Pandiagonal and Knut Vik Sudoku Squares

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In her informative paper on the construction of pandiagonal magic squares, Dame Kathleen Ollerenshaw [1] conjectured that there are no Knut Vik (or Latin pandiagonal) sudoku squares of order 9. In a subsequent letter, Boyer [2] confirmed that there are no such squares of order $2k$ and $3k$ since the nonexistence of Knut Vik squares of such orders was proved by Hedayat [3]. However, this fact does not preclude the existence of pandiagonal sudoku squares. Here we present systematic methods of constructing pandiagonal sudoku squares of order $k^2$ and Knut Vik sudoku squares of order $k^2$ not divisible by 2 or 3.

Definitions

For our purposes, an order-$n$ Latin square matrix has the integer elements 0, 1, …, $n - 1$ on all rows and all columns which thus sum to the index

$$m = n(n - 1)/2.$$  

(1)

Thus, a Latin square also is semi-magic. In a pandiagonal square all $2n$ broken diagonals (with wraparound) in both directions sum to $m$. A Knut Vik square is a Latin square that has the integers 0, 1, …, $n - 1$ on all broken diagonals; thus, it also is pandiagonal. A sudoku square is a Latin square of order $n = k^2$ whose $n$ main, order-$k$ subsquares also contain the integers 0, 1, …, $n - 1$. We define a super-sudoku square as a sudoku square with the additional property that all $n^2$ of its order-$k$ subsquares, including broken ones, sum to $m$.

Pandiagonal Sudoku Squares

By the foregoing definitions, the nonexistence of a Knut Vik sudoku square does not preclude the existence of a pandiagonal sudoku square of the same order. For example, here is a pandiagonal super-sudoku square matrix of order 9:

$$S_9 = \begin{bmatrix}
0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\
3 & 4 & 5 & 6 & 7 & 8 & 0 & 1 & 2 \\
6 & 7 & 8 & 0 & 1 & 2 & 3 & 4 & 5 \\
1 & 2 & 0 & 4 & 5 & 3 & 7 & 8 & 6 \\
4 & 5 & 3 & 7 & 8 & 6 & 1 & 2 & 0 \\
7 & 8 & 6 & 1 & 2 & 0 & 4 & 5 & 3 \\
2 & 0 & 1 & 5 & 3 & 4 & 8 & 6 & 7 \\
5 & 3 & 4 & 8 & 6 & 7 & 2 & 0 & 1 \\
8 & 6 & 7 & 2 & 0 & 1 & 5 & 3 & 4
\end{bmatrix}$$  

(2)

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which is formed by a simple row-wise permutation scheme. This square is not Knut Vik (none of order-9 exist). It can be decomposed into two auxiliary matrices as

\[ S_9 = 3\hat{S}_9 + \hat{S}_9^T, \quad (3) \]

where

\[
\hat{S}_9 = \begin{bmatrix}
0 & 0 & 0 & 1 & 1 & 1 & 2 & 2 & 2 \\
1 & 1 & 1 & 2 & 2 & 2 & 0 & 0 & 0 \\
2 & 2 & 2 & 0 & 0 & 0 & 1 & 1 & 1 \\
0 & 0 & 0 & 1 & 1 & 1 & 2 & 2 & 2 \\
1 & 1 & 1 & 2 & 2 & 2 & 0 & 0 & 0 \\
2 & 2 & 2 & 0 & 0 & 0 & 1 & 1 & 1 \\
0 & 0 & 0 & 1 & 1 & 1 & 2 & 2 & 2 \\
1 & 1 & 1 & 2 & 2 & 2 & 0 & 0 & 0 \\
2 & 2 & 2 & 0 & 0 & 0 & 1 & 1 & 1 \\
\end{bmatrix}, \quad \hat{S}_9^T = \begin{bmatrix}
0 & 1 & 2 & 0 & 1 & 2 & 0 & 1 & 2 \\
0 & 1 & 2 & 0 & 1 & 2 & 0 & 1 & 2 \\
0 & 1 & 2 & 0 & 1 & 2 & 0 & 1 & 2 \\
1 & 2 & 0 & 1 & 2 & 0 & 1 & 2 & 0 \\
1 & 2 & 0 & 1 & 2 & 0 & 1 & 2 & 0 \\
1 & 2 & 0 & 1 & 2 & 0 & 1 & 2 & 0 \\
2 & 0 & 1 & 2 & 0 & 1 & 2 & 0 & 1 \\
2 & 0 & 1 & 2 & 0 & 1 & 2 & 0 & 1 \\
2 & 0 & 1 & 2 & 0 & 1 & 2 & 0 & 1 \\
\end{bmatrix}. \quad (4) \]

The form of \( \hat{S}_9 \) shows that \( S_9 \) is a pandiagonal super-sudoku square.

