POLYNOMIAL APPROXIMATION ON $C^2$-DOMAINS

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Abstract. We introduce appropriate computable moduli of smoothness to characterize the rate of best approximation by multivariate polynomials on a connected and compact $C^2$-domain $\Omega \subset \mathbb{R}^d$. This new modulus of smoothness is defined via finite differences along the directions of coordinate axes, and along a number of tangential directions from the boundary. With this modulus, we prove both the direct Jackson inequality and the corresponding inverse for the best polynomial approximation in $L_p(\Omega)$. The Jackson inequality is established for the full range of $0 < p \leq \infty$, while its proof relies on a recently established Whitney type estimates with constants depending only on certain parameters; and on a highly localized polynomial partitions of unity on a $C^2$-domain which is of independent interest. The inverse inequality is established for $1 \leq p \leq \infty$, and its proof relies on a recently proved Bernstein type inequality associated with the tangential derivatives on the boundary of $\Omega$. Such an inequality also allows us to establish the inverse theorem for Ivanov’s average moduli of smoothness on general compact $C^2$-domains.

1. Introduction and Main Results

1.1. Historical remarks. One of the primary questions of approximation theory is to characterize the rate of approximation by a given system in terms of some modulus of smoothness. It is well known (see, e.g. [14, 22, 42]) that the quality of approximation by algebraic polynomials increases towards the boundary of the underlying domain. As a result, characterization of the class of functions with a prescribed rate of best approximation by algebraic polynomials on a compact domain with nonempty boundary cannot be described by the ordinary moduli of smoothness. Several successful moduli of smoothness were introduced to solve this problem in the setting of one variable. Among them the most established ones are the Ditzian-Totik moduli of smoothness [22] and the average moduli of smoothness of K. Ivanov [28] (see the survey paper [16] for details). The essential idea is that for the same approximation rate one may allow the function to be much less smooth closer to the endpoints of the interval. Successful attempts were also made to solve the problem in more variables, the most notable being the work of K. Ivanov for polynomial approximation on piecewise $C^2$-domains in $\mathbb{R}^2$ [27], and the recent works of Totik for polynomial approximation on general polytopes and algebraic domains [43, 44]; we will describe [27] and [44] in more details below. The following list is not meant to be exhaustive, but we would like to also mention

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several other related works: results for simple polytopes by Ditzian and Totik [22], an announcement of a characterization of approximation classes by Netrusov [41], possibly reduction to local approximation by Dubiner [23], results for simple polytopes for $p < 1$ by Ditzian [15], a new modulus of smoothness and characterization of approximation classes on the unit ball by the first author and Xu [12], a different alternative approach on the unit ball by Ditzian [17, 18], and a strengthening of the rate of polynomial approximation near conic boundary points of general convex domains by Yu. Brudnyi [4].

The main aim in this paper is to introduce a computable modulus of smoothness for functions on $C^2$-domains, for which both the direct Jackson inequality and the corresponding converse hold. As is well known, the definition of such a modulus must take into account the boundary of the underlying domain.

We start with some necessary notations. Let $L^p(\Omega)$, $0 < p < \infty$ denote the Lebesgue $L^p$-space defined with respect to the Lebesgue measure on a compact domain $\Omega \subset \mathbb{R}^d$. In the limit case we set $L^\infty(\Omega) = C(\Omega)$, the space of all continuous functions on $\Omega$ with the uniform norm $\| \cdot \|_\infty$. Given $\xi, \eta \in \mathbb{R}^d$, and $r \in \mathbb{N}$, we define

$$\Delta_\xi^r f(\eta) := \sum_{j=0}^r (-1)^{r+j} \binom{r}{j} f(\eta + j\xi),$$

where we assume that $f$ is defined everywhere on the set $\{\eta + j\xi : j = 0, 1, \ldots, r\}$. For a function $f : \Omega \to \mathbb{R}$, we also define

$$(1.1) \quad \Delta_\xi^r (f, \Omega, \eta) := \begin{cases} \Delta_\xi^r f(\eta), & \text{if } [\eta, \eta + r\xi] \subset \Omega, \\ 0, & \text{otherwise}, \end{cases}$$

where $[x, y]$ denotes the line segment connecting any two points $x, y \in \mathbb{R}^d$. The symmetric versions of these finite differences are

$$\tilde{\Delta}_\xi^r f(\eta) := \Delta_\xi^r f\left(\eta - \frac{r}{2} \xi\right) \quad \text{and} \quad \tilde{\Delta}_\xi^r (f, \Omega, \eta) := \Delta_\xi^r \left(f, \Omega, \eta - \frac{r}{2} \xi\right).$$

The best approximation of $f \in L^p(\Omega)$ by means of algebraic polynomials of total degree at most $n$ is defined as

$$E_n(f)_p = E_n(f)_{L^p(\Omega)} := \inf \left\{ \| f - Q \|_p : Q \in \Pi_n^d \right\},$$

where $\Pi_n^d$ is the space of algebraic polynomials of total degree $\leq n$ on $\mathbb{R}^d$. Given a set $E \subset \mathbb{R}^d$, we denote by $|E|$ its Lebesgue measure in $\mathbb{R}^d$, and define $\text{dist}(\xi, E) := \inf_{\eta \in E} \| \xi - \eta \|$ for $\xi \in \mathbb{R}^d$, (if $E = \emptyset$, then define $\text{dist}(\xi, E) = 1$). Here and throughout the paper, $\| \cdot \|$ denotes the Euclidean norm. Finally, let $\mathbb{S}^{d-1} \subset \mathbb{R}^d$ be the unit sphere of $\mathbb{R}^d$, and let $e_1 = (1, 0, \ldots, 0), \ldots, e_d = (0, \ldots, 0, 1)$ denote the standard canonical basis in $\mathbb{R}^d$.

Next, we describe the work of K. Ivanov [27], where a new modulus of smoothness was introduced to study the best algebraic polynomial approximation for functions of two variables on a bounded domain with piecewise $C^2$ boundary. To avoid technicalities, we always assume that $\Omega \subset \mathbb{R}^d$ is the closure of an open, bounded, and connected domain in $\mathbb{R}^d$ with $C^2$ boundary $\Gamma$ (see Definition [15]). Consider the following metric on $\Omega$:

$$(1.2) \quad \rho_\Omega(\xi, \eta) := \|\xi - \eta\| + \sqrt{|\text{dist}(\xi, \Gamma) - \sqrt{\text{dist}(\eta, \Gamma)}|}, \quad \xi, \eta \in \Omega.$$
For $\xi \in \Omega$ and $t > 0$, set $U(\xi, t) := \{\eta \in \Omega : \rho_\Omega(\xi, \eta) \leq t\}$. For $0 < q \leq p \leq \infty$, the average $(p, q)$-modulus of order $r \in \mathbb{N}$ of $f \in L^p(\Omega)$ was defined in [27] by

\[(1.3) \quad \tau_r(f; \delta)_{p,q} := \left\| w_r(f, \cdot, \delta) \right\|_p, \]

where

\[w_r(f, \xi, \delta)_q := \begin{cases} \left( \frac{1}{|U(\xi, \delta)|} \int_{U(\xi, \delta)} |\Delta_{(\eta-\xi)/r}(f, \Omega, \xi)|^q \, d\eta \right)^{\frac{1}{q}}, & \text{if } 0 < q < \infty; \\ \sup_{\eta \in U(\xi, \delta)} |\Delta_{(\eta-\xi)/r}(f, \Omega, \xi)|, & \text{if } q = \infty. \]

Intuitively, the smoothness is measured through local subdomains $U(\xi, t)$. When $\xi \in \Gamma$, $U(\xi, n^{-1})$ has the (Euclidean) size roughly $n^{-1}$ in the directions parallel to $\Gamma$ at $\xi$ (tangential directions), while in the (orthogonal) direction of the inward normal the size will be roughly $n^{-2}$. Thus, for the same approximation rate, the function is allowed to be less smooth in the inward normal direction. This is natural to expect as we do not worry about the values of the approximating polynomial outside of the domain. On the other hand, one also needs to account for the varying (in arbitrary $C^2$ manner) throughout the domain tangential directions, which is one of the key difficulties.

With the modulus defined in (1.3), the following result was announced without proof in [27] for a bounded domain in the plane with piecewise $C^2$ boundary.

**Theorem 1.1.** [27] Let $\Omega$ be the closure of a bounded open domain in the plane $\mathbb{R}^2$ with piecewise $C^2$-boundary $\Gamma$. If $f \in L^p(\Omega)$, $1 \leq q \leq p \leq \infty$ and $r \in \mathbb{N}$, then

\[(1.4) \quad E_n(f)_p \leq C_{r,\Omega} \tau_r(f, n^{-1})_{p,q}. \]

Conversely, if either $p = \infty$ or $\Omega$ is a parallelogram or a disk and $1 \leq p \leq \infty$, then

\[(1.5) \quad \tau_r(f, n^{-1})_{p,q} \leq C_{r,\Omega} n^{-\tau} \sum_{s=0}^{n} (s+1)^{r-1} E_s(f)_p. \]

It remained open in [27] whether the inverse inequality (1.5) holds for the full range of $1 \leq p \leq \infty$ for more general $C^2$-domains other than parallelograms and disks. The methods developed in this paper allow us to give a positive answer to this question. In fact, we shall prove the Jackson inequality (1.4) for $0 < p \leq \infty$ and the inverse inequality (1.5) for $1 \leq p \leq \infty$ for all compact, connected $C^2$-domains $\Omega \subset \mathbb{R}^d$. Our results apply to higher dimensional domains as well.

Finally, we describe the recent work of Totik [44], where a new modulus of smoothness using the univariate moduli of smoothness on circles and line segments was introduced to study polynomial approximation on algebraic domains. Let $\Omega \subset \mathbb{R}^d$ be the closure of a bounded, finitely connected domain with $C^2$ boundary $\Gamma$. Such a domain is called an algebraic domain if for each connected component $\Gamma'$ of the boundary $\Gamma$, there is a polynomial $\Phi(x_1, \ldots, x_d)$ of $d$ variables such that $\Gamma'$ is one of the components of the surface $\Phi(x_1, \ldots, x_d) = 0$ and $\nabla \Phi(\xi) \neq 0$ for each $\xi \in \Gamma'$. The $r$-th order modulus of smoothness of $f \in C(\Omega)$ on a circle $C \subset \Omega$ is defined as in the classical trigonometric approximation theory by

\[\tilde{\omega}^C_r(f, \theta, t) := \sup_{0 \leq \theta \leq t} \sup_{0 \leq \varphi \leq 2\pi} \left| \tilde{\Delta}_\varphi f_C(\varphi) \right|, \]

\footnote{Both the metric $\rho_\Omega$ and the average moduli of smoothness $\tau_r(f, \xi, t, p)_{p,q}$ were defined in [27] for a more general domain $\Omega \subset \mathbb{R}^2$.}
where we identify the circle $C$ with the interval $[0, 2\pi)$ and $f_C$ denotes the restriction of $f$ on $C$. Similarly, if $I = [a, b] \subset \Omega$ is a line segment and $e \in \mathbb{S}^{d-1}$ is the direction of $I$, then with $d_I(e, z) := \sqrt{\|z - a\|\|z - b\|}$, we may define the modulus of smoothness of $f \in C(\Omega)$ on $I$ as

$$\hat{\omega}^r_I(f, \delta) = \sup_{0 \leq h \leq \delta} \sup_{z \in I} \left| \frac{\Delta^r_{h d_I(e, z)} f}{h^r} \right|.$$ 

Now we define the $r$-th order modulus of smoothness of $f \in C(\Omega)$ on the domain $\Omega$ as

$$(1.6) \quad \hat{\omega}^r(f, \delta)_\Omega = \max \left( \sup_{C_\rho} \hat{\omega}^r_{C_\rho}(f, \delta), \sup_I \hat{\omega}^r_I(f, \delta) \right),$$

where the suprema are taken for all circles $C_\rho \subset \Omega$ of some radius $\rho$ which are parallel with a coordinate plane, and for all segments $I \subset \Omega$ that are parallel with one of the coordinate axes. With this modulus of smoothness, Totik proved

**Theorem 1.2.** \[44\] If $\Omega \subset \mathbb{R}^d$ is an algebraic domain and $f \in C(\Omega)$, then

$$(1.7) \quad E_n(f)_{C(\Omega)} \leq C \hat{\omega}^r(f, n^{-1})_\Omega, \quad n \geq rd,$$

and

$$(1.8) \quad \hat{\omega}^r(f, n^{-1})_\Omega \leq C n^{-r} \sum_{k=0}^n (k + 1)^{r-1} E_k(f)_{C(\Omega)}$$

with a constant $C$ independent of $f$ and $n$.

From the classical inverse inequalities in one variable, and the way the moduli of smoothness $\hat{\omega}^r(f, t)_\Omega$ are defined, one can easily show that the inverse inequality (1.8) in fact holds on more general $C^2$-domains $\Omega$. On the other hand, however, it is much harder to show the direct Jackson inequality (1.7), even on algebraic domains (see \[44\]). Furthermore, it is unclear how to extend the results of Theorem 1.2 to $L^p$ spaces with $p < \infty$.

In this paper, we will introduce a new computable modulus of smoothness on a connected, compact $C^2$-domain $\Omega \subset \mathbb{R}^d$. Our new modulus of smoothness is defined via finite differences along the directions of coordinate axes, and along tangential directions on the boundary. With this modulus, we shall prove a direct Jackson-type inequality for the full range of $0 < p \leq \infty$, and the corresponding inverse for $1 \leq p \leq \infty$. The proof of the Jackson inequality relies on a Whitney type estimate on certain domains of special type which we recently established in \[9\], and a polynomial partition of unity on $\Omega$ which we construct motivated by the ideas of Dzjadyk and Konovalov \[24\]. On the other hand, the proof of the inverse inequality is more difficult. It relies on a new tangential Bernstein inequality on $C^2$-domains, which we recently established in \[8\].

We give some preliminary materials in the next subsection. After that, we define the new modulus of smoothness in Section 1.5. The main results of this paper are summarized in Section 1.6 where we also describe briefly the organization of the rest of the paper.

\[2\] In a private communication, V. Totik kindly showed us that certain quasi-Whitney inequality can be established for the moduli $\omega^r$ on cells of distance $C/n^r$ from the boundary of $\Omega$, which, combined with certain techniques from Section 8 of the current paper, will yield the Jackson inequality (1.7) for the moduli $\omega^r$ on a general $C^2$-domain.
1.2. Preliminaries. We start with a brief description of some necessary notations. Often we will work with domains bounded by graphs of functions, so it will be more convenient to work on the \((d + 1)\)-dimensional Euclidean space \(\mathbb{R}^{d+1}\) rather than the \(d\)-dimensional space \(\mathbb{R}^d\). We shall often write a point in \(\mathbb{R}^{d+1}\) in the form \((x, y)\) with \(x = (x_1, \ldots, x_d) \in \mathbb{R}^d\) and \(y = x_{d+1} \in \mathbb{R}\). Let \(B_r(\xi)\) (resp., \(B_r(\xi)\)) denote the closed ball (resp., open ball) in \(\mathbb{R}^{d+1}\) centered at \(\xi \in \mathbb{R}^{d+1}\) having radius \(r > 0\). A rectangular box in \(\mathbb{R}^{d+1}\) is a set that takes the form \([a_1, b_1] \times \cdots \times [a_{d+1}, b_{d+1}]\) with \(-\infty < a_j < b_j < \infty, \, j = 1, \ldots, d + 1\). We always assume that the sides of a rectangular box are parallel with the coordinate axes. If \(R\) denotes either a parallelepiped or a ball in \(\mathbb{R}^{d+1}\), then we denote by \(cR\) the dilation of \(R\) from its center by a factor \(c > 0\). Given \(1 \leq i \neq j \leq d + 1\), we call the coordinate plane spanned by the vectors \(e_i\) and \(e_j\) the \(i, j\)-plane. Finally, we use the notation \(A_1 \sim A_2\) to mean that there exists a positive constant \(c > 0\) such that \(c^{-1}A_1 \leq A_2 \leq cA_1\).

1.3. Directional moduli of smoothness. The \(r\)-th order directional modulus of smoothness on a domain \(\Omega \subset \mathbb{R}^{d+1}\) along a set \(\mathcal{E} \subset \mathbb{S}^d\) of directions is defined by

\[
\omega^r(f, \Omega; \mathcal{E})_p := \sup_{\xi \in \mathcal{E}} \sup_{0 < u \leq t} \|\Delta_{u \xi}^r f\|_{L^p(\Omega)} = \sup_{\xi \in \mathcal{E}} \sup_{0 < u \leq t} \|\Delta_{u \xi}^r f\|_{L^p(\Omega)}
\]

where \(\Delta_{u \xi}^r f = \Delta_{u \xi}^r f(\xi, \cdot, \cdot)\) is given in (1.11), and \(\Omega := \{\xi \in \Omega : [\xi, \xi + \eta] \subset \Omega\}\) for \(\eta \in \mathbb{R}^{d+1}\). Let

\[
\omega^r(f, \Omega; \mathcal{E})_p := \omega^r(f, \Omega; \mathcal{E} \cup (-\mathcal{E}))_p,
\]

where \(\text{diam}(\Omega) := \sup_{\xi, \eta \in \Omega} |\xi - \eta|\). If \(\mathcal{E} = \mathbb{S}^d\), then we write \(\omega^r(f, t)_{p} = \omega^r(f, t; \mathbb{S}^d)_p\) and \(\omega^r(f, \Omega)_{p} = \omega^r(f, \Omega; \mathbb{S}^d)_{p}\), whereas if \(\mathcal{E} = \{e\}\) contains only one direction \(e \in \mathbb{S}^d\), we write \(\omega^r(f, t; e)_{p} = \omega^r(f, t; e)_{p}\) and \(\omega^r(f, \Omega; e)_{p} = \omega^r(f, \Omega; e)_{p}\). We shall frequently use the following two properties of these directional moduli of smoothness, which can be easily verified from the definition:

(a) For each \(\mathcal{E} \subset \mathbb{S}^d\),

\[
\omega^r(f, \Omega; \mathcal{E})_p = \omega^r(f, \Omega; \mathcal{E} \cup (-\mathcal{E}))_p.
\]

(b) If \(T\) is an affine mapping given by \(T\eta = \eta_0 + T_0\eta\) for all \(\eta \in \mathbb{R}^{d+1}\) with \(\eta_0 \in \mathbb{R}^{d+1}\) and \(T_0\) being a nonsingular linear mapping on \(\mathbb{R}^{d+1}\), then

\[
\omega^r(f, \Omega; \mathcal{E})_p = |\det (T_0)|^{-\frac{r}{d+1}} \omega^r(f \circ T^{-1}, T(\Omega); \mathcal{E}_T)_p,
\]

where \(\mathcal{E}_T = \{\frac{T_0x}{|T_0x|} : x \in \mathcal{E}\}\). Moreover, if \(\xi, e \in \mathbb{S}^d\) is such that \(e = T_0(\xi)\), then for any \(h > 0\),

\[
|\det (T_0)|^{-\frac{r}{d+1}} \frac{\Delta_{he}^r(f, \Omega)}{L^p(\Omega)} = \frac{\Delta_{he}^r(f \circ T^{-1}, T(\Omega))}{L^p(T(\Omega))}, \quad (1.9)
\]

Next, we recall that the analogue of the Ditzian-Totik modulus on \(\Omega \subset \mathbb{R}^{d+1}\) along a direction \(e \in \mathbb{S}^d\) is defined as (see [13; 14]):

\[
\omega^r_{\Omega, e}(f, t; e)_p := \sup_{|h| \leq \min\{t, 1\}} \|\Delta_{h \varphi_{\Omega}(e, \cdot)}^r f(\Omega, \cdot)\|_{L^p(\Omega)}; \quad t > 0,
\]

where

\[
\varphi_{\Omega}(e, \xi) := \max \left\{ \sqrt{l_1 l_2} : l_1, l_2 \geq 0, \, [\xi - l_1 e, \xi + l_2 e] \subset \Omega \right\}, \quad \xi \in \Omega.
\]

For simplicity, we also define \(\varphi_{\Omega}(\delta e, \xi) = \varphi_{\Omega}(e, \xi)\) for \(e \in \mathbb{S}^d\), \(\delta > 0\) and \(\xi \in \Omega\).
1.4. Domains of special type. A set $G \subset \mathbb{R}^{d+1}$ is called an upward $x_{d+1}$-domain with base size $b > 0$ and parameter $L \geq 1$ if it can be written in the form

$$G = \xi + \left\{ (x, y) : \ x \in (-b, b)^d, \ g(x) - Lb < y \leq g(x) \right\}$$

with $\xi \in \mathbb{R}^{d+1}$ and $g \in C^2(\mathbb{R}^d)$. For such a domain $G$, and a parameter $\lambda \in (0, 2]$, we define

$$G(\lambda) := \xi + \left\{ (x, y) : \ x \in (-\lambda b, \lambda b)^d, \ g(x) - \lambda Lb < y \leq g(x) \right\},$$

$$\partial' G(\lambda) := \xi + \left\{ (x, g(x)) : \ x \in (-\lambda b, \lambda b)^d \right\}.$$  

Associated with the set $G$ in (1.12), we also define

$$G^* := \xi + \left\{ (x, y) : \ x \in (-2b, 2b)^d, \ \min_{u \in [-2b, 2b]^d} g(u) - 4Lb < y \leq g(x) \right\}.$$  

For later applications, we give the following remark on the above definition.

