On the Hermite-Hadamard Inequalities for \( h \)-Convex Functions on Balls and Ellipsoids

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Abstract. In this paper, we establish some Hermite-Hadamard type inequalities for \( h \)-convex function on high-dimensional balls and ellipsoids, which extend some known results. Some mappings connected with these inequalities and related results are also obtained.

1. Introduction

The concept of \( h \)-convexity was first introduced by Varošanec [16] in 2007, and then has been studied extensively by many mathematicians, see e.g. [2, 9, 10, 13] and the references therein.

Definition 1. Let \( h : [0, 1] \rightarrow [0, \infty) \) be a given function. We say that \( f : \mathcal{D} \rightarrow \mathbb{R} \), where \( \mathcal{D} \) is a convex subset of \( \mathbb{R}^n \), is \( h \)-convex if for any \( X, Y \in \mathcal{D} \) and \( \alpha \in [0, 1] \),

\[
    f(\alpha X + (1 - \alpha) Y) \leq h(\alpha)f(X) + h(1 - \alpha)f(Y). \tag{1}
\]

This notion unifies and generalizes the known classes of the usual convex functions, \( s \)-convex functions (in the second sense) [3], \( P \)-functions [14] and Godunova-Levin functions [8], which are obtained by putting in (1)

\[
    h(\alpha) = \alpha, \quad h(\alpha) = \alpha^s \quad (0 < s \leq 1), \quad h(\alpha) = 1,
\]

and

\[
    h(\alpha) = \begin{cases} 
      1/\alpha, & 0 < \alpha \leq 1, \\
      0, & \alpha = 0,
    \end{cases}
\]

respectively.

Convexity and its generalizations are very important both in pure mathematics and in applications. One of the significant application involved in convex type functions is the following well-known Hermite-Hadamard inequality.
Theorem A. Let \( f : [a, b] \subset \mathbb{R} \rightarrow \mathbb{R} \) be a convex function. Then
\[
f\left(\frac{a + b}{2}\right) \leq \frac{1}{b - a} \int_a^b f(x) \, dx \leq \frac{f(a) + f(b)}{2}.
\]

In 1999, Dragomir and Fitzpatrick [7] proved the variant of Hermite-Hadamard’s inequality which holds for \( s \)-convex functions in the second sense.

Theorem B. [7] Let \( f : [a, b] \subset \mathbb{R} \rightarrow \mathbb{R} \) be a nonnegative \( s \)-convex function in the second sense with \( 0 < s < 1 \). Then
\[
2^{s-1} f\left(\frac{a + b}{2}\right) \leq \frac{1}{b - a} \int_a^b f(x) \, dx \leq \frac{f(a) + f(b)}{s + 1}.
\]

In 2008, Sarikaya, Saglam and Yildirim obtained the following analogue inequalities for \( h \)-convex functions.

Theorem C. [15] Let \( f : [a, b] \subset \mathbb{R} \rightarrow \mathbb{R} \) be an \( h \)-convex function on \([a, b]\). Then
\[
\frac{1}{2h\left(\frac{1}{2}\right)} \left(\frac{a + b}{2}\right) \leq \frac{1}{b - a} \int_a^b f(x) \, dx \leq \left[f(a) + f(b)\right] \int_0^1 h(x) \, dx.
\]

At the meantime, there is an extensive literature devoted to develop Hermite-Hadamard’s type inequalities to higher-dimensions. For example, some inequalities for convex type functions on rectangles can be found in \([1, 6, 11] \), and on disks can be found in \([4, 5] \). In this paper, we mainly deal with analogue inequalities for \( h \)-convex functions on balls and ellipsoids. Compared to the methods employed on rectangles, which used on balls (ellipsoids) are rather technical.

In the sequel, unless otherwise specified, \( \mathbb{R}^n \) denotes the Euclidean space of dimension \( n \) and \( |E| \) denotes the Lebesgue measure of a measurable set \( E \subset \mathbb{R}^n \), \( d\sigma(x) \) is the usual surface measure \((n \geq 3)\) or the arc length \((n = 2)\) in general. \( B_n(C, r) \) and \( \delta_n(C, r) \) are the \( n \)-dimensional ball and its sphere respectively centered at the point \( C = (c_1, c_2, \ldots, c_n) \in \mathbb{R}^n \) with radius \( r > 0 \). \( E_n(C, R) \) denotes the \( n \)-dimensional ellipsoid centered at the point \( C = (c_1, c_2, \ldots, c_n) \in \mathbb{R}^n \) with semiaxes \( R = (r_1, r_2, \ldots, r_n) \), that is
\[
\frac{(x_1 - c_1)^2}{r_1^2} + \frac{(x_2 - c_2)^2}{r_2^2} + \cdots + \frac{(x_n - c_n)^2}{r_n^2} \leq 1, \quad 0 < r_1, r_2, \ldots, r_n < \infty,
\]
and \( S_n(C, R) \) is the sphere of \( E_n(C, R) \). It is well known that
\[
|B_n(C, r)| = \frac{\pi^{n/2} r^n}{\Gamma\left(\frac{n}{2} + 1\right)}, \quad |\delta_n(C, r)| = \frac{n\pi^{n/2} r^{n-1}}{\Gamma\left(\frac{n}{2} + 1\right)}; 
\]
\[
|E_n(C, R)| = \frac{\pi^{n/2} r_1 \cdots r_n}{\Gamma\left(\frac{n}{2} + 1\right)}, \quad |S_n(C, tR)| = t^{n-1}|S_n(C, R)|, \quad t > 0,
\]
where \( \Gamma(\cdot) \) denotes the Gamma function and \( tR = (tr_1, tr_2, \ldots, tr_n) \).

Throughout the paper, we also assume that the function \( h \) in Definition 1 is always Lebesgue integrable on the interval \([0, 1]\) and satisfies \( h\left(\frac{1}{2}\right) > 0 \).

Now we recall some known results. In 2000, Dragomir [4] proved the Hermite-Hadamard type inequality of convex functions on the disk in \( \mathbb{R}^2 \).

Theorem D. [4] Let \( f : B_2(C, r) \rightarrow \mathbb{R} \) be a convex function on the disk \( B_2(C, r) \). Then
\[
f(C) \leq \frac{1}{\pi r^2} \int_{B_2(C, r)} f(X) \, dX \leq \frac{1}{2\pi r} \int_{S_2(C, r)} f(X) \, d\sigma(X).
\]
Furthermore, Dragomir extended the proceeding result from the disk in $\mathbb{R}^2$ to the ball in $\mathbb{R}^3$ in the same year and obtained the following similar result.

