STRONG CONVERGENCE OF A GBM BASED TAMED INTEGRATOR FOR SDEs AND AN ADAPTIVE IMPLEMENTATION

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ABSTRACT

We introduce a tamed exponential time integrator which exploits linear terms in both the drift and diffusion for Stochastic Differential Equations (SDEs) with a one sided globally Lipschitz drift term. Strong convergence of the proposed scheme is proved, exploiting the boundedness of the geometric Brownian motion (GBM) and we establish order 1 convergence for linear diffusion terms. In our implementation we illustrate the efficiency of the proposed scheme compared to existing fixed step methods and utilize it in an adaptive time stepping scheme. Furthermore we extend the method to nonlinear diffusion terms and show it remains competitive. The efficiency of these GBM based approaches are illustrated by considering some well-known SDE models.

Keywords First keyword · Second keyword · More

1 Introduction

We consider the numerical integration of Stochastic Differential Equations (SDEs)

\[ dX_t = \mu(X_t)dt + \sigma(X_t)dW_t, \quad X_0 = \xi \]  

on finite time interval \([0, T]\) where \(\mu : \mathbb{R}^d \to \mathbb{R}^d\) and \(\sigma : \mathbb{R}^d \to \mathbb{R}^{d \times m}\) and \(W_t\) is \(m\) dimensional Wiener process on probability space \((\Omega, \mathcal{F}, \mathbb{P})\) with filtration \((\mathcal{F}_t)_{t \in [0, T]}\). This has a well established theory (see for example Kloeden and Platen [2011]) for globally Lipschitz drift and diffusion coefficients \(\mu\) and \(\sigma\). We propose and prove convergence of a tamed method for one-sided Lipschitz drift terms \(\mu\) with linear drift and diffusion terms, see (2) and extend the method to equations of the general form (1) in Section 5.2.

There has been great interest recently in methods for SDEs with one-sided Lipschitz drift and we give a brief review of alternative methods and approaches. First we note that when a numerical method is drift implicit then the nonlinearity \(\mu\) can be controlled, see for example Higham et al. [2013], Yao and Gan [2018], but at the expense of a linear solve at each step. Higham et al. [Higham et al. 2002] investigated the strong convergence of explicit Euler-Maruyama (EM) scheme under non-globally Lipschitz coefficients and concluded that if the numerical solution has bounded moments then strong convergence can be obtained. However, Hutzenthaler et al. [Hutzenthaler et al. 2011] proved that the explicit EM scheme produces numerical solution with unbounded moments for superlinearly growing coefficients \(\mu\) and \(\sigma\). Additionally in Hutzenthaler et al. [2012] they proposed an explicit scheme, "tamed EM", based on replacing a superlinearly growing coefficient with its first order bounded approximation at each step in the scheme. For this scheme they then proved moment bounds and strong convergence for SDEs with one sided globally Lipschitz drift coefficient \(\mu\). There follows a number of different taming type schemes such as Sabanis [2013, 2016], Liu and Mao [2013], Izgi and Cetin [2018], Zong et al. [2014]. We mention in particular Wang and Gan [Wang and Gan 2013] who devised a tamed version of Milstein for SDEs with commutative noise that we consider later Section 5.3, and the generalization considered in the work of Kumar and Sabanis [2019].
Another approach to deal with superlinearly growing coefficients is to modify the numerical solution before each iteration. Mao et al., see for example [Li et al. 2018, Mao 2015], devised truncated methods obtaining numerical solution by truncating intermediate terms according to growth rate of \( \mu \) and \( \sigma \) to avoid blow up. In this category, Beyn et al. [Beyn et al. 2016, 2017] introduced the Projected Euler and Milstein methods consisting of the classical EM/Milstein methods with a projection of the solution onto a ball whose radius is proportional to a negative power of step size. To prove strong convergence of proposed schemes, they utilized suitable generalization of the concepts C-stability and B-consistency which require less strict conditions on moments of numerical solutions.

In [Kelly and Lord 2016], Kelly and Lord proposed adaptive EM methods, extended to Milstein in [Kelly et al. 2019], which aim to control the growth of the numerical solution by choosing adaptively a suitable time step. After a specified cutoff a ‘backstop’ fixed step method is used. We later examine in Section 5.3 the numerical performance of adaptive methods based on the schemes here and [Lord and Erdogan 2018].

We propose novel explicit tamed based methods that are in the broad class of exponential integrators and build on the work in [Lord and Erdogan 2018]. This takes advantage of linear terms not only in the drift but also in the diffusion which is not the case in earlier exponential integrators such as [Biscay et al. 1996, Lord and Rougemont 2004], see also the short review in [Lord and Erdogan 2018]. Recent related work on exponential integrators for SDEs includes [Komori et al. 2017] looking at second-order weak convergence for Runge-Kutta methods, [Yang et al. 2019] who propose Magnus type exponential integrators for Stratonovich SDEs and [Debrabant et al. Accepted, 2021] that examines families of Runge-Kutta Lawson schemes and their weak and strong convergence in the absence of a nonlinear drift term. Specifically we examine new explicit tamed based methods for the semilinear SDE

\[
dx_t = (AX_t + F(X_t)) \, dt + \sum_{i=1}^{m} (B_i X_t) \, dW^i_t, \quad X_0 = \xi \in \mathbb{R}^d ,
\]

with \( m, d \in \mathbb{N} \) where \( W^i_t \) are iid Brownian motions, \( F : \mathbb{R}^d \to \mathbb{R}^d \) is one sided Lipschitz (see Assumption 1), and the matrices \( A, B_i \in \mathbb{R}^{d \times d} \) satisfy the following zero commutator conditions

\[
AB_i - B_i A = 0, \quad B_j B_i - B_i B_j = 0 \quad \text{for} \quad i, j = 1 \ldots m .
\]

When these commutator conditions hold then we can exploit the exact flow of (2). SDE’s of this form arise in many areas of engineering and science, and in particular an SDE with diagonal noise satisfies the commutator condition. The standard cubic nonlinear scalar SDE, see (54), is a classic example but we also consider in Section 5.3 examples from the mathematical biology: stochastic Lotka–Volterra and HIV models as well as tumor growth model. In addition the SDEs that arise from the spectral discretization of semilinear Stochastic Partial Differential Equations (SPDEs) often fall into this class.

The method we examine is based on the exact flow of (2) which is expressed in the following integral form

\[
X_{t_{n+1}} = \Phi_{t_{n+1}, t_n} \left( X_{t_n} + \int_{t_n}^{t_{n+1}} \Phi^{-1}_{s, t_n} F(X_s) \, ds \right) ,
\]

where the fundamental matrix

\[
\Phi_{t, t_0} = \exp \left( (A - \frac{1}{2} \sum_{i=1}^{m} B_i^2)(t - t_0) + \sum_{i=1}^{m} B_i(W^i_t - W^i_{t_0}) \right)
\]

is the solution to the linear equation

\[
d\Phi_{t, t_0} = A\Phi_{t, t_0} \, dt + \sum_{i=1}^{m} B_i \Phi_{t, t_0} \, dW^i_t, \quad \Phi_{t_0, t_0} = I_d .
\]

In [Lord and Erdogan 2018] we proposed the numerical scheme EI0

\[
Y_{n+1}^N = \Phi_{t_{n+1}, t_n} \left( Y_{t_n}^N + F(Y_{t_n}^N) \Delta t \right) ,
\]

where \( \Delta t = \frac{T}{N} \) defined by the uniform time partition \( 0 = t_0 < t_1 < t_2 < \ldots < t_N = T \). Here we introduce a tamed version of (7) that we denote TamedEI0

\[
Y_{n+1}^N = \Phi_{t_{n+1}, t_n} \left( Y_{t_n}^N + \tilde{F}(Y_{t_n}^N) \Delta t \right) , \quad Y_{0}^N = \xi
\]

where

\[
\tilde{F}(Y_{t_n}^N) = \frac{F(Y_{t_n}^N)}{1 + \Delta t \| F(Y_{t_n}^N) \|}.
\]
Compared to method in Lord and Erdogan [2018] the taming has changed the scheme and a key step is to obtain moment bounds on the numerical method. It is clear from (8) that the Brownian increment term is included in the $\Phi_{t,s}^{-1}$. As a consequence the convergence analysis needs to be adapted from that in Hutzenthaler et al. [2012], Wang and Gan [2013], Zong et al. [2014], Sabanis [2013] and we need to exploit the boundedness of geometric Brownian motion (GBM). The numerical results of Lord and Erdogan [2018] indicate that $EI0$ was the most efficient of the three schemes considered. This remains the case for taming versions of these and hence we only consider $EI0$ here. We restrict our analysis to the linear diffusion case, as this leads to an improved rate of convergence, however we examine nonlinear diffusion numerically. Our main result in the paper is a strong convergence of the scheme that we state and prove in Section 4 along with some necessary lemmas. In Section 2 we introduce our notation and give a preliminary result on the linear system before in Section 3 stating results for the boundedness of the moments of the scheme (the proofs are given in A). In Section 5.1 we discuss how the scheme (8) can be used as the backstop method for an adaptive time stepping strategy and in Section 5.2 we discuss how (8) can be extended to general systems of the form (1). Finally in Section 5.3 we illustrate convergence and compare the efficiency of methods. We observe that amongst the fixed step methods (8) is the most efficient on the examples with linear diffusion and that the most efficient is an adaptive version. Where the diffusion is no longer linear our proposed schemes remain competitive.

2 Assumptions and preliminary results

We first introduce the notation for the norms we work with. We let $\|\cdot\|$ denote the standard Euclidean norm for vectors in $\mathbb{R}^d$ and also for the subordinate matrix norm for matrices in $\mathbb{R}^{d \times d}$. In addition we let $\|\cdot\|_{T_3}$ denote the subordinate tensor norm of rank 3 in $\mathbb{R}^{d \times d \times d}$ given by

$$
\|J(x)\|_{T_3} = \sup_{\|h_1\| \leq 1, \|h_2\| \leq 1} \|J(x)(h_1, h_2)\|.
$$

Finally we introduce the norm $\|\cdot\|_{L^p(\Omega; \mathbb{R}^d)}$, where $\|v\|_{L^p(\Omega; \mathbb{R}^d)} := \left(\mathbb{E}[\|v\|^p]\right)^{1/p}$. Throughout the paper, we make use of the following time regularity and norm properties of GBM.

