Integrals over Grassmannians and Random permutations

M. Adler*    P. van Moerbeke†

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Abstract

In testing the independence of two Gaussian populations, one computes the distribution of the sample canonical correlation coefficients, given that the actual correlation is zero. The “Laplace transform” of this distribution is not only an integral over the Grassmannian of p-dimensional planes in complex n-space, but is also related to a generalized hypergeometric function. Such integrals are solutions of Painlevé-like equations. They also have expansions, related to random words of length ℓ formed with an alphabet of p letters. Given that each letter appears in the word, the maximal length of the disjoint union of p increasing subsequences of the word clearly equals ℓ. But the maximal length of the disjoint union of p − 1 increasing subsequences leads to a non-trivial distribution. It is precisely this probability which appears in the expansion above.

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*Department of Mathematics, Brandeis University, Waltham, Mass 02454, USA. E-mail: adler@math.brandeis.edu. The support of a National Science Foundation grant # DMS-98-4-50790 is gratefully acknowledged.

†Department of Mathematics, Université de Louvain, 1348 Louvain-la-Neuve, Belgium and Brandeis University, Waltham, Mass 02454, USA. E-mail: vanmoerbeke@geom.ucl.ac.be and @math.brandeis.edu. The support of a National Science Foundation grant # DMS-98-4-50790, a Nato, a FNRS and a Francqui Foundation grant is gratefully acknowledged.
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Random words, longest increasing sequences and mean hook lengths

Consider the set of words
\[ \pi \in S_p^\ell := \{ \text{words } \pi \text{ of length } \ell, \text{ built from an alphabet } \{1, \ldots, p\}\}, \]
with the uniform probability distribution
\[ P^{\ell,p}(\pi) = \frac{1}{p^\ell}. \tag{0.0.1} \]
The RSK correspondence (see section 2.1) between words and pairs of semi-standard and standard tableaux induces a probability measure on partitions
\[ \lambda \in \mathbb{Y}_\ell = \{ \text{partitions } \lambda \in \mathbb{Y} \text{ of weight } |\lambda| = \ell\}, \tag{0.0.2} \]
given by
\[ P^{\ell,p}(\lambda) = \frac{f^\lambda s_\lambda(1^p)}{p^\ell}, \quad |\lambda| = \ell, \tag{0.0.3} \]
where \( s_\lambda \) is the Schur polynomial associated with the partition \( \lambda \),
\[ 1^p = (1, \ldots, 1, 0, 0, \ldots) \quad \text{and} \quad f^\lambda = \#\{\text{standard tableaux of shape } \lambda\}, \tag{0.0.4} \]
with
\[ (\text{support } P^{\ell,p}) \subseteq \mathbb{Y}^{(p)} := \{ \lambda \in \mathbb{Y}_\ell, \text{ such that } \lambda^\top_1 \leq p\}. \]

A subsequence \( \sigma \) of the word \( \pi \) is \( \text{weakly } k\text{-increasing} \), if it can be written as
\[ \sigma = \sigma_1 \cup \sigma_2 \cup \ldots \cup \sigma_k, \tag{0.0.5} \]
where \( \sigma_i \) are disjoint weakly increasing subsequences of the word \( \pi \), i.e., possibly with repetitions. The length of the longest increasing/decreasing subsequences is closely related to the shape of the associated partition, via the RSK correspondence:
\[ d_1(\pi) = \left\{ \begin{array}{l} \text{length of the longest } \text{strictly decreasing subsequence of } \pi \end{array}\right\} = \lambda^\top_1 \]
\[ i_k(\pi) = \left\{ \begin{array}{l} \text{length of the longest } \text{weakly } k\text{-increasing subsequence of } \pi \end{array}\right\} = \lambda_1 + \ldots + \lambda_k \tag{0.0.6} \]
\[ ^1\lambda^\top \] is the dual partition, i.e., obtained by flipping the Young diagram \( \lambda \) about its diagonal. So, \( \lambda^\top_1 \) is the length of the first column of \( \lambda \).
For integer $0 \leq p < n$, consider the fixed rectangular Young diagram of width $n - p$,

$$
\mu = (n - p)^p := (n - p, n - p, \ldots, n - p).
$$

(0.0.7)

Consider a word $\pi \in S_\ell^p$. Then the statement $d_1(\pi) = p$ implies, in particular, that all letters of the alphabet $\{1, \ldots, p\}$ are represented in $\pi$; then automatically $i_p(\pi) = \ell$. The theorem below deals with the first non-trivial quantity $i_{p-1}(\pi)$, given that $d_1(\pi) = p$. Using the standard notation, defined for a general parameter $\beta > 0$,

$$
(a)^{(1/\beta)} := \prod_i (a + \beta(1 - i))^{\lambda_i}, \text{with } (x)_n := x(x+1)\ldots(x+n-1), \ x_0 = 1,
$$

(0.0.8)

we now state Theorem 0.1, which will be established in section 3.2 (note that here symbol (0.0.8) is used for $\beta = 1$):

\textbf{Theorem 0.1} Given the probability (0.0.1) and (0.0.3), the following holds ($h^\kappa$ denotes the product of hook lengths over all boxes of the partition $\kappa$):

\begin{center}
\begin{tikzpicture}
\draw[step=1cm,very thin] (0,0) grid (4,4);% This grid represents the Young diagram.
\draw[very thick] (0,0) rectangle (4,4);% This line represents the entire diagram.
\draw[thick] (0,0) rectangle (2,2);% This line represents the first row.
\draw[thick] (2,0) rectangle (4,4);% This line represents the second row.
\draw[thick] (0,2) rectangle (2,4);% This line represents the third row.
\draw[thick] (4,0) rectangle (4,4);% This line represents the fourth row.
\draw[thick] (0,3) rectangle (1,4);% This line represents the fifth row.
\end{tikzpicture}
\end{center}
\[(0 \leq p < n)\]

\[P^\ell,p(\lambda \supseteq \mu) = P^\ell,p\left( \pi \in S_p^\ell \left| d_1(\pi) = p \text{ and } i_{\ell-1}(\pi) \leq \ell - n + p \right) \right.\]

\[= \frac{\ell!}{p^p} \prod_{1}^{p} \frac{(p-i)!}{(n-i)!} \sum_{\kappa \in \mathbb{Y}_{\ell-p(n-p)} \kappa_1 \leq p} \frac{1}{(h^\kappa)^2 (n^\kappa)}. \quad (0.0.9)\]

More generally, for fixed \(p \leq q < n\), the mathematical expectation (with regard to \(P^\ell,p\)) of the hook length of \(\lambda\), emanating from the vertical strip \(\nu\) (of width \(q - p\)), equals:

\[E^\ell,p\left( I_{\{\lambda \supseteq \mu\}}(\lambda) \prod_{(i,j) \in \lambda \atop n-q < j \leq n-p} h^\lambda_{(i,j)} \right) = \frac{\ell!}{p^p} \prod_{1}^{p} \frac{(q-i)!}{(n-i)!} \sum_{\kappa \in \mathbb{Y}_{\ell-p(n-p)} \kappa_1 \leq p} \frac{1}{(h^\kappa)^2 (n^\kappa)}. \quad (0.0.10)\]

Generating function for the mathematical expectation of the hook length, integrals over Grassmannians and Painlevé V

Theorem 0.2 below involves an integral over the Grassmannian

\[Gr(p, \mathbb{C}^n) = \frac{U(n)}{U(p) \times U(n-p)} =: G/K \quad (0.0.11)\]

of \(p\)-dimensional planes in \(\mathbb{C}^n\) through the origin and Haar measure \(d\mu(Z)\) on \(Gr(p, \mathbb{C}^n)\), expressed in the parametrizing coordinate \(Z\) of

\[\text{Affine } Gr(p, \mathbb{C}^n) = \left\{ \text{span} \left( \frac{I_p}{Z} \right) \left| Z := A_{21}A_{11}^{-1}, A \in G \right. \right\}, \quad (0.0.12)\]

where \(A \in G\) is represented in block form:

\[
\begin{pmatrix}
A_{11} & A_{12} \\
A_{21} & A_{22}
\end{pmatrix}
\]

\[
\begin{pmatrix}
p \\
\end{pmatrix}
\]

\[
\begin{pmatrix}
n-p
\end{pmatrix}
\in G. \quad (0.0.13)
\]

Section 1 will be devoted to the geometry of \(Gr(p, \mathbb{F}^n)\), where \(\mathbb{F} = \mathbb{C}, \mathbb{R}\) or \(\mathbb{H}\), and to the study of integrals over \(Gr(p, \mathbb{F}^n)\).
We also need Jack polynomials, which are the unique symmetric functions orthogonal with respect to a certain \( \alpha \)-dependent inner-product \( \langle \cdot, \cdot \rangle_\alpha \), such that

\[
\langle J^{(\alpha)}_\lambda, J^{(\alpha)}_\mu \rangle = \delta_{\mu \lambda} j^{(\alpha)}_\lambda,
\]

with

\[
j^{(\alpha)}_\lambda = \prod_{(i,j) \in \lambda} \left( \lambda_j^\top - i + \alpha(\lambda_i - j + 1) \right) \left( \lambda_j^\top - i + 1 + \alpha(\lambda_i - j) \right).
\]

Facts about Jack polynomials relevant for this project will be discussed in section 2.3.

Finally, generalized hypergeometric functions \( _2F_1^{(\alpha)} \) are defined by: \( (p, q, n \in \mathbb{C}, \ x = (x_1, x_2, \ldots)) \)

\[
_2F_1^{(\alpha)}(p, q; n; x) := \sum_{\kappa \in \mathbb{Y}} \frac{(p)_\kappa (q)_\kappa}{(n)_\kappa} \frac{J^{(\alpha)}_\kappa(x)}{j^{(\alpha)}_\kappa}. \tag{0.0.15}
\]

In particular,

\[
_2F_1^{(1)}(p, q; n; x) := \sum_{\kappa \in \mathbb{Y}} \frac{(p)_\kappa^{(1)} (q)_\kappa^{(1)}}{(n)_\kappa^{(1)}} \frac{J_\kappa^{(1)}(x)}{h_\kappa} s_\kappa(x). \tag{0.0.16}
\]

The following theorem will be established in section 4.2; it is strongly motivated by certain integrals appearing in the context of testing statistical independence of Gaussian populations, as will be explained in the next paragraph.

**Theorem 0.2** For fixed \( p \leq q \leq n/2 \), the generating function for the mathematical expectation of the hook length \((0.0.10)\) over a strip, with regard to the probability \((0.0.3)\), is given by \( ^2 \)

\(^2\)Remember \( \text{ support } P^{d,p} \subseteq \{ \lambda \in \mathbb{Y}, \text{ such that } |\lambda| = \ell, \ \lambda_1^\top \leq p \} \).
\[
\prod_{i=1}^{p} \frac{(n-i)!}{(q-i)!} x^{-p(n-p)} \sum_{\ell \geq p(n-p)} \frac{(px)^\ell}{\ell!} E_\ell^p \left( \prod_{n \in \lambda \subseteq \mu} I_{\ell(\mu)}(\lambda) \prod_{n \leq \mu \leq \lambda} h_{(i,j)}^{\lambda} \right)
\]

\[
= (c_{n,q,p}^{(1)})^{-1} \int_{Gr(p,C^n)} e^{x \text{Tr}(I+Z^\dagger Z)^{-1}} \det(Z^\dagger Z)^{-(q-p)} d\mu(Z)
\]

\[
= 2F_1^{(1)}(p, q; n; y) \left| \sum_{\ell \geq p(n-p)} \frac{y^\ell}{\ell!} \right.
\]

\[
= \exp \int_0^x \frac{u(y) - p(n-p) + py}{y} dy \quad (0.0.17)
\]

where \(u(x)\) is the unique solution to the initial value problem:

\[
\begin{cases}
  x^2 u''' + xu'' + 6xu + 4uu' + 4Qu' - 2Q'u + 2R = 0 \\
  \text{with } u(x) = p(n-p) - \frac{p(n-q)}{n} x + \ldots + an+1x^{n+1} + O(x^{n+1}) + \ldots, \text{ near } x = 0.
\end{cases} \quad (0.0.18)
\]

with \(a_{n+1}\) specified by the hypergeometric function in (0.0.17) and

\[
\begin{align*}
4Q &= -x^2 + 2(n + 2(p - q))x - (n - 2p)^2 \\
2R &= p(p-q)(x + n - 2p).
\end{align*} \quad (0.0.19)
\]

**Remark:** The constant \(c_{n,q,p}^{(1)}\) in (0.0.17) is the one below for \(\beta = 1:\)

\[
c := c_{n,q,p}^{(1)} := \prod_{i=1}^{p} \frac{\Gamma(i\beta + 1)\Gamma(\beta(n-q-i+1))\Gamma(\beta(q-i+1))}{\Gamma(\beta+1)\Gamma(\beta(n-i+1))}.
\]

**Testing Statistical Independence of Gaussian Populations**

To summarize section 5, consider \(p+q\) normally distributed random variables \((X_1, \ldots, X_p)^\top\) and \((Y_1, \ldots, Y_q)^\top\) \((p \leq q)\) with mean zero and covariance matrix \(\Sigma\). According to Hotelling ([24]) (see also Muirhead [35], p106), \((X_1, \ldots, X_p)^\top\) and \((Y_1, \ldots, Y_q)^\top\) can be replaced by a linearly transform of the \(X^\prime s\) and \(Y^\prime s\), so that the covariance matrix takes on the canonical form:

\[
\begin{align*}
\end{align*}
\]
\[ \Sigma = \text{cov} \left( \begin{array}{c} X \\ Y \end{array} \right) = \begin{pmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{12}^\top & \Sigma_{22} \end{pmatrix}, \quad \Sigma_{can} = \begin{pmatrix} I_p & P \\ P^\top & I_q \end{pmatrix}, \]

where

\[
P = \begin{pmatrix} \rho_1 & & \cdots & & \cdots & & \cdots \\ & \ddots & & & & & \\ & & \rho_k & & & & \\ & & & \rho_{k+1} & & & \\ & & & & \ddots & & \\ & & & & & \ddots & \\ & & & & & & \rho_p \end{pmatrix}, \quad k = \text{rank } \Sigma_{12},
\]

is the \( p \times q \)-matrix of canonical correlation coefficients:

\[
1 \geq \rho_1 \geq \rho_2 \geq \ldots \geq \rho_k > 0, \quad \rho_{k+1} = \ldots = \rho_p = 0.
\]

The \( n \) (\( n \geq p + q \)) independent samples \((x_{11}, \ldots, x_{1p}, y_{11}, \ldots, y_{1q})^\top, \ldots, (x_{n1}, \ldots, x_{np}, y_{n1}, \ldots, y_{nq})^\top\) arising from observing \( \left( \begin{array}{c} X \\ Y \end{array} \right) \) lead to a matrix \( \left( \begin{array}{c} x \\ y \end{array} \right) \) of size \((p+q, n)\), having a normal distribution with correlation \( \Sigma \). The roots \( r_1^2, \ldots, r_p^2 \) (sample canonical correlation coefficients) of the equation

\[
\det(xy^\top(yy^\top)^{-1}yx^\top - r^2xx^\top) = 0
\]

are the estimators (maximum likelihood estimators) of the canonical correlation coefficients \( \rho_1^2, \ldots, \rho_p^2 \).

In testing the null hypothesis, versus the alternative hypothesis,

\[
H_0 : \quad \rho_1^2 = \ldots = \rho_p^2 = 0 \quad \text{versus} \quad H_a : \quad (\rho_1^2, \ldots, \rho_p^2) \neq 0
\]

one needs the joint density of the \( r_i^2 \), given that \( \rho_1^2 = \ldots = \rho_p^2 = 0 \); namely, up to a \((q, p, n)\)-dependent normalizing constant, the density is given by

\[
|\Delta_p(r^2)|^{\beta} \prod_{i=1}^p (r_i^2)^{\beta(q-p+1)-1}(1 - r_i^2)^{\beta(n-q-p+1)-1}dr_i^2 \quad (0.0.21)
\]

for \( \beta = 1/2 \). This formula generalizes to the formula above, upon considering random variables \((X_1, \ldots, X_p)^\top\) and \((Y_1, \ldots, Y_q)^\top\), with values in the complex \( \mathbb{C} (\beta = 1) \) and the quaternions \( \mathbb{H} (\beta = 2) \).
Expectation of the ratio of Jack polynomials, integrals over Grassmannians, sample canonical correlation coefficients and PDE’s

Consider now a Poissonized probability on partitions \( \lambda \in \mathcal{Y} \), which depends on a parameter \( x > 0 \):

\[
P_{x,p}(\lambda) = e^{-\beta p x} \frac{J^{(1/\beta)}(1_p)(\lambda)}{\lambda^{|\lambda|}} = e^{-\beta p x} (\beta p x)^{|\lambda|}/|\lambda|! P^{\ell,p}(\lambda), \quad \lambda \in \mathcal{Y},
\]

with \( P^{\ell,p} \) generalizing probability measure (0.0.3), as we shall see in section 3.1,

\[
P^{\ell,p}(\lambda) = \frac{J^{(1/\beta)}(1_p)}{J^{(1/\beta)}(1_n)} \frac{|\lambda|!}{(\beta p)^{|\lambda|}} \quad \text{with} \quad |\lambda| = \ell,
\]

This probability has its support on \( \lambda^1 \leq p \). Many of these probability distributions on partitions have been introduced and extensively studied by Borodin, Kerov, Okounkov, Olshanski and Vershik (see [10, 11, 12, 50, 31]).

The following statement involves an integral suggested again by the statistical theory mentioned earlier.

**Theorem 0.3** For fixed \( p \leq q \leq n/2 \), the following holds (\( \beta = 1/2, 1, 2 \))

\[
I^{(\beta)}_{n,p,q} = \gamma e^{\beta p x} E_{x,p} \left( \frac{1}{\beta |\lambda|} \frac{J^{(1/\beta)}(1^p)(\lambda)}{J^{(1/\beta)}(1^n)} \right)
\]

\[
= \int_{Gr(p, \mathbb{F}^n)} e^{x \text{Tr}(I + Z^1 Z)^{-1}} \det(Z^1 Z)^{-\beta(q-p)} d\mu(Z)
\]

\[
= \int_{[0,1]^p} e^{x \sum_{i=1}^p z_i |\Delta_p(z)|^2 \prod_{i=1}^p z_i^{\beta(q-p) - 1} (1 - z_i)^{\beta(n-q-p+1)-1} } dz_i
\]

\[
= c \ 2 F_1^{(1/\beta)}(\beta p, \beta q; \beta n; y) \sum_{v_i^1 = \frac{\beta}{\beta n}} d_{\theta_{n,p}}
\]

\[
= c \exp \int_0^x v(y) dy.
\]

where \( c = c^{(\beta)}_{n,q,p} \) is as in (0.0.20) and where

- \( d\mu(Z) \) is Haar measure on the space \( Gr(p, \mathbb{F}^n) \) of \( p \)-planes in \( \mathbb{F}^n \), where \( \mathbb{F} := \mathbb{C}, \mathbb{R} \) or the quaternions \( \mathbb{H} \).
• The integral over $[0, 1]^p$, appearing in (0.0.23), is the “Laplace transform” of the distribution of the sample canonical correlation coefficients (0.0.21). This integral is a H"{a}nkel determinant for $\beta = 1$ and a Pfaffian for $\beta = 1/2$ and 2.

• $v(y) := v^{(\beta)}_{n,p,q}(y)$ and $I_p := I^{(\beta)}_{n,p,q}$ satisfies the differential equation (define $\delta_1^{\beta} := 1$ for $\beta = 1$ and := 0 otherwise):

$$4 \left( y^3 v''' + 6y^3 v'^r + (1 + \delta_1^{\beta})(2y^2 v'' + 4y^2 vv' + yv') \right) - yP_0v' + P_1v + P_2$$

$$= \begin{cases} 
0, & \text{for } \beta = 1, \text{ (Painlevé V)} \\
\frac{3}{16} \frac{p(p-1)}{(p+1)(p+2)} y^3 \frac{I_{p-2}I_{p+2}}{I_p^2}, & \text{for } \beta = 1/2, \\
\frac{3}{16^2 p+1} y^3 \frac{I_{p-1}I_{p+1}}{I_p^2}, & \text{for } \beta = 2,
\end{cases}$$

with $P_0$ quadratic and $P_1, P_2$ linear polynomials in $y$, with coefficients depending on $n$ and $r = pq, \quad s = n - 2p - 2q$. (0.0.24)

This statement will be established in section 4.1 and the differential equation part in section 6.1. As a by-product, we show incidentally that the multivariate hypergeometric function $2F_1^{(1)}(p, q; n; y)$ expressed in the $it_i = \sum_{k \geq 1} y_k$-variables are $\tau$-functions for the KP-hierarchy; but also that the function $2F_1^{(1)}(p, q; n; y)$ properly restricted is a solution of Painlevé V. For related questions, see Orlov and Sherbin ([37, 38]). Section 7 gives new differential equations for the spectrum of Wishart matrices and for the sample canonical correlations of Gaussian populations.

