Exact dynamics of XX central spin models

M A Jivulescu¹,², E Ferraro¹, A Napoli¹ and A Messina¹

¹ CNISM and Dipartimento di Scienze Fisiche ed Astronomiche, Università di Palermo, Italy
² Department of Mathematics, ‘Politehnica’ University of Timișoara, Romania

E-mail: maria.jivulescu@mat.upt.ro and messina@fisica.unipa.it

Received 8 January 2009
Accepted for publication 12 January 2009
Published 31 July 2009

Abstract
The dynamical behavior of a star network of spins, wherein each N decoupled spin interacts with a central spin through non-uniform Heisenberg XX interaction, is exactly studied. The time-dependent Schrödinger equation of the spin system model is solved starting from an arbitrary initial state. The resulting solution is analyzed and briefly discussed.

PACS numbers: 02.30.Ik, 73.21.La, 71.70.Jp, 31.30.Gs

1. Introduction
In the last decades, investigations on the properties of coupled-spin systems have gained increasing interest in the quantum community [1]. Specially, the time evolution of a spin star system, that is a single spin coupled to a surrounding environment composed of a finite number of spins noninteracting [2–4] or interacting [5, 6] among them, has been studied in detail. Central spin models provide an appropriate description of quantum information processes such as, for instance, quantum state transfer [7] and quantum cloning [8].

In this paper, we study in detail the dynamics of an XX central spin model that is composed of a localized spin ½, hereafter called central spin, coupled to an environment of N not interacting spins. The Hamiltonian describing our system is

\[ H = \omega \sum_{j=1}^{N} \sigma_j^z + \omega_0 \sigma_A^z + \sum_{j=1}^{N} \alpha_j (\sigma_A^+ \sigma_j^- + \sigma_A^- \sigma_j^+). \]  

The Pauli operators \( \sigma_A^\pm \) refer to the central spin whereas the others, labelled by the index \( j \), refer to the N environmental spins. The central spin Hamiltonian model (1) can be successfully exploited to effectively describe many physical systems like quantum dots [9], two-dimensional electron gases [10] and optical lattices [11]. The Hamiltonian model given by equation (1) is a realization of the so-called Gaudin model whose diagonalization has been derived in the framework of the Bethe ansatz (BA) [12]. Such an approach provides, however, a rather formal solution whose implications for the dynamics of the spin system have not yet been fully explored. In this paper, we solve exactly the Schrödinger equation of motion of the total system starting from an arbitrary completely factorized initial condition. Our successful treatment is strictly related to the circumstance that the component along the \( z \)-axes of the total angular momentum operator \( S_z = \frac{\sigma_A^z}{2} + \frac{1}{2} \sum_{j=1}^{N} \sigma_j^z \equiv \frac{\sigma_A^z}{2} + J_z \) is a constant of motion. Our main result is the derivation of a closed formula for the time dependence of the probability amplitude of finding our spin system in any given state.

2. Exact dynamics of the XX-central spin model
The goal of this section is to derive the exact dynamics of our spin system starting from a completely factorized initial condition wherein the central spin, as well as \( p = 0, \ldots, N - 1 \) of the N surrounding spins are in their respective up state, whereas the others are prepared in their down state. The case corresponding to \( p = N \) is trivial since the corresponding factorized state is an eigenstate of Hamiltonian (1).

2.1. \( p = 0 \)
The initial condition taken into consideration in this subsection is the following one:

\[ |\psi(0)\rangle = |\uparrow_A\rangle |\downarrow \cdot \cdot \cdot \downarrow\rangle, \]

where only the central spin is in the up state. It is easy to convince oneself [4] that, thanks to the conservation
of $S_z$, state \( \langle \uparrow \rangle \) evolves into the state representable by the following normalized superposition $|\psi(t)\rangle = a(t)|\uparrow\rangle \downarrow \cdots \downarrow + \sum_{j=1}^{N} b_j(t)|\downarrow\rangle \downarrow \cdots \uparrow_j \downarrow$, where

$$a(t) = \cos \left( \sqrt{\frac{\Delta}{\sum_{j=1}^{N} \alpha_j^2 + \Delta^2}} \right) t$$

$$b_j(t) = -i \sqrt{\frac{\Delta}{\sum_{j=1}^{N} \alpha_j^2 + \Delta^2}} \sin \left( \sqrt{\frac{\Delta}{\sum_{j=1}^{N} \alpha_j^2 + \Delta^2}} t \right), \quad (3)$$

with $\Delta = \omega - \omega_0$. We underline that starting from such an initial condition, the time evolution is characterized by only one effective frequency, namely $\alpha_{\text{eff}} = \sqrt{\sum_{j=1}^{N} \alpha_j^2 + \Delta^2}$. Thus the spin system fully restores its initial condition with a period $T = 2\pi/\alpha_{\text{eff}}$ and behaves as if its dynamics were governed by an effective Hamiltonian model like that given by equation (1), where $\alpha_j$ is substituted by $\alpha_{\text{eff}}$, independent of $j$.

