ON AUTOMORPHISMS OF FLEXIBLE VARIETIES

S. KALIMAN AND D. UDUMYAN

Abstract. Let $X$ be a flexible variety and $\varphi : C_1 \to C_2$ be an isomorphism of closed one-dimensional subschemes of $X$. We develop criteria which guarantee that $\varphi$ extends to an automorphism of $X$.

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Introduction

Every algebraic variety $X$ in this paper is considered over an algebraically closed field $k$ of characteristic 0. We study the following question:

when can an isomorphism $f : Y_1 \to Y_2$ of subschemes of $X$ be extended to an automorphism of $X$?

For $X \simeq \mathbb{A}^n$ and $Y_i \simeq \mathbb{A}^1$ the answer to this question is known to be always positive in all cases but $n = 3$ (see [AMo], [Su], [Cr], [Je]). For $n = 3$ the answer is unknown but if $k = \mathbb{C}$ then $f$ can be extended to a holomorphic automorphism of $\mathbb{C}^3$ [Ka92]. Furthermore, very general sufficient conditions for extension of isomorphisms of closed subvarieties of $\mathbb{A}^n$ to automorphisms of $\mathbb{A}^n$ where found in [Ka91] and [St] (in a weaker form they were already present in [Je]). For $X \neq \mathbb{A}^n$ the first break-through is due to van Santen (formerly Stampfli) who proved that every isomorphism of smooth polynomial curves in $X \simeq \text{SL}_n(\mathbb{C})$, $n \geq 3$ is extendable to an automorphisms of $X$ [St]. Later jointly with Feller he showed that the same is true when $Y_i \simeq \mathbb{C}$ and $X$ is any connected complex linear algebraic group different from $\mathbb{C}^3$, $\text{SL}_2(\mathbb{C})$ or $\text{PSL}_2(\mathbb{C})$ [FS]. Some extra cases with a positive answer to the extension problem were found in [Ka20]. Say, the answer is positive when $X$ is a nonzero fiber of a non-degenerate quadric form.

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in $\mathbb{A}^m$ (we call such $X$ an $(m-1)$-sphere over $k$) and $Y_i$ is a smooth closed subvariety with $\dim Y_i \leq \frac{m}{3} - 1$.

Starting from dimension 2 affine spaces, linear algebraic groups without nontrivial characters and spheres over $k$ are examples of so-called flexible varieties (recall that a normal quasi-affine variety $X$ is flexible if $\text{SAut}(X)$ acts transitively on the smooth part $X_{\text{reg}}$ of $X$ where $\text{SAut}(X)$ is the subgroup of the group $\text{Aut}(X)$ of algebraic automorphisms of $X$ generated by one-parameter unipotent subgroups). For a smooth flexible variety $X$ every isomorphism of zero-dimensional subvarieties is extendable to an automorphism [AFKKZ]. Furthermore, recall that for a closed subvariety $Y$ of $X$ with defining ideal $I$ in the ring $k[X]$ of regular functions on $X$ its $k$th infinitesimal neighborhood is the subscheme of $X$ with the defining ideal $I^k$ (in particular, an automorphism of such neighborhood is essentially an automorphism of the algebra $k[X]/I^k$). It follows from [AFKKZ, Theorem 4.11 and Remark 4.16-4.17] that any automorphism $g : Z \to Z$ of an $k$th infinitesimal neighborhood $Z$ of a zero dimensional subvariety $Y$ of a smooth flexible variety $X$ is extendable to an automorphism of $X$ provided that $g|_Y = \text{id}_Y$ and $g$ preserves local volume forms modulo $I^k$ (we say that such maps of infinitesimal neighborhoods have Jacobian 1). The second author gave a proof in [Ud] of the similar result in the case when $Y$ is a smooth polynomial curve in a smooth flexible variety $X$ of $\dim X \geq 4$.

In this paper we give a much stronger version of this result (Theorem 5.5) and develop some new important technical tools for the extension problem. In particular, instead of polynomial curves we consider smooth curves in $X$ with trivial normal bundles. (The class of such curves is sufficiently big. It contains, for instance, curves which have only rational irreducible components. Furthermore, consider $X$ as a subvariety of an affine space. Then by the Bertini theorem the intersection of $X$ with $n-1$ general hyperplanes where $\dim X = n$ is a smooth curve. Every irreducible component of this curve has a trivial normal bundle.) As an application we get the following generalization of the Feller-van Santen theorem (Corollary 7.5).

**Theorem 0.1.** Let $X$ be isomorphic to a connected complex linear algebraic group without nontrivial characters and let $\dim X \geq 4$. Let $C_1$ and $C_2$ be smooth polynomial curves in $X$. Then every isomorphism of $k$th infinitesimal neighborhoods of $C_1$ and $C_2$ with Jacobian 1 extends to an automorphism of $X$.

The absence of nontrivial characters is essential - without this assumption this theorem does not hold (see Remark 7.6). If $C_1 = C_2$ we can make a stronger statement in the holomorphic category without assuming that $C_i$ is a polynomial curve or even that $C_i$ is irreducible. Namely, the holomorphic category has the advantage of the Ivarsson-Kutzschebauch theorem [IK] which leads, in particular, to the following (Corollary 8.12).

**Theorem 0.2.** Let $X$ be isomorphic to a connected complex linear algebraic group without nontrivial characters and $\dim X \geq 4$. Suppose that $C$ is a smooth closed curve in $X$ with a trivial normal bundle and defining ideal $I$ in the algebra $\text{Hol}(X)$ of holomorphic functions on $X$. Let $\hat{A} = \frac{\text{Hol}(X)}{I}$ be the algebra of holomorphic functions on
Then every \( \hat{\lambda} \)-automorphism \( \lambda : \frac{1}{f} \to \frac{1}{f} \) with Jacobian 1 extends to a holomorphic automorphism of \( X \).

In the case of \( X \simeq \text{SL}_n(\mathbb{C}) \) we can say even more (Corollary 8.13).

**Theorem 0.3.** Let \( X \) be isomorphic to \( \text{SL}_n(\mathbb{C}) \) where \( n \geq 3 \). Suppose that \( C_1 \) and \( C_2 \) are isomorphic smooth closed curves in \( X \) such that their normal bundles in \( X \) are trivial. Then every isomorphism of \( k \)th infinitesimal neighborhoods of \( C_1 \) and \( C_2 \) with Jacobian 1 extends to a holomorphic automorphism of \( X \).

In the case of spheres over \( k \) the similar result holds in algebraic category (Theorem 7.8).

**Theorem 0.4.** Let \( X \) be an \((m - 1)\)-sphere over \( k \) where \( m \geq 6 \). Let \( C_1 \) and \( C_2 \) be isomorphic smooth closed curves in \( X \) such that their normal bundles are trivial. Then every isomorphism of \( k \)th infinitesimal neighborhoods of \( C_1 \) and \( C_2 \) with Jacobian 1 extends to an automorphism of \( X \).

The paper is organized as follows. In Section 1 we collect technical tools developed in [Ka20] that are important for this paper. Section 2 contains some criterion which enables us to deal with curves that are not once-punctured (recall that a curve \( C \) is once-punctured if for any completion \( \bar{C} \) of \( C \) the set \( \bar{C} \setminus C \) is a singleton). The advantage of a once-punctured curve (like \( C \)), which was exploited in [FS] and [Ka20], is that every morphism of this curve to a quasi-affine algebraic variety is automatically proper. The criterion we developed guarantees that for any closed curve \( Z \) in a flexible variety \( X \) and a morphism \( \varphi : X \to Q \) to an affine variety \( Q \) there is an automorphism \( \alpha \) of \( X \) such that \( \varphi \circ \alpha |_Z : Z \to Q \) is proper. In section 3 using this criterion we describe the structure of normal bundles of smooth curves in a large class of flexible varieties. Actually, we show later (Corollary 6.6) that if a normal bundle of a smooth closed curve in such variety is trivial then every nonvanishing section of this bundle is induced by a locally nilpotent vector field (cf., [AFKKZ, Corollary 4.3]). Section 4 contains some facts about special linear groups and their subgroups generated by elementary transformations over commutative rings. In Sections 5 we prove the main theorem (Theorem 5.5) that states that under the assumption on Jacobians and some other mild assumptions automorphisms of \( k \)th infinitesimal neighborhoods of smooth close irreducible curves in a flexible variety \( X \) whose normal bundles are trivial extend to automorphisms of \( X \). In section 6 we remove the assumption of irreducibility of these curves in the main theorem. As applications we prove Theorems 0.1 and 0.4 in Section 7. Section 8 contains Theorems 0.2 and 0.3.

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1. **Preliminaries**

In this section we present some technical tools developed in [Ka20] which require the following definitions.
Definition 1.1. (1) Given an irreducible algebraic variety $\mathcal{A}$ and a map $\varphi: \mathcal{A} \to \text{Aut}(X)$ (where $\text{Aut}(X)$ is the group of algebraic automorphisms of $X$) we say that $(\mathcal{A}, \varphi)$ is an algebraic family of automorphisms of $X$ if the induced map $\mathcal{A} \times X \to X$, $(\alpha, x) \mapsto \varphi(\alpha).x$ is a morphism (this definition was introduced in [Ra]).

(2) If we want to emphasize additionally that $\varphi(\mathcal{A})$ is contained in a subgroup $G$ of $\text{Aut}(X)$ then we say that $\mathcal{A}$ is an algebraic $G$-family of automorphisms of $X$.

(3) In the case when $\mathcal{A}$ is a connected algebraic group and the induced map $\mathcal{A} \times X \to X$ is not only a morphism but also an action of $\mathcal{A}$ on $X$ we call this family a connected algebraic subgroup of $\text{Aut}(X)$.

Definition 1.2. We call a subgroup $G$ of $\text{Aut}(X)$ algebraically generated if it is generated as an abstract group by a family $\mathcal{G}$ of connected algebraic subgroups of $\text{Aut}(X)$.

Definition 1.3. (1) A nonzero derivation $\delta$ on the ring $A$ of regular functions on a quasi-affine algebraic variety $X$ is called locally nilpotent if for every $0 \neq a \in A$ there exists a natural $n$ for which $\delta^n(a) = 0$. This derivation can be viewed as a vector field on $X$ which we also call locally nilpotent. The set of all locally nilpotent vector fields on $X$ will be denoted by $\text{LND}(X)$. The flow of $\delta \in \text{LND}(X)$ is an algebraic $G_a$-action on $X$, i.e., the action of the group $(k, +)$ which can be viewed as a one-parameter unipotent group $U$ in the group $\text{Aut}(X)$ of all algebraic automorphisms of $X$. In fact, every $G_a$-action is a flow of a locally nilpotent vector field (e.g., see [Fr]).

(2) Note that the kernel $\text{Ker}\delta$ of $\delta$ coincides with the ring $k[X]^G_a$ of $G_a$-invariants in $k[X]$. The morphism $\tau : X \to \text{Spec}k[X]^G_a$ induced by the natural embedding $k[X]^G_a \hookrightarrow k[X]$ is called the categorical quotient morphism of the $G_a$-action and $\text{Spec}k[X]^G_a$ is always quasi-affine [Wil]). When $\text{Spec}k[X]^G_a$ is affine (i.e., $k[X]^G_a$ is finitely generated) we call $\tau$ an algebraic quotient morphism but it is not always the case by Nagata’s counterexample (e.g., see [DF]). Hence, sometimes we use instead of $\tau$ partial quotient morphisms. Namely, a morphism $\varrho : X \to Q$ into a normal affine variety $Q$ is called a partial quotient associated with $\delta$ if the ring $\varrho^*(k[Q])$ is contained in $\text{Ker}\delta$ and general fibers of $\varrho$ are isomorphic to $\mathbb{A}^1$. Partial quotient morphisms always exist by virtue of the Rosenlicht Theorem (e.g., see [PV, Theorem 2.3] and [FKZ, Definition 2.16]).

Remark 1.4. Definition 1.3 (2) implies that if $\delta$ is nonzero then there exists a dense open smooth subvariety $Q_0$ of $Q$ such that for $X_0 = \varrho^{-1}(Q_0)$ the morphism $\varrho|_{X_0} : X_0 \to Q_0$ is smooth and surjective (in fact, one can assume that $X_0$ is naturally isomorphic to $Q_0 \times \mathbb{A}^1$ by [KaMi]) and $\delta$ does not vanish on $X_0$.

Definition 1.5. (1) For every locally nilpotent vector fields $\delta$ and each function $f \in \text{Ker}\delta$ from its kernel the field $f\delta$ is called a replica of $\delta$. Note that such replica is automatically locally nilpotent.

(2) Let $\mathcal{N}$ be a set of locally nilpotent vector fields on $X$ and $G_\mathcal{N} \subset \text{Aut}(X)$ denote the group generated by all flows of elements of $\mathcal{N}$. We say that $G_\mathcal{N}$ is generated by $\mathcal{N}$.

(3) A collection of locally nilpotent vector fields $\mathcal{N}$ is called saturated if $\mathcal{N}$ is closed under conjugation by elements in $G_\mathcal{N}$ and for every $\delta \in \mathcal{N}$ each replica of $\delta$ is also contained in $\mathcal{N}$.
Definition 1.6. Let $X$ be a normal quasi-affine algebraic variety of dimension at least 2, $\mathcal{N}$ be a saturated set of locally nilpotent vector fields on $X$ and $G = G_{\mathcal{N}}$ be the group generated by $\mathcal{N}$. Then $X$ is called $G$-flexible if for any point $x$ in the smooth part $X_{\text{reg}}$ of $X$ the vector space $T_xX$ is generated by the values of locally nilpotent vector fields from $\mathcal{N}$ at $x$ (which is equivalent to the fact that $G$ acts transitively on $X_{\text{reg}}$ [AFKKZ]). In the case of $G = \text{SAut}(X)$ we call $X$ flexible without referring to $\text{SAut}(X)$ (recall that $\text{SAut}(X)$ is the subgroup of $\text{Aut}X$ generated by all one-parameter unipotent subgroups).

Notation 1.7. Given a morphism $\kappa : X \rightarrow P$ of algebraic varieties we denote by $\text{Aut}(X/P)$ the subgroup of $\text{Aut}(X)$ consisting of those automorphisms $\alpha$ for which $\kappa = \kappa \circ \alpha$. We let $\text{SAut}(X/P) = \text{Aut}(X/P) \cap \text{SAut}(X)$.

The following Collective Transversality Theorem is one of the main technical tools in [Ka20].

Theorem 1.8. ([Ka20, Theorem 1.4], cf. [AFKKZ, Theorem 1.15]) Let $X$ and $P$ be smooth irreducible algebraic varieties and $\kappa : X \rightarrow P$ be a smooth morphism. Let a group $G \subseteq \text{Aut}(X/P)$ be algebraically generated by a system $\mathcal{G}$ of connected algebraic subgroups closed under conjugation in $G$. Suppose that the restriction of the $G$-action to $\kappa^{-1}(p)$ is transitive for every $p \in P$.

Then there exist subgroups $H_1, \ldots, H_m \in \mathcal{G}$ such that for any locally closed reduced subschemes $Y$ and $Z$ in $X$ one can find a dense open subset $U = U(Y, Z) \subseteq H_1 \times \ldots \times H_m$ such that every element $(h_1, \ldots, h_m) \in U$ satisfies the following:

(i) $\dim(Y \cap (h_1 \cdot \ldots \cdot h_m).Z) \leq \dim(Y \times_P Z) + \dim P - \dim X$.

In particular, when $\dim Y \times_P Z \leq \dim Y + \dim Z - \dim P$ one has

(ii) $\dim(Y \cap (h_1 \cdot \ldots \cdot h_m).Z) \leq \dim Y + \dim Z - \dim X$.

Furthermore, suppose that the inequality $\dim Y \times_P Z \leq \dim Y + \dim Z - \dim P$ holds and also that $Z, Y \times_P Z$, and $Y \times_P X$ are smooth. Then

(iii) $(h_1 \cdot \ldots \cdot h_m).Z$ meets $Y$ transversally.

The crucial step in the proof of Theorem 1.8 is the following.

Proposition 1.9. Let the assumption of Theorem 1.8 hold. Then there is a sequence $H_1, \ldots, H_m$ in $\mathcal{G}$ such that for a suitable open dense subset $U \subseteq H_m \times \ldots \times H_1$, the map

(1) $\Phi : H_m \times \ldots \times H_1 \times X \rightarrow X \times_P X$ \hspace{1em} \text{with} \hspace{1em} (h_m, \ldots, h_1, x) \mapsto ((h_m \cdot \ldots \cdot h_1).x, x)$

is smooth on $U \times X$.

Namely, we have the following.

Proposition 1.10. ([Ka20, Proposition 1.10]) If a sequence $H_1, \ldots, H_m \in \mathcal{G}$ satisfies the conclusions of Proposition 1.9 then it satisfies also the conclusions of Theorem 1.8. Furthermore, for any element $H$ of $\mathcal{G}$ the sequence $H_1, \ldots, H_m, H$ (resp. $H, H_1, \ldots, H_m$) satisfies the conclusions of Proposition 1.9 and Theorem 1.8 as well.
We shall need the following notions which, unfortunately, were not introduced in [Ka20].

