ISOMETRY GROUPS AMONG TOPOLOGICAL GROUPS

PIOTR NIEMIEC

Abstract. It is shown that a topological group \( G \) is topologically isomorphic to the isometry group of a (complete) metric space iff \( G \) coincides with its \( G_δ \)-closure in the Rajkov completion of \( G \) (resp. if \( G \) is Rajkov-complete). It is also shown that for every Polish (resp. compact Polish; locally compact Polish) group \( G \) there is a complete (resp. proper) metric \( d \) on \( X \) inducing the topology of \( X \) such that \( G \) is isomorphic to \( \text{Iso}(X, d) \) where \( X = \ell_2 \) (resp. \( X = [0, 1]^{\omega}; X = [0, 1]^{\omega} \setminus \{\text{point}\}) \). It is demonstrated that there are a separable Banach space \( E \) and a nonzero vector \( e \in E \) such that \( G \) is isomorphic to the group of all (linear) isometries of \( E \) which leave the point \( e \) fixed. Similar results are proved for an arbitrary complete topological group.

1. Introduction

In [13] Gao and Kechris proved that every Polish group is isomorphic to the (full) isometry group of some separable complete metric space. Later Melleray [18] and Malicki and Solecki [15] improved this result in the context of compact and, respectively, locally compact Polish groups by showing that every such group is isomorphic to the isometry group of a compact and, respectively, a proper metric space. (A metric space is proper iff each closed ball in this space is compact). All their proofs were complicated and based on the techniques of the so-called Katětov maps. Very recently [20] we involved quite a new method to characterize groups of homeomorphisms of a locally compact Polish space which coincide with the isometry groups of the space with respect to some proper metrics. As a consequence, we showed that every (separable) Lie group is isomorphic to the isometry group of another Lie group equipped with some proper metric and that every finite-dimensional [locally] compact Polish group is isomorphic to the isometry group of a finite-dimensional [proper locally] compact metric space. One of the aims of this paper is to give Gao’s-Kechris’, Melleray’s and Malicki’s-Solecki’s results more “explicit” and unified form:

1.1. Theorem. Let \( G \) be a Polish group.
(a) There is a complete compatible metric \( d \) on \( \ell_2 \) such that \( G \) is isomorphic to \( \text{Iso}(\ell_2, d) \).
(b) If \( G \) is compact, there is a compatible metric \( d \) on the Hilbert cube \( Q \) such that \( G \) is isomorphic to \( \text{Iso}(Q, d) \).
(c) If \( G \) is locally compact, there is a proper compatible metric \( d \) on \( Q \setminus \{\text{point}\} \) such that \( G \) is isomorphic to \( \text{Iso}(Q \setminus \{\text{point}\}, d) \).

We shall also prove the following

1.2. Theorem. For every Polish group \( G \) there exist a separable real Banach space \( E \) and a nonzero vector \( e \in E \) such that \( G \) is isomorphic to the group of all linear isometries of \( E \) (endowed with the pointwise convergence topology) which leave the point \( e \) fixed.

2010 Mathematics Subject Classification. Primary 37B05, 54H15; Secondary 57N20.
Key words and phrases. Polish group; isometry group; Hilbert cube; Hilbert space; Hilbert cube manifold; Rajkov-complete group; isometry group of a Banach space.
Our methods can be adapted to general settings and give a characterization of topological groups which are isomorphic to isometry groups of complete as well as incomplete metric spaces. To this end, we recall that a topological group $G$ is Rajkov-complete (or upper complete) iff it is complete with respect to the upper uniformity (see e.g. the remarks on page 1581 in [28] or [21] for more information on uniformities on topological groups). In other words, $G$ is upper complete if every net $\{x_\sigma\}_{\sigma \in \Sigma} \subset G$ satisfying the following condition (C) is convergent in $G$.

\[(C)\text{ For every neighbourhood } U \text{ of the neutral element of } G \text{ there is } \sigma_0 \in \Sigma \text{ such that both } x_\sigma x_\sigma^{-1} \text{ and } x_\sigma^{-1} x_\sigma \text{ belong to } U \text{ for any } \sigma, \sigma' \geq \sigma_0.\]

Equivalently, the net $\{x_\sigma\}_{\sigma \in \Sigma}$ satisfies (C) if both the nets $\{x_\sigma\}_{\sigma \in \Sigma}$ and $\{x_\sigma^{-1}\}_{\sigma \in \Sigma}$ are fundamental with respect to the left uniformity of $G$. We call Rajkov-complete groups briefly complete. With this terminology we follow Uspenskij [28]. The class of all complete topological groups coincides with the class of all absolutely closed topological groups (a topological group is absolutely closed if it is closed in every topological group containing it as a topological subgroup). It is well-known that for every topological group $G$ there exists a unique (up to topological isomorphism) complete topological group containing $G$ as a dense subgroup (see e.g. [21]). This complete group is called the Rajkov completion of $G$ and we shall denote it by $\overline{G}$.

Less classical are topological groups which we call $G_\delta$-complete. To define them, let us agree with the following general convention. Whenever $\tau$ is a topology on a set $X$, $\tau_3$ stands for the topology on $X$ whose base is formed by all $G_\delta$-sets (with respect to $\tau$) in $X$. (In particular, $G_\delta$-sets in $(X, \tau)$ are open in $(X, \tau_3)$.) Subsets of $X$ which are closed or dense in the topology $\tau_3$ are called $G_\delta$-closed and $G_\delta$-dense, respectively (see e.g. [5]).

It may be easily verified that if $(G, \tau)$ is a topological group, so is $(G, \tau_3)$. We now introduce

1.3. Definition. A topological group $G$ is $G_\delta$-complete if $(G, \tau_3)$ is a complete topological group (where $\tau$ is the topology of $G$).

Equivalently, a topological group $G$ is $G_\delta$-complete iff $G$ is $G_\delta$-closed in $\overline{G}$. The class of all $G_\delta$-complete groups is huge (see Proposition 4.3 below) and contains all complete as well as metrizable topological groups (more detailed discussion on this class is included in Section 4). However, there are topological groups which are not $G_\delta$-complete (see Example 4.4 below).

$G_\delta$-complete groups turn out to characterize isometry groups of metric spaces, as shown by

1.4. Theorem. Let $G$ be a topological group.

(A) The following conditions are equivalent:

- (A1) there exists a metric space $(X, d)$ such that $G$ is isomorphic to $\text{Iso}(X, d)$,
- (A2) $G$ is $G_\delta$-complete.

Moreover, if $G$ is $G_\delta$-complete, the space $X$ witnessing (A1) may be chosen so that $w(X) = w(G)$.

(B) The following conditions are equivalent:

- (B1) there exists a complete metric space $(X, d)$ such that $G$ is isomorphic to $\text{Iso}(X, d)$,
- (B2) $G$ is complete.

Moreover, if $G$ is complete, the space $X$ witnessing (B1) may be chosen so that $w(X) = w(G)$.

(By $w(X)$ we denote the topological weight of a topological space $X$.) A generalization of Theorems 1.1 and 1.2 has the following form:
1.5. Proposition. Let $G$ be a complete topological group of topological weight not greater than $\beta \geq \aleph_0$.

(a) There is a complete compatible metric $\rho$ on $H_\beta$ such that $G$ is isomorphic to $\text{Iso}(H_\beta, \rho)$, where $H_\beta$ is a real Hilbert space of (Hilbert space) dimension equal to $\beta$.

(b) There are an infinite-dimensional real Banach space $E$ whose topological weight is equal to $\beta$ and a nonzero vector $e \in E$ such that $G$ is isomorphic to the group of all linear isometries of $E$ which leave the point $e$ fixed.

As an immediate consequence of Theorem 1.4 and Proposition 1.5 we obtain

1.6. Corollary. Let $H$ be a Hilbert space of Hilbert space dimension $\beta \geq \aleph_0$ and let $\mathcal{G} = \{\text{Iso}(H, \rho) : \rho$ is a complete compatible metric on $H\}$. Then, up to isomorphism, $\mathcal{G}$ consists precisely of all complete topological groups of topological weight not exceeding $\beta$.

The paper is organized as follows. In Section 2 we give a new proof of the Gao-Kechris theorem mentioned above. We consider our proof more transparent, more elementary and less complicated. The techniques of this part are adapted in Section 3 where we demonstrate that every closed subgroup of the isometry group of a [complete] metric space $(X, d)$ is actually (isomorphic to) the isometry group of a certain [complete] metric space, closely related to $(X, d)$. This theorem is applied in the next part, where we establish basic properties of the class of all $G_\delta$-complete groups and prove Theorem 1.4. Fifth part contains proofs of Theorem 1.2, Proposition 1.5, point (a) of Theorem 1.1 and Corollary 1.6. The last section is devoted to the proofs of points (b) and (c) of the third of these results.

Notation and terminology. In this paper $\mathbb{N} = \{0, 1, 2, \ldots\}$ (and it is equipped with the discrete topology). All isomorphisms between topological groups are topological, all topological groups are Hausdorff and all isometries between metric spaces are, by definition, bijective. All normed vector spaces are assumed to be real. The topological weight of a topological space $X$ is denoted by $w(X)$ and it is understood as an infinite cardinal number. Isometry groups (and all their subsets) of metric as well as normed vector spaces are endowed with the pointwise convergence topology, which makes them topological groups. A Polish space (resp. group) is a completely metrizable separable topological space (resp. group). A metric on a topological space is compatible iff it induces the topology of the space. It is proper if all closed balls with respect to this metric are compact (in the topology induced by this metric). Whenever $(X, d)$ is a metric space, $a \in X$ and $r > 0$, $B_X(a, r)$ and $\bar{B}_X(a, r)$ stand for, respectively, the open and the closed $d$-balls with center at $a$ and of radius $r$. The Hilbert cube, that is, the countable infinite Cartesian power of $[0, 1]$, is denoted by $Q$ and $\ell_2$ stands for the separable Hilbert space. A map means a continuous function.

2. The Gao-Kechris theorem revisited

This part is devoted to the proof of the Gao-Kechris theorem [13] mentioned in the introductory part and stated below. Another proof may be found in [18].

2.1. Theorem. Every Polish group is isomorphic to the isometry group of a certain separable complete metric space.

For the purpose of this and the next section, let us agree with the following conventions. For every nonempty collection $\{X_s\}_{s \in S}$ of topological spaces, $\bigsqcup_{s \in S} X_s$ denotes the topological disjoint union of these spaces. In particular, whenever the notation $\bigsqcup_{s \in S} X_s$ appears, the sets $X_s$ ($s \in S$) are assumed to be pairwise disjoint.
(the same rule for the symbol ‘∪’). Further, for a function \( f: X \to X \), an integer number \( n \geq 1 \) and an arbitrary point \( w \), \( f^\otimes: X^n \to X^n \) is the \( n \)-th Cartesian power of \( f \) (that is, \( f^\otimes(x_1, \ldots, x_n) = (f(x_1), \ldots, f(x_n)) \) and \( f \times w: X \times \{w\} \to X \times \{w\} \) sends \((x, w)\) to \((f(x), w)\) for any \( x \in X \). Similarly, if \( d \) is a metric on \( X \), \( d^\otimes \) denotes the maximum metric on \( X^n \) induced by \( d \), i.e.
\[
d^\otimes((x_1, \ldots, x_n), (y_1, \ldots, y_n)) = \max_{j=1, \ldots, n} d(x_j, y_j);
\]
and \( d \times w \) is the metric on \( X \times \{w\} \) such that \((d \times w)((x, w), (y, w)) = d(x, y)\). Finally, for a topological space \( V \) and a map \( v: V \to V \) we put \( \hat{V} = (V \times \mathbb{N}) \cup (\bigcup_{n=2}^\infty V^n) \cup \mathbb{N} \) and define \( \hat{\nu}: \hat{V} \to V \) by the rules: \( \hat{\nu}(x, m) = (v(x), m) \), \( \hat{\nu}(m) = v^\otimes \) and \( \hat{\nu}(m) = m \) for any \( x \in V \), \( m \in \mathbb{N} \) and \( n \in \mathbb{N} \setminus \{0\} \). To avoid repetitions, for a metric space \((X, d)\) and arbitrary sets \( A, B \subset \mathbb{N} \) and \( C \subset \mathbb{N} \setminus \{0, 1\} \), let us say a metric \( \varrho \) on \((X \times A) \cup (\bigcup_{j \in C} X^j) \cup B(\subset \hat{X}) \) preserves \( d \) if the following three conditions are satisfied:
\[(AX1) \quad \varrho \text{ coincides with } d \times m \text{ on } X \times \{m\} \text{ for each } m \in A, \]
\[(AX2) \quad \varrho \text{ coincides with } d^\otimes \text{ on } X^n \text{ for each } n \in C, \]
\[(AX3) \quad \varrho(x, y) \geq 1 \text{ whenever } x \text{ and } y \text{ belong to distinct members of the collection } \{X \times \{m\}: m \in A\} \cup \{X^n: n \in C\} \cup \{\{k\}: k \in B\}. \]

Observe that (AX1)–(AX3) imply that

\[(AX4) \quad \text{if } \varrho \text{ preserves } d, \text{ then: \bullet } \varrho \text{ is compatible; \bullet } \varrho \text{ is complete if so is } d. \]

The main result of this section is the following

**2.2. Proposition.** Let \((X, d)\) be a separable bounded complete metric space and let \( G \) be a closed subgroup of \( \text{Iso}(X, d) \). There exists a metric \( \varrho \) on \( \hat{X} \) such that \( \varrho \) preserves \( d \) and the function

\[ G \ni u \mapsto \hat{u} \in \text{Iso}(\hat{X}, \varrho) \]

is a well defined isomorphism of topological groups.

The proof of Proposition 2.2 will be preceded by a few auxiliary results. The first of them is a kind of folklore and we leave its (simple) proof to the reader.

**2.3. Lemma.** Let \( \{(X_s, d_s)\}_{s \in S} \) be a nonempty family of metric spaces such that for \( A = \bigcap_{s \in S} X_s \) we have:

- \( X_s \cap X_{s'} = A \) and \( d_s|_{A \times A} = d_{s'}|_{A \times A} \) for any two distinct indices \( s \) and \( s' \) of \( S \),
- \( A \) is nonempty and closed in \( (X_s, d_s) \) for each \( s \in S \).

Let \( X = \bigcup_{s \in S} X_s \) and let \( d: X \times X \to [0, \infty) \) be given by the rules:

- \( d \) coincides with \( d_s \) on \( X_s \times X_s \) for every \( s \in S \),
- \( d(x, y) = \inf\{d_s(x, a) + d_s(a, y): a \in A\} \) whenever \( x \in X_s \setminus X_{s'} \) and \( y \in X_{s'} \setminus X_s \) for distinct indices \( s \) and \( s' \).

