Ergodicity for Time Changed Symmetric Stable Processes

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Abstract

In this paper we study the ergodicity and the related semigroup property for a class of symmetric Markov jump processes associated with time changed symmetric α-stable processes. For this purpose, explicit and sharp criteria for Poincaré type inequalities (including Poincaré, super Poincaré and weak Poincaré inequalities) of the corresponding non-local Dirichlet forms are derived. Moreover, our main results, when applied to a class of one-dimensional stochastic differential equations driven by symmetric α-stable processes, yield sharp criteria for their various ergodic properties and corresponding functional inequalities.

Keywords: symmetric stable processes, time-change, Poincaré type inequalities, non-local Dirichlet forms

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1 Introduction and Main Results

Given a conservative symmetric Markov process which is not ergodic such as Brownian motion or symmetric stable process on $\mathbb{R}^d$, is it possible to turn it into an ergodic process by a time-change? It is known that transience and recurrence are invariant under time-change (see [8, Theorem 5.2.5]). Thus one can never turn a transient process into a recurrent process. In fact, after a time-change a transient conservative process may have finite lifetime. In [7], extension of such a time changed process has been investigated for transient reflected Brownian motion. In this paper, we investigate various ergodic properties of recurrent symmetric stable processes under suitable time-change.

For $\alpha \in (0, 2)$, let $X$ be a rotationally symmetric $\alpha$-stable process on $\mathbb{R}^d$. It is well known that its infinitesimal generator is the fractional Laplacian $\Delta^{\alpha/2} := -(-\Delta)^{\alpha/2}$ on $\mathbb{R}^d$, which enjoys the following expression

$$\Delta^{\alpha/2} u(x) = \int_{\mathbb{R}^d \setminus \{0\}} \left( u(x+z) - u(x) - \nabla u(x) \cdot z I_{\{|z| \leq 1\}} \right) \frac{C_{d,\alpha}}{|z|^{d+\alpha}} dz.$$ 

Here $C_{d,\alpha} = \frac{\alpha^{d-1} \Gamma((d+\alpha)/2)}{\pi^{d/2} \Gamma(1-\alpha/2)}$ is the normalizing constant so that the Fourier transform $\widehat{\Delta^{\alpha/2} u}(\xi)$ of $\Delta^{\alpha/2} u$ is $-|\xi|^{\alpha} \widehat{u}(\xi)$. Observe that

$$\lim_{\alpha \uparrow 2} \frac{C_{d,\alpha}}{ \alpha^{\alpha - 1} (d+\alpha)/2} = \frac{d \Gamma(d/2)}{\pi^{d/2}}.$$

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The Dirichlet form \((\mathcal{E}, \mathcal{F})\) of \(X\) on \(L^2(\mathbb{R}^d; dx)\) is given by
\[
\mathcal{E}(f, g) = \frac{1}{2} \int_{\mathbb{R}^d \times \mathbb{R}^d} (f(x) - f(y))(g(x) - g(y)) \frac{C_{d,\alpha}}{|x - y|^{d+\alpha}}
dx dy,
\]
\[
\mathcal{F} = \{ f \in L^2(\mathbb{R}^d; dx) : \mathcal{E}(f, f) < \infty \}.
\]
Note that for \(f \in C_c^2(\mathbb{R}^d)\),
\[
\mathcal{E}(f, g) = -\int_{\mathbb{R}^d} g(x) \Delta^{\alpha/2} f(x) dx \quad \text{for every } g \in \mathcal{F}.
\]
It is known that \(X\) is recurrent if and only if \(\alpha \geq d\). In other words, the symmetric \(\alpha\)-stable process \(X\) is recurrent if and only if \(d = 1\) and \(\alpha \in [1, 2)\). In the latter case, \(X\) is recurrent with Lebesgue measure as its symmetrizing measure but it does not have a stationary probability distribution. On the other hand, we know that a time-change of \(X\) does not change its transience and recurrence (see [8, Theorem 5.2.5]) but will change its symmetrizing measure. Let \(\tau\) be a positive and locally bounded measurable function on \(\mathbb{R}^d\) so that \(1/\tau\) is \(L^1(\mathbb{R}^d; dx)\) locally integrable. Let \(\mu(dx) = \frac{1}{\tau(x)} dx\) and \(A_t := \int_0^t 1/a(X_s) ds\), which is a positive continuous additive functional of \(X\). Define
\[
\tau_t = \inf\{ s > 0 : A_s > t \}
\]
and set \(Y_t = X_{\tau_t}\). It is known (cf. [8, Theorem 5.2.2]) that \(Y\) is a \(\mu\)-symmetric strong Markov process on \(\mathbb{R}^d\).

From now on, we assume \(d = 1\), \(\alpha \in [1, 2)\) and \(\alpha\) is a positive and locally bounded measurable function on \(\mathbb{R}\) so that \(\mu\) defined above is a probability measure (that is, \(\int_{\mathbb{R}} \alpha(x)^{-1} dx = 1\)). As we noted above, the time changed process \(Y\) is a pointwise recurrent \(\mu\)-symmetric Markov process on \(\mathbb{R}\) and so every point in \(\mathbb{R}\) has positive \(\mathcal{E}\)-capacity. In particular, \(\mu\) is a reversible probability measure (and hence an invariant probability measure) of \(Y\). In fact, we have

**Proposition 1.1.** \(\mu\) is the unique invariant measure of \(Y\) and for every \(x \in \mathbb{R}\) and \(f \in C_b(\mathbb{R})\),
\[
\lim_{t \to \infty} \frac{1}{t} \int_0^t \mathbb{E}_x[f(Y_s)] ds = \int_{\mathbb{R}} f(x) \mu(dx).
\]

The goal of this paper is to study the ergodicity of \(Y\); that is, the way of the marginal distribution of \(Y\) converging to its equilibrium distribution \(\mu\). For this, we need to describe the Dirichlet form of the time changed process \(Y\).

Let \(\mathcal{F}_e\) be the extended Dirichlet space of \((\mathcal{E}, \mathcal{F})\) for the symmetric \(\alpha\)-stable process \(X\). By [8 (6.5.4)],
\[
\mathcal{F}_e = \{ u : \text{Borel measurable with } |u| < \infty \text{ a.e. and } \mathcal{E}(u, u) < \infty \}.
\]
It follows from [8 Corollaries 3.3.6 and 5.2.12] that \((\mathcal{F}_e, \mathcal{E})\) is also the extended Dirichlet space of the time changed process \(Y\). Thus the Dirichlet form for process \(Y\) on \(L^2(\mathbb{R}; \mu)\) is \((\mathcal{E}, \mathcal{F}^\mu)\), where
\[
\mathcal{F}^\mu = \mathcal{F}_e \cap L^2(\mathbb{R}; \mu) = \{ u \in L^2(\mathbb{R}; \mu) : \mathcal{E}(u, u) < \infty \}.
\]
Since the space \(C_c^\infty(\mathbb{R})\) of smooth function with compact support is a core for the Dirichlet form \((\mathcal{E}, \mathcal{F})\) of \(X\), by [8 Theorem 5.2.8], \(C_c^\infty(\mathbb{R})\) is also a core for \((\mathcal{E}, \mathcal{F}^\mu)\). It in particular implies that \((\mathcal{E}, \mathcal{F}^\mu)\) is a regular Dirichlet form on \(L^2(\mathbb{R}; \mu)\). By abusing the notation a little bit, we also denote the \(L^2\)-generator of \(X\) by \(\Delta^{\alpha/2}\). Then the \(L^2\)-generator of \(Y\) (or equivalently, of \((\mathcal{E}, \mathcal{F}^\mu)\)) is \(a\Delta^{\alpha/2}\).

Let
\[
P_t f(x) = \mathbb{E}_x(f(Y_t)), \quad f \in \mathbb{B}_b(\mathbb{R})
\]
be the semigroup of \(Y\) (or equivalently, associated with \((\mathcal{E}, \mathcal{F}^\mu)\)).
1.1 Main Results

We now present results on various ergodic properties of $Y$. For a function $f$ defined on $\mathbb{R}$, we use $\mu(f)$ to denote the integral $\int_{\mathbb{R}} f(x) \mu(dx)$ whenever it is well defined. We first consider the case of $1 < \alpha < 2$.

**Theorem 1.2.** Suppose $\alpha \in (1, 2)$. For $r > 0$, set

$$
\Phi(r) := \inf_{|x| \geq r} \frac{a(x)}{(1 + |x|)^\alpha}, \quad \Phi_0(r) := \inf_{|x| \leq r} \frac{a(x)}{(1 + |x|)^\alpha}
$$

and

$$
K(r) := \sup_{|x| \leq r} a(x)^{-1}, \quad k(r) := \inf_{|x| \leq r} a(x)^{-1}, \quad K_0(r) := \frac{K(r)^{1+1/\alpha}}{k(r)^2}.
$$

Assume in addition that $a$ is locally bounded between two positive constants. Then the following holds.

(i) If

$$
\lim_{r \to \infty} \Phi(r) > 0,
$$

then the following Poincaré inequality

$$
\mu(f - \mu(f))^2 \leq C \mathcal{E}(f, f), \quad f \in \mathcal{F}^\mu
$$

holds for some constant $C > 0$. Equivalently, for every $f \in L^2(\mathbb{R}; \mu)$ and $t > 0$,

$$
\|P_t f - \mu(f)\|_{L^2(\mathbb{R}; \mu)} \leq e^{-t/C} \|f\|_{L^2(\mathbb{R}; \mu)},
$$

(ii) If

$$
\lim_{r \to \infty} \Phi(r) = \infty,
$$

then the following super Poincaré inequality

$$
\mu(f^2) \leq r \mathcal{E}(f, f) + \beta(r)\mu(|f|^2), \quad r > 0, f \in \mathcal{F}^\mu
$$

holds with

$$
\beta(r) = C_1 \left(1 + r^{-1/\alpha} K_0 \circ \Phi^{-1}(C_2/r)\right)
$$

for some constants $C_1$ and $C_2 > 0$. Consequently, if $\int_0^\infty \beta(r)/r \, dr < \infty$ for some $t > 0$, then there is a constant $C_3 > 0$ so that for every $f \in L^2(\mathbb{R}; \mu)$ and $t > 1$,

$$
\sup_{x \in \mathbb{R}} |P_t f(x) - \mu(f)| \leq C_3 e^{-t/C} \|f\|_{L^2(\mathbb{R}; \mu)},
$$

where $C > 0$ is a constant appearing in (1.5). In this case, we have

$$
\sup_{x \in \mathbb{R}} \|P_t(x, \cdot) - \mu\|_{TV} := \sup_{x \in \mathbb{R}} \sup_{0 \leq f \leq 1} |P_t f(x) - \mu(f)| \leq C_3 e^{-t/C}, \quad t > 1,
$$

where for a signed measure $\nu$, $\|\nu\|_{TV}$ is denoted by its variation, and $P_t(x, \cdot)$ is the probability distribution of $Y_t$ with $Y_0 = x$. 

