A Combinatorial Approach to Diffeomorphism Invariant Quantum Gauge Theories

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Abstract

Quantum gauge theory in the connection representation uses functions of holonomies as configuration observables. Physical observables (gauge and diffeomorphism invariant) are represented in the Hilbert space of physical states; physical states are gauge and diffeomorphism invariant distributions on the space of functions of the holonomies of the edges of a certain family of graphs. Then a family of graphs embedded in the space manifold (satisfying certain properties) induces a representation of the algebra of physical observables. We construct a quantum model from the set of piecewise linear graphs on a piecewise linear manifold, and another manifestly combinatorial model from graphs defined on a sequence of increasingly refined simplicial complexes. Even though the two models are different at the kinematical level, they provide unitarily equivalent representations of the algebra of physical observables in separable Hilbert spaces of physical states (their s-knot basis is countable). Hence, the combinatorial framework is compatible with the usual interpretation of quantum field theory.

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Quantum gauge theories can be described using the holonomies along the edges of a regular lattice as basic configuration observables. This idea was introduced by Wilson [3] in the '70s and is now the basis of the modern lattice gauge theory. In diffeomorphism invariant gauge theories (like gravity using Ashtekar variables [2,1] or Yang-Mills coupled to gravity), the use of Wilson loops as primary observables of the theory led to the discovery of an interesting relation between quantum gauge theories and knot theory [4].

Twenty years after the early works, the notion of Wilson loops was extended and serves as a rigorous foundation of quantum gauge field theory [5]. The modern approach rests on the following idea: Begin by considering “the family of all the possible lattice gauge theories” defined on graphs whose edges are embedded in the base space. Then use a projective structure to organize the repeated information from graphs that share edges. For a manageable theory, the precise definition of “the family of all the possible lattice gauge theories” had to avoid situations where two different edges intersect each other an infinite number of times. The first solution to this problem [6] led to the framework referred in this article as the analytic category; by restricting the set of allowed graphs $\Gamma_\omega$ to contain only graphs with piecewise analytic edges, one acquires a controllable theory. In the analytic category the diffeomorphisms are restricted to be analytic accordingly. After a subtle analysis, it was possible to sacrifice part of the simplicity of the results of the analytic case and extend the theory to the smooth category [7].

While the foundations were solidifying, the theory also produced its first kinematical results for quantum gravity (the canonical quantization of gravity expressed in terms of Ashtekar variables). Regularized expressions for operators measuring the area of surfaces and volume of regions were developed [8]. These operators were also diagonalized and its eigenvectors were found to be labeled by spin networks (one-dimensional objects). In other words, a picture of polymer-like geometry arises from quantum gravity [9]. A polymer-like geometry is predicted from a theory whose foundations require space to be an analytic manifold. This peculiar situation was the main motivation for the work presented in this article.

In this article we present two quantum models: the combinatorial and the piecewise linear (PL) categories. The intention is to keep a simple framework that minimizes background structure and is suited to a polymer-like geometry, but that can still recover the classical macroscopic theory. Both models are based on the projective techniques used for the analytic and smooth categories; again, the difference relies on the family of graphs $\Gamma$ considered and the corresponding “diffeomorphisms.”

In the the piecewise linear category we fix a piecewise linear structure in the space manifold to specify the elements of the family of graphs $\Gamma_{\text{PL}}$ that define the Hilbert space. A piecewise linear structure on a manifold $\Sigma$ can be specified by a division of the manifold into cells with a fixed affine structure (flat connection). Also it can be specified by a triangulation, that is, a fixed homeomorphism $\varphi: \Sigma \to \Sigma_0$ where $\Sigma_0 \subset \mathbb{R}^{2n+1}$ is a n-dimensional polyhedron with a fixed decomposition into simplices. An element of $\Gamma_{\text{PL}}$ is a graph whose edges are piecewise linear according to the fixed PL structure. This seems to be far from a background-free situation, but a PL structure is much weaker than an analytic structure; the same PL structure can be specified by any refinement of the original triangulation. Furthermore, we
will prove that in three (or less) dimensions different choices of PL structures yield unitarily equivalent representations of the algebra of physical observables. This result is of particular interest for 3 + 1 (2 + 1 or 1 + 1) quantum models of pure gravity or of gravity coupled to Yang-Mills fields. To avoid confusion, we stress that the piecewise linear spaces used in this approach are not directly related to the ones used in Regge calculus. In simplified theories of gravity, like 2 + 1 gravity and BF theory, the lattice dual to the one induced by one of our piecewise linear spaces can be successfully related to a Regge lattice \([10]\). On the other hand, our approach contains a treatment based in cubic lattices as a particular case; the difference with the usual lattice gauge theory is that the continuum limit is taken by considering every lattice instead of just one.

The manifestly combinatorial model has two main ingredients: simplicial complexes that describe geometry in combinatorial fashion, and a refinement mechanism that makes it capable to describe field theories. If we use a simplicial complex as the starting point of our combinatorial approach, the resulting model would be appropriate to describe topological field theories, but we want to generate a model for gauge theories with local degrees of freedom. A way to achieve this goal is to replace physical space (the base space) with a sequence of simplicial complexes \(K_0, K_1, \ldots\) that are finer and finer. Our combinatorial model for quantum gauge theory is based in the family of graphs defined using our combinatorial representation of space.

Even though the PL and the combinatorial categories are closely related, the resulting kinematical Hilbert spaces \(\mathcal{H}_{\text{kin,PL}}\) and \(\mathcal{H}_{\text{kin,c}}\) are dramatically different. While the combinatorial Hilbert space \(\mathcal{H}_{\text{kin,c}}\) is separable (admits a countable basis), \(\mathcal{H}_{\text{kin,PL}}\) (like the Hilbert space constructed from the analytic category) is much bigger.

Physically, what we need is a Hilbert space to represent physical (gauge and “diffeomorphism” invariant) observables; such Hilbert space can be constructed by “averaging” the states of the kinematic Hilbert space to produce physical states. An encouraging result is that the two models produce unitarily equivalent representations of the algebra of physical observables in the naturally isomorphic separable Hilbert spaces \(\mathcal{H}_{\text{diff,PL}}, \mathcal{H}_{\text{diff,c}}\). Separability in the combinatorial case is no surprise, and that both spaces of physical states (PL and combinatorial) are isomorphic follows from the fact that every knot-class of piecewise linear graphs has a representative that fits in our combinatorial representation of space.

Two aspects of the loop approach to gauge theory are enhanced in its combinatorial version. On the mathematical-physics side, other approaches to quantum gravity coming from topological quantum field theory \([11]\) are much closer to the combinatorial category than they are to the analytic or smooth categories. On the practical side, the loop approach to quantum gauge theory is at least as attractive; a powerful computational technique comes built into this approach. Given any state in the Hilbert space of the continuum we can express it, to any desired accuracy, as a finite linear combination of states that come from the Hilbert space of a lattice gauge theory. Therefore, the matrix elements of every bounded operator can be computed, to any desired accuracy, in the Hilbert space of a lattice gauge theory. In this respect, the combinatorial picture presented in this article is favored because it is best suited for a computer implementation.

We organize this article as follows. Section \([1]\) reviews the general procedure to construct the kinematical Hilbert space in the continuum starting from a family of lattice gauge theories. Then, in section \([11]\), we carry out the procedure in the combinatorial and PL
frameworks. In section IV, we construct the physical Hilbert space. We treat separately the PL and combinatorial categories. Then we prove that the combinatorial and PL frameworks provide unitarily equivalent representations of the algebra of physical observables. We also prove that the mentioned algebra of physical observables is independent of the background PL structure when the dimension of the space manifold is three or less. A summary, an analysis of some problems from the combinatorial perspective and a comparison with the analytic category are the subjects of the concluding section.

II. FROM QUANTUM GAUGE THEORY IN THE LATTICE TO THE CONTINUUM VIA THE PROJECTIVE LIMIT: A REVIEW

A connection on a principal bundle is characterized by the group element that it assigns to every possible path in the base space. Historically, this simple observation led to treat the set of holonomies for all the loops of the base space as the basic configuration observables to be promoted to operators.

Now we start the construction of a kinematical Hilbert space for quantum gauge theories. To avoid extra complications, we only treat cases with a compact base space $\Sigma$ and we restrict our attention to trivializable bundles over $\Sigma$. For convenience, we start with a fixed trivialization. In the modern approach (Baez, Ashtekar et al [6]) the concept of paths or loops has been extended to that of graphs $\gamma \subset \Sigma$ whose edges, in contrast with their predecessors, are allowed to intersect.

A graph $\gamma$ is, by definition, a finite set $E_\gamma$ of oriented edges and a set $V_\gamma$ of vertices satisfying the following conditions:

- $e \in E_\gamma$ implies $e^{-1} \in E_\gamma$.
- The vertex set is the set of boundary points of the edges.
- The intersection set of two different edges $e_1, e_2 \in E_\gamma$ ($e_1 \neq e_2, e_1 \neq e_2^{-1}$) is a subset of the vertex set.

