Extractive Contest Design

Tomohiko Kawamori

Abstract

We consider contest success functions (CSFs) that extract contestants’ prize values. In the common-value case, there exists a CSF extractive in any equilibrium. In the observable-private-value case, there exists a CSF extractive in some equilibrium; there exists a CSF extractive in any equilibrium if and only if the number of contestants is greater than or equal to three or the values are homogeneous. In the unobservable-private-value case, there exists no CSF extractive in some equilibrium. When extractive CSFs exist, we explicitly present one of them.

Keywords: contest success function; extraction of values; common or private values; observable or unobservable values; aggregate effort equivalence across equilibria

JEL classification codes: C72; D72
1 Introduction

In the literature on contests, formalized by Tullock (1980), the design of contest success functions (CSFs) that maximize the aggregate effort of contestants is a key topic. Many papers have examined this issue. Maximization of aggregate effort provides CSFs with a positive foundation. In the rent-seeking interpretation, the contest designer (politician) intends to maximize the contestans’ efforts. This is because if the efforts are political contributions, he/she obtains monetary benefits from these efforts, and if the efforts are political lobbying, he/she flaunts his/her power through the efforts. Thus, he/she should determine the CSF as it maximizes the efforts.

This paper considers the design of CSFs that extract the contestans’ values. Most of the existing literature has investigated the maximization of aggregate effort in a class of CSFs that satisfy some restrictions. Owing to these restrictions, the aggregate effort is generically less than the highest value of the contestans, i.e., the optimal CSF does not extract contestans’ values. Instead, this paper considers the design of CSFs without restrictions and examines the extraction of contestans’ values.

We define extractiveness of CSFs. In a contest, contestans make an effort, and the winner of a prize is determined according to a probability distribution that depends on the efforts. A function that maps effort tuples to winning probability distributions is called a CSF. We consider the design of CSFs that extract the contestans’ prize values through the contestans’ efforts. We say that a CSF is extractive if under this CSF, there exists a Nash equilibrium such that the aggregate effort is equal to the maximum value (the maximum of the contestans’ prize values). We also say that a CSF is strictly extractive if this CSF is extractive, and under this CSF, in every Nash equilibrium, the aggregate effort is equal to the maximum value. We consider pure-strategy Nash equilibria.

We examine extractiveness of CSFs, focusing on whether the contestans’ prize
values are common or private and whether they are observable or unobservable by
the contest designer. First, we consider the case where these values are common. We
consider the CSF such that the winning probabilities are proportional to the $\frac{n}{a-1}$th
power of the efforts, where $a$ is in $\{2, 3, \ldots, n\}$ ($n$ is the number of contestants). We
show that the CSF with $a = 2$ is strictly extractive for all common values. Because
this CSF does not depend on common values, this result holds regardless of whether
they are observable or unobservable. For every common value, we also show that
the CSF with $a \geq 3$ is extractive, but not strictly extractive. Thus, aggregate effort
equivalence across Nash equilibria holds for $a = 2$, but not for $a \geq 3$. Second, we
consider the case where the values are private and observable. For every value tuple,
we present an extractive CSF. For every value tuple, we show that there exists a
strictly extractive CSF if and only if the number of contestants is greater than or
equal to 3 or the values are homogeneous. When strictly extractive CSFs exist, we
present one of them. The aggregate effort equivalence across Nash equilibria holds
if and only if the number of contestants is greater than or equal to 3 or the values
are homogeneous. Third, we consider the case where the values are private and
unobservable. We show that there exists no CSF that is extractive for all value
tuples. Therefore, observability matters in the private-value case, but not in the
common-value case.

Several papers have presented CSFs that are reduced to CSFs extractive in the
unobservable-common-value case. In the 2-contestant case, Nti (2004) (Epstein et al.
(2013); Ewerhart (2017a), resp.) presented a CSF that maximizes the aggregate
effort in a class of CSFs (Section 4 (Subsection 4.2; Proposition 6, resp.)). In the
$n$-contestant common-value case, Michaels (1988) did so (Subsection 2.1). Each CSF
in Nti (2004), Epstein et al. (2013) and Ewerhart (2017a) in the common-value case
(the CSF in Michaels (1988), resp.) is the CSF such that the winning probabilities
are proportional to the 2nd ($\frac{n}{n-1}$th, resp.) power of the efforts, and it is in the
unobservable-common-value case because it does not depend on the common value.
The CSF using the 2nd power is strictly extractive in the 2-contestant case. We
show that this CSF is strictly extractive in the $n$-contestant case. The CSF using the
$\frac{n}{n-1}$th power is extractive. We show that if $n \geq 3$, this CSF is not strictly extractive.
Pérez-Castrillo and Verdier (1992) derived Nash equilibria under the CSF such that the winning probabilities are proportional to the $r$th power of the efforts in the $n$-contestant case (Proposition 4). The result of Pérez-Castrillo and Verdier (1992) implies that CSFs such that the winning probabilities are proportional to the $\frac{a}{n-1}$th power of the efforts ($2 \leq a \leq n$) are extractive. We show that if $a \geq 3$, this CSF is not strictly extractive.

Several papers have presented CSFs that are extractive but not strictly extractive in the observable-value case. In the 2-contestant common-value case, Glazer (1993) presented a CSF such that a certain contestant wins if his/her effort is equal to his/her value, and the other contestant wins otherwise (Subsection 3.1). In the 2-contestant case (the $n$-contestant case, resp.), Nti (2004) (Franke et al. (2018), resp.) presented a CSF such that a contestant with the maximum value wins if his/her effort is greater than or equal to his/her value, and a contestant with the second highest value, which may be equal to the maximum value, wins otherwise (Proposition 2 (Proposition 4.7, resp.)). These CSFs are extractive. However, they are not strictly extractive, because there exists a Nash equilibrium such that every contestant’s effort is zero. Nti (2004) suggested that under a modified CSF such that the effort threshold is slightly lowered, the Nash equilibrium such that every contestant’s effort is zero is removed. However, under the modified CSF, in a unique Nash equilibrium, the aggregate effort is slightly smaller than the maximum value. Meanwhile, in the 3-or-more-contestant or homogeneous-value case, we present strictly extractive CSFs. In any Nash equilibrium under such CSFs, the aggregate effort is exactly equal to the maximum value. In the 2-heterogeneous-contestant case, we show that there exists no strictly extractive CSF. This implies that the contest designer cannot design a strictly extractive CSF even though he/she can fully use the information of the values.

