Local gravity test of unified models of inflation and dark energy in $f(R)$ gravity

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We study local-gravity tests on unified models of inflation and dark energy in $f(R)$ gravity. In this paper, we consider three unified models, which are combination of the known dark energy models and the same inflationary term. During inflation, all unified models reduce to the model $f(R) = R + \alpha R^n$, which is a generalization of the $R^2$ inflation model proposed by Starobinsky. From the observational constraint on the tensor-to-scalar ratio and the spectral index, we obtain $1.977 < n < 2.003$. We then investigate local-gravity tests for these unified models. The inflationary term becomes important at extremely high curvature in the action. However, we point out that it can be important at much lower curvature for the scalaron’s potential and then its mass. Investigating how it works in the chameleon screening mechanism, we nonetheless find that the inflationary term does not relax or tighten the constraints on the three unified models in relevant density scales.

I. INTRODUCTION

There have been growing evidences that the Universe experiences two accelerated expansion eras: inflation [1, 3] and the late-time acceleration [4–6]. However, in spite of many attempts, the origin(s) of both cosmic accelerations has/have not been identified yet.

Both for inflation and the late-time acceleration, models are classified into two categories: dark energy (DE) and modified gravity (MG). In the first approach, one introduces exotic matter components such as the cosmological constant or a scalar field, so-called dark energy (DE) for the late-time acceleration. If the exotic component has an equation of state with $w \equiv P/\rho < -1/3$, it drives the accelerated expansion of the Universe. The other approach is to modify the theory of gravity from general relativity (GR) [7–9]. A simple family of the modified gravity is $f(R)$ theories [10, 11], where the Ricci scalar $R$ in the Einstein-Hilbert action is replaced by a nonlinear function $f(R)$. In $f(R)$ gravity, by choosing an appropriate function $f(R)$, the expansion of the Universe is accelerated without introducing any exotic matter components.

An example of an inflationary model in $f(R)$ theories is the Starobinsky model $f(R) = R + \alpha R^2$ [12, 13]. The Starobinsky model is known as the first inflationary model and its prediction gives a good fit to the Planck data for the spectral index $n_s$ and the tensor-to-scalar ratio $r$ [3, 14]. On the other hand, it is known that some other forms of $f(R)$ can drive the late-time acceleration [15, 16].

On this ground, it has been considered whether a single $f(R)$ model can describe both the early and late accelerated expansion eras in a unified way [19,21]. Such unified models should not only be viable as a model for the two accelerations but also satisfy a local-gravity test. In $f(R)$ theories, the law of gravity in a local system is modified as well as that on cosmological scales. In more words, there is an extra scalar degree of freedom with a universal coupling to matter, which is called scalaron [13]. Therefore, it mediates a fifth force and can violate local-gravity constraints. A possible resolution for this problem is given by the chameleon screening mechanism [22, 23], which makes the range of the fifth force short in high-density regions and an $f(R)$ model compatible with the local-gravity constraints. In fact, for some $f(R)$ models of the late-time acceleration [24, 25], the chameleon screening mechanism is known to work without spoiling their success in cosmology.

A subject of this paper is to discuss whether the local-gravity constraints above on the corresponding DE models can be applied to the unified models. A typical unified model is constructed by adding an inflationary higher-curvature term to the function $f(R)$ of a DE model. This higher-curvature term is naively expected to be negligible in a low dense region but we show that it can be important even at a very low density for the scalaron’s potential. Therefore, it should be reanalyzed for unified models whether the local-gravity constraints are unchanged from the corresponding DE models or not.

The unified models should consistently describe the evolution of the Universe from inflation to the current accelerated expansion. We give constraints on the unified models from the inflationary era, i.e. the constraints from the primordial spectral index $n_s$ and the tensor-to-scalar ratio $r$. In addition, we analyze the chameleon screening mechanism with the inflationary term for the three unified models.

This paper is organized as follows. First, in the section II we show basic equations of $f(R)$ theories used in this paper. In the section III we introduce the unified models and their corresponding form in the Einstein frame. In the section IV the inflationary constraints are derived for the unified models, which is extended from our previous
paper \[26\]. In the section \[5\] the local-gravity test is studied for the unified models. Finally, the section \[VI\] is devoted to the conclusion.

In the following, we set \(8\pi G = M_{\text{pl}}^{-2} = 1\), where \(G\) is the gravitational constant and \(M_{\text{pl}}\) is the reduced Planck mass.

II. BASIC EQUATIONS

First, let us briefly show some basic equations of a general \(f(R)\) model used in this paper. (See, e.g., Ref. \[17\] for a more extensive review.)

The action of \(f(R)\) gravity is given by

\[
S = \frac{1}{2} \int d^4x \sqrt{-g} F(R) + \int d^4x \sqrt{-\tilde{g}} \mathcal{L}_m(g_{\mu\nu}, \Phi_m),
\]

where \(\mathcal{L}_m\) is a matter Lagrangian density and a matter field \(\Phi_m\) is assumed to minimally couple to the metric \(g_{\mu\nu}\).

