CLASSIFICATION OF FUSION CATEGORIES OF DIMENSION $pq$

PAVEL ETINGOF, SHLOMO GELAKI, AND VIKTOR OSTRIK

1. Introduction

A fusion category over $\mathbb{C}$ is a $\mathbb{C}$-linear semisimple rigid tensor category with finitely many simple objects and finite dimensional spaces of morphisms, such that the neutral object is simple (see [ENO]). To every fusion category, one can attach a positive number, called the Frobenius-Perron (FP) dimension of this category ([ENO, Section 8]). It is an interesting and challenging problem to classify fusion categories of a given FP dimension $D$. This problem is easier if $D$ is an integer, and for integer $D$ its complexity increases with the number of prime factors in $D$. Specifically, fusion categories of FP dimension $p$ or $p^2$ where $p$ is a prime were classified in [ENO, Section 8]. The next level of complexity is fusion categories of FP dimension $pq$, where $p < q$ are distinct primes. In this case, the classification has been known only in the case when the category admits a fiber functor, i.e. is a representation category of a Hopf algebra ([GW, EG1]).

In this paper we provide a complete classification of fusion categories of FP dimension $pq$, thus giving a categorical generalization of [EG1]. As a corollary we also obtain the classification of semisimple quasi-Hopf algebras of dimension $pq$. A concise formulation of our main result is:

Theorem 1.1. Let $\mathcal{C}$ be a fusion category over $\mathbb{C}$ of FP dimension $pq$, where $p < q$ are distinct primes. Then either $p = 2$ and $\mathcal{C}$ is a Tambara-Yamagami category of dimension $2q$ ([TY]), or $\mathcal{C}$ is group-theoretical in the sense of [ENO].

The organization of the paper is as follows.

In Section 2 we recall from [ENO] and [O] some facts about fusion categories that will be used below.

In Section 3 we classify fusion categories $\mathcal{C}$ of dimension $pq$ which contain objects of non-integer dimensions. They exist only for $p = 2$ and are the Tambara-Yamagami categories [TY]; there are four such categories for each $q$.

In Section 4 we classify fusion categories of dimension $pq$ where all simple objects are invertible. This reduces to computing the cohomology groups $H^3(G, \mathbb{C}^*)$, where $G$ is a group of order $pq$.

In Section 5 we deal with the remaining (most difficult) case, when $\mathcal{C}$ has integer dimensions of simple objects, but these dimensions are not all equal to 1. We show, by generalizing the methods of [EG1], that in this case $q - 1$ is divisible by $p$, and the simple objects of $\mathcal{C}$ are $p$ invertible objects and $\frac{q - 1}{p}$ objects of dimension $p$.

In Section 6, we classify fusion categories $\mathcal{C}$ whose simple objects are $p$ invertible objects and $\frac{q - 1}{p}$ objects of dimension $p$. Namely, we show that they are group-theoretical in the sense of [ENO], which easily yields their full classification.

As a by-product, the method of Section 6 yields a classification of finite dimensional semisimple quasi-Hopf (in particular, Hopf) algebras whose irreducible
representations have dimensions 1 and \( n \), such that the 1-dimensional representations form a cyclic group of order \( n \). All such quasi-Hopf algebras turn out to be group-theoretical. This is proved in Section 7. We also classify fusion categories whose invertible objects form a cyclic group of order \( n > 1 \) and which have only one non-invertible object of dimension \( n \).

We note that all constructions in this paper are done over the field of complex numbers.

Acknowledgments. The first author was partially supported by the NSF grant DMS-9988796. The first author partially conducted his research for the Clay Mathematics Institute as a Clay Mathematics Institute Prize Fellow. The second author’s research was supported by Technion V.P.R. Fund - Dent Charitable Trust- Non Military Research Fund, and by THE ISRAEL SCIENCE FOUNDATION (grant No. 70/02-1). The third author’s work was partially supported by the NSF grant DMS-0098830.

2. Preliminaries

Let \( \mathcal{C} \) be a fusion category over \( \mathbb{C} \) and let \( K(\mathcal{C}) \) denote its Grothendieck ring. Let \( \text{Irr}(\mathcal{C}) \) be the (finite) set of isomorphism classes of simple objects in \( \mathcal{C} \) and let \( V \in \text{Irr}(\mathcal{C}) \).

Definition 2.1. \[ \text{ENO} \]

(i) The Frobenius-Perron (FP) dimension of \( V \), \( \text{FPdim}(V) \), is the largest positive eigenvalue of the matrix of multiplication by \( V \) in \( K(\mathcal{C}) \).

(ii) The FP dimension of \( \mathcal{C} \) is \( \text{FPdim}(\mathcal{C}) = \sum_{V \in \text{Irr}(\mathcal{C})} \text{FPdim}(V)^2 \).

Let \( Z(\mathcal{C}) \) be the Drinfeld center of \( \mathcal{C} \). Then by Proposition 8.12 in \[ \text{ENO} \], \( \text{FPdim}(Z(\mathcal{C})) = \text{FPdim}(\mathcal{C})^2 \).

If \( \mathcal{C} \) is a full tensor subcategory of \( \mathcal{D} \) then \( \text{FPdim}(\mathcal{D})/\text{FPdim}(\mathcal{C}) \) is an algebraic integer, so in particular, if the two dimensions are integers then the ratio is an integer (\[ \text{ENO} \], Proposition 8.15). Moreover, if \( \mathcal{M} \) is a full module subcategory of \( \mathcal{D} \) over \( \mathcal{C} \), then the same is true about \( \text{FPdim}(\mathcal{M})/\text{FPdim}(\mathcal{C}) \), where \( \text{FPdim}(\mathcal{M}) \) is the sum of squares of the Frobenius-Perron dimensions of simple objects of \( \mathcal{M} \) (\[ \text{ENO} \], Remark 8.17).

By Proposition 8.20 in \[ \text{ENO} \], if \( \mathcal{C} = \bigoplus_{g \in G} C_g \) is faithfully graded by a finite group \( G \) then \( \text{FPdim}(C_g) \) are equal for all \( g \in G \) and \( |G| \) divides \( \text{FPdim}(\mathcal{C}) \).

There is another notion of a dimension for fusion categories \( \mathcal{C} \); namely, their global dimension \( \text{dim}(\mathcal{C}) \) (see \[ \text{ENO} \], Section 2). The global dimension of \( \mathcal{C} \) may be different from its FP dimension. However, categories \( \mathcal{C} \) for which these two dimensions coincide are of great interest. They are called pseudounitary \[ \text{ENO} \]. One of the main properties of pseudounitary categories is that they admit a unique pivotal structure. This extra structure allows one to define categorical dimensions of simple objects of \( \mathcal{C} \), which by Proposition 8.23 in \[ \text{ENO} \], coincide with their FP dimensions. One instance in which it is guaranteed that \( \mathcal{C} \) is pseudounitary is when \( \text{FPdim}(\mathcal{C}) \) is an integer (Proposition 8.24 in \[ \text{ENO} \]). If \( \mathcal{C} \) has integer FP dimension, then the dimensions of simple objects in \( \mathcal{C} \) are integers and square roots of integers (\[ \text{ENO} \], Proposition 8.27).

