Strict continuity of the transition semigroup for the solution of a well-posed martingale problem

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September 12, 2019

Abstract

In this note we connect the notion of solutions of a martingale problem to the notion of a strongly continuous and locally equi-continuous semigroup on the space of bounded continuous functions equipped with the strict topology. This extends the classical connection of semigroups to Markov processes that was used successfully in the context of compact spaces to the context of Polish spaces.

In addition, we consider the context of locally compact spaces and show how the transition semigroup on the space of functions vanishing at infinity can be extended to the space of bounded continuous functions.

Keywords: Markov processes; semigroups and generators; martingale problem; strict topology
MSC2010 classification: 60J25; 60J35

1 Introduction

A key technique in the study of (Feller) Markov processes $t \mapsto \eta(t)$ on compact spaces is the use of strongly continuous semigroups $\{S(t)\}_{t \geq 0}$ and their generators. In this setting, the transition semigroup of conditional expectations

$$S(t)f(x) := \mathbb{E}[f(\eta(t))|\eta(0) = x], \quad x \in \mathcal{X},$$

maps $C(\mathcal{X})$ into $C(\mathcal{X})$ and is strongly continuous. The infinitesimal generator $\lambda = \partial_t S(t)|_{t=0}$ encodes the behaviour of the Markov process. These techniques have been used since the 50’s e.g. to study the convergence of processes via the convergence of generators [5,14,15,16,24].

For compact spaces, the cornerstone that allows the application of these functional analytic techniques to probability theory, and Markov process theory in particular, is the Riesz-representation theorem which states that the continuous dual space of $(C(\mathcal{X}),|\cdot|)$ is the space of Radon measures $\mathcal{M}(\mathcal{X})$ and that $C(\mathcal{X})$ equips the space of measures with the weak topology.

Already in the context of a locally compact space this effective connection breaks down. Consider for example Brownian motion on $\mathbb{R}$. The semigroup of transition operators is not strongly continuous on $C_b(\mathbb{R})$ as can be seen by considering the function $f(x) = \sin(x^2)$.

In the locally compact setting, two analytic ways to resolve this issue are

(1) weakening the topology on $C_b(\mathcal{X})$ and work with a locally convex topology instead of a Banach topology;

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(2) restricting the space of functions to $C_0(X)$, as the continuous dual space of $(C(X),|\cdot|)$ also equals $M(X)$.

Even though $C_b(X)$ clearly has the benefit of having more test-functions to work with, the loss of the Banach property has lead the literature on Markov process to generally opt for the use of $C_0(X)$ instead. The vague topology induced by $C_0(X)$ on $M(X)$, however, is defective. It does not see whether mass vanishes at infinity. Thus using strongly continuous semigroups on $C_0(X)$ to study Markov processes on a locally compact space comes at the cost of ad-hoc control of tightness.

It was soon realized by Stroock and Varadhan [20–22] followed by [16, 17] that a, now widely used, probabilistic technique called

(3) the martingale problem

is a third way to study Markov processes via its infinitesimal characterization. We briefly give the definition of the martingale problem. Let $\mathcal{D}(A)$ be al linear operator on $C_b(X)$. Then a process $\eta$ solves the martingale problem if

$$t \mapsto f(\eta(t)) - f(\eta(0)) - \int_0^t Af(\eta(s))ds$$

is a martingale with respect to the filtration $\mathcal{F}_t := \sigma(\eta(s) \mid s \leq t)$ for all $f \in \mathcal{D}(A)$. It can be shown that the solution of a well-posed martingale problem corresponds to a Markov process.

As this method uses probabilistic methods, it does not suffer from functional analytic problems and works in the general context of Polish spaces. It can, therefore, also not benefit from functional analytic techniques that can speed up proofs by providing a strong framework.

The appropriate topology that corresponds to the weakening described in (1) in our trichotomy of options is called the strict topology. This topology was first introduced by Buck [2] in the 50’s for locally compact $X$ and was studied in the 70’s by Sentilles [18] in the context of more general spaces, including Polish spaces.

The essential property of the strict topology is that the continuous dual space of $(C_b(X),\beta)$ is the space of Random measures. This way, the strict topology is the proper extension of the uniform topology from the compact to the Polish setting and has much greater potential to be applicable to the study of probability measures than the uniform topology. The topology also satisfies various other desirable properties that are known from the compact setting, see Appendix A.

