SPATIAL REALISATIONS OF KMS STATES ON THE $C^*$-ALGEBRAS OF HIGHER-RANK GRAPHS

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Abstract. Several authors have recently been studying the equilibrium or KMS states on the Toeplitz algebras of finite higher-rank graphs. For graphs of rank one (that is, for ordinary directed graphs), there is a natural dynamics obtained by lifting the gauge action of the circle to an action of the real line. The algebras of higher-rank graphs carry a gauge action of a higher-dimensional torus, and there are many potential dynamics arising from different embeddings of the real line in the torus. Previous results show that there is nonetheless a “preferred dynamics” for which the system exhibits a particularly satisfactory phase transition, and that the unique KMS state at the critical inverse temperature can then be implemented by integrating vector states against a measure on the infinite path space of the graph. Here we obtain a similar description of the KMS state at the critical inverse temperature for other dynamics. Our spatial implementation is given by integrating against a measure on a space of paths which are infinite in some directions but finite in others. Our results are sharpest for the algebras of rank-two graphs.

1. Introduction

There has recently been renewed interest in the KMS states of dynamical systems associated to directed graphs [15, 10, 12, 4] and their higher-rank analogues [33, 34, 11, 13]. For systems based on the Toeplitz algebra of the graph, there is a simplex of KMS$_\beta$ states at each inverse temperature $\beta$ larger than a critical value $\beta_c$; under additional hypotheses on the graph, this simplex collapses to a single KMS state at inverse temperature $\beta_c$. This last state often factors through a state of the graph algebra of the graph, which is then the only KMS state of the graph algebra.

Both the Toeplitz algebra and graph algebra of a directed graph $E$ carry a natural gauge action of the circle $T$ which lifts via $t \mapsto e^{it}$ to a natural dynamics, and the results in [15, 10, 12] are about this dynamics for finite $E$ (more general dynamics have been studied in [7, 4, 14], for example). The critical inverse temperature $\beta_c$ is given in terms of the spectral radius $\rho(A)$ of the vertex matrix $A$ of the graph by $\beta_c = \ln \rho(A)$ (this goes back to [6]).

For a higher-rank graph $\Lambda$ of rank $k$, the gauge action is an action of the $k$-torus $T^k$, and to get a dynamics we have to choose an embedding of the real line $\mathbb{R}$ in $T^k$. The graph $\Lambda$ has $k$ vertex matrices $\{A_j : 1 \leq j \leq k\}$, and if the embedding is given by $t \mapsto e^{itr}$ for some $r \in (0, \infty)^k$, the critical inverse temperature is $\beta_c = \max_{j} \{r_j^{-1} \ln \rho(A_j)\}$. For $\beta > \beta_c$, the dynamics on the Toeplitz algebra again admits a simplex of KMS$_\beta$ states [11 Theorem 6.1]. At $\beta = \beta_c$ it matters what $r$ is. The best results in [11] and [13] concern a preferred dynamics in which $r = (\ln \rho(A_1), \ldots, \ln \rho(A_k))$, and for which we have $\beta_c = 1 = r_j^{-1} \ln \rho(A_j)$ for all $j$. Under strong irreducibility hypotheses on the graph, there is then a unique KMS$_1$ state on the Toeplitz algebra and on the graph algebra [11].
Theorem 7.2; for more general graphs, uniqueness requires aperiodicity of the graph \[13, \text{ Corollary 10.3}\].

Graph and Toeplitz algebras have large commutative subalgebras \(D\) generated by range
projections, and we expect, both from previous studies \[7, 19, 18\] and from general results
in \[21\], that KMS states should be given by integrating vector states against measures on
the spectrum of \(D\). For \(\beta > \beta_c\), this is indeed the case: the KMS
\(\beta\) states are constructed in \[11, \S 6\] as weighted sums of vector states in the Toeplitz representation on the finite-path space, and the weights give atomic measures with the required property (see Remark \[4.3\]). For the preferred dynamics, where \(\beta_c = 1\), Proposition 10.2 of \[13\] describes a measure on the infinite-path space such that the KMS \(1\) state is an integral of vector states for the
infinite-path representation, and indeed that result was needed in \[13\] to prove existence
for periodic graphs. So finding such measures seems an interesting and potentially useful
enterprise. In this paper we construct suitable measures for other dynamics, in which
\(\beta_c = r^{-j} \ln \rho(A_j)\) for some but not all \(j\).

Suppose that \(\Lambda\) is a finite \(k\)-graph. The finite-path space is just \(\Lambda\) itself in the discrete
topology. The infinite-path space \(\Lambda^\infty\) consists of functors \(x\) from a model graph based
on \(\mathbb{N}^k\) into \(\Lambda\) \[17, \S 2\], and has a compact Hausdorff topology (because \(\Lambda\) is finite). Both
path spaces sit naturally in the spectrum of \(D\), but they are not all of it by any means:
there are many ways to go to infinity in \(\mathbb{N}^k\), and \(\Lambda^\infty\) is the part of the boundary in which
we have gone to infinity in every direction. We will focus on \(K := \{j : r^{-j} \ln \rho(A_j) = \beta_c\}\),
and our measures will live on the part of the boundary where we have gone to infinity
in the directions in \(K\), and not in the others. We work with some concretely defined
semi-infinite path spaces, instead of working explicitly inside the spectrum of \(D\).

We begin with a section on preliminary material. We briefly review facts about KMS
states and results from Perron-Frobenius theory that we later rely on. We then set
out our conventions for higher-rank graphs and their vertex matrices, and discuss the
Toeplitz-Cuntz-Krieger algebra and \(C^*\)-algebra of a higher-rank graph. At the end of
\[2.4\] we discuss the dynamics \(\alpha^r\) which we will be using throughout the paper. In \[3\] we
investigate the full path space \(W_\Lambda\) of a higher-rank graph \(\Lambda\), building on the recent work
of Webster \[31\]. In particular, we discuss the semi-infinite path spaces, and realisations of
certain subsets as inverse limits which we will use to build measures. We then discuss the
semi-infinite path representations that we use in our spatial realisations of KMS states.

We begin our analysis of KMS states in \[4\] by looking at KMS \(\beta\) states on Toeplitz algebras above the critical inverse temperature. The main analysis of these states remains that of \[11, \text{ Theorem 6.1}\], but we make some minor improvements to the general results. Then
in Remarks \[4.3\] and \[4.4\] we motivate our later analysis by describing a spatial realisation for
\(\beta > \beta_c\), and examining why it breaks down at \(\beta = \beta_c\). Our main results are formulated
in Theorem \[5.1\] and most of \[5\] is devoted to its proof. In \[6\] we consider another spatial
construction of KMS states which works when the set \(K := \{j : r^{-j} \ln \rho(A_j) = \beta_c\}\) is a
singleton, and in particular for any non-preferred dynamics on the Toeplitz algebra of a
2-graph. This is itself of some interest, since many of the most interesting examples of
higher-rank graph algebras are those of 2-graphs \[5, 22, 23\].

For the preferred dynamics, the KMS states on the Toeplitz algebra at critical inverse
temperatures factor through the quotient map onto the graph algebra. Our KMS states
factor through the quotient which imposes the Cuntz-Krieger relations for degrees in \(\mathbb{N}^K\)
(Proposition \[4.2\]). This quotient looks rather like the relative graph algebras of Muhly and Tomforde \[20\], but not at first sight like the relative higher-rank graph algebras of Sims \[29\]. In Appendix \[A\] we confirm that it is one of Sims’ relative algebras. In our final
Appendix \[B\] we reconcile our results with Neshveyev’s general machine for computing
KMS states on groupoid algebras [21]. Unfortunately, to do this we need an appropriate groupoid model for the Toeplitz algebras, and this does not seem to be explicitly available in the literature. So we provide one here, by adapting results of Yeend [35], and then show that our measure is the quasi-invariant measure predicted by Neshveyev’s theorem.

2. Background material

2.1. KMS states. Suppose that \((A, \alpha)\) is a dynamical system consisting of an action \(\alpha\) of \(\mathbb{R}\) on a \(C^*\)-algebra \(A\). As in [2, 23], we say that \(a \in A\) is analytic for \(\alpha\) if the function \(t \mapsto \alpha_t(a)\) extends to an analytic function \(z \mapsto \alpha_z(a)\) on \(\mathbb{C}\) (and then that extension is automatically unique). A state \(\phi\) on \(A\) is a KMS state with inverse temperature \(\beta\) (or a \(KMS_\beta\) state) of \((A, \alpha)\) if

\[
\phi(ab) = \phi(b\alpha_{\beta}(a))
\]

for all analytic elements \(a, b\). Proposition 8.12.3 of [23] implies that it suffices to check the KMS condition on a set of analytic elements which span a dense subspace of \(A\). In this paper, we are only interested in \(KMS_\beta\) states with inverse temperature \(\beta \in (0, \infty)\).

The following simple lemma will be handy when we want to normalise our dynamics. It says that changing the unit of time does not affect the behaviour of the system in any material way.

Lemma 2.1. Suppose that \(\alpha : \mathbb{R} \to \text{Aut} A\) is an action of \(\mathbb{R}\) on a \(C^*\)-algebra \(A\) and that \(\phi\) is a \(KMS_\beta\) state of \((A, \alpha)\). Let \(d \in (0, \infty)\) and define \(\alpha' : \mathbb{R} \to \text{Aut} A\) by \(\alpha'_t = d\alpha_{td}\). Then \(\phi\) is a \(KMS_{d^{-1}\beta}\) state of \((A, \alpha')\).

2.2. The Perron-Frobenius theorem. Let \(X\) be a finite set. A matrix \(A \in M_X(\mathbb{C})\) is irreducible if for all \(v, w \in X\) there exists \(n \in \mathbb{N}\) such that \(A^n(v, w) \neq 0\). We say that a matrix is positive (non-negative) if all its entries are positive (non-negative).

Let \(A \in M_X([0, \infty))\) be an irreducible non-negative matrix. The Perron-Frobenius theorem says that the spectral radius \(\rho(A)\) of \(A\) is an eigenvalue of \(A\) with a 1-dimensional eigenspace and a positive eigenvector (see, for example, [30, Theorem 1.5]). We call the unique positive eigenvector with eigenvalue \(\rho(A)\) and unit 1-norm the unimodular Perron-Frobenius eigenvector of \(A\). A vector \(\epsilon \in [0, \infty)^X\) is subinvariant for \(A\) and \(t \in \mathbb{R}\) if \(A\epsilon \leq te\). The subinvariance theorem [30, Theorem 1.6] says that if a vector \(\epsilon \in [0, \infty)^X\) is subinvariant for \(A\) and a positive real number \(t\), then all the entries of \(\epsilon\) are positive and \(t \geq \rho(A)\); moreover, \(t = \rho(A)\) if and only if \(A\epsilon = te\).

2.3. Higher-rank graphs and their vertex matrices. Let \(k \in \mathbb{N}\) with \(k \geq 1\). We write \(e_1, \ldots, e_k\) for the generators of \(\mathbb{N}^k\) and \(n_i\) for the \(i^{th}\) coordinate of \(n \in (\mathbb{N} \cup \{\infty\})^k\). For \(m, n \in (\mathbb{N} \cup \{\infty\})^k\) we write \(m \leq n\) if and only if \(m_i \leq n_i\) for \(1 \leq i \leq k\).

Let \(\Lambda\) be a \(k\)-graph with vertex set \(\Lambda^0\) and degree functor \(d : \Lambda \to \mathbb{N}^k\), as in [17]. We say that \(\Lambda\) is finite if \(\Lambda^0 := d^{-1}(n)\) is finite for all \(n \in \mathbb{N}^k\). Except in the appendices, we consider only finite \(k\)-graphs in this paper. For \(v, w \in \Lambda^0\) and \(n \in \mathbb{N}^k\), we write, for example,

\[
v\Lambda := \{\lambda \in \Lambda : r(\lambda) = v\}\quad\text{and}\quad v\Lambda^n w := \{\lambda \in \Lambda^n : r(\lambda) = v, s(\lambda) = w\}.
\]

We say that \(\Lambda\) has no sources if \(v\Lambda^n \neq \emptyset\) for every \(v \in \Lambda^0\) and \(n \in \mathbb{N}^k\).

Example 2.2. Let \(\Omega_k^p := \mathbb{N}^k\), \(\Omega_k := \{(p, q) \in \mathbb{N}^k \times \mathbb{N}^k : p \leq q\}\), define \(r, s : \Omega_k \to \Omega_k^p\) by \(r(p, q) := p\) and \(s(p, q) := q\), define composition by \((p, q)(q, r) = (p, r)\), and define \(d : \Omega_k \to \mathbb{N}^k\) by \(d(p, q) := q - p\). Then \(\Omega_k\) is a \(k\)-graph with no sources.

For \(n \in (\mathbb{N} \cup \{\infty\})^k\), we denote by \(\Omega_{k, n}\) the subgraph \(\{(p, q) \in \mathbb{N}^k \times \mathbb{N}^k : q \leq n\}\) of \(\Omega_k\).
For $1 \leq i \leq k$, let $A_i$ be the the matrix in $M_{N}(\mathbb{N})$ with entries $A_i(v, w) = |v\lambda^e w|$; we call the $A_i$ the vertex matrices of $\Lambda$. Since $(A_iA_j)(v, w) = |v\Lambda^e w|$, the factorisation property of $\Lambda$ implies that $A_iA_j = A_jA_i$, and we define

$$A^n := \prod_{i=1}^{k} A_i^{n_i} \quad \text{for } n \in \mathbb{N}^k.$$  

We say that $\Lambda$ is coordinatewise irreducible if vertex matrix $A_i$ is irreducible for $1 \leq i \leq k$. If $\Lambda$ is coordinatewise irreducible, then [11, Lemma 2.1] implies that the unimodular Perron-Frobenius eigenvectors of the $A_i$ are all equal, and we call it the common Perron-Frobenius eigenvector of $\Lambda$. We write $\rho(\Lambda)$ for the vector $(\rho(A_1), \ldots, \rho(A_k))$.

We visualise $k$-graphs as coloured graphs, by choosing $k$ different colours $c_1, \ldots, c_k$, and viewing paths in $\Lambda^{e_i}$ as edges of colour $c_i$. (See [26, Chapter 10] for a discussion of how this relates to the factorisation property, and [7, §3] for details of the relationship between a $k$-graph and its underlying coloured graph.) When $k = 2$, we view edges in $\Lambda^{e_1}$ as blue, and edges in $\Lambda^{e_2}$ as red.

2.4. The Toeplitz-Cuntz-Krieger $C^*$-algebra of a $k$-graph. Let $\Lambda$ be a finite $k$-graph with no sources. For $\mu, \nu \in \Lambda$, we write $\Lambda^{\min}(\mu, \nu)$ for the set of $(\eta, \zeta)$ in $\Lambda \times \Lambda$ such that $\mu\eta = \nu\zeta$ and $d(\mu\eta) = d(\mu) \lor d(\nu)$. As in [11, 27], a Toeplitz-Cuntz-Krieger $\Lambda$-family consists of partial isometries $\{T_\lambda : \lambda \in \Lambda\}$ such that

(T1) $\{T_v : v \in \Lambda^0\}$ are mutually orthogonal projections;
(T2) $T_\mu T_\mu^* = T_\nu$ whenever $s(\lambda) = r(\mu)$;
(T3) $T_\lambda^* T_\lambda = T_{s(\lambda)}$ for all $\lambda \in \Lambda$;
(T4) for all $v \in \Lambda^0$ and $n \in \mathbb{N}^k$, we have

$$T_v \geq \sum_{\lambda \in v\Lambda^n} T_\lambda T_\lambda^*;$$

(T5) for all $\mu, \nu \in \Lambda$, we have

$$T_\mu^* T_\nu = \sum_{(\eta, \zeta) \in \Lambda^{\min}(\mu, \nu)} T_\eta T_\zeta^*,$$

where by convention the sum over the empty set is 0.

