ERGODIC PROPERTIES OF FOLDING MAPS ON SPHERES

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Abstract. We consider the trajectories of points on \( S^{d-1} \) under sequences of certain folding maps associated with reflections. The main result characterizes collections of folding maps that produce dense trajectories. The minimal number of maps in such a collection is \( d + 1 \).

1. Introduction. The subject of this paper is the reconstruction of full radial symmetry from partial information. A function on \( \mathbb{R}^d \) is radial if and only if it is symmetric under reflection about arbitrary hyperplanes through the origin. Since the orthogonal group \( O(d) \) acts transitively and faithfully on \( \mathbb{R}^d \), an equivalent statement is that the reflections generate \( O(d) \).

It is well-known that a finite set of reflections suffices to generate a dense subgroup of \( O(d) \). In the plane, the composition of two reflections is a rotation by twice the enclosed angle. If the angle is incommensurable with \( \pi \), then the multiples of the rotation are dense in the circle. Likewise in dimension \( d > 2 \), the reflections at \( d \) hyperplanes in general position generate a dense subgroup of the orthogonal group. From any given starting point, the subset of points that can be reached by a suitable composition of these reflections is dense in the centered sphere that contains the point. The corresponding random walk is almost surely equidistributed on the sphere.

Here we study the corresponding issues for a family of piecewise isometries that fold the sphere across a hyperplane onto a hemisphere. We label a folding map by the unit vector in the direction of the target hemisphere, and consider sequences
of such maps indexed by a given set of directions $G \subset S^{d-1}$. The principal aim of this article is to answer the question: When can a dense subset of points in $S^{d-1}$ be reached from an arbitrary starting position by applying folding maps with directions chosen from $G$? The question came up in prior work on the convergence of random sequences of symmetrizations [8]. Our motivation will be discussed in Section 3.

It turns out that a pair of obvious necessary conditions on the set of directions — one geometric and one algebraic — is also sufficient (Theorem 1). If a set of directions meets the conditions, then the only functions on $\mathbb{R}$ — one geometric and one algebraic — is also sufficient (Theorem 1). If a set of directions meets the conditions, then the only functions on $\mathbb{R}$ that increase under composition with each of the corresponding folding maps are radial. Furthermore, the random walk generated by randomly alternating these maps is uniquely ergodic (Theorem 2). The invariant measure gives positive mass to all non-empty open subsets, but does not agree, in general, with the uniform measure on $S^{d-1}$.

2. Main results. We begin with some definitions. Let $S^{d-1}$ be the standard sphere, viewed as the set of unit vectors in $\mathbb{R}^d$. The geodesic distance $d(x, y)$ on $S^{d-1}$, given by the enclosed angle between $x$ and $y$, is related to the chordal distance in $\mathbb{R}^d$ by

$$|x - y|^2 = 2 - 2 \cos d(x, y).$$

The symbol $O(d)$ refers to the group of orthogonal linear transformations on $\mathbb{R}^d$, and $SO(d)$ to the orientation-preserving subgroup.

The uniform probability measure on the sphere induced by Lebesgue measure on $\mathbb{R}^d$ is denoted by $\sigma$. It will be used as a reference measure throughout the paper. By a null set we mean a subset $A \subset S^{d-1}$ with $\sigma(A) = 0$. All sets and functions under consideration are understood to be Borel measurable, and measures are assumed to be regular Borel measures.

For a direction $u \in S^{d-1}$, let $R_u x = x - (2x \cdot u) u$ be the reflection of a point $x \in \mathbb{R}^d$ about the orthogonal hyperplane $u^\perp$. Clearly, $R_u = R_{-u}$. Let $H_u = \{ x \in \mathbb{R}^d \mid x \cdot u > 0 \}$ be the open positive half-space associated with $u$, and define

$$F_u(x) = \begin{cases} x, & \text{if } x \in H_u, \\ R_u x, & \text{otherwise.} \end{cases}$$

We call $F_u$ the folding map in the direction of $u$, as it folds $\mathbb{R}^d$ across the crease at $u^\perp$ onto the closed nonnegative half-space. It is nonlinear and idempotent ($F_u^2 = F_u$). Since $R_u$ is isometric, $F_u$ is non-expansive

$$|F_u(x) - F_u(y)| \leq |x - y|,$$

with strict inequality if $x$ and $y$ lie on opposite sides of $u^\perp$.

The two-point symmetrization of a real-valued function $\phi$ on $S^{d-1}$ is the equimeasurable rearrangement defined by

$$S_u \phi(x) = \begin{cases} \max\{\phi(R_u x), \phi(x)\}, & \text{if } x \in H_u, \\ \min\{\phi(R_u x), \phi(x)\}, & \text{otherwise.} \end{cases}$$

Note that $S_u \phi = \phi$, if and only if $\phi \circ F_u \geq \phi$.

Let $(u_n)_{n \geq 1}$ in $S^{d-1}$ be a sequence of directions in $S^{d-1}$. We study the trajectory of a point $x$ in $\mathbb{R}^d$ or $S^{d-1}$ under the corresponding sequence of folding maps, given by

$$x_0 = x, \quad x_n = F_{u_n} x_{n-1} \text{ for } n \geq 1.$$ (1)
Under what conditions on the sequence \((u_n)\) are these trajectories dense in \(S^{d-1}\)? Our main result shows that it suffices to choose the directions from a small subset \(G \subset S^{d-1}\).

**Theorem 1** (Transitivity). Let \(G \subset S^{d-1}\) be a set of directions. If

(C1) the open half-spaces \(H_u\) with \(u \in G\) cover \(S^{d-1}\), and

(C2) the reflections \(\{R_u \mid u \in G\}\) generate a dense subgroup of \(O(d)\),

then there exists a sequence \((u_n)_{n \geq 1}\) in \(G\) such that for every starting point \(x \in S^{d-1}\),

the trajectory defined by Eq. (1) is dense in \(S^{d-1}\).

Both assumptions on \(G\) are clearly necessary for the existence of even one dense trajectory. In Section 4, we show that the geometric condition (C1) is equivalent to the origin lying in the interior of the convex hull of \(G\) in \(\mathbb{R}^d\) (Proposition 4.1(c)). In particular, \(G\) must contain at least \(d + 1\) directions that span \(\mathbb{R}^d\). Condition (C2) can be replaced by an explicit sufficient condition (Proposition 4.2). Any set of directions that spans \(\mathbb{R}^d\) can be augmented by one more direction to satisfy both conditions (Proposition 4.3).

As for the conclusion, we claim that there is a sequence of directions that produces dense trajectories simultaneously for all starting points on the sphere. A useful consequence of Theorem 1 is the following characterization of radial functions in terms of two-point symmetrizations.

**Corollary 1.** Let \(\phi\) be a continuous function on \(\mathbb{R}^d\), and let \(G \subset S^{d-1}\) satisfy (C1) and (C2). Then

\[
S_u \phi = \phi \quad \text{for all } u \in G \iff \phi \text{ is radial.}
\]

The corollary improves upon the statement that for any non-radial function there exist two-point symmetrizations that push the function towards its symmetric decreasing rearrangement. This observation appears as a key step in Baernstein and Taylor’s proof of Riesz’ rearrangement inequality on the sphere [3, Theorem 2 and p. 252], and many other classical results (see also [4, p. 226], [6, Lemma 6.5], [9, Lemma 2.8], and [22, Lemma 3.11]). The corollary extends directly to measurable functions modulo null sets.

