The maximum queue length for heavy-tailed service times in the M/G/1 FB queue

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Abstract
This paper treats the maximum queue length \( M \), in terms of the number of customers present, in a busy cycle in the M/G/1 queue. The distribution of \( M \) depends both on the service time distribution and on the service discipline. Assume that the service times have a logconvex density and the discipline is Foreground Background (FB). The FB service discipline gives service to the customer(s) that have received the least amount of service so far. It is shown that under these assumptions the tail of \( M \) is bounded by an exponential tail. This bound is used to calculate the time to overflow of a buffer, both in stable and unstable queues.

1 Introduction

In a stochastic process often the extreme values rather than the usual values are of great interest. Large values of the queue length may ask for extraordinary measures such as the allocation of auxiliary storage space. In finite waiting room systems a natural question is: what is the probability that the queue length will

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exceed a specific buffer size in a certain time period? Unlike the workload process, the queue length process is determined by the service discipline.

In this paper we study the maximum queue length $M$, measured in number of customers present in the system, in a busy period. The busy period maximum was studied by Cohen [5], who gave an integral representation of the distribution of $M$ in the M/G/1 FIFO queue. For the M/M/1 queue with load $\rho < 1$, relation (2.50) in Cohen [5] yields a simple expression for the exceedance probabilities:

\[
P(M > n) = \rho^n (1 - \rho) / (1 - \rho^{n+1}).
\]  

(1)

There is a growing interest in models with heavy-tailed service times, since statistical data analysis has provided convincing evidence of heavy-tailed traffic characteristics in high-speed communication networks. For heavy-tailed service times, the performance of the FIFO discipline decreases and it is natural to consider non-FIFO service disciplines.

In this paper the service times have a log-convex density $f$, i.e. log $f$ is convex. Both heavy-tailed distributions and light-tailed distributions may occur. Righter [8] has shown that for queues with log-convex densities the Foreground Background (FB) service discipline minimises the queue length, see Theorem 2 below. The FB service discipline gives service to the customers that have received the least amount of service so far. If there are $n$ such customers, then they are served simultaneously at rate $1/n$. When the age of a customer is the amount of service he has received, the FB discipline gives service to the ‘youngest’ customer(s) present in the system.

Our main result is that in the M/G/1 queue with a log-convex density operating under the FB service discipline the tail of the maximum queue length $M$ in a busy period is bounded by an exponential tail:

\[
P(M > n) \leq \rho^n,
\]  

(2)

where $\rho$ is the load of the system.

Interestingly, the upper bound does not depend on the precise form of the distribution. Note furthermore that (2) is comparable to the precise value of the exceedance probabilities for exponential service times in (1).

By the regenerative structure of the queue length process, the maximum queue length over a busy period is related to the maximum over the time interval $[0, t]$ for $t \to \infty$, see the survey article Asmussen [1]. Using the upper bound (2) we show that in case of service times with heavy-tailed log-convex densities,
the time to overflow of a buffer is of another order in the FB queue than in the FIFO queue. This illustrates the idea that using the FB discipline instead of FIFO may increase the performance of the queue considerably in case of heavy tails.

The class of log-convex densities contains both heavy-tailed distributions like Pareto distributions (with density \( f(x) = (1 + x)^{-\alpha} \) for \( \alpha > 1 \)) and Weibull distributions with tail \( 1 - F(x) = \exp(-x^\beta) \) for \( \beta \in (0, 1) \) and light-tailed distributions like the gamma distribution with density \( f(x) = c(\alpha, \beta)x^\beta \exp(-\alpha x) \) for \( \beta \geq 0, \alpha, c(\alpha, \beta) > 0 \). Equivalently, distributions with a log-convex density may be characterised by a decreasing likelihood ratio and have been called DLR distributions in the literature.

Borst et al. [4] and Zwart and Boxma [11] discuss service disciplines in the case of heavy-tailed service times. For more results on the FB discipline, also known as FBPS or LAST, see Kleinrock [7] and the survey paper Yashkov [10].