A Toeplitz-Cuntz-Krieger $\Lambda$-family is a Cuntz-Krieger $\Lambda$-family if in addition we have

(CK) $T_v = \sum_{\lambda \in v\Lambda^n} T_\lambda T_\lambda^*$ for all $v \in \Lambda^0$ and $n \in \mathbb{N}^k$.

The Toeplitz algebra $\mathcal{TC}^*(\Lambda)$ of a $k$-graph $\Lambda$ is generated by a universal Toeplitz-Cuntz-Krieger $\Lambda$-family $\{t_\lambda : \lambda \in \Lambda\}$, and the standard arguments show that

$$\mathcal{TC}^*(\Lambda) = \overline{\text{span}} \{t_\mu t_\nu^* : \mu, \nu \in \Lambda\}.$$  

The Cuntz-Krieger algebra $C^*(\Lambda)$ is the quotient of $\mathcal{TC}^*(\Lambda)$ by the ideal generated by

$$\{t_v - \sum_{\lambda \in v\Lambda^n} t_\lambda t_\lambda^* : v \in \Lambda, n \in \mathbb{N}^k\}.$$  

The universal property gives a gauge action $\gamma$ of $\mathbb{T}^k$ on $\mathcal{TC}^*(\Lambda)$ such that $\gamma_z(t_\lambda) = z^{d(\lambda)}t_\lambda$ (using multi-index notation, so that $z^n = \prod_{i=1}^{k} z_i^n$, for $z = (z_1, \ldots, z_k) \in \mathbb{T}^k$ and $n \in \mathbb{Z}^k$).
3. Semi-infinite path spaces

Let $\Lambda$ be a finite $k$-graph, let $n \in \mathbb{N}^k$ and consider the graph $\Omega_{k,n}$ of Example 2.2. Then each $\lambda \in \Lambda^n$ gives a functor $x_{\lambda} : \Omega_{k,n} \to \Lambda$, as follows. Take $p \leq q \leq n$, use the factorisation property to see that there are unique paths $\lambda' \in \Lambda^p$, $\lambda'' \in \Lambda^{q-p}$ and $\lambda''' \in \Lambda^{n-q}$ such that $\lambda = \lambda'\lambda''\lambda'''$, and then define $x_{\lambda}(p,q) := \lambda(p,q) := \lambda''$. The map $\lambda \mapsto x_{\lambda}$ is a bijection from $\Lambda^n := d^{-1}(n)$ onto the set of degree-preserving functors from $\Omega_{k,n}$ to $\Lambda$. We use this bijection to identify the two sets, and this identification motivates the definitions of infinite and semi-infinite paths.

Now let $n \in (\mathbb{N} \cup \{\infty\})^k$. Then we denote by $\Lambda^n$ the set of $k$-graph morphisms from $\Omega_{k,n}$ to $\Lambda$. (When $n \in \mathbb{N}^k$ we had already identified the set of $k$-graph morphisms from $\Omega_{k,n}$ to $\Lambda$ with $\Lambda^n := d^{-1}(n)$ in the paragraph above.) We write $d(x) = n$ whenever $x \in \Lambda^n$. As usual, we write $\Lambda^\infty$ for the infinite-path space $\bigwedge_{n=1}^{\infty} \Lambda^n$ and call its elements infinite paths.

We consider the path space

$$W_\Lambda := \bigcup_{n \in (\mathbb{N} \cup \{\infty\})^k} \Lambda^n.$$

For each $\lambda \in \Lambda$ and finite subset $G$ of $s(\lambda)\Lambda$ we write

$$Z(\lambda) := \{x \in W_\Lambda : x(0,d(\lambda)) = \lambda\}$$

and

$$Z(\lambda \setminus G) := Z(\lambda) \setminus (\bigcup_{\alpha \in G} Z(\lambda\alpha)).$$

Theorems 3.1 and 3.2 of [31] show that the $Z(\lambda \setminus G)$ form a basis for a locally compact Hausdorff topology on $W_\Lambda$ (see also [24, §2] and [8, §3]). Webster shows in the proof of [31] Theorem 3.2] that $Z(v)$ is compact for $v \in \Lambda^0$. Since $\Lambda$ is finite, $W_\Lambda = \bigcup_{v \in \Lambda^0} Z(v)$ is compact. Then we also have:

**Lemma 3.1.** Let $\lambda \in \Lambda$ and $G$ be a finite subset of $s(\lambda)\Lambda$. Then $Z(\lambda \setminus G)$ is compact in $W_\Lambda$.

**Proof.** Since $Z(r(\lambda))$ is compact, it suffices to show $Z(\lambda)$ and $Z(\lambda \setminus G)$ are closed. Let $\{x_n\} \subset Z(\lambda)$ and $x_n \to x$. Then $x_n = \lambda y_n$ for $y_n \in Z(s(\lambda))$. Since $Z(s(\lambda))$ is compact, there is a convergent subsequence $y_{n_i} \to y \in Z(s(\lambda))$. Now it is easy to see that $x_{n_i} = \lambda y_{n_i} \to \lambda y$. Since $W_\Lambda$ is Hausdorff, $x = \lambda y$ and $Z(\lambda)$ is closed.

Similarly, let $\{x_n\} \subset Z(\lambda \setminus G)$ and $x_n \to x$. Then $x_n \in Z(\lambda)$. Since $W_\Lambda$ is compact, $Z(\lambda)$ is closed, and then $x \in Z(\lambda)$. Suppose, by way of contradiction, that $x \in Z(\lambda\alpha)$ for some $\alpha \in G$. Since $Z(\lambda\alpha)$ is open, we have $x_n \in Z(\lambda\alpha)$ eventually, a contradiction. Hence $Z(\lambda \setminus G)$ is closed.

We consider a nonempty subset $K$ of $\{1, \ldots, k\}$, and set $J := \{1, \ldots, k\} \setminus K$. We view $\mathbb{N}^k$ as $\mathbb{N}^J \times \mathbb{N}^K$, and for $n \in (\mathbb{N} \cup \{\infty\})^k$, we write $n = (n_J, n_K)$ where $n_J \in (\mathbb{N} \cup \{\infty\})^J$ and $n_K \in (\mathbb{N} \cup \{\infty\})^K$. For $m \in \mathbb{N}^J$, we define

$$\Lambda^{m,\infty_K} := \{x \in W_\Lambda : d(x)_J = m \text{ and } d(x)_K = \infty_K\}$$

and

$$\partial K \Lambda := \bigcup_{m \in J} \Lambda^{m,\infty_K}.$$

We call elements of $\partial K \Lambda$ semi-infinite paths.

For $m \in \mathbb{N}^J$ and $n \leq p \in \mathbb{N}^K$, we define $r_{n,p} : \Lambda^{m,p} \to \Lambda^{m,n}$ by $r_{n,p}(\lambda) = \lambda(0, (m,n))$. Then the factorisation property implies that for $n \leq p \leq q \in \mathbb{N}^K$ we have $r_{n,p} \circ r_{p,q} = r_{n,q}$. So when we view the finite sets $\Lambda^{m,n}$ as topological spaces with the discrete topology, $\{\Lambda^{m,n}, r_{n,p}\}$ is an inverse system of compact Hausdorff spaces. Then the inverse limit

$$\lim_{\leftarrow n \in \mathbb{N}^K} \Lambda^{m,n}$$

is a compact Hausdorff space, which we can realise concretely as the subspace of the product $\prod_{n \in \mathbb{N}^K} \Lambda^{m,n}$ consisting of the elements $\{\lambda^n\}_{n \in \mathbb{N}^K}$ satisfying $r_{n,p}(\lambda^n) = \lambda^n$ for $n \leq p$. 


Proposition 3.2. Let $m \in \mathbb{N}^I$. For each \( \{\lambda^n\} \in \varprojlim_{n \in \mathbb{N}^K} \Lambda^{m,n} \) there is a path \( x^\lambda \in \Lambda^{m,\infty_K} \) such that \( x^\lambda(0, (m, n)) = \lambda^n \) for all \( n \in \mathbb{N}^K \), and then \( \phi : \{\lambda^n\} \mapsto x^\lambda \) is a homeomorphism of \( \varprojlim_{n \in \mathbb{N}^K} \Lambda^{m,n} \) onto the subset \( \Lambda^{m,\infty_K} \) of \( W_\Lambda \). In particular, \( \Lambda^{m,\infty_K} \) is compact Hausdorff.

For the proof, we need a lemma.

Lemma 3.3. The cylinder sets \( \{ Z(\lambda) \cap \Lambda^{m,\infty_K} : \lambda \in \Lambda, d(\lambda)_J = m \} \) are a basis for the relative topology on \( \Lambda^{m,\infty_K} \subset W_\Lambda \).

Proof. Suppose that \( x \in \Lambda^{m,\infty_K} \) and \( Z(\tau \setminus G) \) is a basic open neighbourhood of \( x \) in \( W_\Lambda \) — in other words, \( \tau \in \Lambda \), \( G \) is a finite subset of \( s(\tau)\Lambda \), and

\[
x \in Z(\tau \setminus G) = Z(\tau) \setminus \left( \bigcup_{\alpha \in G} Z(\tau_\alpha) \right).
\]

We have to find \( \lambda \in \Lambda \) such that \( d(\lambda)_J = m \), \( x \in Z(\lambda) \) and

\[
Z(\lambda) \cap \Lambda^{m,\infty_K} \subset Z(\tau) \setminus \left( \bigcup_{\alpha \in G} Z(\tau_\alpha) \right).
\]

We note first that \( x \in Z(\tau) \) implies that \( d(\tau)_J \leq m \). Next, we observe that

\[
\Lambda^{m,\infty_K} \cap Z(\tau_\alpha) \neq \emptyset \implies d(\tau_\alpha)_J \leq m.
\]

Now we take

\[
n = \left( \bigvee_{\alpha \in G} d(\tau_\alpha)_J \leq m d(\tau_\alpha)_K \right) \lor d(\tau)_K.
\]

Then since \( x \notin Z(\tau_\alpha) \) for all \( \alpha \in G \), we have

\[
x \in Z(x(0, (m, n))) \cap \Lambda^{m,\infty_K} \subset Z(\tau) \setminus \left( \bigcup_{\alpha \in G} Z(\tau_\alpha) \right),
\]

as required. \( \Box \)

Proof of Proposition 3.2. The existence of \( x^\lambda \) is established as in [17, Remarks 2.2] (which covers the case \( J = \emptyset \)). The factorisation property implies that for every \( x \in \Lambda^{m,\infty_K} \), \( \{\lambda^n\} \) belongs to \( \varprojlim_{n \in \mathbb{N}^K} \Lambda^{m,n} \), and that \( x \mapsto \{x(0, (m, n))\} \) is a set-theoretic inverse for \( \phi \).

Let \( \sigma \in \Lambda \) such that \( d(\sigma)_J = m \). Then \( \phi^{-1}(Z(\sigma)) \) is the intersection of \( \varprojlim_{n \in \mathbb{N}^K} \Lambda^{m,n} \) with the product set

\[
\left\{ \{\mu^n\} \in \prod_{n \in \mathbb{N}^K} \Lambda^{m,n} : \mu^n = \sigma(0, (m, n)) \text{ for } n \leq d(\sigma)_K \right\},
\]

and is therefore open in \( \varprojlim_{n \in \mathbb{N}^K} \Lambda^{m,n} \). Since \( \{Z(\sigma) \cap \Lambda^{m,\infty_K} : d(\sigma)_J = m\} \) is a basis for the topology on \( \Lambda^{m,\infty_K} \) by Lemma 3.3, it follows that \( \phi \) is continuous. Since the inverse limit is compact and \( W_\Lambda \) is Hausdorff [31, Theorem 3.2], \( \phi \) is a homeomorphism onto its range \( \Lambda^{m,\infty_K} \). Since \( \varprojlim_{n \in \mathbb{N}^K} \Lambda^{m,n} \) is compact so is \( \Lambda^{m,\infty_K} \). \( \Box \)

Next we build a Toeplitz-Cuntz-Krieger \( \Lambda \)-family \( \{T_\lambda : \lambda \in \Lambda \} \) on \( \ell^2(\partial K\Lambda) \), and then the universal property of \( TC^*(\Lambda) \) gives a representation \( \pi_T : TC^*(\Lambda) \to B(\ell^2(\partial K\Lambda)) \). Since \( \pi_T \) depends on the partition \( \{1, \ldots, k\} = J \sqcup K \), we denote it by \( \pi^K \), and we call it the semi-infinite path representation for \( K \). We prove that \( \pi^K \) factors through the quotient of \( TC^*(\Lambda) \) by the ideal \( I^K \) generated by

\[
\left\{ t_v - \sum_{e \in \varepsilon} t_e t_{e^i}^* : v \in \Lambda^0, i \in K \right\}.
\]

Remark 3.4. The quotient \( TC^*(\Lambda)/I^K \) is an interesting example of the relative Cuntz-Krieger algebras \( C^*(\Lambda; \mathcal{E}) \) of [29, §3], and \( \pi^K \) is one of the boundary-path representations in that paper. Since this last observation can only be directed to those familiar with [29], we have put the details in Appendix A.
Proposition 3.5. Let $\Lambda$ be a finite $k$-graph with no sources. Let $J \sqcup K$ be a nontrivial partition of $\{1, \ldots, k\}$. Let $\{\xi_{m,x} : m \in \mathbb{N}^J, x \in \Lambda^{m,\infty_k} \}$ be the usual orthonormal basis of point masses in

$$\ell^2(\partial K\Lambda) = \bigoplus_{m \in \mathbb{N}^J} \ell^2(\Lambda^{m,\infty_k}).$$

For $\lambda \in \Lambda$, let $T_\lambda$ be the operator on $\ell^2(\partial K\Lambda)$ such that

$$T_\lambda \xi_{m,x} = \begin{cases} \xi_{m+d(\lambda)J,x} & \text{if } s(\lambda) = r(x) \\ 0 & \text{otherwise.} \end{cases}$$

Then $\{T_\lambda : \lambda \in \Lambda\}$ is a Toeplitz-Cuntz-Krieger $\Lambda$-family such that

$$\sum_{e \in v\Lambda^i} T_e T^*_e = T_v \quad \text{for } v \in \Lambda^0 \text{ and } i \in K.$$

Proof. Let $\lambda \in \Lambda$. Then the adjoint $T^*_\lambda$ is characterised by

$$T^*_\lambda \xi_{m,x} = \begin{cases} \xi_{m-d(\lambda)J,x,d(\lambda)\infty} & \text{if } x(0, d(\lambda)) = \lambda \text{ and } m \geq d(\lambda)J \\ 0 & \text{otherwise.} \end{cases}$$

Now it is easy to see that $T_\lambda T^*_\lambda = T_\lambda$, so $T_\lambda$ is a partial isometry. Let $v, w \in \Lambda^0$. Then $T_v$ is the projection onto $\text{span}\{\xi_{m,x} : x \in \partial K\Lambda, r(x) = v\}$, and $T_v T_w = 0$ unless $v = w$. Thus $\{T_v : v \in \Lambda^0\}$ is a set of mutually orthogonal projections, and we have proved (T1).

