CLASSIFICATION OF SINGULAR $\mathbb{Q}$-ACYCLIC SURFACES WITH
SMOOTH LOCUS OF NON-GENERAL TYPE

KAROL PALKA

Abstract. We classify singular $\mathbb{Q}$-acyclic surfaces with smooth locus of non-general type. We
analyze their weighted boundaries and completions, the existence and uniqueness of $C^1$- and $C^*$-
ruleds and give constructions. In case such a surface contains a non-quotient singularity or is
non-rational it is isomorphic to a quotient of an affine cone over a projective curve by an action of
a finite group. The dimension of a family of homeomorphic but non-isomorphic singular $\mathbb{Q}$-acyclic
surfaces having the same weighted boundary, singularities and Kodaira dimension can be arbitrarily
big.

We work with algebraic varieties defined over $\mathbb{C}$.

1. Main result

A $\mathbb{Q}$-homology plane is a normal surface with Betti numbers of $\mathbb{C}^2$. As for any open surface,
one of its basic invariants is the logarithmic Kodaira dimension (see [Iit82]). Smooth $\mathbb{Q}$-homology
planes of non-general type, i.e. having Kodaira dimension smaller than two, have been classified,
see [Miy01, §3.4] for summary and for what is known in the case of general type. The main
result of this paper is the classification of singular $\mathbb{Q}$-homology planes with smooth locus of non-
general type. A lot of attention has been given to understand these surfaces in special cases (see
[MS91, GM92, PS97, DR04, KR07]), let us mention explicitly at least the role of the contractible
ones in proving the linearizability of $\mathbb{C}^*$-actions on $\mathbb{C}^3$ (see [KR99]). To our knowledge, in the
available literature on this subject it is always assumed that the planes are logarithmic, by what is
meant that each singular point is of quotient type, i.e. is analytically of type $\mathbb{C}^2/G$ for some finite
subgroup $G < GL(2, \mathbb{C})$. For surfaces this is a strong assumption and one of our goals was to avoid
it. The final classification is obtained in a series of results, the main lines of division depend on the
existence of $C^1$- or $C^*$-fibrations and on the Kodaira dimension of the smooth locus (see 4.5, 4.8
for the case of smooth locus of negative Kodaira dimension; 5.4 for the non-logarithmic case; sec.
5.3, 5.11 and 6.12 for the case of $C^*$-ruled smooth locus). If a $C^1$- or a $C^*$-fibration of the smooth
locus exists we analyze its uniqueness (cf. 4.5, 5.3). Using the notion of a balanced completion (cf.
2.16) we analyze possible balanced boundaries and completions of singular $\mathbb{Q}$-homology planes.

We obtain in particular the following structure theorem (see 3.2(iv), 3.4(i)-(ii), 3.5, and 3.6, 5.8):

Theorem 1.1.

(1) Each singular $\mathbb{Q}$-homology plane is affine and birationally equivalent to a product of a curve
with an affine line. Up to isomorphism there exist exactly two exceptional singular $\mathbb{Q}$-homology
planes, for which the smooth locus is not of general type and it admits no $C^1$- and no $C^*$-
fibration. They have Kodaira dimension and the Kodaira dimension of the smooth loci equal
zero.

(2) If a singular $\mathbb{Q}$-homology plane is non-rational or if it contains a non-quotient singularity then
it is a quotient of an affine cone over a smooth projective curve by an action of a finite group
acting freely off the vertex of the cone.

Both exceptional planes contain unique, cyclic, singular points. Since a cone over a curve has
negative Kodaira dimension and since its smooth locus is not of general type, basic properties of

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the Kodaira dimension imply that the same holds for the above mentioned quotients. We show that even if the quotient is a rational surface, the singularity may be both a rational as well as a non-rational singularity in the sense of Artin (cf. [5.7]).

The following result is of independent interest (the second part is a counterpart of a similar example in case of smooth $\mathbb{Q}$-homology planes in [PZ94]):

**Proposition 1.2.**

1. If a singular $\mathbb{Q}$-homology plane has more than one singular point then its smooth locus is not of general type. Moreover either the smooth locus is affine-ruled or the plane has exactly two singular points, both of type $A_1$ (2.3).

2. Families of homeomorphic but non-isomorphic singular $\mathbb{Q}$-homology planes with the same singularities, weighted boundary and Kodaira dimension can have arbitrarily large dimension (4.7).

Our methods rely heavily on the theory of open algebraic surfaces for which [Miy01] is the basic reference. Important results concerning the classification of $\mathbb{Q}$-homology planes with quotient singularities (mainly analysis of the affine part of fibrations in case they exist) were obtained by Miyanishi and Sugie in [MS91], we explain the relation to our results below. We now give a more detailed overview of successive steps of proofs.

We denote a singular $\mathbb{Q}$-homology plane by $S'$ and its smooth locus by $S_0$. First we prove some basic topological results, whose simpler versions for logarithmic $\mathbb{Q}$-homology planes were known before. In absence of restriction on the type of singularities arguments get more complicated. Once we prove that the Neron-Severi group of the smooth locus is torsion, we apply Fujita’s argument to show that the affiness of $S'$ is a consequence of $\mathbb{Q}$-acyclicity (cf. [2.2]). It was proved in [FS97] that a logarithmic singular $\mathbb{Q}$-homology plane is rational. We complete this result by showing that in general $S'$ is birationally equivalent to a product of an affine line with a curve. Next by general structure theorems for open surfaces (cf. [Miy01], 2.2.1, 2.5.1, 2.6.1) we know that if $\pi(S_0) = -\infty$ or 1 then $S_0$ is $\mathbb{C}^1$- or $\mathbb{C}^*$-ruled. If the smooth locus of a singular $\mathbb{Q}$-homology plane is of non-general type and admits no $\mathbb{C}^1$- and no $\mathbb{C}^*$-rulings then the plane is called exceptional. Under the assumption that singularities are topologically rational we have proved in [Pal00] that there are exactly two such surfaces up to isomorphism. Here we show that the mentioned assumption can be omitted, establishing in this way (1).

We prove that if $S'$ is not logarithmic then it is of very special kind, namely there is a unique $\mathbb{C}^*$-ruling of $S_0$ and it does not extend to a $\mathbb{C}^*$-ruling of $S'$, this implies in particular that $\pi(S') = -\infty$ (cf. [3.6]). We analyze this $\mathbb{C}^*$-ruling and classify all non-logarithmic $S'$’s (cf. 5.4). In particular, to reconstruct them we use a contractibility criterion deduced from a proof of Nakai’s criterion (cf. [2.4], 2.6). Knowing that a non-logarithmic $S'$ admits a $\mathbb{C}^*$-action with a unique fixed point, we infer (1) from [Pin77].

To obtain more detailed description of singular $\mathbb{Q}$-homology planes (and to eventually give their general construction) we need to understand their boundaries and completions. We introduce the notion of a balanced weighted boundary and a balanced completion of an open surface (which is a more flexible version of the notion of a ‘standard graph’ from [PZ07] and has its origin in the paper of [Dai03] together with some normalizing conditions for them (cf. 2.14)). We show that every $\mathbb{Q}$-acyclic surface admits up to isomorphism one or two strongly balanced boundaries, hence such a boundary is a useful invariant of the surface (see cf. 6.11(1) for summary). Next we reprove and extend the description of case $\pi(S_0) = -\infty$ given in [MS91], 2.7-2.9] analyzing balanced completions (cf. [3]).

Having done the above, the problem of classification of singular $\mathbb{Q}$-homology planes with smooth locus not of general type is then reduced to cases when $S'$ is logarithmic, $\pi(S_0) \geq 0$ and $S_0$ is $\mathbb{C}^*$-ruled. In this situation the ruling extends to a $\mathbb{C}^*$-ruled of $S'$. This is exactly what is assumed in [MS91], (2.11-2.16), where one can find necessary conditions for a $\mathbb{C}^*$-ruled surface to be $\mathbb{Q}$-acyclic, a description of singular fibers and formulas (which in fact need some corrections) for $\pi(S_0)$ in terms of these fibers. We correct these formulas (6.8) and give necessary and sufficient conditions (6.11), which allow us to give general constructions (sec. 6.3) and to describe balanced completions (6.11(1)).
The description of balanced boundaries and completions is a vital ingredient in the analysis of the number of possible \( C^1 \)-rulings of \( S' \) in case \( S' \) is \( C^1 \)-ruled (cf. 1.4) and of possible \( C^* \)-rulings of \( S_0 \) and \( S' \) in case \( S' \) is not affine-ruled (cf. 5.12). The last computation has an interesting application. Namely, using it we are able to compute the number of topologically contractible curves contained in a singular \( \mathbb{Q} \)-homology plane in cases where it was not known up to now (see 6.13).

We should also mention that by a classification of open surfaces of some type we mean giving a general construction (i.e. under which each surface of given type can be obtained), describing completions, weighted boundaries and understanding to what extent are these unique. We do not make attempts to describe moduli spaces, which one is forced to do if one wants to give a classification up to isomorphism. Indeed, there are arbitrarily large-dimensional families of non-isomorphic singular \( \mathbb{Q} \)-homology planes having the same homeomorphism type, weighted boundary, Kodaira dimensions of the surface and of its smooth locus, the same number and type of singularities (cf. 4.7).

Let us finally mention the remaining piece of the classification, the difficult case of singular \( \mathbb{Q} \)-homology planes with smooth locus of general type, which due to its different nature is mainly untouched in this paper. By 1.2(1) in this case \( S' \) has a unique singular point, which is of quotient type. There are some partial results concerning these surfaces (GM92, DP89). The main result of KR07 can be restated as saying that if \( \varpi(S') = -\infty \) and \( \varpi(S_0) = 2 \) then \( S' \) cannot be contractible. In a forthcoming paper we generalize this theorem by showing that there are simply no singular \( \mathbb{Q} \)-homology planes of negative Kodaira dimension with smooth locus of general type (cf. PK10).

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2. Preliminaries

2.1. Divisors and pairs. Let $T = \sum t_i T_i$ be an snc-divisor on a smooth complete surface (hence projective by the theorem of Zariski) with distinct irreducible components $T_i$. We follow the notational conventions of [Miy01] and [Pal98, §1, §2]. We write $T = \sum T_i$ for a reduced divisor with the same support as $T$ and denote the branching number of $T_i$ by $\beta_T(T_i) = T_i \cdot (T - T_i)$. A component $T_i \subseteq T$ is branching if $\beta_T(T_i) \geq 3$. If $T$ contains a branching component then it is branched. The determinant of $-Q(T)$, where $Q(T)$ is the intersection matrix of $T$, is denoted by $d(T)$, $d(0) = 1$ by definition. Considering $T$ as a topological subspace of a complex surface with its Euclidean topology it is easy to check that if $\text{Supp} \ T$ is connected then

$$T \approx \bigvee_{i=1}^{n} T_i \lor |DG(T)|,$$

where $|DG(T)|$ is a geometric realization of a dual graph of $T$. In particular, $b_1(T) = \sum_{i=1}^{n} b_1(T_i) + b_1(|DG(T)|)$.

If $T$ is a chain (i.e., it is reduced and its dual graph is linear) then writing it as a sum of irreducible components $T = T_1 + \ldots + T_n$ we always assume that $T_i \cdot T_{i+1} = 1$ for $1 \leq i \leq n - 1$. If $T$ is a chain and some tip (a component with $\beta \leq 1$), say $T_1$, is fixed to be the first one then we distinguish between $T$ and $T = T_n + \ldots + T_1$. We write $T = [-T_2^n, \ldots, -T_n^1]$ in case $T$ is a rational chain. If $T$ is a rational chain with $T_i^2 \leq -2$ for each $i$ we say that $T$ is admissible. If $D$ is some fixed reduced snc-divisor which is not an admissible chain and $T$ is a twig of $D$ (a rational chain not containing branching components of $D$ and containing one of its tips) then we always assume that the tip of $D$ is the first component of $T$. For an admissible (ordered) chain we define

$$e(T) = \frac{d(T - T_1)}{d(T)} \text{ and } \bar{e}(T) = e(T^t).$$

In general $e(T)$ and $\bar{e}(T)$ are defined as the sums of respective numbers computed for all maximal admissible twigs of $T$.

An snc-pair $(X, D)$ consists of a complete surface $X$ and a reduced snc-divisor $D$ contained in the smooth part of $X$. We write $X - D$ for $X \setminus D$ in this case. The pair is a normal pair (smooth pair) if $X$ is normal (resp. smooth). If $X$ is a normal surface then an embedding $\iota: X \to \overline{X}$, where $(\overline{X}, \overline{X} \setminus X)$ is a normal pair, is called a normal completion of $X$. If $X$ is smooth then $\overline{X}$ is smooth and $(\overline{X}, D, \iota)$ is called a smooth completion of $X$. We often identify $X$ with $X - D$ by $\iota$ and neglect $\iota$ in the notation. A morphism of two completions $\iota_j: X \to \overline{X}_j$, $j = 1, 2$ is a morphism $f: \overline{X}_1 \to \overline{X}_2$, such that $\iota_2 = f \circ \iota_1$.

Let $\pi: (X, D) \to (X', D')$ be a birational morphism of normal pairs. We put $\pi^{-1}D' = \pi^*D'$, i.e. $\pi^{-1}D'$ is the reduced total transform of $D'$. If $\pi$ is a blowup then we call it subdivisional (sprouting) for $D'$ if its center belongs to two (one) components of $D'$. In general we say that $\pi$ is subdivisional for $D'$ (and for $D$) if for any component $T$ of $D'$ we have $\beta_{D'}(T) = \beta_D(\pi^{-1}T)$.

The exceptional locus of a birational morphism between two surfaces $\eta: X \to X'$, denoted by $\text{Exc}(\eta)$, is defined as the locus of points in $X$ for which $\eta$ is not a local isomorphism.

A $b$-curve is a smooth rational curve with self-intersection $b$. A divisor is snc-minimal if all its $(-1)$-curves are branching.

**Definition 2.1.** A birational morphism of surfaces $\pi: X \to X'$ is a connected modification if it is proper, $\pi(\text{Exc}(\pi))$ is a smooth point on $X'$ and $\text{Exc}(\pi)$ contains a unique $(-1)$-curve. In case $\pi$ is a morphism of pairs $\pi: (X, D) \to (X', D')$ and $\pi(\text{Exc}(\pi)) \subseteq D'$ we call it a connected modification over $D'$.

Note that since a connected modification has the exceptional locus containing a unique $(-1)$-curve, it can be decomposed into a sequence of blowdowns $\sigma_n \circ \ldots \circ \sigma_1$ such that for $i \leq n - 1$ the center of $\sigma_i$ belongs to the exceptional divisor of $\sigma_{i+1}$. A sequence of blowdowns (and its reversing sequence of blowups) whose composition is a connected modification will be called connected sequence of blowdowns (blowups).
Lemma 2.2. Let $A$ and $B$ be $\mathbb{Q}$-divisors on a smooth complete surface, such that $Q(B)$ is negative definite and $A \cdot B_i \leq 0$ for each irreducible component $B_i$ of $B$. Denote the integral part of a $\mathbb{Q}$-divisor by $\lfloor \cdot \rfloor$.

(i) If $A + B$ is effective then $A$ is effective.
(ii) If $n \in \mathbb{N}$ and $n(A + B)$ is a $\mathbb{Z}$-divisor then $h^0(n(A + B)) = h^0([nA])$.

Proof. (i) We can assume that $A$ and $B$ are $\mathbb{Q}$-divisors and $B$ is effective and nonzero. Write $B = \sum b_i B_i$, where $B_i$ are distinct irreducible components of $B$. Choose $b_i^t \in \mathbb{N}$, such that the sum $\sum b_i^t$ is the smallest possible among divisors $\sum b_i^t B_i$, such that $A + \sum b_i^t B_i$ is effective. If $b_i^t > 0$ for some $i$ then $(A + \sum b_i^t B_i) \cdot (\sum b_i B_i) \leq (\sum b_i B_i)^2 < 0$ by the assumptions. Hence $\text{Supp}(A + \sum b_i^t B_i)$ contains some $B_i$, a contradiction with the definition of $b_i^t$. Thus $A$ is effective.

(ii) Let $\{ R \}$ denote the fractional part of a $\mathbb{Q}$-divisor $R$, i.e. $\{ R \} = R - \lfloor R \rfloor$. Let $T$ be some effective divisor, such that $n(A + B) \sim T$. Then $nA \sim T - nB$ as $\mathbb{Q}$-divisors. Since $T - nB$ is effective by (i), the coefficient of each irreducible component of $[T - nB]$ is bounded below by the coefficient of the same component in $-\{ T - nB \}$. Since $[T - nB]$ is a $\mathbb{Z}$-divisor and the coefficients of components in $\{ T - nB \}$ are fractional and positive, $[T - nB]$ is effective. Moreover, $\{ T - nB \} - \{ nA \}$ being a $\mathbb{Z}$-divisor is equal to 0, so the rational function giving the equivalence of $n(A + B)$ and $T$ gives an equivalence of $[T - nB]$ and $[nA]$.

\[ \square \]

2.2. Singularities and contractibility. Let $\hat{E}$ be the reduced exceptional divisor of the (unique) minimal good (i.e. such that $\hat{E}$ is an snc-divisor) resolution of a singular point on a normal surface $X$. Then $\hat{E}$ is connected and $Q(\hat{E})$ is negative definite. Recall that a point $q \in X$ is of quotient type if there exists an analytical neighborhood $N \subseteq X$ of $q$ and a small (i.e. not containing any pseudo-reflections) finite subgroup $G$ of $GL(2, \mathbb{C})$, such that $(N, q)$ is analytically isomorphic to $(\tilde{N}/G, 0)$ for some ball $\tilde{N}$ around 0 \in $\mathbb{C}^2$. Then $G = \pi_1(N \setminus \{ q \})$. Note that by a result of Tsunoda (1938) for normal surfaces quotient singularities are the same as log-terminal singularities. For a singular point $q \in X$ of quotient type it is known (Brieskorn 1968) that $G$ is cyclic if and only if $\hat{E}$ is an admissible chain and that $G$ is non-cyclic if and only if it is non-abelian if and only if $\hat{E}$ is an admissible fork (rational snc-minimal fork with three twigs and with negative definite intersection matrix, cf. [Miy01, 2.3.5]), in each case $d(\hat{E}) = |G/[G, G]|$. In case $\hat{E}$ is a fork, we will say that $\hat{E}$ is of type $(d_1, d_2, d_3)$ if the maximal twigs of $\hat{E}$ have $d( )$ equal to $d_1, d_2, d_3$. Quotient singularities are rational, as the first direct image of the structure sheaf of their resolutions vanishes. It follows from [Art66, 1] that for a rational singularity $\hat{E}$ is a rational tree, hence rational singularities are topologically rational, which by definition means that $\hat{C}(\hat{E}) = 0$. This notion is a bit stronger than the quasirationality in the sense of Abhyankar (cf. Abhyankar 1979), for which only the rationality of components of $\hat{E}$ is required.

Example 2.3. Let $V \subseteq \mathbb{C}^3$ be given by $x^2 + y^3 + z^7 = 0$. Then the blowup of $V$ in 0 has an exceptional line contained in the singular locus, hence is not normal. Since the blowup of a normal surface with rational singularity remains normal by [Lip69, 8.1], $0 \in V$ is not a rational singularity. On the other hand, it is topologically rational.

More generally, let $V(p_1, p_2, p_3) \subseteq \mathbb{C}^3$ be a Pham-Brieskorn surface given by the equation $x_1^{p_1} + x_2^{p_2} + x_3^{p_3} = 0$, where $p_1, p_2, p_3 \geq 2$. This surface is contractible (note it has a $\mathbb{C}^\ast$-action with the singularity as the unique fixed point) and it is known that $0 \in V(p_1, p_2, p_3)$ is a topologically rational singularity if and only if one of $p_1, p_2, p_3$ is coprime with two others or \( \frac{1}{p_1}, \frac{1}{p_2}, \frac{1}{p_3} \) are integers coprime in pairs. (In [Ore93] and [Z03, 0.1] the above is stated as a condition for quasirationality, but in both cases the graph of the resolution is a tree by looking at the proof or by using [DW74]). On the other hand, the rationality of 0 \in V(p_1, p_2, p_3) is by [Z03, 2.21] equivalent to each of the following conditions: (i) $\sum_{i=1}^{3} \frac{1}{p_i} > 1$, (ii) $0 \in V(p_1, p_2, p_3)$ is of quotient type, (iii) $\pi(V \setminus \{ 0 \}) = -\infty$.

We have the following corollary from the Nakai criterion.
Lemma 2.4. Let $A$ and $B$ be effective snc-divisors on a smooth complete surface $X$ having disjoint supports. If for every irreducible curve $C$ on $X$ either $C \subseteq B$ or $A \cdot C > 0$ then for sufficiently large and sufficiently divisible $n$ one has:

(i) $[nA]$ has no base points,
(ii) $\varphi_{[nA]}$ is birational and contracts exactly the curves in $B$,
(iii) $\text{Im } \varphi_{[nA]}$ is normal, projective and is isomorphic to $\text{Proj } \bigoplus_{n \geq 0} H^0(\mathcal{O}_X(nA))$.

Proof. (i) Repeating part of the proof of Nakai’s criterion (cf. [Har77, V.1.10]) we get that $\mathcal{O}(nA)$ is generated by global sections for $n \gg 0$. For (ii) and (iii) see for example [Rei87, 2.3, 2.4]. See also [Sch00, 3.4] for contractibility criterion for normal surfaces not involving effectiveness.

Definition 2.5. Let $(\overline{X}, D)$ be a smooth completion of a smooth surface $X$ and let $\text{NS}(\overline{X})$ be the Neron-Severi group of $\overline{X}$ consisting of numerical equivalence classes of divisors. The Neron-Severi group $\text{NS}(X)$ of $X$ is defined as the cokernel of the natural map $\mathbb{Z}[D] \rightarrow \text{NS}(X)$, where $\mathbb{Z}[D]$ is a free abelian group generated by irreducible components of $D$. We denote $\text{NS}(X) \otimes \mathbb{Q}$ by $\text{NS}_Q(X)$.

Remark. The above definition does not depend on a smooth completion of $X$ (cf. [Fuj82, 1.19]). Contrary to the case when $X$ is complete, in general $\text{NS}(X)$ can have torsion.

Corollary 2.6. Let $A$ and $B$ be effective snc-divisors on a smooth complete surface $X$ having disjoint supports. Assume that $A$ is connected, $Q(B)$ is negative definite and $\text{NS}_Q(X - A - B) = 0$. Then there exists a normal affine surface $Y$ and a morphism $\zeta : X - A \rightarrow Y$ contracting connected components of $B$, such that $\zeta : X - A - B \rightarrow Y - \zeta(B)$ is an isomorphism.

Proof. Smooth complete surface is projective by the theorem of Zariski. Since $\text{NS}_Q(X - A - B) = 0$, there exists a divisor $H = H_A + H_B$ with $H_A \subseteq A$ and $H_B \subseteq B$, which is numerically equivalent to an ample divisor on $X$. Then $H$ is ample, because ampleness is a numerical property by Nakai’s criterion. To use [2.4], we need to show that there exists an effective divisor $F$, such that $\text{Supp } F = \text{Supp } A$ and $F \cdot C > 0$ for all irreducible curves $C \not\subseteq B$. To deal with curves $C \subseteq A$ we use Fujita’s argument ([Fuj82, 2.4]). Let $U$ consist of all effective divisors $T$, such that $T \subseteq A$ and $T \cdot T_i > 0$ for any prime component $T_i$ of $T$. Writing $H_A = H_+ - H_-$, where $H_+, H_-$ are effective and have no common component, we see that $U$ is nonempty because $H_+ \in U$. Suppose $F$ is an element of $U$ with maximal number of components. For an irreducible curve $C \not\subseteq F$ satisfying $C \cdot F > 0$ one would get $tF + C \in U$ for $t > \max(0, -C^2)$, hence $\text{Supp } F = \text{Supp } A$ by the connectedness of $A$.

