ORBIFOLDS OF LATTICE VERTEX ALGEBRAS
UNDER AN ISOMETRY OF ORDER TWO

BOJKO BAKALOV AND JASON ELSINGER

Abstract. Every isometry $\sigma$ of a positive-definite even lattice $Q$ can be lifted to an automorphism of the lattice vertex algebra $V_Q$. An important problem in vertex algebra theory and conformal field theory is to classify the representations of the $\sigma$-invariant subalgebra $V_Q^\sigma$ of $V_Q$, known as an orbifold. In the case when $\sigma$ is an isometry of $Q$ of order two, we classify the irreducible modules of the orbifold vertex algebra $V_Q^\sigma$ and identify them as submodules of twisted or untwisted $V_Q$-modules. The examples where $Q$ is a root lattice and $\sigma$ is a Dynkin diagram automorphism are presented in detail.

1. Introduction

The notion of a vertex algebra introduced by Borcherds \cite{B1} provides a rigorous algebraic description of two-dimensional chiral conformal field theory (see e.g. \cite{BPZ, Go, DMS}), and is a powerful tool for studying representations of infinite-dimensional Lie algebras. The theory of vertex algebras has been developed in \cite{FLM2, K2, FB, LL, KRR} among other works. One of the spectacular early applications of vertex algebras was Borcherds’ proof of the moonshine conjectures about the Monster group (see \cite{B2, Ga}), which used essentially the Frenkel–Lepowsky–Meurman construction of a vertex algebra with a natural action of the Monster on it \cite{FLM2}.

The FLM vertex algebra was constructed in three steps (see \cite{FLM2, Ga}). First, one constructs a vertex algebra $V_Q$ from any even lattice $Q$ (see \cite{B1, FLM2}) by generalizing the Frenkel–Kac realization of affine Kac–Moody algebras in terms of vertex operators \cite{FK}. Second, given an isometry $\sigma$ of $Q$, one lifts it to an automorphism of $V_Q$ and constructs the so-called $\sigma$-twisted $V_Q$-modules \cite{D2, FFR} that axiomatize the properties of twisted vertex operators \cite{KP, LL, FLM1}. Then the FLM vertex algebra is defined as the direct sum of the $\sigma$-invariant
part $V_Q^\sigma$ and one of the $\sigma$-twisted $V_Q$-modules, in the special case when $\sigma = -1$ and $Q$ is the Leech lattice.

In general, if $\sigma$ is an automorphism of a vertex algebra $V$, the $\sigma$-invariants $V^\sigma$ form a subalgebra of $V$ known as an orbifold, which is important in conformal field theory (see e.g. [DVVV] among many other works). Every $\sigma$-twisted representation of $V$ becomes untwisted when restricted to $V^\sigma$. It is a long-standing conjecture that all irreducible $V^\sigma$-modules are obtained by restriction from twisted or untwisted $V$-modules. Under certain assumptions, this conjecture has been proved recently by M. Miyamoto [M1, M2, M3], and he has also shown that the vertex algebra $V^\sigma$ is rational, i.e., every (admissible) module is a direct sum of irreducible ones.

In this paper, we are concerned with the case when $Q$ is a positive-definite even lattice and $\sigma$ is an isometry of $Q$ of order two. We classify and construct explicitly all irreducible modules of the orbifold vertex algebra $V_Q^\sigma$, and we realize them as submodules of twisted or untwisted $V_Q$-modules. In the important special case when $\sigma = -1$, the classification was done previously by Dong–Nagatomo and Abe–Dong [DN, AD]. Our approach is to restrict from $V_Q^\sigma$ to $V_L^\sigma$ where $L$ is the sublattice of $Q$ spanned by eigenvectors of $\sigma$. The subalgebra $V_L^\sigma$ factors as a tensor product $V_L^+ \otimes V_L^-$, where $L_\pm$ consists of $\alpha \in Q$ with $\sigma\alpha = \pm\alpha$, respectively. By the results of [DLM2, ABD, DJL], every $V_L^\sigma$-module is a direct sum of irreducible ones. We describe explicitly the irreducible $V_L^\sigma$-modules using [FHL, DN, AD], and then utilize the intertwining operators among them [A1, ADL] to determine the irreducible $V_Q^\sigma$-modules. A new ingredient is the introduction of a sublattice $\bar{Q}$ of $Q$ such that $V_Q^\sigma = V_{\bar{Q}}^\sigma$ and $\sigma$ has order 2 as an automorphism of $V_{\bar{Q}}$ (while in general $\sigma$ has order 4 on $V_Q$).

Our work was motivated by the examples when $Q$ is the root lattice of a simply-laced simple Lie algebra and $\sigma$ is a Dynkin diagram automorphism, which are related to twisted affine Kac–Moody algebras [K1]. We also consider the case when $\sigma$ is the negative of a Dynkin diagram automorphism, which is relevant to the Slodowy generalized intersection matrix Lie algebras [S, Be].

The paper is organized as follows. In Section 2, we briefly review lattice vertex algebras and their twisted modules, and the results for $\sigma = -1$ that we need. Our main results are presented in Section 3. The examples when $Q$ is a root lattice of type ADE are discussed in detail in Section 4.
2. Vertex algebras and their twisted modules

In this section, we briefly review lattice vertex algebras and their twisted modules. We also recall the case when \( \sigma = -1 \), which will be used essentially for the general case. Good general references on vertex algebras are \([FLM2, K2, FB, LL, KRR]\).

2.1. Vertex algebras. Recall that a vertex algebra is a vector space \( V \) with a distinguished vector \( 1 \in V \) (vacuum vector), together with a linear map (state-field correspondence)

\[
Y(\cdot, z) : V \otimes V \to V[[z]][z^{-1}] .
\]

Thus, for every \( a \in V \), we have the field \( Y(a, z) : V \to V[[z]] \). This field can be viewed as a formal power series from \( (\text{End } V)[[z, z^{-1}]] \), which involves only finitely many negative powers of \( z \) when applied to any vector.

The coefficients in front of powers of \( z \) in this expansion are known as the modes of \( a \):

\[
Y(a, z) = \sum_{n \in \mathbb{Z}} a(n) z^{-n-1}, \quad a(n) \in \text{End } V .
\]

The vacuum vector \( 1 \) plays the role of an identity in the sense that

\[
a_{(-1)} 1 = 1 a = a , \quad a(n) 1 = 0 , \quad n \geq 0 .
\]

In particular, \( Y(a, z) 1 \in V[[z]] \) is regular at \( z = 0 \), and its value at \( z = 0 \) is equal to \( a \).

The main axiom for a vertex algebra is the Borcherds identity (also called Jacobi identity \([FLM2]\)) satisfied by the modes:

\[
\sum_{j=0}^{\infty} \binom{m}{j} (a_{(n+j)} b)_{(k+m-j)} c = \sum_{j=0}^{\infty} \binom{n}{j} (-1)^j a_{(m+n-j)} (b_{(k+j)} c)
\]

\[
- \sum_{j=0}^{\infty} \binom{n}{j} (-1)^{j+n} b_{(k+n-j)} (a_{(m+j)} c) ,
\]

where \( a, b, c \in V \). Note that the above sums are finite, because \( a(n) b = 0 \) for sufficiently large \( n \).

We say that a vertex algebra \( V \) is (strongly) generated by a subset \( S \subset V \) if \( V \) is linearly spanned by the vacuum \( 1 \) and all elements of the form \( a_1(n_1) \cdots a_k(n_k) 1 \), where \( k \geq 1, a_i \in S, n_i < 0 \).
2.2. Twisted representations of vertex algebras. A representation of a vertex algebra $V$, or a $V$-module, is a vector space $M$ endowed with a linear map $Y(\cdot, z) : V \otimes M \to M((z))$ (cf. (2.1), (2.2)) such that the Borcherds identity (2.3) holds for $a, b \in V, c \in M$ (see [FB, LL, KRR]).

Now let $\sigma$ be an automorphism of $V$ of a finite order $r$. Then $\sigma$ is diagonalizable. In the definition of a $\sigma$-twisted representation $M$ of $V$ [FFR, D2], the image of the above map $Y$ is allowed to have nonintegral rational powers of $z$, so that

$$Y(a, z) = \sum_{n \in \mathbb{Z} + \frac{1}{r}} a(n) z^{-n-1}, \quad \text{if } \sigma a = e^{-2\pi i p} a, \quad p \in \frac{1}{r} \mathbb{Z},$$

where $a(n) \in \text{End} M$. The Borcherds identity (2.3) satisfied by the modes remains the same in the twisted case, provided that $a$ is an eigenvector of $\sigma$. An important consequence of the Borcherds identity is the locality property [DL, Li]:

$$(z - w)^N [Y(a, z), Y(b, w)] = 0$$

for sufficiently large $N$ depending on $a, b$ (one can take $N$ to be such that $a(n)b = 0$ for $n \geq N$).

The following result provides a rigorous interpretation of the operator product expansion in conformal field theory (cf. [Go, DMS]) in the case of twisted modules.

**Proposition 2.1** ([BM]). Let $V$ be a vertex algebra, $\sigma$ an automorphism of $V$, and $M$ a $\sigma$-twisted representation of $V$. Then

$$(2.5) \quad \frac{1}{k!} \partial_z^k \left( (z - w)^N Y(a, z) Y(b, w)c \right) \bigg|_{z = w} = Y(a_{(N-k)}b, w)c$$

for all $a, b \in V, c \in M, k \geq 0$, and sufficiently large $N$. Conversely, (2.4) and (2.5) imply the Borcherds identity (2.3).

Recall from [FHL] that if $V_1$ and $V_2$ are vertex algebras, their tensor product is again a vertex algebra with

$$Y(v_1 \otimes v_2, z) = Y(v_1, z) \otimes Y(v_2, z), \quad v_i \in V_i.$$ 

Furthermore, if $M_i$ is a $V_i$-module, then the above formula defines the structure of a $(V_1 \otimes V_2)$-module on $M_1 \otimes M_2$ (see [FHL]). This is also true for twisted modules.

**Lemma 2.2.** For $i = 1, 2$, let $V_i$ be a vertex algebra, $\sigma_i$ an automorphism of $V_i$, and $M_i$ a $\sigma_i$-twisted representation of $V_i$. Then $M_1 \otimes M_2$ is a $(\sigma_1 \otimes \sigma_2)$-twisted module over $V_1 \otimes V_2$. 
Proof. By Proposition 2.1, it is enough to check (2.4) and (2.5) for $a = a_1 \otimes a_2$ and $b = b_1 \otimes b_2$, given that they hold for the pairs $a_1, b_1 \in V_1$ and $a_2, b_2 \in V_2$. This is done by a straightforward calculation. □

2.3. Lattice vertex algebras. Let $Q$ be an integral lattice, i.e., a free abelian group of finite rank equipped with a symmetric nondegenerate bilinear form $\langle \cdot, \cdot \rangle : Q \times Q \to \mathbb{Z}$. We will assume that $Q$ is even, i.e., $|\alpha|^2 = \langle \alpha, \alpha \rangle \in 2\mathbb{Z}$ for all $\alpha \in Q$. We denote by $\mathfrak{h} = \mathbb{C} \otimes_\mathbb{Z} Q$ the corresponding complex vector space considered as an abelian Lie algebra, and extend the bilinear form to it.

The Heisenberg algebra $\hat{\mathfrak{h}} = \mathfrak{h}[t, t^{-1}] \oplus \mathbb{C}K$ is the Lie algebra with brackets

\begin{equation}
[a_m, b_n] = m\delta_{m,-n}(a|b)K, \quad a_m = at^m,
\end{equation}

where $K$ is central. Its irreducible highest-weight representation

$\mathcal{F} = \text{Ind}_{\mathfrak{h}[t] \subseteq \mathbb{C}K} \mathbb{C} \cong S(\mathfrak{h}[t^{-1}] t^{-1})$

on which $K = 1$ is known as the (bosonic) Fock space.