For (T2), fix $\lambda, \mu \in \Lambda$ with $s(\lambda) = r(\mu)$. Then

$$T_\lambda T_\mu \xi_{m,x} = \begin{cases} T_\lambda \xi_{m+d(\mu)J,\mu x} & \text{if } s(\mu) = r(x) \\ 0 & \text{otherwise} \end{cases}$$

$$= \begin{cases} \xi_{m+d(\mu)J+d(\lambda)J,\lambda\mu x} & \text{if } s(\mu) = r(x) \\ 0 & \text{otherwise} \end{cases}$$

$$= T_{\lambda\mu} \xi_{m,x},$$

since $d(\lambda)J + d(\mu)J = d(\lambda\mu)J$. To see (T3), take $\lambda \in \Lambda$. Then

$$T^*_\lambda T_\lambda \xi_{m,x} = \begin{cases} T^*_\lambda \xi_{m+d(\lambda)J,\lambda x} & \text{if } s(\lambda) = r(x) \\ 0 & \text{otherwise} \end{cases}$$

$$= \begin{cases} \xi_{m,x} & \text{if } s(\lambda) = r(x) \\ 0 & \text{otherwise} \end{cases}$$

$$= T_{s(\lambda)} \xi_{m,x}.$$

We want to use (T5) to prove (T4), so we next establish (T5). Let $\mu, \nu \in \Lambda$ and $\xi_{m,x}, \xi_{n,y} \in \ell^2(\partial K\Lambda)$. We will show that

$$\langle T^*_\mu T_v \xi_{m,x} | \xi_{n,y} \rangle = \left( \sum_{(n,\zeta) \in \Lambda^{\min(\mu,\nu)}} T_{\eta} T^*_\zeta \xi_{m,x} | \xi_{n,y} \right).$$

We consider three cases. First, suppose that $\Lambda^{\min(\mu,\nu)} = \emptyset$. Then $\mu x \neq \nu y$ and

$$\langle T^*_\mu T_v \xi_{m,x} | \xi_{n,y} \rangle = \langle T_v \xi_{m,x} | T_\mu \xi_{n,y} \rangle = \langle \xi_{m+d(\nu)J,\nu x} | \xi_{n+d(\mu)J,\mu y} \rangle = 0.$$

The empty sum on the right-hand-side of (3.3) is by definition zero, and hence (3.3) holds in this first case.
Second, suppose that $\Lambda^{\min}(\mu, \nu) \neq \emptyset$ and $(T^*_\mu T \xi_{m,x} \mid \xi_{n,y}) \neq 0$. Then

$$0 \neq (T^*_\mu T \xi_{m,x} \mid \xi_{n,y}) = (T \xi_{m,x} \mid T^*_\mu \xi_{n,y}) = (\xi_{m+d(\nu),j} \mid \xi_{n+d(\mu),j}).$$

Thus $\xi_{m+d(\nu),j} = \xi_{n+d(\mu),j}$, and (T4) holds. Also

$$\xi_{m+d(\nu),j} = (T^*_\mu T \xi_{m,x} \mid \xi_{n,y}) = 1.\!
$$

To see (T4), let $\xi_{m+d(\nu),j} = \xi_{n+d(\mu),j}$, and (T5) implies that

$$m + d(\nu) = n + d(\mu), \quad (m,x) = (n,y).$$

Let $\sigma = m(0, d(\mu) \cup d(\nu))$ and $\tau = m(0, d(\mu) \cup d(\nu) - d(\nu))$. Then

$$\mu \sigma = (m(y), 0, d(\mu) \cup d(\nu)) = (\nu x, y, d(\nu)) = \nu \eta.$$

So $(\sigma, \tau) \in \Lambda^{\min}(\mu, \nu)$. Then $x = \tau x'$ and $y = \sigma y'$ for some $x', y'$. But $\nu \tau x' = \nu x = \mu \sigma y'$ and $\nu \tau = \mu \sigma$, so $x' = y'$. Also $m + d(\nu) = n + d(\mu)$ and $\nu \tau = \mu \sigma$ imply that $m - d(\tau) = n - d(\sigma)$. So we have

$$(T^*_\tau T \xi_{m,x} \mid \xi_{n,y}) = (T^*_\mu T \xi_{m,x} \mid \xi_{n,y}) = 1,$$

as required.

Third, suppose that $\Lambda^{\min}(\mu, \nu) \neq \emptyset$ and $(T^*_\mu T \xi_{m,x} \mid \xi_{n,y}) = 0$. We argue by contradiction. Suppose that the right-hand-side of (3.3) is not zero. Then there exists $(\sigma, \tau) \in \Lambda^{\min}(\mu, \nu)$ such that $(T^*_\mu T \xi_{m,x} \mid \xi_{n,y}) \neq 0$. This implies that

$$0 \neq (T^*_\tau T \xi_{m,x} \mid \xi_{n,y}) = (\xi_{m-d(\sigma),j, x, d(\tau), \infty} \mid \xi_{n-d(\tau),j, y, d(\sigma), \infty}).$$

So $\xi_{m-d(\sigma),j, x, d(\tau), \infty} = \xi_{n-d(\tau),j, y, d(\sigma), \infty}$, and there exists a semi-infinite path $x'$ such that $x = \tau x'$ and $y = \sigma x'$. Since $\mu \sigma = \nu \eta$, we get $\nu x = \nu \tau x' = \mu \sigma y'$. Also $m - d(\tau) = n - d(\sigma)$ implies that $m + d(\nu) = n + d(\mu)$, and $(\xi_{m-d(\nu),j, x, d(\tau), \infty} \mid \xi_{n-d(\nu),j, y, d(\sigma), \infty}) = 1$, which contradicts $(T^*_\mu T \xi_{m,x} \mid \xi_{n,y}) = 0$. Thus we have proved our third case, and we have verified that (T5) holds.

To see (T4), let $v \in \Lambda^0$ and $n \in \mathbb{N}^k$. Let $\lambda, \mu \in v \Lambda^0$ such that $\lambda \neq \mu$. Since $\lambda$ and $\mu$ have the same degree they cannot have a common extension, and hence $\Lambda^{\min}(\lambda, \mu) = \emptyset$. Then (T5) forces $T^*_\lambda (T^*_n T^*_n) = T^*_n T^*_n = 0$. By (T2), $T_v T^*_n T^*_n = T^*_n T^*_n$, that is, $T_v \geq T^*_n T^*_n$. Thus $T_v \geq \sum_{\lambda \in v \Lambda^0} T^*_\lambda T^*_n T^*_n$ which is (T4). We have now shown that $\{T^*_\lambda : \lambda \in \Lambda\}$ is a Toeplitz-Cuntz-Krieger $\Lambda$-family.

Finally, let $v \in \Lambda^0$, $i \in K$ and $x \in \partial^k \Lambda$. Let $e \in v \Lambda^e$. If $r(x) = 0$, then $T^*_e T^*_e \xi_{m,x} = 0 = T^*_v \xi_{m,x}$, and hence $(\sum_{f \in v \Lambda^e} T f T^*_f) \xi_{m,x} = 0 = T^*_v \xi_{m,x}$. So suppose $r(x) = v$. Since $d(e)_j = 0$,

$$T^*_e T^*_e \xi_{m,x} = \begin{cases} T^*_e \xi_{m,x} (d(e), \infty) & \text{if } x(0, e_i) = e \\ 0 & \text{otherwise} \end{cases}$$

$$= \begin{cases} \xi_{m,x} & \text{if } x(0, e_i) = e \\ 0 & \text{otherwise.} \end{cases}$$

Since $d(x)_i = \infty$ we have $T^*_e T^*_e \xi_{m,x} = \xi_{m,x}$ for exactly one edge $e = x(0, e_i)$, and hence $(\sum_{f \in v \Lambda^e} T f T^*_f) \xi_{m,x} = T^*_e \xi_{m,x}$. Thus $\sum_{f \in v \Lambda^e} T f T^*_f = T^*_e$, and we are done. \qed
4. KMS states on Toeplitz algebras

For \( r \in (0, \infty)^k \), define \( \alpha^r : \mathbb{R} \to \text{Aut} \mathcal{T}C^*(\Lambda) \) in terms of the gauge action by \( \alpha^r_t = \gamma_{e^{itr}}. \) Then for \( \mu, \nu \in \Lambda, \)
\[
\alpha^r_t(t^*_\mu t^*_\nu) = e^{itr-(d(\mu)-d(\nu))}t^*_\mu t^*_\nu
\]
is the restriction of the analytic function \( z \mapsto e^{itr-(d(\mu)-d(\nu))}t^*_\mu t^*_\nu. \) Thus to see that a state is a KMS\( _\beta \) state for \( (\mathcal{T}C^*(\Lambda), \alpha^r) \), it suffices to check the KMS condition on pairs of elements of the form \( t^*_\mu t^*_\nu. \)

The following result is an improvement on [11, Corollary 4.3], which requires \( \Lambda \) to be coordinatewise irreducible.

**Proposition 4.1.** Let \( \Lambda \) be a finite \( k \)-graph with no sources, and suppose that all the coordinate graphs \( (\Lambda_0, \Lambda_\varepsilon, r, s) \) have cycles. Let \( r \in (0, \infty)^k \) and \( \beta \in [0, \infty). \) If there is a KMS\( _\beta \) state of \( (\mathcal{T}C^*(\Lambda), \alpha^r) \), then \( \beta r_i \geq \ln \rho(A_i) \) for \( 1 \leq i \leq k. \)

**Proof.** Suppose that \( \phi \) is a KMS\( _\beta \) state of \( (\mathcal{T}C^*(\Lambda), \alpha^r) \), and fix \( i. \) Since \( (\Lambda_0, \Lambda_\varepsilon, r, s) \) is not a cycle, there is a strongly connected component \( C \) of this coordinate graph such that the \( C \times C \) block \( A_C \) of \( A_\varepsilon \) has \( \rho(A_C) = \rho(A_\varepsilon) > 0. \) (To see this, consider a Seneta decomposition of \( A_\varepsilon \), as described in [12, §2,3].) Then \( A_C \) is irreducible, and the directed graph \( E_C = (\Lambda_0, \mathcal{CA}^\times C, r, s) \) is strongly connected with vertex matrix \( A_C. \) The set \( \{t_v : v \in C\} \cup \{t_e : e \in \mathcal{C}\Lambda^\times C\} \) is a Toeplitz-Cuntz-Krieger \( E_C \)-family in \( \mathcal{T}C^*(\Lambda) \), and hence there is a homomorphism \( \pi \) of \( \mathcal{T}C^*(E_C) \) into \( \mathcal{T}C^*(\Lambda) \) such that \( \pi(s_e) = t_e \) for all \( e \in \mathcal{C}\Lambda^\times C. \) This homomorphism is equivariant for the action \( \alpha \) studied in [10] and the action \( \alpha' \) on \( \mathcal{T}C^*(\Lambda) \) defined by \( \alpha'_i = \alpha^r_{r_i^{-1}}. \) Lemma 2.1 implies that \( \phi \) is a KMS\( _{r_i \beta} \) state of \( (\mathcal{T}C^*(\Lambda), \alpha'). \) Thus \( \pi \circ \phi \) is a KMS\( _{r_i \beta} \) state of \( (\mathcal{T}C^*(E_C), \alpha) \), and since \( E_C \) is strongly connected, it follows from [10, Theorem 4.3(c)] that \( r_i \beta \geq \ln \rho(A_C) = \ln \rho(A_\varepsilon). \)

When \( \beta \) is strictly larger than all the numbers \( r_i^{-1} \ln \rho(A_i) \), Theorem 6.1 of [11] gives a \((|\Lambda^0| - 1)\)-dimensional simplex of KMS\( _\beta \) states of \( (\mathcal{T}C^*(\Lambda), \alpha^r). \) When \( \beta \) is strictly less than any of the \( r_i^{-1} \ln \rho(A_i) \), Proposition 4.1 implies that there are no KMS\( _\beta \) states at all. So the behaviour of the KMS\( _\beta \) states changes dramatically as the inverse temperature \( \beta \) passes through the value
\[
\beta_c := \max_i \{r_i^{-1} \ln \rho(A_i)\};
\]
we call \( \beta_c \) the critical inverse temperature. In this paper, we are interested in what happens at \( \beta = \beta_c. \)

Recall from Lemma 2.1 that scaling the time \( t \) does not effectively change the behaviour of KMS states. So in our case, replacing the vector \( r \) by a scalar multiple will not change things significantly. We choose to use the unique multiple that has
\[
\beta_c := \max_i \{r_i^{-1} \ln \rho(A_i)\} = 1,
\]
and then we are interested in the KMS\( _1 \) states. To emphasise: even if we forget to say so locally, the restriction (4.1) is in force throughout the rest of the paper. Thus we have \( r_i = \ln \rho(A_i) \) for \( i \) in some nonempty subset \( K \) of \( \{1, \ldots, k\} \), and \( r_i > \ln \rho(A_i) \) for \( i \in J := \{1, \ldots, k\} \setminus K. \) For the preferred dynamics studied in [11, §7] and [13], we have \( K = \{1, \ldots, k\} \), but here we are thinking primarily about the case where \( K \) is a proper subset of \( \{1, \ldots, k\}. \)

**Proposition 4.2.** Suppose that \( \Lambda \) is a finite \( k \)-graph with no sources, and that \( r \in (0, \infty)^k. \)
We suppose that \( r_i \geq \ln \rho(A_i) \) for all \( i \), and that
\[
K := \{i \in \{1, \ldots, k\} : r_i = \ln \rho(A_i)\}
\]
is nonempty.

(a) There exist KMS$_1$ states of $(TC^*(\Lambda), \alpha^\beta)$.

(b) Every KMS$_1$ state of $(TC^*(\Lambda), \alpha^\beta)$ factors through the ideal generated by

$$\{t_v - \sum_{e \in i \Lambda^i} t_et_e^* : v \in \Lambda^0, i \in K, A_i \text{ is irreducible}\}.$$

(c) Suppose that the coordinates of $r$ are rationally independent and that there exists $i \in K$ such that $A_i$ is irreducible. Let $\kappa$ be the unimodular Perron-Frobenius eigenvector of $A_i$. Then [11, Theorem 6.1] describes the KMS$_\beta$ representation of $\Lambda$ on $\ell^2(A_i)$ for the usual orthonormal basis of $\Lambda$. To see this, suppose $\phi$ which implies that $\Lambda$ is a finite $k$-graph with no sources and $\beta > \beta_c = 1$. Let $(T, Q)$ be the path representation of $\Lambda$ on $\ell^2(\Lambda)$ described at the end of [11, §2.2], and $\pi_{T, Q}$ the corresponding representation of $TC^*(\Lambda)$. Write $\{h_\lambda : \lambda \in \Lambda\}$ for the usual orthonormal basis of $\ell^2(\Lambda)$. Then [11, Theorem 6.1] describes the KMS$_\beta$ states of $(TC^*(\Lambda), \alpha^\beta)$, as follows: associated to each $\epsilon \in [0, \infty)^{\Lambda^0}$ satisfying a constraint $\epsilon \cdot y = 1$, there is a KMS$_\beta$ state $\phi_\epsilon$ such that

$$\phi_\epsilon(a) = \sum_{\lambda \in \Lambda} (\pi_{T, Q}(a) h_\lambda \mid h_\lambda) e^{-\beta r(d(\lambda))} \epsilon_s(\lambda).$$

(See [11, page 279], where the weight $e^{-\beta r(d(\lambda))} \epsilon_s(\lambda)$ was denoted $\Delta_\lambda$.) We now define a measure $\nu_\beta$ on $\Lambda$ by $\nu_\beta(\{\lambda\}) = e^{-\beta r(d(\lambda))} \epsilon_s(\lambda)$. Since $\phi_\epsilon(1) = 1$, $\nu_\beta$ is a probability measure, and then

$$\phi_\epsilon(a) = \int_{\Lambda} (\pi_{T, Q}(a) h_\lambda \mid h_\lambda) d\nu_\beta(\lambda).$$

Proof. Choose a decreasing sequence $\{\beta_n\} \subset (1, \infty)$ such that $\beta_n \to 1$. Since $\beta_n > 1 \geq e^{-r_1 \ln \rho(A_i)}$ for all $i$, [11, Theorem 6.1] implies that there is at least one KMS$_{\beta_n}$ state $\phi_n$ of $(TC^*(\Lambda), \alpha^\beta)$. Since the state space of $TC^*(\Lambda)$ is weak$^*$-compact, we may assume by passing to a subsequence that $\phi_n \to \phi$. Then $\phi$ is a KMS$_1$ state of $(TC^*(\Lambda), \alpha^\beta)$ by [2, Proposition 5.3.23]. This gives (i).

