OPTIMAL ERROR ESTIMATES FOR FRACTIONAL STOCHASTIC PARTIAL DIFFERENTIAL EQUATION WITH FRACTIONAL BROWNIAN MOTION

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Abstract. In this paper, we consider the numerical approximation for a class of fractional stochastic partial differential equations driven by infinite dimensional fractional Brownian motion with hurst index \( H \in \left( \frac{1}{2}, 1 \right) \). By using spectral Galerkin method, we analyze the spatial discretization, and we give the temporal discretization by using the piecewise constant, discontinuous Galerkin method and a Laplace transform convolution quadrature. Under some suitable assumptions, we prove the sharp regularity properties and the optimal strong convergence error estimates for both semi-discrete and fully discrete schemes.

1. Introduction. In recent years, there has been considerable interest in studying fractional stochastic partial differential equations (SPDEs, in short) due to their applications in various scientific and technological areas including physics, biology, telecommunications, turbulence, and engineering (see, for example, Mehaute [21]). For example, the classical heat equation \( \partial_t u(t, x) = \Delta u(t, x) \), is used for modeling heat diffusion in homogeneous media, while the fractional heat equation \( \partial_t^\alpha u(t, x) = \Delta u(t, x) \), describes heat propagation in inhomogeneous media. Moreover, stochastic partial differential equations driving by infinite dimensional fractional Brownian motion also is a recent research direction in probability theory and its applications, see, for examples [4, 18, 8, 7, 30].

On the other hand, numerical analysis of stochastic partial differential equations is currently an active area of research due to the need of practical applications, and there are a large number of literatures on numerical methods for spdes. So far, most of the work done on the stochastic evolution equations has dealt with

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the usual time derivative and the driving noise is a standard Cylindrical Wiener process, see, for examples, Yan [35], Kruse [15, 16], Jentzen and Kloeden [10, 11], Wang [33], Zhang [37] and the references therein. As we all know, it is difficult to obtain the exact analytical solutions of (stochastic) partial differential equations, so it is necessary to study the numerical solution of such equations. Some surveys and literatures can be found in Jin et al. [12], McLean and Mustapha [19, 20], Li and Yang [17]. However, in contrast to the extensive studies on fractional Brownian motion, there has been little systematic investigation on fractional stochastic partial differential equations driven by fractional Brownian motions (see, Kamrani and Jamshidi [13] and Wang et al [34]).

Motivated by all these papers, in this paper we study the sharp regularity properties of fractional stochastic partial differential equations with infinite dimensional fractional Brownian motion and obtain the optimal convergence rates of the numerical approximations on this solution. Given a real separable Hilbert space $(U, \langle \cdot, \cdot \rangle, \| \cdot \|)$, and let $A : \mathcal{D}(A) \subset U \to U$ be a densely defined, linear unbounded, positive self-adjoint operator with compact inverse (see, Pazy [25] on some details). Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P})$ be a filtered probability space and $\{W^H(t)\}_{t \in [0,T]}$ be a standard cylindrical fractional Brownian motion with hurst index $H > \frac{1}{2}$ (for more details, see Section 2). We will consider the equation

\[
\begin{aligned}
D^\alpha_t X(t) + AX(t) &= I^{1-\alpha}_t [\Psi W^H(t)], \quad \alpha \in (0,1), \quad t \in [0,T], \\
X(0) &= x_0,
\end{aligned}
\]

where $\Psi : U \to U$ is deterministic mapping, $I^{1-\alpha}_t$ is the fractional integral operator defined by

\[
I^{1-\alpha}_t f(t) := \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-r)^{-\alpha} f(r)dr, \quad \alpha \in (0,1),
\]

and the fractional differential operator $D^\alpha_t$ is understood in the Caputo sense (see, Caputo [2])

\[
D^\alpha_t u(t,x) = \frac{1}{\Gamma(1-\alpha)} \int_0^t \partial_r u(r,x) \frac{dr}{(t-r)^\alpha}, \quad \alpha \in (0,1).
\]

In the past years, many researchers have investigated the numerical approximation of deterministic fractional partial differential equations by different methods. For example, an implicit finite difference method has been used in [5, 26, 36] while a standard Galerkin finite element method has been employed in [19, 20]. Recently, Li and Yang [17] studied the numerical approximation for fractional SPDEs driven by Q-Wiener process, where they used the standard continuous finite element method and the piecewise constant, discontinuous Galerkin method to obtain a fully discrete implicit scheme and get the the corresponding strong convergence error estimates, respectively. However, to the best knowledge of the authors, there is little work on the numerical approximation for SPDEs driven by fractional Brownian motion. In this paper, our aim is to discuss the numerical approximation for fractional SPDEs driven by fractional Brownian motion and obtain optimal strong convergence error estimates for both semi-discrete and fully discrete schemes under some suitable assumptions. The rest of the paper is organized as follows. In Section 2, we recall some properties of the stochastic integration with respect to $W^H$ and give the solution representation of (1) by means of the Mittag-Leffler function. In Section 3, we obtain the sharp regularity of the equation (1). Section 4 is devoted
to study the optimum error estimates both in space and time with smooth initial data.

2. Preliminaries. In this section, we briefly recall the definition of the stochastic integration with respect to \( W^H \) and the representation of the solution to equation (1) by using the Mittag-Leffler function. For convenience, we let \( C \) stand for a positive constant depending only on the subscripts, while its value may be different in different appearance.

Throughout this paper we assume that \( U \) is a real separable Hilbert space with the inner \( \langle \cdot, \cdot \rangle \) and the norm \( \| \cdot \| \). Moreover, let \( \mathcal{L}(U) \) be the space of bounded linear operators from \( U \) to \( U \) endowed with the usual operator norm \( \| \cdot \|_{\mathcal{L}(U)} \) and let \( \mathcal{L}_2(U) \subset \mathcal{L}(U) \) be the space of all Hilbert-Schmidt operator equipped with the inner product and norm

\[
\langle T_1, T_2 \rangle_{\mathcal{L}_2(U)} = \sum_{j=1}^{\infty} \langle T_1 e_j, T_2 e_j \rangle, \quad \| T \|_{\mathcal{L}_2(U)} = \left( \sum_{j=1}^{\infty} \| T e_j \|^2 \right)^{\frac{1}{2}},
\]

where \( \{ e_n \}_{n \in \mathbb{N}} \) is a complete orthonormal basis of \( U \). Moreover, \( \langle \cdot, \cdot \rangle_{\mathcal{L}_2(U)} \) and \( \| \cdot \|_{\mathcal{L}_2(U)} \) are independent of the choice \( \{ e_n \}_{n \in \mathbb{N}} \) (see, DaPrato and Zabczyk [6]).

2.1. Cylindrical fractional Brownian motion. Recall that a real valued Gaussian process \( \beta = \{ \beta(t), 0 \leq t \leq T \} \) defined on \( (\Omega, \mathcal{F}, (\mathcal{F}_t), P) \) is called a fractional Brownian motion (fBm, in short) with Hurst index \( H \in (0, 1) \) if \( \beta(0) = 0, \ E\beta(t) = 0 \) and

\[
E[\beta(t)\beta(s)] = \frac{1}{2}[u^{2H} + s^{2H} - |t - s|^{2H}]
\]

for all \( t, s \in [0, T] \). For \( H = 1/2, \beta^H \) coincides with the standard Brownian motion \( B \). \( \beta^H \) is neither a semimartingale nor a Markov process unless \( H = 1/2 \). Some surveys and complete literatures for fBm could be found in Biagini et al [1], Hu [9], Mishura [22], Nourdin [23], Nualart [24], Tudor [31] and the references therein. A cylindrical fBm \( \{ W^H(t), 0 \leq t \leq T \} \) is a \( U \)-valued, \( \mathcal{F}_t \)-adapted fractional Brownian motion defined on \( (\Omega, \mathcal{F}, (\mathcal{F}_t), P) \) with the representation

\[
W^H(t) = \sum_{n=1}^{\infty} u_n^H(t)e_n, \quad t \in [0, T],
\]

where \( \{ u_n^H(t) \}_{t \in [0, T]} \) is a sequence of mutually independent real-valued standard fBms each with the same Hurst index \( 0 < H < 1 \).

We now define the stochastic integration of an integrand \( \Phi : [0, T] \to \mathcal{L}_2(U) \) with respect to cylindrical fBm \( W^H \), denoted by

\[
I(\Psi(s)) := \int_0^T \Psi(s)dW^H(s).
\]

There are different ways to define \( I(\Psi(s)) \), here we adopt the method in Duncan et al [8] and recall an inequality of the integrands.

