ON \textit{f}-EIKONAL HELICES AND \textit{f}-EIKONAL SLANT HELICES IN RIEMANNIAN MANIFOLDS

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Abstract. In this paper, we define \textit{f}-eikonal helix curves and \textit{f}-eikonal \(V_n\)-slant helix curves in a \(n\)-dimensional Riemannian manifold. Also, we give the definition of harmonic curvature functions related to \(\textit{f}\)-eikonal helix curves and \(\textit{f}\)-eikonal \(V_n\)-slant helix curves in a \(n\)-dimensional Riemannian manifold. Moreover, we give characterizations for \(\textit{f}\)-eikonal helix curves and \(\textit{f}\)-eikonal \(V_n\)-slant helix curves by making use of the harmonic curvature functions.

1. Introduction

The helices share common origins in the geometries of the platonic solids, with inherent hierarchical potential that is typical of biological structures. The helices provide an energy-efficient solution to close-packing in molecular biology, a common motif in protein construction, and a readily observable pattern at many size levels throughout the body. The helices are described in a variety of anatomical structures, suggesting their importance to structural biology and manual therapy [13].

A general helix in Euclidean 3-space \(E^3\) is defined by the property that tangent makes a constant angle with a fixed straight line (the axis of general helix). A classical result stated by Lancret in 1802 and first proved by de Saint Venant in 1845 ([8] and [14]) is: A necessary and sufficient condition that a curve be a general helix is that the ratio of curvature to torsion be constant. In [11], Özdamar and Hacısalihoğlu defined harmonic curvature functions \(H_i\) \((1 \leq i \leq n-2)\) of a curve \(\alpha\) in \(n\)-dimensional Euclidean space \(E^n\). They generalized inclined curves (general helix) in \(E^3\) to \(E^n\) and then gave a characterization for the inclined curves in \(E^n\). Then, Izumiya and Takeuchi defined a new kind of helix (slant helix) and they gave a characterization of slant helices in Euclidean 3-space \(E^3\) [6]. Kula and Yaylı have studied spherical images of tangent indicatrix and binormal indicatrix of a slant helix and they showed that the spherical images are spherical helix [7]. In 2008, Önder et al. defined a new kind of slant helix in Euclidean 4-space \(E^4\) which is called \(B_2\)-slant helix and they gave some characterizations of these slant helices in Euclidean 4-space \(E^4\) [10]. And then in 2009, Gök et al. defined a new kind of slant helix in Euclidean \(n\)-space \(E^n\), \(n > 3\), which they called \(V_n\)-slant helix and they gave some characterizations of these slant helices in Euclidean \(n\)-space [3]. On the other hand, Camcı et al. give some characterizations for a non-degenerate curve to be a generalized helix by using its harmonic curvatures [11].

Let \(M\) be a Riemannian manifold, where \(\langle , \rangle\) is the metric. Let \(f : M \to \mathbb{R}\) be a function and let \(\nabla f\) be its gradient, i.e., \(df(X) = \langle \nabla f, X \rangle\). We say that \(f\) is eikonal if it satisfies: \(\|\nabla f\| = \text{constant}\) [2]. \(\nabla f\) is used many areas of science such as mathematical physics and geometry. So, \(\nabla f\) is very important subject. For example, the Riemannian condition \(\|\nabla f\|^2 = 1\) (for non-constant \(f\) on connected \(M\)) is precisely the eikonal equation of geometrical optics. Thus on a connected \(M\), a non-constant real valued \(f\) is Riemannian iff \(f\) satisfies this eikonal equation. In the geometrical optical interpretation, the level sets of \(f\) are interpreted as wave fronts. The characteristics of the eikonal equation (as a partial differential equation), are then the solutions of the gradient flow equation for \(f\) (an ordinary differential equation), \(x' = \text{grad} f(x)\), which are geodesics of \(M\) orthogonal to the level sets of \(f\), and which are parametrized by arc length. These geodesics can be interpreted as light rays orthogonal to the wave fronts [12].

In this work, we introduced \(\textit{f}\)-eikonal helices and \(\textit{f}\)-eikonal slant helices in a \(n\)-dimensional Riemannian manifold \(M^n\). Moreover, in \(M^n\), we give the definition of harmonic curvature functions. Also, we give new characterizations for \(\textit{f}\)-eikonal helix curves and \(\textit{f}\)-eikonal \(V_n\)-slant helix curves by making use of the harmonic curvature functions.
2. Preliminaries

In this section, we give some basic definitions from differential geometry.

**Definition 2.1.** Let $\alpha = \alpha(s)$ be a smooth curve parametrized by its arc length $s$ in $n$-dimensional Riemannian manifold $M^n$. If there exist orthonormal frame fields $\{V_1, ..., V_n\}$ along $\alpha$ and positive functions $k_1(s), ..., k_{n-1}(s)$ satisfying the following system of ordinary equations

$$\nabla_{\alpha}V_i(s) = -k_{i-1}(s)V_{i-1}(s) + k_i(s)V_{i+1}(s), \quad i = 1, ..., n,$$

(2.1)

where $V_0 = V_{n+1} = 0$ and $\nabla_{\alpha}$ denotes the Riemannian connexion along $\alpha$, then the curve $\alpha$ is called a Frenet curve of proper $n$-curvatures of the curve $\alpha$. A Frenet curve $\alpha$ is called a Frenet curve of order $n$.

**Definition 2.2.** Let $M$ be a Riemannian manifold, where $\langle , \rangle$ is the metric. Let $f: M \to \mathbb{R}$ be a function and let $\nabla f$ be its gradient, i.e., $df(X) = \langle \nabla f, X \rangle$. We say that $f$ is eikonal if it satisfies:

$$\|\nabla f\| = \text{constant.}$$

[1]

3. $f$-eikonal helix curves and their harmonic curvature functions

In this section, we define $f$-eikonal helix curves and we give characterizations for a $f$-eikonal helix curve in $n$-dimensional Riemannian manifold $M^n$ by using harmonic curvature functions of the curve.