**Lemma 1.** Let $0 < t - s < 1$ and consider the stochastic matrix $\Phi_{t,s}$ given in (5) with $t_0 = s$. Then for each $F_s$ measurable random variable $v \in L^p(\Omega, \mathbb{R}^d)$ with $p \geq 2$

$$
\|\Phi_{t,s}v - v\|_{L^p(\Omega, \mathbb{R}^d)} \leq C_p|t - s|^{1/2},
$$

(10)

$$
\|\Phi_{t,s}^{-1}v - v\|_{L^p(\Omega, \mathbb{R}^d)} \leq C_p|t - s|^{1/2}.
$$

(11)

**Proof.** For (10), by definition of GBM, $U_t = \Phi_{t,s}U_s$ solves

$$
U_t = U_s + \int_s^t A U_r dr + \int_s^t \sum_{i=1}^m B_i U_r dW_i^r.
$$

(12)

Taking the $p$th power of the Euclidean norm we have

$$
\|U_t - U_s\|^p \leq 2^{p-1} \left|\int_s^t & A & dr \right|^p + 2^{p-1} \left|\int_s^t \sum_{i=1}^m & B_i & dW_i^r \right|^p.
$$

By taking the expectation, using the subordinate matrix norm inequality, Jensen’s inequality and the moment inequality in [Mao, 2007, Theorem 1.7.1] we find

$$
\|U_t - U_s\|^p_{L^p(\Omega, \mathbb{R}^d)} \leq 2^{p-1}(t - s)^{p-1} \int_s^t \|A\| \|U_r\|^p_{L^p(\Omega, \mathbb{R}^d)} dr

+ 2^{p-1}C(p, m)(t - s)^{p/2-1} \int_s^t \sum_{i=1}^m \|B_i\| \|U_r\|^p_{L^p(\Omega, \mathbb{R}^d)} dr.
$$

By boundedness of $U_t$, the solution to (12), in $L^p(\Omega, \mathbb{R}^d)$, we obtain (10).

To prove (11) we note that the operator $U_t = \Phi_{t,s}^{-1}U_s$ satisfies the linear SDE

$$
dU_t = (-A + \sum_{i=1}^m B_i^2)U_t dt - \sum_{i=1}^m B_i U_t dW_i^r
$$

(13)

and the desired result follows as for (10).
We now prove a preliminary lemma that shows the subordinate matrix norm $\|\Phi_{t,s}\|_{L^p(\Omega, \mathbb{R}^d \times \mathbb{R}^d)}$ of the operator $\Phi_{t,s}$ acting on the space $L^p(\Omega, \mathbb{R}^d)$ is well defined and bounded on a finite time interval.

**Lemma 2.** Suppose $p \geq 2$, $s < t$ and let $\Phi_{t,s}$ be given in (5) with $t_0 = s$.

i) For any $\mathcal{F}_s$ measurable random variable $v$ in $L^p(\Omega, \mathbb{R}^d)$

$$\|\Phi_{t,s}v\|_{L^p(\Omega, \mathbb{R}^d)} \leq \exp \left( \|A\| + \frac{p-1}{2} \sum_{i=1}^{m} \|B_i\|^2 \right) (t-s) \|v\|_{L^p(\Omega, \mathbb{R}^d)}.$$ 

ii) For any $\mathcal{F}_t$ measurable random variable $v$ in $L^p(\Omega, \mathbb{R}^d)$

$$\|\Phi_{t,s}v\|_{L^p(\Omega, \mathbb{R}^d)} \leq K \|v\|_{L^p(\Omega, \mathbb{R}^d)},$$

where $K_p = \|\Phi_{t,s}\|_{L^p(\Omega, \mathbb{R}^d)} < \infty$ for $1/p = 1/r + 1/q$ with $r, q > 1$.

**Proof.**

i) By definition $U_t = \Phi_{t,s}v$ solves (12) where $v$ is an $\mathcal{F}_s$ measurable random variable in $L^p(\Omega, R^d)$. By Itô’s lemma,

$$\|U_t\|^p = \|v\|^p + \int_s^t p \|U_r\|^{p-2} \langle U_r, AU_r \rangle dr + \frac{1}{2} \int_s^t \left( p \|U_r\|^{p-2} \sum_{i=1}^{m} \|B_iU_r\|^2 + p(p-2) \|U_r\|^{p-4} \sum_{i=1}^{m} (B_iU_r, U_r)^2 \right) dr + \int_s^t p \|U_r\|^{p-2} \sum_{i=1}^{m} (B_iU_r, U_r) dW_i^t. \quad (14)$$

By applying the Cauchy-Schwarz inequality, taking the expectation and using mean zero property of Itô integrals we have

$$\mathbb{E}[\|U_t\|^p] \leq \mathbb{E}[\|v\|^p] + \left( p \|A\| + \frac{p}{2} \sum_{i=1}^{m} \|B_i\|^2 + \frac{p(p-2)}{2} \sum_{i=1}^{m} \|B_i\|^2 \right) \int_s^t \mathbb{E}[\|U_r\|^p] dr.$$ 

An application of the standard Gronwall inequality, gives

$$\mathbb{E}[\|U_t\|^p] \leq \exp \left( p \left( \|A\| + \frac{p-1}{2} \sum_{i=1}^{m} \|B_i\|^2 \right) (t-s) \right) \mathbb{E}[\|v\|^p]$$

and hence the result.

ii) By applying the subordinate norm inequality to $\|\Phi_{t,s}v\|$, taking the expected value of both sides, and finally applying Hölder’s inequality in $L^p(\Omega, \mathbb{R}^d)$, we have the inequality

$$\mathbb{E}[\|\Phi_{t,s}v\|^{p}]^{1/p} \leq \mathbb{E}[\|\Phi_{t,s}\|^{p}]^{1/p} \leq \mathbb{E}[\|\Phi_{t,s}\|^{r}]^{1/r} \mathbb{E}[\|v\|^{q}]^{1/q}. \quad (15)$$

It remains to show that $\mathbb{E}[\|\Phi_{t,s}\|^{r}] < \infty$. Note that

$$\|\Phi_{t,s}\|^r \leq \exp \left( \left( r A - \frac{1}{2} \sum_{i=1}^{m} B_i^2 \right)(t-s) \right) \exp \left( r \sum_{i=1}^{m} \|B_i\| \|W_i^t - W_i^s\| \right). \quad (17)$$

By taking expected value and considering the independence of random values $|W_i^t - W_i^s|$, for $i = 1, 2, \ldots, m$, we have

$$\|\Phi_{t,s}\|_{L^r(\Omega, \mathbb{R}^d)} \leq \exp \left( \left( r A - \frac{1}{2} \sum_{i=1}^{m} B_i^2 \right)(t-s) \right) \prod_{i=1}^{m} \mathbb{E} \left[ \exp \left( r \|B_i\| \|W_i^t - W_i^s\| \right) \right]$$

$$= \exp \left( \left( r A - \frac{1}{2} \sum_{i=1}^{m} B_i^2 \right)(t-s) \right) \prod_{i=1}^{m} \exp(r^2 \|B_i\|^2 (t-s)) \left( 1 + \text{erf} \left( \frac{r \|B_i\|}{\sqrt{2}} \right) \right)$$

$$< \infty$$
Assumption 1. Let $F$ and its Jacobian $DF$ be continuously differentiable. Furthermore let there exist positive constants $K \geq 1$ and $c, q \geq 1$ such that $\forall x, y \in \mathbb{R}^d$

1. $(x - y, F(x) - F(y)) \leq K \|x - y\|^2$
2. $\|DF(x)\| \leq K(1 + \|x\|^q)$
3. $\|D^2F(x)\|_{T_3} \leq K(1 + \|x\|^q)$.

We note that Beyn et al. [2017] and Kumar and Sabanis [2019] both examine the case where $F$ is only continuously differentiable and there is no condition on $D^2F$. We use this condition as in Wang and Gan [2013] to get a martingale to apply Doob’s maximal inequality. It may be possible to weaken this assumption on $F$ but would require an alternative approach to the proof. Our proof is for the case of a linear diffusion, however in Section 5.2 we extend to the nonlinear case and we comment on the expected rate of convergence there.

Our focus in the analysis is on obtaining an order 1 rate of convergence and hence we analyse the special diffusion term in (2) and only consider more general diffusion terms (such as in Wang and Gan [2013], Beyn et al. [2017], Kumar and Sabanis [2019]) in our numerical experiments.

3 Boundedness of moments of the numerical scheme

The main idea of the proof of the moment bounds is adapted from that in Hutzenthaler et al. [2012], Wang and Gan [2013], Zong et al. [2014], Sabanis [2013]. We highlight here (and in the proofs in A) the differences required to examine (8). In the standard way we start by establishing the boundedness of numerical solutions on a set $\Omega^N$ where increments of the noise are controlled. Then, by using standard inequalities on the measure of complement set, we obtain the boundedness of $p$th moments of numerical solution. We make extensive use of Lemma 2 and the key property of the taming factor that

$$\|\bar{F}(Y^N_n)\| = \frac{\|F(Y^N_n)\|}{1 + \Delta t \|F(Y^N_n)\|} \leq \|F(Y^N_n)\|.$$  \hspace{1cm} (18)

We introduce appropriate sub events of $\Omega$. We let $\Omega^N_0 = \Omega$, then for $n \in \{1, 2, \ldots, N\}$

$$\Omega^N_n = \left\{ \omega \in \Omega, \sup_{0 \leq k \leq n-1} D^N_k(\omega) \leq N^{1/2\varepsilon}, \sup_{0 \leq i \leq n} \left\| \sum_{i=1}^n B_i \Delta W_{t_k} \right\| \leq 1 \right\},$$  \hspace{1cm} (19)

and introduce the dominating stochastic process, $D^N_n = (\lambda + \|\xi\|) e^\lambda$ and

$$D^N_n = (\lambda + \|\xi\|) e^\lambda \sup_{0 \leq u \leq n} \prod_{k=1}^{n-1} \left\| \Phi_{t_{k+1}, t_k} \right\|, \hspace{1cm} n \in \{1, 2, \ldots, N\}.$$  \hspace{1cm} (20)

where

$$\lambda = e^{1+T\|A^{-\frac{1}{2}} \sum_{i=1}^m B_i\|} (1 + 4TK + 2T \|F(0)\| + 2K)^2.$$

The first result shows we can dominate the numerical solution on the set $\Omega^N_n$.

Lemma 3. Let $Y^N_n$ be given by (8). For all $n = 0, 1, \ldots, N$ we have

$$1_{\Omega^N_n} \|Y^N_n\| \leq D^N_n.$$

The second result then bounds the dominating process.

Lemma 4. For all $p \geq 1$, $\sup_{N \in \mathbb{N}} E \sup_{0 \leq n \leq N} \left[ D^N_n \|Y^N_n\|^p \right] < \infty$.

For the complement we have from Hutzenthaler et al. [2012] Lemma 3.6 the following:

Lemma 5. For all $p \geq 1$, $\sup_{N \in \mathbb{N}} (N^p \|\xi\|^p) \mathbb{P}(\Omega^N_N^c) < \infty$.

