Direct Estimation of Sizes of Higher-Order Graphs

John C. Collins$^{a,b}$, Andreas Freund$^{a}$

$^a$Department of Physics, Penn State University, 104 Davey Lab., University Park, PA 16802, U.S.A.

$^b$CERN — TH division, CH-1211 Geneva 23, Switzerland. *

Abstract

With the aid of simple examples we show how to make simple estimates of the sizes of higher-order Feynman graphs. Our methods enable appropriate values of renormalization and factorization scales to be made. They allow the diagnosis of the source of unusually large corrections that are in need of resummation.
I. INTRODUCTION

The starting point for this paper is formed by the following observations:

• The only (known) systematic method for calculating scattering in QCD is perturbation theory. (Lattice Monte-Carlo methods work in Euclidean space-time, and are excellent for calculating static quantities such as masses from first principles. But they are essentially useless when a calculation in real Minkowski space-time is needed.)

• In field theory, calculations beyond low orders of perturbation theory are computationally complex, both because the calculations of individual graphs are hard and because there are many different graphs.

• Hence it is important to make the most efficient use of low-order calculations.

Since the coupling in practical calculations is not very weak, the accuracy of predictions can be ruined by uncalculated higher-order terms. It follows that there is a need to estimate the sizes of the errors. For this one wants quick estimates of terms in perturbation theory. The computational complexity of the estimates should increase as little as possible with the size of the graphs. Indeed, our aim is that one only calculates integrals of the form

\[ \int_1^u dx x^n \ln^p x. \] (1)

With suitable methods:

• One can determine good values for renormalization and factorization scales, by asking how to minimize the error estimates.

• When the estimates get substantially larger than some appropriate “natural” size, one would get a diagnosis of a need for resummation of classes of higher-order corrections. The diagnosis would include an explanation of the large terms and thus indicate the physics associated with the resummation.

In this paper, we explain how to start such a program. It builds on work first reported in Ref. [1]. Our methods treat properties of the integrands of Feynman graphs, and are therefore directly sensitive to the physics of the process being discussed. Some other treatments of these issues discuss the problems in terms of the mathematics of series expansions in general, without asking what is causing the graphs to be the sizes they are. A particular exception is the work of Brodsky, Lepage, and Mackenzie [2]. They use heavy quark loops to probe the actual momentum scales that dominate in a particular calculation; this is then used to provide a suitable value for the renormalization/factorization scale. But we believe that our methods provide a more direct route to answering the question of why the scales are what they are and why a calculation gives a particular order of magnitude. The issues addressed by methods involving the Borel transform and Padé summation address complementary issues [3].

---

1 See also the more recent work of Brodsky and Lu [4] and of Neubert [5].
FIG. 1. One-loop self-energy graph.

Now, most cross sections in QCD cannot be directly computed by perturbation theory; this can only be used to compute the short-distance coefficients that appear in the factorization theorem. So we will also treat the specific problems that arise in estimating the sizes of the coefficient functions. These functions have the form of a sum over Feynman graphs (typically massless), with subtractions to cancel some infra-red (IR) divergences. Remaining IR divergences are cancelled between different graphs or between different final-state cuts. Two characteristic features appear. First, we can obtain estimates for sums of particular sets of graphs, but not for the individual graphs, which are divergent. Secondly, the coefficient functions are not in fact genuine functions. They are normally singular generalized functions (or distributions) and an estimate can only be made for the integral of a coefficient function with a smooth test function.

We show how to estimate the sizes of graphs by a direct examination of the integrands. An important part of our technique is an implementation of subtractions directly in the integrands, both for the subtractions that implement counterterms for ultra-violet (UV) renormalization and for the IR subtractions that are used in short-distance coefficient functions. In order to explain our ideas, we will examine two examples: (1) the one-loop self-energy graph in $\phi^3$ theory, and (2) a particular set of graphs for the Wilson coefficient for deep-inelastic scattering. Our estimates are in the form of approximations to ordinary integrals that are absolutely convergent. This is in contrast to the original integrals, which are typically divergent in the absence of a regulator. Thus a by-product of our work will be algorithms for computing graphs numerically in Minkowski space-time, which may have relevance to work such as Ref. [6]. As an illustration of how estimations can be carried out, even analytically, for a measurable quantity, we will estimate the size of the Wilson coefficient for the structure functions $F_T$ and $F_L$ in the last section.

II. EUCLIDEAN SELF-ENERGY IN $\phi^3$ THEORY

In this section, we give a representation of the one-loop self-energy graph of Fig. 1 in $\phi^3$ theory in four dimensions. A particular renormalization scheme is used, which we relate to ordinary $\overline{\text{MS}}$ renormalization. Then we show how to estimate the size of the graph from elementary integrals and hence how to choose the renormalization scale suitably.

A. Renormalization

The only complication in computing the graph of Fig. 1 is its UV divergence, given that we choose to work in Euclidean space. We give a representation of the graph that is an absolutely convergent integral over the loop momentum itself in four dimensions and in which
the renormalization is explicitly “minimal”. This last term means that the counterterm for the (logarithmic) divergence is independent of the mass and the external momentum. It is a very useful property when one wants to take zero-mass limits, etc. For a general divergence, the counterterm would be polynomial in masses and external momenta.

Our representation of the graph is:

\[ I(p) = \frac{1}{2} \frac{g^2}{(2\pi)^4} \int d^4 k \left\{ \frac{1}{(k^2 + m^2) [(p - k)^2 + m^2]} - \frac{\theta(k > \mu_c)}{k^4} \right\}. \]

(2)

We recognize in the integrand a term that is given by the usual Feynman rules (in Euclidean space-time), and a subtraction term. The subtraction term is the negative of the asymptote of the first term as \( k \to \infty \), so we term our procedure “renormalization by subtraction of the asymptote”. A cut-off is applied to prevent the subtraction term giving an IR divergence at \( k = 0 \); the cut-off does not affect the \( k \to \infty \) behavior and therefore does not affect the fact that the UV divergence is cancelled. To see that Eq. (2) is equivalent to standard renormalization, one simply applies a UV regulator, after which each term can be integrated separately. The first term is the unrenormalized graph and the second term is a \( p \)-independent counterterm.