An important special case of fusion categories with integer FP dimension are categories in which the FP dimensions of all simple objects are integers. It is well known (see e.g. Theorem 8.33 in \[ \text{ENO} \]) that this happens if and only if
the category is equivalent to \( \text{Rep}(H) \) where \( H \) is a finite dimensional semisimple quasi-Hopf algebra.

By Corollary 8.30 in [ENO], if \( \text{FPdim}(C) \) is equal to a prime number \( p \), then \( C \) is equivalent \( \text{Rep}(\text{Fun}(\mathbb{Z}/p\mathbb{Z})) \) with associativity defined by a cocycle \( \xi \) representing \( \omega \in H^3(\mathbb{Z}/p\mathbb{Z}, \mathbb{C}^*) = \mathbb{Z}/p\mathbb{Z} \).

An important class of fusion categories with integer FP dimensions of simple objects is the class of \textit{group theoretical fusion categories}, introduced and studied in [ENO], [O]. These are categories associated with quadruples \( (G, B, \xi, \psi) \), where \( G \) is a finite group, \( B \) is a subgroup of \( G \), \( \xi \in Z^3(G, \mathbb{C}^*) \), and \( \psi \in C^2(B, \mathbb{C}^*) \) such that \( \xi_B = d\psi \). Namely, let \( \text{Vec}_{G, \xi} \) be the category of finite dimensional \( G \)-graded vector spaces with associativity defined by \( \xi \). Let \( \text{Vec}_{G, \xi}(B) \) be the subcategory of \( \text{Vec}_{G, \xi} \) of objects graded by \( B \). Consider the twisted group algebra \( A := \mathbb{C}^\psi[B] \). It is an associative algebra in \( \text{Vec}_{G, \xi}(B) \), since \( \xi_B = d\psi \). Then \( \text{C}(G, B, \xi, \psi) \) is defined to be the category of \( A \)-bimodules in \( \text{Vec}_{G, \xi} \). Such a category is called \textit{group-theoretical}.

Note that the data \( (\xi, \psi) \) is not uniquely determined by the category. Namely, there are two transformations of \( (\xi, \psi) \) which leave the category unchanged:

1) \( \xi \to \xi + d\phi, \psi \to \psi + \phi_B, \phi \in C^2(G, \mathbb{C}^*) \); and
2) \( \xi \to \xi, \psi \to \psi + d\eta, \eta \in C^1(B, \mathbb{C}^*) \).

Thus the essential data is the cohomology class of \( \xi \) (which must vanish when restricted to \( B \)) and an element \( \psi \) of a principal homogeneous space (torsor) \( T_\xi \) over the group \( \text{Coker}(H^2(G, \mathbb{C}^*) \to H^2(B, \mathbb{C}^*)) \) (ENO, Remark 8.39).

Proposition 8.42 in [ENO] gives a simple characterization of group-theoretical fusion categories. Namely, a fusion category \( C \) is group-theoretical if and only if it is dual to a pointed category (= category whose all simple objects are invertible) with respect to some indecomposable module category over \( C \). For the definitions of all unfamiliar terms we refer the reader to [ENO], [O].

3. Categories with non-integer dimensions

Throughout the paper we will consider a fusion category \( C \) over \( \mathbb{C} \) of FP dimension \( pq \), where \( p \) and \( q \) are distinct primes, such that \( p < q \). As explained in Section 2, such a category is pseudo-unitary, and hence admits a canonical pivotal structure, in which categorical dimensions of objects coincide with their FP dimensions. Thus from now on we will refer to FP dimensions simply as “dimensions”.

By Proposition 8.27 in [ENO], the dimensions of simple objects in \( C \) may be integers or square roots of integers.

**Theorem 3.1.** If \( C \) contains a simple object whose dimension is not an integer then \( p = 2 \) and \( C \) is equivalent to a Tambara-Yamagami category of dimension \( 2q \) [TY].

**Proof.** Let \( C_{ad} \) be the full tensor subcategory of \( C \) generated by the constituents in \( X \otimes X^* \in C \), for all simple \( X \in C \). By Proposition 8.27 in [ENO], the dimensions of objects in \( C_{ad} \) are integers. Since \( C \) has objects whose dimension is not an integer, \( C \neq C_{ad} \). Thus by Proposition 8.15 in [ENO], the dimension \( d \) of \( C_{ad} \) is an integer dividing \( pq \) and less than \( pq \), so it is either 1 or \( p \) or \( q \). If \( d = 1 \) then all objects of \( C \) are invertible, contradiction. Thus \( d = p \) or \( q \).

By Proposition 8.30 in [ENO], \( C_{ad} \) is \( \text{Rep}(\text{Fun}(\mathbb{Z}/d\mathbb{Z})) \) with associativity defined by a 3-cocycle (i.e. the simple objects are \( \chi^i, i = 0, \ldots, d - 1 \)).
Let \( C' \) be the full subcategory of \( C \) consisting of objects with integer dimension. We claim that it is a tensor subcategory of \( C \). Indeed, if \( X \oplus Y \) has integer dimension, then so do \( X \) and \( Y \) (sum of square roots of positive integers is an integer only if each summand is an integer).

Thus, \( C' \) is a proper tensor subcategory of \( C \) containing \( C_{ad} \). So by Proposition 8.15 in [ENO], \( C' \) coincides with \( C_{ad} \).

Now, let \( L \in C \) be a simple object, such that \( L \notin C_{ad} \). Then one has \( L \otimes L^* = \bigoplus_i \chi^i \). Indeed, since \( \dim(L) > 1 \), \( L \otimes L^* \) must contain simple objects other than \( 1 \) (which it contains with multiplicity 1). But \( L \otimes L^* \in C_{ad} \), so the other constituents could only be \( \chi^i \), and they can only occur with multiplicity 1.

Thus, the dimension of \( L \) is \( \sqrt{d} \), and \( \chi^i \otimes L = L \otimes \chi^i = L \).

Let us now show that \( L \) is unique. Let \( L, M \) be two such simple objects (in \( C \) but not in \( C_{ad} \)). Then \( M \otimes L^* \) has dimension \( d \), so it lies in \( C_{ad} \). Hence, \( \chi^i \) occurs in \( M \otimes L^* \) for some \( i \). So \( \chi^i \otimes L = M \), and hence \( L = M \).

