Creep and depinning in disordered media

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Elastic systems driven in a disordered medium exhibit a depinning transition at zero temperature and a creep regime at finite temperature and slow drive $f$. We derive functional renormalization group equations which allow to describe in detail the properties of the slowly moving states in both cases. Since they hold at finite velocity $v$, they allow to remedy some shortcomings of the previous approaches to zero temperature depinning. In particular, they enable us to derive the depinning law directly from the equation of motion, with no artificial prescription or additional physical assumptions, such as a scaling relation among the exponents. Our approach provides a controlled framework to establish under which conditions the depinning regime is universal. It explicitly demonstrates that the random potential seen by a moving extended system evolves at large scale to a random field and yields a self-contained picture for the size of the avalanches associated with the deterministic motion. At finite temperature $T > 0$ we find that the effective barriers grow with length scale as the energy differences between neighboring metastable states, and demonstrate the resulting activated creep law $v \sim \exp \left(-C f^{-\mu}/T\right)$ where the exponent $\mu$ is obtained in a $\epsilon = 4-D$ expansion ($D$ is the internal dimension of the interface). Our approach also provides quantitatively a new scenario for creep motion as it allows to identify several intermediate length scales. In particular, we unveil a novel “depinning-like” regime at scales larger than the activation scale, with avalanches spreading from the thermal nucleus scale up to the much larger correlation length $R_V$. We predict that $R_V \sim T^{-\sigma} f^{-\lambda}$ diverges at small drive and temperature with exponents $\sigma, \lambda$ that we determine.

\section{I. INTRODUCTION}

Understanding the statics and dynamics of elastic systems in a random environment is a longstanding problem with important applications for a host of experimental systems. Such problems can be split into two broad categories: (i) propagating interfaces such as magnetic domain walls, fluid invasion in porous media, or epitaxial growth; (ii) periodic systems such as vortex lattices, charge density waves, or Wigner crystals of electrons. In all these systems the basic physical ingredients are identical: the elastic forces tend to keep the structure ordered (flat for an interface and periodic for lattices), whereas the impurities locally promote the wandering. From the competition between disorder and elasticity emerges a complicated energy landscape with many metastable states. This results in glassy properties such as hysteresis and history dependence of the static configuration. In the dynamics, one expects of course this competition to have important consequences on the response of the system to an externally applied force.

To study the statics, the standard tools of statistical mechanics could be applied, leading to a good understanding of the physical properties. Scaling arguments and simplified models showed that even in the limit of weak disorder, the equilibrium large scale properties of disordered elastic systems are governed by the presence of impurities. In particular, below four (internal) dimensions, displacements grow unboundedly with the distance, resulting in rough interfaces and loss of strict translational order in periodic structures. To go beyond simple scaling arguments and obtain a more detailed description of the system is rather difficult and at present only main two methods, each with its own shortcomings, have been developed. The first one is to perform a perturbative renormalization group calculation on the disorder, and is valid in $4 - \epsilon$ dimensions to first order in $\epsilon$. In this functional renormalization group (FRG) approach\textsuperscript{1,2}, the whole correlation function of the disorder is renormalized. The occurrence of glassiness is signalled by a non-analyticity appearing at a finite length scale during the flow, specifically a cusp in the force correlator. This yields non trivial predictions for the roughness exponents of interfaces. Another approach relies on the replica method to study either the mean field limit (i.e. large number of components) or to perform a gaussian variational approximation of the physical model. Using this variational approach both for manifolds\textsuperscript{3,9,11} and for periodic systems\textsuperscript{8,13} correlation functions and thermodynamic properties could be obtained. It confirms the existence of glassy properties, with energy fluctuations growing as $L^{\theta}$ where $\theta$ is a positive exponent. To obtain the glass phase in this method, one must break the replica symmetry. At a qualitative level, this is in good agreement with the physical intuition of such systems as being composed of many low lying metastable states separated by high barriers. As was clearly shown in the case of periodic manifolds, the correlation functions can be obtained by both the FRG and variational approach and are found to be in very reasonable agreement, bridging the gap between the two methods.\textsuperscript{3,9} Taken together, these two approaches thus provide a very coherent picture for the statics.

In particular they allow to understand that although disorder leads to glassy features in both the manifold and the
periodic systems, these two types of problems belong to quite different universality classes in other respects, such as the large distance behaviour of the correlations.

These properties have drastic consequences for the dynamics of driven systems in the case, important in practice, where an elastic description holds (i.e. when plastic deformations can be neglected). Determining the response to an externally applied force is not only an interesting theoretical question, but also one of the most important experimental issues. Indeed in most systems the velocity \( v \) versus force \( f \) characteristics is directly measurable and is simply linked to the transport properties (voltage-current for vortices, current-voltage for CDW and Wigner crystals, velocity-applied magnetic field for magnetic domain walls). In the presence of disorder it is natural to expect that, at zero temperature, the system remains pinned and only polarizes under the action of a small applied force, i.e. moves until it locks on a local minimum of the tilted energy landscape. At larger drive, the system follows the force \( f \) and acquires a non-zero asymptotic velocity \( v \). In the simplest cases, the effect of disorder at large velocity is washed out and one recovers the viscous flow, as in the pure case. In the thermodynamic limit, it is believed that there exists a threshold force \( f_c \) separating both states, and that a dynamical transition occurs at \( f_c \), called depinning, where the velocity is continuously switched on, like an order parameter of a second order transition in an equilibrium system, leading to a \( v-f \) characteristics such as the one shown in Figure 1.

![Typical force–velocity characteristics](image)

**FIG. 1.** Typical force–velocity characteristics, exhibiting pinning at \( T = 0 \) with a threshold force \( f_c \) and creep at \( T > 0 \), \( f < f_c \). At large drive, the system flows as if there were no disorder.

An estimate of \( f_c \) can be obtained via scaling arguments or with a criterion for the breakdown of the large velocity expansion. Beyond \( f_c \), if one describes the depinning as a conventional dynamical critical phenomenon, the important quantities to determine are of course the depinning exponent \( \beta \) giving the velocity \( v \sim (f - f_c)^\beta \) and the dynamical exponent \( z \) which relates space and time as \( t \sim v^z \).

An even more challenging question, and experimentally at least as relevant, is the response at finite temperature \( T > 0 \). In the most naive description, the system can now overcome barriers via thermal activation, leading to a thermally assisted flow and a linear response at small force of the form \( v \sim e^{-\Delta f/T} f \), where \( \Delta \) is some typical barrier. It was realized that because of the glassy nature of the static system, the motion is actually dominated by barriers which diverge as the drive \( f \) goes to zero, and thus the flow formula with finite barriers is incorrect. Well below the threshold critical force, the barriers are very high and thus the motion, usually called “creep” is extremely slow. Scaling arguments, relying on strong assumptions such as the scaling of energy barriers and the use of statics properties to describe an out of equilibrium system, were used to infer the small \( f \) response. This led to a non linear response, characteristic of the creep regime, of the form \( v \sim \exp(-\zeta f^{-\mu}/T) \) where \( \mu = (D-2+2\zeta)\zeta/(D-2+2\zeta) \) and \( \zeta \) is the roughening exponent for the static \( D \)-dimensional system.

Obtaining a detailed experimental confirmation of this behaviour is a non trivial feat, in reasons of the range in velocity required. Although in vortex systems these highly non linear flux creep behaviours have been measured ubiquitously, it is rather difficult to obtain clean determination of the exponents, given the many regimes of lengthscales which characterize type II superconductors. In some recent measurement, some agreement with the creep law in the Bragg glass regime was obtained.

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Given the phenomenological aspect of these predictions and the uncontrolled nature of the assumptions made, both for the creep and for the depinning, it is important to derive this behavior in a systematic way from the equation of motion. Less tools are available than for the statics, and averages over disorder should be made using dynamical methods. Fortunately, it is still possible to use a functional renormalization group (FRG) approach for the dynamical problem. Such an approach has been used at \( T = 0 \) to study depinning. It allowed for a calculation of the depinning exponents, in \( D = 4 - \epsilon \). However this approach is still rather unsatisfactory. The FRG flow used in is essentially the static one, the finite velocity being only invoked to remove - by hand - some ambiguities and to cutoff the flow, with no real controlled way to show that this is the correct procedure. Furthermore in these approaches it is also necessary to assume, instead of deriving them from the FRG, some scaling relations in order to obtain the exponents. Another rather problematic point is that, with no additional input, the method of would yield three universality classes for the depin-
ning: two universality classes depending on the nature of the disorder (random bond versus random field) for manifolds and one for periodic systems, while numerics and physical arguments suggested that only two (random field and periodic) universality classes could exist. In addition, since this is also intrinsically a $T = 0$ (and $v = 0$) approach, it can not be used to tackle the creep behavior.

We propose here a single theory for describing all the regimes of a moving elastic system, including depinning and the non-zero temperature regimes. Our FRG equations contain from the start the finite velocity and finite temperature. They thus allow to address questions which are beyond the reach of either approximate scaling theories, or $v = 0$ FRG flow. For the depinning we are able to determine the conditions required for the existence of a universal depinning behavior, as well as computing the depinning exponents (and estimating $f_c$). We show in particular that only two universality classes exist (out of the three) for the depinning since we explicitly find that random bond systems flow to the random field universality class. We can also extract from our equations the characteristic lengthscales of the depinning. The main advantage of our approach is of course to address the finite $T$ small $v$ regime as well. The method allows to derive the creep formula directly and thus allows to confirm the assumptions made on the scaling of the energy barriers. In addition we show that the creep is followed by a depinning-like regime and determine its characteristic lengthscales. A short account of some of these results was presented in Ref. 22.

The paper is organized as follows: In Section II we present the equation of motion and the types of disorder studied here. Section III is devoted to a brief review of scaling arguments and a summary of useful results from perturbation theory, presented in Appendix A. Section IV contains the field theoretical formulation of the problem and the associated renormalization group flow equations, derived in Appendix B. The static case is studied in Subsection IV C. In the next sections, we study the depinning and creep regimes. Both sections contain the outline of the derivation and a physical discussion. Appendix C is devoted to the effect of a small velocity on the FRG. We conclude in Section VII, referring to an extension of our work proposed in Appendix F. In Appendix A we fix the notations used throughout the paper.

II. ELASTICITY AND DISORDER

Elastic systems are extended objects which “prefer” to be flat or well ordered. We are dealing with two different types of elastic systems which however can be treated in the same way. On the one hand, interfaces, i.e. surfaces with a stiffness that makes local distortions energetically expensive, on the other hand, lattices with elastic displacements allowed about a regularly ordered configuration.

The first type is the easiest to visualize. The interface is assumed to have no overhangs and is thus described by a height function $u_r$ defined at each point $r$ (see Figure 2). Its energy is proportional to its area $\int_r \sqrt{1 + |\nabla u|^2}$ and in the elastic limit $|\nabla u| \ll 1$, reduces to

$$H_{01}[u] = \int_r \frac{c}{2} |\nabla u|^2$$

relative to the flat $u_r = 0$ configuration (notations are defined in the Appendix A). We denote by $c$ the stiffness, or elastic constant.

![FIG. 2. Top: an interface with height field $u_r$ above $r$. We denote by $r$ the (internal) coordinates along the interface and by $z$ the height coordinates. Bottom: a lattice with reference positions $R$ and displacements $u_R$ from $R$.](image)

Periodic structures, such as flux line lattices or charge density waves (CDW), can be described by the same type of elastic Hamiltonian. For each point (or line) in the elastic periodic system one can introduce a (vector) displacement field $u_R$ that gives the shift from the reference position $R$ (see Figure 3). The elastic energy for small displacements is given by a quadratic form in the differences $u_R - u_{R'}$ between neighboring points and thus can be written as in (2.1) in a continuum description ($r$ being a generic point in space). When $u$ has more than one component, $c$ should be understood as a tensor (see Appendix E).

To take the quenched disorder into account in such systems it is necessary to express the energy of the above elastic structure in the presence of impurities. The coupling to a substrate or to local fields is easily written for interface models and is more subtle for lattices. Quite generally the coupling to disorder leads to an energy:

$$H_{\text{dis}} = \int_r V(r, u_r)$$

which gives rise to a pinning force $F(r, u) = -\partial_u V(r, u)$ acting on the displacement $u_r$. Depending on the micro-
scopic origin of the disorder term $V$, the coupling \((2.2)\) leads to quite different physics. In the case of interfaces \((2.2)\) originates from

\[
H_{\text{dis}} = \int_{r,z} V(r,z) \rho(r,z) \tag{2.3}
\]

\[
\rho(r,z) = \int_{\kappa} \delta(z-u_r) \tag{2.4}
\]
in terms of the density $\rho(r,z)$. One then usually distinguishes two cases: either “random bond” (RB) when $V(r,z)$ is short range (random exchange for magnetic domain walls), or “random field” (RF) as discussed below, where $V(r,z)$ has long range correlations.

In the case of periodic structures, the density $\rho(r)$ can be expressed using the set of vectors $\kappa$ of the reciprocal lattice and \((2.2)\) originates from

\[
H_{\text{dis}} = \int_{F} W(r) \rho(r) \tag{2.5}
\]

\[
\rho(r) \approx \rho_{0} \sum_{\kappa} e^{i\kappa \cdot (r-u_r)} \tag{2.6}
\]

where $\rho_{0}$ the average density. The potential $W$ is random, of short range $r_f$ (e.g. point impurities for a vortex lattice or a CDW). We call this case “random periodic” (RP).

In both cases, using \((2.3,2.4,2.5,2.6)\) and \((2.2)\) one obtains for the correlations of $V$ in \((2.2)\)

\[
(V(r,u) - V(r',u'))^2 = -2\delta_{rr'} R(u-u') \tag{2.7}
\]

where $R(u)$ is a periodic function with the periodicity $a$ of the lattice in the periodic (RP) case. The $\delta$ function is cutoff at the microscopic scale $r_f$.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure3.png}
\caption{up: RB case, down: RF case. Right: correlator of the potential, left: correlator of the force.}
\end{figure}

For an interface, $R(u)$ has the shape shown in Figure 3. In that case, the width $r_f$ of $R(u)$ is typically given by the width of the interface or the size of impurities. The force resulting from such a random bond disorder has correlations

\[
\mathcal{F}(r,u) \mathcal{F}(r',u') = \delta_{rr'} \Delta(u-u') \tag{2.8}
\]

as shown in Figure 3 where

\[
\Delta(u) = -R'(u) \tag{2.9}
\]

The signature of such a RB disorder for the interface is that $\int \Delta = 0$ since $R'(u)$ decreases to zero at infinity.

Another type of disorder occurs in the case of interfaces separating two phases, like e.g. a domain wall in a disordered magnet. A random field couples differently to the two phases on the right and left of the interface, thus the energy resulting from the coupling to disorder involves an integral in the bulk of the system and not just at the interface position. The correlation of the force can still be expressed by \((2.8)\) and $\Delta$ still decreases to zero above a scale $r_f$ as shown on Figure 3. Contrarily to the RB case, $\int \Delta$ does not vanish. For a single component displacement field $u$, the RF, of correlator \((2.8)\), is still formally the derivative of a potential $V(r,u) = -\int du' F(r,u')$. The correlations of this fictitious potential are of the form \((2.7)\) with $R(u) = -\int_{0}^{u} du' \int_{0}^{u} du'' \Delta(u'')$, and one has $R(u) \simeq -\frac{1}{2} [u] \int \Delta$ for $|u| \gg r_f$ which can be visualized as a random walk (where $u$ plays the role of “time” and the random field strength $\int \Delta = -2R(\infty)$ is the “diffusion constant”). Contrarily to the RB for which $R(u)$ is short range, $R(u)$ for the RF grows at large $u$ as shown on Figure 3.

In this paper we study the overdamped driven motion of such elastic systems which obey

\[
\eta \partial_{t} u_{rt} = \kappa \nabla^{2} u_{rt} + F(r, u_{rt}) + \zeta_{rt} + \tilde{f} \tag{2.10}
\]

where $\eta$ is a friction, $f$ is the external driving force density and $\zeta_{rt}$ a Langevin noise. The correlation $\langle \zeta_{rt} \zeta_{rt'} \rangle = 2\eta T \delta_{rr'} \delta_{tt'}$ defines as usual a temperature $T$ for this out of equilibrium system. The long time behavior of \((2.10)\) at zero drive $\tilde{f} = 0$, reduces to the thermodynamics at temperature $T$. In \((2.10)\) the bare pinning force $F(r,u)$ is gaussian with zero average and correlator given by \((2.8)\).

We will consider three universality classes for $\Delta$ corresponding to an interface in a random potential (RB), in a random field (RF) or a periodic system in a random potential (RP). Physical realizations of such disorders would be respectively a random anisotropy for a magnetic domain wall, the random field Ising model and vortex lattices or CDW.

It is also useful to rewrite \((2.10)\) in the co-moving frame at average velocity $v = \langle \partial_{t} u_{rt} \rangle$. In the remainder of this paper, we switch to $u_{rt} \rightarrow u_{rt} + vt$ and thus study the following equation of motion

\[
\left\{ \begin{array}{l}
\eta \partial_{t} u_{rt} = 0 \\
(\eta \partial_{t} - e \nabla^{2}) u_{rt} = F(r, vt + u_{rt}) + \zeta_{rt} + \tilde{f}
\end{array} \right. \tag{2.11}
\]

where $\tilde{f} = f - \eta v$ is the average pinning force and $r$ belongs to a $D$-dimensional internal space. From now on we specialize to an unidimensional displacement field $u_r$ as would be the case for an interface model or a single $Q$ CDW. This simpler case already captures the main
physics at small velocity, investigated here. Extensions to many-component systems will be briefly discussed.

Before giving a quantitative treatment using renormalization group, let us review the qualitative arguments which have been given previously to describe the physics originating from (2.10).

III. PRELIMINARY ARGUMENTS

1. Statics

In the absence of drive, (2.10) is equivalent to the equilibrium problem at temperature $T$. The state of the system results from the competition between elasticity, pinning and thermal fluctuations. The physics of such problems can be investigated by a host of methods, and here we only recall the salient points. Temperature does not play an important role as will become clear and we begin with the $T = 0$ case.