Finally we obtain bounded moments for the numerical scheme (8).

Theorem 1. Let $Y^N_n$ be given by (8). Then, for all $p \in [1, \infty)$

$$\sup_{N \in \mathbb{N}} \left[ \sup_{0 \leq n \leq N} E \left[ \|Y^N_n\|^p \right] \right] < \infty.$$

The proofs of Theorem 1 and relevant lemmas are contained in A.
4 Strong convergence of TamedEI0

The aim of this section is to prove the following strong convergence result

\[
\left( \mathbb{E} \left[ \sup_{t \in [0, T]} \| X_t - \tilde{Y}_t \|^p \right] \right)^{\frac{1}{p}} \leq C_{p,T} \Delta t,
\]

where \( C_{p,T} \) is a constant independent of \( \Delta t \), see Theorem 2 in Section 4.1. Before we state and prove this result we require a number of preliminary lemmas.

4.1 Preliminary Lemmas

Lemma 6. Let Assumption 7 hold. Then for all \( p \geq 1 \), we have

\[
\sup_{t \in [0, T]} \| X_t \|_{L^p(\Omega, \mathbb{R}^d)} < \infty, \quad \sup_{t \in [0, T]} \| F(X_t) \|_{L^p(\Omega, \mathbb{R}^d)} < \infty.
\]

Proof. These follow from the boundedness of moments, see [Mao, 2007, Theorem 4.1] and [Wang and Gan, 2013, Lemma 3.4].

Now we define a continuous form \( \tilde{Y}_t \) of the numerical solution by introducing

\[
\hat{t} = t_n, \text{ for } t_n \leq t < t_{n+1},
\]

and setting

\[
\tilde{Y}_t = \Phi_{t,0} \xi + \Phi_{t,0} \int_0^t \Phi_{s,0}^{-1} \tilde{F}(\tilde{Y}_s) ds.
\]

Note that \( \tilde{Y}_t \) agrees with the approximation \( Y_n^N \) at \( t = t_n \).

The following lemma provides an integral representation for the continuous version of numerical solution.

Lemma 7. Let \( \tilde{Y}_t \) be the interpolated numerical solution given in (22). Then the differential for this solution is given by

\[
d\tilde{Y}_t = \left( A\tilde{Y}_t + \Phi_{t,0} \tilde{F}(\tilde{Y}_t) \right) dt + \sum_{i=1}^m B_i \tilde{Y}_t dW_i^i.
\]

Proof. By definition of GBM

\[
d\Phi_{t,0}^{-1} = (-A + \sum_{i=1}^m B_i^2) \Phi_{t,0}^{-1} dt - \sum_{i=1}^m B_i \Phi_{t,0}^{-1} dW_i^i.
\]

Let us set \( d\tilde{Y}_t = \mu dt + \sum_{i=1}^m \sigma_i dW_i^i \) and seek the appropriate \( \mu \) and \( \sigma_i \). The representation for matrix-vector product gives us

\[
d \left[ \Phi_{t,0}^{-1} \tilde{Y}_t \right] = \Phi_{t,0}^{-1} \left[ (-A + \sum_{i=1}^m B_i^2) \tilde{Y}_t + \mu - \sum_{i=1}^m B_i \sigma_i \right] dt + \Phi_{t,0}^{-1} \left[ - \sum_{i=1}^m B_i \tilde{Y}_t + \sigma_i \right] dW_i^i.
\]

On the other hand, the linearly interpolated continuous solution is given by

\[
\tilde{Y}_t = \Phi_{t,0} \xi + \Phi_{t,0} \int_0^t \Phi_{s,0}^{-1} \tilde{F}(\tilde{Y}_s) ds.
\]

By comparison of the above expression with (24) after multiplying through by \( \Phi_{t,0} \), we find

\[
\Phi_{t,0}^{-1} \left[ (-A + \sum_{i=1}^m B_i^2) \tilde{Y}_t + \mu - \sum_{i=1}^m B_i \sigma_i \right] = \Phi_{t,0}^{-1} \tilde{F}(\tilde{Y}_t)
\]

and

\[
\Phi_{t,0}^{-1} \left[ - \sum_{i=1}^m B_i \tilde{Y}_t + \sigma_i \right] = 0.
\]

Finally, we have that \( \mu = A\tilde{Y}_t + \Phi_{t,0} \tilde{F}(\tilde{Y}_t) \) and \( \sigma_i = B_i \tilde{Y}_t \).
Lemma 8. Let Assumption 7 hold. Then for all $p \geq 1$, we have the following:

i) For $Y_n^N$

$$\sup_{N \in \mathbb{N}} \sup_{0 \leq n \leq N} \mathbb{E} \left[ \left\| F(Y_n^N)^p \right\| \right] < \infty$$

and

$$\sup_{N \in \mathbb{N}} \sup_{0 \leq n \leq N} \mathbb{E} \left[ \left\| D^2 F(Y_n^N)^p \right\| \right] < \infty.$$

ii) For $\tilde{Y}_t$

$$\sup_{t \in [0, T]} \left\| \tilde{Y}_t \right\|_{L^p(\Omega, \mathbb{R}^d)} < \infty,$$

and

$$\sup_{t \in [0, T]} \left\| F(\tilde{Y}_t) \right\|_{L^p(\Omega, \mathbb{R}^d)} < \infty.$$

Proof. Theorem 1 and the polynomial growth condition on $DF$ and $D^2F$ from Assumption 7 imply the estimates in i).

For the estimates in ii) we consider the one step continuous extension

$$\tilde{Y}_t = \Phi_{t, \hat{t}} \left( \tilde{Y}_{\hat{t}} + \int_{\hat{t}}^t \tilde{F}(\tilde{Y}_s) ds \right)$$

where $\hat{t}$ is as defined in (21). Therefore, by Theorem 1 we see $\left\| \tilde{Y}_t \right\|_{L^p(\Omega, \mathbb{R}^d)} = \left\| Y_n^N \right\|_{L^p(\Omega, \mathbb{R}^d)}$ and for $p \geq 2$

$$\left\| \tilde{Y}_t \right\|_{L^p(\Omega, \mathbb{R}^d)} \leq e^{(\|A\| + \frac{1}{2} \sum_{i=1}^d \|B_i\|^2)(t-\hat{t})} \left( \left\| \tilde{Y}_{\hat{t}} \right\|_{L^p(\Omega, \mathbb{R}^d)} + \int_{\hat{t}}^t \left\| \tilde{F}(\tilde{Y}_s) \right\|_{L^p(\Omega, \mathbb{R}^d)} ds \right) < \infty.$$  

By Hölder’s inequality we obtain this for $p \geq 1$. Similarly, the polynomial growth condition on $DF$ and the Mean Value Theorem implies the final estimate. \qed

Lemma 9. Let Assumption 7 hold on $F$. Then for all $p \geq 2$, $t - \hat{t} < 1$,

$$\left\| \tilde{Y}_t - \tilde{Y}_{\hat{t}} \right\|_{L^p(\Omega, \mathbb{R}^d)} \leq C_p (t - \hat{t})^{1/2}$$

and

$$\left\| F(X_t) - F(X_{\hat{t}}) \right\|_{L^p(\Omega, \mathbb{R}^d)} \leq C_p (t - \hat{t})^{1/2}.$$  

Proof. The time difference in one step continuous form of the numerical solution can be written as

$$\tilde{Y}_t - \tilde{Y}_{\hat{t}} = (\Phi_{t, \hat{t}} - I) \tilde{Y}_{\hat{t}} + (t - \hat{t}) \Phi_{t, \hat{t}} \tilde{F}(\tilde{Y}_{\hat{t}})$$

since

$$\tilde{Y}_t = \Phi_{t, \hat{t}} \left( \tilde{Y}_{\hat{t}} + \int_{\hat{t}}^t \tilde{F}(\tilde{Y}_s) ds \right).$$

Taking norm on the space $L^p(\Omega, \mathbb{R}^d)$, we have,

$$\left\| \tilde{Y}_t - \tilde{Y}_{\hat{t}} \right\|_{L^p(\Omega, \mathbb{R}^d)} \leq \left\| \Phi_{t, \hat{t}} \tilde{Y}_{\hat{t}} \right\|_{L^p(\Omega, \mathbb{R}^d)} + (t - \hat{t}) \left\| \Phi_{t, \hat{t}} \tilde{F}(\tilde{Y}_{\hat{t}}) \right\|_{L^p(\Omega, \mathbb{R}^d)}.$$  

Using (10) for the first term above we obtain the first estimate of the lemma. For the second estimate on the exact solution, see [Wang and Gan, 2013, Lemma 3.5]. \qed

Lemma 10. Let Assumption 7 hold on $F$. Consider the GBM operator $\Phi_{t, \hat{t}}$ from (5) (with $t_0 = \hat{t}$) and its inverse $\Phi_{t, \hat{t}}^{-1}$ where $t - \hat{t} < 1$. Then for $p \geq 2$

i) $\left\| \Phi_{t, \hat{t}} F(X_t) - F(X_{\hat{t}}) \right\|_{L^p(\Omega, \mathbb{R}^d)} \leq C_p (t - \hat{t})^{1/2}$

ii) $\left\| \Phi_{t, \hat{t}}^{-1} F(X_t) - F(X_{\hat{t}}) \right\|_{L^p(\Omega, \mathbb{R}^d)} \leq C_p (t - \hat{t})^{1/2}$. 

7
Proof. Since \(X_t\) is not \(\mathcal{F}_t\) measurable, we are not able to apply Lemma 10 directly. By adding and subtracting \(F(X_i)\), we have
\[
\| \Phi_{t,i} F(X_i) - F(X_i) \|_{L^p(\Omega, \mathbb{R}^d)} \leq \| \Phi_{t,i} F(X_i) - F(X_i) \|_{L^p(\Omega, \mathbb{R}^d)} + \| F(X_i) - F(X_i) \|_{L^p(\Omega, \mathbb{R}^d)}.
\] (28)

The second term on RHS above is bounded as desired due to Lemma 9. Let us consider the first term. By a Taylor expansion of GBM near \(X_t\), we obtain the result.

Applying the moment inequality in [Mao, 2007, Theorem 1.7.1], we obtain the result.

