CONSTRUCTION AND EXTENSION OF NEAR-HOMOGENEOUS TOURNAMENTS

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Abstract. A homogeneous tournament is a tournament with $4t+3$ vertices that every arc is contained in exactly $t+1$ cycles of length 3. A generalized class of graph is that of near-homogeneous tournaments with $4t+1$ vertices, in which every arc is contained in $t$ or $t+1$ cycles of length 3. In this paper, we raise a new method to construct near-homogeneous tournaments with $4t+1$ vertices. We also extend the definition of near-homogeneous tournament to those tournaments with an even number of vertices. We prove a few results that support the rationality of our definition. Finally we construct some infinite classes of near-homogeneous tournaments with even number of vertices as we define.

Keywords: homogeneous tournament, near-homogeneous tournament, doubly regular tournament, skew Hadamard matrix

2010 MSC: 05C20, 05B20

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1. INTRODUCTION

We consider strict digraphs, i.e. digraphs without loops or parallel arcs in this paper. A tournament $T$ is a digraph consists of a set of vertices $V(T)$ and a set of arcs (edges with direction) $A(T)$ such that there is exactly one arc between every two vertices $u$ and $v$. If there is an arc from $u$ to $v$, we say that $u$ is an in-neighbor of $v$ and $v$ is an out-neighbor of $u$, and denote it by $u \to v$. The number of in-neighbors of $u$ is called the in-degree of $u$, denoted by $d^-(u)$. The number of out-neighbors of $u$ is called the out-degree of $u$, denoted by $d^+(u)$. The set of out-neighbors of $u$ is called the out-neighborhood of $u$, and the set of in-neighbors

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1. INTRODUCTION

We consider strict digraphs, i.e. digraphs without loops or parallel arcs in this paper. A tournament $T$ is a digraph consists of a set of vertices $V(T)$ and a set of arcs (edges with direction) $A(T)$ such that there is exactly one arc between every two vertices $u$ and $v$. If there is an arc from $u$ to $v$, we say that $u$ is an in-neighbor of $v$ and $v$ is an out-neighbor of $u$, and denote it by $u \to v$. The number of in-neighbors of $u$ is called the in-degree of $u$, denoted by $d^-(u)$. The number of out-neighbors of $u$ is called the out-degree of $u$, denoted by $d^+(u)$. The set of out-neighbors of $u$ is called the out-neighborhood of $u$, and the set of in-neighbors

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of a vertex its **semi-degrees**. A **regular tournament** \( T \) is then a tournament in which \( d^+(u) = d^-(u) \) for every vertex \( u \in V(T) \). Let \( V' \) be a subset of \( V(T) \), \( (V') \) stands for the sub-tournament induced by \( V' \), i.e. the sub-tournament with vertex set \( V' \) and the arc set containing all the arcs among vertices in \( V' \). For the terms and concepts not defined in this paper, the reader is referred to [3].

Regularity can be viewed as a kind of symmetry in a graph, in the sense that the neighborhood of every vertex looks alike. There are two equivalent ways to figure out subset of regular tournaments that are even more symmetric. One way is to require a regular tournament to be doubly regular. **Doubly regular tournaments** are regular tournaments that the number of common out-neighbors remains the same value for every vertex pair \( u \) and \( v \). Thus, the in-neighbors or the out-neighbors of any vertex in a doubly regular tournament induce a regular sub-tournament. Alternatively, we can demand every arc in a tournament \( T \) is contained in a constant number of 3-cycles, which gives us the definition of **homogeneous tournaments**. In [11], Reid and Brown showed that doubly regularity and homogeneity are actually equivalent. It is commonly known that a homogeneous tournament \( T \) must have \( n = 2k + 1 = 4t + 3 \) vertices, and each arc \( uv \) is contained in \( t + 1 \) 3-cycles. Actually, the number of common in-neighbors of \( u \) and \( v \), the number of common out-neighbors of \( u \) and \( v \), the number of \( (u, v) \)-2-paths and the number of \( (v, u) \)-2-paths are equal to \( t \), \( t \), \( t + 1 \) respectively, where an \( (x, y)-p\)-**path** denotes a path containing \( p \) arcs from vertex \( x \) to vertex \( y \), and the number of \( (x, y)-2\)-paths is exactly the number of 3-cycles containing \( uv \), which we name it as the **3-cyclic-index** of the arc \( uv \).

A **Hadamard matrix** \( H \) of order \( 4m \) is a \( 4m \times 4m \) matrix of \(+1\)'s and \(-1\)'s such that \( HH^T = 4mI \), where \( I \) is the \( 4m \times 4m \) identity matrix. Further, \( H \) is **skew** if \( H + H^T = 2I \). Skew Hadamard matrices are closely related to the construction of various combinatorial designs, and thus of special interests to the research community (for examples see [6, 5, 9, 7]). In [11], Reid and Brown showed by constructive proofs that the existence of a homogeneous tournament with \( n \) vertices is equivalent to the existence of a skew Hadamard matrix of order \( n + 1 \). Hence the various constructions of skew Hadamard matrices (surveyed in [6]) also gives constructions of homogeneous tournaments. On the other hand, homogeneous tournament is a special subset of directed strongly regular graphs (DSRG for short), another topic of rich problems and results, a survey of which can be found in [4].

For tournaments with \( n \neq 3 \) (mod 4) vertices, we naturally ask for a similar symmetry, which is the main theme of the current paper. In [12], Tabib generalized the concept of homogeneity to tournaments with \( n = 4t + 1 \) vertices and defined **near-homogeneous tournaments** as follows.

**Definition 1.1.** A tournament \( T \) of order \( n = 4t + 1 \), \( t \geq 1 \) is **near-homogeneous** if any arc of \( T \) is contained in \( t \) or \( t + 1 \) 3-cycles.

Based on a result in [1], Astié and Dugat in [2] figured out an exhaustive algorithm to find a particular kind of near-homogeneous tournaments; through assistance of computer programming, they found six new near-homogeneous tournaments, for orders \( p = 29, 53, 173, 229, 293 \) and 733. Later in [10], Moukouelle put forward a method to construct a near-homogeneous tournament of order \( 16t + 13 \), given any homogeneous tournament of order \( 4t + 3 \). In this paper, we will present a new method to construct a near-homogeneous tournament of order \( 8t + 5 \), given
any homogeneous tournament of order $4t + 3$. Next, we extend the definition of near-homogeneous tournament to those with an even number of vertices. We not only setup a definition of such tournaments, but also prove its rationality by investigating all tournaments with relatively small order (listed in [8]) as well as careful theoretical analysis that leads to some theorems to support the definition. Finally we provide some methods to construct a few classes of infinite graphs that satisfy our definition.

