SOME GENERAL FIXED-POINT THEOREMS FOR NONLINEAR MAPPINGS CONNECTED WITH ONE CAUCHY THEOREM

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Abstract. In this work, using a new geometrical approach we study to the existence of the fixed-point of mappings that independence of the smoothness, and also of their single-values or multi-values. This work proved the theorems that generalize in some sense the Brouwer and Schauder fixed-point theorems, and also such type results in multi-valued cases. One can reckon this approach is based on the generalization of the one theorem Cauchy and on the convexity properties of sets. As the used approach is based on the geometry of the image of the examined mappings that are independent of the topological properties of the space we could to prove the general results for almost every vector space. The general results we applied to the study of the nonlinear equations and inclusions in VTS, and also by applying these results are investigated different concrete nonlinear problems. Here provided also sufficient conditions under which the conditions of the theorems will fulfill.

1. Introduction

The aim of this work is to show the existence of the fixed-point of the mapping one can study without usual conditions such as some smoothness of the examined mapping and also the compactness of the subset into which acts the examined mapping. Namely, we show the existence of the fixed-point of the nonlinear mapping (in single-valued and multivalued cases) one can study by using certain geometrical conditions and without the above-mentioned conditions. This approach allows studying nonlinear equations and inclusions with the mappings acting from one vector topological space \( VTS \) to another \( VTS \).

In the \( 1 \)–dimension case, one can assume to this question was studied by Cauchy proving the theorem on the “means” value of the continuous functions, that after is generalized by Hadamard to the functions that act between of \( 1 \)–dimension vector spaces and with the condition the image of some given connected subset is the connected subset. Later can be to see, how need to formulate the mentioned theorem for this remark. In particular, these results answer the question of the existence of the fixed point. Later Poincare proved in the \( 2 \)–dimension case the existence of the fixed-point for some continuous mapping under sufficiently severe constraints and in this article [39], he proposed to prove a result that can show when the continuous mappings can have a fixed-point, moreover result of such type is very important. By the question posed Poincare many mathematicians began to study the existence of the fixed-point of the mappings. This problem was solved

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in various variants by Brouwer, who was proved the well-known Brouwer fixed-point theorem for the finite-dimensional cases, and by Banach, who was proved the well-known Banach fixed-point theorem for the contractions operator on the Banach spaces. For brevity, we won’t cite other results relative to this problem (for more see, e.g. [2-11, 13, 18, 28, 32, 35, 38, ], etc.). The problem of the existence of the fixed-point of mappings in the infinite-dimensional case was investigated in many works (see, e.g. [2-10, 28, 31, 32, 36, 39] and their references), where were used various approaches. Schauder generalized the Brouwer theorem to the infinite-dimensional case and later this result were generalized in the joint work by Leray and Schauder. There exist some generalization of the Banach theorem in the case when the operator is nonexpansive. We should be noted the fixed-point theorems of Schauder and Fan-Kakutani were had certain generalizations, but in this work are obtained the generalizations of these and also the fixed-point theorems of Brouwer, Kakutani in the other sense.

This work is proved the theorems that generalize in some sense of the Cauchy theorem (and some other results) to the finite-dimensional and the infinite-dimensional cases moreover, are proved some new fixed-point theorems. All obtained here results are based on the geometry of the image of the examined mappings. The used here approach connected, in some sense, with the topological method, more exactly with the Brouwer and Schauder theorems. In this work is generalized also the lemma that was called ”acute-angled lemma” that is the variant of the Brouwer theorem. It should be noted the ”acute-angled lemma” together with Galerkin methods successfully is applied under investigations of the nonlinear partial differential equations and inequations (see, e.g. [14, 15, 30, 32, 36, 38, 40, 42], etc.).

The obtained general results are applied to the study of the solvability of the nonlinear equations and inclusions in VTS, in particular, to the investigations of the boundary value problems for nonlinear equations and inclusions. Here provided also sufficient conditions, which show when the conditions of the theorems are fulfilled.

So, here considered a nonlinear mapping $f : D \subseteq X \rightarrow Y$, where $X$, $Y$ are the VTS and is investigated the question: under which conditions a given $y \in Y$ belongs to the image $f(G)$ of some subset $G \subseteq D(f)$? It is clear this question is equivalent to the question on the solvability of the equation $f(x) = y$, and also the inclusion $f(x) \ni y$ depending on the single-valued or multi-valued of mapping $f$. For the study, the question has used the approach based on the geometrical structure of the image $f(G)$ of the given subset $G \subseteq D(f)$ that apriori isn’t connected with any smoothness of the mapping $f$. Therefore, this approach one can call a geometrical approach.

Now we will lead the simple variant of the main fixed-point theorem of this work on the Hilbert space.

**Theorem 1.** (Fixed-point theorem) Let $X$ be a Hilbert space, $B^X_r(0) \subset X$ is the closed ball and the mapping $f$ acting in $X$ be such that $f(B^X_r(0)) \subseteq B^X_r(0)$. Then if the image $f_1(B^X_r(0)) \subset X$ be a open (or closed) convex set then there exists such $x_0 \in B^X_r(0)$ that $f(x_0) = x_0$ (or $f(x_0) \ni x_0$ if $f$ is the multi-valued), where $f_1(x) \equiv x - f(x) \text{ for } \forall x \in B^X_r(0)$.

In this article, the above-posed questions in multi-dimensional (Section 2) and infinite-dimensional (Section 3) cases are investigated. The case of the reflexive Banach spaces is studied separately in Section 4. Section 5 the nonlinear equations and inclusions in Banach spaces by application of the obtained general results are
investigated. Section 4 some sufficient conditions for fulfillment the conditions of the theorems are obtained, and Section 7 leads the concrete examples of problems.

2. Some Generalization of the Brouwer Fixed-Point Theorem and its implications

In the beginning we will provide some known results that are necessary for next (see, [8, 9, 13, 16, 17, 19, 21–27, 35]).

We denote by $B^R_r (x_0)$ the closed ball in $R^n$, $n \geq 1$ and by $S^R_r (x_0)$ the sphere (boundary of the ball $B^R_r (x_0)$) with a center $x_0 \in R^n$ and radius $r > 0$.

**Theorem 2.** (see, [3, 15]) Let $f$ acted in $R^n$ and for some $r > 0$ on the closed ball $B^R_r (0) \subseteq D (f)$ satisfies conditions: 1) $f$ is continuous; 2) the inequality $\langle f (x), x \rangle \geq 0$ for any $x \in S^R_r (0)$. Then there exists, at least, one element $x_1 \in B^R_r (0)$ such that $f (x_1) = 0$. (For the proof see, [3, 15].)

Well-known that the closed ball $B^R_r (0)$ one can change with the closed convex absorbing subset that is homeomorphic to the ball. Below will be shown that this theorem is the generalization to the finite-dimension case of one theorem Cauchy.

**Theorem 3.** (see, [21]) Any two nonintersecting nonempty convex sets of linear space can separate if either one of these has a nonempty interior or they be subsets of finite-dimension space.

**Theorem 4.** (see, [22]) If $M$ is the closed convex subset of the locally convex linear topological space $X$ and $x_0 \notin M$ then there exists a nonzero linear continuous functional $x_0^* \in X^*$ that separates $M$ and $x_0$, i.e. will found constants $c > 0$, $c_0 > 0$ such that

$$\sup \{ \langle x, x_0^* \rangle | x \in M \} \leq c_0 - c < c_0 = \langle x_0, x_0^* \rangle,$$

where $\langle o, o \rangle$ is the dual form for the pair $(X, X^*)$.

**Theorem 5.** (see, [35]) Let $X$ be a real vector topological space, $M$ be an open convex subset in $X$ and $N$ be a convex subset in $X$, and also $M \cap N = \emptyset$. Then there exist such linear continuous functional $x_0^* \in X^*$ in $X$ and real number $\alpha \in R^1$ that $\langle x, x_0^* \rangle \geq \alpha$ for $\forall x \in M$ and $\langle x, x_0^* \rangle < \alpha$ for $\forall x \in N$.

As, below will use also some concepts from the theory of vector spaces (see, [5, 7, 12, 13, 25, 35, ] ) therefore here we will reduce these. Let $X$ be a vector space, $L \subset X$ called a linear manifold in $X$ (or an affine subspace of $X$), if $L$ is a certain shift of some subspace $X_0$ of $X$, i.e. there exists $x_0 \in X$ such that $L = X_0 + x_0$; $L \subset X$ called a hyperplane in $X$ if $L$ is a maximal affine subspace of $X$ that is different of $X$: $L \equiv \{ x \in X | \langle x, x_0^* \rangle = \alpha \}$ for some $\alpha \in R^1$ and some nonzero linear continuous functional $x_0^* \in X^*$. If the convex set $M \subset X$ is the subset of affine subspace generated over $M$ then the totality of the interior of the set $M$ is called relatively interior elements of $M$ from $X$ and denoted by $riM$. A convex set $K$ from the vector space $X$ is called a convex cone with a vertex on zero of $X$ if $K$ invariant relative to all homothety, i.e. $x \in K \implies \alpha x \in K$, for $\forall \alpha \in R^1_+$, if $0 \in K \subset X$ then $K$ is called a pointed cone.
Let $\mathbb{R}^n (n \geq 1)$ $n-$dimension Euclid space, $f$ be nonlinear mapping acting in $\mathbb{R}^n$, $B_r^R(x_0) \subset \mathbb{R}^n$ be a closed ball with a center $x_0$ and a radius $r > 0$ and $S_r^R(x_0)$ be its boundary, denote by $D(f) \subseteq \mathbb{R}^n$ the domain of $f$.

