Harmonic scaling laws and underlying structures

Ji-Feng Yang
Department of Physics, East China Normal University, Shanghai 200062, China

Abstract

Based on the effective field theory philosophy, a universal form of the scaling laws could be easily derived with the scaling anomalies naturally clarified as the decoupling effects of underlying physics. In the novel framework, the conventional renormalization group equations and Callan-Symanzik equations could be reproduced as special cases and a number of important and difficult issues around them could be clarified. The underlying theory point of view could envisage a harmonic scaling law that help to fix the form of the loop amplitudes through anomalies, and the heavy field decoupling can be incorporated in this underlying theory approach in a more unified manner.

I. INTRODUCTION

According to the effective field theory (EFT) point of view [1], the insensitivity of the 'low energy' physics to 'high energy' details allows us to parametrize the ill-definedness in EFTs in artificial regularization schemes and then remove them through renormalization program [2]. The necessary presence of renormalization procedure and the insensitivity of the effective theories to the underlying one leads to the renormalization group equation (RGE) [3–6] and anomalous scaling laws encoded in the Callan-Symanzik equations (CSE) [7,6]. But all these equations are prescriptions dependent, and therefore suffer from various shortcomings associated with the prescriptions. The prescription dependence problem might be severe in nonperturbative contexts [8,9]. Even in perturbative regime it is in fact not a simple task to remove the prescription dependence, especially in QCD processes [10]. Moreover, the interpretation of RGE as independence of renormalization points seems unsatisfactory.

In this report we wish perform a pedagogical but general derivation of the EFT scaling laws based on existence of underlying structures to clarify a number of important issues around RGE and CSE and to uncover the physical significance behind the prescription dependence. We first introduce low energy expansion for operators with underlying structures in section two as a warm up. In section three, we perform the formal derivation of the scaling laws for vertex functions in any EFT (renormalizable or not) by only assuming that their
underlying theory definitions exists. The emergence of RGE and CSE and the problems associated with prescriptions will be discussed in section four. Section five will be devoted to a 'harmonic' understanding of the scaling anomalies and a novel use of the scaling laws illustrated with some simple examples. The decoupling issue is discussed in the underlying theory perspective in section six. The whole report will be summarized in section seven.

II. LOW ENERGY EXPANSION OF OPERATORS WITH UNDERLYING STRUCTURES

It is not difficult to convince ourselves that there should general nontrivial structures\footnote{The true underlying structures remain unknown. But for our purpose, it is enough to postulate that they exist and make the extremely short distances physics well defined.} underlying the simple point-particle fields in EFTs. Therefore, for each EFT operator (elementary or not) there should be a corresponding one with nontrivial underlying structures. In long distances or at low energy scales (the decoupling limit), the underlying structures are effectively 'invisible' and the operators reduce (via expansion) to the EFT ones for point particles. But in very short distances or at extremely high energy scales, such underlying structures should become 'visible' and dictate the short distance dynamics.

Using operator and Hilbert state vector formalism, the above scenario means the following expansion equations:

\[
\hat{O}(\{x\};\{g\};\{\sigma\})|\text{LES}\rangle = \hat{O}(\{x\};\{g\})|\text{LES}\rangle + \mathcal{O}(1/\Lambda_{\{\sigma\}}); \\
\prod_{i}\hat{O}_i(\{x_i\},\{g\};\{\sigma\})|\text{LES}\rangle = \prod_{i}^{\hat{O}_i}(\{x_i\},\{g\})|\text{LES}\rangle + \mathcal{O}(1/\Lambda_{\{\sigma\}}); \\
\hat{O}(\{x\};\{\sigma\})|\text{LES}\rangle = \sum_{i}^{c_i(\{g\})}\hat{O}_i^\prime(\{x, \{g\}\})|\text{LES}\rangle + \mathcal{O}(1/\Lambda_{\{\sigma\}}),
\]

where \(x, \{g\}\) and \(\hat{O}_j\) denote respectively the spacetime coordinates (we do not address quantum gravity for simplicity), the EFT parameters and operators, while \(|\text{LES}\rangle\) refers to a generic EFT state and \(\Lambda_{\{\sigma\}}\) refers to the smallest scale among the underlying parameters \(\{\sigma\}\). The first two equations could be viewed as a generalization of the OPE a la Wilson [11] and the third one as Witten’s heavy quark decoupling [12].

However, for the states of extremely high energy scales, the above three equations naturally break down, then we must employ the operators with underlying structures, otherwise we get UV ill-definedness. This is what we usually met in Feynman amplitudes: the intermediate (free particle) states summation (loop momentum integration) extends to infinite regions where we must employ nontrivial underlying structures to make the loop integration finite. Only after the loop integrations are safely carried out could we perform the expansion with respect to the underlying scales, that is, the decoupling limit operation and EFT loop integration generally do not commute. For each loop that is originally ill defined in EFTs, such limit operation performed after integration should lead to a definite nonlocal component and a finite local component. The coefficients in the local component would contain additional constants arising from the decoupling limit that should be definite.
In conventional renormalization prescriptions these constants would be replaced by various prescription dependent cutoff or subtraction scales. For the following derivation, we only need that the underlying structures and hence the finite constants from the decoupling limit exist in ‘right’ places. The good thing is we only need their existence.

III. CANONICAL SCALING WITH UNDERLYING STRUCTURES

Now let us consider a general complete vertex function (1PI) $\Gamma(n)(p, g; \{\sigma\})$ that is well defined in the underlying theory with $[p], [g]$ denoting the external momenta and the Lagrangian couplings (including masses) in a EFT and $\{\sigma\}$ denoting the underlying parameters or constants. Now it is easy to see that such a vertex function must be a homogeneous function of all its dimensional arguments, that is

$$\Gamma(n)(\lambda p, \lambda^d g; \{\lambda^d \sigma\}) = \lambda^d \Gamma(n)(p, g; \{\sigma\}) \quad (4)$$

where $d\ldots$ refers to the canonical mass dimension of any parameters involved.

In the decoupling limit ($L_{\{\sigma\}} \equiv \lim_{\Lambda_{\{\sigma\}} \to \infty}$) that the underlying structures look vanishingly small, there will necessarily arise finite constants $\{\bar{c}\}$ besides the EFT couplings and masses, then Eq.(4) becomes

$$\Gamma(n)(\lambda p, \lambda^d g; \{\lambda^d \bar{c}\}) \equiv L_{\{\sigma\}} \Gamma(n)(\lambda p, \lambda^d g; \{\lambda^d \sigma\})$$

$$= \lambda^d L_{\{\sigma\}} \Gamma(n)(p, g; \{\sigma\}) \equiv \lambda^d L_{\{\sigma\}} \Gamma(n)(p, g; \{\sigma\}). \quad (5)$$

Note that $\{\bar{c}\}$ only appear in the loop diagrams or in the pure quantum corrections, as 'agents' for the hidden participation of the underlying structures. From Eqs.(1, 2, 3), such constants could only appear or reside in the local component of the vertex functions involved, which leads to local operators. We will further clarify these constants later.

The differential form for Eq.(4) reads

$$\{\lambda \partial_{\lambda} + \sum d_g g \partial_g + \sum d_\sigma \sigma \partial_\sigma - d_{\Gamma(n)}\} \Gamma(n)([\lambda p], [g]; \{\sigma\}) = 0. \quad (6)$$

Noting that the operation $\sum d_g g \partial_g$ inserts the trace of the stress tensor ($\Theta$) for EFTs in concern, Eq.(6) could be rewritten as the following inhomogeneous form,

$$\{\lambda \partial_{\lambda} + \sum d_\sigma \sigma \partial_\sigma - d_{\Gamma(n)}\} \Gamma(n)([\lambda p], [g]; \{\sigma\}) = i\Gamma(n)([0; \lambda p], [g]; \{\sigma\}). \quad (7)$$

Obviously the constants with zero canonical mass dimensions do not contribute to the scaling behavior. Eq.(6) or Eq.(7) is just the most general underlying theory version of the EFT scaling laws. The only distinction with naive EFT scaling laws is the canonical scaling contribution from the underlying structures ($\sum d_\sigma \sigma \partial_\sigma$) as the only source for EFT scaling anomalies in the low energy limit.