**Remark 1.3.** In the above definition, we may choose the base size $b$ as small as we wish, and we may also assume the parameter $L$ in (1.12) satisfies

$$L \geq L_0 := 4\sqrt{d} \max_{x \in [-2b, 2b]^d} \| \nabla g(x) \| + 1,$$

since otherwise we may consider a subset of the form

$$G_0 = \xi + \left\{ (x, y) : \ x \in (-b_0, b_0)^d, \ g(x) - L_0 b_0 < y \leq g(x) \right\}$$

with $L_0 = Lb/b_0$ and $b_0 \in (0, b)$ being a sufficiently small constant. Unless otherwise stated, we will always assume that the condition (1.13) is satisfied for each upward $x_{d+1}$-domain.

We may define an upward $x_j$-domain $G \subset \mathbb{R}^{d+1}$ and the associated sets $G(\lambda), \partial' G(\lambda), G^*$ for $1 \leq j \leq d$ in a similar manner, using the reflection

$$\sigma_j(x) = (x_1, \ldots, x_{j-1}, x_{d+1}, x_j - x_j, \ldots, x_d, x_j), \ x \in \mathbb{R}^{d+1}.$$  

Indeed, $G \subset \mathbb{R}^{d+1}$ is an upward $x_j$-domain with base size $b > 0$ and parameter $L \geq 1$ if $E := \sigma_j(G)$ is an upward $x_{d+1}$-domain with base size $b$ and parameter $L$, in which case we define

$$G(\lambda) = \sigma_j(E(\lambda)), \ \partial' G(\lambda) = \sigma_j(\partial' E(\lambda)), \ G^* = \sigma_j(E^*).$$

We can also define a downward $x_j$-domain and the associated sets $G(\lambda), \partial' G(\lambda)$, using the reflection with respect to the coordinate plane $x_j = 0$:

$$\tau_j(x) := (x_1, \ldots, x_{j-1}, -x_j, x_{j+1}, \ldots, x_{d+1}), \ x \in \mathbb{R}^{d+1}.$$  

Indeed, $G \subset \mathbb{R}^{d+1}$ is a downward $x_j$-domain with base size $b > 0$ and parameter $L \geq 1$ if $H := \tau_j(G)$ is an upward $x_j$-domain with base size $b$ and parameter $L \geq 1$, in which case we define

$$G(\lambda) = \tau_j(H(\lambda)), \ \partial' G(\lambda) = \tau_j(\partial' H(\lambda)), \ G^* = \tau_j(H^*).$$

We say $G \subset \mathbb{R}^{d+1}$ is a domain of special type if it is an upward or downward $x_j$-domain for some $1 \leq j \leq d+1$, in which case we call $\partial' G(\lambda)$ the essential boundary of $G(\lambda)$, and write $\partial' G = \partial' G(1)$ and $\partial' G^* = \partial' G(2)$.  

Definition 1.4. Let $\Omega \subset \mathbb{R}^{d+1}$ be a bounded domain with boundary $\Gamma = \partial \Omega$, and let $G \subset \Omega$ be a domain of special type. We say $G$ is attached to $\Gamma$ if $\overline{G} \cap \Gamma = \partial G^*$ and there exists an open rectangular box $Q$ in $\mathbb{R}^{d+1}$ such that $G^* = Q \cap \partial \Omega$.

$C^2$-domains. In this paper, we shall mainly work on $C^2$-domains, defined as follows:

Definition 1.5. A bounded domain $\Omega \subset \mathbb{R}^{d+1}$ is called $C^2$ if there exist numbers $\delta > 0$, $M > 0$ and a finite cover of the boundary $\Gamma := \partial \Omega$ by connected open sets $\{U_j\}_{j=1}^J$ such that: (i) for every $x \in \Omega$ with $\text{dist}(x, \Gamma) < \delta$, there exists an index $j$ such that $x \in U_j$, and $\text{dist}(x, \partial U_j) > \delta$; (ii) for each $j$ there exists a Cartesian coordinate system $(\xi_{j,1}, \ldots, \xi_{j,d+1})$ in $U_j$ such that the set $\Omega \cap U_j$ can be represented by the inequality $\xi_{j,d+1} \leq f_j(\xi_{j,1}, \ldots, \xi_{j,d})$, where $f_j : \mathbb{R}^d \to \mathbb{R}$ is a $C^2$-function satisfying $\max_{1 \leq i, k \leq d} \|\partial_i \partial_k f_j\| \leq M$.

1.5. New moduli of smoothness on $C^2$ domains. Let $\Omega \subset \mathbb{R}^{d+1}$ be the closure of an open, connected, bounded $C^2$-domain in $\mathbb{R}^{d+1}$ with boundary $\Gamma = \partial \Omega$. In this section, we shall give the definition of our new moduli of smoothness on the domain $\Omega$.

The definition requires a tangential modulus of smoothness $\overline{\omega}^*_{G^*}(f, t)$ on a domain $G \subset \Omega$ of special type, which is described below. We start with an upward $x_{d+1}$-domain $G$ given in (1.12) with $\xi = 0$. Let

$$\xi_j(x) := e_j + \partial_1 g(x)e_{d+1} \in \mathbb{R}^{d+1}, \quad j = 1, \ldots, d, \quad x \in (-2b, 2b)^d.$$ 

Clearly, $\xi_j(x)$ is the tangent vector to the essential boundary $\partial^* G^*$ of $G^*$ at the point $(x, g(x))$ that is parallel to the $x_jx_{d+1}$-coordinate plane. Given a parameter $A_0 > 1$, we set

$$(1.14) \quad G^t := \{\xi \in G : \text{dist}(\xi, \partial^* G) \geq A_0 t^2\}, \quad 0 \leq t \leq 1.$$ 

We then define the $r$-th order tangential modulus of smoothness $\overline{\omega}^*_{G^*}(f, t)$, $0 < t \leq 1$, of $f \in L^p(\Omega)$ by

$$(1.15) \quad \overline{\omega}^*_{G^*}(f, t) := \left( \sup_{0 \leq r \leq t} \left( \int_{G^t} \left[ \frac{1}{(tb)^d} \int_{I_x(tb)} |\Delta^r_{\xi_i}(f, \Omega, (x, y))|^p \, du \right] \, dx \, dy \right)^{\frac{1}{p}} \right)^{\frac{1}{r}},$$

where $I_x(t) := \{u \in (-b, b)^d : \|u - x\| \leq t\}$, and we use $L^\infty$-norm to replace the $L^p$-norm when $p = \infty$. For $t > 1$, we define $\overline{\omega}^*_{G^*}(f, t) = \overline{\omega}^*_{G^*}(f, 1)$. Next, if $G \subset \Omega$ is a general domain of special type, then we define the tangential moduli $\overline{\omega}^*_{G^*}(f, t)$ through the identity,

$$\overline{\omega}^*_{G^*}(f, t) = \overline{\omega}^*_{T(G^*)}(f \circ T^{-1}, t),$$

where $T$ is a composition of a translation and the reflections $\sigma_j, \tau_j$ for some $1 \leq j \leq d + 1$ which takes $G^*$ to an upward $x_{d+1}$-domain of the form (1.12) with $\xi = 0$.

To define the new moduli of smoothness on $\Omega$, we also need the following covering lemma, which was proved in [8, Section 2].

Lemma 1.6 ([8, Proposition 2.7]). There exists a finite cover of the boundary $\Gamma = \partial \Omega$ by domains of special type $G_1, \ldots, G_m \subset \Omega$ that are attached to $\Gamma$. In addition, we may select the domains $G_j$ in such a way that the size of each $G_j$ is as small as we wish, and the parameter of each $G_j$ satisfies the condition (1.13).

Now we are in a position to define the new moduli of smoothness on $\Omega$.
Definition 1.7. Given $0 < p \leq \infty$, the $r$-th order modulus of smoothness of $f \in L^p(\Omega)$ is defined by

\begin{equation}
\omega^r_{\Omega}(f, t)_p := \omega^r_{\Omega, \varphi}(f, t)_p + \omega^r_{\Omega, \tan}(f, t)_p,
\end{equation}

where

\begin{equation}
\omega^r_{\Omega, \varphi}(f, t)_p := \max_{1 \leq j \leq d+1} \omega^r_{\Omega, \varphi}(f, t; e_j)_p \quad \text{and} \quad \omega^r_{\Omega, \tan}(f, t)_p := \sum_{j=1}^{m_0} \tilde{\omega}^r_{f, \xi, t}(f, t)_p.
\end{equation}

Here $G_1, \ldots, G_{m_0} \subset \Omega$ are the domains of special type from Lemma 1.6.

Note that the second term on the right hand side of (1.16) is defined via finite differences along certain tangential directions of the boundary $\Gamma = \partial \Omega$. As a result, we call $\omega^r_{\Omega, \tan}(f, t)_p$ the tangential part of the $r$-th order modulus of smoothness on $\Omega$. More specifically, in (1.15), which is the main component of the tangential modulus, for each point $(x, y) \in G_1$ where a finite difference is computed, we find the “closest” (measuring only along $(d+1)$-st coordinate) boundary point of the domain $(x, g(x))$ and take the direction of the tangent vector $\xi_j(x)$ from the $x, x_{d+1}$-coordinate plane. Such directions “follow” (are “parallel” to) the boundary and allow to capture the required smoothness information from the function in the tangential directions as the $C^2$-smoothness of the boundary ensures that $(x, y) + rt\xi_j(x) \in \Omega$ when necessary. (Observe that we need to be perfectly orthogonal to the boundary) and is a rather straightforward generalization of the one-dimensional modulus for the segment.

Comparing the above with the moduli in (1.3), one can see that the point sets where the finite differences are computed in (1.10) are from the local subdomains resembling $U(\xi, t)$ (see the discussion after (1.3) for the boundary case). However, only more specific directions of the finite differences are needed in (1.10) and those directions are easily expressed through the decomposition into the domains of special type (they are $\xi_j(x)$ as well as the coordinate directions).

The modulus from (1.4) is similar in the interior (non-tangential) directions also computing the finite differences along segments. However, for the tangential directions, $\tilde{\omega}^r(f, \delta)_{\Omega}$ uses finite differences along arcs of circles which are inside the domain and parallel to one of the coordinate axes. It is not hard to observe that the “size” of such circular finite differences matches that for the linear finite differences for the other two moduli: roughly $t \approx \sin t$ in the tangential and $t^2 \approx 1 - \cos t$ in the interior directions near the boundary.

We conclude this subsection with the following remark.

Remark 1.8. The moduli of smoothness defined in Definition 1.7 rely on the parameter $A_0$ in (1.14). To emphasize the dependence on this parameter, we often write

\begin{equation}
\tilde{\omega}^r_{\xi}(f, t; A_0)_p := \tilde{\omega}^r_{\xi}(f, t)_p, \quad \omega^r_{\varphi}(f, t; A_0)_p := \omega^r_{\varphi}(f, t)_p, \quad \omega^r_{\tan}(f, t; A_0)_p := \omega^r_{\tan}(f, t)_p.
\end{equation}

By the Jackson theorem (Theorem 1.9) and the univariate Remez inequality (see 10), it can be easily shown that given any two parameters $A_1, A_2 \geq 1$,

\begin{equation}
\omega^r_{\Omega}(f, t; A_1)_p \sim \omega^r_{\Omega}(f, t; A_2)_p, \quad t > 0, \quad 0 < p \leq \infty.
\end{equation}
1.6. **Summary of main results.** In this subsection, we shall summarize the main results of this paper. As always, we assume that \( \Omega \) is the closure of an open, connected and bounded \( C^2 \)-domain in \( \mathbb{R}^{d+1} \). For simplicity, we identify with \( L^\infty(\Omega) \) the space \( C(\Omega) \) of continuous functions on \( \Omega \).

The main aim of this paper is to prove the Jackson type inequality and the corresponding inverse inequality for the modulus of smoothness \( \omega_r^p(f,t) \) defined in (1.16), as stated in the following two theorems.

**Theorem 1.9.** If \( r, n \in \mathbb{N} \), \( 0 < p \leq \infty \), and \( f \in L^p(\Omega) \), then
\[
E_n(f)_{L^p(\Omega)} \leq C\omega_r^p(f,n^{-1})_p,
\]
where the constant \( C \) is independent of \( f \) and \( n \).

**Theorem 1.10.** If \( r, n \in \mathbb{N} \), \( 1 \leq p \leq \infty \), and \( f \in L^p(\Omega) \), then
\[
\omega_r^p(f,n^{-1})_p \leq C\sum_{j=0}^n (j+1)^{r-1}E_j(f)_{L^p(\Omega)},
\]
where the constant \( C \) is independent of \( f \) and \( n \).

As an example of application of the above, we obtain the following relation between approximation and smoothness classes (for further details in the classical settings, see, for example [14, Sect. 2.10, 7.9, 8.7]).

**Corollary 1.11.** Suppose \( 1 \leq p \leq \infty \), \( 0 < q \leq \infty \), and \( 0 < \alpha < r \). For \( f \in L^p(\Omega) \),
\[
(i) \quad E_n(f)_{L^p(\Omega)} = O(n^{-\alpha}), \quad n = 1, 2, \ldots, \text{ if and only if } \omega_r^p(f,t)_p = O(t^\alpha), \quad t > 0.
\]
\[
(ii) \quad \sum_{n=1}^\infty [n^\alpha E_n(f)_{L^p(\Omega)}]^q n^{-1} < \infty \text{ if and only if } \int_0^\infty \frac{[t^{-\alpha}\omega_r^p(f,t)_p]^q}{t} dt < \infty.
\]

An implication of this corollary is that the corresponding smoothness classes (for example, the class of functions \( f \in L^p(\Omega) \) satisfying \( \omega_r^p(f,t)_p = O(t^\alpha) \), \( t > 0 \)) do not depend on the particular choices of parameters \( A_0, L, b \) and the decomposition into the domains of special type.

Note that the Jackson inequality stated in Theorem 1.9 holds for the full range of \( 0 < p \leq \infty \).

Now let us describe two main ingredients in the proof of the direct Jackson theorem: multivariate Whitney type inequalities on certain domains (not necessarily convex); and localized polynomial partitions of unity on \( C^2 \)-domains.

The Whitney type inequality gives an upper estimate for the error of local polynomial approximation of a function via the behavior of its finite differences. A useful multivariate Whitney type inequality was established by Dekel and Leviatan [13] on a convex body (compact convex set with non-empty interior) \( G \subset \mathbb{R}^{d+1} \) asserting that for any \( 0 < p \leq \infty \), \( r \in \mathbb{N} \), and \( f \in L^p(G) \),
\[
E_{r-1}(f)_{L^p(G)} \leq C(p,d,r)\omega^p(f,G)_p.
\]

It is remarkable that the constant \( C(p,d,r) \) here depends only on the three parameters \( p, d, r \), but is independent of the particular shape of the convex body \( G \). However, the Whitney inequality (1.17) is NOT enough for our purpose because our domain \( \Omega \) is not necessarily convex, and the definition of our local moduli of smoothness (Definition 5.1) uses local finite differences along a finite number of
Corollary 1.14. If \(d \geq 1\) and the full range of \(0 < q \leq p \leq \infty\) and \(r \in \mathbb{N}\), then
\[
E_n(f)_p \leq C_{r,\Omega} \tau_r \left( f, \frac{c_0}{m} \right)_{p,q}.
\]

As mentioned in the introduction, Corollary 1.14 for \(1 \leq p \leq \infty\) and \(d = 1\) was announced in [27] for a piecewise \(C^2\)-domain \(\Omega \subset \mathbb{R}^2\).

We shall prove the corresponding inverse theorem for the average moduli of smoothness \(\tau_r(f, t)_{p,q}\) as well:
Theorem 1.15. If \( r \in \mathbb{N}, \ 1 \leq q \leq p \leq \infty \) and \( f \in L^p(\Omega) \), then
\[
\tau_r(f, n^{-1})_{p,q} \leq C_r n^{-r} \sum_{s=0}^{n} (s+1)^{r-1} E_s(f)_p.
\]

In the case when \( \Omega \subset \mathbb{R}^2 \) (i.e., \( d = 1 \)), Theorem 1.15 was announced without detailed proofs in [27] for the case \( p = \infty \) and the case when \( 1 \leq p \leq \infty \) and \( \Omega \) is a parallelogram or a disk.

The rest of the paper is organized as follows. Section 2 is devoted to the statements of the required Bernstein and Whitney-type inequalities obtained in [8] and [9]. Sections 3–5 contain the proof of the Jackson theorem (Theorem 1.9). In Section 6, we compare our moduli of smoothness \( \omega_r(\Omega)_r(f, t)_{p} \) with the average moduli \( \tau(f, n^{-1})_{p,q} \). The main result of Section 6 is stated in Theorem 1.13. Finally, in Section 7, we prove the inverse theorems as stated in Theorem 1.10 and Theorem 1.15.

2. Tools

In this section we collect several necessary ingredients which we established recently in [8] and [9]. A useful domain covering result Lemma 1.6 has already been stated.

2.1. Equivalence of different metrics. Let \( \rho_{\Omega} : \Omega \times \Omega \rightarrow [0, \infty) \) be the metric on \( \Omega \) given in (1.2). As in [8], we introduce another metric \( \hat{\rho}_G \) on a domain \( G \) of special type, which is equivalent to the restriction of \( \rho_{\Omega} \) on \( G \) if \( G \subset \Omega \) is attached to \( \Gamma := \partial \Omega \). Let \( G \subset \mathbb{R}^{d+1} \) be an \( x_d \)-upward domain with base size \( b \in (0, 1) \) and parameter \( L > 0 \):
\[
G := \varsigma + \{(x, y) : x \in (-b, b)^d, \ g(x) - Lb < y \leq g(x)\}, \ \varsigma \in \mathbb{R}^{d+1},
\]
where \( g \) is a \( C^2 \)-function on \( \mathbb{R}^d \). Then
\[
G^* := \varsigma + \{(x, y) : x \in (-2b, 2b)^d, \ \min_{u \in [-2b, 2b]^d} g(u) - 4Lb < y \leq g(x)\}
\]
and we define a metric \( \hat{\rho}_G : \overline{G^*} \times \overline{G^*} \rightarrow (0, \infty) \) by
\[
\hat{\rho}_G(\varsigma + \xi, \varsigma + \eta) := \max \left\{ ||\xi_x - \eta_x||, \ \sqrt{g(\xi_x) - \xi_y} - \sqrt{g(\eta_x) - \eta_y} \right\}
\]
for all \( \xi = (\xi_x, \xi_y), \eta = (\eta_x, \eta_y) \in \overline{G^*} - \varsigma \). We can define the metric \( \hat{\rho}_G \) on a more general \( x_j \)-domain \( G \subset \mathbb{R}^{d+1} \) (upward or downward) in a similar way.

We will use the following equivalence of the metric \( \hat{\rho}_G \) and the restriction of \( \rho_{\Omega} \) on \( G \) when \( G \subset \Omega \) is attached to \( \Gamma = \partial \Omega \).

Proposition 2.1 ([8 Proposition 3.1]). If \( G \subset \Omega \) is a domain of special type attached to \( \Gamma \), then
\[
\hat{\rho}_G(\xi, \eta) \sim \rho_{\Omega}(\xi, \eta), \ \xi, \eta \in G
\]
with the constants of equivalence depending only on \( G \) and \( \Omega \).
2.2. Whitney type inequality.