**Definition 3.1.** Let $\alpha(s): I \subset \mathbb{R} \to M^n$ be a Frenet curve of proper $n$-curvatures in $n$-dimensional Riemannian manifold $M^n$ and let the functions $k_i(s)$ $(i = 1, ..., n-1)$ be curvatures of the curve $\alpha$. Harmonic curvatures of the curve $\alpha$ are defined by $H_i: I \subset \mathbb{R} \to \mathbb{R}$ along $\alpha$ in $M^n$, $i = 1, ..., n-2$, such that

$$H_i = \frac{k_1}{k_2}, \quad H_i = \{V_1 [H_{i-1}] + k_iH_{i-2}\} \frac{1}{k_{i+1}}$$

for $i = 2, ..., n-2$, where $V_1$ is the unit vector tangent vector field of the curve $\alpha$.

**Definition 3.2.** Let $M^n$ be a Riemannian manifold with the metric $\langle , \rangle$ and let $\alpha(s)$ be a Frenet curve with the unit tangent vector field $V_1$ in $M^n$. Let $f: M^n \to \mathbb{R}$ be an eikonal function along curve $\alpha$, i.e. $\|\nabla f\| = \text{constant}$ along the curve $\alpha$. If the function $\langle \nabla f, V_i \rangle$ is non-zero constant along $\alpha$, then $\alpha$ is called a $f$-eikonal helix curve. And, $\nabla f$ is called the axis of the $f$-eikonal helix curve $\alpha$.

**Example 3.1.** We consider the Riemannian manifold $M^3 = \mathbb{R}^3$ with the Euclidean metric $\langle , \rangle$. Let

$$f: M^3 \to \mathbb{R}$$

be a function defined on $M^3$. Then, the curve

$$\alpha: I \subset \mathbb{R} \to M^3$$

$$s \to \alpha(s) = (\cos \frac{s}{\sqrt{2}}, \frac{s}{\sqrt{2}}, \sin \frac{s}{\sqrt{2}})$$

is a $f$-eikonal helix curve on $M^3$.

Firstly, we will show that $f$ is a eikonal function along the curve $\alpha$. If we compute $\nabla f$, we find $\nabla f$ as

$$\nabla f = (2x, 1, 2z).$$

So, we get

$$\|\nabla f\| = \sqrt{4(x^2 + z^2) + 1}.$$ 

And, if we compute $\|\nabla f\|$ along the curve $\alpha$, we find

$$\|\nabla f\|_\alpha = \sqrt{5} = \text{constant.}$$

That is, $f$ is a eikonal function along the curve $\alpha$.

Now, we will show that the function $\langle \nabla f, V_i \rangle$ is non-zero constant along the curve $\alpha$. Since

$$\nabla f|_\alpha = (2\cos \frac{s}{\sqrt{2}}, 1, 2\sin \frac{s}{\sqrt{2}})$$

and

$$V_i = (-\frac{1}{\sqrt{2}} \sin \frac{s}{\sqrt{2}} \frac{1}{\sqrt{2}} \frac{1}{\sqrt{2}} \cos \frac{s}{\sqrt{2}}),$$

we get

$$\langle \nabla f, V_i \rangle = (2\cos \frac{s}{\sqrt{2}}, 1, 2\sin \frac{s}{\sqrt{2}})$$

and

$$V_i = (-\frac{1}{\sqrt{2}} \sin \frac{s}{\sqrt{2}} \frac{1}{\sqrt{2}} \frac{1}{\sqrt{2}} \cos \frac{s}{\sqrt{2}}).$$


we obtain
\[ (\nabla f|_\alpha, V_1) = \frac{1}{\sqrt{2}} = \text{constant} \]
along the curve \( \alpha \). Consequently, \( \alpha \) is a \( f \)-eikonal helix curve on \( M \).

**Theorem 3.1.** Let \( M^n \) be a \( n \)-dimensional Riemannian manifold with the metric \( \langle , \rangle \) and complete connected smooth without boundary. Let \( M^n \) be isometric to a Riemannian product \( N \times \mathbb{R} \). Let us assume that \( f : M^n \to \mathbb{R} \) be a non-trivial affine function (see main Theorem in [5]) and \( \alpha(s) \) be a Frenet curve of proper \( n \) in \( M^n \). If \( \alpha \) is a \( f \)-eikonal helix curve with the axis \( \nabla f \), then the system
\[ (V_{i+2}, \nabla f) = H_i (V_i, \nabla f), \quad i = 1, \ldots, n-2 \quad (3.1) \]
holds, where \( \{V_1, \ldots, V_n\} \) and \( \{H_1, \ldots, H_{n-2}\} \) are the Frenet frame and the Harmonic curvatures of \( \alpha \), respectively.

