Self-organized criticality and synchronization in pulse coupled relaxation oscillator systems; the Olami, Feder and Christensen and the Feder and Feder model

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Abstract

We reexamine the dynamics of the Olami, Feder and Christensen (OFC) model. We show that, depending on the dissipation, it exhibits two different behaviors and that it can or cannot show self-organized criticality (SOC) and/or synchronization. We also show that while the Feder and Feder model perturbed by a stochastic noise is SOC and has the same exponent for the distribution of avalanche sizes as the OFC model, it does not show synchronization. We conclude that a relaxation oscillator system can be synchronized and/or SOC and that therefore synchronization is not necessary for criticality in these models.

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I. INTRODUCTION

The absence of characteristic scales in numerous natural phenomena has motivated the introduction of the concept of Self-Organized-Criticality (SOC) [1]. According to this concept, scale invariance would be “naturally” and dynamically generated in out of equilibrium systems. Up to now, it is not known if a few basic mechanisms are at work in SOC or if each scale invariant phenomenon is a particular case. The dynamics of most of the known SOC models is an avalanche dynamics due, in many cases, to a threshold dynamics at the microscopic level. For instance, in the sandpile model, once the sand height exceeds the threshold on one site, it relaxes at that site and being redistributed on the neighbors, can trigger an avalanche of relaxations. In this kind of models, the avalanches have no characteristic scale if the local state variable, the sand in our example, is conserved. However, this conservation is not in general necessary for criticality. In particular, the whole class of coupled relaxation oscillator systems exhibiting SOC do not require conservation. In this case, the critical exponent \( \nu \) of the avalanche distribution \( P(s) \sim s^{-\nu} \) can depend continuously on the dissipation. In the following, we study two of these systems, the Olami, Feder and Christensen (OFC) model and variants of the Feder and Feder (FF) model and we propose an explanation of their dynamics. In the OFC model, we show in particular that, depending on \( \nu \), there are two qualitatively different regimes of the dynamics in the thermodynamical limit. For small \( \nu \) (low dissipation) the dynamics is dominated by large avalanches and is highly non trivial. On the contrary, for large \( \nu \) (high dissipation), the dynamics is dominated by small avalanches. We show in this case that, as \( \nu \) increases, due to the spatial distribution of large avalanches, the dynamics becomes more and more trivial, in the thermodynamical limit — but on the boundaries —, even if the multi-site avalanches are still distributed as power laws. In both cases, the system exhibits also synchronization, the stability of which depends on the dissipation. When the system is almost conservative, the synchronization is almost absent and, when \( \alpha \) is small, it exhibits very stable synchronization.
II. THE MODEL

The definition of a lattice relaxation oscillator system is the following. The free, i.e. uncoupled, evolution of the state variable $E_i$ of the oscillator at site $i$ is given by $E_i = f(\phi_i)$ where $\phi_i$ is the phase of the oscillator $i$, $\phi_i = at \mod 1/a$ where $t$ is the time and $1/a$ the period. For convenience, we call $E_i$ the local stress. $f$ is a continuously increasing function on $[0,1]$ such that $f(0) = 0$ and $f(1) = 1 = E_c$. $E_c$ is called the threshold. Thus $E_i$ increases up to the threshold and then relaxes instantaneously to 0. For uncoupled oscillators, once $f$ is given, $E_i(t)$ is a periodic function with fixed frequency. These oscillators are usually called Integrate and Fire oscillators. They are widely used in biology for the modelization of biological rhythms (see for instance [2]). In the following, $f$ is supposed linear except stated otherwise. The interaction between the oscillators is chosen such that when a site relaxes to zero ($E_i \geq E_c$) it emits a pulse $\Delta_i$ that increments all its neighbors:

$$E_i \rightarrow 0$$
$$E_j \rightarrow E_j + \Delta_i \quad \text{where } j = \text{nearest neighbors of } i$$

The oscillator $j$ is therefore phase advanced and relaxes also to zero if it exceeds its threshold. Therefore an avalanche of relaxations can occur that stops only when no more oscillator is unstable.

Several studies of systems of pulse coupled Integrate and Fire oscillators on a lattice have recently been performed and a large variety of behaviors have been observed depending on the boundary conditions [3, 4], on the shape of the function $E_i(\phi)$ [5] and on the choice of the coupling $\Delta_i$ [7, 10]. In this article, we study the dynamics of the two dimensional OFC model [7, 1, 11, 13] which is a discretized and simplified version of the Burridge-Knopoff model of earthquakes [14]. It is a model of pulse coupled Integrate and Fire oscillators with an especially rich dynamics. In this model the pulse $\Delta_i$ is proportional to $E_i$ just before the relaxation: $\Delta_i = \alpha E_i$, $E_i \geq E_c$ and $\alpha \leq 1/4$ is a parameter that takes care of the dissipation. This model was shown to display SOC for a wide range of parameters $\alpha$ [7, 9, 12, 13]. It has also a complex spatio-temporal behavior with clustering of almost periodic sequences of large
avalanches and a multi-fractal distribution in time of the avalanche sizes [11]. Let us recall the principal results on the OFC model relevant for our study. The largest simulations have been performed by Grassberger [3] and Middleton and Tang [4] who have found power law distributions for the avalanche sizes for a wide range of $\alpha$, including values as small as 0.07 [3] and 0.05 [3]. For sufficiently large $\alpha$ the SOC behavior is well established and confirmed by the finite size scaling. For small $\alpha$, the conclusions are not so firmly established since the cut-off of the distribution does not obey the correct finite size scaling. This is probably due to the transients that are extremely long and forbid to be in the asymptotic regime [3]. Grassberger has also suggested that there exists a critical value $\alpha_c$ of $\alpha$, $\alpha_c \sim 0.18$, below which the interior of the system would almost be decoupled from the boundaries. According to this author, the bulk would then behave as in the case of periodic boundary conditions, where all the avalanches are of size one and occur perfectly periodically. Grassberger also noticed that most of the multi-site avalanches are triggered near the boundaries of the system.