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1 Integrals over Grassmannians

Consider the Grassmannian $Gr(p, \mathbb{F}^n)$ of $p$-planes through the origin in $\mathbb{F}^n$, where $\mathbb{F} = \mathbb{C}$, $\mathbb{R}$ or $\mathbb{H}$ (= quaternions). Let $G$ be the group of matrices $A$, with entries in $\mathbb{F}$, such that $A^{-1} = A^\dagger$, with $A^\dagger := \bar{A}^\top$. Matrices $A \in G$ will be represented as block matrices
\[
\begin{pmatrix}
A_{11} & A_{12} \\
A_{21} & A_{22}
\end{pmatrix}
\begin{array}{c}
\overset{p}{\leftarrow} \\
\overset{n-p}{\rightarrow}
\end{array}
\in G.
\tag{1.0.1}
\]

The main statement of this section is theorem 1.1, where it is assumed, without loss of generality, that $n \geq 2p$. The values of $\beta$ are related to $\mathbb{C}$, $\mathbb{R}$ and $\mathbb{H}$, as follows:
\[
\begin{align*}
Gr(p, \mathbb{C}^n) : & \quad \beta = 1 \\
Gr(p, \mathbb{R}^n) : & \quad \beta = 1/2 \\
Gr(p, \mathbb{H}^n) : & \quad \beta = 2
\end{align*}
\]

The geometry of the symmetric spaces $G/K$ and $K \backslash G/K$ has been studied by Helgason [22, 23]. In his recent Princeton thesis, Dueñez (16) has systematically studied integrals over symmetric spaces. Explicit information on this subject is not readily available in the literature; therefore we explain the theory in the Grassmannian case and the useful aspects for our purposes.

**Theorem 1.1** Consider the two parametrization of $Gr(p, \mathbb{F}^n)$

Affine $Gr(p, \mathbb{F}^n) = \left\{ \text{span} \left( \begin{pmatrix} I_p \\ Z \end{pmatrix} \right) \bigg| Z := Z(A) = A_{21}A_{11}^{-1}, \ A \in G \right\}$, \tag{1.0.2}

and its invariant measures $d\mu(Z)$. Then, for $p \leq q \leq n/2$, we have
\[
\int_{Gr(p, \mathbb{F}^n)} e^{x \text{Tr}(I+Z^\dagger Z)^{-1}} \det(Z^\dagger Z)^{-\beta(q-p)}d\mu(Z)
\]
\[
= \int_{[0,1]^p} e^{x \sum_{i=1}^p z_i} |\Delta_p(z)|^{2\beta} \prod_{1}^{p} z_i^{\beta(q-p+1)-1}(1-z_i)^{\beta(n-q-p+1)-1}dz_i.
\tag{1.0.3}
\]

\footnote{Given $a_i \in \mathbb{R}$, we define for $a = a_0 + ia_1 \in \mathbb{C}$, $\bar{a} = a_0 - ia_1$ and for $a = a_0 + a_1 e_1 + a_2 e_2 + a_3 e_3 \in \mathbb{H}$, $\bar{a} := a_0 - a_1 e_1 - a_2 e_2 - a_3 e_3$.}
This is the “Fourier transform” of the joint density of the sample canonical correlations \((z_1, \ldots, z_p) = (r_1^2, \ldots, r_p^2)\), for \(\beta = 1/2, 1, 2\), for the real, complex and quaternionic cases, given that the canonical correlation coefficients are zero. (see (0.0.21) and section 5)

Consider the following block matrix

\[
I_{p,q} = \begin{pmatrix} I_p & O \\ O & -I_q \end{pmatrix}.
\]

**Theorem 1.2** An alternative description for \(Gr(p, \mathbb{F}^n)\) is given by

\[
Gr(p, \mathbb{F}^n) \simeq S := \{ M = AI_{p,n-p}A^{-1}I_{p,n-p} \mid A \in G \} \quad (1.0.4)
\]

The \(n \times n\) matrices \(M\) have a \(n-2p\)-dimensional eigenspace corresponding to the eigenvalue 1, so that \(M\) can be decomposed into \(M = M_0 \otimes M_1\) (with \(M_1\) corresponding to the 1-eigenspace), with \(Tr(M - M_0) = n - 2p\). Then we have

\[
e^{px/2} \int_S e^{xTrM_0/4} \det \left( \frac{I - M_0}{I + M_0} \right)^{\beta(p-q)} d\mu(M) = \int_{[0,1]^p} e^{x \sum_1^p z_i} |\Delta_p(z)|^{2\beta} \prod_1^p z_i^{\beta(q-p+1)-1}(1 - z_i)^{\beta(n-q-p+1)-1} dz_i.
\]  

**Theorem 1.3** Considering the parametrization

\[
T_{Id}Gr(p, \mathbb{F}^n) = \{ Z \mid \text{arbitrary } (n-p) \times p \text{ matrix, with } Z_{ij} \in \mathbb{F} \}, \quad (1.0.6)
\]

we have

\[
\int_{Z \in T_{Id}Gr(p, \mathbb{F}^n)} e^{-x Tr Z^\dagger Z} \det(Z^\dagger Z)^{\beta p} d\mu(Z), \quad E \subset \mathbb{R}^+
\]

\[
\int_{E^p} |\Delta_p(u)|^{2\beta} \prod_1^p e^{-xu_iu_i^{\beta(n-p+1)-1}} du_i.
\]  

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For $\beta = 1/2$ and $x = \frac{1}{2\pi}$, this is an integral of the joint density (Wishart density) (see section 5 and Muirhead [35], p.107) of the eigenvalues $u_1, \ldots, u_p$ of the matrix $A = Z^*Z$, where $Z$ is a $n \times p$ matrix ($p \leq n$), with Gaussian density centered at 0 and covariance $\lambda I_p$, namely the density

$$c_{n,p}(2\pi\lambda)^{-np/2}e^{-\frac{1}{2\pi}TrZ^*Z} \prod_{1 \leq i \leq n} \prod_{1 \leq j \leq p} dz_{ij}.$$ 

**Proposition 1.4** Then

$$\text{Gr}(p, \mathbb{C}^n) = \frac{U(n)}{U(p) \times U(n-p)} =: G/K$$

$$\text{Gr}(p, \mathbb{R}^n) = \frac{SO(n)}{SO(p) \times SO(n-p)} =: G/K$$

$$\text{Gr}(p, \mathbb{H}^n) = \frac{Sp(n)}{Sp(p) \times Sp(n-p)} =: G/K,$$

and the affine part can be parametrized as follows:

**Affine** $\text{Gr}(p, \mathbb{F}^n) = \left\{ \text{span} \left( \begin{pmatrix} I_p \\ Z \end{pmatrix} \right) \mid Z = Z(A) = A_{21}A_{11}^{-1}, A \in G \right\}$

and

$$K \setminus G/K = \left\{ \text{span} \left( \begin{pmatrix} I_p \\ Z \end{pmatrix} \right) \mid Z = \begin{pmatrix} -\tan \theta_1 & O \\ \vdots & \ddots & \vdots \\ O & -\tan \theta_p \\ O_{n-2p,p} \end{pmatrix}, \begin{array}{l} \text{with } 0 \leq \theta_i \leq \pi \end{array} \right\};$$

(1.0.9)

also \footnote{Setting $\text{Id} = \begin{pmatrix} 1_p \\ 0 \end{pmatrix}$.}

$$T_{Id}\text{Gr}(p, \mathbb{F}^n) = \{Z \mid \text{arbitrary } (n-p) \times p \text{ matrix, with } Z_{ij} \in \mathbb{F} \}. \quad (1.0.10)$$
Setting, for the respective cases of $Gr(p, \mathbb{F}^n)$ and $T_{Id}Gr(p, \mathbb{F}^n)$,

$$\{y_1, \ldots, y_p\} = \text{spectrum} \left( \frac{1 - Z^\dagger Z}{1 + Z^\dagger Z} \right) = (\cos 2\theta_1, \ldots, \cos 2\theta_p)$$

$$\{u_1, \ldots, u_p\} := \text{spectrum} (Z^\dagger Z)$$

with $-1 \leq y_i \leq 1$, and $0 \leq u_i < \infty$, Haar measure on $Gr(p, \mathbb{F}^n)$ and $T_{Id}Gr(p, \mathbb{F}^n)$ reads, setting $k = n - 2p$, (Weyl integration formulae)

$$Gr(p, \mathbb{F}^n): \quad d\mu(Z) = |\Delta_p(y)|^{2\beta} \prod_{i=1}^{p} (1 - y_i)^{\beta_k + (\beta - 1)} (1 + y_i)^{\beta - 1} dy_i dK$$

$$T_{Id}Gr(p, \mathbb{F}^n): \quad d\mu(Z) = |\Delta_p(u)|^{2\beta} \prod_{i=1}^{p} u_i^{\beta_k + (\beta - 1)} du_i dK, \quad (1.0.11)$$

leading to the table:

| $G/K$ | induced measure $d\mu$ on $K \setminus G/K$ | $d\mu$ on $T_{Id}G/K$ |
|-------|------------------------------------------|-------------------|
| $Gr(p, \mathbb{C}^n)$ | $|\Delta_p(y)|^2 \prod_{i=1}^{p} (1 - y_i)^k dy_i$ | $|\Delta_p(u)|^2 \prod_{i=1}^{p} u_i^k du_i$ |
| $Gr(p, \mathbb{R}^n)$ | $|\Delta_p(y)|^2 \prod_{i=1}^{p} (1 - y_i)^{\frac{k}{2}(k-1)} (1 + y_i)^{-\frac{1}{2}} dy_i$ | $|\Delta_p(u)|^2 \prod_{i=1}^{p} u_i^{\frac{1}{2}(k-1)} du_i$ |
| $Gr(p, \mathbb{H}^n)$ | $|\Delta_p(y)|^4 \prod_{i=1}^{p} (1 - y_i)^{2k+1} (1 + y_i)^{k} dy_i$ | $|\Delta_p(u)|^4 \prod_{i=1}^{p} u_i^{2k+1} du_i$ |

**Table 1**

In the other description (1.0.4) of $Gr(p, \mathbb{F}^n)$, given by

$$Gr(p, \mathbb{F}^n) \simeq S := \left\{ M = A I_{p,n-p} A^{-1} I_{p,n-p} \mid A \in G \right\}$$

an appropriate left action of $B \in K$ on $A \in G$, amounting to conjugation in $S$, leads to the matrix in the torus $\mathfrak{A} \subset G$,

$$(BA) I_{p,n-p} (BA)^{-1} I_{p,n-p} = B (AI_{p,n-p} A^{-1} I_{p,n-p}) B^{-1}$$

$$= \begin{pmatrix}
\Re e^{2i\Theta_p} & \Im e^{2i\Theta_p} \\
-\Im e^{2i\Theta_p} & \Re e^{2i\Theta_p}
\end{pmatrix} \begin{pmatrix}
O \\
I_{n-2p}
\end{pmatrix} \in \mathfrak{A},$$

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with $\Theta_p := \text{diag}(\theta_1, \ldots, \theta_p)$.

Proof of Proposition 1.4: The $p$ columns of the $n \times p$ matrix $A \left( \begin{array}{c} I_p \\ O \end{array} \right)$, with $A \in G$, and $O := O_{n-p,p}$ (a zero matrix of size $(n-p, p)$) span a $p$-dimensional plane in $\mathbb{F}^n$, so that

$$Gr(p, \mathbb{F}^n) = \left\{ \text{span } A \left( \begin{array}{c} I_p \\ O \end{array} \right), \text{ with } A \in G \right\}.$$  

The right action of $B \in K$ on $A \in G$ acts on the $p$-plane $A \left( \begin{array}{c} I_p \\ O \end{array} \right)$ as:

$$AB \left( \begin{array}{c} I_p \\ O \end{array} \right) = A \left( \begin{array}{cc} B_{11} & O \\ O & B_{22} \end{array} \right) \left( \begin{array}{c} I_p \\ O \end{array} \right) = A \left( \begin{array}{c} B_{11} \\ O \end{array} \right) = A \left( \begin{array}{c} I_p \\ O \end{array} \right) B_{11}$$

and therefore it has no effect on that plane

$$\text{span } AB \left( \begin{array}{c} I_p \\ O \end{array} \right) = \text{span } A \left( \begin{array}{c} I_p \\ O \end{array} \right) B_{11} = \text{span } A \left( \begin{array}{c} I_p \\ O \end{array} \right),$$

since multiplication to the right by $B_{11}$ merely replaces the $p$ columns of $A \left( \begin{array}{c} I_p \\ O \end{array} \right)$ by $p$ linear combination. Then the $n \times p$ matrix $V := A \left( \begin{array}{c} I_p \\ O \end{array} \right)$ satisfies

$$V^\dagger V = \left( \begin{array}{cc} I_p \\ O \end{array} \right) A^\dagger A \left( \begin{array}{c} I_p \\ O \end{array} \right) = \left( \begin{array}{cc} I_p \\ O \end{array} \right) I_n \left( \begin{array}{c} I_p \\ O \end{array} \right) = I_p.$$  \hspace{1cm} (1.0.12)

Conversely, we show that

$$\text{span } A_1 \left( \begin{array}{c} I_p \\ O \end{array} \right) = \text{span } A_2 \left( \begin{array}{c} I_p \\ O \end{array} \right)$$  \hspace{1cm} (1.0.13)

implies $A_2^{-1} A_1 \in K = K_1 \times K_2$. Indeed, (1.0.13) holds if and only if

$$A_1 \left( \begin{array}{c} I_p \\ O \end{array} \right) = A_2 \left( \begin{array}{c} I_p \\ O \end{array} \right) g,$$

with an invertible $p \times p$ matrix $g$.  \hspace{1cm} (1.0.14)
Then we prove \( g \in K_1 \). Indeed, from (1.0.13), the matrices \( V_i := A_i \begin{pmatrix} I_p \\ O \end{pmatrix} \) satisfy \( V_i^\dagger V_i = I_p \) and so \( V_1 = V_2 g \) implies \( g^\dagger g = (V_2 g)^\dagger V_2 g = V_1^\dagger V_1 = I_p \).

Multiplying (1.0.14) to the left with \( A_2^{-1} \) yields

\[
A_2^{-1} A_1 \begin{pmatrix} I_p \\ O \end{pmatrix} = \begin{pmatrix} g \\ O \end{pmatrix}
\]

and thus \( G \ni A_2^{-1} A_1 = \begin{pmatrix} g & * \\ O & h \end{pmatrix} \);

the fact that the latter matrix is in \( G \), implies \(* = 0\) and \( g \in K_1, h \in K_2 \).

This means that \( A_2^{-1} A_1 \in K \) and so \( Gr(p, \mathbb{F}^n) \cong G/K \).

To describe Affine \( Gr(p, \mathbb{F}^n) \), notice that a plane \( A \begin{pmatrix} I_p \\ O \end{pmatrix} \) for which \( \det A_{11} \neq 0 \), can be expressed as

\[
G/K \ni \text{span} \begin{pmatrix} I_p \\ O \end{pmatrix} = \text{span} \begin{pmatrix} A_{11} \\ A_{21} \end{pmatrix} = \text{span} \begin{pmatrix} A_{11} \\ A_{21} \end{pmatrix} A_{11}^{-1} = \text{span} \begin{pmatrix} I_p \\ Z(A) \end{pmatrix},
\]

where \( Z(A) := A_{21} A_{11}^{-1} \) is a \((n-p) \times p\) matrix. Also notice that \( Z(A) \) is unchanged upon multiplying \( A \) to the right with \( B \in K \).

The left action of \( B \in K \) on \( A \in G \) has the following effect on

\[
\text{Affine } G/K \rightarrow \text{Affine } G/K : \begin{pmatrix} I_p \\ Z \end{pmatrix} \mapsto \begin{pmatrix} I_p \\ B_{22} Z B_{11}^{-1} \end{pmatrix}, \quad (1.0.15)
\]

because in

\[
BA \begin{pmatrix} I_p \\ O \end{pmatrix} = \begin{pmatrix} B_{11} & O \\ O & B_{22} \end{pmatrix} \begin{pmatrix} A_{11} \\ A_{21} \end{pmatrix} = \begin{pmatrix} B_{11} A_{11} \\ B_{22} A_{21} \end{pmatrix},
\]

we have \( Z(BA) = (B_{22} A_{21})(B_{11} A_{11})^{-1} = B_{22} Z(A) B_{11}^{-1} \). Picking arbitrary matrices \( B_{11} \in K_1, B_{22} \in K_2 \), the \((n-p) \times p \) \((n \geq 2p)\) matrix \( Z(A) \) can be “diagonalized”, namely

\[
Z(BA) = B_{22} Z(A) B_{11}^{-1} = \begin{pmatrix} \alpha_1 & \cdots & O \\ \vdots & \ddots & \vdots \\ O & \cdots & \alpha_p \end{pmatrix}.
\]
Now we use the fact that the $p \times p$ matrix $Z^\dagger Z$ is “self-adjoint” and positive definite\footnote{since $v^\dagger Z^\dagger Zv = (Zv)^\dagger Zv = \sum_{i=1}^{p} |(Zv)_i|^2 > 0$, for $v \in \mathbb{F}^p \setminus 0$.} and so, setting $\alpha_i =: -\tan \theta_i$,

\[
(Z(BA))^\dagger Z(BA) = \begin{pmatrix}
\alpha_1^2 & 0 & \cdots & 0 \\
0 & \ddots & \ddots & \vdots \\
0 & \cdots & \alpha_p^2 & 0 \\
O & \cdots & O & O
\end{pmatrix} = \begin{pmatrix}
\tan^2 \theta_1 & O \\
O & \ddots & \ddots & \vdots \\
O & \cdots & \tan^2 \theta_p & 0 \\
O & \cdots & O & O
\end{pmatrix}
\]
(1.0.17)

Therefore, by the left action of $K$ on $G$, the $p$-plane in $\mathbb{F}^n$ can be represented by the span of the columns of the following matrix, which by taking the linear combination of the columns, each multiplied with $\cos \theta_i$ reads

\[
\begin{pmatrix}
I_p \\
-\tan \theta_1 \\
\vdots \\
O
\end{pmatrix} = \text{span} \begin{pmatrix}
\cos \theta_1 & O \\
O & \ddots & -\sin \theta_p \\
-\sin \theta_1 & \cdots & O \\
O & \cdots & -\sin \theta_p
\end{pmatrix},
\]
and so

\[
K \setminus G/K \simeq \left\{ \text{span} \begin{pmatrix}
\Re e^{i\Theta_p} \\
-\Im e^{i\Theta_p} \\
O_{n-2p,p}
\end{pmatrix}, \ 0 \leq \theta_i \leq \pi \right\}.
\]

Similarly, $Z \in T_{Id}Gr(p, \mathbb{F}^n)$ can be diagonalized by the action (1.0.16) on $T_{Id}Gr(p, \mathbb{F}^n)$; i.e.,

\[
Z \mapsto B_{22}ZB_{11}^{-1} = \begin{pmatrix}
z_1 & O \\
O & \ddots \\
O & \cdots & z_p
\end{pmatrix} \in a^+, \quad (1.0.18)
\]

where the $z_i$ are linearized versions of the $\tan \theta_i$’s and where $a^+$ is a fixed Weyl chamber in the Cartan of $\mathfrak{p}$.
By the Weyl integration formula, the measure induced on $K \backslash G/K$, via the Haar measure and the embedding $G/K \hookrightarrow G$, is given by (in the compact case) (see Helgason [22], p. 188)

$$d\mu(H) = \prod_{\alpha \in \Sigma^+} |\sin \alpha(iH)|^{m_\alpha} dH, \quad H \in a_*,$$

where $g = \mathfrak{t} + \mathfrak{p}$, with compact real form $u = \mathfrak{t} + \mathfrak{p}_*$, $\mathfrak{p}_* = i\mathfrak{p}$ and with $a_*$ a maximal abelian subspace of $\mathfrak{p}$, where $\Sigma^+$ is the set of roots having positive values on the fixed Weyl chamber $a^+$ of $a_*$ and where the root space $g_\alpha$ has dimension $m_\alpha$ for any restricted root $\alpha$:

$$g_\alpha := \{X \in g \mid [H, X] = \alpha(H)X, \text{ for all } H \in a\},$$

and we also have the induced measure on $T_{Id}(K \backslash G/K)$ ([22], p. 195)

$$d\mu = \prod_{\alpha \in \Sigma^+} \alpha(H)^{m_\alpha} dH, \quad H \in a^+.$$

The roots and multiplicities $m_\alpha$ are as follows ($n \geq 2p$):

| $Gr(p, \mathbb{C}^n)$ | $Gr(p, \mathbb{R}^n)$ | $Gr(p, \mathbb{H}^n)$ | $\alpha \in \Sigma^+$ |
|----------------------|----------------------|----------------------|----------------------|
| 2                    | 1                    | 4                    | $i(\varepsilon_j + \varepsilon_k)$ |
| 2                    | 1                    | 4                    | $i(\varepsilon_j - \varepsilon_k)$ |
| 1                    | 0                    | 3                    | $2i\varepsilon_\ell$          |
| $2(n - 2p)$          | $n - 2p$             | $4(n - 2p)$          | $i\varepsilon_\ell$          |

with $1 \leq j < k \leq p$, $1 \leq \ell \leq p$, yielding Table 1, upon setting

$$H = i(\theta_1, \ldots, \theta_p), 0 \leq \theta_i \leq \pi,$$

(1.0.19)

and we check table 1 for $Gr(p, \mathbb{H}^n)$. Setting $k = n - 2p$ and $y_j = \cos 2\theta_j$, which is very natural in view of (1.0.20).
for $K\backslash G/K$:

\[
d\mu = \prod_{\alpha \in \Sigma^+} |\sin \alpha(iH)|^{m_\alpha} dH
\]

\[
= \prod_{1 \leq j < k \leq p} |\sin(\theta_j - \theta_k)\sin(\theta_j + \theta_k)|^4 \prod_{j=1}^p |\sin 2\theta_j|^3 |\sin^2 \theta_j|^{2(n-2p)} d\theta_j
\]

\[
= \prod_{1 \leq j < k \leq p} \frac{1}{2}(\cos 2\theta_j - \cos 2\theta_k)^4 \prod_{j=1}^p (1 - \cos^2 2\theta_j) \left(\frac{1 - \cos 2\theta_j}{2}\right)^{2k/2} \frac{1}{2} d\cos 2\theta_j
\]

\[
= 2^{-p(2p+k-1)} \Delta(y)^4 \prod_{j=1}^p (1 + y_j)(1 - y_j)^{2k+1} dy_j
\]

(1.0.20)

for $T_{id}K\backslash G/K$:

\[
d\mu = \prod_{\alpha \in \Sigma^+} \alpha(H)^{m_\alpha} dH
\]

\[
= c \prod_{1 \leq j < k \leq p} (v_j - v_k)^4 (v_j + v_k)^4 \prod_{j=1}^p v_j^{4k+3} dv_j
\]

\[
= c \prod_{1 \leq j < k \leq p} (v_j^2 - v_k^2)^4 \prod_{j=1}^p (v_j^2)^{2k+1} \frac{1}{2} dv_j^2
\]

\[
= c 2^{-p} \Delta^4(u) \prod_{j=1}^p u_j^{2k+1} du_j,
\]

(1.0.21)

setting $u_j = v_j^2$, where in the above we made the identification (1.0.6) of Proposition 1.3 and put $Z$ in the normal form (1.0.16), so that $Z^\dagger Z = \text{diag}(u_1, \ldots, u_p)$.