**Theorem 6.** Let the subset $G \subset \mathbb{R}^n$ belong to $D(f)$ and the following conditions are fulfilled: 1) $f(G)$ is a convex subset in $\mathbb{R}^n$; 2) there exists a subspace $X$ of $\mathbb{R}^n$ with the dimension $0 < k \leq n$ such that for any $x_0 \in S_1^R(0) \cap X$ there exists such $x_1 \in G \cap X$ that

\[
\{ \langle y, x_0 \rangle \mid y \in f(x_1) \cap X \} \cap R_+^1 \neq \emptyset, \quad R_+^1 = (0, \infty),
\]

holds (here the $R_+^1$ can be substituted by $R^1_+$. Then $0 \in f(G)$, i.e. $\exists \hat{x} \in G$ that $0 \in f(\hat{x})$ (if $f$ single-valued then $f(\hat{x}) = 0$).

**Remark 1.** We will formulate this result for $\mathbb{R}^1$ in the following form. Let the mapping $f$ acting in $\mathbb{R}$ and the image $f(G)$ of some bounded subset $G \subset \mathbb{R}^1$ is the connected subset of $\mathbb{R}^1$ then a point $C \in \mathbb{R}^1$ belong to $f(G)$ and consequently there is point $c \in G$ such that $C \in f(c)$ if there exist such points $a, b \in G \ (a < b)$ that the inequations $(f(a) - C) \cdot (-1) \geq 0$ and $(f(b) - C) \cdot (1) \geq 0$ are fulfilled. The proof follows from the connectivity of $f(G)$.

This result is generalized of the results Cauchy and Hadamard as one-dimension vector space is equivalent to $\mathbb{R}^1$.

Before the proof of Theorem 6 we will prove some particular variants of this, which have independent interest. In beginning, we bring a simple variant.

**Lemma 1.** Let for some $r > 0$ the image $f(B_r^R(0))$ of the ball $B_r^R(0)$ is closed (or opened) convex set and is fulfilled the inequation \(\{ \langle y, x \rangle \mid y \in f(x) \} \cap (0, \infty) \neq \emptyset\) for $\forall x \in S_r^R(0)$ then $0 \in f(B_r^R(0))$, i.e. $\exists x_1 \in B_r^R(0)$ such that $0 \in f(x_1)$. (if $f$ single-valued then $f(x_1) = 0$).

**Proof.** The proof we bring from inverse. Let $0 \notin f(B_r^R(0)) \equiv M$. According to condition $M$ is closed or opened convex set in $\mathbb{R}^n$ then due to the separation theorem of convex subsets there exists a linear bounded functional $\mathfrak{F} \in \mathbb{R}^n$ separating of $M$ and zero under the conditions of the lemma. Since $B_r^R(0)$ is the absorbing set of $\mathbb{R}^n$, therefore one can assume that functional $\mathfrak{F}$ belongs to $S_r^R(0)$. Whence we get $\langle y, \mathfrak{F} \rangle < 0$ for $\forall y \in f(\mathfrak{F})$ that contradicts the condition of the lemma. Lemma proved.

Analogously is proved the following result.

**Lemma 2.** Let $f(B_r^R(0))$ is the convex set and on the sphere $S_r^R(0)$ is fulfilled the inequation \(\{ \langle y, x \rangle \mid y \in f(x) \} \cap (0, \infty) \neq \emptyset\) for $\forall x \in S_r^R(0)$ then $0 \in f(B_r^R(0))$.

**Lemma 3.** Let $f(B_r^R(x_0))$ is the convex set and there exists mapping $g$ acting in $\mathbb{R}^n$ such that $g(S_r^R(x_0))$ is the boundary of an absorbing subset of $\mathbb{R}^n$. Then if for $\forall x \in S_r^R(x_0)$ the expression

\[
\{ \langle y, z \rangle \mid \forall y \in f(x), \ \forall z \in g(x) \} \cap (0, \infty) \neq \emptyset
\]

holds then $0 \in f(B_r^R(x_0))$.

**Remark 2.** It is clear that if $f$ single-valued then expression (2.3) one can rewrite as

\[
\{ \langle f(x), z \rangle \mid \forall z \in g(x) \} \cap (0, \infty) \neq \emptyset.
\]
Proof. (Lemma 3) As above the proof we bring from inverse. Let \( 0 \notin f \left( B_{R}^{R} (x_0) \right) \) then repeating of previous argue we get there is such point (a linear bounded functional) \( z_0 \in S_{i}^{R} (0) \subset R^n \) that \( (y, z_0) \leq 0 \) for \( \forall y \in f (x) \) and \( \forall x \in B_{R}^{R} (x_0) \) due to the condition on convexity of \( f \left( B_{R}^{R} (x_0) \right) \) and Theorem 3 on separation of the convex sets. But this contradicts the condition of the lemma, consequently, Lemma proved.

Corollary 1. Let the mapping \( f \) acting in \( R^n \) such that for an subspace \( X \) dimension \( k \leq n \) the \( f \left( B_{R}^{R} (x_0) \right) \cap X \) is convex set and there exists mapping \( g \) acting in \( R^n \) such that \( g \left( S_{i}^{R} (x_0) \cap X \right) \) is the boundary of an absorbing subset of \( X \). Then if for \( \forall x \in S_{i}^{R} (x_0) \cap X \) the expression (2.3) is fulfilled then \( 0 \in f \left( B_{R}^{R} (x_0) \right) \).

The proof follows from the proof of the Lemma 3 more exactly, the above reasoning enough to conduct for the convex set \( f \left( B_{R}^{R} (x_0) \right) \cap X \) (that is subset of \( f \left( B_{R}^{R} (x_0) \right) \)) in the subspace \( X \), since \( X \) is also the \( k \)-dimensional space.

Proof. (of Theorem 6) As to see from the proofs the above Lemmas the selection of the subset from the domain of the examined functions isn’t essential but essentially the convexity of its image. So, if \( X \) is \( R^n \) then the proof follows from Lemmas 1 and 2. Therefore, let \( X = R^k \), \( 1 \leq k < n \). Assume \( 0 \notin f (G) \) and the affine space generated over the convex set \( f (G) \) is the hyperplane \( L \subset R^n \) and show that under condition 2 of the theorem \( L \) couldn’t be the hyperplane different of the subspace of \( R^n \). Assume \( 0 \notin L \), i.e. \( L \) isn’t a subspace then there exist \( x_0 \in f (G) \) and subspace \( X_0 \), \( \dim X_0 < n \) such that \( L = X_0 + x_0 \). Consequently, \( X_0 \) is the subspace generated over \( f (G) - x_0 \). Since \( 0 \notin L \) there is an element (point) \( z \) on \( S_{i}^{R} (0) \) such that \( (y, z) < 0 \) for \( \forall y \in L \), due to the separation theorems for the finite-dimensions spaces ([see,18]) and, consequently, for \( \forall y \in f (G) \subset L \) but according to condition 2 of Theorem 6 there is such point \( y_0 \in f (G) \) that \( (y_0, z) > 0 \). Whence, \( L \) is the subspace \( X_0 \) of \( R^n \), i.e. \( X_0 \subset X \subset R^n \). On the other hand, condition 2 satisfies for \( \forall z \in S_{i}^{R} (0) \), therefore, \( X_0 = X \), i.e. \( X \) is the \( k \)-dimension subspace of \( R^n \). Thus we arrive at the case considered in Lemma 3. Then using the result of this lemma we get \( 0 \in f (G) \).

From Theorem 6 follows the correctness of the following result.

Corollary 2. Let \( f \) acting in \( R^n \) and there is such subset \( G \) in the image \( \text{Im} f \) that satisfy the following conditions:

i) \( G \) is the convex set; ii) There exists such subspace \( X \subset R^n \) with the dimension \( k \leq n \) that for \( \forall z \in S_{i}^{R} (0) \cap X \) there exists \( y \in G \) such that \( (z, y) > 0 \) or \( (z, y) < 0 \). Then \( 0 \in \text{Im} f \).

It is clear that one can the above inequation rewrite in the form

(2.4)

\[
\left\{ (z, y) \mid \forall z \in S_{i}^{R} (0) \cap X, \exists x \in D (f), \text{ for some } y \in G \cap f (x) \right\} \cap (0, \infty) \neq \varnothing.
\]

Theorem 7. Let \( f \) acting in \( R^n \) and the ball \( B_{R}^{R} (x_0) \) with center on \( x_0 \) and the radius \( r > 0 \) belong the domain of \( f \). Let \( f \left( B_{R}^{R} (x_0) \right) \subset B_{R}^{R} (x_0) \) and for some subspace \( R^k \), \( k : 1 \leq k \leq n \), takes place \( f \left( B_{R}^{R} (x_0) \cap R^k \right) \subset B_{R}^{R} (x_0) \cap R^k \), and let \( f_1 \) is the operator in the form \( f_1 (x) \equiv lx - f (x) \) for \( \forall x \in B_{R}^{R} (x_0) \). Assume \( f_1 \left( B_{R}^{R} (x_0) \right) \) is the convex subset then the mapping \( f \) has a fixed point in \( B_{R}^{R} (x_0) \).
The proof immediately follows from the above-mentioned results, therefore here it doesn’t will be provide. Since, according to conditions of Theorem 7 imply the fulfillment the conditions of Theorem 6 for the mapping \( f_1 \) on the ball \( B_{r^2}^R(x_0) \).

\[ \text{Remark 3. Theorem 7 is the Fixed-point Theorem, where doesn’t condition onto mapping \( f \) that usually assumed in theorems of such type (such as its smoothness, single-value, or multi-value). Consequently, this theorem can consider as the generalization of such type theorems in the above sense.} \]

Now we provide an example that shows the rigor of inequation in condition 2 of the above results is essential. For simplicity assume \( n = 2 \), i.e., mapping \( f \) acting in \( \mathbb{R}^2 \) and the image of ball \( B_{r^2}^R(0) \subset D(f) \) \((r > 0)\) is
\[ f \left( B_{r^2}^R(0) \right) = B_{r^2}^R(0) \cap \{ x = (x_1, x_2) \in \mathbb{R}^2 \mid x_2 > 0 \} \cup \left[ \left( \frac{r}{2}, 0 \right) , (r, 0) \right]. \]

