On observational predictions from multidimensional gravity

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We discuss possible observational manifestations of static, spherically symmetric solutions of a class of multidimensional theories of gravity, which includes the low energy limits of supergravities and superstring theories as special cases. We discuss the choice of a physical conformal frame to be used for the description of observations. General expressions are given for (i) the Eddington parameters $\beta$ and $\gamma$, characterizing the post-Newtonian gravitational field of a central body, (ii) $p$-brane black hole temperatures in different conformal frames and (iii) the Coulomb law modified by extra dimensions. It is concluded, in particular, that $\beta$ and $\gamma$ depend on the integration constants and can be therefore different for different central bodies. If, however, the Einstein frame is adopted for describing observations, we always obtain $\gamma = 1$. The modified Coulomb law is shown to be independent of the choice of a 4-dimensional conformal frame. We also argue the possible existence of specific multidimensional objects, T-holes, potentially observable as bodies with mirror surfaces.

1. Introduction

The observed physical world is fairly well described by the conventional 4-dimensional picture. On the other hand, in theoretical physics, whose basic aims are to construct a “theory of everything” and to explain why our universe looks as we see it and not otherwise, most of the recent advances are connected with models in dimensions greater than four: Kaluza-Klein type theories, 10-dimensional superstring theories, M-theory and their further generalizations. Even if such theories (or some of them) successfully explain the whole wealth of particle and astrophysical phenomenology, there remains a fundamental question of finding direct observational evidence of extra dimensions, which is of utmost importance for the whole human world outlook.

Observational “windows” to extra dimensions are actively discussed for many years. Thus, well-known predictions from extra dimensions are variations of the fundamental physical constants on the cosmological time scale\(^1\)\(^2\). Such constants are, e.g., the effective gravitational constant $G$ and the fine structure constant $\alpha$. There exist certain observational data on $G$ stability on the level of $\Delta G/G \sim 10^{-11} \div 10^{-12}$ y$^{-1}$\(^3\)\(^4\), which restrict the range of viable cosmological models. Very recently some evidence was obtained from quasar absorption spectra, testifying the variability of $\alpha$: $\Delta \alpha/\alpha \sim -0.72 \cdot 10^{-5}$ over the redshift range $0.5 < z < 3.5$\(^5\) (the minus sign means smaller $\alpha$ in the past).

Some effects connected with waves in small compactified extra dimensions are also discussed\(^6\): it is argued that such excitations can behave as particles with a large variety of masses and contribute to dark matter or to cross-sections of usual particle interactions.

Other possible manifestations of extra dimensions are monopole modes in gravitational waves, various predictions for standard cosmological tests and generation of the cosmological constant\(^7\), and numerous effects connected with local field sources, some of them being the subject of the present paper. These include, in particular, deviations from the Newton and Coulomb laws\(^8\)\(^9\)\(^10\)\(^11\) and properties of black holes.

It had been conventional, starting from the pioneering papers of Kaluza and Klein, to suppose that extra dimensions, if any, are not directly observable due to their tiny size and compactness. For about two years, however, an alternative picture, connected with the so-called “brane world” models, is being actively developed. This trend rests on the suggestion advanced in 1982–83\(^12\)\(^13\) that we may live in a domain wall, or brane, of 3 spatial dimensions, embedded in a higher-dimensional space, which is unobservable directly since our brane world is located in a potential “trench” and/or most of the types of matter are concentrated on this brane. The recent boom was apparently launched by the works of Randall and Sundrum\(^15\) who showed, in particular, a way of obtaining Newtonian gravity on the brane from a multidimensional model. Since their publication hundreds of papers have appeared, with a diversity of specific models and predictions. We will not try here to review this vast trend since it seems too

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early to make conclusions: while one is preparing a survey, tens of new works, drastically changing the picture, can appear. We would only mention an opportunity suggested by Maia and Silveira [16] well before the present outburst. These authors argued that near a black hole (BH) particles may gain sufficient gravitational energy to overcome the potential barrier confining them to four dimensions and can thus run away from our world.

In this paper we discuss in some detail predictions from extra dimensions connected with local sources of gravity. Assuming that extra dimensions of variable size inside and around such sources (e.g., stars, galaxies, black holes) can affect various physical phenomena, one should apply multidimensional theories to describe these phenomena.

In Sec. 2 and 3 we present some exact static, spherically symmetric solutions of a generalized field model [17]–[23], associated with charged p-branes and motivated by the bosonic sector of the low-energy field approximation of superstring theories, M-theory and their generalizations [22]–[27]. Our model, however, is not restricted to known superstring theories, M-theory and their generalizations [22]–[27]. Our model, however, is not restricted to known theories since it assumes arbitrary dimensions of factor spaces, arbitrary ranks of antisymmetric forms and an arbitrary number of scalar fields.

Then, in Sec. 4, we discuss the choice of the conformal frame (CF) in which observational predictions should be formulated. Since at present there is no generally accepted unified theory, in our approach we use a generalized model with arbitrary p-branes in diverse dimensions and study the physical applications on the basis of exactly solvable models. Thus we do not fix the underlying fundamental theory and have no reason to prescribe a particular frame, therefore all further results are formulated in an arbitrary frame.

In Sec. 5 we derive the post-Newtonian (PN) approximation of the above solutions in order to designate possible traces of extra dimensions and p-branes in the comparatively weak gravitational fields of the great majority of planetary and stellar systems, including binary pulsars. This section generalizes the results of previous papers [23]–[27]. For black holes, apart from the PN parameters which determine the motion of test bodies in their sufficiently far neighbourhood, there is one more potentially observable parameter, the Hawking temperature \( T_h \), which is obviously important for small (e.g., primordial) black holes rather than those of stellar or galactic mass range. We discuss the expressions for \( T_h \) for a variety of black hole solutions and their applicability in different conformal frames.

Sec. 6 and 7 describe such consequences of these (and many other) field models as the Coulomb law violation and the possible existence of new, purely multidimensional objects, \( T \)-holes [30, 31]. The results are briefly discussed in Sec. 8.

2. \( D \)-dimensional action and minisuperspace representation

2.1. The model

The starting point is, as in Refs. [7]–[23], the model action for \( D \)-dimensional gravity with several scalar dilatonic fields \( \varphi^a \) and antisymmetric \( n_s \)-forms \( F_s \):

\[
S = \frac{1}{2\kappa^2} \int d^D z \sqrt{|g|} \left\{ R[g] - \delta_{ab} g^{MN} \partial_M \varphi^a \partial_N \varphi^b - \sum_{s \in S} \frac{n_s}{n_s!} e^{2\lambda_s \varphi^a} F_s^2 \right\},
\]

(1)

in a (pseudo-)Riemannian manifold \( M = \mathbb{R}_u \times \mathbb{R}_0 \times \ldots \times \mathbb{R}_n \) with the static, spherically symmetric metric

\[
ds_D^2 = g_{MN} dz^M dz^N = e^{2\alpha u} du^2 + \sum_{i=0}^n e^{2\beta_i} ds_i^2
\]

\[= e^{2\alpha u} du^2 + e^{2\beta_0} d\Omega^2 - e^{2\beta_1} dt^2 + \sum_{i=2}^n e^{2\beta_i} ds_i^2.\]

(2)

Here \( u \) is a radial coordinate ranging in \( \mathbb{R}_u \subseteq \mathbb{R} \); \( ds_0^2 = d\Omega^2 \) is the metric on a unit \( d_0 \)-dimensional sphere \( S^{d_0} ; t \in \mathbb{R}_1 = \mathbb{R}_t \) is time; the metrics \( g^i = ds_i^2 \) of the “extra” factor spaces \( (i \geq 2) \) are assumed to be \( u \)-independent, Ricci-flat and can have arbitrary signatures \( \varepsilon_i = \text{sign} g^i \); \( |g| = |\det g_{MN}| \) and similarly for subspaces; \( F_s^2 = F_s \times \ldots \times M_{n_s} = \sum_{i=2}^{n_s} \lambda_{sa} \); \( \lambda_{sa} \) are coupling constants; \( \eta_s = \pm 1 \) (to be specified later); \( s \in S, a \in A \), where \( S \) and \( A \) are some finite sets.

This formulation admits both spacelike and timelike extra dimensions. Models with multiple timelike dimensions were considered in a number of papers, e.g., [23] and more recently in [24, 26].

The “scale factors” \( e^{\beta_i} \) and the scalars \( \varphi^a \) are assumed to depend on \( u \) only.
The $F$-forms should also be compatible with spherical symmetry. A given $F$-form may have several essentially (non-permutatively) different components; such a situation is sometimes called “composite $p$-branes”. For convenience, we will nevertheless treat essentially different components of the same $F$-form as individual (“elementary”) $F$-forms. A reformulation to the composite ansatz, if needed, is straightforward.

