PROVING ERGODICITY VIA DIVERGENCE OF ERGODIC SUMS

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Abstract. A classical fact in ergodic theory is that ergodicity is equivalent to almost everywhere divergence of ergodic sums of all nonnegative integrable functions which are not identically zero. We show two methods, one in the measure preserving case and one in the nonsingular case, which enable one to prove this criteria by checking it on a dense collection of functions and then extending it to all nonnegative functions. The first method, Theorem 1, is then used in a new proof of a folklore criterion for ergodicity of Poisson suspensions which does not make any reference to Fock spaces. The second method, Theorem 2, which involves the double tail relation is used to show that a large class of nonsingular Bernoulli and inhomogeneous Markov shifts are ergodic if and only if they are conservative. In the last section we discuss an extension of the Bernoulli shift result to other countable groups including \( \mathbb{Z}^d \), \( d \geq 2 \) and discrete Heisenberg groups.

1. Introduction

Given a non-singular system \( (X, \mathcal{B}, \mu, T) \), one of the major challenges is to prove ergodicity. In the finite measure preserving case, a common approach is to establish that for every \( f \in L^1(X, \mu) \), for \( \mu \)-a.e. \( x \in X \),

\[
\frac{1}{n} \sum_{k=0}^{n-1} f \circ T^k(x) = \frac{1}{n} S_n(f)(x) \xrightarrow{n \to \infty} \int_X f \, d\mu.
\]

It then follows from the pointwise ergodic theorem that \( T \) is ergodic. The maximal inequality, which states that there exists \( C > 0 \), such that for all \( f \in L^1(X, \mu) \) and \( t > 0 \),

\[
\mu \left( \sup_{n \in \mathbb{N}} \left| \frac{S_n(f)}{n} \right| > t \right) \leq \frac{C |f|_{L^1}}{t}
\]

is used in the classical proof of the pointwise ergodic theorem in order to establish the almost everywhere convergence for all \( f \in L^1(X, \mu) \) from the knowledge of a.e. convergence for a dense set of integrable functions. This principle lies in the heart of the Hopf method, which is a method of proving ergodicity for many smooth systems by showing that for a dense collection of continuous functions \( f \),

\[
\lim_{n \to \infty} \frac{S_n(f)}{n} = \int_X f \, d\mu, \quad \mu - a.e.,
\]

See [16] and references therein for a description of the Hopf method and some references of where it has been used for proving ergodicity. In the case of infinite \( \sigma \)-finite measure preserving systems, one can replace the pointwise ergodic theorem with Hopf’s ratio ergodic theorem by fixing a well chosen positive integrable function \( g \in L^1(X, \mu) \) and then showing that for all \( f \in L^1(X, \mu) \),

\[
\frac{S_n(f)}{S_n(g)} \xrightarrow{n \to \infty} \frac{\int_X f \, d\mu}{\int_X g \, d\mu}, \quad \mu - a.e.
\]

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Again by a maximal inequality it is enough to establish this convergence for a dense class of \( f \) in \( L^1(X,\mu) \) and this is the starting point in Coudene’s extension of the Hopf method for some infinite measure preserving systems \([2]\). See also \([12,13]\) for other cases where ergodicity is proved via the ratio ergodic theorem. A similar method can be done in the case of non-singular systems by replacing the ratio ergodic theorem with Hurewicz ergodic theorem. Indeed, this is used in \([10,15]\) to show that a non-singular \( K \)-system is ergodic if and only if it is conservative.

Another criteria for ergodicity is that for every \( 0 \leq f \in L^1(X,\mu) \) with \( \int_X f d\mu > 0 \),

\[
S_n(f) \xrightarrow{n \to \infty} \infty, \quad \mu - a.e.
\]

In this note we first make use of this well known ergodicity criteria for proving ergodicity in two cases, namely Poisson suspensions and (in-homogenous) Markov shifts. Given a standard probability space \((X,\mathcal{B},\mu)\), we say that a collection of sets \( A \subset \mathcal{B} \) is dense in \( \mathcal{B} \) if for every \( B \in \mathcal{B} \) and \( \epsilon > 0 \) there exists \( A \in \mathcal{A} \) with \( \mu(A) > 0 \) and

\[
\mu(A \cap B) \geq (1-\epsilon)\mu(A).
\]

**Theorem 1.** Let \((X,\mathcal{B},\mu)\) be a standard probability space, \( T : X \to X \) a measure preserving system, and \( A \subset \mathcal{B} \) a collection of sets which is dense in \( \mathcal{B} \). If there exists \( \alpha > 0 \) such that for all \( A \in \mathcal{A} \) there exists two subsequences \( n_j \to \infty \) and \( N_n \to \infty \) such that

\[
\liminf_{n \to \infty} \left( \frac{1}{N_n} \sum_{k=0}^{N_n-1} 1_A \circ T^{\alpha k} \right) \geq \alpha \mu(A), \quad \mu - a.e.,
\]

then \( T \) is ergodic.

We use this criteria to show a new proof of the classical fact that a measure preserving Poisson suspension \((X^*,\mathcal{B}^*,\mu^*,S_\ast)\) is ergodic if and only if there exists no absolutely continuous invariant probability measure for \((X,\mathcal{B},\mu,S)\). Our proof does not involve any use of the Fock Space structure and therefore it might be useful for proving ergodicity of other more complicated point processes. We also show that a simple argument shows that \((X^*,\mathcal{B}^*,\mu^*,S_\ast)\) is indeed weak mixing if and only if it is ergodic, thus this direct method recovers the full statement of the classical fact.

In the case where \( T \) is invertible and nonsingular, \( T \) is ergodic if and only if for all \( 0 \leq f \in L^1(X,\mu) \) with \( \int_X f d\mu > 0 \),

\[
\tilde{T}_n(f) = \sum_{k=0}^{n-1} \frac{d(\mu \circ T^{-k})}{d\mu} f \circ T^{-k} \xrightarrow{n \to \infty} \infty, \quad \mu - a.e.
\]

Given a finite or countable set \( F \), a closed shift invariant subset \( X \subset F^\mathbb{Z} \) is called a subshift. The double tail relation of \( X \) is a Borel subset of \( X \times X \) defined by

\[
\mathcal{T} = \{(x,y) \in X \times X : \exists n \in \mathbb{N}, \forall |k| > n, \ x_k = y_k\}.
\]

The set \( \mathcal{T} \) is in addition an equivalence relation on \( X \). Given a probability measure \( \mu \) on \( X \) for which the shift \( T \) is non-singular, the symbolic system \((X,\mathcal{B},\mu,T)\) is double tail trivial if for all \( A \in \mathcal{B} \), \( \mu(T(A)) = 0 \) or \( \mu(X \setminus T(A)) = 0 \), where

\[
\mathcal{T}(A) = \cup_{x \in A} [x] = \{y \in X : \exists x \in A, (x,y) \in \mathcal{T}\}.
\]

See \([11]\) and the references therein for examples of double tail trivial processes.

\(^4\)It is enough to consider \( \{f = 1_A : A \in \mathcal{B}, \ 0 < \mu(A) < \infty\} \).
Theorem 2. Let \((X, \mathcal{B}, \mu, T)\) be a conservative, non-singular subshift which is double tail trivial and \(\mathcal{A} \subset \mathcal{B}\) the collection of finite union of cylinder sets in \(\mathcal{B}\). If there exists \(L : T \rightarrow (0, \infty)\) such that for \(\mu \times \mu\) almost all \((x, y) \in T\), for all \(n \in \mathbb{N}\),

\[
L(x, y)^{-1} \frac{d(\mu \circ T^{-n})}{d\mu}(y) \leq \frac{d(\mu \circ T^{-n})}{d\mu}(x) \leq L(x, y) \frac{d(\mu \circ T^{-n})}{d\mu}(y),
\]

then \(T\) is ergodic.

In Section 5 we use this theorem to show an ergodicity criterion for two natural symbolic models, non-singular Bernoulli shifts which are shifts of independent not necessarily identically distributed random variables and in-homogenous Markov (chains) shifts which are fully supported on a topologically mixing subshift of finite type. We give a short discussion on how the latter implies a certain hurdle for a natural approach towards a variant on a classical question of Bowen on the existence of a measure preserving \(C^1\) Anosov diffeomorphism of \(\mathbb{T}^2\) which is not ergodic. Finally we extend the result on Bernoulli shifts for countable groups which have a version of the Hurewicz’s ratio ergodic theorem.

We end the introduction with a description of the result in the case of non-singular Bernoulli shifts. A non-singular Bernoulli shift is a quadruple \(\left(\{1, \ldots, N\}^\mathbb{Z}, \mathcal{B}, \mu, T\right)\) where \(\mu = \prod_{k \in \mathbb{Z}} \mu_k\) is a product measure on \(\{1, \ldots, N\}^\mathbb{Z}\), \(T\) is the shift map on \(\{1, \ldots, N\}^\mathbb{Z}\) defined by

\[
(Tx)_i = x_{i+1}
\]

and \(\mu \sim \mu \circ T\) (i.e. the shift is \(\mu\)-non-singular). By Kakutani’s theorem, non-singularity of the shift is equivalent to

\[
(1.1) \quad \sum_{k \in \mathbb{Z}} \sum_{j=1}^{N} \left( \sqrt{\mu_k(j)} - \sqrt{\mu_{k-1}(j)} \right)^2 < \infty.
\]

In the case where in addition there exists a probability distribution \(P\) on \(\{1, \ldots, N\}\) such that \(\mu_k = P\) for all \(k < 0\), the Bernoulli shift is a \(K\)-automorphism in the sense of Silva and Thieullen [15], hence it is ergodic if and only if it is conservative. We prove that ergodicity is equivalent to conservativity for general, not necessarily half stationary, Bernoulli shifts satisfying a natural non-degeneracy condition.

Theorem 3. If a non-singular Bernoulli shift \(\left(\{1, \ldots, N\}^\mathbb{N}, \mathcal{B}, \prod_{k \in \mathbb{Z}} \mu_k, T\right)\) is conservative and

\[
L = \sup_{k \in \mathbb{Z}} \frac{\max_{j \in \{1, \ldots, N\}} (\mu_k(\{j\}))}{\min_{j \in \{1, \ldots, N\}} (\mu_k(\{j\}))} < \infty,
\]

then it is ergodic.