For the proof of Theorem 1 in Section 5, we consider subsets of the sphere that are positively invariant under the folding maps indexed by a set \(G\). Assuming \(G\) satisfies (C1), every positively invariant subset of the sphere is invariant, up to null sets, under the corresponding reflections (Lemma 5.1). Condition (C2) precludes the existence of non-trivial invariant compact subsets of the sphere (Lemmas 5.2 and 5.3). For any given starting point, the union of all possible trajectories under sequences of folding maps indexed by \(G\) is a positively invariant subset. We argue that this union is dense, and then select the desired dense trajectory from it. Our construction yields a sequence of directions \((u_n)\) with the following property: For any \(\varepsilon > 0\), there exists a number \(N\) such that every trajectory intersects every ball of radius \(\varepsilon\) in \(S^{d-1}\) within the first \(N\) steps, see Eq. (5).

In the last two sections, we consider sequences of directions \((U_n)_{n \geq 1}\) that are chosen independently at random from a probability distribution \(\mu\) on \(S^{d-1}\), that is,

\[
P(U_n \in A_n, \text{for } n = 1, \ldots, N) = \prod_{n=1}^{N} \mu(A_n)
\]
for any Borel sets \( A_1, \ldots, A_N \subset S^{d-1} \). The trajectory defined by
\[
X_0 = x, \quad X_n = F_{U_n}X_{n-1} \text{ for } n \geq 1
\]
will be called the random walk generated by \( \mu \) with starting point \( x \). We are most interested in examples where \( \mu \) is concentrated on a finite set of directions.

Let \( G \subset S^{d-1} \) be the support of \( \mu \), that is, the smallest closed subset of full \( \mu \)-measure. If \( G \) satisfies the hypotheses of Theorem 1, then the random walk is almost surely dense (Corollary 5.5). Our second theorem strengthens this observation. To state the result, define a linear transformation on measures by
\[
T_\mu \# \nu(A) = \int_{S^{d-1}} \nu(F_u^{-1}(A)) \, d\mu(u)
\]
for \( A \subset S^{d-1} \), and every Borel measure \( \nu \). By definition, \( T_\mu \# \) maps the probability distribution of \( X_n \) to the distribution of \( X_{n+1} \) for each \( n \geq 0 \).

**Theorem 2** (Unique ergodicity). Let \( \mu \) be a regular Borel probability measure on \( S^{d-1} \) whose support satisfies (C1) and (C2), and let \( T_\mu \# \) be given by Eq. (3). There is a unique regular Borel probability measure \( \rho \) on \( S^{d-1} \) with \( T_\mu \# \rho = \rho \). The support of \( \rho \) is the entire sphere \( S^{d-1} \).

We refer to \( \rho \) as the invariant measure for the random walk. By uniqueness, it assigns zero or one to every invariant subset. The invariant measure governs the behavior of the random trajectories \((X_n)\) through Birkhoff’s ergodic theorem.

**Corollary 2.** Let \( A \subset S^{d-1} \) be a Borel set. Under the assumptions of Theorem 2,
\[
\lim_{N \to \infty} \frac{1}{N} \left| \{ n \leq N \mid X_n \in A \} \right| = \rho(A)
\]
almost surely for \( \rho \)-almost every starting point \( x \in S^{d-1} \).

Thus, typical trajectories are equidistributed according to the invariant measure \( \rho \). Since \( \rho \) gives positive measure to every non-empty open subset, this provides another proof that almost all trajectories are dense.

The proof of Theorem 2 is given in Section 6. To construct the invariant measure, we study the Markov chain associated with the random walk from Eq. 2 through its adjoint action on functions, defined by
\[
\phi_n(x) = E_x(\phi(X_n)),
\]
where \( x \) is the starting point of the random walk. Under the assumptions of Theorem 2, we show that for each continuous function \( \phi \) on the sphere, the sequence \( \phi_n \) converges uniformly to a constant \( \bar{\phi} \) (Proposition 6.2). Writing \( \phi = \int \phi \, d\rho \) for a suitable measure \( \rho \) on the sphere, we show that \( \rho \) is uniquely determined by the map \( \phi \mapsto \bar{\phi} \), and invariant under the transformation \( T_\mu \# \). Since \( \bar{\phi} \) lies strictly between the maximum and minimum of \( \phi \) unless \( \phi \) is constant, we find that \( \rho(A) > 0 \) for every non-empty open set \( A \).

In Section 7 we consider the relationship between the invariant measure \( \rho \) and the uniform probability measure \( \sigma \) on \( S^{d-1} \). If the directions \((U_n)\) are uniformly distributed on the sphere, then \( \rho \) is uniform as well. In that case, Corollary 2 implies that the random trajectories are almost surely equidistributed on the sphere. More generally, the uniform measure is invariant if and only if the distribution of \( U_n \) is even under the antipodal map \( u \mapsto -u \) (Proposition 7.4). It is an open question whether the invariant measure is always mutually absolutely continuous with respect to the uniform measure.
3. Motivation and related work. Symmetrizations are equimeasurable rearrangements of sets and functions which are commonly used for proving that certain optimization problems have radially symmetric solutions. For instance, the sharp constants in the Young, Sobolev, and Hardy-Littlewood-Sobolev inequalities are assumed among radial functions, the ground state of the hydrogen atom is symmetric decreasing, and the membrane of given area with the lowest fundamental frequency is shaped like a disk.

A standard technique for establishing geometric inequalities is to approximate full radial symmetrization by a sequence of simpler symmetrizations. Two-point symmetrization has been used in this way to prove the isoperimetric inequality on spheres [5], and sharp inequalities for path integrals [9, 22, 23]. The convergence of random sequences of symmetrizations has received some attention in the literature, most notably in the work of Klartag [19] on Steiner symmetrizations of convex sets. Convergence of two-point symmetrizations is less well studied. Very recently, De Keyser and Van Schaftingen have considered random symmetrization processes with time correlations [11]. Open questions in the area include precise conditions for convergence, and bounds on the rate of convergence.

To explain the relationship with our results, let \( \phi \) be a nonnegative continuous function with compact support on \( \mathbb{R}^d \). Consider a sequence \((\phi_n)\) of functions obtained from \( \phi \) by iterated Steiner symmetrization according to a sequence of directions. Any limit point \( \psi \) of \((\phi_n)\) has at least some reflection symmetries, that is, \( \psi \circ R_u = \psi \) for all \( u \) in a non-empty set of directions \( G \subset S^{d-1} \). If \( G \), which is determined by the sequence of Steiner symmetrizations, satisfies (C2), it follows that \( \psi \) is radial. Then one can identify \( \psi \) as the symmetric decreasing rearrangement of \( \phi \) and conclude that the entire sequence of symmetrizations converges to \( \psi \) [8, Corollary 2.3b]. This provides a sharp condition for an i.i.d. sequence of Steiner symmetrizations to converge to the symmetric decreasing rearrangement.

If, instead, \( \phi_n \) is obtained by iterated two-point symmetrization at hyperplanes in \( \mathbb{R}^d \) that do not contain the origin, then any limit point \( \psi \) of the sequence has the property that \( S_u \psi = \psi \), that is, \( \psi \circ F_u \geq \psi \) for all \( u \) in a non-empty set of directions \( G \subset S^{d-1} \). If \( G \) satisfies (C1) and (C2), then according to Corollary [1] the function \( \psi \) is radial. In combination with [8, Theorem 2.2] this yields a sharp condition for the convergence of i.i.d. sequences of two-point symmetrizations, the first result of this type. Our hope is that a full analysis of the random walk defined in Eq. (2) would shed light on the best achievable rate of convergence. We note that for non-convex sets, the strongest known lower bound on the rate of convergence of Steiner symmetrizations was proved by comparison with two-point symmetrization [8, Corollary 5.4].