Generally an edge $e \in E_\gamma$, is considered to be an equivalence class of not self-intersecting curves, under orientation preserving reparametrizations. Formally, $e := [e'(I) \subset \Sigma]$ such that $e'(I) \approx I$, where we denoted the unit interval by $I = [0, 1]$. Composition of edges $e, f$ is defined if they intersect only at the final point of the initial edge and the initial point of the final edge $e'(I) \cap f'(I) = e'(1) = f'(0)$. Then the composition is defined by $f \circ e := [f' \circ e'(I)]$; and given an edge $e := [e']$ the edge defined by paths with the opposite orientation is denoted by $e^{-1} := [e'^{-1}]$.

The idea of considering “every possible path” in the base space to construct the space of generalized connections has to be made precise. Different choices in the class of edges that form the family of graphs considered lead to the different categories –analytic, smooth, PL and combinatoric– of this general approach to diffeomorphism invariant quantum gauge theories. We denote a generic family of graphs by $\Gamma$, and the analytic, smooth and combinatoric families by $\Gamma_\omega, \Gamma_\infty, \Gamma_{PL}$ and $\Gamma_C$.  

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A connection on a graph assigns a group element to each of the $2N_1$ graph’s edges. Therefore, we can identify the space of connections $A_\gamma$ of graph $\gamma$ with $G^{N_1}$. An element $A \in A_\gamma$ is represented by $(A(e_1), A(e_1^{-1}) = A(e_1)^{-1}, \ldots, A(e_{N_1}), A(e_{N_1}^{-1}) = A(e_{N_1})^{-1})$, where $A(e_i) \in G$.

The collection of the spaces $A_\gamma$ for every graph $\gamma \in \Gamma$ gives an over-complete description of the space of generalized connections in the category specified by $\Gamma$. For example, $\Gamma_\omega$ determines the analytic category and $\Gamma_C$ specifies the combinatoric category.

It is possible to organize all the repeated information by means of a projective structure. We say that graph $\gamma$ is a refinement of graph $\gamma'$ ($\gamma \geq \gamma'$) if the edges of $\gamma'$ are “contained” in edges of $\gamma$; more precisely, if $e \in \gamma$ then either $e = e_1$ or $e = e_1 \circ \ldots \circ e_n$ for some $e_1, \ldots, e_n \in \gamma$. Given any two graphs related by refinement $\gamma \geq \gamma'$ there is a projection $p_{\gamma \to \gamma'} : A_\gamma \to A_{\gamma'}$

$$(A(e_1), A(e_2), \ldots, A(e_{N_1})) \xrightarrow{p_{\gamma \to \gamma'}} (A'(e_1) = A(e_2)A(e_1), A'(e_2), \ldots, A'(e_{N_1}))$$

(2.1)

where $e = e_1 \circ e_2, e \in \gamma', e_1, e_2 \in \gamma$.

The projection map and the refinement relation have two properties that will allow us to define $\overline{A}$ as “the space of connections of the finest lattice.” First, we can easily check that $p_{\gamma \to \gamma'} \circ p_{\gamma' \to \gamma''} = p_{\gamma \to \gamma''}$. Second, equipped with the refinement relation “$\geq$”, the set $\Gamma$ is a partially ordered, directed set; i.e. for all $\gamma, \gamma'$ and $\gamma''$ in $\Gamma$ we have:

$$\gamma \geq \gamma ; \quad \gamma \geq \gamma' \quad \text{and} \quad \gamma' \geq \gamma \Rightarrow \gamma = \gamma' ; \quad \gamma \geq \gamma' \quad \text{and} \quad \gamma' \geq \gamma'' \Rightarrow \gamma \geq \gamma'' ;$$

(2.2)

and, given any $\gamma', \gamma'' \in \Gamma$, there exists $\gamma \in \Gamma$ such that

$$\gamma \geq \gamma' \quad \text{and} \quad \gamma \geq \gamma'' .$$

(2.3)

This last property, that $\Gamma$ is directed, is the only non trivial property; it will be proved for the PL and the combinatoric categories in the next section. The projective limit of the spaces of connections of all graphs yields the space of generalized connections $\overline{A}$

$$\overline{A} := \left\{ (A_\gamma)_{\gamma \in \Gamma} \in \prod_{\gamma \in \Gamma} A_\gamma : \gamma' \geq \gamma \Rightarrow p_{\gamma \to \gamma'} A_\gamma = A_\gamma \right\}. \quad (2.4)$$

That is, the projective limit is contained in the cartesian product of the spaces of connections of all graphs in $\Gamma$, subject to the consistency conditions stated above. There is a canonical projection $p_\gamma$ from the space $\overline{A}$ to the spaces $A_\gamma$ given by,

$$p_\gamma : \overline{A} \to A_\gamma, \quad p_\gamma((A_\gamma)_{\gamma \in \Gamma}) := A_\gamma.$$  

(2.5)

With this projection, functions $f_\gamma$ defined on the space $A_\gamma$ can be pulled-back to $\text{Fun}(\overline{A})$. Such functions are called cylindrical functions. The sup norm

$$||f||_\infty = \sup_{A \in A_\gamma} |f(A)|$$

(6.2)

can be used to complete the space of cylindrical functions. As result we get the Abelian $C^*$ algebra usually denoted by $\text{Cyl}(\overline{A})$; to simplify the notation, in the rest of the article we will
denote this algebra by $\text{Cyl}_\square$, where $\square = \omega, \infty, \text{PL}, C$ labels the family of graphs defining the space of cylindrical functions considered.

The uniform generalized measure $\mu_0 : \text{Cyl}_\square \rightarrow C$, sometimes called the Ashtekar-Lewandowski measure, is induced in $\mathcal{A}$ by the uniform (Haar) measure on the spaces $\mathcal{A}_\gamma = G^{N_1}$. Other gauge invariant measures are available; when they are diffeomorphism invariant they induce “generalized knot invariants” (see [13]). Finally, we define the kinematical Hilbert space to be the completion of $\text{Cyl}(\mathcal{A})$ on the norm induced by the (strictly positive) generalized measure $\mu_0$

$$\mathcal{H}_{\text{kin}} := L^2(\bar{\mathcal{A}}, d\mu_0).$$

(2.7)

This construction yields a cyclic representation of the algebra of cylindrical functions, the so called connection representation. Given a function defined on a lattice $\gamma$, for example the trace of the holonomy $T_\alpha$ along a loop $\alpha$ contained in $\gamma$, the corresponding operator $\hat{T}_\alpha$ will act by multiplication on states $\Psi_{\gamma} \in \mathcal{H}_{\text{kin}}$:

$$(\hat{T}_\alpha \cdot \Psi_{\gamma})(\bar{A}) := T_\alpha(\bar{A})\Psi_{\gamma}(\bar{A}).$$

(2.8)

A complete set of Hermitian momentum operators on the Hilbert space $L^2(G_e, d\mu_{\text{Haar}})$ of a graph with a single edge $e$ come from the left $L_e(f)$ and right invariant $R_e(f)$ vector fields on $G_e$ as labeled by $f \in \text{Lie}(G_e)$. These momentum operators are compatible with the projective structure [13]; thus, the set of momentum operators

$$X_{\alpha,e}(f) = \begin{cases} L_e(f) & \text{if edge } e \text{ goes out of vertex } \alpha \\ -R_e(f) & \text{if edge } e \text{ comes into vertex } \alpha \end{cases}$$

(2.9)

is a complete set of Hermitian momentum operators on $\mathcal{H}_{\text{kin}}$ when we use the generalized measure $\mu_0$. In regularized expressions of operators involving the triad, the place of the triad is taken by the vector fields $X$; therefore, the measure $\mu_0$ incorporates the physical reality conditions.

Our main goal is to construct a Hilbert space where we can represent the algebra of physical (gauge and diffeomorphism invariant) observables. Because it is customary we will proceed in steps; in this section we deal with the issue of gauge invariance and in the next with that of diffeomorphism invariance. If we had chosen to generate the space of states invariant under both symmetries simultaneously we would arrive at the same result.

A finite gauge transformation takes the holonomy $A_{e_1}$ to $g(\alpha)A_{e_1}g(\beta)^{-1}$ (where edge $e_1$ goes from vertex $\alpha$ to vertex $\beta$). Then a quantum gauge transformation is given by the unitary transformation

$$G(g)\Psi_{\gamma}(A_{e_1}, \ldots A_{e_n}) := \Psi_{\gamma}(g(\alpha)A_{e_1}g(\beta)^{-1}, \ldots g(\mu)A_{e_n}g(\nu)^{-1})$$

(2.10)

Gauge transformations are just generalizations of right and left translations in the group. This implies that they are generated by left and right invariant vector fields. Given a graph $\gamma$, $C_\alpha(f)$ generates gauge transformations at vertex $\alpha$. Therefore gauge invariance of $\Psi_{\gamma} = \Psi_{\gamma}(A_{e_1}, \ldots A_{e_n})$ at vertex $\alpha$ means that it lies in the kernel of the Gauss constraint

$$C_\alpha(f) \cdot \Psi_{\gamma} := \sum_{e \rightarrow \alpha} X_{e}^f \cdot \Psi_{\gamma} = 0,$$

(2.11)
where the sum is taken over all the edges $e$ that start at vertex $\alpha$. Because it is a real linear combination of the momentum operators (2.9), the Gauss constraint is essentially self-adjoint on $H_{\text{kin}}$.