Now, we are going to consider the order of three remainder terms arising from Itô-Taylor expansions. First we introduce the remainder term \(R_{GBM}(t, \tilde{t})\) from the Itô-Taylor expansion of GBM near \(t = \tilde{t}\):
\[
(\Phi_{t,i} - I) \tilde{Y}_i = \sum_{i=1}^{m} B_i \tilde{Y}_i (W_i - W_i) + R_{GBM}(t, \tilde{t}).
\] (30)

Lemma 11. Let Assumption 7 hold at \(t - \tilde{t} < 1\). Then for \(p \geq 2\),
\[
\| R_{GBM}(t, \tilde{t}) \|_{L^p(\Omega, \mathbb{R}^d)} < C_p(t - \tilde{t}).
\]

Proof. By the Itô-Taylor expansion of GBM, we have
\[
R_{GBM}(t, \tilde{t}) = \sum_{i,j=1}^{m} B_i B_j \int_{\tilde{t}}^{t} \int_{\tilde{t}}^{z} \Phi_{z,i} \tilde{Y}_i dW_i(z) dW_j(s) + \int_{\tilde{t}}^{t} A \Phi_{t,i} \tilde{Y}_i ds
\]
\[+ \sum_{i=1}^{m} B_i A \int_{\tilde{t}}^{t} \int_{\tilde{t}}^{z} \Phi_{z,i} \tilde{Y}_i dz dW_i s.\] (31)

Applying the moment inequality in [Mao, 2007, Theorem 1.7.1], we obtain the result.

The second remainder term comes from deterministic Taylor expansion of \(F(\tilde{Y}_i)\) near \(F(\tilde{Y}_i)\):
\[
F(\tilde{Y}_i) - F(\tilde{Y}_i) = DF(\tilde{Y}_i) (\tilde{Y}_i - \tilde{Y}_i) + R_F(t, \tilde{t}).
\] (32)

Lemma 12. Let Assumption 7 hold. Then, for \(t - \tilde{t} < 1\) and \(p \geq 1\)
\[
\| R_F(t, \tilde{t}) \|_{L^p(\Omega, \mathbb{R}^d)} < C_p(t - \tilde{t})
\]
Proof. By defining a variable \( Z := \hat{Y}_t + \theta (\bar{Y}_t - \hat{Y}_t) \), we have the remainder term in integral form,

\[
R_F(t, \hat{t}) = \int_0^1 \left( 1 - \theta \right) D^2 F (Z) \left( \hat{Y}_t - \hat{Y}_{\hat{t}}, \bar{Y}_t - \hat{Y}_t \right) d\theta.
\]

Taking the norm, using Assumption 1, Jensen’s and Hölder’s inequality

\[
\| R_F(t, \hat{t}) \|_{L^p(\Omega, \mathbb{R}^d)} \leq \int_0^1 \left( 1 - \theta \right) \left\| D^2 F (Z) \left( \hat{Y}_t - \hat{Y}_{\hat{t}}, \bar{Y}_t - \hat{Y}_t \right) \right\|_{L^p(\Omega, \mathbb{R})} d\theta.
\]

\[
\leq \int_0^1 \left\| D^2 F (Z) \right\|_{L^p(\Omega, \mathbb{R})} \left\| \hat{Y}_t - \hat{Y}_{\hat{t}} \right\|^2_{L^p(\Omega, \mathbb{R})} d\theta.
\]

\[
\leq K \left\| 1 + \| \hat{Y}_t \|^q + \| \hat{Y}_{\hat{t}} \|^q \right\|_{L^{2p}(\Omega, \mathbb{R})} \left\| \hat{Y}_t - \hat{Y}_{\hat{t}} \right\|^2_{L^{2p}(\Omega, \mathbb{R})}
\]

\[
= K \left[ 1 + \| \hat{Y}_t \|^q + \| \hat{Y}_{\hat{t}} \|^q \right] \| \hat{Y}_t - \hat{Y}_{\hat{t}} \|^2_{L^{2p}(\Omega, \mathbb{R})} \]

\[
\leq C_p (t - \hat{t}),
\]

where for the final inequality Lemma 9 is applied.

Thirdly we define a remainder term \( R \) from an Itô-Taylor expansion of \( F(\hat{Y}_t) \) near \( F(\bar{Y}_t) \)

\[
R(t, \hat{t}) = F(\hat{Y}_t) - F(\bar{Y}_t) - \sum_{i=1}^m B_i \hat{Y}_t \left( W_i^t - W_i^\hat{t} \right).
\]

The following lemma determines the order of the term \( R \).

Lemma 13. Let Assumption 7 hold. Then for \( p \geq 2 \),

\[
\| R(t, \hat{t}) \|_{L^p(\Omega, \mathbb{R}^d)} < C_p (t - \hat{t}).
\]

Proof. The substitution of (26) into (32) gives us

\[
F(\hat{Y}_t) - F(\bar{Y}_t) = DF(\hat{Y}_t) \left( \Phi_{t, \hat{t}} - I \right) \hat{Y}_t + (t - \hat{t}) DF(\hat{Y}_t) \Phi_{t, \hat{t}} \hat{F}(\hat{Y}_t) + R_F(t, \hat{t}).
\]

Therefore we can re-express \( R \) as follows

\[
R(t, \hat{t}) = DF(\hat{Y}_t) \left( R_{GBM}(t, \hat{t}) + (t - \hat{t}) \hat{F}(\hat{Y}_t) \right) + R_F(t, \hat{t})
\]

Similar to [Wang and Gant 2013, Lemma 3.6], we can determine the order of each term above.

Taking the norm and using Lemmas 11 and 12 with Assumption 1 we find

\[
\| R(t, \hat{t}) \|_{L^p(\Omega, \mathbb{R}^d)} \leq \| DF(\hat{Y}_t) R_{GBM}(t, \hat{t}) \|_{L^p(\Omega, \mathbb{R}^d)} + | t - \hat{t} | \| DF(\hat{Y}_t) \hat{F}(\hat{Y}_t) \|_{L^p(\Omega, \mathbb{R}^d)}
\]

\[
+ \| R_F(t, \hat{t}) \|_{L^p(\Omega, \mathbb{R}^d)} \leq C_p (t - \hat{t}).
\]

4.2 Strong convergence

We now state and prove our main convergence result.

Theorem 2. Let Assumption 7 hold. Let \( X_t \) be the solution of (2) and \( \hat{Y}_t \) the solution from TamedEI0, 8. Then there exists a family of real numbers \( C_{p, T} \leq 0 \), independent of \( \Delta t \) with \( p \geq 1 \) such that

\[
\left( E \left[ \sup_{t \in [0, T]} \| X_t - \hat{Y}_t \|^p \right] \right)^{\frac{1}{p}} \leq C_{p, T} \Delta t.
\]
Proof. We prove the result for $p \geq 4$ and then from Hölder’s inequality, we can conclude the desired result for $1 \leq p < 4$.

By considering (23), the difference between the exact and numerical solution is given by

$$d(X_t - \tilde{Y}_t) = \left( A(X_t - \tilde{Y}_t) + F(X_t) - \Phi_{s,2} \hat{F}(\tilde{Y}_t) \right) dt + \sum_{i=1}^{m} B_i(X_t - \tilde{Y}_t) dW_i^t. \quad (36)$$

Applying the Itô formula we get

$$d\left( \|X_t - \tilde{Y}_t\|^2 \right) = \left( 2 \langle X_t - \tilde{Y}_t, A(X_t - \tilde{Y}_t) \rangle + 2 \langle X_t - \tilde{Y}_t, F(X_t) - \Phi_{s,2} \hat{F}(\tilde{Y}_t) \rangle \right) dt + 2 \sum_{i=1}^{m} \langle X_t - \tilde{Y}_t, B_i(X_t - \tilde{Y}_t) \rangle dW_i^t. \quad (37)$$

By the Cauchy–Schwarz inequality, and using that the error at $t = 0$ is zero, we find for all $t \in [0, T]$

$$\|X_t - \tilde{Y}_t\|^2 \leq 2 \|A\| \int_0^t \|X_s - \tilde{Y}_s\|^2 ds + 2 \int_0^t \langle X_s - \tilde{Y}_s, F(X_s) - \Phi_{s,2} \hat{F}(\tilde{Y}_s) \rangle ds$$

$$\sum_{i=1}^{m} \|B_i\|^2 \int_0^t \|X_s - \tilde{Y}_s\|^2 ds + 2 \sum_{i=1}^{m} \int_0^t \langle X_s - \tilde{Y}_s, B_i(X_s - \tilde{Y}_s) \rangle dW_i^t$$

$$:= \int_0^t I_1(s, \hat{s}) ds + 2 \int_0^t I_2(s, \hat{s}) ds + \int_0^t I_3(s, \hat{s}) ds + 2 \sum_{i=1}^{m} \int_0^t I_4(i)(s, \hat{s}) dW_s^i. \quad (38)$$

For the term $I_2$, we add and subtract the terms $F(\tilde{Y}_s)$ and $\hat{F}(\tilde{Y}_s)$ to get

$$I_2(s, \hat{s}) = \langle X_s - \tilde{Y}_s, F(X_s) - \Phi_{s,2} \hat{F}(\tilde{Y}_s) \rangle$$

$$= \langle X_s - \tilde{Y}_s, F(X_s) - F(\tilde{Y}_s) \rangle + \langle X_s - \tilde{Y}_s, F(\tilde{Y}_s) - \hat{F}(\tilde{Y}_s) \rangle$$

$$+ \langle X_s - \tilde{Y}_s, \hat{F}(\tilde{Y}_s) - \Phi_{s,2} \hat{F}(\tilde{Y}_s) \rangle$$

$$=: I_{2,1}(s, \hat{s}) + I_{2,2}(s, \hat{s}) + I_{2,3}(s, \hat{s}). \quad (39)$$

