Congruences for the coefficients of the Gordon and McIntosh mock theta function $\xi(q)$

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Abstract Recently Gordon and McIntosh introduced the third order mock theta function $\xi(q)$ defined by

$$\xi(q) = 1 + 2 \sum_{n=1}^{\infty} \frac{q^{6n^2 - 6n + 1}}{(q; q^6)_n(q^5; q^6)_n}.$$ 

Our goal in this paper is to study arithmetic properties of the coefficients of this function. We present a number of such properties, including several infinite families of Ramanujan–like congruences.

Keywords congruence · generating function · mock theta function

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1 Introduction

In his last letter to Hardy in 1920, Ramanujan introduced the notion of a mock theta function. He listed 17 such functions having orders 3, 5, and 7. Since then, other mock theta functions have been found. Gordon and McIntosh [7], for example, introduced many additional such functions, including the following of order 3:

$$\xi(q) = 1 + 2 \sum_{n=1}^{\infty} \frac{q^{6n^2 - 6n + 1}}{(q; q^6)_n(q^5; q^6)_n},$$

(1)
where we use the standard $q$-series notation:

$$(a; q)_0 = 1,$$

$$(a; q)_n = (1 - a)(1 - aq) \cdots (1 - aq^{n-1}), \forall n \geq 1,$$

$$(a; q)_\infty = \lim_{n \to \infty} (a; q)_n, |q| < 1.$$ 

Arithmetic properties of the coefficients of mock theta functions have received a great deal of attention. For instance, Zhang and Shi [15] recently proved seven congruences satisfied by the coefficients of the mock theta function $\beta(q)$ introduced by McIntosh. In a recent paper, Brietzke, da Silva, and Sellers [5] found a number of arithmetic properties satisfied by the coefficients of the mock theta function $V_0(q)$, introduced by Gordon and McIntosh [6]. Andrews et al. [2] prove a number of congruences for the partition functions $p_\omega(n)$ and $p_\nu(n)$, introduced in [1], associated with the third order mock theta functions $\omega(q)$ and $\nu(q)$, where $\omega(q)$ is defined below and

$$\nu(q) = \sum_{n=0}^{\infty} \frac{q^{n(n+1)}}{(-q; q^2)_{n+1}}.$$ 

In a subsequent paper, Wang [14] presented some additional congruences for both $p_\omega(n)$ and $p_\nu(n)$.

This paper is devoted to exploring arithmetic properties of the coefficients $p_\xi(n)$ defined by

$$\sum_{n=0}^{\infty} p_\xi(n)q^n = \xi(q). \quad (2)$$

It is clear from (1) that $p_\xi(n)$ is even for all $n \geq 1$. In Sections 4 and 5, we present other arithmetic properties of $p_\xi(n)$, including some infinite families of congruences.

2 Preliminaries

McIntosh [12, Theorem 3] proved a number of mock theta conjectures, including

$$\omega(q) = g_3(q, q^2) \quad \text{and} \quad (3)$$

$$\xi(q) = q^2 g_3(q^3, q^6) + \frac{(q^2; q^2)_\infty^4}{(q; q^3)_\infty^2 (q^6; q^6)_\infty}, \quad (4)$$

where

$$g_3(a, q) = \sum_{n=0}^{\infty} \frac{(-q; q)_n q^{n(n+1)/2}}{(a; q)_{n+1} (a^{-1} q; q)_{n+1}}$$

and $\omega(q)$ is the third order mock theta functions given by

$$\omega(q) = \sum_{n=0}^{\infty} \frac{q^{2n(n+1)}}{(q^2 q^4^2)_{n+1}^2}.$$
It follows from (1), (3), and (4) that
\[ \xi(q) = q^2 \omega(q^3) + \frac{(q^2; q^2)_\infty^4}{(q; q)_\infty^2(q^2; q^2)_\infty}. \] (5)

Throughout the remainder of this paper, we define \( f_k := (q^k; q^k)_\infty \) in order to shorten the notation. Combining (5) and (2), we have
\[ \sum_{n=0}^{\infty} p_k(n)q^n = q^2 \omega(q^3) + \frac{f_4^4}{f_2^4 f_6}. \] (6)

We recall Ramanujan’s theta functions
\[ f(a, b) := \sum_{n=\infty}^{\infty} a^{n(n+1)2} b^{n(n-1)2}, \text{ for } |ab| < 1, \]
\[ \phi(q) := f(q, q) = \sum_{n=\infty}^{\infty} q^{n^2} = \frac{f_5^5}{f_1^2 f_4^2}, \text{ and } \] (7)
\[ \psi(q) := f(q, q^3) = \sum_{n=0}^{\infty} q^{n(n+1)/2} = \frac{f_2^2}{f_1}. \] (8)

The function \( \phi(q) \) satisfies many identities, including (see [3, (22.4)])
\[ \phi(-q) = \frac{f_2^2}{f_4}. \] (9)

In some of the proofs, we employ the classical Jacobi’s identity (see [4, Theorem 1.3.9])
\[ f_1^3 = \sum_{n=0}^{\infty} (-1)^n(2n + 1)q^{n(n+1)/2}. \] (10)

We note the following identities which will be used below.

