SEMIPROJECTIVITY WITH AND WITHOUT A GROUP ACTION

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Abstract. The equivariant version of semiprojectivity was recently introduced by the first author. We study properties of this notion, in particular its relation to ordinary semiprojectivity of the crossed product and of the algebra itself.

We show that equivariant semiprojectivity is preserved when the action is restricted to a cocompact subgroup. Thus, if a second countable compact group acts semiprojectively on a $C^*$-algebra $A$, then $A$ must be semiprojective. This fails for noncompact groups: we construct a semiprojective action of $\mathbb{Z}$ on a nonsemiprojective $C^*$-algebra.

We also study equivariant projectivity and obtain analogous results, however with fewer restrictions on the subgroup. For example, if a discrete group acts projectively on a $C^*$-algebra $A$, then $A$ must be projective. This is in contrast to the semiprojective case.

We show that the crossed product by a semiprojective action of a finite group on a unital $C^*$-algebra is a semiprojective $C^*$-algebra. We give examples to show that this does not generalize to all compact groups.

Equivariant semiprojectivity was introduced in [Phi12], by applying the usual definition of semiprojectivity to the category of unital $G$-algebras ($C^*$-algebras with actions of the group $G$) with unital $G$-equivariant $^*$-homomorphisms. See Definition 1.1 below. The purpose of [Phi12] was to show that certain actions of compact groups on various specific $C^*$-algebras are semiprojective. In particular, it is shown that any action of a second countable compact group on a finite dimensional $C^*$-algebra is semiprojective, and that for $n < \infty$, quasifree actions of second countable compact groups on the Cuntz algebras $\mathcal{O}_n$ are semiprojective.

In this paper we study equivariant semiprojectivity more abstractly. We also introduce equivariant projectivity and carry out a parallel study of it. We extend the definition to allow actions by general locally compact groups, and we consider the nonunital version of equivariant semiprojectivity.

From the work in [Phi12], it is not even clear whether a semiprojective action of a noncompact group can exist. One reason for skepticism was that the trivial...
action of $\mathbb{Z}$ on $\mathbb{C}$ is not semiprojective, as was shown by Blackadar ([Bla12]). We give a wide reaching generalization of this result in Corollary 6.5 by showing that if the trivial action of a group on a (nonzero) $C^*$-algebra is semiprojective then the group must be compact.

There are, however, many nontrivial semiprojective (and even projective) actions of noncompact groups. Indeed, given a countable discrete group $G$ and a semiprojective $C^*$-algebra $A$, we show in Proposition 2.4 that the free Bernoulli shift action of $G$ on the free product $\ast_{g \in G} A$ is equivariantly semiprojective.

Our main motivation was to understand how equivariant semiprojectivity (with group action) is related to semiprojectivity (without group action). The following question naturally occurs:

**Question 0.1.** Assume that $(G, A, \alpha)$ is an equivariantly semiprojective $G$-algebra (Definition 1.1 below). Is $A$ semiprojective in the usual sense?

We give a positive answer in Corollary 3.12 under the assumption that $G$ is compact. If we drop this assumption, then the answer to the question may be negative. Indeed, in Example 3.13 we construct a semiprojective action of $\mathbb{Z}$ on a nonsemiprojective $C^*$-algebra.

**Question 0.1** is a special case of a more natural question:

**Question 0.2.** Assume that $(G, A, \alpha)$ is a $G$-algebra that is equivariantly semiprojective (equivariantly projective), and let $H \leq G$ be a closed subgroup. Is the restricted $H$-algebra $(H, A, \alpha|_H)$ equivariantly semiprojective (equivariantly projective)?

The two main results of this paper answer this question positively under certain natural assumptions on the factor space $G/H$. In the semiprojective case, we get a positive answer (Theorem 3.11) if $H$ is cocompact, that is, if $G/H$ is compact. In the projective case, we get a positive answer (Theorem 4.23) if $H$ is compact or cocompact, or if $G$ is a [SIN]-group (meaning that the left and right uniformities on $G$ agree) and $H$ is arbitrary. These conditions are much less restrictive than in the semiprojective case.

This paper is organized as follows. In Section 1 we give the definition of equivariant semiprojectivity (Definition 1.1). We also introduce equivariant projectivity (Definition 1.2), and we investigate the relation between the unital and nonunital versions of these definitions (Lemma 1.5, Lemma 1.6, and Proposition 1.7).

In Section 2 we introduce free Bernoulli shifts, and we show in Proposition 2.4 that the free Bernoulli shifts constructed from semiprojective $C^*$-algebras provide examples of semiprojective actions of countable discrete groups. We also study the orthogonal Bernoulli shift of $G$ on $\bigoplus_G A = C_0(G, A)$ for (semi)projective $C^*$-algebras $A$. It turns out to be a much harder problem to determine when this action is semiprojective, and we give a positive answer, for a semiprojective $C^*$-algebra $A$, only for finite cyclic groups of order $2^n$. See Proposition 2.10.

In Section 3 we study the semiprojective case of Question 0.2. We give a positive answer (Theorem 3.11) when $G/H$ is compact. It follows (Corollary 3.12) that a second countable compact group can only act semiprojectively on a $C^*$-algebra that is semiprojective in the usual sense. We show that this is not true in general, by constructing in Example 3.13 a semiprojective action of $\mathbb{Z}$ on a nonsemiprojective $C^*$-algebra. The main ingredient in this section is the induction functor...
which assigns to each $H$-algebra an induced $G$-algebra. We show that this functor is exact (Proposition 3.7) and continuous (Proposition 3.8).

In Section 4 we study Question 0.2 in the projective case. The main result is Theorem 4.23 which gives a positive answer in considerable generality. In particular, restriction to compact subgroups preserves equivariant projectivity. This shows that every equivariantly projective $C^*$-algebra is (nonequivariantly) projective (Corollary 4.24), which is in contrast to the semiprojective case (Example 3.13).

The main technique of this section is an induction functor which uses uniformly continuous functions; see Definition 4.16. In Theorem 4.18 we show that this functor is exact if $H$ is compact or if $G$ is a [SIN]-group. To prove this, we need conditions under which uniformly continuous functions to a quotient $C^*$-algebra can be lifted to uniformly continuous functions, and in Theorem 4.8 we provide a satisfying answer that might also be of independent interest.

In Section 5, we study semiprojectivity of crossed products. In Theorem 5.1, we show that for a discrete group $G$ whose group $C^*$-algebra is semiprojective, unital semiprojectivity of an action $\alpha: G \to \text{Aut}(A)$ on a unital $C^*$-algebra implies semiprojectivity of the crossed product $A \rtimes_\alpha G$. Example 5.2 shows that this can fail when the group is compact but not finite. At the end of Section 5, we give counterexamples to several other plausible relations between equivariant semiprojectivity for finite groups and semiprojectivity, and state further open problems.

In Section 6, we study semiprojectivity of fixed point algebras. We show that for a saturated semiprojective action of a finite group $G$ on a unital $C^*$-algebra $A$, the fixed point algebra $A^G$ is semiprojective (Proposition 6.2). We show in Example 6.1 that this does not generalize to compact groups. For a semiprojective action of a noncompact group, we show in Theorem 6.4 that the fixed point algebra is trivial. Thus, the trivial action of a noncompact group on a nonzero $C^*$-algebra is never semiprojective. We therefore obtain a precise characterization of when the trivial action of a group is (semi)projective (Corollary 6.5).

We use the following terminology and notation in this paper. By a topological group we understand a group $G$ together with a Hausdorff topology such that the map $(s,t) \mapsto s \cdot t^{-1}$ is jointly continuous. We mainly consider locally compact topological groups. For such a group, we denote its Haar measure by $\mu$. By the Birkhoff-Kakutani theorem (see Theorem 1.22 of [MZ55]), $G$ is metrizable if and only if it is first countable. Moreover, in that case, the metric $d$ may be chosen to be left invariant, that is, $d(rs, rt) = d(s,t)$ for all $r, s, t \in G$. We will always take our metrics to be left invariant. We usually require $G$ to be second countable.

For a topological group $G$, by a $G$-algebra we understand a triple $(G, A, \alpha)$ in which $A$ is a $C^*$-algebra and $\alpha: G \to \text{Aut}(A)$ is a continuous action of $G$ on $A$. Continuity means that for each $a \in A$ the map $s \mapsto \alpha_s(a)$ is continuous. (Such an action is also called strongly continuous.)

By a $G$-morphism between two $G$-algebras $(G, A, \alpha)$ and $(G, B, \beta)$ we mean a $G$-equivariant $*$-homomorphism, that is, a $*$-homomorphism $\varphi: A \to B$ such that $\beta_s \circ \varphi = \varphi \circ \alpha_s$ for each $s \in G$. We say that a $G$-algebra $(G, A, \alpha)$ is separable if $A$ is a separable $C^*$-algebra and $G$ is second countable (hence also metrizable).

Given a $G$-algebra $(G, A, \alpha)$, we denote by $A^G$ its fixed point algebra

$$A^G = \{ a \in A : \alpha_s(a) = a \text{ for all } s \in G \}$$

(even when $G$ is not compact), and by $A \rtimes_\alpha G$ the (maximal) crossed product of $(G, A, \alpha)$. 

If $\mathcal{A}$ is a $\mathcal{C}^*$-algebra, we denote by $\mathcal{A}^+$ its unitization (adding a new identity even if $\mathcal{A}$ already has an identity). We let $\tilde{\mathcal{A}}$ be $\mathcal{A}^+$ when $\mathcal{A}$ is not unital and be $\mathcal{A}$ when $\mathcal{A}$ is unital. If $\mathcal{A}$ is a $G$-algebra, then $\mathcal{A}^+$ and $\tilde{\mathcal{A}}$ are both $G$-algebras in an obvious way.

Subalgebras of $\mathcal{C}^*$-algebras are always assumed to be $\mathcal{C}^*$-subalgebras, and ideals are always closed and two sided.

We use the convention $\mathbb{N} = \{1, 2, \ldots\}$.

1. Equivariant Semiprojectivity and Equivariant Projectivity

In this section we recall the definition of equivariant semiprojectivity. We also give a nonunital version, and we will see in Lemmas 1.5 and 1.6 and Proposition 1.7 how the two variants are related. We also introduce equivariant projectivity.

The unital case of the following definition is Definition 1.1 of [Phi12].

**Definition 1.1.** A separable $G$-algebra $(G, \mathcal{A}, \alpha)$ is called *equivariantly semiprojective* if whenever $(G, \mathcal{C}, \gamma)$ is a $G$-algebra, $J_1 \subset J_2 \subset \cdots$ is an increasing sequence of $G$-invariant ideals in $\mathcal{C}$, $J = \bigcup_{n=1}^\infty J_n$, $\pi_n : \mathcal{C}/J_n \to \mathcal{C}/J$ is the quotient $^*$-homomorphism for $n \in \mathbb{N}$, and $\varphi : \mathcal{A} \to \mathcal{C}/J$ is a $G$-morphism, then there exist $n \in \mathbb{N}$ and a $G$-morphism $\psi : \mathcal{A} \to \mathcal{C}/J_n$ such that $\pi_n \circ \psi = \varphi$.

When no confusion can arise, we say that $\mathcal{A}$ is equivariantly semiprojective, or that $\alpha$ is semiprojective.

We say that a separable unital $G$-algebra $(G, \mathcal{A}, \alpha)$ is *equivariantly semiprojective in the unital category* if the same condition holds, but under the additional assumption that $\mathcal{C}$ and $\varphi$ are unital, and the additional requirement that one can choose $\psi$ to be unital.

The lifting problem of the definition means that in the right diagram that appears below [Definition 1.2] the solid arrows are given, and $n$ and $\psi$ are supposed to exist which make the diagram commute.

**Definition 1.2.** A $(G, \mathcal{A}, \alpha)$ is called *equivariantly projective* if whenever $(G, \mathcal{C}, \gamma)$ is a $G$-algebra, $J$ is a $G$-invariant ideal in $\mathcal{C}$ with quotient $^*$-homomorphism $\pi : \mathcal{C} \to \mathcal{C}/J$, and $\varphi : \mathcal{A} \to \mathcal{C}/J$ is a $G$-morphism, then there exists a $G$-morphism $\psi : \mathcal{A} \to \mathcal{C}$ such that $\pi \circ \psi = \varphi$.

When no confusion can arise, we say that $\mathcal{A}$ is equivariantly projective, or that $\alpha$ is projective.

We say that a unital $G$-algebra is *equivariantly projective in the unital category* if the same condition holds, but under the additional assumption that $\mathcal{C}$ and $\varphi$ are unital, and the additional requirement that one can choose $\psi$ to be unital.
The lifting problem of the definition means that the left diagram on the right can be completed. Again, the solid arrows are given, and $\psi$ is supposed to exist which makes the diagram commute.

When working with semiprojectivity and projectivity, it is often convenient, in the notation of Definition 1.1 and Definition 1.2, to require that the map $\varphi$ be an isomorphism.

This can also be done in the equivariant case. The proof follows that of Proposition 2.2 of [Bla04]. We give the proof since [Bla04] is a survey article and its proof omits some details.

Lemma 1.3. Let $A$ be a $C^*$-algebra, let $B \subset A$ be a $C^*$-subalgebra, and let $I_1 \subset I_2 \subset \cdots \subset A$ be ideals. Then $B \cap \bigcup_{n=1}^{\infty} I_n = \bigcup_{n=1}^{\infty} (B \cap I_n)$.

Proof. We have $\bigcup_{n=1}^{\infty} (B \cap I_n) \subset B \cap \bigcup_{n=1}^{\infty} I_n$, so $\bigcup_{n=1}^{\infty} (B \cap I_n) \subset B \cap \bigcup_{n=1}^{\infty} I_n$.

For the reverse, let $b \in B$ and suppose that $b \notin \bigcup_{n=1}^{\infty} (B \cap I_n)$. Let $\rho$ be the norm of the image of $b$ in $B/\bigcup_{n=1}^{\infty} (B \cap I_n)$. For $n \in \mathbb{N}$, let $\kappa_n: B \to B/\bigcup_{n=1}^{\infty} (B \cap I_n)$ be the quotient maps. Then $\|\kappa_n(b)\| \geq \rho$ for all $n \in \mathbb{N}$. The inclusion $\iota: B \to A$ induces injective $^*$-homomorphisms $\iota_n: B/\bigcup_{n=1}^{\infty} (B \cap I_n) \to A/I_n$ such that $\pi_n \circ \iota = \iota_n \circ \kappa_n$. Since $\iota_n$ is isometric, we have

$$\|\pi_n (\iota_n \circ \kappa_n)(b)\| = \|\kappa_n(b)\| \geq \rho,$$

whence $\operatorname{dist}(b, I_n) \geq \rho$. This is true for all $n \in \mathbb{N}$, so $b \notin \bigcup_{n=1}^{\infty} I_n$. \qed

Proposition 1.4. Let $(G, A, \alpha)$ be a $G$-algebra (separable for the statements involving semiprojectivity). As for usual (semi)projectivity, the definitions of equivariant semiprojectivity [Definition 1.1] and of equivariant projectivity [Definition 1.2] for $(G, A, \alpha)$, in both the unital and nonunital categories, are unchanged if, in the notation of these definitions, we require one or both of the following:

1. $\varphi$ is injective.
2. $\varphi$ is surjective.

Proof. We give the proof for equivariant semiprojectivity in the unital category. The other cases are similar but slightly simpler.

Throughout, let the notation be as in [Definition 1.1].

We first prove the result for the restriction (1). So assume that $C, J_1 \subset J_2 \subset \cdots \subset C, J$, quotient maps $\pi_n: C/J_n \to C/J$, and $\varphi: A \to C/J$, all as in [Definition 1.1] are given. The following diagram shows the algebras and maps to
be constructed:

\[
\begin{array}{c}
A \oplus C \xrightarrow{\rho} C \\
A \oplus C/J \xrightarrow{\rho_n} C/J_n \\
A \xrightarrow{\mu} A \oplus C/J \xrightarrow{\rho} C/J.
\end{array}
\]

 Equip \(A \oplus C\), \(A \oplus C/J\), and \(A \oplus C/J_n\) for \(n \in \mathbb{N}\), and \(A \oplus C/J\) with the direct sum actions of \(G\). Let \(\rho: A \oplus C \to C\), \(\rho_n: A \oplus C/J_n \to C/J_n\) for \(n \in \mathbb{N}\), and \(\rho_\infty: A \oplus C/J \to C/J\) be the projections on the second summand. Define \(\mu: A \to A \oplus C/J\) by \(\mu(a) = (a, \varphi(a))\) for \(a \in A\). Then \(\mu\) is a unital injective \(G\)-morphism such that \(\rho_\infty \circ \mu = \varphi\). By hypothesis, there are \(n \in \mathbb{N}\) and a unital \(G\)-morphism \(\nu: A \to A \oplus C/J_n\) such that \((\id_A \oplus \pi_n) \circ \nu = \mu\). Then the map \(\psi = \rho_n \circ \nu\) is a unital \(G\)-morphism such that \(\pi_n \circ \nu = \varphi\). This completes the proof of (1).

We now prove that the condition is equivalent when both restrictions (2) and (1) are applied. It follows that the condition is also equivalent when only (2) is applied. So let the notation be as before, and assume in addition that \(\varphi\) is injective. The following diagram shows the algebras and maps to be constructed:

\[
\begin{array}{c}
D \xrightarrow{\rho} C \\
D/I_n \xrightarrow{\rho_n} C/J_n \\
A \xrightarrow{\mu} D/I \xrightarrow{\rho_\infty} C/J.
\end{array}
\]

Let \(\pi: C \to C/J\) be the quotient map. Set \(D = \pi^{-1}(\varphi(A))\) and \(I = D \cap J\). For \(n \in \mathbb{N}\) set \(I_n = D \cap J_n\) and let \(\kappa_n: D/I_n \to D/I\) be the quotient map. Then \(\bigcup_{n=1}^\infty I_n = I\) by \textbf{Lemma 1.3}.

