Infrared degrees of freedom of Yang-Mills theory in the Schrödinger representation

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We set up a new calculational framework for the Yang-Mills vacuum transition amplitude in the Schrödinger representation. After integrating out hard-mode contributions perturbatively, we perform a gauge invariant gradient expansion of the ensuing soft mode action which renders a subsequent saddle point expansion for the vacuum overlap manageable. The standard “squeezed” approximation for the vacuum wave functional then allows for an essentially analytical treatment of physical amplitudes. Moreover, it leads to the identification of dominant and gauge invariant classes of gauge field orbits which play the role of gluonic infrared (IR) degrees of freedom. Those emerge as a rich variety of (mostly solitonic) solutions to the saddle point equations which are characterized by a common relative gauge orientation of the underlying gluon fields. We discuss their scale stability, guaranteed by a virial theorem, and other general properties including topological quantum numbers and action bounds. We then find important saddle point solutions explicitly and examine their physical impact. Some of them are related to tunneling solutions of the classical Yang-Mills equation, i.e. to instantons and merons, while others appear to play unprecedented roles. A remarkable new class of IR degrees of freedom comprises vortex and knot solutions of Faddeev-Niemi type, potentially related to glueballs.

I. INTRODUCTION

The strong couplings among soft QCD gluons manifest themselves in a variety of complex long-distance phenomena. Most of them are thoroughly entwined with the vacuum state, as illustrated by such prominent examples as quark confinement, spontaneous chiral symmetry breaking, vacuum tunneling processes, the ensuing $\theta$ structure as well as large gluon condensates. Despite the apparent diversity of these and other effects, however, the essence of the underlying dynamics is often expected to involve just a few soft gluonic modes.

The quest for these long-wavelength excitations began shortly after the inception of QCD and has inspired the development of various kinds of vacuum models, based e.g. on glueball condensation \cite{1}, Gaussian stochastic processes \cite{2}, gluonic domains \cite{3,4} and instanton \cite{5} as well as meron \cite{6} ensembles. Over the last decade, lattice simulations increasingly assisted in the search for predominant infrared (IR) gluon fields, mostly by means of numerical “filtering” \cite{7} and gauge-fixing \cite{8} techniques. These simulations are now beginning to generate quantitative insights into the role of instantons and their size distribution \cite{9}, and into the classic confinement scenarios based on (gauge-projected abelian) monopole \cite{10} or center vortex \cite{11,12} condensation.

Nevertheless, at present no mechanism involving soft vacuum gluons can be uniquely or systematically related to QCD, and many crucial questions regarding the underlying fields, their stability, gauge-independent physical interpretation, mutual interactions, relations to other vacuum fields etc. remain unanswered. Analytical progress in this realm has been particularly slowed by the inevitable gauge dependence of the generally rather complex classical gluon field configurations on which most of the existing proposals are based. Similarly, approaches which reformulate non-Abelian gauge theory in terms of gauge-independent loop variables \cite{13} or resolve the gauge constraints explicitly (e.g. in Coulomb gauge \cite{14}), are often technically too involved for direct practical applications.

In the present paper, we circumvent such complications by developing an approach in which not the contributions of single gauge fields but rather those of gauge invariant classes are treated jointly. These classes gather contributions from dominant gauge field orbits to low-energy Yang-Mills amplitudes and thus represent collective gluonic IR degrees of freedom. Technically, they are the saddle points of a soft mode action for gauge invariant matrix fields and therefore also provide the principal input for a systematic saddle point expansion of soft Yang-Mills amplitudes.

Manifest gauge invariance is maintained throughout all calculations by working in the Hamiltonian formulation of Yang-Mills theory in the “coordinate” Schrödinger representation and by making use of explicit gauge projection operators. The Schrödinger picture is adopted mainly because it restricts gauge transformations to a fixed reference time, thereby effectively decoupling them from the dynamical time evolution, and because it often renders the impact of topological gluon properties particularly transparent (even without recourse to the semi-classical approximation) \cite{15}. For the reasons already alluded to, we will focus on gluonic effects and work in pure gauge theory without quarks.

The individual IR degrees of freedom, i.e. the solutions of the saddle point equations, turn out to comprise a diverse range of specific features. Additionally, they have several important properties in common, including stability against scale transformations (an indispensable prerequisite for the saddle point expansion which is ensured by a
virial theorem) and characteristic topological properties inherited from the gauge group. The topology will turn out to be particularly useful for establishing relations between specific saddle-point families and the instanton and meron solutions of the classical Yang-Mills equation.

Our approach maintains explicitly traceable links between the soft collective fields and the underlying gluon fields. The resulting IR dynamics is at an intermediate level of complexity, somewhere inbetween the microscopic theory itself and effective theories (e.g. for Polyakov loops) which just share the symmetries of the fundamental dynamics while coupling parameters have to be fitted to experimental (or lattice) data. Our soft-mode Lagrangian, in contrast, follows uniquely from the adopted vacuum wave functional and combines a reasonable amount of transparency with accessibility to essentially analytical treatment.

The paper is organized as follows: in Sec. II we recapitulate the definition of the vacuum overlap amplitude in the Schrödinger picture. We then implement a gauge-projected vacuum wave functional on the basis of the Gaussian approximation and rewrite the overlap in terms of a bare action which had previously emerged in a variational context. In Sec. III we take advantage of known 1-loop results to integrate out the hard-mode contributions to the bare action perturbatively. By means of a controlled derivative expansion, the renormalized soft-mode action density is then transformed it into a local Lagrangian which lends itself to direct analytical treatment. In Sec. IV we build on these results by establishing the IR-sensitive saddle point expansion for the functional integral over the soft modes and by deriving the saddle point equations whose nontrivial solutions constitute the new gluonic IR degrees of freedom.

Important generic properties of these IR variables are established in Sec. V including their scale stability due to a virial theorem, three topological quantum numbers and a lower bound of Bogomol’nyi type on their action. In Sec. VI, several classes of the more symmetric and most important saddle point solutions are found explicitly. They comprise topological soliton solutions of hedgehog type, which are related to classical solutions of the Yang-Mills equation, and solutions which carry different types of topological information and seem to have no obvious counterparts in classical Yang-Mills theory. One of the most interesting solution classes consists of solitonic links, twisted links and knots. Those emerge from a generalization of Faddeev-Niemi theory which turns out to be embedded in our soft-mode Lagrangian. In Sec. VII we classify all hedgehog soliton solutions, find their most important representatives explicitly, and establish the role of the regular solutions as mainly summarizing contributions from instanton and meron gauge orbits to the vacuum overlap. In Sec. VIII finally, we collect our principal results, comment on evaluating the contributions of the gluonic IR degrees of freedom to relevant amplitudes, and suggest directions for future work.

II. VACUUM OVERLAP AMPLITUDE

The vacuum overlap amplitude of SU (N) Yang-Mills theory (without matter fields) in the Schrödinger ”coordinate” representation \cite{13} reads

\[ Z' := \langle 0, t_+ | 0, t_- \rangle = \int \mathcal{D}\vec{A} \Psi'_0 \left( \vec{A}, t_+ \right) \Psi_0 \left( \vec{A}, t_- \right). \tag{1} \]

The vacuum wave functional (VWF) \( \Psi_0 \) depends on half of the canonical variables, i.e. on the static gauge fields \( \vec{A}(\vec{x}) \). Its gauge invariance, like that of any other physical state and wave functional, is dictated by Gauß' law. This crucial requirement can be imposed on a given functional by simply projecting out its gauge-singlet component, i.e. by integration over the (compact) gauge group \( \text{SU}(N) \). Starting from an approximate and therefore generally gauge dependent wave functional \( \psi_0 \), one then obtains the associated VWF

\[ \Psi_0 \left( \vec{A} \right) = \sum_n e^{iQ\theta} \int d\mu \left( U^{(Q)} \right) \psi_0 \left( A^{U^{(Q)}} \right) =: \int DU \psi_0 \left( A^U \right) \tag{2} \]

(\( d\mu \) is the invariant Haar measure of the gauge group, \( Q \) is the homotopy degree or winding number of the group element \( U^{(Q)} \), and \( \theta \) is the vacuum angle) for which Gauß' law is manifest. The vacuum energy has been set to zero.

After interchanging the order of integration over gauge fields and gauge group in Eq. (1), it becomes obvious that a gauge group volume can be factored out of \( Z' \), i.e.

\[ Z' = \int DU_+ \int DU_- \int D\vec{A} \Psi_0^{\ast} \left[ A^{U_+} \right] \psi_0 \left[ A^{U_-} \right] =: Z \int DU_- , \tag{3} \]

since the \( \vec{A} \)-integral is gauge invariant. In fact, the group volume \( Z' / Z \) is left over when the two un-normalized gauge projectors in the matrix element \( Z' \) are multiplied into one. The integrand of the remaining integral over the gauge group is naturally rewritten as a Boltzmann factor, i.e.

\[ Z = \int DU \exp \left( -\Gamma_b \left[ U \right] \right), \tag{4} \]
which defines the 3-dimensional Euclidean bare action $\Gamma_b$ as a functional of the “relative” gauge orientation $U \equiv U^{-1} U_+$ only. Owing to the gauge invariance of the gluon field measure, $\Gamma_b$ is gauge invariant as well and takes the explicit form

$$\Gamma_b[U] = -\ln \int D\bar{A}\psi_0^* \left[ A^U \right] \psi_0 \left[ \bar{A} \right].$$

This action describes the dynamical correlations which the gauge projection of the functional $\psi_0$ in Eq. 2 has generated. Hence it would become trivial, i.e. $U$-independent, if $\psi_0$ were gauge invariant by itself. More specifically, $\Gamma_b[U]$ gathers all those contributions to $Z$ whose approximate vacua $\psi_0$ at $t = \pm \infty$ differ by the relative gauge orientation $U$. The variable $U$ thus represents the contributions of a specifically weighted ensemble of all gluon field orbits to the vacuum overlap and is gauge invariant by construction.

To proceed in an analytically tractable fashion, we now adopt the standard Gaussian approximation

$$\psi_0^{(G)}[\bar{A}] = \exp \left[ -\frac{1}{2} \int d^3 x \int d^3 y A^a_b(\bar{x}) G^{-1ab}(\bar{x} - \bar{y}) A^b_i(\bar{y}) \right]$$

for the unprojected VWF [55], which has the decisive advantage of allowing integrals over $\bar{A}$ to be done exactly. As expected from a ground state wave functional, $\psi_0^{(G)}$ has no nodes. It describes a “squeezed” state, i.e. an oscillator-type extension of the unstable coherent gluon states [18] and thus the simplest natural candidate for the vacuum functional. In fact, Eq. 6 turns into the exact ground state for $U(1)$ gauge theory (up to color factors) if the “covariance” $G^{-1}$ is taken to be the inverse of the static vector propagator. Several additional properties indicate that Gaussian VWFs with suitably adapted covariances capture crucial features of the Yang-Mills dynamics as well. Indeed, with an appropriate choice for $G^{-1}$ (see below) the wave functional [60] becomes exact at high momenta and incorporates asymptotic freedom. Moreover, it is known from variational analyses that Gaussian VWFs generate a dynamical mass gap and possibly confinement [19, 20]. (Mass generation and most other features of 2+1 dimensional compact photodynamics are also reproduced [21].) Additional support for the Gaussian approximation will emerge from our results below.

After specializing the expression [59] for the bare action to $\psi_0^{(G)}$, the functional integral over the gluon fields becomes Gaussian and can readily be carried out. The result takes the form of a 3-dimensional, bilocal nonlinear sigma model [22],

$$\Gamma_b[U] = \frac{1}{2g_b^2} \int d^3 x \int d^3 y L^a_i(\bar{x}) L^b_i(\bar{y}) \delta^{ab}.$$

(Above we have omitted a term of higher order in the small bare coupling $g_b$ which vanishes at the saddle points in which we will be interested below.) The $U$-dependence enters $\Gamma_b$ both via the one-forms

$$L^a_i(\bar{x}) = U^\dagger(\bar{x}) \partial_i U(\bar{x}) =: L^a_i(\bar{x}) \frac{x^a}{2i},$$

i.e. the Lie-algebra valued, left-invariant Maurer-Cartan “currents” (with real components $L^a_i$), and through higher-order corrections to the bilocal operator

$$D^{ab} = \left[ (G + G^U)^{-1} \right]^{ab} \approx \frac{1}{2} G^{-1} \delta^{ab} + ...$$

where $G^U = G^{ab}(\bar{x} - \bar{y}) U^\dagger(\bar{x}) T^a U(\bar{y}) T^b U^\dagger(\bar{y})$, $T^a = \lambda^a/2$ and $G^{-1ab} = G^{-1} \delta^{ab}$. The above reformulation of the Yang-Mills vacuum overlap on the basis of a gauge-invariant Gaussian VWF was employed in Ref. 22 as the starting point for a variational approach [57]. Alternatively, it can be obtained from a saddle point evaluation of the functional integral [22] in Eq. 5 which becomes exact for the Gaussian VWF [57].