2.2. $p = 1, 2, \ldots, N - 1$

Our aim is now to treat the more complicated dynamics of the XX central spin system starting from an arbitrary initial condition of the form

$$|\psi(0)\rangle = |\uparrow\rangle |\downarrow \cdots \uparrow_i \downarrow \cdots \uparrow_{p} \rangle,$$  \hspace{1cm} (5)

where $p$ ($p \neq 0$) of the $N$ uncoupled spins, labelled by $i_1, \ldots, i_p$, are in their up state $|\uparrow\rangle$, while the remaining $N - p$ spins are in their down state $|\downarrow\rangle$. Since $[S_z, H] = 0$, we claim that at any time instant $t$ the system evolves into a normalized superposition of $\left( \begin{array}{c} N \\ p \end{array} \right)$ states wherein the central spin, as well as $p$ among the $N$, are up and $\left( \begin{array}{c} N - p \\ p \end{array} \right)$ states wherein the central spin is down and $p + 1$ spins among the $N$ are up. Thus, starting from the initial condition (5), the vector state of the system evolves within a finite-dimensional subspace whose dimension is $C_N^p + C_N^{p+1} = C_N^{p+1}$. Starting from this property we proceed to write down effectively the evolved state of the system. To represent it, we exploit the set of $p$-tuples, that is the set of all the subsets of $p$ elements from the first $N$ natural numbers, $S_p = \{(i_1, i_2, \ldots, i_p), 1 \leq i_1 < \cdots < i_p \leq N \}$. It is well known that the number of all $p$-tuples from $N$ numbers is exactly $C_N^p$. Therefore, establishing a bijection between the set $S_p$ and the set of the states $|\uparrow\rangle |\downarrow \cdots \uparrow_i \downarrow \cdots \uparrow_{p} \rangle$, as well as between the set $S_{p+1}$ and the set $|\uparrow\rangle |\downarrow \cdots \uparrow_j \downarrow \cdots \uparrow_{p+1} \rangle$, it is possible to represent the state of the system at the generic time instant as follows:

$$|\psi(t)\rangle = \sum_{S_p} a_{\{i_1, i_2, \ldots, i_p\}}(t) |\uparrow\rangle |\downarrow \cdots \downarrow \rangle |\uparrow_{i_1} \cdots \uparrow_{i_p} \rangle + \sum_{S_{p+1}} b_{\{j, i_1, \ldots, i_{p+1}\}}(t) |\uparrow\rangle |\downarrow \cdots \downarrow \rangle |\uparrow_j \cdots \uparrow_{p+1} \rangle,$$ \hspace{1cm} (6)

where the $p$-tuple $(i_1, i_2, \ldots, i_p) \in S_p$ identifies the probability amplitude $a_{\{i_1, i_2, \ldots, i_p\}}(t)$ of finding central spin $A$ and exactly the spins $i_1, i_2, \ldots, i_p$ among the $N$ around spins in their respective up state. Analogously, the $(p + 1)$-tuple $(j, i_1, i_2, \ldots, i_{p+1}) \in S_{p+1}$ provides the probability amplitude $b_{\{j, i_1, \ldots, i_{p+1}\}}(t)$ of finding out the spin system in the particular state with the central spin down and exactly the environmental spins $j, i_1, \ldots, i_p$ up. In order to get explicit equations for $|a_{\{i_1, i_2, \ldots, i_p\}}(t)\rangle$ and $|b_{\{j, i_1, \ldots, i_{p+1}\}}(t)\rangle$ we start from the time-dependent Schrödinger equation, introducing an appropriate mathematical notation useful to represent the transformations undergone by the states appearing in expression (6) by the application of Hamiltonian (1). For this reason, we define two families of mappings $|O_t\rangle |\uparrow\rangle \rightarrow |S_{p+1}\rangle$, so that to decouple the system of equations (1) and (2), we start from the time-dependent Schrödinger equation (9) and (10) of physical and mathematical interest. Thus, to proceed further we follow a standard procedure by which we succeed in converting the above system into two decoupled systems for each unknown set $|a_{\{i_1, i_2, \ldots, i_p\}}(t)\rangle$.
and \( \{ b_{i_1,j_1,...,j_{p+1}}(t) \} \). In this way, we obtain two systems of coupled equations for the amplitudes \( a_{i_1,j_1,...,j_p}(t) \) and \( b_{i_1,j_1,...,j_{p+1}}(t) \), respectively. We have indeed