There are many known results analogous to Theorem [1] for other collections of maps associated with linear isometries of spheres. For example, Crouch and Silva Leite [26] found pairs of one-parameter subgroups of \( SO(d) \) which generate all of \( SO(d) \), and produced upper bounds on the number of elements from each subgroup required to generate any element (see also Levitt and Sussmann [20] for related results). An example of this phenomenon is the Euler angles decomposition, a formula for writing any element of \( SO(3) \) as a product of three rotations about the \( x \) and \( y \) axes. Rosenthal [25] has established bounds on the rate of convergence to the steady state for random walks generated by conjugacy classes of planar rotations on \( SO(d) \). Porod [24] has obtained analogous results for random walks on \( O(d), U(d) \) and \( Sp(d) \) generated by the conjugacy class of reflections.
All of the above results can be translated into statements about trajectories on \( S^{d-1} \) via the standard actions of the groups. Historically, they were preceded by results on the symmetric group \( S_n \), with transpositions playing the role of reflections. Dixon [13] proved that the probability that two randomly selected elements of the symmetric group \( S_n \) generate the whole group approaches \( \frac{3}{4} \) as \( n \to \infty \). Diaconis and Shahshahani [12] studied random walks on the symmetric group generated by random transpositions and developed methods for estimating the rate of convergence.

Since a folding map agrees with the identity map on one half-space, and with a reflection on its complement, it is a piecewise isometry. The dynamics of piecewise isometries have been studied in various contexts by numerous authors, for example see the survey by Goetz [17]. Different from interval exchange maps, our folding maps are neither one-to-one nor onto.

On an abstract level, our results are motivated by classical theorems about directed graphs, and more generally Markov chains on discrete state spaces. A graph is **transitive** if every vertex \( x \) can be connected to every other vertex \( y \) by a path of directed edges. Finding a minimal subset of edges that still connects all vertices is known to be a hard problem, particularly if the graph has cycles. As a practical alternative, Aho, Garey, and Ullmann developed the theory of transitive reductions, which are graphs on the same vertex set with the smallest possible number of edges [1]. In our setting, the points \( x \in S^{d-1} \) play the role of vertices, and a pair \((x, u)\) with \( u \in G \) plays the role of a directed edge connecting \( x \) to \( F_u x \). Edge paths correspond to trajectories, and transitivity means that all trajectories are dense. Theorem 1 gives necessary and sufficient conditions for transitivity, Proposition 4.3 constructs a transitive reduction, and Proposition 5.4 establishes the presence of cycles. The random walk defined in Eq. (2) corresponds to the Markov chain defined by a weighted directed graph, with \( \mu \) playing the role of the edge weights. Theorem 2 and Corollary 2 concern the steady-state of the Markov chain.

Another parallel can be drawn between Eq. (2) and shift dynamics. Suppose that the probability density \( \mu \) is concentrated on a finite set \( G \) of directions. For a given initial point, \( x \in S^{d-1} \), if we omit from the underlying sequence of directions those terms where the folding map acts as the identity (rather than by reflection), we obtain a random sequence in \( G \) where only certain transitions are admissible. The conditions for admissibility, however, depend on the current location of the random walk in \( S^{d-1} \); for this reason the sequence of directions is not simply a subshift of finite type.

### 4. Conditions (C1) and (C2)

We now discuss the hypotheses of the main theorems. The geometric condition (C1) can be expressed in a number of different forms. Note that only parts (c) and (d) of Proposition 4.1 are needed subsequently.

**Proposition 4.1.** Let \( G \subset S^{d-1} \). The following are equivalent:

(a) The open half-spaces \( H_u \) with \( u \in G \) cover \( S^{d-1} \);

(b) \( G \) is not contained in any closed hemisphere of \( S^{d-1} \);

(c) the convex hull of \( G \) in \( \mathbb{R}^d \) contains the origin in its interior.

Let \( G \) be the support of a Borel probability measure \( \mu \). Then the above conditions hold if and only if

(d) \( 0 < \mu(H_x) < 1 \) for all \( x \in S^{d-1} \).
\begin{proof}
\((a)\Rightarrow(b)\): If \(G\) satisfies (a), then for every \(x \in \mathbb{S}^{d-1}\), there is a direction \(u \in G\) such that \(-x \in H_u\). This means that \(x \cdot u < 0\), implying that \(G\) is not contained in the closed hemisphere \(\{ u \in \mathbb{S}^{d-1} \mid x \cdot u \geq 0 \}\). Since \(x\) was arbitrary, this shows (b).

\((b)\Rightarrow(c)\): If (c) does not hold, then there is a hyperplane through the origin that does not meet the interior of the convex hull of \(G\). Therefore, \(G\) is contained in a closed half-space, which intersects \(\mathbb{S}^{d-1}\) in a closed hemisphere, contradicting (b).

\((c)\Rightarrow(a)\): Let \(C\) be the convex hull of \(G\) in \(\mathbb{R}^d\). Given a point \(x \in \mathbb{S}^{d-1}\), consider the linear functional defined by \(\ell(y) = x \cdot y\) on \(\mathbb{R}^d\). If the origin is an interior point of \(C\), then \(\ell\) takes both positive and negative values on \(C\). Since \(\ell\) assumes its maximum at an extreme point of \(C\), and the extreme points of \(C\) are contained in the closure of \(G\), we conclude that \(\ell(u) > 0\) for at least one \(u \in G\). Thus \(x \in H_u\), establishing (a).

\((a)\Leftrightarrow(d)\): Let \(\mu\) be a Borel probability measure on \(\mathbb{S}^{d-1}\), and \(x \in \mathbb{S}^{d-1}\). Assuming (a), the support of \(\mu\) contains a direction \(u\) with \(x \in H_u\), that is, \(u \in H_x\). Since \(H_x\) is open, it follows that \(\mu(H_x) > 0\), proving the first inequality in (d). Replacing \(x\) with \(-x\) shows that \(1 - \mu(H_x) \geq \mu(H_{-x}) > 0\), proving the second inequality in (d). The converse implication is obvious. \end{proof}

Denote by \(\langle G \rangle := \{ R_{u_n} \ldots R_{u_1} \in O(d) \mid n \geq 0, u_1, \ldots, u_n \in G \}\) the subgroup generated by the reflections \(R_u\) at hyperplanes with unit normals \(u \in G\). The algebraic condition (C2) can fail in three ways: If \(\langle G \rangle\) lies in a lower-dimensional subgroup, if it splits into two subgroups that act on orthogonal subspaces, or if \(\langle G \rangle\) is a finite Coxeter subgroup of \(O(d)\). Thus, assumptions (1) and (2) in the following proposition are also necessary. In dimension \(d > 2\), assumption (3) is stronger than necessary, because only integer multiples of \(\frac{\pi}{2}\), \(\frac{\pi}{4}\), and \(\frac{\pi}{3}\) can appear as angles between elements of a finite Coxeter subgroup that acts on \(\mathbb{R}^d\) \cite{Felikson}. Theorem 9 and Proof of Lemma 4.2]. For \((d+1)\)-element sets \(G \subset \mathbb{S}^d\), a precise condition for generating finite Coxeter groups was obtained by Felikson \cite{Felikson}; for more general sets of reflections in \(\mathbb{R}^d\) this is an open problem. A set of conditions slightly stronger than (1), (2), and (3) was used by Eggleston \cite{Eggleston} proof of Theorem 46] and Klain \cite{Klain} after Corollary 5.4] to establish convergence to balls for sequences of Steiner symmetrizations.

\begin{proposition}
Let \(G \subset \mathbb{S}^{d-1}\). If (1) \(G\) spans \(\mathbb{R}^d\), (2) \(G\) is not a union of two non-empty mutually orthogonal subsets, and (3) not all angles between directions are commensurable with \(\pi\), then the subgroup \(\langle G \rangle\) is dense in \(O(d)\).
\end{proposition}

\begin{proof}
We argue by induction over the dimension. For \(d = 1\) there is nothing to show. For \(d = 2\), by (1) and (3) there are directions \(u, v \in G\) such that the angle \(d(u, v)\) is incommensurable with \(\pi\). Since the composition \(R_u R_v\) generates a dense subgroup of rotations in \(SO(2)\), it follows that \(\langle \{ u, v \} \rangle\) is dense in \(O(2)\).