Suppose an irreducible curve $C \not\subseteq B$ satisfies $C \cdot F = 0$. Since $F \in U$, we have $C \not\subseteq F$. We can choose some reduced divisor $F' \subseteq F$, such that irreducible components of $F' + B$ give a basis of $\text{NS}_Q(X)$. Let us write $C \equiv \sum \alpha_i F_i + B^+ - B^-$, where $F_i \subseteq F'$, the divisors $B^+, B^- \subseteq B$ are effective and have no common component. For each $j$ we have $C \cdot F_j = 0$, so $(\sum_i \alpha_i F_i) \cdot F_j = C \cdot F_j = 0$, hence $\sum_i \alpha_i F_i = 0$ because $d(F') \neq 0$. We have $(B^+)^2 = B^+ \cdot C + B^+ \cdot B^- \geq 0$, so $B^+ = 0$. Thus the divisor $C - B^-$ is nonzero, effective and numerically trivial, a contradiction. Let $\zeta = \varphi_{[nF]}$ for $n$ as in lemma 2.4. Then $\zeta : X - A \rightarrow \text{Im } \zeta$ contracts connected components of $B$. We have also $nF = \zeta^* H$, where $H$ is a very ample divisor on $\text{Im } \zeta$, which implies that $\text{Im } \zeta$ is affine.

Remark. Note that any divisor with negative definite intersection matrix can be contracted in the analytical category by the theorem of Grauert (cf. [Gra62]). However, in general it is a more subtle problem if this can be done in the algebraic category (see [Art66] for results concerning rational singularities).

2.3. Minimal models. By the Castelnuovo criterion a smooth projective surface $X$ is minimal if and only if there is no irreducible curve $L$ on $X$ for which $K_X \cdot L < 0$ and $L^2 < 0$, which is equivalent to $L$ being a $(-1)$-curve. Similarly, we can say that a smooth pair $(X, D)$ is relatively minimal if and only if there is no irreducible curve $L$ on $X$ for which $K_X \cdot L < 0$ and $L^2 < 0$. In case $L \not\subseteq D$ this implies that $L$ is a $(-1)$-curve intersecting $D$ in at most one point and transversally. However, if $L \subseteq D$ then the conditions are equivalent to $L^2 < 0$ and $\beta_D(L) = L \cdot (D - L) < 2(1 - p_a(L))$ and hence to $L$ being a smooth rational curve with negative self-intersection and branching number
\( \beta_D(L) < 2 \). Contraction of such an \( L \) immediately leads out of the category of smooth pairs, as in particular any tip of any admissible maximal twig of \( D \) would have to be contracted. Thus one repeats the definition of a relatively minimal pair for pairs \((X, D)\) consisting of a normal projective surface and reduced Weil divisor (cf. [Miy01, 2.4.3]). Then the relatively minimal model of a given pair (which can be singular and not unique) is obtained by successive contractions of curves satisfying the above conditions. To go back to the smooth category one can translate the conditions for \((X, D)\) to be relatively minimal in terms of the properties of its minimal resolution. This leads to the notion of an \textit{almost minimal pair}, which we recall now for the convenience of the reader (cf. [Miy01, 2.3.11]).

First, for any smooth pair \((X, D)\) we define the \textit{bark of} \( D \). For non-connected \( D \) bark is a sum of barks of its connected components, so we will assume \( D \) is connected. If \( D \) is an snc-minimal resolution of a quotient singularity (i.e. \( D \) is an admissible chain or an admissible fork) then we define \( \text{Bk} D \) as a unique \( \mathbb{Q} \)-divisor with \( \text{Supp} \text{Bk} D \subseteq D \), such that

\[
(K_X + D - \text{Bk} D) \cdot D_i = 0 \text{ for each component } D_i \subseteq D.
\]

In other case let \( T_1, \ldots, T_s \) be all the maximal admissible twigs of \( D \). (If \( \pi(X - D) \geq 0 \) and \( D \) is snc-minimal then all rational maximal twigs of \( D \) are admissible, cf. [Fuj82, 6.13]). In this case we define \( \text{Bk} D \) as a unique \( \mathbb{Q} \)-divisor with \( \text{Supp} \text{Bk} D \subseteq \bigcup T_j \), such that

\[
(K_X + D - \text{Bk} D) \cdot D_i = 0 \text{ for each component } D_i \subseteq \bigcup_{j=1}^s T_j.
\]

The definition implies that \( \text{Bk} D \) is an effective \( \mathbb{Q} \)-divisor with negative definite intersection matrix and its components can be contracted to quotient singular points. In fact all components of \( \text{Bk} D \) in its irreducible decomposition have coefficients smaller than 1, so \( \text{Bk} D \) is effective and \( \text{Supp} D = \text{Supp} \text{Bk} D \).

A smooth pair \((X, D)\) is \textit{almost minimal} if for every curve \( L \) on \( X \) either \((K_X + \text{Bk} D) \cdot L \geq 0 \) or \((K_X + \text{Bk} D) \cdot L < 0 \) but the intersection matrix of \( \text{Bk} D + L \) is not negative definite. Consequently, the almost minimal model of a given pair \((X, D)\) can be obtained by successive contractions of curves \( L \) for which \((K_X + \text{Bk} D) \cdot L < 0 \) and \( \text{Bk} D + L \) is negative definite. These are the non-branching \((-1)\)-curves in \( D \) and \((-1)\)-curves \( L \not\subseteq D \) for which \( \text{Bk} D \cdot L < 1 \) and \( \text{Bk} D + L \) is negative definite. Minimalization does not change the Kodaira dimension. One shows that \((X, D)\) is almost minimal if and only if after taking the contraction \( \epsilon : (X, D) \to (\overline{X}, \overline{D}) \) of connected components of \( \text{Bk} D \) to singular points the pair \((\overline{X}, \overline{D})\) is relatively minimal. Moreover, if \((X, D)\) is almost minimal and \( \pi(X - D) \geq 0 \) then \( K_X + \text{Bk} D \) is nef and negative definite parts of the Zariski decomposition of \( K_X + D \).

Let \( L \not\subseteq D \) be an irreducible curve for which \( \text{Bk} D \cdot L < 1 \) and \( \text{Bk} D + L \) is negative definite. It is known that \( L \) intersects \( D \) transversally, in at most two points, each connected component of \( D \) at most once. Both points of intersection belong to \( \text{Supp} \text{Bk} D \). For more properties of barks the reader is referred to [Miy01, §2.3].

### 2.4. Rational rulings

By a \textit{rational ruling} of a normal surface we mean a surjective morphism of this surface onto a smooth curve, for which a general fiber is a rational curve. If its general fiber is isomorphic to \( \mathbb{P}^1 \) it is called a \( \mathbb{P}^1 \)-ruling.

**Definition 2.7.** If \( p_0 : X_0 \to B_0 \) is a rational ruling of a normal surface then by a \textit{completion of} \( p_0 \) we mean a triple \((X, D, p)\), where \((X, D)\) is a normal completion of \( X_0 \) and \( p : X \to B \) is an extension of \( p_0 \) to a \( \mathbb{P}^1 \)-ruling with \( B \) being a smooth completion of \( B_0 \). We say that \( p \) is a \textit{minimal completion} of \( p_0 \) if \( D \) is \( p \)-minimal, i.e. if \( p \) does not dominate any other completion of \( p_0 \).

For any rational ruling \( p_0 \) as above there is a completion \((X, D, p)\). Let \( f \) be a general fiber of \( p \). We call \( p_0 \) a \( \mathbb{C}^1 \)-ruling (a \( \mathbb{C}^{(n+1)} \)-ruling) if \( f \cdot D = 1 \) (if \( f \cdot D = n + 1 \)). We always write \( K_X \) for the canonical divisor on a complete surface \( X \). Recall that the \textit{arithmetic genus} of \( D \) is \( p_a(D) = \frac{1}{2}D \cdot (K_X + D) + 1 \). Any fiber of a \( \mathbb{P}^1 \)-ruling has vanishing arithmetic genus and self-intersection. The following well-known lemma shows that these conditions are also sufficient.
Lemma 2.8. Let $F$ be a connected snc-divisor on a smooth complete surface $X$. If $p_a(F) = F^2 = 0$ then there exists a $\mathbb{P}^1$-ruling $p: X \to B$ and a point $b \in B$ for which $p^*b = F$.

Proof. The proof given in [BHPVdV04, V.4.3] after minor modifications works with the above assumptions.

Lemma 2.9. Let $F$ be a singular fiber of a $\mathbb{P}^1$-ruling of a smooth complete surface. By [Fuj82, §4] one has:

(i) $F$ is a rational snc-tree containing a $(-1)$-curve,
(ii) each $(-1)$-curve of $F$ intersects at most two other components of $F$,
(iii) after the contraction of a $(-1)$-curve contained in $F$ the number of $(-1)$-curves in the induced fiber is not greater than the one for $F$, unless $F = [2,1,2]$,
(iv) $F$ is produced from a smooth 0-curve by a sequence of blowups.

Suppose further that $F$ as above has a unique $(-1)$-curve $C$. Let $B_1, \ldots, B_n$ be the branching components of $F$ written in order in which they are created in the (connected) sequence of blowups as in (iv) and let $B_{n+1} = C$. We can write $-F = T_1 + T_2 + \ldots + T_{n+1}$, where the divisor $T_i$ is a reduced chain consisting of all components of $F$ intersected $i$ times not later than $B_i$. We call $T_i$ the $i$-th branch of $F$. If $J$ is an irreducible vertical curve then we denote its multiplicity in the fiber containing it by $\mu_F(J)$ (or $\mu(J)$ if $F$ is fixed).

Lemma 2.9. With the notation as above one has:

(v) $\mu(C) > 1$ and there are exactly two components of $F$ with multiplicity one. They are tips of the fiber and belong to the first branch,
(vi) if $\mu(C) = 2$ then either $F = [2,1,2]$ or $C$ is a tip of $F$ and then $-F - C = [2,2,2]$ or $-F - C$ is a $(-2)$-fork of type $(2,2,n)$,
(vii) if $F$ is branched then the connected component of $-F - C$ not containing curves of multiplicity one is a chain (possibly empty).

Note that if $D$ and $p$ are as in [2.7] then $D$ is $p$-minimal if and only if each non-branching $(-1)$-curve contained in $D$ is horizontal.

Notation 2.10. Recall that having a fixed $\mathbb{P}^1$-ruling of a smooth surface $X$ and a divisor $D$ we define

$$\Sigma_{X-D} = \sum_{F \not\in D} (\sigma(F) - 1),$$

where $\sigma(F)$ is the number of $(X-D)$-components of a fiber $F$ (cf. [Fuj82, 4.16]). The horizontal part $D_h$ of $D$ is a divisor without vertical components, such that $D - D_h$ is vertical. The numbers $h$ and $\nu$ are defined respectively as $\#D_h$ and as the number of fibers contained in $D$. We will denote a general fiber by $f$.

With the notation as above the following equation is satisfied (cf. loc. cit. or [Pal09, 2.2]):

$$\Sigma_{X-D} = h + \nu + b_2(X) - b_2(D) - 2.$$

We call a connected component of $F \cap I^0_D$ a $D$-rivet (or rivet if this makes no confusion) if it meets $D_h$ at more than one point or if it is a node of $D_h$.

Definition 2.11. Let $(X,D,p)$ be a completion of a $\mathbb{C}^*$-ruling of a normal surface $X$. We say that the original ruling $p_0 = p |_{X-D}$ is twisted if $D_h$ is a 2-section. If $D_h$ consists of two sections we say that $p_0$ is untwisted. A singular fiber $F$ of $p$ is columnar if and only if it is a chain not containing singular points of $X$ and which can be written as $-F = A_n + \ldots + A_1 + C + B_1 + \ldots + B_m$ with a unique $(-1)$-curve $C$, such that $D_h$ intersects $F$ exactly in $A_n$ and $B_m$, in each once and transversally. The chains $A = A_1 + \ldots + A_n$ and $B = B_1 + \ldots + B_m$ are called adjacent chains.

Remark. By [KR07, 2.1.1] and the fact that $d(A)$ and $d(A - A_1)$ are coprime we get easily that $e(A) + e(B) = 1$ and $d(A) = d(B) = \mu_F(C)$. In fact we have also $\bar{e}(B) + \bar{e}(A) = 1$ (see [Fuj82, 3.7]).
By abuse of language we call $p$ twisted or untwisted depending on the type of $p_0$. Twisted and untwisted $\mathbb{C}^*$-rulings are called respectively gyozas (Chinese dumpling) and sandwich in \cite{PUJ82}. The following easy lemma describes singular fibers with at most one $(X - D)$ -component (see 7.5-7.7 loc. cit. for a proof).

**Lemma 2.12.** Let $(X, D, p)$ be as in \cite{FJ2}. Assume that $D$ is $p$-minimal and let $F$ be a singular fiber of $p$. One has:

(i) if $\sigma(\mathcal{F}) = 0$ then $F = [2, 1, 2]$, $p$ is twisted and $F$ contains a branching point of $p_1 D_n$,

(ii) if $\sigma(\mathcal{F}) = 1$ and $F$ does not contain a $D$-rivet then either $F$ is columnar or $p$ is twisted and $F$ contains a branching point of $p_1 D_n$.

(iii) if $\sigma(\mathcal{F}) = 1$ and $F$ contains a $D$-rivet then $D_n$ meets $F$ in two different points.

### 2.5. Completions and boundaries.

**Definition 2.13.** A pair $(D, w)$ consisting of a complete curve $D$ and a rationally-valued function $w$ defined on the set of irreducible components of $D$ is called a weighted curve. If $(X, D)$ is a normal pair then $(D, w)$ with $w$ defined by $w(D_i) = D_i^2$ is a weighted boundary of $X - D$.

**Definition 2.14.** Let $(X, D)$ be a normal pair.

(i) Let $L$ be a 0-curve which is a non-branching component of $D$. Make a blowup of a point $c \in L$, such that $c \in L \cap (D - L)$ in case $\beta_D(L) = 2$ and contract the proper transform of $L$. The resulting pair $(X', D')$, where $D'$ is the reduced direct image of the total transform of $D$ is called an elementary transformation of $(X, D)$. The pair $\Phi = (\Phi^o, \Phi^s)$ consisting of an assignment $\Phi^o : (X, D) \mapsto (X', D')$ together with the resulting rational mapping $\Phi^s : X \dashrightarrow X'$ is called an elementary transformation over $D$. $\Phi$ is inner (for $D$) if $\beta_D(L) = 2$ and outer (for $D$) if $\beta_D(L) = 1$. The point $c \in L$ is the center of $\Phi$.

(ii) For a sequence of (inner) elementary transformations $\Phi_i : (X_i, D_i) \mapsto (X_{i+1}, D_{i+1}), i = 1, \ldots, n - 1$ we put $\Phi^o = (\Phi^o_1, \ldots, \Phi^o_{n-1})$, $\Phi^s = (\Phi^s_1, \ldots, \Phi^s_{n-1})$ and we call $\Phi = (\Phi^o, \Phi^s)$ an (inner) flow in $D_1$. We denote it by $\Phi : (X_1, D_1) \mapsto (X_n, D_n)$.

Note that $\Phi^s = (\Phi^s_1, \ldots, \Phi^s_{n-1})$ induces a rational mapping $X_1 \dashrightarrow X_n$, which we also denote by $\Phi^s$. There exist a largest open subset of $X_1$ on which $\Phi^s_1$ is a morphism, the complement of this subset is called the support of $\Phi$. Clearly, $\text{Supp(}\Phi_1 \subseteq D_1$. If $\text{Supp(}\Phi = \emptyset$ then $\Phi$ is a trivial flow.

In general, a weighted curve $(D, w)$ determines the weighted dual graph of $D$. If $(D, w)$ is a weighted boundary coming from a fixed normal pair $(X, D)$ we omit the weight function $w$ from the notation. Note that for $\Phi$ as above $D_1$ and $D_n$ are isomorphic as curves. They have the same dual graphs, but usually different weights of components.

**Example 2.15.** Let $T = [0, 0, a_1, \ldots, a_n]$. Then each chain of type $[0, b, a_1, \ldots, a_n]$, $[a_1, \ldots, a_{k-1}, a_k - b, 0, b, a_{k+1}, \ldots, a_n]$ or $[a_1, \ldots, a_n, b, 0]$ where $1 \leq k \leq n$ and $b \in \mathbb{Z}$, can be obtained from $T$ by a flow. This follows easily from the observation that an elementary transformation changes the chains $[w, x, 0, y - 1, z]$ and $[w, x - 1, 0, y, z]$ one into another. Looking at the dual graph we see the weights can ‘flow’ from one side of a 0-curve to another, including the possibility that they vanish ($b = 0$ or $b = a_k$). If they do then again the weights can flow through the new zero.

**Definition 2.16.** A rational chain $D = [a_1, \ldots, a_n]$ is balanced if $a_1, \ldots, a_n \in \{0, 2, 3, \ldots\}$ or if $D = [1]$. A reduced snc-divisor whose dual graph contains no loops (snc-forest) is balanced if all rational chains contained in $D$ which do not contain branching components of the divisor are balanced. A normal pair $(X, D)$ is balanced if $D$ is balanced.

Recall that if $(X_i, D_i)$ for $i = 1, 2$ are normal pairs such that $X_1 - D_1 \cong X_2 - D_2$ then $D_1$ is a forest if and only if $D_2$ is a forest.

**Proposition 2.17.** Any normal surface which admits a normal completion with a forest as a boundary has a balanced completion. Two such completions differ by a flow.

As we discovered after completing the proof, the above proposition in a more general version was proved in a graph theoretic context in \cite{FKZ07} (see Theorem 3.1 and Corollary 3.36 loc. cit.). We
leave therefore our more direct arguments to be published elsewhere. In fact, some key observations were done earlier in [Dai03] (see 4.23.1, 3.2, 5.2 loc. cit.). Let us restate some definitions from [FKZ07] on the level of pairs.

**Definition 2.18.** Let \((X, D)\) be a normal pair and assume \(D\) is an snc-forest.

(i) Connected components of the divisor which remains after subtracting all non-rational and all branching components of \(D\) are called the **segments** of \(D\).

(ii) \(D\) is **standard** if for each of its connected components either this component is equal to \([1]\) or all its segments are of types \([0],[0,0,0]\) or \([0^k,a_1,\ldots,a_n]\) with \(k\in\{0,1\}\) and \(a_1,\ldots,a_n\geq 2\).

(iii) If \(D_0 = [0,0,a_1,\ldots,a_n]\) with \(a_i \neq 0, i = 1,\ldots,n\) is a segment of \(D\) then a **reversion** of \(D_0\) is a nontrivial flow \(\Phi : (X,D) \to (X',D')\) with support in \(D_0\), which is inner for \(D_0\) and for which \(D' - (\Phi^\ast)(D-D_0) = [a_1,a_2,\ldots,a_n,0,0]\).

The condition that \(\Phi\) is introduced for the following reason: we want the reversion to transform the two zeros 'to the other end' of the chain, and the condition in necessary to force this in case \(D\) is symmetric, i.e when \([a_1,\ldots,a_n]^t = [a_1,\ldots,a_n]\). Standard chains are called **canonical** in [Dai03]. Note that the Hodge index theorem implies that if \((X,D)\) is a smooth pair and \(D\) is a forest then it cannot have segments of type \([0^{2k+1}]\) or \([0^{2k},a_1,\ldots,a_n]\) for \(k > 1\) and can have at most one such segment with \(k = 1\).

Clearly, not every balanced forest is standard, but by a flow one can easily change it to such. Now it follows from 2.17 that if \(D\) and \(D'\) are two standard boundaries of the same surface and \(D\) is a chain then either \(D\) and \(D'\) are isomorphic as weighted curves or \(D'\) is the reversion of \(D\). Unfortunately, the notion of a standard boundary in not as restrictive as one may wish and the difference between two standard boundaries can be more than just a reversion of some segments. An additional ambiguity is related to the existence of segments of type \([0^{2k+1}]\). Namely, if \([0^{2k+1}]\) is a segment of \(D\) then one can change by a flow the self-intersections of the components of \(D\) intersecting the segment. For example, all rational forks with dual graph

```
  -2     b     -2

  0
```

for some \(b \in \mathbb{Z}\) can appear as standard boundaries of the same surface.\(^1\) Let us therefore introduce the following more restrictive conditions, which will be sufficient for the needs of this paper:

**Definition 2.19.** A balanced snc-forest \(D\) is **strongly balanced** if and only if it is standard and either \(D\) contains no segments of type \([0],[0,0,0]\) or for at least one of such segments there is a component \(B \subseteq D\) intersecting it, such that \(B^2 = 0\). A normal pair \((X,D)\) for which \(D\) is a forest is strongly balanced if \(D\) is strongly balanced.

### 3. Topology and Singularities

#### 3.1. Homology

Let \(S'\) be a **singular** \(Q\)-homology plane. Let \(\epsilon : S \to S'\) be a good resolution and \((\widehat{S}, D)\) a smooth completion of \(S\). Denote the singular points of \(S'\) by \(p_1,\ldots,p_q\) and the smooth locus by \(S_0\). We put \(\widehat{E}_i = \epsilon^{-1}(p_i)\) and assume that \(\widehat{E} = \widehat{E}_1 + \widehat{E}_2 + \ldots + \widehat{E}_q\) is snc-minimal. Define \(M\) as the boundary of the closure of \(\text{Tub}(\widehat{E})\), where \(\text{Tub}(\widehat{E})\) is a tubular neighborhood of \(\widehat{E}\). The construction of \(\text{Tub}(\widehat{E})\) can be found in [Mum61]. We can assume that \(M\) is a disjoint sum of \(q\) closed oriented 3-manifolds. We write \(H_i(X,A)\) for \(H_i(X,A,Q)\) and \(b_i(X,A)\) for \(\dim H_i(X,A)\).

Let us mention that the results we obtain below are generalizations of similar results obtained in the logarithmic case by Miyanishi and Sugie. However, restriction to quotient singularities is a strong assumption, which makes the considerations much easier, even if at the end we prove that not so many non-logarithmic \(Q\)-homology planes exist.

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\(^1\)This observation was missed in [FKZ07] and the corollary 3.33 loc. cit. is false. See [FKZ09] for corrections. In [Dai03], Solution to problem 5, p. 45, this ambiguity is implicitly taken into account without restricting to balanced divisors.
Proposition 3.1. Let $j_{\hat{\mathcal{E}}}: \hat{\mathcal{E}} \to S$, $j_{\mathcal{M}}: M \to S_0$, $i_D: D \to \mathcal{S}$ and $i_{D\cup\hat{\mathcal{E}}}: D \cup \hat{\mathcal{E}} \to \mathcal{S}$ be the inclusion maps. One has:

(i) $H_1(M, \mathbb{Z}) = H_1(\hat{\mathcal{E}}, \mathbb{Z}) \oplus K$ for some finite group $K$ of order $d(\hat{\mathcal{E}})$,

(ii) $H_k(j_{\hat{\mathcal{E}}})$ and $H_k(j_{\mathcal{M}})$ are isomorphisms for positive $k$,

(iii) $D$ is connected, $H_1(i_D)$ is an isomorphism and $b_1(D) = b_1(\hat{\mathcal{E}})$,

(iv) $H_2(i_D j_{\hat{\mathcal{E}}})$ is an isomorphism,

(v) $H_k(S', \mathbb{Z}) = 0$ for $k \neq 0, 1$,

(vi) $\pi_1(\gamma): \pi_1(S) \to \pi_1(S')$ is an epimorphism, it is an isomorphism if $b_1(\hat{\mathcal{E}}) = 0$,

(vii) if $b_1(\hat{\mathcal{E}}) = 0$ then $|d(D)| = |d(\hat{\mathcal{E}})| \cdot |H_1(S', \mathbb{Z})|^2$.

Proof. (i) By [Mum61] there is an exact sequence

$$0 \to K \to H_1(M, \mathbb{Z}) \xrightarrow{\gamma} H_1(\hat{\mathcal{E}}, \mathbb{Z}) \to 0,$$

where $K$ is a finite group of order $d(\hat{\mathcal{E}})$ and $r$ is induced by the composition of embedding of $M$ into the closure of $\text{Tab}(\hat{\mathcal{E}})$ with retraction onto $\hat{\mathcal{E}}$. Since $H_1(\hat{\mathcal{E}}, \mathbb{Z})$ is free abelian, it follows that $H_1(M, \mathbb{Z}) = H_1(\hat{\mathcal{E}}, \mathbb{Z}) \oplus K$.