\[
\begin{align*}
\ddot{a}_{i_1,j_1,...,j_p}(t) & = \left( \Delta^2 + \sum_{j=p+1}^{N} \alpha_j^2 \right) a_{i_1,j_1,...,j_p}(t) \\
& + \sum_{r,s=1, r \neq s}^{N} \alpha_r \alpha_s a_{b}(0,i_1,...,j_p)(t), \quad (11)
\end{align*}
\]

\[
\begin{align*}
\ddot{b}_{i_1,j_1,...,j_{p+1}}(t) & = \left( \Delta^2 + \sum_{i=1}^{p+1} \alpha_i^2 \right) b_{i_1,j_1,...,j_{p+1}}(t) \\
& + \sum_{r,s=1, r \neq s}^{N} \alpha_r \alpha_s b_{O_x}(i_1,...,j_{p+1})(t), \quad (12)
\end{align*}
\]

Equation (11) (equation (12)) defines a linear system of \( C^p_N \) (\( C^p_{N+1} \)) coupled second-order differential equations in the variables \( a_{i_1,j_1,...,j_p}(t) \) (\( b_{i_1,j_1,...,j_{p+1}}(t) \)). The \( C^p_N \) (\( C^p_{N+1} \)) amplitudes \( a_{i_1,j_1,...,j_p}(t) \) (\( b_{i_1,j_1,...,j_{p+1}}(t) \)) may be ordered in accordance with lexicographical prescription, that is \( a(i_1',...,i_{p+1}') \) (\( b(i_1',...,i_{p+1}') \)) follows \( a(i_1,...,i_p) \) (\( b(i_1,...,i_{p+1}) \)) if \( i_1 = i_1' \), \( i_2 = i_2' \), \( ..., i_{m-1} = i_{m-1}' \), \( i_m < i_m' \) with \( m = 1, 2, ..., p \) (\( m = 1, 2, ..., p+1 \)). Therefore, equation (11) (equation (12)) admits the matrix representation

\[
x(t) = -\mathcal{M} x(t), \quad (13)
\]

where \( x(t) = a(t) \) (\( b(t) \)) is the lexicographically ordered vector of the probability amplitudes \( a_{i_1,j_1,...,j_p}(t) \) (\( b_{i_1,j_1,...,j_{p+1}}(t) \)) and \( \mathcal{M} = A, B \) is the corresponding companion matrix. In accord with equation (11), the matrix elements of \( \mathcal{A} = (A_{mn})_{1 \leq |m,n| \leq C^p_N} \) are given by

\[
A_{mn'} = \begin{cases} 
\Delta^2 + \sum_{j=p+1}^{N} \alpha_j^2, & \text{if } \text{card}(\{m-m'\}) = 0 \, (\Leftrightarrow m = m'), \\
\alpha_{m-n'} \alpha_{m'-n}, & \text{if } \text{card}(\{m-m'\}) = 1, \\
0, & \text{otherwise},
\end{cases} \quad (14)
\]

where \( m = (i_1, i_2, ..., i_p) \) and \( \text{card}(\{m-m'\}) \) is the total number of the elements in the difference set \( \{m-m'\} \). In a similar manner, denoting now the \( (p+1) \)-tuples by \( q = (j_1, j_2, ..., j_{p+1}) \), the matrix \( \mathcal{B} = (B_{qq'})_{1 \leq q,q' \leq C^p_{N+1}} \) is defined by

\[
B_{qq'} = \begin{cases} 
\Delta^2 + \sum_{i=1}^{p+1} \alpha_i^2, & \text{if } \text{card}(\{q-q'\}) = 0 \, (\Leftrightarrow q = q'), \\
\alpha_{q-q'} \alpha_{q'-q}, & \text{if } \text{card}(\{q-q'\}) = 1, \\
0, & \text{otherwise}.
\end{cases} \quad (15)
\]

It is quite simple to notice that matrices \( \mathcal{A} \) and \( \mathcal{B} \) are symmetric. For example, if \( p = 1 \) (\( p = N - 2 \)) the matrix elements of \( \mathcal{A}, \mathcal{B} \) in \( M_x[C] \) assume the following simple form:

\[
\mathcal{A}_{ij} = \begin{cases} 
\Delta^2 + \sum_{i=1}^{N} \alpha_i^2, & \text{if } i = j, \\
\alpha_i \alpha_j, & \text{if } i \neq j,
\end{cases} \quad (16)
\]