Evidently, the integral is absolutely convergent, and can therefore be computed by any appropriate numerical method. (It can also be evaluated analytically. But this is not interesting to us, since we wish to obtain methods that work for integrals that are too complicated for purely analytic methods to be convenient or useful.)

The counterterm is in fact the most general one that is independent of \( m \) and \( p \), since any other renormalization counterterm can differ only by a finite term added to the integral that is independent of \( m \) and \( p \), and a change of the cut-off \( \mu_c \) is equivalent to adding such a term. We can relate the counterterm to the commonly used \( \overline{\text{MS}} \) one simply by computing the counterterm alone, with dimensional regularization:

\[ \text{standard pre-factor} \times \int_{|k| > \mu_c} d^n k \frac{1}{k^4}. \]

(3)

The result is that setting \( \mu_c \) equal to the scale \( \mu \) of the \( \overline{\text{MS}} \) scheme gives exactly \( \overline{\text{MS}} \) renormalization. In general, we would find that \( \mu_c \) would be a factor times \( \mu \), or equivalently that we should set \( \mu_c = \mu \) and then add a specific finite counterterm to the graph.

Notice that the integral to relate our renormalization scheme to the \( \overline{\text{MS}} \) scheme is algorithmically simpler to compute analytically than the original integral. There is always the possibility of adding finite counterterms. Moreover, the precise form of the cut-off is irrelevant to the general principles. One can, for example, change the sharp cut-off function \( \theta(k > \mu_c) \) to a smooth function \( f(k/\mu_c) \) that obeys \( f(\infty) = 1 \) and \( f(0) = 0 \). Such a function would probably be better in numerical integration.

Of course, our method as stated is specific to one-loop graphs. But it is an idea that has been generalized \[ \text{to higher orders.} \]

---

2 See \[ \text{for a previous account. A formalization of such ideas (to all orders of perturbation theory) was given earlier by Ilyin, Imashev and Slavnov \[ , and later by Kuznetsov and Tkachov \[ .} \]

---
We now show how to estimate the size of the integral Eq. (2). To give ourselves a definite case, let us choose $p^2 \lesssim m^2$. We obtain the estimate as the sum of contributions from $k < m$ and from $k > m$. Since the renormalization counterterm is designed to subtract the $k \to \infty$ behavior of the unrenormalized integrand, we regard it as a $\delta$-function at infinity and therefore to be associated completely with the $k > m$ term in our estimate.

In the region $k < m$, our estimate is obtained by replacing each propagator by $1/m^2$ so that

$$\text{Contribution from } k < m \simeq \frac{g^2}{32\pi^4} \int_{k<m} d^4 k \frac{1}{m^4} \approx \frac{g^2}{64\pi^2}. \quad (4)$$

This factor is the product of $g^2/32\pi^4$ for the prefactor and $\pi^2/2$ for the volume of a unit 4-sphere. As advertised, we have had to calculate no integral that is more complicated than a simple power of $k$. The approximation of replacing the propagators by $1/m^2$ leads us to an over-estimate of the integral, but not by a great factor, since we are in a region of small momentum.

The estimate for $k > m$ is obtained by replacing the propagators by their large $k$ asymptote:

$$\text{Contribution from } k > m \simeq \frac{g^2}{32\pi^4} \int_{k>m} d^4 k \left[ \frac{1}{k^4} - \frac{\theta(k > \mu_c)}{k^4} \right] = \frac{g^2}{32\pi^4} \left[ \int_{\mu}^\infty \frac{dk}{k} - \int_{m}^\infty \frac{dk}{k} \right]. \quad (5)$$

Since we are taking the difference of two terms, we must be careful about the errors, which are of order

$$\int_{k>m} d^4 k \frac{m^2}{k^6} = \frac{g^2}{32\pi^2}. \quad (6)$$

To understand the structure of the result, let us examine how the original integral (2) appears after integrating over the angle of $k$:

$$\frac{g^2}{16\pi^2} \int_0^\infty \frac{dk}{k} \left[ A(k, p, m) - \theta(k > \mu_c) \right]. \quad (7)$$

The function $A$ is the angular average of $k^4$ times the two propagators. It approaches 0 as $k \to 0$, so that the integral is convergent there, and it approaches unity as $k \to \infty$, which would give a UV divergence were it not for the subtraction.

We can represent this situation by the graphs of Fig. 2. The transition region for the unsubtracted integrand, where it changes from being 0 to 1, is around $k = m$, to within a factor of 2 or so; this is evident by examining the denominators. By setting $\mu_c$ to be $m$ to within a factor of 2, we achieve the following:

- The integrand is less than unity everywhere.

---

B. Estimate
• It is concentrated in a shell of thickness of order $m$ around $k = m$.

Thus we can say that the natural size of the graph is the product of

• The prefactor $g^2/32\pi^4$.
• The surface area of a unit 4-sphere: $2\pi^2$.
• The width of the important region, about unity, in units of $m$.
• A factor of $m$ to the dimension of the integral, i.e., 1.
• A factor less than unity, say $1/2$, to allow for the fact that, after subtraction, the integrand in Eq. (7) is smaller than unity and that there is a cancellation between negative and positive pieces.

That is, the natural size is $g^2/32\pi^2$, if $\mu$ is reasonably close to $m$.

If $\mu$ is not close to $m$, then we get a long plateau in the integrand—see Fig. 2. The height of the plateau is unity, and its length is $\ln(\mu/m)$ to about $ \pm 1$ in units of $\ln k$. This clearly gives a larger-than-necessary size, and an optimal choice of the MS scale is around $m$.