Thus, \( p = 2, d = q \), and the dimension of \( C \) is \( 2q \). Moreover, the simple objects of \( C \) are \( q \) invertible objects and one object of dimension \( \sqrt{q} \). Hence \( C \) is a Tambara-Yamagami category, as desired.

\[ \square \]

4. Categories with 1-dimensional simple objects

It is well known [K] that fusion categories with 1-dimensional simple objects are classified by pairs \((G, \omega)\), where \( G \) is a finite group, and \( \omega \in H^3(G, \mathbb{C}^*) \). Namely, the category \( \mathcal{C}(G, \omega) \) attached to such a pair is the category of representations of the function algebra \( \text{Fun}(G) \) with associativity defined by a cocycle \( \xi \) representing \( \omega \). Thus, in order to classify such categories of dimension \( pq \), it is sufficient to classify pairs \((G, \omega)\) with \( |G| = pq \).

There are two kinds of groups of order \( pq \): the cyclic group \( G = \mathbb{Z}/pq\mathbb{Z} \), and the nontrivial semidirect product \( G = \mathbb{Z}/p\mathbb{Z} \rtimes \mathbb{Z}/q\mathbb{Z} \) (which exists and is unique if and only if \( q - 1 \) is divisible by \( p \)). In the first case, it is well known that \( H^3(G, \mathbb{C}^*) = \mathbb{Z}/pq\mathbb{Z} \). So it remains to consider the second case, \( G = \mathbb{Z}/p\mathbb{Z} \rtimes \mathbb{Z}/q\mathbb{Z} \), where \( q - 1 \) is divisible by \( p \), and the action of \( \mathbb{Z}/p\mathbb{Z} \) on \( \mathbb{Z}/q\mathbb{Z} \) is nontrivial.

**Lemma 4.1.** For \( i > 0 \) we have \( H^i(G, \mathbb{C}^*) = H^i(\mathbb{Z}/pq\mathbb{Z}, \mathbb{C}^*) \oplus H^i(\mathbb{Z}/q\mathbb{Z}, \mathbb{C}^*)/\mathbb{Z}/p\mathbb{Z} \).

**Proof.** The Hochschild-Serre spectral sequence has second term \( E_2^{ij} = H^j(\mathbb{Z}/p\mathbb{Z}, H^i(\mathbb{Z}/q\mathbb{Z}, \mathbb{C}^*)) \). For any \( j > 0 \), \( H^j(\mathbb{Z}/q\mathbb{Z}, \mathbb{C}^*) \) is a \( q \)-group and thus \( E_2^{ij} = 0 \) if both \( i,j \) are nonzero. Moreover, all differentials are zero since they would map \( q \)-groups to \( p \)-groups. Thus the spectral sequence collapses and the lemma is proved. \[ \square \]

**Corollary 4.2.** One has \( H^3(G, \mathbb{C}^*) = H^3(\mathbb{Z}/p\mathbb{Z}, \mathbb{C}^*) = \mathbb{Z}/p\mathbb{Z} \) if \( p \neq 2 \) and \( H^3(G, \mathbb{C}^*) = H^3(\mathbb{Z}/2\mathbb{Z}, \mathbb{C}^*) \oplus H^3(\mathbb{Z}/q\mathbb{Z}, \mathbb{C}^*) = \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/q\mathbb{Z} \) if \( p = 2 \).

**Proof.** By Lemma 4.1, it is enough to check that the action of \( \mathbb{Z}/p\mathbb{Z} \) on \( H^3(\mathbb{Z}/q\mathbb{Z}, \mathbb{C}^*) \) is nontrivial for odd \( p \) and trivial for \( p = 2 \). For this, observe that \( H^3(\mathbb{Z}/q\mathbb{Z}, \mathbb{C}^*) = H^2(\mathbb{Z}/q\mathbb{Z}, \mathbb{Z}) \otimes H^1(\mathbb{Z}/q\mathbb{Z}, \mathbb{C}^*) = (H^1(\mathbb{Z}/q\mathbb{Z}, \mathbb{C}^*))^{\otimes 2} = (\text{Hom}(\mathbb{Z}/q\mathbb{Z}, \mathbb{C}^*))^{\otimes 2} \) as \( \mathbb{Z}/p\mathbb{Z} \)-modules, and the claim is proved. \[ \square \]

5. Categories with integer dimensions, not all equal to 1

In this section we will prove the following result.
Theorem 5.1. Let $\mathcal{C}$ be a fusion category of dimension $pq$ with integer dimensions of simple objects, not all equal to 1. Then $q - 1$ is divisible by $p$, and the simple objects of $\mathcal{C}$ are invertible or have dimension $\frac{q - 1}{p}$ objects of dimension $p$.

The proof of this theorem occupies the rest of the section.

In the proof we will assume that $pq \neq 6$, because in the case $pq = 6$ the theorem is trivial.

By Theorem 8.33 in [ENO], $\mathcal{C}$ is equivalent to $\text{Rep}(H)$, where $H$ is a finite dimensional semisimple quasi-Hopf algebra. Let $Z(\mathcal{C})$ be the Drinfeld center of $\mathcal{C}$, then $Z(\mathcal{C})$ is equivalent to $\text{Rep}(D(H))$, where $D(H)$ is the double of $H$ [HN].

Lemma 5.2. The simple objects in $\text{Rep}(D(H))$ have dimension 1, $p$ or $q$.

Proof. Since $\mathcal{C}$ is pivotal, $Z(\mathcal{C})$ is modular, and the result follows from Lemma 1.2 in [EG2] (see also Proposition 3.3 in [ENO]). \qed

Lemma 5.3. $D(H)$ admits nontrivial 1-dimensional representations; i.e., the group of grouplike elements $G(D(H)^*)$ of the coalgebra $D(H)^*$ is nontrivial.

Proof. Assume the contrary. Let $m$ be the number of $p$-dimensional representations of $D(H)$, and $n$ the number of its $q$-dimensional representations. Then by Lemma 82 one has $1 + mp^2 + nq^2 = p^2q^2$, which implies that $m > 0$.

Let $V$ be a $p$-dimensional irreducible representation of $D(H)$. Then $V \otimes V^*$ is a direct sum of the trivial representation $\mathbb{C}$, a $p$-dimensional irreducible representations of $D(H)$ and $b$ $q$-dimensional irreducible representations of $D(H)$. Therefore, we have: $p^2 = 1 + ap + bq$. Clearly $b > 0$. Let $W$ be a $q$-dimensional irreducible representation of $D(H)$ such that $W \subset V \otimes V^*$. Since $0 \neq \text{Hom}_{D(H)}(V \otimes V^*, W) = \text{Hom}_{D(H)}(V, W \otimes V)$, we have that $V \subset W \otimes V$. Since $W \otimes V$ has no 1-dimensional constituent (because $\dim V \neq \dim W$), $\dim(W \otimes V) = pq$ and $W \otimes V$ contains a $p$-dimensional irreducible representation of $D(H)$, from dimension counting it follows that $W \otimes V = V_1 \oplus \cdots \oplus V_q$ where $V_i$ is a $p$-dimensional irreducible representation of $D(H)$ with $V_1 = V$.