**Lemma 1** The following 2-dissection identities hold.
\[ \frac{1}{f_1^4} = \frac{f_8^4}{f_2^4 f_8^4} + 2q \frac{f_2^2 f_8^2 f_8^4}{f_2^4 f_8^4}, \] (11)
\[ f_2^2 = \frac{f_2^2 f_8^5}{f_2^2 f_1^2 f_4^2} - 2q \frac{f_2^4 f_8^4}{f_8}, \] (12)
\[ \frac{1}{f_1^4} = \frac{f_4^4}{f_2^2 f_8^4} + 4q \frac{f_2^4 f_8^4}{f_2^4 f_8^4}, \] (13)
\[ \frac{f_8}{f_1} = \frac{f_4 f_6 f_{16} f_{24}}{f_2 f_{16} f_{24}} + q \frac{f_6 f_2^2 f_{48}}{f_5 f_{16} f_{24}}, \] (14)
\[ \frac{f_3^2}{f_1} = \frac{f_4 f_6 f_{12} f_{24}^2 + 2q f_2 f_8 f_{24}}{f_2 f_{12} f_{24}^2}, \] (15)
\[
\frac{f_3^3}{f_3} = \frac{f_1^3}{f_1} - 3q \frac{f_2^3 f_3^3}{f_4 f_5^2} \tag{16}
\]
\[
\frac{f_3}{f_1^2} = \frac{f_4^6 f_6^6}{f_2^2 f_1^2} + 3q \frac{f_2^2 f_6 f_1^2}{f_2^2} \tag{17}
\]
\[
\frac{1}{f_1 f_3} = \frac{f_2^2 f_4^2}{f_2^2 f_1 f_6 f_2^2} + q \frac{f_1^2 f_2^2}{f_2^2 f_1 f_6 f_2^2} \tag{18}
\]

**Proof** By Entry 25 (i), (ii), (v), and (vi) in [3, p. 40], we have
\[
\phi(q) = \phi(q^4) + 2q \psi(q^8), \tag{19}
\]
\[
\phi(q)^2 = \phi(q^2)^2 + 4q \psi(q^4)^2. \tag{20}
\]

Using (7) and (8) we can rewrite (19) in the form
\[
\frac{f_5^2}{f_5^2} = \frac{f_8^2}{f_2 f_5^2} + 2q f_4^2,
\]
from which we obtain (11) after multiplying both sides by \(\frac{f_4^2}{f_5^2}\). Identity (12) can be easily deduced from (11) using the procedure described in Section 30.10 of [9].

By (7) and (8) we can rewrite (20) in the form
\[
\frac{f_4^{10}}{f_4^{10}} = \frac{f_4^{10}}{f_2 f_4^2} + 4q f_8^4,
\]
from which we obtain (13).

Identities (14), (15), and (18) are equations (30.10.3), (30.9.9), and (30.12.3) of [9], respectively. Finally, for proofs of (16) and (17) see [13, Lemma 4].

The next lemma exhibits the 3-dissections of \(\psi(q)\) and \(1/\phi(-q)\).

**Lemma 2** We have
\[
\psi(q) = \frac{f_6 f_2}{f_5 f_1 18} + q \frac{f_2^3}{f_9}, \tag{21}
\]
\[
\frac{1}{\phi(-q)} = \frac{f_4 f_6^5}{f_5 f_6 f_1 18} + 2q \frac{f_6 f_1 f_2^3}{f_5^2} + 4q^2 \frac{f_6 f_1 f_3^3}{f_5^3}. \tag{22}
\]

**Proof** Identity (21) is equation (14.3.3) of [9]. A proof of (22) can be seen in [11].
3 Dissections for $p_\xi(n)$

This section is devoted to proving the 2-, 3-, and 4-dissections of (2). We begin with the 2-dissection.

**Theorem 1** We have
\[
\sum_{n=0}^{\infty} p_\xi(2n+1)q^{n+1} = \frac{f_6^5 f_{12}^5}{f_3^8 f_{24}^8} - f(q^{12}) + 4q \frac{f_2^2 f_4^2}{f_1 f_3 f_4}, \quad \text{and} \quad (23)
\]
\[
\sum_{n=0}^{\infty} p_\xi(2n)q^n = q \frac{f_6^8 f_{24}^8}{f_3^8 f_{12}^8} - q^4 \omega(-q^6) + \frac{f_4^5}{f_1 f_3 f_4^5}. \quad (24)
\]

**Proof** We start with equation (4) of [2]:
\[
f(q^8) + 2q\omega(q) + 2q^3\omega(-q^4) = F(q),
\]
where $f(q)$ is the mock theta function
\[
f(q) = \sum_{n=0}^{\infty} \frac{q^{n^2}}{(-q; q)_n^2}
\]
and
\[
F(q) = \frac{\phi(q)\phi(q^2)^2}{f_2^4} = f_2 f_6^4.
\]
Thus,
\[
f(q^{24}) + 2q^3\omega(q^3) + 2q^6\omega(-q^{12}) = F(q^3).
\]
Using (5), it follows that
\[
\sum_{n=0}^{\infty} p_\xi(n)q^{n+1} = F(q^3) - f(q^{24}) - 2q^3\omega(-q^{12}) + 2q \frac{f_2^4}{f_1 f_6^5}. \quad (25)
\]
By (11), we have
\[
F(q^3) = \frac{f_6^8 f_{24}^8}{f_6^8 f_{24}^8} + 2q^3 \frac{f_6^8 f_{12}^8}{f_6^8 f_{24}^8},
\]
which along with (11) allows us to rewrite (25) as
\[
\sum_{n=0}^{\infty} p_\xi(n)q^{n+1} = f_6^8 f_{24}^8 \frac{f_6^8 f_{12}^8}{f_6^8 f_{24}^8} + 2q^3 \frac{f_6^8 f_{12}^8}{f_6^8 f_{24}^8} - f(q^{24}) - 2q^3\omega(-q^{12})
\]
\[+ 2q \frac{f_6^5}{f_2 f_6 f_{16}^5} + 4q^2 \frac{f_2^4 f_{16}^4}{f_2 f_6 f_{16}^5}.
\]
Thus,
\[
\sum_{n=0}^{\infty} p_\xi(2n+1)q^{2n+2} = \frac{f_6^5 f_{24}^5}{f_6^5 f_{24}^5} - f(q^{24}) + 4q^2 \frac{f_2^4 f_{16}^4}{f_2 f_6 f_{16}^5}, \quad \text{and} \quad (26)
\]
\[
\sum_{n=0}^{\infty} p_\xi(2n)q^{2n+1} = \frac{f_6^5 f_{12}^5}{f_6^5 f_{24}^5} - q^3 \omega(-q^{12}) + q \frac{f_6^4}{f_2 f_6 f_{16}^5}. \quad (27)
\]
Dividing (27) by \( q \) and replacing \( q^2 \) by \( q \) in the resulting identity and in (26), we obtain (23) and (24).