Let \(\rho: D \to C\) be the inclusion. Then \(\rho\) drops to a \(\ast\)-homomorphism \(\rho_n: D/I_n \to C/J_n\) for every \(n \in \mathbb{N}\), and to a \(\ast\)-homomorphism \(\rho_\infty: D/I \to C/J\). All these maps are injective. Clearly the range of \(\varphi\) is contained in \(\rho_\infty(D/I)\), so there is a \(\ast\)-homomorphism \(\mu: A \to D/I\) such that \(\rho_\infty \circ \mu = \varphi\). This \(\ast\)-homomorphism is injective because \(\varphi\) is and surjective by the definition of \(D\). The hypothesis implies that there are \(n \in \mathbb{N}\) and \(\nu: A \to D/I_n\) such that \(\kappa_n \circ \nu = \mu\). Then the map \(\psi = \rho_n \circ \nu\) satisfies \(\pi_n \circ \nu = \varphi\).

It is a standard result in the theory of semiprojectivity (contained in Lemma 14.1.6 and Theorem 14.1.7 of \textbf{Lor97b}) that for a nonunital \(C\ast\)-algebra \(A\) the following are equivalent:

1. \(A\) is semiprojective.
2. \(\tilde{A}\) is semiprojective in the unital category.
(3) \( \tilde{A} \) is semiprojective.

In the equivariant case, the equivalence of all three conditions holds when the group \( G \) is compact. The proof of the analog of the implications from (1) to (2) and from (2) to (3) breaks down when the trivial action of \( G \) on \( C \) is not semiprojective in the nonunital category, but the remaining implications hold in general. The trivial action on \( C \) is always semiprojective in the unital category, but we will show in Corollary 6.5 that it is semiprojective in the nonunital category only if \( G \) is compact.

Lemma 1.5. Let \((G, A, \alpha)\) be a separable \( G \)-algebra, with \( A \) nonunital. Then \( A \) is equivariantly (semi)projective if and only if \( A \) is equivariantly (semi)projective in the unital category.

Proof. We give the proof for equivariant semiprojectivity. The proof for equivariant projectivity is similar but easier. We use the notation of Definition 1.1.

Since \( A \) is nonunital, we have \( \hat{A} = A^+ \).

First assume that \( A \) is equivariantly semiprojective, and that \( C \) and \( \varphi : A^+ \to C/J \) are unital. By equivariant semiprojectivity of \( A \), there are \( n \in \mathbb{N} \) and \( \psi_0 : A \to C/J_n \) such that \( \pi_n \circ \psi_0 = \varphi|A \). Then the formula \( \psi(a + \lambda \cdot 1_{A^+}) = \psi_0(a) + \lambda \cdot 1_{C/J_n} \), for \( a \in A \) and \( \lambda \in \mathbb{C} \), defines a \( G \)-morphism \( \psi : A^+ \to C/J \) such that \( \pi_n \circ \psi = \varphi \).

We have shown that \( A \) is equivariantly semiprojective in the unital category.

Now assume that \( A^+ \) is equivariantly semiprojective in the unital category, and in the notation of Definition 1.1 take \( C \) and \( \varphi : A \to C/J \) to be not necessarily unital. We have obvious isomorphisms \( C^+/J_n \cong (C/J_n)^+ \) for \( n \in \mathbb{N} \) and \( C^+/J \cong (C/J)^+ \). (We add a new unit even if \( C \) is already unital.) Let \( \nu_n : C^+/J_n \to C \) for \( n \in \mathbb{N} \), and \( \nu_\infty : C^+/J \to C \), be the maps associated with the unitizations. Define a unital \( G \)-morphism \( \varphi^+ : A^+ \to C^+/J \) by \( \varphi^+(a + \lambda \cdot 1_{A^+}) = \varphi(a) + \lambda \cdot 1_{C^+/J} \), for \( a \in A \) and \( \lambda \in \mathbb{C} \). For \( n \in \mathbb{N} \), similarly define \( \pi_n^+ : C^+/J_n \to C^+/J \), giving \( \nu_\infty \circ \pi_n^+ = \nu_n \). By hypothesis, there are \( n \in \mathbb{N} \) and \( \psi_0 : A^+ \to C^+/J_n \) such that \( \pi_n^+ \circ \psi_0 = \varphi^+ \).

We claim that \( \psi_0(A) \subset C/J_n \). We have
\[
\nu_n \circ \psi_0 = \nu_\infty \circ \pi_n^+ \circ \psi_0 = \nu_\infty \circ \varphi^+,
\]
which vanishes on \( A \). The claim follows. So \( \psi = \psi_0|A : A \to C/J_n \) is a \( G \)-morphism such that \( \pi_n \circ \psi = \varphi \).

Lemma 1.6. Let \((G, A, \alpha)\) be a separable \( G \)-algebra, with \( A \) unital. If \( A \) is equivariantly semiprojective, then \( A \) is equivariantly semiprojective in the unital category. If \( G \) is compact, then the converse also holds.

Proof. The proof is essentially the same as that of Lemma 14.1.6 of [Lor97b]. In the first paragraph of the proof there, \( B_t \) should be \( C_t \) and it is \( 1 - \varphi_t(1) \), not \( \varphi_t(1) - 1 \), that is a projection. In the second paragraph of the proof there, we need the equivariant version of Lemma 14.1.5 of [Lor97b]; it follows from Corollary 1.9 of [Phi12].

For compact groups, we now obtain the analog of the equivalence of the first two parts in Theorem 14.1.7 of [Lor97b].

Proposition 1.7. Let \( G \) be a second countable compact group, and let \( A \) be a separable \( G \)-algebra. Then \( A \) is equivariantly semiprojective if and only if \( \hat{A} \) is.

Proof. Combine Lemma 1.5 and Lemma 1.6.
The paper [Phi12] contains many examples of equivariantly semiprojective $C^*$-algebras. In particular, it is shown that for a semiprojective $C^*$-algebra $A$ and a second countable compact group $G$, the trivial action of $G$ on $A$ is semiprojective (Corollary 1.9 of [Phi12]). In the same way one may prove the analog for the projective case, and we include the short argument for completeness. The following lemma is an immediate consequence of Lemma 1.6 of [Phi12].

**Lemma 1.8.** Let $G$ be a compact group and let $\pi: A \to B$ be a surjective $G$-morphism of $G$-algebras. Then the restriction of $\pi$ to the fixed point algebras is surjective, that is, $\pi(A^G) = B^G$.

**Lemma 1.9.** Let $G$ be a second countable compact group, let $A$ be a projective $C^*$-algebra, and let $\iota: G \to \text{Aut}(A)$ be the trivial action. Then $(G, A, \iota)$ is equivariantly projective.

**Proof.** By [Proposition 1.4] it is enough to show that any surjective $G$-morphism $\pi: C \to A$ has an equivariant right inverse $\psi$. Since $G$ acts trivially on $A$, we have $A^G = A$, and then $\pi(C^G) = A$ by Lemma 1.8. We can now use projectivity of $A$ to get a $\gamma$-homomorphism $\gamma: A \to C^G$ such that $\pi \circ \gamma = \text{id}$. Let $\psi$ be the composition of $\gamma$ with the inclusion of $C^G$ in $C$.

**Remark 1.10.** The statement of Lemma 1.8 can fail if $G$ is not compact. In fact, for every (second countable) noncompact group $G$, we construct in the proof of Theorem 6.3 a surjective $G$-morphism $\pi: A \to B$ such that $\pi(A^G) \neq B^G$.

So far, we have only seen semiprojective actions of compact groups. In the next section we will show that every countable discrete group even admits projective actions.

## 2. Free and orthogonal Bernoulli shifts

In this section, we introduce for every countable discrete group $G$ and $C^*$-algebra $A$ a natural action, called the free Bernoulli shift, of $G$ on the full free product $\ast_{g \in G} A$. We show [Proposition 2.4] that this action is (semi)projective if $A$ is (nonequivariantly) semiprojective.

We also investigate the orthogonal Bernoulli shift of $G$ on $\bigoplus_{g \in G} A$, that is, the translation action of $G$ on $C_0(G, A)$. It seems to be much more difficult to determine when this action is (semi)projective. In [Proposition 2.10] we give a positive answer for the special case that $G$ is finite cyclic of order $2^n$.

We use the following notation, roughly as before Remark 3.1.2 of [Lor97], for the universal $C^*$-algebra on countably many contractions.

**Notation 2.1.** Set

$$\mathcal{F}_\infty = C^*\langle z_1, z_2, \ldots | \|z_j\| \leq 1 \text{ for } j \in \mathbb{N} \rangle,$$

the universal $C^*$-algebra on generators $z_1, z_2, \ldots$ with relations $\|z_j\| \leq 1$ for $j \in \mathbb{N}$.

Let $G$ be a countable discrete group. Set $P_G = \ast_{g \in G} \mathcal{F}_\infty$. For $s \in G$ let $\iota_s: \mathcal{F}_\infty \to \mathcal{F}_\infty$ be the map which sends $\mathcal{F}_\infty$ to the copy of $\mathcal{F}_\infty$ in $P_G$ indexed by $s$. We identify $P_G$ with

$$C^*\langle \{z_{s,k}: s \in G \text{ and } k \in \mathbb{N} \} | \|z_{s,k}\| \leq 1 \text{ for } s \in G \text{ and } k \in \mathbb{N} \rangle,$$

in such a way that $\iota_s(z_{k}) = z_{s,k}$ for $s \in G$ and $k \in \mathbb{N}$.
**Lemma 2.2.** Let $A$ be a $C^*$-algebra and let $G$ be a discrete group. For $s \in G$ let $\iota_{A,s}: A \to *_{g \in G} A$ be the map which sends $A$ to the copy of $A$ in $*_{g \in G} A$ indexed by $s$. Then there exists a unique action $\tau^A: G \to \text{Aut}(*_{g \in G} A)$ such that $\tau^A_g(\iota_{A,s}(a)) = \iota_{A,gs}(a)$ for all $g,s \in G$ and $a \in A$. For every $*$-homomorphism $\varphi: A \to B$ between $C^*$-algebras $A$ and $B$, the corresponding $*$-homomorphism $*_{g \in G} \varphi: *_{g \in G} A \to *_{g \in G} B$ is equivariant. Moreover, for every $G$-algebra $(G,C,\gamma)$ and every $*$-homomorphism $\varphi: A \to C$, there is a unique $G$-morphism $\psi: *_{g \in G} A \to C$ such that $\psi \circ \iota_{A,s} = \gamma_s \circ \varphi$ for all $s \in G$.

**Proof.** This is immediate. $\square$

**Definition 2.3.** Let $A$ be a separable $C^*$-algebra and let $G$ be a countable discrete group. The action $\tau^A$ of Lemma 2.2 is called the free Bernoulli shift based on $A$. If $A = F_\infty$, so that $*_{g \in G} A = P_G$, we call it the universal free Bernoulli shift, and denote it by $\tau$.

Following Notation 2.1, we have $\tau_s(z_{t,k}) = z_{st,k}$ for $s,t \in G$ and $k \in \mathbb{N}$.

Any separable $C^*$-algebra $A$ is a quotient of $F_\infty$. This is just the fact that $A$ contains a countable set of contractive generators. Similarly, the action $\tau^A: G \to \text{Aut}(P_G)$ is universal for all $G$-actions, that is, for every separable $G$-algebra $A$ there exists a surjective $G$-morphism $P_G \to A$.

**Proposition 2.4.** Let $A$ be a separable $C^*$-algebra and let $G$ be a countable discrete group. If $A$ is (semi)projective, then the free Bernoulli shift of $G$ based on $A$ is (semi)projective.

**Proof.** We give the proof when $A$ is projective. The semiprojective case is very similar, but has bigger diagrams.

Let $(G,B,\beta)$ and $(G,D,\delta)$ be $G$-algebras, let $\pi: B \to D$ be a surjective $G$-homomorphism, and let $\varphi: *_{g \in G} A \to D$ be a $G$-morphism. We find a $G$-morphism $\psi: *_{g \in G} A \to B$ such that $\pi \circ \psi = \varphi$.

Since $A$ is projective, there is a $*$-homomorphism $\psi_0: A \to B$ such that $\pi \circ \psi_0 = \varphi \circ \iota_A$. By universality of $*_{g \in G} A$ (Lemma 2.2), there is a $*$-homomorphism $\psi: *_{g \in G} A \to B$ such that $\pi \circ \psi = \psi_0$ for all $s \in G$. The following diagram shows some of the maps:

$$
\begin{array}{ccc}
A & \xrightarrow{\psi} & *_{g \in G} A \\
\downarrow{\iota_A} & & \downarrow{\psi} \\
B & \xrightarrow{\pi} & D
\end{array}
$$

It remains to show that $\pi \circ \psi = \varphi$ and that $\psi$ is $G$-equivariant.

Let $t \in G$. Using equivariance of $\pi$ at the second step and equivariance of $\varphi$ at the fourth step, we get

$$
\pi \circ \psi \circ \iota_{A,t} = \pi \circ \beta_t \circ \psi_0 = \delta_t \circ \pi \circ \psi_0 = \delta_t \circ \varphi \circ \iota_1 = \varphi \circ \tau_t \circ \iota_1 = \varphi \circ \iota_{A,t}.
$$

Since this is true for all $t \in G$, and since $\bigcup_{t \in G} \iota_{A,t}(A)$ generates $*_{g \in G} A$, it follows that $\pi \circ \psi = \varphi$.

To see that $\psi$ is equivariant, let $s,t \in G$. We compute:

$$
\beta_s \circ \psi \circ \iota_{A,t} = \beta_s \circ \beta_t \circ \psi_0 = \beta_s \circ \psi_0 = \psi \circ \iota_{A,st} = \psi \circ \tau_s \circ \iota_{A,t}.
$$

For the same reason as in the previous paragraph, it follows that $\beta_s \circ \psi = \psi \circ \tau_s$, and so $\psi$ is $G$-equivariant. $\square$
Remark 2.5. Let $G$ be countable discrete group. The universal $G$-algebra $P_G$ is (nonequivariantly) projective, since it is isomorphic to $F_\infty$. We can use this to show that if $\alpha: G \to \mathrm{Aut}(A)$ is a projective action, then $A$ must be projective. This is a special case of Corollary 4.24.

Using the universal property of $P_G$ and separability of $A$, we can find a surjective $G$-morphism $\rho: P_G \to A$. Since $\alpha$ is equivariantly projective, we can find a $G$-morphism $\lambda: A \to P_G$ such that $\rho \circ \lambda = \mathrm{id}_A$. If now $\varphi: A \to C/J$ is a *-homomorphism, there is a *-homomorphism $\psi: P_G \to C$ which lifts $\varphi \circ \rho$. Then $\psi \circ \lambda$ lifts $\varphi$.

A more involved argument, which we do not give here, gives a similar result for finite groups and semiprojectivity.

We now turn to what we call the orthogonal Bernoulli shift.

Definition 2.6. Let $A$ be a separable $C^*$-algebra and let $G$ be a countable discrete group. The orthogonal Bernoulli shift based on $A$ is the action $\sigma^A: G \to \mathrm{Aut}(C_0(G, A))$ given by $\sigma^A_t(a)(t) = a(s^{-1}t)$ for $a \in C_0(G, A)$ and $s, t \in G$.

We think of $C_0(G, A)$ as $\bigoplus_{g \in G} A$. Then the automorphism $\sigma^A_t$ sends the summand indexed by $t \in G$ to the summand indexed by $st$.

By analogy with equivariant semiprojectivity of actions of compact groups on finite dimensional $C^*$-algebras (Theorem 2.6 of [Phi12]) and projectivity of $C_0((0,1])$, it seems reasonable to hope that the orthogonal Bernoulli shift based on $C_0((0,1])$ is projective whenever $G$ is finite. This seems difficult to prove; we have been able to do so only for $G = \mathbb{Z}_{2^n}$, the finite cyclic group of order $2^n$. (See Proposition 2.10.)

We start with some lemmas.

Lemma 2.7. Let $n \in \mathbb{N}$. Let $B$ be a $C^*$-algebra, let $\beta \in \mathrm{Aut}(B)$ satisfy $\beta^{2^n} = \mathrm{id}_B$, let $I$ be a $\beta$-invariant ideal in $B$, let $\pi: B \to B/I$ be the quotient map, and let $\alpha \in \mathrm{Aut}(B/I)$ be the induced automorphism. If $x \in B/I$ is selfadjoint and satisfies $\alpha(x) = -x$, then there is a selfadjoint element $y \in B$ such that $\beta(y) = -y$ and $\pi(y) = x$.

Proof. First lift $x$ to a selfadjoint element $b \in B$. Put

$$y = \frac{1}{2^n} \left[ b - \beta(b) + \beta^2(b) - \beta^3(b) + \cdots - \beta^{2^n-1}(b) \right].$$

Then $y$ is selfadjoint, $\beta(y) = -y$, and $y$ is a lift of $x$. \hfill \Box

Lemma 2.8. Let $n \in \mathbb{N}$. Let $B$ be a $C^*$-algebra, let $\beta \in \mathrm{Aut}(B)$ satisfy $\beta^{2^n} = \mathrm{id}_B$, let $I$ be a $\beta$-invariant ideal in $B$, let $\pi: B \to B/I$ be the quotient map, and let $\alpha \in \mathrm{Aut}(B/I)$ be the induced automorphism. Let $h_1, h_2 \in B/I$ be positive orthogonal elements such that $\alpha(h_1) = h_2$ and $\alpha(h_2) = h_1$. Then there exist positive orthogonal elements $k_1, k_2 \in B$ such that

$$\pi(k_1) = h_1, \quad \pi(k_2) = h_2, \quad \beta(k_1) = k_2, \quad \text{and} \quad \beta(k_2) = k_1.$$

Proof. Put $x = h_1 - h_2$. Since $x$ is selfadjoint and $\alpha(x) = -x$, we can, by Lemma 2.7, lift it to a selfadjoint element $y \in B$ with $\beta(y) = -y$. Let $k_1$ be the positive part of $y$, that is, $k_1 = \frac{1}{2}(y + |y|)$. Then $\pi(k_1) = \frac{1}{2}(x + |x|) = h_1$. Put $k_2 = \beta(k_1)$. Routine calculations show that $k_2$ is the negative part of $y$, and thus orthogonal to $k_1$. Essentially the same calculations show that $\beta(k_2) = k_1$. It is clear that $\pi(k_2) = h_2$. \hfill \Box
Proposition 2.9. Let $n \in \mathbb{N}$. Let $B$ be a C*-algebra, let $\beta \in \text{Aut}(B)$ satisfy $\beta^{2^n} = \text{id}_B$, let $I$ be a $\beta$-invariant ideal in $B$, let $\pi : B \rightarrow B/I$ be the quotient map, and let $\alpha \in \text{Aut}(B/I)$ be the induced automorphism. Let $h_1, h_2, \ldots, h_{2^n} \in B/I$ be orthogonal positive elements such that $\alpha(h_m) = h_{m+1}$ for $m = 1, 2, \ldots, 2^n - 1$, and such that $\alpha(h_{2^n}) = h_1$. Then they can be lifted to orthogonal positive elements $k_m \in B$ for $m = 1, 2, \ldots, 2^n$ such that $\beta(k_m) = k_{m+1}$ for $m = 1, 2, \ldots, 2^n - 1$ and such that $\beta(k_{2^n}) = k_1$. Moreover, if $\|h_m\| \leq 1$ for $m = 1, 2, \ldots, 2^n$, then we can require that $\|k_m\| \leq 1$ for $m = 1, 2, \ldots, 2^n$.