That the strict topology can be used for a systematic study of Markov processes and their semigroups was shown by [7, 25]. Semigroup theory in the context of topologies like the strict topology has been developed in the literature on so called bi-continuous semigroups, see e.g. [6,13], for more references on this area see also [11] in which a more topological, but to some extent equivalent, point of view is taken.

An application of the strict topology in the study of properties of the propagation of stochastic order can be found in [12].

This note aims at the resolution of the apparent trichotomy between (1), (2) and (1). The two main results of this note are:

- Theorem 3.1 the transition semigroup of a collection of solutions to a well-posed martingale problem is strongly continuous and locally equi-continuous for the strict topology. The generator of this semigroup extends the operator of the martingale problem. This allows us to unify approach (1) and (2).
- Theorem 4.1 shows that in the context of locally compact spaces the transition semigroup on $C_0(X)$ can be extended, given that mass is conserved, to $C_b(X)$. Thus, giving a partial merge of (1) and (2), modulo the known issue of having to avoid mass running of to infinity.
2 Preliminaries: Strongly continuous semigroups for the strict topology and the martingale problem

\( X \) is a Polish space. \( C_b(X) \) denotes the space of continuous and bounded functions on \( X \). If \( X \) is locally compact, \( C_0(X) \) is the space of continuous functions that vanish at infinity. \( \mathcal{M}(X) \) is the space of Radon measures on \( X \). \( \mathcal{P}(X) \) is the subset of probability measures.

2.1 The martingale problem

**Definition 2.1** (The martingale problem). Let \( A : D(A) \subseteq C_b(X) \to C_b(X) \) be a linear operator. For \( (A, D(A)) \) and a measure \( \nu \in \mathcal{P}(X) \), we say that \( \mathcal{P} \in \mathcal{P}(D_\infty(\mathbb{R}^+)) \) solves the martingale problem for \( (A, \nu) \) if \( \mathcal{P} \eta(0) - 1 = \nu \) and if for all \( f \in D(A) \)

\[
M_f(t) := f(\eta(t)) - f(\eta(0)) - \int_0^t Af(\eta(s))ds
\]

is a mean 0 martingale with respect to its natural filtration \( \mathcal{F}_t := \sigma(\eta(s) | s \leq t) \) under \( \mathcal{P} \).

We denote the set of all solutions to the martingale problem, for varying initial measures \( \nu \), by \( \mathcal{M}_A \). We say that uniqueness holds for the martingale problem if for every \( \nu \in \mathcal{P}(X) \) the set \( \mathcal{M}_\nu := \{ \mathcal{P} \in \mathcal{M}_A | \mathcal{P} \eta(0)^{-1} = \nu \} \) is empty or a singleton. Furthermore, we say that the martingale problem is well-posed if this set contains exactly one element for every \( \nu \).

2.2 The strict topology

The strict topology on \( C_b(X) \) is defined in terms of the compact-open topology \( \kappa \) on \( C_b(X) \). This locally convex topology is generated by the semi-norms \( p_K(f) := \sup_{x \in K} |f(x)| \), where \( K \) ranges over all compact sets in \( X \).

The strict topology \( \beta \) on the space bounded continuous functions \( C_b(X) \) is generated by the semi-norms

\[
p_{K_n, a_n}(f) := \sup_n a_n \sup_{x \in K_n} |f(x)|
\]

varying over non-negative sequences \( a_n \) converging to 0 and sequences of compact sets \( K_n \subseteq X \).

**Remark 2.2.** We refer the reader to the discussion of the strict and sub-strict topology in [18], where it is shown that these two topologies coincide for Polish spaces. Because the definition of the sub-strict topology is more accessible, we use this as a characterisation of the strict topology in our context.

**Remark 2.3.** The strict topology can equivalently be given by the collection of semi-norms

\[
p_g(f) := |fg|
\]

where \( g \) ranges over the set

\[
\{ g \in C_b(X) | \forall \alpha > 0 : \{ x : |g(x)| \geq \alpha \} \text{ is compact in } X \}.
\]

See [2] and [25].
2.3 Semigroups

We call the family of linear operators \( \{T(t)\}_{t \geq 0} \) on \( C_b(X) \) a **semigroup** if \( T(0) = 1 \) and \( T(t)T(s) = T(t+s) \) for \( s, t \geq 0 \). A family of \( \beta \) to \( \beta \) continuous operators \( \{T(t)\}_{t \geq 0} \) is called a **strongly continuous semigroup** if \( t \mapsto T(t)f \) is \( \beta \) continuous. We call \( \{T(t)\}_{t \geq 0} \) a **locally equi-continuous** family if for every \( t \geq 0 \) and \( \beta \)-continuous semi-norm \( p \) on \( C_b(X) \) there exists a continuous semi-norm \( q \) such that \( \sup_{s \leq t} p(T(s)f) \leq q(x) \) for every \( f \in E \). Finally, if \( \{T(t)\}_{t \geq 0} \) is a locally equi-continuous strongly continuous semigroup we say that \( \{T(t)\}_{t \geq 0} \) is a **SCLE** semigroup.