The next theorem exhibits the 3-dissection of (2).

**Theorem 2** We have

\[
\sum_{n=0}^{\infty} p_\xi(3n)q^n = \frac{f_2}{f_1} \frac{f_4}{f_3} f_5, \quad (28)
\]

\[
\sum_{n=0}^{\infty} p_\xi(3n+1)q^n = 2\frac{f_3 f_6}{f_1}, \quad \text{and} \quad (29)
\]

\[
\sum_{n=0}^{\infty} p_\xi(3n+2)q^n = \omega(q) + \frac{f_6^4 f_2}{f_6 f_9}. \quad (30)
\]

**Proof** In view of (8), we rewrite (6) as

\[
\sum_{n=0}^{\infty} p_\xi(n)q^n = q^2 \omega(q^3) + \frac{\psi(q)^2}{f_6}.
\]

Using (21), we obtain

\[
\sum_{n=0}^{\infty} p_\xi(n)q^n = q^2 \omega(q^3) + \frac{f_6 f_9^4 f_2}{f_3 f_1^2} + 2q f_9 f_{18} + q^2 \frac{f_6^4 f_2}{f_6 f_9}.
\]

(31)

Extracting the terms of the form \( q^{3n+r} \) on both sides of (31), for \( r \in \{0, 1, 2\} \), dividing both sides of the resulting identity by \( q^r \) and then replacing \( q^3 \) by \( q \), we obtain the desired results.

We close this section with the 4-dissection of (2).

**Theorem 3** We have

\[
\sum_{n=0}^{\infty} p_\xi(4n)q^n = 4q^2 \frac{f_6}{f_3 f_6} - q^2 \omega(-q^3) + \frac{f_6^4 f_2}{f_1 f_3 f_1^2 f_2}, \quad (32)
\]

\[
\sum_{n=0}^{\infty} p_\xi(4n+1)q^n = 2q \frac{f_6}{f_3} \frac{f_3 f_1}{f_2} + 2 \frac{f_6^4 f_2}{f_2 f_3 f_1^2 f_2}.
\]

(33)

\[
\sum_{n=0}^{\infty} p_\xi(4n+2)q^n = \frac{f_6^2}{f_3^2 f_1^2} + \frac{f_6^4 f_2^2}{f_1 f_3 f_1^2 f_2}, \quad \text{and} \quad (34)
\]

\[
2\sum_{n=0}^{\infty} p_\xi(4n+3)q^{n+1} = \frac{f_6^2}{f_3^2 f_1^2} - f(q^6) + 4q \frac{f_6^4 f_2^2}{f_1^2 f_3^2 f_6}.
\]

(35)
Proof In order to prove (32), we use (13) and (18) to obtain the even part of (24), which is given by

\[ \sum_{n=0}^{\infty} p_{\xi}(4n)q^{2n} = 4q^4 \frac{f_6^9}{f_6^3 f_1^2} - q^4 \omega(-q^6) + \frac{f_4^4 f_1^5}{f_6^2 f_6^6 f_2^{12}}. \]

Replacing \( q^2 \) by \( q \) we obtain (32).

Using (13) and (18) we can extract the odd part of (23):

\[ 2 \sum_{n=0}^{\infty} p_{\xi}(4n+1)q^{2n+1} = 4q^4 \frac{f_6^3 f_2^2}{f_6^3} + 4q \frac{f_4^4 f_1^5}{f_6^2 f_6^6 f_2^{12}}. \]

After simplifications we arrive at (33).

Next, extracting the odd part of (24) with the help of (13) and (18) yields

\[ \sum_{n=0}^{\infty} p_{\xi}(4n+2)q^{2n+1} = q \frac{f_6^9 f_2^4}{f_6^3 f_2^{12}} + q \frac{f_1^4 f_2^4}{f_6^2 f_6^6 f_2^{12}}, \]

which, after simplifications, gives us (34).

In order to obtain (35), we use (13) and (18) in (23) to extract its even part:

\[ 2 \sum_{n=0}^{\infty} p_{\xi}(4n+3)q^{2n+2} = \frac{f_1 f_3^7}{f_6^3 f_2^{12}} - f(q^{12}) + 4q^2 \frac{f_4^4 f_2^4}{f_6^2 f_6^6 f_2^{12}}. \]

Replacing \( q^2 \) by \( q \) in this identity, we obtain (35).

4 Arithmetic properties of \( p_{\xi}(n) \)

Our first observation provides a characterization of \( p_{\xi}(3n) \) (mod 4).

