Initial data for gravity
coupled to scalar, electromagnetic and Yang-Mills fields

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Abstract

We give ansatze for solving classically the initial value constraints of general relativity minimally coupled to a scalar field, electromagnetism or Yang-Mills theory. The results include both time-symmetric and asymmetric data. The time-asymmetric examples are used to test Penrose’s cosmic censorship inequality. We find that the inequality can be violated if only the weak energy condition holds.

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The problem of finding solutions of the initial value constraints ("initial data") of general relativity in vacuum, or coupled to matter, is of interest from both mathematical and physical viewpoints \cite{1}. The mathematical questions concern existence and uniqueness of solutions for various classes of initial data and topology of Cauchy surfaces. The more physical questions concern cosmic censorship, and numerical evolution of physically relevant initial data.

There has been much work on construction of initial data sets using analytical and numerical techniques, as well as combinations of both. Analytical methods initiated by Lichnerowitz, and further developed by York, are now standard material \cite{2}. Combinations of analytic and numerical techniques continue to be investigated, particularly for problems of physical interest.

In vacuum, perhaps the simplest initial data is the spherically symmetric "throat" connecting two asymptotically flat regions, which has a generalization to \(N\) throats with masses and charges \cite{3}. There is also a wormhole solution due to Misner \cite{4}, where the wormhole is a handle on flat space. Other multi-black hole solutions, having time and inversion symmetry, have also been derived \cite{4}, with subsequent generalization by Bowen and York \cite{6}. In more recent work, a variation on the theme of analytic-numerical data for \(N\) black holes with arbitrary momenta and spin numerically is given in \cite{7}.

Most of the analytic solutions of the initial data problem are in vacuum. Analytic solutions with specific matter coupling are less studied. Such solutions are potentially useful, both for their intrinsic value, and for providing starting points for numerical time evolution schemes. As an example of the latter, analytic initial data for scalar field coupling may be useful for further study of the spherically symmetric collapse problem \cite{8}, which has been a subject of recent interest. This collapse problem has also been studied for matter couplings other than the scalar field \cite{9}, but always with numerically generated initial data.

A second reason for seeking analytic initial data sets is their potential usefulness for probing the cosmic censorship conjecture. One statement of this conjecture is that trapped surfaces in a spacetime always lie inside event horizons. This has led Penrose to suggest an "initial data test" for weak cosmic censorship \cite{10}. The test is an inequality relating the area of the outermost trapped surface \(A(S)\), the area of the event horizon \(A_{EH}\), and the ADM mass \(M\) of asymptotically flat initial data, all for matter satisfying reasonable energy conditions. The inequality is

\[
A(S) \leq A_0 \leq A_{EH} \leq 16\pi M^2, \tag{1}
\]

where \(A_0\) is the area of a surface that encloses the outermost marginally trapped surface \(S\). The first and second inequalities are reasonable from a physical viewpoint because, as matter collapses the region containing trapped surfaces gets larger, and in the long time static (or stationary) limit \(A(S) \to A_{EH}\). The third inequality allows the possibility that all the mass does not end up in the black hole. The content of the cosmic censorship test is this: If initial data exists such that the area of the outermost trapped surface is larger than \(16\pi\) times the square of the ADM mass of the data, then weak cosmic censorship is violated.

Trapped surfaces may be present on an initial data surface for certain ranges of the initial data parameters. The boundary separating trapped and untrapped regions on the initial
data surface is the apparent horizon. On this horizon the outward expansion of light rays vanishes. In terms of initial data, a closed spatial 2-surface $S$ with normal $s^a$ is outer trapped if
\[(q^{ab} - s^a s^b)(K_{ab} + D_a s_b) = 0,\] (2)
where $K_{ab}$ is the extrinsic curvature of the initial data surface, and $q_{ab}$ is its metric. This equation expresses the vanishing on $S$ of the outward null expansion.

Eqn. (1) may be viewed as providing a relation between local and global information in the sense that the apparent horizon is a solution of a differential equation involving the (local) phase space variables, whereas the ADM mass is an integral of a phase space function over the boundary of the initial data surface. While it is true that the apparent horizon is also an embedded 2-surface (and hence ‘global’), its shape and size are subject to change due to local matter flows.

There have been a number of successful tests of the inequality (1). The first were for null shells with flat interior, in certain special cases [10,12]. However, it has been shown recently that the inequality holds generally for null shells with flat interior [13], (see also [14]). For arbitrary matter satisfying the dominant energy condition, it was proven by Jang and Wald [15] that time-symmetric initial data (zero extrinsic curvature) satisfies this inequality. The status of the inequality for the time-asymmetric cases is in general open (with the exception of null shells with flat interior mentioned above [16,17]).

With the above motivation, it is the purpose of this paper to present classes of analytic initial data sets for certain matter couplings, discuss some of their properties, and use them as probes of Penrose’s inequality.

The matter couplings discussed are to the massless scalar field, electromagnetism, or Yang-Mills theory. It is based on the conformally flat ansatz for the spatial metric [1]. The next section gives solutions for a class of time-symmetric cases, followed in Section III by time-asymmetric data for scalar field coupling. This section also contains results pertaining to Penrose’s inequality for time-asymmetric data: the inequality is satisfied for all the examples considered, with the exception of cases where only the weak energy condition holds.

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1. There is also an alternative terminology in the literature: only the outermost marginally trapped surface is called the apparent horizon. Here we refer to all marginally trapped surfaces as apparent horizons, as in Ref. [11].