We wish to show that for any $i = 1, \ldots, q$, $V_i = V$. Suppose on the contrary that this is not true for some $i$. Then $V \otimes V_i^*$ has no 1-dimensional constituent, hence it must be a direct sum of $p$ $p$-dimensional irreducible representations of $D(H)$. Therefore, $W \otimes (V \otimes V_i^*)$ has no 1-dimensional constituent. But, $(W \otimes V) \otimes V_i^* = (V_1 \oplus \cdots \oplus V_q^*)^* = \mathbb{C} \oplus \cdots$, which is a contradiction.

Therefore, $W \otimes V = qV$. Hence $\text{Hom}_{D(H)}(V \otimes V^*, W)$ is a q-dimensional space, i.e. $p^2 = \dim(V \otimes V^*) \geq q^2$, contradiction. \qed

Lemma 5.4. (i) The natural map $\tau : G(D(H)^*) \rightarrow G(H^*)$ is injective.

(ii) $|G(D(H)^*)| = p$ or $q$, and thus $\tau$ is an isomorphism.

Proof. Assume the contrary. Then there is a non-trivial cyclic subgroup $L$ in $G(D(H)^*)$ which maps trivially to $G(H^*)$. This means that the category $\mathcal{C}$ is faithfully $L^\gamma$-graded, by Proposition 5.10 in [ENO]. So, $\mathcal{C}$ is a direct sum of $\mathcal{C}_\gamma$, $\gamma \in L^\gamma$, and the dimension of $\mathcal{C}_\gamma$ is $s := pq/|L|$, which is 1 or $p$ or $q$. If $s = 1$, all simple objects of $\mathcal{C}$ are invertible, which is a contradiction. If $s = p$ or $s = q$ then $\mathcal{C}_0$ is a fusion category of prime dimension. So by Proposition 8.30 in [ENO], $\mathcal{C}_0 = \mathcal{C} / s \mathbb{Z}$, and the $\mathcal{C}_\gamma$‘s are module categories over it. If $\mathcal{C}_\gamma$ has a non-1-dimensional object $V$ then $\gamma \otimes V = V \otimes \gamma = V$ for $\chi \in \mathcal{C}_0$ (otherwise, dim($\mathcal{C}_\gamma$)
Lemma 5.6. desired.

\[\chi\] is a 1-dimensional representation of \(D\). By Lemma 5.4 (i), this is zero if \(\chi\) is a 1-dimensional representation of \(D\). Therefore we have that \(\chi\) is a 1-dimensional representation of \(D\), which is a contradiction. So, by Lemma 5.4 (ii), \(m = p\), as desired.

In this case, \(\dim(J(\chi)) = q\), and \(J(\chi)\) must be \(\chi\) plus sum of \(p\)-dimensional simple modules, whose number is then \((q - 1)/p\). Thus, \(q - 1\) is divisible by \(p\), as desired.

\[\square\]

Lemma 5.6. \(|G(D(H)^*)| = p\), and \(q - 1\) is divisible by \(p\).

Proof. Set \(m := |G(D(H)^*)|\). Let \(D \subset C\) be the subcategory generated by the invertible objects in \(C\) (it is of dimension \(m\)). Then we have \(C \subset C \boxtimes D^{op} \subset C \boxtimes C^{op}\). Taking the dual of this sequence with respect to the module category \(C\) (see [ENO], Sections 5 and 8), we get a sequence of surjective functors \(Z(C) \to E \to C\), where the dimension of \(E\) is \(mpq\). We can think of the category \(E\) as representations of a quasi-Hopf subalgebra \(B \subset D(H)\), containing \(H\), of dimension \(mpq\).

Let \(\chi\) be a 1-dimensional representation of \(D(H)\). Let \(J(\chi) := D(H) \otimes_B \chi\) be the induced module. By Schauenburg’s freeness theorem [SI] (see also [ENO], Corollary 8.9), it has dimension \(pq/m\).

For any two 1-dimensional representations \(\chi, \chi'\) of \(D(H)\), we have

\[\Hom_{D(H)}(J(\chi), \chi') = \Hom_B(\chi, \chi').\]

By Lemma 5.4 (i), this is zero if \(\chi \neq \chi'\) and \(C\) if \(\chi = \chi'\). Thus the only 1-dimensional constituent of \(J(\chi)\) as a \(B\)-module is \(\chi\), and it occurs with multiplicity 1.

Assume that \(m = q\). Then the dimension of \(J(\chi)\) is \(p\). Since other constituents of \(J(\chi)\) can only have dimensions 1, \(p, q\), and \(p < q\), we get that \(J(\chi)\) is a sum of characters \(\chi'\) of \(D(H)\), which is a contradiction. So, by Lemma 5.4 (ii), \(m = p\), as desired.

In this case, \(\dim(J(\chi)) = q\), and \(J(\chi)\) must be \(\chi\) plus sum of \(p\)-dimensional simple modules, whose number is then \((q - 1)/p\). Thus, \(q - 1\) is divisible by \(p\), as desired.

\[\square\]

Lemma 5.6. Let \(V, U\) be \(p\)-dimensional representations of \(D(H)\). If \(V \otimes U\) contains a 1-dimensional representation \(\chi\) of \(D(H)\), then it contains another 1-dimensional representation.

Proof. Without loss of generality we can assume that \(\chi\) is trivial and hence that \(U = V^*\). Otherwise we can replace \(U\) with \(U \otimes \chi^{-1}\).

Suppose on the contrary that \(V \otimes V^*\) does not contain a non-trivial 1-dimensional representation. Then \(V \otimes V^*\) is a direct sum of the trivial representation \(C\), \(p\)-dimensional irreducible representations of \(D(H)\) and \(q\)-dimensional irreducible representations of \(D(H)\). Therefore we have that \(p^2 = 1 + ap + bq\). Since by Lemma 5.4 (i), \(q - 1\) is divisible by \(p\), we find that \(b + 1\) is divisible by \(p\). So \(b \geq p - 1\), and hence \(p^2 \geq 1 + (p - 1)q\), i.e. \(p + 1 \geq q\). Thus \(p = 2, q = 3\), and \(pq = 6\). But we assumed that it is not the case, so we have a contradiction.

\[\square\]

Lemma 5.7. The algebra \(D(H)\) has \(p^2 - p\) \(q\)-dimensional irreducible representations and \((q^2 - 1)/p\) \(p\)-dimensional irreducible representations. Moreover, the direct sums of 1-dimensional and \(p\)-dimensional irreducible representations form a tensor subcategory \(\mathcal{F}\) in \(\text{Rep}(D(H))\) of dimension \(pq^2\).

Proof. Let \(a\) and \(b\) be the numbers of \(p\)-dimensional and \(q\)-dimensional irreducible representations of \(D(H)\). Then by Lemma 5.4 (i), \(ap^2 + bq^2 = p^2q^2 - p\). This
equation clearly has a unique nonnegative integer solution \((a, b)\). By Lemma 5.6 (ii), this solution is \(a = (q^2 - 1)/p, b = p^2 - p\).

