A SPECTRAL EQUIVALENCE FOR JACOBI MATRICES

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Abstract. We use the classical results of Baxter and Gollinski-Ibragimov to prove a new spectral equivalence for Jacobi matrices on $l^2(\mathbb{N})$. In particular, we consider the class of Jacobi matrices with conditionally summable parameter sequences and find necessary and sufficient conditions on the spectral measure such that $\sum_{k=n}^\infty b_k$ and $\sum_{k=n}^\infty (a_k^2 - 1)$ lie in $l^2_1 \cap l^1$ or $l^1_s$ for $s \geq 1$.

1. Introduction

Let us begin with some notation. We study the spectral theory of Jacobi matrices, that is semi-infinite tridiagonal matrices

$$J = \begin{pmatrix} b_1 & a_1 & 0 & 0 \\ a_1 & b_2 & a_2 & 0 \\ 0 & a_2 & b_3 & \ddots \\ 0 & 0 & \ddots & \ddots \end{pmatrix}$$

where $a_n > 0$ and $b_n \in \mathbb{R}$. In this paper we make the overarching assumption that the sequences $b_n$ and $a_n^2 - 1$ are conditionally summable. We may then define

$$\lambda_n := -\sum_{k=n+1}^\infty b_k$$

$$\kappa_n := -\sum_{k=n+1}^\infty (a_k^2 - 1)$$

for $n = 0, 1, \ldots$.

Let $d\nu$ be the spectral measure for the pair $(J, \delta_1)$, where $\delta_1 = (1, 0, 0, \ldots)^t$, and assume that $d\nu$ is not supported on a finite set of points (we call such measures nontrivial). Let

$$m(z) := \langle \delta_1, (J - z)^{-1} \delta_1 \rangle = \int \frac{d\nu(x)}{x - z}$$

be the associated $m$-function, defined for $z \in \mathbb{C}\setminus\text{supp}(\nu)$.

Recall that

$$\{\beta_n\} \in l^p_\mathbb{A} \quad \text{if} \quad ||\beta||_{l^p_\mathbb{A}} := \sum_n |n|^s |\beta_n|^p < \infty.$$ 

Throughout, let $\mathbb{A}$ denote either of the algebras $l^2_1 \cap l^1$ or $l^1_s$ where $s \geq 1$, and $\mathbb{A}$ the set of (complex valued) functions on the circle $\partial \mathbb{D}$ whose Fourier coefficients lie in $\mathbb{A}$. Notice that every $f \in \mathbb{A}$ has $l^1$ Fourier coefficients so is continuous. If $f$ is a function on $[-2, 2]$, we write $f \in \mathbb{A}$ if $f(2 \cos \theta) \in \mathbb{A}$. Finally, we will say that $d\nu \in V$ if
(1) $J$ has finitely-many eigenvalues and they all lie in $\mathbb{R} \setminus [-2,2]$

(2) $d\nu$ is absolutely continuous on $[-2,2]$ and may be written there as

$$(\sqrt{2} + x)^l(\sqrt{2} - x)^rv_0(x)dx$$

where $l, r \in \{ \pm 1 \}$ and $\log v_0 \in \mathfrak{A}$.

Our main result is$^1$:

**Theorem 1.1.** Let $J$ be a Jacobi matrix. The following are equivalent:

1. The sequences associated to $J$ by (1.1) obey $\lambda, \kappa \in \hat{\mathfrak{A}}$
2. $d\nu \in \mathcal{V}$.

The main ingredient in the proof will be the following versions of the Strong Szegö Theorem and Baxter’s Theorem$^2$.

**Theorem 1.2 (Golinskii-Ibragimov).** Let $d\mu$ be a nontrivial probability measure on $\partial \mathbb{D}$ with Verblunsky parameters $\{\alpha_n\} \subseteq \mathbb{D}$. The following are equivalent:

1. $\alpha \in l^2$
2. $d\mu = w\frac{d\theta}{2\pi}$ and $(\log w)\wedge \in l^2$.

**Theorem 1.3 (Baxter).** Let $d\mu(\theta)$ be a nontrivial probability measure on $\partial \mathbb{D}$ with Verblunsky parameters $\{\alpha_n\}$, and let $s \geq 0$. The following are equivalent:

1. $\alpha \in l^1_s$
2. $d\mu = w\frac{d\theta}{2\pi}$ and $(\log w)\wedge \in l^1_s$.

In Section 2 we develop relations between the Jacobi and Verblunsky parameters, in Section 3 we discuss the relationship between measures and $m$-functions, in Section 4 we prove some results about adding and removing eigenvalues, and in Section 5 we prove Theorem 1.1. To motivate the results of Sections 2 and 4 we outline the proof now. Let $J^{(N)}$ be the Jacobi matrix obtained from $J$ by removing the first $N$ rows and columns. To prove the forward direction, we choose $N$ large enough that $\sigma(J^{(N)}) \subseteq [-2,2]$ so the Verblunsky parameters exist. The results of Section 2 then allow us to apply Theorems 1.2 and 1.3 to an operator differing from $J^{(N)}$ in the first row and column to see that this operator has a spectral measure with the correct form. We conclude the proof by using the results of Sections 3 and 4 to show that the conditions on the spectral measure are unaffected by changing the top row and column of the operator, or by adding back on the removed rows and columns. To prove the reverse implication we essentially run this argument backward.

It is a pleasure to thank Rowan Killip for his helpful advice.

2. **The Geronimus Relations**

Given a nontrivial probability measure $d\mu$ on $\partial \mathbb{D}$ that is invariant under conjugation, define a nontrivial probability measure $d\nu$ on $[-2,2]$ by

$$\int_{-2}^{2} g(x)d\nu(x) = \int_{0}^{2\pi} g(2\cos \theta)d\mu(\theta).$$

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$^1$A similar result is proved in [6], but with $\hat{\mathfrak{A}}$ replaced by $l^2$. While the techniques of that paper extend to handle the case discussed here, they are quite lengthy and involved. Our aim is to provide a proof of this simpler result that is both general and short.