2. In terms of the gravitational momentum $\tilde{\pi}^{ab} = \sqrt{q}(K^{ab} - K q^{ab})$, where $K = K^{ab} q_{ab}$, this equation is $\tilde{\pi}^{ab} s_a s_b = \sqrt{\pi} D_a s^a$.

3. The vanishing of the inward null expansion is obtained by changing the sign of $s^a$ in (2). This corresponds to a white hole situation.
Consider Einstein gravity minimally coupled to a massless scalar field with no self-interaction. In the Hamiltonian formulation, the phase space variables are the canonically conjugate pairs $(\phi, \tilde{P})$ for the scalar field, and $(q_{ab}, \tilde{\pi}^{ab})$ for the gravitational field. The initial value constraints are

$$\mathcal{H} \equiv \frac{1}{\sqrt{q}} G_{abcd} \tilde{\pi}^{cd} - \sqrt{q} (3) R + \frac{1}{\sqrt{q}} \tilde{P}^2 + \sqrt{q} q^{ab} \partial_a \phi \partial_b \phi = 0,$$

$$C^a \equiv \partial_b \tilde{\pi}^{ba} + \tilde{P} q^{ab} \partial_b \phi = 0.$$  

where $G_{abcd} = (g_{ae} g_{bd} + g_{ad} g_{bc} - g_{ab} g_{cd})/2$ is the DeWitt supermetric, $D_a$ is the covariant derivative of $q_{ab}$, and $\tilde{}$ denotes densities of weight one.

Consider the topology of the spatial slice $\Sigma$ to be $\mathbb{R}^3$, with the rectangular coordinates $x, y, z$. The usual time-symmetric conformal ansatz for the vacuum constraints is

$$\tilde{\pi}^{ab} = 0; \quad q_{ab} = \psi^4 \delta_{ab},$$

where $\delta_{ab}$ is the flat Euclidean 3-metric. With this ansatz the diffeomorphism constraint is identically satisfied and the scalar constraint $\mathcal{H}$ becomes the Laplace equation. The simplest one throat (or Schwarzschild) solution of the Hamiltonian constraint is

$$\psi = 1 + \frac{m}{2r},$$

where $m$ is a constant (the ADM mass) and $r^2 = x^2 + y^2 + z^2$. To avoid the singular behavior at $r = 0$ this solution is viewed as being on $\mathbb{R}^3 - \{0\}$. Evolution of this data gives the Schwarzschild solution. This result generalizes to $N$ black holes [3] with masses $m_i$ at the coordinate locations $r_i$ on $\mathbb{R}^3 - \{\text{points excised for each black hole}\}$:

$$\psi = 1 + \sum_{i=1}^{N} \frac{m_i}{2|r - r_i|}.$$  

Consider now the following ansatz for the Einstein-scalar field theory. Again with the spatial slice $\Sigma \approx \mathbb{R}^3$, set

$$\tilde{P} = 0, \quad \tilde{\pi}^{ab} = 0, \quad q_{ab} = \psi^4 \delta_{ab},$$

with arbitrary initial scalar field $\phi(x, y, z, t = 0)$. The Ricci scalar of $q_{ab}$ is

$$(3) R = -8 \nabla^2 \psi / \psi^5.$$  

With this, the Hamiltonian constraint (3) becomes

$$\left( \nabla^2 + \frac{1}{8} \delta^{ab} \partial_a \phi \partial_b \phi \right) \psi(x, y, z) = 0.$$  

This is the Schrödinger equation in three dimensions with potential $V(x, y, z)$ and eigenvalue $E$ related by
\[ \frac{2m}{\hbar^2}[E - V(x, y, z)] = \frac{1}{8} \delta^{ab} \partial_a \phi \partial_b \phi. \] (11)

Since the r.h.s. is positive or zero, the physical solutions of the Hamiltonian constraint must satisfy \( E - V \geq 0 \). This leads to an infinite class of solutions, derived from any \( V \) and \( E \), if the scalar field is allowed to be arbitrary, and possibly singular on the initial data slice. However, since regular initial data is more physically relevant, the “wavefunctions” \( \psi \) of interest, for example for matter collapse, will mainly be those which do not vanish anywhere. This criteria appears to exclude normalizable wavefunctions, where \( \psi \) vanishes somewhere. A further restriction is that the scalar field must be real; there are \( E \) and \( V \) for which this is not the case for all \( r \). These conditions are however not drawbacks to the analogy with the Schrodinger equation, as it is still possible to remove the normalizability boundary condition to get many instances of non-singular initial data sets with real \( \phi \), as we see below.

(For a free massless scalar field, the Hamiltonian constraint(10) can also be converted into the Poisson equation \( \nabla^2 \psi = -V_a V_b \delta^{ab} \), where the vector field \( V_a \) is freely specified; the initial scalar field is determined from any solution \( \psi \) using \( \partial_a \phi = V_a / \sqrt{\psi} \). Also, neither this nor the previous case gives a linear equation for \( \psi \) if the scalar field is massive or has an arbitrary potential term.)

It is straightforward to obtain some simple explicit examples of regular solutions (i.e. nowhere vanishing \( \psi \)), and a physical picture.