The paper is organised as follows. In Section 2 we establish the notation and prove the basic proposition. The inequality (2) is proved in Section 3 and a sharper bound depending on the Laplace transform of the service-time distribution is established. In Section 4 we use (2) to obtain asymptotics for the maximum queue length \( M(t) \) over the interval \( [0, t] \) for \( t \to \infty \) and apply these to study the overflow time of the buffer in a stable queue. In Section 5 we describe a coupling that allows us to apply the results of Section 4 to the time to overflow in an unstable queue.

2 Preliminaries

In this section we establish a relation between the maximum queue length in the M/G/1 FB queue and in a related queue. This relation is the basis of the theorems in the next section.

Consider an M/G/1 FB queue with Poisson arrival rate \( \lambda \) and i.i.d. service times \( B_1, B_2, \ldots \) with distribution function \( F \). Let \( B \) denote a generic service time and let the load \( \rho = \lambda EB \) satisfy \( \rho < 1 \). Assume the queue is empty at time 0 and let \( M \) be the maximum queue length in the first busy period. In order to determine an upper bound for the tail probabilities \( P(M > n) \), we compare the M/G/1 FB queue with a queue with the same arrivals and service times, but with an alternative service discipline, FB∗ say, defined as follows:
the customer that starts the busy period has the lowest priority, and is
only served when there are no other customers present in the queue

• the other customers are served according to the FB discipline.

Let \( M^* \) denote the maximum queue length in a busy period in the M/G/1 FB* queue. The basic idea of this paper is contained in the following proposition that relates the systems FB and FB*.

**Proposition 1** For the discipline FB* above

\[ P(M^* > n) = 1 - Ee^{-\lambda P(M > n-1)}B. \]

**Proof** By definition of the discipline FB*, the busy period consists of two (disjoint) types of periods: periods in which only the customer that started the busy period is served and periods during which only other customers are served. The latter we call sub-busy periods. Let the random variable \( K \) denote the number of sub-busy periods in the busy period. The sub-busy periods are distributed as M/G/1 FB busy periods with one additional customer that is not served. Let the maximum queue lengths in the sub-busy periods be \( M_1 + 1, M_2 + 1, \ldots, M_K + 1 \). These are i.i.d. and have the same distribution as \( M + 1 \).

![Figure 1: A realisation of the busy period in the M/G/1 FB* queue with \( K = 2 \).](image)

Conditional on the first service time \( B_1 = x \), the number of sub-busy periods \( K \) is a Poisson distributed random variable with parameter \( \lambda x \). Indeed, \( K \) is the number of times the service of the first customer is interrupted. By the memoryless property of the arrival process, the times between two interruptions are independent and \( \exp(\lambda) \) distributed.

Given that \( K = k \) and \( B_1 = x \) the maximum queue length \( M^* \) is the maximum of the \( k \) independent variables \( M_1 + 1, \ldots, M_k + 1 \) and hence

\[ P(M^* \leq n \mid K = k, B_1 = x) = (P(M < n))^k. \]

Since \( K \) is Poisson distributed for given \( x \) we find

\[ P(M^* \leq n, K = k \mid B_1 = x) = (P(M < n))^k \frac{(\lambda x)^k}{k!} e^{-\lambda x}. \]
Summing over $k$ yields a simple expression

$$P(M^* \leq n \mid B_1 = x) = \sum_{k=0}^{\infty} (P(M < n))^k \frac{(\lambda x)^k}{k!} e^{-\lambda x} = e^{-\lambda x P(M > n - 1)}$$

and

$$P(M^* \leq n) = \int_{0}^{\infty} e^{-\lambda x P(M > n - 1)} dF(x) = E e^{-\lambda P(M > n - 1) B}.$$

The result now follows. \qed

3 Theorems

In this section we derive exponential upper bounds for the tails of the maximum queue length $M$ by combining Proposition 1 with Theorem 2 below.

The following theorem is Theorem 13.D.8 of Righter [8] in a form adapted to the present setting. It states that for service time distributions with a log-convex density, the FB discipline is optimal for minimising the queue length in a strong sense.

As is usual, we restrict attention to service disciplines that cannot look into the future. This means that each instant the choice of the customer to be served is measurable with respect to the filtration $\{F_t, t \geq 0\}$ where $F_t$ is the $\sigma$-algebra generated by the arrival and departure times and ages of the customers up to time $t$.