Then \( d \) is a well defined metric on \( X \) with the following property. Whenever \( f_s \in \text{Iso}(X_s, d_s) \) (\( s \in S \)) are maps such that \( f_s|_A = f_{s'}|_A \) and \( f_s(A) = A \) for any \( s, s' \in S \), then their union \( f := \bigcup_{s \in S} f_s \) (that is, \( f = f_s \) on \( X_s \)) is a well defined function such that \( f \in \text{Iso}(X, d) \).

The above result will be the main tool for constructing the metric \( \varrho \) appearing in Proposition 2.2.

In the next two results, \((X, p)\) is a complete nonempty metric space with

\[(2-1) \quad p < 1. \]

**2.4. Lemma.** Let \( J \subset \mathbb{N} \setminus \{0\} \) be a finite set such that \( n = \text{card}(J) > 1 \). There is a metric \( \lambda \) on \( F := [X \times (J \cup \{0\})] \cup X^n \) which has the following properties:
(a) \( \lambda \) preserves \( p \) and \( \lambda \leq 5 \),
(b) for every \( u \in \text{Iso}(X,p) \), \( \tilde{u}|_{p} \in \text{Iso}(F,\lambda) \),
(c) if \( g \in \text{Iso}(F,\lambda) \) is such that \( g(X \times \{j\}) = X \times \{j\} \) for each \( j \in J \cup \{0\} \), then \( g = \tilde{u}|_{p} \) for some \( u \in \text{Iso}(X,p) \).

**Proof.** With no loss of generality, we may assume that \( J = \{1, \ldots, n\} \). Let \( A = \{(x_{1}, \ldots, x_{n}) \in X^{n}: x_{1} = \ldots = x_{n}\} \) and \( \lambda_{0} \) be a metric on \( X_{0} := (X \times \{0\}) \cup A \) such that \( \lambda_{0}^{\prime} \) coincides with \( p \times 0 \) on \( X \times \{0\} \), with \( p^{\circ} \) on \( A \), and \( \lambda_{0}^{\prime}((x,0),(a,\ldots,a)) = 1 + p(x,a) \) for \( x \in X \) and \( (a,\ldots,a) \in A \). Now apply Lemma 2.3 for \( \{(X_{0},\lambda_{0}^{\prime}),(X^{n},p^{\circ})\} \) to obtain a metric \( \lambda_{0} \) on \( (X \times \{0\}) \cup X^{n} \) which extends both \( \lambda_{0}^{\prime} \) and \( p^{\circ} \). Observe that \( \lambda_{0} \) preserves \( p \), \( \lambda_{0} \leq 5 \) by (2-1), \( \tilde{u}|_{(X \times \{0\}) \cup X^{n}} \in \text{Iso}(X \times \{0\}) \cup X^{n}, \lambda_{0} \) for each \( u \in \text{Iso}(X,p) \) and for arbitrary \( x,x_{1},\ldots,x_{n} \in X \):

\[
\lambda_{0}((x,0),(x_{1},\ldots,x_{n})) = 1 \iff x_{1} = \ldots = x_{n} = x.
\]

Further, for \( j \in J \) let \( \lambda_{j} \) be a metric on \( (X \times \{j\}) \cup X^{n} \) such that \( \lambda_{j} \) coincides with \( p \times j \) on \( X \times \{j\} \), with \( j \) on \( X^{n} \) and \( \lambda_{j}((x,j),(x_{1},\ldots,x_{n})) = 1 + p(x,j) \) for any \( x,x_{1},\ldots,x_{n} \in X \) (\( \lambda_{j} \) is indeed a metric thanks to (2-1)). Similarly as before, notice that \( \lambda_{j} \) preserves \( p \), \( \lambda_{j} \leq 2 \) and for any \( x,x_{1},\ldots,x_{n} \in X \):

\[
\lambda_{j}((x,j),(x_{1},\ldots,x_{n})) = 1 \iff x_{j} = x.
\]

Now again apply Lemma 2.3 for the family \( \{(X_{j},\lambda_{j}): j \in J \cup \{0\}\} \) to obtain a metric \( \lambda \) on \( F \) which extends each of \( \lambda_{j} \) (\( j \in J \cup \{0\} \)). It follows from the construction and Lemma 2.3 that points (a) and (b) are satisfied. We turn to be.

Let \( g \) be as there. Let \( u: X \to X \) be such that \( u \times 0 = g|_{X \times \{0\}} \) and similarly, for \( j \in J \) let \( u_{j}: X \to X \) be such that \( u_{j} \times j = g|_{X \times \{j\}} \). Finally, put \( f = g|_{X^{n}}: X^{n} \to X^{n} \). Since \( \lambda \) preserves \( p \), \( u \in \text{Iso}(X,p) \). So, we only need to check that \( u_{1} = \ldots = u_{n} = u \) and \( f = u^{\circ} \). Let \( \pi_{j}: X^{n} \to X \) be the projection onto the \( j \)-th coordinate (\( j = 1,\ldots,n \)). For any \( x = (x_{1},\ldots,x_{n}) \in X^{n} \) and \( j \in J \) we have, by (2-3):

\[
1 = \lambda((\pi_{j}(x,j),x)) = \lambda(g(\pi_{j}(x,j)),g(x)) = \lambda(((u_{j} \circ \pi_{j})(x),j),f(x))
\]

and therefore, again by (2-3), \( u_{j} \circ \pi_{j} = \pi_{j} \circ f \). Consequently, \( f(x_{1},\ldots,x_{n}) = (u_{1}(x_{1}),\ldots,u_{n}(x_{n})) \). Finally, for any \( z \in X \) we have, by (2-2):

\[
1 = \lambda((z,0),(z,\ldots,z)) = \lambda(g(z,0),g(z,\ldots,z)) = \lambda((u(z),0),(u_{1}(z),\ldots,u_{n}(z))\).
\]

and hence, again by (2-2), \( u_{1}(z) = \ldots = u_{n}(z) = u(z) \). \( \square \)

**2.5. Lemma.** Let \( G \) be a subgroup of \( \text{Iso}(X,p) \) and let \( z \in X^{n} \) and \( J \subset \mathbb{N} \setminus \{0\} \) be such that \( \text{card}(J) = n \leq 1 \). Let \( D \) denote the closure (in \( X^{n} \)) of the set \( \{u^{\circ}(z): u \in G\} \). There exists a metric \( \mu \) on \( F := [X \times (J \cup \{0\})] \cup X^{n} \cup \{n-1\} \) which has the following properties:

(a) \( \mu \) preserves \( p \) and \( \mu \leq 11 \),

(b) \( \tilde{u}|_{p} \in \text{Iso}(F,\mu) \) for every \( u \in G \),

(c) for any \( g \in \text{Iso}(F,\mu) \) there is \( u \in \text{Iso}(X,p) \) such that \( g = \tilde{u}|_{p} \) and \( u^{\circ}(z) \in D \).

**Proof.** Without loss of generality, we may assume that \( J = \{1,\ldots,n\} \). Let \( \lambda \) be as in Lemma 2.4 (so, \( \lambda \) is a metric on \( F \setminus \{n-1\} \)). Let \( c_{0},\ldots,c_{n+1} \) be such that

\[
5 < c_{0} < c_{1} \ldots < c_{n+1} < 6.
\]

Put \( A = [X \times (J \cup \{0\})] \cup D \) and denote by \( \mu_{0} \) the metric on \( A \setminus \{n-1\} \) such that \( \mu_{0} \) coincides with \( \lambda \) on \( A \), \( \mu_{0}((x,j),n-1) = c_{j} \) for \( x \in X \) and \( j = 0,\ldots,n \), and \( \mu(y,n-1) = c_{n+1} \) for \( y \in D \) (\( \mu_{0} \) is a metric thanks to point (a) of Lemma 2.4, (AX3) and (2-4)). Now apply Lemma 2.3 for the family \( \{(A \cup \{n-1\},\mu_{0}),(F \setminus \{n-1\},\lambda)\} \)
to obtain a metric $\mu$ which extends both $\mu_0$ and $\lambda$. We infer from (2-4) and point (a) of Lemma 2.4 that point (a) is satisfied. Further, since $u^\circ(D) = D$ for each $u \in G$ and thanks to point (b) of Lemma 2.4, condition (b) is fulfilled as well (see Lemma 2.3). We turn to (c).

Let $g \in \text{Iso}(F, \mu)$. Since $n - 1$ is a unique point $q \in F$ such that $\mu(q, x) = c_0$ and $\mu(q, y) = c_1$ for some $x, y \in F$ (since $\lambda \leq 5 < c_0 < c_1$), we conclude that $g(n - 1) = n - 1$. Further, observe that for each $x \in X^n$, $\mu(x, n - 1) \geq c_{n+1}$, because of (AX3) and (2-4). Consequently, $X \times \{j\} = \{x \in F; \mu(x, n - 1) = c_j\}$ for $j = 0, 1, \ldots, n$. Thus, we see that $g(X \times \{j\}) = X \times \{j\}$ for such $j$'s. Since $g|_{F \setminus \{n-1\}} \in \text{Iso}(F \setminus \{n-1\}, \lambda)$ and $g(n-1) = n-1$, point (c) of Lemma 2.4 implies that there is $u \in \text{Iso}(X, p)$ such that $g = \hat{u}|_{F}$. Finally, $(z) = u^\circ(z) \in X^n$ and for $y \in X^n$, $\mu(y, n - 1) = c_{n+1}$ iff $y \in D$ (by (2-4) and (AX3)) which gives $u^\circ(z) \in D$ and finishes the proof. \hfill \Box

Proof of Proposition 2.2. Let $r \geq 1$ be such that $d < r$. Put $p = \frac{4}{3}d < 1$ and notice that $\text{Iso}(X, p) = \text{Iso}(X, d)$. Let $X_0 = \{x_n; n \geq 1\}$ be a dense subset of $X$. Let $J_1, J_2, \ldots$ be pairwise disjoint sets such that $\bigcup_{n=1}^{\infty} J_n = \mathbb{N}\setminus\{0\}$ and $\text{card}(J_n) = n+1$. For each $n \geq 2$ put $z_n = (x_1, \ldots, x_n) \in X^n$, $F_n = [X \times (J_{n-1} \cup \{0\})] \sqcup X^n \sqcup \{n-1\}$ and let $D_n$ be the closure (in $X^n$) of $\{u^\circ(z_n); u \in G\}$. Further, let $\mu_n$ be a metric on $F_n$ obtained from Lemma 2.5 (applied for $z_n$ and $J_{n-1}$). Now apply Lemma 2.3 for the collection $\{(F_n, \mu_n); n \geq 2\}$ to get a metric $\lambda_0$ on $\hat{X} \setminus \{0\}$ which extends each of $\mu_n (n \geq 2)$. In particular, $\lambda_0$ preserves $p$ and $\lambda_0 \leq 22$. Finally, we extend the metric $\lambda_0$ to a metric $\lambda$ on $\hat{X}$ in such a way that for $k \geq 0$, $\lambda(x, 0) = c_{k+1}$ for $x \in X \times \{k\}$, $\lambda(x, 0) = c_{k+2}$ for $x \in X^{k+2}$ and $\lambda(k+1, 0) = c_{k+3}$ where the numbers

\[c_{0,1}, c_{0,2}, c_{0,3}, c_{1,1}, c_{1,2}, c_{1,3}, \ldots\]

and belong to (22, 23) ($\lambda$ is a metric thanks to (AX3)). It follows from Lemma 2.3 and point (b) of Lemma 2.5 that $\hat{u} \in \text{Iso}(\hat{X}, \lambda)$ for any $u \in G$. It is clear that the function $G \ni u \mapsto \hat{u}$ is a group homomorphism and a topological embedding. We shall now show that it is also surjective.

Let $g \in \text{Iso}(\hat{X}, \lambda)$. Since $\emptyset$ is a unique point $q \in \hat{X}$ such that $\lambda(q, x) = c_{0,1}$ and $\lambda(q, y) = c_{0,2}$ for some $x, y \in \hat{X}$, we see that $g(0) = 0$. Consequently, $g(X \times \{k\}) = X \times \{k\}$, $g(X^{k+2}) = X^{k+2}$ and $g(k+1) = k+1$ for each $k \geq 0$, by (2-5). So, taking into account that $g|_{F_n} \in \text{Iso}(F_n, \mu)$, point (c) of Lemma 2.5 yields that there is $u \in \text{Iso}(X, p)$ such that $g = \hat{u}$ and $u(z_n) \in D_n$. The latter condition implies that there are elements $u_1, u_2, \ldots$ of $G$ which converge pointwise to $u$ on $X_0$. We now infer from the density of $X_0$ in $X$ that $u = \lim_{n \to \infty} u_n$ and in fact $u \in G$, by the closedness of $G$.

To end the proof, it suffices to put $g = r\lambda$. \hfill \Box

Proof of Theorem 2.1. Let $(H, \cdot)$ be a Polish group. First we involve a standard argument, used e.g. by Melleray [18] in his proof of this theorem: take a left invariant metric $d_0 \leq 1$ on $H$ and denote by $(X, d)$ the completion of $(H, d_0)$. Then, of course, $X$ is separable and for every $h \in H$ there is a unique $u_h \in \text{Iso}(X, d)$ such that $u_h(x) = hx$ for $x \in H$. Observe that the function $H \ni h \mapsto u_h \in \text{Iso}(X, d)$ is a group homomorphism as well as a topological embedding. Therefore its image $G$ is isomorphic to $H$. Since $G$ is a Polish subgroup of a Polish group, $G$ is closed in $\text{Iso}(X, d)$. Now it suffices to apply Proposition 2.2 and to use (AX4) to deduce the completeness of the metric obtained by that result. \hfill \Box
3. Closed subgroups of isometry groups

In this section we generalize the ideas of the previous part to the context of all isometry groups. Our aim is to show that a closed subgroup of the isometry group of a metric space is isomorphic to the isometry group of another metric space. We have decided to discuss the separable case separately, because in that case the proofs are more transparent and easier. Actually all tools were prepared in the previous section, except the following one:

3.1. Lemma. Let $X$ be a set with $\text{card}(X) \neq 2$ and $I \subset (0, \infty)$ be a nondegenerate interval. There is a metric $d : X \times X \to I \cup \{0\}$ such that the identity map of $X$ is the unique member of $\text{Iso}(X, d)$.