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(iii) If
\[ \lim_{r \to \infty} \Phi_0(r) = 0, \] (1.10)
then the following weak Poincaré inequality holds: for every \( r > 0 \), there is
\[ \alpha(r) = C_4 \inf \left\{ \frac{1}{\Phi_0(s)} : \mu(B(0,s)) \geq \frac{1}{1 + r} \right\} \] (1.11)
for some constant \( C_4 > 0 \) independent of \( r \) so that
\[ \mu(f^2) \leq \alpha(r) \mathcal{E}(f,f) + r\|f\|_\infty^2 \quad \text{for every } f \in \mathcal{F}^\mu \text{ with } \mu(f) = 0. \] (1.12)

Consequently,
\[ \|P_t f - \mu(f)\|_{L^2(\mathbb{R};\mu)}^2 \leq \xi(t) \left( \|f\|_{L^2(\mathbb{R};\mu)}^2 + \|f\|_\infty^2 \right) \]
for every \( f \in L^2(\mathbb{R};\mu) \) and \( t > 0 \), where \( \xi(t) = 2\inf\{r > 0 : -\alpha(r) \log r \leq 2t\} \). Note that \( \lim_{t \to \infty} \xi(t) = 0 \).

As a direct consequence of (1.7), we get that the following defective Poincaré inequality
\[ \mu(f^2) \leq c_1 \mathcal{E}(f,f) + c_2 \mu(|f|^2), \quad f \in \mathcal{F}^\mu \] (1.13)
holds for some constants \( c_1, c_2 > 0 \). Since the Dirichlet form \((\mathcal{E}, \mathcal{F}^\mu)\) is irreducible, i.e. \( \mathcal{E}(f,f) = 0 \) implies \( f \) is a constant function, it follows from [19, Corollary 1.2] (see also [15, Theorem 1]) that the defective Poincaré inequality (1.13) is equivalent to the Poincaré inequality (1.4). Therefore, the super Poincaré inequality (1.7) is stronger than the Poincaré inequality (1.4). Clearly, for \( t > 1 \) the uniform ergodicity (1.9) is also stronger than ergodicity (1.5) under stationary distribution. The super Poincaré inequality (1.7) is equivalent to the uniform integrability of the semigroup \((P_t)_{t \geq 0}\), and also the absence of the essential spectrum of its generator if the semigroup \((P_t)_{t \geq 0}\) has an asymptotic density, see, e.g., [17, Theorems 6.1, 2.1 and 5.1]. It often implies the ultracontractivity of \( P_t \) for some \( t > 0 \). See [18, Chapter 3] for more details about the applications of super Poincaré inequality. Clearly the Poincaré inequality (1.4) implies that the weak Poincaré inequality (1.12) holds with \( \alpha(r) = C \). On the other hand, under condition (1.10), it is easy to see that \( \Phi(r) = 0 \) for all \( r > 0 \), and the function \( \alpha(r) \) defined in (1.11) tends to \( \infty \) as \( r \to 0 \). The weak Poincaré inequality (1.12) characterizes the \( L^2 \)-convergence rates of the semigroup \((P_t)_{t \geq 0}\) slower than exponential. See, e.g., [18, Chapter 4] for more information on weak Poincaré inequality and its consequences.

Theorem 1.2 is sharp in many situations; see Examples 1.3 and 1.4 below. Note that the inequalities (1.4) and (1.7) are related to the weighted Poincaré inequalities for non-local Dirichlet forms studied in [6]. However, applying [6, Proposition 1.7] with \( d = 1 \) to our case, one only gets that the Poincaré inequality (1.4) holds when
\[ \lim_{|x| \to \infty} \frac{a(x)}{(1 + |x|)^{1+\alpha}} > 0, \]
which is stronger than condition (1.3) and is far from being optimal. See also Examples 1.3 and 1.4 below. Furthermore, according to [4, Table 5.1, p. 100], Theorem 1.2 is also optimal for the case when \( \alpha = 2 \), which corresponds to the limiting one-dimensional Brownian case. See Appendix for more details on this Brownian case.

We now present some examples to illustrate our main results. The proofs of the claims made in these examples are given in Section 3.
Example 1.3. Suppose $\alpha \in (1, 2)$. Let $a(x) = C_\gamma (1 + |x|)^\gamma$ with $\gamma > 1$ such that $\int a(x)^{-1} \, dx = 1$.

(i) The Poincaré inequality (1.12) holds for some constant $C > 0$ if and only if $\gamma \geq \alpha$.

(ii) The super Poincaré inequality (1.7) holds for some function $\beta : (0, \infty) \to (0, \infty)$ if and only if $\gamma > \alpha$. In this case, there exists a constant $c > 0$ such that the super Poincaré inequality (1.7) holds with
\[
\beta(r) = c \left(1 + r^{-\left(\frac{1}{\alpha} + \frac{2\gamma}{\gamma - \alpha}\right)}\right), \quad r > 0,
\]
which is equivalent to
\[
\|P_t\|_{L^1(\mathbb{R}; \mu)} \leq c_0 \left(1 + t^{-\left(\frac{1}{\alpha} + \frac{2\gamma}{\gamma - \alpha}\right)}\right), \quad t > 0
\] (1.14)
for some constant $c_0 > 0$. Since $\beta^{-1}(r)/r$ is integrable at infinity, uniform strong ergodicity (1.9) holds.

(iii) If $\gamma \in (1, \alpha)$, then the weak Poincaré inequality (1.12) holds with
\[
\alpha(r) = c \left(1 + r^{-\left(\alpha - \gamma\right)/(\gamma - 1)}\right), \quad r > 0
\]
for some constant $c > 0$. Consequently, there exists a constant $c_0 > 0$ such that
\[
\|P_t - \mu\|_{L^\infty(\mathbb{R}; \mu)} \leq c_0 t^{-\left(\gamma - 1\right)/(\alpha - \gamma)}, \quad t > 0.
\]
The rate function $\alpha$ given above is sharp in the sense that (1.12) does not hold if
\[
\lim_{r \to 0} r^{\left(\alpha - \gamma\right)/(\gamma - 1)} \alpha(r) = 0.
\]

Example 1.4. Suppose $1 < \alpha < 2$. Let $a(x) = C_{\alpha, \gamma} (1 + |x|)^\alpha \log^\gamma (e + |x|)$ with $\gamma \in \mathbb{R}$ such that $\int a(x)^{-1} \, dx = 1$.

(i) The Poincaré inequality (1.12) holds for some constant $C > 0$ if and only if $\gamma \geq 0$.

(ii) The super Poincaré inequality (1.7) holds for some function $\beta : (0, \infty) \to (0, \infty)$ if and only if $\gamma > 0$. In this case, there exists a constant $c > 0$ such that the super Poincaré inequality (1.7) holds with
\[
\beta(r) = \exp \left(c(1 + r^{-1/\gamma})\right), \quad r > 0.
\]
Consequently, when $\gamma > 1$,
\[
\|P_t\|_{L^1(\mathbb{R}; \mu)} \leq \exp \left(c_0 (1 + t^{-1/(\gamma - 1)})\right), \quad t > 0
\] (1.15)
holds for some constant $c_0 > 0$. Moreover, the uniform strong ergodicity (1.9) holds when $\gamma > 1$. The rate function $\beta$ above is sharp in the sense that (1.7) does not hold if
\[
\lim_{r \to 0} r^{1/\gamma} \log \beta(r) = 0.
\]
In particular, the following log-Sobolev inequality
\[
\mu(f^2 \log f^2) \leq C \mathcal{E}(f, f) \quad \text{for } f \in \mathcal{F}^\mu \text{ with } \mu(f^2) = 1
\] (1.16)
holds for some constant $C > 0$ if and only if $\gamma \geq 1$. In this case, we have
\[
\text{Ent}_\mu(P_t f) \leq \text{Ent}_\mu(f) e^{-4t/C}, \quad t > 0, f \in L^2(\mathbb{R}; \mu) \text{ with } f > 0,
\] (1.17)
where $\text{Ent}_\mu(f) = \mu(f \log f) - \mu(f) \log \mu(f)$.
(iii) If $\gamma < 0$, then the weak Poincaré inequality (1.12) holds with
\[
\alpha(r) = c \left( 1 + \log^{-\gamma}(1 + r^{-1}) \right), \quad r > 0
\]
for some constant $c > 0$. Consequently, there exist constants $c_1$ and $c_2 > 0$ such that
\[
\|P_t - \mu\|_{L^\infty(\mathbb{R}; \mu)} \leq \exp \left( c_1 - c_2 t^{1/(1-\gamma)} \right), \quad t > 0.
\]

The rate function $\alpha$ defined above is sharp in the sense that (1.12) does not hold if
\[
\lim_{r \to 0} \log(1 + r^{-1}) \alpha(r) = 0.
\]

We now turn to the case of $\alpha = 1$. For $1 < \delta < \beta$ and $r > 0$, define
\[
\Psi_\beta(r) := \inf_{|x| \geq r} a(x), \quad \Psi_{\beta,\delta}(r) := \Psi_\beta(r)(1 + r)^{\beta - \delta}, \quad K_0(r) := \left( \frac{K(r)}{k(r)} \right)^2.
\]

**Theorem 1.5.** Suppose that $\alpha = 1$. If there exists a constant $\beta > 1$ such that
\[
\lim_{r \to \infty} \Psi_\beta(r) > 0,
\]
then for every $\delta \in (1, \beta)$ there are positive constants $C_1$ and $C_2$ so that the super Poincaré inequality (1.7) holds with
\[
\beta(r) = C_1 \left( 1 + r^{-1} K_0 \circ \Psi_{\beta,\delta}^{-1}(C_2/r) \right).
\]

In particular, the Poincaré inequality (1.4) holds with some constant $C > 0$; or equivalently, the process $Y$ is $L^2(\mathbb{R}; \mu)$-exponentially ergodic, i.e. for every $f \in L^2(\mathbb{R}; \mu)$ and $t > 0$,
\[
\|P_t f - \mu(f)\|_{L^2(\mathbb{R}; \mu)} \leq e^{-t/C} \|f\|_{L^2(\mathbb{R}; \mu)}.
\]

To illustrate Theorem 1.5, we reconsider Example 1.3 for the case of $\alpha = 1$.

**Example 1.6 (Continuation of Example 1.3 with $\alpha = 1$).** Let $\alpha = 1$, and $a(x) = C_\beta(1 + |x|)^\gamma$ with $\gamma > 1$ such that $\int a(x)^{-1} dx = 1$. Then, for the Dirichlet form $(\mathcal{E}, \mathcal{F}^\mu)$, the super Poincaré inequality (1.7) holds with
\[
\beta(r) \leq c \left( 1 + r^{-(1+\frac{2\gamma}{1+\gamma})} \right), \quad r > 0
\]
for any $\delta \in (1, \gamma)$ and some constant $c = c(\delta) > 0$; and equivalently,
\[
\|P_t\|_{L^1(\mathbb{R}; \mu) \to L^\infty(\mathbb{R}; \mu)} \leq c_0 \left( 1 + t^{-(1+\frac{2\gamma}{1+\gamma})} \right), \quad t > 0
\]
holds for some constant $c_0 = c_0(\delta) > 0$. Consequently, the uniform strong ergodicity (1.9) holds.