Notation:
- \(a_n \preceq b_n\) means that \(\limsup_{n \to \infty} \frac{a_n}{b_n} \leq 1\).
- For \(a, b \in \mathbb{R}\) and \(c > 0\) we write \(a = b \pm c\) if \(|a - b| < c\).
- For \(a, b > 0\) and \(L > 1\), \(a = bL^{\pm \epsilon}\) means \(bL^{-\epsilon} \leq a \leq bL^\epsilon\).

2. Proof of Theorem 1

Suppose \(T\) is not ergodic, then there exists a \(T\) invariant set \(B \in \mathcal{B}\) with \(\mu(B), \mu(X \setminus B) > 0\). As \(\mathcal{A}\) generates \(\mathcal{B}\) there exists \(A \in \mathcal{A}\) with

\[
\mu(A \cap B) \geq (1 - \alpha \mu(X \setminus B)) \mu(A).
\]
By the assumptions of Theorem 1 there exists \( n_j \to \infty \) and \( N_j \to \infty \) such that for \( \mu \) almost every \( x \in X \),

\[
\liminf_{n \to \infty} \left( \frac{1}{N_n} \sum_{k=0}^{N_n-1} 1_A \circ T^{n_k}(x) \right) \geq \alpha \mu(A).
\]

In addition, by \( T \) invariance of \( B \), for every \( x \in X \setminus B \) and \( n \in \mathbb{N} \)

\[
(2.1) \quad \sum_{k=0}^{n-1} 1_A \circ T^{n_k}(x) = \sum_{k=0}^{n-1} 1_{A \setminus B} \circ T^{n_k}(x).
\]

By Fatou’s lemma,

\[
\alpha \mu(A) \mu(X \setminus B) \leq \int_{X \setminus B} \liminf_{n \to \infty} \left( \frac{1}{N_n} \sum_{k=0}^{N_n-1} 1_A \circ T^{n_k}(x) \right) \, d\mu
\]

\[
\leq \liminf_{n \to \infty} \int_{X \setminus B} \left( \frac{1}{N_n} \sum_{k=0}^{N_n-1} 1_A \circ T^{n_k}(x) \right) \, d\mu
\]

\[
\leq \liminf_{n \to \infty} \int_{X \setminus B} \left( \frac{1}{N_n} \sum_{k=0}^{N_n-1} 1_{A \setminus B} \circ T^{n_k}(x) \right) \, d\mu
\]

\[
\leq \mu(A \setminus B) < \alpha \mu(A) \mu(X \setminus B)
\]

This is a contradiction, hence \( T \) is ergodic.

3. Folklore criteria for ergodicity of Poisson suspensions

Let \((X, B, \mu)\) be a standard \( \sigma \)-finite measure space and \((X^*, B^*, \mu^*)\) its associated Poisson point process. That is, \( X^* \) is the collection of all countable subsets of \( X \) (or counting measures), \( B^* \) the \( \sigma \)-algebra generated by

\[
\{ \nu \in X^* : N(A)(\nu) = n \}
\]

with \( A \in B \) with \( 0 < \mu(A) < \infty \) and \( n \in \mathbb{N} \cup \{0, \infty\} \), where

\[
N(A)(\nu) = |\nu \cap A|.
\]

Finally, the measure \( \mu^* \) is the unique measure such that for all pairwise disjoint sets \( A_1, A_2, ..., A_n \in B \), the random variables \( \{N(A_i)\}_{i=1}^n \) are independent and for each \( A \in B \) with \( \mu(A) < \infty \), \( N(A) \) is Poisson distributed with parameter \( \mu(A) \), that is, for all \( k \in \mathbb{N} \cup \{0\} \),

\[
\mu^*(N(A) = k) = \frac{e^{-\mu(A)} \mu(A)^k}{k!}.
\]

Given a measure preserving transformation \( T : (X, B, \mu) \to (X, B, \mu) \), its Poisson suspension is a probability preserving map \( T_* : (X^*, B^*, \mu^*) \to (X^*, B^*, \mu^*) \) defined by

\[
T_* \{ \{x\}_{x \in \nu} \} = \{ Tx \}_{x \in \nu}.
\]

In what follows we will write

\[
B_{\text{fin}} = \{ B \in B : 0 < \mu(B) < \infty \}.
\]

**Theorem 4.** Let \((X, B, \mu, T)\) be a \( \sigma \)-finite measure preserving system. Then \( T_* \) is ergodic if and only if \( T \) has no absolutely continuous invariant probability measure.
If $T$ has an absolutely continuous invariant probability (a.c.i.p.), then it is immediate that $T_*$ is not ergodic as in that case there is a set $A \in B_{fin}$ with $T^{-1}A = A$. For all $k \in \mathbb{N}$ the sets

$$[N(A) = K] = \{ \nu \in X^* : N(A)(\nu) = k \}$$

are $T_*$ invariant sets of positive, non-full $\mu^*$-measure. Our proof of ergodicity of $T_*$ when $T$ has no a.c.i.p. is by establishing the conditions of Theorem 1 with the collection of sets

$$A^* = \left\{ \bigcap_{i=1}^L [N(A_i) = k_i] : L \in \mathbb{N}, \{A_i\}_{i=1}^L \subset B_{fin} \text{ (pairwise disjoint)}, \{k_i\}_{i=1}^L \subset \mathbb{N} \cup \{0\} \right\}$$

and $\alpha = 1$.

**Lemma 5.** Let $(X,B,\mu,T)$ be a $\sigma$-finite measure preserving system. If there exists no absolutely continuous invariant probability measure, then for all $L \in \mathbb{N}$, $A_1,A_2..A_L \in B_{fin}$, there exists a strictly increasing subsequence $n_k \to \infty$ such that for all $\alpha,\beta \in \{1,2,\ldots,L\}$,

$$\lim_{j-l \to \infty} \mu\left( A_\alpha \cap T^{-n_j-n_l} A_\beta \right) = 0.$$

A subset $K \subset \mathbb{N}$ has full Banach density if

$$\lim_{n \to \infty} \frac{|K \cap [1,n]|}{n} = 1.$$

In what follows we will use the well known fact that if $a_n \geq 0$ satisfies

$$\lim_{n \to \infty} \sum_{i=1}^n a_i = 0,$$

then for all $\epsilon > 0$, the sequence $K^\epsilon = \{ n \in \mathbb{N} : 0 \leq a_n < \epsilon \}$ has full Banach density. Another trivial consequence of the definition of full Banach density is that if $K_1,K_2,\ldots,K_N \subset \mathbb{N}$ are sets of full Banach density then $\bigcap_{i=1}^N K_i$ has full Banach density.

**Proof.** Let $A_1,A_2,\ldots,A_L \in B_{fin}$. We construct $n_k \to \infty$ by an inductive procedure. As $T$ is $\mu$ measure preserving and there exists no a.c.i.p., given a finite set $F \subset \mathbb{N}$ and $\{B_\alpha\}_{\alpha \in F} \subset B_{fin}$, by the pointwise ergodic theorem, for all $\alpha \in F$,

$$\frac{1}{n} S_n \left( 1_{B_\alpha} \right) \xrightarrow{n \to \infty} 0, \text{ a.e.}$$

By the dominated convergence theorem for all $\alpha,\beta \in F$,

$$\frac{1}{n} \sum_{k=0}^{n-1} \mu \left( A_\alpha \cap T^{-k} A_\beta \right) = \int_{A_\alpha} \left( \frac{1}{n} S_n \left( 1_{A_\beta} \right) \right) d\mu \xrightarrow{n \to \infty} 0.$$

We conclude, using the previous discussion on sets of full Banach density, that for all $\epsilon > 0$, the set

$$K^\epsilon = \{ n \in \mathbb{N} : \forall \alpha,\beta \in F, \mu \left( A_\alpha \cap T^{-n} A_\beta \right) < \epsilon \}$$

is of full Banach density.

Taking first $F = \{1,\ldots,L\}$ and for $\alpha \in F$, $B_\alpha = A_\alpha$, we can choose $n_1 \in \mathbb{N}$ such that for all $\alpha,\beta \in \{1,\ldots,L\}$,

$$\mu \left( A_\alpha \cap T^{-n_1} A_\beta \right) < \frac{1}{2}.$$

Assume that we have chosen a sequence $n_0 = 0$ and $n_1,\ldots,n_k \in \mathbb{N}$ such that for all $0 \leq l < j \leq k$ and $\alpha,\beta \in F$,

$$\mu \left( T^{-n_l} A_\alpha \cap T^{-n_j} A_\beta \right) = \mu \left( A_\alpha \cap T^{-n_l} A_\beta \right) < 2^{-j}.$$
Looking at $F_k = \{1, 2, \ldots, kL\}$ and

$$B_s = T^{-n_j}A_\alpha \text{ for } s = jL + \alpha,$$

we conclude that the set

$$\left\{ n \in \mathbb{N} : \forall \alpha, \beta \in F_k, \mu(B_\alpha \cap T^{-n}B_\beta) < 2^{-(k+1)} \right\}$$

is of full Banach density. In particular there exists $n_{k+1} > n_k$, such that for all $\alpha, \beta \in \{1, \ldots, L\}$, and $0 \leq j < k + 1$

$$\mu(T^{-n_j}A_\alpha \cap T^{-n_{k+1}}A_\beta) = \mu(B_{jL+\alpha} \cap T^{-n_{k+1}}B_\beta) < 2^{-(k+1)}$$
as was required. This concludes the proof of the lemma. \(\square\)

