Collective and relative variables for a classical Klein-Gordon field

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Abstract

In this paper a set of canonical collective variables is defined for a classical Klein-Gordon field and the problem of the definition of a set of canonical relative variables is discussed.

This last point is approached by means of a harmonic analysis in momentum space. This analysis shows that the relative variables can be defined if certain conditions are fulfilled by the field configurations.

These conditions are expressed by the vanishing of a set of conserved quantities, referred to as supertranslations since as canonical observables they generate a set of canonical transformations whose algebra is the same as that which arises in the study of the asymptotic behaviour of the metric of an isolated system in General Relativity [9].

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I. INTRODUCTION.

In the search for a theory of $N$ relativistic interacting particles a possible way is the use of constrained dynamics $[1]$. This is a phase-space approach based on a set of mass constraints, one for each particle, in involution among themselves, or first class following Dirac $[2]$, where the masses are potential masses depending on the relative coordinates of the particles as well on their momenta.

Since each constraint can be interpreted as a Hamilton-Jacobi equation $[3]$ and their first class character as the corresponding integrability condition, one describes the dynamics in terms of $N$ independent time parameters, with no necessity of introducing gauge fixing conditions.

This approach has many conceptual advantages but, from the point of view of model building, apart from the case of $N = 2$ $[4]$, shows great difficulties if we try to fulfill the cluster decomposition (also called separability) $[5]$.

So it is desirable to have a one-time formulation, preserving the relativistic invariance in a controlled way, where the cluster decomposition property could be more easily satisfied.

A way to do this was suggested many years ago by Dirac $[2]$. He suggested to introduce a foliation of Minkowski spacetime by a family of space-like surfaces; the parameter of such a family can be regarded as a unique time parameter, and can be used for a one-time formulation of the dynamics of an $N$-particle system. The various quantities which characterize each space-like surface of the family become dynamical variables, which must be added to the phase-space of the system.

A great simplification can be obtained with suitable gauge fixings which restrict the surfaces to be space-like planes: in the case of an isolated system there is the possibility of using the conservation of the total momentum of the system to define, in an intrinsic way, a family of space-like planes orthogonal to the total momentum $P_\mu$. It amounts in using as time the one that would be measured in the rest frame of the total momentum.

This approach has been developed by Lusanna $[6]$: following this line it becomes of
interest to consider fields besides particles to take account of their interactions, for instance electromagnetic interactions.

In order to completely work out a correct canonical framework of the isolated system of matter and fields (treated as in [6]), it becomes necessary to define ”center of mass” and relative variables for the fields too: by using this canonical collective-relative variables the complete Dirac-Bergmann [7] reduction of the whole gauge system is possible.

It is well known that the definition of a center of mass in special relativity presents various difficulties [8]. It is not possible to have a sound definition of a center of mass coordinate adapted to the first class constraints of the model under study, which is at the same time canonically conjugated to the total momentum and covariant. Nevertheless, for the purpose discussed here, it can be sufficient to define ”some” canonical variable $X^\mu$ conjugated to the momentum, renouncing for the moment to an explicit physical meaning analogous to the Newtonian one. We will reserve the name collective coordinates for this canonical coordinate and for the total momentum; the other independent canonical coordinates will be referred to as relative.

Moreover, there will be the problem of defining such variables for an infinite-dimensional system like a field. We propose in this paper an approach to this problem and show, in the simplest case of a real scalar field, that it is possible to define a canonical set of collective and relative variables with some restrictions that appear to be as integrability conditions: our recipe for the definition of collective variables seems to work only within a particular class of field configurations; there will be the necessity to understand if these restrictions are of a general nature (and are eventually met whenever one tries to define collective and relative variables for a relativistic field), or only affect our particular way to work out the variable $X^\mu$.

Anyway, when collective and relative variables are singled out, the symplectic Lorentz algebra splits into the direct sum of two copies of $\mathfrak{so}(1, 3)$, the first involving only relative variables, the second depending only on $X$ and $P$; this gives rise to a decomposition of the Poincaré algebra in a direct sum of a collective inhomogeneous Lorentz algebra, and a
homogeneous relative one. In this way we get a definition of the total conserved intrinsic angular momentum of the system.

The path we have chosen for the definition of this canonical basis was the use of a canonical modulus-phase set in momentum space. This is not of course the only way and probably not the most convenient, indeed the relation between the new variables and the old ones is very involved. Nevertheless, as far as the authors know, no attempt of this kind is present in the literature.

In Section II we discuss a possible choice of the collective canonical variables. In order to define the variable $X^\mu$ conjugated to the total momentum $P^\mu$, we introduce a partially arbitrary function $F$, which is a function of $(P \cdot k)$ and $P^2$, where $k$ is the momentum space variable of the field. This is quite analogous to the classical particle case, where the center of mass position is defined in terms of a set of weights, which are arbitrary apart from a normalization condition. The transformation properties of $X^\mu$ are discussed.

In Section III we give a discussion of the relative variables, which are required to be scalar fields. Their complete definition requires the full analysis of the Laplace-Beltrami differential operator defined on the mass-hyperboloid in momentum space, which has been exhaustively studied in [9].

The Poisson algebra of the new variables with the infinitesimal generators of Lorentz transformations is studied, and it is shown the splitting of the Poincaré algebra in collective and relative parts. This decomposition was expected in view of the analogy with the classical particle case.

Section IV is finally devoted to the discussion of the consistency conditions of the method, that single out the field configurations for which this all seems to work: they form a very well defined family of configurations selected for purely mathematical reasons at this level; anyway we think their physical content could be deeper, and it would deserve further study.

These consistency conditions essentially involve those quantities $P_{l,m}[^\Phi, \Pi]$ referred to as supertranslations (see [9] and references therein), that appeared for the first time in a general relativistic context.
The definitions and notations of the Klein-Gordon field are given in Appendix A. In Appendix B some useful Poisson brackets are worked out, while in Appendix C the Laplace-Beltrami operator needed by the whole study is discussed.

In Appendix D we give some details of the calculation of the Green function of the Laplace-Beltrami operator, which is necessary for the canonical transformation to the new variables and for its inverse. Finally, in Appendix E we construct the Poisson algebra of the Poincaré group in terms of the new relative set.

\section*{II. THE COLLECTIVE VARIABLES.}

\textbf{A. Definition of the collective variables.}

In this Section we are interested in the search of a variable conjugated to the total 4-momentum of the field (see Appendix A),

$$P^\mu = \int \bar{\eta}k^\mu \bar{a}(k)a(k). \quad (2.1)$$

(we will use the symbol $f(k)$ for simplicity, instead of $f(\vec{k})$).

For a field $\Phi(x)$ solution of the Klein-Gordon equation $P^\mu$ is a timelike four-momentum \cite{10}. This $P^\mu$ is finite if

$$|a(k)| \simeq \left| \vec{k} \right|^{-\frac{\sigma}{2}}, \quad \sigma > 0, \text{ for } \left| \vec{k} \right| >> m,$$

$$|a(k)| \simeq \left| \vec{k} \right|^{-\frac{2+\epsilon}{2}}, \quad \epsilon > 0, \text{ for } \left| \vec{k} \right| << m, \quad (2.2)$$

and this condition allows the other Poincaré generators (as given in \text{[A19]} and \text{[A20]}) to be finite too.

A possible approach to the problem of finding some $X$ conjugated to $P$ is to look for a field $\phi(k)$ conjugated to the square modulus of $a(k)$

$$I(k) = \bar{a}(k)a(k), \quad (2.3)$$

that is
\{I(k), \phi(k')\} = \Omega(k) \delta^3(\vec{k} - \vec{k}'). \tag{2.4}

The general solution for \(\phi\) is the phase

\[ \phi(k) = \frac{1}{2i} \ln \frac{a(k)}{\bar{a}(k)}, \tag{2.5} \]

(where the principal value of the logarithm is meant) plus an arbitrary function of \(I(k)\). The inverse is

\[ a(k) = \sqrt{I(k)} \exp i\phi(k), \tag{2.6} \]

and the Poisson bracket \(\{I(k), \phi(k')\}\) is exactly what we needed.

Relation (2.4) has been criticized by Dubin et al. [11]. These authors observed that, if we interpret the square modulus \(I(k)\) and the phase \(\phi(k)\) as distributions in \(S'(\mathbb{R}^3)\) (the space of tempered distributions on \(\mathbb{R}^3\)), due to a discontinuity in any possible definition of the phase, one necessarily gets a non canonical term added to the right hand side of Eq.(2.4). This noncanonical term is a distribution having the form of Dirac comb with support on the values of \(k\) which give \(\phi(k)\) equal to some integer multiple of \(2\pi\), depending on the definition of the original function \(\phi\).

We will adopt the point of view that the phase is a function. In this case only the classical derivatives of the phase will appear in the Poisson bracket, and no anomalous term.

Let us now define a collective coordinate, conjugated to the total momentum \(P^\mu\).

It is well known that no center of mass coordinate in the usual sense exists in relativistic theories [8]; in any case we may define a coordinate conjugated to \(P^\mu\). Indeed we may choose

\[ X^\mu = \int \tilde{dk} \phi(k) \frac{\partial}{\partial P_\mu} F(P, k), \tag{2.7} \]

where \(F(P, k)\) is a scalar function of \(P\) and \(\vec{k}\), i.e. a function of \(P^2\) and \((P \cdot k)\), where

\[ k^0 = \omega(k) = \sqrt{\vec{k}^2 + m^2}. \]

The function \(F(P, k)\) is the analogous of the weights required in order to define the center of mass position in particle physics. They undergo a normalization condition to render the center of mass coordinate conjugate to the total momentum.
The conditions we put on the function $F$ are:

a)-normalization:

$$\int \tilde{d}kk^\mu F(P, k) = P^\mu,$$  \hspace{1cm} (2.8)

which can be satisfied by any scalar function, since the integral on the left hand side is, by covariance, equal to $P^\mu$ times a function of $P^2$. Dividing by this function we may identically satisfy the normalization (2.8).

b)-Reality:

$$\overline{F}(P, k) = F(P, k).$$  \hspace{1cm} (2.9)

c)-Integrability and differentiability. The existence of the integral in Eq.(2.7) depends on the behaviour of the phase $\phi(k)$, which will be discussed in the sequel. We will assume for $F$ the necessary behaviour for $|\vec{k}| \to \infty$ and for $|\vec{k}| \to 0$.

A simple choice of $F$ could be $F \propto e^{-(P \cdot k)}$, to be properly normalized. Since both $P^\mu$ and $k^\mu$ are time-like and future directed, this choice provides a smoothing factor at infinity in momentum space.

Using the Eq.(2.8) we find

$$\{P^\mu, X^\nu\} = \int \tilde{d}kk^\mu \frac{\partial}{\partial P^\nu} F((P \cdot k), P^2) = \eta^{\mu\nu}.$$  \hspace{1cm} (2.10)

Since

$$\{\phi(k), P^\mu\} = -k^\mu,$$  \hspace{1cm} (2.11)

we have also

$$\{X^\mu, X^\nu\} = 0.$$  \hspace{1cm} (2.12)

Indeed

$$\{X^\mu, X^\nu\} =$$

$$= \int \tilde{d}k\rho \frac{\partial}{\partial P^\mu} F((P \cdot k), P^2) \int \tilde{d}k'\phi(k') \frac{\partial^2}{\partial P_\rho \partial P_\nu} F((P \cdot k'), P^2) - (\mu \leftrightarrow \nu) =$$

$$= \eta^\mu_\rho \int \tilde{d}k\phi(k) \frac{\partial^2}{\partial P_\rho \partial P_\nu} F((P \cdot k'), P^2) - (\mu \leftrightarrow \nu) = 0.$$
Let us observe that a redefinition of the phase $\phi(k)$ by

$$\phi(k) \rightarrow \phi(k) + 2\pi N(k), \quad N(k) \in \mathbb{Z},$$

(2.13)

produces a canonical transformation on the variables $P^\mu$ and $X^\mu$, under which $P^\mu$ is unchanged and $X^\mu$ is transformed in

$$X'^\mu = X^\mu + \frac{\partial}{\partial P_\mu} \mathcal{G}(P^2),$$

(2.14)

where

$$\mathcal{G}(P^2) = 2\pi \int \tilde{d}k N(k) F((P \cdot k), P^2).$$

(2.15)

B. Transformation properties of $P^\mu$ and $X^\mu$.

The field $\Phi(x)$ is, by hypothesis, a scalar field, that is, if $U(\Lambda, a)$ represents the Poincaré transformation $(\Lambda, a)$ on the field $\Phi$ we have

$$(U(\Lambda, a) \Phi)(x) = \Phi(\Lambda^{-1}(x - a)).$$

(2.16)

This yields the following transformation on the modulus and phase

$$(U(\Lambda, a) \Lambda)(k) = I(\Lambda^{-1}k), \quad (U(\Lambda, a) \phi)(k) = \phi(\Lambda^{-1}k) + (k \cdot a).$$

(2.17)

Denoting with a prime the corresponding transformed quantities, we may easily verify that under the transformation (2.16) we have

$$P'^\mu = \Lambda^\mu_{\nu} P^\nu, \quad X'^\mu = \Lambda^\mu_{\nu} X^\nu + a^\mu.$$  

(2.18)

Our $X$ transforms exactly as the spacetime coordinate of an event in $M_4$.