The $n_s$-forms $F = F_{[n_s]} = dA_s$, where $A_s$ is a potential $(n_s - 1)$-form, are naturally classified as electric ($F_{el}^I$) and magnetic ($F_{mI}^I$) ones, both associated with a certain subset $I = \{i_1, \ldots, i_k\}$ ($i_1 < \ldots < i_k$) of the set of numbers labeling the factor spaces: \{\} = I_0 = \{0, \ldots, n\}. By definition, the potential $A_I$ of an electric form $F_{el}$ carries the coordinate indices of the subspaces $M_i, i \in I$ and is $u$-dependent (since only a radial component of the field may be nonzero), whereas a magnetic form $F_{mI}$ is built as a form dual to a possible electric one associated with $I$. Thus nonzero components of $F_{mI}$ carry coordinate indices of the subspaces $M_i, i \in \overline{I} = I_0 \setminus I$. One can write:

$$n_{el} = \text{rank } F_{el} = d(I) + 1, \quad n_{mI} = \text{rank } F_{mI} = D - \text{rank } F_{el} = d(\overline{I})$$

where $d(I)$ are the dimensions of the subspaces $M_I = M_{i_1} \times \ldots \times M_{i_k}$. The index $s$ will be used to jointly describe the two types of forms, so that $\{e, m\}$.

$$S = \{s\} = \{e_{I_s} \cup \{m_{I_s}\}\}.$$ (4)

We will make some more natural assumptions:

(i) The branes only “live” in extra dimensions, i.e., $0 \not\in I_s, \forall s$.

(ii) The energy density of each $F$-form is positive: $-T^{I_I}_I(F_s) > 0, \forall s$.

When all extra dimensions are spacelike, the second requirement holds if, as usual, in $[\text{19}]$ $\eta_s = 1$ for all $s$. In more general models, with arbitrary $\varepsilon_I$, the requirement $-T^{I_I}_I > 0$ holds if

$$\eta_{el} = -\varepsilon(I)\varepsilon(I), \quad \eta_{mI} = -\varepsilon(\overline{I})\varepsilon(\overline{I}),$$

$$\varepsilon(I) \equiv \prod_{i \in I} \varepsilon_i, \quad \varepsilon(I) = \begin{cases} 1, & \mathbb{R}_I \subset M_I, \\ -1, & \text{otherwise} \end{cases}$$

where $\mathbb{R}_I$ is the time axis. If $\varepsilon(I) = 1$, we are dealing with a true electric or magnetic form, directly generalizing the familiar Maxwell field; otherwise the $F$-form behaves as an effective scalar or pseudoscalar in the physical subspace. $F$-forms with $\varepsilon(I) = -1$ will be called quasiscalar.

Several electric and/or magnetic forms (with maybe different coupling constants $\lambda_{sa}$) can be associated with the same $I$ and are then labelled by different values of $s$. (We sometimes omit the index $s$ by $I$ when this cannot cause confusion.)

The forms $F_s$ are associated with $p$-branes as extended sources of the spherically symmetric field distributions, where the brane dimension is $p = d(I_s) - 1$, while $d(I_s)$ is the brane worldvolume dimension.

The following example illustrates the possible kinds of $F$-forms.

Example 1: $D = 11$ supergravity, representing the low-energy limit of M-theory \[23\]. The action (6) for the bosonic sector of this theory (truncated by omitting the Chern-Simons term) does not contain scalar fields ($\varphi^n = \lambda_{sa} = 0$) and the only $F$-form is of rank 4, whose various nontrivial components $F_s$ (elementary $F$-forms, to be called simply $F$-forms according to the above convention) are associated with electric 2-branes [for which $d(I_s) = 3$] and magnetic 5-branes [such that $d(I_s) = 6$, see (4)]. The action has the form

$$S = \frac{1}{2k^2} \int_\mathbb{M} d^{11}z \sqrt{|g|} \left[ R[g] - \frac{1}{4!} F^3[^3] \right].$$

Let us put $d_0 = 2$ and ascribe to the external space-time coordinates the indices $M = u, \theta, \phi, t$ ($\theta$ and $\phi$ are the spherical angles), and let $M = 2, \ldots, 8$ refer to the extra dimensions. Furthermore, let the extra factor spaces $M_i, i = 2, \ldots, 8$, be one-dimensional and coincide with the respective coordinate axes. The number $i = 1$ is ascribed to the time axis, $M_1 = \mathbb{R}_t$, as stated previously. Then different kinds of forms can be exemplified as follows:

- $F_{\Delta 23}$ is a true electric form, $I = \{123\}; \quad \overline{I} = \{045678\}$.
- $F_{\phi 23}$ is a true magnetic form, $I = \{145678\}; \quad \overline{I} = \{023\}$.
- $F_{\psi 23}$ is an electric quasiscalar form, $I = \{234\}; \quad \overline{I} = \{015678\}$.
- $F_{\phi 012}$ is a magnetic quasiscalar form, $I = \{345678\}; \quad \overline{I} = \{012\}$.

3There is an exception: two components, having only one noncoinciding index, cannot coexist since in this case there emerge nonzero off-block-diagonal components of the energy-momentum tensor (EMT) $T^s_M$, while the Einstein tensor in the l.h.s. of the Einstein equations is block-diagonal. See more details in Ref. [19].
2.2. The target space $\mathbb{V}$

Under the above assumptions, the system is well described using the so-called $\sigma$ model representation \([9]\), to be briefly outlined here as applied to static, spherically symmetric systems.

Let us choose, as in \([32]\) and many later papers, the harmonic coordinate in $\mathbb{M}$\((\nabla^M \nabla_M u = 0)\), such that

\[
\alpha(u) = \sum_{i=0}^{n} d_i \beta^i \equiv d_0 \beta^0 + \sigma_1(u), \quad \sigma_1(u) \overset{\text{def}}{=} \sum_{i=1}^{n} d_i \beta^i. \tag{8}
\]

The Maxwell-like field equations for $F_s$ may be integrated in a general form. Indeed, for an electric form $F_s$\((s = eI)\) the field equations due to \([3]\)

\[
\partial_a \left( F_{s\mu M_3...M_m} \sqrt{|g|} e^{2\lambda_s \varphi^s} \right) = 0, \tag{9}
\]

where $m = d(I) + 1$, are easily integrated to give

\[
F_{s\mu M_3...M_m} = Q_s e^{-\alpha_0 - \sigma_0 - 2\lambda_s \varphi^s} \varepsilon_{M_3...M_m} / \sqrt{|g_I|} \quad \Rightarrow \quad \frac{1}{m!} F_{s}^{2} = \varepsilon(I) Q_s^2 e^{-2\sigma(I) - 2\lambda_s \varphi^s}. \tag{10}
\]

where $\sqrt{|g_I|} = \lambda_s \varphi^s$, $\varepsilon_-$ and $\varepsilon_+$ are Levi-Civita symbols, where $|g_I| = \prod_{\alpha I} |g_I|$, and $Q_s = \text{const}$ are charges. In a similar way, for a magnetic $m$-form $F_s$\((s = mI, m = d(T_s))\), the field equations and the Bianchi identities $dF_s = 0$ lead to

\[
F_{s\mu M_1...M_d(T)\varepsilon} = Q_s \varepsilon_{M_1...M_d(T)\varepsilon} / \sqrt{|g_T|} \quad \Rightarrow \quad \frac{1}{m!} F_{s}^{2} = \varepsilon(T) Q_s^2 e^{-2\sigma(T) + \lambda_s \varphi^s} \tag{11}
\]

We use the notations

\[
\sigma_i = \sum_{j=i}^{n} d_j \beta^j(u), \quad \sigma(I) = \sum_{i \in I} d_i \beta^i(u). \tag{12}
\]

Consequently, in the r.h.s. of the Einstein equations due to \([1]\), $R^N_M - \frac{1}{2} \delta^N_M R = T^N_M$, the energy-momentum tensor ($\text{EMT}$) $T^N_M$ takes the form

\[
e^{2\alpha} T^N_M = -\frac{1}{2} \sum_{s} \epsilon_s Q_s^2 e^{2\sigma(I) - 2\chi_s \varphi^s} \text{diag}(+1, [+1]_I, [-1]_I) + \frac{1}{2} (\lambda_s^2)^2 \text{diag}(+1, [-1]_I) \tag{13}
\]

where the first place on the diagonal belongs to $u$ and the symbol $[f]_j$ means that the quantity $f$ is repeated along the diagonal for all indices referring to $M_j, j \in J$; $\sigma(I) \overset{\text{def}}{=} \sum_{i \in I} d_i \beta^i$; the sign factors $\epsilon_s$ and $\chi_s$ are

\[
\epsilon_{el} = -\eta_{el} \varepsilon(I), \quad \epsilon_{ml} = \eta_{ml} \varepsilon(T); \quad \chi_{el} = +1, \quad \chi_{ml} = -1, \tag{14}
\]

so that $\chi_s$ distinguishes electric and magnetic forms.

