Thermodynamics of Schwarzschild black hole surrounded by quintessence with generalized uncertainty principle

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Abstract In this manuscript, we consider a deformation on the Heisenberg algebra and investigate the effects on the thermodynamics of the Schwarzschild black hole that is surrounded by quintessence matter. To this end, we obtain the temperature, entropy and heat capacity functions of the black hole by using the standard laws of thermodynamic according to the considered deformation. We show that upper and lower bound values appear on these functions based on the quintessence and deformed algebra. Then, we derive the corrected density of quintessence matter and the black hole’s equation of state functions. We compare these results with the standard Schwarzschild black hole with and without quintessence with the graphical methods and interpret the quantum deformation effects.

1 Introduction

Black holes are probably the most mysterious objects in the universe. In recent years, we have observed an increase in the studies which examine black hole physics and thermodynamics in the modern cosmology literature \cite{1–16}. The basic assumption in these studies is that: “black holes could be taken as a thermal system, so the well-known thermodynamics laws could be used to interpret their nature”. Such a correlation is originally presented by Hawking and Bekenstein a half-century ago \cite{17–19}. According to them, black holes are thermodynamic objects that emit radiation from their event horizon with a characteristic temperature associated with their surface gravity. Furthermore, the entropy functions of these objects are linearly proportional to the event horizon areas in Planck units, and regarding the second law of thermodynamics their surface areas do not decrease. With these revolutionist propositions, they examined some properties of the black holes, such as Hawking temperature and mass functions, within the framework of statistical mechanics, and they concluded that the first law of thermodynamics was not violated \cite{20}.

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The recent reliable observational astronomical evidence shows that the universe expands at an accelerating rate [21–23]. To explain this acceleration, the presence of dark energy is postulated. In the current model, dark energy is considered as another form of energy with negative pressure and is estimated to exist approximately in seventy percent of the total energy density of the universe. The simplest way to model this acceleration mathematically is given by the definition of a cosmological constant [24]. However, its experimental value is much smaller than its theoretically expected value [25]; therefore, alternative models have been put forward. These models are based on dynamic scalar fields, and they differ from each other by a parameter value which indicates the ratio of pressure to the energy density of the dark energy [26–31]. For a detailed discussion and comparison among these models, we refer the readers to have a look at the review given in [32].

A dark energy surrounding the black hole is thought to have important effects on black hole thermodynamics. There are dozens of studies examining this scenario in the literature. For example, in 2003, Kiselev took the quintessence matter, which is among the black energy candidates, into account and investigated the Schwarzschild black hole thermodynamics, assuming that it surrounds the black hole [33]. In 2008, Chen et al. discussed the Hawking radiation in a d-dimensional spherically symmetric static black hole with the presence of quintessence matter [34]. Later, Wei and Chu explored the thermodynamics properties of the Reissner–Nördstrom (NR) black hole surrounded by quintessence matter [35]. In 2013, Wei, this time with Ren, studied the thermodynamics of the NR black hole in de Sitter quintessence spacetime [36]. The same year, Fernando examined Narai type black holes with quintessence matter [37,38]. Three years later, in a series of work Ghaderi and Malakolkalami, presented the effects of quintessence matter on the thermodynamic functions of the Bardeen [39], Schwarzschild, and RN black holes [40]. In 2019, Shahjalal considered some quantum correction to the Schwarzschild metric and compared the thermodynamics of the black hole with the presence and absence of the quintessence matter [41]. Last year, Haldar and Biswas showed that the thermodynamic volume of Bardeen anti-de Sitter black hole is not equal to the geometric volume in the presence of quintessence matter [42]. Very recently, Ndogmo et al. computed various thermodynamic quantities of a rotating nonlinear magnetic-charged black hole embedded in the quintessence matter and discussed the phase transition [43].