$^2$The version of the Strong Szegö Theorem we use is due to [2] and [3]. The version of Baxter’s Theorem is due to [4]. For relevant definitions see, for instance, [7].
Similarly, given such a measure \( d\nu \), one can define a measure \( d\mu \) by
\[
\int_0^{2\pi} h(\theta)d\mu(\theta) = \int_{-2}^{2} h(\arccos(x/2))d\nu(x)
\]
when \( h(-\theta) = h(\theta) \). It is clear that \( d\mu \) is a nontrivial probability measure that is invariant under conjugation.

This sets up a one-to-one correspondence between the set of nontrivial probability measures on \([-2, 2]\) and the set of nontrivial probability measures on \( \partial \mathbb{D} \) invariant under conjugation. We call the map \( d\mu \mapsto d\nu \) the Szegő mapping and denote it by \( d\nu = Sz(d\mu) \). If the two measures are absolutely continuous with respect to Lebesgue measure we will write \( d\mu(\theta) = w(\theta)\frac{d\theta}{2\pi} \) and \( d\nu(x) = v(x)dx \). In this case we have
\[
w(\theta) = 2\pi|\sin(\theta)|v(2\cos(\theta))
\]
(2.1)
\[
v(x) = \frac{1}{\pi\sqrt{4-x^2}}w(\arccos(x/2)).
\]

The connection between \( \alpha \), and \( \alpha, b \) is given by

**Theorem 2.1** (The Geronimus Relations \[2\]). Let \( d\mu \) be a nontrivial probability measure on \( \partial \mathbb{D} \) that is invariant under conjugation, and let \( d\nu = Sz(d\mu) \). Then for all \( n \geq 0 \)
\[
a_{n+1}^2 = (1 - \alpha_{2n-1})(1 - \alpha_{2n}^2)(1 + \alpha_{2n+1})
\]
\[
b_{n+1} = (1 - \alpha_{2n-1})\alpha_{2n} - (1 + \alpha_{2n-1})\alpha_{2n-2}.
\]

Since \( a_n > 0 \), there is no ambiguity in which sign to choose for the square root in (2.2). Unless otherwise noted we take \( \alpha_{-1} = -1 \). The value of \( \alpha_{-2} \) is irrelevant since it is multiplied by zero.

From (2.2) we see that decay of the \( \alpha \)'s determines decay of the \( a \)'s and \( b \)'s. However, given sequences \( a, b \) it is difficult to determine whether the corresponding \( \alpha \) sequence even exists\(^3\), and then whether decay of \( a, b \) is passed to \( \alpha \). The rest of this section is devoted to resolving these problems. We begin with the simple

**Lemma 2.2.** Let \( p = 1, 2, s \geq 1, \beta, \gamma \in l^p_s \), and define a sequence \( \eta_n := \sum_{k=n}^{\infty} \beta_k \gamma_k \). Then\(^4\) \( \eta \in l^p_s \) and \( \|\eta\|_{l^p_s} \leq \|\beta\|_{l^p_s} \|\gamma\|_{l^p_s} \).

**Proof.** First consider the \( l^2_s \) case. By Cauchy-Schwarz and the definition of \( \|\cdot\|_{l^2_s} \) we have
\[
\|\eta\|^2 = \sum_{n=1}^{\infty} n^s \left( \sum_{k=n}^{\infty} |\beta_k \gamma_k|^2 \right) \leq \sum_{n=1}^{\infty} n^s \left( \sum_{k=n}^{\infty} |\beta_k|^2 \right) \left( \sum_{k=n}^{\infty} |\gamma_k|^2 \right)
\]
\[
\leq \|\beta\|^2 \sum_{n=1}^{\infty} \sum_{k=n}^{\infty} |\gamma_k|^2 = \|\beta\|^2 \sum_{n=1}^{\infty} n |\gamma_n|^2 \leq \|\beta\|^2 \sum_{n=1}^{\infty} n |\gamma_n|^2 \leq \|\beta\|^2 \sum_{n=1}^{\infty} n |\gamma_n|^2 \leq \|\beta\|^2 \|\gamma\|^2.
\]

\(^3\)The existence of \( \alpha \) is equivalent to \( \sigma(J) \subseteq [-2, 2] \). See, for instance, [1].

\(^4\)We do not expect such a result for \( l^p_s \) if \( 0 \leq s < 1 \), as can be seen by considering \( \beta_n = \gamma_n = \delta_N(n) \).
For $l^1_s$ we replace the use of Cauchy-Schwarz by
\[
\sum |\beta_k\gamma_k| \leq \left(\sum |\beta_k|\right)\left(\sum |\gamma_k|\right)
\]
then argue as above.

We can use Lemma 2.2 to show that decay of the $\alpha$'s is inherited by $\lambda$ and $\kappa$.

**Lemma 2.3.** Let $\{\alpha_n\} \subseteq [-1, 1]$ and $\alpha \in l^p_s$ for $p = 1, 2$ and $s \geq 1$. Define
\[
K(\alpha)_n = \sum_{k=n}^{\infty} \alpha_{2k} + \alpha_{2k-1}\alpha_{2k+1} - \alpha_{2k}^2 \alpha_{2k-1} - \alpha_{2k}\alpha_{2k-1}\alpha_{2k+1}
\]
and
\[
L(\alpha)_n = \sum_{k=n}^{\infty} \alpha_{2k-1}(\alpha_{2k} + \alpha_{2k-2}).
\]
Then $L(\alpha), K(\alpha) \in l^p_s$ with
\[
\|L(\alpha)\|_{l^p_s} + \|K(\alpha)\|_{l^p_s} \leq C\|\alpha\|_{l^p_s}^2
\]
for some $C > 0$. If $\alpha \in l^1_s \cap l^1$, then $\lambda, \kappa \in l^1_s$ as well.