**Definition 1.11.** Let \( \kappa : X \to P \), \( G \) and \( \mathcal{G} \) satisfy the assumptions of Theorem 1.8 and \( \dim X = n \). Consider \( (X \times_P X) \setminus \Delta \) (where \( \Delta \) is the diagonal), the complement \( T'X \) to the zero section in the tangent bundle of \( X \) and the frame bundle \( \text{Fr}(X) \) bundle of \( TX \) (i.e., the fiber of \( \text{Fr}(X) \) over \( x \in X \) consists of all bases of \( T_xX \)). Projectivization of \( TX \) replaces \( \text{Fr}(X) \) with a bundle \( \text{PFr}(X) \) whose fiber over \( x \) consists all ordered \( n \)-tuples of points in the projectivization \( \mathbb{P}^{n-1} \) of \( T_xX \) that are not contained in the same hyperplane of \( \mathbb{P}^{n-1} \). Then we have natural \( G \)-actions on all these objects. Let \( Y \) be either \( X \), or \( (X \times_P X) \setminus \Delta \), or \( T'X \), or \( \text{PFr}(X) \). Suppose that the \( G \)-action is transitive on every fiber of \( Y \) over \( P \). Then we say that an algebraic \( G \)-family \( \mathcal{A} \) of automorphisms of \( X \) is a regular \( G \)-family for \( Y \) over \( P \) if

1. \( \mathcal{A} = H_m \times \ldots \times H_1 \) where each \( H_i \) belongs to \( \mathcal{G} \);  
2. for a suitable open dense subset \( U \subseteq H_m \times \ldots \times H_1 \), the map 

\[
\Psi : H_m \times \ldots \times H_1 \times Y \to Y \times_P Y \quad \text{with} \quad (h_m, \ldots, h_1, x) \mapsto ((h_m \cdot \ldots \cdot h_1), y, y)
\]

is smooth on \( U \times Y \).

Of course, by Proposition 1.9 the assumption about transitivity implies that regular families exist. Note also that if \( \mathcal{A} \) is a regular \( G \)-family, say, for \( (X \times_P X) \setminus \Delta \) over \( P \) and \( \mathcal{B} \) is a regular \( G \)-family for \( T'X \) over \( P \) then by Proposition 1.10 \( \mathcal{A} \times \mathcal{B} \) (resp. \( \mathcal{B} \times \mathcal{A} \)) is a regular \( G \)-family for both \( (X \times_P X) \setminus \Delta \) and \( T'X \) over \( P \). This implies the existence of algebraic \( G \)-families \( \mathcal{A} \) that are regular for all four varieties \( X \), \( (X \times_P X) \setminus \Delta \), \( T'X \) and \( \text{PFr}(X) \) over \( P \). We call such \( \mathcal{A} \) a perfect \( G \)-family over \( P \) (if \( P \) is a singleton then we just say that \( \mathcal{A} \) is a perfect \( G \)-family).

**Remark 1.12.** It is worth observing the following property of regular \( G \)-families. Let \( K \) be a finite subset of \( Y \) and for every \( y \in K \) let \( Z_y \) be a dense open subvariety of \( Y \). Note that \( U_y = \Psi^{-1}(Z_y \times y) \) is a nonempty open subset of \( U \) since \( \Psi \) is smooth. Hence, \( \bigcap_{y \in Y} U_y \) is open and nonempty. Therefore, perturbing \( \alpha \in U \) we can always guarantee that \( \alpha(y) \in Z_y \) for every \( y \in K \), i.e., each \( \alpha(y) \) is general in \( Y \).

**Example 1.13.** Let \( X \) be a smooth \( G \)-flexible variety where \( G \) is generated by a saturated set \( \mathcal{N} \subset \text{LND}(X) \). Then the \( G \)-action is \( m \)-transitive for every \( m > 0 \) [AFKKZ, Theorem 2.2]. Hence, the natural action of \( G \) on \( (X \times_P X) \setminus \Delta \) is transitive. Similarly, by [AFKKZ, Theorem 4.14 and Remark 4.16] \( G \) acts transitively on \( T'X \) and on \( \text{PFr}(X) \) (but not on \( \text{Fr}(X) \)). Hence, \( X \) admits a perfect \( G \)-family.

**Theorem 1.14.** ([Ka20, Theorem 4.2(i)]) Let \( X \) and \( P \) be smooth algebraic varieties and \( Q \) be a normal algebraic variety. Let \( \varrho : X \to Q \) and \( \tau : Q \to P \) be dominant morphisms such that \( \kappa : X \to P \) is smooth for \( \kappa = \tau \circ \varrho \). Suppose that \( Q_0 \) is a nonempty open smooth subset of \( Q \) so that for \( X_0 = \varrho^{-1}(Q_0) \) the morphism \( \varrho|_{X_0} : X_0 \to Q_0 \) is smooth. Let \( G \subset \text{Aut}(X/P) \) be an algebraically generated group acting 2-transitively on each fiber of \( \kappa : X \to P \) and \( Z \) be a locally closed reduced subvariety in \( X \).
(i) Let \( \dim Z \times_P Z \leq 2 \dim Z - \dim P \) and \( \dim Q \geq \dim Z + m \) where \( m \geq 1 \). Then there exists an algebraic \( G \)-family \( \mathcal{A} \) of automorphisms of \( X \) such that for a general element \( \alpha \in \mathcal{A} \) one can find a constructible subset \( R \) of \( \alpha(Z) \cap X_0 \) of dimension \( \dim R \leq \dim Z - m \) for which \( \varrho(R) \) and \( \varrho(\alpha(Z) \setminus R) \) are disjoint and the restriction \( \varrho|_{\alpha(Z) \cap X_0 \setminus R} : (\alpha(Z) \cap X_0) \setminus R \to Q_0 \) of \( \varrho \) is injective.

In particular, if \( \dim Q \geq 2 \dim Z + 1 \) and \( Z'_\alpha \) is the closure of \( Z'_\alpha = \varrho \circ \alpha(Z) \) in \( Q \) then for a general element \( \alpha \in \mathcal{A} \) the map \( \varrho|_{\alpha(Z) \cap X_0} : \alpha(Z) \cap X_0 \to Z'_\alpha \cap Q_0 \) is a bijection, while in the case of a pure-dimensional \( Z \) and \( \dim Q \geq \dim Z + 1 \) the morphism \( \varrho|_{\alpha(Z) \cap X_0} : \alpha(Z) \cap X_0 \to Z'_\alpha \cap Q_0 \) is birational.

(ii) Suppose that \( X \) is \( G \)-flexible where \( G \) is generated by a saturated set \( \mathcal{N} \subset \text{LND}(X) \) and \( P_0 = \tau(Q_0) \). Let \( \dim Z \times_P Z \leq 2 \dim Z - \dim P, 2 \dim Z + 1 \leq \dim Q \) and \( \dim T(Z_0/P_0) \leq \dim Q - \dim P \). Then the algebraic family \( \mathcal{A} \) from (i) can be chosen so that for a general element \( \alpha \in \mathcal{A} \) the morphism \( \varrho|_{\alpha(Z) \cap X_0} : \alpha(Z) \cap X_0 \to Z'_\alpha \cap Q_0 \) is bijective and it induces an injective map of the tangent bundle of \( \alpha(Z) \cap X_0 \) into the tangent bundle of \( Q_0 \).

Remark 1.15. (1) We shall need below the case when \( P \) is a singleton. In particular, the assumption \( \dim Z \times_P Z \leq 2 \dim Z - \dim P \) can be omitted.

(2) If \( X \) is a smooth \( G \)-flexible variety then the \( G \)-action on \( X \) is \( m \)-transitive for every \( m \geq 1 \) [AFKKZ, Corollary 2.2]. In particular, when \( P \) is a singleton the assumption that \( G \) acts 2-transitively on \( X \) can be also omitted in this case.

(3) It follows from the proof that any regular \( G \)-family for \( (X \times_P X) \setminus \Delta \) over \( P \) can serve as \( \mathcal{A} \) in Theorem 1.14 (i), whereas for Theorem 1.14 (ii) any \( G \)-family regular for both \( (X \times_P X) \setminus \Delta \) and \( T'X \) over \( P \) works. Thus, whenever we apply Theorem 1.14 below we use a perfect \( G \)-family. In particular, for every \( H \in \mathcal{G} \) the families \( H \times \mathcal{A} \) and \( \mathcal{A} \times H \) also satisfy the conclusions of Theorem 1.14.

Another fact from [Ka20] (also based on Theorem 1.8) which we shall need later is the following.

Theorem 1.16. ([Ka20, Theorem 6.1]) Let \( X \) be a smooth quasi-affine algebraic variety, \( \mathcal{N} \) be a saturated set of locally nilpotent vector fields on \( X \), and \( G \subset \text{SAut}(X) \) be the group generated by \( \mathcal{N} \) such that \( X \) is \( G \)-flexible. Let \( \varrho : X \to Q \) be a partial quotient morphism associated with a nontrivial \( \delta \in \mathcal{N} \), \( Z \) be a locally closed reduced subvariety of \( X \) of codimension at least 2 and \( K \) be a finite subset of \( Z \) such that \( \dim T_{z_0}Z \leq \dim Q \) for every \( z_0 \in K \). Then there exists a connected algebraic \( G \)-family \( \mathcal{A} \) of automorphisms of \( X \) such that for a general element \( \alpha \in \mathcal{A} \) and the closure \( Z'_\alpha \) of \( Z'_\alpha = \varrho \circ \alpha(Z) \) in \( Q \) one can find a neighborhood \( V'_\alpha \) of \( \varrho(\alpha(K)) \) in \( Z'_\alpha \) such that for \( V_\alpha = \varrho^{-1}(V'_\alpha) \cap \alpha(Z) \) the morphism \( \varrho|_{V_\alpha} : V_\alpha \to V'_\alpha \) is an isomorphism.

Remark 1.17. (1) The proof of Theorem 1.16 (see [Ka20, page 550]) implies that \( \mathcal{A} \) is of the form \( \mathcal{A} = U_\kappa \times \mathcal{A}_1 \) where \( U_\kappa \) is a fixed family, while \( \mathcal{A}_1 = H_1 \times \ldots \times H_m \) is any regular \( G \)-family for \( X \) that contains an element \( \alpha_0 \in G \) with the following property:

\(^1\)This family \( U_\kappa \) is a product of several one-parameter unipotent subgroup depending on \( \delta \) and \( K \) but not on \( Z \) [Ka20, Formula (20)].
for every \( z \in \alpha_0(K) \) and \( q = g(z) \) the induced map \( g_* : T_z \alpha(Z) \to T_q Q \) is injective. Note that by Remark 1.12 such \( \alpha_0 \) exists automatically if one requires that \( A_i \) is a regular \( G \)-family for \( \text{PFr}(X) \). Thus, from now on whenever we apply Theorem 1.16 we suppose that \( A_i \) is a perfect family.

(2) Another important feature of Theorem 1.16 we want to emphasize is that by construction the varieties \( V_\alpha \) depend regularly on \( \alpha \) and \( V'_\alpha \) is contained in \( Q_0 \subset Q \) as in Remark 1.4.

In conclusion of this section we state the following simple fact which will be crucial in further considerations.

**Lemma 1.18.** ([KaKu08, Corollary 2.8]) Let \( \delta \) be a locally nilpotent vector field on a variety \( X \), and let \( p \in X \) be a point. Assume that \( f \in \text{Ker} \delta \subset k[X] \) and \( f(p) = 0 \). If \( \Phi = \exp(f \delta) \) is the automorphism associated with the replica \( f \delta \) then

\[
(3) \quad d_p \Phi(w) = w + df(w)\delta(p) \quad \text{for all} \quad w \in T_p X.
\]

2. Properness

**Definition 2.1.** Let \( \delta_1 \) and \( \delta_2 \) be locally nilpotent vector fields on an affine algebraic variety \( X \) and \( \varphi_i : X \to Q_i \) be partial quotient morphisms associated with \( \delta_i \), \( i = 1, 2 \).

(1) The pair \( \delta_1, \delta_2 \) will be called suitable if \( \varphi_1 \) and \( \varphi_2 \) can be chosen so that the morphism \( \tau = (\varphi_1, \varphi_2) : X \to Q_1 \times Q_2 \) is proper.

(2) The pair \( \delta_1, \delta_2 \) will be called semi-compatible if \( \text{Ker} \delta_1 \) and \( \text{Ker} \delta_2 \subset k[X] \) are finitely generated (i.e., \( \varphi_1 \) and \( \varphi_2 \) are algebraic quotient morphisms), \( \tau : X \to \tau(X) \) is birational and finite where \( \tau(X) \) is closed in \( Q_1 \times Q_2 \). In particular, every semi-compatible pair of locally nilpotent vector fields is automatically suitable.

It is useful to keep in mind the following criterion of semi-compatibility.

**Proposition 2.2.** ([KaKu08, Proposition 3.4]) Let \( \delta_1 \) and \( \delta_2 \) be locally nilpotent vector fields on an affine algebraic variety \( X \) such that their kernels \( \text{Ker} \delta_1 \) and \( \text{Ker} \delta_2 \subset k[X] \) are finitely generated. Then these vector fields are semi-compatible if and only if the span of \( \text{Ker} \delta_1 \cdot \text{Ker} \delta_2 \) contains a nontrivial ideal of \( k[X] \).

Though the next fact will not be used later in the paper we include it as a natural property of suitability.

**Proposition 2.3.** (cf. [KaKu08, Proposition 3.6]) Let \( \delta_1 \) and \( \delta_2 \) be suitable locally nilpotent vector fields on an affine algebraic variety \( X \) with finitely generated kernels \( \text{Ker} \delta_1 \) and \( \text{Ker} \delta_2 \). Suppose that \( \Gamma \) is a finite group acting on \( X \) so that each \( \delta_i \) is \( \Gamma \)-invariant, i.e., \( \delta_i \) induces a locally nilpotent vector field \( \delta_i' \) on the quotient \( X' = X/\Gamma \). Then \( \delta_1' \) and \( \delta_2' \) are suitable on \( X' \).

**Proof.** Let \( \varphi_i : X \to Q_i \) and \( \tau \) be as in Definition 2.1 and let \( K_1 \subset \text{SAut}(X) \) be the flow of \( \delta_i \). Since by the assumption the \( K_i \)-action on \( X \) commutes with the \( \Gamma \)-action we have a \( \Gamma \)-action on \( Q_i \) such that \( \varphi_i \) is \( \Gamma \)-equivariant. We have also the induced morphisms

\(^2\)The assumption that the kernels are finitely generated was, unfortunately, missed in [KaKu08].
$g_i' : X' \to Q'_i := Q_i/\Gamma$. Every morphism $\varphi' : X' \to Z$ constant on the orbits of $K_i$ generates a $\Gamma$-equivariant morphism $\varphi : X \to Z$ (where $Z$ is equipped with a trivial $\Gamma$-action). Since $g_i$ is an algebraic quotient morphism, $\varphi$ is a composition of $g_i$ and a morphism $\psi : Q_i \to Z$. Since $g_i$ is $\Gamma$-equivariant, so is $\psi$. Hence, $\psi$ factors through $Q'_i$ which shows that $g_i' : X' \to Q'_i$ is the algebraic quotient morphism. Consider the following commutative diagram

$$
\begin{array}{ccc}
X & \xrightarrow{\tau} & Q_1 \times Q_2 \\
\downarrow \kappa & & \downarrow \kappa' \\
X' & \xrightarrow{\tau'} & Q'_1 \times Q'_2
\end{array}
$$

where $\tau' = (g'_1, g'_2)$, $\kappa$ is the quotient morphism of the $\Gamma$-action on $X$ and $\kappa'$ is the quotient morphism of the natural $\Gamma \times \Gamma$-action on $Q_1 \times Q_2$. Since $\tau$ and $\kappa'$ are proper morphisms, so is $\kappa' \circ \tau = \tau' \circ \kappa$. Since $\kappa$ is surjective this implies that $\tau'$ is proper which is the desired conclusion. \hfill $\square$

The next simple observation is crucial.

**Theorem 2.4.** Let $\{\delta_\beta | \beta \in B\}$ (where $B$ is an index set) be an infinite collection of locally nilpotent vector fields on affine algebraic variety $X$ such that for every $\beta \neq \gamma \in B$ the pair $(\delta_\beta, \delta_\gamma)$ is suitable for appropriate partial quotient morphisms $\varrho_\beta : X \to Q_\beta$ and $\varrho_\gamma : X \to Q_\gamma$. Suppose that $C$ is a closed (but not necessarily irreducible) curve in $X$. Then there is a finite subset $T$ of $B$ such that for every $\beta \in B \setminus T$ the morphism $\varrho_\beta|_C : C \to Q$ is proper.

**Proof.** Suppose that $\bar{C}$ is a completion of $C$ such that it is smooth at every point of $\bar{C} \setminus C$. By [Ha, Chapter II, Theorem 4.7] the fact that a morphism $\theta : C \to Y$ is not proper is equivalent to the fact that $\theta$ extends regularly to some point $c \in \bar{C} \setminus C$. Assume that $\varrho_\beta|_C : C \to Q$ extends to $c \in \bar{C} \setminus C$. Then for every $\beta \neq \gamma \in B$ the morphism $\varrho_\gamma|_C : C \to Q$ cannot be extended to $c$. Indeed, otherwise the morphism $\tau|_C : C \to Q \times Q$ extends to $c$ where $\tau = (\varrho_\beta, \varrho_\gamma) : X \to Q \times Q$. That is, $\tau$ is not proper by [Ha, Chapter II, Corollary 4.8] which contradicts suitability. Thus, there is at most a finite number of elements $\beta$ of $B$ such $\varrho_\beta|_C$ extends to some $c \in \bar{C} \setminus C$. This yields the desired conclusion. \hfill $\square$

**Definition 2.5.** Let $X$ be an affine algebraic variety, $B$ be an algebraic family of automorphism of $X$ with $\dim B \geq 1$ and $\delta_\beta = \beta_*(\delta)$ for some $\delta \in \text{LND}(X)$. Suppose that for general elements $(\beta, \gamma) \in B \times B$ the pair $(\delta_\beta, \delta_\gamma)$ is suitable. Then we say that $X$ is parametrically suitable. If we want to emphasize that $B$ is contained in some group $G \subset \text{Aut}(X)$ we say that $X$ is parametrically $G$-suitable.