**Proof.** Since \( \{V_1, \ldots, V_n\} \) is the orthonormal frame of the curve \( \alpha \) in \( M^n \), \( \nabla f \) can be expressed in the form
\[ \nabla f = \lambda_1 V_1 + \ldots + \lambda_n V_n. \quad (3.2) \]
Doing dot product with \( V_1 \) in each part of (3.2), we get
\[ \langle \nabla f, V_1 \rangle = \lambda_1 = \| \nabla f \| \cos (\theta) = \text{constant} \quad (3.3) \]
since \( \alpha \) is a \( f \)-eikonal helix curve. If we take the derivative in each part of (3.3) in the direction \( V_1 \) in \( M^n \), then we have
\[ \langle \nabla V_1 \nabla f, V_1 \rangle + \langle \nabla f, \nabla V_1 V_1 \rangle = 0. \quad (3.4) \]
On the other hand, from Lemma 2.3 (see [12]), \( \nabla f \) is parallel in \( M^n \). That is, \( \nabla V_1 \nabla f = 0 \). Hence, by using (3.4) and Frenet formulas, we obtain
\[ k_1 \langle \nabla f, V_2 \rangle = 0. \quad (3.5) \]
And, since \( k_1 \) is positive function, from (3.5), we get
\[ \langle \nabla f, V_2 \rangle = 0. \quad (3.6) \]
By taking the derivative in each part of (3.6) in the direction \( V_1 \) in \( M^n \), we can write the equality
\[ \langle \nabla V_1 \nabla f, V_2 \rangle + \langle \nabla f, \nabla V_1 V_2 \rangle = 0. \quad (3.7) \]
And, since \( \nabla V_1 \nabla f = 0 \), by using (3.7) and Frenet formulas, we obtain
\[ -k_1 \langle \nabla f, V_1 \rangle + k_2 \langle \nabla f, V_3 \rangle = 0. \quad (3.8) \]
Therefore, from (3.8), we have
\[ \langle \nabla f, V_3 \rangle = \lambda_3 = \frac{k_1}{k_2} \langle \nabla f, V_1 \rangle. \quad (3.9) \]
Moreover, since \( H_1 = \frac{k_1}{k_2} \), from (3.9), we can write
\[ \langle \nabla f, V_4 \rangle = H_1 \langle \nabla f, V_1 \rangle. \]
It follows that the equality (3.1) is true for \( i = 1 \). According to the induction theory, let us assume that the equality (3.1) is true for all \( k \), where \( 1 \leq k \leq i \) for some positive integers \( i \). Then, we will prove that the equality (3.1) is true for \( i + 1 \). Since the equality (3.1) is true for some positive integers \( i \), we can write
\[ (V_{i+2}, \nabla f) = H_i \langle V_i, \nabla f \rangle \quad (3.10) \]
for some positive integers \( i \). If we take derivative in each part of (3.10) in the direction \( V_1 \) in \( M^n \), we have
\[ \langle \nabla V_1 V_{i+2}, \nabla f \rangle + (V_{i+2}, \nabla V_1 \nabla f) = V_1 [H_i \langle V_i, \nabla f \rangle]. \quad (3.11) \]
And, by using (3.11) and Frenet formulas, we get
\[ -k_{i+1} \langle V_{i+1}, \nabla f \rangle + k_{i+2} \langle V_{i+3}, \nabla f \rangle + (V_{i+2}, \nabla V_1 \nabla f) = V_1 [H_i \langle V_i, \nabla f \rangle]. \quad (3.12) \]
Moreover, \( \nabla V_1 \nabla f = 0 \). Hence, from (3.12), we can write
\[ -k_{i+1} \langle V_{i+1}, \nabla f \rangle + k_{i+2} \langle V_{i+3}, \nabla f \rangle = V_1 [H_i \langle V_i, \nabla f \rangle]. \quad (3.13) \]
And so, we obtain
\[ (V_{i+3}, \nabla f) = \{ V_1 [H_i \langle V_i, \nabla f \rangle] + k_{i+1} \langle V_{i+1}, \nabla f \rangle \} \frac{1}{k_{i+2}}. \quad (3.14) \]
On the other hand, since the equality (3.1) is true for \( i - 1 \) according to the induction hypothesis, we have

\[
\langle V_{i+1}, \nabla f \rangle = H_{i-1} \langle V_1, \nabla f \rangle.
\]  

(3.15)

Therefore, by using (3.14) and (3.15), we get

\[
\langle V_{i+3}, \nabla f \rangle = \{V_1 [H_i] + k_{i+1} H_{i-1}\} \cdot \frac{1}{k_{i+2}} \cdot \langle V_1, \nabla f \rangle.
\]  

(3.16)

Moreover, we obtain

\[
H_{i+1} = \{V_1 [H_i] + k_{i+1} H_{i-1}\} \cdot \frac{1}{k_{i+2}}.
\]  

(3.17)

for \( i + 1 \) in the Definition 3.1. So, we have

\[
\langle V_{i+3}, \nabla f \rangle = H_{i+1} \langle V_1, \nabla f \rangle
\]  by using (3.16) and (3.17). It follows that the equality (3.1) is true for \( i + 1 \). Consequently, we get

\[
\langle V_{i+2}, \nabla f \rangle = H_i \langle V_1, \nabla f \rangle
\]  for all \( i \) according to induction theory. This completes the proof. 

□

Theorem 3.2. Let \( M^n \) be a n-dimensional Riemannian manifold with the metric \( (, \) \) and complete connected smooth without boundary. Let \( M^n \) be isometric to a Riemannian product \( N \times \mathbb{R} \). Let us assume that \( f : M^n \to \mathbb{R} \) be a non-trivial affine function (see main Theorem in [5]) and \( \alpha(s) \) be a Frenet curve of proper \( n \) in \( M^n \). If \( \alpha \) is a \( f \)-eikonal helix curve with the axis \( \nabla f \), then the axis of the curve \( \alpha \):

\[
\nabla f = \|\nabla f\| \cos(\theta) (V_1 + H_1 V_3 + ... + H_{n-2} V_n),
\]

where \( \{V_1, ..., V_n\} \) and \( \{H_1, ..., H_{n-2}\} \) are the Frenet frame and the Harmonic curvatures of \( \alpha \), respectively.

Proof. Since \( \alpha \) is a \( f \)-eikonal helix curve, we can write

\[
\langle \nabla f, V_1 \rangle = \text{constant}.
\]  

(3.18)

If we take the derivative in each part of (3.18) in the direction \( V_1 \) in \( M^n \), then we have

\[
\langle \nabla V_1 \nabla f, V_1 \rangle + \langle \nabla f, \nabla V_1 V_1 \rangle = 0.
\]  

(3.19)

On the other hand, from Lemma 2.3 (see [12]), \( \nabla f \) is parallel in \( M^n \). That’s why, \( \nabla V_1 \nabla f = 0 \). Then, we obtain

\[
k_1 \langle \nabla f, V_2 \rangle = 0.
\]  

(3.20)

by using (3.19) and Frenet formulas. Since \( k_1 \) is positive function, (3.20) implies that

\[
\langle \nabla f, V_2 \rangle = 0.
\]

Hence, we can write the axis of \( \alpha \) as

\[
\nabla f = \lambda_1 V_1 + \lambda_3 V_3 + ... + \lambda_n V_n.
\]  