We say that the linear map \( A \subseteq C_b(X) \times C_b(X) \) is the **generator** of a SCLE semigroup \( \{T(t)\}_{t \geq 0} \) if

\[
g = \lim_{t \downarrow 0} \frac{T(t)f - f}{t} \quad \Leftrightarrow \quad (f, g) \in \mathcal{D}(A).
\]

Generally, we will write \( g = Af \) if \( (f, g) \in A \). If \( A, B \subseteq C_b(X) \times C_b(X) \) are two operators such that \( A \subseteq B \), we say that \( B \) is an **extension** of \( A \).

For more information on **SCLE** semigroups, see \([11]\).

**Remark 2.4.** In the context of Banach spaces a strongly continuous semigroup is automatically locally equi-continuous.

3 The transition semigroup is strongly continuous and locally equi-continuous with respect to the strict topology

In the setting that the martingale problem is well-posed, we obtain a strengthened version of Theorems 4.4.2 and 4.5.11, \([9]\), showing that the transition semigroup of the solution is strongly continuous for the strict topology. For a overview of results on SCLE semigroups relevant for the results to follow, see Section \([23]\).

**Theorem 3.1.** Let \( A \subseteq C_b(X) \times C_b(X) \) and let the martingale problem for \( A \) be well-posed.

- Suppose that \( \mathcal{D}(A) \) contains an algebra and vanishes nowhere, i.e. for each \( x \in X \) there is \( f \in \mathcal{D}(A) \) such that \( f(x) \neq 0 \).
- Suppose that for all compact \( K \subseteq \mathcal{P}(X) \), \( \epsilon > 0 \) and \( T > 0 \), there exists a compact set \( K' = K'(K, \epsilon, T) \) such that for all \( P \in M_A \), we have
  \[
  P[\eta(t) \in K' \mid \text{for all } t < T, \eta(0) \in K] \geq (1 - \epsilon)P[\eta(0) \in K]. \tag{3.1}
  \]

Then the measures \( P \in M_A \) correspond to strong Markov processes with a \( \beta \)-**SCLE** semigroup \( \{S(t)\}_{t \geq 0} \) on \( C_b(X) \) defined by \( S(t)f(x) = Ef[\eta(t)] | \eta(0) = x] \). The generator of \( \{S(t)\}_{t \geq 0} \) is an extension of \( A \).

**Proof of Theorem 3.1.** The proof that the solutions are strong Markov and correspond to a semigroup

\[ S(t)f(x) = Ef[\eta(t)] | \eta(0) = x] \]

that maps \( C_b(X) \) into \( C_b(X) \) follows as in the proof of (b) and (c) of Theorem 4.5.11 \([9]\) and the proof of (b) of Theorem 4.4.2 \([9]\). We are left to show that \( \{S(t)\}_{t \geq 0} \) is SCLE for \( \beta \), which we do in Lemmas 3.2 and 3.3 below. That the generator of the semigroup extends follows from Proposition 3.4. \( \square \)

**Lemma 3.2.** Let \( \{S(t)\}_{t \geq 0} \) be the semigroup introduced in Theorem 3.1. The family \( \{S(t)\}_{t \geq 0} \) is locally equi-continuous for \( \beta \).
By the almost sure convergence of \( \{S(t)\}_{t \leq T} \) is \( \beta \) equi-continuous by using Theorem 3.4 (c) and (d). Pick a sequence \( f_n \) converging to \( f \) with respect to \( \beta \). It follows that \( \sup_n |f_n| \leq \infty \), which directly implies that \( \sup_n \sup_{t \leq T} |S(t)f_n| < \infty \).