Although the nonlinear sigma model [60] is easier to handle than the original Yang-Mills theory, its exact non-perturbative treatment remains beyond analytical reach [24]. Nevertheless, the parametric enhancement of the action [4] by the factor $g_b^2$ suggests that a useful approximation may be obtained from a saddle point expansion of the functional integral [4]. In order to render this approximation practical, however, one has to deal with the nonlocality of the bare action [7] which encumbers the identification and evaluation of the saddle points. We will show in the following section that this can be efficiently accomplished by combining a renormalization group evolution of the bare action (which removes the explicit UV modes) with a subsequent derivative expansion to transform the IR dynamics into an approximately local soft-mode action.
III. SOFT GRADIENT EXPANSION

For the reasons outlined in the introduction, we are mainly interested in soft Yang-Mills amplitudes with external momenta \(|\vec{p}_i|\) smaller than a typical hadronic scale \(\mu\) (the lowest glueball mass, for example). This restricted focus permits us to recast the bare action \(\mathcal{S}\) into a form which only retains soft field modes explicitly and which can be systematically approximated by a local Lagrangian. The present section describes the derivation and some useful features of this soft-mode action.

Since the action \(\mathcal{S}\) incorporates asymptotic freedom (for proper choices of \(G\), see below), the bare coupling \(g_b\) is small at the large cutoff scale \(\Lambda_{\text{UV}}\) where the theory is originally defined. The hard modes of the \(U\) field with momenta \(|\vec{k}| > \mu\) can therefore be integrated out of the functional integral \(\mathcal{Z}\) perturbatively, down to values of the infrared scale \(\mu\) where the renormalized coupling \(g(\mu)\) ceases to be much smaller than unity. In practice, this may be done for instance by Wilson’s momentum-shell technique \([22]\), after factorizing \(U\) into contributions from high- and low-frequency modes. To one-loop order, the resulting renormalization of the action just amounts to the replacement of the bare coupling \(g_b\) by the running coupling \(g(\mu)\). This was confirmed in Ref. \([20]\) where the one-loop coupling was obtained as

\[
g(\mu) = g_b + \frac{g_b^3 N_c}{(2\pi)^3} \ln \frac{\Lambda_{\text{UV}}}{\mu} + O(g_b^5) \tag{10}\]

(for \(G(k) = k^{-1}\) at \(k > \mu\)). The scaling behavior of \(g(\mu)\) makes asymptotic freedom explicit and reaffirms that the Gaussian VWF reproduces the qualitative UV behavior of Yang-Mills theory. In fact, Eq. (10) equals the one-loop Yang-Mills coupling up to a small correction factor \(1/11\) which arises from the absence of transverse gluons and could be avoided by introducing an anisotropic component for \(G^{-1}\) \([23]\). The one-loop integration over the high-momentum modes was found to be reliable down to \(\mu \approx 1.3\) GeV \([20]\), which provides a useful benchmark for numerical estimates. In the valid range of \(\mu\) values Eq. (7) turns into the renormalized soft-mode action

\[
\Gamma[U_\mathcal{S}] = \frac{1}{4g^2(\mu)} \int d^3 x \int d^3 y L_{\mathcal{S},\mathcal{S}}(\vec{x}) G^{-1}(\vec{x} - \vec{y}) L_{\mathcal{S},\mathcal{S}}(\vec{y}) \tag{11}\]

where the subscript “\(\mathcal{S}\)” indicates that \(U\) contains only \(\mu\) modes.

The action (11) is still nonlocal. However, this nonlocality is substantially weaker than in the bare action \(\mathcal{S}\) since the soft \(U\) fields in the integrand vary too little to resolve details of \(G^{-1}\) over distances smaller than \(\mu^{-1}\). This observation can be turned into a controlled, local approximation scheme for the soft-mode action (11) by exploiting the fact that the gradients of \(U_\mathcal{S}\) are bounded by the IR gluon mass scale,

\[
\left| \partial U_\mathcal{S} \right| < \mu. \tag{12}\]

Indeed, this bound suggests to expand the nonlocality of \(G^{-1}\) into derivatives \(\partial/\mu\) which will act upon \(U_\mathcal{S}\) after partial integration. Using the isotropy of \(G_{ij}^{-1} = G^{-1} \delta_{ij}\) (as mentioned above, an anisotropic component could be allowed in principle and would lead to somewhat more general expressions), one has

\[
G^{-1}(\vec{x} - \vec{y}) = \mu \left[ c_0 + c_1 \frac{\partial^2}{\mu^2} + c_2 \left( \frac{\partial^2}{\mu^2} \right)^2 + c_3 \left( \frac{\partial^2}{\mu^2} \right)^3 + \ldots \right] \delta^3(\vec{x} - \vec{y}). \tag{13}\]

The dimensionless constants \(c_i\) encode the low-momentum behavior of \(G^{-1}\) and could, e.g., be determined variationally. For our present purposes, however, it will be sufficient to adopt the standard expression \(G^{-1}(\vec{k}) = \sqrt{\vec{k}^2 + \mu^2}\) which approximates the solution of Schwinger-Dyson equations and variational estimates \([19, 20]\) and incorporates both asymptotic freedom and a dynamical mass gap. The corresponding \(c_i\) can be read off directly from the Fourier transform

\[
G^{-1}(\vec{x} - \vec{y}) = \sqrt{-\vec{\partial}^2 + \mu^2} \delta^3(\vec{x} - \vec{y}) = -\frac{1}{2\pi^2} \frac{\mu^2 K_2(\mu|x-y|)}{|x-y|^2} \tag{14}\]

\((K_2\) is a McDonald function \([27]\), i.e. \(c_0 = 1, c_1 = -1/2, c_2 = -1/8\) etc.

The combination of the above results leads to the intended reformulation of the nonlocal dynamics (7). As anticipated, the bilocal action density for the soft modes in Eq. (11) becomes a (“quasi”-) local Lagrangian \(\mathcal{L}(\vec{x})\) and the action takes the familiar form

\[
\Gamma[U_\mathcal{S}] = \int d^3 x \mathcal{L}(\vec{x}). \tag{15}\]
The Lagrangian is an expansion into powers of $\mu^{-1} \partial U_<$ and therefore belongs to the class of generalized nonlinear sigma models. When expressed in terms of the Cartan-Maurer currents $L_{<,i}$, it reads

$$\mathcal{L}(\vec{x}) = -\frac{\mu}{2g^2(\mu)} \text{tr} \left\{ L_{<,i}(\vec{x}) L_{<,i}(\vec{x}) + \frac{1}{2g^2(\mu)} \partial_i L_{<,j}(\vec{x}) \partial_i L_{<,j}(\vec{x}) - \frac{1}{8g^4(\mu)} \partial^2 L_{<,i}(\vec{x}) \partial^2 L_{<,i}(\vec{x}) + \ldots \right\}.$$  \hspace{1cm} (16)

We have omitted total derivatives $\Delta \mathcal{L}$ from the higher-order terms of $\mathcal{L}$ since they do not affect the field equations. Nevertheless, they may generate non-vanishing surface terms due to infinite-action configurations which are generally irrelevant for the saddle point expansion. To lowest order,

$$\Delta \mathcal{L}(\vec{x}) = \frac{1}{8g^2(\mu)^2} \text{tr} \left\{ \partial^2 [L_{<,i}(\vec{x}) L_{<,i}(\vec{x})] + \ldots \right\}.$$  \hspace{1cm} (17)

The gradient expansion in Eq. (16) is controlled by increasing powers of the parametrically small $(\partial U_</\mu)^2$. For practical purposes it can therefore be reliably truncated, at an order which is determined by the desired accuracy of the approximation to the exact action. Below, we will be interested in specific field configurations (saddle points) which are generally sufficient approximation (at the few percent level) to the full action (11).

The first term in the Lagrangian (16) has the standard form of a 3-dimensional nonlinear $\sigma$-model (or principal chiral model). The second one, with four derivatives acting on $U_<$, is reminiscent of a similar term in the Skyrme model. However, the identity

$$\partial_i L_j \partial_j L_i = \frac{1}{2} [L_i, L_j]^2 + \partial_i L_j \partial_j L_i$$  \hspace{1cm} (18)

shows that the four-derivative contribution to Eq. (16) contains, besides the commutator or Skyrme term, a part which qualitatively alters the character of the Euler-Lagrange equations. While the commutator generates only second-order terms to the field equations (see below), the piece without equivalent in the Skyrme model leads to additional fourth-order terms which allow for new types of solution. (Several families of topological soliton solutions from the Lagrangian (16) will nevertheless turn out to resemble static Skyrmions.)

We end this section by emphasizing that the construction of an analogous gradient expansion in terms of the original gauge fields $A_<$ would require the (residual) gauge freedom to be completely fixed and thus give rise to all the associated conceptual and calculational complications. (Otherwise a “soft” gauge field could just be turned into a “hard” one by a suitable gauge transformation and would spoil the “convergence” of the derivative expansion.) The locality and structural simplicity of the soft-mode Lagrangian (16) can therefore be regarded as a benefit of reformulating the dynamics in terms of the gauge invariant $U$ field variables.

IV. SADDLE POINT EXPANSION

Our next task will be to set up the saddle point (or, more specifically, steepest descent) expansion of the functional integral over the soft modes,

$$Z = \int DU_< \exp \left(-\Gamma [U_<] \right),$$  \hspace{1cm} (19)

where the vacuum overlap $Z$ serves as the prototype for similar integrals in other soft amplitudes. This expansion is based on the IR saddle point fields $\bar{U}_i(\vec{x})$, i.e. the local minima of the soft-mode action (14). The search for these minima is simplified by the fact that all finite-action $U$-field configurations, including most saddle points, fall into disjoint topological classes (cf. Sec. V B and below). Since fields which carry different topological charges - for now summarily denoted by $Q$ - cannot be continuously deformed into each other, the local variation of the action may be performed in each topological sector separately. This amounts to solving the saddle point equations

$$\frac{\delta \Gamma [U_<]}{\delta U_< (\vec{x})} \bigg|_{U_<=\bar{U}_i^{(Q)}} = 0$$  \hspace{1cm} (20)

at fixed $Q$. 
To leading order, the saddle-point expansion for the vacuum overlap \( Z \) can then be assembled by summing the contributions from the solutions \( \bar{U}_i^{(Q)} \) of Eq. (20), i.e.

\[
Z \simeq \sum_{Q \in \mathbb{Z}_i} F_i \left[ \bar{U}_i^{(Q)} \right] \exp \left( -\Gamma \left[ \bar{U}_i^{(Q)} \right] \right),
\]

(21)

where the pre-exponential factors \( F_i \) are generated by zero-mode contributions which typically arise if continuous symmetries of the action are broken by the solutions. The sum over the saddle points, labeled by \( i \), symbolically includes integrals with the appropriate measure when the saddle points come in continuous families. Functional integrals for more complex amplitudes, including the gluonic Green functions, receive contributions from the same saddle points and are obtained by differentiating \( Z \) with respect to suitably implemented sources. Higher-order corrections to the approximation (21) can be systematically calculated from the (nondegenerate) fluctuations around the solutions \( \bar{U}_i^{(Q)} \). The reliability of the leading-order approximation (21) increases with the action values of the saddle points because \( \Gamma [\bar{U}] \gg 1 \) prevents the saddle point contributions from being rendered insignificant by the fluctuations of \( O(1) \) around them. In our case, compliance with this criterion is reinforced by the overall factor \( g^{-2}(\mu) \gg 1 \) (for \( \mu \geq 1.3 - 1.5 \text{ GeV} \)) in the Lagrangian (10), which parametrically enhances the action.