Let us now prove the second statement. We wish to show that if \(V\) and \(U\) are two irreducible representations of \(D(H)\) of dimension \(p\) then \(V \otimes U\) is a direct sum of 1-dimensional irreducible representations of \(D(H)\) and \(p\)-dimensional irreducible representations of \(D(H)\) only. Indeed, by Lemma 5.6 either \(V \otimes U\) does not contain any 1-dimensional representation or it must contain at least two different 1-dimensional representations. But if it contains two different 1-dimensional representations, then (since the 1-dimensional representations of \(D(H)\) form a cyclic group of order \(p\)) \(V \otimes U\) contains all the \(p\) 1-dimensional representations of \(D(H)\).

We conclude that either \(p^2 = mp + nq\) or \(p^2 = p + mp + nq\). At any rate \(n = 0\), and the result follows.

Therefore, the subcategory \(\text{Rep}(D(H))\) generated by the 1 and \(p\)-dimensional irreducible representations of \(D(H)\) is the representation category of a quotient quasi-Hopf algebra \(A\) of \(D(H)\) of dimension \(pq^2\).

**Lemma 5.8.** The composition map \(H \to D(H) \to A\) is injective. Thus \(H\) is a quasi-Hopf subalgebra in \(A\).

**Proof.** Assume that the composition map is not injective. Then the image of this map is a nontrivial quotient of \(H\). The image definitely contains the subalgebra \(A_0\) in \(A\) corresponding to the invertible objects. This quasi-Hopf subalgebra is \(p\)-dimensional, while \(H\) is \(pq\)-dimensional, so by Schauenburg’s theorem \[S2\], see also [ENO], Proposition 8.15, the image must coincide with \(A_0\). On the other hand, by Schauenburg freeness theorem \[S1\], \(D(H)\) is a free left \(H\)-module of rank \(pq\). Since the projection \(D(H) \to A\) is a morphism of left \(H\)-modules, we find that \(A\) is generated by \(pq\) elements as a left \(A_0\)-module. Hence, the dimension of \(A\) is at most \(p^2q\). On the other hand, we know that this dimension is \(pq^2\), a contradiction. □

**Lemma 5.9.** Any irreducible representation \(V\) of \(H\) which is not 1-dimensional has dimension \(p\).

This lemma clearly completes the proof of the theorem.

**Proof.** It is clear from Lemma 5.8 that this dimension is at most \(p\) (as any simple \(H\)-module occurs as a constituent in a simple \(A\)-module). On the other hand, we claim that \(V\) is stable under tensoring with 1-dimensional representations. Indeed, assume not, and let \(W\) be an irreducible representation of \(A\) whose restriction to \(H\) contains \(V\). Since \(W\) is \(p\)-dimensional, and contains all \(\chi^j \otimes V\) (where \(\chi\) is a non-trivial 1-dimensional representation of \(H\)), by dimension counting we get that \(p \geq p \dim(V)\), a contradiction.

But now by Remark 8.17 in [ENO], the dimension of \(V\) is divisible by \(p\). We are done. □

### 6. Categories with integer dimensions

In this section we will prove that any fusion category of dimension \(pq\) with integer dimensions of objects is group theoretical, and will classify such categories. Before we do so, we need to prove two lemmas.

Lemma 6.1. Let $\mathcal{C}$ be a fusion category and let $A \in \mathcal{C}$ be an indecomposable semisimple algebra. Then for any right $A$–module $M$ and left $A$–module $N$ one has $\text{FPdim}(M \otimes_A N) = \frac{\text{FPdim}(M)\text{FPdim}(N)}{\text{FPdim}(A)}$.

Proof. Let $M_i, i \in I$ be the collection of simple right $A$–modules and let $N_j, j \in J$ be the collection of simple left $A$–modules. It is clear that it is enough to prove the lemma for $M = M_i$ and $N = N_j$. Note that the vector $\text{FPdim}(M_i)$ (resp. $\text{FPdim}(N_j)$) is the Frobenius-Perron eigenvector (see [ENO]) for the module category of right $A$–modules (resp. left $A$–modules). For any left $A$–module $N$ the vector $\text{FPdim}(M_i \otimes_A N)$ is also the Frobenius-Perron eigenvector and thus is proportional to $\text{FPdim}(M_i)$. Similarly, for any right $A$–module $M$ the vector $\text{FPdim}(M \otimes_A N_j)$ is proportional to $\text{FPdim}(N_j)$. Thus $\text{FPdim}(M_i \otimes_A N_j) = \alpha \text{FPdim}(M_i)\text{FPdim}(N_j)$ for some constant $\alpha$. Finally, by choosing $M = N = A$ we find out that $\alpha = 1/\text{FPdim}(A)$. The lemma is proved. 

Let $n > 1$ be an integer. Let $\mathcal{C}$ be a fusion category and $\chi \in \mathcal{C}$ be a nontrivial invertible object such that $\chi^n = 1$. Assume that $\mathcal{C}$ contains a simple object $V$ such that $\chi \otimes V = V \otimes \chi = V$. This implies that $A := 1 \oplus \chi \oplus \chi^2 \oplus \ldots \oplus \chi^{n-1}$ has a unique structure of a semisimple algebra in $\mathcal{C}$. Indeed, the existence of $V$ implies that there is a fiber functor (= module category with one simple object $V$) on the category generated by $\{\chi^i\}$, i.e. the 3-cocycle of this category is trivial and thus it is the representation category of the Hopf algebra $\text{Fun}(\mathbb{Z}/n\mathbb{Z})$. Then the dual to this Hopf algebra is the algebra $A$.

Assume additionally that for any simple object $X$ of $\mathcal{C}$ we have either $\text{FPdim}(X) = n$ or $X$ is isomorphic to $\chi^i$ for some $i$.

Lemma 6.2. Let $M$ be a simple $A$–bimodule such that $\text{Hom}_\mathcal{C}(V, M) \neq 0$. Then $M = V$ as an object of $\mathcal{C}$. In particular $M$ is invertible in the tensor category of $A$–bimodules.

Proof. Assume first that $M$ is a simple right $A$–module such that $\text{Hom}_\mathcal{C}(V, M) \neq 0$. Then $\text{Hom}_A(V \otimes A, M) = \text{Hom}_\mathcal{C}(V, M) \neq 0$ and hence $M$ is a direct summand of $V \otimes A = V^{\oplus n}$. On the other hand it is obvious that the object $V$ has $n$ different structures of an $A$–module. Thus we have proved that any simple right (and similarly left) $A$–module $M$ with $\text{Hom}_\mathcal{C}(V, M) \neq 0$ is isomorphic to $V$ as an object of $\mathcal{C}$.