We could construct the space of connections modulo gauge transformations of a graph $A\gamma/G$. Then, using the same projective machinery, we could construct the Hilbert space $L^2(A\gamma/G, d\nu_0)$. It is easy to see that the space of gauge invariant functions of $L^2(A\gamma/G, d\nu_0)$ is naturally isomorphic to $H'_{\text{kin}} = L^2(A\gamma/G, d\nu_0)$ if the measure $\nu_0$ is the one induced by $\mu_0$. The space $H'_{\text{kin}}$ of gauge invariant functions is spanned by spin network states. Spin network states are cylindrical functions $S_{\vec{\gamma}, j(e), c(v)}(A)$ labeled by an oriented graph (a graph $\gamma$ plus a choice of either $e \in E_{\gamma}$ or $e^{-1} \in E_{\gamma}$, for every edge in $\gamma$, to belong to the oriented graph $\vec{\gamma}$) whose edges and vertices are colored. The “colors” $j(e)$ on the edges $e \in E_{\gamma}$ assign a non-trivial irreducible representation of the gauge group to the edges. And the “colors” $c(v)$ on the vertices $v \in V_{\gamma}$ assign to each vertex a gauge invariant contractor (intertwining operator) that has indices in the representations determined by the colored edges that meet at the vertex. The spin network states is defined by

$$S_{\vec{\gamma}, j(e), c(v)}(A) = \prod_{e \in E_{\vec{\gamma}}} \pi_{j(e)}[A(e)] \cdot \prod_{v \in V_{\vec{\gamma}}} c(v),$$

(2.12)

where '.' stands for contraction of all the indices of the matrices attached to the edges with the indices of the intertwiners attached to the vertices. In the inner product that the uniform measure $\mu_0$ induces in $H'_{\text{kin}}$ two spin network states are orthogonal if they are not labeled by the same (unoriented) graph or if their edge’s colors are different. For calculational purposes it is convenient to choose an orthonormal basis for $H'_{\text{kin}}$ by normalized spin network states with special labels of the intertwining operators assigned to the vertices; see [14].

III. PL AND COMBINATORIC CATEGORIES

In this section we construct two quantum models using the general framework outlined above. First the family of piecewise linear (PL) graphs is introduced. Then we prove that it is a partially ordered, directed set. As a result, the algebra of functions of the connection defined by the PL graphs has a cyclic representation in the Hilbert space $H_{\text{kin}_{PL}}$. The second subsection briefly reviews some elements of combinatoric topology while constructing the family of combinatoric graphs. In this case, the resulting algebra of functions is represented in the separable Hilbert space $H_{\text{kin}_{C}}$. While at this level the two quantum models yield completely different Hilbert spaces, in the section [14] we will prove that the corresponding spaces of “diffeomorphism” invariant states are naturally isomorphic.

A. The PL category

To specify the elements of the family of graphs $\Gamma_{PL}$ that define the Hilbert space of the PL category we need a fixed piecewise linear structure on space $\Sigma$. A piecewise linear structure on a manifold $\Sigma$ can be specified by a division of the manifold into cells with a fixed affine structure (flat connection). Also it can be specified by a triangulation, that
is, a fixed homeomorphism $\varphi : \Sigma \to \Sigma_0$ where $\Sigma_0$ is a $n$-dimensional polyhedron with a fixed decomposition into simplices. To be more explicit, we can use the fact that every $n$-dimensional polyhedron can be embedded in $R^{2n+1}$ and consider from the beginning $\Sigma_0 \subset R^{2n+1}$. Then $\Sigma_0$ can be decomposed into a collection of convex cells (geometrical simplices).

A geometric simplex in $R^{2n+1}$ is simply the convex region defined by its set of vertices $\{s_0, \ldots, s_k\}$, $s_i \in R^{2n+1}$

$$\Delta(\{s_0, \ldots, s_k\}) = \{s = \sum_{i=0}^{k} t_i s_i\}$$

(3.1)

where $t_i \in [0, 1]$ and $\sum_{i=0}^{k} t_i = 1$. The triangulation of $\Sigma_0$ fixes an affine structure in its cells, namely, a PL structure. Using the local affine coordinate systems $t_i$, we can decide which curves are straight lines inside any cell. Then a piecewise linear curve in $\Sigma_0$ is a curve that is straight inside every cell except for a finite set of points; in this set of points and in the points where it crosses the boundaries of the cells the curve bends, but is continuous.

A piecewise linear graph $\gamma \in \Gamma_{PL}$ is a graph (according to the definition given in the previous section) such that every edge $e \in E_{\gamma}$ is piecewise linear.

In the previous section we gave a natural partial order ("refinement relation", $\geq$) for any family of graphs. Our task is now to prove that the partially ordered set $\Gamma_{PL}$ is a projective family; once we prove this property, the general procedure outlined in the previous section gives us the Hilbert space of the PL category.

The only non-trivial property to prove is that the family of graphs $\Gamma_{PL}$ is directed. For instance, according to the definition of a graph given in last section, the family of all the graphs with piecewise smooth edges is not directed. In this case, two edges of different graphs can intersect an infinite number of times; such two graphs would only accept a common refinement with an infinite number of edges, that according to our definition is not a graph.

We will construct a graph $\gamma_3$ that refines two given graphs $\gamma_1$ and $\gamma_2$.

A trivial property of PL edges lies in the heart of our construction; due to its importance, it is stated as a lemma.

**Lemma 1** Given two edges of different graphs $e_1 \in \gamma_1$ and $e_2 \in \gamma_2$, we know that $e_1 \cap e_2$ has finitely many connected components. These connected components are either isolated points or piecewise linear segments.

Now we start our construction. First we note that every graph $\gamma$ is refined by a graph $\gamma'$ constructed from $\gamma$ simply by adding a finite number of vertices $v \in V'$ in the interior of its edges (and by splitting the edges in the points where a new vertex sits).

Because of lemma [1] we know that given two graphs $\gamma_1, \gamma_2 \in C_{PL}$ we can refine each of them trivially by adding finitely many new vertices to form the graphs $\gamma'_1 \geq \gamma_1, \gamma'_2 \geq \gamma_2$ that satisfy the following property. Every edge $e_1 \in E_{\gamma'_1}$ falls into one of the three categories given bellow:

- $e_1$ does not intersect any edge of $\gamma'_2$.
- $e_1$ is also an edge of $\gamma'_2$; $e_1 \in E_{\gamma'_2}$.
• \( e_1 \) intersects an edge \( e_2 \) of \( \gamma'_2 \) at vertices (one or two) of both graphs \( e_1 \cap e_2 \subset V_{\gamma'_1}, e_1 \cap e_2 \subset V_{\gamma'_2} \).

A direct consequence of these properties is the following:

**Lemma 2** The graph \( \gamma_3 \) defined by \( E_{\gamma_3} = E_{\gamma'_1} \cup E_{\gamma'_2} \) and \( V_{\gamma_3} = V_{\gamma'_1} \cup V_{\gamma'_2} \) is a refinement of \( \gamma'_1 \) and \( \gamma'_2 \). By the properties of the partial ordering relation it follows that \( \gamma_3 \) is also a refinement of the original graphs \( \gamma_3 \geq \gamma_1, \gamma_3 \geq \gamma_2 \); thus the family of piecewise linear graphs \( \Gamma_{PL} \) is a projective family.

In the light of lemma 2, the rest of the construction is a simple application of the general framework described in the previous section. There is a canonical projection \( p_\gamma \) from the space of generalized connections \( \mathcal{A}_{PL} \) to the spaces of connections \( A_\gamma \) on graphs \( \gamma \in \Gamma_{PL} \) given by,

\[
p_\gamma : \mathcal{A}_{PL} \to A_\gamma, \quad p_\gamma((A_\gamma)_{\gamma \in \Gamma_{PL}}) := A_\gamma.
\]

This projective structure is the main ingredient that yields the Hilbert space of the connection representation in the PL category. Below we state our result concisely.

**Theorem 1** The completion (in the sup norm) of the family of functions \( p_\gamma^* f_\gamma(\bar{A}) \), defined by graphs \( \gamma \in \Gamma_{PL} \), is an Abelian \( C^* \) algebra \( \text{Cyl}_{PL} \). A cyclic representation of \( \text{Cyl}_{PL} \) is provided by the Hilbert space

\[
\mathcal{H}_{\text{kin}_{PL}} := L^2(\mathcal{A}_{PL}, d\mu_0).
\]

that results after completing \( \text{Cyl}_{PL} \) in the norm provided by the Ashtekar-Lewandowski measure \( \mu_0 \).

In the manner described in the previous section we can also consider the space of gauge invariant states and obtain \( \mathcal{H}'_{\text{kin}_{PL}} \) that is is spanned by spin network states labeled by piecewise linear graphs.