We shall prove Lemma 3.1 at the end of the section. Now we generalize the concepts involved in Section 3. Let $\beta$ be an infinite cardinal number and let $D_\beta$ denote a fixed discrete topological space of cardinality $\beta$. For a metrizable space $X$ and a function $f : X \to X$ let $\tilde{X}_\beta = [\bigcup_{n=1}^\infty (X^n \times D_\beta)] \cup D_\beta$ and let $\tilde{f}_\beta : \tilde{X}_\beta \to \tilde{X}_\beta$ be given by: $\tilde{f}_\beta = f^\oplus \times \xi$ on $X^n \times \{\xi\}$ and $\tilde{f}_\beta(\xi) = \xi$ for any $n \geq 1$ and $\xi \in D_\beta$.

(Notice that $w(\tilde{X}_\beta) = \beta$ provided $\beta \geq w(X)$.) For any $J \subset \mathbb{N} \setminus \{0\}$ and $A, B \subset D_\beta$, we say a metric $\varrho$ on $[\bigcup_{n\in J}(X^n \times A)] \cup B \subset \tilde{X}_\beta$ preserves a compatible metric $d$ on $X$ if the following two conditions are fulfilled:

(PR1) $\varrho$ coincides with $d^\oplus \times \xi$ on $X^n \times \{\xi\}$ for any $n \in J$ and $\xi \in A$.

(PR2) $\varrho(x, y) \geq 1$ whenever $x$ and $y$ belong to distinct members of the collection $\{X^n \times \{\xi\} : n \in J, \xi \in A\} \cup \{\{\eta\} : \eta \in B\}$.

As before, we see that (A4) is satisfied.

A counterpart of Proposition 2.2 in the general case is the following

3.2. Theorem. Let $\beta$ be an infinite cardinal number and $(X, d)$ be a nonempty bounded metric space such that $w(X) \leq \beta$. For any closed subgroup $G$ of $\text{Iso}(X, d)$ there exists a metric $\varrho$ on $\tilde{X}_\beta$ such that $\varrho$ preserves $d$ and the function

$$(3-1) \quad G \ni u \mapsto \tilde{u}_\beta \in \text{Iso}(\tilde{X}_\beta, \varrho)$$

is a well defined isomorphism of topological groups.

Proof. It follows from the proof of Proposition 2.2 that we may assume $d < 1$. Let $Z$ be a dense set in $X$ such that $\text{card}(Z) \leq \beta$. Fix arbitrary $\theta \in D_\beta$ and write $D_\beta \setminus \{\theta\}$ in the form $\bigcup_{n=0}^\infty S_n$ where $\text{card}(S_n) = \beta$ for any $n$ and

$$(3-2) \quad S_n \cap S_m = \emptyset \quad (n \neq m).$$

The set $S_0$ will be employed in the last part of the proof. For simplicity, put $S_* = \bigcup_{n=1}^\infty S_n$ and $S_* := \tilde{X}_\beta \setminus S_0 = [\bigcup_{n\geq 1}(X^n \times D_\beta)] \cup S_*$. It follows from (3-2) that for any $\xi \in D_\beta \setminus \{\theta\}$ there is a unique number $n(\xi) \in \mathbb{N}$ such that $\xi \in S_{n(\xi)}$. Further, for every $n \geq 1$ there are a surjection $\kappa_n : S_n \to Z^n$ and a bijection $\tau_n : S_n \to D_\beta$. Take a collection $\{J_\xi : \xi \in S_*\}$ such that for any $\xi, \eta \in S_*$:

(S1) $\text{card}(J_\xi) = n(\xi)$,

(S2) $J_\xi \cap J_\eta = \emptyset$ whenever $\xi \neq \eta$,

(S3) $\bigcup_{\xi \in S_*} J_\xi = D_\beta \setminus \{\theta\}$.

We deduce from (S1) and Lemma 2.5 that for each $\xi \in S_*$ there exists a metric $\mu_\xi$ on $F_\xi := [X \times (J_\xi \cup \{\theta\})] \cup (X^{n(\xi)} \times \{\tau_n(\xi)\})$ such that $\mu_\xi$ has the following properties:

(D1) $\mu_\xi$ preserves $d$ and $\mu_\xi \leq 11$,

(D2) $\tilde{u}_\beta | F_\xi \in \text{Iso}(F_\xi, \mu_\xi)$ for every $u \in G$,

(D3) for any $g \in \text{Iso}(F_\xi, \mu_\xi)$ there is $u \in \text{Iso}(X, d)$ such that $g = \tilde{u}_\beta | F_\xi$ and $u^\oplus(\kappa_n(\xi))$ belongs to the closure $B_\xi$ of $\{f^\oplus(\kappa_n(\xi)) : f \in G\}$ in $X^n$. 

ISOMETRY GROUPS AMONG TOPOLOGICAL GROUPS 7
Observe that (S2)–(S3), (D1) and the bijectivity of \( \tau_n \) imply that
\[(3-3) \quad F_ξ \cap F_η = X \times \{ θ \}\]
for distinct \( ξ, η ∈ S_*, \) and \( X \times \{ θ \} \) is closed in \( (F_ξ, µ_ξ) \) (cf. (PR2)). Moreover, it follows from (D1) that we may apply Lemma 2.3 for the family \( \{(F_ξ, µ_ξ): \ ξ ∈ S_*\} \).

Let \( µ \) be the metric on \( \bigcup _{ξ ∈ S_*} F_ξ = X_* \) obtained by that lemma. Then
\[(3-4) \quad \hat u_β \big|_{X_*} ∈ \text{Iso}(X_*, µ) \quad (u ∈ G)\]
(see (D2) and the last claim in Lemma 2.3) and
\[(3-5) \quad µ \text{ preserves } d \text{ and } µ ≤ 22\]
(by (D1)). What is more,
(i) \( g ∈ \text{Iso}(X_*, µ) \) is such that \( g(A) = A \) for any \( A ∈ S := \{ X^n × \{ ξ \}: n ≥ 1, ξ ∈ D_β \} \cup \{ \{ ξ \}: \ ξ ∈ S_* \} \), then \( g = \hat u_β \big|_{X_*} \) for some \( u ∈ G \).

Let us briefly show (i). If \( g \) is as there, then \( g(F_ξ) = F_ξ \) for any \( ξ \neq θ \). So, we infer from (D3) that \( g = (u_ξ) β \) on \( F_ξ \) for some \( u_ξ ∈ \text{Iso}(X, d) \) with
\[(3-6) \quad (u_ξ) β(κ_n(ξ)) ∈ B_ξ\]
where \( n = n(ξ) \). We conclude from (3-3) that \( u := u_ξ \) is independent of the choice of \( ξ \neq θ \). Consequently, \( g = \hat u_β \big|_{X_*} \). To end the proof of (i), it remains to check that \( u \in G \). Since \( κ_n \)’s are surjective, (3-6) yields that \( (u(z_1), \ldots, u(z_n)) \) belongs to the closure (in \( X^n \)) of \( \{(f(z_1), \ldots, f(z_n)): f ∈ G\} \) for any \( n ≥ 1 \) and \( z_1, \ldots, z_n ∈ Z \).

But this, combined with the fact that the function \( \text{Iso}(X, d) \ni f \mapsto f|Z ∈ X^Z \) is an embedding (when \( X^Z \) is equipped with the pointwise convergence topology), yields that \( u \) belongs to the closure of \( G \) in \( \text{Iso}(X, d) \). But \( G \) is a closed subgroup and we are done.

By Lemma 3.1, there is a metric
\[(3-7) \quad λ: S_0 × S_0 → \{ 0 \} ∪ [1, 2]\]
for which \( \text{Iso}(S_0, λ_0) = \{ \text{id}_{S_0} \} \) (\( \text{id}_{S_0} \) is the identity map on \( S_0 \)). Let \( S \) be as in (i).

Since \( \text{card}(S) = β = \text{card}(S_0) \), there is a one-to-one function \( v: S → \{ 11, 12 \}^{S_0} \).

We define a metric \( g \) on \( X_* ∪ S_0 = \hat X_β \setminus \{ θ \} \) by the rules:
- \( g = µ \) on \( X_* \times X_* \),
- \( g = λ \) on \( S_0 × S_0 \),
- \( g(ξ, η) = v(A)[η] \) for \( ξ ∈ X_* \) and \( η ∈ S_0 \), where \( A ∈ S \) is such that \( ξ ∈ A \) (such \( A \) is unique).

That \( g \) is indeed a metric it follows from (3-7), (3-5), axiom (PR2) for \( µ \) and the fact that for any \( η ∈ S_0, g(ξ, η) \) is constant on each member of \( S \). Finally, we extend the metric \( g \) to \( \hat X_β \) by putting \( g(ξ, θ) = 22 \) for \( ξ ∈ X_* \) and \( g(ξ, θ) = 23 \) for \( ξ ∈ S_0 \). Direct calculations show that \( g \) is indeed a metric on \( \hat X_β \) and that \( g \) preserves \( d \). It remains to check that (3-1) is a well defined surjection (cf. the proof of Proposition 2.2).

We infer from (3-4) and the fact that \( g(ξ, η) \) is constant on each member of \( S \) for any \( η ∈ S_0 \cup \{ θ \} \) that the function (3-1) is well defined. Now let \( g ∈ \text{Iso}(\hat X_β, g) \).

Since \( θ \) is a unique point \( ω ∈ \hat X_β \) such that \( \text{card}(\{ ξ ∈ \hat X_β: g(ξ, ω) = 23\}) > 1 \), we obtain \( g(θ) = θ \). Consequently, \( g(X_*) = X_* \) and \( g(S_0) = S_0 \). The latter yields that
\[(3-8) \quad g|_{S_0} = \text{id}_{S_0}\]
(because \( g \) extends \( λ \)). Now if \( ξ, η ∈ X_* \) are arbitrary, the injectivity of \( v \) and the definition of \( g \) imply that \( g(ξ, ·) = g(η, ·) \) on \( S_0 \) iff \( ξ \) and \( η \) belong to a common member of \( S \). But this, combined with (3-8), allows us to conclude that \( g(A) = A \) for any \( A ∈ S \). Now an application of (i) (recall that \( g \) extends \( µ \)) provides us the
existence of $u \in G$ for which $g = \tilde{u}_\beta$ on $X_*$. Since $g(\xi) = \xi$ for $\xi \notin X_*$, we see that $g = \tilde{u}_\beta$, which finishes the proof. \hfill \square

3.3. Remark. Under the notation and the assumptions of Theorem 3.2, if $M \geq 1$ is such that $d \leq M$ and $\varepsilon > 0$ is arbitrary, the metric $g$ appearing in the assertion of that theorem may be chosen so that $g \leq M + \varepsilon$. Indeed, the above proof provides the existence of a bounded metric $g$, say $g \leq C$ where $M < C < \infty$. Now it suffices to replace $g$ by $\omega \circ g$, where $\omega: [0, C] \to [0, M + \varepsilon]$ is affine on $[0, M]$ and $[M, C]$, and $\omega(0) = 0$, $\omega(M) = M$ and $\omega(C) = M + \varepsilon$.

To complete the proof of Theorem 3.2, we need to show Lemma 3.1. But that result immediately follows from the following much stronger

3.4. Proposition. Let $a$ and $b$ be two reals such that

\[(3-9) \quad 0 < a < b \leq 2a.\]

For every set $X$ having more than 5 points there is a metric $d: X \times X \to \{0, a, b\}$ such that

\[(3-10) \quad \text{Iso}(X, d) = \{\text{id}_X\}.\]

Proof. First of all, observe that any function $d: X \times X \to \{0, a, b\}$ which is symmetric and vanishes precisely on the diagonal of $X$ is automatically a complete metric, which follows from (3-9). So, we only need to take care of (3-10). For the same reason, we may (and do) assume, with no loss of generality, that $a = 1$ and $b = 2$. We shall make use of transfinite induction with respect to $\beta = \text{card}(X) > 5$. Everywhere below in this proof, for $x \in X$, by $S(x)$ we denote the set of all $y \in X$ with $d(x, y) = 1$. Since we have to define a metric taking values in $\{0, 1, 2\}$, it is readily seen that it suffices to describe the sets $S(x)$ ($x \in X$).

First assume $\beta = n \geq 6$ is finite. We may assume that $X = \{1, \ldots, n\}$. Our metric $d$ is defined by the following rules: $S(1) = \{2\}$, $S(2) = \{1, 3, 4, 5\}$, $S(3) = \{2, 4\}$, $S(4) = \{2, 3, 5\}$, $S(5) = \{2, 4, 6\}$, $S(n) = \{n - 1\}$ and $S(j) = \{j - 1, j + 1\}$ if $5 < j < n$. Take $g \in \text{Iso}(X, d)$ and observe that:

- $g(2) = 2$, since 2 is the only point $j \in X$ such that $\text{card}(S(j)) = 4$;
- $g(1) = 1$, because 1 is the unique point $j \in X$ for which $S(j) = \{2\}$;
- $g(3) = 3$, since 3 is the only point $j \in X$ such that $\text{card}(S(j)) = 2$ and $2 \in S(j)$;
- $g(4) = 4$, because 4 is the unique point $j \in X$ for which $2 \neq j \in S(3)$;
- $g(5) = 5$, since 5 is the only point $j \in X$ such that $j \in S(4) \setminus \{2, 3\}$.

Now it is easy to check, using induction, that $g(j) = j$ for $j = 6, \ldots, n$.

When $\beta = \aleph_0$, we may assume $X = \mathbb{N}$. Define a metric $d: \mathbb{N} \times \mathbb{N} \to \{0, 1, 2\}$ by $d(n, m) = \min(|m - n|, 2)$. It is left to the reader that $\text{Iso}(\mathbb{N}, d) = \{\text{id}_\mathbb{N}\}$ (use induction to show that $g(n) = n$ for any $n \in \mathbb{N}$ and $g \in \text{Iso}(\mathbb{N}, d)$). Below we assume that $\beta > \aleph_0$ is such that for every infinite $\alpha < \beta$ the proposition holds for an arbitrary set $X$ of cardinality $\alpha$. For simplicity, for any uncountable cardinal $\gamma$ we denote by $I_\gamma$ the set of all cardinals $\alpha$ for which $\aleph_0 \leq \alpha < \gamma$. To get the assertion, we consider three cases.