---

2Mathematically, this could be seen from the expansion of the finite amplitudes $\Gamma(n)([p], [g]; \{\sigma\})$ in terms of the ratios $\frac{[p,g]}{\Lambda_{\{\sigma\}}}$ as they tend to zero, which will give rise to a local polynomial.
To see this point more clearly, let us examine the low energy limit of the above scaling laws in terms of the agent constants \( \{ \hat{c} \} \) that can be derived from Eq.(5) read

\[
\{ \lambda \partial_\lambda + \sum d_c \hat{c} \partial_c + \sum d_g g \partial_g - d_{\Gamma(n)} \} \Gamma^{(n)}([\lambda p], [g]; \{ \hat{c} \}) = 0. \tag{8}
\]

Comparing this equation with Eq.(6), we have,

\[
L_{\{\sigma\}} \{ \sum d_\sigma \sigma \partial_\sigma \} \Gamma^{(n)}([p], [g]; \{ \sigma \}) = \{ \sum d_c \hat{c} \partial_c \} \Gamma^{(n)}([p], [g]; \{ \hat{c} \}), \tag{9}
\]

provided the underlying structures scenario is valid. We should stress that Eqs.(6, 7, 8, 9) are valid not only order by order but also graph by graph. Since the agent constants only appear in the local components of each loop that is not well defined in EFTs, the variations in these agent constants naturally induce the insertions of local operators \( (I_{O_i}) \) in the corresponding loop graphs. Therefore we arrive at the following decoupling theorem,

\[
L_{\{\sigma\}} \{ \sum d_\sigma \sigma \partial_\sigma \} = \sum d_c \hat{c} \partial_c = \sum_i \delta_{O_i}([g]; \{ \hat{c} \}) I_{O_i}, \tag{10}
\]

where \( \delta_{O_i} \) must be functions of the EFT coupling and masses and the agent constants: \( \delta_{O_i} = \delta_{O_i}([g]; \{ \hat{c} \}) \). That means, in the decoupling limit of the underlying structures’ contribution to the scaling can be expanded in terms of EFT operators.

As some of these operators appear in EFT Lagrangian with coupling constants (denoted as \([g]\)) as coefficients, then their insertions can be realized via \( \sum g d_g g \partial_g \). The kinetic operator insertion for EFT field \( \phi \) will be denoted as \( \hat{I}_\phi \). The rest are not present in Lagrangian, they will be denoted as \( O_N \). Therefore, we can write,

\[
\sum_i \delta_{O_i}([g]; \{ \hat{c} \}) I_{O_i} = \sum_g \delta_g([g]; \{ \hat{c} \}) g \partial_g + \sum_\phi \delta_\phi([g]; \{ \hat{c} \}) \hat{I}_\phi + \sum_{O_N} \delta_{O_N}([g]; \{ \hat{c} \}) \hat{I}_{O_N}, \tag{11}
\]

Now with Eqs.(9, 11) we can turn the decoupling theorem in Eq.(10) and the full scaling of Eq.(8) into the following forms,

\[
\{ \sum_c d_c \hat{c} \partial_c - \sum_{O_N} \delta_{O_N} \hat{I}_{O_N} - \sum_g \delta_g g \partial_g - \sum_\phi \delta_\phi \hat{I}_\phi \} \Gamma^{(n)}([\lambda p], [g]; \{ \hat{c} \}) = 0, \tag{12}
\]

\[
\{ \lambda \partial_\lambda + \sum_{O_N} \delta_{O_N} \hat{I}_{O_N} + \sum_g (d_g + \delta_g) g \partial_g + \sum_\phi \delta_\phi \hat{I}_\phi - d_{\Gamma(n)} \} \Gamma^{(n)}([\lambda p], [g]; \{ \hat{c} \}) = 0. \tag{13}
\]

Here Eqs.(12, 13) are only true for the complete sum of all graphs (or up to a certain order). It is obvious that \( \delta_{O_i} \) are just the anomalous dimensions and comprise all the scaling anomalies, which just come from the decoupling limit of the canonical scaling behaviors of the underlying structures, as stated in Eqs.(9). These two general equations can describe both the elementary vertex functions’ scaling and the ones for composite operators.

In the underlying theory, Eqs.(6, 7, 8, 12) and (13) must be identities. But in practice, the underlying structures and hence the agent constants \( \{ \hat{c} \} \) are unknown. Then Eq.(8) or (13) could serve as the constraints for these constants. Later, we will illustrate this point on some simple one-loop amplitudes in section five.
IV. NOVEL PERSPECTIVE OF RGE AND CSE

In this section we limit our attention to a special type of theories: the one without the scaling anomalies \( \sum_{i(O_N)} \delta_{O_N}([g]; \{c\}) \hat{I}_{O_N} \), i.e., the renormalizable theories in conventional terminology. Without these operators, Eqs. (12) and (13) become simpler,

\[
\{ \sum_{\epsilon} d_\epsilon \delta \partial_\epsilon - \sum_g \delta_g g \partial_g - \sum_\phi \delta_\phi \hat{\Gamma}_\phi \} \Gamma^{(n)}([\lambda p], [g]; \{\epsilon\}) = 0; \\
\{ \lambda \partial_\lambda + \sum_{\epsilon} d_\epsilon \delta \partial_\epsilon + \sum_g \delta_g g \partial_g - d_{\Gamma^{(n)}} \} \Gamma^{(n)}([\lambda p], [g]; \{\epsilon\}) \\
= \{ \lambda \partial_\lambda + \sum_g (\delta_g + \delta_g) g \partial_g + \sum_\phi \delta_\phi \hat{\Gamma}_\phi - d_{\Gamma^{(n)}} \} \Gamma^{(n)}([\lambda p], [g]; \{\epsilon\}) = 0. 
\]

These are just the underlying theory versions for RGE and CSE for renormalizable theories. The significant differences between the underlying theory versions and the original ones will be discussed later in this section. To facilitate the comparisons we will turn these equations into more familiar forms. For this purpose, we note that all the agent constants could be parametrized in terms of a single scale \( \bar{\mu} \) as \( \{\bar{\mu}; \{\bar{c}_0\}\} \) with each \( \bar{c}_0 \) (≡ \( \bar{c}_0 \)) being dimensionless. In conventional renormalization programs, they are first predetermined through renormalization conditions, finally transformed into the physical parameters [14] or optimized somehow [10,15].

We should stress the most striking difference between \( [g] \) and \( \{\epsilon\} \): The EFT parameters \( [g] \) or the tree vertices alone can not make EFTs well defined. There must be underlying structures or their agents \( \{\epsilon\} \) for characterizing the quantum ”paths” of EFTs.

A. RGE and CSE as decoupling theorems

Mathematically, an ‘invariance’ for the complete vertex functions is encoded in the decoupling theorem of Eq.(14): the finite variation in the agent constants could be absorbed by the EFT parameters. This amounts to a finite ‘renormalization’ invariance of the EFTs, i.e., the renormalization ‘group’. But this invariance is only valid provided: (1) the finite renormalization originates from a homogeneous scaling of all the dimensional parameters; (2) after transformation no scale is comparable to the underlying scales, otherwise it breaks down as a decoupling theorem. This is just the underlying theory version of RGE with nontrivial validity conditions. These premises seem to be overlooked conventionally.

In Eq.(14) and (15) the insertion of kinetic operators appears unfamiliar. To remove it let us make use of the well known facts that the kinetic insertion only rescales the line momenta in the graphs: for the fermions, this is \( \hat{p} \to (1 + c_\phi) \hat{p} \); for bosons, \( p^2 \to (1 + c_\phi)p^2 \). Since vertices are joined by lines (flows of momenta), each line’s rescaling in turn leads to the rescaling of the vertices pair at the two ends of the line. Thus a vertex is typically rescaled as \( \Pi_i (1 + c_{\psi_i})^{-1/2} \Pi_j (1 + c_{\phi_j})^{-1/2} \). For an \( n \)-point 1PI vertex function there must be \( n \) external momenta that are not subject to the rescaling, thus to compensate this an overall rescaling of the complete \( n \)-point vertex functions must be introduced. Thus \( \sum_\phi \delta_\phi \hat{\Gamma}_\phi \) lead to the following consequences:

\[
\delta_g \to \bar{\delta}_g \equiv (\delta_g - n_{g;\phi} \frac{\delta_\phi}{2} - n_{g;\psi} \frac{\delta_\psi}{2}), \quad \Gamma^{(n_\phi,n_\psi)} \to (1 + c_\psi)^{n_\psi/2}(1 + c_\phi)^{n_\phi/2}\Gamma^{(n_\phi,n_\psi)}.
\]
with $n_{\phi\phi}$ and $n_{\psi\psi}$ being respectively the number of bosonic and fermionic field operators contained in the vertex with coupling $g$. Then the above equations take the following forms:

\[
\{\bar{\mu}\partial_{\bar{\mu}} - \sum_{g} \delta_{g} g \partial_{g} - \sum_{\phi} n_{\phi} \frac{\delta_{\phi}}{2} - \sum_{\psi} n_{\psi} \frac{\delta_{\psi}}{2}\} \Gamma^{(n_{\phi,n_{\psi}})}([p],[g];\{\bar{\mu};(\bar{c}_{0})\}) = 0; \tag{17}
\]

\[
\{\lambda \partial_{\lambda} + \sum_{g} D_{g} g \partial_{g} + \sum_{\phi} n_{\phi} \frac{\delta_{\phi}}{2} + \sum_{\psi} n_{\psi} \frac{\delta_{\psi}}{2} - d_{T}(n_{\phi,n_{\psi}})\} \Gamma^{(n_{\phi,n_{\psi}})}([\lambda p],[g];\{\bar{\mu};(\bar{c}_{0})\}) = 0, \tag{18}
\]

with $D_{g} \equiv \delta_{g} + d_{g}$. Now Eqs. (17) and (18) take the familiar forms. One might take the $[g]$ as our finite ”bare” constants as no infinity subtraction is needed in whole derivation.

For a concrete example, let us consider QED where the conventional RGE and CSE in Feynman gauge read respectively [6]

\[
\{\mu \partial_{\mu} - \gamma_{mR} m_{R} \partial_{mR} + g \partial_{A} - n_{\phi} \gamma_{A} - n_{\psi} \gamma_{\psi}\} \Gamma^{(n_{A,n_{\psi}})}([p],m_{R},\alpha_{R}) = 0; \tag{19}
\]

\[
\{\lambda \partial_{\lambda} + (1 + \gamma_{mR}) m_{R} \partial_{mR} - \beta \partial_{\alpha} - n_{\lambda} \gamma_{A} - n_{\phi} \gamma_{\phi} - d_{T}(n_{A,n_{\psi}})\} \Gamma^{(n_{A,n_{\psi}})}([\lambda p],m_{R},\alpha_{R}) = 0, \tag{20}
\]

in a renormalization prescription. To write down the new versions we note that in QED there are only three independent series of agent constants: $[c_{m}(m,e;\bar{\mu},\bar{c}_{m,0})]$,$ [c_{e}(m,e;\bar{\mu},\bar{c}_{e,0}) = c_{\psi}(m,e;\bar{\mu},\bar{c}_{\psi,0})]$, and $[c_{A}(m,e;\bar{\mu},\bar{c}_{A,0})]$ for $m\bar{\psi}\psi, e\bar{\psi}A\psi, i\bar{\psi}\gamma\partial\psi$ and $\frac{1}{2}F^{2}$ thanks to gauge invariance. Then Eqs. (14, 15, 17, 18) imply the following equations:

\[
\{\bar{\mu}\partial_{\bar{\mu}} - \delta_{m} m \partial_{m} - \delta_{e} e \partial_{e} - \delta_{\lambda} \hat{\lambda}_{A} - \delta_{\phi} \hat{\phi}_{\psi}\} \Gamma^{(n_{A,n_{\psi}})}([p],m,e;\{c_{m},c_{e},c_{A},c_{\psi}\}) = 0; \tag{21}
\]

\[
\{\lambda \partial_{\lambda} + m \partial_{m} + \bar{\mu} \partial_{\bar{\mu}} - d_{T}(n_{A,n_{\psi}})\} \Gamma^{(n_{A,n_{\psi}})}([\lambda p],m,e;\{c_{m},c_{e},c_{A},c_{\psi}\}) = 0;
\]

\[
\{\lambda \partial_{\lambda} + (1 + \delta_{m}) m \partial_{m} + \delta_{e} e \partial_{e} + \delta_{\lambda} \hat{\lambda}_{A} + \delta_{\phi} \hat{\phi}_{\psi} - d_{T}(n_{A,n_{\psi}})\} \Gamma^{(n_{A,n_{\psi}})}([\lambda p],m,e;\{c_{m},c_{e},c_{A},c_{\psi}\}) = 0.
\]

\[
\times \Gamma^{(\nu)}([\lambda p],m,e;\{c_{m},c_{e},c_{A},c_{\psi}\}) = 0. \tag{22}
\]

Thus the forms are in conformity after noting the following correspondence:

\[
\gamma_{mR} \sim \delta_{m} \equiv (\delta_{m} - \delta_{\psi}); \quad \beta(\alpha_{R})/\alpha_{R} = 2\gamma_{A} \sim \delta_{A} \sim -2\delta_{e}; \quad 2\gamma_{\psi} \sim \delta_{\psi}. \tag{23}
\]

Here we note again that the first version of scaling law in Eq. (22) (i.e., $\{\lambda \partial_{\lambda} + m \partial_{m} + \bar{\mu} \partial_{\bar{\mu}} - d_{T}(n_{A,n_{\psi}})\} \Gamma^{(n_{A,n_{\psi}})} = 0$, $\sum d_{\psi}\phi\partial_{\psi} = \bar{\mu} \partial_{\bar{\mu}}$ as $\sum d_{\psi}\phi\partial_{\psi} = 0$) are also valid order by order and graph by graph. This property could lead to a useful means for determining the constants to certain degree, see section five.

Similarly the full scaling laws for the generating functional read,

\[
\{\sum_{\bar{\psi}} d_{\psi}\phi\partial_{\psi} - \sum_{g} \delta_{g} g \partial_{g} - \sum_{\phi} \delta_{\phi} \hat{\phi}\} \Gamma^{1PI}([\phi],[g];\{\bar{\psi}\}) = 0; \tag{24}
\]

\[
\left\{ \sum_{\phi} \int d^{D} x [d_{\phi} - x \cdot \partial_{x}] \phi(x) \frac{\delta}{\delta \phi(x)} + \sum_{g} d_{g} g \partial_{g} + \sum_{\bar{\psi}} d_{\psi}\phi\partial_{\psi} - D \right\} \Gamma^{1PI}([\phi],[g];\{\bar{\psi}\}) = \left\{ \sum_{\phi} \int d^{D} x [d_{\phi} - x \cdot \partial_{x}] \phi(x) \frac{\delta}{\delta \phi(x)} + \sum_{g} D_{g} g \partial_{g} + \sum_{\bar{\psi}} \delta_{\psi} \hat{\phi}\partial_{\psi} - D \right\} \Gamma^{1PI}([\phi],[g];\{\bar{\psi}\}) = 0. \tag{25}
\]
with $D$ denoting the spacetime dimension. Again this is correct for any consistent EFTs.

We note that the operator trace anomalies could be readily read from Eq. (25). For QED this is simply the following

$$g_{\mu\nu} \Theta^{\mu\nu} = (1 + \delta_m) m \bar{\psi} \psi + \frac{1}{4} \delta \Lambda F^{\mu\nu} F_{\mu\nu} - \delta \psi i \bar{\psi} \slashed{D} \psi = (1 + \bar{\delta}_m) m \bar{\psi} \psi + \frac{1}{4} \delta \Lambda F^{\mu\nu} F_{\mu\nu}. \quad (26)$$

In the last step we have used the motion equation. Taking Eq. (23) into account, this agrees exactly with that given in Ref. [16] in form. In unrenormalizable theories, the trace anomalies would contain an infinite sum of local composite operators.

**B. Implications from the underlying structures’ perspective**

Now it is time for us to discuss the consequences following from the general parametrization of the scaling laws derived in the underlying structures perspective. We will focus on renormalizable EFTs. To proceed, let us rewrite Eqs. (14) and (15) with a variant parametrization of both the EFT parameters ($[\bar{\phi}]$) and the agent constants ($\{\bar{c}\}$),

$$\{ \sum \bar{c}_{\bar{c}} \bar{c}_{\bar{c}} - \sum \delta_{\bar{g}} \bar{g} \partial_{\bar{g}} - \sum \delta_{\bar{\phi}} \bar{\phi} \partial_{\bar{\phi}} \} \Gamma^{(\alpha)}([\bar{p}], [\bar{g}]; \{\bar{c}\}) = 0, \quad (27)$$

$$\{ \lambda \partial_{\lambda} + \sum \bar{g}_{\bar{g}} \partial_{\bar{g}} + \sum \bar{c}_{\bar{c}} \bar{c}_{\bar{c}} - d_{\Gamma^{(\alpha)}} \} \Gamma^{(\alpha)}([\lambda \bar{p}], [\bar{g}]; \{\bar{c}\}) = 0. \quad (28)$$

Mathematically, one can formally reproduce the RGE and CSE in any renormalization prescription by appropriately defining $\{\bar{c}\}$ and $[\bar{g}]$. Here a prescription is characterized by the parametrization $\{\bar{c}\}$ or $\{\bar{\mu}; (\bar{c}_0)\}$, unlike the conventional approaches [10]. Furthermore, with each of $\{\bar{c}\}$ taken as mathematically independent (not merely fixed within a prescription 'orbit'), Eq. (27) could also describe any consistent prescription variations: the unified form of the St"ukelberg-Petermann equations [3,17]. Thus, Eqs. (27, 28) (or Eqs. (14, 15)), are simpler and more universal than any conventional version of RGE and CSE, with which we could clarify a number of important and difficult issues around renormalization prescriptions and the applications of RGE and CSE.