The positive energy requirement \([3]\) that fixes the input signs $\eta_s$, can be written as follows using the notations \([14]\):

\[
\epsilon_s = \varepsilon(I_s). \tag{15}
\]

Thus $\epsilon_s = 1$ for true electric and magnetic forms $F_s$ and $\epsilon_s = -1$ for quasiscalar forms.

Due to \([13]\), the combination \((I_s) + (\theta_s)\) of the angular coordinates on $S^{d_0}$, has a Liouville form, $\ddot{\alpha} - 3\dot{\beta} = (d_0 - 1)^2 e^{2\sigma - 2\beta^0}$ (an overdot means $d/du$), and is integrated giving

\[
e^{2\sigma - \alpha} = (d_0 - 1) s(k, u), \quad s(k, u) \overset{\text{def}}{=} \begin{cases} \frac{k^{-1} \sinh ku}{u}, & k > 0, \\ u, & k = 0, \\ \frac{k^{-1} \sin ku}{k u}, & k < 0. \end{cases} \tag{16}
\]

where $k$ is an integration constant. Another integration constant is suppressed by properly choosing the origin of $u$. With \([13]\) the $D$-dimensional line element may be written in the form

\[
ds_T^2 = \frac{e^{-2\sigma_T / \Omega}}{[ds(k, u)]^2} \left\{ \frac{du^2}{[ds(k, u)]^2} + d\Omega^2 \right\} - e^{2\beta^1} dt^2 + \sum_{i=2}^{n} e^{2\beta^i} ds_i^2, \tag{17}
\]
\( \bar{d} \equiv d_0 - 1 \). The range of the \( u \) coordinate is \( 0 < u < u_{\text{max}} \), where \( u = 0 \) corresponds to spatial infinity while \( u_{\text{max}} \) may be finite or infinite depending on the form of a particular solution.

The remaining set of unknowns \( \beta^i(u), \varphi^a(u) \), \( i = 1, \ldots, n, \ a \in \mathcal{A} \) can be treated as a real-valued vector function \( x^A(u) \) (so that \( \{ A \} = \{ 1, \ldots, n \} \cup \mathcal{A} \) in an \( (n+|\mathcal{A}|) \)-dimensional vector space \( \mathbb{V} \) (target space). The field equations for \( x^A \) can be derived from the Toda-like Lagrangian

\[
L = G_{AB} \dot{x}^A \dot{x}^B - V_Q(y) = \sum_{i=1}^{n} d_i (\dot{\beta}^i)^2 + \frac{\dot{\sigma}^2}{d_0 - 1} + \delta_{ab} \dot{\varphi}^a \dot{\varphi}^b - V_Q(y),
\]

\[
V_Q(y) = - \sum_s \epsilon_s Q_s^2 e^{2y_s},
\]

with the “energy” constraint

\[
E = G_{AB} \dot{x}^A \dot{x}^B + V_Q(y) = \frac{d_0}{d_0 - 1} k^2 \text{sign } k,
\]

where the IC \( k \) has appeared in \([16]\). The nondegenerate symmetric matrix

\[
(G_{AB}) = \begin{pmatrix} d_i d_j / \bar{d} + d_i \delta_{ij} & 0 \\ 0 & \delta_{ab} \end{pmatrix}
\]

specifies a positive-definite metric in \( \mathbb{V} \); the functions \( y_s(u) \) are defined as scalar products:

\[
y_s = \sigma(I_s) - \chi_s \bar{\lambda}_s \
\]

\[
(Y_{s,A}) = \begin{pmatrix} d_i \delta_{I_s} & -\chi_s \lambda_{sa} \end{pmatrix},
\]

where \( \delta_{I_s} = 1 \) if \( i \in I \) and \( \delta_{I_s} = 0 \) otherwise. The contravariant components and scalar products of the vectors \( \bar{Y}_s \) are found using the matrix \( G^{AB} \) inverse to \( G_{AB} \):

\[
(Y_{s}^A) = \begin{pmatrix} \delta_{I_s} - \frac{d(I)}{D - 2} & -\chi_s \lambda_{sa} \end{pmatrix};
\]

\[
Y_{s,A} Y^A_s \equiv \bar{Y}_s \bar{Y}_s' = d(I_s \cap I_{s'}) - \frac{d(I_s)d(I_{s'})}{D - 2} + \chi_s \chi_{s'} \bar{\lambda}_s \lambda_{s'}.
\]

The equations of motion in terms of \( \bar{Y}_s \) read

\[
\dot{x}^A = \sum_s q_s Y_s^A e^{2y_s}, \quad q_s \equiv \epsilon_s Q_s^2.
\]

### 3. Some exact solutions. Black holes

#### 3.1. Exact solutions: orthogonal systems (OS)

The integrability of the Toda-like system \([18]\) depends on the set of vectors \( \bar{Y}_s \), each \( \bar{Y}_s \) consisting of input parameters of the problem and representing one of the \( F \)-forms, \( F_s \), with a nonzero charge \( Q_s \), in other words, one of charged \( p \)-branes.

In many cases general or special solutions to Eqs. \([21]\) are known. The simplest case of integrability takes place when \( \bar{Y}_s \) are mutually orthogonal in \( \mathbb{V} \) \([21]\), that is,

\[
\bar{Y}_s \bar{Y}_{s'} = \delta_{s s'} \bar{Y}_s^2, \quad \bar{Y}_s^2 = d(I) [1 - d(I)/(D - 2)] + \bar{\lambda}_s^2 > 0
\]

where \( \bar{\lambda}_s = \sum_a \lambda_{sa}^2 \). Then the functions \( y_s(u) \) obey the decoupled Liouville equations \( \dot{y}_s = \epsilon_s Q_s^2 Y_s^2 e^{2y_s} \), whence

\[
e^{-2y_s(u)} = \begin{cases} Q_s^2 \bar{Y}_s^2 \bar{\lambda}_s^2, & \epsilon_s = 1, \\
Q_s^2 Y_s^2 \bar{\lambda}_s^2 \cosh^2 [h_s(u + u_s)], & \epsilon_s = -1, \quad h_s > 0,
\end{cases}
\]
where \( h_s \) and \( u_s \) are integration constants and the function \( s(.,.) \) has been defined in (16). For the sought-for functions \( x^A(u) \) and the “conserved energy” \( E \) we then obtain:

\[
x^A(u) = \sum_s \frac{Y^A_s}{Y^s_{\mu}} y_s(u) + c^A u + \xi^A, \tag{27}
\]

\[
E = \sum_s \frac{h^2_s \text{sign} \ h_s}{Y^2_{\mu}} + c^2 = \frac{d_0}{d_0 - 1} k^2 \text{sign} k, \tag{28}
\]

where the vectors of integration constants \( c \) and \( \xi \) are orthogonal to all \( \hat{Y}_s \): \( c^A Y_{s,A} = \xi^A Y_{s,A} = 0 \), or

\[
c^A d_i \delta_{i\mu} - c^\alpha \lambda_{s\alpha} = 0, \quad \xi^A d_i \delta_{i\mu} - \xi^\alpha \lambda_{s\alpha} = 0. \tag{29}
\]

### 3.2. Exact solutions: block-orthogonal systems (BOS)

The above OS solutions are general for input parameters \((D, d_i, Y_s)\) satisfying Eq. (23): there is an independent charge attached to each (elementary) \( F \)-form. One can, however, obtain less general solutions for more general sets of input parameters, under less restrictive conditions than (25). Namely, assuming that some of the functions \( y_s(u) \) (21) coincide, one obtains the so-called BOS solutions (22), where the number of independent charges coincides with the number of different functions \( y_s(u) \).