Several papers have shown the extraction of values in mixed-strategy Nash equilibria. Hillman and Riley (1989) (Che and Gale (1998); Baye et al. (1993); Baye et al. (1996), resp.) showed that in the all-pay auction, if the highest two values are equal, the expected aggregate effort is equal to the maximum value in any mixed-strategy Nash equilibrium (Proposition 1; the second last paragraph in Section 3 (equation (9); Theorem 1, resp.)). Ewerhart (2017b) showed it in a modified all-pay auction.
(Proposition 1). Alcalde and Dahm (2010) showed that under CSFs that satisfy certain conditions, there exists a mixed-strategy Nash equilibrium such that if the highest two values are equal, the expected aggregate effort is equal to the maximum value (Theorem 3.2). We show the extraction of values in pure-strategy Nash equilibria, even if the highest two values are not equal.

Several papers have considered maximizing aggregate effort in a class of CSFs. CSFs with the following devices have been examined: concave technologies\(^3\) in the lottery contest (Dasgupta and Nti (1998)); concave technologies and power technologies\(^4\) in the lottery contest (Nti (2004)); power technologies in the lottery contest (Michaels (1988)); biases multiplying efforts in the lottery contest (Franke et al. (2013)); biases multiplying efforts with power technologies in the lottery contest and biases multiplying efforts in the all-pay auction (Epstein et al. (2013)); biases multiplying efforts in the lottery contest and all-pay auction (Franke et al. (2014a)); head starts added to efforts in the lottery contest and all-pay auction (Franke et al. (2014b)); biases multiplying efforts given a power technology in the lottery contest (Ewerhart (2017a)); biases multiplying efforts and head starts added to efforts in the lottery contest and all-pay auction (Franke et al. (2018)). Fang (2002) compared the simple lottery contest with the simple all-pay auction. Owing to restrictions on the forms of CSFs, the maximized aggregate effort is not equal to the maximum value except for the aforementioned results. In our paper, because no restrictions are imposed on the forms of CSFs, the values are extracted.

Several papers have considered the aggregate effort under asymmetric information. In Kirkegaard (2012), Pérez-Castrillo and Wettstein (2016), Matros and Possajennikov (2016), Drugov and Ryvkin (2017) and Olszewski and Siegel (2020), the values or productivities of the efforts are private information. In our paper, contestants know the contestants’ values; the contest designer knows them in the observable-value case, but not in the unobservable-value case.

The contributions of this paper are as follows. In the observable-private-value

\(^3\)A concave technology is a concave function that transforms efforts. Winning probabilities are determined based on the transformed efforts.

\(^4\)A power technology is a power function that transforms efforts.

\(^5\)Epstein et al. (2011) considered the same class of CSF, but a different objective of the contest designer, which is the weighted sum of the aggregate effort and welfare.
case, we present strictly extractive CSFs in the 3-or-more-contestant or homogeneous-
value subcase, where we only use the pure-strategy Nash equilibria, whereas we show
that there exists no strictly extractive CSF in the other subcase. In the unobservable-
private-value case, we show that there exists no extractive CSF. In the common-value
case, we show that the CSF with the 2nd-power technology is strictly extractive in
the multi-contestant contest, and the CSF with the $\frac{a}{a-1}$-th-power technology ($a \geq 3$)
is not strictly extractive. We demonstrate that for extractive or strictly extractive
CSFs to exist, observability of values matters in the private-value case, but not in the
common-value case. The framework in this paper could serve as a general framework
for investigating the extraction of values in contests.

The remainder of this paper is organized as follows. Section 2 describes the model.
Section 3 presents the results. Section 4 concludes the paper. The proofs of all the
propositions are provided in the appendix.

## 2 Model

For any sets $X$, $Y$ and $I$, any $f : X \rightarrow Y^I$ and any $x \in X$ and $i \in I$, let $f_i(x)$ be the
value of $f(x)$ for $i$.

Let $N$ be a finite set such that $|N| \geq 2$: $N$ is the set of contestants. Let
$n := |N|$. Let $X := \mathbb{R}_{\geq 0}^N$: $X$ is the set of tuples of contestants’ efforts. Let
$\Delta := \{p \in \mathbb{R}_{\geq 0}^N \mid \sum_{i \in N} p_i = 1\}$: the set of tuples of contestants’ success probabili-
ties (for any $p \in \Delta$ and any $i \in N$, $p_i$ is the probability of contestant $i$’s winning).
Let $F := \{f \mid f : X \rightarrow \Delta\}$: $F$ is the set of contest success functions (CSFs). Let
$V := \mathbb{R}_{\geq 0}^N$: the set of tuples of contestants’ prize values. For any $f \in F$ and any $v \in V$,
let $u_f^v : X \rightarrow \mathbb{R}^N$ such that for any $x \in X$ and any $i \in N$, $u_f^v (x) = f_i(x) v_i - x_i$;
$u_f^v (x)$ is contestant $i$’s utility from effort tuple $x$ ($f_i(x) v_i$ is the expected value that
he/she obtains, and $x_i$ is the cost of his/her effort).