**Theorem E.** [5] Let $f : B_2(C, r) \to \mathbb{R}$ be a convex function on $B_2(C, r)$. Then

$$f(C) \leq \frac{1}{|B_2(C, r)|} \int_{B_2(C, r)} f(X)dX \leq \frac{1}{|B_3(C, r)|} \int_{b_3(C, r)} f(X)d\sigma(X).$$

In 2014, Matłoka [12] generalized Theorem D for $h$–convex functions on disks and established the corresponding Hermite-Hadamard inequality.

**Theorem F.** [12] Let $f : B_2(C, r) \to \mathbb{R}$ be an $h$–convex function on $B_2(C, r)$. Then

$$\frac{1}{2h\left(\frac{1}{2}\right)} f(C) \leq \frac{1}{\pi r^2} \int_{B_2(C, r)} f(X)dX \leq F(r) \frac{1}{2\pi} \int_{b_2(C, r)} f(X)d\sigma(X),$$

where

$$F(r) = \frac{2}{r^2} \int_0^r t h\left(\frac{1}{r}\right) dt \left[ 1 + \frac{2}{\pi} \int_0^2 \frac{1}{(t^2 + 1)^{1/2}} dt - 2 \int_0^2 \frac{1}{t^2 + 1} dt \right].$$

As a consequence of Theorem F, the author obtained the variant Hermite-Hadamard inequality for $s$-convex functions.

**Theorem G.** [12] Let $f : B_2(C, r) \to \mathbb{R}$ be an $s$–convex function in the second sense on $B_2(C, r)$ with $0 < s < 1$. Then

$$\frac{s^2}{2} f(C) \leq \frac{1}{\pi r^2} \int_{B_2(C, r)} f(X)dX \leq \frac{1}{\pi r} \frac{2^s(s + 1)}{2^s(s + 1)(s + 2) - 4} \int_{b_2(C, r)} f(X)d\sigma(X).$$

**Remark 1.** Taking the changing of variable $\frac{1}{r} = \nu$ in Theorem F, we have

$$F(r) = \frac{2}{1 - 4h\left(\frac{1}{2}\right)} \frac{\int_0^1 v h(v)dv}{\int_0^1 v h(1 - v) dv},$$

which implies that $F(r)$ is independent of the radius $r$.

**Remark 2.** There was a mistake in Theorem F. The condition

$$1 - 4h\left(\frac{1}{2}\right) \int_0^1 th(1 - t) dt > 0$$

is necessary for the second inequality in (4). We will prove the assertion by contradiction. Suppose that

$$1 - 4h\left(\frac{1}{2}\right) \int_0^1 th(1 - t) dt \leq 0.$$

Then $F(r) \leq 0$. Choosing $f > 0$, we yield that

$$\int_{B_2(C, r)} f(X)dX > 0, \quad \int_{b_2(C, r)} f(X)d\sigma(X) > 0,$$

which is a contradiction with $F(r) \leq 0$.

According to proceeding argument, the second inequality in (5) of Theorem G is valid under the additional assumption of

$$2^s(s + 1)(s + 2) > 4.$$
With these motivations, one of the purposes of this paper is to establish analogues of Hermite-Hadamard inequalities for \( h \)-convex functions on \( n \)-dimensional convex bodies—balls and ellipsoids. Now we are in a position to state our results.

**Theorem 1.** Let \( f : B_n(C, r) \to \mathbb{R} \) be an \( h \)-convex function on \( B_n(C, r) \). Suppose that \( h \) satisfies

\[
1 - 2nh\left(\frac{1}{2}\right) \int_0^1 t^{n-1}h(1-t)dt > 0. \tag{6}
\]

Then

\[
\frac{1}{2h\left(\frac{1}{2}\right)} f(C) \leq \frac{1}{|B_n(C, r)|} \int_{B_n(C, r)} f(X)dX \leq \frac{\mathcal{K}(n)}{|\partial B_n(C, r)|} \int_{\partial B_n(C, r)} f(X)d\sigma(X), \tag{7}
\]

where

\[
\mathcal{K}(n) = \frac{n \int_0^1 t^{n-1}h(t)dt}{1 - 2nh\left(\frac{1}{2}\right) \int_0^1 t^{n-1}h(1-t)dt}. \tag{8}
\]

It is not difficult to see that (6) is always true if \( h(t) = t \). In fact, we have

\[
1 - 2nh\left(\frac{1}{2}\right) \int_0^1 t^{n-1}h(1-t)dt = 1 - n \int_0^1 (t^{n-1} - t^n)dt = \frac{n}{n+1} > 0.
\]

On the other hand, a direct calculation shows that \( \mathcal{K}(n) = 1 \). These observations imply that

**Corollary 1.** If \( f : B_n(C, r) \to \mathbb{R} \) be a convex function, then

\[
f(C) \leq \frac{1}{|B_n(C, r)|} \int_{B_n(C, r)} f(X)dX \leq \frac{1}{|\partial B_n(C, r)|} \int_{\partial B_n(C, r)} f(X)d\sigma(X).
\]

Particularly, Corollary 1 reduces to Theorem D and Theorem E if \( n = 2 \) and \( n = 3 \) respectively. If \( h(t) = t^s \), \( 0 < s < 1 \), then integration by parts tells us that

\[
\int_0^1 t^{n-1}h(1-t)dt = \int_0^1 t^{n-1}(1-t)^sdt = \int_0^1 (t-1)^{n-1}t^sdt = \frac{(n-1)!}{(s+1)(s+2) \cdots (s+n)} \tag{9}
\]

Combining (9) and Theorem 1, we arrive at the Hermite-Hadamard inequality of \( s \)-convex functions on the ball.