**Remark 2.6.** (1) Let the notations of Definition 2.5 hold and $\varrho : X \to Q$ be a partial quotient morphism associated with $\delta$. For every automorphism $\alpha$ of $X$ denote by $\varrho_\alpha : X \to Q$ an associated partial quotient morphism of $\alpha_* (\delta)$. By definition $\delta_\alpha(f) = \delta(f \circ \alpha)$ for every $f \in k[X]$. Hence, if $g = f \circ \alpha \in \text{Ker } \delta$ then $f = g \circ \alpha^{-1} \in \text{Ker } \delta$. This implies that with an appropriate choice of partial quotient morphisms we can always suppose that $\varrho_\alpha = g \circ \alpha^{-1}$. We see, therefore, that if a pair $(\delta_\beta, \delta_\gamma)$ is suitable then
the pair \((\delta_{\alpha\beta}, \delta_{\alpha\gamma})\) is also suitable, since \((q_{\alpha\beta}, q_{\alpha\gamma}) = (q_{\beta}, q_{\gamma}) \circ \alpha^{-1}\). In particular, \(B\) in Definition 2.5 can be replaced by \(\alpha B\) and, therefore, we can always assume that \(B\) contains the identity automorphism.

(2) One can modify the notion of suitability and consider it not for pairs but, say, for triples of locally nilpotent vector fields on \(X\). That is, for a connected nonconstant algebraic family \(B\) of automorphism of \(X\) and general triple \((\beta, \gamma, \chi)\) in \(B \times B \times B\) one can require that the morphism \((q_{\beta}, q_{\gamma}, q_{\chi}) : X \to Q \times Q \times Q\) is proper. Then the straightforward adjustment of the proof of Theorem 2.4 shows that \(q_{\beta}|_{C} : C \to Q\) is proper for a general \(\beta \in B\). In this case we also call \(X\) parametrically suitable.

(3) Though we use partial quotients morphisms in the definition of (parametric) suitability in all applications below \(\delta\) (and, consequently, \(\delta_{\beta}\)) admits an algebraic quotient morphism.

The following is one of the main criteria for parametric suitability.

**Proposition 2.7.** Let \(X\) be a smooth complex affine algebraic variety, \(F \simeq \text{SL}_2(\mathbb{C})\) be a subgroup of a group \(G \subset \text{Aut}(X)\) such that the natural \(F\)-action on \(X\) is fixed point free and non-degenerate\(^3\). Then \(X\) is parametrically \(G\)-suitable.

**Proof.** Fixing some basis \(\bar{u} = (u_1, u_2)\) in \(\mathbb{C}^2\) with the natural action of \(\text{SL}_2(\mathbb{C})\) we get a unipotent subgroup \(H_1\) (resp. \(H_2\)) of upper (resp. lower) triangular matrices in \(\text{SL}_2(\mathbb{C}) \simeq F\). The \(\mathbb{G}_a\)-action induced by \(H_1\) (resp. \(H_2\)) on \(X\) is the flow of a locally nilpotent vector field \(\delta_1\) (resp. \(\delta_2\)). By [Had] there exists an algebraic quotient morphism \(q : X \to Q\) associated with \(\delta_1\). The central fact we use now is that \(\delta_1\) and \(\delta_2\) are semi-compatible and even compatible in the terminology of [DDK, Theorem 12]. However, \(H_1\) and \(H_2\) (resp. \(\delta_1\) and \(\delta_2\)) depends on the choice of \(\bar{u}\). In particular, for every \(\alpha \in \text{SL}_2(\mathbb{C})\) the groups \(\alpha H_1 \alpha^{-1}\) and \(\alpha H_2 \alpha^{-1}\) present upper and lower triangular matrices in an appropriate basis (note that this conjugation sends \(\delta_1\) to \(\alpha_s(\delta_1)\)). Consider \(B = \text{SL}_2\) and let us show that for general \(\alpha \neq \beta \in B\) the locally nilpotent vector fields \(\alpha_s(\delta_1)\) and \(\beta_s(\delta_1)\) are semi-compatible, or, equivalently, in an appropriate basis \(\alpha H_1 \alpha^{-1}\) and \(\beta H_1 \beta^{-1}\) corresponds to upper and lower unipotent matrices respectively. The latter is equivalent to the similar claim about the groups \(\beta^{-1} \alpha H_1 \alpha^{-1}\beta\) and \(H_1\).

We can suppose that \(\alpha : (u_1, u_2) \mapsto (u_1, u_2 + au_1)\) and \(\beta : (u_1, u_2) \mapsto (u_1, u_2 + bu_1)\). Then the direct computation shows that \(\beta^{-1} \alpha H_1 \alpha^{-1}\beta\) consists of matrices of the form

\[
\begin{bmatrix}
1 + (a - b)t & t \\
-(a - b)^2t & 1 - (a - b)t
\end{bmatrix}.
\]

Another direct computation shows that for the automorphism \(\gamma : (u_1, u_2) \mapsto (u_1 - (a - b)^{-1}u_2, u_2)\) one has \(\gamma H_2 \gamma^{-1} = \beta^{-1} \alpha H_1 \alpha^{-1}\beta\). Hence, \(\beta^{-1} \alpha H_1 \alpha^{-1}\beta\) and \(H_1\) corresponds to upper and lower unipotent matrices in an appropriate basis since \(\gamma H_2 \gamma^{-1}\) and \(H_1\) = \(\gamma H_1 \gamma^{-1}\) do. Thus, \(\alpha_s(\delta_1)\) and \(\beta_s(\delta_1)\) are suitable for general \(\alpha \neq \beta \in B\) which yields the desired conclusion. \(\Box\)

\(^3\)An action of a linear algebraic group is non-degenerate if general orbits have the same dimension as the group has.
Example 2.8. Let $H$ be a complex semi-simple Lie group with Lie algebra different from $\mathfrak{sl}_2$. Let $R$ be a proper closed reductive subgroup of $H$. Then the homogeneous space $H/R$ is parametrically suitable. Indeed, by [DDK, Theorem 24 and the proof of Corollary 25] $H/R$ admits a non-degenerate fixed point free $\text{SL}_2(\mathbb{C})$-action. Hence, we are done by Theorem 2.9.

**Theorem 2.9.** Let $X$ be a smooth affine algebraic variety equipped with a fixed point non-degenerate $\text{SL}_2(k)$-action. Then $X$ is parametrically suitable.

**Proof.** If $k$ is a universal domain (i.e., it is an algebraically closed field of characteristic 0 with an infinite transcendence degree over $\mathbb{Q}$) then one can just refer to [Ek]. To allow a finite transcendence degree one can argue like this. By the “finite extension principle” there exist an algebraically closed subfield $k_0$ of $k$ of finite transcendence degree over $\mathbb{Q}$, an algebraic variety $X_0$ over $k_0$ and an $\text{SL}_2(k_0)$-action on $X_0$ such that the $\text{SL}_2(k)$-action on $X$ is obtained by the base extension $\otimes_{\text{Spec} k_0} \text{Spec} k$. Consider subgroups $H_1^0$ and $B_0^0 = H_2^0$ of upper and lower unipotent triangular matrices in $\text{SL}_n(k_0)$ and a locally nilpotent vector field $\delta_1^0$ associated with $H_1^0$. Then for $\beta^0 \in B_0^0$ and $\delta_{\beta^0} = \beta_{\beta^0}(\delta_1^0)$ one has an associated partial quotient morphism $\delta_{\beta^0} : X^0 \to Q^0$.

By the Lefschetz principle we can view $k_0$ as a subfield of $\mathbb{C}$. The base extension $\otimes_{\text{Spec} k_0} \text{Spec} \mathbb{C}$ leads to a non-degenerate fixed point free $\text{SL}_2(\mathbb{C})$-action on $X^\mathbb{C}$, a subgroup $B^\mathbb{C}$ of $\text{SL}_2(\mathbb{C})$ and morphisms $\delta_{\beta^0}^\mathbb{C} : X^\mathbb{C} \to Q^\mathbb{C}$. As we showed in the proof of Theorem 2.9 for $\beta^\mathbb{C} \neq \gamma^\mathbb{C} \in B$ the morphism $\tau^\mathbb{C} = (\delta_{\beta^0}^\mathbb{C}, \delta_{\gamma^0}^\mathbb{C}) : X^\mathbb{C} \to Q^\mathbb{C} \times Q^\mathbb{C}$ is proper. Hence, every morphism $\tau^0 = (\delta_{\beta^0}^0, \delta_{\gamma^0}^0) : X^0 \to Q^0 \times Q^0$ is proper for general $(\beta^0, \gamma^0) \in B^0 \times B^0$. Let $\omega^0$ be the generic point of $B^0 \times B^0$ (i.e., $\omega^0 = \text{Spec} K^0$ where $K^0$ is the field of rational functions on $B^0 \times B^0$). Consider the morphism $\Theta^0 : B^0 \times B^0 \times X^0 \to B^0 \times B^0 \times Q^0 \times Q^0, (\beta^0, \gamma^0, x^0) \mapsto (\beta^0, \gamma^0, \delta_{\beta^0}^0(x^0), \delta_{\gamma^0}^0(x^0))$ over $B^0 \times B^0$. Then the statement that $\tau^0$ is proper for general $(\beta^0, \gamma^0) \in B^0 \times B^0$ is equivalent to the statement that the restriction of $\Theta^0$ to a morphism over $\omega^0$ is proper.

Applying the base extension $\otimes_{\text{Spec} k_0} \text{Spec} k$ to the above construction one gets unipotent one-parameter subgroups $H_1$ and $B = H_2$ in $\text{SL}_2(k)$, $\delta_1 \in \text{LND}(X)$ associated with $H_1$, partial quotient morphisms $\delta_1^\beta : X \to Q$ associated with $\beta(\delta_1)$, $\beta \in B$ and $\Theta : B \times B \times X \to B \times B \times Q \times Q, (\beta, \gamma, x) \mapsto (\beta, \gamma, \delta_1^\beta(x), \delta_1^\gamma(x)).$ Since base extension preserves properness by [Ha, Chapter II, Corollary 4.8] the restriction of the latter morphism to a morphism over $\omega$ (where $\omega$ is the generic point of $B \times B$) is proper and we have the desired conclusion. \qed

3. NORMAL BUNDLES IN FLEXIBLE VARIETIES

**Setting 3.1.** In this section (and whenever we speak about flexible varieties later) $X$ is a normal $G$-flexible quasi-affine variety with $\dim X = n \geq 4$ where the group $G \subset \text{SAut}(X)$ is generated by a saturated set $\mathcal{N} \subset \text{LND}(X)$. We suppose that $\delta \in \mathcal{N}$ is nonzero and $\varrho : X \to Q$ is a partial quotient morphism associated with $\delta$. For any $\alpha \in \text{Aut}(X)$ we let $\delta_\alpha = \alpha_* (\delta)$ and $\varrho_\alpha = \varrho \circ \alpha^{-1} : X \to Q$ (i.e., $\varrho_\alpha$ is a partial quotient morphism associated with $\delta_\alpha$). We consider also a smooth closed (but not
necessarily irreducible) curve $C$ in $X$ such that $C \subset X_{\text{reg}}$ and the natural projection $\text{pr} : TX|_C \to N_X C$ to the normal bundle $N_X C$ of $C$ in $X$.

**Proposition 3.2.** Let Setting 3.1 hold and $K$ be a finite subset of $C$. Then there exists a perfect $G$-family $A_1$ such that for general $\alpha_1, \ldots, \alpha_{n-1} \in A := A_1 \times U_n$ (where $U_n$ is as in Remark 1.17) there is an open neighborhood $C^*$ of $K$ in $C$ for which the following is true

(i) For every $i = 1, \ldots, n - 1$ the restriction of $\varrho_{\alpha_i}$ yields an isomorphism between $C^*$ and $C_i^* := \varrho_{\alpha_i}(C^*)$.

(ii) the normal bundle $N_X C^*$ is trivial and $\text{pr}(\delta_{\alpha_1}), \ldots, \text{pr}(\delta_{\alpha_{n-1}})$ generate $N_X C^*$ as a module over $k[C^*].$

**Proof.** Statement (i) follows from Theorem 1.16 applied to $X_{\text{reg}}$ and the fact that $\varrho_{\alpha} = g \circ \alpha^{-1}$. By Remark 1.12 we can always suppose that for every $x \in K$ and each $i = 1, \ldots, n - 1$ the vector $\delta_{\alpha_i}(x)$ is general in $T_x X$. In particular, we can suppose that the vectors $\delta_{\alpha_1}(x), \ldots, \delta_{\alpha_{n-1}}(x)$ are linearly independent modulo $T_x C$. Taking, if necessary, a smaller neighborhood $C^*$ of $K$ such that for every $x \in C^*$ the similar fact holds we get (ii). Hence, we are done. \hfill \square

It turns out that under the following mild assumption we can get a much stronger version of Proposition 3.2.

**Convention 3.3.** We suppose that further in this section that the following conditions hold.

(1) For $Q_0 \subset Q$ is as in Remark 1.4 the variety $g^{-1}(Q \setminus Q_0)$ is of codimension at least 2 in $X$ (in particular, the zero locus of $\delta$ is at least of codimension 2).

(2) $X$ is affine and parametrically $G$-suitable.

**Remark 3.4.** The condition (1) cannot be achieved for some flexible smooth Gizatullin surfaces but in higher dimensions the authors do not know examples of smooth affine flexible varieties when (1) does not hold for an appropriate $\delta$. For instance, condition (1) is true when $X$ is a linear algebraic group without nontrivial characters. Another example is a complex sphere given by $z_1^2 + \ldots + z_m^2 = 1$ where $z = (z_1, \ldots, z_m)$ is a coordinate system on $\mathbb{A}^m$. Then $\sigma = z_2 \frac{\partial}{\partial z_1} - z_1 \frac{\partial}{\partial z_2}$ is a locally nilpotent vector field on this sphere whose zero locus has codimension 2. The kernel of $\sigma$ is generated by $z_3, \ldots, z_m$ and, therefore, the quotient space is smooth and (1) holds.

**Proposition 3.5.** Let Setting 3.1 and Convention 3.3 hold. Then one can choose a perfect $G$-family $A_1$ so that for general $\alpha \in A := A_1 \times B$ (where $B$ is as in Definition 2.5)

(i) the morphism $\varrho_{\alpha}|_C : C \to Q$ is proper;

(ii) its restriction yields an isomorphism between $C$ and $\varrho_{\alpha}(C)$ and $\varrho_{\alpha}(C) \subset Q_0$ (in particular, $\delta_{\alpha}$ does not vanish on $C$).

**Proof.** Theorem 1.8 (applied to $X_{\text{reg}}$) implies that for a general $\alpha$ in any perfect family contained in $\text{SAut}(X_{\text{reg}}) = \text{SAut}(X)$ the curve $\alpha^{-1}(C)$ does not meet $g^{-1}(Q \setminus Q_0)$ since the latter is of codimension at least 2. Since $\varrho_{\alpha} = g \circ \alpha^{-1}$ this yields (iii). Let
$\mathcal{A}_1$ now be a prefect $G$-family as in Theorem 1.14(ii) where $G \subset \text{SAut}(X_{\text{reg}})$, i.e., for general $\alpha \in \mathcal{A}_1$ the morphism $g_\alpha|_C : C \to Q$ is an injective immersion. Note that if $C$ is once-punctured then $g_\alpha|_C : C \to Q$ is automatically proper and, in particular, $g_\alpha(C)$ is closed in $Q$. Hence, in this case $g_\alpha|_C$ yields an isomorphism between $C$ and $g_\alpha(C)$. If $C$ is not once-punctured consider a family $\mathcal{B} \subset G$ of automorphisms of $X$ as in Definition 2.5. Recall that by Remark 1.15 the replacement of $\mathcal{A}_1$ by $\mathcal{A} = \mathcal{A}_1 \times \mathcal{B}$ leaves the conclusions of Theorem 1.14 valid. On the other hand, for general element $(\alpha, \beta) \in \mathcal{A}_1 \times \mathcal{B}$ the map $g_{\alpha\beta}|_C : C \to Q$ factors through the isomorphism $C \to \alpha^{-1}(C)$ and the map $g_\beta : \alpha^{-1}(C) \to Q$. Perturbing $\beta$ we makes the latter map proper which yields the properness of $g_{\alpha\beta}|_C : C \to Q$ and the desired conclusion. \hfill $\Box$

**Lemma 3.6.** Let Setting 3.1 and Convention 3.3 hold and $\mathcal{A}$ be a prefect $G$-family as in Proposition 3.5. Let $Z_\alpha$ be the zero locus of $\delta_\alpha$ and $L_\alpha$ be the line bundle induced by $\delta_\alpha$ on $X \setminus Z_\alpha$. Suppose that $K$ is a finite subset of $C$ and $X_k$ is an open subset of $X$ containing $C$. Let $B_k \subset TX_k$ be a vector bundle of rank $k \leq n - 2$. For general $\alpha$ in $\mathcal{A}$ one has the following.

(i) There exists a closed subvariety $Y_k \subset X$ of dimension at most $k$ such that $B_k$ and $L_\alpha$ generate a vector bundle $B_{k+1}$ of rank $k + 1$ on $X_{k+1} = X_k \setminus (Y_k \cup Z_\alpha)$ and $C \subset X_{k+1}$.

(ii) If $k \leq n - 3$ and $\text{pr}(B_k)$ yields a vector subbundle of $N_X C$ of rank $k$ then $\text{pr}(B_{k+1})$ yields a vector subbundle of $N_X C$ of rank $k + 1$.

(iii) If $\text{pr}(B_{n-2})$ is a vector subbundle of $N_X C$ of rank $n - 2$ then there exists a finite set $H_\alpha \subset C \setminus K$ such that $\text{pr}(B_{n-1})$ yields $N_X(C \setminus H_\alpha)$. Furthermore, $\bigcap_{\alpha \in \mathcal{A}} H_\alpha = \emptyset$.