By expanding the right-hand side of (2.2) one obtains
\[
\kappa_n = \alpha_{2n-1} + K(\alpha)_n
\]
(2.3)
\[
\lambda_n = \alpha_{2n-2} + L(\alpha)_n.
\]
So the above result may be interpreted as saying that when $\alpha \in l^p_s$ or $l^1 \cap l^1_s$ then so are $\lambda$ and $\kappa$, with norms depending on that of $\alpha$.

**Proof.** To see the $l^p_s$ statement holds, we use the bound $|a| + |b|^2 \leq 2(|a|^2 + |b|^2)$, the hypothesis that $|a_n| \leq 1$, and repeated applications of Lemma 2.2.

To prove $K \in l^1$ if $\alpha \in l^1_s \cap l^1$ we use
\[
\|K\|_{l^1} = \sum_{n=0}^{\infty} \left| \sum_{k=n}^{\infty} \alpha_{2k} + \alpha_{2k-1}\alpha_{2k+1} - \alpha_{2k}^2 \alpha_{2k-1} - \alpha_{2k}\alpha_{2k-1}\alpha_{2k+1} \right|
\leq \sum_{n=0}^{\infty} \left| \sum_{k=n}^{\infty} \alpha_{2k} + \alpha_{2k-1}\alpha_{2k+1} - \alpha_{2k}^2 \alpha_{2k-1} - \alpha_{2k}\alpha_{2k-1}\alpha_{2k+1} \right|
\leq \sum_{k=1}^{\infty} (k-1)|\alpha_{2k} + \alpha_{2k-1}\alpha_{2k+1} - \alpha_{2k}^2 \alpha_{2k-1} - \alpha_{2k}\alpha_{2k-1}\alpha_{2k+1}|
\lesssim \|\alpha\|_{l^p_s}^2.
\]
The proof that $L \in l^1$ is similar.

Given two sequences $\lambda, \kappa$, we now investigate when there exists a sequence $\alpha$ solving (2.2). The main step is the following technical bound.

**Lemma 2.4.** Let $p = 1, 2$, $s \geq 1$, and $\lambda, \kappa \in l^p_s$ be given. Define a map $F$ by
\[
F(\beta)_{2n-1} = \lambda_n + L(\beta)_n \quad \text{and} \quad F(\beta)_{2n} = \kappa_n + K(\beta)_n.
\]
Then $F : l^p \to l^p_s$, and for any $\beta, \gamma \in l^p_s$ we have
\[
\|F(\beta) - F(\gamma)\|_{l^p_s} \leq C'(\|\beta\|_{l^p_s} + \|\gamma\|_{l^p_s})^{1/2}\|\beta - \gamma\|_{l^p_s}^2
\]
for some $C' > 0$. 
Proof. By Lemma 2.4, the range of $F$ is as stated. Now let $\beta, \gamma \in l_p^p$. We’ll bound the sum over odd values of $n$, the proof for even values of $n$ is analogous. For $p = 2$ we have

$$
\|F(\beta) - F(\gamma)\|^2 = \|L(\beta) - L(\gamma)\|^2
$$

$$
= \sum_{n=1}^{\infty} (2n - 1)^2 \left| \sum_{k=2n-1}^{\infty} \beta_{2k-1}(\beta_{2k} + \beta_{2k-2}) - \gamma_{2k-1}(\gamma_{2k} + \gamma_{2k-2}) \right|^2
$$

$$
\lesssim \sum_{n=1}^{\infty} (2n - 1)^2 \left( \left( \sum_{k=2n-1}^{\infty} |\beta_{2k} + \beta_{2k-2}| \cdot |\beta_{2k-1} - \gamma_{2k-1}| \right)^2 + \left( \sum_{k=2n-1}^{\infty} |\gamma_{2k-1} - \beta_{2k-1}| \cdot |(\beta_{2k} + \beta_{2k-2}) - (\gamma_{2k} + \gamma_{2k-2})| \right)^2 \right)
$$

where the inequality was obtained by adding and subtracting the term

$$
\gamma_{2k-1}(\beta_{2k} + \beta_{2k-2}).
$$

By Lemma 2.2 we can replace the sums in $k$ by norms to bound

$$
\|F(\beta) - F(\gamma)\|^2 \lesssim \left( \|\beta_{2k} + \beta_{2k-2}\|^2 \cdot \|\beta_{2k-1} - \gamma_{2k-1}\|^2 + \|\gamma_{2k-1}\|^2 \cdot \|\beta_{2k} + \beta_{2k-2} - (\gamma_{2k} + \gamma_{2k-2})\|^2 \right)
$$

$$
\lesssim \left( \|\beta\|^2 + \|\gamma\|^2 \right) \|\beta - \gamma\|^2
$$

as claimed. The proof for $p = 1$ is similar and simpler. \(\Box\)

Proposition 2.5. Given $\lambda, \kappa \in \hat{A}$ with small enough norms, there exists a sequence $\alpha \in \hat{A}$ solving $2.3$.

Proof. Let $\| \cdot \|$ denote the norm on $\hat{A}$, and let $C$ be the universal constant arising in Lemma 2.4 $\|L(\beta)\| + \|K(\beta)\| \leq C\|\beta\|^2$. Let $0 < \varepsilon < \frac{1}{2C}$ be chosen momentarily, and suppose that $\|\lambda\|, \|\kappa\| \leq \varepsilon(1/2 - C\varepsilon)$. Then by considering the even and odd terms separately we see $\|F(\beta)_{\text{odd}}\| \leq \|\lambda\| + \|L(\beta)\| \leq \varepsilon/2$ if $\|\beta\| \leq \varepsilon$, and similarly $\|F(\beta)_{\text{even}}\| \leq \varepsilon/2$. So $F$ maps the $\varepsilon$-ball in $\hat{A}$ back to itself. By Lemma 2.4, it is Lipschitz on the $\varepsilon$-ball with Lipschitz constant $\sqrt{2C\varepsilon}$, where $C$ is the universal constant arising in Lemma 2.4. So if $\varepsilon$ is small enough, the Banach Fixed Point Theorem provides a unique fixed point $\alpha$ of $F$ with $\|\alpha\| < \varepsilon$. From the definition of $F$ we see this fixed point solves $2.3$ with the prescribed $\lambda$ and $\kappa$. \(\Box\)

3. CONNECTING $d\nu$ AND $m$

In the next section we will begin to add and remove eigenvalues of $J$. It is more convenient to recast the criterion on $d\nu$ from Theorem 1.1 in terms of its associated $m$-function, which we do in this section.