**Corollary 2.** Let \( f : B_n(C, r) \to \mathbb{R} \) be an \( s \)-convex function in the second sense on \( B_n(C, r) \). If \( 0 < s < 1 \) and it satisfies

\[
2^s(s+1)(s+2) \cdots (s+n) > 2n! \tag{10}
\]

then

\[
\frac{2^s}{2} f(C) \leq \frac{1}{|B_n(C, r)|} \int_{B_n(C, r)} f(X)dX \leq \frac{\mathcal{K}_1}{|\partial B_n(C, r)|} \int_{\partial B_n(C, r)} f(X)d\sigma(X),
\]

where

\[
\mathcal{K}_1 = \frac{n2^s(s+1)(s+2) \cdots (s+n-1)}{2^s(s+1)(s+2) \cdots (s+n) - 2n!}. \tag{11}
\]
Furthermore, we will extend the above results to more general convex sets, i.e. ellipsoids.

**Theorem 2.** Let \( f : E_n(C, R) \to \mathbb{R} \) be an \( h \)-convex function on the ellipsoid \( E_n(C, R) \). Suppose that \( h \) satisfies (6). Then

\[
\frac{1}{2h(\frac{1}{2})} \int_{E_n(C, R)} f(X) dX \leq \frac{1}{|E_n(C, R)|} \int_{E_n(C, R)} f(X) dX \leq \frac{\mathcal{K}_1(n)}{|S_n(0, 1)|} \int_{S_n(0, 1)} f(\bar{X}) d\sigma(X'),
\]

where \( \mathcal{K}(n) \) is as in Theorem 1,

\[ X' = (x_1', x_2', \ldots, x_n') \in \delta_n(0, 1), \quad \bar{X} = (\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_n) \quad \text{and} \quad \bar{x}_j = r_j x'_j + c_j, \quad j = 1, 2, \ldots, n. \]

Furthermore, if \( f \geq 0 \), we have

\[
\frac{1}{|E_n(C, R)|} \int_{E_n(C, R)} f(X) dX \leq \frac{\mathcal{F}(R)}{|S_n(C, R)|} \int_{S_n(C, R)} f(X) d\sigma(X),
\]

where

\[
\mathcal{F}(R) = \frac{|S_n(C, R)|}{|S_n(0, 1)|} \frac{\Gamma(\frac{n}{2} + 1)}{n \pi^{\frac{n}{2}}} \mathcal{K}(n),
\]

and

\[ r = \min\{r_1, r_2, \ldots, r_n\}. \]

It follows from Theorem 2 and the similar arguments as in Corollary 1 and Corollary 2 that

**Corollary 3.** If \( f : E_n(C, R) \to \mathbb{R} \) be a convex function, then

\[
f(C) \leq \frac{1}{|E_n(C, R)|} \int_{E_n(C, R)} f(X) dX \leq \frac{\Gamma(\frac{n}{2} + 1)}{n \pi^{\frac{n}{2}}} \int_{S_n(0, 1)} f(\bar{X}) d\sigma(X'),
\]

where \( \bar{X} \) are as in Theorem 2.

Especially, if \( f \) is a nonnegative convex function on \( E_n(C, R) \), then

\[
\frac{1}{|E_n(C, R)|} \int_{E_n(C, R)} f(X) dX \leq \frac{\Gamma(\frac{n}{2} + 1)}{n \pi^{\frac{n}{2}} \rho^{n-1}} \int_{S_n(0, 1)} f(X) d\sigma(X).
\]

If taking \( h(t) = t^s \), we derive from Theorem 2 and Corollary 2 that

**Corollary 4.** Let \( f : E_n(C, R) \to \mathbb{R} \) be an \( s \)-convex function in the second sense on the ellipsoid \( E_n(C, R) \) and \( \mathcal{K}_1 \) be the constant defined by (11). If \( 0 < s < 1 \) and (10) holds, then

\[
\frac{\Gamma(\frac{n}{2} + 1)}{\pi^{\frac{n}{2}}} \frac{2^s n(s + 1)(s + 2) \cdots (s + n - 1)}{2^s(s + 1)(s + 2) \cdots (s + n) - 2n!} \mathcal{K}_2 \leq \frac{1}{|E_n(C, R)|} \int_{E_n(C, R)} f(X) dX \leq \frac{\Gamma(\frac{n}{2} + 1)}{n \pi^{\frac{n}{2}}} \mathcal{K}_1,
\]

where \( \bar{X} \) are as in Theorem 2 and

\[
\mathcal{K}_2 = \frac{\Gamma(\frac{n}{2} + 1)}{\pi^{\frac{n}{2}}} \frac{2^s n(s + 1)(s + 2) \cdots (s + n - 1)}{2^s(s + 1)(s + 2) \cdots (s + n) - 2n!} \mathcal{K}_1.
\]
Furthermore, if \( f \geq 0 \), we have
\[
\frac{1}{|E_n(C, R)|} \int_{E_n(C, R)} f(X) dX \leq \frac{\tilde{F}(R)}{|S_n(C, R)|} \int_{S_n(C, R)} f(X) d\sigma(X),
\]
where
\[
\tilde{F}(R) = \frac{\Gamma (\frac{d}{2} + 1) |S_n(C, R)|}{n! \pi^{\frac{d}{2} - 1}} \frac{2^n n(s + 1)(s + 2) \cdots (s + n - 1)}{2^{s + 1}(s + 2) \cdots (s + n) - 2n!} = \frac{|S_n(C, R)| \Gamma (\frac{d}{2} + 1)}{n! \pi^{\frac{d}{2}}} K_1.
\]

The second purpose in this paper is to provide some applications of the Hermite-Hadamard inequalities for \( h \)-convex functions. In [4] and [5], Dragomir studied some properties of the mappings connected to the Hermite-Hadamard type inequality of convex function on disks and balls. In [12], Matłoka considered the similar mappings connected to the \( h \)-convex function on disks.

