Proportional-Integral Projected Gradient Method for Infeasibility Detection in Conic Optimization

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Abstract—A constrained optimization problem is primal infeasible if its constraints cannot be satisfied, and dual infeasible if the constraints of its dual problem cannot be satisfied. We propose a novel iterative method, named proportional-integral projected gradient method (PIPG), for detecting primal and dual infeasibility in convex optimization with quadratic objective function and conic constraints. The iterates of PIPG either asymptotically provide a proof of primal or dual infeasibility, or asymptotically satisfy a set of primal-dual optimality conditions. Unlike existing methods, PIPG does not compute matrix inverse, which makes it better suited for large-scale and real-time applications. We demonstrate the application of PIPG in quasiconvex and mixed-integer optimization using examples in constrained optimal control.

I. INTRODUCTION

Conic optimization is the minimization of a quadratic objective function subject to conic constraints:

\[
\begin{aligned}
\text{minimize} & \quad \frac{1}{2} z^T P z + q^T z \\
\text{subject to} & \quad H z - g \in K, \quad z \in \mathbb{D},
\end{aligned}
\]

where \( z \in \mathbb{R}^n \) is the optimization variable, symmetric positive semidefinite matrix \( P \in \mathbb{R}^{n \times n} \) and vector \( q \in \mathbb{R}^n \) define the quadratic objective function, matrix \( H \in \mathbb{R}^{m \times n} \) and vector \( g \in \mathbb{R}^m \) are constraint parameters, cone \( K \subseteq \mathbb{R}^m \) and set \( \mathbb{D} \subseteq \mathbb{R}^n \) are nonempty, closed, and convex. Optimization (1) includes many common convex optimization problems as special cases, including linear, quadratic, second-order cone, and semidefinite programming [1, 2].

Optimization (1) is primal infeasible if there exists no \( z \in \mathbb{D} \) such that \( H z - g \in K \), it is dual infeasible if the constraints of its dual problem cannot be satisfied. If optimization (1) is not primal infeasible, the infeasibility of its dual problem implies that the primal optimal value is unbounded from below [3].

Given optimization (1), infeasibility detection is the problem of providing a proof of primal or dual infeasibility if it is the case [3]. In particular, a proof of primal infeasibility is the existence of a hyperplane that separates cone \( K \) from the set \( \{ H z - g \mid z \in \mathbb{D} \} \). A proof of dual infeasibility is the existence of a vector \( \tau \in \mathbb{R}^n \) that satisfies the following conditions for all \( z \in \mathbb{D} \) [4]:

\[
P \tau = 0, \quad q^T \tau < 0, \quad H \tau \in K, \quad z + \tau \in \mathbb{D}.
\]

Infeasibility detection is necessary for adjusting the parameters in pathological optimization problems [5], and an integral subproblem in quasiconvex optimization [2, 6] and mixed-integer optimization [7, 8].

Infeasibility detection via Douglas-Rachford splitting method (DRS) has recently attracted increasing attention [9, 10, 11, 3]. DRS detects infeasibility of optimization (1) by computing either a nonzero solution of the homogeneous self-dual embedding [10], or the minimal-displacement vector [9, 11, 3]. Compared with traditional infeasibility detection methods [12, 13], DRS has empirically achieved up to three-orders-of-magnitude speedups in computation for relatively large-scale problems [10].

A challenge of DRS-based infeasibility detection is computing matrix inverse [10, 9, 11, 3]. Such computation becomes expensive as the size of optimization (1) increases, or when optimization (1) is solved repeatedly with different constraint parameters in real-time [14, 15]. While such matrix inverse is not necessary for problems with only linear constraints [16, 17], whether it is necessary in general for optimization (1) is, to our best knowledge, still an open question.

We propose a novel infeasibility detection method for optimization (1), named proportional-integral projected gradient method (PIPG). The iterates of PIPG either asymptotically satisfy a set of primal-dual optimality conditions or diverge. In the latter case, the difference between the consecutive iterates converges to a nonzero vector, known as the minimal displacement vector, that proves primal or dual infeasibility.

PIPG is the first infeasibility detection method for optimization (1) that avoids computing matrix inverse. All existing methods compute matrix inverse as a subroutine of either interior point methods [12, 13] or DRS [9, 10, 11, 3]. In contrast, PIPG only computes matrix multiplication and projections onto the set \( \mathbb{D} \) and the cone \( K \) in optimization (1). As a result, PIPG allows implementation for large-scale and real-time problems using limited computational resources.

PIPG provides an efficient method for quasiconvex and mixed-integer optimization problems. We demonstrate the application of PIPG in these problems using two examples in constrained optimal control: minimum-time landing and nonconvex path-planning of a quadrotor.

The rest of the paper is organized as follows. After some basic results in convex analysis and monotone operator theory, Section II reviews the existing results on DRS-based infeasibility detection. Section III introduces PIPG along with its convergence properties. Section IV demonstrates the application of PIPG in constrained optimal control problems. Finally, Section V concludes and comments on future work directions.
II. Preliminaries and related work

We will review some basic results in convex analysis and monotone operator theory, and some existing results on infeasibility detection using Douglas-Rachford splitting method.

A. Notations and preliminaries

We let \( \mathbb{N} \), \( \mathbb{R} \), and \( \mathbb{R}_+ \) denote the set of positive integer, real numbers, and non-negative real numbers, respectively. For a vector \( z \in \mathbb{R}^n \) and integers \( i,j \in \{1 \leq i < j \leq n \} \), we let \( [z]_i \) denote the \( i \)-th element of vector \( z \), and \( [z]_{i,j} \in \mathbb{R}^{n \times n} \) denote the vector composed of the entries of \( z \) between (and including) the \( i \)-th entry and the \( j \)-th entry. For two vectors \( z, z' \in \mathbb{R}^n \), we let \( \langle z, z' \rangle \) denote their inner product, and \( \|z\| := \sqrt{\langle z, z \rangle} \) denote the \( l_2 \) norm of \( z \). We let \( I_z \) denote the \( n \times n \) identity matrix and \( 0_{m \times n} \) denote the \( m \times n \) zero matrix. When the dimensions of these matrices are clear from the context, we omit their subscripts. For a matrix \( H \in \mathbb{R}^{m \times n} \), \( H^T \) denotes its transpose, \( \|H\| \) denotes its largest singular value. For a square matrix \( A \in \mathbb{R}^{n \times n} \), \( \exp(A) \) denotes the matrix exponential of \( A \). For a set \( S \), we let \( -S := \{ z \mid -z \in S \} \). Given two sets \( S_1 \) and \( S_2 \), we let \( S_1 \times S_2 \) denote their Cartesian product. We say function \( f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\} \) is convex if \( f(az + (1 - a)z') \leq a f(z) + (1 - a)f(z') \) for all \( a \in [0, 1] \). We say set \( D \subseteq \mathbb{R}^n \) is convex if \( az + (1 - a)z' \in D \) for any \( a \in [0, 1] \) and \( z, z' \in D \). We say set \( \mathcal{K} \subseteq \mathbb{R}^m \) is a convex cone if \( \mathcal{K} \) is a convex set and \( \alpha w \in \mathcal{K} \) for any \( w \in \mathcal{K} \) and \( \alpha \in \mathbb{R}_+ \).