The map $z \mapsto E(z) := \frac{z + z^{-1}}{2}$ is a conformal mapping of $\mathbb{D}$ to $\mathbb{C} \cup \{\infty\} \setminus [-2, 2]$ sending $0$ to $\infty$ and $\pm 1$ to $\pm 2$. Define the $M$-function associated to $d\nu$ as

$$
M(z) = -m(E(z)) = -m(z + z^{-1}) = \int \frac{zd\nu(x)}{1 - xz + z^2}.
$$
We have introduced the minus sign so that $M$ is Herglotz; that is $M$ is analytic on $\mathbb{D}$ and maps $\mathbb{C}_+$ to itself (as $E \mapsto z$ maps the upper half-plane to the lower half-disc).

The $M$-function encodes all the spectral information of $J$ (see, for instance, [5, 9]). The poles of $M$ in $(-1,1)$ are related to the eigenvalues of $J$ off $[-2,2]$ by the map $z \mapsto E$. We can recover the entries of $J$ from the continued fraction expansion of $M(z)$ near infinity (see [5])

$$M(z) = \frac{1}{z + \frac{-b_1}{z + \frac{-b_2}{z + \ldots}}}$$

We will say a Jacobi matrix is resonant at $\lambda, \kappa$ if

$$\lim_{z \uparrow 1} |M(z)| = \infty,$$

and we will say $J$ is nonresonant at $E = 2$ otherwise. We define resonance at $E = -2$ similarly. If $J$ is resonant at both $E = -2$ and $E = 2$ we will say $J$ is doubly-resonant.

Throughout what follows, we make frequent use of a theorem of Wiener and Levy. For convenience we recall it here (a proof can be found in [10]):

**Theorem 3.1 (Wiener-Levy).** Let $\mathcal{B}$ be a commutative Banach algebra, $x \in \mathcal{B}$, and $F$ analytic in a neighborhood of $\sigma(x)$. Then $F(x) \in \mathcal{B}$ can be naturally defined so that $F \mapsto F(x)$ is an algebra homomorphism of the functions analytic in a neighborhood of $\sigma(x)$ into $\mathcal{B}$.

Recall that if $f \in \mathfrak{A}$ then its spectrum is its range. So this shows that if $F$ is analytic in a neighborhood of the range of $f \in \mathfrak{A}$, then $F(f) \in \mathfrak{A}$ too.

We enrich the algebra framework a bit further by allowing functions that are only locally in the algebra. Given $\theta_0 \in [0,2\pi)$, we’ll say that $f \in \mathfrak{A}_{loc}(\theta_0)$ if there is a smooth bump $\chi$ on $\partial \mathbb{D}$ equalling one in a neighborhood of $\theta_0$ such that $\chi f \in \mathfrak{A}$. Given an open interval $I \subseteq \partial \mathbb{D}$, we will say $f \in \mathfrak{A}_{loc}(I)$ if $f \in \mathfrak{A}_{loc}(\theta_0)$ for all $\theta_0 \in I$.

Notice that the bumps $\chi$ are in $\mathfrak{A}$, so if $f \in \mathfrak{A}_{loc}(\theta_0)$ for all $\theta_0 \in [0,2\pi)$, then by choosing a partition of unity on $[0,2\pi)$ with small enough supports we see $f \in \mathfrak{A}$ too.

We now transfer criterion on $d\nu$ to criterion on $M$. We will write $M \in \mathcal{M}$ if

1. For all intervals $I \subseteq \partial \mathbb{D}$ avoiding $z = \pm 1$ we have $M \in \mathfrak{A}_{loc}(I)$ and $\text{Im } M \neq 0$ on $I$.
2. For $z_0 \in \{-1,1\}$, there is a $\partial \mathbb{D}$-neighborhood $I$ of $z_0$ and a smooth bump $\chi$ supported on $I$ and equalling one near $z_0$, such that either

$$\chi(\theta) M(\theta) = \chi(\theta) \frac{G(\theta)}{\sin(\theta)}$$

or

$$\chi(\theta) M(\theta) = \chi(\theta) \left( c + \sin(\theta) G(\theta) \right)$$

where $c \in \mathbb{R}$, $G \in \mathfrak{A}$, and $\text{Im } G \neq 0$ on $I$.

**Proposition 3.2.** For any Jacobi matrix $J$, $d\nu \in \mathcal{V}$ if and only if $M \in \mathcal{M}$.

In particular, to prove Theorem 1.1 it suffices to prove that $\lambda, \kappa \in \mathfrak{A}$ if and only if $M \in \mathcal{M}$. In the proof of Proposition 3.2 we will make frequent use of the following lemma.
Lemma 3.3. Let $H$ be the Hilbert transform on $\partial \mathbb{D}$. Let $f$ be a smooth function on $\partial \mathbb{D}$, and let $A_f$ represent the operator $g(\theta) \mapsto f(\theta)g(\theta)$. If $\eta$ is a measure on $\partial \mathbb{D}$, then $[A_f,H]\eta$ is a smooth function (where $[A,B] = AB - BA$ is the usual commutator bracket).

We will not prove this here. It is a fairly standard result from Harmonic Analysis.

Proof of Proposition 3.2. Recall that Lebesgue almost everywhere

$$(3.2) \quad \frac{d\nu}{dx}(x) = \lim_{\varepsilon \downarrow 0} \frac{1}{\pi} \text{Im} m(x + i\varepsilon).$$

Using this and the definition of $M$, it is easy to see that $M \in M$ implies $d\nu \in \mathcal{V}$.