The permutation scheme used to construct the pandiagonal super-sudoku square \( S_9 \) can be generalized to construct pandiagonal super-sudoku squares of order \( n = k^2 \). Also, the simple form of \( \hat{S}_9 \) can be generalized to \( \hat{S}_n \) \( (n = k^2) \) with elements \( \hat{S}_{ij}^{(n)} \) given by

\[
\hat{S}_{ij}^{(n)} = \left( i - 1 + \left\lfloor \frac{j - 1}{k} \right\rfloor \right) \mod k, \quad i, j = 1, 2, \ldots, n, \quad n = k^2. \quad (5) \]

Then

\[ S_n = k\hat{S}_n + \hat{S}_n^T, \quad (6) \]

is a pandiagonal super-sudoku square of the same permutation form as \( S_9 \). The pandiagonal and super-sudoku properties of \( S_n \) can be verified using formulas given by Nordgren [4,5].

In addition, we observe that a natural pandiagonal magic square \( M_n \) with elements 0, 1, \ldots, \( n^2 - 1 \) can be constructed from a pandiagonal super-sudoku square \( S_n \) of odd order \( n \) by the auxiliary square formula

\[ M_n = nS_n + S_n R, \quad (7) \]

where \( R \) is the reflection matrix with 1 on its cross diagonal (top-right to bottom-left) and 0 for all other elements. Here, \( S_n \) and its reflection about its vertical centerline, \( S_n R \), are seen to be orthogonal (as defined in [1,4]). For example, for \( n = 9 \), from (2) and (7), we obtain

\[
M_9 = \begin{bmatrix}
8 & 16 & 24 & 32 & 40 & 48 & 56 & 64 & 72 \\
29 & 37 & 45 & 62 & 70 & 78 & 5 & 13 & 21 \\
59 & 67 & 75 & 2 & 10 & 18 & 35 & 43 & 51 \\
15 & 26 & 7 & 39 & 50 & 31 & 63 & 74 & 55 \\
36 & 47 & 28 & 69 & 80 & 61 & 12 & 23 & 4 \\
66 & 77 & 58 & 9 & 20 & 1 & 42 & 53 & 34 \\
25 & 6 & 17 & 49 & 30 & 41 & 73 & 54 & 65 \\
46 & 27 & 38 & 79 & 60 & 71 & 22 & 3 & 14 \\
76 & 57 & 68 & 19 & 0 & 11 & 52 & 33 & 44 \\
\end{bmatrix} \quad (8) \]

in which all 3 by 3 subsquares, including broken ones, also sum to the index 360.
Knut Vik Sudoku Squares

Boyer [2] gives a Knut Vik sudoku square of order 25, the lowest order for which such a square exists. However, he does not indicate the method of constructing it nor how to construct such squares of higher order. On decomposing Boyer’s square, we find a general method of constructing a Knut Vik sudoku square \( V_n \) from auxiliary squares using

\[
V_n = kV'_n + V''_n
\]  

(9)

for order \( n = k^2 \) not divisible by 2 or 3. The first auxiliary matrix \( V'_n \) has the same submatrix \( A_k \) for its \( n \) sudoku subsquares, where \( A_k \) is the order-\( k \) submatrix with integers 0, 1, \ldots, \( k-1 \) on its main diagonal which is then replicated via a chess knight’s move of right two/down one (with wraparound), e.g., for \( n = 25, k = 5 \)

\[
V'_{25} = \begin{bmatrix}
A_5 & A_5 & A_5 & A_5 & A_5 \\
A_5 & A_5 & A_5 & A_5 & A_5 \\
A_5 & A_5 & A_5 & A_5 & A_5 \\
A_5 & A_5 & A_5 & A_5 & A_5 \\
A_5 & A_5 & A_5 & A_5 & A_5 \\
\end{bmatrix}, \quad A_5 = \begin{bmatrix}
0 & 4 & 3 & 2 & 1 \\
2 & 1 & 0 & 4 & 3 \\
4 & 3 & 2 & 1 & 0 \\
1 & 0 & 4 & 3 & 2 \\
3 & 2 & 1 & 0 & 4 \\
\end{bmatrix}
\]  

(10)

The form of the second auxiliary matrix \( V''_n \) is illustrated for \( n = 25, k = 5 \) by

\[
V''_5 = \begin{bmatrix}
B & E & C & F & D \\
C & F & D & B & E \\
D & B & E & C & F \\
E & C & F & D & B \\
F & D & B & E & C \\
\end{bmatrix},
\]  

(11)

where the submatrices \( B, C, D, E, F \) replicate in a chess knight’s move of left two/down one (with wraparound) and are given by

\[
B = \begin{bmatrix}
0 & 3 & 1 & 4 & 2 \\
2 & 0 & 3 & 1 & 4 \\
4 & 2 & 0 & 3 & 1 \\
1 & 4 & 2 & 0 & 3 \\
3 & 1 & 4 & 2 & 0 \\
\end{bmatrix}, \quad C = \begin{bmatrix}
2 & 0 & 3 & 1 & 4 \\
4 & 2 & 0 & 3 & 1 \\
1 & 4 & 2 & 0 & 3 \\
3 & 1 & 4 & 2 & 0 \\
0 & 3 & 1 & 4 & 2 \\
\end{bmatrix}, \quad \cdots.
\]  

(12)

Here, \( B \) is formed by starting its first column with the first column of \( A_5 \) and replicating via a chess bishop’s move of down one/right one (with wraparound). Next, \( C \) is formed by shifting the rows of \( B \) up one row (top row to bottom) and similarly for \( C \Rightarrow D \Rightarrow E \Rightarrow F \) which form the first column of \( V''_5 \). The second column of \( V''_5 \) starts with submatrix \( E \) which has the element 1 in its upper-left corner. The third column of \( V''_5 \) starts with submatrix \( C \) which has the element 2 in its upper-left corner and so on. The submatrices in each column progress downward in the same order as in the first column (with wraparound).