Let \( f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\} \) be a convex function. The \textit{proximal operator} of function \( f \) is a function given by

\[
\text{prox}_f(z) := \arg\min_{z'} f(z') + \frac{1}{2} \|z' - z\|^2 \quad (2)
\]

for all \( z \in \mathbb{R}^n \). Let \( \mathcal{D} \subseteq \mathbb{R}^n \) be a nonempty closed convex set. The \textit{projection} of \( z \in \mathbb{R}^n \) onto set \( \mathcal{D} \) is given by

\[
\pi_D(z) := \arg\min_{z' \in \mathcal{D}} \|z - z'\| \quad (3)
\]

The \textit{point-to-set distance} from \( z \) to set \( \mathcal{D} \) is given by

\[
d(z, \mathcal{D}) := \min_{z' \in \mathcal{D}} \|z - z'\| \quad (4)
\]

The \textit{normal cone} of set \( \mathcal{D} \) at \( z \) is given by

\[
N_D(z) := \{ y \in \mathbb{R}^n | \langle y, z' - z \rangle \leq 0, \forall z' \in \mathcal{D} \} \quad (5)
\]

The \textit{recession cone} of set \( \mathcal{D} \) is a nonempty closed convex cone given by

\[
\text{rec} \mathcal{D} := \{ y \in \mathbb{R}^n | y + z \in \mathcal{D}, \forall z \in \mathcal{D} \} \quad (6)
\]

The \textit{indicator function} of set \( \mathcal{D} \) is defined as

\[
\delta_D(z) := \begin{cases} 
0, & \text{if } z \in \mathcal{D}, \\
+\infty, & \text{otherwise}.
\end{cases} \quad (7)
\]

for all \( z \in \mathbb{R}^n \). The \textit{support function} of set \( \mathcal{D} \) is given by

\[
\sigma_D(z) := \sup_{y \in \mathcal{D}} \langle y, z \rangle \quad (8)
\]

for all \( z \in \mathbb{R}^n \). Let \( \mathcal{K} \subseteq \mathbb{R}^m \) be a nonempty closed convex cone. The \textit{recession cone} of cone \( \mathcal{K} \) is itself, i.e., \( \text{rec}\mathcal{K} = \mathcal{K} \) [18, Cor. 6.50]. The \textit{polar cone} of \( \mathcal{K} \) is a closed convex cone given by

\[
\mathcal{K}^\circ := \{ w \in \mathbb{R}^m | \langle w, y \rangle \leq 0, \forall y \in \mathcal{K} \} \quad (9)
\]

One can verify that \( (\mathcal{K}^\circ)^\circ = \mathcal{K} \) [19, Cor. 6.21]. We will use the following results on polar cone.

\textbf{Lemma 1.} [18, Thm. 6.30] If \( \mathcal{K} \subseteq \mathbb{R}^n \) is a nonempty closed convex cone, then \( w = \pi_K[w] + \pi_{\mathcal{K}^\circ}[w] \) and \( \langle \pi_K[w], \pi_{\mathcal{K}^\circ}[w] \rangle = 0 \) for all \( w \in \mathbb{R}^m \).

We say function \( T : \mathbb{R}^p \to \mathbb{R}^p \) is a \( \gamma \)-averaged operator for some \( \gamma \in (0, 1) \) if and only if the following condition holds for all \( \xi_1, \xi_2 \in \mathbb{R}^p \) [18, Prop. 4.35]:

\[
\|T(\xi_1) - T(\xi_2)\|^2 - \|\xi_1 - \xi_2\|^2 \\
\leq \frac{\gamma - 1}{\gamma} \|\xi_1 - T(\xi_1) - \xi_2 + T(\xi_2)\|^2. \quad (10)
\]

We will use the following result on averaged operators.

\textbf{Lemma 2.} [3, Lem. 5.1] Let \( T : \mathbb{R}^p \to \mathbb{R}^p \) be a \( \gamma \)-averaged operator for some \( \gamma \in (0, 1) \). Let \( \xi^1 \in \mathbb{R}^p \) and \( \xi^{j+1} := T(\xi^j) \) for all \( j \in \mathbb{N} \). Then there exists \( \xi \in \mathbb{R}^p \), known as the minimal-displacement vector of operator \( T \), such that

\[
\lim_{j \to \infty} \frac{\xi^j}{j} = \lim_{j \to \infty} \xi^{j+1} - \xi = \xi. \]

B. Infeasibility detection via Douglas-Rachford splitting method

Douglas-Rachford splitting method (DRS) is an iterative method for optimization problems of the following form:

\[
\begin{aligned}
\text{minimize}_\xi & \quad \ell(\xi) + r(\xi), \\
\text{subject to} & \quad H \xi - g \in \mathcal{K}.
\end{aligned} \quad (11)
\]

where \( \ell : \mathbb{R}^p \to \mathbb{R} \cup \{+\infty\} \) and \( r : \mathbb{R}^p \to \mathbb{R} \cup \{+\infty\} \) are convex functions. DRS uses the following iteration, where \( j \in \mathbb{N} \) is the iteration counter and \( \alpha \in (0, 2) \) is the step size:

\[
\xi^{j+1} = \xi^j + \alpha (\text{prox}_\alpha(2\text{prox}_r(\xi^j) - \xi^j) - \text{prox}_r(\xi^j)). \quad (12)
\]

One can show that equation (12) is equivalent to \( \xi^{j+1} = T(\xi^j) \) for some \( \frac{\alpha}{2} \)-averaged operator \( T \) [20].

In the following, we will review two different ways of infeasibility detection using DRS. For simplicity, we assume that DRS is terminated when a maximum number of iterations, denoted by \( k \) is reached. We also let \( \epsilon \in \mathbb{R}_+ \) be such that numbers within the interval \( [-\epsilon, \epsilon] \) are treated as approximately zero.

1) Homogeneous self-dual embedding method: The first way of infeasibility detection via DRS considers the following special case of optimization (1):

\[
\begin{aligned}
\text{minimize}_z & \quad q^T z, \\
\text{subject to} & \quad H z - g \in \mathcal{K}.
\end{aligned} \quad (13)
\]

The homogeneous self-dual embedding of optimization (13) is given by the following set of linear equations and conic inclusions [12, 13, 10]:

\[
\begin{aligned}
v &= Su, & u &\in \mathcal{K}, & v &\in -\mathcal{K}^\circ.
\end{aligned} \quad (14)
\]
where
\[
S = \begin{bmatrix}
0 & H^T & q \\
-H & 0 & g \\
-q^T & -g^T & 0
\end{bmatrix}, \quad K = \mathbb{R}^n \times (-K^c) \times \mathbb{R}_+.
\] (15)

Here \(-K^c\) is also known as the dual cone of \(K\). Notice that the conditions in (14) are satisfied if and only if \(\xi = [u^T \ v^T]^\top\) is an optimal solution of optimization (11) with
\[
\ell(u, v) = \delta_{\mathbb{R}K}(-K)(u, v), \quad r(u, v) = \delta_{\mathbb{S}u_\mathbb{K}}(u, v),
\] (16)
where \(\delta_{\mathbb{S}u_\mathbb{K}} = \mathbb{S}u_\mathbb{K}\) is the indicator function of set \(\mathbb{S}u_\mathbb{K}\) if \(\delta_{\mathbb{S}u_\mathbb{K}}(u, v)\). If \(u, v \in \mathbb{R}^{m+n+1}\) satisfy (14), then one can detect the primal and dual infeasibility of optimization (13) as follows. Let
\[
z = [u]_1:n, \quad w = [u]_{n+1:n+m}, \quad \tau = [u]_{m+n+1}, \quad \kappa = [v]_{m+n+1}.
\]
If \(\tau > 0\) and \(\kappa = 0\), then \(\tau z\) satisfies the optimality conditions of optimization (13). If \(\tau = 0\) and \(\langle q, v \rangle < 0\), then one can construct a proof of primal infeasibility. If \(\tau = 0\) and \(\langle q, z \rangle < 0\), then one can construct a proof of dual infeasibility. See [10, Sec. 2.3] for a detailed discussion.

Algorithm 1 summarizes the above infeasibility detection method, where the optimization (11) is solved using DRS. In this case, the DRS iteration in (12) is equivalent to line 2 of Algorithm 1 [10, Sec. 3.2]. Further, with a proper choice of initial values of \(u^1, v^1\) and a sufficiently large iteration number \(k\), the vectors \(u^k\) and \(v^k\) approximately satisfy (14) without both being zero vectors [10, Sec. 3.4]. Notice that line 1 computes the inverse of a symmetric matrix in \(\mathbb{R}^{(m+n+1) \times (m+n+1)}\).