A subsystem of size $R$, with displacement $w(R) = \sqrt{\langle u_R^2 \rangle - \langle u_0 \rangle^2}$, is submitted to a typical elastic force density $f_{el} = cw(R)/R^2$ and to a typical pinning force density $f_{pin} = \sqrt{\Delta(0)/R^D}$. Balancing these quantities, one obtains that elasticity wins at large scales for $D > 4$, resulting in a flat interface with a priori bounded displacements. In $D < 4$, systems of size $R$ smaller than the Larkin length

$$R_c = \left( \frac{c^2 R^2}{\Delta(0)} \right)^{\frac{1}{D-4}}$$

wander as predicted by the Larkin model.

$$w(R) \sim r_f \left( \frac{R}{R_c} \right)^{\frac{4-D}{4}}$$

(3.1)

(3.2)

At larger scales $R > R_c$, the system wanders further than the correlation length $r_f$ of the disorder. This simple picture breaks down and the system can be viewed as made of Larkin domains of size $R_c$, which are independently pinned. First order perturbation theory confirms this picture below the Larkin length. The static equilibrium (equal-time) correlation function at $T = 0$ is (see Appendix B)

$$\langle u_{-q,\ell} u_{q,\ell} \rangle = \frac{\Delta(0)}{(c q^2)^2}$$

(3.3)

The wandering computed from (3.3)

$$\frac{1}{2} \langle u_{r,t}^2 \rangle \sim \frac{\Delta(0)}{c^2} S_4 r^\epsilon / \epsilon$$

(3.4)

for $D = 4 - \epsilon$ gives back (3.2), and we recover the scaling expression (3.1) by equating the wandering to $r_f^2$.

We used that $\int_0^1 \frac{1 - \exp (-q^2 r^D)}{q^2 \pi} = A_D r^4 - D$ for $2 < D < 4$ with $A_{D=4-\epsilon} = -\pi^{\epsilon/2 - 2} \Gamma[-\epsilon/2]/16 \sim S_4 / \epsilon$ when $\epsilon \to 0^+$.

The remarkable feature is that $T = 0$ perturbation theory (either using replicas or equilibrium dynamics) gives that (2.10) is exact to all orders in $\Delta$ and is identical to the correlation in the Larkin model. Indeed, the naive perturbation series organizes as if the pinning energy were simply expanded in $u$ (thus the pinning force is independent of $u$ with $F(r) F(r') = \Delta(0) \delta(r - r')$, resulting in a gaussian model.

In fact, due to the occurrence of multiple minima beyond $R_c$, this perturbative result is incorrect at large scale. It can be shown, for example on discrete systems, that if a configuration $u^{GS}$ which minimizes $H[u] = \int_r \left[ \frac{1}{2} (\nabla u)^2 + V(u, r) \right]$ is defined on a volume larger than $R_c$, then the Hessian $\delta^2 H_{u_n u_m}[u^{GS}]$ becomes singular. Such instability appears clearly in a functional renormalization group (FRG) treatment of the problem which proves that $\Delta$ becomes nonanalytic beyond the length $R_c$, as will be discussed below. It can also be seen within variational or mean-field treatments using replica that replica symmetry breaking (RSB) is necessary to describe the physics beyond the Larkin length $R_c$. Using either replicas with RSB or the FRG it is possible to describe the physics at all scales and to obtain the correct roughness exponent $\zeta_{eq}$ defined by

$$w(R) \sim r_f \left( \frac{R}{R_c} \right) \zeta_{eq}$$

where the value of $\zeta_{eq}$ depends on the statics universality class. Since disorder induces unbounded displacements, the system is rough and the temperature is always formally irrelevant in $D > 2$. It is described by a $T = 0$ fixed point, characteristic of a glass phase.

2. Depinning

An elastic system does not necessarily move under the action of a driving force. The disorder leads to the existence of a threshold force $f_c$ at $T = 0$ as shown on Figure 1. A simple dimensional estimate of $f_c$ can be obtained by computing the sum of the independent pinning forces acting on the Larkin domains $(R/R_c)^D \sqrt{\Delta(0)/R_c^D}$ and balancing it with the driving force acting on the same volume $R^D f$. This gives

$$f_c \sim \frac{c r_f}{R_c^2}$$

(3.5)

Another estimate of $f_c$ comes from the large velocity expansion [14] of the equation of motion (2.11) (from the criterion $f \simeq \eta v$). It coincides with (3.7).
For \( f \geq f_c \) the system moves with a small velocity, and it has been proposed \( \nu \) that depinning can be described in the framework of standard critical phenomena, with the velocity as an order parameter. This leads to the assumption of two independent critical exponents \( \zeta \) and \( \nu \), defined through the correlation function in the comoving frame (in the stationary state for \( f \rightarrow f^+ \))

\[
\left( u_{r,t} - u_{0,0} \right)^2 = r^{2 \zeta} C(t/r^2) \tag{3.8}
\]

\( C(x) \rightarrow \text{cst for } x \rightarrow 0 \) and \( C(x) \sim x^{2(\zeta/\nu)} \) for \( x \rightarrow \infty \). The dynamical roughening exponent \( \zeta \) close to the threshold a priori differs from its equilibrium value \( \zeta_{\text{eq}} \). Several related exponents can be also introduced such as: (i) the depinning exponent \( \beta \); (ii) the correlation length exponent \( \nu \) describing the divergence of the length \( \xi \) defined from the equal-time velocity-velocity correlation function. They satisfy

\[
v \sim (f - f_c)^\beta \tag{3.9}
\]

\[
\xi \sim (f - f_c)^{-\nu} \tag{3.10}
\]

Numerically \( \xi \) the motion of the system looks like a deterministic succession of avalanches of size \( \xi \) with characteristic time \( \tau \sim (f - f_c)^{-2\nu} \). From the argument \( u \sim v \tau \) and the statistical tilt symmetry \( \zeta_{\text{eq}} \) (see below), the exponents \( \beta \) and \( \nu \) are usually determined from \( \zeta \), \( \nu \) by the scaling relations

\[
\nu = \frac{1}{2 - \zeta} = \frac{\beta}{\nu} \tag{3.11}
\]

To obtain these exponents analytically, one needs to perform an FRG analysis of the equation of motion. This will be discussed in more details in Section 3.

3. Creep

At finite temperature \( T > 0 \), motion occurs at any drive. For low temperatures and very small drive \( f \ll f_c \) one expects the motion to be very slow, and thus, although it is a dynamical problem, a qualitative understanding can be obtained by considering thermal activation over barriers determined from statics arguments. An original estimate \( \xi_{\text{eq}} \) of such barriers led to linear, albeit activated, response. However the effects linked to the glassy nature of the problem were understood at a qualitative level using scaling arguments.

The argument proceed as follows: systems larger than \( R_c \) have a (static) roughness \( \omega(R) \sim r_f \left( \frac{R}{R_c} \right)^\zeta_{\text{eq}} \) and hence the energy has typical fluctuations of order

\[
E(R) \sim U_c \left( \frac{R}{R_c} \right)^{D-2+2\zeta_{\text{eq}}} \tag{3.12}
\]

with \( U_c = \epsilon R_c^{D-2}r_f^2 \) the energy scale of a Larkin domain. Assuming that the energy landscape is characterized by a unique energy scale, and thus that the energy differences between neighbouring metastable states is the same as the energy barrier separating them as schematically shown in Figure 3.1, one obtains that the barriers height scale with an exponent \( D - 2 + 2\zeta_{\text{eq}} \).

\[
U_c \left( \frac{R}{R_c} \right)^{D-2+2\zeta_{\text{eq}}} = f R^D r_f \left( \frac{R}{R_c} \right)^{\zeta_{\text{eq}}} \tag{3.13}
\]

The maximum of (3.13), obtained at \( R_{\text{opt}} \approx R_c \left( \frac{f}{f_c} \right)^{-\frac{\zeta_{\text{eq}}}{D}} \), gives via Arrhenius law the largest time spent in the valley by the thermally activated system and thus yields the velocity

\[
v \sim \exp \left[ \frac{U_c}{T} \left( \frac{f}{f_c} \right)^{-\zeta_{\text{eq}}/D} \right] = \mu = \frac{D-2+2\zeta_{\text{eq}}}{2-\zeta_{\text{eq}}} \tag{3.14}
\]

known as the creep motion, characterized by the stretched exponential with exponent \( \mu \). Note that the effective barrier given by the above formula vanishes at a scale \( R_0 \sim R_c \left( \frac{f}{f_c} \right)^{-\frac{\zeta_{\text{eq}}}{D}} \) which diverges as fast as \( R_{\text{opt}} \), the typical size of a thermally activated excitation.

\[
R_{\text{opt}} \sim R_c \left( \frac{f}{f_c} \right)^{-\frac{\zeta_{\text{eq}}}{D}} \tag{3.15}
\]

FIG. 4. Energy landscape, with many metastable states in the valleys, differing by \( \Delta E \), and barriers \( U \) between them.

Since the motion is very slow, it is usually argued that the effect of the drive is just to tilt the energy landscape, and the effective barrier becomes

\[
U_c \left( \frac{R}{R_c} \right)^{D-2+2\zeta_{\text{eq}}} = \frac{f R^D r_f}{R_c} \left( \frac{R}{R_c} \right)^{\zeta_{\text{eq}}} \tag{3.16}
\]

FIG. 5. Effective barrier and motion by nucleation.

This elegant scaling argument leading to the creep formula relies however on strong assumptions and does not yield any information on the detailed behavior, in particular on what happens after the thermal jumps. The fact that static barriers and valleys scale with the same exponent is already a non-trivial hypothesis about the structure of the infinite-dimensional energy landscape.
Refined simulations of a directed polymer \( D = 1 \) in \( N = 1 \) and \( N = 2 \) are consistent with the “equal scaling assumption” for this particular case, but a general proof is still lacking. The second, and more delicate hypothesis is the validity of the Arrhenius description: (i) the system being out of equilibrium, it is not clear that dynamical barriers can be determined purely from the statics; (ii) one assumes that the motion is dominated by a typical barrier. These assumptions can turn incorrect for some specific problems. For example, in the case of a point moving in a one-dimensional random potential, the \( v-f \) characteristics at low drive is not that of Arrhenius type. Although this \( 0 + 1 \) case is peculiar since the particle has no freedom to pass aside impurities (it is dominated at \( T = 0 \) by the highest slope of the potential and at finite \( T \) by the rare highest barriers), one should also address the question of the distribution of barriers in higher dimensions.

IV. DYNAMICAL ACTION AND RENORMALIZATION

A. Formalism and exact relations

Let us now study the equation of motion (2.11) using a full FRG treatment. This will enable us to describe the physics at all lengthscales and in particular the depinning and creep regime.

A natural framework for computing perturbation theory in off-equilibrium systems is the dynamical formalism. After exponentiating the equation of motion (2.10) using a response field \( \hat{u} \), the average over thermal noise and disorder can safely be done and yields the simple (“unshifted”) action

\[
S_{\text{uns}}(u, \hat{u}) = \int \left[ i \hat{u}_r \cdot (\eta \partial_t - c \nabla^2) u_r - f \int i \hat{u}_r \right] - \frac{1}{2} \int i \hat{u}_r \delta_{r r'} \Delta(u_r - u_{r'})
\]

Disorder and thermal averages \( \langle A[u] \rangle = \langle A[u] \rangle_{\text{uns}} \) of any observable \( A[u] \) can be computed with the weight \( e^{-S_{\text{uns}}} \). Furthermore, response functions to an external perturbation \( h_r \) added to the right hand side of (2.10) are simply given by correlations with the response field: \( \langle A[u] i \hat{h}_r \rangle = \frac{\delta}{\delta h_r} \langle A[u] \rangle \). It can be checked that causality is satisfied: \( \langle A[u_{r'}] i \hat{h}_r \rangle \) vanishes. In the time-continuum, the response to a perturbation at time \( t \) of an observable depending on \( u_t \) is ill-defined. We choose Itô convention for the equation of motion, which ensures that equal time response functions, and hence any diagram occurring in perturbation theory containing a loop of response functions, vanish. The continuum field theory necessarily breaks down at small scales and it becomes necessary to cut off the integrals over the modes at large \( q \), using a large wave vector \( \Lambda \). A full summary of the formalism can be found in Appendix A.

It proves more convenient to work in the comoving frame (i.e. with equation (2.11)). The corresponding action is

\[
S(u, \hat{u}) = \int_\mathbb{R} \left[ i \hat{u}_r (\eta \partial_t - c \nabla^2) u_r - \eta T \int i \hat{u}_r i \hat{u}_r \right] - f \int i \hat{u}_r - \frac{1}{2} \int i \hat{u}_r i \hat{u}_{r'} \Delta(u_r - u_{r'})
\]

where the field \( u \) satisfies \( \langle \partial_t u_{r r'} \rangle = \langle \partial_t u_{r r'} \rangle = 0 \). This condition fixes \( \tilde{f} \equiv f - \eta v \) in (2.12). This quantity is the (macroscopic) pinning force, since it shifts the viscous law \( f = \eta v \) by the amount of \( \tilde{f} \).

Several exact relations can be derived directly from (2.12). For any static field \( h_r \), vanishing at infinity

\[
S(u, \tilde{h}) = S(u, \hat{u}) - \int \left[ i \tilde{h}_r \right] + \frac{1}{c} \nabla^2 \hat{u}
\]

Performing the change of variable \( u + \frac{1}{c} \nabla^{-2} h \) gives

\[
\int D u \ D \tilde{h} \ e^{-S(u, \hat{u})} = \int D u \ D \tilde{h} \ e^{-S(u, \tilde{h}) - \int i \tilde{h}_r h_r}
\]

Applying \( \frac{\delta}{\delta h_r} |_{h=0} \) yields the exact relation

\[
\int \mathcal{R}_{g f} = \frac{1}{c q^2}
\]

where we denote by \( \mathcal{R}_{g f} \) the exact response function. This symmetry, known as statistical tilt symmetry, ensures that the elasticity is not corrected during the renormalization.

Another important relation can be derived from

\[
\frac{d}{df} \langle \partial_t u_{r r'} \rangle_{\text{uns}} = \int \mathcal{R}_{g f} \langle \partial_t u_{r r'} \rangle_{\text{uns}}
\]

This leads to the identity between the macroscopic mobility and the slope of the \( v-f \) characteristics at any drive and any temperature:

\[
\frac{d}{df} v(f) = \lim_{\omega \to 0} -i \omega \mathcal{R}_{q=0, \omega}
\]

This exact result can also checked explicitly in the case of a particle moving in a one-dimensional environment.

To extract the physical properties from the action (2.12) it is necessary to build a perturbative approach in the disorder. A particularly simple case occurs when the velocity is very large. In that case the disorder operator in the action can be formally replaced by
\[-\frac{1}{2} \int_{rt'} i \dot{u}_{rt} \dot{u}_{rt'} \Delta(v(t - t')) \]  
(4.6)

since one may neglect the \(u_{rt} - u_{rt'}\) compared to \(v(t - t')\). This trick suppresses the non-linearity and the remaining action is quadratic. Furthermore, at large velocity, \(\Delta(v(t - t'))\) can be replaced by \(\delta(v(t - t'))\) if \(\Delta = \frac{1}{2} \delta(t - t') \int \Delta\) and the disorder operator transforms into a temperature operator (because it becomes local in time \(t = t'\)). The resulting action is the dynamical action associated to the Edwards-Wilkinson equation\(^4\) describing the motion of an elastic system in a purely thermal noise

\[\eta \dot{u}_{rt} = c \nabla^2 u_{rt} + \nu_{rt} \]  
(4.7)

with \(\langle \nu_{rt} \nu_{rt'} \rangle = 2g(T + T_{ew}) \delta_{rt'} \delta_{tt'}\), Langevin noise\(^3\) of additional temperature \(T_{ew} = \int \Delta / 2\eta v\).

Note that at \(T = 0\) the results at large \(v\) coincide with the perturbative expansion in powers of the disorder. The equal-time correlation function in the driven system with force \(f\) crosses over from the static \(\frac{\eta v}{\eta r}\) Larkin behavior at small scale to a thermal \(\frac{1}{q^2}\) behavior at larger scale

\[\frac{u_{rt}}{u_{r't'}} \simeq \left\{ \begin{array}{ll} \frac{\Delta(0)}{2\eta v} & \text{for } q^2 \gg \frac{\eta v}{\eta r} \\ \frac{\eta v}{q^2} & \text{for } q^2 \ll \frac{\eta v}{\eta r} \end{array} \right. \]  
(4.8)

with the same \(T_{ew}\), generated at length scales \(r \gg \sqrt{\eta r / \eta v}\).

### B. Renormalization

We renormalize the theory using Wilson’s momentum-shell method. As the cutoff \(\Lambda_1 = \epsilon c^{-1}\) is reduced, corresponding to a growing microscopic scale \(R_l = \epsilon^l / \Lambda\) in real space, the parameters of the effective action for slow fields (whose modes \(q\) are smaller than \(\Lambda_1\)) are computed by integration over the fast part of the fields (whose modes \(q\) lie between \(\Lambda_1\) and \(\Lambda\)). This iterative integration gives rise to flow equations, better expressed in terms of the reduced quantities

\[\tilde{\Delta}_l(u) = \frac{S_D \Lambda_l^D}{(c \Lambda_l^D)^2} \Delta_l(u e^{\zeta l})\]  
\(\tilde{T}_l = \frac{S_D \Lambda_l^D}{c \Lambda_l^D e^{2 \zeta l}} T_l\]  
(4.9)

\[\lambda_l = \frac{\eta v}{c \Lambda_l^D e^{\zeta l}}\]  
\[\tilde{f}_0 = f - \eta_0 v\]

where \(S_D\) is the surface of the unit sphere in \(D\) dimensions divided by \((2\pi)^D\). The exponent \(\zeta\) is for the moment arbitrary and will be fixed later so that the reduced parameters flow next to appropriate fixed points. In one case (RB) we will need a \(l\)-dependent \(\zeta\), and it is understood that everywhere the rescaling factors \(e^{\zeta l}\) (appearing e.g. in (4.9)) should then be replaced by \(\exp \int_0^l dl' \zeta_l\).

The reduced quantities \(\tilde{\Delta}, \tilde{T}\) are homogeneous to \(u^2\) and \(\lambda\) to \(u\). The parameter \(\lambda_1\), which plays a crucial role below, can simply be expressed as the following ratio

\[\frac{v \tau(R)}{\delta u(R)} = \frac{\lambda(R)}{r_f} \]  
(4.10)

of the distance (along \(u\)) travelled by the center of mass of the interface during \(\tau(R)\) and the roughness \(\delta u(R) = r_f (R / R_c)^\xi\). We have defined \(\tau(R) = \eta(R) R^2 / c\) as the characteristic relaxation time in the model renormalized up to scale \(R\).