For any $f \in F$ and any $v \in V$, $(N, X, u_f^v)$ is a strategic-form game: $N$ is the set
of players, $X$ is the set of strategy tuples, and $u_f^v$ is the function that maps each
strategy tuple to its payoff tuple. For any $f \in F$ and any $v \in V$, let $E_f^v$ be the
set of pure-strategy Nash equilibria in $(N, X, u_f^v)$. In the following, we refer to a
pure-strategy Nash equilibrium simply as a Nash equilibrium.
Let $\hat{V} := \{v \in V \mid (\forall i, j \in N) v_i = v_j\}$: the set of value tuples such that all contestants have a common value. For any $v \in V$, let $m^v := \max_{i \in N} v_i$ and $M^v := \arg\max_{i \in N} v_i$: $m^v$ is the maximum of contestants’ prize values, and $M^v$ is the set of contestants with the maximum value.

3 Results

We refer to the case where the domain of the value tuples is $\hat{V}$ ($V$, resp.) as the common-value case (private-value case, resp.). We also refer to the case where value tuples are observable (unobservable, resp.) by the contest designer, i.e., CSFs can (cannot, resp.) depend on value tuples the observable-value case (unobservable-value case, resp.). We seek CSFs under which the equilibrium aggregate effort is equal to the maximum value in the common-value case, the observable-private-value case and the unobservable-private value case, respectively. We say that in the private-value case, if $v \in V$ satisfies that for any $i, j \in N$, $v_i = v_j$, we say that $v$ is homogeneous. The value tuples are observable by the contestants.

3.1 Bound of aggregate effort

For any CSF and any value tuple, in any Nash equilibrium, the aggregate effort is less than or equal to the maximum value.

**Proposition 1.** Let $f \in F$ and $v \in V$. Let $x^* \in E^{fv}$. Then, $\sum_{i \in N} x_i^* \leq m^v$.

We say that a CSF is extractive if in some Nash equilibrium, the aggregate effort is equal to the maximum value. We say that a CSF is strictly extractive if it is extractive and in any Nash equilibrium, the aggregate effort is equal to the maximum value.

**Definition 1.** Let $f \in F$ and $v \in V$. $f$ is extractive for $v$ if there exists $x^* \in E^{fv}$ such that $\sum_{i \in N} x_i^* = m^v$. $f$ is strictly extractive for $v$ if $f$ is extractive for $v$ and for all $x^* \in E^{fv}$, $\sum_{i \in N} x_i^* = m^v$. 

7
3.2 Common-value case

We consider the case where contestants have a common value. For any \( a \in \mathbb{N} \) such that \( 2 \leq a \leq n \), let \( f^a \in F \) such that for any \( i \in N \) and any \( x \in X \),

\[
f_i(x) = \begin{cases} \frac{x^{\frac{a}{a-1}}}{\sum_{j \in N} x_j} & \text{if } (\exists j \in N) x_j > 0 \\ \frac{1}{n} & \text{otherwise.} \end{cases}
\]

Under \( f^a \), winning probabilities are proportional to the \( \frac{a}{a-1} \)-th power of the efforts.

There exists a CSF strictly extractive for all common values.

**Proposition 2.** There exists \( f \in F \) that is strictly extractive for all \( v \in \hat{V} \).

**Remark 1.** A purely logical consequence of this proposition is that for all \( v \in \hat{V} \), there exists \( f \in F \) that is strictly extractive for \( v \). Thus, this proposition implies that whether the contest designer can observe the common value, there exists a strictly extractive CSF. Furthermore, regardless of the observability, there exists an extractive CSF.

**Remark 2.** In the proof, such CSF \( f \) is constructed as \( f = f^2 \).

**Remark 3.** As seen in the proof, under \( f = f^2 \) and any \( v \in \hat{V} \), for any \( x \in X \), \( x \in Efv \) if and only if for some \( A \in 2^N \) such that \( |A| = a \), for any \( i \in A \), \( x_i = \frac{m^{v}_{a}}{a} \) and for any \( i \in N \setminus A \), \( x_i = 0 \).

**Remark 4.** Under \( f = f^2 \) and any \( v \in \hat{V} \), the aggregate effort equivalence across Nash equilibria holds.

For all common values, \( f^a \ (3 \leq a \leq n) \) is extractive but not strictly extractive.

**Proposition 3.** Let \( a \in \mathbb{N} \) such that \( 3 \leq a \leq n \). Let \( v \in \hat{V} \). Then, \( f^a \) is extractive for \( v \) and not strictly extractive for \( v \).

**Remark 5.** As seen in the proof, under \( f = f^a \) and any \( v \in \hat{V} \), for any \( x \in X \), \( x \in Efv \) if (i) for some \( A \in 2^N \) such that \( |A| = a \), for any \( i \in A \), \( x_i = \frac{m^{v}_{a}}{a} \) and for any \( i \in N \setminus A \), \( x_i = 0 \). The aggregate effort in a strategy tuple satisfying (ii) is \( (a-1) \frac{m^{v}(a-2)}{(a-1)^2} = (a-1) \frac{m^{v}_a(a-2)}{(a-1)^2} < v \). However, for any \( a \in \mathbb{N} \) such that \( 3 \leq a_n \leq n \) and \( \lim_{n \to \infty} a_n = \infty \), \( \lim_{n \to \infty} \frac{m^{v}_a(a-2)}{(a-1)^2} = v \).
Remark 6. Under \( f = f^a \) and any \( v \in \hat{V} \), the aggregate effort equivalence across Nash equilibria does not hold.

Under \( f^a \) \((2 \leq a \leq n)\), each contestant’s effort \( x \) is transformed into \( x^{\frac{a}{a-1}} \), and the winning probabilities are determined in proportion to the transformed efforts. As \( a \) is larger, elasticity of the transformed effort \( x^{\frac{a}{a-1}} \) to effort \( x \), \( \frac{dx^{\frac{a}{a-1}}}{dx}x = \frac{a}{a-1} \), is smaller; thus, in the Nash equilibrium such that the aggregate effort is equal to the maximum value, the effort of each active contestant is smaller, but the number of active contestants is larger.