**Theorem H.** [12] Define the mapping \( \hat{S} : [0, 1] \rightarrow \mathbb{R} \) by
\[
\hat{S}(t) = \frac{1}{\pi r^2} \int_{B_2(C, r)} f(tX + (1 - t)C) dX.
\]
If \( f \) is an \( h \)-convex function on the disk \( B_2(C, r) \), then
(i) the function \( \hat{S} \) is an \( h \)-convex function on \( [0, 1] \),
(ii) for any \( t \in (0, 1) \),
\[
\frac{f(C)}{2h \left( \frac{1}{2} \right)} \leq \hat{S}(t) \leq \hat{S}(1) \left[ h(t) + 2h \left( \frac{1}{2} \right) h(1 - t) \right].
\]

**Theorem I.** [12] Define the mapping \( \delta : [0, 1] \rightarrow \mathbb{R} \) by
\[
\delta(t) = \begin{cases} 
\frac{1}{2\pi r} \int_{B_2(C, r)} f(X) d\sigma(X), & \text{if } t \in (0, 1), \\
\frac{f(C)}{C}, & \text{if } t = 0.
\end{cases}
\]
If \( f \) is an \( h \)-convex function on the disk \( B_2(C, r) \), then
(i) the function \( \delta \) is an \( h \)-convex function on \( [0, 1] \),
(ii) for any \( t \in (0, 1) \), \( \delta(t) \leq \mathcal{F}(r) \delta(0), \)
(iii) for any \( t \in (0, 1) \),
\[
\frac{f(C)}{2h \left( \frac{1}{2} \right) \mathcal{F}(r)} \leq \delta(t) \leq \delta(0) \left[ h(t) + 2h \left( \frac{1}{2} \right) h(1 - t) \mathcal{F}(r) \right].
\]

**Remark 3.** According to Remark I and using the notation in (8), we can rewrite (ii) and (iii) in Theorem I as the following explicit forms, respectively,
(ii') for any \( t \in (0, 1) \), \( \delta(t) \leq \mathcal{K}(2) \delta(t), \)
(iii') for any \( t \in (0, 1) \),
\[
\frac{f(C)}{2h \left( \frac{1}{2} \right) \mathcal{K}(2)} \leq \delta(t) \leq \delta(0) \left[ h(t) + 2h \left( \frac{1}{2} \right) h(1 - t) \mathcal{K}(2) \right].
\]

**Remark 4.** There was a mistake in Theorem I. By checking the proof of Theorem I in [12] and the statement of Remark 2, the condition
\[
1 - 4h \left( \frac{1}{2} \right) \int_0^1 th(1 - t) dt > 0
\]
is necessary for (ii) and (iii) in Theorem 1.

Now, we will prove some properties of these two mappings assuming that the function $f$ is $h$–convex on ellipsoids. Correspondingly, the associated properties of balls are also obtained.

**Theorem 3.** Define the mapping $\tilde{\mathcal{G}} : [0, 1] \to \mathbb{R}$ by

$$\tilde{\mathcal{G}}(t) = \frac{1}{|E_n(C, R)|} \int_{E_n(C, R)} f(X + (1 - t)C)dX.$$  

If $f$ is an $h$–convex function on the ellipsoid $E_n(C, R)$, then

(i) the function $\tilde{\mathcal{G}}$ is an $h$–convex function on $[0, 1]$,

(ii) for any $t \in (0, 1]$,

$$\frac{f(0)}{2h(\frac{1}{2})} \leq \tilde{\mathcal{G}}(t) \leq \tilde{\mathcal{G}}(1) \left[h(t) + 2h(\frac{1}{2})h(1-t)\right].$$  

(15)

As a consequence of the proceeding theorem, we have the following results.

**Corollary 5.** Define the mapping $\tilde{H} : [0, 1] \to \mathbb{R}$ by

$$\tilde{H}(t) = \frac{1}{|B_n(C, r)|} \int_{B_n(C, r)} f(X + (1 - t)C)dX.$$  

If $f$ is an $h$–convex function on the ball $B_n(C, r)$, then the mapping $\tilde{H}$ enjoys the same properties as $\tilde{\mathcal{G}}$ in Theorem 3.

If we choose $n = 2$ in Corollary 5, then it reduces to Theorem 3.

**Theorem 4.** Define the mapping $\mathcal{G} : [0, 1] \to \mathbb{R}$ by

$$\mathcal{G}(t) = \begin{cases} \frac{1}{|S_n(C, t)R|} \int_{S_n(C, t)R} f(X)d\sigma(X), & t \in (0, 1], \\ f(C), & t = 0. \end{cases}$$

If $f$ is an $h$–convex function on the ball $B_n(C, r)$ and (6) holds, then

(i) the function $\mathcal{G}(t)$ is an $h$–convex function on $[0, 1]$,

(ii) for any $t \in (0, 1]$,

$$\frac{f(0)}{2h(\frac{1}{2})K(n)} \leq \mathcal{G}(t) \leq \mathcal{G}(1) \left[h(t) + 2h(\frac{1}{2})h(1-t)K(n)\right].$$  

(16)

No virtue of Remark 3 and Remark 4, it is obviously that Theorem 4 generalizes Theorem 1.

**Theorem 5.** Define the mapping $\tilde{\mathcal{G}} : [0, 1] \to \mathbb{R}$ by

$$\tilde{\mathcal{G}}(t) = \begin{cases} \frac{1}{|S_n(C, t)R|} \int_{S_n(C, t)R} f(X)d\sigma(X), & t \in (0, 1], \\ f(C), & t = 0. \end{cases}$$

If $f$ is an $h$–convex function on the ellipsoid $E_n(C, R)$ and (6) holds, then

(i) the function $\tilde{\mathcal{G}}(t)$ is an $h$–convex function on $[0, 1]$,

(ii) when $f \geq 0$, for any $t \in (0, 1]$, $\tilde{\mathcal{G}}(t) \leq \tilde{F}(R)\tilde{\mathcal{G}}(t)$,

(iii) when $f \geq 0$, for any $t \in (0, 1]$,

$$\frac{f(C)}{2h(\frac{1}{2})\tilde{F}(R)} \leq \tilde{\mathcal{G}}(t) \leq \tilde{\mathcal{G}}(1) \left[h(t) + 2h(\frac{1}{2})h(1-t)\tilde{F}(R)\right],$$  

(17)

where $\tilde{F}(R)$ is defined by (14), i.e.

$$\tilde{F}(R) = \frac{|S_n(C, R)|}{\Gamma(\frac{n}{2} + 1)} \frac{\Gamma(\frac{n}{2} + 1)}{nr^{\frac{n}{2}}}K(n) \quad \text{and} \quad r = \min\{r_1, r_2, \ldots, r_n\}.$$
2. Proof of The Theorems

2.1. Proof of Theorem 1

(i) A changing of variables yields that

\[ \int_{B_n(C,r)} f(X) dX = \int_{B_n(C,r)} f(2C - X) dX. \]

Since \( f(C) = f\left(\frac{X}{2} + \frac{2C - X}{2}\right) \), then

\[
 f(C) = \frac{1}{|B_n(C,r)|} \int_{B_n(C,r)} f\left(\frac{X}{2} + \frac{2C - X}{2}\right) dX \\
 \leq \frac{1}{|B_n(C,r)|} \int_{B_n(C,r)} \left[ h\left(\frac{1}{2}\right) f(X) + h\left(\frac{1}{2}\right) f(2C - X) \right] dX \\
 = \frac{2h\left(\frac{1}{2}\right)}{|B_n(C,r)|} \int_{B_n(C,r)} f(X) dX.
\]

In this way we obtain the first part of (7).