The details of the renormalization procedure can be found in Appendix \(\Box\). The flow equations read:

\[\partial \tilde{\Delta}(u) = (\epsilon - 2\zeta) \tilde{\Delta}(u) + \zeta u \tilde{\Delta}'(u) + T \tilde{\Delta}''(u) \]  
(4.11)

\[+ \int_{s>0,s'>0} e^{-s-s'} \left[ \tilde{\Delta}''(u) \tilde{\Delta}(s' - s) - \tilde{\Delta}(u + (s' - s)) \right] - \tilde{\Delta}'(u - s' \lambda) \tilde{\Delta}'(u + s) + \tilde{\Delta}'((s' + s) \lambda) \left( \tilde{\Delta}'(u - s' \lambda) - \tilde{\Delta}'(u + s) \right) \]  
\[\partial \ln \lambda = 2 - \epsilon - \int_{s>0} e^{-s} \tilde{\Delta}''(s \lambda) \]  
\[\partial \ln T = \epsilon - 2 - 2\zeta + \int_{s>0} e^{-s} \lambda \tilde{\Delta}'''(s \lambda) \]  
\[\partial \tilde{f} = e^{-(2-\zeta)l} c \Lambda_0^2 \int_{s>0} e^{-s} \tilde{\Delta}'(s \lambda) \]  
where \(\epsilon = 4 - D\) and \(\partial\) denotes \(\frac{\partial}{\partial r}\). This complicated set of equations require a few comments: (i) as for the statics it is necessary to renormalize the whole function \(\Delta\), instead of just keeping few couplings as in standard field theory, (ii) the elasticity \(c\) is not renormalized \(\partial c = 0\) due to the statistical tilt symmetry; (iii) our equations correctly show that no temperature can be generated at \(v = 0\) since the fluctuation dissipation theorem holds at equilibrium.

Setting both \(T = 0\) and \(v = 0\) in (4.11) gives back the simplified set of equations used in Refs. \(\Box\) setting only \(v = 0\) also yields equations found in Ref. \(\Box\). But compared to the previous FRG approaches of the depinning transition, our equations correctly take into account the effect of the velocity on the flow itself (instead of being treated simply as a cutoff as in Ref. \(\Box\)). Other attempts\(^4\) to incorporate velocity and temperature in the FRG equations did not obtain the first equation giving the renormalization of the disorder at \(T > 0\) and \(v > 0\). To be able to tackle the full dynamical problem and study the depinning and the creep regime, one cannot avoid keeping track of the velocity and of the temperature in the flow, as will become clear later, since they yield non-trivial effects which are unreachable by simple scaling arguments.

Our flow equations allow in principle to compute the whole \(v-f\) characteristics at low temperature. In the following we analyse them in the three regimes corresponding to the statics \((v = 0)\), to the depinning at zero tem-

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\(\Box\)
perature \((T = 0, f \sim f_c)\) and the creep regime \((T > 0, f \sim 0)\).

C. Statics: the cusp

At zero velocity, our approach is a dynamical formulation of the equilibrium problem. It thus allows to recover the known results about the statics, avoiding the use of replicas. The standard derivation of the statics using the FRG consists in writing a replicated Hamiltonian for the elastic system pinned in a random potential with correlator \((V(r, u) - V(r', u'))^2 = -2\delta^D(r - r')R(u - u')\). After averaging over \(V\) the replicated action read

\[
S[\tilde{n}] = \frac{1}{2T} \sum_a \int_r |\nabla u_c^a|^2 - \frac{1}{2T^2} \sum_{ab} \int_r R(u_c^a - u_c^b)
\]

(4.12)

where \(a, b\) are the \(n\) replica indices. Performing an FRG analysis of (4.12) yields for the flow of \(R\) and \(T\) (remarkably independent of \(n\)):

\[
\begin{align*}
\partial \tilde{R}(u) &= (\epsilon - 4\zeta)\tilde{R}(u) + \zeta u\tilde{R}'(u) + T\tilde{R}''(u) \\
\partial \ln \tilde{T} &= \epsilon - 2 - 2\zeta
\end{align*}
\]

(4.13)

with \(\tilde{R}(u) = e^{-4\zeta S_0 A^0_D/(\epsilon a_T^D)} R(ue^{c\xi})\) and \(\tilde{T} = e^{-2\zeta S_0 A^0_D/(\epsilon a_T^D)} T\), which are the same redefinitions as (4.9) with the correlator \(\Delta\) of the force is related to \(R\) by (2.9). It is easy to see that (4.13) coincides with our equations (4.11) when \(v = 0\) which read

\[
\begin{align*}
\partial \Delta(u) &= (\epsilon - 2\zeta)\Delta(u) + \zeta u\Delta'(u) + \tilde{T}\Delta''(u) + \Delta''(u) (\Delta(0) - \Delta(u)) - \Delta'(u)^2 \\
\partial \ln \tilde{T} &= \epsilon - 2 - 2\zeta
\end{align*}
\]

(4.14)

Thus the two methods give the same results for the static and equal-time physical quantities. The additional information conveyed by the flow of the friction \(\eta\) in the dynamical formalism is discussed later in \[VA\] and \[VB\].

The temperature in the static system is an irrelevant operator, since it decreases exponentially fast with \(l\). One thus commonly restricts to the \(T = 0\) version of the above equations. In that case, as is obvious from the closed equation

\[
\partial \Delta''(0) = \epsilon \Delta''(0) - 3\Delta''(0)^2
\]

(4.15)

the curvature \(\Delta''(0) < 0\) (see Figure \[B\]) of the correlator, for any initial condition, blows up at a finite length scale for \(D < 4\)

\[
l_c = \frac{1}{\epsilon} \ln \left( 1 + \frac{\epsilon}{3|\Delta''(0)|} \right)
\]

(4.16)

which corresponds to

\[
R_c = e^{\epsilon/\Lambda} \simeq \left( \frac{\epsilon^2}{3S_D|\Delta''(0)|} \right)^{1/\epsilon} \sim \left( \frac{\epsilon^2}{S_D \Delta(0)} \right)^{1/\epsilon}
\]

(4.17)

when approximating \(|\Delta''(0)|\) by \(\Delta(0)/\epsilon^2\). One thus recovers the Larkin length (3.5). The blowup of the curvature of \(\Delta\) corresponds to the generation of a cusp singularity: \(\Delta\) becomes non-analytic at the origin and acquires for \(l > l_c\) a non-zero \(\Delta'(0^+)^2 < 0\). However, the flow equation for the running non-analytic correlator still makes sense. The non-analyticity just signals the occurrence of metastable states. A well-defined fixed point function \(R^*(u)\) exists for each of the RB, RF, RP cases when a suitable \(\zeta\) is chosen.

In the RP case, \(\zeta = \zeta_{eq} = 0\) so as to conserve the period \(a\), and the fixed point is given by

\[
\Delta^*(ax) = \frac{ea^2}{6} \left( \frac{1}{6} - x(1-x) \right)
\]

(4.18)

for \(x \in [0,1]\). In the RF case, \(\zeta = \zeta_{eq} = \epsilon/3\) so as to conserve the RF strength \(f\) \(\Delta\) and the fixed point is given by

\[
x^2 = y - 1 - \ln y
\]

(4.19)

where \(y = \Delta^*(u)/\Delta^*(0), x = u\sqrt{\epsilon/(3\Delta^*(0))}\) and \(\Delta^*(0) \simeq 0.5e^{1/3}(f\Delta_0)^{2/3}\) (see \[E\]).

In the RB case, it has been shown by numerical integration of the fixed point equation that \(\zeta = \zeta_{eq} \simeq 0.2083\epsilon\) yields a physical fixed point, for which no analytical expression is available.

Despite the irrelevance of the temperature, this operator has important transient effects during the flow, even if we are left asymptotically with the \(T = 0\) cuspy fixed point. It can be shown (see Appendix \[B\]) that the temperature hinders the flow from becoming singular at a finite scale. The running correlator evolves smoothly towards its cuspy fixed point and remains analytic, as was also noticed in Ref. \[47\]. As shown in Appendix \[B\], the rounding due to temperature occurs in a boundary layer of width proportional to \(T\) around the origin. This is confirmed by the existence of a well-defined expansion in \(T\) (see Appendix \[D\]). This effect is missed by simple perturbation theory that would naively suggest that the rounding occurs on a width proportional to \(\sqrt{T}\).

Indeed the correlation function is proportional to \(T\) and smooths \(\Delta\) by \(\Delta_\kappa \rightarrow \Delta_\kappa e^{-T\kappa^2}\). Although not crucial for the statics this rounding has drastic consequences for the creep as analysed in Section \[VI\].

Let us return on the differences between the static and dynamical formalisms. Within the static approach (4.13) in the \(T \to 0\) limit, despite the occurrence of the cusp at \(l_c\), the RG equation for \(R(u)\) still makes sense after \(l_c\) and
flows to a fixed point controlled by $\epsilon = 4 - D$. However the physical meaning of the cusp is delicate. On the other hand, the use of the dynamical formalism allows to put $T = 0$ from the beginning but adds to the problem a time dimension and the corresponding parameter, the friction $\eta$. In this dynamical version of the problem, the cusp has strong physical consequences which are more immediate: after $l_c$, the cusp generates infinite corrections to the friction. This feature marks the onset of a non-zero threshold force at scales larger than the Larkin length and signals that an infinite time is needed to go from one metastable state to another. Metastability thus appears very clearly in the dynamical formulation of the statics problem.

A simple physical picture of the cusp in the statics at $T = 0$ was given in Ref. 13. The renormalized potential $V_{\text{ren}}(r, u)$ at scales $R > R_c$ develops “shocks” (i.e. discontinuities of the force $-\partial_u V_{\text{ren}}(r, u)$ of typical magnitude $f_{\text{disc}}(R)$ at random positions). Let us now extend this description to draw the link with the critical force and to include thermal effects.

The force correlator for small $u - u'$ is dominated by the configurations with a shock present between $u$ and $u'$:

\[
(F_{\text{ren}}(r, u) - F_{\text{ren}}(r, u'))^2 \sim f_{\text{disc}}(R)^2 \frac{dp}{du}|u-u'| \tag{4.20}
\]

where $dp/du$ denotes the probability to find a shock between $u$ and $u + du$. Identifying the rhs with $R^{-2}\Delta_{\text{ren}}(0)(u-u')$ one finds, using the rescalings (4.12), that the discontinuity in the force has the following scale dependence

\[
f_{\text{disc}}(R) \sim f_c \left( \frac{R}{R_c} \right)^{(2-\zeta)} \equiv f_c^{\text{eff}}(R) \tag{4.21}
\]

and can thus be identified with an “effective critical force” $f_c^{\text{eff}}(R)$ at scale $R$, which will play a role in the following (see Subsection VI C). At $R = R_c$, $f_c^{\text{eff}}(R)$ reduces to the true critical force $f_c$.

The renormalized problem at scale $R$ being the one of an interface in a potential $V_{\text{ren}}(r, u)$ with the above characteristics, one can now easily understand the result that the cusp of $\Delta_l(u)$ is rounded on a width $T_l/x$ at $T > 0$. Extending the previous argument, one expects a rounding of a shock if the barrier between $u$ and $u'$ is of order $T$. Since near a shock the potential is linear of slope $f_{\text{disc}}(R)$, the barrier is $f_{\text{disc}}(R)|u-u'|$, and the thermal rounding should thus occur in a boundary layer of width $u$ given by

\[
f_{\text{disc}}(R) u R^D \sim T \tag{4.22}
\]

Using the rescalings (4.3), this is indeed equivalent to the expression $T_l/x$ for the width of the boundary layer in rescaled variables found in Appendix D.

V. DEPINNING

At $T = 0$ and $v \to 0$, our flow equations give a self-contained picture of the depinning transition. Thanks to our formalism, the problem is reduced to the mathematical study of (4.11), which although complicated, requires no additional physical assumptions. To focus on the depinning transition, we must analyze the solutions of these equations in the regime of small velocity where, using (1.9), $\lambda_{t=0}$ is small. We will examine the various regimes in the RG flow keeping in mind that $\lambda_t$ increases monotonically with $l$.

The equations (4.11) involve averages over a range $u \sim \lambda_t$ and thus one naturally expects that, at least at the beginning of the flow, $\Delta_l(u)$ remains close to the $v = 0$ solution. The two functions will differ in a boundary layer around $u = 0$ of width denoted by $\rho_l$. Although the precise form of the solution for $|u| < \rho_l$ (e.g. whether the cusp persists at $v > 0$) is very hard to obtain analytically, fortunately most of our results will not depend on such details. As we discuss below, the main issue will be to decide whether $\rho_l \ll \lambda_t$ or not, which is a well-posed mathematical question.

Let us start by analyzing the flow up to the Larkin scale $l_c$ of the statics, at which the cusp occurs and the corrections to the friction become singular in the $v = 0$ flow. Here at $v \gtrsim 0$ one enters at $l_c$ a regime where $\Delta_l$ is close to its fixed point (see Appendix D). Within the boundary layer, the effect of the velocity is to decrease the singularities of the statics. As shown in Appendix D, the blow-up of the curvature $\hat{\Delta}''(0)$ is slowed down by the velocity as

\[
\partial \hat{\Delta}''(0) = \epsilon \hat{\Delta}''(0) - 3 \hat{\Delta}''(0)^2 - 9\lambda^2 \hat{\Delta}''(0) \hat{\Delta}^{iv}(0) + O(\lambda^4)
\]

and the same is true for the friction

\[
\partial \ln \eta = -3 \hat{\Delta}''(0) - 3\lambda^2 \hat{\Delta}^{iv}(0) + O(\lambda^4)
\]

If the blurring of the singularity results in a suppression of the cusp, i.e. if $\Delta_l$ remains analytic, one should wonder whether the $v \simeq 0$ flow can really remain close to $\Delta^*$ since the convergence to the fixed point is crucially dependent on the existence of the non-analyticity and in particular on the term $-\hat{\Delta}'(0^+)^2$ in the flow of $\Delta(0)$ in (4.14). A hint that $\Delta_l$ can stabilize for a while at $v > 0$ is obtained by noting that one has (see (4.37))

\[
\partial \Delta_l(0) = (\epsilon - 2\zeta) \hat{\Delta}(0) - \int_{s>0} e^{-s} \left( \int_{s'>0} e^{-s'} \frac{\hat{\Delta}(\lambda s') - \hat{\Delta}(\lambda s)}{\lambda} \right)^2 \tag{4.23}
\]

which has indeed the correct sign to give the same effect.

Hence it is natural to expect for $l > l_c$ that $\Delta_l(u)$ has reached everywhere a fixed point form except in the boundary layer. The correction to the friction, crucial to determine the $v$-$f$ characteristics, reads.
\[ \partial_t \ln \eta_t = - \int_{s > 0} e^{-s} s \tilde{\Delta}'(s \lambda_l) \]  
(5.1)

and thus depends on the values of \( \tilde{\Delta}(u) \) for \( u \sim \lambda_l \). To estimate this expression, one must know whether the width \( \rho_l \) of the boundary layer is smaller than \( \lambda_l \) or not.

To summarize these preliminary remarks, the flow in the Larkin regime \( l < l_c \) is similar to the \( v = 0 \) flow and \( \tilde{\Delta}(u) \) is close to \( \Delta^* \) except for \( |u| < \rho_l \). We will now analyze in detail the flow for \( l > l_c \) under the assumption that

\[ \rho_l \ll \lambda_l \]  
(5.2)

As mentioned above, the validity of (5.2) can in principle be established by a mathematical or a numerical analysis of our equations. It turns out that (5.2) leads to the most physically reasonable results. The alternative case will be discussed below.

### A. Derivation of the depinning law

For \( l > l_c \), called the depinning regime, and relying on (5.2), the flow of \( \eta \) becomes

\[ \partial_t \ln \eta_t \approx - \Delta^{*''}(0^+) \]  
(5.3)

The friction is renormalized downwards with a non-trivial exponent \( -\Delta^{*''}(0^+) = -\zeta \) with \( \zeta = \frac{4}{5} \) for the RF case (see Appendix E.1) and \( \zeta = 0 \) for the RP case (see Appendix E.2). For the random bond one would naively take the static \( \zeta_{eq} \). However our flow equations show that during the Larkin regime, the form of the disorder correlator evolves to a RF, and thus \( \zeta = \frac{4}{5} \) also in this case. This non trivial effect of the transformation for the dynamical properties of a RB into a RF is discussed in details in Section 5.2.

Since \( \lambda_l \) keeps on growing in the depinning regime, the assumption that \( \tilde{\Delta}(u) \) can be replaced by \( \Delta^* \) will cease to be valid. This occurs when \( \lambda_l \) reaches the range \( \tau_f(l) \) of \( \Delta_l(u) \), correlation length of the running disorder. This defines a scale \( l_V = \ln(\hbar R_V) \) given by \( \lambda_l V = f \tau_f(l_V) \).

Above this scale, one enters a regime where the corrections due to disorder are simply washed out by the velocity, since the integrals over \( s,s' \) in (4.11) average completely over the details of \( \tilde{\Delta}(u) \). One thus enters the Edwards Wilkinson regime. Perturbation theory (4.8) shows that the interface is flat for these large scales for \( D > 2 \), the disorder leading only to the effective temperature \( T_{eff} \).