Proof. The proof (except for the last statement) is by induction on $n$. The case $n = 1$ is Lemma 2.8. Let $n > 1$, suppose that we have shown the statement to hold for all natural numbers $l < n$ and all choices of $B, \beta, I,$ and $h_1, h_2, \ldots, h_{2^l}$, and let $B, \beta, I,$ and $h_1, h_2, \ldots, h_{2^n}$ be as in the statement.

Set $a_1 = h_1 + h_3 + \cdots + h_{2^n-1}$ and $a_2 = h_2 + h_4 + \cdots + h_{2^n}$.

Then $\alpha(a_1) = a_2, \alpha(a_2) = a_1,$ and $a_1a_2 = 0$. So, by Lemma 2.8, we can lift $a_1$ and $a_2$ to orthogonal positive elements $b_1, b_2$ such that $\beta(b_1) = b_2$ and $\beta(b_2) = b_1$.

The hereditary subalgebra $b_1Bb_1 \subset B$ is $\beta^2$-invariant and it is easy to check that $\pi(b_1Bb_1) = \alpha_1(B/I)\alpha_1$. Apply the induction hypothesis with $b_1Bb_1$ in place of $B$, with $\beta^2$ in place of $\beta$, with $b_1Bb_1 \cap I$ in place of $I$, and with $h_1, h_3, \ldots, h_{2^n-1}$ in place of $h_1, h_2, \ldots, h_{2^n}$. We obtain orthogonal positive elements $k_1, k_3, \ldots, k_{2^n-1} \in b_1Bb_1$ such that $\pi(k_m) = h_m$ and $\beta^2(k_m) = k_{m+2}$ for $m = 1, 3, 5, \ldots, 2^n - 1$, and such that $\beta^2(k_{2^n-1}) = k_1$. Set $k_m = \beta(k_{m-1}) \in b_2Bb_2$ for $m = 2, 4, 6, \ldots, 2^n$. Then $\pi(k_m) = h_m$ also for $m = 2, 4, 6, \ldots, 2^n$. It is clear that $\beta(k_m) = k_{m+1}$ for $m = 1, 2, \ldots, 2^n - 1$ and that $\beta(k_{2^n}) = k_1$. It only remains to check that the elements $k_m$ are orthogonal for $m = 1, 2, \ldots, 2^n$. The only case needing work is $k_l$ and $k_m$ when one of $l$ and $m$ is even and the other is odd. But then one of $k_l$ and $k_m$ is in $b_1Bb_1$ and the other is in $b_2Bb_2$, so the desired conclusion follows from $b_1b_2 = 0$.

It remains to prove the last statement. Let $x_1, x_2, \ldots, x_{2^n} \in B$ be the elements produced in the first part. Let $f : [0, \infty) \rightarrow [0, 1]$ be the function $f(t) = \min(t, 1)$ for $t \geq 0$. Then set $k_m = f(x_m)$ for $m = 1, 2, \ldots, 2^n$.  

\begin{flushright} \Box \end{flushright}

Proposition 2.10. Let $A$ be a separable (semi)projective C*-algebra, let $n \in \mathbb{N}$, and let $\sigma : \mathbb{Z} \rightarrow \text{Aut}(\bigoplus_{m=1}^n A)$ be the orthogonal Bernoulli shift of Definition 2.6.

If $n$ is a power of 2, then $\sigma$ is (semi)projective.

Proof. We give the proof when $A$ is projective. The semiprojective case is analogous, but requires Lemma 1.3.

By Proposition 1.4, it is enough to show that for every $G$-algebra $(G, B, \beta)$ and every surjective $G$-morphism $\pi : B \rightarrow \bigoplus_{m=1}^n A$, there exists a $G$-morphism $\psi : \bigoplus_{m=1}^n A \rightarrow B$ such that $\pi \circ \psi = \text{id}_A$.

Let $\alpha = \sigma_1$, the automorphism corresponding to the generator $1 \in \mathbb{Z}$, and similarly let $\gamma = \beta_1 \in \text{Aut}(B)$. For $m = 1, 2, \ldots, n$, let $t_m : A \rightarrow \bigoplus_{m=1}^n A$ be the map that sends $A$ to the summand in $\bigoplus_{m=1}^n A$ indexed by $m$, and let $t_m : A \rightarrow \bigoplus_{m=1}^n A \rightarrow A$ be the surjection onto the summand indexed by $m$. Then $a = \sum_{m=1}^n (t_m \circ \rho_m)(a)$ for every $a \in \bigoplus_{m=1}^n A$.

Let $h$ be a strictly positive element in $A$. For $m = 1, 2, \ldots, n$, set $h_m = t_m(h)$. Then $h_1, h_2, \ldots, h_n$ are orthogonal positive elements in $\bigoplus_{m=1}^n A$ such that $\alpha(h_m) = h_{m+1}$ for $m = 1, 2, \ldots, n - 1$, and such that $\alpha(h_n) = h_1$. By Proposition 2.9 they
can be lifted to orthogonal positive elements \( k_m \in B \) for \( m = 1, 2, \ldots, 2^n \) such that 
\[
\gamma(k_m) = k_{m+1} \quad \text{for} \quad m = 1, 2, \ldots, n - 1 \quad \text{and such that} \quad \gamma(k_n) = k_1.
\]

Let \( D = k_1 B k_1 \). Then \( \pi(D) = \iota_1(A) \). Since \( A \) is projective, there exists an 
*-homomorphism \( \psi_1 : A \to D \) such that \( \pi \circ \psi_1 = \iota_1 \). Define \( \psi : \bigoplus_{m=1}^n A \to B \) by
\[
\psi(a) = \sum_{m=1}^n \left( \gamma^{m-1} \circ \psi_1 \circ \rho_m \right)(a).
\]
It is easily checked that \( \psi \) has the desired properties. \( \square \)

**Question 2.11.** Consider the orthogonal Bernoulli shift \( G \to \text{Aut} \left( \bigoplus G C_0((0,1]) \right) \) of Definition 2.6. For which groups \( G \) obtain a positive result when \( H \) is second countable and compact group can only act semiprojectively on a 
\( C^* \)-algebra? Consider the orthogonal Bernoulli shift \( \psi \). It is easily checked that

In particular, is it projective for \( G = \mathbb{Z}_n \) for all \( n \in \mathbb{N} \)? Is it projective for \( G = \mathbb{Z} \)?

### 3. Equivariant Semiprojectivity of Restrictions to Subgroups

Let \( \alpha : G \to \text{Aut}(A) \) be a semiprojective action. In this section we investigate semiprojectivity of the restriction of \( \alpha \) to a subgroup \( H \leq G \). In Theorem 3.11 we obtain a positive result when \( H \) is cocompact. It follows (Corollary 3.12) that a second countable compact group can only act semiprojectively on a \( C^* \)-algebra that is semiprojective in the usual sense. Some condition on \( H \) is necessary. For instance, in Example 3.13 we construct a semiprojective action of \( \mathbb{Z} \) on a nonsemiprojective \( C^* \)-algebra.

To obtain these results, we use the induction functor, which assigns in a natural way to each \( H \)-algebra an induced \( G \)-algebra. We will show that this functor preserves exact sequences (Proposition 3.7) and behaves well with respect to direct limits (Proposition 3.8).

We begin by recalling the definition of the induction functor, from the beginning of Section 2 of [KW99] or the beginning of Section 6 of [Ech10]. In Definition 3.2 below, one easily checks that the action defined on the algebra \( \text{Ind}^G_H(A) \) is continuous, so that \( (G, \text{Ind}^G_H(A), \text{Ind}^G_H(\alpha)) \) is in fact a \( G \)-algebra, and that \( \text{Ind}^G_H \) really is a functor.

**Definition 3.1.** For a locally compact group \( G \), we let \( C_G \) denote the category whose objects are \( G \)-algebras and whose morphisms are \( G \)-equivariant *-homomorphisms (also called \( G \)-morphisms).

**Definition 3.2.** Let \( H \leq G \) be a closed subgroup, and let \( (H, A, \alpha) \) be an object in \( C_H \). We define an object \( (G, \text{Ind}^G_H(A), \text{Ind}^G_H(\alpha)) \) in \( C_G \) as follows. We take
\[
\text{Ind}^G_H(A) = \left\{ f \in C_0(G, A) : \begin{array}{l} \alpha_h(f(s h)) = f(s) \text{ for all } s \in G \text{ and } h \in H \\ \text{and } s H \mapsto \| f(s) \| \text{ is in } C_0(G/H) \end{array} \right\}.
\]
The induced action \( \text{Ind}^G_H(\alpha) : G \to \text{Aut} \left( \text{Ind}^G_H(A) \right) \) is given by
\[
(\text{Ind}^G_H(\alpha)_s)(f)(t) = f(s^{-1} t)
\]
for \( f \in \text{Ind}^G_H(A) \) and \( s, t \in G \). If \( A \) and \( B \) are \( H \)-algebras and \( \varphi : A \to B \) is an \( H \)-morphism, then the induced \( G \)-morphism \( \text{Ind}^G_H(\varphi) : \text{Ind}^G_H(A) \to \text{Ind}^G_H(B) \) is given by
\[
\text{Ind}^G_H(\varphi)(f)(s) = \varphi(f(s))
\]
for \( f \in \text{Ind}^G_H(A) \) and \( s \in G \).
The induction functor is often defined on a different category than that considered here. The objects are still $G$-algebras, but the morphisms are equivariant Hilbert bimodules. We refer to Section 6 of \[Ech10\] and to \[EKQR00\] for more details.

We next recall the definition of a $C_0(X)$-algebra. See Section 4.5 of \[Phi87\], Definition 1.5 of \[Kas88\], or Definition 2.6 of \[Bln96\]. We recall that if $A$ is a $C^*$-algebra, then $M(A)$ is its multiplier algebra and $Z(A)$ is its center.

**Definition 3.3.** Let $X$ be a locally compact Hausdorff space. A $C_0(X)$-algebra is a $C^*$-algebra $A$ together with a $^*$-homomorphism $\eta: C_0(X) \to Z(M(A))$, called the structure map, such that

$$\{\eta(f)a: f \in C_0(X) \text{ and } a \in A\}$$

is dense in $A$.

We will usually write $fa$ or $f \cdot a$ instead of $\eta(f)a$ for the product of a function $f \in C_0(X)$ and an element $a \in A$. For an open set $U \subset X$, we set

$$A(U) = \{fa: f \in C_0(U) \text{ and } a \in A\},$$

which is an ideal of $A$. (See [Proposition 3.4](#).) For a closed subset $Y \subset X$, we denote by $A(Y)$ the quotient $A(A(X) \setminus Y)$.

For $x \in X$ we write $A(x)$ for $A(\{x\})$, and this $C^*$-algebra is called the fiber of $A$ at $x$. Given $a \in A$, we denote its image in the fiber $A(x)$ by $a(x)$, and we define $\hat{a}: X \to [0, \infty)$ by $\hat{a}(x) = ||a(x)||$ for $x \in X$. We call $A$ a continuous $C_0(X)$-algebra if $\hat{a}$ is continuous for each $a \in A$.

If $A$ and $B$ are $C_0(X)$-algebras and $\varphi: A \to B$ is a $^*$-homomorphism, then $\varphi$ is said to be a $C_0(X)$-morphism if $\varphi(f \cdot a) = f \cdot \varphi(a)$ for all $f \in C_0(X)$ and $a \in A$.

We recall the following facts about $C_0(X)$-algebras.

**Proposition 3.4.** Let $X$ be a locally compact Hausdorff space and let $A$ be a $C_0(X)$-algebra with structure map $\eta: C_0(X) \to Z(M(A))$. Then:

1. $A = \{\eta(f)a: f \in C_0(X) \text{ and } a \in A\}$.
2. If $U \subset A$ is open then $A(U)$ is an ideal in $A$.
3. For $a \in A$, the function $\hat{a}$ is upper semicontinuous and vanishes at infinity.
4. For $a \in A$, we have $||a|| = \sup_{x \in X} \hat{a}(x)$.

**Proof.** Part 1 is Proposition 1.8 of \[Bln96\]. (This is essentially the Cohen Factorization Theorem.)

For 2, it follows from Corollary 1.9 of \[Bln96\] that $A(U)$ is a closed $C_0(X)$-submodule of $A$. It now easily follows that $A(U)$ is an ideal.

Part 3 is Proposition 1.2 of \[Rie89\].

Part 4 is Proposition 2.8 of \[Bln96\].

We refer to Section 2 of \[Bln96\] for more details on $C_0(X)$-algebras.

**Proposition 3.5.** Let $G$ be a locally compact group, let $H \leq G$ be a closed subgroup, and let $(H, A, \alpha)$ be an $H$-algebra. Define $\eta: C_0(G/H) \to Z(M(\text{Ind}_H^G(A)))$ by

$$(\eta(g)f)(s) = g(sH) \cdot f(s)$$

for $g \in C_0(G/H)$, $f \in \text{Ind}_H^G(A)$, and $s \in G$. This map makes $\text{Ind}_H^G(A)$ a continuous $C_0(G/H)$-algebra. Moreover:
(1) If $(H, B, \beta)$ is a second $H$-algebra, and $\varphi: A \to B$ is an $H$-morphism, then $\Ind_H^G(\varphi)$ is a morphism of $C_0(G/H)$-algebras.

(2) For every $x \in G$, the map $\text{ev}_x: \Ind_H^G(A) \to A$, which evaluates a function in $\Ind_H^G(A)$ at $x$, defines an isomorphism from $\Ind_H^G(A)(xH)$ to $A$.

In particular, the fibers of $\Ind_H^G(A)$ as a $C_0(G/H)$-algebra are all isomorphic to $A$. However, the isomorphism is not canonical. In the proof below, the isomorphism for the fiber at $xH \in G/H$ depends on the choice of the coset representative $x$.

**Proof of Proposition 3.3.** It is easy to check that $\eta$ makes $\Ind_H^G(A)$ a continuous $C_0(G/H)$-algebra, and we omit the details. The proof of (1) is immediate. It remains to prove (2). We abbreviate $\Ind_H^G(A)$ to $\text{Ind}(A)$.

Let $x \in G$. We show that $\text{ev}_x$ is surjective. It is immediate that

$$\ker(\text{ev}_x) = \text{Ind}(A)(G/H \setminus \{xH\}),$$

so this will complete the proof.

Since $\text{ev}_x$ is a $^*$-homomorphism, it is enough to show that it has dense image in $A$. So let $a \in A$ and let $\varepsilon > 0$. We want to find $f \in \text{Ind}_H^G(A)$ such that $\|f(x) - a\| < \varepsilon$.

Let $\mu$ denote the Haar measure of $H$. Since the action is continuous, there exists an open neighborhood $U \subset H$ of the identity element $1 \in H$, with compact closure, such that $\|\alpha_s(a) - a\| \leq \tfrac{\varepsilon}{2}$ for all $s \in U$. Let $\chi: G \to [0, \infty)$ be a nonzero continuous function with $\text{supp}(\chi) \subset U$. By scaling, we may assume $\int_H \chi\,d\mu = 1$. We define a function $f: G \to A$ by

$$f(s) = \int_H \chi(x^{-1}sht) \cdot \alpha_t(a)\,d\mu(t)$$

for $s \in G$. The integral exists for all $s$, since the integrand is continuous and has compact support. We will now check that $f$ has the desired properties.

For $s \in G$ and $h \in H$ we have, using left invariance of $\mu$ at the last step,

$$\alpha_h(f(sh)) = \alpha_h \left( \int_H \chi(x^{-1}sht) \cdot \alpha_t(a)\,d\mu(t) \right) = \int_H \chi(x^{-1}sht) \cdot \alpha_{ht}(a)\,d\mu(t) = f(s).$$

The function $sH \mapsto \|f(s)\|$ has compact support, so $f \in \text{Ind}(A)$. Moreover,

$$\|f(x) - a\| = \left\| \int_H \chi(t) \cdot \alpha_t(a)\,d\mu(t) - \int_H \chi(t) \cdot a\,d\mu(t) \right\|$$

$$\leq \int_H \chi(t)\|\alpha_t(a) - a\|\,d\mu(t) \leq \frac{\varepsilon}{2} < \varepsilon.$$

This completes the proof that $\text{ev}_x$ is surjective. \hfill \Box

The following result is similar to Lemma 3.2 of [TW12]. It is Lemma 2.1(iii) of [Dad09], but the proof given there assumes that $X$ is compact.

**Lemma 3.6.** Let $A$ be a $C_0(X)$-algebra with structure map $\eta: C_0(X) \to Z(M(A))$. Assume $B \subset A$ is a $C^*$-subalgebra satisfying the following two conditions:

(1) For each $x \in X$, the set $\{b(x): b \in B\}$ exhausts the fiber $A(x)$.

(2) $\eta(C_0(X))B \subset B$, that is, $B$ is invariant under multiplication by functions in $C_0(X)$.

Then $A = B$. 

Proof. It suffices to show that \( B \) is dense in \( A \). Let \( a \in A \) and let \( \varepsilon > 0 \). Using Proposition 3.4(1), choose \( f \in C_0(X) \) and \( a_0 \in A \) such that \( fa_0 = a \). Choose \( g \in C_c(X) \) such that \( \|f - g\| < \varepsilon/(2\|a_0\|) \). Then \( \|ga_0 - a\| < \frac{\varepsilon}{2} \) and \( (ga_0)(x) = 0 \) for \( x \in X \setminus \text{supp}(g) \).