One should not expect to get an exact value for the scale $\mu$. A physical quantity in the exact theory is independent of $\mu$, and any finite-order calculation differs from the correct value by an amount whose precise value is necessarily unknown until one has done a more accurate calculation. If one is able to estimate the size of the error, as we are proposing, then an appropriate value of $\mu$ is one that minimizes the error. Given the intrinsic imprecision of an error estimate, there is a corresponding imprecision in the determination of $\mu$. One can expect to identify, without much work, an appropriate scale $\mu$ to within a factor 2, and, with
a bit more work, to within perhaps 50%. These estimates just come from asking where the transition region in the integrand is, and by then obtaining an answer by simple examination of the integrand. But one cannot enter into a religious argument of the wrong kind as to whether the correct scale is $1.23m$ as opposed to $1.24m$, for example. By definition an error estimate is approximate.

Since we have not yet investigated how to estimate even higher-order graphs, we are making the reasonable conjecture that the properties of graphs do not change rapidly with order. Then our estimate that $\mu$ should be close to $m$ will ensure that higher-order graphs are of the order of their natural size.

### C. Implication for QCD

The same arguments applied to similar graphs in QCD show that the natural expansion parameter in QCD is

$$\frac{\alpha_s}{4\pi} \times \text{(group theory)} \times \text{(factor for multiplicity of graphs)}. \quad (8)$$

These arguments rely on being able to show that in the dominant part of the range of integration all lines have approximately a particular virtuality and that the relevant range of integration is a corresponding volume of momentum space.

In the general case we cannot expect to get a much smaller result, but we can expect that in some situations the properties of the dominant integration region(s) will not be so good. So what we need to do next is to analyze more interesting graphs in QCD. This we will do in the next section.

In general, when we get corrections in QCD that are substantially larger than the natural size given above, it must be either because the integrand is excessively large, or because there is no single natural scale, or because we have not chosen a good scale. If the integrand is especially large relative to the natural unit, or if there is no single natural scale, then we should investigate in more detail the reasons, and derive something like a resummation of higher-order corrections. In precisely such situations, one does indeed have to compute high-order graphs. At the same time, there is no need to compute the complete graphs in all their gory detail, but only their simple parts.

---

3 Brodsky and Lu obtain very precise estimates of a suitable scale. Their rationale is the elimination of IR renormalons in the relations between IR-safe observables. This is a concern with very high-order perturbation theory, an issue that we do not address.

4 Our use of the word “natural” may suggest that we are proposing to estimate higher-order corrections simply by multiplying the appropriate power of the natural expansion parameter by the number of graphs. This is not what we mean. We are arguing first that the sizes of graphs can actually be estimated fairly simply, and that the natural expansion parameter is a useful unit for these estimates. Secondly, we show that in the most favorable cases, graphs are less than or about unity in these natural units. Finally, we argue, in the next section, that general kinematic arguments about the physics of a graph are useful in diagnosing cases where graphs are large in natural units.
FIG. 3. The cuts of a one-loop graph for the Wilson coefficient for deep-inelastic scattering.

### III. WILSON COEFFICIENT FOR DEEP-INELASTIC SCATTERING

The factorization formula for the leading-twist part of a deep-inelastic structure function $F(x)$ is

$$F(x) = \int \frac{d\xi}{\xi} f(\xi) \hat{F}(x/\xi).$$  \hspace{1cm} (9)

Here, $f(\xi)$ is a parton density, and $\hat{F}$ is the short-distance coefficient (“Wilson coefficient”). We have suppressed the indices for the different structure functions ($F_1$, $F_2$, etc.) and for the parton flavor. The coefficient function $\hat{F}(x/\xi)$ is obtained from Feynman graphs for scattering on a parton target with momentum $\xi p$, where $p$ is the momentum of the hadron target. Subtractions for initial-state collinear singularities are applied to the Wilson coefficient and the massless limit is taken.

As an example, we will examine the contribution to the Wilson coefficient from the diagrams in Fig. 3. These diagrams are the two possible cuts of a particular uncut one-loop graph, and since we will need to use a cancellation of final-state interactions, we must consider the sum of the two cut graphs as a single unit. Note:

- We will apply a subtraction to cancel the effect of initial-state collinear interactions where the incoming quark splits into a quark–gluon pair which are moving almost parallel to the incoming particle.\(^5\)

- There will be soft-gluon interactions and collinear final-state interactions. These will cancel after the sum over cuts. (In a more general situation, a sum over a gauge-invariant set of graphs is necessary to get rid of all soft-gluon interactions.)

\(^5\) According to the factorization theorem, the subtractions cancel all the sensitivity to small momenta, i.e., to the initial-state collinear interactions. The subtractions are of the form of terms in the perturbative expansion of the distribution of a parton in a parton convoluted with lower-order terms in the coefficient function. (See, for example, [10,13] for details.) Of course, both the partonic cross section and the subtraction term have to be properly renormalized. We will call the subtraction terms eikonal because of the particular rules involved in their calculation, for graphs such as Fig. 3.
FIG. 4. Space-time structure of deep-inelastic scattering, in the center-of-mass frame of the virtual photon and the struck quark. The solid line is the almost (light-like) world line of the incoming quark. The dashed line is the world line of the single struck quark in the lowest-order (Born) graph for the hard scattering.

- The Wilson coefficient is a distribution (or generalized function) rather than an ordinary function of $x/\xi$. Thus it is useful to discuss only the size of the coefficient after it is integrated with a test function, but not the size of the unintegrated coefficient function. The parton density $f(\xi)$ provides a ready-made test function that has a physical interpretation.

- The first graph of Fig. 3, which has a virtual gluon, has a 4-dimensional integral, but the second graph, with a real gluon, has only a 3-dimensional integral. Thus the cancellations associated with the sum over cuts can only be seen after doing at least a 1-dimensional integral.