We also know that \( f_n \to f \) uniformly on compact sets. We prove that this implies the same for \( S(t)f_n \) and \( S(t)f \) uniformly in \( t \leq T \). Fix \( \varepsilon > 0 \) and a compact set \( K \subseteq \mathbb{X} \), and let \( K \) be the set introduced in Equation (4.1) for \( T \). Then we obtain that

\[
\sup_{t \leq T} \sup_{x \in K} |S(t)f(x) - S(t)f_n(x)| \\
\leq \sup_{t \leq T} \sup_{x \in K} \mathbb{E}_x |f(\eta(t)) - f_n(\eta(t))| \\
\leq \sup_{t \leq T} \sup_{x \in K} \mathbb{E}_x \left[ |f(\eta(t)) - f_n(\eta(t))| \mathbb{1}_{[\eta(t) \in K]} \right] \\
\quad \quad \quad \quad + \left( |f(\eta(t)) - f_n(\eta(t))| \mathbb{1}_{[\eta(t) \in K^c]} \right) \\
\leq \sup_{t \leq T} \sup_{y \in K} |f(y) - f_n(y)| \sup_n |f_n - f| \varepsilon .
\]

As \( n \to \infty \) this quantity is bounded by \( \sup_n |f_n - f| \varepsilon \) as \( f_n \) converges to \( f \) uniformly on compacts. As \( \varepsilon \) was arbitrary, we are done. \( \square \)

**Lemma 3.3.** Let \( \{S(t)\}_{t \geq 0} \) be the semigroup introduced in Theorem 3.4. Then \( \{S(t)\}_{t \geq 0} \) is \( \beta \) strongly continuous.

For the proof, we recall the notion of a weakly continuous semigroup. A semigroup is **weakly continuous** if for all \( f \in C_b(\mathbb{X}) \) and \( \mu \in M(\mathbb{X}) \) the trajectory \( t \mapsto \langle S(t)f, \mu \rangle \) is continuous in \( \mathbb{R} \).

**Proof of Lemma 3.3.** We first establish that it suffices to prove weak continuity of the semigroup by an application of Proposition 3.5 in [11]. To apply this proposition, note that we have completeness and the strong Mackey property of \( (C_b(\mathbb{X}), \beta) \) by Theorem 3.1 and local equi-continuity of the semigroup \( \{S(t)\}_{t \geq 0} \) by Lemma 3.2.

Thus, we establish weak continuity. Let \( f \in C_b(\mathbb{X}) \) and \( \mu \in M(\mathbb{X}) \). Write \( \mu \) as the Hahn-Jordan decomposition: \( \mu = c^+ \mu^+ - c^- \mu^- \), where \( c^+, c^- \geq 0 \) such that \( \mu^+, \mu^- \in \mathcal{P}(\mathbb{X}) \). It thus suffices to show that \( t \mapsto \langle S(t)f, \mu^+ \rangle \) and \( t \mapsto \langle S(t)f, \mu^- \rangle \) are continuous. Clearly, it suffices to do this for either of the two.

Let \( \mathbb{P} \) be the unique solution to the martingale problem for \( A \) started in \( \mu^+ \). It follows by Theorem 4.3.12 in [5] that \( \mathbb{P}[X(t) = X(t-)] = 1 \) for all \( t > 0 \), so \( t \mapsto X(t) \) is continuous \( \mathbb{P} \) almost surely. Fix some \( t > 0 \), we show that our trajectory is continuous at this specific \( t \).

\[
\left| \langle S(t)f, \mu^+ \rangle - \langle S(t+h)f, \mu^+ \rangle \right| \leq \mathbb{E}_x |f(\eta(t)) - f(\eta(t+h))| .
\]

By the almost sure convergence of \( X(t+h) \to X(t) \) as \( h \to 0 \), and the boundedness of \( f \), we obtain by the dominated convergence theorem that this difference converges to \( 0 \) as \( h \to 0 \). As \( t \geq 0 \) was arbitrary, the trajectory is continuous at all \( t \geq 0 \). \( \square \)

**Proposition 3.4.** Let \( \{S(t)\}_{t \geq 0} \) be the semigroup introduced in Theorem 3.4 and let \( \hat{A} \) be the generator of this semigroup. Then \( \hat{A} \) is an extension of \( A \).

**Proof.** Let \( f \in \mathcal{D}(A) \), we prove that \( f \in \mathcal{D}(\hat{A}) \). We again use the characterisation of \( \beta \) convergence as given in Theorem 3.1 (d). From this point onward, we write \( g := Af \) to ease the notation.