It isn’t difficult to see that for \( \forall z \in S_1^R(0) \) there exists an \( y \in f \left( B_{r^2}^R(0) \right) \) such that \( \langle z, y \rangle \geq 0 \) but \( 0 \notin f \left( B_{r^2}^R(0) \right) \). In particular, such mapping \( f : B_{r^2}^R(0) \rightarrow \mathbb{R}^2 \) one can define e.g. by following way
\[ f(x_1, x_2) = \begin{cases} (y_1, y_2) = (x_1, -x_2), & -r \leq x_1 \leq r, -r \leq x_2 < 0 \\ (y_1, y_2) = (x_1, x_2), & -r \leq x_1 \leq r, 0 < x_2 \leq r \\ (y_1, y_2) = \left( \frac{r}{2}, r \right) + r, 0), & -r \leq x_1 \leq 0, x_2 = 0 \\ (y_1, y_2) = \left( \frac{r}{2}, x_1 \right) + 1, 0 < x_1 < \frac{r}{2}, x_2 = 0 \\ (y_1, y_2) = (x_1, 0), & -\frac{r}{2} \leq x_1 \leq r, x_2 = 0 \\ (y_1, y_2) = (0, 0), & x_2 = 0 \end{cases}. \]

The essentialness of the condition that image \( f(G) \), (for \( G \subseteq D(f) \)) of the examined mapping is the convex set is obviously. Indeed, if assume that in the above example we have \( f \left( B_{r^2}^R(0) \right) = B_{r^2}^R(0) \setminus B_{r/2}^R(0) \) and \( f(0) = k \) then condition 2 satisfies but \( B_{r^2}^R(0) \setminus B_{r/2}^R(0) \cup \{ k \} \) isn’t a convex set and obviously \( 0 \notin B_{r^2}^R(0) \setminus B_{r/2}^R(0) \cup \{ k \} \).

It is clear that Lemma \( \Xi \) is the generalization of the "acute-angle" lemma (see, e.g. \([3, 12, 15, 16, 40, 42]\) in the above-mentioned sense. Consequently, it is also the generalization of the Brauwer fixed-point theorem (see, e.g. \([3, 6, 15, 32, 34]\)) since it is equivalent to "acute-angle" lemma, and also the Kakutani fixed-point theorem (see, e.g. \([13, 16, 19, 26]\)) for the multi-valued case.

3. Generalization of Theorem \( \Xi \) to Infinite-Dimension Cases and their Corollaries

In this section, we will generalize the results of the previous section to the general spaces. Such generalization is possible since the convexity concept is independent of the dimension and the topology of the space. In the beginning, we will generalize results to the case of the Hausdorff VTS. Let \( f : D(f) \subseteq X \rightarrow Y \) be some mapping in general nonlinear.

Let \( X \) and \( Y \) be locally convex \( VTS \) (LVTS) and \( X^* \) and \( Y^* \) be their dual spaces, respectively. Denote by \( \partial U (\partial U^*) \) the boundary of a convex closed bounded absorbing set \( U \left( U^* \right) \) in the appropriate space (see, \([33, 3]\)). In particular, any ball

\[^1\text{This is chosen for simplicity. In reality, here is sufficient to choose this set as the boundary of closed balanced absorbing set that will be seen from the further discuss.}\]
exists $x$ in the case ii) holds then set that for the image convex set in $Y$. Then if for any linear continuous functional $y^* \in \partial U^* \subset Y^*$ there exists $x \in G$ such that the following inequation

$$\{\langle y^*, y \rangle \mid y \in f(x) \} \cap (0, \infty) \neq \emptyset \text{ in the case i); } \{\langle y^*, y \rangle \mid y \in f(x) \} \cap [0, \infty) \neq \emptyset \text{ in the case ii) holds then } 0 \in f(G).$$

Proof. The proof we bring from inverse. Let $0 \not\in f(G)$ then there exists such linear continuous functional $y^*_0 \in \partial U^*$ that $\langle y^*_0, y \rangle \leq 0$ for any $y \in f(G)$ in the case i), due to convexity of $f(G)$ according to the separation theorems of the convex sets in the LVTS (see, e.g. [15, 16, 17, 19]). By the analogous reasoning to the previously, we get that in case ii) according to the separation theorems of the convex sets in the LVTS there exists such linear continuous functional $y^*_0 \in \partial U^*$ that $\langle y^*_0, y \rangle < 0$ for any $y \in f(G)$. Later on, by using the reasoning lead in the previous section, we obtain the correctness $0 \in f(G)$ due to the obtained contradiction with the conditions of the theorem. \qed

It should be noted that such type result is correct and also for the Hausdorff VTS.

Consider the case when $X$ and $Y$ be vector spaces $(VS)$ and $f : D(f) \subseteq X \rightarrow Y$ be some mapping.

Theorem 9. Let mapping $f : D(f) \subseteq X \rightarrow Y$ and for some $G \subseteq D(f)$ the image $f(G)$ be convex subset in $Y$. Assume there exist such subspace $Y_1$ that for any subspace $Y_0$ with $\text{codim}_Y Y_0 = 1$ the following expressions fulfill

$$f(G) \cap (Y_1)^+_{Y_0} \neq \emptyset \text{ and } f(G) \cap (Y_1)^-_{Y_0} \neq \emptyset. \quad (3.1)$$

Then zero of the space $Y$ belongs to $f(G) \subseteq Y$, i.e. there exists such $x_0 \in G$ that $0 \in f(x_0)$. \[\text{Proof.} \text{ Let } 0 \not\in f(G). \text{ According to the convexity of } f(G) \text{, there exist the subspace } Y_2 \subseteq Y_1 \text{ relative to which the set } f(G) \text{ is the convex set with nonempty } C-\text{interior (see, }[22]\text{) moreover, sufficiently choose the affine space generated over } f(G) \text{, as due to conditions of Theorem on the } f(G) \text{ it will be the subspace of } Y \text{.} \]

Consequently, without loss of generality can be account that $Y_2 \equiv Y_1$. For simplicity, in the beginning, assume $Y_1 = Y$. Then we obtain there exists such subspace $Y_3 \subseteq Y$ with respect to which $f(G)$ belongs to one of half-spaces $Y_3^{(1)}$ or $Y_3^{(2)}$ (see, [23, 24]) and $f(G) \cap Y_3 = \emptyset$, according to the separation theorem of the convex subsets in $VS$ (see, [21]). But this contradicts the condition \footnote{Brief notes on some properties of the linear spaces. Let $Y$ be $VS$. Each hyperplane $L$ of $Y$ is equivalent to a subspace $Y_0$ of $Y$ with $\text{codim}_Y Y_0 = 1$ it separates the space to two half-space, which can be denoted as $Y_0^+$ and $Y_0^-$. The $C-$interior point is defined as ensuing way: a point $y_0$ of the subset $U \subseteq Y$ called $C-$interior point if for $\forall y \in U$ there exists $\varepsilon > 0$ such that for all $\delta : 0 < |\delta| < \varepsilon$ the inclusion $y_0 + \delta y \in U$ holds.} this shows the correctness of the state of the theorem in the case when $Y_1 = Y$.

Let now $Y_1 \subseteq Y$ and denote the mapping $f_0(x) = f(x) \cap Y_1$ for $\forall x \in G$ then $f_0(G) = f(G) \cap Y_1$. Clear that $f_0(G)$ is convex set in $Y_1$. Consequently, one can
repeat the above reasoning for the mapping $f_0(x)$ and the space $Y_1$ as independent $VS$ from $Y$ that again will give the same result as previous, which contradicts the condition \[\text{3.1}\].

Thus we obtain the correctness of the state of the Theorem \[\text{9}\] $\square$

From above-mentioned theorems enthrone the correctness of the following fixed-point theorem.

**Corollary 3. (Fixed-Point Theorem)** Let the mapping $f$ acts into the space $X$ that is (a) $VS$ or (b) $LVTS$ and the convex subset $G \subseteq D(f)$. Let $f : G \longrightarrow G$ and denote by $f_1 : G \longrightarrow G$ the mapping $f_1 = Id - f$ ($f_1(x) = Id x - f(x)$ for $\forall x \in G$). Assume the mapping $f_1$ on the set $G$ satisfies condition \[\text{3.1}\], in the case (a); the condition of the Theorem \[\text{8}\] in the case (b). Then the mapping $f$ in the set $G$ has a fixed-point, i.e. there exists $x_0 \in G$ such that $x_0 \in f(x_0)$ (if $f$ is single-value mapping then $f(x_0) = x_0$).

The proof is obvious. If this result compares with the Schauder and Fan-Kakutani fixed-point theorems (see, e.g. [13, 17, 22, 23, 39]) then can be to see it generalized these in above-mentioned sense.

It isn’t difficult to see the correctness of following result.

**Corollary 4.** Let the mapping $f$ acting from $LVTS X$ to $LVTS Y$ on some subset $G \subseteq D(f)$ satisfies the following condition: there exists a subspace $Y_0$ of $Y$ ($Y_0 \subseteq Y$) such that $f(G) \cap Y_0$ is a convex set, moreover either (a) is open or closed, or with the nonempty interior with respect to $Y_0$. Then if for each $y^* \in \partial U^* \cap Y_0^* \subseteq Y^*$ there exists such element $x \in G$ that the expression

\[\begin{align*}
\text{(a)} \quad & \{(y, y^*) \mid y \in f(x) \cap Y_0\} \cap [0, \infty) \neq \emptyset; \\
\text{(b)} \quad & \{(y, y^*) \mid y \in f(x) \cap Y_0\} \cap [0, \infty) \neq \emptyset
\end{align*}\]

holds. Then $0 \in Y$ belongs to $f(G)$, i.e. $0 \in f(G)$.