Indeed, suppose (22) that the set \( S \) splits into several non-intersecting non-empty subsets,

\[
S = \bigcup_{\omega} S_\omega, \quad |S_\omega| = m(\omega), \tag{30}
\]

such that the vectors \( \hat{Y}_\mu(\omega) \) \((\mu(\omega) \in S_\omega)\) form mutually orthogonal subspaces \( \forall \omega \subseteq \mathbb{V} \):

\[
\hat{Y}_\mu(\omega) \hat{Y}_{\nu(\omega')} = 0, \quad \omega \neq \omega'. \tag{31}
\]

Then the corresponding result from (22) can be formulated as follows:

**BOS solution.** Let, for each fixed \( \omega \), all \( \hat{Y}_\nu \in \mathbb{V}_\omega \) be linearly independent, and let there be a vector \( \hat{Y}_\omega = \sum_{\mu \in S_\omega} a_\mu \hat{Y}_\mu \) with \( a_\mu \neq 0 \) such that

\[
\hat{Y}_\mu(\omega) \hat{Y}_\nu(\omega) = 0, \quad \forall \mu \in S_\omega. \tag{32}
\]

Then one has the following solution to the equations of motion (24), (13):

\[
x^A = \sum_\omega \frac{Y^A_\omega}{Y^s_{\mu}} y_s(u) + c^A u + \xi^A, \tag{33}
\]

\[
e^{-2y_\omega} = \begin{cases} 
\hat{q}_\omega Y^s_\omega h^2 \omega, u + u_\omega, & \hat{q}_\omega > 0, \\
|\hat{q}_\omega| Y^2_\omega h^2 \omega \cosh^2 \text{[h}_\omega(u + u_\omega)], & \hat{q}_\omega < 0, \quad h_\omega > 0;
\end{cases} \tag{34}
\]

\[
E = \sum_\omega \frac{h^2_s \text{sign} \ h_s}{Y^2_\omega} + c^2 = \frac{d_0}{d_0 - 1} k^2 \text{sign} k, \tag{35}
\]

where \( h_\omega, u_\omega, c^A \) and \( \xi^A \) are integration constants; \( c^A \) and \( \xi^A \) are constrained by the orthogonality relations (22) (the vectors \( c \) and \( \xi \) are orthogonal to each individual vector \( Y_s \in \mathbb{V} \)); the function \( s(.,.) \) has been defined in (16).

Eqs. (32) form a set of linear algebraic equations with respect to the “charge factors” \( a_\mu = e_\mu Q^2_\mu/\hat{q}_\mu \neq 0 \), satisfying the condition \( \sum_\omega a_\mu = 1 \). The existence of a solution to (32) guarantees that \( \hat{q}_\omega \neq 0 \). On the other hand, if a solution to (32) gives \( a_\mu = 0 \) for some \( \mu \in S_\omega \), this means that the block cannot contain such a \( p \)-brane, and then the consideration may be replaced without it.

The function \( y_s(u) \) is equal to \( y_\mu(\omega)(u) = Y_\mu(\omega), A x^A \), which is, due to (32), the same for all \( \mu \in S_\omega \). The BOS solution generalizes the OS one, (24), (26); the latter is restored when each block contains a single \( F \)-form.

Both kinds of solutions are asymptotically flat, and it is natural to normalize the functions \( y_s(u) \) and \( y_\mu(u) \) by the condition \( y_s(0) = 0 \) or \( y_\mu(0) = 0 \), so that the constants \( u_s \) and \( u_\omega \) are directly related to the charges.

Other solutions to the equations of motion are known, connected with Toda chains and Lie algebras [34, 35].

\(^4\text{Geometrically, the vector } \hat{Y}_\omega \text{ solving Eqs. (24) is the altitude of the pyramid formed by the vectors } \hat{Y}_\mu, \mu \in S_\omega \text{ with a common origin. The condition } a_\mu > 0 \text{ means that this altitude is located inside the pyramid, while } a_\mu = 0 \text{ means that the altitude belongs to one of its faces.}\)
3.3. Black-hole solutions

Black holes (BHs) are distinguished among other spherically symmetric solutions by the existence of horizons instead of singularities in the physical 4-dimensional space-time \( M_{\text{phys}} \); the extra dimensions and scalar fields are also required to be well-behaved on the horizon to provide regularity of the \( D \)-dimensional metric. Thus BHs are described by the above solutions under certain constraints upon the input and integration constants. The no-hair theorem of Ref. [33] states that BHs are incompatible with quasiscalar \( F \)-forms. This means that all \( \epsilon_s = 1 \), hence, in particular, in the above BOS solution (32)–(35), \( \tilde{Q}_s > 0 \) and all \( a_e > 0 \). Furthermore, requiring that all the scale factors \( e^{\beta_1} \) (except \( e^{\beta_1} = \sqrt{|g_{tt}|} \) which should tend to zero) and scalars \( \varphi^a \) tend to finite limits as \( u \to u_{\text{max}} \), we get [32]:

\[
h_{\omega} = k > 0, \quad \forall \omega; \quad c^A = k \sum_{\omega} Y_{\omega}^{-2} Y_{\omega}^A - k\delta_i^A
\]

where \( A = 1 \) corresponds to \( i = 1 \) (time). The constraint (28) then holds automatically. The value \( u = u_{\text{max}} = \infty \) corresponds to the horizon. The same condition for the OS solution (26)–(29) is obtained by replacing \( \omega \to s \).

Under the asymptotic conditions \( \varphi^a \to 0 \), \( \beta^i \to 0 \) as \( u \to 0 \), after the transformation

\[
e^{-2k u} = 1 - \frac{2k}{d^d}, \quad \bar{d} \overset{\text{def}}{=} d_0 - 1
\]

the metric (17) for BHs and the corresponding scalar fields may be written as

\[
d_{\text{phys}}^2 = \left( \prod_{\omega} H_{\omega}^{A_{\omega}} \right) \left[ -dt^2 \left( 1 - \frac{2k}{d^d} \right) \prod_{\omega} H_{\omega}^{-2/Y_{\omega}^2} \right.
\]

\[
+ \left( \frac{d r^2}{1 - 2k/(d^d)} + r^2 d\Omega^2 \right) + \sum_{i=2}^{n} ds_i^2 \prod_{\omega} H_{\omega}^{A_i} \right];
\]

\[
A_{\omega} \overset{\text{def}}{=} \frac{2}{Y_{\omega}^2} \sum_{\mu \in S_\omega} a_\mu d(I_\mu)/D - 2; \quad \bar{A}_{\omega} \overset{\text{def}}{=} \frac{2}{Y_{\omega}^2} d(I_\omega)/D - 2; \quad \bar{A}_{\omega} \overset{\text{def}}{=} \frac{2}{Y_{\omega}^2} \delta I_\omega;
\]

\[
\varphi^a = - \sum_{\omega} \frac{1}{Y_{\omega}^2} \ln H_{\omega} \sum_{\mu \in S_\omega} a_\mu \lambda_{\mu a} \overset{\text{OS}}{=} - \sum_{s} \frac{\lambda_{s a}}{Y_s^2} \ln H_s,
\]

where \( \overset{\text{OS}}{=} \) means “equal for OS, with \( \omega \to s \)”, and \( H_{\omega} \) are harmonic functions in \( \mathbb{R}_+ \times S_d^d \):

\[
H_{\omega}(r) = 1 + P_{\omega}/(\bar{d}r^2), \quad P_{\omega} \overset{\text{def}}{=} \sqrt{k^2 + \bar{q}_{\omega} Y_{\omega}^2} - k.
\]  

The subfamily (36), (38)–(40) exhausts all BOS BH solutions with \( k > 0 \); the OS ones are obtained in the special case of each block \( S_\omega \) consisting of a single element \( s \). The only independent integration constants remaining in BH solutions \( k \), related to the observed mass (see below), and the brane charges \( Q_s \).

**Example 2.** The simplest, single-brane BH solutions are described by (38), (40) where all sums and products in \( s \) consist of a single term. These solutions are well known [38]. the metric (3) for, e.g., \( D = 11 \) supergravity [6] can be presented as

\[
d_{\text{phys}}^2 = H^{d(I)/9} \left[ -\frac{1 - 2k/(\bar{d}r^2)}{H} \ dt^2 + \left( \frac{dr^2}{1 - 2k/(\bar{d}r^2)} + r^2 d\Omega^2 \right) + H^{-1} ds_{\text{on}}^2 + ds_{\text{off}}^2 \right]
\]

where \( H = H(r) = 1 + P/\bar{d}r^2 \). \( P = \sqrt{k^2 + 2Q^2} - k \), \( \bar{d} = d_0 - 1 \); \( ds_{\text{on}}^2 \) and \( ds_{\text{off}}^2 \) are \( r \)-independent “on-brane” and “off-brane” extra-dimension line elements, respectively; the dimension \( d_0 \) of the sphere \( M_0 \) varies from 2 to 7 for \( d(I) = 3 \) (an electric brane) and from 2 to 4 for \( d(I) = 6 \) (a magnetic brane). In particular, the cases of maximum \( d_0 \), when off-brane dimensions are absent, correspond in the extremal near-horizon limits to the famous structures \( AdS_4 \times S^7 \) (electric) and \( AdS_7 \times S^4 \) (magnetic). All these BH solutions are stable under linear spherically symmetric perturbations [23]; though, small multidimensional BHs, whose horizon size is of the order of the compactification length, are known to possess the Gregory-Laflamme instability [10] related to distortions in extra dimensions.
The above relations describe non-extremal BHs. Extremal ones, corresponding to minimum BH mass for given charges (the so-called BPS limit), are obtained in the limit $k \to 0$. The same solutions follow directly from under the conditions $h_{\omega} = k = c^4 = 0$. For $k = 0$, the solution is defined in the whole range $r > 0$, while $r = 0$ in many cases corresponds to a naked singularity rather than an event horizon, so that we no more deal with a black hole. However, in many other important cases $r = 0$ is an event horizon of extremal Reissner-Nordström type, with an AdS near-horizon geometry and even the global metric turns out to be regular, as it happens for the $AdS_4 \times S^7$ and $AdS_7 \times S^4$ structures mentioned in the previous paragraph [23].