2. Construction of near-homogeneous tournaments with $n = 4t + 1$ vertices

In attempt to find new near-homogeneous tournaments, we figure out the following original method to construct near-homogeneous tournaments from homogeneous tournaments.

Let $T$ be a given homogeneous tournament with $4t + 3$ vertices. Let $T^*$ be the complement of $T$, i.e. $T^*$ is obtained from $T$ by reversing the direction of every arc. For any vertex $u \in T$, we denote the corresponding vertex of $u$ in $T^*$ be $u^*$. For each vertex $u \in T$, denote the set of in-neighbors and the set of out-neighbors of $u$ in $T$ as $I(u)$ and $O(u)$, respectively; for each vertex $u^* \in T^*$, denote the set of in-neighbors and the set of out-neighbors of $u$ in $T$ as $I(u^*)$ and $O(u^*)$, respectively. Arbitrarily select a vertex $x$ of $T$, delete $x$ from $T$ and delete $x^*$ from $T^*$. Then, add a new vertex $z$.

Construct a tournament $H$ with the vertex set $V(H) = O(x) \cup I(x) \cup O(x^*) \cup I(x^*) \cup z$, and the arc set defined by the out-neighborhood of every vertex as below:

a. For each $u \in O(x)$, the out-neighborhood of $u$ in $H$ is $O(u) \cup (O(u^*) - \{x^*\}) \cup \{z\}$;

b. For each $u \in I(x)$, the out-neighborhood of $u$ in $H$ is $(O(u) - \{x\}) \cup O(u^*) \cup \{u^*\}$;

c. For each $u^* \in O(x^*)$, the out-neighborhood of $u^*$ in $H$ is $I(u) \cup O(u^*)$;

d. For each $u^* \in I(x^*)$, the out-neighborhood of $u^*$ in $H$ is $(O(u^*) - \{x^*\}) \cup (I(u) - \{x\}) \cup \{z\} \cup \{u\}$;

e. The out-neighborhood of $z$ in $H$ is $I(x) \cup O(x^*)$.

Theorem 2.1. The tournament $H$ constructed as above is a near-homogeneous tournament of order $8t + 5$.

Proof. First, we prove that $H$ is regular. $H$ contains $2(4t + 3) - 1 = 8t + 5$ vertices. We verify that the out-neighborhood of every vertex $u$ in $H$ is of order $4t + 2$, thus $d^+(u) = d^-(u) = 4t + 2$.

For any $u \in O(x)$, the set of out-neighbors of $u$ is $O(u) \cup (O(u^*) - \{x^*\}) \cup \{z\}$, which has order $(2t + 1) + 2 + 1 = 4t + 2$. For any $u \in I(x)$, the set of out-neighbors of $u$ is $(O(u) - \{x\}) \cup O(u^*) \cup \{u^*\}$, which has order $2t + (2t + 1) + 1 = 4t + 2$.

For any $u^* \in O(x^*)$, the set of out-neighbors of $u^*$ is $I(u) \cup O(u^*)$, which has order $2(2t + 1) = 4t + 2$. For any $u^* \in I(x^*)$, the set of out-neighbors of $u^*$ is $O(u^*) - \{x^*\} \cup (I(u) - x) \cup \{z\} \cup \{u\}$, which has order $2(2t + 1) + 1 + 1 = 4t + 2$. Finally, the set of out-neighbors of $z$ is $I(x) \cup O(x^*)$, which has order $2(2t + 1) = 4t + 2$.

Next, to prove that $H$ is near-homogeneous, we only need to verify that every pair of vertices $u$ and $v$ have $2t$ or $2t + 1$ common out-neighbors. For, without lose of generality we may assume that $u \rightarrow v$. Since $H$ is a tournament and $v$ is of out-degree $4t + 2$, the number of 3-cycles that contains the arc $uv$ is then $4t + 2 - 2t = 2t + 2$ or $4t + 2 - (2t + 1) = 2t + 1$. We will divide our discussion into 14 cases according to the vertex subset that $u$ and $v$ belongs to. Note that $T^*$ is also
homogeneous, and in a homogenous tournament of order $4t + 3$, every two distinct vertices have $t$ common in-neighbors and $t$ common out-neighbors. Furthermore for any arc $uv$ in $T$, there are $t$ $(u, v)$-paths of length $2$ in $T$, i.e. $|O(u) \cap I(v)| = t$.

To improve the readability of our proof, we will write it in a relatively formal form and if the vertex $u$ or $v$ is from $T^*$, we write it as $u^*$ and $v^*$.

Case 1. $v = z$ and $u \in O(x)$.

The out-neighborhood of $v$ in $H$ is $I(x) \cup O(x^*)$, and the out-neighborhood of $u$ in $H$ is $O(u) \cup (O(u^*) - \{x^*\}) \cup \{z\}$ where $u^* \in I(x^*)$.

$T$ is homogeneous $\implies xu$ is contained in $t + 1$ 3-cycles in $T \implies |O(u) \cap I(x)| = t + 1$.

$T^*$ is homogeneous $\implies u^*$ and $x^*$ have $t$ common out-neighbors $\implies |(O(u^*) - \{x^*\}) \cap O(x^*)| = t$.

$|O(u) \cap I(x)| = t + 1$ and $|(O(u^*) - \{x^*\}) \cap O(x^*)| = t \implies u$ and $v$ have $t + 1 + t = 2t + 1$ common out-neighbors in $H$.

Case 2. $v = z$ and $u \in I(x)$.

The out-neighborhood of $v$ in $H$ is $I(x) \cup O(x^*)$, and the out-neighborhood of $u$ in $H$ is $(O(u) - \{x\}) \cup O(u^*) \cup \{u\}$ where $u^* \in O(x^*)$.

$T$ is homogeneous $\implies (I(x))$ is regular $\implies |(O(u) - \{x\}) \cap I(x)| = t$.

$T^*$ is homogeneous $\implies (O(x^*))$ is regular $\implies |O(u^*) \cap O(x^*)| = t$.

$u^* \in O(x^*) \implies v \rightarrow u^*$.

$|O(u) \cap I(x)| = t$, $|(O(u^*) - \{x\}) \cap O(x^*)| = t$, $u \rightarrow u^*$ and $v \rightarrow u^*$ $\implies u$ and $v$ have $2t + 1$ common out-neighbors in $H$.

Case 3. $v = z$ and $u^* \in O(x^*)$.

The out-neighborhood of $v$ in $H$ is $I(x) \cup O(x^*)$, and the out-neighborhood of $u^*$ in $H$ is $I(u) \cup O(u^*)$ where $u \in I(x)$.

$T$ is homogeneous $\implies (I(x))$ is regular $\implies |I(u) \cap I(x)| = t$.

