Currency target zone modeling: An interplay between physics and economics

Sandro Claudio Lera, Didier Sornette

Abstract

We study the performance of the euro/Swiss franc exchange rate in the extraordinary period from September 6, 2011 and January 15, 2015 when the Swiss National Bank enforced a minimum exchange rate of 1.20 Swiss francs per euro. Within the general framework built on geometric Brownian motions (GBM) and based on the analogy between Brownian motion in finance and physics, the first-order effect of such a steric constraint would enter a priori in the form of a repulsive entropic force associated with the paths crossing the barrier that are forbidden. Non-parametric empirical estimates of drift and volatility show that the predicted first-order analogy between economics and physics are incorrect. The clue is to realise that the random walk nature of financial prices results from the continuous anticipations of traders about future opportunities, whose aggregate actions translate into an approximate efficient market with almost no arbitrage opportunities. With the Swiss National Bank stated commitment to enforce the barrier, traders’ anticipation of this action leads to a vanishing drift together with a volatility of the exchange rate that depends on the distance to the barrier. This effect is described by Krugman’s model [P.R. Krugman. Target zones and exchange rate dynamics. The Quarterly Journal of Economics, 106(3):669-682, 1991]. We give direct quantitative empirical evidence that Krugman’s theoretical model provides an accurate description of the euro/Swiss franc target zone. Motivated by the insights from the economical model, we revise the initial economics-physics analogy and show that, within the context of hindered diffusion, the two systems can be described with the same mathematics after all. Using a recently proposed extended analogy in terms of a colloidal Brownian particle embedded in a fluid of molecules associated with the underlying order book, we derive that, close to the restricting boundary, the dynamics of both systems is described by a stochastic differential equation with a very small constant drift and a linear diffusion coefficient. As a side result, we present a simplified derivation of the linear hydrodynamic diffusion coefficient of a Brownian particle close to a wall.

Keywords: Exchange rate dynamics, target zone, order book fluid, econophysics

PACS: 89.65.Gh, 05.40.Jc; 89.75.-k

1. Introduction

Perhaps not apparent at first glance, physics and economics have been life-long companions during their mutual development of concepts and methods emerging in both fields. There has been much mutual enrichment and cross-fertilization. Since the beginning of the formulation of the scientific approach in the physical and natural sciences, economists have taken inspiration from physics, in particular in its success in describing natural regularities and processes [1]. Already in 1776, Adam Smith formulated his “Inquiry into the Nature and Causes of the Wealth of Nations” inspired by the “Philosophiae Naturals Principia Mathematica” (1687) of Sir Isaac Newton, which specifically stresses the notion of causative forces. Reciprocally, physics has been inspired several times by observations in economics. A prominent example of this kind is the theory of Brownian motion and random walks. In order to model the apparent random walk motion of bonds and stock options in the Paris stock market, mathematician Louis Bachelier [2] developed in his thesis the mathematical theory of diffusion. He solved the parabolic diffusion equation five years before Albert Einstein [3] established the theory of Brownian motion based on the same diffusion equation, also underpinning the theory of random walks. These two works have ushered research on mathematical descriptions of fluctuation phenomena in statistical physics, of quantum fluctuation processes in elementary particles-fields physics, on the one hand, and of financial prices on the other hand, both anchored in the random walk model and Wiener process.

Here, we will extend this analogy even further and describe a restricted Brownian motion both from a physical and an economical perspective, paying close attention to the interplay between the two fields. The purpose of this paper is thereby twofold. On the one hand, we provide novel evidence to support the famous, yet often times refuted Krugman target zone model. Since this aspect is, however, primarily of interest for the economics community, a more detailed report of these results will be published elsewhere [4]. More importantly, here, we use the example of a restricted Brownian motion to point out several new observa-
tions made about the interplay between natural and social sciences, which are relevant in particular for the physicist aspiring to do interdisciplinary research in economics and finance.