Algorithm 1 DRS for optimization (13) via homogeneous self-dual embedding

**Input:** \(k, \epsilon\) initial values \(u^1, v^1\).

**Output:** \(z^*, \text{"Primal infeasible"}, \text{"Dual Infeasible"}\)

1: for \(j = 1, 2, \ldots, k - 1\) do
2: \(\hat{u}^{j+1} = (I + S)^{-1} (u^j + v^j)\)
3: \(u^{j+1} = \pi_K[\hat{u}^{j+1} - v^j]\)
4: \(v^{j+1} = v^j - \hat{u}^{j+1} + w^j\)
5: end for
6: if \([u^k]_{m+n+1} < \epsilon\) then
7: \(\hat{z}^* = \frac{1}{\epsilon} [u^k]_1:n\).
8: else if \([v^k]_{m+n+1} < \epsilon\) then
9: \(\hat{z}^* = \frac{1}{\epsilon} [v^k]_1:n\).
10: return \("\text{Primal infeasible}"\)
11: end if
12: if \(\langle q, [u^k]_1:n \rangle < -\epsilon\) then
13: return \("\text{Dual infeasible}"\)
14: end if
15: end if
16: end for
17: return \(z^* = z^k\)

**2) Minimal-displacement vector method:** Another way to detect the primal and dual infeasibility of optimization (11) via DRS is to compute the minimal-displacement vector of DRS. In particular, let
\[
\Pi = [H^T \ I]^\top, \quad \mathbb{D} = \{g + y \mid y \in K\} \times \mathbb{D}.
\]
Then one can rewrite optimization (11) by as an instance of optimization (11) with \(\xi = [z^T \ y^T]^\top\) and
\[
\ell(z, y) = \frac{1}{2} z^T P z + q^T z + \delta_{\Pi_{z=\mathbb{D}}}(z, y), \quad r(z, y) = \delta_{\Pi_{z=\mathbb{D}}}(y),
\]
where \(\delta_{\Pi_{z=\mathbb{D}}}(z, y)\) is the indicator function of set \(\{z, y \mid \Pi z = y\}\). In addition, when DRS is applied to the above instance of optimization (11), the difference between the consecutive iterates converges to a limit, known as the minimal-displacement vector of DRS. If this limit is zero, the iterates of DRS asymptotically satisfies a set of primal-dual optimality conditions. If this limit is nonzero, the iterates of DRS asymptotically provide a proof for either primal or dual infeasibility [3, 21].

Algorithm 2 summarizes the above infeasibility detection method, where the DRS iteration in (12) simplifies to line 2 of [21, Sec. 4]. With a sufficient large iteration number \(k\), the vectors \(w^k - w^{k-1}\) and \(v^k - v^{k-1}\) together give an approximation of the aforementioned minimal-displacement vector. Notice that line 1 computes the inverse of a symmetric matrix in \(\mathbb{R}^{n \times n}\).

**Algorithm 2 DRS for optimization (11) via minimal-displacement vector**

**Input:** \(k, \alpha, \epsilon\) initial values \(z^1, y^1\).

**Output:** \(z^*, \text{"Primal infeasible"}, \text{"Dual Infeasible"}\)

1: for \(j = 1, 2, \ldots, k - 1\) do
2: \(\hat{y}^j = \pi_{\Pi}[y^j]\)
3: \(w^j = y^j - \hat{y}^j\)
4: \(\hat{z}^j = (I + P + \Pi^T \Pi)^{-1} (z^j - q + \Pi^T (2\hat{y}^j - y^j))\)
5: \(z^{j+1} = z^j + \alpha (\hat{z}^j - z^j)\)
6: \(y^{j+1} = y^j + \alpha (\Pi \hat{z}^j - \hat{y}^j)\)
7: end for
8: if \(\max(\|w^k - w^{k-1}\|, \|z^k - z^{k-1}\|) \leq \epsilon\) then
9: return \(z^* = z^k\)
10: else
11: if \(\|w^k - w^{k-1}\| > \epsilon\) then
12: return \("\text{Primal infeasible}"\)
13: end if
14: if \(\|v^k - v^{k-1}\| > \epsilon\) then
15: return \("\text{Dual infeasible}"\)
16: end if
17: end if

III. Proportional-integral projected gradient method

We now introduce proportional-integral projected gradient method (PIPG), a primal-dual infeasibility detection method for optimization (1). Provided that an optimal primal-dual solution exists, PIG ensures that both the primal-dual gap and the constraint violation converge to zero along certain...
sequences of averaged iterates [22, 23]. In the following, we will show that the iterates of PIPG can also provide proof of primal or dual infeasibility.

Algorithm 3 summarizes PIPG for infeasibility detection, where \( k \in \mathbb{N} \) is the maximum number of iteration, \( \alpha > 0 \) is the step size, and \( \epsilon \in \mathbb{R}_+ \) is chosen such that numbers in interval \([-\epsilon, \epsilon]\) are treated as approximately zero. Unlike Algorithm 1 and Algorithm 2, Algorithm 3 does not compute matrix inverse. Furthermore, using Lemma 1 one can show that the projections in Algorithm 3 are no more difficult to compute than those in Algorithm 2.

Algorithm 3 PIPG for optimization (1)

Input: \( k, \alpha, \epsilon \), initial values \( z^1, v^1 \).
Output: \( z^*, \) “Primal infeasible”, “Dual infeasible”.
1: for \( j = 1, 2, \ldots, k-1 \) do
2: \( w^{j+1} = \pi_{p\Gamma}[v^j + \alpha (H z^j - g)] \)
3: \( z^{j+1} = z^j - \alpha (P z^j + q + H^T w^{j+1}) \)
4: \( v^{j+1} = v^j + \alpha (H (z^{j+1} - z^j)) \)
5: end for
6: if \( \max(\|w^k - w^{k-1}\|, \|z^k - z^{k-1}\|) \leq \epsilon \) then
7: return \( z^* = z^k \)
8: else
9: if \( \|w^k - w^{k-1}\| > \epsilon \) then
10: return “Primal infeasible”
11: end if
12: if \( \|z^k - z^{k-1}\| > \epsilon \) then
13: return “Dual infeasible”
14: end if
15: end if

Proposition 1. Suppose that Assumption 1 and Assumption 2 hold. Then line 24 in Algorithm 3 implies that
\[
\begin{pmatrix}
  z^{j+1} \\
  w^{j+1}
\end{pmatrix} = T \begin{pmatrix}
  z^j \\
  w^j
\end{pmatrix},
\]
where \( T : \mathbb{R}^{m+n} \to \mathbb{R}^{m+n} \) is a \( \gamma \)-averaged operator, where \( \gamma \in (\frac{1}{2}, 1) \) is given in Assumption 2.
Proof. See Appendix II

Remark 1. In order to satisfy Assumption 2, one needs to estimate the values of \( \|H\| \) and \( \|P\| \). The following power iteration ensures that \( \|z^j\| \) quickly converges to \( \|H\| \) as the number of iteration \( j \) increases:
\[
z^{j+1} = \frac{1}{\|z^j\|}H^T H z^j,
\]
where the entries in vector \( z^j \in \mathbb{R}^m \) are randomly generated. Notice that the above power iteration does not compute matrix inverse or decomposition; see [27] for further details. The computation of \( \|P\| \) is similar.

Due to Proposition 1 and Lemma 2 one can show that both \( \|z^k - z^{k-1}\| \) and \( \|w^k - w^{k-1}\| \) computed in Algorithm 3 converge to certain limits as \( k \) increases. To show that these limits yield information regarding optimality and infeasibility, we first introduce the following results.

Lemma 3. Suppose that Assumption 1 holds and the sequence \( \{w^j, z^j, v^j\}_{j \in \mathbb{N}} \) is computed recursively using line 24 in Algorithm 3 where \( \alpha > 0 \). Let \( \lambda \geq \|P\| \) and \( \nu \geq \|H\| \). Then, for all \( j \geq 3 \),
\[
d(-P z^j - q - H^T w_j | N_D(z^j)) \leq \left( \frac{1}{\alpha} + \lambda \right) \|z^j - z^{j-1}\|, \\
d(H z^j - g | N_K(z^j)) \leq \frac{1}{\alpha} \|w^j - w^{j-1}\| + \nu \|z^j - z^{j-1}\| + \|z^{j-1}\|.
\]
Proof. See Appendix III

Lemma 4. Suppose that Assumption 1 and Assumption 2 hold and the sequence \( \{w^j, z^j, v^j\}_{j \in \mathbb{N}} \) is computed recursively using line 24 in Algorithm 3. Then there exists \( \pi \in \text{rec}D \) and \( w \in K^c \) such that \( \lim_{j \to \infty} z^j = \pi \) and \( \lim_{j \to \infty} w^j = w \). Further,
\[
H \pi \in K, \quad P \pi = 0, \quad \langle g, \pi \rangle = -\frac{1}{\alpha} \|\pi\|^2, \\
\sigma_D(-H^T w) + \langle g, w \rangle = -\frac{1}{\alpha} \|w\|^2.
\]
Proof. See Appendix III

Equipped with Lemma 3 and Lemma 4 we summarize our main theoretical contribution in the following theorem.