The results in Nti (2004), Epstein et al. (2013) and Ewerhart (2017a) imply that \( f^2 \) is strictly extractive for all \( v \in \hat{V} \) when \( n = 2 \). Our paper shows that \( f^2 \) is strictly extractive for all \( v \in \hat{V} \) when \( n \geq 3 \). The results in Michaels (1988) imply that \( f^n \) is extractive for all \( v \in \hat{V} \). Our paper shows that when \( n \geq 3 \), for any \( v \in \hat{V} \), \( f^n \) is not strictly extractive for \( v \). The results in Pérez-Castrillo and Verdier (1992) imply that \( f^a \) is extractive for all \( v \in \hat{V} \). Our paper shows that when \( a \geq 3 \), for any \( v \in \hat{V} \), \( f^a \) is not strictly extractive for \( v \).

### 3.3 Observable-private-value case

We consider the case where contestants have private values, and the contest designer can observe them and design CSFs dependent on them.

For all value tuples, there exists a CSF extractive for this value tuple.

**Proposition 4.** Let \( v \in V \). Then, there exists \( f \in F \) that is extractive for \( v \).

**Remark 7.** In the proof, such CSF \( f \) is constructed as follows. In the case where \( n \geq 3 \), for some distinct \( i,j,k \in N \) such that \( i \in M^v \), for any \( x \in X \), if \( x_i = m^v \), then \( f_i (x) = 1 \); if \( x_i \neq m^v \) and \( x_j > 0 \), then \( f_j (x) = 1 \); if \( x_i \neq m^v \) and \( x_j = 0 \), then \( f_k (x) = 1 \). In the case where \( n = 2 \) and \( v \in \hat{V} \), \( f \) is \( f^2 \) defined in Subsection 3.2. In the case where \( n = 2 \) and \( v \notin \hat{V} \), for some \( i \in M^v \), for any \( x \in X \), \( f_i (x) = 1_{x_i=m^v} \).

For all value tuples, if the number of contestants is greater than or equal to 3 or the values of the contestants are homogeneous, there exists a CSF strictly extractive for this value tuple, and otherwise, there does not.
Proposition 5. Let \( v \in V \). Then, there exists \( f \in F \) that is strictly extractive for \( v \) if and only if \( n \geq 3 \) or \( v \in \hat{V} \).

Remark 8. In the proof, such \( f \) is constructed as follows. In the case where \( n \geq 3 \) and the case where \( n = 2 \) and \( v \in \hat{V} \), \( f \) is one in the corresponding case in Remark 7.

Remark 9. Aggregate effort equivalence across Nash equilibria holds if and only if \( n = 3 \) or \( v \in \hat{V} \).

The above CSFs make the contestant with the maximum value win with certainty (or probability \( \frac{1}{2} \)) if his/her effort is equal to the maximum value (or \( \frac{1}{2} \) of the maximum value), in order that the aggregate effort is equal to the maximum value. In the 3-or-more-contestant or homogeneous-value case, the above CSFs are designed as they exclude the Nash equilibrium such that every contestant’s effort is zero. In the other case, it is impossible to design an extractive CSF that excludes such a Nash equilibrium even though the contest designer can fully use the information on the contestants’ values.

In the observable-value case, Glazer (1993), Nti (2004) and Franke et al. (2018) presented a CSF that is extractive, but this is not strictly extractive. For any \( v \in V \), when \( n \geq 3 \) or \( v \in \hat{V} \), our paper presents a CSF that is strictly extractive for \( v \); when \( n = 2 \) and \( v \notin \hat{V} \), our paper shows that there exists no CSF that is strictly extractive for \( v \).

3.4 Unobservable-private-value case

We consider the case where contestants have private values, and the contest designer cannot observe them and must design CSFs independent of them.

There exists no CSF that is extractive for all value tuples.

Proposition 6. There exists no \( f \in F \) that is extractive for all \( v \in V \).

Remark 10. An immediate consequence of this proposition is that there exists no \( f \in F \) that is strictly extractive for all \( v \in V \).

Under any CSF, if for some value tuple, the aggregate effort in a Nash equilibrium is equal to the maximum value, then for some other value tuple, it must be less than...
the maximum value.

In the literature, it has not been examined whether there exists a CSF that is extractive for all $v \in V$. Our paper provides a negative answer to this question.

## 4 Conclusion

Table 1 summarizes the results of this paper, where $\phi^E$ is a formula meaning that $f$ is extractive for $v$, and $\phi^{SE}$ is a formula meaning that $f$ is strictly extractive for $v$. Whether the values are common or private is represented by whether the domain of the values is $\hat{V}$ or $V$. Whether the values are observable or unobservable is represented by whether the order of the quantifiers is $(\forall v) (\exists f)$ or $(\exists f) (\forall v)$. In the common-value case, regardless of the observability, there exist extractive and strictly extractive CSFs. In the observable-private-value case, there exists an extractive CSF, but there does not always exist a strictly extractive CSF. In the unobservable-private-value case, there exists neither extractive nor strictly extractive CSF. In the common-value case, we also present a class of extractive CSFs that can be used to control the number of active contestants.

|            | Observable ($\forall v \ (\exists f)$) | Unobservable ($\exists f \ (\forall v)$) |
|------------|--------------------------------------|----------------------------------------|
| Common $\hat{V}$ | $(\forall v \in \hat{V}) (\exists f \in F) \phi^E$ | $(\exists f \in F) (\forall v \in \hat{V}) \phi^E$ |
|            | $(\forall v \in \hat{V}) (\exists f \in F) \phi^{SE}$ | $(\exists f \in F) (\forall v \in \hat{V}) \phi^{SE}$ |
| Private $V$ | $(\forall v \in V) (\exists f \in F) \phi^E$ | $\neg (\exists f \in F) (\forall v \in V) \phi^E$ |
|            | $(\exists f \in F) (\forall v \in \hat{V}) \phi^{SE}$ | $\neg (\exists f \in F) (\forall v \in V) \phi^{SE}$ |

