1. Introduction

Theories of gravity with the Lagrangian \( L = f(R) \), where \( f \) is a certain function of the scalar curvature \( R \), are one of the well-known and important generalizations of Einstein’s general relativity (in which \( L = R \)). Curvature-nonlinear corrections to the Einstein theory are known to emerge due to quantum effects of material fields [1]. Different choices of \( f(R) \) have been used for solving cosmological problems, in particular, corrections to the Einstein Lagrangian, proportional to \( R^2 \), for describing inflation in the early Universe [2], and those of the form \( R^{-n} \), where \( n > 1 \), for explaining the present-day accelerated expansion of the Universe [3]. Of no lesser importance are the effects of \( f(R) \) theories for local configurations, such as, e.g., galaxies and black holes. There have been attempts to explain the galactic rotation curves by \( f(R) \)-induced modifications of Newton’s law [4] and extensive studies of black hole properties in this class of theories [5] (see also references cited in the above-mentioned papers).

There is a well-known conformal mapping from the manifold \( M_J \) with the metric \( g_{\mu\nu} \), where an \( f(R) \) theory is initially formulated (it is called the Jordan conformal frame, or Jordan picture), to the manifold \( M_E \) with the metric \( g_{\mu\nu} = g_{\mu\nu}/F(x) \) (the Einstein picture), in which the equations of the original theory turn into the equations of general relativity with a scalar field \( \phi \) with a certain potential \( V(\phi) \) (see, e.g., [6] and references therein). If the conformal factor \( F(x) \) is everywhere regular, then the basic physical properties of the manifolds \( M_J \) and \( M_E \) coincide since, in such transformations, a flat asymptotic in \( M_J \) maps to a flat asymptotic in \( M_E \), a horizon to a horizon, a centre to a centre. Using this transformation, some general properties of vacuum static, spherically symmetric solutions of an arbitrary \( f(R) \) theory have been established [7]. However, of special interest are the cases when a singularity in \( M_E \) maps (due to the properties of \( F(x) \)) to a regular surface in \( M_J \), and then \( M_J \) may be continued in a regular manner beyond this surface (this phenomenon has been named a conformal continuation [7]), and the global properties of the manifold \( M_J \) can be much richer than those of \( M_E \). The new region may, in particular, contain a horizon or another spatial infinity.

From a more general viewpoint, the possible existence of conformal continuations may mean that the observed Universe is only a region of a real, greater Universe which should be described in another, more fundamental conformal frame. Detailed discussions of the physical meaning and role of different conformal frames may be found in Refs. [8, 9].

Necessary and sufficient conditions for the existence of conformal continuations (CC) in static, spherically symmetric solutions of scalar-tensor theories of gravity have been obtained in Ref. [10]. In this paper, a similar problem is solved for \( f(R) \) theories is space-times of arbitrary dimension \( D \geq 3 \), and two specific examples are considered.

2. Field equations

Consider a theory of gravity with the gravitational field action

\[
S_{HOG} = \int d^Dx\sqrt{|g|f(R)}
\]  (1)
where \( f \) is a function of the scalar curvature \( R \) calculated for the metric \( g_{\mu\nu} \) of a space-time \( \mathcal{M}_3 = \mathcal{M}_3[g] \). In accord with the weak field limit \( f \sim R \) at small \( R \), we assume \( f(R) > 0 \) and \( f_R \equiv df/dR > 0 \), at least in a certain range of \( R \) including \( R = 0 \), but admit \( f_R < 0 \) and maybe \( f < 0 \) in general. The vacuum field equations in the theory (1) are fourth-order in \( g_{\mu\nu} \):

\[
(R^\mu_\mu + \nabla_\mu \nabla^\nu - \delta^\nu_\mu \Box) f_R - \frac{1}{2} \delta^\nu_\mu f(R) = 0, \tag{2}
\]

where \( \Box = g^{\mu\nu} \nabla_\mu \nabla_\nu \) and \( f_R \equiv df/dR \).