$T^*$ is homogeneous $\implies (O(x^*))$ is regular $\implies |O(u^*) \cap O(x^*)| = t$.

$|I(u) \cap I(x)| = t$ and $|O(u^*) \cap O(x^*)| = t \implies u$ and $v$ have $2t$ common out-neighbors in $H$.

Case 4. $v = z$ and $u^* \in I(x^*)$.

The out-neighborhood of $v$ in $H$ is $I(x) \cup O(x^*)$, and the out-neighborhood of $u^*$ in $H$ is $(O(u^*) - \{x^*\}) \cup (I(u) - x) \cup \{z\} \cup \{u\}$ where $u \in O(x)$.

$T$ is homogeneous $\implies u$ and $x$ have $t$ common in-neighbors in $T \implies |I(x) \cap (I(u) - x)| = t$.

$T^*$ is homogeneous $\implies u^*$ and $x^*$ have $t$ common out-neighbors in $T^* \implies |(O(u^*) - \{x^*\}) \cap O(x^*)| = t$.

$|I(x) \cap I(u)| = t$ and $|O(u^*) \cap O(x^*)| = t \implies u$ and $v$ have $2t$ common out-neighbors in $H$.

Case 5. $u, v \in O(x)$.

The out-neighborhood of $u$ in $H$ is $O(u) \cup (O(u^*) - \{x^*\}) \cup \{z\}$ where $u^* \in I(x^*)$, and the out-neighborhood of $v$ in $H$ is $O(v) \cup (O(v^*) - \{x^*\}) \cup \{z\}$ where $v^* \in I(x^*)$.

$T$ is homogeneous $\implies u$ and $v$ have $t$ common out-neighbors in $T \implies |O(u) \cap O(v)| = t$.

$T^*$ is homogeneous $\implies u^*$ and $v^*$ have $t$ common out-neighbors in $T^*$ including $x^* \implies |(O(u^*) - \{x^*\}) \cap (O(v^*) - \{x^*\})| = t - 1$.

$|O(u) \cap O(v)| = t$, $|(O(u^*) - \{x^*\}) \cap (O(v^*) - \{x^*\})| = t - 1$ and $z$ is a common out-neighbor of $u$ and $v \implies u$ and $v$ have $2t$ common out-neighbors in $H$.

Case 6. $u, v \in I(x)$.
The out-neighborhood of $u$ in $H$ is $(O(u) - \{x\}) \cup O(u^*) \cup \{u^*\}$ where $u^* \in O(x^*)$, and the out-neighborhood of $v$ in $H$ is $(O(v) - \{x\}) \cup O(v^*) \cup \{v^*\}$ where $v^* \in O(x^*)$.

$T$ is homogeneous $\implies u$ and $v$ have $t$ common out-neighbors in $T$ including $x$ $\implies |(O(u) - \{x\}) \cap (O(v) - \{x\})| = t - 1$.

$T^*$ is homogeneous $\implies u^*$ and $v^*$ have $t$ common out-neighbors in $T^* \implies |O(u^*) \cup O(v^*)| = t$.

Without lose of generality, we assume that $u^* \to v^*$ in $T^*$ $\implies v^*$ is a common out-neighbor of $u$ and $v$.

$|(O(u) - \{x\}) \cap (O(v) - \{x\})| = t - 1, |O(u^*) \cup O(v^*)| = t$ and $v^*$ is a common out-neighbor of $u$ and $v$ $\implies u$ and $v$ have $(t - 1) + t + 1 = 2t$ common out-neighbors in $H$.

Case 7. $u \in O(x), v \in I(x)$.

The out-neighborhood of $u$ in $H$ is $O(u) \cup (O(u^*) - \{x^*\}) \cup \{z\}$ where $u^* \in I(x^*)$, and the out-neighborhood of $v$ in $H$ is $(O(v) - \{x\}) \cup O(v^*) \cup \{v^*\}$ where $v^* \in O(x^*)$.

$T$ is homogeneous $\implies u$ and $v$ have $t$ common in-neighbors in $T$ not including $x^*$ $\implies |(O(u) - \{x\}) \cap O(v^*)| = t$.

$T^*$ is homogeneous $\implies u^* \lor v^*$ have $t$ common out-neighbors in $T^*$ not including $x^*$ $\implies |(O(u^*) - \{x^*\}) \cap O(v^*)| = t$.

If $u^* \to v^*$ then $v^*$ is a common out-neighbor of $u$ and $v$, else $v^*$ is not a common out-neighbor of $u$ and $v$.

$|O(u) \cap (O(v) - \{x\})| = t, |(O(u^*) - \{x^*\}) \cap O(v^*)| = t$ and $v^*$ may or may not be a common out-neighbor of $u$ and $v$ $\implies u$ and $w$ have $2t$ or $2t + 1$ common out-neighbors in $H$.

Case 8. $u^*, v^* \in O(x^*)$.

The out-neighborhood of $u^*$ is $I(u) \cup O(u^*)$ where $u \in I(x)$, and the out-neighborhood of $v^*$ is $I(v) \cup O(v^*)$ where $v \in I(x)$.

$T$ is homogeneous $\implies u$ and $v$ have $t$ common in-neighbors in $T$ not including $x^*$ $\implies |I(u) \cap I(v)| = t$.

$T^*$ is homogeneous $\implies u^* \lor v^*$ have $t$ common out-neighbors in $T$ not including $x^*$ $\implies |O(u^*) \cap O(v^*)| = t$.

$|I(u) \cap I(v)| = t$ and $|O(u^*) \cup O(v^*)| = t$ $\implies u^* \lor v^*$ have $2t$ common out-neighbors in $H$.

Case 9. $u^*, w^* \in I(x^*)$.

The out-neighborhood of $u^*$ is $(O(u^*) - \{x^*\}) \cup (I(u) - \{x\}) \cup \{z\} \cup \{u\}$ where $u \in O(x)$, and the out-neighborhood of $v^*$ is $(O(v^*) - \{x^*\}) \cup (I(v) - \{x\}) \cup \{z\} \cup \{v\}$ where $v \in O(x)$.

$T$ is homogeneous $\implies u$ and $v$ have $t - 1$ common in-neighbors besides $x$ $\implies |(I(u) - \{x\}) \cap (I(v) - \{x\})| = t - 1$.

$T^*$ is homogeneous $\implies u^* \lor v^*$ have $t - 1$ common out-neighbors besides $x^*$ $\implies |(O(u^*) - \{x^*\}) \cap (O(v^*) - \{x^*\})| = t - 1$.

Without lose of generality we may assume that $u \to v$, i.e. $u \in I(v) \implies v^* \to u \implies u$ is a common out-neighbor of $u^*$ and $v^*$.