**Proof.** Recall that by $G$-flexibility and [AFKKZ, Theorem 4.14] $G$ acts transitively on $T'X$ where $T'X$ is the complement to the zero section in $TX$. For every subset $S$ of $TX$ let $S' = S \cap T'X$. Note that $\dim B'_k = n + k$, $\dim B'_k|_C = k + 1$ and $\dim L'_\alpha = n + 1$. By Theorem 1.8 for general $\alpha \in \mathcal{A}$ the variety $L'_\alpha$ meets $B'_k|_C$ along a subvariety $R$ of dimension $(k + 1) + (n + 1) - 2n = k + 2 - n \leq 0$. However, this dimension cannot be zero since for each $v \in R \subset TX$ every vector proportional to $v$ is also in $R$, i.e., $R$ is empty. Again by Theorem 1.8 $L'_\alpha$ meets $B'_k$ transversely along a subvariety $\tilde{Y}$ of dimension at most $(n + k) + (n + 1) - 2n = k + 1$. Hence, if $p : TX \to X$ is the natural projection then $Y_{\tilde{\alpha}} = p(\tilde{Y})$ does not meet $C$ and has dimension $\dim Y_{\tilde{\alpha}} = \dim \tilde{Y} - 1 \leq k$, while $B_k$ and $L_\alpha$ generate a rank $k$ vector subbundle $B_{k+1}$ of the tangent bundle of $X_{k+1}$. Note that $C \subset X_{k+1}$ since $C \cap Z_\alpha = \emptyset$ by Proposition 3.5 (iii). Thus, we have (i).

The claim in (ii) that $\text{pr}(B_k)$ is a vector subbundle of $N_X C$ of rank $k$ is equivalent to the fact that the constructible set $\bigcup_{x \in C}(B_k(x) + T_x C)$ is a bundle $\tilde{B}_k$ of rank $k + 1$ over $C$. We note that since $\dim B'_k + \dim L'_\alpha - \dim TX = (k + 2) + (n + 1) - 2n = k + 3 - n \leq 0$, by Theorem 1.8 for general $\alpha \in \mathcal{A}$ the variety $L'_\alpha$ meets $B'_k$ transversely along a variety $\tilde{H}$ of dimension at most zero. However, arguing as before we see that $\tilde{H} = \emptyset$ which implies (ii).

In (iii) the same argument shows that $\tilde{H}$ may have dimension 1. Therefore, $H_\alpha := p(\tilde{H})$ is a finite set and $\text{pr}(B_{n-1})$ yields $N_X(C \setminus H_\alpha)$. The transversality argument implies also that for general $\alpha \in \mathcal{A}$ the variety $L'_\alpha$ does not meet the fiber of $\tilde{B}_{n-2}$ over
a fixed point \( c \in C \). This implies that \( \bigcap_{\alpha \in \mathcal{A}} H_\alpha = \emptyset \) and, consequently, \( K \cap H_\alpha = \emptyset \) for general \( \alpha \) which concludes the proof. \( \Box \)

**Proposition 3.7.** Let Setting 3.1 and Convention 3.3 hold and \( K \) be a finite subset of \( C \). Then there exists a perfect \( G \)-family \( \mathcal{A} \) (as in Proposition 3.5) such that for \( \bar{\alpha} = (\alpha_1, \ldots, \alpha_{n-1}) \) in a dense open subset \( U \) of \( A^{n-1} \) the following is true.

1. For every \( i = 1, \ldots, n - 1 \) the map \( g_{\alpha_i} : C \to Q \) is proper and generates an isomorphism between \( C \) and \( g_{\alpha_i}(C) \subset Q_0 \).
2. The span of \( \delta_{\alpha_1}, \ldots, \delta_{\alpha_{n-2}} \) over \( k[C] \) is a subbundle \( B_{n-2} \subset TX|_C \) such that \( pr(B_{n-2}) \) is a subbundle of \( N_X C \) whose rank is \( n - 2 \).
3. Let \( H_{\alpha_{n-1}} \) be as in Lemma 3.6 and \( C^* = C \setminus H_{\alpha_n} \). Then \( K \subset C^* \) and the span of \( pr(B_{n-2}) \) and \( pr(\delta_{\alpha_{n-1}}) \) over \( k[C^*] \) is the normal bundle \( N_X C^* \).
4. Furthermore, the vector \( \{\delta_{\alpha_{n-1}}(x)| x \in K\} \) is a general element of \( \prod_{x \in K} T_x X \).
5. \( N_X C \) is naturally isomorphic to \( pr(B_{n-2}) \oplus L \) where \( L \) is a line subbundle of \( TX|_C \).

**Proof.** The existence of a perfect \( G \)-family \( \mathcal{A} \) satisfying (1) is provided by Proposition 3.5. Let \( B_0 \) be the zero subbundle of \( TX \). Then applying Lemma 3.6 to \( B_0 \) and consequent vector bundles \( B_k \) we get (2) and (3).

By [AFKKZ, Theorem 4.14 and Remark 4.16] the stabilizer subgroup \( G_0 \subset G \) of \( K \) acts transitively on \( \prod_{x \in K} T'_x X \) where \( T'_x X = T_x X \setminus \{0\} \). Thus, given a perfect \( G \)-family \( \mathcal{A} \) we can always enlarge it so that it contains a subfamily \( \mathcal{A}_0 \) acting trivially on \( K \) whose induced action on \( \prod_{x \in K} T'_x X \) is transitive. In particular, we can find an element \( \alpha \in \mathcal{A}_0 \) with prescribed values of \( \delta_\alpha \) at the points of \( K \). Recall that in Proposition 3.5 we replace \( \mathcal{A} \) by \( \mathcal{A} \times \mathcal{B} \) where \( \mathcal{B} \) as in Definition 2.5. By Remark 2.6 we can suppose that \( \mathcal{B} \) contains the identity automorphism \( e \). Hence, for a general element \( (\alpha, e) \in \mathcal{A} \times \mathcal{B} \) near \( (\alpha, e) \) the values of \( \delta_{\alpha \circ \beta} \) at the points of \( K \) yield a general element of \( \prod_{x \in K} T'_x X \). Thus, we have (4).

Now let \( v_i = pr(\delta_{\alpha_i}) \). Then each \( v_i \) is a non-vanishing section of \( N_X C \) and by (3) \( v_{n-1} \) meets \( pr(B_{n-2}) \) over \( H_{\alpha_{n-1}} = \{x_1, \ldots, x_m\} \subset C \). Consider a neighborhood \( U_i \) of \( x_i \) in \( C \) such that the normal bundle \( N_X U_i \) is trivial, i.e., \( v_1, \ldots, v_{n-2} \) and some section \( u_i \) is a basis of \( N_X U_i \). Then locally \( v_{n-1} = g_i u_i + f_{i,1} v_1 + \ldots + f_{i, n-2} v_{n-2} \) where \( g_i, f_{i,1}, \ldots, f_{i, n-2} \) are regular functions on \( U_i \). Consider regular functions \( f_1, \ldots, f_{n-2} \) on \( C \) such that \( f_j f_{j,i} \) is divisible by \( g_i \) for every \( i \) and \( j \). Consider also the projectivization \( D \) of \( N_X C \) which is a \( \mathbb{P}^{n-2} \)-bundle over \( C \). It contains a \( \mathbb{P}^{n-3} \)-bundle \( D_0 \) corresponding to \( pr(B_{n-2}) \). Note that \( v_{n-1} - (f_1 v_1 + \ldots + f_{n-2} v_{n-2}) \) defines a section of \( D \) over \( C \setminus H_{\alpha_{n-1}} \) which extends to a section over \( C \) that does not meet \( D_0 \). This section yields the desired line bundle \( L \) which concludes the proof. \( \Box \)

**Remark 3.8.** (1) Recall that by [Sel] every vector bundle \( W \) of rank \( k \) over \( C \) can be presented as \( W_0 \oplus W_1 \) where \( W_0 \) is a trivial bundle of rank \( k - 1 \) and \( W_1 \) is a line bundle. Proposition 3.7 (5) tells additionally that for \( W = N_X C \) the bundle \( W_0 \) has a basis of sections generated by locally nilpotent vector fields.
(2) Note that by construction $\mathcal{A}$ in Proposition 3.7 is independent from the choice of $\delta$ in Setting 3.1 and it is only $U$ which changes with a change of $\delta$. Similarly, in Proposition 3.5 $\mathcal{A}$ is also independent of $\delta$.

**Corollary 3.9.** Let the notations and conclusions of Proposition 3.7 hold and $v_i = \text{pr}(\delta_a)$, $i = 1, \ldots, n - 1$. Suppose that $N_C X$ and, therefore, $L$ are trivial bundles and $v_0$ is a nonvanishing section of $L$. Then there is $q \in A$ such that $v_{n - 1} = qv_0 + \sum_{i=1}^{n-2} a_i v_i$ where $a_1, \ldots, a_{n-2}$ are regular functions on $C$ with $q$ and all $a_i$ having simple zeros on $C$.

**Proof.** Consider the natural projection $N_X C \to L$. Let $\tilde{v}_0 = qv_0$ be the image of $v_{n-1}$. Then $v_{n-1} - \tilde{v}_0 \in \text{pr}(B_{n-2})$ which yields an equality $v_{n-1} = \tilde{v}_0 + \sum_{i=1}^{n-2} a_i v_i$. The claim about zeros $q$ (which are zeros of $\tilde{v}_0$) follows from the transversality of the intersection of $L_{a_{n-1}}$ and $B_{n-2}$ which is provided by Theorem 1.8. The same argument works for zeros of $a_i$. Thus, we get the desired conclusion. \hfill $\Box$

4. $\text{SL}_n(A)$ over a commutative ring $A$

In this section we collect some facts about matrices over commutative rings and prove Proposition 4.5 which is an essential step for our main result.

**Notation 4.1.** Let $A$ be a commutative ring, $n \geq 3$ and $\text{SL}_n(A)$ be the special linear group with entries from $A$, whereas $\text{Mat}_n(A)$ be the ring of $(n \times n)$-matrices with entries from $A$. By $I \in \text{SL}_n(A)$ we denote the identity matrix and by $e_{ij}$, $1 \leq i, j \leq n$ the matrix in $\text{Mat}_n(A)$ with all zero entries but the entry in position $(i, j)$ which is 1. By $\mathbb{E}_n(A)$ we denote the subgroup of $\text{SL}_n(B)$ generated all elementary matrices of the form $I + ae_{ij}$, $1 \leq i \neq j \leq n$ where $a \in A$. Let $\mathfrak{q}$ be any ideal in $A$. By $\text{SL}_n(A, \mathfrak{q})$ we denote the kernel of the natural group homomorphism $\text{SL}_n(A) \to \text{SL}_n(A/\mathfrak{q})$, by $\mathbb{F}_n(A, \mathfrak{q})$ we denote the subgroup of $\mathbb{E}_n(A)$ generated by all elementary transformations of the form $I + ae_{ij}$ where $a \in \mathfrak{q}$ and by $\mathbb{E}_n(A, \mathfrak{q})$ the normal closure of $\mathbb{F}_n(A, \mathfrak{q})$ in $\mathbb{E}_n(A)$.

We need the following result of Bass, Milnor and Serre [BMS, Theorem 4.1] (see also [Ba, Theorem 4]).

**Theorem 4.2.** Let Notation 4.1 hold, $A$ be a Dedekind domain and $n \geq 3$. Then

\begin{equation}
\mathbb{E}_n(A, \mathfrak{q}) = [\text{SL}_n(A), \text{SL}_n(A, \mathfrak{q})].
\end{equation}

Another important fact unfortunately omitted in [Ud] is the following “Tits’ theorem” in [Ni].

**Theorem 4.3.** Let Notation 4.1 hold. Then $\mathbb{E}_n(A, \mathfrak{q}^2)$ is contained in $\mathbb{F}_n(A, \mathfrak{q})$.

Let us reproduce two facts from [Ud, Lemma 2.2.1 and Theorem 2.2.2].

**Lemma 4.4.** ([Ud, Theorem 2.2.2]) Let Notations 4.1 hold, $\mathfrak{q} = qA$, $q \in A$ be a proper principal ideal in $A$ and $D \in \text{Mat}_n(A)$ be a matrix which is invertible in $\text{Mat}(A[1/2])$.

(i) For some $m > 0$ one has $D^{-1}\text{SL}_n(A, \mathfrak{q})D \supset \text{SL}_n(A, \mathfrak{q}^m)$.

(ii) Let $A$ be a Dedekind ring. Then the group $D^{-1}\mathbb{E}_n(A, \mathfrak{q})D$ contains $\mathbb{E}_n(A, \mathfrak{q}^l)$ for some $l > 0$. 

Lemma 4.6. Let Notation 4.1 hold, \( A \) be a Dedekind ring and let \( q \) and \( q' \in A \) generate comaximal principal ideal \( q \) and \( q' \subset B \). Let a matrix \( D' \) plays the same role for \( q' \) as \( D \) for \( q \) in Lemma 4.4. Suppose that \( H \) is a subgroup of \( SL_n(A) \) that contains all elements from \( SL_n(A) \cap (D^{-1}F_n(A,q,D)) \) and \( SL_n(A) \cap ((D')^{-1}F_n(A,q',D')) \). Then \( H \) contains \( E_n(A) \).

Proof. By Theorem 4.3 \( H \) contains the subgroups \( SL_n(A) \cap (D^{-1}E_n(A,q^2,D)) \) and \( SL_n(A) \cap ((D')^{-1}E_n(A,q'^2,D')) \). Hence, by Lemma 4.4 \( H \) contains \( E_n(A,q^2l) \) and \( E_n(A,q'^2l) \) for some \( l > 0 \). Now the desired conclusion follows from the Nullstellensatz. □

The rest of this section will be needed only for the holomorphic case in Section 7.

Lemma 4.6. Let Notation 4.1 hold, \( \sigma \in \text{Mat}_n(A) \) be a matrix of the form \( q(I - \sum_{i=2}^n \frac{a_i}{q}e_{ii}) \) where \( q \in A \setminus \{0\} \) and each \( a_i \in A \). Suppose that \( p \) is an ideal in \( A \). Let \( E \) be a subgroup of \( SL_n(A) \) that contains

(i) all elementary matrices of the form \( \tau = I + ce_{ij} \) where \( c \in A, 2 \leq j \neq i \leq n \) and

(ii) all matrices \( \sigma \tau \sigma^{-1} \) where \( \tau = I + bq^2e_{i1}, b \in p \) and \( 2 \leq i \leq n \).

Then for every \( 2 \leq k \neq l \leq n \) the group \( E \) contains the following matrices

\[
I + a_i bq^2e_{k1} + a_k a_l bq^2e_{kk} + a_l^2 bqe_{kl} - a_k bq^2e_{i1} - a_k a_l bqe_{il} - a_l^2 bqe_{lk}.
\]

Proof. Note that \( \sigma^{-1} = q^{-1}(I + \sum_{i=2}^n \frac{a_i}{q}e_{ii}) \in \text{Mat}_n(\text{Frac}(A)) \). Let \( \alpha = I + bq^2e_{k1} - baq^2e_{i1} \). Then \( \sigma \alpha \sigma^{-1} = I + \sigma^{-1}(baq^2e_{k1} - baq^2e_{i1}) \sigma = I + \sigma^{-1}(baq^2e_{k1}) \sigma - \sigma^{-1}(baq^2e_{i1}) \sigma \). Consider

\[
\sigma^{-1}(baq^2e_{k1}) \sigma = bq^2e_{k1} + (\sum_{i=2}^n \frac{a_i}{q}e_{ii}) bq^2e_{k1} - bq^2e_{k1}(\sum_{i=2}^n \frac{a_i}{q}e_{ii}) =
\]

\[
-bq^2e_{k1} - a_k bqe_{11} + a_k bqe_{kk} - a_k^2 bqe_{k1} + \sum_{i=2,i \neq k}^n a_i bqe_{ki} - \sum_{i=1,i \neq k}^n a_k a_i b_{1i} =
\]

\[
bq^2e_{k1} - a_k bqe_{11} + a_k bqe_{kk} - a_k^2 bqe_{k1} + \sum_{i=2,i \neq k}^n a_i bqe_{ki} - \sum_{i=1,i \neq k}^n a_k a_i b_{1i}.
\]

Replacing \( b \) by \( a_i b \) we get \( \sigma^{-1}(baq^2e_{k1}) \sigma \) and then, exchanging the role of \( k \) and \( l \) we get \( \sigma^{-1}(baq^2e_{i1}) \sigma \). Hence, one has

\[
\sigma \sigma^{-1} = I + a_i bq^2e_{k1} - a_k a_l bq^2e_{kk} + a_k a_l bqe_{kl} - a_k a_l^2 bqe_{k1} + a_l^2 bqe_{kl} - a_k a_l^2 bqe_{k1} + \beta
\]

\[
= I + a_i bq^2e_{k1} + a_k a_l bqe_{kk} + a_l^2 bqe_{kl} - a_k bqe_{k1} - a_k a_l bqe_{il} - a_l^2 bqe_{lk} + \beta
\]

\[
= I + a_i bq^2e_{k1} + a_k a_l bqe_{kk} + a_l^2 bqe_{kl} - a_k bqe_{k1} - a_k a_l bqe_{il} - a_l^2 bqe_{lk} + \beta
\]
where $\beta$ is a sum of terms of the form $\sum_{i=2, i\neq k,l} c_i e_{ki} + d_i e_{il} + f_i e_{1i}$ where $c_i, d_i, f_i \in A$. One can rewrite $\sigma \alpha \sigma^{-1}$ now as
\[(I + a t b q^2 e_{ki} + a k a b q e_{kk} + a t^2 b q e_{kl} - a k b q^2 e_{1l} - a k a b q e_{1l} - a b q^2 e_{1l})(I + \beta).\]
Since $I + \beta \in \mathcal{E}$ we have the desired conclusion. \qed

**Example 4.7.** Consider the case of $(k, l) = (2, 3)$ in Lemma 4.6. Then
\[
\sigma \alpha \sigma^{-1} = \begin{bmatrix}
1 & 0 & 0 & 0 \\
-a_3 b q^2 & 1 + a_2 a_3 b q & a_3^2 b q & 0 \\
a_2 b q^2 & -a_3^2 b q & 1 - a_2 a_3 b q & \bar{0} \\
0 & 0 & 0 & I_{n-2}
\end{bmatrix}
\]
where $I_{n-2} \in \text{SL}_{n-3}(A)$ is the identity matrix. Removing the first row and the first column we get the matrix
\[
\gamma = \begin{bmatrix}
1 + a_2 a_3 b q & a_3^2 b q & \bar{0} \\
-a_3^2 b q & 1 - a_2 a_3 b q & \bar{0} \\
0 & 0 & I_{n-2}
\end{bmatrix}
\]
which has determinant 1.