(i) Solutions in spherical symmetry: Given a radial wavefunction \( \psi(r) \) with eigenvalue \( E \) for a potential \( V(r) \), the initial scalar field \( \phi(r, t = 0) \) from (11) is

\[
\phi(r, t = 0) = \sqrt{\frac{16m}{\hbar^2}} \int_0^r dr' \sqrt{E - V(r')}.
\] (12)

Thus, the exact initial scalar field which solves the Hamiltonian constraint is proportional to the phase of the WKB wavefunction for any potential \( V(r) \) and energy \( E \), with the restriction that \( E - V(r) \geq 0 \) to get real \( \phi(r, t = 0) \).

A particular solution in spherical symmetry: consider the scalar field pulse defined by

\[
\phi(r, t = 0) = \begin{cases} 
0 & \text{if } 0 \leq r < r_0 \\
\sqrt{2} \ln r & \text{if } r_0 \leq r \leq r_1 \\
0 & \text{if } r > r_1
\end{cases}
\] (13)

where \( r_0, r_1 \) are parameters giving the pulse width. One solution for \( \psi(r) \) is

\[
\psi(r, t = 0) = \begin{cases} 
1 & \text{if } 0 < r < r_0 \\
A/\sqrt{r} + B \ln r / \sqrt{r} & \text{if } r_0 \leq r \leq r_1 \\
C + D/2r & \text{if } r > r_1
\end{cases}
\] (14)

Here we have chosen the 3-metric to be flat in the “inner” region, Schwarzschild like in the “outer” region, and determined by the scalar field shell in the middle. The coefficients \( A, B, C \) and \( D \) are determined by continuity of \( \psi(r) \) and its first derivative at the interfaces \( r = r_0 \) and \( r = r_1 \) to be

\[
A = \sqrt{r_0} \left( 1 - \frac{1}{2} \ln r_0 \right); \quad B = \frac{\sqrt{r_0}}{2}.
\] (15)
\[ C = \frac{1}{\sqrt{\alpha}} \left[ 1 - \frac{1}{4} \ln \alpha \right], \quad D = r_0 \sqrt{\frac{\alpha}{2}} \ln \alpha, \]  

where \( \alpha \equiv r_1/r_0 > 0 \) gives the “width” of the pulse. The ADM mass of this solution \( M = CD \) is a function of the “distance” \( r_0 \) of the pulse from the origin and the number \( \alpha \). It is possible to have a non-flat vacuum in the inner region by setting \( \psi = 1 + m/2r \) in the inner region. The data then has the additional parameter \( m \), and is singular.

In spherical symmetry with vanishing extrinsic curvature, the apparent horizon equation (14) simplifies to \( g^{ab} \partial_a R \partial_b R = 0 \), where \( R = r \psi^2(r) \) is the radial coordinate of the nested spheres and \( g^{ab} \) is the spacetime metric. For the conformally flat ansatz for the 3-metric, this leads to the equation

\[ \psi + 2r \psi' = 0. \]  

Thus, apparent horizons exist on the initial data surface if there are real solutions \( r = r_{AH} \) of this equation, with \( r_{AH} \) lying in the appropriate region(s) for each \( \psi \). For the solution (14), there are no apparent horizons in the region of non-vanishing scalar field because (17) reduces to \( 2B/\sqrt{r} = 0 \).

There are other examples where this is not the case, and apparent horizons are present. One interesting case is a solution which is asymptotically flat without matching to Schwarzschild exterior. Consider

\[ \phi = \frac{\sqrt{3}C}{r}, \]  

where \( C \) is a constant. The corresponding solution of the Hamiltonian constraint is

\[ \psi = A \cos \left( \frac{C}{2r} \right) + B \sin \left( \frac{C}{2r} \right). \]  

This solution is asymptotically flat. For large \( r \), the constant \( A \) is a conformal factor which may be set to unity. Comparing with Schwarzschild shows that the ADM mass \( M \) of this data (with \( A = 1 \)) is \( M = BC \). It has a curvature singularity at \( R = r \psi^2 = 0 \). It is interesting to note that there are an infinite number of zeroes of \( \psi \) and a corresponding number of apparent horizons near these zeroes, even though the scalar field is singular only at \( r = 0 \). Surprisingly, for this data there are regions of parameter space where Penrose’s inequality appears to be violated. One such region is at and near the point \( B = 0.01 \) and \( C = 2.00 \) (with \( A = 1 \)), where \( 2M - R_{AH} = -0.286 \). However, this does not provide a counterexample of cosmic censorship because the conformal factor \( \psi \) goes to zero outside the horizon (18). We note that Penrose’s inequality may still be tested, and is satisfied, for the apparent horizon that lies in the region connected to spatial infinity; ie. as one comes in from infinity, the horizon is encountered before any zero of \( \psi \).

It is possible to get solutions with a finite number of apparent horizons by patching flat space in the inner region, from \( r = 0 \) to some \( r = r_1 \). Set \( r_1 = 1 \), and \( \phi = 0 \) and \( \psi = 1 \) for \( 0 \leq r < 1 \). Then, with \( \phi \) as in (18) for \( r \geq 1 \), the solution is

\[ \psi(r) = A(C) \left[ \cos \left( \frac{C}{2r} \right) + \tan \left( \frac{C}{2} \right) \sin \left( \frac{C}{2r} \right) \right] \quad \text{for } r \geq 1, \]
where

\[ A(C) = \left[ \cos(C/2) + \tan(C/2) \sin(C/2) \right]^{-1}. \]  

(21)

The ADM mass of this one parameter \( (C) \) configuration is

\[ M = A^2(C) \, C \tan \left( \frac{C}{2} \right). \]  

(22)

Now, depending on the value of \( C \), there are zero or a finite number of apparent horizons in the matter region. It is also possible to construct a solution with a finite number of horizons by patching Schwarzschild rather than flat data in the inner region. It is again possible to choose parameters such that a horizon is encountered before a zero of \( \psi \), as \( r \) is varied inward from infinity.