Given $B = \bigcap_{j=1}^L [N(A_j) = k_j]$ a (Poissonian) cylinder set we write $S(B) = \bigcup_{i=1}^L A_j$. \(\textbf{Lemma 6.}\) If $B, C \in \mathcal{B}^*$ are cylinder sets then

$$|\mu^*(B \cap C) - \mu^*(B) \mu^*(C)| \leq 2 \mu(S(B) \cap S(C)).$$

\textit{Proof.} Note that as $\mu^*$ is a probability measure we can assume that $\mu(S(B) \triangle S(C)) < 1$. Write $B = \bigcap_{j=1}^L [N(A_j) = k_j]$ and $D = \bigcap_{j=1}^L [N(A_j \setminus S(C)) = k_j]$. Note that

$$B \triangle D \subset [N(S(B) \cap S(C)) > 0].$$

Thus

$$\mu^*(B \triangle D) \leq 1 - \mu^*(N(S(B) \cap S(C)) = 0)$$

$$= 1 - \exp(-\mu(S(B) \cap S(C))) \leq \mu(S(B) \cap S(C)).$$

As $S(C) \cap S(D) = \emptyset$ by the independence property of the Poisson process,

$$\mu^*(D \cap C) = \mu^*(D) \mu^*(C) = \mu^*(B) \mu^*(C) \pm \mu(S(B) \cap S(C)).$$

Similarly

$$|\mu^*(B \cap C) - \mu^*(D \cap C)| \leq \mu^*(B \triangle D).$$

This shows that

$$|\mu^*(B \cap C) - \mu^*(B) \mu^*(C)| \leq 2 \mu(S(B) \cap S(C)).$$

\(\square\)

\textbf{Corollary 7.} For all $A_1, A_2, \ldots, A_L \in \mathcal{B}_{fim}$ and $k_1, k_2, \ldots, k_L \in \mathbb{N} \cup \{0\}$, there exists a subsequence $n_j \to \infty$ such that writing $B = \bigcap_{j=1}^L [N(A_j) = k_j] \in \mathcal{B}^*$, for all $0 \leq l < l'$,

$$|\mu^*(T^{-n_l}B \cap T^{-n_{l'}}B) - \mu^*(B)|^2 \leq 2^{-(j-l)}.$$ 

\textit{Proof.} For all cylinder sets $B$ and $n \in \mathbb{Z}$,

$$S(T_{-n}B) = T^nS(B).$$

By Lemma 5 there exists $n_j \to \infty$ such that for all $1 \leq l < j$,

$$\mu(S(T_{-n_j}B) \cap S(T_{-n}B)) = \mu(T^{-n_j}S(B) \cap T^{-n}S(B))$$

$$= \mu(S(B) \cap T^{-(n_j-n)}S(B)) \leq 2^{-(j+1-l)} \mu^*.$$

The conclusion follows from Lemma 6 as $T_{*}$ is $\mu^*$ preserving. \(\square\)
Proof of 4. Note that as $T$ preserves $\mu$, if $T$ has an a.c.i.p., then there exists a set a set $A \in \mathcal{B}$ such that $T^{-1}A = A \mod \mu$ and $\mu(A) < \infty$. In that case for each $K \in \mathbb{N}$, the set $[N(A) = k]$ is $T$ invariant and of positive measure. As for $K \neq K'$, 

$$[N(A) = K] \cap [N(A) = K'] = \emptyset,$$

this is a contradiction to ergodicity.

In the other direction assume $T$ has no absolutely continuous invariant probability measure. The collection $\mathcal{A}^*$ generates $\mathcal{B}^*$. We show that the conditions of Theorem 1 hold for all $B \in \mathcal{A}^*$. Let $B \in \mathcal{A}^*$. By Corollary 7, there exists a sequence $n_j \to \infty$ such that for all $l < j$,

$$\int_{X^*} 1_B \circ T_{n_j} \circ T_{n_j}^t \, d\mu^* = \mu^* (B \cap T_{n_j}^{-l} B) \leq \left( 1 + 2^{-(j-l)} \right) \mu^* (B)^2.$$

By this, for all $N \in \mathbb{N}$,

$$\int_{X^*} \left( \sum_{j=0}^{N-1} 1_B \circ T_{n_j}^t \right)^2 \, d\mu^* = \sum_{j=0}^{N-1} \int_{X^*} 1_B \circ T_{n_j}^t \, d\mu^* + 2 \sum_{0 \leq l < j < N} \int_{X^*} 1_B \circ T_{n_j}^t 1_B \circ T_{n_l}^t \, d\mu^*$$

$$= N \mu^* (B) + 2 \sum_{0 \leq l < j < N} \left( 1 + 2^{-(j-l)} \right) \mu^* (B)^2$$

$$= N^2 (\mu^* (B))^2 + O(N)$$

$$= \left( \int_{X^*} \sum_{j=0}^{N-1} 1_B \circ T_{n_j}^t \, d\mu^* \right)^2 + O(N).$$

This shows that

$$Var \left( \frac{1}{N} \sum_{j=0}^{N-1} 1_B \circ T_{n_j}^t \right) = O \left( \frac{1}{N} \right) \xrightarrow{N \to \infty} 0.$$

A classical application of Chebychev’s inequality then shows that

$$\frac{1}{N} \sum_{j=0}^{N-1} 1_B \circ T_{n_j}^t \to \mu^* (B), \text{ in } \mu^* \text{ measure.}$$

It then follows that there exists $N_n \to \infty$ such that

$$\frac{1}{N_n} \sum_{j=0}^{N_n-1} 1_B \circ T_{n_j}^t \xrightarrow{n \to \infty} \mu^* (B), \mu^* - \text{almost everywhere.}$$

We have shown that for all $B \in \mathcal{A}^*$ we have $n_j \to \infty$ and $N_n \to \infty$ as in the conditions of Theorem 1 and therefore $T_*$ is ergodic.

\[\square\]

3.0.1. Ergodicity implies weak mixing. For $T_*$ a Poisson suspension over a measure preserving transformation $T$, it is known that if $T_*$ is ergodic then $T_*$ is weak mixing. We show an argument which gives the weak mixing result.

A probability preserving transformation $(\Omega, C, \nu, R)$ is weak mixing if $R \times R$ is ergodic. Weak mixing implies ergodicity and there are several (standard) equivalent definitions of the weak mixing property. Among them is the spectral condition, $R$ is weakly mixing if and only if there are no functions $f \in L^2 (\Omega, \nu)$
with $\int f\,d\nu = 0$ and $\lambda \in \mathbb{C}$ with $|\lambda| = 1$ which satisfy

$$f \circ R = \lambda f.$$  

**Proposition 8.** Let $(\Omega, \mathcal{C}, \nu, R)$ be a probability preserving transformation. If for all $f \in L^2(\Omega, \nu)$, there exists $n_j \to \infty$ such that

$$\int f \circ R^{n_j} f\,d\nu \xrightarrow{j \to \infty} \int f^2\,d\nu$$

as $j \to \infty$, then $R$ is weak mixing.

**Proof.** Assume that the conditions of the proposition are satisfied and $R$ is not weak mixing. Then there exists a non-constant $f \in L^2(\Omega, \nu)$ with $\int f\,d\nu = 0$ and $\lambda \in \mathbb{C}$ with $|\lambda| = 1$ such that

$$f \circ R = \lambda f.$$  

By the conditions of the proposition, there exists $n_j \to \infty$ such that

$$0 = \lim_{j \to \infty} \int f \circ R^{n_j} f\,d\nu = \lim_{j \to \infty} \lambda^{n_j} \int |f|^2\,d\nu.$$

This can happen only if $f \equiv 0$, which is a contradiction. \qed

As $(X, \mathcal{B}, \mu)$ is a standard $\sigma$-finite measure space, there exists a countable collection of sets $Z \subset \mathcal{B}_{fin}$ such that for all $A \in \mathcal{B}_{fin}$ and $\epsilon > 0$, there exists $C$ which is a finite union of sets in $Z$ such that $\mu(A \Delta C) < \epsilon$. In the case $X = \mathbb{R}$ and $\mu$ the Lebesgue measure one can take for example $Z$ to be the collection of intervals with rational endpoints. Denote by $\mathcal{F}$ the collection of finite unions of sets in $Z$ and define

$$\mathcal{A}(Z) = \left\{ \bigcap_{i=1}^{L} [N(A_i) = k_i] : L \in \mathbb{N}, \{A_i\}_{i=1}^{L} \subset \mathcal{F}, \{k_i\}_{i=1}^{L} \subset \mathbb{N} \cup \{0\} \right\}.$$  

In what follows, the fact that $\mathcal{F}$ and hence $\mathcal{A}(Z)$ are countable will be useful.

**Lemma 9.** The collection of simple functions of the form $\sum_{i=1}^{L} c_i 1_{A_i}^\star$ with $\{A_i^\star\}_{i=1}^{L} \subset \mathcal{A}(Z)$ is dense in $L^2(X^\star, \mu^\star)$.

**Proof.** Firstly, the collection of simple functions with $\{A_i^\star\}_{i=1}^{L} \subset \mathcal{A}^\star$ is dense in $L^2(X^\star, \mu^\star)$.