**Proposition 4.8.** Let the assumptions of Lemma 4.6 hold and $\text{SL}_{n-1}(A) = E_{n-1}(A)$. Then $\mathcal{E}$ contains of matrices of the form $I + a t b q^2 e_{k1} - a k b q^2 e_{11}$.

**Proof.** We show only that $I + a_3 b q^2 e_{21} - a_2 b q^2 e_{31} \in \mathcal{E}$ since the general case of $(k, l) \neq (2, 3)$ is similar. Hence, we use the notations of Example 4.7. Consider a block matrix of the form
\[
\bar{\beta} = \begin{bmatrix}
1 & 0 \\
0 & \beta
\end{bmatrix} \in \text{SL}_n(A).
\]
Then
\[
\sigma \alpha \sigma^{-1} \bar{\beta} = \begin{bmatrix}
1 & 0 \\
\bar{c} & \gamma \beta
\end{bmatrix}
\]
where $\bar{c}$ is an $n$-column with the first two entries $a_3 b q^2 e_{21}$ and $-a_2 b q^2 e_{31}$, whereas the rest of the entries are zeros. Thus, taking $\beta = \gamma^{-1} \in \text{SL}_{n-1}(A) = E_{n-1}(A)$ we see that $I + a_3 b q^2 e_{21} - a_2 b q^2 e_{31} \in \mathcal{E}$ which concludes the proof. \qed

5. **Main Theorem. I**

In this section we present a strengthened version of the main result in [Ud]. Let us start with terminology.

**Definition 5.1.** (1) Let $C_1$ and $C_2$ be closed subvarieties of a smooth quasi-affine variety $X$ with defining ideals $I_1$ and $I_2$ in $k[X]$. Let $G$ be a subgroup of $\text{Aut}(X)$, i.e., $G$ acts on $k[X]$. Suppose that for some $\varphi \in G$ one has $\varphi(I_1) = I_2$. Then $\varphi(I_1^k) = I_2^k$ for every $k \in \mathbb{N}$ and, therefore, $\varphi$ generates an isomorphism $\varphi_k : \frac{k[X]}{I_1^k} \rightarrow \frac{k[X]}{I_2^k}$. We say such $\varphi_k$ is a $G$-induced isomorphism.

(2) Let $\theta : \frac{k[X]}{I_1^k} \rightarrow \frac{k[X]}{I_2^k}$ be a ring homomorphism. Note that one has $\theta(\frac{I_2^k}{I_1^k}) = \frac{I_2}{I_1}$ since $\theta$ sends nilpotent elements to nilpotent ones. Hence, $\theta$ generates isomorphisms
Lemma 5.2. Let $X$ be a smooth quasi-affine algebraic variety equipped with a volume form $\omega$ such that every two points of $X$ can be joined by a chain of polynomial curves (which is the case when $X$ is smooth flexible). Let $\alpha : X \to X$ be an automorphism. Then the Jacobian of $\alpha$ is constant. Furthermore, if $\alpha \in SAut(X)$ then its Jacobian is 1.

Proof. The action of $\alpha$ transforms $\omega$ into a volume form $\omega_\alpha$. The Jacobian $f_\alpha = \frac{\omega_\alpha}{\omega}$ of this automorphism is an invertible function on $X$ and $f_e = 1$ where $e$ is the identity automorphism. Since the restriction of $f_\alpha$ to every polynomial curve is constant by the fundamental theorem of algebra, we see that $f_\alpha$ is constant. Thus, for $\alpha \in SAut(X)$ the Jacobian $f_\alpha$ can be viewed as a value of an invertible function $f$ on $SAut(X)$. By the definition any two elements $\alpha_1, \alpha_2 \in SAut(X)$ can be joined by a chain of polynomial curves (corresponding to one-parameter unipotent groups). Hence, $f$ is constant and, consequently, $f = 1$ which is the desired conclusion. \hfill $\square$

To see how the above condition on Jacobians can be reformulated for automorphisms of infinitesimal neighborhoods of closed subvarieties of $X$ we need to remind the following well-known fact.

Lemma 5.3. Let $Y$ be a smooth quasi-affine algebraic variety of dimension $m$ and $Z$ be a smooth closed curve in $Y$ with a defining ideal $J$ in $k[Y]$. Then any section of the conormal bundle of $Z$ in $Y$ can be presented by a regular function on $Y$. Furthermore, suppose that the conormal bundle is trivial. Then there exists an open neighborhood $U$ of $Z$ in $Y$ and regular functions $u_1, \ldots, u_{m-1} \in k[Y]$ such that $Z$ is a strict complete intersection in $U$ given by $u_1|_U = \ldots = u_{m-1}|_U = 0$.

Proof. Let $J$ be the ideal sheaf on $Y$ induced by $J$. Consider a section $s : Z \to J^*$ of the conormal bundle. By definition there is an open affine cover $\{Z_i\}$ of $Z$ such that $s$ is induced by section $s_i : Z_i \to J$ for which the cocycle $\{(s_i - s_j)|_{Z_i \cap Z_j}\}$ has coefficients in $J^2$. Since $Z$ is affine and $J^2$ is coherent we can suppose that $s_i - s_j = 0$ by Serre’s theorem B. Hence, $s$ is induced by a global regular function on $Y$. In the case of a trivial $N_YZ$ we have $m - 1$ such functions $u_1, \ldots, u_{m-1}$ which generate $J$ over $k[Z]$. By construction the set $V$ of their common zeros contain $Z$ and any other irreducible component of $V$ is disjoint from $Z$. Thus, $U = (Y \setminus V) \cup Z$ is the desired neighborhood. \hfill $\square$
Definition 5.4. Let the notations of Definition 5.1 hold, \( X \) be an affine algebraic variety and each \( C_j \) be a smooth closed curve in \( X \) with a trivial normal bundle such that \( C_j \subset X_{\text{reg}} \). By Lemma 5.3 \( C_j \) is a strict complete intersection in some open \( U_j \subset X \) given by \( u_{1j} = \ldots = u_{n-1,j} = 0 \) where \( u_{1j}, \ldots, u_{n-1,j} \) are regular functions on \( X \). One can see now that the ring \( \frac{k[X]}{I_j} \) is naturally isomorphic to \( A_j \oplus \bigoplus_{l=1}^{k-1} \frac{I_l}{I_j} \) where each summand \( \frac{I_l}{I_j} \) can be treated as the space of homogeneous polynomials in \( u_{1j}, \ldots, u_{n-1,j} \) of degree \( l \) over \( A_j \). In particular, for any homomorphism \( \theta: \frac{k[X]}{I_1} \to \frac{k[X]}{I_2} \) of algebras one can consider the Jacobi matrix \[ \left[ \frac{\partial \theta(u_{1j})}{\partial u_{n+2}} \right]_{l,m=1}^{n-1}. \] In the case of \( I_1 = I_2 \) and \( u_{i1} = u_{i2} \) the determinant of this matrix is independent of the choice of coordinates \( u_{1j}, \ldots, u_{n-1,j} \) and we say that \( \theta \) has Jacobian 1 if the determinant is equal to 1 modulo \( I_1^k \).

In this terminology the main strengthened result from [Ud] is the following.

Theorem 5.5. ([Ud, Theorem 2.5.19]) Let \( X \) be a normal \( G \)-flexible quasi-affine variety of dimension \( n \geq 4 \) where \( G \subset \text{SAut}(X) \) is a group generated by a saturated set \( \mathcal{N} \subset \text{LND}(X) \). Let \( \theta \) be an associated partial quotient morphism of \( \delta \) and \( C \) be a smooth closed curve in \( X \) such that \( C \subset X_{\text{reg}} \). Suppose also that

(\( \bullet \)) \( C \) is irreducible and either \( C \) is once-punctured or Convention 3.3 is valid.

Let \( I \) be the defining ideal of \( C \) in the algebra \( k[X] \) of regular functions on \( X \) and \( A = \frac{k[X]}{I} \simeq k[C] \). Suppose that the normal bundle \( N_XC \) of \( C \) in \( X \) is trivial (i.e., it is a free \( A \)-module of dimension \( n - 1 \)). Let \( \lambda: \frac{1}{f_1} \to \frac{1}{f_2} \) be an \( A \)-automorphism with Jacobian 1 (in particular, the automorphism \( \lambda_1: \frac{1}{f_1} \to \frac{1}{f_2} \) generated by \( \lambda \) belongs to \( \text{SL}_{n-1}(A) \)). Then one has the following

(i) If \( \lambda_1 \) belongs to \( E_{n-1}(A) \) then \( \lambda_1 \) is \( G \)-induced.

(ii) If \( E_{n-1}(A) = \text{SL}_{n-1}(A) \) (which true in the case when \( C \) is a smooth polynomial curve) then \( \lambda \) is \( G \)-induced.

Before the proof we need to consider several facts the first of which follows immediately from Formula (3) in Lemma 1.18.

Lemma 5.6. (cf. [Ud, page 47]) Let Setting 3.1 and the notations of Proposition 3.2 (resp. Proposition 3.7) hold. Suppose that \( C_i := \mathcal{g}_i(C) \) for some \( i \in \{1, \ldots, n-1\} \).

For every \( \sigma \in \text{LND}(X) \) let \( \bar{\sigma} = \sigma|_C \). Suppose that \( f_j \) and \( h_j, j = 1, \ldots, n-1 \) are regular functions on \( Q \) such that each \( f_j \) vanishes on \( C_i \).

(i) Then the flow of the locally nilpotent vector field \( \mathcal{g}_i(h_j f_j) \delta_{\alpha_i} \), leaves every point of \( C \) fixed (i.e., it preserves \( TC \)) and it generates an automorphism of the bundle \( TX|_C \) (and, hence, of \( N_XC \)) such that \( \delta_{\alpha_i} \mapsto \bar{\delta}_{\alpha_i} \) and \( \delta_{\alpha_k} \mapsto \bar{\delta}_{\alpha_k} + t(\mathcal{g}_i(h_j)\delta_{\alpha_k}(\mathcal{g}_i(h_j)(f_j)))|_C \delta_{\alpha_i} \) for \( 1 \leq k \neq i \leq n-1 \) where \( t \) is the time parameter.

(ii) For \( j \neq l \) the locally nilpotent vector fields \( \mathcal{g}_i(h_j f_j) \delta_{\alpha_i} \) and \( \mathcal{g}_i(h_l f_l) \delta_{\alpha_i} \) commute, i.e., the vector field \( \sigma = (\sum_{j=1, j \neq i}^{n-1} \mathcal{g}_i(h_j f_j)) \delta_{\alpha_i} \) is locally nilpotent. In particular, the
flow of \(\sigma\) induces an automorphism of \(TX|_C\) such that \(\tilde{\delta}_{\alpha_i} \mapsto \tilde{\delta}_{\alpha_i}\) and

\[
\tilde{\delta}_{\alpha_k} \mapsto \tilde{\delta}_{\alpha_k} + t \left( \sum_{j=1,j \neq i}^{n-1} \alpha_k^* (h_j) \delta_{\alpha_k} (\alpha_i^*(f_j)) \right) |_C \delta_{\alpha_i}
\]

for \(1 \leq k \neq i \leq n - 1\).

**Notation 5.7.** Let the notations of Theorem 5.5 and Lemma 5.6 hold. We also suppose that \(K\) is as in Proposition 3.2 (resp. Proposition 3.7). Choose functions \(f_1, \ldots, f_{i-1}, f_{i+1} \ldots f_{n-2}\) from Lemma 5.6 so that at every point of \(\alpha_i(K)\) their differential are linearly independent. Replace \(C^*\) in Proposition 3.2 (resp. Proposition 3.7) with a smaller open subset of \(C\) such that the same is true for every point of this new \(C^* \supset K\). Then the matrix \(T = [\delta_{\alpha_k} (\alpha_i^*(f_j))]_{j,k=1,j \neq i}^{n-1}\) is invertible over \(C^*\). Suppose now that \(a\) is a nonzero regular function on \(C\) that vanishes on \(C \setminus C^*\) only but with a sufficiently high multiplicity such that the function \(a \det T\) is regular on \(C\).

Then using the argument with the classical adjoint as in Lemma 4.4 we get the following.

**Lemma 5.8.** Let \(\vec{b}\) be any \((n-1)\)-column vector whose entries are regular functions on \(C\). Then there exists another \((n-1)\)-column vector \(\vec{c}\) with entries in \(k[C]\) such that \(T \vec{c} = a \vec{b}\). In particular, \(T(a \vec{c}) = a^2 \vec{b}\).

To use Lemma 5.8 for automorphisms of \(N_XC\) as in Lemma 5.6 (ii) we need the entries of \(a \vec{c}\) to be pullbacks \(\alpha_i^* (h_j)|_C\) of regular functions \(h_j|_C\) on \(C\) which is guaranteed by the next fact.

**Proposition 5.9.** ([Ud, Theorem 2.3.1]) Let \(\tau : C \rightarrow \bar{C}\) be a proper birational morphism of affine curves such that its restriction to an open subset \(C^*\) of \(C\) is an isomorphism. Then there exists \(m > 0\) such that every regular functions \(a \in k[C]\) vanishing on \(C \setminus C^*\) with multiplicity at least \(m\) is a pullback of a regular function on \(\bar{C}\).

Recall that \(\alpha_i|_C : C \rightarrow Q\) is not only birational but also proper because either \(C\) is once-punctured or the conclusions of Proposition 3.5 are valid by virtue of conditions (\#) and (\##). Hence, Proposition 5.9 is applicable and choosing the column \(a^2 \vec{b}\) in Lemma 5.8 so that its \(k_0\)-entry is \(a^2\) whereas the rest of them are zeros we find now regular functions \(h_1, \ldots, h_{i-1}, h_{i+1}, \ldots, h_{n-1} \in k[Q]\) for which one has \(\sum_{j=1,j \neq i}^{n-1} (\alpha_i^*(h_j) \delta_{\alpha_k} (\alpha_i^*(f_j))) |_C = a^2\) and \(\sum_{j=1,j \neq i}^{n-1} (\alpha_i^*(h_j) \delta_{\alpha_k} (\alpha_i^*(f_j))) |_C = 0\) for \(k \neq k_0\). Thus, we have the following.

**Proposition 5.10.** For every \(1 \leq k \neq i \leq n - 1\) a locally nilpotent vector field \(\sigma\) in Lemma 5.6 can be chosen so that its flow induces an automorphism of \(TX|_C\) such that \(\tilde{\delta}_{\alpha_i} \mapsto \tilde{\delta}_{\alpha_i}\) for \(1 \leq k \neq k_0 \leq n - 1\) and \(\tilde{\delta}_{\alpha_{k_0}} \mapsto \tilde{\delta}_{\alpha_{k_0}} + ta^2 d \tilde{\delta}_{\alpha_i}\) where \(t\) is a time parameter, \(a\) is a fixed regular function on \(C\) that vanishes on \(C \setminus C^*\) only but with sufficiently high multiplicity and \(d\) is any given regular function on \(C\). In particular, automorphisms of \(N_XC\) that have this form are \(G\)-induced.
Corollary 5.11. Let $pr : TX|_C \rightarrow NX_C$ be the natural projection, $A = k[X]$ and $w_j = pr(\hat{\delta}_{\alpha_j})$ for $j = 1, \ldots, n-1$. Then $w_1, \ldots, w_{n-1}$ generate a free $A$-module $M$ whose localization over $C^*$ coincides with $NX_C^*$. Furthermore, there is an automorphism $\alpha \in G$ of $X$ such that it induces an automorphism of $M$ given by $w_k \mapsto w_k$ for $1 \leq k \neq k_0 \leq n - 1$ and $w_{k_0} \mapsto w_{k_0} + t\alpha dw_i$.

Proof. The first statement follows from Propositions 3.2 and 3.7. The second one follows from application of $pr$ to the automorphism of $TX|_C$ in Proposition 5.10. \qed

Letting index $i$ in Proposition 5.10 run from 1 to $n - 1$ and taking a smaller neighborhood $C^*$ of $K$ (where $K$ is as in Notations 5.7), if necessary, we get the next result.

Corollary 5.12. Let Notation 5.7 hold and $M$ be as in Corollary 5.11 (i.e., every $A$-automorphism of $M$ can be presented as a matrix from $\text{Mat}_{n-1}(A)$). Then $C^* \supset K$ can be chosen so that for some principal ideal $q = qA \subset A$ whose zero locus is $C \setminus C^*$ every element of $F_{n-1}(A, q)$ is $G$-induced (where $\text{Mat}_{n-1}(A)$ and $F_{n-1}(A, q)$ are as in Notation 4.1).