Higher-order Knut Vik sudoku squares can be constructed in a similar manner with the elements of \( V_n \) given by

\[
V_n (i, j) = k ((2i - j - 1) \mod k) + \left(2i + 2 \left\lfloor \frac{i - 1}{k} \right\rfloor - 2j + \left\lfloor \frac{j - 1}{k} \right\rfloor \right) \mod k,
\]  

(13)

\[i, j = 1, 2, \ldots, n, \quad n = k^2, \quad n \mod 2 \neq 0, \quad n \mod 3 \neq 0.\]

The order-\( k \) sudoku squares again form a Knight’s move pattern of left two/down one (with wraparound). A detailed analysis shows that \( V_n \) from (13) is a Knut Vik sudoku.
square. As in the construction of pandiagonal magic squares given in [1] and [4], the knight’s move patterns of $A_k$ and $V_n''$, as embodied in (13), do not work when the order $n = k^2$ is a multiple of 2 or 3, i.e., when a Knut Vik square does not exist.

An order-25, Knut Vik sudoku square of the form given by Boyer [2] follows from (12). Also, a natural pandiagonal magic square can be constructed from a Knut Vik sudoku square using (7). Example pandiagonal and Knut Vik sudoku squares constructed by the methods presented here are given in the Appendix.

References

[1] K. Ollerenshaw, Constructing pandiagonal magic squares of arbitrarily large size, Mathematics Today, 42, 23-29, 66-69 (2005).

[2] C. Boyer, Magic squares, Mathematics Today, 43, 70 (2006).

[3] A. Hedayat, Complete solution to the existence and nonexistence of Knut Vik designs and orthogonal Knut Vik designs, Journal of Combinatorial Theory (A) 22, 331-337 (1977).

[4] R.P. Nordgren, New constructions for special magic squares, International Journal of Pure and Applied Mathematics 78, 133-154 (2012).

[5] R.P. Nordgren, On properties of special magic squares, Linear Algebra and its Applications 437, 2009-2025 (2012).
Appendix: Examples

Here we present more examples of pandiagonal and Knut Vik sudoku squares constructed using Excel© and Maple© in the Scientific WorkPlace© program. The special properties of these squares are verified using defining formulas from [4] or [5] which are reviewed next.

Definitions

First we define two permutation matrices that are used in what follows. Let $R$ denote the $n \times n$ reflection matrix with 1 on its cross diagonal (top-right to bottom-left) and 0 for all other elements. For a given order-$n$ matrix $M$, the operation $RM$ reflects the elements of $M$ about its horizontal centerline and $MR$ reflects the elements of $M$ about its vertical centerline.

Let $K$ denote the order-$n$ shifter matrix that has all elements 0 except $K(1, n) = 1$ (upper right corner) and $K(i, i - 1) = 1$, $i = 2, 3, \ldots, n$ (diagonal below the main diagonal). It can be defined by

$$K(i, j) = \delta \left( (i - j + n - 1) \mod n \right), \quad i, j = 1, 2, \ldots, n,$$

where $\delta (k)$ is the Dirac delta for integers, defined as

$$\delta (k) = \begin{cases} 1 & \text{if } k = 0 \\ 0 & \text{if } k \neq 0 \end{cases}$$

and in Maple© as the function

$$\delta (k) = \text{Heaviside} \left( k + \frac{1}{2} \right) - \text{Heaviside} \left( k - \frac{1}{2} \right).$$

The operation $KM$ shifts rows of $M$ down one (and bottom row to top) while $MK$ shifts columns of $M$ one to the left (and first column to last). Power operations $K^i M$ and $MK^i$ give rise to repeated shifts. The following identities can be easily verified:

$$K^n = I, \quad K^{-i} = K^{n-i}, \quad i = 1, 2, \ldots, n,$$

$$\sum_{i=1}^{n} K^i = \sum_{i=1}^{n} K^{-i} = U.$$  \hspace{1cm} (17)

An order-$n$ matrix $M$ is semi-magic if the sums of all its rows and all its columns equal the same index $m$, i.e., if

$$Mu = (u^T M)^T = m u \quad \text{or} \quad MU = UM = mU,$$

where $u$ is an order-$n$ column vector with all elements 1 and $U$ is an order-$n$ matrix with all elements 1. The matrix $M$ is magic if in addition to (18) its main diagonal and cross diagonal also sum to $m$, i.e., if

$$\text{tr} [M] = \text{tr} [RM] = m.$$  \hspace{1cm} (19)