To describe $G/K$ in a second way, remember $g = \mathfrak{k} + \mathfrak{p}$, with $\mathfrak{k}, \mathfrak{p}$ the ± eigenspaces of a lie algebra involution $\sigma$. The latter lifts to the group as an involution $\sigma$, which commutes with inversion, i.e., $(g^\sigma)^{-1} = (g^{-1})^\sigma$. Use $\sigma$ to define the following embedding

\[
i : G \hookrightarrow G : g \mapsto \iota(g) = g(g^\sigma)^{-1},
\]

(1.0.22)

which induces a natural injective map

\[
i : G/K \hookrightarrow G : g \mapsto \iota(g).
\]
Indeed, \( \iota(g_1) = \iota(g_2) \) is equivalent to \( g_1(g_1^{-1}) = g_2(g_2^{-1}) \), which amounts to \( g_2 g_1 = (g_2 g_1)^{-1} = (g_2^{-1} g_1^{-1}) \), meaning that \( g_2^{-1} g_1 \in K \).

From the polar decomposition \( G = KAK \), we have that every \( g \in G \) can be decomposed into

\[
g = k_1^{-1}A(\theta)k_2^{-1}, \quad k_1, k_2 \in K, \quad A(\theta) \in A,
\]

with \( \theta \) the torus coordinates, such that \( A(\theta)A(\theta') = A(\theta + \theta') \), \( A(0) = I \) and \( A^{-1}(\theta) = A(-\theta) \). Since \( A = \exp a \), with \( a \in i \mathfrak{p} \), we have \( A(\theta)^\sigma = A^{-1}(\theta) \).

From (1.0.22) the torus \( A \) embeds into \( S \) as follows:

\[
\iota(A(\theta)) = A(\theta)(A(\theta)^\sigma)^{-1} = A(\theta)A(\theta) = A(2\theta).
\]

Moreover the polar decomposition \( g = k_1^{-1}A(\theta)k_2^{-1} \) yields conjugation in \( S \) by \( k_1 \):

\[
\iota(g) = k_1^{-1}(k_1 g k_2 k_2^{-1} g^{-1} k_1^{-1}) k_1
= k_1^{-1}(k_1 g k_2 k_2^{-1} g^{-1}) k_1
= k_1^{-1} k_1^{-1} (k_1 g k_2) k_1
= k_1^{-1} \iota(A(\theta)) k_1
= k_1^{-1} A(2\theta) k_1
\]

(1.0.23)

Specializing to the Grassmannian case, we have the involution

\[
\sigma : G \to G : g \mapsto g^\sigma := I_{p,n-p} g I_{p,n-p}
\]

with

\[
K = \{ \text{fixed points in } G \text{ of the involution } \sigma \}
= \{ g \in G \text{ such that } g I_{p,n-p} = I_{p,n-p} g \}.
\]

Setting

\[
\Theta_p := \text{diag}(\theta_1, \ldots, \theta_p),
\]

we have that a maximal abelian subspace of the Lie algebra is given by

\[
a = \left\{ a(\theta) = \begin{pmatrix} O_p & \Theta_p & O \\ -\Theta_p & O_p & O \\ O & I_{n-2p} \end{pmatrix}, \quad 0 \leq \theta < 2\pi \right\}
\]

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and by exponentiation, we find the torus

\[ A = \left\{ A(\theta) = \begin{pmatrix} \Re e^{i\Theta_p} & \Im e^{i\Theta_p} \\ -\Im e^{i\Theta_p} & \Re e^{i\Theta_p} \end{pmatrix} O, \quad 0 \leq \theta < 2\pi \right\} \in G. \]

The spectrum of \( A(\theta) \) is easily seen to be

\[ (e^{i\theta_1}, \ldots, e^{i\theta_p}, e^{-i\theta_1}, \ldots, e^{-i\theta_p}, 1, \ldots, 1) \quad (1.0.24) \]

To connect with the previous description, given \( g \in G \), we pick \( k_1, k_2 \) such that \( k_1gk_2 = A(\theta) \) and by the previous discussions, we have

\[ Z^\dagger(g)Z(g) = Z^\dagger(k_1gk_2)Z(k_1gk_2) = Z^\dagger(A(\theta))Z(A(\theta)) = \text{diag} (\tan^2 \theta_1, \ldots, \tan^2 \theta_p). \]

The embedding \( \mathfrak{A} \hookrightarrow Gr(p, \mathbb{F}^n) \) is then given by

\[ \mathfrak{A} \hookrightarrow Gr(p, \mathbb{F}^n) : A(\theta) \mapsto A(\theta) \begin{pmatrix} I_p \\ O_{n-p,p} \end{pmatrix} = \begin{pmatrix} \Re e^{i\Theta_p} \\ -\Im e^{i\Theta_p} \end{pmatrix}, \]

and so the \( \theta_i \)'s in the two discussions are identical.

\[ \text{Proof of Theorems 1.1, 1.2 and 1.3:} \quad \text{In order to compute integral (1.0.3), recall in the above description of } K\backslash G/K, \]

\[ H = i(\theta_1, \ldots, \theta_p). \]

Since the integrand is invariant under the left action of \( K \) on \( G \), which induces on \( Z \) the map \( Z \to B_{22}ZB_{11}^{-1} \), use (1.0.17), from which it follows that, upon using \( \tan^2 \theta_i = \frac{1 - \cos 2\theta_i}{1 + \cos 2\theta_i} = \frac{1 - y_i}{1 + y_i} \), and \( \cos^2 \theta_i = (1 + y_i)/2 \), we have

\[ Z^\dagger Z = \text{diag}(\tan^2 \theta_1, \ldots, \tan^2 \theta_p) = \text{diag} \left( \frac{1 - y_1}{1 + y_1}, \ldots, \frac{1 - y_p}{1 + y_p} \right). \]
and
\[(I + Z^\dagger Z)^{-1} = \begin{pmatrix} \cos^2 \theta_1 & \cdots & O \\ \vdots & \ddots & \vdots \\ O & \cdots & \cos^2 \theta_p \end{pmatrix} = \frac{1}{2} \text{diag}(1 + y_1, \ldots, 1 + y_p).\]

Hence, setting \(z_i = \cos^2 \theta_i\) in the last identity below, we have \(y_i = 2z_i - 1\), using (1.0.11) and picking an appropriate normalizing constant \(c\):
\[
\int_{Gr(p, \mathbb{F}^n)} e^{x \text{Tr}(I + Z^\dagger Z)^{-1}} \det(Z^\dagger Z)^{-\beta(q-p)} d\mu(Z)
\]
\[
= c \int_{[-1,1]^p} e^{\frac{x}{2} \sum_{i=1}^p (1+y_i) |\Delta_p(y)|^{2\beta} \prod_{i=1}^p \left(1 - \frac{y_i}{1+y_i}\right)^{\beta(p-q)} (1 - y_i)^{\beta(n-2p)} (1 - y_i^2)^{\beta-1} dy_i
\]
\[
= c \int_{[0,1]^p} e^{x \sum_{i=1}^p y_i |\Delta_p(z)|^{2\beta} \prod_{i=1}^p (1 - z_i)^{\beta(n-p-q+1) - 1} z_i^{\beta(q-p+1) - 1} dz_i,
\]
upon setting \(y_i = 2z_i - 1\).

Finally, using the Weyl integration formula (1.0.21) on the tangent space and identifying the \((z_1, \ldots, z_p)\) in (1.0.18) with a point in \(a^+\), and setting \(u_i = z_i^2 \geq 0, 1 \leq i \leq p\), we find
\[
\int_{Z \in T_{id} Gr(p, \mathbb{F}^n)\text{ with spectrum } (Z^\dagger Z)\in E} e^{-x \text{Tr} Z^\dagger Z} \det(Z^\dagger Z)^{\beta p} d\mu(Z),
\]
\[
= \int_{E_p} |\Delta_p(u)|^{2\beta} \prod_{i=1}^p e^{-x u_i} u_i^{\beta(n-p+1)-1} du_i,
\]
establishing Theorem 1.3.

Finally, to prove Theorem 1.2, notice that, according to (1.0.23), matrices \(M\) in \(S\) can be diagonalized to matrices \(A(2\theta)\), with spectrum
\[
(e^{2i\theta_1}, \ldots, e^{2i\theta_p}, e^{-2i\theta_1}, \ldots, e^{-2i\theta_p}; 1, \ldots, 1).
\]

So \(M\) decomposes into \(M = M_0 \otimes M_1\), with \(M_1\) being the \(n-2p\)-dimensional eigenspace corresponding to the eigenvalue 1 and so
\[
\prod_{i=1}^p \frac{1 - y_k}{1+y_k} = \prod_{i=1}^p \frac{1 - \cos 2\theta_k}{1 + \cos 2\theta_k} = \prod_{i=1}^p \frac{(1 - e^{2i\theta_k})(1 - e^{-2i\theta_k})}{(1 + e^{2i\theta_k})(1 + e^{-2i\theta_k})} = \det \frac{I - M_0}{I + M_0}.
\]
\[\blacksquare\]
2 Jack polynomials

2.1 Young diagrams and Schur polynomials

Standard references to this subject are MacDonald, Sagan, Stanley, Stanton and White [34, 41, 44, 45]. To set the notation, we remind the reader of a few basic facts.

- A partition of \( n = |\lambda| := \lambda_1 + \ldots + \lambda_\ell \) (with \( n = |\lambda| \) called the weight) is represented by a Young diagram \( \lambda \) with \( \lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_\ell \geq 0 \). A dual Young diagram \( \lambda^\top = (\lambda_1^\top \geq \lambda_2^\top \geq \ldots) \) is the diagram obtained by flipping the diagram \( \lambda \) about its diagonal.

- A semi-standard Young tableau of shape \( \lambda \) is an array of positive integers \( a_{i,j} \) placed at \((i,j)\) in the Young diagram \( \lambda \), which are non-decreasing from left to right and strictly increasing from top to bottom.

- A standard Young tableau of shape \( \lambda \) is an array of integers \( 1, \ldots, n \) placed in the Young diagram, which are strictly increasing from left to right and from top to bottom.

- The Schur polynomial \( s_\lambda \) associated with a Young diagram \( \lambda \) is a symmetric function in the variables \( x_1, x_2, \ldots \), (finite or infinite), where \( n = |\lambda| \) and defined by (for notation \( f^\lambda \), see the next point)

\[
s_\lambda(x_1, x_2, \ldots) = \sum_{\{a_{i,j}\} \text{ semi-standard tableaux } \lambda} \prod_{(i,j) \in \lambda} x_{a_{i,j}} = f^\lambda x_1 \ldots x_n + \ldots
\]

- The hook length of the \( i, j \)th box is defined by \( h^\lambda_{ij} := \lambda_i + \lambda_j^\top - i - j + 1 \). Also define

\[
h^\lambda := \prod_{(i,j) \in \lambda} h^\lambda_{ij} = \frac{\prod_{m=1}^{m}(m + \lambda_i - i)!}{\Delta_m(m + \lambda_1 - 1, \ldots, m + \lambda_m - m)}, \quad \text{for } m \geq \lambda_1^\top
\]

- The number of standard Young tableaux of a given shape \( \lambda = (\lambda_1 \geq\)

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... \geq \lambda_m) is given by

\begin{align*}
\lambda^\lambda & = \# \{\text{standard tableaux of shape } \lambda\} \\
& = |\lambda|! \cdot s_\lambda(x) \bigg|_{\sum_k x_k = 1, \sum_k x_k = 0 \text{ for } i \geq 2} \\
& = \text{coefficient of } x_1 x_2 \ldots x_n \text{ in } s_\lambda(x) \\
& = |\lambda|! \cdot \frac{1}{h^\lambda} = |\lambda|! \cdot \det \left( \frac{1}{(\lambda_i - i + j)!} \right) \\
& = |\lambda|! \cdot \frac{\Delta_m(m + \lambda_1 - 1, \ldots, m + \lambda_m - m)}{\prod_{i=1}^m (m + \lambda_i - i)!}, \quad \text{for } m \geq \lambda^\top_1. \\
\end{align*}

\begin{equation}
(2.1.2)
\end{equation}

\bullet \text{ The number of semi-standard Young tableaux of a given shape } \lambda, \text{ with numbers 1 to } k \text{ for } k \geq 1:

\begin{align*}
\# \bigg\{ \text{semi-standard tableaux of shape } \lambda \bigg\} \\
& \text{filled with numbers from 1 to } k \\
& = \prod_{(i,j) \in \lambda} \frac{1}{h_{i,j}^\lambda} \cdot \frac{\Delta_k(k + \lambda_1 - 1, \ldots, k + \lambda_k - k)}{\prod_{i=1}^{k-1} i!}, \quad \text{when } k \geq \lambda^\top_1, \\
& = \left\{ \begin{array}{ll}
0, & \text{when } k < \lambda^\top_1, \\
\end{array} \right.
\end{align*}

\begin{equation}
(2.1.3)
\end{equation}

using the fact that

\begin{equation}
(2.1.4)
\end{equation}
• **Robinson-Schensted-Knuth correspondence**: Given

\[ S_n = \text{group of permutations of } \{1, \ldots, n\} \]
\[ S^k_n = \{\text{words of length } n \text{ built from the set } \{1, \ldots, k\}\} \]

the following 1-1 correspondences hold:

\[ S_n \rightarrow \left\{ (P, Q), \text{where } P \text{ and } Q \text{ are two standard Young tableaux of same shape } \lambda, \text{ with } |\lambda| = n \text{ and taken from } \{1, \ldots, n\} \right\} \]

\[ S^k_n \rightarrow \left\{ (P, Q), \text{where } P \text{ and } Q \text{ have same shape } \lambda, \text{ with } |\lambda| = n \right\} \]

- *P* is semi-standard, filled with numbers from 1 to \(k\), and
- *Q* is standard, filled with numbers from 1 to \(n\)

It follows that for given \(n\) and \(k\), we have

\[ \sum_{\lambda \text{ with } |\lambda| = n} (f^\lambda)^2 = n! \quad (2.1.5) \]

\[ \sum_{\lambda \text{ with } |\lambda| = n} f^\lambda \cdot s_\lambda(1^k) = k^n \quad (2.1.6) \]

• **Increasing and decreasing sequences**

According to Greene [20], given a word \(\pi \in S^k_n\), mapped, via the RSK correspondence, into \((P, Q)\) of shape \(\lambda = (\lambda_1, \ldots, \lambda_\ell)\), then for any \(k\),

\[ \lambda_1 + \ldots + \lambda_k = \begin{cases} \text{length of the longest weakly } \\ \text{k-increasing subsequence} \end{cases} \]

\[ \lambda^\top_1 + \ldots + \lambda^\top_k = \begin{cases} \text{length of the longest strictly } \\ \text{k-decreasing subsequence} \end{cases} \]

### 2.2 Some useful formulae on hook length

Remembering the notation (0.0.8), we have the following statement:
Lemma 2.1 Given a partition \( \lambda \supseteq \mu := (n-p)^p \) with \( \lambda^\top_1 = p, 0 \leq p < n \), then

\[
\begin{align*}
    h^\lambda &= h^\mu h^{\lambda\setminus\mu}(n)_{\lambda\setminus\mu} \quad (p)_{\lambda\setminus\mu}, \quad (2.2.1) \\
    h^\mu &= \prod_{i=1}^{p} (n-i)! \\
    s_{\mu}(1^p) &= 1, \quad (2.2.2) \\
    \frac{s_{\lambda}(1^p)}{h^\lambda} &= \frac{1}{h^\mu (h^{\lambda\setminus\mu})^2} \frac{(p)_{\lambda\setminus\mu}}{(n)_{\lambda\setminus\mu}}. \quad (2.2.3)
\end{align*}
\]

Proof: Setting \( \kappa = \lambda \setminus \mu \)

\[
\begin{align*}
    \frac{h^\lambda}{h^{\lambda\setminus\mu}} &= \prod_{(i,j) \in \mu} h^\lambda_{(i,j)} \\
    &= \prod_{i=1}^{p} \prod_{j=1}^{n-p} (n+1+\kappa_i - i - j) \\
    &= \prod_{i=1}^{p} (n+\kappa_i - i) \ldots (n+1 + \kappa_i - i - n + p) \\
    &= h^\mu \prod_{i=1}^{p} \frac{(p+1+\kappa_i - i) \ldots (n+1 + \kappa_i - i)}{(p+1 - i) \ldots (n - i)} \\
    &= h^\mu \prod_{i=1}^{p} \frac{(n-i+1)_{\kappa_i}}{(p-i+1)_{\kappa_i}}
\end{align*}
\]

\(^6\)Here \( (a)_\lambda := (a)_{\lambda}^{(1)} = \prod_i (a+1 - i)_{\lambda_i}. \)
\[ h^\mu = [(n - 1)(n - 2) \ldots p][(n - 2) \ldots (p - 1)] \ldots [(n - p) \ldots 1] \]

\[ = \frac{(n - 1)!(n - 2)! \ldots (n - p)!}{(p - 1)!(p - 2)! \ldots 1!} \]

\[ = \frac{\prod_{i=1}^{p} (n - i)!}{\prod_{i=1}^{p-1} i!}. \quad (2.2.4) \]

Then
\[ s_\mu(1^p) = \frac{\prod_{(i,j) \in \mu} (j - i + p)}{h^\mu} = \frac{\prod_{i=1}^{p} (n - i)!}{h^\mu \prod_{i=1}^{p-1} i!} = 1, \quad \text{using (2.1.4)}. \]

Finally, using in the second identity below formula (2.1.4) and in the fourth identity \( \prod_{i=1}^{p} (n - i + \kappa_i)! = \prod_{i=1}^{p} (n - i)! \prod_{i=1}^{p} (n - i + 1) \), one computes
\[ s_\lambda(1^p) = \frac{\prod_{(i,j) \in \lambda} (j - i + p)}{h^\lambda} = \frac{\prod_{i=1}^{p} (p + \lambda_i - i)!}{(h^\lambda)^2 \prod_{i=1}^{p-1} i!} = \frac{1}{(h^\lambda)^2} \prod_{i=1}^{p-1} i!, \]

\[ = \frac{1}{(h^\lambda)^2} \prod_{i=1}^{p-1} i! \prod_{i=1}^{p} (n - i + \kappa_i)!, \quad \text{since } \lambda_i = n - p + \kappa_i \]

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\[
\begin{align*}
= & \frac{1}{(h^\lambda)^2} \prod_{i=1}^{p} \frac{(n-i)!}{\prod_{i=1}^{p} (n-i+1)_{\kappa_i}} \\
= & \frac{h^\mu}{(h^\mu h^{\lambda \setminus \mu})^2} \prod_{i=1}^{p} \frac{((p-i+1)(\lambda \setminus \mu)_i)^2}{(n-i+1)(\lambda \setminus \mu)_i}, \text{ using (2.2.1) and (2.2.2)}
\end{align*}
\]
ending the proof of Lemma 2.1.