On the other hand, it is a very well-known fact that quantum physics deals with the physical properties of nature at the atomic scale which classical physics cannot. Quantum mechanics is the mathematical toolbox that is used to explain the aspects of nature at atomic and subatomic scales. One of the mystery of this formalism is the operator concept. According to the Heisenberg algebra, the operation order of the position and momentum operators on the wavefunction is important, and it differs from each other up to a constant factor. If one takes the quantum theory of gravity into account, the Heisenberg algebra has to be extended with quantum gravitational corrections, which lead the generalization of the Heisenberg algebra. As a natural consequence of this, the Heisenberg uncertainty principle (HUP), which is a measure of the observable values of the position and momentum operators, must be replaced by the generalized (gravitational) uncertainty principle (GUP). It is worth noting that such an extension is not unique; therefore, different GUP predictions exist. Each of the non-equivalent algebraic deformations produces different new features, such as measurable minimal length and maximum momentum [44,45].

In the literature, we observe that the GUP is frequently employed in the papers that examine black hole thermodynamics [46–59]. However, to the best of our knowledge, no one has studied the black hole thermodynamics surrounded by the quintessence matter within the framework of the deformed Heisenberg algebra. Since the GUP formalism introduces new features, such as the existence of a minimal length and a maximal observable momentum,
values, we believe that this novelty could play an important role on the interpretation of the thermodynamics of black holes surrounded by the quintessence matter.

With this motivation, we prepare the manuscript by using natural units. The outline is as follows: In Sect. 2, we introduce the metric of the considered black hole surrounded by the quintessence matter. Then, in Sect. 3, we briefly describe the deformed Heisenberg algebra and derive the temperature, heat capacity and entropy functions, respectively. After we examine these thermodynamic functions in three particular values of the quintessence state-parameter, we obtain the GUP-corrected energy matter density and equation of state functions. We analyze the findings and conclude the manuscript in the final section.

2 Schwarzschild black hole surrounded by quintessence

We start by considering the Kiselev’s paper [33], which is based on the derivation of the exact solution of Einstein equations of a static spherically symmetric black hole that is surrounded by the quintessence energy matter. In that work, Kiselev expressed the general form of metric to be in the form of:

\[ ds^2 = -N(r) dt^2 + \frac{1}{N(r)} dr^2 + r^2 d\Omega_2^2, \]  

where, for the Schwarzschild black hole, \( N(r) \) can be taken as:

\[ N(r) = 1 - \frac{2M}{r} - \frac{\alpha \omega_q}{r^{3\omega_q+1}}. \]  

Here, \( M \) denotes the mass of the black hole, \( \omega_q \) is the quintessential state parameter and \( \alpha \omega_q \) is the positive normalization factor that depends on the density of quintessence matter (For simplicity, hereafter, we will denote it without the subscript). In the range of \(-1 < \omega_q < -1/3\), the quintessence successfully explains the accelerated expansion of the universe. Note that, when \( \alpha \) is taken to be equal to zero, the metric reduces to the ordinary Schwarzschild black hole metric.

Kiselev also showed that in this quintessence matter model the energy–momentum tensor has the following components:

\[ T_t^t = T_r^r = -\rho, \]  

\[ T_\theta^\theta = T_\phi^\phi = \frac{\rho}{2} (3\omega_q + 1), \]  

where the pressure, \( P_q \), and the matter–energy density, \( \rho_q \), are written in the form of:

\[ P_q = \omega_q \rho_q, \]  

\[ \rho_q = -\frac{3}{2} \frac{\alpha \omega_q}{r^{3(\omega_q+1)}}. \]  

If we consider the quintessence state parameter and normalization factor values that are given above, then we can conclude that the matter–energy density takes only positive values, while the pressure of the quintessence matter takes only negative values.

After the introduction of these facts, we intend to clarify the event horizon radius. In order to determine them, we set the following condition:

\[ N(r)|_{r=r_H} = 0, \]
We classify the solutions in three particular cases:

- For $\omega_q = -2/3$, two event horizon radii appear, namely inner, $r_{in}$, and outer, $r_{out}$, as follows:

$$ r_{in} = \frac{1 - \sqrt{1 - 8M\alpha}}{2\alpha}, \quad r_{out} = \frac{1 + \sqrt{1 - 8M\alpha}}{2\alpha}. $$

It is worth noting that the outer horizon is commonly called as the quintessence horizon, alike to the cosmological horizon in the de Sitter spacetime [20,41,60,61].