Definition 2.2. Given \( \xi \in \mathbb{S}^{d-1} \), we say \( G \subset \mathbb{R}^{d+1} \) is a regular \( \xi \)-directional domain with parameter \( L \geq 1 \) if there exists a rotation \( \rho \in SO(d+1) \) such that
(i) \( \rho(0, \ldots, 0, 1) = \xi \), and \( G \) takes the form
\[
G := \rho\left( \{(x, y) : x \in D, \ g_1(x) \leq y \leq g_2(x)\} \right),
\]
where \( D \subset \mathbb{R}^d \) is compact and \( g_i : D \to \mathbb{R} \) are measurable;
(ii) there exist an affine function (element of \( \Pi^{d+1}_1 \)) \( H : \mathbb{R}^d \to \mathbb{R} \) and a constant \( \delta > 0 \) such that \( S \subset G \subset S_L \), where
\[
\rho^{-1}(S) := \{(x, y) : x \in D, \ H(x) - \delta \leq y \leq H(x) + \delta\},
\]
\[
\rho^{-1}(S_L) := \{(x, y) : x \in D, \ H(x) - L\delta \leq y \leq H(x) + L\delta\}.
\]
In this case, we say \( S \) is the base of \( G \).

For \( r \in \mathbb{N} \), \( 0 < p \leq \infty \) and a nonempty set \( E \subset \mathbb{S}^d \), we define the directional Whitney constant by
\[
w_r(\Omega; E)_p := \sup \{ E_{(d+1)(r-1)}(f)_{L^p(\Omega)} : f \in L^p(\Omega), \ \omega^r(f, \Omega; E)_p \leq 1 \}.
\]

We remark that the above definition differs from the corresponding definition in \([9]\) by using approximation from the wider space \( \Pi^{d+1}_{d+1}(r-1) \) instead of certain “directional” polynomial space \( \Pi^{d+1}_{d+1}(E) \), see \([9]\) Prop. 1.1(ii)]. This results in smaller Whitney constants which are subject to the same upper bound as in the next lemma which is sufficient for our purposes here.

Lemma 2.3 (\([9]\) Lemma 2.5). Let \( G \subset \mathbb{R}^{d+1} \) be a regular \( \xi \)-directional domain with parameter \( L \geq 1 \) and base \( S \) as given in Definition 2.2 for some \( \xi \in \mathbb{S}^d \). Let \( E \subset \mathbb{S}^d \) be a set of directions containing \( \xi \). Assume that \( K \) is a measurable subset of \( \mathbb{R}^{d+1} \) such that \( S \subset K \cap G \) and \( w_r(K; E)_p < \infty \) for some \( r \in \mathbb{N} \), \( 0 < p \leq \infty \). Then
\[
w_r(G \cup K; E)_p \leq C_{p,r} L^{r-1+2/p}(1 + w_r(K; E)_p),
\]
where the constant \( C_{p,r} \) depends only on \( p \) and \( r \).

2.3. Bernstein inequality. If \( P \) is an algebraic polynomial of one variable of degree \( \leq n \), then by the univariate Bernstein inequality ([14], p. 265]), we have that for any \( b > 0 \) and \( \alpha > 1 \),
\[
\left\| (\sqrt{b^{-1}t} + n^{-1})^i P^{(i+j)}(t) \right\|_{L^p([0,b],dt)} \leq C_\alpha n^{i+2j} b^{-(i+j)} \| P \|_{L^p([0,ab])}.
\]

Let \( G \subset \mathbb{R}^{d+1} \) be an \( x_{d+1} \)-upward domain with base size \( b > 0 \) and parameter \( L \geq 1 \) given by
\[
G := \left\{ (x, y) \in \mathbb{R}^{d+1} : x \in (-b, b)^d, \ g(x) - Lb < y \leq g(x) \right\},
\]
where \( g : \mathbb{R}^d \to \mathbb{R} \) is a \( C^2 \)-function satisfying that \( \min_{x \in [-2b, 2b]^d} g(x) = 4Lb \). Denote for each \( \mu \in (0, 2] \)
\[
G(\mu) := \{(x, y) : x \in (-\mu b, \mu b)^d, \ g(x) - \mu Lb < y \leq g(x) \}.
\]
For \( (x, y) \in G(2) \), we define
\[
\delta(x, y) := g(x) - y \quad \text{and} \quad \varphi_n(x, y) := \sqrt{\delta(x, y)} + \frac{1}{n}, \quad n = 1, 2, \ldots.
\]
The Bernstein type inequality on the domain $G$ is formulated in terms of certain tangential derivatives along the essential boundary $\partial G$ of $G$, whose definition is given as follows. For $x_0 \in [-2a, 2a]^d$, let

$$\xi_j(x_0) := e_j + \partial_j g(x_0)e_{d+1}, \quad j = 1, \ldots, d$$

be the tangent vector to $\partial G$ at the point $(x_0, g(x_0))$ that is parallel to the $x_j x_{d+1}$-coordinate plane. We denote by $\partial^d \xi_j(x_0)$ the $\ell$-th order directional derivative along the direction of $\xi_j(x_0)$:

$$\partial^d \xi_j(x_0) := (\xi_j(x_0) \cdot \nabla)^\ell = \sum_{i=0}^{\ell} \binom{\ell}{i} (\partial_j g(x_0))^i \partial^{|i-1|} \partial^d \xi_{j+1},$$

where $j = 1, 2, \ldots, d$ and $x_0 \in [-2b, 2b]^d$. Thus, for $(x, y) \in G$ and $f \in C^1(G)$,

$$\partial^d \xi_j(x_0) f(x, y) = \sum_{i=0}^\ell \binom{\ell}{i} (\partial_j g(x))^i (\partial^{|i-1|} \partial^d \xi_i f)(x, y), \quad 1 \leq j \leq d.$$

We also need to deal with certain mixed directional derivatives. Let $N_0$ denote the set of all nonnegative integers. For $\alpha = (\alpha_1, \ldots, \alpha_d) \in N_0^d$, we set $|\alpha| = \alpha_1 + \alpha_2 + \cdots + \alpha_d$, and define

$$D^\alpha_{\text{tan}, x_0} = \partial^{\alpha_1}_{\xi_1(x_0)} \partial^{\alpha_2}_{\xi_2(x_0)} \cdots \partial^{\alpha_d}_{\xi_d(x_0)}, \quad x_0 \in [-2b, 2b]^d.$$

Finally, we are ready to state the required result.

**Theorem 2.4.** [S] Corollary 5.2 Let $\lambda \in (1, 2]$ and $\mu > 1$ be two given parameters. If $0 < p \leq \infty$ and $f \in \Pi_n^{d+1}$, then for any $\alpha \in N_0^d$, and $i, j = 0, 1, \ldots,$

$$\left\| \varphi_n(\xi)^i \max_{u \in \Xi_{n, \mu, \lambda}(\xi)} \| D^\alpha_{\text{tan}, u} \partial^{|i+j|}_{d+1} f(\xi) \|_{L^p(G; d\xi)} \right\|_{L^p(G(\lambda))} \leq c \mu n |\alpha| + |j-i| |f|_{L^p(G(\lambda))},$$

where

$$\Xi_{n, \mu, \lambda}(\xi) := \left\{ u \in [-\lambda b, \lambda b]^d : \|u - \xi\| \leq \mu \varphi_n(\xi) \right\}, \quad \xi = (\xi_x, \xi_y).$$

3. POLYNOMIAL PARTITIONS OF THE UNITY

3.1. Polynomial partitions of the unity on domains of special type. The main purpose in this section is to construct a localized polynomial partition of the unity on a domain $G \subset \mathbb{R}^{d+1}$ of special type. Without loss of generality, we may assume that $G$ is an upward $x_{d+1}$-domain given in (1.12) with $\xi = 0$, small base size $b > 0$ and parameter $L = b^{-1}$. Namely,

$$G := \{(x, y) : x \in [-b, b]^d, \ g(x) - 1 \leq y \leq g(x)\},$$

where $b \in (0, (2\sqrt{d})^{-1})$ is a sufficiently small constant and $g$ is a $C^2$-function on $\mathbb{R}^d$ satisfying that $\min_{x \in [-b, b]^d} g(x) \geq 4$.

Our construction of localized polynomial partition of the unity relies on a partition of the domain $G$, which we now describe. Given a positive integer $n$, let $\Lambda_n^d := \{0, 1, \ldots, n-1\}^d \subset \mathbb{Z}^d$ be an index set. We shall use boldface letters $i, j, \ldots$ to denote indices in the set $\Lambda_n^d$. For each $i = (i_1, \ldots, i_d) \in \Lambda_n^d$, define

$$\Delta_i := [t_{i_1}, t_{i_1+1}] \times \cdots \times [t_{i_d}, t_{i_d+1}] \quad \text{with} \quad t_i = -b - \frac{2i}{n},$$
Then \( \{ \Delta_i \}_{i \in \Lambda_n^d} \) forms a partition of the cube \([-b, b]^d\). Next, let \( N := N_n := \ell_1 n \) and \( \alpha := 1/(2\sin^2 \frac{\pi}{2\ell_1}) \), where \( \ell_1 \) is a sufficiently large positive integer such that \( \alpha \) satisfies
\[
\alpha \geq 5d \max_{x \in [-4b, 4b]^d} (|g(x)| + \max_{1 \leq i, j \leq d} |\partial_i \partial_j g(x)|).
\]
Let \( \{ \alpha_j := 2\alpha \sin^2 \left( \frac{\pi}{2\ell_j} \right) \}_{j=0}^N \) denote the Chebyshev partition of the interval \([0, 2\alpha] \) of order \( N \) such that \( \alpha_n = 1 \). Then \( \{ \alpha_j \}_{j=0}^n \) forms a partition of the interval \([0, 1] \).

Finally, we define a partition of the domain \( G \) as follows:
\[
G = \left\{ (x, y) : x \in [-b, b]^d, \ g(x) - y \in [0, 1] \right\} = \bigcup_{i \in \Lambda_n^d} \bigcup_{j=0}^{n-1} I_{i, j},
\]
where
\[
I_{i, j} := \left\{ (x, y) : x \in \Delta_i, \ g(x) - y \in [\alpha_j, \alpha_{j+1}] \right\}.
\]

Note that \( \Lambda_n^d \times \{0, \ldots, n-1\} = \Lambda_n^{d+1} \).

With the above notation, we have

**Theorem 3.1.** For any \( m \geq 2 \), there exists a sequence of polynomials \( \{q_{i, j} : (i, j) \in \Lambda_n^{d+1}\} \) of degree at most \( C(m, d)n \) on \( \mathbb{R}^{d+1} \) such that
\[
\sum_{(i, j) \in \Lambda_n^{d+1}} q_{i, j}(x, y) = 1 \quad \text{for all} \ (x, y) \in G,
\]
and for each \( (x, y) \in I_{k, l} \) with \( (k, l) \in \Lambda_n^{d+1} \),
\[
|q_{i, j}(x, y)| \leq \frac{C_{m,d}}{\left(1 + \max\{\|i - k\|, |j - l|\}\right)^m}.
\]

Theorem 3.1 is motivated by [24, Lemma 2.4], but some important details of the proof were omitted there. In this section, we shall give a complete and simpler proof of the theorem.

Recall that we write \( \xi \in \mathbb{R}^{d+1} \) in the form \( \xi = (\xi_x, \xi_y) \) with \( \xi_x \in \mathbb{R}^d \) and \( \xi_y \in \mathbb{R} \).

**Remark 3.2.** Recall that in [24] we introduced the following metric on the domain \( G \): for \( \xi = (\xi_x, \xi_y) \) and \( \eta = (\eta_x, \eta_y) \in G \),
\[
\hat{\rho}_G(\xi, \eta) = \max\left\{\|\xi_x - \eta_x\|, \left|\sqrt{g(\xi_x)} - \sqrt{g(\eta_x)}\right|, \sqrt{g(\xi_y)} - \sqrt{g(\eta_y)}\right\}.
\]
It can be easily seen that if \( \xi \in I_{i,j} \) and \( \eta \in I_{k,l} \), then
\[
1 + n\hat{\rho}_G(\xi, \eta) \sim 1 + \max\{\|i - k\|, |j - l|\}.
\]
This implies that
\[
|q_{i, j}(\xi)| \leq \frac{C_{m,d}}{(1 + n\hat{\rho}_G(\xi, \omega_{i,j}))^m}, \quad \forall \xi \in G, \ \forall \omega_{i,j} \in I_{i,j}.
\]

**Remark 3.3.** If \( r \in \mathbb{N} \) and \( n \geq 10r \), then the polynomials \( q_{i, j} \) in Theorem 3.1 can be chosen to be of total degree \( \leq n/r \). Indeed, this can be obtained by invoking Theorem 3.1 with \( c(m, d)n/r \) in place of \( n \), relabeling the indices, and setting some of the polynomials to be zero.

For the proof of Theorem 3.1 we need two additional lemmas, the first of which is well known.
Lemma 3.4. [24] Theorem 1.1] Given any parameter $\ell > 1$, there exists a sequence of polynomials \( \{u_j\}_{j=1}^{n} \) of degree at most $2n$ on $\mathbb{R}$ such that $\sum_{j=0}^{n-1} u_j(x) = 1$ for all $x \in [-1, 1]$ and

$$|u_j(\cos \theta)| \leq \frac{C_{\ell}}{(1 + n|\theta - \frac{i\pi}{n}|)^{\ell}}, \quad \theta \in [0, \pi], \quad j = 0, \ldots, n - 1.$$ 

The second lemma gives a polynomial partition of the unity associated with the partition $\{\Delta_j : j \in \Lambda_n^d\}$ of the cube $[-b, b]^d$.

Lemma 3.5. Given any parameter $\ell > 1$, there exists a sequence of polynomials $\{v^d_j\}_{j \in \Lambda_n^d}$ of total degree $\leq 2dn$ on $\mathbb{R}^d$ such that for all $x \in [-b, b]^d$, $\sum_{j \in \Lambda_n^d} v^d_j(x) = 1$ and

$$|v^d_j(x)| \leq \frac{C_{\ell,d}}{(1 + n\|x - x_j\|)^{\ell}}, \quad j \in \Lambda_n^d,$$

where $x_j$ is an arbitrary point in $\Delta_j$.

This lemma is probably well known, but for completeness, we present a proof below.

Proof. Without loss of generality, we may assume that $d = 1$ and $b = \frac{1}{2}$. The general case can be deduced easily using tensor products of polynomials in one variable. Let $\{u^d_j\}_{j=0}^{n-1}$ be a sequence of polynomials of degree at most $2n$ as given in Lemma 3.4 with $2\ell$ in place of $\ell$. Noticing that for $u \in [-1, 1]$ and $v \in [-\frac{1}{2}, \frac{1}{2}]$,

$$|u - v| \leq |\arccos u - \arccos v| \leq \pi|u - v|,$$

we obtain

$$(3.3) \quad |u_j(x)| \leq \frac{C_{\ell}}{(1 + n|x - \cos \frac{i\pi}{n}|)^{2\ell}}, \quad x \in \left[-\frac{1}{2}, \frac{1}{2}\right].$$

Next, we define a sequence of polynomials $\{v^1_j\}_{j=0}^{n-1}$ of degree at most $2n$ on $[-\frac{1}{2}, \frac{1}{2}]$ as follows:

$$v^1_j(x) = \sum_{i \colon s_i \cos \frac{i\pi}{n} \leq s_{i+1}} u_i(x),$$

where $0 \leq i \leq n - 1$, $s_0 = -2$, $s_n = 2$, $s_j = t_j = -\frac{1}{2} + \frac{j}{n}$ for $1 \leq j \leq n - 1$, and we define $v^1_j(x) = 0$ if the sum is taken over the empty set. Clearly, $\sum_{i=0}^{n-1} v^1_i(x) = 1$ for all $x \in [-\frac{1}{2}, \frac{1}{2}]$. Furthermore, using (3.3), we have

$$|v^1_j(x)| \leq \frac{C_{\ell}}{(1 + n|x - s_j|)^{\ell}} \sum_{i=0}^{n-1} \frac{1}{(1 + n|x - \cos \frac{i\pi}{n}|)^{2\ell}} \leq \frac{C_{\ell}}{(1 + n|x - s_j|)^{\ell}},$$

where the last step uses (3.3). This completes the proof. \(\square\)

We are now in a position to prove Theorem 3.1.

Proof of Theorem 3.1

Set

$$M := d \max_{1 \leq i, j \leq d} \max_{x \in [-b, b]^d} |\partial_i \partial_j g(x)| + 1.$$ 

For each $i \in \Lambda_n^d$, let $x_i \in \Delta_i$ be an arbitrarily fixed point in the cube $\Delta_i$, and define

$$f_i(x) := g(x_i) + \nabla g(x_i) \cdot (x - x_i) + \frac{M}{2} \|x - x_i\|^2.$$
By Taylor’s theorem, it is easily seen that for each \(x \in [-b, b]^d\),
\[
(3.4) \quad f_i(x) - M\|x - x_i\|^2 \leq g(x) \leq f_i(x).
\]
Since \(0 < b < (2\sqrt{d})^{-1}\), this implies that for each \(i \in \Lambda_n^d\),
\[
G \subset \left\{(x, y) : \; x \in [-b, b]^d, \; 0 \leq f_i(x) - y \leq M + 1\right\}.
\]
Recall that \(\{\alpha_j\}_{j=0}^N\) is a Chebyshev partition of \([\alpha_0, \alpha_N] = [0, 2\alpha]\) of degree \(N = 2\ell_n\), \(\alpha_n = 1\) and according to (3.1), \(\alpha \geq 4M + 1\). Thus,
\[
G \subset \bigcup_{i \in \Lambda_n^d} \bigcup_{j=0}^{N-1} \left\{(x, y) : \; x \in \Delta_i, \; \alpha_j \leq f_i(x) - y \leq \alpha_{j+1}\right\}.
\]

Next, using Lemma 3.4 we obtain a sequence of polynomials \(\{u_j\}_{j=0}^{N-1}\) of degree at most \(4\ell_n\) on \([0, 2\alpha]\) such that \(\sum_{j=0}^{N-1} u_j(t) = 1\) for all \(t \in [0, 2\alpha]\), and
\[
(3.5) \quad |u_j(t)| \leq \frac{C_m}{(1 + n\|t - \alpha_j\|)^{4m}}, \quad t \in [0, 2M] \subset [0, 2\alpha].
\]
Similarly, using Lemma 3.5 we may obtain a sequence of polynomials \(\{v_j\}_{i \in \Lambda_n^d}\) of total degree \(\leq n\) on the cube \([-b, b]^d\) such that \(\sum_{j \in \Lambda_n^d} v_j(x) = 1\) for all \(x \in [-b, b]^d\), and
\[
(3.6) \quad |v_j(x)| \leq \frac{C_m}{(1 + n\|x - x_j\|)^{4m}}, \quad x \in [-b, b]^d.
\]
Define a sequence \(\{q_{i,j}^*: \; i \in \Lambda_n^d, \; 0 \leq j \leq N - 1\}\) of auxiliary polynomials as follows:
\[
(3.7) \quad q_{i,j}^*(x, y) := u_j(f_i(x) - y)v_1(x).
\]
It is easily seen from (3.5) and (3.6) that for each \((x, y) \in G\),
\[
(3.8) \quad |q_{i,j}^*(x, y)| \leq \frac{C_m}{(1 + n\|x - x_j\|)^{4m}(1 + n\|\sqrt{f_i(x)} - y - \sqrt{\alpha_j}\|)^{4m}}.
\]

We claim that for each \((x, y) \in G\),
\[
(3.9) \quad |q_{i,j}^*(x, y)| \leq \frac{C_m}{(1 + n\|x - x_j\|)^{2m}(1 + n\|g(x) - y - \sqrt{\alpha_j}\|)^{2m}}.
\]

Note that (3.9) follows directly from (3.8) if \(6M\|x - x_j\| > \|\sqrt{g(x)} - y - \sqrt{\alpha_j}\|\). Thus, for the proof of (3.9), it suffices to prove that the equivalence
\[
(3.10) \quad |\sqrt{f_i(x)} - y - \sqrt{\alpha_j}| \sim |\sqrt{g(x)} - y - \sqrt{\alpha_j}|,
\]
holds under the assumption
\[
(3.11) \quad 6M\|x - x_j\| \leq |\sqrt{g(x)} - y - \sqrt{\alpha_j}|.
\]
Indeed, if \(\sqrt{f_i(x)} - y + \sqrt{g(x)} - y \leq 2M\|x - x_j\|\), then (3.11) implies
\[
\sqrt{\alpha_j} \geq 4M\|x - x_j\| \geq 2\max\{\sqrt{f_i(x)} - y, \sqrt{g(x)} - y\},
\]
and hence
\[
|\sqrt{f_i(x)} - y - \sqrt{\alpha_j}| \sim \sqrt{\alpha_j} \sim |\sqrt{g(x)} - y - \sqrt{\alpha_j}|.
\]
On the other hand, if \( \sqrt{f(x)} - y + \sqrt{g(x)} - y > 2M\|x - x_i\| \), then by (3.11) and (3.4), we have

\[
\left| \sqrt{f(x)} - y - \sqrt{g(x)} - y \right| = \frac{|f(x) - g(x)|}{\sqrt{f(x)} - y + \sqrt{g(x)} - y}
\]

\[
\leq \frac{M\|x - x_i\|^2}{2M\|x - x_i\|} = \frac{1}{2}\|x - x_i\| \leq \frac{1}{12M} |\sqrt{g(x)} - y - \sqrt{\alpha_j}|,
\]

which in turn implies (3.10). This completes the proof of (3.9).