(ii) Let $k > 0$. We look at the reduced homology exact sequence of the pair $(S, \hat{\mathcal{E}})$. The pairs $(S, \hat{\mathcal{E}})$ and $(S', \text{Sing } S')$ are 'good CW-pairs' (see [Hat02, Thm 2.13]), so for $k \neq 1$ we have $H_k(S, \hat{\mathcal{E}}) = H_k(S', \text{Sing } S') = 0$ and then $H_k(j_{\hat{\mathcal{E}}}) = H_k(S)$ induced by $j_{\hat{\mathcal{E}}}$ is an isomorphism for $k > 1$. Now $b_1(S, \hat{\mathcal{E}}) = b_1(S', \text{Sing } S') = b_0(\hat{\mathcal{E}}) - 1$, so $b_1(S) = b_1(\hat{\mathcal{E}})$ and $H_1(j_{\hat{\mathcal{E}}})$ is also an isomorphism. Since $H_k(j_{\hat{\mathcal{E}}})$ are epimorphisms, the Mayer-Vietories sequence for $S = S_0 \cup \text{Tab}(\hat{\mathcal{E}})$ splits into exact sequences:

$$0 \to H_k(M) \to H_k(S_0) \oplus H_k(\hat{\mathcal{E}}) \to H_k(S) \to 0.$$

Since $H_k(j_{\hat{\mathcal{E}}})$ is injective, $H_k(j_{\mathcal{M}})$ is injective by exactness, so it is an isomorphism.

(iii)-(iv) By (ii) $b_3(S) = b_3(\hat{\mathcal{E}}) = 0$, so the homology exact sequence of the pair $(\mathcal{S}, S)$ yields $H_4(\mathcal{S}, S) \cong H_4(\mathcal{S})$, hence $H^0(D) = H_4(\mathcal{S}, S) = \mathbb{Q}$ by the Lefschetz duality, which implies the connectedness of $D$. The components of $\hat{\mathcal{E}}$ are numerically independent because $d(\hat{\mathcal{E}}) \neq 0$, hence they are independent in $H_2(\mathcal{S})$, which implies that the inclusion $i_{\hat{\mathcal{E}}}: \hat{\mathcal{E}} \to \mathcal{S}$ induces a monomorphism on $H_2$. By (ii) we can write the exact sequence of the pair $(\mathcal{S}, S)$ as:

$$\ldots \to 0 \to H_3(\mathcal{S}) \to H_3(\mathcal{S}, S) \to H_2(\hat{\mathcal{E}}) \to H_2(\mathcal{S}) \to \ldots .$$

Now $H_2(i_{\hat{\mathcal{E}}})$ is a monomorphism, so by the Lefschetz and Poincare duality $b_1(D) = b_3(\mathcal{S}, S) = b_3(\mathcal{S}) = b_1(\mathcal{S})$. On the other side $b_1(\mathcal{S}, D) = b_3(S) = 0$, so $H_1(i_D)$ is an isomorphism.

Since $H_1(i_D)$ is an isomorphism, the homology exact sequence of the pair $(\mathcal{S}, D \cup \hat{\mathcal{E}})$ yields an exact sequence:

$$0 \to H_3(\mathcal{S}) \to H_3(\mathcal{S}, D \cup \hat{\mathcal{E}}) \xrightarrow{\delta} H_2(\mathcal{S}, D \cup \hat{\mathcal{E}}) \xrightarrow{\gamma} H_2(\mathcal{S}) \to H_2(D, \mathcal{S} \cup \hat{\mathcal{E}}) \to H_1(\hat{\mathcal{E}}) \to 0.$$

We have $b_2(\mathcal{S}, D \cup \hat{\mathcal{E}}) = b_2(\mathcal{S}_0) = b_2(M)$ by (ii) and $b_2(M) = b_1(M) = b_1(\hat{\mathcal{E}})$ by (i), so $\gamma$ is an epimorphism. We need to prove that $b_1(D) = b_1(\hat{\mathcal{E}})$ and $\text{Ker } \gamma = 0$. Note that $b_2(\mathcal{S}) = b_2(D \cup \hat{\mathcal{E}}) - \dim \text{Im } \delta$ and $\dim \text{Im } \delta = b_3(\mathcal{S}, D \cup \hat{\mathcal{E}}) - b_3(\mathcal{S}) = b_3(S) - b_3(S_0) = b_2(\mathcal{E}) - b_1(S_0) = 0$, so $b_2(D \cup \hat{\mathcal{E}}) = b_2(\mathcal{S}) = b_1(\mathcal{E}) - b_1(D)$, so $b_2(D \cup \hat{\mathcal{E}}) = b_2(\mathcal{S}) = b_1(\mathcal{E}) - b_1(D)$, this implies that $\gamma$ is an isomorphism.

If $b_1(\mathcal{E}) = 0$ then $b_3(\mathcal{S}, D \cup \hat{\mathcal{E}}) = b_3(S_0) = b_1(S_0) = 0$, so $\gamma$ is a monomorphism. We can therefore assume that $\tilde{\mathcal{E}}$ is not a rational forest, in particular $S'$ is not logarithmic. Note that since $\gamma$ is an epimorphism, $S'$ is affine by [2.1], so we can use [3.3] below to infer that $\pi(S_0) \neq 2$. Suppose $\pi(S_0) = 1$, then $S_0$ is $\mathbb{C}^*$-ruled (cf. [Kaw73, 2.3]). Since modifications over $D + \tilde{\mathcal{E}}$ do not change $b_1(D)$ and $b_1(\tilde{\mathcal{E}})$, we can assume that this ruling extends to $\mathcal{S}$. The divisor $D$ is not vertical, otherwise $Q(D + \tilde{\mathcal{E}})$ would be semi-negative definite, which contradicts the Hodge index theorem. On the other hand, $\tilde{\mathcal{E}}$ is not vertical because is not a rational forest, so each of $D$ and $\tilde{\mathcal{E}}$ contains a unique section. Then $b_1(D) = b_1(\tilde{\mathcal{E}})$, so we are done. We can now assume $\pi(S_0) \leq 0$. Suppose $\pi(S) = 0$, then $\pi(S_0) = 0$. 
Put $F = D + \hat{E} - E_0$, where $E_0$ is a connected component of $\hat{E}$ with $b_1(E_0) \neq 0$. Let $(\hat{S}, \hat{F} + \hat{E}_0)$ be the almost minimal model of $(S, D + \hat{E}) = (\hat{S}, F + E_0)$ with $\hat{F}$ and $\hat{E}_0$ being the direct images of $F$ and $E_0$. The divisors $\hat{F}$ and $\hat{E}_0$ are disjoint, so $K_{\hat{S}} + \hat{F} - Bk \hat{F} + \hat{E}_0 - Bk \hat{E}_0 \equiv (K_{\hat{S}} + F + E_0)^+ \equiv 0$. Since by 2.2(ii) $h^0(n(K_{\hat{S}} + \hat{F} - Bk \hat{F})) = h^0(n(K_{\hat{S}} + F)) \geq h^0(n(K_{\hat{S}} + F))$, we have $K_{\hat{S}} + \hat{F} - Bk \hat{F} \geq 0$, so $E_0 = Bk \hat{E}_0$, which contradicts $b_1(\hat{E}_0) = b_1(E_0) \neq 0$. We get $\pi(S) = -\infty$, so $S$ is $\mathbb{C}$-ruled by $\mathbb{M}_0$. Consider an extension of this ruling to $\mathbb{S}$ and a divisor $T = \sum_i d_i D_i + \sum_j e_j E_j$ with distinct irreducible components $D_i \subseteq D$, $E_j \subseteq \hat{E}$, such that $T \equiv 0$. To finish the proof that $\text{Ker} \gamma = 0$ it is enough to show that $T = 0$. Suppose $T \neq 0$. Using negative definiteness of $Q(\hat{E})$ we see that each $e_j$ vanishes, otherwise $0 > (\sum_j e_j E_j)^2 = T \cdot (\sum_j e_j E_j)$. Intersecting $T$ with a fiber we see that the horizontal component of $D$ does not occur in the sum $T = \sum_j d_j D_j$, therefore $T$ is vertical. It follows that $\text{Supp}T$ contains at least one fiber, otherwise $T^2 < 0$. However, then the equality $T \cdot \hat{E} = 0$ implies that $\hat{E}$ is vertical, a contradiction with $b_1(\hat{E}) \neq 0$.

(v) Let $k \in \{3, 4\}$. The groups $H_k(S', \mathbb{Z}) \cong H_k(S, \hat{E}, \mathbb{Z})$ are torsion, so the exact sequence of the pair $(S, \hat{E})$ gives $H_k(S, \hat{E}, \mathbb{Z}) \cong H_k(S, \mathbb{Z})$. By the universal coefficient formula and Lefschetz duality $H_k(S, \mathbb{Z}) \cong H^{k+1}(S, \mathbb{Z}) \cong H_{k-1}(\mathbb{S}, D, \mathbb{Z}) = 0$. Vanishing of $H_2(S', \mathbb{Z})$ is more subtle. The generalization of Andreotti-Frankel theorem to the singular case proved by Karchyauskas says that an affine variety $X$ of complex dimension $n$ has the homotopy type of a $CW$-complex of real dimension not greater than $n$ (see [GMSS] for proofs and generalizations). In particular, $H_n(X, \mathbb{Z})$ is a free abelian group. We showed in the proof of (iii)-(iv) that $S'$ is affine, so we get $H_2(S', \mathbb{Z}) = 0$.

(vi) Choose points $y, x_1, \ldots, x_q \in S$, such that $y \in S_0$ and $x_i \in \hat{E}_i$. Let $\alpha_1, \ldots, \alpha_q$ be smooth paths in $S$ joining $y$ with $x_i$. We can choose $\alpha_i$ in such a way that they meet transversally in $y$, $\alpha_1 \setminus \{y\}$ are disjoint, $R = \bigcup \alpha_i \setminus \{x_i\}$ is contained in $S_0$ and meets $\hat{E}$ transversally. Let $N$ be a tubular neighborhood of $\hat{E} \cup R$ in $S$. Then $\epsilon(N) \subseteq S'$ is a contractible neighborhood of $\text{Sing} S'$. Put $H = \pi_1(N \setminus (R \cup \hat{E}))$. Clearly, $\epsilon$ identifies $N \setminus (R \cup \hat{E})$ with $\epsilon(N) \setminus (\text{Sing} S' \cup \epsilon(R))$, so since $\pi_1(S_0 \setminus R) \cong \pi_1(S_0)$, by van Kampen’s theorem $\pi_1(S') \cong \pi_1(S_0) * \pi_1(N)$ and $\pi_1(S') \cong \pi_1(S_0) * \{1\}$. We have $\pi_1(N) = \pi_1(\hat{E}_1) * \ldots * \pi_1(\hat{E}_q)$ and each $\pi_1(\hat{E}_i)$ is contained in the kernel of $\pi_1(\epsilon)$. If $b_1(\hat{E}) = 0$ then $\hat{E}$ is a rational forest, so $\pi_1(N) = \{1\}$ and we get $\pi_1(S') \cong \pi_1(S)$.

(vii) Let $M_D = \partial \text{Tub}(D)$ be the boundary of a (closure of a) tubular neighborhood of $D$. We can assume that $M_D$ is a 3-manifold disjoint from $M$. By (iii) $b_1(D) = 0$ and by (iv) the components of $D$ are independent in $H_2(\mathbb{S})$, so $D$ is a rational tree with $d(D) \neq 0$. Then using the presentation given in [Mum61] we get that $H_1(M_D)$ is a finite group of order $|d(D)|$. By Poincare duality $H_3(M_D, \mathbb{Z})$ (and similarly $H_2(M_D, \mathbb{Z})$) are trivial. Consider the reduced homology exact sequence of the pair $(K, M_D)$, where $K = \mathbb{S} \setminus (\text{Tub}(D) \cup \text{Tub}(\hat{E}))$:

$$0 \to H_2(K, \mathbb{Z}) \to H_2(K, M_D, \mathbb{Z}) \to H_1(M_D, \mathbb{Z}) \to H_1(K, \mathbb{Z}) \to H_1(K, M_D, \mathbb{Z}) \to 0.$$

By the Lefschetz duality (cf. [Hat02], 3.43] $H_i(K, M_D, \mathbb{Z}) \cong H^{4-i}(K, M, \mathbb{Z}) = H^{4-i}(S', \text{Sing} S', \mathbb{Z})$, which for $i > 1$ implies that $H_i(K, M_D, \mathbb{Z}) \cong H^{4-i}(S', \mathbb{Z}) \cong H_{3-i}(S', \mathbb{Z})$ by the universal coefficient formula. This gives an exact sequence:

$$0 \to H_2(K, \mathbb{Z}) \to H_1(S', \mathbb{Z}) \to H_1(M_D, \mathbb{Z}) \to H_1(K, \mathbb{Z}) \to H_2(S', \mathbb{Z}) \to 0.$$
(ii) \( \chi(S_0) = 1 - q, \) \( \chi(S) = \# \hat{E} + 1 - b_1(\hat{E}), \) \( \chi(\overline{S}) = \# D + \# \hat{E} + 2 - 2b_1(\hat{E}), \)

(iii) \( \Sigma_{S_0} = h + \nu - 2 \) and \( \nu \leq 1, \)

(iv) \( S' \) is affine and \( NS_Q(S_0) = 0, \)

(v) \( d(D) < 0, \)

(vi) if \( \pi_1(S') = \{1\} \) then \( S' \) is contractible.

**Proof.** Part (i) follows from (1.1(i)-(ii)). Then (ii) is a consequence of (1.3) and the equality \( \chi(S_0) = \chi(S') - q = 1 - q. \) By (1.4) \( H_2(i_{D\cup \hat{E}}) \) is surjective, so \( NS_Q(S_0) = 0 \) and then by (2.6) \( S' \) is affine, which gives (iv). Since \( H_2(i_{D\cup \hat{E}}) \) is injective, the Hodge index theorem implies that the signature of \( Q(D + \hat{E}) \) is \( (1-\#(D + \hat{E}^+), \) hence \( d(\hat{E})d(D) = d(D + \hat{E}) < 0 \), which proves (v). For (iii) note that since \( b_2(\overline{S}) = b_2(D \cup \hat{E}), \) Fujita’s equation (sec. 2.4) yields \( \Sigma_{S_0} = h + \nu - 2. \) If \( \nu > 1 \) then the numerical equivalence of fibers of a \( \mathbb{P}^1 \)-ruling gives a numerical dependence of components of \( D + \hat{E}, \) hence \( d(D + \hat{E}) = 0, \) a contradiction with (v). If \( \pi_1(S') = \{1\} \) then by (3.1(v) and the Hurewicz theorem all homotopy groups of \( S' \) vanish, so Whitehead’s theorem implies (vi). \( \Box \)

3.2. Birational type and logarithmicity. By [PS97, Theorem 1.1] it is known that singular \( \mathbb{Q} \)-homology planes which have at most quotient singularities are rational. We will see that this is not true in general. We describe the birational type of \( S' \) and prove some general properties of its singular locus.

**Lemma 3.3.** Let \( S_0 \) be the smooth locus of a singular \( \mathbb{Q} \)-homology plane \( S'. \)

(i) If \( \pi(S_0) = 2 \) then \( S' \) is logarithmic and \( \# \text{Sing} S' = 1. \)

(ii) If \( \pi(S_0) = 0 \) or \( 1 \) then either \( \# \text{Sing} S' = 1 \) or \( \# \text{Sing} S' = 2 \) and \( \hat{E}_1 = \hat{E}_2 = [2]. \)

**Proof.** Let \( (S_m, D_m) \) be the almost minimal model of \( (\overline{S}, D + \hat{E}). \) Since \( S' \) is affine, the almost minimal model \( S_m - D_m \) of \( S_0 \) is isomorphic to an open subset of \( S_0 \) satisfying \( \chi(S_m - D_m) \leq \chi(S_0) = 1 - q \) (see [Palo09, 2.8]). For each connected component of \( D_m \) being a connected component of \( BkD_m \) (hence contractible to quotient singularity) denote the local fundamental group of the respective singular point \( P \) by \( G_P \) and the set of such points by \( Q. \) By the Kobayashi inequality (see [Lan03] or [Palo09, 2.5(ii)] for a generalization, which we use here) \( \frac{1}{3}(K_{S_m} + D_m)^2 \leq \chi(S_m - D_m) + \sum_{P \in Q} \frac{1}{G_P} \leq 1 - q + \frac{\# Q}{2} \leq 1 - q/2. \) If \( \pi(S_0) = 2 \) then we get \( q = 1 \) and \( 0 < \sum_{P \in Q} \frac{1}{|G_P|}, \) so there is a unique singular point on \( S' \) and it is of quotient type. If \( \# \text{Sing} S' > 1 \) and \( \pi(S_0) \geq 0 \) then we get \( q = 2 \) and \( 1 \leq 1/|G_{P_1}| + 1/|G_{P_2}|, \) so \( |G_{P_1}| = |G_{P_2}| = 2. \) \( \Box \)

**Proposition 3.4.** With the notation as above one has:

(i) \( \overline{S} \) is \( \mathbb{P}^1 \)-ruled over a curve of genus \( \frac{1}{3}b_1(D) = \frac{1}{3}b_1(\hat{E}), \)

(ii) if \( \pi(S') \geq 0 \) then \( S' \) is rational and has topologically rational singularities (cf. 3),

(iii) both \( \hat{E} \) and \( D \) are forests with at most one nonrational component,

(iv) if \( \hat{E} \) consists only of \( (-2) \)-curves then \( \pi(S') = \pi(S_0). \)

**Proof.** (i)-(iii) We have \( b_1(\hat{E}) = b_1(D) = b_1(\overline{S}) \) by (3.1(iii)), so if \( b_1(\hat{E}) = 0 \) then we are done. We can therefore assume that \( b_1(\hat{E}) \neq 0. \) Suppose \( \pi(S) = -\infty. \) Then \( S \) is affine-ruled (i.e. \( \mathbb{C}^1 \)-ruled), because \( D \) is connected, so we need only to prove (iii). Let \( \overline{S} \to B \) be a \( \mathbb{P}^1 \)-ruling extending the affine ruling of \( S. \) Then \( D \) is a tree and has a unique nonrational component as the horizontal section. Since \( b_1(\hat{E}) \neq 0, \) \( \hat{E} \) has a horizontal component \( E_0. \) Clearly, \( g(E_0) \geq g(B), \) so \( b_1(E_0) \geq b_1(B). \) However, \( b_1(B) = b_1(D) = b_1(\hat{E}), \) so \( b_1(E_0) = b_1(\hat{E}) = E_0 \) is the unique horizontal component of \( \hat{E}, \) hence \( \hat{E} \) is a forest. Thus we can assume that \( \pi(S_0) \geq \pi(S) \geq 0. \) Suppose \( \pi(S_0) = 1. \) Then, since \( S' \) does not contain complete curves, by [Kaw79, 2.3] \( S_0 \) is \( \mathbb{C}^* \)-ruled and this ruling does not extend to \( S' \) (\( \hat{E} \) would be a rational forest then). Thus some resolution of \( S' \) (not necessarily the minimal resolution \( S \)) is affine-ruled, which implies \( \pi(S) = -\infty, \) a contradiction. By (3.1(i) \( \pi(S_0) \neq 2. \) Thus we are left with the case \( \pi(S) = \pi(S_0) = 0. \) We argue as in the proof of (3.1(iii)-(iv) that \( b_1(\hat{E}) = 0, \) a contradiction.
(iv) We have to prove that \( \pi(S_0) \leq \pi(S) \). If \( \hat{E} \) consists of \((-2)\)-curves then \((K + D) \cdot E_i = 0\) for each irreducible component \( E_i \) of \( \hat{E} \). If \( T \) is an effective divisor linearly equivalent to \( n(K + D + \hat{E}) \) then, since \( Q(\hat{E}) \) is negative definite, \( T - n\hat{E} \) is effective by 2.2 and we are done.

Before stating a theorem strengthening the proposition 3.4(ii) we need another corollary from proposition 3.1, which strengthens [Pal09, 1.4].

**Lemma 3.5.** If the smooth locus \( S_0 \) of a singular \( \mathbb{Q} \)-homology plane \( S' \) is not of general type then either \( S_0 \) is \( \mathbb{C}^1 \)- or \( \mathbb{C}^* \)-ruled or \( S' \) is up to isomorphism one of two exceptional surfaces described in [Pal09, 4.4]. In the last case \( \pi(S') = \pi(S_0) = 0 \) and \( S' \) has a unique singular point, which is of type \( A_1 \) or \( A_2 \).

**Proof.** By general structure theorems for open surfaces if \( \pi(S_0) = -\infty \) or 1 then \( S_0 \) is \( \mathbb{C}^1 \)- or \( \mathbb{C}^* \)-ruled (cf. [Kaw79, 2.3] and section 4). We can therefore assume that \( \pi(S_0) = 0 \). By [Pal09, 1.4] we have only to consider the case when singularities of \( S' \) are not topologically rational, which means that \( b_1(\hat{E}) \neq 0 \). Suppose \( b_1(\hat{E}) \neq 0 \). Then \( \hat{E} \) is connected by 3.3 and \( b_1(D) \neq 0 \) by 3.1(iii). Let \((\hat{S}, \hat{D} + \hat{E})\) be the almost minimal model of \((\hat{S}, D + \hat{E})\). Then by [Fuj82, 8.8] \( \hat{D} \) and \( \hat{E} \) are disjoint smooth elliptic curves. By 3.4(i) \( \hat{S} \) is \( \mathbb{P}^1 \)-ruled over a smooth elliptic curve, so Lüroth theorem implies that every rational curve in \( \hat{S} \) is vertical. In particular, \((-1)\)-curves contracted in the process of minimalization are vertical, hence the number of horizontal components of \( D + \hat{E} \) and \( \hat{D} + \hat{E} \) is the same. For a general fiber \( f \) we get \(-2 + f \cdot (D + \hat{E}) = f \cdot K_{\hat{S}} + f \cdot \hat{D} + f \cdot \hat{E} = f \cdot Bk(\hat{D} + \hat{E}) = 0\), because all components contained in \( \text{Supp} \, Bk(\hat{D} + \hat{E}) \) are rational, hence vertical. Thus \( f \cdot (D + \hat{E}) = 2 \), so \( S_0 \) is \( \mathbb{C}^* \)-ruled. \( \Box \)

**Theorem 3.6.** If a singular \( \mathbb{Q} \)-homology plane is not logarithmic then its smooth locus has a unique \( \mathbb{C}^* \)-rueling. This \( \mathbb{C}^* \)-rueling does not extend to a ruling of the whole surface. Moreover, the Kodaira dimension of the surface is negative and the Kodaira dimension of the smooth locus is zero or one.

**Proof.** We will assume that \( \pi(S_0) \geq 0 \), it will be shown in the next section that \( S' \) is necessarily logarithmic in case \( \pi(S_0) = -\infty \). By 3.3 \( \hat{E} \) is connected and \( \pi(S_0) \leq 1 \). By 3.5 we can assume that \( S_0 \) is \( \mathbb{C}^* \)-ruled. We will first show that this ruling cannot be extended to a ruling of \( S' \). Consider a minimal completion \((\hat{S}, D + \hat{E}, \pi)\) of a \( \mathbb{C}^* \)-rueling of \( S_0 \). It is enough to show that \( \hat{E}_h \neq 0 \). Suppose \( \hat{E}_h = 0 \). Then \( D_h \) consists either of two 1-sections or of one 2-section. In particular, it can intersect only these fiber components which have multiplicity one or two and in the second case \# \( D_h \) = 1 and the point of intersection is a branching point of \( \pi|_{D_h} \). The exceptional divisor \( \hat{E} \) is vertical, so \( \hat{S} \) and \( D \) are rational by 3.4(i). Let \( F \) be a singular fiber containing \( \hat{E} \) and let \( D_h \) be the divisor of \( D \)-components of \( F \). By 3.2(iii) we have \( \nu \leq 1 \) and \( \Sigma_S = \# D_h + \nu - 2 \leq 1 \), so \( \sigma > 1 \) for at most one fiber of \( \pi \). We obtain successive restrictions on \( F \) eventually leading to a contradiction. We use 2.9 without comments.

