Absolute Parallelism Geometry: Developments, Applications and Problems

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Abstract

Absolute parallelism geometry is frequently used for physical applications. It has two main defects, from the point of view of applications. The first is the identical vanishing of its curvature tensor. The second is that its autoparallel paths do not represent physical trajectories. The present work shows how these defects were treated in the course of development of the geometry. The new version of this geometry contains simultaneous non-vanishing torsion and curvatures. Also, the new paths discovered in this geometry do represent physical trajectories. Advantages and disadvantages of this geometry are given for each stage of its development. Physical applications are just mentioned without giving any details.

1 Introduction

"To understand nature, one has to start with geometry and end with physics"

This statement summarizes the geometrization philosophy, introduced by Albert Einstein at the beginning of the 20th century. Einstein succeeded in applying this philosophy to construct his theory of general relativity (GR). He started with Riemannian geometry and ended with a successful theory for gravity.

After the success of this theory, Einstein tried to construct a theory unifying gravity and electromagnetism, using the same philosophy. He realized the fact that Riemannian Geometry is not the appropriate candidate for his aim. This geometry is relatively limited and it is just sufficient to describe gravity alone. It has only a unique affine connection, a unique curvature tensor and two path equations (the geodesic and the null geodesic). The building blocks of this geometry are ten components of the metric tensor (for n=4) which are just sufficient to describe gravity as stated above. So, he started to look for another geometry, wider than the Riemannian one. Einstein started his first attempt in this context in 1928 by using Absolute Parallelism (AP) geometry (some authors prefer to call it distant parallelism or teleparallelism or fernparallelismus). Due to the long history of development of this geometry, about seventy five years, a full review of its applications and developments cannot be given in such a small number of pages. In what follows, the
main lines of this subject are discussed in four stages. Readers interested in the details
are referred to the references listed at the end of this paper.

2 The First Stage (1928-1951)

In this stage the geometric structure of the AP-space can be summarized as follows (cf.
[1],[2]): An absolute parallelism space is an n-dimensional manifold each point of which
is labeled by n-independent variables $x^\nu (\nu = 1, 2, 3, \ldots n)$ and at each point we define n-linearly independent contravariant vectors $\lambda^\mu (i = 1, 2, 3, \ldots, n, \text{denotes the vector number and } \mu = 1, 2, 3 \ldots n \text{ denotes the coordinate component})$ subject to the condition,

$$\frac{\partial \lambda^\mu_i}{\partial x^\nu} = 0,$$

where the stroke denotes absolute differentiation to be defined later. Equation (2.1) is the
condition for the absolute parallelism. The covariant components of $\lambda^\mu_i$ are defined such
that:

$$\lambda^\mu_i \lambda_\nu = \delta^\mu_\nu,$$  \hspace{2cm} (2.2)

and

$$\lambda_i^\nu \lambda_j = \delta_{ij}.$$  \hspace{2cm} (2.3)

Using these vectors, the following second order symmetric tensors are defined:

$$g^\mu\nu \overset{\text{def}}{=} \lambda^\mu_i \lambda^\nu_i,$$  \hspace{2cm} (2.4)

$$g_{\mu\nu} \overset{\text{def}}{=} \lambda_{\mu i} \lambda_{\nu i},$$  \hspace{2cm} (2.5)

consequently,

$$g^{\mu\alpha} g_{\nu\alpha} = \delta^\mu_\nu.$$  \hspace{2cm} (2.6)

These second order tensors can serve as metric tensors of Riemannian space, associated
with the AP-space, when needed. This type of geometry admits, at least, four affine
connections. The first is non-symmetric connection given as a direct solution of the AP-
connection, i.e.

$$\Gamma^\alpha_{\mu\nu} = \lambda^\alpha_i \lambda_{\mu i, \nu}.$$  \hspace{2cm} (2.7)

The second is its dual $\hat{\Gamma}^\alpha_{\mu\nu} (= \Gamma^\alpha_{\nu\mu})$ since (2.7) is non-symmetric. The third one is the
symmetric part of (2.7), $\Gamma^\alpha_{\mu\nu}$. The fourth is Christoffel symbol defined using (2.4),(2.5) (as a consequence of a metricity condition of the associated Riemannian space). The
torsion tensor[3] is twice the skew symmetric part of the affine connection (2.7), i.e.

