CHARACTERIZATIONS OF RECURRENCE AND POISSON STABILITY OF FLOWS ON SURFACES

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Abstract. The long-time behavior of orbits is one of the most fundamental properties in dynamical systems. Poincaré studied the Poisson stability, to capture a property whether points return arbitrarily near the initial positions after a sufficiently long time. Birkhoff introduced and studied the concept of recurrent points. We show that the recurrence and Poisson stability of flows on surfaces are topological properties of the orbit spaces. In fact, a flow on a compact connected surface is Poisson stable (resp. recurrent) if and only if the Kolmogorov quotient of the orbit space satisfies $T_1$ (resp. $T_{1/2}$) separation axiom. Using such characterizations, we characterize Hausdorff separation axiom for orbit spaces and their Kolmogorov quotients of flows on compact connected surfaces.

1. Introduction

The long-time behavior of orbits is one of the most fundamental properties in dynamical systems. In [12, 13], Poincaré studied the Poisson stability, to capture a property whether points return arbitrarily near the initial positions after a sufficiently long time. In [4], Birkhoff introduced and studied the concepts of non-wandering points and recurrent points, by introducing the concepts of $\omega$-limit set and $\alpha$-limit set of a point. Cherry showed that the set of orbits in the closure of a non-closed recurrent orbit of a flow on a manifold contains uncountably many Poisson orbits [5]. Athanassopoulos characterized a flow that is either irrational or Denjoy on a closed surface by using non-closed Poisson stable orbits [2]. In this paper, we show that the recurrence and Poisson stability of flows on surfaces are topological properties of the orbit spaces. In fact, the recurrence and Poisson stability of flows on surfaces are characterized using separation axiom. More precisely, to state the main results, we recall some concepts as follows. A topological space is $T_{1/2}$ if any singletons are closed or open. A topological space is $S_{1/2}$ (resp. $S_1$, $T_1$) if the Kolmogorov quotient is $T_{1/2}$ (resp. $T_1$, $T_2$). Then we have the following topological characterizations of recurrence and Poisson stability.

Theorem A. A flow on a compact connected surface is recurrent if and only if the orbit space is $S_{1/2}$.

Theorem B. A flow on a compact connected surface is Poisson stable if and only if the orbit space is $S_1$. 

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Using the characterizations, we characterize Hausdorff separation axiom for flows on surfaces as follows.

**Theorem C.** The orbit space of a flow on a compact connected surface is $S_2$ if and only if the flow is non-wandering and the singular point set either is the whole surface or consists of at most two centers.

**Theorem D.** The orbit space of a flow on a compact connected surface is $T_2$ if and only if the flow consists of closed points and the singular point set either is the whole surface or consists of at most two centers.

The present paper consists of five sections. In the next section, as preliminaries, we introduce fundamental concepts. In §3, we topologically characterize recurrence for flows on surfaces. In §4, Poisson stability for flows on surfaces is characterized topologically. In the final section, these characterizations imply the topological characterizations of separations axioms for flows on surfaces.

2. Preliminaries

2.1. **Topological notion.** A surface is a two-dimensional paracompact manifold with or without boundary. By a continuum, we mean a nonempty compact connected metrizable space. A subset $C$ in a topological space $X$ is separating if the complement $X - C$ is disconnected.

2.1.1. **Separation axioms.** A point $x$ of a topological space $X$ is $T_0$ (or Kolmogorov) if for any point $y \neq x \in X$ there is an open subset $U$ of $X$ such that $|\{x, y\} \cap U| = 1$, where $|A|$ is the cardinality of a subset $A$. A point of a topological space is $T_{1/2}$ if its singleton is either closed or open. A point of a topological space is $T_1$ if its singleton is closed. A topological space is $T_{1/2}$ (resp. $T_1$) if each point is $T_{1/2}$ (resp. $T_1$).

2.1.2. **$T_0$-tification of a topological space.** Let $X$ a topological space with the specialization order. Then the set $\hat{X} := \{\hat{x} \mid x \in X\} = \{\{y \in X \mid \overline{\{y\}} = \overline{\{x\}}\} \mid x \in X\}$ of classes is a decomposition of $X$ and is a $T_0$ space as a quotient space, which is called the $T_0$-tification (or Kolmogorov quotient) of $X$. A topological space is $S_2$ (resp. $S_1$, $S_{1/2}$) if the $T_0$-tification is $T_2$ (resp. $T_1$, $T_{1/2}$).