In terms of modulus and phase, the generators of the Poincaré algebra are

$$P^\mu = \int \tilde{d}k k^\mu I(k),$$

(2.19)
\[ M_{ij} = \int \tilde{dk} I(k) \left( k^i \frac{\partial}{\partial k^j} - k^j \frac{\partial}{\partial k^i} \right) \phi(k), \]  
(2.20)

\[ M_{0j} = x_0 P_j + M'_{0j}, \quad M'_{0j} = - \int \tilde{dk} I(k) \omega(k) \frac{\partial}{\partial k^j} \phi(k). \]  
(2.21)

For the existence of these generators, and in order to be able to perform an integration by parts in the expression of \( M_{\mu\nu} \), we have to require the following asymptotic behaviour of the fields \( I(k) \) and \( \phi(k) \), as \( |\vec{k}| \to \infty \) and as \( |\vec{k}| \to 0 \)

\[ |I(k)| \approx |\vec{k}|^{-3-\sigma}, \quad |\phi(k)| \approx |\vec{k}| \quad \text{as} \quad |\vec{k}| \to \infty, \]  
(2.22)

\[ |I(k)| \approx |\vec{k}|^{-3+\epsilon}, \quad |\phi(k)| \approx |\vec{k}|^\tau, \quad \tau > -\epsilon \quad \text{as} \quad |\vec{k}| \to 0, \]  
(2.23)

for any \( \sigma, \epsilon > 0 \), which are coherent with (2.2), so that the behaviour of \( I(k) \) and \( \phi(k) \) (2.22) and (2.23) is consistent with the existence of the generators of the Poincaré generators (2.19), (2.20) and (2.21). The asymptotic behaviour of \( \phi(k) \) is determined by the transformation rule Eq.(2.17), while around \( |\vec{k}| = 0 \) the behaviour (2.23) is worked out in order to keep \( M_{ij} \) and \( M_{0j} \) finite.

Under the action of the Poincaré generators the modulus and phase transform as

\[ \{ P^\mu, I(k) \} = 0, \]  
(2.24)

\[ \{ P^\mu, \phi(k) \} = k^\mu, \]  
(2.25)

\[ \{ M^{ij}, I(k) \} = \left( k^i \frac{\partial}{\partial k^j} - k^j \frac{\partial}{\partial k^i} \right) I(k), \]  
(2.26)

\[ \{ M^{ij}, \phi(k) \} = \left( k^i \frac{\partial}{\partial k^j} - k^j \frac{\partial}{\partial k^i} \right) \phi(k), \]  
(2.27)

\[ \{ M^0j, I(k) \} = \omega(k) \frac{\partial}{\partial k^j} I(k), \]  
(2.28)

\[ \{ M^0j, \phi(k) \} = \omega(k) \frac{\partial}{\partial k^j} \phi(k), \]  
(2.29)
which can be collected in the form

$$\{M_{\mu\nu}, I(k)\} = D_{\mu\nu} I(k),$$

(2.30)

$$\{M_{\mu\nu}, \phi(k)\} = D_{\mu\nu} \phi(k),$$

(2.31)

where

$$D_{\mu\nu} = (\eta^{h\mu} k^\nu - \eta^{h\nu} k^\mu) \frac{\partial}{\partial k^h},$$

(2.32)

and \(k_0 = \omega(k)\).

The differential operator \(D_{\mu\nu}\) satisfies the following \(\mathfrak{so}(1,3)\) algebra

$$[D_{\mu\nu}, D_{\rho\lambda}] = \eta_{\mu\rho} D_{\nu\lambda} + \eta_{\nu\lambda} D_{\mu\rho} - \eta_{\mu\lambda} D_{\nu\rho} - \eta_{\nu\rho} D_{\mu\lambda}.\)

(2.33)

In order to find the transformation properties of the collective variables \(P^\mu\) and \(X^\mu\) under the action of the Poincaré’s generators, we must before develop two relations involving the function \(F((P \cdot k), P^2)\).

Let us call \(F_{/1}\) and \(F_{/2}\) the derivatives of the function \(F\) with respect to the first and second argument respectively. From the form of \(F\) we get

$$\frac{\partial}{\partial k^i} F = \left( P_i - \frac{k_i P_0}{\omega(k)} \right) F_{/1},$$

(2.34)

$$\frac{\partial}{\partial P_i} F = k_i F_{/1} + 2 P_i F_{/2},$$

(2.35)

$$\frac{\partial}{\partial P^0} F = \omega(k) F_{/1} + 2 P^0 F_{/2}.$$  

(2.36)

From these relations it’s easy to get

$$\left( k_j \frac{\partial}{\partial k^i} - k_i \frac{\partial}{\partial k^j} \right) F = \left( P_i \frac{\partial}{\partial P_j} - P_j \frac{\partial}{\partial P_i} \right) F,$$

(2.37)

and

$$\omega(k) \frac{\partial}{\partial k^j} F = \left( P_j \frac{\partial}{\partial P^0} - P^0 \frac{\partial}{\partial P_j} \right) F.$$  

(2.38)
With the use of these equations it can be shown (see Appendix B) that

\[ \{ M'_{\mu\nu}, X_\rho \} = - (\eta_{\mu\rho} X_\nu - \eta_{\nu\rho} X_\mu) . \] (2.39)

In conclusion we have shown how it is possible to define a set of collective variables. In particular the variable \( X^\mu \) transforms as a four-vector, as stated in equation (2.18).

III. THE RELATIVE VARIABLES.

In this Section we will give a discussion of the problem of defining a set of relative variables. In order to have a guiding line for such search we will take a suggestion from the example of the relativistic string model, which was discussed from the point of view of a canonical formulation in [12]. We shall need the mathematical framework that has been developed in [9]; we will see that a set of canonical relative variables can be defined at the expense of certain restrictions on the field configuration.

A. Subsidiary variables.

We first introduce a set of subsidiary variables \( \hat{I}(k) \) and \( \hat{\phi}(k) \) such that

\[ \int \tilde{d}k k^\mu \hat{I}(k) = 0, \] (3.1)

\[ \int \tilde{d}k \hat{\phi}(k) \frac{\partial}{\partial P^\mu} F((P \cdot k), P^2) = 0, \] (3.2)

that is, we require the contribution to the collective variables from these new fields to be zero. This is analogous to the definition of the relative variables for the relativistic string model in the quoted reference [12] (see Eq.(2.20) of that paper).

A simple choice for \( \hat{I}(k) \) and \( \hat{\phi}(k) \) seems to be

\[ \hat{I}(k) = I(k) - F((P \cdot k), P^2), \] (3.3)

\[ \hat{\phi}(k) = \phi(k) - (k \cdot X). \] (3.4)
However, these variables are not canonical: they satisfy

$$\{ \hat{I}(k), \hat{\phi}(k) \} = \Delta(k, k'), \quad (3.5)$$

where

$$\Delta(k, k') = \Omega(k) \delta^3(k - k') - \hat{k}^\mu \frac{\partial}{\partial P^\mu} F((P \cdot k), P^2), \quad (3.6)$$

with the following properties:

$$\int \tilde{d}k' \Delta(k, k') \Delta(k', k'') = \Delta(k, k''), \quad (3.7)$$

$$\int \tilde{d}k' k^\mu \Delta(k, k') = 0, \quad (3.8)$$

$$\int \tilde{d}k' \Delta(k, k') \frac{\partial}{\partial P^\mu} F((P \cdot k'), P^2) = 0. \quad (3.9)$$

They have some Poisson brackets good for relative variables

$$\{ P^\mu, \hat{\phi}(k) \} = 0, \quad \{ P^\mu, \hat{I}(k) \} = 0, \quad \{ X^\mu, \hat{I}(k) \} = 0; \quad (3.10)$$

however

$$\{ X^\mu, \hat{\phi}(k) \} \neq 0. \quad (3.11)$$

So they are not canonical relative coordinates; nevertheless, we will go on using them since, as can be ascertained with some algebra, they have the interesting property of separating the generators $M_{\mu\nu}$ of the Lorentz group in a collective part, depending solely on $X$ and $P$, and a second part, depending on $\hat{I}(k)$ and $\hat{\phi}(k)$, as:

$$M_{\mu\nu} = L_{\mu\nu}[X, P] + S_{\mu\nu}[\hat{I}, \hat{\phi}]. \quad (3.12)$$

In spite of Eq. (3.11), due to the condition (3.1), these two parts are in involution, that is their Poisson bracket is zero: this means that the part $S_{\mu\nu}[\hat{I}, \hat{\phi}]$ of $M_{\mu\nu}$ depending solely on subsidiary variables is what must be referred to as the relative part.
B. The relative variable $\mathcal{H}(k)$.

Two new fields $\mathcal{H}(k)$ and $\mathcal{K}(k)$, going to play the role of canonical relative variables, can be defined in order to fulfill the integral constraints on $\hat{I}(k)$ and $\hat{\phi}(k)$:

$$\int d\kappa \kappa \hat{I}(k) = 0, \quad \int d\kappa \hat{\phi}(k) \frac{\partial}{\partial P_\mu} F((P \cdot k), P^2) = 0. \quad (3.13)$$

Let us deal with the first constraint on $\hat{I}(k)$, obtained putting $\mu = 0$:

$$\int d^3k \hat{I}(k) = 0; \quad (3.14)$$

if the quantity $\int d^3k \hat{I}(k)$ is zero, it can be thought of as resulting from the integration of some divergence:

$$\hat{I}(k) = \vec{\nabla} \cdot \vec{J}(k), \quad \lim_{|\vec{k}| \to +\infty} \vec{J}(k) = 0. \quad (3.15)$$

This relationship (3.15) can now be used in the other 3 constraints on $\hat{I}(k)$, for $\mu = i$: by replacing $\hat{I}(k)$ with $\vec{\nabla} \cdot \vec{J}(k)$ inside the condition

$$\int d^3k \hat{I}(k) = 0$$

one gets:

$$\int \frac{d^3k}{\omega(k)} k^i \partial_h J^h(k) = 0, \quad (3.16)$$

which can be re-written as

$$\int d^3k \left[ \partial_h \left( \frac{k^i}{\omega(k)} J^h(k) \right) - J^h(k) \partial_h \left( \frac{k^i}{\omega(k)} \right) \right] = 0. \quad (3.16)$$

The integral of the first term is zero under the hypothesis (3.15), so that the constraints on $\hat{I}(k)$ with $\mu = i$ are equivalent to:

$$\int J^h(k) \partial_h \left( \frac{k^i}{\omega(k)} \right) d^3k = 0. \quad (3.17)$$

One can suppose that there exist some field $\tilde{\mathcal{H}}(k)$ so that
\[- J^h(k) \partial_h \left( \frac{k^i}{\omega(k)} \right) = \partial_i \tilde{H}(k) ; \quad (3.18)\]

in this way a suitable ultraviolet behaviour of \( \tilde{H}(k) \) allows (3.17) to take place. By performing the calculations indicated in (3.18), making the scalar product with \( \vec{k} \) of both r.h.s. and l.h.s., exploiting again the definition (3.15) and the expression of \( \omega(k) \) in terms of \( \vec{k} \), one finally gets:

\[ \hat{I}(k) = \tilde{D} \tilde{H}(k) , \quad (3.19) \]

with

\[ \tilde{D} = -\frac{\omega(k)}{m^2} \left[ m^2 \vec{\nabla} \cdot \vec{\nabla} + 4 \left( \vec{k} \cdot \vec{\nabla} \right) + \left( \vec{k} \cdot \vec{\nabla} \right)^2 \right] . \quad (3.20) \]

If the new field \( \tilde{H}(k) \) defined as in (3.19) is introduced, the subsidiary variable \( \hat{I}(k) \) immediately fulfills the constraints (3.13).

There is one problem with using this \( \tilde{H}(k) \) as a relative variable: it is not scalar under Poincaré transformations. In fact, the differential operator \( \tilde{D} \) does not behave like a scalar, while \( \hat{I}(k) \) does (so that from Eq.(3.19) the non-scalar behaviour of \( \tilde{H}(k) \) is evident).