2. Target zone arrangements

On September 6, 2011, the Swiss National Bank (SNB) announced that it would enforce a minimum exchange rate of 1.20 Swiss francs (CHF) per euro (EUR), in response to the European debt crisis and a continuously weakening euro. With a level around 1.6 CHF at the introduction of the euro in 1999 and a peak above 1.67 CHF on October 2007, the EUR/CHF has been floating freely until it dived to the record low of CHF 1.0070 per euro on August 9, 2011. The Swiss National Bank intervened massively leading to a fast rebound of the euro. On September 6, the SNB announced officially that it would defend the minimum exchange rate of CHF 1.20 by all means (buying euros and selling Swiss francs in unlimited amounts as deemed necessary). The SNB held this policy until January 15, 2015, leading to an exchange rate levelling off between 1.20 and 1.24 CHF per euro, exhibiting a dynamics that is spectacularly different from what is observed for a freely floating currency pair, as can be seen in figure 1.

Such a restriction of an exchange rate is known in the finance literature as a target zone arrangement. The central bank announces that the exchange rate between the domestic and a foreign currency as determined by the free market will not be let to exceed some pre-defined lower boundary (an upper boundary is analogous, two simultaneous boundaries lead to slight modifications that are not further relevant for this discussion). As long as the exchange rate is well above the target zone boundary, the exchange rate is allowed to fluctuate freely, controlled only by supply and demand on the foreign-exchange market (FX). Once the exchange rate approaches the defended limit from above, the central bank intervenes by selling its domestic currency to buy foreign exchange, thus increasing supply of the domestic and demand of the foreign currency. This pushes the exchange rate back above its lower limit.

The most general ansatz to model the dynamics of a financial time series is represented by

$$\frac{dx}{dt} = f(s, t) + g(s, t) \cdot \eta(t)$$

where $s$ is the logarithm of the exchange rate (EUR/CHF exchange rate in this article), with $\eta$ a Gaussian white noise and $f, g$ are two functions representing respectively the drift or expected return and the volatility (standard deviation). This ansatz is general since higher order derivatives would imply temporal correlations, thus violating the no-arbitrage condition at the heart of the efficient market hypothesis (EMH) [5–7]. (Interestingly, it has been pointed out by several physicists that exactly the use of second order differential equations can have its merits even in finance, see [8–10].)

Dealing with exchange rates requires a priori to pay attention to the so-called triangular arbitrage [11], as occurs when an additional currency, say US dollar (USD) is included. We then have to consider the relations between the triplet of exchange rates EUR/CHF, CHF/USD and USD/EUR. The triangular (no-) arbitrage condition is that the number of CHF for 1 EUR should be equal to the product of the number of CHF per 1 USD times the number of USD per 1 EUR. And a similar condition extends to cycles involving more currencies of arbitrary lengths. In practice, deviation from triangular (no-)arbitrage conditions may occur only at small time scales and disappears very fast as a result of the action of traders taking advantage of inconsistent cross-rates. Since our work is focused on the EUR/CHF exchange rate close to a boundary imposed only on it and other currencies are not directly targeted by the action of the central bank, we neglect such influence. An a priori justification is that the statistical properties found for exchange rate price series are in general similar if not undistinguishable from those of equities. Our results support this simplifying assumption.

We follow the usual convention that the exchange rate denotes the amount of domestic currency that is needed to buy one unit of foreign currency. Pure Brownian motion is recovered for $f = 0$ and $g$ constant, whereas pure geometric brownian motion (GBM) [2, 12] denotes the special case of $f$ and $g$ being constant, which embodies EMH that financial markets incorporate information so effectively that the resulting price trajectory is akin to a random walk with no possible arbitrage. In order to respect causality for the correct calculation of investments performance, this stochastic equation is understood in the Itô-sense. In the remainder of this paper, we will investigate what is the predicted shape of $f$ and $g$ from different points of view.

3. Target zone modeling I: A first physicist’s approach

Starting from the structure (1), we investigate the nature of the minimal ingredients needed to capture the abnormal dynamics observed in figure 1. The aberrant trajectory of the EUR/CHF log-exchange rate $s(t)$ is clearly embodied by the visible existence of the barrier at $s = \xi \equiv \log(1.20)$ and the tendency for $s(t)$ to remain very close to it between September 2011 and January 2015. The simplest direct application of the GBM model to this situation is to assume that $s(t)$ continues to follow a simple random walk but now constrained to remain above the impenetrable cap at $\xi$. Since such a situation is not intrinsic to finance, but could just as well correspond to a (one-dimensional) physical Brownian particle that is restricted by a wall at $\xi$, we can use this analogy to employ well known mathematical tools from physics. Putting a wall constraint on a random walk is known to induce an effective entropic force acting on the particle, resulting from the reduction of path configurations by reflecting all random walks that would cross the wall [13, 14]. In 1 + 1 dimensions (one spatial dimension and one temporal dimension), the corresponding entropic repulsive force can be shown to derive from an effective long-range entropic potential $V_{\text{ENT}} = C/(s - \xi)$, where $C > 0$ is a constant [15, 16]. Intuitively, this self-similar long-range potential is associated with the relationship between the average distance to
Figure 1: We show the Euro/Swiss franc (EUR/CHF) exchange rate between January 1, 2010 and June 30, 2015. On September 6, 2011, the Swiss National Bank (SNB) officially announced its decision to enforce a minimum of 1.20 Swiss francs per euro by buying euros and selling Swiss francs in unlimited amounts if necessary.