2.3 Jack polynomials

Define symmetric polynomials

\[
p_\lambda(x_1, x_2, \ldots) := p_{\lambda_1} p_{\lambda_2} \cdots = \sum_{i} x_{i}^{\lambda_i} \sum_{i} x_{i}^{\lambda_2} \cdots
\]

\[
m_\lambda(x_1, x_2, \ldots) := \sum_{\text{permutations of } x_1, \ldots, x_n} x_{1}^{\lambda_1} \cdots x_{n}^{\lambda_n},
\]

and the dominance ordering between partitions:

\[
\mu \leq \lambda \text{ means } : \sum_{1}^{\ell} \mu_i \leq \sum_{1}^{\ell} \lambda_i, \text{ for all } \ell.
\]

Given that \( \lambda \) has \( m_i = m_i(\lambda) \) parts equal to \( i \), define the inner-product \( \langle , \rangle \) on the vector space of all symmetric functions of bounded degree (to be explained)

\[
\langle p_\lambda, p_\mu \rangle = \delta_{\lambda \mu} (1^{m_1}2^{m_2} \cdots)m_1!m_2! \cdots \alpha^{\lambda^\top}.
\]

Jack polynomials are the unique symmetric functions \( J_\lambda^{(\alpha)} \) satisfying

(i) \( \langle J_\lambda^{(\alpha)}, J_\mu^{(\alpha)} \rangle = 0 \), if \( \lambda \neq \mu \),

(ii) \( J_\lambda^{(\alpha)} = \sum_{\mu \leq \lambda} v_{\lambda \mu}(\alpha)m_\mu \),

(iii) If \(|\lambda| = n\), then

\[
J_\lambda^{(\alpha)} = n!x_1 \cdots x_n + \ldots
\]
It follows that
\[ v_{\lambda\lambda} = \prod_{(i,j) \in \lambda} (\lambda_j^\top - i + 1 + \alpha(\lambda_i - j)). \]

**Special cases:**

*Jack polynomials* for \( \alpha = 1 \) are proportional to Schur polynomials, namely
\[ J^{(1)}_\lambda = h^\lambda s_\lambda. \]

*Zonal polynomials* are given by
\[ Z^{(\beta)}_\lambda = J^{(1/\beta)}_\lambda, \quad \text{with } \beta = 1, 1/2, 2. \]

They have the remarkable property that for \( G = O(n), U(n) \) or \( U(n, \mathbb{H}) := \{ g \mid gg^\top = I \} \),
\[ \int_G J^{(\alpha)}_\lambda(\sigma \tau k^{-1}) dk = \frac{J^{(\alpha)}_\lambda(\sigma) J^{(\alpha)}_\lambda(\tau)}{J^{(\alpha)}_\lambda(1^n)}, \quad (2.3.1) \]
for all
\[ \sigma, \tau \in \Sigma = \begin{cases} \{ \text{real symmetric matrices} \} & \text{for } \alpha = 2 \\ \{ \text{Hermitian matrices} \} & \text{for } \alpha = 1 \\ \{ \text{quaternionic matrices, with } \sigma = \bar{\sigma}^\top \} & \text{for } \alpha = 1/2. \end{cases} \]

The function \( J^{(\alpha)}_\lambda(\tau) \) is a symmetric function of the (real) spectrum of \( \tau \).

**Orthogonality:**
\[ \langle J^{(\alpha)}_\lambda, J^{(\alpha)}_\mu \rangle = \delta_{\mu \lambda} j^{(\alpha)}_\lambda, \quad (2.3.2) \]
where
\[ j^{(\alpha)}_\lambda = \prod_{(i,j) \in \lambda} (\lambda_j^\top - i + \alpha(\lambda_i - j + 1)) (\lambda_j^\top - i + 1 + \alpha(\lambda_i - j)) \]
\[ = \begin{cases} (h^\lambda)^2 & \text{for } \alpha = 1 \\ k^{2\lambda} & \text{for } \alpha = 2 \\ h^{2\lambda^\top} / 2^{2|\lambda|} & \text{for } \alpha = 1/2. \end{cases} \quad (2.3.3) \]
Special values:

For arbitrary $n$, we have

$$J_{\alpha}^{(\alpha)}(1^n) = \prod_{(i,j) \in \lambda} (n - (i - 1) + \alpha(j - 1)),$$

where $1^n = (1, \ldots, 1, 0, 0, \ldots)$

$$= \alpha^{\vert \lambda \vert} \prod_{(i,j) \in \lambda} \left( \frac{1}{\alpha} (n - i + 1) + j - 1 \right) = \alpha^{\vert \lambda \vert} \left( \frac{n}{\alpha} \right)^{(\alpha)}$$

$$= \left\{ \begin{array}{ll} \alpha^{\vert \lambda \vert} \prod_{i=1}^{m} \frac{\Gamma\left(\frac{1}{\alpha}(n-i+1)+\lambda_i\right)}{\Gamma\left(\frac{1}{\alpha}(n-i+1)\right)} > 0, & \text{for all } m \geq \lambda_i^T, \text{ if } n \geq \lambda_i^T, \\ 0, & \text{if } n < \lambda_i^T \end{array} \right. \ \ (2.3.4)$$

and so, for $\alpha = 1$,

$$s_{\lambda}(1^n) = \frac{1}{h^\lambda} J_{\lambda}^{1}(1^n)$$

$$= \left\{ \begin{array}{ll} \frac{1}{h^\lambda} \prod_{i=1}^{m} \frac{(n-i+\lambda_i)!}{(n-i)!} \geq 0, & \text{for all } m \geq \lambda_i^T, \text{ if } n \geq \lambda_i^T, \\ 0, & \text{if } n < \lambda_i^T. \end{array} \right. \ \ (2.3.5)$$

The last identity in (2.3.4) is obtained by taking the product over the $i$th row of $\lambda$ and using \((x+n)(x+n-1)\ldots x = \frac{\Gamma(x+n+1)}{\Gamma(x)}\). When the Gamma functions blow up, the formulas must be understood as limits. Also

$$J_{\lambda}(x)\vert_{\sum_i x_i^{\lambda_i} = \delta_{1,u}} = u^{|\lambda|}$$

$$s_{\lambda}(x)\vert_{\sum_i x_i^{\lambda_i} = \delta_{1,u}} = \frac{u^{|\lambda|}}{h^\lambda}.$$

Expansion of $(x_1 + x_2 + \ldots)^n$:

$$\frac{(x_1 + x_2 + \ldots)^n}{n!} = \alpha^n \sum_{|\lambda|=n} J_{\lambda}^{(\alpha)}(x). \ \ (2.3.6)$$
Then also
\[
(x_1 + x_2 + \ldots)^n = \sum_{|\lambda|=n} C_{\lambda}^{(a)}(x), \quad \text{with } C_{\lambda}^{(a)}(x) = \frac{|\lambda|!}{j_{\lambda}^{(a)}} J_{\lambda}^{(a)}(x). \quad (2.3.7)
\]

**Cauchy identity:**
\[
\prod_{i,j \geq 1} (1 - x_i y_j)^{-1/\alpha} = \sum_{\lambda \in \mathcal{Y}} \frac{J_{\lambda}^{(a)}(x) J_{\lambda}^{(a)}(y)}{j_{\lambda}^{(a)}}; \quad (2.3.8)
\]
in particular, for \( \alpha = 1 \)
\[
\prod_{i,j \geq 1} (1 - x_i y_j)^{-1} = \sum_{\lambda \in \mathcal{Y}} s_{\lambda}(x) s_{\lambda}(y). \quad (2.3.9)
\]

**Hypergeometric functions**

Generalized hypergeometric functions \(2F_1^{(a)}\) are defined by:
\[
2F_1^{(a)}(p, q; n; x) := \sum_{\kappa \in \mathcal{Y}} \frac{(p)_\kappa^{(a)}(q)_\kappa^{(a)}}{(n)_\kappa^{(a)}} \alpha_{|\kappa|} J_{\kappa}^{(a)}(x). \quad (2.3.10)
\]

For \( \alpha = 1 \), using \( J_{\kappa}^{(1)} = h\kappa s_{\kappa} \) and \( j_{\kappa}^{(1)} = (h\kappa)^2 \), we have
\[
2F_1^{(1)}(p, q; n; x) := \sum_{\kappa \in \mathcal{Y}} \frac{(p)_\kappa^{(1)}(q)_\kappa^{(1)}}{(n)_\kappa^{(1)}} s_{\kappa}(x) \frac{h\kappa}{(h\kappa)^2}, \quad (2.3.11)
\]
and so, upon restriction,
\[
2F_1^{(1)}(p, q; n; x) \bigg|_{\sum_i x_i^2 = \delta_{11}} = \sum_{\kappa \in \mathcal{Y}} \frac{(p)_\kappa^{(1)}(q)_\kappa^{(1)}}{(n)_\kappa^{(1)}} \alpha_{|\kappa|} \frac{h\kappa}{(h\kappa)^2}. \quad (2.3.12)
\]

**Generalized Selberg formula:**

Kaneko [30] computes the following integrals, subjected to the condition that \( a, b > \beta(p - 1) \) (see also MacDonald [34] and Kadell [29]):
\[
\int_{[0,1]^p} J^{(1/\beta)}_\lambda(x) \Delta_p(x) \left| \Delta_p(x) \right|^{2\beta} \prod_{i=1}^p (1 - x_i)^{a-\beta(p-1)-1} x_i^{b-\beta(p-1)-1} dx_i
\]
\[
= J^{(1/\beta)}_\lambda(1^p) \prod_{i=1}^p \frac{\Gamma(i\beta + 1) \Gamma(a + \beta(1 - i)) \Gamma(\lambda_i + b + \beta(1 - i))}{\Gamma(\beta + 1) \Gamma(\lambda_i + a + b + \beta(1 - i))}. \tag{2.3.13}
\]

Setting \(u = (u_1, \ldots, u_m)\), we have the following representation in terms of the hypergeometric function:
\[
\int_{[0,1]^p} \prod_{1 \leq i \leq p} (1 - x_i u_k)^{-\beta} \left| \Delta_p(x) \right|^{2\beta} \prod_{i=1}^p (1 - x_i)^{a-\beta(p-1)-1} x_i^{b-\beta(p-1)-1} dx_i
\]
\[
= \, _2F_1^{(1/\beta)}(\beta p, b, a + b; u) \prod_{i=1}^p \frac{\Gamma(i\beta + 1) \Gamma(a + \beta(1 - i)) \Gamma(b + \beta(1 - i))}{\Gamma(\beta + 1) \Gamma(a + b + \beta(1 - i))}. \tag{2.3.14}
\]

### 3 Probability measures on partitions

#### 3.1 Probability measure on the set \(\mathbb{Y}\) of all partitions

In view of formula (2.3.8), define the (not necessarily positive) probability measure on the space \(\mathbb{Y}\) of Young diagrams, depending on \(x, y\) and \(\alpha\): (see [11, 12, 30, 31])
\[
P(\lambda) := \frac{J^{(\alpha)}_\lambda(x) J^{(\alpha)}_\lambda(y)}{J^{(\alpha)}_\lambda \prod_{i,j} (1 - x_i y_j)^{-1/\alpha}}, \quad \lambda \in \mathbb{Y}. \tag{3.1.1}
\]

In particular, evaluating \(P(\lambda)\) along the locus
\[
\mathcal{L} = \left\{ \begin{array}{l}
\text{for } x = (x_1, x_2, \ldots) \text{ such that } \sum_{\ell} x_{\ell}^i = \delta_i u, \text{ we have } J^{(\alpha)}_\lambda(x) = u^{\lambda} \\
\text{for } y = (1, 1, \ldots, 1, 0, 0, \ldots) = 1^p, \text{ we have } J^{(\alpha)}_\lambda(1^p) \text{ as in (2.3.4)}
\end{array} \right\} \tag{3.1.2}
\]

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and using
\[ \prod_{i,j \geq 1} (1 - x_i y_j)^{-1} = \prod_{j=1}^p \prod_{i \geq 1} (1 - x_i y_j)^{-1} \bigg|_{y_j=1} = \prod_{i \geq 1} (1 - x_i)^{-p} = e^{pu}, \]
we obtain the genuine \((\geq 0)\) probability measure for \(u > 0\) on the space \(\mathbb{Y}\), depending on \(u \in \mathbb{R}\) and the integer \(p > 0\),
\[ P_{u,p}(\lambda) := P(\lambda) \bigg|_{\mathcal{L}} = e^{-pu/|\lambda|} \frac{j_{p}^{(\alpha)}(1^p)}{j_{\lambda}^{(\alpha)}} = e^{-pu/\alpha} \frac{(pu/\alpha)^{|\lambda|}}{|\lambda|!} P^{\ell,p}(\lambda), \text{ with } \ell := |\lambda|, \tag{3.1.3} \]
with
\[
(\text{support } P_{u,p}) \subseteq \mathbb{Y}^{(p)} := \{ \lambda \in \mathbb{Y} \text{ such that } \lambda^\top 1 \leq p \}.
\]
Probability (3.1.3) can be viewed as a Poissonized probability of
\[ P^{\ell,p}(\lambda) = \frac{j_{\lambda}^{(\alpha)}(1)p!}{j_{p}^{(\alpha)}(p/\alpha)^\ell}, \text{ for } \lambda \in \mathbb{Y}^{(p)}_\ell. \tag{3.1.4} \]
Probability \(P_{u,p}(\lambda)\) is \(\geq 0\), because, from (2.3.4), \(J_{\lambda}^{(\alpha)}(1^p) > 0\), for \(p \geq \lambda_1^\top\) and = 0 otherwise.
In particular, setting \(\alpha = 1\) and using \(J_{\lambda}^{(1)} = h^\lambda s_\lambda\) and \(j_{\lambda}^{(1)} = (h^\lambda)^2\), (3.1.3) leads to
\[ P_{u,p}(\lambda) \bigg|_{\alpha = 1} = \frac{u^{|\lambda|} J_{\lambda}^{(1)}(1^p)}{e^{pu} j_{p}^{(1)}} = \frac{u^{|\lambda|} s_\lambda(1^p)}{e^{pu} h^\lambda} = e^{-pu/\alpha} \frac{(pu/\alpha)^{|\lambda|}}{|\lambda|!} P^{\ell,p}(\lambda) \bigg|_{\alpha = 1}, \quad \lambda \in \mathbb{Y}^{(p)}, \tag{3.1.5} \]
where by (2.1.2)
\[ P^{\ell,p}(\lambda) \bigg|_{\alpha = 1} = \frac{f_{\lambda}^\lambda s_\lambda(1^p)}{p^{|\lambda|}}, \quad \lambda \in \mathbb{Y}^{(p)}_\ell. \tag{3.1.6} \]
Probability (3.1.6) will be considered next.
\[ \prod_{i \geq 1} (1 - x_iz)^{-1} = \exp \left( \sum_{\ell=1}^\infty \frac{x^\ell}{\ell} \sum_{i \geq 1} x_i^\ell \right). \]
3.2 Probability measures on the set \( \mathcal{Y}_\ell \) of partitions of \( \ell \) and random words

From (3.1.6), setting in this section \( P_{\ell,p} := \left. P_{\ell,p} \right|_{\alpha=1} \),

\[
P_{\ell,p}(\lambda) = \frac{f^\lambda s_\lambda(1^p)}{p^{\left|\lambda\right|}}, \quad \text{for } \lambda \in \mathcal{Y}_\ell^{(p)}
\]

\[
= \frac{\ell!}{p^\ell} \Delta_p(h)^2 \prod_{i=1}^{p-1} \frac{1}{i! \prod_{j=1}^p h_j!}, \quad \text{by (2.1.2) and (2.1.3)},
\]

\[
= \text{probability on Young diagrams } \lambda \in \mathcal{Y}_\ell^{(p)} \text{ coming from the uniform distribution on } S^\ell_p, \text{ via the RSK correspondence},
\]

(3.2.1)

where \( h := (h_1, ..., h_p) \) with \( h_i := p + \lambda_i - i \), and where

\[
S^\ell_p := \{ \text{words of length } \ell, \text{ built from an alphabet } \{1, ..., p\}\},
\]

with \( \left| S^\ell_p \right| = p^\ell \). As already pointed out, this is a probability, firstly because of the connection with the word problem, secondly because of (3.1.5). As already pointed out, this probability was considered in \([10, 11, 12, 50, 31]\) and also in the context of random words, by Tracy and Widom \((48)\).

**Proposition 3.1** Given rectangular Young diagrams \( \mu = (n-p)^p \supseteq \mu' = (n-q)^p \), with \( p \leq q < n \), and \( \ell \geq p(n-p) \), the expectation equals

\[
E^{\ell,p} \left( I_{\lambda \supseteq \mu} (\lambda) \frac{h^\lambda \setminus \mu'}{h^\lambda \setminus \mu} \right)
\]

\[
= \frac{\ell!}{p^\ell} \prod_{i=1}^p \frac{(q-i)!}{(n-i)!} \sum_{\kappa \in \mathcal{Y}_{\ell-p(n-p)}} \frac{1}{(h^\kappa)^2} \prod_{i=1}^p \frac{(p-i+1)\kappa_i (q-i+1)\kappa_i}{(n-i+1)\kappa_i}. \quad (3.2.2)
\]

**Proof:** From (2.2.1) with \( \lambda \mapsto \lambda \setminus \mu' \), (so \( n \mapsto q, \ p \mapsto p \)) we have, upon setting \( \kappa = \lambda \setminus \mu \),

\[
\prod_{(i,j) \in \mu \setminus \mu'} h^\lambda_{(i,j)} \frac{h^\lambda \setminus \mu'}{h^\lambda \setminus \mu} = h^\mu \setminus \mu' \prod_{i=1}^p \frac{(q-i+1)\kappa_i}{(p-i+1)\kappa_i}. \quad (3.2.3)
\]
Then combining (3.2.3) and (2.2.3),

\[ \frac{s_\lambda(1^p)}{h^\lambda h^{\lambda\mu'}} = \frac{h^{\mu\mu'}}{h^\mu} \frac{1}{(h^{\lambda\mu})^2} \prod_{i=1}^p \frac{(q-i+1)(p-i+1)_{\kappa_i}}{(n-i+1)_{\kappa_i}}. \]

In particular, setting \( \lambda = \mu \), and using (2.2.2),

\[ \frac{s_\lambda(1^p)}{h^\lambda h^{\lambda\mu}} \bigg|_{\lambda=\mu} = \frac{h^{\mu\mu'}}{h^\mu} = \prod_{i=1}^p \frac{(q-i)!}{(n-i)!}, \]

and therefore

\[ \frac{s_\lambda(1^p)}{h^\lambda h^{\lambda\mu}} = \left( \frac{s_\lambda(1^p)}{h^\lambda h^{\lambda\mu}} \right) \bigg|_{\lambda=\mu} \frac{1}{(h^{\lambda\mu})^2} \prod_{i=1}^p \frac{(p-i+1)_{\kappa_i}(q-i+1)_{\kappa_i}}{(n-i+1)_{\kappa_i}}, \] (3.2.4)

Then, taking the expectation \( E^{\ell,p} \) with regard to the probability measure \( P^{\ell,p} \), defined in (3.2.1),

\[ E^{\ell,p}\left(I_{\{\lambda \supseteq \mu\}}(\lambda) \prod_{n-q<j \leq n-p} h_{(i,j)}^\lambda \right) = \sum_{\lambda \in \mathcal{Y}_\ell \atop \lambda \supseteq \mu} \frac{f^\lambda}{p^\ell} \frac{s_\lambda(1^p)}{h^\lambda h^{\lambda\mu}}, \]

\[ = \frac{\ell!}{p^\ell} \sum_{\lambda \in \mathcal{Y}_\ell \atop \lambda \supseteq \mu} \frac{s_\lambda(1^p)}{h^\lambda h^{\lambda\mu}}, \text{ using } f^\lambda = \frac{\lambda!}{h^\lambda} \text{ as in (2.1.2)}, \]

\[ = \frac{\ell!}{p^\ell} \prod_{i=1}^p \frac{(q-i)!}{(n-i)!} \sum_{\lambda \in \mathcal{Y}_\ell \atop \lambda \supseteq \mu} \frac{1}{(h^{\lambda\mu})^2} \prod_{i=1}^p \frac{(p-i+1)_{(\lambda\mu)^i}(q-i+1)_{(\lambda\mu)^i}}{(n-i+1)_{(\lambda\mu)^i}}, \]

using (3.2.4)

\[ = \frac{\ell!}{p^\ell} \prod_{i=1}^p \frac{(q-i)!}{(n-i)!} \sum_{\kappa \in \mathcal{Y}_{\ell-p(n-p)} \atop \kappa_i \leq p} \frac{1}{(h^{\kappa})^2} \prod_{i=1}^p \frac{(p-i+1)_{\kappa_i}(q-i+1)_{\kappa_i}}{(n-i+1)_{\kappa_i}}. \]

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4 Expressing integrals as mathematical expectation on partitions and on random words

4.1 Expressing an integral on $Gr(p, \mathbb{F}^n)$ as a mathematical expectation on partitions

Remembering the probability (3.1.3)

$$P_{x,p}(\lambda) = e^{-\beta px} \frac{\beta^{\lambda|J_{\lambda}(1/p)}(1)^p}{J_{\lambda}(1/\beta)} \lambda \in \mathcal{Y}, \quad (4.1.1)$$
onumber

on partitions $\mathcal{Y}$, with support on $\lambda^\top_1 \leq p$, the following statement holds:

**Theorem 4.1** For fixed $p \leq q \leq n/2$, the following holds ($\beta = 1/2, 1, 2$)

$$c^{-1} \int_{Gr(p, \mathbb{F}^n)} e^{x \text{Tr}(I+Z\check{Z})^{-1}} \det(Z\check{Z})^{-\beta(q-p)} d\mu(Z) \quad (4.1.2)$$

$$= 2 F_1^{(1/\beta)}(\beta p; \beta q; \beta n; y) \bigg| \sum_i y_i = \frac{2}{\beta} \delta_{ti} \quad (4.1.3)$$

$$= e^{\beta px} E_{x,p} \left( \frac{1}{\beta |\lambda| J_{\lambda}(1/\beta)(1^n)} \right) \quad (4.1.4)$$

where $c := c^{(\beta)}_{n,q,p}$ is as in (0.0.20).