This is less catastrophic than one could think, indeed it is easily shown that

\[ \tilde{D} = -\frac{1}{m^2} \left( m^2 \Delta - 3 \right) \omega(k) , \quad \Delta = \vec{\nabla} \cdot \vec{\nabla} + \frac{2}{m^2} \left( \vec{k} \cdot \vec{\nabla} \right) + \frac{1}{m^2} \left( \vec{k} \cdot \vec{\nabla} \right)^2 . \quad (3.21) \]

The new operator \( \Delta \) is scalar under Poincaré transformations, and has a clear geometrical meaning, as we shall show few lines below.

Due to Eq.(3.21) one can introduce one new scalar differential operator \( D \) and one new scalar field \( \mathcal{H} \)

\[ D = 3 - m^2 \Delta , \quad \mathcal{H}(k) = \frac{\omega(k)}{m^2} \tilde{H}(k) \quad (3.22) \]

such that

\[ \hat{I}(k) = D \mathcal{H}(k) \quad (3.23) \]
identically gives rise to the constraints on $\hat{I}(k)$ as in Eqs. (3.13). See Appendix C for the study of the operator $D$.

The Poincaré-scalar nature of the operator $D$ is clear: in fact $D$ is naturally defined in terms of the Laplace-Beltrami operator on the mass shell submanifold $H^1_3$ (referred to as $H^1_3$ and studied in [13]), that has equation

$$k^\mu k_\mu = m^2, \text{ and } k^0 > 0.$$  

(3.24)

This Laplace-Beltrami operator on $H^1_3$ turns out to be that $\Delta$ introduced in Eq. (3.21).

$\Delta$ is the only invariant differential operator of the second order on $H^1_3$ (no invariant first order differential operator exists), and it is defined as

$$\Delta = -\frac{1}{\sqrt{\hat{\eta}}} \frac{\partial}{\partial k^i} \hat{\eta}^{ij} \sqrt{\hat{\eta}} \frac{\partial}{\partial k^j}$$

(3.25)

in terms of $H^1_3$ geometry, where $\hat{\eta} = \det \hat{\eta}_{ij}$ and where $\hat{\eta}_{ij}$ is the embedding-induced metric on the mass shell submanifold [9], [13]. Explicitly, it has the form given in (3.24).

$\Delta$ is invariant under a Lorentz transformation $\Lambda$, that is represented as

$$k^i \rightarrow k'^i = \Lambda^i_j k^j + \Lambda^i_0 \omega(k),$$

(3.26)

on $\vec{k} \in H^1_3$, and is formally self-adjoint with respect to the invariant measure $\tilde{dk}$. It is an elliptic operator, and has the property

$$\Delta k^\mu = \frac{3}{m^2} k^\mu.$$  

(3.27)

Then the differential operator $D$

$$D = 3 - m^2 \Delta$$

(3.28)

is formally self-adjoint too with respect to the scalar product of $L_2(\tilde{dk})$, and Eq. (3.27) suggests that the four functions

$$k^\mu(k) : \begin{cases} k^0(k) = \omega(k), \\ k^i(k) = k^i \end{cases}$$

(3.29)
are all among its null modes. Thus, assuming (3.23) it is possible to get
\[ \int \tilde{d}kk^\mu \hat{I}(k) = \int \tilde{d}kk^\mu D\mathcal{H}(k) = \int \tilde{d}k\mathcal{H}(k)Dk^\mu = 0 \] (3.30)
identically (self-adjointness of \( D \) cancels any boundary term in (3.30) by definition).

However, the assumption (3.23) is not in general consistent. Its consistency rests upon a set of conditions that \( \hat{I}(k) \) must satisfy, which will be thoroughly discussed in Section IV. Let us for the present assume Eq.(3.23) and develop its consequences: we postpone the discussion of its mathematical consistency just for simplicity, since here we are more interested in the physical picture than in even important mathematical conditions.

After this construction the field \( I(k) \) can be expressed as a non linear function of \( P^\mu \) and as a linear function of the field \( \mathcal{H}(k) \), that is
\[ I(k) = D\mathcal{H}(k) + F((P \cdot k), P^2), \] (3.31)
while the inverse transformation reads:
\[ P^\mu = \int \tilde{d}kk^\mu I(k), \]
\[ \mathcal{H}(k) = \int \tilde{d}k'G(k, k') \left[ I(k') - F((P \cdot k'), P^2) \right] \] (3.32)
(where in the second equation \( P^\mu \) must be considered a functional of \( I(k) \) and where \( G(k, k') \) is the Green function of the operator \( D \), which is discussed in the Appendix D).

Let us finally show that the new field \( \mathcal{H}(k) \) is a relative variable, since we can easily check that its Poisson brackets with \( X \) and \( P \) are zero:
\[ \{ \mathcal{H}(k), P^\mu \} = 0, \quad \{ \mathcal{H}(k), X^\mu \} = 0. \] (3.33)
In fact, since
\[ \{ \phi(k'), \hat{I}(k'') \} = -\Delta(k'', k'), \] (3.34)
we get
\[ \{ X^\mu, \mathcal{H}(k) \} = -\int \tilde{d}k' \int \tilde{d}k'' G(k, k') \frac{\partial F((P \cdot k'), P^2)}{\partial P^\mu} \Delta(k'', k') = 0, \] (3.35)
from Eq.(3.3). Besides, \( \mathcal{H}(k) \) and \( P^\mu \) have zero Poisson brackets since they are both functionally dependent from the same field \( I(k) \) only.
C. The second relative variable $K(k)$.

Here the canonical variables conjugated to the relative field $H(k)$ are introduced. We have already defined the field $\hat{\phi}(k)$ in Eq. (3.4) as a preliminary choice for the relative phases

$$\hat{\phi}(k) = \phi(k) - (k \cdot X);$$  \hspace{1cm} (3.36)

now we start from the fact that the subsidiary variable $\hat{\phi}(k)$ satisfies the integral constraint:

$$\int \tilde{d}k \hat{\phi}(k) \frac{\partial}{\partial P^\mu} F((P \cdot k), P^2) = 0;$$  \hspace{1cm} (3.37)

We’ll define some relative field $K(k)$ canonically conjugated to $H(k)$ so that (3.37) is an identity when $\hat{\phi} = \hat{\phi}[K, ...]$. We choose to relate the new variable to the old one in a linear way

$$\hat{\phi}(k) = \int \tilde{d}k' A(k', k) K(k'),$$  \hspace{1cm} (3.38)

so that matching (3.38) with (3.37) one gets:

$$\int \tilde{d}k' K(k') \int \tilde{d}k A(k', k) \frac{\partial}{\partial P^\mu} F((P \cdot k), P^2) = 0, \forall K.$$  \hspace{1cm} (3.39)

In order to have a null integral for every possible form of the function $K$ one has to admit

$$\int \tilde{d}k A(k', k) \frac{\partial}{\partial P^\mu} F((P \cdot k), P^2) = 0,$$  \hspace{1cm} (3.40)

and since we’ve already noted that

$$\int \tilde{d}k \Delta (k'', k) \frac{\partial}{\partial P^\mu} F((P \cdot k), P^2) = 0,$$

one can always get a solution for (3.40) with the position:

$$A(k', k) = \int \tilde{d}k'' R(k', k'') \Delta (k'', k).$$  \hspace{1cm} (3.41)

The relative variable $K(k)$ has been constructed so that

$$\hat{\phi}(k) = \int \tilde{d}k' \int \tilde{d}k'' K(k') R(k', k'') \Delta (k'', k),$$  \hspace{1cm} (3.42)
and the distribution $R(k', k'')$ can be identified by requiring the field $\hat{\phi}(k)$ to have the right Poisson bracket with $\hat{I}(k)$; it is very easy to check that:

$$\{\hat{I}(k), \hat{\phi}(k')\} = \Delta(k, k') \Rightarrow D k' R(k', k'') = \Omega(k') \delta^3(\vec{k}' - \vec{k}'').$$

The conclusion is that the coefficients $R(k', k'')$ must be a Green function for our operator $D$, so that we can choose to write:

$$\hat{\phi}(k) = \int d\vec{k}' \int d\vec{k}'' K(k') G(k', k'') \Delta(k'', k). \quad (3.43)$$

It is easily found that two different inversions of (3.43) are possible

$$K(k) = D\hat{\phi}(k), \quad K(k) = D\phi(k): \quad (3.44)$$

this fact suggests that the transformation $\phi \rightarrow K$ is some sort of projection, which will be seen to be equivalent to a limiting assumption on the form of $\phi(k)$ in order for the transformation $\Phi, \Pi \rightarrow X, P, \mathcal{H}, K$ to be consistent.

Eqs.(3.44) give us concrete information about the ultraviolet behaviour of $K(k)$:

$$|K(k)| \simeq |\vec{k}|^{1-\epsilon}, \quad \text{for any } \epsilon > 0, \quad (3.45)$$

since any linear dependence of $k^\mu$ in $\phi(k)$ is annihilated by $D$.

The two fields $\mathcal{H}$ and $K$ are canonically conjugated, as we may easily verify by observing that the Green function $G$ is symmetric in its arguments. So one has:

$$\{\mathcal{H}(k), K(k')\} = \Omega(k) \delta^3(\vec{k} - \vec{k}'). \quad (3.46)$$

Moreover, the Poisson bracket of $\mathcal{H}(k)$ with $P^\mu$ and $X^\mu$ is zero. Indeed

$$\{P^\mu, K(k)\} = D\{P^\mu, \phi(k)\} = D \int d\vec{k}' k'^\mu \{I(k'), \phi(k)\} = D k^\mu = 0, \quad (3.47)$$

and
\[ \{X^\mu, \mathcal{K}(k)\} = D_k \int \tilde{d}k' \phi(k') \left\{ \frac{\partial F((P \cdot k'), P^2)}{\partial P_\mu}, \phi(k) \right\} = \]

\[ = D_k \int \tilde{d}k' \phi(k') \frac{\partial^2 F((P \cdot k'), P^2)}{\partial P_\mu \partial P_\nu} \{P_\nu, \phi(k)\} = \]

\[ = \left[ \int \tilde{d}k' \phi(k') \frac{\partial^2 F((P \cdot k'), P^2)}{\partial P_\mu \partial P_\nu} \right] D_k k_\nu = 0, \]
since \( k_\nu \in \ker D. \)

As we shall stress discussing the consistency conditions of the whole transformation, Eqs. (3.44) should be assumed as the definition of \( \mathcal{K}(k) \); thus, it would be more "mathematically correct" to start from the first of (3.44) and show that with the inversion (3.43) the variable \( \hat{\phi}(k) \) has the nice properties it has to have. The choice of the present exposition simply underlines the physical derivation of \( \hat{\phi} = \hat{\phi} [\mathcal{K}, \ldots] \) directly from the relations that must hold for \( \hat{\phi}(k) \).

D. Field variables and Lorentz algebra in terms of the new variables.

The new canonical set has the following nonzero Poisson brackets

\[ \{P^\mu, X^\nu\} = \eta^{\mu\nu}, \]

\[ \{\mathcal{H}(k), \mathcal{K}(k')\} = \Omega(k) \delta^3 \left( \vec{k} - \vec{k}' \right), \]

and the other possible combinations all vanish.

These variables, except \( X^0 \), are constants of motion, since they have zero Poisson bracket with the Hamiltonian \( P^0 \). Thus the canonical basis we were looking for \( \{P^\mu, X^\mu, \mathcal{H}(k), \mathcal{K}(k)\} \) is some set of Hamilton-Jacobi data.

The canonical transformation from \( \{I(k), \phi(k)\} \) is

\[ P^\mu = \int \tilde{d}k k^\mu I(k), \]
\[ X^\mu = \int \tilde{d}k \frac{\partial F((P \cdot k), P^2)}{\partial P_\mu} \phi(k), \]  
(3.52)

\[ \mathcal{H}(k) = \int \tilde{d}k' G(k, k') \left[ I(k') - F((P \cdot k'), P^2) \right], \]  
(3.53)

\[ K(k) = D\phi(k), \]  
(3.54)

and the inverse reads:

\[ I(k) = D\mathcal{H}(k) + F((P \cdot k), P^2), \]  
(3.55)

\[ \phi(k) = (k \cdot X) + \int \tilde{d}k' \int \tilde{d}k'' K(k') G(k', k'') \Delta(k'', k) \]  
(3.56)

We may give the original field variables in terms of the new variables as:

\[ \Phi(x) = \]  
\[ = \int \tilde{d}k \sqrt{D\mathcal{H}(k) + F((P \cdot k), P^2)} \left[ e^{i(k \cdot (X-x)) + \frac{i}{2} \int \tilde{d}k' \int \tilde{d}k'' K(k') G(k', k'') \Delta(k'', k)} + c.c \right], \]  
(3.57)

\[ \Pi(x) = \]  
\[ = -i \int \tilde{d}k \omega(k) \sqrt{D\mathcal{H}(k) + F((P \cdot k), P^2)} \left[ e^{i(k \cdot (X-x)) + \frac{i}{2} \int \tilde{d}k' \int \tilde{d}k'' K(k') G(k', k'') \Delta(k'', k)} - c.c \right]. \]  
(3.58)

It’s particularly interesting to study the Poincaré algebra in terms of the new variables: the generators of the group naturally decompose into collective and relative parts, as much as in the case of a system of particles, so our main goal has been achieved.