**Theorem 4** For all \( n \geq 0 \), we have

\[ p_{\xi}(3n) \equiv \begin{cases} 1 \pmod{4}, & \text{if } n = 0, \\ 2 \pmod{4}, & \text{if } n \text{ is a square}, \\ 0 \pmod{4}, & \text{otherwise}. \end{cases} \]

**Proof** By (28), using (9) and the fact that \( f_k^4 \equiv f_{2k}^2 \pmod{4} \) for all \( k \geq 1 \), it follows that

\[ \sum_{n=0}^{\infty} p_{\xi}(3n)q^n = \frac{f_2 f_2^4}{f_1^2 f_6} = \frac{f_2}{f_2} \frac{f_2^4}{f_2^2} = \frac{f_2}{f_2} \phi(-q) \pmod{4}. \]

By (7), we obtain

\[ \sum_{n=0}^{\infty} p_{\xi}(3n)q^n \equiv \sum_{n=-\infty}^{\infty} (-1)^n q^{n^2} \equiv 1 + 2 \sum_{n=1}^{\infty} q^{n^2} \pmod{4}, \]

which completes the proof.
Theorem 4 yields an infinite family of Ramanujan–like congruences modulo 4.

**Corollary 1** For all primes \( p \geq 3 \) and all \( n \geq 0 \), we have

\[
p_{\xi}(3(pn + r)) \equiv 0 \pmod{4},
\]

if \( r \) is a quadratic nonresidue modulo \( p \).

**Proof** If \( pn + r = k^2 \), then \( r \equiv k^2 \pmod{p} \), which contradicts the fact that \( r \) is a quadratic nonresidue modulo \( p \).

Since \( \gcd(3, p) = 1 \), among the \( p - 1 \) residues modulo \( p \), we have \( \frac{p-1}{2} \) residues \( r \) for which \( r \) is a quadratic nonresidue modulo \( p \). Thus, for instance, the above corollary yields the following congruences:

\[
\begin{align*}
p_{\xi}(9n + 6) & \equiv 0 \pmod{4}, \\
p_{\xi}(15n + k) & \equiv 0 \pmod{4}, \text{ for } k \in \{6, 9\}, \\
p_{\xi}(21n + k) & \equiv 0 \pmod{4}, \text{ for } k \in \{9, 15, 18\}, \\
p_{\xi}(33n + k) & \equiv 0 \pmod{4}, \text{ for } k \in \{6, 18, 21, 30\}.
\end{align*}
\]

**Theorem 5** For all \( n \geq 0 \), we have

\[
p_{\xi}(3n + 1) \equiv \begin{cases} 2 & \text{if } 3n + 1 \text{ is a square}, \\ 0 & \text{otherwise.} \end{cases} \quad \text{(mod 4)}
\]

**Proof** From Theorem 2,

\[
\sum_{n=0}^{\infty} p_{\xi}(3n + 1)q^n = 2 \frac{f_3f_6}{f_1}. \tag{36}
\]

So we only need to consider the parity of

\[
\frac{f_3f_6}{f_1}.
\]

Note that

\[
\frac{f_3f_6}{f_1} \equiv \frac{3}{f_1} = \sum_{n=0}^{\infty} a_3(n)q^n \pmod{2},
\]

where \( a_3(n) \) is the number of 3-core partitions of \( n \) (see [10, Theorem 1]). Thanks to [8, Theorem 7], we know that

\[
a_3(n) \equiv \begin{cases} 1 & \text{if } 3n + 1 \text{ is a square}, \\ 0 & \text{otherwise.} \end{cases} \pmod{2}
\]

This completes the proof.

Theorem 5 yields an infinite family of congruences modulo 4.
Corollary 2 For all primes \( p > 3 \) and all \( n \geq 0 \), we have
\[
p(3pn + r + 1) \equiv 0 \pmod{4},
\]
if \( 3r + 1 \) is a quadratic nonresidue modulo \( p \).

Proof If \( 3(pm + r) + 1 = k^2 \), then \( 3r + 1 \equiv k^2 \pmod{p} \), which would be a contradiction with \( 3r + 1 \) being a quadratic nonresidue modulo \( p \).

For example, the following congruences hold for all \( n \geq 0 \):
\[
p(15n + k) \equiv 0 \pmod{4}, \text{ for } k \in \{7, 13\},
p(21n + k) \equiv 0 \pmod{4}, \text{ for } k \in \{10, 13, 19\},
p(33n + k) \equiv 0 \pmod{4}, \text{ for } k \in \{7, 10, 13, 19, 28\}.
\]

We next turn our attention to the arithmetic progression \( 4n + 2 \) to yield an additional infinite family of congruences.

Theorem 6 For all \( n \geq 0 \), we have
\[
p(4n + 2) \equiv \begin{cases} 
2 & \text{for } n = 6k(3k \pm 1), \\
0 & \text{otherwise}.
\end{cases} \pmod{4}.
\]

Proof From (34), we obtain
\[
\sum_{n=0}^{\infty} p(4n + 2)q^n \equiv \frac{f_6^2}{f_3^2 f_1^2} + \frac{f_{12}^2}{f_3^2 f_6^2} \equiv 2 \frac{f_6^2}{f_3^2} \equiv 2 f_1^2 \equiv 2 (\pmod{4}). \tag{37}
\]

Using Euler’s identity (see [9, Eq. (1.6.1)])
\[
f_1 = \sum_{n=-\infty}^{\infty} (-1)^n q^{n(3n-1)/2}, \tag{38}
\]

we obtain
\[
\sum_{n=0}^{\infty} p(4n + 2)q^n \equiv 2 \sum_{n=-\infty}^{\infty} (-1)^n q^{6n(3n-1)} \pmod{4},
\]

which concludes the proof.

Theorem 6 yields an infinite family of congruences modulo 4.