For each point \( x \in \text{supp}(g) \), choose \( b_x \in B \) such that \( b_x(x) = (ga_0)(x) \). By Proposition 3.4(3), there is an open set \( U_x \subseteq X \) with \( x \in U_x \) such that for all \( y \in U_x \) we have \( \|b_x(y) - (ga_0)(y)\| < \frac{\varepsilon}{2} \). Choose \( x_1, x_2, \ldots, x_n \in \text{supp}(g) \) such that the sets \( U_{x_1}, U_{x_2}, \ldots, U_{x_n} \) cover \( \text{supp}(g) \). Choose \( h_1, h_2, \ldots, h_n \in C_c(X) \) such that for \( k = 1, 2, \ldots, n \) we have \( \text{supp}(h_k) \subseteq U_{x_k} \) and \( 0 \leq h_k \leq 1 \), and such that \( \sum_{k=1}^n h_k \leq 1 \) and is equal to 1 on \( \text{supp}(g) \). Set \( b = \sum_{k=1}^n h_k b_{x_k} \). Then \( b \in B \). We claim that \( \|b - ga_0\| \leq \frac{\varepsilon}{2} \). This will imply that \( \|b - a\| < \varepsilon \), and complete the proof.

It suffices to show that \( \|b(y) - (ga_0)(y)\| \leq \frac{\varepsilon}{2} \) for \( y \in X \). Set \( b_0 = 1 - \sum_{k=1}^n h_k \). Then \( ga_0 = \sum_{k=0}^n h_k g a_0 \). Set \( b_k = b_{x_k} \) and \( U_k = U_{x_k} \) for \( k = 1, 2, \ldots, n \), and set \( b_0 = 0 \) and \( U_0 = X \setminus \text{supp}(g) \). Then for \( k = 0, 1, \ldots, n \), we have \( \|b_k(y) - (ga_0)(y)\| < \frac{\varepsilon}{2} \) whenever \( h_k(y) \neq 0 \). Using this fact at the second step, we have

\[
\|b(y) - (ga_0)(y)\| \leq \sum_{k=0}^n h_k(y) \|b_k(y) - (ga_0)(y)\| \leq \sum_{k=0}^n h_k(y) \cdot \frac{\varepsilon}{2} \leq \frac{\varepsilon}{2}.
\]

This proves the claim, and completes the proof.

The following result is Lemma 3.8 of [KW99], but the proof given in [KW99] does not address surjectivity of \( \text{Ind}^G_H(\pi) \).

**Proposition 3.7.** Let \( G \) be a locally compact group, and let \( H \subseteq G \) be a closed subgroup. Then the induction functor \( \text{Ind}^G_H : C_H \rightarrow C_G \) is exact, that is, given an \( H \)-equivariant short exact sequence of \( H \)-algebras

\[
0 \rightarrow I \rightarrow A \overset{\pi}{\rightarrow} B \rightarrow 0,
\]

the induced \( G \)-equivariant sequence of \( G \)-algebras

\[
0 \rightarrow \text{Ind}^G_H(I) \overset{\text{Ind}^G_H(\iota)}{\rightarrow} \text{Ind}^G_H(A) \overset{\text{Ind}^G_H(\pi)}{\rightarrow} \text{Ind}^G_H(B) \rightarrow 0
\]

is also exact.

**Proof.** To simplify the notation, we abbreviate \( \text{Ind}^G_H \) to \( \text{Ind} \).

We may think of \( I \) as an \( H \)-invariant ideal in \( A \), so that \( \iota \) is just the inclusion. It follows that \( \text{Ind}(I) \) may be considered as an ideal in \( \text{Ind}(A) \), and then \( \text{Ind}(\iota) \) is also just the inclusion morphism.

It is straightforward to check that the sequence is exact in the middle, that is, \( \ker(\text{Ind}(\pi)) = \text{Ind}(I) \subseteq \text{Ind}(A) \). Thus, it remains to check that \( \text{Ind}(\pi) \) is surjective. Following Proposition 3.5, we consider \( \text{Ind}(A) \) and \( \text{Ind}(B) \) as \( C_0(G/H) \)-algebras. We want to apply Lemma 3.6.

Condition 2 of Lemma 3.6 follows immediately from Proposition 3.5(1).

Let us verify condition 1. For \( x \in G \), let \( \text{ev}^A_x : \text{Ind}(A) \rightarrow A \) and \( \text{ev}^B_x : \text{Ind}(B) \rightarrow B \) be the evaluation maps at \( x \). By Proposition 3.5(2), these maps are surjective and implement the isomorphisms \( \text{Ind}(A)(xH) \cong A \) and \( \text{Ind}(B)(xH) \cong B \). We have
\[ \text{ev}^B \circ \text{Ind}(\pi) = \pi \circ \text{ev}^A, \] that is, the following diagram commutes:

\[
\begin{array}{ccc}
\text{Ind}(A) & \xrightarrow{\text{ev}^A} & A \\
\downarrow \text{Ind}(\pi) & & \downarrow \pi \\
\text{Ind}(B) & \xrightarrow{\text{ev}^B} & B.
\end{array}
\]

Since \( \text{ev}^A \) and \( \pi \) are surjective, it follows that the image of \( \text{Ind}(\pi) \) exhausts each fiber of \( \text{Ind}(B) \). This verifies condition (1) of Lemma 3.6. So \( \text{Ind}(\pi) \) is surjective. \( \square \)

**Proposition 3.8.** Let \( G \) be a locally compact group, and let \( H \leq G \) be a closed subgroup. Then the induction functor \( \text{Ind}^G_H : C_H \to C_G \) is continuous, that is, given an \( H \)-equivariant direct system

\[ A_1 \to A_2 \to A_3 \to \cdots, \]

there is a natural isomorphism

\[ \text{Ind}^G_H(\lim \to A_k) \cong \lim \to \text{Ind}^G_H(A_k). \]

**Proof.** To simplify the notation, we abbreviate \( \text{Ind}^G_H \) to \( \text{Ind} \). As explained in Proposition 3.5 we consider the induced algebras as \( C_0(G/H) \)-algebras.

Denote the connecting \( H \)-morphisms by \( \phi^m_n : A_m \to A_n \) for \( m \leq n \). Let \( A = \lim \to A_k \), and denote the \( H \)-morphisms to the direct limit by \( \phi^\infty : A_m \to A \). Denote the induced \( G \)-morphisms by \( \theta^m_n : \text{Ind}(A_m) \to \text{Ind}(A_n) \), and let \( B = \lim \to \text{Ind}(A_k) \), together with \( G \)-morphisms \( \theta^\infty : \text{Ind}(A_m) \to B \).

The maps \( \phi^\infty_k \) induce \( G \)-morphisms \( \text{Ind}(\phi^\infty_k) : \text{Ind}(A_k) \to \text{Ind}(A) \), and these induce a \( G \)-morphism \( \psi \) from the direct limit \( B \) to \( \text{Ind}(A) \). The situation is shown in the following commutative diagram:

\[
\begin{array}{cccccccc}
\text{Ind}(A_1) & \xrightarrow{\theta^1_2} & \text{Ind}(A_2) & \to & \cdots & \to & \lim \to \text{Ind}(A_k) & = B \\
\downarrow \text{Ind}(\phi^\infty_1) & & & & & & \downarrow \psi \\
\text{Ind}(\phi^\infty) & & & & & & \text{Ind}(A).
\end{array}
\]

To show that \( \psi \) is surjective, we apply Lemma 3.6.

To verify condition (2) of Lemma 3.6 let \( b \in B \) and \( f \in C_0(G/H) \) be given. We will show that for every \( \varepsilon > 0 \) there exists \( c \in B \) such that \( ||f \cdot \psi(b) - \psi(c)|| < \varepsilon \). Fix \( \varepsilon > 0 \). By properties of the direct limit, there exist \( k \in \mathbb{N} \) and \( a \in \text{Ind}(A_k) \) such that \( ||b - \theta^\infty_k(a)|| < \varepsilon/||f|| \). One checks that \( c = \theta^\infty_k(f \cdot a) \) has the desired properties.

To verify condition (1) of Lemma 3.6 we need to show that every fiber of \( \text{Ind}(A) \) is exhausted by the image of \( \psi \). We denote by \( \text{ev}^k_x : \text{Ind}(A_k) \to A_k \) and \( \text{ev}^\infty : \text{Ind}(A) \to A \) the evaluation maps at \( x \in G \). Then it is enough to show that \( \text{ev}^\infty_x \circ \psi \) is surjective for every \( x \in G \).

For each \( k \in \mathbb{N} \), we have

\[ \text{ev}^\infty_x \circ \psi \circ \theta^\infty_k = \text{ev}^\infty_x \circ \text{Ind}(\phi^\infty_k) = \phi^\infty_k \circ \text{ev}^k_x. \]

Since \( \text{ev}^k_x : \text{Ind}(A_k) \to A_k \) is surjective (by Proposition 3.5(2)), the image of \( \text{ev}^\infty_x \circ \psi \) contains the image of \( \phi^\infty_k \). Thus, the image of \( \text{ev}^\infty_x \circ \psi \) contains \( \bigcup_{k=1}^\infty \text{ran}(\phi^\infty_k) \), which is dense in \( A \) by properties of the direct limit. It follows that the image of
\(\psi\) exhausts each fiber of \(\text{Ind}(A)\). We have verified the conditions of Lemma 3.6 so we have shown that \(\psi\) is surjective.

To show that \(\psi\) is injective, let \(b \in B\), and suppose that \(\psi(b) = 0\). Let \(\varepsilon > 0\); we show that \(\|b\| < \varepsilon\). By properties of \(B\) as a direct limit, there exist \(k \in \mathbb{N}\) and \(a \in \text{Ind}(A_k)\) such that \(\|b - \theta^\infty_k(a)\| < \frac{\varepsilon}{3}\). For \(n \geq k\), let \(f_n \in C_0(G/H)\) be defined by \(f_n(sH) = \|\theta^\infty_k(a)(sH)\|\). One checks that \((f_n)_{n \in \mathbb{N}}\) is a nonincreasing sequence of functions such that \(\lim_{n \to \infty} f_n(sH) < \frac{\varepsilon}{3}\) for each \(s \in G\). For \(n \in \mathbb{N}\), define a continuous function \(g_n\) on the one point compactification \((G/H)^+\) by \(g_n(sH) = \max(f_n(sH), \frac{\varepsilon}{n})\) for \(s \in G\) and \(g_n(\infty) = \frac{\varepsilon}{n}\). The functions \(g_n\) decrease pointwise to the constant function with value \(\frac{\varepsilon}{n}\). Since \((G/H)^+\) is compact, Dini’s Theorem (Proposition 11 in Chapter 9 of [Roy88]) implies that the convergence is uniform. So there exists \(n \geq k\) such that \(\|f_n\| < \frac{\varepsilon}{2n}\). Then \(\|\theta^\infty_k(a)\| = \|f_n\| < \frac{\varepsilon}{2n}\) by Proposition 3.4[4], and thus also \(\|\theta^\infty_k(a)\| < \frac{2\varepsilon}{n}\). It follows that \(\|b\| < \varepsilon\), as desired.

This completes the proof that \(\psi\) is an isomorphism. \(\square\)

**Lemma 3.9.** Let \(G\) be a locally compact group, and let \(H \leq G\) be a closed subgroup. For any \(H\)-algebra \(A\), let \(\text{ev}_1^A : \text{Ind}_H^G(A) \to A\) denote the action on \(A\). Let \(\gamma = \text{Ind}_H^G(\alpha)\) be the induced action of \(G\) on \(\text{Ind}_H^G(A)\). For \(f \in \text{Ind}_H^G(A)\) and \(h \in H\), we have, using the definition of \(\text{Ind}_H^G(A)\) at the third step,

\[
\text{ev}_1^A(\gamma h(f)) = (\gamma h(f))(1) = f(h^{-1}) = \alpha h(f(1)) = \alpha h(\text{ev}_1^A(f)),
\]

as desired. \(\square\)

**Lemma 3.10.** Let \(G\) be a locally compact group, and let \(H \leq G\) be a closed subgroup such that \(G/H\) is compact. Let \((G, A, \alpha)\) be a \(G\)-algebra, and let \((H, B, \beta)\) be an \(H\)-algebra. Let \(\varphi : A \to B\) be an \(H\)-morphism. Then there is a \(G\)-morphism \(\eta : A \to \text{Ind}_H^G(B)\) such that \(\eta(a)(s) = \varphi(\alpha_s^{-1}(a))\) for all \(a \in A\) and \(s \in G\).

**Proof.** We only have to prove that the formula for \(\eta(a)\) defines an element of \(\text{Ind}_H^G(B)\) and that the resulting map from \(A\) to \(\text{Ind}_H^G(B)\) is \(G\)-equivariant. Let \(a \in A\).

For the first, since \(G/H\) is compact, the function \(sH \mapsto ||\eta(a)(s)||\) is obviously in \(C_0(G/H)\). Let \(s \in G\) and \(h \in H\). Then

\[
\beta_h(\eta(a)(sh)) = \beta_h(\varphi(\alpha_s^{-1}(a))) = \varphi(\alpha_h \circ \alpha_{h^{-1}s^{-1}}(a)) = \eta(a)(s),
\]

as desired.

For the second, let \(\gamma = \text{Ind}_H^G(\alpha)\) be the action of \(G\) on \(\text{Ind}_H^G(B)\). Let \(s, t \in G\). Then

\[
\gamma_s(\eta(a))(t) = \eta(a)(s^{-1}t) = \varphi(\alpha_{t^{-1}}(\alpha_s(a))) = \eta(\alpha_s(a))(t),
\]

as desired. \(\square\)

**Theorem 3.11.** Let \(G\) be a locally compact group, and let \(H \leq G\) be a closed subgroup such that \(G/H\) is compact. Let \((G, A, \alpha)\) be a \(G\)-algebra. If \(\alpha\) is equivariantly semiprojective, then \(\alpha|_H\) is equivariantly semiprojective.
Proof. Let \((H, C, \gamma)\) be an \(H\)-algebra. To simplify the notation, we abbreviate \(\text{Ind}_G^H\) to \(\text{Ind}\). The maps to be introduced are shown in the diagram near the end of the proof. Let \(J_0 \subset J_1 \subset \cdots\) be \(H\)-invariant ideals in \(C\), let \(J = \bigcup_{n=0}^\infty J_n\), let 
\[
\kappa: C \to C/J, \quad \kappa_n: C \to C/J_n, \quad \text{and} \quad \pi_n: C/J_n \to C/J
\]
be the quotient maps, and let \(\varphi: A \to C/J\) be an \(H\)-morphism. Then

\[
\text{Ind}(J) = \bigcup_{n=0}^\infty \text{Ind}(J_n)
\]

by Proposition 3.8. Moreover, Proposition 3.7 allows us to identify the quotients \(\text{Ind}(C)/\text{Ind}(J_n)\) with \(\text{Ind}(C/J_n)/\text{Ind}(J)\) and \(\text{Ind}(C)/\text{Ind}(J)\) with \(\text{Ind}(C/J)\), with quotient maps

\[
\text{Ind}(\kappa): \text{Ind}(C) \to \text{Ind}(C)/\text{Ind}(J), \quad \text{Ind}(\kappa_n): \text{Ind}(C) \to \text{Ind}(C)/\text{Ind}(J_n),
\]

and

\[
\text{Ind}(\pi_n): \text{Ind}(C)/\text{Ind}(J_n) \to \text{Ind}(C)/\text{Ind}(J).
\]

Let \(\eta: A \to \text{Ind}(C)/\text{Ind}(J)\) be as in Lemma 3.10. Since \(\alpha\) is equivariantly semiprojective, there exist \(n \in \mathbb{N}\) and a \(G\)-morphism \(\lambda: A \to \text{Ind}(C)/\text{Ind}(J_n)\) such that \(\text{Ind}(\pi_n) \circ \lambda = \eta\). We now have the following commutative diagram, with the horizontal maps on the right being as in Lemma 3.9.

\[
\begin{array}{ccc}
\text{Ind}(C) & \xrightarrow{\text{ev}_1^C} & C \\
| & | & | \\
\text{Ind}(\kappa_n) & \xrightarrow{\text{ev}_1^{C/J_n}} & C/J_n \\
| & | & | \\
\text{Ind}(\pi_n) & \xrightarrow{\text{ev}_1^{C/J}} & C/J \\
\end{array}
\]

It is easy to check that \(\text{ev}_1^{C/J} \circ \eta = \varphi\). Therefore the map \(\psi = \text{ev}_1^{C/J} \circ \lambda\) is an \(H\)-morphism from \(A\) to \(C/J_n\) such that \(\pi_n \circ \psi = \varphi\). □

Corollary 3.12. Let \(G\) be a compact group, and let \(A\) be a \(G\)-algebra that is equivariantly semiprojective. Then \(A\) is (nonequivariantly) semiprojective.

In Theorem 3.11, some condition on \(G/H\) is necessary, as the following example shows.

Example 3.13. Let \(A = C(S^1)\) be the universal \(C^*\)-algebra generated by a unitary, and consider the free Bernoulli shift \(\tau: \mathbb{Z} \to \text{Aut}(\ast_x C(S^1))\) of Definition 2.3. This action is semiprojective by Proposition 2.4, but its restriction to the trivial subgroup is not.

Thus, \(\mathbb{Z}\) can act semiprojectively on nonsemiprojective \(C^*\)-algebras. This is in contrast to the projective case, discussed in Remark 4.25. An analogous example can be constructed for any infinite countable discrete group in place of \(\mathbb{Z}\).
4. Equivariant projectivity of restrictions to subgroups

In this section we study the projective analog of the question of Section 3. Given a projective action $\alpha : G \to \text{Aut}(A)$, we show in Theorem 4.23 that the restriction of $\alpha$ to a subgroup $H \leq G$ is also projective in considerable generality. The condition we have to put is either a restriction on the subgroup (namely that $H$ or $G/H$ is compact) or that $G$ is a [SIN]-group, in which case $H$ can be arbitrary. A [SIN]-group is a topological group for which the right and left uniform structures agree. See the paragraph before Theorem 4.18. We do not know if these hypotheses can be removed.