The family of systems indexed by \( 0 < l < \infty \) have all the same velocity \( v \) and the same slope \( df/dv(v) \). However they have lesser and lesser singular behavior \( f(v) \). We can thus iterate the FRG flow up to a point where the theory can solved perturbatively (e.g. above \( l_V \)). For the depinning, one can simply use the fact that the renormalized action at \( l = \infty \) is gaussian and its friction \( \eta_{\infty} \) is, from (4.5) equal to the slope \( df/dv \) of the depinning characteristics. Using the flow of \( \lambda_l \) in (4.11), the expressions for \( \lambda_{l_V} \), \( \lambda_c \) with (4.4) and (4.8) lead to

\[ \frac{\lambda_{l_V}}{\lambda_{l_c}} \approx \left\{ \exp \left[ (2 - \zeta) \beta(l_{V} - l_c) \right] \right\} \]  
(5.4)

with

\[ \beta = 1 - \Delta^{*''}(0^+)/2 - \zeta \]  
(5.5)

which will turn to be the depinning exponent and we have defined a characteristic force \( F_c = c r f / R_c^2 \). Note that \( F_c \) is not exactly the critical force \( f_c \). Solving (5.4) gives

\[ \frac{R_{l_V}}{R_{l_c}} \approx \left( \frac{F_c}{\eta_{l_c} v} \right)^{2 - 2 \zeta} \]  
(5.6)

\[ \frac{\eta_{l_V}}{\eta_{l_c}} \approx \left( \frac{\eta_{l_c} v}{F_c} \right)^{2 - 1} \]  
(5.7)

Since the system at \( l_V \) is nearly pure, one has \( \eta_{l_V} \approx \eta_{\infty} \) and, integrating over \( v \) the derivative \( \frac{df}{dv} = \eta_{\infty} \approx \eta_{l_V} \), one gets

\[ \frac{\eta_{l_c} v}{F_c} = \left( \frac{f - f(v = 0^+)}{F_c} \right)^{\beta} \]  
(5.8)

which shows that the depinning is characterized by an exponent \( \beta \) and a pinning force \( f_c = f(v = 0^+) \) (yet to be determined).

The flow of \( \tilde{f}_l \) allows to fix the value of \( f_c \). Instead of just computing \( f_c \) we also show that the integration of the flow of \( \tilde{f}_l \) provides a second way to derive the depinning law (5.8). Indeed, as discussed below, in our formalism the term proportional to \( v \) which was problematic in the previous approaches cancels naturally.

In the theory renormalized up to \( l_{V} \), the short scale cutoff is \( R_{l_V} \) and one can use first order perturbation theory. One has (see [B3] and [B11]).

\[ \tilde{f}_{l_{V}} = - \int_{l} \Delta'_{l_{V}}(vt) R_{l_{V}} \]  
(5.9)

Since in the renormalized theory the disorder is close to \( \Delta^* \) (with the rescalings [1.9]), and the friction hidden in the response function is such that \( \lambda_{l_{V}} \) matches the range of \( \Delta_{l_{V}} \), the velocity disappears from (5.5) which gives

\[ \tilde{f}_{l_{V}} \approx e^{-(2-\zeta)(l_{V} - l_{c})} A e \]  
(5.10)

where \( A \) is some constant and the only \( v \)-dependent quantity is \( l_{V} \). To connect \( \tilde{f}_{l_{V}} \) to the initial parameters, one has to integrate the flow \( \tilde{f}_{l_{V}} - \tilde{f}_0 = \int_{l_{V}} dl \partial_l \tilde{f}_l \).

Expanding \( \partial_l \tilde{f}_l \) in (4.11) at small velocity and using \( \Delta'(s\lambda) = \Delta'(0^+) + s\lambda \Delta''(s\lambda) + O(\lambda^2) \) one recognizes in the second term the correction to \( \eta \). Thus for \( l_c < l < l_{V} \) one has
\[ \partial_t \tilde{f}_t \simeq e^{-(2-\zeta)t}c \Lambda_0^2 \Delta^r(0^+) - v \partial_t \eta_t \] (5.11)

where we dropped the sub-dominant terms in velocity. The integration of the flow gives

\[ \tilde{f}_{tV} - \tilde{f}_0 \simeq -f_c \left(1 - e^{-(2-\zeta)(tV - \ell_c)}\right) - v(\eta_{tV} - \eta_0) \] (5.12)

where we defined \( f_c = c \Lambda_0^2 e^{-(2-\zeta)\ell_c} \Delta^r(0^+)/ (2-\zeta) \). Injecting \( \tilde{f}_0 = f - \eta_0 v \) and (5.11), we note that quite remarkably the \( \eta_{tV} \) cancel each other. We are left with

\[ f - f_c \simeq e^{-(2-\zeta)(tV - \ell_c)} \left( A c - f_c \right) + \eta_{tV} v \] (5.13)

We already know (5.7) that \( \eta_{tV} \sim v^{\frac{1}{\beta} - 1} \) and (5.6) \( \eta_{tV} v \sim v^{\frac{1}{2} - 1} \) thus both terms r.h.s of (5.13) scale like \( v^{1/\beta} \). This leads to the following result to lowest order in \( \epsilon \):

\[ f_c \simeq \begin{cases} \frac{\epsilon}{1-\frac{\zeta}{2}} & \text{RF} \\ \frac{\epsilon}{2} & \text{RP} \end{cases} \] (5.14)

\[ v \sim (f - f_c)^\beta \] (5.15)

\[ \beta = \begin{cases} 1 - \frac{2\epsilon}{\zeta} & \text{RF} \\ 1 - \frac{\epsilon}{\zeta} & \text{RP} \end{cases} \] (5.16)

where we used the fact that \( \zeta = \epsilon/3 \) (RF or RB) or \( \zeta = 0 \) (RP) and the link between \( \chi = |\Delta^r(0^+)| \) and \( f \) (RF) or \( a \) (RP) stated in Appendix A. In addition, we assumed that \( \eta_\epsilon \) has a regular behavior when \( v \to 0 \), a non-trivial point which we discuss in Subsection V.C.

### B. Discussion

The approach of the previous Subsection V.A allows us to obtain the characteristics of the \( T = 0 \) depinning. We extract the depinning exponent \( \beta \), the pinning force \( f_c \) and the characteristic lengthscales from the equation of motion without any additional physical hypothesis or scaling relation. Although the depinning problem the exponents \( \beta \) and the critical force were determined in previous studies, our method is an improvement in several ways.

To get the depinning exponent and critical force, two main derivations exist in the literature. One of them extends the static FRG formalism to the out of equilibrium depinning problem at zero temperature using an “expansion” around an unknown mean-field solutions. Instead of directly looking at the renormalized correlator of the disorder, the method deals with the time correlation of the force, \( C(v(t-t')) \) in Ref. 23. This procedure does not allow for a precise enough calculation of the \( \nu f \) characteristics to demonstrate the cancellation of the \( \eta_{tV} \) term (in our equation (5.12)). In order to obtain a depinning exponent \( \beta \) different from its “mean-field” value \( \beta = 1 \), it is necessary in Ref. 23 to neglect \( \nu f \) in the small \( v \) limit a term proportional to \( v \) against a term proportional to \( v^{1/\beta} \) with \( \beta < 1 \). Our method, that directly uses averaging over the disorder and properly takes into account the velocity in the flow of the renormalized action, allows to show explicitly the needed cancellation.

The other analytical study of depinning does not consider the renormalization before the Larkin length and assumes that the singularity is fully developed beyond this lengthscale. This amounts to take as a starting point the equation of motion at zero velocity with a cuspy correlator for the force, and the Larkin length as the microscopic cutoff. Since the anomalous exponent of the friction is \( -\Delta^r(0^+) \) which is ill defined for a cuspy correlator, one is forced in this method to argue that it should be replaced by \( -\Delta^r(0^+) \) which is finite. This prescription and scaling relations linking the roughness, the depinning and the time exponent \( 2 - \Delta^r(0^+) \), allows to extract the depinning exponent. In our method, the ambiguities that existed in Ref. 23 to write the flow of \( \eta \) beyond \( \ell_c \) when using the zero velocity equations, and the trick \( 0 \to 0^+ \) becomes a well-defined mathematical property of our finite velocity RG equations: if \( \Delta^r \) is confirmed, our approach directly shows that the \( -\Delta^r(0^+) \) prescription is the correct one and allows to prove directly the scaling relations, instead of assuming them, to obtain the exponent.

Furthermore the occurrence of the asymptotic Edwards Wilkinson regime in Ref. 24 has to be put by hand as a cut in the \( \nu \) RG flow. The important correlation length \( R_V \) (denoted \( L_V \) in Ref. 23) at which this regime takes place is thus not well under control and has to be estimated from dimensional analysis. In our case the depinning regime is naturally cut when our \( \lambda \) which tells how fast the system runs on the disorder, reaches the range of the flowing correlator. The scale \( R_V \) at which it occurs, and above which the non-linearities are washed out, can clearly be identified with the correlation length of the moving interface (or more precisely, of the velocity-velocity correlation). The physical interpretation of (5.17), i.e.,

\[ R_V \sim R_c \left( \frac{f_c}{\eta_{tV}} \right)^{\frac{1}{\beta}} \] (5.17)

with \( z = 2 - \frac{\epsilon \zeta}{3} \) is the following: \( R_V \) is the scale at which “avalanches” occur in the driven deterministic system. The motion proceeds in a succession of such processes, where pieces of interface of typical size \( R_V \) depin over a distance \( r_f(R_V / R_c)^z \) during a time \( r_f(R_V / R_c)^z v \).

In addition to providing a clean derivation of the depinning exponents and of the critical force, our equations contain new physics that was unreachable by the previous methods.

Although in principle one would expect three universality classes (RF,RB,RP) for the depinning exponent, it was conjectured by Narayan and Fisher that the roughness exponent \( \beta \) of the system at the depinning transition
for RB or RF is equal to the roughness exponent of the static RF case, $\zeta = \epsilon/3$. This result cannot be obtained by the approach of Narayan and Fisher or that of Nattermann et al. since these authors did not include the velocity in their RG analysis, and simply treated the small $v$ limit as $v = 0$. On the contrary, our flow equations for the correlator shows directly that a RB disorder does indeed evolve during the flow towards a RF disorder, leaving only two different universality classes (RF,RP) for the dynamics against three for the statics (RB,RF,RP). Such evolution is shown in Figure 6, where an initial RB becomes dynamically a RF. In Appendix G we show that $\rho(\Delta) = 2\eta + \mathcal{O}(\lambda^2)$, which measures the RF strength of the disorder, grows as

$$\partial \int \Delta = 2\lambda^2 \int \Delta \nu^2 + \mathcal{O}(\lambda^4)$$

where we have used $\zeta = \epsilon/3$. This ensures that a moving system, even at arbitrary small velocity, sees an effective random field at large scale.

C. Open questions

Our FRG equations prompt for several remarks and questions. In the previous sections, we have examined the consequences of the property (5.2) and established in that case that the values of the exponents were the ones proposed in Refs. 23,27,31. Although we consider it as unlikely, we have not been able to rule out the possibility that either $\rho_l \sim \lambda_l$ (or even worse, $\rho_l > \lambda_l$) and thus we should examine the consequences of a violation of property (5.2). If $\rho_l \sim \lambda_l$, it is not excluded a priori that there exists another “fixed point” behavior (e.g. with a scaling function of $u/\lambda_l$). However in that case, the exponents should differ from the standard ones (unless some hidden and rather mysterious sum rule would fix the value of the integral in (5.1)). In the absence of an identified fixed point, it is not clear whether universality would hold. Again this crucial point (5.2) can be definitely answered by an appropriate integration of (4.11).

Thus the present approach, which clearly takes $v$ into account, identifies as (5.2) the condition under which the trick used in Refs. 23,27 gives the correct exponents.

Another intriguing point concerns the continuity between the $v = 0$ and the $v \to 0$ problems. Indeed, to derive the depinning law (5.14) we have assumed that $\eta_l$ remains finite as $v \to 0$. However, we should recall that in the non-driven case ($v = 0$ and $f = 0$), $\eta_l$ diverges at $l_c$ and thus $\eta_l = a\exp(l_c)$. If there is any continuity in the RG flow as $v \to 0$ then $\eta_l \to \infty$ in this limit. In that case the consequence would be (see (5.8) a modification of the exponent $\beta \to \beta/(1 - \alpha)$ if $\eta_l \sim v^{-\alpha}$ (or weaker logarithmic multiplicative corrections). We would then find for the depinning a different result from the conventional one. Since we are unable to solve analytically accurately enough the equation for $\eta$ around $l_c$, one should resort to a numerical solution of our flow equations (4.11) to resolve this question. Using (4.11) it is necessary to check that $\eta_l$ does not diverge as $v \to 0$ like a power of $v$ so as to recover the standard depinning exponent (5.3). The question is of particular importance since, if really a finite-scale behavior, occurring near $R_c$, would control the macroscopic asymptotic behavior, then again one could wonder whether universality would hold.

Therefore, the description of depinning in terms of a standard critical phenomenon may be risky. Indeed as clearly appears in our FRG approach, since the fixed point at $v = 0$ is characterized by a whole function $\Delta^*$ (i.e. an infinite number of marginal directions in $D = 4$) rather than a single coupling constant (as in usual critical phenomena) the effect of an additional relevant perturbation, here the velocity, can be more complex due the feedback of $v$ itself on the shape of the function during the flow. This is particularly clear in the RB case which dynamically tranforms into RF.

VI. CREEP

We now deal with the non-zero temperature case. The system can jump over any energy barrier and overcome the pinning forces, thus it moves with $v > 0$ for any drive $f > 0$ and never gets pinned. Let us now show how our equations (4.11) allow to investigate the creep regime that occurs when the system moves very slowly with $f \ll f_c$, at low temperature.
A. Derivation from FRG

As for the depinning, we are interested in infinitesimal velocities. The bare $\lambda_0$ is thus very small. The main difference compared to Section 7.2 is that the temperature is now finite as well. The main effect of $T$ is to round the cusp in the flow. Since we are interested in extremely small velocities, we will consider $\lambda_0$ as the smallest quantity to start with. A non-zero temperature gives thus rise to a new regime in the RG flow, where the rounding of the cusp is due to temperature and not to velocity. This leads to the following regimes in the FRG flow shown on Figure 7. We will examine the various regimes in the RG flow keeping in mind that again, $\lambda_t$ increases monotonically with $l$.

[Diagram showing characteristic scales and regimes for creep motion]

Just as in the previous case, we expect a Larkin regime for $0 < l < l_c$ with small corrections. Above $l_c$, the disorder reaches a regime where scaling is imposed by the temperature. Indeed since $\lambda_t \ll \tilde{T}_l/\chi$ one can forget about the velocity and the FRG equations are very similar to the $v = 0$ case. In Appendix B we show that the temperature rounds the cusp on a boundary layer $u \sim \tilde{T}_l/\chi$ and we obtain the explicit scaling form (6.1)

$$\tilde{\Delta}_l(u) \simeq \tilde{\Delta}_l(0) - \tilde{T}_l f(u\chi/\tilde{T}_l)$$

$$f(x) = \sqrt{1 + x^2 - 1}$$

$$\chi = |\Delta^*'(0^+)|$$

which in the statics holds at all scales larger than a scale of order $l_c$. Here, because we focus on $v \to 0$, the scanning scale $\lambda_{l_t}$ is smaller than the width of the boundary layer, and the flow of the friction reads in this regime

$$\partial \ln \eta_t \simeq -\tilde{\Delta}''_l(0) \simeq \chi^2/\tilde{T}_l$$

(6.2)

The temperature being irrelevant by power counting, the initial flow of $\tilde{T}$ is

$$\partial \ln \tilde{T} = -\theta$$

(6.3)

since the anomalous correction to $\tilde{T}$ vanish as $\lambda \to 0$. Here and in the following, $\theta = D - 2 + 2\zeta_{eq}$ denotes the energy fluctuation exponent of the static problem. Together with (6.2), it shows that the friction grows extremely fast, like $\exp e^{-\tilde{T}}$. This is the thermal regime where motion only occurs via thermal activation over barriers. The velocity is so small that the center of mass motion is unimportant and the temperature essentially flows as in the $v = 0$ problem. We have determined the flow in its initial stages, and we now determine the scale at which this behavior ceases to hold.

The flow equation (6.3) for $\eta_t$ together with the scaling function (6.1) for $u \sim \tilde{T}_l/\chi$ shows that (6.2) holds only until the new scale $l_T = \ln \Lambda R_T$ defined as

$$\lambda_t \sim T_l/\chi$$

(6.4)

For $l < l_T$ the temperature remains the main source of rounding of the cusp. Above that scale one must take the velocity into account.

In fact, this simple picture is not complete since, before reaching $l_T$ another phenomenon occurs, leading to another lengthscale. In the thermal regime the correction to $\tilde{T}$ due to disorder competes with the simple exponential decay and (6.3) breaks down. This physically expresses that motion in a disordered landscape generates a thermal noise (provided some thermal noise is already present). Using (6.1), one has $\partial \ln \tilde{T} \simeq -\theta + 6\chi^4\lambda^2/\tilde{T}_l^3$ at small $\lambda$. Thus the correction to $\tilde{T}$ reverts at a scale $l_s = \ln \Lambda R_s$ such that $\lambda_s \sim \tilde{T}_l/\chi^2$. Note that $l_s < l_T$. Above $l_s$ the temperature does not decrease any more due to heating by motion. One can show using (4.11) that $\tilde{T}$ saturates and does not vary much until the scale $l_T$. We call this intermediate regime $l_s < l < l_T$ the saturation regime. We checked it using a numerical integration of the flow in this regime with the scaling form of the disorder (6.1). Analytically, if we suppose that after $l_s$, the correction of $\tilde{T}$ due to disorder dominates $-\theta$, then one would have in this regime an invariant of the flow $\partial \tilde{T} \left( \tilde{T}_l^2 - 6\chi^2\lambda^2 \right) \approx 0$. If this were true, it is clear that the flow could never realize the condition $\lambda_t \sim T_l/\chi$, possibility that is excluded on physical basis and by the numerics shown in Figure 8.

[Diagram showing the integration of the flow using (6.1) after $l_c$ and reduced variables $x_t \equiv \chi \lambda_t/\tilde{T}_l$ and $y_t \equiv \chi^2 \lambda_t^2/\tilde{T}_l$. The dotted lines indicate the set of points where $\partial_t x_t = 0$ or $\partial_t y_t = 0$. Some trajectories are displayed, with an arrow showing the direction of growing $l$. The initial conditions for creep are close to the origin, and closer to the x-axis.]