Now we will reduce examples showing the essentialness of the conditions of the above-proved theorems. Let $X$ be a reflexive Banach space and $Y = X^*$, i.e. dual space to the $X$. Assume $f : D(f) \subseteq X \longrightarrow X^*$ and $B_{r_0}(x_0) \subseteq D(f)$, moreover $f(B_{r_0}(x_0)) = B_{r_0}(x_0) \subseteq X^*$, where $r_0 > 0$ and the centers $x_0 \in X, x_0^* \in X^*$ of these balls such that $\|x_0\|_X, \|x_0^*\|_{X^*} > r_0$. Let the mapping $f$ is such as the duality mapping between of the dula spaces, more exactly, for $\forall x \in B_{r_0}(x_0)$ fulfill the ensuing expressions $f(x) = x^* \in B_{r_0}^X(x_0^*)$ and $\langle f(x), x \rangle = \langle x^*, x \rangle = \|x\|_X \cdot \|x^*\|_{X^*} > 0$. Clearly, condition 1 satisfies, but condition 2 doesn’t be satisfied as there exist such $\tilde{x} \in S_{r_0}^X(0)$ for which doesn’t be exists $\tilde{x} \in B_{r_0}^X(x_0)$ satisfying $\langle f(\tilde{x}), \tilde{x} \rangle \geq 0$, consequently, the claims of the theorems don’t take place. The essentialness of the convexity of the image is obvious.

We will prove one result in the case when the space $Y$ is $LVTS$, in some sense, which is sufficient for fulfilling the conditions of Theorem \[\text{8}\] for simplicity, in the case of the single-valued mappings.

**Proposition 1.** Let $X, Y$ be $LVTS$, and $f : D(f) \subseteq X \longrightarrow Y$ is the single-valued mapping. Let the image $f(G)$ for some subset $G \subseteq D(f)$ is connected open or closed body in $Y$. Then for each fixed element $y \in \text{int } f(G)$, there exists such subset of $G$ on which for the mapping $f_1(x) = f(x) - y$ the conditions of Theorem \[\text{8}\] are fulfills.
Proof. Indeed, from conditions of proposition imply for each point \( y \in f(G) \) there exists an open or closed convex neighborhood \( V(y) \subseteq f(G) \) contains this point. Then by defining the preimage \( f^{-1}(V(y)) \subseteq G \) as \( G_1 \subseteq G \) it is enough to consider the mapping \( f_1 \) on the subset \( G_1 \), i.e. \( f_1 : G_1 \to V(y) \).

We won’t consider the cases of more general spaces.

4. On Mappings acted in Reflexive Banach Spaces

In this section, we will investigate mappings acted from one Banach space to another. It is clear, the results obtained in Section 3 remain correct, and in this case. We will study this case separately since well-known that the geometry of the reflexive Banach spaces was studied sufficiently complete that allows proving more exact results. Therefore, these results can be more applicable for in the detail studying various problems.

So, in this section whole, we assume \( X \) and \( Y \) be the reflexive spaces with the strongly convex norms jointly with their dual spaces (see, e.g. [7, 15, 22, 37], etc.). As well-known, each reflexive Banach space can be renormalized in such a way this space and its dual will the strongly convex spaces (see, [37], and also [7, 15]). Consequently, in what follows we will account that all of the examined reflexive Banach spaces are strongly convex spaces, without loss of generality.

Let \( X \) be strongly convex reflexive Banach space jointly with its dual space \( X^* \). For simplicity in the beginning assume \( Y = X^* \) and the mapping \( f \) acts from \( X \) to \( X^* \). Then the main result of this section will be formulated as follows.

**Theorem 10.** Let \( f : D(f) \subseteq X \to X^* \) be some mapping and \( G \subseteq D(f) \). Assume \( f(G) \) be convex subset in \( X^* \) and there exists such subspace \( X_0^* \subseteq X^* \) that belongs to the affine space \( X^*_f(G) \) generating over \( f(G) \) and either \( \text{co} \dim X \cdot X_0^* \geq 1 \) or \( 0 \in X^*_{f(G)} \); Moreover, there exists such \( X_1 \subseteq X \) that \( X_0^* \subseteq X_1^* \), \( \text{co} \ dim X \cdot X_0^* \geq 0 \) and for \( \forall x_0 \in S_X^X(0) \cap X_1 \) the inequation

\[
\{ (x^*, x_0) | \exists x \in G \& x^* \in f(x) \cap X_1^* \} \cap (0, \infty) \neq \emptyset
\]

holds. Then \( 0 \in f(G) \), i.e. \( \exists x_1 \in G \implies 0 \in f(x_1) \) (if the mapping \( f \) is singl-value then \( f(x_1) = 0 \)).

For the proof of this theorem is necessary some auxiliary results, therefore, previously need to prove these results. We start with a simple variant of Theorem 10

**Lemma 4.** Let \( f : D(f) \subseteq X \to X^* \) be some mapping and \( G \subseteq D(f) \). Assume \( f(G) \) be convex subset in \( X^* \) and there exists such subspace \( X_0^* \subseteq X^* \) that belongs to the affine space \( X^*_f(G) \) generated over \( f(G) \) and \( \text{co} \ dim X \cdot X_0^* \geq 1 \). Then if for \( \forall x_0 \in S_X^X(0) \) there exists \( x \in G \) such that \( \{ (f(x), x_0) \} \cap (0, \infty) \neq \emptyset \) then \( 0 \in f(G) \).

Proof. The proof ensues from the Theorem if here the condition for the case (i) of this theorem fulfills. Assume this condition isn’t fulfilled then we will use the following result that will be proved later.

³In what follow we will denote for briefness by \( \{ (f(x), x_0) \} \) the set \( \{ (y, x_0) | y \in f(x) \} \), where \( (y, x_0) \in (X^*, X) \).
Lemma 5. Let \( f : D(f) \subseteq X \rightarrow Y \) be some mapping and \( G \subseteq D(f) \subseteq X \), where the spaces \( X, Y \) be LVTS. Let \( U^* \subset Y^* \) be a closed bounded balanced absorbing set in \( Y^* \) with the boundary \( \partial U^* \). Then if for \( \forall y^* \in \partial U^* \) there exists \( x \in G \) such that \( \{ \langle f(x), y^* \rangle \} \cap (0, \infty) \neq \emptyset \) fulfills then the affine space \( Y_{f(G)} \) generated over \( f(G) \) is everywhere dense affine subspace in the space \( Y \). □

Proof. Continuation of the proof of Lemma 4. According to Lemma 5, under the condition of Lemma 4, \( X_{f(G)}^* \) is, at least, everywhere dense linear subspace in \( X^* \). Let \( X_{f(G)}^1 \) be a subspace of \( X^* \) that belongs to \( X_{f(G)}^* \). Then \( f(G) \cap X_{f(G)}^1 \) is the convex set with nonempty interiors in \( X_{f(G)}^1 \) (see, e.g. [5, 17, 21, 22]). As \( X \) is the reflexive Banach space then \( X_{f(G)}^1 \) also is the reflexive space. Obviously that under the conditions of Lemma 4 on \( X_{f(G)}^1 \) for \( \forall y^* \in \partial U^* \cap X_{f(G)}^1 \) there exists \( x \in G \) such that \( \{ \langle f(x), y^* \rangle \} \cap (0, \infty) \neq \emptyset \) fulfills (for the proof see, [32]). Thus, we get that with respect to \( X_{f(G)}^1 \) for the examined mapping all conditions of the Theorem 8 are fulfilled, therefore using this theorem the correctness of the claim of Lemma 4 is obtained.

It is remains to prove of the Lemma 5.

Proof. (of the Lemma 5) The proof we bring from inverse. Let \( Y_{f(G)} \) isn’t the everywhere dense affine subspace in the space \( Y \). Denote the closure of \( Y_{f(G)} \) by \( Y_1 \equiv \overline{Y_{f(G)}} \). According to assumption \( Y_1 \subset Y \) and \( Y_1 \neq Y \) moreover, which is the closed convex affine subspace. Then there exists \( y^* \in Y^* \) such that \( \langle y^*, Y_1 \rangle < 0 \), which is contradicts to the condition of Lemma 5. □

Now we provide one result using that one can to prove the Lemma 4 by another way.

Proposition 2. Let \( X \) be strongly convex reflexive Banach space jointly with its dual space \( X^* \), and \( \Omega \subset X \) be bounded convex set. Then if each of one-dimension subspace \( L \) from \( X \) intersects \( \Omega \), i.e., \( L \cap \Omega \neq \emptyset \) then either \( 0 \in \Omega \) or \( \Omega \) is such convex body that \( 0 \) of \( X \) belongs to the closure of \( \Omega \), i.e., \( 0 \in \Omega \).

It is clear that under the condition of this result the affine space generated over \( \Omega \) contains a linear subspace of \( X \).

Lemma 6. Let conditions of the Theorem 10 fulfilled, where the expression \( \langle f, 1 \rangle \) fulfilled in the following form. Let \( X_1 \subset X \) be a subspace and \( \dim X_1, X_{f(G)}^1 \geq 0 \). Then if for \( \forall \alpha \in S_1^X (0) \cap X_1 \) there exists \( x \in G \) such that \( \{ \langle f(x), X^1, \alpha \rangle \} \cap (0, \infty) \neq \emptyset \) holds. Then \( 0 \in f(G) \).

The proof of this lemma analogously to the proof of the Lemma 4, therefore we don’t provide it.

Proof. (of the Theorem 10) This proof follows from lemmas 4 and 6. Indeed Lemma 4 shows correctness of the Theorem 10 from one side, and Lemma 6 shows correctness of the Theorem 10 from in another side. Consequently, the Theorem 10 complete proved.