For (ii), suppose $i \in K$ and $A_i$ is irreducible. Let $\phi$ be a KMS$_1$ state, and for $v \in \Lambda^0$, set $m_v^\phi = \phi(t_v)$. Then [11, Proposition 4.1] implies that $m^\phi = (0, 1]$ is a probability measure on $\Lambda^0$ such that $(1 - e^{-r_1 \ln \rho(A_i)}) m_v^\phi \geq 0$. Since $r_1 = \ln \rho(A_i)$, we get $A_i m^\phi \leq e^{r_1 m^\phi} = \rho(A_i) m^\phi$. Since $A_i$ is irreducible and $\rho(A_i)$ is the Perron-Frobenius eigenvalue of $A_i$, the subinvariance theorem [30, Theorem 1.6] implies that $A_i m^\phi = \rho(A_i) m^\phi$. Thus $m^\phi$ is the unimodular Perron-Frobenius eigenvector of $A_i$. Now

$$\phi\left(\sum_{e \in i \Lambda^i} t_e^* t_e\right) = \sum_{e \in i \Lambda^i} e^{-r_1 \ln \rho(A_i)} = \rho(A_i)^{-1} \sum_{w \in \Lambda^0} |v\Lambda^i w| \phi(t_w) = \rho(A_i)^{-1} (A_i m^\phi)_w = m^\phi = \phi(t_v),$$

and (ii) follows from the general lemma [10, Lemma 2.2].

For (iii), suppose $\phi$ and $\psi$ are KMS$_1$ states. Let $i \in K$ such that $A_i$ is irreducible. The argument of (ii) then implies that $m^\phi = m^\psi$ is the unimodular Perron-Frobenius eigenvector $\kappa$ of $A_i$. Since $r$ has rationally independent coordinates, [11, Proposition 3.1(b)] says that for all $\mu, \nu \in \Lambda$,

$$\phi(t_v t_v^*) = \delta_{\mu, \nu} e^{-r d(\mu)} \phi(t_s(\mu)) = \delta_{\mu, \nu} e^{-r d(\mu)} \kappa_s(\mu) = \psi(t_v t_v^*),$$

which implies that $\phi = \psi$. \qed

Remark 4.3. Above the critical inverse temperature, it is quite easy to find a spatial implementation of the sort we were using the calculations in [11]. To see this, suppose that $\Lambda$ is a finite $k$-graph with no sources and $\beta > \beta_c = 1$. Let $(T, Q)$ be the path representation of $\Lambda$ on $\ell^2(\Lambda)$ described at the end of [11, §2.2], and $\pi_{T, Q}$ the corresponding representation of $TC^*(\Lambda)$. Write $\{h_\lambda : \lambda \in \Lambda\}$ for the usual orthonormal basis of $\ell^2(\Lambda)$. Then [11, Theorem 6.1] describes the KMS$_\beta$ states of $(TC^*(\Lambda), \alpha^\beta)$, as follows: associated to each $\epsilon \in [0, \infty)^{\Lambda^0}$ satisfying a constraint $\epsilon \cdot y = 1$, there is a KMS$_\beta$ state $\phi_\epsilon$ such that

$$\phi_\epsilon(a) = \sum_{\lambda \in \Lambda} (\pi_{T, Q}(a) h_\lambda \mid h_\lambda) e^{-\beta r(d(\lambda))} \epsilon_s(\lambda).$$

(See [11, page 279], where the weight $e^{-\beta r(d(\lambda))} \epsilon_s(\lambda)$ was denoted $\Delta_\lambda$.) We now define a measure $\nu_\beta$ on $\Lambda$ by $\nu_\beta(\{\lambda\}) = e^{-\beta r(d(\lambda))} \epsilon_s(\lambda)$. Since $\phi_\epsilon(1) = 1$, $\nu_\beta$ is a probability measure, and then

$$\phi_\epsilon(a) = \int_{\Lambda} (\pi_{T, Q}(a) h_\lambda \mid h_\lambda) d\nu_\beta(\lambda).$$
Remark 4.4. The spatial realisation (4.3) breaks down at \( \beta = \beta_c \). We illustrate the problems by considering the case \( k = 1 \). For this brief discussion, we resume the notation of [10], where the dynamics is normalised to give \( \beta_c = \ln \rho(A) \). As \( \beta \) decreases to \( \beta_c \), the factor \( e^{-\beta |\lambda|} \) converges to \( \rho(A)^{-1}|\lambda| \), so at first sight the measures \( \nu_\beta \) converge. However, the constraint \( \epsilon \cdot w = 1 \) involves the vector \( w \) of [10, Theorem 3.1(a)], which depends on \( \beta \). In particular, the \( \epsilon \) which is a multiple of the point mass \( \delta_v e^{-\beta \lambda} \). The argument in the first paragraph of the proof of [10, Theorem 3.1(a)], and in particular the calculation (3.2), shows that the convergence of the series defining \( y \) is equivalent to that of the series \( \sum_{n=0}^{\infty} e^{-\beta n} A^n \). But this series does not converge for \( \beta = \ln \rho(A) \); if it did, it would give an inverse for \( 1 - \rho(A)^{-1} A \), which is not invertible because \( \rho(A) \) is an eigenvalue of \( A \) (by Perron-Frobenius theory). So there is no multiple of \( \delta_v e^{-\beta \lambda} \) to which we can apply [10, Theorem 3.1(b)].

Geometrically, the simplex \( \Sigma_{\beta} \) shrinks towards the origin as \( \beta \rightarrow \beta_c \), and hence the measures \( \nu_\beta \) satisfy \( \nu_\beta(\{\lambda\}) \rightarrow 0 \) for each fixed \( \lambda \). So as \( \beta \rightarrow \beta_c \), the mass distribution of the probability measure \( \nu_\beta \) is spreading out.

We can still get a KMS\( \beta \) state of \( TC^*(E) \) by taking limits of KMS\( \beta \) states as \( \beta \rightarrow \beta_c \), but this state is not spatially realisable on \( \ell^2(E^*) \). Indeed, if \( A \) is irreducible, this state factors through the quotient map \( TC^*(E) \rightarrow C^*(E) \) [10, Theorem 4.3], and the usual faithful representation of \( C^*(E) \) is on the infinite-path space \( \ell^2(E^*) \). (The case \( k = 1 \) is different from \( k > 1 \); the spectrum of the commutative subalgebra \( D \) discussed in the introduction is \( E^* \cup E^{\infty} \) [22].)

5. A spatial realisation of a KMS state

We summarise our main results as follows. The map \( \Phi^\gamma \) appearing in (5.3) is the expectation of \( TC^* (\Lambda) \) onto the fixed-point algebra \( TC^* (\Lambda)^\gamma \) obtained by averaging over the gauge action \( \gamma \) of \( \mathbb{T}^k \).

**Theorem 5.1.** Let \( \Lambda \) be a finite coordinatewise-irreducible \( k \)-graph with no sources, and let \( \kappa \) be the common unimodular Perron-Frobenius eigenvector of the vertex matrices \( A_i \). Let \( J \uplus K \) be a nontrivial partition of \( \{1, \ldots, k\} \), and suppose that \( r \in (0, \infty)^k \) satisfies

\[
 r_j > \ln \rho(A_j) \quad \text{for } j \in J \quad \text{and} \quad r_i = \ln \rho(A_i) \quad \text{for } i \in K.
\]

Set \( C_J := \prod_{j \in J} (1 - e^{-r_j} \rho(A_j)) \).

(a) For each \( m \in \mathbb{N}^J \) there is a measure \( \nu^m \) on \( \Lambda^{m, \infty_K} \) such that, for \( n \in \mathbb{N}^K \) and \( \lambda \in \Lambda^{m,n} \),

\[
 \nu^m(Z(\lambda) \cap \Lambda^{m,\infty_K}) = e^{-r \cdot d(\lambda)} C_J \kappa_s(\lambda) = e^{-r \cdot (m,n)} C_J \kappa_s(\lambda).
\]

(b) Let \( m \in \mathbb{N}^J \), \( n \in \mathbb{N}^K \), and \( \lambda \in \Lambda^{m,n} \). Then

\[
 \sum_{l \in \mathbb{N}^J} \nu^l(Z(\lambda) \cap \Lambda^{l,\infty_K}) = \sum_{l \geq m} \nu^l(Z(\lambda) \cap \Lambda^{l,\infty_K}) = e^{-r \cdot d(\lambda)} \kappa_s(\lambda).
\]

(c) Let \( \pi^K : TC^* (\Lambda) \rightarrow B(\ell^2(\partial^K \Lambda)) \) be the semi-infinite path representation of \( \Lambda \). Then there is a bounded functional \( \phi \) on \( TC^* (\Lambda) \) such that

\[
 \phi(a) := \sum_{m \in \mathbb{N}^J} \int (\pi^K(\Phi^\gamma (a)) \xi_{m,x}) d\nu^m(x) \quad \text{for } a \geq 0,
\]

and \( \phi \) is a KMS\( \beta \) state of \( (TC^* (\Lambda), \alpha^\gamma) \) satisfying

\[
 \phi(t_\sigma t_\tau^* = \delta_{\sigma,\tau} e^{-r \cdot d(\sigma)} \kappa_s(\sigma) \quad \text{for } \sigma, \tau \in \Lambda.
\]

---

1 Applied to an irreducible block of \( A \) if \( A \) is not itself irreducible.
The state $\phi$ factors through the quotient by the ideal generated by
\[
\left\{ t_v - \sum_{e \in \Lambda^e} t_{e} t_{e}^* : v \in \Lambda^0, i \in \mathbb{K} \right\},
\]
and we have $\phi(t_v - \sum_{e \in \Lambda^e} t_{e} t_{e}^*) \neq 0$ for all $j \in J$ and $v \in \Lambda^0$.

(d) If $r$ has rationally independent coordinates, then the state $\phi$ of (c) is the only
KMS$_1$ state of $(T \mathcal{C}^*(\Lambda), \alpha^r)$.

Before we start the proof of Theorem 5.1, we prove a couple of lemmas. The next lemma
describes a standard construction of measures on inverse limits. It is a mild generalisation
of Lemma 6.1 (where the partially ordered set is $\mathbb{N}$ and the measures are probability
measures), and the proof given there carries over.

**Lemma 5.2.** Let $I$ be a directed partially ordered set with smallest element $0$. For $i, j \in I$
let $X_i$ be a compact space and $r_{ij} : X_j \to X_i$ be a surjection. Let $(X_\infty, \pi_i)$ be the inverse
limit of the system $(\{X_i\}, \{r_{ij}\})_{i,j \in I}$. Suppose that we have Borel measures $\mu_i$ on $X_i$
such that $\mu_0$ is finite and
\[
\int (f \circ r_{ij}) \, d\mu_i = \int f \, d\mu_i \quad \text{for } i \leq j \text{ and } f \in C(X_i).
\]
Then there is a unique finite Borel measure $\mu$ on $X_\infty$ such that
\[
\int f \circ \pi \, d\mu = \int f \, d\mu_i \quad \text{for } f \in C(X_i).
\]

**Lemma 5.3.** Let $\Lambda$ be a finite $k$-graph with no sources and $J \sqcup K$ be a nontrivial
partition of $\{1, \ldots, k\}$. Let $\pi^K$ be the semi-infinite path representation of $T \mathcal{C}^*(\Lambda)$ from
Let $\sigma, \tau \in \Lambda$ and $m \in \mathbb{N}$ and $m \in \mathbb{N}^J$. Then for $x \in \Lambda_{m,\infty K}$, we have
\[
(\pi^K(t_{\sigma} t_{\tau}^*) \xi_{m,x} | \xi_{m,x}) = \delta_{d(\tau),d(\sigma)} x Z(\tau) \cap Z(\sigma) \cap \Lambda_{m,\infty K}(x).
\]
Let $f : \partial^K \Lambda \to \mathbb{R}$ be the function defined by $f(x) = (\pi^K(t_{\sigma} t_{\tau}^*) \xi_{m,x} | \xi_{m,x})$ for $x \in \Lambda_{m,\infty K}$.
Then $f$ is Borel and its restriction to $\Lambda_{m,\infty K}$ is continuous.

**Proof.** Let $x \in \Lambda_{m,\infty K}$. Then
\[
(\pi^K(t_{\sigma} t_{\tau}^*) \xi_{m,x} | \xi_{m,x}) = (T^*_{\sigma} \xi_{m,x} | T^*_{\tau} \xi_{m,x}) = \begin{cases}
1 & \text{if } d(\tau) = d(\sigma) \text{ and } x \in Z(\tau) \cap Z(\sigma) \cap \Lambda_{m,\infty K} \\
0 & \text{otherwise}.
\end{cases}
\]
So either $f = 0$ or $f$ is the characteristic function of the Borel set $Z(\tau) \cap Z(\sigma) \cap \partial^K \Lambda$. In
either case, $f$ is Borel.

By Proposition 5.2, $\Lambda_{m,\infty K}$ is compact. Since $Z(\tau) \cap Z(\sigma)$ is compact and open in
$W_\Lambda$ by Lemma 6.1, its intersection with the compact set $\Lambda_{m,\infty K}$ is compact and open in
$\Lambda_{m,\infty K}$. So the restriction of $f$ to $\Lambda_{m,\infty K}$ is continuous.