First assume $\beta$ is not limit, i.e. $\beta$ is the immediate successor of an infinite cardinal $\alpha$. We may assume that $X$ is the union of three pairwise disjoint sets $X', X' \times Y$ and $\{a\}$ where $\text{card}(X') = \alpha$ and $\text{card}(Y') = \beta$. It follows from the transfinite induction hypothesis that there is a metric $d': X' \times X' \to \{0, 1, 2\}$ such that $\text{Iso}(X', d') = \{\text{id}_{X'}\}$. Since $\beta \leq \omega$, there is a one-to-one function $\mu: X' \times Y \to \{1, 2\}^{X'}$ such that

\[(3-11) \quad [\mu(x, y)](x) = 1 \quad (x \in X', \ y \in Y)\]
(such a function $\mu$ may easily be constructed by transfinite induction with respect to an initial well order on $X' \times Y$). We now define a metric $d$ on $X$ (with values in $\{0, 1, 2\}$) by the rules:

(d1) $d = d'$ on $X' \times X'$,
(d2) $d((x,y),(x',y')) = 1$ if $(x,y)$ and $(x',y')$ are distinct elements of $X' \times Y$,
(d3) $d((x,y),x') = [\mu(x,y)](x')$ if $x,x' \in X'$ and $y \in Y$, and
(d4) $d(x,a) = 1$ and $d(x,\alpha) = 2$ for any $x \in X'$ and $y \in Y$.

Observe that $S(x) \supset \{x\} \times Y$ and $S(x,\alpha) \supset (X' \times Y) \setminus \{(x,y)\}$ for any $x \in X'$ and $y \in Y$ (thanks to (3-11) and (d2)–(d3)); and

\[
S(\alpha) = X'
\]

(by (d4)). We infer from these facts that $\alpha$ is a unique point $x \in X$ such that $\text{card}(S(x)) = \alpha$. Consequently, if $\alpha \in \text{Iso}(X,d)$, then $g(\alpha) = a$ and $g(X') = X'$ (because of (3-12)). So, $g|_{X'} \in \text{Iso}(X',d')$ and therefore $g(x) = x$ for any $x \in X'$.

Finally, if $x,x' \in X'$ and $y \in Y$ are arbitrary, then $g(x,y) \notin X'$ and $d(g(x,y),x') = d(x,y),x')$, which yields that $\mu(g(x,y)) = \mu(x,y)$. So, $g(x,y) = (x,y)$ (since $\mu$ is one-to-one) and we are done.

Now we assume that $\beta$ is limit and $\text{card}(I_\beta) < \beta$. For simplicity, put $I = I_\beta$.

Let $\{X_\alpha\}_{\alpha \in I}$ be a family of pairwise disjoint sets such that

\[
X_\alpha \cap I = \emptyset \quad \text{and} \quad \text{card}(X_\alpha) = \alpha < \beta \quad (\alpha \in I).
\]

Note that the set $X_\alpha = \bigsqcup_{I \in I} X_\alpha$ is of cardinality $\beta$ and therefore we may assume $X = \{\emptyset\} \cup I \cup X_\alpha$ (recall that this notation means that $\omega \notin I \cup X_\alpha$). It follows from the transfinite induction hypothesis that there are metrics $d_I : I \times I \to \{0, 1, 2\}$ and $d_\alpha : X_\alpha \times X_\alpha \to \{0, 1, 2\}$ ($\alpha \in I$) for which the groups $\text{Iso}(I,d_I)$ and $\text{Iso}(X_\alpha,d_\alpha)$ are trivial. We define a metric $d$ on $X$ as follows:

(d1') $d = d_I$ on $I \times I$ and $d = d_\alpha$ on $X_\alpha \times X_\alpha$ for any $\alpha \in I$,
(d2') $d((x,y),x') = 1$ if $x$ and $y$ belong to different members of the collection $\{X_\alpha\}_{\alpha \in I}$,
(d3') $d(\alpha,x) = 2$ for $x \in X_\alpha$ and $d(\alpha,x) = 1$ for $x \in X_\alpha \setminus X_\alpha$ ($\alpha \in I$),
(d4') $d(\alpha,\omega) = 1$ and $d(x,\omega) = 2$ for any $\alpha \in I$ and $x \in X_\alpha$.

Observe that for any $\alpha \in I$ and $x \in X_\alpha$, $S(\alpha) \supset X_\alpha \setminus X_\alpha$ (cf. (d3')) and $S(x) \supset X_\alpha \setminus X_\alpha$ as well (cf. (d2')). At the same time, $S(\omega) = I$ (by (d4')) and hence $\omega$ is a unique point $x \in X$ such that $\text{card}(S(x)) < \beta$ (cf. (3-13)). Consequently, if $g \in \text{Iso}(X,d)$, then $g(\omega) = \omega$, $g(I) = I$ and $g(X_\alpha) = X_\alpha$. Then $g|_I \in \text{Iso}(I,d_I)$ (cf. (d1')) and hence $g(\alpha) = \alpha$ for each $\alpha \in I$. Now use (d3') to conclude that $g(X_\alpha) = X_\alpha$ for any $\alpha \in I$. So, according to (d1'), $g|_{X_\alpha} \in \text{Iso}(X_\alpha,d_\alpha)$ for every $\alpha \in I$ and consequently $g(x) = x$ for all $x \in \bigcup_{\alpha \in I} X_\alpha = X_\alpha$, and we are done.

Finally, assume $\beta$ is limit and $\text{card}(I_\beta) = \beta$. Then we may assume $X = I_\beta$.

Since

\[
\text{card}(I_\alpha) \leq \alpha < \beta
\]

whenever $\alpha \in X$, for every $\alpha \in X$ there is a cardinal $\gamma(\alpha) \in X$ such that

\[
\text{card}(\{\xi : \alpha < \xi \leq \gamma(\alpha)\}) = \alpha.
\]

Now define a metric $d : X \times X \to \{0, 1, 2\}$ by the following rule: if $\alpha_0 \leq \alpha_1 < \alpha_2 < \beta$, then $d(\alpha_1,\alpha_2) = 1$ if $\alpha_2 \leq \gamma(\alpha_1)$. It is easy to check that then $\text{card}(S(\alpha)) = \alpha$ for any $\alpha \in X$ (thanks to (3-14) and (3-15)) and hence the identity map is a unique isometry on $(X,d)$.

\[\square\]

4. Models for $\mathcal{G}_\beta$-complete groups

We begin this section with a useful characterization of $\mathcal{G}_\beta$-complete groups.
4.1. Proposition. For a topological group $G$ all conditions stated below are equivalent.

(I) $G$ is $G_3$-complete.

(II) $G$ is isomorphic to a $G_3$-closed subgroup of a complete topological group.

(III) $G$ is $G_3$-closed in every topological group which contains $G$ as a topological subgroup.

(IV) Every net $\{x_\sigma\}_{\sigma \in \Sigma}$ of elements of $G$ satisfying the following condition (CC) is convergent in $G$.

(CC) Every sequence $U_1, U_2, \ldots$ of neighbourhoods of the neutral element of $G$ there exist points $y, z \in G$ and a sequence $\sigma_1, \sigma_2, \ldots \in \Sigma$ such that both $x_{\sigma_1}^{-1}y$ and $x_{\sigma_2}z^{-1}$ belong to $U_n$ whenever $n \geq 1$ and $\sigma \geq \sigma_n$.

(V) Every net $\{x_\sigma\}_{\sigma \in \Sigma}$ of elements of $G$ satisfying the following condition (CC') is convergent in $G$.

(CC') For every continuous left-invariant pseudometric $d$ on $G$ there are points $y, z \in G$ such that $\lim_{\sigma \in \Sigma} d(x_\sigma, y) = \lim_{\sigma \in \Sigma} d(x_\sigma^{-1}, z^{-1}) = 0$.

Proof. Everywhere below $\tau$ is the topology of $G$ and $e$ is its neutral element.

First assume $G$ is a $G_3$-closed subgroup of a complete group $H$. We want to show that $G$ is $G_3$-complete. Let $\{x_\sigma\}_{\sigma \in \Sigma} \subset G$ be a net which satisfies condition (C) with respect to the topology $\tau_3$. Then it satisfies this condition with respect to $\tau$ as well. Since $H$ is complete, there is $y \in H$ such that $\lim_{\sigma \in \Sigma} x_\sigma = y$. It suffices to check that $y$ belongs to the $G_3$-closure of $G$ in $H$. Take a $G_3$-subset of $H$ containing $y$ and write $Ay^{-1}$ in the form $Ay^{-1} = \bigcap_{n=1}^{\infty} U_n$ where each $U_n$ is an open in $H$ and contains $e$. Let $V_1, V_2, \ldots$ be a sequence of open (in $H$) neighbourhoods of $e$ such that the closure (in $H$) of $V_n$ is contained in $V_{n-1} \cap U_n$ for each $n$. Let $V_0 = H$. Then $F := \bigcap_{n=1}^{\infty} V_n$ is a closed $G_3$-subset of $H$ and $e \in F \subset Ay^{-1}$. It follows from our assumption about the net that there is $\sigma_0 \in \Sigma$ such that $x_{\sigma_0}^{-1} \in F$ for any $\sigma \geq \sigma_0$. We now infer from the closedness of $F$ in $H$ that $x_{\sigma_0}y^{-1} \in F$ as well and consequently $x_{\sigma_0} \in A$, which shows that $y$ belongs to the $G_3$-closure of $G$. This proves that (II) is followed by (I). Conversely, if $G$ is a $G_3$-complete subgroup of a topological group $K$ and $O$ denotes the topology of $K$, then $\tau_3$ coincides with the topology (on $G$) of a subspace inherited from $(K, \tau_3)$. It follows now from the completeness of $(G, \tau_3)$ that $G$ is closed in $(K, \tau_3)$ or, equivalently, that $G$ is $G_3$-closed in $K$, which proves that (III) is implied by (I). Since (II) obviously follows from (III), in this way we have shown that conditions (I), (II) and (III) are equivalent. We shall now show that (II) is equivalent to (IV) and then that (IV) is equivalent to (V).

If (II) is fulfilled, then $G$ is $G_3$-closed in $\overline{G}$. Let $\{x_\sigma\}_{\sigma \in \Sigma}$ be a net of elements of $G$ which satisfies condition (CC). Then it fulfils condition (C) as well and hence there is $w \in G$ such that $\lim_{\sigma \in \Sigma} x_\sigma = w$. It suffices to check that $w \in G$ or, equivalently, that $w$ belongs to the $G_3$-closure of $G$. To this end, take any $G_3$-subset $A$ of $\overline{G}$ which contains $w$. Write $Aw^{-1} = \bigcap_{n=1}^{\infty} V_n$ where $V_1, V_2, \ldots$ are open neighbourhoods of $e$. For each $n \geq 1$ take a neighbourhood $U_n$ of $w$ with $U_n = U_{n-1}$ and $U_n \cdot U_n \subset V_n$. Now let $y, z$ and $\sigma_1, \sigma_2, \ldots$ be as in (CC), applied for the sequence $U_1 \cap G, U_2 \cap G, \ldots$. Fix for a moment $n \geq 1$. Choose $\sigma \geq \sigma_n$ such that $x_{\sigma}^{-1}w \in U_n$. Then $zw^{-1} = (x_{\sigma}^{-1}z^{-1})^{-1}(x_{\sigma}^{-1}w^{-1}) \subset U_n \cdot U_n \subset V_n$. So, $zw^{-1} \in \bigcap_{n=1}^{\infty} V_n = Aw^{-1}$, which implies that $w \in A$. Consequently, $A \cap G \neq \emptyset$ and we are done.

The converse implication goes similarly: when (IV) is satisfied, we show that $G$ is $G_3$-closed in $\overline{G}$. Let $w \in \overline{G}$ belong to the $G_3$-closure of $G$. Then, of course, $w$ is in the closure of $G$ and thus there is a net $\{x_\sigma\}_{\sigma \in \Sigma} \subset G$ which converges to $w$. To prove that $w \in G$, it is enough to verify that (CC) is fulfilled. To this end, fix a sequence $U_1, U_2, \ldots$ of neighbourhoods of $e$ and choose its open symmetric neighbourhoods $V_1, V_2, \ldots$ such that $V_n \cdot V_n \subset U_n$ ($n \geq 1$). We conclude from the fact that $w$
is in the $G_\delta$-closure of $G$ that there is $y \in G$ such that $y \in \bigcap_{n=1}^{\infty} (V_n w \cap w V_n)$. Fix $n \geq 1$. There is $\sigma_n \in \Sigma$ such that both $x_\sigma w^{-1}$ and $w^{-1} x_\sigma$ belong to $V_n$ for $\sigma \geq \sigma_n$. Then, for such $\sigma$'s, $x_\sigma y^{-1} = (x_\sigma w^{-1})(yw)^{-1} \subset V_n \cdot V_n^{-1} \subset U_n$ and $x_\sigma^{-1} y = (w^{-1} x_\sigma)^{-1} (w y)^{-1} \subset V_n^{-1} \cdot V_n \subset U_n$ as well. This shows that (CC) is satisfied for $z = y$, and we are done.

Point (V) is easily implied by (IV) (for a fixed continuous left-invariant pseudometric $d$ and a net satisfying (CC) apply (CC) for $U_n = \{x \in G : d(x, e) < 2^{-n}\}$). The converse implication follows from the well-known fact that for an arbitrary sequence $U_1, U_2, \ldots$ of neighbourhoods of $e$ there exists a left-invariant pseudometric $d$ on $G$ such that $\{x \in G : d(x, e) < 2^{-n}\} \subset U_n$ for every $n \geq 1$ (see e.g. the proof of the Kakutani-Birkhoff theorem on the metrizability of topological groups presented in [6, Theorem 6.3]).

4.2. Remark. The proof of Proposition 4.1 shows that points (IV) and (V) of that result may be weakened by assuming that every net satisfying condition (CC) or (CC') is $G_\delta$-closed. However, to prove Theorem 1.4 we need (IV) in its present form.

Now we can give many examples of $G_\delta$-complete groups. We inform that by the Cartesian product of a family $\{G_s\}_{s \in S}$ of topological groups we mean the 'full' Cartesian product $\prod_{s \in S} G_s$ of them and by the direct product of this family we mean the topological subgroup $\bigoplus_{s \in S} G_s$ of $\prod_{s \in S} G_s$ consisting of all its finitely supported elements.

4.3. Proposition. Each of the following topological groups is $G_\delta$-complete.

(a) A $G_\delta$-closed subgroup of a $G_\delta$-complete group. A complete group.

(b) The Cartesian as well as the direct product of arbitrary collection of $G_\delta$-complete groups.

(c) A topological group which is the countable union of its subgroups each of which is $G_\delta$-complete.

(d) A topological group which is $\sigma$-compact as a topological space. In particular, all countable topological groups are $G_\delta$-complete.

(e) A topological group in which singletons are $G_\delta$. In particular, metrizable groups are $G_\delta$-complete.

(f) $G = \text{Iso}(X, d)$ for an arbitrary metric space $(X, d)$. Moreover, $w(G) \leq w(X)$, and $G$ is complete provided $(X, d)$ is complete.

Proof. In each point we involve Proposition 4.1.