1.2 Application: one-dimensional SDEs driven by symmetric stable processes

As a direct application of our main results, we consider the following one dimensional stochastic differential equation (SDE) driven by a symmetric $\alpha$-stable process $X$ on $\mathbb{R}$:
\[
dZ_t = \sigma(Z_{t-}) dX_t,
\]
where $\alpha \in (1,2)$ and $\sigma : \mathbb{R} \to \mathbb{R}$ is a locally $1/\alpha$-Hölder continuous and strictly positive function. Then, the SDE (1.19) has a unique strong solution (see the proof of Theorem 1.7 in Section 3). Denote by $Z$ this unique solution. The process $Z$ has strong Markov property, and its infinitesimal generator is given by

$$
\mathcal{L}u(x) = \int_{\mathbb{R}\setminus\{0\}} \left( u(x + \sigma(x)z) - u(x) - u'(x)\sigma(x)z1_{\{|z|\leq1\}} \right) \frac{C_{1,\alpha}}{|z|^{1+\alpha}} \, dz.
$$

Letting $v = \sigma(x)z$, we find that

$$
\mathcal{L}u(x) = \sigma(x)\int_{\mathbb{R}\setminus\{0\}} \left( u(x + v) - u(x) - u'(x)v1_{\{|v|\leq1\}} \right) \frac{C_{1,\alpha}}{|v|^{1+\alpha}} \, dv = \sigma(x)^\alpha \Delta^{\alpha/2}.
$$

According to Theorem 1.7, we have the following statement for the exponential ergodicity of the process $Z$.

**Theorem 1.7.** Let $Z$ be the unique solution to the SDE (1.19).

(i) Suppose that

$$
\liminf_{|x| \to \infty} \frac{\sigma(x)}{|x|} > 0.
$$

Then the process $Z$ is exponentially ergodic. More explicitly, $\mu(dx) := \sigma(x)^{-\alpha} \frac{dx}{\sigma(x)}$ is a unique reversible probability measure for the process $Z$, and there is a constant $\lambda_0 > 0$ such that for any $x \in \mathbb{R}$

$$
\|P_t(x,\cdot) - \mu\|_{TV} \leq C(x)e^{-\lambda_0 t},
$$

where $P_t(x,\cdot)$ is the probability distribution of $Z_t$ with $Z_0 = x$ and $C(x)$ is a positive measurable function on $\mathbb{R}$.

(ii) Furthermore, if

$$
\liminf_{|x| \to \infty} \frac{\sigma(x)}{|x|^\gamma} > 0.
$$

holds for some constant $\gamma > 1$, then the process $Z$ is uniformly strongly ergodic, i.e. there are constants $C_1$ and $\lambda_1 > 0$ such that

$$
\sup_{x \in \mathbb{R}} \|P_t(x,\cdot) - \mu\|_{TV} \leq C_1 e^{-\lambda_1 t}.
$$

To the best of our knowledge, Theorem 1.7 is new. Recall that the symmetric $\alpha$-stable process $X$ is recurrent for $\alpha > 1$, but it is not ergodic for any $\alpha > 1$. Theorem 1.7 shows that the solution of a stochastic differential equation driven by $X$ can be exponentially ergodic under some growth condition such as (1.21) on the multiplicative coefficient, and can be uniformly strongly ergodic under (1.22). On the other hand, according to Theorem 1.5(i) and (the proof of) Theorem 1.7(i), Theorem 1.7(i) also holds for $\alpha = 1$ if the condition (1.21) is replaced by (1.22).

The remainder of this paper is organized as follows. In Section 2 we give a proof of Proposition 1.1 and present some preliminaries of non-local Dirichlet forms corresponding to time-change of pure jump symmetric Lévy processes. The proofs of all the results and examples mentioned above are presented in Section 3, where the sharp drift condition for truncated fractional Laplacian is also given.
2 Lévy processes and their time-change

We begin with a proof of Proposition 1.1.

Proof of Proposition 1.1. Observe that since $\alpha \in [1, 2)$, every singleton is non-polar for symmetric $\alpha$-stable process $X$ on $\mathbb{R}$, and hence for $Y$. So it follows from [S] Lemma 3.5.5(ii) that any non-negative invariant function $h$ (that is, $P_t h = h$ for every $t > 0$) is constant. This implies that the invariant $\sigma$-algebra $\mathcal{I} := \{A : 1_A$ is an invariant function$\}$ is trivial. So by [I] Theorem 1.1 (taking $g = 1$ there),

$$\lim_{t \to \infty} \frac{1}{t} \int_0^t P_s f(x) ds = \int_{\mathbb{R}} f(y) \mu(dy)$$

for every bounded non-negative function $f$ on $\mathbb{R}$ and for every $x \in \mathbb{R}$. Suppose $\nu$ is an invariant probability measure of $Y$; that is, $\nu = \nu P_t$ for every $t > 0$. Integrating the above display with respect to $\nu$ and applying bounded convergence theorem, we get

$$\int_{\mathbb{R}} f(y) \nu(dy) = \int_{\mathbb{R}} f(y) \mu(dy),$$

from which we conclude $\nu = \mu$. \hfill \Box

Suppose that $\rho$ is a nonnegative measurable function on $\mathbb{R}$ such that $\rho(0) = 0$, $\rho(z) = \rho(-z)$ and \int(1 + z^2)\rho(z) dz < \infty. Let $\tilde{X}$ be the symmetric pure jump Lévy process on $\mathbb{R}$ with Lévy measure $\rho(z)dz$, and denote by $(\tilde{E}, \tilde{F})$ its Dirichlet form on $L^2(\mathbb{R}; dx)$. By [S] (2.2.16),

$$\tilde{E}(f, g) = \frac{1}{2} \int_{\mathbb{R} \times \mathbb{R}} (f(x) - f(y))(g(x) - g(y))\rho(x - y) dx dy$$

and

$$\tilde{F} = \{f \in L^2(\mathbb{R}; dx) : \tilde{E}(f, f) < \infty\}.$$

Moreover, we know from [S] §2.2.2, $(\tilde{E}, \tilde{F})$ is a regular Dirichlet form on $L^2(\mathbb{R}; dx)$. Using smooth mollifier, we see that both $C_c^\infty(\mathbb{R})$ and $C^2_c(\mathbb{R})$ are cores for $(\tilde{E}, \tilde{F})$. Denote by $(A, D(A))$ the $L^2$-generator of $\tilde{X}$ (or equivalently, of $(\tilde{E}, \tilde{F})$). It is easy to verify that $C^2_c(\mathbb{R}) \subset D(A)$ and for $u \in C_c^2(\mathbb{R})$,

$$Au(x) = \int_{\mathbb{R} \setminus \{0\}} \left( u(x + z) - u(x) - u'(x)z \mathbb{1}_{\{|z| \leq 1\}} \right) \rho(z) dz. \quad (2.1)$$

Observe that $Au \in L^2(\mathbb{R}; dx)$ for $u \in C^2_c(\mathbb{R})$. This is because if we use $\tilde{f}$ to denote the Fourier transform of $f$, then $\tilde{A}u(\xi) = \tilde{\psi}(\xi)\tilde{\nu}(\xi)$, where

$$\tilde{\psi}(\xi) = \int_{\mathbb{R}} (1 - \cos(\xi y))\rho(y) dy \leq 2 \int_{\mathbb{R}} (1 + (\xi y)^2)\rho(y) dy \leq 2(1 + \xi^2) \int_{\mathbb{R}} (1 + y^2)\rho(y) dy \leq c_0(1 + \xi^2).$$

It follows that $\|Au\|_2^2 = \|\tilde{A}u\|_2^2 \leq c_0^2(1 + |\xi|^2)\|\tilde{\nu}\|_2^2 \leq 2c_0^2(\|u\|_2^2 + \|u''\|_2^2) < \infty$.

Let $a$ be a positive and locally bounded measurable function on $\mathbb{R}$ so that $a(x)^{-1}$ is locally integrable. Set $\mu(dx) = a(x)^{-1} dx$. Then $\mu$ is a smooth measure. Let $A_t := \int_0^t \frac{1}{a(X_s)} ds$, which is a positive continuous additive functional of $\tilde{X}$. Define

$$\tau_t = \inf\{s > 0 : A_s > t\}$$

and set $\tilde{Y}_t = \tilde{X}_{\tau_t}$. It follows from [S] Theorems 5.2.2 and 5.2.8 that $\tilde{Y}$ is a $\mu$-symmetric strong Markov process on $\mathbb{R}^d$, whose associated Dirichlet form $(\tilde{E}, \tilde{F})$ is regular on $L^2(\mathbb{R}; \mu)$ having $C^2_c(\mathbb{R})$.
as its core. In view of [8 Corollary 5.2.12], \( \mathcal{F}^\mu \) is given by \( \mathcal{F}^\mu = \mathcal{F}_e \cap L^2(\mathbb{R}; \mu) \), where \( \mathcal{F}_e \) is the extended Dirichlet space of \((\mathcal{E}, \mathcal{F})\). We know from [8 (2.2.18) and (6.5.4)] that

\[
\mathcal{F}_e := \left\{ u \in L^1_{\text{loc}}(\mathbb{R}) \cap S' : \mathcal{E}(u, u) < \infty \right\}
\]

(2.2)

when \( \tilde{X} \) is transient, and

\[
\mathcal{F}_e = \{ u : \text{Borel measurable with } |u| < \infty \text{ a.e. and } \mathcal{E}(u, u) < \infty \}
\]

(2.3)

when \( \tilde{X} \) is recurrent. Here \( S' \) denotes the space of tempered distributions. Clearly the \( L^2 \)-infinitesimal generator of \( Y \) is \( \mathcal{L} = a\mathcal{A} \). Abusing the notation a little bit, we also use \( \mathcal{A}u \) to denote the function defined by (2.1) whenever its right hand side is well defined. The same remark applies to operator \( \mathcal{L} = a\mathcal{A} \).

