Massive and massless phases in self-dual $\mathbb{Z}_N$ spin models: some exact results from the thermodynamic Bethe ansatz

Patrick Dorey, Roberto Tateo and Kevin E. Thompson

Department of Mathematical Sciences,
University of Durham, Durham DH1 3LE, England

Abstract

The generally accepted phase diagrams for the discrete $\mathbb{Z}_N$ spin models in two dimensions imply the existence of certain renormalisation group flows, both between conformal field theories and into a massive phase. Integral equations are proposed to describe these flows, and some properties of their solutions are discussed. The infrared behaviour in massless and massive directions is analysed in detail, and the techniques used are applied to a number of other models.

*e-mail: P.E.Dorey, Roberto.Tateo, K.E.Thompson *durham.ac.uk
1 Introduction

A planar vector model in which an $O(2)$ symmetry has been broken down to a $\mathbb{Z}_N$ subgroup can be defined by the reduced Hamiltonian

$$H[\{\theta_i\}] = \sum_{\langle ij \rangle} V(\theta_i - \theta_j) - h \sum_i \cos N \theta_i .$$

(1.1)

The pairs $\langle ij \rangle$ are nearest-neighbour sites of the two-dimensional lattice on which the angular spins $\theta_i$ live; these interact via a 2$\pi$-periodic, even function $V(\theta)$. The strength of the explicit symmetry-breaking is governed by the coupling $h$.

When $h=0$, continuous symmetry is regained and this forbids the formation of an ordered phase, no matter how low the temperature $T$ becomes. Instead, the model undergoes a Kosterlitz-Thouless transition as $T$ decreases through some critical value $T_c$. Below this temperature, the completely disordered high-temperature phase, in which the typical spin configuration is dominated by vortices, is replaced by a massless phase in which the vortices are suppressed and correlations decay algebraically. For $h \neq 0$, this picture must change: since residual symmetry is discrete there is no obstruction to its spontaneous breaking at low enough temperatures, and a rigorous result of Fröhlich and Lieb asserts that this does indeed happen [1]. Despite this fact, for $N \geq 5$ the massless phase is not completely lost: José et al [2] showed that the $\mathbb{Z}_N$ perturbation is irrelevant above a temperature $T_{c'}$ which, for $N \geq 5$, lies below $T_c$. A small non-zero value of $h$ cannot change the $h=0$ critical behaviour until $T$ falls below $T_{c'}$, and so the Kosterlitz-Thouless phase survives as an intermediate stage between massive high- and low-temperature regimes.

Another situation amenable to analysis is the limit $h \to \infty$: this serves to pin the spins to $N$ discrete values, and results in a ‘pure’ $\mathbb{Z}_N$ model. Universality would suggest the continuing validity of the picture just outlined, and this expectation was confirmed by Elitzur et al in 1979 [3]. However, both this and the earlier work of José et al concentrated on the Villain form [4] of the interaction $V(\theta)$, and it is important to know how much influence this choice has on the nature of the phases and transitions exhibited by the model. For the pure $\mathbb{Z}_N$ case, it is feasible to investigate this question directly, since the full phase space is finite ($[N/2]$) dimensional: a particular system is specified by the $N$ reduced energy differences $V_r = V(2\pi r/N) - V(0)$, subject to the condition $V_r = V_{N-r}$. The general...
features of the resulting phase diagrams have been the subject of much attention since the early 1980’s, motivated both by the intrinsic interest of the problem and by the analogies between such two-dimensional spin systems and certain four-dimensional gauge theories.

For each \( N \), a duality transformation can be defined as follows. First parametrise the couplings as \( x_r = \exp(-V_r) \). Then the summation over the spins \( \theta_i \) can be replaced by one over dual spins \( \tilde{\theta}_\tilde{i} \) defined on the sites \( \tilde{i} \) of the dual lattice, so long as the set of couplings \( \{x_r\} \) is simultaneously replaced by the dual set \( \{\tilde{x}_r\} \):

\[
\tilde{x}_r = \left( 1 + \sum_{s=1}^{N-1} x_s e^{2\pi i r s/N} \right) \left/ \left( 1 + \sum_{s=1}^{N-1} x_s \right) \right.
\]

The effect is to exchange the low and high temperature fixed points (\( x_r \equiv 0 \) and \( x_r \equiv 1 \) respectively), while leaving an \([N/4]\)-dimensional hyperplane invariant – systems on this hyperplane are said to be self-dual.

Setting \( \sigma^k_i = e^{i k \theta_i} \) and \( \mu^l_\tilde{i} = e^{i l \tilde{\theta}_\tilde{i}} \), the various phases can be characterised by the order parameters \( \langle \sigma^k \rangle \) and their duals, the ‘disorder parameters’ \( \langle \mu^l \rangle \). In a completely ordered phase, \( \langle \sigma^k \rangle \neq 0, \langle \mu^l \rangle = 0 \); in a completely disordered phase \( \langle \sigma^k \rangle = 0, \langle \mu^l \rangle \neq 0 \); while both vanish in a Kosterlitz-Thouless phase. In addition, if \( N \) is not prime there are ‘partially ordered’ phases where expectation values of certain powers \( \langle \sigma^k \rangle \) or \( \langle \mu^l \rangle \) are nonzero even though others vanish. However such phases, and the corresponding partial-ordering transitions, are found away from the region that will be of interest below.

The way that the three phases which are important for this paper fit together can be illustrated with the case \( N=5 \), shown in figure 1. The transition from the ordered to the disordered phase can happen in one of two ways: either via an intermediate massless phase, as along the Villain line (b), or else directly through a single first-order transition, as exemplified by the behaviour along the Potts line (a). Clearly, the points \( C \) and \( C' \) on the figure are rather special: they mark the bifurcations of the line of first-order transitions into pairs of Kosterlitz-Thouless transitions, and the beginnings of the massless regions. These points are thought to be special for other reasons too. In 1982 Fateev and Zamolodchikov identified particular sets of self-dual Boltzman weights for the general \( \mathbb{Z}_N \) model which satisfy the star-triangle relations, and conjectured that they were precisely the critical points \( C \) and \( C' \) for \( N=5 \), or their generalisations to higher \( N \). Subsequently, in 1985, they found for each \( N \) a \( \mathbb{Z}_N \)-symmetric conformal field theory that was
a natural candidate for the field-theory limit of these same special points \[9\].

Figure 1: Phases for the general $\mathbb{Z}_5$ model: the Kosterlitz-Thouless regions 1 and 2 are massless, while regions 3 (ordered) and 4 (completely disordered) are massive. The line $AB$ is self-dual; $a$ labels the line of Potts models, and $b$ is the Villain line.

To verify these claims, two distinct questions must be addressed. First, the criticality of the Fateev-Zamolodchikov Boltzmann weights should be established and their critical exponents matched with those of the corresponding $\mathbb{Z}_N$-symmetric conformal field theory: this has now been done, in both numerical \[10\] and analytical \[11\] studies, with complete agreement with expectations. However, on its own this information is not enough to locate the models relative to the opening of the first-order transition into a massless phase. To answer this second question, neighbouring regions of the phase diagram must be examined. Alcaraz \[10\] found numerical results consistent with the conjecture for the $\mathbb{Z}_5$ case, but beyond this, direct evidence from the lattice model is hard to come by. But the question can also be studied in the continuum, using the techniques pioneered by A.B.Zamolodchikov in his work on perturbed conformal field theories \[12\]. If the system described by the $\mathbb{Z}_N$-symmetric conformal field theories indeed lie on the border of a Kosterlitz-Thouless phase, then a suitable perturbation will shift it either into a massive phase, or else into the massless Kosterlitz-Thouless
The conformal theory should admit perturbations $S_{\text{CFT}} \rightarrow S_{\text{CFT}} + \lambda \int \epsilon d^2x$ which provoke both massive and massless renormalisation group flows, according to the sign of the coupling $\lambda$. The perturbing operator $\epsilon$ should be $\mathbb{Z}_N$-symmetric, since the initial shift of the Hamiltonian is within the phase diagram of the $\mathbb{Z}_N$ models; and taking $\epsilon$ to be self-dual will ensure that any massive direction of the perturbation is onto the surface of first-order transitions. The massless direction, shifting the model into a Kosterlitz-Thouless phase, should lead to a flow from $c = 2(N-1)/(N+2)$, the central charge of the $\mathbb{Z}_N$-symmetric conformal field theory, to $c = 1$, the central charge of the various one-component Gaussian models.

If $\epsilon$ can be chosen so that the perturbed theory is integrable, then past experience of a method known as the Thermodynamic Bethe Ansatz (TBA) suggests that there should exist a set of integral equations encoding the exact flow of the ground-state energy $E(R)$ of the system on a cylinder of circumference $R$, from $R=0$ all the way through to $R=\infty$. Equations of this type were initially derived for theories for which a conjectured exact S-matrix was already available [13, 14], but it has since proved possible to make educated guesses for equations to describe many ground-state energy flows even in the absence of direct derivations [15, 16].

The purpose of this paper is to propose just such sets of equations, pertinent to self-dual flows from the $\mathbb{Z}_N$-symmetric conformal field theory into either a massive or a Kosterlitz-Thouless phase. The systems themselves are given in the next section, while section 3 shows that at small $R$ they match a suitable perturbation already shown by Fateev to be integrable [17]. The discussion then turns to the large $R$ behaviour, starting with the massless direction. In section 4.1, this is shown to be compatible with a self-dual flow into the Kosterlitz-Thouless region. We were able to extract some exact infrared information directly from the TBA, and this is explained in section 4.2. The technique is applied in a number of contexts, with results that reflect the connections between the various models associated with $c=1$ conformal field theories. In the massive direction, described in section 4.3, a detailed examination of the asymptotics allows us to conjecture a pattern of massive kinks linking $N+1$ degenerate vacua, as expected on a surface of first-order transitions where phase coexistence occurs. The concluding section 5 outlines some questions that might merit further study.

A number of the more technical details have been relegated to a series of appendices: the integral kernels, Y-systems and dilogarithm sum rules used in the main text can be found in appendices A, B and C, while numerical results for the ultraviolet asymptotics are reported in appendix D.
2 TBA equations

In this section, the equations will be introduced, leaving a detailed study of their properties until later. Some are already in the literature \cite{18}--\cite{21}. Since these motivate the subsequent proposals, the section begins with a review of this material.

2.1 Known cases

The initial goal is to find sets of equations consistent with movement from $c = 2(N-1)/(N+2)$ to $c = 1$. For $N$ even such behaviour has been seen before. Each $W_{SO(m)}$ series of coset conformal theories allows for interpolating flows, analogous to the $\phi_{13}$-induced flows in the minimal series. The final step,\n
$$\frac{SO(m)^{(2)} \times SO(m)^{(1)}}{SO(m)^{(3)}} + \lambda \phi_{1,1,\text{Adj}} \rightarrow \frac{SO(m)^{(1)} \times SO(m)^{(1)}}{SO(m)^{(2)}}$$ (2.1)\n
involves a change of central charge $2(2m-1)/(2m+2) \rightarrow 1$, which fits the bill for $N=2m$. This was noted by Fateev \cite{17}, and our only contribution here will be to exploit the TBA systems that have since been proposed: $d_{n+1}$-related for $m=2n+2$ \cite{18} and $b_n$-related for $m=2n+1$ \cite{19,20} ($n$ has been picked in this way for later convenience: in particular, the choice means that the associated sine-Gordon models always have $n-1$ breathers). The lack of an explicit $\mathbb{Z}_{2m}$ symmetry in the models (2.1) is not necessarily a problem, particularly as attention is being restricted to the flow of the ground-state energy. The real test will come later, when a more detailed look is taken at the predicted asymptotic behaviours of $E(R)$ at small and large values of $R$.

Thus, for $N=4n+4$ we conjecture that the ground-state energy will be as in ref. \cite{18}:

$$RE(R) = -\frac{1}{2\pi} \sum_{\alpha=1,2} \int_{-\infty}^{\infty} d\theta \nu^{(\alpha)}(\theta) L_i^{(1)}(\theta)$$ (2.2)

where, both here and subsequently,

$$L_i^{(\alpha)}(\theta) = \log(1 + e^{-\varepsilon_i^{(\alpha)}(\theta)}) .$$

The functions $\varepsilon_i^{(\alpha)}(\theta)$, $i = 1 \ldots n+1$, are to be found as the solutions to the
following set of coupled integral equations:

\[
\varepsilon_i^{(\alpha)}(\theta) = \nu_i^{(\alpha)}(\theta) - \sum_{j=1}^{n+1} \left[ \phi_{ij} \ast L_j^{(\alpha)}(\theta) - \psi_{ij} \ast L_j^{(\bar{\alpha})}(\theta) \right] \quad (\alpha = 1, 2),
\]

(2.3)

with \(\bar{\alpha} = 3 - \alpha\), and \(\ast\) denoting the convolution

\[
f \ast g(\theta) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(\theta') g(\theta - \theta') d\theta'.
\]

The kernels \(\phi_{ij}\) and \(\psi_{ij}\) can be extracted from equation (A.8) of appendix A on setting \(h = 2n\). They can equally be defined using the \(\text{d}_{n+1}\) Toda S-matrices, though this fact will only briefly be relevant below. The ‘energy terms’ \(\nu_i^{(\alpha)}(\theta)\) are

\[
\nu_i^{(1)}(\theta) = \nu_i^{(1)}(-\theta) = \frac{1}{2} M_i \text{Re} \theta \quad (i = 1 \ldots n+1),
\]

(2.4)

and the numbers \(M_i\) are given by (A.3),(A.4), again with \(h = 2n\). The full system has a \(\mathbb{Z}_2\) symmetry under \(\alpha \to \bar{\alpha}, \theta \to -\theta\). This transformation has a simple interpretation, as follows. The pseudoenergies \(\varepsilon_i^{(1)}\) and \(\varepsilon_i^{(\bar{1})}\) can be associated with right and left moving massless ‘particles’ [Bar], with S-matrix elements

\[
S_{ij}^{LL}(\theta) = S_{ij}^{RR}(\theta) = S_{ij}(\theta), \quad S_{ij}^{RL}(\theta) = S_{ij}^{LR}(\theta) = (T_{ij}(\theta))^{-1}
\]

(with \(S_{ij}\) and \(T_{ij}\) as in appendix A). The \(\mathbb{Z}_2\) transformation reverses their spatial momenta (by sending \(\theta\) to \(-\theta\)) and swaps right and left movers; it therefore implements parity. In the rest of the paper, this sort of language will often be used, even though the general problem of associating S-matrices to the flows is being left for future work.

Were the theory conformal, \(E(R)\) would be given in terms of the central charge \(c\) by the relation \(E(R) = -\pi c/6R\) [22]. For a non-conformal theory an ‘effective central charge’ \(c(R)\) can still be defined, as

\[
E(R) = -\frac{\pi}{6R} c(R),
\]

after which the ultraviolet and infrared central charges can be extracted as the \(R \to 0\) and \(R \to \infty\) limits of \(c(R)\). These limits follow from (2.2) and (2.3)

\[\text{1}^{\text{Strictly speaking, any bulk terms must also be subtracted before the infrared result is valid. The TBA does this automatically.}}\]
via by-now standard manipulations, and are indeed as claimed. For the moment the only point to note is the way that the $R \to \infty$ limit works: $L_i^{(2)}(\theta)$ becomes vanishingly small at any values of $\theta$ for which $L_i^{(1)}(\theta)$ is non-zero, and vice versa, the two halves of (2.3) decouple, and the equations become

$$\varepsilon_i^{(\alpha)}(\theta) = \nu_i^{(\alpha)}(\theta) - \sum_{j=1}^{n+1} \phi_{ij} \ast L_j^{(\alpha)}(\theta), \quad (\alpha = 1, 2). \tag{2.5}$$

Noting that the $R$-dependence can be removed by shifts in $\theta$ of $\pm \log R$, these ‘kink systems’ can be recognised as two copies of the scale-independent TBA equations associated with the minimal $d_{n+1}$-related S-matrices, and also, more suggestively, with the sine-Gordon model at the reflectionless points

$$\beta^2 = \frac{8\pi}{n+1} \equiv \frac{32\pi}{N}. \tag{2.6}$$

For $N=4n+2$ the story is much the same. The underlying massless scattering theory is not expected to be diagonal, and as a consequence the TBA system includes some auxiliary functions, in addition to those $\varepsilon_i^{(\alpha)}$ which appear explicitly in the formula for the ground-state energy. These functions, often referred to as ‘magnonic’ pseudoenergies, make themselves felt via their appearance in the coupled set of integral equations that all the pseudoenergies must satisfy together. For the ($b_n$-related) case in hand, three such functions must be introduced. It will be convenient to label them as $\varepsilon_n^{(0)}$, $\varepsilon_n^{(2)}$ and $\varepsilon_n^{(4)}$, with the remaining non-magnonic pseudoenergies being $\varepsilon_i^{(1)}$ and $\varepsilon_i^{(3)}$, $i = 1 \ldots n$. (This notation is in line with the idea that it is particle number $n$ that should be blamed for the magnonic terms in the equations, $n$ being the label of the soliton in the related sine-Gordon model, while $1 \ldots n-1$ label the diagonally-scattering sine-Gordon breathers.) The appropriate $b_n$-related TBA system can then be written as

$$RE(R) = -\frac{1}{2\pi} \sum_{\alpha=1,3}^{n} \int_{-\infty}^{\infty} d\theta \nu_i^{(\alpha)}(\theta) L_i^{(1)}(\theta) \tag{2.7}$$

with

$$\varepsilon_i^{(\alpha)}(\theta) = \nu_i^{(\alpha)}(\theta) - \sum_{j=1}^{n} \left[ \phi_{ij} \ast L_j^{(\alpha)}(\theta) - \psi_{ij} \ast L_j^{(\tilde{\alpha})}(\theta) \right]$$

$$- \delta_{i,n} \sum_{\beta=0}^{4} \phi_{1\beta} \ast L_n^{(3)}(\theta), \quad (\alpha = 1, 3);$$

7
\[ \varepsilon_{n}^{(\alpha)}(\theta) = -\frac{4}{\beta=0} [\alpha_{\beta}] \phi_{1} \ast L_{n}^{(\beta)}(\theta), \quad (\alpha = 0, 2, 4). \] (2.8)

Setting \( \tilde{\alpha} = 4 - \alpha \), the energy terms are again given by equation (2.4). The masses \( M_{i} \), and the kernels \( \phi_{i,j}, \psi_{i,j}, \) and \( \phi_{1} \), can be found in appendix A, with \( h = 2n-1 = N/2 - 2 \). Parity symmetry is implemented as \( \alpha \rightarrow \tilde{\alpha}, \theta \rightarrow -\theta \).

In the \( R \rightarrow \infty \) limit, the interesting behaviour is confined to kink systems near \( \theta = \pm \log R \), and again the equations separate, into one set in which the index \( \alpha \) takes the values 0, 1, 2 and a second in which it takes the values 2, 3, 4. As before, the resulting equations could have been found by examining the ultraviolet limit of a sine-Gordon TBA system, this time at the coupling

\[ \beta^{2} = \frac{8\pi}{n + 1/2} = \frac{32\pi}{N}. \] (2.9)

Having observed this connection, we can hope to use it in reverse to construct new TBA systems for the remaining \( (N \text{ odd}) \) cases, using as an input the known sine-Gordon TBA systems at yet further values of the coupling \( \beta \). The question is, just which values? Looking at equations (2.8) and (2.9), a natural guess comes to mind, and to confirm it we now recall the one other case where an exact conjecture for the ground-state energy has previously appeared, namely the \( N=5 \) flow from \( c=8/7 \) to \( c=1 \). This was found by Ravanini et al in the course of an exhaustive study of TBA systems of a particular type, and reads:

\[ RE(R) = -\frac{1}{2\pi} \int_{-\infty}^{\infty} d\theta \left[ \nu^{(1)}(\theta)L^{(1)}(\theta) + \nu^{(6)}(\theta)L^{(6)}(\theta) \right], \] (2.10)

where the functions \( \varepsilon^{(1)}(\theta) \) and \( \varepsilon^{(6)}(\theta) \) appearing in \( L^{(1)} \) and \( L^{(6)} \) are coupled, together with four auxiliary functions \( \varepsilon^{(2)} \ldots \varepsilon^{(5)} \), in the following system of equations:

\[ \varepsilon^{(\alpha)}(\theta) = \nu^{(\alpha)}(\theta) - \sum_{\beta=1}^{6} [e_{\alpha}^{\beta}] \phi_{\ast} L^{(\beta)}(\theta) \quad (\alpha = 1 \ldots 6). \] (2.11)

Here \( \nu^{(\alpha)}(\theta) = \frac{1}{2} M R e^{\theta} \delta_{1\alpha} + \frac{1}{2} M R e^{-\theta} \delta_{6\alpha}, \phi(\theta) = 1/\cosh \theta, \) and \([e_{\alpha}^{\beta}]\) is the incidence matrix of the \( e_{6} \) Dynkin diagram, labelled so that 1 and 6 are the two extremal nodes. The \( \mathbb{Z}_{2} \) symmetry of this diagram, combined with \( \theta \rightarrow -\theta \), exchanges left and right movers and thus implements parity. The
$R \to \infty$ limit removes all trace of $\varepsilon^{(6)}$ from the equations satisfied by $\varepsilon^{(1)}$, and vice versa; thus in this limit

$$\varepsilon^{(\alpha)}(\theta) = \nu^{(\alpha)}(\theta) - \sum_{\beta=1}^{5} [\delta_{\alpha\beta}] L^{(3)}(\theta), \quad (\alpha = 1 \ldots 5),$$

with $[\delta_{\alpha\beta}]$ the incidence matrix of the $d_5$ Dynkin diagram. (An analogous equation determines the infrared form of $\varepsilon^{(6)}(\theta)$.) Just as in the earlier cases, the final TBA system could equally have emerged in a discussion of the thermodynamics of the sine-Gordon model. This time, the coupling to take is $\beta^2 = 32\pi/5$ [23], precisely the value predicted by equations (2.6) and (2.9).