Proof of Theorem 5.5 (i). Suppose that $\bar{v} = (v_1, \ldots, v_{n-1})$ is a basis of $NX_C$ as a free $A$-module. Let $\bar{w} = (w_1, \ldots, w_{n-1})$ where $w_i$ is as in Corollary 5.11. That is, $\bar{w}$ is a basis of a free $A$-submodule $M$ of $NX_C$ as in Corollary 5.12. Then for some matrix $D \in \text{Mat}_{n-1}(A)$ we have $\bar{w} = D\bar{v}$. Since the localization $M[\frac{1}{q}]$ of $M$ over $C^*$ coincides with $NX_C^* = NX_C[\frac{1}{q}]$ by Corollary 5.11, the matrix $D$ is invertible as an element of $\text{Mat}_{n-1}(A[\frac{1}{q}])$. Let $K' = C \setminus C^*$. Then by Proposition 3.2 (resp. 3.7) and Corollary 5.12 we can find general elements $\alpha'_1, \ldots, \alpha'_{n-1}$ of some perfect $G$-family $A$ such that for the free $A$-module $M'$ with the basis $\bar{w}' = (pr(\hat{\delta}_{\alpha'_1}), \ldots, pr(\hat{\delta}_{\alpha'_{n-1}}))$ there exists a principal ideal $q' = q'A$ such that its zero locus is disjoint from $K'$ and every element of $F_{n-1}(A, q')$ is $G$-induced. Furthermore, $\bar{w}' = D'\bar{v}$ for some matrix $D'$ similar to $D$. By construction, automorphisms of $NX_C$ whose matrices in basis $\bar{v}$ belong to $\text{SL}_n(A) \cap (D^{-1}F_n(A, q)D)$ or $\text{SL}_n(A) \cap ((D')^{-1}F_n(A, q')D')$ are $G$-induced. Hence, by Proposition 4.5 every element of $E_n(A)$ is $G$-induced. Switching to dual spaces we see that every $A$-automorphism $\lambda_1 : \frac{1}{p} \rightarrow \frac{1}{p}$ that belongs to $E_n(A)$ is $G$-induced which concludes the proof. \qed

The proof of (ii) of Theorem 5.5 is done by induction on $k$ with the first step of induction supplied by (i). For the second step we consider only the case of $k = 3$ since the general case differs only by the length of notations (see [Ud] for the general case). That is, under the assumptions of Theorem 5.5 (ii) we are going to show that a $A$-automorphism $\lambda : \frac{1}{p} \rightarrow \frac{1}{p}$ with Jacobian 1 is $G$-induced.

Notation 5.13. We suppose further in this section that the assumptions of Theorem 5.5 hold and $\bar{u} = (u_1, \ldots, u_{n-1})$ is a basis of $\frac{1}{p}$ over $A$. Every $A$-automorphism $\lambda$ of the ring $\frac{1}{p}$ is given by its values on generators of $\frac{1}{p}$. That is, it is given by the maps $u_i \mapsto l_i(\bar{u}) + q_i(\bar{u})$, $i = 1, \ldots, n-1$ where each $l_i$ (resp. $q_i$) is a linear (resp. quadratic) form. Letting $L(\bar{u}) = (l_1(\bar{u}), \ldots, l_{n-1}(\bar{u}))$ and $Q(\bar{u}) = (q_1(\bar{u}), \ldots, q_{n-1}(\bar{u}))$ we see that
λ sends \( \bar{u} \) to \( L(\bar{u}) + Q(\bar{u}) \) where \( L(\bar{u}) = L \cdot \bar{u} \) with \( L \) being an invertible matrix. The automorphism \( \lambda_1 : \frac{I}{F} \to \frac{I}{F} \) induced by \( \lambda \) is given exactly by this matrix \( L \).

The next fact is nothing but a direct computation.

**Lemma 5.14.** Let \( \lambda^i : \frac{I}{F} \to \frac{I}{F}, i = 1, 2 \) be \( A \)-automorphisms of \( \frac{I}{F} \) given by \( \bar{u} \mapsto L^i(\bar{u}) + Q^i(\bar{u}) \). Then the composition \( \lambda = \lambda^1 \circ \lambda^2 \) is given by \( \bar{u} \mapsto L(\bar{u}) + Q(\bar{u}) \) where \( L = L^1 L^2 \) and \( Q(\bar{u}) = L^1(Q^2(\bar{u})) + Q^1(L^1(\bar{u})) \).

**Notation 5.15.** The assumption that the Jacobian of \( \lambda \) from Notation 5.13 is 1 implies that \( L \in \text{SL}_{n-1}(A) \) and \( \text{div} \, Q = \frac{\partial q_1}{\partial u_1} + \ldots + \frac{\partial q_{n-1}}{\partial u_{n-1}} = 0 \) (i.e., see \([\text{AFKKZ}, \text{proof of Lemma 4.13}]\)). Consider the space \( \mathcal{F} \) of all \((n-1)\)-tuples of quadratic forms over \( A \) of divergence zero and the group \( H \) of \( A \)-automorphisms of \( \frac{I}{F} \) given by \( \bar{u} \mapsto \text{Id} \cdot \bar{u} + Q(\bar{u}) \) where \( \text{Id} \) is the identity matrix and \( Q \in \mathcal{F} \). Suppose that \( H_0 \) is the subgroup of \( H \) that consists of \( G \)-induced automorphisms and \( \mathcal{R} \) is set of quadratic forms \( Q \in \mathcal{F} \) corresponding to the elements of \( H_0 \). Let \( \mathcal{F}^0 \) be a maximal \( A \)-module contained in \( \mathcal{R} \).

Now Lemma 5.14 implies the following.

**Proposition 5.16.** Suppose that \( H = H_0 \) (or, equivalently, \( \mathcal{F} = \mathcal{F}^0 \)) and every \( A \)-automorphism \( \mu \in \text{SL}_{n-1}(A) \) of the conormal bundle \( \frac{I}{F} \) is \( G \)-induced. Then the statement (ii) of Theorem 5.5 is true.

The next facts are also consequences of Lemma 5.14.

**Lemma 5.17.** (1) Let \( \lambda^i : \frac{I}{F} \to \frac{I}{F}, i = 1, 2 \) be \( A \)-automorphisms of \( \frac{I}{F} \) given by \( \bar{u} \mapsto \text{Id} \cdot \bar{u} + Q^i(\bar{u}) \), i.e., \( \lambda^i \in H \). Then the composition \( \lambda = \lambda^1 \circ \lambda^2 \) is given by \( \bar{u} \mapsto \text{Id} \cdot \bar{u} + (Q^1 + Q^2)(\bar{u}) \) (i.e., we have an isomorphism of the multiplicative group \( H \) and the additive group \( \mathcal{F} \)).

(2) Let \( \gamma \) be a \( A \)-automorphism of \( \frac{I}{F} \) such that the automorphism of \( \frac{I}{F} \) generated by \( \gamma \) given by a matrix \( L \in \text{SL}_{n-1}(A) \). Then the composition \( \gamma^{-1} \lambda^1 \gamma \in H \) and it is given by \( \bar{u} \mapsto \text{Id} \cdot \bar{u} + Q(\bar{u}) \) where \( Q(\bar{u}) = L^{-1}(Q^1(L(\bar{u}))) \) (in particular, one has a natural \( \text{SL}_{n-1}(A) \)-action on \( H \) (resp. \( \mathcal{F} \)).

We shall need also the following well-known fact from representation theory.

**Theorem 5.18.** Let \( F \) be the space of \((n-1)\)-tuples of homogeneous \( k \)-forms in variables \( \bar{u} = (u_1, \ldots, u_{n-1}) \) over a field \( K \) such that the divergence of every \( q \in F \) is zero. Then \( F \) is an irreducible \( \text{SL}_{n-1}(K) \)-module under the \( \text{SL}_{n-1}(K) \)-action given by \( q(\bar{u}) \mapsto L^{-1}(q(L(\bar{u}))) \) where \( L \in \text{SL}_{n-1}(K) \) and \( q \in F \).

**Proof.** Denote by \( V \) the \( K \)-vector space with the basis \( \bar{u} \) and by \( V^* \) its dual. Then the space of homogeneous \( k \)-forms in \( \bar{u} \) can be treated as \( \text{Sym}^k V \) and the space \( F \) as \( V^* \otimes \text{Sym}^k V \) (cf. \([\text{AFKKZ}, \text{Lemma 4.12(a) and Formula (13))}]\)). In these notations divergence is the contraction map \( V^* \otimes \text{Sym}^k V \to \text{Sym}^{k-1} V \) and \( F \) is its kernel which is invariant under the natural \( \text{SL}_{n-1}(k) \)-action.

Using now notations from \([\text{FuHa}]\) we observe that \( V^* \cong \bigwedge^{n-2} V \) is the natural space of the irreducible representation \( \Gamma_{\lambda_{n-1}, k} \) of \( \text{SL}_{n-1}(k) \), while for \( \text{Sym}^k V \) the corresponding irreducible representation is \( \Gamma_{k, 0, \ldots, 0} \) (i.e., we have irreducible representations...
of $\text{SL}_{n-1}(k)$ with the highest weights $(1,1,\ldots,1)$ and $(k,0,\ldots,0)$ respectively). By [FuHa, Proposition 15.25] $\text{Sym}^k V \otimes \bigwedge^{n-2} V$ is the sum of two irreducible representations $\Gamma_{k-1,0,\ldots,0}$ and $\Gamma_{k,0,\ldots,0,1}$. The first of them is, of course, the space $\text{Sym}^{k-1} V$ and the second one is, therefore, the kernel of the contraction homomorphism. Hence, $F$ is irreducible and we are done. □

**Lemma 5.19.** Let Notation 5.13 hold (in particular, every automorphism of $\frac{1}{\tau}$ is given by a matrix from $\text{GL}_{n-1}(A)$). Let automorphisms of $\frac{1}{\tau}$ given by elements of $\text{SL}_{n-1}(A)$ be $G$-induced. Suppose that for every $x \in C$ there exist a neighborhood $V \subset C$ and $\sigma \in \mathcal{N}$ such that an associated partial quotient morphism $\tau : X \to P$ induces an isomorphism $\tau|_V : V \to \tau(V)$ and $\tau(V)$ is contained in the smooth part of $P$. Then every $A$-automorphism $\lambda : \frac{1}{\tau} \to \frac{1}{\tau}$ with Jacobian $1$ is $G$-induced.

**Proof.** As we mentioned we consider the case of $k = 3$. Note that Lemma 5.17 implies that $\mathcal{F}^0$ is not just a maximal $A$-module contained in $\mathcal{R}$ but it is the only maximal $A$-module in $\mathcal{R}$. Furthermore, since by the assumption every automorphism $\gamma$ as in Lemma 5.17 is $G$-induced, $\mathcal{F}^0$ is invariant under the $\text{SL}_{n-1}(A)$-action on $\mathcal{F}$ from Lemma 5.17. For every $A$-module $M$ denote by $M(\mu)$ the localization of $M$ with respect to a maximal ideal $\mu \subset A$. Recall that two submodules $M$ and $N$ of an $A$-module coincide if $M(\mu) = N(\mu)$ for every maximal ideal $\mu$ [AM, Proposition 3.9]. Hence, by Proposition 5.16 it suffices to prove that $\mathcal{F}(\mu) = \mathcal{F}^0(\mu)$. Let $k$ be the residue field of $\mu$ and, hence, $\mathcal{F}(\mu)$ is the space of $(n-1)$-tuples of homogeneous $k$-forms in variables $\bar{u} = (u_1,\ldots,u_{n-1})$ over a field $k$ such that the divergence of every $q \in \mathcal{F}(\mu)$ is zero. By Theorem 5.18 it is enough to prove that $\mathcal{F}(\mu)$ is not zero. By Nullstellensatz $\mu$ is the defining ideal of a point $x \in C$. Choose a regular function $f$ on $P$ that vanishes on $\tau(C)$ and has a nonzero differential at $\tau(x)$. Then consideration in local coordinates implies that the flow of the vector field $\tau^*(f^2)\sigma$ produces the desired nonzero element of $\mathcal{F}(\mu)$, which concludes the proof. □

**Proof of Theorem 5.5 (ii).** By Propositions 3.2 and 3.7 letting $\sigma = \delta_{\alpha\beta}$ we see that $\sigma$ satisfies the assumptions of Lemma 5.19. Since $\text{E}_{n-1}(A) = \text{SL}_{n-1}(A)$ and by Theorem 5.5(i) every element of $\text{E}_{n-1}(A)$ is $G$-induced we see that all assumptions of Lemma 5.19 are valid. This implies the desired conclusion. □

In conclusion of this section we want to pose the following problem.

**Question.** Does Theorem 5.5 remain true if the assumption $n \geq 4$ is replaced by $n \geq 3$?

6. Main Theorem. II

The main aim of this section is to show that Theorem 5.5 is valid without the assumption that $C$ is irreducible.

**Notation 6.1.** Further, in this section we suppose that Setting 3.1 holds. That is, $C$ is a smooth closed curve in a $G$-flexible affine variety $X$ where $G$ is generated by a saturated set $\mathcal{N} \subset \text{LND}(X)$, $0 \neq \delta \in \mathcal{N}$ is associated with a partial quotient morphism
\( \rho : X \to Q \). We also let \( \alpha_1, \ldots, \alpha_{n-1} \) to be general elements of some perfect family \( \mathcal{A} \) as in Propositions 3.2 or 3.7. Let \( Z \) be a closed subvariety of \( X \) disjoint from \( C \). We denote by \( N_{Z,k} \) the subset of \( N \) that consists of those elements of \( N \) that vanish on \( Z \) with multiplicity at least \( k \). Note that this implies that the subgroup \( G_{Z,k} \) of \( G \) generated by the flows of elements of \( N_{Z,k} \) acts trivially on the \( k \)th infinitesimal neighborhood of \( Z \). It is also important to remember that by [FKZ] \( X \setminus Z \) is \( G_{Z,k} \)-flexible provided \( \text{codim}_X Z \geq 2 \).

The next preliminary observation will be important for us.

**Remark 6.2.** By construction an automorphism \( \alpha \in G \) whose restriction to \( C \) induces \( \lambda_1 \) (resp. \( \lambda \)) in Theorem 5.5 is a composition of automorphisms of the form \( g^*_\alpha(hf)\delta_\alpha \) from Lemma 5.6 where \( f, h \in k[Q] \) are such that \( f \) vanishes on \( C_i = g_\alpha(C) \). Furthermore, this construction requires only the knowledge of \( f \) and \( h \) on the \( k \)th infinitesimal neighborhood of \( C_i \) in \( Q \) and we do not care about the behavior of \( f \) and \( h \) on \( Q \setminus C_1 \).

Hence, one can require additionally that \( f \) and \( h \) vanish with a given multiplicity on a given closed subvariety of \( Q \) disjoint from \( C_i \).

We need also the following technical result.

**Lemma 6.3.** Let Notations 6.1 hold, condition (\#) from Convention 3.3 be valid and \( \text{codim}_X Z \geq 3 \). Let \( \mathcal{A} \) be an algebraic \( G \)-family. Suppose that \( W_\alpha = g_\alpha^{-1}(g_\alpha(Z)) \) where \( \alpha \in \mathcal{A} \) and \( g_\alpha(Z) \) is the closure of \( g_\alpha(Z) \) in \( Q \).

Then there exists \( \gamma \in G_{Z,1} \) such that for general \( \alpha \in \mathcal{A} \) the curve \( \gamma(C) \) does not meet \( W_\alpha \).

**Proof.** Let \( \mathcal{X} = \mathcal{A} \times X \) and \( \pi : \mathcal{X} \to \mathcal{A} \) be the natural projection. Consider \( \mathcal{C} = \mathcal{A} \times C \), \( Z = \mathcal{A} \times Z \) and the closure \( W \) of \( \bigcup_{\alpha \in \mathcal{A}} \alpha \times W_\alpha \) in \( \mathcal{X} \). Note that for general \( \alpha \in \mathcal{A} \) one has \( \pi^{-1}(\alpha) \cap W = W_\alpha \). Let \( G_{Z,1} \) act on \( \mathcal{X} \) by \( \gamma.(\alpha, x) = (\alpha, \gamma(x)) \). Note that this action preserves \( Z \) and it is transitive on every fiber of \( \pi|_{\mathcal{X}\setminus Z} \) by [FKZ]. By Theorem 1.8 for general \( \gamma \) in some algebraic \( G_{Z,1} \)-family \( \gamma \mathcal{C} \) meets \( W \) along a subvariety \( R \) of dimension at most \( \dim W + \dim \mathcal{C} - \dim \mathcal{X} = \dim \mathcal{A} + \dim W_\alpha + 1 - \dim X \) where \( \alpha \) is a general element of \( \mathcal{A} \). Note that by condition (\#) \( \dim W_\alpha \leq \dim X - 2 \) since \( \dim Z \leq \dim X - 3 \). Hence, \( \dim R < \dim \mathcal{A} \) and for general \( \alpha \) the set \( R \cap \pi^{-1}(\alpha) = W_\alpha \cap \gamma(C) \) is empty. \( \square \)

**Theorem 6.4.** (1) Theorem 5.5 is valid if assumption (\bullet) is replaced by the following: each irreducible component of \( C \) is once-punctured or Convention 3.3 holds.

(2) Furthermore, let Convention 3.3 hold and \( Z \) be as in Notation 6.1. Suppose also that \( X_{\text{sing}} \subset Z \) and \( \text{codim}_X Z \geq 3 \). Then the automorphisms \( \lambda_1 \) and \( \lambda \) in Theorem 5.5 (modified as in (1)) are \( G_{Z,1} \)-induced for every \( l \geq 1 \).

**Proof.** For (1) we use induction on the number of irreducible components of \( C \). Suppose that \( C' \) is an irreducible component of \( C \) and \( C'' = C \setminus C' \). Then \( \lambda \) generates an automorphism \( \lambda' \) (resp. \( \lambda'' \)) of the \( k \)th infinitesimal neighborhood of \( C' \) (resp. \( C'' \)). Suppose that \( F \) is a smooth closed curve \( X \) disjoint from \( C \). Let us assume as a step of induction that \( \lambda' \) is induced by \( \Phi'' \in G \) such that \( \Phi'' \) induces also the identity automorphism of the \( k \)th infinitesimal neighborhood of \( C' \cup F \). Suppose first that \( C' \) is once-punctured. Recall that for \( E = C'' \cup F \) the variety \( X \setminus E \) is \( G_{E,k} \)-flexible by
[FKZ]. Hence, by Theorem 5.5 the automorphism λ is induced by Φ′ ∈ G_{E,k} where by construction Φ′ induces the identity automorphism of the kth infinitesimal neighborhood of E. Hence, Φ = Φ′ ◦ Φ″ induces λ and its restriction to the kth infinitesimal neighborhood of F is the identity automorphism which concludes induction in this case.