Following [FK, B1], we consider a 2-cocycle $\varepsilon : Q \times Q \to \{\pm 1\}$ such that

\begin{equation}
\varepsilon(\alpha, \alpha) = (-1)^{|\alpha|^2(|\alpha|^2 + 1)/2}, \quad \alpha \in Q,
\end{equation}

and the associative algebra $\mathbb{C}[Q]$ with basis $\{e^\alpha\}_{\alpha \in Q}$ and multiplication

\begin{equation}
e^\alpha e^\beta = \varepsilon(\alpha, \beta)e^{\alpha+\beta}.
\end{equation}

Such a 2-cocycle $\varepsilon$ is unique up to equivalence and can be chosen to be bimultiplicative. In addition,

\begin{equation}
\varepsilon(\alpha, \beta)\varepsilon(\beta, \alpha) = (-1)^{\langle \alpha, \beta \rangle}, \quad \alpha, \beta \in Q.
\end{equation}

The lattice vertex algebra associated to $Q$ is defined as $V_Q = \mathcal{F} \otimes \mathbb{C}_\varepsilon[Q]$, where the vacuum vector is $1 = 1 \otimes e^0$. We let the Heisenberg algebra act on $V_Q$ so that

$$a_n e^\beta = \delta_{n,0}(a|\beta)e^\beta, \quad n \geq 0, \quad a \in \mathfrak{h}.$$ 

The state-field correspondence on $V_Q$ is uniquely determined by the generating fields:

\begin{equation}
Y(a_{-1}1, z) = \sum_{n \in \mathbb{Z}} a_n z^{-n-1}, \quad a \in \mathfrak{h},
\end{equation}

\begin{equation}
Y(e^\alpha, z) = e^{\alpha z^{\alpha_0}} \exp\left(\sum_{n < 0} a_n z^{-n}/-n\right) \exp\left(\sum_{n > 0} a_n z^{-n}/-n\right),
\end{equation}

where $z^{\alpha_0} e^\beta = z^{\langle \alpha|\beta \rangle} e^\beta$. 
Notice that \( \mathcal{F} \subset V_Q \) is a vertex subalgebra, which we call the Heisenberg vertex algebra. The map \( h \to \mathcal{F} \) given by \( a \mapsto a_{-1}\mathbf{1} \) is injective. From now on, we will slightly abuse the notation and identify \( a \in h \) with \( a_{-1}\mathbf{1} \in \mathcal{F} \); then \( a_{(n)} = a_n \) for all \( n \in \mathbb{Z} \).

2.4. **Twisted Heisenberg algebra.** Every automorphism \( \sigma \) of \( h \) preserving the bilinear form induces automorphisms of \( \hat{h} \) and \( \mathcal{F} \), which will be denoted again as \( \sigma \). As before, assume that \( \sigma \) has a finite order \( r \). The action of \( \sigma \) can be extended to \( \hat{h}[[t^{1/r}, t^{-1/r}] \oplus \mathbb{C}K \) by letting

\[
\sigma(at^m) = \sigma(a)e^{2\pi i m}, \quad \sigma(K) = K, \quad a \in h, \ m \in \frac{1}{r}\mathbb{Z}.
\]

The \( \sigma \)-twisted Heisenberg algebra \( \hat{h}_\sigma \) is defined as the set of all \( \sigma \)-invariant elements (see e.g. [KP, Le, FLM1]). In other words, \( \hat{h}_\sigma \) is spanned over \( \mathbb{C} \) by \( K \) and the elements \( a_m = at^m \) such that \( \sigma a = e^{-2\pi i m}a \). This is a Lie algebra with bracket (cf. (2.6))

\[
[a_m, b_n] = m\delta_{m,-n}(a|b)K, \quad a, b \in h, \ m, n \in \frac{1}{r}\mathbb{Z}.
\]

Let \( \hat{h}_\sigma^\geq \) (respectively, \( \hat{h}_\sigma^\leq \)) be the abelian subalgebra of \( \hat{h}_\sigma \) spanned by all elements \( a_m \) with \( m \geq 0 \) (respectively, \( m < 0 \)).

The \( \sigma \)-twisted Fock space is defined as

\[
\mathcal{F}_\sigma = \text{Ind}_{\hat{h}_\sigma^\geq \oplus \mathbb{C}K}^\hat{h}_\sigma \mathbb{C} \cong S(\hat{h}_\sigma^\leq),
\]

where \( \hat{h}_\sigma^\geq \) acts on \( \mathbb{C} \) trivially and \( K \) acts as the identity operator. Then \( \mathcal{F}_\sigma \) is an irreducible highest-weight representation of \( \hat{h}_\sigma \), and has the structure of a \( \sigma \)-twisted representation of the vertex algebra \( \mathcal{F} \) (see e.g. [FLM2, KRR]). This structure can be described as follows. We let \( Y(1, z) \) be the identity operator and

\[
Y(a, z) = \sum_{n \in \mathbb{Z}} a_n z^{-n-1}, \quad a \in h, \ \sigma a = e^{-2\pi ip}a, \ p \in \frac{1}{r}\mathbb{Z},
\]

and we extend \( Y \) to all \( a \in h \) by linearity. The action of \( Y \) on other elements of \( \mathcal{F} \) is then determined by applying several times the product formula (2.5). More explicitly, \( \mathcal{F} \) is spanned by elements of the form \( a_{m_1}^1 \cdots a_{m_k}^k \mathbf{1} \) where \( a^j \in h \), and we have:

\[
Y(a_{m_1}^1 \cdots a_{m_k}^k \mathbf{1}, z)c = \prod_{j=1}^k \partial_{z_j}^{N-1-m_j} \left( \prod_{j=1}^k (z_j - z)^N Y(a_1^1, z_1) \cdots Y(a_k^k, z_k)c \right) \bigg|_{z_1 = \cdots = z_k = z}
\]
for all \( c \in \mathcal{F}_\sigma \) and sufficiently large \( N \). In the above formula, we use the divided-power notation \( \partial^{(n)} = \partial^n / n! \).

### 2.5. Twisted representations of lattice vertex algebras

Let \( \sigma \) be an automorphism (or isometry) of the lattice \( Q \) of finite order \( r \), so that

\[
(\sigma \alpha | \sigma \beta) = (\alpha | \beta), \quad \alpha, \beta \in Q.
\]

The uniqueness of the cocycle \( \varepsilon \) and (2.13), (2.9) imply that

\[
\eta(\alpha + \beta) \varepsilon(\sigma \alpha, \sigma \beta) = \eta(\alpha) \eta(\beta) \varepsilon(\alpha, \beta)
\]

for some function \( \eta : Q \to \{\pm 1\} \).

**Lemma 2.3.** Let \( L \) be a sublattice of \( Q \) such that \( \varepsilon(\sigma \alpha, \sigma \beta) = \varepsilon(\alpha, \beta) \) for \( \alpha, \beta \in L \). Then there exists a function \( \eta : Q \to \{\pm 1\} \) satisfying (2.14) and \( \eta(\alpha) = 1 \) for all \( \alpha \in L \).

**Proof.** First observe that, by (2.7) and (2.13), (2.14) for \( \alpha = \beta \), we have \( \eta(2\alpha) = 1 \) for all \( \alpha \in Q \). Since, by bimultiplicativity, \( \varepsilon(2\alpha, \beta) = 1 \), we obtain that \( \eta(2\alpha + \beta) = \eta(\beta) \) for all \( \alpha, \beta \). Therefore, \( \eta \) is defined on \( Q/2Q \). If \( \alpha_1, \ldots, \alpha_\ell \) is any \( \mathbb{Z} \)-basis for \( Q \), we can set all \( \eta(\alpha_i) = 1 \) and then \( \eta \) is uniquely extended to the whole \( Q \) by (2.14). We can pick a \( \mathbb{Z} \)-basis for \( Q \) so that \( d_1 \alpha_1, \ldots, d_\ell \alpha_\ell \) is a \( \mathbb{Z} \)-basis for \( L \), where \( m \leq \ell \) and \( d_i \in \mathbb{Z} \). Then the extension of \( \eta \) to \( Q \) will satisfy \( \eta(\alpha) = 1 \) for all \( \alpha \in L \). \( \Box \)

In particular, \( \eta \) can be chosen such that

\[
\eta(\alpha) = 1, \quad \alpha \in Q \cap h_0,
\]

where \( h_0 \) denotes the subspace of \( h \) consisting of vectors fixed under \( \sigma \). Then \( \sigma \) can be lifted to an automorphism of the lattice vertex algebra \( V_Q \) by setting

\[
\sigma(a_n) = \sigma(a)_n, \quad \sigma(e^\alpha) = \eta(\alpha) e^{\sigma \alpha}, \quad a \in h, \alpha \in Q.
\]

**Remark 2.4.** The order of \( \sigma \) is \( r \) or \( 2r \) when viewed as an automorphism of \( V_Q \), where \( r \) is the order of \( \sigma \) on \( Q \). The set \( \overline{Q} \) of \( \alpha \in Q \) such that \( \sigma^r(e^\alpha) = e^\alpha \) is a sublattice of \( Q \) of index 1 or 2, and we have \( (V_Q)^{\sigma^r} = V_{\overline{Q}} \).

We will now recall the construction of irreducible \( \sigma \)-twisted \( V_Q \)-modules (see [KP, Le, D2, BK]). Introduce the group \( G = \mathbb{C}^\times \times \exp h_0 \times Q \) consisting of elements \( c e^h U_\alpha (c \in \mathbb{C}^\times, h \in h_0, \alpha \in Q) \) with
multiplication

\[ e^h e^{h'} = e^{h+h'}, \]

\[ e^h U_\alpha e^{-h} = e^{(h|\alpha)} U_\alpha, \]

\[ U_\alpha U_\beta = \varepsilon(\alpha, \beta) B^{-1}_{\alpha,\beta} U_{\alpha+\beta}, \]

where

\[ B_{\alpha,\beta} = r^{-(\alpha|\beta)} \prod_{k=1}^{r-1} (1 - e^{2\pi i k/r})^{(\sigma^k \alpha|\beta)}. \]

Let

\[ C_\alpha = \eta(\alpha) U_{-\sigma\alpha}^{-1} U_\alpha e^{2\pi i (b_\alpha + \pi_0 \alpha)}, \quad b_\alpha = \frac{1}{2} (|\pi_0 \alpha| - |\alpha|^2), \]

where \( \pi_0 \) is the projection of \( h \) onto \( h_0 \). Then \( C_\alpha C_\beta = C_{\alpha+\beta} \) and all \( C_\alpha \) are in the center of \( G \). We define \( G_\sigma \) to be the quotient group \( G/\{C_\alpha\}_{\alpha \in Q} \).

Consider an irreducible representation \( \Omega \) of \( G_\sigma \). Such representations are parameterized by the set \( (Q^*/Q)^\sigma \) of \( \sigma \)-invariants in \( Q^*/Q \), i.e., by \( \lambda + Q \) such that \( \lambda \in Q^* \) and \( (1 - \sigma) \lambda \in Q \) (see [BK, Proposition 4.4]). Furthermore, the action of \( \exp h_0 \) on \( \Omega \) is semisimple:

\[ \Omega = \bigoplus_{\mu \in \pi_0(Q^*)} \Omega_\mu, \]

where

\[ \Omega_\mu = \{ v \in \Omega \mid e^h v = e^{(h|\mu)} v \quad \text{for} \quad h \in h_0 \}. \]

Then \( \mathcal{F}_\sigma \otimes \Omega \) is an irreducible \( \sigma \)-twisted \( V_Q \)-module with an action defined as follows. We define \( Y(a, z) \) for \( a \in h \) as before, and for \( \alpha \in Q \) we let

\[ Y(e^\alpha, z) = \exp \left( \sum_{n \in \mathbb{Z}_{<0}} \frac{\alpha_n}{n} z^{-n} \right) \exp \left( \sum_{n \in \mathbb{Z}_{>0}} \frac{\alpha_n}{n} z^n \right) \otimes U_\alpha z^{b_\alpha + \pi_0 \alpha}. \]

Here the action of \( z^{\pi_0 \alpha} \) is given by \( z^{\pi_0 \alpha} v = z^{(\pi_0 \alpha|\mu)} v \) for \( v \in \Omega_\mu \), and \( (\pi_0 \alpha|\mu) \in \frac{1}{r} \mathbb{Z} \). The action of \( Y \) on all of \( V_Q \) can be obtained by applying the product formula (2.15).