**Theorem 2 (Righter)** Let $\pi$ be a service discipline and let $N_{FB}(t)$ and $N_{\pi}(t)$ denote the queue lengths at time $t$ in $G/GI/1$ queues with disciplines $FB$ and $\pi$, respectively, under the same interarrival and service time distribution. If the service time distribution has a log-convex density, then there exist processes $\{\tilde{N}_{FB}(t), t \geq 0\}$ and $\{\tilde{N}_{\pi}(t), t \geq 0\}$ such that $\{\tilde{N}_{FB}(t), t \geq 0\} \overset{d}{=} \{N_{FB}(t), t \geq 0\}$, $\{\tilde{N}_{\pi}(t), t \geq 0\} \overset{d}{=} \{N_{\pi}(t), t \geq 0\}$ and

$$P(\tilde{N}_{FB}(t) \leq \tilde{N}_{\pi}(t), t \geq 0) = 1.$$

On applying Theorem 2 to Proposition 1 it is seen that $P(M > n) \leq P(M^* > n)$. So we find

**Theorem 3** Let $M$ be the maximum queue length in the busy period in the $M/G/1$ FB queue. If the service time distribution has a log-convex density, then
the exceedance probabilities \( r_n := P(M > n) \) satisfy

\[
r_{n+1} \leq 1 - E e^{-\lambda r_n B}, \quad n = 0, 1, 2, \ldots.
\] (3)

**Theorem 4** Let \( M \) denote the maximum queue length in the busy period in the \( M/G/1 \ FB \) queue with workload \( \rho = \lambda E B < 1 \). If the service times have a log-convex density, then \( r_n = P(M > n) \) satisfies

\[
r_n \leq \rho^n, \quad n = 0, 1, \ldots.
\] (4)

In fact, \( r_0 = 1, \quad r_1 = 1 - E e^{-\lambda B} \) and

\[
r_n \leq \rho r_{n-1}, \quad n \geq 1.
\] (5)

**Proof** By Theorem 3 we have \( r_{n+1} \leq \phi(r_n) \), where \( \phi(x) = 1 - E \exp(-\lambda B x) \).

Furthermore \( \phi(0) = 0 \) and

\[
\phi'(x) = E \lambda B \exp(-\lambda x B) \leq \lambda E B = \rho < 1.
\] (6)

Hence \( r_{n+1} \leq \phi(r_n) = \int_0^{r_n} \phi'(u) du \leq \rho r_n \).

This is (5), which implies (4) by induction since \( r_0 = 1 \). The probability \( r_1 \) may be computed exactly. The first interarrival time \( U \) is \( \exp(\lambda) \)-distributed and

\[
r_1 = P(M \geq 2) = P(B_1 > U) = 1 - P(U \geq B_1) = 1 - \int_0^\infty e^{-\lambda x} dF(x) = 1 - E e^{-\lambda B}.
\]

This proves the theorem. \( \square \)

The approximation \( 1 - E \exp(-\lambda B x) \approx \rho x \) in the proof of Theorem 4 is quite good for \( x \) close to zero, since \( \rho \) is the slope of \( \phi(x) = 1 - E \exp(-\lambda B x) \) in \( x = 0 \).

The following corollary to Theorem 4 gives even sharper upper bounds for \( P(M > n) \).

**Corollary 5** Let \( q_0 = 1 \) and set \( q_{n+1} = 1 - E e^{-\lambda B q_n} \) for \( n \geq 1 \). Then

\[
P(M > n) \leq q_n \leq (1 - E e^{-\lambda B}) \rho^{n-1}, \quad n \geq 1.
\]

**Proof** By Theorem 4 induction and the fact that \( 1 - E \exp(-\lambda B x) \) is increasing in \( x \), we have for \( n \geq 0 \)

\[
P(M > n) \leq 1 - E e^{-\lambda P(M>n-1)B} \leq 1 - E \exp(-\lambda q_{n-1} B) = q_n.
\]
Inequality (6) and induction yield \( q_{n+1} \leq \rho q_n \leq \rho^n q_1 = (1 - Ee^{-\lambda B})\rho^n \).