For the converse, assume that we can write

$$d\nu(x) = \sum_{j=1}^{N} c_j \delta(x - \lambda_j) + \left(\sqrt{2 + x}\right)^l \left(\sqrt{2 - x}\right)^r \psi_0(x) dx$$

where $\lambda_j \in \mathbb{R} \setminus [-2,2]$, $c_j \in [0,1]$, $l, r \in \{\pm 1\}$, and $\log \psi_0 \in \mathfrak{A}$.

Let $\{\psi_1, \psi_2\}$ be a partition of unity of $[-2,2]$ subordinate to the cover $\{[-2,1/2), (-1/2,2]\}$, with $\psi_1$ equalling one near $E = -2$ and $\psi_2$ equalling one near $E = 2$. Extend $\psi_1$ and $\psi_2$ to be zero outside $[-2,2]$. In this way we may write the $m$-function as

$$m(z) = \sum_{j=1}^{N} \frac{c_j}{\lambda_j - z} + \int \frac{1}{x - z} \psi_1(x) d\nu(x) + \int \frac{1}{x - z} \psi_2(x) d\nu(x)$$

$$= s(z) + l(z) + r(z).$$

By our choice of $\psi_1$ and $\psi_2$, and because $\lambda_j \in \mathbb{R} \setminus [-2,2]$, we have that $l(z)$ is smooth on $(1,2)$, $r(z)$ is smooth on $[-2,1)$, and $s(z)$ is smooth on $[-2,2]$.

We can now write

$$M(z) = S(z) + L(z) + R(z)$$

where

$$S(z) = -s(z + z^{-1})$$

and similarly for $L$ and $R$. Finally, we let

$$N(z) = M(z) - S(z) = L(z) + R(z).$$

As we have removed all the poles from $M$, we see that $N$ is analytic in $\mathbb{D}$. Moreover, because $S$ is smooth on $\partial \mathbb{D}$, it is clear that $M \in M$ if $N \in M$, which we now prove.

We will first show that condition (1) holds for $N$. Let $I_1$ be an interval in $\partial \mathbb{D}$ avoiding $z = \pm 1$, and let $I$ a slightly larger interval still avoiding $\pm 1$. Let $\chi$ be a smooth bump supported on $I$ equalling one on $I_1$.

By (3.2) and the assumption that $d\nu \in \mathcal{V}$ we see three things: $\sin(\theta)$ is a measure on $\partial \mathbb{D}$, $\chi(\theta)\sin(\theta)\text{Im}N(\theta)$ is a measure on $\partial \mathbb{D}$, $\chi(\theta)\sin(\theta)\text{Im}N(\theta) \in \mathfrak{A}$, and $\text{Im}N$ is nonzero on $I$.

By Lemma 3.3 we see

$$\chi(\theta)H[\sin(\theta)\text{Im}N(\theta)] = H[\chi(\theta)\sin(\theta)\text{Im}N(\theta)] + f$$

where $f \in C^\infty$. As $\chi(\theta)\sin(\theta)\text{Im}N(\theta) \in \mathfrak{A}$ and $H$ is a contraction in $\mathfrak{A}$ ($H$ multiplies the Fourier coefficients by 0 or $\pm i$, so it is a contraction in any space determined only by Fourier coefficients), we see that

$$\chi(\theta)H[\sin(\theta)\text{Im}N(\theta)] \in \mathfrak{A}.$$
too. But it is easy to see \((z - z^{-1})N(z)\) is analytic, so

\[
H[\sin(\theta) \text{Im} N(\theta)] = \text{Im}(z - z^{-1})N(z) = -\sin(\theta) \text{Re} N(\theta).
\]

Combining these we find

\[
\chi(\theta) \text{Re} N(\theta) = \chi(\theta) \frac{g(\theta)}{\sin(\theta)}
\]

for some \(g \in \mathbb{A}\).

Now we prove that (2) holds. We will only consider the case \(z_0 = 1\), the other case being similar. Let \(I_1\) be a \(\partial \mathbb{D}\)-interval around \(z_0\) to be chosen momentarily, let \(I\) be a slightly larger interval, and let \(\chi\) be a smooth bump supported on \(I\) equalling one on \(I_1\).

By (3.2) and \(d\nu \in \mathcal{V}\), we have two cases to consider. Suppose first that

\[
\chi(\theta) \text{Im} N(\theta) = \chi(\theta) g(\theta) \sin(\theta)
\]

for some \(g \in \mathbb{A}\). Then arguing as in the proof of (1) shows

\[
\chi(\theta) \text{Re} N(\theta) = \chi(\theta) h(\theta) \frac{\sin(\theta)}{\sin(\theta)}
\]

for some \(h \in \mathbb{A}\), so (2) holds in this case.

For the second case suppose

\[
\chi(\theta) \text{Im} N(\theta) = \chi(\theta) \sin(\theta) g(\theta)
\]

for some \(g \in \mathbb{A}\). As \(L\) is smooth near \(z_0\) we have

\[
\chi(\theta) \text{Re} L(\theta) = \chi(\theta) \left( \text{Re} L(0) + \sin(\theta) \chi(\theta) \frac{\text{Re} L(\theta) - \text{Re} L(0)}{\sin(\theta)} \right)
\]

(3.3)

\[
= \chi(\theta) \left( \text{Re} L(0) + \sin(\theta) h_1(\theta) \right).
\]

where \(h_1 \in \mathbb{A}\) if \(I\) is chosen small enough.