**Proposition 2.1.** Assume that \( \rho(z) = 0 \) for any \( z \in \mathbb{R} \) with \( |z| \leq 1 \), and that there exists a constant \( \delta \in (0, 1] \) such that \( \int_{\{|z| > 1\}} |z|^\delta \rho(z) \, dz < \infty \). Set

\[
\mathcal{H}_\delta := \left\{ \varphi \in \mathcal{B}(\mathbb{R}) : \text{there is some } C > 0 \text{ such that} \right. \]

\[
|\varphi(x) - \varphi(y)| \leq C|x - y|^\delta \text{ for any } x, y \in \mathbb{R} \text{ with } |x - y| \geq 1 \left\}
\]

Then \( \mathcal{L}\varphi \) exists pointwise as a locally bounded function for \( \varphi \in \mathcal{H}_\delta \). Moreover for any bounded measurable function \( f \) on \( \mathbb{R} \) with compact support and for \( \varphi \in \mathcal{H}_\delta \),

\[
\int_{\{|x-y| > 1\}} |f(x) - f(y)| \varphi(x) - \varphi(y)| \rho(x - y) \, dx \, dy < \infty
\]

and

\[
-\int_{\mathbb{R}} f(x) \mathcal{L}\varphi(x) \mu(dx) = \frac{1}{2} \int_{\{|x-y| > 1\}} ((f(x) - f(y))(\varphi(x) - \varphi(y)) \rho(x - y) \, dx \, dy.
\]

**Proof.** Since \( \rho(z) = 0 \) for all \( |z| \leq 1 \),

\[
\mathcal{A}u(x) = \int_{\{|z| > 1\}} (u(x + z) - u(x)) \rho(z) \, dz.
\]

Let \( \varphi \in \mathcal{H}_\delta \). Then

\[
|\mathcal{A}\varphi(x)| \leq \int_{\{|z| > 1\}} |\varphi(x + z) - \varphi(x)| \rho(z) \, dz
\]

\[
\leq C \int_{\{|z| > 1\}} |z|^\delta \rho(z) \, dz =: c < \infty,
\]

so \( \mathcal{A}\varphi \) is well-defined pointwise on \( \mathbb{R} \) and is bounded. Consequently, \( \mathcal{L}\varphi(x) = a(x)\mathcal{A}\varphi(x) \) is pointwisely well-defined on \( \mathbb{R} \) and is locally bounded. It follows that for every bounded function \( f \) with compact support on \( \mathbb{R} \), due to the symmetry of \( \rho(z) \),

\[
\int_{\{|x-y| > 1\}} |f(x) - f(y)| |\varphi(x) - \varphi(y)| \rho(x - y) \, dx \, dy
\]

\[
\leq 2 \int_{\mathbb{R}} |f(x)| \left( \int_{\{|y| \leq |x-y| > 1\}} |\varphi(y) - \varphi(x)| \rho(x - y) \, dy \right) \, dx
\]

\[
= 2 \int_{\mathbb{R}} |f(x)| \left( \int_{\{|z| \leq |x-z| > 1\}} |\varphi(x + z) - \varphi(x)| \rho(z) \, dz \right) \, dx < \infty.
\]
Again by the symmetry of $\rho(z)$,
\[
- \int f \mathcal{L} \varphi \, d\mu = \int f(x) A \varphi(x) \, dx \\
= - \int f(x) \, dx \int_{\{|y-x|>1\}} (\varphi(y) - \varphi(x)) \rho(x-y) \, dy \\
= \frac{1}{2} \int_{\{|y-x|>1\}} (f(x) - f(y))(\varphi(x) - \varphi(y)) \rho(x-y) \, dy \, dx.
\]
The proof is completed. \hfill \Box

3 Proofs

Let $\alpha \in (0, 2)$ and $d = 1$. We first consider the symmetric Lévy process $\tilde{X}$ whose Lévy measure is $C_{1,0}|z|^{-(1+\alpha)}1_{\{|z|>1\}}dz$. Denote by $(\tilde{E}, \tilde{F})$ the Dirichlet form of $\tilde{X}$ on $L^2(\mathbb{R}; dx)$. We know from Section 2 that
\[
\tilde{E}(f, f) = \frac{1}{2} \int_{\mathbb{R} \times \mathbb{R}} (f(x) - f(y))^2 \frac{C_{1,\alpha}}{|x-y|^{1+\alpha}} 1_{\{|x-y|>1\}} \, dx \, dy
\]
and
\[
\tilde{F} = \left\{ f \in L^2(\mathbb{R}; dx) : \tilde{E}(f, f) < \infty \right\}.
\]

Moreover, it follows from (2.2) and (2.3) that the extended Dirichlet space $\tilde{\mathcal{F}}_e$ of $(\tilde{E}, \tilde{F})$ is given by
\[
\tilde{\mathcal{F}}_e = \left\{ f \in L^1_{\text{loc}}(\mathbb{R}; dx) : \tilde{E}(f, f) < \infty \right\}.
\]

Suppose that $a$ is a positive and locally bounded measurable function on $\mathbb{R}$ such that $\int_{\mathbb{R}} a(x)^{-1} \, dx = 1$. Define $\mu(dx) := a(x)^{-1} \, dx$. Let $\tilde{Y}_t = \tilde{X}_{\tau_t}$ be the time-change of $\tilde{X}$, where
\[
\tau_t := \inf \left\{ s > 0 : \int_0^s a(\tilde{X}_r)^{-1} \, dr > t \right\}.
\]

We know from Section 2 that the time changed process $Y$ is $\mu$-symmetric and its associated Dirichlet form is given by $(\tilde{E}, \tilde{F}^\mu)$, which is regular on $L^2(\mathbb{R}; dx)$ with core $C^2_c(\mathbb{R})$. Here
\[
\tilde{F}^\mu = \tilde{\mathcal{F}}_e \cap L^2(\mathbb{R}; \mu) = \left\{ f \in L^2(\mathbb{R}; \mu) : \tilde{E}(f, f) < \infty \right\}.
\]

Let $(\Delta_{>1}^{\alpha/2}, \mathcal{D}(\Delta_{>1}^{\alpha/2}))$ and $(\mathcal{L}^{(\alpha)}_{>1}, \mathcal{D}(\mathcal{L}^{(\alpha)}_{>1}))$ be the $L^2$-infinitesimal generator of $\tilde{X}$ and $\tilde{Y}$, respectively. It follows from Section 2 that $\mathcal{L}^{(\alpha)}_{>1} = a\Delta_{>1}^{\alpha/2}$ and $C^2_c(\mathbb{R}) \subset \mathcal{D}(\Delta_{>1}^{\alpha/2}) \cap \mathcal{D}(\mathcal{L}^{(\alpha)}_{>1})$. Moreover, for $u \in C^2_c(\mathbb{R})$,
\[
\Delta_{>1}^{\alpha/2} u(x) := \int_{\{|z|>1\}} (u(x+z) - u(x)) \frac{C_{1,\alpha}}{|z|^{1+\alpha}} \, dz.
\]

To prove Theorem 1.2(i), we need two lemmas. We begin with a local Poincaré inequality for $\mathcal{E}$. We use $\int_A f(x) \mu(dx)$ to denote $\int_A f(x) \mu(x)/\mu(A)$.

Lemma 3.1. For any $r > 0$ and $f \in C^2_c(\mathbb{R})$,
\[
\int_{B(0, r)} \left( f(x) - \int_{B(0, r)} f(x) \, \mu(dx) \right)^2 \mu(dx) \leq \frac{2^\alpha K(r)^2 r^\alpha}{C_{1,\alpha} K(r)} \mathcal{E}(f, f).
\]
Proof. For any \( r > 0 \) and \( f \in C^2_c(\mathbb{R}) \), by the Cauchy-Schwarz inequality,

\[
\int_{B(0,r)} \left( f(x) - \int_{B(0,r)} f(y) \mu(dy) \right)^2 \mu(dx) \\
\leq \int_{B(0,r)} \left( \int_{B(0,r)} (f(x) - f(y))^2 \mu(dy) \right) \mu(dx) \\
\leq \frac{2^{1+\alpha} r^{1+\alpha}}{C_{1,\alpha} \mu(B(0,r))} \int_{B(0,r) \times B(0,r)} (f(x) - f(y))^2 \frac{C_{1,\alpha}}{|x-y|^{1+\alpha}} a(x)^{-1} a(y)^{-1} dxdy \\
\leq \frac{2^\alpha K(r)^2 r^\alpha}{C_{1,\alpha} k(r)} E(f,f).
\]

\( \square \)

In the following lemma, we set \( V(x) = 1 + |x|^\theta \) for \( \theta \in (0,1) \). It is clear that \( V \in \mathcal{H}_\theta \), where \( \mathcal{H}_\theta \) is defined in Proposition 2.1.

Lemma 3.2. For any \( \alpha \in (1,2) \), there exists a constant \( \theta \in (0,1) \) small enough such that for the function \( V \) defined above, \( \mathcal{L}^{(\alpha)}_\theta V \) is well defined and there exists a constant \( r_0 > 0 \) large enough such that for all \( x \in \mathbb{R} \),

\[
\mathcal{L}^{(\alpha)}_\theta V(x) \leq \frac{\theta \pi C_{1,\alpha} \cot(\alpha \pi/2)}{2\alpha} \frac{a(x)}{(1 + |x|)^\alpha} V(x) \mathbb{I}_{B(0,r_0)}(x) + \frac{2C_{1,\alpha}}{\alpha - \theta} \sup_{|x| \leq r_0} a(x) \mathbb{I}_{B(0,r_0)}(x). \tag{3.1}
\]

Proof. Let \( \theta \in (0,1) \) be a constant which is determined later. First, according to the fact that \( |x+z|^\theta \leq |x|^\theta + |z|^\theta \) for any \( x, z \in \mathbb{R} \), we have

\[
|\Delta^{\alpha/2}_\theta V(x)| \leq \int_{|z| \geq 1} |V(x+z) - V(x)| \frac{C_{1,\alpha}}{|z|^{1+\alpha}} dz \\
\leq \int_{|z| \geq 1} |z|^\theta \frac{C_{1,\alpha}}{|z|^{1+\alpha}} dz = \frac{2C_{1,\alpha}}{\alpha - \theta} < \infty.
\]

Then, by the local boundedness of \( a \), we know that \( \mathcal{L}^{(\alpha)}_\theta V(x) \) is a well defined and locally bounded measurable function. Hence, it suffices to verify (3.1) for \( |x| \) large enough.

For this, one can follow the proof of [16, Theorem 1.2] (see [16, (3.33), (3.35)–(3.38)] for more details) and obtain that

\[
\limsup_{|x| \to \infty} |x|^{\alpha-\theta} \Delta^{\alpha/2}_\theta V(x) = \frac{\theta C_{1,\alpha}}{\alpha} E(\alpha,\theta), \tag{3.2}
\]

where

\[
E(\alpha,\theta) := \theta \sum_{i=1}^{\infty} C^2 \frac{2}{2i - \alpha} - \frac{2}{\theta} \\
+ \frac{\alpha_2 F_1(-\theta, \alpha - \theta, 1 + \alpha - \theta; -1) + \alpha_2 F_1(-\theta, \alpha - \theta, 1 + \alpha - \theta; 1)}{\theta (\alpha - \theta)}.
\]

\( C_i := \frac{\theta (\theta - 1) \cdots (\theta - i + 1)}{i!} \)
and \( {}_2F_1(a, b, c; z) \) is the Gauss hypergeometric function which can be calculated by the formula as follows

\[
{}_2F_1(a, b, c; z) = \sum_{n=0}^{\infty} \frac{(a)_n(b)_nz^n}{(c)_nn!},
\]

\( (r)_0 = 1, \ (r)_n = r(r+1) \cdots (r+n-1), \ n \geq 1. \)