**Proof of Theorem 5.1.** We construct the measure $\nu^m$ using Lemma 5.2. For $n \in \mathbb{N}^K$, we
give $\Lambda_{m,n}$ the discrete topology, and let $\nu_{m,n}$ be the measure on $\Lambda_{m,n}$ such that
\[
\nu_{m,n}(\{\lambda\}) = e^{-r \cdot (m,n)} C_{J,K}(\lambda) \text{ for } \lambda \in \Lambda_{m,n}.
\]
For $p, n \in \mathbb{N}^K$ such that $p \geq n$, we define $r_{n,p} : \Lambda_{m,p} \to \Lambda_{m,n}$ by $r_{n,p}(\lambda) = \lambda(0, (m,n))$;
since $\Lambda$ has no sources, each $r_{n,p}$ is a surjection.
We claim that \( \int f \circ r_{n,p} \, d\nu^{m,p} = \int f \, d\nu^{m,n} \) for all \( f \in C(\Lambda^{m,n}) \). Since the characteristic functions of singletons span \( C(\Lambda^{m,n}) \), it suffices to consider \( f = \chi_{\{\lambda\}} \). A quick calculation shows that

\[
\chi_{\{\lambda\}} \circ r_{n,p} = \sum_{\alpha \in s(\lambda) \Lambda^{0,p-n}} \chi_{\{\lambda \alpha\}},
\]

and hence

\[
(5.5) \quad \int \chi_{\{\lambda\}} \circ r_{n,p} \, d\nu^{m,p} = \sum_{\alpha \in s(\lambda) \Lambda^{0,p-n}} \nu^{m,p}(\{\lambda \alpha\})
= e^{-r(m,p)} C_J \sum_{w \in \Lambda^0} A^{(0,p-n)}(s(\lambda), w) \kappa_w
= e^{-r(m,p)} C_J \rho(\Lambda)^{(0,p-n)} \kappa_s(\lambda)
= e^{-r(m,p)} C_J \kappa_s(\lambda)
= \int \chi_{\{\lambda\}} \, d\nu^{m,n},
\]

as claimed.

Since \( \Lambda^{m,0} \) is finite, \( \nu^{m,0} \) is trivially a finite measure, and Lemma \( (5.2) \) gives a unique measure \( \nu^m \) on \( \Lambda^{m,\infty K} \) such that, for \( \lambda \in \Lambda^{m,n} \),

\[
\nu^m(Z(\lambda) \cap \Lambda^{m,\infty K}) = \int \chi_{\{\lambda\}} \circ \pi_n \, d\nu^m = \int \chi_{\{\lambda\}} \, d\nu^{m,n}
= \nu^{m,n}(\{\lambda\}) = e^{-r_d(\lambda)} C_J \kappa_s(\lambda),
\]

and we have proved (iii).

For (iv), we fix \( m \in \mathbb{N}^j \), \( n \in \mathbb{N}^K \) and \( \lambda \in \Lambda^{m,n} \). We first observe that for \( l \in \mathbb{N}^j \), we have

\[
Z(\lambda) \cap \Lambda^{l,\infty K} \neq \emptyset \implies l \geq m \text{ in } \mathbb{N}^j,
\]

and this immediately implies the first equality in (5.2). So we suppose that \( l \geq m \) in \( \mathbb{N}^j \). Then repeating the first few steps in the calculation (5.5) gives

\[
(5.6) \quad \nu^l(Z(\lambda) \cap \Lambda^{l,\infty K}) = \sum_{\alpha \in s(\lambda) \Lambda^{l-m,0}} \nu^l(Z(\lambda \alpha) \cap \Lambda^{l,\infty K})
= e^{-r(l,n)} C_J \rho(\Lambda^{(l-m,0)K}) \kappa_s(\lambda)
= e^{-r(l,n)} C_J \rho(\Lambda^{(l-m,0)K}) \kappa_s(\lambda).
\]

Summing over \( l \in \mathbb{N}^j \), writing \( l = m + p \) and remembering that \( e^{-r_j \rho(A_j)} < 1 \) for \( j \in J \) gives

\[
\sum_{l \in \mathbb{N}^j} \nu^l(Z(\lambda) \cap \Lambda^{l,\infty K}) = e^{-r(m,n)} C_J \left( \sum_{p \in \mathbb{N}^j} e^{-r(p,0) \rho(\Lambda^{(p,0)})} \kappa_s(\lambda) \right)
= e^{-r_d(\lambda)} C_J \left( \prod_{j \in J} (1 - e^{-r_j \rho(A_j)})^{-1} \right) \kappa_s(\lambda)
= e^{-r_d(\lambda)} \kappa_s(\lambda).
\]

This gives (iv).
For (13), we consider \( a \in \mathcal{T}C^*(\Lambda) \). Lemma 5.3 implies that \( x \mapsto (\pi^K(a)\xi_{m,x} \mid \xi_{m,x}) \) is continuous on \( \Lambda^{m,\infty}_N \), so the integrals make sense, and we next have to show that the sum on the right-hand side of (5.3) converges absolutely. For \( m \in \mathbb{N}_J \) we compute
\[
\nu^m(\Lambda^{m,\infty}_N) = \sum_{\lambda \in \Lambda^{m,0}} \nu^m(Z(\lambda) \cap \Lambda^{m,\infty}_N)
\]
using (5.6) with \( l = m \) as well as the techniques of (5.9), finding that
\[
\nu^m(\Lambda^{m,\infty}_N) = e^{-r(m,0)\rho(\Lambda)(m,0)}C_J.
\]
Since \( j \in J \) implies \( 1 > e^{-r}\rho_0(A_j) \) we get \( \sum_{m \in \mathbb{N}_J} e^{-r(m,0)\rho(\Lambda)(m,0)}C_J = 1 \). Since the integrands all have absolute value at most \( \|a\| \), the series in (5.3) converges absolutely, with sum at most \( \|a\| \). So there is a functional \( \phi \) satisfying (5.3), and this functional has norm at most one. For \( a \geq 0 \) all the summands in (5.3) are non-negative, and hence \( \phi \) is positive. Equation (5.2) implies that \( \phi(1) = 1 \), and hence \( \phi \) is a state.

Next we show that \( \phi \) satisfies (5.4). Fix \( \sigma \in \Lambda \) and \( x \in \Lambda^{m,\infty}_N \). If \( m \geq d(\sigma)_J \), then Lemma 5.3 gives
\[
(\pi^K(t_\sigma t_\sigma^*)\xi_{m,x} \mid \xi_{m,x}) = \chi_{Z(\sigma) \cap \Lambda^{m,\infty}_N}(x);
\]
otherwise, it is 0. Thus (13) gives half of (5.4):
\[
\phi(t_\sigma t_\sigma^*) = \sum_{m \geq d(\sigma)_J} \nu^m(Z(\sigma) \cap \Lambda^{m,\infty}_N) = e^{-r(d(\sigma))K_{\sigma}(\sigma)}.
\]

Now take a pair \( \sigma, \tau \in \Lambda \). If \( d(\sigma) \neq d(\tau) \), then \( \Phi^\gamma(t_\sigma t_\sigma^*) = 0 \) and \( \phi(t_\sigma t_\sigma^*) = 0 \). So suppose that \( d(\sigma) = d(\tau) \), \( \sigma \neq \tau \) and \( \phi(t_\sigma t_\sigma^*) \neq 0 \). Then there exists \( m \in \mathbb{N}_J \) and \( x \in \Lambda^{m,\infty}_N \) such that
\[
(\pi^K(t_\sigma t_\tau^*)\xi_{m,x} \mid \xi_{m,x}) = (\pi^K(t_\tau^*)\xi_{m,x} \mid \pi^K(t_\sigma)\xi_{m,x}) \neq 0.
\]
But then there exists \( y \) such that \( x = \sigma y = \tau y \) and \( \sigma = x(0, d(\sigma)) = x(0, d(\tau)) = \tau \). This gives the other half of (5.4).

In view of (5.4), it follows from Proposition 3.1(b)] that \( \phi \) is a KMS\(_1\) state and hence that \( \phi \) is a KMS\(_1\) state of \( (\mathcal{T}C^*(\Lambda), \alpha) \). Since \( \Lambda \) is coordinatewise irreducible, all the \( A_i \) are irreducible, and with \( K = \{i\} \), Proposition 4.2[3] implies that \( \phi \) factors through the ideal generated by \( t_{e} - \sum_{e \in v} t_{e}t_{e}^* : v \in \Lambda^0 \).

To complete the proof of part (c), we take \( v \in \Lambda^0 \) and \( j \in J \), and consider the value of \( \phi \) on the projection \( t_v - \sum_{e \in v} t_{e}t_{e}^* \). We compute, using (5.2) and (5.4),
\[
\phi\left(t_v - \sum_{e \in v} t_{e}t_{e}^*\right) = \phi(t_v) - \sum_{e \in v} \phi(t_{e}t_{e}^*)
\]
\[
= \sum_{m \in \mathbb{N}_J} \nu^m(Z(v) \cap \Lambda^{m,\infty}_N) - \sum_{m \geq e_j} \nu^m(Z(e) \cap \Lambda^{m,\infty}_N)
\]
\[
= \sum_{m \in \mathbb{N}_J, m_j = 0} \nu^m(Z(v) \cap \Lambda^{m,\infty}_N) + \sum_{m \geq e_j} \nu^m(Z(v) \cap \Lambda^{m,\infty}_N)
\]
\[
- \sum_{e \in \Lambda^0} \nu^m(Z(e) \cap \Lambda^{m,\infty}_N)
\]
\[
= \sum_{m \in \mathbb{N}_J, m_j = 0} \nu^m(Z(v) \cap \Lambda^{m,\infty}_N)
\]
\[
\geq \nu^0(Z(v) \cap \Lambda^{0,\infty}_N)
\]
\[
= C_J K_v \text{ by (5.1).}
\]
Since $\kappa$ is the common Perron-Frobenius eigenvector for the $A_i$, it has positive entries, and (i) follows.

Item (ii) follows from Proposition 4.2(c).

\begin{corollary}
Let $\phi$ be the KMS$_1$ state of $(\mathcal{T}C^*(\Lambda), \alpha^t)$ from Theorem 5.7(a). Then there is a probability measure $\mu$ on $\partial K\Lambda = \bigcup_{m \in \mathbb{N}^J} \Lambda^{m,\infty K}$ such that
\begin{equation}
\phi(a) = \int (\pi K(\Phi^t(a))\xi_{m,x} | \xi_{m,x}) \, d\mu(x) \quad \text{for } a \in \mathcal{T}C^*(\Lambda).
\end{equation}

In particular, for $\sigma \in \Lambda^{m,n}$ we have
\begin{equation}
\phi(t_\sigma t_\sigma^* x) = \mu(Z(\sigma) \cap \partial K\Lambda).
\end{equation}

\begin{proof}
By Theorem 5.1(a), for $m \in \mathbb{N}^J$ we have finite Borel measures $\nu^m$ on $\Lambda^{m,\infty K}$ which we can view as measures on $\partial K\Lambda = \bigcup_{m \in \mathbb{N}^J} \Lambda^{m,\infty K}$ with support in $\Lambda^{m,\infty K}$. Then by Theorem 5.1(b) we have
\begin{equation}
\sum_{m \in \mathbb{N}^J} \nu^m(\partial K\Lambda) = \sum_{m \in \mathbb{N}^J} \nu^m(\Lambda^{m,\infty K}) = \sum_{m \in \mathbb{N}^J, v \in \Lambda^0} \nu^m(Z(v) \cap \Lambda^{m,\infty K})
= \sum_{v \in \Lambda^0} \sum_{m \in \mathbb{N}^J} \nu^m(Z(v) \cap \Lambda^{m,\infty K}) \quad \text{(by Tonelli’s Theorem)}
= \sum_{v \in \Lambda^0} \kappa_v = 1.
\end{equation}

Since we are viewing the $\nu^m$ as measures on $\partial K\Lambda$, they define the functionals on $C(\partial K\Lambda)$ with norm $\|\nu^m\| = \nu^m(\partial K\Lambda)$. Thus (5.9) implies that the series $\sum_{m \in \mathbb{N}^J} \nu^m$ converges in $C(\partial K\Lambda)^*$ to a positive functional on $C(\partial K\Lambda)$ of norm 1, which by the Riesz representation theorem is given by a probability measure $\mu$ on $\partial K\Lambda$. Then for $f \in C(\partial K\Lambda)$, we have
\begin{equation}
\int f \, d\mu = \sum_{m \in \mathbb{N}^J} \int f \, d\nu^m.
\end{equation}

Let $\sigma, \tau \in \Lambda$. Then Lemma 5.3 shows that $x \mapsto (\pi K(t_\sigma t_\tau^*)\xi_{m,x} | \xi_{m,x})$ is Borel on $\partial K\Lambda$, and hence it is $\mu$-measurable. Then (5.10) and the formula for $\phi$ in (5.3) give
\begin{equation}
\int (\pi K(t_\sigma t_\tau^*)\xi_{m,x} | \xi_{m,x}) \, d\mu(x) = \sum_{m \in \mathbb{N}^J} \int (\pi K(t_\sigma t_\tau^*)\xi_{m,x} | \xi_{m,x}) \, d\nu^m(x) = \phi(t_\sigma t_\tau^*).
\end{equation}

Now (5.7) follows by continuity and (5.8) follows by taking $\sigma = \tau$.
\end{proof}

\begin{remark}
Our proof uses that $K$ is not all of $\{1, \ldots, k\}$, so that $J$ is nonempty. A similar result for the case $J = \emptyset$ is proved in [13] Proposition 10.2. However, it is easier to construct the measure when $J = \emptyset$, because then we can view $\Lambda^\infty$ as the inverse limit $\varprojlim_{n \in \mathbb{N}^k} \Lambda^n$ of the finite path spaces (see the proof of [13] Proposition 8.1). Note that [13] Proposition 10.2 applies to a broader class of graphs.
\end{remark}

\begin{remark}
Theorem 7.1 of [13] says that the KMS$_1$ state is unique if and only if the graph is aperiodic. At least for graphs with one vertex, the “rationally independent” hypothesis in Theorem 5.1(b) is linked to aperiodicity. If $\Lambda$ has one vertex, $N_1$ blue edges and $N_2$ red edges, and if $\ln N_1 / \ln N_2$ is irrational, then $\Lambda$ is aperiodic [5 Corollary 3.2]. However, even if $\ln N_1 / \ln N_2$ is rational, then $\Lambda$ can be aperiodic. There is a detailed discussion of this question in [5], and a necessary and sufficient condition is described in [5] Theorem 3.1. (The proof of this in [5] is algebraic: when $[\Lambda^0] = 1$, the path space $\Lambda$ is a semigroup, and one can study the graph by studying the algebraic properties of this semigroup. There is an alternative graph-based proof in the appendix to [5].)
\end{remark}
6. Better results for 2-graphs

The formula \((5.3)\) for the KMS state \(\psi\) in Theorem 5.1 involves the expectation \(\Phi^x\) onto the core \(\mathcal{TC}^*(\Lambda)^\gamma\). It did not appear in the corresponding formula in [13, Proposition 10.2], so one naturally wonders whether it is necessary here. We have been able to answer this when \(|K| = 1\): formula \((5.2)\) implies that \(\psi\) is the state \(\phi\) of Theorem 5.1. (Of course, this is the only nontrivial possibility for \(K\) when \(k = 2\).) For \(|K| \geq 2\), the state \(\psi\) is not necessarily supported on the diagonal \(\text{span}\{t_n t_\sigma^*\}\), and hence need not be the state in Theorem 5.1 (see Remark 6.2).

**Proposition 6.1.** Resume the notation of Theorem 5.1, and suppose in addition that \(K = \{\varnothing\}\) and that the directed graph \((\Lambda^0, \Lambda^*, r, s)\) is not a cycle. Then there is a bounded functional \(\psi\) on \(\mathcal{TC}^*(\Lambda)\) such that

\[ \psi(a) := \sum_{m \in \mathbb{N}^J} \int_{\Lambda^{m,\infty,K}} (\pi^K(a) \xi_{m,x} \mid \xi_{m,x}) \, d\nu^m(x) \quad \text{for } a \geq 0, \]

and \(\psi\) is a KMS\(\_\) state of \((\mathcal{TC}^*(\Lambda), \alpha^r)\) satisfying

\[ \psi(t_n t_\sigma^*) = \delta_{\sigma, \tau} e^{-r d(\sigma) \kappa_s(\sigma)} \quad \text{for } \sigma, \tau \in \Lambda. \]

The state \(\psi\) factors through the quotient by the ideal generated by \(t_v - \sum_{e \in v\Lambda^*} t_e t_e^*\), and we have \(\psi(t_v - \sum_{e \in v\Lambda^*} t_e t_e^*) \neq 0\) for all \(v \in J\) and \(v \in \Lambda^0\).

