Cohomology of line bundles on horospherical varieties

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Received: 10 May 2019 / Accepted: 24 October 2019 / Published online: 10 December 2019
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Abstract
A horospherical variety is a normal algebraic variety where a connected reductive algebraic group acts with an open orbit isomorphic to a torus bundle over a flag variety. In this article we study the cohomology of line bundles on complete horospherical varieties. The main tool in this article is the machinery of Grothendieck–Cousin complexes, and we also prove a Künneth-like formula for local cohomology.

Keywords Grothendieck–Cousin complexes · Horospherical varieties · Local cohomology

Mathematics Subject Classification 14M27 · 14L30

1 Introduction
Let $G$ be a connected reductive algebraic group over the field of complex numbers $\mathbb{C}$ and let $H$ be a closed subgroup of $G$. A homogeneous space $G/H$ is said to be horospherical if $H$ contains the unipotent radical of a Borel subgroup of $G$, or equivalently, $G/H$ is isomorphic to a torus bundle over a flag variety $G/P$. A normal $G$-variety is called horospherical if it contains a dense open $G$-orbit isomorphic to a horospherical homogeneous space $G/H$. Toric varieties and flag varieties are horospherical varieties, and horospherical varieties form an interesting class of spherical varieties generalizing both of these. A spherical variety is a normal $G$-variety such that a Borel subgroup $B$ of $G$ acts with a dense open orbit. One of the main importance of horospherical varieties is that, any spherical variety has a degeneration to a horospherical variety, such a degeneration being called a horospherical contraction (see for instance [29, Proposition 7.10]). The advantage of degenerating to a horospherical one is
that their underlying combinatorial description is easier to understand as compared to general spherical varieties.

Let $\mathcal{L}$ be a line bundle on a normal $G$-variety $X$. Up to replacing $G$ by a finite cover, we can assume that $G = C \times [G, G]$, where $C$ is a torus and $[G, G]$ is a simply-connected semisimple group, hence one can assume that $\mathcal{L}$ is $G$-linearized. If $X$ is complete, then the cohomology groups $H^i(X, \mathcal{L})$ are finite dimensional representations of $G$ (see [16, Theorem 11.6] and [5, page 589]). If $X$ is a flag variety, then Borel–Weil–Bott Theorem describes these cohomology groups (see [1, Section 10] or [9]). If $X$ is a toric variety, then the cohomology groups can be described using combinatorics of its associated fan (proved by Demazure in [8], see also [7, Chapter 9] for more details). For spherical varieties, in [5], Brion has given a bound on the multiplicities of irreducible components of these modules. In [4], if $X$ is projective spherical and $\mathcal{L}$ is generated by global sections, Brion proved that the cohomology groups $H^i(X, \mathcal{L})$ for $i > 0$ vanish. In [25–28], Tchoudjem studied these cohomology groups in the case where $X$ is a compactification of an adjoint semisimple group, a wonderful compactification of a reductive group, a wonderful variety of minimal rank and a complete symmetric variety respectively.

In this article we consider the cohomology of line bundles on complete horospherical varieties. We prove that the cohomology groups can be expressed as a product of the cohomology groups of line bundles on flag varieties and on toric varieties. Our main tool in this article is the machinery of Grothendieck–Cousin complexes (see [16] for more details), and we also prove a Künneth-like formula for local cohomology:

**Theorem A** Let $X$ and $Y$ be irreducible, Cohen–Macaulay affine schemes over an algebraically closed field $k$. Let $Z_1 \subset X$ and $Z_2 \subset Y$ be two locally complete intersections subschemes, of respective co-dimensions $l_1$ and $l_2$. Let $L_1$ and $L_2$ be two invertible sheaves respectively on $X$ and $Y$. Let $p_1 : X \times Y \to X$ and $p_2 : X \times Y \to Y$ be the projections and let $L_1 \boxtimes L_2 := p_1^* L_1 \otimes p_2^* L_2$. Then we have the following isomorphism

$$H^{l_1+l_2}_{Z_1 \times Z_2}(X \times Y, L_1 \boxtimes L_2) \simeq H^{l_1}_{Z_1}(X, L_1) \otimes_k H^{l_2}_{Z_2}(Y, L_2).$$

Let $S$ and $F$ be two smooth complete connected schemes over an algebraically closed field $k$ such that $F$ is rational and $\text{Pic}(F)$ is a projective $\mathbb{Z}$-module, and let $E \to S$ be a Zariski-locally trivial fibration with fibre $F$. These conditions ensure the existence of an isomorphism $\text{Pic}(E) \simeq \text{Pic}(S) \oplus \text{Pic}(F)$ (see Proposition 2). Let us fix once and for all such an isomorphism. Let $\mathcal{L}$ be a line bundle on $E$. Then we can identify $\mathcal{L}$ with $\mathcal{L}_1 \otimes \mathcal{L}_2$ for some line bundles $\mathcal{L}_1$ and $\mathcal{L}_2$ on $S$ and $F$ respectively. Then the Künneth-like formula given by Theorem A is crucial for proving the following theorem, which provides, under mild hypotheses, a spectral sequence to compute the cohomology of line bundles on the total space of a Zariski-locally trivial fibration.

**Theorem B** Assume that $S$ is stratified by locally closed affine irreducible schemes $\{Z^1_i\}_{i \in I}$, and that $F$ is stratified by locally closed affine irreducible schemes $\{Z^2_j\}_{j \in J}$ such that:

(i) there are affine open subsets $U^1_i \subset S$ and $U^2_j \subset F$ containing respectively $Z^1_i$ and $Z^2_j$, in which they are locally complete intersections.

(ii) $E$ is stratified by the $\{Z^1_i \times Z^2_j\}$, and there are open embeddings $U^1_i \times U^2_j \to E$. Then we can compute $H^n(E, \mathcal{L}_1 \otimes \mathcal{L}_2)$ with the following converging spectral sequence

$$E_2^{p,q} = H^p(S, \mathcal{L}_1) \otimes_k H^q(F, \mathcal{L}_2) \Rightarrow H^n(E, \mathcal{L}_1 \otimes \mathcal{L}_2).$$
If further, either $L_1$ or $L_2$ has its cohomology concentrated in a single degree, this spectral sequence collapses, and we get

$$H^n(E, L_1 \otimes L_2) \simeq \bigoplus_{p+q=n} H^p(S, L_1) \otimes_k H^q(F, L_2). \quad (1)$$

Now assume that $S$ and $F$ are equipped with respective actions of factorial reductive groups $G_1$ and $G_2$. The next step is to understand when the isomorphism (1) inherits a $G_1 \times G_2$-module or $g_1 \times g_2$-module structure, where $g_i$ denotes the Lie algebra of $G_i$ for $i = 1, 2$.

**Theorem C** Assume $G_1 \times G_2$ acts on $E$, $G_1$ acts on $S$ and trivially on $F$, and $G_2$ acts on $F$ and trivially on $S$, such that the morphism $E \to S$ is $G_2$ invariant and $G_1$ equivariant, and the inclusions of the fibers $F \to E$ are $G_2$-equivariant and $G_1$-invariant. Assume that $L_i$ are $G_i$-linearized (hence $G_1 \times G_2$-linearized because the action of the other group $G_j$ is trivial).