2.1.3. **Cantor manifolds.** A separable metrizable space $X$ whose small inductive dimension is $n > 0$ is an $n$-dimensional Cantor manifold if the complement $X - L$ for any closed subset $L$ of $X$ whose small inductive dimension is less than $n - 1$ is connected. In [18], Urysohn showed that any $n$-dimensional topological manifold is an $n$-dimensional Cantor manifold.

2.2. **Notion of dynamical systems.** By a flow, we mean a continuous $\mathbb{R}$-action on a surface. Let $v: \mathbb{R} \times S \to S$ be a flow on a compact surface $S$. Then $v_t := v(t, \cdot)$ is a homeomorphism on $S$. For a point $x$ of $S$, we denote by $O(x)$ the orbit of $x$ (i.e. $O(x) := \{v_t(x) \mid t \in \mathbb{R}\}$), $O^+(x)$ the non-negative orbit (i.e. $O^+(x) := \{v_t(x) \mid t \geq 0\}$), and $O^-(x)$ the non-positive orbit (i.e. $O^-(x) := \{v_t(x) \mid t \leq 0\}$). A subset of $S$ is said to be invariant (or saturated) if it is a union of orbits. An invariant subset is minimal if it has no non-empty proper invariant subsets. A point $x$ of $S$ is singular if $x = v_t(x)$ for any $t \in \mathbb{R}$, is periodic if there is a positive number $T > 0$ such that $x = v_T(x)$ and $x \neq v_t(x)$ for any $t \in (0, T)$, and if closed if it is either singular or periodic. Denote by $\text{Sing}(v)$ (resp. $\text{Per}(v), \text{Cl}(v)$)
the set of singular (resp. periodic, closed) points. A point is wandering if there are its neighborhood \( U \) and a positive number \( N \) such that \( v_t(U) \cap U = \emptyset \) for any \( t > N \). Then such a neighborhood is called a wandering domain. A point is non-wandering if it is not wandering (i.e. for any its neighborhood \( U \) and for any positive number \( N \), there is a number \( t \in \mathbb{R} \) with \( |t| > N \) such that \( v_t(U) \cap U \neq \emptyset \)).

For a point \( x \in S \), define the \( \omega \)-limit set \( \omega(x) \) and the \( \alpha \)-limit set \( \alpha(x) \) of \( x \) as follows: \( \omega(x) := \bigcap_{n \in \mathbb{R}} \{v_t(x) \mid t > n\} \), \( \alpha(x) := \bigcap_{n \in \mathbb{R}} \{v_t(x) \mid t < n\} \). A point \( x \) of \( S \) is Poisson stable (or strongly recurrent) if \( x \in \omega(x) \cap \alpha(x) \). A point \( x \) of \( S \) is recurrent if \( x \in \omega(x) \cup \alpha(x) \). Denote by \( R(v) \) the set of non-closed recurrent points. The closure of a non-closed recurrent orbit is called a \( Q \)-set (or quasi-minimal set). An orbit is singular (resp. periodic, closed, non-wandering, recurrent, Poisson stable) if it consists of singular (resp. periodic, closed, non-wandering, recurrent, Poisson stable) points. A flow is non-wandering (resp. Poisson stable, recurrent) if each point is non-wandering (resp. Poisson stable, recurrent). A flow is pointwise almost periodic if any orbit closures are minimal sets. Notice that a flow is pointwise almost periodic if and only if the orbit space is \( S_1 \).

2.2.1. Positive and negative asymptotically stability. A compact invariant subset \( \mathcal{M} \) of a flow on a topological space \( X \) is positively asymptotically stable if, for each neighborhood \( U \) of \( \mathcal{M} \), there is a neighborhood \( V \) of \( \mathcal{M} \) with \( \bigcup_{x \in V} O^+(x) \subseteq U \) such that \( \{y \in X \mid \omega(y) \subseteq \mathcal{M}\} \) is a neighborhood of \( \mathcal{M} \). Similarly, a compact invariant subset \( \mathcal{M} \) of a flow on a topological space \( X \) is negatively asymptotically stable if, for each neighborhood \( U \) of \( \mathcal{M} \), there is a neighborhood \( V \) of \( \mathcal{M} \) with \( \bigcup_{x \in V} O^-(x) \subseteq U \) such that \( \{y \in X \mid \alpha(y) \subseteq \mathcal{M}\} \) is a neighborhood of \( \mathcal{M} \).