**B. The combinatoric category**

In this subsection we introduce the family of combinatoric graphs that leads to a manifestly combinatoric approach to quantum gauge theory. The construction of combinatoric graphs uses as a corner stone the same stone that serves as the combinatoric foundation of topology. Thus, our construction provides a quantum/combinatoric model for physical space, the space where physical processes take place.

Simplicial complexes appear first as the combinatoric means of capturing the topological information of a topological space \( X \). By definition, a simplicial complex \( K \) is a set of finite sets closed under formation of subsets, formally:

\[
x \in K \text{ and } y \subset x \Rightarrow y \in K.
\]

A member of a simplicial complex \( x \in K \) is called an \( n \)-simplex if it has \( n + 1 \) elements; \( n \) is the dimension of \( x \). Generically, the set of which all simplices are subsets is called the vertex set and denoted by \( \Lambda \). Some examples of simplicial complexes are given in figure 1.
The simplices are the sets; in the figures, what we draw are the geometric realizations $\Delta_x$ of the abstract simplices $x$. b) A two dimensional complex is a set of simplices of dimension smaller or equal to two. In this case the complex $K = \{\{p\}, \{q\}, \{r\}, \{s\}, \{p,q\}, \{q,r\}, \{r,p\}, \{s,p\}, \{s,q\}, \{s,r\}, \{p,q,r\}, \{p,q,s\}, \{q,r,s\}, \{r,p,s\}, \{p,q,r,s\}\}$ represents a sphere $S^2$. Figure (1b) is the geometric realization $\|K^1\|$ of the one dimensional subcomplex of $K$ given by $K^1 = \{\{p\}, \{q\}, \{r\}, \{s\}, \{p,q\}, \{q,r\}, \{r,p\}, \{s,p\}, \{s,q\}, \{s,r\}\}$.

c) The vertices of the barycentric subdivision $Sd(K)$ are the simplices of $K$. For example if $K = \{\{p\}, \{q\}, \{p,q\}\}$ then $Sd(K) = \{\{\{p\}\}, \{\{p,q\}\}, \{\{q\}\}, \{\{p\}, \{p,q\}\}, \{\{p,q\}, \{q\}\}\}$. Given an open cover $U(\Lambda) = \{U_\lambda : \lambda \in \Lambda\}$ of a topological space $X$ the information about the relative position of the open sets $U_1, U_2, ... \in U(\Lambda)$ is the combinatoric information that the nerve $K(\Lambda)$ of $U(\Lambda)$ casts. The simplicial complex $K(\Lambda)$ is the set of all finite subsets of $\Lambda$ such that

$$\bigcap_{\lambda \in \Lambda} U_\lambda \neq \emptyset$$

Using the information encoded in the $K(\Lambda)$ one can often recover the topological space $X$. More precisely, every open cover $U(\Lambda)$ of $X$ admits a refinement $U'(\Lambda')$ such that the geometric realization (to be defined below) of its nerve is homeomorphic to $X$, $|K(\Lambda')| \approx X$. This is the sense in which simplicial complexes constitute a combinatoric foundation of topology.

A simplicial complex stores topological information combinatorially, but the same information can be encoded in a geometric fashion (see [10]). The geometric realization $|K|$ of a simplicial complex $K = K(\Lambda)$, is the subset of $R^\Lambda$ given by $|K| := \bigcup_{x \in K} \Delta_x$ where $\Delta_x$ is a geometrical simplex represented as a segment of a plane of codimension one, embedded in $R^x$; more precisely,

$$\Delta_x := \left\{ s := (s_\lambda : \lambda \in \xi) \in \Gamma^\times : \sum_{\lambda \in \xi} s_\lambda = 1 \right\}$$
where $I = [0, 1]$ is the unit interval. The topology of $|K|$ is determined by declaring all its geometrical simplices $\Delta_n$ to be closed sets.

Our main purpose is to find a combinatoric analog of a generalized connection. We need to find the appropriate concept of the space of all combinatoric graphs; then a generalized connection will be an assignment of group elements to the edges of the graphs. We could fix a simplicial complex $K$ to represent the base space and consider that a combinatoric graph is a one-dimensional subcomplex $\gamma \subset K$. The resulting model would properly describe topological field theories, but we want to generate a model for gauge theories with local simplicial complexes. Given a simplicial complex $K$, a simplicial complex $K$ connection will be an assignment of group elements to the edges of the graphs. We could fix $\Lambda = K$ to find the appropriate concept of the space of all combinatoric graphs; then a generalized $\Lambda = K$. Then, the simplices of $\Sigma_d$ are the finite subsets $X \subset \Lambda$ that satisfy

$$x, y \in X \Rightarrow x \subset y \text{ or } y \subset x \quad (3.7)$$

A geometric representation of the operation baricentric subdivision $\Sigma_d$ is given in figure 1.

Our approach to quantum gauge theory replaces the base space $\Sigma$ with a sequence of simplicial complexes $\{K, \Sigma_0(K), \ldots, \Sigma_d(K), \ldots\}$ such that $|K| \approx \Sigma_0$, where $\Sigma_0$ is a compact Hausdorff three dimensional manifold. This concept of space leads to the definition of combinatoric graphs.

A combinatoric graph $\gamma \in \Gamma_C$ is simply a graph, according to the definition given in the previous section, where the set of vertices $V_\gamma$ and the set of edges $E_\gamma$ are restricted to be subsets of the set of points $V(K)$ and the set of oriented paths $E(K)$.

In the combinatoric representation of space, a point $p \in V(K)$ is represented by an equivalence class of sequences of the kind $\{p_n, p_{n+1} = \Sigma_d(p_n), p_{n+2} = \Sigma_d^2(p_n), \ldots\}$ of zero-dimensional simplices $p_n \in \Sigma_d^n(K)$, $p_{n+1} \in \Sigma_d^{n+1}(K)$, etc. Notably one single element of the sequence determines the whole sequence. Two sequences $\{p_n, p_{n+1} = \Sigma_d(p_n), \ldots\}$ $\{q_m, q_{m+1} = \Sigma_d(q_m), \ldots\}$ are equivalent if all their elements coincide, $p_s = q_s \in \Sigma_d^s(K)$ for all $s \geq \max(n, m)$.

The definition of oriented paths follows the same idea, but is a little more involved. First we will define paths, then oriented paths, and composition of oriented paths. A path $e \in P(K)$ is an equivalence class of sequences $\{e_n, e_{n+1} = \Sigma_d(e_n), \ldots\}$ of one dimensional subcomplexes $e_n \subset \Sigma_d^n(K)$ such that the geometric realizations of its elements are homeomorphic to the unit interval $|e_n| \approx I$. Again, two sequences $\{e_n, e_{n+1} = \Sigma_d(e_n), \ldots\}, \{f_m, \ldots\}$ are equivalent if all their elements coincide $e_s = f_s \in \Sigma_d^s(K)$ for all $s \geq \max(n, m)$.

An oriented path $e \in E(K)$ is a path $e' \in P(K)$ and a sequence of relations that order the vertices of each of the one-dimensional subcomplexes $e_n$ in the path. We denote the initial point of a path by $e(0) \in V(K)$ and it is defined by the class of the sequence of initial

1 Here the term vertex refers to a zero-dimensional simplex in the one of the one-dimensional subcomplexes $e_n$ in the path $e$. It should not be confused with a vertex $v \in V_\gamma$ of a combinatoric graph.
vertices \( e(0) = [\{e_n(0), e_{n+1}(0) = Sd(e_n(0)), \ldots \}] \in V(K) \); the final point of a combinatoric path is denoted by \( e(1) \in V(K) \). Composition of two oriented paths \( e, f \in E(K) \) is possible if they intersect only at the final point of the initial path and the initial point of the final path \( [\{e_n \cap f_n, e_{n+1} \cap f_{n+1}, \ldots \}] = e(1) = f(0) \); it is denoted by \( f \circ e \in E(K) \) and is defined by

\[
 f \circ e = \{((f \circ e)_n = f_n \cup e_n, (f \circ e)_{n+1} = Sd((f \circ e)_n), \ldots \} \tag{3.8}
\]

and the obvious sequence of ordering relations.

Given an oriented path \( e \in E(K) \) its inverse \( e^{-1} \in E(K) \) is defined by the same path \( e' \in P(K) \) and the opposite orientation. Notice that the composition relation is not defined for \( e \) and \( e^{-1} \); it is possible to define combinatoric curves that behave like usual curves, but it is not necessary for the purpose of this article.

Once the set of edges \( E \) is endowed with the composition operation, the rest of our construction is almost a simple application of the general framework reviewed in the previous section. The only gap to be filled is proving that the family of combinatoric graphs \( \Gamma_C \) is directed.

To prove the directedness in the PL case we used the finiteness property stated in lemma \( \text{[3]} \), an adapted statement of this same property holds trivially in the combinatoric case.