In the following, we shall confirm the existence of two domains of $\alpha$ corresponding to two qualitatively different behaviors of the system. By using a very crude approximation, we shall relate the existence of these two regimes to the existence of a critical value $\nu_c$ of the exponent $\nu$ of the distribution $P(s)$, separating the domain ($\nu < \nu_c$), where, statistically, the large avalanches penetrate in the bulk, from the domain, ($\nu > \nu_c$), where they do not. In our approximation, $\nu_c = 2$, a value for which $\alpha$ is indeed close to 0.18, see Fig.1.

Christensen proposed that the behavior of this model arises from a tendency of the oscillators to synchronize their relaxations [3]. Systems of Integrate and Fire oscillators can indeed evolve towards large scale synchronization as shown in models with a global, all to all, coupling [15,16,17] and in lattice models with a strongly convex function $E(\phi)$ [6,18]. The question of the relationship between SOC and synchronization in the OFC model was already addressed by Middleton and Tang who studied the dynamics of this model for $\alpha = 0.07$ and $\alpha = 0.15$ and who argued on theoretical grounds that there should exist a close relationship between these two forms of organization [3]. However, to the best of our
knowledge there has been no report of the direct observation of synchronization in the OFC model nor on its possible relevance for the criticality of the OFC model. In this article, we reexamine the question of this relationship between SOC and synchronization and propose a “phase diagram” for all values of $\alpha$ of the OFC model.

**III. THE ROLE OF THE SPATIAL DISTRIBUTION OF AVALANCHES**

In our opinion, the role of the spatial distribution of avalanches – that is their localisation as a function of their size – has been underestimated in SOC models. Let us now show why and why it is likely to be crucial for the OFC model. For all values of $\alpha$, it has been observed that “large” avalanches are preferentially triggered near the boundaries [3], see Fig.2. For tractability, we make the simplifying approximation, capturing the essential features of the model, that they are all triggered on the boundaries of the system. Let us now show, by calculating the percentage of large avalanches, $s \in [\bar{s}(p), s_{\text{max}}]$, accounting for a percentage $p$ of the relaxations of a site in the bulk, that the system exhibits two different regimes depending on $\nu$. For computational simplicity, we assume a sharp cut-off of the distribution $P(s)$ at the value $s_{\text{max}}$ where the power law is no longer valid, see Fig.3. We thus neglect the tail of the largest avalanches, an hypothesis that we show is correct, at least for $\alpha$ small.

The scaling of $s_{\text{max}}$ with the system size $L$ is not exactly known [13,3,19,21]. In the numerical simulations available, it seems that $s_{\text{max}}$ scales for $\alpha > 0.18$ slightly faster than $L^2$ [13]. This is of course impossible, in the limit of large $L$, since, as we have verified numerically, there is no multiple relaxations of the oscillators inside an avalanche for $\alpha \lesssim 0.24$, i.e. for non conservative systems (see also [20]). As indicated in [3,20], this means that the true asymptotics is not exactly reached with the system sizes studied. We suppose in the following that the true asymptotic behavior is given by $s_{\text{max}} \sim L^\rho$. For $0.15 < \alpha \lesssim 0.24$, the most reasonable assumption [21] is $\rho \simeq 2$. For smaller $\alpha$, the determination of $\rho$ is extremely difficult due to very long transients. We shall see however, that our results are independent of its precise value.
With our approximation, an elementary calculation shows that

\[ \bar{s}(p) = [s_{\text{max}}^{2-\nu}(1-p) + p]^{1/(2-\nu)} \]  