By [BK, Theorem 4.2], every irreducible \( \sigma \)-twisted \( V_Q \)-module is obtained in this way, and every \( \sigma \)-twisted \( V_Q \)-module is a direct sum of irreducible ones. In the special case when \( \sigma = 1 \), we get Dong’s Theorem that the irreducible \( V_Q \)-modules are classified by \( Q^*/Q \) (see [D1]). Explicitly, they are given by:

\[ V_{\lambda+Q} = \mathcal{F} \otimes C e^{[Q]} e^\lambda, \quad \lambda \in Q^*. \]
When the lattice $Q$ is written as an orthogonal direct sum of sublattices, $Q = L_1 \oplus L_2$, we have a natural isomorphism $V_Q \cong V_{L_1} \otimes V_{L_2}$. The following lemma shows that if $L_1$ and $L_2$ are $\sigma$-invariant, there is a correspondence of irreducible twisted modules (cf. [FHL] and Lemma 2.2).

**Lemma 2.5.** Let $Q$ be an even lattice, $Q = L_1 \oplus L_2$ an orthogonal direct sum, and $\sigma$ an automorphism of $Q$ such that $\sigma(L_i) \subseteq L_i$ $(i = 1, 2)$. Set $\sigma_i = \sigma|_{L_i}$. Then every irreducible $\sigma$-twisted $V_Q$-module $M$ is a tensor product, $M \cong M_1 \otimes M_2$, where $M_i$ is an irreducible $\sigma_i$-twisted $V_{L_i}$-module.

**Proof.** This follows from the classification of irreducible $\sigma$-twisted $V_Q$-modules described above. Indeed, $Q = L_1 \oplus L_2$ gives rise to a decomposition of the Heisenberg Lie algebra as a direct sum $\mathfrak{h} = \mathfrak{h}_1 \oplus \mathfrak{h}_2$. We also have a similar decomposition for the corresponding twisted Heisenberg algebras. Then for the twisted Fock spaces, we get $\mathcal{F}_\sigma = \mathcal{F}_{\sigma_1} \otimes \mathcal{F}_{\sigma_2}$. Similarly, the group $G$ is a direct product of its subgroups $G_1$ and $G_2$ associated to the lattices $L_1$ and $L_2$, respectively. \[\square\]

2.6. **The case $\sigma = -1$.** Now we will review what is known in the case when $\sigma = -1$, which will be used essentially in our treatment of the general case. In this subsection, we will denote the even integral lattice by $L$ instead of $Q$. As before, let $\mathfrak{h} = \mathbb{C} \otimes \mathbb{Z} L$ be the corresponding complex vector space.

Observe that $\mathfrak{h}_0 = 0$, $\pi_0 = 0$, and we can assume that $\eta(\alpha) = 1$ for all $\alpha \in L$. Hence, the automorphism $\sigma$ acts on $\mathcal{F}_L$ by

$$\sigma(h^1_{(-n_1)} \cdots h^k_{(-n_k)} e^\alpha) = (-1)^k h^1_{(-n_1)} \cdots h^k_{(-n_k)} e^{-\alpha}$$

for $h^i \in \mathfrak{h}$, $n_i \in \mathbb{Z}_{\geq 0}$ and $\alpha \in L$. The group $G$ consists of $cU_\alpha$ ($c \in \mathbb{C}^\times, \alpha \in L$), and its center consists of $cU_\alpha$ with $\alpha \in 2L^* \cap L$. Then $G_\sigma$ is the quotient of $G$ by $\{U_\alpha^{-1}U_{-\alpha}\}_{\alpha \in L}$. The twisted Heisenberg algebra is $\hat{\mathfrak{h}}_\sigma = \mathfrak{h}[t, t^{-1}]^{1/2} \odot \mathbb{C}K$.

For any $G_\sigma$-module $T$, define $V^T_L = \mathcal{F}_\sigma \otimes T$, where $\mathcal{F}_\sigma$ is the twisted Fock space (cf. (2.12)). By [DY2], every irreducible $\sigma$-twisted $V_L$-module is isomorphic to $V^T_L$ for some irreducible $G_\sigma$-module $T$ on which $cU_0$ acts as $cI$, where $I$ is the identity operator. Such modules $T$ can be described equivalently as $G$-modules, on which $cU_0 = cI$ and $U_\alpha = U_{-\alpha}$. The irreducible ones are determined by the central characters $\chi$ such that $\chi(U_\alpha) = \chi(U_{-\alpha})$ for $\alpha \in 2L^* \cap L$. We have:

$$U^2_\alpha = U_\alpha U_{-\alpha} = \epsilon(\alpha, -\alpha) B^{-1}_{\alpha, -\alpha} U_0,$$
which implies that
\begin{equation}
\chi(U_\alpha) = s(\alpha)e^{\pi i|\alpha|^2(1/2+1/2^{|\alpha|^2})},
\end{equation}
where \(s(\alpha) \in \{\pm 1\}\) satisfies \(s(\alpha + \beta) = s(\alpha)s(\beta)\). All such maps \(s\) have the form
\[s(\alpha) = (-1)^{(\alpha|\mu)}\]
for some \(\mu \in (2L^* \cap L)^*\). The corresponding central characters \(\chi\) will be denoted as \(\chi_\mu\), and the corresponding \(G_\sigma\)-module \(T\) as \(T_\mu\).

We define an action of \(\sigma\) on \(V_L^T\) by
\begin{equation}
\sigma(h_{(-n_1)}^1 \cdots h_{(-n_k)}^k t) = (-1)^k h_{(-n_1)}^1 \cdots h_{(-n_k)}^k t
\end{equation}
for \(h^i \in h, n_i \in \frac{1}{2} + \mathbb{Z}_{\geq 0}\) and \(t \in T\). The eigenspaces for \(\sigma\) are denoted \(V_L^{T,\pm}\). Then we have
\begin{equation}
\sigma(Y(a,z)v) = Y(\sigma a, z)(\sigma v), \quad a \in V_L, \ v \in V_L^T,
\end{equation}
which implies that \(\sigma\) is an automorphism of \(V_L^\sigma\)-modules. In particular, \(V_L^{T,\pm}\) are \(V_L^\sigma\)-modules. Similarly, we define an action of \(\sigma\) on the untwisted \(V_L\)-modules \(V_{\lambda+L}\) by (2.18) for \(\alpha \in \lambda + L\). Note that \(\sigma V_{\lambda+L} \subseteq V_{-\lambda+L}\). Hence, if \(\lambda \in L^*, 2\lambda \in L\), then the eigenspaces \(V_{\lambda+L}^\pm\) are \(V_L^\sigma\)-modules. On the other hand, if \(2\lambda \not\in L\), then \(V_{\lambda+L} \cong V_{-\lambda+L}\) as \(V_L^\sigma\)-modules.

**Theorem 2.6 (DN, AD).** Let \(L\) be a positive-definite even lattice and \(\sigma = -1\) on \(L\). Then any irreducible admissible \(V_L^\sigma\)-module is isomorphic to one of the following:
\[V_{\lambda+L}^\pm (\lambda \in L^*, 2\lambda \in L), \quad V_{\lambda+L} (\lambda \in L^*, 2\lambda \not\in L), \quad V_L^{T,\pm},\]
where \(T\) is an irreducible \(G_\sigma\)-module.

Next, we will discuss intertwining operators between the irreducible \(V_L^\sigma\)-modules. For a vector space \(U\), denote by
\[U\{z\} = \left\{ \sum_{n \in \mathbb{Q}} v(n)z^{-n-1} \Big| v(n) \in U \right\}\]
the space of \(U\)-valued formal series involving rational powers of \(z\). Let \(V\) be a vertex algebra, and \(M_1, M_2, M_3\) be \(V\)-modules, which are not necessarily distinct. Recall from [FHL] that an **intertwining operator** of type \(\begin{pmatrix} M_3 \\ M_1 & M_2 \end{pmatrix}\) is a linear map \(\mathcal{Y}: M_1 \otimes M_2 \to M_3\{z\}\), or equivalently,
\[\mathcal{Y}: M_1 \to \text{Hom}(M_2, M_3)\{z\}, \quad v \mapsto \mathcal{Y}(v, z) = \sum_{n \in \mathbb{Q}} v(n)z^{-n-1}, \quad v(n) \in \text{Hom}(M_2, M_3)\]
such that \( v(n)u = 0 \) for \( n \gg 0 \), and the Borcherds identity \((2.3)\) holds for \( a \in V, b \in M_1 \) and \( c \in M_2 \) with \( k \in \mathbb{Q} \) and \( m, n \in \mathbb{Z} \). The intertwining operators of type \( \left( \begin{array}{c} M_3 \\ M_1 M_2 \end{array} \right) \) form a vector space denoted \( V_{M_1 M_2}^{M_3} \). The fusion rule associated with an algebra \( V \) and its modules \( M_1, M_2, M_3 \) is \( N_{M_1 M_2}^{M_3} = \dim V_{M_1 M_2}^{M_3} \).

The fusion rules for \( V^\sigma_L \) and its irreducible modules were calculated in \([A1, ADL]\) to be either zero or one. In order to present their theorem, we first introduce some additional notation. We assume that \( \lambda \in L^* \) is such that \( 2\lambda \in L \), and we let

\[
\pi_{\lambda, \mu} = (-1)^{|\lambda| |\mu|^2}, \quad \lambda, \mu \in L^*,
\]

\[
c_\chi(\lambda) = (-1)^{(|\lambda| |2\lambda|)} \varepsilon(\lambda, 2\lambda) s(2\lambda).
\]

For any central character \( \chi \) of \( G_\sigma \), let \( \chi^{(\lambda)} \) be the central character defined by

\[
\chi^{(\lambda)}(U_\alpha) = (-1)^{(\alpha|\lambda)} \chi(U_\alpha),
\]

and set \( T^{(\lambda)}_\chi = T^{(\lambda)}_{\chi^{(\lambda)}} \). Note that when \( \chi = \chi_\mu \) and \( T = T_\mu \), we have \( \chi^{(\lambda)}_\mu = \chi_{\lambda+\mu} \) and \( T^{(\lambda)}_\mu = T^{(\lambda+\mu)}_\chi \).

The following theorem is a special case of Theorem 5.1 from \([ADL]\).

**Theorem 2.7** \([ADL]\). Let \( L \) be a positive-definite even lattice and \( \lambda \in L^* \cap \frac{1}{2} L \). Then for two irreducible \( V_L^\sigma \)-modules \( M_2, M_3 \) and for \( \epsilon \in \{\pm\} \), the fusion rule of type \( \left( \begin{array}{c} M_3 \\ V_{\lambda+L}^\sigma M_2 \end{array} \right) \) is equal to 1 if and only if the pair \( (M_2, M_3) \) is one of the following:

\[
\begin{align*}
(V_{\mu+L}, V_{\lambda+\mu+L}), & \quad \mu \in L^*, \ 2\mu \notin L, \\
(V_{\mu+L}^{\epsilon_1}, V_{\lambda+\mu+L}^{\epsilon_2}), & \quad \mu \in L^*, \ 2\mu \in L, \ \epsilon_1 \in \{\pm\}, \ \epsilon_2 = \epsilon_1 \epsilon \pi_{\lambda, 2\mu}, \\
(V_L^{T_\chi, \epsilon_1}, V_L^{T_\chi^{(\lambda)}, \epsilon_2}), & \quad \epsilon_1 \in \{\pm\}, \ \epsilon_2 = c_\chi(\lambda) \epsilon_1 \epsilon.
\end{align*}
\]

In all other cases, the fusion rules of type \( \left( \begin{array}{c} M_3 \\ V_{\lambda+L}^\epsilon M_2 \end{array} \right) \) are zero.