Lemma 2.1 (Duncan et al [8]). Let \( f \in L^p(0, T; \mathbb{R}) \) for \( p > \frac{1}{H} \) be a deterministic function. Then there exists a constant \( C_T > 0 \) such that

\[
\int_0^T \int_0^T f(u)f(v)\phi(u, v)dudv \leq C_T \| f \|_{L^p(0, T; \mathbb{R})}^2,
\]

where \( \phi(u, v) = \alpha_H|u - v|^{2H - 2} \) with \( \alpha_H = H(2H - 1) \).
Let $\mathcal{E}$ be the family of $U$-valued step functions, defined by

$$\mathcal{E} := \left\{ g : g(s) = \sum_{i=0}^{n-1} g_i 1_{[t_i, t_{i+1})}(s), 0 = t_0 < t_1 < ... < t_n = T, g_i \in U \right\}.$$ 

For $f \in \mathcal{E}$, we define

$$\int_0^T f(s)dw^H(s) := \sum_{i=0}^{n-1} g_i (w^H(t_{i+1}) - w^H(t_i)).$$

It is easy to see that $\mathbb{E} \left( \int_0^T f(s)dw^H(s) \right) = 0$, and

$$\mathbb{E} \left[ \left\| \int_0^T f(s)dw^H(s) \right\|^2 \right] = \int_0^T \int_0^T (f(u), f(v))\phi(u, v)dudv \leq C_{p,T}\|f\|_{L^p(0,T;U)}^2.$$ 

Since $\mathcal{E}$ is dense in $L^p(0,T;U)$, the stochastic integration defined above can be uniquely extended to $L^p(0,T;U)$. To define the integral $I(\Psi(s))$, we also need the next assumptions:

$$\forall x \in U, \ \Psi(\cdot)x \in L^p(0,T;U)$$

and

$$\int_0^T \int_0^T \|\Psi(u)\|_{\mathcal{H}(U)}\|\Psi(v)\|_{\mathcal{H}(U)}\phi(u, v)dudv < \infty.$$ 

Under these assumptions, we define the stochastic integral $I(\Psi(s))$ by

$$I(\Psi(s)) = \int_0^T \Psi(s)dw^H(s) := \sum_{n=1}^{\infty} \int_0^T \Psi(s)e_n dw^H(s), \quad (2)$$

where the summation is in $L^2(\Omega)$ sense. The series in (2) is a zero mean, $U$-valued Gaussian random variable because of

$$\mathbb{E} \left[ \left\| \int_0^T \Psi(s)e_n dw^H(s) \right\|^2 \right] = \sum_{n=1}^{\infty} \mathbb{E} \left[ \left\| \int_0^T \Psi(s)e_n dw^H(s) \right\|^2 \right]$$

$$= \sum_{n=1}^{\infty} \int_0^T \int_0^T \langle \Psi(u)e_n, \Psi(v)e_n \rangle\phi(u,v)dudv$$

$$= \int_0^T \int_0^T \langle \Psi(u), \Psi(v) \rangle_{\mathcal{H}(U)}\phi(u,v)dudv$$

$$\leq \int_0^T \int_0^T \|\Psi(u)\|_{\mathcal{H}(U)}\|\Psi(v)\|_{\mathcal{H}(U)}\phi(u,v)dudv < \infty.$$ 

2.2. The representation of a solution to (1). In order to give the representation, we need some assumptions.

**Assumption 2.1.** The operator $A : \mathcal{D}(A) \subset U \to U$ is a densely defined, linear unbounded, positive self-adjoint operator with compact inverse.

Under this assumption, there exists an increasing sequence of real numbers $\{\lambda_n\}_{n \in \mathbb{N}}$, and an orthonormal basis $\{e_n\}_{n \in \mathbb{N}}$ of $U$ such that $Ae_n = \lambda_n e_n$ and

$$0 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n \leq \cdots \to \infty, \quad n \to \infty.$$
Then, we can define the fractional powers of $A$, i.e. $A^{\gamma}$, $\gamma \in \mathbb{R}$ as follows:

$$A^{\gamma}x = \sum_{j=1}^{\infty} \lambda_j^{\gamma} \langle x, e_j \rangle e_j$$

with

$$x \in D(A^{\gamma/2}) := \left\{ x \in U, \| A^{\gamma/2}x \|^2 = \sum_{j=1}^{\infty} \lambda^2_j \langle x, e_j \rangle^2 < \infty \right\}.$$ 

Let $\dot{U}^{\gamma} = D(A^{\gamma/2})$ and its norm is denoted by

$$\| x \|_{\gamma} = \| A^{\gamma/2}x \| = \left( \sum_{j=1}^{\infty} \lambda^2_j \langle x, e_j \rangle^2 \right)^{1/2}, \quad x \in \dot{U}^{\gamma}.$$ 

Consider the operator $E(t)$ by

$$E(t)x = \sum_{j=1}^{\infty} E_{\alpha,1}(-\lambda_j t^\alpha) \langle x, e_j \rangle e_j, \quad x \in \dot{U}^{\gamma},$$

where $E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(k\alpha+\beta)}$, $z \in \mathbb{C}$, $\beta \in \mathbb{R}$ is the Mittag-Leffler function. By using time fractional Duhamel’s principle (see, for example, Umarov [32]), we can define the solution $X(t)$ of (1) as follows

$$X(t) = E(t)x_0 + \int_0^t E(t-s)\Psi dW^H(s). \quad (3)$$

The function $E_{\alpha,\beta}(z)$ and the operator $E(t)$ admit the following properties.

**Lemma 2.2 ([3, 28]).** If $0 < \beta_1 < 1$, $\beta_2 \geq \beta_1$, then there exists a constant $C_{\beta_1,\beta_2} > 0$, such that

$$0 < E_{\beta_1,\beta_2}(-x^{\beta_1}) \leq C_{\beta_1,\beta_2} \frac{1}{1 + x^{\beta_1}}, \quad x > 0.$$ 

In particular, we have

$$\frac{1}{1 + \Gamma(1-\beta)x} \leq E_{\beta,1}(-x) \leq \frac{1}{1 + \Gamma(1+\beta)^{-1}x}, \quad x > 0, \quad 0 < \beta < 1. \quad (4)$$

**Lemma 2.3 ([27]).** For any $\alpha, \lambda > 0$, and $m \in \mathbb{N}$, we have

$$\frac{d^m}{dt^m} E_{\alpha,1}(-\lambda t^\alpha) = -\lambda^m t^{-\alpha} E_{\alpha,\alpha-m+1}(-\lambda t^\alpha), \quad t > 0.$$ 

**Lemma 2.4 ([17]).** For any $\alpha \in (0, 1)$, $s \in [0, 1]$, we have

$$\| A^{\alpha} E(t) \| \leq Ct^{\alpha}.$$ 

3. **Sharp regularity results.** In this section, we prove the regularity properties of equation (1) in both time and space. Keeping the notations in Section 2. The following assumption will be used.

**Assumption 3.1.** The following assumptions hold:

(a) for some $\beta \in (1/\alpha - 2H, 1/\alpha]$, we have

$$\| A^{\frac{\alpha - 1}{\alpha}} \Psi \|_{\mathcal{L}^2(U)} < \infty;$$

(b) let $x_0 : \Omega \to U$ be $\mathcal{F}_0/\mathcal{B}(U)$-measurable and satisfy $x_0 \in L^2(\Omega, U_{2H+\alpha\beta-1}).$
Example 3.1. Consider the following fractional stochastic heat equation:

\[
\begin{align*}
D_t^\alpha X(t, x) &= \Delta X(t, x) + I_t^{1-\alpha}[W^H(t)], \quad t \in (0, 1), \ x \in (0, 1), \\
X(t, 0) &= X(t, 1) = 0, \quad t \in (0, 1), \\
X(0, x) &= \sin(\pi x).
\end{align*}
\] (5)

Here \( U = L^2(0, 1), \ \Psi = I_U \) and \(-A\) is the Laplacian operator with Dirichlet boundary conditions. It is well known that

\[
\lambda_j = \pi^2 j^2, \ e_j(x) = \sqrt{2}\sin(j\pi x), \ j \geq 1, \ x \in (0, 1).
\]

Obviously, condition (b) of Assumption 3.1 holds. Moreover, we choose \( \beta < \frac{1}{\alpha} - \frac{1}{2} \), then

\[
\left\| A^{\frac{\alpha-1}{\alpha}} \Psi \right\|_{L^2(U)}^2 = \pi^2 \sum_{j=1}^\infty j^{2(\alpha-1)} < \infty.
\]

Thus, Assumption 3.1 is satisfied.