$|I(u) - \{x\}) \cap (I(v) - \{x\})| = t - 1, |(O(u^*) - \{x^*\}) \cap (O(v^*) - \{x^*\})| = t - 1, u \lor z$ are two common out-neighbors of $u^*$ and $v^*$ $\implies u^* \lor v^*$ have $(t - 1) + (t - 1) + 2 = 2t$ common out-neighbors in $H$.

Case 10. $u^* \in O(x^*), v^* \in I(x^*)$. 
The out-neighborhood of $u^*$ in $H$ is $I(u) \cup O(u^*)$ where $u \in I(x)$, and the out-neighborhood of $v^*$ in $H$ is $(O(v^*) - \{x^*\}) \cup (I(v) - \{x\}) \cup \{z\} \cup \{v\}$ where $v \in O(x)$.

$T$ is homogeneous $\iff u$ and $v$ have $t$ common in-neighbors in $T$ not including $x$ $\implies |I(u) \cap (I(v) - \{x\})| = t$.

$T^*$ is homogeneous $\iff u^*$ and $v^*$ have $t$ common out-neighbors in $T^*$ not including $x^*$ $\implies |O(u^*) \cap (O(v^*) - \{x^*\})| = t$.

If $v \to u$ in $T$, i.e. $v \in I(u)$ then $u^* \to v$ and $v$ is a common out-neighbor of $u^*$ and $v^*$, else $v$ is not a common out-neighbor of $u^*$ and $v^*$.

$|I(u) \cap (I(v) - \{x\})| = t$, $|O(u^*) \cap (O(v^*) - \{x^*\})| = t$ and $v$ may or may not be a common out-neighbor of $u^*$ and $v^*$ $\implies u^*$ and $v^*$ have $2t$ or $2t + 1$ common out-neighbors in $H$.

Case 11. $u \in O(x)$, $v^* \in O(x^*)$.

The out-neighborhood of $u$ is $O(u) \cup (O(u^*) - \{x^*\}) \cup \{z\}$ where $u^* \in I(x^*)$, and the out-neighborhood of $v^*$ is $I(v) \cup O(v^*)$ where $v \in O(x)$.

$T$ is homogeneous $\iff u \to v$ then $|O(u) \cap I(v)| = t$, and if $v \to u$ then $|O(u) \cap I(v)| = t + 1$, where $x \notin O(u) \cap I(v)$.

$T^*$ is homogeneous $\iff u^*$ and $v^*$ have $t$ common out-neighbors not including $x^*$ $\implies |(O(u^*) - \{x^*\}) \cap O(v^*)| = t$.

$|O(u) \cap I(v)| = t$ or $t + 1$ and $|(O(u^*) - \{x^*\}) \cap O(v^*)| = t \implies u$ and $v^*$ have $2t$ or $2t + 1$ common out-neighbors in $H$.

Case 12. $u \in I(x)$, $v^* \in I(x^*)$.

The out-neighborhood of $u$ is $(O(u) - \{x\}) \cup O(u^*) \cup \{u^*\}$ where $u^* \in O(x^*)$, and the out-neighborhood of $v^*$ is $(O(v^*) - \{x^*\}) \cup (I(v) - \{x\}) \cup \{z\} \cup \{v\}$ where $v \in O(x)$.

Suppose that $u \to v$, then $v^* \to u^*$. $T$ is homogeneous $\implies$ there are $t$ paths of length 2 from $u$ to $v$ including $uxv$ $\implies |(O(u) - \{x\}) \cap ((I(v) - \{x\})| = t - 1$.

Furthermore $v$ and $u^*$ are common out-neighbors of $u$ and $v^*$.

Suppose that $v \to u$, then $u^* \to v^*$. $T$ is homogeneous $\implies$ there are $t + 1$ paths of length 2 from $u$ to $v$ including $uxv$ $\implies |(O(u) - \{x\}) \cap ((I(v) - \{x\})| = t$.

Furthermore $v$ and $u^*$ are not common out-neighbors of $u$ and $v^*$.

$T^*$ is homogeneous $\iff u^*$ and $v^*$ have $t$ common out-neighbors not including $x^*$ $\implies |O(u^*) \cap (O(v^*) - \{x^*\})| = t$.

Therefore, $u^*$ and $v^*$ have $(t - 1) + 2 + t = 2t + 1$ or $t + t = 2t$ common out-neighbors in $H$.

Case 13. $u \in O(x)$, $v^* \in I(x^*)$.

The out-neighborhood of $u$ is $O(u) \cup (O(u^*) - \{x^*\}) \cup \{z\}$ where $u^* \in I(x^*)$, and the out-neighborhood of $v^*$ is $(O(v^*) - \{x^*\}) \cup (I(v) - \{x\}) \cup \{z\} \cup \{v\}$ where $v \in O(x)$.

If $v^* = u^*$, then the common out-neighbors of $v^*$ and $u$ are precisely $z$ and the 2$t$ vertices in $O(v^*) - \{x^*\}$. Thus, $u$ and $v^*$ have $2t + 1$ common out-neighbors in $H$.

Now assume that $v^* \neq u^*$.

Suppose that $u \to v$, then $v$ is a common out-neighbor of $u$ and $v^*$. $T$ is homogeneous $\iff$ there are $t$ paths of length 2 from $u$ to $v$, not including $uxv$ $\implies |O(u) \cap ((I(v) - \{x\}) \cap \{v\})| = t + 1$. 

Suppose that $v \to u$, then $v$ is not a common out-neighbor of $u$ and $v^*$. $T$ is homogeneous $\iff$ there are $t + 1$ paths of length 2 from $u$ to $v$, not including $uxv$ $\implies |O(u) \cap I(v) - \{x\}| = t + 1$.

$T^*$ is homogeneous $\implies u^*$ and $v^*$ have $t$ common out-neighbors in $T^*$ including $x^* \implies |(O(u^*) - \{x^*\}) \cap (O(v^*) - \{x^*\})| = t - 1$.

Since $z$ is also a common out-neighbor of $u$ and $v^*$, in either case we conclude that $u$ and $v^*$ have $(t + 1) + (t - 1) + 1 = 2t + 1$ common out-neighbors in $H$.

Case 14. $u \in I(x), v^* \in O(x^*)$.

The out-neighborhood of $u$ is $(O(u) - \{x\}) \cup O(u^*) \cup \{u^*\}$ where $u^* \in O(x^*)$, and the out-neighborhood of $v^*$ is $I(v) \cup O(v^*)$ where $v \in I(x)$.