Therefore, there are two different cases depending on \( \nu \). If \( 1 < \nu < 2 \), then \( \bar{s}(p) \sim s_{\text{max}} \) and the fraction of avalanches accounting for the percentage \( p \) of relaxations goes to zero as \( s_{\text{max}}^{1-\nu} \) and therefore as \( L^{\rho(1-\nu)} \). This means that the dynamics is completely dominated by large avalanches and that, in the thermodynamical limit, a vanishing percentage of very large avalanches is responsible for a finite fraction of the relaxations. For \( \rho = 2 \), these avalanches sweep a finite fraction of the whole lattice, and therefore contribute to a finite fraction of the relaxations of the sites in the bulk. Therefore, in this case, even if the avalanches are triggered near the boundaries, we expect the dynamics of the bulk to be non trivial, even in the thermodynamical limit, since the avalanches are able to penetrate into the system. If true, this would mean that the critical behavior of this system would be highly non-trivial since it would depend, even in the limit \( L \to \infty \), on the boundary conditions through the avalanches triggered near the boundaries. Although the true spatial distribution of avalanches is more complicated than what we have supposed, and \( \rho \) could be slightly different from 2, we have been able to verify numerically, at least for \( \alpha \geq 0.2 \) and for lattice sizes up to \( L = 150 \), that the relaxations in the bulk are mostly due to the largest avalanches. Let us finally notice, that for general SOC models with \( \nu < 2 \), the tails of the distributions \( P(s) \) could actually dominate the dynamics, see however [20]. Our conclusion for the previous case \( 1 < \nu < 2 \) would however be unchanged if we had taken into account this tail.

On the contrary, if \( \nu > 2 \), then, in the limit \( L \to \infty \), \( \bar{s}(p) \sim p^{\frac{1}{2-\nu}} \) independently of \( L \). This means that the dynamics is dominated by small avalanches. In a first step, let us suppose that the avalanches are compact, two dimensional objects. The avalanches that are able to reach a site in the bulk have, in this case, a size \( s \) of the order of \( L^2 \), \( s \in [\eta L^2, s_{\text{max}}] \) with \( \eta \) less than one and of the same order. Since the dynamics is dominated by the small avalanches, the average number of relaxations per site, during \( N \) avalanches, scales as \( N/L^2 \).
Then the fraction $f$ of relaxations of a site in the bulk, due only to large avalanches, behaves as:

$$f \sim s_{\text{max}}^{1-\nu} - (\eta L^2)^{1-\nu} \frac{1}{1/L^2} \sim L^{\rho(2-\nu)}. \quad (3)$$

Therefore, we conclude that for $\nu > 2$ and with our hypothesis, a site in the bulk relaxes almost never in a large avalanche when $L \to \infty$. It is not difficult to show that this result is unchanged if the avalanches are one dimensional objects starting on the boundaries. In this case, one finds in the most favorable case for the penetration of avalanches in the bulk, $\rho = 2$, that $f$ behaves as:

$$f \sim L^{2-\nu}. \quad (4)$$

In this case also, the avalanches are not able to penetrate into the system and the conclusions are the same as before.

As shown numerically for $\alpha = 0.07$ and $\alpha = 0.15$ in [5], the toppling rate $r$ of any site in the bulk of the system behaves as a function of the distance $y$ from the boundary as:

$$r(y) = (1 - 4\alpha)^{-1} + cy^{-\eta} \quad (5)$$

where $c$ is a constant and $\eta$ an exponent that depends on $\alpha$. The relaxation rate is therefore almost constant, at the thermodynamical limit, for a site in the bulk. We now show that this result combined with Eq.(3) or (4) is incompatible with a power law distribution of avalanche sizes for small avalanches. Provided $\nu > 2$, it is easy to show that, whatever the spatial distribution of the avalanches, the number $N$ of avalanches necessary for all the sites in a band on the boundaries of finite width $d$ to relax once, scales as $N \sim L$. On the other hand, since a site in the bulk has almost the same relaxation rate as a site at distance $d$ from the boundary, for $d$ large enough, it relaxes also once during these $N$ avalanches. This is not possible if we assume a power law distribution of avalanche sizes valid from $s = 1$ to $s = s_{\text{max}}$, since the number of sites in the bulk that relax in small avalanches, i.e. mostly in avalanches of size 1, varies as $L^2$. We expect therefore a large excess of avalanches of size 1,
a smaller of avalanches of size 2, etc... This has been observed by Grassberger for $\alpha = 0.05$
and we have verified numerically for $\alpha = 0.1$, $L = 95, 130$, with $10^9$ avalanches, where the
system is in the steady state, that this is indeed correct. In fact, all the avalanches in the
bulk are not of size one and it happens (rarely) that large avalanches are also triggered in
the bulk. However, it is clear on Fig.3 that the distribution $P(s)$ has indeed an excess of
avalanches of size one and two and that the power law really begins for $s \geq 3$. The fact
that only two points are not aligned on the Log-Log curve should not be underestimated
since the dynamics is precisely dominated by these small avalanches for $\nu > 2$. Moreover,
Grassberger has also observed for $\alpha = 0.05$, that the relative weight of avalanches of size one
increases with $L$. For $\alpha = 0.1$, $L = 95$ and $10^{10}$ avalanches, the pure power law fit of $P(s)$
shown in Fig.3 predicts that 37% of the relaxations should be due to avalanches of size one,
while our simulations show that 75±5% of the relaxations in the bulk are due to avalanches
of size one, see Fig.4. This is confirmed by a simulation on a system of size $L = 130$, where
it is however more difficult to obtain a good statistics. When $\nu$ increases, i.e. $\alpha$ decreases,
we expect that the relative weight of the avalanches of size one will get bigger and bigger.
We conclude that, although the system seems to exhibit a SOC behavior, the boundaries
and the bulk are largely decoupled for $\nu > 2$. The fact that $s = 1$ and $s = 2$ do not fit
the power law, unfortunately turns out to be crucial and spoils the critical behavior in the
thermodynamical limit. We have confirmed this decoupling numerically for $\alpha = 0.1$ and our
arguments strongly suggest that it is not a transient effect.