(ii) The translation invariance of Lebesgue measure shows that

\[
\frac{1}{|B_n(C,r)|} \int_{B_n(C,r)} f(C) dX = \frac{1}{|B_n(0,0,r)|} \int_{B_n(0,0,r)} f(X + C) dX. \tag{18}
\]

Taking the spherical change of the unit sphere \( \delta_n(0,1) \)

\[
 X' = \begin{cases} 
 x'_1 = \cos \varphi_1, \\
 x'_2 = \sin \varphi_1 \cos \varphi_2, \\
 x'_3 = \sin \varphi_1 \sin \varphi_2 \cos \varphi_3, \\
 \vdots \\
 x'_{n-1} = \sin \varphi_1 \sin \varphi_2 \cdots \sin \varphi_{n-2} \cos \varphi_{n-1}, \\
 x'_n = \sin \varphi_1 \sin \varphi_2 \cdots \sin \varphi_{n-2} \sin \varphi_{n-1}, 
\end{cases} \tag{19}
\]

where

\[ 0 \leq \varphi_1, \ldots, \varphi_{n-2} \leq \pi, \ 0 \leq \varphi_{n-1} \leq 2\pi, \]

we have

\[
\int_{B_n(0,0,r)} f(X + C) dX \\
= \int_0^r \int_{B_n(0,1)} f(tX' + C) t^{n-1} d\sigma(X') dt \\
= \int_0^r \int_{B_n(0,1)} f\left(\frac{t}{r} (rX' + C) + \left(1 - \frac{t}{r}\right) C\right) t^{n-1} d\sigma(X') dt \\
\leq \int_0^r \int_{B_n(0,1)} \left[ h\left(\frac{t}{r}\right) f(rX' + C) + h\left(1 - \frac{t}{r}\right) f(C) \right] t^{n-1} d\sigma(X') dt \\
= \left( \int_0^r t^{n-1} h\left(\frac{t}{r}\right) dt \right) \left( \int_{B_n(0,1)} f(rX' + C) d\sigma(X') \right) \\
+ f(C) \delta_n(0,1) \int_0^r h\left(1 - \frac{t}{r}\right) t^{n-1} dt. \tag{20}
\]
On the other hand, the change of variable formula tells us that
\[
\frac{1}{r^n} \int_0^r t^{n-1} h\left(\frac{t}{r}\right) dt = \int_0^1 t^{n-1} h(t) dt,
\]
(21)
and
\[
\frac{1}{r^n} \int_0^r t^{n-1} h\left(1 - \frac{t}{r}\right) dt = \int_0^1 t^{n-1} h(1 - t) dt,
\]
(22)
and
\[
\int_{\delta_n(0,1)} f(rX' + C) d\sigma(X') = \frac{1}{r^n} \int_{\delta_n(C,r)} f(X) d\sigma(X).
\]
(23)

Then, by (2), (18)-(23) and the first inequality in (7),
\[
\begin{align*}
&\frac{1}{|B_n(C,r)|} \int_{B_n(C,r)} f(X) dX \\
\leq &\ n \int_0^1 t^{n-1} h(t) dt \frac{1}{|B_n(C,r)|} \int_{\delta_n(C,r)} f(X) d\sigma(X) + n \int_0^1 t^{n-1} h(1 - t) dt f(C) \\
\leq &\ n \int_0^1 t^{n-1} h(t) dt \frac{1}{|B_n(C,r)|} \int_{\delta_n(C,r)} f(X) d\sigma(X) \\
&+ 2n \int_0^1 t^{n-1} h(1 - t) dt h\left(\frac{1}{2}\right) \frac{1}{|B_n(C,r)|} \int_{\delta_n(C,r)} f(X) dX.
\end{align*}
\]

Recalling
\[
1 - 2nh\left(\frac{1}{2}\right) \int_0^1 t^{n-1} h(1 - t) dt > 0,
\]
we have
\[
\frac{1}{|B_n(C,r)|} \int_{B_n(C,r)} f(X) dX \leq \frac{n \int_0^1 t^{n-1} h(t) dt}{1 - 2nh\left(\frac{1}{2}\right) \int_0^1 t^{n-1} h(1 - t) dt} \frac{1}{|B_n(C,r)|} \int_{\delta_n(C,r)} f(X) d\sigma(X),
\]
which completes the proof. \(\Box\)

2.2. Proof of Theorem 2

Since the proof for the left part of (12) follows the same procedure as in Theorem 1 (i), we omit the details. Now, we will focus on proving the right part of (12). Let \(X' = (x'_1, x'_2, \ldots, x'_n)\) be the spherical transformation of the unit sphere \(\delta_n(0,1)\) defined by (19). Suppose that \(\bar{X} = (\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_n)\) and \(\bar{x}_j = r_j x'_j + c_j, \ j = 1, 2, \ldots, n.\)

That is
\[
\begin{align*}
\bar{x}_1 &= r_1 \cos \varphi_1 + c_1, \\
\bar{x}_2 &= r_2 \sin \varphi_1 \cos \varphi_2 + c_2, \\
\bar{x}_3 &= r_3 \sin \varphi_1 \sin \varphi_2 \cos \varphi_3 + c_3, \\
&\vdots \\
\bar{x}_{n-1} &= r_{n-1} \sin \varphi_1 \sin \varphi_2 \sin \varphi_3 \cdots \sin \varphi_{n-2} \cos \varphi_{n-1} + c_{n-1}, \\
\bar{x}_n &= r_n \sin \varphi_1 \sin \varphi_2 \sin \varphi_3 \cdots \sin \varphi_{n-1} \sin \varphi_{n-1} + c_n,
\end{align*}
\]
Thus, for any $X = (x_1, x_2, \ldots, x_n) \in E_n(C, R)$, there is $0 \leq t \leq 1$ such that