$$\Lambda^\alpha_{\mu\nu} \overset{\text{def}}{=} \Gamma^\alpha_{\mu, \nu} - \Gamma^\alpha_{\nu\mu}.$$  \hspace{2cm} (2.8)

Another third order tensor (contortion) is defined by,

$$\gamma^\alpha_{\mu\nu} \overset{\text{def}}{=} \lambda^\alpha_i \lambda_{i, \mu; \nu},$$  \hspace{2cm} (2.9)
where the semicolon is used for covariant differentiation using Christoffel symbol. The two tensors are related by,

\[ \gamma_{\mu\nu}^\alpha = \frac{1}{2}(\Lambda_{\mu,\nu}^\alpha - \Lambda_{\nu,\mu}^\alpha - \Lambda_{\mu,\nu}^\alpha). \]  

(2.10)

A basic vector could be obtained by contraction of the above tensors,

\[ C_\mu \overset{\text{def}}{=} \Lambda_{\mu}^\alpha = \gamma_{\mu}^\alpha. \]  

(2.11)

One of the advantages of AP-geometry is that for any tensor \( T^{\alpha}_{\beta\gamma} \) defined in the AP-space one can construct a set of scalars \( T_{ijk} \),

\[ T_{ijk} \overset{\text{def}}{=} \lambda_i^\alpha \lambda_j^\beta \lambda_k^\gamma T^{\alpha}_{\beta\gamma}. \]  

(2.12)

If \( T^{\alpha}_{\beta\gamma} \) is the contortion (2.9) then the corresponding scalars are known in the literature as Ricci coefficients of rotation [4].

The curvature tensor, in this stage, is defined by,

\[ B_{\mu\nu\sigma}^{\alpha} \overset{\text{def}}{=} \Gamma_{\mu,\nu,\sigma}^{\alpha} - \Gamma_{\mu,\nu,\sigma}^{\alpha} + \Gamma_{\nu,\mu,\sigma}^{\alpha} - \Gamma_{\nu,\mu,\sigma}^{\alpha} - \Gamma_{\mu,\nu}^{\alpha} \equiv 0. \]  

(2.13)

This tensor vanishes identically because of (2.1). The autoparallel path equation can be written in the form,

\[ \frac{d^2x^\mu}{d\lambda^2} + \Gamma_{\alpha\beta}^\mu \frac{dx^{\alpha}}{d\lambda} \frac{dx^{\beta}}{d\lambda} = 0. \]  

(2.14)

Robertson in 1932 developed the theory of groups of motion in AP-spaces [5]. He constructed three AP-structures, one with spherical symmetry and the other two with homogeneity and isotropy suitable for cosmological applications. These structures are important for physical applications.

The only application of this geometry was Einstein’s series of papers e.g.[3],[6], to construct a field theory, in an attempt to unify gravity with electromagnetism. Unfortunately, this attempt failed for the following reasons. The first reason is that the solution of the unified field equations in the case of spherical symmetry [6] did not reduce to the Schwarzschild solution when electromagnetism was switched off. The second reason is the non-physical consequence of axially symmetric solution of theory [7]. The third reason was that autoparallel paths do not represent physical trajectories of any test particle (cf.[8]). It is worth of mention that Levi-Civita had tried to simplify Einstein’s unified field equations, but he really wrote a different theory [4].

It may be of interest to give a brief comparison between the AP-geometry and the Riemannian one in this stage. Table 1 summarizes the main features of each geometry. The objects displayed in this table are those important for applications. It is clear from this table that the AP-geometry is more wider, from the point of view of applications, than the Riemannian one. For example, for \( n = 4 \), the number of independent components of the building blocks of Riemannian geometry is ten, which is just sufficient to describe gravity. On the other hand, the corresponding number in the AP-geometry is sixteen. So, the extra degrees of freedom of the AP-geometry could be used to represent other physical fields together with gravity.