Now we consider \(\text{Re} R\) on \(I\). By assumption, \(\text{Im} R\) is continuous on \(\partial \mathbb{D}\) and hence defines a measure. Also, \(R\) is analytic in \(\mathbb{D}\), so

\[
\chi(\theta) \text{Re} R(\theta) = -\chi(\theta) H[\text{Im} R(\theta)]
\]

\[
= H[-\chi(\theta) \text{Im} R(\theta)] + f_1
\]

\[
= H[-\chi(\theta) \sin(\theta) g(\theta)] + H[\chi(\theta) \text{Im} L(\theta)] + f_1 + f_2
\]

\[
= \chi(\theta) \sin(\theta) h_2(\theta) + H[\chi(\theta) \text{Im} L(\theta)] + f_1 + f_2
\]

where the first equality follows from analyticity, the second and fourth from Lemma (so \(f_1, f_2 \in C^\infty\)), and the third from writing \(R = M - L\). As before, \(h_2 \in \mathbb{A}\) because \(g \in \mathbb{A}\). Because \(L\) is smooth near \(\theta = 0\), \(\chi(\theta) \text{Im} L(\theta)\) is smooth on all of \(\partial \mathbb{D}\) if \(I\) is chosen small enough. In particular,

\[
f := f_1 + f_2 + H[\chi(\theta) \text{Im} L(\theta)]
\]
is smooth as well. Thus
\[
\chi(\theta)f(\theta) = \chi(\theta)\left(f(0) + \sin(\theta)\chi(\theta)\frac{f(\theta) - f(0)}{\sin(\theta)}\right)
\]
with \(h_3 \in \mathbb{R}\), and so
\[
(3.4) \quad \chi(\theta) \Re R(\theta) = \chi(\theta)\left(f(0) + \sin(\theta)(h_2(\theta) + h_3(\theta))\right).
\]
Combining (3.3) and (3.4) shows (2) holds.

\[\square\]

4. \(m\)-functions and eigenvalues

In this section we derive some properties of \(\mathcal{M}\).

**Proposition 4.1.** Let \(J\) a Jacobi matrix and let \(J^{(1)}\) be the operator obtained by removing the first row and column (from the top and left). Let \(M\) and \(M^{(1)}\) be the \(M\)-functions corresponding to \(J\) and \(J^{(1)}\). Then \(M \in \mathcal{M}\) if and only if \(M^{(1)} \in \mathcal{M}\).

**Proof.** We will show that \(M^{(1)} \in \mathcal{M}\) implies \(M \in \mathcal{M}\), the other direction being similar.

By (3.1) we have
\[
M(\theta) = \frac{1}{2\cos \theta - b_1 - a_1^2 M^{(1)}(\theta)}
\]
\[
\Im M = \frac{a_1}{2\cos \theta - b_1 - a_1^2 M^{(1)}}^2 \Im M^{(1)}.
\]
Let \(I\) be an arc of \(\partial \mathbb{D}\) missing \(\theta = 0, \pi\). As \(\Im M^{(1)} \neq 0\) on \(I\), we see
\[
2\cos \theta - b_1 - a_1^2 M^{(1)} \neq 0
\]
on \(I\). By assumption
\[
2\cos \theta - b_1 - a_1^2 M^{(1)} \in \mathbb{C}_{loc}(I),
\]
so by Theorem 3.1 we have \(\Im M \in \mathbb{C}_{loc}(I)\) and is nonzero there, so part (1) of the definition of \(\mathcal{M}\) holds.

Next we will show that if \(M^{(1)}\) has the form required in part (2) of the definition, then so does \(M\). By hypothesis, we may assume that \(M^{(1)}(\theta) = c + (\sin \theta)^k g(\theta)\) on some neighborhood \(I\) of \(\theta_0\), where \(c \in \mathbb{R}, k \in \{\pm 1\}\), and \(g \in \mathbb{C}_{loc}(I)\) with \(\Im g \neq 0\) there.

**Case 1:** Suppose \(k = -1\). Then by subsuming the \(c\) into \(g\) we can write
\[
M(\theta) = \frac{1}{2\cos \theta - b_1 - a_1^2 \frac{1}{\sin \theta} g(\theta)}
\]
\[
= \frac{1}{(2\cos \theta - b_1)(\sin \theta) - a_1^2 g(z)}
\]
\[
= (\sin \theta)G(\theta).
\]
Next we compute $G$ where $H$ nonvanishing in a neighborhood $A$. Both the numerator and the denominator of $G$ are in $\mathcal{A}_{\text{loc}}(I)$. As $\text{Im} g \neq 0$ we see that the denominator is nonzero too. By Theorem 3.1 we have $G \in \mathcal{A}_{\text{loc}}(I)$. As

$$\text{Im} G = \left| \frac{1}{(2 \cos \theta - b_1)(\sin \theta) - a_1^2 g(z)} \right|^2 \text{Im}(-a_1^2 g)$$

we see $\text{Im} G \neq 0$ on $I$.

**Case 2:** Suppose $k = 1$. Then we can write

$$M(\theta) = \frac{1}{(2 \cos \theta - b_1 - a_1^2 c) - (\sin \theta) a_1^2 g(\theta)} = \frac{1}{H(\theta) - (\sin \theta) a_1^2 g(\theta)}.$$

If $H(\theta_0) = 0$, then because it is a real trigonometric polynomial we can factor $H(\theta) = (\sin \theta) h(\theta)$ for some real $h \in \mathcal{A}_{\text{loc}}(I)$. Then $h - a_1^2 g \in \mathcal{A}_{\text{loc}}(I)$ and $\text{Im}(h - a_1^2 g) = \text{Im}(-a_1^2) \neq 0$, so

$$M(\theta) = \frac{1}{\sin \theta} \frac{1}{h - a_1^2 g} = \frac{1}{\sin \theta} G(\theta)$$

where $G \in \mathcal{A}_{\text{loc}}(I)$ and has nonvanishing imaginary part.

If $H(\theta_0) = \frac{1}{1}$ for some constant $c \in \mathbb{R} \setminus \{0\}$, then as before we can write $H(\theta_0) - H(\theta) = (\sin \theta) h(\theta)$ for some real $h \in \mathcal{A}_{\text{loc}}(I)$. Then

$$M(\theta) - c = \frac{1}{H(\theta) - (\sin \theta) a_1^2 g(\theta)} - \frac{1}{H(\theta_0)} = c \frac{H(\theta_0) - H(\theta)}{H(\theta) - (\sin \theta) a_1^2 g(\theta)}$$

$$= (\sin \theta) c \frac{h(\theta) - a_1^2 g(\theta)}{H(\theta) - (\sin \theta) a_1^2 g(\theta)}$$

$$= (\sin \theta) G(\theta).$$

Both the numerator and the denominator of $G$ are in $\mathcal{A}_{\text{loc}}(I)$. The denominator is nonvanishing in a neighborhood $I' \subseteq I$ of $\theta_0$. Thus, $G \in \mathcal{A}_{\text{loc}}(I')$ by Theorem 3.1.