The conformal mapping \( \mathcal{M}_3 \rightarrow \mathcal{M}_E \) with

\[
g_{\mu\nu} = F(\phi)\overline{g}_{\mu\nu}, \quad F = |f_R|^{-2/(D-2)}, \tag{3}
\]

transforms the “Jordan-frame” action (1) into the Einstein-frame action

\[
S = \int d^Dx \sqrt{|g|} [\Box f + (\partial \phi)^2 - 2V(\phi)] \tag{4}
\]

where

\[
\phi = \pm \sqrt{\frac{D - 1}{D - 2}} \log |f_R|, \tag{5}
\]

\[
2V(\phi) = |f_R|^{-2/(D-2)}(R|f_R| - f). \tag{6}
\]

The field equations due to (1) after this substitution turn into the field equations due to (4). Let us write them down for static, spherically symmetric configurations, taking the metric \( g_{\mu\nu} \) in the form

\[
ds_E^2 = \overline{g}_{\mu\nu} dx^\mu dx^\nu = A(\rho) dr^2 - \frac{d\rho^2}{A(\rho)} - r^2(\rho) d\Omega_2^2, \tag{7}
\]

where \( d\Omega_2^2 \) is the linear element on a sphere \( \mathbb{S}^2 \) of unit radius, and \( \phi = \phi(\rho) \). Three independent combinations of the Einstein equations can be written as

\[
(A_p^2\rho)_\rho = -(4/3)\rho^2 V; \tag{8}
\]

\[
dr^2/\rho = -\phi^2; \tag{9}
\]

\[
A(r^2)_{\rho\rho} - r^2 A_{\rho\rho} + (\overline{d} - 2)r (2A_{\rho r} - A_{rr}) = 2(\overline{d} - 1); \tag{10}
\]

where the subscript \( \rho \) denotes \( d/d\rho \). The scalar field equation \( (Ar^2\phi')' = r^2 V_{\phi} \) follows from the Einstein equations.

Given a potential \( V(\phi) \), (8)–(10) is a determined set of equations for the unknowns \( r, A, \phi \).

The metric \( g_{\mu\nu} = F\overline{g}_{\mu\nu} \) will be taken in a form similar to (7):

\[
ds_J = g_{\mu\nu} dx^\mu dx^\nu = A(q) dt^2 - \frac{dq^2}{A(q)} - r_s^2(q) d\Omega_2^2, \tag{11}
\]

The quantities in (11) and (7) are related by

\[
A(q) = FA(\rho), \quad r_s^2(q) = Fr^2(\rho), \quad dq = \pm F dp. \tag{12}
\]

Three different combinations of Eqs. (2) have the form

\[
df_r r'' + fr'' = 0, \tag{13}
\]

\[
\left[ \frac{\overline{d} - 1}{r_s^2} + \frac{\overline{d} + 2}{2} r_s r_s'B' + \frac{1}{2} r_s^2 B' \right] f_R + \frac{1}{2} r_s^2 B' f_R' = 0, \tag{14}
\]

\[
\left[ B'' r_s^2 + (\overline{d} + 4) B' r_s r_s' + 2(\overline{d} + 1) B (r_s r_s'' + r_s r_{s'}^2) \right] f_R - [2(\overline{d} + 1) B r_s r_s' + B' r_s^2] f_R' + f = 0, \tag{15}
\]

where the prime stands for \( d/dq \) and \( B(q) \equiv A/r_s^2 \). One can notice that Eqs. (14) and (15) [which are the difference \( (\overline{d}) - (\overline{1}) \) and the \( (\overline{1}) \) component of Eqs. (2)] are only third-order with respect to \( g_{\mu\nu} \) while Eq. (13) [the difference \( (\overline{1}) - (\overline{1}) \)] is fourth-order. Eq. (13) is a consequence of (23) and (24).

In both metrics (7) and (11) we have chosen the “quasiglobal” radial coordinates \([11]\) \( (\rho \quad q \), respectively), which are convenient for describing Killing horizons: near a horizon \( \rho = \rho_h \), the function \( A(\rho) \) behaves as \( (\rho - \rho_h)^k \) where \( k \) is the horizon order: \( k = 1 \) corresponds to a simple, Schwarzschild-type horizon, \( k = 2 \) to a double horizon, like that in an extremal Reissner-Nordström black hole etc. The function \( A(q) \) plays a similar role in the metric (11).
3. Conformal continuations: necessary conditions and properties

Let us consider the possible situation when the metric $\mathcal{g}_{\mu\nu}$ is singular at some value of $\rho$ while the metric $g_{\mu\nu}$ at the corresponding value of $q$ is regular. In such a case $\mathcal{M}_1$ can be continued in a regular manner through this surface (to be denoted $\mathcal{S}_{\text{trans}}$), i.e., by definition [7, 10], we have a conformal continuation (CC).