Corollary 3 Let \( p > 3 \) be a prime and \( r \) an integer such that \( 2r + 1 \) is a quadratic nonresidue modulo \( p \). Then, for all \( n \geq 0 \),
\[
p(4pn + r + 2) \equiv 0 \pmod{4}.
\]

Proof If \( pn + r = 6k(3k \pm 1) \), then \( r \equiv 18k^2 \pm 6k \pmod{p} \). Thus, \( 2r + 1 \equiv (6k \pm 1)^2 \pmod{p} \), which contradicts the fact that \( 2r + 1 \) is a quadratic nonresidue modulo \( p \).
Thanks to Corollary 3, the following example congruences hold for all \( n \geq 0 : 
\begin{align*}
p_\xi(20n + j) &\equiv 0 \pmod{4}, \text{ for } j \in \{6, 14\}, 
p_\xi(28n + j) &\equiv 0 \pmod{4}, \text{ for } j \in \{6, 10, 26\}, 
p_\xi(44n + j) &\equiv 0 \pmod{4}, \text{ for } j \in \{6, 14, 26, 34, 38, 42\}, 
p_\xi(52n + j) &\equiv 0 \pmod{4}, \text{ for } j \in \{10, 14, 22, 30, 38, 42\}. 
\end{align*}

We now provide a mod 8 characterization for \( p_\xi(3n) \).

**Theorem 7** For all \( n \geq 0 \), we have

\[
p_\xi(3n) \equiv \begin{cases} 
1 & \text{if } n = 0, \\
6(-1)^k & \text{if } n = k^2, \\
4 & \text{if } n = 2k^2, n = 3k^2, \text{ or } n = 6k^2, \\
0 & \text{otherwise}.
\end{cases} \pmod{8}
\]

**Proof** By (28), using (7) and (9), we have

\[
\sum_{n=0}^{\infty} p_\xi(3n)q^n = \frac{f_3^6 f_2^4 f_3^4}{f_3^8 f_6^8} \equiv \left( \frac{f_3^2}{f_2} \right)^3 \left( \frac{f_3^2}{f_3} \right)^2 \equiv \phi(-q)^3 \phi(q)^3 \\
\equiv \left( \sum_{n=-\infty}^{\infty} (-1)^n q^{n^2} \right)^3 \left( \sum_{n=-\infty}^{\infty} (-1)^n q^{3n^2} \right)^2 \\
\equiv \left( 1 + 2 \sum_{n=1}^{\infty} (-1)^n q^{n^2} \right)^3 \left( 1 + 2 \sum_{n=1}^{\infty} (-1)^n q^{3n^2} \right)^2 \pmod{8}
\]

which yields

\[
\sum_{n=0}^{\infty} p_\xi(3n)q^n \equiv 1 + 6 \sum_{n=1}^{\infty} (-1)^n q^{n^2} + 4 \left( \sum_{n=1}^{\infty} (-1)^n q^{n^2} \right)^2 \\
+ 4 \sum_{n=1}^{\infty} (-1)^n q^{3n^2} + 4 \left( \sum_{n=1}^{\infty} (-1)^n q^{3n^2} \right)^2 \pmod{8}.
\]

Since

\[
\left( \sum_{n=1}^{\infty} (-1)^n q^{n^2} \right)^2 \equiv \sum_{n=1}^{\infty} q^{2n^2} \pmod{2},
\]

we have

\[
\left( \sum_{n=1}^{\infty} (-1)^n q^{3n^2} \right)^2 \equiv \sum_{n=1}^{\infty} q^{6n^2} \pmod{2}.
\]
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Therefore

$$\sum_{n=0}^{\infty} p_\xi(3n)q^n \equiv 1 + 6 \sum_{n=1}^{\infty} (-1)^n q^{n^2} + 4 \sum_{n=1}^{\infty} q^{2n^2} + 4 \sum_{n=1}^{\infty} (-1)^n q^{3n^2} + 4 \sum_{n=1}^{\infty} q^{6n^2} \quad (\text{mod } 8),$$

which completes the proof.

As with the prior results, Theorem 7 provides an effective way to yield an infinite family of congruences modulo 8.

**Corollary 4** Let $p$ be a prime such that $p \equiv \pm 1 \pmod{24}$. Then,

$$p_\xi(3(pm + r)) \equiv 0 \pmod{8},$$

if $r$ is a quadratic nonresidue modulo $p$.

**Proof** Since $p \equiv \pm 1 \pmod{8}$ and $p \equiv \pm 1 \pmod{12}$, it follows that 2 and 3 are quadratic residues modulo $p$. Thus, $p, 2r, 3r$, and $6r$ are quadratic nonresidues modulo $p$. Indeed, according to the properties of Legendre’s symbol, for $j \in \{1, 2, 3, 6\}$, we have

$$\left( \frac{jr}{p} \right) = \left( \frac{j}{p} \right) \left( \frac{r}{p} \right) = -1.$$

It follows that we cannot have $3(pm + r) = jk^2$, for some $k \in \mathbb{N}$ and $j \in \{1, 2, 3, 6\}$. In fact, $3(pm + r) = jk^2$ would imply $3(pm + r) \equiv 3r \equiv jk^2 \pmod{p}$. However, for $j = 1, 2, 3, 6$, this would imply that $3r, 6r, r$, or $2r$, respectively, is a quadratic residue modulo $p$, which would be a contradiction since 2, 3, and 6 are quadratic residues modulo $p$. The result follows from Theorem 7.