(i) Assume that the subsets $Z^1_i$ and $U^1_i$ are $G_1$-stable, that the subsets $Z^2_j$ and $U^2_j$ are $G_2$-stable. Then the isomorphism (1) is an isomorphism of $G_1 \times G_2$-modules.

(ii) Assume that the base field $k$ is of characteristic 0. Then the isomorphism (1) is a $g_1 \times g_2$-module.

These results have been motivated by the case of smooth toroidal complete horospherical varieties over $\mathbb{C}$, that falls under this technical framework. Let $X$ be such a variety. By [21, prop. 2.2 and ex. 2.3], there is a parabolic subgroup $P \subset G$ and a torus $T' = \text{Aut}^G(X)$, such that there is a Zariski-locally trivial fibration $X \to G/P$, whose fibre is a toric $T'$-variety $Y$ (more details are provided in Sect. 5). Fix an isomorphism

$$\text{Pic}^{G \times T'}(X) \simeq \text{Pic}^G(G/P) \oplus \text{Pic}^{T'}(Y)$$

Let $L \in \text{Pic}^{G \times T'}(X)$, that we write as $L_1 \otimes L_2$ with $L_1 \in \text{Pic}^G(G/P)$ and $L_2 \in \text{Pic}^{T'}(Y)$ using the previously fixed isomorphism. Let $g$ be the Lie algebra of $G$. Using the previous results, we show

**Theorem D** There is an isomorphism of $T \times T'$-modules and of $g \times T'$-modules

$$H^n(X, L_1 \otimes L_2) \simeq \bigoplus_{p+q=n} H^p(G/P, L_1) \otimes_k H^q(Y, L_2)$$

**Remark 1** (i) The $T \times T'$-module and $g \times T'$-module structures on these cohomology groups are compatible. That is, they coincide as $(t \times T')$-modules, where $t$ is the Lie algebra of $T$.

(ii) By Borel–Weil–Bott Theorem, at most one of the terms in the previous direct sum is nonzero.

(iii) We give a more concrete description of these cohomology groups in Corollary 2.

(iv) We also expect this isomorphism to be an isomorphism of $G \times T'$-modules, but we cannot prove it using our method.

The case of other complete horospherical varieties is then straightforward: let $X$ be a complete horospherical variety. Then there is a $G$-equivariant smooth complete toroidal $\tilde{X}$ and a proper birational map $\pi : \tilde{X} \to X$ (see more detail in Sect. 5.4). By equivariance, $\tilde{X}$ is also horospherical. Then we can compute with Theorem D the cohomology groups of line bundles on $X$ as well.
Corollary E  Let $L$ be a line bundle on $X$. Then we have $H^i(X, L) = H^i(\tilde{X}, \pi^* L)$ for all $i \geq 0$.

This paper is organized as followed. Section 2 is devoted to proving Theorem A. In Sect. 3, we shall recall the Cousin complexes and show how to compute, under nice assumptions, the cohomology groups of line bundles on some locally trivial fibrations. The behaviour of the obtained isomorphisms with respect to actions of algebraic groups will be investigated in Sect. 4, and we shall focus on the case of horospherical varieties in Sect. 5. In the last section, we give a more combinatorial description of these cohomology groups. It should be noted that, unless mentioned otherwise, results of Sects. 2, 3, and 4 are characteristic free.

2 A Künneth-like formula for local cohomology

We start by recalling a few results on local cohomology functors. We refer the reader to [14, Exposé I] and [16, Sections 7 and 8] for definitions.

In Sects. 2, 3, 4, for simplicity and unless noted otherwise, all schemes will be supposed to be Noetherian over an algebraically closed field $\mathbb{k}$. Let $X = \text{Spec}(A)$ be an affine scheme, let $\text{QCoh}(X)$ be the category of quasi-coherent $\mathcal{O}_X$-modules, and let $Z \subset X$ be a closed subscheme.

Proposition 1 Let $F \in \text{QCoh}(X)$, and $p \geq 0$. The functor $H^p_Z(F) : \text{QCoh}(X) \to \text{QCoh}(X)$ is additive and commutes with direct limits.

Proof Since we assume $X$ to be Noetherian, all open $U \subset X$ are quasi-compact. Then to prove the proposition, one can use the same arguments as in [17, Section 2], where this is done for $H^p(X, \bullet)$. One should recall that, as for ordinary sheaf cohomology, one can compute $H^p_Z(X, \bullet)$ using a flabby resolution. \hfill $\Box$

We can now prove the following useful lemma.

Lemma 1 Let $F, G \in \text{QCoh}(X)$. Let $t^p_{F,G} : H^p_Z(G) \otimes \mathcal{O}_X F \to H^p_Z(G \otimes \mathcal{O}_X F)$ be the natural map (see [13, 7.2.2] for a construction), and let $l$ be the co-dimension of $Z$ in $X$.

(i) If $F$ is a flat $\mathcal{O}_X$-module, $t^p_{F,G}$ is an isomorphism for all $p \geq 0$.

(ii) Assume that $X$ is Cohen-Macaulay. If $G$ is a flat $\mathcal{O}_X$-module, and if $Z$ is a locally complete intersection, $t^l_{F,G}$ is an isomorphism.

Proof Proof of (i): We first recall the following facts:

(1) If $T : \text{left}A - \text{Mod} \to \text{Ab}$ is an additive functor, and if $L$ is a free $A$-module, then

$$T(A) \otimes_A L \to T(L)$$

is an isomorphism (see [13, Lemma 7.2.4])

(2) (Lazard’s theorem) Let $M$ be a $A$-module. Then $M$ is flat if and only if it is the colimit of a directed system of free finite $A$-modules. Then by flatness of $F$, $F(X)$ is flat over $A$, hence $F(X)$ is a directed colimit of free $A$-modules of finite rank, and $F$ is a directed colimit of free $\mathcal{O}_X$-modules of finite rank. Now since both $H^p_Z(G \otimes \mathcal{O}_X \bullet)$ and global sections commute with direct limits, it is enough to show the isomorphism for global sections of free $\mathcal{O}_X$-modules, which is exactly (1).