However, as stressed in the beginning of last subsection, the 'invariance' in Eq. (27) or (14) is valid only for the complete vertex functions under two conditions, i.e., it is a restricted invariance, in contrast to the conventional convictions. Once truncated, this invariance only makes sense up to the truncated order, and the unaccounted higher orders' contribution must be under control or smaller, which turns out to be a criterion for renormalization prescription. Then we could see the possible problems with the conventional prescriptions: (1) for the full EFT, it is not warranted that an arbitrary variation of the agent constants could lead to a prescription that is perfectly equivalent to the original physical underlying theory parametrization; (2) for truncated series, it is not true that an arbitrary subtraction prescription could qualify for the criterion noted above. It is known that certain prescriptions fail in unstable EFT sectors [18] or are flawed somehow like IR [19] or threshold divergences in the renormalization constants, so are the RGE and CSE defined there.

On general grounds, this could be understood as follows: the physical information is parametrized in the functional dependence upon all the domain variables: $([p], [g]; \{c\})$, especially on the physical momenta. All the physical threshold effects and other momenta
behavior of all the EFT observables (or all the vertex functions) must be preserved or re-
produced in the variation of the agent constants and the EFT parameters. In homogeneous
rescaling of all the domain variables in a physical parametrization, the momentum behavior
is preserved or 'invariant'. But once a variation (for example, one that is not homogeneous
in all the variables) in the agent constants does affect the momentum behaviors up to the
truncated order, the perturbative invariance is violated in effect.

We could also view from the reverse angle: since the momenta does not vary with pre-
scription, then different definition of the EFT parameters amounts to expanding the theory
in terms of different the EFT parameters. Then we need to ask whether the expansions
at different places of the parameters space are equivalent to each other. To approach the
answer, firstly we note that there could be troubles like IR singularities mentioned above.
Secondly we should be clear that the true physical parametrization is not done before con-
fronting with physical data [14], as pure theoretical definition from the underlying theory
is unavailable. Then the definition in a prescription ('renormalized' couplings and masses)
should not preset any special momentum behavior through subtractions in momentum space.
Otherwise, it might preclude the later confrontation with physical data or boundaries.

Thus, in the stage of theoretical calculation, any parametrization of the EFT parameters
and agent constants ([g]; {c}) or ([g]; {c}) only takes or should take symbolic meaning before
confronting with physical data or boundaries. This observation applies to any calculational
programs. From the underlying theory point of view, this determinacy is very natural. Of
course in some EFTs, the 'classical' parameters could be readily determined, then problem
is to determine the agent constants3.

Now let us remark on a technical aspect of the conventional subtraction prescriptions
and the RGE and CSE defined there: a number of different disguises for each EFT coupling
constant g are involved [20], which are in turn the infinite bare coupling gB, the renormalized
coupling (prescription dependent) gR, the running coupling g(λ) due to momentum rescaling
(p → λp), renormalization point independent coupling g (but still prescription dependent),
the physically determined coupling gphys, and the effective charge geff(q2) that is in fact a
physical form factor. The former four are either infinite or prescription dependent, hence
unphysical and will not appear in the physical parametrization after confronting with phys-
cal data, at least in principle. In other words, the subtraction programs seem to introduce
inconveniences or make things unnecessarily complicated.

In the parametrization adopted here, we only need [g] and {c} that could first be taken
as unspecified in the stage of theoretical calculations, then fixed somehow through physical
boundaries or data, at most we could introduce a running one [g(λ)] from physical rescaling
of [g] based on Coleman’s bacteria analogue [21], therefore formulation is significantly sim-
plified. This is because the solution to Eq.( 15) can be found as the solution to the following
equation,

\[
\{ \lambda \partial_\lambda + \sum_g [d_g + \delta_g([g]; \{c\}] \bar{g} \partial_g + \sum_\phi \delta_\phi([g]; \{c\}) \hat{I}_\phi - d_{\Gamma^{(n)}} \} \Gamma^{(n)}([\lambda p], [\bar{g}]; \{c\}) = 0 \quad (29)
\]

3This is like what we do in solving the differential equations in quantum mechanics or classical
electrodynamics: to determine the unknown constants in the solutions, one must impose concrete
physical boundary conditions in terms of given masses and couplings.
by introducing a running $\bar{g}$ for each EFT parameter $g$ $(d_{\bar{g}} = d_g)$ that satisfies the following kind of equation,

$$\lambda \partial_\lambda \{\bar{g}(\{g\}; \lambda)/\lambda^{d_{\bar{g}}}\} = -\{d_g + \delta_g(\{\bar{g}\}; \{\bar{c}\})\}\bar{g}(\{g\}; \lambda)/\lambda^{d_g},$$

(30)

with the natural boundary condition: $\bar{g}(\{g\}; \lambda)|_{\lambda=1} = g$ for each EFT parameter. If we take the undetermined EFT parameters $\{g\}$ as given by the underlying theory, then we could take them as finite ‘bare’ parameters that are also physical. The finite ‘renormalization’ constants can be defined as $z_g(\{g\}; \lambda) \equiv \bar{g}(\{g\}; \lambda)/g$.

For other operators (kinetic or composite) that have nonzero anomalous dimensions, say $\hat{O}_i$, we could first introduce for each of them a fictitious coupling, say $C_i$, then the finite renormalization constants could be defined as $z_{\hat{O}_i} \equiv \bar{C}_i(\lambda)/C_i$. Similarly we could introduce the finite ‘renormalization’ constant matrix for ‘mixing’ operators [22]. The finite ‘renormalization’ constants thus obtained for the field operator could naturally satisfy the constraint imposed by the Källen-Lehmann spectral representation. As a result, in the underlying theory point of view, the ‘renormalization’ constants are finite and could be introduced afterwards as byproducts, not as compulsory components. Therefore no complications associated with the infinities manipulations are needed. We suspect that this scenario might be very helpful in the complicated EFTs, e.g., the Standard model, especially in its unstable sector and the CKM matrix. Again we stress that the running should not be extrapolated to extremely high scales where the underlying structures’ decoupling might fail, an important warning that is usually overlooked in the applications of the conventional RGE or CSE, as no underlying structures are conceptually incorporated there.

By now we have not been specific in how to obtain the finite loop amplitudes except the existence of the underlying structures that is employed as a postulate. In next section, we will briefly explain a differential equation approach for computing loop amplitudes based on the underlying structures’ scenario [13].