If $v^* = u^*$, then the common out-neighbors of $v^*$ and $u$ are precisely the $2t + 1$ vertices in $O(u^*)$, thus $u$ and $v^*$ have $2t + 1$ common out-neighbors in $H$.

Now assume that $v^* \neq u^*$.

If $u \to v$, then $v^* \to u^*$ and $u^*$ is a common out-neighbor of $u$ and $v^*$.

$T$ is homogeneous $\implies$ there are $t$ paths of length 2 from $u$ to $v$, not including $uxv$ $\implies |O(u) - \{x\}| \cap I(v)| = t$.

$T^*$ is homogeneous $\implies u^*$ and $v^*$ have $t$ common out-neighbors in $T^*$, not including $x^* \implies |O(u^*) \cap O(v^*)| = t$.

Totally, $u$ and $v^*$ have $1 + t + t = 2t + 1$ common out-neighbors in $H$.

If $v \to u$, then $u^* \to v^*$ and $u^*$ is not a common out-neighbor of $u$ and $v^*$.

$T$ is homogeneous $\implies$ there are $t + 1$ paths of length 2 from $u$ to $v$, not including $uxv$ $\implies |(O(u) - \{x\}) \cap I(v)| = t + 1$.

$T^*$ is homogeneous $\implies u^*$ and $v^*$ have $t$ common out-neighbors in $T^*$, not including $x^* \implies |O(u^*) \cap O(v^*)| = t$.

Totally, $u$ and $v^*$ have $(t + 1) + t = 2t + 1$ common out-neighbors in $H$.

Summarizing all cases, every pair of vertices in $H$ have $2t$ or $2t + 1$ common out-neighbors. Therefore, $H$ is near-homogeneous and the theorem is proved. \(\square\)

3. Near-homogeneous tournaments of even order

A tournament $T$ on $n = 2k$ vertices cannot be regular. For convenience, we call a vertex with in-degree $d^+$ and out-degree $d^-$ a $(d^+, d^-)$-vertex. $T$ is called almost regular if each vertex $v \in V$ is a $(k, k - 1)$-vertex or a $(k - 1, k)$-vertex. We will define near-homogeneous tournaments of even order among almost regular tournaments.

If we classify the arcs in a tournament by the degrees of its endvertices, then in a regular tournament, there is only one class of arcs, whose both endvertices are $(\frac{k-1}{2}, \frac{k-1}{2})$-vertices. Nonetheless, in an almost regular tournament $T$ with $n = 2k$ vertices, there are as many as four types of arcs, named as below:

Class A: the arcs $(u, v)$ with $(d^+(u), d^-(u)) = (d^+(v), d^-(v)) = (k, k - 1)$.

Class B: the arcs $(u, v)$ with $(d^+(u), d^-(u)) = (d^+(v), d^-(v)) = (k + 1, k)$.

Class C: the arcs $(u, v)$ with $(d^+(u), d^-(u)) = (k, k - 1), (d^+(v), d^-(v)) = (k - 1, k)$.

Class D: the arcs $(u, v)$ with $(d^+(u), d^-(u)) = (k - 1, k), (d^+(v), d^-(v)) = (k, k - 1)$.

Doubly regularity or homogeneity can be viewed as a way to further classify the arcs. By definition, the 3-cyclic-index of every arc in a homogeneous tournament remains a constant $t$, and the 3-cyclic-index of an arc in a near-homogeneous tournaments on $4t + 1$ vertices can be $t$ or $t + 1$, therefore the arcs can be further
Table 1. Different types of common neighbors of vertex pairs and their numbers counted by 3-cyclic-index

| arc $uv$ | number of common out-neighbors of $u, v$ | number of common in-neighbors of $u, v$ | number of vertices $w$ satisfying $u \rightarrow w \rightarrow v$ | number of vertices $w$ satisfying $v \rightarrow w \rightarrow u$ (3-cyclic-index) |
|----------|----------------------------------------|----------------------------------------|------------------------------------------------|----------------------------------------------------------------------------------|
| Class A  | $k - \lambda_A$                       | $k - \lambda_A - 1$                   | $\lambda_A - 1$                               | $\lambda_A$                                                                     |
| Class B  | $k - \lambda_B - 1$                   | $k - \lambda_B$                       | $\lambda_B - 1$                               | $\lambda_B$                                                                     |
| Class C  | $k - \lambda_C - 1$                   | $k - \lambda_C - 1$                   | $\lambda_C$                                   | $\lambda_C$                                                                     |
| Class D  | $k - \lambda_D$                       | $k - \lambda_D$                       | $\lambda_D - 2$                               | $\lambda_D$                                                                     |

classified into two categories in a near-homogeneous tournament. In the case that the tournament is almost regular, because the arcs have been classified into four categories, it sounds reasonable to ask whether the 3-cyclic-index can remain constant among each class of arcs at first. Our first result will show that this could only happen when $n \equiv 2 \pmod{4}$.

For further discussion, we need to calculate the number of arcs in each class in an almost regular tournament.

There are $k$ $(k, k-1)$-vertices, so there are $\binom{k}{2}$ arcs of Class $A$. Similarly, there are $\binom{k}{2}$ arcs of Class $B$.

Now, for each $(k, k-1)$-vertex $u$, there are $k$ arcs sending out from $u$. Hence, in total there are $k^2$ arcs sending out from $(k, k-1)$-vertices, each of which is either in Class $A$ or in Class $C$. Therefore, there are $k^2 - \binom{k}{2} = \frac{k(k+1)}{2} = \frac{(k+1)^2}{2}$ edges in Class $C$.

Similarly, for each vertex $(k-1, k)$-vertex $u$ there are $k-1$ arcs sending out from $u$. Hence, in total there are $k(k-1)$ arcs sending out from $(k-1, k)$-vertices, each of which is either in Class $B$ or in Class $D$. Hence, there are $k(k-1) - \binom{k}{2} = \frac{k(k-1)}{2} = \binom{k}{2}$ edges in Class $D$.

**Theorem 3.1.** If the 3-cyclic-index of each class of arcs remains a constant in an almost regular tournament $T$ with $n$ vertices, then $n = 4t + 2$ for certain integer $t \geq 1$. Furthermore, $\lambda_C = t$ and $\lambda_B = \lambda_D = t + 1$, where $\lambda_A$, $\lambda_B$, $\lambda_C$, and $\lambda_D$ denote the 3-cyclic-index of arcs of class $A, B, C, D$, respectively.

