Construction asymptotic solution while studying electrovortex flow in hemispherical container using Stokes approximation

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Abstract. The flow generated in the conductive medium with the electromagnetic force appearing when non-uniform electric current interacts with the own magnetic field was considered. The problem was solved analytically using Stokes approximation in a hemispherical geometry. Also numerical solution was obtained and comparing with the oldest mode of analytical one was carried out. The numerical and asymptotic results are quite similar.

1. Introduction

The electrovortex flow is caused by the force appearing as a result of the interaction between non-uniform electrical current and the own magnetic field of this current passing through the conductive medium. Such problems are very important to study the processes in metallurgical engineering, for example to study electrical melting of metals or electrical welding. Nowadays most of such problems are solved numerically. However, some of the problems are described by linear differential equations or can be linearized. So the analytical solutions can be constructed. Such solutions can be used to check the numerical ones and to estimate some parameters of the flow.

We consider the volume between two concentrical hemispherical electrodes. The electrical current passes from the inner electrode to the outer one. The current produces the magnetic field and this field while interacting with the conducting liquid produces the Ampere's force that creates the vortex flow in the liquid.

We describe the flow for small Reynolds numbers, using so-called Stokes approximation. We can neglect the convective term in the Navier – Stokes equation, because it is proportional to the second power of the velocity. So the problem becomes linear and we can construct the analytical solution. This problem has been of interest for several decades. The analytical solution of the equations with infinitely small electrode was constructed by Sozou and Pickering [1]. The numerical methods were used by Sozou and Pickering [2], Shatrov and Gerbeth [3] and some other authors. Also, in various aspects, this problem was studied experimentally and theoretically in Institute of Physics of University of Latvia [4], Joint Institute for High Temperatures [5,6], University of Leoben (Austria) [7] and others.
To solve the problem we used the variables “vector potential – vorticity”. We constructed the analytical solution of the problem using the eigenfunctions of this problem. After that we took the first mode of the solution and compared it with the numerical results.

2. Main equations
The motion of the liquid is described by the Navier–Stokes equation:

$$\frac{\partial \mathbf{V}}{\partial t} + (\mathbf{V}, \nabla)\mathbf{V} = -\frac{1}{\rho} \nabla p + \nu \nabla^2 \mathbf{V} + \frac{1}{\rho} \mathbf{F}, \quad (1)$$

where \( \mathbf{V} \) is the velocity of the liquid, \( \rho \) is the density of the liquid, \( \nu \) is the viscosity coefficient, \( p \) is the pressure and \( \mathbf{F} \) is the electromagnetic force. We use the so-called electrodynamic approximation, according to which the force is connected only with the magnetic field:

$$\mathbf{F} = [\mathbf{J}, \mathbf{B}],$$

where \( \mathbf{J} \) is the density of the current. It can be described as:

$$\mathbf{J} = \frac{I}{2\pi r^2} \mathbf{e}_r,$$

where \( I \) is the current. The magnetic field can be found from the equation:

$$[\nabla, \mathbf{B}] = \mu_0 \mathbf{J}.$$  

This equation can be rewritten in projection to the \( r \)-coordinate:

$$\frac{1}{r \sin \theta} \frac{\partial (B_\theta \sin \theta)}{\partial \theta} = \frac{I}{2\pi r^2}.$$

So the magnetic field will be:

$$B_\phi = -\mu_0 \frac{I(1-\cos \theta)}{2\pi \sin \theta}.$$  

We assume that the velocities are quite small, so we can neglect the convective term \( (\mathbf{V}, \nabla)\mathbf{V} \) of the Navier–Stokes equation (1). We will find the stationary solutions, so the equation will be:

$$0 = -\frac{1}{\rho} \nabla p + \nu \nabla^2 \mathbf{V} - \frac{\mu_0}{\rho} \frac{I^2(1-\cos \theta)}{4\pi^2 r^3 \sin \theta} \mathbf{e}_\theta.$$  

The main problem is connected with the pressure that usually is unknown. We can remove it if we take the curl of the both parts of the equation:

$$0 = \nu \nabla[\nabla, \mathbf{V}] - \frac{\mu_0}{\rho} \frac{I^2(\cos \theta - 1)}{2\pi^2 r^3 \sin \theta} \mathbf{e}_\phi.$$  

We can introduce the vorticity variable:

$$\omega = [\nabla, \mathbf{V}]$$

It can be assumed that:

$$\omega = \omega \mathbf{e}_\phi.$$  

We can also introduce the vector potential \( \psi \) of the velocity:
\[
\mathbf{V} = [\nabla, \psi].
\]

For the vector potential we have the same approximation:

\[
\psi = \psi e_\phi.
\]

As for the boundary conditions, we can use the following (\(a\) is the radius of the inner electrode, \(b\) is the radius of the outer electrode):

\[
\hat{\omega} \big|_{r=a} = \hat{\omega} \big|_{r=b} = \omega = \psi \big|_{r=a} = \psi \big|_{r=b} = 0.
\]