As an example, we note that, for $p = 23$ and all $n \geq 0$, we have

$$p_\xi(69n + k) \equiv 0 \pmod{8}, \text{ for } k \in \{15, 21, 30, 33, 42, 45, 51, 57, 60, 63, 66\}.$$

**Theorem 8** For all $n \geq 0$, we have

$$p_\xi(12n + 4) \equiv p_\xi(3n + 1) \pmod{8}.$$

**Proof** Initially we use (14) to extract the odd part on both sides of (29). The resulting identity is

$$\sum_{n=0}^{\infty} p_\xi(6n + 4)q^n = 2f_2^2 f_3 f_2 f_4 f_{24} \quad (\text{mod } 8).$$

(39)

Using (15) in (39), we obtain

$$\sum_{n=0}^{\infty} p_\xi(12n + 4)q^n = 2f_6^2 f_3 f_5 f_6 = 2f_1^2 f_3 f_5 f_6 f_2^2 f_{12} \equiv 2f_1 f_6 \pmod{8}.$$

The result follows using (29).
Now we present complete characterizations of $p_ξ(48n + 4)$ and $p_ξ(12n + 1)$ modulo 8.

**Theorem 9** For all $n \geq 0$, we have

$$p_ξ(48n + 4) \equiv p_ξ(12n + 1) \equiv \begin{cases} 2(-1)^k \pmod{8}, & \text{if } n = k(3k \pm 1), \\ 0 \pmod{8}, & \text{otherwise}. \end{cases}$$

**Proof** The first congruence follows directly from Theorem 8. Replacing (14) in (29), we obtain

$$\sum_{n=0}^{\infty} p_ξ(3n + 1)q^n = 2f_4f_6f_{16}f_{24} \frac{f_2^2f_8f_{12}f_{48}}{f_2^2f_8f_{12}f_{48}}.$$ 

Extracting the terms of the form $q^{2n}$, we have

$$\sum_{n=0}^{\infty} p_ξ(6n + 1)q^{2n} = 2f_4f_6f_{16}f_{24} \frac{f_2^2f_8f_{12}f_{48}}{f_2^2f_8f_{12}f_{48}},$$

which, after replacing $q^2$ by $q$, yields

$$\sum_{n=0}^{\infty} p_ξ(6n + 1)q^n = 2f_2f_5f_8f_{12} \frac{f_2f_5f_8f_{12}}{f_2f_5f_8f_{12}}. \quad (40)$$

Now we use (15) to obtain

$$\sum_{n=0}^{\infty} p_ξ(12n + 1)q^n = 2f_3^3f_6^4 \frac{f_1f_4}{f_1f_4}$$

$$\equiv 2f_2 \equiv 2 \sum_{n=-\infty}^{\infty} (-1)^n q^{(3n-1)} \pmod{8}, \quad (by \ (38))$$

which completes the proof.

Theorem 9 also provides an effective way to yield an infinite family of congruences modulo 8.

**Corollary 5** For all primes $p > 3$ and all $n \geq 0$, we have

$$p_ξ(48(pn + r) + 4) \equiv p_ξ(12(pn + r) + 1) \equiv 0 \pmod{8},$$

if $12r + 1$ is a quadratic nonresidue modulo $p$.

**Proof** Let $p > 3$ be a prime and $12r + 1$ a quadratic nonresidue modulo $p$. If $pn + r = k(3k \pm 1)$, then $r \equiv 3k^2 \pm k \pmod{p}$, which implies that $12r + 1 \equiv (6k \pm 1)^2 \pmod{p}$, a contradiction. The result follows from Theorem 9.
5 Additional congruences

In this section, we prove several additional Ramanujan–like congruences that are not included in the results of the previous section.

**Theorem 10** For all $n \geq 0$, we have

\[ p_\xi(24n + 19) \equiv 0 \pmod{3}, \quad (41) \]
\[ p_\xi(27n + 18) \equiv 0 \pmod{3}, \quad \text{and} \quad (42) \]
\[ p_\xi(72n + 51) \equiv 0 \pmod{3}. \quad (43) \]

**Proof** Using (15) we can now 2-dissect (40) to obtain

\[
\sum_{n=0}^{\infty} p_\xi(6n + 1)q^n = 2\frac{f_4 f_6}{f_2 f_4} + 4q \cdot \frac{f_6 f_8 f_{12}}{f_2^2},
\]
from which we have

\[
\sum_{n=0}^{\infty} p_\xi(12n + 7)q^{2n+1} = 4q \cdot \frac{f_6 f_8 f_{12}}{f_2^2}.
\]

Now, dividing both sides of the above expression by $q$ and replacing $q^2$ by $q$, we obtain

\[
\sum_{n=0}^{\infty} p_\xi(12n + 7)q^n = 4 \cdot \frac{f_3 f_6 f_6}{f_1^2}.
\]

(44)

Using (17) we rewrite (44) as

\[
\sum_{n=0}^{\infty} p_\xi(12n + 7)q^n = 4 \cdot \frac{f_3^2 f_6^2 f_6}{f_1^2} + 12q \cdot \frac{f_1^2 f_6 f_{12}^2}{f_2^2}.
\]

Taking the odd parts on both sides of the last equation, we are left with

\[
\sum_{n=0}^{\infty} p_\xi(24n + 19)q^n = 12 \cdot \frac{f_3^2 f_6^2 f_6}{f_1^2},
\]
which proves (41).