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Proof of (ii): We start by proving that the functor $\mathcal{H}^l_Z(G \otimes_{O^X} \bullet)$ is right exact. Let
\[ 0 \to \mathcal{F}_1 \to \mathcal{F}_2 \to \mathcal{F}_3 \to 0 \]
be an exact sequence of quasi-coherent $O_X$-modules. Since $G$ is flat, this gives a long exact sequence of local cohomology sheaves
\[ \ldots \to \mathcal{H}^l_Z(G \otimes_{O_X} \mathcal{F}_1) \to \mathcal{H}^l_Z(G \otimes_{O_X} \mathcal{F}_2) \to \mathcal{H}^l_Z(G \otimes_{O_X} \mathcal{F}_3) \to \mathcal{H}^{l+1}_Z(G \otimes_{O_X} \mathcal{F}_1). \]
But the conditions on $Z$ ensure the vanishing of $\mathcal{H}^{l+1}_Z(G \otimes_{O_X} \mathcal{F}_1)$ (see [14, Lemma 3.12, III]), hence the right-exactness. Now pick an exact sequence $\mathcal{L}' \to \mathcal{L} \to \mathcal{F} \to 0$ with $\mathcal{L}$ and $\mathcal{L}'$ being free $O_X$-modules. Then we have a commutative diagram
\[
\begin{array}{c}
\mathcal{H}^l_Z(G \otimes_{O_X} \mathcal{L}') \longrightarrow \mathcal{H}^l_Z(G \otimes_{O_X} \mathcal{L}) \longrightarrow \mathcal{H}^l_Z(G \otimes_{O_X} \mathcal{F}) \longrightarrow 0 \\
\downarrow \downarrow \downarrow \\
\mathcal{H}^l_Z(G \otimes_{O_X} \mathcal{L}') \longrightarrow \mathcal{H}^l_Z(G \otimes_{O_X} \mathcal{L}) \longrightarrow \mathcal{H}^l_Z(G \otimes_{O_X} \mathcal{F}) \longrightarrow 0.
\end{array}
\]
Since $\mathcal{L}$ and $\mathcal{L}'$ are free, the first two vertical maps are isomorphisms, so is the third one. $\square$

Remark 2 Any invertible sheaf $\mathcal{L}$ on $X = \text{Spec}(A)$ is flat, see [20, Theorem 7.10].

Before proving our Künneth-like formula for local cohomology, let us recall a few known results on local cohomology (which hold in much more generality, see [16, Section 8.9 and 11]).

Lemma 2 Let $\mathcal{F} \in QCoh(X)$.

(i) If $Z_2 \subset Z_1$ are two closed subschemes of $X$ such that $Z_1 \setminus Z_2$ is affine, then
\[ H^l_{Z_1/Z_2}(X, \mathcal{F}) \simeq H^0(X, \mathcal{H}^l_{Z_1/Z_2}(\mathcal{F})). \]

(ii) Let $Z_2 \subset Z_1, W_2 \subset W_1$ be closed subschemes, and let $S_1 = W_1 \cap Z_1, S_2 = (W_1 \cap Z_2) \cup (W_2 \cap Z_1)$. Then we have a spectral sequence
\[ E^p_q = \mathcal{H}^p_{W_1/W_2}(\mathcal{H}^q_{Z_1/Z_2}(\mathcal{F})) \Rightarrow \mathcal{H}^*_{S_1/S_2}(\mathcal{F}). \]
In particular, when $Z_2 = W_2 = \emptyset$, we have
\[ E^p_q = \mathcal{H}^p_{W_1}(\mathcal{H}^q_{Z_1}(\mathcal{F})) \Rightarrow \mathcal{H}^*_{W_1\cap Z_1}(\mathcal{F}). \]

(iii) Let $Z_2 \subset Z_1$ be two closed subschemes of $X$. Let $Y$ be scheme, and $p : X \times Y \to X$ be the first projection. There is an isomorphism
\[ p^*\mathcal{H}^l_{Z_1/Z_2}(\mathcal{F}) \simeq \mathcal{H}^l_{Z_1\times Y/Z_2\times Y}(p^*\mathcal{F}). \]

Proof For proof of (i) see [16, Theorem 9.5 (d)]. For proof of (ii) see [16, Lemma 8.5 (d)]. For proof of (iii) see [16, Proposition 11.5]. $\square$

We are now ready to prove Theorem A.

Proof of Theorem A First let us notice that we have
\[ H^p_{Z_1 \times Z_2}(X \times Y, L_1 \boxtimes L_2) = H^0(X \times Y, \mathcal{H}^a_{Z_1 \times Z_2}(L_1 \boxtimes L_2)) \]
\[ \mathcal{C} \text{ Springer} \]
Let $\mathcal{L}_i := p^*_i L_1$. Since $\mathcal{L}_1$ and $\mathcal{L}_2$ are locally free, and by the conditions on $Z_i$’s, the spectral sequence

$$E_2^{p,q} = H^p_{Z_1 \times Y}(H^q_{X \times Z_2}(L_1 \boxtimes L_2)) \Rightarrow H^p_{Z_1 \times Z_2}(L_1 \boxtimes L_2)$$

collapses, and it may only be nonzero for $p = l_1$ and $q = l_2$. Here we are using that $L_1 \boxtimes L_2$ is also locally free, and that $Z_{1} \times Z_{2}$ is locally generated by $l_{1} + l_{2}$ elements. Using Lemma 1, we compute

$$H^0_{Z_1 \times Y}(H^q_{X \times Z_2}(L_1 \boxtimes L_2)) \cong H^p_{Z_1 \times Y}(L_1 \otimes_{O_{X \times Y}} H^q_{X \times Z_2}(L_2)) \quad \text{(by Lemma 1(i))}$$

Taking global sections yields the wanted isomorphism. $\square$

### 3 Locally trivial fibrations and cousin complexes

Let $S$ and $F$ be smooth complete connected schemes, and let $E \to S$ be a (Zariski) locally trivial fibration, with fiber $F$. Moreover, assume that the Picard group $\text{Pic}(F)$ of $F$ is a projective $\mathbb{Z}$-module, and that $F$ is rational. We start with the following useful description of the Picard group $\text{Pic}(E)$ of $E$.

**Proposition 2** We have an (noncanonical) isomorphism $\text{Pic}(E) \simeq \text{Pic}(S) \oplus \text{Pic}(F)$

**Proof** Using [10, Proposition 2.3], we have an exact sequence

$$H^0(F, O^*_F)/k^* \to \text{Pic}(S) \to \text{Pic}(E) \to \text{Pic}(F) \to 0.$$  

By the hypotheses on $F$, $H^0(F, O^*_F) = k^*$. Since $\text{Pic}(F)$ is projective, we get $\text{Pic}(E) \simeq \text{Pic}(S) \oplus \text{Pic}(F)$. This completes the proof of the proposition. $\square$

Let $\mathcal{L}$ be a line bundle on $E$. By fixing an isomorphism $\text{Pic}(E) \simeq \text{Pic}(S) \oplus \text{Pic}(F)$, we can identify $\mathcal{L}$ with $\mathcal{L}_1 \otimes \mathcal{L}_2$ for some line bundles $\mathcal{L}_1$ and $\mathcal{L}_2$ on $S$ and $F$ respectively. The aim of this section is to compute the cohomology groups $H^n(E, \mathcal{L})$. To do that, we shall need the formalism of Cousin complexes.

Let $X$ be a Noetherian scheme, let $\mathcal{F}$ be an $O_X$-module, and let $\{Z\} := \{0 = Z_{n+1} \subset Z_n \subset \cdots \subset Z_1 \subset Z_0 = X\}$ be a filtration by closed subschemes. For simplicity, let us assume that $X$ is irreducible. Then one can construct (see [16, Lemma 7.8]) the Cousin complex of $\mathcal{F}$ relatively to the filtration $\{Z\}$

$$\text{Cousin}_{[Z]}(\mathcal{F}) := \{0 \to H^0(X, \mathcal{F}) \to H^0_{Z_0/Z_1}(X, \mathcal{F}) \to H^1_{Z_1/Z_2}(X, \mathcal{F}) \to H^2_{Z_2/Z_3}(X, \mathcal{F}) \to \cdots \}$$

and its sheaf analogue, denoted by $\text{Cousin}_{[Z]}(\mathcal{F})$. Kempf showed in [16, Theorem 9.6, 10.3 and 10.5] that under some conditions, these Cousin complexes can be used to compute the cohomology groups $H^n(X, \mathcal{F})$.