Finally, we define for \( i \in \Lambda_n^d \),

\[
q_{i,j}(x, y) = \begin{cases} q_{i,j}^*(x, y), & \text{if } 0 \leq j \leq n - 2, \\ \sum_{k=n-1}^{N-1} q_{i,k}^*(x, y), & \text{if } j = n - 1. \end{cases}
\]

Clearly, each \( q_{i,j} \) is a polynomial of degree at most \( Cn \). Since for any \( (x, y) \in G \) the polynomial \( u_j \) in the definition (3.7) is evaluated at the point \( f_i(x) - y \), which lies in the interval \( [0, M + 1] \subset [\alpha_0, \alpha_N] \), it follows that for any \( (x, y) \in G \),

\[
\sum_{i \in \Lambda_n^d} \sum_{j=0}^{n-1} q_{i,j}(x, y) = \sum_{i \in \Lambda_n^d} \sum_{j=0}^{N-1} q_{i,j}^*(x) = \sum_{i \in \Lambda_n^d} u_j(f_i(x) - y) = 1.
\]

To complete the proof, by (3.9), it remains to estimate \( q_{i,j} \) for \( j = n - 1 \). Note that for \( j \geq n \),

\[
\sqrt{\alpha_j} - \sqrt{g(x)} - y \geq \sqrt{\alpha_n} - \sqrt{g(x)} - y \geq 0.
\]

Thus, using (3.9), and recalling that \( m \geq 2 \), we obtain that

\[
|q_{i,n-1}(x)| \leq \frac{C_m}{(1 + n\|x - x_i\|)^{2m}(1 + n|\sqrt{g(x)} - y - \sqrt{\alpha_n}|)^m} \cdot \sum_{j=n}^{N} \frac{1}{(1 + n|\sqrt{g(x)} - y - \sqrt{\alpha_j}|)^m}
\]

\[
\leq \frac{C_m}{(1 + n\|x - x_i\|)^{2m}(1 + n|\sqrt{g(x)} - y - \sqrt{\alpha_n}|)^m}.
\]

This completes the proof. \( \square \)

### 3.2. Polynomial partitions of the unity on general \( C^2 \)-domains

In this section, we shall extend Theorem 3.3 to the \( C^2 \)-domain \( \Omega \). We will use the metric \( \rho_\Omega \) defined by (1.2). Our goal is to show the following theorem:

**Theorem 3.6.** Given any \( m > 1 \) and any positive integer \( n \), there exist a finite subset \( \Lambda \) of \( \Omega \) and a sequence \( \{ \varphi_\omega \}_{\omega \in \Lambda} \) of polynomials of degree at most \( C(m)n \) on the domain \( \Omega \) satisfying

(i) \( \rho_\Omega(\omega, \omega') \geq \frac{1}{n} \) for any two distinct points \( \omega, \omega' \in \Lambda \);

(ii) for every \( \xi \in \Omega \), \( \sum_{\omega \in \Lambda} \varphi_\omega(\xi) = 1 \) and

(iii) for any \( \xi \in \Omega \) and \( \omega \in \Lambda \),

\[
|\varphi_\omega(\xi)| \leq C_m(1 + n\rho_\Omega(\xi, \omega))^{-m}.
\]

**Remark 3.7.** Recall that for \( \xi \in \Omega \) and \( \delta > 0 \), we defined \( U(\xi, \delta) = \{ \eta \in \Omega : \rho_\Omega(\xi, \eta) \leq \delta \} \). By [3 Corollary 3.3(ii)], we have

\[
\left| U\left( \xi, \frac{1}{n} \right) \right| \sim \frac{1}{n^{d+1}} \left( \frac{1}{n} + \sqrt{\text{dist}(\xi, \Gamma)} \right), \quad \xi \in \Omega.
\]
Proof of Theorem 3.6. For convenience, we say a subset $K \subset \Omega$ admits a polynomial partition of the unity of degree $Cn$ with parameter $m > 1$ if there exist a finite subset $\Lambda \subset \Omega$ and a sequence $\{\varphi_\omega\}_{\omega \in \Lambda}$ of polynomials of degree at most $Cn$ such that $\rho_\Omega(\omega, \omega') \geq \frac{1}{n}$ for any two distinct points $\omega, \omega' \in \Lambda$, $\sum_{\omega \in \Lambda} \varphi_\omega(x) = 1$ for every $x \in K$ and $|\varphi_\omega(x)| \leq C(1 + n\rho_\Omega(x, \omega))^{-m}$ for every $x \in K$ and $\omega \in \Lambda$, in which case $\{\varphi_\omega\}_{\omega \in \Lambda}$ is called a polynomial partition of the unity of degree $Cn$ on the set $K$. According to Theorem 3.1 Remark 3.2 and Proposition 2.4 if $G \subset \Omega$ is a domain of special type attached to $\Gamma$ or if $G = Q$ is a cube such that $4Q \subset \Omega$, then for any $m > 1$, $G$ admits a polynomial partition of the unity of degree $Cn$ with parameter $m$.

Our proof relies on the decomposition in Lemma 4.5. Let $\{\Omega_i\}_{i=1}^J$ be the sequence of subsets of $\Omega$ given in Lemma 4.5. For $1 \leq j \leq J$, let $H_j = \bigcup_{i=1}^{J_j} \Omega_i$. Assume that for some $1 \leq j \leq J - 1$, $H_j$ admits a polynomial partition $\{u_{\omega_i}\}_{i=1}^{J_j}$ of the unity of degree $Cn$ with parameter $m > 1$. By induction and Lemma 4.5 it will suffice to show that $H_{j+1}$ also admits a polynomial partition of the unity of degree $Cn$ with parameter $m > 1$. For simplicity, we write $H = H_j$ and $K = \Omega_{j+1}$. Without loss of generality, we may assume that $\gamma_0 = S_{G, \lambda_0}$ with $\lambda_0 \in (\frac{1}{2}, 1)$ and $G \subset \Omega$ a domain of special type attached to $\Gamma$. The case when $K = Q$ is a cube such that $4Q \subset \Omega$ can be treated similarly, and in fact, is simpler.

By Theorem 3.1 $G$ admits a polynomial partition $\{u_{\omega_j}\}_{j=n_0+1}^{n_0+n_1}$ of the unity of degree $Cn$ with parameter $m > 1$. Recall $H \cap G$ contains an open ball of radius $\gamma_0 \in (0, 1)$. Let $L > 1$ be such that $\Omega \subset B_L[0]$, and let $\theta := \frac{2\pi}{L} \in (0, 1)$. According to Lemma 4.10 there exists a polynomial $R_n$ of degree at most $Cn$ such that $0 \leq R_n(\xi) \leq 1$ for $\xi \in B_L[0]$, $1 - R_n(\xi) \leq \theta^n$ for $\xi \in K$ and $R_n(\xi) \leq \theta^n$ for $x \in \Omega \setminus G$. We now define

$$w_j(\xi) = \begin{cases} u_{\omega_j}(\xi)(1 - R_n(\xi)), & \text{if } 1 \leq j \leq n_0, \\ u_{\omega_j}(\xi)R_n(\xi), & \text{if } n_0 + 1 \leq j \leq n_0 + n_1. \end{cases}$$

Clearly, each $w_j$ is a polynomial of degree at most $Cn$ on $\mathbb{R}^{d+1}$. Since polynomials are analytic functions and $H \cap G$ contains an open ball of radius $\gamma_0$, it follows that

$$\sum_{j=1}^{n_0+n_1} w_j(\xi) = R_n(\xi) + 1 - R_n(\xi) = 1, \quad \forall \xi \in \mathbb{R}^{d+1}.$$

Next, we prove that for each $1 \leq j \leq n_0 + n_1$,

$$|w_j(\xi)| \leq C(1 + n\rho_\Omega(\xi, \omega_j))^{-m}, \quad \forall \xi \in H \cup K.$$  

(3.12)

Indeed, if $1 \leq j \leq n_0$, then for $\xi \in H$,

$$|w_j(\xi)| \leq |u_{\omega_j}(\xi)| \leq C(1 + n\rho_\Omega(\xi, \omega_j))^{-m},$$

whereas for $\xi \in K \subset G$,

$$|w_j(\xi)| \leq \theta^n \|u_{\omega_j}\|_{L^\infty(B_L[0])} \leq C\theta^n \left(\frac{10L}{\gamma_0}\right)^n \|u_{\omega_j}\|_{L^\infty(H \cap G)} \leq C2^{-n} \leq C_m(1 + n\rho_\Omega(\xi, \omega_j))^{-m},$$

where the second step uses Lemma 4.5. Similarly, if $n_0 < j \leq n_0 + n_1$, then for $\xi \in G$,

$$|w_j(\xi)| \leq |u_{\omega_j}(\xi)| \leq C_m(1 + n\rho_\Omega(\xi, \omega_j))^{-m},$$
whereas for \( \xi \in H \setminus G \),

\[
|w_j(\xi)| \leq \theta^n \|w\|_{L^\infty(B_{\xi}(0))} \leq C\theta^n \left( \frac{10L}{\theta_0} \right)^n \leq C2^{-n} \leq C_m(1 + n\rho_3(\xi, \omega_j))^{-m}.
\]

Thus, in either case, we prove the estimate \( \text{(3.12)} \).

Finally, we write the set \( H \) of degree \( cn \) by \( [8, \text{Corollary 3.3(iii)}] \) we have that \# \( \Omega \) \( \leq \frac{1}{n} \) for each \( \omega \in \Lambda \). We then define

\[
\varphi_\omega(\xi) := \sum_{j: \omega_j \in I_\omega} w_j(\xi), \quad \xi \in H \cup G, \quad \omega \in \Lambda,
\]

where \( 1 \leq j \leq n_0 + n_1 \). Clearly, each \( \varphi_\omega \) is a polynomial of degree at most \( Cn \) and

\[
\sum_{\omega \in \Lambda} \varphi_\omega(\xi) = \sum_{j=1}^{n_0+n_1} w_j(\xi) = 1, \quad \forall \xi \in H \cup G.
\]

On the other hand, we recall that \( \rho_3(\omega_i, \omega_j) \geq \frac{1}{n} \) if \( 1 \leq i \neq j \leq n_0 \) or \( n_0 + 1 \leq i \neq j \leq n_0 + n_1 \). Thus, by the standard volume estimates and Remark 3.7 (or directly by \([8, \text{Corollary 3.3(iii)}]\)) we have that \( \#I_\omega \leq C(\Omega, m) \) for each \( \omega \in \Lambda \), where \#I denotes the cardinality of a set \( I \). It then follows from \( \text{(3.12)} \) that

\[
|\varphi_\omega(\xi)| \leq C(1 + n\rho_3(\xi, \omega))^{-m}, \quad \xi \in H \cup G, \quad \omega \in \Lambda.
\]

Thus, we have shown that the set \( H \cup K \) admits a polynomial partition of the unity of degree \( cn \) with parameter \( m \), completing the induction.

\( \square \)

**Remark 3.8.** The above proof implies \( \#\Lambda = O(n^{d+1}) \); recall that \( \Omega \subset \mathbb{R}^{d+1} \).

4. Geometric reduction near the boundary

Our main goal in this section is to show that the Jackson inequality in Theorem 1.9 can be deduced from the following Jackson-type estimates on domains of special type.

**Theorem 4.1.** If \( 0 < p \leq \infty \), \( c > 0 \) is arbitrary fixed, and \( G \subset \Omega \) is an upward or downward \( x_j \)-domain attached to \( \Gamma \) for some \( 1 \leq j \leq d+1 \), then

\[
E_n(f)_{L^p(G)} \leq C \left[ \omega_G(f, \frac{c}{n})_p + \omega_{G,x}(f, \frac{c}{n}, (e_j)_p) \right],
\]

where the constant \( C \) is independent of \( f \) and \( n \).

The proof of Theorem 4.1 will be given in Section 5.1. In this section, we will show how Theorem 1.9 can be deduced from Theorem 4.1. The idea of our proof is close to that in \([44, \text{Chapter 7}]\).

4.1. Lemmas and geometric reduction. We need a series of lemmas, the first of which gives a well known Jackson type estimate (see \([15, \text{Theorem 1.1}]\)) on a rectangular box (recall that we always assume that the sides of such boxes are parallel to the coordinate axes).

**Lemma 4.2.** Let \( B \) be a compact rectangular box in \( \mathbb{R}^{d+1} \). Assume that \( f \in L^p(B) \) if \( 0 < p < \infty \) and \( f \in C(B) \) if \( p = \infty \). Then for \( 0 < p \leq \infty \),

\[
\inf_{P \in \Pi^{d+1}_n} \|f - P\|_{L^p(B)} \leq C \max_{1 \leq j \leq d+1} \omega_{B,x}(f, \frac{1}{n}, (e_j)_p),
\]

where \( \omega_{B,x}(f, \frac{1}{n}, (e_j)_p) \) is the modulus of continuity of \( f \) with respect to \( x_j \) at the point \( \frac{1}{n} \) on the interval \( [0, 1] \) in the \( x_j \)-direction.
where $C$ is independent of $f$.

Our second lemma is a simple observation on domains of special type. Recall that unless otherwise stated we always assume that the parameter $L$ of a domain of special type satisfies the condition (1.13).

**Lemma 4.3.** Let $G \subset \Omega$ be an (upward or download) $x_j$-domain of special type attached to $\Gamma$ for some $1 \leq j \leq d + 1$. Then for each parameter $\mu \in (\frac{1}{2}, 1]$, there exists an open rectangular box $Q_\mu$ in $\mathbb{R}^{d+1}$ such that

$$\partial' G(\mu) \subset S_{G, \mu} := Q_\mu \cap \Omega \subset G(\mu) \quad \text{and} \quad \overline{\cup_{\mu}} \subset Q_1 \quad \text{provided} \ \mu < 1.$$  

**Proof.** Without loss of generality, we may assume that $G$ is given in (1.12) with $\xi = 0$. Let $g_{\max} := \max_{x \in [-b, b]^d} g(x)$ and $g_{\min} := \min_{x \in [-b, b]^d} g(x)$. Using (1.13), we have

$$g_{\max} - g_{\min} \leq 2\sqrt{\max_{x \in [-b, b]^d} \|\nabla g(x)\|} \leq \frac{1}{2} Lb.$$  

Thus, given each parameter $\mu \in (\frac{1}{2}, 1]$, we may find a constant $a_{1, \mu}$ such that

$$g_{\max} - \mu Lb < a_{1, \mu} < g_{\min}.$$  

We may choose the constant $a_{1, \mu}$ in such a way that $a_{1, 1} = a_{1, \mu}$ if $\mu < 1$. On the other hand, since $G$ is attached to $\Gamma$, we may find an open rectangular box $Q$ of the form $(-2b, 2b)^d \times (a_1, a_2)$ such that $G^* = Q \cap \Omega$, where $a_1, a_2$ are two constants and $a_2 > g_{\max}$. Let $a_{2, 1} = a_2$ and let $a_{2, \mu}$ be a constant so that $g_{\max} < a_{2, \mu} < a_2$ for $\mu \in (\frac{1}{2}, 1)$. Now setting

$$Q_\mu := (-\mu b, \mu b)^d \times (a_{1, \mu}, a_{2, \mu}) \quad \text{and} \quad S_{G, \mu} := Q_\mu \cap \Omega,$$

we obtain (4.1). \hfill \Box

**Remark 4.4.** Note that (1.1) implies that $\text{proj}_j(Q_\mu) = \text{proj}_j(G(\mu))$ for $\mu \in (\frac{1}{2}, 1]$, where $\text{proj}$ denotes the orthogonal projection onto the coordinate plane $x_j = 0$.

Now let $G_1, \ldots, G_{m_0} \subset \Omega$ be the domains of special type in Lemma 1.6. Note that for every domain $G$ of special type, its essential boundary can be expressed as $\partial' G = \bigcup_{n=1}^{\infty} \partial' G(1 - n^{-1})$. Since $G$ is compact and each $\partial' G_j$ is open relative to the topology of $\Gamma$, there exists $\lambda_0 \in (\frac{1}{2}, 1)$ such that $\Gamma = \bigcup_{j=1}^{m_0} \partial' G_j(\lambda_0)$. For convenience, we call $S \subset \Omega$ an admissible subset of $\Omega$ if either $S = S_{G_j, \lambda_0}$ for some $1 \leq j \leq m_0$ or $S$ is an open cube in $\mathbb{R}^{d+1}$ such that $4S \subset \Omega$.

Our third lemma gives a useful decomposition of the domain $\Omega$.

**Lemma 4.5.** There exists a sequence $\{\Omega_s\}_{s=1}^{J}$ of admissible subsets of $\Omega$ such that

$$\Omega = \bigcup_{j=1}^{J} \Omega_j, \quad \text{and} \quad \Omega_s \cap \Omega_{s+1} \text{ contains an open ball of radius } \gamma_0 > 0 \text{ in } \mathbb{R}^{d+1} \text{ for each } s = 1, \ldots, J - 1,$$

where the parameters $J$ and $\gamma_0$ depend only on the domain $\Omega$.

To state the fourth lemma, let $\{\Omega_s\}_{s=1}^{J}$ be the sequence of sets in Lemma 4.5 and let $H_m := \bigcup_{j=1}^{m} \Omega_j$ for $m = 1, \ldots, J$. For $1 \leq j \leq J$, define $\tilde{\Omega}_j = G_j$ if $\Omega_j = S_{G_j, \lambda_0}$ for some $1 \leq i \leq m_0$; and $\tilde{\Omega}_j = 2Q$ if $\Omega_j$ is an open cube $Q$ such that $4Q \subset \Omega$.

**Lemma 4.6.** If $0 < p < \infty$ and $1 \leq j < J$, then there exist constants $c_0, C > 1$ depending only on $p$ and $\Omega$ such that

$$E_{c_0}(f)_{L^p(H_{j+1})} \leq C \max\{E_{0}(f)_{L^p(\tilde{\Omega}_{j+1})}, E_{0}(f)_{L^p(H_{j})}\}.$$
We also need a technical inequality which directly follows from the definition (1.10) and from the growth properties of the one-dimensional Ditzian-Totik modulus [22 (4.1.3), p. 38] and [20 (5.7)]. For any fixed \( c > 0 \)
\[
\omega^r_{\Omega, \varphi}(f, t)_p \leq C\omega^r_{\Omega, \varphi}(f, ct)_p, \quad t > 0,
\]
where \( C \) is independent of \( f \) and \( t \).

Now we take Theorem 1.9, Lemma 4.5 and Lemma 4.6 for granted and proceed with the proof of Theorem 1.9.