### 2.2 New massless TBA systems

By now it is natural to suppose that a system of equations appropriate to the $N=7$ flow will have something to do with the sine-Gordon TBA at $\beta^2 = 32\pi/7$. In its ultraviolet limit, this latter system becomes

$$\begin{align*}
\varepsilon^{(1)}(\theta) &= \nu^{(1)}(\theta) - \phi_3*(L^{4}(\theta) + L^{5}(\theta)) - \phi_4*L^{(2)}(\theta) - \phi_5*L^{(3)}(\theta) \\
\varepsilon^{(2)}(\theta) &= \phi_2*(K^{(3)}(\theta) - L^{(1)}(\theta)) \\
\varepsilon^{(3)}(\theta) &= \phi_2*(K^{(2)}(\theta) + K^{(4)}(\theta) + K^{(5)}(\theta)) ; \quad \varepsilon^{(4)}(\theta) = \phi_2*K^{(3)}(\theta) \\
\varepsilon^{(5)}(\theta) &= \phi_2*K^{(3)}(\theta)
\end{align*}$$

where

$$L^{(\alpha)} = \log(1 + e^{-\varepsilon^{(\alpha)}}), \quad K^{(\alpha)} = \log(1 + e^{\varepsilon^{(\alpha)}}), \quad \nu^{(1)}(\theta) = \frac{1}{2}MRe^\theta,$$

and the kernels are given by equation (A.12) with $h = 3/2$. (One way to obtain this system is to Fourier transform the $Y$-system given in ref. [24], divide through by appropriate hyperbolic cosines, and then transform back.)

The scale $R$ enters into these equations in a trivial way, and can be removed by a shift in $\theta$. This is as it should be, since the ultraviolet limit has already been taken. The scale-dependent sine-Gordon TBA could be recovered on redefining $\nu^{(1)}(\theta) = MR \cosh \theta$, a manoeuvre that leaves the $R \to 0$ limit unchanged but forces $c_{SG}(\infty) = 0$, as appropriate for a massive theory. However, here the idea is different – we want (2.13) to describe the infrared destination of some massless flow, rather than the ultraviolet limit of a massive one. To this end, the scale-invariant system, thought of
as describing only leftmoving particles, should be augmented with an extra piece to describe the right-movers. This new piece will re-introduce a scale dependence, but in such a way that the system (2.13) is recovered as $R \to \infty$. Furthermore, the augmented system must treat left and right movers on an equal footing, with a $\mathbb{Z}_2$ symmetry exchanging them. Remarkably, the system (2.13) is such that both of these requirements can be satisfied in a non-trivial way. On noticing the $\mathbb{Z}_2$ symmetry already exhibited by the mutual couplings of the magnonic pseudoenergies $\varepsilon^{(2)} \ldots \varepsilon^{(5)}$, it is natural to extend this to the full system with the addition of a single right-moving pseudoenergy $\varepsilon^{(6)}(\theta)$:

\begin{align*}
\varepsilon^{(1)}(\theta) &= \nu^{(1)}(\theta) - \phi_3^*(L^{(4)}(\theta) + L^{(5)}(\theta)) - \phi_4^*L^{(2)}(\theta) - \phi_5^*L^{(3)}(\theta) \\
\varepsilon^{(2)}(\theta) &= \phi_2^*(K^{(3)}(\theta) - L^{(1)}(\theta)) \\
\varepsilon^{(3)}(\theta) &= \phi_2^*(K^{(2)}(\theta) + K^{(4)}(\theta) + K^{(5)}(\theta)) \quad ; \quad \varepsilon^{(4)}(\theta) = \phi_2^*K^{(3)}(\theta) \\
\varepsilon^{(5)}(\theta) &= \phi_2^*(K^{(3)}(\theta) - L^{(6)}(\theta)) \\
\varepsilon^{(6)}(\theta) &= \nu^{(6)}(\theta) - \phi_3^*(L^{(4)}(\theta) + L^{(2)}(\theta)) - \phi_4^*L^{(5)}(\theta) - \phi_5^*L^{(3)}(\theta)
\end{align*}

(2.14)

Taking $\nu^{(6)} = \frac{1}{2}MR e^{-\theta}$ gives the desired left-right symmetry, under $\varepsilon^{(1)} \leftrightarrow \varepsilon^{(6)}$, $\varepsilon^{(2)} \leftrightarrow \varepsilon^{(5)}$, $\theta \to -\theta$. If the ground state energy is

$$RE(R) = -\frac{\pi}{6} c(R) = -\frac{1}{2\pi} \int_{-\infty}^{\infty} d\theta \left[ \nu^{(1)}(\theta)L^{(1)}(\theta) + \nu^{(6)}(\theta)L^{(6)}(\theta) \right],$$

(2.15)

then the result $c(\infty) = 1$ follows, as intended, from the result $c_{SG}(0) = 1$ for the sine-Gordon model – the determining equations reduce to (2.13) in both cases. More interesting is the $R \to 0$ limit of the extended system (2.14), and this confirms that the discussion thus far has been on the right track. Using the dilogarithm sum rules given in appendix C, it can be checked that $c(0) = 4/3$, exactly as required.

Once this case has been understood, the general procedure is clear. Take the sine-Gordon TBA at $\beta^2 = 32\pi/N$, and look for a $\mathbb{Z}_2$ symmetry of its magnonic part, which is not a symmetry of the full system. Now add pseudoenergies so that this becomes a symmetry of the whole, and then check that this enlarged system does indeed give the desired value, $2(N-1)/(N+2)$, for the ultraviolet central charge. For $N$ even the magnonic structure is rather trivial and the results are already contained in equations (2.2), (2.3) and (2.7), (2.8). However the idea also works for every odd $N$; the resulting systems will now be given.
When \( N=4n+1 \), the magnonic structure of the sine-Gordon TBA is identical to that at \( N=5 \), as given in equation (2.12) – the only differences come from the appearance of \( n-1 \) breathers in the spectrum in addition to the fundamental soliton-antisoliton doublet. This leads to the following proposal for an \( N=4n+1 \) TBA system:

\[
RE(R) = -\frac{1}{2\pi} \sum_{i=1}^{n} \int_{-\infty}^{\infty} d\theta \nu_{i}^{(\alpha)}(\theta)L_{i}^{(1)}(\theta)
\]  

with

\[
\varepsilon_{i}^{(\alpha)}(\theta) = \nu_{i}^{(\alpha)}(\theta) - \sum_{j=1}^{n} \left[ \phi_{ij} \ast L_{j}^{(\alpha)}(\theta) - \psi_{ij} \ast L_{j}^{(\bar{\alpha})}(\theta) \right]
\]

\[- \delta_{i,n} \sum_{\beta=1}^{6} I_{\alpha\beta}^{(v)} \phi_{2} \ast L_{n}^{(\beta)}(\theta), \quad (\alpha=1,6, \bar{\alpha} = 7-\alpha); \]

\[
\varepsilon_{n}^{(\alpha)}(\theta) = \sum_{\beta=1}^{6} I_{\alpha\beta}^{(v)} \phi_{2} \ast L_{n}^{(\beta)}(\theta), \quad (\alpha = 2 \ldots 5).
\]

The energy terms are \( \nu_{i}^{(1)}(\theta) = \nu_{i}^{(1)}(-\theta) = \frac{1}{2}M_{i}Re^{\theta} \), with the \( M_{i} \) given by eqs. (A.3),(A.4) with \( h=N/2-2 \), and the kernels are given by equations (A.8)–(A.12).

Finally, for \( N=4n+3 \) the sine-Gordon magnonic structure mimics that for \( N=7 \), \( \beta^{2} = 32\pi/7 \). The system can therefore be found simply by tacking the appropriate number of breather-like pseudoenergies onto equations (2.14), (2.15). With energy terms and kernels extracted from appendix A as before, it reads

\[
RE(R) = -\frac{1}{2\pi} \sum_{i=1}^{n} \int_{-\infty}^{\infty} d\theta \nu_{i}^{(\alpha)}(\theta)L_{i}^{(\alpha)}(\theta)
\]  

with

\[
\varepsilon_{i}^{(1)}(\theta) = \nu_{i}^{(1)}(\theta) - \sum_{j=1}^{n} \left[ \phi_{ij} \ast L_{j}^{(1)}(\theta) - \psi_{ij} \ast L_{j}^{(6)}(\theta) \right]
\]

\[- \delta_{i,n} \left[ \phi_{3} \ast L_{n}^{(4)}(\theta) + L_{n}^{(5)}(\theta) + \phi_{4} \ast L_{n}^{(2)}(\theta) + \phi_{5} \ast L_{n}^{(3)}(\theta) \right]
\]

\[
\varepsilon_{n}^{(2)}(\theta) = \phi_{2} \ast (K_{n}^{(3)}(\theta) - L_{n}^{(1)}(\theta))
\]

\[
\varepsilon_{n}^{(3)}(\theta) = \phi_{2} \ast (K_{n}^{(2)}(\theta) + K_{n}^{(4)}(\theta) + K_{n}^{(5)}(\theta)) \quad \varepsilon_{n}^{(4)}(\theta) = \phi_{2} \ast K_{n}^{(3)}(\theta)
\]
\begin{align}
\varepsilon_n^{(5)}(\theta) &= \phi_2^* (K_n^{(3)}(\theta) - L_n^{(6)}(\theta)) \\
\varepsilon_i^{(6)}(\theta) &= \nu_i^{(6)}(\theta) - \sum_{j=1}^{n} \left[ \phi_{ij}^* L_j^{(6)}(\theta) - \psi_{ij}^* L_j^{(1)}(\theta) \right] \\
&\quad - \delta_{i,n} \left[ \phi_3^* (L_n^{(4)}(\theta) + L_n^{(2)}(\theta)) + \phi_4^* L_n^{(5)}(\theta) + \phi_5^* L_n^{(3)}(\theta) \right].
\end{align}

(2.19)

2.3 Modifications for massive flows

In their ultraviolet limits, \( MR \ll 1 \), all of the TBA equations again simplify, though in a rather different manner to the infrared situation already described. In a central region \( -\log(1/MR) \ll \theta \ll \log(1/MR) \) the energy terms \( \nu_{i}^{(\alpha)}(\theta) \) can be ignored and the pseudoenergies are approximately constant. Then, in the neighbourhoods of \( \theta = \pm \log(1/MR) \), the equations reduce to kink systems of a new type, different from those encountered in the infrared in that they continue to involve all of the pseudoenergies. They are obtained from the original TBA equations by neglecting either the energy terms proportional to \( MR e^\theta \) (this near \( \theta = -\log(1/MR) \)), or else the terms proportional to \( MR e^{-\theta} \) (this near \( \theta = +\log(1/MR) \)). If the solutions to these two kink systems are denoted \( \varepsilon_i^L(\alpha) \) and \( \varepsilon_i^R(\alpha) \) respectively, then as \( R \to 0 \)

\[
\varepsilon_i^{(\alpha)}(\theta) \sim \begin{cases} 
\varepsilon_i^L(\alpha)(\theta) & (\theta \approx -\log \frac{1}{MR}) \\
\varepsilon_i^R(\alpha)(\theta) & (\theta \approx +\log \frac{1}{MR}) 
\end{cases}
\]

(2.20)

Once these two kink systems have decoupled, all scale dependence is lost from the equations determining the effective central charge, and with the aid of the dilogarithm identities listed in appendix C, the value of \( c(0) \) can be extracted. The result is \( 2(N-1)/(N+2) \); and this, together with the result \( c(\infty) = 1 \) already found, is exactly as would be expected of a flow from the Fateev-Zamolodchikov multicritical point into the Kosterlitz-Thouless phase. Subsequent sections will give further evidence for this interpretation. But first, the picture should be completed with a set of ansätze for the opposite perturbation, into the massive phase. There is a standard procedure to go from a massless to a massive TBA system: simply remove all energy terms \( \nu_{i}^{(5)} \) proportional to \( \frac{1}{2} MR e^{-\theta} \), and in the remaining terms \( \nu_{i}^{(1)} \) replace \( \frac{1}{2} MR e\theta \) with \( MR \cosh \theta \) throughout. Up to a relabelling of the nodes \( \alpha \to \tilde{\alpha} \) for the kink system near \( \theta = -\log(1/MR) \), the leading ultraviolet behaviour of the equations is unchanged by this manoeuvre, and so the result \( c(0) = 2(N-1)/(N+2) \) is preserved. However, the decoupling previously
induced by the energy terms in the large $R$ limit no longer occurs, and instead one finds $c(\infty) = 0$, as required for a massive flow. These systems are thus good candidates to describe the situation for the opposite sign of the coupling constant, and their properties will also be investigated below. It is interesting that after this final step, the sine-Gordon scaffolding, by which we levered ourselves up from $c=1$ to $c=2(N-1)/(N+2)$, has been completely removed.

Without a prior knowledge of the $\mathbb{Z}_N$ conformal field theories, the notion to reinterpret suitable sine-Gordon TBA systems as the infrared limiting forms of larger sets of equations might conceivably have led to their rediscovery. It is therefore worth investigating whether there are any other values of $\beta^2$ at which this might work. Curiously, it seems that the points $\beta^2 = 32\pi/N$ already captured really are special in this regard – for no other values of the coupling constant does the magnonic part of the sine-Gordon TBA system exhibit a $\mathbb{Z}_2$ symmetry with the desired properties. The only other option is to aim for an enlarged system in which the left-right $\mathbb{Z}_2$ symmetry acts trivially on the magnonic pseudoenergies. Formally, this can be done at any value of $\beta^2$: in the repulsive regime ($\beta^2 > 4\pi$; no breathers) it leads to sausage models [25] in every case. In the attractive regime the picture is not so straightforward. For example, the most natural choice of kernels to link the doubled-up nodes, namely the functions $\psi_{ij}$ successfully used above, does not seem to yield any sensible (that is, rational) values for the ultraviolet central charges. This is probably related to the ‘intolerable’ properties that the massless sausage S-matrices acquire as the attractive regime is entered [25].

3 Ultraviolet asymptotics

If the functions $E(R)$ defined in the last section really are the ground-state energies of certain perturbed conformal field theories, then their behaviours at small $R$ should be consistent with the results of conformal perturbation theory. For a unitary theory perturbed by a primary operator $\epsilon$ of dimensions $(\Delta, \Delta)$, this predicts the expansion

$$E^{(\text{pert})}(\lambda, R) = -\frac{\pi c}{6R} + \frac{2\pi}{R} \sum_{m=2}^{\infty} B_m t^m .$$

The coupling $\lambda$ appears in the dimensionless parameter $t = -2\pi \lambda (R/2\pi)^{2-2\Delta}$, while the conveniently-normalised coefficients $B_m$ are given in terms of the
connected correlation functions of the perturbing field $\epsilon$ on the plane:

$$B_m = -\frac{1}{(2\pi)^{m-1} m!} \int \langle \epsilon(1,1)\epsilon(z_1,\bar{z}_1)\ldots\epsilon(z_{m-1},\bar{z}_{m-1}) \rangle C \prod_{k=1}^{m-1} \frac{d^2 z_k}{(z_k \bar{z}_k)^{1-\Delta}} .$$  \hspace{1cm} (3.2)

The conformal mapping $z = e^{-2\pi w/R}$, from the $w$-cylinder of circumference $R$ to the infinite $z$-plane, is responsible for the appearance of the conformal anomaly $c$ in (3.1) and also for the measure used in (3.2); see for example ref. [13]. If $2\Delta \geq 1$, then a finite number of terms, up to order $n \leq 1/(1-\Delta)$, suffer from ultraviolet divergences which must be regularised: the usual prescription is to analytically continue in $\Delta$ from a region where the relevant integrals converge. Sometimes even this fails to render the answer finite, a sign of an irregular term in the expansion of $E^{(\text{pert})}$.

The constraints of conformal invariance are such that $B_2$ and $B_3$ are completely fixed by the values of $\Delta$ and the operator product coefficient $C_{\epsilon\epsilon\epsilon}$, irrespective of any other details of the theory. For later reference, they are [26]

$$B_2 = -\frac{1}{4} \frac{\gamma(\Delta)^2}{\gamma(2\Delta)} , \quad B_3 = -\frac{C_{\epsilon\epsilon\epsilon}}{48} \frac{\gamma(\Delta/2)^3}{\gamma(3\Delta/2)} , \hspace{1cm} (3.3)$$

where $\gamma(x) = \Gamma(x)/\Gamma(1-x)$. For the particular models under discussion, the values of $C_{\epsilon\epsilon\epsilon}$ were found by Zamolodchikov and Fateev to be [9]:

$$C_{\epsilon\epsilon\epsilon}^{(N)} = \frac{2}{35} \frac{\Pi^{(N)}(7)\Pi^{(N)}(2)^3}{\Pi^{(N)}(4)} \sqrt{\frac{\Gamma(1+\frac{5}{N+2})\Gamma(1-\frac{5}{N+2})}{\Gamma(1+\frac{5}{N+2})\Gamma(1-\frac{5}{N+2})}} \hspace{1cm} (3.4)$$

where $\Pi^{(N)}(j) = \prod_{k=1}^{j} \frac{\Gamma(1+\frac{k}{N+2})}{\Gamma(1-\frac{k}{N+2})}$.

The perturbative expansion is expected to have a non-zero radius of convergence, and the function that it defines should behave at large $R$ as $E^{(\text{pert})}(\lambda, R) \sim \mathcal{E}(\lambda) R$, $\mathcal{E}(\lambda)$ being the bulk contribution to the ground-state energy. Whilst this is subtracted off in the TBA, this is not the case in conformal perturbation theory and $\mathcal{E}(\lambda)$ may be nonzero. To take this into account, the predictions of the last section should be compared not with $E^{(\text{pert})}$ directly, but rather with the subtracted quantity

$$E(\lambda, R) = E^{(\text{pert})}(\lambda, R) - \mathcal{E}(\lambda) R . \hspace{1cm} (3.5)$$

Finally, note that the functions $E(R)$ of the last section all in fact depend on $R$ and $M$, where $M$ is a mass scale, part of the infrared specification.
of the model. (In the massive direction, $M$ will turn out to be the mass of a certain asymptotic one-particle state; in the massless direction it sets the crossover scale.) Conformal perturbation theory sees instead $R$ and the coupling constant $\lambda$. However, on dimensional grounds there must be a relation between $\lambda$ and $M$ of the form

$$\lambda = \kappa M^y,$$

where $y = 2 - 2\Delta$ is the dimension of $\lambda$, and $\kappa$ is some dimensionless constant relating the short and long distance descriptions of the theory. Later we will conjecture an exact expression for $\kappa$, but for the moment it must be considered as another unknown.