□

The correctness of the following statements immediately ensues from the above-mentioned results.
Corollary 5. (Fixed-Point Theorem) Let $f$ be a mapping acting in the reflexive Banach space $X$ and $B_r^X(x_0) \subseteq D(f) \subseteq X$ be a closed ball. Assume $f$ map $B_r^X(x_0)$ into itself, where $r > 0$ is a number. Let $f_1$ be a mapping defined as $f_1(x) \equiv x - f(x)$ for $\forall x \in B_r^X(x_0)$. Then if the image $f_1(B_r^X(x_0))$ of the ball $B_r^X(x_0)$ is a convex set either open (closed) or $f_1(B_r^X(x_0)) \subset \text{int } B_r^X(x_0)$ and $\text{int } f_1(B_r^X(x_0)) \neq \emptyset$ then the mapping $f$ has a fixed point in the ball $B_r^X(x_0)$.

Corollary 6. (Fixed-Point Theorem) Let $f$ be a mapping acting in the reflexive Banach space $X$ and $B_r^X(x_0) \subseteq D(f) \subseteq X$ be a closed ball. Assume $f$ map $B_r^X(x_0)$ into itself, where $r > 0$ is a number. Let $f_1$ be a mapping defined as $f_1(x) \equiv x - f(x)$ for $\forall x \in B_r^X(x_0)$. Then if the image $f_1(B_r^X(x_0))$ of the ball $B_r^X(x_0)$ is a convex set and $f_1(B_r^X(x_0)) \subset \text{int } B_r^X(x_0)$ moreover, the affine space generated over $f_1(B_r^X(x_0))$ contains some subspace of $X$ then the mapping $f$ has a fixed point in the ball $B_r^X(x_0)$.

Remark 4. Since the closed convex body is homeomorphic to the closed ball the cited above corollaries one can transfer to the case of the closed convex body of the Banach space.

5. On solvability of the nonlinear equations and inclusions

In the beginning, we will provide the results, which in some sense are corollaries of results from the above sections, therefore we will lead them simplified variants.

Let $X$ and $Y$ be $\text{LVTS}$, and $f$ be a mapping acting from $X$ to $Y$.

Theorem 11. Let $y \in Y$ be an element. If there exists such subset $G \subseteq D(f) \subseteq X$ that $f(G)$ is a convex subset of $Y$ satisfying the condition i) or ii) of Theorem 8. Then if for $\forall y^* \in \partial U^* \cap Y^*$ there exists such $x \in G$ that fulfills the corresponding inequation

$$
i) \quad \langle f(x) - y, y^* \rangle \cap (0, \infty); \quad \text{or} \quad \langle f(x) - y, y^* \rangle \cap [0, \infty)
$$

then $y \in f(G)$, i.e. $\exists x_1 \in G \Rightarrow y \in f(x_1) \quad (f(x_1) = y)$.

For the proof sufficient to noted that if denote by $f_1$ the mapping $f_1(\cdot) \equiv f(\cdot) - y$ on the subset $G$ then it isn’t difficult to see that for the mapping $f_1 : G \rightarrow Y$ all conditions of Theorem 8 are fulfilled, consequently its claim fulfills also.

Remark 5. We should be noted the condition "for $\forall y^* \in \partial U^* \cap Y^*$ there exists such $x \in G$ " be some relation between $\partial U^*$ and some subset $G_0$ of $G$, therefore one can denote it as a mapping $g$ acting from $X$ to $Y^*$ such that for each $y^* \in \partial U^*$ $\Rightarrow g(y^*) = x \in G_0$ moreover, $g(\partial U^*) = G_0$, or it inverse $g^{-1}(G_0) = \partial U^*$. Since under investigation of the concrete problem it is necessary to seek the above-mentioned relation unlike of the brought general results. The existence of such mapping allows obtaining the necessary a priori estimates. Therefore in what follows, we will lead results according to this remark.

Corollary 7. Let the mapping $f$ act from Banach space $X$ to the Banach space $Y$, where (iii) $Y \equiv X^*$ or (iv) $Y \equiv X \equiv X^\ast$. Then the equation (the inclusion) $f(x) = y$ ($f(x) \ni y$) is solvable for any $y \in Y$ fulfilling conditions:

1) There exists such subset $G \subseteq D(f) \subseteq X$ that $f(G)$ is a convex subset with nonempty interior in $Y$;

2) For $\forall x \in \partial U^* \cap Y^*$ takes place the expression (iii) $\langle f(x) - y, x \rangle \cap (0, \infty) \neq \emptyset$; (iv) $\langle f(x) - y, J(x) \rangle \cap (0, \infty) \neq \emptyset$, where $J$ is the duality operator: $J : X \rightarrow X^\ast$. 
Now we will prove certain results that generalize the well-known theorems from articles [15, 32, 40], etc. in the sense that the examined mapping isn’t continuous or completely continuous and the subset on which this mapping satisfies the needed condition may be arbitrary.

So, let \( Y \) be a semi-reflexive \( LVTS \) (see, e.g. [5]), \( S \) be a weakly complete "reflexive" \( pn \)-space (see, [32, 44] and references in these), \( X \) be a separable \( VTS \), moreover \( X \subset S \) and is everywhere dense in \( S \) (or \( Y \) and \( S \) be semi-reflexive \( LVTS \) ), \( X_m \) be an \( m \)-dimension linear subspace of \( X \), generated over first \( m \) elements from total systems of \( X \).

3) Let \( f : S \to Y \) be a bounded and weakly closed mapping. Let \( G \subset X \) be such neighborhood of zero the space \( X \) that for each \( m = 1, 2, \ldots \) the set \( G \cap X_m = G_m \) is closed neighborhood of zero the space \( X \) and \( f (G_m) \) is closed convex subset in \( Y \).

**Theorem 12.** Let the condition 3) is fulfilled and \( A : X \to Y^* \) be a linear continuous operator. Then each \( y \in Y \) belongs to the subset \( f (G) - \ker A^* \) in other words \( y + z \in f (G) \) i.e. there exist \( x_0 \in G \) such that \( f (x_0) = y + z \) if operator \( A : X \to Y^* \) such that the inequation

\[
(f (x) - y, Ax) \geq 0, \quad x \in \partial G_m, \; m = 1, 2, \ldots
\]

holds for each \( m \), where \( z \in \ker A^* \subset Y \).

For the proof is sufficiently noted the proof is led applying the Galerkin method, which usually is applied in such type cases (see, [15, 40, 44], etc), however here instead of "acute-angle lemma" is used its generalization proved in Section 2.

**Remark 6.** The weakly closedness of the mapping in the multivalued case be understood as: if a sequence \( \{x_\alpha\} \) from \( D (f) \) weakly converges to \( x \in D (f) \) and correspondent sequence \( \{f (x_\alpha)\} \) weakly converges to the subset \( Y \subset Y \) then \( Y = f (x) \). Consequently, the claim of Theorem 12 in this case will be as: each \( y \in Y \) belongs to the subset \( f (G) - \ker A^* \) in other words \( y + z \in f (G) \) i.e. there exist \( x_0 \in G \) such that \( y + z \in f (x_0) \) if operator \( A : X \to Y^* \) such that the inequation (5.1) holds for each \( m \), where \( z \in \ker A^* \subset Y \).

Now consider equations, and also inclusions in Banach spaces. Let spaces \( Y, S \) be such as above, \( X \) be a reflexive separable Banach space everywhere dense in \( S \) and \( f : S \to Y \) be a bounded mapping. Consider the following conditions.

c) Let there exists such \( r_0 > 0 \) that for any closed ball \( B^X_r (0) \subset X \) \((r > r_0 > 0)\) there exists such neighborhood \( G_r \) of \( 0 \in S \) that \( B^X_r (0) \subset G_r, G_r \cap B^X_{r_1} (0) \subset B^X_{r_1} (0) \) and \( f (G_r) \) be open (or closed) convex set in \( Y \), where \( r_1 (r) \geq r (r_1 \text{ dependent only on } r) \);

d) There exists such linear bounded operator \( A : X \to Y^* \) that the following expression fulfills

\[
(f (x), Ax) \geq \varphi ([x]_S) [x]_S \quad \& \quad \varphi ([x]_S) \nearrow \infty \quad \text{at} \quad [x]_S \nearrow \infty
\]

where \( \varphi : R^1_+ \to R^1 \) be a continuous function and \([x]_S\) be \( p \)-norm in \( S \);

---

4Let \( X \) and \( Y \) be \( LVTS \) and \( f \) be a mapping that acts from \( X \) to \( Y \). A mapping \( f \) be called bounded if the image of each bounded subset of \( X \) is a bounded subset of \( Y \).

5Let \( X \) and \( Y \) be \( LVTS \) and \( f \) be a mapping that acts from \( X \) to \( Y \). A mapping be called weakly closed if a sequence \( \{x_\alpha\} \) from \( D (f) \) weakly converges to \( x \in D (f) \) and sequence \( \{f (x_\alpha)\} \) converges to \( y \in Y \) then \( y = f (x) \).
e) There exists such nonlinear operator \( g : X \subseteq S \to Y^\ast \) that \( \frac{g(x)}{\|g(x)\|_{Y^*}} = \hat{g}(x) \) satisfies the condition \( \hat{g}(X) = S_1^{Y^*}(0) \) and on \( X \) fulfills the expression

\[
\langle f(x), \hat{g}(x) \rangle \geq \varphi(\langle x \rangle) \cdot \|x\|_S \quad \& \quad \varphi(\langle x \rangle) \to \infty \quad \text{at} \quad \|x\|_S \to \infty,
\]

where \( \varphi \) and \( \langle x \rangle \) are same as in the condition d).