2.2.2. Orbit classes and orbit class spaces of flows. For a flow \( v \) on a topological space \( X \) and for an invariant subset \( T \subseteq X \), the orbit space \( T/v \) of \( T \) is a quotient space \( T/\sim \) defined by \( x \sim y \) if \( O(x) = O(y) \) (resp. \( \overline{O(x)} = \overline{O(y)} \)). Notice that an orbit space \( T/v \) is the set \( \{O(x) \mid x \in T\} \) as a set. The (orbit) class \( \hat{O} \) of an orbit \( O \) is the union of orbits each of whose orbit closure corresponds to \( \overline{O} \) (i.e. \( \hat{O} = \{y \in X \mid \overline{O(y)} = \overline{O}\} \)). Moreover, the orbit class space \( T/\hat{v} \) is the set \( \{\hat{O}(x) \mid x \in T\} \) with the quotient topology. Note that the orbit class space is a \( T_0 \)-ification of the orbit space.

2.2.3. Topological properties of orbits. An orbit is proper if it is embedded, locally dense if its closure has a nonempty interior, and exceptional if it is neither proper nor locally dense. A point is proper (resp. locally dense, exceptional) if its orbit is proper (resp. locally dense, exceptional). Denote by \( LD(v) \) (resp. \( E(v), P(v) \)) the union of locally dense orbits (resp. exceptional orbits, non-closed proper orbits). Then \( S = \text{Cl}(v) \cup P(v) \cup LD(v) \cup E(v) \), where \( \cup \) denotes a disjoint union. Note that an orbit on a paracompact manifold (e.g. a surface) is proper if and only if it has a neighborhood in which the orbit is closed. This implies that a non-recurrent point is proper and so that a non-proper point is recurrent. In \[5\] Theorem VI, Cherry showed that the closure of a non-closed recurrent orbit \( O \) of a flow on a manifold contains uncountably many non-closed Poisson orbits whose closures are \( \overline{O} \). This means that each non-closed recurrent orbit of a flow on a manifold has no neighborhood in which the orbit is closed, and so is non-proper. In particular, a non-closed proper orbit is non-recurrent. Therefore the union \( P(v) \) of non-closed
proper orbits is the set of non-recurrent points and that \( R(v) = \text{LD}(v) \cup E(v) \). Hence we have a decomposition \( S = \text{Cl}(v) \cup P(v) \cup R(v) \).

3. Topological characterization of recurrence

We have the following property.

**Lemma 3.1.** Then the following are equivalent for a flow \( v \) on a compact connected surface \( S \):

1. The flow \( v \) is non-wandering.
2. There are finitely many orbits \( O_1, O_2, \ldots, O_k \subset S \) with \( S - \text{Cl}(v) \subseteq \bigcup_{i=1}^{k} O_i \).

In the second case, we can choose \( O_1, O_2, \ldots, O_k \) such that the orbit classes \( \hat{O}_1, \hat{O}_2, \ldots, \hat{O}_k \) are connected components of the set \( R(v) \) of non-closed recurrent points.

**Proof.** Let \( v \) be a flow on a compact connected surface \( S \). Suppose that there are finitely many non-closed orbits \( O_1, O_2, \ldots, O_k \subset S \) with \( S - \text{Cl}(v) \subseteq \bigcup_{i=1}^{k} O_i \). Then \( S = \text{Cl}(v) \cup \bigcup_{i=1}^{k} O_i \). Since any point in \( S - \text{Cl}(v) \) is contained in the closure \( \overline{O}_j \) for some \( j \in \{1, 2, \ldots, k\} \), the flow \( v \) is non-wandering.