**Proposition 3** Assume that the $Z_i \setminus Z_{i+1}$ are all affine, that $Z_i$ is of codimension $i$ in $X$ for all $n \geq i \geq 0$, and that $\mathcal{F}$ is locally free. Then we have an isomorphism of complexes

$$\text{Cousin}_{[Z]}(\mathcal{F}) \simeq H^0(X, \text{Cousin}_{[Z]}(\mathcal{F}))$$

and $H^n(X, \mathcal{F})$ is the $n$-th homology group of $\text{Cousin}_{[Z]}(\mathcal{F})$.

We can prove now Theorem B

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Proof of Theorem B

Let us start by fixing some notations. Let

\[
\overline{S}_l := \bigcup_{\text{codim}(Z_i^1) \geq l} Z_i^1, \quad F_l := \bigcup_{\text{codim}(Z^2_j) \geq l} Z_j^2, \quad \text{and} \quad \overline{E}_l := \bigcup_{\text{codim}(Z_i^1 \times Z_j^2) \geq l} Z_i^1 \times Z_j^2
\]

and let \( S_l = \overline{S}_l \setminus \overline{S}_{l+1}, \) \( F_l = \overline{F}_l \setminus \overline{F}_{l+1}, \) and \( E_l = \overline{E}_l \setminus \overline{E}_{l+1}. \)

By using twice the excision formula, we can rewrite the cohomology groups occurring in the Cousin complex of \( L_1 \otimes L_2 \) in the following way:

\[
H^{l}_{E_l/E_{l+1}}(E, L_1 \otimes L_2) \simeq H^{l}_{E_l}(E \setminus \overline{E}_{l+1}, L_1 \otimes L_2) \simeq \bigoplus_{p+q=l} \bigoplus_{\text{codim}(Z_i^1)=p} \bigoplus_{\text{codim}(Z_j^2)=q} H_{Z_i^1 \times Z_j^2}^l(E \setminus \overline{E}_{l+1}, L_1 \otimes L_2)
\]

where \( Z_i^1 \times Z_j^2 \) denotes \( Z_i^1 \setminus \overline{Z}_j^2, Z_i^1 \times Z_j^2 \). The boundary maps

\[
\partial^l_E : H_{Z_i^1 \times Z_j^2 / \partial(Z_i^1 \times Z_j^2)}^l(E, L_1 \otimes L_2) \to H_{Z_j^2 / \partial(Z_i^1 \times Z_j^2)}^{l+1}(E, L_1 \otimes L_2)
\]

are zero whenever \( Z_i^1 \times Z_j^2 \not\subset \partial(Z_i^1 \times Z_j^2) \). But a stratum \( Z_i^1 \times Z_j^2 \) of co-dimension \((l + 1)\) lies \( \partial(Z_i^1 \times Z_j^2) \) exactly when it is of the form \( Z_i^1 \times Z_j^2 \) or \( Z_i^1 \times Z_j^2 \), with either \( Z_i^1 \subset \partial Z_i^1 \) of co-dimension \( \text{codim}(Z_i^1) + 1 \) or \( Z_j^2 \subset \partial Z_j^2 \) of co-dimension \( \text{codim}(Z_j^2) + 1 \).

We shall now introduce the following bi-complex:

\[
K_{p,q} := \bigoplus_{\text{codim}(Z_i^1)=p} \bigoplus_{\text{codim}(Z_j^2)=q} H^{p+q}_{Z_i^1 \times Z_j^2}(U_i^1 \times U_j^2, L_1 \otimes L_2) \simeq \bigoplus_{\text{codim}(Z_i^1)=p} \bigoplus_{\text{codim}(Z_j^2)=q} H^p_{Z_i^1}(U_i^1, L_1) \otimes_k H^q_{Z_j^2}(U_j^2, L_2) \text{ by Theorem A.}
\]

Notice that with the same kind of arguments than for \((*)\), \( \bigoplus_{\text{codim}(Z_i^1)=p} H^p_{Z_i^1}(U_i^1, L_1) \) is the \((p + 1)\)-th term of the Cousin\(_{S_i}(L_1)\), and denote by \( \partial^p_{S_i, j} \), the induced map on \( H^p_{Z_i^1}(U_i^1, L_1) \).

For the same reasons, \( \bigoplus_{\text{codim}(Z_j^2)=q} H^q_{Z_j^2}(U_j^2, L_2) \) is the \((q + 1)\)-th term of the Cousin\(_{F_i}(L_2)\), and denote by \( \partial^q_{F, j} \), the induced map on \( H^q_{Z_j^2}(U_j^2, L_2) \). Define the boundary maps on \( K_{p,q} \) by:

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\[\begin{align*}
\cdots & \rightarrow K^{p,q+1} \xrightarrow{\partial^p_{h,q}} K^{p+1,q} \rightarrow \cdots \\
\cdots & \rightarrow K^{p,q} \xrightarrow{\partial^p_{c,q}} K^{p+1,q} \rightarrow \cdots \\
\cdots & \rightarrow K^{p,q} \xrightarrow{\partial^p_{h,q}} K^{p+1,q} \rightarrow \cdots \\
\cdots & \rightarrow K^{p,q+1} \xrightarrow{\partial^p_{h,q+1}} K^{p+1,q+1} \rightarrow \cdots
\end{align*}\]

- the horizontal boundary maps \(\partial^p_{h,q} : K^{p,q} \rightarrow K^{p+1,q}\) are given by \(\bigoplus_{\text{codim}(Z^i_1) = p} (\partial^p_{S,i} \otimes 1)\);
- the vertical boundary maps \(\partial^p_{c,q} : K^{p,q} \rightarrow K^{p,q+1}\) are given by \(\bigoplus_{\text{codim}(Z^j_2) = q} (1 \otimes \partial^q_{F,j})\).

The previous argument on the boundary maps \(\partial'_E\) of Cousin \(\{E_i\}\) shows that

\[\text{Tot}(K) = \text{Cousin}_{\{E_i\}}(L_1 \otimes L_2)\]

where \(\text{Tot}(K)\) denotes the total complex of \(K\). Since \(K^{p,q}\) vanishes for \(p < 0\) or \(q < 0\), the bi-complex \(\text{C}_{H^1}(K)\) obtained from \(K\) by taking homology of columns gives rise to a spectral sequence whose \(E^2\)-page is given by

\[E^2_{p,q} = H_p(\text{C}_{H^1}(K))\]

converging to \(H_*(\text{Tot}(K))\). But by Proposition 3, we have

\[H_n(\text{Tot}(K)) = H_n(\text{Cousin}_{\{E_i\}}(L_1 \otimes L_2)) = H^n(E, L_1 \otimes L_2)\]

and

\[H_p(\text{C}_{H^1}(K)) = H_p(\bigoplus_{\text{codim}(Z^i_1) = *} H^*_Z(U^1_i, L_1) \otimes_k H^q(F, L_2)) = H^p(S, L_1) \otimes_k H^q(F, L_2)\]

If either \(L_1\) or \(L_2\) has its cohomology concentrated in a single degree, then the last claim is obvious.

\[\square\]

### 4 Equivariance

Let \(G_1\) and \(G_2\) be two reductive algebraic groups. In this section, we shall take into account actions of \(G_1 \times G_2\) in the previous results. We start by showing the following lemma:

**Lemma 3** With notations and settings of Lemma 1, if \(G_1\) acts on \(X\) in such a way that \(Z\) is \(G_1\)-stable, and that both \(F\) and \(G\) are \(G_1\)-linearized, then the isomorphism \(t^p_{F,G}\) given by Lemma 1 is \(G_1\)-equivariant.