Consider now the situation when C′ is not once-punctured but Convention 3.3 holds. Recall that by Proposition 3.7 φ_{α_i} : C ∪ F → Q is a proper morphism which yields an isomorphism C ∪ F → C_i ∪ F_i where C_i = φ_{α_i}(C) and F_i = φ_{α_i}(F). In particular, F_i, C'_i = φ_{α_i}(C_i) and C''_i = φ_{α_i}(C''_i) are closed disjoint subvarieties of Q. By Theorem 5.5 there is an automorphism Ψ ∈ G that induces λ and by Remark 6.2 Ψ is a composition of automorphisms of the form φ_{α_i}(hf)δ_{α_i} restricted to C′ where h and f are regular functions of Q such that f vanishes on C′. Consider ˜f and ˜h ∈ k[Q] such that in the kth infinitesimal neighborhood of C′ in Q they coincide with f and h respectively but they vanish on C''_i ∪ F_i with multiplicity at least k. By Remark 6.2 replacing each element φ_{α_i}(hf)δ_{α_i} in the composition with φ_{α_i}(hf)δ_{α_i} we get a new automorphism Φ′ that induces the same λ. However, Φ′ is already an element of G_{E,k} where E = C''∪F. Hence, Φ = Φ′ ◦ Φ″ induces λ and its restriction to the kth infinitesimal neighborhood of F is the identity automorphism which concludes induction in this case together with the first statement.

For (2) we consider W_α = φ_{α_i}^{-1}(φ_{α_i}^{-1}(Z)) as in Lemma 6.3 where α runs over A as in Notation 6.2. That is, W_α is the preimage of the closed set φ_{α_i}(Z) under the morphism φ_{α_i} : X → Q. By Lemma 6.3 we can suppose that C and W_α are disjoint. Since φ_{α_i}|C : C_i → Q is proper (by Proposition 3.5) this implies that C_i and φ_{α_i}(Z) are disjoint. By Remark 6.2 an automorphism Φ that induces λ (resp. λ) is a composition of elements of the form φ_{α_i}(hf)δ_{α_i}. As before we can suppose that h and f vanish on φ_{α_i}(Z) with multiplicity l. This yields (2) and concludes the proof.

Remark 6.5. If all components of C are once-punctured then there is no need to assume that codim_X Z ≥ 3 in Theorem 6.4 (2). It suffices to assume that codim_X Z ≥ 2 since in this case one can replace X with X \ Z (which is G_{Z,F}-flexible by [FKZ]) and get the desired conclusion from Theorem 6.4 (1).

Corollary 6.6. Let the assumptions of Theorem 6.4 hold and Convention 3.3 be valid. For every nonvanishing section v of N_X C there exists σ ∈ N such that v = pr(σ) where pr : TX|_C → N_X C is the natural projection. In particular, N_X C admits a basis v_1, ..., v_{n-1} over A such that each v_i = pr(σ_i) where σ_1, ..., σ_{n-1} ∈ N.

Proof. The second statement is especially simple. Indeed, by Proposition 3.7(5) we can suppose that v_i = pr(σ_i) where σ_i ∈ N for i ≤ n - 2. Then one can consider an A-automorphism θ of N_X C given by v_{n-1} → v_1, v_1 → -v_{n-1} and v_i → v_i for 2 ≤ i ≤ n - 2. Since every element of E_{n-1}(A) is G-induced by Theorem 6.4 and θ ∈ E_n(A) we see that v_{n-1} = pr(σ_{n-1}) where σ_{n-1} ∈ N.

For the first statement note that v = n-1 \sum a_i v_i where a_1, ..., a_{n-1} ∈ A have no common zeros. Since n - 1 ≥ 3, by the Bass stable range theorem [Ba] for some b_i ∈ A, i ≥ 1 the ideal generated by a'_1, ..., a'_{n-2} is A where a'_i = a_i - b_i a_{n-1}, i = 1, ..., n-2.
Hence, applying consequently elements of \( E_{n-1}(A) \) we can send \( v \) first to \( \sum_{i=1}^{n-2} a_i' v_i \), then to \( v_{n-1} + \sum_{i=1}^{n-2} a_i' v_i \) and then to \( v_{n-1} \). This yields the desired conclusion. \( \square \)

7. Applications

**Notation 7.1.** In this section we suppose that Setting 3.1 holds. Let \( C_1 \) and \( C_2 \) be smooth curves in \( X \) with defining ideals \( I_1 \) and \( I_2 \) in the algebra \( k[X] \) of regular functions on \( X \). We suppose that \( X \) possesses a volume form \( \omega \) (i.e., \( \omega \) is a nonvanishing section of the canonical bundle on \( X \)) and that each conormal bundle \( I_j \) of \( C_j \) in \( X \) is trivial. Hence, by Lemma 5.3 \( I_j \) is generated by some functions \( u_{1,j}, \ldots, u_{n-1,j} \) and \( k[X]_j \simeq A_j \oplus \bigoplus A_l^j \) is a graded algebra where \( A_j = k[X]_j \) and each summand \( A_l^j \) can be viewed as the space of homogeneous polynomials in \( u_{1,j}, \ldots, u_{n-1,j} \) over \( A_j \) of degree \( l \). For \( k \in \mathbb{N} \) we denote by \( \theta : k[X]_1 \to k[X]_2 \) an isomorphism of algebras.

**Definition 7.2.** Let Notation 7.1 hold. Since \( N_XC_1 \) is trivial the existence of \( \omega \) implies the existence of a volume form on \( C_1 \). Fix volume forms \( \omega_i \) on \( C_i \) such that \( \theta^* \omega_1 = \omega_2 \) where the isomorphism \( \theta : C_2 \to C_1 \) is induced by \( \theta \). Then one can require that \( \omega \mid_{C_j} \) coincides with \( \omega_j \wedge du_{1,j} \wedge \ldots \wedge du_{n-1,j} \). Under this requirement the determinant of the matrix \( \left[ \frac{\partial \theta(u_{i,j})}{\partial u_{m,j}} \right]_{l,m=1}^{n-1} \) is well-defined modulo \( I_2^k \) (i.e., it is independent of the choice of coordinates \( u_{1,j}, \ldots, u_{n-1,j} \)). Hence, we say again that \( \theta \) has Jacobian 1 if the determinant of \( \left[ \frac{\partial \theta(u_{i,j})}{\partial u_{m,j}} \right]_{l,m=1}^{n-1} \) is equal to 1 modulo \( I_2^k \).

The next fact was is a straightforward consequence of Theorem 6.4.

**Theorem 7.3.** Let Notation 7.1 hold, the assumptions of Theorem 6.4 be satisfied and \( X \) possess a volume form. Let \( \text{Aut}^1(X) \subset \text{Aut}(X) \) be the subgroup of automorphisms of \( X \) that have Jacobian 1. Suppose that \( \mathcal{F} \) is a family of closed curves in \( X \) with trivial normal bundles such that each \( C \in \mathcal{F} \) is contained in \( X_{\text{reg}} \) and every isomorphism between \( C_1 \in \mathcal{F} \) and \( C_2 \in \mathcal{F} \) extends to an automorphism from \( \text{Aut}^1(X) \) (resp. \( \text{SAut}(X) \)). Then every isomorphism \( \theta : \frac{k[X]}{I_1} \to \frac{k[X]}{I_2} \) with the Jacobian 1 is \( \text{Aut}^1(X) \)-induced (resp. \( \text{SAut}(X) \)-induced).

**Proof.** Let \( \theta_1 : \frac{k[X]}{I_1} \to \frac{k[X]}{I_2} \) be an isomorphism generated by \( \theta \). By the assumption \( \theta_1^{-1} \) is generated by some automorphism \( \Phi \in G \) such that \( \Phi(C_2) = C_1 \). By Lemma 5.2 \( \Phi \) generates an isomorphism \( \varphi : \frac{A}{I_2} \to \frac{A}{I_1} \) with Jacobian 1. Hence, \( \varphi \circ \theta : \frac{k[X]}{I_1} \to \frac{k[X]}{I_2} \) has Jacobian 1 and by Theorem 6.4 it is generated by an automorphism \( \Psi \in G \). Hence, \( \theta \) is induced by \( \Phi^{-1} \circ \Psi \) which is the desired conclusion. \( \square \)

**Example 7.4.** Let Notation 7.1 hold and \( \varphi : C_1 \to C_2 \) be any isomorphism of smooth polynomial curves.

(1) Consider the case of \( X = \text{SL}_n(k), n \geq 3 \). By van Santen’s theorem [St] (see also [Ka20]) there is an automorphism \( \alpha \in \text{Aut}^1(X) \) such that \( \alpha|_{C_1} = \varphi \) and also \( X \) has a volume form.
(2) More generally, let $X$ be a connected complex algebraic group of dimension at least 4 without nontrivial characters. The absence of characters implies that the natural action of $\text{SAut}(X)$ on $X$ is transitive and, therefore, $X$ is a flexible variety ([AFKKZ, Theorem 0.1]). It has also a left invariant volume form. The Feller-van Santen theorem [FS] states that there is an automorphism $\alpha \in \text{Aut}^1(X)$ such that $\alpha|_{C_i} = \varphi$.

(3) Let $X$ be an $(m - 1)$-sphere over $k$ (i.e., a nonzero fiber of a non-degenerate quadratic form $f$ on $\mathbb{A}^m$) where $m \geq 6$. Then $X$ is flexible and by [Ka20, Theorem 8.3] there is an automorphism $\alpha \in \text{SAut}(X)$ such that $\alpha|_{C_i} = \varphi$. By the adjunction formula there is a volume form $\omega$ on $X$ such that $df \wedge \omega$ is the standard volume form on $\mathbb{A}^m$.

Thus, we have the following.

**Corollary 7.5.** The conclusions of Theorem 7.3 are valid for all varieties in Example 7.4 (1)-(3) in the case when $\mathcal{F}$ is the family of smooth polynomial curves contained in $X$.

**Remark 7.6.** (i) The absence of nontrivial characters is essential for Example 7.4 (2). Without this assumption the statement is not true. Indeed, in the presence of nontrivial characters a linear algebraic group $X$ is isomorphic as an algebraic variety to $X_0 \times (\mathbb{C}^*)^m$ where $m > 0$ and $X_0$ does not admit nonconstant invertible functions. In particular, the image $\pi(C_j)$ of a polynomial curve $C_j \subset X$ under the natural projection $\pi : X \to (\mathbb{C}^*)^m$ is a point $p_j$. The absence of nonconstant invertible functions on $X_0$ implies that every automorphism $\Phi$ of $X$ induces an automorphism $\varphi$ of $(\mathbb{C}^*)^m$. In particular, $\pi_\ast \circ \Phi_\ast = \varphi_\ast \circ \pi_\ast$. However, one easily can find an isomorphism $\lambda : N_XC_1 \to N_XC_2$ for which $\pi_\ast \circ \lambda \neq \varphi_\ast \circ \pi_\ast$. In particular, such isomorphism is not induced by an automorphism of $X$.

(ii) By virtue of the Lefschetz principle Corollary 7.5 as well as the Feller-van Santen theorem are valid if one replaces the ground field $\mathbb{C}$ by a universal domain $k$ [Ek].

Actually, for Example 7.4 (3) Corollary 7.5 can be strengthened. To show this we need next fact.

**Proposition 7.7.** Let $X$ be an $(m - 1)$-sphere over $k$ where $m \geq 6$. Then $X$ is parametrically suitable.

**Proof.** Parametric suitability follows immediately from Example 2.8 since $X$ is a homogeneous space of $\text{SO}(m)$. However, this argument involves a nontrivial result from [DDK]. Hence, we give another rather trivial argument where the sake of notations we consider the case of $m = 6$. Then one can treat $X$ as a hypersurface in $\mathbb{A}^6$ whose coordinate form is $z_1z_2 + z_3z_4 + z_5z_6 = 1$. Consider the commuting locally nilpotent vector fields $\delta_1 = z_4\frac{\partial}{\partial z_1} - z_2\frac{\partial}{\partial z_3}$, $\delta_2 = z_2\frac{\partial}{\partial z_5} - z_6\frac{\partial}{\partial z_1}$ and $\delta_3 = z_6\frac{\partial}{\partial z_2} - z_4\frac{\partial}{\partial z_5}$ on $X$. The kernel of $\sigma = a_1\delta_1 + a_2\delta_2 + a_3\delta_3$, $a_i \in k$ is generated by $z_2, z_4, z_6$ and $a_3z_1 + a_2z_3 + a_1z_5$. Choose the algebraic family $B \subset \text{SO}(6)$ such that for every $\beta \in B$ the homomorphism

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4It is not stated explicitly in [FS] that the Jacobian of $\alpha$ is 1. However, by virtue of Lemma 5.2 this follows from the construction in [FS].
Theorem 7.8. Let $X$ be an $(m - 1)$-sphere over $k$ where $m \geq 6$. Let $C_1$ and $C_2$ be isomorphic smooth closed curves in $X$ with trivial normal bundles and $I_j$ be the defining ideal of $C_j$ in $k[X]$. Suppose that $E_{m-2}(k[C_j]) = \text{SL}_{m-2}(k[C_j])$. Then every automorphism $\lambda : \frac{1}{7\pi} \to \frac{1}{7\pi}$ with Jacobian 1 is $\text{SAut}(X)$-induced.

Proof. As we mentioned in Example 7.4 (3) $X$ is flexible, it has a volume form and every isomorphism of smooth closed curves in $X$ extends to an automorphism from $\text{SAut}(X)$ by [Ka20, Corollary 8.4]. The vector fields $\delta_1, \delta_2, \delta_3$ on $X$ from the proof of Proposition 7.7 have zero loci of codimension 2 and smooth algebraic quotient morphisms.

Hence, condition (#) from Setting 3.1 is satisfied, while condition (###) follows from Proposition 7.7. Therefore, all assumptions of Theorem 7.3 are satisfied which yields the desired conclusion.

Corollary 7.9. Let $X$ be an $(m - 1)$-sphere over $k$ where $m \geq 6$. Let $C_1$ and $C_2$ be isomorphic smooth closed curves in $X$ such that their irreducible components are rational. The every isomorphism of $k$th infinitesimal neighborhoods of $C_1$ and $C_2$ with Jacobian 1 extends to an automorphism of $X$.

Proof. By Theorem 7.8 it suffices to show that $N_X C_j$ is a trivial vector bundle and $E_{m-2}(k[C_j]) = \text{SL}_{m-2}(k[C_j])$. By [Sc1] $N_X C_j$ is isomorphic to $T \oplus L$ where $T$ is a trivial vector bundle and $L$ is a line bundle on $C_j$. Note that $L$ is trivial since the Picard group of a smooth affine rational curve is zero. Hence, $N_X C_j$ is trivial.

In the case when every irreducible component $C'$ of $C_j$ is a smooth polynomial curve the equality $E_{m-2}(k[C_j]) = \text{SL}_{m-2}(k[C_j])$ follows from the fact that $k[C']$ is a Euclidean domain. Hence, in the general case of smooth rational irreducible components of $C_j$ the similar equality follows from [Lam, Corollary 5.4] and we are done.

The next example demonstrates an application of Theorem 6.4 (2).

Example 7.10. Let $Z$ be a closed subvariety of $X = \mathbb{A}^n$ of dimension $\dim Z \leq n - 3$ where $n \geq 4$. Recall that $X \setminus Z$ is flexible by [Wi], [Gr, Sec. 2.15, p. 72, Exercise (b')] (see also [FKZ]) and $X$ has a volume form inherited from $\mathbb{A}^n$. Therefore, we have the assumptions and conclusions of Theorem 6.4 (2) in this case. Furthermore, if $n \geq 5$ then by [Ka20, Theorem 7.1] any isomorphism $\varphi : C_1 \to C_2$ of two smooth closed curves in $X$ disjoint from $Z$ extends to an automorphism of $X \setminus Z$ which is an element of $\text{SAut}(X \setminus Z)$. Hence, arguing as in Theorem 7.3 we see that every isomorphism of $k$th infinitesimal neighborhood of $C_1$ and $C_2$ extends to an automorphism from $\text{SAut}(X \setminus Z)$.
We want to mention one more example which was surprisingly unknown.

**Example 7.11.** Let \( X \simeq \mathbb{A}^n = \mathbb{A}^2 \times \mathbb{A}^{n-2}, n \geq 4. \) Suppose that the first factor is equipped with a coordinate system \((z_1, z_2)\). Let \( \delta = \frac{\partial}{\partial z_1} \) and \( B = \text{SL}_2(k) \) act on the first factor, i.e., \( \delta_\beta = \beta_* (\delta) \) is of the form \( a \frac{\partial}{\partial z_1} + b \frac{\partial}{\partial z_2} \) for \( \beta \in B \). Then Proposition 2.2 implies that for general \( \beta \neq \gamma \in B \) the pair \((\delta_\beta, \delta_\gamma)\) is semi-compatible and, thus, we have parametric suitability. Every isomorphism of smooth closed curves extends to an automorphism of \( X \) ([Cr] and [Je]) which belong to \( \text{SAut}(X) \). Hence, the assumptions and the conclusions of Theorem 7.3 are valid. It is also worth mentioning that the assumption that Jacobian is 1 can be replaced in this case by the assumption that the Jacobian is any nonzero constant.

8. Holomorphic case

In this section we shall show that in the complex case the assumption that \( \text{SL}_{n-1}(A) = E_{n-1}(A) \) in Theorem 5.5 (ii) (resp. 6.4) can be omitted if we consider automorphisms of the normal bundle of \( C \) induced by holomorphic automorphisms of \( X \).