First, \( \sup_t \left\| \frac{S(t)f(x) - f(x)}{t} \right\| \leq |g| \) as

\[
\frac{S(t)f(x) - f(x)}{t} = \mathbb{E}_x \left[ \frac{f(\eta(t)) - f(x)}{t} \right] = \mathbb{E}_x \left[ \frac{1}{t} \int_0^t g(\eta(s))ds \right] .
\]
Second, we show that we have uniform convergence of \( \frac{S(t)f - f}{t} \) to \( g \) as \( t \downarrow 0 \) on compacts sets. So pick \( K \subseteq \mathcal{X} \) compact. Now choose \( \varepsilon > 0 \) arbitrary, and let \( \hat{K} := \hat{K}(K, \varepsilon, 1) \) as in \( \text{[6.1]} \).

\[
\sup_{x \in K} \left| \frac{S(t)f(x) - f(x)}{t} - g(x) \right| \\
\leq \sup_{x \in K} \mathbb{E}_x \left[ \left| \frac{1}{t} \int_0^t g(\eta(s)) - g(x) \, ds \right| \right] \\
\leq \sup_{x \in K} \mathbb{E}_x \left[ \left| \frac{1}{t} \int_0^t g(\eta(s)) - g(x) \, ds \right| \right] \\
+ \sup_{x \in K} \mathbb{E}_x \left[ \left| \frac{1}{t} \int_0^t g(\eta(s)) - g(\eta(s)) \, ds \right| \right] \\
\leq \sup_{x \in K} \mathbb{E}_x \left[ \left| \frac{1}{t} \int_0^t g(\eta(s)) - g(x) \, ds \right| \right] + 2\varepsilon |g| . \tag{3.2}
\]

Thus, we need to work on the first term on the last line.

The function \( g \) restricted to the compact set \( \hat{K} \) is uniformly continuous. So let \( \varepsilon' > 0 \), chosen smaller then \( \varepsilon' \), be such that if \( d(x,y) < \varepsilon' \), \( x,y \in \hat{K} \), then \( |g(x) - g(y)| \leq \varepsilon' \).

By Lemma 4.5.17 in [5], the set \( \{ \mathbb{P}_x \mid x \in K \} \) is a weakly compact set in \( \mathcal{P}(\mathcal{D}_X(\mathbb{R}^+) \).

So by Theorem 3.7.2 in [6], we obtain that there exists a \( \delta = \delta(\varepsilon') > 0 \) such that

\[
\sup_{x \in K} \mathbb{P}_x \left[ \eta \in \mathcal{D}_X(\mathbb{R}^+) \mid \sup_{s \leq \delta} d(\eta(0), \eta(s)) < \varepsilon' \right] > 1 - \varepsilon' > 1 - \varepsilon .
\]

Denote \( S_\delta := \{ \eta \in \mathcal{D}_X(\mathbb{R}^+) \mid \sup_{s \leq \delta} d(\eta(0), \eta(s)) < \varepsilon' \} \), so that we can summarize the equation as \( \sup_{x \in K} \mathbb{P}_x[S_\delta] > 1 - \varepsilon \).

We reconsider the term that remained in equation \( \text{[3.2]} \).

\[
\sup_{x \in K} \mathbb{E}_x \left[ \left| \frac{1}{t} \int_0^t g(\eta(s)) - g(x) \, ds \right| + 2\varepsilon |g| \right] \\
\leq \sup_{x \in K} \mathbb{E}_x \left[ \left| \frac{1}{t} \int_0^t g(\eta(s)) - g(\eta(s)) \, ds \right| + 4\varepsilon |g| \right] .
\]

On the set \( \{ \eta(s) \in \hat{K} \mid s \leq 1 \} \cap S_\delta \), we know that \( d(\eta(s), x) \leq \eta \) as long as \( s \leq \delta \).

Thus by the uniform continuity of \( g \) on \( \hat{K} \), we obtain \( |g(\eta(s)) - g(x)| \leq \varepsilon \) if \( s \leq \delta \).

Hence:

\[
\sup_{t \leq 1} \sup_{x \in K} \left| \frac{S(t)f(x) - f(x)}{t} - g(x) \right| \leq \varepsilon + 4\varepsilon |g| .
\]

As \( \varepsilon > 0 \) was arbitrary, it follows that \( f \in \mathcal{D}(\hat{A}) \) and \( Af = \hat{A}f \).

\[
\square
\]

4  Transition semigroup for a process on a locally compact space

In the context of a locally compact space \( \mathcal{X} \) the transition semigroup is usually strongly continuous on the space \( C_b(\mathcal{X}) \). We have seen however that \( (C_b(\mathcal{X}), \beta) \) is a natural space to consider semigroup theory. In the following theorem, we will show that we can go forward and backward between the two perspectives.