The one sided global Lipschitz of $F$ gives

$$I_{2,1}(s, \hat{s}) \leq K \|X_s - \tilde{Y}_s\|^2. \quad (39)$$

Noting that

$$\hat{F}(\tilde{Y}_s) = F(\tilde{Y}_s) - \frac{\Delta t}{1 + \Delta t} \left( F(\tilde{Y}_s) \|F(\tilde{Y}_s)\| \right)$$

yields

$$I_{2,2}(s, \hat{s}) = \langle X_s - \tilde{Y}_s, F(\tilde{Y}_s) - F(\tilde{Y}_s) \rangle + \langle X_s - \tilde{Y}_s, \frac{\Delta t}{1 + \Delta t} \|F(\tilde{Y}_s)\| \rangle$$

$$\leq \langle X_s - \tilde{Y}_s, F(\tilde{Y}_s) - F(\tilde{Y}_s) \rangle + 1 + \frac{\Delta t}{1 + \Delta t} \|F(\tilde{Y}_s)\|^4 \quad (40)$$

where the Cauchy–Schwarz inequality and the inequality $2ab \leq a^2 + b^2$ for the second term are applied. Returning to (37), we have

$$\|X_t - \tilde{Y}_t\|^2 \leq \left( 2 \|A\| + \sum_{i=1}^{m} \|B_i\|^2 + 2K \right) \int_0^t \|X_s - \tilde{Y}_s\|^2 ds$$

$$+ 2 \int_0^t \langle X_s - \tilde{Y}_s, F(\tilde{Y}_s) - F(\tilde{Y}_s) \rangle ds + 2 \int_0^t \langle X_s - \tilde{Y}_s, \hat{F}(\tilde{Y}_s) - \Phi_{s,2} \hat{F}(\tilde{Y}_s) \rangle ds$$

$$\|X_t - \tilde{Y}_t\|^2 \leq \left( 2 \|A\| + \sum_{i=1}^{m} \|B_i\|^2 + 2K \right) \int_0^t \|X_s - \tilde{Y}_s\|^2 ds$$

$$+ 2 \int_0^t \langle X_s - \tilde{Y}_s, F(\tilde{Y}_s) - F(\tilde{Y}_s) \rangle ds + 2 \int_0^t \langle X_s - \tilde{Y}_s, \hat{F}(\tilde{Y}_s) - \Phi_{s,2} \hat{F}(\tilde{Y}_s) \rangle ds$$

$$+ \Delta t^2 \int_0^t \|F(\tilde{Y}_s)\|^4 ds + 2 \sum_{i=1}^{m} \int_0^t \langle X_s - \tilde{Y}_s, B_i(X_s - \tilde{Y}_s) \rangle dW_s^i. \quad (41)$$
By taking supremum, we have
\[
\sup_{s \in [0, T]} \|X_s - \bar{Y}_s\|^2 \leq \left( 2 \|A\| + \sum_{i=1}^{m} \|B_i\|^2 + 2K \right) \int_0^T \|X_s - \bar{Y}_s\|^2 \, ds
+ 2 \left( \int_0^T \langle X_r - \bar{Y}_r, F(\bar{Y}_r) - F(\bar{Y}_r) \rangle \, dr \right)
+ 2 \sup_{s \in [0, T]} \int_0^s \langle X_r - \bar{Y}_r, \tilde{F}(\bar{Y}_r) - \Phi_{r \tau} \tilde{F}(\bar{Y}_r) \rangle \, dr
+ \Delta t^2 \int_T^T \|F(\bar{Y}_s)\|^4 \, ds
+ 2 \sup_{s \in [0, T]} \sum_{i=1}^{m} \int_0^s \langle X_r - \bar{Y}_r, B_i(X_r - \bar{Y}_r) \rangle \, dW_r^i.
\] (42)

Working in the space \(L^{p/2}(\Omega, \mathbb{R})\) for \(p \geq 4\), using that \(\|X\|^2 \chi_{L^{p/2}(\Omega, \mathbb{R})} = \|X\|^2 \chi_{L^{p}(\Omega, \mathbb{R}^d)}\) for a random variable in \(\mathbb{R}^d\), we have
\[
\left\| \sup_{s \in [0, T]} \|X_s - \bar{Y}_s\|^2 \right\|_{L^{p/2}(\Omega, \mathbb{R})} \leq \left( 2 \|A\| + \sum_{i=1}^{m} \|B_i\|^2 + 2K \right) \int_0^T \|X_s - \bar{Y}_s\|^2 \chi_{L^{p}(\Omega, \mathbb{R}^d)} \, ds
+ 2 \left( \int_0^T \langle X_r - \bar{Y}_r, F(\bar{Y}_r) - F(\bar{Y}_r) \rangle \, ds \right)
+ 2 \sup_{s \in [0, T]} \int_0^s \langle X_r - \bar{Y}_r, \tilde{F}(\bar{Y}_r) - \Phi_{r \tau} \tilde{F}(\bar{Y}_r) \rangle \, ds
+ \Delta t^2 \int_T^T \|F(\bar{Y}_s)\|^4 \, ds + 2 \left( \int_0^T \langle X_r - \bar{Y}_r, B_i(X_r - \bar{Y}_r) \rangle \, dW_r^i \right)^{1/2}. \] (43)

We consider the last term separately. By applying the triangle inequality, the inequality in [Hutzenthaler et al. 2012, Lemma 3.7], Cauchy–Schwarz inequality, Hölder inequality and arithmetic-geometric mean inequality, we have
\[
2 \left\| \sup_{s \in [0, T]} \sum_{i=1}^{m} \int_0^s \langle X_r - \bar{Y}_r, B_i(X_r - \bar{Y}_r) \rangle \, dW_r^i \right\|_{L^{p/2}(\Omega, \mathbb{R})} \leq 2 \left\| \sup_{s \in [0, T]} \sum_{i=1}^{m} \int_0^s \langle X_r - \bar{Y}_r, B_i(X_r - \bar{Y}_r) \rangle \, dW_r^i \right\|_{L^{p/2}(\Omega, \mathbb{R})}^{1/2}
\leq p \sum_{i=1}^{m} \left( \int_0^T \|X_s - \bar{Y}_s\| \|B_i(X_s - \bar{Y}_s)\|^2 \chi_{L^{p}(\Omega, \mathbb{R}^d)} \, ds \right)^{1/2}
\leq p \sum_{i=1}^{m} \left( \int_0^T \|X_s - \bar{Y}_s\| \|B_i(X_s - \bar{Y}_s)\|_{L^{p}(\Omega, \mathbb{R}^d)} \, ds \right)^{1/2}
\leq p \sum_{i=1}^{m} \left( \int_0^T \|X_s - \bar{Y}_s\|^2 \|B_i(X_s - \bar{Y}_s)\|^2 \chi_{L^{p}(\Omega, \mathbb{R}^d)} \, ds \right)^{1/2}
\leq \sup_{s \in [0, T]} \|X_s - \bar{Y}_s\|_{L^{p}(\Omega, \mathbb{R}^d)} \sum_{i=1}^{m} \left( \int_0^T \|B_i(X_s - \bar{Y}_s)\|^2 \chi_{L^{p}(\Omega, \mathbb{R}^d)} \, ds \right)^{1/2}
\leq \frac{1}{4} \left\| \sup_{s \in [0, T]} \|X_s - \bar{Y}_s\|_{L^{p}(\Omega, \mathbb{R})} \right\|^2 + p^2 m \sum_{i=1}^{m} \int_0^T \|B_i(X_s - \bar{Y}_s)\|^2 \chi_{L^{p}(\Omega, \mathbb{R}^d)} \, ds.
\]
Substitution of this result into (43) results in

\[
\frac{3}{4} \left\| \sup_{s \in [0,T]} \| X_s - \tilde{Y}_s \| \right\|_{L^{p/2}(\Omega, \mathbb{R})}^2 = \frac{3}{4} \left\| \sup_{s \in [0,T]} \| X_s - \tilde{Y}_s \| \right\|_{L^{p}(\Omega, \mathbb{R})}^2 \\
\leq \left( 2 \| A \| + \sum_{i=1}^{m} \| B_i \| (1 + p^2 m) + 1 + 2K \right) \int_{0}^{T} \| X_s - \tilde{Y}_s \|_{L^{p}(\Omega, \mathbb{R})}^2 \, ds \\
+ \Delta^2 \int_{0}^{T} \| F(\tilde{Y}_s) \|_{L^{p}(\Omega, \mathbb{R})}^2 \, ds + 2 \left\| \sup_{s \in [0,T]} \int_{0}^{s} \langle X_r - \tilde{Y}_r, F(\tilde{Y}_r) - F(\tilde{Y}_r) \rangle \, dr \right\|_{L^{p/2}(\Omega, \mathbb{R})} \\
+ 2 \left\| \sup_{s \in [0,T]} \int_{0}^{s} \langle X_r - \tilde{Y}_r, \tilde{F}(\tilde{Y}_r) - \Phi_{r, \tilde{r}} \tilde{F}(\tilde{Y}_r) \rangle \, dr \right\|_{L^{p/2}(\Omega, \mathbb{R})} \\
:= T_1 + T_2 + T_3 + T_4. 
\]

We need to expand the third and fourth terms. Starting with third term by applying Cauchy–Schwarz inequality and arithmetic-geometric mean inequality, we have

\[
T_3 = 2 \left\| \sup_{s \in [0,T]} \int_{0}^{s} \langle X_r - \tilde{Y}_r, F(\tilde{Y}_r) - F(\tilde{Y}_r) \rangle \, dr \right\|_{L^{p/2}(\Omega, \mathbb{R})} \\
\leq T_{3,1} + 2 \left\| \sup_{s \in [0,T]} \int_{0}^{s} \langle X_r - \tilde{Y}_r, R(r, \tilde{r}) \rangle \, dr \right\|_{L^{p/2}(\Omega, \mathbb{R})} \\
\leq T_{3,1} + \int_{0}^{T} \| X_r - \tilde{Y}_r \|_{L^{p}(\Omega, \mathbb{R})}^2 \, dr + \int_{0}^{T} \| R(r, \tilde{r}) \|_{L^{p}(\Omega, \mathbb{R})}^2 \, dr
\]

(45)

where \( R \) is the remainder term explicitly defined in (44) and

\[
T_{3,1} = 2 \left\| \sup_{s \in [0,T]} \int_{0}^{s} \langle X_r - \tilde{Y}_r, D F(\tilde{Y}_r) \left( \sum_{i=1}^{m} B_i \tilde{Y}_r \right) (W_i^r - W_i^\tilde{r}) \rangle \, dr \right\|_{L^{p/2}(\Omega, \mathbb{R})}.
\]

To determine the order of \( T_{3,1} \) we expand \( X_r \) and \( \tilde{Y}_r \) as follows

\[
X_r = \Phi_{r, \tilde{r}} \left( X_{\tilde{r}} + \int_{\tilde{r}}^{r} \Phi_{r, \tilde{r}}^{-1} F(X_{\tau}) \, d\tau \right), \quad \tilde{Y}_r = \Phi_{r, \tilde{r}} \left( \tilde{Y}_{\tilde{r}} + \int_{\tilde{r}}^{r} \tilde{F}(\tilde{Y}_{\tau}) \, d\tau \right)
\]

and take the difference to obtain

\[
X_r - \tilde{Y}_r = \Phi_{r, \tilde{r}} \left( X_{\tilde{r}} - \tilde{Y}_{\tilde{r}} \right) + \Phi_{r, \tilde{r}} \int_{\tilde{r}}^{r} \left[ \Phi_{r, \tilde{r}}^{-1} F(X_{\tau}) - F(X_{\tau}) \right] \, d\tau \\
+ \Phi_{r, \tilde{r}} \int_{\tilde{r}}^{r} \left[ F(X_{\tau}) - \tilde{F}(\tilde{Y}_{\tau}) \right] \, d\tau.
\]

(46)