Lemma 3.1. Let \( H < \alpha < 1 \). Then, under assumptions 2.1 and 3.1, there exists a constant \( C = C_{\alpha,H} > 0 \), such that for \( \rho \in [0, \frac{2H}{\alpha}] \), \( 0 < s < t, \ x \in U \),

\[
\int_s^t \int_s^t \langle A^\rho E(t-u)x, A^\rho E(t-v)x \rangle \phi(u,v) dudv \leq C(t-s)^{2(H-\alpha\rho)} \|x\|.
\]

Proof. Observe that \( x = \sum_{j=1}^\infty \langle x, e_j \rangle e_j \), we have

\[
\begin{align*}
\int_s^t \int_s^t \langle A^\rho E(t-u)x, A^\rho E(t-v)x \rangle \phi(u,v) dudv \\
= \sum_{j=1}^\infty \lambda_j^{2\rho} \langle x, e_j \rangle^2 \int_s^t \int_s^t E_{\alpha,1}(-\lambda_j(t-u)\alpha)E_{\alpha,1}(-\lambda_j(t-v)\alpha) \phi(u,v) dudv \\
= \sum_{\lambda_j(t-s)^\alpha \leq 1} \lambda_j^{2\rho} \langle x, e_j \rangle^2 \int_s^t \int_s^t E_{\alpha,1}(-\lambda_j(t-u)\alpha)E_{\alpha,1}(-\lambda_j(t-v)\alpha) \phi(u,v) dudv \\
+ \sum_{\lambda_j(t-s)^\alpha > 1} \lambda_j^{2\rho} \langle x, e_j \rangle^2 \int_s^t \int_s^t E_{\alpha,1}(-\lambda_j(t-u)\alpha)E_{\alpha,1}(-\lambda_j(t-v)\alpha) \phi(u,v) dudv \\
:= I_1 + I_2.
\end{align*}
\]

We firstly estimate the term \( I_1 \), according to lemma 2.2,

\[
I_1 \leq C_{\alpha} \sum_{\lambda_j(t-s)^\alpha \leq 1} \lambda_j^{2\rho} \langle x, e_j \rangle^2 \int_s^t \int_s^t \phi(u,v) dudv
\]

\[
= C_{\alpha,H} \sum_{\lambda_j(t-s)^\alpha \leq 1} \lambda_j^{2\rho} \langle x, e_j \rangle^2 (t-s)^{2H}
\]

\[
\leq C_{\alpha,H}(t-s)^{2H-2\alpha\rho} \sum_{\lambda_j(t-s)^\alpha \leq 1} \langle x, e_j \rangle^2.
\]
We now turn to estimate $I_2$, using a change of variable and combine with lemma 2.2, we have

\[
I_2 = \sum_{\lambda_j (t-s) > 1} \lambda_j^{2\rho} \langle x, e_j \rangle^2 \int_0^t \int_0^{t-s} E_{\alpha,1}(-\lambda_j t^\alpha) E_{\alpha,1}(-\lambda_j s^\alpha) \phi(u, v) du dv
\]

\[
= \sum_{\lambda_j (t-s) > 1} \lambda_j^{2\rho-2H/\alpha} \langle x, e_j \rangle^2 \cdot \int_0^{t-s} \int_0^{t-s} E_{\alpha,1}(-u^\alpha) E_{\alpha,1}(-v^\alpha) \phi(u, v) du dv
\]

\[
\leq C_\alpha \sum_{\lambda_j (t-s) > 1} \lambda_j^{2\rho-2H/\alpha} \langle x, e_j \rangle^2 \cdot \int_0^\infty \int_0^\infty (1 + \Gamma(1 + \alpha)^{-1} u^\alpha)^{-1} (1 + \Gamma(1 + \alpha)^{-1} v^\alpha)^{-1} \phi(u, v) du dv
\]

\[
\leq C_{\alpha,H} (t-s)^{2H-2\alpha \rho} \sum_{\lambda_j (t-s) > 1} \langle x, e_j \rangle^2,
\]

where we have used the fact that for $H < \alpha < 1$ (see Nualart [24]),

\[
\int_0^\infty \int_0^\infty (1 + \Gamma(1 + \alpha)^{-1} u^\alpha)^{-1} (1 + \Gamma(1 + \alpha)^{-1} v^\alpha)^{-1} \phi(u, v) du dv
\]

\[
\leq C_H \left[ \int_0^1 \left( \frac{1}{1 + (\Gamma(1 + \alpha)^{-1} u^\alpha)^{1/H}} \right) du \right]^{2H}
\]

\[
\leq C_H \left[ 1 + \int_1^\infty \left( \frac{1}{1 + (\Gamma(1 + \alpha)^{-1} u^\alpha)^{1/H}} \right) du \right]^{2H} < \infty.
\]

Combining the above two estimates, we complete the proof. \qed

**Lemma 3.2.** Let $\rho \in [-1, 0]$. Then, we have

\[
\| A^\rho (E(t) - E(s)) \| \leq C_\alpha (t-s)^{-\alpha \rho}
\]

for all $0 < s < t$.

**Proof.** For $x \in U$, we have

\[
\| A^\rho (E(t) - E(s)) \| x^2
\]

\[
= \sum_{j=1}^{\lambda_j (t-s) \leq 1} \lambda_j^{2\rho} |E_{\alpha,1}(-\lambda_j t^\alpha) - E_{\alpha,1}(-\lambda_j s^\alpha)|^2 \langle x, e_j \rangle^2
\]

\[
= \sum_{\lambda_j (t-s) > 1} \lambda_j^{2\rho} |E_{\alpha,1}(-\lambda_j t^\alpha) - E_{\alpha,1}(-\lambda_j s^\alpha)|^2 \langle x, e_j \rangle^2
\]

\[
:= \Lambda_1 + \Lambda_2.
\]

For the term $\Lambda_1$, by the proof of Lemma 4.3 in Li and Yang [17] we have

\[
|E_{\alpha,1}(-\lambda_j t^\alpha) - E_{\alpha,1}(-\lambda_j s^\alpha)|^2 \leq C_\alpha \lambda_j^2 (t-s)^{2\alpha},
\]
which implies that
\[
\Lambda_1 \leq C_\alpha (t-s)^{2\alpha} \sum_{\lambda_j (t-s)^\alpha \leq 1} \lambda_j^{2\alpha+2} \langle x, e_j \rangle^2 \\
\leq C_\alpha (t-s)^{-2\alpha q} \sum_{\lambda_j (t-s)^\alpha \leq 1} \langle x, e_j \rangle^2.
\]

On the other hand, Lemma 2.2 deduces to
\[
\Lambda_2 \leq C_\alpha \sum_{\lambda_j (t-s)^\alpha > 1} \lambda_j^{2\alpha} \langle x, e_j \rangle^2 \leq C_\alpha (t-s)^{-2\alpha q} \sum_{\lambda_j (t-s)^\alpha > 1} \langle x, e_j \rangle^2.
\]

The proof is completed. \qed

**Proposition 1.** Let \( H < \alpha < 1 \). Then, under assumptions 2.1 and 3.1, the stochastic evolution
\[
\Xi(t) := \int_0^t E(t-s)\Psi dW^H(s), \quad t \in [0,T]
\]
is well-defined in \( L^2(\Omega, U) \) and admits the following temporal and space regularity:

(i) for all \( t \in [0,T] \), we have
\[
\|\Xi(t)\|_{L^2(\Omega, U_{(2H+\alpha\beta-1)/\alpha})} \leq C_{\alpha,H} \| A^{\frac{\alpha\beta-1}{2\alpha}} \Psi \| \mathcal{L}_2(U);
\]

(ii) for any \( \rho \in [0, \frac{2H+\alpha\beta-1}{\alpha}] \), the following temporal regularity holds:
\[
\|\Xi(t) - \Xi(s)\|_{L^2(\Omega, U_{\rho})} \leq C_{\alpha,H} |t-s|^{2H+\alpha\beta-1-\alpha\rho} \| A^{\frac{\alpha\beta-1}{2\alpha}} \Psi \| \mathcal{L}_2(U).
\]