the wall and the long wavelengths of the random walks that are suppressed by the rigid impenetrable barrier.

It is easy to see that the two ingredients “Brownian motion” and “restricting wall” cannot account for the sustained proximity of the particle (exchange rate) to the wall (figure 1). We need to add at least one more ingredient to account for this fact, which, in the economic picture, comes from the strong economic “pressure” on the euro resulting from the European crisis, which led to the introduction of the 1.20-cap in the first place. The simplest assumption is to assume a constant physical pressure that pushes the particle towards the wall, corresponding to the linear potential $V_{\text{ECO}} = F \cdot (s - s_{\text{eq}})$ with a constant $F > 0$. Together, this yields the following total potential

$$ V \equiv V_{\text{ENT}} + V_{\text{ECO}} = \frac{C}{s - s_{\text{eq}}} + F \cdot (s - s_{\text{eq}}). $$

(2)

depicted in figure 2. The equilibrium position at which expression (2) finds its minimum is $s_{\text{eq}} = s + \sqrt{C/F}$; unsurprisingly, the stronger the pressure $F$ on the euro, the closer is the equilibrium exchange rate to the barrier. Expanding (2) around $s_{\text{eq}}$, using $f \equiv -dV/ds$ and inserting this into (1) gives to leading orders

$$ \frac{ds}{dt} = 3 \frac{F^2}{C} \left( s - s_{\text{eq}} \right)^2 - 2 \sqrt{\frac{F^3}{C}} \left( s - s_{\text{eq}} \right) + g \cdot \eta(t). $$

(3)

With equation (3), we have derived a model aimed at capturing the constrained EUR/CHF dynamics using only a minimal number of ingredients. Theoretically, one predicts from (3) a volatility scaling as $(s_{\text{eq}} - 1.20)^{3/2}$ and a skewness scaling as $(s_{\text{eq}} - 1.20)^2$, as can be derived without solving (3) using a path integral formalism and an expansion in terms of Feynman diagrams [17] (see [18] for the detailed calculations). One way to test the naive hypothesis (3) would be by calculating the empirical moments from the data and comparing to the theoretical results. Instead, we choose a more direct test and determine $f$ and $g$ empirically from the data.

Figure 2: Random trajectory (the fluctuating, continuous line on the left) of a one-dimensional Brownian particle moving in a potential $V(s)$ (continuous line on the right). This potential is the sum of an attractive potential (dashed line) and a repulsive potential (dotted line). From the most simplified physical perspective, one would expect the EUR/CHF exchange rate between September 2011 and January 2015 to be controlled by such force potentials.
4. Empirical estimation of drift and volatility

We test the hypothesis (3) by extracting the terms \( f \) and \( g \) directly from the empirical data, using the definition [19]

\[
f(s, t) \equiv \lim_{\tau \to 0} \frac{1}{\tau} \mathbb{E} [s(t + \tau) - s(t)]
\]

(4)

\[
g(s, t) \equiv \lim_{\tau \to 0} \frac{1}{\tau} \mathbb{E} [(s(t + \tau) - s(t))^2]
\]

(5)

with \( \mathbb{E} [\cdot] \) the theoretical expectation operator. Assume that we are given a discrete time series consisting of \( N \) data points \( s_1, s_2, \ldots, s_N \), which is a discrete realisation of a stochastic process \( \{s(t)\}_{t \geq 0} \) (for instance a dataset of historical exchange rates). The temporal distance between two succeeding data points \( s_i \) and \( s_{i+1} \) is equal to \( \tau (0 < \tau \ll 1) \) and assumed independent of \( i \). Under the additional assumption that the process is stationary, \( f(s, t) = f(s), g(s, t) = g(s) \), a parameter-free approach to extract \( f \) and \( g \) directly from this time series is obtained by slicing up the value range \([\text{min}, s_i, \text{max}, s_j]\) of the time series into \( K \) bins \( B_i, i = 1, \ldots, K \) and approximating \( f \) and \( g \) in each bin according to [20]

\[
f(s') \approx \frac{1}{\tau} \sum_{i \in B_i} (s_{i+1} - s_i)
\]