Suppose now that the proposition holds in dimension \(d - 1\), where \(d > 2\). If \(G \subset \mathbb{S}^{d-1}\) satisfies (1), (2), and (3), then it contains a subset \(G'\) that spans some hyperplane \(v^\perp\) in \(\mathbb{R}^d\) and also satisfies (2) and (3). Let \(S_v\) be the orthogonal group on \(v^\perp\). By the inductive hypothesis, \(\langle G' \rangle\) is dense in \(S_v\). Note that \(S_v\) has a unique pair of fixed points at \(\pm v\), and acts transitively on the unit sphere in \(v^\perp\).
By (1) and (2) there is a direction \( u \in G \) that is linearly independent but not orthogonal to \( G' \). In particular, \( w = R_u v \) is linearly independent of \( v \). Therefore the conjugate subgroup \( S_w = R_u S_v R_u^{-1} \) is different from \( S_v \). Intersecting \( S_v \) and \( S_w \) with \( SO(d) \), we obtain two distinct subgroups conjugate to \( SO(d-1) \times \{1\} \) in \( \langle G \rangle \). But \( SO(d) \) has no non-trivial compact subgroup that properly contains a copy of \( SO(d-1) \) \cite{21} Lemma 4. It follows that \( \langle G \rangle \) contains \( SO(d) \). Since it also contains orientation-reversing elements, \( \langle G \rangle = O(d) \).

Our next results concern finite subsets of \( S^{d-1} \).

**Proposition 4.3.** The minimal number of directions in \( G \subset S^{d-1} \) required to satisfy (C1) and (C2) is \( d + 1 \).

**Proof.** Condition (C1) requires at least \( d + 1 \) points, the minimal number of extreme points for a convex set with interior in \( \mathbb{R}^d \).

To construct a set of \( d + 1 \) points that satisfies both (C1) and (C2), we start from a basis of unit vectors \( u_1, \ldots, u_d \) for \( \mathbb{R}^d \). The convex cone generated by this basis has non-empty interior. We choose a unit vector \( u_{d+1} \) whose antipode \( -u_{d+1} \) lies in the interior of the cone, and, moreover, \( u_{d+1} \) encloses an angle with \( u_1 \) that is not a rational multiple of \( \pi \), and set \( G = \{u_1, \ldots, u_{d+1}\} \). Then (C1) holds by Proposition 4.1(c), and (C2) by Proposition 4.2.

We do not have a good characterization of minimal subsets (under inclusion) satisfying (C1) and (C2). Such minimal subsets may contain more than \( d + 1 \) directions. For example, both conditions hold for \( G = \{u_1, -u_1, \ldots, u_d, -u_d\} \), where \( u_1 \) and \( u_2 \) enclose an angle incommensurable with \( \pi \) and \( u_1, \ldots, u_d \) form a basis of unit vectors in \( \mathbb{R}^d \), but every proper subset is contained in a closed hemisphere and thus fails (C1).

There are sets in \( S^1 \) satisfying (C1) and (C2) that have no minimal subsets. For example, the set \( G = \{\pm e^{i\pi/k} \mid k \geq 1\} \subset S^1 \) satisfies (C1) and (C2), and so does every infinite even subset, while every finite subset generates a finite dihedral subgroup of \( O(1) \). Our next result shows that in higher dimensions, minimal subsets always exist, and are finite.

**Proposition 4.4.** Let \( d > 2 \). If \( G \subset S^{d-1} \) satisfies (C1) and (C2), then there is a finite subset \( G' \subset G \) that also satisfies both conditions.

**Proof.** If \( G \) is finite, there is nothing to show. If \( G \) is infinite, we choose a small \( d \)-dimensional simplex inside the convex hull of \( G \) in \( \mathbb{R}^d \) that contains the origin in its interior. By the Krein-Milman theorem, each vertex of the simplex is a convex combination of a finite subset of \( G \). The convex hull of the union of these subsets contains the origin in its interior. By Proposition 4.1(c), it satisfies condition (C1). To obtain \( G' \), we include up to two more elements of \( G \) in the union in such a way that \( G' \) does not split into mutually orthogonal subsets, and that it contains a pair of vectors enclosing an angle that is not an integer multiple of \( \frac{\pi}{4} \), \( \frac{\pi}{6} \), or \( \frac{\pi}{3} \). In dimension \( d > 2 \), this implies that \( G' \) is not contained in a finite Coxeter group. Thus (C1) and (C2) both hold for \( G' \).

5. **Invariant subsets and dense trajectories.** This section is dedicated to the proof of Theorem 1 and Corollary 1. Let \( F_u \) and \( R_u \) be the folding map and the reflection defined by a direction \( u \in S^{d-1} \). By definition, a set \( A \subset \mathbb{R}^d \) is positively 

\[ F_u(A) \subset A; \]
if \( R_u A = A \) then \( A \) is invariant under \( R_u \). We say that \( A \) is almost positively invariant if \( F_u A \setminus A \) is a null set with respect to the uniform probability measure \( \sigma \); if the symmetric difference \( A \triangle R_u A \) is a null set then \( A \) is almost invariant.

We consider sets that are positively invariant under the folding maps indexed by a non-empty set of directions \( G \subset S^{d-1} \). Under condition (C1), positively invariant subsets are almost invariant.

**Lemma 5.1.** Let \( G \subset S^{d-1} \) be a subset that is not contained in any closed hemisphere. If \( A \subset S^{d-1} \) is almost positively invariant under \( F_u \) for all \( u \in G \), then it is almost invariant under \( R_u \) for all \( u \in G \).

**Proof.** If \( A \) is almost positively invariant under \( F_u \), then \( (R_u A \cap H_u) \setminus A \) is a null set. In that case,

\[
\int_A x \cdot u \, d\sigma(x) = \int_{A \cap H_u} x \cdot u \, d\sigma(x) - \int_{(R_u A) \cap H_u} x \cdot u \, d\sigma(x) \geq 0.
\]

Equality holds only if \( (A \cap H_u) \setminus R_u A \) is a null set, in which case \( A \) is almost invariant under \( R_u \).

Fix \( u_0 \in G \). By Proposition 4.1 the origin is an interior point of the convex hull of \( G \). This means that for \( \varepsilon > 0 \) sufficiently small, \(-\varepsilon u_0\) can be represented as a convex combination

\[
-\varepsilon u_0 = \sum_{i=1}^n \alpha_i u_i \tag{4}
\]

with \( u_i \in G \) and \( \alpha_i \geq 0 \) for \( 1 \leq i \leq n \). It follows that

\[
\varepsilon \int_A x \cdot u_0 \, dx + \sum_{i=1}^n \alpha_i \int_A x \cdot u_i \, d\sigma(x) = 0.
\]

By the positive invariance of \( A \), all summands are nonnegative. Since \( \varepsilon > 0 \), the integral \( \int x \cdot u_0 \, d\sigma(x) \) vanishes, and therefore \( A \) is almost invariant under \( R_{u_0} \). \( \square \)

Under condition (C2), the sphere has no non-trivial almost invariant subsets.

**Lemma 5.2.** Let \( G \subset S^{d-1} \). If \( \langle G \rangle \) is dense in \( O(d) \), then the only subsets of \( S^{d-1} \) that are almost invariant under the reflections \( \{ R_u \mid u \in G \} \) are null sets and their complements.