**Claim 1.** The \((-1)\)-curves of \( F \) are \( S_0 \)-components.

Suppose \( F \) contains a \((-1)\)-curve \( D_0 \subset D \). By the \( \pi \)-minimality of \( D \) the divisor \( D_h \) intersects \( D_0 \), so either \( \mu(D_0) = 1 \) or \( \mu(D_0) = 2 \). Moreover, \( D_0 \) can be a tip of \( F \) only if \( D_h \) intersects it in two distinct points. In particular, we see that \( D_v \) contains components of multiplicity one and does not contain more \((-1)\)-curves. We have \( \Sigma_{S_0} = 0 \). Indeed, if \( \Sigma_{S_0} = 1 \) then \# \( D_h \) = 2 and \( \nu = 1 \), so by simply connectedness of \( D \) at most one horizontal component of \( D \) intersects \( D_0 \). However, in this case \( \mu(D_0) = 1 \), so \( D_0 \) is a tip of \( F \), a contradiction. The unique \( S_0 \)-component \( C \) of \( F \) is exceptional, otherwise \( D_0 \) would be the unique \((-1)\)-curve of \( F \), which would imply that \( F = [2, 1, 2] \) with no place for \( \hat{E} \). Clearly, there are no more \((-1)\)-curves in \( F \). Let us make a connected sequence of blowdowns starting from \( D_0 \) until the number of \((-1)\)-curves decreases. Since \( \hat{E} \cap D = \emptyset \), in this process we do not touch \( C + \hat{E} \) (first we would touch \( C \), and then \( C \) becomes a 0-curve). Let \( F' \) be the image of \( F \), we can write \( F' - C = D' + \hat{E} \), where \( D' \) is the image of \( D_v \). Since \( C + \hat{E} \) is not touched, \( D' \neq 0 \). We know that \( D_v \) contains a component of multiplicity one, so the same is true for \( D' \). By 2.9(vii) \( \hat{E} \) is a chain, a contradiction.
Claim 2. $F$ contains two $(-1)$-curves.

Suppose $F$ has a unique $(-1)$-curve $C$. Write $F - C = A + B$, where $A$ and $B$ are disjoint, connected, and $B$ is a chain (possibly empty). By our assumption on $E$ we have $E \subseteq A$, hence $B$ can contain only $S_0$- and $D$-components. Note that by $[3,4]$ each $S_0$-component intersects $D$. By connectedness of $D$ this implies that either $B \cdot D_h > 0$ or $B = 0$. If $B \neq 0$ we get that $B$ contains a curve with $\mu \leq 2$, so then $F$ consist of two branches with the first being equal to $[2, k, 2]$ for some $k > 1$, hence $\hat{E}$ is an admissible fork of type $(2,2,n)$, a contradiction. Thus $B = 0$. If $\mu(C) \leq 2$ then again $\hat{E}$ would be an admissible fork, so we get $\mu(C) > 2$. If follows that $D \cdot C = D_1 \cdot C$ for some $D$-component $D_1$. Since $D$ is connected, there is a chain $T \subseteq D_v$ containing $D_1$ and some $D$-component $D_2$ with $\mu(D_2) \leq 2$. Since $\hat{E}$ is not a chain, $D_2$ cannot belong to the first branch of $F$ because then $T$ would contain all branching components of $F$. Moreover, it follows that $D_2$ belongs to the second branch and $\hat{E}$ is an admissible fork of type $(2,2,n)$, a contradiction.

Claim 3. Both $(-1)$-curves of $F$ intersect $\hat{E}$.

Let $C_1$ and $C_2$ be the $(-1)$-curves of $F$. They are unique $S_0$-components of $F$ because $\sigma(F) = 2$. Now $D_h$ consists of two 1-sections, which can intersect $F$ only in components of multiplicity one. Suppose one of $C_i$’s, say $C_2$, does not intersect $\hat{E}$. Then $D_v \neq 0$, because $C_2$ has to intersect some component of $F$. As in (1) we make a connected sequence of blowdowns starting from $C_2$ until there is only one $(-1)$-curve left, we denote the image of $F$ by $F'$. Again in this process we do not touch $C_1 + \hat{E}$, so we can write $F' - C_1 = D' + \hat{E}$, where $D'$ is the image of $D_v$. Since $D'$ intersects the image of $D_h$, it contains a component of multiplicity one. It follows that $\hat{E}$ is a chain, a contradiction.

Claim 4. There are no $D$-components in $F$.

We can write $F - C_1 - C_2 = \hat{E} + D' + D''$, where $D_v = D' + D''$, $D'$ and $D''$ are connected and disjoint. Suppose $D' \neq 0$. One of $C_i$’s, say $C_1$, intersects $D'$. Contract $C_2$ and subsequent $(-1)$-curves until the number of $(-1)$-curves decreases. Clearly, $C_1 + D'$ is not touched in this process. Denote the image of $F$ by $F'$ and let $U$ be the image of $D'' + C_2 + \hat{E}$. Now $F'$ is a fiber with a unique $(-1)$-curve and since both $C_2 + D''$ and $C_1 + D'$ intersect $D_h$, we infer that both $U$ and $C_1 + D'$ contain components of multiplicity one. Thus $F'$ is a chain. Consider the reverse sequence of blowups recovering $F$ from $F'$. The fiber $F$ is not a chain, so a branching curve is produced. It follows that $D'' + C_2$ contains no curves of multiplicity one, so $D_h \cdot (D'' + C_2) = 0$, a contradiction.

The last claim implies that $D_h$ intersects both $C_i$’s, so they have multiplicity one, hence are tips of $F$. It follows that $F$ is a chain. Thus $\hat{E}$ is a chain, a contradiction. This finishes the proof that no $\mathbb{C}^*$-ruling of $S_0$ can be extended to a ruling of $S'$. We see also that $S$ is affine-ruled, hence $\pi(S') = -\infty$.

We only need to show that there is at most one non-extendable $\mathbb{C}^*$-ruling of $S_0$. Suppose $S_0$ has another $\mathbb{C}^*$-ruling. There is a modification $\sigma : (\overline{S}, \overline{D} + \overline{E}) \to (\overline{S}, \overline{D} + \overline{E})$, where $\sigma_* \overline{D} = D$ and $\sigma_* \overline{E} = \overline{E}$, such that this ruling extends to a $\mathbb{P}^1$-ruling of $\overline{S}$. Let $F'$ be a smooth fiber of this extension and suppose $F \cdot F' \neq 0$ for a smooth fiber $F$ of $\pi$. Writing the numerical class of $F$ in terms of the components of $\overline{D} + \overline{E}$ we see that $\overline{D}_h$ appears with nonzero coefficient (otherwise we get $F^2 < 0$), so the components of $F + \overline{D} - \overline{D}_h + \overline{E}$ generate $NS_0(\overline{S})$. Writing $F' \equiv V + \alpha \overline{E}_h + \beta F$, where $\text{Supp} V \subseteq \text{Supp}(\overline{D} - \overline{D}_h + \overline{E} - \overline{E}_h)$, we see that $V^2 \leq 0$ and $\alpha = F \cdot F' > 0$. Taking a square we get $0 = (V + \alpha \overline{E}_h)^2 + 2\alpha \beta$, so $\beta \geq 0$. Since by the second $\mathbb{C}^*$-ruling of $S_0$ is non-extendable, we have $F' \cdot \overline{D} = F' \cdot \overline{D}_h = 1$ and $F' \cdot \overline{E} = F' \cdot \overline{E}_h = 1$, so multiplying the equality by $F'$ we get $0 = \alpha + \alpha \beta \geq \alpha$, a contradiction. \hfill $\Box$

Non-logarithmic singular $\mathbb{Q}$-homology planes are classified in [5,4].
4. Smooth locus of negative Kodaira dimension

In this section we assume that \( \overline{\pi}(S_0) = -\infty \), which implies \( \overline{\pi}(S') = -\infty \). This case was analyzed in [MS91, 2.5-2.8], where (under the assumption of logarithmicity which is in fact redundant) a structure theorem was given. We recover this result. In particular, this completes the proof of 3.6.

To obtain more information we analyze possible completions instead of \( S' \) alone.

The boundary of \( S \) is connected and \( \overline{\pi}(S) = -\infty \), so by the structure theorem (see [Rus81] or [Miy01, 2.2.1]) \( S \) is affine-ruled. Let \( (\overline{S}, D, p) \) be a minimal completion of the affine ruling of \( S \).

4.1. Affine-ruled \( S' \).

Lemma 4.1. If \( S_0 \) is affine-ruled then \( S' \) is rational and there exists exactly one fiber of \( p \) contained in \( D \) (see Fig. 1). Each other singular fiber has a unique \((-1)\)-curve, which is an \( S_0 \)-component. The singularities of \( S' \) are cyclic.

Proof. The section of \( p \) contained in \( D + \overline{E} \) is in fact contained in \( D \), otherwise \( D \) would be contained in some fiber and \( Q(D) \) would be negative definite. Then \( \overline{E} \) is vertical, so it is a rational forest, which implies that \( D \) is a rational tree and \( \overline{S} \) and the base of \( p \) are rational by [3.4(i)]. We have \( \Sigma S_0 = \nu - 1 \) and \( \nu \leq 1 \) by [3.2(iii)], so \( \Sigma S_0 = 0 \) and there is exactly one fiber \( F_{\infty} \) contained in \( D \), which is smooth by the \( p \)-minimality of \( D \). Each singular fiber of \( p \) contains exactly one \((-1)\)-curve. Indeed, if \( D_0 \subseteq D \) is a vertical \((-1)\)-curve then by the \( p \)-minimality of \( D \) it intersects \( D_h \) and two \( D \)-components. But then \( \mu(D_0) > 1 \), a contradiction with the equality \( D_h \cdot f = 1 \) which holds for any fiber \( f \). Fixing a singular fiber \( F \) we have exactly one \((-1)\)-curve \( C \subseteq F \), which is the unique \( S_0 \)-component of \( F \), hence has \( \mu(C) > 1 \). There are exactly two components of multiplicity one in \( F \), they are tips of \( F \) and \( D_h \) intersects one of them. By [2.9(vii)] the connected component of \( F - C \) not contained in \( D \) is a chain. Thus \( S' \) has only cyclic singularities. \( \square \)

![Figure 1. Affine-ruled S'](image)

Construction 4.2. Let \( \mathbb{F}_1 = \mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-1)) \) be the first Hirzebruch surface with (unique) projection \( \overline{p}: \mathbb{F}_1 \to \mathbb{P}^1 \). Denote the section coming from the inclusion of the first summand by \( D'_h \), then \( D'_h^2 = -1 \). Choose \( n + 1 \) distinct points \( x_\infty, x_1, \ldots, x_n \in D'_h \) and let \( F_\infty \) be the fiber containing \( x_\infty \). For each \( i = 1, \ldots, n \) starting from a blowup of \( x_i \) create a fiber \( F_i \) over \( \overline{p}(x_i) \) containing a unique \((-1)\)-curve \( C_i \). Let \( D_i \) be the connected component of \( F_i - C_i \) intersecting \( D_h \), the proper transform of \( D'_h \). By renumbering we can assume there is \( m \leq n \), such that \( C_i \) is a tip of \( F_i \) if and only if \( i > m \). Assume also that \( m \geq 1 \) (for \( m = 0 \) we would get a smooth surface). For \( i \leq m \) put \( \overline{E}_i = F_i - D_i - C_i \). Clearly, each \( \overline{E}_i \) is a chain. Let \( \overline{S} \) be the resulting surface and let \( \overline{p}: \overline{S} \to \mathbb{P}^1 \) be the induced \( \mathbb{P}^1 \)-ruling. Put \( D = F_\infty + D_h + \sum_{i=1}^{n} D_i \), \( S = \overline{S} - D \) and \( \overline{E} = \sum_{i=1}^{m} \overline{E}_i \). Let \( S \to S' \) be the morphism contracting \( \overline{E}_i \)'s.

Let us state a general lemma, which will be used later too.
Lemma 4.3. Let $(\overline{S}, T)$ be a smooth pair and let $p: \overline{S} \rightarrow \mathbb{P}^1$ be a $\mathbb{P}^1$-ruling. Assume the following conditions are satisfied:

(i) there exists a unique connected component $D$ of $T$ which is not vertical,
(ii) $D$ is a rational tree,
(iii) $\Sigma_{\overline{S}-T} = h + \nu - 2$ (cf. [2.10]),
(iv) $d(D) \neq 0$.

Then the surface $S'$ defined as the image of $\overline{S} - D$ after contraction of connected components of $T - D$ to points is a rational normal $\mathbb{Q}$-acyclic surface and $p$ induces a rational ruling of $S'$. Conversely, if $p': S' \rightarrow B$ is a rational ruling of a rational $\mathbb{Q}$-homology plane $S'$ (not necessarily singular) then any completion $(\overline{S}, T, p)$ of the restriction of $p'$ to the smooth locus of $S'$ has the above properties.

Proof. Since the base of $p$ is rational, $\overline{S}$ is rational. Put $\widehat{E} = T - D$. Since $\widehat{E}$ is vertical and since $\widehat{E} \cap D = \emptyset$, $Q(\widehat{E})$ is negative definite and $b_1(\widehat{E}) = 0$. Fujita’s equation $\Sigma_{\overline{S}-T} = h + \nu - 2 + b_2(\overline{S}) - b_2(D + \widehat{E})$ gives $b_2(\overline{S}) = b_2(T)$, so by (iv) the inclusion $T \rightarrow \overline{S}$ induces an isomorphism on $H_2$. By 2.6 $S'$ is normal and affine, in particular $b_4(S') = b_3(S') = 0$. Since $b_1(D) = 0$, the exact sequence of the pair $(\overline{S}, D)$ together with the Lefschetz duality give $b_2(S) = b_2(\overline{S}, D) = b_2(\overline{S}) - b_2(D) = b_2(\widehat{E})$. Since $b_1(\widehat{E}) = 0$, we get from the exact sequence of the pair $(S, \widehat{E})$ that $b_2(S') = b_2(S, \widehat{E}) = b_2(S) - b_2(\widehat{E}) = 0$. As $\chi(S') = \chi(\overline{S}) - \chi(D \cup \widehat{E}) + b_0(\widehat{E}) = b_0(D) = 1$, we get $b_1(S') = b_2(S') = 0$, so $S'$ is $\mathbb{Q}$-acyclic. Conversely, taking a lifting of the ruling to a resolution we get $b_1(\widehat{E}) = 0$, as $\widehat{E}$ is vertical. Then the base is rational by [3.4] and the necessity of the above conditions follows from 3.2.

Remark 4.4. Let $p: \overline{S} \rightarrow \mathbb{P}^1$ be as in 4.2 and for a fiber $F$ denote the greatest common divisor of multiplicities of all $S$-components of $F$ by $\mu_S(F)$. By [3.1](vi) $H_1(S', \mathbb{Z}) = H_1(S, \mathbb{Z})$ and by [Fu82, 4.19, 5.9] $H_1(S, \mathbb{Z}) = \bigoplus_{i=1}^n \mathbb{Z}[\mu_S(F_i)]$. It is easy to see that $\mu_S(F_i) = \mu(C_i)/d(E_i)$, where $d(E_i) = d(0) = 1$ if $i > m$. In particular, $S'$ is a $\mathbb{Z}$-homology plane if and only if $m = n$ and each $F_i$ is a chain. In fact then $\pi_1(S)$ vanishes, so $S'$ is contractible.

Theorem 4.5. The surface $S'$ constructed in 4.2 is an affine-ruled singular $\mathbb{Q}$-homology plane. Conversely, each singular $\mathbb{Q}$-homology plane admitting an affine ruling can be obtained by construction 4.2. Its strongly balanced boundary is unique if it is branched and is unique up to reversion if it is a chain. The affine ruling of $S'$ is unique if and only if its strongly balanced boundary is not a chain.

Proof. By definition $\widehat{E}_i$’s are admissible chains, so $S'$ is normal and has only cyclic singularities. We have $d(D) = -\prod_i d(D_i)$ (cf. [KR99, 2.1.1]), so $d(D) \neq 0$, hence $S'$ is a singular $\mathbb{Q}$-homology plane by 4.3. The last part of the statement almost follows from 4.1. It remains to note that by a flow (cf. 2.13) we can change freely the self-intersection of the horizontal boundary component without changing the rest of $D$, so we can assume that the construction starts with a negative section on $\mathbb{F}_1$, which removes unnecessary ambiguity. (We could for example start with $D'_h$ equal to the negative section on $\mathbb{F}_n$, so that the resulting boundary would be strongly balanced, cf. 2.19.) The uniqueness of a strongly balanced boundary follows from 2.17.

We now consider the uniqueness of an affine ruling. Let $(V_i, D_i, p_i)$ be two minimal completions of two affine rulings of $S'$ (cf. 2.7). In particular, both $D_i$ contain a 0-curve $F_{\infty,i}$ as a tip. We can assume both $D_i$ are standard (cf. 2.18). Suppose $D_i$ is not a chain. Then $D_2$ is not a chain and $D_1, D_2$ are isomorphic as weighted curves (cf. 2.17). Let $T_i$ be the unique maximal twig of $D_i$ containing a 0-curve, write $T_i = [0, a_1, \ldots, a_n]$ with $[a_1, \ldots, a_n]$ admissible. By 2.17 there is a flow $\Phi: (V_i, D_1) \sim (V_2, D_2)$. Since $D_i$ is branched, $\Phi^*$ is an isomorphism on $V_i - T_i$. Moreover, one can easily show by induction on $n$ that $\Phi^*$ extends to an isomorphism of $V_i - F_{\infty,1}$ and $V_2 - F_{\infty,2}$. For $i = 1, 2$ let $f_i$ be some fiber of $p_i$ different than $F_{\infty,i}$. Since $\Phi^*(f_i)$ is disjoint from $F_{\infty,2}$, we get $\Phi^*(f_1) \cdot f_2 = 0$, so $p_1$ and $p_2$ agree on $S'$. Suppose now that $(V_1, D_1)$ is a standard completion of $S'$ with $D_1 = [0, 0, a_1, \ldots, a_n]$. We have $a_1 \neq 0$, otherwise looking at the ruling induced by the 0-tip we get that $n = 1$ and then $d(D_1) = 0$, a contradiction. Thus we
can assume that \([a_1, \ldots, a_n]\) is admissible and nonempty. Let \((V_2, D_2)\) be another completion of \(S'\) with \(D_2\) being a reversion of \(D\). The 0-tip \(T_i\) of each \(D_i\) induces an affine ruling on \(S'\). Let \((V, D)\) be a minimal normal pair dominating both \((V_i, D_i)\), such that both affine rulings extend to \(\mathbb{P}^1\)-ruled families of non-isomorphic singular \(\mathbb{Q}\)-homology plane which are images of contracted \(\sigma^*_1 T_1 \cdot \sigma^*_2 T_2 \neq 0\), where \(\sigma_1: (V, D) \rightarrow (V_i, D_i)\) are the dominations. Suppose \(\sigma^*_1 T_1 \cdot \sigma^*_2 T_2 = 0\). Let \(H\) be an ample divisor on \(V\) and let \((\lambda_1, \lambda_2) \neq (0, 0)\) be such that \(\overline{T} \cdot H = 0\) for \(\overline{T} = \lambda_1 \sigma^*_1 T_1 + \lambda_2 \sigma^*_2 T_2\). We have 
\[(\sigma^*_1 T_1)^2 = T_1^2 = 0, \text{ so } \overline{T}^2 = 2\lambda_1 \lambda_2 \sigma^*_1 T_1 \cdot \sigma^*_2 T_2 = 0, \text{ hence } \overline{T} \equiv 0 \text{ by the Hodge index theorem.}\]
However, \(D\) has a non-degenerate intersection matrix, because \(d(D) = d(D_1) \neq 0\), so \(\overline{T}\) is a zero divisor. Then \(\sigma^*_1 T_1 = [0]\), otherwise \(\sigma^*_1 T_1\) and \(\sigma^*_2 T_2\) would contain a common \((-1)\)-curve, which contradicts the minimality of \((V, D)\). It follows that \(\sigma_1\) (and \(\sigma_2\)) are identities. This contradicts the fact that the reversion for nonempty \([a_1, \ldots, a_n]\) is a nontrivial transformation of the completion (even if \([a_1, \ldots, a_n]^t = [a_1, \ldots, a_n]\)).

**Example 4.6.** Let \((V, D, \iota)\) be an snc-minimal completion \((\iota\) is an embedding) of an affine-ruled singular \(\mathbb{Q}\)-homology plane \(S'\) as above, for which \(D\) is branched. The only change of \(D\) which can be made by a flow is a change of the weight of \(D_h\). Let us assume that \(D_h^2 = -1\). If we now make an elementary transformation \((V, D) \rightarrow (V_x, D_x)\) with a center \(x \in F_\infty \setminus D_h\) then \(D\) becomes strongly balanced (cf. 2.19). Denote the resulting completion by \((V_x, D_x, \iota_x)\) and let \(F_{\infty,x}\) be the new fiber at infinity. The isomorphism type of the weighted boundary \(D_x\) does not depend on \(x\), but the completions are different for different \(x\). Moreover, in general even the isomorphism type of the pair \((V_x, D_x)\) depends on \(x\). To see this suppose \((V_x, D_x) \cong (V_y, D_y)\). As the isomorphism maps \(F_{\infty,x}\) to \(F_{\infty,y}\), we get an automorphism of \((V, D)\) mapping \(x\) to \(y\). Taking a minimal resolution \(\overline{S} \rightarrow V\), contracting all singular fibers to smooth fibers without touching \(D_h\) and then contracting \(D_h\) we see that for \(x \neq y\) this automorphism descends to a nontrivial automorphism of \(\mathbb{P}^2\) fixing points which are images of contracted \(S_0\)-components and of \(D_h\). In general such an automorphism does not exist.

**Example 4.7.** Repeating the construction \[4.2\] in a special case we will now obtain arbitrarily high-dimensional families of non-isomorphic singular \(\mathbb{Q}\)-homology planes with negative Kodaira dimension of the smooth locus and the same homeomorphism type. This type of examples is quite intrusive, for smooth \(\mathbb{Q}\)-homology planes it was considered in [FZ94, 4.16]. Put \(m = 2\) and \(n = N+2\) for some \(N > 0\) and let \(\overline{S}, D, \overline{E}\) etc. be created as in the construction above, so that \(D_1 = [3]\), \(D_2 = [2]\) and \(D_i = [2, 2, 2]\) for \(3 \leq i \leq n\). Then \(\overline{E}_1 = [2, 2]\) and \(\overline{E}_2 = [2]\) (see Fig. 2).

\[
\begin{array}{cccccccc}
F_x & F_1 & F_2 & F_1 & F_2 & F_1 & D_h \\
\end{array}
\]

**Figure 2.** Singular fibers in example \[4.7\]

Denoting the contraction of \(\sum_{i=3}^n C_i\) by \(\sigma: \overline{S} \rightarrow V\) we can factor the contraction \(\overline{S} \rightarrow F_1\) (which reverses the construction) as the composition \(\overline{S} \xrightarrow{\overline{y}} V \xrightarrow{\sigma'} F_1\). Put \(y_i = \sigma(C_i)\) and \(y = (y_3, \ldots, y_n)\). While \(\sigma'^{-1}\) is determined uniquely by the choice of \((x_1, \ldots, x_n)\), \(\sigma^{-1}\) and the resulting surface \(\overline{S}\) (and hence \(S'\)) can depend on the choice of \(y\). Let us write \(\overline{S}_y\) and \(S'_y\) to indicate this dependence. For \(3 \leq i \leq n\) let \(D_i^0\) be the open subset of the middle component of \(D_i\) remaining
after subtracting two points belonging to other components of \( D_i \). Put \( U = D_1^0 \times \ldots \times D_n^0 \cong \mathbb{C}^{N-1} \).