(ii) **Scalar field wormhole:** Away from spherical symmetry, the situation is virtually unchanged. Consider perturbing Misner’s wormhole solution [4] by the presence of a compact scalar field pulse. The mathematical problem is one of matching an ‘interior’ vacuum solution to a pulse solution, which in turn is matched to an ‘exterior’ vacuum solution. (Vacuum solution means a solution of the constraint equations without matter sources.) The former and latter are just Misner’s \( \psi \), with different parameters in each region to account for the mass in the scalar field. The solution in the intermediate region depends on the scalar field pulse, which determines the potential in the Schrödinger equation analogy. The details are a matching problem not unlike the one in spherical symmetry considered above. Evolution of such initial data would be of interest for seeing how presence of matter affects the gravitational radiation from the two black hole problem in the so called “close” limit [19].

**Electromagnetism:** There are similar results for coupling to electromagnetism, with phase space variables \( (A_a, \tilde{E}^a) \) and Hamiltonian density \( (\tilde{E}^a \tilde{E}^b + \tilde{B}^a \tilde{B}^b)q_{ab}/2\sqrt{q}, \) \( (\tilde{B}^a = \epsilon^{abc} \partial_b A_c) \). Using the same conformal ansatz for the metric and conjugate momentum, \( q_{ab} = \psi^4 \delta_{ab} \) and \( \tilde{\pi}^{ab} = 0 \), set

\[ \tilde{E}^a = \psi^2 \tilde{e}^a, \quad \tilde{B}^a = \psi^2 \tilde{b}^a \]  

(23)

where \( \tilde{e}^a(x, y, z) \) and \( \tilde{b}^a(x, y, z) \) are arbitrarily specified. Then the Hamiltonian constraint becomes

\[ \left[ \nabla^2 + \frac{1}{16} (\tilde{e}^a \tilde{e}^b \delta_{ab} + \tilde{b}^a \tilde{b}^b \delta_{ab}) \right] \psi = 0, \]  

(24)

identical in form to the scalar field coupling case [19]. Thus, given the freely specified fields \( \tilde{e}^a \) and \( \tilde{b}^a \), one solves the Hamiltonian constraint for \( \psi \), and then determines the physical electric and magnetic fields via [23]. Note that this ansatz is not the same as the one given by Misner and Wheeler [3], which gives charged black hole data; their metric ansatz is \( g_{ab} = (\psi^2 - \chi^2)^2 \delta_{ab} \), from which the scalar constraint gives Laplace equations for \( \psi \) and \( \chi \).

As for the scalar field case, there is an alternative specification of the electric and magnetic fields which converts the Hamiltonian constraint into the Poisson equation with source given implicitly by these fields. This arises by setting

\[ \tilde{E}^a = \psi^{3/2} \tilde{e}^a, \quad \tilde{B}^a = \psi^{3/2} \tilde{b}^a, \]  

(25)
and gives
\[ \nabla^2 \psi = -\frac{1}{16} (\tilde{e}^a e^b \delta_{ab} + \tilde{b}^a \tilde{b}^b \delta_{ab}) \] (26)
for the Hamiltonian constraint.

Of course, neither (23) or (25) solve the diffeomorphism constraint unless either the electric field or vector potential \( A_a \) is set to zero. Furthermore, there is also the Gauss law for the electric field. One obvious solution to all the initial value constraints is obtained by setting \( \tilde{E}^a = 0 \), and solving the scalar constraint (24) or (26) for purely magnetic initial matter. In this context, one can again consider special cases, such as spherical or toroidal symmetry, or a wormhole.

Purely electric solutions, other than the monopole, are more difficult to find without imposing further symmetries, since it is difficult to simultaneously solve the vacuum Gauss law constraint and obtain a linear equation for the scalar constraint. However, this is possible if symmetries are imposed. As an example consider the case of one translational symmetry and seek solutions \( \psi = \psi(x, y) \). For two arbitrary functions \( u(x, y) \) and \( v(x, y) \), set
\[ \tilde{e}^a = \tilde{e}^{abc} \partial_b u \partial_c v, \] (27)

where \( \tilde{e}^{abc} \) is the metric independent Levi-Civita tensor density. Then the electric field given by (23) satisfies Gauss’s law
\[ \partial_a \tilde{E}^a = 2 \psi^2 \tilde{e}^{abc} \partial_b u \partial_c v = 0 \] (28)
because \( u, v \) and \( \psi \) depend only on two coordinates. The electric field lines are tangent to the curves defined by the intersection of the surfaces \( u = \text{constant} \) and \( v = \text{constant} \). In the present case, the only non-vanishing component of \( E^a \) is \( E^z(x, y) \).

Finally, we point out a class of solutions for which both initial electric and magnetic fields are non-zero. This is again in the context of one translational symmetry. Given arbitrary \( u(x, y) \), \( v(x, y) \) and \( b^a(x, y) \), define \( e^a(x, y) \) according to (27). As before, since the solutions of the Hamiltonian constraint are \( \psi = \psi(x, y) \), Gauss’s law is automatically satisfied. The diffeomorphism constraint
\[ \partial_a \tilde{\pi}^{ab} + \tilde{E}^b (\partial_a A_b - \partial_b A_a) = 0 \] (29)
reduces to
\[ \psi^2 \tilde{e}_{abc} \tilde{E}^b \tilde{B}^c = 0. \] (30)
Now, since \( \tilde{E}^a \) has only a \( z \)-component, restricting \( \tilde{B}^a \) to have only a \( z \)-component solves this constraint. Furthermore, \( \partial_a \tilde{B}^a = \partial_z (\psi^2 (x, y) b^z (x, y)) = 0 \) so there are no magnetic sources.