Other families of solutions, mentioned at the end of the previous section, also contain BH subfamilies. The most general BH solutions are considered in Ref. [37].

4. 4-dimensional conformal frames

To discuss possible observational manifestations of the above solutions, it is necessary to specify the 4-dimensional physical metric. (Here and henceforth we put $d_0 = 2$.) A straightforward choice of the $M_{\text{phys}}$ (The 4-dimensional part of the original metric $g_{\mu\nu}$ used in [1], while $e^{\sigma z}$, defined in [12], is the volume factor of all extra dimensions.

The choice of a physical CF in non-Einsteinian theories of gravity is widely discussed, but the discussion is mostly restricted to the 4-dimensional metric — see e.g. [11] and numerous references therein. There are arguments in favour of the Einstein frame, and the most important ones, applicable to higher order and scalar-tensor theories (and many multiscalar-tensor theories obtainable from multidimensional gravity) are connected with the positivity of scalar field energy and the existence of a classically stable ground state [41, 42]; though, these requirements are violated if quantum effects are taken into account [8].

In our view, however, the above arguments could be convincing if we were dealing with an “absolute”, or “ultimate” theory of gravity. If, on the contrary, the gravitational action is obtained as a certain limit of a more fundamental unification theory, theoretical requirements like the existence of a stable ground state should be addressed to this underlying theory rather than its visible manifestation. In the latter, the notion of a physical CF should be only related to the properties of instruments used for measuring lengths and time intervals. Moreover, different sets of instruments (different measurement systems [2]) are described, in general, by different CFs.

Therefore, for any specific underlying theory that leads to the action (1) in a weak field limit, two CFs are physically distinguished: one, which may be called the fundamental frame, where the theory is originally formulated and another one, the observational frame, or the atomic system of measurements (the 4-A frame), providing the validity of the weak equivalence principle (or geodesic motion) for ordinary matter in 4 dimensions. The fundamental frame is specified in the original space-time where the theory is formulated and is a natural framework for discussing such issues as space-time singularities, horizons, topology, etc. (what happens as a matter of fact). On the other hand, the 4-A frame is necessary for formulating observational predictions (what we see), and its choice depends on how fermions are introduced in the underlying theory [3, 33]. The reason is that as long as clocks and other instruments used in observations and measurements consist of fermionic matter, the basic atomic constants are invariable in space and time by definition. For instance, the modern definition of reference length is connected with a certain spectral line, determined essentially by the Rydberg constant and, basically, by the electron and nucleon masses.

The (4-dimensional section of the) fundamental frame and the 4-A frame are, generally speaking, different, and none of them necessarily coincides with the 4-E frame, which represents the gravitational system of measurements [3].

If the underlying theory is string theory, the fundamental frame is realized by the so-called “string metric” (see e.g. [45, 46]), connected with $g_{MN}$ of Eq. (4) (the D-E metric) by a dilatonic-dependent conformal factor. On the other
hand, to distinguish the observational frame, we have to take into account that even for a fixed underlying theory, such frames may be different for different particular cosmological models. Thus, for the case of string theory, new results on the equivalence of quantum and some classical dilatonicbrane-worlds in string and Einstein frames have been obtained in Ref. [11]. For a more general context of string theory, let us recall that, in the effective field-theoretic limit of string theory in 10 dimensions, the Lagrangian is presented in a form similar to (1) with some quadratic fermion terms do not contain the dilaton field ([22], Eq. (13.1.49)). If those terms are associated with matter, then, by analogy, it is reasonable for illustration purposes to write the matter Lagrangian simply as an additional term in the brackets of Eq. (1).

In the observational frame, the matter part of the 4-dimensional action in terms of the corresponding metric \( g^*_{\mu\nu} \) should be simply \( \int d^4 x \sqrt{g} L_m \). Then \( g^*_{\mu\nu} \) is related to \( g_{\mu\nu} \) in the following way [33]:

\[
g^*_{\mu\nu} = e^{-\sigma/2} g_{\mu\nu}.
\]

In what follows, since we do not fix a particular underlying theory, we leave the 4-dimensional CF arbitrary and only single out some results corresponding to the choices [22] and [23].

5. Post-Newtonian parameters. Black-hole observables

One can imagine that some real astrophysical objects (stars, galaxies, quasars, black holes) may be described (perhaps approximately) by some solutions of multidimensional theory of gravity, i.e., are essentially multidimensional objects, whose structure is affected by charged \( p \)-branes. (It is in this case unnecessary to assume that the antisymmetric form fields are directly observable, though one of them may manifest itself as the electromagnetic field.)

The post-Newtonian (PN) (weak gravity, slow motion) approximation of these multidimensional solutions then determines the predictions of the classical gravitational effects: gravitational redshift, light deflection, perihelion advance and time delay (see [1, 7]). Observational restrictions on the PN parameters will then determine the admissible limits of theoretical models.

For spherically symmetric configurations, the PN metric is conventionally written in terms of the Eddington parameters \( \beta \) and \( \gamma \) in isotropic coordinates, in which the spatial part is conformally flat [4]:

\[
ds_{\text{PN}}^2 = -(1 - 2V + 2\beta V^2)dt^2 + (1 + 2\gamma V)(d\rho^2 + \rho^2 d\Omega^2)
\]

where \( d\Omega^2 \) is the metric on \( S^2 \), \( V = GM/\rho \) is the Newtonian potential, \( G \) is the Newtonian gravitational constant and \( M \) is the active gravitating mass.

Observations in the Solar system lead to tight constraints on the Eddington parameters [1]:

\[
\begin{align*}
\gamma &= 0.99984 \pm 0.0003, \\
\beta &= 0.9998 \pm 0.0006.
\end{align*}
\]

The first restriction is a result of over 2 million VLBI observations [17]. The second one follows from the \( \gamma \) data and an analysis of lunar laser ranging data. In this case a high precision test based on the calculation of the combination \( (4\beta - \gamma - 3) \), appearing in the Nordtvedt effect [4], is used [8].

For the multidimensional theory under consideration, the metric (44) should be identified with the asymptotics of the 4-dimensional metric from a solution in the observational (4-A) frame. Preserving its choice yet undetermined, we can write according to (47) with \( d_0 = 2 \):

\[
ds_4 = e^{2f(u)} \left\{ -e^{2\beta t} dt^2 + \frac{e^{-2\sigma}}{s^2(k, u)} \left[ \frac{du^2}{s^2(k, u)} + d\Omega^2 \right] \right\}
\]

where \( f(u) \) is an arbitrary function of \( u \), normalized for convenience to \( f(0) = 0 \). Recall that by our notations \( \sigma_1 = \beta^2 + \sigma_2 \), the function \( s(k, u) \) is defined in Eq. (44), and spatial infinity takes place at \( u = 0 \). The choice of the frame [13] means \( f = \sigma_2/4 \). The 4-E frame (42) corresponds to \( f = \sigma_2/2 \).

Passing to isotropic coordinates in (47) with the relations

\[
\frac{d\rho}{\rho} = -\frac{du}{s(k, u)}, \quad \frac{du^2}{s^2(k, u)} + d\Omega^2 = \frac{1}{\rho^2}(d\rho^2 + \rho^2 d\Omega^2),
\]

one finds that for small \( u \) (large \( \rho \))

\[
\frac{1}{\rho} = u \left[ 1 - \frac{u^2}{4} k^2 \text{sign} k + O(u^4) \right].
\]
so that \( u = 1/\rho \) up to cubic terms, and the decomposition in powers of \( 1/\rho \) up to \( O(\rho^{-2}) \), needed for comparison with (14), precisely coincides with the \( u \)-decomposition near \( u = 0 \).

Using this circumstance, it is easy to obtain for the mass and the Eddington parameters corresponding to (17):

\[
GM = -\beta' - f'; \quad \beta = 1 + \frac{1}{2} \beta'' + f'' \quad \text{and} \quad \gamma = 1 + \frac{2f' - \sigma_2}{GM},
\]

(49)

where \( f' = df/du \bigg|_{u=0} \) and similarly for other functions. The expressions (49) are quite general, being applicable to asymptotically flat, static, spherically symmetric solution of any theory where the EMT has the property \( T^\rho_\rho + T^\phi_\phi = 0 \), which leads to the metric (17). They apply, in particular, to all solutions of the theory (4) under the conditions specified, both mentioned and not mentioned above and those yet to be found.