**Proof.** The first two paragraphs of the proof of Theorem 5.1(c) apply almost verbatim. For the other half of \((6.2)\), we fix \(\tau \neq \sigma \in \Lambda\), and aim to get an estimate for \(\psi(t_n t_\sigma^*)\). First we have to look at the integrands. For \(a\) of the form \(t_n t_\sigma^*\) they all take only the values 0 and 1. We need to fix \(m\) and look at the semi-infinite paths \(x \in \Lambda^{m,\infty,K}\) such that

\[ (\pi^K(t_n t_\sigma^*) \xi_{m,x} \mid \xi_{m,x}) = \left( \xi_{m - d(\tau), j, x(d(\tau), s)} \mid \xi_{m - d(\sigma), j, x(d(\sigma), s)} \right) = 1. \]

If \(x\) is such a path, then \(d(\tau)_j = d(\sigma)_j \leq m\), and there is a semi-infinite path \(y\) such that \(x = \tau y = \sigma y\). Say \(d(\tau) = (d(\tau)_j, n)\) and \(d(\sigma) = (d(\sigma)_j, p)\) for some \(n, p \in \mathbb{N}^K\). We cannot have \(p = n\), because then \(d(\tau) = d(\sigma)\) and \(\sigma y = \tau y\) imply \(\sigma = \tau\). Since \(|K| = 1\), \(\mathbb{N}^K \cong \mathbb{N}\) is totally ordered, so we may suppose that \(p > n\) (otherwise swap \(p\) and \(n\)). Then \(\sigma y = \tau y\) implies that \(\sigma(0, (d(\tau)_j, n)) = \tau \). So \(\sigma y = \tau \lambda y\) where \(\lambda = \sigma((d(\tau)_j, n), (d(\tau)_j, p))\).

Thus \(\lambda = \lambda y\), and \(y = \lambda \lambda y\). By induction we have \(y = \lambda^N y\) for all \(N \geq 1\), and \(x \in Z(\tau \lambda^N) \cap \Lambda^{m,\infty,K}\) for all \(N \geq 1\). Thus

\[ \{ x \in \Lambda^{m,\infty,K} : (\pi^K(t_n t_\sigma^*) \xi_{m,x} \mid \xi_{m,x}) = 1 \} \subset \cap_{N=1}^\infty Z(\tau \lambda^N) \cap \Lambda^{m,\infty,K}, \]

and we can estimate

\[ \psi(t_n t_\sigma^*) = \sum_{m \in \mathbb{N}^J} \nu^m \left( \{ x \in \Lambda^{m,\infty,K} : (\pi^K(t_n t_\sigma^*) \xi_{m,x} \mid \xi_{m,x}) = 1 \} \right) \]

\[ \leq \sum_{m \geq d(\tau)_j} \nu^m \left( \cap_{N=1}^\infty Z(\tau \lambda^N) \cap \Lambda^{m,\infty,K} \right) \]

\[ = \sum_{m \geq d(\tau)_j} \lim_{N \to \infty} \nu^m (Z(\tau \lambda^N) \cap \Lambda^{m,\infty,K}). \]

Of course, we want to pull the limit through the infinite sum. So we note that

\[ Z(\tau \lambda^N) \cap \Lambda^{m,\infty,K} = \bigcup_{\mu \in s(\lambda) \Lambda^{m-d(\tau)_j,0}} Z(\tau \lambda^N \mu) \cap \Lambda^{m,\infty,K}. \]
Then using (5.1) we get
\[ \nu^i(Z(\tau\lambda^N) \cap \Lambda^{m,\infty K}) = \sum_{\mu \in s(\lambda)\Lambda^{m-d(\tau)}J,0} \nu^m(Z(\tau\lambda^N \mu) \cap \Lambda^{m,\infty K}) \]
\[ = \sum_{\mu \in s(\lambda)\Lambda^{m-d(\tau)}J,0} e^{-r(d(\tau,\mu)} C_{\lambda Ks(\mu)} \]
\[ = e^{-r(m,n+N(p-n))} C_{\lambda} \sum_{w \in \Lambda^0} A^{(m-d(\tau),0)}(s(\lambda), w) \kappa_w \]
\[ = e^{-r(m,n+N(p-n))} C_{\lambda} (A^{(m-d(\tau),0)} \kappa_s(\lambda)) \]
\[ = e^{-r(m,n+N(p-n))} C_{\lambda} \rho(\Lambda)(\kappa_s(\lambda)) \]
\[ = e^{-r\ln\rho(\Lambda)(m,0)} \rho(\Lambda)^{-d(\tau),n+N(p-n))} C_{\lambda Ks(\lambda)} \]
since \( e^{r_i} = \rho(A_i) \) for \( i \in K \)
\[ \leq e^{-r\ln\rho(\Lambda)(m,0)} C_{\lambda Ks(\lambda)} \] since \( \rho(A_i) \geq 1 \).

Since \( r_i > \ln\rho(A_i) \) for \( j \in J \), the series \( \sum_{m \in \mathbb{N}^J} e^{-r\ln(\Lambda)(m,0)} \) converges. Thus the dominated convergence theorem and (6.3) imply that
\[ \psi(t_\sigma t_\sigma^*) = \lim_{N \to \infty} \sum_{m \in \mathbb{N}^J} \nu^m(Z(\tau\lambda^N) \cap \Lambda^{m,\infty K}) \]

Equation (5.2) implies that the sum on the right-hand side of (6.4) is
\[ \sum_{m \in \mathbb{N}^J} \nu^m(Z(\tau\lambda^N) \cap \Lambda^{m,\infty K}) = e^{-r(\ln(\Lambda)(m,0))} \kappa_s(\lambda) \]
and this goes to 0 as \( N \to \infty \) because \( p - n > 0 \) and (for the i such that \( K = \{i\} \)) \( r_i = \rho(A_i) > 1 \) because \( (\Lambda^0, \Lambda^\gamma, r, s) \) is not a cycle. Thus (6.4) implies that \( \psi(t_\sigma t_\sigma^*) = 0 \), and this completes the proof of (6.2).

Now the last two paragraphs in the proof of Theorem 5.1(c) carry over to this situation.

**Remark 6.2.** The formula (6.1) defines a state \( \psi \) on \( TC^*(\Lambda) \) for every \( K \) with \( |K| \geq 1 \). The formula (6.2) says that when \( |K| = 1 \), this state is supported on the diagonal \( D := \text{span}\{T_\sigma T_\sigma^* : \sigma \in \Lambda\} \). We claim that this is not necessarily the case if \( |K| \geq 2 \). To see this, suppose that \( |K| \geq 2 \) and that the graph \( \Lambda^K := (\Lambda^0, d^{-1}(\Lambda^K), r, s) \) is periodic. We write \( \{t^K_\lambda\} \) for the universal Toeplitz-Cuntz-Krieger family in \( TC^*(\Lambda^K) \). Then
\[ \{T^K_\lambda := t_\mu : \mu \in \Lambda^K\} \]
is a Toeplitz-Cuntz-Krieger \( \Lambda^K \)-family in \( TC^*(\Lambda) \), and hence gives a homomorphism \( \pi_{TK} : TC^*(\Lambda^K) \to TC^*(\Lambda) \) such that \( \pi_{TK}(t^K_\lambda) = t_\lambda \). We are going to use the results of [13] to compute the values of the KMS state \( \psi \) on elements of the form \( \pi_{TK}(t^K_\mu)^* \) for \( \mu, \nu \in \Lambda^K \).

So suppose \( \lambda, \mu \in \Lambda^K \) (so that \( d(\lambda) = d(\mu) = 0 \)). By (6.1), our state satisfies
\[ \psi \circ \pi_{TK}(t^K_\mu)^* = \sum_{m \in \mathbb{N}^J} \int_{\Lambda^{m,\infty K}} (T_\lambda T_\mu^* \xi_{m,x} \mid \xi_{m,x}) d\nu^m(x). \]

For each \( x \in \Lambda^{m,\infty K} \), the vector \( T_\lambda T_\mu^* \xi_{m,x} \) is either a basis element or 0. Thus the integrands on the right-hand side of (6.5) are all non-negative. Thus we have
\[ \psi \circ \pi_{TK}(t^K_\mu)^* \geq \int_{\Lambda^{0,\infty K}} (T_\lambda T_\mu^* \xi_{0,x} \mid \xi_{0,x}) d\nu^0(x) \geq 0. \]
Recall that $C_j := \prod_{j \in J} (1 - e^{-r} \rho(A_j))$. Then, because $d(\lambda)_J = 0$, the formula (5.1) shows that the measure $\nu^0$ satisfies

$$\nu^0(Z(\lambda) \cap \Lambda^{0,\infty}) = e^{-r \cdot d(\lambda)} C_J h_{s(\lambda)}(\lambda) = \rho(\Lambda^K)^{-d(\lambda)_K} C_J h_{s(\lambda)}(\lambda).$$

Now we compare our formula for $\psi$ with that of the state in [13, Proposition 10.2] for the preferred dynamics on the graph $\Lambda^K$. Since $\Lambda^K$ has vertex matrices $\{A_i : i \in K\}$, $\Lambda$ and $\Lambda^K$ have the same unimodular Perron-Frobenius eigenvector. The model graph $\Omega_{2|K}$ for infinite paths in $\Lambda^K$ sits inside the model graph $\Omega_{k,(0,\infty K)}$ for paths in $\Lambda^{0,\infty K}$, and the map $x \mapsto x|_{[\Omega|K]}$ is a homeomorphism $h$ of $\Lambda^{0,\infty K}$ onto $(\Lambda^K)^{\infty}$. The homeomorphism $h$ carries the set $Z(\lambda) \cap \Lambda^{0,\infty K}$ into the cylinder set $Z(\lambda) \subset (\Lambda^K)^{\infty}$. Comparing (6.6) with the formula (8.3) for $M$ in [13] shows that $\nu^0$ is the measure $C_j h_{s} M$ pulled over from $(\Lambda^K)^\infty$. The underlying bijection of $(\Lambda^K)^\infty$ onto $\Lambda^{0,\infty K}$ gives a unitary isomorphism $V$ of $\ell^2((\Lambda^K)^\infty)$ onto the summand $H_0 := \ell^2(\Lambda^{0,\infty K})$ of $\ell^2(\partial^K(\Lambda)) = \bigoplus_{m \in \mathbb{N}} \ell^2(\Lambda^{m,\infty K})$. This isomorphism $V$ maps the usual basis $\{h_x\}$ for $\ell^2((\Lambda^K)^\infty)$ into the basis $\{\xi_{0,x}\}$, and intertwines the usual infinite-path representation (denoted $\pi_S$ in [13, §10]) and $\pi_{T^K}|_{H_0}$. Thus

$$\psi(t_\lambda t_\mu^*) = \psi \circ \pi_{T^K} (t_\lambda (t_\mu^*)) \geq \int_{\Lambda^{0,\infty K}} (T_\lambda T_\mu^* \xi_{0,x} | \xi_{0,x}) \, d\nu^0(x)$$

$$= C_j \int_{(\Lambda^K)^\infty} (\pi_S(t_\lambda t_\mu^*) h_x | h_x) \, dM(x),$$

which is, modulo the nonzero scalar $C_j$, the formula for the KMS state of $(\mathcal{T} C^*(\Lambda^K), \alpha^\beta)$ in [13, Proposition 10.2], which we will denote here by $\psi^K$.

One of the main points made in [13, §10] was that, when $d(\lambda) - d(\mu)$ belongs to the periodicity group $\text{Per}(\Lambda^K)$, the state $\psi^K$ does not vanish on $t_\lambda (t_\mu^*)$. Hence our estimate (6.7) shows that $\psi(t_\lambda t_\mu^*)$ does not vanish either. This settles the claim we made above.

7. 2-GRAPHS WITH A SINGLE VERTEX

Here we illustrate our results by applying them to a 2-graph with a single vertex. Such graphs were first studied by Kribs and Power [16], and their $C^*$-algebras, have been extensively studied by Davidson and Yang [5, 33, 34]. Yang in particular has made a convincing case that these $C^*$-algebras should be viewed as higher-rank analogues of the Cuntz algebras, and share many of their properties.

We suppose that $\Lambda$ is a 2-graph with one vertex, $N_1 > 0$ blue edges and $N_2 > 0$ red edges. Such graphs are always coordinatewise irreducible, and their spectral radii are $\ln N_1$ and $\ln N_2$. Our conventions at (4.11) say that the dynamics is given by a vector $r \in (0,\infty)^2$ such that $r_i^{-1} \ln \rho(A_i) = 1$ for one $i \in \{1,2\}$ and $r_j^{-1} \ln \rho(A_j) \geq 1$ for $j \neq i$. We may as well suppose that $r_2^{-1} \ln \rho(A_2) = 1$ and $r_1^{-1} \ln \rho(A_1) \geq 1$ (otherwise swap the colours).

We first consider inverse temperatures $\beta$ satisfying $\beta > 1$. Then we have $\beta r_i > \ln N_i$ for both $i$, and [11] Theorem 6.1] implies that there is a single KMS $\beta$ state. By [11] (6.2), the vector $y$ is the real number given by

$$y_v = \sum_{n=0}^{\infty} e^{-\beta (r_1 \ln N_2 - n |\Lambda^n|}$$

$$= \sum_{n=0}^{\infty} e^{-\beta r_1 n_1 N_2^{-\beta n_2} N_1^{n_1} N_2^{n_2}}$$
\[
\begin{align*}
\phi_e(t_t^*) &= \delta_{\mu,\nu} e^{-r_1 d(\mu)} N_2^{-d(\mu)}.
\end{align*}
\]

The measure \( \nu_\beta \) on \( \Lambda \) implementing this (see Remark 4.3) satisfies
\[
\nu_\beta(\{\lambda\}) = e^{-r_1 d(\lambda)} N_2^{-d(\lambda)} (1 - N_1 e^{-r_1})(1 - N_2^{-1}).
\]

Next we suppose that \( r_1^{-1} \ln N_1 = r_2^{-1} \ln N_2 = 1 \) and consider \( \beta = 1 \). Then \( K = \{1,2\} \) and \( \alpha^* \) is the preferred dynamics studied in [11]. The common unimodular Perron-Frobenius eigenvector of the vertex matrices is \( \kappa = 1 \), and hence the argument in the first paragraph of the proof of [11] Theorem 7.2] gives a KMS1 state \( \phi \) of \( TC^*(\Lambda,\alpha^*) \) such that
\[
\phi(t_t^*) = \delta_{\mu,\nu} N_1^{-d(\mu)} N_2^{-d(\mu)}
\]
(as observed in Remark 7.3 of [11]). The measure \( M \) implementing this state in [13, Proposition 10.2] satisfies
\[
M(Z(\lambda)) = N_1^{-d(\lambda)} N_2^{-d(\lambda)}
\]
(see [13 Proposition 8.1]). We do not need rational independence of \( r_1 = \ln N_1 \) and \( r_2 \ln N_2 \) to get existence of the dynamics studied in Remark 7.3 of [11], but we do need it to deduce from [11] Theorem 7.2] that this is the only KMS1 state of \( TC^*(\Lambda,\alpha^*) \), and that it factors through a state of \( C^*(\Lambda,\alpha^*) \). However, we now know from Theorem 7.1 of [13] that there is a unique KMS1 state if and only if \( \Lambda \) is aperiodic, and rational dependence is only a sufficient condition for that (see Remark 5.6).