Theorem 1. Suppose that Assumption 1 and Assumption 2 hold and the sequence \( \{w^j, z^j, v^j\}_{j \in \mathbb{N}} \) is computed recursively using line 24 in Algorithm 3. Then there exist \( \pi \in \text{rec}D \) and \( w \in K^c \) such that \( \lim_{j \to \infty} z^j = \pi \) and \( \lim_{j \to \infty} w^j = w \). Further,
\[
H \pi \in K, \quad P \pi = 0, \quad \langle g, \pi \rangle = -\frac{1}{\alpha} \|\pi\|^2, \\
\sigma_D(-H^T w) + \langle g, w \rangle = -\frac{1}{\alpha} \|w\|^2.
\]
Further, the following three statements hold.

1. If \( ||\vec{\pi}|| = ||\vec{w}|| = 0 \), then
   \[
   \lim_{j \to \infty} (Pz^j - q - H^Tw^j|N_D(z^j)) = 0,
   \]
   \[
   \lim_{j \to \infty} (Hz^j - g|N_{K^c}(w^j)) = 0.
   \]

2. If \( ||\vec{w}|| > 0 \), then
   \[
   \inf_{z \in D} \langle Hz - g, \vec{w} \rangle = \sup_{z \in K} \langle y, \vec{w} \rangle.
   \]

3. If \( ||\vec{\pi}|| > 0 \), then
   \[
   H\vec{z} \in K, \quad P\vec{z} = 0, \quad \langle q, \vec{\pi} \rangle < 0.
   \]

Proof. Lemma 2 implies that the limits in (20) hold for some \( \vec{z} \in rec D \) and \( \vec{w} \in K^c \). Further, the first statement is due to Lemma 3 and (20). The second and the third statement are due to (19) and the fact that \( \sigma_D(-H^Tw) = \inf_{z \in D} \langle Hz, \vec{w} \rangle = \sup_{z \in K} \langle y, \vec{w} \rangle = 0 \) when \( \vec{w} \in K^c \). The latter fact is due to the definition of polar cone in (9).

We now discuss the implications of the three statements in Theorem 1. First, the two limits in (21) imply that a set of primal-dual optimality conditions are satisfied asymptotically. To see this implication, let \( z^* \in D \) and \( w^* \in K^c \) be such that
\[
d(-Pz^* - q - H^Tw^*|N_D(z^*)) = d(Hz^* - g|N_{K^c}(w^*)) = 0.
\]

Since the point-to-set distance is nonnegative, the above conditions are equivalent to the following primal-dual optimality conditions of optimization (1) [19, Ex. 11.52]:

\[
-Pz^* - q - H^Tw^* \in N_D(z^*),
\]
\[
Hz^* - g \in N_{K^c}(w^*). \tag{24b}
\]

Provided that the constraints in optimization (1) satisfy certain qualification, the conditions in (24) imply that \((z^*, w^*)\) is an optimal primal-dual solution of optimization (1) [19, Cor. 11.51]. We note that even if the conditions in (21) hold, optimization (1) can still be infeasible, i.e., its constraints cannot be satisfied. In this case, however, an infinitesimal perturbation of vector \( g \) will make optimization (1) either feasible or infeasible. See [3, Sec. 6.3] for an example.

Second, the strict inequality in (22) implies that \( \langle \vec{\pi}, \vec{w} \rangle = 0 \) is a hyperplane that separates cone \( K \) from the set \( \{Hz - g|z \in D\} \). See Fig. 1 for an illustration. As a result, optimization (1) is infeasible.

Third, if there exists \( z \in D \) such that \( Hz - g \in K \), then the conditions in (23) imply that the optimal value of optimization (1) is unbounded. In particular, let \( y = z + \vec{\pi} \), then (23) and the fact that \( \vec{\pi} \in rec D \) and \( Hz \in K \) imply the following:
\[
y \in D, \quad Hy - g \in K, \quad \frac{1}{2}y^TPy + q^Ty < \frac{1}{2}z^TPz + q^Tz.
\]

Hence the value of the objective function in optimization (1) strictly decreases along direction \( \vec{\pi} \) without violating the constraints. By repeating a similar process we can show the optimal value of optimization (1) is indeed unbounded from below. Furthermore, the dual problem of optimization (1) is infeasible [3, 21].

IV. APPLICATIONS IN CONSTRAINED OPTIMAL CONTROL

We demonstrate the application of PIPG in constrained optimal control problems. In particular, we will show how to use PIPG to compute minimum-time landing trajectory in Section IV-A and reduce the number of binary variables in nonconvex path-planning problem in Section IV-B.

Fig. 1: An illustration of the separating hyperplane.

Fig. 2: A custom-made quadrotor designed by the Autonomous Control Laboratory.

We consider the optimal control of the custom-made quadrotor designed by the Autonomous Control Laboratory (ACL) at University of Washington, which we referred to as the ACL quadrotor. See Fig. 2 for an illustration and [28] for a detailed description of the ACL quadrotor. For simplicity, we let all problem parameters in the following (e.g., mass of the quadrotor, maximum thrust magnitude) be unitless.

The dynamics of an ACL quadrotor is given by
\[
\frac{d}{ds}x(s) = A_c x(s) + B_c u(s) + h_c \tag{25}
\]
for all time \( s \in \mathbb{R}_+ \), where \( x(s) \in \mathbb{R}^d \) denotes the position and velocity of the center of mass of the quadrotor at time \( s \), \( u(s) \in \mathbb{R}^3 \) denotes the thrust vector provided by the propellers of the quadrotor at time \( s \). In addition,
\[
A_c = \begin{bmatrix} 0_{3 \times 3} & I_3 \\ 0_{3 \times 3} & 0_{3 \times 3} \end{bmatrix}, \quad B_c = \begin{bmatrix} 0_{3 \times 3} \\ 0_{3 \times 3} \end{bmatrix}, \quad h_c = \begin{bmatrix} 0_{5 \times 1} \\ 9.8_{-1} \end{bmatrix},
\]
where 0.35 is the total mass of the quadrotor, 9.8 is the gravity constant.

We can further simplify dynamics (25) by assuming the thrust vector does not change value within each sampling time period of length \( \Delta \in \mathbb{R}_+ \). In particular, suppose that
\[
u(s) = u(t\Delta), \quad t\Delta \leq s < (t + 1)\Delta \tag{26}
\]
for all $t \in \mathbb{N}$. Let $x_t := x(t\Delta)$ and $u_t := u(t\Delta)$ for all $t \in \mathbb{N}$. Then dynamics equation (25) is equivalent to

$$x_{t+1} = Ax_t + Bu_t + h$$

for all $t \in \mathbb{N}$, where

$$A = \exp(A_\Delta), \quad B = \left(\int_0^\Delta \exp(A_s)ds\right) B_c,$$

$$h = \left(\int_0^\Delta \exp(A_s)ds\right) h_c.$$  

Due to the physical limits of its onboard motors, we consider the following constraints on the thrust vector of the quadrotor. The magnitude of the thrust vector is lower bounded by $\rho_1 = 2$ and upper bounded by $\rho_2 = 5$. Further, the direction of the thrust vector is confined within an ice cream cone with half-angle angle $\theta = \pi/4$. These constraints can be written as $u_t \in \mathbb{U}$ for all $t \in \mathbb{N}$, where

$$\mathbb{U} := \{u \in \mathbb{R}^3 \mid \|u\| \cos \theta \leq [u]_3, [u]_3 \geq \rho_1, \|u\| \leq \rho_2\}.$$  

Notice that we approximate the lower bound on $\|u\|$ using a linear inequality constraint $[u]_3 \geq 2$ so that set $\mathbb{U}$ remains convex. See Fig. 3 for an illustration.