**Proof of Theorem 1.9.** Applying Lemma 4.6 \( J - 1 \) times and recalling \( H_J = \Omega \), we obtain
\[
E_{c,n}(f)_{L^p(\Omega)} \leq C \max_{1 \leq j \leq J} E_n(f)_{L^p(\hat{\Omega}_j)},
\]
where \( C, c_1 > 1 \) depend only on \( p \) and \( \Omega \). If \( \Omega_j = S_{G_{\Omega}, \lambda_0} \) for some \( 1 \leq i \leq m_0 \), then \( \hat{\Omega}_j = G_1 \), and by Theorem 4.1
\[
E_n(f)_{L^p(\hat{\Omega}_j)} \leq \max_{1 \leq j \leq m_0} E_n(f)_{L^p(G_i)} \leq C\omega^r_{\Omega}(f, c/c_1n)_p.
\]
If \( \Omega_j = Q \) is a cube such that \( 4Q \subset \Omega \), then \( \hat{\Omega}_j = 2Q \) and by Lemma 4.2
\[
E_n(f)_{L^p(2Q)} \leq C \max_{1 \leq j \leq d+1} \omega^r_{S_{Q}, \varphi}(f, n^{-1}; e_j)_p \leq C\omega^r_{\Omega, \varphi}(f, n^{-1})_p,
\]
where the last step uses the fact that for any \( S \subset \Omega \)
\[
\max_{1 \leq j \leq d+1} \omega^r_{S, \varphi}(f, t; e_j)_p \leq C\omega^r_{\Omega, \varphi}(f, t)_p.
\]
By (4.2), \( \omega^r_{\Omega, \varphi}(f, n^{-1})_p \leq C\omega^r_{\Omega, \varphi}(f, c/(c_1n))_p \), thus, in either case, we have
\[
E_n(f)_{L^p(\hat{\Omega}_j)} \leq C\omega^r_{\Omega}(f, c/c_1n)_p.
\]
Theorem 1.9 then follows from the estimate (4.3). \( \square \)

**Remark 4.7.** It is clear from the proof that a slightly stronger version of Theorem 1.9 is true. Namely, for arbitrary \( c > 0 \), under the same hypotheses we obtain
\[
E_n(f)_{L^p(\Omega)} \leq C\omega^r_{\Omega}(f, c/n)_p, \quad \text{where} \quad C \text{ depends only on } \Omega, \ r, \ p \text{ and } c.
\]
While it would be desirable to simply use the growth condition of the type (4.2) directly for our modulus \( \omega^r_{\Omega}(f, t)_p \), it appears that establishing an analog of (4.2) for the tangential component of \( \omega^r_{\Omega}(f, t)_p \) is not immediate. We hope to obtain this in a future work.

To complete the reduction argument in this section, it remains to prove Lemma 4.5 and Lemma 4.6.

4.2. **Proof of Lemma 4.5.** The proof of Lemma 4.5 is inspired by [43, p. 17] but written in somewhat different language. Let \( S_j = S_{G_{\Omega}, \lambda_0} \) for \( 1 \leq j \leq m_0 \). Note that \( S_j \) is an open neighborhood of \( \partial G_{\Omega}(\lambda) \) relative to the topology of \( \Omega \). Since \( \partial G_{\Omega}(\lambda_0) \subset S_j \subset \Omega \) and \( S_j \) is open relative to the topology of \( \Omega \) for each \( 1 \leq j \leq m_0 \), there exists \( \varepsilon > 0 \) such that
\[
\Gamma_{\varepsilon} := \{ \xi \in \Omega : \ dist(\xi, \Gamma) < 16\sqrt{d+1}\xi \} \subset \bigcup_{j=1}^{m_0} S_j.
\]
Let us cover the remaining set $\Omega \setminus \Gamma_\varepsilon$ by finitely many open cubes $Q_j$, $j = m_0 + 1, \ldots, M_0$ of side length $\varepsilon$ such that $4Q_j \subset \Omega$ for each $j$. Thus, setting $E_j = S_j$ for $1 \leq j \leq m_0$, and $E_j = Q_j$ for $m_0 < j \leq M_0$, we have $\Omega = \bigcup_{j=1}^{M_0} E_j$. The required sets $\Omega_s$, $s = 1, \ldots, J$, will be selected from the family of the sets $\{E_j\}_{j=1}^{M_0}$, with possibly choosing the same set multiple times, so that each intersection $\Omega_s \cap \Omega_{s+1}$, $s = 1, \ldots, J-1$, contains a non-empty open ball.

First, note that if $E_j \cap E_{j'} \neq \emptyset$ for some $1 \leq j, j' \leq M_0$, then $E_j \cap E_{j'}$ must contain a nonempty open ball in $\mathbb{R}^{d+1}$. Indeed, since $E_j \cap E_{j'}$ is open relative to the topology of $\Omega$, there exists an open set $V$ in $\mathbb{R}^{d+1}$ such that $V \cap \Omega = E_j \cap E_{j'} \neq \emptyset$.

Since $\Omega$ is the closure of an open set in $\mathbb{R}^{d+1}$, the set $V \cap \Omega$ must contain an interior point of $\Omega$.

Next, we set $A = \{E_1, \ldots, E_{M_0}\}$. We say two sets $A, B$ from the collection $A$ are connected with $E_1$. Assume that $A \neq B$. We obtain a contradiction as follows. Let $H := \bigcup_{E \in B} E$. Then a set $E$ from the collection $A$ is connected with $E_1$ (i.e., $E \in B$) if and only if $E \cap H \neq \emptyset$. Since $A \neq B$, there exists $E \in A$ such that $E \cap H = \emptyset$, which in particular, implies that $H$ is a proper subset of $\Omega = \bigcup_{A \in A} A$. Since $\Omega$ is a connected subset of $\mathbb{R}^{d+1}$, $H$ must have nonempty boundary relative to the topology of $\Omega$. Let $x_0$ be a boundary point of $H$ relative to the topology of $\Omega$. Since $H$ is open relative to $\Omega$, $x_0 \in \Omega \setminus H = \bigcup_{A \in A} A \setminus H$. Let $A_0$ be such that $x_0 \in A_0$. Then $A_0$ is an open neighborhood of $x_0$ relative to the topology of $\Omega$, and hence $A_0 \cap H \neq \emptyset$, which in turn implies $A_0 \in B$ and $A_0 \subset H$. But this is impossible as $x_0 \notin H$.

4.3. Proof of Lemma 4.6. We now turn to the proof of Lemma 4.6. The proof relies on three additional lemmas. The first one is similar to [43, Lemma 14.3], however, we could not follow the conclusion of its proof in [43], where some averaging argument appears to be missing. Our proof below uses a multivariate Nikol’skii inequality which simplifies the transition to the multivariate case.

Lemma 4.8. If $B$ is a ball in $\mathbb{R}^{d+1}$ and $\lambda > 1$, then for each $P \in \Pi_n^{d+1}$ and $0 < q \leq \infty$,

$$
\|P\|_{L^q(\lambda B)} \leq C_{d,q}(5\lambda)^{n+\frac{d+1}{q}} \|P\|_{L^q(B)}.
$$

Proof. By dilation and translation, we may assume that $B = B_1[0]$. \[(4.4)\] with the explicit constant $C_{d}(\lambda)^n$ was proved in [43, Lemma 4.2] for $q = \infty$. For $q < \infty$, we have

$$
\|P\|_{L^q(\lambda B)} \leq C_{d}\lambda^{\frac{d+1}{q}} \|P\|_{L^q(\lambda B)} \leq C_{d}\lambda^{\frac{d+1}{q}} \|P\|_{L^q(B)}
$$

$$
\leq C_{d,q}(5\lambda)^{n+\frac{d+1}{q}} \|P\|_{L^q(B)} \leq C_{d,q}(5\lambda)^{n+\frac{d+1}{q}} \|P\|_{L^q(B)}.
$$
where we used Hölder’s inequality in the first step, (4.3) for the already proven case $q = \infty$ in the second step, and Nikol’skii’s inequality for algebraic polynomials on the unit ball (see [7] or [21 Section 7]) in the third step.

The second lemma is probably well known. It can be proved in the same way as in [43 Lemma 4.3].

**Lemma 4.9.** Let $I$ be a parallelepiped in $\mathbb{R}^d$. Then given parameters $R > 1$ and $\theta, \mu \in (0, 1)$, there exists a polynomial $P_n$ of degree at most $C(\theta, \mu, R, d)n$ such that $0 \leq P_n(\xi) \leq 1$ for $\xi \in B_R[0], 1 - P_n(\xi) \leq \theta^n$ for $\xi \in \mu I$, and $P_n(\xi) \leq \theta^n$ for $\xi \in B_R[0] \setminus I$, where $\mu I$ denotes the dilation of $I$ from its center by a factor $\mu$.

As a consequence of Lemma 4.9 we have

**Lemma 4.10.** Let $G \subset \Omega$ be a domain of special type attached to $\Gamma$, and $S_{G,\mu} := \Omega \cap Q_\mu$ be as defined in Lemma 4.3 with $\mu \in (\frac{1}{2}, 1]$. Let $R \geq 1$ be such that $Q_1 \cap \Omega \subset B_R[0]$. Then given $\lambda \in (\frac{1}{2}, 1)$ and $\theta \in (0, 1)$, there exists a polynomial $P_n$ of degree at most $C(d, \theta, R, G, \lambda)n$ with the properties that $0 \leq P_n(\xi) \leq 1$ for $\xi \in B_R[0], 1 - P_n(\xi) \leq \theta^n$ for $\xi \in S_{G,\lambda}$ and $P_n(\xi) \leq \theta^n$ for $\xi \in \Omega \setminus S_{G,1}$.

**Proof.** Since $\lambda < 1$ and $Q_\lambda$ is an open rectangular box such that $\overline{Q_\lambda} \subset Q_1$, it follows by Lemma 4.9 that there exists a polynomial $P_n$ of degree at most $Cn$ such that $0 \leq P_n(\xi) \leq 1$ for all $\xi \in B_R[0], 1 - P_n(\xi) \leq \theta^n$ for all $\xi \in Q_\lambda$ and $P_n(\xi) \leq \theta^n$ for all $\xi \in B_R[0] \setminus Q_1$. To complete the proof, we just need to observe that

$$\Omega \setminus S_{G,1} = \Omega \setminus (Q_1 \cap \Omega) = \Omega \setminus Q_1 \subset B_R[0] \setminus Q_1.$$

We are now in a position to prove Lemma 4.6.

**Proof of Lemma 4.6.** The proof is essentially a repetition of that of [43 Lemma 4.1] or [44 Lemma 3.3] for our situation. Let $R > 1$ be such that $\Omega \subset B_R[0]$, and set $\theta := \min\{\frac{2}{R^2}, \frac{1}{2}\}$. Write $H = H_j$ and $S = \Omega_{j+1}$. Without loss of generality, we may assume that $S = S_{G,\lambda_0}$ for some domain $G$ of special type attached to $\Gamma$. (The case when $S$ is a cube $Q$ such that $4Q \subset \Omega$ can be proved similarly using Lemma 4.9 instead of Lemma 4.10.) Then $S_{G,\lambda_0} \cap H$ contains a ball $B$ of radius $r_0$.

By Lemma 4.10 there exists a polynomial $R_n$ of degree $\leq C(d, R, G)$ such that $0 \leq R_n(x) \leq 1$ for all $x \in B_R[0], 1 - R_n(x) \leq \theta^{-n}$ for $x \in \Omega \setminus S_{G,1}$ and $1 - R_n(x) \leq \theta^{-n}$ for $x \in S_{G,\lambda_0}$. Let $P_1, P_2 \in \Pi_n^{d+1}$ be such that

$$E_n(f)_{L^p(S_{G,1})} = \|f - P_1\|_{L^p(S_{G,1})} \quad \text{and} \quad E_n(f)_{L^p(H)} = \|f - P_2\|_{L^p(H)}.$$

Define

$$P(x) := R_n(x)P_1(x) + (1 - R_n(x))P_2(x) \in \Pi_{cn}^{d+1}.$$

Then

$$E_n(f)_{L^p(H_{j+1})} \leq \|f - P\|_{L^p(H \cup S_{G,\lambda_0})} \leq \|f - P\|_{L^p(H \cup S_{G,1})} + \|f - P\|_{L^p(H \setminus S_{G,1})} + \|f - P\|_{L^p(S_{G,\lambda_0})}.$$
First, we can estimate the term $\|f - P\|_{L^p(H \cap S_{G,1})}$ as follows:

$$
\|f - P\|_{L^p(H \cap S_{G,1})} = \|R_n(f - P) + (1 - R_n)(f - P)\|_{L^p(H \cap S_{G,1})}
\leq C_p \max \left\{ \|f - P_1\|_{L^p(S_{G,1})}, \|f - P_2\|_{L^p(H)} \right\}
\leq C_p \max \left\{ E_n(f)_{L^p(S_{G,1})}, E_n(f)_{L^p(H)} \right\}.
$$

Second, we show

$$
\|f - P\|_{L^p(H \setminus S_{G,1})} \leq C_{p,\gamma_0,R} \max \left\{ E_n(f)_{L^p(S_{G,1})}, E_n(f)_{L^p(H)} \right\}.
$$

Indeed, we have

$$
\|f - P\|_{L^p(H \setminus S_{G,1})} = \|f - P_2 + R_n(P_2 - P_1)\|_{L^p(H \setminus S_{G,1})}
\leq C_p E_n(f)_{L^p(H)} + C_p \theta^n \|P_1 - P_2\|_{L^p(\Omega)}.
$$

However, by Lemma 4.3,

$$
\|P_1 - P_2\|_{L^p(\Omega)} \leq \|P_1 - P_2\|_{L^p(B_R[0])} \leq C \left( \frac{5R}{\gamma_0} \right)^{n+\frac{d+1}{p}} \|P_1 - P_2\|_{L^p(B)}
\leq C \left( \frac{5R}{\gamma_0} \right)^{n+\frac{d+1}{p}} \|P_1 - P_2\|_{L^p(H \cap S_{G,1})}
\leq C(R, d, \gamma_0, p) \theta^{-n} \max \left\{ E_n(f)_{L^p(S_{G,1})}, E_n(f)_{L^p(H)} \right\}.
$$

Thus, combining (4.6) with (4.7), we obtain (4.5).

Finally, we estimate the term $\|f - P\|_{L^p(S_{G,\lambda_0})}$ as follows:

$$
\|f - P\|_{L^p(S_{G,\lambda_0})} = \|f - P_1 + (1 - R_n)(P_1 - P_2)\|_{L^p(S_{G,\lambda_0})}
\leq C_p \|f - P_1\|_{L^p(S_{G,1})} + C_p \theta^n \|P_1 - P_2\|_{L^p(\Omega)}
\leq C_{p,\gamma_0,R} \max \left\{ E_n(f)_{L^p(S_{G,1})}, E_n(f)_{L^p(H)} \right\},
$$

where the last step uses (4.7).

Now putting the above estimates together, and noticing $S_{G,1} \subset G = \hat{\Omega}_{j+1}$, we complete the proof of Lemma 4.6.

5. The direct Jackson theorem

5.1. Jackson inequality on domains of special type. We will first prove the Jackson inequality. Theorem 4.1 on a domain $G$ of special type that is attached to $\Gamma = \partial \Omega$. Without loss of generality, we may assume that

$$
G := \{(x, y) : x \in (-b, b)^d, \ g(x) - 1 \leq y \leq g(x)\},
$$

where $b \in (0, (2\sqrt{d})^{-1})$ is the base size of $G$, and $g$ is a $C^2$-function on $\mathbb{R}^d$ satisfying that $\min_{x \in [-b, b]^d} g(x) \geq 4$. We may choose the base size $b$ to be sufficiently small so that

$$
\max_{x \in [-b, b]^d} \|\nabla g(x)\| \leq \frac{1}{200bd} \quad \text{and} \quad \|\nabla^2 g\|_{L^\infty([-b, b]^d)} \leq \frac{1}{1600b^2}.
$$

We first recall some notations from Section 3.1 and Section 1.3. Given $n \in \mathbb{N}$, the partition $\{\Delta_i\}_{i \in \Lambda_n^d}$ of the cube $[-b, b]^d$ is defined by

$$
\Delta_i := [t_{i_1}, t_{i_1+1}] \times \cdots \times [t_{i_d}, t_{i_d+1}] \quad \text{with} \quad t_i = \left(-1 + \frac{2i}{n}\right)b,
$$

$\chi_i$ be the characteristic function of $\Delta_i$, and $\chi_i \psi_i$ be the product of $\chi_i$ and $\psi_i$. Then, the Jackson inequality is obtained by

$$
\|f - P\|_{L^p(G)} \leq C_{p, \gamma_0, R} \max \left\{ E_n(f)_{L^p(S_{G,1})}, E_n(f)_{L^p(H)} \right\}.
$$
where $\Lambda_n^d := \{0, 1, \ldots, n-1\}^d \subset \mathbb{Z}^d$ is the index set. For simplicity, we also set $t_i = -b$ for $i < 0$, and $t_i = b$ for $i > n$, and therefore, $\Delta_i$ is defined for all $i \in \mathbb{Z}^d$.

Next, the sequence,

$$\alpha_j := 2\alpha \sin^2 \left(\frac{j\pi}{2N}\right), \quad j = 0, 1, \ldots, N := 2\ell_1 n,$$

forms a Chebyshev partition of the interval $[0, 2\alpha]$, where $\alpha := 1/(2\sin^2 \frac{\pi}{2N})$, and $\ell_1$ is a fixed large positive integer for which (3.1) is satisfied. Note that $\alpha_n = 1$, and

$$\frac{4j\alpha}{N^2} \leq \alpha_j - \alpha_{j-1} \leq \frac{\pi^2 j\alpha}{N^2}, \quad j = 1, \ldots, N.$$

Finally, a partition of the domain $G$ is defined as

$$G = \left\{(x, y) \colon \begin{array}{l} x \in [-b, b]^d, \quad g(x) - y \in [0, 1]\end{array}\right\} = \bigcup_{(i, j) \in \Lambda_n^d} I_{i,j},$$

where

$$I_{i,j} := \{(x, y) \colon \begin{array}{l} x \in \Delta_i, \quad g(x) - y \in [\alpha_j, \alpha_{j+1}]\end{array}\}.$$

Next, we introduce a few new notations for this section. Without loss of generality, we assume that $n \geq 50$. By (5.2), we can select $10 \leq m_0, m_1 \leq n/5$ to be two fixed large integer parameters satisfying

$$m_1 \geq \frac{32\ell_1^2 m_0^3 b^2}{\alpha} \|\nabla^2 g\|_{L^\infty([-b, b]^d)}.$$

We define, for $i \in \Lambda_n^d$,

$$\Delta_i^* := [t_{i_1-m_0}, t_{i_1+m_0}] \times [t_{i_2-m_0}, t_{i_2+m_0}] \times \cdots \times [t_{i_d-m_0}, t_{i_d+m_0}],$$

and for $(i, j) \in \Lambda_n^{d+1},$

$$I_{i,j}^* := \{(x, y) \colon \begin{array}{l} x \in \Delta_i^*, \quad \alpha_{j-m_{1}}^* \leq g(x) - y \leq \alpha_{j+m_{1}}^*\end{array}\},$$

where $\alpha_j^* = \alpha_j$ if $0 \leq j \leq n$, $\alpha_j^* = 0$ if $j < 0$ and $\alpha_j^* = 1$ if $j > n$. Let $x_i^*$ be an arbitrarily given point in the set $\Delta_i^*$. Denote by $\zeta_k(x_i^*)$ the unit tangent vector to the boundary $\Gamma$ at the point $(x_i^*, g(x_i^*))$ that is parallel to the $x_k x_{d+1}$-plane and satisfies $\zeta_k(x_i^*) \cdot e_k > 0$ for $k = 1, \ldots, d$; that is, $\zeta_k(x_i^*) := \frac{\partial g(x_i^*)}{\partial x_k} / \sqrt{1 + \partial^2 g(x_i^*)}$. Set

$$E(x_i^*) := \left\{\zeta_{1}(x_i^*), \ldots, \zeta_{d}(x_i^*)\right\}, \quad i \in \Lambda_n^d.$$

By Taylor’s theorem, we have

$$\left| g(x) - H_i(x) \right| \leq M_0 n^{-2}, \quad \forall x \in \Delta_i^*,$$

where

$$H_i(x) := g(x_i^*) + \nabla g(x_i^*) \cdot (x - x_i^*), \quad x \in \mathbb{R}^d,$$

and $M_0 := 8m_0^2 b^2 \|\nabla^2 g\|_{L^\infty([-b, b]^d)} + C_d A_0$. Here we recall that $A_0$ is the parameter in (1.4). Thus, setting

$$S_{i,j} := \left\{(x, y) \colon x \in \Delta_i^*, H_i(x) - \alpha_{j+m_{1}}^* + \frac{M_0}{n^2} \leq y \leq H_i(x) - \alpha_{j-m_{1}}^* - \frac{M_0}{n^2}\right\}$$

and

$$S_{i,j}^* := \left\{(x, y) \colon x \in \Delta_i^*, H_i(x) - \alpha_{j+m_{1}}^* - \frac{M_0}{n^2} \leq y \leq H_i(x) - \alpha_{j-m_{1}}^* + \frac{M_0}{n^2}\right\},$$
we have

\begin{equation}
S_{i,j} \subset I_{i,j}^* \subset S_{i,j}^c, \quad (i, j) \in \Lambda^d_{n+1}.
\end{equation}

On the other hand, it is easily seen from (5.3), (5.5) and (5.4) that \( S_{i,j} \neq \emptyset \) and

\[ \alpha_j^{* + m_1} - \alpha_j^{* - m_1} - \frac{2M_0}{n^2} \sim \frac{j + M_0}{n^2}. \]

Thus, \( S_{i,j} \) and \( S_{i,j}^c \) are two nonempty compact parallelepipeds with the same set \( \mathcal{E}(x^*_i) \cup \{e_{d+1}\} \) of edge directions and comparable side lengths.