Now let $M$ be a simple right $A$–module such that $M = V$ as an object of $\mathcal{C}$. Let us calculate $\text{Hom}(M, M)$. By example 3.19 in [EO], $\text{Hom}(M, M) = (M \otimes_A \ast M)\ast$ and by Lemma 6.1, $\text{FPdim}(\text{Hom}(M, M)) = n$. Clearly, $1 \subset \text{Hom}(M, M)$ and thus $\text{Hom}(M, M) = 1 \oplus \chi \oplus \ldots \oplus \chi^{n-1}$. In particular $\chi \otimes M = M$ as right $A$–modules. Choose such an isomorphism; after normalizing we can consider it as a structure of a left $A$–module on $M$ commuting with the structure of a right $A$–module. In other words, $M$ has $n$ different structures of an $A$–bimodule (for a fixed right $A$–module structure). Thus altogether we constructed $n^2$ different structures of an $A$–bimodule on the object $V$. Finally, any simple $A$–bimodule $M$ with $\text{Hom}_\mathcal{C}(V, M) \neq 0$ is a direct summand of $A \otimes V \otimes A = V^{\oplus n^2}$. The lemma is proved. 

Now we are ready to state and prove the main result of this section.
Theorem 6.3. Let $p < q$ be primes. Then any fusion category $C$ of Frobenius-Perron dimension $pq$ with integer dimensions of simple objects is group-theoretical. More specifically, it is equivalent to one in the following list:

(i) A category with 1-dimensional simple objects (these are described in Section 4).

(ii) $\text{Rep}(G)$ where $G = \mathbb{Z}/p\mathbb{Z} \rtimes \mathbb{Z}/q\mathbb{Z}$ is a non-abelian group.

(iii) If $p = 2$, the category $C(G, \mathbb{Z}/2\mathbb{Z}, \xi, \psi)$ (see [ENO], Section 8.8; [O]) where $G$ is the nonabelian group $\mathbb{Z}/2\mathbb{Z} \rtimes \mathbb{Z}/q\mathbb{Z}$, and $\xi \in Z^3(G, \mathbb{C}^*) = \mathbb{Z}/2q\mathbb{Z}$ is a cocycle which represents a cohomology class of order $q$, and $\psi$ is determined by $\xi$. In this case $C$ is not a representation category of a Hopf algebra.

Proof. Assume that not all simple objects of $C$ are invertible. To prove the first statement, observe that by Theorem 5.1, the assumptions of Lemma 6.2 are satisfied. Hence Lemma 6.2 applies. Thus, any simple $A$-bimodule $M$ containing a $p$-dimensional representation $V$ is invertible. On the other hand, it is clear that any simple $A$-bimodule which involves only $\chi^i$ must be isomorphic to $A$. Thus, any simple $A$-bimodule is invertible. In other words, the dual category $\text{C}^*_\text{Rep}(A) = A-\text{bimod}$ has only invertible simple objects. So $C$ is group-theoretical, as desired.

Let us now prove the second statement. The category $\text{C}^*_\text{Rep}(A)$ is of the form $C(G, \omega)$, where $G$ is a group of order $pq$, and $\omega \in H^3(G, \mathbb{C}^*)$. So $C$ is of the form $C(G, B, \xi, \psi)$, where $B$ is a subgroup of $G$, $\xi$ a 3-cocycle representing $\omega$ and $\psi$ is a 2-cocycle on $B$.

It is easy to check that if the category $C(G, B, \xi, \psi)$ has non-1-dimensional simple objects then $G$ is the nonabelian group $\mathbb{Z}/p\mathbb{Z} \rtimes \mathbb{Z}/q\mathbb{Z}$, and $B = G$ or $B = \mathbb{Z}/p\mathbb{Z}$.

Further, the cocycle $\xi$ must be trivial on $B$, and $\psi$ is determined by $\xi$ up to equivalence (see Lemma 6.1). If $B = G$ then this implies that we can set $\xi = 1$, and we are in case (ii). Suppose that $B = \mathbb{Z}/p\mathbb{Z}$. If $p$ is odd, then Lemma 6.2 implies that we can set $\xi = 1$. In this case $C$ is the representation category of the Kac algebra attached to the exact factorization $G = (\mathbb{Z}/p\mathbb{Z})(\mathbb{Z}/q\mathbb{Z})$. It is easy to see that this Kac algebra is isomorphic to the group algebra of $G$ as a Hopf algebra, so we are still in case (ii).

If $p = 2$ and $\xi = 1$ in cohomology, we are in case (ii) as well, for the same reason. If $p = 2$ and $\xi \neq 1$ in cohomology, then we are in case (iii). It follows from [O] that in this case $C$ does not admit fiber functors. We are done. \qed

Remark 6.4. Let $\xi$ be a 3-cocycle on $\mathbb{Z}/q\mathbb{Z}$ representing a nontrivial cohomology class. Since $\mathbb{Z}/2\mathbb{Z}$ acts trivially on $H^3(\mathbb{Z}/q\mathbb{Z}, \mathbb{C}^*)$, and since $2$ is relatively prime to $q$, $\xi$ can be chosen to be invariant under $\mathbb{Z}/2\mathbb{Z}$. Let $\Phi$ be an associator in $\text{Fun}(\mathbb{Z}/q\mathbb{Z})^3$ corresponding to $\xi$. Then $(\mathbb{C}[\mathbb{Z}/2\mathbb{Z}] \rtimes \text{Fun}(\mathbb{Z}/q\mathbb{Z}), \Phi)$, with the usual coproduct, is a finite dimensional semisimple quasi-Hopf algebra $H$. Then $\text{Rep}(H)$ is a category from case (iii), and any category of case (iii) (there are two of them up to equivalence) is obtained in this way.

Remark 6.5. Theorem 6.3 implies in particular the classification of semisimple quasi-Hopf algebras of dimension $pq$, where $p$ and $q$ are distinct primes.

Remark 6.6. In the case $pq = 6$, Theorem 6.3 was proved by T. Chmutova. Namely, she discovered that besides categories whose simple objects are invertible, there are exactly three 6-dimensional categories with integer dimensions of simple objects: the category of representations of $S_3$ (case (ii)) and two additional categories with the same Grothendieck ring (case (iii)).
7. Categories with simple objects of dimension 1 and n

Let \( n > 1 \) be an integer. Let \( N \) be a finite group with a fixed-point-free action of \( \mathbb{Z}/n\mathbb{Z} \) and let \( \omega \in H^3(N, \mathbb{C}^*) \) be an invariant class under the \( \mathbb{Z}/n\mathbb{Z} \)-action. Since \( n \) and \( |N| \) are coprime there exists a 3-cocycle \( \xi \) representing \( \omega \) and invariant under the \( \mathbb{Z}/n\mathbb{Z} \)-action. Let \( \Phi \) be an associator in \( \text{Fun}(N)^{\otimes 3} \) corresponding to \( \xi \). Then \( (\mathbb{C}[\mathbb{Z}/n\mathbb{Z}] \otimes \text{Fun}(N), \Phi) \), with the usual coproduct, is a finite dimensional semisimple quasi-Hopf algebra \( H \). It is easy to see that any simple \( H \)-module has dimension 1 or \( n \). The following theorem gives an abstract characterization of quasi-Hopf algebras constructed in such a way.