Since the trivial subgroup is compact, it follows that every equivariantly projective $C^\ast$-algebra is (nonequivariantly) projective. See Corollary 4.24. This is in contrast to the semiprojective case. See Example 3.13.

To obtain the results in this section, we use a different induction functor, which considers uniformly continuous functions; see Definition 4.16. To show that this functor is exact, we need a criterion for when uniformly continuous functions into quotient $C^\ast$-algebras can be lifted to uniformly continuous functions. In Theorem 4.8, we solve this problem in some generality, and we think that this result might also be of independent interest.

There are several equivalent ways to define a uniform space. We will mostly use the concept of a uniform cover to define a uniformity on a set. We refer to Isbell’s book [Isb64] for the theory of uniform spaces. The basic definitions are in Chapter I. The definition of a uniformity is before item 6 in Chapter I of [Isb64].

If $\mathcal{U}$ and $\mathcal{V}$ are covers of a space $X$, we write $\mathcal{V} \leq \mathcal{U}$ to mean that $\mathcal{V}$ refines $\mathcal{U}$.

**Definition 4.1.** Let $(X,d)$ be a metric space. For $\varepsilon > 0$ and $x \in X$, define $U_{\varepsilon}(x) = \{y \in X : d(x,y) < \varepsilon\}$. The **basic uniform covers** of $X$ are the collections $B(\varepsilon) = \{U_{\varepsilon}(x) : x \in X\}$ for $\varepsilon > 0$. A cover $\mathcal{U}$ of $X$ is called **uniform** if there exists $\varepsilon > 0$ such that $B(\varepsilon) \leq \mathcal{U}$.

The proof of the following result is essentially contained in items 1–3 in Chapter I of [Isb64]. One should note that if $(X,d)$ is a metric space, $\varepsilon_1,\varepsilon_2 > 0$, and $\mathcal{U}_1$ and $\mathcal{U}_2$ are covers of $X$ such that $B(\varepsilon_1) \leq \mathcal{U}_1$ and $B(\varepsilon_2) \leq \mathcal{U}_2$, then $B(\min(\varepsilon_1,\varepsilon_2))$ refines both $\mathcal{U}_1$ and $\mathcal{U}_2$, so that the collection of uniform covers in [Definition 4.1] is downwards directed. Uniformly continuous functions are defined after Theorem 11 in Chapter I of [Isb64], and equiuniformly continuous families of functions are defined before item 19 in Chapter III of [Isb64]. The usual notion for functions on metric spaces is just that a family $F$ of functions from $(X_1,d_1)$ to $(X_2,d_2)$ is equiuniformly continuous if for every $\varepsilon > 0$ there is $\delta > 0$ such that whenever $x,y \in X_1$ satisfy $d_1(x,y) < \delta$, then for all $f \in F$ we have $d_2(f(x),f(y)) < \varepsilon$.

**Proposition 4.2.** Let $(X,d)$ be a metric space. Then the collection of uniform covers in [Definition 4.1] is a uniform structure on $X$. Moreover, for any two metric spaces $(X_1,d_1)$ and $(X_2,d_2)$, the uniformly continuous functions and the equiuniformly continuous families of functions from $X_1$ to $X_2$ are the uniformly continuous functions and the equiuniformly continuous families as traditionally defined in terms of the metrics.

The following theorem is the key result. We warn that the term “subordinate” is used in [Isb64] with a meaning inconsistent with its standard meaning in the context of ordinary partitions of unity.
Theorem 4.3 (Theorem 11 in Chapter IV of [Isb64]). Let \( X \) be a uniform space and let \( U \) be a uniform cover of \( X \). Then there is an equiuniformly continuous (but not necessarily locally finite) partition of unity \( (h_U)_{U \in U} \) such that \( h_U(x) = 0 \) for all \( U \in U \) and \( x \in X \setminus U \).

We recall the following standard definition.

Definition 4.4. Let \( X \) be a set, and let \( U \) be a cover of \( X \). The order of \( U \), denoted \( \text{ord}(U) \), is the least number \( n \in \mathbb{N} \cup \{0\} \) such that whenever \( U_0, U_1, \ldots, U_n \in U \) are distinct, then \( U_0 \cap U_1 \cap \cdots \cap U_n = \emptyset \). We take \( \text{ord}(U) = \infty \) if no such \( n \) exists.

Equivalently, \( \text{ord}(U) \) is the largest number \( n \) such that there are \( n \) distinct elements of \( U \) which have nonempty intersection. We warn the reader that some authors use a different convention, in which what we defined above is order \( n - 1 \). For example, see page 111 of [Pea75]. We are following the convention implicitly used in our reference [SSG93].

The first part of the following definition is found at the very beginning of Chapter V of [Isb64], where the term “large dimension” is used. The second part is Definition 1.7 of [SSG93].

Definition 4.5. Let \( X \) be a uniform space. Then the large uniform dimension of \( X \), denoted \( \Delta d(X) \), is the least \( n \in \{-1, 0, 1, 2, \ldots, \infty\} \) such that for every uniform open cover \( U \) of \( X \) there is a uniform open cover \( V \) of \( X \) of order at most \( n + 1 \) which refines \( U \). (We take \( \Delta d(\emptyset) = -1 \).)

We say that \( X \) is uniformly finitistic if for every uniform open cover \( U \) of \( X \) there is a uniform open cover \( V \) of \( X \) of finite order which refines \( U \).

An equivalent condition for being uniformly finitistic is that there exists a base for the uniformity consisting of uniform covers of finite order.

If a uniform space \( X \) is locally compact and paracompact (in the induced topology), then its covering dimension is bounded by its large uniform dimension, that is, \( \dim(X) \leq \Delta d(X) \). To see this, let \( \text{locdim}(X) \) be the local covering dimension of \( X \) (Definition 5.1.1 of [Pea75]). Proposition 5.3.4 of [Pea75] gives \( \dim(X) = \text{locdim}(X) \). For a locally compact Hausdorff space \( X \), let \( X^+ \) denote the one point compactification of \( X \). It is a standard result that \( \text{locdim}(X) = \dim(X^+) \); for instance, this is easily deduced from Propositions 3.5.6, 5.2.1, 5.2.2, and 5.3.4 of [Pea75]. It follows from Theorems V.5 and VI.2 of [Isb64] that for every compactification \( \gamma X \) of \( X \) we have \( \dim(\gamma X) \leq \Delta d(X) \). Thus, if \( X \) is locally compact and paracompact, we may combine these results to obtain

\[
\dim(X) = \text{locdim}(X) = \dim(X^+) \leq \Delta d(X),
\]
as desired.

The concept of being finitistic was first defined for topological spaces, where it means that every open cover can be refined by an open cover of finite order. This definition is implicit in [Swa59], although the term “finitistic” was only later introduced by Bredon on page 133 of his book [Bre72].

In general, for a uniform space there is no connection between being finitistic and uniformly finitistic. Example (d) after Definition 1.7 of [SSG93] gives a uniformly finitistic space which is not finitistic. Example 2.4 of [Isb59] gives a discrete uniform space, hence obviously finitistic, with a uniform open cover having no uniform open refinement of finite order, thus not uniformly finitistic.
**Notation 4.6.** Let $X$ be a topological space and let $A$ be a $C^*$-algebra. We denote by $C_b(X, A)$ the $C^*$-algebra of all bounded continuous functions from $X$ to $A$, with the supremum norm. If $X$ is a uniform space, we let $C_u(X, A) \subset C_b(X, A)$ denote the subset consisting of all bounded uniformly continuous functions from $X$ to $A$.

**Proposition 4.7.** Let $X$ be a uniform space and let $A$ be a $C^*$-algebra. Then $C_u(X, A)$ is a $C^*$-algebra.

*Proof.* It is easy to check that $C_u(X, A)$ is closed under the algebraic operations. That it is norm closed in $C_b(X, A)$ follows from Corollary 32 in Chapter III of [Vid69].

The following theorem is in some sense a dual version of Theorem 1 of [Vid69], on the problem of extending uniformly continuous maps from subspaces. We do not know whether it is necessary that $X$ be uniformly finitistic. Its proof serves as a simpler model for the proof of Theorem 4.18.

**Theorem 4.8.** Let $\pi: A \to B$ be a surjective $*$-homomorphism between two $C^*$-algebras, and let $X$ be a uniformly finitistic space. Then the induced $*$-homomorphism $\kappa: C_u(X, A) \to C_u(X, B)$ is surjective.

*Proof.* It is enough to show that $\kappa$ has dense range.

Given $b \in C_u(X, B)$ and $\varepsilon > 0$, we will construct $a \in C_u(X, A)$ such that $\|\pi \circ a - b\| < \varepsilon$. We may clearly assume $b \neq 0$. Let $\mathcal{U}$ be a uniform cover of $X$ such that whenever $U \in \mathcal{U}$ and $x, y \in U$, then $\|b(x) - b(y)\| < \frac{\varepsilon}{2}$. Since $X$ is uniformly finitistic, we may assume $\mathcal{U}$ has finite order. Set $n = \text{ord}(\mathcal{U})$.

Let $(h_U)_{U \in \mathcal{U}}$ be an equiuniformly continuous partition of unity for $\mathcal{U}$ as in Theorem 4.3. Equiuniform continuity in our situation means that for every $\rho > 0$ there exists a uniform open cover $\mathcal{V}$ of $X$ such that whenever $V \in \mathcal{V}$ and $x, y \in V$, then for all $U \in \mathcal{U}$ we have $|h_U(x) - h_U(y)| < \rho$.

For each $U \in \mathcal{U}$ choose a point $x_U \in U$, and let $a_U \in A$ be a lift of $b(x_U)$ with $\|a_U\| = \|b(x_U)\|$. For $x \in X$, there are at most $n$ sets $U \in \mathcal{U}$ such that $x \in U$, and $h_U(x)$ can be nonzero only for these sets. Therefore the sum in the following definition of a function $a: X \to A$ is finite at each point:

$$a(x) = \sum_{U \in \mathcal{U}} h_U(x) \cdot a_U$$

for $x \in X$. Since $\sum_{U \in \mathcal{U}} h_U(x) = 1$, it further follows that $\|a\| \leq \|b\|$, so that $a$ is bounded.

We claim that $a$ is uniformly continuous. We follow an argument in the proof of Theorem 1 of [Vid69]. Let $\rho > 0$. We must find a uniform open cover $\mathcal{V}$ of $X$ such that whenever $V \in \mathcal{V}$ and $x, y \in V$, we have $\|a(x) - a(y)\| < \rho$. Set $\rho_0 = \rho/(2n\|b\|)$. Let $\mathcal{V}$ be a uniform open cover which witnesses equiuniform continuity of $(h_U)_{U \in \mathcal{U}}$ as above, but with $\rho_0$ in place of $\rho$. Let $V \in \mathcal{V}$ and let $x, y \in V$. Set

$$\mathcal{U}_0 = \{ U \in \mathcal{U}: x \in U \text{ or } y \in U \}.$$

Then $\text{card}(\mathcal{U}_0) \leq 2n$. Therefore

$$\|a(x) - a(y)\| = \left\| \sum_{U \in \mathcal{U}_0} (h_U(x) - h_U(y)) \cdot a_U \right\|$$

$$\leq 2n \cdot \|b\| \cdot \max_{U \in \mathcal{U}_0} |h_U(x) - h_U(y)| < 2n\|b\|\rho_0 = \rho.$$
The claim is proved.

It remains to prove that \( \| \pi \circ a - b \| < \varepsilon \). Let \( x \in X \). Then \( \| \pi(a_U) - b(x) \| < \frac{\varepsilon}{2} \) whenever \( h_U(x) \neq 0 \). Therefore

\[
\| (\pi \circ a)(x) - b(x) \| = \left\| \sum_{U \in \mathcal{U}} h_U(x)(\pi(a_U) - b(x)) \right\| \leq \sum_{U \in \mathcal{U}} h_U(x) \| \pi(a_U) - b(x) \| < \frac{\varepsilon}{2}.
\]

So \( \| \pi \circ a - b \| \leq \frac{\varepsilon}{2} < \varepsilon \), as desired. \( \square \)

**Remark 4.9.** The proof of Theorem 4.8 can easily be adapted to the case of bounded continuous maps. More precisely, if \( \pi: A \to B \) is a surjective \(*\)-homomorphism of \( C^*\)-algebras, and \( X \) is a paracompact space, then the method of proof shows that the induced \(*\)-homomorphism \( C_b(X,A) \to C_b(X,B) \) is surjective. This is a \( C^*\)-algebraic version of the Bartle-Graves Selection Theorem, Theorem 4 of [BG52], which treats the case in which \( A \) and \( B \) are arbitrary Banach spaces. The \( C^*\)-algebraic version is much easier to prove since the image of a \(*\)-homomorphism is always closed.

Since a \( C^*\)-algebra is paracompact, one may also formulate the theorem as follows. Let \( \pi: A \to B \) be a surjective \(*\)-homomorphism between \( C^*\)-algebras. Then there exists a continuous function \( \sigma: B \to A \) (not necessarily linear) such that \( \pi \circ \sigma = \text{id}_B \) (that is, \( \sigma \) is a section), and such that there is a constant \( M \) such that \( \| \sigma(a) \| \leq M \cdot \| a \| \) for all \( a \in A \). This also appears in Theorem 2 of [Lor97a]. To get this statement from the surjectivity of \( C_b(X,A) \to C_b(X,B) \), take \( X = \{ b \in B : \| b \| = 1 \} \), lift the function \( f \) (which is in \( C_b(X,B) \)) to a bounded function \( g: X \to A \), and take \( \sigma(b) = \| b \| \cdot g(\| b \|^{-1}b) \) for \( b \in B \setminus \{ 0 \} \) and \( g(0) = 0 \).

**Definition 4.10.** Let \( G \) be a locally compact group, and let \( H \leq G \) be a closed subgroup. Let \( q: G \to G/H \) be the quotient map. For a nonempty open subset \( U \subset G \) with \( 1 \in U \), define \( \mathcal{B}_{G,H}(U) = \{ q(Us) : s \in G \} \), the open cover of \( G/H \) by the images in \( G/H \) of the right translates of \( U \). Define the right uniformity on \( G/H \) to consist of all open covers \( \mathcal{U} \) of \( G/H \) such that there is a nonempty open subset \( U \subset G \) with \( 1 \in U \) for which \( \mathcal{B}_{G,H}(U) \leq \mathcal{U} \), and call such covers the right uniform covers.

We define the left uniformity on \( G \) and left uniform covers of \( G \) analogously, using the covers by the left translates \( \{ sU : s \in G \} \) for nonempty open subsets \( U \subset G \) with \( 1 \in U \).

We do not define a left uniformity on \( G/H \) since the images in \( G/H \) of the left uniform covers in \( G \) will in general not define a uniformity.

Taking \( H = \{ 1 \} \), we see that the inversion map \( s \mapsto s^{-1} \) is uniformly continuous if and only if the right and left uniformities on \( G \) agree. However, for fixed \( t \in G \), both the left translation map \( s \mapsto ts \) and the right translation map \( s \mapsto st \) are uniformly continuous in the right uniformity (and also in the left uniformity).

Uniform structures on topological groups are discussed on pages 20–22 of [HR79], but from the point of view of neighborhoods of the diagonal rather than uniform open covers.

Clearly the map \( q: G \to G/H \) is uniformly continuous when both spaces are given the right uniformity. In fact, the right uniformity on \( G/H \) is the quotient uniformity, as defined before item 5 in Chapter II of [Isb64], of the right uniformity on \( G \). We do not need this fact, so we omit the proof.
Let $G$ be a metrizable topological group. Then $G$ has a left invariant metric determining its topology, by Theorem 1.22 of [MZ55], and analogously it also has a right invariant metric. It is easy to check that the uniformity induced by any right invariant metric (as in Proposition 4.2) is equal to the right uniformity of $X$.

**Definition 4.10.**

Given a locally compact group $G$ and a closed subgroup $H$, it is shown in Lemma 2 of [Ank89] that every (left) uniform cover of $H \setminus G$ can be refined by a cover of finite order. In the following result we adapt the proof to ensure that the refining cover is uniform. We formulate the result for the space of right cosets.

**Proposition 4.11.** Let $G$ be a locally compact group, and let $H \leq G$ be a closed subgroup. Then $G/H$ is uniformly finitistic (with respect to the right uniformity).

**Proof.** Let $\mu$ be a right Haar measure on $G$. We will use the notation from Definition 4.10. In particular, we let $q: G \to G/H$ be the quotient map. The basic uniform covers of $G/H$ are defined to be $B_{G,H}(U) = \{q(U s): s \in G\}$ for open neighborhoods $U$ of $1 \in G$. Given such a set $U$, we will construct a uniform cover $W$ of $G/H$ that refines $B_{G,H}(U)$ and that has finite order.

First, without loss of generality, we may assume that $U$ has compact closure in $G$ and that $U = U^{-1}$. Let $W$ be an open neighborhood of $1 \in G$ such that $W^3 \subset U$ and such that $W = W^{-1}$.

As in the proof of Lemma 2 in [Ank89], we let $X$ be a maximal subset of $G$ such that the sets $q(Wx)$ for $x \in X$ are pairwise disjoint. Set $W = \{q(W^3 x): x \in X\}$. We show that it has the desired properties.

We claim that $W$ is refined by $B_{G,H}(W)$. To prove the claim let $g \in G$ be given. By maximality of $X$, there exists $x \in X$ such that $q(Wg)$ and $q(Wx)$ are not disjoint. Thus, there are $w_1, w_2 \in W$ and $h_1, h_2 \in H$ such that $w_1 g h_1 = w_2 h_2$. Then $g = w_1^{-1} w_2 x h_2 h_1^{-1}$ and so $q(g) \in q(W^2 x)$. Therefore $q(Wg) \subseteq q(W^3 x)$. This proves the claim.

Hence, $W$ is uniform and it clearly refines the given cover $B_{G,H}(U)$.