Conversely, suppose that \( v \) is non-wandering. By [20] Lemma 2.4 and Theorem 2.5, since \( \text{Sing}(v) \) is closed, we obtain \( S = \text{Cl}(v) \cup \partial P(v) \cup \text{LD}(v) = \text{Cl}(v) \cup \text{LD}(v) \). Then \( R(v) = \text{LD}(v) \). By the Maier theorem [10, 9] (cf. Remark 2 [1]), total number of \( Q \)-sets for \( v \) are finite. From [20] Proposition 2.2, there are finitely many locally dense orbits \( O_1, O_2, \ldots, O_k \subset S \) with \( S - \text{Cl}(v) \subseteq \text{LD}(v) = \bigcup_{i=1}^{k} O_i \) and \( \text{LD}(v) = \bigcup_{i=1}^{k} \hat{O}_i \) such that \( \overline{O}_i \cap \overline{O}_j = \overline{O}_i \cap \overline{O}_j \subseteq \text{Sing}(v) \cup \partial P(v) \) for any \( i \neq j \in \{1, 2, \ldots, k\} \). Then \( \hat{O}_i \cap \hat{O}_j = \overline{O}_i \cap \overline{O}_j = \emptyset \) for any \( i \neq j \in \{1, 2, \ldots, k\} \). This means that the orbit classes \( \hat{O}_1, \hat{O}_2, \ldots, \hat{O}_k \) are connected components of the set \( R(v) \) of non-closed recurrent points.

We have the following characterization of recurrence for flows on surfaces.

**Theorem 3.2.** Then the following are equivalent for a flow on a compact connected surface:

1. The flow is recurrent.
2. Each orbit is closed or locally dense.
3. The orbit class space is \( T_{1/2} \).
4. The orbit space is \( S_{1/2} \).

**Proof.** Since the orbit class space is the \( T_0 \)-ification of the orbit space, assertions (3) and (4) are equivalent. By definition of recurrence, assertion (2) implies assertion (1). Let \( v \) be a flow on a compact connected surface \( S \). Suppose that \( S/\hat{v} \) is \( T_{1/2} \). This means that each orbit class is closed or open. Therefore each orbit closure is either a minimal set or a locally dense \( Q \)-set. This implies that each orbit is recurrent.

Conversely, suppose that \( v \) is recurrent. By definition of non-wandering property and recurrence, the flow \( v \) is non-wandering. Since the union \( P(v) \) is the set of non-recurrent points, we have \( P(v) = \emptyset \). By [20] Lemma 2.3, we obtain \( E(v) = \emptyset \). Then \( S = \text{Cl}(v) \cup \text{LD}(v) \). This means that each orbit is closed or locally dense. [20] Lemma 2.3 implies that \( \text{LD}(v) \cap \text{Cl}(v) = \emptyset \) and so that \( \text{LD}(v) \) is open. By Lemma 3.1, there are finitely many non-closed recurrent orbits \( O_1, O_2, \ldots, O_k \)
with \( \text{LD}(v) = \bigsqcup_{k=1}^k \hat{O}_k \) such that the orbit classes \( \hat{O}_1, \hat{O}_2, \ldots, \hat{O}_k \) are connected components of \( \text{LD}(v) \). Since any connected components are closed, the finiteness of \( \text{LD}(v) \) implies that the orbit classes \( \hat{O}_1, \hat{O}_2, \ldots, \hat{O}_k \) are open in the open subset \( \text{LD}(v) \) and so open in \( S \). This implies that the orbit classes of any non-closed points are open. Therefore assertion (3) holds. \( \square \)

Theorem 4 is followed from Theorem 3.2

4. Topological characterization of Poisson stability

We will show that Poisson stability and pointwise almost periodicity for flows on compact surfaces are equivalent. To demonstrate this, we show the following statements.

**Lemma 4.1.** For a flow \( v \) on a connected closed surface \( S \), there is a flow \( w \) whose singular point set is totally disconnected on a surface \( T \) which is a disjoint union of closed surfaces such that the restriction \( v|_{S - \text{Sing}(v)} \) is topologically equivalent to the restriction \( w|_{T - \text{Sing}(w)} \).

**Proof.** Fix a Riemannian metric on \( S \). Since the singular point set is closed, the complement \( S_0 := S - \text{Sing}(v) \) is open and so a surface with at most finitely many genus. Let \( S_{mc} \) be the metric completion of \( S_0 \). Collapsing each connected component of \( S_{mc} - S_0 \) into a singleton, let \( S_{mc} \) be the resulting space. Define a flow \( v_{mc} \) on \( S_{mc} \) as follows: \( v_{mc}|_{S_0} = v|_{S_0} \) and the difference \( S_{mc} - S_0 \) is the set \( \text{Sing}(v_{mc}) \) of singular points of \( v_{mc} \). We show that \( v_{mc} \) is desired. Indeed, by [14, Theorem 3], there is a surface \( T \) which is a disjoint union of closed surfaces such that the surface \( S_0 \) is homeomorphic to the resulting surface from \( T \) by removing a closed totally disconnected subset. Then the surface \( S_{mc} \) is homeomorphic to \( T \). By construction, the singular point set \( \text{Sing}(v_{mc}) \) of \( v_{mc} \) is totally disconnected and the restriction \( v_{mc}|_{S_0} \) is topologically equivalent to the restriction \( v_{mc}|_{S_0} \). \( \square \)

**Lemma 4.2.** Each non-minimal locally dense Q-set of a flow on a compact connected surface contains orbits that are not Poisson stable.