**Lemma 3** The intersection of two one dimensional subcomplexes \( e_n, f_n \subset Sd^n(K) \), defining the paths \( e, f \in P(K) \) respectively, has finitely many connected components. These connected components are either isolated zero-dimensional simplices or one-dimensional subcomplexes homeomorphic to the unit interval. That is,

\[
e_n \cap f_n = \bigcup_{i=1}^{N} p(i)_n \cup \bigcup_{j=1}^{M} g(j)_n \tag{3.9}
\]

where \( p(i)_n \subset Sd^n(K) \) is a zero-dimensional simplex and \( I \approx g(i)_n \subset Sd^n(K) \). In addition, \( p(i)_n \cap p(j)_n = p(i)_n \cap g(j)_n = g(i)_n \cap g(j)_n = \emptyset \) for all \( i \neq j \).

By defining the appropriate notion of union and intersection of classes of sequences we can state the result as

\[
e \cap f = \bigcup_{i=1}^{N} p(i) \cup \bigcup_{j=1}^{M} g(j) \tag{3.10}
\]

where \( p(i) \in V(K), g(j) \in P(K), \) and \( p(i) \cap p(j) = p(i) \cap g(j) = g(i) \cap g(j) = \emptyset \) for all \( i \neq j \).

Therefore, the construction of a graph \( \gamma_3 \in \Gamma_C \) that refines two given graphs \( \gamma_1, \gamma_2 \in \Gamma_C \) is just an adaptation of the construction given for the piecewise linear case.

Using lemma \( \text{[3]} \) it is easy to prove that given two graphs \( \gamma_1, \gamma_2 \in C_C \) we can refine each of them trivially by adding finitely many new vertices; forming graphs \( \gamma'_1 \geq \gamma_1, \gamma'_2 \geq \gamma_2 \) such that every edge \( e_1 \in E_{\gamma'_1} \) falls in one of the three categories \( \text{[III A]}, \text{[II A]}, \text{[I A]} \) itemized in the previous subsection.

From the previous construction the following lemma is evident.
Lemma 4 Let $\gamma_3$ be the graph defined by
\[ V_{\gamma_3} := V_{\gamma_1} \cup V_{\gamma_2} \subset V(K) \text{ and } E_{\gamma_3} := E_{\gamma_1} \cup E_{\gamma_2} \subset E(K). \]
$\gamma_3$ is a refinement of $\gamma_1'$ and $\gamma_2'$. By the properties of the partial ordering relation it follows that $\gamma_3$ is also a refinement of the original graphs $\gamma_3 \geq \gamma_1', \gamma_3 \geq \gamma_2'$; thus the family of combinatoric graphs $\Gamma_C$ is a projective family.

Following the general framework described in the previous section we will complete the construction of our combinatoric/quantum model for gauge theory. There is a canonical projection $p_{\gamma}$ from the space of generalized connections $A_C$ to the spaces of connections $A_{\gamma}$ on graphs $\gamma \in \Gamma_C$ given by,
\[ p_{\gamma} : A_C \rightarrow A_{\gamma}, \quad p_{\gamma}((A_{\gamma'})_{\gamma' \in \Gamma_C}) := A_{\gamma}. \quad (3.11) \]
These projections are the key ingredient that yields the Hilbert space of the connection representation in the combinatoric category. Below we state our result concisely.

Theorem 2 The completion (in the sup norm) of the family of functions $p_{\gamma}^* f_{\gamma}(\bar{A})$, defined by graphs $\gamma \in \Gamma_C$, is an Abelian $C^*$ algebra $\text{Cyl}_C$. A cyclic representation of $\text{Cyl}_C$ is provided by the Hilbert space
\[ \mathcal{H}_{\text{kinc}} := L^2(\mathcal{A}_C, d\mu_0). \quad (3.12) \]
that results after completing $\text{Cyl}_C$ in the norm provided by the Ashtekar-Lewandowski measure $\mu_0$.

As described in the previous section we can consider the space of gauge invariant states and get $\mathcal{H}'_{\text{kinc}}$ that is is spanned by spin network states labeled by combinatoric graphs.

The constructions, given in this and the previous subsection, of the Hilbert spaces for the piecewise linear and the combinatoric categories were similar. Despite the parallelism, the resulting Hilbert spaces are completely different. A property that marks the difference is the size of these Hilbert spaces.

Theorem 3 The Hilbert space $\mathcal{H}'_{\text{kinc}}$ is separable.

Proof – We will prove that the spin network basis is countable in the combinatoric case.
We did not describe precisely the spin network basis, but we stated that two spin network states $S_{\gamma,j(e),c(e)}^1(A)$, $S_{\delta,j(e),c(e)}^2(A)$ are orthogonal if $\gamma \neq \delta$ or if their edge’s colors are different.
Let $L_{\gamma,j(e)}$ be the space spanned by all the spin network states with labels $\gamma, j(e)$. Our task is bound $n = \dim(L_{\gamma,j(e)})$. We know that $n$ is less than the number of labels that we would get by assigning not one integer but three integers to the graphs edges. The first integer $j(e)$ labels the irreducible representation assigned to $e$, and the other two $m_L(e), m_R(e)$ determine basis vectors in the vector space selected by $j(e)$. With these basis vectors sitting at both ends of every edge we can label any set of (generally non gauge invariant) contractors for the vertices.

Thus, the spin network basis is countable if the set of finite subsets of
\[ E(K) \times \mathbb{N} \quad (3.13) \]
is countable. Then to prove the theorem we just have to show that the set $E(K)$ is countable, which in turn reduces to prove that the set of paths $P(K)$ is countable.

A path $e \in P(K)$ is determined by a sequence of one-dimensional subcomplexes that are all related by baricentric subdivision. Therefore, a path $e \in P(K)$ can be specified by just one one-dimensional subcomplex of an appropriate $Sd^n(K)$. A particular one-dimensional subcomplex can be described by specifying which of the one-dimensional and zero-dimensional simplices belong to it. We can use the set $\{0, 1\}$ to specify which simplex belong or does not belong to a particular subcomlpex.

Therefore, there is an onto map

$$M : \bigcup_{n=1}^{\infty} Sd^n(K) \times \{0, 1\} \rightarrow P(K)$$

(3.14)

since a countable union of finite sets is countable and each $Sd^n(K)$ is finite, we have proved that $P(K)$ is countable. □

IV. PHYSICAL OBSERVABLES AND PHYSICAL STATES

In this section we construct the Hilbert space of physical states of our model for quantum gauge theory; where we can represent the algebra of physical (gauge and “diffeomorphism” invariant) observables. Our quantization procedure follows the same steps as in the analytic category; that is, it follows (a refined version of) the algebraic quantization program [2,17]. When we deal with theories with extra constraints, like gravity, we need to solve these extra constraints to find the space of physical states.

Since the issue of “diffeomorphism” invariance acquires quite different faces in the PL and combinatoric categories, we tackle it first for the PL category. Then we find the space of physical states of the combinatoric category and prove that it is separable and isomorphic to the space of physical states of the PL category.

A. “diffeomorphism” invariance in the PL category

Any operator can be defined by specifying its action on the space of cylindrical functions $Cyl$ and then using continuity to extend it to the whole Hilbert space $\mathcal{H}_{\text{kin}}$. This is what we did to define the unitary operators induced by the gauge symmetry and it is what we will do in this section to define quantum “diffeomorphisms.”

Our piecewise linear framework is based on the family of graphs $\Gamma_{\text{PL}}$ selected by a fixed piecewise linear structure in $\Sigma$. Therefore, the role of “diffeomorphisms” is played by piecewise linear homeomorphisms. It is important to note that the space of such maps can be defined as

$$\text{Hom}_{\text{PL}}(\Sigma) := \{h \in \text{Hom}(\Sigma) : h(\Gamma_{\text{PL}}) = \Gamma_{\text{PL}}\}.$$  

(4.1)

The unitary operator $\hat{U}_h : \mathcal{H}_{\text{kinPL}} \rightarrow \mathcal{H}_{\text{kinPL}}$ induced by a piecewise linear homeomorphism $h$ is determined by its action on cylindrical functions

$$\hat{U}_h \cdot \Psi_{\gamma}(A) := \Psi_{h^{-1}(\gamma)}(A).$$

(4.2)
In contrast with our treatment of gauge invariance, the space of diffeomorphism invariant states is not the kernel of any Hermitian operator; the reason is that the one-dimensional subgroups of the diffeomorphism group induce one-parameter families of unitary transformations that are not strongly continuous in our Hilbert space \( \mathcal{H}_{\text{kin}} \). Another important difference is that the space of “diffeomorphism” invariant states cannot be made a subspace of the Hilbert space \( \mathcal{H}_{\text{kin}} \), the solutions are true distributions, i.e., they lie in a subspace of the topological dual of \( \text{Cyl}^*_{\text{PL}} \).

A distribution \( \tilde{\phi} \in \text{Cyl}^*_{\text{PL}} \) is “diffeomorphism” invariant if
\[
\tilde{\phi}[\hat{U}_h \circ \psi] = \tilde{\phi}[\psi] \quad \forall \ h \in \text{Hom}_{\text{PL}}(\Sigma) \quad \text{and} \quad \psi \in \text{Cyl}_{\text{PL}}.
\]

We can construct such distributions by “averaging” over the group \( \text{Hom}_{\text{PL}}(\Sigma) \). The infinite size of \( \text{Hom}_{\text{PL}}(\Sigma) \) makes a precise definition of the group average procedure very subtle. Here we follow the procedure used for the analytic category \( \text{G} \).