By adding and subtracting \( \Phi_{r, \tilde{r}} F(X_{\tau}) \), we have

\[
X_r - \tilde{Y}_r = \Phi_{r, \tilde{r}} \int_{\tilde{r}}^{r} \left[ F(X_{\tau}) - F(X_{\tau}) \right] \, d\tau + \Phi_{r, \tilde{r}} \int_{\tilde{r}}^{r} \left[ \Phi_{r, \tilde{r}}^{-1} F(X_{\tau}) - F(X_{\tau}) \right] \, d\tau \\
+ \Phi_{r, \tilde{r}} \left( X_{\tilde{r}} - \tilde{Y}_{\tilde{r}} + (r - \tilde{r}) F(X_{\tilde{r}}) - (r - \til{r}) \til{F}(\til{Y}_{\til{r}}) \right).
\]

(47)
Expanding $T_{3,1}$ using (47) we have

$$T_{3,1} \leq 2 \sup_{s \in [0,T]} \left\| \int_{0}^{s} \left( \Phi_{r,\hat{r}} \int_{\hat{r}}^{r} \left[ F(X_{t}) - F(X_{\hat{r}}) \right] dt, \Theta_{r,\hat{r}} \right) dr \right\|_{L^{p/2}(\Omega, \mathbb{R})} + 2 \sup_{s \in [0,T]} \left\| \int_{0}^{s} \left( \Phi_{r,\hat{r}} \int_{\hat{r}}^{r} \left[ \Phi_{r,\hat{r}}^{-1} F(X_{t}) - F(X_{\hat{r}}) \right] dt, \Theta_{r,\hat{r}} \right) dr \right\|_{L^{p/2}(\Omega, \mathbb{R})} + 2 \sup_{s \in [0,T]} \left\| \int_{0}^{s} \left( \Phi_{r,\hat{r}} (X_{r} - \hat{Y}_{r}), \Theta_{r,\hat{r}} \right) dr \right\|_{L^{p/2}(\Omega, \mathbb{R})}$$

where

$$\Theta_{r,\hat{r}} = DF(\hat{Y}_{r}) \left( \sum_{i=1}^{m} B_{i} \hat{Y}_{r} \right) \left( W_{r}^{i} - \hat{W}_{r}^{i} \right)$$

and

$$\zeta_{r,\hat{r}} = (r - \hat{r}) F(X_{r}) - (r - \hat{r}) \hat{F}(\hat{Y}_{r}).$$

By definition and by order property of Brownian increment in $L^{p}(\Omega, \mathbb{R}^{d})$ and polynomial growth condition on $DF$ and bounded moments of numerical solutions, it is straightforward to write

$$\left\| \Theta_{r,\hat{r}} \right\|_{L^{p}(\Omega, \mathbb{R}^{d})} \leq C_{p} \sqrt{r - \hat{r}}. \quad (48)$$

By applying Cauchy–Schwarz and Hölder inequalities respectively, we have

$$T_{3,1,1} \leq \int_{0}^{T} \left\| \Phi_{r,\hat{r}} \right\|_{L^{p}(\Omega, \mathbb{R})} \left\| F(X_{t}) - F(X_{\hat{r}}) \right\|_{L^{2p}(\Omega, \mathbb{R}^{d})} \left\| \Theta_{r,\hat{r}} \right\|_{L^{p}(\Omega, \mathbb{R}^{d})} dr \leq C_{p,T} \Delta t^{2}$$

where we have used Lemma[3ii, Lemma[9] equation (45) and boundedness of the terms stated in Lemmas[6] and[8] in a similar way, but considering the inequality (ii) given in Lemma[10] we have

$$T_{3,1,2} \leq 2 \int_{0}^{T} \left\| \Phi_{r,\hat{r}} \int_{\hat{r}}^{r} \left[ \Phi_{r,\hat{r}}^{-1} F(X_{t}) - F(X_{\hat{r}}) \right] dt \right\|_{L^{p}(\Omega, \mathbb{R}^{d})} \left\| \Theta_{r,\hat{r}} \right\|_{L^{p}(\Omega, \mathbb{R}^{d})} dr \leq C_{p,T} \Delta t^{2}.$$

By adding and subtracting, $\zeta_{r,\hat{r}}$ to the first term of the inner product in $T_{3,1,3}$, we have

$$T_{3,1,3} = 2 \sup_{s \in [0,T]} \left\| \int_{0}^{s} \left( \Phi_{r,\hat{r}} \zeta_{r,\hat{r}}, \Theta_{r,\hat{r}} \right) dr \right\|_{L^{p/2}(\Omega, \mathbb{R})} \leq 2 \sup_{s \in [0,T]} \left\| \int_{0}^{s} \left( \Phi_{r,\hat{r}} \zeta_{r,\hat{r}} - \zeta_{r,\hat{r}}, \Theta_{r,\hat{r}} \right) dr \right\|_{L^{p/2}(\Omega, \mathbb{R})} + 2 \sup_{s \in [0,T]} \left\| \int_{0}^{s} \left( \zeta_{r,\hat{r}}, \Theta_{r,\hat{r}} \right) dr \right\|_{L^{p/2}(\Omega, \mathbb{R})}$$

The first term above is of order $\Delta t^{2}$ as in $T_{3,1,1}$ and $T_{3,1,2}$. The second term, which is denoted by $J_{4}$ in [Wang and Gan 2013], is determined as $O(\Delta t^{2})$ in [Wang and Gan 2013], Eq. (3.45) to Eq. (3.56) by use of Doob’s maximal inequality and a Burkholder-Davis-Gundy type inequality for discrete-time martingales. Therefore, we find $T_{3,1,1} < C \Delta t^{2}$.
We now examine $T_{3,1,4}$. By adding and subtracting $X_t - \hat{Y}_t$ to the first term of the inner product in $T_{3,1,4}$, we have

$$T_{3,1,4} \leq 2 \left| \sup_{s \in [0,T]} \int_0^s \langle \Phi_{r,t} (X_t - \hat{Y}_t) - (X_t - \hat{Y}_t), \Theta_{r,t} \rangle dr \right|_{L^p(\Omega, \mathbb{R})} + 2 \left| \sup_{s \in [0,T]} \int_0^s \langle (X_t - \hat{Y}_t), \Theta_{r,t} \rangle dr \right|_{L^p(\Omega, \mathbb{R})} =: T_{3,1,4,1} + T_{3,1,4,2}.$$

Applying Hölder’s inequality and the arithmetic-geometric mean inequality to $T_{3,1,4,1}$

$$T_{3,1,4,1} \leq \frac{1}{4} \sup_{s \in [0,T]} \left| X_s - \hat{Y}_s \right|_{L^p(\Omega, \mathbb{R})}^2 + C \Delta t^2.$$

The term $T_{3,1,4,2}$, which is denoted by $J_5$ in [Wang and Gan, 2013, Lemma 3.7], is also bounded in a similar way to $T_{3,1,4,1}$.

Finally putting these bounds together for the third term in (44), we have

$$T_3 \leq C_{p,T} \Delta t^2 + \frac{1}{4} \sup_{s \in [0,T]} \left| X_s - \hat{Y}_s \right|_{L^p(\Omega, \mathbb{R})}^2 + C_{p,T} \int_0^T \left| X_s - \hat{Y}_s \right|_{L^p(\Omega, \mathbb{R})}^2 ds. \tag{49}$$

The term $T_4$ in (44) remains to be considered. By the expansion (40), we see that

$$T_4 = 2 \left| \sup_{s \in [0,T]} \int_0^s \langle X_t - \hat{Y}_t, \hat{F}(\hat{Y}_t) - \Phi_{r,t} \hat{F}(\hat{Y}_t) \rangle dr \right|_{L^p(\Omega, \mathbb{R})} \leq 2 \left| \sup_{s \in [0,T]} \int_0^s \left( X_t - \hat{Y}_t - \left( \sum_{i=1}^m B_{t} \hat{Y}_t \right) \right) \langle W_t^i - W_t^i \rangle dr \left|_{L^p(\Omega, \mathbb{R})} \right| + 2 \left| \sup_{s \in [0,T]} \int_0^s \langle X_t - \hat{Y}_t - R_{GBM}(r, \hat{r}) \rangle dr \right|_{L^p(\Omega, \mathbb{R})}.$$

Denoting the first term by $T_{4,1}$ and using the Cauchy-Schwarz inequality

$$T_4 \leq T_{4,1} + \int_0^T \left| X_s - \hat{Y}_s \right|_{L^p(\Omega, \mathbb{R})}^2 ds + \int_0^T \left| R_{GBM}(s, \hat{s}) \right|_{L^p(\Omega, \mathbb{R})}^2 ds \tag{50}$$

where $R_{GBM}$ is remainder term which is explicitly defined in (31) and its order is determined in Lemma [11]. If the expansion of $X_t - \hat{Y}_t$ given in (47) is substituted in $T_{4,1}$, by following same steps for $T_3$, we obtain exactly the same bound as for $T_3$. Inserting these bounds for $T_3$ and $T_4$ into (44), and applying the Gronwall inequality completes the proof for $p \geq 4$.

5 Extensions of the scheme and numerical results

5.1 An Adaptive GBM based scheme with TamedEI0 as a backstop method

The drawback of taming methods is the use of the modified drift functions at every step, even in the case that the numerical solution remains small. As noted in the introduction, in the recent literature, this issue is handled in several ways. We examine the adaptive methods proposed by Kelly et al. [Kelly and Lord 2016, Kelly et al. 2019] that employ both usual and tamed time stepping schemes. If the numerical solution stays in the region determined by the admissibility condition, usual time stepping scheme is applied. Otherwise, a tamed version of the time stepping scheme is chosen (termed a backstop method).
In our numerical experiments in Section 5.3 we observe an adaptive strategy employing EI0 introduced in [Lord and Erdogan 2018] and the TamedEI0 as a backstop performs remarkably well.

The scheme we examine numerically is given by

\[ Y_{n+1}^{adp} = \Phi_{t_{n+1},t_n} \left( Y_n^{adp} + F(Y_n^{adp})h_{n+1} \mathbb{1}_{\{h_{n+1} > h_{\min}\}} + \tilde{F}(Y_n^{adp}) \mathbb{1}_{\{h_{n+1} = h_{\min}\}} \right) \]  

where

\[ h_{n+1}(Y_n) = \max \left\{ h_{\min}, \min \left\{ h_{\max} \frac{\max \| Y_n \|}{h_{\min}} \right\} \right\} \]

and \( t_n = \sum_{i=1}^{n} h_i \) with fixed ratio \( \frac{h_{\max}}{h_{\min}} = \rho \).

To prove convergence of the scheme (51), one would need to follow the steps in [Kelly and Lord 2016, Kelly et al. 2019]. The major issue for (51) would be to establish suitable one-step error bounds as in [Kelly et al. 2019, Section 3] (rather than the final time bounds here) and this is the subject of future work.