**Proof** We recall here the \(G_1\)-linearization of \(\mathcal{H}_Z^p(F)\) given by Kempf: Let \(\rho : G_1 \times X \rightarrow X\) be the action map, and let \(\pi\) denote the second projection. The \(G_1\)-linearization of \(\mathcal{F}\) gives an isomorphism \(\rho^* \mathcal{F} \rightarrow \pi^* \mathcal{F}\). Since \(Z\) is \(G_1\)-stable, we get a map \(\rho^*(\mathcal{H}_Z^p(F)) \rightarrow \mathcal{H}_{G_1 \times Z}^p(\pi^* \mathcal{F})\), and composing it with the isomorphism \(\mathcal{H}_{G_1 \times Z}^p(\pi^* \mathcal{F}) \rightarrow \pi^*(\mathcal{H}_Z^p(F))\) gives the required natural \(G_1\)-linearization on \(\mathcal{H}_Z^p(F)\), see [16, Lemma 11.3]. The commutativity of the diagram (whose horizontal arrows are isomorphisms thanks to the linearizations, and all vertical arrows are isomorphisms by flatness)

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Lemma 3 is enough to prove equivariant analogues of Theorems A and B. Let us keep the notations and settings of Theorem A. Assume that $G_1$ acts on $X$ and that $Z_1$ is $G_1$-stable, that $G_2$ acts on $Y$ and that $Z_2$ is $G_2$-stable, and that the $L_i$ are $G_i$-linearized. Equip $L_1 \boxtimes L_2$ with the induced $G_1 \times G_2$-linearization. Then we have:

**Proposition 4** The isomorphism given in Theorem A is an isomorphism of $G_1 \times G_2$-modules.

**Proof** By Lemma 3, the isomorphism

$$\rho^*\mathcal{H}_Z^0(\mathcal{F} \otimes \mathcal{G}) \longrightarrow \mathcal{H}_Z^0(\pi^*(\mathcal{F} \otimes \mathcal{G})) \longrightarrow \pi^*\mathcal{H}_Z^0(\mathcal{F} \otimes \mathcal{G})$$

is enough to conclude the lemma. \hfill $\Box$

Now take the notations and settings of Theorem B. As in the introduction, replacing $G_i$ by a finite cover we assume that $G_i$ is factorial for $i = 1, 2$. Assume $G_1 \times G_2$ acts on $E$, $G_1$ acts on $S$ and trivially on $F$, and $G_2$ acts on $F$ and trivially on $S$, such that the morphism $E \to S$ is $G_2$ invariant and $G_1$ equivariant, and the inclusions of the fibers $F \to E$ are $G_2$-equivariant and $G_1$-invariant. Assume that the subsets $Z_i^1$ and $U_i^1$ are $G_1$-stable, that the subsets $Z_i^2$ and $U_i^2$ are $G_2$-stable, and that the $L_i$ are $G_i$-linearized (hence $G_1 \times G_2$-linearized because the action of the other group $G_j$ is trivial). Note that we have an exact sequence

$$0 \to \text{Pic}^{G_1}(S) \to \text{Pic}^{G_1 \times G_2}(E) \to \text{Pic}^{G_2}(F) \to 0 \quad (2)$$

(where Pic$^G$ (−) denotes the $G$-equivariant Picard group) since it sits in the following commutative diagram with exact columns whose first and last rows are exact:

$$
\begin{array}{cccc}
0 & 0 & 0 \\
\downarrow & \downarrow & \downarrow \\
0 & \mathcal{X}(G_1) & \mathcal{X}(G_1 \times G_2) & \mathcal{X}(G_2) & 0 \\
\downarrow & \downarrow \iota_E & \downarrow \iota_F \\
\text{Pic}^{G_1}(S) & \text{Pic}^{G_1 \times G_2}(E) & \text{Pic}^{G_2}(F) \\
\downarrow & \downarrow p_E & \downarrow p_F \\
0 & \text{Pic}(S) & \text{Pic}(E) & \text{Pic}(F) & 0 \\
\downarrow & \downarrow & \downarrow \\
0 & 0 & 0 & 0 & 0
\end{array}
$$

Recall that we have fixed a section $s : \text{Pic}(F) \to \text{Pic}(E)$. We also have a natural section $s_X : \mathcal{X}(G_2) \to \mathcal{X}(G_1 \times G_2)$ sending a character $\chi$ to the character $\tilde{\chi}(g_1, g_2) = \chi(g_2)$. Since Pic$(F)$ is projective module, so is Pic$^{G_2}(F)$, and we get a (non-canonical) section Pic$^{G_2}(F) \to \text{Pic}^{G_1 \times G_2}(E)$. Let us fix such a section $\tilde{s}$. Then it satisfies $\tilde{s} \circ \iota_F = \iota_E \circ s_X$ and $p_E \circ \tilde{s} = s \circ p_F$. Then the above short exact sequence (2) splits and by using $s_X$ and $\tilde{s}$, we have an isomorphism

$$\text{Pic}^{G_1 \times G_2}(E) \simeq \text{Pic}^{G_1}(S) \oplus \text{Pic}^{G_2}(F). \quad (3)$$

This induces a $G_1 \times G_2$-linearization on $L_1 \boxtimes L_2$. 

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Remark 3 All $G_1 \times G_2$-linearized line bundles on $E$ occur that way: let $\mathcal{L}$ be such a line bundle. By (3), we can write $\mathcal{L} = \mathcal{L}_1 \otimes \mathcal{L}_2$, where $\mathcal{L}_1$ is $G_1$-linearized line bundle on $S$ and $\mathcal{L}_2$ is $G_2$-linearized line bundle on $F$.

Proof of Theorem C The proof of (1) is a direct consequence of [16, Theorem 11.6 (c)]. Let us now prove (2).

For an affine algebraic group $G$, let $\hat{G}$ denote the formal completion of $G$ at the identity (see [19, Section 1.2 §2] for more details). The $G_1 \times G_2$-linearization of $\mathcal{L}_1 \otimes \mathcal{L}_2$ gives rise to a $G_1 \times G_2$-linearization on $\mathcal{L}_1 \otimes \mathcal{L}_2$. Then by using the natural map $\hat{G}_1 \times \hat{G}_2 \to G_1 \times G_2$, we get $\hat{G}_1 \times \hat{G}_2$-linearization. Hence by [16, Lemma 11.1(a)], the cohomology groups $H^{i_1 + i_2}_{Z_1^i \times Z_2^j}(U^1 \times U^2, L_1 \otimes L_2)$, $H^{i_1}_{Z_1^i}(U^1, L_1)$ and $H^{i_2}_{Z_2^j}(U^2, L_2)$ have respectively a natural structure of $\hat{G}_1 \times \hat{G}_2$, $\hat{G}_1$ and $\hat{G}_2$-module, that we explicitly describe here.