By these definitions, a magic square also is semi-magic and a semi-magic square may or may not be magic. Subscripts are used to denote special classes of $M$. The matrix $M_N$ is natural if its elements are integers in the numerical sequence $0, 1, \ldots, n^2 - 1$. The
The natural property of $M_N$ can be verified by sorting its elements into numerical order or from checking that
\[ M_N - K^i M_N K^j = \Omega, \quad i, j = 1, 2, \ldots, n, \quad (i = j \neq n), \quad (20) \]
where $\Omega$ is any order-$n$ matrix that has no zero elements. The index of a natural magic or semi-magic square is
\[ m_N = n(n^2 - 1)/2. \quad (21) \]

An order-$n$ square matrix $M_P$ is pandiagonal if all its broken diagonals (of $n$ elements) in both directions sum to the index $m$, i.e., if
\[ \text{tr} [K^i M_P] = \text{tr} [K^i R M_P] = m, \quad i = 1, 2, \ldots, n. \quad (22) \]

Also, the pandiagonal property of $M_P$ can be verified from the identities
\[ \sum_{i=1}^{n} K^i M_P K^i = nU \quad \text{and} \quad \sum_{i=1}^{n} K^i M_P K^{-i} = mU. \quad (23) \]

A square that is pandiagonal and magic is called panmagic.

An order-$n$ Latin square matrix $M_L$ has the integer elements $0, 1, \ldots, n-1$ on all rows and all columns which thus sum to the index
\[ m = n(n - 1)/2. \quad (24) \]
Thus, a Latin square also is semi-magic. The Latin square property of $M_L$ can be verified by sorting its rows and columns into numerical order or from checking that
\[ M_L - K^i M_L = \Omega, \quad M_L - M_L K^i = \Omega, \quad i = 1, 2, \ldots, n-1. \quad (25) \]

An order-$n$ Knut Vik square $V$ is a Latin square that has the integers $0, 1, \ldots, n-1$ on all broken diagonals; thus, it also is pandiagonal. The Knut Vik diagonal property of $V$ can be verified by sorting its diagonals into numerical order or by checking that
\[ V - K^i V K^i = \Omega, \quad V - K^i V K^{-i} = \Omega, \quad i = 1, 2, \ldots, n-1. \quad (26) \]

A sudoku square $S$ is a Latin square of order $n = k^2$ whose $n$ main, order-$k$ subsquares also contain the integers $0, 1, \ldots, n-1$. The sudoku property of a square can be verified by sorting the elements of its sudoku subsquares or by checking a matrix equation formed by first defining the order-$n$ permutation matrix $H$ in terms of submatrices $K_k$ and $O_k$ as
\[
H = \begin{bmatrix}
K_k & O_k & \cdots & O_k \\
O_k & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \ddots \\
O_k & O_k & \cdots & K_k
\end{bmatrix},
\]
where $K_k$ is an order-$k$ shifter matrix and $O_k$ is an order-$k$ matrix with all elements 0. Then, $S$ is a Sudoku square if
\[ S - H^i S H^j = \Omega, \quad i, j = 1, 2, \ldots, k, \quad (i = j \neq k). \quad (28) \]

We define a super-sudoku square as a sudoku square with the additional property that all $n^2$ of its order-$k$ subsquares, including broken ones, sum to $m$. The super-sudoku sum property of $S$ can be verified from
\[ (I + K + K^2 + \cdots + K^{k-1}) S (I + K + K^2 + \cdots + K^{k-1}) = mU. \quad (29) \]
Pandiagonal Sudoku Squares

For \( n = 9 \), by (5) and (9), we have

\[
S_9(i, j) = 3 \left( (i - 1 + \left\lfloor \frac{j - 1}{3} \right\rfloor) \mod 3 \right) + \left( (j - 1 + \left\lfloor \frac{i - 1}{3} \right\rfloor) \mod 3 \right) \equiv s_9, \ i, j = 1, 2, \ldots, 9,
\]

(30)

\[
s_9 = \begin{bmatrix}
0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\
3 & 4 & 5 & 6 & 7 & 8 & 0 & 1 & 2 \\
6 & 7 & 8 & 0 & 1 & 2 & 3 & 4 & 5 \\
1 & 2 & 0 & 4 & 5 & 3 & 7 & 8 & 6 \\
4 & 5 & 3 & 7 & 8 & 6 & 1 & 2 & 0 \\
7 & 8 & 6 & 1 & 2 & 0 & 4 & 5 & 3 \\
2 & 0 & 1 & 5 & 3 & 4 & 8 & 6 & 7 \\
5 & 3 & 4 & 8 & 6 & 7 & 2 & 0 & 1 \\
8 & 6 & 7 & 2 & 0 & 1 & 5 & 3 & 4
\end{bmatrix}
\]

and (14) gives

\[
K_9(i, j) = \delta ((i - j + 9 - 1) \mod 9) \equiv k_9, \ i, j = 1, 2, \ldots, 9,
\]

(31)

\[
k_9 = \begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0
\end{bmatrix}
\]

The pandiagonal condition (23) for \( s_9 \) can be evaluated using Maple\textsuperscript{®} as follows:

\[
\sum_{r=1}^{9} \left[ x^r \right]_{x=k_9} \left[ x \right]_{x=s_9} = \sum_{r=1}^{9} \left[ x^r \right]_{x=k_9} \left[ x \right]_{x=s_9} \left[ x^{9-r} \right]_{x=k_9} = 36U
\]