**Proof.** Let \( v \) be a flow on a compact connected surface \( S \) and \( M \) a non-minimal locally dense Q-set of \( v \). Since Poisson stability is invariant under taking doubles of manifolds with boundary, taking the double of \( M \) if necessary, we may assume that \( M \) is closed. Fix any locally dense orbit \( O \) with \( \overline{O} = M \subseteq \text{LD}(v) \). By [20, Proposition 2.2], we have \( \overline{O} \cap \text{Per}(v) = \emptyset \) and \( \hat{O} = \overline{O \setminus (\text{Sing}(v) \cup P(v))} \). [20, Lemma 2.3] implies that \( E(v) \cap \overline{\text{Cl}(v) \cup \text{LD}(v)} = \emptyset \) and so that the union \( \overline{P(v) \cup E(v)} \) is a neighborhood of \( E(v) \). Since \( O \cap (P(v) \cup E(v)) = \emptyset \), we have \( \overline{O} \cap E(v) = \emptyset \). Therefore \( \overline{O} \subseteq \text{Sing}(v) \cup \overline{P(v) \cup \text{LD}(v)} = \hat{O} \subseteq \text{LD}(v) \).

Assume that \( \overline{O} \) consists of Poisson stable orbits. Since \( P(v) \) is the set of non-recurrent points, we have \( \overline{O} \cap P(v) = \emptyset \). Then \( \overline{O} \subseteq \text{Sing}(v) \cup \text{LD}(v) \). [20, Lemma 2.3] implies that \( \text{LD}(v) \cap \overline{\text{Cl}(v) \cup E(v)} = \emptyset \) and so that the union \( \overline{P(v) \cup \text{LD}(v)} \) is a neighborhood of \( \text{LD}(v) \). Since both \( \overline{O} \) and \( \overline{\text{LD}(v) \cup P(v)} \) are neighborhoods of \( \hat{O} \subseteq \text{LD}(v) \), the intersection \( U := \overline{O} \cap (\overline{\text{LD}(v) \cup P(v)}) \subseteq \text{LD}(v) \) is an invariant neighborhood of \( \hat{O} \). Then \( U = \overline{O} \cap \text{LD}(v) = \overline{\text{Sing}(v)} = \hat{O} \subseteq \text{LD}(v) \) and so that \( U = \hat{O} \) is open. Since \( \overline{U} \setminus \text{Sing}(v) = U = M \cap \text{LD}(v) \subseteq \text{LD}(v) \), non-minimality implies that \( U \) is an open surface such that \( \emptyset \neq \partial U = \overline{U} - U \subseteq \text{Sing}(v) \).

By Lemma 4.1, there is a flow \( w \) with a nonempty totally disconnected singular
Theorem 4.3. The following statements are equivalent for a flow \( v \) on a compact connected surface \( S \):

1. The flow \( v \) is Poisson stable.
2. Either the flow \( v \) is minimal or all orbits are closed.
3. Either \( P(v) \cup LD(v) = \emptyset \) or \( S = LD(v) \).
4. The orbit class space \( S/\hat{v} \) is \( T_1 \).
5. The orbit space \( S/v \) is \( S_1 \).
6. The flow \( v \) is pointwise almost periodic.

Proof. Since the orbit class space is a \( T_0 \)-ification of the orbit space, assertions (4) and (5) are equivalent. By definition of pointwise almost periodicity, assertions (4) and (6) are equivalent. Let \( v \) be a flow on a compact connected surface \( S \). If \( v \) is minimal, then \( v \) is Poisson stable, \( S = LD(v) \), and the orbit class space \( S/\hat{v} \) is a singleton and so \( T_1 \). If \( v \) is identical, then \( v \) is Poisson stable, \( S = S/v = S/\hat{v} \) is \( T_1 \), and \( P(v) \cup LD(v) = \emptyset \). Thus we may assume that \( v \) is not trivial. By definition of orbit class spaces, assertion (2) implies assertion (5). Since any closed orbits are Poisson stable, assertion (2) implies assertion (1).