Next we compute

$$\text{Im} G(\theta) = c (\text{Re} H - (\sin \theta) a_1^2 \text{Re} g) + a_1^2 (\sin \theta) \text{Im} g (\text{Re} h - a_1^2 \text{Re} g)$$

$$\frac{|H - (\sin \theta) a_1^2 g|^2}{|H|}.$$
Proof. By (3.1) we have
\[ M(\theta) = \frac{1}{2 \cos \theta - b_1 - a_1^2 M(1)(\theta)}. \]

Similarly, if \( \tilde{J}^{(1)} = J^{(1)} \) then we have
\[ \tilde{M}(\theta) = \frac{1}{2 \cos \theta - \tilde{b}_1 - \tilde{a}_1^2 M(1)(\theta)} = \frac{1}{2 \cos \theta - \tilde{b}_1 - \tilde{a}_1^2 M(1)(\theta)}. \]

Combining these one finds
\[ \tilde{M}(\theta) = \frac{a_1^2}{a_1^2 M(\theta)^{-1} - \Delta(\theta)} \]

where \( \Delta(\theta) = (\delta a)(2 \cos \theta) - \delta ab, \delta a = \tilde{a}_1^2 - a_1^2, \) and \( \delta ab = \tilde{a}_1^2 \tilde{b}_1 - a_1^2 b_1. \)

As \( M \in \mathcal{M} \), in some \( \partial D \)-neighborhood \( I_+ \) of \( \theta = 0 \) we can write
\[ M(\theta) = c_+ + (\sin \theta)^{k_+} g_+(\theta) \]

for some \( c_+, k_+ \in \{ \pm 1 \}, \) and \( g_+ \in \mathcal{A}_{loc}(I_+) \) with \( \text{Im} g_+ \neq 0. \) Similarly, in a neighborhood \( I_- \) of \( \theta = \pi \) we can write
\[ M(\theta) = c_- + (\sin \theta)^{k_-} g_-(\theta). \]

By (4.1) we see that to make \( \tilde{J} \) doubly-resonant, we must choose \( \tilde{a}_1 \) and \( \tilde{b}_1 \) so that \( \tilde{a}_1^2 M(\theta)^{-1} - \Delta(\theta) = 0 \) at \( \theta = 0, \pi \). There are four cases depending on the various combinations of \( k_- \) and \( k_+ \). When \( k_- = k_+ = -1 \) we just choose \( \tilde{a}_1 = a_1 \) and \( \tilde{b}_1 = b_1 \). When \( k_- = 1 \) and \( k_+ = -1 \)

\[ \tilde{a}_1^2 = a_1^2 \left( \frac{4 c_-}{4 c_- + 1} \right) \quad \text{and} \quad \tilde{b}_1 = 2 \left( \frac{2 b_1 c_- + 1}{4 c_- + 1} \right). \]

When \( k_- = -1 \) and \( k_+ = 1 \)
\[ \tilde{a}_1^2 = a_1^2 \left( \frac{4 c_+}{4 c_+ - 1} \right) \quad \text{and} \quad \tilde{b}_1 = 2 \left( \frac{2 b_1 c_+ + 1}{4 c_+ - 1} \right). \]

When \( k_- = k_+ = 1 \)
\[ \tilde{a}_1^2 = a_1^2 \left( \frac{4 c_- c_+}{4 c_- c_+ - c_- + c_+} \right) \quad \text{and} \quad \tilde{b}_1 = 2 \left( \frac{2 b_1 c_- c_+ + c_- + c_+}{4 c_- c_+ - c_- + c_+} \right). \]

Of course, we must check that \( \tilde{a}_1^2 > 0 \) so that \( \tilde{J} \) really is a Jacobi matrix. This amounts to showing that \( c_- < -1/4 \) and \( c_+ > 1/4 \). As \( J \) has no eigenvalues off \([-2, 2]\),
\[ m(E) = \int_{-2}^{2} \frac{d\nu(x)}{x - E} \]

where \( d\nu \) is the spectral measure corresponding to \( J \). For \( t \in [-2, 2] \) and \( E > 2, t - E \geq -4 \). So because \( d\nu \) is a probability measure that is not a point mass at \( t = 2 \) we have
\[ M(\theta = 0) = \lim_{E \downarrow 2} -m(E) > 1/4. \]

Similar arguments show \( M(\theta = \pi) < -1/4. \)

It remains to show that \( \tilde{J} \) has no eigenvalues off \([-2, 2]\), or equivalently that \( \tilde{M} \) has no poles on \((-1, 1)\). As \( J \) has no eigenvalues off \([-2, 2]\), \( M \) is analytic on \( D \).
So by (4.1) it suffices to show that \( f(E) := \tilde{a}^2 + m(E)(\delta aE - \delta ab) \neq 0 \) for \( |E| > 2 \). As \( J \) has no eigenvalues off \([-2, 2]\) we have

\[
\frac{d}{dE}m(E) = \int_{-2}^{2} \frac{d\nu(x)}{(x-E)^2} > 0.
\]

Since \( \delta aE - \delta ab \) is linear in \( E \), \( f \) is monotone in \( E \) for \( |E| > 2 \). By our choice of \( \tilde{a}^2 \) and \( \tilde{b} \), we have \( f(\pm 2) = 0 \), and so \( f(E) \neq 0 \) for \( |E| > 2 \), as required. \( \square \)

5. **Proof of Theorem 1.1**

As a final preliminary, we recall the definition of the Carathéodory function associated to \( d\mu = \frac{w}{2\pi} d\theta \):

\[
F(z) = \int_{0}^{2\pi} \frac{e^{i\theta} + z}{e^{i\theta} - z} d\mu(\theta).
\]