(In the case when \( f : X \to Y \) in the above expression instead \( p \)-norm needs to take \( \|x\|_X \).

**Theorem 13.** Let conditions c) and d) fulfill. Then for any \( y \in Y \) satisfying condition

\[
(5.2) \quad \sup \left\{ \frac{\langle y, Ax \rangle}{\|x\|_S} \mid x \in X \right\} < \infty
\]

there exists such \( x_0 \) and \( y_0 \in \text{ker } A^* \cap Y \) that \( f(x_0) = y + y_0 \).

**Proof.** For the proof is sufficient to note that according to condition d) there exists ball \( B_X^X(0) \) such that on \( S_{r_0}^X(0) \) fulfills inequation \( \|y, Ax\| \leq \varphi(\langle x \rangle) \cdot \|x\|_S \) due to (5.2). Therefore one can use the condition c). According to condition c) there exist such neighborhood \( G_{r_0} \) of \( 0 \) and ball \( B_X(0) \) that on \( S_{r_0}^X(0) \) fulfills inequation \( \langle f(x) - y, Ax \rangle \geq 0 \). Consequently, conditions of Theorem 11 satisfy then its claim satisfies also.

**Theorem 14.** Let conditions c) and e) fulfill. Then for any \( y \in Y \) satisfying condition

\[
(5.3) \quad \sup \left\{ \frac{\langle y, \hat{g}(x) \rangle}{\|x\|_S} \mid x \in X \right\} < \infty
\]

there exists such \( x_0 \) that \( f(x_0) = y \).

The proof of this theorem is analogous to the above-provided proof.

**Remark 7.** We should be noted one can prove theorems of the previous theorems type in the case when \( Y \), \( X \) be Banach spaces moreover, \( Y \) is reflexive space and \( f : X \to Y \) be a bounded mapping, as these are led by the analogous way, therefore we do not to adduce their here.

We provide now one result for the equation with the odd operator.

**Theorem 15.** Let \( f \) acts from reflexive Banach space \( X \) to its dual space \( X^* \) and is the single-value odd operator. Assume there exists such closed balanced convex neighborhood \( G \subseteq X \) that \( f(G) \) is a convex closed subset of \( X^* \). Then there exist such subset \( G_1 \subseteq G \) and a subspace \( X_0^* \subseteq X^* \) that for each \( x^* \in X_0^* \) satisfying of inequation \( \|x^*\|_{X^*} \leq \|f(x)\|_{X^*}, \forall x \in G_1 \) the equation \( f(x) = x^* \) is solvable in \( G \).

**Proof.** Due to the condition of theorem \( 0 \in f(G) \). Then according to the convexity of the \( f(G) \) the affine space \( X_0^* \) generated over \( f(G) \) is a subspace of \( X^* \), and \( f(G) \) is the convex closed body in the subspace \( X_0^* \). Whence follows that without loss of generality one can suppose that \( X_0^* = X^* \). Since \( X \) and \( X^* \) are the reflexive Banach spaces, one can assume that these are the strongly convex spaces, and the dual mapping \( J : X \leftarrow J^{-1} \to X^* \) is the monotone biecction (one-to-one, see, e.g. [31, 40]).

As \( f(G) \) is the closed convex set there exists a subset \( G_1 \) of \( G \) for which takes place the relations \( f(G_1) = \partial f(G), f^{-1}(\partial f(G)) \supseteq G_1 \). Another hand using the
dual mapping $J$ one can determine a subset $G_0 = J^{-1}(\partial f(G))$. Then there exists such one-to-one mapping $f_0$ acting in $X$ that $f_0 : G_1 \hookrightarrow G_0$ and for each $x \in G_1$ fulfill the equation

$$\langle f(x), f_0(x) \rangle = \langle f(x), J^{-1}(f(x)) \rangle = \|f(x)\|_X \cdot \|J^{-1}(f(x))\|_X.$$  

According to the condition of the theorem, the set $\partial f(G)$ is the boundary of the closed convex absorbing subset of $X^*$ therefore, the subset $G_0$ also will be a boundary of the closed absorbing neighborhood of zero of $X$. Then using of Theorem 16 we obtain the solvability of the equation $f(x) = y$ for any $y \in X^*$ satisfying the inequality $\langle f(x) - y, J^{-1}(f(x)) \rangle \geq 0$. Due to the above reasoning, the equation $f(x) = y$ is solvable for any $y \in X^*$ satisfying the inequality

$$\langle y, J^{-1}(f(x)) \rangle \leq \langle f(x), J^{-1}(f(x)) \rangle = \|f(x)\|_X \cdot \|J^{-1}(f(x))\|_X.$$  

\[ \square \]

6. SOME SUFFICIENT CONDITIONS ON THE CONVEXITY OF THE IMAGE OF MAPPINGS

Let $X, Y$ be the VTS, and $f$ be a mapping acting from $X$ to $Y$.

**Lemma 7.** Let the mapping $f$, acting from LVTS $X$ to LVTS $Y$, some subsets from $D(f) \subseteq X$ translated to connected subsets of $Y$. Let there exist such subsets $G_1 \subseteq G \subseteq D(f)$ that $f(G_1)$, $f(G)$ be connected subsets with the nonempty interiors of $Y$ and such convex subset $M$ of $Y$ that inequations $f(G_1) \subseteq M \subseteq f(G)$ hold. Then there exists such subset $G_2 \subseteq G$ that $f(G_2) = M$ (if the mapping $f$ is multivalued then one can determine the mapping $f_1$ (or restrict this mapping) as follows $f_1(x) = f(x) \cap M$ for each $x \in G_2$).

The proof is obvious.

**Proposition 3.** Let the mapping $f$, acting from LVTS $X$ to LVTS $Y$, be locally homeomorphic from $D(f) \subseteq X$ on $\text{Im}(f) \subseteq Y$. Then if the $D(f)$ contains a subset with a nonempty interior then the mapping $f$ is fulfilled the condition of Lemma 11.

**Lemma 8.** Let $X$ be reflexive Banach space, and $f$ acting from $X$ to its dual space $X^*$, be a potentially mapping with a convex potential $F$ (the Gateaux derivative is $\partial F = f$). Then the image $f(G)$ of each convex subset $G$, generated by potential be a convex subset of $X^*$. (Where a subset $G$, generated by potential understood as $G = \{ x \in X \mid F(x) \leq C \}$, where $C$ is the constant.)

**Proof.** According to the condition on $f$ is the Gateaux derivative of a differentiable convex functional $F$, i.e. $\partial F(x) = f(x)$ for each $x \in \text{int } D(f)$. Well-known that the dual functional $F^*$ to $F$ also is convex functional (see, [12, 16, 19, 25, 26, 27]). Moreover, the following relations

$$(6.1) \ (i) \ \forall x \in \text{dom} F \implies f(x) \in \text{dom} F^*; \ (ii) \ \forall x^* \in \partial F(x) \iff x \in \partial F^*(x^*)$$  

are fulfilled.

Where $\partial F$ is is the subdifferential of the functional $F$ that in this case is the Gateaux derivative of $F$ (see, [16, 19]). Then if $G \subseteq \text{int } \text{dom} F$ is a convex closed subset then the corresponding subset $G^* \subseteq \text{int } \text{dom} F^*$ also is a convex closed subset.

In the other words, the image $f(G)$ of the subset $G$ is the $G^*$, i.e. $f(G) = G^*$.  \[ \square \]

Whence implies the correctness of the following result.
Corollary 8. Let \( X \) be reflexive Banach space, and \( f \) acts from \( X \) to its dual space \( X^* \), then if the mapping \( f \) is the subdifferential of a convex functional then the claim of the Lemma 8 is true.

Now we will lead one result for the case when \( X \) and \( Y \) are the spaces of functions, and the mappings are concrete chosen.

Lemma 9. Let the bounded mapping \( f \) acting in \( \mathbb{R}^1 \) the connected subset translates to connected subset and satisfy the following inequalities

\[
f(\xi) \cdot \xi \geq c_0 |\xi|^{p+1} - \tilde{c}_0; \quad |f(\xi)| \leq c_1 |\xi|^p + \tilde{c}_1, \quad \forall \xi \in \mathbb{R}^1,
\]

for some constants \( c_0, c_1 > 0, \tilde{c}_0, \tilde{c}_1 \geq 0, p > 1 \).

Assume \( A : W_0^{k,p_0}(\Omega) \to L_{p_1}(\Omega) \) is the linear continuous operator satisfying the following inequation

\[
c_2 \|u\|_{W_0^{k,p_0}} - \tilde{c}_2 \leq \|A(u)\|_{L_{p_1}} \leq c_3 \|u\|_{W_0^{k,p_0}} + \tilde{c}_3, \quad \forall u \in W_0^{k,p_0}(\Omega),
\]

where \( \Omega \subset \mathbb{R}^n, n \geq 1 \), is the bounded domain with sufficiently smooth boundary, and \( c_2, c_3 > 0, \tilde{c}_2, \tilde{c}_3 \geq 0, p_0 \geq p \cdot p_1, p, p_1 > 1, k \geq 0 \) are constants.

Then for each ball \( B_{r_1}^{W_0^{k,p_0}}(0) \subset W_0^{k,p_0}(\Omega), r \geq r_0 > 0 \) there exists such subset \( G_r \subset W_0^{k,p_0}(\Omega) \) that \( f_0(G_r) \equiv (f \circ A)(G_r) \equiv f(A(G_r)) \) is a convex subset with the nonempty interior of \( L_{p_1}(\Omega) \) moreover, \( f_0 \left( B_{r_1}^{W_0^{k,p_0}}(0) \right) \subseteq f_0(G_r) \subseteq f_0 \left( B_{r_1}^{W_0^{k,p_0}}(0) \right) \) holds for some \( r_1 \geq r \), where \( r_0 \) is a constant depending at the above constants.