It is useful to visualize the process in space-time, Fig. 4, and to use light-front coordinates $(+, -, T)$ (defined by $V^\pm = (V^0 \pm V^3)/\sqrt{2}$). Our axes are such that the incoming momenta for the hard scattering are:

$$\xi p^\mu = (\xi p^+, 0, 0, 0), \quad q^\mu = \left(-xp^+, \frac{Q^2}{2xp^+}, 0, 0, T\right). \quad (10)$$

Also, we find it convenient to parameterize the gluon momentum in Fig. 3 in terms of two longitudinal momentum fractions, $u$ and $z$, and a transverse momentum $k_T$, as follows:

$$k^\mu = (u\zeta(1-z)p^+, zq^-, k_T). \quad (11)$$

Thus $z$ is exactly the fraction of the total incoming minus component of momentum that is carried off by the gluon, while $u$ is a scaled fraction of the plus component. The scaling is somewhat unobvious, but it has the effect that positive energy constraints on the final state restrict each of $u$ and $z$ to range over the 0 to 1.
Now we summarize how the calculation of the real and virtual graphs and of the subtraction graphs \cite{10}, contributing to the Wilson coefficients, is to be carried out:

- Using the Feynman rules for cut diagrams, we write down the momentum integral with the appropriate $\delta$-functions. Then we contract the trace over Dirac matrices with the appropriate transverse, longitudinal or asymmetric tensor on the photon indices to obtain the structure function $F_1$, $F_2$, etc. For the sake of a definite simple example we choose to contract with $-g_{\mu\nu}$, which in fact gives the combination $3F_1 - F_2/2x$.

- We express the results of the calculations in terms of the light-cone components of $k$ (the internal gluon momentum), of $p$ (the incoming quark momentum), and of $q$ (the incoming photon momentum).

- In the graphs with real gluon emission, we use the two $\delta$-functions to perform the integrals over $k_T$ and over the fractional momentum $\xi$ entering from the parton density. This leaves a 2-dimensional integral.

- In the virtual graphs, we use the one $\delta$-function to perform the integral fractional momentum $\xi$ entering from the parton density. We then perform the $k_T$ integral analytically. Again we have a 2-dimensional integral.

- We change the integration variables to the scaled dimensionless variables $u$ and $z$ defined in Eq. (11). This gives us an overall factor, just like the $g^2/16\pi^2$ in the self energy, times an integral over roughly the unit square in $u$ and $z$.

**A. Real gluon**

For the real gluon graph Fig. 3(a) it is well known that the integrand has the following singularities in the massless limit:

- Initial-state collinear singularity on $z = 0$. 

FIG. 5. Integration region and position of singularities for Fig. 3(a).
• Final-state collinear singularity on $u = 0$.

• Soft singularity $k = 0$ at the intersection of the previous two singularities, i.e., at $u = z = 0$.

(See Fig. 3.) In accordance with the standard recipe for constructing the Wilson coefficient, we subtract the initial-state collinear singularity. The subtraction term itself has a UV divergence, which we choose to cancel by using the same method as we used for the UV divergence of the self-energy graph. This gives:

$$\frac{g^2}{8\pi^2} C_F \int_0^1 dz \int_0^{1-x} du \, f \left( \frac{x}{1-u} \right) \left[ \frac{1-z}{zu} - \theta(z < z_{\text{cut}}) \right].$$

(12)

The cut-off $z_{\text{cut}}$ on the subtraction term is analogous to the cut-off $\mu_c$ we used for the UV counterterm for the self-energy graph. The value of $z_{\text{cut}}$ needed to reproduce the \overline{MS} prescription can be found by a simple calculation from the Feynman rules for parton densities. But we will not use this result here. Rather, we will aim at calculating an appropriate value for $z_{\text{cut}}$ to keep the one-loop correction down to a “normal size” (and, most importantly, whether it is possible to find such an appropriate value at all). This effectively amounts to a choice of factorization scheme. Once a suitable value for $z_{\text{cut}}$ has been obtained, it is a mechanical matter to translate it to a value for $\mu_{\overline{\text{MS}}}$ (or to a value of the scale $\mu$ in any other chosen scheme). The calculation may also result in a need for an extra finite counterterm.

The subtraction in Eq. (12) has evidently accomplished its purpose of cancelling the initial-state singularity. But we are still left with the singularity on the line $u = 0$. This singularity will cancel against a singularity in the virtual graph, as we will now see.

B. Sum of virtual and real graphs

Next we compute the virtual graph of Fig. 3(b), following the same line as in the previous subsection. We will construct an integral in the same variables as the real graph. The reason why we do this is that the cancellation of the divergence at $u = 0$ will be point-by-point in the integrand. This can be seen from the proof by Libby and Sterman [11]. They treat a general case of final-state interactions, of which our example is a particular case. They first treat one integration analytically, with the aid of the mass-shell conditions for the final state, in such a way that the integrations for graphs related by different positions of the final-state cut then have the same dimensions. After that, the cancellation between the different graphs is point-by-point in the integrand.

This implies that we need to perform the $\xi$ and $k_T$ integrals. We do the convolution with $\xi$ by the mass-shell $\delta$-function, which now gives $\xi = x$. Then we do the $k_T$-integral analytically. (This is not the most trivial integral, but it works conveniently with our choice of variables.) An example of the type of integral encountered is:

\footnote{We have chosen the overall normalization of the graphs to be such that the lowest-order Born graph gives just $f(x)$.}
FIG. 6. The integrand of Eq. (14) as a function of $u$ at fixed $z$ when $f(x)$ is a steeply falling function of $x$.

\[
\int_0^\infty \frac{dk_T^2}{[Q^2(z-1)u-k_T^2+i\epsilon][Q^2(u-1)z-k_T^2+i\epsilon][Q^2uz-k_T^2+i\epsilon]}.
\] (13)

In the virtual case, no longer does a positive energy condition restrict the range of $u$ and $z$. Nevertheless, it is convenient to split up the result into a piece inside the square $0 \leq u, z \leq 1$ and a piece from outside the square. It is sufficient to examine the contribution within the square. After subtracting the collinear singularity and adding the result from the real graph we get

\[
\frac{g^2}{8\pi^2}C_F \int_0^1 \int_0^1 du \, dz \left[ f\left(\frac{x}{1-u}\right) - (1-u)f(x) \right] \left[ \frac{1-z}{zu} - \theta(z < z_{cut}) \right] + \text{contribution from outside } 0 \leq u, z \leq 1.
\] (14)

The UV divergence ($k \to \infty$) is from outside the square and is cancelled by subtractions just as for the self-energy graph.