V. CONSTRAINTS ON THE AGENT CONSTANTS: HARMONY NOTION

In the preceding sections, we have seen that the EFT scaling anomalies originate in fact from the normal scaling laws of the indispensable underlying structures. Therefore taking the underlying structures into account there is a perfect harmony in the scaling of everything: the typical EFT parameters and the typical underlying parameters or the agent constants for the latter. This is the harmony notion that is natural only in the presence of the underlying structures. In the scaling of everything, as the underlying structures are not directly observable, their normal contributions (via the agent constants) will therefore be exposed in terms of the low energy variables as anomalies. Since the low energy dynamics (including the anomalies) should be definite over distances large enough than the underlying structures, then from homogeneity or harmony, the agent constants’ scaling are locked with the definite EFT scaling anomalies. Conversely, this locking from harmony could serve as constraints on the agent constants as they are unknown in practice, just like gauge invariance helps to constrain the agent constants before confronting with physical data or before the true underlying structures are discovered. Similar understanding of anomalies from both EFT and underlying theory perspectives is familiar in chiral anomaly [23]. We hope this harmony principle could be generalized to other transformation behaviors of EFTs.
A. Scaling anomalies and agent constants: simple examples

First let us explain our point on two very simple 1-loop vertices in massless QED: electron self-energy $\Sigma^{(1)}$ and vacuum polarization $\Pi^{(1)}_{\mu\nu}$. One can use any regularization to compute the definite part and just leave the local part ambiguous, or use the differential equation method adopted in Ref. [13] based on the existence of nontrivial underlying structures\(^4\) that will be explained later,

$$
\Sigma^{(1)}(p, -p) = -i \frac{e^2 \not{p}}{16\pi^2} \ln \frac{-p^2}{c^2_{\psi(1)}}, \quad \Pi^{(1)}_{\mu\nu}(p, -p) = i \frac{e^2}{12\pi^2} (g_{\mu\nu} p^2 - p_\mu p_\nu) \ln \frac{-p^2}{c^2_{A;(1)}}.
$$

(31)

Here all the agent constants are obvious. These objects have scaling anomalies,

$$
(p \cdot \partial_p - 1)\Sigma^{(1)} = -i \frac{e^2}{8\pi^2} \not{p}, \quad (p \cdot \partial_p - 2)\Pi^{(1)}_{\mu\nu} = i \frac{e^2}{6\pi^2} (p^2 g_{\mu\nu} - p_\mu p_\nu).
$$

(32)

Then, if the agent constants’ contributions are included, the normal scaling behavior or exact scale harmony in Eq. (8) or (15) is restored order by order, graph by graph:

$$
\{p \cdot \partial_p + c_{\psi(1)} \partial_{c_{\psi(1)}} - 1\} \Sigma^{(1)}(1) = 0, \quad \{p \cdot \partial_p + c_{A;1} \partial_{c_{A;1}} - 2\} \Pi^{(1)}_{\mu\nu} = 0.
$$

(33)

This is just harmony we have explained above. Of course the agent constants’ scaling becomes EFT scaling anomalies once expressed in terms of EFT parameters,

$$
\bar{c}_{\psi(1)} \partial_{\bar{c}_{\psi(1)}} \Sigma^{(1)} = i \frac{e^2}{8\pi^2} \not{p}, \quad \bar{c}_{A;1} \partial_{\bar{c}_{A;1}} \Pi^{(1)}_{\mu\nu} = -i \frac{e^2}{6\pi^2} (p^2 g_{\mu\nu} - p_\mu p_\nu).
$$

(34)

Thus the agent constants are locked by the scaling anomalies and must appear as $\ln \frac{1}{p^2}$ to balance the dimension in $\ln p^2$ in the definite nonlocal component. In the operator insertion version (only valid for the complete or truncated vertices), the scaling laws take the following forms up to 1-loop level,

$$
\{p \cdot \partial_p + \delta^{(1)}_{\psi} \hat{\Gamma}_{\psi} - 1\} \Gamma^{(1)}_{\psi} = 0, \quad \{p \cdot \partial_p + \delta^{(1)}_{A} \hat{\Gamma}_{A} - 2\} \Gamma^{(1)}_{\mu\nu} = 0,
$$

(35)

with the anomalous dimensions $\delta^{(1)}_{\psi} = \frac{e^2}{8\pi^2}, \delta^{(1)}_{A} = \frac{e^2}{6\pi^2}$.

The anomalies are independent of mass. To see this we can recalculate the vertices considered above in the massive case. Here we just discuss the vacuum polarization following the parametrization considered by Chanowitz and Ellis [26], the only agent constant appears in the local component,

$$
\Pi^{\mu\nu} = -i \frac{e^2}{12\pi^2} (p^\mu p^\nu - g^{\mu\nu} p^2) \left\{ C(m; \bar{c}_{A}) + 6 \int_0^1 dz z (1 - z) \ln \left[ 1 - \frac{z(1 - z) p^2}{m^2} \right] \right\};
$$

(36)

$$
\Delta^{\mu\nu} = \frac{e^2}{\pi^2} (p^\mu p^\nu - g^{\mu\nu} p^2) \frac{m^2}{p^2} \left[ \frac{2m^2}{p^2 \tau} \ln \frac{1 - \tau}{1 + \tau} - 1 \right], \quad \tau \equiv \sqrt{1 - 4m^2/p^2},
$$

(37)

\(^4\)The use pioneering use of the differential equation method could be found in Ref. [24]. The author is very grateful to the referee for this important information. This method proves powerful in dealing with the notorious overlapping divergences [25].
with $\Delta^{\mu\nu}$ being obtained through the canonical trace operator insertion, or equivalently by $i(m\partial_m)\Pi^{\mu\nu}$. The anomalous scaling equation for the vacuum polarization reads,

$$\{p \cdot \partial_p - 2\} \Pi^{\mu\nu} = i\Delta^{\mu\nu} + i\frac{e^2}{6\pi^2}(g^{\mu\nu}p^2 - p^\mu p^\nu), \quad (38)$$

where as anticipated the anomaly is exactly the same as in the massless case. The agent constant appears in the unknown constant $C(m; \bar{c}_A)$ which is also an unknown function of mass. Let us determine its functional form using the harmony notion.

To this end, we note that Eq.( 38) could be put into the following form as the vacuum polarization tensor must be homogenous functions in all dimensional parameters, $p, m, \bar{c}_A$:

$$\{p \cdot \partial_p + m\partial_m + \bar{c}_A\partial_{\bar{c}_A} - 2\} \Pi^{\mu\nu} = 0. \quad (39)$$

Since $i\Delta^{\mu\nu} = -m\partial_m \Pi^{\mu\nu}$, then from Eq.( 36) we could find after some calculation that $-m\partial_m \Pi^{\mu\nu} = \frac{i e^2}{12\pi^2}(p^\mu p^\nu - g^{\mu\nu}p^2)[m\partial_m C(m; \bar{c}_A) - 2] + i\Delta^{\mu\nu}$, that is $m\partial_m C(m; \bar{c}_A) - 2 = 0$. Then combining this fact with Eq.( 39), we could obtain the following the equation,

$$0 = (m\partial_m + \bar{c}_A\partial_{\bar{c}_A})C(m; \bar{c}_A) = \bar{c}_A\partial_{\bar{c}_A}C(m; \bar{c}_A) + 2, \quad (40)$$

from which we have

$$C(m; \bar{c}_A) = -\ln \frac{\bar{c}_A}{m^2}. \quad (41)$$

From this functional form, we could say that the agent constant is ‘locked’ in the full scaling, the harmony notion. Or each agent constant associated with a ill-defined loop integration must appear in the place similar to Eq.( 41).

It is easy to see that all the EFT vertices should exhibit the same feature. This is quite a natural conclusion following from the underlying theory point of view. In this point of view, any prescription where the subtraction scale (as an agent constant) appears in ways that are inequivalent to the way described by Eq.( 41) seems unnatural. Here we must mention that the dependence on the agent constants in the way specified in Eq.( 41) conventionally leads to the failure of heavy particles decoupling in the running couplings [27], which is not a serious problem but often costs theoretical and computational labors [28]. We will exclusively discuss this issue in the underlying structures’ perspective in section six.

**B. A technical explanation based on differential equation method**

The natural appearance of the dimensional agents in the way dictated in Eq. (41) can also be technically understood in the differential equation method. First we need to briefly review the method solely based on the existence of the underlying structures [13] as follows: For a 1-loop graph $G$ of superficial divergence degree $\omega_G - 1$ [29] with its underlying version amplitude denoted as $\Gamma_G([p], [g]; \{\sigma\})$, we have

$$\partial^\omega_G \!{}_{[p]}L(\sigma)\Gamma_G([p], [g]; \{\sigma\}) = L(\sigma) \partial^\omega_G \!{}_{[p]}\Gamma_G([p], [g]; \{\sigma\})$$

$$= L(\sigma) \int d^d k[\partial^\omega_G \!{}_{[p]} g([p, k], [g]; \{\sigma\})] = \int d^d k[\partial^\omega_G \!{}_{[p]} L(\sigma) g([p, k], [g]; \{\sigma\})]$$

$$= \int d^d k g^{\omega_G}([p, k], [g]; \{\sigma\}) = \Gamma_G^{\omega_G}([p], [g]), \quad (42)$$

11
where \( \partial^{\omega G}_{[p]} \) denotes the differentiation for \( \omega \) times with respect to the external momenta and the resulting amplitude \( \Gamma_G^{\omega G} \) is convergent or definite already in EFT. That means, before knowing the underlying structures, we could at most determine the true decoupling limit of the amplitude \( \Gamma_G \) up to an ambiguous local component (as a polynomial with power index \( \omega_G - 1 \) in EFT parameters, \( N^{(\omega_G-1)}([p],[g]) \)) by performing the indefinite integration on the definite amplitude \( \Gamma_G^{\omega G} \) with respect to the external momenta for \( \omega_G \) times:

\[
\Gamma_G([p],[g];\{\bar{c}_G\}) \equiv L(\sigma)\Gamma_G([p],[g];\{\sigma\}) = [\partial^{\omega G}_{[p]}]^{-1}\Gamma_G^{\omega G}([p],[g])\text{mod}N^{(\omega_G-1)}([p],[g]) \tag{43}
\]

with \([\partial^{\omega G}_{[p]}]^{-1}\) denotes the indefinite integration. To us the agent constants \( \{\bar{c}_G\} \) are unknown. The deduction in Eq. (42) is justified by the existence of the underlying structures make the EFTs well defined [13].