Since

\[
\begin{align*}
{}_2F_1(-\theta, \alpha - \theta, 1 + \alpha - \theta; -1) + {}_2F_1(-\theta, \alpha - \theta, 1 + \alpha - \theta; 1) \\
= \sum_{n=0}^{\infty} \frac{(-\theta)_n(-\theta)_n(-1)_n}{(1+\alpha-\theta)_nn!} + \sum_{n=0}^{\infty} \frac{(-\theta)_n(-\theta)_n1^n}{(1+\alpha-\theta)_nn!} \\
= 2 \sum_{i=0}^{\infty} \frac{(-\theta)_{2i}(\alpha-\theta)_{2i}}{(1+\alpha-\theta)_{2i}} \frac{1}{(2i)!} \\
= 2 + 2 \sum_{i=1}^{\infty} \frac{\theta(\theta-1) \cdots (\theta-2i+1)(\alpha-\theta)}{(2i+\alpha-\theta)} \frac{1}{(2i)!},
\end{align*}
\]

we arrive at

\[
E(\alpha, \theta) = \frac{2}{\alpha - \theta} + 2\alpha \sum_{i=1}^{\infty} \frac{(-1) \cdots (-2i+1)}{(2i)!} \left( \frac{1}{2i-\alpha} + \frac{1}{2i+\alpha-\theta} \right). \tag{3.3}
\]

It is easy to see that the series appearing in (3.3) is absolutely convergent, and so

\[
\begin{align*}
\lim_{\theta \to 0} E(\alpha, \theta) &= \frac{2}{\alpha} + 2\alpha \sum_{i=1}^{\infty} \frac{(-1) \cdots (-2i+1)}{(2i)!} \left( \frac{1}{2i-\alpha} + \frac{1}{2i+\alpha-\theta} \right) \\
&= \frac{2}{\alpha} - \sum_{i=1}^{\infty} \frac{4\alpha}{4i^2-\alpha^2} \\
&= \pi \cot(\pi\alpha/2), \tag{3.4}
\end{align*}
\]

where in the last equality we have used the identity (see [13, Formula 1.421.3, p. 44])

\[
\cot(\pi x) = \frac{1}{\pi x} + \frac{2x}{\pi} \sum_{i=1}^{\infty} \frac{1}{x^2 - i^2}, \ x \in \mathbb{R}.
\]

Since for \( \alpha > 1, \pi \cot(\pi\alpha/2) < 0 \), the desired assertion (3.1) for \( |x| \) large enough follows from (3.2)-(3.4). This completes the proof.

We are now in the position to prove Theorem 1.2(i).

**Proof of Theorem 1.2(i).** Since \( C_c^\infty(\mathbb{R}) \) is the core of \( \mathcal{F}^\mu \), it is enough to prove the desired inequality (1.4) for any \( f \in C_c^\infty(\mathbb{R}) \). Since \( V \geq 1 \), we have by (1.3) and (3.1) that there exists a constant \( r_0 > 0 \) such that

\[
\mathbb{I}_{B(0,r^c)} \leq \frac{1}{C_3 \Phi(r)} \frac{3^\alpha}{V} + \frac{C_4}{C_3 \Phi(r)} \mathbb{I}_{B(0,r_0)}, \quad r \geq r_0,
\]

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where
\[
C_3 = -\frac{\theta \pi C_{1,\alpha} \cot(\alpha \pi/2)}{2\alpha}, \quad C_4 = \frac{2C_{1,\alpha} \sup_{|x| \leq r_0} a(x)}{\alpha - \theta}.
\]

Then, for any \( f \in C_c^\infty(\mathbb{R}) \),
\[
\mu(\mathbb{I}_{B_0(r)} f^2) \leq \frac{1}{C_3 \Phi(r)} \mu \left( f^2 - \frac{c_1 V}{V} \right) + \frac{C_4}{C_3 \Phi(r)} \mu(\mathbb{I}_{B_0(r)} f^2), \quad r \geq r_0.
\]

Note that for any \( x, y \in \mathbb{R} \),
\[
\left( \frac{f^2(x)}{V(x)} - \frac{f^2(y)}{V(y)} \right) (V(x) - V(y)) = f^2(x) + f^2(y) - \left( \frac{V(y)}{V(x)} f^2(x) + \frac{V(x)}{V(y)} f^2(y) \right)
\leq f^2(x) + f^2(y) - 2|f(x)||f(y)|
\leq (f(x) - f(y))^2,
\]
which, together with Proposition 2.1, yields that
\[
\mu \left( f^2 - \frac{c_1 V}{V} \right) \leq \hat{E}(f, f).
\]

Therefore, for any \( f \in C_c^\infty(\mathbb{R}) \),
\[
\mu(\mathbb{I}_{B_0(r)} f^2) \leq \frac{1}{C_3 \Phi(r)} \hat{E}(f, f) + \frac{C_4}{C_3 \Phi(r)} \mu(\mathbb{I}_{B_0(r)} f^2), \quad r \geq r_0.
\]

On the other hand, by Lemma 3.1, for any \( f \in C_c^\infty(\mathbb{R}) \) with \( \mu(f) = 0 \), and \( r \geq r_0 \),
\[
\mu(\mathbb{I}_{B_0(r)} f^2) \leq \frac{C_5 K(r)^{2r^{\alpha}}}{k(r)} \mathcal{E}(f, f) + \frac{1}{\mu(B(0, r))} \left( \int_{B(0, r)} f \, d\mu \right)^2
\leq \frac{C_5 K(r)^{2r^{\alpha}}}{k(r)} \mathcal{E}(f, f) + \frac{1}{\mu(B(0, r))} \left( \int_{B(0, r)} f \, d\mu \right)^2,
\]
where \( C_5 = 2^\alpha / C_{1,\alpha} \), and in the equality above we have used the fact that \( \int_{B(0, r)} f \, d\mu = -\int_{B(0, r)^c} f \, d\mu \).

The Cauchy-Schwarz inequality yields that
\[
\left( \int_{B(0, r)^c} f \, d\mu \right)^2 \leq \mu(B(0, r)^c) \int_{B(0, r)^c} f^2 \, d\mu.
\]

Hence, for any \( f \in C_c^\infty(\mathbb{R}) \) with \( \mu(f) = 0 \), and \( r \geq r_0 \),
\[
\mu(\mathbb{I}_{B_0(r)} f^2) \leq \frac{C_5 K(r)^{2r^{\alpha}}}{k(r)} \mathcal{E}(f, f) + \frac{\mu(B(0, r)^c)}{\mu(B(0, r))} \mu(\mathbb{I}_{B_0(r)} f^2).
\]

Putting it into the right hand side of (3.5) and noting that \( \hat{E}(f, f) \leq \mathcal{E}(f, f) \), we get that for any \( f \in C_c^\infty(\mathbb{R}) \) with \( \mu(f) = 0 \), and \( r \geq r_0 \),
\[
\mu(\mathbb{I}_{B_0(r)} f^2) \leq \left( \frac{1}{C_3 \Phi(r)} + \frac{C_4}{C_3 \Phi(r)} \frac{C_5 K(r)^{2r^{\alpha}}}{k(r)} \right) \mathcal{E}(f, f) + \frac{C_4}{C_3 \Phi(r)} \frac{\mu(B(0, r)^c)}{\mu(B(0, r))} \mu(f^2).
\]
Summing (3.6) and (3.7), we have

$$\mu(f^2) \leq \left( \frac{1}{C_3\Phi(r)} + \left( \frac{C_4}{C_3\Phi(r)} + 1 \right) \frac{C_5 K(r)^2 r^\alpha}{k(r)} \right) \mathcal{E}(f, f) + \left( \frac{C_4}{C_3\Phi(r)} + 1 \right) \frac{\mu(B(0, r))^c}{\mu(B(0, r))} \mu(f^2).$$

Under (1.3), we can choose $r_1 \geq r_0$ such that

$$\left( \frac{C_4}{C_3\Phi(r_1)} + 1 \right) \frac{\mu(B(0, r_1))^c}{\mu(B(0, r_1))} \leq \frac{1}{2},$$

and so we arrive at

$$\mu(f^2) \leq 2 \left( \frac{1}{C_3\Phi(r_1)} + \left( \frac{C_4}{C_3\Phi(r_1)} + 1 \right) \frac{C_5 K(r_1)^2 r_1^\alpha}{k(r_1)} \right) \mathcal{E}(f, f).$$

This gives the desired Poincaré inequality (1.4) with

$$C = 2 \left( \frac{1}{C_3\Phi(r_1)} + \left( \frac{C_4}{C_3\Phi(r_1)} + 1 \right) \frac{C_5 K(r_1)^2 r_1^\alpha}{k(r_1)} \right).$$

The equivalence of the Poincaré inequality and the corresponding bound of $\|P_1 - \mu\|_{L^2(\mathbb{R}, \mu)}$ is well known, see, e.g., [18, Theorem 1.1.1].

To prove Theorem 1.2(ii), we need the following local super Poincaré inequality for $\mathcal{E}$.

**Lemma 3.3.** There exists a constant $C_6 > 0$ such that for any $s, r > 0$ and any $f \in C_c^\infty(\mathbb{R}),$

$$\int_{B(0, r)} f^2(x) \mu(dx) \leq sC_{1,\alpha} \int_{B(0, r) \times B(0, r)} \frac{(f(y) - f(x))^2}{|y - x|^{1+\alpha}} dy \mu(dy)$$

$$+ \frac{C_6 K(r)}{k(r)^2} \left( 1 + (C_{1,\alpha}s)^{-1/\alpha} K(r)^{1/\alpha} \left( \int_{B(0, r)} |f|(x) \mu(dx) \right)^2 \right)$$

$$\leq s\mathcal{E}(f, f) + \frac{C_6 K(r)}{k(r)^2} \left( 1 + (C_{1,\alpha}s)^{-1/\alpha} K(r)^{1/\alpha} \right) \mu(|f|)^2.$$

**Proof.** The second inequality is trivial, and so we only need the consider the first inequality. The Sobolev inequality of dimension $2/\alpha$ for fractional Laplacians in $\mathbb{R}$ holds uniformly on balls, e.g. see [9, Theorem 3.1]. Thus according to [18, Corollary 3.3.4], there exists a constant $c_1 > 0$ such that

$$\int_{B(0, r)} f^2(x) dx \leq s \int_{B(0, r) \times B(0, r)} \frac{(f(y) - f(x))^2}{|y - x|^{2+\alpha}} dy dx + c_1(1 + s^{-1/\alpha}) \left( \int_{B(0, r)} |f(x)| dx \right)^2$$

holds for all $f \in C_c^\infty(\mathbb{R})$ and all $s, r > 0$. Therefore,

$$\int_{B(0, r)} f^2(x) \mu(dx) \leq K(r) \int_{B(0, r)} f^2(x) dx$$

$$\leq sK(r) \int_{B(0, r) \times B(0, r)} \frac{(f(y) - f(x))^2}{|y - x|^{2+\alpha}} dy dx$$

$$+ c_1(1 + s^{-1/\alpha}) K(r) k(r)^{-2} \left( \int_{B(0, r)} |f(x)| \mu(dx) \right)^2.$$