An inner product for the space of solutions is given by the same formula that defines \( \text{Hom}_{\text{PL}}(\Sigma) \) in the norm provided by the inner product defined by

\[
\langle S, \bar{S} \rangle = \sum_{[h] \in \text{GS}(\gamma)} \langle S_{\text{G}_h \gamma}, \bar{S}_{\text{G}_h \gamma} \rangle.
\]

where \( \delta_{[\gamma],[\delta]} \) is non vanishing only if there is a homeomorphism \( h_0 \in \text{Hom}_{\text{PL}}(\Sigma) \) that maps \( \gamma \) to \( \delta \), \( a([\gamma]) \) is a normalization parameter, and \( h \in \text{Hom}_{\text{PL}}(\Sigma) \) is any element in the class of \([h] \in \text{GS}(\gamma)\). The discrete group \( \text{GS}(\gamma) \) is the group of symmetries of \( \gamma \); i.e. elements of \( \text{GS}(\gamma) \) are maps between the edges of \( \gamma \). The group can be constructed from subgroups of \( \text{Hom}_{\text{PL}}(\Sigma) \) as follows: \( \text{GS}(\gamma) = \text{Iso}(\gamma) / \text{TA}(\gamma) \) where \( \text{Iso}(\gamma) \) is the subgroup of \( \text{Hom}_{\text{PL}}(\Sigma) \) that maps \( \gamma \) to itself, and the elements of \( \text{TA}(\gamma) \) are the ones that preserve all the edges of \( \gamma \) separately.

The Hilbert space of physical states \( \mathcal{H}_{\text{diff}} \) is obtained after completing the space spanned by the s-knot states \( \eta(\text{Cyl}_{\text{PL}}) \) in the norm provided by the inner product defined by
\[
(F, G) = \langle \eta(f), \eta(g) \rangle := G[f] + F[\bar{g}] \quad \text{ provided by the inner product defined by}
\]

Define the algebra \( \mathcal{A}_{\text{diff}} \) to be the algebra of operators on \( \mathcal{H}_{\text{kin}} \) satisfying the following
two properties: First, for $O \in \mathcal{A}_{PL}^{\mathcal{diff}}$, both $O$ and $O^\dagger$ are defined on $Cyl_{PL}$ and map $Cyl_{PL}$ to itself. Second, both $O$ and $O^\dagger$ are representable in $\mathcal{H}_{\mathcal{diff}}$ by means of

$$r_{PL}(\hat{O})F = r_{PL}(\hat{O})\eta(f) := \eta(\hat{O}f) \quad (4.6)$$

$\mathcal{A}_{\mathcal{diff}}$ is the analog of the algebra of weak “observables.” Different weak observables can be weakly equivalent; in the same way, many operators of $\mathcal{A}_{\mathcal{diff}}$ are represented by the same operator in $\mathcal{H}_{\mathcal{diff}}$. For example, $r_{PL}(\hat{U}_h) = r_{PL}(1) = 1$. We can define the algebra of classes of operators of $\mathcal{A}_{\mathcal{diff}}$ that are represented by the same operator in $\mathcal{H}_{\mathcal{diff}}$; this algebra is faithfully represented in $\mathcal{H}_{\mathcal{diff}}$ and is called the algebra of physical operators $\mathcal{A}_{\mathcal{diff}}$ \([17]\). Even more, it is easy to prove that every operator on $\mathcal{H}_{\mathcal{diff}}$ is in the image of $r_{PL}(\mathcal{A}_{\mathcal{diff}})$.

The algebra of strong observables (Hermitian operators invariant under gauge transformations and “diffeomorphisms”) sits inside of $\mathcal{A}_{\mathcal{diff}}$ (with the commutator as product); then it is representable in $\mathcal{H}_{\mathcal{diff}}$ faithfully.

Since (\(\square\)) maps any observable to a Hermitian operator in $\mathcal{H}_{\mathcal{diff}}$, this representation implements the reality conditions. In particular (when the space manifold is three dimensional and the gauge group is $SU(2)$), the construction provides a “quantum Husain-Kuchař model” \([8]\), that has local degrees of freedom \([8]\).

An interesting feature of the quantum Husain-Kuchař model (and of any other diffeomorphism invariant quantum gauge theory defined over a compact manifold $\Sigma$ with $\dim(\Sigma) = 1, 2, 3$ following our general framework) is that the choice of background structure is not reflected in the resulting quantum theory. To be precise, fix a piecewise linear structure $PL_0$ on $\Sigma$ and construct the algebra of physical operators $\mathcal{A}_{\mathcal{diff}_0}$ (acting on $\mathcal{H}_{\mathcal{diff}_0}$) that it induces. Given another piecewise linear structure $PL_1$ on $\Sigma$ and a piecewise linear homeomorphism connecting both PL structures $h_1 : \Sigma_{PL_0} \rightarrow \Sigma_{PL_1}$, we get a representation of $\mathcal{A}_{\mathcal{diff}_0}$ in $\mathcal{H}_{\mathcal{diff}_1}$ by $r_{PL_1}(O) = \hat{U}_{h_1}^{-1} O \hat{U}_{h_1}$. In fact, $r_{PL_1} : \mathcal{A}_{\mathcal{diff}_0} \rightarrow \mathcal{A}_{\mathcal{diff}_1}$ is onto and it is independent of $h$. Thus we can label the operators of $\mathcal{A}_{\mathcal{diff}_0}$ by the elements of $\mathcal{A}_{\mathcal{diff}_0}$. Using $\mathcal{A}_{\mathcal{diff}_0}$ as a fiducial abstract algebra, the independence of the background PL structure on $\Sigma$ may be stated as follows.

**Theorem 4** Any piecewise linear structure $PL_1$ on a fixed manifold $\Sigma$ of dimension $\dim(\Sigma) = 1, 2, 3$ defines a representation $r_{PL_1}(\mathcal{A}_{\mathcal{diff}_0})$ of $\mathcal{A}_{\mathcal{diff}_0}$. This representation is independent of the piecewise linear structure, in the sense that, given any two piecewise linear structures $PL_1$ and $PL_2$ on $\Sigma$, the representations $r_{PL_1}(\mathcal{A}_{\mathcal{diff}_0})$ and $r_{PL_2}(\mathcal{A}_{\mathcal{diff}_0})$ are unitarily equivalent.

Proof – In dimensions $\dim(\Sigma) = 1, 2, 3$ it is known \([8]\) that any two PL structures $PL_i$ and $PL_0$ are related by a piecewise linear homeomorphism $h_i : \Sigma_{PL_0} \rightarrow \Sigma_{PL_i}$. This implies that $r_{PL_i}(\mathcal{A}_{\mathcal{diff}_0})$ defined above is a representation of $\mathcal{A}_{\mathcal{diff}_0}$. That the representations induced by $PL_1$ and $PL_2$ are equivalent is trivial, $U_{h_2^{-1} \circ h_1} : \mathcal{H}_{\mathcal{diff}_1} \rightarrow \mathcal{H}_{\mathcal{diff}_2}$; that $U_{h_2^{-1} \circ h_1}$ is the required unitary map and it induces an algebra isomorphism. \(\square\)

**B. Physical observables and physical states in the combinatorial category**

Now our task is to find the analog of knot-classes of combinatoric graphs. In section \([\phantom{\square}]\) we reviewed how is that a simplicial complex $K$ encodes combinatorially topological infor-
mation, and how this information can be displayed in its geometric realization |K|. Then, to decide whether or not two combinatoric graphs \( \gamma, \delta \in \Gamma_C \) belong to the same knot-class we are going to display them in the same space and compare them.

To this end, we fix the sequence of piecewise linear maps

\[
M_n : |Sd^n(K)| \rightarrow |K|
\]

(4.7)
defined by successive application of the canonical map \( M_1 : |Sd(K)| \rightarrow |K| \) that maps the vertices of \( |Sd(K)| \) to the baricenter of the corresponding simplex in \( |K| \). Then, we map every every representative \( \{ \gamma_n, c_{n+1} = Sd(\gamma_n), \ldots \} \) of the combinatoric graph \( \gamma \) in to a sequence

\[
\{M_n(\gamma_n, M_{n+1}(\gamma_{n+1}) = M_n(\gamma_n), \ldots \}
\]

(4.8)
that assigns the same geometric graph \( |\gamma| := M_n(\gamma_n) \) to every integer. Using these maps we are going to define that the combinatoric graphs \( \gamma, \delta \in \Gamma_C \) are “diffeomorphic” if the their corresponding geometrical graphs \( |\gamma|, |\delta| \) are related by a piecewise linear homeomorphism.