### 3. Classification of irreducible modules

In this section, we prove our main result, the classification of all irreducible modules over the orbifold vertex algebra \( V_Q^\sigma \). As before, \( Q \) is a positive-definite even integral lattice and \( \sigma \) is an isometry of \( Q \) of order 2.
3.1. The sublattice $\bar{Q}$. The map $\sigma$ is extended by linearity to the complex vector space $\mathfrak{h} = \mathbb{C} \otimes \mathbb{Z} Q$. We will denote by

\begin{equation}
\pi_\pm = \frac{1}{2} (1 \pm \sigma), \quad \alpha_\pm = \pi_\pm (\alpha)
\end{equation}

the projections onto the eigenspaces of $\sigma$. Introduce the important sublattices

\begin{equation}
L_\pm = \mathfrak{h}_\pm \cap Q, \quad L = L_+ \oplus L_- \subseteq Q,
\end{equation}

where $\mathfrak{h}_\pm = \pi_\pm (\mathfrak{h})$. Note that $\mathfrak{h} = \mathfrak{h}_+ \oplus \mathfrak{h}_-$ is an orthogonal direct sum.

**Lemma 3.1.** We have $\alpha_\pm \in (L_\pm)^* \subseteq L^*$ for any $\alpha \in Q$.

**Proof.** Indeed,

\[(\alpha_+ | \beta) = (\alpha_+ + \alpha_- | \beta) = (\alpha | \beta) \in \mathbb{Z}\]

for all $\alpha \in Q$, $\beta \in L_+ \subseteq Q$. \hfill \Box

Observe that $2\alpha_\pm \in L_\pm$ and $|\alpha_\pm|^2 = \frac{1}{4} |2\alpha_\pm|^2 \in \frac{1}{2} \mathbb{Z}$ for all $\alpha \in Q$.

**Lemma 3.2.** For $\alpha \in Q$, the following are equivalent:

(i) $\sigma^2(e^\alpha) = e^\alpha$,

(ii) $|\alpha_\pm|^2 \in \mathbb{Z}$,

(iii) $(\alpha | \sigma \alpha) \in 2\mathbb{Z}$.

**Proof.** Note that $4 |\alpha_\pm|^2 = |\alpha \pm \sigma \alpha|^2 = 2|\alpha|^2 \pm 2(\alpha | \sigma \alpha)$, so that

\[|\alpha_\pm|^2 = \frac{1}{2} (\alpha | \sigma \alpha) \mod \mathbb{Z}.\]

This shows the equivalence between (ii) and (iii).

Using (2.16), we find $\sigma^2(e^\alpha) = \eta(\alpha) \eta(\sigma \alpha)e^\alpha$. On the other hand, by (2.9), (2.14) and (2.15), we have

\[\eta(\alpha) \eta(\sigma \alpha) = \varepsilon(\alpha, \sigma \alpha) \varepsilon(\sigma \alpha, \alpha) = (-1)^{(\alpha | \sigma \alpha)}.\]

This proves the equivalence between (i) and (iii). \hfill \Box

From now on, we let

\begin{equation}
\bar{Q} = \{ \alpha \in Q \mid (\alpha | \sigma \alpha) \in 2\mathbb{Z} \}.
\end{equation}

It is clear that $\bar{Q}$ is $\sigma$-invariant.

**Lemma 3.3.** The subset $\bar{Q}$ is a sublattice of $Q$ of index 1 or 2.

**Proof.** For $\alpha, \beta \in Q$, we have

\[(\alpha - \beta | \sigma \alpha - \sigma \beta) = (\alpha | \sigma \alpha) + (\beta | \sigma \beta) \mod 2\mathbb{Z},\]

since

\[(\alpha | \sigma \beta) = (\sigma \alpha | \sigma^2 \beta) = (\beta | \sigma \alpha).\]

Now if $\alpha, \beta \in \bar{Q}$ or $\alpha, \beta \notin \bar{Q}$, then $\alpha - \beta \in \bar{Q}$. \hfill \Box
As a consequence of Lemmas 3.1 and 3.2, we obtain:

**Corollary 3.4.** The lattices $\mathbb{Z}\alpha_\pm + L$ are integral for all $\alpha \in \bar{Q}$.

By definition, we have $(V_Q)^\sigma = V_Q$, and

$$V_Q^\sigma = ((V_Q)^{\sigma^2})^\sigma = V_Q^\sigma.$$  

Therefore, we may assume that $|\sigma| = 2$ on $V_Q$ and only work with the sublattice $\bar{Q}$. For simplicity, we use $Q$ instead of $\bar{Q}$ for the rest of this section.

### 3.2. Restricting the orbifold $V_Q^\sigma$ to $V_L^\sigma$.

By [FHL, LL], we have that the subalgebra $V_L$ of $V_Q$ is isomorphic to the tensor product $V_L^+ \otimes V_L^-$, since $L = L_+ \oplus L_-$ is an orthogonal direct sum. Note that $\sigma$ acts as the identity operator on $L_+$ and as $-1$ on $L_-$. Then $V_L^\sigma \cong V_{L_+} \otimes V_{L_-}^+$ is a subalgebra of $V_Q^\sigma$.

**Proposition 3.5.** Every $V_Q^\sigma$-module is a direct sum of irreducible $V_L^\sigma$-modules. In particular, $V_Q^\sigma$ has this form.

**Proof.** It is shown in Theorem 3.16 of [DLM2] that the vertex algebra $V_{L_+}$ is regular, since $L_+$ is positive definite. It is also shown in [A2, ABD, DJL] that the vertex algebra $V_{L_-}^+$ is regular. Since the tensor product of regular vertex algebras is again regular (Proposition 3.3 in [DLM2]), we have that $V_L^\sigma \cong V_{L_+} \otimes V_{L_-}^+$ is also regular. □

In order to obtain a precise description of $V_Q^\sigma$, we will decompose $V_Q$ as a direct sum of irreducible modules over $V_L^\sigma$. This is done in two steps. The first step is to break $V_Q$ as a direct sum of $V_L$-modules, using the cosets of $Q$ modulo $L$. Since the lattice $Q$ is integral, we have $Q \subseteq L^*$ and we can view $Q/L$ as a subgroup of $L^*/L$. It follows that

$$V_Q = \bigoplus_{\gamma + L \in Q/L} V_{\gamma + L},$$

where each $V_{\gamma + L}$ is an irreducible $V_L$-module [D1]. Writing $\gamma = \gamma_+ + \gamma_-$, we get

$$V_{\gamma + L} \cong V_{\gamma_+ + L_+} \otimes V_{\gamma_- + L_-}$$

as modules over $V_L \cong V_{L_+} \otimes V_{L_-}$. Therefore,

$$V_Q \cong \bigoplus_{\gamma + L \in Q/L} V_{\gamma_+ + L_+} \otimes V_{\gamma_- + L_-}$$

as $V_L$-modules. Note that $\gamma_+ + L_+$ and $\gamma_- + L_-$ depend only on the coset $\gamma + L$ and not on the representative $\gamma \in Q$, because $\alpha_{\pm} \in L_{\pm}$ for $\alpha \in L$. 

Since $\sigma \gamma_\gamma = -\gamma_\gamma$ and $2\gamma_\gamma \in L_-\gamma$, it follows that $\sigma$ acts on the $V_{L_-\gamma}$ module $V_{\gamma_\gamma + L_-\gamma}$. The second step is to decompose each module $V_{\gamma_\gamma + L_-\gamma}$ into eigenspaces for $\sigma$, which are irreducible as $V^+_{L_-\gamma}$-modules [AD]. We thus obtain the following description of $V^\sigma_Q$.

**Proposition 3.6.** The orbifold $V^\sigma_Q$ decomposes as a direct sum of irreducible $V^\sigma_L$-modules as follows:

\[ V^\sigma_Q \cong \bigoplus_{\gamma + L \in Q/L} V_{\gamma + L} \otimes V_{\gamma + L}^{\eta(\gamma)} \]

**Proof.** Using (3.5), it is enough to determine the subspace $S_\gamma$ of $\sigma$-invariants in $V_{\gamma + L_+} \otimes V_{\gamma + L_-}$, for a fixed $\gamma \in Q$. As a $V^\sigma_L$-module, $S_\gamma$ is generated by the element (cf. (2.16)):

\[ v_\gamma = e^\gamma + \eta(\gamma)e^{\sigma\gamma} = e^{\gamma + L} \otimes (e^{\gamma} + \eta(\gamma)e^{\gamma - L}) \in V_{\gamma + L_+} \otimes V_{\gamma + L_-}^{\eta(\gamma)} \]

Since $V_{\gamma + L_+} \otimes V_{\gamma + L_-}^{\eta(\gamma)}$ is an irreducible $V^\sigma_L$-module, it must be equal to $S_\gamma$. \[ \square \]

From the study of tensor products in [FHL, LL], all irreducible modules over $V^\sigma_L \cong V^+_{L_+} \otimes V^{+}_{L_-}$ are tensor products of irreducible modules over the factors $V^+_{L_+}$ and $V^+_{L_-}$. By the results of [D1, DN, AD] reviewed in Sections 2.5 and 2.6 we obtain that all irreducible $V^\sigma_L$-modules have the form:

1. $V_{\lambda + L_+} \otimes V_{\mu + L_-}$, where $\lambda \in L^+_{+}$, $\mu \in L_{+}^{-}$, $2\mu \notin L_{-}$,
2. $V_{\lambda + L_+} \otimes V^{+}_{\mu + L_-}$, where $\lambda \in L^+_{+}$, $\mu \in L_{+}^{-}$, $2\mu \notin L_{-}$,
3. $V_{\lambda + L_+} \otimes V^{+}_{T_{+}}$, where $\lambda \in L^+_{+}$,

and $T$ is an irreducible module for the group $G_\sigma$ associated to the lattice $L_-\gamma$. We refer to the $V^\sigma_L$-modules obtained from untwisted $V_L$-modules as orbifold modules of untwisted type and the ones obtained from twisted $V_L$-modules as orbifold modules of twisted type.

### 3.3. Irreducible Modules over $V^\sigma_Q$

In this subsection, we present our main result, the explicit classification of irreducible $V^\sigma_Q$-modules. As a consequence, we will find all of them as submodules of twisted or untwisted $V_Q$-modules. Recall that, by (3.4), we can assume that $Q = \bar{Q}$.

**Theorem 3.7.** Let $Q$ be a positive-definite even lattice, and $\sigma$ be an automorphism of $Q$ of order two such that $(\alpha | \sigma \alpha)$ is even for all $\alpha \in Q$. Then as a module over $V^\sigma_L \cong V^+_{L_+} \otimes V^+_{L_-}$ each irreducible $V^\sigma_Q$-module
is isomorphic to one of the following:

\[(3.8) \bigoplus_{\gamma + L \in Q/L} V_{\gamma + \lambda + L_+} \otimes V_{\gamma + \mu + L_-} \quad (2\mu \not\in L_-),\]

\[(3.9) \bigoplus_{\gamma + L \in Q/L} V_{\gamma + \lambda + L_+} \otimes V_{\gamma + \mu + L_-}^{e\eta(\gamma)} \quad (2\mu \in L_-),\]

\[(3.10) \bigoplus_{\gamma + L \in Q/L} V_{\gamma + \lambda + L_+} \otimes V_{\gamma L_+}^{T(\gamma_+), \epsilon_\gamma},\]

where \(\lambda \in L_+, \mu \in L_-, \epsilon \in \{\pm\}, \chi\) is a central character for the group \(G_\sigma\) associated to \(L_-\), and \(\epsilon_\gamma = e\eta(\gamma)c_\chi(\gamma_-)\).