To prove point (a), use the equivalence between conditions (I) and (III) in Proposition 4.1. We turn to (b). Let $\{G_s\}_{s \in S}$ be a nonempty family of $G_\delta$-complete groups and let $G = \prod_{s \in S} G_s$. Let $x_\sigma = (x_\sigma^{(s)})_{s \in S} \in G$ ($\sigma \in \Sigma$) be a net satisfying condition (CC). It remains to check that for any $t \in S$, the net $\{x_t^{(s)}\}_{s \in S} \subset G_t$ satisfies condition (CC) as well (because then it will be convergent), which is immediate: if $V_1, V_2, \ldots$ is a sequence of neighbourhoods of the neutral element of $G_t$, apply (CC) for the sequence $U_1, U_2, \ldots$ with $U_j := \{(x_t^{(s)})_{s \in S} \in G : x_t^{(s)} \in V_j\}$ ($j \geq 1$) to obtain two points $y, z \in G$ and then use their $t$-coordinates. Now to prove that also $H := \bigoplus_{s \in S} G_s$ is $G_\delta$-complete, it suffices to check that it is $G_\delta$-closed in $G$ by (a). But if $y = (y_s)_{s \in S} \in G \setminus H$, there is a countable (infinite) set $S' \subset S$ such that $y_s \neq e_s$ for any $s \in S'$, where $e_s$ is the neutral element of $G_s$. Then the set $A := \{z_\sigma \in G : \forall s \in S' : z_\sigma \neq e_s\}$ is a $G_\delta$-subset of $G$ which contains $y$ and is disjoint from $H$, which finishes the proof of (b).

Since the proofs of points (c) and (d) are similar, we shall show only (c). Let $G = \bigcup_{n=1}^{\infty} G_n$ where $G_n$ is $G_\delta$-complete for any $n$. Let $y \in G \setminus G$. Then $y \notin G_n$ and $G_n$ is $G_\delta$-closed in $G$. Consequently, there are $G_\delta$-subsets $A_1, A_2, \ldots$ of $G$ containing
y such that $A_n \cap G_\delta = \emptyset$. Then $A := \bigcap_{n=1}^{\infty} A_n$ is also a $G_\delta$-subset of $G$ containing $y$, and $A \cap G = \emptyset$. This shows that $y$ is not in the $G_\delta$-closure of $G$ and we are done.

Further, if all singletons are $G_\delta$ in $G$, then $\tau_3$ is discrete and hence $G$ is $G_\delta$-complete. This proves (e).

Finally, we turn to (f). The second and the third claims of (f) are well-known, but for the sake of completeness we shall prove them too. Let $(X,d)$ be a metric space and $G = \text{Iso}(X,d)$. Let $\{u_\sigma\}_{\sigma \in \Sigma} \subset G$ be a net satisfying condition $(CC')$. Fix $x \in X$ and put $g: G \times G \ni (u,v) \mapsto d(u(x),v(x)) \in [0,\infty)$. Observe that $g$ is a left-invariant continuous pseudometric on $G$. It follows from $(CC')$ that there are $f,g \in G$ such that $\lim_{\sigma \in \Sigma} d(u_{\sigma}(x),f(x)) = \lim_{\sigma \in \Sigma} d(u_{\sigma}^{-1}(x),g(x)) = 0$. We conclude that both the nets $\{u_{\sigma}(x)\}_{\sigma \in \Sigma}$ and $\{u_{\sigma}^{-1}(x)\}_{\sigma \in \Sigma}$ converge in $X$. So, we may define $u,v: X \to X$ by $u(x) = \lim_{\sigma \in \Sigma} u_{\sigma}(x)$ and $v(x) = \lim_{\sigma \in \Sigma} u_{\sigma}^{-1}(x)$ ($x \in X$). It is readily seen that both $u$ and $v$ are isometric. What is more, a standard argument proves that $u \circ v = v \circ u = id_X$ and hence $u \in G$ and $\lim_{\sigma \in \Sigma} u_{\sigma} = u$. So, $G$ is $G_\delta$-complete. Furthermore, if $D$ is a dense subset of $X$ such that $\text{card}(D) = w(X)$, then the function $G \ni g \mapsto g|_D \in X^D$ is a topological embedding (when $X^D$ is equipped with the pointwise convergence topology) and therefore $w(G) \leq w(X^D) \leq w(X)$. Finally, if $(X,d)$ is in addition complete and $\{u_\sigma\}_{\sigma \in \Sigma} \subset G$ is a net satisfying (C), similar argument to that above shows that then for any $x \in X$ the nets $\{u_{\sigma}(x)\}_{\sigma \in \Sigma}$ and $\{u_{\sigma}^{-1}(x)\}_{\sigma \in \Sigma}$ are fundamental in $(X,d)$ and hence converge. It now follows from the previous part of the proof that $\{u_{\sigma}\}_{\sigma \in \Sigma}$ is convergent in $G$, which finishes the proof.

\[ \square \]

4.4. Example. As we announced in the introductory part, not every topological group is absolutely $G_\delta$-closed. Let us briefly justify our claim. Let $S$ be an uncountable set and for each $s \in S$ let $G_s$ be a nontrivial complete group with the neutral element $e_s$. Then $G := \prod_{s \in S} G_s$ is a complete group as well and

\[ G_0 = \{(x_s)_{s \in S} \in G: \quad \text{card}\{s \in S: x_s \neq e_s\} \leq \aleph_0\} \]

is a proper subgroup of $G$ which is $G_\delta$-dense in $G$ (and thus $G_0$ is not $G_\delta$-closed in $G$). Indeed, if $y = (y_s)_{s \in S} \in G$ and $A$ is a $G_\delta$-subset of $G$ containing $y$, then there is a countable set $S_0 \subset S$ such that $\{(x_s)_{s \in S} \in G: \quad \forall s \in S_0: \quad x_s = y_s\} \subset A$; then $z \in G_0 \cap A$ where $z_s = y_s$ for $s \in S_0$ and $z_s = e_s$ otherwise.

Due to a private communication of the author, the same example was found independently by Vladimir Uspenskij.

We are almost ready for proving Theorem 1.4. For the purpose of its proof and the next result, let us introduce auxiliary notations and terminology. Whenever $d$ and $d'$ are two bounded pseudometrics on a common nonempty set $X$, we put

\[ \|d - d'\|_\infty := \sup_{x,y \in X} |d(x,y) - d'(x,y)|. \]

Further, the relation $R := \{(x,y) \in X \times X: \quad d(x,y) = 0\}$ is an equivalence on $X$. Let $\pi: X \to X/R$ be the canonical projection. The function $D: (X/R) \times (X/R) \ni (\pi(x),\pi(y)) \mapsto d(x,y) \in [0,\infty)$ is a well defined metric on $X/R$. We call a triple $(Y,g;\xi)$ a metric space associated with $(X,d)$ if $(Y,g)$ is a metric space isometric to $(X/R,D)$ and $\xi$ is a function of $X$ onto $Y$ such that there is an isometry $g: (X/R,D) \to (Y,g)$ for which $\xi = g \circ \pi$. Observe that then $g(\xi(x),\xi(y)) = d(x,y)$ for any $x,y \in X$.

With use of the following result we shall take care of condition $w(X) = w(G)$ in point (A) of Theorem 1.4.

4.5. Lemma. Let $G$ be a topological group and $\{g_s\}_{s \in S}$ be a collection of bounded continuous left-invariant metrics on $G$. For each $s \in S$, let $(X_s,d_s;\pi_s)$ be a metric
space associated with \((G, \varrho_s)\) chosen so that the sets \(X_s\)'s are pairwise disjoint. There exists a metric \(d\) on \(X := \bigcup_{s \in S} X_s\) with the following properties:

\begin{enumerate}[(D1)]
\item \(\frac{1}{2} \sqrt{d_s(x, y) \leq d(x, y) \leq \sqrt{d_s(x, y)}}\) for any \(x, y \in X_s\) and \(s \in S\),
\item \(d(\pi_s(a), \pi_t(a)) \leq \sqrt{\|g_s - g_t\|_\infty}\) for any \(a \in G\) and \(s, t \in T\),
\item each of the sets \(X_s\) \((s \in S)\) is closed in \((X, d)\),
\item \(d(\pi_s(ah), \pi_t(ah)) = d(\pi_s(g), \pi_t(h))\) for any \(a, g, h \in G\) and \(s, t \in S\).
\end{enumerate}

**Proof.** To simplify arguments, for each \(x \in X\) denote by \(\kappa(x)\) the unique index \(s \in S\) such that \(x \in X_s\). Define a function \(v: X \times X \to [0, \infty)\) as follows:

\[ v(x, y) = \|g_{\kappa(x)} - g_{\kappa(y)}\|_\infty + \inf\{d_{\kappa(x)}(x, \pi_{\kappa(x)}(g)) + d_{\kappa(y)}(\pi_{\kappa(y)}(g), y) : g \in G\}. \]

Observe that:

\begin{itemize}
\item \(v(x, x) = 0\) and \(v(y, x) = v(x, y)\) for any \(x, y \in X\),
\item \(v(x, y) = d_s(x, y)\) for any \(x, y \in X_s\) and \(s \in S\),
\item \(v(\pi_s(g), \pi_t(g)) = \|g_s - g_t\|_\infty\) whenever \(s, t \in S\) and \(g \in G\),
\item \(v(\pi_s(g), \pi_t(h)) \geq \|g_s - g_t\|_\infty\) for all \(s, t \in S\) and \(g, h \in G\),
\item \(v(\pi_s(ah), \pi_t(ah)) = v(\pi_s(g), \pi_t(h))\) for any \(a, g, h \in G\) and \(s, t \in S\).
\end{itemize}

Let us now show that for any \(x_0, x_1, x_2, x_3 \in X\) and each \(\varepsilon > 0\):

\[ (4-1) \max_{j=1,2,3} v(x_{j-1}, x_j) < \varepsilon \implies v(x_0, x_3) < 8\varepsilon. \]

Assume \(v(x_{j-1}, x_j) < \varepsilon\) \((j = 1, 2, 3)\). This means that there are \(a_1, a_2, a_3 \in G\) for which

\[ (4-2) \|g_{\kappa(x_{j-1})} - g_{\kappa(x_j)}\|_\infty + d_{\kappa(x_{j-1})}(x_{j-1}, \pi_{\kappa(x_{j-1})}(a_j)) + d_{\kappa(x_j)}(\pi_{\kappa(x_j)}(a_j), x_j) < \varepsilon. \]

In particular, \(\|g_{\kappa(x_{j-1})} - g_{\kappa(x_j)}\|_\infty < \varepsilon\) for \(j = 1, 2, 3\) and thus

\[ (4-3) \|g_{\kappa(x_0)} - g_{\kappa(x_3)}\|_\infty < 3\varepsilon. \]

For simplicity, for \(j \in \{0, 1, 2, 3\}\) put \(s_j = \kappa(x_j)\) and take \(b_j \in G\) such that \(\pi_{s_j}(b_j) = x_j\). Recall that \(d_s(\pi_s(g), \pi_s(h)) = \|g_s - g_t\|_\infty\) for any \(s \in S\) and \(g, h \in G\). We therefore have

\[ g_{s_j}(b_{j-1}, b_j) \leq g_{s_j}(b_{j-1}, a_j) + g_{s_j}(a_j, b_j) \leq \|g_{s_{j-1}} - g_{s_j}\|_\infty + g_{s_{j-1}}(b_{j-1}, a_j) + g_{s_j}(a_j, b_j) < \varepsilon \]

(by \(4-2\)) and consequently

\[ g_{s_2}(a_0, a_1) + g_{s_2}(a_1, b_1) \leq g_{s_2}(a_0, b_1) + g_{s_2}(b_1, b_2) \leq \|g_{s_2} - g_{s_1}\|_\infty + g_{s_1}(a_0, b_1) + g_{s_1}(b_1, b_2) < 3\varepsilon. \]

Similarly,

\[ g_{s_3}(b_0, b_3) \leq g_{s_3}(b_0, a_2) + g_{s_3}(a_2, b_3) \leq \|g_{s_3} - g_{s_2}\|_\infty + g_{s_2}(b_1, b_2) + g_{s_2}(b_2, b_3) < 5\varepsilon. \]

Finally, by \(4-3\) and \(4-4\) we obtain:

\[ v(x_0, x_3) \leq \|g_{s_0} - g_{s_1}\|_\infty + d_{s_0}(\pi_{s_0}(b_0), \pi_{s_0}(b_0)) + d_{s_1}(\pi_{s_1}(b_1), \pi_{s_1}(b_1)) < 3\varepsilon + g_{s_2}(b_0, b_2) + g_{s_3}(b_2, b_3) < 8\varepsilon \]

and the proof of \(4-1\) is complete. Now let \(f: X \times X \to [0, \infty)\) be given by \(f(x, y) = \sqrt{v(x, y)}\). Below we collect all properties established for \(v\) and translated to the case of the function \(f\):

\begin{enumerate}[(F1)]
\item \(f(x, x) = 0\) and \(f(x, y) = f(y, x) > 0\) for any distinct points \(x\) and \(y\) of \(X\),
\item if \(\varepsilon > 0\) and \(\{f(x, y), f(y, z), f(z, w)\} \subset [0, \varepsilon]\) for some \(x, y, z, w \in X\), then \(f(x, w) \leq 2\varepsilon\) (thanks to \(4-1\)).
\end{enumerate}
(F3) $f(x, y) = \sqrt{d(x, y)}$ and $f(\pi_s(g), \pi_t(g)) = \sqrt{\|g_s - g_t\|_\infty}$ whenever $x, y \in X$, $g \in G$ and $s, t \in S$.

(F4) $f(\pi_s(g), \pi_t(h)) \geq \sqrt{\|g_s - g_t\|_\infty}$ for all $g, h \in G$ and $s, t \in S$.

(F5) $f(\pi_s(a g), \pi_t(a h)) = f(\pi_s(g), \pi_t(h))$ for any $a, g, h \in G$ and $s, t \in S$.

Finally, we define $d : X \times X \to [0, \infty)$ as follows:

$$d(x, y) = \inf\left\{ \sum_{j=1}^{n} f(z_{j-1}, z_j) : n \geq 1, z_0, \ldots, z_n \in X, z_0 = x, z_n = y \right\}.$$  

Lemma 6.2 of [6] asserts that for any function $f : X \times X \to [0, \infty)$ satisfying conditions (F1)–(F2) the function $d$ defined above is a metric on $X$ such that

$$(4-5) \quad \frac{1}{2} f(x, y) \leq d(x, y) \leq f(x, y) \quad (x, y \in X).$$

It follows from (F5) and the formula of $d$ that (D4) is fulfilled, while (D1) and (D2) may easily be deduced from (F3) and (4-5). Finally, (D3) is a consequence of (F4) and (4-5), and we are done. \hfill \Box

Proof of Theorem 1.4. Implications ‘(A1) $\implies$ (A2)’ and ‘(B1) $\implies$ (B2)’ follow from Proposition 4.3. It remains to show the converse implications.