(6)

\[
g(s') \approx \sqrt{\frac{1}{\tau} \sum_{i \in B_i} (s_{i+1} - s_i)^2},
\]

(7)

where \( s' \) denotes the mid point of the \( \ell \)-th bin and the summation is meant over all data points \( s_i \) that lie in the \( \ell \)-th bin. For our application, we download tick by tick data of the EUR/CHF exchange rate, which is then coarse-grained to equally spaced time stamps of 10 seconds \( (\tau = 1/360 \text{ hours}) \) by taking the median. The result is shown in figure 3 for \( K = 100 \) bins. We have verified that the results are robust to different choices of the number of bins \( K \) ranging at least from 20 to 140 as well as with respect to sub-sampling at multiples of the initial time scale \( \tau \) [21], confirming that the procedure (6) and (7) attains a reasonable linear convergence.

Remarkably, we find that \( f \) is essentially constant (and close to 0), in complete contradiction with the constrained random walk entropic argument: there is no entropic or other potential-derived force acting on the particle. The second interesting observation is that it is \( g \) that exhibits a non trivial \( s \) dependence. It turns out that this non trivial behavior of \( g \) is intrinsic to the target zone regime. We have applied the algorithm of Friedrich et al. [20] to EUR/CHF exchange rate data before September 2011. In the period preceding the committed action of the Swiss National Bank, \( g \) remains approximately constant over a large range of values, thus recovering the standard GBM model. The corresponding figure is shown in Appendix A.

5. Target zone modeling II: An economist’s approach

The empirical results from figure 3 clearly rejects the simple naive physical model (3). Before we go on and look for answers in a more sophisticated physical model, let us consider first what is expected from a purely economical point of view.

5.1. The no-arbitrage condition

The fact that \( f(s) \) is essentially zero for all \( s \) reveals an important difference between physical and economical Brownian motion. The exchange rate fluctuations are not due to unconscious random actions as would be the myriads of collisions of fluid molecules on a Brownian particle but due to the decisions of investors trying to extract profit from their investments. The aggregate result of this behavior of extremely motivated and driven agents is the quasi-absence of arbitrage, namely the impossibility to extract an excess return. The no-arbitrage condition is one of the organising principles of financial mathematics and is expressed in general by the condition that the process \( s(t) \) obeying (1) should be a martingale [22]. In a risk neutral framework (which means that investors do not require additional return for being exposed to risks), this translates mechanically into the condition of zero drift \( f(s) = 0 \). In the presence of risk aversion, small values of \( f(s) \) are present to remunerate the investors from their expositions to the risks associated with the fluctuating prices. If there was a well-defined, significant drift, knowledge thereof could immediately be translated into sure gains.

To illustrate that this would be the case in our simplified physical model (3), we simulated synthetic time series with the generating process (3), which has a non-zero \( f(s) \), with parameters chosen to match the empirical volatility. We used the simple strategy of selling (resp. buying) the euro and buying (resp. selling) the Swiss franc whenever \( s > s_{eq} \) (resp. \( s < s_{eq} \)). Including typical transaction costs between 1 and 2 pips (1 pip = 0.0001 is approximately equal to the bid-ask spread of the real EUR/CHF tick data from Sept. 6, 2011 to January 14, 2015), we find this strategy to deliver extremely high, two-digit annualised Sharpe ratios (as a benchmark, it is
5.2. Brief summary of Krugman’s model