The above lines give a brief review of AP-geometry in its first stage.
Table 1: Comparison Between The Riemannian Geometry and AP-Geometry

| Object                        | Riemannian geometry | AP-geometry |
|-------------------------------|---------------------|-------------|
| Building Blocks               | $g_{\mu\nu}$        | $\chi^\mu_i$ |
| Affine Connection             | $\{^\mu_{\alpha\beta}\}$ | $\{^\mu_{\alpha\beta}\}, \Gamma^\alpha_{\mu\nu}, \hat{\Gamma}^\alpha_{\mu\nu}, \Gamma^\alpha_{(\mu\nu)}$ |
| Second Order Symmetric Tensors| $g_{\mu\nu}, R_{\mu\nu}$ | $g_{\mu\nu}, R_{\mu\nu}$ |
| Second Order Skew Tensors     | $-$                  | $\xi_{\mu\nu}, \zeta_{\mu\nu}$ |
| Third Order Tensor            | $-$                  | $\gamma^\alpha_{\mu\nu}, \Lambda^\alpha_{\mu\nu}$ |
| Vectors                       | $-$                  | $C_\mu$ |
| Scalars                       | $R$                  | Many |
| Curvature                     | $R^\alpha_{\beta\gamma\delta} \neq 0$ | $B^\alpha_{\beta\gamma\delta} \equiv 0$ |

In this stage the AP-geometry has two main problems concerning applications: The first is the identical vanishing of its curvature tensor and the second is that its path equations do not represent physical trajectories. After the failure of Einstein’s attempt, the AP-geometry was neglected for about twenty years except one or two papers by A.G. Walker in 1940’s, e.g. [9], obtaining results similar to those of Robertson.

3 The Second Stage (1952-1974)

In this stage the development of AP-geometry can be summarized in the following. Mikhail in 1952 [10] revisited this geometry and constructed the following second order tensors (Table 2) which are important for applications, with the algebraic identity,

$$\eta_{\mu\nu} + \varepsilon_{\mu\nu} - \xi_{\mu\nu} \equiv 0.$$  \hspace{1cm} (3.1)
Table 2: Second Order World Tensors [10]

| Skew-Symmetric Tensors | Symmetric Tensors |
|------------------------|-------------------|
| \( \xi_{\mu\nu} \) def \( = \gamma_{\mu\nu|\alpha} \) | \( \phi_{\mu\nu} \) def \( = C_{\alpha} \Delta_{\mu\nu}^{\alpha} \) |
| \( \zeta_{\mu\nu} \) def \( = C_{\alpha} \gamma_{\mu\nu,\alpha} \) | \( \psi_{\mu\nu} \) def \( = \Delta_{\mu\nu|\alpha}^{\alpha} \) |
| \( \eta_{\mu\nu} \) def \( = C_{\alpha} \Lambda_{\mu\nu}^{\alpha} \) | \( \theta_{\mu\nu} \) def \( = C_{\mu|\nu}^{\alpha} + C_{\nu|\mu}^{\alpha} \) |
| \( \chi_{\mu\nu} \) def \( = \Lambda_{\mu\nu|\alpha}^{\alpha} \) | \( \omega_{\mu\nu} \) def \( = \gamma_{\mu\nu}^{\alpha} \gamma_{\alpha\nu} - \gamma_{\mu\nu}^{\alpha} \gamma_{\alpha\nu} \) |
| \( \varepsilon_{\mu\nu} \) def \( = C_{\mu|\nu}^{\alpha} - C_{\nu|\mu}^{\alpha} \) | \( \sigma_{\mu\nu} \) def \( = \gamma_{\mu\nu}^{\alpha} \gamma_{\alpha\nu} \) |
| \( \kappa_{\mu\nu} \) def \( = \gamma_{\mu\nu}^{\alpha} \gamma_{\alpha\nu} - \gamma_{\mu\nu}^{\alpha} \gamma_{\alpha\nu} \) | \( \alpha_{\mu\nu} \) def \( = C_{\mu} C_{\nu} \) |
| | \( R_{\mu\nu} \) def \( = \frac{1}{2} (\psi_{\mu\nu} - \phi_{\mu\nu} - \theta_{\mu\nu}) + \omega_{\mu\nu} \) |

Where \( \Delta_{\mu\nu}^{\alpha} \) is twice the symmetric part of the contorsion (2.9), \( R_{\mu\nu} \) is Ricci tensor and \( T_{\mu\nu|\sigma} \equiv T_{\mu\nu|\sigma} \).