Secondly, for any $\{A_i\}_{i=1}^{N} \subset \mathcal{B}_{fin}$ there exists an array of sets $\{B_{n,i} : n, i \in \mathbb{N}, 1 \leq i \leq N\}$ such that

$$\max_{1 \leq i \leq N} \mu(A_i \Delta B_{n,i}) \xrightarrow{n \to \infty} 0.$$  

By this, for any $A^\star = \bigcap_{i=1}^{N} [N(A_i) = k_i] \in \mathcal{A}^\star$ writing $B_n^\star = \bigcap_{i=1}^{N} [N(B_{n,i}) = k_i]$, one has

$$\|1_{A^\star} - 1_{B_n^\star}\|_2 = \mu(A^\star \Delta B_n^\star) \xrightarrow{n \to \infty} 0.$$  

The combination of these two observations proves the claim. \qed

**Lemma 10.** Let $(X, \mathcal{B}, m, T)$ be a measure preserving transformation with $m(X) = \infty$ and no absolutely continuous probability measure. For any countable collection of sets $\mathcal{V} \subset \mathcal{B}_{fin}$, there exists $n_j \to \infty$ such that for all $A, B \in \mathcal{V}$

$$(3.1) \quad m\left( A \cap T^{-n_j} B \right) \xrightarrow{j \to \infty} 0.$$  

\footnote{This proposition was communicated to us by J. Aaronson.}
Proof. Let \( \{A_i\}_{i=1}^\infty \) be an enumeration of the sets in \( \mathcal{V} \). We will construct \( n_j \) as follows. As \( T \) has no a.c.i.p. and \( \{1_{A_i}\}_{i=1}^\infty \subset L^1(\mathcal{X},\mathcal{B},m) \), for all \( 1 \leq i, j < \infty \),

\[
\frac{1}{n} \sum_{i=1}^n m(A_i \cap T^{-n}A_j) \xrightarrow{n \to \infty} 0.
\]

Consequently for all \( \epsilon > 0 \) and \( L \in \mathbb{N} \), the set

\[
D(L, \epsilon) = \left\{ n \in \mathbb{N} : \max_{1 \leq i, j \leq L} (m(A_i \cap T^{-n}A_j)) < \epsilon \right\}
\]

is of Banach density 1 since it is an intersection of \( 2^L \) elements of full density. A simple inductive construction gives an increasing subsequence \( n_j \) such that \( n_j \in D(j, 2^{-j}) \). The lemma is proven. \( \Box \)

**Theorem 11.** If \( (X, \mathcal{B}, \mu, T) \) be a \( \sigma \)-finite measure preserving system with no absolutely continuous invariant probability measure, then \( T_* \) is weak mixing.

Proof. Let \( \mathcal{Z} \subset \mathcal{B}_{\text{fin}}, \mathcal{A}(\mathcal{Z}) \subset \mathcal{B}^* \) and \( \mathcal{F} \) be as above and \( n_j \to \infty \) such that for all \( A, B \in \mathcal{F}^3 \), \[ (3.2) \]

\[
m(A \cap T^{-n_j}B) \xrightarrow{j \to \infty} 0.
\]

First we show that for all \( C, D \in \mathcal{A}(\mathcal{Z}) \),

\[
\int_{X^*} l_C \left( 1_D \circ T_+^{n_j} \right) \, dm^* \xrightarrow{j \to \infty} m^*(C) m^*(D).
\]

Indeed, any \( C, D \in \mathcal{A}(\mathcal{Z}) \) are of the form \( C = \bigcap_{i=1}^L [N(A_i) = k_i] \) and \( D = \bigcap_{i=L+1}^{L+M} [N(A_i) = k_i] \) with \( L, M \in \mathbb{N}, \{k_i\}_{i=1}^L \subset \{0,1\} \) and \( \{A_i\}_{i=L+1}^{L+M} \subset \mathcal{F} \). As \( \mathcal{F} \) is closed under finite unions, \( \mathcal{A} = \bigcup_{i=1}^{L+M} A_i \in \mathcal{F} \), thus

\[
m(A \cap T^{-n_j}A) \xrightarrow{j \to \infty} 0.
\]

By Lemma 6

\[
\left| \int_{X^*} l_C \left( 1_D \circ T_+^{n_j} \right) \, dm^* - m^*(C) m^*(D) \right| \xrightarrow{j \to \infty} 0.
\]

Consequently, by Lemma 6 and standard approximation arguments, it then follows that for all \( F, G \in L^2(X^*, m^*) \),

\[
\int_{X^*} F \left( G \circ T_+^{n_j} \right) \, dm^* \xrightarrow{j \to \infty} \left( \int_{X^*} F dm^* \right) \left( \int_{X^*} G dm^* \right).
\]

By Proposition 8 \( T_* \) is weak mixing. \( \Box \)

**Remark 12.** Emmanuel Roy has pointed out to us that in the case of Poisson suspensions weak mixing follows from ergodicity by the following argument. Given a measure preserving \( T : (X, \mathcal{B}, \mu) \to (X, \mathcal{B}, \mu) \) let \( S = T \times \text{Id} : X \times \{0,1\} \to X \times \{0,1\} \) be a two point extension of \( T \) which preserves \( \mu \times (\frac{1}{2} (\delta_0 + \delta_1)) \). Then \( S_* \) is isomorphic to \( T_* \times T_* \). Also \( T \) has no a.c.i.p. if and only if \( S \) has no a.c.i.p. thus ergodicity of \( T_* \) implies ergodicity of \( S_* \cong T_* \times T_* \).

3.1. **Original motivation for the statement of Theorem** \([1]\). A set \( W \) in \( (X, \mathcal{B}, m) \) is weakly wandering for \( T \) if there exists \( n_j \to \infty \) such that \( \{T^{-n_j}W\} \) are pairwise disjoint. If there exists no a.c.i.p., then \( X \) is a countable disjoint union of weakly wandering sets \( \bigcup_{j \in \mathbb{Z}} T^{-n_j}W \), see \([2, 4]\) for discussion on weakly wandering sets. As for all weakly wandering set \( W \) and \( A \in \mathcal{B}, A \cap W \) is weakly wandering, this implies that every finite measure set can be approximated from within by a finite union of weakly wandering sets.

\[3\]Recall that \( \mathcal{F} \) is countable.
Here given $W$ weakly wandering with respect to $n_j \to \infty$ of positive and finite $\mu$ measure and $k \in \mathbb{N}$, the sequence

$$Y_j := 1_{[N(W)=k]} \circ T_n^{n_j}$$

is a sequence of i.i.d. integrable random variables. By the strong law of large numbers,

$$\frac{1}{n} \sum_{j=0}^{n-1} 1_{[N(W)=k]} \circ T_n^{n_j} \xrightarrow{n \to \infty} \mu^* (N(W) = k), \quad \mu^* - a.s.$$ 

Our first attempt was to use this to show the conditions of Theorem 1. The problem for doing this lies in the following: Given $W_1, W_2, \ldots, W_N$ pairwise disjoint weakly wandering sets, does there exists $n_j \to \infty$ such that $\bigcup_{i=1}^{N} W_i$ is weakly wandering along $n_j$?

4. Proof of theorem 2

4.1. Some relevant material from non-singular ergodic theory. This subsection contains several classical statements and definitions from non-singular ergodic theory. The reader is referred to [1] where the statements and their proofs are written. Let $(X, \mathcal{B}, \mu)$ be a standard probability space and $T : X \to X$ a measurable and invertible transformation such that $\mu \circ T$ and $\mu$ have the same collection of null sets ($\mu$ and $\mu \circ T$ are equivalent measures). Let $\hat{T} : L^1 (X, \mathcal{B}, \mu) \to L^1 (X, \mathcal{B}, \mu)$ be the dual operator of $T$ defined by

$$\int_X f \cdot g \circ T \, d\mu = \int_X \left( \hat{T} f \right) g \, d\mu,$$

for all $g \in L^\infty (X, \mathcal{B}, \mu)$ and $f \in L^1 (X, \mathcal{B}, \mu)$. In our case, as $T$ is invertible, for all $n \in \mathbb{Z}$,

$$\hat{T}^n (f)(x) = \frac{d(\mu \circ T^{-n})}{d\mu}(x) f \circ T^{-n}(x).$$

A set $W \in \mathcal{B}$ is wandering if $\{T^n W\}_{n \in \mathbb{Z}}$ are pairwise disjoint. The measurable union of all wandering sets, denoted by $\mathcal{D}(T)$, is called the dissipative part of $T$. Its complement $\mathcal{C}(T) = X \setminus \mathcal{D}(T)$ is called the conservative part of $T$. The decomposition $X = \mathcal{D}(T) \cup \mathcal{C}(T)$ is called the Hopf decomposition of $T$. The map $T$ is conservative if there exists no wandering set $W$ of positive $\mu$ measure, or equivalently $\mathcal{C}(T) = X \mod \mu$. An equivalent definition is that $T$ satisfies the conclusion of the Poincare recurrence theorem, in the sense that for all $A \in \mathcal{B}$ with $\mu(A) > 0$, for almost every $x \in A$

$$\sum_{k=1}^{\infty} 1_A \circ T^k(x) = \infty.$$ 

The conservative part, modulo a null set, is equal to

$$\mathcal{C}(T) = \left\{ x \in X : \sum_{k=1}^{\infty} \frac{d(\mu \circ T^{-n})}{d\mu}(x) = \infty \right\} \mod \mu$$

and $T$ is conservative if and only if

$$\sum_{k=1}^{\infty} \frac{d(\mu \circ T^{-n})}{d\mu}(x) = \infty, \quad \mu - a.e.$$ 

To shorten notation we will write $1$ for the constant function $1(x) = 1$ and

$$\hat{T}^n 1(x) = \frac{d(\mu \circ T^{-n})}{d\mu}(x).$$
Finally, by the Hurewicz ergodic theorem for all $A \in \mathcal{B}$ with $\mu(A) > 0$, for $\mu$ almost every $x \in X$, 
\[
\frac{\sum_{k=0}^{n-1} \hat{T}^n 1_A(x)}{\sum_{k=1}^{n} T^n 1(x)} = \frac{\sum_{k=0}^{n-1} \hat{T}^n 1_A(x)}{\sum_{k=1}^{n} (T^{-n})^k (x)} \to h(1_A, 1)(x)
\]
where $h = h(A) \in L^1(X, \mathcal{B}, \mu)$ satisfies:

- $h \circ T = h$ and $h \geq 0$.
- $\int_X h \psi d\mu = \int_X \psi 1_d \mu$ for all $\psi \in L^\infty(X, \mu)$ satisfying $\psi \circ T = \psi$. Consequently, the set 
\{ $x \in X : h(x) > 0$ \} is of positive $\mu$ measure.

The two bullets simply say that $h = \mathbb{E}(1_A | \mathcal{I})$, where $\mathcal{I}$ is the $\sigma$-algebra of $T$ invariant sets. One way of proving the Hurewicz ergodic theorem goes through the following special case of the maximal inequality.

Write $\hat{T}_n(f) = \sum_{k=0}^{n-1} T^k f$.