Returning now to the equations of the last section, the first piece of analysis, following Zamolodchikov [27], determines the value of $y$. This relies on the fact that any solution to one of the TBA systems above automatically furnishes a solution

$$Y_{i}^{(a)}(\theta) \equiv \epsilon_{i}^{(a)}(\theta)$$

to a set of functional relations known as a Y-system. The Y-systems appropriate to the problem in hand are given in appendix B, and numerically they turn out to entail the following periodicity property for their solutions:

$$Y_{i}^{(a)} \left( \theta + 2\pi \frac{N+2}{N-4} \right) = Y_{i}^{(a)}(\theta).$$

As a result, the $Y_{i}^{(a)}$ have an expansion in powers of $u(\theta) = e^{\frac{N-4}{N+4}\theta}$. For the full TBA system this will be in positive and negative powers, since in the full system energy terms blow up both at $\theta=-\infty$ ($u=0$), and at $\theta=+\infty$ ($u=\infty$). However for the kink systems which govern the ultraviolet limit, energy terms diverge in one direction only and so $Y_{i}^{R(a)} = \exp\epsilon_{i}^{R(a)}$ is regular at $u=0$, and $Y_{i}^{L(a)}$ is regular at $u=\infty$. Therefore $\epsilon_{i}^{R(a)}$ has an expansion in non-negative powers of $u$ about $u=0$, and $\epsilon_{i}^{L(a)}$ an expansion in non-positive powers about $u=\infty$. This information is important, because a part of the corrections to the asymptotic form of each kink solution can be traced to the presence of the other kink at a finite distance in $\theta$, namely $2 \log(1/\text{MR})$. If this is incorporated in an iterative fashion, then the corrections will enter as powers of $u(2 \log(1/\text{MR}))$. In turn, this implies that in the TBA corrections to $RE(R)$ form a regular series:

$$RE(R) = -\frac{\pi \epsilon}{6} + 2 \pi \sum_{m=1}^{\infty} F_{m}(\text{MR})^{2 \frac{N-4}{N+4}} m,$$

(3.8)
modulo further terms, to be mentioned shortly, that can be traced to the subtraction of the bulk term performed in equation (3.5). This agrees with the result (3.1) from conformal perturbation theory so long as 
\[ y = \frac{2(N-4)}{N+2}. \]
This gives a first check on the TBA: the prediction

\[ \Delta = \frac{6}{N+2} \]

for the conformal dimension of the perturbing field. From the work of Zamolodchikov and Fateev [9], there is indeed a suitable scalar operator in the \( \mathbb{Z}_N \)-symmetric conformal theory: sometimes denoted \( \epsilon^{(2)} \), it is both \( \mathbb{Z}_N \)-neutral and self-dual – precisely the two properties that were demanded in the later part of section 1. Furthermore, Fateev [17] has shown that this perturbation is integrable, so the existence of a TBA system describing it is to be expected.

Next we turn to the bulk piece, which, via equation (3.5), should show itself in the ultraviolet as a term \( E(\lambda) R^2 \) in the expansion of \( R E(R) \). Following refs. [13, 28], it is possible to extract the exact value of the coefficient \( E \), as a function of \( M \), from the TBA. The result, which depends on whether the system is massive or massless, is

\[ E_{\text{massive}} = \frac{-M^2 \sin \theta N \sin 2\theta N}{2 \sin 3\theta N}, \quad E_{\text{massless}} = \frac{-M^2 \sin^2 \theta N}{2 \sin 3\theta N}, \quad (3.9) \]

where \( \theta N = \frac{2\pi}{(N-4)} \). This formula can be derived generally for \( N \geq 8 \), and was also checked by specific calculations for \( N=5 \) and \( N=6 \). When \( N=4n+4 \), the value of \( E_{\text{massive}} \) agrees with an earlier result [17] for the massive \( \phi_{1,1,\text{Adj}} \) perturbation of the \( d_{n+1}^{(2)} \times d_{n+1}^{(1)} / d_{n+1}^{(3)} \) coset model; this follows the correspondence (2.1) already mentioned for some of the massless flows.

There are two values of \( N \) for which these formulae break down: if \( N=7 \) or 10, the results are formally infinite. This infinity should cancel against a term in the regular expansion (3.1), to render the overall result finite [29]. For the \( R \)-dependencies to match up, \( m \), the order of the putative cancelling term, must be equal to \( (N+2)/(N-4) \); this is indeed an integer for \( N=7 \) and 10. (Such a resonance is also found in the \( N=5 \) theory, between the bulk term and the term of order \( \lambda^7 \) in the regular expansion, but (3.9) shows that the divergence is anyway absent in this case.) The dominant residual piece can be extracted by evaluating (in the massive case)

\[ \lim_{N \to 2m+4} \left[ -\frac{\sin \theta N \sin 2\theta N}{2 \sin 3\theta N} \left( (MR)^2 - (MR)^{2N+4m} \right) \right] \]
and then setting \( m = (N+2)/(N-4) \) at the end. This gives

\[
\mathcal{E}_{\text{massive}} = -M^2 \frac{\sin \theta_N \sin 2\theta_N}{3 \cos \theta_N} \frac{N-4}{N+2} \log MR .
\] (3.10)

The calculation for the massless case is almost identical: just replace \( \sin 2\theta_N \) by \( \sin \theta_N \) throughout. The two instances of these formulae pertinent to the current discussion give us

\[
\begin{align*}
\mathcal{E}^{(N=7)}_{\text{massive}} &= \frac{1}{4\pi} M^2 \log MR , & \mathcal{E}^{(N=10)}_{\text{massive}} &= \frac{3}{8\pi} M^2 \log MR \\
\mathcal{E}^{(N=7)}_{\text{massless}} &= -\frac{1}{4\pi} M^2 \log MR , & \mathcal{E}^{(N=10)}_{\text{massless}} &= \frac{3}{8\pi} M^2 \log MR 
\end{align*}
\] (3.11)

It is also possible to say something about the nonsingular parts of the bulk terms, which have so far been disregarded: see equation (3.14) below.

These results have been obtained by analytic continuation in \( N \), back to the ‘critical’ values of 7 and 10. But it should also be possible to see the logarithms directly in conformal perturbation theory, at order \( \lambda^3 \) when \( N=7 \) and \( \lambda^2 \) when \( N=10 \). Note how this is already reflected in the relative signs of the massive and massless results in (3.11). To say more, the integrals (3.2) must be re-examined, since their values are needed at precisely the points where the general formulae (3.3) break down. The relevant integrals acquire logarithmic divergences at short distances, though not in fact at long distances: the ‘metric factors’ \( 1/(z_k \overline{z}_k)^{1-\Delta} \) prevent this. To cope with the short-distance divergences, an explicit cutoff \( a \) can be introduced, with the understanding that when the perturbative integrals are done on the cylinder, no two operators should get any closer than \( a \). Transforming this onto the plane, via \( z = e^{2\pi w/R} \), to leading order in \( a \) operators are now required to stay at least \( 2\pi a/R \) apart. This can be implemented by inserting Heaviside step functions \( \theta_{|x|} \equiv \theta(|x| - 2\pi a/R) \) as appropriate. Since the interest is in the limit where all the \( z_k \to 1 \), the metric factors can be ignored and replaced with an explicit cutoff in the infrared. Then the logarithmic divergence in \( B_2 \), which occurs when \( N=10 \) and \( \Delta=1/2 \), is found by

\[
\begin{align*}
B_2 &= -\frac{1}{4\pi} \int_{|z|<1} \langle \epsilon(0,0)\epsilon(z,\overline{z}) \rangle_C |z| z \, d^2z + \text{n.s.} \\
&= -\frac{1}{2} \int_{2\pi a/R}^1 r^{-1} dr + \text{n.s.}
\end{align*}
\]
\[ -\frac{1}{2} \log \frac{R}{2\pi a} + \text{n.s.} \]  

(3.12)

Here, ‘n.s.’ denotes the non-singular terms which do not contribute to the logarithm. Similarly, when \( N=7 \), \( \Delta=2/3 \) and there is a logarithmic divergence in \( B_3 \). This can be extracted as follows:

\[
B_3 = -\frac{1}{24\pi^2} \int_{|x|<1} |y|<1 \langle \epsilon(0,0)\epsilon(x,\vec{r})\epsilon(y,\vec{r}) \rangle C_{\epsilon\epsilon\epsilon}^{(N=7)} \theta_{|x|} \theta_{|y|} \theta_{|x-y|} d^2x d^2y + \text{n.s.} \\
= -\frac{C_{\epsilon\epsilon\epsilon}^{(N=7)}}{24\pi^2} \int_{|x|<1} |y|<1 \theta_{|x|} \theta_{|y|} \theta_{|x-y|} d^2x d^2y + \text{n.s.} \\
= -\frac{C_{\epsilon\epsilon\epsilon}^{(N=7)}}{24\pi^2} \int_{|x|<1} |y|<1 \left[ \int_{|y|<1/|x|} \theta_{|xy|} \theta_{|x-xy|} d^2y \right] d^2x + \text{n.s.}
\]

Now the integral in square brackets, call it \( I(x,a) \), is convergent (and non-zero) at \( x=a=0 \), and so to leading order it can be replaced by \( I(0,0) \), which is equal to \( \pi \gamma \left( \frac{1}{3} \right)^3 \) \[26\]. The remaining integral gives the logarithm, just as in equation (3.12). Collecting the pieces together,

\[
B_3 = -\frac{1}{12} C_{\epsilon\epsilon\epsilon}^{(N=7)} \frac{\gamma}{\frac{1}{3}} \log \frac{R}{2\pi a} + \text{n.s.} ,
\]

(3.13)

where \( C_{\epsilon\epsilon\epsilon}^{(N=7)} \) is given by equation (3.4).

Since these calculations have left the nonsingular parts of \( B_2 \) and \( B_3 \) completely uncontrolled, it would appear that no prediction can be made about the contribution to \( RE(R) \) that is proportional to \( R^2 \) itself. However this is not quite true: assuming that the same regularisation scheme is used in the massless and massive directions, all of the perturbative contributions to \( \mathcal{E}_{\text{massless}}^{(N)} \) and \( \mathcal{E}_{\text{massive}}^{(N)} \) must cancel out when their sum is taken for \( N=7 \), or their difference for \( N=10 \). The residual finite pieces then come entirely from (3.9), and are:

\[
\mathcal{E}_{\text{massive}} \pm \mathcal{E}_{\text{massless}} = \pm \frac{\sqrt{3}}{8} M^2 .
\]

(3.14)

The plus signs apply when \( N=7 \); the minus signs when \( N=10 \).

Returning to the singular terms, substituting equations (3.12) and (3.13) into the general expansion (3.1) gives new expressions for the logarithmically-divergent pieces of the bulk terms. But whereas the earlier expressions,
equations (3.11), were in terms of \( M \), this time the coupling constant \( \lambda \) appears. Equating the alternatives allows the constant \( \kappa \) appearing in the relation (3.6) to be determined exactly, at least for these two values of \( N \).

The results turn out not to depend on whether the perturbations are in the massive or the massless directions, and are:

\[
N=7 : \quad \kappa^3 = \frac{3}{4\pi^3\gamma \left( \frac{1}{3} \right)^3 C^{(N=7)}};
\]
\[
N=10 : \quad \kappa^2 = \frac{3}{8\pi^2}.
\]

(3.15)

There are other values of \( N \) for which \( \kappa \) is known exactly: a special case of a formula due to Fateev gives its value for the massive flow, whenever \( N=4n \) \[30\]. It turns out that when continued to the fractional values \( n=7/4 \) and \( n=5/2 \), the results (3.15) are reproduced. This motivates the conjecture that the analytic continuation of Fateev’s result holds for all values of \( N \), and for both the massive and the massless perturbations:

\[
\text{Any } N : \quad \kappa^2 = \frac{4}{9\pi^2} \gamma \left( \frac{4}{N+2} \right)^2 \gamma \left( \frac{5}{N+2} \right) \left[ \frac{\pi \Gamma \left( \frac{N+2}{N-1} \right)}{\Gamma \left( \frac{2}{N} \right) \Gamma \left( \frac{N}{N-1} \right)} \right]^{\frac{4N+4}{N+2}}.
\]

(3.16)

It is not meant to be obvious that the formulae (3.15) are special cases of this! Further support for the conjecture comes from numerical results, the subject of the remainder of this section.

One reason for undertaking a numerical study of the TBA equations is the lack of any analytic results for the coefficients \( F_m \) in the expansions (3.8). In particular, to be sure that the massive and massless TBA systems really do describe perturbed conformal theories differing only in the sign of a coupling constant \( \lambda \), the relative signs of the \( F_m \)’s predicted for each pair of flows should be examined. Letting \( F_m \) denote the coefficients for the massless flow, and \( \tilde{F}_m \) those for the massive flow, and accepting provisionally the conjecture that the mass scales are equal in the massive and massless directions, the expectation is

\[
F_m = (-1)^m \tilde{F}_m.
\]

(3.17)

An iterative method was used to solve the TBA equations for \( N=5, 6, 7, 9 \) and 10, discretising the \( \theta \)-axis in steps of \( \delta \theta = 0.1 \) and normalising \( M=1 \). Direct iteration fails to converge \[31\]; adding \( \varepsilon_i^{(a)} \) to both sides of
each equation, dividing by two and basing the iteration on the result solves this problem, though convergence is rather slow (up to 300 iterations were used at each value of $R$). Results for $RE(R)$ to about 14 digits precision were obtained at 30 values of $(MR)^y$ between 0.03 and 0.5. The exact constant piece, namely $-\pi(N-1)/3(N+2)$, was then subtracted, along with the bulk terms as predicted by equations (3.3) and (3.11). In addition, for $N=7$ and 10, the unambiguously nonperturbative contributions to the bulk terms, identified using equation (3.14), were subtracted in a symmetrical way. The remainder of $RE(R)$ after all this should in each case have a regular expansion in $(MR)^y$, with coefficients that conform to the general rule (3.17). A fit against the numerical data was consistent with this, and the values of the first few $F_m$ and $\tilde{F}_m$ thereby obtained are reported in appendix D. Note that the TBA system for $N=8$ is simply a doubling-up of Zamoldchikov’s proposal for the tricritical Ising to Ising flow [14]. The expansion coefficients for $N=8$ are therefore easily obtained from the earlier data, and are also included. In our fits, the $m=0$ and $m=1$ coefficients were left free; comparing their measured values with the exact predictions that they be zero gives a check on the numerical accuracy.

Besides the clear agreement of the results with equation (3.17), a number of other features deserve mention.

First, to within the numerical errors the $Z_5$ TBA has $\tilde{F}_3 = F_3 = 0$. This matches with the perturbative expansion: when $N=5$, $\Pi^{(5)}(\tau) = 0$ and, from equation (3.4), $C^{(5)}_{\tau\tau}$ vanishes.

| $N$ | $B_3^2/B_3^2$ | $\tilde{F}_2^2/F_2^2$ | $F_2^2/F_3^2$ |
|-----|----------------|-----------------------|----------------|
| 6   | 0.046260423    | 0.046260427           | 0.046260424    |
| 8   | 0.379827746    | 0.379827746           | 0.379827746    |
| 9   | 14.4489696    | 14.448967             | 14.448991     |

Table 1: Comparison of the $Z_6$, $Z_8$ and $Z_9$ systems with conformal perturbation theory

For $N=6$, 8 and 9 a more delicate test can be done. Independently of the value of $\kappa$, $F_2^3/F_3^2 = \tilde{F}_2^3/F_3^2 = B_2^3/B_3^2$ should hold if the TBA is to be consistent with conformal perturbation theory. Using equations (3.3)
and (3.4) to evaluate $B_2^3/B_3^2$, the numerical results are indeed consistent with this, as table 1 demonstrates.

Finally, the results for $N=7$ and 10 support the formulae (3.11) and (3.14): any errors in the bulk terms subtracted would have destroyed the good agreement that the fits show with the prediction (3.17).

\[ \begin{array}{|c|c|c|c|}
\hline
N & \kappa_{\text{exact}}^2 & \tilde{\kappa}_{\text{num}}^2 & \kappa_{\text{num}}^2 \\
\hline
5 & 0.0235204646 & 0.0235204667 & 0.0235204663 \\
6 & 0.0371477546 & 0.0371477547 & 0.0371477547 \\
7 & 0.0434800500 & 0.0434800501 & 0.0434800501 \\
8 & 0.0442207130 & 0.0442207131 & 0.0442207131 \\
9 & 0.0418094000 & 0.0418094001 & 0.0418094003 \\
10 & 0.0379954439 & 0.0379954451 & 0.0379954487 \\
\hline
\end{array} \]

Table 2: Comparison of exact and numerical values for $\kappa^2$

Once the consistency of the data with conformal perturbation theory has been established, numerical estimates for $\kappa$ can be extracted and checked against the conjecture (3.16). Equating the terms proportional to $R^2y$ in equations (3.1) and (3.8) yields

\[
\kappa_{\text{num}}^2 = \frac{(2\pi)^{2y-2} F_2}{B_2} = -\frac{(2\pi)^{2y}}{\pi^2} \frac{\gamma \left( \frac{4}{N+2} \right)}{\gamma \left( \frac{6}{N+2} \right)} F_2,
\]

and similarly for $\tilde{\kappa}_{\text{num}}^2$. For $N=10$, the second-order ($m=2$) term is obscured by the logarithm, but a prediction for $\kappa$ can be obtained in an analogous way from the third-order behaviour. The numbers obtained in this way are compared with the exact predictions, $\kappa_{\text{exact}}^2$, in table 3: the agreement is very good.
4 Infrared limits

The behaviours of the massless and massive systems become very different as $R$ grows, and they will be described separately. We start with the massless flows.

4.1 Into the Kosterlitz-Thouless phase

The massless systems all exhibit $c(\infty) = 1$, a prerequisite if the picture outlined at the end of section 1 is to be confirmed. For a more detailed check, some information on the possible operator contents of each $\mathbb{Z}_N$ model in its Kosterlitz-Thouless phase is needed. A complete description can be found in ref. [32]: the scaling limit of the model is equivalent to a free boson compactified on a circle of some radius $r$. Adopting the normalisation traditional in conformal field theory, the action can be taken as

$$ S_{\text{IR}}^*[\Phi] = \frac{1}{2\pi} \int d^2x (\partial \mu \Phi)^2. \tag{4.1} $$

The boson $\Phi$ can be split into holomorphic and antiholomorphic components by writing $z = x_1 + i x_2$, $\Phi(z,\bar{z}) = \frac{1}{2}(\phi(z) + \tilde{\phi}(\bar{z}))$, with

$$ \langle \phi(z) \phi(w) \rangle = -\log(z-w) , \quad \langle \bar{\phi}(z) \bar{\phi}(w) \rangle = -\log(\bar{z}-\bar{w}) . $$

Compactifying onto a circle of radius $r$ amounts to the identification $\Phi \equiv \Phi + 2\pi r$. Closer to the discussion of section 1 would have been to work with $\theta(z,\bar{z}) \equiv \Phi(z,\bar{z})/r$, whereupon the prefactor in (4.1) would have become $r^2/2\pi$. This shows that $r^2/\pi$ should be thought of as the inverse temperature.

Invariance under $\Phi \to \Phi + 2\pi r$ and mutual locality restrict the possible vertex operator primary fields to

$$ V_{nm}^+(z,\bar{z}) = \sqrt{2} \cos(p\phi(z) + p\bar{\phi}(\bar{z})) , \quad V_{nm}^-(z,\bar{z}) = \sqrt{2} \sin(p\phi(z) + p\bar{\phi}(\bar{z})) , $$

where $(p,p) = (\frac{n}{2r} + mr, \frac{n}{2r} - mr)$ and $n, m \in \mathbb{Z}$. These have the conformal weights

$$ (\Delta_{nm}, \bar{\Delta}_{nm}) = \left( \frac{1}{2} p^2, \frac{1}{2} p^2 \right) = \left( \frac{1}{2} \left( \frac{n}{2r} + mr \right)^2, \frac{1}{2} \left( \frac{n}{2r} - mr \right)^2 \right) . $$

Included in this collection are the vortex fields $V_{0M}^\pm$, and the $N$-fold symmetry breaking fields $V_{N0}^\pm$. Their conformal weights are:

$$ \Delta_{N0}(r) = \bar{\Delta}_{N0}(r) = \frac{N^2}{8r^2} , \quad \Delta_{0M}(r) = \bar{\Delta}_{0M}(r) = \frac{M^2r^2}{2} . $$
This information allows the of values of $r$ actually realised in the Kosterlitz-Thouless region to be determined. At the low-temperature (large-$r$) border, the fields $V_{N0}^\pm$ become relevant, so $\Delta_{N0}(r_{\text{max}}) = 1$, and $r_{\text{max}} = N/\sqrt{8}$. Similarly the high-temperature boundary is encountered when the least irrelevant vortex fields, $V_{01}^\pm$, become relevant, that is when $\Delta_{01}(r) = 1$, so $r_{\text{min}} = \sqrt{2}$. One further value of $r$ can be identified in the thermodynamic phase space. The duality transformation for the $\mathbb{Z}_N$ model exchanges the $N$-fold symmetry-breaking field with the unit vortex, that is $V_{N0}$ with $V_{01}$. Thus $r_{\text{sd}}$, the compactification radius appropriate to any self-dual system, must satisfy $\Delta_{N0}(r_{\text{sd}}) = \Delta_{01}(r_{\text{sd}})$. Hence, $r_{\text{sd}} = \sqrt{N/2}$. To summarise:

| $r$       | $\Delta_{N0}$ | $\Delta_{01}$ |
|-----------|---------------|---------------|
| $r_{\text{min}} = \sqrt{2}$ | $N^2/16$      | 1             |
| $r_{\text{sd}} = \sqrt{N}$   | $N/4$         | $N/4$         |
| $r_{\text{max}} = N/\sqrt{8}$ | 1             | $N^2/16$      |

Since initial (ultraviolet) perturbation was self-dual, it is the middle row of the table that will be important. The discussion has been couched in the language of continuum-valued spins, and it might appear surprising that a model with discrete spins can mimic such behaviour. Universality and a suitable renormalisation group should anyway suffice, but for a more physical picture of the mechanism involved, see ref. [33].