One method in implementing the above idea is to use the sequence of maps \( M_n \) to induce a map that links the kinematical Hilbert spaces of the combinatoric and PL categories. The map \( M : Cyr_C \rightarrow Cyr_{PL} \) is defined by

\[
M(f_\gamma) := f_{M_n(\gamma_n)} = f_{|\gamma|}.
\]

(4.9)
Now the map \( \eta : Cyr_{PL}(\overline{A/G}) \rightarrow Cyr_{PL}(\overline{A/G}) \) induces a new map \( \eta_C : Cyr_C(\overline{A/G}) \rightarrow Cyr_C(\overline{A/G}) \)

\[
\eta_C := M^* \circ \eta \circ M : Cyr_C \rightarrow Cyr_C^*
\]

(4.10)
that produces “diffeomorphism” invariant distributions in the combinatoric category. Again, we characterize the averaging map by the s-knot states \( s_{[\gamma]} g, j(e), c(v) \in Cyr_C^* \) induced by the combinatoric spin network states \( S_{\tilde{\gamma}, j(e), c(v)} \)

\[
s_{[\gamma]} g, j(e), c(v)[S_{\tilde{\gamma}, j(e), c(v)}] = \eta_C[S_{\tilde{\gamma}, j(e), c(v)}][S_{\tilde{\gamma}, j(e), c(v)}] := s_{[\gamma]} g, j(e), c(v)[S_{\tilde{\gamma}', j(e), c(v)}]
\]

(4.11)
As follows from the above formula, the label \([\gamma]_C \) of the s-knot states is an equivalence class of oriented combinatoric graphs, where \( \tilde{\gamma} \) and \( \tilde{\delta} \) are considered equivalent if there is \( h \in Hom_{PL}(|K|) \) such that \( h([\gamma]) = [\delta] \).

Just as in the PL case, the Hilbert space of physical states \( \mathcal{H}_{diff} \) is obtained after completing the space spanned by the s-knot states \( \eta_C(Cyr_C(\overline{A/G})) \) in the norm provided by the inner product defined by

\[
(F, G) = (\eta_C(f), \eta_C(g)) := G[f] \quad .
\]

(4.12)
It may seem odd that we are constructing the space of “diffeomorphism” states without a family of unitary maps called “diffeomorphisms”. The reason for this peculiarity is behind the very beginning of our construction. We chose to represent space combinatorially with a sequence generated by the simplicial complex \( K \), and we did not consider the sequence
generated by other complex, say $L$, even if it had the same topological information $|K| \approx |L|$. If we had done that, we would have ended with a kinematical Hilbert space that would be made of two copies of the one that we defined here, and these two copies would be linked by “diffeomorphisms”. What we did was to construct every thing above the minimal kinematical Hilbert space. A relevant question is if by shrinking the kinematical Hilbert space we also shrank the space of physical states. Below, we will prove that this is not the case.

Now we state two important characteristics of the spaces of physical states of the combinatoric and PL models.

First, we constructed the space $H_{\text{diff}}^C$ using the map $\eta_C$; the same map can be restricted to give an onto map from the spin network basis of $H'_\text{kinC}$ to the basis of $H_{\text{diff}}^C$. Since the kinematical Hilbert space is separable, we have the following physically interesting result.

**Theorem 5** The Hilbert space $H_{\text{diff}}^C$ is separable.

Second, the map $M^*: \text{Cyl}^r_{\text{PL}} \rightarrow \text{Cyl}^r_C$ can be extended by continuity to link the spaces of physical states of the PL and combinatoric categories. Using this map we can compare these two spaces.

**Theorem 6** The spaces of physical states in the PL and combinatoric categories are naturally isomorphic, $H_{\text{diffPL}} \approx H_{\text{diffC}}$.

**Proof** – If $\gamma^r_{\text{PL}} = |\gamma|$ then $M^*$ identifies the s-knot states that they generate by averaging, in other words, $M^*(s_{[\gamma^r_{\text{PL}}],j(e),c(v)}) = s_{[\gamma]^C,j(e),c(v)}$. From the definition of the inner products and the definition of the combinatoric s-knot states it follows immediately that $M^*$ is an isometry.

Since the spaces of physical states were constructed by completing the vector spaces spanned by the s-knot states, the theorem is a consequence of the following lemma, which will be proved in the appendix.

**Lemma 5** In any knot-class of PL oriented graphs $[\gamma^r_{\text{PL}}]$ there is at least one representative that comes from the geometric representation of a combinatoric oriented graph $|\gamma| \in [\gamma^r_{\text{PL}}]$.

Now we proceed to construct a representation of the algebra of physical operators in the combinatoric category. As in the PL category, we define the algebra $A_{\text{diffC}}'$ to be the algebra of operators on $H_{\text{kinC}}$ that satisfy the following two conditions: First, for $O \in A_{\text{diffC}}'$, both $O$ and $O^\dagger$ are defined on $Cyl^r_C$ and map $Cyl^r_C$ to itself. Second, both $O$ and $O^\dagger$ are representable in $H_{\text{diffC}}$ by means of

$$r_C(\hat{O})F = r_C(\hat{O})\eta_C(f) := \eta_C(\hat{O}f).$$

(4.13)

We are interested in the algebra of classes of operators of $A_{\text{diffC}}'$ that are represented by the same operator in $H_{\text{diffC}}$; this algebra is faithfully represented in $H_{\text{diffC}}$ and is called the algebra of physical operators $A_{\text{diffC}}$. In contrast with the PL case, in the combinatoric framework the “diffeomorphism group” does not have a natural action; for this reason the notion of strong observables can not be intrinsically defined. However, it is easy to prove that in the PL case the subset of $A_{\text{diffPL}}$ consisting of Hermitian operators is, in fact, the algebra
of strong observables (with the commutator as product). Therefore, in the combinatoric category we can regard the algebra of Hermitian operators in $A_{\text{diff}_C}$ as the algebra of strong observables; this algebra is naturally represented in $H_{\text{diff}_C}$.

Since (11.13) maps any observable to a Hermitian operator in $H_{\text{diff}_C}$, this representation implements the reality conditions. In particular (when the space manifold is three dimensional and the gauge group is $SU(2)$), the construction provides another “quantum Husain-Kuchař model” [18]. A natural question is whether the PL and combinatoric models are physically equivalent or not. We saw that the algebra $A_{\text{diff}_C(K)}$ is represented in $H_{\text{diff}_C(K)}$ by $r_{C(K)}$; it is also natural to give the representation $d_K(A_{\text{diff}_C(K)})$ on $H_{\text{diff}_{PL}(K)}$ by $d_K(\hat{O})F_{PL} = d_K(\hat{O})(\eta \circ Mf_C) := \eta \circ M(\hat{O}f_C)$. This two representations are identified by the isomorphism exhibited in (3), more precisely:

**Theorem 7** The representations $r_{C(K)}(A_{\text{diff}_C(K)})$ on $H_{\text{diff}_C(K)}$ and $d_K(A_{\text{diff}_C(K)})$ on $H_{\text{diff}_{PL}(K)}$ of the algebra $A_{\text{diff}_C(K)}$ are unitarily equivalent. In addition if dim$(\Sigma) = 1, 2, 3$ this algebra does not depend on $K$ but only on the topology of $|K| \approx \Sigma$; the combinatoric and PL frameworks (based on the choice of the Ashtekar-Lewandowski measure $\mu_0$ on $H_{\text{kin}}$) provide unitarily equivalent representations of the abstract algebra $A_{\text{diff}_\Sigma}$.

Proof – The unitary equivalence of $r_{C(K)}(A_{\text{diff}_C(K)})$ and $d_K(A_{\text{diff}_C(K)})$ is given by the unitary map $M^* : H_{\text{diff}_{PL}(K)} \to H_{\text{diff}_C(K)}$.

$d_K(A_{\text{diff}_C(K)})$ maps $A_{\text{diff}_C(K)}$ onto the algebra of operators on $H_{\text{diff}_{PL}(K)}$ and the representation is faithful; the same thing happens for the combinatoric model based on a different simplicial complex $L$. From theorem (11) we know that if dim$(\Sigma) = 1, 2, 3$ for any two simplicial complexes $K, L$ such that $|K| \approx |L| \approx \Sigma$ the Hilbert spaces $H_{\text{diff}_{PL}(K)}, H_{\text{diff}_{PL}(L)}$ and the algebras of operators on them are identified (unambiguously) by a unitary map. Since $d_K(A_{\text{diff}_C(K)}), d_L(A_{\text{diff}_C(L)}), d_K(A_{\text{diff}_{PL}(K)}), d_L(A_{\text{diff}_{PL}(L)})$ label the operators on $H_{\text{diff}_{PL}(\Sigma)}$, there is an unambiguous invertible map identifying these algebras. Thus the family of all these equivalent algebras may be regarded as the abstract algebra $A_{\text{diff}_\Sigma}$ and the combinatoric and PL frameworks are procedures that yield unitarily equivalent representations of this abstract algebra. □

From the theorems it follows that the PL and combinatoric frameworks are physically equivalent. They yield representations of the algebra of physical observables in separable Hilbert spaces; hence, maintaining the usual interpretation of quantum field theory [19].