In our case of spherical symmetry, the sphere $\mathcal{S}_{\text{trans}} \in \mathcal{M}_1$ may be either an ordinary sphere, at which both metric coefficients $r, q$ and $A$ are finite (we label such a continuation CC-I), or a Killing horizon at which $r, q$ is finite but $A = 0$ (to be labelled CC-II).

Without loss of generality, we suppose for convenience that at $\mathcal{S}_{\text{trans}}$ the coordinate values are $\rho = 0$ and $q = 0$ and $\rho > 0$ in $\mathcal{M}_E$ outside $\mathcal{S}_{\text{trans}}$. According to (12), we must have, in terms of $\mathcal{g}_{\mu\nu}$,

$$F^{-1} \sim r^2 \rightarrow 0 \quad \text{as} \quad \rho \rightarrow 0,$$

and, in addition, $A(\rho) \sim r^2(\rho)$ for CC-I, whereas for CC-II we must have in $\mathcal{M}_1$: $A(q) \sim q^n$ at small $q$, where $n \in \mathbb{N}$ is the order of the horizon.

Let us use the field equations in $\mathcal{M}_E$ for some further estimates. Eq. (10) may be rewritten in the form

$$\frac{d}{dp} \left( r^D \frac{dB}{dp} \right) = -2(\overline{d} - 1)r^{\overline{d} - 2},$$

where the function $B(\rho) = A/r^2 = B(q) = A/r_s^2$ is invariant under the transformation (3) and should be finite at $\rho = q = 0$. Moreover, since $\mathcal{S}_{\text{trans}}$ is a regular sphere in $\mathcal{M}_1$, $B(q)$ should be a smooth function near $q = 0$, and in its Taylor series expansion the first two terms may be written as

$$B(q) = B_0 + B_1 q^n + \ldots, \quad n \in \mathbb{N},$$

where $B_0 \neq 0$ for CC-I and $B_0 = 0$ for CC-II.

Without knowing the potential $V(\phi)$, we cannot specify the behaviour of the functions $B(\rho)$ and $r(\rho)$, we only have a relation between them given by Eq. (17). There, we should require $1/r^2 = o(1/\rho)$, otherwise (12) would give an infinite value of $q$ at $\rho = 0$. Taking, for simplicity, $r(\rho) \sim \rho^m$, $m > 0$, we then have to require $m < 1/2$. Some general restrictions can be obtained.

Let us first consider 3D gravity, $D = 3$. Then Eq. (17) gives $dB/dp = c_1/r^3, c_1 = \text{const}$. In case $c_1 = 0$ we have $B = \text{const}$ which agrees with (18). If $c_1 \neq 0$, assuming, as before, $r \sim \rho^m$ ($0 < m < 1/2$), we obtain $q \sim \rho^{1-2m}$ and $B(\rho) \approx B_0 + B_1 \rho^{1-3m}$ ($B_0, B_1 = \text{const}$) and have to put $m < 1/3$ in order to have $B - B_0 \rightarrow 0$ as $\rho \rightarrow 0$. Lastly, comparing this expression for $B$ with (18), we arrive at $n = (1-3m)/(1-2m) < 1$, which contradicts (18).

We conclude that for $D = 3$, a CC can only exist if $B = B_0 = 0$.

For $D > 3$, let us again assume $r \sim \rho^m$ ($0 < m < 1/2$), so that $q \sim \rho^{1-2m}$. Then, according to (18), $B - B_0$ behaves at small $\rho$ as $\rho^{m(1-2m)}$. On the other hand, Eq. (17) now gives

$$\frac{d}{dp} \left[ \rho^{Dm+n(1-2m)-1} \right] \sim \rho^{m(\overline{d} - 2)}.$$  

The exponent inside the square brackets is nonzero, since otherwise we would have $n = (1 - Dm)/(1 - 2m) < 1$ for $D > 2$. So, differentiating and comparing the exponents on the two sides of (19), we obtain (recall that $D = \overline{d} + 2$)

$$2 - 4m = n(1 - 2m) \quad \text{whence} \quad n = 2, \quad \text{in agreement with} \quad (18).$$

We conclude that, for $D \geq 4$, a CC is possible with $r \sim \rho^m$ ($0 < m < 1/2$), and in this case the function $B(q)$ behaves near $\mathcal{S}_{\text{trans}}$ as

$$B(q) = B_0 + \frac{1}{4} B_2 q^2 + o(q^2), \quad B_2 \neq 0.$$  

Moreover, substituting $B - B_0$ from (20) to Eq. (17), we see that its left-hand side behaves as $B_2 \rho^{(D-4)m}$, having the sign of $B_2$, whereas its right-hand side has the same $\rho$ dependence but is negative. Therefore we have to put $B_2 < 0$, which means that the function $B(q)$ has a maximum at $q = 0$.