In the observational frame (43) we have

\[
GM = -\frac{1}{4}(3\beta' + \sigma_2'); \quad \beta = 1 + \frac{3\beta'' + \sigma_2''}{8(GM)^2}, \quad \gamma = 1 - \frac{\sigma_2'}{2GM},
\]

(50)

Similar expressions for the 4-E frame (42) are

\[
GM = -\frac{1}{2}(2\beta' + \sigma_2'); \quad \beta = 1 + \frac{2\beta'' + \sigma_2''}{4(GM)^2}, \quad \gamma = 1.
\]

(51)

We thus conclude that the Eddington parameter \( \gamma \) is the same as in general relativity for all p-brane solutions in the general model (4) in the 4-E frame (under the assumptions of Sec. 2).

Expressions of \( \beta \) and \( \gamma \) for specific solutions can be obtained by substituting them to (49) or (50). One may notice, however, that \( \beta \) may be calculated directly from the equations of motion (24), without solving them. This is true for any function \( f \) of the form \( f = \tilde{F} \tilde{x} \) where \( \tilde{F} \in V \) is a constant vector (i.e., \( f \) is a linear combination of \( \beta \) and the scalar fields \( \varphi^a \)):

\[
\beta - 1 = \frac{1}{2(GM)^2} \sum s \epsilon_s Q_s^2 \left( Y_s^2 + \tilde{F} \tilde{Y_s} \right) e^{2\varphi_s(0)}.
\]

(52)

In particular, if \( f = N\sigma_2 \),

\[
\beta - 1 = \frac{1}{2(GM)^2} \left\{ \sum_{s: \epsilon_s = +1} Q_s^2 \left[ 1 - N + \frac{2(N - 1)d(I_s)}{D - 2} \right] e^{2\varphi_s(0)} + \sum_{s: \epsilon_s = -1} Q_s^2 \left( \frac{1 - 2N}{D - 2} d(I_s) \right) e^{2\varphi_s(0)} \right\};
\]

(53)

recall that \( \epsilon_s = 1 \) refers to true electric and magnetic forms, \( \epsilon_s = -1 \) to quasiscalar ones. For \( N = 1/2 \) and \( N = 1/4 \) we obtain the values of \( \beta \) for the frames (42) and (43), respectively.

Explicit expressions for \( M \) and \( \gamma \) (in frames other than 4-E) require the asymptotic form of the solutions.

It is convenient, without loss of generality, to normalize the scale factors at spatial infinity in such a way that \( e^{\beta_i(0)} = 1, \ i = 1, \ldots, n \), so that the real scales of the extra dimensions are hidden in the factor space metrics \( g^i \). In a similar way, one can re-define the dilatonic fields: \( \varphi^a - \varphi^a(0) \rightarrow \varphi^a \), so that \( \varphi^a(0) = 0 \), while the former asymptotic values of \( \varphi^a \) have been actually absorbed in the charges \( Q_s \). Then all \( y_s(0) = 0 \).

For all OS and BOS solutions it then follows that the constants \( \xi^i \) are zero; the constants \( u_\omega, h_\omega \) and \( \tilde{q}_\omega \) in (14) are related by

\[
1 = \begin{cases} 
\tilde{q}_\omega Y_s^2 s^2(h_\omega, u_\omega), & \tilde{q}_\omega > 0, \\
\tilde{q}_\omega |Y_s^2 h_\omega^{-2} \cosh^2(h_\omega u_\omega)|, & \tilde{q}_\omega < 0, \quad h_\omega > 0.
\end{cases}
\]

(54)

In a similar way for OS, according to (24),

\[
1 = \begin{cases} 
Q_s^2 Y_s^2 s^2(h_s, u_s), & \epsilon_s = 1, \\
Q_s^2 Y_s^2 h_s^{-2} \cosh^2(h_s u_s), & \epsilon_s = -1, \quad h_s > 0.
\end{cases}
\]

(55)

In (24), the second line corresponds to a quasiscalar form \( F_s \), while in (54) the second line means that the summed squared charge \( \tilde{q}_\omega \) of the block \( \mathcal{S}_\omega \) is dominated by quasiscalar forms.

In what follows we will only give expressions for BOS solutions; their OS versions are then evident.
Eq. (54) gives
\[ y' = \frac{dy}{du} \Big|_{u=0} = \left\{ \begin{array}{ll} -(\hat{\omega}_e Y^2 + h^2 \text{sign } h) \hat{\omega}, & \hat{\omega} > 0; \\ \pm(h^2 - |\hat{\omega}_e| Y^2) \hat{\omega}, & \hat{\omega} < 0, \end{array} \right. \]  
\[ (56) \]

For OS \( \hat{\omega}_e \) is replaced by \( \epsilon_s Q^2 \), the two lines refer to \( \epsilon_s = 1 \) and \( \epsilon_s = -1 \), respectively.

The quantities needed for calculating \( M \) and \( \gamma \) are
\[ x^{\alpha'} = \sum_\omega Y^{-2} Y^\alpha b'_\omega + e^\alpha, \quad \sigma'_1 = \frac{1}{2} \sum_\omega A_\omega y'_\omega + \sum_{i=1}^n d_i n^i, \]
\[ (57) \]
with \( A_\omega \) defined in (38). The quantity \( \sigma_2 \) can be obtained as \( \sigma_1 - \beta^1 \); as before, \( x^1 = \beta^1, Y^1 = \sum_{\rho \in S_\omega} a_\rho Y^\rho_1 \) and \( Y^1 = \delta(I_\rho) - d(I_\rho)/(D - 2) \). The values of \( M \) and \( \gamma \) are now easily found from (38) in terms of the solution parameters for any given \( f \) of the above general form, \( f = \tilde{F} \tilde{x} \).

In particular, for BH solutions (34) \( \rightarrow \) (35) there is no need to change the coordinates from \( u \) to \( r \) or \( \rho \); it is sufficient to use Eqs. (36) \( \rightarrow \) (37) for the constants. Moreover, BH solutions contain only true electric and magnetic forms, \( \epsilon_s = +1 \) (33) and \( \hat{\omega}_e > 0 \).

Thus, for instance, for BOS BH solutions the quantities \( \beta^{1'} \) and \( \sigma'_2 \) are
\[ \beta^{1'} = -k - \sum_\omega P_\omega \frac{1 - b_\omega}{Y^2}, \quad \sigma'_2 = -\sum_\omega \frac{1 - 2b_\omega}{Y^2} \]
\[ (58) \]
with \( P_\omega \) defined in (40) and \( b_\omega \) defined as \( \sum_{\rho \in S_\omega} a_\rho d(I_\rho)/(D - 2) \). In the OS case \( b_\omega \) becomes \( b_s = d(I_s)/(D - 2) \).

Accordingly, for BHs in CFs with \( f = N \sigma_2 \) we obtain
\[ GM = k + \sum_\omega P_\omega Y^2 \left[ 1 - b_\omega + N(1 - 2b_\omega) \right], \quad \gamma = \frac{1 - 2N}{GM} \sum_\omega P_\omega Y^2 \left( 1 - 2b_\omega \right). \]
\[ (59) \]

Some general observations can be made from the above relations.

- The expressions for \( \beta \) depend on the input constants \( D, d(I_s) \) (hence on \( p \)-brane dimensions: \( p_s = d(I_s) - 1 \)), on the mass \( M \) and on the charges \( Q_s \). For given \( M \), they are independent of other integration constants, emerging in the solution of the Toda system (24), and also on \( p \)-brane intersection dimensions, since they are obtained directly from Eqs. (24) \( \rightarrow \) (36). This means, in particular, that \( \beta \) is the same for BH and non-BH configurations with the same set of input parameters, mass and charges.

- According to (53), all \( p \)-branes give positive contributions to \( \beta \) in both frames (22) and (33), therefore (33) combined with (43) leads to a general restriction on the charges \( Q_s \) for given mass and input parameters.

- The expressions for \( \gamma \) depend, in general, on the integration constants \( h_s \) or \( \hat{\omega}_s \) and \( e^\rho \) emerging from solving Eqs. (24). For BH solution these constants are expressed in terms of \( k \) and the input parameters, so both \( \beta \) and \( \gamma \) depend on the mass, charges and input parameters.

- In the 4-E frame, one always has \( \gamma = 1 \). The same is true for some BH solutions in all frames with \( f = N \sigma_2 \). Indeed, a pair of electric and magnetic \( p \)-branes with equal \( |Q_s| \), corresponding to \( F \)-forms \( F_1 \) and \( F_2 \) of equal rank (in particular, if \( F_1 \) and \( F_2 \) are the electric and magnetic components of the same composite \( F \)-form), always forms a BOS block, with \( a_1 = a_2 = 1/2 \), so that \( b_\omega = 1/2 \), and this pair does not contribute to \( \sigma'_2 \) in (58). Evidently \( \gamma = 1 \) as well for a BOS black hole containing several such dyonic pairs and no other \( F \)-forms. This property was noticed in Ref. (24) for the frame \( f = 0 \).