First assume that $G$ is complete (in this case the proof is much shorter). By a well-known result (see e.g. [28, Theorem 2.1]), there is a bounded metric space $(Y, \bar{g})$ such that $w(G) = w(Y)$ and $G$ is isomorphic to a subgroup $H$ of $\text{Iso}(\bar{Y}, \bar{g})$. Since $\text{Iso}(Y, g)$ is naturally isomorphic to a subgroup of $\text{Iso}(\bar{Y}, \bar{g})$, where $(\bar{Y}, \bar{g})$ is the completion of $(Y, g)$, we may assume that $(\bar{Y}, \bar{g})$ is a complete metric space. Since $G$ is complete, $H$ is a closed subgroup of $(Y, g)$. Now Theorem 3.2 implies that $H$ is isomorphic to $\text{Iso}(X, d)$ where $X = \bar{Y}_{\beta}$ with $\beta = w(Y)$, and $d$ is a metric which preserves $\bar{g}$. Notice that $d$ is complete (by (AX4)), $w(X) = w(G)$ (because $\beta = w(Y) = w(G)$) and $G$ is isomorphic to $\text{Iso}(X, d)$ (being isomorphic to $H$). This proves the remainder of point (B).

We now turn to (A). Assume $G$ is $\mathcal{G}_\delta$-complete. Thanks to Theorem 3.2, it suffices to show that $G$ is isomorphic to a closed subgroup of $\text{Iso}(X, d)$ for a metric space $(X, d)$ of topological weight equal to $w(G)$ (see the previous part of the proof). We shall do this employing Lemma 4.5 and improving a classical argument, presented e.g. in 1st proof of [28, Theorem 2.1].

Let $\mathcal{B}$ be a base of open neighbourhoods of the neutral element $e$ of $G$ such that $\text{card}(\mathcal{B}) \leq w(G)$. Let $S$ be the set of all finite and all infinite sequences of members of $\mathcal{B}$. For any $U \in \mathcal{B}$ there exists a continuous left-invariant metric $\lambda_U$ on $G$ bounded by 1 such that

$$(4-6) \quad \{x \in G : \lambda_U(x, e) < 1\} \subset U.$$  

We leave it as a simple exercise that the family $\{\lambda_U\}_{U \in \mathcal{B}}$ determines the topology of $G$. Now for any $s = (U_j)_{j=1}^N \in S$ (where $N$ is finite or $N = \infty$) let

$$(4-7) \quad g_s := \sum_{j=1}^{N} \frac{1}{2^j} \lambda_{U_j}.$$  

Notice that $g_s$ is a continuous left-invariant metric on $G$ bounded by 1. What is more,

(T) the family $\{g_s\}_{s \in S}$ determines the topology of $G$

(since $g_s = \lambda_U$ for $s = (U, U, \ldots) \in S$). Let $(X_s, d_s; \pi_s) (s \in S)$ as well as $(X, d)$ be as in Lemma 4.5. For each $s \in S$ let $\widehat{g}_s : G \times G \to [0, \infty)$ be given by $\widehat{g}_s(g, h) = \text{dist}(g(s), h(s), X_s).$
We infer from (D3) and the above convergences that there are contractible open set in a normed vector space of the same topological weight as \(G\) equal to the topological weight of the whole space). In the next section we shall improve Theorem 1.4 by showing that \(G\) is dense in \(X_s\), since \(\pi_s\) is continuous. In particular, the closure of \(Z\) contains all points of \(\bigcup_{s \in S_f} X_s\). Fix \(s \notin S_f\) and \(a \in G\). Then \(s\) is of the form \(s = (U_j)_{j=1}^\infty \in S\). Put \(s_n := (U_j)_{j=1}^n \in S_f\) and observe that, by (D2) and (4-7),
\[
\lim_{n \to \infty} \|\phi_n - \phi\|_{\infty} = 0 \quad (n \to \infty).
\]
So, since \(\pi_s(G) = X_s\), the above argument shows that \(Z\) is indeed dense in \(X_s\).

It remains to check that \(G\) is isomorphic to a closed subgroup of \(\text{Iso}(X, d)\). For \(g \in G\) let \(u_g : X \to X\) be such that \(u_g(\pi_s(x)) = \pi_s(gx)\) for any \(s \in S\) and \(x \in G\). Then \(\Phi : G \ni g \mapsto u_g \in \text{Iso}(X, d)\) is a well defined (by (D4)) group homomorphism as well as a topological embedding (thanks to (T')). So, it follows from point (f) of Proposition 4.3 that \(w(X) \geq w(G)\) and hence in fact \(w(X) = w(G)\). We shall check that \(\Phi(G)\) is closed, which will finish the proof. Assume \(\{x_{\sigma}\}_{\sigma \in \Sigma}\) is a net in \(G\) such that the net \(\{u_{x_{\sigma}}\}_{\sigma \in \Sigma}\) converges in \(\text{Iso}(X, d)\) to some \(u \in \text{Iso}(X, d)\). It is enough to prove that the net \(\{x_{\sigma}\}_{\sigma \in \Sigma}\) is convergent in \(G\). Since \(G\) is \(\hat{G}_\delta\)-complete, actually it suffices to verify condition (CC) (see Proposition 4.1). To this end, let \(V_1, V_2, \ldots\) be a sequence of neighbourhoods of \(e\). For any \(n \geq 1\) choose \(U_n \in \mathcal{B}\) for which \(U_n \subset V_n\). Now for \(s := (U_j)_{j=1}^n \in S\) we have
\[
\lim_{\sigma \in \Sigma} d(\pi_s(x_{\sigma}), u(\pi_s(e))) = \lim_{\sigma \in \Sigma} d(u_{x_{\sigma}}(\pi_s(e)), u(\pi_s(e))) = 0,
\]
\[
\lim_{\sigma \in \Sigma} d(\pi_s(x_{\sigma}^{-1}), u^{-1}(\pi_s(e))) = \lim_{\sigma \in \Sigma} d(u_{x_{\sigma}}^{-1}(\pi_s(e)), u^{-1}(\pi_s(e))) = 0.
\]
We infer from (D3) and the above convergences that there are \(y, z \in G\) such that
\[
\lim_{\sigma \in \Sigma} d(\pi_s(x_{\sigma}), \pi_s(y)) = \lim_{\sigma \in \Sigma} d(\pi_s(x_{\sigma}^{-1}), \pi_s(z^{-1})) = 0.
\]
But \(d(\pi_s(a), \pi_s(b)) = \hat{\phi}_s(a, b)\) for any \(a, b \in G\) and thus, thanks to (4-8),
\[
\lim_{\sigma \in \Sigma} \phi_s(x_{\sigma}, y) = \lim_{\sigma \in \Sigma} \phi_s(x_{\sigma}^{-1}, z^{-1}) = 0.
\]
For each \(n \geq 1\) let \(\sigma_n \in \Sigma\) be such that both the numbers \(\phi_s(x_{\sigma}, y) = \phi_s(x_{\sigma}^{-1}, y, e)\) and \(\phi_s(x_{\sigma}^{-1}, z^{-1}) = \phi_s(x_{\sigma}^{-1}, e, z^{-1})\) are less than \(2^{-n}\) for any \(\sigma \geq \sigma_n\). We deduce from the formula of \(s\), (4-7) and (4-6) that \(\{x_{\sigma}^{-1}, x_{\sigma}^{-1}\} \subset U_n(\subset V_n)\) for all \(\sigma \geq \sigma_n\) and we are done.

The above proof provides the existence of a metric space (namely, \(X_\beta\)) whose isometry group is isomorphic to a given \(G_\delta\)-complete group \(G\). This metric space is highly disconnected (since it contains a clopen discrete set whose cardinality is equal to the topological weight of the whole space). In the next section we shall improve Theorem 1.4 by showing that \(G\) is isomorphic to the isometry group of a contractible open set in a normed vector space of the same topological weight as \(G\).

4.6. Corollary. (A) Let \(G\) be a topological group and \(\beta\) be an infinite cardinal number. There exists a complete metric space \((X, d)\) such that \(w(X) = \beta\) and \(\text{Iso}(X, d)\) is isomorphic to \(G\) iff \(G\) is \(G_\delta\)-complete [resp. complete] and \(\beta \geq w(G)\).
(B) A topological group is isomorphic to the isometry group of some separable metric space if and only if it is second countable.

Proof. Both points (A) and (B) follow from Theorems 1.4 and 3.2, (AX4) and, respectively, points (f) and (e) of Proposition 4.3.

We call a metrizable space $X$ zero-dimensional if and only if $X$ has a base consisting of clopen (that is, simultaneously open and closed) sets; and $X$ is strongly zero-dimensional if the covering dimension of $X$ equals 0.

4.7. Corollary. Let $G$ be an infinite metrizable topological group.

(A) If $G$ is discrete, there is a complete compatible (possibly non-left-invariant) metric $d$ on $G$ such that $G$ is isomorphic to $\text{Iso}(G, d)$.

(B) If $G$ is countable and non-discrete, there is a compatible metric $d$ on $F := \mathbb{Q} \cup \mathbb{Z}$ such that $G$ is isomorphic to $\text{Iso}(F, d)$.

(C) If $G$ is totally disconnected (zero-dimensional; strongly zero-dimensional), there is a metric space $(X, d)$ such that $X$ is totally disconnected (resp. zero-dimensional; strongly zero-dimensional) as well, $w(X) = w(G)$ and $\text{Iso}(X, d)$ is isomorphic to $G$.

Proof. Let $d$ be a bounded left-invariant compatible metric on $G$ (if $G$ is discrete, we may additionally assume that $d$ is complete). It is an easy (and well-known) exercise that all left translations on $G$ form a closed subgroup of $\text{Iso}(G, d)$. Consequently, by Theorem 3.2, $G$ is isomorphic to $\text{Iso}(\hat{G}_\beta, \delta)$ where $\beta = w(G)$ and $\delta$ is some metric preserving $d$. Note that if $G$ is discrete, $\hat{G}_\beta$ is homeomorphic to $G$, which proves (A). Further, if $G$ is countable and non-discrete, it is homeomorphic to the space of all rationals (e.g. by Sierpiński’s theorem [23] that every countable metrizable topological space without isolated points is homeomorphic to $\mathbb{Q}$; cf. point (d) of Exercise 6.2.A on page 370 in [12]) and hence $\hat{G}_\beta$ is homeomorphic to $F$, from which we deduce (B). Finally, (C) simply follows from the following remarks: if $G$ is totally disconnected (resp. zero-dimensional; strongly zero-dimensional), so is $\hat{G}_\beta$ (cf. Theorems 1.3.6, 4.1.25 and 4.1.3 in [11]).

4.8. Example. Let $(X, d)$ be an arbitrary metric space and $G$ be a subgroup of $\text{Iso}(X, d)$. It follows from Theorem 1.4 and Proposition 4.1 that $G$ is isomorphic to the isometry group of some metric space if and only if $G$ is $G_\delta$-closed in $\text{Iso}(X, d)$. Let us briefly show that the $G_\delta$-closure of $G$ coincides with the set of all $u \in \text{Iso}(X, d)$ such that for any separable subspace $A$ of $X$ there is $v \in G$ which agrees with $u$ on $A$. Indeed, there is a countable set $D \subset A$ which is dense in $A$. Then the set $F(u, A) := \{ v \in \text{Iso}(X, d) : v|_A = u|_A \}$ coincides with $\{ v \in \text{Iso}(X, d) : v|_D = u|_D \}$. This implies that $F(u, A)$ is $G_\delta$ in $\text{Iso}(X, d)$. Consequently, if $u$ belongs to the $G_\delta$-closure of $G$, then necessarily $G \cap F(u, A) \neq \varnothing$. Conversely, it may be easily shown that for every $G_\delta$-set $P$ containing $u$ there is a countable set $A$ such that $F(u, A) \subset P$ and hence the condition on $u$ under the question is also sufficient.

According to the above remark, Theorem 3.2 may now be generalized as follows: a subgroup $G$ of $\text{Iso}(X, d)$ is isomorphic to the isometry group of some metric space (of the same topological weight as $X$) if and only if $G$ satisfies the following condition. Whenever $u \in \text{Iso}(X, d)$ is such that for any separable subspace $A$ of $X$ there exists $v \in G$ which agrees with $u$ on $A$, then $u \in G$.

We end the section with the concept of $G_\delta$-completions. Similarly as in case of Rajkov completion, any topological group $G$ has a unique $G_\delta$-completion, i.e. $G$ may be embedded in a $G_\delta$-complete group as a $G_\delta$-dense subgroup in a unique way, as shown by
4.9. Proposition. Let $G$ and $K$ be $G_d$-complete groups and $H$ be a $G_d$-dense subgroup of $G$.

(a) Every continuous homomorphism of $H$ into $K$ extends uniquely to a continuous homomorphism of $G$ into $K$.

(b) If $f : H \to K$ is a group homomorphism as well as a topological embedding and $\tilde{G}$ denotes the $G_d$-closure of $f(H)$ in $K$, then there is a unique (topological) isomorphism $F : G \to \tilde{G}$ which extends $f$.

Proof. Let $f : H \to K$ be a continuous homomorphism. Since $H$ is dense in $G$, there is a unique continuous group homomorphism $F : G \to \tilde{G}$ which extends $f$. It suffices to show that $F(G) \subset K$. But this follows from the fact that the preimage of a $G_d$-closed set under a continuous function is $G_d$-closed too. This proves (a). In addition $f$ is a topological embedding, it follows from the above argument that there is a continuous group homomorphism $\tilde{F} : G \to \tilde{G}$ which extends $f^{-1}$. We then readily see that both $\tilde{F} \circ F$ and $F \circ \tilde{F}$ are the identity maps and hence $F$ is an isomorphism (and $\tilde{F} = F^{-1}$), which shows (b). \qed

4.10. Definition. Let $G$ be a topological group. The $G_d$-completion of $G$ is a $G_d$-complete group which contains $G$ as a $G_d$-dense (topological) subgroup. It follows from Proposition 4.9 that the $G_d$-completion is unique up to isomorphism fixing the points of $G$. It is also obvious that any group has the $G_d$-completion.

5. Hilbert spaces as underlying topological spaces

Our first aim of this section is to prove Theorem 1.2 and point (b) of Proposition 1.5. To this end, we recall a classical construction due to Arens and Eells [4] (see also [29, Chapter 2]).