The work of Krugman [23] turns out to be the reference of a large part of the economic target zone literature. According to Krugman, the constrained exchange rate $s$ can be described as

$$ s = m + v + \gamma \frac{\mathbb{E}[ds]}{dt}. \quad (8) $$

By $m$, we denote the (logarithm of the) money supply. As long as $s$ is above the lower boundary $\tilde{s}$, $m$ is supposed to be held constant. Once $s$ touches $\tilde{s}$, the central bank (here the SNB) is supposed to increase the money supply, thus weakening the domestic currency (CHF) relative to the foreign one (EUR), which means that $s$ is pushed away from the lower boundary. By $v$, we denote the (logarithm of) exogenous velocity shocks, i.e. influences on the exchange rate coming from the economic and political environment that cannot be controlled by the national bank. It is assumed that $v$ follows a standard Brownian motion

$$ dv = \sigma \, dW_t \quad (\sigma > 0). \quad (9) $$

The last ingredient to Krugman’s model is the expected change in $s$, $\mathbb{E}[ds]/dt$. It is this term that makes all the difference between the naive physical model and Krugman’s economical model. The reasoning behind this term is that, as $s$ approaches $\tilde{s}$ from above, market participants anticipate the central bank’s intervention and act accordingly. This is different from the unconscious physical particle that, as long as there is no contact (either direct or mitigated through the fluid), behaves as if there was no wall.

The constant $\gamma$ denotes the semi-elasticity of the exchange rate with respect to the instantaneous expected rate of currency depreciation. Equation (8) can be solved with basic stochastic calculus, see [23]. The result reads

$$ s = m + v + Ae^{-\rho v} \quad (10) $$

where $\rho = \sqrt{2}/\gamma \sigma^2$. Denote by $y$ the unique value of $v$ at which $s(y) = \tilde{s}$ (for fixed $m$). Then, the constant $A$ is determined uniquely by demanding that the derivative of $s$ as a function of $v$ vanishes at $\tilde{s}$.

$$ \frac{ds}{dv} \bigg|_{v=\tilde{s}} = 0. \quad (11) $$

This condition is rooted in a no-arbitrage argument known in option pricing as smooth pasting [24]. The final result is depicted in figure 4.

5.3. Assumptions of Krugman’s model

Krugman’s target zone model is based on two crucial assumptions: First, the target zone is perfectly credible. This means that market participants believe at every time that the central bank will stick to its announced target zone. Second, the interventions by the central bank are marginal, meaning the monetary supply is held constant as long as $s$ is within the target zone band. Only when $s$ touches $\tilde{s}$, the monetary supply is increased, just sufficiently to keep $s$ at $\tilde{s}$. These assumptions have been investigated specifically for the EUR/CHF exchange rate between 2011 and 2015 in [25]. It is found that the two assumptions hold sufficiently well so that Krugman’s model can be applied. This sets the EUR/CHF target zone apart from many earlier empirical studies in which Krugman’s model was already challenged on the basis of its assumptions. We refer to [26, 27] for detailed reviews.

5.4. Drift and volatility in Krugman’s model

By applying Itô’s lemma to (10), we derive the following drift $f$ and volatility $g$ in the Krugman framework:

$$ f(v) = \frac{1}{2} A \sigma^2 \rho^2 \, e^{-\rho v} \quad (12) $$

$$ g(v) = \sigma - \sigma A \rho \, e^{-\rho v}. \quad (13) $$

For practical purposes, working with (12) and (13) is cumbersome because $v$ cannot be measured but only estimated [28]. Nevertheless, testing directly the non-linear $s(v)$ relation (10) by estimating $v$ is the method that has been widely applied in the empirical literature. The reported results have then either rejected Krugman’s target zone model entirely or have shown
only a very noisy evidence for (10). We refer again to [26, 27] for a broad overview and to [25] for EUR/CHF specific results.

Our strategy is different. Instead of relying on \( v \), we invert the \( s(v) \) relation (10) locally to lowest order in \( v - \bar{v} \) (it is easy to see that (10) has a well-defined, global inverse \( v(s) \) that, however, has no analytical closed form expression). For \( s \) close to \( \underline{s} \), a second-order Taylor expansion gives

\[
s(f) \approx s + \frac{1}{2} \sigma (f - \bar{f})^2.
\]

(14)

Inverting (14) yields \( f(s) \) and plugging this into (12) and (13), the following expressions for drift and volatility are found:

\[
f(s) = \alpha
\]

(15)

\[
g(s) = \beta \sqrt{s - \underline{s}}
\]

(16)

where \( \alpha = \sigma \sqrt{2\gamma} \), \( \beta = 2^{3/4} \sqrt{\gamma} / \sqrt{4} \). In particular, we note that \( \sqrt{\alpha}/\beta = 1/2 \). There are higher order terms leading to corrections to (15) and (16). It is easy to check that, for our data where \( s < \log(1.26) \), these corrections are negligible.