**Example** Numerical calculations show that the bounds \( q_n \) in Corollary 5 may be significantly better than the bounds in Theorem 4. For service times with the Pareto density \( 3(1 + x)^{-4} \) and arrival rate \( \lambda = 1.8 \), the load is \( \rho = 0.9 \) and 
\[
(1 - E\exp(-\lambda B))\rho^{99}/q_{100} \approx 7.5.
\]

If the service times in one queue are stochastically smaller than those in another queue with the same arrival rate and service discipline, the maximum queue length in the first queue is stochastically smaller than that in the second queue. Hence the upper bounds in this section also hold for service times that are stochastically smaller than a service time with a log-convex density. The next corollary states this idea for the Pareto distribution.

**Corollary 6** Consider an M/G/1 FB queue with arrival rate \( \lambda \) and generic service time \( B \). Let \( M \) denote the maximum queue length in a busy period. Let \( \alpha > 2 \) and \( c > 0 \). If
\[
P(B > x) \leq \frac{1}{(1 + cx)^{\alpha}}, \quad x \geq 0,
\]
then \( P(M > n) \leq \theta^n \) for all \( n \geq 0 \) where \( \theta = \lambda/((\alpha - 1)\alpha c^2)\).

**4 Asymptotics for the maximum queue length over an interval**

In this section we present asymptotics for \( M(t) \), the maximum queue length in the interval \((0, t)\), for \( t \to \infty \). These are applied to calculate the time to overflow in a finite-buffer system.

A stochastic process \( X(t) \) is called a *regenerative process* if there exists a (possibly delayed) renewal process with epochs \( 0 \leq T_0 < T_1 < \cdots \) such that the cycles
\[
\{X(t + T_{i-1})\}_{0 \leq t \leq T_i - T_{i-1}}
\]
are i.i.d. for \( i = 1, 2, \ldots \).
Consider an M/G/1 FB queue with arrival rate $\lambda$, i.i.d. service times with log-convex density $f$, generic service time $B$ and workload $\rho < 1$. The initial state is arbitrary. Since the interarrival times are memoryless, the queue length process is a regenerative process, where a cycle consists of an idle and a busy period. The expected cycle length $\mu$ is given by 

$$
\mu = \lambda^{-1} + EB/(1 - \rho) = \lambda^{-1}(1 - \rho)^{-1}.
$$

Let $M(t)$ denote the queue length over the time interval $[0, t]$. Proposition VI.4.7 in Asmussen [2] then states that for $t \to \infty$

$$
\sup_x |P(M(t) \leq x) - P(M \leq x)^t/\mu| \to 0. \quad (7)
$$

**Example (Buffer overflow)** We are interested in the time to overflow of a buffer of size $d \geq 1$. Denote by $t_{d,p} = \inf\{t : P(M(t) > d) \geq p\}$ the lower $p$th quantile for the (random) time to overflow. Since $t_{d,p} \to \infty$ as $d \to \infty$, we have by (7) that

$$
P(M(t_{d,p}) \leq d) = P(M \leq d)^{t_{d,p}/\mu}(1 + o(1))
$$

as $d \to \infty$. Theorem 4 then yields

$$
t_{d,p} \sim \frac{\mu \log(1 - p)}{\log P(M \leq d)} \geq \frac{\mu \log(1 - p)}{\log(1 - \rho^d)} \quad (8)
$$

where the asymptotic equality holds as $d \to \infty$. The RHS of (8) may be sharpened by using $q_n$ instead of $\rho^n$, where $q_n$ is defined in Corollary 5.

We compare the time to overflow in the FB and the FIFO queue. Set $p = \frac{1}{2}$. Then $t_{d,p}$ is the median of the time to overflow. Suppose the service times have a Pareto distribution with distribution function $1 - (x + 1)^{-3}$ and the arrival rate $\lambda$ is 1.8, so that the load of this queue is $\rho = 0.9$. Time is measured in milliseconds. Let the buffer size $d$ be 1000. Using the asymptotic inequality (8), we find for the FB discipline a value of $t_{d,p}$ larger than $10^{46}$. This is approximately $10^{35}$ years.