Varying Eq. \[1\] with respect to \(g_{\mu\nu}\), we obtain

\[
F(R) R_{\mu\nu}(g) - \frac{1}{2} f(R) g_{\mu\nu} - \nabla_\mu \nabla_\nu F(R) + g_{\mu\nu} \Box F(R) = T^{(m)}_{\mu\nu},
\]

where \(F(R) \equiv df/dR\) and \(T^{(m)}_{\mu\nu}\) is the energy-momentum tensor of the matter field. The trace of Eq. \[2\] gives

\[
3\Box F + FR - 2f = T^{(m)},
\]

where \(T^{(m)} \equiv g^{\mu\nu} T^{(m)}_{\mu\nu}\). This propagating degree of freedom \(F\) corresponds to the scalaron, which is absent in general relativity because it corresponds to \(F = 1\).

It is sometimes more intuitive to use the Einstein frame action as follows \[27, 28\]. The action \[1\] can be rewritten as

\[
S = \int d^4x \sqrt{-g} \left(\frac{1}{2} F R - U(F)\right) + \int d^4x \mathcal{L}_m(g_{\mu\nu}, \Phi_m),
\]

where \(U(F)\) is defined by

\[
U(F) = \frac{FR - f}{2}.
\]

To obtain the action in the Einstein frame, we make the conformal transformation \[11, 27\],

\[
\tilde{g}_{\mu\nu} = F g_{\mu\nu},
\]

where a tilde denotes a quantity in the Einstein frame.

Here, we introduce a scalar field \(\phi\) by

\[
\phi = \sqrt{\frac{3}{2}} \ln F.
\]

Using Eqs. \[6\] and \[7\], the action \[4\] can be rewritten as

\[
S_E = \int d^4x \sqrt{-\tilde{g}} \left[\frac{1}{2} \tilde{R} - \frac{1}{2} \tilde{g}^{\mu\nu} \partial_\mu \partial_\nu \phi - V(\phi)\right] + \int d^4x \mathcal{L}_m(F^{-1}(\phi) \tilde{g}_{\mu\nu}, \Phi_m),
\]

where \(V(\phi)\) is the field potential in the Einstein frame:

\[
V(\phi) = \frac{U}{F^2} = \frac{FR - f}{2F^2}.
\]

In the Einstein frame, the scalar field equation is given by

\[
\Box \phi = \frac{dV_{\text{eff}}}{d\phi},
\]

where the effective potential is defined as

\[
V_{\text{eff}}(\phi) = V(\phi) + \rho^* e^{-\frac{\phi}{\rho^*}},
\]

for a non-relativistic object \[22\]. Here, \(\rho^*\) is a conserved quantity in the Einstein frame, which is related to the energy density \(\rho\) in the Jordan frame as \(\rho^* = e^{-3\phi/\sqrt{\rho}}\rho\).

In relevant cases, the difference between these two densities is negligible because \(\phi \ll 1\). Therefore, we will omit the asterisk hereafter.

III. \(f(R)\) UNIFIED MODELS

Next, we introduce the unified models used in this paper. As the unified models, we consider three known \(f(R)\) dark energy (DE) models with the inflationary term \(\alpha R^n\) added. The \(f(R)\) DE models should satisfy viable conditions that tachyonic and ghost instabilities are absent. The conditions can be written in terms of \(F\) as

\[
F > 0, \quad F_{,R} \equiv dF/dr > 0, \quad \text{for} \quad r \geq r_0,
\]

where \(r_0\) is the present value of the Ricci scalar. (See, e.g., Ref. \[17\] for more details on these issues.)

A. Model 1: Power-law DE + \(R + \alpha R^n\) model

First, we introduce the following unified model:

\[
f(R) = R + \alpha R^n - \beta R^p,
\]

with \(n > 1, 0 < p < 1\) and \(\alpha, \beta > 0\). This model is an extension of the model proposed by Artymowski and Lalak \[21\], \(f(R) = R + \alpha R^n - \beta R^{2-n}\). For \(\alpha = 0\), this model corresponds to the Power-law DE model \[15, 18\], which satisfies the conditions \[12\].
In the absence of the $\beta$ term, the field potential has a minimum $V = 0$ at $R = 0$, which corresponds to $\phi = 0$ (see Fig. 1). Therefore, the expansion of the Universe is not accelerated without introducing matter. In contrast, when the $\beta$ term is introduced, a minimum appears at $R = R_{\text{min}} \simeq [\beta(2 - p)]^{1/(1 - p)} > 0$ and $V > 0$ at the minimum, while $R = 0$ cannot be reached now because the field value $\phi$ diverges at $R = R_{F=0} \simeq [\beta p]^{1/(1 - p)} > 0$ (see Fig. 2). As in the original model [21], this potential energy drives the late-time acceleration.

FIG. 1. Potential in the Einstein frame for $R^2$ inflation model. We show parametric representations $V(R)$ and $\phi(R)$ in the top and the bottom, respectively. In this model, the potential minimum is given by $V = 0$ at $\phi = 0$. The potential as a function of the field value, $V(\phi)$, is shown in Ref. 17.

Thus, the second condition $F_R > 0$ [12] is satisfied for $n > 1$, $0 < p < 1$ and $\alpha, \beta > 0$. On the other hand, the first condition $F > 0$ is violated for $R < R_{F=0} \simeq [\beta p]^{1/(1 - p)}$. However, the curvature $R$ never dynamically reaches to this region because there is an infinite potential barrier at $R = R_{F=0}$.