As the amplitude \( \Gamma_G^{\omega G}([p],[g]) \) is finite (with negative scaling dimension) then it should take the following form (under appropriate Feynman parametrization)

\[
\Gamma_G^{\omega G}([p],[g]) \sim \int_0^1 [dx]f(\{x\})\frac{N([p],[g]|\{x\})}{D([p],[g]|\{x\})}, \tag{44}
\]

where \( N \) and \( D \) are respectively the polynomials in terms of external momenta and/or masses with the help of Feynman parameters \( \{x\} \), with the fraction \( N/D \) scales as \( 1/p \). Then after the indefinite integration is done a logarithmic factor will necessarily appear,

\[
[\partial^{\omega G}_{[p]}]^{-1}\Gamma_G^{\omega G}([p],[g]) \sim \int_0^1 [dx]f(\{x\})P([p],[g]|\{x\})\ln \left[ \frac{Q([p],[g]|\{x\})}{\bar{c}_{\text{scale}}^2} \right], \tag{45}
\]

with \( P \) and \( Q \) being again two polynomials in EFT dimensional parameters, with the latter being usually quadratic in momenta. The agent constant (the scale \( \bar{c}_{\text{scale}} \)) naturally appears to balance the dimension of \( Q \). If one trades the constant scale in the logarithmic factor with an EFT mass parameter that appear at least in one of the lines in the loop, say \( m \), so that the nonpolynomial function of momenta only involve EFT parameters, then the agent scale would only appear in the polynomial part

\[
-\int_0^1 [dx]f(\{x\})P([p],[g]|\{x\})\ln \frac{\bar{c}_{\text{scale}}^2}{m^2} \tag{46}
\]

just as we saw from scale anomaly analysis. The constant factor multiplying this logarithmic function is the anomalous dimension. This logarithmic factor is the key to understand heavy field decoupling in the ‘running’ of EFT parameters, as will be shown in next section.

Thus we could see from the foregoing general arguments that the agent parameters are generally dimensional constants or scales, they appear in most situations in the argument of a logarithmic function of external momenta and masses to balance the dimensions\(^5\). They should, in perturbative Feynman diagrammatic language, appear in each loop integration that is ill defined in EFT.

\(^5\)Some agent constants appear not as balancing scales, e.g., in the power law divergent loops [8]. Such agent constants’ variation do not 'renormalize' the EFT parameters.
C. A simple use of the new version CSE: asymptotic freedom

Before closing this section we demonstrate a simple use of the new version scaling law: the reproduction of the asymptotic freedom of QCD [30]. To this end consider the following scaling law satisfied by the two-point gluon correlation function (in Feynman gauge)

\[ \{ \lambda \partial_\lambda + \delta g_s g_s \partial g_s + \sum (1 + \delta_m \partial_m) \} D^c(\lambda p, g_s, m_i; \{ \bar{c} \}) = 0. \] (47)

Taking all quarks as massless (high energy region) and introducing \( \alpha = g_s^2/(4\pi) \), from Eq.( 47) we could obtain the following equation for the effective coupling \( \alpha_{\text{eff}} \equiv \alpha D^c \),

\[ (\lambda \partial_\lambda - \delta_A \alpha \partial_\alpha) \alpha_{\text{eff}}(\lambda p, \alpha; \{ \bar{c} \}) = 0. \] (48)

Here we used that \( \bar{\delta}_g s = -\frac{1}{2}\delta_A \). Truncating \( \delta_A \) at one loop level we have \( \delta_A^{(1)} \alpha = -\frac{1}{2}l_f \alpha \) (\( l_f \) is a flavor dependent number), and there is only one agent scale, \( \bar{c}_{A;(1)} \), which allows us to make the following Taylor expansion,

\[ \alpha_{\text{eff}}^{(1)} = \sum_{n=0}^{\infty} f_n(\alpha) t^n, \quad t \equiv \frac{1}{2} \ln(p^2/\bar{c}_{A;(1)}^2), \] (49)

then substitutes it into Eq.( 48), one can find the following relation,

\[ nf_n = \delta_A^{(1)} \alpha \partial_\alpha f_{n-1}, \quad f_0 = \alpha, \quad \forall n \geq 1. \] (50)

The solution to Eq.( 50) and therefore the solution to Eq.( 48) are easy to find as

\[ f_n = a^n \alpha^{n+1}, \quad \forall n \geq 0, \quad \Rightarrow \alpha_{\text{eff}}^{(1)}(p^2, \alpha; \bar{c}_{A;(1)}^2) = \frac{\alpha}{1 - \frac{1}{2}l_f \alpha \ln(p^2/\bar{c}_{A;(1)}^2)}. \] (51)

For QCD it is well known that \( l_f \) is negative and hence the effective charge in Eq.( 51) exhibit asymptotic freedom. Similar phenomenon could be exhibited by introducing a running coupling for \( \alpha \) as in Eq.( 30): \( \lambda \partial_\lambda \bar{\alpha}(\lambda) - \delta_A^{(1)}(\bar{\alpha}) \bar{\alpha}(\lambda) = 0 \). The solution is well known: \( \bar{\alpha}(\lambda) = \frac{\alpha}{1 - \frac{1}{2}l_f \alpha \ln(\lambda)} \), which fulfills the boundary condition: \( \bar{\alpha}(\lambda)|_{\lambda=1} = \alpha \). Note again we only need the canonical EFT (QCD) coupling \( g_s \) or \( \alpha \). Thus for massless case the effective coupling’s scaling behavior is the same as the running one. For massive case, the two differ in scaling behavior. The effective charge should exhibit the correct threshold behavior while the running one could not possess the correct threshold behavior due to the absence of explicit mass effects in its evolution equation. Thus the running coupling seems to fail the decoupling theorem [27]. This lead us to section six.

VI. APPELQUIST-CARAZZONE DECOUPLING THEOREM AND UNDERLYING STRUCTURES

As is generally shown in subsection five B, in massive sectors of an EFT, the agent constants (scales) generally appear in the simple logarithmic function as parametrized in Eq.( 46). But such parametrization would lead to failure of the decoupling theorem a la Appelquist-Carazzone [27] in the beta function or anomalous dimensions [28]. In this section we shall show how to resolve the problem in the underlying structures’ perspective.
A. Decoupling and repartition

First let us note that, in the underlying theory perspective, the EFT parameters and the underlying parameters are grouped or partitioned into two separate sets by the reference scale that naturally appear in any physical processes (e.g., center energy in a scattering) according to the relative magnitudes: one effective \([g]\), one underlying \([\sigma]\). The decoupling of a heavy field in an EFT could only alter the EFT contents, not the well-definedness of the whole theory. Physically, the heavy field could no longer be excited in the EFT processes. Mathematically, in the decoupling limit when an EFT mass scale \(M_H\) become infinitely large for the rest EFT scales, the mass itself becomes an underlying parameter. That means the decoupling induces a new partition between the effective and underlying parameters, with the union of the two sets kept conserved in the course of decoupling:

\[
\begin{align*}
([g]; \{\sigma\})_{\text{EFT with } M_H} &\implies ([g']'; \{\sigma\}')_{\text{EFT without } M_H}; \\
[g] \cup \{\sigma\} &\implies [g']' \cup \{\sigma\}', \quad [g] \equiv [g]/[M_H], \quad \{\sigma\}' \equiv \{\sigma\} \cup [M_H].
\end{align*}
\]

This repartition yields a new EFT that differs from the original one by a very massive field, hence a new set of agent constants \(\{\bar{c}'\}\) is generated from this repartition.