All this was obtained by comparing the metrics $g_{\mu\nu}$ and $\mathcal{g}_{\mu\nu}$, without specifying a theory in which the CC takes place, and for both kinds of transitions, CC-I and CC-II. Both kinds of transitions are thus possible for $D > 3$, and, in particular, in CC-II $\mathcal{S}_{\text{trans}}$ is a double horizon connecting two $T$ regions (since $B = A/r_s^2$ is negative at both sides of $\mathcal{S}_{\text{trans}}$).

In 3D gravity only CC-I are admissible: a horizon, at which $B = 0$ but $B \neq 0$ in its neighbourhood, is inconsistent with the condition $B = \text{const}$.

Now, for the theory (1), the CC conditions can be made more precise. A transition surface $\mathcal{S}_{\text{trans}}$ should correspond to values of $R$ at which the function $F(\phi)$ introduced in the transformation (3) tends to infinity, i.e.,
where \( f_R = 0 \). In this case, according to (16), near \( \rho = 0 \) we have \( f_R^{-2/(d-1)} \sim r^{-2} \), whence \( f_R \sim r^d \). Substituting this to the expression for \( \phi \) in (5), we obtain \( \pm \phi \approx \sqrt{d(d+1)\ln r} \). Then, excluding \( \phi \) from Eq. (9) and integrating it under the initial condition \( r(0) = 0 \), we finally obtain

\[
\rho \approx \text{const} \cdot \rho^{1/D} \quad \text{as} \quad \rho \to 0.
\]

(21)

This result justifies our previous assumption \( r \sim \rho^m \) [moreover, the value \( m = 1/D \) is in the allowed range \((0,1/2)\)]. According to (12), we also obtain

\[
F \sim \rho^{-2/D}, \quad q \sim \rho^{1-2/D} \quad \text{as} \quad \rho \to 0.
\]

(22)

We can now discuss how the curvature \( R \) in \( M_4 \) is changing across \( S_{\text{trans}} \). We know that if \( R = R_0 \) at \( S_{\text{trans}} \), then \( f_R(R_0) = 0 \), but we do not know which is the first nonzero derivative of \( f(R) \) at \( R = R_0 \). An inspection shows that if \( f_{RR}(R_0) \neq 0 \), i.e., \( f_R \sim R - R_0 \sim \rho^{1/D} \sim q \) near \( S_{\text{trans}} \), therefore \( dR/dq \neq 0 \): the curvature changes smoothly and passes from the range \( R < R_0 \) to the range \( R > R_0 \) or vice versa. On the other hand, if we assume that \( f_R \sim (R - R_0)^p, \ p > 1 \), then \( dR/dq \to \infty \) as \( q \to 0 \), i.e., there is no smooth transition. We conclude that a CC is only possible at such values of \( R \) where \( f(R) \) has an extremum with \( f_{RR} \neq 0 \).

The above results hold for both CC-I and CC-II, if any. For CC-I we know, in addition, that \( A(\rho) \sim r^2(\rho) \sim \rho^{2/D} \) at small \( \rho \).

Summing up, we have the following necessary conditions and properties of a CC at \( \rho = q = 0 \) for a static, spherically symmetric configuration in the theory (1) in \( D \geq 3 \) dimensions:

(a) \( f(R) \) has an extremum, at which \( f_R = 0 \) and \( f_{RR} \neq 0 \);

(b) \( dR/dq \neq 0 \) at \( q = 0 \), hence the ranges of the curvature \( R \) are different at the two sides of \( S_{\text{trans}} \);

(c) in the Einstein frame, \( r(\rho) \sim \rho^{1/D} \) as \( \rho \to 0 \);

(d) in the Jordan frame, \( B(q) \) behaves at small \( q \) according to (20) with \( B_2 < 0 \), i.e., has a maximum at \( q = 0 \).