**Theorem.** Let $(X, \mathcal{B}, \mu)$ be a conservative non-singular transformation. Then for all $f \in L^1(X, \mathcal{B}, \mu)$ and $t > 0$,
\[
\mu \left( x \in X : \sup_{n \in \mathbb{N}} \left| \frac{\hat{T}_n(f)}{\hat{T}_n(1)} \right| > t \right) \leq \frac{\|f\|_1}{t}.
\]

The proof of Theorem 3 is done by showing that for all $A \in \mathcal{B}$ with $\mu(A) > 0$,
\[
\lim_{n \to \infty} \hat{T}_n(1_A) = \infty, \quad \mu - a.e.
\]

This is equivalent to ergodicity by [1, Proposition 1.3.2.] and the fact that for a random variable $G : X \to [0, \infty]$, if for all $A \in \mathcal{B}$ with $\mu(A) > 0$,
\[
\int_A G d\mu = \infty,
\]
then $G = \infty \mu$ almost surely.

**4.2. Proof of Theorem 2.** In this section, $F = \{1, \ldots, N\}$, $X \subset F^\mathbb{Z}$ is a subshift, $\mu$ is a probability measure supported on $X$ and $T$ denotes the shift on $F^\mathbb{Z}$. We assume that the measurable equivalence relation
\[
\mathcal{T} = \{(x, y) \in X \times X : \exists n \in \mathbb{N}, x|_{\mathbb{Z}\setminus[-n,n]} = y|_{\mathbb{Z}\setminus[-n,n]} \}
\]
is ergodic. In this section, $A \subset \mathcal{B}$ denotes the collection of finite union of cylinder sets in $\mathcal{B}$ where a cylinder set is denoted by
\[
[b]_k = \left\{ x \in F^\mathbb{Z} : \forall i \in [k, l] \cap \mathbb{Z}, x_i = b_i \right\},
\]
where $b \in F^\mathbb{Z}$, $k, l \in \mathbb{Z}$.

**Proposition 13.** Let $(X, \mathcal{B}, \mu, T)$ be a conservative, non-singular subshift which is double tail trivial. If there exists $L : \mathcal{T} \to (0, \infty)$ such that for $\mu \times \mu$ almost all $(x, y) \in \mathcal{T}$, for all $n \in \mathbb{N}$,
\[
(4.1) \quad L(x, y)^{-1} \frac{d(\mu \circ T^{-n})}{d\mu}(y) \leq \frac{d(\mu \circ T^{-n})}{d\mu}(x) \leq L(x, y) \frac{d(\mu \circ T^{-n})}{d\mu}(y)
\]
then
\[(i) \quad T \text{ is either conservative or dissipative (either } \mathcal{C}(T) = X \text{ mod } \mu \text{ or } \mathcal{D}(T) = X \text{ mod } \mu. \]
\[(ii) \quad \text{If } T \text{ is conservative, then for all } A \in \mathcal{A}, \text{ for } \mu \times \mu \text{ almost all } (x, y) \in \mathcal{T},
\]
\[
\hat{T}_n(1_A)(x) \lesssim L(x, y) \hat{T}_n(1_A)(y).
\]

- Recall that this means that for all $A \in \mathcal{B}$, $\mu(\mathcal{T}(A)) = 0$ or $\mu(X \setminus \mathcal{T}(A)) = 0$.
Proof. Write $\tilde{T} \subset \mathcal{T}$ for the collection of points on which (4.1) holds and recall in what follows that $(\mu \times \mu) \left(\mathcal{T} \setminus \tilde{T}\right) = 0$. For all $(x, y) \in \tilde{T}$, we have for all $n \in \mathbb{N}$,

$$\hat{T}_n(1) (x) = \sum_{k=1}^{n} \frac{d (\mu \circ T^{-k})}{d\mu} (x) = L(x, y)^{\pm 1} \hat{T}_n(1) (y).$$

Consequently the set

$$\mathcal{C}(T) = \left\{ x \in X : \lim_{n \to \infty} \hat{T}_n(1) (x) = \infty \right\}$$

is $\mathcal{T}$ invariant in the sense that $T (\mathcal{C}(T)) = \mathcal{C}(T)$ modulo $\mu$-null sets. By ergodicity of $\mathcal{T}$ either $\mathcal{C}(T) = X \mod \mu$ or $\mathcal{D}(T) = X \setminus \mathcal{C}(T) = X \mod \mu$, showing part (i).

**Proof of (ii) and (iii).** Let $A$ be a cylinder set and $x, y \in X$ such that $(x, y) \in \mathcal{T}$. Suppose that (4.1) holds and $\lim_{n \to \infty} \hat{T}_n(1_A)(x) = \infty$. In this case for all $n \in \mathbb{N}$,

$$\hat{T}_n(1_A)(x) = \sum_{k=0}^{n-1} \frac{d (\mu \circ T^{-n})}{d\mu}(x) 1_A \circ T^{-k}(x) \leq L(x, y) \sum_{k=0}^{n-1} \frac{d (\mu \circ T^{-n})}{d\mu}(y) 1_A \circ T^{-k}(x)$$

This shows, as the left hand side tends to infinity as $n \to \infty$, that

$$\sum_{k=0}^{n-1} \frac{d (\mu \circ T^{-n})}{d\mu}(y) 1_A \circ T^{-k}(x) \underset{n \to \infty}{\sim} \hat{T}_n(1_A)(y).$$

As $(x, y) \in \mathcal{T}$ and $A$ is a cylinder set, there exists $n_0 = n_0(x, y, A)$ such that if $n > n_0$, then $x \in T^n A$ if and only if $y \in T^n A$. This together with (4.2) imply that as $n \to \infty$,

$$\sum_{k=0}^{n-1} \frac{d (\mu \circ T^{-n})}{d\mu}(y) 1_A \circ T^{-k}(x) \sim \sum_{k=0}^{n-1} \sum_{k=0}^{n-1} \frac{d (\mu \circ T^{-n})}{d\mu}(y) 1_A \circ T^{-k}(y) = \hat{T}_n(1_A)(y).$$

We have shown that if $x, y \in X, (x, y) \in \tilde{T}$ and $\lim_{n \to \infty} \hat{T}_n(1_A)(x) = \infty$, then

$$\hat{T}_n(1_A)(x) \lesssim L(x, y) \hat{T}_n(1_A)(y)$$

as $n \to \infty$, thus

$$\lim_{n \to \infty} \hat{T}_n(1_A)(y) = \infty.$$

This implies that the set

$$\tilde{A} = \left\{ x \in X : \lim_{n \to \infty} \hat{T}_n(1_A)(x) = \infty \right\}$$

is $\mathcal{T}$ invariant.

As $T$ is conservative, by the Hurewicz ergodic theorem the set

$$\left\{ x \in X : h(A)(x) > 0 \right\} \subset \tilde{A}$$

is of positive measure. Consequently, by ergodicity of $\mathcal{T}$, $\mu \left( X \setminus \tilde{A} \right) = 0$ proving part (iii). Part (ii) follows from part (iii) and (4.3).

Proof of Theorem 3. Assume in the contrapositive that there exists $B, D \in \mathcal{B}$ of positive $\mu$ measure such that for all $x \in D$,

$$\sum_{n=0}^{\infty} \hat{T}_n(1_B)(x) < \infty.$$
By the ratio ergodic theorem, there exists \( \epsilon > 0 \) for which the set

\[
C = \left\{ x \in X : \lim_{n \to \infty} \frac{\hat{T}_n(1_B)(x)}{\hat{T}_n(1)(x)} = h(B) > 2\epsilon \right\}
\]

satisfies \( \mu(C) > 0 \). Secondly, there exists \( A_n \in \mathcal{A} \) such that

\[
\|1_{A_n} - 1_B\|_1 = \mu(A_n \triangle B) \leq \frac{1}{n^2}.
\]

By the maximal inequality,

\[
\mu\left(x \in X : \sup_{n \in \mathbb{N}} \left| \frac{\hat{T}_n(1_B - 1_{A_k})(x)}{\hat{T}_n(1)(x)} \right| > \epsilon \right) \leq \frac{1}{n^2}\epsilon.
\]

As the right hand side is summable, it follows from the Borel-Cantelli Lemma that the set

\[
A = \left\{ x \in X : \exists K \in \mathbb{N}, \forall k > K, \sup_{n \in \mathbb{N}} \left| \frac{\hat{T}_n(1_B - 1_{A_k})(x)}{\hat{T}_n(1)(x)} \right| < \epsilon \right\}
\]

is of full \( \mu \)-measure. Consequently, the set \( E = \bigcup_{K \in \mathbb{N}} E_K \)

\[
E_K = \left\{ x \in X : \forall k > K, \liminf_{n \to \infty} \frac{\hat{T}_n(1_{A_k})(x)}{\hat{T}_n(1)(x)} > \epsilon \right\},
\]

satisfies

\[
C \cap A \subset C \cap E,
\]

whence

\[
\mu(E) \geq \mu(C) > 0.
\]

To see the set inclusion, notice that for all \( x \in C \),

\[
\lim_{n \to \infty} \frac{\hat{T}_n(1_B)(x)}{\hat{T}_n(1)(x)} > 2\epsilon.
\]

Now, if \( x \in A \cap C \), there exists \( K \) such that for all \( k > K \),

\[
\sup_{n \in \mathbb{N}} \left| \frac{\hat{T}_n(1_B - 1_{A_k})(x)}{\hat{T}_n(1)(x)} \right| < \epsilon.
\]

Thus, for all \( k > K \),

\[
\liminf_{n \to \infty} \frac{\hat{T}_n(1_{A_k})(x)}{\hat{T}_n(1)(x)} \geq \lim_{n \to \infty} \frac{\hat{T}_n(1_B)(x)}{\hat{T}_n(1)(x)} - \sup_{n \in \mathbb{N}} \left| \frac{\hat{T}_n(1_B - 1_{A_k})(x)}{\hat{T}_n(1)(x)} \right| > \epsilon.
\]

and therefore \( x \in E \). As \( \mu(E) > 0 \) and for all \( K \in \mathbb{N}, E_K \subset E_{K+1} \), it follows that for all large \( K \), \( \mu(E_K) > 0 \). By ergodicity of \( T \), for all large \( K \),

\[
(4.4) \quad \mu(T(E_K) \cap D) = \mu(D).
\]