(32)

which verifies that \( s_9 \) is pandiagonal. The sudoku condition (28) can be verified with \( h_9 \) from (27) as

\[
h_9 = \begin{bmatrix}
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0
\end{bmatrix}
\]

(33)

The verification of (28) can be done in Maple\textsuperscript{®} by examining each term in the sum

\[
3 \sum_{p=1}^{3} \sum_{r=1}^{3} \left[ x \right]_{x=s_9} - \left[ x^p \right]_{x=h_9} \left[ x \right]_{x=s_9} \left[ x^r \right]_{x=h_9}
\]

(34)

\footnote{Use the “New Definition” command in Maple\textsuperscript{®} to define \( S_9(i, j) \) and use the “Fill Matrix” command to form it.}
The super-sudoku sum equation (29) can be evaluated using Maple© as

\[
\left[ [1 + x + x^2]_{x=k_9} [x]_{x=s_9} \right] [1 + x + x^2]_{x=k_9} = 36U
\]

which, together with the sudoku property, verifies that \( s_9 \) is super-sudoku.

A natural panmagic square with elements 0, 1, \ldots, 80 can be constructed from (7) as

\[
M_9(i, j) = 9S_9(i, j) + S_9(i, 10 - j) \equiv m_9, \quad i, j = 1, 2, \ldots, 9,
\]

\[
m_9 = \begin{bmatrix}
8 & 16 & 24 & 32 & 40 & 48 & 56 & 64 & 72 \\
29 & 37 & 45 & 62 & 70 & 78 & 5 & 13 & 21 \\
59 & 67 & 75 & 2 & 10 & 18 & 35 & 43 & 51 \\
15 & 26 & 7 & 39 & 50 & 31 & 63 & 74 & 55 \\
36 & 47 & 28 & 69 & 80 & 61 & 12 & 23 & 4 \\
66 & 77 & 58 & 9 & 20 & 1 & 42 & 53 & 34 \\
25 & 6 & 17 & 49 & 30 & 41 & 73 & 54 & 65 \\
46 & 27 & 38 & 79 & 60 & 71 & 22 & 3 & 14 \\
76 & 57 & 68 & 19 & 0 & 11 & 52 & 33 & 44
\end{bmatrix}.
\]

Checking properties of \( M_9(i, j) \), we find that

\[
\sum_{i=1}^{9} M_9(i, j) = 360, \quad j = 1, \ldots, 9,
\]

\[
\sum_{j=1}^{9} M_9(i, j) = 360, \quad i = 1, \ldots, 9,
\]

\[
\sum_{r=1}^{9} \left[ [x^r]_{x=k_9} [x]_{x=m_9} \right] [x^r]_{x=k_9} = 360U,
\]

\[
\sum_{r=1}^{9} \left[ [x^r]_{x=k_9} [x]_{x=m_9} \right] [x^{9-r}]_{x=k_9} = 360U,
\]

\[
\left[ [1 + x + x^2]_{x=k_9} [x]_{x=m_9} \right] [1 + x + x^2]_{x=k_9} = 360U
\]

which verifies that \( M_9 \) is a panmagic square whose 81 3 by 3 subsquares all add to \( m = 360 \). The natural property can be verified from (20) by examining each term in the sum

\[
\sum_{p=1}^{9} \sum_{r=1}^{9} \left( [x]_{x=m_9} - \left[ [x^p]_{x=h_9} [x]_{x=m_9} \right] [x^r]_{x=h_9} \right).
\]

For \( n = 16 \), by (5) and (6), we have

\[
T_{16}(i, j) = \left( i - 1 + \left\lfloor \frac{j - 1}{4} \right\rfloor \right) \mod 4 \equiv t_{16}, \quad i, j = 1, 2, \ldots, 16,
\]
Again, it is clear from the structure of $t_{16}$ and $t_{16}^T$ that $s_{16}$ is a panmagic super-sudoku square as can be verified from the defining formulas as before. Note that the elements of $s_{16}$ follow the same permutation scheme as those of $s_9$ in (30).
A panmagic square can be constructed from $S_{16}(i, j)$ as