(3.21)

Moreover, from (3.21), we get

\[
\lambda_1 = \langle \nabla f, V_1 \rangle
\]
\[
\lambda_3 = \langle \nabla f, V_3 \rangle
\]
\[
\vdots
\]
\[
\lambda_n = \langle \nabla f, V_n \rangle
\]

by using dot product. On the other hand, from Theorem 3.1, we know that

\[
\lambda_1 = \langle \nabla f, V_1 \rangle = \|\nabla f\| \cos(\theta)
\]
\[
\lambda_3 = \langle \nabla f, V_3 \rangle = \|\nabla f\| \cos(\theta) H_1
\]
\[
\vdots
\]
\[
\lambda_n = \langle \nabla f, V_n \rangle = \|\nabla f\| \cos(\theta) H_{n-2}.
\]  

(3.22)
Thus, it can be easily obtained the axis of the curve $\alpha$ as
\[
\nabla f = \|\nabla f\| \cos(\theta) (V_1 + H_1 V_3 + \ldots + H_{n-2} V_n)
\]
by making use of the equality (3.21) and the system (3.22). This completes the proof.

**Theorem 3.3.** Let $M^n$ be a $n$-dimensional Riemannian manifold with the metric $(,)$ and complete connected smooth without boundary. Let $M^n$ be isometric to a Riemannian product $N \times \mathbb{R}$. Let us assume that $f : M^n \to \mathbb{R}$ be a non-trivial affine function (see main Theorem in [1]) and $\alpha(s)$ be a Frenet curve of proper $n$ in $M^n$. If $\alpha$ is a $f$-eikonal helix curve, then $H_{n-2} = 0$ and $H_1^2 + H_2^2 + \ldots + H_{n-1}^2$ is nonzero constant, where \{H_1, ..., H_{n-2}\} is the Harmonic curvatures of $\alpha$.

**Proof.** Let $\alpha$ be a $f$-eikonal helix curve and \{V_1, ..., V_n\} be the Frenet frame of $\alpha$. Then, from Theorem 3.2, we know that
\[
\nabla f = \|\nabla f\| \cos(\theta) (V_1 + H_1 V_3 + \ldots + H_{n-2} V_n).
\]
Therefore, from (3.23), we can write
\[
\langle \nabla f, \nabla f \rangle = \|\nabla f\|^2 \left( \cos^2(\theta) + \sum_{i=1}^{n-2} H_i^2 \cos^2(\theta) \right).
\]
Moreover, by the definition of Riemannian metric, we have
\[
\langle \nabla f, \nabla f \rangle = \|\nabla f\|^2.
\]
Hence, from (3.24) and (3.25), we obtain
\[
\cos^2(\theta) + \sum_{i=1}^{n-2} H_i^2 \cos^2(\theta) = 1.
\]
It follows that
\[
H_1^2 + H_2^2 + \ldots + H_{n-1}^2 = \tan^2(\theta) = \text{constant}.
\]
Now, we will show that $H_{n-2} = 0$. We assume that $H_{n-2} = 0$. Then, for $i = n - 2$ in Theorem 3.1,
\[
\langle V_n, \nabla f \rangle = H_{n-2} \langle V_1, \nabla f \rangle = 0.
\]
If we take derivative in each part of (3.27) in the direction $V_1$ on $M^n$, then we have
\[
\langle \nabla V_1 V_n, \nabla f \rangle + \langle V_n, \nabla V_1 \nabla f \rangle = 0.
\]
On the other hand, from Lemma 2.3 (see [12]), $\nabla f$ is parallel in $M^n$. That’s why, $\nabla V_1 \nabla f = 0$. Hence, we get
\[
-k_{n-1} \langle V_{n-1}, \nabla f \rangle = 0
\]
by using (3.28) and Frenet formulas. So, from (3.29), we deduce that $\langle V_{n-1}, \nabla f \rangle = 0$ since $k_{n-1}$ is positive. For $i = n - 3$ in Theorem 3.1,
\[
\langle V_{n-1}, \nabla f \rangle = H_{n-3} \langle V_1, \nabla f \rangle.
\]
And, since $\langle V_{n-1}, \nabla f \rangle = 0$, $H_{n-3} = 0$.
Continuing this process, we get $H_1 = 0$. Let us recall that $H_1 = \frac{k_1}{k_2}$, thus we have a contradiction. Because, all the curvatures are nowhere zero. As a result, $H_{n-2} \neq 0$. This completes the proof.

**Lemma 3.1.** Let $\alpha(s)$ be a Frenet curve of proper $n$ in $n$-dimensional Riemannian manifold $M^n$ and let $H_{n-2} \neq 0$ be for $i = n - 2$. Then, $H_1^2 + H_2^2 + \ldots + H_{n-2}^2$ is a nonzero constant if and only if $V_1$ and \{H_1, ..., H_{n-2}\} are the unit vector tangent vector field and the Harmonic curvatures of $\alpha$, respectively.