From now on we assume that \( K \) is large enough so that (4.3) holds. For almost every \( y \in D \), there exists \( x \in E \) with \( x \sim y \). As \( A_k \) is a finite union of cylinder sets, by Proposition 16 part (ii), for all \( n \in \mathbb{N} \) and \( k \in \mathbb{N} \), for almost every \( y \in X \),

\[
\frac{\hat{T}_n(1_{A_k})(y)}{\hat{T}_n(1)(y)} \geq \left( \frac{1}{L(x, y)} \right)^2 \frac{\hat{T}_n(1_{A_k})(x)}{\hat{T}_n(1)(x)}.
\]
thus
\[
\liminf_{n \to \infty} \frac{\hat{T}_n(1_A_k)(y)}{T_n(1)(y)} \geq \left(\frac{1}{L(x, y)}\right)^2 \liminf_{n \to \infty} \frac{\hat{T}_n(1_A_k)(x)}{T_n(1)(x)}.
\]

By the definition of \(E\), we have shown that for almost every \(y \in D\), there exists \(L(y) \in \mathbb{N}\) such that for all \(k > K\),
\[
\liminf_{n \to \infty} \frac{\hat{T}_n(1_A_k)(y)}{T_n(1)(y)} \geq \frac{\epsilon}{L(y)}.
\]

By this, there exists \(D' \subset D\) with \(\mu(D') > 0\) and \(L, K \in \mathbb{N}\), such that for all \(y \in D'\) and \(k > K\),
\[
\liminf_{n \to \infty} \frac{\hat{T}_n(1_A_k)(y)}{T_n(1)(y)} > \frac{\epsilon}{L}.
\]

For \(y \in D'\), for all \(n \in \mathbb{N}\) and \(k > K\),
\[
\frac{\epsilon}{L} \hat{T}_n(1)(y) - |\hat{T}_n(1)_{B - 1_A_k}|(y) \leq \hat{T}_n(1_A_k)(y) - |\hat{T}_n(1)_{B - 1_A_k}|(y) \leq \hat{T}_n(1_B)(y) \leq \sum_{j=0}^{\infty} \hat{T}_j(1_B)(y) < \infty, \text{ as } y \in D.
\]

This shows that for all \(y \in D'\) and \(k > K\)
\[
\liminf_{n \to \infty} \frac{|\hat{T}_n(1)_{B - 1_A_k}|(y)}{T_n(1)(y)} \geq \frac{\epsilon}{L}.
\]

This is a contradiction to \(\mu(D') > 0\), since for all \(k > K\)
\[
\mu(D') \leq \mu \left( x \in X : \liminf_{n \to \infty} \frac{|\hat{T}_n(1)_{B - 1_A_k}|(y)}{T_n(1)(y)} \geq \frac{\epsilon}{L} \right) \leq \mu \left( x \in X : \sup_{n \in \mathbb{N}} \frac{|\hat{T}_n(1)_{B - 1_A_k}|(y)}{T_n(1)(y)} \geq \frac{\epsilon}{L} \right) \leq \frac{L}{\epsilon k^2} \to 0.
\]

The result follows.

\[\square\]

5. Examples

5.1. The model of non-singular Bernoulli shifts. Let \(N \in \mathbb{N}\), \(X = \{1, \ldots, N\}^\mathbb{Z}\) and \(\mathcal{B} = \mathcal{B}_X\) the Borel \(\sigma\)-algebra of \(X\) which is generated by the collection of cylinder sets
\[
\{[b]^k_l : k, l \in \mathbb{Z}, b \in X\}.
\]

Given a sequence \((\mu_k)_{k \in \mathbb{Z}}\) of probability measures on \(\{1, \ldots, N\}\), the product measure \(\mu = \prod_{k \in \mathbb{Z}} \mu_k\) is the measure on \(X\) defined by
\[
\mu([b]^k_l) = \prod_{j=l}^{k} \mu_j(b_j).
\]

In other words, \(\mu\) is the distribution of an independent sequence of random variables \((X_n)_{n \in \mathbb{Z}}\), where for all \(n\), \(X_n\) is distributed according to \(\mu_n\). A non-singular Bernoulli shift on \(N\) symbols is the quadruple
(X, B, μ, T) where T is the shift map on X, and μ ∘ T is equivalent to μ (i.e. the shift is a nonsingular transformation).

As μ and μ ∘ T are product measures, the following is a direct consequence of Kakutani’s theorem on equivalence of product measures. From now on, when the measure μ is fixed, we denote by $(T^n)' = \frac{d(μ ∘ T^n)}{dμ}$.

**Proposition 14.** \(\{1, ..., N\}^Z, B, μ = \prod_{k=-\infty}^{\infty} \mu_k, T\) is non-singular if and only if (5.1) holds. In that case there exists \(X' \subset \{1, ..., N\}^Z\) with \(μ (X') = 1\) such that for all \(x \in X'\),

\[
(T^n)'(x) = \prod_{k=-\infty}^{\infty} \frac{P_{k-n}(x_k)}{P_k(x_k)}.
\]

One should note that if for some \(k \in \mathbb{Z}\) there exists \(j \in \{1, ..., N\}\) with \(\mu_k (\{j\}) = 1\), then the shift is non-singular if and only if \(μ = \prod_{k \in \mathbb{Z}} δ(j)\), which is supported on a single point in \(X\). As this case is not interesting, we always assume that for all \(k \in \mathbb{Z}\),

\[
(5.1) \quad M_k = \max_{j \in \{1, ..., N\}} \mu_k (\{j\}) < 1.
\]

In addition, by a similar argument, if \(μ ∘ T \sim μ\) and for some \(k \in \mathbb{Z}\), there exists \(j \in \{1, ..., N\}\) with \(\mu_k (\{j\}) = 0\) then for all \(m \in \mathbb{Z}\),

\[
μ_m (\{j\}) = 0
\]

and we can reduce \(\{1, ..., N\}^Z, B, μ = \prod_{k=-\infty}^{\infty} \mu_k, T\) to \(\{1, ..., N'\}^Z, B, μ' = \prod_{k=-\infty}^{\infty} μ'_k, T\) with \(N' < N\). As our statement holds for all \(N \in \mathbb{N}\), we will henceforth assume that for all \(k \in \mathbb{Z}\),

\[
(5.2) \quad m_k = \min_{j \in \{1, ..., N\}} \mu_k (\{j\}) > 0.
\]

We say that \(x, y \in X\) are double tail equivalent, denoted by \(x \sim y\), if there exists \(N = N(x, y)\) such that for all \(|n| > N\),

\[
x_n = y_n.
\]

**Lemma 15.** Let \(\{1, ..., N\}^Z, B, μ = \prod_{k=-\infty}^{\infty} \mu_k, T\) be a non-singular Bernoulli shift satisfying (5.1) and (5.2). There exists \(X' \in B\) with \(μ (X') = 1\) such that if \(x, y \in X'\) and \(x \sim y\), then for all \(n \in \mathbb{Z}\)

\[
\prod_{k=-N(x,y)}^{N(x,y)} \left( \frac{m_k m_{k-n}}{M_k M_{k-n}} \right) \leq \frac{(T^n)'(x)}{(T^n)'(y)} \leq \prod_{k=-N(x,y)}^{N(x,y)} \left( \frac{M_k M_{k-n}}{m_k m_{k-n}} \right).
\]

In particular, if condition (5.2) holds then for all \(x, y \in X'\) with \(x \sim y\)

\[
L^{-4N(x,y)} \leq \inf_{n \in \mathbb{Z}} \frac{(T^n)'(x)}{(T^n)'(y)} \leq \sup_{n \in \mathbb{Z}} \frac{(T^n)'(x)}{(T^n)'(y)} \leq L^{4N(x,y)}.
\]

The following is a double tail \(\{0, 1\}\)-law. It is certainly not new, a proof is presented here for the sake of completeness.

**Lemma 16.** The double tail relation of a conservative, non-singular Bernoulli shift \(\{1, ..., N\}^Z, B, μ = \prod_{k=-\infty}^{\infty} \mu_k, T\) is ergodic.

**Proof.** It is enough to show that for every set \(B \in B\) with \(μ(B) > 0\), the set

\[
T(B) = \{y \in X : \exists x \in B, (x, y) \in T\}
\]
is of full $\mu$ measure. To see this, let $\epsilon > 0$. As the collection of cylinder sets is dense in $\mathcal{B}$, there exists $C = [c]^m$ such that
\[ \mu (C \cap B) \geq (1 - \epsilon)\mu (C). \]
For $x \in C \cap B$ and $[z]^m = Z$ another cylinder set, the point $y = y(x, Z) \in X$ with
\[ y_i = \begin{cases} x_i, & i \notin [r, m], \\ z_i, & i \in [r, m] \end{cases} \]
satisfies $(x, y) \in \mathcal{T}$. This shows that for all $z \in X$,
\[ \mu ([z]^m \cap T (B)) \geq (1 - \epsilon)\mu ([z]^m) \]
whence as two different $r, m$ cylinders are disjoint we see that
\[ \mu (T (B)) = \sum_{Z = [z]^m} \mu (Z \cap T (B)) \geq 1 - \epsilon. \]
Since $\epsilon$ is arbitrary, we see that $\mu (T (B)) = 1$. \qed

**Proof of Theorem 3.** Let $\left\{ 1, \ldots, N \right\}^Z, \mathcal{B}, \mu, T \right\}$ be a conservative nonsingular Bernoulli shift which satisfies condition (1.2). By Lemma 16 its double tail is ergodic. By Lemma 15 it satisfies the conditions of Theorem 2 hence it is ergodic. \qed

### 5.2. Inhomogeneous Markov shifts supported on topologically mixing subshifts of finite type.

Let $S$ be a finite space and $A = (A(s, t))_{s, t \in S}$ be an $S \times S \{0, 1\}$-valued matrix. The shift invariant set
\[ \Sigma_A = \left\{ x \in S^Z : \forall i \in \mathbb{Z}, A (x_i, x_{i+1}) = 1 \right\} \]
is a subshift of finite type (SFT). It is topologically mixing iff there exists $n \in \mathbb{N}$ such that $A^n$ has all entries positive (i.e. $A$ is primitive).