In the homogeneous rescaling of all dimensional parameters, this repartition of parameters also leads to the repartition in the scaling effects and hence different contents of EFT scaling and anomalies are resulted. Thus the decoupling of a heavy EFT field could be naturally formulated in the underlying structures’ scenario. Symbolically, this repartition means the following rearrangement,

\[
\begin{align*}
\sum_{\sigma} d_{\sigma} \sigma \partial_{\sigma} + \sum_{g} d_{g} g \partial_{g} &\implies \sum_{\sigma}' d_{\sigma} \sigma \partial_{\sigma} + \sum_{g}' d_{g} g \partial_{g}.
\end{align*}
\]

Here the summation is performed for the primed sets defined above. Then the original underlying structures’ decoupling operation (low energy limit) \(L_{\{\sigma\}}\) should be replaced by that with respect to the new set \(\{\sigma\}'(= \{\sigma, M_H\})\) when \(M_H\) becomes much larger than the rest of the EFT scales (especially than the center energy): \(L_{\{\sigma\}'} \equiv \lim_{M_H \to \infty}.\) This in turn leads to the new parametrization in terms of agent constants,

\[
\begin{align*}
L_{\{\sigma\}} \sum_{\sigma} d_{\sigma} \sigma \partial_{\sigma} = \sum_{\bar{c}} d_{\bar{c}} \bar{c} \partial_{\bar{c}},
\end{align*}
\]

Then in the theoretical limit that \(M_H\) becomes extremely large (comparable to the smallest underlying scale), a new set of agent constants \(\{\bar{c}'\}\) appear instead of the original one \(\{\bar{c}\}\). That means the following reorganization,

\[
\sum_{\bar{c}} d_{\bar{c}} \bar{c} \partial_{\bar{c}} + \sum_{g} d_{g} g \partial_{g} \implies \sum_{\bar{c}'} d_{\bar{c}'} \bar{c}' \partial_{\bar{c}'} + \sum_{g} d_{g} g \partial_{g}.
\]

Thus in discussing the decoupling of heavy EFT particles, we should first return to the formulation with the explicit presence of the underlying parameters and the accompanying...
limit operators, i.e., to work with \( \{L(\sigma)^r, \Gamma([p], [g]'; \{\sigma}\} \) and \( \{L(\sigma), \Gamma([p], [g]; \{\sigma}\}) \). Then from the configuration of the parameters described in Eq. (53), the decoupling of a heavy EFT field could be formally be taken as the following two operations: (i) repartitioning the EFT and underlying parameters; (ii) taking the low energy limit with respect to the new underlying parameters, including the heavy EFT field’s mass. This is the mathematical formulation of EFT field decoupling.

B. Practical decoupling of EFT fields

But in practical situation, the mass of the heavy particle is not infinitely large comparing to the rest of the EFT scales. If the energy or momentum transfer scale is not vanishingly small, then the above mathematical decoupling limit does not apply in effect. We must find alternative realization of decoupling in the anomalous dimensions or in the ‘running’ of the EFT parameters (C.f. Eqs. (29) and (30)).

For this purpose, let us review what we really mean by ‘running’. From Eq. (12) or (14) we see that, as the underlying parameters or their agents scale homogeneously while keeping the EFT parameters intact, this partial scaling just leads to the ‘running’ or ‘renormalization’ of the latter. Thus the partition determines the ‘running’ characteristics: what to scale as underlying parameters versus what to scale as EFT parameters. For the EFT parameters, the underlying ones and their agents vary with the same scaling parameter \( \lambda: \{\sigma\} \Rightarrow \{\lambda^{d_\sigma} \sigma\} \) or \( \{\bar{c}\} \Rightarrow \{\lambda^{d_\bar{c}} \bar{c}\}. \) That means once an EFT scale becomes ‘underlying’ one, then it should vary with the same scaling parameter as the underlying parameters or their agents do:

\[
\{\bar{c}, M_H\} \Rightarrow \{\lambda^{d_\bar{c}} \bar{c}, \lambda M_H\}. \]

Then we can understand the decoupling of the heavy field with mass \( M_H \) in EFT in the following sense:

(a). Before decoupling, namely, the momentum transfer \( q^2 \) or other EFT scales is larger than \( M_H^2 \) (that is \( M_H \) is an EFT parameter), as the agent constants \( \{\bar{c}\} \) vary with \( \lambda \) while keeping \( M_H \) and other EFT parameters intact, then the heavy field should generally contribute to the ‘running’ of EFT parameters when the corresponding diagrams containing the heavy line(s) due to the presence of the following variation component as is generally shown in section five B,

\[
\delta \lambda \left[ \ln \frac{\bar{c}^2}{M_H^2} \right] = \ln \frac{[(1 + \delta \lambda)\bar{c}]^2}{M_H^2} - \ln \frac{\bar{c}^2}{M_H^2} \neq 0. \tag{58}
\]

(b). When the momentum transfer scale \( q \) is below the heavy field threshold \( q_{thr} = n M_H, n > 1, M_H \) is no longer an EFT parameter and scales in the same quantitative way as \( \{\bar{c}\} \). This prescription of scaling both \( M_H \) and \( \{\bar{c}\} \) in the same way would yield no contribution to the ‘running’ of the rest EFT parameters:

\[
\delta \lambda \left[ \ln \frac{\bar{c}^2}{M_H^2} \right] = \ln \frac{[(1 + \delta \lambda)\bar{c}]^2}{[(1 + \delta \lambda)M_H]^2} - \ln \frac{\bar{c}^2}{M_H^2} = 0. \tag{59}
\]

That is to say, taking as part of the underlying parameters or their agents, \( M_H \) will cancel out the some agents’ contributions to the ‘running’ as they against each other in the same logarithmic factor. It is then obvious that due to such mechanisms the anomalous dimensions
of the other EFT parameters will alter by a finite amount ($\Delta \bar{\delta}_g$) as the realization of heavy EFT field decoupling:

$$\bar{\delta}_g \Rightarrow \bar{\delta}'_g = \bar{\delta}_g + \Delta \bar{\delta}_g.$$  

(60)

This the resolution of the decoupling issue in the EFT parameters in EFTs with underlying structures. It is not difficult to see that this resolution is more close to the 'subtraction' solution [28]. The difference is that we switch the status of the heavy masses rather than perform subtractions: take the heavy masses as EFT or underlying parameters. Of course for the 'running' parameters we must employ matching conditions across the decoupling thresholds [31]. This part is similar to conventional approaches. This could also be realized by employing appropriate boundary conditions that differ across the thresholds, which should be equivalent to matching. We will further demonstrate such understandings in various concrete applications in the future. We hope our understanding could also be useful in the heavy quark effects in deeply inelastic scattering [32], which is important both theoretically and phenomenologically. This ends our main technical presentation.

VII. DISCUSSION AND SUMMARY

We should emphasize that what we have done is to improve our understanding of the UV ill-definedness of quantum field theories. Based on the well known EFT philosophy, we show that the scaling law analysis incorporating the postulated existence of underlying short distance structures could lead to quite useful insights into the renormalization issue though it is already a textbook stuff. But the conventional machinery for renormalization is quite complicated and not pedagogically easy to go with.

The underlying structures' scenario shows that the conventional complications are just due to our oversimplified guess of the true underlying structures implemented through various artificial regularization methods, which usually yield us UV divergences that should not have appeared in the true parametrization of the underlying structures. Then many complications and superficial unreasonableness associated with infinity manipulation could be removed. Of course the low energy physics should be effectively insensitive to the UV details, which is in fact the philosophical basis for conventional renormalization programs. But if we take a closer look at underlying structures scenario, we could see that a number of important consequences could be readily obtained before knowing the true underlying structures and without employing any sort of infinity manipulation or subtraction. This situation, in our opinion, is the real meaning of insensitivity. More importantly, the essential indeterminacy in any EFT could be easily read off from the unavailability of the underlying parameters, which in turn implies that the agent constants must be finally fixed or determined through physical boundaries or data, which is also necessary in conventional renormalization programs [14]. One important result that might seem trivial is that, though we do not the true values of these agent constants, their positions could be determined through examining the full scaling laws as explained by in section five. The remaining task is to determine their physical values, which are scheme and scale invariants in conventional terminology.