**Proof.** First, we assume that $H_1^2 + H_2^2 + \ldots + H_{n-2}^2$ is a nonzero constant. Consider the functions
\[
H_i = \frac{1}{k_{i+1}} \frac{H_{i+1} + k_i H_{i-1}}{k_{i+1}}
\]
for $3 \leq i \leq n - 2$. (see Definition 3.1). So, from the equality, we can write
\[
k_{i+1} H_i = H_{i+1} + k_i H_{i-1}, \quad 3 \leq i \leq n - 2.
\]
Hence, in (3.30), if we take $i + 1$ instead of $i$, we get
\[
H'_i = k_{i+2} H_{i+1} - k_{i+1} H_{i-1}, \quad 2 \leq i \leq n - 3
\]
together with
\[ H'_1 = k_3 H_2. \tag{3.32} \]
On the other hand, since \( H'_1^2 + H'_2^2 + \ldots + H'_{n-2}^2 \) is constant, we have
\[ H'_1 H'_1 + H'_2 H'_2 + \ldots + H'_{n-2} H'_{n-2} = 0 \]
and so,
\[ H_{n-2} H'_{n-2} = -H'_1 H'_1 - H'_2 H'_2 - \ldots - H'_{n-3} H'_{n-3}. \tag{3.33} \]
By using (3.31) and (3.32), we obtain
\[ H'_1 H'_1 = k_3 H_1 H_2 \tag{3.34} \]
and
\[ H_i H'_i = k_{i+2} H_i H_{i+1} - k_{i+1} H_{i+1} H_i, \quad 2 \leq i \leq n-3. \tag{3.35} \]
Therefore, by using (3.33), (3.34) and (3.35), an algebraic calculus shows that
\[ H_{n-2} H'_{n-2} = -k_{n-1} H_{n-3} H_{n-2}. \]
Since \( H_{n-2} \neq 0 \), we get the relation \( H'_{n-2} = -k_{n-1} H_{n-3} \).
Conversely, we assume that
\[ H'_{n-2} = -k_{n-1} H_{n-3} \tag{3.36} \]
By using (3.36) and \( H_{n-2} \neq 0 \), we can write
\[ H_{n-2} H'_{n-2} = -k_{n-1} H_{n-2} H_{n-3} \]
From (3.35), we have

\[
\begin{align*}
\text{for } i &= n - 3, \quad H_{n-3} H'_{n-3} = k_{n-1} H_{n-3} H_{n-2} - k_{n-2} H_{n-4} H_{n-3} \\
\text{for } i &= n - 4, \quad H_{n-4} H'_{n-4} = k_{n-2} H_{n-4} H_{n-3} - k_{n-3} H_{n-5} H_{n-4} \\
\text{for } i &= n - 5, \quad H_{n-5} H'_{n-5} = k_{n-3} H_{n-5} H_{n-4} - k_{n-4} H_{n-6} H_{n-5} \\
&\quad \vdots \\
\text{for } i &= 2, \quad H_2 H'_2 = k_3 H_2 H_3 - k_3 H_1 H_2
\end{align*}
\]
and from (3.34), we have
\[ H'_1 H'_1 = k_3 H_1 H_2. \]
So, an algebraic calculus show that
\[ H'_1 H'_1 + H'_2 H'_2 + \ldots + H'_{n-3} H'_{n-3} + H'_{n-4} H'_{n-4} + H'_{n-5} H'_{n-5} + H_{n-2} H'_{n-2} = 0. \tag{3.37} \]
And, by integrating (3.37), we can easily say that
\[ H'_1^2 + H'_2^2 + \ldots + H'_{n-2}^2 \]
is a non-zero constant. This completes the proof. \( \square \)

**Corollary 3.1.** Let \( M^n \) be a \( n \)-dimensional Riemannian manifold with the metric \( \langle \cdot, \cdot \rangle \) and complete connected smooth without boundary. Let \( M^n \) be isometric to a Riemannian product \( N \times \mathbb{R} \). Let us assume that \( f : M^n \to \mathbb{R} \) be a non-trivial affine function (see main Theorem in [6]) and \( \alpha (s) \) be a Frenet curve of proper \( n \) in \( M^n \). If \( \alpha \) is a \( f \)-eikonal helix curve, then \( H'_{n-2} = -k_{n-1} H_{n-3} \).

**Proof.** It is obvious by using Theorem 3.3 and Lemma 3.1. \( \square \)
4. f-eikonal $V_n$-slant helix curves and their harmonic curvature functions

In this section, we define $f$-eikonal $V_n$-slant helix curves and we give characterizations for a $f$-eikonal $V_n$-slant helix curve in $n$-dimensional Riemannian manifold $M^n$ by using harmonic curvature functions in terms of $V_n$ of the curve.

**Definition 4.1.** Let $M^n$ be a Riemannian manifold and let $\alpha(s)$ be a Frenet curve with the curvatures $k_i$. Then, harmonic curvature functions of $\alpha$ are defined by $H_i^\prime : I \subset \mathbb{R} \rightarrow \mathbb{R}$ along $\alpha$ in $M^n$, 

$$H_0^\prime = 0, \quad H_i^\prime = \frac{k_{i-1}}{k_{i-2}}, \quad H_i^\prime = \left\{k_{i-1}H_{i-2}^\prime - H_{i-1}^\prime \right\} \frac{1}{k_{i-(i+1)}}$$

for $2 \leq i \leq n-2$.

**Definition 4.2.** Let $M^n$ be a Riemannian manifold with the metric $\langle \cdot, \cdot \rangle$ and let $\alpha(s)$ be a Frenet curve with the orthonormal frame $\{V_1, ..., V_n\}$ in $M^n$. Let $f : M^n \rightarrow \mathbb{R}$ be a eikonal function along curve $\alpha$, i.e. $\|\nabla f\|=\text{constant}$ along the curve $\alpha$. If the function $\langle \nabla f, V_n \rangle$ is non-zero constant along $\alpha$, then $\alpha$ is called a $f$-eikonal $V_n$-slant helix curve. And, $\nabla f$ is called the axis of the $f$-eikonal $V_n$-slant helix curve $\alpha$.

**Theorem 4.1.** Let $M^n$ be a n-dimensional Riemannian manifold with the metric $\langle \cdot, \cdot \rangle$ and complete connected smooth without boundary. Let $M^n$ be isometric to a Riemannian product $N \times \mathbb{R}$. Let us assume that $f : M^n \rightarrow \mathbb{R}$ be a non-trivial affine function (see main Theorem in [1]) and $\alpha(s)$ be a Frenet curve of proper $n$ in $M^n$. If $\alpha$ is a $f$-eikonal $V_n$-slant helix curve with the axis $\nabla f$, then the system

$$\langle V_{n-(i+1)}, \nabla f \rangle = H_i^\prime \langle V_n, \nabla f \rangle, \quad i = 1, ..., n-2$$

holds, where $\{V_1, V_2, ..., V_n\}$ and $\{H_1^\prime, ..., H_{n-2}^\prime\}$ are the Frenet frame and the harmonic curvatures of $\alpha$, respectively.