Comparing (15) and (16) with figure 3, one can check that the data conform very well to Krugman’s theory. For the volatility, we can apply a one parameter least-squares fit which determines \( \beta = (5.42 \pm 0.06) \cdot 10^{-3} \). Another least-squares fit determines \( \alpha = (1.40 \pm 0.8) \cdot 10^{-5} \). Basic error propagation calculations yield \( \sqrt{\alpha}/\beta = 0.68 \pm 0.22 \). Despite the relatively large fluctuations for \( s \gtrsim \log(1.24) \), the data agrees with the theoretical value 1/2 within one standard deviation. Ignoring the large fluctuations around \( s \gtrsim \log(1.24) \) leads to even better correspondence between data and theory. We have also applied a maximum likelihood ratio test of nested hypotheses, which consists in comparing the hypothesis (16) to the more general \( g(s) = \beta(s - \underline{s})^\mu \) with variable \( \mu \). We find a \( p \)-value of 0.74 (above the standard confidence bound 0.05), which means that the extended model with fitted exponent \( \mu \) is not necessary and that expression (16) with the fixed \( \mu = 1/2 \) is sufficient to describe the data. This confirms that Krugman’s target zone model provides a suitable description of the constrained EUR/CHF exchange rate.

We stress that this result is novel also from the perspective of pure economical research. First, by inverting locally the relationship between \( s \) and \( v \), we have derived a way by which the model can be tested directly, without the need of estimating the (usually unknown and unknowable) fundamental value of the exchange rate. Second, the theoretically celebrated Krugman model has been mostly rejected by previous empirical studies. Although target zones are not a new concept and have been particularly popular in the European Monetary System (EMS) from the 1970s to the 1990s, never has there been such a consistent pressure keeping the exchange rate remarkably close to the target zone boundary over large periods of time, as was observed for the EUR/CHF target zone. Another example of a recently introduced target zone is the one of the US dollar/Hong Kong dollar (USD/HKD) exchange rate implemented by the Hong Kong Monetary Authority (HKMA) in 2005 (figure 5). It is easy to see that here, unlike for the EUR/CHF target zone, the pressure is not as consistent and the barrier not as vehemently defended (there are small fluctuations around the target zone limit). This disqualifies many other target zones already due to the restrictive assumptions of Krugman’s target zone model and highlights the special case of the EUR/CHF target zone. Thus, we have shown that Krugman’s model holds after all, but not for arbitrary target zones, but only under extreme and sustained pressure (in this case associated with the European crisis) that pushes continuously the exchange rate very close to the boundary of the target zone, which is rigorously defended by the central bank.

6. Target zone modeling III: Hindered diffusion

We have shown in the previous section that the initial naive physical model failed to account to for the action anticipating traders. In broader terms, we can classify this finding as “the presence of the wall must be felt even away from the wall” at all times and for all random walk realisations. We use this insight to develop a second, more elaborate physical model. We note that the physical Brownian particle can receive the information “there is a wall” not only through direct contact but also through the fluid.

Consider a physical Brownian particle in a fluid. The presence of a wall leads to a modification of the hydrodynamic flow of the molecules trapped between the wall and the Brownian particle. The closer the Brownian particle to the wall, the thinner the lubrication layer between them and the more hindered is the diffusion of the Brownian particle. In physics, it is more common to work with the diffusion coefficient \( D(s) \) which is related to our volatility via \( g = \sqrt{2D} \). In the bulk of a fluid (where the wall is not felt), the diffusion coefficient \( D \) is a constant \( D_0 \). The Einstein-Stokes equation predicts for a spherical particle with radius \( R \)

\[
D_0 = \frac{k_B T}{6\pi \nu R}
\]

(17)

with \( k_B \) the Boltzmann constant, \( T \) the temperature and \( \nu \) the viscosity of the fluid. In presence of a wall at \( s = \underline{s} \), Brenner
found decades before by Lorentz [30] who predicted

\[ \lambda \sim 1 + \frac{9}{8} \frac{R}{s - \frac{R}{2}} \]  

(18)

from which we find, to first order, \( D(s) \sim (s - \frac{R}{2}) \). It follows immediately that, close to the wall, the volatility (\( g = \sqrt{2D} \)) of the particle increases like the square-root of \( s - \frac{R}{2} \), in correspondence with the model (16) from finance (it is clear that the constant pre-factor depends on physical quantities that are unrelated to the economical analogy). Hence, we have shown that there is an analogy between physical and economical Brownian motion after all. Had we relied from the beginning on the correct physical picture of hindered diffusion, we could have predicted the behavior of the exchange rate close to the barrier solely on the basis of physical theories that were known already long before Krugman’s model (up to a scale factor, discussed in the next section).