B. Model 2: Starobinsky DE + $\alpha R^n$ model

Next, we introduce the following unified model:

$$f(R) = R - \mu R_0 \left[ 1 - \left(1 + \frac{R^2}{R_0^2}\right)^{-j} \right] + \alpha R^n,$$  

(18)

where $j, \mu > 0$. For $\alpha = 0$, this model corresponds to the Starobinsky DE model [25], which also satisfies the conditions [12]. For $R \gg R_0$, this model can be approximated by a power-law model as

$$f(R) \simeq R + \mu R_0 \left( \frac{R}{R_0} \right)^{-2j} + \alpha R^n.$$  

(19)

In the model [18], the field potential in the Einstein frame [19] and its first derivative are read as

$$V(\phi) = e^{-\frac{4}{F} \phi} \left[ \mu R_0 + \alpha(n-1)R^n - \mu R_0 \right.\times \left(1 + \frac{R^2}{R_0^2}\right)^{-j} \left(1 + 2j \left(1 + \frac{R^2}{R_0^2}\right)^{-1}\right),$$  

(20)
FIG. 3. Potential in the Einstein frame for Model 2 (red line) and Starobinsky DE model (blue line). We show parametric representations $V(R)$ and $\phi(R)$ in the top and the bottom, respectively. Dashed lines show $\phi < 0$. The potential is positive at the minimum.

and

$$V_{,\phi} = e^{-\frac{\sqrt{6}}{\sqrt{6}} \phi} \left[ R - 2\mu R_0 + \alpha (2-n) R^n + 2\mu R_0 \left( 1 + \frac{R^2}{R_0^2} \right)^{-j} \left( 1 + j \left( 1 + \frac{R^2}{R_0^2} \right)^{-1} \right) \right],$$

(21)

where

$$e^{\frac{\sqrt{6}}{\sqrt{6}} \phi} = F(R) = 1 + \alpha n R^{n-1} - 2\mu j \frac{R}{R_0} \left( 1 + \frac{R^2}{R_0^2} \right)^{-j-1}.$$  

(22)

In the model (18), $F_{,R}$ becomes

$$F_{,R} = \alpha n (n-1) R^{n-2} - 2\mu j \frac{R}{R_0} \left( 1 + \frac{R^2}{R_0^2} \right)^{-j-1} \times \left[ 1 - 2\mu (j+1) \left( 1 + \frac{R^2}{R_0^2} \right)^{-1} \right].$$

(23)

When the DE term is introduced, a minimum appears at $R = R_{\min} \simeq R_0$ and $V > 0$ at the minimum (see Fig. 3). Therefore, the potential energy drives the late-time acceleration in the model.

FIG. 4. The allowed parameter range of $g$ and $b$ in $g$-AB model. This curve is obtained from the stability conditions (12).

C. Model 3: $gR^n$-AB model

Finally, we introduce the following unified model:

$$f(R) = (1 - g) R + g \epsilon_{AB} \ln \left( \frac{\cosh (R/\epsilon_{AB} - b)}{\cosh b} \right) + \alpha R^n,$$

(24)

where $b > 0$, $0 < g < 1/2$ and,

$$\epsilon_{AB} \equiv \frac{R_0}{2g \ln (1 + e^{2b})}.$$  

(25)

This model is an extension of the model proposed by Appleby et al. [19], where the model has $n = 2$. For $\alpha = 0$ and $g = 1/2$, this model corresponds to the Appleby & Battye (AB) model [29], which also satisfies the conditions [12]. The constraints on $g$ and $b$ are obtained from the stability conditions (12). We show the relation between $g$ and $b$ in Fig. 4 (see [19] for more details). For $R \gg R_0$, the model (24) can be approximated by

$$f(R) \simeq R - \frac{R_0}{2} + g \epsilon_{AB} e^{2b} e^{-2R/\epsilon_{AB}} + \alpha R^n.$$  

(26)

In the model (24), the field potential in the Einstein frame (9) and the first derivative of the potential read,

$$V(\phi) = e^{-\frac{\sqrt{6}}{\sqrt{6}} \phi} \left[ \alpha (n-1) R^n + g R \tanh \left( \frac{R}{\epsilon_{AB}} - b \right) 
- g \epsilon_{AB} \log \left( \frac{\cosh (R/\epsilon_{AB} - b)}{\cosh b} \right) \right],$$

(27)

and

$$V_{,\phi} = e^{-\frac{\sqrt{6}}{\sqrt{6}} \phi} \left[ R (1-g) + \alpha (2-n) R^n - g R \tanh \left( \frac{R}{\epsilon_{AB}} - b \right) 
+ 2g \epsilon_{AB} \log \left( \frac{\cosh (R/\epsilon_{AB} - b)}{\cosh b} \right) \right],$$

(28)
which corresponds to the Einstein frame action with $f$ large during inflation. Hence, the function $F$ in the unified models above can be approximated as

$$F(R) \equiv 1 - g + \alpha n R^{n-1} + g \tanh \left( \frac{R}{\epsilon_{AB} - b} \right).$$

In the model \cite{24}, $F_R$ becomes

$$F_R = \alpha n (n-1) R^{n-2} + \frac{g}{\epsilon_{AB}} \operatorname{sech}^2 \left( \frac{R}{\epsilon_{AB} - b} \right).$$

When the DE term is introduced, a minimum appears at $R = R_{\text{min}} \approx b \epsilon_{AB} > 0$ and $V > 0$ at the minimum. Fig. 5.