This implies the first desired assertion by replacing $s$ with $C_{1,\alpha}sK(r)^{-1}$ in the above inequality.
Proof of Theorem 1.2(ii). As in the proof of Theorem 1.2(i), it suffices to consider \( f \in C_c^\infty(\mathbb{R}) \). First, by Lemma 3.3 there exists a constant \( C_7 > 0 \) that depends on \( C_0, C_{1,\alpha} \) and \( K(0+) \) such that
\[
\mu(f^2 1_{B(0,r)}) \leq s\mathcal{E}(f, f) + C_7 K_0(r)(1 + s^{-1/\alpha})\mu(|f|)^2, \quad r, s > 0, f \in C_c^\infty(\mathbb{R})
\]
where
\[
K_0(r) = \frac{K(r)^{1+1/\alpha}}{k(r)^2}.
\]
Combining this with (3.5) and noting that \( \hat{\mathcal{E}}(f, f) \leq \mathcal{E}(f, f) \), there exists a constant \( c_0 > 0 \) such that for any \( r \geq r_0, s > 0 \) and \( f \in C_c^\infty(\mathbb{R}) \),
\[
\mu(f^2) \leq \left( \frac{c_0}{\Phi(r)} + s \left( 1 + \frac{c_0}{\Phi(r)} \right) \right) \mathcal{E}(f, f) + \left( 1 + \frac{c_0}{\Phi(r)} \right) K_0(r)(1 + s^{-1/\alpha})\mu(|f|)^2.
\]
By (1.6), letting \( s_0 = c_0/\Phi(r_0) \) and taking \( r = \Phi^{-1}(c_0/s) \), we have for \( s \in (0, s_0] \) and \( f \in C_c^\infty(\mathbb{R}) \),
\[
\mu(f^2) \leq (2s + s^2)\mathcal{E}(f, f) + (1 + s)(1 + s^{-1/\alpha})K_0 \circ \Phi^{-1}(c_0/s)\mu(|f|)^2.
\]
In particular, for \( s \in (0, s_0 \wedge 1] \) and \( f \in C_c^\infty(\mathbb{R}) \),
\[
\mu(f^2) \leq 3s\mathcal{E}(f, f) + 2(1 + s^{-1/\alpha})K_0 \circ \Phi^{-1}(c_0/s)\mu(|f|)^2.
\]
Replacing \( s \) by \( s/3 \), we get for \( s \in (0, 3(s_0 \wedge 1)] \) and \( f \in C_c^\infty(\mathbb{R}) \),
\[
\mu(f^2) \leq s\mathcal{E}(f, f) + 2 \cdot 3^{1/\alpha}(1 + s^{-1/\alpha})K_0 \circ \Phi^{-1}(3c_0/s)\mu(|f|)^2.
\]
This proves the super Poincaré inequality (1.7) with \( \beta(r) = 2 \cdot 3^{1/\alpha}(1 + r^{-1/\alpha})K_0 \circ \Phi^{-1}(3c_0/r) \) for \( r \in (0, 3(s_0 \wedge 1)] \) and \( \beta(r) = \beta(3(s_0 \wedge 1)) \) when \( r \geq 3(s_0 \wedge 1) \). Combining both estimates for \( \beta \) yields the desired assertion (1.8).

Define \( \Psi(t) = \int_t^\infty \frac{\beta^{-1}(r)}{r} \, dr \) for \( t > \inf \beta(r) \). Thus by [18] Theorem 3.3.14,
\[
\|P_t\|_{L^1(\mathbb{R}; \mu) \to L^\infty(\mathbb{R}; \mu)} \leq 2\Psi^{-1}(t) \quad \text{for } t > 0,
\]
where
\[
\Psi^{-1}(t) = \inf\{r \geq \inf \beta : \Psi(r) \leq t\}.
\]
In particular, we conclude that \( P_t \) has a bounded kernel \( p(t, x, y) \) with respect to \( \mu \) for every \( t > 0 \). Moreover, since every point is visited by the time-changed process \( Y \) and thus having positive capacity, we have by [2] Theorem 3.1 that the symmetric kernel \( p(t, x, y) \) can be chosen in such a way that for every fixed \( y \in \mathbb{R} \) and \( t > 0 \), \( x \to p(t, x, y) \) is bounded and quasi-continuous on \( \mathbb{R} \). Thus for every \( f \in L^2(\mathbb{R}; \mu) \) and \( t > 1 \), we have by the Cauchy-Schwarz inequality and (1.5) that
\[
\sup_{x \in \mathbb{R}} |P_tf(x) - \mu(f)| = \sup_{x \in \mathbb{R}} |P_t(P_{t-1}f - \mu(f))(x)| \\
\leq \sup_{x \in \mathbb{R}} p_2(x, x)^{1/2} \|P_{t-1}f - \mu(f)\|_{L^2(\mathbb{R}; \mu)} \\
\leq ce^{-t/C} \|f\|_{L^2(\mathbb{R}; \mu)}.
\]
This establishes (1.9). The last assertion is a direct consequence of (1.9), see, e.g., [4] Theorem 8.5. \( \square \)

Next, we prove Theorem 1.2(iii).
Proof of Theorem 1.2(iii). Let \( a_0(x) = (1 + |x|)^{\alpha} \) and \( \mu_0(dx) = \frac{a_0(x)^{-1} dx}{\int a_0(x)^{-1} dx} \). According to Theorem 1.2(i), the following Poincaré inequality

\[
\mu_0(f^2) \leq C \mathcal{E}(f, f), \quad f \in \mathcal{F}^\mu, \quad \mu_0(f) = 0
\]

holds for some constant \( C > 0 \). Therefore, for any \( s > 0 \) and \( f \in \mathcal{F}^\mu \),

\[
\int_{B(0,s)} \left( f(x) - \frac{1}{\mu(B(0,s))} \int_{B(0,s)} f(x) \mu(dx) \right)^2 \mu(dx)
\]

\[
= \inf_{a \in \mathbb{R}} \int_{B(0,s)} (f(x) - a)^2 \mu(dx)
\]

\[
\leq \int_{B(0,s)} (f(x) - \mu_0(f))^2 \mu(dx)
\]

\[
\leq c_1 \left( \sup_{|x| \leq s} \frac{a(x)^{-1}}{a_0(x)^{-1}} \right) \int_{B(0,s)} (f(x) - \mu_0(f))^2 \mu_0(dx)
\]

\[
\leq \frac{c_2}{\Phi_0(s)} \mathcal{E}(f, f).
\]

The desired weak Poincaré inequality (1.12) now follows from (1.10) and [18, Theorem 4.3.1]. By [18, Theorem 4.1.3] and its proof, (1.12) implies that

\[
\|P_t f - \mu(f)\|_{L^2_\mu}^2 \leq \eta(t) \left( \|f - \mu(f)\|_{L^2_\mu}^2 + \|f - \mu(f)\|_{L^\infty}^2 \right) \leq 2\eta(t) \left( \|f\|_{L^2_\mu}^2 + \|f\|_{L^\infty}^2 \right)
\]

for every \( f \in \mathcal{F}^\mu \), where

\[
\eta(t) = \inf \{ r > 0 : -\alpha(r) \log r \leq 2t \}.
\]

This completes the proof of Theorem 1.2(iii). \( \square \)

To prove Examples 1.3 and 1.4 for \( n \geq 1 \), let \( g_n \in C^\infty(\mathbb{R}) \) be such that

\[
g_n(x) = \begin{cases} 
0, & |x| \leq n; \\
\in [0,1], & n \leq |x| \leq 2n; \\
1, & |x| \geq 2n,
\end{cases}
\]

and \( |g'_n(x)| \leq 2/n \) for all \( x \in \mathbb{R} \). By (1.1) and (1.2), \( g_n \in \mathcal{F} \cap L^2(\mathbb{R}; \mu) = \mathcal{F}^\mu \). Moreover, there exists a constant \( c > 0 \) independent of \( n \) such that for all \( n \geq 1 \),

\[
\mathcal{E}(g_n, g_n) = C_{1,\alpha} \frac{n^2}{2} \int_{|x| \leq 3n} \int_{|y| \leq 4n} \frac{(g_n(x) - g_n(y))^2}{|x - y|^{1+\alpha}} \, dy \, dx
\]

\[
+ C_{1,\alpha} \frac{n^2}{2} \int_{|x| \leq 3n} \int_{|y| > 4n} \frac{(g_n(x) - g_n(y))^2}{|x - y|^{1+\alpha}} \, dy \, dx
\]

\[
+ C_{1,\alpha} \frac{n^2}{2} \int_{|x| \geq 3n} \int_{|y| \leq 2n} \frac{(g_n(x) - g_n(y))^2}{|x - y|^{1+\alpha}} \, dy \, dx
\]

\[
\leq \frac{2C_{1,\alpha}}{n^2} \int_{|x| \leq 3n} \left( \int_{|x-y| \leq 7n} \frac{|x-y|^2}{|x-y|^{1+\alpha}} \, dy \right) \, dx
\]

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Observe that \( \beta \) is a non-increasing positive function, and so the inequality (1.7) holds for some non-increasing positive function \( \Phi \). Then, \( \Phi^{-1}(r) = (r/C_\gamma)^{1/(\gamma-\alpha)} - 1 \) for \( r > 1 \) and so \( \beta(r) \leq c(1 + r^{-(1/\alpha + 2\gamma/(\alpha - \gamma)}) \) for \( r > 0 \).

Therefore, for any constant \( C > 0 \), the Poincaré inequality (1.4) does not hold.

Proof of Example 1.3. (i) If \( \gamma \geq \alpha \), we have \( \Phi(0) = C_\gamma \). So the Poincaré inequality (1.4) follows from Theorem 1.2(i).

To disprove the Poincaré inequality (1.4) for \( \gamma \in (1, \alpha) \), let \( g_n \) be the function defined by (3.8). It is clear that there are constants \( c_1, c_2 > 0 \) so that

\[
\mu(g_n^2) \geq \frac{c_1}{n^{\gamma-1}} \quad \text{and} \quad \mu(g_n)^2 \leq \frac{c_2}{n^{2(\gamma-1)}} \quad \text{for } n \geq 1.
\]

Combining these with (3.9), we have

\[
\lim_{n \to \infty} \frac{\mathcal{E}(g_n, g_n)}{\mu(g_n^2) - \mu(g_n)^2} \leq \lim_{n \to \infty} \frac{cn^{-\alpha+1}}{c_1 n^{-\gamma+1} - c_2 n^{-2\gamma+2}} = 0.
\]

Therefore, for any constant \( C > 0 \), the Poincaré inequality (1.4) does not hold.

(ii) Let \( \gamma > \alpha \). It is easy to see that \( \Phi(r) = C_\gamma (1 + r)^{\gamma-\alpha} \),

\[
K(r) = C_\gamma^{-1}, \quad k(r) = C_\gamma^{-1} (1 + r)^{-\gamma} \quad \text{and} \quad K_0(r) = C_\gamma^{1-1/\gamma} (1 + r)^{2\gamma}.
\]

Then, \( \Phi^{-1}(r) = (r/C_\gamma)^{1/(\gamma-\alpha)} - 1 \) for \( r > 1 \) and so \( \beta(r) \leq c(1 + r^{-(1/\alpha + 2\gamma/(\alpha - \gamma)}) \) for \( r > 0 \).