It remains to show that $W$ has finite order. Let $x_0, x_1, \ldots, x_k \in X$ be elements such that $\bigcap_{j=0}^{k} q(W^3 x_j) \neq \emptyset$. This means that there are elements $w_0, w_1, \ldots, w_k \in W^3$ and $h_0, h_1, \ldots, h_k \in H$ such that $w_j x_j h_j = w_0 x_0 h_0$ for $j = 1, 2, \ldots, k$. It follows that

$$W x_j h_j h_0^{-1} = W w_j^{-1} w_0 x_0 \subset W^3 x_0,$$

for $j = 1, 2, \ldots, k$.

However, by construction of $X$, the sets $W x_j h_j h_0^{-1}$ for $j = 1, 2, \ldots, k$ are pairwise disjoint. So

$$k \cdot \mu(W) = \mu \left( \bigcup_{j=1}^{k} W x_j h_j h_0^{-1} \right) \leq \mu \left( W^3 x_0 \right) = \mu(W^3).$$

Since $W$ is open and has compact closure, $\mu(W)$ is non-zero and finite. Thus $k \leq \mu(W^3)/\mu(W)$ and so $W$ has finite order.

**Corollary 4.12.** Every locally compact group is uniformly finitistic (with respect to both the right and left uniformity).

**Notation 4.13.** Let $G$ be a topological group and let $A$ be a $C^*$-algebra. We denote by $C_r^*(G, A)$ the $C^*$-algebra of bounded functions $f: G \to A$ which are right uniformly continuous. This is just $C_u(G, A)$ as in Notation 4.6 when $G$ is
equipped with the right uniformity. We further let \( \lambda : G \to \text{Aut}(C_b(G,A)) \) be the (not necessarily continuous) action given by \( \lambda_s(f)(t) = f(s^{-1}t) \) for \( f \in C_b(G,A) \) and \( s,t \in G \).

Left translation is continuous on the right uniformly continuous functions, not the left uniformly continuous functions. The proof is known and not difficult; we give it here primarily to convince the reader that the statement is correct. We start with a preparatory lemma, which we also need for the left uniformity.

**Lemma 4.14.** Adopt Notation 4.13. Let \( f \in C_b(G,A) \). Then \( f \in C_{ru}(G,A) \) if and only if for every \( \varepsilon > 0 \) there is an open set \( V \subset G \) with \( 1 \in V \) such that whenever \( s,t \in G \) satisfy \( st^{-1} \in V \), then \( \| f(s) - f(t) \| < \varepsilon \). Also, \( f \) is left uniformly continuous if and only if for every \( \varepsilon > 0 \) there is an open set \( V \subset G \) with \( 1 \in V \) such that whenever \( s,t \in G \) satisfy \( t^{-1}s \in V \), then \( \| f(s) - f(t) \| < \varepsilon \).

**Proof.** The proofs of the two statements are the same, and we do only the first.

First assume \( f \) is right uniformly continuous. Then there is a nonempty open set \( V \subset G \) with \( 1 \in V \) such that whenever \( s,t,g \in G \) satisfy \( s,t \in Vg \), then \( \| f(s) - f(t) \| < \varepsilon \). If now \( s,t \in G \) satisfy \( st^{-1} \in V \), then \( s \in Vt \) and, since \( 1 \in V \), also \( t \in Vt \). Taking \( g = t \) above, we get \( \| f(s) - f(t) \| < \varepsilon \).

Now assume that \( f \) satisfies the condition of the lemma. Let \( \varepsilon > 0 \), and choose \( V \subset G \) as in this condition. Choose an open subset \( U \subset G \) such that \( 1 \in U \) and \( s,t \in U \) implies \( st^{-1} \in V \). Let \( s,t,g \in G \) satisfy \( s,t \in Ug \). Then \( sg^{-1}, tg^{-1} \in U \), so \( st^{-1} = (sg^{-1})(tg^{-1})^{-1} \in V \). Therefore \( \| f(s) - f(t) \| < \varepsilon \). \( \square \)

**Lemma 4.15.** Let the notation be as in Notation 4.13. Let \( f \in C_b(G,A) \). Then \( s \mapsto \lambda_s(f) \) is continuous if and only if \( f \in C_{ru}(G,A) \).

**Proof.** First assume \( f \) is right uniformly continuous. Let \( \varepsilon > 0 \). It suffices to find an open subset \( V \subset G \) such that \( 1 \in V \) and whenever \( s,t \in V \) and \( g \in G \), then \( \| \lambda_s(f) - \lambda_t(f) \| < \varepsilon \). Choose an open subset \( V \subset G \) as in Lemma 4.14 with \( \frac{\varepsilon}{2} \) in place of \( \varepsilon \). Let \( s \in V \) and \( t \in G \). Then for \( g \in G \) we have \( (t^{-1}g)(s^{-1}t^{-1}g)^{-1} = s \in V \), so

\[
\| \lambda_s(f)(g) - \lambda_t(f)(g) \| = \| f(s^{-1}t^{-1}g) - f(t^{-1}g) \| < \frac{\varepsilon}{2}.
\]

Taking the supremum over \( g \in G \), we get \( \| \lambda_s(f) - \lambda_t(f) \| \leq \frac{\varepsilon}{2} < \varepsilon \).

For the converse, assume that \( s \mapsto \lambda_s(f) \) is continuous. We verify the criterion of Lemma 4.14. Let \( \varepsilon > 0 \). Choose an open subset \( V \subset G \) such that \( 1 \in V \) and whenever \( s \in V \) then \( \| \lambda_s(f) - f \| < \varepsilon \). Let \( s,t \in G \) satisfy \( st^{-1} \in V \). Then

\[
\| f(s) - f(t) \| = \| f(s) - \lambda_{st^{-1}}(f)(s) \| \leq \| f - \lambda_{st^{-1}}(f) \| < \varepsilon.
\]

This completes the proof. \( \square \)

We now give a definition which is very similar to Definition 3.2 but which uses bounded uniformly continuous functions instead of functions vanishing at infinity.

**Definition 4.16.** Let \( G \) be a locally compact group, and let \( H \leq G \) be a closed subgroup. Let \( \alpha : H \to \text{Aut}(A) \) be an action of \( H \) on a \( C^* \)-algebra \( A \). We define a \( C^* \)-algebra \( F^G_H(A) \), with not necessarily continuous action \( F^G_H(\alpha) : G \to \text{Aut} (F^G_H(A)) \), by

\[
F^G_H(A) = \{ f \in C_b(G,A) : \alpha_h(f(sh)) = f(s) \text{ for all } s \in G \text{ and } h \in H \}.
\]
and
\[ (F^G_H(\alpha))_s(f)(t) = f(s^{-1}t) \]
for \( f \in F^G_H(A) \) and \( s, t \in G \). We further define a subalgebra \( \text{UInd}^G_H(A) \subset F^G_H(A) \) by
\[ \text{UInd}^G_H(A) = \{ f \in F^G_H(A) : s \mapsto (F^G_H(\alpha))_s(f) \text{ is continuous} \}, \]
and we take \( \text{UInd}^G_H(\alpha) \) to be the restriction of \( F^G_H(\alpha) \) to this subalgebra.

If \( A \) and \( B \) are \( H \)-algebras and \( \varphi : A \to B \) is an \( H \)-morphism, then the induced \( G \)-morphisms
\[ F^G_H(\varphi) : F^G_H(A) \to F^G_H(B) \quad \text{and} \quad \text{UInd}^G_H(\varphi) : \text{UInd}^G_H(A) \to \text{UInd}^G_H(B) \]
are defined by sending \( f \) in \( F^G_H(A) \) or \( \text{UInd}^G_H(A) \) as appropriate to the function \( s \mapsto \varphi(f(s)) \) for \( s \in G \).

We call \( \text{UInd}^G_H \) the right uniform induction functor.

**Lemma 4.17.** Let \( G \) be a locally compact group, and let \( H \subseteq G \) be a closed subgroup. Let the notation be as in [Definition 4.16](#) and [Definition 3.1](#). Then:

1. \( \text{UInd}^G_H(A) = F^G_H(A) \cap C^*_u(G, A) \).
2. \( F^G_H \) is a functor from the category \( \mathcal{C}_H \) of \( H \)-algebras to the category of \( C^* \)-algebras with not necessarily continuous actions of \( G \).
3. \( \text{UInd}^G_H \) is a functor from \( \mathcal{C}_H \) to \( \mathcal{C}_G \).
4. If \( G/H \) is compact, then \( \text{UInd}^G_H = F^G_H = \text{Ind}^G_H \).

**Proof.** Part (1) follows from [Lemma 4.15](#) and Part (2) is an algebraic calculation. Part (3) follows from part (1), part (2), and the fact that the formula for \( F^G_H(\varphi) \) preserves uniform continuity. Part (4) follows from the observation that the condition in [Definition 3.2](#) that \( sH \mapsto \|f(s)\| \) be in \( C_0(G/H) \), is automatic when \( G/H \) is compact, and the fact that left translation is continuous on \( \text{Ind}^G_H(A) \). \( \square \)

To formulate the next result, recall that a topological group \( G \) is called a \([\text{SIN}]\)-group (for “small invariant neighborhoods”) if every neighborhood of \( 1 \in G \) contains a neighborhood \( V \) of \( 1 \) that is invariant (meaning that \( gVg^{-1} = V \) for all \( g \in G \)). Such groups are also called balanced. It is easy to see that a group is a \([\text{SIN}]\)-group if and only if the left and right uniformities on \( G \) agree (equivalently, the assignment \( g \mapsto g^{-1} \) is uniformly continuous when regarded as a map from \( G \) to itself, both equipped with the right uniformity). The class of \([\text{SIN}]\)-groups includes all groups that are abelian, compact, or discrete. See [Pal78](#) for more on \([\text{SIN}]\)-groups.

**Theorem 4.18.** Let \( G \) be a locally compact group and let \( H \subseteq G \) be a closed subgroup. Assume that at least one of the following conditions is satisfied:

1. \( H \) is compact.
2. \( G/H \) is compact.
3. \( G \) is a \([\text{SIN}]\)-group.

Then the right uniform induction functor \( \text{UInd}^G_H : \mathcal{C}_H \to \mathcal{C}_G \) is exact, that is, given an \( H \)-equivariant short exact sequence of \( H \)-algebras
\[ 0 \to I \xrightarrow{i} A \xrightarrow{\pi} B \to 0, \]
the induced \( G \)-equivariant sequence of \( G \)-algebras
\[ 0 \to \text{UInd}^G_H(I) \xrightarrow{\text{UInd}^G_H(i)} \text{UInd}^G_H(A) \xrightarrow{\text{UInd}^G_H(\pi)} \text{UInd}^G_H(B) \to 0, \]
is also exact.

Remark 4.19. It seems natural to expect that Theorem 4.18 holds in greater
generality. We do not know whether any condition is necessary to show that the
right uniform induction functor is exact.

We need two further lemmas for the proof.

Lemma 4.20. Let $G$ be a locally compact $\text{[SIN]}$-group, and let $f \in C_c(G)$. Then
for every $\varepsilon > 0$ there is an open set $U \subset G$ such that $1 \in U$ and such that whenever
g, h, s, t \in G$ satisfy $s^{-1}t \in U$, then $|f(gsh) - f(gth)| < \varepsilon$.

Proof. Lemma 1.62 of [Wil07] provides an open set $V \subset G$ such that $1 \in V$ and such that whenever
$s, t \in G$ satisfy $s^{-1}t \in V$, then $|f(s) - f(t)| < \varepsilon$. (By Lemma 4.14
this is just left uniform continuity of $f$.) Since $G$ is a $\text{[SIN]}$-group, there is an
invariant open set $U \subset G$ such that $1 \in U$ and $U \subset V$. Now let $g, h, s, t \in G$ satisfy
$s^{-1}t \in U$. Then
\[(gsh)^{-1}(gth) = h^{-1}s^{-1}th \in h^{-1}Uh = U \subset V,
\]
so that $|f(gsh) - f(gth)| < \varepsilon$. \qed

Lemma 4.21. Let $G$ be a locally compact group, let $H \leq G$ be a closed subgroup, let
$\mu$ be a left Haar measure on $H$, and let $L \subset G$ be compact. Then $\sup_{s \in G} \mu(sL \cap H)$
is finite.

Proof. Let $q: G \to G/H$ be the quotient map. Choose a continuous function
$f: G \to [0, 1]$ with compact support and such that $f = 1$ on $L$. For $s \in G$ define
\[g_0(s) = \int_H f(sh) \, d\mu(h).
\]
Then $g_0$ is continuous and satisfies $g_0(sk) = g_0(s)$ for all $s \in G$ and $k \in H$.
Therefore $g_0$ drops to a continuous function $g$ on $G/H$. If $s \notin supp(f)H$, then
$g_0(s) = 0$. Therefore $supp(g) \subset q(supp(f))$, and so is compact. Now
\[
\sup_{s \in G} \mu(sL \cap H) \leq \sup_{s \in G} \int_H f(s^{-1}h) \, d\mu(h) = \sup_{x \in G/H} g(x) < \infty.
\]
This completes the proof. \qed

Proof of Theorem 4.18. To simplify the notation, we abbreviate the functor $\text{UInd}_H^G$
to $\text{UInd}$. As in the proof of Proposition 3.7 it is easy to check that the induced sequence is exact at the left and in the middle. Thus, it remains to check that
$\text{UInd}(\pi): \text{UInd}(A) \to \text{UInd}(B)$ is surjective.

If $G/H$ is compact, the right uniform induction functor agrees with the usual
induction functor by Lemma 4.17, so is exact by Proposition 3.7.

Now assume that $H$ is compact. It is clear from Definition 4.10 and Lemma 4.15
that $\text{UInd}(A)$ is the fixed point algebra of the action $\gamma: H \to \text{Aut}(C_{ru}(G, A))$ given
by $\gamma_h(f)(s) = \alpha_h(f(sh))$ for $f \in C_{ru}(G, A)$, $s \in G$, and $h \in H$, and similarly for $B$.
By Theorem 4.8 and Corollary 4.12 the induced $\ast$-homomorphism $\kappa: C_{ru}(G, A) \to
C_{ru}(G, B)$ is surjective. Since $H$ is compact, Lemma 1.8 implies that the restriction
to the fixed point algebras is also surjective.

For the last case, let us assume that $G$ is a $\text{[SIN]}$-group. Let $\alpha: H \to \text{Aut}(A)$ and
$\beta: H \to \text{Aut}(B)$ denote the actions of $G$. Let $q: G \to G/H$ denote the quotient
map. Let $\mu$ be a left Haar measure on $H$. 


Let \( b \in \text{Ulnd}(B) \) and let \( \varepsilon > 0 \). We construct \( a \in \text{Ulnd}(A) \) such that \( \|\pi \circ a - b\| < \varepsilon \). The function \( b \) is right uniformly continuous by [Lemma 4.17](1). The hypothesis on \( G \) implies that \( b \) is left uniformly continuous. So [Lemma 4.14](2) provides an open neighborhood \( U \) of \( 1 \in G \) such that \( t^{-1}s \in U \) implies \( \|b(s) - b(t)\| < \varepsilon \). Since \( G \) is locally compact, we may assume that \( U \) is compact. Let \( V_0 \) be an open neighborhood of \( 1 \) such that \( \overline{V_0} \subset U \).

We claim that there is a continuous function \( f : G \to [0, \infty) \) such that \( \text{supp}(f) \subset U \), such that for every \( s \in V_0H \) we have

\[
(4.1) \quad \int_H f(\sigma h) \, d\mu(h) = 1,
\]

and such that for every \( s \in G \) we have

\[
(4.2) \quad \int_H f(\sigma h) \, d\mu(h) \leq 1.
\]

We prove the claim. Choose an open set \( Z \subset G \) with \( \overline{V_0} \subset Z \subset \overline{Z} \subset U \), and choose \( f_0 \in C_c(G) \) such that

\[ 0 \leq f_0 \leq 1, \quad \text{supp}(f_0) \subset U, \quad \text{and} \quad f_0|_{\overline{Z}} = 1. \]

Since \( q(\overline{V_0}) \) is compact, \( q(Z) \) is open, and \( q(\overline{V_0}) \subset q(Z) \), there exists \( f_1 \in C_c(G/H) \) such that

\[ 0 \leq f_1 \leq 1, \quad \text{supp}(f_1) \subset q(Z), \quad \text{and} \quad f_1|_{q(\overline{V_0})} = 1. \]

Define a continuous function \( k : G \to [0, \infty) \) by

\[ k(s) = \int_H f_0(\sigma h) \, d\mu(h) \]

for \( s \in G \). For \( s \in Z \), the integrand is equal to 1 on the open set \( H \cap s^{-1}Z \subset H \). This set contains 1, so is nonempty, whence \( k(s) \neq 0 \). Since also \( k(\sigma h) = k(s) \) for all \( s \in G \) and \( h \in H \), we see that \( k(s) \neq 0 \) for all \( s \in ZH \). Therefore the definition

\[ f(s) = \begin{cases} f_1(\sigma h)f_0(s)k(s)^{-1} & s \in ZH \\ 0 & s \in G \setminus q^{-1}(\text{supp}(f_1)) \end{cases} \]

is consistent and, since \( q^{-1}(\text{supp}(f_1)) \subset ZH \), gives a continuous function \( f : G \to [0, \infty) \). For \( s \in G \), by considering the cases \( s \in ZH \) and \( s \notin q^{-1}(\text{supp}(f_1)) \) separately, one checks that \( \int_H f(\sigma h) \, d\mu(h) = f_1(\sigma h) \). So (4.1) holds for \( s \in V_0H \) and (4.2) holds for \( s \in G \). This proves the claim.

Since the left and right uniformities on \( G \) agree, 4.14(g) in Chapter II of [HR79] provides an open neighborhood \( V_1 \) of \( 1 \) such that \( sV_1s^{-1} \subset V_0 \) for all \( s \in G \). This implies, in particular, that

\[
(4.3) \quad HV_1H \subset V_0H.
\]

Now choose an open neighborhood \( V \) of \( 1 \) such that \( s,t \in V \) imply \( s^{-1}t \in V_1 \).