The Eq. (2.19), (2.20), and (2.21) give the generators of the Lorentz algebra in terms of the modulus-phase variables. We want to stress that, in terms of the new collective and relative variables, we get for the generators of the Lorentz group

\[ M_{ij} = L_{ij} [X, P] + S_{ij} [\mathcal{H}, K], \]  
(3.59)
where

\[ L_{ij} [X, P] = X^i P^j - X^j P^i, \]  

(3.60)

\[ S_{ij} [\mathcal{H}, \mathcal{K}] = \int \tilde{d}k \mathcal{H}(k) \left( k^i \frac{\partial}{\partial k^j} - k^j \frac{\partial}{\partial k^i} \right) \mathcal{K}(k). \]  

(3.61)

Moreover, each generator \( L_{ij} [X, P] \) and \( S_{ij} [\mathcal{H}, \mathcal{K}] \) separately satisfies the Lorentz algebra, given by Eq.(A11). The calculus is a boring one, and is sketched in Appendix E.

Analogously we get

\[ M_{0j} = L_{0j} [X, P] + S_{0j} [\mathcal{H}, \mathcal{K}], \]  

(3.62)

where

\[ L_{0j} [X, P] = (x_0 + X_0) P_j - X_j P_0, \]  

(3.63)

\[ S_{0j} [\mathcal{H}, \mathcal{K}] = - \int \tilde{d}k \mathcal{H}(k) \omega(k) \frac{\partial}{\partial k^j} \mathcal{K}(k). \]  

(3.64)

Putting shortly

\[ L_{\mu\nu} [X, P] = M^{(0)}_{\mu\nu}, \quad S_{\mu\nu} [\mathcal{H}, \mathcal{K}] = M^{(1)}_{\mu\nu}, \]

their algebra reads

\[ \left\{ M^{(r)}_{ij}, M^{(s)}_{hk} \right\} = \left[ \delta_{ih} M^{(r)}_{jk} + \delta_{jk} M^{(r)}_{ih} - \delta_{ik} M^{(r)}_{jh} - \delta_{jh} M^{(r)}_{ik} \right] \delta^{(r)(s)}, \]  

(3.65)

\[ \left\{ M^{(r)}_{ij}, M^{(s)}_{0k} \right\} = \left[ \delta_{ik} M^{(r)}_{0j} - \delta_{jk} M^{(r)}_{0i} \right] \delta^{(r)(s)}, \]  

(3.66)

\[ \left\{ M^{(r)}_{0i}, M^{(s)}_{0j} \right\} = -M^{(r)}_{ij} \delta^{(r)(s)}, \]  

(3.67)

where \((r), (s) = 0, 1\).

The canonical decomposition of the angular momenta in a collective \( L_{\mu\nu} [X, P] \) part and a relative one \( S_{\mu\nu} [\mathcal{H}, \mathcal{K}] \) is achieved in this way.
IV. THE CONSISTENCY CONDITIONS.

Now it is necessary to come to the consistency conditions of the whole mechanism elaborated in order to perform the transformation

$$\Phi(x), \Pi(x) \mapsto X^\mu, P^\mu; \mathcal{H}(k), \mathcal{K}(k)$$ (4.1)

for the Klein-Gordon field; the key-equations of our approach are the conditions

$$\int d\tilde{k} k^\mu \hat{I}(k) = 0, \quad \int d\tilde{k} \frac{\partial F(P,k)}{\partial P^\mu} \hat{\phi}(k) = 0$$ (4.2)

on the subsidiary variables, that have been identically fulfilled by putting:

$$\begin{cases}
I(k) = D\mathcal{H}(k) + F(P,k) = \hat{I}(k) + F(P,k), \\
\phi(k) = (k \cdot X) + \int d\tilde{k}' \int d\tilde{k}'' \mathcal{K}(k') G(k',k'') \Delta(k'',k).
\end{cases}$$ (4.3)

One has to identify carefully those functional spaces for \(\Phi\) and \(\Pi\) within which these Eqs.(4.3) are consistent.

The first of Eqs.(4.3) is not in general consistent: we must check the effect that the zero modes of the operator \(D\) may have on it. The analysis of Appendix C, where we study the null space of \(D\), shows that this space has no nontrivial intersection with the \(L_2(d\tilde{k})\) space. A naïve application of the Fredholm alternative theorem could imply that the operator \(D\) is a 1-1 application within \(L_2(d\tilde{k})\). However the usual form of Fredholm theorem applies only to operators which are compact or have compact inverse: if this is the case, the theorem asserts that an operator with empty null space in \(L_2\) is 1-1, and that an equation like the first one in (4.3) is consistent.

Unfortunately our operator \(D\) is not compact, nor its inverse as determined by the Green function studied in Appendix D, and this all can be easily checked by observing that the inverse operator, whose kernel is the Green function, does not change the leading asymptotic term of a function of \(\vec{k}\) as \(|\vec{k}| \to \infty\).

Let us start with checking the consistency conditions of the first of (4.3):

$$\hat{I}(k) = D\mathcal{H}(k).$$ (4.4)
If \( f_0 \) is a function belonging to the kernel of \( D \)
\[
f_0 \in \ker D
\] (4.5)
and has suitable properties so that the integral
\[
\int d\tilde{k} I(k)f_0(k)
\]
is a finite quantity, one can see that an unavoidable consequence of (4.4) is:
\[
\int d\tilde{k} I(k)f_0(k) = \int d\tilde{k} f_0(k) D\mathcal{H}(k).
\] (4.6)

Let us define
\[
T_{\partial}[f, g] = \int g(k) Df(k) d\tilde{k} - \int f(k) Dg(k) d\tilde{k}
\] (4.7)
and then re-express the r.h.s. of Eq.(4.6) as:
\[
\int d\tilde{k} f_0(k) D\mathcal{H}(k) = \int d\tilde{k} \mathcal{H}(k) Df_0(k) + T_{\partial}[\mathcal{H}, f_0] = T_{\partial}[\mathcal{H}, f_0]
\] (4.8)
from Eq.(4.3). Assuming the behaviour of \( \mathcal{H} \) so that the boundary term \( T_{\partial}[\mathcal{H}, f_0] \) vanishes
\[
T_{\partial}[\mathcal{H}, f_0] = 0
\] (4.9)
we have:
\[
\int d\tilde{k} I(k) f_0(k) = 0,
\] (4.10)
that must hold as an integrability condition.

The condition (4.9) can be realized for those zeroes \( f_0 \) of \( D \) that render the integral
\[
\int d\tilde{k} I f_0
\] a finite quantity with a suitable assumption on \( \mathcal{H}(k) \): those zeroes are the only ones for which the whole discussion makes sense. The surface term \( T_{\partial}[f_0, \mathcal{H}] \) is given by
\[
T_{\partial}[f_0, \mathcal{H}] = \int \left[ f_0(k) D\mathcal{H}(k) - \mathcal{H}(k) Df_0(k) \right] d\tilde{k}
\]
\[
= -\frac{1}{2(2\pi)^3} \int d^3k \frac{\partial}{\partial k^i} \left[ \frac{m^2 \delta^{ij} + k^i k^j}{\omega(k)} \right] \left[ f_0(k) \frac{\partial}{\partial k^j} \mathcal{H}(k) - \mathcal{H}(k) \frac{\partial}{\partial k^j} f_0(k) \right],
\]
or, putting $\vec{x} = m^{-1} \vec{k}$ and $r = |\vec{x}|$,

$$T_{\partial} [f_0, H] =$$

$$= -\frac{m^2}{2 (2\pi)^3} \left\{ \lim_{r \to \infty} - \lim_{r \to 0} \right\} \int d^2 \Omega r^2 \sqrt{1 + r^2} \left[ f_0(x) \frac{\partial}{\partial r} H(x) - H(x) \frac{\partial}{\partial r} f_0(x) \right].$$

(4.11)

In order to estimate it, we may assume for $H$ the following asymptotic behaviours:

$$\begin{align*}
H(x) &\sim \alpha (\hat{x}) r^{-1+\varepsilon} \text{ as } r \to 0, \varepsilon > 0, \\
H(x) &\sim \beta (\hat{x}) r^{-3-\sigma} \text{ as } r \to \infty, \sigma > 0,
\end{align*}$$

where $\alpha (\hat{x})$ and $\beta (\hat{x})$ are unknown functions of the direction of $\vec{x}$ only.

This asymptotic behaviour of $H$ is a consequence of the assumptions on $I(k)$, i.e. the existence of the generators of the Poincaré group in Eqs. (2.19), (2.20) and (2.21). The addition of any null mode of $D$ to $H$ has no effect on the surface term.

In Appendix C we determine the following behaviour of the null modes of $D$, see (C13), (C23) and (C24):

$$\begin{align*}
v^{(0)}_{1,-3,l,m}(r, \vartheta, \varphi) &\sim r^l, \\
v^{(0)}_{2,-3,l,m}(r, \vartheta, \varphi) &\sim r^{-l-1}, \forall \varphi, \vartheta
\end{align*}$$

as $r \to 0$, and

$$\begin{align*}
v^{(0)}_{1,-3,l,m}(r, \vartheta, \varphi) &\sim a_l r, \\
v^{(0)}_{2,-3,l=0,1}(r, \vartheta, \varphi) &\sim b_l r, \quad v^{(0)}_{2,-3,l>1}(r, \vartheta, \varphi) &\sim b_l r^{-3}, \forall \varphi, \vartheta
\end{align*}$$

as $r \to \infty$, where $a_l$ and $b_l$ are constants (being the $v$’s those null modes of $D$ as defined in (C25), (C26), (C27) and (C28)). It is easily seen that $T_{\partial} v^{(0)}_{1,-3,l,m}, H]$ is always zero. For $T_{\partial} v^{(0)}_{2,-3,l,m}, H]$ we have the following situation: for $\epsilon > l + 1$ it is zero; it is finite and different from zero if $\epsilon = l + 1$, and is divergent if $\epsilon < l + 1$.

Using as null modes of $D$ the functions $v^{(0)}_{1,-3,l,m}(k)$ and $v^{(0)}_{2,-3,l,m}(k)$ in the integrability condition (1.10) one sees that the first equation (1.3) is consistent under the following condition for $\hat{l}(k)$:
\[ \hat{P}_{l,m} = \int d\tilde{k} \hat{I}(k) v_{1,-3,l,m}^{(0)}(k) = 0 \] (4.12)
and
\[ \hat{Q}_{l,m} = \int d\tilde{k} \hat{I}(k) v_{2,-3,l,m}^{(0)}(k) = 0 \] (4.13)

if \( \hat{I}(k) \) behaves like \( k^{-3+\epsilon} \), for \( \epsilon > l + 1 \), about the origin.

It is very important to observe that conditions (4.13) are absent for \( 0 < \epsilon \leq 1 \), since in this case the integrals diverge, and the procedure worked out from Eq.(4.11) to Eq.(4.8) has no meaning. Instead, both the l.h.s. of this last equation and the surface term are divergent.

The conditions (4.12) and (4.13) make the transformation to collective and relative variables consistent. They determine the quantities
\[ P_{l,m} = \int d\tilde{k} I(k) v_{1,-3,l,m}^{(0)}(k), \quad Q_{l,m} = \int d\tilde{k} I(k) v_{2,-3,l,m}^{(0)}(k) \] (4.14)
in terms of \( P \), like:
\[ P_{l,m} = \int d\tilde{k} F(P,k) v_{1,-3,l,m}^{(0)}(k), \quad Q_{l,m} = \int d\tilde{k} F(P,k) v_{2,-3,l,m}^{(0)}(k). \] (4.15)
The first of Eqs.(4.15) is an identity for \( l = 0, 1 \), see Eq.(2.8). Apart from this case these equations show that \( P_{l,m} \)'s and \( Q_{l,m} \)'s are functionals of \( P_\mu \), and exclude the possibility of using them as other independent canonical degrees of freedom.

In particular the condition given by Eq.(4.12) can be referred to as no-supertranslation condition: in fact it renders impossible the use of the \( P_{l,m} \)'s (which are referred to as ”supertranslations” in [9]) as canonical variables independent with respect to \( P \).