We see that the singularity at $u = 0$ for fixed $z$ has cancelled, even at the intersection of the two singular lines. However, how good the cancellation is depends on the test function. If $f(x)$ is slowly varying, the cancellation is good over a wide range of $u$. But if $f(x)$ is steeply falling as $x$ increases, the cancellation is good only over a narrow range of $u$, and we are left with an integrand that behaves like $1/u$ for larger $u$. For fixed $z$ we then have an integrand like that in Fig. 6.

This kind of behavior is quite typical when singularities are cancelled between different graphs with different final states. The integrand is a non-trivial distribution, and how good the cancellation of the singularity is depends on properties of the test function. The

---

7 In (13) the upper limit on $u$ is $1-x$, but we replace the limit by 1 in when we copy the formula into (14). This change is innocuous since the limit $1-x$ arises from the fact that the parton density $f(\xi)$ is 0 when $\xi > 1$, and hence that $f(x/(1-u))$ is 0 when $u > 1-x$. The limits on $u$ that result from positivity of energy of the two final-state lines in Fig. 3(a) are just $0 < u < 1$. 

---
cancellation only occurs after integration over a range of final states. This is in distinct contrast with the case of the initial-state singularity for which we have constructed an explicit subtraction. The cancellation of the \( z \to 0 \) singularity occurs \textit{before} the integration with the test function.

Immediately we also get complications in computing the typical size of the graph. In particular, when \( f(x) \) is steeply falling, we should therefore expect a size for the integral that is much larger than the natural size we defined earlier.

\section*{C. Estimate of size}

First, as a benchmark case, let us assume that \( f(x/(1 - u)) \) is slowly and smoothly varying when \( u \) increases from 0 to 1/2, and that \( z \) is around 1/2. Then the size of \( (14) \) can be estimated as

\[
\frac{g^2}{8\pi^2} \times \text{(group theory)} \times \text{(area of unit square in \((u, z)\))} \times f(x) \simeq \frac{g^2}{6\pi^2} f(x). \tag{15}
\]

A way of obtaining this result with the same method as we used for the self-energy is to write

\[
\frac{g^2}{(2\pi)^4} C_F \times \left( \text{range of } k, \text{ i.e., } 2\pi^2 Q^4 \right) \times \left( \text{size of integrand, } \frac{f(x)}{Q^4} \right). \tag{16}
\]

Given that the lowest-order graph is \( f(x) \), and that the integrand varies in sign, all this implies that the contribution of this graph (with all the cuts and subtractions) is probably somewhat smaller than \( g^2/6\pi^2 \) times the lowest-order graph. In other words, the simplest estimate for the graphs, Eq. (8), is a valid estimate in this situation.

There are a modest number of graphs, so this result would have very nice implications for the good behavior of perturbation theory: the real expansion parameter of QCD would be \( \alpha_s/\pi \), which is a few per cent in many practical situations.

Unfortunately, the conclusion is vitiated when \( f(x) \) is steep, as is often the case. Consider the parton density factor times the \( 1/u \) factor, relative to the lowest-order factor \( f(x) \). In the limit \( u \to 0 \), this is

\[
f \left( \frac{x}{1-u} \right) - f(x) \sim \frac{xf'(x)}{f(x)}. \tag{17}
\]

\footnote{The reason for using ‘\( u = 1/2 \)’ in these criteria rather than, say, \( u = 1 \) is the same as for using \( \mu = m \) in the calculation of the self-energy. It is a rough attempt to optimize the errors without using the details of the integrand, since \( u = 1/2 \) is midway between the singularity at \( u = 0 \) and the approximate edge of the region of integration. The integration region for the real graph extends to \( u = 1 - x \), while that for the virtual graph extends beyond \( u = 1 \). Analogous reasoning applies to \( z \) and the relation between the real graph and its eikonal approximation even though, in this case, one has to look at the whole integrand since both test functions are of the form \( f \left( \frac{x}{1-u} \right) \).}
This factor should be of order unity, if the previous estimate of the size, Eq. (15), is to be valid.

But if the logarithmic derivative \( x f'(x)/f(x) \) is much bigger than 1, then we have to change our estimates. Consider a typical ansatz for a parton density:

\[
f(x) \propto (1 - x)^6,
\]

for which

\[
\left| x f' \right| = \frac{6x}{1 - x}.
\]

This is 6 when \( x = 1/2 \), and goes to infinity as \( x \to 1 \). Clearly our estimate in Eq. (15) is bad. Moreover the \( u \to 0 \) estimate, Eq. (17), is only approximately valid when \( f(x) \) does not change by more than a factor 2 (roughly). Once \( u \) gets larger than the inverse of the logarithmic derivative, the \( f(x/(1 - u)) \) term in Eq. (14) is no longer important, and we get the result pictured in Fig. 6: we have basically a \( 1/u \) form with a cut-off at small \( u \). This is a recipe for a large logarithm, with the argument of the logarithm being the large logarithmic derivative.

To make the estimate, it is convenient to define

\[
\delta u = -\frac{f(x)}{2f'(x) x}.
\]

From Eq. (17), we see that \( \delta u \) is approximately the change in \( u \) to make \( f(x/(1 - u)) \) fall by a factor of 2. We interpret \( \delta u \) as the value of \( u \) at which the final-state cancellations become “bad”.

Next we note that for normal parton densities \( f(x) \) is a decreasing function of \( x \). So a simple useful estimate can be made by making the following approximation:

\[
f \left( \frac{x}{1 - u} \right) \simeq \begin{cases} 
  f(x) \left( 1 - \frac{u}{2\delta u} \right) & \text{if } u < 2\delta u, \\
  0 & \text{if } u > 2\delta u.
\end{cases}
\]

Therefore, in Eq. (14), we can replace \( \int_0^1 \frac{du}{u} f(x/(1 - u)) \) by \( \int_0^{2\delta u} \frac{du}{u} f(x) \).