Exploiting the well-known solution of a matrix second-order initial-value equation [13] like equation (13) yields

\[
x(t) = \exp \left( \sum_{k=0}^{\infty} \frac{(-1)^k t^{2k}}{(2k)!} \right) x(0) \\
+ \sum_{k=0}^{\infty} \frac{(-1)^k t^{2k+1}}{(2k+1)!} x(t). \quad (17)
\]

Moreover, if \( \mathcal{X} \) is a nonsingular matrix, taking into account that \( \mathcal{X} \) admits nonsingular square roots, solution (17) may be rewritten in the following closed form:

\[
x(t) = \cos \left( \sqrt{\mathcal{X}} t \right) x(0) + \sin \left( \sqrt{\mathcal{X}} t \right) \left( \sqrt{\mathcal{X}} \right)^{-1} x(0). \quad (18)
\]

Practically, the possibility of exploiting solution (17)/(18) depends of course on our ability of diagonalizing matrices \( \mathcal{A} \) and/or \( \mathcal{B} \). The dimensions \( C^p_N \) and \( C^p_{N+1} \) and the structure of the two matrices \( \mathcal{A} \) and \( \mathcal{B} \), respectively, anyway make an analytical treatment difficult. From this point of view, solution (17) might play a formal role only. However, it is always possible to recourse to the numerical diagonalization of matrices \( \mathcal{A} \) and/or \( \mathcal{B} \). We may wonder in this case about the efficiency of this numerical treatment in comparison with other numerical procedures [14, 15]. We observe, however, that such treatments get effective results when \( N \) is confined to values within 10, more or less, mainly due to exponentially increasing resources required from such computations. Overcoming these limitations on \( N \) thus becomes a mandatory target to improve the quality of the results achieved from numerical approaches. The calculation scheme introduced by us is based on numerical diagonalization of matrices \( \mathcal{A} \) and \( \mathcal{B} \) and presents the advantage that it enables the successful treatment of central spin models possessing a large number of bath spins, at least for small or large polarization, that is for small or large spin models. Indeed, in these cases the dimensions of the companion matrices \( \mathcal{A} \) and \( \mathcal{B} \) are such that the performance of numerical simulations is not obstructed by computational obstacles. We believe that our diagonalization procedure might be of help to investigate the behavior of our system as a function of other initial conditions.

3. Conclusions

Over the last few years, a whole variety of methods have been applied to study decoherence phenomena in the central spin models. The appropriate version of the BA has allowed one to integrate the Hamiltonian model, but the results are rather formal and practically very difficult to handle. In this paper, we have analyzed the exact dynamics of a central spin nonuniformly coupled through the Heisenberg interaction to a surrounding environment composed of \( N \) spins. Considering arbitrary initial conditions we have determined a numerically manipulative general solution from which information about the full dynamics of our spin models may be extracted.
Acknowledgments

M A J acknowledges financial support from Fondazione Don Giuseppe Puglisi ‘E se ognuno fa qualcosa’.

References

[1] Awschalom D, Loss D and Samarth N 2002 Semiconductor Spintronics and Quantum Computation (Berlin: Springer)
[2] Woods L M, Reinecke T L and Rajagopal A K 2008 Phys. Rev. B 77 073313
[3] Breuer H P, Burgarth D and Petruccione F 2004 Phys. Rev. B 70 045323
[4] Ferraro E, Napoli A, Jivulescu M A and Messina A 2008 Eur. Phys. J. Special Topics 160 157
[5] Deng X L and Fang X M 2008 J. Phys. B: At. Mol. Phys. 41 025503
[6] Hamdouni Y and Petruccione F 2007 Phys. Rev. B 76 174306
[7] Bose S 2003 Phys. Rev. Lett. 91 207901
[8] Chiara G De, Fazio R, Macchiavello C, Montanaro S and Palma G M 2005 Phys. Rev. A 72 012328
[9] Imamoglu A et al 1999 Phys. Rev. Lett. 83 4204
[10] Privman V, Vagner I D and Kventsel G 1998 Phys. Lett. A 239 141
[11] Sorensen A and Mølmer K 1999 Phys. Rev. Lett. 83 2274
[12] Gaudin M 1976 J. Physique 10 1087
[13] Horn R A and Johnson C R 1991 Topics in Matrix Analysis (Cambridge: Cambridge University Press)
[14] Jing J and Ma H R 2007 Phys. Rev. E 75 016701
[15] Schliemann J, Khatskii A and Loss D 2003 J. Phys.: Condens. Matter 15 R1909