**V. DISCUSSION AND COMPARISON**

In this paper we have presented two models for quantum gauge field theory. We proved that the two models represented the algebra of physical observables in separable Hilbert spaces $H_{\text{diff}_{PL}}$ and $H_{\text{diff}_C}$; furthermore, we proved that the two models where physically equivalent in the sense that they gave rise to unitarily equivalent representations of the algebra of physical observables. The equivalence of the two models is a good feature, but we may still ask if by choosing a different background structure (like a different PL structure for our base space manifold) we could have arrived at a physically different model. In contrast to the analytic case, this problem has been thoroughly studied (see for example [20]). For example, in dimensions dim$(\Sigma) = 1, 2, 3$ any two PL structures, like any two
differential structures, of a fixed topological manifold $\Sigma$ are known to be equivalent in the sense that they are related by a PL homeomorphism (diffeomorphism). Then, if the base space is three dimensional (like in canonical quantum gravity) all the different choices of background structure would yield unitarily equivalent representations of the algebra of physical (gauge and diffeomorphism invariant) observables (the unitary map given by a quantum “diffeomorphism”).

Our quantum models are not equivalent to the ones created in the analytic category \[6\]; for instance, in the analytic category the physical Hilbert space is not separable. The reason for this size difference is not that the family of piecewise analytic graphs is too big; the kinematic Hilbert space of the PL category is also not separable. In a separate paper \[21\] we show that concept of knot-classes that should be used in the piecewise analytic category is with respect to the group of maps defined by

$$P_{\text{diff}}(\Sigma) := \{ h \in \text{Hom}(\Sigma) : h(\Gamma_\omega) = \Gamma_\omega \} \quad .$$

(5.1)

In the appendix we show how to adapt the proof of lemma \[3\] to show that every (modified) knot-class of piecewise analytic graphs has a representative induced by a combinatorial graph. Then, theorem \[3\] and theorem \[7\] have analogs proving that the the Hilbert space of physical states of the piecewise analytic category is also separable and that the representation of the algebra of physical observables given by the piecewise analytic category is unitarily equivalent to the one provided by the combinatorial framework.

We can expect (the author does) that a more satisfactory understanding of field theory may arise from this combinatorial picture of quantum geometry. The bridge between three-dimensional quantum geometry and a smooth macroscopic space-time is the missing ingredient to complete this picture of quantum field theory. Three unsolved problems prevent us from building this bridge. Dynamics in quantum gravity is only partially understood \[22\]. The emergence of a four-dimensional picture from solutions to the constraints has just begun to be explored \[23\]. And the statistical mechanics needed to find the semiclassical/macroscopic behavior of the theory of quantum geometry is also at its developing stage \[24\].

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APPENDIX

First we will prove Lemma 4, and then, indicate how the proof can be extended to link our models and the refined version of the analytic category that was mentioned in section V.

Given an oriented PL graph $\gamma_{PL} \subset |K|$ we will construct an oriented combinatoric graph $\gamma$ and a piecewise linear homeomorphism (PL map) $h : |K| \rightarrow |K|$ such that $h(|\gamma|) = \gamma_{PL}$.

The construction has four steps.

1. Let $\gamma'_{PL}$ be a refinement of $\gamma_{PL}$ such that for every $\Delta(x_n) \in |K|$ $e \in E_{\gamma'_{PL}}$ implies that $e \cap \Delta(x_n)$ is empty or linear according to the affine coordinates given by $\Delta(x_n)$.

2. Find $n$ such that $M_n(|Sd^n(K)|)$ separates the vertices of $\gamma'_{PL}$ to lie in different geometric simplices $M_n(\Delta(x_n))$, where $\Delta(x_n) \in |Sd^n(K)|$. Namely, we chose $n$ as big as necessary to accomplish a fine enough refinement of $|K|$, where $v_1, v_2 \in M_n(\Delta(x_n))$ for two different vertices of the PL graph $v_1, v_2 \in V_{\gamma'_{PL}}$ does not happen.

3. Let $h_1 : |K| \rightarrow |K|$ be the PL map that fixes the vertices of $M_n(|Sd^n(K)|)$ and sends the new vertices $M_{n+1}(v(\Delta(x_n)))$ of $M_{n+1}(|Sd^{n+1}(K)|)$ to

   a) $v \in V_{\gamma'_{PL}}$ if $v$ lies in the interior of $M_n(\Delta(x_n))$; symbolically, $v \in (M_n(\Delta(x_n)))^o$.

   b) the baricenter of $M_n(\Delta(x_n))$ if there is no $v \in V_{\gamma'_{PL}}$ such that $v \in (M_n(\Delta(x_n)))^o$.

4. Find $m$ such that $h_1(M_{n+m}(|Sd^{n+m}(K)|))$ separates the edges of $\gamma_{PL}$. Stated formally, find $m \geq 1$ such that $\gamma_{PL} \cap h_1(M_{n+m}(\Delta(x_{n+m})))^o$ has one connected component or it is empty.

5. Let $h = h_2 \circ h_1 : |K| \rightarrow |K|$, where $h_2$ is the PL map that fixes the vertices of $h_1(M_{n+m}(|Sd^{n+m}(K)|))$ and sends the new vertices $h_1(M_{n+m+1}(v(\Delta(x_{n+m}))))$ of $h_1(M_{n+m+1}(|Sd^{n+m+1}(K)|))$ to

   a) the baricenter of $\gamma_{PL} \cap h_1(M_{n+m}(\Delta(x_{n+m})))$ if $\gamma_{PL} \cap (h_1(M_{n+m}(\Delta(x_{n+m}))))^o \neq \emptyset$.

   b) the baricenter of $h_1(M_{n+m}(\Delta(x_{n+m})))$ if $\gamma_{PL} \cap (h_1(M_{n+m}(\Delta(x_{n+m}))))^o = \emptyset$.

From the construction of $h \circ M_{n+m} : |Sd^{n+m}(K)| \rightarrow |K|$ it is immediate that $(h \circ M_{n+m})^{-1}(\gamma_{PL}) = |\gamma_{n+m}|$ if $\gamma_{n+m} \subset Sd^{n+m}(K)$ is defined by

- The zero-dimesional simplex $p \in Sd^{n+m}(K)$ belongs to $\gamma_{n+m}$ if $(h \circ M_{n+m})^{-1}(\gamma_{PL}) \cap |p| \neq \emptyset$.

- The one-dimesional simplex $e \in Sd^{n+m}(K)$ belongs to $\gamma_{n+m}$ if $(h \circ M_{n+m})^{-1}(\gamma_{PL}) \cap |e|^o \neq \emptyset$.

Then the obvious orientation of $\gamma_{n+m}$ defines the oriented combinatoric graph $\gamma$ and the pair $h, \gamma$ satisfies

$$ h(|\gamma|) = \gamma_{PL} \quad (5.2) $$
To link the combinatoric and the analytic categories we need to fix a map $N_0 : |K| \to \Sigma_{P_\omega}$ that assigns a piecewise analytic curve in $\Sigma_{P_\omega}$ to every PL curve of $|K|$. Then the map $N : CyI_C \to CyI_\omega$ defined by

$$N(f) := f_{N_0 \circ M_n(|\gamma|)} = f_{N_0(|\gamma|)}$$

links the kinematical Hilbert spaces, and the map $N^* : CyI^*_\omega \to CyI^*_C$ links the spaces of physical states of the analytic and combinatoric categories. As it was argued in section IV $N^*$ is an isometry between $H_{diff_\omega}$ and $H_{diff_C}$, which means that the two Hilbert spaces are isomorphic if every knot-class of piecewise analytic graphs $[\gamma_\omega]$ has at least one representative that comes from a combinatoric graph $N_0(|\gamma|) \in [\gamma_\omega]$.

An extension of the lemma proved in this appendix solves the issue. Given a piecewise analytic graph $\gamma_\omega \subset \Sigma_{P_\omega}$ we can construct a combinatoric graph $\gamma$ and a piecewise analytic map $\phi$ such that $\phi \circ N_0(|\gamma|) = \gamma_\omega$. First find a refinement $\gamma'_\omega$ of $\gamma_\omega$ such that its edges are analytic according to the domains of analycity of $\Sigma_{P_\omega}$. Then define a graph in $|K|$ by $\alpha = N_0^{-1}(|\gamma'_\omega|)$ and do steps (2), (3) and (4) using $\alpha$ instead of $\gamma_{PL}$. At this moment $N_0 \circ h_1 \circ M_{n+m}(|Sd^{n+m}(K)|)$ separates the edges of $\gamma'_\omega$; we only need to find a replacement for step (5). Our strategy is to find a map of the form $\phi = \phi_2 \circ N_0 \circ h_1$ to solve the problem. This would be achieved if the piecewise analytic diffeomorphism $\phi_2$ fixes the mesh given by $N_0 \circ h_1 \circ M_{n+m}(|Sd^{n+m}(K)|)$ and at the same time matches the mesh given by $N_0 \circ h_1 \circ M_{n+m+1}(|Sd^{n+m+1}(K)|)$ and the graph $\gamma'_\omega$. The map $\phi_2$ needs to send every cell $N_0 \circ h_1 \circ M_{n+m}(\Delta(x_{n+m}))$ to itself and match the graph with analytic edges. An explicit construction would be cumbersome, but the existence of such a piecewise analytic map is clear. After this is completed, the construction of the combinatoric graph follows the instructions given above to link the combinatoric and PL categories.
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