Pure $c=1$ behaviour is only expected in the very far infrared. For any finite value of $R$, there will be corrections to scaling stemming from irrelevant operators compatible with the symmetry of the model [34]. Apart from $TT$ and its descendants, the most relevant of these when the theory is self-dual have conformal dimensions $(N/4, N/4)$. The model is therefore described by the continuum action

$$S = S_{\text{IR}}^* + \mu_1 \int \psi_{N/4} d^2 x + \mu_2 \int T T d^2 x + \text{(further terms)},$$

(4.2)

where $S_{\text{IR}}^*$ is the action (4.1), and $\psi_{N/4}$ denotes an equal (self-dual) combination of the unit vortex and $N$-fold symmetry-breaking fields. The further terms are not optional in the continuum theory: the perturbation theory is non-renormalisable, and so infinitely many counterterms are needed. Nevertheless, it is possible to extract some non-trivial predictions: an asymptotic expansion for $RE(R)$, in various powers of $R$ and $\mu_1, \mu_2, \ldots$, can be obtained after a mapping from the cylinder to the plane [15].
The fact that $\psi_{N/4}$ is primary means that its contributions only start at order $\mu^2 R^2$, and the charge conservation selection rule for vertex operator correlation functions restricts subsequent terms to even powers. For this operator, $y = (2-2\Delta) = 2-N/2$, and so its leading correction to $RE(R)$ is generally proportional to $\mu^2 R^{4-N}$.

By contrast, $T\bar{T}$ is a secondary operator and has a non-vanishing one-point function on the cylinder. Consequently, despite its renormalisation group eigenvalue $y$ being equal to 4, the corrections due to $T\bar{T}$ start early, with a term proportional to $\mu^2 R^{-2}$.

Evidence for $\psi_{N/4}$ can be found in the influence of the magnonic pseudoenergies on the large $R$ solutions of the massless TBA equations, using a small extension of an argument given by Zamolodchikov [16]. In the $R\rightarrow\infty$ limit, the pseudoenergies acquire the form of a pair of sine-Gordon kink systems, one located near $\theta=-\log MR$, and one near $\theta=\log MR$. In the central region between the kinks, the non-magnonic pseudoenergies are dominated by their energy terms, and play no direct rôle in the equations: the functions $L_n^{(1)}$ and $L_n^{(3)}$, which would otherwise couple to the reduced system of magnonic pseudoenergies, are doubly-exponentially suppressed. Their dominant effect comes instead via the ‘tails’ of the relevant kernel functions, which only suffer a single-exponential decay.

Thus for $N = 4n+2$ (when the TBA system (2.8) applies) the magnonic pseudoenergy $\varepsilon_n^{(2)}(\theta)$ sees $\varepsilon_n^{(1)}(\theta)$ and $\varepsilon_n^{(3)}(\theta)$ via the convolutions $\phi_1*L_n^{(1)}(\theta)$ and $\phi_1*L_n^{(3)}(\theta)$. In the central region these expand as odd series in $e^{\pm h\theta}$ (recall that $\phi_1(\theta)=h/\cosh h\theta$), implying that $Y_n^{(2)}(\theta)=e^{\varepsilon_n^{(2)}(\theta)}$ satisfies the equation

$$Y_n^{(2)}(\theta - \frac{i\pi}{4h})Y_n^{(2)}(\theta + \frac{i\pi}{4h}) = 1. \tag{4.3}$$

(Note that this is just the truncation of the full Y-system to the central magnonic node.) Therefore $Y_n^{(2)}(\theta+\frac{i\pi}{2h}) = Y_n^{(2)}(\theta)$. Repeating the argument used in the ultraviolet limit, this periodicity implies that the leading ‘magnonic’ corrections to $RE(R)$ are powers of $(MR)^{-2h} = (MR)^{4-N}$.

For $N = 4n+1$ and $N = 4n+3$, the magnonic pseudoenergies form a $d_4$-type system with kernel $\phi_2(\theta) = 2h/\cosh 2h\theta$. The pseudoenergies $\varepsilon_n^{(1)}(\theta)$ and $\varepsilon_n^{(3)}(\theta)$ are felt via $\phi_2*L_n^{(1)}(\theta)$ and $\phi_2*L_n^{(3)}(\theta)$. These expand in the central region as odd powers of $e^{\pm 2h\theta}$, and such terms cancel out when shifted by $\pm i\pi/4h$ and summed. As a result they do not impede the construction of
the Y-systems. For \( N = 4n+1 \), the system is

\[
Y_n^{(\alpha)}(\theta - \frac{\pi}{4n}) Y_n^{(\alpha)}(\theta + \frac{\pi}{4n}) = \prod_{\beta=2}^{5} \left( 1 + Y_n^{(\beta)}(\theta) \right)^{-l^{[4]}_{\alpha\beta}}, \tag{4.4}
\]

while for \( N = 4n+3 \),

\[
Y_n^{(\alpha)}(\theta - \frac{\pi}{4n}) Y_n^{(\alpha)}(\theta + \frac{\pi}{4n}) = \prod_{\beta=2}^{5} \left( 1 + Y_n^{(\beta)}(\theta)^{-1} \right)^{-l^{[4]}_{\alpha\beta}}. \tag{4.5}
\]

The index \( \alpha \) runs from 2 to 5 in both cases. Again, these are just truncations of the full Y-systems. They imply the periodicity \( Y_n^{(\alpha)}(\theta + 2\frac{\pi}{2n}) = Y_n^{(\alpha)}(\theta) \) \[27\], after which the argument runs just as for \( N = 4n+2 \), and again predicts corrections to scaling as powers of \((MR)^4-N\).

Finally, to the case \( N=4n+4 \). There are no magnonic pseudoenergies in the TBA system, and so all corrections to scaling must come from the ‘direct’ interaction between the non-magnonic pseudoenergies, to be discussed shortly. This would seem to be a problem, since the general idea will be to attribute these corrections to \( TT \) and descendants. Two points can be made. First, contributions from \( \psi_{N/4} \), expected to appear as powers of \((MR)^{-4n} \), will anyway be badly tangled up with those from \( TT \) and its descendants. However this point applies equally when \( N = 4n+2 \), where at least for \( N = 6 \) the entanglement manifests itself in a logarithm at order \((MR)^{-2} \) (see below). Such a logarithm is not found at order \((MR)^{-4} \) in the \( \mathbb{Z}_8 \) theory: in fact the coefficient of this term fits perfectly the hypothesis of a pure \( TT \) perturbation. (Note, coefficients up to high order can be obtained for this case by doubling the (exact) results for the tricritical Ising to Ising flow obtained by Zamolodchikov \[16\].) The resolution comes on re-examining the formula (3.3) for the coefficients in conformal perturbation theory: when \( N=4n, \Delta=n \) and is a positive integer, and \( B_2 \) vanishes. Thus signs of \( \psi_{N/4} \) anyway appear anomalously late in the series. Without any control of the counterterms, it is not possible to be any more precise than this, but at least the proposal has not turned out to be inconsistent.

So much for \( \psi_{N/4} \). For \( TT \), the analysis can be pushed a little further. To eliminate the corrections already discussed, it is convenient to replace the TBA equations with a set in which the magnonic pseudoenergies associated with the two kink systems are separated, by doubling them up and coupling one copy only to \( \varepsilon^{(1)}_n \), and the other only to \( \varepsilon^{(1)}_n \). Since the magnonic interaction only cuts in at order \((MR)^{4-N} \), the approximation should be
good at least up to this point. For $N=4n+4$, no approximation is needed at all. The kinks near $\theta=\log MR$ and $\theta=\log MR$ now only interact via the tails of the kernel functions $\psi_{ij}(\theta)$. In Zamolodchikov’s original paper on the massless TBA [16] an iterative procedure was used to account for such an interaction. It turns out that in all of the subsequently-introduced TBA systems for the flows $g^{(2)} \times g^{(1)}/g^{(3)} \rightarrow g^{(1)} \times g^{(1)}/g^{(2)}, \; g \in ade$ [35, 38], the first two corrections to infrared scaling also have a simple form. Only the case $g = d_{n+1}$ will be directly relevant here, but the general result will be recorded as it does not seem to have been given elsewhere. The TBA systems were given in their full generality in ref. [18], and involve kernel functions $\phi_{ij}(\theta)$ and $\psi_{ij}(\theta)$, related to the corresponding purely elastic S-matrix elements in the way described for the $d_{n+1}$ case after equation (2.3).

Using some tricks previously employed in the (ultraviolet) calculation of the bulk terms for certain massive flows from the $g^{(1)} \times g^{(1)}/g^{(2)}$ cosets [31], the final result is

$$\frac{1}{2\pi} RE(R) = -\frac{1}{24} c(\infty) + \frac{C_1}{(MR)^2} + \frac{C_2}{(MR)^4} + O\left((MR)^{-6}\right) \quad (4.6)$$

where $c(\infty)$ is the central charge of the infrared limit. The coefficients $C_1$ and $C_2$ are

$$C_1 = -\frac{c(\infty)^2}{12} \frac{\pi}{3 M_1^2} \psi_{11}^{(1)}, \quad C_2 = -\frac{c(\infty)^3}{6} \left( \frac{\pi}{3 M_1^2} \psi_{11}^{(1)} \right)^2 \quad (4.7)$$

where $M_1$ is the mass of the lightest particle in the theory, and $\psi_{11}^{(1)}$ is the first coefficient of an expansion of the kernel $\psi_{ij}(\theta)$:

$$\psi_{ij}(\theta) = -\sum_{s=1}^{\infty} \psi_{ij}^{(s)} e^{-s|\theta|}.$$\n
These coefficients are only non-zero when $s$ taken modulo $h$ is an exponent of the relevant non-affine algebra $g$. An explicit expression is

$$\psi_{ij}^{(s)} = \frac{h}{\sin \frac{\pi}{h}} q_i^{(s)} q_j^{(s)}, \quad (4.8)$$

where $q_i^{(s)}$ and $q_j^{(s)}$ are components of a unit-normalised eigenvector of the Cartan matrix of $g$ with eigenvalue $2-2\cos \frac{\pi}{h}s$. This can be derived from the general S-matrix formulae of ref. [36], using the identity

$$\sum_{p=0}^{h-1} (\lambda_i, w^{-p}\phi_j) \sin(2p+1+u_{ij}) \frac{\pi}{h} = \frac{h}{2\sin \frac{\pi}{h}} \frac{q_i^{(s)} q_j^{(s)}}{\psi_{11}^{(1)}} \quad (4.9)$$
which is the Fourier transform of equation (2.14) of ref. [37] (See refs. [36, 37] for an explanation of the notation used in this last formula.)

The results (4.6), (4.7) can be compared with the perturbative expansion of the action $S^*_\text{IR} + \mu_2 \int T\bar{T}d^2x$ [13]:

$$\frac{1}{2\pi} RE(R) = -\frac{1}{12} c(\infty) + \frac{(2\pi)^3 \mu_2}{R^2} \left( \frac{c(\infty)}{24} \right)^2 - \frac{(2\pi)^6 \mu_2^2}{R^4} \left( \frac{c(\infty)}{24} \right)^3 + O\left( \frac{\mu_2^3}{R^6} \right).$$

(4.10)

This much of the expansion is uncontaminated by counterterms, and therefore gives the unambiguous prediction $C_2^2 / C_2 = -c(\infty)/24$ if the TBA is to be compatible with conformal perturbation theory. It is easy to verify that this does indeed hold when the coefficients are given by (4.7). (This was checked numerically by Martins, for $g = a_2$, in ref. [35].) A comparison of the two series also allows the renormalised coupling $\mu_2$ to be expressed in terms of the crossover scale $M$:

$$\mu_2 = -\frac{2M^2}{\pi^2 M_1^2} \psi_{11}^{(1)} M^{-2}. \quad (4.11)$$

Note that for any $g \in \text{ade}$, this $\mu_2$ is negative: for a unitary flow with $T\bar{T}$ contributing the leading infrared correction, this has to be the case, since otherwise the expansion (4.10) would contradict Zamolodchikov’s $c$-theorem.

Specialising now to $g = d_{n+1}$, we obtain results relevant to the self-dual $\mathbb{Z}_N$ flows with $N = 4n+4$. For other values of $N$, the kernel $\psi_{nn}(\theta)$ is more complicated. Nevertheless, the identity

$$\psi_{ij}^{(1)} = \frac{M_i M_j}{M_1^2} \psi_{11}^{(1)} \quad (i, j = 1 \ldots n)$$

(4.11)

(4.12)
These formulae break down when \( N = 5 \) and \( N = 6 \). This is not too surprising: the whole derivation fails for \( N \leq 7 \), since the direct interaction kernels \( \psi_{ij}(\theta) \) are zero in these cases. As the numerical results below show, \( 1/R^2 \) corrections do nonetheless appear. At a qualitative level, these can be seen to come, exceptionally, from the magnonic kernels (though for \( N = 7 \) it is necessary to invoke the thus-far disregarded kernels \( \phi_3, \phi_4 \) and \( \phi_5 \)). However, for exact results it is more fruitful to suppose that an analytic continuation of the formulae valid for larger values of \( N \) makes sense. For \( N = 7 \) this is easily done, and gives a result that checks well with the numerical data. For \( N = 5 \) and \( N = 6 \), a little more work is needed: in the same spirit as for the bulk term logarithms, the ‘\( T \bar{T} \)’ divergences given by (4.12) should balance against divergences in other parts of the expansion, allowing the final result to remain finite. Indeed, at these two values of \( N \) an additional contribution proportional to \( R^{-2} \) comes from the term of order \( \mu^2/(N-4) \) in the perturbative expansion, and this has the potential to cancel the pole in \( C_1 \) (furthermore, the regularised \( B_2 \) itself diverges when \( N = 6 \)). To find the residual finite term, set \( m = 2/(N-4) \) and evaluate

\[
\lim_{N \to 4+} \frac{1}{2m} \left[ -1 + \frac{\pi}{12} \sin \frac{2\pi}{N-4} \left( \frac{1}{(MR)^2} - \frac{1}{(MR)^{(N-4)m}} \right) \right] = \frac{(N-4)}{36} \log MR \left\{ \frac{2}{(MR)^2} \right\}.
\]

This gives the corrected coefficients

\[
C_1^{(N=5)} = \frac{1}{36} \log MR + \text{const}, \quad C_1^{(N=6)} = -\frac{1}{18} \log MR + \text{const}.
\]

These values can also be extracted directly from the TBA, as explained in section 4.2 below. To deal with \( C_2 \), two additional terms must be taken into account, at perturbative orders \( \mu_1^{2m} \) and \( \mu_1^m \mu_2 \); both become proportional to \( R^{-4} \) when \( N = 5 \) or 6. In terms of \( x = N - (4+\frac{2}{m}) \), they should cancel the \( x \to 0 \) divergences in

\[
C_2(x) = -\frac{2}{27m^4} \left[ \frac{1}{x^2} + \frac{m}{x} + \frac{3m^2}{2} + \frac{m^4 \pi^2}{12} + \ldots \right].
\]

To achieve this, \( a(x) \) and \( b(x) \) must be found such that

\[
\frac{1}{x^2} \left( \frac{1}{(MR)^4} + \frac{mx}{(MR)^4} - \frac{a(x)}{(MR)^{4+2mx}} - \frac{b(x)}{(MR)^{4+mx}} \right)
\]

is finite at \( x = 0 \), determining \( a(x) = -1 + \alpha x + O(x^2) \), \( b(x) = 2 + (m-\alpha)x + O(x^2) \). In turn this fixes that part of the finite residue proportional to
log² MR. The upshot is that the earlier expression for C₂ should be modified to

\[ C_{2}^{(N=5,6)} = -\frac{(N-4)^2}{54} \log² MR + \text{const.} \log MR + \text{const}'. \]  (4.15)

Infrared logarithms appearing to a single power have previously been observed numerically, in the interpolating flows between the low-lying minimal models \( \mathcal{M}_p \rightarrow \mathcal{M}_{p-1} \) \([28, 38]\), and also analytically, in a massless sausage model \([25]\). The method used in section 4.2 to extract the \( Z_N \) logarithms directly from the TBA can also be applied to these other flows, and the calculations are described in second half of that section.

We also investigated the infrared behaviour of the TBA systems numerically, both to verify the results derived above and to provide information on coefficients that we have not been able to extract exactly. For \( R \) ranging from 120 to 11000, \( \frac{1}{R} RE(R) \) was found for \( N = 5, 6 \) and 7. The iterative procedure was as described in the last section, and again the normalisation \( M=1 \) was adopted.

First, we estimated the exponents of the leading correction terms by finding the limiting slopes of plots of \( \log(\frac{1}{R} RE(R) + \frac{1}{12}) \) against \( \log R \), with the results \(-0.9993, -1.9967 \) and \(-2.0010 \) respectively (to be compared with the predictions of \(-1, -2, -2 \)). Thus reassured that at least the leading behaviour was as expected, we then fitted the data to expansions in \( R \) and \( \log R \) of the predicted forms, leaving all the coefficients unconstrained. Numerical precision was not as good as in the ultraviolet, possibly reflecting the asymptotic nature of the infrared expansions. The following fits were obtained for \( \frac{1}{R} RE(R) + \frac{1}{12} \):

\[
\begin{align*}
N=5 & : -\frac{0.0177380}{R} + \frac{0.027733 \log R}{R^2} - \frac{0.01951}{R^2} + \frac{0.097}{R^3} + \ldots \\
N=6 & : -\frac{0.0555546 \log R}{R^2} + \frac{0.033585}{R^2} - \frac{0.0715 \log^2 R}{R^4} + \ldots \\
N=7 & : -\frac{0.1007662}{R^2} + \frac{0.153}{R^3} + \ldots
\end{align*}
\]  (4.16)

In each case, the omitted constant piece (measuring the difference between the numerical and exact values for \( c(\infty)/12 \)) was zero to 11 significant figures. For \( N=5 \) and 6, the coefficients of \( R^{-2} \log R \) match well with the predicted values of \( 0.02777\ldots \) and \(-0.05555\ldots \) respectively, while for \( N=7 \) the
formula (4.12) predicts the coefficient of $R^{-2}$ to be $-0.1007663\ldots$, again in good agreement with the numerical results. The other predictions are more difficult to verify: if the expansions really are asymptotic, then results at larger and larger values of $R$ are needed before higher terms can be captured, and one rapidly runs up against the limitations in numerical accuracy. For $N=7$, a coefficient of $0.24369\ldots$ is predicted for the $R^{-1}$ term, while for $N=5$ and $N=6$ the coefficients of $R^{-4}\log^2 R$ should be $-0.0185185\ldots$ and $-0.074074\ldots$ respectively. The agreement for $N=6$ is reasonable, and improved when a restricted fit (using the exact values for $c(\infty)$ and $C_1$) was performed, the estimate for this coefficient changing to $-0.0743\ldots$. For the other cases, no such convergence was observed for any coefficients beyond those reported above. This is not particularly surprising, since for $N=5$ there are more unknowns to fix before the term of interest is reached, and for $N=7$ it is anyway smaller, lacking the $\log^2 R$ factor. Greater numerical precision will be needed before any more can be said.

Nevertheless, in all instances where we have reliable numerical results, they agree with the exact predictions. Since some of these latter were derived using an additional physical assumption of analyticity in $N$, this lends further credibility to the whole scenario.

One further remark can be made before leaving the Kosterlitz-Thouless phase. If the action defined by (4.2) and (4.1) is converted into the normalisations of appendix A by setting $\Phi = \sqrt{\pi} \varphi$, then the $N$-fold symmetry breaking part of the perturbation becomes proportional to $\cos(\sqrt{2N\pi} \varphi)$, and formally matches the sine-Gordon Lagrangian (A.1) at $\beta^2 = 2N\pi$. Toda-type duality (see for example ref. [39]) maps $\beta^2$ to $64\pi^2 / \beta^2$ in these normalisations. Thus $\beta^2 = 2N\pi$ is Toda-dual to $\beta^2 = 32\pi / N$, the value already picked out in section 2. This gives an alternative perspective on these special values of the sine-Gordon coupling constant.