Hayashi and Bragman [11] derived the irreducible decomposition of torsion tensor which can be written as,

\[
\Lambda_{\alpha\mu\nu} = \frac{2}{3} (t_{\alpha\mu\nu} - t_{\alpha\nu\mu}) + \frac{1}{3} (g_{\alpha\mu} C_{\nu} - g_{\alpha\nu} C_{\mu}) + \epsilon_{\alpha\mu\nu\sigma} a_{\sigma}, \quad (3.2)
\]

where

\[
t_{\alpha\mu\nu} \text{ def } = \frac{1}{2} (\Lambda_{\alpha\mu\nu} + \Lambda_{\alpha\nu\mu}) + \frac{1}{6} (g_{\nu\alpha} C_{\mu} + g_{\mu\nu} C_{\alpha}) - \frac{1}{3} g_{\alpha\mu} C_{\nu}, \quad (3.3)
\]

\[
a_{\mu} \text{ def } = \frac{1}{6} \epsilon_{\mu\alpha\beta\gamma} \Lambda^{\alpha\beta\gamma}, \quad (3.4)
\]

and \( \epsilon_{\mu\alpha\beta\gamma} \) is the Levi-Civita tensor.

Although the development of the AP-geometry in this stage was not as big as in its first stage, its applications were carried out in many diverse problems. For example, McCrea and Mikhail [10],[12], have used this geometry to modify GR in order to account for continuous creation of matter in the Universe. Bergman and Thomson [13] have used it to treat spin and angular momentum in GR. Bilby et al. [14] have used the geometry in studying dislocations in crystals. Utiyama [15], Kibble [16] and Sciama [17] started some attempts for gauging gravity using AP-geometry. Møller [18], [19] has used the geometry.
to solve the problem of localization of energy in GR, a problem that is impossible to be solved in the framework of Riemannian geometry. Mikhail [20] has constructed a pure geometric unified field theory using the AP-geometry. Hehl [21] continued the attempts of Utiyama, Kibble and Sciama to construct a gauge field theory for gravity.

## 4 The Third Stage (1975-1994)

Many authors believe that, because of (2.13), the AP-space is a flat one. In this stage it is shown that AP-spaces are, in general, curved. The problem of curvature in AP-spaces is a problem of definition. In any affinely connected space there are, at least, two methods for defining the curvature tensor. The first method is by replacing Christoffel symbol, in the definition of curvature tensor of Riemannian geometry, by the connection defined in the space. The second method is to define curvature as a measure of non-commutation of covariant (absolute in the present case) differentiation using the connection of the space. It is known that, the two methods give identical results in case of Riemannian space. But the situation is different for spaces with non-symmetric connections. The two methods are not identical.

The application of the second method for non-symmetric geometries implies a problem. That is, we usually use an arbitrary vector in order to study the non-commutation of covariant differentiation, and the resulting expression is not free from this vector. Fortunately, this problem is solved for the AP-spaces [22]. We can replace the arbitrary vector by the vectors defining the structure of AP-spaces. In this case we can define the following curvature tensors:

\[
\lambda^\mu_{\nu\sigma} - \lambda^\mu_{\nu\sigma} \overset{\text{def}}{=} \lambda^\alpha B^\mu_{\alpha\nu\sigma}, \quad (4.1)
\]

\[
\lambda^\mu_{\nu\sigma} - \lambda^\mu_{\nu\sigma} \overset{\text{def}}{=} \lambda^\alpha L^\mu_{\alpha\nu\sigma}, \quad (4.2)
\]

\[
\lambda^\mu_{\nu\sigma} - \lambda^\mu_{\nu\sigma} \overset{\text{def}}{=} \lambda^\alpha N^\mu_{\alpha\nu\sigma}, \quad (4.3)
\]

\[
\lambda^\mu_{\nu\sigma} - \lambda^\mu_{\nu\sigma} \overset{\text{def}}{=} \lambda^\alpha R^\mu_{\alpha\nu\sigma}, \quad (4.4)
\]

here we use the stroke, a (+) sign and (-) sign to characterize absolute differentiation using the connection (2.7) and its dual, respectively. We use the stroke without signs to characterize absolute differentiation using the symmetric part of (2.7), while the semicolon is used to characterize covariant differentiation using the Christoffel symbols. The curvature tensors defined by (4.1), (4.2), (4.3) and (4.4) are in general non-vanishing except the first one, which vanishes (because of the AP-condition). From the application point of view, one of the two problems of the AP-geometry, the curvature problem, is partially solved. There is a net of relations between the different contractions of these tensors [22]. Also there is another net of relations between the absolute derivative of different geometric objects.

A Lagrangian, built using these curvature tensors, has been used to construct a field theory unifying gravity and electromagnetism [22], [23]. When this theory was linearized
both Newton’s theory of gravitation and Maxwell’s theory of electromagnetism were obtained. Also an interpretation of Lorentz condition, used usually in solving Maxwell’s equations, is given [24].