Observe that \( \beta^{-1}(r)/r \) is integrable near infinity. Thus (1.9) holds. Moreover, by [18, Theorem 3.3.15(2)], the corresponding bound (1.14) on \( \|P_t\|_{L^1(R^d) \to L^\infty(R^d)} \) holds.

We next show that when \( \gamma \in (1, \alpha] \), the super Poincaré inequality (1.7) fails. Indeed, if the inequality (1.7) holds for some non-increasing positive function \( \beta(r) \), then, applying it to the function \( g_n \) defined in (3.8), we get

\[
\frac{c_3}{n^{\gamma-1}} \leq \frac{cr}{n^{\alpha-1}} + \frac{c_4 \beta(r)}{n^{2(\gamma-1)}}, \quad r > 0, n \geq 1
\]

for some constants \( c, c_3, c_4 > 0 \). Since \( \gamma \in (1, \alpha] \), one has

\[
c_3 \leq \lim_{n \to \infty} \left( \frac{cr}{n^{\alpha-1}} + \frac{c_4 \beta(r)}{n^{2(\gamma-1)}} \right) \leq cr \quad \text{for every } r > 0,
\]
which is impossible.

(iii) Let $\gamma \in (1, \alpha)$. Then $\Phi_0(r) = C_\gamma (1 + r)^{\gamma - \alpha}$, and so the desired weak Poincaré inequality follows from Theorem 1.2(iii). According to [18, Theorem 4.1.5(2)], we have the claimed bound for $\|P_t - \mu\|_{L^\infty(\mathbb{R}^d) \to L^2(\mathbb{R}^d)}$. On the other hand, for function $g_n$ given by (3.8), we have $\|g_n\|_\infty = 1$, $\mu(g_n^2) - \mu(g_n)^2 \geq c_1 n^{-\gamma \alpha}$ for some constant $c_1 > 0$. Hence, according to (1.12) and (3.9),

$$c_1 n^{-\gamma \alpha} \leq c \alpha(r) n^{-(\alpha - 1)} + 2r, \quad r > 0.$$ 

Taking $r = r_n := \frac{c_1}{4n} \log n$, which goes to zero as $n \to \infty$, we get that

$$\lim_{n \to \infty} r_n^{(\alpha - \gamma)/(-\gamma + 1)} \alpha(r_n) > 0.$$ 

Thus, the weak Poincaré inequality (1.12) does not hold if $\lim_{r \to 0} r^{(\alpha - \gamma)/(-\gamma + 1)} = 0$. □

**Proof of Example 1.4.** (i) If $\gamma \geq 0$, we have $\Phi(0) = C_{\alpha, \gamma}$ and so the Poincaré inequality (1.4) follows from Theorem 1.2(i).

To prove that the Poincaré inequality (1.4) does not hold for $\gamma < 0$, we take function $g_n$ defined by (3.9). It is clear that

$$\mu(g_n^2) \geq \frac{c_1}{n^{\alpha - 1} \log n}, \quad \mu(g_n)^2 \leq \frac{c_2}{n^{2(\alpha - 1)} \log^{2\gamma} n}$$

hold for $n$ large enough and some constants $c_1, c_2 > 0$ (independent of $n$). Combining these with (3.9), we arrive at

$$\lim_{n \to \infty} \frac{\mathcal{E}(g_n, g_n)}{\mu(g_n^2)} \leq \lim_{n \to \infty} \frac{c_1 n^{-\alpha + 1} \log^{-\gamma} n - c_2 n^{-2\alpha + 2} \log^{-2\gamma} n}{c_1 n^{\alpha - 1} \log n} = 0$$

provided $\gamma < 0$. This implies that the Poincaré inequality (1.4) does not hold for any constant $C > 0$.

(ii) Let $\gamma > 0$. It is easy to see that $\Phi(r) = C_{\alpha, \gamma} \log \gamma(e + r)$,

$$K(r) = C_{\alpha, \gamma}^{-1}, \quad K(r) = \frac{C_{\alpha, \gamma}^{-1}(1 + r)^{-\alpha} \log^{-\gamma} (e + r)}{r} \quad \text{and} \quad K_0(r) = C_{\alpha, \gamma}^{-1}(1 + r)^{2\alpha} \log^{2\gamma} (e + r).$$

Then $\Phi^{-1}(r) = \exp \left( (r/C_{\alpha, \gamma})^{1/\gamma} \right) - e$ and so the super Poincaré inequality (1.7) holds with

$$\exp \left( c_1 (1 + r^{-1/\gamma}) \right) \leq \beta(r) \leq \exp \left( c_2 (1 + r^{-1/\gamma}) \right), \quad r > 0$$

for some constant $c_1, c_2 > 0$. When $\gamma > 1$, we get (1.15) from [18, Theorem 3.3.15(1)]. Moreover, $\beta^{-1}(r)/r$ is integrable at infinity if and only if $\gamma > 1$. In this case, the uniform strong ergodicity (1.9) holds.

Next, we prove that if $\gamma \leq 0$, then for any $\beta : (0, \infty) \to (0, \infty)$ the super Poincaré inequality (1.7) does not hold. Indeed, if the inequality (1.7) holds, then, applying it to the function $g_n$ of (3.9),

$$\frac{e_3}{n^{\alpha - 1} \log \gamma n} \leq \frac{cr}{n^{\alpha - 1}} + \frac{c_4 \beta(r)}{n^{2(\alpha - 1)} \log^{2\gamma} n}, \quad r > 0, n \geq 1$$

holds for some constants $c, e_3, c_4 > 0$. Since $\gamma \leq 0$, we get that

$$c_3 \leq \lim_{n \to \infty} \left( \frac{cr}{\log^{-\gamma} n} + \frac{c_4 \beta(r)}{n^{\alpha - 1} \log^{-\gamma} n} \right) \leq cr, \quad r > 0.$$
Letting \( r \to 0 \) we get that \( c_3 \leq 0 \), which is impossible.

Consider the function \( g_n \) defined by (3.8). It is easy to see that

\[
\mu(g_n^2) \geq \frac{c_5}{n^{\alpha-1} \log^\gamma(e + n)}, \quad \mu((g_n)^2) \leq \frac{c_6}{n^{2(\alpha-1)} \log^2\gamma(e + n)}
\]

hold for all \( n \geq 1 \) and some constants \( c_5, c_6 > 0 \). Combining these with (3.3) and (1.7), we have

\[
\frac{c_5}{\log^\gamma(e + n)} \leq cr + \frac{c_6\beta(r)}{n^{\alpha-1} \log^2\gamma(e + n)}, \quad r > 0.
\]

Taking \( r = r_n := \frac{c_5}{2 \log^\gamma(e + n)} \), we derive

\[
\beta(r_n) \geq \frac{c_5}{2c_6} n^{\alpha-1} \log^\gamma(e + n), \quad n \geq 1.
\]

Therefore,

\[
\liminf_{n \to \infty} r_n^{1/\gamma} \log \beta(r_n) \geq \alpha - 1 > 0.
\]

According to [18 Corollary 3.3.4(1)], the super Poincaré inequality with \( \beta(r) = \exp(c(1 + r^{-1})) \) for some \( c > 0 \) is equivalent to the following defective log-Sobolev inequality

\[
\mu(f^2 \log f^2) \leq C_1 \mathcal{E}(f, f) + C_2 \quad \text{for } f \in \mathcal{F}^\mu \text{ with } \mu(f^2) = 1,
\]

(3.10)

where \( C_1 \) and \( C_2 \) are two positive constants. Since the symmetric Dirichlet form \( (\mathcal{E}, \mathcal{F}^\mu) \) is conservative and irreducible, it follows from [19 Corollary 1.3] that the defective log-Sobolev inequality (3.10) is also equivalent to log-Sobolev inequality (1.16). It is well known that the log-Sobolev inequality implies the entropy of the semigroup \( P_t \) decays exponentially (1.17), see, e.g., [3 Corollary 1.1].

(iii) Let \( \gamma < 0 \). Then there exist two constants \( c_7 \) and \( c_8 > 0 \) such that for all \( r > 0 \),

\[
\Phi_0(r) = c_7 \log^\gamma(1 + r), \quad \mu(B(0, r)^c) = c_8 r^{-(\alpha-1)} \log^{-\gamma}(1 + r).
\]

Then the desired weak Poincaré inequality follows from Theorem 1.2(iii), and so the corresponding bound for \( \|P_t - \mu\|_{L^\infty(\mathbb{R}^d; \nu)} \to L^2(\mathbb{R}^d; \mu) \) follows from [18 Theorem 4.1.5(1)]. On the other hand, for function \( g_n \) defined in (3.8), we have \( \|g_n\|_{\infty} = 1 \) and

\[
\mu(g_n^2) - \mu(g_n)^2 \geq c_9 n^{-(\alpha-1)} \log^{-\gamma}(1 + n)
\]

for some constant \( c_9 > 0 \). Hence, according to (1.12) and (3.9),

\[
c_9 n^{-(\alpha-1)} \log^{-\gamma}(1 + n) \leq c\alpha(r)n^{-(\alpha-1)} + 2r, \quad r > 0.
\]

Taking \( r = r_n := \frac{c_9 n^{-(\alpha-1)} \log^{-\gamma}(1 + n)}{4} \) which goes to zero as \( n \to \infty \), we get that

\[
\liminf_{n \to \infty} \log^\gamma(1 + r_n^{-1})\alpha(r_n) = \alpha - 1 > 0.
\]

Thus, the weak Poincaré inequality (1.12) fails if \( \lim_{r \to 0} \log^\gamma(1 + r^{-1})\alpha(r) = 0 \).

We next prove Theorem 1.5.
Proof of Theorem 1.5. We only need to consider \( f \in C_c^\infty(\mathbb{R}) \). First, according to Lemma 3.3, there exists a constant \( c_1 > 0 \) such that

\[
\mu(f^2 \mathbb{1}_{B(0,r)}) \leq s \mathcal{E}(f, f) + c_1 K_0(r)(1 + s^{-1}) \mu(|f|^2), \quad r, s > 0, f \in C_c^\infty(\mathbb{R})
\]

On the other hand, for any \( 1 < \delta < \beta \), let \( V \) be the function defined in Lemma 3.2. Then, by Lemma 3.2 and \([14, \text{Theorem 2.1}(1)]\), we know that

\[
\mathcal{L}^{(\delta)}_s V(x) \leq -c_2 \Psi_\delta(|x|)(1 + |x|)^{\beta - \delta} V(x) \mathbb{1}_{B(0,r_0)}(x) + c_3 \mathbb{1}_{B(0,r_0)}(x)
\]

for some positive constants \( c_2, c_3 \) and \( r_0 \) depending on \( \delta \). Combining this with the argument of Theorem 1.2(ii), and noting that \( \mathbb{E} \) and \( \mathbb{P} \) are time-continuous, it follows from \([1, \text{Theorem 1.1}] \) or \([12, \text{Theorem 1}]\) that (1.19) has a unique strong solution

\[
Z_t = \mathbb{1}_{(0, s)} \geq \mathbb{1}_{(0, s)} Z_0 \quad \text{for all } t \geq 0.
\]

Theorem 1.2(ii), and noting that \( \text{by taking } \mathbb{E} \) and \( \mathbb{P} \) are time-continuous, it follows from \([1, \text{Theorem 1.1}] \) or \([12, \text{Theorem 1}]\) that (1.19) has a unique strong solution

\[
Z_t = \mathbb{1}_{(0, s)} \geq \mathbb{1}_{(0, s)} Z_0 \quad \text{for all } t \geq 0.
\]

The required assertion follows from the inequality above and by taking \( \beta(s) = \beta(3(s_0 \wedge 1)) \) for all \( s \geq 3(s_0 \wedge 1). \)

The proof of Example 1.6 is similar to that of Example 1.3(ii), we omit the details here. We now present the proof of Theorem 1.7.