Using the radius of the outer electrode \(b\) as a scale we can obtain scales for the current density: \(I/b^2\), and velocity: \(I \left( \frac{\mu_0}{\rho} \right)^{1/2}\). We can also introduce the number \(A = \frac{I}{2\pi Y} \left( \frac{\mu_0}{\rho} \right)^{1/2}\), (corresponding to the Reynolds number) and the equations will be the following:

\[
\begin{aligned}
&\left\{ \frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \frac{\partial \omega}{\partial r} \right) + \frac{1}{r^2} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial \omega}{\partial \theta} \right) - \frac{\omega}{r^2 \sin^2 \theta} = \frac{A(\cos \theta - 1)}{r^2 \sin \theta}; \\
&\left\{ \frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \frac{\partial \psi}{\partial r} \right) + \frac{1}{r^2} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial \psi}{\partial \theta} \right) - \frac{\psi}{r^2 \sin^2 \theta} = -\omega. 
\end{aligned}
\]

### 3. Asymptotic solution

The problem has the eigenfunctions [8, 9]:

\[
u_{l_k}(r, \theta) = r^{-1/2} \left( J_{2l+1/2} \left( r \lambda_{l_k} \right) - g_l(\lambda_{l_k}, a) Y_{2l+1/2} \left( r \lambda_{l_k} \right) \right) P_{2l}^l(\cos \theta),
\]

where \(J_{2l+1/2}\) and \(Y_{2l+1/2}\) are the Bessel functions, \(P_{2l}^l\) are the associated Legendre functions, function \(g_l\) can be described as:

\[
g_l(\lambda_{l_k}, a) = \frac{J_{2l+1/2} \left( a \lambda_{l_k} \right)}{Y_{2l+1/2} \left( a \lambda_{l_k} \right)}.
\]

and \(\lambda_{l_k}\) can be found as the route of the equation:

\[
J_{2l+1/2} \left( b \lambda_{l_k} \right) - g_l(\lambda_{l_k}, a) Y_{2l+1/2} \left( b \lambda_{l_k} \right) = 0.
\]

So we can rewrite the functions as:

\[
\begin{aligned}
\hat{\omega}(r, \theta) &= \sum_{l,k} \Omega_{l_k} u_{l_k}(r, \theta); \\
\psi(r, \theta) &= \sum_{l,k} \Psi_{l_k} u_{l_k}(r, \theta); \\
&\frac{\cos \theta - 1}{r^2 \sin \theta} = \sum_{l,k} F_{l_k} u_{l_k}(r, \theta).
\end{aligned}
\]

So the equations can be rewritten as:

\[
\sum_{l,k} \lambda_{l_k} \Omega_{l_k} u_{l_k}(r, \theta) = \sum_{l,k} A F_{l_k} u_{l_k}(r, \theta);
\]
\[
\sum_{l,k} \lambda_{lk} \Psi_{lk}(r, \theta) = -\sum_{l,k} \Omega_{lk} u_{lk}(r, \theta)
\]

We can take the first mode \( u_{11}(r, \theta) \), so we can obtain the system:

\[
-\lambda_{11} \Omega_{11} = AF_{11};
\]

\[
-\lambda_{11} \Psi_{11} = -\Omega_{11};
\]

where \( \lambda_{11} = 33.21 \) (in our units \( b = 1 \)). For the coefficients we have [8]:

\[
F_{11} = -A(5.69 - 11.19a);
\]

\[
\Omega_{11} = A(0.171 - 0.337a);
\]

\[
\Psi_{11} = A(0.00516 - 0.0101a).
\]

The functions will be the following:

\[
\omega(r, \theta) = Ar^{-1/2} \left( 0.171 - 0.337a \right) \left[ J_{2l+1/2} (r \lambda_{lk}^{1/2}) - \frac{\lambda_{lk}^{1/2}}{r} Y_{2l+1/2} (r \lambda_{lk}^{1/2}) \right] P_{2l}^1 (\cos \theta);
\]

\[
\psi(r, \theta) = Ar^{-1/2} \left( 0.00516 - 0.0101a \right) \left[ J_{2l+1/2} (r \lambda_{lk}^{1/2}) - \frac{\lambda_{lk}^{1/2}}{r} Y_{2l+1/2} (r \lambda_{lk}^{1/2}) \right] P_{2l}^1 (\cos \theta);
\]

The velocity can be found as the curl:

\[
\mathbf{V} = [\nabla, \psi \phi].
\]

It is shown on figure 1.

**Figure 1.** Asymptotic solution of the equations.
4. Numerical solution
We have solved system of equations (2) numerically too using the time marching method [10]. The finite difference method was used. After that the curl of the vector potential was calculated to find the velocity. It is shown on figure 2. As we can see, the results are quite similar.

Figure 2. Numerical solution of the equations

5. Conclusions
We have studied the electrovortex flow using different approaches. The first one is connected with analytical solution for the equations using eigenfunctions of the problem. We took the first mode and show the vector graph of the velocity. We have also modelled the problem numerically and the results were nearly the same.

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7. References
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