Suppose that \( P(v) \cup LD(v) = \emptyset \). By [20, Lemma 2.3], we have \( E(v) = \emptyset \) and so \( S = Cl(v) \). This means that assertion (3) implies assertion (2).

Suppose that \( v \) is Poisson stable. Since Poisson stable orbits are recurrent, Lemma 3.2 implies that \( S = Cl(v) \cup LD(v) \). [20, Lemma 2.3] implies that \( LD(v) \cap Cl(v) = \emptyset \) and so that \( LD(v) \) is open. By Lemma 4.2, Poisson stability implies that each locally dense Q-set of \( v \) is minimal. This implies that the orbit closure of any non-closed recurrent point consists of non-closed recurrent points and so that the finite union \( LD(v) \) of Q-sets is closed. Non-minimality implies \( LD(v) = \emptyset \) and so \( S = Cl(v) \).

Suppose that the orbit class space \( S/\hat{v} \) is \( T_1 \). Then each orbit closure is a minimal set and so \( v \) is non-wandering. Since the closure of a non-recurrent orbit is not minimal, we have \( P(v) = \emptyset \). By [20, Lemma 2.4], the periodic point set \( Per(v) \) is open and \( S = Cl(v) \cup LD(v) \). [20, Lemma 2.3] implies that \( LD(v) \cap Cl(v) = \emptyset \) and so that \( LD(v) \) is open. By Lemma 4.1 the union \( LD(v) \) consists of a finite disjoint union of orbit classes. Since each orbit closure is a minimal set, the union \( LD(v) \) is a finite disjoint union of locally dense minimal sets and so is closed. By non-minimality of \( v \) and connectivity of \( S \), we obtain \( LD(v) = \emptyset \) and so \( P(v) \cup LD(v) = \emptyset \).
Theorem 4.3 is followed from Theorem 4.3.

5. CHARACTERIZATIONS OF $T_1$ AND $T_2$ SEPARATION AXIOMS FOR ORBIT SPACES

Theorem 4.3 implies the following characterization of $T_1$ separation axiom for flows on compact connected surfaces.

Corollary 5.1. The orbit space of a flow on a compact connected surface is $T_1$ if and only if the flow is not minimal but Poisson stable.

We show the following equivalence to characterize $T_1$ separation axiom for flows on compact connected surfaces.

Lemma 5.2. The following statements are equivalent for a non-trivial flow with finitely many singular points on a compact connected surface:

1. The flow is non-wandering, and the singular point set consists of at most two centers.
2. The flow is Poisson stable.
3. The orbit space is $T_1$.
4. Each singular point is a center, and there are neither limit cycles nor exceptional Q-sets.
5. The flow consists of periodic orbits and at most two centers.

Proof. Let $v$ be a non-trivial flow with finitely many singular points on a compact connected surface $S$. By non-minimality of $v$, Corollary 5.1 implies that assertions (2) and (3) are equivalent. Assertion (5) implies assertion (3). Recall that each Poisson stable flow is non-wandering. The finiteness of singular points implies that each singular point is isolated. The non-existence of wandering domains implies the non-existence of limit cycles in any case.

Suppose that $v$ is non-wandering and the singular point set $\text{Sing}(v)$ consists of at most two centers. By [20, Lemma 2.4], there are no exceptional Q-sets. This means that assertion (1) implies assertion (4).

Suppose that each singular point is a center and there are neither limit cycles nor exceptional Q-sets. By a generalization of the Poincaré-Bendixon theorem for a flow with finitely many singular points (cf. [11, Theorem 2.6.1]), since any singular points are centers, each of $\omega$-limit set and $\alpha$-limit set of a non-closed point is a locally dense minimal set. The non-minimality implies that there are non-closed points and so that $S = \text{Cl}(v)$. This shows that assertion (4) implies assertion (3).

Suppose that $S/v$ is $T_1$. This means that $S = \text{Cl}(v)$ and so that $v$ is non-wandering. [6, Theorem 3] implies that each singular point is either a center or a multi-saddle. By the non-existence of non-closed orbits, each singular point is a center. By connectivity of $S$, Poincaré-Hopf theorem implies that there are at most two centers. This shows that assertion (3) implies assertions (1) and (5).

We characterize the Hausdorff separation property of the orbit (class) spaces for non-trivial flows on compact connected surfaces. The orientable case of the following result has stated in [19, Theorem 6.6].