(e) For \( D = 3 \), \( B(\rho) = B(q) = \text{const} \);

(f) A CC-II is only possible for \( D \geq 4 \), and \( S_{\text{trans}} \) is then a double horizon connecting two T regions.

4. Sufficient conditions for conformal continuations

4.1. CC-I: continuations through an ordinary sphere

Let us prove that the above necessary conditions for CC-I are also sufficient. In other words, given a theory (1) with a smooth function \( f(R) \) such that \( f_R = 0 \) and \( f_{RR} \neq 0 \) at some \( R = R_0 \), there exists a solution to the field equations which is smooth in a neighbourhood of the sphere \( S_{\text{trans}} (R = R_0) \).

It is sufficient to show that there is a solution to the field equations (13)–(15) in the form of Taylor series near \( q = 0 \). It is convenient to take \( B(q) \) and \( s(q) = r^2(q) \) as the two unknown metric functions and to find them from Eqs. (14) and (15), rewritten as

\[
\left[4(\bar{d} - 1) + (\bar{d} + 2)B'ss' + 2B''s^2 \right]f_R + 2B's^2 f_R' = 0,
\]

(23)

\[
\left[B''s + \frac{1}{2}(\bar{d} + 4)B's' + (\bar{d} + 1)Bs'' \right]f_R - [(\bar{d} + 1)B's' + B's]f_R' + f = 0;
\]

(24)

as before, the prime denotes \( dq/dq \).

We seek a solution in the form

\[
s(q) = \sum_{k=0}^{\infty} \frac{1}{k!} s_k q^k,
\]

(25)

\[
B(q) = \sum_{k=0}^{\infty} \frac{1}{k!} B_k q^k.
\]

(26)

The scalar curvature \( R \) is a known function of \( B, s \) and their derivatives:

\[
R s = \bar{d}(\bar{d} - 1) - B''s^2 - (\bar{d} + 2)B'ss' - (\bar{d} + 1)Bs'' - \frac{1}{4}\bar{d}(\bar{d} + 1)Bs'^2
\]

(27)
It is helpful, however, to treat \( R(q) \) as one more unknown function and then to use (27) as one more field equation. Accordingly, \( R(q) \) is sought for in the form

\[
R(q) = \sum_{k=0}^{\infty} \frac{1}{k!} R_k q^k.
\]  

(28)

Eqs. (23) and (24) contain the function \( f(R) \) from the original action \( (1) \), which can be specified as a Taylor series near \( R = R_0 \),

\[
f(R) = \sum_{n=0}^{\infty} \frac{1}{n!} f_n (R - R_0)^n,
\]  

(29)

with known coefficients \( f_n \). We assume that at small \( R \) the theory is close to GR, i.e., \( f(R) \) is approximately a linear function with nonzero slope, whereas at a CC we have \( f_R = 0 \). Therefore we assume \( R_0 \neq 0 \).

In accord with the necessary conditions and properties of CCs found in the previous section, we require

\[
s_0 > 0, \quad B_0 \neq 0, \quad B_1 = 0, \quad B_2 < 0, \quad R_0 \neq 0; \quad R_1 \neq 0, \quad f_1 = 0, \quad f_2 \neq 0.
\]  

(30)

In the expressions for \( f(R) \) and its derivatives, one has to single out a series into a series; it is still possible, however, to single out in each order of magnitude \( O(q^n) \) a new term, containing the coefficient \( R_k \) with the largest \( k \). One evidently has \( (R - R_0)^n = (R_1 q^n + o(q^n)) \), therefore it is easy to see that the \( R_n \) with greatest \( n \) appear in \( q \)-expansions from the lowest powers of \( R - R_0 \). The function (29) with substituted \( R(q) \) is represented by the following expansion:

\[
f(R) = f_0 + \frac{1}{2} f_2 (R_1 q + \ldots + \frac{R_n}{n!} q^n + \ldots)^2 + \ldots
\]  

(31)

where \( K_{n-1} \) is a certain combination of \( R_0, \ldots, R_{n-1} \), whose explicit form is insignificant for us.