In the following, we will consider optimal control problems with dynamics constraints (27) (where we fix $\Delta = 0.2$) and input constraint set (29). We will solve these optimal control problems using Algorithm 2 (where $\alpha = 1$) and Algorithm 3 (where $\gamma$ satisfies Assumption 2 with $\gamma = 0.9$) and compare their performance. We will also use the following notation:

$$u_{[\tau \to i\tau]} := \begin{bmatrix} u^T_0 & u^T_1 & \ldots & u^T_{i-1} \end{bmatrix}^T,$$

$$x_{[\tau \to i\tau]} := \begin{bmatrix} x^T_1 & x^T_2 & \ldots & x^T_{i-1} \end{bmatrix}^T.$$  

A. Minimum-time landing via quasiconvex optimization

We consider the problem of landing a quadrotor on a charging station in the minimum amount of time possible subject to various state and input constraints. Let $x_0 \in \mathbb{R}^6$ denote the initial state of the quadrotor, and suppose that the charging station is located at the origin of the position coordinate system. Then the minimum landing time is given by $i\Delta$, where $\Delta$ is the sampling time in (26), and $i$ is the smallest integer that makes the following optimization problem feasible (i.e., there exists one solution that satisfies its constraints):

$$\begin{aligned}
\text{minimize} & \quad \frac{1}{2} \sum_{t=0}^{\tau-1} \|u_t\|^2 \\
\text{subject to} & \quad x_{t+1} = Ax_t + Bu_t + h, \ 0 \leq t \leq \tau - 1, \\
& \quad u_t \in \mathbb{U}, \ 0 \leq t \leq \tau - 1, \\
& \quad x_t \in \mathbb{X}, \ 1 \leq t \leq \tau - 1, \\
& \quad x_0 = 0, \ i \leq t \leq \tau - 1,
\end{aligned}$$

where $\tau \in \mathbb{N}$ is large enough such that optimization (31) is feasible if $i = \tau - 1$, set $\mathbb{U}$ is given by (29), and set $\mathbb{X}$ is given by

$$\mathbb{X} = \{r \in \mathbb{R}^3 \mid \|r\| \cos \beta \leq [r]_3 \times \{r \in \mathbb{R}^3 \mid \|r\| \leq \eta\},$$

where $\beta = \pi/4$, $\eta = 5$. The quadratic objective function in (31) penalizes large-magnitude inputs. The constraint $x_t \in \mathbb{X}$ upper bounds the speed of the quadrotor and limits the directions from which the quadrotor approach the charging station. The latter is also known as the approach cone constraint, which is widely used in spacecraft landing [15].

See Fig. 4 for an illustration.

Fig. 4: An illustration of the quadrotor landing.

One can compute the smallest integer $i$ that makes optimization (31) feasible by solving a quasiconvex optimization problem. To this end, we say $\{u_{[0,\tau-1]}, x_{[1,\tau-1]}\}$ is feasible for (31) if $\{u_{[0,\tau-1]}, x_{[1,\tau-1]}\}$ satisfy the constraints in optimization (31). We also define function $f$ using (32) (see the bottom of the next page). Then the smallest integer $i$ that makes optimization (31) feasible is the optimal value of the following optimization:

$$\begin{aligned}
\text{minimize} & \quad f(u_{[0,\tau-1]}, x_{[1,\tau-1]}),
\end{aligned}$$

We will show that optimization (33) is a quasiconvex optimization problem and can be solved using bisection method and infeasibility detection as follows. First, given any $\epsilon \in \mathbb{R}_+$, one can verify that

$$\{(u_{[0,\tau-1]}, x_{[1,\tau-1]})) \mid f(u_{[0,\tau-1]}, x_{[1,\tau-1]})) \leq \epsilon\} \quad \begin{cases}
\{u_{[0,\tau-1]}, x_{[1,\tau-1]}\} \text{ is feasible for (31) with } i = \lfloor \epsilon \rfloor,
\end{cases}$$

where $\lfloor \epsilon \rfloor$ is the largest integer less or equals to $\epsilon$. Since the constraints in (31) are all convex with respect to $u_{[0,\tau-1]}$ and $x_{[1,\tau-1]}$, we conclude that set (34) is convex. In other words, all sublevel sets of function $f$ are convex sets. Therefore, function $f$ is quasiconvex and one can solve optimization
by checking whether the set in (34) is empty, i.e., whether optimization (31) is feasible when \( i = [\epsilon] \), for different choices of \( \epsilon \in \mathbb{R}_+ \) [2, Sec. 4.2.5].

In the following, we consider optimization (33) where \( \tau = 40 \), \( i = \{24, 26\} \), and \( x_0 = [6 \ 6 \ 15 \ 2 \ 2 \ 2]^T \).

To solve the above optimization, we apply Algorithm 2 and Algorithm 3 to the corresponding optimization (31). In particular, Appendix IV gives the transformation from optimization (31) to optimization (1). Fig. 5 illustrates the convergence of \( \|w^j - w^{j-1}\| \) and \( \|z^j - z^{j-1}\| \) computed in Algorithm 2 and Algorithm 3, where the two algorithms show similar convergence. These convergence results show that both \( \|w^j - w^{j-1}\| \) and \( \|z^j - z^{j-1}\| \) converge to zero if \( i = 26 \), and \( \|w^j - w^{j-1}\| \) does not converge to zero if \( i = 24 \). As a result of Theorem 1, we conclude that the optimal value of optimization (33) is between 24 and 26 (solving another instance of optimization (31) will confirm that it is 25).

Fig. 5: Convergence of \( \|w^j - w^{j-1}\| \) and \( \|z^j - z^{j-1}\| \) in Algorithm 2 and Algorithm 3 when applied to optimization (31) with different values of \( i \).

We note that the quasiconvex optimization approach presented in this section has also been used for the minimum time control of overhead cranes [29].

B. Path-planning via mixed-integer optimization

We consider the problem of flying a quadrotor from a initial position to a final position, both located in a L-shaped corridor. See Fig. 6 for an illustration. To avoid collision with the boundary of the corridor, we consider the following state constraints

\[
\begin{align*}
& \mathbf{l}^1 \leq T \mathbf{x}_t \leq \mathbf{r}^1 \quad \text{or} \quad \mathbf{l}^2 \leq T \mathbf{x}_t \leq \mathbf{r}^2, \\
& \text{for all } t, \quad \mathbf{T} = \begin{bmatrix} I_3 & 0_{3 \times 3} \end{bmatrix}, \\
& \mathbf{l}^1 = [0 \ -2 \ 0]^T, \quad \mathbf{r}^1 = [2 \ 9 \ 3]^T, \\
& \mathbf{l}^2 = [2 \ -2 \ 0]^T, \quad \mathbf{r}^1 = [12 \ 0 \ 3]^T.
\end{align*}
\]

Given the initial and final state of the quadrotor (denoted by \( x_0 \in \mathbb{R}^6 \) and \( x_\tau \in \mathbb{R}^6 \), respectively), the above problem can be formulated as follows:

\[
\begin{align*}
\text{minimize} & \quad u_{[0, \tau-1]}^*, x_{[1, \tau-1]}^* \quad \frac{1}{2} \sum_{t=0}^{\tau-1} \|u_t\|^2 \\
\text{subject to} & \quad x_{t+1} = A x_t + B u_t + \mathbf{h}, \quad 0 \leq t \leq \tau - 1, \\
& \quad u_t \in U, \quad 0 \leq t \leq \tau - 1, \\
& \quad x_t \in X, \quad 1 \leq t \leq \tau - 1, \\
& \quad \mathbf{l}^r + b_t (\mathbf{r}^r - \mathbf{l}^r) \leq T x_t, \quad 1 \leq t \leq \tau - 1, \\
& \quad \mathbf{r}^r + b_t (\mathbf{r}^r - \mathbf{l}^r) \geq T x_t, \quad 1 \leq t \leq \tau - 1, \\
& \quad b_t \in \{0, 1\}, \quad 1 \leq t \leq \tau - 1,
\end{align*}
\]

where the notation \( b_{[0, \tau-1]} \) is similar to those in (30), \( U \) is given by (29), and \( X = \mathbb{R}^3 \times \{ r \in \mathbb{R}^3 \, | \, \|r\| \leq \eta \} \) with \( \eta = 5 \). The quadratic objective function in (36) penalizes large-magnitude inputs. The constraint \( x_t \in X \) upper bounds the speed of the quadrotor. We assume \( \tau \) is large enough such that optimization (36) has an optimal solution.