Let us now emphasize that, within our approximation, the critical value of $\nu$ that sep-
arates the two regimes of the model is $\nu = 2$, to which corresponds $\alpha \simeq 0.18$, see Fig.4.
While it seems numerically that this value of $\alpha_c$ is reasonable, it is of course very difficult
to conclude definitely about its value since the spatial distribution of avalanches is certainly
non trivial. For $\alpha = 0.15$, which is smaller but close to 0.18, it is numerically difficult, since
it requires very large lattices, to see the decoupling of the interior of the system. It is, in fact,
even difficult to predict if there exists a sharp separation between two regimes. However,
we think that our analysis clearly shows that the existence of a power law distribution of
the avalanche sizes, which is only a global information on the system, is not sufficient to make sure that the physics is really scale invariant. The point that makes this issue non trivial, is that systems like the OFC model are not obviously translationally invariant, since their physics can depend crucially on their boundaries, even at the thermodynamical limit. Moreover, we have shown that the whole behavior of a model, which is thought to be SOC, can actually be dramatically affected by the fact that the power law distribution $P(s)$ is not verified for only one avalanche size, as it is the case in the OFC model for small $\alpha$ and for $s = 1$.

IV. SYNCHRONIZATION

Let us define more precisely what we mean by synchronization since different behaviors are called synchronized in the litterature. Our definition of synchronization is the following: a cluster of oscillators is said to be perfectly synchronized when all the oscillators of the cluster always relax together in the same avalanche. We say that a lattice is partially synchronized when the whole system is not synchronized, but contains clusters of sites that are synchronized. We shall also be interested in unstable synchronization where the synchronized clusters have a large, but finite, life-time. Note that perfect synchronization is of course incompatible with SOC. It is also important to be aware that our definition of synchronization does not imply that the state variables of synchronized oscillators are degenerate. This is the case for the OFC and FF models where synchronized oscillators are in general non degenerate. Furthermore, synchronization should not be confused with a periodic activity of the lattice, as the one observed for periodic boundary conditions, where the avalanches are all of size one and the period is $L \times L$ [4,5]. Since we are interested in the dynamics of avalanches, we believe that it is our definition of synchronization which is physically relevant.

Let us call cycle the time necessary for the oscillators of the system to relax once. This notion is well-defined for the sites in the interior of the system since they all relax the
same number of times in the same time interval, apart from small fluctuations. We have studied by direct inspection and cycle after cycle, the shapes of the clusters of sites that relax simultaneously. For a wide range of parameters $\alpha$, we have found partial and unstable synchronization of all length scales in the OFC model. To the best of our knowledge, this is the first time that the presence of synchronization and its co-existence with SOC is reported in the OFC model. Let us examine in greater details this behavior when $\alpha$ varies. For $\alpha$ very small the oscillators are almost decoupled so that most of the avalanches are of size one. The largest avalanches, that occur near the borders, are rare and small. The bulk behaves very much like the state described by Grassberger [3] for periodic boundary conditions and $\alpha < 0.18$. In this state, called “ordered” by this author, the avalanches are only of size one and occur periodically one after the other. Since, in the bulk, there are no multi-site avalanches, there is evidently no synchronization.

As $\alpha$ increases in the interval $[0, 0.18]$, larger avalanches occur and synchronization begins to make sense. We have followed cycle after cycle the avalanches to which belongs a given site that has been previously chosen randomly in the bulk. For $\alpha = 0.1$, for instance, most of the avalanches in the bulk are of size one or two. However, it happens that large avalanches – large for this $\alpha$ means typically $s \geq 50$ – are triggered in the interior of the system and are extremely stable. Although we do not have statistics, we have been able to see on each synchronized cluster that, cycle after cycle, avalanches are always triggered by the same site and always for the same phase of this site, and that their life times are usually of hundreds and even of thousands of cycles. We have checked that these results are independent of the test site chosen in the bulk. For $\alpha = 0.15$, the shape of these avalanches are also very stable with, however, smaller life times of tenths of cycles. During these avalanches, only a few oscillators on the border of the avalanches desynchronize. The synchronized clusters evolve either by merging with a neighboring synchronized cluster or by breaking into two clusters. These new clusters are themselves synchronized during many cycles until they evolve by breaking or merging. The system appears as a collection of synchronized clusters together with many sites that topple in avalanches of size one or two. The evolution of these clusters
is the superimposition of a mechanism of merging and breaking and a very slow evolution of the shape of these clusters. Thus, for a given site, the nearest sites around it can belong to a synchronized subcluster that does not break during hundreds of cycles.