**Proof.** The claim can be proved directly, but we give a shorter proof which uses spherical harmonics. Assume that \( A \subset S^{d-1} \) is almost invariant under \( R_u \) for all \( u \in G \). Consider the indicator function of \( A \) as an element of \( L^2(S^{d-1}) \), and expand it in spherical harmonics as

\[
\mathbb{1}_A(x) = \sum_{k=0}^\infty Y_k(x).
\]

Here, \( Y_k \) is orthogonal projection of \( \mathbb{1}_A \) onto the spherical harmonics of degree \( k \), and the series converges in the mean-square sense. Since this representation is unique, each \( Y_k \) is invariant under \( R_u \) for all \( u \in G \). Therefore, \( Y_k \) is invariant under \( \langle G \rangle \), and, by continuity, under the entire group \( O(d) \). But for \( k > 0 \), the action of \( O(d) \) on the spherical harmonics of degree \( k \) fixes only the zero polynomial. It follows that \( Y_k = 0 \) for all \( k > 0 \), and the constant function \( Y_0 = \sigma(A) \) agrees \( \sigma \)-almost everywhere with \( \mathbb{1}_A \). \( \square \)
Lemma 5.3. Assume \( G \subset \mathbb{S}^{d-1} \) satisfies (C1) and (C2). Then no compact subset (and no open subset) of \( \mathbb{S}^{d-1} \) other than \( \emptyset \) and \( \mathbb{S}^{d-1} \) is positively invariant under all foldings \( \{ F_u \mid u \in G \} \).

Proof. Let \( A \subset \mathbb{S}^{d-1} \) be a non-empty compact set that is positively invariant under \( F_u \) for all \( u \in G \). By Lemma 5.1, \( A \) is almost invariant under \( R_u \) for all \( u \in G \), and by Lemma 5.2 either \( A \) or its complement is a null set. We want to exclude the first alternative.

For every \( \varepsilon > 0 \),

\[
A_\varepsilon = \{ x \in \mathbb{S}^{d-1} \mid d(x, A) \leq \varepsilon \}
\]

is a compact set of positive measure. Since \( A \) is positively invariant and the maps \( F_u \) are non-expansive, \( A_\varepsilon \) is positively invariant as well. By Lemmas 5.1 and 5.2 its complement in \( \mathbb{S}^{d-1} \) is a null set; in particular, \( A_\varepsilon \) is dense in \( \mathbb{S}^{d-1} \). By compactness, \( A_\varepsilon = \mathbb{S}^{d-1} \), and hence \( A = \bigcap A_\varepsilon = \mathbb{S}^{d-1} \).

If, on the other hand, \( A \) is a non-empty open set that is positively invariant under \( G \), then its complement is a compact set that is positively invariant under \( -G \). By the first part of the proof, it is empty. \( \Box \)

Proof of Theorem 7 For \( x \in \mathbb{S}^{d-1} \), consider the orbit

\[
G_\ast x = \{ F_{u_1} \ldots F_{u_n} x \mid n \geq 0, u_1, \ldots, u_n \in G \}.
\]

By definition, \( G_\ast x \) contains all trajectories of \( x \) under sequences \( (F_{u_n}) \) with directions \( u_n \in G \). Since \( G_\ast x \) is positively invariant under \( F_u \) for \( u \in G \), its topological closure is a positively invariant non-empty compact subset of \( \mathbb{S}^{d-1} \). By Lemma 5.3 the only such subset is the entire sphere. It follows that \( G_\ast x \) is dense in \( \mathbb{S}^{d-1} \). Hence there exists for every \( \varepsilon > 0 \) and every \( x, y \in \mathbb{S}^{d-1} \) a finite sequence of directions \( u_1, \ldots, u_n \in G \) such that \( d(F_{u_n} \ldots F_{u_1} x, y) < \varepsilon \).

We claim that the sequence can be chosen independently of \( x \) and \( y \). In fact, for every \( \varepsilon > 0 \) there is a finite sequence \( u_1, \ldots, u_N \) in \( G \) such that

\[
\min_{n \leq N} d(F_{u_n} \ldots F_{u_1} x, y) < \varepsilon \quad \forall x, y \in \mathbb{S}^{d-1}.
\]

The sequence is constructed by concatenating a finite number of shorter segments \( S_1, \ldots, S_K \) as follows.

Given \( \varepsilon > 0 \), cover \( \mathbb{S}^{d-1} \) by finitely many open balls \( B_1, \ldots, B_K \) of radius \( \varepsilon/3 \) centered at \( c_1, \ldots, c_K \). To construct \( S_1 \), choose a finite sequence of directions in \( G \) such that the corresponding trajectory starting at \( c_1 \) visits the ball \( B_2 \), and then extend that sequence so that the trajectory visits each of the balls \( B_1, \ldots, B_K \). The segments \( S_k \) for \( 1 < k \leq K \) are constructed inductively. Assuming \( S_1, \ldots, S_{k-1} \) have already been chosen, let \( y_k \) be the final point of the trajectory of \( c_k \) under \( S_1, \ldots, S_{k-1} \). Choose \( S_k \) such that the trajectory of \( y_k \) under \( S_k \) visits each of the balls \( B_1, \ldots, B_K \). Then the trajectory of \( c_k \) under \( S_1, \ldots, S_k \) visits each of the balls.

Let \( u_1, \ldots, u_N \) be the sequence of directions given by \( S_1, \ldots, S_K \). For \( x, y \in \mathbb{S}^{d-1} \), let \( B_i \) and \( B_j \) be the balls containing \( x \) and \( y \), respectively. By construction,

\[
F_{u_n} \ldots F_{u_i} c_i \in B_j
\]

for some \( n \leq N \). Since foldings are non-expansive, the triangle inequality implies

\[
d(F_{u_n} \ldots F_{u_i} x, y) \leq d(x, c_i) + d(F_{u_n} \ldots F_{u_i} c_i, c_j) + d(c_j, y) < \varepsilon.
\]
This establishes Eq. (5). The desired infinite sequence \((u_n)\) is obtained by concatenating the finite sequences constructed above for \(\varepsilon = 2^{-j}\) with \(j \geq 1\).

**Proof of Corollary 4.** Let \(\phi\) be a continuous function on \(\mathbb{R}^d\) such that \(S_u\phi = \phi\) for all \(u \in G\). Then \(\phi \circ F_n \geq \phi\) for all \(u \in G\), that is, \(\phi\) increases along trajectories. We need to show that the restriction of \(\phi\) to each centered sphere \(\{|x| = R\}\) is constant. By scaling, it suffices to consider the case \(R = 1\).

Let \(x, y \in S^{d-1}\) be given. By Theorem 1 there exists an infinite sequence of directions \((u_n)\) such that the trajectory \((x_n)\) defined by Eq. (1) is dense in \(S^{d-1}\). Choose a subsequence \((x_{n_k})\) that converges to \(y\). By monotonicity and continuity,

\[
\phi(x) \leq \lim_{k \to \infty} \phi(x_{n_k}) = \phi(y).
\]

Switching the role of \(x\) and \(y\) yields the reverse inequality \(\phi(y) \leq \phi(x)\). We conclude that \(\phi\) is constant on \(S^{d-1}\).

Theorem 1 says that there exists a sequence of directions in a set \(G\) (satisfying (C1) and (C2)) that generates dense trajectories for all starting points. However, given \(G\), it is not obvious how to explicitly find a dense trajectory. Different from reflections, periodic sequences of foldings never generate dense trajectories.

**Proposition 5.4.** Let \((u_n)_{n \geq 1}\) be a periodic sequence of directions, with \(u_{n+p} = u_n\) for some integer \(p\) and all \(n \geq 1\). Then Eq. (1) has no dense trajectories, and at least one trajectory is periodic. If, in addition, \(G = \{u_1, \ldots, u_p\}\) satisfies (C1) and (C2), then the periodic trajectory is non-constant.