**BH temperature.** BHs are, like nothing else, strong-field gravitational objects, while the PN parameters only describe their far neighbourhood. An important characteristic of their strong-field behaviour, potentially observable and depending on their multidimensional structure, is the Hawking temperature \( T_H \). As with other observables, it is of importance to know the role of conformal frames for its calculation. One can ascertain, however, that this quantity is CF-independent, at least if conformal factors that connect different frames are regular on the horizon.

Indeed, if, in an arbitrary static, spherically symmetric space-time with the metric
\[ g = -e^{2\Gamma(r)} dt^2 + e^{2\Lambda(r)} dr^2 + \text{anything else}, \]
\[ (60) \]
the sphere \( r = r_{\text{hor}} \) is an event horizon, its Hawking temperature can be calculated as

\[
T_h = \frac{1}{2\pi k_B} \lim_{r \to r_{\text{hor}}} e^{\Gamma-A}|d\Gamma/dr|
\]  

(61)

where \( k_B \) is the Boltzmann constant. This expression, which is invariant under reparametrizations of the radial coordinate \( r \), is easily obtained from standard ones [39]. The factor \( e^{\Gamma-A} \) is insensitive to conformal transformations \( g \mapsto e^{2f(r)}g \), whereas \( \Gamma \) is replaced by \( \Gamma + f \). At a horizon, \( \Gamma \to -\infty \), and, if \( r_{\text{hor}} \) is finite (such a coordinate always exists), \( |d\Gamma/dr| \to \infty \). Therefore \( T_h \) calculated according to Eq. (61) will be the same in all frames with different \( f(r) \) provided \( df/dr \) is finite at \( r = r_{\text{hor}} \).

This is precisely the case with the BH metric (38) and any \( f \) formed as a linear combination of \( \beta^i \) (\( i > 1 \) and \( \varphi^a \)). Using the recipe [11], one obtains [22]

\[
T_h = \frac{1}{8\pi k_B} \prod_\omega \left( \frac{2k}{2k + P_\omega}\right)^{1/Y_\omega^2} \frac{Q_S}{8\pi k_B} \prod_s \left( \frac{2k}{2k + P_s}\right)^{1/Y_s^2}
\]

(62)

The physical meaning of \( T_h \) is related to quantum evaporation, a process to be considered in the fundamental frame, while the produced particles with a certain spectrum are usually assumed to be observed at flat infinity, where our CFs do not differ. This means that the \( T_h \) expression should be CF-independent. We have seen that it possesses this property “by construction”. The conformal invariance of \( T_h \) was also discussed in another context in Ref. [1].

All this is true for \( T_h \) in terms of the integration constant \( k \) and the charges \( Q_s \). However, the observed mass \( M \) as a function of the same quantities is frame-dependent, see (49). Therefore \( T_h \) as a function of \( M \) and \( Q_s \) is frame-dependent as well. Thus, for small charges, \( Q_s^2 \ll k^2 \), one easily finds from (62) under the same assumption \( \tilde{f} = N\sigma_2 \):

\[
k = GM - \frac{1}{2GM} \sum_\omega \tilde{q}_\omega [1 - b_\omega + N(1 - 2b_\omega)]
\]

(63)

up to higher order terms in \( \tilde{q}_\omega / (GM)^2 \) \( b_\omega \) was defined in [28]). This expression should be substituted into (62). For larger charges the corresponding expressions are more involved.

The temperature of extremal BHs \( (k = 0) \) only depends on the charges and the input parameters \( Y_\omega^2 \). Moreover, by (62), for some sets of \( Y_\omega^2 \), \( T_h \) can tend to infinity as \( k \to 0 \). This means that the horizon becomes a naked singularity in the extremal limit [22].

6. Coulomb law violation

One of specific potentially observable effects of extra dimensions is Coulomb law violation.

Consider the space-time \( M \) described in Sec. 2, with the metric (2) and \( d_0 = 2 \). Suppose that the electrostatic field of a spherically symmetric source is described by a term \( \propto F^2 e^{0\lambda} \) in the action (1) (where, as before, \( \lambda = \lambda_0 \varphi^a \)), corresponding to a true electric \( m \)-form \( F_{el} \) with a certain set \( I \geq 1 \), or

\[
I = 1 \cup J, \quad J \subset \{2, \ldots, n\}.
\]

As before, from the field equations for \( F_{el} \) we have Eq. (11), so that

\[
\frac{1}{m!} F^2 = \varepsilon(I) Q^2 e^{-2\sigma(T)} e^{2\lambda_0}.
\]

(64)

The observable electromagnetic field \( F_{\mu\nu} \) in 4 dimensions is singled out from \( F_{el} \) as follows:

\[
F_{\mu\nu} M_3 \ldots M_m = F_{\mu\nu}, \quad \text{where} \quad \frac{1}{m!} F^2 = \frac{1}{2} \varepsilon(J) F_{\mu\nu} F^{\mu\nu} e^{-2\sigma(J)}.
\]

(65)

where the indices \( M_3, \ldots, M_m \) belong to \( J \). \( F_{\mu\nu} F^{\mu\nu} \) is written here in terms of the \( g_{\mu\nu} \), the 4-dimensional part of \( g_{MN} \). If, as in Sec. 5, the observable metric is assumed to be \( h_{\mu\nu} = e^{2f} g_{\mu\nu} \), then the squared observed radial electric field strength is

\[
E^2[f] = -h^{tt} h^{uu} (F_{ut})^2 = - e^{-4f} g^{tt} g^{uu} (F_{ut})^2 = - \frac{1}{2} e^{-4f} F_{\mu\nu} F^{\mu\nu}
\]

(66)

since \( F_{ut} = -F_{tu} \) is the only nonzero component of \( F_{\mu\nu} \). From (66) with (15) and (14) we obtain

\[
E^2[f] = Q^2 e^{-4f-4\sigma} \cdot e^{-4\lambda_0+2\sigma(J)-2\sigma(T)}
\]

(67)
where $e^{2\beta_0} = g_{\theta\theta}$, the notations \[12\] are used and $\mathcal{J} = \{2, \ldots, n\} \setminus J$.

One can notice that $e^{2f + 2\beta_0} = h_{\theta\theta} = r^2$ where $r$ is the observable radius of coordinate spheres $t = \text{const}$, $u = \text{const}$. Therefore Eq. \[17\] may be rewritten as

$$E = E[f] = (|Q|/r^2) e^{-2\lambda_2 r + \sigma(J) - \sigma(J)}. \quad (68)$$

This is the modified Coulomb law. The deviations from the conventional Coulomb law are evidently both due to extra dimensions (depending on the multidimensional structure of the $F$-form) and due to the interaction with the scalar fields. This relation (generalizing the one obtained in Ref. \[12\] in the framework of dilaton gravity) is valid for an arbitrary metric of the form (2) \((d_0 = 2)\) and does not depend on whether or not this $F$-form takes part in the formation of the gravitational field.

Eq. \[68\] is exact and — which is remarkable — it is CF-independent. This is an evident manifestation of the conformal invariance of the electromagnetic field in $\mathcal{M}_{\text{phys}}$ even in the present generalized framework.

From the observational viewpoint, the weak gravitational field approximation, in the spirit of the previous section, can be of interest. Consider for simplicity the conventional Maxwell field, i.e., the 2-form $\mathcal{F}$ conformal invariance of the electromagnetic field in $\mathcal{M}_{\text{phys}}$. Therefore Eq. (67) may be rewritten as

$$E = E[f] = (|Q|/r^2) e^{-2\lambda_2 r + \sigma(J) - \sigma(J)}. \quad (68)$$

where expressions for $\sigma'$ and $\sigma_2' = \sigma_1' - \beta_1'$ should be taken from specific solutions of the equations of motion — see Eqs. \[52\], \[57\]. (Note that up to higher-order terms, $u \approx 1/\rho \approx 1/r$ at small $u$.) Since, in general, quantities like $\lambda_2'$ and $\sigma_2'$ are of the order of $k \sim GM$, one can conclude that the Coulomb law violation intensity is of the order of the gravitational field strength characterized by the ratio $GM/r$. Though, unlike $E$, the mass $M$ itself is $f$-dependent, see \[4\].

7. **T-holes**

Consider a simple example of a BH metric, e.g., \[11\] in the case $d_0 = 2$, $Q = 0$, hence $p = 0$ and $H \equiv 1$. It is thus a direct generalization of the Schwarzschild metric, charged $p$-branes are absent and all extra dimensions form a 7-dimensional manifold with the $r$-independent Ricci-flat metric $ds_7^2$, which we will assume to be flat. Let us introduce the following modification: interchange the time $t$ with a selected extra coordinate, say, $v$. One has

$$ds_7^2 = -dt^2 + \frac{dv^2}{1 - 2k/r} + r^2d\Omega^2 + \left(1 - \frac{2k}{r}\right)\eta_v dv^2 + ds_6^2 \quad (70)$$

where $ds_6^2$ is the flat metric of the remaining dimensions and $\eta_v = \pm 1$ depending on whether the coordinate $v$ is spacelike or timelike.