The set of conditions (4.12) and (4.13) must be understood as restrictions on the configuration space of the field, and they single out those configurations for which the transformations (4.1) are possible. The problem is that they are not so easily understandable in terms of restrictions directly settled on \( \Phi(\vec{x},t) \) and \( \Pi(\vec{x},t) \), since they correspond to very complicated constraints on the fields.

Note however that when \( \epsilon \in (0,1] \) only conditions (4.12) make sense, so it’s possible to choose suitably \( I(k) \) identifying the consistency conditions simply with no supertranslation condition.
We can finally state that the transformation (4.4) from $I(k)$ to $\mathcal{H}(k)$ makes sense for a well defined set of field configurations, divided into two subsets. First of all, Eq.(4.4) does not make sense if neither conditions (4.12) nor (4.13) are fulfilled. It makes sense for those configurations with

$$\hat{I}(k) \approx k^{-3+\epsilon}, \text{ for } \epsilon \in (0, 1], \quad |\vec{k}| \simeq 0 \quad (4.16)$$

if they fulfill conditions (4.12), and for those with

$$\hat{I}(k) \approx k^{-3+\epsilon}, \text{ for } \epsilon > l + 1, \quad |\vec{k}| \simeq 0 \quad (4.17)$$

if they fulfill Eq.(4.12) as well as Eq.(4.13).

The condition (4.12) excludes supertranslation generators $P_{l,m}$’s from the context of independent canonical variables in order for the definition of these $\mathcal{H}(k)$ to be consistent: we already know that it is possible to give a symplectic realization of the BMS algebra in terms of the Klein-Gordon phase space [9], but the existence of constraints on them, given by the consistency conditions necessary for the definition of canonical relative variables, is a very new fact. The problem of finding whether no supertranslations condition is a fundamental need of any collective-relative splitting framework, or it only affects the particular path we have chosen, deserves much more study.

An analogous discussion can be presented for the variable $\phi(k)$; however the situation is different, since now the relative variable $\mathcal{K}(k)$ is defined by Eq. (3.44); moreover an $L_2$-like scalar product of $\phi(k)$ with the null modes of $D$ does not exist, due to the linear behaviour of $\phi(k)$ at the infinity; there can be a doubt about the validity of this affirmation for what concerns those null modes with $l = 0, 1$: if one mixes carefully $v_{1,-3,l=0,1}^{(0)}$ and $v_{2,-3,l=0,1}^{(0)}$ one can reach a suitably fine-tuned function that is in ker $D$ and can be integrated in $d\tilde{k}$ with $\phi(k)$: the fact that this fine tuned mixing cannot anyway be stable under Lorentz group rules out this possibility, and it can be seriously stated that a scalar product of $\phi(k)$ with the elements of ker $D$ does not exist.

Nevertheless there is a more subtle consequence of the existence of the null modes of $D$ on the phase variable $\phi(k)$. The definition (3.36) and Eq.(3.42) show that $\phi(k)$ has a
contribution from $\ker D$ consisting of the term $(k \cdot X)$, that is from the solution $\left\{ v_{1,-3,l,m}^{(0)} \right\}$ with $l = 0, 1$, but not from terms with $l \geq 2$, and this is clearly a restriction on the mathematically possible configurations of $\phi(k)$.

On the other hand, since the $\lambda = -3$ representation is indecomposable, Minkowskian products like $(k \cdot X)$ are the only Lorentz-invariant bilinear quantities which can be built up by involving a finite number of elements of $\ker D$, because the $l = 0, 1$ subspace is the only Lorentz-invariant subspace within $\ker D$. The construction of Lorentz-invariant bilinear quantities depending on the whole set of solutions could only be performed introducing series whose convergence remain far from obvious, and are not under control.

V. CONCLUSIONS AND OUTLOOK.

We have outlined an analysis of the problem of how to define collective and relative canonical coordinates for a real Klein-Gordon classical field in Minkowski space.

While the collective coordinates, with the meaning given to them in the introduction, can be almost easily defined, the definition of the relative ones presented a serious difficulty. To obtain a sound definition of these variables we have found it necessary to put a set of restrictions on the field, which are essentially equivalent to the vanishing of the so called supertranslations.

We confined the discussion to the classical case. The problem of quantization will face well known difficulties [14], and it will require a separate work. In any case it is known that what can be well defined, as a quantum operator, is the phase factor $\exp(i\phi)$, and not the phase itself. So not really prohibitive difficulties should appear.

The very next step should be to apply the present analysis to the field defined on a space-like surface, following the line mentioned in Section 1, which goes back to Dirac [2]. We may expect some simplification when working on a family of space-like surfaces foliating Minkowski space, since there the problem of covariance of the canonical variables changes. In particular, on a hyperplane determined by the rest frame of the total momentum, the
four vectors split in a scalar and an irreducible spin 1 part \[1\].

This collective-relative variable separating technique will soon be exploited in that context: even with a very different kind of covariance, an easy application of the present results will single out the right gauge-fixing conditions for the Dirac-Bergman reduction of that problem \[15\].

Other interesting developments will be the application of this method to different relativistic Bose fields (Maxwell as well as Yang-Mills) and to non-relativistic fields (Schrödinger classical field, or fields of fluid-dynamics). It could be interesting even to work out the collective-relative splitting basis for solitonic configurations, while the problem of constructing a similar framework for spinor fields will need special cares \[16\].

Another aspect that should deserve some interest is the possibility of switching on a self-interaction among the relative variables. If this self interaction is completely integrable we could get an integrable model for a self interacting scalar field. This should be a highly non local self-interaction, but it could in principle provide interesting integrable models in Minkowski space.

Even if these are all promising further developments, it’s clear that before starting we should understand better, from a more physical point of view, the consistency conditions (4.12) and (4.13).

When conditions formally identical to Eq.(4.12) appear in the context of asymptotic General Relativity they do in order to render well defined the light-like-infinity Poincaré group, allowing to work out consistently the ADM spin of some gravitational source (a pulsing star, a black hole and so on).

Here the BMS charges \( P_m \)’s appear in an unexpected way, making the problem of understanding their presence even more complicated, and few things about them seem to be clear: first, they must be constrained as in Eq.(4.12) in order for the transformations (4.13) to be consistent; second, they do not concern spacetime symmetries, and can simply be considered generating functionals for canonical maps from an initial set of Hamilton-Jacobi data into another \[1\].
We have not been able to include these \( P_{lm} \)'s in a canonical framework, we have not been able to find a basis in which they play the role of canonical momenta conjugated to some generalized coordinates \( X^{lm} \)'s. We have not been able to modify conditions (4.13) in order to free the \( P_{lm} \)'s from constraints, because this should involve the definition of suitable Lorentz-invariant bilinear quantities depending on the whole basis of \( \ker D \), and we have not been able to build up such quantities.

Anyway we have not an explicit no-go theorem obstructing the possibility of working out a collective-relative splitting canonical set \( \{ X^\mu, P_\nu; H, K; X^{lm}, P_{lm'} \} \), in which the BMS charges are an essential part of the phase space.

At this stage we can only conjecture that, going on studying this particular method of collective-relative splitting, either the no-go theorem will be found (with an important meaning from a physical and group-theoretical point of view), or the BMS-including phase space will be constructed.

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**APPENDIX A: NOTATIONS FOR THE SCALAR FIELD.**

We list here the various definitions concerning the scalar real Klein Gordon field.

The Lagrangian and the Lagrangian density are

\[
L = \int d^3x \mathcal{L}, \quad \mathcal{L} = \frac{1}{2} \left( \partial_\mu \Phi \partial^\mu \Phi - m^2 \Phi^2 \right) ;
\]

(A1)

the field is real.
\( \Phi^* = \Phi, \)  \hspace{1cm} (A2)

(the bar means complex conjugate).

The canonical momentum and the equation of motion are

\[ \Pi(x) = \dot{\Phi}(x) \equiv \partial_0 \Phi(x), \quad (\Box + m^2) \Phi(x) = 0. \]  \hspace{1cm} (A3)

The Noether current associated with a Lorentz transformation is

\[ j^{\mu\nu} = \frac{\partial \mathcal{L}}{\partial (\partial_\mu \Phi)} \partial^\nu \Phi - \eta^{\mu\nu} \mathcal{L} = j^{\nu\mu}, \]  \hspace{1cm} (A4)

\[ j^{\mu 0} = \dot{\Phi} \partial^{\mu} \Phi = \Pi \partial^{\mu} \Phi, \quad (\mu \neq 0), \]  \hspace{1cm} (A5)

\[ j^{00} = \frac{1}{2} \left[ \dot{\Phi}^2 + (\bar{\nabla} \Phi)^2 + m^2 \Phi^2 \right]. \]  \hspace{1cm} (A6)

The Poincaré charges are

\[ P^\mu = \int d^3 x j^{0\mu}(x), \quad M^{\mu\nu} = \int d^3 x \left( x^\mu j^{0\nu} - x^\nu j^{0\mu} \right). \]  \hspace{1cm} (A7)

For the existence of these generators it is sufficient that \( \Phi(\vec{x}, \cdot), \bar{\nabla} \Phi(\vec{x}, \cdot) \in L_2(\mathbb{R}^3). \)

The metric signature is \( \eta \equiv (+, -, -, -). \)

The Poisson brackets are

\[ \{ \Phi(\vec{x}, x^0), \Pi(\vec{x}', x^0) \} = \delta^3(\vec{x} - \vec{x}'), \]  \hspace{1cm} (A8)

and the other equal-\( x^0 \) Poisson brackets vanish.

The Poincaré algebra is

\[ \{ P^\mu, P^\nu \} = 0, \]  \hspace{1cm} (A9)

\[ \{ M^{\mu\nu}, P^\rho \} = P^\nu \eta^{\mu\rho} - P^\mu \eta^{\nu\rho}, \]  \hspace{1cm} (A10)

\[ \{ M^{\mu\nu}, M^{\rho\lambda} \} = -M^{\mu\lambda} \eta^{\nu\rho} - \eta^{\mu\lambda} M^{\nu\rho} + M^{\mu\rho} \eta^{\nu\lambda} + \eta^{\mu\rho} M^{\nu\lambda}. \]  \hspace{1cm} (A11)
The Fourier transform of the field is
\[ \Phi(\vec{x}, t) = \int \tilde{d}k [a(k)e^{-i(k \cdot x)} + c.c.], \tag{A12} \]
where c.c. means "the complex conjugate", and \(( \cdot, \cdot )\) is the usual scalar product between 4-vectors. The component \(k_0\) equals \(\sqrt{\vec{k}^2 + m^2}\), and \(\tilde{d}k\) is the Lorentz invariant measure
\[ \tilde{d}k = \frac{d^3k}{\Omega(k)}, \quad \Omega(k) = (2\pi)^3 2\omega(k), \tag{A13} \]
\[ \omega(k) = k_0 = \sqrt{\vec{k}^2 + m^2}. \]

For the existence of the Fourier transform of the field \(\Phi(x)\) we may require that \(a(k), \vec{\nabla}_k a(k) \in L_2(dk)\).

The canonical momentum is
\[ \Pi(x) = -i \int \tilde{d}k \omega(k)[a(k)e^{-i(k \cdot x)} - c.c.]. \tag{A14} \]

In terms of the Fourier coefficients, the current \(j^{00}\) is
\[ j^{00}(x) = \int \tilde{d}k \int \tilde{d}k' \left\{ - \left[ \omega(k)\omega(k') + \vec{k} \cdot \vec{k'} - m^2 \right] \left[ a(k)a(k')e^{-i(k+k' \cdot x)} + c.c. \right] + \right. \]
\[ + \left. \left[ \omega(k)\omega(k') + \vec{k} \cdot \vec{k'} + m^2 \right] \left[ a(k)a(k')e^{-i(k-k' \cdot x)} + c.c. \right] \right\}. \tag{A15} \]

The Poisson brackets are
\[ \{a(k), \bar{a}(k')\} = -i\Omega(k)\delta^3(\vec{k} - \vec{k'}). \tag{A16} \]

The inverse Fourier transform is
\[ a(k) = \int d^3x e^{i(k \cdot x)}[\omega(k)\Phi(x) + i\Pi(x)]. \tag{A17} \]

In terms of the Fourier coefficients, the Poincaré generators are
\[ P^\mu = \int \tilde{d}k k^\mu \bar{a}(k)a(k), \tag{A18} \]
\[ M_{ij} = M'_{ij} = -i \int \tilde{d}k \bar{a}(k) \left( k^i \frac{\partial}{\partial k^j} - k^j \frac{\partial}{\partial k^i} \right) a(k), \tag{A19} \]
\[ M_{0j} = x_0 P_j + M'_{0j} = x_0 P_j + i \int \tilde{d}k \bar{a}(k)\omega(k) \frac{\partial}{\partial k^j} a(k), \tag{A20} \]
where \(x_0 = t\) is the parameter time, and \(k^i = -k_i\).
APPENDIX B: SOME USEFUL POISSON BRACKETS.