In a similar way, we obtain for the derivatives of \( f(R) \):

\[
f_R = \sum_{k=1}^{\infty} \frac{1}{k!} f_k (R - R_0)^k = f_2 R_1 q + \frac{1}{2} (f_2 R_2 + f_3 R_1^2) q^2 + \ldots + \left( \frac{f_2}{n!} R_n + K_{n-1} \right) q^n + \ldots,
\]

\[
f_R' = f_2 R_1 + (f_2 R_2 + f_3 R_1^2) q + \ldots + \left( \frac{f_2}{n!} R_{n+1} + K_n \right) q^n + \ldots,
\]  

(32)

and so on, with the same meaning of \( K_{n-1}, K_n \).

Now, let us assume that the following constants (initial data) are given:

\[
s_0 > 0, \quad B_0 \neq 0, \quad B_1 = 0, \quad R_0 \neq 0, \quad R_1 \neq 0.
\]  

(33)

A further consideration splits into four cases.

**1. General case:** \( D > 3, f_0 \neq 0 \). We obtain in the order of magnitude \( O(q^0) \):

\[
(23)[0]: \quad \text{holds automatically};
\]

\[
(24)[0]: \quad f_2 R_1 (\overline{d} + 1) B_0 s_1 = f_0;
\]

\[
(27)[0]: \quad R_0 s_0 + B_2 s_0^2 + (\overline{d} + 1) B_0 s_0 s_2 = \overline{d}(\overline{d} - 1) - \frac{1}{4} \overline{d}(\overline{d} + 1) B_0 s_1^2.
\]

From (24)[0] we express \( s_1 \) in terms of known constants while (27)[0] connects \( B_2 \) and \( s_2 \) with known constants including \( s_1 \) just found.

In the order \( O(q^1) \) we obtain

\[
(23)[1]: \quad B_2 s_0^2 = -\overline{d} - 1,
\]

\[
(24)[1]: \quad (\overline{d} + 1)(f_3 R_1^2 + f_2 R_2) B_0 s_1 = 0,
\]

\[
(27)[1]: \quad B_3 s_0 + (\overline{d} + 1) B_0 s_3 = \ldots,
\]

(34)
where the dots in (27)[1] denote a combination of previously known quantities including \( B_2 \) and \( s_2 \). Note that \( B_2 < 0 \) is obtained automatically.

According to (24)[0], we have \( s_1 \neq 0 \), hence (24)[1] leads to \( f_3 R_1^2 + f_2 R_2 = 0 \), so that \( R_2 \) is expressed in terms of \( R_1 \), \( f_2 \) and \( f_3 \). It also means that the quantity \( f_R'' \) (see (32) is zero at \( q = 0 \).

In further orders \( O(q^n), n > 1 \) the equations give

\[(23)[n] : \quad (n + 1) R_1 B_{n+1} = \ldots, \]
\[(24)[n] : \quad (n - 1) R_1 s_0 B_{n+1} + (n - 1)(\overline{d} + 1) R_1 B_0 s_{n+1} - (d + 1) B_0 s_1 R_{n+1} = \ldots, \]
\[(27)[n - 1] : \quad s_0 B_{n+1} + (\overline{d} + 1) B_0 s_{n+1} = \ldots, \]

(34)

where, as before, the dots denote combinations of known quantities and Taylor coefficients from previous terms of the expansions.

We are now ready to formulate an algorithm for consecutively finding all \( B_n, s_n, R_n \). We have seen that using the four equations (24)[0]–(24)[1] we have found the sought-for coefficients up to number 2. Now, assuming that we know these coefficients up to number \( n \), we obtain \( B_{n+1} \) from (23)[n], then, using it, we find \( s_{n+1} \) from (27)[n−1] and lastly \( R_{n+1} \) from (24)[n].

The existence of this algorithm proves the existence of a smooth solution to Eqs. (13)–(15), i.e., of the conformal continuation.

2. \( D > 3, f_0 = 0 \). We obtain the same relations as in case 1 in the orders \( O(1) \) and \( O(q) \), with the only difference that now Eq. (24)[0] leads to \( s_1 = 0 \), to be taken into account in further equations. Eq. (23)[1] gives \( B_2 \), and after that (27)[0] expresses \( s_2 \) in terms of known quantities, while Eq. (24)[1] holds trivially.