In order to prove (42), we use (22) to extract the terms of the form $q^{3n}$ of (28). The resulting identity is

\[
\sum_{n=0}^{\infty} p_\xi(9n)q^{3n} = \frac{f_6^3 f_6^3}{f_1^2 f_{18}^2},
\]
which, after replacing $q^3$ by $q$ and using (8), yields
\[
\sum_{n=0}^{\infty} p_\xi(9n)q^n = \frac{f_3^2 f_4^5}{f_1^1 f_6^6} \equiv \frac{f_2^2 f_5^5}{f_1^1 f_6^6} = \psi(q) \frac{f_3^5}{f_6^3} \pmod{3}.
\]

By (8), we have
\[
\sum_{n=0}^{\infty} p_\xi(9n)q^n \equiv \frac{f_5^5}{f_6^6} \sum_{n=0}^{\infty} q^{n(n+1)/2} \pmod{3}.
\]

Since $n(n+1)/2 \not\equiv 2 \pmod{3}$ for all $n \geq 0$, all terms of the form $q^{3n+2}$ in the last expression have coefficients congruent to 0 (mod 3), which proves (42).

We now prove (43). Replacing (22) in (28) and extracting the terms of the form $q^{3n+2}$, we obtain
\[
\sum_{n=0}^{\infty} p_\xi(9n + 6)q^{3n+2} = 4q^2 \frac{f_3^3}{f_5^3}.
\]

Dividing both sides of (45) by $q^2$ and replacing $q^3$ by $q$, we have
\[
\sum_{n=0}^{\infty} p_\xi(9n + 6)q^n = 4 \frac{f_3^3}{f_5^3}.
\]

Now we use (11) to extract the odd part of (46) and obtain
\[
\sum_{n=0}^{\infty} p_\xi(18n + 15)q^n = 8 \frac{f_3^3 f_5^3 f_8^2}{f_1^1 f_4^4}.
\]

Since $f_3^3 \equiv f_3 \pmod{3}$, we have
\[
\sum_{n=0}^{\infty} p_\xi(18n + 15)q^n \equiv 2 \frac{f_3^3 f_5^3 f_8^2}{f_1^1 f_4^4} \pmod{3}.
\]

Using (15) we obtain
\[
\sum_{n=0}^{\infty} p_\xi(36n + 15)q^n \equiv 2 \frac{f_3^3 f_4^4 f_6^6}{f_1^1 f_{12}^{12}} \pmod{3}.
\]

Since the odd part of (17) is divisible by 3, then the coefficients of the terms of the form $q^{2n+1}$ in $\sum_{n=0}^{\infty} p_\xi(36n + 15)q^n$ are congruent to 0 modulo 3. This completes the proof of (43).

We now prove a pair of unexpected congruences modulo 5 satisfied by $p_\xi(n)$.

**Theorem 11** For all $n \geq 0$, we have
\[
\begin{align*}
p_\xi(45n + 33) & \equiv 0 \pmod{5}, \\
p_\xi(45n + 42) & \equiv 0 \pmod{5}.
\end{align*}
\]
Proof By (46), we have
\[ \sum_{n=0}^{\infty} p_\xi(9n + 6)q^n = 4 \frac{r_3^1}{l_4} = 4 \frac{f_3^1 f_6^3}{f_5} \equiv 4 \frac{f_3^1 f_6^3}{f_5} \mod 5. \]

Thanks to Jacobi’s identity (10) we know
\[ f_3^1 f_6^3 = \sum_{j,k=0}^{\infty} (-1)^{j+k}(2j + 1)(2k + 1)q^{3(j+1)+k(k+1)/2}. \]

Note that, for all integers \( j \) and \( k \), \( 3j(j + 1) \) and \( k(k + 1)/2 \) are congruent to either 0, 1 or 3 modulo 5. The only way to obtain \( 3j(j + 1) + k(k + 1)/2 = 5n + 3 \) is the following:
- \( 3j(j + 1) \equiv 0 \mod 5 \) and \( k(k + 1)/2 \equiv 3 \mod 5 \), or
- \( 3j(j + 1) \equiv 3 \mod 5 \) and \( k(k + 1)/2 \equiv 0 \mod 5 \).

Thus, \( j \equiv 2 \mod 5 \) or \( k \equiv 2 \mod 5 \) in all possible cases, and this means
\[ (2j + 1)(2k + 1) \equiv 0 \mod 5. \]

Therefore, for all \( n \geq 0 \), \( p_\xi(45n + 33) = p_\xi(9(5n + 3) + 6) \equiv 0 \mod 5 \), which is (47).

In order to complete the proof of (48), we want to see when
\[ 3j(j + 1) + k(k + 1)/2 = 5n + 4. \]

Four possible cases arise:
- \( k \equiv 1 \mod 5 \) and \( j \equiv 2 \mod 5 \),
- \( k \equiv 3 \mod 5 \) and \( j \equiv 2 \mod 5 \),
- \( j \equiv 1 \mod 5 \) and \( k \equiv 2 \mod 5 \) or
- \( j \equiv 3 \mod 5 \) and \( k \equiv 2 \mod 5 \).

In all four cases above, either \( j \equiv 2 \mod 5 \) or \( k \equiv 2 \mod 5 \). So
\[ (2j + 1)(2k + 1) \equiv 0 \mod 5 \]
in all these cases. Therefore,
\[ p_\xi(45n + 42) = p_\xi(9(5n + 4) + 6) \equiv 0 \mod 5, \]
which completes the proof of (48).

Next, we prove three congruences modulo 8 which are not covered by the above results.