The family \( \{S'_y\}_{y \in D_3^0 \times U} \rightarrow D_3^0 \times U \) is \( N \)-dimensional. Since there exists a compactly supported auto-diffeomorphism of the pair \( (\mathbb{C}^2, \mathbb{C}^* \times \{0\}) \) mapping \( (p, 0) \) to \( (q, 0) \) for any \( p, q \neq 0 \), the choice of \( y \in D_3^0 \times U \) is unique up to a diffeomorphism fixing irreducible components of \( \sigma_\ast(D + \hat{E} + C_1 + C_2) \). Thus all \( S'_y \) are homeomorphic.

Let \( \pi : \mathcal{X} \rightarrow U \) be the subfamily over \( \{y^0_3\} \times U \). We will show that the fibers of \( \pi \) are non-isomorphic. Suppose that \( S'_{y'} \cong S'_{y''} \) for \( y', y'' \in \{y^0_3\} \times U \). The isomorphism extends to snc-minimal resolutions. By [2.11] there is a flow \( \Phi : \mathcal{S}_y \rightarrow \mathcal{S}_z \), which is an isomorphism outside \( F_{\infty} \). Clearly, \( \Phi^\ast \) fixes \( D_h \setminus \{x_\infty\} \), \( F_1 \) and \( F_2 \), hence restricts to an identity on \( D_h \setminus \{x_\infty\} \) and respects fibers. Since \( C_i \) are unique \((-1)\)-curves of the fibers, they are fixed by \( \Phi^\ast \). It follows that \( \Phi^\ast |_{\mathcal{S} - F_{\infty} - D_h} \) descends to an automorphism \( \Phi_V \) of \( V - F_{\infty} - D_h \) fixing the fibers, such that \( \Phi_V(y) = z_i \). Moreover, \( \Phi_V \) descends to an automorphism \( \Phi_{F_1} \) of \( F_1 - F_{\infty} - D_h \) fixing fibers. If \( (x, y) \) are coordinates on \( F_1 - F_{\infty} - D_h \cong \mathbb{C}^2 \), such that \( x \) is a fiber coordinate then \( \Phi_{F_1}(x, y) = (x, \lambda y + P(x)) \) for some \( P \in \mathbb{C}[x] \) and \( \lambda \in \mathbb{C} \). Introducing successive affine maps for the blowups one can check that in some coordinates \( \Phi_V \) acts on \( D_1^0 \) as \( t \rightarrow \lambda \mu(C_i)t \). Now the requirement \( y_3 = y_3^0 \) fixes \( \lambda^2 = 1 \), so since \( \mu(C_i) = 2 \) for each \( 3 \leq i \leq n \), we get that \( y = z \).

**Remark.** Note that by [4.19, 5.9] for \( S' \) as above \( \pi_1(S') \) is the \( N \)-fold free product of \( \mathbb{Z}_2 \). It follows from [4.2] that given a weighted boundary there exist only finitely many affine-ruled singular \( \mathbb{Z} \)-homology planes with this boundary. That is why in the above example we have used branched fibers \( F_i \) for \( 3 \leq i \leq n \), so that the resulting surfaces are \( \mathbb{Q} \)-, but not \( \mathbb{Z} \)-homology planes.

### 4.2. Non-affine-ruled \( S' \)

**Proposition 4.8.** If a singular \( \mathbb{Q} \)-homology plane has smooth locus of negative Kodaira dimension then it is affine-ruled or isomorphic to \( \mathcal{C}^2/G \) for some small, noncyclic subgroup \( G \leq \text{GL}(2, \mathbb{C}) \). The surfaces \( \mathcal{C}^2/G \) and \( \mathcal{C}^2/G' \) are isomorphic if and only if only if \( G \) and \( G' \) are conjugate in \( \text{GL}(2, \mathbb{C}) \).

**Proof.** We follow the arguments of [KR07, §3]. Assume that \( S' \) is not affine-ruled. Then \( S_0 \) is not affine-ruled. Since \( S' \) is affine, the boundary divisor \( D + \hat{E} \) of \( S_0 \) is not negative definite, so by [Miy01, 2.5.1] \( S_0 \) contains a Platonically \( \mathcal{C}^* \)-fibred open subset \( U \), which is its almost minimal model. Moreover, \( \chi(U) \leq \chi(S_0) \) (cf. [Pal03, 2.8]). The algorithm of construction of the almost minimal model (see [Miy01, 2.3.8, 2.3.11]) implies that \( S_0 - U \) is a disjoint sum of \( s \) curves isomorphic to \( \mathcal{C} \) and \( s' \) curves isomorphic to \( \mathcal{C}^* \) for some \( s, s' \in \mathbb{N} \). It follows that \( 0 = \chi(U) = \chi(S_0) - s = \chi(S') - q - s = 1 - q - s \), so \( s = 0, q = 1 \) and \( s' \leq 1 \). If \( s' \neq 0 \) then the boundary divisor of \( U \) is connected, hence \( U \) and \( S_0 \) are affine-ruled. Thus \( s' = 0 \), \( S_0 = U \) and by [MT84] \( S' \cong \mathcal{C}^2/G \), where \( G \) is a small noncyclic subgroup of \( \text{GL}(2, \mathbb{C}) \). If \( G \) and \( G' \) are two subgroups of \( \text{GL}(2, \mathbb{C}) \), such that \( \mathcal{C}^2/G \cong \mathcal{C}^2/G' \) then the \( \hat{0}_{\mathcal{C}^2/G, (0)} \cong \hat{0}_{\mathcal{C}^2/G', (0)} \), so if \( G \) and \( G' \) are small then they are conjugate by [Pri67, Theorem 2].

### 5. Non-logarithmic \( S' \)

Let us consider a singular \( \mathbb{Q} \)-homology plane \( S' \) with a \( \mathcal{C}^* \)-ruled smooth locus \( S_0 \). Note that \( \pi(S_0) \neq 2 \) by the easy addition theorem [R182, 10.4]. We will assume in this section that this ruling is non-extendable, meaning that it does not extend to a ruling of \( S' \). By [3.4] this is the case if \( S' \) is not logarithmic. Let \( (\mathfrak{S}, D + \hat{E}, p) \) be a minimal completion of such a \( \mathcal{C}^* \)-ruled of \( S_0 \), where \( \hat{E} \) is an exceptional locus of some resolution of singularities of \( S' \). We have \( D_h \neq 0 \), otherwise \( D \) would be vertical, which contradicts the affiness of \( S' \). Since \( p|_{S_0} \) does not extend to a ruling of \( S' \), we have \( \hat{E}_h \neq 0 \), so \( p \) is untwisted and \( S \) is affine-ruled, which gives \( \pi(S') = -\infty \). Let \( N = -\hat{E}_h^2 \) and let \( F_1, F_2, \ldots, F_n \) be all the columnar fibers of \( p \). Let \( E_i \subset F_i \) be connected components of \( \hat{E} - \hat{E}_i \). Let \( C_i \) be the unique \((-1)\)-curve of \( F_i \), put \( \mu_i = \mu(C_i) \). Note that \( \mu_i \) is the denominator of the reduced form of the fraction \( \hat{e}(E_i) \) (cf. remark after [2.11]). Denote the base of \( p \) by \( B \). By [3.1] the rationality of one of \( \mathfrak{S}, \hat{E}, D \) or \( B \) implies the rationality of all others.
Lemma 5.1. Singular fibers of \( p \) are columnar and \( \sum_{i=1}^{n} e_i(E_i) < N \) (see Fig. 3). There exists a linear bundle \( \mathcal{L} \) over \( B \) with \( \deg \mathcal{L} = -N < 0 \) and a proper birational morphism \( S \to \mathbb{P}(\mathcal{O}_B \oplus \mathcal{L}) \), such that \( p \) is induced by the projection of \( \mathbb{P}(\mathcal{O}_B \oplus \mathcal{L}) \) onto \( B \).

Proof. Since \( \hat{E} \cap D = \emptyset, \nu = 0 \) and there are no rivets in \( D + \hat{E} \). By \( \textbf{3.2(iii)} \) \( \Sigma S_0 = 0 \), so every fiber has exactly one \( S_0 \)-component. By \( \textbf{2.12(ii)} \) every singular fiber is columnar. We contract all singular fibers to smooth fibers (i.e. we contract subsequently their \((-1)\)-curves) without touching \( \hat{E}_h \). Denote the image of \( \mathcal{S} \) by \( \hat{S} \) and the image of \( D_h \) by \( \hat{D}_h \). Then \( \hat{E}_h \) is disjoint from \( \hat{D}_h \). Since \( \hat{E}_h^2 = -N < 0 \), we can write \( \hat{S} = \mathbb{P}(\mathcal{O}_B \oplus \mathcal{L}) \) for a line bundle \( \mathcal{L} \) with \( \deg \mathcal{L} = -N < 0 \) (see [Har77, V.2]). Now \( \hat{E}_h \) and \( \hat{D}_h \) are sections coming from the linear summands of the bundle. The matrix \( Q(\mathcal{E}) \) is negative definite, so \( 0 < \det Q(\hat{E}) = d(E_1)d(E_2)\ldots d(E_n)(\hat{E}_h^2 - \sum_{i=1}^{n} e_i(E_i)) \) (cf. [KR07, 2.1.1]), hence \( \sum_{i=1}^{n} e_i(E_i) < N \).

Corollary 5.2. \( S' \) is contractible.

Proof. By \( \textbf{5.1} \) singular fibers of \( p \) are columnar, so in each fiber there is a component of \( \hat{E} \) of multiplicity one, hence by [Fuj82, 5.9, 4.19] the embedding \( \hat{E}_h \to S \) induces an isomorphism \( \pi_1(\hat{E}_h) \to \pi_1(S) \). Thus by \( \textbf{3.1}(v)-(vi) \) and Whitehead’s theorem \( S' \) is contractible. \( \square \)

Construction 5.3. Pick \( n \in \mathbb{N} \) and for each \( i = 1, \ldots, n \) choose a number \( e_i \in \mathbb{Q} \cap (0,1) \). Choose a positive integer \( N \), such that \( \sum_{i=1}^{n} e_i < N \). Let \( B \) be a complete curve of genus \( g(B) \), such that \( g(B) > 0 \) if \( n \) was chosen smaller than 3. Define \( \hat{S} = \mathbb{P}(\mathcal{O}_B \oplus \mathcal{L}) \), where \( \mathcal{L} \) is a line bundle over \( B \) of degree \( \deg \mathcal{L} = -N \). Let \( \hat{p} : \hat{S} \to B \) be the induced \( \mathbb{P}^1 \)-fibration. Denote the sections induced by inclusions of the linear summands \( \mathcal{O}_B \) and \( \mathcal{L} \) by \( \hat{E}_h \) and \( \hat{D}_h \). Then \( \hat{E}_h^2 = -N \) and \( \hat{D}_h^2 = N \). Choose \( n \) distinct points \( x_1, \ldots, x_n \in \hat{D}_h \) and blow up each point once. For each \( i \) make a connected sequence of subdivisional blowups creating over \( \hat{p}(x_i) \) a columnar fiber \( F_i \) with \( e_i(E_i) = \hat{e}_i \). Denote the birational transform of \( \hat{D}_h \) by \( \tilde{D}_h \). Write \( F_i = E_i + C_i + D_i \) where \( C_i = -1, E_i \) and \( D_i \) are connected chains and \( D_i \cap \hat{E}_h = \emptyset \). Let \( \mu_i \) be the multiplicity of \( C_i \) in \( F_i \). Fix a natural order on each \( E_i \) and \( D_i \) treated as twigs of \( \hat{E} = E_1 + \ldots + E_n + \hat{E}_h \) and \( D = D_1 + \ldots + D_n + \hat{D}_h \) respectively. Denote the obtained surface by \( \mathcal{S} \) and the induced \( \mathbb{P}^1 \)-ruling by \( p \). Define \( S = \mathcal{S} - D, S_0 = S - \hat{E} \) and \( S' = S/\hat{E} \) (as a topological space). We will show below that \( NS_Q(S_0) = 0 \), hence by \( \textbf{2.6} \) \( S' \) and the quotient morphism can be realized in the algebraic category.

Remark. The additional assumption that \( g(B) > 0 \) if \( n < 3 \) is justified as follows. If \( g(B) = 0 \) and \( n < 3 \), then \( \hat{E} \) is a chain, so it contracts either to a smooth point or to a cyclic singularity. Moreover, \( \pi_1(S_0) = -\infty \) in this case (see the proof of \( \textbf{5.4} \)). But then \( S_0 \) is affine-ruled (see \( \textbf{1.8} \)), and respective \( S' \)'s were described in \( \textbf{1.5} \).
Theorem 5.4. The surface $S'$ constructed in [5.3] is a contractible surface of negative Kodaira dimension. Moreover, each non-logarithmic $\mathbb{Q}$-homology plane (or equivalently, each non-affine-ruled whose smooth locus admits a non-extendable $\mathbb{C}^*$-ruling) can be obtained by construction [5.3]. The Kodaira dimension of the smooth locus is determined by the sign of the number $\alpha = n - 2 + 2g(B) - \sum_{i=1}^{n} \frac{1}{\mu_i}$ (i.e. $\pi(S_0) = -\infty$ for $\alpha < 0$, $0$ for $\alpha = 0$ and $1$ for $\alpha > 0$). The snc-minimal completion and the pair $(B, L)$ used in the construction are determined uniquely by the isomorphism type of $S'$.

Proof. The assumption that $S'$ is not affine-ruled excludes the case when $g(B) = 0$ and $n \leq 2$, as was done in the construction. It follows from [5.1] and [3.6] that if $S'$ is not affine-ruled but admits a non-extendable $\mathbb{C}^*$-ruling then it can be obtained by construction [5.3]. The matrix $Q(\hat{\mathcal{E}} - \hat{\mathcal{E}}_h)$ is negative definite and $d(\hat{\mathcal{E}}) = d(E_1)d(E_2) \cdots d(E_n)(N - \sum_{i=1}^{n} c_i) > 0$, so by Sylvester’s theorem $Q(\hat{\mathcal{E}})$ is negative definite. We have $d(D) = d(D_1)d(D_2) \cdots d(D_n)(-N + n - \sum_{i=1}^{n} (1 - c_i)) = -d(\hat{\mathcal{E}})$ by the remark after [2.11] so $d(D) \neq 0$. It follows that the classes of irreducible components of $D + \hat{\mathcal{E}}$ are independent in $NS_{\mathbb{Q}}(S)$, hence are a basis because $b_2(S) = \#D + \#\hat{\mathcal{E}}$. We apply [2.3] and infer that $S'$ is normal and affine. By Itaka’s easy addition theorem $\pi(S_0) \leq 1$. The divisor $K_S + D + \sum_{i=1}^{n} C_i$ intersects trivially with all vertical components, so it is numerically equivalent to a multiple of a general fiber $f$. Intersecting with $D_h$ we get $K_S + D + \hat{\mathcal{E}} + \sum_{i=1}^{n} C_i \equiv (2g(B) - 2 + n)f$. Putting $G_i = \frac{1}{f} F_i - C_i$ we get $K_S + D + \hat{\mathcal{E}} \equiv \alpha f + \sum_{i=1}^{n} G_i$. Since $\sum_{i=1}^{n} G_i$ is effective, vertical and has a negative definite intersection matrix, by [2.2] we get $\kappa(K_S + D + \hat{\mathcal{E}}) = \kappa(\alpha f)$, so $\pi(S_0)$ is determined by the sign of $\alpha$ as stated.

Now we check that $S'$ is $\mathbb{Q}$-acyclic (then it is contractible by [5.2]). We know from the above that the map $H_2(D + \hat{\mathcal{E}}) \to H_2(S)$ induced by inclusion is an isomorphism. Clearly, $H_1(D) \to H_1(S)$ and $H_1(\hat{\mathcal{E}}) \to H_1(S)$ are monomorphisms, because they are monomorphisms after composing with $H_1(p)$. The exact sequence of the pair $(D, S)$ gives $b_3(S) = b_3(S) = 0$, $b_2(S) = b_2(S) = b_1(B)$. Then the exact sequence of the pair $(\hat{\mathcal{E}}, S)$ gives $b_1(S') = b_2(S') = b_3(S') = b_4(S') = 0$. Since we assumed that $g(B) > 0$ if $n < 3$, $S'$ is singular.

All non-branching rational curves contained in $D$ have negative self-intersection, so the smooth completion of $S$ is unique up to isomorphism by [2.17] (it is snc-minimal, as $B$ is branching if $g(B) = 0$). Suppose $S'_1 \cong S'_2$ are two surfaces constructed as in [3.6] we will use indices 1, 2 consequently to distinguish between objects appearing in the intermediate steps of the construction. Since all rational non-branching components of $D + \hat{\mathcal{E}}$ have negative self-intersection, the isomorphism extends to an isomorphism of completions $\Phi : (\overline{S}_1, D + \hat{\mathcal{E}}) \to (\overline{S}_2, D + \hat{\mathcal{E}})$. Now the argument from the proof of [3.6] shows that there is at most one non-extendable $\mathbb{C}^*$-ruling of $S_0$, so up to composition with an automorphism of $\overline{S}$ induced by an automorphism of $B$ we can assume that $\Phi$ preserves fibers, so in particular it fixes all components of $\hat{\mathcal{E}} + D$. Then $\Phi$ induces an isomorphism of $B$-schemes $\tilde{S}_1$ and $\tilde{S}_2$. Thus $\mathcal{O}_B \otimes \mathcal{L}_1 \cong (\mathcal{O}_B \otimes \mathcal{L}_2) \otimes \mathcal{E}$ for some line bundle $\mathcal{E}$ of degree zero. It follows that $\text{deg} \mathcal{L}_2 \otimes \mathcal{E} < 0$, so non-vanishing constant sections of $\mathcal{O}_B$ (on the left hand side) are sections of $\mathcal{E}$, which gives $\mathcal{E} \cong \mathcal{O}_B$. Thus $\mathcal{O}_B \otimes \mathcal{L}_1 \cong \mathcal{O}_B \otimes \mathcal{L}_2$ which after taking second exterior power gives $\mathcal{L}_1 \cong \mathcal{L}_2$. \hfill $\Box$

Remark 5.5. Note that if $S'$ is a singular $\mathbb{Q}$-homology plane, such that $\pi(S_0) = -\infty$ and $S'$ is not affine-ruled then, as it was observed in the proof of [3.8] $S_0$ has a Platonic $\mathbb{C}^*$-fibration. It follows from its definition that it cannot be extended to a ruling of $S'$. Thus all $S'$’s of type $C^2/G$ can be also obtained by the construction above.

Corollary 5.6. Let $P \in S'$ be the unique singular point of a singular $\mathbb{Q}$-homology plane $S'$, whose smooth locus has a non-extendable $\mathbb{C}^*$-ruling. Then with the notation as above:

(i) $P$ is a topologically rational singularity if and only if $B \cong \mathbb{P}^1$,

(ii) if $\pi(S_0) = -\infty$ then $g(B) = 0$, $n \leq 3$ and $S'$ is logarithmic. If additionally $n > 2$ (as assumed in the construction) then $(\mu_1, \mu_2, \mu_3)$ is up to order one of the Platonic triples (i.e. triples
(3) if \( \pi(S_0) \geq 0 \) then \( S' \) is not logarithmic,
(iv) \( \pi(S_0) = 0 \) if and only if either
(a) \( g(B) = 1 \) and \( n = 0 \) or
(b) \( g(B) = 0, n = 4 \) and \( \mu_1 = \mu_2 = \mu_3 = \mu_4 = 2 \) or
(c) \( g(B) = 0, n = 3 \) and \((\mu_1, \mu_2, \mu_3)\) is up to order one of \((2, 3, 6), (2, 4, 4), (3, 3, 3)\).

Proof. (ii) If \( \alpha < 0 \) then \( \frac{n}{2} \leq \sum_{i=1}^{n}(1 - \frac{1}{\mu_i}) < 2(1 - g(B)) \), so \( g(B) = 0 \) and \( n \leq 3 \). Suppose \( n = 3 \). Then \( \sum_{i=1}^{3} \frac{1}{\mu_i} > 1 \), so \((\mu_1, \mu_2, \mu_3)\) is up to order one of the Platonic triples.

(iii) If \( S' \) is logarithmic then \( \tilde{E} \) is either a chain or an admissible fork. In the first case \( n \leq 2 \) and in the second \( n = 3 \) and \( \sum_{i=1}^{3} \mu_i = 1 \). In both cases \( \alpha < 0 \), so \( \pi(S_0) = -\infty \).

(iv) Assume \( \alpha = 0 \). For \( n = 0 \) we get \( g(B) = 1 \). Assume \( n > 0 \). We have \( \frac{n}{2} \leq \sum_{i=1}^{n}(1 - \frac{1}{\mu_i}) = 2(1 - g(B)) \), so we get \( g(B) = 0 \) and \( n \in \{3, 4\} \). We have then \( \sum_{i=1}^{3} \frac{1}{\mu_i} = 1 \) if \( n = 3 \) and \( \sum_{i=1}^{4} \frac{1}{\mu_i} = 2 \) if \( n = 4 \), which gives (b) and (c). Conversely, in each case \( \alpha = 0 \).

\[ \text{Example 5.7.} \] Suppose \( n \geq 3, N \geq 1, \bar{e}_1, \ldots, \bar{e}_n \in \mathbb{Q} \cap (0, 1) \), \( \bar{e}_1 + \ldots + \bar{e}_n < N \) and \( B \cong \mathbb{P}^1 \). Let \( P \) be the unique singular point of \( S' \) constructed as in \[5.3\].

(i) If \( N \geq n \) then \( P \in S' \) is a rational singularity.
(ii) If \( N < n - 1 \) then \( P \in S' \) is a topologically rational but not a rational singularity.

\[ \text{Proof.} \] We have \( \sum_{i=1}^{n} \bar{e}_i < n \leq N \). The fundamental cycle \( Z_f \) of \( \tilde{E} \), which is the smallest nonzero effective divisor \( Z_f \subseteq \tilde{E} \), such that \( Z_f \cdot E' \leq 0 \) for each irreducible \( E' \subseteq \tilde{E} \), equals \( \tilde{E} \) in this case. Then \( p_a(Z_f) = 0 \), so \( P \) is a rational singularity by [Art66, Theorem 3].

(ii) Let \( Z = \tilde{E} + \beta \tilde{E}_k \), where \( \beta = \left\lfloor \frac{n}{N} \right\rfloor - 1 \) \( \{x \} \) is defined as the smallest integer not smaller than \( x \). Then \( p_a(Z) = \beta(n - 1 - \frac{1}{\beta + 1} N) \), which is non-negative. Indeed, \( \beta \geq 1 \) and \( (\beta + 1)N < (\frac{n}{N} + 1)N = n + N \leq 2n - 1 \), so \( p_a(Z) \geq 0 \). But if \( p_a(Z) = 0 \) then the equality \( \left\lfloor \frac{n}{N} \right\rfloor \cdot N = 2n - 2 \) gives \( n < N + 2 \), so \( n = N + 1 \), a contradiction. It follows from [Art66, Proposition 1] that \( P \) is not a rational singularity.

\[ \text{Remark.} \] As for (ii) note that for example if \( n > N + 1 \) and \( E_i = [x_i] \) with \( x_i \geq \frac{n}{N} \), not all equal \( \frac{n}{N} \), then the condition \( \frac{n}{N} \) \( \{x \} \) is satisfied. In general the fundamental cycle can be computed using [au72, Proposition 4.1].

Note that \( \tilde{S} = \mathbb{P}(L_{1} \oplus L_{2}) \) admits a natural \( \mathbb{C}^* \)-action fixing precisely the sections coming from the inclusion of linear summands. (Each \( v \in \tilde{S} \) can be written as \( v = [u_1 + u_2] \), where \( u_i \in L_{i} \pi(v) \) and \( \pi \) is the projection onto the base, and then the action can be written as \( t \ast [u_1 + u_2] = [u_1 + tu_2] \). This action lifts to a \( \mathbb{C}^* \)-action on \( \overline{S} \) constructed in \[5.3\] because centers of successive blowups creating \( \overline{S} \) belong to fixed loci of successive liftings. Then by [Pin77, 1.1] we have the following corollary.

\[ \text{Corollary 5.8.} \] Each singular \( \mathbb{Q} \)-homology plane which is non-logarithmic (or more generally, whose smooth locus admits a non-extendable \( \mathbb{C}^* \)-ruled) is a quotient of an affine cone over a smooth projective curve by an action of a finite group acting freely off the vertex of the cone and respecting the set of lines through the vertex.