- For $\omega_q = -1/3$, only one horizon arises

$$ r_H = \frac{2M}{1 - \alpha}. $$

- In the case of $\omega_q = -1$, we obtain the de Sitter–Schwarzschild solution if we take

$$ \alpha = \frac{\Lambda}{3}. $$

Before ending this section, we present a graphical demonstration of these cases. We take $M = 1$ and $\alpha = 0.03$, and we plot $N(r)$ versus distance in Fig. 1.

We observe that $\omega_q = -1/3$ case mimics the ordinary case, since the event horizon radius in each case differs by themselves just by a constant that is extremely small. In the case of $\omega_q = -2/3$, real-valued inner and outer event horizon radii appear since $\alpha \leq 1/(8M)$ condition is satisfied. Moreover, with this particular choice of parameters two horizons occur in the $\omega_q = -1$ case.
3 Thermodynamic features of the black hole

At the energies near to the Planck scale, the conventional concepts of time and space break down. Considering the effect of gravity, the existence of a minimal length scale becomes necessary. However, such a minimal length does not exist in the usual Heisenberg algebra. Therefore, one can take the GUP instead of the HUP into account, which is given with the natural units in the form of [44,45]:

\[(\Delta p) (\Delta x) \geq \frac{1}{2} \left(1 + \beta (\Delta p)^2\right),\]  

(13)

where \(\beta\) is a small nonnegative deformation parameter that is proportional to the Planck length, and it is defined within the generalized Heisenberg algebra of the form of \([x, p] = i(1 + \beta p^2)\). This deformation leads to a minimum uncertainty in the position [44,45].

\[
(\Delta x)_{\text{min}} = \sqrt{\beta}. 
\]  

(14)

It is worth noting that there are other scenarios where the deformation parameter is taken as a negative quantity [62–65]. Moreover, there are very interesting works in which the deformation parameter is regarded as a dynamical variable within a more general perspective instead of being a constant [66,67]. After this notice, we start by solving Eq. (13) with respect to \((\Delta x)\).

We find

\[
\left(\frac{\Delta x}{\beta}\right) \left(1 - \sqrt{1 - \frac{\beta}{(\Delta x)^2}}\right) \leq (\Delta p) \leq \left(\frac{\Delta x}{\beta}\right) \left(1 + \sqrt{1 - \frac{\beta}{(\Delta x)^2}}\right). 
\]  

(15)

We observe that for \(\frac{\beta}{(\Delta x)^2} \ll 1\), the left-hand side of the inequality produces some small corrections to the HUP, while the right-hand side implies an upper bound value to the momentum uncertainty; therefore, \((\Delta p)\) cannot be arbitrarily large.

Hereafter, we study the effects of the GUP on the Schwarzschild black hole black hole surrounded by quintessence. In the semiclassical case, if the entropy, \(S\), is assumed to be a function of the black hole area, \(A\), then, the temperature of the black hole can be expressed with a relation between them as given in: [68]

\[
T = \frac{\kappa}{8\pi} \frac{dA}{dS}, 
\]  

(16)

where \(\kappa\) is the surface gravity at the outer horizon, and in our case it is equal to

\[
\kappa = -\lim_{r \to r_H} \sqrt{-\frac{g^{11}}{g^{00}}} \frac{(g^{00})'}{g^{00}} = \frac{1}{r_H} \left(1 + \frac{3\alpha \omega q}{r_H^3}\right). 
\]  

(17)

In [68], it is shown that if a black hole absorbs a particle then its area changes proportionally with the particle’s mass and size that are associated with the uncertainties of momentum and position. Furthermore, such a minimal change of the area leads to a change in the entropy which cannot be smaller than \(\ln 2\). Therefore, one can write:

\[
\frac{dA}{dS} \simeq \frac{(\Delta A)_{\text{min}}}{(\Delta S)_{\text{min}}} \simeq \frac{\epsilon}{\ln 2} (\Delta X) (\Delta P), 
\]  

(18)

where \(\epsilon\) denotes the calibration factor. We use \(\Delta X \simeq 2r_H\) and employ \((\Delta p)\) from Eq. (15) to evaluate

\[
\frac{dA}{dS} \simeq \frac{4\epsilon r_H^2}{\beta \ln 2} \left(1 - \sqrt{1 - \frac{\beta}{4r_H^2}}\right). 
\]  