The key property that allows to do this transition is that the Markov process under consideration conserves mass, and that the corresponding semigroup maps \( C_0(\mathcal{X}) \) into \( C_0(\mathcal{X}) \).
Therefore, let continuous set is \( \beta \) for this result for \( \beta \) \( \geq 0 \) is strongly continuous.

Conversely, suppose that we have a \( \text{a strong continuous semigroup} \) \( S(t) \) on \( \mathcal{M}(X) \) such that \( S(t) \mathcal{P}(X) \subseteq \mathcal{P}(X) \). Then the semigroup can be extended uniquely to a \( \text{a SCLE semigroup} \) \( S(t) \) on \( \mathcal{M}(X) \).

In this setting, denote by \( (A, D(A)) \) the generator of \( [S(t)]_{t \geq 0} \) on \( \{C_b(X), \beta \} \) and by \( (A, D(A)) \) the generator of \( [S(t)]_{t \geq 0} \) on \( \{C_0(X), \beta \} \). Then \( A \subseteq A \) and \( A \) is the \( \beta \) closure of \( A \).

Before we start with the proof, we note that both the space \( (C_b(X), \beta) \) and \( (C_0(X), \beta') \) have the space of Radon measures as a dual. As such, the space of Radon measures carries two weak topologies. The first one is the one that probabilists call the weak topology, i.e. \( \sigma(\mathcal{M}(X), C_b(X)) \), and the second is the weaker vague topology, i.e. \( \sigma(M(X), C_0(X)) \).

Proof. Proof of the first statement.

For a given time \( t \geq 0 \), the operator \( S(t) \) is continuous on \( \{C_0(X), \beta \} \), because \( S(t) \) is \( \beta \) continuous and therefore maps \( \beta \)-bounded sets into \( \beta \)-bounded sets. Norm continuity of the restriction \( \tilde{S}(t) \) on \( C_0(X) \) then follows by the fact that the bounded sets for the norm and for \( \beta \) coincide.

As \( \{S(t)\}_{t \geq 0} \) is \( (C_b(X), \beta) \) strongly continuous, it is also weakly continuous, in other words, for every Radon measure \( \mu \), we have that

\[
\text{t} \mapsto (S(t)f, \mu)
\]

is continuous for every \( f \in C_b(X) \) and in particular for \( f \in C_0(X) \). Theorem 1.5.8 in Engel and Nagel [3] yields that the semigroup \( \{S(t)\}_{t \geq 0} \) is strongly continuous on \( (C_0(X), \beta') \).

Proof of the second statement.

First note that such a \( \beta \)-continuous extension must be unique by the Stone-Weierstrass theorem, cf. Theorem 4.1 (e), which implies that \( C_0(X) \) is \( \beta \) dense in \( C_b(X) \). We will show that \( \tilde{S}(t) \) is \( \beta \) to \( \beta \) continuous, because we can then extend the operator by continuity to \( C_b(X) \). In fact, we will directly prove the stronger statement that \( \{\tilde{S}(t)\}_{t \geq 0} \) is locally \( \beta \) equi-continuous.

First of all, by the completeness of \( (C_b(X), \beta) \), the fact that \( C_0(X) \) is dense in \( (C_b(X), \beta) \) and 21.4.(5) in [9], we have \( (C_0(X), \beta)' = (C_b(X), \beta)' = \mathcal{M}(X) \) and the equi-continuous sets in \( \mathcal{M}(X) \) with respect to \( (C_0(X), \beta) \) and \( (C_b(X), \beta) \) coincide. It follows by 39.3.(4) in [10] that \( \{\tilde{S}(t)\}_{t \geq 0} \) is locally \( \beta \) equi-continuous if for every \( T \geq 0 \) and \( \beta \) equi-continuous set \( K \subseteq \mathcal{M}(X) \) we have that

\[
\mathcal{G}(K) := \{\tilde{S}(t)^{\prime}(\mu) \mid t \leq T, \mu \in K\}
\]

is \( \beta \) equi-continuous. By Theorem 6.1 (c) in Sentilles [15], it is sufficient to prove this result for \( \beta \) equi-continuous sets \( K \) consisting of non-negative measures in \( \mathcal{M}(X) \). Furthermore, we can restrict to weakly closed \( K \), as the weak closure of a \( \beta \) equi-continuous set is \( \beta \) equi-continuous.