**Proof.** Consider the composition \(F := F_{u_p} \circ \cdots \circ F_{u_1}\) as a map from \(H_{u_p}\) to itself. Since \(F\) is continuous and \(H_{u_p}\) is homeomorphic to a ball in \(\mathbb{R}^{d-1}\), by Brouwer’s fixed point theorem there exists a point \(x \in H_{u_p}\) with \(F(x) = x\). By construction, the trajectory \((x_n)_{n \geq 0}\) of \(x\) is periodic, and in particular not dense. If \(x'_n\) is another trajectory, then the sequence \(d(x_n, x'_n)\) is non-increasing, because folding maps are non-expansive. Let \(r = \lim d(x_n, x'_n)\). If \(r = 0\), then the limit points of \((x'_n)\) are precisely \(x_1, \ldots, x_p\). Otherwise, let \(y\) be a limit point of \((x'_n)_{k \geq 1}\), and let \((y_n)\) be the trajectory of \(y\). By construction, each point \(y_n\) on the trajectory of \(y\) is a limit point of the subsequence \((x'_n)_{k \geq 1}\). Therefore, \(d(x_n, y_n) = r\) for all \(n \geq 0\), that is, \((y_n)\) lies in the union of the \(d-2\)-dimensional subspaces of radius \(r\) centered at the points \(x_1, \ldots, x_p\). Since this is a null set, neither \((y_n)\) nor \((x'_n)\) is dense in \(S^{d-1}\).

For the last claim, suppose that \((x_n)\) is constant. Then the singleton \(\{x\}\) is positively invariant under \(G\). By Lemma 5.3 either (C1) or (C2) fails.

On the other hand, most trajectories are dense under the assumptions of Theorem 1.

**Proposition 5.5** (Random walks are dense). Let \((U_n)\) be an i.i.d. sequence of random directions on \(S^{d-1}\) with distribution \(\mu\). If the support of \(\mu\) satisfies (C1) and (C2) of Theorem 1, then almost surely the trajectory in Eq. (2) is dense in \(S^{d-1}\) for every starting point \(x\).

**Proof.** Let \(\varepsilon > 0\) be given. For \(n \geq 1\), let \(A_{\varepsilon, N}\) be the event that for every pair of points \(x, y \in S^{d-1}\), there exists an integer \(n\) such that the random trajectory starting at \(x\) intersects an open \(\varepsilon\)-neighborhood of \(y\) within some segment \(X_{n+1}, \ldots, X_{n+N}\). By Theorem 1, there exists an \(N < \infty\) and a sequence \((u_n)\) in the support of \(\mu\) such that \(A_{\varepsilon, N}\) occurs on \(X_1, \ldots, X_N\). Since finite segments of trajectories depend continuously on the sequence of directions, the probability of \(A_{\varepsilon, N}\) is strictly positive.
By the Borel-Cantelli lemma, almost surely the event $A_{\varepsilon,N}$ occurs infinitely often. Since $\varepsilon$ was arbitrary, the trajectory is almost surely dense.

6. Random walks and invariant measures. In this section, we prove the ergodicity results in Theorem 2 and Corollary 2. For the construction of the invariant measure, we need some more notation.

Consider for the moment a single random direction $U$ in $S^{d-1}$ with probability distribution $\mu$. The random folding $F_U$ pulls a Borel function $\phi$ on $S^{d-1}$ back to

$$T_\mu \phi(x) = E(\phi(F_U(x)) = \int_{S^{d-1}} \phi(F_u(x)) d\mu(u).$$

(6)

The operator $T_\mu$ is linked to the action on measures defined in Eq. (3) by the change-of-variables formula

$$\int_{S^{d-1}} (T_\mu \phi) d\nu = \int_{S^{d-1}} \phi d(T_\mu \# \nu).$$

Clearly, $T_\mu$ is a positivity-preserving linear operator that fixes constant functions. Since folding maps are non-expansive, $T_\mu$ preserves or improves the modulus of continuity of a continuous function.

The following lemma establishes a useful monotonicity property for the extrema of $\phi$ and $T_\mu \phi$.

**Lemma 6.1.** Let $\mu$ be a probability measure on $S^{d-1}$ whose support satisfies (C1). Define $T_\mu$ by Eq. (6), and let $\phi$ be a continuous function on $S^{d-1}$ with $\max \phi = M$. Then $\max T_\mu \phi \leq M$, and for any $x \in S^{d-1}$

$$T_\mu \phi(x) = M \implies \phi(x) = M.$$

The corresponding statements hold for the minimum of $\phi$.

**Proof.** Left $x \in S^{d-1}$. Since $F_u(x) = x$ for all $u \in H_x$, and $\phi(x) \leq M$ for all $u$, we have

$$T_\mu \phi(x) = \int_{H_x} \phi(x) d\mu(u) + \int_{S^{d-1}\setminus H_x} \phi(R_u x) d\mu(u)$$

$$\leq \phi(x) \mu(H_x) + M \mu(S^{d-1}\setminus H_x)$$

$$\leq M.$$ 

Since $\mu(H_x) > 0$ by Proposition 4.1, equality implies $\phi(x) = M$. 

The random walk is associated with a canonical Markov process on $\Omega = \prod_{n \geq 0} S^{d-1}$, the space of sequences $(X_n)_{n \geq 1}$ on $S^{d-1}$ endowed with the product topology. If $\Phi$ is a function on $\Omega$, we write

$$E_x(\Phi((X_n)_{n \geq 0})) = \int \Phi((F_{U_n} \cdots F_{U_1} x)_{n \geq 0}) d\mu^\otimes$$

for the expected value of $\Phi$ on the random walk $(X_n)$ starting at $X_0 = x$. Here, $\mu^\otimes$ is the product measure that defines the distribution of the sequence $(U_n)$. Correspondingly, the probability of an event $A \subset \Omega$ is denoted by $P_x(A) = E_x(1_A)$.

The canonical Markov process is completely determined by either of the operators $T_\mu$ or $T_\mu \#$. By the Markov property,

$$E_x(\phi(X_n)) = (T_\mu)^n \phi(x).$$

The next result says that $(T_\mu)^n \phi$ converges uniformly to a constant function.
Proposition 6.2. Let $\mu$ be a measure on $\mathbb{S}^{d-1}$ whose support satisfies conditions (C1) and (C2) of Theorem 4.10. For every continuous function $\phi$ on $\mathbb{S}^{d-1}$, there exists a constant $\bar{\phi}$ such that

$$\lim_{n \to \infty} E_x (\phi(X_n)) = \bar{\phi}$$

uniformly in $x$. Moreover,

$$\min \phi \leq \bar{\phi} \leq \max \phi.$$

Both inequalities are strict unless $\phi$ is constant.

Proof. Let $|| \cdot ||$ denote the sup-norm on the space of continuous functions. Set $\psi_n(x) = E_x (\phi(X_n))$. Since $\phi_n = (T_\mu)^n \phi$, its norm $||\phi_n||$ decreases monotonically and the modulus of continuity of $\phi_n$ improves with $n$. By the Arzelà-Ascoli theorem, there exists a subsequence $(\phi_{n_k})_{k \geq 1}$ that converges uniformly to some limiting function, $\bar{\phi}$. After passing to a further subsequence, we may assume that the sequence of gaps $n_k - n_{k-1}$ increases strictly with $k$.

We want to show that $\bar{\phi}$ is constant. Consider the sequence $(\psi_n)$ defined by $\psi_n = (T_\mu)^n \bar{\phi}$. Clearly,

$$||\psi_{m+n} - \phi_{m+n}|| = ||(T_\mu)^m (\bar{\phi} - \phi_n)|| \leq ||\bar{\phi} - \phi_n||$$

for all $m, n \geq 0$. We use the triangle inequality and set $m = n_k - n_{k-1}$, $n = n_k$ to obtain the bound

$$||\psi_{n_k-n_{k-1}} - \bar{\phi}|| \leq ||\psi_{n_k-n_{k-1}} - \phi_{n_k}|| + ||\phi_{n_k} - \bar{\phi}||$$

$$\leq ||\bar{\phi} - \phi_{n_k-1}|| + ||\phi_{n_k} - \bar{\phi}||,$$

which converges to zero by the choice of the subsequence $(n_k)$. It follows that the subsequence $\psi_{n_k-n_{k-1}}$ converges uniformly to $\bar{\phi}$.