**Proof.** The well-posedness of \( \Xi(t) \) is clear for every \( t \in [0,T] \), and we only need to check the temporal and space regularity. We first prove the statement (i). By lemma 3.1 it follows that
\[
\|\Xi(t)\|_{L^2(\Omega, U_{(2H+\alpha\beta-1)/\alpha})}^2 = \int_0^t \int_0^t \langle A^{\frac{2H+\alpha\beta-1}{2\alpha}} E(t-u) \Psi, A^{\frac{2H+\alpha\beta-1}{2\alpha}} E(t-v) \Psi \rangle \mathcal{L}_2(U) \phi(u,v) dudv
\]
\[
= \sum_{j=1}^\infty \int_0^t \int_0^t \langle A^{\frac{H}{2} + \alpha\beta - 1} E(t-u) A^{\frac{\alpha\beta-1}{2\alpha}} \Psi e_j, A^{\frac{H}{2} + \alpha\beta - 1} E(t-v) A^{\frac{\alpha\beta-1}{2\alpha}} \Psi e_j \rangle \phi(u,v) dudv
\]
\[
\leq C_{\alpha,H} \sum_{j=1}^\infty \| A^{\frac{\alpha\beta-1}{2\alpha}} \Psi e_j \|^2 = C_{\alpha,H} \| A^{\frac{\alpha\beta-1}{2\alpha}} \Psi \| \mathcal{L}_2(U).
\]

We now prove the statement (ii). Observe that
\[
\Xi(t) - \Xi(s) = \int_s^t (E(t-u) - E(s-u)) \Psi dW^H(u)
\]
\[
= \int_s^t E(t-s) \Psi dW^H(u) := I_3 + I_4.
\]
To estimate $I_3$, for $0 < s < t$, we have

$$
\|I_3\|^2_{L^2(\Omega, U)} = \left\| \int_0^t A^{t/2} (E(t-u) - E(s-u)) \Psi dW^H(u) \right\|^2_{L^2(\Omega, U)}
$$

$$
= \int_0^t \int_0^s \left\langle A^s (E(t-u) - E(s-u)) \Psi, A^s (E(t-v) - E(s-v)) \Psi \right\rangle \mathcal{F}(u) \phi(u, v) dudv
$$

$$
= \sum_{j=1}^{\infty} \sum_{i=1}^{\infty} \lambda_i^{1-\alpha+\alpha_2} \left\langle A^{\frac{\alpha-1}{2\alpha}} \Psi e_j, e_i \right\rangle^2
$$

$$
\cdot \left\{ \int_0^s \int_0^s \left| E_{\alpha,1} (-\lambda_i(t-u)^\alpha) - E_{\alpha,1} (-\lambda_i(s-u)^\alpha) \right| \right. \left. \cdot \left| E_{\alpha,1} (-\lambda_i(t-u)^\alpha) - E_{\alpha,1} (-\lambda_i(s-u)^\alpha) \right| \phi(u, v) dudv \right. \right.
$$

$$
\leq C_H \sum_{j=1}^{\infty} \sum_{i=1}^{\infty} \lambda_i^{1-\alpha+\alpha_2} \left\langle A^{\frac{\alpha-1}{2\alpha}} \Psi e_j, e_i \right\rangle^2
$$

$$
\cdot \left[ \int_0^s \left| E_{\alpha,1} (-\lambda_i(t-u)^\alpha) - E_{\alpha,1} (-\lambda_i(s-u)^\alpha) \right| \frac{1}{s} du \right]^{2H}
$$

$$
:= C_H \sum_{j=1}^{\infty} \Theta_j.
$$

For the term $\Theta_j$ ($j \geq 1$), by integration term-by-term we have

$$
0 \leq \int_s^t E_{\alpha,1} (-\lambda_i(t-u)^\alpha) du = t E_{\alpha,2} (-\lambda t^\alpha) - s E_{\alpha,2} (-\lambda s^\alpha)
$$

with $0 < s < t$, i.e., $t E_{\alpha,2} (-\lambda t^\alpha) \geq s E_{\alpha,2} (-\lambda s^\alpha)$, $0 < s \leq t$. It follows that

$$
\int_0^t \left[ E_{\alpha,1} (-\lambda(t'-u)^\alpha) - E_{\alpha,1} (-\lambda(t-u)^\alpha) \right] du
$$

$$
= t' E_{\alpha,2} (-\lambda t'^\alpha) - t E_{\alpha,2} (-\lambda t^\alpha) + (t-t') E_{\alpha,2} (-\lambda(t-t')^\alpha)
$$

$$
\leq (t-t') E_{\alpha,2} (-\lambda(t-t')^\alpha)
$$

for any $0 < t' < t$. Then we have

$$
\int_0^s \left| E_{\alpha,1} (-\lambda_i(t-u)^\alpha) - E_{\alpha,1} (-\lambda_i(s-u)^\alpha) \right| du \leq C_0 \frac{t-s}{1+\lambda_i(t-s)^\alpha}.
$$

Thus

$$
\Theta_j \leq C_H \sum_{i=1}^{\infty} \lambda_i^{1-\alpha+\alpha_2} \left\langle A^{\frac{\alpha-1}{2\alpha}} \Psi e_j, e_i \right\rangle^2
$$

$$
\cdot \left[ \int_0^s \left| E_{\alpha,1} (-\lambda_i(t-u)^\alpha) - E_{\alpha,1} (-\lambda_i(s-u)^\alpha) \right| du \right]^{2H}
$$

$$
\leq C_0 \sum_{i=1}^{\infty} \lambda_i^{1-\alpha+\alpha_2} \left\langle A^{\frac{\alpha-1}{2\alpha}} \Psi e_j, e_i \right\rangle^2 \left( \frac{t-s}{1+\lambda_i(t-s)^\alpha} \right)^{2H}
$$

$$
\leq C_0 \sum_{\lambda_i(t-s)^\alpha \leq 1} \lambda_i^{1-\alpha+\alpha_2} \left\langle A^{\frac{\alpha-1}{2\alpha}} \Psi e_j, e_i \right\rangle^2 (t-s)^{2H}
$$
\[ + C_{\alpha,H} \sum_{\lambda_i(t-s)^\alpha > 1} \lambda_i^{1-\frac{\alpha}{2} + \alpha \rho - 2H\alpha} \left\langle A^{\frac{\alpha - 1}{2\alpha}} \Psi e_j, e_i \right\rangle^2 (t-s)^{2H-2H\alpha} \]

This gives the desired estimate
\[ \|I_3\|_{L^2(\Omega, U, \rho)} \leq C_{\alpha,H} (t-s)^{2H+\alpha\beta-1-\alpha\rho} \sum_{j=1}^{\infty} \|A^{\frac{\alpha - 1}{2\alpha}} \Psi e_j\|^2 \]

Finally, we can easily get that
\[ \|I_4\|_{L^2(\Omega, U, \rho)} = \left\| \int_s^t A^\frac{\alpha}{2}\mathcal{E}(u) \Psi dW^H(u) \right\| \]

As an immediate conclusion of Proposition 1 and Lemma 3.2, we can establish the following result. We omit the proof of the existence and uniqueness of the mild solution as it is standard. For more details, we refer the readers to [6, 16].

**Theorem 3.3.** Let \( H < \alpha < 1 \). Then, under assumptions 2.1, 3.1, the equation (1) admits a unique mild solution (3). Moreover, we have
\[ \|X(t)\|_{L^2(\Omega, U(2H+\alpha\beta-1/\alpha), \rho)} \leq C_{\alpha,H} \left[ 1 + \|x_0\|_{L^2(\Omega, U(2H+\alpha\beta-1/\alpha), \rho)} \right], \quad t \in [0, T], \]

and for any \( \rho \in [0, (2H + \alpha\beta - 1)/\alpha] \),
\[ \|X(t) - X(s)\|_{L^2(\Omega, U(\rho))} \leq C_{\alpha,H} \left[ 1 + \|x_0\|_{L^2(\Omega, U(2H+\alpha\beta-1/\alpha), \rho)} \right] |t-s|^{2H+\alpha\beta-1-\alpha\rho}. \]

It is important to note that the equation (1) reduces to the abstract evolution equation driven by Cylindrical fractional Brownian motion, as \( \alpha \to 1 \), which was investigated in Wang et al [34], and Theorem 3.3 coincides with Theorem 3.5 in Wang et al [34]. In order to motivate why we discuss the optimal spatial regularity, we conclude this section with the following example, which is a slight modification in the example from Wang et al [34]. More precisely, denote \( U = L^2(0, 1) \) and let \(-A\) be the Laplacian with Dirichlet boundary conditions.