Let us verify the Eq.(2.39) for $\mu = i$ and $\nu = j$:

$$\{ M'_{ij}, X_\nu \} = - (\eta_{i\nu} X_j - \eta_{j\nu} X_i).$$

From the Poisson brackets (2.4) and the definitions (2.7) and (2.20) we get

$$\{ M'_{ij}, X_\nu \} = - \int d\tilde{k} \frac{\partial}{\partial P^\nu} F (P, k) \left( k_i \frac{\partial}{\partial k^j} - k_j \frac{\partial}{\partial k^i} \right) \phi (k) +$$

$$+ \int d\tilde{k} \int d\tilde{k}' I (k) \phi (k') \frac{\partial^2}{\partial P^\nu \partial P^\nu} F (P, k') \left( k_i \frac{\partial}{\partial k^j} - k_j \frac{\partial}{\partial k^i} \right) k^\rho.$$

Since we assumed a good behaviour of the function $F$, we may integrate by parts the first integrand, and use Eq.(2.37):

$$\{ M'_{ij}, X_\nu \} = - \int d\tilde{k} \phi (k) \frac{\partial}{\partial P^\nu} \left( P_i \frac{\partial}{\partial P^j} - P_j \frac{\partial}{\partial P^i} \right) F (P, k) +$$

$$+ \left( P_i \frac{\partial}{\partial P^j} - P_j \frac{\partial}{\partial P^i} \right) \int d\tilde{k} \phi (k) \frac{\partial}{\partial P^\nu} F (P, k) = - (\eta_{i\nu} X_j - \eta_{j\nu} X_i),$$

where there is a compensation of the last term.

In the same way, using Eq.(2.38) and taking care of the mass shell condition $k^0 = \omega (k)$ in the calculation of $\omega (k) \frac{\partial}{\partial k^\rho} k^\rho$, we get

$$\{ M'_{0ij}, X_\nu \} = - (\eta_{i\nu} X_j - \eta_{j\nu} X_0).$$

APPENDIX C: THE LAPLACE-BELTRAMI OPERATOR.

The Laplace-Beltrami operator of Eq.(3.21) has been studied in a series of papers by Raczka, Limic and Niederle [13], where it is called $-\Delta (H_\lambda)$ (see Eq.(5.10) in the first reference). There it is shown that $\Delta$ has not a discrete spectrum, but only a continuous one, and the basis of its generalized eigenfunctions is determined. That study is the background
of our work [1] too, where the reader will find a complete discussion of the spectrum and of
the properties of this operator.

The zero modes of the operator (3.28) correspond to the

$$\lambda = -3$$

representation of the notations used in [1], that is a non unitary representation of the Lorentz
group, which is reducible but not completely reducible as we have shown in [1].

The value $$\lambda = -3$$ is determined by the conditions, Eq.(3.1)

$$\int \tilde{d}k k^\mu \tilde{l}(k) = 0, \quad \tilde{l}(k) = D\mathcal{H}(k),$$

which was assumed as defining the relative variable $$\mathcal{H}$$. This choice is reinforced by the fact
that this is the unique eigenspace of $$\Delta$$ which allows an invariant subspace with dimension
4 (see [1]), corresponding to a 4-vector, that is $$k^\mu$$: so the requirement of the existence of a
4-vector $$P^\mu$$ as in Eq.(2.1) among our variables selects by itself the choice $$\lambda = -3$$.

Working on the functional space of

$$f : H^1_3 \longrightarrow \mathbb{C}$$

on which $$\Delta$$ and $$D$$ act, the generators of $$SO(1, 3)$$ are represented by the following differential
operators

$$l_{\mu\nu} = iD_{\mu\nu},$$

$$l_{ij} = -\epsilon_{ijk} L_k = i \left(x^i \partial_j - x^j \partial_i\right),$$

$$l_{0j} = -K_j = -i \sqrt{1 + |\vec{x}|^2} \partial_j,$$

where

$$\vec{x} = \frac{\vec{k}}{m}, \quad r = |\vec{x}|$$

and $$D_{\mu\nu}$$ is the operator we defined in Eq.(2.32).

$$D$$ reads

$$D_{\mu\nu}$$
\[ D = -m^2 \Delta + 3 = -\Delta|_x + 3, \quad (C5) \]

where
\[ \Delta|_x = (1 + r^2) \frac{\partial^2}{\partial r^2} + \left( \frac{2}{r} + 3r \right) \frac{\partial}{\partial r} - \frac{J^2}{r^2} \quad (C6) \]

and \( J^2 = |\vec{L}|^2 \) as usually.

Its proper eigenfunctions correspond to the values of \( \lambda \in [1, +\infty) \), or \( \Lambda \in [0, +\infty) \) where
\[ \Lambda = \sqrt{\lambda - 1}, \quad (C7) \]

see Appendix D, taken as a limit from the upper complex half-plane of \( \Lambda \) see [13]. They read:
\[ w_{\lambda,l,m}(r, \theta, \phi) = N_{\lambda l} v_{1,\lambda,l,m}^{(0)} = \]
\[ = N_{\lambda l} r^l \, _2F_1 \left( \frac{l + i + i\Lambda}{2}, \frac{l + i - i\Lambda}{2}; l + \frac{3}{2}; -r^2 \right) Y_{l,m}(\theta, \phi). \quad (C8) \]

Here \(_2F_1\) is the hypergeometric function, \( l = 0, 1, 2, \ldots \), and \(|m| \leq l\) and the normalization factor \( N_{\lambda l} \) is
\[ N_{\lambda l} = \frac{2\pi}{m\sqrt{\Lambda}} \left| \frac{\Gamma\left(l+\frac{3}{2}+i\Lambda\right)\Gamma\left(l+\frac{3}{2}-i\Lambda\right)}{\Gamma(i\Lambda)\Gamma(l+\frac{3}{2})} \right|. \quad (C9) \]

With this normalization \( w_{\lambda,l,m} \) is an orthonormal set with respect to the scalar product of \( L_2(\tilde{d}k) \)
\[ \int \tilde{d}k \bar{w}_{\lambda,l,m}(r, \theta, \phi) w_{\lambda,l,m}(r', \theta', \phi') = \delta_{ll'} \delta_{mm'} \delta(\Lambda - \Lambda'). \quad (C10) \]

This basis is complete in \( L_2(\tilde{d}k) \), that is
\[ \sum_{l \geq 0} \sum_{|m| \leq l} \int_{1}^{\infty} d\lambda w_{\lambda,l,m}(r, \theta, \phi) \bar{w}_{\lambda,l,m}(r', \theta', \phi') = \Omega(k) \delta^3(k - k'). \quad (C11) \]

From now on we define
\[ v_{\lambda,t,m}(r, \theta, \phi) = v_{1,\lambda,t,m}^{(0)} = r^l \, _2F_1 \left( \frac{l + i + i\Lambda}{2}, \frac{l + i - i\Lambda}{2}; l + \frac{3}{2}; -r^2 \right) Y_{l,m}(\theta, \phi). \quad (C12) \]

The expressions written up to now hold for \( \Lambda \in [0, +\infty) \), or \( \lambda \in [1, +\infty) \), which corresponds to the continuous spectrum of \( \Delta \). Since we are interested the case \( \lambda = -3 \), we need the same relations for a generic complex \( \lambda \).

For \( \lambda = -3 \) the normalization factor \( N_{\lambda l} \) becomes zero for \( l \geq 2 \), moreover the functions \( w_{\lambda,t,m} \) are no more normalizable.

1. The space \( \text{ker}(\Delta + \lambda) \).

Let us study the solutions of the equation
\[ (\Delta + \lambda) f = 0, \quad (C13) \]
for generic values of \( \lambda \in \mathbb{C} \).

A fundamental system of solutions, in the neighbourhood of the origin, that is for \( r \approx 0 \), of the Eq. (C13), is
\[ v_{1,\lambda,t,m}^{(0)}(\vec{r}) = u_{1,\lambda,t}(r) Y_{l,m}(\theta, \phi), \]
\[ v_{2,\lambda,t,m}^{(0)}(\vec{r}) = u_{2,\lambda,t}(r) Y_{l,m}(\theta, \phi), \quad (C14) \]
that is:
\[
\begin{align*}
  u_{1,\lambda,t}^{(0)}(r) &= r^l \, _2F_1 \left( \frac{l + 1 + i\Lambda}{2}, \frac{l + 1 - i\Lambda}{2}; l + \frac{3}{2}; -r^2 \right), \\
  u_{2,\lambda,t}^{(0)}(r) &= r^{-(l-1)} \, _2F_1 \left( -\frac{l + i\Lambda}{2}, -\frac{l - i\Lambda}{2}; 1 - l; -r^2 \right), \\
  u_{2,\lambda,t}^{(0)}(r) &= u_{1,\lambda,-l-1}^{(0)}(r). \quad (C15)
\end{align*}
\]
A fundamental system in the neighborhood of the point at infinite \( r \to \infty \) is
\[ v_{1,\lambda,t,m}^{(\infty)}(\vec{r}) = u_{1,\lambda,t}^{(\infty)}(r) Y_{l,m}(\theta, \phi), \]
\[ v_{2,\lambda,t,m}^{(\infty)}(\vec{r}) = u_{2,\lambda,t}^{(\infty)}(r) Y_{l,m}(\theta, \phi), \quad (C16) \]
where
\[
\begin{align*}
\left\{ \begin{array}{l}
 u_{1,\lambda,l}^{(\infty)}(r) = r^{-1-i\Lambda} \, _2F_1 \left( \frac{l+1+i\Lambda}{2}, -\frac{l+i\Lambda}{2}; l+i\Lambda; -\frac{1}{r^2} \right), \\
u_{2,\lambda,l}^{(\infty)}(r) = r^{-1+i\Lambda} \, _2F_1 \left( \frac{l+1-i\Lambda}{2}, -\frac{l-i\Lambda}{2}; 1-i\Lambda; -\frac{1}{r^2} \right).
\end{array} \right.
\end{align*}
\]
(C17)

For \( r \geq 0 \) no other singular points are met. For \( z = -1 \) the hypergeometric series
\[ _2F_1 (\alpha, \beta; \gamma; z) \] is absolutely convergent, since its coefficients satisfy
\[ \text{Re}(\alpha + \beta - \gamma) = -\frac{1}{2}. \quad \text{(C18)} \]

For \( i\Lambda \) integer positive, the solution \( u_{1,\lambda,l}^{(\infty)}(r) \) should be modified, but since we will be interested in the solution \( u_{2,\lambda,l}^{(\infty)}(r) \) for \( \text{Im}\Lambda \geq 0 \), we will not give here the necessary modification.

We are interested in the normalization properties of these solutions in the neighborhood of the point 0 and \( \infty \), with respect to the invariant measure
\[ \tilde{dr} = \frac{d^3r}{\sqrt{1+r^2}} \quad \text{(C19)} \]

The solution \( u_{1,\lambda,l,m}^{(0)} \) is regular and normalizable in the neighborhood of the origin. The solution \( u_{2,\lambda,l,m}^{(0)} \) on the other hand is normalizable in the origin for \( l = 0 \) only. We will discard this solution, even in the case \( l = 0 \), since under the action of a boost it would be transformed in a solution with a different value of \( l \), that is non normalizable [9].

The solution \( u_{1,\lambda,l}^{(0)}(r) \) can be analytically continued to \( r \to \infty \)
\[ u_{1,\lambda,l}^{(0)}(r) = \]
\[ = \frac{\Gamma \left( \frac{3}{2} + l \right) \Gamma (-i\Lambda)}{\Gamma \left( \frac{l+1-i\Lambda}{2} \right) \Gamma \left( \frac{l+1-i\Lambda}{2} \right)} r^{-1-i\Lambda} \, _2F_1 \left( \frac{l+1+i\Lambda}{2}, -\frac{l-i\Lambda}{2}; 1+i\Lambda; -r^{-2} \right) + \quad \text{(C20)} \]
\[ + \frac{\Gamma \left( \frac{3}{2} + l \right) \Gamma (i\Lambda)}{\Gamma \left( \frac{l+1+i\Lambda}{2} \right) \Gamma \left( \frac{l+1+i\Lambda}{2} \right)} r^{-1+i\Lambda} \, _2F_1 \left( \frac{l+1-i\Lambda}{2}, -\frac{i\Lambda+l}{2}; 1-i\Lambda; -r^{-2} \right) \]

and, for a generic value of \( \Lambda \) has a behaviour which is a linear combination of the two power of \( r \), see Eq. 2.10(2) of [17].
\[ r^{-1-i\Lambda}, \quad r^{-1+i\Lambda}, \]  \tag{C21}

which, for a real \( \Lambda \), is normalizable.