**Theorem 12** For all \( n \geq 0 \), we have
\[ p_\xi(16n + 14) \equiv 0 \mod 8, \]
\[ p_\xi(24n + 13) \equiv 0 \mod 8, \]
\[ p_\xi(24n + 22) \equiv 0 \mod 8. \]
Proof Initially we prove (49). From (34) and (7) we have
\[ \sum_{n=0}^{\infty} p\xi(4n+2)q^n \equiv \frac{f_3^2 f_6^2}{f_2^2} + \frac{f_2^2}{f_2} \phi(q)^2 \pmod{8}. \]

Now we can use (11), (12), and (20) to extract the terms involving \( q^{2n+1} \) from both sides of the previous congruence:
\[ \sum_{n=0}^{\infty} p\xi(8n+6)q^{2n+1} \equiv -2q^3 \frac{f_6^2 f_{12}^2}{f_6^2 f_2} + 2q^3 \frac{f_6^{10} f_{12} f_{24}^2}{f_6^2 f_2^2 f_6 f_{12}} + 4q f_6^2 f_{12} f_{24}^2 \pmod{8}. \]

After dividing both sides by \( q \) and then replacing \( q^2 \) by \( q \), we are left with
\[ \sum_{n=0}^{\infty} p\xi(8n+6)q^n \equiv -2q f_6^2 f_{12}^2 + 2q^3 f_6^{10} f_{12} f_{24} + 4q f_6^2 f_{12} f_{24} \pmod{8}. \]

whose odd part is congruent to 0 modulo 8, which implies (49).

In order to prove (50), we use (15) to obtain the even part of identity (40), which is
\[ \sum_{n=0}^{\infty} p\xi(12n+1)q^n = 2 \frac{f_2^2 f_6^2}{f_1 f_{12}}. \]

Now, employing (13), we obtain the odd part of the last identity, which is
\[ \sum_{n=0}^{\infty} p\xi(24n+13)q^n = 8 \frac{f_2^2 f_6^2 f_4^4}{f_1^2 f_6^2}, \]
which implies (50).

Now we prove (51). We employ (15) in (39) to obtain
\[ \sum_{n=0}^{\infty} p\xi(12n+10)q^n = 4 \frac{f_2^2 f_6^2 f_2^2}{f_1 f_6^2}. \]

By (12) and (13), we rewrite (52) in the form
\[ \sum_{n=0}^{\infty} p\xi(12n+10)q^n = 4 \frac{f_2^2 f_6^2 f_2^2}{f_1 f_6^2} \left( \frac{f_4^4}{f_1^2 f_6^2} + 4q f_6^2 f_{12}^2 \right) \left( \frac{f_6 f_{24}^2}{f_1 f_2 f_{24}} - 2q^3 f_6 f_{24}^2 \right), \]
from which we obtain
\[ \sum_{n=0}^{\infty} p\xi(24n+22)q^{2n+1} = 4 \frac{f_2^2 f_6^2}{f_6^2} \left( -2q^3 f_6 f_{12} f_{24} + 4q f_6^2 f_{12} f_{24} \right). \]
Dividing both sides by $q$ and replacing $q^2$ by $q$, we are left with

$$\sum_{n=0}^{\infty} p_\xi(24n + 22) q^n = -8q \frac{f_2^{14} f_6 f_{24}}{f_4^{11} f_8 f_{12}} + 16 \frac{f_2^4 f_4 f_6^3}{f_4^7 f_8 f_{24}}.$$  

which implies (51).

We close this section by proving a congruence modulo 9.

**Theorem 13** For all $n \geq 0$, we have

$$p_\xi(96n + 76) \equiv 0 \pmod{9}. \quad (53)$$

**Proof** We use (21) to extract the terms of the form $q^{3n+1}$ from (32). The resulting identity is

$$\sum_{n=0}^{\infty} p_\xi(12n + 4) q^{3n+1} = 2q \frac{f_6^6 f_9 f_{18}}{f_3^3 f_{12}}.$$  

which, after dividing by $q$ and replacing $q^3$ by $q$, yields

$$\sum_{n=0}^{\infty} p_\xi(12n + 4) q^n = 2 \frac{f_2^6 f_6 f_3}{f_4^2 f_1^2} = 2 \frac{f_2^6 f_6 f_3}{f_4^2 f_1^2} \cdot \frac{1}{f_1 f_1}.$$  

Using (13) and (14), we extract the even part on both sides of the above identity to obtain

$$\sum_{n=0}^{\infty} p_\xi(24n + 4) q^n = 2 \frac{f_2^{13} f_6^2 f_{12}^2}{f_4^7 f_8 f_{24}} + 8q \frac{f_2^3 f_6 f_{24}}{f_4^7 f_8 f_{12}}$$

$$\equiv 2 \frac{f_2^{13} f_6 f_{24}}{f_4^7 f_8 f_{12}} + 8q \frac{f_2^3 f_{24} f_3}{f_8 f_{12} f_3} \pmod{9},$$

which implies (53).
6 Concluding remarks

Computational evidence indicates that $p_\xi(n)$ satisfies many other congruences. The interested reader may wish to consider the following two conjectures.

**Conjecture 1**

$$\sum_{n=0}^{\infty} p_\xi(8n + 3)q^n \equiv 2 \sum_{n=0}^{\infty} q^{3n(n+1)/2} \pmod{3}$$

**Conjecture 2**

$$\sum_{n=0}^{\infty} p_\xi(32n + 12)q^n \equiv 6 \sum_{n=0}^{\infty} q^{3n(n+1)/2} \pmod{9}$$

Clearly, once proven, Conjectures 1 and 2 would immediately lead to infinite families of Ramanujan–like congruences. Moreover, Conjecture 2 would immediately imply Theorem 13 since $96n + 76 = 32(3n + 2) + 12$ while the right–hand side of Conjecture 2 is clearly a function of $q^3$. The same argument would imply that, for all $n \geq 0$,

$$p_\xi(96n + 44) \equiv 0 \pmod{9}$$

since $96n + 44 = 32(3n + 1) + 12$.

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