**Definition 8.1.** Let \( \text{Hol}(X) \) be the algebra of holomorphic functions on a complex affine algebraic variety \( X \) and \( C \) be a smooth closed curve in \( X \) whose defining ideal in \( \text{Hol}(X) \) is denoted by \( \hat{I} \). Suppose that for some holomorphic automorphism \( \psi \) of \( X \) one has \( \psi(I) = I \). As before, \( \psi \) generates an automorphism \( \hat{\psi} : \hat{I} \to \hat{I} \) which we call holomorphically induced.

The aim of this section is the following.

**Theorem 8.2.** Let \( X \) be a normal affine \( G \)-flexible variety of dimension \( n \geq 4 \) where \( G \subset \text{SAut}(X) \) is a group generated by a saturated set \( \mathcal{N} \subset \text{LND}(X) \), \( C \) be a smooth closed curve in \( X \) such that \( C \subset X_{\text{reg}} \) and let \( \delta \in \mathcal{N} \) have a partial quotient morphism \( \varphi : X \to Q \) such that conditions (\#) and (\###) of Convention 3.3 are valid. Let \( \hat{I} \) be the defining ideal of \( C \) in \( \text{Hol}(X) \) and \( \hat{A} = \frac{\text{Hol}(X)}{\hat{I}} \). Suppose that the normal bundle \( \mathcal{N}_X C \) of \( C \) in \( X \) is trivial. Then every \( \hat{A} \)-automorphism \( \lambda : \frac{I}{\hat{I}} \to \frac{I}{\hat{I}} \) with Jacobian 1 is holomorphically induced.

We start the proof with the following fact which is a simple consequence of the Ivarsson-Kutzschebauch theorem [IK].

**Proposition 8.3.** Let \( \hat{A} \) be the ring of holomorphic functions on a smooth affine curve \( C \). Then \( E_n(\hat{A}) = \text{SL}_n(\hat{A}) \) for every \( n \geq 2 \).

**Proof.** Consider \( \lambda \in \text{SL}_n(\hat{A}) \) as a holomorphic map \( \theta : C \to \text{SL}_n(\mathbb{C}) \). Recall that such map is called null-homotopic if it can be continuously deformed to a constant map from \( C \) into \( \text{SL}_n(\mathbb{C}) \). Let \( Z \) be the closure of an open subset of \( C \) such that \( Z \) is compact and it is a strong deformation retract of \( C \). Note that if \( \theta|_Z \) is null-homotopic then so is \( \theta \). By [KaZa96] perturbing \( \theta|_Z : Z \to \text{SL}_n(\mathbb{C}) \) we can suppose that this map is an embedding and we can identify \( Z \) with its image in \( \text{SL}_n(\mathbb{C}) \). Since \( Z \) is an Eilenberg-Maclane space with a free finitely generated fundamental group one can find a bouquet
W of circles in Z such that W \to Z is a weak homotopy equivalence. Hence, gluing Z with a finite union $\bigcup_{i=1}^{m} D_i$ of closed discs with disjoint interiors we get a contractible space $\tilde{Z}$. Since the fundamental group of $SL_n(\mathbb{C})$ is trivial we can choose these discs in $SL_{n-1}(\mathbb{C})$. Furthermore, perturbing them we can suppose that their interiors are disjoint and do not meet Z by the standard transversality argument. Thus, we can view $\tilde{Z}$ as a topological subspace of $SL_n(\mathbb{C})$. The fact that $\tilde{Z}$ is contractible implies now that $\theta|_{Z}$ and, therefore, $\theta$ are null-homotopic. By the Ivarsson-Kutzschebauch theorem \cite{IK} we have $SL_n(A) = E_n(\hat{A})$ which is the desired conclusion.

We'll also need the following adjustment of Lemma 5.3.

**Lemma 8.4.** Let Y and Z be as in Lemma 5.3 and $Z^\ast$ be an open subset of Z such that $Z \setminus Z^\ast$ is the zero locus of a function $g \in k[Z]$. Suppose that $w_1, \ldots, w_{m-1}$ are sections of $N_Y Z$ whose restrictions to $Z^\ast$ yield a basis of $N_Y Z^\ast$ over $k[Z^\ast]$. Then there exist functions $u_1, \ldots, u_{m-1}$ such that $w_i(u_j) = g^k \delta_{ij}$ where $\delta_{ij}$ is the Kronecker symbol and $k \geq 0$.

**Proof.** Extend $g$ to a regular function on Y denoted by the same symbol. Then $Z^\ast$ is closed in the affine variety $Y^\ast = Y \setminus g^{-1}(0)$ and $w_1, \ldots, w_{m-1}$ form a basis of $N_Y Z^\ast$. Choose the dual basis in the conormal bundle which by Lemma 5.3 can be presented by regular functions $u'_1, \ldots, u'_{m-1} \in k[Y^\ast]$. For an appropriate $k \geq 0$ every function $u_i = u'_ig^k$ is regular on Y which yields the desired conclusion.

**Notation 8.5.** Further in this section we assume that Setting 3.1 holds. We suppose also that the assumptions of conclusions of Proposition 3.7 and Corollary 3.9 are valid. In particular, $C \simeq C_j := \mathfrak{g}_{\alpha_j}(C)$ and $N_X C \simeq \text{pr}(B_{n-2}) \oplus L$ where $B_{n-2}$ is generated by $\delta_{\alpha_1}, \ldots, \delta_{\alpha_{n-2}}$. We also have $v_{n-1} = qv_0 + \sum_{i=1}^{n-2} a_iv_i$ where each $a_i \in A = k[C]$, $v_0$ is a nonvanishing section of $L$ and $v_i = \text{pr}(\delta_{\alpha_i})$, $i = 1, \ldots, n-1$.

**Lemma 8.6.** Let Notation 8.5 hold and $v_i^j = \text{pr}_j \circ \mathfrak{g}_{\alpha_j}(\delta_{\alpha_i})$ where $\text{pr}_j : TQ|_{C_j} \to N_Q C_j$ is the natural projection. Then for every $j = 1, \ldots, n-2$ the vector bundle $N_Q C_j$ is generated as $k[C_j]$-module by $v_0^j, v_1^j, \ldots, v_{j-1}^j, v_{j+1}^j, \ldots, v_{n-2}^j$. Furthermore, $v_1^{n-1}, \ldots, v_{n-2}^{n-1}$ generate $N_Q C_{n-1}^\ast$ where $C_{n-1}^\ast = \mathfrak{g}_{\alpha_{n-1}}(C \setminus S)$ and S is the zero locus of $q$.

**Proof.** Let $x \in C$ and $x_i = \mathfrak{g}_{\alpha_i}(x) \in C_i$. It suffices to show that the vectors $v_0^j(x_i), v_1^j(x_i), \ldots, v_{j-1}^j(x_i), v_{j+1}^j(x_i), \ldots, v_{n-2}^j(x_i)$ are linearly independent. Note that the kernel of the natural map

$$TX|_C \xrightarrow{\mathfrak{g}_{\alpha_j}} TQ C \xrightarrow{\text{pr}} N_Q C_j$$

is generated by $TC$ and $\delta_{\alpha_j}$. Hence, dependence of the vectors above implies dependence of $\nu(x), \delta_{\alpha_1}(x), \ldots, \delta_{\alpha_{n-2}}(x)$ modulo $T_x C$ (where the vector field $\nu$ is such that locally $\text{pr}(\nu) = v_0$). This contradicts to the fact that $v_0, v_1, \ldots, v_{n-2}$ generate $N_X C$. Hence, we have the first claim. Since $v_{n-1} = qv_0 + \sum_{i=1}^{n-2} a_i v_i$ we see that $v_1, \ldots, v_{n-1}$ form a basis of $N_X (C \setminus S)$. Hence, the argument before implies that $v_1^{n-1}, \ldots, v_{n-2}^{n-1}$ form a basis of $N_Q C_{n-1}^\ast$ and we are done.
Notation 8.7. We let \( A = \mathbb{C}[C] \) and \( \hat{A} = \text{Hol}(C) \). Treating \( N_X C \) as an \( A \)-module we consider the \( \hat{A} \)-module \( M = N_X Q C \otimes_A \hat{A} \). In particular, \( M \) is a free \( \hat{A} \)-module with basis \( \tilde{v} = (v_1, \ldots, v_{n-2}, v_0) \). Denote by \( M' \) its free \( \hat{A} \)-submodule with basis \( \tilde{v}' = (v_1, v_2, \ldots, v_{n-1}) \). Using these bases we identify \( \hat{A} \)-automorphisms of \( M \) and \( M' \) with matrices from \( \text{GL}_{n-1}(\hat{A}) \) and we use Notation 4.1 to describe such matrices. We also denote by \( \text{Ind}(M) \) (resp. \( \text{Ind}(M') \)) those automorphisms of \( M \) (resp. \( M' \)) that are induced by holomorphic automorphisms of \( X \).

Lemma 8.8. Let Notations 8.5 and 8.7 hold and \( S = q^{-1}(0) \subset C \). Then

(i) every matrix of the form \( I + ce_{ij} \in E_{n-1}(\hat{A}) \) where \( 1 \leq i \leq n-1 \) and \( 1 \leq j \neq i \leq n-2 \) is contained in \( \text{Ind}(M) \);

(ii) \( \text{Ind}(M) \) contains also all matrices of the form \( I + ab\delta e_{k,n-1} - a\delta be_{l,n-1} \in E_{n-1}(\hat{A}) \) where \( 1 \leq k \neq l \leq n-2 \), \( b \in q \) and \( q \subset A \) is an ideal whose zero locus is \( S \).

Proof. By Lemmas 8.4 and 8.6 for every \( 1 \leq j \leq n-2 \) and every \( 1 \leq k \neq j \leq n-1 \) we can find regular functions \( f_1, \ldots, f_{j-1}, f_{j+1}, \ldots, f_{n-2}, f_{n-1} \in \mathbb{C}[Q] \) vanishing on \( C_i \) such that \( v_i^j(f_k)_{|C_i} = \delta_{ik} \) where \( \delta_{ik} \) is the Kronecker symbol. Recall that \( v_i^j = \partial f_j \circ g_{\alpha_i}(\delta_{\alpha_i}) \) for \( j \leq n-2 \). Hence, since the differential \( df \) vanishes on \( TC_j \) one has

\[
\begin{align*}
\delta_{ik} = \partial f_j \circ g_{\alpha_i}(\delta_{\alpha_i})(f_k)|_{C} = \delta_{\alpha_i}(g_{\alpha_i}^j(f_k))|_C = \delta_{ik}.
\end{align*}
\]

Suppose that \( h \) is any holomorphic function on \( Q \). Consider the complete vector field \( g_{\alpha_i}^j(hf_i)\delta_{\alpha_i} \). By Formula (3) in Lemma 1.18 and Formula (5) its flow generates \( \hat{A} \)-automorphisms of \( M \) such that \( v_k \mapsto v_k \) for \( k \neq i \) and \( v_i \mapsto v_i + vh|_{C}v_j, t \in \mathbb{C} \) for \( 1 \leq j \leq n-2 \) and \( 0 \leq i \neq j \leq n-2 \) which yields (i).

Similarly, Lemmas 8.4 and 8.6 imply that one can find some \( l \geq 0 \) and regular functions \( f_1, \ldots, f_{n-2} \) on \( Q \) such that \( v_i^{n-1}(f_k) = q'^l \delta_{ik} \) for \( 1 \leq i \neq k \leq n-2 \). Formulas (3) and (5) again that every automorphism of \( M' \) which is of the form \( I + cq'\delta e_{j,1} \), \( c \in \hat{A} \) is contained in \( \text{Ind}(M') \). Note that one has the following relation \( \tilde{v}' = D\tilde{v} \) with the bases of \( M \) and \( M' \) where \( D \) is a matrix as in Lemma 4.6. Since \( \text{SL}_{n-1}(\hat{A}) = E_{n-1}(\hat{A}) \) by Proposition 8.3 we have (ii) and the desired conclusion by Proposition 4.8.

\( \square \)

Lemma 8.9. Let the assumptions of Lemma 8.8 hold. Then \( E_{n-1}(\hat{A}) \) is contained in \( \text{Ind}(M) \).

Proof. Recall that \( \nu_{n-1} = \text{pr}(\delta_{\alpha_{n-1}}) \) where \( \alpha_{n-1} \) is a general element of some perfect \( G \)-family \( \mathcal{A} \). Choose another general element \( \alpha_{n-1}' \in \mathcal{A} \) and let \( \nu_{n-1}' = \text{pr}(\delta_{\alpha_{n-1}'}) \).

By Corollary 3.9 we have \( \nu_{n-1} = q'v_0 + \sum_{i=1}^{n-2} a_i v_i \). By Lemma 8.8 this yields holomorphically induced automorphisms of the form \( \theta' = I + a'b'\varepsilon_{k1} - a'_b\varepsilon_{l1} \) where \( 2 \leq k \neq l \leq n-1 \) and \( b' \in \hat{A} \) is divisible by \( (q')^l \). By Proposition 3.5(4) we can suppose that at some point \( x \in C \) the vector \( \delta_{\alpha_{n-1}(x)} \) is such that the function \( a_i a_k' - a'_k a_i \) does not vanish at \( x \) and, therefore, \( a_i a_k' - a'_k a_i \) is nonzero function. Let \( b = c a_k' (qq')^l \) and \( b' = -c a_k (qq')^l \) where \( c \in \hat{A} \). Then the holomorphically induced automorphism \( \theta' \) is given by \( I + c (a_i a_k - a'_k a_i)(qq')^l \varepsilon_{k1} \). Thus, \( \text{Ind}(M) \) contains all automorphisms given by \( I + d e_{k1} \) where \( d \) belongs to a principal ideal \( \mathfrak{p} = q'\hat{A} \subset \hat{A} \). Choose now new general
elements $\alpha_{n-1}$ and $\alpha'_{n-1}$ in $A$. By Proposition 3.5(4) we can suppose that for every $x$ in the zero locus $K$ of $p$ the vectors $\delta_{\alpha_{n-1}}(x)$ and $\delta_{\alpha'_{n-1}}(x)$ are such that the new function $(aq_{j}a'_{i} - a'_{i}ja_{k})(qq')^i$ does not vanish at $x$. Hence, $\text{Ind}(M)$ contains every automorphism of the form $I + de_{k_1}$, $d \in q$ where $q \subset J$ is an ideal whose zero locus in $C$ is disjoint from the one of $p$. By the weak Nullstellensatz [Car] $p + q = J$ and, hence, every elementary transformation of the form $I + ae_{k_1}$, $a \in A$ is in $\text{Ind}(M)$. In combination with Lemma 8.8 this implies the desired conclusion. \hfill $\square$

Proof of Theorem 8.2. By Proposition 8.3 and Lemma 8.9 we see now that $\text{Ind}(M)$ contains $\text{SL}_n(J)$. It remains to note that Lemma 5.19 remains valid if one replace $A$ by $J$ and $G$ by the group of holomorphic automorphisms of $X$. Hence, this holomorphic analogue of Lemma 5.19 implies that every $J$-automorphism $\lambda : \frac{1}{\ell} \rightarrow \frac{1}{\ell}$ with Jacobian 1 is holomorphically induced and we are done. \hfill $\square$

Remark 8.10. The fact that $J$ is not a Dedekind domain is the reason why in the proof of Theorem 8.2 instead of Theorems 4.2 and 4.3 we use Proposition 4.8.

Repeating the argument in the proof of Corollary 6.6 we get the following.

Corollary 8.11. Let the assumptions of Theorem 8.2 hold. Suppose that $M = N_XC \otimes_A J$ is as in Notation 8.7 (i.e., $M$ is a holomorphic normal bundle of $C$ in $X$). Then every nonvanishing section of $M$ is induced by a complete holomorphic vector field.

Corollary 8.12. Let $X$ be isomorphic to a connected complex linear algebraic group without nontrivial characters and $\dim X \geq 4$. Suppose that $C$ is a smooth closed curve in $X$ with a trivial normal bundle and defining ideal $I$ in the algebra $\text{Hol}(X)$ of holomorphic functions on $X$. Let $\hat{A} = \frac{\text{Hol}(X)}{I}$ be the algebra of holomorphic functions on $C$. Then every $\hat{A}$-automorphism $\lambda : \frac{1}{\ell} \rightarrow \frac{1}{\ell}$ with Jacobian 1 extends to a holomorphic automorphism of $X$.

Proof. By Theorem 8.2 it suffices to check the validity of conditions (#) and (###) of Convention 3.3. Condition(#) is obvious since multiplication produces free $\mathbb{G}_a$-actions on $X$. Condition(###) follows from the following fact: every connected linear algebraic group $Y$ of dimension $n \geq 2$ without nontrivial characters is parametrically suitable. Indeed, By Mostow’s theorem [Mo] $Y$ is isomorphic (as a variety) to the product $S \times U$ where $U$ is unipotent and $S$ is reductive. The absence of nontrivial characters implies that $S$ is semi-simple. If $S$ is not trivial then we have parametric suitability by Theorem 2.4. Otherwise, $Y \simeq \mathbb{A}^n$ where $n \geq 2$ and the argument in Example 7.11 works which concludes the proof. \hfill $\square$

Corollary 8.13. Let $X$ be isomorphic to $\text{SL}_n(C)$ where $n \geq 3$. Suppose that $C_1$ and $C_2$ are isomorphic smooth closed curves in $X$ with defining ideals $I_1$ and $I_2$ in $k[X]$. Suppose also that the normal bundles $N_XC_j$, $j = 1, 2$ are trivial. Then every isomorphism $\theta : \frac{k[X]}{I_1} \rightarrow \frac{k[X]}{I_2}$ with Jacobian 1 is holomorphically induced.
Proof. By [Ka20, Corollary 11.12] every isomorphism $\varphi : C_1 \to C_2$ extends to a holomorphic isomorphism of $X$ which by construction has Jacobian 1. Hence, it suffices to consider the case when $\varphi$ is the identity map which is a special case of Corollary 8.12. Hence, we are done. $\square$

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