For the explicit solution of the saddle-point equation (20), as well as for part of our general analysis below, it is practically necessary to adopt a parametrization of the \( SU(N) \) group elements \( U \) which allows to work directly with their \( N^2 - 1 \) independent degrees of freedom. We will use the exponential representation

\[
U(x) = \exp \left[ \phi(x) \hat{n}^{a}(x) \frac{\tau^a}{2i} \right]
\]

(22)

for this purpose, where the coefficient vector of the Lie algebra generators is decomposed into its direction, specified by the vector field \( \hat{n}^{a} \) with \( \hat{n}^{a} \hat{n}^{a} = 1 \) (which parametrizes the coset \( SU(N)/H \) where \( H \) is the Cartan subgroup of the gauge group), and its length \( \phi \). For simplicity, we will also specialize our following discussion to \( N = 2 \). The soft-mode Lagrangian (10) can then be rewritten in terms of the unit vector field \( \hat{n}^{a} \) and the spin-0 field \( \phi \) as

\[
\mathcal{L}(U) = \mathcal{L}_{2d}(\phi, \hat{n}) + \mathcal{L}_{4d}(\phi, \hat{n})
\]

(23)

where, for the reasons discussed in the paragraph below Eq. (17), all terms of the gradient expansion with up to four derivatives on the \( U \) fields are retained. The two-derivative part \( \mathcal{L}_{2d} \), i.e. the standard nonlinear \( \sigma \)-model, becomes

\[
\mathcal{L}_{2d} = \frac{\mu}{2g^2(\mu)} \left[ (\partial \phi)^2 + 2(1 - \cos \phi) (\partial_i \hat{n}^{a})^2 \right]
\]

(24)

while the four-derivative contributions turn into

\[
\mathcal{L}_{4d} = \frac{1}{2^3 g^2(\mu) \mu} \left[ (\partial^2 \phi - \sin \phi (\partial_i \hat{n}^{a})^2)^2 + 2(1 + \cos \phi) (\partial_i \phi \partial_i \hat{n}^{a})^2 + 2(1 - \cos \phi) (\varepsilon_{ijk} \partial_j \phi \partial_k \hat{n}^{a})^2
\]

\[
+ 4 \sin \phi \partial_i \phi \partial_j \hat{n}^{a} \partial^2 \hat{n}^{b} + 2(1 - \cos \phi) (\varepsilon^{abc} \hat{n}^{b} \partial^2 \hat{n}^{c})^2 + 2(1 - \cos \phi) (\varepsilon^{abc} \partial_i \hat{n}^{b} \partial_j \hat{n}^{c})^2 \right].
\]

(25)

The general expressions above show that \( \mathcal{L} \geq 0 \), as required for the existence of the functional integral (19). The same remains true if higher orders of the derivative expansion (10) are included (as long as \( \partial U_{\perp}/\mu < 1 \), of course). For the analysis of some generic saddle point solution properties, including their behavior under scale transformations (cf. Sec. IV), it will prove useful that both \( \mathcal{L}_{2d} \) and \( \mathcal{L}_{4d} \) are even individually nonnegative.

By varying the action (15) of the Lagrangian (23) with respect to \( \phi \) and \( \hat{n}^{a} \), the saddle point equation (20) turns into a nonlinear system of four coupled partial differential equations. For \( \phi \) one directly obtains

\[
\left( 4g^2 \mu \right) \frac{\delta \Gamma [\phi, \hat{n}]}{\delta (x)} = \partial^4 \phi - 2 \partial^2 \phi (\partial_i \hat{n}^{a})^2 - 4 \partial_i \phi \partial_i \partial_i \hat{n}^{a} \partial_i \hat{n}^{a} - 4 \cos \phi \left( \partial_i \phi \partial_i \phi \partial_i \hat{n}^{a} \partial_i \hat{n}^{a} + \partial_i \phi \partial_i \phi \partial_i \hat{n}^{a} \partial_i \hat{n}^{a} \right) \partial_j \hat{n}^{a}
\]

\[
+ 2 \sin \phi \left[ (\partial_i \phi \partial_i \hat{n}^{a})^2 - (1 - \cos \phi) (\partial_i \hat{n}^{a} \partial_i \hat{n}^{a})^2 - 2 \partial_i \hat{n}^{a} \partial_j \hat{n}^{a} \partial_i \hat{n}^{b} - (\partial_i \hat{n}^{a} \partial_i \hat{n}^{a})^2 + (\partial_i \hat{n}^{a} \partial_j \hat{n}^{a})^2 \right]
\]

\[
- \sin \phi \cos \phi \left( \partial_i \hat{n}^{a} \partial_j \hat{n}^{b} \right) \left( \partial_j \hat{n}^{b} \right)^2 - \sin \phi \left( \delta^{ab} + \hat{n}^{a} \hat{n}^{b} \right) \partial^2 \hat{n}^{a} \partial^2 \hat{n}^{b} - 2 \mu^2 \left[ \partial^2 \phi - \sin \phi (\partial_i \hat{n}^{a})^2 \right] = 0.
\]

(26)

The three equations for the components of the \( \hat{n} \) field, on the other hand, have to be derived by a constrained variation whose task it is to preserve the unit length of \( \hat{n} \). As a consequence, they can be cast into the succinct form

\[
\left( \delta^{ab} - \hat{n}^{a} \hat{n}^{b} \right) \frac{\delta \Gamma [\phi, \hat{n}]}{\delta \hat{n}^{a}(x)} = 0
\]

(27)
where the projection operator ensures that the action is only affected by those variations $\delta \hat{n}^a$ which maintain orthogonality to $\hat{n}^a$. The evaluation of the functional derivative with respect to $\hat{n}$ yields

$$
(2g^2 \mu) \frac{\delta \Gamma [\phi, \hat{n}]}{\delta \hat{n}^a (\bar{x})} = -(1 - \cos \phi) (\partial^2 \hat{n}^a \hat{n}^c \partial^2 \hat{n}^c) + \partial_i \{ \sin \phi \partial^2 \phi \partial_i \hat{n}^a - \sin \phi \partial_i \phi \partial^2 \hat{n}^a - 2 \cos \phi \partial_i \phi \partial_j \phi \partial_j \hat{n}^a \\
+ \cos \phi \partial_i \phi \partial_k \phi \partial_k \hat{n}^a + \sin \phi \partial_i \phi \partial_k \phi \partial_k \hat{n}^a + \sin \phi \partial_i \phi \left( \partial^2 \hat{n}^a - \hat{n}^a \hat{n}^c \partial^2 \hat{n}^c \right) \\
+ (1 - \cos \phi) \left[ \partial_i \partial^2 \hat{n}^a - \hat{n}^c \partial_i \hat{n}^a \partial^2 \hat{n}^c - \hat{n}^a \partial_i \hat{n}^c \partial^2 \hat{n}^a - (\partial_j \phi)^2 \partial_i \hat{n}^a - 2 \mu^2 \partial_i \hat{n}^a \right] \\
- \sin^2 \phi \partial_i \hat{n}^a (\partial_i \hat{n}^a)^2 - 2 (1 - \cos \phi)^2 \left[ \partial_i \hat{n}^a (\partial_j \hat{n}^a)^2 - \partial_i \hat{n}^c \partial_j \hat{n}^a \right] \} .
$$

The saddle point equations (26) and (27) are independent of the coupling $g$ because it enters the action only through the overall factor $g^{-2}$. In general, their solutions have to be found numerically. However, we will demonstrate below that several nontrivial analytical solutions exist and that further important solution classes with a rather high degree of symmetry can be obtained by solving substantially simplified field equations.

Moreover, essential qualitative solution properties can be derived without solving the saddle point equations explicitly (cf. Sec. V). Each topological charge sector contains at least one action minimum, for example, i.e. one solution of Eqs. (26) and (27). The requirement of finding and including all of them would render the saddle point expansion practically useless. Fortunately, however, this turns out to be unnecessary. Below we will establish lower bounds on the action of the saddle points which are monotonically increasing functions of their topological charges. Hence contributions from saddle points in high topological charge sectors can generally be ignored.

V. GENERAL PROPERTIES OF THE SADDLE POINT SOLUTIONS

Before actually solving the saddle-point equations (26) and (27) in Secs. VI and VII it will be useful to obtain a few general insights into the topological structure and stability properties of the solutions. This is the objective of the present section.

A. Virial theorem

In order to analyze the scaling behavior of the extended saddle point solutions and to establish the underlying virial theorem, we define the scaled fields

$$
\phi_\lambda := \phi (\lambda \bar{x}) , \quad \hat{n}_\lambda^a := \hat{n}^a (\lambda \bar{x})
$$

for real $\lambda \neq 0$ and note that $\mu$ is the only mass scale in the field equations (26) and (27). This immediately implies that the solutions of the saddle point equations with scaled parameter $\mu \rightarrow \lambda^{-1} \mu$ can be obtained by rescaling the original solutions $\left( \phi, \hat{n} \right)$ to $\left( \phi_\lambda, \hat{n}_\lambda \right)$.

In the following, however, we will keep $\mu$ fixed. The scale-transformed extended solutions then cease to solve the field equations, and this simple observation together with two basic properties of the Lagrangian (23) can be turned into a virial theorem. The first step towards its derivation consists in establishing the relation between the actions of scaled and unscaled fields (as long as they stay finite), which can be read off from the 2- and 4-derivative parts of the Lagrangian (23) separately:

$$
\Gamma (\lambda) := \Gamma [\phi_\lambda, \hat{n}_\lambda] = \Gamma_{2d} (\lambda) + \Gamma_{4d} (\lambda) = \frac{1}{\lambda} \Gamma_{2d} (1) + \lambda \Gamma_{4d} (1) .
$$

The second relevant property of the action based on Eq. (24) is its strict positivity for extended, i.e. not translationally invariant field configurations (cf. Sec. IV),

$$
\Gamma_{2d} (1) , \ \Gamma_{4d} (1) > 0 .
$$

The remaining step is to specialize the fields under consideration to the saddle point solutions. Since those extremize the action under arbitrary small variations - which of course include infinitesimal scale transformations - one immediately has $d \Gamma (\lambda) / d \lambda |_{\lambda=1} = 0$ and consequently the virial theorem

$$
\Gamma_{2d} (1) = \Gamma_{4d} (1) .
$$
Eq. (32) clearly exhibits the crucial role of the four-derivative terms \( \Gamma_{4d} \) in guaranteeing the stability of the saddle point solutions: for \( \Gamma_{4d} = 0 \) the remaining nonlinear \( \sigma \)-model action would be minimized by sending \( \lambda \to \infty \) (cf. Eq. (30)), i.e. by the scale collapse of the solutions which Derrick’s theorem predicts \( \Gamma_{2d} = 0 \). Our truncation of the gradient expansion \( O(\mu^2) \) at \( O((\partial U_c/\mu)^2) \) therefore turns out to be the minimal approximation which can support localized, stable soliton solutions \( \Gamma_{2d} = 0 \). Furthermore,

\[
\frac{d^2\Gamma(\lambda)}{d\lambda^2} \bigg|_{\lambda=1} = 2\Gamma_{2d}(1) \geq 0
\]

implies that the scaling extrema are indeed action minima and that solutions with \( \Gamma_{2d} = 0 \) are points of inflection.

Finally, it is worth emphasizing that the coexistence of terms with different numbers of derivatives in the Lagrangian \( (\partial U_c/\mu)^2 \), and thus ultimately the virial theorem (32) and the existence of stable solutions, is brought about by the mass scale \( \mu \) which reflects the short-wavelength quantum fluctuations integrated out in Sec. III. This situation is reminiscent of the classical instanton solutions to the Yang-Mills equation whose typical size scale must likewise be generated by quantum fluctuations.

To summarize, we have established a virial theorem which ensures that the extended solutions of Eqs. (20) and (21) are stable against scale transformations. In our context this is an indispensable property since unstable solutions would prohibit a useful saddle point expansion. As a side benefit, the virial theorem also provides stringent tests for the numerical solutions of the saddle point equations.

**B. Topology**

As a three-dimensional principal chiral model with stabilizing higher-derivative terms, the soft-mode Lagrangian \( \Gamma_{4d} \) allows for topological soliton solutions. In the present section we discuss three topological invariants or charges which such solutions and more general continuous fields \( U \) may carry.