Furthermore, a covariant scheme for classifying AP-spaces, known as "type analysis", is suggested [24]. To clarify the physical meaning of the type analysis we give a trivial example from Riemannian geometry. It is well known that gravity cannot be represented, properly, in flat spaces. So, if we have a Riemannian space with a certain metric and we want to construct a model for gravity using this metric, it is better first to calculate the corresponding curvature tensor for this metric. The result is either a vanishing curvature tensor, or a non-vanishing one. Then, we can classify Riemannian spaces, from the application point of view, to two classes $G_0$ and $GI$, say. The first ($G_0$) is the class with vanishing curvature, which is not appropriate for application concerning gravity. The second ($GI$) is the class with non-vanishing curvature and is the good candidate for application. The same idea is applied in the case of AP-spaces using other tensor together with the curvature one. Table 3 gives a summary of this classification. It is to be considered that this classification scheme "the type analysis" is a covariant one, i.e. independent of coordinate system used. Using the type analysis, one can know, before solving the field equations, the type and strength of the fields that an AP-space is capable of representing.

Table 3: Type Analysis

| Indicator | Field Represented | Type |
|-----------|-------------------|------|
| $F_{\mu\nu} = 0$ | No electromagnetic field. | $F0$ |
| $F_{\mu\nu} \neq 0, \zeta_{\mu\nu} = 0$ | Weak electromagnetic field. | $FI$ |
| $F_{\mu\nu} \neq 0, \zeta_{\mu\nu} \neq 0$ | Strong electromagnetic field. | $FII$ |
| $R^{\alpha}_{\beta\gamma\delta} = 0$ | No gravitational field. | $G0$ |
| $R^{\alpha}_{\beta\gamma\delta} \neq 0, T_{\mu\nu} = 0, \Lambda = 0$ | Weak gravitational field in free space. | $GI$ |
| $R^{\alpha}_{\beta\gamma\delta} \neq 0, T_{\mu\nu} \neq 0, \Lambda = 0$ | Gravitational field within a material distribution. | $GII$ |
| $R^{\alpha}_{\beta\gamma\delta} \neq 0, T_{\mu\nu} \neq 0, \Lambda \neq 0$ | Strong gravitational field within a material distribution. | $GIII$ |
As stated before, the AP-geometry has extra (six) degrees of freedom more than the Riemannian geometry. If these extra degrees of freedom are attributed to electromagnetism \((F)\) \([23], [24]\), and the other ten degrees of freedom are used to represent gravity \((G)\), then possible combinations between \(G\) and \(F\) classes could be listed in the following two groups:

The first group: \(F^0 G^0, F^0 G^I, F^I G^I\).

The second group: \(F^0 G^II, F^I G^II, F^I I G^III\).

It is shown that applications using models from the first group give good agreement with classical field theories of gravitation and electromagnetism \([25], [26]\). Deviation from classical field theories appear when using models from the second group (e.g. \([27]\)). Tensors used for classifications are combinations of tensors given in Table 2. viz,

\[
F_{\mu\nu} \equaldef Z_{\mu\nu} - \xi_{\mu\nu},
Z_{\mu\nu} \equaldef \eta_{\mu\nu} + \zeta_{\mu\nu},
T_{\mu\nu} \equaldef g_{\mu\nu} \Lambda + \omega_{\mu\nu} - \sigma_{\mu\nu},
\Lambda \equaldef \frac{1}{2}(\sigma - \omega)
\]

Hehl et al. \([28]\) have used the AP-geometry in constructing a gauge theory for gravity considering the Poincaré group. Møller in 1978 \([29]\) tried to overcome the singularity problem of GR by using the AP-geometry. Hayashi and Shirafuji in 1979 constructed a microscopic gauge field theory for gravity considering the translation group \([30]\).

The curvature tensors defined above can be written explicitly in terms of torsion or contortion via (2.10), i.e.