**Proof:** For a symmetric function $f(z_1, \ldots, z_p)$, define the integral, depending on $\beta$,

$$\langle f \rangle_{\beta} := \int_{[0,1]^p} f(z_1, \ldots, z_p) |\Delta(z)|^{2\beta} \prod_{i=1}^p z_i^{\beta(q-p+1)-1}(1-z_i)^{\beta(n-p-q+1)-1} dz_i. \quad (4.1.5)$$

Kaneko’s formula (2.3.13) will be used for $a = \beta(n - q)$ and $b = \beta q$ in the sequence of identities below; the inequalities $p \leq q \leq n/2$ imply $q - p + 1, n - p - q + 1 \geq 1$, and so the integral (4.1.5) above makes sense, and we first apply (1.0.3) in the following sequence of identities:
\[
\int_{Gr(p, \mathbb{F}^n)} e^{x \Tr (I + Z^\dagger Z)^{-1}} \det(Z^\dagger Z)^{-\beta(p-q)} d\mu(Z)
\]
\[
= \left\langle \frac{p}{\beta} \prod_{i=1}^{p} e^{xz_i} \right\rangle
\]
\[
= \sum_{\ell=0}^{\infty} x^\ell \left\langle \frac{1}{\beta} \sum_{|\lambda|=\ell} \frac{J^{(1/\beta)}_\lambda(z)}{J^{(1/\beta)}_\lambda(1^p)} \right\rangle_{\beta}, \text{ using (2.3.6)}
\]
\[
= \frac{\prod_{i=1}^{p} \Gamma(i\beta + 1) \Gamma(\beta(n-q-i+1))}{\Gamma(\beta + 1)}
\sum_{\ell=0}^{\infty} \frac{x^\ell}{\beta^\ell} \sum_{|\lambda|=\ell} \frac{J^{(1/\beta)}_\lambda(1^p)}{J^{(1/\beta)}_\lambda(1^n)} \prod_{i=1}^{p} \frac{\Gamma(\lambda_i + \beta(q-i+1))}{\Gamma(\lambda_i + \beta(n-i+1))},
\]
using Kaneko’s formula (2.3.13),
\[
= \frac{\prod_{i=1}^{p} \Gamma(i\beta + 1) \Gamma(\beta(n-q-i+1))}{\Gamma(\beta + 1)} e^{\beta px}
\sum_{\lambda \in \mathbb{Y}} \frac{1}{\beta|\lambda|} P_{x,p}(\lambda) \prod_{i=1}^{p} \frac{\Gamma(\lambda_i + \beta(q-i+1))}{\Gamma(\lambda_i + \beta(n-i+1))}, \text{ using the definition (4.1.1) of probability, with support on } \mathbb{Z}(p),
\]
\[
= c^{(\beta)}_{n,q,p} e^{\beta px} \sum_{\lambda \in \mathbb{Y}} P_{x,p}(\lambda) \frac{J^{(1/\beta)}_\lambda(1^q)}{\beta|\lambda| J^{(1/\beta)}_\lambda(1^n)}, \text{ using formula (2.3.4) and the value (0.0.20) of } c^{(\beta)}_{n,q,p},
\]
yielding (4.1.6).

Finally, looking at the expression to the right of \( * \), we find, using (3.1.2) and (2.3.4), to be precise,

\[
J^{(1/\beta)}_\lambda(y) \Bigg|_{\sum_{i} y_i = \frac{1}{\beta} \delta_{1^p}} = \left( \frac{x}{\beta} \right)^{|\lambda|} \text{ and } J^{(1/\beta)}_\lambda(1^p) = \left( \frac{1}{\beta} \right)^{|\lambda|} (p\beta)^{1/\beta},
\]
the following

\[
\left\langle \prod_{i=1}^{p} e^{x_{zi}} \right\rangle_{\beta} = c_{n,q,p}^{(\beta)} \sum_{\lambda \in \mathcal{Y}} \frac{(\beta p)_{\lambda}(\beta q)_{\lambda}}{(\beta n)_{\lambda}} \frac{1}{\beta^{\lambda}} \frac{J_{\lambda}^{(1/\beta)}(y)}{f_{\lambda}^{(1/\beta)}(y)} \Bigg|_{\sum_{i} y_{i} = \delta_{11}},
\]

by (2.3.4),

\[
= c_{n,q,p}^{(\beta)} 2F_{1}^{(1/\beta)}(\beta p, \beta q; \beta n; y) \Bigg|_{\sum_{i} y_{i} = \delta_{11}},
\]

by (2.3.10),

thus ending the proof of Theorem 4.1.

\textbf{Remark:} Identity (4.1.3) is also an immediate consequence of Kaneko’s formula.

### 4.2 Expressing an integral over \( Gr(p, \mathbb{C}^{n}) \) as a mathematical expectation on partitions and random words

We now specialize the previous section to \( \mathbb{F} = \mathbb{C} \). For fixed integer \( p \geq 1 \) and \( \beta = 1 \), recall the probability (3.1.5), with support in \( \mathcal{Y}^{(p)} \),

\[
P_{x,p}(\lambda) = \frac{x^{\lambda}}{e^{px} h^{\lambda}} = e^{-px} (xp)^{\lambda} |\lambda|! P^{\ell,p}(\lambda), \quad \lambda \in \mathcal{Y}^{(p)},
\]

and the probability (3.1.6) on \( \mathcal{Y}^{(p)} \) coming from the uniform distribution on \( S^{p}_{\ell} \), via the RSK correspondence,

\[
P^{\ell,p}(\lambda) = \frac{f^{\lambda}_{p} s_{\lambda}^{(1/p)}}{p^{\lambda}}, \quad \lambda \in \mathcal{Y}^{(p)}.
\]

For integer \( 0 \leq p \leq n/2 \), consider the fixed rectangular Young diagram \( \mu = (n - p)^{p} \).
Theorem 4.2 For fixed \( p \leq q \leq n/2 \),

\[
\int_{Gr(p,C^n)} e^{x\text{Tr}(I+Z^1Z)^{-1}} \det(Z^1Z)^{-(q-p)} d\mu(Z)
\]

\[
= \left\{ \begin{array}{l}
\tilde{c}_{n,q,p} \frac{1}{x^{(n-p)p}} e^{px} E_{x,p} \left( I_{\{\lambda \geq \mu\}}(\lambda) \prod_{(i,j) \in \lambda \atop n-q < j \leq n-p} h^\lambda_{(i,j)} \right) \\
\tilde{c}_{n,q,p} \frac{1}{x^{(n-p)p}} \sum_{\ell \geq p(n-p)} \frac{(px)^\ell}{\ell!} E_{\ell,p} \left( I_{\{\lambda \geq \mu\}}(\lambda) \prod_{(i,j) \in \lambda \atop n-q < j \leq n-p} h^\lambda_{(i,j)} \right), \\
\tilde{c}_{n,q,p}^{(1)} \sum_{r \geq 0} x^r \sum_{\kappa \subseteq \gamma_r \atop \kappa_I \subseteq p} \frac{1}{(h^\kappa)^2} \frac{(p)_\kappa (q)_\kappa}{(n)_\kappa} \\
\tilde{c}_{n,q,p}^{(1)} 2F_1^{(1)}(p,q;n;y) \bigg|_{\sum_\ell y_\ell = \delta_{1,x}} \\
\end{array} \right.
\]

(4.2.3)

In particular, for \( p = q \), the integral above has two different formulations, as a probability or as a generating function of probabilities,

\[
\int_{Gr(p,C^n)} e^{x\text{Tr}(I+Z^1Z)^{-1}} d\mu(Z)
\]

\[
= \left\{ \begin{array}{l}
\tilde{c}_{n,p,p} \frac{1}{x^{(n-p)p}} e^{px} P_{x,p} (\lambda \geq \mu) \\
\tilde{c}_{n,p,p} \frac{1}{x^{(n-p)p}} \sum_{\ell \geq p(n-p)} \frac{(px)^\ell}{\ell!} P_{\ell,p} (\lambda \geq \mu), \\
\tilde{c}_{n,p,p} \frac{1}{x^{(n-p)p}} \sum_{\ell \geq p(n-p)} \frac{(px)^\ell}{\ell!} P_{\ell,p} \left( \pi \in S^p_{\ell} \bigg| \begin{array}{l} d_1(\pi) = p \text{ and} \\
d_{p-1}(\pi) \leq \ell - n + p \end{array} \right), \\
\tilde{c}_{n,p,p}^{(1)} \sum_{r \geq 0} x^r \sum_{\kappa \subseteq \gamma_r \atop \kappa_I \subseteq p} \frac{1}{(h^\kappa)^2} \frac{(p)_\kappa (q)_\kappa}{(n)_\kappa} \\
\tilde{c}_{n,p,p}^{(1)} 2F_1^{(1)}(p,p;n;y) \bigg|_{\sum_\ell y_\ell = \delta_{1,x}} \\
\end{array} \right.
\]
where $P_{\ell,p}$ denotes the uniform distribution on the set $S_\ell^p$ of words of length $\ell$ from an alphabet $1, \ldots, p$, and

\[ d_1(\pi) = \text{length of longest strictly decreasing subsequence} \]

\[ i_k(\pi) = \text{length of the longest union of } k \text{ disjoint weakly increasing subsequences.} \]

**Remark:** The constant $c_{n,q,p}^{(1)}$ in (4.2.2) and (4.2.3) is the same as $c_{n,q,p}^{(\beta)}$ for $\beta = 1$ (see (0.0.20)) and $\tilde{c}_{n,q,p}$ is a new constant:

\[
(4.2.4)
\]

\[
c_{n,q,p}^{(1)} = \prod_{j=1}^{p} \left( \frac{n-j}{n-q-j, q-j, j} \right)^{-1} \tilde{c}_{n,q,p} = \prod_{i=1}^{p} i! (n-q-i)!.
\]

(4.2.5)

From the fourth expression of (4.2.2), it follows readily that, near $x = 0$,

\[
\frac{1}{(c_{n,q,p})^{-1}} \int_{Gr(p,C^n)} e^{x \text{Tr}(I+Z^\dagger Z)^{-1}} \det(Z^\dagger Z)^{-(q-p)} d\mu(Z)
\]

\[
= 1 + \frac{pq}{n} x + \frac{pq}{4n} \left( \frac{(p+1)(q+1)}{n+1} + \frac{(p-1)(q-1)}{n-1} \right) x^2 + \ldots
\]

(4.2.6)

**Proof:** From the last identity in (2.1.2) with $m = p$, it follows that, since $\lambda^\top_1 \leq p$,

\[
h^\lambda+q-p)_{\lambda} = \prod_{i=1}^{p} \frac{(q+\lambda_i-i)!}{(n+\lambda_i-i)!}.
\]

(4.2.7)

For a partition $\lambda$ such that $\lambda^\top_1 \leq p$ and for $\lambda' = \lambda + kp$, with arbitrary integer $k \geq 0$, we have $s_\lambda(1^p) = s_{\lambda'}(1^p)$, using the last identity (2.1.3). Using these facts, we have, continuing from (4.1.6),

\[
\int_{Gr(p,C^n)} e^{x \text{Tr}(I+Z^\dagger Z)^{-1}} \det(Z^\dagger Z)^{p-q} d\mu(Z)
\]

\[
= \int_{[0,1]^p} \Delta(z)^{2} \prod_{i=1}^{p} e^{z_i} \sum_{\lambda \in \mathbb{Y}} x^{\lambda} S_\lambda(1^p) h^\lambda \prod_{i=1}^{p} \frac{(q+\lambda_i-i)!}{(n+\lambda_i-i)!}
\]

by the 4th identity of (4.1.6) and

\[
J^{(1)}_\lambda = h^\lambda s_\lambda \text{ and } j^{(1)}_\lambda = (h^\lambda)^2
\]

\[
= \tilde{c}_{n,q,p} \sum_{\lambda \in \mathbb{Y}} x^{\lambda} S_\lambda \frac{h^\lambda+q-p)_{\lambda}}{h^\lambda h^\lambda+(n-p)^p}, \text{ using } (4.2.7)
\]

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\[
\begin{align*}
&= \tilde{c}_{n,q,p} \sum_{\lambda' \in \mathcal{V}} x^{\lambda'} \left( \frac{s_{\lambda'}(1^p)}{h^{\lambda'-\frac{n-q}{p}}} \right), \quad \text{setting } \lambda' = \lambda + (n - q)^p, \\
&= \tilde{c}_{n,q,p} \sum_{\lambda' \in \mathcal{V}} x^{\lambda'} \left( \frac{h^{\lambda'-\frac{n-q}{p}}}{h^{\lambda'-\frac{n-p}{p}}} \right), \quad \text{using } s_{\lambda}(1^p) = s_{\lambda'}(1^p) \\
&= \tilde{c}_{n,q,p} \sum_{\lambda' \in \mathcal{V}} x^{\lambda'} \left( \frac{P_{x,p}(\lambda')}{h^{\lambda'-\frac{n-p}{p}}} \right), \quad \text{using } P_{x,p} \text{ defined in (4.2.1)}, \\
&= \tilde{c}_{n,q,p} \sum_{\lambda' \in \mathcal{V}} x^{\lambda'} \left( \frac{E_{x,p}(\lambda')}{h^{\lambda'-\frac{n-p}{p}}} \right). \\
\end{align*}
\]

The last equality in (4.2.3) follows from (4.1.3), while the second to the last one follows from (2.3.12).

Finally, to prove the second formula on the right hand side of (4.2.3), start with equality \( \ast \) in (4.2.8), omitting \( \tilde{c}_{n,q,p} \), and replacing \( \lambda' \) by \( \lambda \),

\[
\begin{align*}
&= \frac{1}{x^p(n-p)} \sum_{\lambda \in \mathcal{V}} x^{\lambda} \left( \frac{s_{\lambda}(1^p) \cdot h^{\lambda'-\frac{n-q}{p}}}{h^{\lambda'-\frac{n-p}{p}}} \right), \quad \text{for fixed } \mu = (n - p)^p \\
&= \frac{1}{x^p(n-p)} \sum_{\lambda \in \mathcal{V}} x^{\lambda} \left( \frac{(px)^{\lambda |} \cdot |\lambda|! \cdot s_{\lambda}(1^p)}{p^{\lambda}} \cdot \prod_{(i,j) \in \lambda} h_{(i,j)}^\lambda \right), \quad \text{using (2.1.2),} \\
&= \frac{1}{x^p(n-p)} \sum_{\ell \geq p(n-p)} \left( \frac{(px)^\ell}{\ell!} \sum_{\lambda \in \mathcal{V}} \left( \frac{\lambda}{\ell^\ell} \cdot \prod_{(i,j) \in \lambda} h_{(i,j)}^\lambda \right) \right), \quad \text{using (2.1.2),} \\
&= \frac{1}{x^p(n-p)} \sum_{\ell \geq p(n-p)} \left( \frac{(px)^\ell}{\ell!} \cdot \left( \prod_{(i,j) \in \lambda} h_{(i,j)}^\lambda \right) \right).
\end{align*}
\]

In particular, setting \( q = p \), the latter equals

\[
\begin{align*}
&= \frac{1}{x^p(n-p)} \sum_{\ell \geq p(n-p)} \left( \frac{(px)^\ell}{\ell!} \cdot P_{x,p}(\lambda \geq \mu) \right) \\
&= \frac{1}{x^p(n-p)} \sum_{\ell \geq p(n-p)} \left( \frac{(px)^\ell}{\ell!} \cdot P_{x,p}(\lambda \geq \mu, \lambda_1^\top = p) \right) \\
&= \frac{1}{x^p(n-p)} \sum_{\ell \geq p(n-p)} \left( \frac{(px)^\ell}{\ell!} \cdot P_{x,p}(\pi \in S_{\ell}^p \mid d_1(\pi) = p, \ i_{p-1}(\pi) \leq \ell - n + p) \right).
\end{align*}
\]
To see the last two equalities, one proceeds as follows. From $\lambda \supseteq \mu = (n - p)^p$ and $P^{\ell,p}(\lambda) = 0$ for $\lambda^\top_1 > p$, it follows that $\lambda^\top_1 = p$ and by Greene’s theorem (see Sagan [41], p. 110), $d_1(\pi) = \lambda^\top_1 = p$. Since also by Greene, $i_k(\pi) = \sum_1^k \lambda_i$, and, in particular $i_p(\pi) = \sum_1^p \lambda_i = \ell$, we have

$$\lambda_p = i_p(\pi) - i_{p-1}(\pi) = \ell - i_{p-1}(\pi).$$

From $\lambda \supseteq \mu = (n - p)^p$, it also follows that $\lambda_p \geq n - p$, and thus $i_{p-1}(\pi) \leq \ell - n + p$. Conversely, if $d_1(\pi) = \lambda^\top_1 = p$, and $i_{p-1}(\pi) \leq \ell - n + p$, then $\sum_1^{p-1} \lambda_i = i_{p-1}(\pi) \leq \sum_1^p \lambda_i - n + p$; hence $\lambda_p \geq n - p$, and so $\lambda \supseteq \mu$.

5 Testing Statistical Independence of Gaussian Populations

The statistical facts, used in this paper and summarized in this section, are due to James [25] and Constantine [13]; see also Muirhead [35].

5.1 The Wishart distribution

Let the $p \times n$ matrix $X$, with $n \geq p$ and $n$ identically distributed independent columns, have the normal distribution

$$(\det 2^p \pi \Sigma)^{-n/2} e^{-\frac{1}{2} \text{Tr} \Sigma^{-1} (X - M)(X - M)^\top}. \tag{5.1.1}$$

Then the $p \times p$ matrix $S = XX^\top$ has the non-central Wishart distribution with $n$ degrees of freedom, with $p \times p$ covariance matrix $\Sigma$ and non-centrality matrix $\Omega = \frac{1}{2} MM^\top \Sigma^{-1}$, namely

$$\Gamma_p(n/2)^{-1} \left( \det 2^p \pi \Sigma \right)^{-n/2} e^{-\frac{1}{2} \text{Tr}(\Omega + \frac{1}{2} \Sigma^{-1} X X^\top)} \left( \det S \right)^{(n-p)}/2 \text{Hypergeometric}_2 F_1 \left( n/2, \frac{1}{2}, n/2, \frac{1}{2} \Sigma^{-1} \Omega S \right),$$

where $S > 0$, where $\Gamma_m$ is the multivariate Gamma function and where\(^8\) for the definition of $C_\lambda$, see (2.3.7))

$$\text{Hypergeometric}_2 F_1 \left( n/2, \frac{1}{2}, n/2, \frac{1}{2} \Sigma^{-1} \Omega S \right) = \sum_{\lambda \in \mathcal{Y}} C_\lambda \left( \frac{1}{2} \Sigma^{-1} \Omega S \right) \left( \frac{n}{2} \right)_{\lambda} \left( \frac{1}{2} \right)_{\lambda} J_\lambda^{(2)} \left( \frac{1}{2} \Sigma^{-1} \Omega S \right).$$

\(^8\)Here $(a)_{\lambda} := \prod_{i=1}^{\lambda}(a + \beta(1-i))_{\lambda_i}$ for $\beta = 1/2$, with $(x)_n := x(x+1)\ldots(x+n-1)$, $x_0 = 1$. 