This procedure gives solutions of all the initial value constraints with electromagnetic coupling; the data is characterized by two arbitrary functions of two coordinates. As there are no electromagnetic sources, this class of solutions may be called “geon” data. This result has extensions to any 3-space with one Killing symmetry.
Yang-Mills theory: The phase space variables are \((A^i_a, \tilde{E}^{ai})\), where \(i = 1 \cdots N^2 - 1\) is the SU(N) Lie algebra index. The Yang-Mills contribution to the Hamiltonian constraint is

\[
H = \frac{1}{2 \sqrt{q}} (\tilde{E}^{ai} \tilde{E}^{bj} + \tilde{B}^{ai} \tilde{B}^{bj}) q_{ab} k_{ij},
\]

where \(k_{ij}\) is the Cartan metric, and \(\tilde{B}^{ai} = \varepsilon^{abc} F^i_{bc}\). \((F^i_{ab} = \partial_a A^i_b - \partial_b A^i_a + C^i_{jk} A^j_a A^k_b\) and \(C^i_{jk}\) are the structure constants of the gauge group). The ansatz that reduces the Hamiltonian constraint to the Schrödinger equation is similar to (23), namely

\[
\tilde{E}^{ai} = \psi^{2e^{ai}} \quad \tilde{B}^{ai} = \psi^{2b^{ai}}.
\]

Thus, given arbitrary \(e^{ai}\) and \(b^{ai}\), finding the conformal factor \(\psi\) is again an elementary problem. However, we must also find solutions of the diffeomorphism constraint and the non-abelian Gauss law

\[
D_a \tilde{E}^{ai} \equiv \partial_a \tilde{E}^{ai} + C^i_{jk} A^j_a \tilde{E}^{ak} = 0.
\]

One class of solutions of all the constraints is obtained by starting with \(A^i_a = 0\). This immediately solves the diffeomorphism constraint (because \(\tilde{\pi}^{ab} = 0\), and the Gauss law becomes \(\partial_a (\psi^2 e^{ai}) = 0\). Now set

\[
\tilde{e}^{ai} = C^i_{jk} \varepsilon^{abc} \partial_b u^j \partial_c v^k
\]

where \(u^i\) and \(v^i\) are \(2(N^2 - 1)\) arbitrary functions. This form does give a “full” solution of the Gauss law for this case (i.e., with \(A^i_a = 0\)). This is because its solutions, without any redundancy, are parametrized by \(2(N^2 - 1)\) functions for SU(N), since there are \(N^2 - 1\) constraints for the \(3(N^2 - 1)\) \(\tilde{E}^{ai}\) fields in three spatial dimensions. With this ansatz Gauss law becomes

\[
\partial_a (\psi^2 \tilde{e}^{ai}) = 2 \psi C^i_{jk} \varepsilon^{abc} \partial_a \psi \partial_b u^j \partial_c v^k.
\]

Then, as for the abelian case, there are solutions to all the constraints if the problem is reduced so that all variables depend on two coordinates.

It is more difficult to find solutions for which both electric and magnetic fields are non-vanishing. This is due to the term quadratic in the phase space variables in the Gauss law. To see this, suppose we give an ansatz specified by an arbitrary gauge field \(a^i_a\) whose magnetic field is \(b^{ai}\), and with \(e^{ai}\) given by

\[
\tilde{e}^{ai}(x) = \sum_a \varepsilon^{abc} \partial_b u^a \partial_c v^a \text{Tr}[U[a_a, \gamma(u^a, v^a)] \tau^i](x),
\]

where \(u^a\) and \(v^a\) are a set of scalars (labelled by indices \(a, b\)), \(\tau^i\) are matrix generators of the group, and

\[
U[a_a, \gamma(u, v)](x) = \text{Pexp} \int_{\gamma} ds A^i_a(\gamma(s)) \tau^i
\]

is the holonomy of the gauge field along the loop \(\gamma(u, v)\), with base point \(x\), determined by the intersection of surfaces \(u^a = \text{constant}\) and \(v^a = \text{constant}\). This is a non-abelian
generalization of (27) [20]. It is straightforward to check that it satisfies the Gauss law for the fields \((e^ai, a^i_a)\). However, because of the conformal factor \(\psi\) in (32), it does not satisfy the Gauss law (33) for the physical fields \((\tilde{E}^ai, \tilde{A}^i_a)\). Indeed, from a solution \(\psi\) of the scalar constraint obtained via (32), we find for (36) that

\[
D_a \tilde{E}^ai = D_a (\psi^2 \tilde{e}^ai) = \psi^2 f^i_{jk} (A^j_a - a^j_a) \tilde{e}^ak + 2 \psi \epsilon^ai \partial_a \psi \neq 0, \tag{38}
\]

where \(A^i_a\) is a gauge field associated with the physical magnetic field \(\tilde{B}^ai = \psi^2 \tilde{b}^ai\).