Consider the left uniform cover \( \mathcal{V} = \{sV : s \in G\} \) of \( G \), and its image \( q(\mathcal{V}) = \{(sVH)/H : s \in G\} \) in \( G/H \). Since the left and right uniformities on \( G \) agree, \( \mathcal{V} \) is a right uniform cover of \( G \), so that \( q(\mathcal{V}) \) is a right uniform cover of \( G/H \). Since \( G/H \) is right uniformly finitistic, there exists a right uniform cover \( W \) of \( G/H \) which refines \( q(\mathcal{V}) \) and has finite order \( n \). Let \( (l_W)_{W \in \mathcal{W}} \) be a right equiuniformly continuous partition of unity on \( G/H \) for \( W \) as in [Theorem 4.3](3). Then the functions \( l_W \circ q \)
define an equiuniformly continuous partition of unity on $G$ such that $(l_W \circ q)(x) = 0$ whenever $W \in \mathcal{W}$ and $x \in G \setminus q^{-1}(W)$.

For each $W \in \mathcal{W}$, choose a point $x_W \in q^{-1}(W)$. Define a continuous function $g_W : G \to [0, \infty)$ by

$$g_W(s) = l_W(sH) \cdot f(x_W^{-1}s).$$

This function vanishes outside the set $x_WU \cap q^{-1}(W)$. In particular, $\text{supp}(g_W)$ is contained in the compact set $x_WU$.

We claim that for every $s \in G$ and $W \in \mathcal{W}$, we have

$$\int_H g_W(sh) \, d\mu(h) = l_W(sH). \quad (4.4)$$

If $s \notin q^{-1}(W)$, then both sides of $(4.4)$ are zero. To prove the claim, we therefore assume $s \in q^{-1}(W)$. Choose $t \in G$ such that $q^{-1}(W) \subset tVH$. Then $s, x_W \in tVH$, so there exist $h, k \in H$ such that $t^{-1}sh, t^{-1}x_Wk \in V$. So $k^{-1}x_W^{-1}sh \in V_1$. It follows from $(4.3)$ that $x_W^{-1}s \in V_0H$, and from $(4.1)$ that

$$\int_H f(x_W^{-1}sh) \, d\mu(h) = 1.$$ The claim follows.

For $W \in \mathcal{W}$, choose $a_W \in A$ such that $\pi(a_W) = b(x_W)$ and $\|a_W\| = \|b(x_W)\|$. We next claim that the definition

$$a(s) = \sum_{W \in \mathcal{W}} \int_H g_W(sh) \cdot \alpha_h(a_W) \, d\mu(h), \quad (4.5)$$

for $s \in G$, gives a well defined function $a : G \to A$. For each $W \in \mathcal{W}$, the integral exists because the integrand is continuous and has compact support. Moreover, for every $s \in G$, from $(4.4)$ we get

$$\left\| \int_H g_W(sh) \cdot \alpha_h(a_W) \, d\mu(h) \right\| \leq l_W(sH) \cdot \|a_W\| \leq l_W(sH) \cdot \|b\|. \quad (4.6)$$

It follows that for each $s \in G$ at most $n$ summands in $(4.5)$ are nonzero. The claim follows. Moreover, $\|a(s)\| \leq \|b\|$ for all $s \in G$.

We claim that $a$ is right uniformly continuous. Since the left and right uniformities agree, it suffices to prove that $a$ is left uniformly continuous. Let $\rho > 0$. By [Lemma 4.21] there is $M > 0$ such that $\mu(tU \cap H) \leq M$ for all $t \in G$. Using equiuniform continuity of $(l_W \circ q)_{W \in \mathcal{W}}$, choose an open neighborhood $Z_1$ of 1 such that for every $W \in \mathcal{W}$ and $s, t \in G$ with $t^{-1}s \in Z_1$, we have

$$|l_W(sH) - l_W(tH)| < \frac{\rho}{4n\|b\| + 1}.$$ Using [Lemma 4.20] choose an open neighborhood $Z_2$ of 1 such that whenever $g, h, s, t \in G$ satisfy $s^{-1}t \in Z_2$, then

$$|f(gsh) - f(gth)| < \frac{\rho}{4M\|b\| + 1}. \quad (4.7)$$

Define $Z_0 = Z_1 \cap Z_2$. 


Now let \( s, t \in G \) satisfy \( s^{-1}t \in \mathbb{Z}_0 \). Then, using \( \|a_W\| \leq \|b\| \) for all \( W \in \mathcal{W} \),
\[
\|a(s) - a(t)\| = \left\| \sum_{W \in \mathcal{W}} \left( \int_H l_W(sH)f(x_W^{-1}sh)\alpha_h(a_W)\,d\mu(h) \right. \right. \\
\left. \left. - \int_H l_W(tH)f(x_W^{-1}th)\alpha_h(a_W)\,d\mu(h) \right) \right\| \\
\leq \|b\| \sum_{W \in \mathcal{W}} \|l_W(sH) - l_W(tH)\| \int_H f(x_W^{-1}sh)\,d\mu(h) \\
+ \|b\| \sum_{W \in \mathcal{W}} l_W(tH) \int_H |f(x_W^{-1}sh) - f(x_W^{-1}th)|\,d\mu(h).
\]
In the first term of the last expression, as in the proof of Theorem 4.8, for any fixed \( s, t \in G \), at most \( 2n \) of the terms are nonzero. Therefore, using (4.2), this term is dominated by
\[
\|b\| \cdot 2n \left( \frac{\rho}{4n\|b\| + 1} \right) \left( \sup_{W \in \mathcal{W}} \int_H f(x_W^{-1}sh)\,d\mu(h) \right) \leq \left( \frac{2n\|b\|\rho}{4n\|b\| + 1} \right) \cdot 1 < \frac{\rho}{2}.
\]
Using \( \sum_{W \in \mathcal{W}} l_W(tH) = 1 \), the choice of \( M \), and (4.7), we see that the second term is dominated by
\[
\|b\| \left( \frac{\rho}{4M\|b\| + 1} \right) \left[ \mu(s^{-1}x_W \mathcal{U} \cap H) + \mu(t^{-1}x_W \mathcal{U} \cap H) \right] \leq \frac{2M\|b\|\rho}{4M\|b\| + 1} < \frac{\rho}{2}.
\]
So \( \|a(s) - a(t)\| < \rho \). This completes the proof of the claim.

We now claim that \( a \in \text{UInd}(A) \). Let \( s \in G \) and let \( k \in H \). Using left invariance of \( \mu \) at the last step, we get
\[
\alpha_k(a(sk)) = \alpha_k \left( \sum_{W \in \mathcal{W}} \int_H g_W(skh) \cdot \alpha_h(a_W)\,d\mu(h) \right) \\
= \sum_{W \in \mathcal{W}} \int_H g_W(skh) \cdot \alpha_{kh}(a_W)\,d\mu(h) = a(s).
\]
The claim is proved.

It remains to show that \( \|\pi \circ a - b\| < \epsilon \). Let \( s \in G \). For \( W \in \mathcal{W} \), we have constructed \( g_W \) such that if \( h \in H \) and \( g_W(sh) \neq 0 \), then \( x_W^{-1}sh \in U \). For such \( h \) we have \( \|b(x_W) - b(sh)\| < \frac{\epsilon}{2} \) by the choice of \( U \). Using \( H \)-equivariance of \( \pi \) for the first equality and \( b \in \text{UInd}(B) \) for the third equality, we then get
\[
\|\pi(\alpha_h(a_W)) - b(s)\| = \|\beta_h(b(x_W)) - b(s)\| = \|b(x_W) - \beta_h^{-1}(b(s))\| \\
= \|b(x_W) - b(sh)\| < \frac{\epsilon}{2}.
\]
Therefore, using (4.4) and \( \sum_{W \in \mathcal{W}} l_W(sH) = 1 \) at the first and last steps,
\[
\|\pi(a(s)) - b(s)\| = \left\| \sum_{W \in \mathcal{W}} \int_H g_W(sh)\big(\pi(\alpha_h(a_W)) - b(s)\big)\,d\mu(h) \right\| \\
\leq \frac{\epsilon}{2} \sum_{W \in \mathcal{W}} \int_H g_W(sh)\,d\mu(h) < \epsilon,
\]
as desired. \( \square \)
Theorem 4.22. Let $G$ be a locally compact group, and let $H \leq G$ be a closed subgroup. Suppose that whenever $\varphi: A \to B$ is a surjective $H$-morphism of $C^*$-algebras, then $\text{UInd}_H^G(\varphi)$ is also surjective. Let $\alpha$ be a projective action of $G$. Then $\alpha|_H$ is also projective.

Proof. Let $\beta: H \to \text{Aut}(B)$ be an action of $H$ on a $C^*$-algebra $B$. The maps to be introduced are shown in the diagram below. Let $J$ be an $H$-invariant ideal in $B$, and let $\kappa: B \to B/J$ be the quotient map. Let $\varphi: A \to B/J$ be an $H$-morphism. Then $\text{UInd}_H^G(\kappa): \text{UInd}_H^G(B) \to \text{UInd}_H^G(B/J)$ is surjective by hypothesis. We can still define $\eta: A \to \text{UInd}_H^G(B)$ by the same formula as in [Lemma 3.10] and it is still a $G$-morphism. It is easy to check that its range, which a priori is in $F_H^G(B)$, is actually in $\text{UInd}_H^G(B)$. Since $\alpha$ is projective, there is a $G$-morphism $\lambda: A \to \text{UInd}_H^G(B)$ such that $\text{UInd}_H^G(\kappa) \circ \lambda = \eta$. We still have $H$-equivariant maps $\text{ev}_i^B: \text{UInd}_H^G(B) \to B$ and $\text{ev}_i^{B/J}: \text{UInd}_H^G(B/J) \to B/J$, given by the same formulas as in [Lemma 3.9] which give the following commutative diagram:

\[
\begin{array}{ccc}
\text{UInd}_H^G(B) & \xrightarrow{\text{ev}_i^B} & B \\
\downarrow{\text{UInd}_H^G(\kappa)} & & \downarrow{\kappa} \\
A & \xrightarrow{\eta} & \text{UInd}_H^G(B/J) \xrightarrow{\text{ev}_i^{B/J}} B/J.
\end{array}
\]

It is easy to check that $\text{ev}_i^{B/J} \circ \eta = \varphi$. Therefore the map $\psi = \text{ev}_i^B \circ \lambda$ is a $H$-morphism from $A$ to $B$ such that $\kappa \circ \psi = \varphi$. This completes the proof that $\alpha|_H$ is projective. \hfill \Box

Theorem 4.23. Let $G$ be a locally compact group, and let $H \leq G$ be a closed subgroup. Assume that at least one of the following conditions is satisfied:

1. $H$ is compact.
2. $G/H$ is compact.
3. $G$ is a $[\text{SIN}]$-group.

Let $\alpha$ be a projective action of $G$. Then $\alpha|_H$ is also projective.

Proof. Combine [Theorem 4.18] and [Theorem 4.22]. \hfill \Box

We point out that [Theorem 4.23] applies whenever $G$ is the product of a discrete group and a locally compact abelian group.

Corollary 4.24. Let $G$ be a locally compact group, and let $A$ be a $G$-algebra which is equivariantly projective. Then $A$ is (nonequivariantly) projective.

Remark 4.25. [Corollary 4.24] implies that there is no projective action of a locally compact group on a nonprojective $C^*$-algebra. This is in contrast to [Example 3.13] where it is shown that the discrete group $\mathbb{Z}$ can act semiprojectively on a $C^*$-algebra which is not semiprojective in the usual sense.

Remark 4.26. The proof of [Theorem 4.22] cannot be generalized to cover semiprojectivity. This is clear from [Example 3.13] The problem is that there is no analog of [Proposition 3.8] for the right uniform induction functor.

Let $\mathbb{N}^\times = \{1, 2, \ldots, \infty\}$ be the one point compactification of $\mathbb{N}$. Set $B = C(\mathbb{N}^\times)$, and for $n \in \mathbb{N}$ set

\[J_n = \{b \in B: b(k) = 0 \text{ for } k \in \{n + 1, n + 2, \ldots, \infty\}\}.\]
Then $\bigcup_{n=1}^{\infty} J_n = C_0(\mathbb{N}) \subset B$. Call this ideal $J$. For $l \in \mathbb{N}$, define $b_l \in B$ by
\[
b_l(j) = \begin{cases} 1 & j = l \\ 0 & j \neq l, \end{cases}
\]
and define $a \in C_0(\mathbb{Z}, B)$ by $a(n) = b_n$ for $n \in \mathbb{N}$ and $a(n) = 0$ for $n \in \mathbb{Z} \setminus \mathbb{N}$. Then $a \in C_0(\mathbb{Z}, J)$, but the distance from $a$ to any element of $\bigcup_{n=1}^{\infty} C_0(\mathbb{Z}, J_n)$ is at least 1, so $a \notin \bigcup_{n=1}^{\infty} C_0(\mathbb{Z}, J_n)$.

We have written everything in terms of bounded continuous functions, but on $\mathbb{Z}$ all continuous functions are uniformly continuous.

5. SEMIPROJECTIVITY OF THE CROSSED PRODUCT ALGEBRA

If $(G, A, \alpha)$ is an equivariantly semiprojective $C^*$-algebra, can we deduce that the crossed product algebra $A \rtimes_\alpha G$ is semiprojective? We show in Theorem 5.1 that the answer is yes when $G$ is finite and $A$ is unital, and in Example 5.2 that the answer can be no when $G$ is compact. We then provide examples to show that the converses of both Theorem 5.1 and Corollary 3.12 are false. We end the section with further open problems.

**Theorem 5.1.** Let $G$ be a discrete group such that $C^*(G)$ is semiprojective, and let $(G, A, \alpha)$ be a unital $G$-algebra which is equivariantly semiprojective in the unital category. Then $A \rtimes_\alpha G$ is semiprojective (in the usual sense).

**Proof.** We will show that $A \rtimes_\alpha G$ is semiprojective in the unital category. Applying Lemma 1.6 with the group being trivial, we conclude that $A \rtimes_\alpha G$ is semiprojective.

We regard $A$ as a subalgebra of $A \rtimes_\alpha G$. Also, for $s \in G$ let $u_s \in A \rtimes_\alpha G$ be the standard implementing unitary, so that $u_s a u_s^* = \alpha_s(a)$ for all $a \in A$. The unitaries $u_s$ induce a $^*$-homomorphism $\omega: C^*(G) \to A \rtimes_\alpha G$.

By assumption, $C^*(G)$ is semiprojective. Thus, Lemma 1.4 of [Phi12] shows that it suffices to prove that $\omega$ is relatively semiprojective in the sense of Definition 1.2 of [Phi12] (but with the group being trivial). Accordingly, let $C$ be a unital $C^*$-algebra, let $J_1 \subset J_2 \subset \cdots$ be ideals in $C$, let $J = \bigcup_{n=1}^{\infty} J_n$, let $\kappa: C \to C/J$, $\kappa_n: C \to C/J_n$, and $\pi_n: C/J_n \to C/J$ be the quotient maps, and let $\lambda: C^*(G) \to C$ and $\varphi: A \rtimes_\alpha G \to C/J$ be unital $^*$-homomorphisms such that $\kappa \circ \lambda = \varphi \circ \omega$.

Define an action $\gamma: G \to \text{Aut}(C)$ by $\gamma_s(c) = \lambda(u_s)c \lambda(u_s)^*$ for $c \in C$ and $s \in G$. Then $(G, C, \gamma)$ is a unital $G$-algebra, and the ideals $J_n$ are $G$-invariant.

One checks that $\varphi|_A: A \to C/J$ is $G$-equivariant. Since $(G, A, \alpha)$ is equivariantly semiprojective (in the unital category), there exists $n \in \mathbb{N}$ and a unital $G$-morphism $\psi_0: A \to C/J_n$ such that $\pi_n \circ \psi_0 = \varphi|_A$. Define $v_s = (\kappa_n \circ \lambda)(u_s)$ for $s \in G$. Then $(v, \psi_0)$ is a covariant representation of $(G, A, \alpha)$ in $C/J_n$, so there exists a unique $^*$-homomorphism $\psi: A \rtimes_\alpha G \to C/J_n$ such that $\psi(u_s) = v_s$ and $\psi|_A = \psi_0$. This $^*$-homomorphism is the one required by the definition of relative semiprojectivity. 

The basic examples of countable discrete groups $G$ that satisfy the hypothesis of Theorem 5.1 that is, such that $C^*(G)$ is semiprojective, are finite groups, $\mathbb{Z}$, and the finitely generated free groups. There is no known characterization of those groups $G$ for which $C^*(G)$ is semiprojective.

In Theorem 5.1 some restriction on $G$ is necessary. Even compactness is not enough.
Example 5.2. Let $G$ be an infinite compact group. It follows from Corollary 1.9 of [Phi12] that the trivial action of $G$ on $\mathbb{C}$ is semiprojective. However, the crossed product is $\mathbb{C} \rtimes G = C^*(G)$, which is an infinite direct sum of matrix algebras, so not semiprojective by Corollary 2.10 of [Bla04].

Theorem 5.1 gives us an easy way of proving that many actions by $\mathbb{Z}$ are not equivariantly semiprojective (in the unital category).

Example 5.3. Let $\theta \in \mathbb{R}$. Let $\alpha : \mathbb{Z} \to \text{Aut}(C(S^1))$ be the action generated by rotation by $\exp(2\pi i \theta)$. Then $\alpha$ is never semiprojective in the unital category, for any value of $\theta$.

If $\theta \notin \mathbb{Q}$, then $A = C(S^1) \rtimes_{\alpha} \mathbb{Z}$ is Morita equivalent to $C((S^1)^2)$. Since both $A$ and $C((S^1)^2)$ are unital and $C((S^1)^2)$ is not semiprojective, it follows from Corollary 2.29 of [Bla85] that $A$ is not semiprojective.

In both cases, it follows from Theorem 5.1 that $\alpha$ is not equivariantly semiprojective.

There are versions of Theorem 5.1 in which one takes the crossed product by only part of the action. As an easy example, consider an action of a product of two groups, and take the crossed product by one of them. We will not explore the possibilities further here.

We end this section with two examples that show that the converses of both Theorem 5.1 and Corollary 3.12 are false, and we give more open problems.

Example 5.4. There is an action $\alpha$ of $\mathbb{Z}_2$ on $O_2$ such that the crossed product $B = O_2 \rtimes_{\alpha} \mathbb{Z}_2$ is not semiprojective. It follows from Theorem 5.1 that this action is not equivariantly semiprojective. Thus, the converse of Corollary 3.12 fails.