(19)
Then, by substituting Eqs. (17) and (19) in Eq. (16), we obtain a relation between the temperature and horizon radius of the black hole in the form of:

$$T = \frac{1}{2\pi} \frac{\epsilon r_H}{\beta \ln 2} \left( 1 + \frac{3\alpha \omega_q}{r_H^{3\omega_q+1}} \right) \left( 1 - \sqrt{1 - \frac{\beta}{4r_H^2}} \right).$$

(20)

In the absence of the quintessence and the GUP deformation, ($\alpha = \beta = 0$), Eq. (20) reduces to $T = \frac{\epsilon}{16\pi r_H \ln 2}$, which should be equal to the Hawking temperature, $T_H = \frac{1}{4\pi r_H}$, [19,69]. Therefore, we determine the calibration factor as $\epsilon = 4 \ln 2$. Then, we rewrite the GUP-corrected Hawking temperature as:

$$T = \frac{2r_H}{\pi \beta} \left( 1 + \frac{3\alpha \omega_q}{r_H^{3\omega_q+1}} \right) \left( 1 - \sqrt{1 - \frac{\beta}{4r_H^2}} \right).$$

(21)

For $\beta = 0$, Eq. (21) reduces to the Hawking temperature of Schwarzschild black hole surrounded by the quintessence in the HUP limit

$$T = \frac{1}{4\pi r_H} \left( 1 + \frac{3\alpha \omega_q}{r_H^{3\omega_q+1}} \right).$$

(22)

Regarding the fact that the temperature of a black hole must be real-valued, we find a constraint on the radius that bounds the radius value from above.

$$r_H \leq \left( \frac{3\omega_q+1}{-3\alpha \omega_q} \right)^{\frac{1}{2}}.$$  

(23)

It is worth noting that in this study $\alpha > 0$ and $\omega_q < 0$, and hence, the horizon radius has a positive root value unless the quintessence state parameter is equal to $-1/3$. For $\omega_q = -1/3$, the horizon radius is not bounded from above. Therefore, we conclude that the greatest possible event horizon is determined by the quintessence matter that surrounds the black hole.

On the other hand, in the GUP-corrected case, according to Eq. (21), we observe that the modified Hawking temperature depends not only on the black hole’s properties but also on the GUP parameter which leads to another constraint as

$$r_H \geq \frac{\sqrt{\beta}}{2}.$$  

(24)

This result indicates that the black hole’s lowest radius value is based on the GUP. Furthermore, in the GUP-corrected case a maximal temperature value arises in the form of:

$$T \leq \frac{1}{\pi \sqrt{\beta}} \left[ 1 + 3\alpha \omega_q \left( \frac{\beta}{4} \right)^{-\frac{3\omega_q+1}{2}} \right].$$

(25)

In the absence of the quintessence matter, we find that the GUP-corrected temperature of the black hole can take values only in the following certain range:

$$0 < T \leq \frac{1}{2\pi \sqrt{\beta}}.$$  

(26)

In Fig. 2, we depict the black hole temperature versus the horizon radius in the HUP and GUP approaches, respectively. In particular, in Fig. 2a, we consider the HUP and GUP cases in the absence of the quintessence matter. We observe that in the ordinary case there is no bound on the temperature and the radius. However, in the GUP case, the radius can take values after
0.35 which corresponds the maximal temperature value 0.22 as one can calculate from Eqs. (24) and (26), respectively.

We summarize all our findings in three particular cases:

1. For \( \omega_q = -1 \), the GUP-corrected temperature of the quintessence-surrounded Schwarzschild black hole can be calculated via:

\[
T_{(\omega_q=-1)} = \frac{2r_H}{\pi \beta} \left( 1 - 3\alpha r_H^2 \right) \left( 1 - \sqrt{1 - \frac{\beta}{4r_H^2}} \right).
\]  

In this case, the event horizon radius can only take values in the range of:

\[
\frac{1}{\sqrt{3\alpha}} \geq r_H \geq \frac{\sqrt{2}}{\beta},
\]

and hence the temperature can vary in between

\[
0 \leq T_{(\omega_q=-1)} \leq \frac{4 - 3\alpha \beta}{4\pi}.
\]

Note that, in the absence of the GUP, there is no lower bound value of the radius; therefore, there is no upper limit value for temperature. We depict this case in Fig. 2b. We find that in the GUP case, the radius is confined in the range of [0.35, 0.82], while the temperature has a maximum value 0.37.