This modification only changes the interpretation of different coordinates without changing the mathematical properties of the metric, it therefore remains to be a solution of the field equations.

The main feature of this configuration is that the physical space-time $\mathcal{M}_{\text{phys}}$ changes its signature at $r = 2k$: it is $(+, --, --, +, +)$ for $r > 2k$ and $(+, ++, --, ++) \text{ at } R < 2k$. This evidently means that the anomalous domains should be characterized with quite unconventional physics whose possible consequences and observational manifestations are yet to be studied. It has been suggested \[30\] to call such domains with an unusual space-time signature time holes or T-holes and the corresponding horizons T-horizons, to be designated $\mathcal{H}_T$.

Evidently each BH configuration of any dimension $D > 4$ has a family of T-hole counterparts (a family since the factor spaces may have different dimensions and signatures, and a $v$-axis like the one in \[70\] may be selected in any of them). Conversely, any T-hole solution has BH counterparts. If a BH possesses an external field, such as the $F_{[4]}$-form field corresponding to the metric \[11\], under a BH—T-hole transition, its true electric or magnetic component may be converted into a quasiscalar one. This happens if the new $t$ coordinate (former $v$) is off-brane, $\mathbb{R}_t \not\subset \mathcal{M}_f$. If the new time axis becomes $t$-axis, the $p$-brane remains true electric or magnetic.

Unlike a BH-horizon, a T-horizon $\mathcal{H}_T$ is not in absolute past or future from a distant observer’s viewpoint, it is visible since it takes a finite time for a light signal to come from it (independently of a conformal gauge since the latter does not affect light propagation).

Thus, in addition to the above family of $p$-brane BH solutions, there is a similar family of T-hole ones.

There are certain problems connected with the compactification of extra dimensions. They can be clearly understood using the simple example \[4\], which may be called the $T$-Schwarzschild metric. Note that if we ignore
the “passive” subspace with the metric $ds_2^2$, the remaining 5-dimensional manifold coincides with the “zero dipole moment soliton” in the terminology of [52].

At $r = 2k$ the signs of $g_{rr}$ and $g_{vv}$ change simultaneously. Moreover, if $\eta_v = -1$, i.e., this compactified direction is timelike at large $r$, the total signature of $\mathbb{M}$ is preserved but in the opposite case, $\eta_v = -1$, it changes by four: two spacelike directions become timelike. However, as one can directly verify, $\mathcal{H}_T$ is not a curvature singularity, either for the $D$-dimensional metric or for its 4-dimensional section.

If $\eta_v = -1$, the surface $r = 2k$ is a Schwarzschild-like horizon in the $(r,v)$ subspace, and there exists an analytic continuation to $R < 2k$ with the corresponding Kruskal picture. However, if some points on the $v$ axis are identified, as should be done to compactify the axis $\mathbb{R}_v$ in the conventional way, then the corresponding sectors are cut out in the Kruskal picture, so that the $T$-domain and $R$-domain sectors join each other only at a single point, the horizon intersection point. This should be probably interpreted as a singularity due to intersection of particle trajectories.

Another thing happens if $\eta_v = 1$. Again a further study is possible using a transition to coordinates in which the metric is manifestly nonsingular at $r = 2k$. Let us perform it for (70) in the vicinity of $\mathcal{H}_T$ (more general T-holes may be treated in a similar way):

$$r \to 2k; \quad r - 2k = (x^2 + y^2)/(8k); \quad v = 4k \arctan(y/x);$$

$$ds_2^2(r,v) \approx \frac{r - 2k}{2k} dv^2 + \frac{2k}{r - 2k} dr^2 = dx^2 + dy^2. \quad (71)$$

Thus the $(r,v)$ surface metric is locally flat near the T-horizon $r = 2k$ which is transformed into the origin $x = y = 0$, while the $v$ coordinate has the character of an angle.

This transformation could also be conducted as a conformal mapping of the complex plane with the aid of the analytic function $ln z, z = x + iy$, as was done in Ref. [53] for some cylindrically symmetric Einstein-Maxwell solutions; then $v$ is proportional to $\arg z$.

Consequently, in the general case the $(r,v)$ surface near $r = 2k$ behaves like a Riemann surface having a finite or infinite (if $v$ varies in an infinite range) number of sheets, with a branch point at $x = y = 0$ (a branch-point singularity [52]). If $\mathbb{R}_v$ is compactified, $v$ is naturally described as an angular coordinate $(0 \leq v < 2\pi l$, where $v = 0$ and $v = 2\pi l$ are identified and $l$ is the compactification radius at the asymptotic $R \to \infty$). $r = 2k$ is then the center of symmetry in the $(r,v)$ surface; the surface itself has the shape of a tube with a constant thickness at $r \to \infty$, becoming narrower at smaller $r$ and ending at $r = 2k$ either smoothly (if the regular center condition $l = 4k$ is satisfied), or with a conical or branch-point singularity (otherwise). This suggests that there is no way to go beyond $r = 2k$.

In the singular case the geodesic completeness requirement is violated on $\mathcal{H}_T$, so it is reasonable to require $l = 4k$, or, more generally, $l = 4kj$ where $j$ is a positive integer, so that $r = 2k$ is a $j$-fold branch point. In this case a radial geodesic, whose projection to the $(r,v)$ surface hits the point $r = 2k$, passes through it and returns to greater radii $r$ but with another value of $v$, thus leaving the particular 4-dimensional section of the $D$-dimensional space-time. However, if the multidimensional quantum wave function of the corresponding particle is $v$-independent, the particle does not disappear from an observer’s sight and can look as if reflected from a mirror. The same true for macroscopic bodies if their energy-momentum is $v$-independent. If, on the contrary, the T-hole appears in a braneworld-like model, such that matter is concentrated at a particular value of $v$, then, being reflected from a T-horizon, matter disappears from the observers’ sight.

A T-hole is an example of a configuration looking drastically different in different conformal frames. If in Eq. (67) the function $f$ is a multiple of $\sigma_2$ (it is natural since $e^{\sigma_2}$ is the volume factor of extra dimensions; an example is (43)), then the 4-metric (67) has a curvature singularity on the T-horizon. Indeed, the factor $e^{2f}$ is then proportional to a certain power of $g_{vv}$ which vanishes there. A consistent description of $\mathcal{H}_T$ requires, however, the full multidimensional picture, where a curvature singularity is absent.

8. Concluding remarks

We have obtained expressions for the Eddington PN parameters $\beta$ and $\gamma$ for a wide range of static, spherically symmetric solutions of multidimensional gravity with the general string-inspired action [4]. The existing experimental limits [15] and [16] on $\beta$ and $\gamma$ constrain certain combinations of the solution parameters. This, however, concerns only the particular system for which the measurements are carried out, in our case, the Sun’s gravitational field. The main feature of the expressions for $\beta$ and $\gamma$ is their dependence not only on the theory (the input constants entering into the action), but on the particular solution (the integration constants). This means that the PN parameters should be different for different self-gravitating configurations. They should not only be different, say, for stars and black holes, but even for different stars if we try to describe their external fields in terms of the model [4].
A feature of interest is the universal prediction of $\beta > 1$ in (53) for both frames (42) and (43). This conclusion does not depend on the system integrability and rests solely on the positivity of energy required.

The predicted deviations of $\gamma$ from unity may be of any sign and depend on many integration constants. It turns out that precisely $\gamma = 1$ in the external field of a BH with equal electric and magnetic charges of the same composite $F$-form or of two forms of equal ranks, or with a few such pairs of charges.

If, however, the 4-dimensional Einstein frame is adopted as the observational one, we have a universal result $\gamma = 1$ for all static, spherically symmetric solutions of the theory (1).

The BH temperature $T_\text{H}$ also carries information about the multidimensional structure of space-time. Being a universal parameter of a given solution to the field equations, $T_\text{H}$ as a function of the observable BH mass and charges is still conformal frame dependent due to different expressions for the mass $M$ in different frames.

One more evident consequence of multidimensional theory is the Coulomb law violation, caused by a modification of the conventional Gauss theorem and also by scalar-electromagnetic interaction. A remarkable property of the modified Coulomb law is its conformal frame independence for any given static, spherically symmetric metric where the electromagnetic field is situated.

In addition to modifications of conventional physical laws, extra dimensions can lead to the existence of a new kind of objects, T-holes, which, as we argue, can probably be observable as bodies with mirror surfaces, at least if the T-horizons are connected with compact spacelike extra dimensions. It is also possible that matter simply escapes from our physical space across the T-hole surface. More detailed predictions can be formulated in specific theories.

Although it seems hard to point out a T-hole formation mechanism which might act in the present Universe, their emergence should have been as probable as that of black holes in the early Universe, when all space-time dimensions were on equal footing.

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