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When $M = 0$, we find the (central) Wishart distribution $W_p(n, \Sigma)$, with $p \leq n$, for the $p \times p$ matrix $S = XX^\top$:

$$
\Gamma_p(n/2)^{-1}(\det 2\Sigma)^{-n/2}e^{-\frac{1}{2}n\Sigma^{-1}S}dS = \prod_{1 \leq i \leq j \leq p} dS_{ij}.
$$

(5.1.2)

### 5.2 The canonical correlation coefficients

In testing the statistical independence of two Gaussian populations, one needs to know the distribution of canonical correlation coefficients. To set up the problem, consider $p + q$ normally distributed random variables $(X_1, ..., X_p)^\top$ and $(Y_1, ..., Y_q)^\top$ ($p \leq q$) with mean zero and covariance matrix

$$
\begin{bmatrix}
X \\
Y
\end{bmatrix} = \Sigma = \begin{pmatrix}
\Sigma_{11} & \Sigma_{12} \\
\Sigma_{21} & \Sigma_{22}
\end{pmatrix}
\begin{smallmatrix}
\uparrow p \\
\uparrow q
\end{smallmatrix}
$$

The method proposed by Hotelling [24] is to find linear transformations $U = L_1X$ and $V = L_2Y$ of $X$ and $Y$ having the property that the correlation between the first components $U_1$ and $V_1$ of the vectors $U$ and $V$ is maximal subject to the condition that $\text{Var} U_1 = \text{Var} V_1 = 1$; moreover, one requires the second components $U_2$ and $V_2$ to have maximal correlation subjected to

$$
\begin{cases}
(i) & \text{Var} U_2 = \text{Var} V_2 = 1 \\
(ii) & U_2 \text{ and } V_2 \text{ are uncorrelated with both } U_1 \text{ and } V_1,
\end{cases}
$$

etc ...

Then there exist $O_p \in O(p)$, $O_q \in O(q)$ such that

$$
\Sigma_1^{-1/2} \Sigma_{12} \Sigma_2^{-1/2} = O_p^\top P O_q
$$

where $P$ has the following form:
\[ P = \begin{pmatrix} \rho_1 & & & & & & & & & & & & & & q \\ \vdots & O & & & & & & & & & & & & & & & \\ \rho_k & & \rho_{k+1} & & & & & & & & & & & & & & & \\ O & & \cdots & & & & & & & & & & & & & & & & \\ & & & & & & & & & & \rho_p \\ \end{pmatrix} \]

\[ p, \ k = \text{rank } \Sigma_{12}, \]

\[ 1 \geq \rho_1 \geq \rho_2 \geq \ldots \geq \rho_k > 0, \ \rho_{k+1} = \ldots = \rho_p = 0 \quad (\text{canonical correlation coefficients}), \]

\[ \rho_i \text{ are solutions (} \geq 0) \text{ of } \det(\Sigma_{11}^{-1}\Sigma_{12}\Sigma_{22}^{-1}\Sigma_{12}^{\top} - \rho^2 I) = 0.\]

Then the covariance matrix of the vectors

\[ U = L_1 X := O_p \Sigma_{11}^{-1/2} X \quad \text{and} \quad V = L_2 Y := O_q \Sigma_{22}^{-1/2} Y \]

has the canonical form (\( \det(\Sigma_{\text{can}}) = \prod_i (1 - \rho_i^2) \))

\[ \text{cov} \left( \begin{pmatrix} U \\ V \end{pmatrix} \right) = \Sigma_{\text{can}} = \begin{pmatrix} I_p & P \\ P^\top & I_q \end{pmatrix}, \]

with

\[ \text{spectrum } \Sigma_{\text{can}} = 1, \ldots, 1, 1 - \rho_1, 1 + \rho_1, \ldots, 1 - \rho_p, 1 + \rho_p \]

and inverse

\[ \Sigma_{\text{can}}^{-1} = \frac{1}{\prod_i (1 - \rho_i^2)^2} \begin{pmatrix} I_p & -P \\ -P^\top & I_q \end{pmatrix}. \]

### 5.3 Distribution of the sample canonical correlations

From here on, we may take \( \Sigma = \Sigma_{\text{can}}. \) The \( n (n \geq p + q) \) independent samples \((x_{11}, \ldots, x_{1p}, y_{11}, \ldots, y_{1q})^\top, \ldots, (x_{n1}, \ldots, x_{np}, y_{n1}, \ldots, y_{nq})^\top, \) arising from observing \( \begin{pmatrix} X \\ Y \end{pmatrix} \) lead to a matrix \( \begin{pmatrix} x \\ y \end{pmatrix} \) of size \( (p + q, n) \), having the normal distribution \( \mathcal{B} \) (p. 79 and p. 539)
\[(2\pi)^{-n(p+q)/2} (\det \Sigma)^{-n/2} \exp \left\{ -\frac{1}{2} \text{Tr} \begin{pmatrix} x^T y^T \end{pmatrix} \begin{pmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{12} & \Sigma_{22} \end{pmatrix}^{-1} \begin{pmatrix} x \\ y \end{pmatrix} \right\} \]

The conditional distribution of a \( p \times n \) matrix \( x \) given the \( q \times n \) matrix \( y \) is also normal:

\[ (\det 2\pi \Omega)^{-n/2} e^{-\frac{1}{2} \text{Tr} \Omega^{-1} (x-Py)(x-Py)^T} \] (5.3.1)

with

\[
\begin{align*}
\Omega &= \Sigma_{11} - \Sigma_{12} \Sigma_{22}^{-1} \Sigma_{21} = \text{diag}(1 - \rho_1^2, \ldots, 1 - \rho_p^2) \\
P &= \Sigma_{12} \Sigma_{22}^{-1}.
\end{align*}
\]

Then the maximum likelihood estimates \( r_i \) of the \( \rho_i \) satisfy the determinantal equation

\[ \det(S_{11}^{-1} S_{12} S_{22}^{-1} S_{12}^T - r^2 I) = 0, \] (5.3.2)

corresponding to

\[ S = \begin{pmatrix} S_{11} & S_{12} \\ S_{12}^T & S_{22} \end{pmatrix} = \begin{pmatrix} xx^T & xy^T \\ yx^T & yy^T \end{pmatrix}, \]

where \( S_{ij} \) are the associated submatrices of the sample covariance matrix \( S \).

**Remark:** The \( r_i \) can also be viewed as \( r_i = \cos \theta_i \), where the \( \theta_1, \ldots, \theta_p \) are the critical angles between two planes in \( \mathbb{R}^n \):

(i) a \( p \)-dimensional plane = span \( \{(x_{11}, \ldots, x_{n1}), \ldots, (x_{1p}, \ldots, x_{np})\} \)

(ii) a \( q \)-dimensional plane = span \( \{(y_{11}, \ldots, y_{n1})^T, \ldots, (y_{1q}, \ldots, y_{nq})\} \).

As we shall see, \( z_i = r_i^2 = \cos^2 \theta_i \) are the precise variables \( z_i \) appearing in section 1.

Since the \((q, n)\)-matrix \( y \) has rank \((y) = q \), there exists a matrix \( H_n \in O(n) \) such that \( yH_n = (y_1 \mid O) \); therefore acting on \( x \) with \( H_n \) leads to

\[ yH = (y_1 \mid O) \uparrow q, \quad xH_n = (u \mid v) \downarrow p. \] (5.3.3)
With this in mind,
\[ S_{12} S_{22}^{-1} S_{12}^\top - r^2 S_{11} \]
\[ = x y^\top (y y^\top)^{-1} y x^\top - r^2 x x^\top \]
\[ = x H (y H)^\top (y H (y H)^\top)^{-1} y H (x H)^\top - r^2 (x H) (x H)^\top \]
\[ = (u \mid v) \begin{pmatrix} y_1^\top \\ O \end{pmatrix} \begin{pmatrix} (y_1 \mid O) \\ y_1^\top \end{pmatrix}^{-1} \begin{pmatrix} y_1 \mid O \end{pmatrix} \begin{pmatrix} u^\top \\ v^\top \end{pmatrix} - r^2 (u \mid v) \begin{pmatrix} u^\top \\ v^\top \end{pmatrix} \]
\[ = (u \mid v) \begin{pmatrix} I_q \\ O \\ 0_{n-q} \end{pmatrix} \begin{pmatrix} u^\top \\ v^\top \end{pmatrix} - r^2 (u \mid v) \begin{pmatrix} u^\top \\ v^\top \end{pmatrix} \]
\[ = u u^\top - r^2 (u u^\top + v v^\top), \]
and so the equation (5.3.2) for the \( r_i \) can be rewritten
\[ \text{det}(u u^\top - r^2 (u u^\top + v v^\top)) = 0. \quad (5.3.4) \]

Then setting the forms (5.3.3) of \( x \) and \( y \) in the conditional distribution (5.3.1) of \( x \) given \( y \), one computes the following, setting \( H := H_n \),
\[ \text{Tr} \, \Omega^{-1} (x - P y) (x - P y)^\top \]
\[ = \text{Tr} \, \Omega^{-1} (x H - P y H) (x H - P y H)^\top \]
\[ = \text{Tr} \, \Omega^{-1} ((u \mid v) - P (y_1 \mid O)) ((u \mid v) - P (y_1 \mid O))^\top \]
\[ = \text{Tr} \, \Omega^{-1} (u - P y_1) (u - P y_1)^\top + \text{Tr} \, \Omega^{-1} v v^\top; \quad \Omega = \text{diag}(1 - \rho_1^2, \ldots, 1 - \rho_p^2); \]
this establishes the independence of the normal distributions \( u \) and \( v \), given the matrix \( y \), with
\[ u \equiv N(P y_1, \Omega), \quad v \equiv N(O, \Omega). \quad P = \text{diag}(\rho_1, \ldots, \rho_p). \]

Hence \( u u^\top \) and \( v v^\top \) are conditionally independent and both Wishart distributed; to be precise:

- The \( p \times p \) matrices \( v v^\top \) are Wishart distributed, given \( y \), with \( n - q \) degrees of freedom and covariance \( \Omega \);

- The \( p \times p \) matrices \( u u^\top \) are non-centrally Wishart distributed, given \( y \), with \( q \) degrees of freedom, with covariance \( \Omega \) and with non-centrality matrix
\[ \frac{1}{2} P y_1 y_1^\top P^\top \Omega^{-1}. \]

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The marginal distribution of the \( q \times q \) matrices \( yy^\top \) are Wishart distributed, with \( n \) degrees of freedom and covariance \( I_q \), because the marginal distribution of \( y \) is normal with covariance \( I_q \).

To summarize, given the matrix \( y \), the sample canonical correlation coefficients \( r_1^2 > \ldots > r_p^2 \) are the roots of

\[
\begin{align*}
(r_1^2 > \ldots > r_p^2) &= \text{roots of } \det(xy^\top(yy^\top)^{-1}yx^\top - r^2xx^\top) = 0 \\
&= \text{roots of } \det(uy^\top - r^2(uu^\top + vv^\top)) = 0 \\
&= \text{roots of } \det(uy^\top(uy^\top + vv^\top)^{-1} - r^2I) = 0.
\end{align*}
\]

Then one shows that, knowing \( uu^\top \) and \( vv^\top \) are Wishart and conditionally independent, the conditional distribution of \( r_1^2 > \ldots > r_p^2 \), given the matrix \( y \) is given by

\[
\pi^{p^2/2}c_{n,p,q}e^{-\frac{1}{2}\Tr Pyy^\top P^\top\Omega^{-1}}\Delta_p(r^2)\prod_{1}^{p}(r_i^2)^{(q-p-1)}(1 - r_i^2)^{\frac{1}{2}(n-q-p-1)}.
\]

\[
\sum_{\lambda \in \mathcal{Y}}\left(\frac{(n/2)_\lambda C_{\lambda}(\frac{1}{2}Pyy^\top P^\top\Omega^{-1})}{(q/2)_\lambda C_{\lambda}(I_p)}\right)\frac{1}{|\lambda|!} C_{\lambda}(R^2),
\]

where

\[
R^2 = \text{diag}(r_1^2, \ldots, r_p^2), \quad c_{n,p,q} = \frac{\Gamma_p(n/2)}{\Gamma_p(q/2)\Gamma_p((n - q)/2)\Gamma_p(p/2)}.
\]

By taking the expectation with regard to \( y \) or, what is the same, by integrating over the matrix \( yy^\top \), which is Wishart distributed, we obtain:

**Theorem 5.1** Let \( X_1, \ldots, X_p, Y_1, \ldots, Y_q \ (p \leq q) \) be normally distributed random variables with zero means and covariance matrix \( \Sigma = \left( \begin{array}{cc} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{array} \right) \).

If \( \rho_1^2, \ldots, \rho_p^2 \) are the roots of \( \det(\Sigma_{11}^{-1}\Sigma_{12}\Sigma_{21}\Sigma_{22}^{-1} - \rho^2I) = 0 \), then the maximum likelihood estimates \( r_1^2, \ldots, r_p^2 \) from a sample of size \( n \ (n \geq p + q) \) are given by the roots of

\[
\det(xy^\top(yy^\top)^{-1}yx^\top - r^2xx^\top) = 0.
\]

Setting

\[
Z := \text{diag}(z_1, \ldots, z_p) = \text{diag}(r_1^2, \ldots, r_p^2) \quad \text{and} \quad P^2 := \text{diag}(\rho_1^2, \ldots, \rho_p^2),
\]

\( c_{n,p,q} \) is a different constant from (0.0.20).
the \( z_i = r_i^2 \) have the following density

\[
\pi^{-p^2} c_{n,p,q} \Delta_p(z) \prod_{i=1}^{p} z_i^{(q-p-1)/2} (1 - z_i)^{(n-q-p-1)/2} dz_i
\]

\[
\cdot \left( \prod_{i=1}^{p} (1 - r_i^2)^{n/2} \right) \sum_{\lambda \in \mathcal{Y}} \left( \begin{array}{c} n/2 \\ \lambda \end{array} \right) \frac{C_{\lambda}(Z) C_{\lambda}(P^2)}{C_{\lambda}(1p) |\lambda|!}. \tag{5.3.5}
\]

**Corollary 5.2** If \( r_1^2 = \ldots = r_p^2 = 0 \), then the joint density of the \( z_i = r_i^2 \) is given by the density appearing in the integral of Theorem 1.1, namely

\[
\pi^{p^2/2} c_{n,p,q} \Delta_p(z) \prod_{i=1}^{p} z_i^{(q-p-1)/2} (1 - z_i)^{(n-q-p-1)/2} dz_i. \tag{5.3.6}
\]

**Remark:** As was shown here, the normal distribution over \( \mathbb{R} \) leads to the density (5.3.6) for the \( z_i \), which corresponds to the case \( \beta = 1/2 \) for (1.0.3). Starting with normal distributions over \( \mathbb{C} \) and \( \mathbb{H} \) leads, in a similar way, to integrals (1.0.3) for the cases \( \beta = 1 \) and 2.

### 6 Differential equations for the Grassmannian integrals and the hypergeometric functions

#### 6.1 Differential equations for the Grassmannian integral

Theorem 6.1 shows that the integral over the Grassmannian \( Gr(p, F^n) \) satisfies Painlevé-like differential equations; for \( \beta = 1 \), this equation is the Painlevé V equation with a specific boundary condition (Theorem 6.2).

**Theorem 6.1** The following holds for the integral

\[
I_p(y) = \int_{Gr(p, F^n)} e^{y \text{Tr}(I + Z^\dagger Z)^{-1} \det(Z^\dagger Z)^{-\beta(q-p)}} d\mu(Z) = c' \exp \int_0^x H(y) dy,
\]

\[(6.1.1)\]
where $H(y) = \frac{d}{dy} \log I_p(y)$ satisfies the differential equation ($H' := \frac{dH}{dy}$ and remember $\delta_1^\beta = 1$ for $\beta = 1$ and $= 0$ otherwise):

$$4 \left( y^3 H''' + 6y^3 H' - (1 + \delta_1^\beta)(2y^2 H'' + 4y^2 HH' + yH^2) \right) - yP_0H' + P_1H + P_2$$

$$= \begin{cases} 0, & \text{for } \beta = 1, \\ \frac{3}{16} \frac{p(p-1)}{(p+1)(p+2)} y^3 \frac{I_{p-2}I_{p+2}}{I_p^2}, & \text{for } \beta = 1/2, \\ \frac{3}{16^2} \frac{p}{p+1} y^3 \frac{I_{p-1}I_{p+1}}{I_p^2}, & \text{for } \beta = 2, \end{cases}\quad (6.1.2)$$

where

| $P_0$ | $P_1$ | $P_2$ |
|-------|-------|-------|
| $\beta = 1$ | $4y^2 - 8sy + 4n^2 - 8$ | $4(sy - n^2)$ | $4r(y+n)$ |
| $\beta = 1/2$ | $4y^2 - 4sy + (n+2)^2 - 8$ | $2sy - n(n-2)$ | $r(2y+n-2)$ |
| $\beta = 2$ | $y^2 - 4sy + 4((n-1)^2 - 2)$ | $2sy - 4n(n+1)$ | $r(2y+4(n+1))$ |

in terms of

$r = pq, \quad s = n - 2p - 2q.\quad (6.1.3)$

**Theorem 6.2** For $\beta = 1$, we have

$$\left( c_{n,q,p}^{(1)} \right)^{-1} \int_{Gr(p,\mathbb{C}^n)} e^x \text{Tr}(I+Z^TZ)^{-1} \det(Z^TZ)^{-(q-p)} d\mu(Z)$$

$$= \prod_{0}^{p} \frac{(n-j)!}{(q-j)!} \sum_{\ell \geq p(n-p)} \frac{p^\ell x^{\ell-p(n-p)}}{\ell!} E^{\ell,p} \left( I\{\lambda \geq \mu\}(\lambda) \prod_{n-q \leq \lambda \leq n-p} h^\lambda_{i,j} \right),$$

$$= \sum_{r \geq 0} \sum_{\kappa \in \mathbb{Y}_r} \frac{1}{(h^\kappa)^2} \frac{(p)_\kappa(q)_\kappa}{(n)_\kappa},$$

$$= 2F_1^{(1)}(p, q; n; y) \bigg|_{\sum_i y_i^{i-\delta_1 x}}$$

$$= \exp \int_0^x \frac{u(y) - p(n-p) + py}{y} dy\quad (6.1.4)$$

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where \( u(x) \) is the unique solution to the initial value problem:

\[
\begin{aligned}
x^2 u''' + xu'' + 6xu^2 - 4uu' + 4Qu' - 2Q'u + 2R &= 0 \\
\text{with } u(x) &= p(n-p) - \frac{p(n-q)}{n} x + \ldots + a_{n+1} x^{n+1} + O(x^{n+1}) + \ldots, \text{ near } x = 0.
\end{aligned}
\]

(6.1.5)

with \( a_{n+1} \) specified (see remark). \( Q \) and \( R \) are polynomials in \( x \):

\[
\begin{aligned}
4Q &= -x^2 + 2(n + 2(p - q)) x - (n - 2p)^2 \\
2R &= p(p - q)(x + n - 2p).
\end{aligned}
\]

(6.1.6)

The third order equation (6.1.5) has a first integral, which is second order in \( u \) and quadratic in \( u'' \),

\[
u''^2 + \frac{4}{x^2} \left( (xu'^2 + Qu' + R)u' - (u'^2 + Q'u' + R')u + \frac{1}{2} Q'u'^2 - \frac{p^2(q - p)^2}{4}\right) = 0
\]

(6.1.7)

Remark: Note that the Painlevé equation (6.1.5) admits a solution

\[
u(x) = p(n-p) - \frac{p(n-q)}{n} x + \sum_{i \geq 2} a_i x^i
\]

(6.1.8)

with the \( a_i \) given by the indicial equation

\[i(i - 1 - n)(i - 1 + n)a_i = g_i(a_0, \ldots, a_{i-1}), \quad i = 1, 2, \ldots ,\]

showing the existence of a free parameter at \( i = n+1 \). However the fact that, according to (4.2.3),

\[
u(x) = p(n-p) - px + \frac{d}{dx} \log \sum_{r \geq 0} x^r \sum_{\kappa \in Y_r} \frac{1}{(h^*)^2} \prod_{1 \leq \kappa_i \leq p} \prod_{1}^{p} \frac{(p - i + 1)_{\kappa_i}(q - i + 1)_{\kappa_i}}{(n - i + 1)_{\kappa_i}}
\]

leads to an explicitly known value for \( a_{n+1} \).