The analogy that we have found here extends a previous result [31] in which it has been shown that the GBM model of financial price fluctuations is deeply anchored in the physics-finance analogy of a colloidal Brownian particle embedded in a fluid of molecules as shown in figure 2 (omitting the previously shown incorrect potentials), where the surrounding molecules reflect the structure of the underlying order book. Here, we have shown that this analogy holds even if the order book fluid is restricted from one side by an upper or lower bound.

7. Equilibrium in physics and in economics

In absence of any external force, what is the stochastic process that describes a physical Brownian particle? Naively, one is led to propose \( ds/dt = g(s) \cdot \eta(t) \). However, this implies not only that we are working in Itô’s interpretation of stochastic calculus, but can furthermore be shown to be inconsistent with convergence towards thermal equilibrium. For a system at equilibrium, the probability density \( p(s,t) \) must have a steady state solution with the canonical form \( p(s) \sim \exp(-H/k_BT) \) with \( H \) the Hamiltonian of the system. If we insist on working in Itô’s interpretation as is customary in finance to ensure causality of financial strategies, one must include an additional drift term \( g(s) \frac{ds}{dt} \) to the stochastic differential equation in order to be consistent with the physical steady state distribution (see section 2.2.3 of [32] and [33] for derivations). From (16), we then derive the following stochastic equation for a Brownian particle close the a wall and in absence of external forces:

\[ \frac{ds}{dt} = g(s) \frac{dg(s)}{ds} + g(s) \cdot \eta(t) = \frac{\beta^2}{2} + \beta \sqrt{s - \frac{R}{2}} \cdot \eta(t). \]  

(19)

Remarkably, the square-root shaped volatility is exactly the function that induces a constant positive drift in agreement with Krugman’s prediction (15). From a purely physical perspective, this result (19) has another interesting implication. It reveals the special role played by the linearly increasing diffusion coefficient. It can be shown that a locally linear diffusion coefficient is the only physically sensible choice. Since this result is not the main concern of our paper, we refer the interested reader to Appendix A for its derivation.

The correspondence between physical hindered diffusion and Krugman’s target zone model is therefore only semi-quantitative in the sense that here \( \sqrt{\text{drift term}}/\beta = 1/\sqrt{2} \). For Krugman, on the other hand, we have derived \( \sqrt{\text{drift term}}/\beta = 1/2 \), thus revealing a key difference between Krugman’s constant drift term and the one resulting from a noise-induced drift of the form (19). We attribute this difference of the numerical values of \( \sqrt{\text{drift term}}/\beta \) to the global condition of thermal equilibrium \( p(s) \sim \exp(-H/k_BT) \), which is absent in finance. Lévy and Roll [34] have recently proposed to impose the constraint that the global market portfolio is mean-variance efficient, i.e., that it obeys the predictions of the Capital Asset Pricing Model (CAPM). This global condition can be shown to lead to a reassessment and an improved estimation of the expected returns of the stocks constituting the global market [35]. But it is not known what could be other consequences, in particular in exchange rate dynamics. Indeed, in finance, the existence of an economic equilibrium distribution similar to Boltzmann, and its relation to detailed balance is highly debated and far from trivial. We refer to [36, 37] for recent discussions of this topic and to [1, 38–41] for further details on the interplay and coevolution of physics and economics in general.

8. Conclusions

This paper has served two purposes. From a technical perspective, we have shown that the constrained EUR/CHF exchange rate is well-described by Krugman’s target zone model [23], which incorporates the traders’ expectations as a fundamental ingredient into its equations. By describing the exchange rate as a colloidal Brownian particle embedded in an “order book fluid”, we could show furthermore that there is
a formal analogy to the physical hindered diffusion problem in the sense that both systems can be described by the same stochastic differential equation. This provides novel empirical support for the recently introduced model of a “financial Brownian particle in a layered order book fluid” [31], which generalises the standard random walk model of financial price fluctuations.