5.2 Application to a larger class of SDEs

The appearance of non-linear \( g_i : \mathbb{R}^d \rightarrow \mathbb{R}^d \) functions in diffusion coefficients in (2) yields a semi-linear SDE

\[ dX_t = (AX_t + F(X_t)) \, dt + \sum_{i=1}^{m} (B_iX_t + g_i(X_t)) \, dW_t^i \]  

where \( X_0 = \xi \in \mathbb{R}^d \) and \( m, \, d \in \mathbb{N} \). We proposed the following scheme in [Lord and Erdogan 2018] for globally Lipschitz \( F \) and \( g_i \) functions

\[ Y_{n+1}^N = \Phi_{t_{n+1},t_n} \left( Y_n + \left( F(Y_n^N) - \sum_{i=1}^{m} B_i g_i(Y_n^N) \right) \Delta t + \sum_{i=1}^{m} g_i(Y_n^N) \Delta W_t^i \right) \]  

and proved strong convergence of order \( \frac{1}{2} \).

When the drift term \( F \) satisfies Assumption 1 it is natural to propose drift tamed versions of (53) by utilizing the modified \( \tilde{F} \) in (9) instead of \( F \) in the above and would expect convergence rate of order 1/2.

5.3 Numerical Results

In this section, we illustrate convergence and compare the methods in this paper with recent methods appearing in the literature. In the following examples, the acronyms GBM and AGBM correspond to the the tamed GBM method TamedEI0 with fixed step size and the adaptive version of GBM of (51) respectively introduced here. First we examine some small dimensional SDEs and then the discretization of an SPDE.

For the SDE examples we denote the tamed Milstein of [Wang and Gan 2013] by TM and the adaptive Milstein method of [Kelly et al. 2019] by AM. We let PM denote the Projected Milstein of [Beyn et al. 2017]. We take 2000 realizations in each case and present error bars based on 20 groups of 100 on the root mean square error (RMSE). In each case we take 10^6 steps to form a reference solution and (other than where an analytic solution was available) we used the Tamed Milstein method to construct this. For efficiency we plot the RMSE against an average clock time from the computations, termed cputime below.

In our final example, based on a spectral discretization of a stochastic PDE we compare the GBM based methods to a semi-implicit tamed and exponential based tamed methods and we examine the effectiveness and reduction in order of the scheme proposed in Section 5.2.

5.4 Ginzburg Landau Equation

Consider the scalar SDE

\[ dX_t = \left( -X_t + \sigma \frac{\partial}{\partial X_t} X_t^3 - X_t^3 \right) \, dt + \sqrt{\sigma} X_t \, dW(t) \]  

with exact solution as given in [Kloeden and Platen 2011]

\[ X_t = \frac{X_0 e^{-t + \sqrt{\pi} W(t)}}{\sqrt{1 + 2X_0^2 \int_0^t e^{-2s + 2\sqrt{\pi} W(s)} \, ds}}. \]
Consider the SDE model for HIV internal dynamics introduced in Cresson et al. [2016], Dalal et al. [2008]

\[
\begin{align*}
    dT &= (\lambda - \mu T - kTV)dt + \sigma_1 T dW^1 \\
    dI &= (kTV - \alpha I)dt + \sigma_2 I dW^1 \\
    dV &= (cI - \gamma V - kTV)dt + \sigma_3 V dW^2
\end{align*}
\]

where the parameters values taken are given in the caption to Fig. 5.5 and \( T_0 = 0.5, I_0 = 0.7 \) and \( V_0 = 0.9 \). To construct the GBM based methods, we take \( A = \text{diag}(-\mu, -\alpha, -\gamma) \) and \( \sum_{i=1}^{2} B_i = \text{diag}(\sigma_1, \sigma_2, \sigma_3) \). In Fig. 5.5 (a) we see the two adaptive methods are more accurate than the fixed step methods and that the overall AGBM is the most accurate and from (b) the most efficient. GBM performs well over a large range of step sizes and this leads to an efficient scheme in (b). We note that the projected method has a drop in the error as \( \Delta t \) is reduced as there is less need for the projection.

5.6 Lotka–Volterra SDE

We consider a stochastic Lotka–Volterra model as studied in Kelly and Lord [2016].

\[
\begin{align*}
    dX &= X(\lambda - \beta Y)dt + \sigma_1 X dW^1 \\
    dY &= Y(\gamma X - \delta)dt + \sigma_2 Y dW^2
\end{align*}
\]

The parameter values taken are given in Fig. 3 and for initial data we took \( X_0 = 5 \) and \( Y_0 = 10 \). We constructed the GBM based methods by taking \( A = \text{diag}(\lambda, -\delta) \) and \( \sum_{i=1}^{2} B_i = \text{diag}(\sigma_1, \sigma_2) \). In Fig. 3 we see GBM has the smallest error amongst the fixed step methods and that AGBM is the most accurate overall - indeed the tamed Milstein reference solution is not sufficiently accurate to observe further convergence. In this example the error in Pmil is large and we are only just starting to see convergence as \( \Delta t \) is reduced. It is interesting to note that the fixed step GBM in this case is more efficient than the adaptive Amil.
5.7 Model of tumor growth

Consider the growth model for tumor cells under the influence of random perturbations which is analysed in Lisei and Julitz [2008]

\[ dp_t = \left( \lambda \ln \frac{\mu}{p_t} - G(v(t)) \right) p_t dt + \sigma p_t dW \]  

(59)

where \( p_t \) the number of cancerous tumor cells and \( v(t) = (1 + \cos(t))^{-1} \) is the dose of the drug at time \( t \), \( G(v(t)) = k_1 v(t)(k_2 + v(t))^{-1} \) is the destroying rate per tumor cell and time unit. To apply the GBM based integrators, we take
Figure 4: Stochastic tumor growth model \(59\) with parameters \(\lambda = 1, \mu = 1, k_1 = k_2 = 1, \sigma = 1.5.\) In (a) RMSE against \(\Delta t\) and reference line with slope 1 in (b) RMSE against cputime.

\[ A = 0, \quad F(p) = \left(\lambda \ln p - G(v)\right) p \text{ and } B = \sigma. \]  

The resulting scheme becomes

\[ p_{n+1} = \exp\left(-\frac{1}{2} \sigma^2 \Delta t + \sigma \Delta W_n\right) (p_n + \Delta t F(p_n)). \]

We take \(p_0 = 0.8\) and \(T = 1.\) In Fig. 4 we see that AGBM is again the most accurate and efficient although in this example Pmil is the leading fixed step method once in the asymptotic regime.

### 5.8 Semi-Linear SPDE

Consider the stochastic reaction-diffusion equation

\[
\begin{align*}
    du &= \left[\frac{\partial^2 u}{\partial x^2} + u - \gamma u^3\right] dt + \left[\beta u + \alpha \frac{1}{1+u^2}\right] dW, \\
    u(x,0) &= \sin(\pi x),
\end{align*}
\]

with \(x \in [0,1]\) subject to zero Dirichlet boundary conditions. We take \(W\) to be a \(Q\)-Wiener process and let the covariance operator \(Q\) have orthonormal eigenfunctions \(g_j(x) = \sqrt{2} \sin(j\pi x)\) and eigenvalues \(\nu_j = \frac{1}{j^2}, j \in \mathbb{N},\) so that

\[ W = \sum_{j \in \mathbb{N}} \frac{1}{j} g_j \beta_j, \]

where \(\beta_j\) are iid Brownian motions.

We applied the spectral Galerkin method in space, described for example in [Jentzen and Röckner, 2015], with \(d = 128\) Fourier components to obtain a system of SDEs. For this large system we took \(M = 500\) samples, taking groups of 20 to estimate the standard deviation in the root mean square error (RMSE) and \(10^5\) steps to construct the reference solution.

When \(\alpha = 0\) this system of SDEs is of the form of (2) and we can apply (8) to the system of SDEs. For \(\alpha \neq 0\) the SDEs are in the form of (52) and we can apply the tamed scheme in 5.2. For \(\alpha = 0\) we benchmark against a semi-implicit tamed Milstein Imp Mil, a semi-implicit adaptive tamed Milstein Adaptive Imp Mil and exponential versions Exp Mil and Adaptive Exp Mil. These schemes are chosen so that mean-square linear stability is not an issue and for SDEs they would all be first order. For \(\alpha \neq 0\) we compare to a semi-implicit EM method Imp EM and an adaptive version Adaptive Imp EM as well as exponential versions ETD and Adaptive ETD. Again mean-square linear stability is not an issue and for SDEs they would be order 1/2.

In Fig. 5 we examine linear noise \((\alpha = 0)\) and we observe convergence of order 1 in all methods and see that Adaptive GBM is the most efficient and GBM the most efficient fixed step method.
Figure 5: SPDE (60) with $\alpha = 0$, $\beta = 1$, $\gamma = 1$ with $M = 500$ samples (a) RMSE against $\Delta t$ and reference line with slope 1 (b) RMSE against cputime.

Figure 6: SPDE (60) with $\alpha = 0.5$, $\beta = 1$, $\gamma = 0.25$ with $M = 500$ samples (a) RMSE against $\Delta t$ and a reference line with slope $\frac{1}{2}$ (b) RMSE against cputime.

We now include nonlinearity in the noise and in Fig. 6 we take $\alpha = 0.5$ and in Fig. 7 we have $\alpha = 1$. For a weaker nonlinearity in Fig. 6 we see the GBM methods are the most competitive. When the noise is strongly nonlinear in Fig. 7 we see all the methods display the same convergence and in efficiency there is a slight advantage to the adaptivity. In Fig. 6 although adaptive methods are more accurate, fixed step size methods are ahead in terms of CPU times.
Figure 7: SPDE (60) with $\alpha = 1$, $\beta = 0.1$, $\gamma = 1$ with $M = 500$ samples (a) RMSE against $\Delta t$ with a reference line with slope $\frac{1}{2}$. (b) RMSE against cputime.

A Proof of results from Section 3

Proof of Lemma 2 On $\Omega_{n+1}^N$ by construction we have that $\|\sum_{i=1}^{m} B_i \Delta W_{t_n}^i \| \leq 1$ for $n = 0, 1, \ldots, N-1$. Therefore, $\| \Phi_{t_{n+1}, t_n} \| < \infty$ for $n = 0, 1, \ldots, N-1$. We prove the Lemma on two subsets of $\Omega_{n+1}^N$.

1) $S_{n+1}^{(1)} := \Omega_{n+1}^N \cap \{ \omega \in \Omega |\|Y^N_n(\omega)\| \leq 1 \}$

2) $S_{n+1}^{(2)} := \Omega_{n+1}^N \cap \{ \omega \in \Omega |1 \leq \|Y^N_n(\omega)\| \leq N^{1/(2c)} \}$.