5.1. Definition. Let $(X, d)$ be a nonempty complete metric space. For every $p \in X$ let $\chi_p : X \to \{0, 1\}$ be such that $\chi_p(x) = 1$ if $x = p$ and $\chi_p(x) = 0$ otherwise. A molecule of $X$ is any function $m : X \to \mathbb{R}$ which is supported on a finite set and satisfies $\sum_{p \in X} m(p) = 0$. Denote by $AE_0(X)$ the real vector space of all molecules of $X$ and for $m \in AE_0(X)$ put

$$\|m\|_{AE} = \inf \{ \sum_{j=1}^{n} |a_j|d(p_j, q_j) : m = \sum_{j=1}^{n} a_j(\chi_{p_j} - \chi_{q_j}) \}.$$ 

Then $\| \cdot \|_{AE}$ is a norm and the completion of $(AE_0(X), \| \cdot \|_{AE})$ is called the Arens-Eells space of $(X, d)$ and denoted by $(AE(X), \| \cdot \|_{AE})$. Moreover, $w(AE(X)) = w(X)$.

It is an easy observation that every isometry $u : X \to Y$ between complete metric spaces $X$ and $Y$ induces a unique linear isometry $AE(u) : AE(X) \to AE(Y)$ such that $AE(u)(\chi_p - \chi_q) = \chi_{u(p)} - \chi_{u(q)}$ for any $p, q \in X$. The following is left to the reader (it is surely well-known, but probably nowhere explicitly stated).

5.2. Lemma. For every complete metric space $(X, d)$, the function

$$\text{(5-1)} \quad \text{Iso}(X, d) \ni u \mapsto AE(u) \in \text{Iso}(AE(X), \| \cdot \|_{AE})$$

is both a group homomorphism and a topological embedding.

The homomorphism appearing in (5-1) is hardly ever surjective. There is however a fascinating result discovered by Mayer-Wolf [17] (cf. Proposition 2.4.5 and Theorem 2.7.2 in [29]) which characterizes all isometries of the space $AE(X)$ under some additional conditions on the metric of $X$. Below we formulate only a special case of it, enough for our considerations.
If \( X \), be the Arens-Eells space of \((X, \sqrt{d})\). Every linear isometry of \( \text{AE}(X) \) onto itself is of the form \( \pm \text{AE}(u) \) where \( u \in \text{Iso}(X, \sqrt{d}) \).

We shall also need quite an intuitive result stated below. Although its proof is not immediate, we leave it to the reader as an exercise.

5.4. Lemma. Let \( X \) be a two-dimensional real vector space, \( \| \cdot \| \) be any norm on \( X \) and let \( a \) and \( b \) be two vectors in \( X \).

(a) If \( B_X(0, 2) \subset B_X(b, 2) \cup B_X(a, 1) \), then \( b = 0 \).
(b) If \( \|a\| = \|b\| = 2 \) and \( B_X(b, 1) \subset B_X(0, 2) \cup B_X(a, 1) \), then \( a = b \).

Proof of Theorem 1.2 and point (b) of Proposition 1.5. For Theorem 1.2 is a special case of point (b) of the proposition, we focus only on the proof of the latter result. It follows from Corollary 4.6 that there is a complete metric space \((Y, g)\) such that \( w(Y) = \beta \) and \( \text{Iso}(Y, g) \) is isomorphic to \( G \). Since \( \text{Iso}(Y, g) = \text{Iso}(Y, \sqrt{\frac{1}{2}d}) \) and the metric \( \sqrt{\frac{1}{2}d} \) is complete (and compatible), we may and do assume that \( g < \frac{1}{2} \). We also assume that \( Y \cap [0, 1] = \emptyset \). Let \( X = Y \cup [0, 1] \). We define a metric \( d \) on \( X \) by the rules:

\[ \begin{align*}
& d(s, t) = |s - t| \text{ for } s, t \in [0, 1], \\
& d(x, y) = g(x, y) \text{ for } x, y \in Y, \\
& d(x, t) = d(t, x) = 1 + t \text{ for } x \in Y \text{ and } t \in [0, 1].
\end{align*} \]

We leave it as an exercise that \( d \) is indeed a metric, that \( d \) is complete and \( w(X) = \beta \).

Notice that for any \( a \in X \) and \( t \in [0, 1] \subset X \):

\( a = 1 \iff \exists b, c \in X: d(a, b) = \frac{3}{4} \land d(a, c) = 2, \)

\( a = t \iff d(a, 1) = 1 - t. \)

The above notices imply that for every \( f \in \text{Iso}(X, d) \) we have \( f(t) = t \) for \( t \in [0, 1] \) and \( f \big|_Y \in \text{Iso}(Y, g) \). It is also easy to see that each isometry of \((Y, g)\) extends (uniquely) to an isometry of \((X, d)\). Hence the function \( \text{Iso}(X, d) \ni f \mapsto f \big|_Y \in \text{Iso}(Y, g) \) is a (well defined) isomorphism. Now let \( (E, \| \cdot \|) = (\text{AE}(X), \| \cdot \|_{\text{AE}}) \) be the Arens-Eells space of \((X, \sqrt{d})\) and let \( e = \chi_1 - \chi_0 \). We see that \( w(E) = \beta \) and \( E \) is infinite-dimensional, since \( X \) is infinite. What is more, it follows from Theorem 5.3 that every linear isometry of \( E \) which leaves the point \( e \) fixed is of the form \( \text{AE}(u) \) for some \( u \in \text{Iso}(X, \sqrt{d}) \). Since \( \text{AE}(u)(e) = e \) for any \( u \in \text{Iso}(X, d) \) (because \( u(0) = 0 \) and \( u(1) = 1 \) for such \( u \)), the notice that \( \text{Iso}(X, d) = \text{Iso}(X, \sqrt{d}) \) and Lemma 5.2 finish the proof. \( \Box \)

Our next aim is to show points (a) of both Theorem 1.1 and Proposition 1.5. In the proof we shall involve the next three results. The first of them is due to Mankiewicz [16]:

5.5. Theorem. Whenever \( X \) and \( Y \) are normed vector spaces, \( U \) and \( V \) are connected open subsets of, respectively, \( X \) and \( Y \), then every isometry of \( U \) onto \( V \) extends to a unique affine isometry of \( X \) onto \( Y \).

We recall that a function \( \Phi: X \to Y \) between real vector spaces \( X \) and \( Y \) is affine if \( \Phi - \Phi(0) \) is linear.

The following result is a consequence e.g. of [7, Theorem VI.6.2] and a famous theorem of Toruńczyk [26, 27] which says that every Banach space is homeomorphic to a Hilbert space.

5.6. Theorem. Every closed convex set in an infinite-dimensional Banach space whose interior is nonempty is homeomorphic to an infinite-dimensional Hilbert space.
Our last tool is the next result which in the separable case was proved by Mogilski [19]. The argument presented by him works also in nonseparable case. This theorem in its full generality may also be briefly concluded from the results of Toruńczyk [26, 27].

5.7. Theorem. Let \( X \) be a metrizable space. If \( X \) is the union of its two closed subsets \( A \) and \( B \) such that each of \( A, B \) and \( A \cap B \) is homeomorphic to an infinite-dimensional Hilbert space \( \mathcal{H} \), then \( X \) itself is homeomorphic to \( \mathcal{H} \).

We are now ready to give

Proof of points (a) of Theorem 1.1 and Proposition 1.5. Again, observe that point (a) of the theorem under the question is a special case of point (b) of the proposition. Therefore we focus only on the latter result. Let \( E \) and \( e \) be as in point (b) of Proposition 1.5. Replacing, if needed, \( e \) by \( 2e/\|e\| \), we may assume that \( \|e\| = 2 \). Denote by \( \mathcal{E} \) the group of all linear isometries which leave \( e \) fixed. Let \( W = B_E(0,2) \cup \bar{B}_E(e,1) \) be equipped with the metric \( p \) induced by the norm of \( E \). Notice that if \( V \in \mathcal{E} \), then \( V(W) = W \) and \( V|_W \in \text{Iso}(W,p) \). Conversely, for each \( g \in \text{Iso}(W,p) \), \( g = V|_W \) for some linear isometry \( V \in \mathcal{E} \). Let us briefly justify this claim. Let \( x = g(0) \) and \( y = g(e) \). Then \( W \subset B_E(x,2) \cup B_E(y,1) \) and consequently \( \bar{B}_X(0,2) \subset \bar{B}_X(x,2) \cup \bar{B}_X(y,1) \) where \( X \) is a two-dimensional linear subspace of \( E \) which contains \( x \) and \( y \). We infer from point (a) of Lemma 5.4 that \( x = 0 \). So, \( \|y\| = 2 \) (since \( g \) is an isometry) and thus \( \bar{B}_Y(e,1) \subset \bar{B}_Y(0,2) \cup \bar{B}_Y(y,1) \) where \( Y \) is a two-dimensional linear subspace of \( E \) such that \( e, y \in Y \). Now point (b) of Lemma 5.4 yields that \( y = e \). We then have \( g(B_E(0,2) \cup B_E(e,1)) = B_E(0,2) \cup B_E(e,1) \). So, an application of Theorem 5.5 gives our assertion: there is a linear (since \( g(0) = 0 \)) isometry \( V \in \mathcal{E} \) which extends \( g \).

Having the above fact, we easily see that the function \( \mathcal{E} \ni V \mapsto V|_W \in \text{Iso}(W,p) \) is an isomorphism. Consequently, \( G \) is isomorphic to \( \text{Iso}(W,p) \). So, to finish the proof, it suffices to show that \( W \) is homeomorphic to \( \mathcal{H}_\beta \). But this immediately follows from Theorems 5.6 and 5.7, since \( w(E) = \beta \) and the sets \( B_E(0,2) \), \( B_E(e,1) \) and \( B_E(0,2) \cap \bar{B}_E(1,2) \) are closed, convex and have nonempty interiors.

Proof of Corollary 1.6. It suffices to apply Proposition 1.5 and point (f) of Proposition 4.3.

The arguments used in the proofs of both points of Proposition 1.5 may simply be employed to show the following

5.8. Corollary. Let \( G \) be a \( G_\beta \)-complete topological group of topological weight not exceeding \( \beta \geq N_\kappa \). There are an infinite-dimensional normed vector space \( E \) of topological weight \( \beta \), a contractible open set \( U \subset E \) and a nonzero vector \( e \in E \) such that the topological groups \( G, \text{Iso}(U,d) \) and \( \text{Iso}(E|e) \) are isomorphic where \( d \) is the metric on \( U \) induced by the norm of \( E \) and \( \text{Iso}(E|e) \) is the group of all linear isometries of \( E \) which leave the point \( e \) fixed.

Proof. Let \( (Y_0,\varrho_0) \) be a metric space such that \( w(Y_0) = \beta \) and \( \text{Iso}(Y_0,\varrho_0) \) is isomorphic to \( G \) (cf. Theorems 1.4 and 3.2). Denote by \( (Y,\varrho) \) the completion of \( (Y_0,\varrho_0) \). Now let \( (X,d), (\text{AE}(X),\|\cdot\|_{\text{AE}}) \) and \( e \) be as in the proof of point (b) of Proposition 1.5. We know that the function

\[ \text{Iso}(X,d) \ni u \mapsto \text{AE}(u) \in \text{Iso}(\text{AE}(X)|e) \]

is an isomorphism. Denote by \( E \) the linear span of the set \( \{\chi_a - \chi_b : a,b \in Y_0 \cup [0,1]\}(\subset \text{AE}(X)) \) (recall that \( X = Y \cup [0,1] \)). Observe that if \( u \in \text{Iso}(X,d) \) is such that \( \text{AE}(u)(E) = E \), then \( u(Y_0 \cup [0,1]) = Y_0 \cup [0,1] \) and consequently
u(Y_0) = Y_0 (see the proof of point (b) of Proposition 1.5). This yields that also the function
\[ \text{Iso}(Y_0, b_0) \ni u \mapsto AE(u)|_E \in \text{Iso}(E|e) \]
is an isomorphism. Now it suffices to put \( U = B_E(0, ||e||) \cup B_E(e, \frac{1}{2}||e||) \) and repeat the proof of point (a) of Proposition 1.5 (involving Lemma 5.4 and Theorem 5.5) to get the whole assertion. \( (U \) is contractible as the union of two intersecting convex sets.) □

Taking into account Corollary 1.6, the following two questions arise:

5.9. Problem. Let \( G \) be a \( G_\delta \)-complete topological group.
- When is \( G \) isomorphic to the isometry group of a completely metrizable metric space?
- If \( G \) is isomorphic to the isometry group of a completely metrizable metric space, is it isomorphic to the isometry group of a (possibly incomplete) metric space homeomorphic to a Hilbert space?

6. Compact and locally compact Polish groups

This section is devoted to the proofs of points (b) and (c) of Theorem 1.1. Our main tool will be the following result, very recently shown by us [20].

6.1. Theorem. Let \( G \) be a locally compact Polish group, \( X \) be a locally compact Polish space and let \( G \times X \ni (g,x) \mapsto gx \in X \) be a continuous proper action of \( G \) on \( X \). Assume there is a point \( \omega \in X \) such that the set \( G.\omega = \{g.\omega: g \in G\} \) is non-open and \( G \) acts freely at \( \omega \) (that is, \( g.\omega = \omega \) implies \( g = \text{the neutral element of } G \)). Then there exists a proper compatible metric \( d \) on \( X \) such that \( \text{Iso}(X,d) \) consists precisely of all maps of the form \( x \mapsto gx \) \((g \in G)\).

We recall that (under the above notation) the action is proper if for every compact set \( K \subset X \) the set \( \{g \in G: g.K \cap K \neq \emptyset\} \) is compact as well \((\text{where } g.K = \{g.x: x \in K\})\).

Our next tool is the following classical result due to Keller [14] (see also [7, Theorem III.3.1]).

6.2. Theorem. Every infinite-dimensional compact convex subset of a Fréchet space is homeomorphic to the Hilbert cube.

We recall that a Fréchet space is a completely metrizable locally convex topological vector space.

We call a function \( u: (X,d) \rightarrow \mathbb{R} \) (where \( (X,d) \) is a metric space) nonexpansive iff \( |u(x) - u(y)| \leq d(x,y) \) for all \( x,y \in X \). The function \( u \) is a Katětov map iff \( u \) is nonexpansive and additionally \( d(x,y) \leq u(x) + u(y) \) for any \( x,y \in X \). Katětov maps correspond to one-point extensions of metric spaces.

We are now ready for proving point (b) of Theorem 1.1.