Fig. 6: An illustration of path-planning in an L-shaped corridor.

Optimization (36) is challenging to solve if the number of binary variables is large. For example, if \( \tau = 22 \), then optimization (36) contains 21 binary variables, and solving it naively requires considering \( 2^{21} \) (more than 2 millions) different values of vector \( b_{[0, \tau-1]} \), each value corresponds to a convex optimization problem.

One can reduce the number of binary variables in optimization (36) using infeasibility detection as follows. Let \( \hat{b} \in \{0, 1\} \) and \( (u^*_{[0, \tau-1]}, x^*_{[1, \tau-1]}, b^*_{[1, \tau-1]}) \) be an optimal solution of optimization (36). If \( b^*_1 = \hat{b} \), then

\[
f(u_{[0, \tau-1]}, x_{[1, \tau-1]}):= \min \{ i \in \mathbb{N} \cup \{ \infty \} \mid (u_{[0, \tau-1]}, x_{[1, \tau-1]}) \text{ is feasible for (31)} \}, \quad \text{where } \min \emptyset := \infty. \tag{32}
\]


\((u^*_0, \tau - 1), x^*_{1, \tau - 1}, b^*_{1, \tau - 1})\) satisfies the constraints of the following optimization:

\[
\begin{aligned}
\text{minimize} & \quad \frac{1}{2} \sum_{t=0}^\tau \|u_t\|^2 \\
\text{subject to} & \quad x_{t+1} = Ax_t + Bu_t + h, \ 0 \leq t \leq \tau - 1, \\
& \quad u_t \in U, \ 0 \leq t \leq \tau - 1, \\
& \quad x_t \in X, \ 1 \leq t \leq \tau - 1, \\
& \quad \tilde{r}_1 + b_t(z^2 - \tilde{r}^1) \leq T x_t, \ 1 \leq t \leq \tau - 1, \\
& \quad \tilde{r}_1 + b_t(z^2 - \tilde{r}^1) \geq T x_t, \ 1 \leq t \leq \tau - 1, \\
& \quad b_1 = b, b_t \in [0, 1], \ 1 \leq t \leq \tau - 1.
\end{aligned}
\] (37)

Compared with (36), here the constraints on \(b_{1, \tau - 1}\) are continuous instead of discrete, and \(v_t \) is fixed instead of being a variable.

If optimization (37) is infeasible (i.e., no solution satisfies its constraints), then \((u^*_0, \tau - 1), x^*_{1, \tau - 1}, b^*_{1, \tau - 1})\) does not satisfy the constraints in (37). Consequently, one must have \(b^*_1 \neq b\). In other words, we can eliminate the binary variable \(b_1\) in optimization (36) by fixing its value to be \(1 - b\).

In the following, we consider optimization (36) where \(\tau = 22\) and

\[
\begin{aligned}
x_0 &= \begin{bmatrix} 1 & 9 & 2.5 & 0 & 0 & 0 \end{bmatrix}^T, \\
x_\tau &= \begin{bmatrix} 12 & -1 & 0.5 & 0 & 0 \end{bmatrix}^T.
\end{aligned}
\]

To eliminate binary variable \(b_1\) in the above optimization, we apply Algorithm 2 and Algorithm 3 to the corresponding optimization (37). Particularly, Appendix [I] gives the transformation from optimization (37) to optimization (1). Fig. 7 illustrates the convergence of \(\|w^{j-1} - w^j\|\) and \(\|z^{j-1} - z^j\|\) computed in Algorithm 2 and Algorithm 3 where the two algorithms show similar convergence.

These convergence results show that both \(\|w^{j-1} - w^j\|\) and \(\|z^{j-1} - z^j\|\) converge to zero if \(b = 0\), and \(\|w^{j-1} - w^j\|\) does not converge to zero if \(b = 1\). As a result of Theorem 1, we conclude that optimization (37) is infeasible if \(b = 1\) and feasible if \(b = 0\). Therefore we can eliminate the binary variable \(b_1\) in optimization (36) by fixing its value to be 0.

By repeating a similar process, we can further reduce the number of binary variables, and conclude that \(b_i = 0\) for all \(i = 1, 2, \ldots, 7\) and \(b_i = 1\) for all \(i = 12, 13, \ldots, 21\) in optimization (36). In other words, by solving at most 42 convex optimization problems, each similar to optimization (37), we can reduce the number of possible values of binary variables \(b_{1,21}\) from \(2^{21}\) (more than 2 millions) to merely \(2^4\) (less than 20)!

V. CONCLUSIONS

We introduce a novel method, named proportional-integral projected gradient method (PIPG), for infeasibility detection in conic optimization. The iterates of PIGP either asymptotically provide a proof of primal or dual infeasibility, or asymptotically satisfy a set of primal-dual optimality conditions. We demonstrate the application of PIGP in quasi-convex and mixed integer optimization.

There are several questions about PIGP that still remain open. For example, does PIGP allow larger step sizes like the primal-dual method in [30]? If the projections in PIGP are computed approximately rather than exactly, how will the results in Theorem 1 change? We aim to answer these questions in our future work.

APPENDIX I

PROOF OF PROPOSITION 1

We will use the following result on projection.

**Lemma 5.** \([18, \text{Thm. 3.16}]\) Let \(D \subset \mathbb{R}^n\) be a nonempty closed convex set. Then \(y = \pi_D[z]\) if and only if \(y \in D\) and \(\langle y - z, z' - y \rangle \geq 0\) for any \(z' \in D\).

**Proof.** Let \(z_j \in D, v_j \in \mathbb{R}^m\) and

\[
\begin{aligned}
w_j^+ := \pi_{D^c}[v_j + \alpha(Hz_j - g)], \\
z_j^+ := \pi_{D}[z_j - \alpha(Pz_j + q + H^T w_j^+)], \\
v_j^+ := w_j^+ + \alpha(H(z_j^+ - z_j)),
\end{aligned}
\] (38a)

for \(j = 1, 2\). In order to show that line 2 in Algorithm 3 is the fixed point iteration of a \(\gamma\)-averaged operator, it suffices to show the following inequality

\[
\begin{aligned}
\|z_2^+ - z_1^+\|^2 + \frac{1 - \gamma}{\gamma} \|z_1 - z_1^+ - z_2 + z_2^+\|^2 \\
+ \|v_2^+ - v_1^+\|^2 + \frac{1 - \gamma}{\gamma} \|v_1 - v_1^+ - v_2 + v_2^+\|^2 \\
\leq \|z_2 - z_1\|^2 + \|v_2 - v_1\|^2
\end{aligned}
\] (39)

To this end, first we apply Lemma 5 to (38a) and (38b) for both \(j = 1\) and \(j = 2\) and obtain the following inequalities:

\[
0 \leq \langle w_1^+ - v_1 - \alpha(Hz_1 - g), w_2^+ - w_1^+ \rangle,
\] (40a)
Second, by completing the square we can show the following
\[ 2\langle P(z_2 - z_1), z_1^+ - z_1 - z_2^+ + z_2 \rangle \leq \frac{2}{\lambda} \| P(z_2 - z_1) \|^2 + \frac{1}{\lambda} \| z_1^+ - z_1 - z_2^+ + z_2 \|^2 . \] (50)

Finally, by multiplying both sides of (44) by 2, both sides of (45) by \( \alpha \), both sides of (48) by \( \frac{2\gamma}{2\gamma - 1} \), both sides of (49) by \( 2\alpha \), both sides of (50) by \( \alpha \), then summing them up together with both sides of (46) and (47), we obtain (39) if \( \frac{\gamma\alpha^2}{2\gamma - 1} + \frac{\alpha}{\gamma} \leq \frac{2\gamma - 1}{\gamma} \).

Combining the above two cases, we conclude that (39) holds for all \( \lambda \in \mathbb{R}_+ \) if
\[ \frac{\gamma\alpha^2}{2\gamma - 1} + \frac{\alpha}{\gamma} \leq \frac{2\gamma - 1}{\gamma} . \]

One can verify that the above inequality holds under Assumption 2, which completes the proof.