**Proof.** Since $\{V_1, ..., V_n\}$ is the orthonormal frame of the curve $\alpha$ in $M^n$, $\nabla f$ can be expressed in the form

$$\nabla f = \lambda_1 V_1 + ... + \lambda_n V_n.$$  

(4.2)

Doing dot product with $V_n$ in each part of (4.2), we get

$$\langle \nabla f, V_n \rangle = \lambda_n = \text{constant}$$

(4.3)

since $\alpha$ is a $f$-eikonal $V_n$-slant helix curve. If we take the derivative in each part of (4.3) in the direction $V_1$ in $M^n$, then we have

$$\langle \nabla V_1 \nabla f, V_n \rangle + \langle \nabla f, \nabla V_1 V_n \rangle = 0.$$  

(4.4)

On the other hand, from Lemma 2.3 (see [12]), $\nabla f$ is parallel in $M^n$. That is, $\nabla V_1 \nabla f = 0$. Hence, by using (4.4) and Frenet formulas, we obtain

$$- k_{n-1} \langle \nabla f, V_{n-1} \rangle = 0.$$  

(4.5)

And, since $k_{n-1}$ is positive function, from (4.5), we get

$$\langle \nabla f, V_{n-1} \rangle = 0.$$  

(4.6)

By taking the derivative in each part of (4.6) in the direction $V_1$ in $M^n$, we can write the equality

$$\langle \nabla V_1 \nabla f, V_{n-1} \rangle + \langle \nabla f, \nabla V_1 V_{n-1} \rangle = 0.$$  

(4.7)

And, since $\nabla V_1 \nabla f = 0$, by using (4.7) and Frenet formulas, we obtain

$$- k_{n-2} \langle \nabla f, V_{n-2} \rangle + k_{n-1} \langle \nabla f, V_n \rangle = 0.$$  

(4.8)

Therefore, from (4.8), we have

$$\langle \nabla f, V_{n-2} \rangle = \frac{k_{n-1}}{k_{n-2}} \langle \nabla f, V_n \rangle.$$  

(4.9)

Moreover, since $H_1^\prime = \frac{k_{n-1}}{k_{n-2}}$, from (4.9), we can write

$$\langle \nabla f, V_{n-2} \rangle = H_1^\prime \langle \nabla f, V_n \rangle.$$  

(4.10)

It follows that the equality (4.1) is true for $i = 1$. According to the induction theory, let us assume that the equality (4.1) is true for all $k$, where $1 \leq k \leq i$ for some positive integers $i$. Then, we will prove that the equality (4.1) is true for $i + 1$. Since the equality (4.1) is true for some positive integers $i$, we can write

$$\langle V_{n-(i+1)}, \nabla f \rangle = H_i^\prime \langle V_n, \nabla f \rangle$$

(4.10)
for some positive integers $i$. If we take derivative in each part of (4.10) in the direction $V_i$ in $M^n$, we have
\[
\langle \nabla V_i V_{n-(i+1)}, \nabla f \rangle + \langle V_{n-(i+1)}, \nabla V_i \nabla f \rangle = V_i \left[ H^*_i \langle V_n, \nabla f \rangle \right].
\] (4.11)

And, by using (4.11) and Frenet formulas, we get
\[
-k_{n-(i+2)} \langle V_{n-(i+2)}, \nabla f \rangle + k_{n-(i+1)} \langle V_{n-i}, \nabla f \rangle + \langle V_{n-(i+1)}, \nabla V_i \nabla f \rangle = V_i \left[ H^*_i \langle V_n, \nabla f \rangle \right].
\] (4.12)

Moreover, $\nabla V_i \nabla f = 0$. Hence, from (4.12), we can write
\[
-k_{n-(i+2)} \langle V_{n-(i+2)}, \nabla f \rangle + k_{n-(i+1)} \langle V_{n-i}, \nabla f \rangle = \langle V_{n-(i+1)}, \nabla V_i \nabla f \rangle.
\] (4.13)

And, from (4.13), we obtain
\[
\langle V_{n-(i+2)}, \nabla f \rangle = \frac{1}{k_{n-(i+2)}} \left\{ -V_i \left[ H^*_i \langle V_n, \nabla f \rangle \right] + k_{n-(i+1)} \langle V_{n-i}, \nabla f \rangle \right\}.
\] (4.14)

On the other hand, since the equality (4.1) is true for $i-1$ according to the induction hypothesis, we have
\[
\langle V_{n-i}, \nabla f \rangle = H^*_{i-1} \langle V_n, \nabla f \rangle.
\] (4.15)

Therefore, by using (4.14) and (4.15), we get
\[
\langle V_{n-(i+2)}, \nabla f \rangle = \frac{1}{k_{n-(i+2)}} \left\{ -V_i \left[ H^*_i \langle V_n, \nabla f \rangle \right] + k_{n-(i+1)} \langle V_{n-i}, \nabla f \rangle \right\}.
\] (4.16)

Moreover, we obtain
\[
H^*_{i+1} = \left\{ k_{n-(i+1)} H^*_{i-1} - H^*_i \right\} \frac{1}{k_{n-(i+2)}}
\] (4.17)

for $i+1$ in the Definition 4.1. So, we have
\[
\langle V_{n-(i+2)}, \nabla f \rangle = H^*_{i+1} \langle V_n, \nabla f \rangle
\]
by using (4.16) and (4.17). It follows that the equality (4.1) is true for $i+1$. Consequently, we get
\[
\langle V_{n-(i+1)}, \nabla f \rangle = H^*_i \langle V_n, \nabla f \rangle
\]
for all $i$ according to induction theory. This completes the proof.