Table 1: Existence of extractive or strictly extractive CSFs

In the unobservable-private-value case, there exists no extractive CSF. In such a case, it is necessary to derive CSFs that maximize the expectation of the aggregate effort under some belief on value tuples. For example, this problem is formalized as follows:

$$
\max_{(f,x) \in F \times X^V} \int_{v \in \hat{V}} \sum_{i \in N} x_i(v) \, dP(v)
\quad \text{s.t. } (\forall v \in V) \, x(v) \in E^{fv} \land x \text{ is measurable},
$$
where $P$ is a cumulative distribution function on $V$ (the designer’s belief on value tuples).

In this paper, we consider pure strategies, but not mixed strategies. Propositions 3 and 4 also hold in mixed strategies because if there exists a pure-strategy Nash equilibrium, it is also a mixed-strategy Nash equilibrium. However, it is not clear whether Propositions 2, 5 and 6 hold in mixed strategies because there might exist mixed-strategy Nash equilibria other than pure-strategy Nash equilibria.
Appendix

**Lemma 1.** Let $f \in F$, $v \in V$, $x^* \in E_{f,v}$ and $i \in N$. Then, $u_i^{f,v}(x^*) \geq 0$, and $f_i(x^*) v_i \geq x_i^*.$

*Proof.* Because $x^* \in E_{f,v}$, $u_i^{f,v}(x^*) \geq u_i^{f,v}(0,x_{-i}^*) = f_i(0,x_{-i}^*) v_i \geq 0$. Thus, $f_i(x^*) v_i \geq x_i^*.$ \hfill \Box

*Proof of Proposition.* By Lemma 1, for any $i \in N$, $x_i^* \leq v_i f_i(x^*) \leq m^v f_i(x^*)$. Hence, $\sum_{i \in N} x_i^* \leq m^v \sum_{i \in N} f_i(x^*) = m^v.$ \hfill \Box

**Lemma 2.** Let $v \in \mathbb{R}_{>0}$ and $a, b \in \mathbb{N}$ such that $2 \leq b \leq a$. Let $x^* := \frac{va(b-1)}{b^2(a-1)}$. Let $u : \mathbb{R}_{\geq 0} \to \mathbb{R}$ such that for any $x \in \mathbb{R}_{\geq 0}$, $u(x) = \frac{x^a}{x^a + (b-1)(x^*)^a} v - x$. Then, $x^* \in \arg \max_{x \in \mathbb{R}_{\geq 0}} u(x)$.

*Proof.* Let $\phi : \mathbb{R}_{\geq 0} \to \mathbb{R}$ such that for any $x \in \mathbb{R}_{\geq 0}$,

$$\phi(x) = -(2b-1) (x^*)^{\frac{a}{a-1}} + x^{\frac{a}{a-1}} \sum_{i=0}^{a-1} (x^*)^{\frac{i}{a-1}}x^{\frac{a-1-i}{a-1}} + b^2 (x^*)^{2a-1}.$$ 

For any $x \in \mathbb{R}_{\geq 0}$,

$$\frac{du(x)}{dx} = \frac{\phi(x) \left(x^{\frac{1}{a-1}} - (x^*)^{\frac{1}{a-1}}\right)}{\left(x^{\frac{a}{a-1}} + (b-1)(x^*)^{\frac{a}{a-1}}\right)^2}.$$ 

Note that $\phi(0) = (b-1)^2 (x^*)^{\frac{2a-1}{a-1}} > 0$, and $\phi(x^*) = -b(2a-b)(x^*)^{\frac{2a-1}{a-1}} < 0$. Then, by the intermediate value theorem, there exists $\bar{x} \in (0, x^*)$ such that $\phi(\bar{x}) = 0$. Let $x \in \mathbb{R}_{\geq 0}$. Because $\phi$ is strictly decreasing, $\phi(x) \geq 0$ if and only if $x \leq \bar{x}$. Thus,

$$\frac{du(x)}{dx} = \begin{cases} 
\leq 0 & \text{if } x \leq \bar{x} \\
\geq 0 & \text{if } \bar{x} < x \leq x^* \\
< 0 & \text{if } x > x^*.
\end{cases}$$

Note that $u(0) = 0 \leq u(x^*)$. Then, for any $x \in \mathbb{R}_{\geq 0}$, $u(x^*) \geq u(x).$ \hfill \Box

**Lemma 3.** Let $a \in \mathbb{N}$ such that $2 \leq a \leq n$. Let $f = f^a$. Then, $f$ is extractive for all $v \in \hat{V}$.
Proof. Let \( v \in \hat{V} \). Abuse \( v \) as the common value \((v = m^v = v_i \text{ for any } i \in N)\). Let \( x^* \in X \) such that for some \( A \in 2^N \) such that \(|A| = a\), for any \( i \in A, x_i^* = \frac{v}{a}\) and for any \( j \in N \setminus A, x_j^* = 0 \). Let \( i \in A \). By Lemma 2 with \( b = a \), for any \( x_i \in \mathbb{R}_{\geq 0} \), \( u_i^{f^v}(x^*) \geq u_i^{f^v}(x_i, x_i^*) \). Let \( j \in N \setminus A \). For any \( x_j \in \mathbb{R}_{\geq 0} \),

\[
u_j^{f^v}(x_j, x_j^*) = \frac{x_j^*}{x_j^* + a \left( \frac{v}{a} \right)^{x_j^*}} v - x_j < \frac{x_j^*}{x_j^* + (a - 1) \left( \frac{v}{a} \right)^{x_j^*}} v - x_j
\]

\[
u_i^{f^v}(x_j, x_j^*) \leq u_i^{f^v}(x^*) = 0 = u_j^{f^v}(x^*) .
\]

Thus, \( x^* \in E^{f^v} \). \( \sum_{i \in N} x_i^* = a \cdot \frac{v}{a} = v \).

\[\square\]

Proof of Proposition 2

By Lemma 3, \( f = f^2 \).