Let $M = \max \phi$. Since max $\psi_n$ is non-increasing in $n$, and a subsequence converges to $\psi_0 = \bar{\phi}$, we must have max $\psi_n = M$ for all $n \geq 0$. By Lemma 6.1, the sets $A_n = \{ x : \psi_n(x) = M \}$ form a decreasing chain. Their intersection $A = \bigcap A_n$ is a non-empty compact set that is positively invariant under $F_u$ for $\mu$-a.e. $u \in \mathbb{S}^{d-1}$. By Lemma 5.2, $A = \mathbb{S}^{d-1}$. But this says that $\psi_n \equiv M$ for all $n$, and the same holds for their limit $\bar{\phi}$. Since the sequence $||\phi_n - \bar{\phi}||$ is non-increasing, the convergence of the full sequence follows from the convergence of the subsequence.

Clearly, $\min \phi \leq \bar{\phi} \leq \max \phi$, since $\min \phi_n$ is non-decreasing and max $\phi_n$ is non-increasing. Let $M = \max \phi$, and consider the decreasing chain of compact subsets $A_n = \{ x : \phi_n(x) = M \}$. Since the intersection $A = \bigcap A_n$ is compact and positively invariant, by Lemma 5.3, it is either equal to $\mathbb{S}^{d-1}$ or empty. In the first case, $\phi \equiv M$ is constant. In the second case, by compactness, $A_n$ is empty and thus max $\phi_n < M$ for some sufficiently large $n$. By monotonicity, $\bar{\phi} \leq \max \phi_n < M$. The same argument shows that min $\phi < \bar{\phi}$ unless $\phi$ is constant. \hfill \Box

Proof of Theorem 4.10. Let $\phi$ be a continuous function on $\mathbb{S}^{d-1}$, and let $(X_n)$ be the random walk defined by Eq. (2). By Proposition 6.2, $E_x (\phi(X_n))$ converge to a constant function, $\bar{\phi}$, uniformly in $x$. It is a fact of Ergodic Theory that this uniform convergence is equivalent to the unique ergodicity of the Markov chain [19, Theorem 4.10]. We show the part of the proof that we need here.
The map \( \phi \mapsto \bar{\phi} \) is linear and continuous with respect to the topology of uniform convergence, and its value on nonnegative functions is nonnegative. By the Riesz-Markov theorem, there is a unique regular Borel measure \( \rho \) such that
\[
\bar{\phi} = \int_{S^{d-1}} \phi \, d\rho.
\]
Since the constant function \( \phi \equiv 1 \) is mapped to itself, \( \rho \) is a probability measure.

We next verify that \( \rho \) is invariant. For every continuous function \( \phi \) on \( S^{d-1} \),
\[
\int_{S^{d-1}} \phi \, d(T_\mu \# \rho) = \int_{S^{d-1}} (T_\mu \phi) \, d\rho = \bar{T_\mu \phi}.
\]
Since \( T_\mu \phi = \lim_{n \to \infty} E_x(T_\mu \phi(X_n)) = \lim_{n \to \infty} E_x(\phi(X_{n+1}) = \bar{\phi} = \int_{S^{d-1}} \phi \, d\rho \),
the measure \( T_\mu \# \rho \) represents the same distribution as \( \rho \). By uniqueness, \( T_\mu \# \rho = \rho \).

It remains to show that the support of \( \rho \) is \( S^{d-1} \). Given an arbitrary non-empty open set \( A \subset S^{d-1} \), let \( \phi \) be a nonnegative continuous function supported on \( A \) that takes values in \([0, 1]\) and does not vanish identically. Then \( \rho(A) \geq \bar{\phi} > 0 \) by the last part of Proposition 6.2. Since \( A \) was arbitrary, the proof is complete.

As an immediate consequence of the uniform convergence proved in Proposition 6.2, we obtain the mixing property
\[
\lim_{n \to \infty} \int_{S^{d-1}} E_x(\phi(X_n)) \psi(x) \, d\rho(x) = \bar{\phi} \bar{\psi}
\]
for every pair of continuous functions \( \phi, \psi \) on \( S^{d-1} \).

**Proof of Corollary 2.** Let \( \mu \) and \( \rho \) be as in Theorem 2. We will show that for every \( \rho \)-integrable function on \( S^{d-1} \),
\[
\lim_{N \to \infty} \frac{1}{N} \sum_{k=1}^{N} \phi(X_k) = \int_{S^{d-1}} \phi \, d\rho
\]
almost surely for \( \rho \)-almost every \( x \in S^{d-1} \). The claim then follows by setting \( \phi = 1_A \).

The proof of Eq. (7) calls for a standard application of Birkhoff’s ergodic theorem to the canonical Markov chain associated with the random walk. See for example [27 Chapter 6]. The invariant measure induces a probability measure \( \rho^* \) on \( \Omega \) by
\[
\rho^*(A) = \int_{S^{d-1}} P_t(A) \, d\rho(x).
\]
This measure is invariant under the left shift \( L((X_n)_{n \geq 0}) = (X_{n+1})_{n \geq 0} \), and every shift-invariant subset \( A \subset \Omega \) has \( \rho^*(A) = 0 \) or \( \rho^*(A) = 1 \). By Birkhoff’s theorem,
\[
\lim_{N \to \infty} \frac{1}{N} \sum_{k=1}^{N} \Phi(L^{k}(X_n)_{n \geq 0}) = \int_{\Omega} \Phi \, d\rho^*
\]
for every \( \rho^* \)-integrable function \( \Phi \) on \( \Omega \) and for \( \rho^* \)-almost every sequence \((X_n)_{n \geq 0}\). In the case where \( \Phi((X_n)_{n \geq 0}) = \phi(X_0) \) depends only on the initial point, we have
\[
\Phi(L^{k}(X_n)_{n \geq 0}) = \phi(X_k), \quad \int_{\Omega} \Phi \, d\rho^* = \int_{S^{d-1}} \phi \, d\rho,
\]
which yields Eq. 7 except for sequences \((X_n)\) in a set \(B \subset \Omega\) of \(\rho^*\)-measure zero. By definition of \(\rho^*\) and Fubini’s theorem, \(P_x(B) = 0\) for \(\rho\)-almost every \(x \in S^{d-1}\). \qed

If \(\phi\) is a continuous function on \(S^{d-1}\), then the functions \(\frac{1}{N} \sum_{n=1}^{N} \phi(X_n)\) are uniformly equicontinuous in \(x\) for all \(N \geq 1\) and every sequence of directions \((U_n)\), see Eq. 2. Since \(S^{d-1}\) is separable, it follows that Eq. 7 almost surely holds for every \(x \in S^{d-1}\).

7. Properties of the invariant measure. Finally, we study the properties of \(\rho\). We find that \(\rho\) is generally not the uniform measure on the sphere.

**Proposition 7.1.** Under the assumptions of Theorem 2, the invariant measure is uniform on \(S^{d-1}\), if and only if \(\mu\) is even under \(u \mapsto -u\).