\[
B^{\alpha}_{\mu\nu\sigma} = R^{\alpha}_{\mu\nu\sigma} + Q^{\alpha}_{\mu\nu\sigma} \equiv 0, \quad (4.5)
\]

\[
L^{\alpha}_{\mu\nu\sigma} = \Lambda^{\alpha}_{\mu\nu|\sigma} - \Lambda^{\alpha}_{\mu\nu} + \Lambda^{\beta}_{\mu\nu} \Lambda^{\alpha}_{\beta\sigma} - \Lambda^{\beta}_{\mu\rho} \Lambda^{\alpha}_{\rho\sigma}, \quad (4.6)
\]

\[
N^{\alpha}_{\mu\nu\sigma} = \Lambda^{\alpha}_{\mu\nu|\sigma} - \Lambda^{\alpha}_{\mu\sigma} + \Lambda^{\beta}_{\mu\nu} \Lambda^{\alpha}_{\beta\sigma} - \Lambda^{\beta}_{\mu\rho} \Lambda^{\alpha}_{\rho\sigma}, \quad (4.7)
\]

\[
Q^{\alpha}_{\mu\nu\sigma} = \gamma^{\alpha}_{\mu\nu|\sigma} - \gamma^{\alpha}_{\mu\sigma} + \gamma^{\beta}_{\mu\sigma} \gamma^{\alpha}_{\beta\nu} - \gamma^{\beta}_{\mu\nu} \gamma^{\alpha}_{\beta\sigma}, \quad (4.8)
\]

It is clear that the vanishing of the torsion will lead to the vanishing of (4.6), (4.7). Also this will lead to vanishing of (4.8) via (2.10) and consequently the vanishing of \(R^{\alpha}_{\mu\nu\sigma}\) via (4.5). This is another defect of the geometry which is connected to the viability of field theories written in AP-spaces \([31], [32]\). This will be clarified in the next section.

## 5 The Fourth Stage (1995-2000)

In this stage, new path equations were discovered in the AP-geometry \([33]\). These equations can be written in the form:

\[
\frac{dU^\mu}{dS} + \{\mu;\nu\} U^\alpha U^\beta = 0, \quad (5.1)
\]
\[
\frac{dW^\mu}{dS_0} + \{^\mu_{\alpha\beta}\} W^\alpha W^\beta = -\frac{1}{2} \Lambda_{(\alpha\beta)}^\mu W^\alpha W^\beta, \tag{5.2}
\]
\[
\frac{dV^\mu}{dS^+} + \{^\mu_{\alpha\beta}\} V^\alpha V^\beta = -\Lambda_{(\alpha\beta)}^\mu V^\alpha V^\beta. \tag{5.3}
\]

This set of equations possesses some interesting features:

1. It gives the effect of the torsion on the curves (paths) of the geometry.
2. This set is irreducible i.e. no one of these equations can be reduced to the other unless the torsion vanishes. This will lead to flat space as mentioned at the end of the last section.
3. The coefficients of the torsion term jump by a step of one-half from one equation to the next.

The last feature is tempting to believe that "paths in this geometry are naturally quantized".

As stated in section 2 the symmetric part of the connection (2.7) is not Christoffel symbol. In some applications it is preferable to have a non-symmetric connection whose symmetric part is the Christoffel symbol. Such connection can be built by adding the torsion to Christoffel symbols [34],

\[
\Omega^\alpha_{\mu\nu} \overset{\text{def}}{=} \{^\alpha_{\mu\nu}\} + \Lambda^\alpha_{\mu\nu}. \tag{5.4}
\]

This will add two affine connections to the geometry of AP-spaces, (5.4) and its dual \(\tilde{\Omega}^\alpha_{\mu\nu} \overset{\text{def}}{=} \Omega^\alpha_{\nu\mu}\). Using these connections we can define the following curvature tensors:

\[
\lambda^\mu_{i\|\sigma} - \lambda^\mu_{i\|\nu} = \lambda^\alpha_M^\mu_{\alpha\nu\sigma}, \tag{5.5}
\]
\[
\tilde{\lambda}^\mu_{i\|\sigma} - \tilde{\lambda}^\mu_{i\|\nu} = \lambda^\alpha_K^\mu_{\alpha\nu\sigma}. \tag{5.6}
\]

We use the double stroke and a (+) sign to characterize this type of absolute differentiation using (5.4), and a (-) sign for its dual.

The situation now is that we have partially solved the curvature problem mentioned at the end of section 2. We have now defined six curvature tensors, one of which vanishes identically while the others do not. From the point of view of applications this solution is partial since all these tensors could be written in terms of the torsion tensor. Consequently, the vanishing of the torsion will reduce the space to a flat one. For this property, it is shown [35] that theories written in AP-spaces, in which the torsion is connected with spin, will not be viable, since such theories will not reduce to GR as the torsion vanishes. This problem will be solved in the following lines.