Let \( v \in \hat{V} \). Abuse \( v \) as the common value \((v = m^v = v_i \text{ for any } i \in N)\). Let \( x^* \in E^{f^v} \). Let \( A := \{i \in N \mid x_i^* > 0\} \) and \( \alpha := |A| \). If \( \alpha = 0 \), for some \( i \in N \), \( u_i^{f^v}(x^*) = \frac{v}{\alpha} < \left( \frac{2n - 1}{2n} \right) u_i^{f^v}(x_i, x_i^*) \), which contradicts that \( x^* \in E^{f^v} \). If \( \alpha = 1 \), for some \( i \in A \), \( u_i^{f^v}(x^*) = v - x_i^* < v - x_i^* = u_i^{f^v}(x_i, x_i^*) \), which contradicts that \( x^* \in E^{f^v} \). Thus, \( \alpha \geq 2 \). Let \( i \in \arg \max_{j \in A} x_j^* \). For any \( k \in A \),

\[
0 = \frac{\partial u_i^{f^v}}{\partial x_k}(x^*) = \frac{2vx_k^* \left( \sum_{l \in A} (x_l^*)^2 - (x_k^*)^2 \right)}{\left( \sum_{l \in A} (x_l^*)^2 \right)^2} - 1.
\]

Thus, for any \( k \in A \setminus \{i\} \) such that \( x_k^* < x_i^* \), \( x_i^* \left( \sum_{l \in A} (x_l^*)^2 - (x_k^*)^2 \right) = x_k^* \left( \sum_{l \in A} (x_l^*)^2 - (x_k^*)^2 \right) \) and \( (x_i^* - x_k^*) \left( \sum_{l \in A \setminus \{i,k\}} (x_l^*)^2 - x_i^* x_k^* \right) = 0 \), \( \sum_{l \in A \setminus \{i,k\}} (x_l^*)^2 = x_i^* x_k^* \). Suppose that for some \( j \in A \setminus \{i\} \), \( x_j^* < x_i^* \) (assumption for contradiction). Then, \( \sum_{l \in A \setminus \{i,j\}} (x_l^*)^2 = x_i^* x_j^* \). For any \( k \in A \setminus \{i,j\} \), \( (x_j^*)^2 \leq \sum_{l \in A \setminus \{i,j\}} (x_l^*)^2 = x_i^* x_j^* \) and thus, \( x_k^* < x_i^* \). Hence, for any \( k \in A \setminus \{i\} \), \( x_k^* < x_i^* \). Thus, for any \( k \in A \setminus \{i\} \), \( x_i^* x_j^* + (x_i^*)^2 = \sum_{l \in A \setminus \{i\}} (x_l^*)^2 = x_i^* x_j^* + (x_k^*)^2 \), \( (x_j^* - x_k^*) \left( x_j^* + x_k^* + x_i^* \right) = 0 \), and \( x_i^* = x_k^* \). Thus, \( \alpha \left( \frac{(a - 2)}{\alpha^2} \right) \left( x_j^* \right)^2 = x_i^* x_j^* \) and \( x_i^* = (\alpha - 2) x_j^* \). Hence, \( 0 = \frac{\partial u_i^{f^v}}{\partial x_j}(x^*) = \frac{2v \left( \alpha - 1 \right) \left( \alpha - 2 \right)}{\left( (a - 3) \alpha + 3 \right)^2} \left( x_j^* \right)^2 - 1 \), \( x_j^* = \frac{2v \left( \alpha - 1 \right) \left( \alpha - 2 \right)}{\left( (a - 3) \alpha + 3 \right)^2} \), and \( \alpha \geq 3 \). Thus, \( u_{f^v}(x^*) = \frac{(x_j^*)^2}{\left( (a - 2) x_j^* \right)^2 + (x_i^*)^2} - \frac{v \left( \alpha - 1 \right) \left( \alpha - 2 \right)}{\left( (a - 3) \alpha + 3 \right)^2} < 0 \), which contradicts Lemma 1. Hence, for any \( j \in A \), \( x_j^* = x_i^* \). Thus, \( 0 = \frac{\partial u_i^{f^v}}{\partial x_i}(x^*) = \frac{2v \left( \alpha - 1 \right) \left( \alpha - 2 \right)}{\alpha^2 x_i^*} - 1 \), and \( x_i^* = \frac{2v \left( \alpha - 1 \right) \left( \alpha - 2 \right)}{\alpha^2} \). Hence, \( u_{f^v}(x^*) = \frac{v \left( 2 \alpha - \alpha \right)}{\alpha^2} \). Thus, by Lemma 1 \( \alpha = 2 \). Hence, \( x_i^* = \frac{v}{2} \). Thus, for any \( j \in A \),
$x_j^* = \frac{v}{2}$. Hence, $\sum_{i \in N} x_i^* = 2 \cdot \frac{v}{2} = v$. \hfill \Box