2. For \( \omega_q = -2/3 \), alike, the black hole temperature turns to the form of

\[
T_{(\omega_q=-2/3)} = \frac{2r_H}{\pi \beta} \left( 1 - 2\alpha r_H \right) \left( 1 - \sqrt{1 - \frac{\beta}{4r_H^2}} \right),
\]

which limits the horizon radius to the following range

\[
\frac{1}{2\alpha} \geq r_H \geq \frac{\sqrt{2}}{\beta},
\]

Therefore, in this case the GUP-corrected temperature range becomes

\[
0 \leq T_{(\omega_q=-2/3)} \leq \frac{1 - \alpha \sqrt{\beta}}{\pi \sqrt{\beta}}.
\]

We illustrate this case in Fig. 2c. We find that in the HUP and GUP cases, there is the same upper limit value, 1, for the horizon radius. In the GUP case, the lower bound radius value, 0.35, exists, while in the HUP case does not. Moreover, in the GUP case, the temperature has a maximum value of 0.29.

3. For \( \omega_q = -1/3 \), Eq. (21) becomes

\[
T_{(\omega_q=-1/3)} = \frac{2r_H}{\pi \beta} \left( 1 - \alpha \right) \left( 1 - \sqrt{1 - \frac{\beta}{4r_H^2}} \right).
\]

However, in this case, an upper bound on the horizon radius does not emerge. The GUP-corrected temperature varies in the range of:

\[
0 < T_{(\omega_q=-1/3)} \leq \frac{1 - \alpha}{\pi \sqrt{\beta}}.
\]

We illustrate this case in Fig. 2d. We observe that there is no upper bound on the horizon radii. However, in the GUP case, there is a lower radius value, 0.35, which corresponds a maximum temperature value of 0.23.
Fig. 2 Temperature versus horizon radius for $\alpha = 0.5$

As a conclusion, we observe that all graphical demonstrations approve the predictions of our analysis. Moreover, these results are in a good agreement with the results given in [46].

Next, we derive the GUP-corrected heat capacity function of the black hole heat by employing the following thermodynamic relation:

$$C = \frac{dM}{dT}. \tag{35}$$

We find

$$C = -\frac{\pi \beta}{4} \sqrt{1 - \frac{\beta}{4r_H}} \left(1 + \frac{3\alpha \omega_q}{r_H^{3\omega_q+1}} \right)^2 \sqrt{1 - \frac{\beta}{4r_H}} + \frac{\beta}{4r_H} \frac{3\alpha \omega_q (1+3\omega_q)}{r_H^{3\omega_q+1}}. \tag{36}$$

It is worth noting that when the heat capacity is equal to zero, then the black hole cannot exchange radiation with its surrounding space. This scenario is known as the black hole remnant. Regarding this fact, we observe that at

$$r_{rem} = \frac{\sqrt{\beta}}{2}, \tag{37}$$

Eq. (36) tends to zero, so that black hole remnant exists. By substituting Eq. (37) in Eq. (21), we obtain the nonzero black hole remnant temperature

$$T_{rem} = \frac{1}{\pi \sqrt{\beta}} \left(1 + 3\alpha \omega_q \left(\frac{2}{\sqrt{\beta}}\right)^{3\omega_q+1}\right), \tag{38}$$
with the ensuing mass
\[
M_{rem} = \frac{\sqrt{\beta}}{4} \left( 1 - \alpha \left( \frac{2}{\sqrt{\beta}} \right)^{3\omega q + 1} \right) .
\] (39)