Our estimate for Eq. (14) is the sum of contributions from the following regions:

- \( 0 < u < 2\delta u \): The value of the integrand is about \(-f(x)/2\delta u \) times a function of \( z \). For good choices of \( z_{\text{cut}} \), the function of \( z \) is less than about unity, but with an indefinite sign. Thus we obtain a contribution of about \( f(x) \) in size.

- \( 2\delta u < u < 1 \): The integrand is now approximately \(-f(x)/u \), again times a mild function of \( z \). We therefore obtain a contribution of about \( f(x) \ln(1/2\delta u) \).

- Exterior of unit square: Here, the only contribution is from the virtual graph, and we have no final-state singularity. Hence the naive estimate of unity is valid. (The precise value, when \( \mu = Q \), is in fact somewhat larger.)
All of these are to be multiplied by the pre-factor \( g^2 C_F/8\pi^2 \). So we obtain a total contribution of

\[
\frac{g^2}{8\pi^2} C_F f(x) \left[ \pm 2 \pm \ln \left( \frac{1}{2\delta u} \right) \right],
\]

where each term represents an estimate, valid up to a factor of 2 or so. The contributions from within the unit square may have either sign, depending on the cut in the collinear subtraction, while the contributions from outside the unit square, from virtual graphs only, have a negative sign. As an explicit indication that our estimates are valid for the sizes but not the signs of the graphs, we have inserted a \( \pm \) sign in front of each term. This estimate assumes that renormalization of the UV divergence of the virtual graph is done at the natural scale \( \mu \simeq Q \), and that renormalization of the parton densities is done so that it corresponds to \( z_{\text{cut}} \) of about 1/2. It also assumes that \( f(x) \) falls steeply enough for \( \delta u \) to be less than about 1, as is typically true.

**D. Interpretation**

It is obvious that there is a logarithmic enhancement in Eq. \((22)\) whenever \( \delta u \) is small. Our calculation is, of course, no more than a redervivation of the standard observation that there are large logarithms in the \( x \to 1 \) region. What our derivation adds is to show that it is not so much the limit \( x \to 1 \) that is causing the problem as the steepness of the parton densities. Moreover we have given a numerical criterion for when the correction begins to be larger than what we called the natural size for higher-order corrections. In other words we have shown how to estimate the constant term that accompanies the logarithm. Moreover this is all presented in the context of a general method for obtaining estimates of the sizes of graphs.

It is perhaps clear that, with sufficient foresight, one could have predicted the large corrections merely from the observation that the derivation of the factorization theorem requires the cancellation of final-state divergences between different final states.

There is in fact another source of large corrections that will make its effect felt in even higher order. This is a mismatch in the scales needed for renormalizing the parton densities. We have renormalized these by using a value of \( z_{\text{cut}} \) that must be about 1/2 to avoid making the contribution of the graph unnecessarily large. In the case of the real graph, we can translate this to a scale of transverse momentum by using the mass-shell condition for the gluon:

\[
k_T^2 = Q^2 u z(1-z) \frac{\xi}{x} = Q^2 u z(1-z) \frac{1}{1-u}.
\]

Evidently, whenever small values of \( u \) are important, small values of \( k_T \) (relative to \( Q \)) will be important. This does not affect our one-loop calculation. But in higher-order correction, the virtuality of some internal lines will be controlled by the value of \( k_T \), and hence there will be mismatches between the scales needed at different steps in the calculation.

Once one has diagnosed the problem, we see that a proper solution lies in more accurately calculating the form of the Wilson coefficient near its singularity. This subject goes under the heading of resummation of large corrections \([9]\).
IV. MORE DETAILED ESTIMATION OF $F_{T,L}$ TO ONE-LOOP ORDER

In the following we will estimate the sizes of the one-loop corrections to the structure functions $F_T$ and $F_L$ without doing actual calculations of Feynman diagrams by giving a recipe of how to construct the estimates from general principles and kinematic considerations. However, we will present the recipe in the context of an actual set of Feynman graphs. By using the calculations of the graphs in the appendix, we will verify that these “simple-minded” estimates are actually valid.

One obtains $F_{T,L}$ from the well-known hadronic tensor $W^{\mu\nu}$ by projecting out the “transverse” and “longitudinal” pieces via:

\[
F_T = -g^{\mu\nu}W^{\mu\nu} = 3F_1 - \frac{F_2}{2x},
\]

\[
F_L = \frac{Q^2 p_\mu p_\nu}{p \cdot q^2} W^{\mu\nu} = -F_1 + \frac{F_2}{2x},
\]

which give, for example,

\[
F_1 = \frac{1}{2} \left[ -g^{\mu\nu} - \frac{Q^2 p_\mu p_\nu}{p \cdot q^2} \right] W^{\mu\nu}.
\]

A. Estimation of $F_L$

The recipe for estimating $F_L$ is the following:

- The singularities we encounter (UV, collinear and soft) are all in the form of a factor times the Born graph, and the Born graph has no longitudinal part. Therefore the one-loop graphs for $F_L$ have no UV, soft or collinear singularities.

- Since the parton densities are falling with increasing $x$, the size of a graph is:

\[
\frac{g^2}{8\pi^2} \times C_F \times f(x) \times \text{range of } u \times \text{range of } z.
\]

- The range of $z$ is 1.

- The range of $u$ is $\delta u$.

It is elementary to show that the self-energy and vertex graphs (whether real or virtual) give a zero contribution to $F_L$. Therefore, our result for $F_L$ is:

\[
F_L = \frac{g^2}{6\pi^2} f(x) \delta u.
\]

Let us now check whether our intuition has guided us in the right way. We use Eq. (A5) for the contribution to the coefficient function for $F_L$. We approximate the $u$ integral by

\[
\int du f \left( \frac{x}{1-u} \right) \simeq \delta u f(x),
\]
which is appropriate for a typical parton density, which falls with increasing \( x \). The \( z \) integral gives a factor \( 1/2 \), and we recover Eq. (27), which we obtained by more general arguments.

