}(R_c) = \frac{\partial X_k}{\partial R}(R_c)
\] (7.5)
Assume now a particular choice of \( \alpha_p \) so that, in the vicinity of \( R_c = 1 \) we have
\[
\frac{\partial X_k}{\partial R}(R_c) \sim \frac{(R_c - 1)^m}{\sum pA_p(R_c)\alpha_p} \sim (R_c - 1)^{m-r}
\] (7.6)
for \( 0 < r \leq m \). Differentiating now (7.2) using (7.5) and (7.6) we arrive at:
\[
\frac{dW}{dR}(R_c) \sim F(R_c)(1 - R_c)^{2r} + G(R_c)V'''(\alpha \cdot x)(1 - R_c)^{m-r}
\]
with \( F \) and \( G \) regular functions at \( R_c = 1 \). It is clear that \( dW/dR \) vanishes at \( R_c = 1 \) for \( 0 < r < m \). If \( r = m \) we need additional tuning of \( h_m \) parameters in the potential so that \( V'''(R_c = 1) = 0 \). Hence, we have explicitly shown how \( n = 2 \) polymer critical points can coalesce with Kazakov critical points of any order \( m \). In general, the possibility of having \( \frac{dX_k}{dR} \neq 0 \) at the critical point rules out the problem for arbitrary \( n \).

In order to find an \( n,m \) critical potential we are led to the following hierarchy of tunnings: we first tune \( \alpha_p \) as in (7.6). Then, for fixed \( \alpha_p \), adjust \( g_p \) to get (7.4) and finally we may tune \( h_m \) in (7.2). It is interesting to note that the relative tuning among different traces in the non-linear potential was essential to achieve the full spectrum of string susceptibilities (4.24) quoted in the general discussion.
8. Appendix B

In this appendix we will calculate \( \Phi(p, p) \) and \( \Phi_4(p) \) needed in the calculation of \( 1/N^2 \)-corrections in chapter 6. The identity

\[
0 = \int \mathcal{D}\phi \frac{d}{d\phi_{ab}} \left\{ (e^{\ell \phi})_{ab} \frac{1}{N} Tr e^{\ell_2 \phi} \exp \left\{ -N^2 \left[ \sum_{n=1}^{s} g_n \left( \frac{1}{N} Tr \phi^{2n} \right) + V(t_4) \right] \right\} \right\}
\]

results in

\[
\sum_{n=1}^{s} 2ng_n \left( \frac{d}{d\ell} \right)^{2n-1} \left\langle \frac{1}{N} Tr e^{\ell_2 \phi} \frac{1}{N} Tr e^{\ell \phi} \right\rangle + \sum_{m=1}^{t} 4mh_m \left( \frac{d}{d\ell} \right)^{3} \left\langle \frac{1}{N} Tr e^{\ell_2 \phi} \left( \frac{1}{N} Tr \phi^4 \right)^{m-1} \frac{1}{N} Tr e^{\ell \phi} \right\rangle = (8.1)
\]

After expanding (8.1) in terms of connected correlation functions, using (6.2) and going to the planar limit one is left with

\[
\sum_{n=1}^{s} 2n\tilde{g}_n \left( \frac{d}{d\ell} \right)^{2n-1} \Phi_{\ell,\ell_2} = 2 \int_{0}^{\ell} d\ell' G_{\ell} \Phi_{\ell-\ell',\ell_2} + \ell_2 G_{\ell+\ell_2}
\]

\[
- 4V''(x_2) \left( \frac{d}{d\ell} \right)^{3} G_{\ell} \left\langle Tr e^{\ell_2 \phi} Tr \phi^4 \right\rangle_c.
\]

The Laplace transform of (8.2) is

\[
2M(p) \sqrt{p^2 - R} \Phi(p, p_2) = Q(p, p_2) + \frac{\partial}{\partial p_2} \left( G(p) - G(p_2) \right) \left( \frac{p_2}{p} - 1 \right)
\]

\[
- 4V''(x_2) \Phi_4(p_2)(p^3 G(p) - p^2 - x_1),
\]

where

\[
Q(p, p_2) = \sum_{n=2}^{s} 2n\tilde{g}_n \sum_{i=1}^{2n-2} p^{2n-2-i} \Phi_i(p_2).
\]