**Theorem 4.2.** Let $M^n$ be a $n$-dimensional Riemannian manifold with the metric $\langle , \rangle$ and complete connected smooth without boundary. Let $M^n$ be isometric to a Riemannian product $N \times \mathbb{R}$. Let us assume that $f : M^n \rightarrow \mathbb{R}$ be a non-trivial affine function (see main Theorem in [5]) and $\alpha (s)$ be a Frenet curve of proper $n$ in $M^n$. If $\alpha$ is a $f$-eikonal $V_n$-slant helix curve with the axis $\nabla f$, then the axis of the curve $\alpha$
\[
\nabla f = \left\{ H^*_{n-2} V_1 + ... + H^*_1 V_{n-2} + V_n \right\} \langle \nabla f, V_n \rangle,
\]
where $\{V_1, V_2, ..., V_n\}$ and $\{H^*_1, ..., H^*_{n-2}\}$ are the Frenet frame and the harmonic curvatures of $\alpha$, respectively.

**Proof.** Since $\alpha$ is a $f$-eikonal $V_n$-slant helix curve, we can write
\[
\langle \nabla f, V_n \rangle = \text{constant}.
\] (4.18)

If we take the derivative in each part of (4.18) in the direction $V_1$ in $M^n$, then we have
\[
\langle \nabla V_1 \nabla f, V_n \rangle + \langle \nabla f, \nabla V_1 V_n \rangle = 0.
\] (4.19)

On the other hand, from Lemma 2.3 (see [12]), $\nabla f$ is parallel in $M^n$. That’s why, $\nabla V_1 \nabla f = 0$. Then, we obtain
\[
-k_{n-1} \langle \nabla f, V_{n-1} \rangle = 0
\] (4.20)

by using (4.19) and Frenet formulas. Since $k_{n-1}$ is positive function, (4.20) implies that
\[
\langle \nabla f, V_{n-1} \rangle = 0.
\]

Hence, we can write the axis of $\alpha$ as
\[
\nabla f = \lambda_1 V_1 + \lambda_2 V_2 + ... + \lambda_{n-2} V_{n-2} + \lambda_n V_n.
\] (4.21)
Moreover, from (4.21), we get
\[
\lambda_1 = \langle \nabla f, V_1 \rangle \\
\lambda_2 = \langle \nabla f, V_2 \rangle \\
\vdots \\
\lambda_{n-2} = \langle \nabla f, V_{n-2} \rangle \\
\lambda_n = \langle \nabla f, V_n \rangle
\]
by using Riemannian product. On the other hand, from Theorem 4.1, we know that
\[
\lambda_1 = \langle \nabla f, V_1 \rangle = H_{n-2}^* \langle \nabla f, V_n \rangle \quad \quad (4.22)
\]
\[
\lambda_2 = \langle \nabla f, V_2 \rangle = H_{n-3}^* \langle \nabla f, V_n \rangle \\
\vdots \\
\lambda_{n-2} = \langle \nabla f, V_{n-2} \rangle = H_2^* \langle \nabla f, V_n \rangle \\
\lambda_n = \langle \nabla f, V_n \rangle
\]
Thus, it can be easily obtained the axis of the curve \(\alpha\) as
\[
\nabla f = \{H_{n-2}^* V_1 + \ldots + H_1^* V_{n-2} + V_n \} \langle \nabla f, V_n \rangle,
\]
by making use of the equality (4.21) and the system (4.22). This completes the proof.

**Theorem 4.3.** Let \(M^n\) be a \(n\)-dimensional Riemannian manifold with the metric \(\langle , \rangle\) and complete connected smooth without boundary. Let \(M^n\) be isometric to a Riemannian product \(N \times \mathbb{R}\). Let us assume that \(f: M^n \to \mathbb{R}\) be a non-trivial affine function (see main Theorem in [5]), and \(\alpha(s)\) be a Frenet curve of proper \(n\) in \(M^n\). If \(\alpha\) is a \(f\)-eikonal \(V_n\)-slant helix curve, then \(H_{n-2}^* \neq 0\) and \(H_2^2 + H_3^2 + \ldots + H_{n-2}^2\) is non-zero constant, where \(\{H_1^*, \ldots, H_{n-2}^*\}\) is the harmonic curvatures of \(\alpha\).

**Proof.** Let \(\alpha\) be a \(f\)-eikonal \(V_n\)-slant helix curve and \(\{V_1, \ldots, V_n\}\) be the Frenet frame of \(\alpha\). Then, from Theorem 4.2, we know that
\[
\nabla f = \{H_{n-2}^* V_1 + \ldots + H_1^* V_{n-2} + V_n \} \langle \nabla f, V_n \rangle.
\]
Therefore, from (4.23), we can write
\[
\langle \nabla f, \nabla f \rangle = \langle \nabla f, V_n \rangle^2 (H_{n-2}^2 + \ldots + H_1^2 + 1) = \langle \nabla f, V_n \rangle^2 (H_{n-2}^2 + \ldots + H_1^2 + 1).
\]
Moreover, by the definition of Riemannian metric, we have
\[
\langle \nabla f, \nabla f \rangle = \|\nabla f\|^2.
\]
According to this Theorem, \(\alpha\) is a \(f\)-eikonal \(V_n\)-slant helix curve. So, \(\|\nabla f\| = \text{constant}\) and \(\langle \nabla f, V_n \rangle = \text{constant}\) along \(\alpha\). Hence, from (4.24), we obtain
\[
H_1^2 + H_2^2 + \ldots + H_{n-2}^2 = \text{constant}.
\]
Now, we will show that \(H_{n-2}^* \neq 0\). We assume that \(H_{n-2}^* = 0\). Then, for \(i = n - 2\) in (4.1),
\[
\langle V_1, \nabla f \rangle = H_{n-2}^2 \langle \nabla f, V_n \rangle = 0.
\]
If we take derivative in each part of (4.25) in the direction \(V_1\) on \(M^n\), then we have
\[
\langle V_1 V_1, \nabla f \rangle + \langle V_1, \nabla V_1 \nabla f \rangle = 0.
\]
On the other hand, from Lemma 2.3 (see [12]), \(\nabla f\) is parallel in \(M^n\). That’s why \(\nabla V_1 \nabla f = 0\). Then, from (4.26), we have \(\langle V_1 V_1, \nabla f \rangle = k_1 \langle V_2, \nabla f \rangle = 0\) by using the Frenet formulas. Since \(k_1\) is positive, \(\langle V_2, \nabla f \rangle = 0\). Now, for \(i = n - 3\) in (4.1),
\[
\langle V_2, \nabla f \rangle = H_{n-3}^* \langle V_n, \nabla f \rangle.
\]
And, since \(\langle V_2, \nabla f \rangle = 0\), \(H_{n-3}^* = 0\). Continuing this process, we get \(H_1^* = 0\). Let us recall that \(H_1^* = \frac{k_{n-1}}{k_{n-2}}\), thus we have a contradiction because all the curvatures are nowhere zero. Consequently, \(H_{n-2}^* \neq 0\). This completes the proof. \(\square\)
Lemma 4.1. Let \(\alpha(s)\) be a Frenet curve of proper \(n\) in \(n\)-dimensional Riemannian manifold \(M^n\) and let \(H^*_{n-2} \neq 0\) for \(i = n - 2\). Then, \(H^2_{1} + H^2_{2} + \ldots + H^2_{n-2}\) is a nonzero constant if and only if \(V_1 \{H^*_1, ..., H^*_{n-2}\} = k_1 H^*_{n-3}\), where \(V_1\) and \(\{H^*_1, ..., H^*_{n-2}\}\) are the unit vector tangent vector field and the Harmonic curvatures of \(\alpha\), respectively.