With the above notations, we introduce the following local modulus of smoothness on \( G \):

**Definition 5.1.** For \( 0 < p \leq \infty \), define the local modulus of smoothness of order \( r \) of \( f \in L^p(G) \) by

\[ \omega^n_{loc}(f, n^{-1})_{L^p(G)} := \left[ \sum_{(i,j) \in \Lambda^d_{n+1}} \left( \omega_r^r(f, I_{i,j}^*; e_{d+1})^p + \omega_r^r(f, S_{i,j}; \mathcal{E}(x^*_i))^p \right) \right]^{1/p}, \]

with the usual change of the \( L^p \)-norm over the set \( (i,j) \in \Lambda^d_{n+1} \) for \( p = \infty \).

In this section, we shall prove the following Jackson type estimate for the above local modulus of smoothness, from which Theorem 4.1 will follow.

**Theorem 5.2.** For \( 0 < p \leq \infty \), and \( f \in L^p(G) \),

\[ E_n(f)_{L^p(G)} \leq C \omega^n_{loc}(f, n^{-1})_{L^p(G)}, \]

where the constant \( C \) is independent of \( f \) and \( n \).

**Remark 5.3.** Note that \( \omega^n_{loc}(f, n^{-1})_{L^p(G)} \) depends on the choice of \( x^*_i \), which is an arbitrary point in \( \Delta^*_i \). It follows from the proof that the constant \( C \) in Theorem 5.2 is independent of the selection of the points \( x^*_i \in \Delta^*_i \).

We divide the rest of this section into two parts. In the first part, we shall assume Theorem 5.2 and show how it implies Theorem 4.1, while the second part is devoted to the proof of Theorem 5.2.

### 5.2. Proof of Theorem 4.1

The aim is to show that Theorem 4.1 can be deduced from Theorem 5.2. Recall that for each \( i \in \Lambda^d_n \), \( \mathcal{E}(x^*_i) \) is the set of unit tangent vectors to \( \partial G^* \) at the point \((x^*_i, g(x^*_i))\), where \( x^*_i \in \Delta^*_i \). Thus, by Definition 5.1, Theorem 5.2 and Remark 5.3, to show Theorem 4.1, it suffices to prove that for any fixed \( c > 0 \)

\begin{equation}
\Sigma_1 := \sum_{(i,j) \in \Lambda^d_{n+1}} \omega_r^r(f, I_{i,j}^*; e_{d+1})^p \leq C \omega^n_{loc}(f, n^{-1})_{L^p(G)}
\end{equation}

and for \( k = 1, \ldots, d \),

\begin{equation}
\Sigma_2(k) := n^d \sum_{(i,j) \in \Lambda^d_{n+1}} \int_{\Delta^*_i} \omega_r^r(f, S_{i,j}; \mathcal{E}(x^*_i))^p dx^*_i \leq C \omega^n_{loc}(f, n^{-1})_{L^p(G)}
\end{equation}

with the usual change of the \( L^p \)-norm in the case of \( p = \infty \).

To prove the estimates \( \Sigma_1 \) and \( \Sigma_2(k) \), we need to use the average modulus of smoothness of order \( r \) on a compact interval \( I = [a_I, b_I] \subset \mathbb{R} \) defined as

\[ w_r(f, t; I)_p := \left( \frac{1}{t} \int_{t/4r}^t \left( \int_{I_{r,h}} |\Delta^*_h f(x)|^p dx \right) dh \right)^{1/p}, \quad 0 < p \leq \infty, \]
with the usual change when $p = \infty$. The average modulus $w_r(f, t; l)_p$ turns out to be equivalent to the regular modulus $\omega^r(f, t)_p := \sup_{0<h \leq t} \|\Delta_h f\|_{L^p(t, h)}$, as is well known.

**Lemma 5.4.** [14, p. 373, p. 185] For $f \in L^p(I)$ and $0 < p \leq \infty$,
\begin{equation}
C_1 w_r(f, t; l)_p \leq \omega^r(f, t)_p \leq C_2 w_r(f, t; l)_p, \quad 0 < t \leq |l|,
\end{equation}
where the constants $C_1, C_2 > 0$ depend only on $p$ and $r$.

A consequence of this equivalence and the growth properties of the usual one-dimensional modulus of smoothness (see, e.g. [14, (7.7) and (7.8) on p. 45, (5.8) on p. 370]) is that for $f \in L^p(I)$, $0 < p \leq \infty$, and any fixed $c \in (0, 1)$
\begin{equation}
w_r(f, t; l)_p \leq C w_r(f, ct; l)_p, \quad 0 < t \leq |l|,
\end{equation}
where $C$ is independent of $f$ and $t$.

For simplicity, we will assume $p < \infty$. The proof below with slight modifications works equally well for the case $p = \infty$.

We start with the proof of (5.9). Using (5.4) and (5.11), we have
\[
\omega^r(f, l^*_i; e_{d+1})_p^p = \sup_{0<h<\frac{c_i(j+1)}{4n}} \int_{I_i^*} \left[ \int_{I_i^*} |g(x) - \alpha_{j-m_1} \Delta_{he_{d+1}}^r (f, l^*_i, (x, y))|^p dy \right] dx
\]
\[
\sim \frac{n^2}{j+1} \int_{0}^{c_i(j+1)} \int_{I_i^*} |\Delta_{he_{d+1}}^r (f, l^*_i, \xi)|^p d\xi dh.
\]
By (1.11), we note that for $\xi = (x, y) \in I_i^* = \frac{c_i(j+1)}{4n} e_{d+1}$,
\[
\varphi^r_{\Omega} (e_{d+1}, \xi) \sim \sqrt{g(x) - y} \sim \frac{j+1}{n}, \quad 0 \leq j \leq n.
\]
Thus, performing the change of variable $h = s \varphi^r_{\Omega} (e_{d+1}, \xi)$ for each fixed $\xi \in I_i^* - \frac{c_i(j+1)}{4n} e_{d+1}$, we obtain
\[
\omega^r(f, l^*_i; e_{d+1})_p^p \leq C n \int_{I_i^*} \left[ \int_{0}^{\frac{c_i}{n}} |\Delta_{he_{d+1}}^{r \varphi^r_{\Omega} (e_{d+1}, \xi)} (f, l^*_i, \xi)|^p ds \right] d\xi.
\]
It then follows that
\[
\Sigma_1 \leq C n \sum_{j=0}^{n-1} \sum_{\xi \in \Lambda^*_n} \int_{0}^{\frac{c_i}{n}} \left[ \int_{I_i^*} |\Delta_{he_{d+1}}^{r \varphi^r_{\Omega} (e_{d+1}, \xi)} (f, \Omega, x)|^p d\xi \right] ds
\]
\[
\leq C n \int_{0}^{\frac{c_i}{n}} \int_{\Omega} |\Delta_{he_{d+1}}^{r \varphi^r_{\Omega} (e_{d+1}, \xi)} (f, \Omega, \xi)|^p d\xi ds \leq C \omega^r_{\Omega, c} (f, c n^{-1}; e_{d+1})_p^p,
\]
where the last step uses (1.12). This proves the estimate (5.9).

The estimate (5.10) can be proved in a similar way. Indeed, by (1.9), (5.11) and (5.12), it is easily seen that
\[
\omega^r(f, S_1; \zeta_k(x_1^*))_p^p \sim n \int_{0}^{\frac{c_i}{n}} \|\Delta_{h_{e_{d+1}} (x_1^*)} (f, S_1)|^p_{L^p(S_1)} dh.
\]
It follows that
\[ \Sigma_2(k) \leq Cn^{d+1} \int_0^\pi \left[ \sum_{(i,j) \in \Delta_0^d} \int_{\Delta_0^d} \| \Delta_{hG_e(x_t)}^r(f, S_{i,j}) \|_{L^p(S_{i,j})}^p \, dx_t \right] \, dh \]
\[ \leq Cn^d \sup_{0 < h \leq \pi} \left[ \sum_{(i,j) \in \Delta_0^d} \int_{S_{i,j}} \int_{\|u-x_t\| \leq \pi} |\Delta_{hG_e(u)}^r(f, S_{i,j}, \xi)|^p \, du \, d\xi \right] \]
\[ \leq Cn^d \sup_{0 < h \leq \pi} \int_{G^n} \int_{\|u-x_t\| \leq \pi} |\Delta_{hG_e(u)}^r(f, G, \xi)|^p \, du \, d\xi \leq C\bar{C}_{\delta}^r(f, \frac{C}{n})^p, \]
where \( G^n := \{ \xi \in G : \text{dist}(\xi, \partial G) \geq \frac{\lambda}{2\pi} \} \). This proves (5.10).

5.3. Proof of Theorem 5.2. The proof relies on several lemmas.

Lemma 5.5. Let \( (i,j) \in \Delta_d^d \). Then for \( 0 < p \leq \infty, r \in \mathbb{N} \) and any \( x_t^i \in \Delta_i^d \),
\[ E_{(d+1)(r-1)}(f, L^p(I_{j,i}^r)) \leq C(p, r, d, \Delta) \left[ \omega^r(f, x_t^i; e_{d+1})^p + \omega^r(f, S_{i,j}^r; \mathcal{E}(x_t^i))^p \right]. \]

Proof. Lemma 5.5 follows directly from (5.8) and Lemma 2.3.

Lemma 5.6. Given \( 0 < p \leq \infty \) and \( r \in \mathbb{N} \), there exist positive constants \( \bar{C} = C(p, r) \) and \( s_1 = s_1(p, r) \) depending only on \( p \) and \( r \) such that for any integers \( 0 \leq k, j \leq N/2 \) and any \( P \in \Pi_r^1 \),
\[ \| P \|_{L^p[a_k, a_{k+1}]} \leq C(p, r)(1 + |j - k|)^{s_1} \| P \|_{L^p[a_k, a_{k+1}]} \cdot \]

Proof. First, we prove that
\[ \| P \|_{L^p(I_2(2x))} \leq L_{p,r} \| P \|_{L^p(I_2(x))}, \quad \forall P \in \Pi_r^1, \quad \forall x \in [0, 2a], \quad \forall t \in (0, 1], \]
where
\[ I_t(x) := \{ y \in [0, 2a] : \sqrt{x - y} \leq \sqrt{2at} \}. \]
To see this, we note that with \( \rho_t(x) = 2at^2 + t\sqrt{2ax} \),
\[ \left[ x - \frac{1}{8}\rho_t(x), x + \frac{1}{8}\rho_t(x) \right] \cap [0, 2a] \subset I_t(x) \subset I_{2t}(x) \subset [x - 4\rho_t(x), x + 4\rho_t(x)], \]
where the first relation can be deduced by considering the cases \( 0 \leq x \leq \alpha t^2 \) and \( \alpha t^2 < x \leq 4 \alpha \) separately. By Lemma 1.23, this implies that with \( I_t = I_t(x) \) and \( J = [x - 4\rho_t(x), x + 4\rho_t(x)] \),
\[ \| P \|_{L^p(I_{2t})} \leq \| P \|_{L^p(J)} \leq C_{p,r} \| P \|_{L^p(\frac{1}{\sqrt{x}} [0, 2a])} \leq C_{p,r} \| P \|_{L^p(I_t)}, \]
which proves (5.13).

Next, we note that the doubling property (5.13) implies that for any \( x, x' \in [0, 2a] \) and any \( t \in (0, 1] \),
\[ \| P \|_{L^p(I_t(x))} \leq L_{p,r} \left( 1 + \frac{\sqrt{x - x'}}{\sqrt{2at}} \right)^{s_1} \| P \|_{L^p(I_t(x'))}, \quad \forall P \in \Pi_r^1, \]
where \( s_1 = (\log L_{p,r})/\log 2 \).

Finally, for each \( 1 \leq k \leq N/2 \), we may write \( [\alpha_k, \alpha_{k+1}] = I_{t_k}(x_k) \) with \( t_k := \sqrt{\alpha_{k+1} - \alpha_k}/2\sqrt{2a} \) and \( x_k := \frac{\sqrt{\alpha_k} + \sqrt{\alpha_{k+1}}}{4} \). Note also that by (5.7),
\[ \sqrt{2} |k - j| \leq \frac{\sqrt{\alpha_j} - \sqrt{\alpha_k}}{2\sqrt{2a}} \leq \frac{\pi |k - j|}{2N}, \quad 0 \leq k, j \leq N/2. \]
Lemma 5.7. Given $A := 2$ where the right hand side of (5.17) is bounded above by
\[ \| P \|_{L^p(I \cap (I_n \cup I_d))} \leq L_{p,r} \left( 1 + \frac{4N|\sqrt{x_j} - \sqrt{x_k}|}{\sqrt{2\alpha \pi}} \right)^{s_1} \| P \|_{L^p(I \cap (I_n \cup I_d))} \]
\[ \leq L_{p,r} (1 + |k - j|)^{s_1} \| P \|_{L^p(\alpha_k, \alpha_{k+1})}. \]
\[ \square \]

For $x = (x_1, \ldots, x_d) \in \mathbb{R}^d$, we set $\|x\|_{\infty} := \max_{1 \leq j \leq d} |x_j|$.

Lemma 5.8. Given $0 < p \leq \infty$ and $r \in \mathbb{N}$, there exist positive constants $C = C(p, r, d)$ and $s_2 = s_2(p, r, d)$ depending only on $p$, $r$ and $d$ such that for any $i, k \in \Lambda_n$ and $Q \in \Pi_d$,
\[ \| Q \|_{L^p(\Delta_k)} \leq C(p, r, d)(1 + \| i - k \|_{\infty})^{s_2} \| Q \|_{L^p(\Delta_k)}. \]

Proof. The proof of Lemma 5.7 is similar to that of Lemma 5.6 and in fact, is simpler. It is a direct consequence of Lemma 4.8 \[ \square \]

Lemma 5.8. Given $0 < p \leq \infty$ and $r \in \mathbb{N}$, there exists a positive number $\ell = \ell(p, r, d)$ such that for any $\ell(q, r, d)$ and $x \in \mathbb{R}^d$ depending only on $p$, $r$ and $\| \nabla^2 g \|_{\infty}$.

Proof. For simplicity, we shall prove Lemma 5.8 for the case of $0 < p < \infty$ only. The proof below with slight modifications works for $p = \infty$.

Writing
\[ \| Q \|_{L^p(I_{i,j})}^p = \int_{\Delta_i} \left[ \int_{\alpha_{j-1}}^{\alpha_j} |Q(x, g(x) - u)|^p \, du \right] dx, \]
and using Lemma 5.6, we obtain
\[ \| Q \|_{L^p(I_{i,j})} \leq C(p, r)(1 + |j - l|)^{s_1} \left[ \int_{\Delta_i} \int_{g(x) - \alpha_{j-1}}^{g(x) - \alpha_{j}} |Q(x, y)|^p \, dy \, dx \right]. \]

Using Taylor’s theorem, we have that
\[ |g(x) - t_4(x)| \leq \frac{A}{2b^2} \| x - x_1 \|_{\infty}^2, \quad \forall x \in [-b, b]^d, \]
where $x_1$ is the center of the cube $\Delta_i$, $t_4(x) := g(x_1) + \nabla g(x_1) \cdot (x - x_1)$, and $A := 2b^2d^2 \| \nabla^2 g \|_{L^\infty([-b, b]^d)}$. Thus, the double integral in the square brackets on the right hand side of (5.17) is bounded above by
\[ \int_{\alpha_{j-1}}^{\alpha_j} \left[ \int_{\Delta_i} |Q(x, t_4(x) - u) + \frac{A}{2n^2} \| x - x_1 \|_{\infty}^2 |^p \, dx \right] du =: I. \]

However, applying Lemma 5.7 to this last inner integral in the square brackets, we obtain
\[ I \leq C(p, r, d)(1 + \| i - k \|_{\infty})^{s_2} \int_{\Delta_k} \left[ \int_{t_4(x) - \alpha_{j-1} + \frac{A}{2n^2}}^{t_4(x) - \alpha_{j} + \frac{A}{2n^2}} |Q(x, u)|^p \, du \right] dx. \]
By \((5.18)\), this last integral in the square brackets on the right hand side of \((5.19)\) is bounded above by

\[
\int_{g(x) - \alpha_{t-1}}^{4g(x) + \alpha_{t-1}} |Q(x, u)|^p \, du = \int_{\alpha_{t-1}}^{4\alpha_{t-1}} \frac{|Q(x, g(x) - y)|^p}{\alpha_{t-1}^{1/2}} \, dy;
\]

which, using Lemma \[4.8\] and the fact that \(\alpha_t - \alpha_{t-1} \geq cn^{-2}\), is controlled above by

\[
C(p, r) \left( A(1 + \|k - i\|_\infty) \right)^{2p+4} \int_{\alpha_{t-1}}^{\alpha_t} |Q(x, g(x) - y)|^p \, dy.
\]

Putting the above together, we prove that

\[
|Q|_{L^p(I_{t,j})}^p \leq C(p, r, d)(1 + |j - l|)^s(1 + \|i - k\|_\infty)^{s^2 + 2r} \|Q\|_{L^p(I_{t,j})}^p.
\]

This leads to the desired estimate \((5.16)\) with \(\ell = (s_1 + s_2 + 2r + 4)/p\). \(\square\)

Now we are in the position to prove Theorem \[5.2\].

**Proof of Theorem \[5.2\].** We shall prove the result for the case of \(0 < p < \infty\) only. The proof below with slight modifications works equally well for the case \(p = \infty\).