Define the stochastic evolution
\[ \mathcal{O}(t) = \int_0^t E(t-s)Q^{1/2}dW^H(s), \quad t \in [0, T]. \]
where \( Q : U \to U \) is defined by
\[
Qe_1 = 0, \quad Qe_j = j^{2/\alpha - 1 - 2\beta}[\ln(j)]^{-2}e_j, \quad j \geq 2, \quad \beta \in (1/\alpha - 2H, 1/\alpha].
\]
Then \[
\|A^{\alpha \beta - 1/2}\|_{L^2(U)} < \infty \quad \text{and} \quad \|A^{\alpha \beta - 1/2}\|_{L^2(U)} = \infty \quad \text{for any} \ \delta > \beta.
\]
On the other hand, for any \( \gamma > 0 \) and \( t > 0 \),
\[
\|\Theta(t)\|^2_{L^2((0,t]^{2H+\alpha\beta-1+\gamma}/\alpha)} = \int_0^t \left( A^{2H+\alpha\beta-1+\gamma/\alpha} E(t-u)Q^{1/2} \right) \|L^2(U)\|_2 \phi(u,v) du dv
\]
\[
= \sum_{i=2}^{\infty} \lambda_i^{2/\alpha - 1 - 2\beta}[\ln(i)]^{-2} \int_0^t \int_0^t E_{\alpha,1}(-\lambda_i u^\alpha)E_{\alpha,1}(-\lambda_i v^\alpha)\phi(u,v) du dv
\]
\[
\geq \sum_{i=2}^{\infty} \lambda_i^{2/\alpha - 1 + \gamma} \frac{2 - \alpha - 0\alpha}{\alpha} [\ln(i)]^{-2} \int_0^t \int_0^t E_{\alpha,1}(-u^\alpha)E_{\alpha,1}(-v^\alpha)\phi(u,v) du dv
\]
\[
= C_{H,\alpha,\pi} \sum_{i=2}^{\infty} \lambda_i^{-1} [\ln(i)]^{-2} = +\infty.
\]

4. **Optimal error estimate of a full-discretization.** In this section, we obtain the optimal convergence rates for strong approximate of the underlying problem, which coincides with the regularity of the solution.

4.1. **Spatial semi-discretization.** We spatially discretize (1) with a spectral Galerkin method (for more details, see Thomée [29]). Let \( N \in \mathbb{N} \) and define a finite dimensional subspace of \( U \) by \( U_N := \text{span}\{e_1, \ldots, e_N\} \). Define the projection \( P_N : U \to U_N \) by
\[
P_N x = \sum_{i=1}^N \langle x, e_i \rangle e_i, \quad \forall \ x \in U, \ \varsigma \in \mathbb{R}.
\]
It is easy to see that
\[
\|(P_N - I)x\| \leq \lambda_N^{-\varsigma} \|x\|, \quad x \in U, \ \alpha \geq 0.
\]
The spectral Galerkin method corresponding to (1) is
\[
\left\{
\begin{array}{l}
P_N X^N(t) + A_N X^N(t) = I_1^{1-\alpha}[P_N \Psi \dot{W}^H(t)], \quad \alpha \in (0, 1), \ t \in [0, T], \\
X^N(0) = P_N x_0,
\end{array}
\right.
\]
where \( A_N : U \to U_N, \ A_N := AP_N \). Then, the mild solution of (6) is
\[
X^N(t) = E_N(t)P_N x_0 + \int_0^t E_N(t-s)P_N \Psi dW^H(s)
\]
with
\[
E_N(t)x = \sum_{j=1}^N E_{\alpha,1}(-\lambda_j t^\alpha)(x, e_j) e_j, \quad x \in U_N.
\]
The following theorem gives the error estimate for the spectral Galerkin discretization (6).
Theorem 4.1. Let $H < \alpha < 1$. Then, under assumptions 2.1 and 3.1, we have
\[
\|X(t) - X^N(t)\|_{L^2(\Omega, U)} \leq C_{\alpha, H} \left[ 1 + \|x_0\|_{L^2(\Omega, U(2H+\alpha - 1)/\alpha)} \right]^{\frac{-2H+\alpha - 1}{\alpha}}
\]
for all $N \in \mathbb{N}$.

Proof. Observe that
\[
\|X(t) - X^N(t)\|_{L^2(\Omega, U)} \leq \|(E(t) - E_N(t)P_N)x_0\|_{L^2(\Omega, U)}
+ \left\| \int_0^t (E(t-s) - E_N(t-s)P_N)\Psi dW^H(s) \right\|_{L^2(\Omega, U)}
= \Lambda_3 + \Lambda_4.
\]

It is easy to check that
\[
\Lambda_3 \leq C_{\alpha} \lambda^{\frac{-2H+\alpha - 1}{\alpha}} \|x_0\|_{L^2(\Omega, U(2H+\alpha - 1)/\alpha)}.
\]

Now, we turn to estimate $\Lambda_4$. Define
\[
F(t)x := (E(t-s) - E_N(t-s)P_N)x
= \sum_{j \geq N+1} E_{\alpha,1}(-\lambda_j t^\alpha)\langle x, e_j\rangle e_j, \ x \in U.
\]

It follows that
\[
\int_0^t \int_0^t \langle F(u)x, F(v)x \rangle \phi(u,v) dudv
= \sum_{j=1}^\infty \int_0^t \int_0^t \langle F(u)x, e_j \rangle \langle F(v)x, e_j \rangle \phi(u,v) dudv
= \sum_{j \geq N+1} \langle x, e_j \rangle^2 \int_0^t \int_0^t E_{\alpha,1}(-\lambda_j u^\alpha)E_{\alpha,1}(-\lambda_j v^\alpha)\phi(u,v) dudv
\leq \lambda_{N+1}^{\frac{-2H+\alpha - 1}{\alpha}} \int_0^t \int_0^t \sum_{j \geq N+1} \lambda_j^{\frac{2H+\alpha - 1}{\alpha}} E_{\alpha,1}(-\lambda_j u^\alpha)E_{\alpha,1}(-\lambda_j v^\alpha)\phi(u,v) dudv
\leq \lambda_{N+1}^{\frac{-2H+\alpha - 1}{\alpha}} \int_0^t \int_0^t \sum_{j=1}^\infty \lambda_j^{\frac{2H+\alpha - 1}{\alpha}} E_{\alpha,1}(-\lambda_j u^\alpha)E_{\alpha,1}(-\lambda_j v^\alpha)\phi(u,v) dudv
= \lambda_{N+1}^{\frac{-2H+\alpha - 1}{\alpha}} \int_0^t \int_0^t \langle A H E(u)A^{-\frac{\alpha}{2\alpha - 1}} x, A H E(v)A^{-\frac{\alpha}{2\alpha - 1}} x \rangle \phi(u,v) dudv
\leq C_{\alpha, H} \lambda_{N+1}^{\frac{-2H+\alpha - 1}{\alpha}} \|x\|_{U_{\alpha, \frac{1}{\alpha}}},
\]
for any $x \in U_{\alpha, \frac{1}{\alpha}}$, which implies that
\[
\Lambda_4^2 = \left\| \int_0^t F(t-s)\Psi dW^H(s) \right\|_{L^2(\Omega, U)}^2
= \int_0^t \int_0^t \langle F(t-u)\Psi, F(t-v)\Psi \rangle \mathcal{Z}_2(U) \phi(u,v) dudv
= \sum_{j=1}^\infty \int_0^t \int_0^t \langle F(t-u)\Psi e_j, F(t-v)\Psi e_j \rangle \phi(u,v) dudv.
\]
Thus, the lemma follows from (8), (9) and (10). □

4.2. Fully discrete error estimate. This subsection devotes to a fully discrete approximate for equation (1). We firstly consider time discretization corresponding to equation (1) and analysis the corresponding error estimate.