**Proof.** Given $\lambda_A$, $\lambda_B$, $\lambda_C$, and $\lambda_D$, for an arc $(u, v)$, the number of common in-neighbors of $u$ and $v$, the number of common out-neighbors of $u$ and $v$, the number of $(u, v)$-2-paths and the number of $(v, u)$-2-paths can be calculated according to the class it belongs to, as shown in Table 1. Suppose an arc $uv$ of Class $C$ is contained in a 3-cycle $uvvu$. Since $u$ is a $(k, k-1)$-vertex and $v$ is a $(k-1, k)$-vertex, then either $vuu$ or $wu$ must be an arc of Class $D$. Therefore a 3-cycle containing an arc of Class $C$ must contain an arc of Class $D$, and vice versa. Thus we have the following observation.

**Observation 3.2.** Any 3-cycle contains an arc of Class $C$ iff it contains an arc of Class $D$. 
This means that the sum of 3-cyclic-index of arcs in Class $C$ equals that of arc in Class $D$, that is:

\[
\binom{k+1}{2}\lambda_C = \binom{k-1}{2}\lambda_D,
\]

hence \( \frac{\lambda_C}{\lambda_D} = \frac{k-1}{k+1} \).

If \( k \) is even then \( \gcd(k-1, k+1) = 1 \), therefore \( \lambda_D \geq k+1 \). However, \( \lambda_D \) is the number of the intersection of the in-neighborhood of a certain vertex and the out-neighborhood of another vertex, and cannot exceed the maximal semi-degree, thus \( \lambda_D \leq k \), a contradiction. Therefore \( k \) must be odd. Assume that \( k = 2t+1 \) then \( n = 2k = 4t + 2 \).

Now we have \( \lambda_C \lambda_D = \frac{k-1}{k+1} \).

Since \( \lambda_D \leq k = 2t+1 \), the only integer solution to this equality is \( \lambda_C = t \) and \( \lambda_D = t+1 \).

Now let’s count the number of 2-paths in \( T \).

Every vertex \( u \in V(T) \) satisfies that \( \{d^+(u), d^-(u)\} = \{k, k-1\} \), so the number of 2-paths with \( u \) as the intermediate vertex is \( k(k-1) \). Since \( |V(T)| = n = 2k \), there are totally \( 2k^2(k-1) \) 2-paths in \( T \).

On the other hand, the number of all kinds of 2-paths are listed in the last two columns of Table 1. Thus we can calculate the number of 2-paths in \( T \) by summing up them, so

\[
2k^2(k-1) = \binom{k}{2}(2\lambda_A - 1) + \binom{k}{2}(2\lambda_B - 1) + \binom{k+1}{2}(2\lambda_C) + \binom{k-1}{2}(2\lambda_D - 2).
\]

Substituting \( \lambda_C = t \) and \( \lambda_D = t+1 \) into (4) we get

\[
\lambda_A + \lambda_B = 2t + 2.
\]

Now let’s count the number of 2-paths with endvertices as \( (k, k-1) \)-vertices and those with endvertices as \( (k-1, k) \)-vertices, respectively. From Table 1 and above discussion, this two numbers must be \( \binom{k}{2}(2\lambda_A - 1) \) and \( \binom{k}{2}(2\lambda_B - 1) \). And from (4) we get

\[
2k^2(k-1) = \binom{k}{2}(2\lambda_A - 1) + \binom{k}{2}(2\lambda_B - 1)
\]

\[
= 2k^2(k-1) - \binom{k+1}{2}(2\lambda_C) - \binom{k-1}{2}(2\lambda_D - 2)
\]

\[
= 2k^2(k-1) - \frac{k(k+1)}{2}(2t) + \frac{k(k-1)}{2}(2t)
\]

\[
= 2t(2t+1)^2
\]

Let the sub-tournament of \( T \) induced by all \( (k, k-1) \)-vertices be \( T_1 \) and the one induced by all \( (k-1, k) \)-vertices be \( T_2 \). A 2-path with endvertices as \( (k, k-1) \)-vertices must either be contained in \( T_1 \) or has an intermediate vertex in \( T_2 \). Since there are \( k = 2t + 1 \) vertices in \( T_1 \), and for every \( u \in V(T_1) \) the number of 2-paths go through it is \( d^+_T(u) d^-_T(u) \leq t^2 \), the total number of 2-paths of the first kind is
at most $t^2(2t + 1)$. For a vertex $w$ in $T_2$, the number of 2-paths of the second kind which go through it is at most $t(t + 1)$, hence the total number of 2-paths of the second kind is at most $t(t + 1)(2t + 1)$. Therefore

$$\left(\frac{k}{2}\right)(2\lambda_A - 1) \leq t^2(2t + 1) + t(t + 1)(2t + 1) = t(2t + 1)^2,$$

from which we get $\lambda_A \leq t + 1$. Similarly, $\lambda_B \leq t + 1$. By (5), equalities must hold, and we have

$$\lambda_A = \lambda_B = t + 1.$$

This completes the proof of the theorem. \hfill \Box

Based on the above result, we make the following definition.

**Definition 3.7.** A tournament with order $4t + 2$ is near-homogeneous if it is almost regular, every arc in class $C$ has 3-cyclic-index $t$, and the other arcs have 3-cyclic-index $t + 1$.

Searching through all almost regular tournaments of order at most 12 in Brendan McKay’s graph database ([8]), we find one tournaments on 6 vertices whose adjacent matrix is as below

\[
\begin{pmatrix}
0 & 1 & 0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 1 \\
1 & 0 & 0 & 0 & 1 & 0 \\
0 & 1 & 0 & 1 & 0 & 1 \\
1 & 1 & 0 & 0 & 1 & 0 \\
1 & 0 & 1 & 1 & 0 & 0 \\
\end{pmatrix},
\]

as well as another one on 10 vertices that both satisfy our definition. We shall see in Section 4 that infinite tournaments satisfying the definition can be constructed. Also note that every arc in the tournaments we define is contained in $t$ or $t + 1$ 3-cycles. In this sense, Definition 3.7 is compatible with the former definition of near-homogeneous tournaments.

Next we handle the case that $n \equiv 0 \pmod{4}$.

Assume that $n = 4t$ for certain $t \geq 1$. We have the following result.

**Theorem 3.8.** Let $T$ be an almost regular tournament with $n = 2k = 4t$ vertices.

(i) The arcs in Class $C$ and the arcs in Class $D$ cannot both have constant 3-cyclic-index;

(ii) If the arcs in Class $A$ and Class $B$ have constant 3-cyclic-index respectively, then the arcs in Class $C$ cannot have constant 3-cyclic-index.

**Proof.** If the 3-cyclic-index remains constants among arcs in Class $C$ and among arcs in Class $D$ respectively, then from (3) and the discussion following it we lead to a contradiction. Therefore (i) holds.