It follows now by 5.8 loc. cit. that the unique singular point of \( S' \) as in \[5.3\] is a rational singularity if and only if \( B \) is rational and \( N > \frac{1}{k}(\sum_{i=1}^{n} k\bar{e}_i) - 2 \) for every natural number \( k > 0 \).

\[ \text{6. \( \mathbb{C}^* \)-ruled} \ S' \]

By \[3.3\] and section \ref{section}, the problem of classification of singular \( \mathbb{Q} \)-homology planes \( S' \) with smooth locus \( S_0 \) of non-general type reduces now to the case when \( S' \) is \( \mathbb{C}^* \)-ruled (or in other words, when \( S_0 \) has a \( \mathbb{C}^* \)-ruled which extends to a \( \mathbb{C}^* \)-ruled of \( S' \)). By \[3.3\] \( S' \) is logarithmic. These are exactly the assumptions made in [MS01, 2.9 - 2.17], where a description of singular fibers and a formula for \( \pi(S_0) \) (identified these times with \( \pi(S') \)) in terms of these fibers can be found. Unfortunately,
in three of four cases (2.14(4), 2.15(2), 2.16(2) loc. cit.) this formula is incorrect. We prove the correct formulas and compute the Kodaira dimension of $S'$. We give a more precise description, we answer questions about uniqueness of $\mathbb{C}$-rulings, of balanced boundaries and completions and we give a general method of construction.

6.1. Properties of $\mathbb{C}$*-rulings of $\mathbb{Q}$-homology planes. Recall that in general, if $p' : X \to B$ is a $\mathbb{C}$*-fibration of a normal surface $X$ then taking a completion of $X$ and an extension of $p'$ to a $\mathbb{P}^1$-ruling in principle, using $\mathbb{E}$ we are able to recognize when $X$ is $\mathbb{Q}$-acyclic (note that in particular $B$ has to be rational). The only condition which requires more explicit formulation is $\mathbb{L}3$(iv), we will state it in a form which is longer, but easier to check. Recall that for a family of subsets $(A_i)_{i \in I}$ of a topological space $Y$ a subset $X \subseteq Y$ separates the subsets $(A_i)_{i \in I}$ (inside $Y$) if and only if each $A_i$ is contained in a closure of some connected component of $Y \setminus X$ and none of these closures contains more than one $A_i$. Recall that by convention a twig of a fixed divisor is ordered so that its tip is the first component. For the definition of $\tilde{c}(T)$ see section 2.1.

Lemma 6.1. Let $(\overline{S}, T, p)$ be a triple satisfying conditions $\mathbb{L}3(i)$-$(iii)$. Assume additionally that $T$ is $p$-minimal and $f \cdot T = 2$ for a general fiber $f$ of $p$. In case $(h, \nu) = (2,0)$ let $D_0$, $F_0$, $B$, $\overline{D}_0$ be respectively some horizontal component of $D$, a unique fiber containing a $D$-rivet, a unique component of $D$ separating $D_0$, $D_0 - D_0$, $\overline{E}$ inside $D \cup F_0$ and a connected component of $D - B$ containing $D_0$. Then $d(D) \neq 0$ if and only if the following conditions are satisfied:

(i) $\nu \leq 1$,

(ii) if $(h, \nu) = (2, 1)$ then both $\overline{S} - T$-components of the fiber with $\sigma = 2$ intersect $D$,

(iii) if $(h, \nu) = (2, 0)$ then $\sum_T \tilde{c}(T) \neq -D_0^0$, where the sum is taken over the set of all maximal twigs of $D_0 - D_0$.

Proof. Clearly, if $d(D) \neq 0$ then $S'$ is a $\mathbb{Q}$-homology plane by $\mathbb{L}3$, which implies (i) and (ii) ($D$ intersects each curve not contained in $D + \overline{E}$ because $S'$ is affine). We will prove that (iii) holds too. Suppose now that the conditions (i)-(iii) are satisfied and $d(D) = 0$. There is a nonzero divisor $G$ with support contained in Supp $D$ which intersects trivially with all components of $D$. Thus $G \equiv 0$, because the intersection pairing on $\overline{S}$ is non-degenerate $(G \in \mathbb{Q}[D] \cap \mathbb{Q}[D]^\perp \subseteq NS(\overline{S}) \otimes \mathbb{Q})$. Note that no connected component of $G$ is vertical. Indeed, if $G'$ is such a connected component then $G^2 = G \cdot G' = 0$, so being connected $G'$ is a multiple of a fiber and then $G - G'$ is nonzero, vertical and not negative definite $(G - G' \equiv -G')$, hence contains a fiber, contradicting the inequality $\nu \leq 1$. In particular, since $G \cdot f = 0$ for a general fiber $f$, we infer that $h = 2$, hence $\sum_{\overline{S} - T} = \nu = 1$.

Consider the case $\sum_{\overline{S} - T} = \nu = 1$. Let $F_0$ be the singular fiber with $\sigma(F_0) = 2$ and write $G = G' + G_{\infty}$, where $G'$ and $G_{\infty}$ have no common components and $G_{\infty} \subseteq F_{\infty}$. Note that the numerical triviality of $G$ implies that there is no component of $F_0$ with exactly one common point with $G'$. Since $G$ has no vertical connected components, it follows that $G'$ has two connected components, each contains a section of $p$. Now let $M$ be a chain of components of $F_0$ not contained in $G'$, which joins the two connected components of $G'$. This chain is unique, because $F_0$ is a tree. Moreover, $M$ is in fact irreducible, otherwise it would contain a component intersecting $G'$ once. For the same reason the divisor $U$ of all components of $F_0$ not contained in $G' + M$ does not intersect $G'$. However, we have $\sigma(F_0) = 2$, so $U$ contains an $\overline{S} - T$-component, which intersects $D$ by (ii), so $M \subseteq D$. This contradicts the fact that $F_0$ does not contain a rivet.

Consider the case $\sum_{\overline{S} - T} = \nu = 0$. Note that there is exactly one fiber $F_0$ containing a $D$-rivet and other singular fibers are columnar by $\mathbb{L}12$(ii), so they contain no components of $\overline{E}$. Moreover, the $p$-minimality of $T$ implies that $F_0$ contains at most two $(-1)$-curves and if it contains two then the one contained in $D$ intersects $D_0$ twice. This shows that $B$ as in the formulation of the lemma exists and is unique. Let $\overline{D}_0$ and $\overline{D}_{\infty}$ be connected components of $D - B$ containing $D_0$ and $D_{\infty} = D_h - D_0$ respectively. Let $U$ be the connected component of $F_0 - B$ containing $\overline{E}$. Since $G$ has no vertical connected components and since no component of $F_0$ can intersect $G$ in one point, we infer that $B \not\subseteq G$, otherwise $G$ would contain the whole $U$, which is in a contradiction with $G \subseteq D$. 

\quad
It follows that \( G \subseteq \overline{D}_0 + \overline{D}_\infty \). Let \( \eta : \widehat{S} \to \overline{S} \) be the contraction of \( U \) (which can be obtained by contractions of \((-1\)-curves in \( U \) and its images). We see that \( \eta \) does not touch \( F_0 - B - U \), so \( \eta_*B \) is either a 0-curve or a unique \((-1\)-curve of the columnar fiber \( \eta_*F_0 \) for the induced \( \mathbb{P}^1 \)-ruled \( \overline{p} : \overline{S} \to \mathbb{P}^1 \). For \( j = 0, \infty \) put \( \widetilde{\eta}_j = \sum_{T \in I_j} \tilde{e}(T) \), where \( I_j \) is the set of maximal twigs of \( \overline{D}_j - D_j \). We have \( d(\widetilde{D}_j) = \prod_{T \in I_j} d(T) \cdot (\overline{D}_j^2 - \widetilde{\eta}_j) \), so we see that the condition (iii) is equivalent to vanishing some (and in fact each, as we show below) of \( d(\widetilde{D}_j) \). By the properties of columnar fibers \( d(\widetilde{D}_0) + d(\widetilde{D}_\infty) = -\prod_{T \in I_0} d(T) \cdot (D_0^2 + \tilde{e}_0 + \tilde{e}_\infty + D_\infty^2) = -\prod_{T \in I_\infty} d(T) \cdot (D_0^2 + D_\infty^2 + n) \), where \( n = \# I_0 = \# I_\infty \). Moreover, when contracting all singular fibers to smooth ones \( D_0 + D_\infty \) is touched \( n \) times and its image consists of two disjoint sections on a Hirzebruch surface, hence \( D_0^2 + D_\infty^2 + n = 0 \) and \( d(\widetilde{D}_0) = -d(\widetilde{D}_\infty) \). Now \( \overline{D}_0 + \overline{D}_\infty \) contains a numerically trivial divisor \( G \), hence \( d(D_0 + \overline{D}_\infty) = 0 \), which gives \( d(D_0) = d(\widetilde{D}_\infty) = 0 \), so we are done. We now see also that if \( d(\widetilde{D}_0) = 0 \) then \( d(\widetilde{D}_\infty) = 0 \) and then \( d(D) = 0 \) by [KR99, 2.1.1(i)].

**Corollary 6.2.** Suppose the pair \((\overline{S}, T)\) satisfies \( \ref{corollary}(i)-(iii) \). It follows from the above lemma that in case of a twisted \( \mathbb{C}^\ast \)-ruled (\( h = 1 \)) the condition \( d(D) \neq 0 \) is satisfied automatically.

We now go back to the classification. Let \( S' \) be a \( \mathbb{C}^\ast \)-ruled singular \( \mathbb{Q} \)-homology plane. We can lift the \( \mathbb{C}^\ast \)-ruled to a \( \mathbb{C}^\ast \)-ruled of the resolution and extend it to a \( \mathbb{P}^1 \)-ruled \( p : \overline{S} \to \mathbb{P}^1 \) of the completion. Assume that \( D + \widehat{E} \) is \( p \)-minimal. Since \( b_1(\widehat{E}) = 0 \), by \( \ref{diagram} \) \( D \) is a rational tree and \( S' \) is logarithmic. We have \( \Sigma_{S_0} = h + \nu - 2 \) and \( \nu \leq 1 \), so \((h, \nu, \Sigma_{S_0}) = (1, 1, 0), (2, 1, 1) \) or \((2, 0, 0) \). Note that the original \( \mathbb{C}^\ast \)-ruled of \( S' \) is twisted with base \( \mathbb{C}^1 \) in the first case, untwisted with the base \( \mathbb{C}^1 \) in the second case and untwisted with the base \( \mathbb{P}^1 \) in the third case.

**Lemma 6.3.** Let \( F_1, \ldots, F_n \) be all the singular fibers of \( p : \overline{S} \to \mathbb{P}^1 \) which are columnar (cf. \( \ref{diagram} \)). Let \( F_\infty \) be the fiber contained in \( D \) if \( \nu = 1 \). There is exactly one more singular fiber \( F_0 \), it contains \( \widehat{E} \). We have also:

\begin{itemize}
  \item[(i)] if \((h, \nu) = (1, 1)\) then \( F_\infty = [2, 1, 2] \), \( \sigma(F_0) = 1 \) and \( F_0 \) and \( F_\infty \) contain branching points of \( p|_{D_h} \),
  \item[(ii)] if \((h, \nu) = (2, 1)\) then \( F_\infty \) is smooth and \( \sigma(F_0) = 2 \),
  \item[(iii)] if \((h, \nu) = (2, 0)\) then \( \sigma(F_0) = 1 \) and \( F_0 \) contains a \( D \)-rivet,
  \item[(iv)] if \( h = 2 \) then the components of \( D_h \) are disjoint.
\end{itemize}

**Proof.** Suppose \((h, \nu) = (1, 1)\). Then \( \Sigma_{S_0} = 0 \), so by \( \ref{diagram} \) every singular fiber different than \( F_\infty \) which is not columnar contains a branching point of \( p|_{D_h} \). Thus \( F_0 \) is unique, because \( D_h \) is rational, \( p|_{D_h} \) has two branching points and one of them is contained in \( F_\infty \) as \( D \) is a tree. The \( p \)-minimality of \( D \) implies that \( F_\infty = [2, 1, 2] \). If \( h = 2 \) then \( \Sigma_{S_0} = \nu \in \{0, 1\} \) and \( \ref{diagram} \) gives (ii), (iii) and the uniqueness of \( F_0 \). Suppose \( h = 2 \) and the components of \( D_h \) have a common point. \( D \) is a tree, so in this case \( \nu = 0 \), which gives \( \sigma(F_0) = 1 \). As \( D \) is snc, the common point belongs to the unique \( S_0 \)-component of \( F_0 \), which has therefore multiplicity one. The connectedness of \( D \) implies that \( F_0 \) contains no \( D \)-components and \( S_0 \) is its unique \((-1\)-curve \( F_0 \) is singular because it contains \( \widehat{E} \)). This contradicts \( \ref{diagram} \)(iv).

In case \( \nu = 0 \) we put \( F_\infty = 0 \). Let \( J \) be a reduced divisors with support equal to \( D \cup F_0 \).

**Lemma 6.4.** The divisor \( J \) is an snc-divisor. Let \( \zeta : (\overline{S}, J) \to (W, \zeta_*J) \) be a composition of contractions of vertical \((-1\)-curves as long as the image of \( J \) is snc. Then \( \zeta_*F_i \) are smooth for \( i = 1, \ldots, n \) and:

\begin{itemize}
  \item[(i)] if \( h = 1 \) then \( \zeta_*F_0 = [2, 1, 2] \), \( (\zeta_*D_h)^2 = 0 \) and one can further contract \( \zeta_*F_0 \) and \( F_\infty \) to smooth fibers in such a way that \( W \) maps to \( \mathbb{P}^1 \) and \( \zeta_*D_h \) maps to a smooth 2-section of the \( \mathbb{P}^1 \)-ruled of \( \mathbb{P}^1 \), disjoint from the negative section,
  \item[(ii)] if \( h = 2 \) then \( \zeta_*F_0 \) is smooth, \( W \) is a Hirzebruch surface and the components of \( \zeta_*D_h \) are disjoint. Moreover, at least one of the components of \( D_h \) has negative self-intersection and changing \( \zeta \) if necessary one can assume that it is not touched by \( \zeta \).
Proof. It is clear that if \( p \in D_h \) is neither a branching point of \( p|D_h \) nor an intersection point of two distinct components of \( D_h \) then \( p \) is a point of normal crossings for \( J \). Thus if \( h = 2 \) then \( J \) is an snc-divisor by (3.3(iv)). If \( h = 1 \) then any point where the crossings of \( J \) are not normal is just a point where three components of \( J \) meet, which can happen only if two components of \( F_0 \) of multiplicity one intersect in a point belonging to \( D_h \). However, as \( D \) is snc, one of them has to be the unique \( S_0 \)-component of \( F_0 \) and by the \( p \)-minimality of \( D \) it has to be a unique \((-1)\)-curve of \( F_0 \) too, which is impossible by (2.3(v)). This shows that \( J \) is an snc-divisor. Since \( F_i \) for \( i = 1, \ldots, n \) are columnar, \( \zeta F_i \) are smooth.

Suppose \( h = 2 \). Write \( D_h = H + H' \). By (3.3) \( H \) and \( H' \) are disjoint. If after some number of contractions in \( F_0 \) they intersect the same component of the fiber then this component has multiplicity one and assuming the fiber is still singular one could find a \((-1)\)-curve not intersecting the images of \( H \) and \( H' \). Thus \( \zeta F_0 \) is smooth and \( \zeta H' \) does not intersect \( \zeta H' \). In fact this argument show also that we can, and we will now, assume that \( H' \) is not touched by \( \zeta \). We will also assume that \( H'^2 \leq H^2 \). Since \( \zeta D_h \) consists of two disjoint sections on a Hirzebruch surface, we have \((\zeta \cdot D_h)^2 = 0 \), so \( D_h^2 \leq 0 \). Suppose \( H'^2 = H^2 \). Then \( \zeta \) does not touch \( D_h \), so \( n = 0 \) and \( H \) and \( H' \) intersect the same component \( B \) of \( F_\infty \). If \( \nu = 1 \) then \( B \) is an \( S_0 \)-component and the second \( S_0 \)-component of \( F_0 \) does not intersect \( D \), a contradiction with the affineness of \( S' \). Thus \( \nu = 0 \) and the condition (3.3) is not satisfied (in other words \( d(D) = 0 \)), a contradiction.

Suppose \( h = 1 \). By the maximality of \( \zeta \) the image of \( D_h \) intersects the unique \((-1)\)-curve of \( \zeta F_0 \), which can happen only if \( \zeta F_0 = [2, 1, 2] \). Now after the contraction of \( F_0 \) and \( F_\infty \) to smooth fibers the image of \( W \) is a Hirzebruch surface \( \mathbb{P}_N \), where \( N \geq 0 \), and the image \( D_h' \) of \( D_h \) is a smooth 2-section. Write \( D_h' \equiv \alpha f + 2H \) where \( H \) is a section with \( H^2 = -N \) and \( f \) is a fiber of the induced \( \mathbb{P}^1 \)-ruling of \( \mathbb{P}_N \). We compute \( p_a(\alpha f + 2H) = \alpha - N - 1 \), so since \( D_h' \) is smooth, its arithmetic genus vanishes, so \( \alpha = N + 1 \). Moreover, \( D_h' D_h = \alpha = 2N \), hence \( D_h' H = H + N = 1 \). Now if \( N = 0 \) then \( \mathbb{P}_N = \mathbb{P}^1 \times \mathbb{P}^1 \) and an elementary transformation with center equal to the point of tangency of \( D_h' \) and the image of \( F_\infty \) (which corresponds to a different choice of components to be contracted in \( F_\infty \)) leads to \( N = 1 \) and \( D_h' H = 0 \).

6.2. Kodaira dimension. For \( i = 1, \ldots, n \) denote the \((-1)\)-curve of \( F_i \) by \( C_i \) and the multiplicity of \( C_i \) by \( \mu_i \). We now prove formulas for the Kodaira dimension of \( S' \) and of \( S_0 \) in terms of the structure of singular fibers of \( p \). We can factor \( \zeta : \mathbb{S} \to W \) into \( \eta : \mathbb{S} \to \mathbb{S} \) and \( \theta : \mathbb{S} \to W \), where \( \theta \) is subvariational for \( \eta_*(J + \sum_{i=1}^n C_i) \) and is maximal such, i.e. for every nontrivial factorization \( \eta = \eta'' \circ \eta' \) the composition \( \theta \circ \eta'' \) is not subvariational for \( \eta''_*(J + \sum_{i=1}^n C_i) \). We denote a general fiber of a \( \mathbb{P}^1 \)-ruling by \( f \).

Remark 6.5. Let \((X, D)\) be a smooth pair and let \( L \) be the exceptional divisor of a blowup \( \sigma : X' \to X \) of a point in \( D \). Then \( K_{X'} + \sigma^{-1} D = \sigma^*(K_X + D) \) if \( \sigma \) is subvariational for \( D \) and \( K_{X'} + \sigma^{-1} D = \sigma^*(K_X + D) + L \) if \( \sigma \) is sprouting for \( D \).

Lemma 6.6. Let \( \eta : \mathbb{S} \to \mathbb{S} \) and \( \theta : \mathbb{S} \to W \) be as above. Then

\[
K_\mathbb{S} + \eta_* J \equiv (n + \nu - 1 - \sum_{i=1}^n \frac{1}{\mu_i}) f + G + \theta^* \frac{1}{2} (U + U'),
\]

where \( G \) is a negative definite effective divisor with support contained in \( \text{Supp}(F_\infty + \sum_{i=1}^n F_i) \) and \( U, U' \) are the \((-2)\)-tips of \((\theta \circ \eta)_* F_0 \) in case \( p \) is twisted and are zero otherwise.

Proof. Let \( V \) be defined as the sum of (four) \((-2)\)-tips of \( F_\infty + \eta_* F_0 \) if \( p \) is twisted and as zero otherwise. We check easily that \( K_W + D_h + F_\infty + \theta_* F_0 \equiv (\nu - 1) f + \frac{1}{2} V \). Indeed, if \( p \) is untwisted this is just \( K_W + D_h + 2f \equiv 0 \) on a Hirzebruch surface and if \( p \) is twisted then it follows from the equivalences \( K_W + D_h + f \equiv 0 \) and \( F_\infty + \theta_* F_0 - \frac{1}{2} V \equiv f \). Since \( \theta_* F_i \equiv f \), by (3.3) we get \( K_\mathbb{S} + \eta_* J + \sum_{i=1}^n C_i \equiv (n + \nu - 1) f + \theta^* \frac{1}{2} V \). For every \( i = 1, \ldots, n \) the divisor \( G_i = \frac{1}{\mu_i} F_i - C_i \) is effective and negative definite, as \( C_i \) is not contained in its support. We get \( K_\mathbb{S} + \eta_* J \equiv (n + \nu - 1) f + \sum_{i=1}^n (G_i - \frac{1}{\mu_i} F_i) + \theta^* \frac{1}{2} V \equiv (n + \nu - 1 - \frac{1}{\mu_i}) f + \sum_{i=1}^n G_i + \theta^* \frac{1}{2} V \), so we are done. □
Remark 6.7. Since $K_{\Sigma'} + D + \hat{E}$ and $K_{\Sigma'} + D$ intersect trivially with a general fiber, we can write $K_{\Sigma'} + D + \hat{E} \equiv \kappa_0 f + G_0$ and $K_{\Sigma'} + D + \hat{E} \equiv \kappa f + G$, where $G_0$ and $G$ are some vertical effective and negative definite divisors and $\kappa_0, \kappa \in \mathbb{Q}$ (if $\kappa (K_{\Sigma'} + D) = -\infty$ then one can first prove it for $K_{\Sigma'} + D + f$, for which $\kappa (K_{\Sigma'} + D + f) \geq 0$, as $D \cdot f > 1$, cf. [Miy01, 2.2.2]). Then by $\mathbb{Z} \pi(S_0)$ and $\pi(S)$ are determined by the signs of numbers $\kappa_0$ and $\kappa$ respectively. More explicitly, $\pi(S_0) = -\infty$, 0, 1 depending if $\kappa_0 <, = > 0$, which we can state in short as $\text{sgn} \pi(S_0) = \text{sgn} \kappa_0$, where $\text{sgn} x$ is the sign function. Analogous remarks hold for $\pi(S)$ and $\kappa$.

It appears that $\kappa$ and $\kappa_0$ depend in a quite involved way on the structure of $F_0$. This dependence can be stated in terms of the properties of $\eta : \hat{S} \to \hat{S}$ defined above as follows. Denote the $S_0$-components of $F_0$ by $C, C'$ (or just $C$ if there is only one) and their multiplicities by $\mu, \bar{\mu}$ respectively. Note that $\mu \geq 2$ if $\sigma(F_0) = 1$, but if $\sigma(F_0) = 2$ then it can happen that $\mu = 1$ or $\bar{\mu} = 1$.

Theorem 6.8. Let $\lambda = n + \nu - 1 - \sum_{i=1}^{n} \frac{1}{\mu_i}$. The numbers $\kappa$ and $\kappa_0$ determining the Kodaira dimensions of a $\mathbb{C}^*$-ruled singular $\mathbb{Q}$-homology plane $S'$ and of its smooth locus $S_0$ defined in [6, 7] are as follows:

(A) Case $(h, \nu) = (1, 1)$. Denote the component of $F_0$ intersecting $D_h$ by $B$.

(i) If $\eta = id$ and $F_0 = [2, 1, 2]$ then $\kappa = \kappa_0 = \lambda - \frac{1}{2}$.

(ii) If $\eta = id$, $B$ is not a tip of $F_0$ and $C \cdot B > 0$ then $(\kappa, \kappa_0) = (\lambda - \frac{1}{2}, \lambda - \frac{1}{2\mu})$.

(iii) If $\eta = id$, $C \cdot B = 0$ and $F_0$ is a chain then $(\kappa, \kappa_0) = (\lambda - \frac{1}{2}, \lambda)$.