From a didactical perspective, we have given a complete example of how economic models can be motivated by ideas and mathematical tools from physics and vice versa. We have also pointed out a fundamental difference between physical and economic hindered diffusion. In physics, we have an additional constraint in terms of a thermal equilibrium based on detailed balance. In finance, the existence of such a global equilibrium is a priori not clear and must be investigated further. It would not be unsurprising, if further motivations or analogies can be found from the study of physical out-of-equilibrium systems.

Acknowledgment

The research performed by one of us (SL), which is reported in this article, was conducted at the Future Resilient Systems at the Singapore-ETH Centre (SEC). The SEC was established as a collaboration between ETH Zurich and National Research Foundation (NRF) Singapore (FI 370074011) under the auspices of the NRF’s Campus for Research Excellence and Technological Enterprise (CREATE) programme.

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Appendix A. Target zone dependence

For the edifice of this paper, it is vital to show that the square-root shaped volatility is intrinsic to the target zone regime from September 2011 to January 2015. Indeed, applying the algorithm of Friedrich et al. [20] to EUR/CHF exchange rate data ranging from 2005 to 2007 and from 2008 to 2010 (figure A.7) shows that g is roughly constant over a large regime of values.

![Figure A.7: Approximation of drift f and volatility g using 10 seconds data of the EUR/CHF exchange rate from 2005 to 2007. This figure is obtained by applying the algorithm by Friedrich et al. [20]. In contrast to the target zone regime, we observe that here g is roughly constant.](image)

We have chosen data ranging over periods of three years in order to have approximately the same amount of data points as during the target zone regime.

Appendix B. Diffusion close to a wall

Working with Itô’s interpretation of stochastic calculus, it can be shown (section 2.2.3 of [32] and [33]) that a Brownian particle with general diffusion coefficient \( D(s) = g(s)^2/2 \) and in absence of any external forces is described by the stochastic differential equation

\[
\frac{ds}{dt} = g(s)\frac{dg(s)}{ds} + g(s) \cdot \eta(t). \tag{B.1}
\]

We want to determine the volatility g(s) of a Brownian particle at position s close to a wall located at \( s = \xi \). This problem was first solved in an exact (but fairly complicated) manner by Brenner [29], stating that the bulk diffusion coefficient (17) must be replaced by \( D_0/\lambda \). Without loss of generality, we set now \( s = 0 \). From Lorentz’ approximate result (18), we infer that close to the wall, \( D(s) = D_0/\lambda \) is, to first order, linear in s.

In this appendix, we want to give a less rigorous but simple heuristic derivation of this result. What is nice about our derivation is that no detailed knowledge about hydrodynamic interactions is required. We make the fairly general approximation that, close to the wall, \( g(s) = \beta s^\gamma \) for some \( \gamma > 0 \) (it is easy to see that \( \lim_{s \downarrow 0} D(s) = 0 \) is a necessary condition). Plugging this into (B.1) gives

\[
\frac{ds}{dt} = \beta^2 \gamma s^{2\gamma-1} + \beta s^\gamma \cdot \eta(t). \tag{B.2}
\]

In the limit \( s \downarrow 0 \), we can distinguish three cases:

\[
\begin{align*}
\text{the drift } g(s) \frac{dg(s)}{ds} & \begin{cases} 
\text{diverges if } \gamma < 1/2, \\
\text{is constant if } \gamma = 1/2, \\
\text{vanishes if } \gamma > 1/2.
\end{cases}
\end{align*}
\]

If \( \gamma < 1/2 \), the particle will be repelled with infinite force and can never touch the wall. Furthermore, placing initially the particle at the wall is ill-defined. If \( \gamma > 1/2 \), the particle, once it has reached the wall, will stay there forever (more precisely, it can be shown that a particle starting from \( s > 0 \) can never exactly reach the wall, but approach it arbitrarily close [42]). Also, a particle placed at the the wall will simply stay there forever. We deduce that \( \gamma = 1/2 \), and hence \( D(s) \sim s \) is the only physically reasonable choice. In this case, a particle starting from \( s \geq 0 \) has non-zero probability to reach the boundary in finite time, upon which it will be repelled. These arguments can be formalised by solving analytically the Fokker-Planck equation corresponding to (B.2) in terms of an eigenfunction expansion [19, 43].