Despite the saturation of the temperature, (6.2) remains true after $l_s$. Thus the friction and $\lambda$ keep on growing and one finally reaches the scale $l_T$ at which the scanning length $\lambda_t$ crosses the boundary layer width $\tilde{T}_l/\chi$. 
Above $l_T$, a rigorous analytical analysis of (4.11) becomes difficult. We however expect, since the velocity controls now the boundary layer, a regime similar to the depinning regime at $T = 0$ to occur. Using the same arguments than for the depinning, one obtains in that regime

$$\partial \ln \eta = -\Delta^{\ast \ast}(0^+)$$

$$\partial \ln \tilde{T} = 2 - D - 2\zeta$$

leading again to a decrease of the temperature, even slightly accelerated by a negative $\mathcal{O}(\epsilon^{4/3})$ exponent. Let us call $l_d$ the depinning scale at which one enters such a depinning regime. From the above discussion it is very reasonable to expect that one goes directly from the saturation to the depinning regime, i.e. $l_d - l_T \sim c s t$. However we cannot strictly rule out the possibility of an intermediate regime (divergent $l_d - l_T$ when $v \to 0$) during which the correction to the friction goes smoothly from positive (as in the thermal and saturation regimes) to negative values (depinning regime). Again, it would be useful to settle this point through a numerical solution of our flow equations. Note that in the RF and RP cases, the exponent $\zeta$ and the fixed point $\Delta^\ast$ in (4.5) are the same as in the statics. However in the RB case, one have used a $l$-dependent $\zeta$ which crosses over between $\zeta_{eq}$ for $l < l_T$ and $\zeta = \epsilon/3$ for $l > l_d$ corresponding to the change from RB to RF fixed points $\Delta^\ast$.

In the depinning regime, motion now proceeds in a similar way than for the one studied in Section 5. Here again at large enough scale, velocity will wash out the disorder for $l > l_V$ with $l_V$ determined by $\lambda_1 \sim r_f(l)$. One then enters the Edwards-Wilkinson regime. Let us now compute from the flow (4.11) the largetscales defined above (see Figure 6).

In the thermal regime $l_c < l < l_s$ one can compute $\lambda_{l_c}^{\lambda_{l_s}}$ either by integrating its flow or by equating the boundary values to their expression. This gives

$$\frac{\lambda_{l_c}}{\lambda_{l_s}} \approx \left\{ \frac{\exp\left[\frac{2}{3} \left\{ (\lambda_{l_c} - \lambda_{l_s}) + \frac{U_c}{T}\right.\right]}{\exp\left[\frac{2}{3} \left\{ (\lambda_{l_s} - \lambda_{l_c}) + \frac{U_c}{T}\right.\right]} \right\}$$

where we defined $f_c \equiv \frac{c_{eq}}{R_c}$ and $U_c \equiv \epsilon^2 R_c^D c_{eq}^2 \frac{C}{R_c^2}$. Expressing the scales as a function of the velocity leads to

$$\left(\frac{R_s}{R_c}\right)^{\theta} \approx \frac{T}{U_c} \ln \left( \frac{T}{U_c} \right)^{3/2} f_c \eta_v$$

$$\frac{\eta_v}{\eta_c} \approx \frac{f_c}{\eta_v} \left( \frac{T}{U_c} \right)^{1/\mu} \ln \left( \frac{T}{U_c} \right)^{3/2} f_c \eta_v$$

with $\mu \equiv \theta/(2 - \zeta_{eq}$).

In the saturation regime $l_s < l < l_T$ we proceed in the same manner and obtain

$$\frac{\lambda_{l_T}}{\lambda_{l_s}} \approx \exp\left[\frac{2}{3} \left\{ (\lambda_{l_T} - \lambda_{l_s}) + \frac{U_c}{T}\right.\right]$$

Thus

$$\frac{R_T}{R_s} \approx 1$$

$$\frac{\eta_T}{\eta_s} \approx \ln \left[ \left( \frac{T}{U_c} \right)^{3/2} \frac{f_c}{\eta_v} \right]^{1/2}$$

Assuming $l_d \sim l_T$, the depinning regime $l_d < l < l_V$ follows directly and

$$\frac{\lambda_{l_d}}{\lambda_{l_V}} \approx \left\{ \exp\left[\frac{2}{3} \left\{ (\lambda_{l_V} - \lambda_{l_d}) + \frac{U_c}{T}\right.\right] \right\}$$

leads to

$$\frac{R_V}{R_d} \approx \left\{ \frac{1}{\epsilon} \ln \left[ \left( \frac{T}{U_c} \right)^{3/2} \frac{f_c}{\eta_v} \right]^{1/2} \right\}$$

with $\beta \equiv \frac{2 - \zeta - \Delta^{\ast \ast}(0^+)}{2 - \zeta}$ the depinning exponent (and $\zeta$ the dynamical roughness exponent).

We are now in a position to compute the characteristics $f(v)$. We fix a small velocity $v$ and solve the flow equations for $\lambda_1$, $\Delta_1$ and $\tilde{T}_1$ up to $l_V$. This allows to relate $\tilde{T}_1$, to the unknown $\tilde{T}_0$. We can now use the fact that at the scale $l_V$, the disorder is essentially washed out and a perturbative calculation of $\tilde{T}_1 \approx \tilde{T}_\infty = 0$ is possible. Solving backwards we determine $\tilde{T}_0$, which is simply $f - \eta v$ where $f$ is the real force applied on the system and $\eta = \eta_0$ the bare friction.

The correction to $\tilde{T}_0$ can not be neglected during the depinning regime, thus, using $\tilde{T}_0 = f - \eta_0 v$, $\tilde{T}_\infty = 0$ and expressing $\int_0^\infty dl \partial T f_1$ one has

$$f - \eta_0 v \approx - \int_0^\infty dl \partial T f_1 \approx \frac{c_{eq}^2 \lambda}{2 - \zeta}$$

In the thermal regime there is essentially no correction to the flow of $f$. Thus (6.13) is controled by the depinning regime and one should integrate essentially between $l_d$ and $l_V$. In fact due to the exponentially decreasing behavior of the integrand in (6.13) the whole integral depends in fact only of the behavior at the scale $l_d$. Assuming that $l_d \sim l_T$, using (6.7,6.9), one see that $e^{\lambda_{l_T} - \lambda_{eq}}/l_d \gg v$ for $v \to 0$ and thus one obtains

$$\frac{\eta v}{f_c} \approx \exp\left[ \frac{U_c}{T} \right] \left( \frac{f_c}{\eta_v} \right)^{-\mu}$$

$$\mu = D - 2 + 2\zeta_{eq}$$

The prefactor in front of the exponential cannot be obtained reliably at this order. Note that for the creep, contrarily to the depinning, the possible divergence of $\eta_v$ when $v \to 0$ (and $T \to 0$) does not affect the argument of the exponential but only the prefactor.
B. Alternative method and open questions

For the depinning it was possible to recover the depinning law using both the integration of the flow of \( f \) and of the friction \( \eta \) and the relation (4.3). Although one can also use in principle this method for the creep it gives poor results in this case. Indeed contrarily to the derivation involving \( f \) one needs here the flow of \( \eta \) in all regimes including the depinning regime \( l > l_d \), where \( \eta \) is still renormalized. Since the renormalization of \( \eta \) goes from large positive growth (first like \( \exp \theta \), then exponentially) in the thermal/saturation regime to negative in the depinning regime (where the system accelerates with subdiffusive \( z < 2 \)) a precise knowledge of the behavior around \( l_T \) would be needed. Unfortunately the lack of precise analytical methods available above \( R_T \) prevents from computing precisely such a crossover. A crude estimate of the flow can thus only give a bound of the exact result. If we use (e.g. in the RF or RP cases) the estimates of each regime, and the perturbative estimate of \( \eta_v \) in the theory at \( l_V \): \( \eta \propto \eta_v + \int_t \Delta_n',(v)l R_{\eta}^{(v)} \sim e^{-(2-\zeta)(l_V-l)}(t) \) (it will appear that \( \eta_v \) diverges faster than \( e^{-(2-\zeta)(l_V-l)} \) when \( v \to 0 \). The product of (6.8, 6.10, 6.12) is equal to \( \eta_v \). Integrated from 0 to \( v \), it yields

\[
\frac{\eta \nu_v}{f_c} \approx \exp \left[ -\frac{U_c}{T} \left( \frac{f}{f_c} \right)^{\nu} \right] \sum_{1+\nu(\frac{1}{g} - 1)} \frac{d \nu}{d \varepsilon} (6.16)
\]

One would thus find, using the \( \eta \) method a non-Arrhenius law for the creep regime. Even if one cannot strictly speaking exclude this result, as discussed above it is most likely an artefact of the approximate integration of the flow, and only a lower bound of the barrier height. Indeed compared to the integration of the flow of \( f \), this procedure is much more sensitive to the neglect of the crossover \( l_T < l < l_d \). A more precise integration of the flow would very likely show a compensation between the latent growth of the friction during the decrease of \( \partial \ln \eta \) (for \( l_T < l < l_d \) and the reduction of the friction occurring in the depinning regime \( l_d < l < l_V \). Note that if \( \Delta_n \) were equal to \( \eta_v \) then, one would recover (6.14). It would be useful to check explicitly on a numerical integration of the flow that such a cancellation does occur and verify that the \( \eta \) method confirms also the result (6.14).

We also note that the precise determination of the lengthscales for \( R > R_T \) depend on obtaining an accurate solution of the RG flow equations. In the previous section, we have obtained the formulas (6.3) and (6.11) under some assumptions about the mathematical form of the solutions of the flow in the region where \( \lambda_T \) and \( T_1 \) cross. These assumptions, discussed in the previous Section, should be checked further, e.g via numerical integration. Although this should not affect the creep exponent derived above, the precise determination of these

length scales is important to ascertain the exact value of the scale \( R_V \) (i.e the avalanche scale discussed below).

C. Discussion

Since our flow equations (4.11) include finite temperature and velocity, they allow for the first time to treat the regime of slow motion at finite temperature, directly from (6.10). As for the depinning we derive directly from the equation of motion the force–velocity law and we obtain new physics.

The first important result is of course the creep formula itself (6.14). Our method allows to prove the main physical assumptions, reviewed in Subsection III 3, needed for the phenomenological estimate, namely: (i) the equal scaling of the barriers and the valleys; (ii) the fact that velocity is dominated by activation over the barriers correctly described by an Arrhenius law. In our derivation such law comes directly from the integration of the flow equations in the thermal regime; (iii) the fact that one can use the static exponents in the calculation of the barriers. This appears directly in the formula (6.14) but can also be seen from the fact that in the thermal regime the velocity can essentially be ignored in the flow equations. We also recover the characteristic lengthscale predicted by the phenomenological estimate. Indeed one can identify the scale \( (6.7, 6.9, 6.10) R_T \sim R_c (f / f_c)^{-1/2 (2-\zeta)} \) as the \( R_{\text{opt}} \) of Subsection III 3.

Our equations allow to obtain additional physics in the very slow velocity regime. In particular, we see that the slow motion consists in two separate regimes. At small lengthscales \( R < R_T \) the motion is controlled by thermal activation over barriers as would occur at \( v = 0 \). This is the regime described by the phenomenological theory of the creep. Qualitatively, the main novel result obtained here is that the thermally activated regime is followed by a depinning regime, as shown by our equations. This leads to the following physical picture: at the length \( R_T \), bundles can depin through thermal activation. When they depin they start an avalanche like process, reminiscent of the case of the avalanche proceeds on larger scales in a deterministically. Thus one is left with a depinning-like motion, and the size of the avalanches is determined by the natural cut of the RG (\( \lambda = r_f \)), i.e., the scale at which the propagating avalanche motion is overcome by the regular motion of the center of mass. One recovers qualitatively and quantitatively some features of the \( T = 0 \) case at intermediate scale. The typical nucleus jumps over an energy barrier \( U_{\text{opt}} \sim U_c (R_T / R_c)^{\theta} \) resulting in \( v \sim \exp \left[ -\frac{U_c}{T} \left( \frac{f}{f_c} \right)^{\nu} \right] \). This jump of a region of size \( R_T \) initiates an avalanche spreading over a much larger size \( R_V \) which we find to be (see (6.7, 6.9, 6.11))

\[
\frac{R_V}{R_c} \sim \left( \frac{U_c}{T} \right)^{\frac{\mu}{\nu}} \left( \frac{R_T}{R_c} \right)^{1+\frac{\mu}{\nu}} (6.17)
\]
with \( \nu = \beta/(z - \zeta) = 1/(2 - \zeta) \) and \( z = 2 - (\epsilon - \zeta)/3 \) the critical exponents of the depinning, and \( \theta \) the energy exponent of the statics. Note that the correlation length \( R_c \) diverges at small drive and temperature as \( R_c \sim T^{-\sigma} f^{-\lambda} \) with \( \sigma = \frac{\beta}{z - \zeta} = \frac{1}{2 - \zeta} \) and \( \lambda = \frac{1}{2 - \zeta} + \frac{\delta}{z - \zeta} \).

To push the analogy further one can consider that the avalanches at small scales \( R > R_T \) are similar to the ones occurring in a regular depinning phenomenon due to an excess driving force \( (f - f_c)^{\alpha} \). Considering a minimal block size \( R_T \) instead of \( R_c \) for this “creepy” depinning, \( R_T/R_c \sim (f - f_c)^{\alpha} \), one obtains for this effective excess force:

\[
\frac{\eta^{\text{eff}}}{f_c^{\text{eff}}} \sim \left( \frac{f - f_c}{f_c} \right)^\beta \frac{T}{U_0} \tag{6.18}
\]

linking the creepy motion at \( T > 0 \) and the threshold depinning at \( T = 0 \). As explained before, there might be an uncertainty in the value of the avalanche exponent, which could be changed by a quantity of \( O(\epsilon) \). To confirm \( (6.17) \), one would need to further check the precise behaviour of the solution of the RG equations for \( R > R_T \).

One can understand qualitatively that the problem at scale \( R > R_T \) looks like depinning according to \( (6.18) \). The tilted barrier (see Subsection 11.3) \( E(R, f) = U_c(R/R_c)^\theta - f R_D \lambda f (R/R_c)^\omega \) to obtain in order to move a region of size \( R \) (all barriers corresponding to smaller scales having been eliminated), vanishes at \( R_0(f) \gtrsim R_T \). For the \( T = 0 \) depinning problem, one can define a scale dependent effective threshold force \( f_c^{\text{eff}}(R) \sim f_c(R/R_c)^{-2 - \zeta} \) such that \( E(R, f_c^{\text{eff}}) = 0 \) (also defined in Subsection 11.3), which corresponds to the force needed to depin scales larger than \( R \) (the true threshold \( f_c = f_c^{\text{eff}}(R_c) \) being controlled in that case by the Larkin length). A possible scaling derivation of \( (6.18) \) is obtained by noting that at \( T > 0 \), non-activated motion at scale \( R \) occurs when the tilted barrier \( E(R, f) \) is of the order of \( T \). This yields a \( T \) dependent effective threshold force such that

\[
\frac{f_c^{\text{eff}}(R) - f_c^{\text{eff}}(R, T)}{f_c^{\text{eff}}(R)} \sim \frac{T}{U_0(R/R_c)^\theta} \tag{6.19}
\]

At \( R = R_0(f) \), one has \( f = f_c^{\text{eff}}(R) \) and \( (6.18) \) is identical to \( (6.18) \) to zeroth order in \( \epsilon \) (i.e. \( \beta = 1 \)). In fact, to apply the above static barrier argument, it might be better to work in the co-moving frame where the velocity of the interface vanishes. This amounts to replace \( f \) by \( f - \eta v \) in the previous argument, and \( E(R_0(f), f, f) = 0 \), \( E(R_0(f), f - \eta v) = T \) gives back \( (6.18) \).

The crossover between thermally activated processes and depinning-like motion can also be recovered by noting that the condition \( \lambda t \sim T \chi \) which appears in the FRG flow can be rewritten as (using \( (4.12) \) and \( (4.10) \)):

\[
f_c^{\text{eff}}(R) \nu T (R) R_D \sim T \tag{6.20}
\]

where the lhs is a natural energy scale involved in the depinning due to driving effect of the center of mass. If it is much larger than \( T \), depinning effects dominate, while if it is smaller, the dynamics is activated.

Finally many open questions still remain. Technically it would be interesting to reconcile the two methods based on \( \eta \) and \( \tilde{J} \) which proved to be equivalent for the study of depinning. In fact although the two methods should formally agree, the comparison at a given order in the RG is more subtle. Indeed \( \frac{d}{d\bar{\epsilon}} \delta f = -\delta \eta \), by integration over \( t \) between 0 and \( \infty \) and derivative with respect to \( v \), gives back \( \eta_\infty = \frac{d}{d\bar{\epsilon}} \) provided that \( f_\infty = 0 \). However, one should notice that in \( \frac{d}{d\bar{\epsilon}} \delta f = -\delta \eta \), the derivative is understood at fixed parameters at the given scale. The occurrence of this hidden dependence in the velocity in the running parameters makes the equivalence between both approaches delicate. However the additional term is of higher order in disorder. Thus, as pointed out above, it is very likely that a careful integration of the flow of \( \eta \) should resolve this discrepancy, but this remains to be explicitly checked.

As for the \( T = 0 \) depinning, the existence of the depinning regime at \( l_d \) depends on the precise form of the boundary layer in the presence of a velocity. Note that the alternative scenario discussed in Section 11.4, e.g. whether or not the depinning regime is universal, would not affect the creep exponents, but only the subleading corrections.

**VII. CONCLUSION**

We examine in this paper the dynamics of disordered elastic systems such as interfaces or periodic structures, driven by an external force. We take into account both the effect of a finite temperature and of a finite velocity to derive the general renormalization group equations describing such systems. We extract the main features of the analytical solution to these equations both in the case of the \( T = 0 \) depinning (shown on Figure 1) and in the “creep” regime (small applied force \( f \) and finite temperature).

Our RG equations, when properly analyzed, allow to recover the depinning law \( v \sim (f - f_c)^{\beta} \) and the depinning exponent \( \beta \) also obtained by other methods. However, contrarily to previous approaches that needed additional physical assumptions, such as scaling relations among exponents or by hand regularisation, our approach is self-contained, all quantities being derived directly from the equation of motion. It thus provides a coherent framework to solve the difficulties and ambiguities encountered in the previous analytical studies. In addition our method allows to establish the universality classes for driven systems. It shows explicitly that a random bond type disorder gives rise close to a random field critical behavior at the depinning. Thus the dynamics is characterized by only two universality classes (random field (RF) for interfaces and random periodic (RP) for periodic systems) instead of three. Since this phenomenon is
an intrinsically dynamical one, it was out of the reach of the previous analytical approaches that used \( v = 0 \) flow equations together with additional physical prescriptions using e.g. the velocity as a cutoff on the \( v = 0 \) RG flow.