The second problem of AP-geometry, mentioned in section 2 still present. Although the new path equations, given above, have some interesting features, these equations do not represent physical trajectories of any particle. So, from the point of view of applications, AP-geometry should be parameterized. Parameterizing this geometry led to the following parameterized geometric objects [36]. Combining linearly the above mentioned connections, we get the following parameterized connection,

\[
\nabla^\mu_{\alpha\beta} = (a + b)\{^\mu_{\alpha\beta}\} + b^\mu_{\alpha\beta} \tag{5.7}
\]
where $a$ and $b$ are parameters. As a consequence of metricity condition, using (5.7), we get

$$a + b = 1.$$  

The set of new paths (5.1), (5.2) and (5.3) can be generalized using (5.7). The result is the following parameterized path equation, [37]

$$\frac{dZ^\mu}{d\tau} + \{^\mu_{\alpha\beta}\} Z^\alpha Z^\beta = -b \Lambda_{(\alpha\beta)}^\mu Z^\alpha Z^\beta,$$  

(5.8)

where $\tau$ is a parameter varying along the path and $Z^\mu$ is the tangent to the path. All curvature tensors defined in this parameterized version of geometry, are non-vanishing. For example if we redefine the curvature (2.13) using the connection (5.7) we get [36]

$$\hat{B}^\alpha_{\mu\nu\sigma} = R^\alpha_{\mu\nu\sigma} + b \hat{Q}^\alpha_{\mu\nu\sigma},$$  

(5.9)

This tensor is, in general non-vanishing although the corresponding one, in the old version of the geometry, vanishes identically.

6 Concluding Remarks

The importance of the new version of AP-geometry, given in the last section, can be summarize in the following points:

1. It is more general than the Riemannian geometry and the conventional AP-geometry. It has a general non-symmetric connection (5.7) giving rise to simultaneous non-vanishing curvature and torsion. This is in contrast to what mentioned by some authors, e.g. [38].

2. If metricity condition is required ($a + b = 1$) then we can take either $a = 1 \Rightarrow b = 0$ and get Riemannian geometry without any need for a vanishing torsion, or we take $a = 0 \Rightarrow b = 1$ to get the conventional AP-geometry.

3. From the application point of view the parameter $b$ and the torsion term appearing in the path equation (5.8) have been connected to some physical interaction [37]. This equation has been used to interpret [39] the discrepancy in the COW-experiment which gives strong evidences for the existence of this interaction on the laboratory scale. Another application [40] gives some evidences for the existence of the interaction on the galactic scale. A third application [41] studies the effect of this interaction on the cosmological scale.

4. The parameterized version of AP-geometry is more suitable for physical applications, especially for constructing theories that require both torsion and curvature to describe different interactions. For example attempts to geometrize strings [42], theories accounting for Dirac fields [43] and theories gauging gravity [44], are among this class of theories.

The following table (Table 4) gives a summary of the important developments, applications and problems of AP-geometry.
Table 4: Stages of Developments, Applications and Problems of AP-geometry

| Stage     | Development                                                                 | Applications                                                                 | Problems                                                                 |
|-----------|-----------------------------------------------------------------------------|------------------------------------------------------------------------------|---------------------------------------------------------------------------|
| 1928-1951 | Basic Structure. Gps of motion in AP-spaces.                                | Gravity and electromagnetism unification [3].                                | Vanishing curvature. Non physical paths.                                  |
| 1952-1974 | Second order tensors and identity.                                          | Unification [20]. Creation of matter [12]. Isospin description [13].         | Vanishing Curvature. Non-physical paths.                                  |
|           |                                                                             | Gauging gravity [16]. Material-energy complex [19]. Dislocations [14].       |                                                                           |
| 1975-1994 | New absolute derivatives. Non-vanishing curvature tensors. Classification of AP-Spaces. | Trials to quantize gravity. Unification [23]. Gauging gravity [44]. Trials to solve the singularity problem [29]. | If $\Lambda^\alpha_{\mu\nu} = 0 \Rightarrow$ all curvature tensors vanish. Non-physical paths. |
| 1995-2000 | New paths New affinity Parameterized AP-geometry.                           | Delay of neutrinos from SN1987A [40]. Interpretation of the discrepancy in the COW-experiment [39]. Quantum paths [37]. |                                                                           |