$$X = t\bar{X} + (1 - t)C := X(t, \varphi_1, \ldots, \varphi_{n-1}).$$

It is not difficult to check that the Jacobian of the transformation $X$ is

$$f_1(t, \varphi_1, \ldots, \varphi_{n-1}) := \left| \frac{\partial (x_1, x_2, \ldots, x_n)}{\partial (t, \varphi_1, \ldots, \varphi_{n-1})} \right| = \det \left| \begin{array}{ccc}
\frac{\partial x_1}{\partial t} & \frac{\partial x_1}{\partial \varphi_1} & \cdots & \frac{\partial x_1}{\partial \varphi_{n-1}} \\
\frac{\partial x_2}{\partial t} & \frac{\partial x_2}{\partial \varphi_1} & \cdots & \frac{\partial x_2}{\partial \varphi_{n-1}} \\
\vdots & \vdots & \ddots & \vdots \\
\frac{\partial x_n}{\partial t} & \frac{\partial x_n}{\partial \varphi_1} & \cdots & \frac{\partial x_n}{\partial \varphi_{n-1}} 
\end{array} \right|$$

$$= r_1r_2 \cdots r_n t^{n-1} (\sin \varphi_1)^{n-2} \cdots (\sin \varphi_{n-2})^2 (\sin \varphi_{n-1}). \quad (24)$$

We infer from (24) that

$$\int_{E_n(C, R)} f(X)dX = \int_0^1 \int_{\varphi_1=0}^{\pi} \cdots \int_{\varphi_{n-2}=0}^{\pi} \int_{\varphi_{n-1}=0}^{2\pi} f\left(t\bar{X} + (1 - t)C\right) f_1(t, \varphi_1, \ldots, \varphi_{n-1}) d\varphi_{n-1} d\varphi_{n-2} \cdots d\varphi_1 dt$$

$$\leq r_1r_2 \cdots r_n \int_0^1 t^{n-1} h(t) dt$$

$$\times \int_{\varphi_1=0}^{\pi} \cdots \int_{\varphi_{n-2}=0}^{\pi} \int_{\varphi_{n-1}=0}^{2\pi} f(\bar{X}) (\sin \varphi_1)^{n-2} \cdots (\sin \varphi_{n-2})^2 (\sin \varphi_{n-1})^2 d\varphi_{n-1} d\varphi_{n-2} \cdots d\varphi_1$$

$$+ r_1r_2 \cdots r_n f(C) \left| (1 - t)C \right| d\sigma \left( \mathcal{C} \right)$$

$$\leq r_1r_2 \cdots r_n \int_0^1 t^{n-1} h(t) dt \int_{\delta_n(0, 1)} f(\bar{X}) d\sigma \left( \mathcal{C} \right)$$

$$+ r_1r_2 \cdots r_n f(C) \left| (1 - t)C \right| d\sigma \left( \mathcal{C} \right)$$

With the aid of (2), (3) and the inequality

$$f(C) \leq \frac{2h(\frac{1}{2})}{|E_n(C, R)|} \int_{E_n(C, R)} f(X)dX,$$

we deduce that

$$\int_{E_n(C, R)} f(X)dX \leq r_1r_2 \cdots r_n \int_0^1 t^{n-1} h(t) dt \int_{\delta_n(0, 1)} f(\bar{X}) d\sigma \left( \mathcal{C} \right)$$

$$+ 2nh \left( \frac{1}{2} \right) \int_0^1 t^{n-1} h(1 - t) dt \int_{E_n(C, R)} f(X)dX.$$

Then, by (6),

$$\frac{1}{|E_n(C, R)|} \int_{E_n(C, R)} f(X)dX \leq \frac{\mathcal{K}(n)}{|\delta_n(0, 1)|} \int_{\delta_n(0, 1)} f(\bar{X}) d\sigma \left( \mathcal{C} \right).$$
This proves the right part of (12).

Now we turn to prove inequality (13). Let

$$A_i(q_1, q_2, \ldots, q_{n-1}) := \left| \frac{\partial (\tilde{x}_1, \ldots, \tilde{x}_{i-1}, \tilde{x}_{i+1}, \ldots, \tilde{x}_n)}{\partial (q_1, q_2, \ldots, q_{n-1})} \right|$$

$$= \det \begin{pmatrix} \frac{\partial \tilde{x}_1}{\partial q_1} & \ldots & \frac{\partial \tilde{x}_{i-1}}{\partial q_1} & \frac{\partial \tilde{x}_i}{\partial q_1} & \ldots & \frac{\partial \tilde{x}_{i+1}}{\partial q_1} \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \frac{\partial \tilde{x}_1}{\partial q_{n-1}} & \ldots & \frac{\partial \tilde{x}_{i-1}}{\partial q_{n-1}} & \frac{\partial \tilde{x}_i}{\partial q_{n-1}} & \ldots & \frac{\partial \tilde{x}_{i+1}}{\partial q_{n-1}} \end{pmatrix}.$$  \tag{25}

Comparing with the Jacobian of transformation of the unit sphere, we easily see that

$$J_2(q_1, q_2, \ldots, q_{n-1}) := \sqrt{\sum_{i=1}^{n} A_i^2(q_1, q_2, \ldots, q_{n-1})}$$

$$\geq r^{n-1} (\sin q_1)^{n-2} \cdot (\sin q_{n-3})^2 (\sin q_{n-2}),$$

where \( r = \min\{r_1, r_2, \ldots, r_n\} \). Since \( f \geq 0 \),

$$\int_{\partial S_t(C,R)} f(X) d\alpha(X)$$

$$= \int_{\theta=0}^{\pi} \cdots \int_{\theta_{n-2}=0}^{\pi} \int_{\theta_{n-1}=0}^{2\pi} f(\tilde{X}) J_2(q_1, q_2, \ldots, q_{n-1}) d\theta_{n-1} d\theta_{n-2} \cdots d\theta_1$$

$$\geq r^{n-1} \int_{\theta_1=0}^{\pi} \cdots \int_{\theta_{n-2}=0}^{\pi} \int_{\theta_{n-1}=0}^{2\pi} f(\tilde{X}) (\sin q_1)^{n-2} \cdot (\sin q_{n-3})^2 (\sin q_{n-2}) d\theta_{n-1} d\theta_{n-2} \cdots d\theta_1,$$

which yields that

$$\int_{\partial S_t(0,1)} f(\tilde{X}) d\sigma'(X') \leq \frac{1}{r^{n-1}} \int_{\partial S_t(C,R)} f(X) d\alpha(X).$$  \tag{27}