Of course one of the great advantages of the present set of RG equations is to allow for the precise study of the small applied force regime at finite \( T \), for which up to now, only phenomenological scaling arguments could be given. Our FRG study confirms the existence of a creep law at small applied force

\[
\frac{\eta v}{f_c} \approx \exp \left[ \frac{U_c}{T} \left( \frac{f}{f_c} \right)^{-\mu} \right]
\]

with a creep exponent related to the static ones \( \mu = (D - 2 + 2\zeta_{eq})/(2 - \zeta_{eq}) \), where \( \zeta_{eq} \) is the statics roughening exponent. It provides a framework to demonstrate, directly from the equation of motion, the main assumptions used in the phenomenological scaling derivation of the creep namely: (i) the existence of a single scaling for both the barriers and the minima of the energy landscape of the disordered system; (ii) the fact that the motion is characterized by an activation (Arrhenius) law over a typical barrier.

![FIG. 9. Schematic picture of the creep process emerging from the present study: while thermally activated motion occurs between scales \( R_c \) (Larkin length) and \( R_T \) (thermal nucleus size), depinning-like motion occurs up to the avalanche size \( R_V \).](image)

In addition, our study unveils a novel “depinning-like regime” within the creep phenomena, not addressed previously, even at the qualitative level since the phenomenological creep arguments did not address what happens after the thermally activated jump of the optimal nucleus. Although the velocity is dominated by the time spent to thermally jump over the barriers, our equations show that the small \( f \) behavior consists in fact of two different regimes. Up to a size \( R_T \) motion can only occur through thermal activation over barriers. This is the regime described by the phenomenological approach to the creep. The optimal nucleus of the scaling estimate is given directly by the RG derivation as \( R_T \sim (1/f)^{1/(2 - \zeta_{eq})} \). Remarkably, another interesting regime exists above this lengthscale (see Figure 9). It emerges directly from our RG equations and can be given the following simple physical interpretation. In some regions of the system, bundles of size \( R_T \) depin due to thermal activation. These small events then trigger much larger ones, and the motion above \( R_T \) proceeds in a deterministic way, much as the \( T = 0 \) depinning. In particular once the initial bundle depins it triggers an avalanche up to a size \( R_V \) which is given by \( R_V/R_T \sim (U_c/T)^{\nu/\beta} (R_T/R_c)^{\nu/\beta} \) where \( \theta \), \( \beta \) and \( \nu \) are the energy, depinning and correlation length exponents respectively.

The present study also raises several interesting questions which deserve further investigation, some of them rely on being able to obtain a more accurate solution of our flow equations. We have shown explicitly how to recover from our equations the conventional depinning law (and the scaling creep exponents). It rests on a mathematical property, likely to hold, but not yet rigorously established, of the solution for the flow of the correlator of the disorder. Such behavior should be checked in details. The equations being quite complicated, a numerical solution, albeit delicate, seems to be appropriate. If the constraint (5.2) on the flow defined in Section VI B were found to be violated, then the conventional picture of the depinning would very likely fail, as we have analyzed in detail. A similar question arises concerning the flow of the friction \( \eta \) as discussed in Section VI C. If the solution of the flow is found to depend on the precise behavior at the Larkin length \( R_c \), it is likely that even universality could be questioned. These issues are a priori less important for the first, thermally activated, part of the creep regime, but because of the existence of a second, depinning-like regime, they would also have consequences for creep. Again, these question depend on the precise form of the flow and can be answered unambiguously by a detailed enough analysis of our equations.

Several applications and extensions of our work can be envisioned. First, extensions to many-dimensional displacement field (of dimension \( N > 1 \)), given in Appendix E would be interesting to study within the methods used here. One could check whether the approximation used in Ref. [22] yields the correct result for the \( N > 1 \) depinning. Second, the effect of additional KPZ non-linearities could be investigated. In particular one could check the usual argument which yields that KPZ terms are unimportant for the depinning since their
coupling constant is proportional to the (small) velocity. Also, extensions to other types of disorder, such as correlated disorder are possible. Finally, it should allow to describe in a systematic way to the thermal rounding of the depinning, i.e. the study of the $v$–$f$ characteristics for $f$ close to the threshold and small $T$. If one assumes that one can simply carry naive perturbation theory in $T$ around the $T = 0$ solution of the RG flow near $f_c$ (i.e. only keeping the contribution beyond $\Gamma(v)$), one is led to an additional term proportional to $T/v^2$, which readily yields the value for the thermal rounding exponent $\rho = 1 + 2\beta$ proposed in Ref. [80] (i.e. a scaling form near $f = f_c$ and small $T$ for the velocity $v \sim T^{\beta/\rho} \Phi \left( \frac{f-f_c}{T^{1/\rho}} \right)$). Although this exponent seems to be consistent with starting values $\lambda \ll T$, its validity could be further checked by solving our RG flow equations at small $T$.

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APPENDIX A: NOTATIONS

Here are some notations and conventions and diagrammatics we use in the text.

The surface of the unit sphere in $D$ dimensions divided by $(2\pi)^D$ is denoted by $S_D = 2(4\pi)^{-D/2}/\Gamma(D/2)$. The thermal average of any observable $A$ is $\langle A \rangle$, the disorder average is $\overline{A}$, and the average with the dynamical action $S[u, \hat{u}]$ is denoted by $\langle A \rangle_S = \overline{A}$. The Fourier transform of a function $h_{rt}$ of $(r, t)$ is $h_{qr} = \int r e^{-iq\cdot r + iw\cdot t} \hat{h}_{q\omega}$ where $\int r \equiv \int dr dt$, and the inversion reads $h_{rt} = \int q e^{iq\cdot r - iw\cdot t} \hat{h}_{q\omega}$, where $\int q \equiv \int \frac{dq}{(2\pi)^D}$.

The Fourier transform of the correlator $\Delta(u)$ is $\Delta_\kappa = \int du e^{-i\kappa u} \Delta(u)$ in general or $\Delta_\kappa = \int_0^\infty du e^{-i\kappa u} \Delta(u)$ in the periodic case. One has thus $\Delta(u) = \int_\kappa^\infty e^{i\kappa u} \Delta_\kappa$, where $\int_\kappa = \int_{\kappa}^{(2\pi)^D}$ or $\frac{1}{N} \sum_\kappa$ in the periodic case. Note that $\Delta_\kappa$ is a real and even function of $\kappa$.

The graphs are made of the following units (see Figure 10): a full line between points $(r,t)$ and $(r',t')$ is a correlation $\langle u_{rt} u_{r't'} \rangle_S$, an oriented line with an arrow from point $(r',t')$ to point $(r,t)$ is a response $\langle u_{rt} \delta u_{r't'} \rangle_S$ (the arrow means that $t > t'$, for the function does not vanish by causality). The vertex is represented as a dashed line linking points $(r,t)$ and $(r,t')$. The dashed line means that both points have the same position $r$. From each point emerges a field. No arrow is needed for the full line or for the dashed line, since they are symmetric with respect to the exchange of their endpoints. The correlation being proportional to $T$ vanishes at $T = 0$.

The graphs renormalizing the disorder (see Figure 13) are made of vertices and responses, and they possess two external $\delta u$ lines. It can be easily seen that arrows are no more necessary since the two external $\delta u$ lines provide an orientation to all the responses of the graph. Indeed, due to causality, each of the external $\delta u$ is root of a tree, whose branches are response functions, which are oriented in the direction of the root.

APPENDIX B: PERTURBATION THEORY

We derive here the direct perturbation theory at $T > 0$ without the use of the MSR formalism. To organize the perturbation series, let us multiply the non-linear part of the equation of motion $F(r, vt + u_{rt})$ by a fictitious small parameter $\alpha$, which will be fixed to one at the end of the calculation. Directly on

$$\left\{ \begin{array}{l} \frac{\partial \hat{u}}{\partial t} - c \nabla^2 \hat{u}_{rt} = \alpha F(r, vt + u_{rt}) + \tilde{f} + \tilde{\zeta}_{rt} + h_{rt} \\
\end{array} \right. \quad (B1)$$

we can formally expand $u = \sum_{n \geq 0} \alpha^n u^{(n)}$, $f - \eta v \equiv \tilde{f} = \sum_{n \geq 0} \alpha^n \tilde{f}^{(n)}$, solve recursively the system (B1), even at non-zero temperature, and compute the $\alpha$-expansion of every observable. Note that we added a source $h_{rt}$ (with no constant uniform part) so as to compute the response function. As the force is gaussian, the expansion of disorder averaged quantities is in powers of $\alpha^2$, and is in fact an expansion in powers of $\Delta$. We denote by $C_{r-r', t-t'} = \langle u_{rt} u_{r't'} \rangle$ the exact correlation and by $R_{r-r', t-t'} = \langle \delta u_{rt} \delta u_{r't'} \rangle$ the exact response functions.

The first iterative steps are $\tilde{f}^{(0)} = \tilde{f}^{(1)} = 0$ and

$$\left\{ \begin{array}{l} \frac{\partial \hat{u}}{\partial t} - c \nabla^2 \hat{u}_{rt}^{(0)} = \tilde{\zeta}_{rt} + h_{rt} \\
\frac{\partial \hat{u}}{\partial t} - c \nabla^2 \hat{u}_{rt}^{(1)} = F(r, vt + u_{rt}^{(0)}) \\
\frac{\partial \hat{u}}{\partial t} - c \nabla^2 \hat{u}_{rt}^{(2)} = \partial_u F(r, vt + u_{rt}^{(0)}) u_{rt}^{(1)} + \tilde{f}^{(2)} \\
\end{array} \right.$$
These results can be extended to any temperature \( T \) \( v > \) FDT at \( T \). Note that \( \mathcal{R} \) and \( \mathcal{C} \) do not verify FDT at \( v > 0 \).

To first order in \( \Delta \) one obtains at \( T = 0 \)

\[
\begin{align*}
  f - \eta v &= -\int \frac{i\kappa \Delta_{\kappa}}{c q^2 - i \kappa \eta v} \\
  C_{q,\omega} &= \frac{1}{(c q^2 + (\eta \omega)^2)} R_{q,\omega} \\
  \mathcal{R}_{q,\omega} &= \frac{1}{(c q^2 + (\eta \omega)^2)} \int t \Delta''(vt) R_{q,t}(1 - e^{i\omega t})
\end{align*}
\]

These results can be extended to any temperature \( T \):

\[
\begin{align*}
  f - \eta v &= -D_1(\omega = 0) \\
  C_{q,\omega} &= C_{q,\omega} + \\
  R_{q,\omega} D_0(\omega) R_{q,-\omega} + \\
  \mathcal{R}_{q,\omega} (D_2(\omega = 0) - D_2(\omega)) C_{q,\omega} + h.c. \\
  \mathcal{R}_{q,\omega} &= \mathcal{R}_{q,\omega} + \\
  R_{q,\omega} (D_2(\omega = 0) - D_2(\omega)) R_{q,\omega},
\end{align*}
\]

where we have introduced the effective vertices, non-local in time, smoothed by the temperature (see Figure 11).

\[
\begin{align*}
  D_0(t) &= \int \Delta_{\kappa} e^{i\kappa v t + (i \kappa)^2 (C_{00} - C_{00})} \\
  D_1(t) &= \int \kappa \Delta_{\kappa} e^{i\kappa v t + (i \kappa)^2 (C_{00} - C_{00})} R_{0t} \\
  D_2(t) &= \int (i \kappa)^2 \Delta_{\kappa} e^{i\kappa v t + (i \kappa)^2 (C_{00} - C_{00})} R_{0t}
\end{align*}
\]

We now want to compute the corrections to the parameters \( c, \eta, \tilde{f}, T, \Delta(u) \) so that \( v, C_{q,\omega}, \mathcal{R}_{q,\omega} \) remain unchanged while the physical (ultra-violet) cutoff \( \Lambda \) on the \( q \) integrations is reduced. To first order in \( \Delta \) and \( T \), one obtains,

\[
\begin{align*}
  \partial c &= 0 \\
  \partial \eta &= -\int t R_{0t} \Delta''(vt) \\
  \partial \tilde{f} &= \int R_{0t} \Delta'(vt) \\
  \partial T &= \frac{1}{\eta} \int t C_{0t} \partial_t \Delta''(vt) \\
  \partial \Delta(u) &= C_{0t} \Delta''(u)
\end{align*}
\]

with \( \partial \equiv -\Lambda \partial_{\Lambda} \) and \( R_{0t}, C_{0t} \) are the on-shell gaussian response and correlation functions, i.e., with modes \( q \) lying only between \( \Lambda - \Delta \Lambda \) and \( \Lambda \).

A completely different way for obtaining the perturbation expansion is presented in Refs. [4, 5], as a first attempt to include thermal fluctuations in the large-velocity expansion of Ref. [18]. It consists in splitting the displacement field into a \( T = 0 \) part and a thermal part. This procedure is probably only true to first order in \( T \) and not controlled at higher \( T \). Instead, the method presented here is really an expansion in disorder at any \( T \).

Although the calculation can in principle be pushed to second order, the method is too cumbersome to do it in practice (see however Ref. [5] at \( T = 0 \)). It is easier to use the formalism of dynamical field theory as shown in Appendix C.

**APPENDIX C: DERIVATION OF THE FLOW AT FINITE VELOCITY AND FINITE TEMPERATURE**

Here we give the details of the renormalization procedure used for the moving system. We use the MSR formalism with action \( S[u, \dot{u}] \) given by (1.2). Having shifted the field \( u_{rt} \) so that its average vanishes \( \langle u_{rt} \rangle = 0 \), we can do perturbation theory with the gaussian part

\[
S_0[u, \dot{u}] = \int_{rt} [i \dot{\tilde{u}}_r (\eta \partial_t - c \nabla^2) u_r - \eta \tilde{f} i \dot{\tilde{u}}_r i \dot{\tilde{u}}_r]
\]

of the action. The gaussian correlation \( C_{rt} \) and response \( R_{rt} \) functions were defined in Appendix C.

The interaction part of the action contains the disorder correlator and also the pinning force \( \tilde{f} = \mathcal{O}(\Delta) \):

\[
S_i[u, \dot{u}] = -\tilde{f} \int_{rt} i \dot{\tilde{u}}_{rt} - \frac{1}{2} \int_{rt'} i \dot{\tilde{u}}_{rt} i \dot{\tilde{u}}_{rt} \Delta(u_{rt} - u_{rt'} + \dot{u})(t - t')
\]
The effective action for slow fields $u$, $\dot{u}$ is given by the following cumulant expansion where the averages are computed within the gaussian part $S_0$ over the fast fields $u^>, \dot{u}^>$:

$$S_\langle[u, \dot{u}] = S_0[u, \dot{u}] + \langle S_1[u + u^>, \dot{u} + \dot{u}^>]\rangle - \frac{1}{2}\langle[S_1[u + u^>, \dot{u} + \dot{u}^>]^2\rangle_c + O(S^3) \quad \text{(C3)}$$

We now turn to the computation of the first and second order terms.

1. First order

To first order, the corrections arise from the graph shown in Figure 12.

FIG. 12. First order RG corrections. The internal lines carry fast fields.

They read

$$\langle S_1[u + u^>, \dot{u} + \dot{u}^>]\rangle = -\bar{J} \int_{rt} \dot{i}u_{rt}$$

$$- \int_{rtt'\kappa} (ik)\Delta_\kappa[u](r,t,t') R_{0t-l}^{rt} i\dot{u}_{rt'}$$

$$- \frac{1}{2} \int_{rtt'\kappa} \Delta_\kappa[u](r,t,t') i\dot{u}_{rt}i\dot{u}_{rt'}$$

with the shorthand notation:

$$\Delta_\kappa[u](r,t,t') \equiv \Delta_\kappa e^{i\kappa(u_{rt} - u_{rt'})} e^{\lambda u_{rt'} (C_{00}^\kappa - C_{0t}^\kappa)}$$

The term (C4) appears to be the sum of a $i\dot{u}\bar{F}[u]$ term and a $i\dot{u}\dot{u}\bar{G}[u]$ term. Let us begin to deal with the first type. A short time expansion of $e^{\lambda u_{rt} (C_{00}^\kappa - C_{0t}^\kappa)}$ yields the following operators

$$- (\int_{rt} \dot{i}u_{rt}) \int_{rt} i\kappa \Delta_\kappa e^{i\kappa u_{rt}} e^{\lambda \lambda u_{rt}} R_{0t}^{rt}$$

which is a correction to $\bar{f}$ and

$$- (\int_{rt} i\dot{u}_{rt}\partial_t u_{rt}) \int_{rt} (ik)^2 \Delta_\kappa e^{i\kappa u_{rt}} e^{\lambda \lambda u_{rt}} t R_{0t}^{rt}$$

which is a correction to $\eta$. The elasticity operator $i\dot{u}\nabla^2 u$ is not corrected and no higher gradients like $i\dot{u}\nabla^2 u$ are generated in the equation of motion. Note also that to this order, no KPZ term $i\dot{u}\dot{u}\nabla^2 u^2$ is generated.

The $i\dot{u}\dot{u}\bar{G}[u]$ term can be rewritten as the sum of

$$\frac{1}{2} \int_{rtt'} \Delta_\kappa e^{i\kappa u_{rt}} e^{\lambda u_{rt} (C_{00}^\kappa - C_{0t}^\kappa)}$$

which has the form of a disorder correlator and yields a correction to $\Delta(u)$, and an operator quasi local in time

$$\int_{rtt'} i\dot{u}_{rt}i\dot{u}_{rt'} \times$$

$$\times \int_{rt} \Delta_\kappa e^{i\kappa u_{rt}} e^{\lambda u_{rt} (C_{00}^\kappa - C_{0t}^\kappa)} \left(1 - \frac{e^{\lambda u_{rt} (C_{00}^\kappa - C_{0t}^\kappa)}}{2}\right)$$

which yields a correction to the $\int_{rt} i\dot{u}_{rt}i\dot{u}_{rt}$ term. The projection of (C7) on this thermal noise operator is

$$\left(\int_{rt} \dot{i}u_{rt}i\dot{u}_{rt}\right) \int_{rt} \Delta_\kappa e^{i\kappa u_{rt}} e^{\lambda u_{rt}} \left(1 - \frac{e^{\lambda u_{rt} (C_{00}^\kappa - C_{0t}^\kappa)}}{2}\right)$$

To obtain the correction to the temperature $T$, one uses $\frac{\partial\delta c}{\partial T} = \frac{\delta T}{\eta} - \frac{\delta c}{\eta}$. An integration by parts of (C4), thanks to FDT for the “pure” $R$ and $C$, yields $\frac{\delta T}{\eta}$.