Now we prove (ii). Assume contrarily the 3-cyclic-index of arcs in Class $A$, Class $B$ and Class $C$ remain a constant, respectively. We use the former notes $\lambda_A$, $\lambda_B$ and $\lambda_C$ to denote these constants. Further we let the set $\Lambda_D = \{\lambda_1, \lambda_2, \ldots, \lambda_r\}$ be all possible 3-cyclic-index of arcs in Class $D$, where $m_i$ arcs in Class $D$ is of 3-cyclic-index $\lambda_i$ for $1 \leq i \leq r$, and $\sum_{i=1}^{r} m_i = \left(\frac{k}{2}\right)$. 

Now equation (4) becomes
\[ 2k^2(k - 1) = \binom{k}{2}(2\lambda_A - 1) + \binom{k}{2}(2\lambda_B - 1) + \binom{k + 1}{2}(2\lambda_C) + \sum_{i=1}^{r} m_i(2\lambda_i - 2). \]

On the other hand, from Observation 3.2, we have
\[ \binom{k + 1}{2}\lambda_C = \sum_{i=1}^{r} m_i\lambda_i. \]

Substituting (10) into (9), we have
\[ 2k^2(k - 1) = \binom{k}{2}(2\lambda_A - 1) + \binom{k}{2}(2\lambda_B - 1) + \binom{k + 1}{2}(2\lambda_C) + \sum_{i=1}^{r} m_i(2\lambda_i - 2). \]
\[ = \binom{k}{2}(2\lambda_A - 1) + \binom{k}{2}(2\lambda_B - 1) + 2\binom{k + 1}{2}\lambda_C + 2 \sum_{i=1}^{r} m_i\lambda_i - 2\sum_{i=1}^{r} m_i \]
\[ = \binom{k}{2}(2\lambda_A - 1) + \binom{k}{2}(2\lambda_B - 1) + 2\binom{k + 1}{2}\lambda_C + 2\binom{k + 1}{2}\lambda_C - 2\binom{k}{2}, \]
from which we get
\[ 2(2t + 1)\lambda_C = (2t - 1)(4t - \lambda_A - \lambda_B + 2). \]
However \(\gcd(2(2t + 1), 2t - 1) = 1\), thus \(\lambda_C\) must be a multiple of \(2t - 1\). As \(\lambda_C\) cannot exceed the maximal semi-degree, we have \(\lambda_C \leq k = 2t\), and so \(\lambda_C = 2t - 1\). Without lose of generality we assume \(\lambda_1 = \max \Lambda_D\). Then we have
\[ \binom{k + 1}{2}\lambda_C = \sum_{i=1}^{r} m_i\lambda_i \leq \sum_{i=1}^{r} m_i\lambda_1 = \binom{k}{2}\lambda_1, \]
and that \(\lambda_1 \geq k + 1\), contradicting that \(\lambda_1\) cannot exceed the maximal semi-degree \(k\). The theorem is then proved by this contradiction. \(\square\)

Inspired by Theorem 3.8, we investigate those tournaments on \(4t\) vertices in which the arcs in Class A, Class B and Class D have constant 3-cyclic-index respectively, but the arcs in Class C have two different 3-cyclic-indices, denote by \(\lambda_1\) and \(\lambda_2\), where \(\lambda_1 < \lambda_2\), \(m_i\) arcs in Class C have 3-cyclic-index \(\lambda_i\) for \(i \in \{1, 2\}\), and \(m_1 + m_2 = \binom{k + 1}{2}\).

Now (4) becomes
\[ 2k^2(k - 1) = \binom{k}{2}(2\lambda_A - 1) + \binom{k}{2}(2\lambda_B - 1) + (2m_1\lambda_1 + 2m_2\lambda_2) + \binom{k}{2}(2\lambda_D - 2). \]

By Observation 3.2 we have
\[ \binom{k}{2}\lambda_D = m_1\lambda_1 + m_2\lambda_2. \]

Substituting (12) into (11) we get
\[ \lambda_A + \lambda_B + 2\lambda_D = 2k + 2. \]

Let the sub-tournament induced by all the \((k, k - 1)\)-vertices be \(T_1\), and the one induced by all the \((k - 1, k)\)-vertices be \(T_2\). A 2-path with endvertices as \((k, k - 1)\)-vertices either lays entirely in \(T_1\) or goes from \(T_1\) to \(T_2\) and then goes back to \(T_1\).
Since there are $k$ vertices in $T_1$, and the number of 2-paths going through any vertex $u \in V(T_1)$ is $d_{T_1}^+(u)d_{T_1}^+(u) \leq \frac{k}{2} \cdot \left(\frac{k}{2} - 1\right) = \frac{k(k-2)}{4}$, so the total number of 2-paths of the first kind is at most $\frac{k^2(k-2)}{4}$. Since there are $k$ arcs between any $v \in V(T_2)$ and $T_1$, the number of 2-paths of the second kind that go through $v$ is at most $(\frac{k}{2})^2$, and the total number of which is at most $k(\frac{k}{2})^2 = \frac{k^3}{4}$. By the number of 2-paths between any two $(k,k-1)$-vertices given in Table 1, we have

$$\binom{k}{2} (2\lambda_{\alpha} - 1) \leq \frac{k^2(k-2)}{4} + \frac{k^3}{4} = t^2(2t - 2) + 2t^3 = 2t^2(2t - 1),$$

from which we get $\lambda_{\alpha} \leq \frac{2t+1}{2}$. Since $\lambda_{\alpha}$ is an integer, $\lambda_{\alpha} \leq t$. Similarly $\lambda_{\beta} \leq t$.

Together with (13) we have $\lambda_D \geq t + 1$. In summary,

$$\max\{\lambda_{\alpha}, \lambda_{\beta}\} \leq t < t + 1 \leq \lambda_D.$$

Hence $\lambda_{\alpha}$, $\lambda_{\beta}$ and $\lambda_D$ take at least two different values.

It can be verified that

(15) $\lambda_{\alpha} = \lambda_{\beta} = t, \lambda_D = t + 1, \lambda_1 = 0, m_1 = 1, \lambda_2 = t, m_2 = \binom{k + 1}{2} - 1$

is a solution to (12) and (13) satisfying the constraint of (14), and as we shall see in the next section, we can construct infinite almost regular tournaments with these parameters. Thus we make the following definition.

**Definition 3.16.** A tournament with $4t$ vertices is near-homogeneous if it is almost regular, and the arcs in Class $D$ take constant 3-cyclic-index $t + 1$, while the arcs in other classes take constant 3-cyclic-index $t$, except one arc in $C$ takes 3-cyclic-index 0.

Though the arcs can have three different 3-cyclic-indices in this class of tournaments, it is almost compatible with the former definition of near-homogeneous tournaments if we view the unique arc of 3-cyclic-index 0 as an exception.