**APPENDIX II**

**PROOF OF LEMMA 3**

We will again use Lemma 5.

**Proof.** Using line 2 in Algorithm 3 we can show the following
\[ z^j = \pi_D[z^{j-1} - \alpha(Pz^{j-1} + q + H^T w^j)], \]
\[ w^j = \pi_{K^c}[w^{j-1} + \alpha(H(2z^{j-1} - z^{j-2}) - g)]. \]

Applying Lemma 5 to the above two projections and using the definition of normal cone in (5) we can show
\[ \frac{1}{\alpha}(z^{j-1} - z^j) - Pz^{j-1} - q - H^T w^j \in N_D(z^j), \]
\[ \frac{1}{\alpha}(w^{j-1} - w^j) + H(2z^{j-1} - z^{j-2}) - g \in N_{K^c}(w^j). \] (51a) (51b)

Hence
\[ d(-Pz^j - q - H^T w^j|N_D(z^j)) \leq \frac{1}{\alpha} \| z^{j-1} - z^j \| - \| Pz^j \|, \]
where the first inequality is due to the definition of distance function in (4) and (51a), the second inequality is due to the triangle inequality. Finally, combining the above inequality with the fact that \( \lambda \| z \| \geq \| Pz \| \) for any \( z \in \mathbb{R}^n \), we obtain the first inequality in Lemma 5.

Similarly, we can show the following
\[ d(Hz^j - g|N_{K^c}(w^j)) \leq \frac{1}{\alpha} \| w^{j-1} - w^j \| + H(z^j - 2z^{j-1} + z^{j-2}) \| , \]
where the first inequality is due to the definition of distance function in (4) and (51b), and the second inequality is due to the triangle inequality. Finally, combining the above inequality with the fact that \( \nu \| z \| \geq \| Hz \| \) for any \( z \in \mathbb{R}^n \), we obtain the second inequality in Lemma 5.

\( \square \)
APPENDIX III

Proof of Lemma 2

Let \( y_j \in \mathbb{R}^n \) for all \( j \in \mathbb{N} \) and there exists \( \gamma \in \mathbb{R}^n \) such that
\[
\lim_{j \to \infty} \gamma_j = \gamma.
\]
Let \( D \subset \mathbb{R}^n \) be a closed convex set.

1. \( y_j - \pi_D[\gamma_j] \in (\text{rec } D)^\circ \)?
2. \( \lim_{j \to \infty} \pi_D[\gamma_j] = \pi_{\text{rec } D}[\gamma] \).
3. \( \lim_{j \to \infty} \gamma_j = \pi_{\text{rec } D}[\gamma] \).
4. \( \lim_{j \to \infty} \pi_D[\gamma_j] = \pi_{\text{rec } D}[\gamma] \).

We will also use the following fact: given sequences \( \{y_j\}_{j \in \mathbb{N}} \) and \( \{z_j\}_{j \in \mathbb{N}} \) where \( x^j, y_j \in \mathbb{R}^n \) for all \( j \), if the limits \( \lim_{j \to \infty} x^j \) and \( \lim_{j \to \infty} z^j \) both exist and are finite-valued, then
\[
\lim_{j \to \infty} \langle y_j, z_j \rangle = \lim_{j \to \infty} \langle y_j, z_j \rangle.
\]

Proof. We start by showing the two limits. From Proposition 1 and Lemma 2 we know that there exists \( \tau \in \mathbb{R}^n \) and \( \overline{w} \in \mathbb{R}^n \) such that
\[
\begin{align*}
\lim_{j \to \infty} x^j_j &= \lim_{j \to \infty} z^j_j - z_j = \tau, \quad (53a) \\
\lim_{j \to \infty} w^j_j &= \lim_{j \to \infty} v^j_j - v_j = \overline{w}, \quad (53b)
\end{align*}
\]

In addition, using line 2 in Algorithm 3 one can show the following
\[
\begin{align*}
v^j_j - v_j &= w^j_j - w_j - v^j_j \quad (54a) \\
v^j_j &= \frac{w^j_j}{\alpha } + \frac{\alpha }{2} H(z_j - z_j).
\end{align*}
\]

Letting \( j \to \infty \) in the above two equations, then using (53a) and (53b) we can show the following
\[
\lim_{j \to \infty} w^j_j = \lim_{j \to \infty} w^j_j + w_j = \overline{w}.
\]

The above equation and (53a) together give the two limits in Lemma 2

Next, we show that \( \tau \in \text{rec } D, \overline{w} \in \mathbb{K}^\circ \) and the conditions in (19) hold. To this end, let
\[
\begin{align*}
z^j_j &= \tau^j_j - \alpha (P \tau^j_j + H^\top \overline{w}^j_j) \quad (55a) \\
\hat{w}^j_j &= \tau^j_j - \alpha (H(2z^j_j - z^j_j - g) + \overline{w}^j_j).
\end{align*}
\]

Then using (53a) and (54) we can show that the following two limits
\[
\begin{align*}
\lim_{j \to \infty} z^j_j &= \tau - \alpha (P \tau + H^\top \overline{w}) \quad (56a) \\
\lim_{j \to \infty} \hat{w}^j_j &= \overline{w} + \alpha H \overline{w}.
\end{align*}
\]

Further, using line 2 in Algorithm 3 we can show that
\[
z^j_j = \pi_D[z^j_j], \quad \hat{w}^j_j = \pi_{\text{rec } K}[\hat{w}^j_j].
\]

Hence, by applying Lemma 6 to sequences \( \{z^j\}_{j \in \mathbb{N}} \) and \( \{\hat{w}^j\}_{j \in \mathbb{N}} \) we can show that
\[
\begin{align*}
z^j_j - z^j_j &= \pi_D[z^j_j], \quad \hat{w}^j_j - \hat{w}^j_j \in \mathbb{K},
\end{align*}
\]

Proof. We start by showing the two limits. From Proposition 1 and Lemma 2 we know that there exists \( \tau \in \mathbb{R}^n \) and \( \overline{w} \in \mathbb{R}^n \) such that
\[
\begin{align*}
\lim_{j \to \infty} x^j_j &= \lim_{j \to \infty} z^j_j - z_j = \tau, \quad (53a) \\
\lim_{j \to \infty} w^j_j &= \lim_{j \to \infty} v^j_j - v_j = \overline{w}, \quad (53b)
\end{align*}
\]

In addition, using line 2 in Algorithm 3 one can show the following
\[
\begin{align*}
v^j_j - v_j &= w^j_j - w_j - v^j_j \quad (54a) \\
v^j_j &= \frac{w^j_j}{\alpha } + \frac{\alpha }{2} H(z_j - z_j).
\end{align*}
\]

Letting \( j \to \infty \) in the above two equations, then using (53a) and (53b) we can show the following
\[
\lim_{j \to \infty} w^j_j = \lim_{j \to \infty} w^j_j + w_j = \overline{w}.
\]

The above equation and (53a) together give the two limits in Lemma 2

Next, we show that \( \tau \in \text{rec } D, \overline{w} \in \mathbb{K}^\circ \) and the conditions in (19) hold. To this end, let
\[
\begin{align*}
z^j_j &= \tau^j_j - \alpha (P \tau^j_j + H^\top \overline{w}^j_j) \quad (55a) \\
\hat{w}^j_j &= \tau^j_j - \alpha (H(2z^j_j - z^j_j - g) + \overline{w}^j_j).
\end{align*}
\]

Then using (53a) and (54) we can show that the following two limits
\[
\begin{align*}
\lim_{j \to \infty} z^j_j &= \tau - \alpha (P \tau + H^\top \overline{w}) \quad (56a) \\
\lim_{j \to \infty} \hat{w}^j_j &= \overline{w} + \alpha H \overline{w}.
\end{align*}
\]

Further, using line 2 in Algorithm 3 we can show that
\[
z^j_j = \pi_D[z^j_j], \quad \hat{w}^j_j = \pi_{\text{rec } K}[\hat{w}^j_j].
\]