$$M_{16}(i, j) = 16S_{16}(i, j) + S_{16}(i, 17 - j) \equiv m_{16} \quad (41)$$

$$m_{16} = \begin{bmatrix}
15 & 30 & 45 & 60 & 75 & 90 & 105 & 120 & 135 & 150 & 165 & 180 & 195 & 210 & 225 & 240 \\
67 & 82 & 97 & 112 & 143 & 158 & 188 & 203 & 218 & 233 & 248 & 7 & 22 & 37 & 52 \\
135 & 150 & 165 & 180 & 195 & 210 & 225 & 240 & 15 & 30 & 45 & 60 & 75 & 90 & 105 & 120 \\
203 & 218 & 233 & 248 & 7 & 22 & 37 & 52 & 67 & 82 & 97 & 112 & 143 & 158 & 188 & 203 \\
28 & 47 & 62 & 13 & 88 & 107 & 122 & 73 & 148 & 167 & 182 & 133 & 208 & 227 & 242 & 193 \\
80 & 99 & 114 & 65 & 156 & 175 & 190 & 141 & 216 & 235 & 250 & 201 & 20 & 39 & 54 & 5 \\
148 & 167 & 182 & 208 & 227 & 242 & 193 & 28 & 47 & 62 & 13 & 88 & 107 & 122 & 73 \\
216 & 235 & 250 & 201 & 20 & 39 & 54 & 5 & 80 & 99 & 114 & 65 & 156 & 175 & 190 & 141 \\
45 & 60 & 15 & 30 & 105 & 120 & 75 & 90 & 165 & 180 & 135 & 150 & 225 & 240 & 195 & 210 \\
97 & 112 & 67 & 82 & 173 & 188 & 143 & 158 & 233 & 248 & 203 & 218 & 37 & 52 & 7 & 22 \\
165 & 180 & 135 & 150 & 225 & 240 & 195 & 210 & 45 & 60 & 15 & 30 & 105 & 120 & 75 & 90 \\
233 & 248 & 203 & 218 & 37 & 52 & 7 & 22 & 97 & 112 & 67 & 82 & 173 & 188 & 143 & 158 \\
62 & 13 & 28 & 47 & 122 & 73 & 88 & 107 & 182 & 133 & 148 & 167 & 242 & 193 & 208 & 227 \\
114 & 65 & 80 & 99 & 190 & 141 & 156 & 175 & 250 & 201 & 216 & 235 & 54 & 5 & 20 & 39 \\
182 & 133 & 148 & 167 & 242 & 193 & 208 & 227 & 62 & 13 & 28 & 47 & 122 & 73 & 88 & 107 \\
250 & 201 & 216 & 235 & 54 & 5 & 20 & 39 & 114 & 65 & 80 & 99 & 190 & 141 & 156 & 175
\end{bmatrix}$$

However, this square is not natural as it contains many duplicate elements. As before, it can be verified that that $m_{16}$ is a panmagic square whose 4 by 4 subsquares all add to $m = 2040$. Higher order-$k^2$ pandiagonal super-sudoku squares can be generated and studied in a similar manner. For such squares, we conjecture that a panmagic square formed from (41) is natural only when $k$ is odd.

**Knut Vik Sudoku Squares**

According to [13], an order-25 Knut Vik sudoku square (the lowest possible order) is generated by

$$V_{25}(i, j) = 5 \begin{pmatrix} (1 - j + 2(i - 1)) \mod 5 \\ + (2(j - 1) + 2(i - 1) + 2[(i - 1)/5] + [(j - 1)/5]) \mod 5 \end{pmatrix}, \quad i, j = 1, 2, \ldots, 25. \quad (42)$$

We also can express $V_{25}$ in terms of order-5 submatrices $A_k$ as

$$V_{25} = \begin{bmatrix} Z_0 & Z_1 & Z_2 & Z_3 & Z_4 \\ Z_2 & Z_3 & Z_4 & Z_0 & Z_1 \\ Z_4 & Z_0 & Z_1 & Z_2 & Z_3 \\ Z_1 & Z_2 & Z_3 & Z_4 & Z_0 \\ Z_3 & Z_4 & Z_0 & Z_1 & Z_2 \end{bmatrix}, \quad (43)$$

where, as indicated by [10], [11], and [12],

$$Z_i = \begin{bmatrix} 0 & 4 & 3 & 2 & 1 \\ 1 & 0 & 4 & 3 & 2 \\ 4 & 3 & 2 & 1 & 0 \\ 1 & 0 & 4 & 3 & 2 \\ 3 & 2 & 1 & 0 & 4 \end{bmatrix} + \begin{bmatrix} i & i + 3 & i + 1 & i + 4 & i + 2 \\ i + 2 & i & i + 3 & i + 1 & i + 4 \\ i + 4 & i + 2 & i & i + 3 & i + 1 \\ i + 1 & i + 4 & i + 2 & i & i + 3 \\ i + 3 & i + 1 & i + 4 & i + 2 & i \end{bmatrix} \mod 5, \quad (44)$$

$i = 0, 1, \ldots, 4.$
The structure of these submatrices shows why $V_{25}$ is a Knut Vik sudoku square. These properties also can be verified by evaluating (25), (26), and (28). However, evaluation of (29) shows that $V_{25}$ is not a super-sudoku square. By (12) or (13) we have the following order-25 Knut Vik sudoku square:

$$
\begin{array}{cccccc}
0 & 23 & 16 & 14 & 7 & 1 \\
12 & 4 & 22 & 15 & 14 & 7 \\
14 & 7 & 0 & 23 & 16 & 10 \\
19 & 12 & 5 & 3 & 21 & 15 \\
22 & 15 & 13 & 6 & 4 & 22 \\
23 & 16 & 14 & 7 & 0 & 23 \\
24 & 10 & 8 & 1 & 24 & 17 \\