Therefore, let \( K \) be an arbitrary weakly closed \( \beta \) equi-continuous subset of the non-negative Radon measures. We show that \( \mathcal{G}(K) \) is relatively weakly compact, as this will imply that \( \mathcal{G}(K) \) is \( \beta \) equi-continuous as \( \beta \) is a strong Mackey topology, cf. Theorem 4.1 (a). This in turn would establish that \( \{\tilde{S}(t)\}_{t \geq 0} \) is locally \( \beta \) equi-continuous.

By Theorem 8.9.4 in [4], we obtain that the weak topology on the positive cone in \( \mathcal{M}(E) \) is metrizable. So, we only need to show sequential relative weak compactness of
Let \( \nu_n \) be a sequence in \( \mathfrak{S}K \). Clearly, \( \nu_n = \tilde{S}(t_n)\mu_n \) for some sequence \( \mu_n \in K \) and \( t_n \leq T \). As \( K \) is \( \beta \) equi-continuous, it is weakly compact by the Bourbaki-Alaoglu theorem, so without loss of generality we restrict to a weakly converging subsequence \( \mu_n \in K \) with limit \( \mu \in K \) and \( t_n \to t \), for some \( t \leq T \).

Now there are two possibilities, either \( \mu = 0 \), or \( \mu \neq 0 \). In the first case, we obtain directly that \( \nu_n = \tilde{S}(t_n)\mu_n \to 0 \) \( \in K \subseteq \mathfrak{S}K \) weakly. In the second case, one can show that

\[
\hat{\mu}_n := \frac{\mu_n}{\langle 1, \mu_n \rangle} \to \frac{\mu}{\langle 1, \mu \rangle} := \hat{\mu}
\]

weakly, and therefore vaguely. As \( (\tilde{S}(t))_{t \geq 0} \) is strongly continuous on \( (C_0(X), \| \cdot \|) \), it follows that \( \tilde{S}'(t_n)\hat{\mu}_n \to \tilde{S}'(t)\hat{\mu} \) vaguely. By assumption, all measures involved are probability measures, so by Proposition 3.4.4 in Ethier and Kurtz \[5\] implies that the convergence is also in the weak topology. By an elementary computation, we infer that the result also holds without the normalising constants: \( \nu_n \to \tilde{S}'(t)\mu \) weakly.

So both cases give us a weakly converging subsequence in \( \mathfrak{S}K \).

We conclude that \( (\tilde{S}(t))_{t \leq T} \) is \( \beta \) equi-continuous. So we can extend all \( \tilde{S}(t) \) by continuity to \( \beta \) continuous maps \( S(t) : C_b(X) \to C_b(X) \). Also, we directly obtain that \( (S(t))_{t \geq 0} \) is locally equi-continuous. The semigroup property of \( (S(t))_{t \geq 0} \) follows from the semigroup property of \( (\tilde{S}(t))_{t \geq 0} \). The last thing to show is the \( \beta \) strong continuity of \( (S(t))_{t \geq 0} \).

By Proposition 3.5 in \[11\] it is sufficient to show weak continuity of the semigroup \( (S(t))_{t \geq 0} \). Pick \( \mu \in M(X) \), and represent \( \mu \) as the Hahn-Jordan decomposition \( \mu = \mu^+ - \mu^- \), where \( \mu^+ , \mu^- \) are non-negative measures. By construction, the adjoints of \( S(t) \) and \( \tilde{S}(t) \) coincide, so \( t \mapsto S'(t)\mu^+ \) and \( t \mapsto S'(t)\mu^- \) are vaguely continuous. The total mass of the measures in both trajectories remains constant by the assumption of the theorem, so by Proposition 3.4.4 in \[5\], we obtain that \( t \mapsto S'(t)\mu^+ \) and \( t \mapsto S'(t)\mu^- \) are weakly continuous. This directly implies that \( (S(t))_{t \geq 0} \) is weakly continuous and thus strongly continuous.

**Proof of the third statement.**

Let \( (A, \mathcal{D}(\hat{A})) \) be the generator of \( (\tilde{S}(t))_{t \geq 0} \) and \( (A, \mathcal{D}(A)) \) the one of \( (S(t))_{t \geq 0} \). As the norm topology is stronger than \( \beta \), it is immediate that \( \hat{A} \subseteq A \).