When $\alpha$ becomes larger than 0.18, but still lower than 1/4, say $\alpha \sim 0.2$, the system shows SOC together with partial and unstable synchronization, see Fig.5. The synchronized clusters are much larger, in the average, than for $\alpha \leq 0.18$, but evolve more rapidly. As seen on Fig.5, a given site and its neighbors can belong to a synchronized subcluster that does not break during hundreds of cycles. We have observed that synchronization occurs on all length scales, from small clusters of a few sites near the borders to clusters that represent a macroscopic fraction of the whole system. By following one particular site $i_0$ in the interior of the system, it appears that the neighboring sites are in general almost perfectly synchronized with $i_0$ and that the level of synchronization decreases with the distance from $i_0$. Finally, when $\alpha$ becomes close to 1/4, say $\alpha = 0.24$, the synchronization disappears almost completely.

Let us now try to understand the behavior of this system. As we argue in the following, its dynamics involves a competition between a tendency towards synchronization, due to the open boundary conditions, randomness and dissipation. Let us first clarify what we mean by randomness in the OFC model. The microscopic rules of this model are entirely deterministic but randomness is anyway present because of the initial conditions, where the phases are taken random. This randomness is maintained in the system since the pulses $\Delta_i$ are proportional to $E_i$: $\Delta_i = \alpha E_i$. It is, of course, partially dissipated when $\alpha$ is less than 1/4 but, as we shall see, the threshold dynamics is able to amplify the effect of a very small noise.

Let us study the synchronization in the OFC model. As shown by Socolar et al. [4] and Middleton and Tang [5], two isolated oscillators of different frequencies, interacting with each other as in the OFC model, synchronize necessarily with a frequency equal to the frequency of the slowest oscillator. These authors argued that since the oscillators on the boundaries receive less pulses by unit time, since they have only three neighbors, than those
far from the borders, their effective frequencies should be lower than those in the interior. According to Middleton and Tang, this should lead to a tendency towards synchronization and should be responsible for the large correlations in the OFC model. However, as we have mentioned previously, the system does not always show SOC nor synchronization. Moreover, its behavior depends strongly on $\alpha$, so that it is not clear whether and how this can produce large scale synchronization and whether this is related to criticality.

As a first step, let us show that for $\alpha$ sufficiently large, the model cannot show stable synchronization. In the following, we assume that for $\alpha < 1/4$, the system, which is highly dissipative, is such that, in the bulk, the system dissipates in the average and on each site, $100\%$ of the stress it receives. Put it differently, there is no transport of stress from the bulk to the boundaries. This assumption can be justified in the following way. In a subsystem of volume $l \times l$, representing the interior of the system of volume $L \times L$, $l$ being of the order of $L$ for large systems, $l \sim L$, the stress to be dissipated during a fixed time $t$ grows as $l^2$. If a fraction of this stress was dissipated on the boundaries, it would topple typically $L-l \propto L$ times from site to site, in large avalanches, to go from the interior to the boundaries. Thus, only a fraction of this stress of order $(1-4\alpha)^L \sim (1-4\alpha)^L$ would arrive on the boundaries. Therefore, since $l^2(1-4\alpha)^L \to 0$ as $L$ increases, nothing of the interior of the system can be dissipated on the boundaries for $\alpha < 1/4$. In this sense, the conservative model $\alpha = 1/4$ – and possibly $\alpha$ very close to $1/4$ – is a very particular case since everything is dissipated on the boundaries in this case. We have verified numerically that for any $\alpha$ not very close to $1/4$ (it would require extremely large lattices to study this case), the dissipation rate is indeed $100\%$ on each site of the interior, see Fig.6. This is expressed for a site $i$ in the interior of the system by:

$$\bar{E}_i(1-4\alpha)r_i = 100\% = 1$$

with $\bar{E}_i$ being in this equation the average stress of the oscillator $i$ just before the relaxation and $r_i$ the relaxation rate of this oscillator. Now, if we suppose that a (large) cluster of sites is permanently synchronized with a site on (or close to) the boundary, the relaxation rate of
which is roughly \((1 - 3\alpha)^{-1}\) since it has three neighbors, see Fig.7, we deduce from (3) that

\[
\bar{E}_i \simeq \frac{1 - 3\alpha}{1 - 4\alpha}
\]  

(7)

This is clearly impossible for \(1 - 4\alpha \to 0\), since \(\bar{E}_i\) diverges in this limit. This means that the synchronization cannot be stable for large \(\alpha\) and this comes from the fact that it becomes harder and harder to synchronize oscillators as their “natural” frequencies become more and more different. In fact, we can obtain a rough estimate of the value of \(\alpha\) above which the synchronization must be unstable. Since the propagation of the avalanche of relaxations occurs as the propagation of a front, there is in the average two back-firings from the sites that relax to the sites that have just relaxed before them in the avalanche. Thus, the value of a site after an avalanche is roughly \(2\alpha\bar{E}_i\). This value cannot be larger than one otherwise the back-firing would trigger another relaxation. We have verified numerically that indeed, for generic values of \(\alpha\), not very close to \(1/4\), there is never multiple relaxations of a site inside an avalanche [22]. Therefore, we deduce that stable synchronization requires necessarily

\[
\frac{1 - 3\alpha}{1 - 4\alpha} < \frac{1}{2\alpha} \implies \alpha < 0.21
\]  