The most fundamental topological classification arises from the fact that all \( U \)-fields with a finite action \( \Gamma \) based on any truncation of the Lagrangian \( (\partial U_c/\mu)^2 \) have to approach the same constant at \( |\vec{x}| \to \infty \). As a higher-dimensional analog of the stereographic projection, this compactifies their domain to a three-sphere \( S^3 \). All \( U \)’s with finite \( \Gamma \) therefore describe continuous maps from \( S^3 \) into the “topologically active” part of the group manifold. For \( SU(N) \) the latter is the trivially embedded subgroup \( SU(2) \sim S^3 \). The resulting maps \( S^3 \to S^3 \) fall into disjoint homotopy classes, the elements of the third homotopy group \( \pi_3 (S^3) = Z \), which are characterized by a topological degree or charge

\[
Q[U] = \frac{1}{24\pi^2} \int d^3x \varepsilon_{ijk} \text{tr} \left\{ U^\dagger \partial_i U U^\dagger \partial_j U U^\dagger \partial_k U \right\}.
\]

(The integrand can be shown to be a total derivative, as expected for a topological “charge density”. For finite-action fields \( Q \in Z \). In terms of the \((\phi, \hat{n})\) parametrization \( (\phi, \hat{n}) \) for \( U \in SU(2) \), Eq. (34) reduces to

\[
Q[\phi, \hat{n}] = \frac{1}{24\pi^2} \int d^3x (\cos \phi - 1) \varepsilon_{ijk} \varepsilon^{abc} \hat{n}^a \partial_i \hat{n}^b \partial_j \hat{n}^c.
\]

Two additional topological quantum numbers of \( U \) are carried solely by its \( \hat{n} \)-field component. The first is the homotopy degree of the maps \( \hat{n}^a \) from the space boundary \( S^2_\infty \) (where \( |\vec{x}| \to \infty \)) into the unit sphere \( S^2_\infty \) on which \( \hat{n}^a \) takes values. Continuous maps of this sort are classified by the elements of the homotopy group \( \pi_2(S^2) = Z \). (The same topology characterizes the magnetic charge of Wu-Yang monopoles \( \pi_2(S^2) = Z \).) An explicit integral representation of their degree is

\[
q_m[\hat{n}] = \frac{1}{8\pi} \int_{\partial R^3} d\sigma i \varepsilon_{ijk} \varepsilon^{abc} \hat{n}^a \partial_j \hat{n}^b \partial_k \hat{n}^c
\]

where the integral extends over the closed surface \( \partial R^3 \to S^2_\infty \) at \( |\vec{x}| \to \infty \). As expected, \( q_m \) is \( \phi \)-independent.

The third topological invariant owes its existence to the fact that all \( \hat{n} \) fields of finite action are required to approach a constant unit vector at spacial infinity. As above, this requirement compactifies \( R^3 \) into \( S^3 \) and thereby turns \( \hat{n} \) into continuous maps \( S^3 \to S^2_\hat{n} \). Such maps carry a Hopf charge \( q_H \) which labels the elements of the homotopy group \( \pi_3(S^2) = Z \), i.e. the Hopf bundle \( \pi_3(S^2) = Z \). An explicit integral representation for \( q_H \) can be constructed by means
of the local isomorphism between the nonlinear $O(3)$ and $CP(1)$ fields which expresses $\hat{n}^a$ in terms of a complex 2-component field $\chi = (\chi_1, \chi_2)$ with unit modulus $\chi^\dagger \chi = \chi_1^2 \chi_1 + \chi_2^2 \chi_2 = 1$ as $\hat{n}^a = \chi^\dagger \tau^a \chi$. Then one has

$$q_H [\hat{n} (\chi)] = \frac{1}{4\pi^2} \int d^3 x \varepsilon_{ijk} (\chi^\dagger \partial_i \chi) (\partial_j \chi^\dagger \partial_k \chi). \quad (37)$$

(A local integral representation for $q_H$ directly in terms of the $\hat{n}$-field does not exist.) In contrast to the topological charges $Q$ and $q_m$ which are of winding-number type, the Hopf invariant $q_H$ is a linking number. The underlying topological structure enables and classifies the link and knot solutions to be encountered in Sec. VI C.

Finally, we recall that the $U$ field topology - as summarized in the conserved topological quantum numbers $Q$, $q_m$ and $q_H$ - characterizes not only the saddle point solutions but a much larger class of continuous field configurations with finite and in some cases even infinite (see below) action.

C. Bogomol’nyi bound

The distribution of the solutions to Eqs. (26) and (27) over a denumerably infinite set of topological charge sectors allows for the existence of more saddle points than could be handled in practical applications of the expansion (34). Hence additional criteria are required to select the most relevant saddle points in a controlled fashion. Such criteria will be established below, in the form of action bounds which are monotonically increasing functions of the absolute topological charge values. These bounds imply that contributions from saddle points with higher topological quantum numbers to functional integrals are increasingly suppressed by the Boltzmann factor exp $(-\Gamma)$ and can be systematically neglected. In the present section we establish the action bound for fields which carry finite values of $Q$. A similar bound for fields with nonvanishing Hopf charge will be obtained in Sec. VI C.

The lower bound on the action of the Lagrangian (23) for any field $U$ with a well-defined homotopy degree (34) can be derived from the Lie-algebra valued expression

$$M_{ij} = a \partial_i L_j + b \varepsilon_{ijk} L_k \quad (38)$$

where $a, b \in R$ and $L_i$ are the components of the Maurer-Cartan one-form (8). With the help of the Maurer-Cartan identity

$$\partial_i L_j - \partial_j L_i = [L_j, L_i] \quad (39)$$

one obtains for its square

$$M_{ij} M_{ij} = a^2 \partial_i L_j \partial_i L_j - 2ab \varepsilon_{ijk} L_j L_k L_k + 2b^2 L_i L_i \quad (40)$$

which obeys the basic inequality

$$tr \{M_{ij} M_{ij}\} \leq 0. \quad (41)$$

(Recall from Eq. (8) that the $L_i$ are expanded into anti-hermitean generators.)

After specializing the bound (41) to $b = 1/\sqrt{2}$ and $a = \pm 1/ (\sqrt{2} \mu)$ (62), multiplying by $-\mu/ (2g^2)$, integrating over $\vec{x}$ and using the integral representation (34) for $Q$, one arrives at

$$-\frac{\mu}{2g^2 (\mu)} \int d^3 x tr \left\{ L_i L_i + \frac{1}{2\mu^2} \partial_i L_j \partial_i L_j \right\} \geq \mp \frac{12\pi^2}{g^2 (\mu)} Q [U]. \quad (42)$$

The more stringent of these inequalities results from the lower (upper) sign on the right-hand side if $Q > 0$ ($Q < 0$). Their left-hand side amounts to the action which is produced by the first two terms (63) of the Lagrangian (16). Hence the expressions (42) for both signs combine into an inequality of Bogomol’nyi type,

$$\Gamma [U] \geq \frac{12\pi^2}{g^2 (\mu)} |Q [U]|. \quad (43)$$

This is the desired lower bound on the action of any sufficiently smooth $U$ field with well-defined degree $Q$. It is saturated by those fields which obey the Bogomol’nyi-type equation

$$\partial_i L_j = \mp \mu \varepsilon_{ijk} L_k, \quad (44)$$
where the lower (upper) sign again refers to $Q > 0$ ($Q < 0$). The equations (44) can be considered as analogs of the self-(anti)-duality equations of Yang-Mills theory and have the interesting consequence that the rescaled Maurer-Cartan currents $\tilde{L}_i := \mp L_i / (2i\mu)$ of minimal-action fields in any $Q$-sector become generators of the $su(2)$ Lie algebra,

$$\left[ \tilde{L}_i, \tilde{L}_j \right] = i\varepsilon_{ijk} \tilde{L}_k. \quad (45)$$

Translationally invariant solutions (cf. Sec. VI A), for example, have $L_i = 0$ and therefore trivially saturate the bound in the $Q = 0$ sector. It remains to be seen whether nontrivial solutions in sectors with larger $|Q|$ exist as well [64].

The large factor multiplying $|Q|$ in the inequality (43) indicates that contributions from saddle points with higher $|Q|$ are strongly suppressed. In fact, they seem safely negligible in most amplitudes which receive nonvanishing contributions from the $Q = 0$ sector. However, one should keep in mind that even contributions with extremely small “fugacities” can sometimes have an important qualitative impact on the partition function. A case in point are the decisive monopole contributions in the 2+1 dimensional Yang-Mills-Higgs model [35]. The physical interpretation of the $|Q| \neq 0$ solutions and their analogs in Yang-Mills theory will be discussed in Sec. VII.

VI. IMPORTANT SADDLE POINT SOLUTION CLASSES

In the following sections we are going to solve the four saddle point equations (26) and (27) explicitly. As already mentioned, our main focus will be on solutions with a relatively high amount of symmetry since their typically smaller action values enhance their contributions to the saddle point expansion. Besides playing a predominant role in most amplitudes, these solutions can often be obtained either analytically or with moderate numerical effort.

A. Translationally invariant solutions

The simplest solutions of the saddle point equations (26) and (27) are the $\vec{x}$-independent matrices

$$U_c = \exp \left[ \phi_c \hat{n}_a^c \frac{x^a}{2i} \right] = \text{const.} \quad (46)$$

where $\phi_c$ and $\hat{n}_a^c$ are both constant. These solutions form the complete vacuum manifold of the dynamics (23), i.e. the set of all fields which attain the absolute action minimum

$$\Gamma [U_c] = 0. \quad (47)$$

Due to a redundancy in the parametrization (22), the subset of vacua in the center of the gauge group is completely $\hat{n}_a^c$-independent:

$$\phi_{c,k} = 2k\pi, \quad U_{c,k} = (-1)^k. \quad (48)$$

In addition, those are the only vacua which do not break the global $U(2)$ symmetry of the Lagrangian (23) spontaneously. A glance at the integral representation (35) shows that none of the $U_c$ carries topological charges $Q \neq 0$.

B. Constant-$\hat{n}$ solutions

Any constant vector $\hat{n}^a$ solves the saddle point equation (27) identically and reduces the other one, Eq. (26) for $\phi$, to the linear field equation

$$\partial^2 \left( \partial^2 \phi - 2\mu^2 \phi \right) = 0. \quad (49)$$

Obviously, the solutions of this equation constitute families of new saddle points which differ by an additive constant and have degenerate action values. (This trivially ensures periodicity in $\phi$.) Alternatively, Eq. (49) and the action of its solutions can be derived from the reduced Lagrangian (50)

$$L^{(\hat{n}=c)} = \frac{1}{2g^2(\mu) \mu} \left[ \left( \partial^2 \phi \right)^2 + 2\mu^2 \left( \partial_i \phi \right)^2 \right]. \quad (50)$$
The presence of the 4th-order term in Eq. (49) allows for solution types which have no equivalent in Skyrme models. (Indeed, the commutator or Skyrme term (cf. Eq. (18)) alone leads to a Laplace equation for \( \phi \), without a mass scale and with only constant regular finite-action solutions on \( R^3 \).) A glance at the integrals (35) - (37) shows that solutions with constant \( \hat{n} \) do not carry any topological quantum numbers, i.e.

\[
Q^{(\hat{n}=c)} = Q_{m}^{(\hat{n}=c)} = Q_{H}^{(\hat{n}=c)} = 0.
\]

All solutions of the linear field equation (49) can be constructed by standard Green function techniques. The perhaps most straightforward approach is to fold the static Klein-Gordon propagator

\[
\Delta (\vec{x} - \vec{y}; m) = - \int \frac{d^3k}{(2\pi)^3} \frac{e^{i\vec{k}(\vec{x}-\vec{y})}}{k^2 + m^2} = - \frac{1}{4\pi} \frac{e^{-|\vec{x}-\vec{y}|}}{|\vec{x}-\vec{y}|}
\]

with a “scalar potential” \( \Phi \) which is defined both to be a solution of the Laplace equation, \( \partial^2 \Phi = 0 \), and to act as the inhomogeneity of the static Klein-Gordon equation

\[
\partial^2 \phi - 2\mu^2 \phi = \Phi.
\]

The field equation (49) is recovered from Eq. (53) by applying the Laplacian to both sides. Since the potential \( \Phi \) plays the role of a source for the \( \phi \) field, inversion of the Klein-Gordon operator immediately yields the general solution

\[
\phi (\vec{x}) = \int d^3y \Delta (\vec{x} - \vec{y}; \sqrt{2\mu}) \Phi (\vec{y}).
\]

Of course, the regular finite-action solutions form but a small subset of those comprised in Eq. (54).