An $S$ valued inhomogeneous Markov shift consists of a sequence of $S \times S$ stochastic matrices $(P_n)_{n \in \mathbb{Z}}$ and a sequence of probability distributions $(\pi_n)_{n \in \mathbb{Z}}$ regarded as row vectors satisfying for all $j \in \mathbb{Z},$
\[ \pi_j P_j = \pi_{j-1}. \]
With this condition the measure $\mu$ defined on the collection of cylinder sets by
\[ \mu \left( \left[ b \right]_k^l \right) = \pi_k \left( b_k \right) \prod_{j=k}^{l-1} P_j \left( b_j, b_{j+1} \right) \]
has a unique extension to a measure $\mu = \mu \left( (P_n)_{n \in \mathbb{Z}}, (\pi_n)_{n \in \mathbb{Z}} \right)$ on all $\mathcal{B}_S$. Writing $X_i : S^Z \to S$ for the projection to the $i$-th coordinate, the sequences $P_n$ and $\pi_n$ have the following interpretation, which is the standard definition of an inhomogeneous Markov chain:
\[ \pi_n(s) = \mu (X_n = s), \]
\[ P_n(s, t) = \mu (X_{n+1} = t | X_n = s) = \mu (X_{n+1} = t | X_n = s, X_{n-1}, \ldots). \]

In this section we assume that the measure $\mu$ is fully supported on $\Sigma_A$, in the sense that for all $n \in \mathbb{Z},$
\[ \text{supp} (P_n) = \{ (s, t) : P_n(s, t) > 0 \} = \{ (s, t) : A(s, t) = 1 \} = \text{supp} (A). \]
Theorem 17. Let $\Sigma_A \subset S^\mathbb{Z}$ be a topologically mixing Markov shift. Assume that $\mu = \mu \left((P_n)_{n \in \mathbb{Z}}, (\pi_n)_{n \in \mathbb{Z}}\right)$ is fully supported on $\Sigma_A$ and

$$\sup_{n \in \mathbb{Z}} \sup_{s \in S} \left( \frac{P_n(s, t)}{P_n(s, t')} : t, t' \in S, P_n(s, t) > 0 \right) = L < \infty.$$  

If the shift $(\Sigma_A, \mathcal{B}_{\Sigma_A}, \mu, T)$ is nonsingular and conservative, then it is ergodic.

A criterion for non-singularity of the shift can be obtained in the following way using the method of [8]; full details and proofs of these statements are also in [14]. Write

$$\mathcal{F}_n = \{ [b]_{-n}^n : b \in \Sigma_A \}$$

for the collection of symmetric cylinder sets and for a measure $\nu$ on $\mathcal{B}_{\Sigma_A}$, write $\nu_n$ for the measure $\nu$ restricted to $\mathcal{F}_n$. For $\mu = \mu \left( (P_n), (\pi_n) \right)$ an inhomogeneous Markov measure, $\mu \circ T$ is the Markov measure with transition matrices $Q_n = P_{n-1}$ and $\bar{\pi}_n = \bar{\pi}_{n-1}$. A necessary condition for $T$ to be nonsingular is that $(\mu \circ T)_n$ and $\mu_n$ are absolutely continuous for all $n \in \mathbb{N}$, which is referred to in [14] as local absolute continuity. This amounts to the condition that for all $n \in \mathbb{Z}$,

$$\mu \left( [b]_{-n}^n \right) > 0 \iff \mu \circ T \left( [b]_{-n}^n \right) > 0.$$ 

In that case one defines

$$Z_n(x) = \frac{d(\mu \circ T)_n}{d\mu_n}(x) = \frac{\pi_{n-1}(x_{-n})}{\pi_n(x_n)} \cdot \prod_{j=-n}^{n-1} \frac{P_{j-1}(x_{j}, x_{j+1})}{P_j(x_{j}, x_{j+1})}.$$ 

The sequence $\{Z_n\}_{n=1}^\infty$ is a martingale with respect to the filtration $\mathcal{F}_n$ and $\mathcal{F}_n \uparrow \mathcal{B}_{\Sigma_A}$. Thus $Z_n$ converges almost surely to a $[0, \infty]$-valued random variable. It then follows that $\mu$ and $\mu \circ T$ are equivalent measures if and only if $Z_n$ converges in $L^1$, which is equivalent to uniform integrability of $\{Z_n\}_{n=1}^\infty$. For a streamlined discussion of necessary and sufficient conditions see [14]. We will only make use of the form of the Radon-Nykodym derivatives which is summarized in the following lemma.

Lemma 18. Let $\mu = \mu \left( (P_n)_{n \in \mathbb{Z}}, (\pi_n)_{n \in \mathbb{Z}} \right)$ be an inhomogeneous Markov chain with state space $S$. If $\mu \circ T \sim \mu$, then there exists $X' \subset S^\mathbb{Z}$ with $\mu(X') = 1$ such that for all $x \in X'$ and $N \in \mathbb{Z}$,

$$\frac{d(\mu \circ T^N)}{d\mu}(x) = \lim_{n \to \infty} \left( \frac{\pi_{n-N}(x_{-n})}{\pi_n(x_n)} \cdot \prod_{j=-n}^{n-1} \frac{P_{j-N}(x_{j}, x_{j+1})}{P_j(x_{j}, x_{j+1})} \right).$$ 

Proposition 19. Under the assumptions of Theorem 17 there exists $L(x, y) : \mathcal{T} \to (0, \infty)$ such that for all $(x, y) \in (X' \times X') \cap \mathcal{T},$

$$L(x, y)^{-1} \frac{d\mu \circ T^N}{d\mu}(y) \leq \frac{d\mu \circ T^N}{d\mu}(x) \leq L(x, y) \frac{d\mu \circ T^N}{d\mu}(y).$$ 

Proof. For two $S \times S$ matrices $A, B$, we write $A \leq B$ if for all $(s, t) \in S \times S$,

$$A(s, t) \leq B(s, t).$$

Firstly, it follows from (5.3) that if $P_n(s, t) > 0$, then

$$P_n(s, t) \geq \frac{1}{L^{|S|}}.$$
Indeed, there are at most $|S|$ elements $t \in S$ such that $P_n(s, t) > 0$. Organizing them as an increasing sequence, we get that for all $s \in S$,

$$\max \left( \frac{P_n(s, t)}{\min(P_n(s, t) : t \in S, P_n(s, t) > 0)} \right) \leq L^{|S|}.$$ 

This implies (5.4). Secondly, as $\mu$ is fully supported, this implies that for all $m \leq n$,

$$P^{(m,n)} := P_m P_{m+1} \cdots P_n \geq L^{-|S(n-m+1)|} A^{n-m+1}.$$ 

Thirdly, as $\mu$ is fully supported and $P_n$ are stochastic matrices, it follows that for all $n \in \mathbb{Z}$ and $s, t \in S$,

$$P_n(s, t) \leq A(s, t)$$

and thus for all $m \leq n$,

$$P^{(m,n)} \leq A^{n-m+1}.$$ 

Finally, let $(x, y) \in (X' \times X') \cap \mathcal{T}$, $N \in \mathbb{Z}$ and $(n(x, y)) \in \mathbb{N}$ such that for all $K \in \mathbb{Z}$ with $|K| > n(x, y)$,

$$x_K = y_K.$$ 

Then, for all $K$ such that $K - N > n(x, y)$,

$$\frac{\pi_{-K-N}(x_K)}{\pi_{-K}(x_K)} \prod_{j=-K}^{K-1} \frac{P_{j-N}(x_j, x_{j+1})}{P_j(x_j, x_{j+1})} = \left( \frac{\pi_{-K-N}(y_K)}{\pi_{-K}(y_K)} \prod_{j=-K}^{K-1} \frac{P_{j-N}(y_j, y_{j+1})}{P_j(y_j, y_{j+1})} \right) \cdot I(x, y),$$

where

$$I(x, y) = \prod_{j=-n(x, y)}^{n(x, y)} \left( \frac{P_{j-N}(x_j, x_{j+1})}{P_{j-N}(y_j, y_{j+1})} \frac{P_j(y_j, y_{j+1})}{P_j(x_j, x_{j+1})} \right).$$

As all elements in the product in $I(x, y)$ are strictly positive (since $x, y \in \Sigma_A$), it follows from (5.5) and (5.6) that

$$I(x, y) \leq L^{|S|n(x, y)} =: L(x, y).$$

We have shown that

$$\frac{\pi_{-K-N}(x_K)}{\pi_{-K}(x_K)} \prod_{j=-K}^{K-1} \frac{P_{j-N}(x_j, x_{j+1})}{P_j(x_j, x_{j+1})} = \left( \frac{\pi_{-K-N}(y_K)}{\pi_{-K}(y_K)} \prod_{j=-K}^{K-1} \frac{P_{j-N}(y_j, y_{j+1})}{P_j(y_j, y_{j+1})} \right) \cdot L(x, y).$$

Taking the limit as $K \to \infty$, we see that

$$\frac{d(\mu \circ T^N)}{d\mu}(x) \leq L(x, y) \frac{d(\mu \circ T^N)}{d\mu}(y).$$

The proof is complete as the roles of $x$ and $y$ are symmetric (thus the lower bound).

In order to prove Theorem [7], it remains to show that the double tail relation $\mathcal{T} \subset \Sigma_A \times \Sigma_A$ is trivial. This is the following proposition.

**Theorem 20.** Under the assumptions of Theorem [7], the double tail relation $\mathcal{T} \subset \Sigma_A \times \Sigma_A$ is trivial.