(8)

The conditions (7) and (8) are of course only necessary for a stable synchronization, they are absolutely not sufficient. In particular, synchronization needs a very well defined and stable spatial repartition of the values of the \(\bar{E}_i\). For this reason, we expect randomness to play a crucial role in the OFC model since randomness is never completely dissipated in this model and since it tends to destabilize synchronized clusters. Randomness is thus surely in competition with synchronization [24]. In fact, we can expect that since \(\bar{E}_i\) is bounded, the relaxation rate \(r_i\) behaves roughly as \((1 - 4\alpha)^{-1}\) in the interior when \(\alpha\) is not too close to \(1/4\), in which case multiple topplings become possible. We even expect that for \(\alpha < 0.18\), this behavior becomes exact since most of the avalanches are of size one. This result is in agreement with the numerical results of [2] for \(\alpha = 0.07, 0.15\) and Eq.(5). For \(\alpha \geq 0.18\), it is given by Eq.(6) with \(\bar{E}_i > 1\) since the sites relax in (large) avalanches. This is confirmed numerically in Fig.7.
V. SYNCHRONIZATION AND STOCHASTICITY

To study separately in a model of relaxation oscillators the tendency towards synchronization and the effect of stochasticity, we have studied different versions of the Feder and Feder (FF) model which is very close to the OFC model and where this separation is possible [8]. This model is identical to the OFC model but for the pulse which is a constant: $\Delta = \alpha E_c$. In this case, stochasticity of the initial conditions is almost completely dissipated since the pulse is independent of the value of the oscillator before the relaxation. The only memory of the initial conditions lies in the hierarchy of values present initially on the lattice and that determine in which order the oscillators relax. In fact, this model has the unpleasant feature for us of having, in the steady state, many sites exactly at the same value that therefore relax exactly at the same time, triggering disjoint avalanches. To remove this degeneracy that does not occur in the OFC model and which is inconsistent with the slow drive limit – the avalanches are supposed instantaneous compared to the drive –, we have added an infinitesimally small frozen disorder $\delta_i$ on the thresholds $E_{ci} = E_c + \delta_i$, $\delta_i \to 0$. In this case the analog of Eq.(6) for the FF model is

$$ (\bar{E}_i - 4\alpha) r_i = 100\% $$

and the condition of stable synchronization analogous to Eq.(7) is

$$ (\bar{E}_i - 4\alpha) \frac{1}{1 - 3\alpha} = 1 \implies \bar{E}_i = 1 + \alpha $$

This condition is clearly non singular as $\alpha$ varies. Moreover, since there is almost no stochasticity, we expect from the argument of Middleton and Tang that the system shows very stable synchronization. This is indeed what we observe: in the steady state the system consists of (almost) perfectly stable clusters of synchronized sites, the relaxation of which are triggered by sites near the borders. We have checked numerically that Eq.(10) is effectively fulfilled and that the frequency of the synchronized clusters is $(1 - 3\alpha)^{-1}$ thus proving that the mechanism of Middleton and Tang works even for macroscopic clusters of sites. A typical
synchronized cluster is nucleated during the transient on the boundaries and grows until it meets another cluster. Since both clusters have the same frequency of relaxation, they do not synchronize together in general. Since this process does not seem to involve any characteristic scale, we expect that the distribution of these synchronized clusters follows a power law. By performing an ensemble average over 4000 realisations – obtained by choosing different initial conditions – we have verified that, for \( \alpha = 0.2 \), the distribution of sizes of the synchronized clusters follows approximately for large avalanches a power law, see Fig.8. Moreover, we have verified that the size \( s_{\text{max}} \) of the largest synchronized clusters vary with \( L \) as \( s_{\text{max}} \sim L^2 \), so that the synchronization of a macroscopic fraction of the system should remain valid in the infinite volume limit. This behavior proves several things. First, in the absence of stochasticity, the system is not critical. We have shown numerically that it is in fact periodic in time since the same avalanches – but for some sites at the edge of the avalanche – occur always at the same phase. Second, the mechanism proposed by Middleton and Tang works very well in this case since the oscillators synchronize with the slowest ones even in macroscopic avalanches and the larger is \( \alpha \) the more rapid is the formation of the clusters, i.e. the more efficient is this mechanism. Since we know that in the OFC model there is only unstable synchronization even for relatively small \( \alpha \) and SOC for \( \alpha \geq 0.18 \) we can expect that stochasticity plays a role in destabilizing synchronization and producing SOC. If this is true, the addition of a stochastic noise in the FF model should also produce SOC. This is indeed what happens \[9\] and we have verified this fact for many values of \( \alpha \) by adding to the pulse received by an oscillator \( E_i \), a very small stochastic noise \( \Delta_i = \alpha E_i + \eta_i \) or equivalently by changing, after each relaxation, the threshold \( E_{ci} \) by a small stochastic amount, \( E_{ci} \to E_{ci} + \eta_i \). In fact, it is possible to go smoothly from the disordered (frozen disorder on the thresholds) to the noisy (stochastic noise) FF model by updating the disorder on the thresholds after each avalanche of only a fraction \( q \) of the sites participating in the avalanche. For \( q = 0 \), we get the disordered FF model while, for \( q = 1 \), we get the noisy FF model. We have performed many simulations and have observed that the system becomes critical already for rather small \( q \), while for \( q \to 0 \) it is impossible to
conclude since the system stays stuck in quasi-stable synchronized situations for a very long time. However, we believe that even in this limit the system should be critical since the ensemble average, previously mentioned for \( q = 0 \), shows that the distribution of avalanche sizes, which is frozen in each realisation, follows a power law when averaged over different realisations. Therefore, for \( q \neq 0 \), the stochasticity breaks the partially synchronized state into a partially and unstable synchronized one that evolves more and more rapidly as \( q \) approaches 1. We conjecture that the behavior of this noisy model changes also at \( \alpha_c \) since finite size scaling is obeyed also only above this value. Moreover, we have checked that the interior of the system is decoupled from the boundary for small values of \( \alpha \), around \( \alpha = 0.07 \), and behaves exactly in the same way as in the OFC model. The level of synchronization decreases of course as \( q \) increases and for \( q = 1 \) becomes very poor. We have also shown numerically that for the same level of dissipation, the noisy FF model and the OFC model have the same exponent for the distribution of avalanche sizes, Fig.9. This was done by direct comparison of the critical exponents that in both cases vary continuously with the dissipation and also by constructing a set of models that interpolate smoothly between these two models.