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For our even matrix integral measure, $\Phi(p, p')$ splits into even and odd parts.

$$\Phi(p, p') = \sum_{m,n=1}^{\infty} p^{-m-1}(p')^{-n-1} \langle Tr\phi^m Tr\phi^n \rangle_c,$$

$$= \sum_{m,n=1}^{\infty} p^{-2m-1}(p')^{-2n-1} \langle Tr\phi^{2m} Tr\phi^{2n} \rangle_c +$$

$$+ \sum_{m,n=1}^{\infty} p^{-2m}(p')^{-2n} \langle Tr\phi^{2m-1} Tr\phi^{2n-1} \rangle_c = \Phi_{\text{even}}(p, p') + \Phi_{\text{odd}}(p, p').$$

We will calculate $\Phi_{\text{even}}$ using (5.4) to obtain $\langle Tr\phi^{2j} Tr\phi^{2k} \rangle_c = -\frac{dx_k}{dg_j}$ for $l = 2$. To get $\Phi_{\text{odd}}$ one can, following [34], expand (8.3). The coefficient of $1/p^{2m}$ will determine $\Phi_{2m-1}(p)$. Since all explicit dependence on $V''$ drops out of this expansion, $\Phi_{\text{odd}}$ will have the same form as in [34]:

$$\Phi_{\text{odd}}(p, p) = \frac{2p^2R - R^2}{8p^2(p^2 - R)^2}.$$

As for the even part, in calculating

$$\Phi_{\text{even}}(p, p) = - \sum_{k,j=1}^{\infty} \frac{dx_k}{dg_j} p^{-2k-2j-2},$$

$$\Phi_{4}(p) = - \sum_{k=1}^{\infty} \frac{dx_k}{dg_2} p^{-2k-1},$$

we will need the following sums:

$$s_1(a) = \sum_{k=1}^{\infty} \frac{k}{k+2} \binom{2k}{k} \left(\frac{a}{4}\right)^{k+2},$$

$$s_2(a) = \sum_{k,j=1}^{\infty} \frac{jk}{j+k} \binom{2j}{j} \binom{2k}{k} \left(\frac{a}{4}\right)^{k+j}.$$
One can evaluate them as follows:

\[
s_1(a) = \sum_{k=1}^{\infty} \binom{2k}{k} \left(\frac{a}{4}\right)^{k+2} - \sum_{k=1}^{\infty} \frac{2}{k+2} \binom{2k}{k} \left(\frac{a}{4}\right)^{k+2},
\]

\[
= \left(\frac{a}{4}\right)^2 \left[\frac{1}{\sqrt{1-a}} - 1\right] - \frac{1}{8} \int_0^a dt \left[\frac{1}{\sqrt{1-t}} - 1\right],
\]

\[
= \frac{8 - 4a - a^2}{48\sqrt{1-a}} - \frac{1}{6},
\]

\[
s_2(a) = \int_0^a dt \frac{1}{t} \left(\sum_{k=1}^{\infty} k \binom{2k}{k} \left(\frac{t}{4}\right)^k\right)^2,
\]

\[
= \int_0^a dt \frac{1}{t} \left(t \frac{\partial}{\partial t} \sum_{k=0}^{\infty} \binom{2k}{k} \left(\frac{t}{4}\right)^k\right)^2,
\]

\[
= \frac{1}{4} \int_0^a dt t(1-t)^{-3} = \frac{1}{8} \left(\frac{a}{1-a}\right)^2.
\]

Now it is easy to show that

\[
\Phi_4(p) = \frac{1}{1+W} \left[\frac{8p^4 - 4Rp^2 - R^2}{4\sqrt{p^2 - R}} - 2p^3\right],
\]

\[
\Phi(p, p) = \frac{R}{4(p^2 - R)^2}
\]

\[-\frac{4W}{9R^4(1+W)} \left[128p^6 + \frac{16p^3(R^2 + 4Rp^2 - 8p^4)}{\sqrt{p^2 - R}} + \frac{R^3(8p^2 + R)}{p^2 - R}\right].
\]
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