One way to solve (38) is to assume, at the outset, that \(\epsilon^ai\) and \(\alpha^i_a\) are parallel in the internal direction, i.e. that \(\tilde{e}^ai = \epsilon^ai \partial^i\) and \(\alpha^i_a = a^i_a \partial^i\) for a fixed Lie algebra vector \(t^i\). This means that \([a^i_a, a^i_b] = 0\), and \(\epsilon^ai\) and \(A^i_a(\psi, a^i_a)\) are also parallel in the internal direction. With these conditions on the initial configuration, the first term in (38) vanishes. The second term vanishes if we assume, as before, that all fields depend on only two of the spatial coordinates \((x, y)\), i.e. that there is a translation Killing symmetry. This means that the YM electric field from (36) has only a \(z\)−component.

The diffeomorphism constraint

\[
\epsilon_{abc} \tilde{E}^{bi} B^{cj} k_{ij} = 0 \tag{39}
\]

is satisfied if the YM magnetic field is also restricted to have only a \(z\)− component. Finally, note that \(D_a \tilde{B}^ai = \partial_a \tilde{B}^zi + [A_z, \tilde{B}^z]^a = 0\) because \(B^zi\) is a function only of \(x, y\), and the commutator vanishes by the ansatz used to solve the Gauss law. Thus there are no magnetic sources.

In this way one can obtain solutions to all the constraints for the time symmetric Yang-Mills problem, with the conformal ansatz for the metric. Such solutions are parametrized by functions of two variables \(u^a\) and \(v^a\). Because of the way the Gauss law is solved, the Yang-Mills matter for this data are arranged into “non-abelian lines of force.” This data is therefore a direct generalization of the electromagnetic data given above.

**Charged scalar field**: For gravitational coupling to both electromagnetism and a charged scalar field, the Hamiltonian constraint has the additional term

\[
\sqrt{q} q^{ab} D_a \phi D_b \phi^* + \tilde{P} \tilde{P}^* / 2 \sqrt{q}, \tag{40}
\]

where \(D_a = \partial_a + A_a\) and * denotes complex conjugation. With the time-symmetric conformal ansatz, and \(\tilde{P} = 0 = A_a\), the Hamiltonian constraint is

\[
8 \nabla^2 \psi + \left( \delta^{ab} \partial_a \phi \partial_b \phi^* + \epsilon^a \epsilon^b \delta_{ab} \right) \psi = 0, \tag{41}
\]

which must be solved with the Gauss law \(\partial_a \tilde{E}^a = \tilde{P} \phi^* + \tilde{P}^* \phi\).

A large class of solutions of this system can be obtained when there is one translational symmetry. As before set \(\epsilon^a = \epsilon^{abc} \partial_b u \partial_c v\), where \(u\) and \(v\) are arbitrary functions of the coordinates \(x, y, z\). \(\phi\) is also arbitrarily. Then the linear equation (11) can be solved for \(\psi\). With a solution \(\psi\), the Gauss law for \(\tilde{E}^a = \psi^2 \epsilon^a\) (with \(\tilde{P} = 0\)) becomes

\[
\epsilon^a \partial_a \psi = \epsilon^{abc} \partial_b u \partial_c v \partial_a \psi = 0. \tag{42}
\]

Thus, if all the functions \(u, v, \psi\) depend on only two coordinates, the Gauss law is identically solved.
III. TIME-ASYMMETRIC DATA

So far we have restricted attention to time-symmetric (i.e. $\pi^{ab} = 0$) initial conditions. We now consider some cases which relax this condition, but still restrict attention to the conformally flat form of the three-metric. The goal of the ansatz is to obtain more general initial data by attempting to convert the Hamiltonian constraint into a solvable equation. This is useful not only for obtaining more general classes of solutions, but also for addressing the cosmic censorship conjecture via the initial data test \[}; as noted in the introduction, the status of this initial data test is open for the general time-asymmetric case.

If $\pi^{ab} \neq 0$, the vector (or spatial diffeomorphism) constraint must also be solved. It is obvious (and well known) that in vacuum an easy way to solve this constraint is to set $\pi^{ab} = Aq^{ab}$, where $A$ is a constant. When matter is present, it is possible to obtain initial data by generalizations of this, as we now describe.

Consider the scalar field case with the spherically symmetric ansatz

$$q_{ab} = \psi^4(r)\delta_{ab}, \quad \pi^{ab} = \psi^{-4}\bar{P}(r) \left(\alpha n^a n^b + \beta \delta^{ab}\right),$$

where $\bar{P}(r)$ is the scalar field momentum density, $n^a = x^a/r$ is the unit radial vector, and $\alpha$ and $\beta$ are constants. The diffeomorphism and Hamiltonian constraint become

$$\left(\alpha + \beta \right)\partial_r(\psi^{-4}\bar{P}) + \frac{2\alpha\psi^{-4}\bar{P}}{r} + \psi^{-4}\bar{P}\partial_r\phi = 0,$$

and

$$\left(\frac{\alpha^2}{2} - \alpha \beta - \frac{3\beta^2}{2} + 1\right)\psi^{-6}\bar{P}^2 + 8\psi\nabla^2\psi + \psi^2\partial_a\phi\partial_b\phi\delta^{ab} = 0.$$

The diffeomorphism constraint gives

$$\bar{P} = \frac{C\psi^4}{r^{2\alpha/\alpha+\beta}} \exp\left[-\frac{\phi}{\alpha + \beta}\right].$$

One class of solutions is obtained by arranging cancellation of the gravitational and scalar momenta. Thus, with $\bar{P}$ arbitrary, set the coefficient of $\bar{P}^2$ to zero. This gives

$$\alpha = \beta \left(1 \pm 2\sqrt{1 - \frac{1}{2\beta^2}}\right).$$

Now, if the scalar field is chosen to fall off sufficiently fast at large $r$, $q_{ab}$ falls off like Schwarzschild at spatial infinity. However, from (47) it is evident that the leading order term of $\pi^{ab}$ cannot be $O(1/r^2)$; the falloff is slower than this. For the special case $\psi = 1$ (flat slice), the required falloff is $r \sim r^{-3/2}$ (see below). Even this violates (47). This means that this data is not asymptotically flat.