Spherically symmetric solutions can be obtained more directly by restricting the angular dependence of \( \phi \), i.e. by substituting the ansatz \( \phi (r) \) with \( r := |\vec{x}| \) into the general equation (53) and ignoring for the moment potential singularities at the origin. This yields the radial equation

\[
r \phi'''' + 4 \phi''' - 2\mu^2 (r \phi'' + 2\phi') = 0
\]

(radial derivatives \( d/dr \) are denoted by a prime) whose four linearly independent solutions can be found analytically:

\[
\phi_1 = c, \quad \phi_2 = \frac{1}{\sqrt{2\mu r}}, \quad \phi_3 = \frac{e^{+\sqrt{2\mu r}}}{\sqrt{2\mu r}}, \quad \phi_4 = \frac{e^{-\sqrt{2\mu r}}}{\sqrt{2\mu r}}.
\]

The subset of regular finite-action solutions is therefore of the form

\[
\bar{\phi}^{(\hat{n}=c)} (r) = c_1 + \frac{c_2}{\sqrt{2\mu r}} \left( 1 - e^{-\sqrt{2\mu r}} \right).
\]

The associated potential

\[
\bar{\Phi}^{(\hat{n}=c)} (r) = (\partial^2 - 2\mu^2) \bar{\phi}^{(\hat{n}=c)} (r) = -2\mu^2 \left( c_1 + \frac{c_2}{\sqrt{2\mu r}} \right)
\]

shows that the expression (57) in fact solves a generalization of the homogeneous field equation (19), with an additional delta-function singularity at the origin. Eq. (57) is therefore a solution of Eq. (19) everywhere except at \( \vec{x} = 0 \) and, strictly speaking, one of its Green functions. A representative of this solution class is drawn in Fig. 1.

After insertion into Eq. (55), based on the Lagrangian (23), and use of the virial theorem (32) one finds the solutions (57) to have the action

\[
\Gamma \left[ \phi^{(\hat{n}=c)}, \hat{n} \right] = \frac{2\pi \mu}{g^2 (\mu)} \int_0^\infty dr \left( r \phi^{(\hat{n}=c)} \right)^2 = \frac{\pi}{2} \frac{c_2^2}{\sqrt{2} g^2 (\mu)}.
\]

This action is not subject to topological bounds and reaches the absolute minimum \( \Gamma = 0 \) for \( c_2 = 0 \) where the constant-\( \hat{n} \) solutions turn into the translationally invariant vacua of Sec. VII A (In contrast to the center elements (55), however, the value of \( \phi \) remains unrestricted here.) Due to their partly very small action values, the constant-\( \hat{n} \) solutions may have a strong impact on the saddle point expansion which should be explored in detail by studying their contributions to suitable amplitudes.
C. Faddeev-Niemi type knot solutions

In addition to the translationally invariant saddle points of Sec. VI A there are other and less trivial solutions of the field equations (26) and (27) with constant $\phi$ fields. Among them, the most intriguing class has the general form

$$\phi_{c,2,k} = (2k + 1) \pi, \quad U_{c,2,k} (\vec{x}) = (-1)^k \imath r^a \hat{n}^a (\vec{x}),$$

which satisfies Eq. (60) identically and carries no topological charge $Q$. In fact, the nontrivial topology of $U_{c,2}$ (as that of any other constant-$\phi$ field configuration) has to reside exclusively in the $\vec{x}$-dependence of its $\hat{n}$ field, whose dynamics is governed by the $(k$-independent) equation

$$\varepsilon^{abc} \partial_i \hat{n}^b \left( 2 \mu^2 \partial_i \hat{n}^c + 2 \left( \partial_j \hat{n}^j \right)^2 \partial_i \hat{n}^c - 4 \partial_j \hat{n}^j \partial_i \hat{n}^d \partial_j \hat{n}^d - \partial^2 \partial_i \hat{n}^c \right) = 0.$$

The field equation (61) follows from the general saddle point equation (27) by substituting $\phi_{c,2,k}$ and simultaneously plays the role of a (static) continuity equation for the conserved $O(3)$ current. Alternatively, it can be obtained by directly varying the reduced Lagrangian

$$\mathcal{L}^{(\phi_{c,2})} (\vec{x}) = \frac{\mu}{g^2 (\mu)} \left[ (\partial_i \hat{n}^a)^2 + \frac{1}{\mu^2} \left( \varepsilon^{abc} \partial_i \hat{n}^b \partial_j \hat{n}^c \right)^2 + \frac{1}{2 \mu^2} \left( \varepsilon^{abc} \hat{n}^b \partial^2 \hat{n}^c \right)^2 \right]$$

which follows from Eq. (28) after specialization to $\phi_{c,2,k}$ and reproduces the action (11) for the $U_{c,2,k}$. Remarkably, the Lagrangian (62) is a generalization of the static Skyrme-Faddeev-Niemi (SFN) model [36, 37]

$$\mathcal{L}^{(SFN)} = \frac{1}{2 \lambda^2} (\partial_i \hat{n}^a)^2 + \frac{c^2}{2} \left( \varepsilon^{abc} \partial_i \hat{n}^b \partial_j \hat{n}^c \right)^2.$$

In contrast to the SFN model, however, which was postulated on the basis of qualitative symmetry and renormalization group arguments [36, 37], our Lagrangian (62) follows uniquely from the Yang-Mills dynamics and the Gaussian approximation to the vacuum wave functional. All coefficients are therefore fixed in terms of the IR scale $\mu$ and the coupling $g (\mu)$, i.e. Eq. (62) does contain free parameters.

Particular solutions of equation (61) are $\hat{n}^a = \text{const.}$, which belong to the class of translationally invariant vacua (cf. Sec. VII B), and $\hat{n}^a = \vec{x}^a$ (except at $r = 0$) which is an example from the “hedgehog” solution family whose detailed discussion will be the subject of the following sections. The $\phi_{c,2}$ hedgehog has infinite action (since $U_{c,2,k}$ develops a monopole-type singularity at $\vec{x} = 0$, cf. Sec. VII C) and its Lagrangian reduces exactly to the Faddeev-Niemi form (63). Eq. (61) does probably also have cylindrically symmetric vortex solutions which are analogs of the “baby Skyrminon” solutions [40] in similar models.

The most interesting and many-faceted solution classes of the field equation (61), however, are expected to be twists, linked loops and knots made of closed fluxtubes. Indeed, an intriguing variety of such topological soliton solutions was found numerically for the SFN part of the Lagrangian (62) in Refs. [37, 41]. These solutions generally lack axial symmetry and carry a finite Hopf charge $\hat{q}_H (c^2) \neq 0$. Moreover, their number and complexity increases strongly with the value of $\hat{q}_H (c^2)$. As in the higher-$|Q|$ solution sectors discussed previously, a practically useful saddle point expansion thus requires an effective means for selecting the relevant contributions in a controlled fashion.

As anticipated in Sec. V C such a means can be provided by establishing that the action $\Gamma (\phi_{c,2})$ based on the Lagrangian (62) is bounded from below by a monotonically increasing function of $|\hat{q}_H|$. Actually, this just requires a straightforward adaptation of a known bound on the SFN action [38]. One combines the obvious inequalities

$$\Gamma (\phi_{c,2}, \hat{n}) \geq \frac{\mu}{g^2 (\mu)} \int d^3 x \left[ (\partial_i \hat{n}^a)^2 + \frac{1}{\mu^2} \left( \varepsilon^{abc} \partial_i \hat{n}^b \partial_j \hat{n}^c \right)^2 \right]$$

$$\geq \frac{2}{g^2 (\mu)} \left[ \int d^3 x (\partial_i \hat{n}^a)^2 \right]^{1/2} \left[ \int d^3 x \left( \varepsilon^{abc} \hat{n}^a \partial_i \hat{n}^b \partial_j \hat{n}^c \right)^2 \right]^{1/2}$$

(64, 65)

(the first one holds because the omitted term in the Lagrangian (62) is manifestly non-negative; the second one is a consequence of $(a - b)^2 \geq 0$ for any real $a, b$) with the Sobolev-type inequality [38]

$$|\hat{q}_H| \lesssim \frac{1}{2 \sqrt{2}} \frac{1}{\sqrt{3} \sqrt{4 \pi}} \int d^3 x \sqrt{\varepsilon^{abc} \hat{n}^a \partial_i \hat{n}^b \partial_j \hat{n}^c} \int d^3 x \left( \varepsilon^{abc} \hat{n}^a \partial_i \hat{n}^b \partial_j \hat{n}^c \right)^2$$

and a simple inequality due to Ward [39],

$$\left[ (\partial_i \hat{n}^a)^2 \right]^{1/2} \geq 2 \left( \varepsilon^{abc} \hat{n}^a \partial_i \hat{n}^b \partial_j \hat{n}^c \right)^2,$$

(66, 67)
to end up with the bound
\[
\Gamma [\phi_{c2}, \hat{n}] \geq \frac{2^{9/2} 3^{3/8} \pi^2}{g^2 (\mu)} |q_{H} [\hat{n}]|^{3/4}.
\] (68)

$2^{9/2} 3^{3/8} \pi^2 \simeq 337.17$ This bound is rather rough and could probably be made more stringent by incorporating the 3rd term of the Lagrangian \[62\] and by refining the Sobolev bound \[66\] which is expected to remain valid with about half of the factor on its right-hand side \[39\].

The probably most important lesson of the present section is a new physical interpretation for Faddeev-Niemi-type knot solutions. In our framework, they reemerge as gauge invariant IR degrees of freedom which represent gluon field ensembles with a nonvanishing “collective” Hopf charge in the vacuum overlap and other amplitudes. This new interpretation may actually put the tentative identification of knot solutions with glueballs, advocated as a natural candidate for gluonic IR degrees of freedom in the $Q\sigma$ color direction. Our $\hat{n}$ field, on the other hand, is manifestly gauge invariant \[67\]. Moreover, glueballs are anyhow natural candidates for gluonic IR degrees of freedom in the $Q = 0$ sector, so that their (perhaps partial or indirect) appearance in the saddle point solution spectrum would not be unexpected.

An additional advantage of our new framework for the knot dynamics is that it allows the investigation of potential relations between the solutions of Eq. \[61\] and specific Yang-Mills fields which may play important roles in the vacuum, including e.g. topologically nontrivial pure-gauge fields in a non-linear maximally Abelian gauge \[43\] and center vortices \[12\]. If such relations exist, they could perhaps be qualitatively traced by analytical methods. A full quantitative survey of the knot solution sector, however, will require a devoted numerical effort \[68\].

\section*{D. Hedgehog solutions}

The existence of Skyrmions \[28\] in nonlinear $\sigma$-models with higher-derivative interactions suggests that our field equations \[25\] and \[27\] have topological soliton solutions of “hedgehog” type,
\[
\hat{n}^a (\vec{x}) = \hat{x}^a, \quad \phi (\vec{x}) = \phi^{(hh)} (r)
\] (69)
($\hat{x}^a \equiv \hat{x}/r$, $r \equiv |\vec{x}|$), as well. Their $SU(2)$ “grand spin” symmetry characterizes the invariance of the corresponding $U$ fields under simultaneous spatial and internal rotations and implies a substantial simplification of their dynamics. (For larger gauge groups $SU(N)$ with $N > 2$, the three components of $\hat{x}$ form the part of the $\hat{n}^a$ field which parametrizes the trivially embedded $SU(2)$ subgroup.) In the present section, we discuss general properties of the hedgehog fields and derive their reduced Lagrangian and saddle point equation. In the subsequent sections \[VI E\] and \[VII\] we find the most important solution classes explicitly and determine their physical interpretation.