In higher orders of magnitude, \( n \geq 2 \), we obtain the following equations:

\[(23)[n] : \quad s_0^2 (n + 1) R_1 B_{n+1} - (n - 1)(\overline{d} - 1) R_n = \ldots, \]
\[(24)[n] : \quad R_1 s_0 B_{n+1} + (\overline{d} + 1) R_1 B_0 s_{n+1} - s_0 B_2 + (\overline{d} + 1) B_0 s_2 R_n = \ldots, \]
\[(27)[n - 1] : \quad s_0 B_{n+1} + (\overline{d} + 1) B_0 s_{n+1} = \ldots. \]

One can see that, as in case 1, there is a recursive algorithm of finding all \( B_n, s_n, R_n \), provided the expression

\[
s_0 B_2 + (\overline{d} + 1) B_0 s_2 = -R_0 + \frac{1}{s_0} \overline{d}(\overline{d} - 1) \]

is nonzero. Indeed, given \( R_{n-1}, B_n, s_n \) from previous orders, we find a combination of \( B_{n+1} \) and \( s_{n+1} \) from (27)[n−1], substitute it into (24)[n] to obtain \( R_n \), then (23)[n] gives us \( B_{n+1} \), and, knowing it, we lastly obtain \( s_{n+1} \) from (27)[n−1].

If the quantity (35) is zero, we generally obtain two different expressions for the combination in the left-hand side of (27)[n−1], i.e., the set of equations is, in general, inconsistent. We can conclude that the sought-for solution exists in all cases with this exception.

3. \( D = 3, f_0 \neq 0 \). Now, a necessary condition for a CC is that \( B = B_0 = \text{const} \), which considerably simplifies the equations. Thus, (23) now becomes trivial, while the other equations lead to consecutive determination of all \( s_n \) and \( R_n \) just as in case 1.

4. \( D = 3, f_0 = 0 \). The situation is like that in case 2 but simpler. Eq. (23) holds trivially; we again obtain \( s_1 = 0 \) from (24)[0], then \( s_2 \neq 0 \) from (27)[0] and \( R_2 \) from (13)[0]. Then, (24)[1] holds automatically, (27)[1] gives us \( s_3 \), and then, for \( n \geq 2 \), knowing \( R_{n-1} \) and \( s_n \), we obtain the next coefficients \( s_{n+1} \) from Eq. (27)[n−1] and \( R_n \) from (24)[n], containing the combination \( R_1 s_{n+1} - s_2 R_{n} \).

Thus the continuation CC-I exists in all cases described by the necessary conditions of the previous section, with the only exception that \( D > 3 \) and the quantity (35) is zero.

4.2. CC-II: continuations through a horizon

For a CC-II, \( B_0 = 0 \), and the expansion of \( B(q) \) in (25) begins with \( k = 2 \). We now specify the following constants:

\[
s_0 > 0, B_0 = B_1 = 0, R_0 \neq 0, R_1 \neq 0. \]

The existence of a smooth solution to Eqs. (13)- (24), i.e., the existence of a conformal continuation, is proved in a way quite similar to CC-I. Recall that a CC-II can only exist for \( D > 3 \). Besides, CC-II are evidently a phenomenon of very special nature because a double horizon is a very special case of a sphere in a spherically symmetric space-time.
Strictly speaking, the above proofs only provide the existence of solutions in the form of asymptotic series, and it is hard to study their convergence in a general form. We would note, however, that equations with analytical coefficients have, in general, analytical solutions, whose Taylor expansions have finite convergence radii near their regular points.

5. Examples

Consider two simple examples of exact solutions to Eqs. (23), (24), (27) with CC-I continuations in space-time with the dimensions \( D = 3 \) and \( D = 4 \).

Three-dimensional example. In case \( D = 3 \), Eq. (23) holds automatically while (24) and (27) (taking into account that \( f'_{R} = f_{RR}R' \)) are written in the form

\[
2B s'' - 2Bs' R' f_{RR} + f = 0, \quad R = -2Bs'' - Bs'^{2}/2s. \tag{36}
\]

These are two equations for three unknowns \( R, s, f \), so one of them may be taken arbitrarily. Let \( s = q^{k} \) (choosing the units accordingly) with \( k \neq 0, 2 \) (since otherwise we would obtain \( R = \text{const} \)). Then the second equation (36) gives \( R = -\frac{1}{2}Bk(5k - 4)q^{k-2} \) while the first one takes the form

\[
4(k - 2)R^{2} f_{RR} - 4(k - 1)R f_{R} + (5k - 4)f = 0. \tag{37}
\]