There is a generalization of this data obtained by requiring that $\partial_r\phi$ be proportional to $\bar{P}$, again with $\psi = 1$ in (43). This requires $\phi \sim \ln r$, which means that $\bar{P} \sim r^{-1}$. Therefore this also does not give asymptotically flat data.
In order to find asymptotically flat data, let us consider the general spherically symmetric ansatz with flat spatial metric, \( \psi = 1 \), or \( g_{ab} = \delta_{ab} \). The general form of \( \tilde{\pi}^{ab} \) is
\[
\tilde{\pi}^{ab} = f(r)n^an^b + g(r)\delta^{ab}.
\]
(48)
The diffeomorphism constraint, with matter current \( J^a = 0 \), is
\[
f(r) = -\frac{1}{r^2} \int dr \left( r^2 \partial_r g \right) + \frac{c}{r^2},
\]
(49)
\((c = \text{constant})\) and the Hamiltonian constraint is
\[
\tilde{\pi}^{ab} \tilde{\pi}_{ab} - \frac{1}{2} (\tilde{\pi}^{a} a)^2 + \rho = \frac{1}{2} (f + g) (f - 3g) + \rho = 0,
\]
(50)
where \( \rho \) is the matter energy density.

In vacuum \((\rho = 0)\), the solution of the constraint equations may be found by taking \( g = r^a \) in the above equations. The solution is
\[
\tilde{\pi}^{ab} = \frac{C}{r^{3/2}} (3n^an^b + \delta^{ab}),
\]
(51)
where \( C \) is a constant. Since this is vacuum data is spherical symmetry, it must be data for the Schwarzschild metric. Its mass \( M \) may be determined in terms of \( C \) by locating the horizon on the initial data surface. With \( s^a = n^a \) in the apparent horizon eqn. (2), and the horizon radius fixed to be \( 2M \), eqn. (2) gives
\[
C = \sqrt{M/2}.
\]
(52)
The corresponding space time metric is [21]
\[
ds^2 = -dt^2 + \left( dr + \sqrt{\frac{2M}{r}} dt \right)^2 + r^2 d\Omega^2.
\]
(53)

The flat slice data for Schwarzschild is unusual in that the normal to the spatial slice is not perpendicular to the normal of the timelike boundary. Because of this, the mass formula is different from the standard ADM integral in that it involves the extrinsic curvature [22]. The above approach provides a direct way of obtaining flat slice data for the Schwarzschild metric.

From (51) it is evident that even with matter, flat slice data must have \( \tilde{\pi}^{ab} \) falling off at least as fast as \( r^{-3/2} \) to maintain asymptotic flatness. Furthermore the positive energy condition \( \rho > 0 \) requires from (50) that
\[
(f + g)(f - 3g) < 0
\]
for all \( r \). It is difficult to find analytical data explicitly that satisfies both these conditions, and (49), without patching to Schwarzschild exterior. Consider for example the electric

\[\text{I thank Ted Jacobson for suggesting the use of flat slices.}\]
monopole. The solution of $\partial_a \tilde{E}^a = 0$ is $\tilde{E}^a = Qn^a/r^2$ (with the point $r = 0$ removed), and the magnetic field $F_{ab} = 0$. Then the condition on $f(r)$ and $g(r)$ from the diffeomorphism constraint is (49), and that from the Hamiltonian constraint is

$$f^2 - 2fg - 3g^2 + \frac{Q^2}{r^4} = 0. \quad (55)$$

These two conditions give a single equation for $f(r)$ (or $g(r)$):

$$f' + \frac{3f}{r} + \frac{\epsilon}{2}X' = 0, \quad (56)$$

where $\epsilon = \pm 1$ and $X = \sqrt{4f^2 + 3E^2}, \ (E = |E^a|)$. The same equation applies to scalar field coupling or other matter coupling with the appropriate replacement of energy density.

It is possible to find solutions of this equation numerically. The large $r$ behaviour is $\sim r^{-3/2}$ as expected, if the matter falls faster than $r^{-3/2}$ at large $r$. For $\epsilon = +1$, $f$ and $g$ are positive for large $r$, so the mass of the data is positive, and there is an apparent horizon. For $\epsilon = -1$ on the other hand, $f$, $g$, and hence the mass are all negative, and there is no apparent horizon. This is surprising because the matter satisfies the dominant energy condition. It suggests a naked singularity like the negative mass Schwarzschild case.