**Proof of Proposition 3.** By Lemma 4, $f$ is extractive for $v$.

Abuse $v$ as the common value ($v = m^v = v_i$ for any $i \in N$). Let $x^*$ be a strategy tuple such that for some $A \in 2^N$ such that $|A| = a - 1$, for any $i \in A$, $x_i^* = \frac{va(a-2)}{(a-1)^2}$ and for any $j \in N \setminus A$, $x_j^* = 0$. Let $i \in A$. By Lemma 2 with $b = a - 1$, for any $x_i \in \mathbb{R}_{\geq 0}$, $u_i^{fv}(x^*) \geq u_i^{fv}(x_i, x_{-i})$. Let $j \in N \setminus A$. For any $x_j \in \mathbb{R}_{\geq 0}$,

$$u_j^{fv}(x_j, x_{-j}^*) = \frac{x_j \left( \frac{1}{a} v - x_j \frac{a}{(a-1)^2} - (a-1) \frac{va(a-2)}{(a-1)^2} \frac{a}{a-1} \right)}{x_j^{a-1} + (a-1) \frac{va(a-2)}{(a-1)^2} \frac{a}{a-1}} = - \frac{(a-1) x_j \left( \frac{a}{a-1} \left( 1 + \frac{a(a-3)+1}{(a-1)^2} \frac{a}{a-1} \right) - 1 \right)}{x_j^{a-1} + (a-1) \frac{va(a-2)}{(a-1)^2} \frac{a}{a-1}} \leq 0 = u_j^{fv}(x^*) .$$

Thus, $x^* \in Ef^v$. $\sum_{i \in N} x_i^* = (a-1) \cdot \frac{va(a-2)}{(a-1)^2} = \frac{(a-1)^2-1}{(a-1)^2} v \neq v$. Thus, $f$ is not strictly extractive for $v$. \hfill \Box

**Lemma 4.** Let $v \in V$. Suppose that $n \geq 3$. Let $f \in F$ such that for some distinct $i, j, k \in N$ such that $i \in M^v$, for any $x \in X$, (i) if $x_i = m^v$, then $f_i (x) = 1$, (ii) if $x_i \neq m^v$ and $x_j > 0$, then $f_j (x) = 1$, and (iii) if $x_i \neq m^v$ and $x_j = 0$, then $f_k (x) = 1$. Then, $f$ is strictly extractive for $v$.

**Proof.** Let $x^* \in X$ such that $x_i^* = m^v$ and for any $l \in N \setminus \{i\}$, $x_l^* = 0$. Then, $\sum_{l \in N} x_l^* = m^v$. It suffices to show that $Ef^v = \{x_i^* \}$.

For any $x_i \in \mathbb{R}_{\geq 0} \setminus \{x_i^* \}$, $u_i^{fv}(x^*) = 0 \geq -x_i = u_i^{fv}(x_i, x_{-i}^*)$. For any $l \in N \setminus \{i\}$ and any $x_l \in \mathbb{R}_{\geq 0} \setminus \{x_l^* \}$, $u_l^{fv}(x^*) = 0 \geq -x_l = u_l^{fv}(x_l, x_{-l}^*)$. Thus, $x^* \in Ef^v$.

Let $x \in X \setminus \{x^* \}$. If $x_i = m^v$, then for some $l \in N \setminus \{i\}$, $x_l > 0$, and $u_l^{fv}(x) = -x_l < 0 = u_l^{fv}(0, x_{-l})$. If $x_i \neq m^v$ and $x_j > 0$, then $u_j^{fv}(x) = v_j - x_j < v_j - \frac{x_j}{2} = 0$. If $x_i \neq m^v$ and $x_j = 0$, then $u_j^{fv}(x) = 0 = v_j - x_j = -v_j$. Thus, $x \notin Ef^v$. Therefore, $Ef^v = \{x_i^* \}$. \hfill \Box
$u^v_j(x, x_j)$. If $x_i \neq m^v$ and $x_j = 0$, then $u^v_j(x) = 0 < v^i_j = u^v_j(x, x_j)$. Thus, $x \notin E^v$.

\[ \square \]

**Lemma 5.** Let $v \in V$. Suppose that $n = 2$ and $v \notin \hat{V}$. (i) Let $f \in F$ such that for some $i \in M^v$, for any $x \in X$, $f_i(x) = 1_{x_i=m^v}$. Then, $f$ is extractive for $v$. (ii) Let $f \in F$. $f$ is not strictly extractive for $v$.

**Proof.** (i) Let $j \in N \setminus \{i\}$. Let $x^* \in X$ such that $x^*_i = m^v$, and $x^*_j = 0$. For any $x_i \in \mathbb{R}_{\geq 0} \setminus \{x^*_i\}$, $u^v_i = x^*_i = m^v \geq f_i(x^*) \leq 0$. Thus, $f_j(x^*) = 0$. Hence, $f_j(x^*) = 0$. Thus, by Lemma 1, $x^*_i = m^v$. Hence, $0 = u^v_i(0, x^*_i) = f_i(0, x^*_i) v_i$. Thus, $f_i(0, x^*_i) = 0$. Hence, $f_i(0, 0) = 0$. Let $y^* \in X$ such that $y^*_i = y^*_j = 0$. Because $x^* \in E^v$ and $x^*_j = y^*_j$, for any $y_i \in \mathbb{R}_{\geq 0}$, $u^v_{y_i} = f_i(0) v_i = 0 = u^v_i(0, x^*_i) \geq u^v_i(y_i, x^*_i) = u^v_i(y_i, y^*_i)$. For any $y_j \in \mathbb{R}_{\geq 0}$, $u^v_{y_j} = f_j(0) v_j = v_j \geq f_j(y_j, y^*_j) v_j - y_j = u^v_j(y_j, y^*_j)$. Thus, $y^* \in E^v$. $y^*_i + y^*_j = 0 \neq m^v$. Thus, $f$ is not strictly extractive for $v$.

\[ \square \]

**Proof of Proposition 4.** The conclusion follows from Proposition 2 and Lemmas 4 and 5.

**Proof of Proposition 5.** The conclusion follows from Proposition 2 and Lemmas 4 and 5.

**Proof of Proposition 6.** Suppose that there exists $f \in F$ that is extractive for all $v \in V$ (assumption for contradiction).