Proof. It isn’t difficult to see that the linear operator \( A \) is surjective (see, e.g. [5, 15, 19, 26, 29]) therefore, it is enough to lead proof for the mapping \( f : L_{p_0}(\Omega) \to L_{p_1}(\Omega) \). Consider the following subset of \( L_{p_0}(\Omega) \)

\[
M_{r_2} \equiv \left\{ u \in L_{p_0}(\Omega) \mid \|f(u)\|_{L_{p_1}} \leq r_2, \; r_2 > r_0 \right\}.
\]

Whence using conditions and providing some estimations we obtain

\[
\exists_0 \|u\|_{L_{p_0}}^p - \tilde{c}_0 \leq \|f(u)\|_{L_{p_1}} \leq \exists_1 \|u\|_{L_{p_0}}^p + \tilde{c}_1.
\]

These inequations show that the image \( f(M_{r_2}) \) of the above-introduced subset \( M_{r_2} \) contains some ball and belongs to another ball from \( L_{p_1}(\Omega) \). Consequently, there exist such subsets \( G_r \) that \( f_0(G_r) \) will be convex subsets with the nonempty interior of \( L_{p_1}(\Omega) \). \( \square \)

We will provide some well-known facts from [5] that are necessary for the next result.

Definition 1. (see, [5]) Let \( K \) be a convex set of the linear space \( LS \) \( X \) containing zero as the \( C \)-interior point. If \( \mu \) is the Minkowski functional of \( K \), then the function defined for all pair \( x, y \in X \) by the equation

\[
\tau(x, y) = \lim_{\alpha \to +0} \frac{1}{\alpha} [\mu(x + \alpha y) - \mu(x)]
\]

called tangential functional of the set \( K \).
We should be noted if $K$ is the convex set as above then $\frac{1}{\alpha} [\mu(x + \alpha y) - \mu(x)]$ is the growing function at real $\alpha > 0$ and the above-mentioned limit exists for all pair $x, y \in X$ (see, [5]). Let $K$ is a subset of the LS $X$ and the point $x \in K$ is the $C$—boundary point of $K$ then the functional $x^* \in X^*$ called the tangent to subset $K$ on the point $x \in K$ if there exists such constant $c$ that $\langle x^*, x \rangle = c$ and $\langle x^*, y \rangle \leq c$ for $\forall y \in K$.

**Theorem 16.** (see, [5]). Let $X$ be a VTS and $K \subset X$ be a closed subset of $X$ possessing interior points. Assume $K$ possesses a nontrivial tangent functional on all points of an everywhere dense subset of the boundary of $K$ then $K$ is a convex set.

For the proof, and also about the correctness of its inverse statement see, [5].

**Corollary 9.** Let a bounded mapping $f$ acting from reflexive Banach space $X$ to its dual space $X^*$ is (i) a monotone hemi-continuous coercive operator (see, [15, 42]). Then $f$ translates a closed convex subset with a nonempty interior of $X$ onto a closed convex subset with a nonempty interior of $X^*$; (ii) a positive homogeneous radially continuous monotone mappings. Then $f$ translates a convex subset of $X$ defined by a functional depending on mapping $f$, for which zero is an interior point, onto a convex subset of $X^*$.

The proof of the (i) follows from Theorem 16, and the proof of the (ii) immediately follows from the presentation of the corresponding functional

$$F(x) \equiv \frac{1}{\alpha + 1} \int_0^1 \langle f(tx), x \rangle \, dt \equiv \frac{1}{\alpha + 1} \langle f(x), x \rangle,$$

where $\alpha$ is an exponent of homogeneity.

### 7. Examples

1. Let $X$ be a reflexive Banach space, $J$ be a duality operator $J : X \rightarrow X^*$, generated by a strongly monotone growing continuous function (see, e.g. [15])

$$\Phi : R_1^+ \rightarrow R_1^+, \quad \Phi(0) = 0, \quad \Phi(\tau) \nearrow \infty \ at \ \tau \nearrow \infty,$$

$\varphi : R_1^+ \rightarrow R_1^+$ be some mapping translating the connected set to connected set (i.e. connected mapping) that satisfies the condition: for each interval $I \subset R_1^+$ there exists sup $\{ \varphi(\tau) \mid \tau \in I \} = \varphi(\tau_I)$, moreover, here maybe $\varphi(\tau_I) = \infty$.

Let $X$ and $X^*$ be strongly convex spaces, $f : X \rightarrow X^*$ be a mapping having the form as in the following equation

(7.1)$$f(x) \equiv \varphi(\|x\|_X) \ J(x) = y, \quad y \in X^*.$$

**Theorem 17.** Under the above conditions the equation (7.1) solvable for any $y \in X^*$ satisfies of the following inequation

$$\|y\|_{X^*} \leq \sup \{ \varphi(r) \Phi(r) \mid r \geq 0 \} = \varphi(r_0) \Phi(r_0).$$

**Proof.** In the beginning note that according to Lemma 8 the image $f(B_X^N(0)) \subset X^*$ of the ball $B_X^N(0) \subset X$ is a convex subset of $X^*$. According to spaces $X$ and $X^*$ be strongly convex, therefore the duality operator $J$ is a bijective mapping (see,
moreover the image of each ball $B_r^X (0) \subset X$ is a ball $B_{r_1}^{X*} (0) \subset X^*$, where $r_1 = \Phi (r_0)$. Whence implies that the mapping $f$ can be represented as
\begin{equation}
\tag{7.2}
f(x) \equiv \varphi (\|x\|_X ) \Phi (\|x\|_X ) x^*, \quad \forall x \in B_r^X (0),
\end{equation}
as $J(x) \equiv \Phi (\|x\|_X ) x^*$, where the functional $x^* \in S_1^{X*} (0) \subset X^*$ is the functional generating the norm $\|x\|_X$.

Thus the image $f \left( S_{r_0}^X (0) \right)$ of each sphere $S_{r_0}^X (0) \subset X$ be a the sphere $S_{r_1}^{X*} (0) \subset X^*$ under mapping $f$. Consequently, since $x^* \in S_1^{X*} (0) \subset X^*$ from (7.2) implies that the image $f \left( B_{r_0}^X (0) \right)$ of each ball $B_{r_0}^X (0) \subset X$ be a the ball $B_{r_1}^{X*} (0) \subset X^*$ with the corresponding radius $r_1 = r_1 (r_0)$.

Then use the Theorem 11 we get that the equation (7.1) is solvable for any $y \in X^*$ satisfying inequation
\[ |\langle y, x \rangle | \leq r \varphi (r) \Phi (r), \quad \forall x \in S_r^X (0), \quad r > 0. \]
In other words, the equation (7.1) is solvable for any $y \in X^*$ satisfying follows there exists such $r > 0$ that $\|y\|_{X^*} \leq \varphi (r) \Phi (r)$.

2. On the bounded domain $\Omega \subset R^n$, $n \geq 1$, with sufficiently smooth boundary $\partial \Omega$ consider the following problem
\begin{equation}
\tag{7.3}
f(u) \equiv -\Delta u + \psi (u) \ni h(x), \quad x \in \Omega, \quad u \mid \partial \Omega = 0,
\end{equation}
where $\Delta$ is the laplacian, $\psi$ be some mapping acting from $H_0^1 \equiv W_0^{1,2} (\Omega)$ (see, [41]) to $L_2 (\Omega)$ such that for each function $u \in H_0^1$ the image $\psi (u)$ is the set of functions $\{v (x)\} \subset L^\infty (\Omega)$ that has the following representation
\[
v(x) = \begin{cases}
1, & x \in \{y \in \Omega \mid u (x) > 0\} \\
\in L^\infty (\Omega), & x \in \Omega_0 \equiv \{y \in \Omega \mid u (x) = 0\} , \\
-1, & x \in \{y \in \Omega \mid u (x) < 0\}
\end{cases}
\]
moreover $|w(x)| \leq 1$ a.e. on $\Omega_0$, and $[-1, 1] \subset \text{Im } \psi$. Here $W_0^{1,2} (\Omega)$ is the Sobolev space of functions and $W^{-1,2} (\Omega)$ is its dual space.

In other words, mapping $f$ acts from $H_0^1$ to $H^{-1} \equiv W^{-1,2} (\Omega)$, in this case, inclusion (7.3) is understood in the following sense the image $f(u)$ of each function $u \in H_0^1$ is a set $-\Delta u + \{v (x)\}$.

**Theorem 18.** Let the above conditions on the problem (7.3) be fulfilled. Then for any $h \in H^{-1}$ there exists a solution $u(x)$ of the problem (7.3), that belongs to the space $H_0^1$.

**Proof.** For the proof enough to show that all condition of the Theorem 11 fulfills for the mapping $f$. We will show that for any $h \in H^{-1}$ there exists such set $G \subset H_0^1$ that $f(G)$ be a convex body and exists such subset of $G$ that be a boundary $\partial G_1$ of an absorbing subset $G_1$ on which takes place the inequation
\[
\langle f(u) - h, u \rangle > 0, \quad \forall u \in \partial G_1 \subset G \subset H_0^1.
\]
Here we will use the Lemma 4, namely here enough to choose such balls $B_{r_1}^{H_0^1} (0)$, $B_{r_2}^{H_0^1} (0) \subset H_0^1$ ($0 < r_1 < r_2$) that satisfy the inequation $f \left( B_{r_1}^{H_0^1} (0) \right) \subset M \subset f \left( B_{r_2}^{H_0^1} (0) \right)$, where $M$ be a convex subset. We should be noted choosing of balls is dependent on the given $h \in H^{-1}$.
In the beginning, is necessary to show the correctness of some inequalities. (Rel- 
itive to the connectivity of the image of the connective set under mappings that
do’t are continuous can be to lead the following result.)