By operating $D_t^{1-\alpha}$ on both sides of (1), we get

$$
\begin{align*}
D_tX(t) + D_t^{1-\alpha}AX(t) &= \Psi W^H(t), \quad \alpha \in (0, 1), \quad t \in [0, T], \\
X(0) &= x_0,
\end{align*}
$$

where $D_t$ denotes the usual derivative with respect to $t$. For a fixed time step size $\Delta t > 0$, we put $t_n = n\Delta t$ and time discretization (11) by using the DG method (see [19, 20]), define a piecewise constant approximation $V(t) \approx X(t)$ by

$$
V^n - V^{n-1} + \int_{t_{n-1}}^{t_n} D_t^{1-\alpha}AV(t)dt = \int_{t_{n-1}}^{t_n} \Psi dW^H(t)
$$

for $n \geq 1$, with $V^0 = x_0$, where $V^n = V(t_n^{-}) = \lim_{t \to t_n^{-}} V(t)$ denotes the one-side limit from below at the $n$th time level. Thus, $V(t) = V^n$, $t_{n-1} < t \leq t_n$. An elementary calculus may prove that

$$
\int_{t_{n-1}}^{t_n} D_t^{1-\alpha}AV(t)dt = (\Delta t)^{\alpha} \sum_{j=1}^{n} \beta_{n-j}AV^j
$$

with

$$
\beta_0 = (\Delta t)^{-\alpha} \int_{t_{n-1}}^{t_n} \frac{(t_n - t)^{\alpha-1}}{\Gamma(\alpha)} dt = \frac{1}{\Gamma(1 + \alpha)};
$$

and

$$
\beta_j = (\Delta t)^{-\alpha} \int_{t_{n-j-1}}^{t_{n-j}} \frac{(t_n - t)^{\alpha-1} - (t_{n-1} - t)^{\alpha-1}}{\Gamma(\alpha)} dt
= \frac{(j + 1)^{\alpha} - 2j^{\alpha} + (j - 1)^{\alpha}}{\Gamma(1 + \alpha)}, \quad j \geq 1.
$$

Following Kovács and Printems [14] (see also [17]), we can give a formulation for the discrete mild solution of (12). Consider the deterministic algorithm

$$
V^n_1 - V^{n-1}_1 + (\Delta t)^{\alpha} \sum_{j=1}^{n} \beta_{n-j}AV^j_1 = 0, \quad n \geq 1, \quad V^0_1 = x_0.
$$

Define $\widehat{V}_1(z) = \sum_{k=0}^{\infty} V_1^k z^k$, $\tilde{\beta}(z) = \sum_{k=0}^{\infty} \beta_k z^k$, then by taking $z$-transform we have

$$
\widehat{V}_1(z) - x_0 - z\widehat{V}_1(z) + (\Delta t)^{\alpha} \tilde{\beta}(z)A(\widehat{V}_1(z) - x_0) = 0.
$$

That is,

$$
\widehat{V}_1(z) = (I + (\Delta t)^{\alpha} \tilde{\beta}(z)A)((1 - z)I + (\Delta t)^{\alpha} \tilde{\beta}(z)A)^{-1}x_0 = \widehat{B}(z)x_0,
$$

where $\widehat{B}(z)x_0 := \sum_{k=0}^{\infty} B_k x_0 z^k = \sum_{k=0}^{\infty} V_1^k z^k$. We can rewrite $\widehat{B}(z)x_0$ as (see (4.4) in [17])

$$
\widehat{B}(z)x_0 = (z((1 - z)I + (\Delta t)^{\alpha} \tilde{\beta}(z)A)^{-1} + I)x_0.
$$
Let $w_0^H = 0$, $w_n^H = \Psi(W^H(t_n) - W^H(t_{n-1}))$, $n \geq 1$, and

$$
\hat{w}^H(z) = \sum_{k=0}^{+\infty} u_k^H z^k, \quad \hat{V}(z) = \sum_{k=0}^{+\infty} V_k z^k.
$$

Taking z-transform for (12), then

$$
\hat{V}(z) - x_0 - z\hat{V}(z) + (\Delta t)\hat{\beta}(z)A(\hat{V}(z) - x_0) = \hat{w}^H(z).
$$

Combining this equality with the expression of $\hat{B}(z)x_0$, we can easily get that

$$
\hat{V}(z) = \hat{B}(z)x_0 + ((1 - z)I + (\Delta t)^\alpha \hat{\beta}(z)A)^{-1}\hat{w}^H(z)
$$

$$
= \hat{B}(z)x_0 + \hat{B}(z)\frac{w^H(z)}{z} - \frac{1}{z}\hat{w}^H(z).
$$

Taking the inverse z-transform, we further get

$$
V^n = B_n x_0 + \sum_{k=0}^{n-1} B_{n-k} u_{k+1}^H
$$

that is

$$
V^n = B_n x_0 + \sum_{k=1}^{n} \int_{t_{k-1}}^{t_k} B_{n-k+1} dW^H.
$$

(13)

**Lemma 4.2.** Let $\ell(u, t_i) = E(t_n - u) - E(t_n - t_i)$ for $i = 0, 1, \ldots, n$. Then, for any $x \in U$, we have

$$
\sum_{i,j=0}^{n-1} \int_{t_j}^{t_{j+1}} \int_{t_i}^{t_{i+1}} \langle \ell(u, t_i)x, \ell(v, t_j)x \rangle \phi(u, v) dudv
$$

$$
\leq C_{\alpha,H}(\Delta t)^{2H+\alpha\beta-1} \left\| A^{\alpha\beta-1} x \right\|^2.
$$

**Proof.** Denote $\zeta(u, i) = |E_{\alpha,1}(-\lambda_m(t_n-u)^\alpha) - E_{\alpha,1}(-\lambda_m(t_n-t_i)^\alpha)|$ for $i = 0, 1, \ldots, n$, and

$$
\Lambda_5 = \sum_{i,j=0}^{n-1} \sum_{\lambda_m(\Delta t)^{\alpha}} \langle A^{\alpha\beta-1} x, e_m \rangle^2 \lambda_m^{-\alpha\beta-1} \int_{t_j}^{t_{j+1}} \int_{t_i}^{t_{i+1}} \zeta(u, i)\zeta(v, j) \phi(u, v) dudv,
$$

and

$$
\Lambda_6 = \sum_{i,j=0}^{n-1} \sum_{\lambda_m(\Delta t)^{\alpha}} \langle A^{\alpha\beta-1} x, e_m \rangle^2 \lambda_m^{-\alpha\beta-1} \int_{t_j}^{t_{j+1}} \int_{t_i}^{t_{i+1}} \zeta(u, i)\zeta(v, j) \phi(u, v) dudv.
$$

It follows that

$$
\sum_{i,j=0}^{n-1} \int_{t_j}^{t_{j+1}} \int_{t_i}^{t_{i+1}} \langle \ell(u, t_i)x, \ell(v, t_j)x \rangle \phi(u, v) dudv
$$

$$
= \sum_{i,j=0}^{n-1} \sum_{m=1}^{\infty} \langle x, e_m \rangle^2 \lambda_m^{-\alpha\beta-1} \int_{t_j}^{t_{j+1}} \int_{t_i}^{t_{i+1}} \zeta(u, i)\zeta(v, j) \phi(u, v) dudv
$$

$$
= \sum_{i,j=0}^{n-1} \sum_{m=1}^{\infty} \langle A^{\alpha\beta-1} x, e_m \rangle^2 \lambda_m^{-\alpha\beta-1} \int_{t_j}^{t_{j+1}} \int_{t_i}^{t_{i+1}} \zeta(u, i)\zeta(v, j) \phi(u, v) dudv
$$