**Theorem 7.1.** Let \( \mathcal{C} \) be a fusion category such that

(i) Invertible objects of \( \mathcal{C} \) form a cyclic group of order \( n \).

(ii) For any simple object \( X \in \mathcal{C} \) either \( \text{FPdim}(X) = 1 \) or \( \text{FPdim}(X) = n \), and \( \mathcal{C} \) contains at least one simple object of FP dimension \( n \).

Then there exists a finite group \( N \neq \{1\} \) with a fixed-point-free action of \( \mathbb{Z}/n\mathbb{Z} \) and a \( \mathbb{Z}/n\mathbb{Z} \)-invariant class \( \omega \in H^3(N, \mathbb{C}^*) \) such that \( \mathcal{C} \) is equivalent to \( \text{Rep}(H) \) where \( H \) is the quasi-Hopf algebra constructed above.

**Proof.** Let \( m \) be the number of simple objects \( X \in \mathcal{C} \) with \( \text{FPdim}(X) = n \). Then \( \text{FPdim}(\mathcal{C}) = n(mn + 1) \).

Let \( V \) be an \( n \)-dimensional simple object of \( \mathcal{C} \). Then \( V \otimes V^* \) contains the neutral object, so by dimension counting it must contain all 1-dimensional objects. Thus \( V \) is stable under tensoring with 1-dimensional objects. Hence, we are in the conditions of Lemma 6.2. Thus we see that the category \( \text{Rep}(A) \) has exactly \( mn + 1 \) simple objects (namely, the regular module and \( n \) structures of an \( A \)-module on each simple noninvertible object of \( \mathcal{C} \)), and the dual category \( \mathcal{C}^{*}_{\text{Rep}(A)} \) has only invertible objects. Thus \( \mathcal{C}^{*}_{\text{Rep}(A)} = \mathcal{C}(G, \omega) \) for some finite group \( G \) and \( \omega \in H^3(G, \mathbb{C}^*) \). Moreover, it is clear from the classification of module categories over \( \mathcal{C}(G, \omega) \) that \( \text{Rep}(A) \), as a module category over \( \mathcal{C}^{*}_{\text{Rep}(A)} \), is of the form \( \mathcal{M}(G, B, \xi, \psi) \) where \( B \subset G \) is a cyclic subgroup of order \( n \) such that \( \xi|_B \) trivial, see \([O]\). Thus, \( \psi \) is determined by \( \xi \) up to equivalence (as \( H^2(\mathbb{Z}/n\mathbb{Z}, \mathbb{C}^*) = 0 \)), and \( \mathcal{C} = (\mathcal{C}^{*}_{\text{Rep}(A)})_{\text{Rep}(A)} = \mathcal{C}(G, B, \xi, \psi) \).

The simple objects in the category \( \mathcal{C}(G, B, \xi, \psi) \) are classified by pairs \((g, \lambda)\) where \( g \in G \) and \( \lambda \) is an irreducible representation of \( B \cap gBg^{-1} \) (see \([O]\)), and the Frobenius-Perron dimension of the simple object corresponding to a pair \((g, \lambda)\) is \(|B : B \cap gBg^{-1}| \dim(\lambda)\). Thus conditions (i), (ii) translate to the following: \( B \cap gBg^{-1} = 1 \) for any \( g \notin B \). In other words, \( G \) is a Frobenius group (see e.g. \([G]\)). Thus \( G \simeq B \rtimes N \) for some normal subgroup \( N \subset G \) and the action of \( B \) on \( N \) is fixed-point-free (see loc. cit.). Furthermore, \( H^3(G, \mathbb{C}^*) = H^3(B, \mathbb{C}^*) \oplus H^3(N, \mathbb{C}^*)^B \) by the same argument as in Lemma 3.11. Clearly the subgroup of \( \omega \in H^3(G, \mathbb{C}^*) \) such that \( \omega|_B = 1 \) is identified with \( H^3(N, \mathbb{C}^*)^B \). Thus, by the Frobenius theorem, \( G = B \rtimes N \). The Theorem is proved. \(\square\)

**Corollary 7.2.** Let \( H \) be a semisimple Hopf algebra with 1-dimensional and \( n \)-dimensional irreducible representations, such that \( G(H^*) \) is a cyclic group of order \( n \). Then \( H = \mathbb{C}[B] \rtimes \text{Fun}(N) \) is the Kac algebra attached to the exact factorization \( G = BN \), where \( B = \mathbb{Z}/n\mathbb{Z}, N \) is a group with a fixed-point-free action of \( B \), and \( G = B \rtimes N \).

**Proof.** In the Hopf algebra case the category \( \mathcal{C} \) admits a fiber functor, i.e. a module category with only one simple object. Thus, by \([O]\), there exists a subgroup \( P \) of
G such that $G = BP$, $\omega|p = 1$. Clearly, $P$ contains $N$, so $\omega = 1$, $\psi = 1$, and we are done. \hfill \Box

**Remark 7.3.** Recall that the famous Thompson’s Theorem states that the group $N$ above is nilpotent (see e.g. [G]).

Now consider the special case of Theorem 7.1 when $m = 1$. Let $X$ denote the non-invertible object and $\chi^i, i = 0, \ldots, n - 1$, denote the invertible objects of $C$. Then obviously the multiplication in the category $C$ is given by

$$\chi \otimes X = X \otimes \chi = X, \quad X \otimes X = (n - 1)X \oplus \chi^0 \oplus \ldots \oplus \chi^{n-1}.$$  

**Corollary 7.4.** Let $C$ be a fusion category such that the invertible objects of $C$ form a cyclic group of order $n > 1$ and $C$ has only one non-invertible object of dimension $n$. Then $n + 1 = p^a$ is a prime power. If $n = 2$ there are three such categories, if $n = 3$ or $7$ there are two such categories, and for all other $n = p^a - 1 > 1$ there is exactly one such category – the category of representations of the semi-direct product $\mathbb{F}_{p^a} \rtimes \mathbb{F}_{p^a}^\ast$.

**Remark 7.5.** We thank R. Guralnick for help in the proof of the corollary.

**Proof.** In this case the group $N$ above is of order $n + 1$ and the group $B$ acts simply transitively on the non-identity elements of $N$. Thus all non-identity elements of $N$ have the same order and hence $N$ is a $p$–group. Consequently, any element of $N$ is conjugated to some central element and hence $N$ is abelian. Henceforth, $N$ is an elementary abelian group of order $q = p^a$. The cyclic group $B$ acts irreducibly on $N$, hence by Schur’s Lemma, $N$ is identified with a one dimensional vector space over the finite field $\mathbb{F}_q$, and $B$ is identified with $GL_1(\mathbb{F}_q) = \mathbb{F}_q^\ast$.

The following statement is well known:

**Lemma 7.6.** Let $V$ be an elementary abelian $p$–group. Consider $H^i(V, \mathbb{C}^\ast)$ as a functor in the variable $V$. Then we have

(i) $H^1(V, \mathbb{C}^\ast) = V^\ast$.