To summarize,

$$\delta c = 0$$

$$\delta f = \int_{rt} i\kappa \Delta_\kappa e^{i\kappa u_{rt}} e^{\lambda u_{rt}} R_{0t}^{rt}$$

$$\delta \eta = - \int_{rtt>0} (ik)^2 \Delta_\kappa e^{i\kappa u_{rt}} e^{\lambda u_{rt}} t R_{0t}^{rt}$$

$$\delta \Delta(u) = \int_{rt} \Delta_\kappa e^{i\kappa u_{rt}}$$

$$\eta \delta T = \int_{rt} K v t \Delta_\kappa e^{i\kappa u_{rt}}$$

The correction $\delta f$ has the same form as the perturbative expression for $f$, with opposite sign and shell-restricted functions $C, R$. Note that $\delta \eta = - \eta \frac{\delta f}{\delta T}$.

In the infinitesimal shell limit, the shell-restricted functions $C^>, R^>$ which are evaluated at $r = 0$ are of order $dl$. The differential flow is thus given by (4,11).

2. Second order

The fast-modes average $\langle S_\kappa^2 \rangle_c$ can be decomposed into one term with $\bar{f}$ in factor plus the rest which does not contain $\bar{f}$. The former vanishes for the following reason: the contraction of the $f \int_{rt} i\dot{u}_{rt}$ with the $u_{rt}$ contained in the vertex operator involves a fast response $R_{0t}^\kappa$. But $\int R_{0t}^\kappa = 0$, since its modes live in the shell. The latter is the connected average of two disorder vertices. We now extract from it a correction to the disorder, i.e., a term which has the form

$$\frac{1}{2} \int_{rtt'} i\dot{u}_{rt}i\dot{u}_{rt'} \delta \Delta(u_{rt} - u_{rt'})$$

The corresponding diagrams are represented in Figure 13.
spectively. Calling $\tau$ of both response functions, denoting $U_r$, expressions, integrated over tree causality, and provides an orientation to these lines: we drew the arrows just for clarity.

Each diagram has two external $i\tilde{u}_{r\ell} i\tilde{u}_{r'\ell'}$ legs, to which corresponds a functional half vertex of $u_{r\ell}$ and $u_{r'\ell'}$ respectively. Calling $r$, $r'$ the (positive) time arguments of both response functions, denoting $U = u_{r\ell} - u_{r'\ell'} + \nu(t - t')$, the diagrams have the following analytical expressions, integrated over $r, r', t, t', \rho, \tau, \tau'$:

\[
\begin{align*}
  a &= -i\tilde{u}_{r\ell}i\tilde{u}_{r'\ell'}\delta_{r', r}\Delta''(U)(U + v(\tau' - \tau))R_{\rho r}^{\rho r}R_{\rho r'}, \\
  b &= -i\tilde{u}_{r\ell}i\tilde{u}_{r'\ell'}\delta_{r, r'}\Delta'(U + v\tau')\Delta'(U - v\tau)R_{\rho r}^{\rho r}R_{\rho r'}, \\
  c &= i\tilde{u}_{r\ell}i\tilde{u}_{r'\ell'}\delta_{r', r}\Delta''(U)(v(\tau' - \tau))R_{\rho r}^{\rho r}R_{\rho r'}, \\
  d &= i\tilde{u}_{r\ell}i\tilde{u}_{r'\ell'}\delta_{r, r'}\Delta'(U + v\tau')\Delta'(-v(\tau' + \tau))R_{\rho r}^{\rho r}R_{\rho r'}. 
\end{align*}
\]

After another short distance expansion of $b$ and $d$, noting that \( \int_{\rho} \bar{R}_{\rho r}^2 R_{\rho r'}^2 = \int_{\rho} \bar{R}_{\rho r}^2 R_{\rho r'}^2 = S_D \Lambda^D e^{-c\Lambda^2(v\tau' + \tau)/\eta \Lambda} \), a proper symmetry counting yields the term of order $\Lambda^2$ of (11).

The results obtained here are consistent with the analysis of Ref. 56.

**APPENDIX D: NEW RESULTS IN THE NON-DRIVEN CASE AT FINITE TEMPERATURE**

We give here a detailed analysis of the functional renormalization group flow at $T > 0$ and zero velocity. The temperature is an irrelevant operator and flows exponentially fast to zero. We show however that the temperature rounds the cusp in a region of size proportional to $T$ around the origin and that in this boundary layer, the disorder correlator takes a super-universal (to lowest order in $\epsilon$) scaling form. In addition we show how to carry a systematic expansion at low $T$. As temperature decreases, the correlator of the disorder becomes more and more pinched, and eventually reaches its zero-temperature cuspy fixed point at infinity.

We show that during the renormalization at $v = 0$ with a flowing temperature $T_l \to 0$, the cusp forms only asymptotically ($l \to \infty$), and $\Delta(u)$ has the following scaling form in the boundary layer $|u| \sim T_l/\chi$

\[
\Delta_l(u) \simeq \Delta_l(0) - T_l f(\nu u/\bar{T_l})
\]

with $f(x) = \sqrt{1 + x^2} - 1$ and where $\chi = |\Delta''(0^+]|$ measures the cusp.

Furthermore, we show that the following expansion in temperature for the solution of the FRG flow holds

\[
(\Delta(u) - \Delta(0) - T)^2 = \sum_{n \geq 2} T^n f_n(u/T)
\]

thus we obtained a fairly complete picture of the solution.

1. The curvature

The flow equation of the value at zero of the disorder correlator is

\[
\partial_l \Delta_l(0) = (\epsilon - 2\zeta)\Delta_l(0) + T_l \Delta''(0)
\]

Since $\Delta_l \to \Delta^*$, the convergence of $\Delta_l(0)$ towards $\Delta^*(0)$ implies that $T_l \Delta''(0)$ also converges. From the fixed point equation

\[
(\epsilon - 3\zeta)\Delta^*(u) + \zeta(u\Delta^*(u))' = \frac{1}{2} (\Delta^*(u) - \Delta^*(0))^{2''}
\]

one has simply $\epsilon - 2\zeta\Delta^*(0) = \Delta''(0^+)^2$, and thus,

\[
- T_l \Delta''(0) \to \Delta''(0^+)^2
\]

2. Scaling function in the boundary layer

We show here that the assumption that the curvature at zero of $\Delta_l$ diverges like a power of the inverse temperature implies that all the derivatives at zero also diverge and that there exists a well defined and particularly simple scaling function in the boundary layer around zero.

Precisely, for any function $T_l$ decreasing to zero and a function $\Delta_l(u)$ such that

\[
\partial_l \Delta_l(u) = (\epsilon - 2\zeta)\Delta_l(u) + \zeta u \Delta_l'(u) + T_l \Delta''(u)
\]

\[
+ \Delta''(u) (\Delta_l(0) - \Delta_l(u)) - \Delta_l'(u)^2
\]

if $\Delta''(0) \sim - (\chi^2/T_l)^\alpha$ for some $\alpha > 0$ and $\chi$, then, defining the functions $f_l(x) = T_l^{-(\alpha + 1)/2} \chi^{-\alpha}$, we obtain that every derivatives of $f_l$ at $x = 0$ converge to the corresponding derivatives of $f(x) = \sqrt{1 + x^2} - 1$, and that $f$ is the only fixed possible fixed point for $f_l$. 

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A simple way to see the convergence to the scaling function $f$ is to write the flow of $f$

$$T_l \Delta''(0) + \frac{1}{2} \chi^{2\alpha} T_l^{1-\alpha} (1 + f)^{2n} =$$
$$T_l \left( \partial_t f_l - 2 f_l + \frac{\alpha + 1}{2} f_l' \right)$$

and eliminate at large $l$ the rhs term which is subdominant (higher order in $T_l$) for $\alpha > 0$, since $T$ has been absorbed in the variable $x$ of $f$. We have used that $\partial_x = 2 - \epsilon + 2\zeta$. Hence the fixed point equation for $f_l$ is

$$\frac{1}{2} (1 + f)^{2n} = 1$$

which has the solution $f(x)$ above since we know that $f(0) = 0$, $f''(0) = 1$ and $f^{(4)}(0) = -3$ is easily checked.

This is confirmed by the study of the flow equations for the successive derivatives $a_n = \Delta^{(2n)}(0)$:

$$\partial a_n = (\epsilon + 2(n - 1)\zeta) a_n + T a_n + \sum_{k=1}^{n} \left( \frac{(2n+1)}{2k} \right) a_k a_{n+1-k}$$

From a trivial recurrence, the hypothesis $\Delta''(0) \sim - (\chi^2/T_l)^\alpha$ implies that $T^{n(\alpha+1)-1} a_n$ converges for any $n$. Moreover the limit $c_n = \lim_{n \to \infty} T^{n(\alpha+1)-1} \chi^{-2n\alpha} a_n$ can be obtained from (D6) and is $c_n = \frac{(1.3...2n-1)^2}{2n-1} = f^{(2n)}(0)$.

To fix the value of $\alpha$ ($\alpha = 1$ as strongly suggested by (D4)), we checked that the only values of $\beta > 0$, $\gamma > 0$ such that $g_l(x) = \frac{1}{T_l} (\Delta_l(T_l^\beta x) - \Delta_l(T_l^\delta x))$ has a meaningful fixed point are $\beta, \gamma = (1, 1)$. For these values, the fixed point is $g(x) = \Delta^{\ast\ast}(0^+) x + \sqrt{1 + (\Delta^{\ast\ast}(0^+)) x^2}$.

### 3. Next order in $T$

The procedure which gives us the leading behavior in the boundary layer controlled by temperature can be extended analytically with arbitrary accuracy in an expansion to any order in $T$. We study

$$\partial_l \Delta_l(u) = (\epsilon - 2\zeta) \Delta_l(u) + \zeta u \Delta'_l(u) + T_l \Delta''(u) - \Delta''(u) (\Delta(u) \Delta_l(0)) - \Delta'_l(u)^2$$
$$\partial_l \ln T_l = -\theta$$

with $\theta = 2 - \epsilon + 2\zeta$. For numerical purposes or for the following analytical computation, it is useful to switch to the function $y(u) = (\Delta(u) - \Delta(0) - T)^2$ which remains quadratic at the origin when $T \to 0$, since $y(u) = T^2 + |T \Delta''(0)| u^2 + O(u^4)$ for $T > 0$ and $y(u) = \Delta''(0^+) u^2 + O(u^4)$ for $T = 0$. This function flows as

$$\partial_l y = 2(\epsilon - 2\zeta) y + \zeta u y' + \sqrt{y} (y'' - y''(0) - 4T)$$

We can replace the scale $l$ dependence of $y_l(u)$ by a $T$ dependence since $T$ and $l$ are linked by $T_l = 2l\theta e^{-\theta l}$. The function $y_T(u)$ can be expanded in

$$y_T(u) = \sum_{n \geq 2} T^n f_n \left( \frac{u}{T} \right)$$

The expansion begins at $n = 2$ since $y_T(0) \equiv T^2$ and we have

$$f_2(0) = 1 \quad f_{n>2}(0) = 0$$

The equation for the $f_n$’s reads

$$\sum_{n \geq 2} T^n (2(\epsilon - 2\zeta) n + \Delta''(0^+) x f_n) =$$
$$\frac{\sum_{n \geq 2} T^n f_n}{\sqrt{n} \sum_{n \geq 2} T^n - \sum_{n \geq 2} T^{-2}(f''_{n} - f''_{n}(0))}$$

One can solve this equation order by order in $T$. It is useful to divide $\Delta$ and $T$ by $\chi^2$ and $u$ by $\chi$. With these rescaled quantities, we have simply

$$y^{\ast\ast} = -2T_l \Delta''(0) \to 2$$

and thus $f''(0) = 2$. If we knew the full behavior of $y^{\ast\ast}_T(0)$, i.e., the $f''_n(0)$’s, we could completely solve the system. Here, we get

$$f_3(x) = 4 \left( 1 - \frac{\epsilon - \zeta}{3} \left( \sqrt{1 + x^2} - 1 - \frac{x^2}{2} \right) \right)$$
$$-4(\epsilon - \zeta) x \left( \frac{1 + x^2}{\sqrt{1 + x^2}} \right)$$
$$+ f_4'(0) \frac{x^2}{2}$$

where we wrote $f_3(x)$ such that the three first lines are functions which vanish and have zero curvature at zero. Note that while $f_3(x)$ is universal, the last term $f_4(x)$ contains an unknown integration constant $f''_4(0)$ which presumably depends on the initial condition of the flow and is thus not universal. Indeed we observed a non-universal $f''_4(0)$ in a numerical integration of the flow of $y_T(u)$.

The procedure can be carried to any order in $T$ and the all the $f_n$’s are accessible. The unknown coefficients of the expansion

$$-2T_l \Delta''(0) = 2 + \sum_{n>0} T^n f_{n+2}(0)$$

are similarly non-universal.

Both Subsections D1 and D2 thus provide a rather convincing and consistent picture for the solution of the $T > 0$, $v = 0$ FRG equations (awaiting a mathematical proof).
APPENDIX E: ANALYTICAL SOLUTIONS AT FIXED TEMPERATURE

We present here the analytical solutions of the fixed point equations for RF and RP at fixed $T$. Thanks to the exact expression of these fixed points, we are able to check the scaling form derived in Appendix B within an “adiabatic” hypothesis where the running correlator at $l$ is identified with the fixed point at $T = T_l$. Our families of fixed temperature fixed points (FTFP) give back the known fixed points at $T = 0$ in both the RF and the RP cases. However, even if we obtain the same form (E6) for the RF $T = 0$ fixed point as in Ref. [8] we disagree with the scaling in $\epsilon$.

1. Random field

We look for a fixed point of

$$\partial_l \Delta_l(u) = (\epsilon - 3\zeta) \Delta_l(u) + \zeta (u \Delta_l(u))' - \frac{1}{3} (\Delta_l(u) - \Delta_l(0) - T)^{2n}$$  (E2)

with fixed $T$ and initial random field condition $\int \Delta_0 > 0$. Since $\partial_l \ln \int du \Delta_l(u) = \epsilon - 3\zeta$, a meaningful fixed point can be obtained only for $\zeta = \epsilon/3$. Fixing the RF strength $\int \Delta_0$ to one, we are led to the following problem: for any $T \geq 0$, find the fixed temperature fixed point function (FTFP) $\Delta(T, u)$ such that

$$\frac{\epsilon}{3} (u \Delta(u))' = \frac{1}{2} \left( \Delta(u) - \Delta(0) - T \right)^{2n}$$  (E3)

$$\int du \Delta(u) = 1$$  (E4)

Integrating (E3) from 0 to $\infty$ yields $T \Delta'(0^+) = 0$, hence the FTFP has a fixed point for $T = 0$ but no null for $T \neq 0$. At $T = 0$, integrating (E3) from 0 to $u$ and dividing by $\Delta(u)$ yields $u = \Delta'(u) - \Delta(0)/\Delta(u)$. Then, integrating again from 0 to $u$ yields the $T = 0$ FTFP, by imposing (E4)

$$\Delta(T = 0, u) = \left( \frac{\epsilon}{3(\int y)^2} \right)^{1/3} y \left( u \left( \frac{\epsilon}{3} y \right)^{1/3} \right)^{1/3}$$  (E5)

where the function $y(x)$ is implicitly defined by

$$\frac{x^2}{2} = y - 1 - \ln y$$  (E6)

Since $y(0) = 1$ one has

$$\Delta(0, 0) = \left( \frac{\epsilon}{3(2\sqrt{2} \int_0^1 \sqrt{y - 1 - \ln y})^2} \right)^{1/3}$$  (E7)

It is easy to compute the number $\int \int_{-\infty}^\infty dx dy(x) = 2 \int_0^1 dy \cdot \frac{dy}{\sqrt{y - 1 - \ln y}} \approx 1.55$. Note the behavior near 0 given by $y(x) = 1 - |x| + \frac{x^2}{2} - \frac{|x|^3}{36} + \mathcal{O}(x^4)$ thus $\partial^2 \Delta(0, 0^+) = \frac{x}{2}$. Note also the gaussian decrease of correlations at infinity $y(x) \sim e^{-x^2}$. An intriguing fact is the scaling of the $T = 0$ fixed point with $\epsilon$: its $n$th derivative at $0^+$ scales like

$$\partial^2 \Delta(0, u = 0^+) \sim e^{(1+n)/3}$$  (E8)

At $T > 0$, there is no cusp ($\partial_u \Delta(T > 0, 0^+) = 0$) and the same double integration of (E3) yields

$$\Delta(T, u) = \Delta(T, 0) y(T, u/\epsilon/(3\Delta(0)))$$  (E9)

with $y(T, x)$ implicitly defined by

$$\frac{x^2}{2} = y - 1 - (1 + \frac{T}{\Delta(0)}) \ln y$$  (E10)

The value of $\Delta(T, 0)$ is determined by condition (E4). Using $\int dx y(x) = \int dy x(y)$, this condition reads

$$\sqrt{\frac{24\Delta(T, 0)^3}{\epsilon}} \int_0^1 dy \sqrt{y - 1 - (1 + \frac{T}{\Delta(T, 0)}) \ln y} = 1$$  (E11)

This equation admits a unique solution $\Delta(T, 0) > 0$ for any $T > 0$. Then there exists a unique FTFP $\Delta(T, u)$ for each $T > 0$. Some of them are displayed in Figure 14. Note that $T = 0$ in (E11) gives back the $T = 0$ non-analytic fixed point $\Delta(T = 0, u)$ (E2). Hence the set of FTFP has a nice $T \rightarrow 0$ limit, even if there is a qualitative difference between the cuspy $T \neq 0$ FTFPs and the analytic $T > 0$ FTFPs.

As is obvious from their analytical expression, or from Figure 14, $\lim_{T \rightarrow 0} \Delta(T, u) = \Delta(0, u)$: the $T = 0$ non-analytic fixed point is approached smoothly by the set of analytic fixed $T$ fixed points. When $T$ approaches zero, the curvature of the FTFPs at the origin goes to $-\infty$ like

$$\lim_{T \rightarrow 0} -T \Delta''(T, 0) = \frac{\epsilon}{3} \Delta(0, 0)$$  (E12)

with $\Delta(0, 0)$ given by (E7).