By combing (12) and (27) we finish the proof of Theorem 2. \hfill \Box

2.3. Proof of Theorem 3

(i) Let \( t_1, t_2 \in [0, 1] \), and \( \alpha, \beta \geq 0, \alpha + \beta = 1 \). Then

$$\tilde{\delta}(t_1 + \beta t_2)$$

$$= \frac{1}{|E_n(C,R)|} \int_{E_n(C,R)} f(\alpha [t_1 X + (1 - t_1)C] + \beta [t_2 X + (1 - t_2)C]) dX$$

$$\leq \frac{h(\alpha)}{|E_n(C,R)|} \int_{E_n(C,R)} f(t_1 X + (1 - t_1)C) dX + \frac{h(\beta)}{|E_n(C,R)|} \int_{E_n(C,R)} f(t_2 X + (1 - t_2)C) dX$$

$$= h(\alpha) \tilde{\delta}(t_1) + h(\beta) \tilde{\delta}(t_2),$$

which means that \( \tilde{\delta} \) is an \( h \)-convex function on \([0, 1]\).

(ii) For any fixed \( t \in (0, 1) \), taking the substitution \( \eta = (\eta_1, \eta_2, \ldots, \eta_n) \), where \( \eta_i = t x_i + (1 - t) c_i \), we have

$$\tilde{\delta}(t) = \frac{1}{|E_n(C,R)|} \int_{E_n(C,R)} f(tX + (1 - t)C) dX$$
Then Theorem 2 gives us that
\[ f(X) \leq \frac{1}{2} h \left( \frac{1}{2} \right) f(C) \leq \tilde{S}(t) \]  

In this way the first part of the inequality (15) is proved.

By the \( h \)-convexity of \( f \) on the ellipsoid and the left-side of (12), we have
\[
\tilde{S}(t) \leq \frac{h(t)}{|E_\alpha(C, R)|} \int_{E_\alpha(C, R)} f(X) dX + h(1 - t) f(C) \\
\leq \frac{h(t)}{|E_\alpha(C, R)|} \int_{E_\alpha(C, R)} f(X) dX + \frac{2h(1 - t) h \left( \frac{1}{2} \right)}{|E_\alpha(C, R)|} \int_{E_\alpha(C, R)} f(X) dX \\
= \left[ h(t) + 2h \left( \frac{1}{2} \right) h(1 - t) \right] \frac{1}{|E_\alpha(C, R)|} \int_{E_\alpha(C, R)} f(X) dX.
\]  

And the definition of \( \tilde{S} \) implies that
\[
\tilde{S}(t) \leq \tilde{S}(1) \left[ h(t) + 2h \left( \frac{1}{2} \right) h(1 - t) \right],
\]
which completes the proof. \( \square \)

2.4. Proof of Theorem 4

Due to (2) and the spherical transformation given by (19), we can deduce that
\[
\tilde{G}(t) = \frac{1}{|\hat{D}_n(0, 1)|} \int_{\hat{D}_n(0, 1)} f(\alpha X' + C) d\sigma(X'). \tag{29}
\]

(i) Let \( t_1, t_2 \in [0, 1] \) and \( \alpha, \beta \geq 0, \alpha + \beta = 1 \). Then, by (29),
\[
\tilde{G}(at_1 + \beta t_2) = \frac{1}{|\hat{D}_n(0, 1)|} \int_{\hat{D}_n(0, 1)} f(\alpha (t_1 X' + C) + \beta (t_2 X' + C)) d\sigma(X') \\
\leq \frac{1}{|\hat{D}_n(0, 1)|} \int_{\hat{D}_n(0, 1)} \left[ h(\alpha) f(t_1 X' + C) + h(\beta) f(t_2 X' + C) \right] d\sigma(X') \\
= h(\alpha) \tilde{G}(t_1) + h(\beta) \tilde{G}(t_2).
\]
2.5. Proof of Theorem 5

This means that $\tilde{G}$ is $h$-convex on $[0, 1]$.

(ii) As a special case of (28), we easily to see that

$$\bar{H}(t) = \frac{1}{|B_n(C,tr)|} \int_{B_n(C,tr)} f(X)dX.$$  

(30)

Thus, according to Theorem 1,

$$\bar{H}(t) \leq \frac{\mathcal{K}(n)}{|B_n(C,tr)|} \int_{B_n(C,tr)} f(X)d\sigma(X) = \mathcal{K}(n) \tilde{G}(t)$$

(31)

holds for all $t \in (0, 1]$.

(iii) With the aid of (30), (31) and the left part of (7), we can arrive at

$$\frac{f(C)}{2h(\frac{1}{2})} \leq \bar{H}(t) \leq \mathcal{K}(n) \tilde{G}(t)$$

(32)

for all $t \in (0, 1]$. Especially,

$$f(C) \leq 2h\left(\frac{1}{2}\right) \mathcal{K}(n) \tilde{G}(1).$$

(33)

On the other hand, (29) provides us that

$$\tilde{G}(t) = \frac{1}{|B_n(0,1)|} \int_{B_n(0,1)} f(t(rX' + C) + (1-t)C)d\sigma(X')$$

$$\leq \frac{1}{|B_n(0,1)|} \int_{B_n(0,1)} \left[ h(t)f((rX' + C) + (1-t)f(C) \right]d\sigma(X')$$

$$= h(t)\tilde{G}(1) + h(1-t)f(C)$$

$$\leq \tilde{G}(1) \left[ h(t) + 2h\left(\frac{1}{2}\right) h(1-t)\mathcal{K}(n) \right],$$

(34)

where the last inequality is obtained by (33). By combining (32) and (34) we finish the proof. \hfill \Box