This was done by building a series of models indexed by a probability \( p \) such that after each avalanche the sites that have just relaxed are chosen with probability \( p \) to be for the next avalanche of the OFC type and \( 1 - p \) to be of the noisy FF type. We have shown by varying \( p \) between 0 and 1 that the slope of the distributions of the sizes (in a Log-Log plot) does not change, see Fig.9. Once again, the OFC models with \( \alpha \) close to 1/4 are particular cases since there is no conservative FF model and therefore no such models in the same universality class.

Let us now come back to the OFC model. It is clear that this model shares a lot of properties with the noisy FF model although the tendency towards synchronization and the level of stochasticity is controlled in the OFC model by the same parameter \( \alpha \). We are now in a position to discuss our scenario about the dynamics of the model.

When \( \alpha \leq 0.18 \), the dynamics is that described by Middleton and Tang. The difference
of frequencies between sites on the border and sites in the interior is not very large so that the synchronized clusters are not very large. On the other hand stochasticity is weak so that the life time of synchronized clusters is large. In the interior, the sites are largely decoupled from the boundaries and relax periodically with period $1 - 4\alpha$.

For $\alpha \geq 0.18$, the situation is rather different. The key observation is that, contrary to the disordered FF model, the sites in the interior of the system cannot synchronize in a stable way with sites on the border. This has two origins. First, stable synchronization is in conflict with the randomness present in the model as in the case $\alpha \leq 0.18$ and as in the noisy FF model. Second, it is anyway impossible, for $\alpha > 0.21$, to synchronize perfectly sites in the interior that have a natural frequency $(1 - 4\alpha)^{-1}$ with those on the borders, the natural frequency of which is close to $(1 - 3\alpha)^{-1}$, because of the dissipation that would not be 100% in this case, if they were synchronized. However, nothing forbids a site to be synchronized during a short time with a site on the boundaries, then to be synchronized with another site of the boundaries and so on. Thus, when $\alpha$ is large but not too close to $1/4$, say $\alpha \sim 0.2$, neighboring sites in the interior are still very well synchronized together and the level of synchronization decreases with the distance, see Fig.5. This is exactly what is seen in the simulations. On the other hand, when $\alpha \to 1/4$ the synchronization time with a site on the boundaries goes to zero since the difference of frequencies between these sites increases rapidly in this limit. Thus, it is normal that the synchronization disappears in this case. Once again, this is what is observed numerically even for values of $\alpha$ such as 0.24.