For a real \( \Lambda \) this is an eigenfunction of the operator \( \Delta \), see \[13\], belonging to the continuous spectrum. Since \( \lambda = 1 + \Lambda^2 \), the spectrum corresponds to \( \lambda \in [1, +\infty) \). The upper half-plane of the complex \( \Lambda \) plane describes all the values of \( \lambda \), and the spectrum is obtained as a limit \( \lambda = y + i0 \), with \( y \in [1, +\infty) \).

For the second fundamental system of Eq.(C17), we have that the solution \( u^{(\infty)}_{1,\lambda,l} \) has the behaviour, for \( r \approx \infty \)

\[ r^{-1+i\text{Re}\Lambda - i\text{Im}\Lambda}, \]  \tag{C22}

and, for \( \text{Im}\Lambda \geq 0 \), is not normalizable.

The solution \( u^{(\infty)}_{2,\lambda,l,m} \) is instead normalizable for \( r \to \infty \) and it will be used in the next Appendix, together with the solution \( u^{(0)}_{1,\lambda,l,m} \), for the determination of the Green function of the \( D \) operator. The behaviour of the solutions (C16) is exhibited in reference \[3\].

\section*{2. The \( \lambda = -3 \) solutions.}

In the case \( \lambda = -3 \), that is \( \Lambda = 2i \), we must use another asymptotic expansion. More in general, for \( \Lambda = ni \), with \( n \) integer, we must use the expansion given in Eq. 2.10(7) of \[17\]; we get, for \( \Lambda = 2i \) and for \( r \to \infty \)

\[ u^{(0)}_{1,-3,l} \simeq r + O(r^{-1}). \]  \tag{C23}

So, this solution is no more normalizable at \( \infty \), but only in 0.

The other solution has a behaviour

\[ u^{(0)}_{2,-3,l} \simeq r + O(r^{-1}), \quad \text{for} \quad l = 0, 1; \quad u^{(0)}_{2,-3,l} \simeq B_l r^{-3}, \quad \text{for} \quad l \geq 2. \]  \tag{C24}

and is singular in the origin as \( r^{-l-1} \) (in \( \text{C24} \) the coefficient \( B_l \) is the same as in \( \text{D7} \), see below).
Let us summarize the fundamental solutions around \( r = 0 \) and \( r \to \infty \) in the particular case of \( \Lambda = 2i \), the case we are interested in.

The fundamental system of solutions around \( r = 0 \) is

\[
v_{1,3,l}(\vec{r})^0 = r^l \, _2F_1 \left( \frac{l-1}{2}, \frac{l+3}{2}; l+3; -r^2 \right) Y_{l,m}(\theta, \phi),
\]

\[
v_{2,3,l}(\vec{r})^0 = r^{l-1} \, _2F_1 \left( \frac{2-l}{2}, -\frac{l+2}{2}; l-2; -r^2 \right) Y_{l,m}(\theta, \phi).
\]  

The fundamental system of solutions around \( r \to \infty \) for \( \lambda = -3 \) reads:

\[
v_{1,3,l}(\vec{r})^{(\infty)} = r \, _2F_1 \left( \frac{l-1}{2}, -\frac{l+2}{2}; l-2; -\frac{1}{r^2} \right) Y_{l,m}(\theta, \phi),
\]

\[
v_{2,3,l}(\vec{r})^{(\infty)} = r^{-3} \, _2F_1 \left( \frac{l+3}{2}, \frac{2-l}{2}; 3; -\frac{1}{r^2} \right) Y_{l,m}(\theta, \phi).
\]

Asymptotic behaviours for these \( v \)'s are those shown in the previous subsection, adapted as \( \Lambda = 2i \).

**APPENDIX D: GREEN FUNCTION OF THE OPERATOR \( D \).**

In this Appendix we study the Green function \( G(\vec{k}, \vec{k}'; \lambda) \) of the operator \( \Delta \) for a generic value of \( \lambda \), belonging to the complex plane cut along the real axis in \( \lambda \in [1, +\infty) \). As in Appendix C it is useful to define \( \lambda = 1 + \Lambda^2 \), with \( \text{Im} \Lambda \geq 0 \). Then we put \( \lambda = -3 \) finding out the particular case we are interested in.

1. **Generic complex \( \lambda \).**

We define the distribution \( G \) as the solution of the equation

\[
(-m^2 \Delta_k - \lambda) G(\vec{k}, \vec{k}'; \lambda) = \Omega(k) \delta^3(\vec{k} - \vec{k}'),
\]

\[
(D1)
\]

where \( \Delta \) is defined in Eq.(3.21) or Eq.(C6), in terms of the variable \( \vec{x} = \frac{\vec{k}}{m} \).

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The solutions of the homogeneous equation have been studied in Appendix C. Following
the usual procedure \[18\], since the origin and the point at \( \infty \) are regular singular points for
the radial equation, we will choose the solutions \( v^{(0)}_{1,\lambda,l,m} \) and \( v^{(\infty)}_{2,\lambda,l,m} \), which are normalizable
in the origin and in the point at infinity respectively. So we will write the Green function as

\[
G(\vec{k}, \vec{k}'; \lambda) = \sum_{l \geq 0} \sum_{|m| \leq l} Y_{l,m}(\hat{k})Y_{l,m}(\hat{k}')G_l(r, r'; \lambda),
\]

(D2)

where \( \hat{k} \) and \( \hat{k}' \) are the directions of \( \vec{k} \) and \( \vec{k}' \), where

\[
r = \frac{1}{m} |\vec{k}|, \quad r' = \frac{1}{m} |\vec{k}'|,
\]

(D3)

and where \( G \) is the radial Green function given by

\[
G_l(r, r'; \lambda) = A_l(\Lambda) \left[ u^{(0)}_{1,\lambda,l}(r)u^{(\infty)}_{2,\lambda,l}(r')\theta(r' - r) + (r \leftrightarrow r') \right].
\]

(D4)

The Green function \( G(\vec{k}, \vec{k}'; \lambda) \) is symmetric as it should be since the operator \( \Delta \) is self-

adjoint.

The radial functions \( u^{(0)}_{1,\lambda,l}(r) \) and \( u^{(\infty)}_{2,\lambda,l}(r') \) are defined in Eq.(C13) and Eq.(C17):

\[
u^{(0)}_{1,\lambda,l}(r) = r^{-2} F_1 \left( \frac{l + 1 + i\Lambda}{2}, \frac{l + 1 - i\Lambda}{2}; l + \frac{3}{2}, -r^2 \right),
\]

(D5)

\[
u^{(\infty)}_{2,\lambda,l}(r') = r'^{-3} F_1 \left( \frac{l + 1 - i\Lambda}{2}, -\frac{l + i\Lambda}{2}; 1 - i\Lambda; -\frac{1}{r'^2} \right),
\]

(D6)

and their asymptotic behaviour was discussed in Appendix C.

The action of the Lorentz generators on the solution \( v^{(0)}_{1,\lambda,l,m} \) has been studied in \[9\]. The action on the solution \( v^{(\infty)}_{2,\lambda,l,m} \) can be obtained from the relation

\[
u^{(\infty)}_{2,\lambda,l} = A_l \nu^{(0)}_{1,\lambda,l}(r) + B_l \nu^{(0)}_{2,\lambda,l}(r),
\]

(D7)

where

\[
A_l = \frac{\Gamma(1 - i\Lambda)\Gamma(-l - \frac{1}{2})}{\Gamma(-\frac{i\Lambda + l}{2})\Gamma(\frac{1 - i\Lambda - l}{2})},
\]

(D8)
\[ B_l = \frac{\Gamma(1 - i\Lambda)\Gamma \left( l + \frac{1}{2} \right)}{\Gamma \left( \frac{l + 1 - i\Lambda}{2} \right)\Gamma \left( \frac{2 - i\Lambda + l}{2} \right)}, \quad (D9) \]

with, for \( \Lambda = 2i \), \( A_l = 0 \) if \( l \geq 2 \).

The radial Green function \( G \) satisfies the equation

\[ (-\Delta_l - \lambda)G_l(r, r'; \lambda) = \Omega(k)\frac{\delta(r - r')}{m^3r^2}, \quad (D10) \]

where the operator \( \Delta_l \) is given in Eq.(C6), with \( J^2 \rightarrow l(l+1) \), and where

\[ \Omega(k) = 2(2\pi)^3m\sqrt{1 + r^2}. \quad (D11) \]

Since \( \text{Im}\Lambda \geq 0 \), the characteristic exponent at the point at \( \infty \) of the solution \( u_{2,\Lambda,t}^\infty(r) \) is (in \( \frac{1}{r} \)) given by

\[ 1 - i\Lambda = 1 + \text{Im}\Lambda - i\text{Re}\Lambda, \quad (D12) \]

and since the other solution \( u_{1,\Lambda,t}^\infty \) has (in \( \frac{1}{r} \)) the exponent

\[ 1 + i\Lambda = 1 - \text{Im}\Lambda + i\text{Re}\Lambda, \quad (D13) \]

see the Appendix C, Eq.(C22), it follows that the logarithmic case does not appear.

The constant \( A_l(\Lambda) \) in the Eq.(D4) is given by

\[ A_l(\Lambda) = -\frac{1}{(1 + r^2)W_l(r)\frac{\Omega(k)}{m^3r^2}}, \quad (D14) \]

where \( W_l(r) \) is the Wronskian

\[ W_l(r) = u_{1,\Lambda,t}^{(0)}(r) \frac{d}{dr}u_{2,\Lambda,t}^{(\infty)}(r) - u_{2,\Lambda,t}^{(\infty)}(r) \frac{d}{dr}u_{1,\Lambda,t}^{(0)}(r), \quad (D15) \]

which turns out to be

\[ W_l(r) = \frac{K}{r^2\sqrt{1 + r^2}}, \quad (D16) \]

with

\[ K = -2 \frac{\Gamma(l + \frac{3}{2})\Gamma(1 - i\Lambda)}{\Gamma \left( \frac{l + 2 - i\Lambda}{2} \right)\Gamma \left( \frac{1 - i\Lambda}{2} \right)} \quad (D17) \]
Collecting everything we have
\[
\mathcal{A}_l(\Lambda) = -\frac{(2\pi)^3}{m^2} \frac{\Gamma\left(\frac{l+2-i\Lambda}{2}\right)\Gamma\left(\frac{l+1-i\Lambda}{2}\right)}{\Gamma(l+\frac{i\Lambda}{2})\Gamma(1-i\Lambda)}.
\] (D18)

The equations \((D2), (D4), (D5), (D6)\) and \((D18)\) give the complete expression of the Green function \(G(\vec{k}, \vec{k}'; \lambda)\).

The so expressed Green function is an analytic function of \(\lambda\) in the complex plane cut along the real axis for \(\lambda \in [0, +\infty)\); it vanishes for \(|\lambda| \to \infty\), when \(r \neq r'\). Its asymptotic behaviour is given by
\[
G_l(r, r'; \lambda) = -\frac{(2\pi)^3}{m^2} \frac{1}{r r'} \frac{1}{\Lambda} \left[ e^{2|\zeta' - \zeta|\Lambda} + (-1)^{l+1} e^{2(\zeta' + \zeta)/\Lambda} \right],
\] (D19)
where
\[
\zeta = 2\ln(r + \sqrt{1 + r^2}), \quad \zeta' = 2\ln(r' + \sqrt{1 + r'^2}).
\] (D20)

In the Eq.(D19) use was made of the Eq.2.3.2(16),(17) of the reference [17].

From the Eq.(D19) we see that \(G\) is exponentially vanishing when \(|\lambda| \to \infty\) for \(r \neq r'\).

In the limit \(r' \to r\) we have
\[
\frac{1}{2\pi i} \int_{C_\infty} d\lambda G_l(r, r'; \lambda) = 2(2\pi)^3 \frac{\sqrt{1 + r^2} \delta(r - r')}{m^2 r^2},
\] (D21)
where \(C_\infty\) is a path of integration at infinity, which avoids the positive real axis. The last equation is useful for the demonstration of the completeness of the eigenfunctions of the Laplace-Beltrami operator.