## 4.2 Exact infrared information from the TBA

This section, something of an interlude from the main development of the paper, discusses some general properties of massless TBA systems. Motivated by the results obtained above, the particular interest is the mechanism by which logarithmic terms can be generated at large $R$. The simplest example of this phenomenon is the flow between the minimal models $\mathcal{M}_5$ and $\mathcal{M}_4$ [28]. The relevant TBA system, proposed by Zamolodchikov [14], reads

\[
\frac{1}{2} M R e^\theta = \varepsilon_1(\theta) + \phi_* L_2(\theta)
\]  

(4.17)
\begin{align}
0 &= \varepsilon_2(\theta) + \phi \ast L_1(\theta) + \phi \ast L_3(\theta) \quad (4.18) \\
\frac{1}{2} M Re^{-\theta} &= \varepsilon_3(\theta) + \phi \ast L_2(\theta) \quad (4.19)
\end{align}

with \( \phi(\theta) = 1/\cosh \theta \), \( L_i = \log(1 + e^{-\varepsilon_i}) \), and

\[
\frac{1}{2\pi} RE(R) = -\frac{1}{12} c(R) = -\frac{1}{4\pi^2} \int_{-\infty}^{\infty} d\theta M Re^{\theta} L_1(\theta) = -\frac{1}{4\pi^2} \int_{-\infty}^{\infty} d\theta M Re^{-\theta} L_3(\theta).
\]

Take the derivative with respect to \( \theta \) of (4.17), multiply by \( -\frac{1}{2\pi} L_1(\theta) \), and integrate from \(-\infty \) to \( \infty \) to find

\[
\frac{1}{2\pi} RE(R) = -\frac{1}{2\pi^2} \int_{-\infty}^{\infty} d\theta \partial_\theta \varepsilon_1(\theta)L_1(\theta) - \frac{1}{2\pi^2} \int_{-\infty}^{\infty} d\theta (\phi \ast \partial_\theta L_2(\theta)) L_1(\theta).
\]

At \( R=\infty \), \( \varepsilon_1 \) solves a kink system obtained from equations (4.17)–(4.19) by omitting all terms involving \( \varepsilon_3 \), and vice versa for \( \varepsilon_3 \). In line with the notation used earlier in the paper, write \( \varepsilon_R^1, \varepsilon_R^2 \) for the solutions to the first kink system, and \( \varepsilon_L^2, \varepsilon_L^3 \) for the solutions to the second. To find the logarithmic correction, which comes from an integral over the central region \( -\log MR \ll \theta' \ll \log MR \) between the two sets of kinks, replace \( L_1 \) with \( L_1^R \) and \( \partial_\theta L_2 \) by \( \partial_\theta L_2^R + \partial_\theta L_2^L \). This gives

\[
\frac{1}{2\pi} RE(R) = -\frac{1}{12} c(\infty) - \frac{1}{2\pi^2} \int_{-\infty}^{\infty} d\theta (\phi \ast \partial_\theta L_2^R(\theta)) L_1^R(\theta).
\]

In the central region \( \varepsilon_2^L(\theta') \) satisfies

\[
0 = \varepsilon_2^L(\theta') + \frac{1}{\pi} \int_{-\infty}^{\infty} d\theta'' \frac{1}{\cosh(\theta' - \theta'')} L_2^L(\theta'')
\]

\[
= \varepsilon_2^L(\theta') + \frac{e^{\theta'}}{\pi} \int_{-\infty}^{\infty} d\theta'' e^{-\theta''} L_3^L(\theta'') + \ldots
\]

\[
= \varepsilon_2^L(\theta') + \frac{\pi c(\infty)}{3MR} e^{\theta'} + \ldots,
\]

and so

\[
\partial_\theta L_2^L(\theta') = \frac{\pi c(\infty)}{6MR} e^{\theta'} + \ldots.
\]

Because of the double-exponential suppression of \( L_1^R(\theta) \) in the central region, \( \phi(\theta-\theta') \partial_\theta L_2^L(\theta') \) can be replaced there by \( 2e^{\theta-\theta'} \partial_\theta L_2^L(\theta') \). It therefore
has a piece constant in $\theta'$, equal to $e^\theta \pi c(\infty)/3MR$. Dividing by $2\pi$ and integrating across the central region for the convolution then produces a factor $\frac{1}{\pi} \log MR$, and equation (4.21) becomes
\[
\frac{1}{2\pi} R E(R) = -\frac{1}{12} c(\infty) - \frac{1}{2\pi^2} \int_{-\infty}^{\infty} d\theta e^{-\theta} \pi c(\infty) \log MR L_0(\theta) + \ldots
\]
\[
= -\frac{1}{12} c(\infty) - \frac{c(\infty)^2}{18(MR)^2} \log MR + \ldots \quad (4.24)
\]

With $c(\infty) = 7/10$, the coefficient of the logarithmic term is $-49/1800 = -0.027222\ldots$, which compares well with the value $-0.02723(2)$ found numerically by Klassen and Melzer in ref. [28].

The calculation goes through essentially unchanged if the TBA system is augmented by equal numbers of rightmoving and leftmoving pseudenergies, so long as they interact only with $\varepsilon_1$ and $\varepsilon_3$ respectively. Their effect is to change the value of $c(\infty)$, and the result (4.24) remains valid. In particular, the $Z_6$ TBA system, a special case of (2.8), can be obtained in this way, simply by adding two magnonic pseudenergies $\varepsilon_0$ and $\varepsilon_4$ to the system (4.17-4.19). (Note, when $N=6$, $\phi_1(\theta)$ is exceptionally equal to $1/\cosh \theta$.) Since $c(\infty)=1$ for this system, the value of $C_1^{(N=6)}$ given by equation (4.14) follows immediately.

A different generalisation is needed to find $C_1^{(N=5)}$, since for $N=5$ the left and right kink systems influence each other via a reduced system containing more than one magnonic pseudoeenergy. This will modify the result (4.24), but if $\partial_0 L_0(\theta')$ retains a piece proportional to $e^{\theta'}$ in the central region, a logarithm will still appear. Formally the question is the same as arises in Zamolodchikov’s treatment of ultraviolet logarithmic terms, described in appendix A of ref. [16], since to leading order the equations are the same as those for a standard TBA. The one change is that instead of energy terms $\frac{1}{2} MRe^{\pm\theta}$, the reduced system is driven by terms $\frac{\pi c(\infty)}{3MR} e^{\pm\theta}$, appearing for those pseudoeenergies which in the full system would be coupled to the remaining right or left moving pseudoeenergies. These are induced in the same manner as already seen in equation (4.22). (An analogous effect was important in the derivation of equations (4.3)–(4.5) of section 4.) By reversing some of the steps of Zamolodchikov’s ultraviolet argument, and then repeating them in the new context, it is possible to derive the following prescription. Suppose that the TBA based on the reduced set of pseudoeenergies would have shown an ultraviolet logarithmic correction to $\frac{1}{2\pi} R E(R)$ equal to $F_{(\log)}(MR/2\pi)^2 \log MR$, had the energy terms been of the usual
form. Then towards the infrared limit of the full problem, \( \partial_\theta L^L_2(\theta') \) develops a piece proportional to \( e^{\theta'} \) in the central region:

\[
\partial_\theta L^L_2(\theta') = \ldots + F^{(UV)}_{(\log)} \frac{\pi c(\infty)}{3MR} e^{\theta'} + \ldots \quad (-\log MR \ll \theta' \ll \log MR).
\]

Note, in contrast to equation (4.23), this will not necessarily be the leading term. Nevertheless it gives the leading logarithmic contribution, feeding through the remainder of the calculation to predict

\[
\frac{1}{2\pi} RE(R) = -\frac{1}{12} c(\infty) + \ldots + \frac{\tilde{C}_1}{(MR)^2} \log MR + \ldots ,
\]

with

\[
\tilde{C}_1 = \frac{1}{9} F^{(UV)}_{(\log)} c(\infty)^2 .
\] (4.25)

The earlier result (4.24) can be recovered on noting that the reduced TBA there is the same as that of the thermally-perturbed Ising model, for which \( F^{(UV)}_{(\log)} = -1/2 \). For the \( \mathbb{Z}_5 \) TBA, the reduced system is based on the \( d_4 \) Dynkin diagram, with driving terms attached one to each node of the fork, and \( F^{(UV)}_{(\log)} = 1/4 \) [40]. Hence \( \tilde{C}_1^{N=5} = 1/36 \), as previously obtained by other means.

The same method can be applied to the sausage models \( SST^{(-)}_\lambda \) at \( \lambda = 1/K, \ K = 3, 4, \ldots \), using the TBA systems given in ref. [23]. These models arrive at \( c(\infty) = 1 \), attracted in the infrared by an operator with conformal dimension \( \Delta_{1R} = 1/(1-\lambda) = K/(K-1) \). The reduced system is based on the (non-affine) \( d_{K-1} \) Dynkin diagram, with (for \( K \geq 4 \)) both left and right moving driving terms coupling to the extremal node of the tail. This entails \( F^{(UV)}_{(\log)} = 0 \) for \( K \) even, and \( F^{(UV)}_{(\log)} = -2/(K-1) \) for \( K \) odd [21]. Hence

\[
\tilde{C}_1^{(\lambda=1/K)} = \left\{ \begin{array}{ll}
0 & (K \text{ even}) \\
-\frac{2}{9(K-1)} & (K \text{ odd})
\end{array} \right..
\] (4.26)

When \( K=3 \) the reduced system is disconnected, and there are two contributions to the logarithmic term. When these are added together, the general formula turns out to be valid in this case as well. In ref. [23] the same result for \( K=3 \) was derived by an indirect route, closer to the analytic continuation argument used elsewhere in this paper. Part of the input was the pair of formulæ

\[
C_1^{(\lambda)} = -\frac{\pi}{18} \tan \frac{\pi}{2\lambda} , \quad \mu_2^{(\lambda)} = -\frac{4}{\pi^2} \tan \frac{\pi}{2\lambda} M^{-2} ,
\] (4.27)
found by putting the model in an external magnetic field. In fact this information is enough to predict the coefficient of the logarithm for all integer $K$, with agreement with the direct TBA derivation. This gives a further check on the results of ref. [25]. Going further (and contrary to equation (5.47) of [25]), arguments similar to those used in section 4 predict an additional $\log^2 MR/(MR)^4$ correction for $K$ odd, with coefficient $-32/27(K-1)^2$. This was checked numerically for $K=3$.

One other special case of the general result (4.25) is of some interest. This covers the TBA systems proposed by Zamolodchikov in ref. [41] to describe the flows $MA^{(+)}(k,l)$:

$$MA^{(+)}(k,l) : M(k,l) \rightarrow M(k,l-k)$$

from the coset models $M(k,l) \equiv a_1^{(k)}a_1^{(l)}/a_1^{(k+l)}$. The pseudoenergies live on an $a_{l-k-1}$ Dynkin diagram, with a rightmoving energy term $\frac{1}{2}MR\cos^2$ on the $k$th node, and a symmetrically-placed lefthemoving term $\frac{1}{2}MR\cos^2$ on the $l$th node. The reduced system operating in the central region sees the $a_{l-k-1}$ Dynkin diagram, with the induced energy terms sitting one at each end. If $l-k-1$ is odd then the ultraviolet version of such a system shows a logarithmic singularity, with $F^{(\log)}_{(UV)} = 2(-1)^{(l-k)/2}/(l-k+2)$. Therefore the flow (4.28) will show a logarithm in the infrared whenever $l-k$ is even. The coefficient is given by (4.25), with $c(\infty) = c(k,l-k) = 3k(l-k)(l+4)/(k+2)(l+2)(l-k+2)$. Setting $k=2$, $l=4$ recovers the $\mathbb{Z}_6$ result, while the cases $k=1$, $l=p-2$ pertain to the flows $MA^{(+)}_p$: $M_p \rightarrow M_{p-1}$ between minimal models. Therefore, for these flows

$$\tilde{C}_1^{(p)} = \begin{cases} 0 & (p \text{ even}) \\ (-1)^{(p-3)/2} \frac{2c(\infty)^2}{9(p-1)} & (p \text{ odd}) \end{cases}$$

where $c(\infty) = 1 - 6/p(p-1)$. An analogous calculation for the massless flows $H_{p-2}^{(\pi)}$: $Z_{p-2} \rightarrow M_{p-1}$ introduced in ref. [40] (where $Z_{p-2}$ denotes the $\mathbb{Z}_{p-2}$-symmetric conformal field theory) finds the same formula (4.29), save for the replacement of the prefactor $(-1)^{(p-3)/2}$ by $-1$. The similarity of the $H_{p-2}^{(\pi)}$ result to the sausage formula (4.26) is not coincidental. As recently stressed in ref. [38], it appears to be consistent to regard the theory $H_{p-2}^{(\pi)}$ as a reduction of the sausage model $SST_1^{(-)}$. Furthermore, in such a case the ground-state energy $E^*(R)$ after reduction should be equal to that of a suitable excited state $E^u_k(R)$ in the unreduced model (this is clearly
explained for the reductions of the sine-Gordon model in ref. [42]). For the reduced ultraviolet and infrared central charges to emerge correctly from this manoeuvre, the excited state must have ultraviolet and infrared scaling dimensions

\[ 2\Delta_k(0) = \frac{1}{12} (c^u(0) - c^r(0)), \quad 2\Delta_k(\infty) = \frac{1}{12} (c^u(\infty) - c^r(\infty)), \]

with \(c^u(0), \ c^u(\infty)\) and \(c^r(0), \ c^r(\infty)\) the ultraviolet and infrared central charges before and after reduction. In infrared perturbation theory, this is reflected in the modification to the \(T\bar{T}\) corrections to scaling for an excited state with infrared scaling dimension \(2\Delta_k(\infty)\) [28]: the result (4.10) still holds, but with \(c^u(\infty)\) replaced by \(c^u(\infty) - 24\Delta_k(\infty) = c^r(\infty)\). Comparing the \(T\bar{T}\) expansions for \(\frac{1}{2\pi} RE_u(R)\) and \(\frac{1}{2\pi} RE^u_k(R)\), this implies that the coefficient of \((MR)^{-2}\) for the excited state can be obtained simply by multiplying the coefficient for the ground state, \(C^u_1\), by \((c^r(\infty)/c^u(\infty))^2\). So long as the mass scale remains the same after reduction, this means that

\[ C^r_1 = \left(\frac{c^r(\infty)}{c^u(\infty)}\right)^2 C^u_1. \]  

(4.30)

A similar result relates \(C^r_2\) to \(C^u_2\). Resonances may occasionally cause these coefficients to diverge. However, so long as the resulting logarithms can be obtained by analytic continuation from non-resonant points, their coefficients must be related in the same way:

\[ \tilde{C}^r_1 = \left(\frac{c^r(\infty)}{c^u(\infty)}\right)^2 \tilde{C}^u_1. \]  

(4.31)

This explains the connection observed above between the logarithmic singularities for \(SST_{1/p}^{(-)}\) and \(H_{p-2}^{(\pi)}\) when \(p\) is odd. Furthermore, combining (4.27) with (4.31) yields a prediction for \(H_{p-2}^{(\pi)}\) when \(p\) is even, namely \(C^{(p)}_1 = 0\).

The flow \(MA_p^{(+)}\) can also be found as a reduction of a theory that flows in the infrared to \(c=1\), this time an imaginary coupled sine-Gordon model (ISG) [43]. The flow to take leaves \(c=1\) by a relevant operator with conformal dimension \(\Delta_{UV} = p/(p+1)\), wanders around in a non-unitary way, and then returns to \(c=1\) via an irrelevant operator of conformal dimension \(\Delta_{IR} = p/(p-1)\) (as usual, there will also be a collection of counterterms). Such models can be defined for all \(p \geq 2\). Unfortunately, we do not know very much about their infrared behaviour, and so we resort instead to the
following trick, which finally brings the discussion back to the TBA sys-
tems introduced in section 2. First, parametrise the arriving dimension
\( \Delta_N = N/4 \) for the self-dual \( \mathbb{Z}_N \) flow as \( \Delta_N = p/(p-1) \), and then rewrite the
formula (4.12) in terms of \( p \): 
\[
C_1^{(N)} = -\pi/(36 \sin \pi\frac{p}{2}(p-1)).
\]
This is singular when \( p \) is an odd integer – precisely the values where the corresponding
coefficient in the ISG should diverge if its reduction is to reproduce the be-
avour of \( \mathcal{M}A_p^{(+)} \). The coefficient of the logarithm arising at these points is,
from equation (4.13), equal to \( \frac{1}{2}\frac{1}{9}\frac{1}{(p-1)} \). Comparing with the
result (4.29) for \( \mathcal{M}A_p^{(+)} \), \( p \) odd, and recalling the relation (4.31), leads to
the conjecture that the coefficient \( C_1 \) for the massless sine-Gordon model is
minus twice the analytic continuation of the \( \mathbb{Z}_N \) value:
\[
C_1^{(\text{ISG})} = \frac{\pi}{18 \sin \frac{\pi}{2}(p-1)}. \tag{4.32}
\]
We have only checked this directly for one case: in ref. [43], a TBA system
was given for the \( p=2 \) ISG, which arrives at \( c=1 \) along operators with the
same dimensions as those for the self-dual \( \mathbb{Z}_8 \) flow. From this system, it is
possible to extract \( C_1 = \pi/18 \), which is indeed the value given by (4.32). Note that the expected monstrous corrections to the ISG TBA, investigated
in ref. [44], have an exponentially small influence in the far infrared and so
do not change this result. At other values of \( p \), the conjecture leads to a
prediction which can at least be checked numerically: for \( \mathcal{M}A_p^{(+)} \) with \( p \)
even, a case previously inaccessible to us, we expect that
\[
C_1^{(p)} = \frac{\pi c(\infty)^2}{18 \sin \frac{\pi}{2}(p-1)} \quad (p \text{ even}). \tag{4.33}
\]
For \( p=4 \) this reproduces the value \( C_1^{(4)} = -\pi/72 \) obtained by Zamolodchikov
for the tricritical Ising to Ising flow [14]. To go further, we extracted values
of \( C_1 \) or \( \bar{C}_1 \) from fits to numerical solutions of the relevant TBA systems,
allowing for the possibility of \( \log^2 MR/(MR)^4 \) terms for \( p \) odd, for \( p = 5 \ldots 10 \). The results are compared with the exact predictions in tables 3
and 4. (The first numerical entry in table 3 was taken from ref. [28].)

The paper [38], by Feverati et al, appeared as this paper was being
written. Amongst other things, it contains a collection of further numerical
results, in particular for the \( H_{p-2}^{(\pi)} \) flows, which are also consistent with the
exact results obtained in this section.
\[
M_p \rightarrow M_{p-1}
\]

| \( M_p \rightarrow M_{p-1} \) | exact         | numeric       |
|-----------------------------|--------------|--------------|
| \( M_5 \rightarrow M_4 \)  | \(-0.027222\ldots\) | \(-0.02723\) |
| \( M_7 \rightarrow M_6 \)  | \(0.027211\ldots\)     | \(0.027211\) |
| \( M_9 \rightarrow M_8 \)  | \(-0.023341\ldots\) | \(-0.0232\) |

Table 3: \( \tilde{C}_1 \) for the flows \( MA_p^{(+)} \), \( p \) odd.

| \( M_p \rightarrow M_{p-1} \) | exact         | numeric       |
|-----------------------------|--------------|--------------|
| \( M_6 \rightarrow M_5 \)  | \(0.111701\ldots\)     | \(0.11166\) |
| \( M_8 \rightarrow M_7 \)  | \(-0.139137\ldots\) | \(-0.13910\) |
| \( M_{10} \rightarrow M_9 \) | \(0.152038\ldots\)     | \(0.151\) |

Table 4: \( C_1 \) for the flows \( MA_p^{(+)} \), \( p \) even.

4.3 Phase coexistence in the massive regime

For the massive flow, a very different picture is expected at large distances. The ultraviolet perturbing operator is self-dual, and so in the massive direction it should move the model onto a surface of first-order transitions. In this region the ordered and disordered phases coexist, and so the possible vacua should form a \( \mathbb{Z}_N \)-multiplet of \( N \) ordered ground states, \( \langle \sigma \rangle = e^{\frac{2\pi i k}{N}} \), and a disordered \( \mathbb{Z}_N \)-singlet with \( \langle \sigma \rangle = 0 \).

Finite-size scaling at a first-order transition is reasonably well-understood. In a cylindrical geometry, dominant configurations consist of a sequence of domain walls stretching across the ‘spacelike’ dimensions of the system. These lift the ground-state degeneracy, replacing it with an energy splitting which in \( d+1 \) dimensions is of order \( R^{d/2-1} \exp(-\sigma R^d) \) [15]. Here \( R^d \) is the spatial volume and \( \sigma \) the surface tension of the domain wall; in \( 1+1 \) dimensions \( \sigma \) is equal to the mass of the corresponding single kink.

When there are \( m > 2 \) different infinite-volume ground states, the standard discussion must be generalised a little. Consider first the situation where all kinks have the same mass, the vacua that they connect being encoded in an incidence matrix \( I_{ab} \). Then the usual instanton-gas argument (see for example ref. [46]) must be supplemented by a diagonalisation of \( I \).
The prefactors of $\mp 1$—minus the eigenvalues of $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$—which distinguish the energies of the lowest-energy symmetric and antisymmetric states in the Ising model are replaced by $-\lambda_0 < -\lambda_1 \leq \ldots \leq -\lambda_{m-1}$, the negatives of the $m$ eigenvalues of $I$. (This can equivalently be understood via the effective quantum-mechanical problem that remains once all transverse degrees of freedom have been integrated out.) In particular, the negative of the (unique) largest eigenvalue controls the leading behaviour of the ground-state energy \[ \text{[15]} \]. Note however that all of the eigenvalues will be seen in the full spectrum, the others appearing in the asymptotics of those energy levels which become degenerate with the ground state in infinite volume. This explains the observation of one instance of this phenomenon made in ref. \[ \text{[47]} \], and it offers the interesting prospect that it might be possible to ‘hear’ the shape of the kink structure in the large-$R$ asymptotics of the degenerating energy levels. We checked that the excited-state TBA systems reported in refs. \[ \text{[48, 28]} \] are consistent with this idea.