References

[1] Eisenhart, L.P. (1926) "Riemannian Geometry", Princeton Univ. Press.
[2] Eisenhart, L.P. (1927) "Non-Riemannian Geometry", American Math. Soc.
[3] Einstein, A. (1929) Sitz. Preuss. Akad. Wiss., 1, 1.
[4] Levi-Civita, T. (1929) Sitz. Preuss. Akad. Wiss., 2, 137.
[5] Robertson, H.P. (1932) Ann. Math. Princeton (2), 33, 496.
[6] Einstein, A. and Mayer, W. (1930) Sitz. Preuss. Akad. Wiss., 1, 110.
[7] McVittie, G.C. (1930) Proc. Ed. Math. Soc., 2, 140.
[8] Rosen, N. (1930) M.Sc. Thesis, M.I.T., p.20.
[9] Walker, A.G. (1940) Quar. J. Math., 11, 81.
[10] Mikhail, F.I. (1952) Ph.D. Thesis, London University.
[11] Hayashi, K. and Bregman, A. (1973) Ann. Phys. (N.Y.), 75, 562.
[12] McCrea, W.H. and Mikhail, F.I. (1956) Proc. Roy. Soc. London, A235, 11.
[13] Bergmann, P.G. and Thomson, R. (1953) Phys. Rev., 89, 400.
[14] Bilby, B.A., Bullough, R. and Smith, E. (1955) Proc. Roy. Soc. London, A231, 263.
[15] Utiyama, R. (1956) Phys. Rev., 101, 1597.
[16] Kibble, T.W.B. (1961) J. Math. Phys. 2, 212.
[17] Sciama, D.W. (1962) In "Recent Developments in General Relativity"
    (Festschrift for Infeld), p.415, Pergamon.
[18] Møller, C. (1961) Math. Fys. Skr. Dan Vid. Selsk., 1, 10.
[19] Møller, C. (1966) Math. Fys. Medd. Dan. Vid. Selsk., 35, 3.
[20] Mikhail, F.I. (1964) Il Nuovo Cimento, 32, 886.
[21] Hehl, F.W. (1973) Gen. Rel. Grav., 4, 333.
[22] Wanas, M.I. (1975) Ph.D. Thesis, Cairo University.
[23] Mikhail, F.I. and Wanas, M.I. (1977) Proc. Roy. Soc. London, A356, 471.
[24] Mikhail, F.I. and Wanas, M.I. (1981) Int. J. Theoret. Phys., 20, 671.
[25] Wanas, M.I. (1981) Il Nuovo Cimento, B 66, 145.
[26] Wanas, M.I. (1985) Int. J. Theoret. Phys., 24, 639.
[27] Wanas, M.I. (1989) Astrophys. Space Sci.154, 165.
[28] Hehl, F.W., Ne’eman, Y., Nitch, J. and Von der Heyde, P. (1978) Phys. Lett. 78B, 102.
[29] Møller, C. (1978) Math. Fys. Medd. Dan. Vid. Selsk., 39, 1.
[30] Hayashi, K. and Shirafuji, T. (1979) Phys. Rev. D19, 3524.
[31] Nitch, J. and Hehl, F.W. (1980) Phys. Lett. 90B, 98.
[32] Muller-Hoissen, F. and Nitch, J. (1983) Phys. Rev. D28, 718.
[33] Wanas, M.I., Melek, M. and Kahil, M.E. (1995) Astrophys. Space Sci., 228, 273.
[34] Wanas, M.I. and Kahil, M.E. (1999) Gen. Rel. Grav., 31, 1921.
[35] Wanas, M.I. and Melek, M. (1995) Astrophys. Space Sci., 228, 277.
[36] Wanas, M.I. (2000) Turk. J. Phys., 24, 473.
[37] Wanas, M.I. (1998) Astrophys. Space Sci., 258, 237.
[38] De Andrade, V.C. and Pereira, J.G. (1998) Gen. Rel. Grav., 30, 263.
[39] Wanas, M.I., Melek, M. and Kahil, M.E. (2000) Gravit. Cosmol., 6, 319, and gr-qc/9812085.
[40] Wanas, M.I., Melek, M. and Kahil, M.E. (2000) Proc. Mg9 At3b.
[41] Wanas, M.I. (2000) Proc-IAU-Symp. # 201.
[42] Hammond, R.T. (1998) Gen. Rel. Grav. Lett., 30, 1803.
[43] Hammond, R.T. (1995) Class. Quantum Grav., 12, 279.
[44] Hehl, F.W., von der Heyde, P., Kerlick, G.D. and Nester, J.M. (1976)
    Rev. Mod. Phys., 48, 393.