The principal topological characteristic of the hedgehog configurations \[69\] is their $\pi_3 (S^3)$ winding number $Q$. For fields of the form \[69\], its integral representation \[65\] reduces to
\[
Q \left[ \phi^{(hh)} \right] = \frac{1}{2\pi} \int_0^\infty (\cos \phi^{(hh)})^r \left( \cos \phi^{(hh)} - 1 \right) dr
\]
\[
= \frac{1}{2\pi} \left[ \sin \phi^{(hh)} (\infty) - \sin \phi^{(hh)} (0) + \phi^{(hh)} (0) - \phi^{(hh)} (\infty) \right].
\] (70)

(As expected from a topological invariant, it depends only on the boundary values of the $\phi$ field.) In Sec. \[V\] we established that finite action fields carry integer values of $Q$, and Eq. \[70\] confirms this explicitly. Indeed, the parametrization \[22\] for hedgehog fields \[69\] implies that well-defined $U$ fields necessitate the boundary condition $\phi^{(hh)} (0) = 2k_1 \pi$ and that finite-action fields must additionally satisfy $\phi^{(hh)} (\infty) = 2k_2 \pi$ (see below) where $k_{1,2}$ are integers. Nevertheless, it is instructive to consider the more general boundary conditions
\[
\phi^{(hh)} (0) = n\pi, \quad \phi^{(hh)} (\infty) = m\pi, \quad Q \left[ \phi^{(hh)} \right] = \frac{n - m}{2}
\] (71)
($n, m$ integer) which admit infinite-action fields with half-integer winding numbers as well (for either $m$ or $n$ odd) \[69\]. We will show in Sec. \[VII\] that hedgehog solutions to the saddle point equations \[26\] under the boundary conditions \[71\], both with finite and infinite action, can indeed be found. The hedgehog solutions with $Q = \pm 1$ will be of particular importance since they probably dominate all $Q \neq 0$ contributions to the saddle point expansion. This follows from the bound \[43\] and from Skyrme-model type arguments \[28\] which suggest that the minimal-action solutions in the $|Q| = 1$ sectors are hedgehogs.
In addition, all hedgehog fields carry one unit of a second topological quantum number, the monopole-type charge $q_m \in \pi_2 (S^2)$. This becomes explicit when evaluating the ($\phi$ independent) integral representation \[(72)\) for $q_m$ with $\hat{n}^a = \hat{x}^a$:

$$q_m^{(hh)} := q_m [\hat{x}] = \frac{1}{4\pi} \int d\sigma_i \frac{\hat{x}_i}{r^2} = 1.$$  

Obviously, $q_m^{(hh)}$ is independent of the boundary conditions for $\phi$ and therefore of $Q$. The field with $q_m^{(hh)} = -1$ (the “anti-monopole”) is obtained by replacing $\hat{x}_i$ by $-\hat{x}_i$, which corresponds to $U \rightarrow U^\dagger$ for fixed $\phi$. Eq. \[(73)\) reveals, finally, that the Hopf charge of all hedgehog configurations vanishes.

The dynamics of the hedgehog fields is governed by the soft-mode Lagrangian \[(73)\). Since $\hat{x}$ is an identical solution of the general field equation \[(74)\) for $\hat{n}$, Eq. \[(74)\) can be directly specialized to $\hat{n}^a = \hat{x}^a$. Hence the integration over angles becomes trivial and the hedgehog action turns into

$$\Gamma \left[ \phi^{(hh)}, \hat{x} \right] = \int_0^\infty dr L^{(hh)} (r) \quad \text{(73)}$$  

where $L^{(hh)}$ is a $\phi^{(hh)}$-dependent radial Lagrangian. After substituting the ansatz \[(75)\) into the full Lagrangian \[(76)\), dropping total derivatives, suppressing the superscript of $\phi^{(hh)}$ and again denoting radial derivatives $d/dr$ by a prime, one arrives at the explicit expression

$$L^{(hh)} (r) = \frac{\pi}{g^2 (\mu)} \left[ \frac{1}{2} (r \phi''')^2 + (3 + \mu^2 r^2) (\phi')^2 + 4\mu^2 (1 - \cos \phi) \right]. \quad \text{(74)}$$

All terms in $L^{(hh)}$ are nonnegative. This has the consequence that each of them must vanish individually at any absolute action minimum. The complete set of hedgehog vacua is therefore \[\phi^{(hh)} (\hat{x}_{c,k}, \hat{n}^{(hh)}) = (2k\pi, \hat{x})\] and forms a subset of the translationally invariant center elements \[(77)\). As anticipated, any finite-action solution of the form \[(78)\) has to approach one of these constant minima when $r \rightarrow \infty$. The constant solutions $\phi^{(hh)} = (2k + 1)\pi$, on the other hand, are maxima of the action. A representative of this type was already encountered in Sec. \[VI\](C).

The radial equation for $\phi^{(hh)}$ can be derived by inserting the ansatz \[(79)\) into the general field equation \[(80)\) or, more directly, by varying the radial Lagrangian \[(81)\). Either way, the result is

$$r^2 \phi''' + 4r \phi'' - 2 \left( 2 + \mu^2 r^2 \right) \phi'' - 4\mu^2 r \phi' + 4\mu^2 \sin \phi = 0, \quad \text{(75)}$$

i.e. an ordinary nonlinear differential equation of fourth order and of Fuchsian type. The associated boundary value problem can be solved numerically with rather moderate computer resources. The exploration of the full solution space is aided by two discrete symmetries of Eq. \[(82)\) which imply that any solution $\phi (r)$ gives rise to the additional solutions $-\phi (r)$ and $\phi (r) + 2\pi n$. The former is a consequence of $\Gamma [U] = \Gamma [U^\dagger]$ while the latter simply reflects the periodicity in the angular variable $\phi$.

Not surprisingly, the field equation \[(83)\) comprises the Gribov equation \[(84)\). It consists of the terms proportional to $\mu^2$ which originate from the nonlinear-$\sigma$-model part of the Lagrangian \[(85)\) and dominate when $\mu$ becomes the largest scale and/or when the higher derivatives become small \[(86)\). The analogy between the nonlinear potential term $\propto \sin \phi$ and a one-dimensional pendulum in a gravitational field \[(87)\) often used to characterize the solution spectrum of the Gribov equation, therefore applies to Eq. \[(83)\) as well. The stable (unstable) equilibrium positions of the “pendulum” are $\phi = \pi (\phi = 0)$, modulo a multiple of $2\pi$ which represents additional full turns. It will be shown in Sec. \[VII\] that this analogy suffices to understand the qualitative behavior of all numerical solutions.

### E. Analytical hedgehog solutions by series expansion

The ($\phi, \hat{n}$) parametrization \[(88)\) of the $U$ field implies that regular solutions $\phi^{(hh)}$ of the radial hedgehog equation \[(89)\) approach a multiple of $2\pi$ at the origin. Their small-$r$ behavior can therefore be determined analytically, either by expanding the nonlinearity of Eq. \[(89)\) into powers of small deviations $\delta \phi (r)$ from the constant action minima $\phi^{(hh)}_{c,k} = 2k\pi$ or by expanding the $r$ dependence of the full solution into a Frobenius series. Similarly, finite-action solutions can be obtained for $r \rightarrow \infty$ by asymptotically expanding around the hedgehog vacua. Inside their regions of validity, these expansions provide useful insights into the qualitative behavior of the hedgehog solutions as well as quantitative checks on the numerical solutions to be found in Sec. \[VII\].
We start by deriving the solutions of the linearized hedgehog equation and the corresponding power series expansion around the origin. Inserting the ansatz

\[ \phi (r) = 2k\pi + \delta \phi (r) + O (\delta \phi^2) \]  

(76)

into the radial saddle point equation \([75]\) and retaining only terms up to first order in \(\delta \phi\), one arrives at the fourth-order linear differential equation

\[ r^2 \delta \phi''' + 4r \delta \phi'' - 2 \left( 2 + \mu^2 r^2 \right) \delta \phi'' - 4\mu^2 r \delta \phi' + 4\mu^2 \delta \phi = 0 \]  

(77)

which can be solved analytically by standard techniques. The general solution is a superposition of four linearly independent base solutions \(\delta \phi_i\),

\[ \delta \phi (r) = \sum_{i=1}^{4} \tilde{\epsilon}_i \delta \phi_i (r), \]  

(78)

whose real, dimensionless coefficients \(\tilde{\epsilon}_i\) remain undetermined and have to be specified by imposing initial or boundary conditions. The base solutions \(\{\delta \phi_i\}\) are

\[ \delta \phi_1 = \mu r, \quad \delta \phi_2 = \frac{1}{\mu^2 r^2}, \quad \delta \phi_3 = \left( 1 + \sqrt{2} \mu r \right) e^{-\sqrt{2} \mu r}/\mu^2 r^2, \quad \delta \phi_4 = \left( 1 - \sqrt{2} \mu r \right) e^{\sqrt{2} \mu r}/\mu^2 r^2. \]  

(79)

The requirements of regularity and uniqueness on the solutions at the origin dictate two of the boundary conditions. The first one, \(\delta \phi (0) = 0\), implies \(\phi (0) = 2k\pi\) and thus ensures uniqueness at \(r = 0\) while the second one, \(\delta \phi'' (0) = 0\), is then imposed by the behavior of the base solutions \([79]\). Accordingly, the small-\(r\) behavior of the general regular solution is restricted to

\[ \phi (r) \xrightarrow{r \to 0} 2n\pi + c_1 \mu r + c_2 \mu^3 r^3 + O (\mu^4 r^4) \]  

(80)

where the constants \(c_{1,2}\) are linear combinations of the \(\tilde{\epsilon}_{1,3,4}\) whose values can e.g. be specified by providing initial data for \(\phi' (0)\) and \(\phi'' (0)\). All higher-order coefficients of the expansion are then fixed.

Alternatively, one can obtain the solutions of the full, nonlinear saddle point equation \([76]\) towards \(r \to 0\) by analytical continuation into a Frobenius series. A somewhat tedious calculation yields

\[ \phi (r) = 2n\pi + \phi_1 r + \phi_3 r^3 + \frac{\mu^2}{14} \left( \phi_3 + \frac{\phi_3^3}{30} \right) r^5 + \frac{\mu^2}{2 \cdot 3^3 7} \left[ \mu^2 \left( \phi_3 + \frac{\phi_3^3}{30} \right) + \frac{1}{2} \phi_1^2 \phi_3 - \frac{1}{5!} \phi_3^5 \right] r^7 + O (r^9) \]  

(81)

where the coefficients \(\phi_1\) and \(\phi_3\) are again left to be determined by initial conditions. Even-derivative derivatives of \(\phi\) (or equivalently the coefficients \(\phi_{2k}\)) vanish at \(r = 0\) while those of odd order, \(\phi_{2k+1}\), can be expressed in terms of \(\phi_1\) and \(\phi_3\). A comparison between Eqs. \((81)\) and \((81)\) shows that the solutions of the exact radial equation start to differ from those of the linearized equation at \(O (r^3)\). Hence the series solution \((81)\) permits a more accurate check of the numerical solutions over a larger radial interval.

Analogous expansions around the constant action minima \(\phi_{c,k}^{(hh)} = 2k\pi\) exist asymptotically, i.e. towards \(r \to \infty\), for all finite-action solutions. Infinite-action solutions of Eq. \((75)\), finally, can be linearized around the constant solutions \(\phi_{2,k}^{(hh)} = (2k + 1) \pi\) which they approach at one or both ends of the radial domain. The resulting equation for \(\delta \phi\) differs from Eq. \((74)\) only in the sign of the \(\delta \phi\) term. Its solutions are linear combinations of generalized hypergeometric functions.

**VII. NUMERICAL ANALYSIS AND PHYSICAL INTERPRETATION**

We now turn to the numerical solution of the hedgehog saddle point equation \([75]\). Due to the periodicity in \(\phi\), the considered range of boundary values can be limited without loss of generality to \(\phi (0) \in [0, 2\pi]\). Regularity at the origin then further specifies \(\phi (0) = 2\pi\) and imposes \(\phi'' (0) = 0\) (see Sec. **VII**). The value of the topological charge \(Q\) fixes a third boundary condition,

\[ \phi (\infty) = 2\pi (1 - Q), \]  

(82)

owing to Eq. \([71]\). Hence all regular hedgehog solutions in a given \(Q\)-sector can be found by just varying the value of a fourth boundary condition. In the following, we use the initial slope \(\beta := \phi' (0)\) for this purpose. At the end of the section, we will also find irregular solutions with \(\phi (0) = \pi\).
A. Instanton classes

We begin our exploration of the hedgehog solution space by searching for the regular finite-action solutions of Eq. (75) which, as established in Sec. VII B, carry integer values of $Q$. The numerical analysis shows (and the pendulum analogy in Sec. VII C will explain) that only one solution of this type exists in each $Q$ sector. In the simplest case, $Q = 0$, this is just the translationally invariant vacuum solution $\phi_{bb} = 2\pi$.