Notice that \( F_L \) is generally rather smaller than what we have termed the natural size, because of the \( \delta u \) factor: i.e., because of the restricted phase space available. There are no enhancements due to final-state singularities.

Our estimation methods can be applied in two ways. One is to estimate the graphs without having explicit expressions for the graphs; one just searches for the possible singularities that would prevent the natural size of a graph from being its actual size. The second way of using the methods is to examine the expressions for graphs with the knowledge of the singularity and subtraction structure and to perform a more direct estimate of the sizes. This is useful since the integrals must often be performed numerically, e.g., whenever one convolutes with parton densities that are only known numerically. Additional information that is now obtained concerns the typical virtualities, etc., of the internal lines of the graphs (compare Neubert’s work [4]). This enables a diagnosis to be made of the extent to which a problem is a multi-scale problem and therefore in need of resummation.

A point that we have not addressed is the estimation of the size of the trace of a string of gamma matrices, for example. In the standard formula for such a trace involves a large number of terms. Nevertheless, it is evident from the above calculations that there are cancellations. The final result is that we obtain, relative to corresponding numbers for a scalar field theory, a small factor (1 to 4) times a standard Lorentz-invariant quantity for the process.

Evidently, more work on this subject is needed. But the issue of the size of the numerator factors from traces, etc., affects completely finite quantities, such as the one-loop coefficient function for \( F_L \), just as much as quantities with divergences that are cancelled. However, it is the latter quantities that have the potential for especially large corrections. Numerator factors result in magnitudes common to all graphs.

\section*{B. Estimation of \( F_T \)}

We have already calculated one graph, Fig. 3, for \( F_T \), so we only need to summarize the method and apply it to the remaining graphs. The general procedure is:

- Since we have singularities in our graphs, they have to be cancelled. In Wilson coefficients such as we are calculating, there are explicit subtractions for the collinear initial-state singularities. Then there are explicit subtractions for UV divergences (both those associated with the interactions and those needed to define the parton densities and that therefore enter into the initial-state subtractions). Finally there are final-state singularities that cancel between real and virtual graphs.

- We consider separately the integrations inside and outside the square \( 0 < u, z < 1 \).

- A term of the order of the natural size arises from outside the square \( 0 < u, z < 1 \). This comes from purely virtual graphs, with their collinear subtraction.

- Inside the square, a real graph without a final-state singularity contributes an amount of the order of the natural size times \( \delta u \). This reflects the restriction on the range of integration imposed by the parton density.
• Similarly a virtual graph without a final-state singularity contributes a term of the natural size.

• Finally, the sum of real and virtual graphs with a final-state singularity contributes a term of the order of the natural size enhanced by a factor $2 + \ln(1/2\delta u)$, just as in our estimate of Fig. 3.

We have the graphs of Fig. 3, their Hermitian conjugates, the cut and uncut self-energy graph of Fig. 4, below, and the ladder graph, Fig. 5. Since $\delta u \lesssim 1$ typically, the ladder graph gives a small contribution, and it is sufficient to multiply the estimate of the Fig. 3 by 3:

$$F_T = \frac{g^2}{2\pi^2} f(x) \left[ 2 + \ln \left( \frac{1}{2\delta u} \right) \right].$$

(29)

To see how this compares with the results from the actual graphs, we use those in the appendix.

The cut and uncut self-energy graphs, Fig. 4 below, have final-state singularities, as can be seen in Eq. (A3). We found it convenient to use $z$ and $k_T$ as integration variables. There is a singularity at $k_T = 0$ in each individual graph. In analogy to the definition of $\delta u$, Eq. (20), we define

$$\delta k_T^2 = -\frac{Q^2 f(x)}{2f'(x)},$$

(30)

and by following the same steps as we applied to the ladder graph, we find the following estimate for the contribution of Fig. 7:

$$-\frac{g^2}{16\pi^2} C_T f(x) \left[ 1 + \ln \left( \frac{\mu^2}{2\delta k_T^2} \right) \right].$$

(31)

We have inserted a $-\$ sign in this estimate; it is fairly easy to see that the coefficient of the logarithm is negative.

For the ladder graph, whose contribution to $F_T$ is in Eq. (A4), there is only an initial-state singularity and that is cancelled by an explicit subtraction. If $z_{\text{cut}}$ is around $1/2$, then we can apply the same reasoning as for the longitudinal part of the ladder graph, and we find an estimate

$$\pm \frac{g^2}{6\pi^2} f(x) \delta u,$$

(32)

where, as in Eq. (22), we use the $\pm$ to indicate that our estimates do not determine the sign of the contribution.

We have already examined the cut and uncut vertex graphs, in Eq. (22). When we multiply this by 2 (to allow for the Hermitian conjugate graphs) and add the self-energy and ladder contributions, Eqs. (31) and (32), we get somewhat less than our original estimate, Eq. (29), provided the renormalization mass $\mu$ is in a reasonable range. This lower value is because the self-energy graph is simpler than the vertex graph.
V. CONCLUSIONS

• We have a systematic method for estimating the sizes of higher-order graphs.

• The natural expansion parameter in QCD is of the order of

$$\frac{g^2}{8\pi^2} \times \text{group theory} \times \text{number of graphs.} \quad (33)$$

In answer to a question asked when this work was presented at a conference, let us observe: The above number may not be the actual expansion parameter, but we argue that this natural size sets a measure of whether actual higher-order corrections are of a normal size or are especially large.

• The method allows an identification of appropriate sizes for renormalization and factorization scales.

• By an examination of the kinematics of graphs, we can identify contributions that are large compared with the estimates based merely on the size of the natural expansion parameter.

• Our method should give a systematic technique of diagnosing the reasons for the large corrections, and hence of indicating where one should work out resummation methods.

• The method gives an algorithm for the numerical integration of graphs, both real and virtual.

This is clearly just the start of a project, and there is also a lot of overlap with work such as that of Catani and Seymour [6], of Brodsky and Lu [3], and of Neubert [4].