The above assumed that all the kinks had the same mass. In general there will be $n \geq 1$ different kinds of kinks, with masses that can be ordered as $M_{\pi(1)} < M_{\pi(2)} < \ldots < M_{\pi(n)}$ for some permutation $\pi$ of $1 \ldots n$. (In the $\mathbb{Z}_N$ models, $\pi(1) = n$ for $N \leq 16$, and $\pi(1) = 1$ thereafter.) For each mass $M_i$, an incidence matrix $I_{ab}^{(i)}$ can be defined to encode the number of kinks of that mass which join each pair of vacua. If these matrices commute, then they can be simultaneously diagonalised to give sets of simultaneous eigenvalues

$$ (\lambda_0^{(1)}, \lambda_0^{(2)}, \ldots, \lambda_0^{(n)}) > (\lambda_1^{(1)}, \lambda_1^{(2)}, \ldots, \lambda_1^{(n)}) \geq \ldots \geq (\lambda_{m-1}^{(1)}, \lambda_{m-1}^{(2)}, \ldots, \lambda_{m-1}^{(n)}), \tag{4.34} $$

The ordering implied here is lexicographic, first by the eigenvalues $\lambda_p^{(\pi(1))}$ for the lowest kink mass, then by those for the second-lowest mass, and so on. The asymptotic behaviour of the $k$th energy level is then

$$ E_k(R) \sim -\lambda_k^{(1)} \Lambda_1(R) - \lambda_k^{(2)} \Lambda_2(R) - \ldots - \lambda_k^{(n)} \Lambda_n(R), \quad k = 0, 1, \ldots, m-1 \tag{4.35} $$

where $\Lambda_i(R)$, a function of order $R^{-1/2} \exp(-M_i R)$, gives the leading energy splitting for a system with just two vacua and kink mass $M_i$. This formula must be interpreted with some caution: if some of the heavier kinks have a mass more than twice that of the lightest, then their leading contributions, included above, will be less important than sub-leading contributions from the lightest kink, which have been omitted. Nevertheless, the functional forms of the $\Lambda_i(R)$ provide clear fingerprints for leading one-kink terms of each type, and match with the terms which emerge at the first step of an
iterative solution of the TBA equations. Thus while not always the leading asymptotic, (4.35) isolates exactly those terms that will be important below.

The result seems to be consistent with a previously-known case. In ref. [49], excited-state TBA systems were proposed for the perturbation of the $\mathbb{Z}_N$-symmetric conformal field theory by its first thermal operator $\epsilon^{(1)}$. In contrast to the $\epsilon^{(2)}$ perturbation that has been the principal concern of this paper, $\epsilon^{(1)}$ is anti-self-dual and moves the model off the self-dual hyperplane. In the low-temperature direction, this leads to a theory of kink scattering. For current purposes it is convenient to group the kinks and antikinks together, even though their scattering is in fact diagonal, so that the model exhibits $[N/2]$ different kink types, with masses $M_j = M \sin(\pi j/N)$, $j = 1 \ldots [N/2]$. A number of different excited-state energies were found in ref. [49]: for those with conformal dimensions $k(N−k)/2N(N+2)$ at short distances, we extracted the large $R$ asymptotics in the way shortly to be described for the self-dual flows. The resulting eigenvalues were $\lambda^{(j)}_{k} = 2 \cos(2\pi jk/N)$, excepting the self-conjugate $(N/2)^{th}$ kink for $N$ even, for which $\lambda^{(N/2)}_{k} = \cos(\pi k)$. If it is assumed that these describe the excited states for untwisted boundary conditions in the low-temperature phase, then a compatible set of incidence matrices $I^{(j)}_{ab}$ can be found on setting $I^{(1)}_{ab} = [\hat{A}_{N−1}]_{ab}$, the incidence matrix of the affine $a^{(1)}_{N−1}$ Dynkin diagram, and then defining $I^{(2)}_{ab} \ldots I^{([N/2])}_{ab}$ via an $SU(2)$-type fusion hierarchy:

$$I^{(j)} I^{(1)} = I^{(j−1)} + I^{(j+1)},$$

(4.36)

with $I^{(0)}$ set equal to $2I$, twice the identity matrix. The factor 2 can be traced to the grouping of kink with antikink, which as before leads to an exceptional case for $N$ even: the matrix $I^{(N/2)}$ obtained from (4.36) must be halved. All of this is in agreement with the usual picture of the kink structure in the ordered phase, and in particular matches the bosonic parts of the supersymmetric solitons studied in [50].

There is a potential problem in more general cases: the assumption that the incidence matrices commute is rather strong, and one can envisage theories of kink scattering for which it fails. (For the example just described, it followed from the $\mathbb{Z}_N$ symmetry of the $N$ degenerate vacua.) In the related context of integrable lattice models, the property is often built in from the start (see for example ref. [51]). However, from the S-matrix perspective it turns out to be a simple consequence of integrability, as shown by the following argument.
Consider the model on the full line, with spatial coordinate $x$, and restrict attention to the sector of those states which interpolate between the vacua $a$ at $x = -\infty$ and $c$ at $x = +\infty$. Irrespective of integrability, the conservation of topological charge implies that this sector is preserved under time evolution. But if the model is integrable, then the sector can be further restricted to the (possibly empty) subspace of two-kink states with masses $M_i \neq M_j$, and rapidities $\theta_i \neq \theta_j$. If we take $\theta_i > \theta_j$, then for the in states, found as $t \to -\infty$, the kink of mass $M_i$ lies to the left of the kink of mass $M_j$. Hence $D_{\text{in}}$, the dimension of the space of in states, is equal to $\sum_b I_{ab}^{(i)} I_{bc}^{(j)}$. Now since $M_i \neq M_j$ and the model is integrable, reflection is ruled out and the space of out states is spanned by states in which the kink of mass $M_i$ is to the right of the kink of mass $M_j$. Thus, $D_{\text{out}} = \sum_b I_{ab}^{(j)} I_{bc}^{(i)}$. But since we can map from one space to the other and back by a unitary matrix $S$ and its inverse, the two dimensions must be equal. That is, $D_{\text{in}} = D_{\text{out}}$, which when taken for all pairs of vacua $a, b$ is exactly the statement $[I^{(i)}, I^{(j)}] = 0$. Note that the argument holds even when some kinks have multiplicities higher than one, and also when there are some kinks of zero topological charge. Both situations will be encountered below.

In this paper attention is being restricted to the ground-state energy. Even so, the formula (4.35) turns out to contain a large amount of information about the kink structure. To extract the asymptotics from the massive TBA equations, an iterative approach can be used [13]. To first approximation the massive pseudoenergies behave as $\varepsilon_i^{(1)}(\theta) \sim \nu_i^{(1)}(\theta) = M_i R \cosh \theta$, while the others take the constant values $\log \mathcal{X}_i^{(\alpha)}$, as given in appendix C. After one iteration,

$$\varepsilon_i^{(1)}(\theta) = M_i R \cosh(\theta) + \sum_{j,\beta} N_{ij}^{\alpha\beta} \log \left( 1 + \mathcal{X}_j^{(\beta)^{-1}} \right), \quad (4.37)$$

where

$$N_{ij}^{\alpha\beta} = \frac{1}{2\pi} \int_{-\infty}^{\infty} \psi_{ij}^{\alpha\beta}(\theta) \, d\theta$$

with $\psi_{ij}^{\alpha\beta}$ a generic notation for the kernel linking $\varepsilon_i^{(\alpha)}$ to $\varepsilon_j^{(\beta)}$ in the TBA system. Hence

$$E(R) = -\frac{1}{2\pi} \sum_i \int_{-\infty}^{\infty} d\theta \ M_i \cosh \theta \log \left[ 1 + e^{-\varepsilon_i^{(1)}(\theta)} \right]$$

$$\sim -\frac{1}{2\pi} \sum_i \int_{-\infty}^{\infty} d\theta \ M_i \cosh \theta \ e^{-M_i R \cosh \theta} \prod_{j,\beta} \left( 1 + \mathcal{X}_j^{(\beta)^{-1}} \right)^{N_{ij}^{\alpha\beta}}.$$
Identifying the factors $\Lambda_i(R) = \frac{1}{2\pi} \int_{-\infty}^{\infty} d\theta M_i \cosh \theta e^{-M_i R \cosh \theta}$, the eigenvalues $\lambda_0^{(i)}$ can be read off:

$$\lambda_0^{(i)} = \prod_{j,\beta} \left( 1 + \lambda_j^{(\beta)} \right)^{N_{ij}}.$$  (4.38)

For the calculations it is actually easier to obtain the correction term in (4.37) by substitution into the relevant Y-system. The final results are remarkably simple: there are $n$ different kink masses $M_i, i = 1 \ldots n$, equal to the corresponding sine-Gordon masses (A.3), (A.4), and for the ground state energy we have

$$\lambda_0^{(i)} = i + 1 \quad (i = 1 \ldots n - 1),$$
$$\lambda_0^{(n)} = \sqrt{N}.$$  (4.39)

It remains to identify a set of commuting, non-negative integer valued, $(N+1) \times (N+1)$, symmetric and $\mathbb{Z}_N$-symmetric matrices $I^{(1)} \ldots I^{(n)}$ for which these form the largest (in the sense of equation (4.34)) set of simultaneous eigenvalues. In fact, the eigenvalues are individually maximal. To see this, first recombine the incidence matrices as $I_{ab}(R) = \sum_i \Lambda_i(R) I_{ab}^{(i)}$. For $R$ large enough, the eigenvector of $I_{ab}(R)$ with the largest eigenvalue is also the simultaneous eigenvector that we are looking for. Furthermore, for $R$ finite $I_{ab}(R)$ is connected (note, the same cannot be assumed for the individual $I_{ab}^{(i)}$). Hence, we can apply the Perron-Frobenius theorem to deduce that this eigenvector is $\mathbb{Z}_N$-symmetric, with all components of like sign. Using the theorem in reverse, this eigenvector must be maximal for the individual incidence matrices. Two of these matrices can be written down almost immediately. Up to a relabelling of the nodes $1 \ldots N$ corresponding to the ordered vacua, they are:

$$I^{(1)} = \begin{pmatrix} \hat{A}_{N-1} & 0 \\ \begin{pmatrix} [\hat{A}_{N-1}]_{ab} \\ 0 \end{pmatrix} & \vdots & \ddots & \vdots \\ \begin{pmatrix} 0 \\ \ddots \end{pmatrix} & \vdots & \ddots & 2 \end{pmatrix}, \\
I^{(n)} = \begin{pmatrix} 0 & 1 \\ 1 & 1 \\ \vdots & \ddots \end{pmatrix}. $$  (4.40)

where $\hat{A}_{N-1}$ is the $N \times N$ incidence matrix of the affine $\tilde{a}_{N-1}^{(1)}$ Dynkin diagram. So long as $N$ is not a perfect square, $I^{(n)}$ is fixed uniquely, while
for $I^{(1)}$ we have to add a connectivity assumption when $N$ is not prime. In fact, any of the other possibilities for $I^{(1)}$ would have the kinks of mass $M_1$ joining non-adjacent ordered vacua. At least in a classical picture of the kinks, this can be ruled out since $M_1$ is minimal among the remaining kink masses. There only remains the possibility for one of the blocks making up $I^{(1)}$ to be equal to zero, and imposing $[I^{(1)}, I^{(n)}] = 0$ rules this out as well.

For the remaining kinks it is not possible to be so definite on the basis of the ground-state energy alone. However, one set of conjectures seems natural. Note first that the forms of $I^{(1)}$ and $I^{(n)}$ are consistent with the idea that the kinks of mass $M_1$ are the (lowest-mass) bound states of a set of fundamental kinks with mass $M_n=M$. This parallels the related sine-Gordon model, where $M_n$ is the mass of the soliton, and $M_1$ the mass of its first bound state – the first breather. If the analogy is to extend, then the kinks of mass $M_j$ in the $\mathbb{Z}_N$ model should be bound states of $j$ of the kinks of mass $M_1$. The allowed topological charges should therefore be found among the paths with $j$ steps on the graph of $I^{(1)}$. Adding this to the other requirements leads to the proposal

$$
I^{(j)} = \begin{pmatrix}
\min(\hat{A}_{N-1}^j, 1) & 0 & \\
0 & \ddots & \\
\vdots & \ddots & j+1
\end{pmatrix}, \quad (j = 1 \ldots n-1). \quad (4.41)
$$

Commutativity with $I^{(n)}$ was used to settle the value of $I^{(j)}_{N+1,N+1}$. An alternative characterisation is the statement that the breather-related incidence matrices form part of an $SU(2)$-type fusion hierarchy: defining $I^{(0)} = I$, the identity matrix, and $I^{(1)}$ as in equation (4.40), the matrices $I^{(2)} \ldots I^{(n-1)}$ are determined by the relation (4.36), just as for the $\varepsilon^{(1)}$ perturbation, though with one more vacuum, and different initial conditions. Note that if $I^{(1)}$ and $I^{(n)}$ commute and have the correct Perron-Frobenius eigenvalues, then the same automatically holds for $I^{(2)} \ldots I^{(n-1)}$, given the fusion property. Some preliminary investigations of excited-state spectra for the $b_n$-related cases are also consistent with (4.41).

Figures 2 and 3 illustrate the general idea with the kink structures for $N=9$ and $N=15$. Tunneling to and from the disordered phase only occurs via the fundamental-kink instantons, while instantons associated with an increasingly intricate structure of higher kinks serve to connect the various ordered phases. Note also the presence of tadpoles in all but the fundamental
set of kinks. These have been seen before [52], and their appearance in the finite-size effects can be understood either semi-classically, as corrections due to instantons of zero topological charge, or else as being due to virtual particle-like excitations (‘breathers’) above the various vacua. Either way, the predicted effect on the large-$R$ asymptotics is the same, and has indeed been observed in some other situations [47]. A more novel feature in this context is that, for $N>16$, some kinks appear in more than one incidence matrix from the set $I^{(1)} \ldots I^{(n)}$. In other words, some pairs of vacua are joined not only by a simple kink of minimal mass, but also by excitations of this kink with higher masses – these can be thought of as breathers with non-zero topological charge.

To see this aspect most clearly, the spectrum of asymptotic one-particle states can be reorganised according to their topological charges. Let $a, a+i$ always label ordered vacua $1 \ldots N$, with $N+1$ the disordered vacuum. With $n$ as ever equal to the integer part of $(N-1)/4$, the prediction is that the massive flow should exhibit:

- for $i = 1, 2, \ldots n-1$: a kink joining each $a$ to $a+i$ with mass $M_i$, and excitations of this kink with masses $M_j$, $j = i+2, i+4, \ldots \leq n-1$;
- kinks joining $N+1$ to each $a$ with mass $M_n$;
• excitations of each ordered vacuum with masses $M_j$, $j = 2, 4, \ldots \leq n-1$;
• excitations of the disordered vacuum with masses $M_1, M_2, \ldots M_{n-1}$.

All appear with unit multiplicity at each mass except for those in the last line, which have multiplicities $2, 3, \ldots n$.

The discussion in this subsection has been based on a physical input to determine how the $\mathbb{Z}_N$ symmetry acts on the vacua: we required one singlet and one $N$-element multiplet. Whenever $N$ factorises, say as $N= PQ$, then it is possible to use an orbifold construction \[53\] to produce alternative patterns of vacua, which might also be described by the self-dual $\mathbb{Z}_N$ TBA systems. The incidence matrices now split into blocks of sizes $P$ and $Q$, rather than the previous $N$ and 1. The basic matrices are:

\[
I^{(1)} = \begin{pmatrix}
\hat{A}_{P-1}^{(1)}_{ab} & 0 \\
0 & \hat{A}_{Q-1}^{(1)}_{cd}
\end{pmatrix}, \quad I^{(n)} = \begin{pmatrix}
0 & 1_{ad} \\
1^t_{cb} & 0
\end{pmatrix}
\]

where $\hat{A}_{P-1}$ and $\hat{A}_{Q-1}$ are the incidence matrices of the affine $a_{P-1}^{(1)}$ and $a_{Q-1}^{(1)}$ Dynkin diagrams, and $1_{ad}$ is a $P \times Q$ matrix with all entries equal to 1. The remaining matrices, $I^{(2)} \ldots I^{(n-1)}$, follow from $I^{(1)}$ via the fusion relation \[L36\], with $I^{(0)} = 1$. Such systems of vacua and kinks could be relevant to first-order transition regions where $\mathbb{Z}_P$ and $\mathbb{Z}_Q$ ordered phases coexist.

5 Conclusions

The checks performed in this paper have been rather exhaustive, and seem to us to establish beyond reasonable doubt that each pair of the self-dual $\mathbb{Z}_N$ TBA systems listed in section 2 does indeed describe a pair of flows from the $\mathbb{Z}_N$-symmetric conformal field theory into massless and massive phases, with the two flows being related by a change in the sign of a coupling constant. Given the earlier results of ref. \[1\], this effectively settles the question posed in the introduction about the location of the Fateev-Zamolodchikov Boltzman weights in the larger phase diagram. In addition, a number of predictions have been obtained which deserve further exploration.
One important outstanding question is to find a collection of S-matrices, both massive and massless, lying behind the TBA systems that have been proposed. In fact, a complete set of S-matrices for the massive flows was suggested some years ago by Fateev [17], but we have not been able to reconcile these with the TBA results reported in section 4.3. In particular, Fateev predicted that for $N$ odd there should be $(N-5)/2$ bound states of the fundamental kinks, all with different masses, while the asymptotics of the TBA systems proposed above predict only $((N-5)/4)$. If the TBA systems are correct, it is very hard to see how about half of the kink masses could be lost from the instanton gas calculation. Given the stringent tests to which these systems have been subject in this paper, it is possible that the resolution will be found in some modification of Fateev’s proposals.

On a more mathematical note, the Y-systems and dilogarithm sum rules found here for $N$ odd appear to be new. Past experience (see for example ref. [54]) would suggest the existence of new fermionic representations for the characters of the $\mathbb{Z}_N$-symmetric conformal field theories, and possibly also an alternative integrable deformation of the Fateev-Zamolodchikov Boltzman weights [8] which, contrary to that discussed by Jimbo et al [11], leaves the $\mathbb{Z}_N$ symmetry unbroken.

Further study of the massless flows should be rewarding: as highlighted in section 4.2, the systems introduced in this paper add to an already-rich collection of flows arriving at $c=1$. The precise relationship between these models needs to be elucidated, and the possibility to extract at least some infrared information exactly should help in this task. Since the various infrared limits can all be regarded as irrelevant perturbations of a single free boson, they should also provide the simplest possible playground to explore more general issues in non-renormalisable field theory.

Finally, there remain many unresolved features of the larger phase diagrams of the $\mathbb{Z}_N$ spin systems [3], beyond the particular self-dual directions out of the Fateev-Zamolodchikov points that have occupied our attention in this paper. We hope that these will provide further opportunities to apply the ideas of perturbed conformal field theory and the TBA to problems of independent statistical-mechanical interest.

Acknowledgements – We would like to thank John Cardy, Günter von Gehlen, Ferdinando Gliozzi, Francesco Ravanini and Robert Weston for useful discussions. PED would also like to thank SPhT Saclay and the Benasque Centre for Physics for hospitality while some of this work was in
progress. PED and KET thank the EPSRC for an Advanced Fellowship and a Research Studentship respectively, and RT thanks the Mathematics Department of Durham University for a postdoctoral fellowship. This work was supported in part by a Human Capital and Mobility grant of the European Union, contract number ERBCHRXCT920069, and in part by a NATO grant, number CRG950751.