For simplicity, we use the Greek letters \(\gamma, \beta, \ldots\) to denote indices in the set \(A_{n}^{d+1}\). By Lemma \[5.3\] for each \(\gamma := (i, j) \in A_{n}^{d+1}\) there exists a polynomial \(s_{\gamma} \in \Pi_{(d+1)(r-1)}^{n}\) such that

\[
\|f - s_{\gamma}\|_{L^p(I_{t,j})} \leq C(p, r, d)W^r(f, I_{t,j}^*),
\]

where

\[
W^r(f, I_{t,j}^*) := \omega^r(f, I_{t,j}^*; e_{d+1}) + \omega^r(f, S_{\gamma}; \mathcal{E}(x_i^*))
\]

Let \(\{q_{\gamma} : \gamma \in A_n^{d+1}\} \subset \Pi_{n/(r(d+1))}^{d+1}\) be the polynomial partition of the unity as given in Theorem \[3.1\] and Remark \[3.3\] with a large parameter \(m > 2d + 2\), to be specified later. Define

\[
P_n(\xi) := \sum_{\gamma \in A_{n}^{d+1}} s_{\gamma}(\xi)q_{\gamma}(\xi) \in \Pi_{n}^{d+1}.
\]

Clearly, it is sufficient to prove that

\[
\|f - P_n\|_{L^p(G)} \leq C\omega^r_{\text{loc}} \left( f, \frac{1}{n} \right)_p.
\]

To show \((5.21)\), we write, for each \(\beta \in A_{n}^{d+1}\),

\[
f(\xi) - P_n(\xi) = f(\xi) - s_{\beta}(\xi) + \sum_{\gamma \in A_{n}^{d+1}} (s_{\beta}(\xi) - s_{\gamma}(\xi))q_{\gamma}(\xi).
\]

It follows by Theorem \[3.1\] that

\[
\|f - P_n\|_{L^p(I_{t,j})}^p \leq C_p\|f - s_{\beta}\|_{L^p(I_{t,j})}^p + C_p \sum_{\gamma \in A_{n}^{d+1}} \|s_{\beta} - s_{\gamma}\|_{L^p(I_{t,j})}^p (1 + \|\beta - \gamma\|_\infty)^{-mp_1},
\]

where \(p_1 := \min\{p, 1\}\). Using \((5.20)\), we then reduce to showing that

\[
\sum_{\beta \in A_{n}^{d+1}} \sum_{\gamma \in A_{n}^{d+1}} \|s_{\beta} - s_{\gamma}\|_{L^p(I_{t,j})}^p (1 + \|\beta - \gamma\|_\infty)^{-mp_1} \leq C\omega^r_{\text{loc}} \left( f, \frac{1}{n} \right)_p.
\]
To show (5.22), we claim that there exists a positive number \( s_3 = s_3(p, d, r) \) such that for any \( \gamma, \beta \in \Lambda_{d+1}^n \),
\[
(5.23) \quad \| s_\gamma - s_\beta \|^p_{L^p(I_n)} \leq C (1 + \| \gamma - \beta \|^\infty)^{s_3 p} \sum_{\eta \in \mathcal{I}_{k_0}(\gamma)} W^p(f, I^*_\eta)^p,
\]
where \( k_0 := 1 + \| \beta - \gamma \|^\infty \), and
\[
\mathcal{I}_t(\gamma) := \{ \eta \in \Lambda_{d+1}^n : \| \gamma - \eta \|^\infty \leq t \} \quad \text{for} \quad \gamma \in \Lambda_{d+1}^n \quad \text{and} \quad t > 0.
\]
For the moment, we assume (5.23) and proceed with the proof of (5.22). Indeed, once such a sequence is constructed, then we have
\[
(5.24) \quad \text{Choosing the parameter } \gamma, \beta \text{ that for any } \| \gamma - \beta \|^\infty \leq \gamma, \beta \in \Lambda_{d+1}^n \quad \text{and} \quad \| \gamma - \beta \|^\infty = 1 + \| \gamma - \beta \|^\infty.
\]
Indeed, once such a sequence is constructed, then we have
\[
\| s_\gamma - s_\beta \|^p_{L^p(I_n)} \leq N_0^{\max(p,1)-1} \sum_{j=1}^{N_0-1} \| s_{\gamma_j} - s_{\gamma_{j+1}} \|^p_{L^p(I_n)},
\]
which, using (5.24) and Lemma 5.8 with \( \ell = \ell(p, r, d) > 0 \), is estimated above by
\[
\leq C N_0^{\max(p,1)-1} (1 + \| \gamma - \beta \|^\infty)^{\ell p} \sum_{j=1}^{N_0-1} \| s_{\gamma_j} - s_{\gamma_{j+1}} \|^p_{L^p(I_n)}.
\]
However, using (5.24) and (5.29), we have that
\[
\| s_{\gamma_j} - s_{\gamma_{j+1}} \|^p_{L^p(I_{\gamma_j})} \leq C_p \left[ \| f - s_{\gamma_j} \|^p_{L^p(I_{\gamma_j})} + \| f - s_{\gamma_{j+1}} \|^p_{L^p(I_{\gamma_{j+1}})} \right]
\]
\[
\leq C(p, r, d) \left[ W^p(f, I^*_\gamma) + W^p(f, I^*_{\gamma_{j+1}}) \right].
\]
Putting the above together, we prove the claim (5.22) with \( s_3 := \ell + 2 \max\{1, \frac{1}{p}\} \).

Finally, we construct the sequence \( \{ \gamma_1, \ldots, \gamma_{N_0} \} \) as follows. Assume that \( \gamma = (k, l) \), and \( \beta = (k', l') \). Without loss of generality, we may assume that \( l \leq l' \). (The case \( l > l' \) can be treated similarly.) Recall that \( \Delta_i := \left\{ x \in \mathbb{R}^d : \| x - x_i \|^\infty \leq \frac{d}{n} \right\} \), where \( x_i \) is the center of the cube \( \Delta_i \). Let \( \{ z_j \}_{j=0}^{n+1} \) be a sequence of points on the line segment \( [x_k, x_{k'}] \) satisfying that \( z_0 = x_k, \ z_{n+1} = x_{k'}, \ |z_j - z_{j+1}|^{\infty} = \frac{d}{n} \) for \( j = 0, 1, \ldots, n_0 - 1 \) and \( \frac{2d}{n} \leq \| z_{j_0} - z_{n_0+1} \|^{\infty} \leq \frac{d}{2d} \), where \( n_0 + 1 \leq \frac{d}{2d} \| k - k' \|^{\infty} \). Let \( i_j \in \Lambda_{d+1}^n \) be such that \( z_j \in \Delta_{i_j} \) for \( 0 \leq j \leq n_0 + 1 \). Since \( \frac{2d}{n} \leq \| z_j - z_{j+1} \|^{\infty} \leq \frac{d}{n} \), the cubes \( \Delta_{i_j} \) are distinct and moreover
\[
(5.25) \quad \Delta_{i_j} \subset 9\Delta_{i_{j+1}}, \quad j = 0, 1, \ldots, n_0.
\]
In particular, this implies that $i_0 = k$ and $i_{n_0 + 1} = k'$. It can also be easily seen from the construction that for $j = 0, \ldots, n_0 + 1$,  
\begin{equation}
\|i_j - k\|_{\infty} \leq \|k - k'\|_{\infty} + 1.
\end{equation}

Next, we order the indices $(i_j, k)$, $0 \leq j \leq n_0 + 1$, $0 \leq k \leq l'$ as follows:
\begin{itemize}
\item $(i_0, l), (i_0, l + 1), \ldots, (i_0, l'), (i_1, l'), (i_1, l' - 1), \ldots, (i_1, l), (i_2, l), \ldots, (i_{n_0 + 1}, l').$
\end{itemize}

We denote the resulting sequence by $\{\gamma_1, \gamma_2, \ldots, \gamma_{N_0}\}$, where
\[ N_0 \leq (1 + |l - l'|)(n_0 + 2) \leq (1 + \|\gamma - \beta\|_{\infty})^2. \]

Clearly, $\gamma_1 = \gamma$, and $\gamma_{N_0} = \beta$. Moreover, by \eqref{5.26}, we have $\|\gamma_j - \gamma\|_{\infty} \leq 1 + \|\gamma - \beta\|_{\infty}$ for $j = 1, \ldots, N_0$, whereas by \eqref{5.26}, $I_{\gamma_j} \subset I_{\gamma_{j+1}}^*$ for $j = 0, \ldots, N_0 - 1$. This completes the proof. \hfill \square

6. Comparison with average moduli

In this section, we shall prove that the moduli of smoothness, defined in \eqref{1.16}, can be controlled from above by Ivanov’s moduli of smoothness, defined in \eqref{1.3}. By Remark \ref{1.8}, it is enough to show

**Theorem 6.1.** There exist a parameter $A_0 > 1$ and a constant $A > 1$ such that for any $0 < q \leq p \leq \infty$,
\[ \omega_{p, r}^\alpha(f, \frac{1}{n}; A_0) \leq C_{r, \alpha}(f, \frac{1}{n} ; p, q), \]
where the constant $C$ is independent of $f$ and $n$.

As a result, using Remark \ref{4.7}, we may establish the Jackson inequality for Ivanov’s moduli of smoothness for any dimension $d \geq 1$ and the full range of $0 < q \leq p \leq \infty$.

**Corollary 6.2.** If $f \in L^p(\Omega)$, $0 < q \leq p \leq \infty$ and $r \in \mathbb{N}$, then
\[ E_n(f) \leq C_{r, \Omega}^\alpha(f, \frac{1}{n} ; p, q). \]

Recall that for $S \subset \mathbb{R}^d$,
\[ S_{rh} := \{\xi \in S : [\xi, \xi + rh] \subset S\}, \quad r > 0, h \in \mathbb{R}^d. \]

The proof of Theorem \ref{6.1} relies on the following lemma, which generalizes Lemma 7.4 of \cite{19}.

**Lemma 6.3.** Let $r \in \mathbb{N}$, $h \in \mathbb{R}^d$ and $\delta_0 \in (0, 1)$. Assume that $(S, E)$ is a pair of subsets of $\mathbb{R}^d$ satisfying that for each $\xi \in S_{rh}$, there exists a convex subset $E^\xi$ of $E$ such that $|E^\xi| \geq \delta_0 |E|$ and $[\xi, \xi + rh] \subset E^\xi$. Then for any $0 < q \leq p < \infty$ and $f \in L^p(E)$, we have
\[ \|\Delta_h^r(f, S, \cdot)\|_{L^p(S)} \leq C(q, d, r) \left( \int_S \left( \frac{1}{|\delta_0 |E|} \int_E |\Delta_{(\eta - \xi)/r}(f, E, \xi)|^q \, d\xi \right)^{\frac{q}{q}} \, d\eta \right)^{\frac{1}{q}}, \]
where the constant $C(q, d, r)$ is independent of $S$, $E$ and $q$ if $q \geq 1$.

Lemma \ref{6.3} was proved in \cite{19} Lemma 7.4 in the case when $p = q$ and $E = S$ is convex. For the general case, it can be obtained by modifying the proof there.
Proof: The proof is based on the following combinatorial identity, which was proved in [19] Lemma 7.3: if $\xi, \eta \in \mathbb{R}^d$ and $f$ is defined on the convex hull of the set $\{\xi, \xi + rh, \eta\}$, then

\[
\Delta^r_{\eta} f(\xi) = \sum_{j=0}^{r-1} (-1)^j \binom{r}{j} \Delta^r f[\xi + jh, \frac{j}{r}(\xi + rh) + \frac{1-\frac{j}{r}}{r}\eta] \\
- \sum_{j=1}^{r} (-1)^j \binom{r}{j} \Delta^r f\left[(1-\frac{j}{r})\xi + \frac{j}{r}\eta, \xi + rh\right],
\]

where we used the notation $\Delta^r f[u, v] := \Delta^r_{r(u-v)/r} f(u)$ for $u, v \in \mathbb{R}^d$.

Since $E^\xi$ is a convex set containing the line segment $[\xi, \xi + rh]$ for each $\xi \in S_{rh}$, we obtain from (6.1) that for $\xi \in S_{rh}$,

\[
|\Delta^r_{\eta} f(\xi)| \leq C_r \max_{0 \leq j \leq r-1} \left( \frac{1}{|E|} \int_{E^\xi} \left| \Delta^r f[\xi + jh, \frac{j}{r}(\xi + rh) + \frac{1-\frac{j}{r}}{r}\eta]\right|^q \, d\eta \right)^\frac{1}{q} \\
+ C_r \max_{1 \leq j \leq r} \left( \frac{1}{|E|} \int_{E^\xi} \left| \Delta^r f\left[(1-\frac{j}{r})\xi + \frac{j}{r}\eta, \xi + rh\right]\right|^q \, d\eta \right)^\frac{1}{q}.
\]

Taking the $L^p$-norm over the set $S_{rh}$ on both sides of this last inequality, we obtain

\[
\left( \int_{S_{rh}} |\Delta^r_{\eta} f(\xi)|^p \, d\xi \right)^\frac{1}{p} \leq C_r \delta_0^{-\frac{1}{q}} \left[ \max_{0 \leq j \leq r-1} I_j(h) + \max_{1 \leq j \leq r} K_j(h) \right],
\]

where

\[
I_j(h) := \left( \int_{S_{rh}} \left( \frac{1}{|E|} \int_{E^\xi} \left| \Delta^r f[\xi + jh, \frac{j}{r}(\xi + rh) + \frac{1-\frac{j}{r}}{r}\eta]\right|^q \, d\eta \right)^\frac{1}{q} \, d\xi \right)^\frac{1}{p},
\]

\[
K_j(h) := \left( \int_{S_{rh}} \left( \frac{1}{|E|} \int_{E^\xi} \left| \Delta^r f\left[(1-\frac{j}{r})\xi + \frac{j}{r}\eta, \xi + rh\right]\right|^q \, d\eta \right)^\frac{1}{q} \, d\xi \right)^\frac{1}{p}.
\]

For the term $I_j(h)$ with $0 \leq j \leq r - 1$, we have

\[
I_j(h) = \left( \int_{S_{rh} + jh} \left( \frac{1}{|E|} \int_{E^{u-jh}} \left| \Delta^r f[u, \frac{j}{r}(u + (r-j)h) + \frac{1-\frac{j}{r}}{r}\eta]\right|^q \, d\eta \right)^\frac{1}{q} \, du \right)^\frac{1}{p} \\
\leq \frac{r^d}{q} \left( \int_{S_{rh} + jh} \left( \frac{1}{|E|} \int_{E^{u-jh}} \left| \Delta^r f[u, v]\right|^q \, dv \right)^\frac{1}{q} \, du \right)^\frac{1}{p},
\]

where we used the change of variables $u = \xi + jh$ in the first step, the change of variables $v = \frac{j}{r}(u + (r-j)h) + \frac{1-\frac{j}{r}}{r}\eta$ and the fact that each set $E^\xi$ is convex in the second step. Since $[u, v] \subset E^{u-jh}$ whenever $u \in S_{rh} + jh$ and $v \in E^{u-jh}$ and since $\Delta^r f[u, v] = \Delta^r f[v, u]$, it follows that

\[
I_j(h) \leq \frac{r^d}{q} \left( \int_{S} \left( \frac{1}{|E|} \int_{E} \left| \Delta^r_{(u-v)/r}(f, E, v)\right|^q \, dv \right)^\frac{1}{q} \, du \right)^\frac{1}{p}.
\]
We claim that for any \(0 < t < \frac{1}{r}\), the change of variables \(u = \xi + rh\) and \(v = \left(1 - \frac{j}{r}\right)(u - rh) + \frac{r}{t}\eta\), we obtain
\[
K_j(h) = \left(\int_{S_{-rh}} \left(\frac{1}{|E|} \int_{E_{-rh}} \Delta^r f \left[(1 - \frac{j}{r})(u - rh) + \frac{r}{t}\eta, \ v\right]^q dv\right)^{\frac{1}{q}} du\right)^{\frac{1}{p}}
\]
\[
\leq r^{d/q} \left(\int_{S_{-rh}} \left(\frac{1}{|E|} \int_{E_{-rh}} \left|\Delta^r f[v, \ u]\right|^q dv\right)^{\frac{1}{q}} du\right)^{\frac{1}{p}}
\]
\[
\leq r^{d/q} \left(\int_{S} \left(\frac{1}{|E|} \int_{E} \left|\Delta^r(u-v)/r(f, E, v)\right|^q dv\right)^{\frac{1}{q}} du\right)^{\frac{1}{p}}.
\]
Putting the above together, we complete the proof. \(\square\)

We are now in a position to prove Theorem 6.1.

**Proof of Theorem 6.1.** We shall prove Theorem 6.1 for \(p < \infty\) only. The case \(p = \infty\) can be deduced by letting \(p \to \infty\). In fact, all the general constants below are independent of \(p\) as \(p \to \infty\).

By Lemma 1.6 there exists \(\delta_0 \in (0, 1)\) such that \(\Omega \setminus \Omega(\delta_0) \subset \bigcup_{j=1}^{m_0} G_j\), where
\[
\Omega(\delta_0) := \{\xi \in \Omega : \text{dist}(\xi, \Gamma) > \delta_0\}.
\]

We claim that for any \(0 < t < \frac{\delta_0}{8 \text{diam}(|\Omega|)^{1/8}}\),
\[
\sup_{\|h\| \leq t} \left\|\Delta_h^r f(h, \cdot)\right\|_{L^p(\Omega(\delta_0))} \leq C_{q, r, \delta_0} \left\|f, A_1 t\right\|_{p, q}.
\]

Indeed, using Fubini’s theorem and Lemma 5.3, we have
\[
\sup_{\|h\| \leq t} \left\|\Delta_h^r f(h, \cdot)\right\|_{L^p(\Omega(\delta_0))} \leq C_d \sup_{\|h\| \leq t} \left\|\Delta_h^r f\right\|_{L^p(\Omega(\delta_0)/2)}.
\]

Let \(\{\omega_1, \ldots, \omega_N\}\) be a subset of \(\Omega(\delta_0/2)\) such that \(\min_{1 \leq i \neq j \leq N} \|\omega_i - \omega_j\| \geq t\) and \(\Omega(\delta_0/2) \subset \bigcup_{j=1}^{N} B_j\), where \(B_j := B_t(\omega_j)\). Using Lemma 6.1, we then have
\[
\sup_{\|h\| \leq t} \left\|\Delta_h^r f\right\|_{L^p(\Omega(\delta_0/2))} \leq C_p \sum_{j=1}^{N} \sup_{\|h\| \leq t} \left\|\Delta_h^r f, 2B_j, \cdot\right\|_{L^p(2B_j)}^p
\]
\[
\leq C_p \sum_{j=1}^{N} \int_{2B_j} \left(\frac{1}{|E_{2B_j}|} \int_{E_{2B_j}} \left|\Delta_h^r f(\xi)\right|^q d\xi\right)^{\frac{1}{q}} d\eta \leq C_{q, r, \delta_0} \left\|f, A_1 t\right\|_{p, q}.
\]

This proves the claim (6.2).

Now using (6.2) and Definition 1.7, we reduce to showing that for each \(x_i\)-domain \(G \subset \Omega\) attached to \(\Gamma\), and a sufficiently large parameter \(A_0\),
\[
\hat{\omega}_G(f, \frac{1}{n}; A_0)_{L^p(G)} \leq C_{\tau}(f, \frac{A_1}{n})_{p, q}
\]
and
\[
\sup_{0 < s \leq \frac{1}{t}} \left\|\Delta_s^r f(h, \cdot)\right\|_{L^p(G)} \leq C_{\tau}(f, \frac{A_1}{n})_{p, q}.
\]

Without loss of generality, we may assume that \(e_i = e_{d+1}\), \(G\) takes the form (5.1) with small base size \(b \in (0, 1)\), and \(n \geq N_0\), where \(N_0\) is a large positive integer depending only on the set \(\Omega\). We follow the same notations as in Section 5.1 with
sufficiently large parameters $m_0$ and $m_1$. Thus, $\{I_{i,j}: (i, j) \in \Lambda_{n,t}^{d+1}\}$ is a partition of $G$, and $S_{i,j} \subset I_{i,j}$ is the compact parallelepiped as defined in (5.6).

We start with the proof of (6.3). Given a parameter $\ell > 1$, we define

$$S_{i,j} := \{ (x, y): x \in (\ell \Delta^*_i) \cap [-2b, 2b]^d, H_i(x) - \alpha_{i,m} + \frac{M_0 - \ell}{n^2} \leq y \leq \leq H_i(x) - \alpha_{i,m} - \frac{M_0 - \ell}{n^2} \},$$

where $\ell \Delta^*_i$ denotes the dilation of the cube $\Delta^*_i$ from its center $x_1$. We choose the parameter $\ell$ sufficiently large so that

(i) for any $\xi = (\xi, \xi_y) \in I_{i,j}$ and $u \in B_{n^{-1}}(\xi_x) \subset \mathbb{R}^d$, $[\xi, \xi + \frac{\epsilon}{n} \zeta_k(u)] \subset S_{i,j}$ for all $1 \leq k \leq d$;

(ii) there exists a constant $c_0 > 0$ such that $I_{i,j} \subset S_{i,j} \subset G^*$ whenever $i \in \Lambda_n^d$ and $j \geq c_0 \ell$.

Furthermore, we may also choose the parameter $A_0$ large enough so that with $\Lambda_{n,t}^{d+1} := \{(i, j) \in \Lambda_{n,t}^{d+1}: c_0 \ell \leq j \leq n\}$,

$$G_n := \{ \xi \in G: \text{dist}(\xi, \partial G) \geq \frac{A_0}{n^2} \} \subset \bigcup_{(i, j) \in \Lambda_{n,t}^{d+1}} I_{i,j}.$$

With the above notation, we have that for any $0 < s \leq \frac{1}{4}$ and $k = 1, \ldots, d$,

$$n^d \int_{G_n} \int_{\|u - \xi_x\| \leq \frac{1}{n^2}} |\Delta^*_{\zeta_k(u)}(f, G^*, \xi)|^p \, du \, d\xi \leq C_d \sum_{(i, j) \in \Lambda_{n,t}^{d+1}} \sup_{\zeta \in S^d} \int_{S_{i,j}} |\Delta^*_{\zeta}(f, S_{i,j}^*, \xi)|^p \, d\xi,$$

which, using Lemma 6.3, is estimated above by

$$C_{q,d,r} \sum_{(i, j) \in \Lambda_{n,t}^{d+1}} \int_{S_{i,j}^*} \left( \int_{S_{i,j}^*} |\Delta^*_{(\eta - \xi)/r} f(\xi)|^q \, d\xi \right)^{\frac{p}{q}} \, d\eta.$$

Recall that for $\xi \in \Omega$ and $t > 0$, we defined $U(\xi, t) = \{ \eta \in \Omega: \rho(t, \xi, \eta) \leq t \}$. Now, by Proposition 2.1, there exists a constant $A_1 > 1$ such that for each $(i, j) \in \Lambda_{n,t}^{d+1}$,

$$U \left( \eta_{i,j}, \frac{1}{nA_1} \right) \subset S_{i,j} \subset U \left( \eta_{i,j}, \frac{A_1}{2n} \right) \text{ for some } \eta_{i,j} \in S_{i,j}^*.$$

Thus, by Remark 3.7, the sum in (6.5) is controlled above by a constant multiple of

$$\int_{\Omega} \left( \frac{1}{|U(\xi, \frac{1}{nA_1})|} \right) \int_{U(\xi, \frac{1}{nA_1})} \left| \Delta^*_{(\eta_x - \xi_x)/r}(f, G, \xi) \right|^q \, d\eta \, d\xi = \tau_r \left( f, \frac{A_1}{n} \right)_{p,q}.$$

This completes the proof of (6.3).