It isn’t difficult to see

$$\langle f (u), u \rangle = \langle - \Delta u + \psi (u), u \rangle = \| \nabla u \|^2_H + \| u \|_{L^1} \geq \| \Delta u \|^2_{H^{-1}},$$

where $$\nabla \equiv \left( \frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n} \right), \quad \Delta \equiv \nabla^* \circ \nabla, \quad H \equiv L_2 (\Omega).$$ Moreover, there exists 
constant $$c \geq 1$$ such that

$$(7.4) \quad \langle f (u), u \rangle = \| \nabla u \|^2_H + \| u \|_{L^1} \leq c \left( \| \Delta u \|^2_{H^{-1}} + \| \psi (u) \|_{H^{-1}} \right)$$

and on each sphere $$S_{H_0^1} (0) \subset H_0^1, \ r > 1$$ for some constants $$c_1, c_2 > 0$$ takes place

$$(7.5) \quad \| \Delta u \|^2_{H^{-1}} \leq c_1 \| f (u) \| \leq \| \Delta u \|^2_{H^{-1}} + \| \psi (u) \|_{H^{-1}} \leq c_2 \| \Delta u \|^2_{H^{-1}}.$$ 

According to the proof of Lemma 8, for the proof that the image $$\bar{B}^{H_1} (0) \subset \ H^{-1}$$ of a ball $$\bar{B}^{H_1} (0) \subset H_0^1, \ r \geq 1$$ be a convex set with the nonempty interior we will use the convexity of the corresponding functional (see, [16, 19]). Since mapping 
$$f$$ is the subdifferential of the convex functional

$$\Phi (u) \equiv \frac{1}{2} \| \nabla u \|^2_{H^n} + \| u \|_{L^1} \equiv \Phi_0 (u) + \Phi_1 (u)$$

due to well-known results (see, [16, 19]) one can assume ball $$\bar{B}^{H_1} (0) \subset H_0^1$$ be an effective set of the functional $$\Phi$$. Consequently, it is sufficient to examine an 
effective set of dual-functional $$\Phi^*$$ of functional $$\Phi$$. Due to sub-differentiability of 
the functional $$\Phi$$, at least on the int dom $$\Phi$$, the inclusions int dom $$\Phi \subseteq \text{Im} (\partial \Phi) \subset \text{dom} \ \Phi^*$$ hold.

As the functional is the sum of functionals $$\Phi_0$$ and $$\Phi_1$$ its dual $$\Phi^*$$ be an infimal 
convolution of functionals $$\Phi_0$$ and $$\Phi_1$$ therefore, it is necessary to define their dual 
functionals. It is known that (see, [30]) under $$v^* \in \text{dom} \Phi_0^*$$ we have $$\Phi_0^* (v^*) \equiv \frac{1}{2} \| v^* \|^2_{H^n}, \$$ moreover

$$\text{dom} \Phi_0^* \equiv \{ v^* \in H^n | \| v^* \|^2_{H^n} \leq r \},$$

and under $$u^* \in \text{dom} \Phi_1^* \subset L^\infty (\Omega)$$ we have $$\Phi_1^* (u^*) \equiv \kappa (u^* | B^{L^\infty} (0)), \$$ where

$$B^{L^\infty} (0)$$ is the closed ball with the radius $$r = 1,$$ with the center in zero of $$L^\infty (\Omega), \$$

$$\kappa (u^* | M)$$ be an indicator function of the set $$M \equiv B^{L^\infty} (0) \equiv \text{dom} \Phi_1^*.$$}

6Let X, Y be linearly connected LVTS, and $$f : D (f) \subseteq X \rightarrow Y$$ be a mapping, $$G \subseteq D (f)$$ be a locally connected subset. Introduce a class of subset of Y and also the corresponding subset of $$R^1$$

$$\mathcal{N} \equiv \{ y^* | \gamma (x, y^*) = \gamma (x, y^*) \in R^1 | y \in \gamma (x) \cup \gamma (x, y^*) \}$$

where $$x_1, x_2 \in G$$ be some points, and $$\gamma (x_1, x_2) \subset G$$ be a curve connecting of these points, 
y* be a linear continuous functional.

Theorem 19. (see, [32]and its references) Let X, Y be linearly connected LVTS, and $$f : D (f) \subseteq X \rightarrow Y$$ be a mapping, $$G \subseteq D (f)$$ be a locally connected subset. Then if for any $$x_1, x_2 \in G$$ there 
eexists such curve $$\gamma (x_1, x_2)$$ that the subset $$\mathcal{N} \equiv \gamma (x_1, x_2)$$ be connected for each linear continuous functional 
y* be a linear continuous functional.

$$\kappa (u^* | M) = \{ 0 \text{ if } u^* \in M, \} \infty \text{ if } u^* \in M$$
Then we get for \( h \in \text{dom}\Phi^* \subset H^{-1} \)
\[
\Phi^* (h) \equiv \Phi^*_0 (h^*) + \Phi^*_1 (u^*) h^* + u^* = h
\]
holds, and also \( \text{dom}\Phi^* = \text{dom}\Phi^*_0 + \text{dom}\Phi^*_1 \), moreover is known \((\Phi_0 \circ \nabla)^* = \nabla^* \circ \Phi_0^* ([19]), in this case \((\Phi_0 \circ \nabla)^* (h^*) = \Phi_0^* (v^*), \nabla^* v^* = h^* ([16]).\)

Thus we obtain
\[
\Phi^*(h) \equiv \inf_{u^*, h^*} \left\{ \frac{1}{2} \|v^*\|^2_{H^2} \mid h^* + u^* = h, \quad h^* = \nabla^* v^*, \quad h^* \in B^{H^{-1}}_1 (0) \subset H^{-1}, u^* \in B^{L^\infty}_1 (0) \subset L^\infty (\Omega) \right\}.
\]

Whence imply that \( \text{int dom}\Phi^* \not= \emptyset \), moreover \( B^{H^{-1}}_1 (0) \subset \text{dom}\Phi^* \). Consequently, \( f \left( B^{H^1}_r (0) \right) \) have nonempty interior, and also other condition of Lemma 8 fulfills for the mapping \( f \) by virtue of inequalities (7.4) and (7.5). Moreover, the \( \text{dom}\Phi^* \) expands according to the growth of the radius \( r \), which proves the correctness of the claim of Theorem 18.

3. Now we will lead one simple example on the application of one of the fixed-point theorems that proved in this work.

Let \( \Omega \subset \mathbb{R}^n (n \geq 1) \) be a bounded domain with sufficiently smooth boundary \( \partial\Omega \), \( H \equiv L_2 (\Omega) \) be a Lebesgue space that is the Gilbert space and \( f : D (f) \subset H \rightarrow H \) be a nonlinear mapping. Assume the mapping \( f \) as acting in \( H \) has the representation \( f (u) \equiv u - \alpha \| u - u_0 \|^2_{H^1} (u - u_0) \), where \( 0 < \alpha \leq 4^{-1} \). We want that \( f : B^{H^1}_1 (u_0) \subset H \rightarrow B^{H^1}_1 (u_0) \) holds. Denote by \( \tilde{u} = u - u_0 \) for any \( u \in B^{H^1}_1 (u_0) \) then \( f (u) \equiv u_0 + \tilde{u} - \alpha \| \tilde{u} \|^2_{H^1} \tilde{u} \). We will show that \( \| f (u) - u_0 \|_H \leq 1 \)
\[
\| f (u) - u_0 \|_H = \| u_0 + \tilde{u} - \alpha \| \tilde{u} \|^2_{H^1} \tilde{u} - u_0 \|_H = \| \left( 1 - \alpha \| \tilde{u} \|^2_{H^1} \right) \tilde{u} \|_H.
\]

Whence according to the definition of \( \tilde{u} \) fulfills \( \| \tilde{u} \|_H = \| u - u_0 \|_H \leq 1 \) then we have
\[
\| f (u) - u_0 \|_H = \| \left( 1 - \alpha \| \tilde{u} \|^2_{H^1} \right) \tilde{u} \|_H = \left( 1 - \alpha \| \tilde{u} \|^2_{H^1} \right) \| \tilde{u} \|_H < 1
\]
due to condition on the \( \alpha \). Hence one can claim for any \( u \in B^{H^1}_1 (0) \) the \( \| f (u) - u_0 \|_H < 1 \), in the other words, \( f \left( B^{H^1}_1 (u_0) \right) \subset B^{H^1}_1 (u_0) \) holds.

Thus, if define the mapping \( f_1 (u) = u - f (u) \) then we get
\[
f_1 (u) = u - f (u) = \alpha \| u - u_0 \|^2_{H^1} (u - u_0) = \alpha \| \tilde{u} \|^2_{H^1} \tilde{u}
\]
cursively, \( f_1 \) be the Gateaux derivative of a convex functional \( F_1 \), i.e. \( f_1 (u) = \partial F_1 (u) \), where \( F_1 (u) = \frac{\alpha}{4} \| u - u_0 \|^2_{H^1} = \frac{\alpha}{4} \| \tilde{u} \|^2_{H^1} \). According to the results of the above section, the image \( f \left( B^{H^1}_1 (u_0) \right) \) of the mapping \( f \) is a convex set with a nonempty interior. On the other hand, the necessary inequality fulfills as for each \( \tilde{u} \in S^{H^1}_1 (0) \) we have
\[
\langle f_1 (u), \tilde{u} \rangle = \langle \alpha \| \tilde{u} \|^2_{H^1} \tilde{u}, \tilde{u} \rangle = \alpha \| \tilde{u} \|^4_{H^1} > 0.
\]

Consequently, the mapping \( f \) possesses in \( B^{H^1}_1 (u_0) \) a fixed-point, i.e. there exists \( u_1 \in B^{H^1}_1 (u_0) \) such that \( f (u_1) = u_1 \).
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