In the absence of quintessence matter, GUP-corrected specific heat function and the remnant temperature reduce to:
\[
C = -\frac{\pi \beta}{4} \sqrt{\frac{1 - \frac{\beta}{4r_H^2}}{1 - \frac{\beta}{4r_H^2}}} ,
\] (40)
\[
T_{rem} = \frac{1}{\pi \sqrt{\beta}} .
\] (41)

Moreover, if we take \( \beta = 0 \), we get the ordinary specific heat function
\[
C = -\frac{\pi \beta}{4} \left( 1 - 3\alpha r_H^2 \right) \sqrt{\frac{1 - \frac{\beta}{4r_H^2}}{1 - \frac{\beta}{4r_H^2}}} + \frac{3\alpha \beta}{2} ,
\] (42)
\[
T_{rem} = \frac{1}{\pi \sqrt{\beta}} \left( 1 - \frac{3\alpha \beta}{4} \right) ,
\] (43)
\[
M_{rem} = \frac{\sqrt{\beta}}{4} \left( 1 - \frac{\alpha \beta}{4} \right) .
\] (44)

• For \( \omega_q = -1 \), Eqs. (36), (38), and (39) reduce to:
\[
C = -\frac{\pi \beta}{4} \left( 1 - 3\alpha r_H^2 \right) \sqrt{\frac{1 - \frac{\beta}{4r_H^2}}{1 - \frac{\beta}{4r_H^2}}} + \frac{3\alpha \beta}{2} ,
\] (45)
\[
T_{rem} = \frac{1}{\pi \sqrt{\beta}} \left( 1 - \frac{3\alpha \beta}{4} \right) ,
\] (46)
\[
M_{rem} = \frac{\sqrt{\beta}}{4} \left( 1 - \frac{\alpha \beta}{4} \right) .
\] (47)

• For \( \omega_q = -2/3 \), Eqs. (36), (38), and (39) become:
\[
C = -\frac{\pi \beta}{4} \left( 1 - 2\alpha r_H \right) \sqrt{\frac{1 - \frac{\beta}{4r_H^2}}{1 - \frac{\beta}{4r_H^2}}} + \frac{\alpha \beta}{2r_H} ,
\] (48)
Fig. 3  Specific heat versus horizon radius for $\alpha = 0.5$

$$T_{rem} = \frac{1 - \alpha}{\pi \sqrt{\beta}},$$  \hspace{1cm} (49)

$$M_{rem} = \frac{(1 - \alpha) \sqrt{\beta}}{4}.$$ \hspace{1cm} (50)

We depict the GUP-corrected specific heat functions for $\omega_q = -1$, $\omega_q = -2/3$ and $\omega_q = -1/3$ in Fig. 3b, Fig. 3c and Fig. 3d, respectively. For $\alpha = \beta = 0.5$, we find the remnant temperatures as 0.37, 0.29 and 0.23 for these cases. We note that these values are the same with the ones given in Eqs. (29), (32) and (34).

Next, we derive the entropy function of the black hole by employing the customary definition, which is given in the form of:

$$S = \int \frac{dM}{T}.$$ \hspace{1cm} (51)

Using Eqs. (8) and (21) in Eq. (51), we find

$$S = \frac{\pi}{2} r_H^2 \left(1 + \sqrt{1 - \frac{\beta}{4r_H^2}}\right) - \frac{\pi \beta}{8} \ln r_H - \frac{\pi \beta}{8} \ln \left(1 + \sqrt{1 - \frac{\beta}{4r_H^2}}\right).$$ \hspace{1cm} (52)

We observe that the quintessence that surrounds the black hole does not affect the black hole’s entropy. Instead, the GUP correction modifies the entropy. In the HUP limit, the entropy reduces to the usual expression, $S = \pi r_H^2 = \frac{A}{4}$. © Springer
Then, we express the GUP-corrected energy–matter density of the quintessence by using the definition given in Eq. (6). We get

$$\rho_q = -\frac{3\alpha\omega_q \omega_q}{2} \left[ \frac{r_H^2}{2} \left( 1 + \sqrt{1 - \frac{\beta}{4r_H^2}} \right) - \frac{\beta}{8} \ln r_H - \frac{\beta}{8} \ln \left( 1 + \sqrt{1 - \frac{\beta}{4r_H^2}} \right) \right]^{-\frac{3\omega_q + 3}{2}} \tag{53}$$

which reduces to

$$\rho_q = -\frac{3\alpha\omega_q \omega_q}{2} \frac{r_H^2}{3\omega_q + 3} \cdot \tag{54}$$

for $\beta = 0$. In the particular values of the quintessence state parameters, the energy–matter density function reduces to the following forms:

- For $\omega_q = -1$, it becomes the same constant that depends only on the normalization factor:

$$\rho_q = \frac{3\alpha(\omega_q=-1)}{2}. \tag{55}$$

- For $\omega_q = -2/3$, GUP-corrected energy density function becomes

$$\rho_q = \alpha(\omega_q=-2/3) \left[ \frac{r_H^2}{2} \left( 1 + \sqrt{1 - \frac{\beta}{4r_H^2}} \right) - \frac{\beta}{8} \ln r_H - \frac{\beta}{8} \ln \left( 1 + \sqrt{1 - \frac{\beta}{4r_H^2}} \right) \right]^{-\frac{1}{2}}, \tag{56}$$

while in the HUP case it reduces to

$$\rho_q = \frac{\alpha(\omega_q=-2/3)}{r_H}. \tag{57}$$

- For $\omega_q = -1/3$, GUP-corrected energy density function appears as:

$$\rho_q = \frac{\alpha(\omega_q=-1/3)}{2} \left[ \frac{r_H^2}{2} \left( 1 + \sqrt{1 - \frac{\beta}{4r_H^2}} \right) - \frac{\beta}{8} \ln r_H - \frac{\beta}{8} \ln \left( 1 + \sqrt{1 - \frac{\beta}{4r_H^2}} \right) \right]^{-\frac{1}{2}}, \tag{58}$$

while in the HUP case it turns to

$$\rho_q = \frac{\alpha(\omega_q=-1/3)}{2r_H^2}. \tag{59}$$

We present the behavior of the energy–matter density of quintessence in Fig. 4. In Fig. 4a and b, we illustrate the GUP and HUP cases, respectively. We observe that for $\omega_q = -1$ the energy–matter density is independent of horizon radius. In Fig. 4c and d, we compare the HUP and GUP cases of $\omega_q = -2/3$ and $\omega_q = -1/3$. We see that energy–matter density decreases as linear inverse and inverse square of horizon radius in the absence of GUP-correction. The presence of GUP correction slightly increases the value of the energy–matter density function, especially for the small values of radius.

Finally, we derive the equation of state of the black hole by taking the relations between the pressure and the quintessence matter–energy density, which are given with Eqs. (5) and (6), into account. Then, we express the mass in terms of the pressure and horizon radius.

$$M = \frac{r_H}{2} + \frac{P_q r_H^3}{3\omega_q^2}. \tag{60}$$
Fig. 4  Energy–matter density of quintessence versus horizon radius

According to $V = \left( \frac{\partial M}{\partial P_q} \right)$, we express the volume in terms of the horizon radius and state parameter [70]:

$$V = \frac{r^3}{3 \omega_q^2}. \quad (61)$$

Then, we use them in Eq. (21), and we get

$$T = \frac{2}{\pi \beta} \left( \frac{2 P_q (3 \omega_q^2 V)^{2/3}}{\omega_q} \right) \left( \frac{1}{1 - \frac{\beta}{4 (3 \omega_q^2 V)^{2/3}}} \right). \quad (62)$$

For $T = 1$ isotherm, we obtain the GUP-corrected equation of state of the black hole in the form of:

$$P_q = \frac{\omega_q}{2 \left( 3 \omega_q^2 V \right)^{2/3}} \left[ 1 - \frac{\pi \beta}{2 \left( 3 \omega_q^2 V \right)^{1/3}} \left( 1 - \frac{1}{\sqrt{1 - \frac{\beta}{4 (3 \omega_q^2 V)^{2/3}}} \right) \right], \quad (63)$$

which reduces to

$$P_q = \frac{\omega_q}{2 \left( 3 \omega_q^2 V \right)^{2/3}} \left[ 1 - 4 \pi \left( 3 \omega_q^2 V \right)^{1/3} \right]. \quad (64)$$
in the HUP limit. In order to obtain a real-valued pressure in the GUP case, the following constraint arises:

\[ V \geq \frac{1}{3\omega_q^2} \left( \frac{\beta}{4} \right)^{\frac{3}{2}}. \]  

(65)

Finally, we summarize these results according to the particular quintessence state parameter values as we have done above.