2.5. Proof of Theorem 5

For any fixed $t \in (0, 1]$, we know that the surface of the ellipsoid can be presented as follows, $X = (x_1, x_2, \ldots, x_n) \in S_n(C,tr)$,

$$X(t, \varphi_1, \varphi_2, \ldots, \varphi_{n-1}) := \begin{cases}
\frac{x_1 = tr_1 \cos \varphi_1 + c_1,}{x_2 = tr_2 \sin \varphi_1 \cos \varphi_2 + c_2,}
\frac{x_3 = tr_3 \sin \varphi_1 \sin \varphi_2 \cos \varphi_3 + c_3,}{\vdots}
\frac{x_{n-1} = tr_{n-1} \sin \varphi_1 \sin \varphi_2 \sin \varphi_3 \cdots \sin \varphi_{n-2} \cos \varphi_{n-1} + c_{n-1},}{x_n = tr_n \sin \varphi_1 \sin \varphi_2 \sin \varphi_3 \cdots \sin \varphi_{n-2} \sin \varphi_{n-1} + c_n,}
\end{cases}$$

(35)

where $0 \leq \varphi_1, \ldots, \varphi_{n-2} \leq \pi$, $0 \leq \varphi_{n-1} \leq 2\pi$. Let

$$B_n(t, \varphi_1, \varphi_2, \ldots, \varphi_{n-1}) := \left| \frac{\partial(x_1, \ldots, x_{n-1}, x_{n+1}, \ldots, x_n)}{\partial(\varphi_1, \varphi_2, \ldots, \varphi_{n-1})} \right|$$
\[ B_i(t, q_1, q_2, \ldots, q_{n-1}) = t^{n-1} A_i(q_1, q_2, \ldots, q_{n-1}) \]

and

\[ J_3(t, q_1, q_2, \ldots, q_{n-1}) := \frac{1}{|S_n(C, I\mathbb{R})|} \int_{\phi_{n-1}=0}^{2\pi} d\phi_{n-1} J_3(t, q_1, q_2, \ldots, q_{n-1}) d\phi_n d\phi_{n-2} \ldots d\phi_1 \]

where \( A_i(q_1, q_2, \ldots, q_{n-1}) \) and \( J_2(q_1, q_2, \ldots, q_{n-1}) \) are presented by (25) and (26) respectively.

In the sequential of the paper, without confusion, we sometimes rewrite the notation \( X(t, q_1, q_2, \ldots, q_{n-1}) \) by \( X(t) \) and \( J_2(q_1, q_2, \ldots, q_{n-1}) \) by \( J_2 \) for the sake of convenience. Therefore, for any \( t \in (0, 1] \), we have

\[ \tilde{\delta}(t) = \frac{1}{|S_n(C, I\mathbb{R})|} \int_{\phi_{n-1}=0}^{2\pi} d\phi_{n-1} f(X(t)) J_3(t, q_1, q_2, \ldots, q_{n-1}) d\phi_n d\phi_{n-2} \ldots d\phi_1 \]

(i) Let \( t_1, t_2 \in [0, 1] \) and \( \alpha, \beta \geq 0, \alpha + \beta = 1 \). Then

\[ \tilde{\delta}(\alpha t_1 + \beta t_2) = \frac{1}{|S_n(C, I\mathbb{R})|} \int_{\phi_{n-1}=0}^{2\pi} d\phi_{n-1} f(X(\alpha t_1 + \beta t_2)) J_2 d\phi_{n-1} d\phi_{n-2} \ldots d\phi_1 \]

(ii) For any given \( t \in (0, 1] \), the identity (28) tells us that

\[ \tilde{\delta}(t) = \frac{1}{|E_n(C, I\mathbb{R})|} \int_{E_n(C, I\mathbb{R})} f(X) dX. \]

Since \( f \geq 0 \), by Theorem 2, we can claim that

\[ \frac{1}{|E_n(C, I\mathbb{R})|} \int_{E_n(C, I\mathbb{R})} f(X) dX \leq \frac{\tilde{F}(IR)}{|S_n(C, I\mathbb{R})|} \int_{S_n(C, I\mathbb{R})} f(X) d\sigma(X). \]

That is

\[ \tilde{\delta}(t) \leq \tilde{F}(IR) \tilde{\delta}(t), \ t \in (0, 1], \]
where
\[ \tilde{\mathcal{F}}(tR) = \frac{|S_n(C, tR)|}{(2t)^n} \Gamma \left( \frac{3}{2} + 1 \right) \frac{K(n)}{n \pi^{n/2}} \]

On the other hand, by (3), we have
\[ \tilde{\mathcal{F}}(tR) = \tilde{\mathcal{F}}(R) \]

This observation yields that
\[ \tilde{S}(t) \leq \tilde{\mathcal{F}}(R) \tilde{\delta}(t) \]

for all \( t \in (0, 1) \). We finish the proof of (ii).

(iii) Since the inequality
\[ \frac{f(C)}{2h(t/2)} \leq \tilde{S}(t) \]
is easily reached by (15) and (ii), next we will pay more attention to proving the right part of (17). Because of
\[ \tilde{\delta}(t) = \frac{1}{|S_n(C, R)|} \int_{\varphi_1 = 0}^{\pi} \cdots \int_{\varphi_{n-1} = 0}^{\pi} f(X(t)) f_2 d\varphi_{n-2} \cdots d\varphi_1 \]

and the \( h \)-convexity of \( f \), we have
\[
\tilde{\delta}(t) = \frac{1}{|S_n(C, R)|} \int_{\varphi_1 = 0}^{\pi} \cdots \int_{\varphi_{n-1} = 0}^{\pi} f(tX(1) + (1-t)C) f_2 d\varphi_{n-2} \cdots d\varphi_1 \\
\leq \frac{1}{|S_n(C, R)|} \int_{\varphi_1 = 0}^{\pi} \cdots \int_{\varphi_{n-1} = 0}^{\pi} \left[ h(t)f(X(1)) + h(1-t)f(C) \right] f_2 d\varphi_{n-2} \cdots d\varphi_1 \\
= \frac{h(t)}{|S_n(C, R)|} \int_{\varphi_1 = 0}^{\pi} \cdots \int_{\varphi_{n-1} = 0}^{\pi} f(X(1)) f_2 d\varphi_{n-2} \cdots d\varphi_1 + h(1-t)f(C) \\
\leq h(t) \tilde{\delta}(1) + 2\tilde{\mathcal{F}}(R) h \left( \frac{1}{2} \right) h(1-t) \tilde{\delta}(1) \\
= \tilde{\delta}(1) \left[ h(t) + 2\tilde{\mathcal{F}}(R) h \left( \frac{1}{2} \right) h(1-t) \right],
\]

which completes the proof. \( \square \)

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