4. CONSTRUCTION OF NEAR-HOMOGENEOUS TOURNAMENT OF EVEN ORDER

In this section we construct infinite families of near-homogeneous tournaments of even order as defined in the last section.

Assume that a homogeneous tournament $T$ of order $4t + 3$ is given, we will construct near-homogeneous tournaments of order $4t + 2$ and $4t + 4$ based on $T$, respectively. Since a lot of constructions of skew Hadamard matrices have been found ([6]), which could be transformed into the construction of homogeneous tournaments ([11]), and particularly some of the constructions produce infinite families of skew Hadamard matrices, we are able to construct infinite near-homogeneous tournaments of order $4t$ or $4t + 2$.

**Theorem 4.1.** Suppose $T$ is a homogeneous tournament with $4t + 3$ vertices. Arbitrarily remove a vertex $x$ from $T$, and obtain a new tournament $T'$. Then $T'$ is a near-homogeneous tournament of order $4t + 2$.

**Proof.** $T'$ is evidently almost regular. Let $O(x)$ be the set of out-neighbors of $x$ and let $I(x)$ be the in-neighbors of $x$. We apply the classification of arcs in the last section. The arcs of Class $A$ in $T'$ are those arcs among $O(x)$. The arcs of Class $B$ in $T'$ are those arcs among $I(x)$. The arcs of Class $C$ in $T'$ are those arcs from $O(x)$ to $I(x)$. Finally the arcs of Class $D$ in $T'$ are those arcs from $I(x)$ to $O(x)$. 

For an arc $uv$ in Class $A$ where $u, v \in O(x)$, $x$ is not contained in any 3-cycle containing $uv$ in $T$, so removing $x$ does not affect the 3-cyclic-index of $uv$. Thus, the 3-cyclic-index of $uv$ in $T'$ is the same as its 3-cyclic-index in $T$, which is $t + 1$. Similarly, an arc $uv$ in Class $B$ with $u, v \in I(x)$ or an arc $uv$ in Class $D$ with $u \in I(x)$ and $v \in O(x)$ both have 3-cyclic-index $t + 1$ in $T'$ as in $T$. For an arc $uv$ in Class $C$ which is from $O(x)$ to $I(x)$, $x$ is contained in precisely one 3-cycle containing $uv$. Thus, removing $x$ decreases the 3-cyclic-index of $uv$ by 1 and $(u, v)$ has 3-cyclic-index $t$ in $T'$.

Thus, the arcs of Class $A, B, D$ in $T'$ have 3-cyclic-index $t + 1$ while the arcs of Class $C$ in $T'$ have 3-cyclic-index $t$, and $T'$ is a near-homogeneous tournament of order $4t + 2$.

**Theorem 4.2.** Let $T$ be a homogeneous tournament with $4t + 3$ vertices. Arbitrarily select a vertex $x$ of $T$. Add a vertex $y$, and let the out-neighbors of $y$ be $x \cup O(x)$, and the in-neighbors of $y$ be $I(x)$. The resulting graph $T'$ is a near-homogeneous tournament of order $4t + 4$.

**Proof.** $T'$ is evidently almost regular. Let $O(x)$ be the set of out-neighbors of $x$ and $I(x)$ be the set of in-neighbors of $x$. The arcs of Class $A$ in $T'$ are those arcs among $I(x) \cup y$. The arcs of Class $B$ in $T'$ are those arcs among $O(x) \cup x$. The arcs of Class $C$ in $T'$ are those arcs from $I(x) \cup y$ to $O(x) \cup x$. Finally, the arcs of Class $D$ in $T'$ are those arcs from $O(x)$ to $I(x)$.

Let $uv$ be an arc in $T'$. Assume that $uv$ is in Class $A$ or Class $B$. Note that adding $y$ does not affect the 3-cyclic-index of arcs between vertices in $I(x)$ or in $O(x) \cup \{x\}$. As for an arc $uy$ from $I(x)$ to $y$, a vertex $w$ forms a 3-cycle with $uy$ if and only if $w$ forms a 3-cycle with $ux$. Hence, $uy$ also has 3-cyclic-index $t + 1$. So, the arcs of Class $A$ and Class $B$ in $T'$ have 3-cyclic-index $t + 1$.

Now assume that $uv$ is in Class $C$. If $uv$ is from $I(x)$ to $O(x) \cup x$ then its 3-cyclic-index is not affected by adding $y$. If $u = y$ and $yv$ is from $y$ to $O(x)$, a vertex $w$ forms a 3-cycle with $yw$ if and only if $w$ forms a 3-cycle with $vx$. Hence, $yw$ has the same 3-cyclic-index with $vx$, which is $t + 1$. If $u = y$ and $v = x$, $yx$ is not contained in any 3-cycle and it is the unique arc in $T'$ with 3-cyclic-index 0.

Finally assume that $uv$ is in Class $D$. Then, $uv$ is from $O(x)$ to $I(x)$, $y$ forms one 3-cycle with $uv$ and thus increases its 3-cyclic-index by 1. Therefore $uv$ is of 3-cyclic-index $t + 2$ in $T'$.

All in all, in $T'$, the arcs of Class $D$ have 3-cyclic-index $t + 2$, and the other arcs have 3-cyclic-index $t + 1$, except one in Class $C$ has 3-cyclic-index 0. Thus $T'$ is a near-homogeneous tournament of order $4t + 4$.

**5. Conclusion**

Homogeneous tournaments are highly symmetric tournaments but their order must be $4t + 3$ for certain integer $t$. In order to find similar symmetry in other kinds of tournaments, near-homogeneity was defined, but still limited to tournaments with order $4t + 1$, and their construction has not been well explored. In this paper, we put forward a new method to construct near-homogeneous tournaments with order $8t + 5$ when a homogeneous tournament of order $4t + 3$ is given. Moreover, we extend the definition of near-homogeneity to tournaments with even order. In particular, the near-homogeneous tournaments on even vertices as we define satisfy the original definition of near-homogenous tournaments that each arc is contained
in $t$ or $t+1$ 3-cycles where $t = \lfloor \frac{n}{4} \rfloor$, except that when $n \equiv 0 \pmod{4}$ there is one arc that contained in no 3-cycle. We also give methods to construct the near-homogeneous tournaments we define.

While the existence, construction and enumeration of homogeneous tournaments of every possible order remain challenging problems, we can find interesting problems concerning near-homogeneous tournaments as well. As we can see, the existing constructions of near-homogeneous tournaments are mostly transformations from given homogeneous tournaments, thus we are interested in the constructions of near-homogeneous tournaments that are not based on homogeneous tournaments. In particular, for near-homogeneous tournaments with an even order, we ask for the existence of tournaments in this class, besides the ones that we construct in section 4.

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