For $Q = 1$ we find the prototypical nontrivial hedgehog solution, depicted in Fig. 2. In order to clarify its physical interpretation, we note that it shares the $\tau_3(S^3)$ homotopy classification, encoded in the topological charge $Q[U]$ of the relative gauge orientation $U = U^{-1}U_+$, with the Yang-Mills instanton [45]. Of course, both also share the saddle point property, as the instanton minimizes the classical Euclidean Yang-Mills action in the $Q = 1$ sector. In order to trace their association further, we inspect the relative gauge orientation $U_{I,YM}$ of a Yang-Mills instanton with size $\rho$. It is of hedgehog form as well, and its $\phi$-dependence (in Euclidean) temporal gauge is known to be [46]

$$
\phi_{I,YM}(r) = -\frac{2\pi r}{\sqrt{r^2 + \rho^2}}
$$

in the parametrization [64]. For a direct comparison with our $Q = 1$ solution, we have included $\phi_{I,YM}$ with $\rho = 2\mu^{-1}$ as the dashed curve in Fig. 2 (and adapted it to our periodicity interval convention by adding $2\pi$). The radial dependence of both can be seen to be surprisingly similar. This implies that the dominant contributions from all $Q = 1$ gauge field orbits to the vacuum overlap have a relative gauge orientation close to that of the Yang-Mills instanton and indicates that the $Q = 1$ hedgehog solution primarily summarizes contributions from the instanton orbit. (Of course, one would not expect exact agreement since our solutions contain scale-symmetry breaking quantum corrections [72] and contributions from other $Q = 1$ gauge fields as well.) Accordingly, and generalizing the above findings to multi-instanton solutions, we will refer to the unique regular finite-action solution of Eq. (75) with integer $Q$ as the “$Q$-instanton class”.

Our hedgehog saddle point equation (75) and its instanton class solutions derive from the gradient-expanded soft-mode Lagrangian [10]. It is instructive to compare this approach with a variational estimate of one-instanton contributions to the bare action [5] in Ref. [17]. By approximately minimizing the bare action with one-parameter families of trial functions similar to the instanton profile [84] and using qualitative scaling properties, it was argued in Ref. [17] that radiative corrections can stabilize the instanton size. Our exact saddle point solutions make the dynamical size stabilization manifest. We have already traced the underlying mechanism to the virial theorem [32] which is independent of most specific features of the soft-mode dynamics and thus overcomes the chronic infrared instability of dilute instanton gases [48] in a rather generic way. For $\mu \approx 1.5$ GeV, the size $\rho \approx 2\mu^{-1}$ [72] of the 1-instanton class solution agrees inside errors with the results of instanton liquid model [51] and lattice [51] simulations. It also assures that the two leading terms of the gradient expansion [13] yield a sufficiently accurate approximation to the instanton action (cf. the comments below Eq. (17)).

Our 1-instanton class profile function $\phi_I(r)$ is rather similar to the one found by approximately minimizing the bare action [6] variationally [17]. This indicates that the bulk of the instanton’s physics and size distribution is generated by soft modes, as one would intuitively expect. Our approach therefore provides a well-adapted and efficient framework for the treatment of these and other vacuum fields. In contrast to variational approaches, furthermore, it allows to systematically find all saddle points exactly (including those which are not of hedgehog form). Already in the hedgehog sector, for example, we will find solutions with more complex and unprecedented shapes than Eq. (84). Since there is little guidance for the choice of suitable trial functions in these and other cases, such solutions would be difficult to find variationally.

According to Eq. (17), all monotonic hedgehog solutions with $Q > 0$ ($Q < 0$) have negative (positive) slopes $\beta = \phi' (0)$ at the origin. The anti-instanton class with $Q = -1$, in particular, results from changing the sign of the instanton boundary value, $\beta_I = -\beta_I$, and can be obtained without further calculation: it simply results from the combined action of the two symmetry transformations $\phi \to -\phi$ and $\phi \to \phi + 4\pi$ on the instanton class solution. Hence the $Q = \pm 1$ instanton classes have degenerate action values, precisely as their Yang-Mills counterparts.

Multi-instanton class solutions are characterized by an integer topological charge $Q \geq 2$. The modulus $|\beta_{Q,I}|$ of their (negative) initial slope grows monotonically with $Q$, i.e.

$$
|\beta_{I,Q'}| > |\beta_{I,Q}| \quad \text{for} \quad Q' > Q.
$$

As in the 1-instanton case, the action-degenerate multi-antiinstanton classes with $Q \leq -2$ can be constructed by flipping the sign of the corresponding multi-instanton classes and adding $4\pi$. The initial slopes of the multi-(anti)instanton solutions are therefore related by

$$
\beta_{I,-Q} = -\beta_{I,Q}.
$$
The $|Q| \geq 2$ hedgehog solutions correspond to special arrangements of the underlying Yang-Mills multiantiinstantons. In fact, the relative gauge orientation $U = U^{-1}_+ U_+$ of a multi-instanton configuration is of hedgehog type only if all individual (anti)instantons are centered at the origin. This raises the question whether $|Q| \geq 2$ Yang-Mills instanton solutions with separated individual positions are at least approximately represented by other solutions of the saddle point equations \[\tag{26}\] and \[\tag{27}\]. Experience from Skyrme-type models, whose analogous $|Q| \geq 2$ Skyrmion solutions are well approximated by rational \[\tag{52}\] or harmonic \[\tag{53}\] maps, might suggest that similar types of non-hedgehog field configurations approximate higher-$Q$ solutions of Eqs. \[\tag{26}\] and \[\tag{27}\] as well. The action of the 1-instanton class solution is large ($\Gamma_1 \sim 220/g^2$ at $\mu = 1$ GeV), in analogy with the large action of typical Yang-Mills instantons, and its direct impact on the saddle point expansion is therefore small \[\tag{74}\]. Moreover, the action bound \[\tag{45}\] implies that higher-$Q$ instanton classes should be irrelevant for most amplitudes, with potentially important exceptions as mentioned in Sec. \[\tag{22}\]. Instanton “liquid” vacuum models (ILMs) \[\tag{6}\] are built on the same premise and suggest that physically far more relevant contributions originate instead from ensembles of instantons and anti-instantons with equal average densities. It would be important to determine whether contributions of this sort are approximately represented by nontrivial $Q = 0$ solutions of Eqs. \[\tag{26}\] and \[\tag{27}\] as well. In any case, our above results imply that they cannot be of hedgehog type.

**B. Meron classes**

We now extend our search to hedgehog solutions with infinite action. Although their relevance for the saddle point expansion is not obvious, one might speculate that their infinite-action suppression could be overcome by some additional mechanism (see below). Our chief motivation for discussing them here, however, derives from their association with the infinite-action meron solutions \[\tag{11}\] of the classical (Euclidean) Yang-Mills equation.

Hedgehog solutions with infinite action are far more the rule than the exception. In fact, all regular ($\phi(0) = 2\pi$) solutions of Eq. \[\tag{11}\] with initial slopes $\beta$ inbetween the discrete set of instanton-class values $\beta_{I,Q}$ have infinite action since they approach one of the constant fields $\phi_M(\infty) = (2k + 1)\pi$ towards spacial infinity. The latter carry a nonzero action density (cf. Eq. \[\tag{71}\]) and furthermore imply that the corresponding, asymptotic $U$ fields $U_M(\|x\| \rightarrow \infty)$ remain angle-dependent. Exactly the same behavior characterizes the relative gauge orientations $U = U^{-1}_+ U_+$ of Yang-Mills merons in temporal gauge, which are of hedgehog form as well. Furthermore, Eq. \[\tag{82}\] shows that solutions with $\phi(\infty) = (2k + 1)\pi$ carry half-integer topological charge $Q$, again as the Yang-Mills merons. In analogy with the instanton-class solutions of the previous section, we will therefore call these solutions “$2Q$-meron classes”.

The profile function $\phi_M(r)$ of a typical 1-meron class solution with $Q = 1/2$ is drawn in Fig. 3. A direct comparison with the corresponding profile of the Yang-Mills meron in temporal gauge,

\[
\phi_{M,YM}(r) = \frac{1}{2} \lim_{\rho \to 0} \phi_{I,YM}(r) = -\pi \theta(r)
\]  

(86)

(where $\theta$ is the step function), is complicated by the fact that the classical meron is pointlike while our solutions incorporate quantum effects which break dilatation symmetry and stabilize their size at a finite value. Such effects are expected to smoothen the singularity of the Yang-Mills meron, too, and probably cause our solution $\phi_M(r)$ to become non-monotonic by overshooting in the transition region. We therefore draw the Yang-Mills meron profile in Fig. 3 (dashed curve) with the finite size $\rho = 2\mu$ of the instanton class solutions (and adapt it to our periodicity interval convention).

Our nomenclature for multi-meron classes with $|Q| > 1/2$ includes only solutions with half-integer topological charge because solutions with even “meron number” and correspondingly integer $Q$ coincide with the $Q$-instanton classes. This is expected since the relative gauge orientation $U_{M,YM}$ of the underlying Yang-Mills multi-merons is of hedgehog type only if all individual merons sit on top of each other. Such configurations, when carrying integer overall values of $Q$, coalesce into the corresponding Yang-Mills instantons and those are represented by the $Q$-instanton class solutions in our framework.

The behavior of the multi-meron class solutions with $Q \geq 3/2$ is qualitatively rather similar to that of the 1-meron class, although size and strength scales may differ substantially. As an example, Fig. 4 shows a typical 3-meron class solution. An important general property of all solutions with half-integer $Q$ is that they come in families of continuously varying sizes. As already alluded to, this is because their size depends on a second mass scale $\beta_M(0)$ (in addition to $\mu$) and because solutions for all values of $\beta_M$ in the finite intervals

\[
\beta_{I,Q} > \beta_{M,Q+1/2} > \beta_{I,Q+1}
\]  

(87)

(where $Q \geq 0$, i.e. $\beta \leq 0$, and $\beta_{I,Q=0} = 0$ are implied) can be found. As in the instanton sector, multi-anti-meron classes with negative $Q$ are obtained from the positive-$Q$ solutions by changing their sign and adding $4\pi$. 

\[
\beta_{I,Q} > \beta_{M,Q+1/2} > \beta_{I,Q+1}
\]  

(87)
The variable mass scale $\beta_M$ in the meron sectors implies that there are infinitely more meron than instanton class solutions. This makes it tempting to speculate that the meron “entropy” contributions to the weight function of functional integrals over $U$ might be able to overcome the infinite-action suppression. If so, it would shed new light on the physical interpretation not only of our solutions but also of the Yang-Mills merons themselves, whose potential role remains controversial. Furthermore, it would suggest a modified saddle point expansion in which action and entropy are minimized jointly. These issues deserve further investigation.

C. Singular hedgehog solutions

Above we have classified all regular hedgehog solutions, i.e. those which satisfy the initial condition $\phi(0) = 2\pi$. We are now going to examine the remaining solution classes of the radial field equation (75). Its members share the alternative initial condition $\phi(0) = \pi$, may display a rather complex spacial structure and are characterized by a monopole-type singularity at the origin, i.e. they solve Eq. (74) everywhere except at $\vec{x} = 0$.

In order to understand the qualitative behavior of both regular and irregular hedgehog solution classes from a common perspective, it is useful to elaborate on the analogy between the hedgehog equation (75) and the pendulum equation which was mentioned in Sec. IV. According to this analogy, the instanton classes correspond to exactly $Q$ full turns of the pendulum, where the sign of $Q$ indicates the direction of the rotation. The pendulum mass starts in the unstable equilibrium position at time $t = \ln r = -\infty$ with just enough initial speed $\beta = \phi'(0)$ to finally end up there again for $t = \ln r = +\infty$. This analogy implies, in particular, that there is exactly one regular hedgehog solution for each integer $Q$ and that the constant $\beta = 2\pi$ is the only regular solution with $Q = 0$. The meron class solutions start from the unstable equilibrium position as well. However, their initial velocity $\beta$ is insufficient for completing all turns in full. The last turn remains uncompleted, i.e. the pendulum swings back, oscillates around and finally settles into the stable equilibrium position. Hence all meron class solutions have half-integer $Q$ and are non-monotonic.