A $\hat{G}$-linearization of a sheaf $\mathcal{F}$ of $\mathcal{O}_X$-modules on a scheme $X$ is given by an inverse system of morphisms $\mathcal{F} \to k[G]/m^i \otimes_k \mathcal{F}$ (where $m$ denotes the ideal of regular functions vanishing at the identity). Since $k[G]/m^i$ are finite dimensional vector spaces and the local cohomology functor commutes with direct sums, we have that for any closed subsets $Z_2 \subset Z_1 \subset X$, there is an inverse system $H^p_{Z_1/Z_2}(X, \mathcal{F}) \to k[G]/m^i \otimes_k H^p_{Z_1/Z_2}(X, \mathcal{F})$, which is precisely the required linearization.

Now since the linearization construction is detailed, it is immediate that the isomorphism

$$H^{i_1 + i_2}_{Z_1^i \times Z_2^j}(U^1 \times U^2, L_1 \otimes L_2) \to H^{i_1}_{Z_1^i}(U^1, L_1) \otimes_k H^{i_2}_{Z_2^j}(U^2, L_2)$$

is an isomorphism of $\hat{G}_1 \times \hat{G}_2$-modules. Notice that for example we can rewrite the $H^{i_1}_{Z_1^i}(U^1, L_1) = H^{i_1}_{Z_1^i \cap Z_1^i}(S, L)$, hence these modules are exactly the direct summands of the modules occurring in the Cousin complexes. Using [16, Lemma 11.1(d)], the Cousin complex of $L_1 \otimes L_2$ relatively $S \otimes [E]$ is a complex of $\hat{G}_1 \times \hat{G}_2$-modules. Hence the isomorphism (1) is an isomorphism of $\hat{G}_1 \times \hat{G}_2$-modules. Since we are in characteristic 0, this gives us an isomorphism of Lie algebra $\mathfrak{g}_1 \times \mathfrak{g}_2$-modules (see [23, II.5 Theorem 3]).

5 Application to horospherical varieties

5.1 Preliminaries on horospherical varieties

In this subsection we recall some basics on definition and properties of horospherical varieties. For more details we refer to [21, Section 2]. Let $G$ be a connected reductive algebraic group over the field of complex numbers. A $G$-variety is an integral separated scheme of finite type over the field of complex numbers, with an algebraic action of $G$. Let $B$ be a Borel subgroup of $G$.

Definition 1 A spherical variety is a normal $G$-variety $X$ such that it has a dense open $B$-orbit.

These varieties have many interesting properties, for example, they have only finitely many $B$-orbits (see [2, Théorème 2]). Here, we consider a special class of spherical varieties, called horospherical varieties.

Definition 2 Let $H$ be a closed subgroup of $G$. Then $H$ is said to be horospherical if it contains the unipotent radical of a Borel subgroup of $G$.

We also say that the homogeneous space $G/H$ is horospherical if $H$ is horospherical. Up to conjugation, we can assume that $H$ contains $U$, the maximal unipotent subgroup of $G$.

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contained in the Borel subgroup $B$. Let $P$ be the normalizer $N_G(H)$ of $H$ in $G$. It is a parabolic subgroup of $G$, and $T' := P/H = \text{Aut}^G(G/H)$ is a torus (see [21, Proposition 2.2 and Example 2.3]). Then it is clear that $G/H$ is a torus bundle over the flag variety $G/P$ with fibers $P/H$.

**Definition 3** Let $X$ be a normal $G$-variety. We say $X$ is horospherical if it contains an open $G$-orbit which is a horospherical homogeneous space.

By Bruhat decomposition it is clear that horospherical varieties are spherical. A geometric characterization can be given as follows (see for example, [21, Example 2.3] and [22, Corollary 3.3.8]): let $X$ be a normal $G$-variety. Then $X$ is horospherical if there exists a parabolic subgroup $P$ of $G$, and a toric $T'$-variety $Y$ such that there is a diagram of $G \times T'$-equivariant morphisms (where $T'$ acts on $Y$ via the $P$ action.)

$$
\begin{align*}
G \times^P Y & \to X \\
\downarrow^p & \\
G/P &
\end{align*}
$$

where $G \times^P Y := (G \times Y)/P$, the action of $P$ is given by $p \cdot (g, y) = (gp, p^{-1} \cdot y)$, $\pi$ is birational and proper, and $p$ is a Zariski-locally trivial fibration with fibers isomorphic to $Y$. The classification of horospherical varieties can be given by using colored fans, see [21, Théorème 2.5] for precise details.

### 5.2 Divisors of horospherical varieties

In this subsection we recall some properties of divisors of a spherical variety $X$. We start by the following result from [3, Section 2.2] (see also [29, Proposition 17.1]):

**Proposition 5** Any divisor of $X$ is linearly equivalent to a $B$-stable divisor.

Let $X$ be a horospherical variety. Now we consider the $B$-stable prime divisors of $X$. We denote by $X_1, X_2, \ldots, X_r$ the $G$-stable divisors of $X$. The other $B$-stable divisors (i.e. those that are not $G$-stable) are the closures of $B$-stable divisors of $G/H$, which are called the colors of $G/H$. These are the inverse images under the torus fibration $G/H \to G/P$ of the Schubert divisors of the flag variety $G/P$. Now we recall the description of the Schubert divisors of $G/P$. Fix a maximal torus $T$ of $B$. Then we denote by $S$, the set of simple roots of $G$ with respect to $B$ and $T$. Let $W$ be the Weyl group of $G$. Also denote by $S_P$ the subset of $S$ of simple roots of $P$, that is simple roots of the Levi factor $L_P$ of $P$. Denote by $W_P$ the subgroup of $W$ generated by simple reflections $\{s_\alpha : \alpha \in S_P\}$. and denote by $W^P$, the quotient $W/W_P$. We identify $W_P$ with the set of minimal length representatives in $W$ and we denote by $w_0^P$ the longest element of $W^P$. By Bruhat decomposition of $G$ (see for example [24, Section 8.3]), we have the following decomposition of $G/P$:

$$
G/P = \bigsqcup_{w \in W^P} BwP/P
$$

Further the dimension of a cell $BwP/P$ equals the length of $w$. In particular, the length of $w_0^P$ is the dimension of $G/P$ and irreducible $B$-stable divisors of $G/P$ are the closures of the cells $Bw_0^P S_\alpha P/P$ with $\alpha \in S\setminus S_P$. The irreducible $B$-stable divisors of $G/P$ are called Schubert divisors and are indexed by the subset of simple roots $S\setminus S_P$. Hence the $B$-stable irreducible divisors of $G/H$ are of the form $Bw_0^P S_\alpha P/H$, which we denote by $D_\alpha$ for $\alpha \in S\setminus S_P$. Then the following holds:
Corollary 1 Any divisor $D$ of $X$ is equivalent to a linear combination of $X_i$’s and $D_\alpha$ for $\alpha \in S \setminus R$. That is,

$$D \sim \sum_{\alpha \in S \setminus R} d_\alpha D_\alpha + \sum_{i=1}^r d_i X_i.$$ 

A horospherical variety is called toroidal if it has no color containing a $G$-orbit. We summarize the description of toroidal varieties that will be used later with the following proposition (and we refer to [21, Proposition 2.2 and Example 2.3], [22, Corollary 3.3.8] and [18, Theorem 7.1] for its proof):

**Proposition 6** Let $X$ be a toroidal horospherical variety, and let $H$ be the stabilizer of a point in the dense open $B$-orbit of $X$.