**Proof.** Let \(W\) be an irreducible \(V_{Q_0}^\sigma\)-module. Then \(W\) is a \(V_{L_+}^\sigma\)-module by restriction and, by Proposition 3.5, \(W\) is a direct sum of irreducible \(V_{L_+}^\sigma\)-modules. Suppose \(A \subseteq W\) is an irreducible \(V_{L_+}^\sigma\)-module, and define \(A(\gamma)\) from \(A\) as follows. If

\[A = V_{\lambda + L_+} \otimes V_{\mu + L_-}, \quad V_{\lambda + L_+} \otimes V_{\mu + L_-}^\epsilon, \quad \text{or} \quad V_{\lambda + L_+} \otimes V_{L_+}^{T(\gamma_+), \epsilon},\]

then

\[A(\gamma) = V_{\lambda + \gamma + L_+} \otimes V_{\mu + \gamma} \otimes V_{\mu + L_-}^{e\eta(\gamma)}, \quad V_{\lambda + \gamma + L_+} \otimes V_{\mu + \gamma + L_-}^{e\eta(\gamma)}, \quad \text{and} \quad V_{\lambda + \gamma + L_+} \otimes V_{\gamma L_+}^{T(\gamma_+), \pm \epsilon_\gamma},\]

respectively, where \(\epsilon \in \{\pm\}\) and \(\epsilon_\gamma = e\eta(\gamma)c_\chi(\gamma_-)\). We will consider separately the untwisted and twisted types.

Let \(A\) be of untwisted type, i.e., one of the modules \(V_{\lambda + L_+} \otimes V_{\mu + L_-}\) for \(2\mu \not\in L_-\), or \(V_{\lambda + L_+} \otimes V_{\mu + L_-}^\epsilon\) for \(2\mu \in L_-\). Let \(B \subseteq W\) be another irreducible \(V_{L_+}^\sigma\)-module that is possibly of twisted type. By Proposition 3.6, \(V_{Q_0}^\sigma\) is a direct sum of irreducible \(V_{L_+}^\sigma\)-modules generated by the vectors \(v_\gamma\) from (3.7), where \(\gamma \in Q\). By restricting the field \(Y(v_\gamma, z)\) to \(A\) and then projecting onto \(B\), we obtain an intertwining operator of \(V_{L_+}^\sigma\)-modules of type \(B_{V_{\gamma L_+}^\sigma(A)}\). From the study of intertwining operators in [ADL], we have that the intertwining operator \(Y(v_\gamma, z)\) can be written as the tensor product

\[Y(v_\gamma, z) = Y(e^{\gamma_+}, z) \otimes Y(e^{\gamma_-} + \eta(\gamma)e^{-\gamma_-}, z),\]

where \(Y(e^{\gamma_+}, z)\) is an intertwining operator of type \(V_{\lambda + L_+}^{V_{\gamma + L_+}}\) and \(Y(e^{\gamma_-} + \eta(\gamma)e^{-\gamma_-}, z)\) is an intertwining operator of type \(V_{\gamma + L_-}^{V_{\gamma + L_-}}\) or of type \(V_{\gamma + L_-}^{V_{\gamma + L_-}}\). By [DL], the fusion
rules for $Y(e^{\gamma_+}, z)$ are zero unless $\lambda' = \lambda + \gamma_+$. Since $\gamma_- \in L_- \cap \frac{1}{2} L_-$ and $|\lambda|^2 \in \mathbb{Z}$, we have that $\pi_{\gamma_-, 2\lambda} = 1$ (cf. (2.22)). Hence the fusion rules for $Y(e^{\gamma_-} + \eta(\gamma)e^{-\gamma_-}, z)$ are zero unless $\mu' = \mu + \gamma_-$, by Theorem 2.7. Therefore, for $\gamma + L \in Q/L$, we have $B = A^{(\gamma)}$. Hence $A \subseteq W$ implies that $\bigoplus_{\gamma \in Q/L} A^{(\gamma)} \subseteq W$. Since $W$ is irreducible, we obtain that

$$W = \bigoplus_{\gamma + L \in Q/L} A^{(\gamma)}.$$

Now let $A = V_{\lambda + L_+} \otimes V_{L_-}^{T_{\chi_{\gamma_-}}} \pm$ and $B \subseteq W$ be another irreducible $V_Q^\sigma$-module that is possibly of untwisted type. As above, the field $Y(v_{\gamma}, z)$ gives rise to an intertwining operator of $V_Q^\sigma$-modules of type $Y_{\gamma + L}^{v} A^{(\gamma)}$ and can be written as the tensor product

$$Y(v_{\gamma}, z) = Y(e^{\gamma_+}, z) \otimes Y(e^{\gamma_-} + \eta(\gamma)e^{-\gamma_-}, z),$$

where $Y(e^{\gamma_+}, z)$ is an intertwining operator of type $V_{\lambda + L_+}^{\gamma + L_+}$ and $Y(e^{\gamma_-} + \eta(\gamma)e^{-\gamma_-}, z)$ is an intertwining operator of type $V_{\gamma + L_-}^{\eta(\gamma)}$. As with the untwisted type, the fusion rules for $Y(e^{\gamma_+}, z)$ are zero unless $\lambda' = \lambda + \gamma_+$. By Theorem 2.7, the action of $Y(e^{\gamma_-}, z)$ on $V_{L_-}^{T_{\chi_{\gamma_-}}}$ is determined by computing $c_{\chi}(\gamma_-)$ (cf. (2.23)) and is zero unless $\gamma' = \chi(\gamma_-)$. Since the lattice $\mathbb{Z}\gamma_- + L_-$ is integral (by Corollary 3.4), the map $\varepsilon$ can be extended to this lattice with values $\pm 1$. Therefore $\varepsilon(\gamma_-, 2\gamma_-) = \varepsilon(\gamma_-, \gamma_-)^2 = 1$ and (2.23) becomes $c_{\chi}(\gamma_-) = s(2\gamma_-)$; see (2.19). Hence the eigenspace of each summand in the $V_Q^\sigma$-module may change depending on the signs of each $U_{2\gamma_-}$. Therefore, $B = A^{(\gamma)}$ and

$$W = \bigoplus_{\gamma + L \in Q/L} A^{(\gamma)}.$$

This completes the proof. \qed

**Theorem 3.8.** Let $Q$ be a positive-definite even lattice, and $\sigma$ be an automorphism of $Q$ of order two such that $(\alpha|\sigma\alpha)$ is even for all $\alpha \in Q$. Then every irreducible $V_Q^\sigma$-module is a submodule of a $V_Q$-module or a $\sigma$-twisted $V_Q$-module.

**Proof.** Let us consider first the untwisted case. By Theorem 3.7, any irreducible $V_Q^\sigma$-module $W$ of untwisted type is given by (3.8) or (3.9) when considered as a $V_Q$-module by restriction. For a fixed $\gamma \in Q$, the
nonzero fusion rules for $Y(e^{\gamma+}, z)$ and $Y(e^{\gamma-} + \eta(\gamma)e^{-\gamma-}, z)$ are equal to 1 and the intertwining operators in [ADL] are given by the usual formula (2.11) up to a scalar multiple. Using that

$$(\gamma|\lambda + \mu) - (\sigma\gamma|\lambda + \mu) = (2\gamma_-|\lambda + \mu) = (2\gamma_-|\mu) \in \mathbb{Z},$$

we have that for some $m \in \mathbb{Z}$,

$$Y(v, z)e^{\lambda+\mu} = Y(e^{\gamma} + \eta(\gamma)e^{\sigma\gamma}, z)e^{\lambda+\mu}$$

$$= z^{(\gamma|\lambda+\mu)}(E(\gamma, z)e^{\gamma+\lambda+\mu} + \eta(\gamma)z^m E(\sigma\gamma, z)e^{\sigma\gamma+\lambda+\mu}),$$

where

$$E(\alpha, z) = \exp\left(\sum_{n<0} \alpha_n \frac{z^{-n}}{-n}\right) \exp\left(\sum_{n>0} \alpha_n \frac{z^{-n}}{-n}\right)$$

contains only integral powers of $z$. This implies that $(\lambda + \mu|\gamma) \in \mathbb{Z}$ for all $\gamma \in Q$, i.e., $\lambda + \mu \in Q^*$. Then for the untwisted $V_Q$-module $V_{\lambda+\mu+Q}$, we have that

$$V_{\lambda+\mu+Q} = \bigoplus_{\gamma+L \in Q/L} V_{\gamma+\lambda+\mu+L}$$

$$\cong \bigoplus_{\gamma+L \in Q/L} V_{\gamma+\lambda+\mu+L} \otimes V_{\gamma-+\mu+L_-}$$

as a direct sum of irreducible $V_L$-modules. Using the intertwining operators in [ADL], we see that $W$ is a submodule of the restriction of $V_{\lambda+\mu+Q}$ to $V_{\sigma}^\gamma$.

Now we consider the twisted case. By Theorem 3.7, any irreducible $V_{\sigma}^\gamma$-module $W$ of twisted type is given by (3.10) when considered as a $V_{\sigma}^\gamma$-module by restriction. Then, for $\gamma \in Q$, the nonzero fusion rules for $Y(e^{\gamma+}, z)$ and $Y(e^{\gamma-} + \eta(\gamma)e^{-\gamma-}, z)$ are equal to 1 and the intertwining operators in [ADL] are given by the usual formula (2.17) up to a scalar multiple. Since these scalars can be absorbed in $U_\gamma$, we will have (2.17) without loss of generality. Therefore, the action of $Y(e^{\gamma}, z)$ on $W$ can be determined, so that its modes are linear maps

$$V_{\lambda+\mu+L} \otimes V_{L_-}^{T_{\alpha, \epsilon}} \rightarrow V_{\gamma+\lambda+\mu+L} \otimes V_{L_-}^{T_{\alpha, \epsilon}} \quad (\epsilon = \{\pm\}),$$

where $\epsilon_\gamma = \epsilon c_\chi(\gamma)\eta(\gamma)$ (cf. (3.10)). Hence the $\sigma$-twisted $V_Q$-module is given by

$$\bigoplus_{\gamma+L \in Q/L} V_{\gamma+\lambda+\mu+L} \otimes V_{L_-}^{T_{\alpha, \epsilon}}$$

and its restriction to $V_{\sigma}^\gamma$ contains $W$. \qed
4. Root lattices and Dynkin diagram automorphisms

In this section, we present examples of the lattice $Q$ being a root lattice of type ADE, corresponding to the simply-laced simple Lie algebras. We use the classification from [D1, DN, AD] and the construction of twisted modules from Section 2.5 to construct explicitly all irreducible $V_Q^\sigma$-modules. In each case, a correspondence between the two constructions is shown. In order to apply Theorems 3.7 and 3.8, we first calculate $\bar{Q}$ and $L$. Then the twisted $V_L$-modules are found. When necessary, the intertwiners from [ADL] are used to construct the $V_Q^\sigma$-modules. The untwisted and twisted $V_{\bar{Q}}$-modules are calculated using $\bar{Q}^*/\bar{Q}$ and its $\sigma$-invariant elements. For more details and additional examples, the reader is referred to [E].

4.1. $A_2$ root lattice with a Dynkin diagram automorphism.

Consider the $A_2$ simple roots $\{\alpha_1, \alpha_2\}$, the associated root lattice $Q = \mathbb{Z}\alpha_1 + \mathbb{Z}\alpha_2$, and the Dynkin diagram automorphism $\sigma: \alpha_1 \leftrightarrow \alpha_2$. The case of $A_n$ root lattice for even $n$ is similar and is treated in [E].

Set $\alpha = \alpha_1 + \alpha_2$ and $\beta = \alpha_1 - \alpha_2$. Then

$$|\alpha|^2 = 2, \quad |\beta|^2 = 6, \quad (\alpha|\beta) = 0,$$

and

$$L_+ = \mathbb{Z}\alpha, \quad L_- = \mathbb{Z}\beta, \quad \bar{Q} = L = L_+ \oplus L_-.$$

Therefore,

$$V_Q^\sigma = V_{\bar{Q}}^\sigma = V_L^\sigma \cong V_{\mathbb{Z}\alpha} \otimes V_{\mathbb{Z}\beta}^\pm.$$

Using the results of [FHL, D1, DN], we obtain a total of 20 distinct irreducible $V_L^\sigma$-modules:

$$V_{\mathbb{Z}\alpha} \otimes V_{\mathbb{Z}\beta}^\pm,$$

$$V_{\mathbb{Z}\alpha} \otimes V_{\mathbb{T}_1+\mathbb{Z}\beta}^\pm,$$

$$V_{\mathbb{Z}\alpha} \otimes V_{\mathbb{T}_2+\mathbb{Z}\beta}^\pm,$$

$$V_{\mathbb{Z}\alpha} \otimes V_{\mathbb{T}_1+\mathbb{T}_2+\mathbb{Z}\beta}^\pm,$$

$$V_{\mathbb{Z}\alpha} \otimes V_{\mathbb{T}_1+\mathbb{T}_2+\mathbb{Z}\beta}^\pm,$$

where $T_1, T_2$ denote the two irreducible modules over the group $G_\sigma$ associated to the lattice $\mathbb{Z}\beta$. They are 1-dimensional and on them $U_\beta = \pm i/64$ by (2.19).