**Proof.** Recall that \(\sigma\) denotes the uniform probability measure on \(S^{d-1}\). We want to show that

\[ T_{\mu} \# \sigma = \sigma \quad \iff \quad \mu(A) = \mu(\neg A) \quad \text{for all Borel sets } A \subset S^{d-1}. \]

\(\Rightarrow\): Suppose \(T_{\mu} \# \sigma = \sigma\). Then, for every \(v \in S^{d-1}\) and \(\varepsilon > 0\),

\[ \int_{S^{d-1}} \sigma(F^{-1}_u(B_\varepsilon(v))) \, d\mu(u) = \sigma(B_\varepsilon). \]

By definition of the folding map, the integrand is given by

\[ \sigma(F^{-1}_u(B_\varepsilon(v))) = 2\sigma(B_\varepsilon(v) \cap H_u). \]

We divide by \(\sigma(B_\varepsilon)\) and take \(\varepsilon \to 0\),

\[ \lim_{\varepsilon \to 0^+} \frac{\sigma(F^{-1}_u(B_\varepsilon(v)))}{\sigma(B_\varepsilon)} = \lim_{\varepsilon \to 0^+} \frac{2\sigma(B_\varepsilon(v) \cap H_u)}{\sigma(B_\varepsilon)} = \begin{cases} 2 & \text{if } u \cdot v > 0, \\ 1 & \text{if } u \cdot v = 0, \\ 0 & \text{otherwise}. \end{cases} \]

Integrating both sides over \(H_u\) and using that \(T_{\mu} \# \sigma = \sigma\), we obtain by dominated convergence

\[ 1 = \lim_{\varepsilon \to 0^+} \int_{S^{d-1}} \frac{\sigma(F^{-1}_u(B_\varepsilon(v)))}{\sigma(B_\varepsilon)} \, d\mu(u) = 2\mu(H_v) + \mu(\partial H_v). \]

It follows that \(\mu(H_v) = \mu(H_{-v})\) for all \(v \in S^{d-1}\). By Lemma 7.2, which is proved below, \(\mu\) is even.

\(\Leftarrow\): Conversely, if \(\mu(A) = \mu(\neg A)\) for all \(A \subset S^{d-1}\), then

\[ T_{\mu} \# \sigma(A) = \int \sigma(F^{-1}_u(A)) \, d\mu(u) \]

\[ = \frac{1}{2} \int \sigma(F^{-1}_u(A)) + \sigma(F^{-1}_{-u}(A)) \, d\mu(u) \]

\[ = \frac{1}{2} \int \sigma(A) + \sigma(R_u A) \, d\mu(u) \]

\[ = \sigma(A). \]

In the second line, we have used that \(\mu\) is even to change the variable \(u\) to \(-u\) in half of the integral. The third line follows, since \(F^{-1}_{\pm u}(A)\) contains a copy of \(A \cap H_{\pm u}\) together with its mirror image. In the last step we have exploited the reflection invariance of \(\sigma\). \qed
Lemma 7.2. Let $\mu$ be a regular Borel measure on $\mathbb{S}^{d-1}$. If

$$\mu(\mathcal{H}_u) = \mu(\mathcal{H}_{-u})$$

for all hemispheres $\mathcal{H}_u$ with $u \in \mathbb{S}^{d-1}$, then $\mu$ is even.

Proof. Assume for the moment that Lemma 7.2.

Let $1198$ A. BURCHARD, G. R. CHAMBERS AND A. DRANOVSKI

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where

We conclude that

Integral is constant on the equatorial sphere

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Z

In the second step, we have applied the divergence theorem on the sphere. Since

$\mu(\mathcal{H}_u) = \mu(\mathcal{H}_{-u})$ for all $u \in \mathbb{S}^{d-1}$ and $Y_k(u) = (-1)^k Y_k(-u)$, the component $\gamma_k Y_k$ must vanish for each odd $k$ by the uniqueness of the expansion. We claim that $\gamma_k \neq 0$ for odd $k$, forcing $Y_k = 0$.

A key point of the Funk-Hecke formula is that the constant $\gamma_k$ depends only on the variable $x_d$. For $k > 0$, we use that $Z_k$ is an eigenfunction of the spherical Laplacian with an eigenvalue $-\lambda_k < 0$ to obtain

$$\int_{\{x_d > 0\}} Z_k \, d\sigma = -\lambda_k^{-1} \int_{\{x_d > 0\}} \Delta Z_k \, d\sigma = \lambda_k^{-1} \int_{\{x_d = 0\}} -\partial x_d Z_k \, d\sigma.$$ 

In the second step, we have applied the divergence theorem on the sphere. Since $Z_k$ depends only on the variable $x_d$, the normal derivative that appears in the last integral is constant on the equatorial sphere $\{x_d = 0\}$.

For $k$ odd, $Z_k$ vanishes on $\{x_d = 0\}$. Since the eigenvalue-eigenvector equation for $Z_k$ is a homogeneous linear second-order ordinary differential equation in $x_d$, the normal derivative cannot vanish simultaneously, and so the integral is non-zero. It follows that

$$\gamma_k = \frac{1}{Z_k(e_d)} \int_{\{x_d > 0\}} Z_k \, d\sigma \neq 0 \quad (k \text{ odd}).$$

We conclude that

$$\mu(A) = \sum_{k \geq 0 \text{ even}} \int_A Y_k \, d\sigma = \mu(-A)$$

for all Borel sets $A \subset \mathbb{S}^{d-1}$, proving the claim when $\mu$ has a smooth density.

Otherwise, we approximate it with smooth measures $\mu_\varepsilon$, defined by

$$\mu_\varepsilon(A) = \int_{\text{SO}(d)} \mu(QA) \psi_\varepsilon(Q) \, d\sigma(Q),$$

where $\psi_\varepsilon$ is a smooth probability density supported on an $\varepsilon$-neighborhood of the identity in $\text{SO}(d)$, and $\sigma$ is the uniform measure. Let $u \in \mathbb{S}^{d-1}$. Since $Q(\mathcal{H}_u) = \mathcal{H}_{Qu} = -QH_{-u}$ by linearity, the assumption on $\mu$ implies that

$$\mu(QH_u) = \mu(QH_{-u}) \quad (Q \in \text{SO}(d)).$$
Therefore, \( \mu_\varepsilon(H_u) = \mu_\varepsilon(H_{-u}) \) for all \( u \in S^{d-1} \). By the first part of the proof, \( \mu_\varepsilon \) is even. Taking \( \varepsilon \to 0 \) we see that \( \mu \) is even as well.

We conclude with some open problems.

**Question 7.3.** Given a measure \( \mu \) on \( S^{d-1} \) whose support satisfies (C1) and (C2), let \( \rho \) be the invariant measure from Theorem 3. Is it true that

\[
\rho(A) = 0 \iff \sigma(A) = 0.
\]

We suspect that the answer is affirmative. To motivate this, we write Eq. (3) as

\[
T \mu \#
u(A) = \int_{S^{d-1}} \int_{S^{d-1}} 1_A(F_u(x)) \, d\mu(u) \, d\nu(x),
\]

and observe that \( T \mu \#
u \) is absolutely continuous with respect to \( \sigma \), whenever either \( \mu \) or \( \nu \) is absolutely continuous. In particular, the absolutely continuous component of the invariant measure \( \rho \) is itself invariant. By the uniqueness part of Theorem 2, \( \rho \) is either absolutely continuous or purely singular. If \( \mu \) has an absolutely continuous component in its Lebesgue decomposition, then \( \rho \) is absolutely continuous. If \( \mu \) is singular, nothing is known. If \( \rho \) were singular as well, then the trajectories of the random walk would accumulate on a set of measure zero.

Another problem is to characterize how quickly the state of the Markov chain associated with the random walk in Eq. (2) converges to the steady-state.

**Question 7.4.** Given a measure \( \mu \) on \( S^{d-1} \) whose support satisfies (C1) and (C2), fix \( x \in S^{d-1} \), and define the random walk \( X_n \) by Eq. (2). At what asymptotic rate does the distribution of \( X_n \) converge to \( \rho \) as \( n \to \infty \)? Is there a cut-off phenomenon?

One approach is to analyze the convergence of \( \phi_n = (T_n)^n \phi \) to its limit \( \bar{\phi} = \int \phi \, d\rho \) for continuous functions \( \phi \) on \( S^{d-1} \). In the case where \( \mu = \sigma \) is the uniform measure on the sphere, computations similar to those in [8, Proposition 5.2] and [7] suggest that the difference \( ||\phi_n - \bar{\phi}|| \) decreases with \( n^{-1} \) in the number of steps. Can this rate of convergence be improved by a judicious choice of \( \mu \), perhaps supported on a finite set?

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