2 & 20 & 18 & 11 & 9 & 0 \\
3 & 21 & 19 & 12 & 5 & 0 \\
4 & 22 & 15 & 13 & 6 & 0 \\
7 & 0 & 23 & 16 & 14 & 7 \\
10 & 8 & 1 & 24 & 17 & 10 \\
11 & 9 & 2 & 20 & 18 & 13 \\
12 & 5 & 3 & 21 & 19 & 12 \\
13 & 6 & 4 & 22 & 15 & 14 \\
14 & 7 & 0 & 23 & 16 & 10 \\
15 & 13 & 6 & 4 & 22 & 16 \\
16 & 14 & 7 & 0 & 23 & 17 \\
17 & 10 & 8 & 1 & 24 & 18 \\
18 & 11 & 9 & 2 & 20 & 19 \\
19 & 12 & 5 & 3 & 21 & 15 \\
20 & 18 & 11 & 9 & 0 & 23 \\
21 & 19 & 12 & 5 & 3 & 22 \\
22 & 15 & 13 & 6 & 4 & 21 \\
23 & 16 & 14 & 7 & 0 & 24 \\
24 & 10 & 8 & 1 & 24 & 17 \\
25 & 0 & 23 & 16 & 14 & 7 \\

1 & 24 & 17 & 10 & 8 & 2 \\
3 & 21 & 19 & 12 & 5 & 4 \\
4 & 22 & 15 & 13 & 6 & 0 \\
7 & 0 & 23 & 16 & 14 & 7 \\
10 & 8 & 1 & 24 & 17 & 10 \\
11 & 9 & 2 & 20 & 18 & 13 \\
12 & 5 & 3 & 21 & 19 & 12 \\
13 & 6 & 4 & 22 & 15 & 14 \\
14 & 7 & 0 & 23 & 16 & 10 \\
15 & 13 & 6 & 4 & 22 & 16 \\
16 & 14 & 7 & 0 & 23 & 17 \\
17 & 10 & 8 & 1 & 24 & 18 \\
18 & 11 & 9 & 2 & 20 & 19 \\
19 & 12 & 5 & 3 & 21 & 15 \\
20 & 18 & 11 & 9 & 0 & 23 \\
21 & 19 & 12 & 5 & 3 & 22 \\
22 & 15 & 13 & 6 & 4 & 21 \\
23 & 16 & 14 & 7 & 0 & 24 \\
24 & 10 & 8 & 1 & 24 & 17 \\
25 & 0 & 23 & 16 & 14 & 7 \\

0 & 23 & 16 & 14 & 7 & 1 \\
12 & 4 & 22 & 15 & 14 & 7 \\
14 & 7 & 0 & 23 & 16 & 10 \\
19 & 12 & 5 & 3 & 21 & 15 \\
22 & 15 & 13 & 6 & 4 & 22 \\
23 & 16 & 14 & 7 & 0 & 23 \\
24 & 10 & 8 & 1 & 24 & 17 \\

2 & 20 & 18 & 11 & 9 & 0 \\
3 & 21 & 19 & 12 & 5 & 0 \\
4 & 22 & 15 & 13 & 6 & 0 \\
7 & 0 & 23 & 16 & 14 & 7 \\
10 & 8 & 1 & 24 & 17 & 10 \\
11 & 9 & 2 & 20 & 18 & 13 \\
12 & 5 & 3 & 21 & 19 & 12 \\
13 & 6 & 4 & 22 & 15 & 14 \\
14 & 7 & 0 & 23 & 16 & 10 \\
15 & 13 & 6 & 4 & 22 & 16 \\
16 & 14 & 7 & 0 & 23 & 17 \\
17 & 10 & 8 & 1 & 24 & 18 \\
18 & 11 & 9 & 2 & 20 & 19 \\
19 & 12 & 5 & 3 & 21 & 15 \\
20 & 18 & 11 & 9 & 0 & 23 \\
21 & 19 & 12 & 5 & 3 & 22 \\
22 & 15 & 13 & 6 & 4 & 21 \\
23 & 16 & 14 & 7 & 0 & 24 \\
24 & 10 & 8 & 1 & 24 & 17 \\
25 & 0 & 23 & 16 & 14 & 7 \\

3 & 21 & 19 & 12 & 5 & 4 \\
4 & 22 & 15 & 13 & 6 & 0 \\
7 & 0 & 23 & 16 & 14 & 7 \\
10 & 8 & 1 & 24 & 17 & 10 \\
13 & 6 & 4 & 22 & 15 & 14 \\
16 & 14 & 7 & 0 & 23 & 17 \\
19 & 12 & 5 & 3 & 21 & 15 \\
22 & 15 & 13 & 6 & 4 & 22 \\

9 & 2 & 20 & 18 & 11 & 5 \\
12 & 5 & 3 & 21 & 19 & 12 \\
15 & 13 & 6 & 4 & 22 & 16 \\
18 & 11 & 9 & 2 & 20 & 19 \\
21 & 19 & 12 & 5 & 3 & 22 \\
24 & 10 & 8 & 1 & 24 & 17 \\

16 & 14 & 7 & 0 & 23 & 17 \\
17 & 10 & 8 & 1 & 24 & 18 \\
19 & 12 & 5 & 3 & 21 & 15 \\

\end{array}
$$

As noted after (13), this square is essentially the same as the one given by Boyer [2]. It differs only by a shift of columns and the addition of 1 to each element.