FIG. 5. Potential in the Einstein frame for Model 3 (red line) and AB model (blue line). We show parametric representations $V(R)$ and $\phi(R)$ in the top and the bottom, respectively. Dashed lines show $\phi < 0$. The potential is positive at the minimum.

where $e^{\frac{2}{3}n \phi} = F(R)$

$$= 1 - g + \alpha n R^{n-1} + g \tanh \left( \frac{R}{\epsilon_{AB} - b} \right). \quad (29)$$

IV. INFLATIONARY CONSTRAINTS

Next, we derive an inflationary constraint on the unified models. Since inflation is driven without introducing any matter in $f(R)$ theories, we neglect the matter in this section. In addition, the Ricci scalar $R$ is sufficiently large during inflation. Hence, the function $f(R)$ in all the unified models above can be approximated as

$$f(R) \simeq R + \alpha R^n, \quad (31)$$

which corresponds to the Einstein frame action with

$$V(\phi) = V_0 e^{-\frac{2}{3}n \phi} \left[ e^{\frac{2}{3}n \phi} - 1 \right]^{\frac{n}{n-1}}, \quad (32)$$

where $V_0 \equiv \alpha (n-1)/[2(\alpha n)^{n/(n-1)}]$. Since the comoving curvature perturbations and tensor perturbations are invariant under the conformal transformation \cite{9 \cite{30} \cite{31}}, the spectral index $n_s$ and the tensor-to-scalar ratio $r$ can be directly evaluated in the Einstein frame, where the action is equivalent to that in a single-field slow-roll inflationary model. Therefore, they are given as,

$$r = 16 \epsilon_V, \quad n_s = 1 - 6 \epsilon_V + 2 \eta_V, \quad (33)$$

in terms of the slow-roll parameters,

$$\epsilon_V \equiv \frac{1}{2} \left( \frac{V_{\phi\phi}}{V} \right)^2, \quad \eta_V \equiv \frac{V_{\phi\phi\phi}}{V}. \quad (34)$$

In the model \cite{31}, the slow-roll parameters are evaluated as \cite{32},

$$\epsilon_V = \frac{4 E_k^2 (2 - n)^2}{3 [2(n - 1) E_k - n]^2}, \quad (35)$$

$$\eta_V = \frac{4 (2 - n) [2(n - 1) E_k^2 - n E_k + n]}{3 [2(n - 1) E_k - n]^2}, \quad (36)$$

where $E_k \equiv e^{4(2-n)N_k/(3n)}$ and $N_k$ is e-folds from the horizon crossing to the end of inflation. They reduce to the well-known results,

$$\epsilon_V = \frac{3}{4 n E_k^2}, \quad \eta_V = - \frac{1}{N_k^2}, \quad (37)$$

in the limit $n \to 2$.

We compare these predictions with the Planck data \cite{3}:

$$r < 0.01, \quad n_s = 0.9659 \pm 0.0041 \quad (95 \% \text{ CL}). \quad (38)$$

As a result, the index $n$ in the model \cite{31} is constrained as

$$N_k = 50 : 1.977 < n < 1.991 \quad (95 \% \text{ CL}), \quad (39)$$

$$N_k = 60 : 1.991 < n < 2.003 \quad (95 \% \text{ CL}). \quad (40)$$

Note that this constraint is slightly relaxed from our previous result $1.965 < n < 2$ \cite{26}. In the original model $f(R) = R + \alpha R^n - \beta R^{2-n}$, $p$ is fixed to be $p = 2 - n$. Therefore, the viable condition $0 < p < 1$ restricts $n$ as $1 < n < 2$. In the extended model \cite{13}, this constraint is absent.

In addition, we can estimate the model parameter $\alpha$ from the amplitude of the scalar power spectrum,

$$\mathcal{P}_R = \frac{H_E^2}{8 \pi^2 \epsilon_V} \simeq \frac{N_k^2}{144 \pi^2 \alpha}, \quad (41)$$

where $H_E$ is the Hubble parameter in the Einstein frame and we used $n \simeq 2$ in the second equation. Comparing this with the CMB observation $\mathcal{P}_R \sim 10^{-10}$ \cite{3}, we find

$$\alpha \sim 10^{10}, \quad (42)$$

for $N_k = 50 - 60$. 
V. LOCAL-GRAVITY CONSTRAINT

Here, we consider the local-gravity constraint on the unified models \([13], [19]\) and \([26]\) following the analysis in Refs. \([15, 19]\). In this section, we work in the Einstein frame, where it is more clear that the extra degree of freedom, i.e. scalaron, mediates the fifth force as

\[
\bar{F}_\phi = \frac{M}{\sqrt{6}} \nabla \phi. \tag{43}
\]