Note that almost everywhere

\[
\lim_{r \uparrow 1} \Re F(re^{i\theta}) = w(\theta)
\]
and if \( M \) has no poles in \( \mathbb{D} \) then

\[
M(z) = \frac{-F(z)}{z - 1}.
\]

**Proof of Theorem 1.1** Suppose \( \lambda, \kappa \in \mathcal{A} \). Choose \( N \) so large that \( \lambda(J(N)), \kappa(J(N)) \in \mathcal{A} \) with small enough norms to apply Proposition 4.3. This produces a sequence \( \tilde{\alpha} \in \mathcal{A} \) solving (2.3), but may not have \( \tilde{\alpha}_1 = -1 \). If we change \( \tilde{\alpha}_1 \) to be \( -1 \), this changes the top row and column of \( J(N) \) producing a matrix \( \tilde{J} \). As \( \tilde{J} \) has a sequence of Verblunsky parameters (namely \( \tilde{\alpha} \)), we see \( \sigma(\tilde{J}) \subseteq [-2, 2] \).

Let \( \tilde{M} \) be the \( M \)-function associated to \( \tilde{J} \). We will show \( \tilde{M} \in \mathcal{M} \). Let \( d\tilde{\mu} \) be the measure on \( \partial \mathbb{D} \) corresponding to \( \tilde{\alpha} \). As \( \tilde{\alpha} \in \mathcal{A} \), we may apply Theorems 1.2 and 1.3 to find \( d\tilde{\mu} = \tilde{w} d\theta \) and \( \log \tilde{w} \in \mathcal{A} \). As \( \tilde{w} = \Re \tilde{F} \) we see that \( \log(\Re F) \in \mathcal{A} \) too, so by Theorem 5.1 \( \Re F \in \mathcal{A} \) and is nonvanishing. As \( \Re f \mapsto \Im f \) is a contraction in \( \mathcal{A} \) we see \( \tilde{F} \in \mathcal{A} \) and is nonvanishing. By (5.1), we have

\[
\tilde{M}(\theta) = i \frac{1}{\sin \theta} \frac{\tilde{F}(\theta)}{2}
\]
\[
\Im \tilde{M} = \frac{1}{\sin \theta} \frac{\tilde{w}(\theta)}{2}.
\]

In particular, \( \tilde{M} \in \mathcal{M} \), as claimed.

By Proposition 4.2, \( M(J^{(N)}) \in \mathcal{M} \), and by repeated applications of Proposition 4.1, we have that \( M \in \mathcal{M} \). Finally, \( J \) and \( \tilde{J} \) differ by a finite-rank perturbation. Since \( J \) has no eigenvalues off \([-2, 2]\) and a finite-rank perturbation can only produce a finite number of eigenvalues in each spectral gap, \( J \) has only finitely-many eigenvalues and they all lie in \( \mathbb{R} \setminus [-2, 2] \). By Proposition 4.2 we have \( d\nu \in \mathcal{V} \).

Now consider the converse. As \( J \) has only finitely-many eigenvalues off \([-2, 2]\), the Sturm Oscillation Theorem guarantees we can choose \( N \) large enough that \( J^{(N)} \) has no eigenvalues off \([-2, 2]\). By Propositions 4.1, 4.2, and 4.3, there is a unique
doubly-resonant Jacobi matrix $\tilde{J}$ with $\tilde{J}^{(1)} = J^{(N+1)}$, $\tilde{M} \in \mathcal{M}$, and no eigenvalues off $[-2, 2]$. As above,

$$\tilde{M} = i \frac{1}{\sin \theta} \frac{\tilde{F}(\theta)}{2}$$

so $\tilde{F} \in \mathfrak{A}$ and $\tilde{w}$ is nonvanishing. By Theorems 1.2 and 1.3 we have $\tilde{\alpha} \in \hat{\mathfrak{A}}$, where $\tilde{\alpha}$ is the sequence of Verblunsky parameters corresponding to $\tilde{J}$. But then $\lambda(\tilde{J}), \kappa(\tilde{J}) \in \hat{\mathfrak{A}}$ by Lemma 2.3. As $\lambda(J), \kappa(J)$ differ from $\lambda(\tilde{J})$ and $\kappa(\tilde{J})$ by only finitely-many terms, we have $\lambda(J), \kappa(J) \in \hat{\mathfrak{A}}$ too. $\square$

References

[1] D. Damanik, R. Killip, *Half-line Schrödinger operators with no bound states*, Acta Math. 193 (2004), no. 1, 31–72.
[2] Ya. L. Geronimus, *Polynomials Orthogonal on a Circle and Their Applications*, Amer. Math. Soc. Translation 104, AMS, Providence, RI, 1954.
[3] B. L. Golinskii, I. A. Ibragimov, *On Szegő’s limit theorem*, Math. USSR Izv. 5 (1971), 421–444.
[4] I. A. Ibragimov, *A theorem of Gabor Szegő*, Mat. Zametki 3 (1968), 693–702.
[5] R. Killip, B. Simon, *Sum rules for Jacobi matrices and their applications to spectral theory*, Ann. of Math. 158 (2004), 253–321.
[6] E. Ryckman, *A Strong Szegő Theorem for Jacobi matrices*, preprint.
[7] B. Simon, *Orthogonal Polynomials on the Unit Circle*, American Mathematical Society Colloquium Publications 54, Parts 1 & 2, AMS, Providence, RI, 2005.
[8] T. J. Stieltjes, *Recherches sur les fractions continues*, Ann. Fac. Sci. Toulouse Math. (6) 4 (1995), no. 3, J76–J122, and no. 4, A5–A47. Also published in Memoires presentes par divers savants à l’Academie des sciences de l’Institut National de France, vol. 33, 1-196.
[9] G. Teschl, *Jacobi Matrices and Completely Integrable Nonlinear Lattices*, Mathematical Surveys and Monographs 72, AMS, Providence, RI, 2000.
[10] A. Zygmund, *Trigonometric Series: Vols. I, II*, Second edition, Cambridge University Press, London-New York, 1968.

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