It remains to prove (6.4). First, by the $C^2$ assumption of the domain $\Omega$ (see, e.g. [15]), there exists a constant $r_0 \in (0, 1)$ such that for each $\xi = (\xi_x, \xi_y) \in G$, there exists a closed ball $B_\xi \subset G^*$ of radius $r_0 \in (0, 1)$ that touches the boundary $\Gamma$ at the point $\gamma(\xi) := (\xi_x, g(\xi_x))$. Given a large parameter $A$, we define

$$E_\xi := \{ \eta \in B_\xi: \text{dist}(\eta, T_\xi) \leq \frac{A}{n^2} \}, \quad \xi \in G,$$
where $T_{\gamma}$ denotes the tangent plane to $\Gamma$ at the point $\gamma(\xi)$. Clearly, $E_{\xi} \subset G^*$ is convex,

\begin{equation}
U(\gamma(\xi), \frac{c_1}{n}) \subset E_{\xi} \subset U(\gamma(\xi), \frac{c_2}{n}),
\end{equation}

where the constants $c_1, c_2 > 0$ depend only on $G$ and the parameter $A$. Next, recall that $S_{i,j}^*$ is the compact parallelepiped defined in (5.7). By definition, there exists a positive integer $j_0$ depending only on $G$ such that $S_{i,j}^* \subset G^*$ whenever $j_0 < j \leq n$. Furthermore, according to Proposition 2.4, we have that

\begin{equation}
\sup_{\xi \in S_{i,j}^*} \|\xi - \gamma(\xi)\| \leq \frac{c_3}{n^2}, \quad \text{for } 0 \leq j \leq j_0,
\end{equation}

and

\begin{equation}
U\left(\eta_{i,j}, \frac{c_4}{n}\right) \subset I_{i,j}^* \subset S_{i,j}^* \cap G^* \subset U\left(\eta_{i,j}, \frac{c_5}{n}\right), \quad \forall (i,j) \in \Lambda_{n}^{d+1},
\end{equation}

for some point $\eta_{i,j} \in I_{i,j}$, where $c_3, c_4, c_5$ are positive constants depending only on the set $G$. By (6.8), we may choose the parameter $A$ in (6.7) large enough so that if $0 \leq j \leq j_0$ and $\xi \in I_{i,j}^*$, then $[\xi, \gamma(\xi)] \subset E_{\xi}$. Note that if $\xi \in I_{i,j}^*$ with $0 \leq j \leq j_0$, then by (6.9) and (6.8),

$$\rho_3(\eta_{i,j}, \gamma(\xi)) \leq \frac{c_6}{n},$$

where $c_6 > 0$ is a constant depending only on $G$. Now we define, for $(i,j) \in \Lambda_{n}^{d+1}$,

$$E_{i,j} = \begin{cases} S_{i,j}^*, & \text{if } j_0 < j \leq n, \\
U(\eta_{i,j}, \frac{c_4+c_5}{n}), & \text{if } 0 \leq j \leq j_0.
\end{cases}$$

Thus, $E_{i,j} \subset G^*$, and by (6.7), (6.8) and (6.9), we have that for $0 \leq j \leq j_0$.

\begin{equation}
\bigcup_{\xi \in I_{i,j}^*} E_{\xi} \subset \bigcup_{\xi \in I_{i,j}^*} U(\gamma(\xi), \frac{c_2}{n}) \subset E_{i,j}.
\end{equation}

Thus, setting $e = e_{d+1}$, and using Lemma 5.4, we have

$$\sup_{0 < s \leq \frac{1}{n}} \|\Delta^r \Delta^r_{x,\gamma(\xi),e}(f, G, \cdot)\|_{L^p(G)} \leq C_n \int_{0}^{\frac{1}{n}} \int_{G} |\Delta^r \Delta^r_{x,\gamma(\xi),e}(f, G, \xi)|^p \, d\xi \, ds \leq C \sum_{(i,j) \in \Lambda_{n}^{d+1}} \sup_{0 < s \leq \frac{1}{n}} \int_{I_{i,j}^*} |\Delta^r_{x,e}(f, I_{i,j}^*, \xi)|^p \, d\xi.$$

However, by (6.7), (6.10) and Lemma 5.3 this last sum can be estimated above by a constant multiple of

$$\sum_{(i,j) \in \Lambda_{n}^{d+1}} \int_{I_{i,j}^*} \left(\frac{1}{|E_{i,j}|} \int_{E_{i,j}} |\Delta^r_{(\eta - \xi)/r, f, \Omega, \xi}|^q \, d\eta\right)^{\frac{p}{q}} \, d\xi \leq C \sum_{(i,j) \in \Lambda_{n}^{d+1}} \int_{I_{i,j}^*} \left(\frac{1}{|U(\xi, \frac{A_1}{n})|} \int_{U(\xi, \frac{A_1}{n})} |\Delta^r_{(\eta - \xi)/r, f, \Omega, \xi}|^q \, d\eta\right)^{\frac{p}{q}} \, d\xi \leq C_{r}(f, A_1 \frac{A_1}{n})_{p,q},$$

where $A_1 := 2(c_2 + c_5 + c_6)$. This completes the proof. \hfill $\square$
7. Inverse inequality for \(1 \leq p \leq \infty\)

The main purpose in this section is to show Theorem 7.1, the inverse theorem. By Theorem 1.13, \(\omega_{0,q}'(f,t)_p \leq C_{p,q} \tau_r(f,A_t)_p\) for \(1 \leq q \leq p \leq \infty\), where \(\tau_r(f,t)_p\) is the \((q,p)\)-averaged modulus of smoothness given in (1.3). Thus, it is sufficient to prove

**Theorem 7.1.** If \(r \in \mathbb{N}, A > 0, 1 \leq q \leq p \leq \infty\) and \(f \in L^p(\Omega)\), then

\[
\tau_r(f, An^{-1})_p \leq C_{r,A} n^{-r} \sum_{s=0}^{n} (s + 1)^{-1} E_\tau(f)_p.
\]

Here we recall that \(L^p(\Omega)\) denotes the space \(L^p(\Omega)\) for \(p < \infty\) and the space \(C(\Omega)\) for \(p = \infty\).

The proof of Theorem 7.1 relies on two lemmas. To state these lemmas, we recall that \(L_p(\Omega)\) and \(L_\infty(\Omega)\) be such that \(2\leq r \leq \infty\). Then by (7.1), we have

\[
\|w_r(f,\cdot, An^{-1})_p\|_{L^p(G)} \leq C_{r,p,A} n^{-r} \sum_{s=0}^{n} (s + 1)^{-1} E_\tau(f)_p.
\]

**Lemma 7.2.** Let \(G \subset \Omega\) be a domain of special type attached to \(\Gamma\). If \(r \in \mathbb{N}, A > 0, 1 \leq q \leq p \leq \infty\) and \(f \in L^p(\Omega)\), then

\[
(7.1) \quad \|w_r(f,\cdot, An^{-1})_p\|_{L^p(G)} \leq C_{r,p,A} n^{-r} \sum_{s=0}^{n} (s + 1)^{-1} E_\tau(f)_p.
\]

**Proof.** By monotonicity, it is enough to consider the case \(q = p\). It is easily seen from the definition that

\[
(7.2) \quad \|w_r(f,\cdot, t)_p\|_p \leq C_{p,r} \|f\|_p.
\]

Without loss of generality, we may assume that

\[
G := \{(x, y) : x \in (-b, b)^d, \ g(x) - 1 < y \leq g(x)\},
\]

where \(b > 0\) and \(g \in C^2(\mathbb{R}^d)\). We may also assume that \(n \geq N_0\), where \(N_0\) is a sufficiently large positive integer depending only on \(\Omega\), since otherwise (7.1) follows directly from the inequality \(\|w_r(f,\cdot, t)_p\|_p \leq CE_\tau(f)_p\), which can be obtained from (7.2).

For \(0 \leq k \leq n\), let \(P_k \in \Pi_d^{k+1}\) be such that \(\|f - P_k\|_{L^p(\Omega)} = E_k(f)_{L^p(\Omega)}\). Let \(m \in \mathbb{N}\) be such that \(2^{m-1} \leq n < 2^m\). Then by (7.2), we have

\[
\|w_r(f,\cdot, An^{-1})_p\|_{L^p(G)} \leq \|w_r(f - P_{2^m},\cdot, An^{-1})_p\|_{L^p(G)} + \|w_r(P_{2^m},\cdot, An^{-1})_p\|_{L^p(G)}.
\]

Thus, for the proof of (7.1), it suffices to show that for each \(P \in \Pi_d^{k+1}\),

\[
(7.3) \quad \|w_r(P,\cdot, An^{-1})_p\|_{L^p(G)} \leq Cn^{-r} k^r \|P\|_{L^p(\Omega)},
\]

where here and below \(C\) and constants in the equivalences may depend on \(A\).
To show (7.3), we first recall the following partition of the domain $\mathcal{G}$ constructed in Section 3.1: $\mathcal{G} = \bigcup_{i \in \Lambda^d_n} \bigcup_{j=0}^{n-1} I_{i,j}$, where

$$I_{i,j} := \left\{ (x, y) : x \in \Delta_i, \ g(x) - y \in [\alpha_j, \alpha_{j+1}] \right\}$$

and

$$i = (i_1, \ldots, i_d) \in \Lambda^d_n := \{0, 1, \ldots, n - 1\}^d \subset \mathbb{Z}^d,$$

$$\Delta_i := [t_i, t_i + 1] \times [t_i, t_i + 1] \times \cdots \times [t_i, t_i + 1] \quad \text{with} \quad t_i = -b + \frac{2i}{n},$$

$$\alpha_j := \sin^2\left(\frac{\pi}{2\ell_1 n}\right)/\sin^2\left(\frac{\pi}{2\ell_1}\right), \quad j = 0, 1, \ldots, \ell_1 n,$$

with $\ell_1 > 1$ being a large integer parameter.

As in Section 5.1, we also define for any two given integer parameters $m_0, m_1 > 1$,

$$\Delta_i^* = \Delta_{i,m_0}^* := [t_i - m_0, t_i + m_0] \times [t_i - m_0, t_i + m_0] \times \cdots \times [t_i - m_0, t_i + m_0],$$

$$I_{i,j}^* := I_{i,j,m_0,m_1}^* := \left\{ (x, y) : x \in \Delta_i^*, \ \alpha_{j-m_0}^* \leq g(x) - y \leq \alpha_{j+m_0}^* \right\},$$

where $\alpha_j^* = \alpha_j$ if $0 \leq j \leq n$, $\alpha_j^* = 0$ if $j < 0$ and $\alpha_j^* = 2$ if $j > n$. By Proposition 2.1 we may choose the parameters $m_0, m_1$ large enough so that

$$U(\xi, An^{-1}) \subset I_{i,j}^* \quad \text{whenever} \ \xi \in I_{i,j}.$$

Note that for $(i, j) \in \Lambda^d_n$ and $(x, y) \in I_{i,j}^*$,

$$\alpha_j \sim \frac{j^2}{n^2} \sim \delta(x, y) := g(x) - y, \quad j \geq 1,$$

$$\alpha_{j+1} - \alpha_j \leq C \frac{j+1}{n^2} \leq C \frac{1}{n} \varphi_n(x, y) := \frac{1}{n} \left(1 + \sqrt{\delta(x, y)}\right). \quad (7.5)$$

Now we turn to the proof of (7.3). Let $P \in \Pi^d_n$ and $1 \leq p < \infty$. Then using Remark 3.7, Proposition 2.1 and (7.3), we have

$$\left\| w_r(P_{\cdot}, An^{-1}) \right\|_{L^p(G)}^p \leq C \sum_{(i,j) \in \Lambda^d_n} \int_{I_{i,j}^*} \int_{I_{i,j}^*} |\Delta_{(\eta - \xi)/r}(P(\Omega, \xi))| |\partial_{(\eta - \xi)/r}^r f(\xi + r^{-1}(\eta - \xi)(1 + t_1 + \cdots + t_r))| \, dt_1 \cdots dr_r \, d\eta \, d\xi,$$

where $I_{i,j}^*(\xi) = \{ \eta \in I_{i,j}^* : [\xi, \eta] \in \Omega \}$. Note that by Hölder’s inequality,

$$|\Delta_{(\eta - \xi)/r}(f, \Omega, \xi)| \leq \int_{[0,1]} |\partial_{(\eta - \xi)/r}^r f(\xi + r^{-1}(\eta - \xi)(1 + t_1 + \cdots + t_r))| \, dt_1 \cdots dr_r \leq C \int_{0}^{1} |\partial_{\eta - \xi}^r f(\xi + t(\eta - \xi))| \, dt.$$

Thus,

$$\left\| w_r(P_{\cdot}, An^{-1}) \right\|_{L^p(G)}^p \leq C \sum_{(i,j) \in \Lambda^d_n} \int_{I_{i,j}^*} \int_{I_{i,j}^*} \int_{I_{i,j}^*} \int_{0}^{1} \left| \partial_{\eta - \xi}^r P(\xi + t(\eta - \xi)) \right| \, dt \, d\eta \, d\xi. \quad (7.6)$$

To estimate the sum in this last equation, we shall use the Bernstein inequality stated in Theorem 2.4. For convenience, given a parameter $\mu > 1$, and two
nonnegative integers \( l_1, l_2 \), we define
\[
M_{\mu,n}^{l_1,l_2} f(\xi) := \max_{u \in \Xi_{n,\mu}(\xi)} \max_{\xi \in \mathbb{R}^{d+1}} \left| (z_\zeta(u) \cdot \nabla)^{l_1} \partial_{d+1}^{l_2} f(\xi) \right|,
\]
where \( z_\zeta(u) = (\zeta, \partial_\zeta g(u)) \), and
\[
\Xi_{n,\mu}(\xi) := \left\{ u \in [-2a, 2a]^d : \|u - \xi\| \leq \mu \varphi_n(\xi) \right\}.
\]
We choose the parameter \( \mu \) large enough so that \( \Delta_{4m_0} \subseteq \Xi_{n,\mu}(\xi) \) for any \( \xi \in I_{i,j}^* \).

By Theorem 2.4 we have
\[
\| \varphi_n^{l_1,l_2} \|_{L^p(G^*)} \leq C \| \xi \|_{L^p(\Omega)}, \quad \forall \xi \in \Pi_{k}^{d+1}.
\]

Now fix temporarily \( \xi = (\xi_x, \xi_y) \in I_{i,j} \) and \( \eta = (\eta_x, \eta_y) \in I_{i,j}^* \). Then \( \|\xi_x - \xi_y\| \leq \frac{\xi}{n} \), and
\[
\|g(\eta_x) - g(\xi_x)\| \leq \alpha_{j,m_1}^+ - \alpha_{j,m_1}^- \leq c_1 \frac{\varphi_n(\xi)}{n},
\]
where the last step uses (7.5). Thus, setting \( \zeta = n(\eta_x - \xi_x) \), we have \( \|\zeta\| \leq c \) and we may write \( \eta - \xi \) in the form
\[
\eta - \xi = \frac{1}{n} \left( \zeta, \partial_\zeta g(u) + s \varphi_n(\xi) \right),
\]
with
\[
s = \frac{n(\eta_y - \xi_y) - \partial_\zeta g(u)}{\varphi_n(\xi)} \in [-c_1, c_1].
\]
It follows that
\[
\partial_{\eta - \xi} = \frac{1}{n} \left( \partial_{z_\zeta(u)} + s \varphi(\xi) \partial_{d+1} \right),
\]
where \( z_\zeta(u) = (\zeta, \partial_\zeta g(u)) \). This implies that for any \( (x, y) \in I_{i,j}^* \),
\[
|\partial_{\eta - \xi}^r P(x,y)| \leq C n^{-r} \max_{0 \leq k \leq r} \varphi_n(\xi) |\partial_{z_\zeta(u)}^{r-k} \partial_{d+1}^k P(x,y)|
\leq C n^{-r} \max_{0 \leq k \leq r} \varphi_n(\xi) M_{\mu,n}^{-r,k} P(x,y).
\]
Thus, setting
\[
P_*(x,y) := \max_{0 \leq k \leq r} \varphi_n(\xi) M_{\mu,n}^{-r,k} P(x,y),
\]
we obtain from (7.3) that
\[
\left\| w_r(P_*, A^{-1}) \right\|_{L^p(G)} \leq C n^{-r p} \sum_{(i,j) \in A_n^G} \int_{I_{i,j}^*} \int_{I_{i,j}^*(\xi)} \int_0^1 \int_{I_{i,j}^*(\xi)} \|P_*(\xi + t(\eta - \xi))\|^p dt d\eta d\xi
\leq C n^{-r p} \sum_{(i,j) \in A_n^G} \int_{I_{i,j,2m_0,2m_1}} \|P_*(\eta)\|^p d\eta \leq C n^{-r p} \|P_\|_{L^p(G^*, (2))} \leq C \left( \frac{k}{n} \right)^r \|P\|_{L^p(\Omega)},
\]
where the last step uses (7.7). This proves (7.3) for \( 1 \leq p < \infty \).

Finally, (7.3) for \( p = \infty \) can be proved similarly. This completes the proof of Lemma 7.2. \( \square \)
Lemma 7.3. Let $\varepsilon \in (0, 1)$ and $\Omega^\varepsilon := \{\xi \in \Omega : \text{dist}(\xi, \Gamma) > \varepsilon\}$. If $r \in \mathbb{N}$, $A > 0$, $1 \leq q \leq p \leq \infty$ and $f \in L^p(\Omega)$, then
\[
\left\|w_r(f, \cdot, An^{-1})_p\right\|_{L^p(\Omega^\varepsilon)} \leq C_{r,p,A} n^{-r} \sum_{s=0}^{n} (s + 1)^{r-1} E_s(f)_{L^p(\Omega)}.
\]

Proof. The proof is similar to that of Lemma 7.2, and in fact, is simpler. It relies on the following Bernstein inequality,
\[
\|\partial^\beta P\|_{L^p(\Omega^\varepsilon)} \leq C_k |\beta| \|P\|_{L^p(\Omega)}, \quad \forall P \in \Pi_{k}^{d+1}, \quad \forall \beta \in \mathbb{Z}_+^{d+1},
\]
which is a direct consequence of the univariate Bernstein inequality (2.2). □

Now we are in a position to prove Theorem 7.1.

Proof of Theorem 7.1. By monotonicity, it suffices to consider the case $p = q$. By Lemma 1.6 there exist $\varepsilon \in (0, 1)$ and domains $G_1, \ldots, G_{m_0} \subset \Omega$ of special type attached to $\Gamma$ such that
\[
\Gamma^\varepsilon := \{\xi \in \Omega : \text{dist}(\xi, \Gamma) \leq \varepsilon\} \subset \bigcup_{j=1}^{m_0} G_j.
\]
Setting $\Omega^\varepsilon := \Omega \setminus \Gamma^\varepsilon$, we have
\[
\tau_r(f, An^{-1})_{p,p} \leq \sum_{j=1}^{m_0} \left\|w_r(f, \cdot, An^{-1})_p\right\|_{L^p(G_j)} + \left\|w_r(f, \cdot, An^{-1})_p\right\|_{L^p(\Omega^\varepsilon)},
\]
which, using Lemma 7.2 and Lemma 7.3 is estimated above by a constant multiple of
\[
n^{-r} \sum_{s=0}^{n} (s + 1)^{r-1} E_s(f)_{L^p(\Omega)}.
\]
This completes the proof. □

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