Knut Vik sudoku squares $V_n$ of higher order $n = k^2$ ($k$ not divisible by 2 or 3) can be constructed in a similar manner as for $n = 25$. An order-49 of this type has the form:

$$
V_{49} = \begin{bmatrix}
W_0 & W_1 & W_2 & W_3 & W_4 & W_5 & W_6 \\
W_2 & W_3 & W_4 & W_5 & W_6 & W_5 & W_1 \\
W_4 & W_5 & W_6 & W_0 & W_1 & W_2 & W_3 \\
W_6 & W_0 & W_1 & W_2 & W_3 & W_4 & W_5 \\
W_1 & W_2 & W_3 & W_4 & W_5 & W_6 & W_0 \\
W_3 & W_4 & W_5 & W_6 & W_0 & W_1 & W_2 \\
W_5 & W_6 & W_0 & W_1 & W_2 & W_3 & W_4
\end{bmatrix},
$$

(45)

---

2The full 49 by 49 square can be constructed from (13) in Excel®.
where the submatrices $W_i$ are given by (13) as

$$
W_i = 7 \begin{bmatrix}
    0 & 6 & 5 & 4 & 3 & 2 & 1 \\
    2 & 1 & 0 & 6 & 5 & 4 & 3 \\
    4 & 3 & 2 & 1 & 0 & 6 & 5 \\
    6 & 5 & 4 & 3 & 2 & 1 & 0 \\
    1 & 0 & 6 & 5 & 4 & 3 & 2 \\
    3 & 2 & 1 & 0 & 6 & 5 & 4 \\
    5 & 4 & 3 & 2 & 1 & 0 & 6
\end{bmatrix}
$$

\[
W_i = \begin{bmatrix}
    \lambda i + 1 & \lambda i + 2 & \lambda i + 3 & \lambda i + 4 & \lambda i + 5 & \lambda i + 6 \\
    \lambda i + 2 & \lambda i + 1 & \lambda i + 4 & \lambda i + 5 & \lambda i + 3 & \lambda i + 6 \\
    \lambda i + 4 & \lambda i + 2 & \lambda i + 1 & \lambda i + 4 & \lambda i + 5 & \lambda i + 3 \\
    \lambda i + 6 & \lambda i + 4 & \lambda i + 2 & \lambda i + 1 & \lambda i + 4 & \lambda i + 5 \\
    \lambda i + 3 & \lambda i + 6 & \lambda i + 4 & \lambda i + 2 & \lambda i + 1 & \lambda i + 4 \\
    \lambda i + 1 & \lambda i + 6 & \lambda i + 4 & \lambda i + 2 & \lambda i + 1 & \lambda i + 4 \\
\end{bmatrix} \mod 7 \tag{46}
\]

\[
i = 0, 1, \ldots, 6,
\]

i.e.,

$$
W_0 = \begin{bmatrix}
    0 & 47 & 38 & 29 & 27 & 18 & 9 \\
    16 & 7 & 5 & 45 & 36 & 34 & 25 \\
    32 & 23 & 14 & 12 & 3 & 43 & 41 \\
    48 & 39 & 30 & 21 & 19 & 10 & 1 \\
    8 & 6 & 46 & 37 & 28 & 26 & 17 \\
    24 & 15 & 13 & 4 & 44 & 35 & 33 \\
    40 & 31 & 22 & 20 & 11 & 2 & 42 \\
\end{bmatrix}, \ldots,
$$

$$
W_6 = \begin{bmatrix}
    6 & 46 & 37 & 28 & 26 & 17 & 8 \\
    15 & 13 & 4 & 44 & 35 & 33 & 24 \\
    31 & 22 & 20 & 11 & 2 & 42 & 40 \\
    47 & 38 & 29 & 27 & 18 & 9 & 0 \\
    7 & 5 & 45 & 36 & 34 & 25 & 16 \\
    23 & 14 & 12 & 3 & 43 & 41 & 32 \\
    39 & 30 & 21 & 19 & 10 & 1 & 48 \\
\end{bmatrix} \tag{47}
$$

The algorithm (13) for Knut Vik sudoku squares can be compared with the following algorithm for Knut Vik squares of prime order $n > 3$ due to Euler as noted by Hedayat [3]:

$$
\hat{V}_n(i, j) = (\lambda i + j) \mod n, \quad \lambda \neq 0, 1, n - 1 \tag{48}
$$

For nonprime $n$ not divisible by 2 or 3, Hedayat gives additional restrictions on $\lambda$. The squares generated by (48) are cyclic, whereas $V_n$ generated by (13) are not.

The $k$ by $k$ sudoku subsquares of $V_n$ are natural and semi-magic. In addition, a natural panmagic square follows from $V_n$ using an equation of the form (17) since $V_n$ and $V_n R$ are orthogonal as can be shown by an extension of Brée’s orthogonality criterion as derived by Nordgren [4]. For this extension, note that in each column of sudoku squares each number follows a path that starts in the first column and progresses down $k + 1$, right two (with wraparound in the column). Furthermore, the sudoku squares themselves follow a knight’s move pattern. Therefore, as in [4], analysis shows that the paths for each number pair in $V_n$ and $V_n R$ intersect only once and orthogonality follows.