Finally, suppose that \( r_1^{-1} \ln N_1 < r_2^{-1} \ln N_2 = 1 \) and \( \beta = 1 \). Since \( |K| = 2 \) has a single element, Theorem 5.1 gives us a KMS1 state \( \phi \) of \( (C^*(\Lambda,\alpha^*) \) satisfying
\[
\phi(t_t^*) = \delta_{\mu,\nu} e^{-r_1 d(\mu)} N_2^{-d(\mu)}.
\]

Since \( \Lambda^0 = \{v\} \), Theorem 5.1 implies that \( \phi \) factors through the quotient by the ideal generated by the single element \( t_v = \sum_{e \in \Lambda^2} t_e^* e \). The measure \( \mu \) giving the spatial realisation of \( \phi \) in Corollary 5.4 is supported on the semi-infinite path space \( \tilde{\partial}[2](\Lambda) \), and there is given by
\[
\mu(Z(\lambda) \cap \tilde{\partial}[2](\Lambda) = e^{-r_1 d(\mu)} N_2^{-d(\mu)}.
\]

If \( r_1^{-1} \ln N_2 \) is irrational, then Theorem 5.1 implies that \( \phi \) is the only KMS1 state. If \( r_1^{-1} \ln N_2 \) is rational, then we have no information. It seems unlikely that the uniqueness is still connected with the periodicity of the graph \( \Lambda \): periodicity requires that \( \ln N_2/\ln N_1 \) is rational, which seems quite unrelated to rationality of \( r_1^{-1} \ln N_2 \).

**Appendix A. Relative graph algebras**

When \( E \) is a row-finite graph and \( V \) is a subset of \( E^0 \) which contains no sources, the *relative graph algebra* \( C^*(E,V) \) is the quotient of \( TC^*(E) \) by the ideal generated by
\[
\left\{ t_v - \sum_{e \in E^1} t_e^* : v \in V \right\}.
\]
These were introduced by Muhly and Tomforde [20] as a tractable family of relative Cuntz-Pimsner algebras: indeed, each \( C^*(E, V) \) is isomorphic to the graph algebra of a graph obtained by adding an extra vertex \( v' \) for each \( v \in E^0 \setminus V \) and an edge \( e' \) from \( v' \) to \( v \) for each \( e \in vE \) [20, Theorem 3.7].

The algebra in Proposition 3.5 looks like a relative graph algebra, but it doesn’t at first sight look much like the relative graph algebras of Sims [29]. This is mostly because he was interested primarily in extending theory to cover the finitely aligned higher-rank graphs of [28], and adjusting constructions and definitions to accommodate them is a complicated business. For the graphs of interest to us, things simplify, and it is quite easy to see that our algebra is indeed a relative graph algebra in his sense (Proposition A.2).

It is trickier to see that our semi-infinite path representation is one of his boundary-path representations (Proposition A.4), but the calculation might provide an instructive example for anyone interested in the satiation process of [29].

We discuss a row-finite \( k \)-graph \( \Lambda \) with no sources. We add the notation

\[
\text{MCE}(\lambda, \mu) = \{ \sigma \in \Lambda : d(\sigma) = d(\lambda) \lor d(\mu) \text{ and } \sigma \text{ has the form } \lambda \alpha = \mu \beta \};
\]

thus \((\alpha, \beta) \in \Lambda^{\text{min}}(\lambda, \mu)\) if and only if \((\lambda \alpha, \mu \beta) \in \text{MCE}(\lambda, \mu)\). A subset \( E \) of \( \Lambda \) is finite exhaustive if it is finite, if there is a vertex \( v \) (denoted \( r(E) \)) such that \( E \subset v\Lambda \), and if for every \( \mu \in v\Lambda \), there exists \( \lambda \in E \) such that \( \text{MCE}(\mu, \lambda) \neq \emptyset \). Notice that any finite subset of \( v\Lambda \) containing a finite exhaustive set \( F \) is itself finite exhaustive, and that any set of the form \( \{ \lambda(0, n_\lambda) : \lambda \in F, n_\lambda \leq d(\lambda) \} \) is finite exhaustive. For the graphs we are considering, each \( v\Lambda^\kappa \) is finite exhaustive.

Sims’ constructions use collections \( \mathcal{F} \) of finite exhaustive sets. For each such \( \mathcal{F} \), the relative Cuntz-Krieger algebra \( C^*(\Lambda; \mathcal{F}) \) is the quotient of \( TC^*(\Lambda) \) by the ideal generated by \( \Lambda \)

(A.1) \[
\left\{ \prod_{\lambda \in E} (q_{r(E)} - t_{\lambda} t_{\lambda}^*) : E \in \mathcal{F} \right\}.
\]

When we take the quotient of \( TC^*(\Lambda) \) by the relations (A.1), we impose other relations of the same form. The main point made in [29] is that these relations are those corresponding to finite exhaustive sets in the “satiation” of \( \mathcal{F} \), which we helpfully define as the smallest satiated set (of finite exhaustive sets) that contains \( \mathcal{F} \). The concept of satiated set is described in Definition 4.1 of [29]; that the satiation has the required properties is the content of [29, Corollary 5.6] and the uniqueness theorems in [29, §6].

Remark A.1. The definition of a Toeplitz-Cuntz-Krieger family in [29] is sufficiently different from the one in [27, 11] (and ours in (2.4)) that one might want to be reassured that these papers are all about the same algebras. The projections \( T_v \) in [29] are denoted by \( Q_v \) in [27, 11], and then the relations (T1), (T2) and (T5) in (2.4) are the same as (TCK1), (TCK2) and (TCK3) in [29, §3]. It may appear that (T3) and (T4) are additional relations, but this is not so. The relation (T3) is subsumed in (TCK3) by interpreting \( \Lambda^{\text{min}}(\lambda, \lambda) = \{ s(\lambda) \} \). Similarly, by interpreting empty sums in (TCK3) as 0, we find that \( T_{\lambda} T_{\mu} = \delta_{\lambda, \mu} T_{s(\mu)} \), and hence \( \{ T_{\lambda} T_{\mu}^* : d(\lambda) = n \} \) is a set of mutually orthogonal projections. From (TCK2) we deduce that if \( r(\lambda) = v \) then

\[
T_v(T_{\lambda} T_{\mu}^*) = T_{v \lambda} T_{\mu}^* = T_{\lambda} T_{v \mu} = (T_{\lambda} T_{\mu}^*) T_v,
\]

and hence \( T_v \geq T_{v \mu} \) (by [26, Proposition A.1], for example). Since \( \text{MCE}(\lambda, \mu) = \text{MCE}(\mu, \lambda) \), the relation (TCK3) also implies that for any \( \lambda \) and \( \mu \) in \( \Lambda \) we have

\[
(T_{\lambda} T_{\mu}^*)(T_{\mu} T_{\mu}^*) = T_{\lambda} \left( \sum_{(\alpha, \beta) \in \Lambda^{\text{min}}(\lambda, \mu)} T_{\alpha} T_{\beta}^* \right) = \sum_{\sigma \in \text{MCE}(\lambda, \mu)} T_{\sigma} T_{\sigma}^* = (T_{\mu} T_{\mu}^*)(T_{\lambda} T_{\lambda}^*).
\]
Now the orthogonality of the $T_\lambda T_\lambda^*$ implies that

$$T_v \geq \sum_{\lambda \in \Lambda^e} T_\lambda T_\lambda^*,$$

which is (T4).

**Proposition A.2.** Let $\Lambda$ be a row-finite $k$-graph with no sources. Suppose that $J \sqcup K$ is a partition of $\{1, \ldots, k\}$ and

$$\mathcal{E} = \{v\Lambda^{e_i} : v \in \Lambda^0 \text{ and } i \in K\}.$$

Then $C^*(\Lambda; \mathcal{E})$ is the quotient of $\mathcal{T}C^*(\Lambda)$ by the ideal generated by

$$\left\{ t_v - \sum_{e \in v\Lambda^{e_i}} t_v t_e^* : v \in \Lambda^0 \text{ and } i \in K \right\}.$$

**Proof.** Since $t_v \geq t_v t_e^*$ for $e \in v\Lambda^{e_i}$, and $(t_v t_e^*)(t_f t_f^*) = 0$ for $e \neq f$ in $v\Lambda^{e_i}$ (by (T4) or by Remark A.1 if you prefer the definitions in [29]), we have

$$\prod_{e \in v\Lambda^{e_i}} (t_v - t_v t_e^*) = t_v - \sum_{e \in v\Lambda^{e_i}} t_v t_e^*,$$

and the result follows. \qed

In [29, §4], Sims constructs a family of representations of his relative graph algebras, and this construction requires that we work with satiated collections of finite exhaustive sets. So to apply his construction to the algebra of Proposition A.2, we need to identify the satiation of the set $\mathcal{E}$ in (A.2).

**Proposition A.3.** Let $\Lambda$ be a row-finite $k$-graph with no sources. Let $J \sqcup K$ be a partition of $\{1, \ldots, k\}$ and $\mathcal{E}$ as in (A.2). Then the satiation $\overline{\mathcal{E}}$ of $\mathcal{E}$ consists of the sets which contain a finite exhaustive subset lying in $d^{-1}(\mathbb{N}^K)$.

**Proof.** We denote by $\mathcal{F}$ the collection of finite subsets $G$ of $\Lambda$ such that $G \subset v\Lambda$ for some $v \in \Lambda^0$, and such that $G$ contains a finite exhaustive subset of $d^{-1}(\mathbb{N}^K)$. Then we need to prove that $\mathcal{F} = \overline{\mathcal{E}}$. First we show that $\mathcal{F}$ is satiated by verifying the axioms (S1–4) of [29, Definition 4.1].

Axiom (S1) is clear. For (S2), we take $G \in \mathcal{F}$, $\mu \in r(G)\Lambda \setminus G\Lambda$, and write $d(\mu) = (m, n) \in \mathbb{N}^I \times \mathbb{N}^K$. Choose a finite exhaustive subset $F$ of $G$ lying in $d^{-1}(\mathbb{N}^K)$; by (S1) it suffices to show that

$$\text{Ext}(\mu, F) := \{\alpha \in r(F)\Lambda : \text{there are } \lambda \in F, \beta \in \Lambda \text{ such that } (\alpha, \beta) \in \Lambda^{\text{min}}(\mu, \lambda)\}$$

is in $\mathcal{F}$, because $\text{Ext}(\mu, G)$ is larger. Suppose $\lambda \in F$ and $(\alpha, \beta) \in \Lambda^{\text{min}}(\mu, \lambda)$. Then $d(\mu) \lor d(\lambda) = (m, \max(n, d(\lambda)))$. If $n \geq d(\lambda)$, then $\alpha = s(\mu)$ trivially has degree in $\mathbb{N}^K$; if $n \leq d(\lambda)$, then $d(\alpha) = (0, n - d(\lambda)) \in \mathbb{N}^K$. Since $\text{Ext}(\mu, F)$ is finite exhaustive [29, Lemma 2.3], we have $\text{Ext}(\mu, F) \in \mathcal{F}$. Thus $\mathcal{F}$ has property (S2).

For (S3), it again suffices to consider $F \in \mathcal{F} \cap d^{-1}(\mathbb{N}^K)$. Then any set

$$\{\lambda(0, n_\lambda) : \lambda \in F, n_\lambda \neq 0, n_\lambda \leq d(\lambda)\}$$

of initial segments is finite exhaustive and lies in $\mathcal{F}$, as required. Next we check (S4), which involves choosing a subset $G'$ of $G$ and replacing each $\lambda \in G'$ with a set of the form $\lambda G_\Lambda$ where $G_\Lambda \subset s(\lambda)\Lambda$ is finite exhaustive. By (S1) it suffices to work with $G$ and all $G_\Lambda$ contained in $d^{-1}(\mathbb{N}^K)$. But then the new set $(G' \setminus G') \cup \left(\bigcup_{\lambda \in G'} \lambda G_\Lambda\right)$ is in $d^{-1}(\mathbb{N}^K)$. We have now shown that $\mathcal{F}$ is satiated.

For $i \in K$, the sets $v\Lambda^{e_i}$ are finite exhaustive and lie in $\mathcal{F}$, so $\overline{\mathcal{E}} \subset \mathcal{F}$. To see that $\overline{\mathcal{E}} = \mathcal{F}$, we take a satiated set $\mathcal{G}$ containing $\mathcal{E}$, and show that $\mathcal{F} \subset \mathcal{G}$. Take $G \in \mathcal{F}$.
Then for each \( i \in K \), the set \( r(G) \Lambda^e_i \) belongs to \( G \). Thus by (S2) and (S4), so does the set \( \bigcup_{e \in r(G) \Lambda^e_i} \text{Ext}(e, G) \). For each \( \mu \in \text{Ext}(e, G) \) either \( e\mu \) is in \( G \) or \( e\mu \) has the form \( \lambda f \) for some \( \lambda \in G \) with \( d(\lambda) \wedge e_i = 0 \) and \( f \in s(\lambda) \Lambda^e_i \). Now removing all such \( f \) from \( \lambda f \in r(G) \Lambda^e_i \) gives us \( G \) back, and by (S3) gives us another element of the satiated set \( G \). Thus \( G \in \tilde{G} \).

Now Corollary 5.6 of [29] implies that \( C^\ast(\Lambda; \tilde{\mathcal{F}}) = C^\ast(\Lambda; \tilde{\mathcal{E}}) = C^\ast(\Lambda; \mathcal{F}) \). Thus \( C^\ast(\Lambda; \mathcal{E}) \) has an \( \mathcal{F} \)-compatibility boundary-path representation as in [29] Lemma 4.6.

**Proposition A.4.** The set \( \partial(\Lambda; \mathcal{F}) \) of \( \mathcal{F} \)-compatibility boundary paths is the same as the set \( \partial^R \Lambda \), and the semi-infinite path representation is the \( \mathcal{F} \)-compatibility boundary-path representation of Sims in [29].

**Proof.** Suppose \( x : \Omega_{k,m} \to \Lambda \) is in \( \partial(\Lambda; \mathcal{F}) \). The collection \( \mathcal{F} \) contains every \( v\Lambda^e_i \) with \( i \in K \). We can apply the definition of boundary path in [29] Definition 4.3 with \( n = m \) and \( E = v\Lambda^e_i \) only if \( m + e_i \leq m \). Hence \( m_i = \infty \) for all \( i \in K \), and \( \partial(\Lambda; \mathcal{F}) \subset \partial^R \Lambda \).

Conversely, suppose \( x \in \Lambda^{p,\infty_k} \) for some \( p \in \mathbb{N}^d \), that \( n \leq (p, \infty_k) \) and that \( G \in \mathcal{F} \) has \( r(G) = x(n) \). Then \( G \) contains a finite exhaustive set \( F \) lying in \( d^{-1}(\mathbb{N}^k) \), and there exists \( \lambda \in F \) with \( \lambda = x(n, n + d(\lambda)) \). Hence \( x \) is an \( \mathcal{F} \)-compatibility boundary path.