Now consider the same queue with the FIFO discipline. Occasionally customers with very large service times arrive. Such customers may cause a buffer overflow. For large values of $d$, the probability that a customer with service time at least $d/\lambda + 3\sigma$, with $\sigma = \sqrt{d/\lambda}$, will cause a buffer overflow is larger than 0.99. The probability that such a customer arrives during the interval $[0, t]$ is

$$
p(t) = 1 - \exp(-\lambda t(1 - F(d/\lambda + 3\sqrt{d/\lambda}))).
$$

Solving $p(t_1) = 1/2$ we find $t_1 < 10^8$, approximately one day.
For the M/G/1 FB queue with load $\rho = 0.9$ and Weibull service time distribution $F(x) = 1 - \exp(-x^\beta)$, the value of $t_0$ is larger than $10^{48}$ for $\beta = 1/4$ and larger than $10^{46}$ for $\beta = 1/2$. For the FIFO queue with the same service time distribution and load, $t_1$ is smaller than $10^3$ for $\beta = 1/4$ and approximately $10^{20}$ for $\beta = 1/2$.

5 Buffer overflow in unstable queues

In this section we apply the results of section 4 to study the time to overflow of the buffer in unstable queues.

When the condition $\rho \leq 1$ is violated, the workload asymptotically grows with rate $\rho - 1$. Balkema and Verwijmeren [3] showed that in the overloaded M/G/1 queue under the FIFO discipline the queue length asymptotically grows linearly with rate $\lambda$, where $\lambda$ is the arrival intensity. The queue length in the FB queue grows linearly as well, but with a smaller rate. By the priority rule of the FB discipline, customers with service time less than the critical value $c^*$, where $c^* = \inf\{ c : \lambda E(B \wedge c) \geq 1 \}$, will a.s. leave the queue, since they are not hindered by customers with long service times. Customers with service time larger than $c^*$ have a positive probability of being stuck in the queue forever. Here we are interested in the time to overflow of a buffer of size $d$ in an overloaded queue.

Consider a queue with heavy-tailed service times and a small arrival rate. Assume the stability condition $\rho < 1$ is violated. This happens because of the mass in the far right tail of the service time distribution. By modifying the right tail, we shall obtain an alternative service-time distribution that does satisfy the condition $\rho < 1$. The maximum queue length in the alternative queue is comparable to that in the original queue for very long periods of time if the arrival process is light. We shall construct the alternative service times in a way such that they have a log-convex density. Then Theorem 4 and (7) are be applied to bound the tail of the maximum queue size.

For the comparison between the two queues we use a coupling argument described in Subsection 5.1 below. Subsection 5.2 illustrates these ideas with an example.

Extreme value behaviour of $M$ in the M/G/1 FIFO queue was found by Cohen [4] under the assumption of the existence of an exponential moment. Heavy-traffic limits in this case were considered by Iglehart [6]. Serfozo [9] studied the behaviour of $M(t)$ both in stable and unstable GI/G/1 FIFO queues.
5.1 The coupling argument

**Theorem 7** Let $M_F(t)$ and $M_G(t)$ be the maximum queue lengths over the interval $[0, t]$ in two $M/G/1$ FB queues with service-time distribution functions $F$ and $G$, and the same arrival rate $\lambda$. Assume $F \wedge p = G \wedge p$ for some $p \in (0, 1)$. Then

$$M_F(t) \leq_{st} M_G(t) + K_p(t)$$

where $K_p(t)$ is a Poisson process with rate $\lambda(1 - p)$.