In this analysis, we study the configuration of the scalar field in the finite density region, which is governed by Eq. \((10)\):

\[
\Box \phi = \frac{dV_{\text{eff}}}{d\phi}; \quad V_{\text{eff}}(\phi) = V(\phi) + \rho e^{-\phi \sqrt{\phi}}. \tag{44}
\]

The minimum of the effective potential \(\phi = \phi_{\text{min}}\) depends on the density \(\rho\) through

\[
\frac{dV_{\text{eff}}}{d\phi} = V_{\phi} - \rho e^{-\frac{\phi_{\text{min}}}{\sqrt{\rho}}} = 0, \tag{45}
\]

and the scalaron’s mass also depends on \(\rho\) through

\[
m_\phi^2 \equiv \frac{d^2V_{\text{eff}}(\phi_{\text{min}})}{d\phi^2} = V_{\phi\phi} + \rho e^{-\frac{\phi_{\text{min}}}{\sqrt{\rho}}}, \tag{46}
\]

at the minimum \(\phi = \phi_{\text{min}}\). These imply that the range of the fifth force changes according to the density in its environment. This is known as the chameleon screening mechanism \([22, 23]\). In the following, we see if the unified models \([13], [19]\) and \([26]\) can evade the local-gravity constraint by the chameleon screening mechanism.

A. Chameleon screening mechanism

First, we briefly review the chameleon screening mechanism for the fifth force sourced by a star, modeling it as a spherically symmetric non-relativistic object with a constant density \(\rho_c\). It is assumed that the star is surrounded by baryons and dark matter with a homogeneous density \(\rho_G \simeq 10^{-24}\text{g/cm}^3\) \([22]\). In this analysis, we focus on the vicinity of the star and neglect the cosmic expansion.

Assuming a static and spherically symmetric profile, the field equation reduces to

\[
d^2\phi + \frac{2}{\tilde{r}} \frac{d\phi}{d\tilde{r}} - \frac{dV_{\text{eff}}}{d\phi} = 0. \tag{47}
\]

In the outer region, the scalaron relaxes to the minimum \(\phi_G \equiv \phi_{\text{min}}(\rho_G)\) as

\[
\phi \simeq \phi_G + \frac{Q e^{-m_G\tilde{r}}}{\tilde{r}}, \tag{48}
\]

where \(m_G \equiv m_\phi(\rho_G)\). Here, \(Q\) is the scalar charge and determined by matching to an inner solution. Inside the star, the effective potential has the minimum at \(\phi_c \equiv \phi_{\text{min}}(\rho_c)\) with a mass \(m_c \equiv m_\phi(\rho_c)\). When the Compton wavelength \(\lambda_c \sim 1/m_c\) is much shorter than the radius of the star \(\tilde{r}_c\),

\[
\lambda_c \ll \tilde{r}_c \quad (m_c \tilde{r}_c \gg 1), \tag{49}
\]

the scalaron stays near \(\phi = \phi_c\) in the inside of the star and has a thin-shell profile. In this case, the scalar charge is given by \([22]\),

\[
\frac{Q}{M_c} = \frac{3}{4\sqrt{6}} \left(\frac{\Delta \tilde{r}_c}{\tilde{r}_c}\right), \tag{50}
\]

in the unit of the mass of the star \(M_c\). Here, the thin-shell parameter \(\Delta \tilde{r}_c/\tilde{r}_c\) is given as,

\[
\Delta \tilde{r}_c = \phi_G - \phi_c; \quad \Phi_c \equiv \frac{M_c}{8\pi\tilde{r}_c}. \tag{51}
\]

Therefore, the fifth force \([43]\) is screened when the thin-shell parameter is small enough.

When the inflationary term is absent, the \(\phi_G\) term is dominant and the models \([13], [19]\) and \([26]\) approximately give,

\[
\left|\frac{\Delta \tilde{r}_c}{\tilde{r}_c}\right| \simeq \begin{cases} \frac{2b}{c} \left(\frac{\rho_c}{\rho_0}\right)^{p-1} & \text{Power-law model} \quad [13] \\ \frac{4b}{c} \left(\frac{\rho_c}{\rho_0}\right)^{-2j-1} & \text{Starobinsky DE model} \quad [19] \\ \left(\frac{g_0^{2b} \rho_G}{\tilde{r}_c}\right)^{-\mu} & \text{g-AB model} \quad [26] \end{cases} \tag{52}
\]

where we have introduced the dimensionless parameter,

\[
\tilde{\beta} \equiv \beta R_0^{-1}. \tag{53}
\]

Here, the parameters \(\tilde{\beta}\) and \(\mu\) are estimated to be order of unity to explain the present value of the dark energy density parameter, \(\Omega_{DE} \simeq 0.7\). In addition, from Fig. \([4]\) \(b > 0\) under the constraint on \(g\). From these expressions, one can see that the thin-shell parameter quickly decreases as \(\rho_G\) increases in the latter two models but only slowly in the first model. Reflecting this fact, the power-law model is tightly constrained by local-gravity tests while the other two are viable. In the table \([4]\), we summarize the solar system constraints for the model parameters in these DE models \([13, 23]\). Here, the constraint is not shown for the \(g\)-AB model because the solar system constraint does not restrict the parameter region under the stability conditions (see Fig. \([4]\)).