On the other hand, by Dong’s Theorem [D1], the irreducible $V_L$-modules have the form $V_{\lambda+L}$ ($\lambda + L \in L^*/L$). If $\sigma(\lambda + L) = \lambda + L$, the module $V_{\lambda+L}$ breaks into eigenspaces $V_{\lambda+L}^\pm$ of $\sigma$; otherwise, $\sigma$ is
an isomorphism of \( V_L^\sigma \)-modules \( V_{\lambda+L} \to V_{\sigma(\lambda)+L} \). Thus, there are 12 distinct irreducible \( V_L^\sigma \)-modules of untwisted type:

\[
\begin{align*}
V_L^\pm, & \quad V_{\frac{\lambda}{2}+L}^\pm, & \quad V_{\frac{\lambda}{2}+\frac{\beta}{2}+L}^\pm, \\
V_{\frac{\lambda}{2}+L}^\pm, & \quad V_{\frac{\lambda}{2}+\frac{\beta}{2}+L}^\pm, & \quad V_{\frac{\lambda}{2}+\frac{\beta}{2}+L}^\pm.
\end{align*}
\]

The correspondence of these modules to those above is given by:

\[
V_{m\alpha+n\beta+L} \cong V_{m\alpha+L+} \otimes V_{n\beta+L-} \quad (m, n \in \mathbb{Q}).
\]

Now we will construct the \( \sigma \)-twisted \( V_L \)-modules using Section 2.5. Consider the irreducible modules over the group \( G_\sigma \) associated to the lattice \( L \). One finds that on them \( U_\beta = \pm i/64 \) and \( U_\alpha \) acts freely. Hence, such modules can be identified with the space \( P = \mathbb{C}[q, q^{-1}] \), where \( U_\alpha \) acts as a multiplication by \( q \). Then on \( P \) we have:

\[
e^{\pi i(0)} = s_1, \quad U_\alpha = q, \quad U_\beta = s_2 \frac{i}{64},
\]

where the signs \( s_1, s_2 \in \{\pm\} \) are independent. The corresponding four \( G_\sigma \)-modules will be denoted as \( P_{(s_1, s_2)} \). The irreducible \( \sigma \)-twisted \( V_L \)-modules have the form \( F_\sigma \otimes P_{(s_1, s_2)} \), where \( F_\sigma \) is the \( \sigma \)-twisted Fock space (see (2.12)). Next, we restrict the \( \sigma \)-twisted \( V_L \)-modules to \( V_L^\sigma \). The automorphism \( \sigma \) acts on each \( P_{(s_1, s_2)} \) as the identity operator, since

\[
\sigma(q^n) = \sigma(U_\alpha^n \cdot 1) = U_\alpha^n \cdot 1 = q^n.
\]

We obtain 8 irreducible \( V_L^\sigma \)-modules of twisted type:

\[
F_\sigma^\pm \otimes P_{(s_1, s_2)}, \quad s_1, s_2 \in \{\pm\},
\]

where \( F_\sigma^\pm \) are the \( \pm 1 \)-eigenspaces of \( \sigma \). We have the following correspondence among irreducible \( V_L^\sigma \)-modules of twisted type:

\[
\begin{align*}
F_\sigma^\pm \otimes P_{(+,+)} & \cong V_{Z\alpha} \otimes V_{Z\beta}^{T_1,\pm}, & F_\sigma^\pm \otimes P_{(-,+)} & \cong V_{Z\alpha} \otimes V_{Z\beta}^{T_1,\pm}, \\
F_\sigma^\pm \otimes P_{(+,-)} & \cong V_{Z\alpha} \otimes V_{Z\beta}^{T_2,\pm}, & F_\sigma^\pm \otimes P_{(-,-)} & \cong V_{Z\alpha} \otimes V_{Z\beta}^{T_2,\pm}.
\end{align*}
\]

4.2. \( A_2 \) root lattice with the negative of a Dynkin diagram automorphism. Consider now the negative Dynkin diagram automorphism \( \phi = -\sigma \). Keeping the notation from the previous subsection, we have: \( L_+ = Z\beta, \ L_- = Z\alpha \), and

\[
V_Q^\phi \cong V_{Z\beta} \otimes V_{Z\alpha}^+.
\]

Furthermore, \( L_+/L_+ = \mathbb{Z}_6/\mathbb{Z}_3 \) and \( (L_+/L_-)^\phi = L_+/L_- = \mathbb{Z}_2/\mathbb{Z}_2 \). Thus, using the results of \([FHL, \text{DI}, \text{DN}]\), we obtain a total of 48 distinct irreducible \( V_Q^\phi \)-modules:

\[
\begin{align*}
V_{\frac{\lambda}{2}+Z\beta} \otimes V_{Z\alpha}^+, & \quad V_{\frac{\lambda}{2}+Z\beta} \otimes V_{\frac{\lambda}{2}+Z\alpha}^+, & \quad V_{\frac{\lambda}{2}+Z\beta} \otimes V_{Z\alpha}^{T_j,\pm},
\end{align*}
\]
where \( i = 0, \ldots, 5 \) and \( j = 1, 2 \).

### 4.3. \( A_3 \) root lattice with a Dynkin diagram automorphism.

Consider the \( A_3 \) simple roots \( \{\alpha_1, \alpha_2, \alpha_3\} \), the root lattice \( Q = \mathbb{Z}\alpha_1 + \mathbb{Z}\alpha_2 + \mathbb{Z}\alpha_3 \), and the Dynkin diagram automorphism \( \sigma: \alpha_1 \leftrightarrow \alpha_3, \alpha_2 \leftrightarrow \alpha_2 \). The case of \( A_n \) root lattice for odd \( n \) is similar and is treated in [E].

Set \( \alpha = \alpha_1 + \alpha_3 \) and \( \beta = \alpha_1 - \alpha_3 \). Then \( \bar{Q} = Q \) and

\[
L_+ = \mathbb{Z} \alpha + \mathbb{Z} \alpha_2, \quad L_- = \mathbb{Z} \beta, \quad Q/L = \{L, \alpha_1 + L\}.
\]

Hence, by Proposition 3.6,

\[
V^\sigma_Q \cong (V_{L+} \otimes V^\pm_{\mathbb{Z}\alpha}) \oplus (V^\pm_{\mathbb{Z}\beta} \otimes V_{L+} \otimes V^\pm_{\mathbb{Z}\beta}).
\]

It follows from Theorems 2.7 and 3.7 that the irreducible \( V^\sigma_Q \)-modules are given by:

\[
\begin{align*}
(V_{L+} \otimes V^\pm_{\mathbb{Z}\alpha}) & \oplus (V^\pm_{\mathbb{Z}\beta} \otimes V^\pm_{\mathbb{Z}\beta}), \\
(V_{L+} \otimes V^\pm_{\mathbb{Z}\beta}) & \oplus (V^\pm_{\mathbb{Z}\alpha} \otimes V^\pm_{\mathbb{Z}\beta}), \\
(V^\pm_{\mathbb{Z}\alpha} \otimes V^\pm_{\mathbb{Z}\beta}) & \oplus (V^\pm_{\mathbb{Z}\beta} \otimes V^\pm_{\mathbb{Z}\beta}), \\
(V_{L+} \otimes V^\pm_{\mathbb{Z}\alpha}) & \oplus (V^\pm_{\mathbb{Z}\beta} \otimes V^\pm_{\mathbb{Z}\beta}), \\
(V_{L+} \otimes V^\pm_{\mathbb{Z}\beta}) & \oplus (V^\pm_{\mathbb{Z}\beta} \otimes V^\pm_{\mathbb{Z}\beta}), \\
(V_{L+} \otimes V^\pm_{\mathbb{Z}\alpha}) & \oplus (V^\pm_{\mathbb{Z}\beta} \otimes V^\pm_{\mathbb{Z}\beta}), \\
(V_{L+} \otimes V^\pm_{\mathbb{Z}\beta}) & \oplus (V^\pm_{\mathbb{Z}\beta} \otimes V^\pm_{\mathbb{Z}\beta}), \\
(V_{L+} \otimes V^\pm_{\mathbb{Z}\alpha}) & \oplus (V^\pm_{\mathbb{Z}\beta} \otimes V^\pm_{\mathbb{Z}\beta}).
\end{align*}
\]

where \( U_\beta = (-1)^j/16 \) on \( T_j \) for \( j = 1, 2 \) (see (2.19)).

On the other hand, by Dong’s Theorem [D1], the irreducible \( V_Q \)-modules have the form \( V^\lambda_{\lambda+Q} \) (\( \lambda + Q \in Q^*/Q \)). We have

\[
Q^*/Q = \{Q, \lambda_1 + Q, 2\lambda_1 + Q, 3\lambda_1 + Q\},
\]

where

\[
\lambda_1 = \frac{1}{4}(3\alpha_1 + 2\alpha_2 + \alpha_3).
\]

Note that \( Q \) and \( 2\lambda_1 + Q = \frac{\alpha}{2} + Q \) are \( \sigma \)-invariant, while \( \sigma(\lambda_1 + Q) = 3\lambda_1 + Q \). Thus, there are 5 distinct irreducible \( V^\sigma_Q \)-modules of untwisted type:

\[
V^\pm_Q, \quad V^\pm_{\frac{\alpha}{2}+Q}, \quad V_{\lambda_1+Q}.
\]

We have the following correspondence:

\[
\begin{align*}
V^\pm_Q & \cong (V_{L+} \otimes V^\pm_{\mathbb{Z}\beta}) \oplus (V^\pm_{\mathbb{Z}\beta} \otimes V^\pm_{\mathbb{Z}\beta}), \\
V^\pm_{\frac{\alpha}{2}+Q} & \cong (V_{L+} \otimes V^\pm_{\mathbb{Z}\beta}) \oplus (V^\pm_{\mathbb{Z}\beta} \otimes V^\pm_{\mathbb{Z}\beta}), \\
V_{\lambda_1+Q} & \cong (V^\pm_{\mathbb{Z}\beta} \otimes V^\pm_{\mathbb{Z}\beta}) \oplus (V^\pm_{\mathbb{Z}\beta} \otimes V^\pm_{\mathbb{Z}\beta}).
\end{align*}
\]
We now construct the $\sigma$-twisted $V_Q$-modules using Section 2.5. Consider the irreducible modules over the group $G_\sigma$ associated to the lattice $Q$. We find from (2.19) that on them $U_\beta = \pm 1/16$, and there are two such modules, $P_1$ and $P_2$, corresponding to $U_\beta = -1/16$ and $U_\beta = 1/16$, respectively. For both $j = 1, 2$, we can identify $P_j = \mathbb{C}[q, q^{-1}, p, p^{-1}]$ so that

$$U_\alpha^1 = q, \quad U_\alpha^2 = p(-1)^{\frac{j}{16}}.$$ Then on $P_j$ we have:

$$U_\alpha^3 = (-1)^{j+1}q, \quad U_\alpha = (-1)^{j+1}4q^2, \quad U_\beta = \frac{(-1)^j}{16}.$$ The automorphism $\sigma$ acts on each of these modules: $\sigma$ is the identity operator on $P_1$, while on $P_2$ we have $\sigma = (-1)^{\frac{d}{16}}$. Hence, $P_2$ decomposes into two eigenspaces $P_2^\pm$ with eigenvalues $\pm 1$. The irreducible $\sigma$-twisted $V_Q$-modules have the form $F_\sigma \otimes P_j$ for $j = 1, 2$, where $F_\sigma$ is the $\sigma$-twisted Fock space (see (2.12)). Since $F_\sigma$ itself decomposes into $\pm 1$-eigenspaces of $\sigma$, we obtain 4 distinct irreducible $V_Q$-modules of twisted type:

$$(F_\sigma \otimes P_1)^\pm \cong (V_{L^+} \otimes V_{Z_\beta}^{T_1, \pm}) \oplus (V_{\frac{\alpha}{2} + L^+} \otimes V_{Z_\beta}^{T_1, \pm}),$$

$$(F_\sigma \otimes P_2)^\pm \cong (V_{L^+} \otimes V_{Z_\beta}^{T_2, \pm}) \oplus (V_{\frac{\alpha}{2} + L^+} \otimes V_{Z_\beta}^{T_2, \pm}).$$