We also checked that the $\Delta(T, u)$ converge when $T \rightarrow 0$ to the zero temperature fixed point with the predicted scaling form (D1)

$$\frac{\Delta(T, 0) - \Delta(T, T^x)}{T} \rightarrow \frac{\chi}{\sqrt{1 + x^2} - 1}$$  (E13)

where $\chi = |\partial_u \Delta(0, 0^+)|$ is given by the $T = 0$ FTFP equation $\chi^2 = \frac{\epsilon}{3} \Delta(0, 0)$. Some of the RF fixed $T$ fixed points are shown on the right bottom quarter of Figure 14, including the cuspy (highest) $T = 0$ fixed point. Absorbing $\epsilon$ in $T$ and $\Delta$, we chose to plot the non-trivial solution to the most reduced problem

$$\frac{1}{3} (u \Delta(u))' = \frac{1}{2} (\Delta(u) - \Delta(0) - T)^{2n}$$  (E14)

$$\int du \Delta(u) = 1$$  (E15)
To restore $\epsilon$ and $\int \Delta$, one simply has to note that the “dimensions” are $T \equiv \Delta \equiv \epsilon^{1/3} \left( \int \Delta \right)^{2/3}$ and $u \equiv \epsilon^{-1/3} \left( \int \Delta \right)^{1/3}$. The left bottom of Figure 14 shows $-T \Delta''(T,0)$ as a function of $T$. This combination has a finite limit ($\approx 0.17$) when $T \to 0$.

2. Random periodic

In the Random Periodic case, the conservation of the period $a$ of $\Delta$ requires $\zeta = 0$. After a suitable rescaling, $u \to u/a$, $\Delta \to \Delta/(ea^2)$ and $T \to T/(ea^2)$, the fixed point equation reads for the 1-periodic function $\Delta(u)$

$$\Delta(u) = -\frac{1}{2} \left( \Delta(u) - \Delta(0) - T \right)^{2u} \quad (E16)$$

and is easily solved by quadrature, by analogy with a particle’s position $X(u) = (\Delta(u) - \Delta(0) - T)^2$ at time $u$ in a potential $V(X) = 4X^{3/2}/3 - 2(\Delta(0) + T)X$ verifying $X''(u) = -V'(X(u))$. The quadrature leads to the reciprocal function $u(X)$, parametrized by $\Delta(0)$ and $T$, as a sum of two elliptic functions. Then, imposing the solution $\Delta(u)$ be 1-periodic fixes $\Delta(0)$ as a function of $T$.

The result is:

- for $T \geq (2\pi)^{-2}$ the only solution is $\Delta(u) \equiv 0$

- for $0 < T < (2\pi)^{-2}$ another solution arises, which resembles a cosine function of linearly vanishing amplitude when $T \to (2\pi)^{-2}$. This non-trivial solution has no cusp but becomes pinched as $T$ decreases (growing curvature $|\Delta''(0)|$ and higher harmonics). As can be seen on the analytical expression (not given here), $-T \Delta''(0) \to 1/36$. In particular, it remains finite when the temperature vanishes.

- eventually for $T \to 0$, the non-trivial solution uniformly tends to the zero temperature fixed point

$$\Delta(u) = \frac{1}{6} \left( \frac{1}{6} - u(1 - u) \right) \quad (E17)$$

The temperature $(2\pi)^{-2}$ in our units is exactly the critical temperature $T_g$ of the random-field XY-model and the fixed points near $T_g$ reproduce the line of fixed points of this problem (since we worked to second order, it is only an approximation). Indeed in $D = 2$, the naive dimension of the temperature is zero and our FTFP has a direct physical meaning. Note that another random gradient term becomes relevant in $D = 2$ but does not feed back on the flow of $\Delta(u)$.

We can now use the exact FTFPs to check that an adiabatic hypothesis is consistent with the scaling form (D1). Indeed, one can numerically check that the correlator with a flowing temperature has the FTFPs have the scaling (D1) as $T \to 0$. To conclude about the problem with a flowing temperature $T_i \to 0$, it appears from these observations that no cusp occurs at finite scale for $T_0 > 0$. The cusp forms only asymptotically ($l \to \infty$), with

$$\lim_{l \to \infty} -T_l \Delta_l''(0) = \Delta(0,0)$$

given by (E7) and it obeys a scaling form in the boundary layer $|u| < T_l/\chi$

$$\Delta_l(u) \simeq \Delta_l(0) - T_l f(u\chi/T_l), \quad f(x) = \sqrt{1 + x^2} - 1$$

We note that the precise form of the flow of the tem-
perature (i.e. the value of $\theta$) only affects subdominant behavior (i.e. the function $f_3(x)$ in Subsection 5.3).

**APPENDIX F: MULTI-DIMENSIONAL CASE**

We give here a possible extension of the FRG to a multi-dimensional displacement field. This study generalizes the approach of Ref. 52 by including the effect of $v > 0$ and $T > 0$ in the flow. For periodic structures, a similar study of the multi-dimensional displacement field was shown in Ref. 54 to yield novel effects.

In a $D + N$ dimensional space, we distinguish between the internal or longitudinal space of dimension $D$, to which $r$ belongs, and the transverse space of dimension $N$, to which $u$ belongs. The elastic energy of an interface without overhangs defined by a height function $u_r$ is quadratic in $\nabla u$ of the form $\frac{1}{2} \int_r c_{ij}^{\mu\nu} (\partial_\mu u_i^r) (\partial_\nu u_j^r)$.

The disorder: the random bond (RB) case corresponds to a random potential $V(r,u)$, with correlations $(V(r,u) - V(r',u'))^2 = -2\delta^{D}(r - r')R_{\parallel}(u - u')$. Function $R(u)$ is even, vanishes at $u = 0$ and goes to a negative constant for $|u| \gg r_f$. The random field (RF) case corresponds to a force $F^2(r,u)$ with correlations $F^2(r,u)F^2(r',u') = \delta^{D}(r - r')\Delta^{ij}(u - u')$, where the $\Delta^{ij}(u)$ vanish for $|u| \gg r_f$. A RF gives rise to a RF via $F_i = -\partial V$ and the correlators are related by $\Delta^{ij}(u) = -\partial^{ij}R(u)$. Note that this type of correlator deriving from a RB has $\int d^N u \Delta^{ij}(u) = 0$. Finally, the random periodic case (RP) occurs when $u$ is defined up to a discrete set of translations forming a lattice of points $P$, e.g. when $u$ is a phase, defined up to $2\pi$ shifts. In this case, the disorder is periodic and one has $\Delta^{ij}(u) = \Delta^{ij}(u + P)$ for any $P$ of the lattice (or $R(u) = R(u + P)$).

The overdamped dynamics is given by

$$\eta_j \partial_t u_i^r + c_{ij}^{\mu\nu} (\partial_\mu u_i^r) + F^{ij}(r,u_{rt}) + \xi^{ij}_{rt} + f^i + h_{rt}^i$$

where $\eta$ is the friction tensor and $\zeta$ a Langevin noise, with correlations $\langle \xi^{ij}_{rt} \xi^{jk}_{r't'} \rangle = 2\eta(T) \delta^{D}(r - r')\delta(t - t')$. The tensor $T$ stands for the temperature(s) of this out of equilibrium system. We added a driving force $f_i^r$ perpendicular to the interface and a source field $h_{rt}^i$, as an external excitation.

Without assuming any symmetry, let $C^{ij}_{rt}$ and $R^{ij}_{rt}$ be the gaussian correlation and response functions. We obtain by the same procedure as for the $N = 1$ case the following first order corrections due to disorder

$$\delta C^{ij}_{rt} = 0$$

$$\delta f^{ij} = -\int e^{i\kappa \cdot (c_{00} - C_{00})}. i\kappa \cdot u^t \Delta_{\kappa}^{ij} \kappa R_{\kappa-}^{ij}$$

$$\delta R^{ij}_{rt} = -\int e^{i\kappa \cdot (c_{00} - C_{00})}. i\kappa \cdot u^t \Delta_{\kappa}^{ij} \kappa R_{\kappa-}^{ij}$$

$$\delta \langle T \rangle^{ij} = \frac{1}{2} \int e^{i\kappa \cdot u^t} (e^{i\kappa \cdot (C_{00} - C_{00})}. i\kappa - e^{i\kappa \cdot C_{00}}) \Delta_{\kappa}^{ij}$$

$$\delta \Delta^{ij}(u) = \int \Delta^{ij} e^{i\kappa \cdot C_{00}. i\kappa + u^t}$$

Using $(\Delta^{\alpha \beta})^* = \Delta^{\alpha \beta}, \Delta^{\alpha \beta} = \Delta^{\beta \alpha}$, we write the on-shell corrections as (with the matrix product $AB = A_{\alpha,\gamma} B_{\gamma \beta}$)

$$\delta C^{ij}_{rt} = 0$$

$$\delta f^{ij} = -\int i\kappa \cdot \int R_{\kappa-}^{ij}. \Delta_{\kappa}^{ij} e^{i\kappa \cdot u^t}$$

$$\delta \eta^{ij} = -\int i\kappa \cdot \int t_\kappa R_{\kappa-}^{ij}. \Delta_{\kappa}^{ij} e^{i\kappa \cdot u^t}$$

$$\delta \langle T \rangle^{ij} = -\int i\kappa \cdot \int R_{\kappa-}^{ij}. \Delta_{\kappa}^{ij} e^{i\kappa \cdot u^t}$$

$$\delta \Delta_{\kappa} = i\kappa. C_{00}. i\kappa \Delta_{\kappa}$$

The second order correction to $\Delta$ reads

$$\delta \Delta^{ij}(u) = \int \int R^{m k t} R^{m l t}$$

$$\left[ (\Delta^{kl}(u + v + (\tau' - \tau)) \right] \partial^{m} \partial^{n} \Delta^{ij}(u)$$

$$-\partial^{m} \Delta^{il}(u + v + \tau) \partial^{n} \Delta^{kj}(u + v + \tau')$$

$$-\partial^{m} \Delta^{kl}(u + v + \tau) \partial^{n} \Delta^{il}(u + v + \tau')$$

$$+\partial^{m} \Delta^{kl}(u + v + \tau) \partial^{n} \Delta^{il}(u + v + \tau')$$

Note that each of the first three terms are symmetric under $i \leftrightarrow j, u \leftrightarrow -u$ and that the fourth is exchanged with the fifth under this symmetry. Then $\Delta$ remains a correlator.

Of course this second order correction to $\Delta$ gives back the expression already computed for a $D + 1$ interface if $N = 1$. At zero velocity, one gets the second derivative of the flow equation of Balents and Fisherd. If we assume that $\Delta(u)$ depends only on the component of $u$ parallel to the velocity and send $v$ to zero, then our expression reduces to the equations of Ertas and Kardaried.

To simplify the analysis, let us rely on the assumed symmetries of the system. If we suppose that the initial problem is rotationnally invariant, i.e. has $O(N)$ symmetry, then the elasticity tensor $c$, the friction tensor $\eta$ and the temperature tensor $T$ are only scalars and the force–force correlator $\Delta$ is covariant, i.e., for any $R$ such that $R^r. R = 1$, $R^r. \Delta(u). R = \Delta(R.u)$.

During the flow, we expect from physical grounds that the running terms of the action will conserve their symmetries but the velocity $v$ which is fixed once for all selects a particular direction in transverse space. The interesting symmetries are given by the little group of the velocity, i.e., the transformations $R$ such that $R^r. R = 1$ and $R^r. v = v$. Then one may decompose the tensors on a basis involving $v$ (one has only two fractions, temperatures, response and correlation functions and five $\Delta_i$’s, functions of $(u^2, v^2, u.v)$).

Unfortunately, the full problem cannot be easily decoupled, even with the simplifications pointed out above.
No closed equation e.g. for the correlator restricted to displacements aligned with the velocity $\Delta(u \parallel v)$, has been found, and the problem even at zero temperature seems involved. The simplification used in Ref. 32 consists in assuming that $\Delta$ does not depend on the transverse coordinates. This assumption reduces the problem to the $N = 1$ case, and it would be interesting to solve at finite $T$ the behavior of transverse coordinates along the lines of our analysis.

**APPENDIX G: THE FLOW OF THE DISORDER CORRELATOR AT SMALL VELOCITY**

The effect of a small velocity on the FRG flow is mainly restricted to the boundary layer of width $\rho_l$ about the origin. Analytically, it is rather difficult to give an estimate of $\rho_l$ or to decide how $\tilde{\Delta}(u)$ precisely behaves in the boundary layer $|u| \sim \rho_l$. It is however possible to simplify the formidable second order correction to the disorder correlator, displayed in (4.11), and to obtain analytically several results, giving some hints about this behavior.

The $O(\Delta^2)$ term in (4.11) is written under a form involving two integrations over $s, s'$, reflecting the presence of two response functions integrated over time. After some integrations by part, the $O(\Delta^2)$ term becomes:

$$
\tilde{\Delta}''(u) \int_{s>0} e^{-s} \left( \tilde{\Delta}(\lambda s) - \frac{\tilde{\Delta}(u + \lambda s) + \tilde{\Delta}(u - \lambda s)}{2} \right) + \int_{s>0} e^{-s} \frac{\tilde{\Delta}(u + \lambda s) - \tilde{\Delta}(u)}{\lambda} \int_{s>0} e^{-s} \frac{\tilde{\Delta}(u - \lambda s) - \tilde{\Delta}(u)}{\lambda} - \int_{s>0} e^{-s} \tilde{\Delta}'(\lambda s) \tilde{\Delta}(u + \lambda s) + \tilde{\Delta}(u - \lambda s) - 2\tilde{\Delta}(u)
$$

Integrating over $u$, this correction becomes

$$
\int_0^{\infty} du \tilde{\Delta}'(u) \int_{s>0} e^{-s} (2 - s) \frac{\tilde{\Delta}(u + \lambda s) - \tilde{\Delta}(u - \lambda s)}{\lambda}
$$

For any non-crazy function $\Delta$, this expression is positive. Assuming that $\Delta$ has no cusp, it can be safely expanded and we can check that it is of order $\lambda^2$:

$$
\partial \int \tilde{\Delta} = (\epsilon - 3\zeta) \int \tilde{\Delta} + 2\lambda^2 \int \tilde{\Delta}'^2 + O(\lambda^4)
$$

Thus at $v \neq 0$, the integral of $\tilde{\Delta}$ grows during the flow, whereas it was conserved in the statics.

Using (G1), one can also compare the flow of $\tilde{\Delta}$ at small velocity to the cuspy $v = 0$ flow. In particular, one observes that the effect of the velocity is to reduce the blow-up of the curvature $\tilde{\Delta}''(0)$

$$
\partial \tilde{\Delta}''(0) = \epsilon \tilde{\Delta}''(0) - \tilde{\Delta}''(0) \int_{s>0} e^{-s} \frac{\tilde{\Delta}(\lambda s) - \tilde{\Delta}(0)}{\lambda^2} - \int_{s>0} e^{-s} \left( \frac{\tilde{\Delta}'(\lambda s)}{\lambda} \right)^2
$$

which has the right sign (using Cauchy inequality) to slow down the exponential growth of $\Delta_l(0)$.

Obtaining a numerical integration of the flow is a highly non-trivial question, since all the interesting properties occur close to the origin yielding inaccurate results in real space. In Fourier space, the number of harmonics to be retained is huge if one wants to focus on the quasi-cuspy behavior ($\Delta^*_c \sim \kappa^{-2}$ at the cuspy fixed point). However, we obtained, at least at the beginning of the flow (up to $l_c$) with small initial velocity, the curve shown in Figure 3. The initial condition was a RB disorder (full line). It is obvious on the snapshot (dotted line) close to $l_c$ that the flow transformed the RB into a RF.

**APPENDIX H: BEFORE THE LARKIN LENGTH**

We show here that at the scale $l_c$, $\tilde{\Delta}_l(u)$ is very close to the static zero-temperature fixed point $\Delta^*_c(u)$. This can be checked numerically, even in the presence of a small temperature and small velocity. Analytically, one cannot obtain an exact integration of the flow, but we can compare $\Delta_l(u)$ to the known $\Delta^*_c(u)$ by the following arguments.

Let us take e.g. the RF case for which $\zeta = \epsilon/3$. For weak disorder one obtains from the integration of (4.15):

$$
e^{-\xi_l} = \left(1 + \frac{\epsilon}{3|\tilde{\Delta}_0(0)|}\right)^{1/3} \approx r_f \left(\frac{\epsilon}{\sqrt{\Delta}}\right)^{1/3}
$$

where we have used $|\tilde{\Delta}_0''(0)| \approx r_f/\epsilon^{2/3} \approx \int \tilde{\Delta}/\epsilon^{2/3}$. We prove in Appendix E.1 that $\chi = |\Delta^*_c(0^+)|$ verifies $\chi \approx \epsilon^{2/3} \left(\int \tilde{\Delta}\right)^{2/3}$ and that $\Delta^*_c(0) \approx \epsilon^{1/3} \left(\int \tilde{\Delta}\right)^{2/3}$. Thus the range $r_f^* \approx \Delta^*_c(0)/|\Delta^*_c(0^+)|$ of $\Delta^*$ verifies

$$
r_f^* \approx r_f e^{-\xi_l} \approx \chi/\epsilon
$$
To determine the range of $\tilde{\Delta}_l(u)$ we use the fact that at the beginning of the flow one can neglect the nonlinear term in the flow equation (H1). We are left with $\tilde{\Delta}(u) = e^{(r-2\zeta)l} \tilde{\Delta}_0(e^{bl})$ and thus the range of $\tilde{\Delta}_l(u)$ is simply

$$r_f(l_c) \simeq r_fe^{-\zeta l_c} \quad \text{(H3)}$$

A comparison of (H2) and (H3) shows that the two ranges are similar. Furthermore, in the RF case $\int \Delta$ is conserved by the flow at $v = 0$, and thus the similarity of ranges shows that the shape of $\tilde{\Delta}_l$ is close to the shape of $\Delta^*$ (same integral, same range).

Similarly in the RP case it is also true that $\tilde{\Delta}_l$ resembles $\Delta^*$, but now $\chi = \epsilon_a/6$ as can be seen on the fixed point $\Delta^*(u) = \frac{\epsilon}{6} \left( \frac{u^2}{6} - u(a-u) \right)$.

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