Proof of Theorem 6.1: Define

\[
\bar{I}_p(t) := \int_{[-1,1]^p} |\Delta_p(y)|^{2\beta} \prod_{1}^{p} e^{\sum_{t_i y_{i_k}} (1 - y_{i_k})^a (1 + y_{i_k})^b} dy_k.
\]

(6.1.9)
and the locus

\[ \mathcal{L} := \{ t_1 = x \neq 0, \text{ all other } t_i = 0 \}. \] (6.1.10)

Using (1.0.3), and setting \( a = \beta(n - q - p + 1) - 1, b = \beta(q - p + 1) - 1 \) in (6.1.9), the linear change of variables \( y_k := 2z_k - 1 \) leads to:

\[
\left. \tilde{I}_p(t) \right|_{\mathcal{L}} = \int_{[-1,1]^p} |\Delta_p(y)|^{2\beta} \prod_{1}^{p} e^{\gamma y_k (1 - y_k)} \beta^{(n-q-p+1)-1}(1 + y_k) \beta^{(q-p+1)-1} dy_k
\]

\[
\left. = e_p(\beta)e^{-px} \int_{[0,1]^p} |\Delta_p(z)|^{2\beta} \prod_{1}^{p} e^{2\gamma x_k z_k} \beta^{(q-p+1)-1}(1 - z_k) \beta^{(n-q-p+1)-1} dz_k \right.
\]

\[
\left. = e_p(\beta)e^{-px} \int_{Gr(p,F^n)} e^{2\gamma Tr(1+Z^*Z)\beta^{-1}} \det(Z^*Z)^{-\beta(q-p)} d\mu(Z) \right.
\]

\[
\left. = e_p(\beta)e^{-px} I_p(x), \right. \] (6.1.11)

with

\[ e_p(\beta) = 2^{p^{\beta n - \beta p + \beta - 1}}. \] (6.1.12)

According to the appendix, the integral \( \tilde{I}_p(t) \) satisfies the Virasoro constraints (8.0.10), with \( a = \beta(n - q - p + 1) - 1, b = \beta(q - p + 1) - 1, \) and thus

\[ b_0 = a - b = \beta(n - 2q), \quad b_1 = a + b = \beta(n - 2p + 2) - 2, \]

and

\[ \sigma_1 = \beta n, \quad \sigma_2 = \beta(n - 1) + 1. \]

These expressions and their first \( t_1 \)- and \( t_2 \)- derivatives, evaluated along the locus \( \mathcal{L} \) read as follows: \( (F(t) := F_p(t) := \log \tilde{I}_p(t)) \)

\[
0 = \left. \frac{\mathcal{J}_1^{(2)}}{I_p} \right|_{\mathcal{L}} = \left. \left( t_1 \frac{\partial}{\partial t_2} + \sigma_1 \frac{\partial}{\partial t_1} \right) F_p + p(b_0 - t_1) \right|_{\mathcal{L}}
\]

\[
0 = \left. \frac{\mathcal{J}_2^{(2)}}{I_p} \right|_{\mathcal{L}} = \left. \left( t_1 \frac{\partial}{\partial t_3} + \sigma_2 \frac{\partial}{\partial t_2} + (b_0 - t_1) \frac{\partial}{\partial t_1} + \beta \frac{\partial^2}{\partial t_1^2} \right) F_p \right.
\]

\[
\left. + \beta \left( \frac{\partial F_p}{\partial t_1} \right)^2 - \frac{p}{2} (\sigma_1 - b_1) \right|_{\mathcal{L}}
\]

\[
0 = \left. \frac{\partial}{\partial t_1} \mathcal{J}_1^{(2)} \right|_{\mathcal{L}} = \left. \left( t_1 \frac{\partial^2}{\partial t_2^2} + \frac{\partial}{\partial t_2} + \sigma_1 \frac{\partial^2}{\partial t_1^2} \right) F_p \right|_{\mathcal{L}} - p
\]
From (8.0.2), it follows that

\[
0 = \left. \frac{\partial}{\partial t_1} \left[ I_p \right] \right|_{\mathcal{L}} = \left( t_1 \frac{\partial^2}{\partial t_2 \partial t_1} + \sigma_2 \frac{\partial^2}{\partial t_2 \partial t_1} + (b_0 - t_1) \frac{\partial^2}{\partial t_3} + \frac{\partial}{\partial t_1} \right)
\]

\[
+ \beta \frac{\partial^3}{\partial t_1^3} \left. F_p \right|_{\mathcal{L}} + 2 \beta \frac{\partial F_p}{\partial t_1} \frac{\partial^2 F_p}{\partial t_1^2} \left. \right|_{\mathcal{L}}.
\]

\[
0 = \left. \frac{\partial}{\partial t_1} \left[ I_p \right] \right|_{\mathcal{L}} = \left( t_1 \frac{\partial^2}{\partial t_2^2} + \sigma_1 \frac{\partial^2}{\partial t_1 \partial t_2} + 2 \left( \frac{\partial}{\partial t_3} - \frac{\partial}{\partial t_1} \right) \right) F_p \left. \right|_{\mathcal{L}}.
\]

The five equations above form a (triangular) linear system in five unknowns

\[
\left. \frac{\partial F_p}{\partial t_2} \right|_{\mathcal{L}}, \quad \left. \frac{\partial F_p}{\partial t_3} \right|_{\mathcal{L}}, \quad \left. \frac{\partial^2 F_p}{\partial t_1 \partial t_2} \right|_{\mathcal{L}}, \quad \left. \frac{\partial^2 F_p}{\partial t_1 \partial t_3} \right|_{\mathcal{L}}, \quad \left. \frac{\partial^2 F_p}{\partial t_2^2} \right|_{\mathcal{L}},
\]

which can be expressed in terms of

\[
\frac{\partial F_p}{\partial t_1}, \quad \frac{\partial^2 F_p}{\partial t_1^2}, \quad \frac{\partial^3 F_p}{\partial t_1^3}.
\]

Setting \( t_1 = x \) and \( F_n' = \partial F_n/\partial x \), these expressions are

\[
\left. \frac{\partial F_n}{\partial t_2} \right|_{\mathcal{L}} = -\frac{1}{t_1} \left( \sigma_1 F_n' - p(t_1 - b_0) \right)
\]

\[
\left. \frac{\partial F_n}{\partial t_3} \right|_{\mathcal{L}} = \frac{1}{t_1} \left( -2\beta t_1 (F_n'' + F_n'^2) + 2(t_1^2 - b_0 t_1 + \sigma_1 \sigma_2) F_n' \right.
\]

\[
\left. + p((\sigma_1 - 2\sigma_2 - b_1) t_1 + 2b_0 \sigma_2) \right)
\]

\[
\left. \frac{\partial^2 F_n}{\partial t_1 t_2} \right|_{\mathcal{L}} = \frac{1}{t_1} \left( -\sigma_1 t_1 F_n'' + \sigma_1 F_n' + b_0 p \right)
\]

\[
\left. \frac{\partial^2 F_n}{\partial t_1 t_3} \right|_{\mathcal{L}} = \frac{1}{2t_1^2} \left( -2\beta t_1^2 (F_n'' + 2F_n'F_n'' - F_n'^2) + 2t_1(t_1^2 - b_0 t_1 + \sigma_1 \sigma_2 + \beta) F_n'' \right.
\]

\[
\left. + 2(b_0 t_1 - 2\sigma_1 \sigma_2) F_n' + p((2\sigma_2 - \sigma_1 + b_1) t_1 - 4\sigma_2 b_0) \right)
\]

\[
\left. \frac{\partial^2 F_n}{\partial t_2^2} \right|_{\mathcal{L}} = \frac{1}{t_1^3} \left( \sigma_1^2 + 2\beta t_1 F_n'' + 2\beta t_1 F_n'^2 + (2b_0 t_1 - 2\sigma_1 \sigma_2 - \sigma_1^2) F_n' \right.
\]

\[
\left. + p((-\sigma_1 + 2\sigma_2 + b_1) t_1 - b_0(\sigma_1 + 2\sigma_2)) \right).
\]

(6.1.13)

From (8.0.2), it follows that
\[ \tilde{I}_p(t) = \begin{cases} p!\tau_p(t) & p \text{ even}, \quad \beta = 1/2 \\
p!\tau_p(t) & p \text{ arbitrary}, \quad \beta = 1 \\
p!\tau_{2p}(t/2) & p \text{ arbitrary}, \quad \beta = 2, \end{cases} \quad (6.1.14) \]

where in all three cases \( \tau_p(t) \) is a \( \tau \)-function satisfying the KP and Pfaff-KP equations (8.0.4). Substituting (6.1.14) in the equation (8.0.4) and evaluating along the locus \( L \) leads to the following equations:

\[
\left( \frac{\partial}{\partial t_1} \right)^4 + 3 \left( \frac{\partial}{\partial t_2} \right)^2 - 4 \frac{\partial^2}{\partial t_1 \partial t_3} \right) F_p + 6 \left( \frac{\partial^2}{\partial t_1^2} F_p \right)^2 = 12 \frac{p(p-1)}{(p+1)(p+2)} \frac{\tilde{I}_{p-2}\tilde{I}_{p+2}}{I_p^2} (1 - \delta_1^\beta) 
\]

for \( \beta = 1/2, 1 \),

\[
\left( \frac{\partial}{\partial t_1} \right)^4 + 3 \left( \frac{\partial}{\partial t_2} \right)^2 - 4 \frac{\partial^2}{\partial t_1 \partial t_3} \right) F_p + 6 \left( \frac{\partial^2}{\partial t_1^2} F_p \right)^2 = \frac{3}{4} \frac{p}{p+1} \frac{\tilde{I}_{p-1}\tilde{I}_{p+1}}{I_p^2} 
\]

for \( \beta = 2 \). \quad (6.1.15)

Then, using \( F(t) = \log \tilde{I}_p(t) \), with \( \tilde{I}_p \) as in (6.1.9) and (6.1.11),

\[
H(y) := \frac{d}{dy} \log \int_{Gr(p, \mathbb{R}^n)} e^{y Tr(I+Z^\dagger Z)^{-1}} \det(Z^\dagger Z)^{-\beta(q-p)} d\mu(Z) 
= \frac{d}{dy} \log(e^{yp/2}) 
= \int_{[-1,1]^p} |\Delta_p(z)|^{2\beta} \prod_{1}^{p} e^{y/2(1-z_{1})^{\beta(n-q-p+1)-1}(1+z_{1})^{\beta(q-p+1)-1} dz_{1}} 
= \frac{d}{dy} F_p \left( \frac{y}{2}, 0, 0, \ldots \right) + \frac{p}{2} 
= \frac{1}{2} \left( \frac{\partial F_p}{\partial t_1} \left( \frac{y}{2}, 0, 0, \ldots \right) + p \right) \quad (6.1.16) 
\]

satisfies the differential equation, upon substituting the derivative (6.1.13) into (6.1.15) \( \left( H' := \frac{dH}{dy} \right) \);
4 \left( y^3 H'' + 6y^3 H' + (1 + \delta_1^\beta)(2y^2 H'' + 4y^2 HH' + yH^2) \right) - yP_0H' + P_1H + P_2 =
\begin{cases}
0, & \text{for } \beta = 1, \\
\frac{p(p-1)}{(p+1)(p+2)}y^3 \left. \frac{\bar{I}_{p-2}\bar{I}_{p+2}}{I_p^2} \right|_\mathcal{L}, & \text{for } \beta = 1/2, \\
\frac{3}{16} \frac{p}{p+1} y^3 \left. \frac{\bar{I}_{p-1}\bar{I}_{p+1}}{I_p^2} \right|_\mathcal{L}, & \text{for } \beta = 2,
\end{cases}
\] where \( P_0, P_1, P_2 \) are polynomials in \( y \), with coefficients depending on \( r = pq \) and \( s = n - 2p - 2q \), given by table (6.1.3).

From \[ \bar{I}_p(t) \big|_\mathcal{L} = c_p^{(\beta)} e^{-px} I_p(x) = 2^{(\beta n - \beta p + \beta - 1)} e^{-px} I_p(x), \]
it follows that for \( \beta = 1/2 \) and 2,
\[ \left. \frac{\bar{I}_{p-2}\bar{I}_{p+2}}{I_p^2} \right|_\mathcal{L} = 2^{-4} \frac{I_{p-2}\bar{I}_{p+2}(x)}{I_p^2(x)} \quad \text{and} \quad \left. \frac{\bar{I}_{p-1}\bar{I}_{p+1}}{I_p^2} \right|_\mathcal{L} = 2^{-4} \frac{I_{p-1}(x)I_{p+1}(x)}{I_p^2(x)}, \]
thus establishing (6.1.2).

**Proof of Theorem 6.2:** In particular, for \( \beta = 1 \), from Theorem 4.2, Theorem 6.1 and (6.1.5),
\[ u(x) := x \frac{\partial}{\partial x} \log E_{x,p} \left( I_{\{\lambda \geq \mu\}}(\lambda) \prod_{(i,j) \in \lambda} h_{(i,j)}^\lambda \right) = x \frac{\partial}{\partial x} \log \left( e^{-px} x^{p(n-p)} \int_{\text{Gr}(p,\mathbb{C}^n)} e^{x \text{Tr}(I + Z^1Z)^{-1}} \det(Z^1Z)^{-q-p} d\mu(Z) \right) = -px + p(n-p) + xH(x) \]
satisfies the differential equation (6.1.5). From \( I_p(x) = 1 + \frac{px}{n} x + \ldots \), as in (4.2.6), it has the behavior near \( x = 0 \), spelled out in (6.1.5), namely
\[ u(x) = p(n-p) - \frac{p(n-q)}{n} x + \ldots \quad (6.1.17) \]
Equation (6.1.7) follows from Cosgrove and Scoufis ([14]) (see [3]), and the constant is obtained by setting $x = 0$ in the equation and using the Taylor expansion (6.1.16) of $u(x)$ about $x = 0$.

6.2 Hypergeometric functions, KP hierarchy and Painlevé equations

Theorem 6.3 The hypergeometric function for $\beta = 1$ and $p, q, n \in \mathbb{C}$, expressed in $t_i$-variables,

$$\tau(t) := \left. \binom{p}{q} (p, q; n; u) \right|_{\sum_{k=1}^{m} u_k^i = it_i},$$  \hspace{1cm} (6.2.1)

satisfies the Hirota bilinear relation as a function of $t_1, t_2, \ldots$, namely for all $t, t' \in \mathbb{C}^\infty$,

$$\oint_{z=\infty} e^{\sum_{i=1}^{\infty} (t_i - t'_i)z^i} \tau(t - [z^{-1}]) \tau(t' + [z^{-1}])dz = 0. \hspace{1cm} (6.2.2)$$

In particular, $\tau(t)$ satisfies the KP hierarchy\footnote{Given a polynomial $p(t_1, t_2, \ldots)$, define the customary Hirota symbol $p(\partial) f \circ g := p(\partial_1, \partial_2, \ldots) f(t + y) g(t - y) \big|_{y=0}$. The $s_k$'s are the elementary Schur polynomials $e^{\sum_{i=1}^{k} t_i z^i} := \sum_{i \geq 0} s_i(t) z^i$ and for later use, set $s_k(\partial) := s_k(\frac{\partial_1}{\partial t_1}, \frac{\partial_2}{\partial t_2}, \ldots)$.} ($k = 0, 1, 2, \ldots$):

$$\left( s_{k+4} \left( \frac{\partial}{\partial t_1}, \frac{1}{2} \frac{\partial}{\partial t_2}, \frac{1}{3} \frac{\partial}{\partial t_3}, \ldots \right) - \frac{1}{2} \frac{\partial^2}{\partial t_1 \partial t_{k+3}} \right) \tau \circ \tau = 0. \hspace{1cm} (6.2.3)$$

Proof: Using

$$1 - a = e^{-\sum_{i=1}^{\infty} \frac{z^i}{t_i}},$$

one has the following formula

$$\prod_{\substack{1 \leq j \leq p \\text{ and } 1 \leq k \leq m}} (1 - z_j u_k)^{-\beta} \bigg|_{t_i = \frac{1}{2} \sum_{k=1}^{m} u_k^i} = \prod_{\substack{1 \leq j \leq p \\text{ and } 1 \leq k \leq m}} e^{\beta \sum_{i=1}^{\infty} \frac{u_k^i z^i}{t_i}}$$

$$= \prod_{1 \leq j \leq p} e^{\beta \sum_{i=1}^{\infty} \frac{z_j^i}{t_i}} \prod_{1 \leq k \leq m} e^{\beta \sum_{i=1}^{\infty} \frac{1}{t_i} \sum_{k=1}^{m} u_k^i}$$

$$= \prod_{1 \leq j \leq p} e^{\beta \sum_{i=1}^{\infty} \frac{t_i z_j^i}{t_i}}. \hspace{1cm} (6.2.4)$$
On the one hand, setting \( a = \beta(n - q) \) and \( b = \beta q \) in (2.3.14) using the new variables \( t_i \), using the constant (0.0.20) and (6.2.4), the hypergeometric function equals an integral for integer \( p \geq 1 \),

\[
\begin{align*}
c_{n,q,p}^{(\beta)} & \ 2F_1^{(1/\beta)}(\beta p, \beta q, \beta n; u_1, ..., u_m) \\
& = \int_{[0,1]^p} \prod_{1 \leq k \leq p} (1 - z_k u_j)^{-\beta} |\Delta_p(z)|^{2\beta} \prod_{\ell=1}^p z_\ell^{\beta(q-p+1)-1} (1 - z_\ell)^{\beta(n-p-q+1)-1} dz_\ell \\
& = \int_{[0,1]^p} |\Delta_p(z)|^{2\beta} \prod_{\ell=1}^p e^{\beta \sum_{i=1}^m t_i z_\ell} z_\ell^{\beta(q-p+1)-1} (1 - z_\ell)^{\beta(n-p-q+1)-1} dz_\ell. \quad (6.2.5)
\end{align*}
\]

According to (8.0.4), this integral is a solution of the Hirota bilinear relation for \( \beta = 1, q, n \in \mathbb{R} \), integer \( p > 1 \), such that \( q - p + 1 \) and \( n - p - q + 1 > 0 \).

On the other hand, the hypergeometric function, is also defined by (0.0.16), for \( p, q, n \in \mathbb{R} \) and \( \beta = 1 \):

\[
\begin{align*}
2F_1(p, q, n; u_1, ..., u_m) \\
& \bigg|_{t_i = \frac{1}{2} \sum_{k=1}^m u_k} = \sum_{\lambda,\mu} \frac{(p)_\lambda (q)_\mu}{(n)_\lambda (n)_\mu} s_{\lambda}(t) =: \sum_{\lambda \in \mathcal{Y}} c_\lambda s_{\lambda}(t). \quad (6.2.6)
\end{align*}
\]

For integer \( p \geq 1 \), the integral (6.2.5) was shown to be a solution of the Hirota bilinear relations and so the coefficients

\[
c_\lambda \bigg|_{\text{integer } p \geq 1}
\]

satisfy Plücker relations. They are homogeneous quadratic relations in a finite number of the \( c_\lambda \)'s for \( \lambda \in \mathcal{Y} \) and they characterize the KP \( \tau \)-functions; so, we have for every integer \( p \geq 1 \),

\[
0 = \sum_{\lambda,\mu} \alpha_{\lambda,\mu} c_\lambda c_\mu, \quad \alpha_{\lambda,\mu} = \pm 1
\]

\[
= \sum_{\lambda,\mu} \alpha_{\lambda,\mu} \frac{(p)_\lambda (q)_\mu}{(n)_\lambda (n)_\mu}
\]

\[
= : \frac{1}{Y(n)} \sum_i X_i(n, q)p^i. \quad (6.2.7)
\]

where \( X_i(n, q) \) are polynomials in \( n, q \) and \( \sum_i \) is a finite sum. Setting \( p = \) distinct integers \( p_k \) in sufficient number, one solves the homogeneous linear
system (6.2.7) in the $X_i(n,q)$ with coefficients $p^i_k$, whose determinant is a Vandermonde, therefore non-zero. This implies that $X_i(n,q) = 0$ for all $n,q \in \mathbb{R}$ and so, we also have

$$\sum_i X_i(n,q)p^i = 0, \quad \text{for all } p \in \mathbb{R},$$

implying the Plücker relations (6.2.6) for $p,q,n \in \mathbb{R}$. One can extend the argument further by analytic continuation to $p,q,n \in \mathbb{C}$.

**Theorem 6.4** The hypergeometric function

$$H(x) = \frac{d}{dx} \log 2F_1^{(1)}(p,q;n;y) \bigg|_{\sum \nu^{(i)} = \delta_{1,x}}$$

satisfies the Painlevé $V$ equation:

$$4 \left( x^3H^{'''} + 6x^3H^{''} + 2(2x^2H'' + 4x^2HH' + xH^2) \right) - xP_0H' + P_1H + P_2 = 0$$

(6.2.9)

with $P_0, P_1, P_2$ as in (6.1.3).

**Proof:** For integer $p,q$, with $0 \leq p, q \leq n/2$, we have by (2.3.12),

$$2F_1^{(1)}(p,q;n;u) \bigg|_{\sum \nu^{(i)} = \delta_{1,x}} = 1 + \sum_{r \geq 1} b_rx^r$$

$$= \sum_{r \geq 0} x^r \sum_{\lambda \in \mathcal{P}} \frac{1}{(h^\lambda)^2} \frac{(p)_\lambda(q)_\lambda}{(n)_\lambda}, \quad p,q,n \in \mathbb{C}$$

with $b_r$ rational in $p,q,n$, and so

$$H(x) := \frac{d}{dx} \log 2F_1^{(1)}(p,q;n;y) \bigg|_{\sum \nu^{(i)} = \delta_{1,x}} = \sum_{r \geq 0} c_rx^r,$$

also with $c_r$ rational in $p,q,n$. Putting this expression into the left hand side of the Painlevé $V$ equation leads to

$$\sum_{r=0}^{\infty} d_rx^r, \quad d_r = \text{rational in } p,q,n, \text{ with universal coefficients}.$$
Differential equations for the Wishart and canonical correlations distributions

Remember from section 1, the tangent space to the Grassmannians at $Id = \left( \begin{array}{cc} I_p \\ O \end{array} \right)$ is given by

$$T_{Id}Gr(p, \mathbb{F}^n) = \{ Z \mid \text{arbitrary} \ (n - p) \times p \text{ matrix with values in } \mathbb{F} \}.$$ 

The subgroup $K = \left\{ B = \left( \begin{array}{cc} B_{11} & O \\ O & B_{22} \end{array} \right), B_{11} \in K_1, B_{22} \in K_2 \right\}$ acts on $T_{Id}Gr(m, \mathbb{F}^n)$ as

$$Ad_B(Z) = B_{22}ZB_{11}^{-1},$$

for which the spectrum $(\lambda_1, \ldots, \lambda_m)$ of $Z^\dagger Z$ is invariant under the action of $K$ and $\lambda_i \geq 0$, since the matrix $Z^\dagger Z$ is positive definite.