A Sine-Gordon and kernel data

The sine-Gordon model can be defined by the Lagrangian

$$\mathcal{L} = \frac{1}{2} (\partial_{\mu} \varphi)^2 + \frac{m^2}{\beta^2} \cos \beta \varphi$$  \hspace{1cm} (A.1)

and results in a family of scattering theories, conveniently parametrised by either \( p \) or \( h \), where

$$p = \frac{2}{h} = \frac{\beta^2}{8\pi - \beta^2}.$$  \hspace{1cm} (A.2)

It will also be useful to define \( N = 2h+4 = 32\pi/\beta^2 \), and to set \( n \) equal to the integer part of \((N-1)/4\). The spectrum contains a soliton-antisoliton doublet \((s, \bar{s})\) and, for \( h > 2 \), \( n-1 \) soliton-antisoliton bound states, or breathers. If \( s \) and \( \bar{s} \) have mass \( M_s \), then the breather masses are

$$M_k = 2M \sin \frac{\pi k}{h} = 2M \sin \frac{2\pi k}{N-4}, \hspace{0.5cm} k = 1, 2, \ldots n-1.$$  \hspace{1cm} (A.3)

The soliton labels \( s \) and \( \bar{s} \) will often be replaced by \( n \) and \( n+1 \), and accordingly

$$M_n = M_{n+1} = M.$$  \hspace{1cm} (A.4)

Scattering among solitons and antisolitons is generally non-diagonal; the amplitudes can be found in [55]. However, once a breather is involved the scattering becomes diagonal. By analogy with a notation used for the affine Toda theories, define the blocks

$$(x)(\theta) = \frac{\sinh(\frac{\theta}{2} + \frac{\pi x}{2h})}{\sinh(\frac{\theta}{2} - \frac{\pi x}{2h})}; \hspace{0.5cm} \{x\} = (x-1)(x+1).$$

Then

$$S_{jk} = \prod_{l=j-k+1}^{j+k-1} \{l\}\{h-l\} \hspace{0.5cm} (j, k = 1 \ldots n-1),$$  \hspace{1cm} (A.5)
and
\[ S_{kn} = S_{k,n+1} = (-1)^{k} \prod_{l=1}^{h/2+k-1} \{l\} \quad (k = 1 \ldots n-1). \]  

(A.6)

The couplings relevant for the \( b_n \) and \( d_{n+1} \) series of TBA systems correspond to \( h = 2n-1 \) and \( h = 2n \) respectively – the dual Coxeter numbers of \( b_n \) and \( d_{n+1} \). When \( h = 2n \), the entire S-matrix is diagonal with the remaining S-matrix elements given by

\[
\begin{align*}
\text{n even} & : S_{nn} = S_{n+1,n+1} = \prod_{l=1}^{2n-3} \{l\} ; \quad S_{n,n+1} = \prod_{l=3}^{2n-1} \{l\} ; \\
\text{n odd} & : S_{nn} = S_{n+1,n+1} = \prod_{l=1}^{2n-1} \{l\} ; \quad S_{n,n+1} = -\prod_{l=3}^{2n-3} \{l\} .
\end{align*}
\]  

(A.7)

In this case, the various prefactors of \(-1\) (which anyway make no difference to the definitions below) can be omitted to leave the minimal version of the \( d_{n+1} \) Toda S-matrix.

The TBA systems in the text involve certain kernels. The first of these are related to the logarithmic derivatives of the diagonal sine-Gordon S-matrix elements:

\[ \phi_{jk} = -i \frac{d}{d\theta} \log S_{jk} ; \quad \psi_{jk} = -i \frac{d}{d\theta} \log T_{jk}. \]  

(A.8)

Here at least one of the indices \( j \) and \( k \) must have a 'breather' value, \( 1,2,\ldots,n-1 \). In the definition of \( \psi_{jk} \), the function \( T_{jk} \) is obtained by replacing each block \( \{x\} \) in (A.5–A.6) by \( (x) \). The kernels \( \psi_{jk} \) do not appear in the TBA equations for the sine-Gordon model itself, but rather they are part of the conjectures that are the main topic of this paper. Their forms are a natural generalisation of previously-known examples [18].

For the \( d_{n+1} \)-related models the definition can be extended immediately to cover the remaining cases when both \( j \) and \( k \) take the values \( n \) or \( n+1 \). Otherwise, the non-diagonal scattering of the solitons means that the associated kernels are more elaborate. Define an integer \( \rho \) by \( N = 4n+\rho \), and then set

\[ \chi_{\rho}(\theta) = \frac{2h}{\rho \cosh \frac{2h}{\rho} \theta} . \]  

(A.9)
This function has the property that
\[ 
\chi_{\rho} \ast f(\theta + \frac{\pi \rho}{4h}) + \chi_{\rho} \ast f(\theta - \frac{\pi \rho}{4h}) = f(\theta). 
\]
The soliton-soliton S-matrix consists of a ‘scalar’ piece multiplying a non-diagonal matrix. Associated with the first factor is a pseudoenergy \( \varepsilon_n(\theta) \), with interaction kernels in the TBA given by
\[
\phi_{nn}(\theta) = \chi_{\rho} \ast \phi_{n,n-1}(\theta) \quad ; \quad \psi_{nn}(\theta) = \chi_{\rho} \ast \psi_{n,n-1}(\theta). \quad (A.10)
\]
(For \( N \leq 7 \), \( \phi_{n,n-1} \) and \( \psi_{n,n-1} \) are not defined and we set \( \phi_{nn} = \psi_{nn} = 0 \).) In addition, depending on the continued-fraction expansion of \( 2/h \), the non-diagonal part induces a number of fictitious ‘magnonic’ particles [56], together with their attendant kernels. We will only need those which arise when \( N \) is an integer; these can be obtained from the Y-systems of ref. [24], and are
\[
\phi_1(\theta) = \frac{h}{\cosh h \theta} \quad (A.11)
\]
\[
\phi_2(\theta) = \frac{2h}{\cosh 2h \theta} \quad , \quad \phi_4(\theta) = \frac{8h \cosh \frac{2h}{3} \theta}{3(4 \cosh^2 \frac{2h}{3} \theta - 3)}
\]
\[
\phi_3(\theta) = \frac{2h}{\cosh \frac{2h}{3} \theta} \quad , \quad \phi_5(\theta) = \frac{8h \cosh \frac{2h}{3} \theta}{\sqrt{3}(4 \cosh^2 \frac{2h}{3} \theta - 1)} \quad . \quad (A.12)
\]
As above, \( h \) is equal to \( N/2 - 2 \).

B \ Y-systems

This section records the self-dual \( \mathbb{Z}_N \) Y-systems. Their derivation from the TBA equations is rather involved, but follows the same lines as the simpler cases explained in ref. [21]. In particular, the energy terms are eliminated using the following variant of the Perron-Frobenius mass property of the ade-related theories:
\[
2 \cos \left( \frac{\pi}{n} \right) M_i = \sum_j \delta_{ij}^{[n-1]} M_j \quad (i = 1, \ldots, n-2); 
\]
\[
2 \cos \left( \frac{\pi}{n} \right) M_{n-1} = M_{n-2} + 2 \cos \left( \frac{(4 - \rho) \pi}{4n} \right) M_n; 
\]
\[
2 \cos \left( \frac{3 \pi}{n} \right) M_n = M_{n-1}.
\]
where \( N = 4n + \rho \), as in the last section, and \( l_{ij}^{[a_{n-1}]} \) is the incidence matrix of the \( a_{n-1} \) Dynkin diagram.

The results are \( \mathbb{Z}_2 \)-symmetrised versions of the sine-Gordon Y-systems constructed in ref. [24]. Figure 4 shows a diagrammatic representation, following refs. [18, 24]. In order to write them down in a manageable form, define

\[
Y_i^{(\alpha)}[\theta, r] = Y_i^{(\alpha)}(\theta - \frac{\pi r}{h}) Y_i^{(\alpha)}(\theta + \frac{\pi r}{h})
\]

\[
Y_i^{(\alpha)}\{\theta, r\} = \left( 1 + Y_i^{(\alpha)}(\theta - \frac{\pi r}{h}) \right) \left( 1 + Y_i^{(\alpha)}(\theta + \frac{\pi r}{h}) \right)
\]

\[
V_i^{(\alpha)}\{\theta, r\} = \left( 1 + Y_i^{(\alpha)}(\theta - \frac{\pi r}{h}) \right)^{-1} \left( 1 + Y_i^{(\alpha)}(\theta + \frac{\pi r}{h}) \right)^{-1}
\]

With this notation in place, the Y-systems for the four cases can be written as follows.

**1) \( N = 4n + 1 \)**

Nodes \((i < n - 1, \alpha = 1, 6)\), with \( \tilde{\alpha} = 7 - \alpha \):

\[
Y_i^{(\alpha)}[\theta, 1] = \left( 1 + Y_i^{(\tilde{\alpha})}(\theta) \right)^{[a_{n-1}]} \prod_{j=1}^{n-1} \left( 1 + Y_j^{(\alpha)}(\theta) \right)^{l_{ij}^{[a_{n-1}]}}, \quad (B.1)
\]

Nodes \((n-1, \alpha = 1, 6)\):

\[
Y_{n-1}^{(1)}[\theta, 1] = \left( 1 + Y_{n-1}^{(6)}(\theta) \right)^{-1} \left( 1 + Y_{n-2}^{(1)}(\theta) \right) \left( 1 + Y_{n-2}^{(4)}(\theta) \right) \left( 1 + Y_{n-2}^{(5)}(\theta) \right)
\]

\[
\times Y_n^{(1)}\{\theta, 3/4\} Y_n^{(2)}\{\theta, 2/4\} Y_n^{(3)}\{\theta, 1/4\}
\]

\[
Y_{n-1}^{(6)}[\theta, 1] = \left( 1 + Y_{n-1}^{(1)}(\theta) \right)^{-1} \left( 1 + Y_{n-2}^{(6)}(\theta) \right) \left( 1 + Y_{n-2}^{(4)}(\theta) \right) \left( 1 + Y_{n-2}^{(2)}(\theta) \right)
\]

\[
\times Y_n^{(6)}\{\theta, 3/4\} Y_n^{(5)}\{\theta, 2/4\} Y_n^{(3)}\{\theta, 1/4\}
\]

Nodes \((n, \alpha = 1 \ldots 6)\):

\[
Y_n^{(\alpha)}[\theta, 1/4] = \left( 1 + Y_{n-1}^{(\alpha)}(\theta) \right) \prod_{\beta} \left( 1 + Y_n^{(\beta)}(\theta) \right)^{-1} Y_n^{[a_{n-1}]} \]

with \( Y_{n-1}^{(\alpha)}(\theta) \equiv 0 \) when \( \alpha \neq 1, 6 \).

**2) \( N = 4n + 2 \) \((b_n)\)**

Nodes \((i < n - 1, \alpha = 1, 3)\): equation \([B.1]\) holds with \( \tilde{\alpha} = 4 - \alpha \).
Figure 4: Diagrammatic representations of the $\mathbb{Z}_N$ Y-systems. The left/right columns represent the massive/massless theories respectively.
In this case the scope of equation (B.1), with \( \tilde{\alpha} \)

\[
\text{Nodes } (n-1, \alpha=1, 3) :
\]

\[
\mathbf{Y}_{n-1}^{(\alpha)}[\theta, 1] = (1 + Y_{n-1}^{(\alpha)}(\theta)^{-1}) (1 + Y_{n-2}^{(\alpha)}(\theta)) (1 + Y_{n-1}^{(\alpha-1)}(\theta))
\]

\[
\times (1 + Y_{n}^{(\alpha+1)}(\theta))^\alpha \{ \theta, 1/2 \}
\]

\[
\text{Nodes } (n, \alpha=0 \ldots 4) \text{ (with } Y_{n-1}^{(\alpha)}(\theta) \equiv 0 \text{ when } \alpha \neq 1, 3) :
\]

\[
\mathbf{Y}_{n}^{(\alpha)}[\theta, 1/2] = (1 + Y_{n-1}^{(\alpha)}(\theta)) \prod_{\beta} (1 + Y_{n}^{(\beta)}(\theta)^{-1})^{-1\alpha_{\beta}^{\beta}}
\]

3) \( N = 4n+3 \)

\[
\text{Nodes } (i<n-1, \alpha=1, 3) : \text{ equation } [3.1] \text{ holds with } \tilde{\alpha} = 7 - \alpha .
\]

\[
\text{Nodes } (n-1, \alpha=1, 6) :
\]

\[
\mathbf{Y}_{n-1}^{(1)}[\theta, 1] = (1 + Y_{n-2}^{(1)}(\theta)) (1 + Y_{n}^{(2)}(\theta)) (1 + Y_{n-1}^{(6)}(\theta)^{-1})^{-1} \{ \theta, 1/4 \}
\]

\[
\mathbf{Y}_{n-1}^{(6)}[\theta, 1] = (1 + Y_{n-2}^{(6)}(\theta)) (1 + Y_{n}^{(5)}(\theta)) (1 + Y_{n-1}^{(1)}(\theta)^{-1})^{-1} \{ \theta, 1/4 \}
\]

\[
\text{Nodes } (n, \alpha = 0 \ldots 6) :
\]

\[
\mathbf{Y}_{n}^{(1)}[\theta, 3/4] = (1 + Y_{n-1}^{(1)}(\theta)) (1 + Y_{n}^{(4)}(\theta)^{-1})^{-1}(1 + Y_{n}^{(5)}(\theta)^{-1})^{-1}
\]

\[
\times V_{n}^{(2)}\{ \theta, 2/4 \} V_{n}^{(3)}\{ \theta, 1/4 \}
\]

\[
\mathbf{Y}_{n}^{(2)}[\theta, 1/4] = (1 + Y_{n}^{(3)}(\theta)) (1 + Y_{n}^{(1)}(\theta)^{-1})^{-1}
\]

\[
\mathbf{Y}_{n}^{(3)}[\theta, 1/4] = (1 + Y_{n}^{(4)}(\theta)) (1 + Y_{n}^{(2)}(\theta)) (1 + Y_{n}^{(5)}(\theta))
\]

\[
\mathbf{Y}_{n}^{(4)}[\theta, 1/4] = (1 + Y_{n}^{(3)}(\theta))
\]

\[
\mathbf{Y}_{n}^{(5)}[\theta, 1/4] = (1 + Y_{n}^{(3)}(\theta)) (1 + Y_{n}^{(6)}(\theta)^{-1})^{-1}
\]

\[
\mathbf{Y}_{n}^{(6)}[\theta, 3/4] = (1 + Y_{n-1}^{(6)}(\theta)) (1 + Y_{n}^{(4)}(\theta)^{-1})^{-1}(1 + Y_{n}^{(2)}(\theta)^{-1})^{-1}
\]

\[
\times V_{n}^{(5)}\{ \theta, 2/4 \} V_{n}^{(3)}\{ \theta, 1/4 \}
\]

4) \( N = 4n+4 \) \((d_{n+1})\)

In this case the scope of equation (B.1), with \( \tilde{\alpha} = 3 - \alpha \), can be extended to cover all of the nodes \((i=1 \ldots n+1, \alpha=1, 2)\) :

\[
\mathbf{Y}_{i}^{(\alpha)}[\theta, 1] = (1 + Y_{i}^{(\tilde{\alpha})}(\theta)^{-1})^{-1} \prod_{j=1}^{n+1} (1 + Y_{j}^{(\alpha)}(\theta))^{d_{ij}}
\]
In the above formulae, $l_{ij}^{[a_n-1]}$, $l_{ij}^{[d_n+1]}$, $l_{a\beta}^{[e_5]}$ and $l_{a\beta}^{[e_6]}$ are the incidence matrices of the corresponding Dynkin diagrams.

We checked numerically the periodicity implied by these Y-systems up to $N=30$, and found

$$Y_i^{(\alpha)} \left( \theta + i\pi \frac{N+2}{N-4} \right) = Y_i^{(\bar{\alpha})} (\theta) ,$$

which in turn implies the result (3.7) quoted in the main text.

**C Dilogarithm sum rules**

The sum rules are conveniently given in terms of certain limits of the functions $Y_i^{(\alpha)}(\theta) = e^{\varepsilon_i^{(\alpha)}(\theta)}$. With the $\varepsilon_i^{(\alpha)}(\theta)$ the solution of a massless system, define

$$\Upsilon_i^{(\alpha)} = \lim_{MR \to 0} Y_i^{(\alpha)}(\theta) \quad (\theta \text{ finite}) ,$$

$$\lambda_i^{(\alpha)} = \lim_{\theta \to \infty} Y_i^{(\alpha)}(\theta) \quad (MR \text{ finite}) ,$$

$$\zeta_i^{(\alpha)} = \lim_{MR \to \infty} Y_i^{(\alpha)}(\theta) \quad (\theta \text{ finite}) .$$

(For the pseudoenergies which solve the massive systems, the $\Upsilon_i^{(\alpha)}$ and $\lambda_i^{(\alpha)}$ are unchanged while $\zeta_i^{(\alpha)} = \lambda_i^{(\alpha)}$.) The numbers defined by (C.1) furnish stationary solutions to the Y-systems of appendix B, subject to the following constraints: all $\Upsilon_i^{(\alpha)}$ are finite; $\lambda_i^{(1)} = \infty \ \forall i$ with the rest finite; and $\zeta_i^{(1)} = \zeta_i^{(1)} = \infty \ \forall i$ with the rest finite. The infinite quantities do not contribute to the sum rules. To find the values of the others, start with the following ansatz for the $a_{n-1} \times a_2$ tail:

$$\Upsilon_i^{(\alpha)} = \frac{\sin((i+3)\eta) \sin(i\eta)}{\sin((2\eta)) \sin(\eta)} ,$$

$$\lambda_i^{(1)} = (i+2)i ,$$

with $i = 1 \ldots n-1$. The value of $\eta$ can be found by examining the final nodes $(n, \alpha)$; a positive solution results on setting $\eta = \frac{\pi}{N+2}$. This also gives values
in good agreement with the direct numerical solution of the TBA equations. For the remaining nodes the expressions are more complicated, and will be given case-by-case.

1) \( N = 4n + 1 \)

\[
\begin{align*}
\Upsilon_n^{(1)} &= \Upsilon_n^{(6)} = \frac{\sin^2(n\eta)}{\sin((2n + 2)\eta) \sin(2\eta)}, \quad \Upsilon_n^{(2)} = \Upsilon_n^{(5)} = \frac{(\Upsilon_n^{(1)})^2}{1 + \Upsilon_n^{(1)} - (\Upsilon_n^{(1)})^2} \\
\Upsilon_n^{(4)} &= \frac{(\Upsilon_n^{(1)})^{3/2}(1 + \Upsilon_n^{(1)})^{1/2}}{(1 + \Upsilon_n^{(1)} - (\Upsilon_n^{(1)})^2)} \\
\Upsilon_n^{(3)} &= \frac{1 + 2\Upsilon_n^{(1)} + (\Upsilon_n^{(1)})^2 - 2(\Upsilon_n^{(1)})^2 - 2\Upsilon_n^{(1)}(\Upsilon_n^{(1)})^2 - (\Upsilon_n^{(1)})^3}{1 + \Upsilon_n^{(1)} - (\Upsilon_n^{(1)})^2} \\
n \chi_{n}^{(6)} &= \frac{n^2}{2n + 1}, \quad \chi_{n}^{(5)} = \frac{n^2}{(3n + 1)(n + 1)} \\
n \chi_{n}^{(4)} &= \chi_{n}^{(2)} = \frac{n}{3n + 1}, \quad \chi_{n}^{(3)} = \frac{n^2}{8n^2 + 1 + 6n} \\
n \mathcal{Z}_{n}^{(4)} &= \mathcal{Z}_{n}^{(2)} = \mathcal{Z}_{n}^{(5)} = \frac{1}{3}, \quad \mathcal{Z}_{n}^{(3)} = \frac{1}{8}.
\end{align*}
\]

2) \( N = 4n + 2 \)

\[
\begin{align*}
\Upsilon_n^{(1)} &= \Upsilon_n^{(3)} = \frac{\sin^2(n\eta)}{\sin((2n + 2)\eta) \sin(2\eta)}, \quad \Upsilon_n^{(0)} = \Upsilon_n^{(4)} = \frac{\sin(n\eta)}{\sin((n + 2)\eta)} \\
\Upsilon_n^{(2)} &= \frac{\sin^2(n\eta)}{\sin^2((n + 2)\eta)} \\
n \chi_{n}^{(3)} &= \frac{n^2}{2n + 1}, \quad \chi_{n}^{(2)} = \chi_{n}^{(4)} = \frac{n}{n + 1}, \quad \chi_{n}^{(0)} = 1 \\
n \mathcal{Z}_{n}^{(0)} &= \mathcal{Z}_{n}^{(2)} = \mathcal{Z}_{n}^{(4)} = 1.
\end{align*}
\]

3) \( N = 4n + 3 \)

\[
\begin{align*}
\Upsilon_n^{(1)} &= \Upsilon_n^{(6)} = \frac{\sin^2(n\eta)}{\sin((2n + 2)\eta) \sin(2\eta)} \\
\Upsilon_n^{(4)} &= \frac{\sin((n + 2)\eta) \sin((n + 1)\eta) + \sin^2((2n + 2)\eta)}{\sin^2((n + 2)\eta)} \\
\Upsilon_n^{(3)} &= (\Upsilon_n^{(4)})^2 - 1, \quad \Upsilon_n^{(2)} = \Upsilon_n^{(5)} = \frac{(\Upsilon_n^{(4)})^2 - 1}{(1 + \Upsilon_n^{(4)})^{1/2}} - 1
\end{align*}
\]