Proof of Theorem 1.7. (i) Since the coefficient \( a \) in the SDE (1.19) is locally \( 1/\alpha \)-Hölder continuous, it follows from [11, Theorem 1.1] or [12, Theorem 1] that (1.19) has a unique strong solution \((Z_t)_{t \geq 0}\) up to the explosion time

\[
\tau = \inf\{t > 0 : Z_t \notin \mathbb{R}\}.
\]

According to Lemma 3.2 and [14, Theorem 2.1(1)], \( \tau = \infty \). This is, the SDE (1.19) has a unique strong solution.

Let \((P_t)_{t \geq 0}\) be the semigroup of the process \((Z_t)_{t \geq 0}\). It follows from (1.20) that \( Z \) is a time-change of the symmetric \( \alpha \)-stable process, and \( \mu \) is its symmetrizing and unique invariant probability measure (see Proposition 1.1).
In view of \((1.20), (1.21)\) and Theorem \(1.2\) ii), the semigroup \((P_t)_{t \geq 0}\) is \(L^2(\mathbb{R}; \mu)\) exponentially ergodic, which is equivalent to the desired assertion for the exponential ergodicity of \((P_t)_{t \geq 0}\) in the total variation norm, see, e.g., \([1, \text{Theorem 8.8}]\).

(ii) Since \(\alpha \in (1,2)\), symmetric \(\alpha\)-stable process \(X\) on \(\mathbb{R}\) is pointwise recurrent, so is \(Z\). This in particular implies that the process \(Z\) is Lebesgue irreducible, that is, for any \(x \in \mathbb{R}\) and any Borel set \(B\) with positive Lebesgue measure, \(P_x(\sigma_B < \infty) > 0\), where \(\sigma_B = \inf\{t \geq 0 : X_t \in B\}\). Let \(B\) be any Borel set in \(\mathbb{R}\). Denote by \((P_t)_{t \geq 0}\) the transition semigroup of \(Y\). Then for each \(t > 0\), \(\mu(P_t 1_B) = \mu(B) = 0\) and so \(P_t 1_B = 0\) \(\mu\)-a.e. on \(\mathbb{R}\). Since \(P_t 1_B\) is \(\mathcal{E}\)-quasi-continuous, it follows that \(P_t 1_B(x) = 0\) for every \(x \in \mathbb{R}\). This shows that for every \(x \in \mathbb{R}\) and \(t > 0\), \(P_t(x, dy)\) is absolutely continuous with respect to \(\mu\). We claim that if \(\mu(B) > 0\), then \(P_t(x, B) > 0\) for every \(x \in \mathbb{R}\) and \(t > 0\). Suppose there are \(x_0 \in \mathbb{R}\) and \(t_0 > 0\) so that \(P_{t_0}(x_0, B) = 0\). Then \(P_{t_0/2}(P_{t_0/2} 1_B)(x_0) = 0\) and so \(P_{t_0/2} 1_B = 0\) \(\mu\)-a.e. on \(\mathbb{R}\). The latter would imply that \(\mu(B) = \mu(P_{t_0/2} 1_B) = 0\), which is absurd. This proves the claim, which in particular implies that \(Z\) is aperiodic.

On the other hand, let \(\varphi \in C^2(\mathbb{R})\) be a nonnegative function such that \(\varphi(x) = |x|\) for \(|x| \geq 1\), and \(\varphi(x) \leq |x|\) for \(|x| \leq 1\). Under \((1.22)\), define the function \(V(x) = 2 - (1 + \varphi(x))^{-\theta}\) for some constant \(\theta \in (0, \gamma - \alpha)\) to be determined later. It is easy to see that \(LV\) exists pointwise as a locally bounded function. Furthermore, according to the proof of \([16, \text{Theorem 1.1(ii)}]\) (also see \([16, (3.24)-(3.28)]\) for more details), we know that

\[
\limsup_{|x| \to \infty} (1 + |x|)^{\alpha + \theta} LV(x) = \frac{\theta C_{1,\alpha}}{\alpha} E(\alpha, -\theta),
\]

where \(E(\alpha, -\theta)\) is defined by \((3.3)\). By \((3.4)\), one can choose \(\theta \in (0, \gamma - \alpha)\) small enough such that

\[
\limsup_{|x| \to \infty} (1 + |x|)^{\alpha + \theta} LV(x) \leq \frac{\theta C_{1,\alpha} \pi}{2\alpha} \cot(\pi\alpha/2).
\]

Combining \((1.20), (1.22)\) with all the conclusions above, we get that there exist \(c_1, c_2\) and \(r_0 > 0\) such that

\[
LV \leq -c_1 V + c_2 1_{B_0(0,r_0)}.
\]

As mentioned above, the process \(Z\) is Lebesgue irreducible and aperiodic. Therefore, according to \((3.11)\) and \([10, \text{Theorem 5.2(c)}]\), the process \(Z\) is uniformly strongly ergodic.

4 Appendix: ergodicity of time changed Brownian motions in dimension one

Let \(Z\) be a Brownian motion on \(\mathbb{R}\), and \(a\) be a positive and locally bounded measurable function on \(\mathbb{R}\) such that \(\mu(dx) := \frac{1}{a(x)} dx\) is a probability measure (that is, \(\int_\mathbb{R} a(x)^{-1} dx = 1\)). For any \(t > 0\), define

\[
A_t = \int_0^t \frac{1}{a(B_s)} ds, \quad \tau_t = \inf\{s > 0 : A_s > t\},
\]

and set \(Y_t = B_{\tau_t}\). Following Section 1, we know that the time changed Brownian motion \(Y\) is a recurrent \(\mu\)-symmetric diffusion process on \(\mathbb{R}\), and \(\mu\) is the unique invariant probability measure of \(Y\). The Dirichlet form \((\mathcal{E}, \mathcal{F})\) of \(Y\) is

\[
\mathcal{E}(f,f) = \frac{1}{2} \int_\mathbb{R} |f'(x)|^2 dx.
\]
where $\mathcal{F}^{\mu}$ is defined by (1.2). In this appendix, we present results on various ergodic properties of $Y$ in terms of Poincaré type inequalities. For readers’s convenience, we also present the sketch of the proof below.

**Theorem 4.1.** For any $r > 0$, set

$$
\Phi(r) := \inf_{|x| \geq r} \frac{a(x)}{(1 + |x|)^2}, \quad \Phi_0(r) := \inf_{|x| \leq r} \frac{a(x)}{(1 + |x|)^2}
$$

and

$$
K(r) := \sup_{|x| \leq r} a(x)^{-1}, \quad k(r) := \inf_{|x| \leq r} a(x)^{-1}, \quad K_0(r) := \frac{K(r)^{3/2}}{k(r)^2}.
$$

Then the following holds.

(i) The Poincaré inequality (1.4) holds for some constant $C > 0$ if and only if

$$
\limsup_{|x| \to \infty} \frac{1}{|x|} \int_{\{|s| \geq |x|\}} \frac{ds}{a(s)} < \infty. \quad (4.1)
$$

In particular, if $\lim_{r \to \infty} \Phi(r) > 0$, then (4.1) is satisfied.

(ii) The super Poincaré inequality (1.7) holds if and only if

$$
\limsup_{|x| \to \infty} \frac{1}{|x|} \int_{\{|s| \geq |x|\}} \frac{ds}{a(s)} = 0. \quad (4.2)
$$

Moreover, in this case (1.7) holds with

$$
\beta(r) = C_1 \left(1 + r^{-1/2} K_0(\Phi^{-1}(C_2/r))\right) \quad (4.3)
$$

with some constants $C_1$ and $C_2 > 0$. In particular, if $\lim_{r \to \infty} \Phi(r) = \infty$, then (1.2) is satisfied.

(iii) If $\lim_{r \to \infty} \Phi_0(r) = 0$, then the weak Poincaré inequality

$$
\mu(f^2) \leq \alpha(r) \mathcal{E}(f, f) + r \|f\|_\infty^2 \quad \text{for every } r > 0, f \in \mathcal{F}^{\mu}, \mu(f) = 0
$$

holds with

$$
\alpha(r) = C \inf \left\{ \frac{1}{\Phi_0(s)} : \mu(B(0, s)) \geq \frac{1}{1 + r} \right\}
$$

for some constant $C > 0$ independent of $r$.

**Sketch of proof for Theorem 4.1.** The assertion (i) immediately follows from criterion for the Poincaré inequality of one-dimensional diffusion processes, see, e.g., [4, Table 5.1, p. 100]. Having (i) at hand, one can follow the proof of Theorem 1.2(iii) to obtain the assertion (iii).

The first statement of assertion (ii) is a direct consequence of criterion for the super Poincaré inequality of one-dimensional diffusion processes, also see [4, Table 5.1, p. 100]. To verify (4.3), we first note that the classical Sobolev inequality for Laplacian holds uniformly on balls. Then, by [13, Corollary 3.3.4(2)], there exists a constant $c_1 > 0$ such that for all $f \in C_c^\infty(\mathbb{R})$ and $r, s > 0$,

$$
\int_{B(0, r)} f(x)^2 dx \leq s \int_{B(0, r)} f'(x)^2 dx + c_1 (1 + s^{-1/2}) \left( \int_{B(0, r)} |f(x)| dx \right)^2.
$$
Therefore, we can conclude that there is a constant $c_2 > 0$ such that for all $f \in C_c^\infty(\mathbb{R})$ and $r, s > 0$,

$$\int_{B(0,r)} f(x)^2 \mu(dx) \leq s \int_{B(0,r)} f'(x)^2 dx + c_2 K_0(r)(1 + s^{-1/2}) \left( \int_{B(0,r)} |f(x)| \mu(dx) \right)^2.$$  

Let $\varphi \in C^\infty(\mathbb{R})$ such that $\varphi(x) = |x|$ for all $|x| \geq 1$. For $\theta \in (0,1)$, set $V(x) = \varphi(x)^\theta$. Then, there are constants $c_3, c_4$ and $r_0 > 0$ such that

$$a(x)V''(x) \leq -c_3 \frac{a(x)}{(1 + |x|)^2} V(x) + c_4 \mathbb{1}_{B(0,r_0)}(x), \quad x \in \mathbb{R}.$$  

This, together with the local super Poincaré inequality above, gives us the desired $\beta$ given by (4.3).

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