Let $v \in V$ such that for some $i \in N$, for any $j \in N \setminus \{i\}$, $v_i > v_j$. Let $x^* \in E^v$ such that $\sum_{j \in N} x^*_j = v_i$. By Lemma 1, $v_i = \sum_{j \in N} x^*_j \geq \sum_{j \in N} v_j f_j(x^*)$. Thus, $f_i(x^*) = 1$, and for any $i \in N \setminus \{i\}$, $f_j(x^*) = 0$. Hence, by Lemma 1, for any $j \in N \setminus \{i\}$, $x^*_j = 0$, and $x^*_i = v_i$.

Let $v, w \in \mathbb{R}^N_{\geq 0}$ such that for some $i \in N$, $v_i = 1$ and $w_i = 2$, and $v_j < v_i$ and $w_j < w_i$ for any $j \in N \setminus \{i\}$. Then, by the assumption for contradiction, there exist $x^* \in E^v$ and $y^* \in E^w$ such that $\sum_{j \in N} x^*_j = v_i$ and $\sum_{j \in N} y^*_j = w_i$. Thus, $x^*_i = 1$
and $y_i^* = 2$, and for any $j \in N \setminus \{i\}$, $x_j^* = y_j^* = 0$. Moreover, $f_i(x^*) = f_i(y^*) = 1$.

Thus, $u^w_i(1, y_i^* - 1) = 2f_i(1, y_i^*) - 1 = 2f_i(x^*) - 1 = 1 > 0 = u^w_i(y^*)$, which contradicts that $y^* \in E^w$. \[\square\]
References

J. Alcalde and M. Dahm. Rent seeking and rent dissipation: A neutrality result. 
*Journal of Public Economics*, 94:1–7, 2010.

M. R. Baye, D. Kovenock, and C. G. de Vries. Rigging the lobbying process: An 
application of the all-pay auction. *American Economic Review*, 83:289–294, 1993.

M. R. Baye, D. Kovenock, and C. G. de Vries. The all-pay auction with complete 
information. *Economic Theory*, 8:291–305, 1996.

Y.-K. Che and I. L. Gale. Caps on political lobbying. *American Economic Review*, 
88:643–651, 1998.

D. J. Clark and C. Riis. Contest success functions: an extension. *Economic Theory*, 
11:201–204, 1998.

A. Dasgupta and K. O. Nti. Designing an optimal contest. *European Journal of 
Political Economy*, 14:587–603, 1998.

M. Drugov and D. Ryvkin. Biased contests for symmetric players. *Games and 
Economic Behavior*, 103:116–144, 2017.

G. S. Epstein, Y. Mealem, and S. Nitzan. Political culture and discrimination in 
contests. *Journal of Public Economics*, 95:88–93, 2011.

G. S. Epstein, Y. Mealem, and S. Nitzan. Lotteries vs. all-pay auctions in fair and 
biased contests. *Economics and Politics*, 25:48–60, 2013.

C. Ewerhart. Revenue ranking of optimally biased contests: The case of two players. 
*Economics Letters*, 157:167–170, 2017a.

C. Ewerhart. Contests with small noise and the robustness of the all-pay auction. 
*Games and Economic Behavior*, 105:195–211, 2017b.

H. Fang. Lottery versus all-pay auction models of lobbying. *Public Choice*, 112: 
351–371, 2002.
J. Franke, C. Kanzow, W. Leininger, and A. Schwartz. Effort maximization in asymmetric contest games with heterogeneous contestants. *Economic Theory*, 52:589–630, 2013.

J. Franke, C. Kanzow, W. Leininger, and A. Schwartz. Lottery versus all-pay auction contests: A revenue dominance theorem. *Games and Economic Behavior*, 83:116–126, 2014a.

J. Franke, W. Leininger, and C. Wasser. Revenue maximizing head starts in contests. Technical Report 524, Ruhr Economic Paper, 2014b.

J. Franke, W. Leininger, and C. Wasser. Optimal favoritism in all-pay auctions and lottery contests. *European Economic Review*, 104:22–37, 2018.

A. Glazer. On the incentives to establish and play political rent-seeking games. *Public Choice*, 75:139–148, 1993.

A. L. Hillman and J. G. Riley. Politically contestable rents and transfers. *Economics and Politics*, 1:17–39, 1989.

H. Jia, S. Skaperdas, and S. Vaidya. Contest functions: Theoretical foundations and issues in estimation. *International Journal of Industrial Organization*, 31:211–222, 2013.

R. Kirkegaard. Favoritism in asymmetric contests: Head starts and handicaps. *Games and Economic Behavior*, 76:226–248, 2012.

A. Matros and A. Possajennikov. Tullock contests may be revenue superior to auctions in a symmetric setting. *Economics Letters*, 142:74–77, 2016.

R. Michaels. The design of rent-seeking competitions. *Public Choice*, 56:17–29, 1988.

K. O. Nti. Maximum efforts in contests with asymmetric valuations. *European Journal of Political Economy*, 20:1059–1066, 2004.

W. Olszewski and R. Siegel. Performance-maximizing large contests. *Theoretical Economics*, 15:57–88, 2020.
D. Pérez-Castrillo and D. Wettstein. Discrimination in a model of contests with incomplete information about ability. *International Economic Review*, 57:881–914, 2016.

J. D. Pérez-Castrillo and T. Verdier. A general analysis of rent-seeking games. *Public Choice*, 73:335–350, 1992.

S. Skaperdas. Contest success functions. *Economic Theory*, 7:283–290, 1996.

G. Tullock. Efficient rent seeking. In J. M. Buchanan, R. D. Tollison, and G. Tullock, editors, *Toward a Theory of the Rent-Seeking Society*, pages 97–112. Texas A&M University Press, 1980.