Proof. First, we assume that \(H^2_{1} + H^2_{2} + \ldots + H^2_{n-2}\) is a nonzero constant. Consider the functions

\[ H^*_i = \left\{ k_{n-i}H^*_{n-2} - H^*_{i-1} \right\}\frac{1}{k_{n-(i+1)}} \]

for \(3 \leq i \leq n - 2\). So, from the equality, we can write

\[ k_{n-(i+1)}H^*_i = k_{n-i}H^*_{n-2} - H^*_{i-1}, \quad 3 \leq i \leq n - 2. \tag{4.27} \]

Hence, in (4.27), if we take \(i + 1\) instead of \(i\), we get

\[ H^*_{i} = k_{n-(i+1)}H^*_{i-1} - k_{n-(i+2)}H^*_{i+1}, \quad 2 \leq i \leq n - 3 \tag{4.28} \]

together with

\[ H^*_{i} = -k_{n-3}H^*_{2} \tag{4.29} \]

On the other hand, since \(H^2_{1} + H^2_{2} + \ldots + H^2_{n-2}\) is constant, we have

\[ H^*_1 H^*_{i} + H^*_2 H^*_{i} + \ldots + H^*_n H^*_{n-2} = 0 \]

and so,

\[ H^*_{n-2} H^*_{n-2} = -H^*_{1} H^*_{i} - H^*_{2} H^*_{i} - \ldots - H^*_{n-3} H^*_{n-3}. \tag{4.30} \]

By using (4.28) and (4.29), we obtain

\[ H^*_1 H^*_{i} = -k_{n-3}H^*_1 H^*_2 \tag{4.31} \]

and

\[ H^*_i H^*_{i} = k_{n-(i+1)}H^*_{i-1} - k_{n-(i+2)}H^*_{i+1}, \quad 2 \leq i \leq n - 3. \tag{4.32} \]

Therefore, by using (4.30), (4.31) and (4.32), a algebraic calculus shows that

\[ H^*_{n-2} H^*_{n-2} = k_1 H^*_{n-3} H^*_{n-2}. \]

Since \(H^*_{n-2} \neq 0\), we get the relation \(H^*_{n-2} = k_1 H^*_{n-3}\).

Conversely, we assume that

\[ H^*_{n-2} = k_1 H^*_{n-3}. \tag{4.33} \]

By using (4.33) and \(H^*_{n-2} \neq 0\), we can write

\[ H^*_{n-2} H^*_{n-2} = k_1 H^*_{n-2} H^*_{n-3} \]

From (4.32), we have

\[
\begin{align*}
\text{for } i &= n - 3, \\
H^*_n H^*_n &= k_2 H^*_{n-4} H^*_{n-3} - k_1 H^*_{n-3} H^*_n \\
\text{for } i &= n - 4, \\
H^*_n H^*_n &= k_3 H^*_{n-5} H^*_{n-4} - k_2 H^*_{n-4} H^*_n \\
\text{for } i &= n - 5, \\
H^*_n H^*_n &= k_4 H^*_{n-6} H^*_n - k_3 H^*_n H^*_{n-5} \\
&\quad \vdots \\
\text{for } i &= 2, \\
H^*_n H^*_n &= k_{n-3} H^*_1 H^*_2 - k_{n-4} H^*_3 H^*_n \\
\text{and from (4.31), we have } \\
H^*_1 H^*_1 &= -k_{n-3} H^*_2 H^*_2.
\end{align*}
\]

So, an algebraic calculus shows that

\[ H^*_1 H^*_1 + H^*_2 H^*_2 + \ldots + H^*_n H^*_{n-5} + H^*_n H^*_{n-4} - H^*_{n-3} H^*_{n-3} + H^*_n H^*_{n-2} = 0. \tag{4.34} \]

And, by integrating (4.34), we can easily say that

\[ H^2_{1} + H^2_{2} + \ldots + H^2_{n-2} \]

is a nonzero constant. This completes the proof.
Corollary 4.1. Let $M^n$ be a $n$-dimensional Riemannian manifold with the metric $(\cdot, \cdot)$ and complete connected smooth without boundary. Let $M^n$ be isometric to a Riemannian product $N \times \mathbb{R}$. Let us assume that $f : M^n \to \mathbb{R}$ be a non-trivial affine function (see main Theorem in [5]) and $\alpha(s)$ be a Frenet curve of proper $n$ in $M^n$. If $\alpha$ is a $f$-eikonal $V_n$-slant helix curve, then $H_{n-2}^{*} = k_1 H_{n-3}^{*}$.

Proof. It is obvious by using Theorem 4.3 and Lemma 4.1. □

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