For the irregular solutions, on the other hand, the pendulum starts at the stable equilibrium position $\phi = \pi$. When not provided with sufficient initial speed $\beta$ to complete a full turn, it just performs damped oscillations around $\phi = \pi$. The corresponding solution has $Q = 0$ and is depicted in Fig. 5. When $|\beta|$ is sufficiently large, however, the pendulum can perform $Q$ full turns before settling into the stable equilibrium position. As a consequence, all these solutions have integer $Q \neq 0$ and infinite action (cf. Eq. (74)). Pursuing the analogy further, one would also expect irregular solutions which have an initial value $\phi'(0) = 0$ exactly as needed to end up at the unstable equilibrium position when $r \to \infty$ (after possibly completing a number of full turns). Such configurations would carry a half-integer topologically charge $Q$ and a finite action. For obvious reasons they turn out to be highly sensitive to variations of the initial condition, however, and therefore difficult to establish numerically.

In contrast to the instanton and meron classes, the singular hedgehog solutions do not seem to have obvious analogs among the solutions of the classical Yang-Mills equation. As opposed to the instanton (meron) classes, furthermore, the irregular integer-$Q$ (half-integer-$Q$) solutions have infinite (finite) action. This allows for nontrivial hedgehog solutions with $Q = 0$, which are necessarily irregular at the origin. The physical interpretation of all irregular hedgehog solutions and their relevance for the saddle point expansion remain to be clarified.

VIII. SUMMARY AND CONCLUSIONS

The main results of this paper are a practicable saddle point expansion for the Yang-Mills vacuum overlap amplitude in terms of gauge invariant, local matrix fields and the identification of new gluonic IR degrees of freedom in this framework. After adopting a gauge-projected Gaussian approximation to the vacuum wave functional, the IR degrees of freedom can be obtained explicitly as the saddle points of a soft-mode action which gather contributions from dominant gluon field families to soft Yang-Mills amplitudes and thus represent collective properties of the Yang-Mills dynamics. Since their gauge invariant definition makes no reference to specific amplitudes, furthermore, the IR degrees of freedom are universal. They provide both new structural insights into the organization of the low-energy Yang-Mills dynamics and the principal input for a systematic saddle point expansion of soft amplitudes.

Our survey of the saddle point solution space uncovered a diverse spectrum of IR degrees of freedom which carry several topological charges with associated Bogomol'nyi-type action bounds and obey a virial theorem which guarantees their scale stability. Solutions with a relatively high degree of symmetry were obtained either analytically or with modest numerical effort. Since solutions of this type are generally characterized by small action values and hence play a dominant role in the saddle point expansion, we have investigated their properties in some detail. Besides translationally invariant vacua and analytical solutions with a fixed relative gauge orientation, we have found topological solitons of hedgehog, (vortex) link and knot types.
Some of the IR degrees of freedom have a transparent physical interpretation directly in terms of the underlying gluon fields. The contributions from the gauge orbits of Yang-Mills instanton and merons, in particular, are gathered by saddle point fields of hedgehog type which share their (integer or half-integer) topological charge and represent vacuum tunneling processes in the Hamiltonian formulation of non-Abelian gauge theory. Although our saddle point solutions contain quantum effects and potentially relevant contributions from other gauge fields, those in the instanton class turn out to reproduce the relative gauge orientation between the in- and out-vacua of the Yang-Mills instanton rather closely. The finite extent of our meron solution classes, on the other hand, is generated by quantum effects which smoothen the singularities of the classical, pointlike Yang-Mills merons. Nevertheless, our meron classes turn out to share the infinite action of their Yang-Mills counterparts.

Among all those IR fields which carry one unit of topological (instanton) charge, the single (anti-) instanton classes are expected to attain the minimal action value. As a consequence of the action bound, they should therefore dominate the saddle point expansion in all topological charge sectors. Similar configurations emerged in a variational treatment along with qualitative arguments in favor of their size stabilization. In our approach, this stabilization is manifest in the exact instanton class solutions themselves. In fact, their size turns out to be fixed at about twice the inverse IR gluon mass scale and agrees inside errors with instanton liquid model and lattice results. The underlying virial theorem and the soft gluon mass generation therefore provide new insight into the mechanism by which the chronic infrared diseases of dilute Yang-Mills instanton gases are overcome.

The sizes of the meron class solutions with half-integer topological charge turn out to be of a more complex origin. Besides the dynamical gluon mass they depend on a second, variable mass scale which is encoded in a boundary condition. Hence meron classes exist within large and continuous size ranges and consequently form a far more extensive solution family than the instanton classes. This opens up the hypothetical possibility for their entropy to overcome their infinite action suppression in functional integrals. Such a mechanism would not only help to clarify the physical impact of our solutions but also shed new light on the still controversial role of the Yang-Mills merons themselves. In addition to the instanton and meron classes, finally, there exists a third class of hedgehog solutions which contains a monopole-type singularity at the origin. These irregular solutions can carry half-integer and integer (including zero) topological charges as well, and generally have infinite action. Hence their potential physical relevance seems to depend on the existence of additional mechanisms which could both smoothen their singularity and overcome their infinite-action suppression.

Several other remarkable families of IR degrees of freedom turn out to be represented by topological solitons of Faddeev-Niemi type, i.e. by (potentially twisted) links and knots. In fact, our saddle point dynamics contains a specific generalization of the Faddeev-Niemi Lagrangian and shows explicitly how it is embedded in the Gaussian approximation to the Yang-Mills vacuum wave functional. This puts Faddeev-Niemi theory into a new perspective, as the effective dynamics of dominant sets of gauge field orbits with a collective Hopf charge, and provides the underlying unit-vector field with a manifestly gauge invariant meaning. The latter would well accord with the tentative interpretation of knot solutions as glueballs by Faddeev, Niemi and coworkers. This and other interpretations could be tested in our framework by directly evaluating the impact of the knot saddle points on suitable amplitudes, e.g. on glueball correlation functions.

More generally, the saddle point expansion allows the systematic calculation of contributions from all relevant IR degrees of freedom to functional integrals which represent soft Yang-Mills amplitudes. First calculations of this type, focusing on fundamental vacuum properties including gluon condensates and the topological susceptibility, are underway. In addition, our approach makes it possible to analyze the gauge field content of any IR variable individually by applying standard functional techniques to the integrals over gluon fields with which they are associated. Investigations of this sort would not only provide further structural insight into specific IR degrees of freedom and their physical role but may also shed new light on the dynamical mechanisms by which soft gauge fields organize themselves into collective degrees of freedom.

The diverse topological properties of the IR variables demonstrate that the gauge-projected Gaussian wave functional not only captures the homotopy structure of the gauge group but also implements "derivative" topologies which further characterize the saddle point families. Due to the typical robustness of such topological properties, the related results are expected to remain at least qualitatively valid beyond the Gaussian approximation. Moreover, the saddle point expansion engenders the means to test this expectation quantitatively, by mapping out limitations of the underlying vacuum wave functional in comparison with lattice data. Extensions of our framework to suitable supersymmetric gauge theories would even permit analytical tests of this sort, e.g. by tracing vestiges of the monopole-based confinement mechanism in the vacuum functional. The insights gained from such investigations may also provide specific guidance for the development of improved collective-mode actions and consequently generate systematic corrections to the IR variables.

Our IR saddle point expansion can be extended in several directions. A first important task would be a more exhaustive survey of the saddle-point solution space which should encompass potentially relevant approximate solutions. The extension to QCD proper requires the generalization to the gauge group $SU(3)$ and the implementation of
quarks into the Gaussian wave functional, both of which pose no conceptual problems. Most topological properties, in particular, reside in the trivially embedded $SU(2)$ subgroup of the full color group and will remain unchanged. A sufficiently complete treatment of the quark-gluon interactions and their impact on the effective action, however, appears to be more challenging.

Our approach opens up a variety of directions for future research. Besides those already mentioned, it would for example be interesting to explore relations between the gauge-invariant IR degrees of freedom and gauge-dependent gluonic structures (monopoles, vortices, branes etc.) and amplitudes (e.g. the 2-dimensional nonlocal gluon condensate and Green functions) which appear in gauge-fixed formulations. Another useful endeavor would be the calculation of those higher-dimensional vacuum condensates which provide the principal input for the operator product expansion of glueball correlators. Duality sum rules could then link the contributions from different IR degrees of freedom with the low-lying glueball spectrum \[54\], e.g. as a precursor and complement to a direct saddle-point evaluation of glueball correlation functions.

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Figure captions:

Fig. 1: An example of the $\hat{n} = \text{const.}$ solution class. As in all following figures, we display the solution at $\mu = 1 \text{ GeV}$. Other values of $\mu$ can immediately be accommodated by scaling the $r$-axis. Our conventions for the periodicity interval of $\phi$ restrict its initial value to $\phi(0) \in [0, 2\pi]$.

Fig. 2: The canonical 1-instanton class solution with $Q = 1$.

Fig. 3: A typical meron-class solution with $Q = 1/2$.

Fig. 4: A 3-meron class solution with $Q = 3/2$.

Fig. 5: An example for a nontrivial $Q = 0$ hedgehog solution with monopole-type behavior at the origin.
Although this Lagrangian is bilinear in $\phi$, it can only be saturated on a hyperspherical domain $S^3$, for example $S^3_{\mu}$.

The enhancement of the soft-mode action is somewhat weaker than that of the bare action  since $g(\mu) \geq g_0$.

The signs of the higher-order terms in the derivative expansion alternate. One might therefore suspect that their contributions could destabilize the extended solutions. However, this would just indicate an incorrect truncation of the gradient expansion. Indeed, if a localized solution is small enough to be significantly affected by higher-derivative terms, those would have to be added to the field equations in the first place. (A somewhat analogous situation is encountered in one-loop Coleman-Weinberg potentials: their physically relevant local minima are stable only as long as fluctuations remain small, i.e. as long as the underlying truncation of the loop expansion is justified.)

Several other variational and related Schwinger-Dyson equation studies on the basis of the Gaussian trial state have been performed in gauge-fixed formulations.

For our exploratory purposes we do not retain a possible longitudinal contribution to $\mathcal{G}^{-1}$ which was discussed in Ref. [23]. A minimal gauge-invariant extension of the exponent in Eq. (3) characterizes the ground state of Yang-Mills theory in 2+1 dimensions [17].

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solutions contribute as well.

[69] The half-integer winding numbers cannot be associated with the degree of a map. The latter is defined for maps between compact manifolds only, which in our context implies finite action.

[70] In the transition region between the boundary values, however, the $\mu$-independent terms of Eq. (169) are not negligible. This makes stable soliton solutions possible.

[71] In this context, it is interesting to recall that the analogous scale symmetry breaking due to 1-loop corrections around Yang-Mills instantons is unable to stabilize their size distribution.

[72] This analogy becomes more explicit after replacing $r$ with the logarithmic variable $t = \ln (r/r_0)$.

[73] In this context, it is interesting to recall that the analogous scale symmetry breaking due to 1-loop corrections around Yang-Mills instantons is unable to stabilize their size distribution.

[74] Note that the 1-instanton class does not saturate the action bound (15) and hence does not solve the Bogomol'nyi-type equation (14). This is in constrast to the Yang-Mills instanton which is the absolute minimum of the Euclidean Yang-Mills action in the $Q = 1$ sector and therefore self-dual.
\phi^{(n=c)}(r) = \frac{1 - \exp(-2^{1/2} \mu r)}{(2^{1/2} \mu r)}
\[ \phi_I, \text{YM}(r) = 2\pi \left[ 1 - \frac{r}{(r^2 - 4)^{1/2}} \right] \]
\[ \phi_M, YM(r) = \pi \left[ 2 - \frac{r}{(r^2 - 4)^{1/2}} \right] \]
3-Meron-class solution ($Q=3/2$)
Singular hedgehog solution (Q=0)