B. Constraints in the unified models

As the figures \([24]\) indicate, the inflationary term can affect the scalaron’s potential even for curvatures lower than the inflationary scale. Here, therefore, we see how
the local-gravity constraints can change when the inflationary term $\alpha R^n$ is added. For later convenience, we make $\alpha$ dimensionless as

$$\dot{\alpha} = \alpha R_0^{n-1},$$

which is estimated to be

$$\dot{\alpha} \sim 10^{-110}.$$  

from the normalization \(^{(42)}\).

1. **Model 1: Power-law DE + $R + \alpha R^n$ model**

First, we consider the model \(^{(13)}\):

$$f(R) = R + \dot{\alpha} R_0 \left( \frac{R}{R_0} \right)^n - \beta R_0 \left( \frac{R}{R_0} \right)^p,$$  

where the parameters are made dimensionless as in Eqs. \(^{(53)}\) and \(^{(54)}\). From the table \(\text{I}\), this model is tightly constrained by the solar-system observations. Here, we verify whether the inflationary term can evade this constraint.

The minimum of the effective potential in this model is determined by

$$\frac{R}{R_0} + \dot{\alpha} (2-n) \left( \frac{R}{R_0} \right)^n - \beta (2-p) \left( \frac{R}{R_0} \right)^p = \frac{\rho e^{-\frac{\phi}{\sqrt{6}}} - \dot{\phi}}{R_0}.$$  

(57)

with

$$e^{\frac{\phi}{\sqrt{6}}} = F(R) = 1 + \dot{\alpha} n \left( \frac{R}{R_0} \right)^{n-1} - \beta p \left( \frac{R}{R_0} \right)^{p-1}.$$  

(58)

As mentioned earlier, the coefficients are roughly estimated to be,

$$\dot{\alpha} \sim 10^{-110}, \quad \beta \sim 1.$$  

(59)

On the other hand, for $\rho = O(10^{-24}) \text{ g/cm}^3$, the right hand side is roughly estimated to be,

$$\frac{\rho e^{-\frac{\phi}{\sqrt{6}}}}{R_0} \sim 10^4 - 10^{28}.$$  

(60)

where we have used $R_0 \simeq 12 H_0^2 = 4 \rho_{\text{crit}} \sim 10^{-28} \text{ g/cm}^3$. Therefore, at the minimum in the relevant region, the first term in Eq. \(^{(57)}\) is dominant and the curvature scale is given by

$$\frac{R_{\text{min}}}{R_0} \sim \frac{\rho}{R_0} \sim 10^{28} \left( \frac{\rho}{1 \text{ g/cm}^3} \right).$$  

(61)

The corresponding field value at the minimum is determined by Eq. \(^{(58)}\). For the typical values \(^{(59)}\) and \(^{(61)}\), $\phi_{\text{min}} \ll 1$ and hence

$$2 \frac{\phi_{\text{min}}}{\sqrt{6}} \sim \dot{\alpha} n \left( \frac{R_{\text{min}}}{R_0} \right)^{n-1} - \frac{\beta p}{R_0} \left( \frac{R_{\text{min}}}{R_0} \right)^{p-1}.$$  

(62)

Therefore, the modified terms are important to determine the field value $\phi_{\text{min}}$. Among these two terms, the $\alpha$ term is negligible unless the index $p$ is extremely small,

$$p < \frac{n\alpha}{\beta}, \quad R_{\text{min}} \sim 10^{54 - p} \left( \frac{\rho}{1 \text{ g/cm}^3} \right)^{n-p},$$  

(63)

or unless the local density is sufficiently large,

$$\rho \gg \frac{p\beta}{n\alpha} \left( \frac{R_{\text{min}}}{R_0} \right)^{\frac{1}{n-p}} \sim \left( \rho_{\text{inf}} \frac{1-p}{\rho_{\text{crit}}} \right)^\frac{1}{n-p}.$$  

(64)

where $\rho_{\text{inf}}$ is the energy density of inflation, $\rho_{\text{inf}} \simeq \rho_{\text{inf}} \equiv \rho_{\text{crit}}^{-1/(n-1)} R_0$. In this case, the field value is given in terms of the density as

$$2 \frac{\phi_{\text{min}}}{\sqrt{6}} \sim -\beta p \left( \frac{R_{\text{min}}}{R_0} \right)^{p-1} \sim -\beta p \left( \frac{\rho}{R_0} \right)^{p-1}.$$  

(65)

We can find that a similar approximation can be applied to the effective mass. Under this approximation, the effective mass is estimated to be,

$$m_\phi^2 \simeq R_0 \left[ \frac{1}{3p(1-p)\beta} \left( \frac{\rho}{R_0} \right)^{2-p} - \frac{5\rho}{6R_0} \right],$$  

(66)

For small values of $p$, the first term gives the dominant contribution,

$$m_\phi^2 \simeq R_0 \left[ \frac{1}{3p(1-p)\beta} \left( \frac{\rho}{R_0} \right)^{2-p} \right] \sim \frac{H_0^2}{p} \left( \frac{\rho}{\rho_{\text{crit}}} \right)^{2-p}.$$  

(67)

and the thin-shell parameter is unchanged from Eq. \(^{(52)}\). From the arguments above, we found that the $\alpha$ term is irrelevant to the local-gravity analysis under the inflationary and DE constraints. Therefore, the local-gravity constraints in the model \(^{(13)}\) reduce to those for the DE model \(^{(10)}\),

$$f(R) = R - \beta R^p.$$  

(68)

Contrary to the statement in Ref. \(^{(21)}\), where the model with $p = 2 - n$ was analysed, the power-law unified model is not viable for the local-gravity constraints.