The scaling laws followed from the rescaling of both EFT and underlying parameters take a number of different forms. The most general one, described in Eq.( 8) or ( 15) is
valid both at the level of individual graph and at the complete vertex functions and their truncations. This envisages a harmony notion in full scaling in EFTs with underlying structures. This form or parametrization is not given effectively in conventional approaches. The scaling of the agent constants alone is shown to lead to the ‘running’ or finite ‘renormalization’ of the EFT parameters. The equations describing this partial scaling arise as the natural consequences of taking the decoupling limit of the underlying scale parameters, which affords the RGE a physical interpretation. The most general form of this decoupling theorem contain the Stückelberg-Peterman RGE that describes also transformation across different renormalization prescriptions. This byproduct could not be obtained within any conventional generalization of Callan-Symanzik equations. Moreover, we have shown that the decoupling of heavy EFT fields could also be realized in a simple manner without performing subtractions. It also follow from the insight provided by deep exploration of the existence of the underlying structures for EFTs.

Of course, we have here focused on more theoretical and logical aspects of the underlying structures’ scenario. These equations and the byproducts thus obtained might have further consequences and productive applications. In the meantime, we must stress that our investigations here are just primary, further elaboration and development are also necessary and worthwhile. We hope our primary attempts here have demonstrated the nontriviality of the EFT philosophy.

In summary, we derived several new versions of the scaling laws in any EFT assuming the existence of nontrivial underlying structures without the need of infinity manipulations and the associated complications. The new equations are more general than the conventional ones and provide more physical insights in our understanding of renormalization and EFTs. The decoupling theorem a la Applequist and Carazzone is still valid with the underlying structures appropriately accounted. When the mass is not extremely large, the decoupling could be implemented in a simple manner.

ACKNOWLEDGEMENT

I wish to thank W. Zhu for his helpful discussions on scalings. This project is supported in part by the National Natural Science Foundation under Grant No.s 10075020 and 10205004, and by the ECNU renovation fund for young researcher under Grant No. 53200179.
REFERENCES

[1] M.E. Peskin and D.V. Schroeder, *An Introduction to Quantum Field Theory* (Addison-Wesley, MA, 1995), Chap. 8; S. Weinberg, *The Quantum Theory of Fields* (Cambridge University Press, Cambridge, 1995), Vol. I, Chap. 12.

[2] Here we mean the BPHZ program, see, e.g., N.N. Bogoliubov and D.V. Shirkov, *An Introduction to the Theory of Quantized Fields* (4th edition, Wiley, NY, 1980).

[3] E.C.G. St"ukelberg and A. Petermann, Helv. Phys. Acta 26, 499 (1953).

[4] M. Gell-Mann and F.E. Low, Phys. Rev. 95, 1300 (1954).

[5] G. ’t Hooft and M. Veltman, Nucl. Phys. B44, 189 (1972).

[6] S. Weinberg, Phys. Rev. D8, 3479 (1973).

[7] C.G. Callan, Jr., Phys. Rev. D2, 1541 (1970); K. Symanzik, Com. Math. Phys. 18, 227 (1970).

[8] See, for example, J.-F. Yang and J.-H. Ruan, Phys. Rev. D65, 125009 (2002)[arXiv:hep-ph/0201255] and references therein.

[9] For examples in nucleon EFTs, see, e.g., S.R. Beane, P.F. Bedaque, W.C. Haxton, D.R. Phillips and M.J. Savage, in *Boris Ioffe Festschrift*, ed. M. Shifman (World Scientific, Singapore, 2001)[arXiv:nucl-th/0008064].

[10] P.M. Stevenson, Phys. Rev. D23, 2916 (1981); G. Grunberg, Phys. Rev. D29, 2315 (1984); S.J. Brodsky, G.P. Lepage and P.B. Mackenzie, Phys. Rev. D28, 228 (1983).

[11] K.G. Wilson, Phys. Rev. 179, 1499 (1969).

[12] E. Witten, Nucl. Phys. B104, 445 (1976). Here by the decoupling notion we mean that there should appear no infinities due to the postulated existence of the underlying structures, not exactly the same as the original notion defined by T. Appelquist and J. Carazzone in [27].

[13] J.-F. Yang, arXiv:hep-th/9708104; invited talk in: *Proceedings of the XIth International Conference ‘Problems of Quantum Field Theory98’*, eds. B.M. Barbashov, G.V. Efimov and A.V. Efremov (Publishing Department of JINR, Dubna, 1999), p.202[arXiv:hep-th/9901138]; arXiv:hep-th/9904055.

[14] G. Sterman, *An Introduction to Quantum Field Theory* (Cambridge University Press, Cambridge, England, 1993), p.299.

[15] D.T. Barclay, C.J. Maxwell and M.T. Reader, Phys. Rev. D49, 3480 (1994); C.J. Maxwell and A. Mirjalili, Nucl. Phys. B577, 209 (2000); C.J. Maxwell and S.J. Burby, Nucl. Phys. B609, 193 (2001).

[16] S.L. Adler, J.C. Collins and A. Duncan, Phys. Rev. D15, 1712 (1977); J.C. Collins, A. Duncan and S.D. Joglekar, Phys. Rev. D16, 438 (1977).

[17] H.J. Lu and S.J. Brodsky, Phys. Rev. D48, 3310 (1993).

[18] See, e.g., S. Willenbrock and G. Valencia, Phys. Lett. B259, 373 (1991); R.G. Stuart, *ibid.* B262, 113 (1991); A. Sirlin, Phys. Rev. Lett. 67, 2127 (1991); T. Bhattacharya and S. Willenbrock, Phys. Rev. D47, 4022 (1993); M. Passera and A. Sirlin, Phys. Rev. D58, 113010 (1998); B.A. Kniehl and A. Sirlin, Phys. Rev. Lett. 81, 1373 (1998).

[19] J.C. Collins, Phys. Rev. D10, 1213 (1974).

[20] R. Coquereaux, Ann. Phys. 125, 401 (1980).

[21] S. Coleman, *Aspects of Symmetry*, (Cambridge University Press, Cambridge, 1985), Chap. 3; J.-F. Yang, arXiv:hep-ph/0212208.
[22] H. Kluberg-Stern and J.B. Zuber, Phys. Rev. D12, 54 (1975); N.K. Nielsen, Nucl. Phys. B97, 527, B120, 212 (1977).
[23] G. ’t Hooft, in: Recent Developments in Gauge Theories, eds. G. ’t Hooft et al (Plenum, New York, 1980), p.135.
[24] K. Symanzik, in: Jasic (ed.), Lectures on High Energy Physics, Zagreb 1961 (Gordon and Breach, NY, 1965); T.T. Wu, Phys. Rev. 125, 1436 (1962); J.G. Taylor, Suppl.al Nuovo Cimento 1, 857 (1963); R.W. Johnson, J. Math. Phys. 11, 2161 (1970).
[25] W.E. Caswell and A.D. Kennedy, Phys. Rev. D25, 392 (1982).
[26] M.S. Chanowitz and J. Ellis, Phys. Rev. D7, 2490 (1973).
[27] T. Appelquist and J. Carazzone, Phys. Rev. D11, 2262 (1975).
[28] S. Weinberg, Phys. Lett. B91, 51 (1980); L.J. Hall, Nucl. Phys. B178, 75 (1981); B. Ovrut and H. Schnitzer, Nucl. Phys. B179, 381 (1981), B189, 509(1981); W. Bernreuther and W. Wetzel, ibid, B197, 228 (1982); W. Wetzel, ibid, B196, 259 (1982).
[29] S. Weinberg, Phys. Rev. 118, 838 (1960).
[30] D.J. Gross and F. Wilczek, Phys. Rev. Lett. 26, 1343 (1973) ; H.D. Politzer, Phys. Rev. Lett. 26, 1346 (1973).
[31] G. Rodrigo and A. Santamaria, Phys.Lett. B313, 441 (1993); K.G. Chetyrkin, B.A. Kniehl and M. Steinhauser, Phys. Rev. Lett. 79, 2184 (1997); G. Rodrigo, A. Pich and A. Santamaria, Phys. Lett. B424, 367 (1998).
[32] see, e.g., M.A. Aivazis, J.C. Collins, F.I. Olness and W.K. Tung, Phys. Rev. D50, 3102 (1994); M. Buza, Y. Matiounine, J. Smith, R. Migneron and W. L. van Neerven, Nucl. Phys. B472, 611 (1996); R.S. Thorne and R.G. Roberts, Phys. Rev. D57, 6871 (1998); M. Krämmer, F.I. Olness and D.E. Soper, ibid, D62, 096007 (2000).