Since \( \ell^2(\partial(\Lambda; \mathcal{F})) = \ell^2(\partial^R \Lambda) \), the formula in Definition 4.5 of [29] shows that the boundary-path family \( S_\mathcal{F} \) on \( \ell^2(\partial(\Lambda; \mathcal{F})) \) is the family in Proposition 3.5. \( \square \)

### Appendix B. A Groupoid Model for the Toeplitz Algebra of a Higher-Rank Graph

In [21], Neshveyev studies KMS states of \( C^\ast \)-algebras of \( \pi \)-algebras arising from continuous \( \mathbb{R} \)-valued cocycles on the groupoids. Both the Toeplitz algebra and the Cuntz-Krieger algebra of a \( k \)-graph admits such groupoid models. Here we claim that the measure \( \mu \) constructed in Corollary 5.1 is the measure on the unit space of the groupoid of the Toeplitz algebra predicted by [21] Theorem 1.3].

The groupoid of the Toeplitz algebra of a \( k \)-graph. We start by using Yeend’s work [35] to show that there is a groupoid model for the Toeplitz algebra of a \( k \)-graph. With the exception of the two appendices, we have worked with finite \( k \)-graphs only. Here we consider a row-finite \( k \)-graph since the extra generality does not cause any technical problems and may be of independent interest.

Let \( \Lambda \) be a row-finite \( k \)-graph. By Theorems 3.1 and 3.2 of [31], the collection of \( Z(\Lambda \setminus G) \) defined at (3.1) form a basis for a locally compact Hausdorff topology on the path space \( W_\Lambda \). For \( x \in W_\Lambda \) and \( n \leq d(x) \), we define \( \sigma^n(x) \in \Lambda^{d(x)-n} \) by \( \sigma^n(x)(p, q) = (x+n, q+n) \); this gives a partially defined shift map \( \sigma \) on \( W_\Lambda \). The set

\[
\mathcal{G} := \{(x, p - q, y) : x, y \in W_\Lambda, p \leq d(x), q \leq d(y) \text{ and } \sigma^p(x) = \sigma^q(y)\}
\]

is a groupoid with range and source maps \( r(x, g, y) = (x, 0, x) \) and \( s(x, g, y) = (y, 0, y) \), partially defined multiplication \( (x, g, y)(y, h, z) = (x, g+h, z) \) and inverses \( (x, g, y)^{-1} = (y, -g, x) \).

For \( \lambda, \eta \in \Lambda \) with \( s(\lambda) = s(\eta) \), we define

\[
Z(\lambda * \eta) = \{(x, d(\lambda) - d(\eta), y) \in \mathcal{G} : x \in Z(\lambda), y \in Z(\eta) \text{ and } \sigma^{d(\lambda)}(x) = \sigma^{d(\eta)}(y)\}
\]

For finite \( G \subset s(\lambda) \Lambda \), we set

\[
Z(\lambda * \eta \setminus G) = Z(\lambda * \eta) \setminus \bigcup_{x \in G} Z(\lambda \tau * \eta \tau).
\]

There is an alternative groupoid model based on an inverse-semigroup action in [8].
We identify the unit space $G^{(0)}$ with $W_{\Lambda}$ via $(x, 0, x) \mapsto x$. The corresponding identification of $C_0(W_{\Lambda})$ with $C_0(G^{(0)})$ sends $\chi_{Z(\lambda)}$ to $\chi_{Z(\lambda \epsilon)}$

The following proposition follows from Yeend’s results for topological higher-rank graphs [35].

**Proposition B.1.** Let $\Lambda$ be a row-finite $k$-graph. The sets $Z(\lambda \epsilon \eta \setminus G)$ form a basis of compact open sets for a locally compact second-countable Hausdorff topology on $G$ under which it is an étale topological groupoid. The set $\Lambda^\infty$ is a closed $G$-invariant subset of $G^{(0)}$. There is an isomorphism $\pi : TC^*(\Lambda) \to C^*(G)$ such that $\pi(t_\lambda) = \chi_{Z(\lambda \epsilon \eta)}$, and $\pi$ restricts to an isomorphism of $\text{span}\{1, t_\lambda^* : \lambda \in \Lambda\}$ onto $C_0(G^{(0)})$.

Let $q_\infty : C^*(G) \to C^*(G|_{\Lambda^\infty})$ be the quotient map induced by restriction of functions. Then $q_\infty \circ \pi$ factors through an isomorphism $\tilde{\pi} : C^*(\Lambda) \to C^*(G|_{\Lambda^\infty})$.

**Proof.** Since $\Lambda$ is row-finite, it is finitely aligned as in [28]. Thus, as a topological $k$-graph with the discrete topology, it is compactly aligned by [35] Remark 2.4. By Proposition 3.6 of [35], the sets $Z(\lambda \epsilon \eta \setminus G)$ form a basis for a second-countable Hausdorff topology on $G$. Since $\Lambda$ is compactly aligned, it follows from [35] Proposition 3.15 that the $Z(\lambda \epsilon \eta)$ are compact, and then that the topology is locally compact. The closed subsets $Z(\lambda \epsilon \eta \setminus G) \subset Z(\Lambda)$ are therefore also compact. By [35] Theorem 3.16, $G$ is an étale topological groupoid.

To see that $\Lambda^\infty$ is closed, we show that the complement is open. Let $x \in W_{\Lambda} \setminus \Lambda^\infty$. Choose $n \in \mathbb{N}^k$ such that $n_i > d(x)_i$ for some $i$. Since $\Lambda$ is row-finite, $x(0)\Lambda^n$ is finite and $Z(x(0) \setminus x(0)\Lambda^n) \subset W_{\Lambda} \setminus \Lambda^\infty$ is an open neighbourhood of $x$. Thus $\Lambda^\infty$ is closed.

To see that $\Lambda^\infty$ is $G$-invariant, let $x \in \Lambda^\infty$. Then

$$r(s^{-1}(\{x\})) = \{y \in W_{\Lambda} : \text{there are } m, n \in \mathbb{N}^k \text{ with } m \leq d(y) \text{ and } \sigma^m(y) = \sigma^n(x)\}.$$  

But each such $\sigma^m(y) = \sigma^n(x)$ is in $\Lambda^\infty$, and hence $y \in \Lambda^\infty$ as well. Thus $r(s^{-1}(\{x\})) \subset \Lambda^\infty$, and so $\Lambda^\infty$ is $G$-invariant.

A straightforward calculation shows that $\{\chi_{Z(\lambda \epsilon \eta \setminus G)} : \lambda \in \Lambda\}$ is a Toeplitz-Cuntz-Krieger $\Lambda$-family. Thus there is a homomorphism $\pi : TC^*(\Lambda) \to C^*(G)$ such that $\pi(t_\lambda) = \chi_{Z(\lambda \epsilon \eta \setminus G)}$. It is easy to verify that $\pi(t_\lambda t_\lambda^*) = \chi_{Z(\lambda)}$.

Each

$$\chi_{Z(\lambda \epsilon \eta \setminus G)} = \pi\left(t_\lambda \left(\prod_{\tau \in G} (t_{\tau(s_\tau)} - t_{\tau t_\lambda^*}) t_{\eta}\right)\right)$$

belongs to the range of $\pi$, so the Stone-Weierstrass theorem implies that $\pi$ has dense range and hence is surjective. Since $G$ is étale, $C_0(G^{(0)})$ embeds in $C^*(G)$. Thus each $\pi(t_v) = \chi_{Z(v)}$ is nonzero, and for a finite $G \subset v\Lambda \setminus \{v\}$, we have

$$\pi\left(\prod_{\lambda \in G} (t_v - t_{\lambda t_\lambda^*})\right) = \chi_{Z(v \setminus G)} \neq 0.$$  

Now [27] Theorem 8.1, applied to the product system $E_\Lambda$ of graphs as defined in [27] Example 3.1, implies that $\pi$ is injective. Thus $\pi : TC^*(\Lambda) \to C^*(G)$ is an isomorphism.

For $v \in \Lambda^\Lambda$ and $n \in \mathbb{N}^k$, we have $q_\infty \circ \pi(t_v - \sum_{\lambda \in v \Lambda^\Lambda} t_\lambda t_\lambda^*) = \chi_{Z(v \setminus v \Lambda^\Lambda \Lambda \setminus v)} = 0$. So $q_\infty \circ \pi$ factors through a homomorphism $\tilde{\pi} : C^*(\Lambda) \to C^*(G|_{\Lambda^\infty})$ such that $\tilde{\pi}(s_\lambda)$ is the characteristic function $\chi_{Z(\lambda \epsilon \eta \setminus G)}$. This is precisely the homomorphism of [17] Corollary 3.5(i)], and hence is an isomorphism.

[\[\]The measure of Corollary 5.4 and Neshveyev’s theorem. Let $r \in (0, \infty)^k$. There is a locally constant cocycle $c : G \to \mathbb{R}$ given by $c(x, n, y) = r \cdot n$. This cocycle induces a dynamics $\alpha^c : \mathbb{R} \to \text{Aut} C^*(G)$ such that $\alpha^c_t (f)(x, n, y) = e^{itc(x, n, y)} f(x, n, y)$ for $f \in C^*(G)$.\]
$C_c(\mathcal{G})$. The isomorphism $\pi : \mathcal{T}C^*(\Lambda) \to C^*(\mathcal{G})$ of Proposition 3.1 intertwines $\alpha^c$ and the dynamics $\alpha^r$ we have been using.

For $x \in \mathcal{G}^{(0)}$, write $\mathcal{G}_x^c$ for the stability subgroup $\{(x,n,x) \in \mathcal{G} : n \in \mathbb{Z}^k\}$ and $\mathcal{G}_x$ for the subset $\{(y,n,x) \in \mathcal{G} : y \in \Lambda_\alpha, n \in \mathbb{Z}^k\}$ of $\mathcal{G}$. Theorem 1.3 of [21] describes the KMS$_1$ states of $(C^*(\mathcal{G}),\alpha^c)$ in terms of pairs $(\mu',\psi)$ consisting of a quasi-invariant probability measure $\mu'$ on $\mathcal{G}^{(0)}$ with Radon-Nikodym cocycle $e^{-c}$ and a $\mu'$-measurable field $\psi = (\psi_x)_{x \in \mathcal{G}^{(0)}}$ of states $\psi_x : C^*(\mathcal{G}_x^c) \to \mathbb{C}$ such that for $\mu'$-almost all $x \in \mathcal{G}^{(0)}$ we have

$$\psi_x(u_g) = \psi_{r(l)}(u_{g'nh^{-1}})$$

for all $g \in \mathcal{G}_x^c$ and $h \in \mathcal{G}_x$.

Now let $\phi$ be the KMS$_1$ state of $\mathcal{T}C^*(\Lambda)$ from Theorem 5.1(c) and $\mu$ the measure from Corollary 5.4. By [21, Theorem 1.3], $\phi \circ \pi^{-1}$ is implemented by a unique pair $(\mu',\psi)$ as above. Burrowing into the proof of [21 Theorem 1.1] shows that $\mu'$ is the measure on $\mathcal{G}^{(0)}$ implementing the functional $\phi \circ \pi^{-1}|_{C^0(\mathcal{G}^{(0)})}$. For $\lambda \in \Lambda$ we have

$$\mu'(Z(\lambda)) = \phi \circ \pi^{-1}(X_{Z(\lambda)}) = \phi(t_\Lambda t_\lambda^*) = \mu(Z(\lambda))$$

by (B.8). Thus $\mu'$ is the extension to $W_\Lambda$ of the measure $\mu$ on $\partial^\Lambda \Lambda$.

Henceforth we view $\mu$ as a measure on $W_\Lambda$, and then Lemma B.2 applies to this $\mu$. The lemma says that the quasi-invariance of the measure $\mu$ has consequences for the dynamics $\alpha^r$ and forces it to have support in $\partial^\Lambda \Lambda$ where $K = \{ r : r_1 = \ln \rho(A_i) \}$.

**Lemma B.2.** Suppose that $\Lambda$ is a finite coordinatewise irreducible $k$-graph. Suppose that $\mu$ is a nonzero quasi-invariant probability measure on $\mathcal{G}^{(0)} = W_\Lambda$ with Radon-Nikodym cocycle $e^{-c}$.

(a) Then $r \geq \ln \rho(\Lambda) := (\ln \rho(A_1), \ldots, \ln \rho(A_k))$.

(b) Let $l \in \{1, \ldots, k\}$. If $\mu(\bigcup_{n_l = \infty} \Lambda^n) \neq 0$, then $r_l = \ln \rho(A_l)$. In particular, if $\mu(\Lambda^\infty) \neq 0$ then $r = \ln \rho(\Lambda)$.

**Proof.** Let $v \in \Lambda^0$ and $\lambda \in v\Lambda^{e^r}$. Then $Z(\lambda) \subset Z(v)$. Since $Z(\lambda * s(\lambda))$ is a bisection of $\mathcal{G}$ with $r(Z(\lambda * s(\lambda))) = Z(\lambda)$ and $s(Z(\lambda * s(\lambda))) = Z(s(\lambda))$, the quasi-invariance of $\mu$ gives

$$\mu(Z(\lambda)) = e^{-d(\lambda;r)} \mu(Z(s(\lambda))) = e^{-r_l} \mu(Z(s(\lambda))).$$

Let $i \in \{1, \ldots, k\}$. Then

$$\mu(Z(v)) \geq \mu(\bigcup_{w \in \Lambda^0} \bigcup_{\lambda \in v\Lambda^{e^r} \cap w} Z(\lambda)) = \sum_{w \in \Lambda^0} \sum_{\lambda \in v\Lambda^{e^r} \cap w} \mu(Z(\lambda))$$

$$= \sum_{w \in \Lambda^0} e^{-r_l} A_l(v, w) \mu(Z(w)).$$

Set $m^+ := (\mu(Z(v))) \in [0, \infty)^{\Lambda^0}$. Then (B.1) says that $m^+$ satisfies $e^{r_l}m^+ \geq A_m m^+$. Since $m^+ \in [0, \infty)^{\Lambda^0}$ is nonzero and $A_m$ is irreducible, the subinvariance theorem 3.10 implies that $e^{r_l} \geq \rho(A_l)$. Thus $r \geq \ln \rho(\Lambda)$, giving (iii).

For (i), suppose that $X := \bigcup_{n_l = \infty} \Lambda^n$ has nonzero measure. Set

$$m := (\mu(Z(v) \cap X))_{v \in \Lambda^0}.$$

Since $\mu(X) \neq 0$, there exists $u \in \Lambda^0$ such that $m_u > 0$, and hence $m \geq 0$. Let $v \in \Lambda^0$. Since $x \in Z(v) \cap X$ implies $d(x) = \infty$ we have

$$\mu(Z(v) \cap X) = \mu(\bigcup_{w \in \Lambda^0} \bigcup_{\lambda \in v\Lambda^{e^r} \cap w} Z(\lambda) \cap X) = \sum_{w \in \Lambda^0} e^{-r_l} A_l(v, w) \mu(Z(w) \cap X).$$

Thus $e^{r_l}m = A_m$. By definition of spectral radius, $e^{r_l} \leq \rho(A_l)$. Now (iii) gives $e^{r_l} = \rho(A_l)$ and $r_l = \ln \rho(A_l)$.

Next observe that $\Lambda^\infty = \bigcap_i \bigcup_{n_i = \infty} \Lambda^n$. If $n_l > \ln \rho(A_l)$ then $\mu(\bigcup_{n_l = \infty} \Lambda^n) = 0$, and hence $\mu(\Lambda^\infty) = 0$ as well. This gives (iii). \qed
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