It is easier to find analytic solutions, with matter satisfying $\rho > 0$, if the data is taken to have Schwarzschild exterior. An example is provided by taking the interior and exterior $\tilde{\pi}^{ab}$ given by $g = \alpha + \beta r$, (which implies $f = -\beta r/3$ (49)), and (51) respectively. Matching $\tilde{\pi}^{ab}$ at $r = 1$ gives

$$\alpha = 10\sqrt{\frac{M}{2}}, \quad \beta = -9\sqrt{\frac{M}{2}}, \quad (57)$$

and interior energy density

$$\rho = 15M(1-r)(5-3r), \quad r \leq 1. \quad (58)$$

This energy density may be interpreted as arising from a scalar field by setting $\rho = \tilde{P}^2$ and $\phi = 0$, or from electromagnetism by setting $\rho = \tilde{E}^a\tilde{E}^b\delta_{ab}, \tilde{B}^a = 0$, with the Gauss law solved as in the last section.

The apparent horizon equation for (48) reduces to $f(r) + g(r) = 2/r$, and for this example gives

$$r_{AH} = \frac{5}{6} \left( 1 \pm \sqrt{1 - \frac{12}{25}\sqrt{\frac{2}{M}}} \right) \quad (59)$$

This shows that an apparent horizon forms for $M = 2(12/25)^2$ with non-zero radius $r = 5/6$. Using (59), it is possible to show that Penrose’s inequality is satisfied for all $M$. Other examples, with $f$ and $g$ polynomials also satisfy the inequality.

If the condition (49) relating $f(r)$ and $g(r)$ is not imposed, then $J^a \neq 0$; the only restrictions on these functions are now those arising from energy conditions. An example of flat slice asymptotically flat data with $J^a \neq 0$ is obtained by setting
Then the diffeomorphism and Hamiltonian constraints give

\[ J^a = n^a \frac{2\beta}{r^3}, \quad \rho = \frac{(\alpha + 3\beta)(\beta - \alpha)}{2r^4} \]

respectively. This data satisfies the weak energy condition for all \( r \) if \( \alpha \) and \( \beta \) are such that \( \rho > 0 \), but violates the dominant energy condition \( \rho \geq (J^a J_a)^{1/2} / (r > (\alpha + 3\beta)(\beta - \alpha)/4\beta) \). The mass of the data is zero because the large \( r \) falloff of \( \tilde{\pi}^{ab} \) is faster than the \( r^{-3/2} \) required to get a non-zero mass from the relevant surface integral \[22\]. (Note that it is possible to get a zero mass even if there is a non-zero energy density, provided the energy density gives a metric whose leading order behavior is \( q_{ab} \sim \delta_{ab} + O(1/r^2) \); i.e. there is no \( 1/r \) term to be captured by the surface integral. Conversely, a form of \( q_{ab} \) and \( \tilde{\pi}^{ab} \) such that the ADM mass is manifestly zero, may be used to deduce an energy density and matter current. This is exactly what is done above for flat slice data. Although this seems counterintuitive, it is possible: a metric with zero ADM mass but non-zero energy density is the Reissner-Nordstrom metric with \( M = 0 \) and electric charge \( Q \neq 0 \); while there are no horizons in this case, there are for the above flat slice example.)

The apparent horizon equation is \( \alpha - \beta = 2r \), showing that there are no horizons for \( \beta \geq \alpha \geq 0 \). Therefore Penrose’s inequality holds even though the dominant energy condition does not. However there are other examples where this is not the case. Consider e.g. \( \beta = -\alpha < 0 \), for which \( \rho = 2\alpha^2/r^4 \); there is a horizon at \( r = \alpha \). Thus, the inequality can be violated if only the weak energy condition holds.

More generally, from (48) and (50), the energy condition \( \rho \geq (J^a J_a)^{1/2} \) is

\[ \frac{1}{2}(f + g)(3g - f) = c \left( f' + \frac{2f}{r} + g' \right) \geq 0, \]

for constant \( c \geq 1 \). Simple ansatze such as \( f \sim g \), or \( f = gh \) for some function \( h \), lead for several analytically solvable cases, to data which is not asymptotically flat. However, for any given asymptotically flat form for \( f \) or \( g \), it is relatively straightforward to study the resulting ordinary differential equation numerically to probe Penrose’s inequality.

IV. SUMMARY

We have given a number of ansatze for solving the initial value constraints of general relativity with matter couplings. These include the massless scalar field, electromagnetic and Yang-Mills fields. Large classes of solutions are obtained in each case, both for time-symmetric and asymmetric situations. For the scalar field case, explicit initial data sets are given for pulses. These may be used as staring points for numerical integration. For electromagnetic and Yang-Mills fields, the data may be interpreted as the initial geometry due to flux lines of the electric field, because of the way the Gauss law is solved in each of these cases. These results go beyond the multi-point sources considered in earlier works.

For the time-asymmetric cases considered, Penrose’s inequality holds if matter satisfies the dominant energy condition. However, violations of the inequality can occur if only the weak energy condition holds.
The solutions given in this paper may be used as a possible starting point for extensions away from spherical symmetry. This would be useful not just for the data that can be obtained, but also for providing new tests of cosmic censorship via Penrose’s inequality. This is a potentially important direction because almost all results pertaining to cosmic censorship are in spherical symmetry.

Several generalizations of the ansatze given here arise if a fixed direction $S^a$ is specified. Then $\tilde{\pi}^{ab}$ can be constructed out of $S^a$, $n^a$, and the metric. One may even take $S^a$ to be a non-constant divergence free vector field specified similarly to the electric field which solves Gauss’s law. These and similar extensions are presently being studied.

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Note added: After this work was submitted and posted, I learned that Penrose’s inequality has been previously studied in spherical symmetry: it was proved under certain conditions in [23], and more generally in [24]. I thank E. Malec and S. Hayward for bringing these references to my attention.
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