(iv) If $\eta = id$ and $B$ is a tip of $F_0$ then $(\kappa, \kappa_0) = (\lambda - \frac{1}{2}, \lambda - \frac{1}{\mu})$.

(v) If $\eta \neq id$ then $\kappa = \kappa_0 = \lambda$.

(B) Case $(h, \nu) = (2, 1)$.

(i) If $\eta = id$ and $C^2 = \hat{C}^2 = -1$ then $(\kappa, \kappa_0) = (\lambda - 1, \lambda - \frac{1}{\text{min}(\mu, \bar{\mu})})$.

(ii) If $\eta = id$ and $C^2 \neq -1$ or $\hat{C}^2 \neq -1$ then $\kappa = \kappa_0 = \lambda - \frac{1}{\text{min}(\mu, \bar{\mu})}$.

(iii) If $\eta \neq id$ then, assuming that $C$ is the $S_0$-component disjoint from $\hat{E}$, $\kappa = \kappa_0 = \lambda - \frac{1}{\mu}$.

(C) Case $(h, \nu) = (2, 0)$. Then $\kappa = \kappa_0 = \lambda$.

Proof. (A) The unique $S_0$-component $C$ of $F_0$ is a $(−1)$-curve. Indeed, otherwise the $p$-minimality of $D$ implies that $B$ is the only $(−1)$-curve in $F_0$ and it intersects two other $D$-components of $F_0$, which gives $F_0 = [2, 1, 2] \subset D$ with no place for $C$. It is now easy to check that the list of cases in (A) is complete. As $C^2 = -1$, $F_0 - C$ has at most two connected components. We see also that the only case when $\hat{E}$ is not connected is when $F_0$ contains no $D$-components, which is possible only if $C = B$ and $F_0 = [2, 1, 2]$. Since $C$ is the unique $(−1)$-curve in $F_0$, $\zeta = \theta \circ \eta$ has at most one center on $\zeta, F_0$, so by symmetry we can and will assume that it does not belong to $U'$ (cf. [7, 0]). Suppose $\eta \neq id$. The maximality of $\theta$ implies that the center of $\eta$ belongs to a unique component of $\eta_\ast J$ and $D_h$ does not intersect components contracted by $\eta$. Then the mentioned component is a proper transform of a $D$-component, so $\eta_\ast (C + \hat{E}) = 0$ by the connectedness of $\hat{E}$. If we now factor $\eta$ as $\eta = \sigma \circ \eta'$, where $\sigma$ is a sprouting blowdown for $\eta_\ast J$ then by [17, 3.5] we get $K + \sigma^{-1} \eta_\ast J \equiv \lambda f + G + \sigma^* \theta^* (U' + U') + Exc(\sigma)$, where $Exc(\sigma)$ is the exceptional $(−1)$-curve contracted by $\sigma$ and $K$ is a canonical divisor on a respective surface. Since $\eta_\ast (C + \hat{E}) = 0$, each component of $C + \hat{E}$ will appear with positive integer coefficient in $\eta^* Exc(\sigma)$, which leads to $K_{\Sigma'} + \eta^{-1} \eta_\ast J \equiv \lambda f + G + G_0$, where $G_0$ is a vertical effective and negative definite divisor for which $G_0 - \hat{E} - C$ is still effective. Since $\eta^{-1} \eta_\ast J = J = D + \hat{E} + C$, we get $\kappa = \kappa_0 = \lambda$. We can now assume that $\eta = id$, so

$$K_{\Sigma'} + D + \hat{E} + C \equiv \lambda f + G + \frac{1}{2}(U' + \theta^* U).$$

This can be written as $K_{\Sigma'} + D \equiv (\lambda - \frac{1}{2}) f + G + \frac{1}{2}(U' + F_0 + \theta^* U - 2C - 2\hat{E})$. All components of $F_0$ appear in $U' + F_0 + \theta^* U$ with coefficients bigger than 1, so $U' + F_0 + \theta^* U - 2C - 2\hat{E}$ is effective and negative definite, as its support does not contain the $\hat{E}$-component which is a proper transform of $U$. This gives $\kappa = \lambda - \frac{1}{2}$. We now compute $\kappa_0$. If $F_0 = [2, 1, 2]$ then $\theta^* U = U$ and $\hat{E} = U + U'$, so...
$K_\Sigma + D \equiv (\lambda - \frac{1}{2})f + G$ and we get $\kappa_0 = \lambda - \frac{1}{2}$. Suppose $B$ is a tip of $F_0$. Since $\mu(B) = 2$, $F_0$ is a fork with two $(-2)$-tips as maximal twigs (cf. [2.9](vi)) and $\theta^*U = U$ ($U$ and $U'$ are components of $\tilde{E}$). The divisor $G_0 = \frac{1}{2}(U + U') + \frac{1}{2}F_0 - C$ is vertical effective and its support does not contain $C$. Writing $K_\Sigma + D + \tilde{E} \equiv (\lambda - \frac{1}{2})f + G + G_0$ we infer that $\kappa_0 = \lambda - \frac{1}{2}$, hence (iv). Consider the case (ii). Since $B$ is not a tip of $F_0$, $F_0$ is a chain. The assumption $B \cdot C > 0$ implies that $B^2 \not= -1$ and $\theta^*U = C + \tilde{E}$. We get $K_\Sigma + D + \tilde{E} \equiv (\lambda - \frac{1}{2})f + G + \frac{1}{2}(U' + \tilde{E} + \frac{1}{2}F_0 - C)$ and $U' + \tilde{E} + \frac{1}{2}F_0 - C$ is effective with support not containing $C$. This gives $\kappa_0 = \lambda - \frac{1}{2}\eta$. We are left with the case (iii). As in (ii) $F_0$ is a chain and we have now $K_\Sigma + D + \tilde{E} \equiv \lambda f + G + \frac{1}{2}(U' + \theta^*U - 2C)$. Since $B \cdot C = 0$, $U' + \theta^*U - 2C$ is effective and does not contain $B$, so $\kappa_0 = \lambda$.

(B) Suppose $\eta \not= id$. Note that $\eta_*F_0$ contains a proper transform of one of $C$, $\tilde{C}$, otherwise $F_0$ would contain a $D$-rivet. It follows that $\eta$ is a connected modification and its center lies on a birational transform of a $D$-component (the $S_0$-component contracted by $\eta$ to intersect $D$). Thus $\eta_*F_0$ is a chain intersected by $D_h$ in two different tips and containing $C$. Since $D \cap \tilde{E} = 0$, we get $\eta_*\tilde{C} + \tilde{E} = 0$. Writing $\eta = \sigma \circ \eta'$, where $\sigma$ is a sprouting blowdown, we see that $\eta^*Exc(\sigma)$ is an effective negative definite divisor which does not contain $C$ in its support and for which $\eta^*Exc(\sigma) - \tilde{C} - \tilde{E}$ is effective. By [6.6] we have $K + \sigma^{-1}\eta_*D + C \equiv \lambda f + G + Exc(\sigma)$, where $K$ is a canonical divisor on a respective surface. It follows from [6.8] and arguments analogous to these from part (A) that $\kappa = (\kappa_0 = \lambda - \frac{1}{2}\eta$. We can now assume that $\eta = id$. By [5.7] $K_\Sigma + D + C + \tilde{E} + \tilde{C} \equiv \lambda f + G$, which implies $\kappa_0 = \lambda - \frac{1}{2}(\min(\mu, \tilde{\mu})$. Writing $K_\Sigma + D + \tilde{C} \equiv (\lambda - \frac{1}{2})f + G + \frac{1}{2}(F_0 - \alpha(C + \tilde{E} + \tilde{C}))$ we see that $\kappa = \lambda - \frac{1}{2}\alpha$, where $\alpha$ is the lowest multiplicity of a component of $\tilde{C} + \tilde{E} + \tilde{C}$ in $F_0$. Note that $C + \tilde{E} + \tilde{C}$ is a chain. Now if for example $C^2 \not= -1$ then $F_0$ is columnar and factoring $\theta$ into blowdowns we see that $\tilde{E}$ is contracted before $C$, hence $\alpha = \mu \leq \tilde{\mu}$. Suppose $C^2 = \tilde{C}^2 = -1$ and let $\theta'$ be the composition of successive contractions of $(-1)$-curves in $F_0$ different than $C$. Now either $\theta'_*F_1 \cap D = \emptyset$ or $\theta'_*F_0$ is columnar. Both imply that $C + \tilde{E}$ contains a component of multiplicity one, hence $\alpha = 1$.

(C) $C$ is a $(-1)$-curve. Indeed, $D \cap F_0$ contains at most one $(-1)$-curve and if it does then by the $p$-minimality of $D$ it intersects both components of $D_h$ and has multiplicity one, so there is another $(-1)$-curve in $F_0$. We infer that $F_0 - C$ has two connected components, one is $\tilde{E}$ and the second contains a rivet. The existence of a rivet in $F_0$ implies that $\eta \not= id$, so $\eta_*\tilde{C} + \tilde{E} = 0$. Factoring out a sprouting blowdown from $\eta$ as above we get $K + \sigma^{-1}\eta_*D \equiv \lambda f + G + Exc(\sigma)$. The divisor $\eta^*Exc(\sigma) - C - \tilde{E}$ is effective and does not contain all components of $F_0$, so by [6.8] $\kappa = \kappa_0 = \lambda$.

Remark. Note that in case (B)(iii) it is not true in general that $\mu = \min(\mu, \tilde{\mu})$. Note also that by [B.1] for any $Q$-homology plane we have $H_i(S', Z) = 0$ for $i > 1$ and $|H_1(S', Z)| = \sqrt{d(D)/d(\tilde{E})}$. For $C^*$-ruled $S'$ more explicit computations are done in [MS91]. For example, by 2.17 loc. cit. a $C^*$-ruled of a $Z$-homology plane with $\pi(S_0) \geq 0$ is always untwisted and has base $\mathbb{P}^1$.

**Corollary 6.9.** Let $S'$ be a $C^*$-ruled singular $Q$-homology plane and let $D$ be a $p$-minimal boundary for an extension $p$ of this ruling to a normal completion. Then $\pi(S_0) = 0$ exactly in the following cases:

(i) $n = 0$ and $F_0$ is of type (A)(iii) or (A)(v),

(ii) $n = 1$, $\mu = \mu_1 = 2$, $F_0$ contains no $D$-components and is of type (A)(ii) or (A)(iv),

(iii) $p$ is untwisted with base $\mathbb{C}^1$, $n = 1$, $\mu_1 = 2$, $\min(\mu, \tilde{\mu}) = 2$ and some connected component of $F_0 \cap D$ is a $(-2)$-curve,

(iv) $p$ is untwisted with base $\mathbb{C}^1$, $n = 2$, $\mu_1 = \mu_2 = 2$, and some $S_0$-component of $F_0$ intersects $D_h$,

(v) $p$ is untwisted with base $\mathbb{P}^1$, $n = 2$ and $\mu_1 = \mu_2 = 2$.

**Proof.** Note that $n - \sum_{i=1}^{n} \frac{1}{\mu_i} \geq \frac{3}{2}$, because $\mu_i \geq 2$ for each $i$. Suppose $p$ is twisted. Then $\mu \geq 2$, so by [5.8] $\lambda \geq \kappa_0 \geq \lambda - \frac{1}{2} \geq \frac{n-1}{2} \geq \frac{n}{2}$. If $n = 0$ then $\lambda = 0$, which gives $\kappa_0 = 0$ exactly in cases (A)(iii) and
(A)(v). If \( n = 1 \) then \( \kappa_0 = \lambda - \frac{1}{2} = 0 \), which is possible in case (A)(i) if \( \mu_1 = 2 \) and in case (A)(iv) if \( \mu = \mu_1 = 2 \). In both cases \( D_h \) intersects the \( S_0 \)-component, so \( F_0 \) contains no \( D \)-components. If \( p \) is untwisted with base \( \mathbb{P}^1 \) then \( n - 1 \geq \lambda = \kappa_0 \geq \frac{n}{2} - 1 \), so \( n = 2 \) (\( \lambda = \frac{1}{\mu_1} < 0 \) for \( n = 1 \)) and \( \kappa_0 = 1 - \frac{1}{\mu_1} - \frac{1}{\mu_2} \), which vanishes only if \( \mu_1 = \mu_2 = 2 \). Assume now that \( p \) is untwisted with base \( \mathbb{C}^1 \). We have \( n > \kappa_0 \geq \lambda - 1 \geq \frac{n}{2} - 1 \), so \( n \in \{1, 2\} \). There are no \((-1)\)-curves in \( D \cap F_0 \) by the \( p \)-minimality of \( D \), so at least one \( S_0 \)-component, say \( C \), is a \((-1)\)-curve. We can also assume that \( C \) is contracted by \( \eta \) in case \( \eta \neq id \) and that \( \mu \geq \bar{\mu} \) in case \( \eta = id \). Then \( \kappa_0 = \lambda - \frac{1}{\mu} \). The composition \( \xi \) of successive contractions of all \((-1)\)-curves in \( F_0 - \tilde{C} \) and its images is a connected modification. Suppose \( n = 2 \). The inequalities above give \( \lambda = 1 \), so \( \mu_1 = \mu_2 = 2 \) and \( \bar{\mu} = 1 \). Then \( \xi_*F_0 = [0] \) and since \( \xi \) is a connected modification, \( \tilde{C} \) is a \((-1)\)-curve. Now if \( \mu = 1 \) then \( \mu < \bar{\mu} \), so by our assumption \( \eta \neq id \). But then \( \mu > 1 \), because \( C^2 = -1 \) and \( C \) intersects \( \tilde{E} \) and \( D \). This contradiction ends the proof of (iii).

\[ \square \]

6.3. Construction. Lemmas 6.4 and 13 give a practical method of reconstructing all \( \mathbb{C}^* \)-ruled \( \mathbb{Q} \)-homology planes. We summarize it in the following discussion. We denote irreducible curves and their proper transforms by the same letters.

Construction 6.10.

Case 1. Twisted ruling. Let \( D_h, x_0, x_\infty \) be a smooth conic on \( \mathbb{P}^2 \) and a pair of distinct points on it. Let \( L_0, L_\infty \) be tangents to \( D_h \) at \( x_0, x_\infty \) respectively and let \( L_i \) for \( i = 1, \ldots, n, n \geq 0 \) be different lines (different than \( L_0, L_\infty \)) through \( L_0 \cap L_\infty \). Blow at \( L_0 \cap L_\infty \) once and let \( p: \mathbb{F}_1 \to \mathbb{P}^1 \) be the \( \mathbb{P}^1 \)-ruling of the resulting Hirzebruch surface. Over each of \( p(L_0) \), \( p(L_\infty) \) blow on \( D_h \) twice creating singular fibers \( F_0 = \{2, 1, 2\} \) and \( F_\infty = \{2, 1, 2\} \). For each \( i = 1, \ldots, n \) by a connected sequence of blowups subvadisional for \( L_i + D_h \) create a column fiber \( F_i \) over \( p(L_i) \) and denote its unique \((-1)\)-curve by \( C_i \). By some connected sequence of blowups with a center on \( F_0 \) create a singular fiber \( F_0 \) and denote the newly created \((-1)\)-curve by \( C \) (if the sequence is empty define \( C \) as the \((-1)\)-curve of \( F_0 \)). Denote the resulting surface by \( S \), put \( T = D_h + F_\infty + (F_1 - C_1) + \ldots + (F_n - C_n) + F_0 - C \) and construct \( S' \) as in 13. \( S' \) is a \( \mathbb{Q} \)-homology plane (singular if only \( T \) is not connected), because conditions 13(i)-(iii) are satisfied by construction and (iv) by 6.2. To see that each \( S' \) admitting a twisted \( \mathbb{C}^* \)-ruled can be obtained in this way note that by the \( p \)-minimality of \( D \) even if \( F_0 \) contains two \((-1)\)-curves \( C \) and \( B \subseteq D \) then \( B \) is not a tip of \( F_0 \) and \( \zeta \) does not touch it, so in each case the modification \( F_0 \to \zeta_*F_0 \) induced by \( \zeta \) is connected and we are done by 6.4.

Case 2. Untwisted ruling with base \( \mathbb{C}^1 \). Let \( x_0, x_1, x_n, x_\infty, y \in \mathbb{P}^2, n \geq 0 \) be distinct points, such that all besides \( y \) lie on a common line \( D_1 \). Let \( L_i \) be a line through \( x_i \) and \( y \). Blow \( y \) once and let \( D_2 \) be the negative section of the \( \mathbb{P}^1 \)-ruling of the resulting Hirzebruch surface \( p: \mathbb{F}_1 \to \mathbb{P}^1 \). For each \( i = 0, 1, \ldots, n \) by a connected sequence of blowups (which can be empty if \( i = 0 \)) with the first center \( x_i \) and subvadisional for \( D_1 + L_i \) create a column fiber \( F_i \) (if \( i = 0 \)) over \( p(x_i) \) and denote its unique \((-1)\)-curve by \( C_i \) if \( i \neq 0 \) and by \( \tilde{C} \) if \( i = 0 \) (put \( \tilde{C} = L_0 \) if the sequence \( p(x_0) \) is empty). Choose a point \( z \in F_0 \) which lies on \( D_1 + F_0 - \tilde{C} \) and by a nonempty connected sequence of blowups with the first center \( z \) create some singular fiber \( F_0 \) over \( p(x_0) \), let \( C \) be the new \((-1)\)-curve. Denote the resulting surface by \( S \), put \( T = D_1 + D_2 + L_\infty + (F_1 - C_1) + \ldots + (F_n - C_n) + F_0 - C - \tilde{C} \) and construct \( S' \) as in 13. \( S' \) is a \( \mathbb{Q} \)-homology plane by 1.1, as 1.1(ii) is satisfied by the choice of \( z \). To see that all \( S' \) admitting an untwisted \( \mathbb{C}^* \)-ruled with base \( \mathbb{C}^1 \) can be obtained in this way note that changing the completion of \( S' \) by a flow if necessary we can assume that one of the components
of $D_h$ is a $(-1)$-curve. Note also that, $D \cap F_0$ contains no $(-1)$-curves and, as it was shown in the proof of $\S 8$, $\eta$ contracts at most one of $C, \tilde{C}$. Then we are done by $\S 4$.

Case 3. Untwisted ruling with base $\mathbb{P}^1$. Let $D_2$ be the negative section of the $\mathbb{P}^1$-ruling of a Hirzebruch surface $p : \mathbb{F}_N \to \mathbb{P}^1, N > 0$. Let $x_0, x_1, \ldots, x_n, n \geq 0$ be points on some section $D_1$ of $p$ disjoint from $D_2$. For each $i = 0, 1, \ldots, n$ by a connected sequence of blowups (which can be empty if $i = 0$) with the first center $x_i$ and subdivisinal for $D_1 + p^{-1}(p(x_i))$ create a column fiber $F_i$ ($\bar{F}_0$ if $i = 0$) over $p(x_i)$ and denote its unique $(-1)$-curve by $C_i$ if $i \neq 0$ and by $B$ if $i = 0$ (put $B = p^{-1}(p(x_0))$ if the sequence over $p(x_0)$ is empty). Assume that the intersection matrix of at least one of two connected components of $D_1 + D_2 + (F_1 - C_1) + \ldots + (F_n - C_n) + (\bar{F}_0 - B)$ is non-degenerate. By a connected sequence of blowups starting from a sprouting blowup for $S$ with center on $B = 0$ in $4.3$. Note that $p$-bruch surface $S$ contains a non-cyclic singularity if and only if one of the following holds:

(i) $S_0$ has a $C^*$-ruling which does not extend to a ruling of $S'$ and either $S'$ is non-logarithmic or is isomorphic to $\mathbb{C}^2/G$ for a small finite noncyclic subgroup of $GL(2, \mathbb{C})$,

(ii) $S'$ has a twisted $C^*$-ruling, the fiber isomorphic to $\mathbb{C}^1$ contains a singular point of Dynkin type $\tilde{D}_k$ for some $k \geq 4$,

(iii) $\overline{\pi}(S_0) = 2$ and the unique singular point of $S'$ is of quotient type but non-cyclic.

Proof. (1) Suppose $S'$ has at least two different balanced completions. These differ by a flow, which in particular implies that the boundary contains a non-branching rational component $F_\infty$ with zero self-intersection. Then $F_\infty$ is a fiber of a $\mathbb{P}^1$-ruling $p$ of a balanced completion $(V, D)$. We can assume that $F_\infty$ is not contained in any maximal twig of $D$, otherwise by a flow we can ‘move’ the 0-curve to be a tip of a new boundary, where it gives an affine ruling of $S'$. Since $F_\infty$ is non-branching, the induced ruling restricts to an untwisted $C^*$-ruling of $S'$. Because $F_\infty$ is not contained in any maximal twig of $D_2$, it follows from the connectedness of the modification $\eta$ (see the proof of $\S 8$) that $n > 0$, so this restriction has more than one singular fiber. Moreover, both components of $D_h$ are branching in $D$. Since $F_\infty$ is the only non-branching 0-curve in $D$, centers of elementary transformations lie on the intersection of the fiber at infinity with $D_h$. If $D$ is strongly balanced then one of the components of $D_h$ is a 0-curve, hence there are at most two strongly balanced completions. Conversely, suppose $S'$ has an untwisted $C^*$-ruling with base $\mathbb{C}^1$ and $n > 0$ and let $(V, D, p)$ be a completion of this ruling. As $S'$ is not affine-ruled, the horizontal components $H, H'$ of $D$ are branching, so $(V, D)$ is balanced and we can assume $H^2 = 0$. Since $H, H'$ are proper transforms of two disjoint sections on a Hirzebruch surface, we have $H^2 + H'^2 + n \leq 0$, so $H^2 \neq 0$ and we can obtain a different strongly balanced completion of $S'$ by a flow which makes $H$ into a 0-curve.

By $\S 3.4$, $\S 4.1$ and $\S 4.5$ (2) and (3) hold if $\overline{\pi}(S_0) = 2$ or $-\infty$ (smooth part of $\mathbb{C}^2/G$ has a non-extendable $C^*$-ruling). By $\S 3.3$ and $\S 5.4$ they hold also if $S'$ is exceptional or non-logarithmic. In other cases $S'$ is $C^*$-ruled. If this ruling is untwisted then it follows from the proof of $\S 8$ that $S'$ has
a unique singular point and it is a cyclic singularity. In the twisted case, since \( \hat{E} \subseteq F_0 \), we see that if \( \hat{E} \) is not connected then \( F_0 \) is of type \((A)(i)\) and if \( \hat{E} \) is not a chain then \( F_0 \) is of type \((A)(iv)\).

**Remark.** The set of isomorphism classes of strongly balanced completions that a given surface admits is an invariant of the surface, which can easily distinguish between many \( \mathbb{Q} \)-acyclic surfaces. As for now an example of type \((3)(iii)\) is not known.

### 6.5. Number of \( \mathbb{C}^* \)-rulings.

Recall that a \( \mathbb{C}^* \)-ruling of \( S_0 \) is extendable if it extends to a ruling (morphism) of \( S' \). We now consider the question of uniqueness of \( \mathbb{C}^* \)-rulings of \( S_0 \) and \( S' \). We neglect affine-ruled \( \mathbb{Q} \)-homology planes, as if \( S' \) admits an affine and a \( \mathbb{C}^* \)-ruling it is the affine ruling which gives more information on \( S' \) (uniqueness of these was considered in [1.5]). Two rational rulings of a given surface are considered the same if they differ by an automorphism of the base. Recall that \( \pi(S_0) = -\infty \) if and only if either \( S' \) is affine-ruled or \( S' \cong \mathbb{C}^2/G \) for a small finite noncyclic subgroup of \( GL(2,\mathbb{C}) \).

**Theorem 6.12.** Let \( S' \) be a singular \( \mathbb{Q} \)-homology and let \( p_1, \ldots, p_r, r \in \mathbb{N} \cup \{ \infty \} \) be all different \( \mathbb{C}^* \)-rulings of its smooth locus \( S_0 \).