**Proof** Let $N_F(t)$ and $N_G(t)$ be the queue lengths at time $t$. We prove that

$$N_F(t) \leq_{st} N_G(t) + K_p(t) \quad (9)$$

where $K_p(t)$ is a Poisson process with rate $\lambda(1 - p)$. Using the fact that $K_p(t)$ is non-decreasing, the theorem then follows from

$$M_F(t) = \max_{s \leq t} \{N_F(s)\} \leq_{st} \max_{s \leq t} \{N_G(s) + K_p(s)\} \leq M_G(t) + K_p(t).$$

The coupling is standard: let $U_1, U_2, \ldots$ be a sequence of i.i.d. random variables distributed uniformly on $[0, 1]$. Define the sequences $B_1, B_2, \ldots$ and $B_1^*, B_2^*, \ldots$ by setting $B_k = F^{-1}(U_k)$ and $B_k^* = G^{-1}(U_k)$ for $k = 1, 2, \ldots$. Then $B_k$ and $B_k^*$ have distribution functions $F$ and $G$ respectively, for all $k$, and

$$B_k \wedge p \equiv B_k^* \wedge p, \quad k = 1, 2, \ldots.$$  

Now let $Q$ and $Q^*$ be the $M/G/1$ FB queues with the same arrival process with rate $\lambda$ and service times introduced above. Let $N(t)$ and $N^*(t)$ be the queue lengths at time $t$ and let $N_p(t)$ and $N_p^*(t)$ be the part of the queue lengths formed by customers younger than $F^{-1}(p)$. By the coupling of the service times and the fact that the FB discipline favours customers younger than $F^{-1}(p)$ over customers older than $F^{-1}(p)$, we have $N_p(t) = N_p^*(t)$. Let $K_p(t)$ be the Poisson arrival process of customers with service time larger than $F^{-1}(p)$. Then $N(t) \leq N_p(t) + K_p(t)$ and $N_p^*(t) \leq N^*(t)$. Hence

$$N_F(t) \overset{d}{=} N(t) \leq N^*(t) + K_p(t) \overset{d}{=} N_G(t) + K_p(t).$$

We conclude the proof by observing that the rate of $K_p(t)$ is $\lambda(1 - p)$. \qed

Note that the random variables $M_G(t)$ and $K_p(t)$ in Theorem 7 are dependent.
5.2 Example

In this subsection we compute an upper bound for the time to overflow of an M/G/1 FB queue with Pareto service density \( f(x) = 1/(x+1)^2 \), so that \( \rho = \infty \).

For \( a > 1 \) define

\[
  g_a(x) = \begin{cases} 
    \frac{1}{(x+1)^2}, & x \leq a \\
    \frac{1}{(a+1)^2}e^{-(x-a)/(a+1)}, & x > a.
  \end{cases}
\]

Then \( f(x) = g_a(x) \) for \( x \leq a \) and it may be shown that \( g_a \) is a probability density which is continuous and log-convex. Now let \( M(t) \) and \( M_a(t) \) denote the maximum queue lengths over the interval \([0,t]\) of two M/G/1 FB queues with service-time densities \( f \) and \( g_a \) respectively, and with the same arrival rate \( \lambda \). Setting \( p = 1 - (a + 1)^{-1} \), we have by Theorem 7

\[
  M(t) \leq_{st} M_a(t) + K_a(t)
\]

where \( K_a(t) \) is a Poisson process with rate \( \lambda/(1 + a) \). Hence for any \( x_1, x_2 \geq 0 \) such that \( x_1 + x_2 = x \) we have

\[
  P(M(t) > x) \leq P(M_a(t) > x_1) + P(K_a(t) \geq x_2).
\]

Setting \( a = 10^{40} \) and \( \lambda = 0.01 \), we find for the queue with density \( g_a \) that \( \rho = \lambda EB = \lambda(\log(a + 1) + 1) = 0.9 \ldots \). Since for this queue \( \rho < 1 \) and \( g_a \) is log-convex, we may apply (7) and (8) to \( M_a(t) \). For a buffer of size \( d = 1000 \) and \( p = 0.01 \) we find that \( t_{d-1,p} = 3 \cdot 10^{42}(1 + \varepsilon) \), where the error \( \varepsilon = \varepsilon(d,p) \) is determined by the asymptotic inequality in [8]. Setting \( x_1 = d - 1 \) and \( x_2 = 1 \) in [11] then yields that the probability of a buffer overflow in \([0,t]\) with \( t = 10^{40} \) is smaller than 0.02.

We conclude that for unstable M/G/1 FB queues, the time to overflow may be very large, provided that the arrival rate is low.

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