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\(^1\) Strictly speaking, this value depends on $n$ but the deviation is irrelevant under the constraint in Sec. \(^{(14)}\) $n - 2 < O(10^{-2})$.  

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\(\text{TABLE I. The solar-system constraints on the model parameters} \cite{15, 53}\)  

| DE models       | constraint          |
|-----------------|---------------------|
| Power-law model | $p < 10^{-10}$      |
| Starobinsky DE model | $j > 0.9$       |
| $\rho$-AB model | -                   |
2. Model 2: Starobinsky DE $+ \alpha R^n$ model

The above argument also showed that, in the model \cite{13}, the inflationary term can affect the local-gravity constraints for the density,

$$\rho \gg \rho_{th} \equiv \left( \rho_{inf} \rho_{crit}^{n-1} \right)^{\frac{1}{n-2}}, \quad (69)$$

which is higher than the density relevant to the solar-system constraints but can be much lower than the inflationary scale. From this observation, next, we see how the effect of the inflationary term appears in the viable models \cite{18} and \cite{26}.

First, we consider the model \cite{18}. In a high density region, it can be approximated by the power-law model \cite{19}.

$$f(R) = R + \mu R_0 \left( \frac{R}{R_0} \right)^{2j} + \alpha R_0 \left( \frac{R}{R_0} \right)^n. \quad (70)$$

Therefore, the analysis is parallel to the model \cite{13} with the replacements $\beta \to -\mu$ and $p \to -2j$ ($j > 0.9$), i.e. a possible value of the index is different. Taking into this fact, the threshold density can be estimated as

$$\rho_{th} \equiv \left( \frac{\rho_{inf}}{\rho_{crit}} \right)^{\frac{1}{n-2}} \quad (71)$$

This shows that the inflationary term can be important even for a very low value of the density when $j$ is large.

We estimate the effective mass,

$$m_{\phi}^2 = \frac{1}{6} \left( FR - 6f \left( \frac{FR - 6f}{FR} + \frac{2}{FR} \right) \right)_{R=R_{min}} \quad (72)$$

without ignoring the $\alpha$ term. The first term is dominated by the GR term and estimated to be,

$$\frac{FR - 6f}{6FR^2} \bigg|_{R=R_{min}} \simeq -\frac{5R_{min}}{6} \simeq -\frac{5\rho}{6}. \quad (73)$$

The second term $1/(3FR)$ is determined by the inflationary/DE terms and larger than the first term for a large value of the density. Therefore, in terms of the Compton wavelength $\lambda_\phi = m_{\phi}^{-1}$, it is given by a sum of the inflationary and DE pieces as

$$\lambda_\phi^2 \simeq 3FR, \quad \lambda_\phi \sim \frac{n(n-1)}{H_{inf}^2} \left( \frac{\rho}{\rho_{inf}} \right)^{n-2} + \frac{\mu j(2j+1)}{2H_0^2} \left( \frac{\rho}{4\rho_{crit}} \right)^{-2j-2}$$

$$\sim j^2 H_0^{-2} \left( \frac{\rho}{\rho_{crit}} \right)^{-2j-2} \left[ 1 + \mathcal{O}(1) \left( \frac{\rho}{\rho_{th}} \right)^{2j+n} \right], \quad (74)$$

where we have introduced the inflationary Hubble scale $H_{inf}^2 \equiv \rho_{inf}/3$.

This result shows that the Compton wavelength first rapidly decreases as the density increases but becomes approximately constant for $\rho > \rho_{th}$ with $n \simeq 2$ (see Figs. \cite{6} - \cite{8}). The Compton wavelength can be enhanced by the factor $(\rho/\rho_{inf})^{n-2}$ for $n < 2$. However, it is not large enough under the inflationary constraint \cite{40}. Therefore, the thin-shell condition \cite{49} is kept even when the inflationary term is added. \cite{7}

Then, let us see next how the thin-shell parameter \cite{51} is modified. With the inflationary term, the scalar field is related to the curvature/density as

$$\frac{2\phi_{min}}{\sqrt{6}} \simeq \frac{\alpha n}{2} \left( \frac{R_{min}}{R_0} \right)^{n-1} - 2\mu j \left( \frac{R_{min}}{R_0} \right)^{-2j-1}$$

$$\sim \left( \frac{\rho}{\rho_{crit}} \right)^{-2j-1} \left[ 1 + \mathcal{O}(1) \left( \frac{\rho}{\rho_{th}} \right)^{2j+n} \right]. \quad (75)$$

This shows that $|\phi_{min}|$ increases as $\rho$ increases for $\rho > \rho_{th}$ in contrast that it decreases when the inflationary term is absent. Hence, the $\phi_c$ term can be also important in Eq. \cite{51}.