- For \( \omega_q = -1 \), Eq. (63) becomes
  \[ P_q = -\frac{1}{2} \left( \frac{3V}{2} \right)^{2/3} \left[ 1 - \pi \beta \frac{1}{2 \left( \frac{3V}{2} \right)^{1/3}} \left( 1 - \sqrt{1 - \frac{\beta}{\frac{4(3V)^{2/3}}{2}}} \right) \right], \]

and in the HUP limit it reduces to
  \[ P_q = -\frac{1}{2} \left( \frac{3V}{2} \right)^{2/3} \left[ 1 - 4\pi \left( \frac{3V}{2} \right)^{1/3} \right]. \]

(66)

- For \( \omega_q = -2/3 \), Eq. (63) turns to the form of
  \[ P_q = -\frac{1}{3} \left( \frac{3V}{2} \right)^{2/3} \left[ 1 - \pi \beta \frac{1}{2 \left( \frac{3V}{2} \right)^{1/3}} \left( 1 - \sqrt{1 - \frac{\beta}{\frac{4(3V)^{2/3}}{2}}} \right) \right], \]

and in the HUP limit it gives
  \[ P_q = -\frac{1}{3} \left( \frac{3V}{2} \right)^{2/3} \left[ 1 - 4\pi \left( \frac{3V}{2} \right)^{1/3} \right]. \]

(68)

- For \( \omega_q = -1/3 \), Eq. (63) becomes
  \[ P_q = -\frac{1}{6} \left( \frac{3V}{2} \right)^{2/3} \left[ 1 - \pi \beta \frac{1}{2 \left( \frac{3V}{2} \right)^{1/3}} \left( 1 - \sqrt{1 - \frac{\beta}{\frac{4(3V)^{2/3}}{2}}} \right) \right], \]

and in the HUP limit we find
  \[ P_q = -\frac{1}{6} \left( \frac{3V}{2} \right)^{2/3} \left[ 1 - 4\pi \left( \frac{3V}{2} \right)^{1/3} \right]. \]

(70)

In Fig. 5, we plot the P–V isotherm. In Fig. 5a, we demonstrate the HUP case results. We observe that while the volume increases the pressure shows a slower decrease for higher quintessence state parameter. In Fig. 5a, b and c, we demonstrate the effects of the GUP correction on the equation of state of the case for \( \omega_q = -1 \), \( \omega_q = -2/3 \) and \( \omega_q = -1/3 \), respectively. We find that with the GUP corrections the pressure of the black hole achieves smaller values. Moreover, we observe that the volume of the black hole is bounded from below in the GUP case as predicted above. On the other hand, the GUP corrections lose their effect in the higher volume values.
In this manuscript, we examine the effects of the generalized uncertainty principle on the thermodynamics of the Schwarzschild black hole surrounded by quintessence. We find that two conditions arise which bound the event horizon radius from above and below. We observe that the upper bound depends on the quintessence, while the lower bound depends on the deformed algebra. We see that these constraints are effective by setting limits on temperature values. Then, we study the GUP-corrected heat capacity and entropy functions of the black hole by considering the laws of thermodynamics. We observe a nonzero remnant temperature in the presence of the deformed algebra. Moreover, we find logarithmic terms in the GUP-corrected entropy. Then, we examine the equation of state of the black hole. We find that the quantum effects slightly increase the energy–matter density function. Finally, we investigate the pressure–volume isotherm of the black hole. We observe the effects of deformed algebra on pressure more dominantly in smaller volumes. We compare the ensuing results through graphical methods in the context of the deformed and undeformed Heisenberg algebra. Moreover, we demonstrate the effect of the GUP correction on the thermal properties of the black hole for three particular values of the quintessence state parameter and make comparison among them.

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