$$
= \Lambda_5 + \Lambda_6.
$$
On the other hand, by the fact
\[
\ln(1 + x) \leq x^\delta, \quad x \geq 0, \quad 0 < \delta \leq 1,
\]
we have
\[
\zeta(u, i) = \left| \int_{t_{n-i}}^{t_n-u} \lambda_m v^{\alpha-1} E_{\alpha, \alpha}(-\lambda_m v^\alpha) dv \right| \\
\leq C_\alpha \int_{t_{n-i}}^{t_n-u} \frac{\lambda_m v^{\alpha-1}}{1 + \lambda_m v^\alpha} dv \\
= C_\alpha |\ln(1 + \lambda_m (t_n - t_i)^\alpha) - \ln(1 + \lambda_m (t_n - u)^\alpha)| \\
\leq C_\alpha \ln \left[ 1 + \lambda_m (u - t_i)^\alpha \right] \\
\leq C_\alpha |\lambda_m (\Delta t)^\alpha|^\delta \left[ 1 + \lambda_m (t_n - u)^\alpha \right]^{-\delta},
\]
where \( t_i \leq u \leq t_{i+1} \) and \( 0 < \delta \leq 1 \). Combining this with a change of variables, we see that
\[
A_5 \leq C_\alpha \sum_{i,j=0}^{n-1} \sum_{\lambda_m(\Delta t)^\alpha \leq 1} \left\langle A^{\frac{\alpha-1}{2\alpha}} x, e_m \right\rangle^2 \lambda_m^{-\frac{\alpha-1}{\alpha} + 2\delta} (\Delta t)^{2\alpha} \\
\cdot \int_{t_{i+1}}^{t_{i+1}} \int_{t_i}^{t_n} \left[ 1 + \lambda_m (t_n - u)^\alpha \right]^{-\delta} \left[ 1 + \lambda_m (t_n - v)^\alpha \right]^{-\delta} \phi(u, v) dudv \\
= C_\alpha \sum_{\lambda_m(\Delta t)^\alpha \leq 1} \left\langle A^{\frac{\alpha-1}{2\alpha}} x, e_m \right\rangle^2 \lambda_m^{-\frac{\alpha-1}{\alpha} + 2\delta - \frac{3H}{\alpha}} (\Delta t)^{2\alpha} \\
\cdot \int_0^{t_n} \int_0^{t_n} \left( \frac{1}{1 + u^\alpha} \right)^\delta \left( \frac{1}{1 + v^\alpha} \right)^\delta \phi(u, v) dudv \\
\leq C_\alpha \sum_{\lambda_m(\Delta t)^\alpha \leq 1} \left\langle A^{\frac{\alpha-1}{2\alpha}} x, e_m \right\rangle^2 \lambda_m^{-\frac{\alpha-1}{\alpha} + 2\delta - \frac{3H}{\alpha}} (\Delta t)^{2\alpha} \\
\cdot \int_0^\infty \int_0^\infty \left( \frac{1}{1 + u^\alpha} \right)^\delta \left( \frac{1}{1 + v^\alpha} \right)^\delta \phi(u, v) dudv \\
\leq C_{\alpha, H} (\Delta t)^{2H + \alpha\beta - 1} \sum_{\lambda_m(\Delta t)^\alpha \leq 1} \left\langle A^{\frac{\alpha-1}{2\alpha}} x, e_m \right\rangle^2
\]
with \( \frac{H}{\alpha} < \delta < 1 \). For the term \( A_6 \), in a similar argument as Lemma 3.1, we get
\[
A_6 \leq \sum_{\lambda_m(\Delta t)^\alpha \geq 1} \left\langle A^{\frac{\alpha-1}{2\alpha}} x, e_m \right\rangle^2 \lambda_m^{-\frac{\alpha-1}{\alpha}} \\
\cdot \int_0^{t_n} \int_0^{t_n} E_{\alpha, 1}(-\lambda_m (t_n - u)^\alpha) E_{\alpha, 1}(-\lambda_m (t_n - v)^\alpha) \phi(u, v) dudv \\
\leq C_{\alpha, H} (\Delta t)^{2H + \alpha\beta - 1} \sum_{\lambda_m(\Delta t)^\alpha \geq 1} \left\langle A^{\frac{\alpha-1}{2\alpha}} x, e_m \right\rangle^2.
\]
Thus, the lemma follows from (14). \( \square \)
Lemma 4.3. Let $H < \alpha < 1$. Then, under the assumptions 2.1, 3.1, we have

$$\sum_{i,j=0}^{n-1} \int_{t_j}^{t_{j+1}} \int_{t_i}^{t_{i+1}} \langle (E(t_n - t_i) - B_{n-1})x, (E(t_n - t_j) - B_{n-j})x \rangle \phi(u,v) du dv \leq C_{\alpha,H}(\Delta t)^{2H+\alpha \beta - 1} \| A^{\frac{\alpha-1}{2\alpha}} x \|^2.$$ 

for all $x \in U$.

Proof. By (13) in McLean and Mustapha [20], we have

$$E(t_n - t_i)x - B_{n-i}x = \sum_{n=1}^{\infty} \delta^{n-i}(\lambda_m(\Delta t)^\alpha)(x,e_j)e_j,$$  \hspace{1cm} (15)

where the function $\delta^n(z)$ is defined in McLean and Mustapha [20] such that

$$|\delta^n(\mu)| \leq C\min\{(\mu n)^2, (\mu n)^{-1}\}, \hspace{1cm} 0 < \mu < \infty.$$  \hspace{1cm} (16)

It follows that

$$\sum_{i,j=0}^{n-1} \int_{t_j}^{t_{j+1}} \int_{t_i}^{t_{i+1}} \langle (E(t_n - t_i) - B_{n-1})x, (E(t_n - t_j) - B_{n-j})x \rangle \phi(u,v) du dv$$

$$= \sum_{i,j=0}^{n-1} \sum_{k=1}^{\infty} \int_{t_j}^{t_{j+1}} \int_{t_i}^{t_{i+1}} \delta^{n-i}(\lambda_k(\Delta t)^\alpha)\delta^{n-j}(\lambda_k(\Delta t)^\alpha) \phi(u,v) du dv$$

$$= \sum_{\lambda_k(\Delta t)^\alpha \leq 1} \sum_{i,j=0}^{n-1} \langle x, e_k \rangle^2 \int_{t_j}^{t_{j+1}} \int_{t_i}^{t_{i+1}} \delta^{n-i}(\lambda_k(\Delta t)^\alpha) \delta^{n-j}(\lambda_k(\Delta t)^\alpha) \phi(u,v) du dv$$

$$+ \sum_{\lambda_k(\Delta t)^\alpha > 1} \sum_{i,j=0}^{n-1} \langle x, e_k \rangle^2 \int_{t_j}^{t_{j+1}} \int_{t_i}^{t_{i+1}} \delta^{n-i}(\lambda_k(\Delta t)^\alpha) \delta^{n-j}(\lambda_k(\Delta t)^\alpha) \phi(u,v) du dv$$

$$:= \Lambda_7 + \Lambda_8.$$

Clearly, we have

$$\Lambda_7 \leq C \sum_{\lambda_k(\Delta t)^\alpha \leq 1} \sum_{i,j=0}^{n-1} \langle A^{\frac{\alpha-1}{2\alpha}} x, e_k \rangle^2 \lambda_k \frac{\alpha-1}{2\alpha} \int_{t_j}^{t_{j+1}} \int_{t_i}^{t_{i+1}} \frac{\phi(u,v)}{(n-i)(n-j)} du dv$$

$$= C \sum_{\lambda_k(\Delta t)^\alpha \leq 1} \sum_{i,j=1}^{n} \langle A^{\frac{\alpha-1}{2\alpha}} x, e_k \rangle^2 \lambda_k \frac{\alpha-1}{2\alpha} \int_{t_{j-1}}^{t_j} \int_{t_{i-1}}^{t_i} \frac{\phi(u,v)}{(u + \Delta t)(v + \Delta t)} du dv$$

$$\leq C \sum_{\lambda_k(\Delta t)^\alpha \leq 1} \langle A^{\frac{\alpha-1}{2\alpha}} x, e_k \rangle^2 \lambda_k \frac{\alpha-1}{2\alpha} (\Delta t)^2 \int_0^{\infty} \int_0^{\infty} \frac{\phi(u,v) du dv}{(u + 1)(v + 1)}$$

$$\leq C_{\alpha,H}(\Delta t)^{2H+\alpha \beta - 1} \sum_{\lambda_k(\Delta t)^\alpha \leq 1} \langle A^{\frac{\alpha-1}{2\alpha}} x, e_k \rangle^2 \int_0^{\infty} \int_0^{\infty} \frac{\phi(u,v) du dv}{(u + 1)(v + 1)}$$

$$\leq C_{\alpha,H}(\Delta t)^{2H+\alpha \beta - 1} \sum_{\lambda_k(\Delta t)^\alpha \leq 1} \langle A^{\frac{\alpha-1}{2\alpha}} x, e_k \rangle^2 \frac{(\lambda_k(\Delta t)^\alpha)^{-2}}{2}.$$  \hspace{1cm} (15)

since $|\delta^n(\mu)| \leq C\frac{\mu}{n}$ for all $0 < \mu < \infty$. For the term $\Lambda_8$, by (16), we also have that

$$\Lambda_8 \leq C \sum_{\lambda_k(\Delta t)^\alpha > 1} \sum_{i,j=0}^{n-1} \langle x, e_k \rangle^2 \frac{(\lambda_k(\Delta t)^\alpha)^{-2}}{2}.$$
This shows that

\[ \text{Lemma 4.1 in [17]} \]

and the lemma follows.

\[ \square \]