(ii) $H^2(V, \mathbb{C}^\ast) = \wedge^2 V^\ast$.

(iii) There is an exact sequence of $GL(V)$-modules $0 \to S^2 V^\ast \to H^3(V, \mathbb{C}^\ast) \to \wedge^3 V^\ast \to 0$.

Here $S^\ast V^\ast$ is the symmetric algebra of the space $V^\ast$; that is, the algebra generated by $v \in V^\ast$ subject to the relations $v_1 v_2 = v_2 v_1$ for any $v_1, v_2 \in V^\ast$. Similarly, $\wedge^\ast V^\ast$ is generated by $v \in V^\ast$ subject to the relations $v^2 = 0$.

**Proof.** Items (i) and (ii) are well known. We prove (iii). Recall that $H^3(V, \mathbb{C}^\ast) = H^4(V, \mathbb{Z})$. It follows from the Kunneth formula that $H^{>0}(V, \mathbb{Z})$ is annihilated by the multiplication by $p$. Thus an exact sequence $0 \to \mathbb{Z} \to \mathbb{Z} \to \mathbb{Z}/p\mathbb{Z} \to 0$ for any $i \geq 1$ gives an exact sequence $0 \to H^i(V, \mathbb{Z}) \to H^i(V, \mathbb{Z}/p\mathbb{Z}) \to H^{i+1}(V, \mathbb{Z}) \to 0$. It is well known (see e.g. [B]) that

$$H^i(V, \mathbb{Z}/p\mathbb{Z}) = \begin{cases} S^2 V^\ast \otimes \wedge^i V^\ast & \text{for } p > 2 \\ S^i V^\ast & \text{for } p = 2 \end{cases}$$

Thus for $p > 2$ we have $H^3(V, \mathbb{Z}/p\mathbb{Z}) = V^\ast \otimes V^\ast \otimes \wedge^3 V^\ast$ and for $p = 2$, $H^3(V, \mathbb{Z}/p\mathbb{Z}) = S^3 V^\ast$. For $p > 2$ one observes that the image of $\wedge^2 V^\ast = H^3(V, \mathbb{Z})$ lies inside $V^\ast \otimes V^\ast$, since the scalar matrices act by different characters on $\wedge^2 V^\ast$ and $\wedge^3 V^\ast$. Also, $V^\ast \otimes V^\ast / \wedge^2 V^\ast = S^2 V^\ast$ and we are done. For $p = 2$ we get that there
is an embedding $\wedge^2 V^* \subset S^3 V^*$ and $H^3(V, \mathbb{Z}) = S^3 V^*/\wedge^2 V^*$. Consider the
obvious surjection $S^3 V^* \rightarrow \wedge^3 V^*$. Since $\wedge^2 V^*$, $\wedge^3 V^*$ are simple non-isomorphic
$GL(V)$–modules, the submodule $\wedge^2 V^*$ is in the kernel of this surjection. On the
other hand, it is easy to see that the kernel is identified with $V^* \otimes V^*$ via the map $x \otimes y \mapsto x^2 y$. Finally, one observes that $V^* \otimes V^*$ has a unique copy of the simple module $\wedge^2 V^*$ spanned by tensors of the form $x \otimes y + y \otimes x$, and $V^* \otimes V^*/\wedge^2 V^* = S^2 V^*$.

The lemma is proved. \hfill \Box

Now, one deduces easily that in our situation $H^3(N, \mathbb{C}^*)_B$ is nontrivial if and only
if $q = 3, 4, 8$. Indeed, let $\alpha$ be a generator of $F_q^*$. Then the operator of multiplication
by $\alpha$ in the vector space $V^* = F_q$ has eigenvalues $\alpha, Fr(\alpha) = \alpha^p, \ldots, Fr^{a-1}(\alpha) = \alpha^{p^{a-1}}$. The eigenvalues for the action on $S^2 V^*$ (resp. $\wedge^3 V^*$) are $\alpha^{p^i+p^j}$, $0 \leq i \leq j \leq a - 1$ (resp. $\alpha^{p^i+p^j+p^k}$, $0 \leq i < j < k \leq a - 1$). Thus we have an eigenvalue 1
on $S^2 V^*$ (resp. $\wedge^3 V^*$) if and only if $p^i + p^j = p^a - 1$ for some $0 \leq i \leq j \leq a - 1$
(resp. $p^i + p^j + p^k = p^a - 1$ for $0 \leq i < j < k \leq a - 1$), and the statement follows.
In all cases the space $H^3(N, \mathbb{C}^*)_B$ is one dimensional over the prime field and the
case $q = 3$ was already considered in Theorem 6.3. Finally, note that the category $C := \text{Rep}(F_q^*) \rtimes F_q$ satisfies the conditions of the corollary. This completes the proof
of the corollary. \hfill \Box

References

[B] D. Benson, Representations and Cohomology, Cambridge Studies in Advanced Mathematics, 1991.

[EG1] P. Etingof and S. Gelaki, Semisimple Hopf algebras of dimension $pq$ are trivial, Journal of Algebra 210 (1998), 664–669.

[EG2] P. Etingof and S. Gelaki, Some properties of finite-dimensional semisimple Hopf algebras, Mathematical Research Letters 5 (1998), 191–197.

[ENO] P. Etingof, D. Nikshych and V. Ostrik, On fusion categories, preprint, math.QA/0203060

[EO] P. Etingof and V. Ostrik, Finite tensor categories, preprint, math.QA/0301027

[G] D. Gorenstein, Finite Groups, Harper & Row, Publishers, New York-London 1968.

[GW] S. Gelaki and S. Westreich, On semisimple Hopf algebras of dimension $pq$, Proceedings of the AMS, 128 (2000), no. 1, 39–47.

[HN] F. Hausser and F. Nill, Doubles of quasi-quantum groups, Comm. Math. Phys. 199 (1999), no. 3, 547–589.

[K] C. Kassel, Quantum Groups, Springer, New York, 1995.

[O] V. Ostrik, Boundary conditions for holomorphic orbifolds, math.QA/0202130

[S1] P. Schauenburg, A quasi-Hopf algebra freeness theorem, math.QA/0204141

[S2] P. Schauenburg, Quotients of finite quasi-Hopf algebras, math.QA/0204337

[TY] D. Tambara and S. Yamagami, Tensor categories with fusion rules of self-duality for finite
abelian groups, J. Algebra 209 (1998), no. 2, 692–707.

Department of Mathematics, Massachusetts Institute of Technology, Cambridge, MA 02139, USA
E-mail address: etingof@math.mit.edu

Department of Mathematics, Technion-Israel Institute of Technology, Haifa 32000, Israel
E-mail address: gelaki@math.technion.ac.il

Department of Mathematics, Massachusetts Institute of Technology, Cambridge, MA 02139, USA
E-mail address: ostrik@math.mit.edu