(i) $P := N_G(H)$ is a parabolic subgroup of $G$, and $T' := P / H = \text{Aut}^G(X)$ is a torus.

(ii) There is a toric $T'$-variety $Y$ such that $X = G \times^P Y$, and the natural map $X = G \times^P Y \to G / P$ is a Zariski-locally trivial fibration, where $P$ acts on $Y$ via $P \to T'$.

(iii) $G \times T'$ acts on $X = G \times^P Y$ via

$$(g, pH)[h, y] = [gh, p \cdot y] \quad (4)$$

for all $g, h \in G$ and $y \in Y$.

5.3 Local structure of toroidal horospherical varieties

In this subsection, we recall the local structure of a toroidal horospherical variety $X$. We keep the notations of the previous subsection. Let $P^-$ be the parabolic subgroup opposite to $P$, and let $L_P = P \cap P^-$ be the Levi factor of $P$ containing $T$. Notice that there is a quotient map $L_P \to T'$, whose kernel is denoted by $L_0$. We denote by $\mathcal{D}(X)$ the set of colors of $X$, and define

$$X_0 := X \setminus \bigcup_{D \in \mathcal{D}(X)} D$$

Then $X_0$ is a $P$-stable open subset such that $X$ is covered by the $G$-translates of $X_0$, and the local structure theorem describes precisely $X_0$ (see [6, Proposition 3.4] and also see [29, Theorem 29.1] for proofs):

**Proposition 7** (i) There exists a closed $L_P$-stable subvariety $Z$ of $X_0$, fixed pointwise by $L_0$, such that

$$X_0 \simeq P^- \times^L Z \simeq R^u(P^-) \times Z$$

where $R^u(P^-)$ is the unipotent radical of $P^-$. Moreover, $Z$ is isomorphic to the toric $T'$-variety $Y$, it is defined by the same fan as $X$, and any $G$-orbit intersect $Z$ along a unique $T'$-orbit.

(ii) The action of $B \times T'$ on $X = G \times^P Y$ (see 4) stabilizes $X_0$ and, the above isomorphism is $B \times T'$-equivariant (where the action on $R^u(P^-) \times Z$ is the product action).

Note that in particular, the fibration $P^- \times^L Z$ is trivial, and that any $G$-stable divisor of $G \times^P Y$ is of the form $G \times^P D'$, where $D'$ is a $T'$-stable divisor of $Y$. Also, this implies that $Y$ is smooth if and only if $X$ is.

From now on, we assume that $G \simeq C \times [G, G]$, where $C$ is a torus, and $[G, G]$ is semisimple and simply-connected. This condition ensures that any invertible sheaf on $X$
admits a $G$-linearization, see [29, Theorem C.4]. We also assume that $X$ is complete (and so is $Y$). Since $Y$ is complete toric variety, $\mathcal{P}(Y)$ projective $\mathbb{Z}$-module (see [7, Proposition 4.2.5]). Then we can apply (3) in our setting and we get the following isomorphism:

$$\text{Pic}^{G \times T'}(X) \simeq \text{Pic}^{G}(G/P) \oplus \text{Pic}^{T'}(Y)$$

(5)

5.4 Proof of Theorem D and of Corollary E

Proof of Theorem D Recall that $X$ is a smooth complete toroidal horospherical variety. Let $T \subset B$ be a maximal torus of $G$. We need to stratify $X$ in such a way that we can apply Theorems B and C. This is done by looking at the stratification of $X$ by $B \times T'$-orbits. The $B \times T'$-orbits are of the form $BwP \times P C_\sigma$, where $w \in W^P$ and $C_\sigma$ is the $T'$-orbit of $Y$ associated to a cone $\sigma$ (living in the fan defining $Y$).

To check that this stratification satisfies the conditions needed to apply Theorem B, we use the local structure theorem (Proposition 7). Let $U_w = w(w_0^P)^{-1}Bw_0^P P / P \subset G/P$ and $U_\sigma = \text{Spec}(k[M \cap \sigma^{-1}]) \subset Y$. Let $p : X \to G/P$ be the projection map. Then $BwP \times P C_\sigma$ is closed in $U_w P \times P U_\sigma$, which is open in $p^{-1}(U_w) = w(w_0^P)^{-1}X_0$. But $p^{-1}(U_w) \simeq U_w \times Y$ by Proposition 7. Hence the $B \times T'$-orbits are isomorphic to $BwP / P \times C_\sigma$, and they are closed in an open subset of $X$ of the form $U_w \times U_\sigma$. Both $BwP / P$ and $C_\sigma$ are locally complete intersections, of respective dimensions $l(w)$ and $\text{dim}(\sigma)$. Hence we can apply Theorem B.

By Borel–Weil–Bott theorem, the cohomology of line bundles on $G/P$ is concentrated in at most one degree, so the spectral sequence collapses and yields the wanted decomposition.

Remark that $U_w$ is $T$-stable, and that $U_\sigma$ is $T'$-stable. By Proposition 6 (iii), the action of $G \times T'$ on $X$ falls in the setting of Theorem C, which proved that the decomposition is compatible with the respective actions of $T \times T'$ and $g \times T'$. $\square$

Now let $X$ be a complete horospherical variety. Using Luna-Vust theory, there is a correspondence between $G$-isomorphism classes of spherical varieties $X$ containing $G/H$ as an open $G$-orbit, and combinatorial objects called strictly convex colored fans (we refer to [12] and [18] for clear presentations of the theory, and in particular to [18, Theorem 4.5] for this key fact). Morphisms between spherical varieties can also be described using these colored fans (see [12, Theorem 8.5]).

We can start by constructing a $G$-equivariant proper birational map $\hat{X} \to X$ called the decoloration morphism, which consists at the level of colored fans in forgetting the colors (see [22, Corollary 3.3.7]). The variety $\hat{X}$ obtained that way is complete, toroidal and horospherical, hence $\hat{X} = G \times P \hat{Y}$ for a certain toric $T'$-variety $\hat{Y}$. We then construct a $G$-equivariant resolution of singularities $\tilde{X} \to \hat{X}$, which corresponds to subdivising the underlying fan (see [7, Theorem 11.1.9]). Note that using Proposition 7 (1), this also corresponds to a resolution of singularities $\tilde{Y} \to \hat{Y}$, so that $\tilde{X} = G \times P \tilde{Y}$. The composition of these two maps gives a $G$-equivariant proper birational map $\pi : \tilde{X} \to X$, such that $X$ is complete, smooth, toroidal and horospherical.