53
\[ X_n^{(6)} = \frac{n^2}{2n+1}, \quad X_n^{(5)} = \frac{n(3n+2)}{(n+1)^2} \]
\[ X_n^{(3)} = \frac{(4n+3)(2n+1)}{(n+1)^2}, \quad X_n^{(4)} = \frac{3n+2}{n+1} \]
\[ Z_n^{(4)} = Z_n^{(2)} = Z_n^{(5)} = 3, \quad Z_n^{(3)} = 8. \]

4) \( N = 4n+4 \)

\[ \Upsilon_s^{(\alpha)} = \Upsilon_{\pi}^{(\alpha)} = \frac{\sin(n\eta)}{2\sin(\eta)\cos((n+1)\eta)} \quad (\alpha = 1,2), \]
\[ \mathcal{X}_s^{(2)} = \mathcal{X}_{\pi}^{(2)} = n. \]

These numbers enter into the central charge calculation via the arguments of Rogers' dilogarithm functions

\[ L(x) = -\frac{1}{2} \int_0^x dy \left[ \frac{\ln y}{1-y} + \frac{\ln(1-y)}{y} \right]. \]

The following sum rules were verified numerically, for all cases up to \( N=30 \):

\[ c_\Upsilon = \frac{6}{\pi^2} \sum_{i,\alpha} L \left( \frac{1}{1 + \Upsilon_i^{(\alpha)}} \right) = \frac{2(N-1)}{N+2} + \begin{cases} 4 & \text{for } N = 4n+1 \\ 5/2 & \text{for } N = 4n+2 \\ 2 & \text{for } N = 4n+3 \\ 1 & \text{for } N = 4n+4 \end{cases} \]

\[ c_\mathcal{X} = \frac{6}{\pi^2} \sum_{i,\alpha} L \left( \frac{1}{1 + \mathcal{X}_i^{(\alpha)}} \right) = \begin{cases} 4 & \text{for } N = 4n+1 \\ 5/2 & \text{for } N = 4n+2 \\ 2 & \text{for } N = 4n+3 \\ 1 & \text{for } N = 4n+4 \end{cases} \]

\[ c_Z = \frac{6}{\pi^2} \sum_{i,\alpha} L \left( \frac{1}{1 + Z_i^{(\alpha)}} \right) = \begin{cases} 3 & \text{for } N = 4n+1 \\ 3/2 & \text{for } N = 4n+2 \\ 1 & \text{for } N = 4n+3 \\ 0 & \text{for } N = 4n+4 \end{cases} \]

In fact, the identities found were more general than this: in the spirit of ref. [57], we checked that they also hold if the constants \( \mathcal{X}, \Upsilon \) and \( Z \) are replaced by any non-stationary solutions to the Y-systems of the last section, so long as an additional average over a full period is performed for each result. These periods are \( 2(N+2) \) for the \( \Upsilon \)'s, \( N \) for the \( \mathcal{X} \)'s, and 8, 4 and 8 (when \( \rho = 1,2 \) and 3) for the \( Z \)'s.
The sum rules are not obviously relevant when \( N = 4n + 3 \), since for these TBA systems the kernels given in the text are not all symmetric. However, at the expense of some additional complexity, the systems can be rewritten in a symmetrical way, after which the central charge calculation goes through as usual. The forms given earlier were selected as the simplest versions which are also suitable for direct numerical solution.

With this detail out of the way, the ultraviolet and infrared central charges are

\[
\begin{align*}
  c_{UV} &= c_T - c_X = \frac{2(N-1)}{(N+2)}; \\
  c_{IR} &= c_X - c_Z = 1.
\end{align*}
\]  

(D) Regular UV expansion coefficients

This appendix tabulates fits of the numerical solutions to the TBA equations at small \( R \) to the expansions

\[
RE(R) + \frac{\pi c}{6} - \text{(bulk piece)} = 2\pi \sum_{m=0}^{\infty} F_m (MR)^{2N-4} \frac{1}{N+2^m}.
\]

The scale \( M \) was set equal to 1, and \( c \) to \( 2(N-1)/(N+2) \). The bulk pieces, and some details of the numerical procedures used, can be found in section 3 of the main text. For completeness, coefficients for \( N = 8 \), obtained simply by doubling those reported by Zamolodchikov for the tricritical Ising to Ising flow in ref. [16], have also been included.
### $N=5$:

| m  | $\vec{F}_m$ (massive) | $F_m$ (massless) |
|----|-----------------------|-------------------|
| 0  | $1.98 \times 10^{-13}$ | $-3.86 \times 10^{-14}$ |
| 1  | $-7.78 \times 10^{-12}$ | $3.23 \times 10^{-12}$ |
| 2  | $0.0112031704$      | $0.0112031702$    |
| 3  | $-3.37 \times 10^{-09}$ | $1.35 \times 10^{-09}$ |
| 4  | $0.00038484$         | $0.00038477$      |
| 5  | $0.019215$           | $-0.019217$       |
| 6  | $0.02056$            | $0.02055$         |
| 7  | $0.0004$             | $-0.0006$         |

### $N=6$:

| m  | $\vec{F}_m$ (massive) | $F_m$ (massless) |
|----|-----------------------|-------------------|
| 0  | $2.15 \times 10^{-18}$ | $4.10 \times 10^{-17}$ |
| 1  | $1.01 \times 10^{-13}$ | $1.22 \times 10^{-13}$ |
| 2  | $0.026663544239$      | $0.026663544230$  |
| 3  | $0.0202428666$        | $-0.0202428658$   |
| 4  | $0.00088616$          | $0.00088614$      |
| 5  | $-0.0061087$          | $0.0061090$       |
| 6  | $-0.002800$           | $-0.002803$       |
| 7  | $0.00033$             | $-0.00031$        |
$N = 7:\begin{array}{ccc}
\hline
m & \bar{F}_m \text{ (massive)} & F_m \text{ (massless)} \\
\hline
0 & 3.98 \times 10^{-15} & 8.02 \times 10^{-16} \\
1 & 2.42 \times 10^{-14} & -5.84 \times 10^{-15} \\
2 & 0.04302032556 & 0.04302032557 \\
3 & 0.0128965993 & -0.0128965995 \\
4 & 0.000972018 & 0.000972029 \\
5 & 0.00120282 & -0.00120286 \\
6 & 0.0000682 & 0.0000690 \\
7 & -0.000085 & 0.000077 \\
\hline
\end{array}$

$N = 8:\begin{array}{ccc}
\hline
m & \bar{F}_m \text{ (massive)} & F_m \text{ (massless)} \\
\hline
2 & 0.06589308648 & 0.06589308648 \\
3 & -0.02744520208 & -0.02744520208 \\
4 & 0.00073623262 & 0.00073623264 \\
5 & 0.00102222 & -0.00102216 \\
6 & -0.00016842 & -0.00016851 \\
7 & -0.00004770 & 0.00004710 \\
8 & -0.0000268 & -0.0000226 \\
\hline
\end{array}$
| $N=9:$ | $m$ | $\tilde{F}_m$ (massive) | $F_m$ (massless) |
|---|---|---|---|
| 0 | 7.27 × 10^{-14} | 2.89 × 10^{-13} |
| 1 | $-4.21 \times 10^{-12}$ | $-1.97 \times 10^{-11}$ |
| 2 | 0.124757776 | 0.124757776 |
| 3 | $-0.01159266$ | 0.01159265 |
| 4 | 0.0004886 | 0.0004887 |
| 5 | 0.000574 | $-0.000575$ |
| 6 | 0.000088 | 0.000089 |
| 7 | $-0.000028$ | 0.000025 |

| $N=10:$ | $m$ | $\tilde{F}_m$ (massive) | $F_m$ (massless) |
|---|---|---|---|
| 0 | 1.17 × 10^{-14} | $-2.94 \times 10^{-14}$ |
| 1 | $-8.58 \times 10^{-13}$ | 2.40 × 10^{-12} |
| 2 | 0.0247185731 | 0.0247185730 |
| 3 | $-0.006211643$ | 0.006211644 |
| 4 | 0.00031116 | 0.00031115 |
| 5 | 0.00030406 | $-0.00030402$ |
| 6 | 0.0000402 | 0.0000401 |
| 7 | $-0.0000137$ | 0.0000138 |
References

[1] J.Fröhlich and E.H.Lieb, ‘Phase transitions in anisotropic lattice spin systems’, Comm. Math. Phys. 60 (1979) 233.

[2] J.V.José, L.P.Kadanoff, S.Kirkpatrick and D.R.Nelson, ‘Renormalization, vortices, and symmetry-breaking perturbations in the two-dimensional planar model’, Phys. Rev. B16 (1977) 1217.

[3] S.Elitzur, R.B.Pearson and J.Shigemitsu, ‘Phase structure of discrete Abelian spin and gauge systems’, Phys. Rev. D19 (1979) 3698.

[4] J.Villain, ‘Theory of one- and two-dimensional magnets with an easy magnetisation plane. II. The planar, classical, two-dimensional magnet’, J. Physique. 36 (1975) 581.

[5] E.Domany, D.Mukamel and A.Schwimmer, ‘Phase diagram of the \( \mathbb{Z}_5 \) model on a square lattice’, J. Phys. A13 (1980) L311; J.L.Cardy, ‘General discrete planar models in two dimensions’, J. Phys. A13 (1980) 1507; F.C.Alcaraz and R.Köberle, ‘Duality and the phases of \( \mathbb{Z}_N \) spin systems’, J. Phys. A13 (1980) L153; F.C.Alcaraz and R.Köberle, ‘The phases of two-dimensional spin and four-dimensional gauge systems with \( \mathbb{Z}_N \) symmetry’, J. Phys. A14 (1981) 1169; G.von Gehlen and V.Rittenberg, ‘Hunting for the central charge of the Virasoro algebra: six- and eight-state spin models’, J. Phys. A19 (1986) 2439; R.Badke, ‘A Monte Carlo renormalisation group study of the two-dimensional discrete cubic model: a hint for superconformal invariance’, J. Phys. A20 (1987) 3425; G.Schütz, ‘Finite-size scaling spectra in the six-states quantum chains’, J. Phys. A22 (1989) 731.

[6] F.Y.Wu and Y.K.Wang, ‘Duality transformations in a many-component spin model’, J. Math. Phys. 17 (1976) 439.

[7] E.Domany and E.K.Riedel, ‘Two-dimensional anisotropic \( N \)-vector models’, Phys. Rev. B19 (1979) 5817.

[8] V.A.Fateev and A.B.Zamolodchikov, ‘Self-dual solutions of the star-triangle relations in \( \mathbb{Z}_N \) models’, Phys. Lett. A92 (1982) 37.
[9] A.B.Zamolodchikov and V.A.Fateev, ‘Nonlocal (parafermion) currents in two-dimensional conformal quantum field theory and self-dual critical points in $\mathbb{Z}_N$-symmetrical statistical systems’, Sov. Phys. JETP 62 (1985) 215.

[10] F.C.Alcaraz, ‘The critical behaviour of self-dual $\mathbb{Z}_N$ spin systems: finite-size scaling and conformal invariance’, J. Phys. A20 (1987) 2511.

[11] M.Jimbo, T.Miwa and M.Okado, ‘Solvable lattice models with broken $\mathbb{Z}_N$ symmetry and Hecke’s indefinite modular forms’, Nucl. Phys. B275 (1986) 517; G.Albertini, ‘Fateev-Zamolodchikov spin chain: excitation spectrum, completeness and thermodynamics’, Int. J. Mod. Phys. A9 (1994) 4921.

[12] A.B.Zamolodchikov, ‘Integrable Field Theory from Conformal Field Theory’, Proceedings of the Taniguchi Symposium, Kyoto (1988).

[13] A.B.Zamolodchikov, ‘Thermodynamic Bethe Ansatz in Relativistic Models. Scaling 3-state Potts and Lee-Yang Models’, Nucl. Phys. B342 (1990) 695.

[14] T.R.Klassen and E.Melzer, ‘Purely elastic scattering theories and their ultraviolet limits’, Nucl. Phys. B338 (1990) 485.

[15] A.B.Zamolodchikov, ‘Thermodynamic Bethe Ansatz for RSOS scattering theories’, Nucl. Phys. B358 (1991) 497.

[16] A.B.Zamolodchikov, ‘From tricritical Ising to critical Ising by Thermodynamic Bethe Ansatz’, Nucl. Phys. B358 (1991) 524.

[17] V.A.Fateev, ‘Integrable deformations in $\mathbb{Z}_N$-symmetrical models of the conformal quantum field theory’, Int. J. Mod. Phys. A6 (1991) 2109.

[18] F.Ravanini, ‘Thermodynamic Bethe Ansatz for $\mathcal{G}_k \otimes \mathcal{G}_l / \mathcal{G}_{k+l}$ coset models perturbed by their $\phi_{1,1,\text{Adj}}$ operator’, Phys. Lett. B282 (1992) 73.

[19] A.Kuniba and T.Nakanishi, ‘Spectra in conformal field theories from the Rogers dilogarithm’, Mod. Phys. Lett. A7 (1992) 3487.

[20] R.Tateo, ‘The sine-Gordon model as $\frac{SO(n) \times SO(n)}{SO(n)_{12}}$ perturbed coset theory and generalizations’, Int. J. Mod. Phys. A10 (1995) 1357.
[21] F.Ravanini, R.Tateo and A.Valleriani, ‘Dynkin TBAs’, Int. J. Mod. Phys. A8 (1993) 1707.

[22] H.W.J.Blöte, J.L.Cardy and M.P.Nightingale, ‘Conformal invariance, the central charge, and universal finite-size amplitudes at criticality’, Phys. Rev. Lett. 56 (1986) 742; I.Affleck, ‘Universal term in the free energy at a critical point and the conformal anomaly’, Phys. Rev. Lett. 56 (1986) 746.

[23] P.Fendley and K.Intriligator, ‘Scattering and thermodynamics in integrable N=2 theories’, Nucl. Phys. B380 (1992) 265.

[24] R.Tateo, ‘New functional dilogarithm identities and sine-Gordon Y-systems’, Phys. Lett. B355 (1995) 157.

[25] V.A.Fateev, E.Onofri and A.B.Zamolodchikov, ‘The sausage model (integrable deformations of O(3) sigma model)’, Nucl. Phys. B406 (1993) 521.

[26] Vl.S.Dotsenko and V.A.Fateev, ‘Four-point correlation functions and the operator algebra in 2D conformal invariant theories with central charge c ≤ 1’, Nucl. Phys. B251 (1985) 691; Appendix B.

[27] A.B.Zamolodchikov, ‘On the thermodynamic Bethe ansatz equations for the reflectionless ADE scattering theories’, Phys. Lett. B253 (1991) 391.

[28] T.R.Klassen and E.Melzer, ‘Spectral flow between conformal field theories in (1+1) dimensions’, Nucl. Phys. B370 (1992) 511.

[29] D.A.Huse and M.E.Fisher, ‘The decoupling point of the axial next-nearest-neighbour Ising model and marginal crossover’, J. Phys. C15 (1982) L585; J.L.Cardy and G.Mussardo, ‘Universal properties of self-avoiding walks from two-dimensional field theory’, Nucl. Phys. B410 (1993) 451.

[30] V.A.Fateev, ‘The exact relations between the coupling constants and the masses of particles for the integrable perturbed conformal field theories’, Phys. Lett. B324 (1994) 45.

[31] T.R.Klassen and E.Melzer, ‘The thermodynamics of purely elastic scattering theories and conformal perturbation theory’, Nucl. Phys. B350 (1991) 635.
[32] G.von Gehlen, V.Rittenberg and G.Schütz, ‘Operator content of $n$-state quantum chains in the $c=1$ region’, J. Phys. A21 (1988) 2805.

[33] M.B.Einhorn, R.Savit and E.Rabinovici, ‘A physical picture for the phase transitions in $\mathbb{Z}_N$ symmetric models’, Nucl. Phys. B170 (1980) 16.

[34] J.L.Cardy, ‘Operator content of two-dimensional conformally invariant theories’, Nucl. Phys. B270 (1986) 186.

[35] M.J.Martins, ‘The thermodynamic Bethe ansatz for deformed $W_{N-1}$ conformal field theories’, Phys. Lett. B277 (1992) 301.

[36] P.E.Dorey, ‘Root systems and purely elastic S-matrices, I & II’, Nucl. Phys. B358 (1991) 654; Nucl. Phys. B374 (1992) 741.

[37] P.E.Dorey, ‘Partition functions, intertwiners and the Coxeter element’, Int. J. Mod. Phys. A8 (1993) 193.

[38] G.Feverati, E.Quattrini and F.Ravanini, ‘Infrared behaviour of massless integrable flows entering the minimal models from $\phi_{31}$’, Bologna preprint DFUB-95-09, hep-th/9512104.

[39] H.W.Braden, E.Corrigan, P.E.Dorey and R.Sasaki, ‘Affine Toda field theory and exact S-matrices’, Nucl. Phys. B338 (1990) 689.

[40] V.A.Fateev and Al.B.Zamolodchikov, ‘Integrable perturbations of $\mathbb{Z}_N$ parafermion models and the $O(3)$ sigma model’, Phys. Lett. B271 (1991), 91.

[41] Al.B.Zamolodchikov, ‘TBA equations for integrable perturbed $SU(2)_k \times SU(2)_l / SU(2)_{k+l}$ coset models’, Nucl. Phys. B366 (1991) 122.

[42] Al.B.Zamolodchikov, ‘Painlevé III and 2D polymers’, Nucl. Phys. B432 (1994) 427.

[43] P.Fendley, H.Saleur and Al.B.Zamolodchikov, ‘Massless flows, I & II’, Int. J. Mod. Phys. A8 (1993) 5717; Int. J. Mod. Phys. A8 (1993) 5751.

[44] Al.B.Zamolodchikov, ‘Thermodynamics of imaginary coupled sine-Gordon. Dense polymer finite-size scaling function’, Phys. Lett. B335 (1994) 436.
[45] V.Privman and M.E.Fisher, ‘Finite-size effects at first-order transi-
tions’, J. Stat. Phys. 33 (1983) 385;
E.Brézin and J.Zinn-Justin, ‘Finite size effects in phase transitions’,
Nucl. Phys. B257 (1985) 867;
G.Münster, ‘Tunneling amplitude and surface tension in $\phi^4$-theory’,
Nucl. Phys. B324 (1989) 630;
K.Jansen and Y.Shen, ‘Tunneling and energy splitting in Ising models’,
Nucl. Phys. B393 (1993) 658.

[46] S.Coleman, ‘The uses of instantons’, Erice Lectures 1977, reprinted in
‘Aspects of symmetry’ (CUP 1985).

[47] F.Ravanini, M.Stanishkov and R.Tateo, ‘Integrable perturbations of
CFT with complex parameter: the $M(3/5)$ model and its generalisa-
tions’, Int. J. Mod. Phys. A11 (1996) 677, [hep-th/9411085].

[48] M.J.Martins, ‘Complex excitations in the thermodynamic Bethe ansatz
approach’, Phys. Rev. Lett. 67 (1991) 39.

[49] P.Fendley, ‘Excited state thermodynamics’, Nucl. Phys. B374 (1992)
667.

[50] P.Fendley, S.D.Mathur, C.Vafa and N.P.Warner, ‘Integrable deforma-
tions and scattering matrices for the $N=2$ supersymmetric discrete se-
ries’, Phys. Lett. B243 (1990) 257.

[51] J.-B.Zuber, ‘Graphs and reflection groups’, Saclay preprint SPHT-95-
089 (July 1995), [hep-th/9507057].

[52] A.B.Zamolodchikov, ‘S-matrix of the subleading magnetic perturbation
of the tricritical Ising model’, preprint PUPT 1195 (August 1990);
F.A.Smirnov, ‘Exact S-matrices for $\phi_{12}$-pertubated minimal models of
conformal field theory’, Int. J. Mod. Phys. A6 (1991) 1407.

[53] P.Fendley and P.Ginsparg, ‘Noncritical orbifolds’, Nucl. Phys. B324
(1989) 549.

[54] W.Nahm, A.Recknagel and M.Terhoeven, ‘Dilogarithm identities in
conformal field theory’, Mod. Phys. Lett. A8 (1993) 1835;
R.Kedem, T.R.Klassen, B.M.McCoy and E.Melzer, ‘Fermionic sum rep-
resentations for conformal field theory characters’, Phys. Lett. B307
(1993) 68.
[55] A.B.Zamolodchikov and Al.B.Zamolodchikov, ‘Factorized S-matrices in two dimensions as the exact solutions of certain relativistic quantum field theory models’ Ann. Phys. 120 (1979) 253.

[56] M.Takahashi and M.Suzuki, ‘One-dimensional anisotropic Heisenberg model at finite temperature’, Prog. Theor. Phys. 48 (1972) 2187.

[57] F.Gliozzi and R.Tateo, ‘ADE functional dilogarithm identities and integrable models’, Phys. Lett. B348 (1995) 84.