Given $E \subseteq [0, \infty)$, consider the following probability:

$$P_{n,p}^{(\beta)}(E) := P(\text{all } \lambda_i \in E) = c_{p,n} \int_{Z \in T_{Id}Gr(p, \mathbb{F}^n)} e^{-Tr V(Z^\dagger Z)} d\mu(Z)$$

$$= \frac{\int_{E^p} |\Delta_p(z)|^{2\beta} \prod_{i=1}^n e^{-V(z_i)} z_i^{\beta(n-2p+1)-1} dz_i}{\int_{F^p} |\Delta_p(z)|^{2\beta} \prod_{i=1}^n e^{-V(z_i)} z_i^{\beta(n-2p+1)-1} dz_i}, \quad (7.0.1)$$

where $d\mu(Z)$ is Haar measure (1.0.11) on $T_{Id}Gr(p, \mathbb{F}^n)$ and where $\beta = 1/2, 1, 2$ correspond to $\mathbb{R}, \mathbb{C}, \mathbb{H}$ respectively. The function $e^{-V(z)}$ will be either

(i) Laguerre: $e^{-V(z)} = e^{-bz} z^a, \ F = \mathbb{R}^+, \ \beta = 1/2, 1, 2$

(ii) Jacobi: $e^{-V(z)} = (1 - z)^a z^b, \ F = [0, 1], \ \beta = 1/2, 1, 2$

(iii) Gaussian: $e^{-V(z)} = e^{-z^2}, \ F = \mathbb{R}^+, \ \beta = 1, n = 2p$

We shall only consider (i) and (ii), the Gaussian distribution being as in [2].
7.1 Laguerre probability on the tangent space to $Gr(p, \mathbb{F}^n)$ and the Wishart distribution

**Theorem 7.1** For $e^{-V(y)} := e^{-by^a}$, the probability (7.0.1) defined on $Z \in \text{the tangent space (at the identity) to the symmetric space } Gr(p, \mathbb{F}^n)$, leads to the following probability on the (positive) spectrum $(\lambda_1, \ldots, \lambda_p)$ of $Z^\dagger Z$,

$$P_{p,n}^{(\beta)}(\max_i \lambda_i \leq x) := c_{p,n} \int_{Z \in \text{spectrum} \, (Z^\dagger Z) \subset [0,x]} e^{-b \text{Tr} Z^\dagger Z (\det Z^\dagger Z)^a} d\mu(Z)$$

$$= \frac{\int_{[0,x]^p} |\Delta_p(y)|^{2\beta} \prod_i e^{-by_i a + \beta(n-2p)+\beta-1} dy_i}{\int_{[0,\infty)^p} |\Delta_p(y)|^{2\beta} \prod_i e^{-by_i a + \beta(n-2p)+\beta-1} dy_i}$$

(7.1.1)

Then

$$f(x) = x \frac{d}{dx} \log P_{p,n}^{(\beta)}(\max_i \lambda_i \leq x)$$

satisfies

- **for $\beta = 1$:**

  $$x^2 f''' + xf'' + 6xf'^2 - 4f f' - (2hf' - h'f) = 0 \quad \text{(Painlevé V)}$$

  with

  $$2h = (a - n - 2p - bx)^2 - 4nbx.$$

- **for $\beta = \frac{1}{2}$:**

  $$Q \left( \frac{P_{n-4p+2}^{(\beta)} P_{n+4p+2}^{(\beta)}}{(P_{n,p}^{(\beta)})^2} - 1 \right) - \left( 3f + \frac{b^2 x^2}{2\beta} - bQ_0 x - 3Q_1 \right) f$$

  $$= x^3 f''' - x^2 f'' + 6x^2 f'^2 - x \left( 8f + \frac{b^2 x^2}{2\beta} - 2bQ_0 x - Q_2 \right) f',$$

  (7.1.2)

where

$$\alpha := a + \beta(n - 2p) + \beta - 1,$$
and
\[
Q = \begin{cases} 
\frac{3}{4}p(p-1)(p+2\alpha)(p+2\alpha+1), & \text{for } \beta = 1/2 \\
\frac{3}{2}p(2p+1)(2p+\alpha)(2p+\alpha-1), & \text{for } \beta = 2 
\end{cases}
\]
\[
Q_2 = 6\beta p^2 - \frac{\alpha^2}{2\beta} + 6\alpha p + 4(1-\beta)\alpha
\]
\[
Q_1 = 2\beta p^2 + 2\alpha p + (1-\beta)\alpha, \quad Q_0 = p + \frac{\alpha}{2\beta}.
\]

(7.1.3)

**Corollary 7.2** Consider a matrix \(A\), which is Wishart \(W_p(n, \frac{1}{2b} I_p)\)-distributed with eigenvalues \(\lambda_1, \ldots, \lambda_p\) (see Muirhead, p. 107). Then
\[
P_{n,p}^W(x) := P_{n,p}^W(\max_i \lambda_i \leq x), \quad f := x \frac{d}{dx} \log P_{n,p}^W(x)
\]
satisfy the equation
\[
3 \frac{p(p-1)n(n-1)}{4} \left( \frac{P_{n+2,p+2}^W - P_{n-2,p-2}^W}{(P_{n,p}^W)^2} - 1 \right) - (3f + b^2 x^2 - bxQ_0 - 3Q_1)f
\]
\[
= x^3 f''' - x^2 f'' + 6x^2 f' - x(8f + (bx - Q_0)^2 - 4Q_1)f',
\]
with
\[
Q_0 = \frac{1}{2}(n + p - 1) \quad \text{and} \quad 4Q_1 = (n - 1)(4p + 1) - p.
\]

**Proof:** According to (5.1.2), the Wishart distribution \(W_p(n, \frac{1}{2b} I_p)\) of the \(p \times p\) matrix \(A\) is given by
\[
P_{n,p}^W(dA) = \Gamma_p(n/2)^{-1}b^{p^2/2}e^{-b\text{Tr}A}\left(\det A\right)^{(n-p-1)/2}\prod_{1 \leq i \leq j \leq p} dA_{ij}.
\]
(7.1.4)
and so the joint probability \(P_{n,p}^W(\max_i \lambda_i \leq x)\) is precisely formula (7.1.1), with \(a = p/2\) and \(\beta = 1/2\),
\[
P_{n,p}^W(\max_i \lambda_i \leq x) = c_{n,p,b} \int_{[0,x]^p} |\Delta_p(y)| \prod_{1}^{p} e^{-by_i} y_i^{(n-p-1)} dy_i
\]
\[
= P_{n,p}^{(\beta)}(\max_i \lambda_i \leq x) \bigg|_{\beta=1/2}. \quad 60
\]
Therefore \( P^W_{n,p}(x) \) also satisfies the inductive differential equation (7.1.2); we only need to check that

\[
P^W_{n\pm 2,p\pm 2}(x) = c \int_{[0,x]^{p\pm 2}} |\Delta_{p\pm 2}(y)|^{p\pm 2} \prod_{1} e^{-by_i y_i^{1/2}}(n-p-1) dy_i
\]

\[
= c \int_{[0,x]^{p\pm 2}} |\Delta_{p\pm 2}(y)|^{p\pm 2} \prod_{1} e^{-by_i y_i^{1/2}}(n-p-2) dy_i
\]

\[
= P^{(1/2)}_{n\pm 4,p\pm 2}(\max_{i} \lambda_i \leq x) \text{ for } a = p/2.
\]

\( Q, Q_0, Q_1, Q_2 \) can immediately be computed by setting \( \beta = 1/2, \delta = 1 \) and \( 2\alpha = n - p - 1 \) in (7.1.3).

### 7.2 Jacobi probability on the tangent space to \( Gr(p, \mathbb{F}^n) \) and the sample canonical correlation distribution

**Theorem 7.3** For \( e^{-V(y)} := (1 - y)^a y^b \), the probability (7.0.1) defined on \( Z \in T \) the tangent space (at the identity) to the symmetric space \( Gr(p, \mathbb{F}^n) \), leads to the following probability on the (positive) spectrum of \( Z^1 Z \),

\[
P^{(\beta)}_{n,p}(\max_i \lambda_i \leq \frac{x + 1}{2}) := c_{p,n} \int_{\text{spectrum } (Z^1 Z) \subset [0, \frac{x + 1}{2}]} e^{-\text{Tr} V(Z^1 Z)} d\mu(Z)
\]

\[
= \frac{\int_{[0,\frac{x + 1}{2}]} |\Delta_p(y)|^{2\beta} \prod_{1}^{p} (1 - y_i)^a y_i^{b + \beta(n - 2p + 1) - 1} dy_i}{\int_{[0,1]} |\Delta_p(y)|^{2\beta} \prod_{1}^{p} (1 - y_i)^a y_i^{b + \beta(n - 2p + 1) - 1} dy_i};
\]

it satisfies the following differential equations, upon setting

\[
f(x) := (1 - x^2) \frac{d}{dx} \log P^{(\beta)}_{n,p}(\max_i \lambda_i \leq \frac{x + 1}{2}),
\]

- for \( \beta = 1 \):

\[
2(x^2 - 1)^2 f''' + 4(x^2 - 1)(x f'' - 3f'^2) + (16xf - u(x^2 - 1) - 2sx - r) f'
\]

\[
- f(4f - ux - s) = 0
\]
\[
\begin{align*}
\text{for } \beta &= \left\{ \frac{1}{2} : \right. \\
Q \left( \frac{P^{(\beta)}_{n+2,p+2} P^{(\beta)}_{n-2,p-2}}{(P^{(\beta)}_{n,p})^2} - 1 \right) &= 4(u+1)(x^2 - 1)^2 \left( -u(x^2 - 1)f''' + (12f - ux - 3s)f'' + 6u(u-1)f' \right) \\
&\quad - (x^2 - 1)f' \left( 24f(u+3)(2f - s) + 8fu(5u-1)x - u(u+1)(ux^2 + 2sx + 8) + Q_2 \right) \\
&\quad + f \left( 48f^3 + 48f^2(ux + 2x - s) + 2f \left( 8u^2x^2 + 2ux^2 - 12usx - 24sx + Q_4 \right) \\
&\quad - u(u+1)x(3ux^2 + sx - 2ux - 3u) + Q_3x - Q_1s \right), \\
\end{align*}
\]

where

\[
\begin{align*}
b_0 &= a - b - \beta(n-2p+1) + 1, \quad b_1 = a + b + \beta(n-2p+1) - 1 \\
r &= \frac{2}{\beta} \left( b_0^2 + (b_1 + 2 - 2\beta)^2 \right) \quad s = \frac{2}{\beta} b_0 (b_1 + 2 - 2\beta) \\
u &= \frac{2}{\beta} (2\beta p + b_1 + 2 - 2\beta)(2\beta p + b_1),
\end{align*}
\]

and

\[
\begin{align*}
Q &= \frac{3}{16} \left( (s^2 - ur + u^2)^2 - 4(rs)^2 - 4us^2 - 4s^2 + u^2r \right) \\
Q_1 &= 3s^2 - 3ur - 6r + 2u^2 + 23u + 24 \\
Q_2 &= 3us^2 + 9s^2 - 4u^2r + 2ur + 4u^3 + 10u^2 \\
Q_3 &= 3us^2 + 6s^2 - 3u^2r + u^3 + 4u^2 \\
Q_4 &= 9s^2 - 3ur - 6r + u^2 + 22u + 24 = Q_1 + (6s^2 - u^2 - u). \tag{7.2.4}
\end{align*}
\]

**Corollary 7.4** Let \( A = ZZ^\top \) have the Wishart distribution \( W_{p+q}(n, \Sigma) \)-distribution. Break up the matrices \( \Sigma \) and \( A \), as follows:

\[
\begin{align*}
\Sigma &= \begin{pmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{12}^\top & \Sigma_{22} \end{pmatrix} \downarrow p & A &= \begin{pmatrix} A_{11} & A_{12} \\ A_{12}^\top & A_{22} \end{pmatrix} \downarrow q
\end{align*}
\]
Assume the eigenvalues $\rho_1^2, \ldots, \rho_p^2$ of $\Sigma_{11}^{-1}\Sigma_{12}\Sigma_{22}^{-1}\Sigma_{12}^\top$ all zero. Then the probability distribution of the eigenvalues $r_1^2, \ldots, r_p^2$ of $A_{11}^{-1}A_{12}A_{22}^{-1}A_{12}^\top$ (sample canonical correlation coefficients) is given by

$$
P_{n,p,q}^C(x) = P(0 \leq r_i^2 \leq \frac{1+x}{2} \text{ for } 1 \leq i \leq p)$$

$$= \int_{[0,1+x/2]^p} |\Delta_p(z)| \prod_{i=1}^p z_i^{\frac{1}{2}(q-p-1)}(1-z_i)^{\frac{1}{2}(n-p-q-1)} dz_i$$

$$= \int_{[0,1]^p} |\Delta_p(z)| \prod_{i=1}^p z_i^{\frac{1}{2}(q-p-1)}(1-z_i)^{\frac{1}{2}(n-p-q-1)} dz_i$$

and

$$f(x) := (1-x^2) \frac{d}{dx} \log P_{n,p,q}^C(x)$$

satisfy the inductive PDE:

$$Q \left( \frac{P_{n+4,p+2,q+2}^C}{P_{n-4,p-2,q-2}^C} - 1 \right) = \left\{ \text{same expression as the right hand side of (7.2.3)} \right\}$$

with $Q_1, Q_2, Q_3$ being symmetric polynomials of $p, q$, given by (7.2.4), where

$$u = n(n-2)$$

$$r/4 = \frac{n^2}{2} - n(p+q) + p^2 + q^2$$

$$s = (n-2p)(n-2q)$$

$$Q = 48pq(p-1)(q-1)(n-p)(n-q)(n-p-1)(n-q-1).$$

Remark: For instance

$$Q_1 = 24(n-1)(p^2 + q^2 - n(p+q)) + 48pq(n^2 - n(p+q) + pq)$$

$$- (n-6)(n-1)^2(n+4).$$

Proof: The proof follows immediately from Corollary 5.2 and Theorem 7.3.

8 Appendix: The Pfaff-KP hierarchy and Virasoro constraints

Consider weights of the form $\rho(z)dz := e^{-V(z)}dz$ on an interval $F = [A, B] \subseteq \mathbb{R}$, with rational logarithmic derivative and subjected to the following bound-
ary conditions:
\[-\frac{\rho'}{\rho} = V' = \frac{g}{f} = \sum_{i=0}^{\infty} b_i z^i, \quad \lim_{z \to A,B} f(z) \rho(z) z^k = 0 \text{ for all } k \geq 0, \quad (8.0.1)\]

**Theorem 8.1** The multiple integrals

\[I_p(t; \beta) = \int_{F^p} |\Delta_p(z)|^{2\beta} \prod_{k=1}^{p} \left( e^{\sum_{1}^{\infty} t_i z^i} \rho(z_k) dz_k \right), \text{ for } p > 0\]

\[= \begin{cases} p! \tau_p(t) & p \text{ even, } \beta = 1/2 \\ p! \tau_p(t) & p \text{ arbitrary, } \beta = 1 \\ p! \tau_{2p}(t/2) & p \text{ arbitrary, } \beta = 2 \end{cases} \quad (8.0.2)\]

with \(I_0 = 1\), satisfy

(i) the following Virasoro constraints for all \(k \geq -1:\)

\[\sum_{i \geq 0} \left( a_i \beta^{(2)}_{k+i,p}(t, p) - b_i \beta^{(1)}_{k+i+1,p}(t, p) \right) I_p(t; \beta) = 0, \quad (8.0.3)\]

in terms of the coefficients \(a_i, b_i\) of the rational function \((- \log \rho)’\)

(ii) The Pfaff-KP hierarchy: (see footnote 14 for notation)

\[\left( s_{k+4}(\bar{\partial}) - \frac{1}{2} \frac{\partial^2}{\partial t_1 \partial t_{k+3}} \right) \tau_p \circ \tau_p = (1 - \delta_1^\beta) s_k(\bar{\partial}) \tau_{p+2} \circ \tau_{p-2} \quad (8.0.4)\]

\(p \text{ even, } k = 0, 1, 2, \ldots \).

of which the first equation reads \((p \text{ even})\)

\[\left( \left( \frac{\partial}{\partial t_1} \right)^4 + 3 \left( \frac{\partial}{\partial t_2} \right)^2 - 4 \frac{\partial^2}{\partial t_1 \partial t_3} \right) \log \tau_p + 6 \left( \frac{\partial^2}{\partial t_1^2} \log \tau_p \right)^2 \]

\[= 12 \frac{\tau_{p-2} \tau_{p+2}}{\tau_p^2} (1 - \delta_{\beta,1}). \quad (8.0.5)\]

(iii) More generally, the functions \(\tau(t)\) satisfy the Hirota bilinear relations for all \(t, t' \in \mathbb{C}^\infty\) and \(m, p\) positive integers (see footnote 10 for notation)

- \(\beta = 1\)

\[\int_{z = \infty} \tau_p(t - [z^{-1}]) \tau_p(t' + [z^{-1}]) e^{\sum_{i}^{\infty} (t_i - t'_i) z^i} dz = 0, \quad (8.0.6)\]
• $\beta = 1/2$ and 2

$$
\int_{z=\infty}^{\infty} \tau_{2p}(t - [z^{-1}])\tau_{2m+2}(t' + [z^{-1}])e^{\sum_{i=0}^{\infty}(t_i-t'_i)z^i}z^{2p-2m-2}dz + \int_{z=0}^{0} \tau_{2p+2}(t + [z])\tau_{2m}(t' - [z])e^{\sum_{i=1}^{\infty}(t'_i-t_i)z^{-i}}z^{2p-2m}dz = 0, \quad (8.0.7)
$$

Example (Jacobi $\beta$-integral)

This case is particularly important, because it covers the integrals in Theorems 0.2 and 0.3. The weight and the $a_i$ and $b_i$, as in (8.0.1), are given by

$$
\rho(z) := e^{-V} = (1 - z)^a(1 + z)^b, \quad V' = \frac{a - b + (a + b)z}{1 - z^2}
$$

$a_0 = 1, a_1 = 0, a_2 = -1, b_0 = a - b, b_1 = a + b$, and all other $a_i, b_i = 0$.

The integrals

$$
I_p = \int_{E_p} |\Delta_p(z)|^{2\beta} \prod_{k=1}^{p} (1 - z_k)^a(1 + z_k)^b e^{\sum_{i=1}^{\infty}t_i z_k^i}dz
$$

satisfy the Virasoro constraints ($k \geq -1$):

$$
J^{(2)}_{-1}I_p = \left( \beta \mathcal{J}^{(2)}_{k+2,p} - \beta \mathcal{J}^{(2)}_{k,p} + b_0 \beta \mathcal{J}^{(1)}_{k+1,p} + b_1 \beta \mathcal{J}^{(1)}_{k+2,p} \right) I_p = 0. \quad (8.0.9)
$$

Introducing $\sigma_i = (2p - i - 1)\beta + i + 1 + b_1$.

Then introducing the function $F_p := \log \tau_p(t)$, the two first Virasoro constraints for $m = 1, 2$ divided by $\tau_p$ are given by

$$
\frac{\mathcal{J}^{(2)}_{-1}I_p}{\tau_p} = \left( \sum_{i \geq 1} i t_i \frac{\partial}{\partial t_{i+1}} - \sum_{i \geq 2} i t_i \frac{\partial}{\partial t_{i-1}} + \sigma_1 \frac{\partial}{\partial t_1} \right) F_p + p(b_0 - t_1) = 0
$$

$$
\frac{\mathcal{J}^{(2)}_{0}I_p}{\tau_p} = \left( \sum_{i \geq 1} i t_i \left( \frac{\partial}{\partial t_{i+2}} - \frac{\partial}{\partial t_i} \right) + b_0 \frac{\partial}{\partial t_1} + \beta \frac{\partial^2}{\partial t_1^2} + \sigma_2 \frac{\partial}{\partial t_2} \right) F_p
$$

$$
+ \beta \left( \frac{\partial F_p}{\partial t_1} \right)^2 - \frac{p}{2}(\sigma_1 - b_1) = 0.
$$

(8.0.10)
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