ACKNOWLEDGEMENTS

This work was supported in part by the U.S. Department of Energy under grant number DE-FG02-90ER-40577.

APPENDIX A: THE REMAINING LONGITUDINAL AND TRANSVERSE CONTRIBUTIONS TO THE WILSON COEFFICIENT

1. Transverse part of the self-energy

The real part of the self-energy is computed along the same lines as mentioned in the main body of the text, except that we chose to integrate over $u$ instead of $k_T$ as a matter of convenience. The graph and its possible cuts are shown in Fig. 7. As far as the virtual part is concerned, one notes first that the general structure of the quark self-energy for zero mass is [12]:

$$\Sigma(\hat{p}) = B(0)\hat{p}, \quad (A1)$$
FIG. 7. (a) Real self-energy graph, (b) virtual self-energy graph. There is a second virtual graph that is the conjugate of graph (b).

FIG. 8. The ladder graph for the Wilson coefficient.

where $B(0)$ is computed via:

$$B(0) = \frac{1}{4} \left. \text{Tr} \left( \gamma^+ \frac{\partial \Sigma(\hat{p})}{\partial \hat{p}^+} \right) \right|_{\hat{p}=0}. \quad (A2)$$

The result for the virtual graph is then given by simply multiplying the Born result by $B(0)$ and convoluting with the test function. The complete result according to our prescription is:

$$\frac{g^2}{8\pi^2} C_F \left[ \int_0^1 dz \int_0^{k_{T,\text{max}}^2} d\mathbf{k}_T^2 \frac{z}{k_T^2} f \left( \frac{x}{y} \right) - \int_0^1 dz \int_0^{\mu^2} d\mathbf{k}_T^2 \frac{z}{k_T^2} f(x) \right], \quad (A3)$$

where $\frac{1}{y} = 1 + \frac{k_T^2}{Q^2}$, after having made the change of variable $k_T^2 \to k_T^2 z(1 - z)$, which also gives $k_{T,\text{max}}^2 = \frac{Q^2(1-x)}{x}$. Renormalization by subtraction of the asymptote is used in our formula with a cut-off on the momentum integral for the virtual graph of $\mu^2$. As one can see there are no initial-state singularities simply because propagator corrections do not induce initial-state singularities as vertex corrections do. The final-state singularities are still there; thus one needs to look at both the real and the virtual graphs together.

2. The transverse part of the ladder diagram

The calculation of the ladder graph has been carried out as outlined in the main text, yielding:

$$\frac{g^2}{8\pi^2} C_F \int_0^{1-x} du \int_0^1 dz \frac{u}{z(1-u)} f \left( \frac{x}{1-u} \right) \left[ 1 - \theta(z - z_{\text{cut}}) \right]. \quad (A4)$$
The second term, with its $\theta$-function, is the collinear subtraction, whose UV divergence is cancelled by subtraction of the asymptote. The above formula assumes that $z_{\text{cut}} < 1$. If $z_{\text{cut}} > 1$, then we must extend the $z$ integration in the second term beyond the limit $z = 1$, of course.

3. Longitudinal part of the vertex correction, self-energy and ladder diagram

The calculation has been carried out as outlined in the main text and yields the following for the ladder graph:

$$\frac{g^2}{4\pi^2} C_F \int_0^{1-x} du \int_0^1 dz (1 - z) f \left( \frac{x}{1 - u} \right).$$

The real and virtual graphs give 0 for both the self-energy and the vertex correction. There is no virtual graph for the ladder diagram.
REFERENCES

[1] J.C. Collins, ‘Choosing the renormalization/factorization scale’, in Proceedings of the Jet Studies Workshop, Durham, U.K., December 9–15, 1990, J. Phys. G 17, 1549 (1991).
[2] S.J. Brodsky, G.P. Lepage, and P.B. Mackenzie, Phys. Rev. D28, 228 (1983).
[3] S.J. Brodsky and H.J. Lu, Phys. Rev. D51, 3652 (1995).
[4] M. Neubert, Phys. Rev. D52, 5924 (1995).
[5] See for example, J. Ellis, E. Gardi, M. Karliner, and M.A. Samuel, Phys. Lett. B366, 268 (1996), and references therein.
[6] S. Catani and M.H. Seymour, ‘A general algorithm for calculating jet cross-sections in NLO QCD’, preprint CERN-TH/96-029, e-Print Archive: [hep-ph/9605323](http://arxiv.org/abs/hep-ph/9605323).
[7] V.A. Ilyin, M.S. Imashev and D.A. Slavnov, Teor. Mat. Fiz. 52, 177 (1982); D.A. Slavnov, Teor. Mat. Fiz. 62, 335 (1985).
[8] A.N. Kuznetsov and F.V. Tkachov, ‘Multiloop Feynman Diagrams and distribution theory, (III) UV renormalization in momentum space’, preprint NIKHEF-H/90-17.
[9] G. Sterman, Nucl. Phys. B281, 310 (1987); S. Catani and L. Trentadue, Nucl. Phys. B327, 323 (1989) and B353, 183 (1991).
   For a review of recent progress, see S. Catani, ‘Higher Order QCD Corrections in Hadron Collisions: Soft Gluon Resummation and Exponentiation’, to be published in the Proceedings of High-energy Physics International Euroconference on Quantum Chromodynamics (QCD 96), Montpellier, France, 4–12 July 1996, e-Print Archive: [hep-ph/9610413](http://arxiv.org/abs/hep-ph/9610413).
[10] J.C. Collins and D.E. Soper, Nucl. Phys. B194, 445 (1982).
[11] S. Libby and G. Sterman, Phys. Rev. D18, 3242, 4737 (1978).
[12] S. Pokorski, ‘Gauge Field Theories’ (Cambridge University Press, 1987).
[13] J.C. Collins, D.E. Soper and G. Sterman, ‘Factorization of Hard Processes in QCD’, in ‘Perturbative Quantum Chromodynamics’, ed. by A.H. Mueller (World Scientific, 1989).