**Proof.** Let $N \in \mathbb{N}$ such that $A^N > 0$. Since for all $n \in \mathbb{Z}$, if $P_n(s, t) > 0$ then

$$P_n(s, t) > L^{-|S|}.$$
we see that for any path \( s = s_0, s_1, ..., s_N = t \) such that
\[
\prod_{i=0}^{N-1} P_{n+i} (s_i, s_{i+1}) > 0
\]
we have
\[
\prod_{i=0}^{N-1} P_{n+i} (s_i, s_{i+1}) \geq L^{-|S|^N}.
\]

As \( A^N > 0 \) and \( \mu \) is fully supported on \( \Sigma_A \) it follows that for all \( s, t \in S \) there exists a path \( s = s_0, s_1, ..., s_N = t \) such that
\[
\prod_{i=0}^{N-1} P_{n+i} (s_i, s_{i+1}) \geq L^{-|S|^N}.
\]

Let \( B = [b]_{-n}^n \) and \( C = [c]_{-n}^n \) be arbitrary symmetric \( n \)-cylinders of positive \( \mu \) measure. We claim that
\[
(5.7) \quad \mu(\mathcal{T}(B) \cap C) \geq |S|^{-1} L^{-2|S|^N} \mu(C).
\]

In order to prove this, take \( s \in S \) such that
\[
\pi_{-n-N} (s) \geq |S|^{-1}.
\]

Such states exist as \( \pi_n, \pi_\cdot \) are probability distributions on \( S \). By the first part, there exist a path \( s_{-n-N} = s, s_{-n-(N-1)}, ..., s_{-n} = b_{-n} \) such that
\[
\prod_{i=-n+N}^{-n-1} P_i (s_i, s_{i+1}) > L^{-|S|^N}
\]
Similarly, there exists a path \( s_n = b_n, ..., s_{n+N} = s \) such that
\[
\prod_{i=n}^{n+N-1} P_i (s_i, s_{i+1}) > L^{-|S|^N}.
\]

Defining a symmetric \( n + N \) cylinder \( B' = [b']_{-n+N}^n \) via
\[
b' = \begin{cases} 
  b_i & i \in [-n, n] \\
  s_i & i \in [-n - N, n + N] \setminus [-n, n],
\end{cases}
\]
it follows that \( B' \subset B \) and
\[
\frac{\mu(B')}{\mu(B)} = \frac{\pi_{-n-N} (s)}{\pi_{-n} (b_n)} \left( \prod_{i=-n+N}^{-n-1} P_i (s_i, s_{i+1}) \right) \left( \prod_{i=n}^{n+N-1} P_i (s_i, s_{i+1}) \right) \geq |S|^{-1} L^{-|S|^N}.
\]

An identical argument constructs a symmetric \( n + N \) cylinder \( C' = [c']_{-n+N}^n \) with \( C' \subset C \),
\[
c_{-n-N}, c_{n+N} = s
\]
and
\[
\frac{\mu(C')}{\mu(C)} \geq |S|^{-1} L^{-|S|^N}.
\]
For every \( x \in B' \), define \( R(x) \in C' \) by
\[
R(x)_i = \begin{cases} 
  x_i, & i \notin [-n - N, n + N] \\
  c_i, & i \in [-n - N, n + N].
\end{cases}
\]
The map \( R : B' \to C' \) is bijective and for all \( x \in B' \), \( (x, R(x)) \in T \). Thus
\[
\mu (T(B) \cap C) \geq \mu (C') \geq |S|^{-1} L^{-|S|N} \mu (C),
\]
proving \( (5.7) \). Another feature of \( R \) is that for all \( x \in B' \),
\[
\frac{d\mu \circ R}{d\mu}(x) = \frac{\mu (C')}{\mu (B')}.
\]
As a consequence, if \( A \subset B' \), then writing \( \epsilon = |S|^{-1} L^{-|S|N} \),
\[
\mu (T(A) \cap C) \geq \mu (R(A))
\]
\( (5.8) \)
\[
= \frac{\mu (A)}{\mu (B')} \mu (C') \geq \frac{\mu (A)}{\mu (B)} \mu (C).
\]
Now let \( D \in \mathcal{B}_{\Sigma_A} \) be a tail invariant set. If \( \mu (D) > 0 \) and \( \mu (\Sigma_A \setminus D) > 0 \), then there exists \( n \in \mathbb{N} \) and two cylinder sets \( B = [b]_{-n}^n, C = [c]_{-n}^n \) such that
\[
\mu (D \cap B) \geq \left( 1 - \frac{\epsilon}{2} \right) \mu (B) \quad \text{and} \quad \mu (D \cap C) < \frac{\epsilon^2}{4} \mu (C).
\]
As
\[
\mu (B') \geq \epsilon \mu (B),
\]
it then follows that
\[
\mu (D \cap B') \geq \frac{\epsilon}{2} \mu (B).
\]
Consequently, by \( (5.8) \),
\[
\mu (D \cap C) = \mu (T(D) \cap C)
\]
\[
\geq \mu (T(D \cap B') \cap C)
\]
\[
\geq \epsilon \frac{\mu (D \cap B')}{\mu (B)} \mu (C) \geq \frac{\epsilon^2}{2} \mu (C).
\]
This is a contradiction, hence for every \( D \in \mathcal{B}_{\Sigma_A} \) which is \( T \)-invariant either \( \mu (D) = 0 \) or \( \mu (\Sigma_A \setminus D) = 0 \).

Proof of Theorem 17. By Proposition 19 and Theorem 20, the shift \( (\Sigma_A, \mathcal{B}_{\Sigma_A}, \mu, T) \) satisfies the conditions of Theorem 2 hence if it is conservative, then it is ergodic.

5.2.1. Relation to a closed relative of question 97 from Rufus Bowen’s notebook. Rufus Bowen has asked the following question.

Problem 21. Is there a nonergodic volume preserving \( C^1 \) Anosov diffeomorphism of \( \mathbb{T}^2 \)?

The following variant of this problem is still open.

Problem 22. Is there a nonergodic, conservative \( C^1 \) Anosov diffeomorphism of \( \mathbb{T}^2 \)?
We say that $G$ is ergodic. Therefore, the shift is nonsingular if and only if for all $(5.9)$

By Kakutani’s dichotomy [14, P. 528, Thm 3], examples of Hurewicz RET groups are $\mathbb{Z}$ (Hurewicz’s theorem), $\mathbb{Z}^d$ [5] with $F_n = [-n, n]^d$ and discrete Heisenberg groups $H^d(\mathbb{Z})$ [7]. Hochman has shown a connection between the Hurewicz property for amenable groups and existence of Følner sequences which satisfy the Besicovitch covering property. See [6] for these definitions and precise statements.

5.3 Bernoulli shifts on groups with the ratio ergodic theorem property. A countable group $G$ satisfies the Ratio-Ergodic-Theorem (RET) property if there exists an increasing sequence of finite subsets $F_n \subset G, \bigcup_n F_n = G$ such that for any non-singular, conservative $G$-action $G \acts (X, \mathcal{B}, \mu)$ with $\mu(X) = 1$, for all $f \in L^1(X, \mathcal{B}, \mu)$ and for $\mu$ almost every $x \in X$,

$$R_n(f, 1)(x) := \sum_{g \in F_n} \frac{d\mu T_g}{d\mu}(x)f \circ T_g(x) \sum_{g \in F_n} \frac{d\mu T_g}{d\mu}(x) \xrightarrow{n \to \infty} h(f, 1)(x)$$

where $h = h(f, 1)$ satisfies

- If $f \geq 0$, then $h \geq 0$.
- For all $g \in G$, $h \circ T_g = h$.
- $\int_X hkd\mu = \int_X k1_Ad\mu$ for all $k \in L^\infty(X, \mu)$ satisfying for all $g \in G, k \circ T_g = k$.

We say that $G$ satisfies the Hurewicz Maximal inequality property, henceforth abbreviated as $G$ is a Hurewicz group, if in addition there exists $C > 0$ such that for all $f \in L^1(X, \mathcal{B}, \mu)$ and $\epsilon > 0$,

$$\mu\left(\sup_{n \in \mathbb{N}} |R_n(f, 1)| > \epsilon \right) \leq C\frac{|f|_1}{\epsilon}.$$  

Examples of Hurewicz RET groups are $\mathbb{Z}$ (Hurewicz’s theorem), $\mathbb{Z}^d$ [5] with $F_n = [-n, n]^d$ and discrete Heisenberg groups $H^d(\mathbb{Z})$ [7]. Hochman has shown a connection between the Hurewicz property for amenable groups and existence of Følner sequences which satisfy the Besicovitch covering property. See [6] for these definitions and precise statements.

Given a countable group $G$ and $N \in \mathbb{N}$, the Bernoulli action of $G$ on $\{1, \ldots, N\}^G$ is defined by

$$(T_g(x))_h = x_{g^{-1}h}.$$  

By Kakutani’s dichotomy [14] P. 528, Thm 3], $h \circ T_g$ and $\mu$ are equivalent if and only if

$$\sum_{h \in G} \sum_{j=1}^N \left(\sqrt{\mu_h(j)} - \sqrt{\mu_{g^{-1}h}(j)}\right)^2 < \infty.$$  

Therefore, the shift is nonsingular if and only if for all $g \in G$, equation $(5.9)$ holds.

**Theorem 23.** Let $G$ be a countable Hurewicz RET group. If a non-singular Bernoulli shift $(\{1, \ldots, N\}^G, \mathcal{B}, \prod_{g \in G} \mu_g, (T_g)_{g \in G})$ is conservative and

$$L = \sup_{g \in G} \max_{j \in \{1, \ldots, N\}} \left(\mu_g(\{j\})\right) \min_{j \in \{1, \ldots, N\}} \left(\mu_g(\{j\})\right) < \infty,$$

then it is ergodic.
The proof of this theorem is identical to the proof of Theorem 3 once one replaces the relation $\mathcal{T}$ with the $(F_n)_n$ homoclinic relation

$$\text{HOM} = \{(x, y) \in \{1, \ldots, N\}^G : \exists n \in \mathbb{N}, x|_{G \setminus F_n} = y|_{G \setminus F_n}\}.$$ 

Here $F_n$ is the sequence from the definition of Hurewicz-RET group. The use of the Ratio and Maximal ergodic theorems is similar.

**Problem.** Is Theorem 23 true for a general countable amenable group $G$?

This problem is interesting as there are currently very few groups which are known to be RET and Hurewicz. Hochman has shown that there are abelian groups (hence amenable) which are not RET.

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