We have the following correspondence:

$$(F_\sigma \otimes P_1)^\pm \cong (V_{L^+} \otimes V_{Z_\beta}^{T_1, \pm}) \oplus (V_{\frac{\alpha}{2} + L^+} \otimes V_{Z_\beta}^{T_1, \pm}),$$

$$(F_\sigma \otimes P_2)^\pm \cong (V_{L^+} \otimes V_{Z_\beta}^{T_2, \pm}) \oplus (V_{\frac{\alpha}{2} + L^+} \otimes V_{Z_\beta}^{T_2, \pm}).$$

4.4. **$A_3$ root lattice with the negative of a Dynkin diagram automorphism.** Consider now the negative Dynkin diagram automorphism, $\phi = -\sigma$. With the notation from the previous subsection, we have:

$$L_+ = \mathbb{Z}_\beta, \quad L_- = \mathbb{Z}_\alpha + \mathbb{Z}_\alpha_2, \quad (L_-^*/L_-)^\phi = L_-^*/L_-,$$

and

$$V_Q^\phi \cong (V_{Z_\beta} \otimes V_{L^+}) \oplus (V_{\frac{\alpha}{2} + Z_\beta} \otimes V_{\frac{\alpha}{2} + L^+}).$$
By Theorem 3.7, the irreducible $V_{Q}^{\sigma}$-modules of untwisted type are given by:

\[
(V_{L_+} \otimes V_{Z_{\beta}}^{\pm}) \oplus (V_{\frac{n}{2}+Z_{\beta}}^{\pm} \otimes V_{\frac{n}{2}+L_-}),
\]
\[
(V_{L_+} \otimes V_{\frac{n}{2}+Z_{\beta}}^{\pm}) \oplus (V_{\frac{n}{2}+Z_{\beta}} \otimes V_{L_-}^{\pm}),
\]
\[
(V_{\frac{n}{2}+Z_{\beta}}^{\pm} \otimes V_{\frac{n}{2}+Z_{\beta}}^{\pm} \oplus (V_{\frac{n}{2}+Z_{\beta}} \otimes V_{\frac{n}{2}+Z_{\beta}}^{\pm}),
\]
\[
(V_{L_+} \otimes V_{\frac{n}{2}+Z_{\beta}}^{\pm} \oplus (V_{\frac{n}{2}+Z_{\beta}} \otimes V_{\frac{n}{2}+Z_{\beta}}^{\pm}),
\]
\[
(V_{L_+} \otimes V_{\frac{n}{2}+Z_{\beta}}^{\pm} \oplus (V_{\frac{n}{2}+Z_{\beta}} \otimes V_{\frac{n}{2}+Z_{\beta}}^{\pm}),
\]
and the ones of twisted type are:

\[
(V_{L_+} \otimes V_{T_{1,\pm}}^{\pm}) \oplus (V_{\frac{n}{2}+Z_{\beta}} \otimes V_{T_{1,\pm}}^{\pm}),
\]
\[
(V_{L_+} \otimes V_{T_{2,\pm}}^{\pm}) \oplus (V_{\frac{n}{2}+Z_{\beta}} \otimes V_{T_{2,\pm}}^{\pm}).
\]

4.5. $D_{n}$ root lattice with a Dynkin diagram automorphism. Consider the $D_{n}$ simple roots $\{\alpha_{1}, \ldots, \alpha_{n}\}$, where $n \geq 4$, the root lattice $Q$, and the Dynkin diagram automorphism $\sigma: \alpha_{n-1} \leftrightarrow \alpha_{n}$ and $\alpha_{i} \leftrightarrow \alpha_{i}$ for $i = 1, \ldots, n - 2$. This case is similar to the $A_{3}$ case discussed in Section 3.3 so we will be brief (see [1] for details).

Set $\alpha = \alpha_{n-1} + \alpha_{n}$ and $\beta = \alpha_{n-1} - \alpha_{n}$. Then

\[
L_+ = \mathbb{Z}\alpha + \sum_{i=1}^{n-2} \mathbb{Z}\alpha_{i}, \quad L_- = \mathbb{Z}\beta, \quad Q = \tilde{Q},
\]

and

\[
Q/L = \{L, \alpha_{n-1} + L\}.
\]

Hence, by Proposition 3.6

\[
V_{Q}^{\sigma} \cong (V_{L_+} \otimes V_{Z_{\beta}}^{\pm}) \oplus (V_{\frac{n}{2}+L_+} \otimes V_{\frac{n}{2}+Z_{\beta}}^{\pm}).
\]

By Theorems 2.7 and 3.7, the irreducible $V_{Q}^{\sigma}$-modules are given by:

\[
(V_{L_+} \otimes V_{Z_{\beta}}^{\pm}) \oplus (V_{\frac{n}{2}+L_+} \otimes V_{\frac{n}{2}+Z_{\beta}}^{\pm}),
\]
\[
(V_{L_+} \otimes V_{\frac{n}{2}+Z_{\beta}}^{\pm}) \oplus (V_{\frac{n}{2}+L_+} \otimes V_{\frac{n}{2}+Z_{\beta}}^{\pm}),
\]
\[
(V_{\frac{n-1}{2}+L_+} \otimes V_{\frac{n}{2}+Z_{\beta}}^{\pm}) \oplus (V_{\frac{n+1}{2}+L_+} \otimes V_{\frac{n}{2}+Z_{\beta}}^{\pm}),
\]
\[
(V_{L_+} \otimes V_{T_{1,\pm}}^{\pm}) \oplus (V_{\frac{n}{2}+L_+} \otimes V_{T_{1,\pm}}^{\pm}),
\]
\[
(V_{L_+} \otimes V_{T_{2,\pm}}^{\pm}) \oplus (V_{\frac{n}{2}+L_+} \otimes V_{T_{2,\pm}}^{\pm}),
\]

where $\theta = \frac{1}{2} \sum_{i=0}^{n-1} \alpha_{2i+1}$ and $U_{\beta} = (-1)^{j}/16$ on $T_{j}$ for $j = 1, 2$.

On the other hand, note that

\[
Q^{*}/Q = \left\{ Q, \frac{\alpha}{2} + Q, \lambda_{n-1} + Q, \lambda_{n} + Q \right\},
\]
where
\[ \lambda_{n-1} = \frac{1}{2} \left( \alpha_1 + 2\alpha_2 + \cdots + (n-2)\alpha_{n-2} + \frac{1}{2}n\alpha_{n-1} + \frac{1}{2}(n-2)\alpha_n \right) \]
and \( \lambda_n = \sigma(\lambda_{n-1}) \). Using Dong’s Theorem \([D1]\), we obtain 5 distinct irreducible \( V_Q^\sigma \)-modules of untwisted type:
\[ V_Q^\pm, \quad V_{\alpha^2+1+Q}^\pm, \quad V_{\lambda_{n-1}+Q}^\pm \]

We have the following correspondence:
\[ V_Q^\pm \cong (V_{L_+} \otimes V_{Z_{2\beta}}^\pm) \oplus (V_{\frac{\alpha}{2}+L_+} \otimes V_Z^\pm) ; \]
\[ V_{\alpha^2+1+Q}^\pm \cong (V_{L_+} \otimes V_{Z_{2\beta}}^\pm) \oplus (V_{\frac{\alpha}{2}+L_+} \otimes V_Z^\pm) ; \]
\[ V_{\lambda_{n-1}+Q}^\pm \cong (V_{\alpha^2+1+L_+} \otimes V_{\frac{\alpha}{2}+Z_{2\beta}}) \oplus (V_{\frac{\alpha}{2}+\alpha+L_+} \otimes V_Z^\pm). \]

We now construct the irreducible \( \sigma \)-twisted \( V_Q \)-modules using Section 2.5. There are two irreducible \( G_\sigma \)-modules, \( P_1 \) and \( P_2 \), and both can be identified with \( \mathbb{C}[p_1^{\pm 1}, \ldots, p_{n-1}^{\pm 1}] \) as vector spaces. The action of \( G_\sigma \) on \( P_j \) is determined by:
\[ U_{\alpha_i} = p_i(-1)^{p_{i+1}} \frac{\partial}{\partial p_{i+1}}, \quad i = 1, \ldots, n-2, \]
\[ U_{\alpha_{n-1}} = p_{n-1}, \quad U_{\alpha_n} = (-1)^{p_{n-1}}; \]
\[ U_{\alpha} = (-1)^{p_{n-1}} 4p_{n-1}^2, \quad U_{\beta} = \frac{(-1)^j}{16}. \]

The automorphism \( \sigma \) acts as the identity operator on \( P_1 \), and as \( (-1)^{j+1} 4p_6 \) on \( P_2 \). We obtain 4 distinct irreducible \( V_Q^\sigma \)-modules of twisted type:
\[ F_\sigma^\pm \otimes P_1, \quad (F_\sigma^\pm \otimes P_2)^\pm = (F_\sigma^\pm \otimes P_2^+) \oplus (F_\sigma^\pm \otimes P_2^-), \]
and we have the following correspondence:
\[ (F_\sigma \otimes P_1)^\pm \cong (V_{L_+} \otimes V_{T_1Z_{\beta}}^\pm) \oplus (V_{\frac{\alpha}{2}+L_+} \otimes V_{Z_{\beta}}^\pm) ; \]
\[ (F_\sigma \otimes P_2)^\pm \cong (V_{L_+} \otimes V_{T_2Z_{\beta}}^\pm) \oplus (V_{\frac{\alpha}{2}+L_+} \otimes V_{Z_{\beta}}^\pm). \]

4.6. \( E_6 \) root lattice with a Dynkin diagram automorphism.
Consider the \( E_6 \) Dynkin diagram with the simple roots \( \{\alpha_1, \ldots, \alpha_6\} \) labeled as follows:

\[ \begin{array}{cccccc}
  & \alpha_6 & \alpha_5 & \alpha_4 & \alpha_3 & \alpha_2 & \alpha_1 \\
 \end{array} \]
Let $Q$ be the root lattice, and $\sigma$ be the Dynkin diagram automorphism $\alpha_1 \leftrightarrow \alpha_5$, $\alpha_2 \leftrightarrow \alpha_4$, with fixed points $\alpha_3$ and $\alpha_6$. Set

\[
\alpha^1 = \alpha_1 + \alpha_5, \quad \beta^1 = \alpha_1 - \alpha_5, \quad \alpha^2 = \alpha_2 + \alpha_4, \quad \beta^2 = \alpha_2 - \alpha_4.
\]

Then we have: $Q = \bar{Q},$

\[
L_+ = \mathbb{Z}\alpha^1 + \mathbb{Z}\alpha^2 + \mathbb{Z}\alpha_3 + \mathbb{Z}\alpha_6, \quad L_- = \mathbb{Z}\beta^1 + \mathbb{Z}\beta^2,
\]
and

\[
Q/L = \{L, \alpha_1 + L, \alpha_2 + L, \alpha_1 + \alpha_2 + L\}.
\]