We have indeed, from the Cauchy theorem
\[
\frac{1}{2\pi i} \int_{C_\infty} \mathcal{G}(r, r'; \lambda) d\lambda = -\frac{1}{2\pi i} \int_{\text{cut}} \mathcal{G}(r, r'; \lambda) d\lambda = -\frac{1}{2\pi i} \int_1^{\infty} [\mathcal{G}_l](r, r'; \lambda) d\lambda,
\] (D22)
where \([\mathcal{G}_l]\) is the discontinuity across the cut of \(\mathcal{G}\). Explicitly
\[
[\mathcal{G}_l](r, r'; \lambda) = \mathcal{G}_l(r, r'; \lambda + i0) - \mathcal{G}_l(r, r'; \lambda - i0) = -\frac{8\pi^3}{m^2} \frac{1}{\Lambda} \left| \frac{\Gamma\left(l+1+i\Lambda\right)\Gamma\left(l+1-i\Lambda\right)}{\Gamma(i\Lambda)\Gamma(l+i\Lambda)} \right|^2
\] (D23)
\[
r^l \Gamma\left(l+\frac{1+i\Lambda}{2}\right) \frac{l+\frac{3}{2} - r^2}{\Gamma\left(l+\frac{3}{2}\right)} F_1\left(l+\frac{1+3i\Lambda}{2}, l+\frac{3}{2}, -r^2\right) r'^l \Gamma\left(l+\frac{1+i\Lambda}{2}\right) \frac{l+\frac{3}{2} - r'^2}{\Gamma\left(l+\frac{3}{2}\right)} F_1\left(l+\frac{1+3i\Lambda}{2}, l+\frac{3}{2}, -r'^2\right),
\]
where \( \Lambda \in [0, +\infty) \).

If we define

\[
 w_{\lambda,l,m}(\vec{k}) = N_{\lambda,l} u_{\lambda,l,m}^{(0)}(\vec{r}) = N_{\lambda,l} r^l \binom{(l+1 + i\Lambda, l+1 - i\Lambda; \frac{3}{2}; -r^2)}{2} Y_{l,m}(\hat{\vec{k}}),
\]

where \( \hat{\vec{k}} \) is the direction of \( \vec{k} \), and

\[
 N_{\lambda,l} = \frac{2\pi}{m\sqrt{\Lambda}} \left| \frac{\Gamma(\frac{l+2-i\Lambda}{2})\Gamma(\frac{l+11-i\Lambda}{2})}{\Gamma(i\Lambda)\Gamma(l+\frac{3}{2})} \right|,
\]

we have the completeness of the eigenfunctions \( w_{\lambda,l,m} \)

\[
 \sum_{l \geq 0} \sum_{|m| \leq 0} \int_1^\infty d\lambda w_{\lambda,l,m}(\vec{k}) \bar{w}_{\lambda,l,m}(\vec{k}') = \Omega(k) \delta^3(\vec{k} - \vec{k}').
\]

So we have recovered the result of [13] by the use of the Green function.

The eigenfunctions \( w_{\lambda,l,m} \) are an orthogonal set with respect to the invariant measure \( \tilde{dk} \):

\[
 \int \tilde{dk} \bar{w}_{\lambda,l,m}(\vec{k}) w_{\lambda',l',m'}(\vec{k}) = \delta(\lambda - \lambda')\delta_{l,l'}\delta_{m,m'}.
\]

The action of the Lorentz generators on this basis is studied in [9].

In terms of these eigenfunction the Green function has the expression

\[
 G(\vec{k}, \vec{k}'; \lambda) = \int_1^\infty d\lambda' \sum_{lm} w_{\lambda',l,m}(\vec{k}) \bar{w}_{\lambda',l,m}(\vec{k}') \frac{\delta(\lambda - \lambda')}{\lambda' - \lambda}.
\]

2. The \( \lambda = -3 \) case.

The Green function of the operator \( D \) is obtained by putting \( \lambda = -3 \) in Eq.(D4), as

\[
 G(k, k'; -3) = G(\vec{k}, \vec{k}'; -3),
\]

which is real and symmetric. Here we give again the explicit expressions of the radial parts of null modes of \( \Delta - 3 \) whose behaviours make them good to build up Green’s functions as in (D4):
These are put together as follows:

\[ G(\vec{k}, \vec{k'}) = \sum_{l \geq 0} \sum_{|m| \leq l} Y_{l,m}(\hat{k}) Y_{l,m}(\hat{k'}) G_l(r, r'; -3), \]

\[ G_l(r, r'; -3) = \]

\[ = -\frac{(2\pi)^3 \Gamma\left(\frac{l+4}{2}\right) \Gamma\left(\frac{l+3}{2}\right)}{m^2 \Gamma\left(l + \frac{3}{2}\right) \Gamma(3)} \left[ u_{1,-3,l}^{(0)}(r) u_{2,-3,l}^{(\infty)}(r') \theta(r' - r) + u_{1,-3,l}^{(0)}(r') u_{2,-3,l}^{(\infty)}(r) \theta(r - r') \right] \]

\[ (D31) \]

**APPENDIX E: LORENTZ GENERATORS AND THEIR ALGEBRA.**

In this Appendix we give some detail of the calculation of the algebra of the Lorentz generators.

Let us limit ourselves to \( M_{ij} \), which is given in Eq.(2.20)

\[ M_{ij} = \int \tilde{d}k I(k) \left( k^i \frac{\partial}{\partial k^j} - k^j \frac{\partial}{\partial k^i} \right) \phi(k). \]

\[ (E1) \]

Expressing \( I(k) \) and \( \phi(k) \) in terms of the auxiliary variables as in Eq.(3.1) and Eq.(3.2) we get

\[ M_{ij} = \int \tilde{d}k \tilde{I}(k) \left( k^i \frac{\partial}{\partial k^j} - k^j \frac{\partial}{\partial k^i} \right) \tilde{\phi}(k) + \]

\[ + \int \tilde{d}k I(k) \left( k^i \frac{\partial}{\partial k^j} - k^j \frac{\partial}{\partial k^i} \right) (k \cdot X) + \]

\[ + \int \tilde{d}k F((P \cdot k), P^2) \left( k^i \frac{\partial}{\partial k^j} - k^j \frac{\partial}{\partial k^i} \right) \tilde{\phi}(k). \]

\[ (E2) \]
The last term is zero: indeed, if we integrate by parts and remember the asymptotic
behaviour of $F$ for $|\vec{k}| \to \infty$, which allows to neglect the surface term at infinity, we get,
using Eq.(2.37)
\[-\int d\vec{k} \hat{\phi}(k) \left( k^i \frac{\partial}{\partial k^j} - k^j \frac{\partial}{\partial k^i} \right) F((P \cdot k), P^2) =
\]
\[-\left[ P^j \int d\vec{k} \hat{\phi}(k) \frac{\partial F((P \cdot k), P^2)}{\partial P^i} - (i \leftrightarrow j) \right] = 0,
\]
where use was made of Eq.(3.2).

The second term of Eq.(E2) is
\[-\int d\vec{k} I(k) \left( k^i X^j - k^j X^i \right) = - \left( P^i X^j - P^j X^i \right),
\]
where there is not a contribution from $X^0$ since $\omega(k)$ is scalar.

Finally, the first term, using the definitions of $\hat{I}(k)$ and $\hat{\phi}(k)$ can be written
\[
\int d\vec{k} (D\mathcal{H}(k)) \left( k^i \frac{\partial}{\partial k^j} - k^j \frac{\partial}{\partial k^i} \right) \hat{\phi}(k) = \int d\vec{k} \mathcal{H}(k) \left( k^i \frac{\partial}{\partial k^j} - k^j \frac{\partial}{\partial k^i} \right) \mathcal{K}(k),
\]
since $D$ is self-adjoint and commutes with $l_{ij}$ of Eq.(C3)

Collecting these terms we get the decomposition given in Eq.(3.59).

An analogous calculation gives the decomposition (3.62).
REFERENCES

[1] For a review see for instance: F.M. Lev, Rivista del Nuovo Cimento, 16, 1 (1993).

[2] P.A.M. Dirac,”Lectures on Quantum Mechanics”, Belfer Graduate school of Science, Yeshiva University, New York, 1964.

[3] G.Longhi, in Proceedings of the Workshop ”Relativistic Action at a Distance: Classical and Quantum Aspects”, ed. J. Llosa, Lecture Notes in Physics, Vol. 162, Springer,1981. D.Dominici, G.Longhi, J.Gomis and J.M.Pons, J.Math.Phys., 25, 2439 (1984).

[4] I.T.Todorov, Report No.Comm. JINR E2-10125, Dubna (1976) (unpublished); Ann.Inst.H.Poincare’, 28A, 207 (1978).
A.Komar, Phys.Rev., D18, 1881, 1887, 3017 (1978); D19, 2908 (1979).
G.Longhi and L.Lusanna, Phys.Rev., D34, 3707 (1986) and references quoted therein.

[5] L.L.Foldy, Phys.Rev., 122, 275 (1962); Phys.Rev., 15,3044 1977);
L.L.Foldy and R.A.Krajcik, Phys.Rev., 32, 1025 (1974); D10, 1777 (1974); D12, 1700 (1975).
S.N.Sokolov, Dokl.Akad.Nauk. SSSR 233, 575 (1977), (english translation: Sov.Phys.Dokl., 22, 198 (1977); Teor.Mat.Fiz., 36, 193 (1978), (english translation: Theor.Math.Phys.(USSR), 36,682 (1978).
S.N.Sokolov and A.N.Shatnii, Theor.Math.Phys., 37, 1029 (1979).
H. Sazdjian, Ann.Phys.(NY), 136, 136 (1981).
F. Rohrlich, Phy.Rev.,D23, 1303 (1981).

[6] L.Lusanna, Int.J.Mod.Phys., A12, 645 (1997); D. Alba and L. Lusanna, ”The Lienard-Wiechert Potential of Charged Scalar Particles and their Relation to Scalar Electrodynamics in the Rest-Frame Instant Form”, Firenze Preprint [hep-th 9705155], May 1997; ”The Classical Relativistic Quark Model in the Rest-Frame Wigner-Covariant Coulomb Gauge”, Firenze preprint [hep-th 9705156], May 1997.
[7] J.L. Anderson, P.G. Bergmann, Phys. Rev., 83, 1018 (1951); P.G. Bergmann, J. Goldber, Phys. Rev., 98, 531 (1955).

[8] C. Møller, Ann.Inst.H.Poincare’, 11, 251 (1949); C. Møller, ”The Theory of Relativity”, Oxford 1960, Ch.VI. For a review of the various definitions of a ”center of mass” coordinate see: M. Pauri, ”Invariant Localization and Mass-spin Relations in the Hamiltonian Formulation of Canonical Relativistic Dynamics”, booklet (IFPR-T-019, Parma, 1971) unpublished; M. Pauri, ”Canonical (Possibly Lagrangian) Realizations of the Poincare’ Group with Increasing Mass-spin Trajectories”, Lecture Notes in Physics, 135, edited by K.B. Wolf, Springer Verlag, Berlin 1980. See also: C.W. Misner, K.S. Thorne and J.A. Wheeler, ”Gravitation”, Freeman and Co., New York, 1973, §5.

[9] G. Longhi, M. Materassi, ”A canonical realization of the BMS algebra”, hep-th/9803128, to appear on the ”Journal of Mathematical Physics”.

[10] R. Jost, ”The General Theory of Quantized Fields”, Am. Math. Soc., Providence 1965.

[11] D.A. Dubin, M.A.Hennigs and T.B.Smith, Publ.RIMS Kyoto Univ., 30, 479-532 (1994).

[12] F. Colomo, G. Longhi and L. Lusanna, Mod.Phy.Lett., A5, 17 (1990); Int.J.Mod.Phys., A5, 3347 (1990).

[13] R. Raczka, N. Limic and J. Niederle, J.Math.Phys, 7, 1861 (1966);7, 2026 (1966); 8, 1079 (1967). See also: A.O. Barut and R. Raczka, ”Theory of Group Representations and Applications”, Polish Scient. Pub., Warszawa 1977, Cap. 15.

[14] For a review of the quantum phase problem see: R. Lynch, Phys.Rep., 256, 367 (1995). For the point of view of the topological quantization see: C.J.Isham, ”Topological and global aspects of quantum theory”, in Relativity, Groups and Topology II, B.S. Dewitt and R. Stora Eds., North-Holland, Amsterdam, 1059-1290, 1984; see also: M. Gotay, ”On Quantizing $T^1S^1$“, preprint Un. of Hawai’i”, 1966.

[15] L. Lusanna and M. Materassi, The canonical collective-relative variable decomposition
for a Klein-Gordon field in the rest frame Wigner-covariant instant form, in preparation.

[16] F. Bigazzi, L. Lusanna, "Dirac Fields on Space-like Hypersurfaces, their Rest-Frame Description and Dirac Observables", hep-th/9807054

[17] A. Erdélyi et al., "Higher Transcendental Functions", Vol. I, R.E.Krieger Pub.Comp., 1953, Reprint Edition 1985.

[18] I. Stakgold, "Green’s Functions And Boundary Value Problems", Pure and Applied Mathematics, University of Delaware, Wiley and Sons, 1979.