Proof of point (b) of Theorem 1.1. Let \( G \) be a compact Polish group. Take a left invariant metric \( g \leq 1 \) on \( G \) and equip the space \( X = G \times [0,1] \) with the metric \( d \) where \( d((x,s),(y,t)) = \max(g(x,y),|t-s|) \). For \( g \in G \) denote by \( \psi_g \) the function \( X \ni (x,t) \mapsto (g^{-1}x,t) \in X \). Notice that \( \psi_g \in \text{Iso}(G,X) \) for any \( g \in G \). Let \( \Delta \) be the space of all nonexpansive maps of \( (X,d) \) into \([0,1]\) endowed with the supremum metric. Observe that \( \Delta \) is a convex set in the Banach space of all real-valued maps on \( X \). What is more, \( \Delta \) is infinite-dimensional, since \( X \) is infinite, and \( \Delta \) is compact, by the Ascoli type theorem. So, we infer from Theorem 6.2 that \( \Delta \) is homeomorphic to \( Q \). Further, \( \Phi_g(u) := u \circ \psi_g \in \Delta \) for any \( g \in G \) and \( u \in \Delta \) (because \( \psi_g \) is isometric). It is also easily seen that the function \( G \times \Delta \ni (g,u) \mapsto \Phi_g(u) \in \Delta \) is a
(proper—since both $G$ and $\Delta$ are compact) continuous action of $G$ on $\Delta$. Finally, the function $\omega: X \ni (x, t) \mapsto d((x, t), (e, 1)) \in \mathbb{R}$ belongs to $\Delta$ (since $d \leq 1$) where $e$ is the neutral element of $G$. Observe that the set $K := \{ \omega \circ \psi_g: g \in G \}$ has empty interior in $\Delta$ (since $\frac{1}{4} + (1 - \frac{1}{4})\omega \circ \psi_g \in \Delta \setminus K$ for any $n \geq 1$) and
\begin{equation}
\omega \circ \psi_g = \omega \iff g = e.
\end{equation}
Now Theorem 6.1 yields that there is a compatible metric $\lambda$ on $\Delta$ such that
Iso($\Delta, \lambda$) = $\{ \Phi_g: g \in G \}$. To end the proof, note that the function $G \ni g \mapsto \Phi_g \in$ Iso($\Delta, \lambda$) is an isomorphism of topological groups, by (6-1).

Our last aim is to prove point (c) of Theorem 1.1. To this end, we need more information on Hilbert cube manifolds. We begin with Toruńczyk's [25] definition of a Z-set (this notion is due to Anderson [2], but his original definition is different; both the definitions are however equivalent in ANR's). A closed set $K$ in a metric space $X$ is a Z-set iff every map of $Q$ into $X$ may uniformly be approximated by maps of $Q$ into $X \setminus K$.

One of the deepest results in infinite-dimensional topology is Anderson's theorem on extending homeomorphisms between Z-sets [2]. Below we formulate it only in the Hilbert cube settings, it holds however in a much more general context. (For the discussion on this topic consult [7, Chapter V]; see also [3] and [8]).

6.3. Theorem. Every homeomorphism between two Z-sets in the Hilbert cube $Q$ is extendable to a homeomorphism of $Q$ onto itself.

The result stated below is a kind of folklore in Hilbert cube manifolds theory. We present its short proof, because we could not find it in the literature.

6.4. Theorem. The spaces $Q \times [0, \infty)$ and $Q \setminus \{ \text{point} \}$ are homeomorphic.

Proof. Since $Q \setminus \{ \text{point} \}$ is a Hilbert cube manifold, it follows from Schori's theorem [22] (see also [9]; compare with [7, Theorem IX.4.1]) that $(Q \setminus \{ \text{point} \}) \times Q$ is homeomorphic to $Q \setminus \{ \text{point} \}$. Now the assertion follows from Theorem 6.3 since $(Q \times [0, 1]) \setminus (Q \times [0, 1]) = Q \setminus \{ 1 \}$ is a Z-set in $Q \times [0, 1]$ homeomorphic to the Z-set $\emptyset$ in $Q \setminus \{ \text{point} \}$.

6.5. Lemma. Let $(X, d)$ be a nonempty separable metric space and let $E(X)$ be the set of all Katětov maps on $(X, d)$ equipped with the pointwise convergence topology.

(i) For any $a \in X$ and $r > 0$ the set $\{ f \in E(X): f(a) \leq r \}$ is compact (in $E(X)$).

(ii) $E(X) \times Q$ is homeomorphic to $Q \setminus \{ \text{point} \}$.

Proof. Point (i) follows from the Ascoli type theorem, since $E(X)$ consists of non-expansive maps and for any $f \in E(X)$ and $x \in X, f(x) \in [0, d(x, a) + f(a)]$.

We turn to (ii). First of all, $E(X)$ is metrizable, because of the separability of $X$ and the nonexpansivity of members of $E(X)$. Further, thanks to Theorem 6.4, it suffices to show that $E(X) \times Q$ is homeomorphic to $Q \times [0, \infty)$. Fix $a \in X$ and let $\omega \in E(X)$ be given by $\omega(x) = d(a, x)$. For each $n \geq 1$ let $K_n = \{ f \in E(X): f(a) \in [n-1, n] \}$ and $Z_{n-1} = \{ f \in E(X): f(a) = n-1 \}$. We infer from (ii) that $K_n$ and $Z_{n-1}$ are compact. It is also easily seen that both they are convex nonempty sets ($\omega + n - 1 \in Z_{n-1} \subset K_n$). Since $K_n \times Q$ and $Z_{n-1} \times Q$ are affinely homeomorphic to convex subsets of Fréchet spaces, Theorem 6.2 yields that both these sets are homeomorphic to $Q$. Let $h_{n-1}: Z_{n-1} \times Q \to Q \times \{ n - 1 \}$ be any homeomorphism. We claim that $Z_{n-1} \cup Z_n$ is a Z-set in $K_n$. This easily follows from the fact that the maps $K_n \ni f \mapsto (1 - \frac{1}{4})f + \frac{1}{4}(\omega + n - \frac{1}{4}) \in K_n$ send $K_n$ into $K_n \cup (Z_{n-1} \cup Z_n)$ and converge uniformly (as $k \to \infty$) to the identity map of $K_n$. Since $Q \times \{ n - 1, n \}$ is a Z-set in $Q \times [n - 1, n]$, Theorem 6.3 provides us the existence of a homeomorphism
we search. It is clear that $H_n: K_n \times Q \to Q \times [n-1,n]$ which extends both $h_{n-1}$ and $h_n$. We claim that the union $H: E(X) \times Q \to Q \times [0,\infty)$ of all $H_n$’s $(n \geq 1)$ is the homeomorphism we search. It is clear that $H$ is a well defined bijection. Finally, notice that the interiors (in $E(X)$) of the sets $\bigcup_{j=1}^{n} K_j$ $(n \geq 1)$ cover $X$ and hence $H$ is indeed a homeomorphism.

Proof of point (c) of Theorem 1.1. Let $G$ be a locally compact Polish group. By a theorem of Struble [24] (see also [1]), there exists a proper left invariant compatible metric $d$ on $G$. Let $E(G)$ be the space of all Katetov maps on $(G, d)$ endowed with the pointwise convergence topology. By Lemma 6.5, $L := E(G) \times Q$ is homeomorphic to $Q \setminus \{\text{point}\}$. So, it suffices to show that there is a proper compatible metric $\varrho$ on $L$ such that $\text{Iso}(L, \varrho)$ is isomorphic to $G$. For any $g \in G$ and $(f,q) \in L$ let $g.(f,q) = (f_g,q) \in L$ where $f_g(x) = f(g^{-1} x)$ (since $d$ is left invariant, $f_g \in E(G)$ for each $f \in E(G)$). As in the proof of point (b) of the theorem, we see that the function $G \times L \ni (g,x) \mapsto g.x \in L$ is a continuous action of $G$ on $L$. It is also clear that each $G$-orbit (i.e. each of the sets $G.x$ with $x \in L$) has empty interior. Similarly as in point (b) we show that there is $\varpi \in L$ such that $G$ acts freely at $\varpi$ (for example, $\varpi = (u,q)$ with arbitrary $q \in Q$ and $u(x) = d(x,e)$ where $e$ is the neutral element of $G$). So, by virtue of Theorem 6.1, it remains to check that the action is proper (see the proof of point (b) of the theorem). To this end, take any compact set $W$ in $L$. Then there is $r > 0$ such that $W \subset \{f \in E(G): f(e) \leq r \} \times Q$. Note that the set $\{g \in G: g.W \cap W \neq \emptyset\}$ is closed and contained in $D \times Q$ where

$$D = \{g \in G: \exists f \in E(G): f(e) \leq r \land f(g^{-1}) \leq r\}$$

and therefore it is enough to show that $D$ has compact closure in $G$. But if $g \in D$, and $f \in E(G)$ is such that $f(e) \leq r$ and $f(g^{-1}) \leq r$, then $d(g,e) = d(e,g^{-1}) \leq f(e) + f(g^{-1}) \leq 2r$. This yields that $D \subset B_G(e,2r)$ and the note that $d$ is proper finishes the proof.

6.6. Remark. Van Dantzig and van der Waerden [10] proved that the isometry group of a connected locally compact metric space $(X, d)$ (possibly with non-proper or incomplete metric) is locally compact and acts properly on $X$. It follows from our result [20] that then there exists a proper compatible metric $\varrho$ on $X$ such that $\text{Iso}(X, d) = \text{Iso}(X, \varrho)$. In particular,

$$\{\text{Iso}(Q \setminus \{\text{point}\}, d): d \text{ is a compatible metric}\} =$$

$$= \{\text{Iso}(Q \setminus \{\text{point}\}, \varrho): \varrho \text{ is a proper compatible metric}\}$$

and hence if we omit the word proper in point (c) of Theorem 1.1, we will obtain an equivalent statement.

As we mentioned in the introductory part, each [locally] compact finite-dimensional Polish group is isomorphic to the isometry group of a [proper locally] compact finite-dimensional metric space. Taking this, and Corollary 4.7, into account, the following question may be interesting.

6.7. Problem. Is every finite-dimensional metrizable (resp. Polish) group isomorphic to the isometry group of a finite-dimensional (resp. separable complete) metric space?

References

[1] H. Abels, A. Manoussos, G. Noskov, Proper actions and proper invariant metrics, J. London Math. Soc. (2) 83 (2011), 619–636.
[2] R.D. Anderson, On topological infinite deficiency, Mich. Math. J. 14 (1967), 365–383.
[3] R.D. Anderson and J. McCharen, On extending homeomorphisms to Fréchet manifolds, Proc. Amer. Math. Soc. 25 (1970), 283–289.
[4] R. Arens and J. Eells, *On embedding uniform and topological spaces*, Pacific J. Math. **6** (1956), 397–403.
[5] A.V. Arhangel’skii, *The Hewitt-Nachbin completion in topological algebra. Some effects of homogeneity*, Appl. Categ. Structures **10** (2002), 267–278.
[6] S.K. Berberian, *Lectures in Functional Analysis and Operator Theory*, Graduate Texts in Mathematics 15, Springer-Verlag, New York, 1974.
[7] Cz. Bessaga and A. Pełczyński, *Selected topics in infinite-dimensional topology*, PWN – Polish Scientific Publishers, Warszawa, 1975.
[8] T.A. Chapman, *Deficiency in infinite-dimensional manifolds*, General Topol. Appl. **1** (1971), 263–272.
[9] T.A. Chapman, *Lectures on Hilbert cube manifolds*, C.B.M.S. Regional Conference Series in Math. No 28, Amer. Math. Soc., 1976.
[10] D. van Dantzig and B.L. van der Waerden, *Über metrisch homogene Räume*, Abh. Math. Sem. Hamburg **6** (1928), 367–376.
[11] R. Engelking, *Dimension Theory*, PWN – Polish Scientific Publishers, Warszawa, 1978.
[12] R. Engelking, *General Topology. Revised and completed edition (Sigma series in pure mathematics, vol. 6)*, Heldermann Verlag, Berlin, 1989.
[13] S. Gao and A.S. Kechris, *On the classification of Polish metric spaces up to isometry*, Mem. Amer. Math. Soc. **161** (2003), viii+78.
[14] O.H. Keller, *Die Homoiomorphie der kompakten konvexen Mengen in Hilbertschen Raum*, Math. Ann. **105** (1931), 748–758.
[15] M. Malicki and S. Solecki, *Isometry groups of separable metric spaces*, Math. Proc. Cambridge Phil. Soc. **146** (2009), 67–81.
[16] P. Mankiewicz, *On extension of isometries in normed linear spaces*, Bull. Acad. Pol. Sci. Sér. Sci. Math. **20** (1972), 367–371.
[17] E. Mayer-Wolf, *Isometries between Banach spaces of Lipschitz functions*, Israel J. Math. **38** (1981), 58–74.
[18] J. Melleray, *Compact metrizable groups are isometry groups of compact metric spaces*, Proc. Amer. Math. Soc. **136** (2008), 1451–1455.
[19] J. Mogilski, *CE-decomposition of ℓ₂-manifolds*, Bull. Acad. Pol. Sci. Sér. Sci. Math. **27** (1979), 309–314.
[20] P. Niemiec, *Isometry groups of proper metric spaces*, submitted to Trans. Amer. Math. Soc. (http://arxiv.org/abs/1201.5675).
[21] W. Roelcke and S. Dierolf, *Uniform Structures on Topological Groups and Their Quotients*, McGraw Hill, New York, 1981.
[22] R. Schori, *Topological stability for infinite-dimensional manifolds*, Compos. Math. **23** (1971), 87–100.
[23] W. Sierpiński, *Sur une propriété topologique des ensambles dénombrables denses en soi*, Fund. Math. **1** (1920), 11–16.
[24] R.A. Struble, *Metrics in locally compact groups*, Compos. Math. **28** (1974), 217–222.
[25] H. Toruńczyk, *Remarks on Anderson’s paper “On topological infinite deficiency”*, Fund. Math. **66** (1970), 393–401.
[26] H. Toruńczyk, *Characterizing Hilbert space topology*, Fund. Math. **111** (1981), 247–262.
[27] H. Toruńczyk, *A correction of two papers concerning Hilbert manifolds*, Fund. Math. **125** (1985), 89–93.
[28] V.V. Uspenskij, *On subgroups of minimal topological groups*, Topology Appl. **155** (2008), 1580–1606.
[29] N. Weaver, *Lipschitz Algebras*, World Scientific, 1999.

Piotr Niemiec, Instytut Matematyki, Wydział Matematyki i Informatyki, Uniwersytet Jagielloński, ul. Lojasiewicza 6, 30-348 Kraków, Poland
E-mail address: piotr.niemiec@uj.edu.pl