Proposition 5.3. Let $v$ be a non-trivial flow on a compact connected surface $S$. The following statements are equivalent:

1. The orbit space $S/v$ is $T_2$.
2. The orbit class space $S/\hat{v}$ is $T_2$ (i.e. $v$ is $R$-closed).
(3) The flow $v$ consists of periodic orbits and at most two centers.

(4) The flow $v$ is non-wandering and the singular point set consists of at most two centers.

(5) The flow $v$ is Poisson stable and each singular point is isolated.

In any case, the Euler characteristic of $S$ is non-negative and the orbit space $S/v$ is either a closed interval or a circle.

Proof. Lemma 5.2 implies that assertions (3)–(5) are equivalent. From Theorem 4.3 and Corollary 5.1, by assertion (2), the orbit space $S/v$ is $T_1$ for any cases. Let $v$ be a non-trivial flow on a compact connected surface $S$ whose orbit space $S/v$ is $T_1$. This means that $S = \text{Cl}(v) = \text{Sing}(v) \sqcup \text{Per}(v)$ and so assertions (1) and (2) are equivalent. The closedness of the singular point set implies that the union $\text{Per}(v)$ is open. Since $v$ is non-trivial, there is a periodic orbit $O$. Let $C$ be the connected component of $\text{Per}(v)$ that contains $O$.

Suppose that $v$ consists of periodic orbits and at most two centers. [20, Corollary 2.9] implies that each connected component of $\text{Per}(v)$ is either an annulus, a torus, a M"obius band, or a Klein bottle whose orbit space is an interval or a circle and whose boundary consists of singular points and one-sided periodic orbits. By the Poincaré-Hopf theorem, the Euler characteristic of $S$ is non-negative. Moreover, each connected component of the boundary $\partial C$ is a center and so the complement $S - \text{Sing}(v)$ is connected. This implies $S = C \sqcup \text{Sing}(v)$. Since the restriction $C/v$ is an interval or a circle, the orbit space $S/v$ is either a closed interval or a circle, and so is $T_2$.

Conversely, suppose that $S/v$ is $T_2$. Then $S = \text{Sing}(v) \sqcup \text{Per}(v)$ and so $\partial \text{Sing}(v) = \partial \text{Per}(v)$. Each boundary component of $\text{Per}(v)$ is a singular point and so is each boundary component of $\text{Sing}(v)$. By definition of dimension, the dimension of $\text{Sing}(v)$ is at most one and so $\text{Sing}(v) = \partial \text{Sing}(v)$. This means that each connected component of $\text{Sing}(v)$ is a singleton. Since a connected surface is a Cantor manifold (cf. [8, Theorem 2.1]), the complement $\text{Per}(v) = S - \text{Sing}(v)$ is connected. Therefore $\text{Per}(v) = C$, and it contains no singular points. This implies that $C$ is a surface whose Euler characteristic is zero and so either an annulus, a torus, a M"obius band, or a Klein bottle. Then the whole surface $S = \text{Per}(v) = \overline{C} = C \sqcup \partial C$ is the union of periodic orbits and at most two centers. $\square$

Proposition 5.3 implies the following characterizations of $T_2$ separation axiom for orbit spaces and orbit class spaces of flows on compact connected surfaces.

5.1. Proof of Theorem C Let $v$ be a flow compact connected surface $S$. If $v$ is minimal, then $v$ is non-wandering, there are no singular points, and $S/\hat{v}$ is a singleton and so is $T_2$. If $v$ is identical, then $v$ is non-wandering, the singular point set is the whole surface, and $S/\hat{v}$ is the original surface $S$ and so is $T_2$. Thus we may assume that $v$ is non-trivial. Proposition 5.3 implies the assertion.

5.2. Proof of Theorem D Let $v$ be a flow compact connected surface $S$. Suppose that $S/v$ is $T_2$. Then $v$ is not minimal. Proposition 5.3 implies that either $v$ is identical or $v$ consists of closed points and at most two centers. This means that $v$ consists of closed points and the singular point set either is the whole surface or consists of at most two centers.

Conversely, suppose that $v$ consists of closed points and that the singular point set either is the whole surface or consists of at most two centers. Then $v$ is not
minimal. Since the orbit spaces of the identical flows are the original surfaces and so is $T_2$, we may assume that $v$ is non-trivial. Proposition 5.3 implies $S/v$ is $T_2$.

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