Its solution

\[
f = C_{1} R^{x_{1}} + C_{2} R^{x_{2}}, \quad x_{1,2} = \frac{2k - 3 \pm \sqrt{-k^{2} + 2k + 1}}{2(k - 2)} \tag{38}
\]

is real for \((1 - \sqrt{2}) \leq k \leq (1 + \sqrt{2})\). Let us specify the numerical values of the constants: let there be \( k = 12/5 \), then \( x_{1} = 5/2, x_{2} = 2, R = -(48/5)Bq^{2/5} \). Let us take, further, \( C_{1} = (2/5)(-48B/5)^{-3/2}, C_{2} = (48B/5)^{-1} \) (recall that \( B > 0 \)). Then the condition \( f_{R} = 0 \), corresponding to the transition sphere \( \mathbb{S}_{\text{trans}} \) in a CC, holds at \( q = q_{\text{trans}} = 1 \).

The metric in the Jordan and Einstein pictures has the form

\[
ds_{J}^{2} = Bq^{12/5} dt^{2} - B^{-1} q^{-12/5} dq^{2} - q^{12/5} d\Omega^{2}, \quad ds_{E}^{2} = q^{4/5}(q^{1/5} - 1)^{2} ds_{J}. \tag{39}
\]

The Jordan metric is singular at \( q = 0 \), while the Einstein metric is singular at \( q = 0 \) as well as at \( q = q_{\text{trans}} = 1 \). Thus the single manifold \( M_{J} \) corresponds to two manifolds \( M_{E1} \) and \( M_{E2} \): one is described by the values \( q > 1 \), the other by \( 0 < q < 1 \). Let us present the expressions for the scalar field and its potential in the Einstein picture:

\[
\phi = \pm \sqrt{2} \ln |q^{3/5} - q^{2/5}|, \quad V = B(2.4q^{4/5} - 2.88q)/(q^{3/5} - q^{2/5})^{3}. \tag{40}
\]

Both quantities are monotonic in the regions \( 0 < q < 1 \) and \( q > 1 \), so that the function \( V(\phi) \) is determined.

Four-dimensional example. One of the solutions to Eqs. (23), (24), (27) at \( D = 4 \) is given by the functions

\[
f = -aeR + 2c\sqrt{R} = 2c/q - ac/q^{2}, \quad s = q^{2}, \quad B = (3q - 2a)/6q^{3}, \quad R = 1/q^{2}, \tag{41}
\]

where \( a, c \sim \text{const} > 0 \). Let for convenience \( a = 1, c = 1 \) (choosing the appropriate units). Then \( f_{R} = 0 \) at \( q = q_{\text{trans}} = 1 \).

The Jordan and Einstein metrics are

\[
ds_{J}^{2} = \left( \frac{1}{2} - \frac{1}{3q} \right) dt^{2} - \left( \frac{1}{2} - \frac{1}{3q} \right)^{-1} dq^{2} - q^{2} d\Omega^{2}, \quad ds_{E}^{2} = |q - 1| ds_{J}. \tag{42}
\]

Thus the Jordan metric has a form close to Schwarzschild’s, it is singular at the centre \( q = 0 \) and has a horizon at \( q = 2/3 \). Its asymptotic is non-flat due to a solid angle deficit equal to \( 2\pi \), i.e., it has the same nature as the asymptotic of a global monopole (as can be easily seen by changing the coordinates from \( t \) and \( q \) to \( \tilde{t} = t/\sqrt{2} \) and \( \tilde{q} = q/\sqrt{2} \)). In \( M_{E} \), the metric is singular at \( q = 0 \) and \( q = 1 \) and contains a horizon at \( q = 2/3 \). The manifold \( M_{J} \) again has two Einstein counterparts \( M_{E1} \) and \( M_{E2} \): separately for \( q > 1 \) and \( q < 1 \). The first of them has a non-flat asymptotic as \( q \rightarrow \infty \) and a naked singularity at the centre \( (q = 1) \), the other has two singular centres at \( q = 0 \) and \( q = 1 \), separated by a horizon at \( q = 2/3 \).

The scalar field and its potential in \( M_{E} \) have the form

\[
\phi = \pm \sqrt{3/2} \ln |q - 1|, \quad V = -\frac{3}{2} q^{-1} (q - 1)^{-2}. \tag{43}
\]

The above examples are of methodological nature and demonstrate essential distinctions between the descriptions of the theory in the Jordan and Einstein pictures when there is a conformal continuation.
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