\footnote{On the other hand, from the viewpoint of the inflationary model, the DE term makes the scalaron's mass light in a low dense region and fifth-force constraints relevant.}
Before proceeding the estimation, it should be noted that the fifth force becomes short range even in the outside region for $\rho_\text{th} < \rho_G$. Therefore, the relevant case is $\rho_G < \rho_\text{th} < \rho_c$. In this case, the thin-shell parameter is estimated to be

$$\left| \frac{\Delta \tilde{r}_c}{\tilde{r}_c} \right| \approx \frac{2}{\Phi_c} \left[ 2 \mu j \left( \frac{\rho_G}{R_0} \right)^{-2j-1} + \tilde{\alpha} n \left( \frac{\rho_c}{R_0} \right)^{n-1} \right],$$

where now the $\rho_c$-dependent term can be non-negligible. The additional term is roughly estimated to be

$$\frac{2 \tilde{\alpha} n}{\Phi_c} \left( \frac{\rho_c}{R_0} \right)^{n-1} \sim \frac{1}{\Phi_c} \left( \frac{\rho_c}{\rho_\text{inf}} \right)^{n-1} \sim \frac{1}{H^2_\text{inf} \tilde{r}_c^2} \left( \frac{\rho_c}{\rho_\text{inf}} \right)^{n-2},$$

where we have estimated the gravitational potential as $\Phi_c \sim \rho_c \tilde{r}_c^2$. For a fixed size of the object $\tilde{r}_c$, the parameter can be enhanced by the factor $(\rho_c/\rho_\text{inf})^{n-2}$ for $n < 2$. However, it is not large enough under the inflationary constraint \([40]\).

3. Model 3: $gR^n$-AB model

Finally, we consider the model \([24]\). In a high density region, it can be approximated by the exponential model \([26]\)

$$f(R) \simeq R - \frac{R_0}{2} + g e_{\text{AB}} e^{2b} e^{-2R/\rho_{\text{AB}}} + \tilde{\alpha} R_0 \left( \frac{R}{R_0} \right)^n.$$ (78)

As in the other models, the curvature scale at the potential minimum can be estimated as

$$\frac{R_{\text{min}}}{R_0} \sim \frac{\rho}{R_0}.$$ (79)

The corresponding field value is

$$\frac{2 \tilde{\alpha} n}{\Phi_c} \left( \frac{\rho_c}{R_0} \right)^{n-1} \sim \tilde{\alpha} n \left( \frac{R_{\text{min}}}{R_0} \right)^{n-1} - 2 g e^{2b} e^{-2R_{\text{min}}/\rho_{\text{AB}}} \ln(1 + e^{2b}) e^{-2\rho/\rho_{\text{AB}}},$$ (80)

and the Compton wavelength is

$$\lambda_\phi^2 \simeq \frac{n(n-1)}{H^2_\text{inf}} \left( \frac{\rho}{\rho_\text{inf}} \right)^{n-2} + \frac{2g e^{2b} \ln(1 + e^{2b})}{H^2_0} e^{-2\rho/\rho_{\text{AB}}}. $$ (81)

We show the relation between the Compton wavelength $\lambda_\phi = m_\phi^{-1}$ and the density $\rho$ with $n = 2$ in Figs. \[12\]. The DE term decreases and the inflationary term becomes relevant for a lower value of the density than the other models. However, the Compton wavelength is asymptotic to the same small value as in the other models. Therefore, the thin-shell condition \([49]\) is kept even when the inflationary term is added.
VI. CONCLUSION

In this paper, we studied cosmological and local-gravity tests on unified models of inflation and dark energy in $f(R)$ gravity for three unified models: the power-law DE model, the Starobinsky DE model, and the $g$-AB DE model with the inflationary $R^n$ term.

From the observation of the primordial fluctuations by the Planck satellite, we have obtained a constraint on the index $n$ and found that it should close to two: $|n-2|<\mathcal{O}(0.01)$. Moreover, the amplitude of the fluctuations determines the scale of the inflationary term.

Next, we studied the local-gravity test in the unified models. In contrast to a naive expectation, we found that the inflationary term can be relevant to the analysis even for a density much lower than the inflationary scale. Then, we reanalyzed the local-gravity tests of each DE model by carefully incorporating the inflationary term.

First, the power-law DE model has been tightly constrained by the solar-system observations. Therefore, the main concern in this model is whether the inflationary term can affect the local-gravity analysis for the densities in the solar system. As a result, we found that the threshold density cannot be low enough to affect the analysis. Thus, this unified model is also tightly constrained.

Second, the other two models have a large viable parameter region as DE models. Therefore, for these models, we studied whether the fifth force is still well screened for objects with various values of the density in the unified models. In these models, the threshold density can be very low and then the inflationary term dominates in the object while the DE term does in the environment. We reanalyse the local gravity constraints in this case and found that the corrections to the scalaron’s mass and thin-shell parameter are negligible under the inflationary constraint on the index $n$.

In conclusion, the local-gravity constraints on the DE models can be applied to the unified models. However, this does not mean that the inflationary term can be negligible for any other models. It is noted that the inflationary term, or a higher curvature term more generally, can be relevant to an analysis even in a low dense region. This would be important when one considers inflationary models other than the $R^n$ model.

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