1. If \( \pi(S_0) = 2 \) or if \( S' \) is exceptional (hence \( \pi(S_0) = 0 \), cf. [2.3]) then \( r = 0 \).
2. If \( \pi(S_0) = 1 \) or if \( S' \) is non-logarithmic then \( r = 1 \).
3. If \( \pi(S_0) = -\infty \) and \( S' \) is not affine-ruled then \( r \geq 1 \) and \( p_1 \) is non-extendable. Moreover, \( r \neq 1 \) only if the fork which is an exceptional divisor of the snc-minimal resolution of \( S' \) is of type \((2,2,k)\) and in this case:
   - if \( k \neq 2 \) then \( r = 2 \), \( p_2 \) is twisted and has a unique singular fiber, which is of type \((A)(iv)\),
   - if \( k = 2 \) then \( r = 4 \), \( p_2, p_3, p_4 \) are twisted and all have unique singular fibers, which are of type \((A)(iv)\).
4. Assume that \( \pi(S_0) = 0 \), \( S' \) is logarithmic and not exceptional. Then all \( p_i \) extend to \( \mathbb{C}^* \)-rulings of \( S' \) and the following hold:
   - if the dual graph of \( D \) is
     \[
     \begin{array}{ccc}
     -2 & -1 & k \\
     \mid & & \mid \\
     -2 & -2 & \\
     \end{array}
     \]
     with \( k \leq -2 \) then \( r = 1 \) and \( p_1 \) is twisted.
   - if the dual graph of \( D \) is
     \[
     \begin{array}{ccc}
     -2 & -1 & -1 \\
     \mid & & \mid \\
     -2 & -2 & \\
     \end{array}
     \]
     then \( r = 2 \) and \( p_1, p_2 \) are twisted.
   - if the dual graph of \( D \) is
     \[
     \begin{array}{ccc}
     -2 & k & 0 \\
     \mid & & \mid \\
     -2 & -2 & \\
     \end{array}
     \]
     then \( r = 3 \), \( p_1, p_2 \) are twisted and \( p_3 \) is untwisted with base \( \mathbb{C}^1 \).
   - in all other cases \( r = 2 \), \( p_1 \) is twisted and \( p_2 \) is untwisted.

**Proof.** (1) By definition exceptional \( \mathbb{Q} \)-homology planes are not \( \mathbb{C}^* \)-ruled. If \( S_0 \) is of general type then \( S_0 \) is not \( \mathbb{C}^* \)-ruled by Iitaka’s easy addition formula [Iit82, 10.4].

(2) If \( S' \) is non-logarithmic then the \( \mathbb{C}^* \)-ruling of \( S' \) is unique by [3.1]. Assume now that \( \pi(S_0) = 1 \). Let \( (\mathcal{S}, D) \) be some normal completion of the snc-minimal resolution \( S \to S' \). Denote the exceptional divisor of the resolution by \( \hat{E} \). By [Fuj82, 6.11] for some \( n > 0 \) the base locus of \( |n(K_\mathcal{S} + D + \hat{E})^+| \)
is empty and the linear system gives a $\mathbb{P}^1$-ruled of $S$ which restricts to a $C^*$-ruled of $S_0$ (cf. also [Miy01, 2.6.1]). Consider another $C^*$-ruled of $S_0$. Modifying $S$ if necessary we can assume that it extends to a $\mathbb{P}^1$-ruled of $S$. Let $f'$ be a general fiber of this extension. Then $f' \cdot (K_S + D + \tilde{E}) = f' \cdot (K_S + D + \tilde{E})^+ + f' \cdot (K_S + D + \tilde{E})^- = 0$. However, $(K_S + D + \tilde{E})^+$ is effective and $(K_S + D + \tilde{E})^-$ is numerically effective, so $f' \cdot (K_S + D + \tilde{E})^+ = f' \cdot (K_S + D + \tilde{E})^- = 0$, hence the rulings are the same.

(3), (4) First we need to understand how to find all twisted $C^*$-ruled of a given $S'$. Consider a twisted $C^*$-ruled of $S'$ and let $(\hat{V}, \hat{D}, \hat{p})$ be a minimal completion of this ruling. By the $\hat{p}$-minimality of $D$, $D_h$ is the only component of $\hat{D}$ which can be a non-branching $(-1)$-curve, so there is a connected modification $(\hat{V}, \hat{D}) \to (V, D)$ with snc-minimal $D$. Let $\hat{D}_0 \subseteq \hat{D}$ be the $(-1)$-curve of the fiber at infinity (cf. 6.3). Note that $D$ is not a chain, otherwise $S'$ is affine-ruled. Let $D_0 \subseteq D$ be the image of $D_0$ and let $T$ be the connected component of $D - D_0$ containing the image of the horizontal component (which is a point if the modification is nontrivial). In this way a twisted $C^*$-ruled of $S'$ determines a pair $(D_0, T)$ (with $D_0 + T$ contained in a boundary of some snc-minimal completion), such that $\beta_D(D_0) = 3$, $D_0^2 \geq -1$, $T$ is a connected component of $D - D_0$ containing the image of the horizontal section and both connected components of $D - D_0 - T$ are $(-2)$-curves. Conversely, if we have an snc-minimal normal completion $(V, D)$ and a pair as above, we make a connected modification $(\hat{V}, \hat{D}) \to (V, D)$ over $D$ by blowing successively on the intersection of the total transform of $T$ with the proper transform of $D_0$ until $D_0$ becomes a $(-1)$-curve. The $(-1)$-curve together with the transform of $D - T - D_0$ induce a $\mathbb{P}^1$-ruled of $V'$ and constitute the fiber at infinity for this ruling (cf. 2.8). The restriction to $S'$ is a twisted $C^*$-ruled.

Suppose $\pi(S_0) = -\infty$. Since $S_0$ is not affine-ruled, $S' \cong C^2/G$ for a finite noncyclic small subgroup $G < GL(2, \mathbb{C})$. Let $(V, D)$ be an snc-minimal normal completion of $S'$ and let $\pi: S \to V$ be a minimal resolution with exceptional divisor $\tilde{E}$. We saw in the proof of 1.8 that $S_0$ admits a Platonic $C^*$-ruled, which extends to a $\mathbb{P}^1$-ruled of $S$. Moreover, $D$ and $\tilde{E}$ are forks for which $D_h$ and $\tilde{E}_h$ are the unique branching components of $D$ and $E$ respectively. In particular, the $C^*$-ruled does not extend to a ruling of $S'$ and as non-branching components of $D$ have negative self-intersections, $(\pi(S), D + \tilde{E})$ is a unique snc-minimal smooth completion of $S_0$ (and hence $(V, D)$ is a unique snc-minimal normal completion of $S'$). It follows from the proof of 1.4 that the non-extendable $C^*$-ruled of $S_0$ is unique. Suppose there is a $C^*$-ruled of $S_0$ which does extend to $S'$. Since $\tilde{E}$ is not a chain, it follows from the proof of 1.8 that this ruling is twisted. Since maximal twigs of $\tilde{E}$ and $D$ are adjacent chains of columnar fibers, we see that a maximal twig of $D - D_h$ is a $(-2)$-curve if and only if the respective maximal twig of $\tilde{E} - \tilde{E}_h$ is a $(-2)$-curve. Moreover, we have $0 < d(\tilde{E})$, so $\tilde{E}_h^2 \leq -2$ and since $\tilde{E}_h^2 + D_h^2 = -3$ (cf. 5.3), we have $D_h^2 \geq -1$. Therefore, $S'$ admits a twisted $C^*$-ruled if and only if $\tilde{E}$ is a fork of type $(2, 2, k)$ for some $k \geq 2$. If $k = 2$ then the choice of $(D_0, T)$ as above is unique and if $k = 2$ then there are three such choices. Note that if $(V', D', p)$ is a minimal completion of such a ruling then $D'$ is a fork, so since $\kappa_0 < 0$, we have $n = 0$ and $F_0$ is of type $(A)(iv)$ (cf. the proof of 1.8). This gives (3).

We can now assume that $\pi(S_0) = 0$, $S'$ is logarithmic and not exceptional. By 5.3(iii) $S_0$ is $C^*$-ruled and each $C^*$-ruled of $S_0$ extends to a $C^*$-ruled of $S$. Let $r \in \{1, 2, \ldots \} \cup \{\infty\}$ be the number of all different (up to automorphism of the base) $C^*$-ruleds of $S'$ and let $(V_i, D_i, p_i)$ for $i \leq r$ be their minimal completions. Minimality implies that non-branching $(-1)$-curves in $D_i$ are $p_i$-horizontal. We add consequently an upper index $(i)$ to objects defined previously for any $C^*$-ruled when we refer to the ruling $p_i$. If $p_i$ is untwisted we denote the horizontal components of $D_h^{(i)}$ by $H^{(i)}$, $H^{(i)}$.

Suppose $p_{\min}$ is untwisted with base $\mathbb{P}^1$. Then $F^{(i)}_0$ contains a rivet and by 5.9 $n^{(i)} = 2$, so $D_1$ does not contain non-branching $b$-curves with $b \geq -1$. Then $(V_1, D_1)$ is balanced and $S'$ does not admit an untwisted $C^*$-ruled with base $C^1$, as it does not contain non-branching $0$-curves (cf. 3.3). By 3.3 each component of $D_h^{(1)}$ has $\beta_{D_1} = 3$ and intersects two $(-2)$-tips of $D_1$. Note that $\zeta^{(1)}$ (cf. 3.4) touches $D_h^{(1)}$ two times if both components of $D_h^{(1)}$ intersect the same horizontal
component of $F_0^{(1)}$ and three times if not. By \[3.4\] and by the properties of Hirzebruch surfaces we get $-3 \leq (D_h^{(1)})^2 \leq -2$. In particular, one of the components of $D_h^{(1)}$, say $H^{(1)}$, has $(H^{(1)})^2 \geq -1$, so by the discussion about twisted $\mathbb{C}^*$-rulings above $H^{(1)}$ together with two $(−2)$-tips of $D_1$ gives rise to a twisted $\mathbb{C}^*$-ruling $p_2$ of $S'$. Since $H^{(1)}$ together with two $(−2)$-tips of $D_1$ intersecting it are contained in a fiber of $p_2$, $(H^{(1)})^2 \leq -2$. Thus $p_2$ is the only twisted ruling of $S'$, because $H^{(1)}$ is the only possible choice for a middle component of the fiber at infinity of a twisted ruling. Suppose $r \geq 3$. Then $p_3$ is untwisted with base $\mathbb{P}^1$. Since $D_1$ does not contain non-branching 0-curves, any flow in $D_1$ is trivial, so $V_3 = V_1$. Since $p_3$ and $p_1$ are different after restriction to $S'$, the $S_0$-components $C^{(1)}$, $C^{(3)}$ contained respectively in $F_0^{(1)}$, $F_0^{(3)}$ are different. As they both intersect $\overline{E}$, they are contained in the same fiber of $p_2$, a contradiction with $\Sigma^{(2)} = 0$. Note that since $D$ contains no non-branching 0-curves, $D$ is not of type (iii). Since $n^{(1)} = 2$, $D$ contains at least seven components, so $D$ is not of type (i) or (ii).

We can now assume that each untwisted $\mathbb{C}^*$-ruling of $S'$ has base $\mathbb{C}^1$. Suppose $p_1$ is such a ruling. By \[3.3\] both horizontal components of $D_1$ have $\beta_{D_1} = 3$ and one of them, say $H^{(1)}$, intersects two $(−2)$-tips $T$ and $T'$ of $D_1$. In particular, $D_1$ is snc-minimal. Since $F^{(1)}_\infty = \{0\}$, changing $V_1$ by a flow if necessary we can assume that $H^{(1)}$ is a $(−1)$-curve. Then $F^{(2)}_\infty = T + 2H^{(1)} + T'$ induces a $\mathbb{P}^1$-ruling $p_2 : V_1 \to \mathbb{P}^1$, which is a twisted $\mathbb{C}^*$-ruling after restricting it to $S'$. Suppose $r \geq 3$. If $p_3$ is untwisted then its base is $\mathbb{C}^1$ and changing $V_3$ by a flow if necessary we can assume that $V_3 = V_1$. But then $F^{(1)}_\infty = F^{(3)}_\infty$, because $D_1$ contains only one non-branching 0-curve, so $p_1$ and $p_3$ have a common fiber and hence cannot be different after restriction to $S'$, a contradiction. Thus $p_3$ is twisted. By the discussion above $p_3$ can be recovered from a pair $(D_0, T)$ on some snc-minimal completion of $S'$. All such completions of $S'$ differ from $(V_1, D_1)$ by a flow, which is an identity on $V_1 = F^{(1)}_\infty$, hence the birational transform of $D_0$ on $V_1$ is either $H^{(1)}$ or $H^{(1)}$. Since the restrictions of $p_1$ and $p_2$ to $S'$ are different, it is $H^{(1)}$. It follows that $r = 3$ and $D_1 = H^{(1)}$ has two $(−2)$-tips as connected components, hence the dual graph of $D_1$ is as in (iii). Conversely, if $S'$ has a boundary as in (iii) then besides the untwisted $\mathbb{C}^*$-ruling induced by the 0-curve it has also two twisted rulings, each with one of the branching components as the middle component of the fiber at infinity.

We can finally assume that all $\mathbb{C}^*$-rulings of $S'$ are twisted. Let $(V, D)$ be a balanced completion of $S'$. Since $S'$ does not admit untwisted $\mathbb{C}^*$-rulings, $D$ does not contain non-branching 0-curves, so $(V, D)$ is a unique snc-minimal completion of $S'$. Thus to find all twisted $\mathbb{C}^*$-rulings of $S'$ we need to determine all pairs $(D_0, T)$, such that $D_0 + T \subseteq D$, $D_0^2 \geq -1$, $\beta_T(D_0) = 3$ and $D = T - D_0$ consists of two $(−2)$-tips. Let $(D_0, T)$ and $(D_0', T')$ be two such pairs. Suppose $D_0 \neq D_0'$ and, say, $D_0^2 \geq D_0'^2$. We have $D_0 \cdot D_0' \neq 0$, otherwise the chain $D - T'$, which is not negative definite, would be contained (and not equal, since $\nu \leq 1$) in a fiber of the twisted ruling associated with $(D_0, T)$, which is impossible. Then $D$ has six components and we check that $d(D) = 16((D_0^2 + 1)(D_0'^2 + 1) - 1)$, so $(D_0^2 + 1)(D_0'^2 + 1) \leq 0$, because $d(D) < 0$. Then $D_0^2 = -1$ and $D_0'$ is a 2-section of the twisted ruling associated with $(D_0, T)$. Since $\beta_T(D_0') = 3$, by \[3.3\] for this ruling $n = 1$, $D_0'$ is a $(−1)$-curve and $D$ has dual graph as in (ii). Conversely, it is easy to see that $S'$ with such a boundary has two twisted $\mathbb{C}^*$-rulings. Therefore, we can assume that the choice of $D_0$ for a pair $(D_0, T)$ as above is unique. Let $p_1$ be a twisted $\mathbb{C}^*$-ruling associated with some pair $(D_0, T)$. Suppose $n^{(1)} = 0$. By \[3.4\] $\zeta \cdot D_h^{(1)}$ is a 0-curve, so $F = \zeta^* \cdot D_h^{(1)}$ induces a $\mathbb{P}^1$-ruling $p$ of $V$. If $\zeta$ touches $D_h^{(1)}$ then $F$ contains the $S_0$-component of $F_0^{(1)}$, so $F \not\subseteq D$ and $p$ restricts to an untwisted $\mathbb{C}^*$-ruling of $S'$ with base $\mathbb{P}^1$. If $\zeta$ does not touch $D_h^{(1)}$ then $p$ restricts to a $\mathbb{C}^*$-ruling of $S'$ with base $\mathbb{C}^1$. This contradicts the assumption. By \[3.9\] we get that $n^{(1)} = 1$, $F_0^{(1)}$ contains no $D_1$-components and $\mu_1 = 2$. In particular, $D_1 = D$. Moreover, as $n^{(1)} = 1$, by \[3.4\] $(D_h^{(1)})^2 \leq -1$, so $D$ has a dual graph as in (i) or (ii). Conversely, if $D$ is of type (i) or (ii) then $r = 2$ if $k = -1$ and $r = 1$ if $k \leq -2$. □

The theorem \[5.12\] has interesting consequences. Namely, it is known that $\mathbb{Q}$-homology planes (may be smooth) with smooth locus of general type do not contain topologically contractible curves. In fact the number $\ell \in \mathbb{N} \cup \{\infty\}$ of contractible curves on a $\mathbb{Q}$-homology plane $S'$ is known except
two cases, when $S'$ is non-logarithmic or when $S'$ is singular and $\pi(S_0) = 0$ (cf. Pal10, 10.1 and references there). Clearly, in the first case $\ell = \infty$ by 5.8. The following theorem is the missing piece of information.

**Corollary 6.13.** If a singular $\mathbb{Q}$-homology plane has smooth locus of Kodaira dimension zero then it contains one or two contractible curves in case the smooth locus admits a $\mathbb{C}^*$-ruling and does not contain contractible curves if not.

**Proof.** We can assume that $S'$ is logarithmic. Suppose $S'$ contains a topologically contractible curve $L$. Since $S'$ is rational, $\text{Pic} \, S_0 = \text{Coker}(\text{Pic}(D + \hat{E}) \to \text{Pic} \, S)$ is torsion, so the class of $L$ in $\text{Pic} \, S_0$ is torsion. Then there exists a morphism $f : S_0 - L \to \mathbb{C}^*$ and taking its Stein factorization we get a $\mathbb{C}^*$-ruling of $S_0 - L$, which (as $\pi(S_0) \neq -\infty$) extends to a $\mathbb{C}^*$-ruling of $S_0$. Since $S_0$ is logarithmic, each $\mathbb{C}^*$-ruling of $S_0$ extends in turn to a $\mathbb{C}^*$-ruling of $S'$ (cf. 5.4). Therefore $L$ is vertical for some $\mathbb{C}^*$-ruling of $S'$, hence exceptional $\mathbb{Q}$-homology planes do not contain contractible curves. It follows from 5.9 that if the ruling is twisted or untwisted with base $\mathbb{P}^1$ then the vertical contractible curve is unique and is contained in the unique singular non-columnar fiber. For an untwisted ruling with base $\mathbb{C}^1$ there are at most two such curves. In particular, in cases (4)(i) and (ii) of the theorem 5.12 $L$ needs to intersect the horizontal component of the boundary, so we get respectively $\ell = 1$ and $\ell = 2$. In case (4)(iii) the unique vertical contractible curves for the twisted rulings $p_1$ and $p_3$ are distinct and do not intersect the horizontal components of respective rulings, hence are both vertical for the untwisted ruling $p_3$, so $\ell = 2$. In the remaining case (4)(iv) we have $r = 2$, $p_1$ is twisted and $p_2$ is untwisted. We can assume that the base of $p_2$ is $\mathbb{C}^1$ and the unique non-columnar singular fiber contains two contractible curves, $L_1$ and $L_2$, otherwise $\ell \leq 2$ from the above remarks and we are done. Since the twisted ruling is unique, there is exactly one horizontal component $H$ of $D^{(2)}$ which meets two $(-2)$-tips of $D^{(1)}$ (together with these tips it induces the twisted ruling). Clearly, only one $L_i$ can intersect $H$, so the second one is vertical for $p_1$ and we get $\ell \leq 2$ is this case as well.

### 6.6. $S'$ of negative Kodaira dimension.

As another corollary from 6.8 we give below a detailed description of singular $\mathbb{Q}$-homology planes of negative Kodaira dimension. We assume that $\pi(S_0) \neq 2$, but as we show in [PK10] a singular $\mathbb{Q}$-homology plane of negative Kodaira dimension cannot have smooth locus of general type, so the following classification is in fact complete.

**Theorem 6.14.** Let $S'$ be a singular $\mathbb{Q}$-homology plane of negative Kodaira dimension and let $S_0$ be its smooth locus. If $\pi(S_0) \neq 2$ then exactly one of the following holds:

(i) $\pi(S_0) = -\infty$, $S'$ is affine-ruled or isomorphic to $\mathbb{C}^2/G$ for a small finite non-cyclic subgroup $G < GL(2, \mathbb{C})$,

(ii) $\pi(S_0) \in \{0, 1\}$, $S'$ is non-logarithmic and is isomorphic to a quotient of an affine cone over a smooth projective curve by an action of a finite group acting freely off the vertex of the cone,

(iii) $\pi(S_0) \in \{0, 1\}$, $S'$ has an untwisted $\mathbb{C}^*$-ruling with base $\mathbb{C}^1$ and two singular fibers, one of them consists of two $\mathbb{C}^1$'s meeting in a cyclic singular point, after taking a resolution and completion the respective completed singular fiber is of type $(B)/(i)$ with $\mu, \bar{\mu} \geq 2$ (see Fig. 4, cf. 6.8).

![Figure 4](image-url)

**Figure 4.** Untwisted $\mathbb{C}^*$-ruling, $\pi(S') = -\infty$
Proof. Since $\pi(S') \neq 0$, $S_0$ is either $\mathbb{C}$- or $\mathbb{C}^*$-ruled by 3.3. By 3.8 and by the results of section 1 we can assume that $S'$ has a $\mathbb{C}^*$-ruling and $\pi(S_0) \geq 0$. Let $(V, D, p)$ be a minimal completion of this ruling. We use 3.8. If $p$ is twisted then $0 > \kappa_0 \geq \lambda - \frac{1}{2} \geq \frac{2}{n-1}$, so $n = \lambda = 0$. The inequalities $\kappa < 0$ and $\kappa_0 \geq 0$ can be satisfied only in case (A) and then $D^2 = 0$ by 3.7, so $D_\lambda$ induces an untwisted $\mathbb{C}^*$-ruling of $S'$. Suppose $p$ is untwisted. Since $\kappa \neq \kappa_0$, $p$ has base $\mathbb{C}$ and is of type (B)(i). Since $0 > \kappa = \lambda - 1 \geq \frac{2}{n-1} - 1$, we get $n \leq 1$, but for $n = 0$ we get $\kappa_0 < \lambda < 0$, so in fact $n = 1$. Then $0 \leq \kappa_0 = 1 - \frac{1}{\mu_1} - \frac{1}{\min(\mu_1, \bar{\mu})}$, hence $\min(\mu_1, \bar{\mu}) \geq 2$. \qed

By 3.1 $H_1(S', \mathbb{Z})$ vanishes for $i > 1$. If $S'$ is of type $\mathbb{C}^2/G$ or of type (ii) then it is contractible, $H_1(S', \mathbb{Z})$ for affine-ruled $S'$ was computed in 4.4. For completeness we now compute the fundamental group of $S'$ of type (iii), which by 3.1(vi) is the same as $\pi_1(S)$. Let $E_0$ be a component of $\tilde{E}$ intersecting $C$. Contract $C$ and successive vertical $(-1)$-curves until $C$ is the only $(-1)$-curve in the fiber ($C$ cannot become a 0-curve, because it does not intersect $D_h$), denote this contraction by $\theta$. Let $\theta'$ be the contraction of $\theta, F_0$ and $F_1$ to smooth fibers. Put $U = S_0 \setminus (C \cup \tilde{C})$ and let $\gamma_1, \gamma, t \in \pi_1(U)$ be the vanishing loops of the images of $F_1$, $F_0$ under $\theta' \circ \theta$ and of some component of $D_h$ (cf. [Fuj82, 4.17]). We need to compute the kernel of the epimorphism $\pi_1(U) \to \pi_1(S)$. Since $\theta$ does not touch $C$, $\theta, E_0 \neq 0$ and $\theta, F_0$ is columnar. Using 7.17 loc. cit. one can show by induction on the number of components of a columnar fiber that since $E_0 \cdot C \neq 0$, the vanishing loops of $E_0$ and $C$, which are of type $\gamma a^t b$ and $\gamma c^t d$, satisfy $ad - bc = \pm 1$. Thus $\gamma$ and $t$ are in the kernel, hence $\pi_1(S) = (\gamma_1 : \gamma^t) \cong \mathbb{Z}_{\mu_1}$. In particular, $S'$ is not a $\mathbb{Z}$-homology plane.

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Karol Palka: Institute of Mathematics, University of Warsaw, ul. Banacha 2, 02-097 Warsaw, Poland

Institute of Mathematics, Polish Academy of Sciences, ul. Śniadeckich 8, 00-956 Warsaw, Poland

E-mail address: palka@impan.pl