**Theorem 4.4.** Under assumption 2.1, 3.1 and let \( H < \alpha < 1 \), then

\[
\|X(t_n) - V^n\|_{L^2(\Omega, U)} \leq C_{\alpha, H} \left[ 1 + \|x_0\|_{L^2(\Omega, \mathbb{R}^{2H+\alpha\beta-1})} \right] (\Delta t)^{\frac{2H+\alpha\beta-1}{2}}.
\]

**Proof.** We have

\[
X(t_n) - V^n = (E(t_n) - B_n)x_0 + \sum_{k=1}^{n} \int_{t_{k-1}}^{t_k} (E(t_n - s) - B_{n-k+1}) \Psi ds + \sum_{k=1}^{n} \int_{t_{k-1}}^{t_k} (E(t_n - s) - B_{n-k+1}) \Psi ds + \sum_{k=1}^{n} \int_{t_{k-1}}^{t_k} (E(t_n - s) - B_{n-k+1}) \Psi ds.
\]

(17)

Then, we have

\[
\|A_1\|^2_{L^2(\Omega, U)} \leq 2\sum_{k=1}^{n} \int_{t_{k-1}}^{t_k} (E(t_n - s) - E(t_n - [s])) \Psi ds \leq 2\sum_{k=1}^{n} \int_{t_{k-1}}^{t_k} (E(t_n - s) - E(t_n - [s])) \Psi ds \leq 2\sum_{k=1}^{n} \int_{t_{k-1}}^{t_k} (E(t_n - s) - E(t_n - [s])) \Psi ds \leq 2\sum_{k=1}^{n} \int_{t_{k-1}}^{t_k} (E(t_n - s) - E(t_n - [s])) \Psi ds.
\]

(18)
Similarly, by Lemma 4.3 we also have

\[ \text{Observe that} \]

Proof.

Thus, we have gotten the desired estimate

\[ \| A^{\infty} \| \leq C_{\alpha,H} (\Delta t)^{2H+\alpha-1} \left\| A^{\frac{\alpha-1}{2\alpha}} \right\|_{L^2(U)}^2. \]

By Lemma 4.2, it follows that

\[ \Lambda_{10,1} := \sum_{i,j=0}^{n-1} \int_{t_j}^{t_{j+1}} \int_{t_i}^{t_{i+1}} \langle \Xi_2(u)\Psi, \Xi_2(v)\Psi \rangle_{L^2(U)} \phi(u,v)dv \]

\[ = \sum_{i,j=0}^{n-1} \int_{t_j}^{t_{j+1}} \int_{t_i}^{t_{i+1}} \langle \ell(u, t_i)\Psi, \ell(v, t_j)\Psi \rangle_{L^2(U)} \phi(u,v)dv \]

\[ \leq C_{\alpha,H} (\Delta t)^{2H+\alpha-1} \left\| A^{\frac{\alpha-1}{2\alpha}} \right\|_{L^2(U)}^2. \]

Similarly, by Lemma 4.3 we also have

\[ \Lambda_{10,2} := \sum_{i,j=0}^{n-1} \int_{t_j}^{t_{j+1}} \int_{t_i}^{t_{i+1}} \langle \Xi_2(u)\Psi, \Xi_2(v)\Psi \rangle_{L^2(U)} \phi(u,v)dv \]

\[ \leq C_{\alpha,H} (\Delta t)^{2H+\alpha-1} \left\| A^{\frac{\alpha-1}{2\alpha}} \right\|_{L^2(U)}^2. \]

Thus, we have gotten the desired estimate

\[ \| \Lambda_{10} \|_{L^2(U)}^2 \leq \Lambda_{10,1} + \Lambda_{10,2} \leq C_{\alpha,H} (\Delta t)^{2H+\alpha-1} \left\| A^{\frac{\alpha-1}{2\alpha}} \right\|_{L^2(U)}^2, \]

and the Theorem follows from (17) and (18).

Now, we consider a fully discrete approximation for equation (1). Let

\[ V^0_N - V^{-1}_N + (\Delta t)^\alpha \sum_{j=1}^{n} \beta_{n-j} A_N V^j_N = \int_{t_{n-1}}^{t_n} P_N (s) dW^H(s), \quad n \geq 1, \]

and \( V^0_N = P_N x_0 \). Similarly, we have

\[ V^n_N = B_{n,N} P_N x_0 + \sum_{k=0}^{n-1} \int_{t_k}^{t_{k+1}} B_{n-k+1,N} P_N (s) dW^H(s), \]

where \( B_{n,N} = P_N B_n \). The following Theorem gives the convergence results for the space discretization and for time discretization.

**Theorem 4.5.** Let \( H < \alpha < 1 \). Then, under the assumptions 2.1 and 3.1, we have

\[ \| X(t_n) - V^n_N \|_{L^2(U)} \leq C_{\alpha,H} \left( 1 + \| x_0 \|_{L^2(U)} \right) \left( (\Delta t)^{2H+\alpha-1} + \lambda_{n+1,^{\frac{\alpha}{2}}} \right). \]

Proof. Observe that \( X(t_n) - V^n_N = J_1 + J_2 \), where

\[ J_1 = E(t_n)x_0 - E_N(t_n)P_N x_0 + \int_{0}^{t_n} (E(t_n - s)\Psi - E(t_n - s) P_N \Psi) dW^H(s) \]

and

\[ J_2 = \sum_{i,j=0}^{n-1} \int_{t_j}^{t_{j+1}} \int_{t_i}^{t_{i+1}} \langle \ell(u, t_i)\Psi, \ell(v, t_j)\Psi \rangle_{L^2(U)} \phi(u,v)dv \]

\[ \leq C_{\alpha,H} (\Delta t)^{2H+\alpha-1} \left\| A^{\frac{\alpha-1}{2\alpha}} \right\|_{L^2(U)}^2. \]
and
\[ J_2 = E_N(t_n)P_Nx_0 - B_{n,N}P_Nx_0 + \sum_{k=0}^{n-1} \int_{t_k}^{t_{k+1}} (E_N(t_n - s) - B_{n-k+1,N})P_N\Psi dW^H(s) \]
\[ = P_N(E(t_n) - B_n)x_0 + \sum_{k=0}^{n-1} \int_{t_k}^{t_{k+1}} P_N(E(t_n - s) - B_{n-k+1})\Psi dW^H(s). \]

On the one hand, the estimation
\[ \|J_1\|_{L^2(\Omega,U)} \leq C_{\alpha,H} \left[ 1 + \|x_0\|_{L^2(\Omega,U)} \right] \lambda^{-\frac{2H+\alpha\beta-1}{2}} \]
is a direct consequence of Theorem 4.1. On the other hand, it follows from Theorem 4.4 and the contractive property of \( P_N \) that
\[ \|J_2\|_{L^2(\Omega,U)} \leq C_{\alpha,H} \left[ 1 + \|x_0\|_{L^2(\Omega,U)} \right] (\Delta t)^{\frac{2H+\alpha\beta-1}{2}}. \]

Therefore the proof is completed.

5. Numerical results. In this section, we present a numerical example to illustrate previous convergence rates. Consider Example 3.1:
\[
\begin{align*}
D^\alpha X(t,x) &= \Delta X(t,x) + I^\alpha_t[W^H(t)], \quad t \in (0,1], \quad x \in (0,1), \\
X(t,0) &= X(t,1) = 0, \quad t \in (0,1], \\
X(0,x) &= \sin(\pi x).
\end{align*}
\]

From now on, we fix \( H = \frac{3}{5} \). Observe that Assumption 3.1 is satisfied with \( \beta < \frac{1}{\alpha} - \frac{1}{2} \). The fBm is simulated by the Cholesky method. We calculate the ‘exact’ solution at \( t = 1 \) using full discretization with \( \Delta t_{\text{exact}} = 1/2^8 \) and \( N_{\text{exact}} = 2^{12} \). We then use \( N = 2^4, 2^5, 2^6, 2^7, 2^8 \) to obtain the approximate solutions. In Figure 1, we chose \( \alpha = \frac{6}{7} \) to get the errors \( \|X(1) - X^N(1)\|_{L^2(\Omega,V)} \), and one can see the errors decrease with slope 0.9. In Figure 2, we consider the convergence rates for different \( \alpha \). We notice that the convergence order is higher when \( \alpha \) is closer to 1. This is consistent with the theoretical results. The temporal errors can be simulated in a similar way.
Figure 2. Convergence rates for the differential $\alpha$

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