We follow [Izu04]; also see Section 6 of [Bla04]. Take $\alpha$ to be as in Lemma 4.7 of [Izu04] or, more generally, as in Theorem 4.8(3) of [Izu04] with the groups $\Gamma_0$ and $\Gamma_1$ chosen so that at least one of them is not finitely generated, and also such that $O_2 \rtimes_{\alpha} \mathbb{Z}_2$ satisfies the Universal Coefficient Theorem. The action $\alpha$ is outer, so $B$ is simple by Theorem 3.1 of [Kis81] and purely infinite by Corollary 4.6 of [JO98]. Therefore it is a Kirchberg algebra (a separable purely infinite simple nuclear $C^*$-algebra). It does not have finitely generated K-theory, so $B$ is not semiprojective by Corollary 2.11 of [Bla04].

Example 5.5. Let $\hat{\alpha} : \mathbb{Z}_2 \to \text{Aut}(B)$ be the dual of the action $\alpha$ of Example 5.4. Then

$$B \rtimes_{\hat{\alpha}} \mathbb{Z}_2 \cong M_2 \otimes O_2 \cong O_2,$$

which is semiprojective. However, $B$ was shown in Example 5.4 not to be semiprojective. So Corollary 3.12 implies that $\hat{\alpha}$ is not equivariantly semiprojective. This shows that the converse of Theorem 5.1 fails.

Example 5.4 also shows if $A$ is semiprojective and $\alpha : G \to \text{Aut}(A)$ is an action of a finite group on $A$, then $(G, A, \alpha)$ need not be equivariantly semiprojective. However, we have neither a proof nor a counterexample for the following question.

Question 5.6. Let $G$ be a finite cyclic group of prime order, and let $(G, A, \alpha)$ be a $G$-algebra. Suppose that $A$ and $A \rtimes_{\alpha} G$ are both semiprojective. Does it follow that $(G, A, \alpha)$ is equivariantly semiprojective?
If \( \alpha: G \to \text{Aut}(A) \) is semiprojective, then Theorem 3.11 implies that for any subgroup \( H \leq G \), the action \( \alpha|_H \) is also semiprojective. Thus, by Theorem 5.1, the crossed product \( A \rtimes_{\alpha|_H} H \) is semiprojective. If \( G \) has proper subgroups, one must therefore probably also consider these intermediate crossed product algebras.

At a conference in August 2010, George Elliott asked if there is a relation between equivariant semiprojectivity and the Rokhlin property. The following question addresses what seems to be a plausible connection.

**Question 5.7.** Let \( G \) be a finite group, and let \((G, A, \alpha)\) be a unital \(G\)-algebra. Suppose that \( A \) is (nonequivariantly) semiprojective and \( \alpha \) has the Rokhlin property. Does it follow that \((G, A, \alpha)\) is equivariantly semiprojective?

Even if this is false in general, it might be true if \( A \) is simple, or using an equivariant version of a weak form of semiprojectivity.

6. **Semiprojectivity of the fixed point algebra**

In this section we study the analog of the question of Section 5 for the fixed point algebra. That is, given an equivariantly semiprojective \( C^*\)-algebra \((G, A, \alpha)\), can we deduce that the fixed point algebra \( A^G \) is semiprojective?

In Proposition 6.2 we give a positive answer when \( G \) is finite, \( A \) is unital, and the action is saturated. We do not know whether one can drop the conditions that \( A \) be unital or that the action be saturated.

Some conditions are necessary. In Example 6.1 we give a semiprojective action of a compact (but not finite) group on a unital \( C^*\)-algebra such that the fixed point algebra is not semiprojective.

In Theorem 6.4 we show that if a noncompact group acts semiprojectively then the fixed point algebra is trivial. This gives a positive answer to the question, but more interestingly it shows that the trivial action of a noncompact group is never semiprojective. We can therefore give a precise characterization when the trivial action of a group is (semi)projective (Corollary 6.5).

Let \( G \) be a second countable compact group and let \( \alpha: G \to \text{Aut}(A) \) be a semiprojective action. Example 5.2 shows that the crossed product \( A \rtimes_\alpha G \) need not be semiprojective, but in that example the fixed point algebra is semiprojective.

In general, though, the fixed point algebra also need not be semiprojective.

**Example 6.1.** Let \( \alpha: S^1 \to \text{Aut}(O_2) \) be the gauge action on the Cuntz algebra \( O_2 \), defined on the standard generators \( s_1 \) and \( s_2 \) by \( \alpha_\zeta(s_j) = \zeta s_j \) for \( \zeta \in S^1 \) and \( j = 1, 2 \). This action is equivariantly semiprojective by Corollary 3.12 of [Phi87]. However, the fixed point algebra is the \( 2^\infty \) UHF algebra, which is not semiprojective, for example by Corollary 2.14 of [Bla04].

We obtain a positive result when the group is finite and the action is saturated in the sense of Definition 7.1.4 of [Phi87]. Saturation is a quite weak noncommutative analog of freeness; see the discussion at the beginning of Section 5.2 of [Phi09].

**Proposition 6.2.** Let \( G \) be a finite group, let \( A \) be a unital, separable \( C^*\)-algebra, and let \( \alpha: G \to \text{Aut}(A) \) be a saturated action of \( G \) on \( A \). If \( \alpha \) is semiprojective, then \( A^G \) is semiprojective.

**Proof.** By definition, saturation implies that \( A^G \) is strongly Morita equivalent to \( A \rtimes_\alpha G \). Theorem 5.1 tells us that \( A \rtimes_\alpha G \) is semiprojective, so \( A^G \) is semiprojective by Corollary 2.29 of [Bla85].
Finiteness is needed, since the gauge action in [Example 6.1] is saturated. (In fact, it follows from Theorem 5.11 of [Pho09] that this action is hereditarily saturated.) However, we don’t know whether saturation is needed.

**Question 6.3.** Let $G$ be a finite group, let $A$ be a unital $C^*$-algebra, and let $\alpha: G \to \text{Aut}(A)$ be an arbitrary semiprojective action of $G$ on $A$. Does it follow that $A^G$ is semiprojective?

If $G$ is compact and $A$ unital, then $A^G$ is isomorphic to a unital corner in $A \rtimes_\alpha G$, for example by Theorem II.10.4.18 of [Bla06]. If we knew that semiprojectivity passes to arbitrary unital corners (an open problem), we would get a positive answer to **Question 6.3**.

**Theorem 6.4.** Let $(G, A, \alpha)$ be a separable equivariantly semiprojective $G$-algebra, and assume that $G$ is not compact. Then $A^G = \{0\}$.

**Proof.** We manufacture an equivariant lifting problem in several steps.

Step 1: The action $\alpha: G \to \text{Aut}(A)$ induces an action $\overline{\alpha}: G \to \text{Aut}(M_2 \otimes A)$ by acting trivially on $M_2$, that is, $\overline{\alpha}_s(x \otimes a) = x \otimes \alpha_s(a)$ for $x \in M_2$, $a \in A$, and $s \in G$. Let $(e_{j,k})_{j,k=1,2}$ be the standard system of matrix units for $M_2$. Let $\lambda \mapsto \mathcal{A}_\lambda \in M_2$ be a continuously differentiable path of unitaries from the identity $u_0 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ to $u_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. Continuous differentiability is required for convenience; it gives us $M \in [0, \infty)$ such that for all $\lambda_1, \lambda_2 \in [0, 1]$ we have $\|u_{\lambda_1} - u_{\lambda_2}\| \leq M|\lambda_1 - \lambda_2|$. For $\lambda \in [0, 1]$ define $\varphi_\lambda: A \to M_2 \otimes A$ by $\varphi_\lambda(a) = u_{\lambda}e_{1,1}u_1^* \otimes a$ for $a \in A$. Thus $\varphi_0(\alpha)(a) = e_{1,1} \otimes a$ and $\varphi_1(a) = e_{2,2} \otimes a$ for $a \in A$. Also, for $\lambda_1, \lambda_2 \in [0, 1]$ and $a \in A$, we have

$$
\|\varphi_{\lambda_1}(a) - \varphi_{\lambda_2}(a)\| \leq 2\|u_{\lambda_1} - u_{\lambda_2}\| \cdot \|a\| \leq 2M|\lambda_1 - \lambda_2| \cdot \|a\|.
$$

(6.1)

It is immediate that

$$
\overline{\alpha}_s \circ \varphi_\lambda = \varphi_\lambda \circ \alpha_s.
$$

(6.2)

for $s \in G$ and $\lambda \in [0, 1]$.

Step 2: Let $G^+ = G \cup \{\infty\}$ denote the one point compactification of $G$. Let $D$ be the $C^*$-algebra

$$
D = \{ f \in C(G^+, M_2 \otimes A): f(\infty) \in \mathbb{C}e_{2,2} \otimes A \}.
$$

For $s \in G$, we take $s \cdot \infty = \infty$. This gives an extension of the action of $G$ on itself by translation to a continuous action of $G$ on $G^+$. We define an action $\beta$ of $G$ on $D$ by $\beta_s(f)(t) = \overline{\alpha}_s(f(s^{-1}t))$ for $f \in D$, $s \in G$, and $t \in G^+$. Since $G$ is not compact, the fixed point algebra of this action consists of the constant functions taking values in $\mathbb{C}e_{2,2} \otimes A^G$.

Step 3: For $k = 1, 2, \ldots$, define “stretching” maps $\sigma_k: [0, \infty) \to [0, 1]$ by

$$
\sigma_k(\lambda) = \min(\lambda/k, 1)
$$

for $\lambda \in [0, \infty)$. We may extend these to maps from $[0, \infty]$ by setting $\sigma_k(\infty) = 1$ for $k \in \mathbb{N}$. For $\lambda_1, \lambda_2 \in [0, \infty)$, we have

$$
|\sigma_k(\lambda_1) - \sigma_k(\lambda_2)| \leq \frac{|\lambda_1 - \lambda_2|}{k}.
$$

(6.3)

Step 4: Recall that a metric is called *proper* if every closed bounded set is compact. By the main theorem of [Str74], there is a proper left invariant metric $d$ which generates the topology of $G$. For $t \in G$ let $d_0(t) = d(t, 1)$ denote the distance
from \( t \) to the identity \( 1 \in G \), and extend this function to \( G^+ \) by setting \( d_0(\infty) = \infty \).

Since \( d \) is proper, the map \( d_0: G^+ \to [0, \infty] \) is continuous.

Using left invariance of \( d \), for \( s, t \in G \) we get \(|d_0(s^{-1}t) - d_0(t)| \leq d_0(s)\). Therefore, for \( k \in \mathbb{N} \),

\[
|\sigma_k(d_0(s^{-1}t)) - \sigma_k(d_0(t))| \leq \frac{d_0(s)}{k}. 
\]

(6.4)

For \( k \in \mathbb{N} \), we define a \(*\)-homomorphism \( \omega_k: A \to D \) by

\[
\omega_k(a)(t) = \varphi_{\sigma_k(d_0(t))}(a)
\]

for \( a \in A \) and \( t \in G^+ \). Then for \( s \in G \) we have, using density of \( G \) in \( G^+ \) and (6.2) at the second step, (6.1) at the third step, and (6.4) at the fourth step,

\[
\|\beta_s(\omega_k(a)) - \omega_k(\alpha_s(a))\| = \sup_{t \in G^+} \|\pi_s(\varphi_{\sigma_k(d_0(s^{-1}t))}(a)) - \varphi_{\sigma_k(d_0(t))}(\alpha_s(a))\|
\]

(6.5)

\[
= \sup_{t \in G} \|\varphi_{\sigma_k(d_0(s^{-1}t))}(\alpha_s(a)) - \varphi_{\sigma_k(d_0(t))}(\alpha_s(a))\|
\]

\[
\leq \sup_{t \in G} 2M|\sigma_k(d_0(s^{-1}t)) - \sigma_k(d_0(t))| \cdot \|\alpha_s(a)\|
\]

\[
\leq \frac{2M\|a\|d_0(s)}{k}.
\]

In particular, we have

\[
\lim_{k \to \infty} \|\beta_s(\omega_k(a)) - \omega_k(\alpha_s(a))\| = 0.
\]

Moreover, for \( k \in \mathbb{N} \) and \( a \in A \), we have \( \omega_k(a)(1) = e_{1,1} \otimes a \), so, using Step 2,

\[
\text{dist}(\omega_k(a), D^{G^+}) \geq \inf_{b \in A} \|e_{1,1} \otimes a - e_{2,2} \otimes b\|
\]

(6.6)

\[
\geq \inf_{b \in A} \|(e_{1,1} \otimes 1)(e_{1,1} \otimes a - e_{2,2} \otimes b)\| = \|a\|.
\]

Step 5: Let \( E \) be the sequence algebra \( E = l^\infty(\mathbb{N}, D) \). Let \( \gamma: G \to \text{Aut}(E) \) denote the (not necessarily continuous) coordinatewise action of \( G \) on \( E \); that is, for \( s \in G \) and \((x_k)_{k \in \mathbb{N}} \in E \) we set \( \gamma_s((x_k)_{k \in \mathbb{N}}) = (\beta_s(x_k))_{k \in \mathbb{N}} \). Let \( F \subset E \) be the \( G^\ast\)-subalgebra

\[
F = \{ x \in E: s \mapsto \gamma_s(x) \text{ is continuous} \}.
\]

Then \( F \) is \( \gamma \)-invariant, and we also use \( \gamma \) to denote the restricted action \( \gamma: G \to \text{Aut}(F) \). By construction, this action is continuous.

For \( n \in \mathbb{N} \) define

\[
J_n = \{(x_k)_{k \in \mathbb{N}} \in E: x_k = 0 \text{ for } k \geq n \} \subset E.
\]

Then \( J_1 \subset J_2 \subset \cdots \) is an increasing sequence of invariant ideals, and the ideal \( J = \bigcup_{n=1}^\infty J_n \) is equal to \( C_0(\mathbb{N}, D) \subset l^\infty(\mathbb{N}, D) \). Clearly \( J \subset F \). We can identify \( F/J \) with the set of elements of \( l^\infty(\{n+1, n+2, \ldots\}, D) \) on which the coordinatewise action of \( G \) is continuous.

For \( n \in \mathbb{N} \), the action of \( G \) on \( F \) drops to \( F/J_n \), and the action also drops to \( F/J \). Let \( \pi_n: F/J_n \to F/J \) be the natural quotient \( G \)-morphism. We have

\[
F^G = l^\infty(\mathbb{N}, D^G),
\]

and one checks by direct computation that \((F/J_n)^G = F/J_n^G\), which we identify with \( l^\infty(\{n+1, n+2, \ldots\}, D^G)\).
Step 6: For each $a \in A$, consider the sequence $\omega(a) = (\omega_1(a), \omega_2(a), \ldots) \in E$ constructed in Step 4. We claim that $\omega(a) \in F$. To see this, let $a \in A$ and let $s, t \in G$. Then, using $\ref{65}$ and $\|\omega_k\| = 1$ at the third step,

$$\|\gamma_s(\omega(a)) - \gamma_t(\omega(a))\| = \sup_{k \in \mathbb{N}} \|\beta_s(\omega_k(a)) - \beta_t(\omega_k(a))\|$$

$$= \sup_{k \in \mathbb{N}} \|\beta_{t^{-1}s}(\omega_k(a)) - \omega_k(\alpha_{t^{-1}s}(a)) + \omega_k(\alpha_{t^{-1}s}(a) - a)\|$$

$$\leq \sup_{k \in \mathbb{N}} \frac{2M\|a\|d_0(t^{-1}s)}{k} + \|\alpha_{t^{-1}s}(a) - a\|$$

$$= 2M\|a\|d(s, t) + \|\alpha_s(a) - \alpha_t(a)\|.$$ 

Since $\alpha$ is a continuous action, this proves the claim.

Step 7: Define a *-homomorphism $\overline{\psi}: A \to F/J$ by sending $a \in A$ to the image of $\omega(a)$ in the quotient $F/J$. It follows from $\ref{6.9}$ that $\overline{\psi}$ is a $G$-morphism.

Suppose now that $A$ is equivariantly semiprojective. Then there are $n \in \mathbb{N}$ and a $G$-morphism $\psi: A \to F/J_n$ such that $\pi_n \circ \psi = \overline{\psi}$.

Fix an element $a \in A^G$. We want to show $a = 0$. Since $\psi$ is $G$-equivariant, $\psi(a) \in (F/J_n)^G$. Identify $(F/J_n)^G$ with $l^\infty(\{n + 1, n + 2, \ldots\}, D^G)$ as at the end of Step 5, and write $\psi(a) = (\psi_{n+1}(a), \psi_{n+2}(a), \ldots)$. Then, using $\ref{6.7}$ at the last step,

$$\|\pi_n(\psi(a)) - \overline{\psi}(a)\| = \left\|\pi_n((\psi_{n+1}(a), \psi_{n+2}(a), \ldots) - (\omega_{n+1}(a), \omega_{n+2}(a), \ldots))\right\|$$

$$= \lim_{k \to \infty} \inf \|\psi_k(a) - \omega_k(a)\|$$

$$\geq \inf_{k \in \{n+1, n+2, \ldots\}} \text{dist}(\omega_k(a), D^G) \geq \|a\|.$$ 

For $a \neq 0$ this contradicts $\pi_n(\psi(a)) = \overline{\psi}(a)$. Thus $A^G = \{0\}$. 

\begin{corollary}
Let $A$ be a nonzero separable C*-algebra, and let $G$ be a second countable locally compact group. Then the trivial action of $G$ on $A$ is (semi)projective if and only if $A$ is (semi)projective and $G$ is compact.
\end{corollary}

\begin{proof}
If $G$ is not compact and $(G, A, \alpha)$ is equivariantly semiprojective, then Theorem 6.3 implies that $A^G = \{0\}$. If $\alpha$ is trivial, it follows that $A = \{0\}$.

Now suppose $G$ is compact. If $A$ is (semi)projective, then it follows from Corollary 1.9 of \cite{Phi12} and Lemma 1.9 that the trivial action of $G$ on $A$ is (semi)projective. The converse follows from Corollary 4.24 and Corollary 3.12.
\end{proof}

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