We will show that \( \mathcal{D}(\hat{A}) \) is a core for \( (A, \mathcal{D}(A)) \), i.e. \( \mathcal{D}(\hat{A}) \) is dense in \( \mathcal{D}(A) \) for the \( \beta \)-graph topology on \( \mathcal{D}(A) \). This will follow by a variant of Proposition II.1.7 in \[9\] which proves for Banach spaces but which also holds for the strict topology. Thus, it suffices to prove \( \beta \) density of \( \mathcal{D}(\hat{A}) \) in \( C_b(X) \) and that \( S(t)\mathcal{D}(\hat{A}) \subseteq \mathcal{D}(\hat{A}) \).

The first claim follows because \( \mathcal{D}(\hat{A}) \) is norm, hence \( \beta \), dense in \( C_0(X) \) by Theorem II.1.4 in \[3\] and because \( C_0(X) \) is \( \beta \) dense in \( C_b(X) \) by the Stone-Weierstrass theorem, cf Theorem A.1(e). The second claim follows because \( S(t)\mathcal{D}(\hat{A}) = \tilde{S}(t)\mathcal{D}(\hat{A}) \subseteq \mathcal{D}(\hat{A}) \) by e.g. Lemma II.1.3 in \[3\] or Lemma 5.2 in \[11\].

We conclude that \( \mathcal{D}(\hat{A}) \) is a core for \( \mathcal{D}(A) \). As \( A \) is \( \beta \) closed, it follows that \( \hat{A} \) is the \( \beta \) graph-closure of \( \hat{A} \).

**A Properties of the strict topology**

The strict topology is the ‘right’ generalisation of the norm topology on \( C(X) \) for compact metric \( X \) to the more general context of Polish spaces. To avoid further scattering of results, we collect some of the main properties of \( \beta \).

**Theorem A.1.** Let \( X \) be Polish. The locally convex space \( (C_b(X), \beta) \) satisfies the following properties.

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8
(a) \((C_b(X), \beta)\) is complete, strong Mackey (i.e. all weakly compact sets in the dual are equi-continuous) and the continuous dual space coincides with the space of Radon measures on \(X\) of bounded total variation.

(b) \((C_b(X), \beta)\) is separable.

(c) For any locally convex space \((F, \tau_F)\) and \(\beta\) to \(\tau_F\) sequentially equi-continuous family \(\{T_i\}_{i \in I}\) of maps \(T_i : (C_b(X), \beta) \to (F, \tau_F)\), the family \(I\) is \(\beta\) to \(\tau_F\) equi-continuous.

(d) The norm bounded and \(\beta\) bounded sets coincide. Furthermore, on norm bounded sets \(\beta\) and \(\kappa\) coincide.

(e) Stone-Weierstrass: Let \(M\) be an algebra of functions in \(C_b(X)\). If \(M\) vanishes nowhere and separates points, then \(M\) is \(\beta\) dense in \(C_b(X)\).

(f) Arzelà-Ascoli: A set \(M \subseteq C_b(X)\) is \(\beta\) compact if and only if \(M\) is norm bounded and \(M\) is an equi-continuous family of functions.

(g) \((C_b(X), \beta, \leq)\), where \(\leq\) is defined as \(f \leq g\) if and only if \(f(x) \leq g(x)\) for all \(x \in X\), is locally convex-solid.

(h) Dinu’s theorem: If \(\{f_\alpha\}_{\alpha}\) is a net in \(C_b(X)\) such that \(f_\alpha\) increases or decreases point-wise to \(f \in C_b(X)\), then \(f_\alpha \rightarrow f\) for the strict topology.

Note that (d) implies that a sequence \(f_n \beta \rightarrow f\) if and only if \(\sup_n |f_n| < \infty\) and \(f_n \kappa \rightarrow f\).

Proof. (a) and (c) follow from Theorems 9.1 and 8.1 in [18], Theorem 7.4 in [27], Corollary 3.6 in [26] and Krein’s theorem [9, 24.5.(4)]. (b) follows from Theorem 2.1 in [23]. (d) follows by Theorems 4.7, 2.4 in [18] and 2.2.1 in [28]. (e) is proven in Theorem 2.1 and Corollary 2.4 in [8]. (f) follows by the Arzelà-Ascoli theorem for the compact-open topology, Theorem 8.2.10 in [4], and (d). To conclude, (g) and (h) follow from Theorems 6.1 and 6.2 in [15].

Acknowledgment
The author thanks Moritz Schauer for discussions on the topic of the strict topology and its applicability to probability theory.

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