Proof of Corollary E First note that, $X$ being a spherical variety it has rational singularities (see for example [22, Corollary 2.3.4]). Then we have $R^q\pi_* \mathcal{O}_{\tilde{X}} = 0$ for all $q > 0$. By projection formula (see [15, Chapter III, Ex. 8.3]), we can see that $R^q\pi_* \pi^*\mathcal{L} = 0$ for all $q > 0$. Hence the Leray spectral sequence

$$H^p(X, R^q\pi_* \pi^*\mathcal{L}) \Rightarrow H^*(\tilde{X}, \pi^*\mathcal{L})$$

degenerates and we get $H^i(X, \mathcal{L}) = H^i(\tilde{X}, \pi^*\mathcal{L})$ for all $i \geq 0$. $\square$
Remark 4 We do not expect that our methods easily extend to study the cohomology groups of line bundles for a general spherical variety. The main issue is that our approach uses a stratification for horospherical varieties by Borel orbits and these orbits have a nice affine neighbourhoods. But for general spherical varieties, Borel orbits are more wild and they do not have a nice structure as in the case of horospherical varieties.

6 Description of cohomology groups

In this section we give a more explicit version of Theorem D for $X = G \times_P Y$, involving the combinatorics of the flag variety $G/P$ and of the toric variety $Y$. Let us keep the previous notations, in particular those of the Sect. 5.2. Recall Lemma 1: for any divisor $D$ on $X$, we have

$$D \sim \sum_{\alpha \in S \setminus S_P} d_\alpha D_\alpha + \sum_{i=1}^r d_i X_i.$$ 

Let $E_1 := \sum_{\alpha \in S \setminus S_P} d_\alpha (Bw_0^P s_\alpha P / P)$. Then $p^*E_1 = \sum_{\alpha \in S \setminus S_P} d_\alpha D_\alpha$. On the other hand, since $X$ is toroidal, the $G$-stable divisors $X_i$ of $X$ are of the form $G \times_P Y_i$, where $Y_i$ is a $T'$-stable divisor on $Y$. Let $E'_2 = \sum_{i=1}^r d_i Y_i$. Then we can rewrite Theorem D as

$$H^n(X, \mathcal{O}_X(D)) \simeq \bigoplus_{p+q=n} H^p(G \times_P \mathcal{O}_{G/P}(E_1)) \otimes_k H^q(Y, \mathcal{O}_Y(E'_2)).$$

Let us now describe more precisely the cohomology groups $H^p(G \times_P \mathcal{O}_{G/P}(E_1))$ and $H^q(Y, \mathcal{O}_Y(E'_2))$. Let $\sigma_\alpha$ be the fundamental weight associated to a simple root $\alpha$. Let $\rho$ be the half-sum of all positive roots, and define $w^*\lambda := w(\lambda + \rho) - \rho$ for any $w \in W$ and $\lambda$ in the weight lattice $\Delta$ of $T'$. Note that the line bundle $\mathcal{O}_{G/P}(E_1)$ is the homogeneous line bundle on $G/P$ corresponding to the weight $\sum_{\alpha \in \Lambda} d_\alpha \sigma_\alpha$. Then the Borel–Weil–Bott theorem states that $H^i(G \times_P \mathcal{O}_{G/P}(E_1)) = 0$ unless there exists a $w \in W_P$ of length $i$ such that $w^*(\sum_{\alpha \in \Lambda} d_\alpha \sigma_\alpha)$ lies in the interior of the dominant chamber, and in that case $H^i(G \times_P \mathcal{O}_{G/P}(E_1))$ is the dual of the irreducible highest weight $G$-module $V_{w^*(\sum_{\alpha \in \Lambda} d_\alpha \sigma_\alpha)}$.

On the other hand, we can describe the cohomology of line bundles on the smooth projective toric fibre by a result of Demazure (we refer to [7, Chapter 9] or [11, Section 3.4] for more details). Let $N$ be the lattice of one-parameter subgroups of the torus $T' = P/H$ and let $M$ be the character lattice of $T'$. Let $M_\mathbb{R} := M \otimes \mathbb{R}$ and $N_\mathbb{R} := N \otimes \mathbb{R}$. Then we have a natural bi-linear pairing $\langle - , - \rangle : M_\mathbb{R} \times N_\mathbb{R} \to \mathbb{R}$. Let $\Delta$ be the fan of the toric variety $Y$. By completeness of $X$, $\Delta$ is a complete fan in $N_\mathbb{R}$, that is $\cup_{\sigma \in \Delta^o} N_\mathbb{R}$. We denote by $\Delta(l)$ the set of $l$-dimensional cones in $\Delta$. Then we can parametrize $T'$-invariant prime divisors of $Y$ by the set $\Delta(1)$. Since the cohomology spaces $H^i(Y, \mathcal{O}_Y(E'_2))$ are $T'$-modules, these spaces have a weight decomposition:

$$H^i(Y, \mathcal{O}_Y(E'_2)) = \bigoplus_{m \in M} H^i(Y, \mathcal{O}_Y(E'_2))_m$$

and by a theorem of Demazure its degree $m$ part is described as follows: Let $Z_{E'_2}(m) := \{v \in N_\mathbb{R}, \langle m, v \rangle \geq \psi_{E'_2}(v)\}$, where $\psi_{E'_2}$ is the support function corresponding to the divisor $E'_2$ (for definition of the support function of a divisor we refer to [7, Theorem 4.2.12]). Then we have

$$H^i(Y, \mathcal{O}_Y(E'_2))_m = H^i_{Z_{E'_2}(m)}(N_\mathbb{R}, \mathbb{C}).$$
where $H^i_{Z_{E_2}^r}(m)(N_{\mathbb{R}}, \mathbb{C}) = H^i(N_{\mathbb{R}} \setminus Z_{E_2}^r(m); \mathbb{C})$ is the relative singular cohomology group (see [7, Theorem 9.1.3]). Now we can reformulate our result as follows:

**Corollary 2**

Let $X$ be a smooth complete toroidal horospherical variety. Then

(i) If there is no $w \in W_P$ such that $w \ast (\sum_{\alpha \in I} d_\alpha \varpi_\alpha)$ lies in the interior of the dominant chamber, then $H^n(X, \mathcal{O}_X(p^* E_1 + E_2)) = 0$ for all $n$.

(ii) If there exists $w \in W_P$ such that $w \ast (\sum_{\alpha \in I} d_\alpha \varpi_\alpha)$ lies in the interior of the dominant chamber, such a $w$ is unique. As a $T \times T'$-module and as a $g \times T'$-module we have

$$H^{l(w) + q}(X, \mathcal{O}_X(p^* E_1 + E_2)) = (V_w \ast (\sum_{\alpha \in I} d_\alpha \varpi_\alpha))^* \otimes \bigoplus_{m \in M} H^q_{Z_{E_2}^r}(m)(N_{\mathbb{R}}, \mathbb{C})$$

and $H^i(X, \mathcal{O}_X(p^* E_1 + E_2)) = 0$ for $i < l(w)$.

**Acknowledgements**

We would like to thank Michel Brion for valuable discussions and many critical comments. We also thank the anonymous referee for numerous comments and suggestions. The first author would also like to thank Max Planck Institute for Mathematics (Bonn) for the postdoctoral fellowship, and for providing very pleasant hospitality.

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