These results prove that synchronization is not necessary for SOC in the OFC model. In fact, the example of the noisy FF model, which is in the same universality class as the OFC model for $\alpha < 1/4$ and that shows almost no synchronization, was already an indication of this fact. Moreover, it is easy to build another model of relaxation oscillators coupled à la OFC that, by construction, does not show synchronization: the random neighbor OFC model in which a site that relaxes increments 4 sites chosen randomly. We have checked that this system is critical for sufficiently large $\alpha$ but does not show of course any synchronization, see Fig.10.
VI. CONCLUSION

In conclusion, we have shown that the OFC model can exhibit partial and unstable synchronization ($\alpha < 0.24$) with ($\alpha \geq 0.18$) or without ($\alpha \leq 0.18$) being critical, and that it can be critical without showing synchronization ($\alpha \geq 0.24$). While it is probable that in this system, the tendency towards synchronization is indeed related to the mechanism that builds long range correlations, it is not necessary for criticality that the system actually shows synchronization and it is even possible to build another model - the noisy FF model - that belongs to the same universality class without showing synchronization for any value of $\alpha$. During the completion of this work, we received an article of Lise and Jensen [25] who performed the same simulation as us on the random neighbor OFC model and who also present a theoretical argument according to which the model is critical only for $\alpha$ above 0.22, which is well verified numerically. Although we agree with their conclusion about the criticality of this model in the absence of synchronization, we disagree with the fact that this random neighbor version of the model is a mean field approximation of the OFC model, or at least we believe that it is not proved. Strictly speaking, to be a mean field approximation requires that this random neighbor version of the model shares the same critical exponents with the original model above the critical dimension. This is not proved up to now, nor is it proved that in dimensions higher than two, there exists a domain of $\alpha$ where SOC and synchronization coexist. Moreover, it is interesting to see that in fact both models – at least for the two dimensional OFC model – are critical without synchronization for the same values of $\alpha$. Therefore, we conclude that up to now there is no “contradiction” between the behaviors of the two models in this domain of parameters. Let us finally remark that the open boundary conditions in the random neighbor model do not play the same role as in the OFC model and that there is not a unique way to implement them. Therefore, as in the noisy FF model or the OFC model for $\alpha > 0.21$, the random neighbor OFC model is a SOC model of coupled relaxation oscillators that does not show synchronization, but up to now its relationship with the OFC model is far from clear.
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FIG. 1. Plot of the exponents $\nu$ of the distributions of avalanche sizes $P(s) \sim s^{-\nu}$ in the FF and OFC models for different values of $\alpha$. Up to $\alpha = 0.23$, the noisy FF and OFC models have the same exponents. For $\nu = 2$, $\alpha$ is roughly 0.18.

FIG. 2. Ratio between the number of times a site triggers an avalanche and the number of times it relaxes as a function of the distance $y$ of the site to the nearest border. When this ratio is one, all the avalanches at that site are of size one. When it is small, most of the avalanches are large. For $\alpha = 0.2$, the error bars are small. For $\alpha = 0.1$, the error bars increase with $y$ since, in the center of the lattice, the number of sites used for averaging is far smaller than near the boundaries. For $\alpha = 0.1$, most avalanches in the bulk are of size one. For $\alpha = 0.2$ on the contrary, the avalanches are large in the center of the lattice, independently of the lattice size.
Fig. 3. Distribution $P(s)$ of avalanche sizes for $L = 95$ and $\alpha = 0.1$. The excess of avalanches of size 1 and 2 is clearly seen. The dotted line is the power law fit $s^{-2.5}$. $s_{\text{max}}$ is the size above which $P(s)$ is no longer fitted by a power law.

Fig. 4. Plot of the percentage of relaxations of a site, due to avalanches of size one, as a function of the distance $y$ of this site from the nearest border. The data were obtained with $10^{10}$ and $14.10^6$ avalanches for respectively $\alpha = 0.1$ and $\alpha = 0.2$. A pure SOC behavior for $\alpha = 0.1$ (power law fit of Fig.2) would predict an average of 37% of relaxations in avalanches of size one.
FIG. 5. Map of 60 successive avalanches involving the site of coordinates (30,30) for $\alpha = 0.2$ and $L = 65$. The grey level is an indication of the number of relaxations inside these avalanches. The blackest spot corresponds to a cluster of sites that have always relaxed together during this sequence.

FIG. 6. The local dissipation rate – on each site – as a function of the distance of the site to the border of the system. Three values of $\alpha$ have been studied: $\alpha = 0.1, 0.2, 0.24$. For a sufficiently large distance, the dissipation is always 1.
FIG. 7. Relaxation rate $r_i$ of the oscillator $i$ as a function of the distance of $i$ to the border. For $\alpha = 0.1$ and $\alpha = 0.15$, $r_i = (1 - 4\alpha)^{-1}$ in the bulk while for $\alpha = 0.2$, $r_i$ is slightly less than $(1 - 4\alpha)^{-1}$.

FIG. 8. Average ensemble over 4000 realisations of the avalanche sizes of the disordered FF model for $\alpha = 0.2$ and $L = 55$. For the avalanche of sizes in $[100, 2800]$ the curve is approximatively $P(s) \sim s^{-0.8 \pm 0.04}$.

FIG. 9. Distribution of the avalanche sizes for models that interpolate between the OFC and noisy FF models for $\alpha = 0.2$ and $L = 90$. The different curves correspond to different percentages of sites that are chosen of the OFC or FF type.
FIG. 10. Plot of the distribution of avalanche sizes for the random neighbor OFC model for 5600 sites. As shown in the inset for $\alpha = 0.23$, ordinary finite size scaling is well verified since the curves obtained for $65^2$, $75^2$ and $95^2$ sites, can be superimposed by a simple shift.
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