Hence, by applying Lemma 6 to sequences \( \{z^j\}_{j \in \mathbb{N}} \) and \( \{\hat{w}^j\}_{j \in \mathbb{N}} \) we can show that
\[
\begin{align*}
z^j_j - z^j_j &= \pi_D[z^j_j], \quad \hat{w}^j_j - \hat{w}^j_j \in \mathbb{K},
\end{align*}
\]
Combining (65) and (66) gives the following
\[ 0 = \lim_{j \to \infty} \frac{1}{j} \langle z^j - z^{j-1}, z^j - z^{j-1} \rangle. \]  
(67)

Second, one can show the following
\[
\begin{align*}
&\lim_{j \to \infty} \frac{1}{j} \langle z^j - z^{j-1}, z^j - z^{j-1} \rangle \\
&= \lim_{j \to \infty} \langle z^j - z^{j-1} + \alpha (P z^{j-1} + q + H^T w^j), z^{j-1} \rangle \\
&= \|w\|^2 + \langle q, \tau \rangle + \alpha \lim_{j \to \infty} \frac{1}{j} \langle P z^{j-1} + H^T w^j, z^{j-1} \rangle \\
&\geq \|z\|^2 + \langle q, \tau \rangle + \alpha \lim_{j \to \infty} \frac{1}{j} \langle H^T w^j, z^{j-1} \rangle,
\end{align*}
\]
where the first step is due to (55a), the second step is due to (53a) and (53a) and the last step is due to the positive semidefiniteness of matrix \( P \). Third, we can show that following
\[
\begin{align*}
0 &= \lim_{j \to \infty} \frac{1}{j} \langle w^j - w^{j-1}, w^j \rangle \\
&= \lim_{j \to \infty} \langle w^j - w^{j-1} - \alpha (H (2 z^{j-1} - z^{j-2}) - g), w^j \rangle \\
&= \|w\|^2 + \langle q, \tau \rangle - \langle H \bar{\tau}, \bar{w} \rangle - \alpha \lim_{j \to \infty} \frac{1}{j} \langle H^T w^j, z^{j-1} \rangle,
\end{align*}
\]  
(69)
where the first equality is due to (58a), the second equality is due to (55b), and the last equality is due to (52a) and (54). By summing up both sides of (55a), (67), (68) and (69) then multiplying the resulting inequality by \( \alpha \), we obtain the following
\[ \sigma_\mathcal{D}(-H^T \bar{w}) + \langle q, \tau \rangle + \langle \bar{w}, g \rangle + \frac{1}{\alpha} \|\bar{\tau}\| + \frac{1}{\alpha} \|\bar{w}\|^2 \leq 0, \]
where we also used (60), (61), and the fact that \( \alpha \sigma_\mathcal{D}(-H^T \bar{w}) = \sigma_\mathcal{D}(-\alpha H^T \bar{w}) \) when \( \alpha > 0 \). Combining the above inequality with the one in (64), we conclude that the inequality in (64) holds as an equality. As a result, both inequalities in (65) also hold as equalities, which completes the proof.

**APPENDIX IV**

**TRANSFORMATION FROM OPTIMAL CONTROL PROBLEMS TO CONIC OPTIMIZATION PROBLEMS**

We let \( \mathbf{1}_n \) and \( \mathbf{0}_n \) denote the \( n \)-dimensional vectors of all 1’s and all 0’s, respectively. Let \( \text{diag}(c) \) denote the diagonal matrix whose diagonal elements are given by vector \( c \). Further, we let
\[
\begin{align*}
\mathbf{A} &= \begin{bmatrix} I_{\tau (\tau - 1)} \\ \mathbf{0}_{\tau \times (\tau - 1)} \end{bmatrix}, \\
\mathbf{B} &= -I_\tau \otimes \mathbf{B}, \quad \mathbf{C} = I_\tau \otimes \begin{bmatrix} 0 & 1 \end{bmatrix}, \\
\mathbf{D} &= I_{\tau - 1} \otimes \begin{bmatrix} I_3 & 0_{3 \times 3} \\ -I_3 & 0_{3 \times 3} \end{bmatrix}, \quad \mathbf{E} = I_{\tau - 1} \otimes \begin{bmatrix} \Sigma_1 - \Sigma_2 \\ \Sigma_2 - \Sigma_1 \end{bmatrix},
\end{align*}
\]
and
\[
\begin{align*}
H_1 &= \begin{bmatrix} \mathbf{A} \\ \mathbf{0}_{\tau \times (\tau - 1)} \end{bmatrix}, \\
g_1 &= \begin{bmatrix} (Ax_0 + h)^T (x_{2r-1}\mathbf{0})^T (x_{2r-1} + h)^T \rho \mathbf{1}_r^T \end{bmatrix}^T, \\
H_2 &= \begin{bmatrix} \mathbf{D} \\ 0_{(\tau - 1) \times 3r} \end{bmatrix}, \quad g_2 = \mathbf{1}_{\tau - 1} \otimes \begin{bmatrix} \mathbf{1}_r \\ -\mathbf{1}_r \end{bmatrix}, \\
P_1 &= \text{diag} \left( \begin{bmatrix} \mathbf{0}_{(\tau - 1)^T} \\ \mathbf{1}_{3r} \end{bmatrix} \right), \quad q_1 = \mathbf{0}_{3r - 6}, \\
P_2 &= \text{diag} \left( \begin{bmatrix} \mathbf{0}_{(\tau - 1)^T} \\ \mathbf{1}_{3r} \end{bmatrix} \right), \quad q_2 = \mathbf{0}_{10r - 7}.
\end{align*}
\]
Further, we let
\[
\begin{align*}
\mathcal{X}_1 &= \{ r \in \mathbb{R}^3 \mid \|r\| \cos \beta \leq \|r\| \leq \eta \}, \\
\mathcal{X}_2 &= \mathbb{R}^3 \times \{ r \in \mathbb{R}^3 \mid \|r\| \leq \eta \}, \\
\mathcal{U}_1 &= \{ u \in \mathbb{R}^3 \mid \|u\| \leq 3 \|u\|, \|u\| \leq \rho_2 \}, \\
\mathcal{K}_1 &= \{ \mathbf{0}_{6r} \} \times \mathcal{X}_2.
\end{align*}
\]
With the above definitions, we are ready to transform the optimal control problems in Section IV into space cases of conic optimization (1). Optimization (31) is the special case of (1) where
\[
\begin{align*}
z &= \begin{bmatrix} x_1 \ldots x_{\tau - 1} u_0 \ldots u_{\tau - 1} \end{bmatrix}^T, \\
p &= \mathbf{P}_1, \quad q = q_1, \quad H = \mathbf{S}_1 H_1, \quad g = \mathbf{S}_1 g_1, \quad \mathcal{K} = \mathcal{K}_1, \\
\mathcal{D} &= (\mathcal{X}_1)^{\tau - 1} \times (\mathcal{U}_1)^{\tau - 1},
\end{align*}
\]
where \( \mathbf{S}_1 \) is a diagonal matrix with positive diagonal entries such that the rows in matrix \( H \) have unit \( \ell_2 \) norm. And optimization (31) is the special case of (1) where
\[
\begin{align*}
z &= \begin{bmatrix} x_1 \ldots x_{\tau - 1} u_0 \ldots u_{\tau - 1} b_1 \ldots b_{\tau - 1} \end{bmatrix}^T, \\
H &= \mathbf{S}_2 \begin{bmatrix} H_1 \\ H_2 \end{bmatrix}, \quad g = \mathbf{S}_2 \begin{bmatrix} g_1 \\ g_2 \end{bmatrix}, \\
p &= \mathbf{P}_2, \quad q = q_2, \quad \mathcal{K} = \mathcal{K}_1 \times \mathcal{X}_2^{\tau - 1}, \\
\mathcal{D} &= (\mathcal{X}_2)^{\tau - 1} \times (\mathcal{U}_1)^{\tau - 1} \times \{1 - b\} \times [0,1]^n - 2^2,
\end{align*}
\]
where \( \mathbf{S}_2 \) is a diagonal matrix with positive diagonal entries such that the rows in matrix \( H \) have unit \( \ell_2 \) norm. Notice that, in both problems, we use matrix \( \mathbf{S}_1 \) and \( \mathbf{S}_2 \) to ensure the constraints parameters \( H \) and \( g \) are well-conditioned.

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