Hence, by Proposition 3.6,

\[
V_Q^\sigma \cong (V_{L_+} \otimes V_{L_-}^\perp) \oplus (V_{\frac{\alpha^1}{2} + L_+} \otimes V_{\frac{\beta^1}{2} + L_-})
\]

\[
\oplus (V_{\frac{\alpha^2}{2} + L_+} \otimes V_{\frac{\beta^2}{2} + L_-}) \oplus (V_{\frac{\alpha^1 + \alpha^2}{2} + L_+} \otimes V_{\frac{\beta^1 + \beta^2}{2} + L_-}).
\]

There are 3 distinct irreducible $V_Q^\sigma$-modules of untwisted type:

\[
(V_{L_+} \otimes V_{L_-}^\perp) \oplus (V_{\frac{\alpha^1}{2} + L_+} \otimes V_{\frac{\beta^1}{2} + L_-})
\]

\[
\oplus (V_{\frac{\alpha^2}{2} + L_+} \otimes V_{\frac{\beta^2}{2} + L_-}) \oplus (V_{\frac{\alpha^1 + \alpha^2}{2} + L_+} \otimes V_{\frac{\beta^1 + \beta^2}{2} + L_-}),
\]

\[
(V_{L_+} \otimes V_{\mu_2 + L_-}) \oplus (V_{\frac{\alpha^1}{2} + L_+} \otimes V_{\mu_2 + \frac{\beta^1}{2} + L_-}) \oplus (V_{\frac{\alpha^2}{2} + L_+} \otimes V_{\frac{\alpha^1 + \alpha^2}{2} + L_-})
\]

where $\mu_2 = \frac{1}{3}(\beta^1 + 2\beta^2)$. To describe the ones of twisted type, notice that the group $G_\sigma$ associated to $L_-$ is abelian and its characters $\chi$ are determined by $\chi(U_{\beta^i})$ where $i = 1, 2$. The latter are given by (2.19) with $s(\beta^i) = s_i \in \{\pm 1\}$. Thus we obtain 8 distinct irreducible $V_Q^\sigma$-modules of twisted type:

\[
(V_{L_+} \otimes V_{L_-}^{T_{X^1}}) \oplus (V_{\frac{\alpha^1}{2} + L_+} \otimes V_{L_-}^{T_{X^1, \pm s_1}})
\]

\[
\oplus (V_{\frac{\alpha^2}{2} + L_+} \otimes V_{L_-}^{T_{X^1, \pm s_2}}) \oplus (V_{\frac{\alpha^1 + \alpha^2}{2} + L_+} \otimes V_{L_-}^{T_{X^1, \pm s_1 s_2}}).
\]

We now describe the irreducible $\sigma$-twisted $V_Q^\sigma$-modules using Section 2.5. The irreducible modules over the group $G_\sigma$ associated to $Q$ can be identified with the vector space $P = \mathbb{C}[q_1^\pm, q_2^\pm, q_3^\pm, q_4^\pm]$, so that:

\[
U_{\alpha_1} = q_1(-1)^\sigma_{q_2}, \quad U_{\alpha_2} = q_2(-1)^\sigma_{q_3},
\]

\[
U_{\alpha_3} = q_3(-1)^\sigma_{q_4}, \quad U_{\alpha_6} = q_4.
\]
In fact, there are 4 such modules depending on the signs of $U_{\beta i}$; for each $s_1, s_2 \in \{\pm 1\}$, we have:

$$
U_{\alpha 4} = -s_2q_2(-1)^{\frac{q_2}{q_2}}, \quad U_{\alpha 5} = -s_1q_1(-1)^{\frac{q_1}{q_2}},
$$

$$
U_{\beta i} = s_i 4q_i^2, \quad U_{\beta i} = s_i \frac{1}{16}.
$$

The corresponding $G_{\sigma}$-module will be denoted $P_\chi$ where $\chi = (s_1, s_2)$. The automorphism $\sigma$ acts on $P_\chi$ by:

$$
\sigma(q_i) = -s_i q_i, \quad i = 1, 2.
$$

The irreducible $\sigma$-twisted $V_Q$-modules have the form $F_\sigma \otimes P_\chi$, and they give rise to 8 irreducible $V_Q^\sigma$-modules of twisted type:

$$
(F_\sigma \otimes P_\chi)^\pm, \quad \chi = (s_1, s_2), \quad s_1, s_2 \in \{\pm 1\}.
$$

They correspond to the above modules of twisted type with the same characters $\chi$.

Acknowledgements

The first author is grateful to Victor Kac and Ivan Todorov for valuable discussions and collaboration on orbifolds of lattice vertex algebras.

References

[A1] T. Abe, Fusion rules for the charge conjugation orbifold. J. Algebra 242 (2001), 624–655.

[A2] T. Abe, Rationality of the vertex operator algebra $V_L^+$ for a positive definite even lattice $L$. Math. Z. 249 (2005), 455–484.

[ABD] T. Abe, G. Buhl, and C. Dong, Rationality, regularity, and $C_2$-cofiniteness. Trans. Amer. Math. Soc. 356 (2004), 3391–3402.

[AD] T. Abe and C. Dong, Classification of irreducible modules for the vertex operator algebra $V_L^+$: general case. J. Algebra 273 (2004), 657–685.

[ADL] T. Abe, C. Dong, and H. Li, Fusion rules for the vertex operator algebra $M(1)$ and $V_L^+$. Comm. Math. Phys. 253 (2005), 171–219.

[BK] B. Bakalov and V.G. Kac, Twisted modules over lattice vertex algebras. In: “Lie theory and its applications in physics V,” 3–26, World Sci. Publishing, River Edge, NJ, 2004; math.QA/0402315.

[BM] B. Bakalov and T. Milanov, W-constraints for the total descendant potential of a simple singularity. Compositio Math. 149 (2013), 840–888.

[BPZ] A.A. Belavin, A.M. Polyakov, and A.B. Zamolodchikov, Infinite conformal symmetry in two-dimensional quantum field theory. Nuclear Phys. B 241 (1984), 333–380.

[Be] S. Berman, On generators and relations for certain involutory subalgebras of Kac–Moody Lie algebras. Comm. Algebra 17 (1989), 3165–3185.

[B1] R.E. Borcherds, Vertex algebras, Kac–Moody algebras, and the Monster. Proc. Nat. Acad. Sci. USA 83 (1986), 3068–3071.
R.E. Borcherds, *Monstrous moonshine and monstrous Lie superalgebras*. Invent. Math. **109** (1992), 405–444.

P. Di Francesco, P. Mathieu, and D. Sénéchal, *Conformal field theory*. Graduate Texts in Contemporary Physics, Springer–Verlag, New York, 1997.

R. Dijkgraaf, C. Vafa, E. Verlinde, and H. Verlinde, *The operator algebra of orbifold models*. Comm. Math. Phys. **123** (1989), 485–526.

C. Dong, *Vertex algebras associated with even lattices*. J. Algebra **161** (1993), 245–265.

C. Dong, *Twisted modules for vertex algebras associated with even lattices*. J. Algebra **165** (1994), 91–112.

C. Dong, C. Jiang, and X. Lin, *Rationality of vertex operator algebra $V_L^+$: higher rank*. Proc. London Math. Soc. **104** (2012), 799–826.

C. Dong and J. Lepowsky, *Generalized vertex algebras and relative vertex operators*. Progress in Math., 112, Birkhäuser Boston, 1993.

C. Dong, H. Li, and G. Mason, *Modular-invariance of trace functions in orbifold theory and generalized Moonshine*. Comm. Math. Phys. **214** (2000), 1–56.

C. Dong, H. Li, and G. Mason, *Regularity of rational vertex operator algebra*. Adv. Math. **312** (1997), 148–166.

C. Dong and K. Nagatomo, *Representations of vertex operator algebra $V_L^+$ for rank 1 lattice $L$*. Comm. Math. Phys. **202** (1999), 169–195.

J.R. Elsinger, *Classification of orbifold modules under an automorphism of order two*. Ph.D. dissertation, 2014, NC State University; http://www.lib.ncsu.edu/resolver/1840.16/9824

A.J. Feingold, I.B. Frenkel, and J.F.X. Ries, *Spinor construction of vertex operator algebras, triality, and $E_8^{(1)}$*. Contemporary Math., 121, Amer. Math. Soc., Providence, RI, 1991.

E. Frenkel and D. Ben-Zvi, *Vertex algebras and algebraic curves*. Math. Surveys and Monographs, 88, Amer. Math. Soc., Providence, RI, 2001; 2nd ed., 2004.

I.B. Frenkel, Y.-Z. Huang, and J. Lepowsky, *On axiomatic approaches to vertex operator algebras and modules*. Mem. Amer. Math. Soc. **104** (1993), no. 494.

I.B. Frenkel and V.G. Kac, *Basic representations of affine Lie algebras and dual resonance models*. Invent. Math. **62** (1980), 23–66.

I. B. Frenkel, J. Lepowsky, and A. Meurman, *Vertex operator calculus, in “Mathematical aspects of string theory,” 150–188*, Adv. Ser. Math. Phys., 1, World Sci. Publishing, Singapore, 1987.

I. B. Frenkel, J. Lepowsky, and A. Meurman, “Vertex operator algebras and the Monster,” Pure and Appl. Math., 134, Academic Press, Boston, 1988.

T. Gannon, *Moonshine beyond the Monster. The bridge connecting algebra, modular forms and physics*. Cambridge Monographs on Math. Phys., Cambridge Univ. Press, Cambridge, 2006.

P. Goddard, *Meromorphic conformal field theory*. In: “Infinite-dimensional Lie algebras and groups,” 556–587, Adv. Ser. Math. Phys., 7, World Sci. Publishing, Teaneck, NJ, 1989.
[K1] V.G. Kac, *Infinite-dimensional Lie algebras*. 3rd ed., Cambridge Univ. Press, Cambridge, 1990.

[K2] V.G. Kac, *Vertex algebras for beginners*. University Lecture Series, 10, Amer. Math. Soc., Providence, RI, 1996; 2nd ed., 1998.

[KP] V.G. Kac and D.H. Peterson, 112 constructions of the basic representation of the loop group of $E_8$. In: “Symposium on anomalies, geometry, topology,” 276–298, World Sci. Publ., Singapore, 1985.

[KRR] V.G. Kac, A.K. Raina, and N. Rozhkovskaya, *Bombay lectures on highest weight representations of infinite dimensional Lie algebras*. 2nd ed., Advanced Ser. in Math. Phys., 29, World Sci. Pub. Co. Pte. Ltd., Hackensack, NJ, 2013.

[KT] V.G. Kac and I.T. Todorov, *Affine orbifolds and rational conformal field theory extensions of $W_{1+\infty}$.* Comm. Math. Phys. 190 (1997), 57–111.

[Le] J. Lepowsky, Calculus of twisted vertex operators, *Proc. Nat. Acad. Sci. USA* 82 (1985), 8295–8299.

[LL] J. Lepowsky and H. Li, *Introduction to vertex operator algebras and their representations*. Progress in Math., 227, Birkhäuser Boston, Boston, MA, 2004.

[Li] H. Li, *Local systems of twisted vertex operators, vertex operator superalgebras and twisted modules*. In: “Moonshine, the Monster, and related topics,” 203–236, Contemp. Math., 193, Amer. Math. Soc., Providence, RI, 1996.

[M1] M. Miyamoto, A $\mathbb{Z}_3$-orbifold theory of lattice vertex operator algebra and $\mathbb{Z}_3$-orbifold constructions. In: “Symmetries, integrable systems and representations,” 319–344, Springer Proc. Math. Stat., 40, Springer, Heidelberg, 2013; [arXiv:1003.0237](https://arxiv.org/abs/1003.0237).

[M2] M. Miyamoto, $C_2$-cofiniteness of cyclic-orbifold models. Preprint [arXiv:1306.5051](https://arxiv.org/abs/1306.5051).

[M3] M. Miyamoto, Flatness and semi-rigidity of vertex operator algebras. Preprint [arXiv:1104.4675](https://arxiv.org/abs/1104.4675).

[S] P. Slodowy, *Beyond Kac–Moody algebras, and inside*. In: “Lie algebras and related topics,” 361–371, CMS Conf. Proc., 5, Amer. Math. Soc., Providence, RI, 1986.