First, on $S_{n+1}^{(1)}$, we have from (63) and the triangle inequality that

$$\|Y^N_{n+1}\| \leq \|\Phi_{t_{n+1}, t_n}\| \left( \|Y^N_n\| + \Delta t \|F(Y^N_n)\| \right).$$

Since $\|Y^N_n\| \leq 1$ on $S_{n+1}^{(1)}$, and by the taming inequality (18), we have that

$$\|Y^N_{n+1}\| \leq \|\Phi_{t_{n+1}, t_n}\| \left( 1 + \Delta t \|F(Y^N_n)\| \right).$$

Adding and subtracting $F(0)$ and applying the triangle inequality we get

$$\|Y^N_{n+1}\| \leq \|\Phi_{t_{n+1}, t_n}\| \left( 1 + \Delta t \|F(Y^N_n) - F(0)\| + \Delta t \|F(0)\| \right).$$

Applying the Mean Value Theorem for $F$ and using the growth bound on the gradient of $F$ from Assumption 1 we get

$$\|F(Y^N_n) - F(0)\| = \|DF(\theta Y^N_n) Y^N_n\| \leq K \left( 1 + \|Y^N_n\|^\gamma \right) \|Y^N_n\|,$$

where $0 < \theta < 1$. Thus, from (64) and due to the fact that $\Delta t < T$, we have

$$\|Y^N_{n+1}\| \leq \|\Phi_{t_{n+1}, t_n}\| \left( 1 + TK \left( 1 + \|Y^N_n\|^\gamma \right) \|Y^N_n\| + T \|F(0)\| \right).$$

Now using that $\|Y^N_n\| \leq 1$ and $\|\sum_{i=1}^{m} B_i \Delta W_{t_n}^i \| \leq 1$, we get on $S_{n+1}^{(1)}$

$$\|Y^N_{n+1}\| \leq \|\Phi_{t_{n+1}, t_n}\| \left( 1 + 2TK + T \|F(0)\| \right) \leq \lambda.$$

Secondly: on the set $S_{n+1}^{(2)}$, we start from (8) by squaring the norm,

$$\|Y^N_{n+1}\|^2 \leq \|\Phi_{t_{n+1}, t_n}\|^2 \|Y^N_n + \Delta t \tilde{F}(Y^N_n)\|^2$$

$$\leq \|\Phi_{t_{n+1}, t_n}\|^2 \left( \|Y^N_n\|^2 + 2\Delta t^2 \|F(Y^N_n)\|^2 + 2\Delta t(Y^N_n, F(Y^N_n)) \right).$$
where we have again used (18) on the last two terms. By the polynomial growth bound on $D^2 F$ (see Assumption 1) on $S^2_{n+1} (1 \leq \|Y^N_n (\omega)\| \leq N^{(1/2c_2)})$ we have
\[ \| F(Y^N_n) \|^2 \leq \left( \| F(Y^N_n) - F(0) \| + \| F(0) \| \right)^2 \leq N \left( 2K + \| F(0) \| \right)^2 \| Y^N_n \|^2. \]

The one sided Lipschitz condition on $F$ (Assumption 1) and Cauchy–Schwarz inequality give that
\[ \langle Y^N_n, F(Y^N_n) \rangle \leq \langle Y^N_n, F(Y^N_n) - F(0) \rangle + \langle Y^N_n, F(0) \rangle \leq (K + \| F(0) \|) \| Y^N_n \|^2, \]
where we have used that $1 \leq \| Y^N_n \|$. As a result, and since $\Delta t = T/N$, we see that (67) becomes
\[
\| Y^N_{n+1} \|^2 \leq \| \Phi_{t_{n+1},t_n} \|^2 \| Y^N_n \|^2 \left( 1 + \frac{2}{N} \left( (2TK + T \| F(0) \|)^2 + (TK + T \| F(0) \|) \right) \right) \\
\leq \| \Phi_{t_{n+1},t_n} \|^2 \| Y^N_n \|^2 (1 + 2\lambda/N) \leq \| Y^N_n \|^2 e^{2\lambda N}. (68)
\]

The base case of induction for $n = 0$ is obvious by initial condition on $\Omega^0 = \Omega$. Let $l \in \{0, 1, \ldots, N-1\}$ and assume $\| Y_n (\omega) \| \leq D^N_n (\omega)$ holds for all $n \in 0, 1, \ldots, l$ where $\omega \in \Omega^N_n$. We now prove that
\[ \| Y_{l+1} (\omega) \| \leq D^N_{l+1} (\omega) \]
for all $\omega \in \Omega^N_{t_{l+1}}$. For all $\omega \in \Omega^N_{t_{l+1}}$, we have $\| Y_n (\omega) \| \leq D^N_n (\omega) \leq N^{1/(2c_2)}$, $n \in 0, 1, \ldots, l$ by induction hypothesis and $\Omega^N_{t_{l+1}} \subseteq \Omega^N_{t_{l+1}}$. For any $\omega \in \Omega^N_{t_{l+1}}$, $\omega$ belongs to $S^{(1)}_{n+1}$ or $S^{(2)}_{n+1}$. For inductive argument we define a random variable
\[ \tau^N_{l+1} (\omega) := \max \left\{ -1 \cup \{ n \in \{0, 1, \ldots, l - 1\} \mid \| Y_n (\omega) \| \leq 1 \} \right\} \]
as done in [Hutzenthaler et al. 2012]. This definition implies that $1 \leq \| Y_n (\omega) \| \leq N^{1/(2c_2)}$ for all $n \in \{ \tau^N_{l+1} (\omega) + 1, \tau^N_{l+2} (\omega), \ldots, l \}$. By estimation (68)
\[ \| Y^N_{l+1} (\omega) \| \leq \| \Phi_{t_{l+1},t_l} (\omega) \| \| Y^N_l (\omega) \| e^{\lambda N} \leq \ldots \leq \| Y^N_{\tau^N_{l+1} (\omega) + 1} (\omega) \| \prod_{n=\tau^N_{l+1} (\omega) + 1}^l \| \Phi_{t_{n+1},t_n} (\omega) \| e^{\lambda N} \leq \| Y^N_{\tau^N_{l+1} (\omega) + 1} (\omega) \| e^{\lambda} \sup_{u \in \{0, 1, \ldots, l+1\}} \prod_{n=u}^l \| \Phi_{t_{n+1},t_n} (\omega) \| \]
By considering (66), the following completes the induction step and proof.
\[ \| Y^N_{l+1} (\omega) \| \leq (\lambda + \| \xi (\omega) \|) e^{\lambda} \sup_{u \in \{0, 1, \ldots, l+1\}} \prod_{n=u}^l \| \Phi_{t_{n+1},t_n} (\omega) \| \]

**Proof of Lemma 2** By Hölder’s inequality and independence of the Brownian increments
\[
\sup_{N \in \mathbb{N}} \sup_{n \in \{0, 1, \ldots, N\}} D^N_n \left\|_{\mathbb{L}^p (\Omega, \mathbb{R})} \right. \leq e^{\lambda} (\lambda + \| \xi \|_{\mathbb{L}^2p (\Omega, \mathbb{R})}) \left( \sup_{N \in \mathbb{N}} \prod_{k=0}^{N-1} \| \Phi_{t_{k+1},t_k} \|_{\mathbb{L}^{2p} (\Omega, \mathbb{R})} \right) \leq e^{\lambda} (\lambda + \| \xi \|_{\mathbb{L}^2p (\Omega, \mathbb{R})}) \left( \sup_{N \in \mathbb{N}} \prod_{k=0}^{N-1} \mathbb{E} \left( \| \Phi_{t_{k+1},t_k} \|_{\mathbb{L}^{2p} (\Omega, \mathbb{R})}^2 \right)^{\frac{p}{2}} \right). \]

By Lemma 2 we obtain the result.
Proof of Lemma 5. For all $N \in \mathbb{N}$ and $q \in [1, \infty)$, we have
\[
P([\Omega_N^N]) \leq \mathbb{P}[\sup_{0 \leq k \leq N-1} D_k^N > N^{1/2e}] + N\mathbb{P}\left[\sum_{i=1}^{m} B_i W_i^2 \right] > 1]
\leq \mathbb{E}\left[\sup_{0 \leq k \leq N-1} |D_k^N|^q\right] N^{-q/(2e)} + N\mathbb{E}\left[\sum_{i=1}^{m} B_i W_i^2 \right] > \sqrt{N}
\leq \mathbb{E}\left[\sup_{0 \leq k \leq N-1} |D_k^N|^q\right] N^{-q/(2e)} + m^{p-1} \max_{i \in \{1, 2, \ldots, m\}} \|B_i\| m \mathbb{E}\left[\|W_i^1\|^{1} \right] N^{1-q/2}
\]
by definition of $\Omega_N^N$ in [19] and subadditivity of probability measure and Markov inequality. The bounded moments of $D_n^N$ is given in [4]. For given $p, q$ is chosen such that $(N^p \mathbb{P}[|\Omega_N^N|])$ becomes bounded for any $N \in \mathbb{N}$.

Proof of Theorem 7. The iterated numerical solution is given by
\[
Y_n^N = \Phi_{t_n, \theta} \xi + \sum_{k=0}^{n-1} \Phi_{t_{n-k}, t_k} \tilde{F}(Y_k^N) \Delta t.
\]
By taking the norm in the space $L^p(\Omega, \mathbb{R}^d)$, applying the triangle inequality and noticing $\|\tilde{F}(Y_k^N) \Delta t\|_{L^p(\Omega, \mathbb{R}^d)} \leq 1$, we have
\[
\|Y_n^N\|_{L^p(\Omega, \mathbb{R}^d)} \leq \|\Phi_{t_n, \theta} \xi\|_{L^p(\Omega, \mathbb{R}^d)} + \left\|\sum_{k=0}^{n-1} \Phi_{t_{n-k}, t_k} \tilde{F}(Y_k^N) \Delta t\right\|_{L^p(\Omega, \mathbb{R}^d)} \leq C(\|\xi\| + N)
\]
where $C = \sup_{s,t \in [0,T]} \|\Phi_{s,t}\|_{L^p(\Omega, \mathbb{R}^d)}$. The existence of $N$ on the RHS of the inequality (70) appears to be an obstacle in completing the proof. However, following the techniques of [Hutzenthaler et al. 2012, equation (59)]
\[
\sup_{N \in \mathbb{N}} \sup_{0 \leq n \leq N} \|1(\Omega_N^N) Y_n^N\|_{L^p(\Omega, \mathbb{R}^d)} < \infty
\]
and using Lemmas 3 and 4
\[
\sup_{N \in \mathbb{N}} \sup_{0 \leq n \leq N} \|1(\Omega_N^N) Y_n^N\|_{L^p(\Omega, \mathbb{R}^d)} \leq \sup_{N \in \mathbb{N}} \sup_{0 \leq n \leq N} \|D_n^N\|_{L^p(\Omega, \mathbb{R}^d)} < \infty.
\]
Combining (71) and (72) completes the proof.

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