Casimir energy in the compact QED on the lattice

Oleg Pavlovsky† and Maxim Ulybyshev‡

Institute for Theoretical Problems of Microphysics, Moscow State University, Moscow, Russia
† E-mail: ovp@goa.bog.msu.ru ‡ E-mail: ulybyshev@goa.bog.msu.ru

Abstract
A new method based on the Monte-Carlo calculation on the lattice is proposed to study the Casimir effect in the compact lattice $U(1)$ theory with Wilson action. We have studied the standard Casimir problem with two parallel plane surfaces (mirrors) and oblique boundary conditions on those as a test of our method. Physically, this boundary conditions may appear in the problem of modelling of the thin material films interaction and are generated by additional Chern-Simons boundary term. This approach for the boundary condition generation is very suitable for the lattice formulation of the Casimir problem due to gauge invariance. This method can be simply generalized on the case of more complicated geometries and other gauge groups.

Keywords: Lattice gauge theory, quantum electrodynamics, Casimir effect.

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1 Introduction and general motivation

During the last few years Casimir effect has attracted much attention due to the great experimental and theoretical progress in studying of this phenomenon. This macroscopic quantum effect plays crucial role in nanophysics, micro-mechanics, quantum optics, condensed matter physics, material science, and also it is very important for different models of boundary states in hadron physics, heavy-ion collisions, and cosmology.

Nowadays, many theoretical methods for calculation of Casimir effect are proposed. Unfortunately, most of these methods are based on the fixed boundary conditions or external potentials. Such a simplification can lead to problems in gauge invariance, renormalizability and locality. Moreover, there are some problems with different thermal corrections to Casimir force: different methods of calculations of the Casimir effect predict different corrections. Another part of the problem is calculation of Casimir force for complicated forms of boundary surfaces. Typically, various approximate methods (like the proximity force approximation method [1], [2]) are used in the case of curved surfaces, but it is still unclear now whether those approximations are correct and for what tasks they can be applied. And finally, analytic methods for Casimir effect calculation are very complicated and strongly dependent on the shape of surfaces. Practically, Casimir effect was studied analytically only for cases of plane, spherical and cylindrical surface forms. More complicated tasks have not been studied very well by this moment but such cases typically appear in experiments.

Another very interesting and important problem is the calculation of Casimir effect for other gauge groups (for example, SU(2) and SU(3)) and for fermion fields; both are very important for particle physics and for cosmology.

Based on arguments discussed above, it seems a very important task now to create a general method for calculation of Casimir effects which would work well for different shapes of boundary
surfaces, for different fields, at non-zero temperature and density and under other external factors. It means that such a method should be formulated very generally for working in different coupling regimes and different external conditions. And we believe that only direct lattice calculations in quantum field theory can meet all these requirements.

In our paper we consider the simplest Casimir problem with two parallel planes as a test of our method. The existence of the analytical answer for this problem is the additional motivation for such a choice of bound surfaces. This analytical answer assists us in fitting procedure for our numerical results.

A crucial obstacle on the way of the realization of any Casimir problem on the lattice is the following. If the Casimir energy is a reaction of the vacuum on the presence of the boundary, what is ”the boundary” in terms of lattice formalism? In other words, what is an observable quantity corresponding to such boundary? In fact, the answer is non-trivial and we devote the first part of the paper to this question. In the second part, we discuss the lattice algorithms and numerical results. And the last part is a conclusion.

2 Chern-Simons boundary conditions and Casimir effect

Casimir effect is a reaction of the vacuum on boundary condition. The spectrum of vacuum fluctuations depends on the boundary conditions. Changing of the boundary conditions leads to changing of the spectrum of vacuum fluctuations and so to generating of the corresponding Casimir force on the boundary. In the standard quantum field theory formalism, such changing of spectrum of vacuum fluctuations can be described, for example, by means of Green function method [1]. This approach is a very powerful tool for studying many essential Casimir tasks [1]. Unfortunately, the application of this analytical method to the case of more complicated shape of the boundary surfaces is not so easy due to calculation difficulties. Our aim is the creation of the numerical method for the Casimir effect calculation directly from the quantum field theory action. The lattice formalism looks very attractive for this role but manifestly we can not base in our approach on the separation of vacuum modes corresponding with boundary from the full spectrum of vacuum fluctuation. In lattice formalism we work in Euclidian space and deal with full spectrum of vacuum fluctuations and can not easily snatch out vacuum fluctuations corresponding to some boundary conditions. We need some very delicate approach for separation of vacuum fluctuation modes that preserve gauge invariance of our lattice formalism. Fortunately, such an approach to Casimir problem was proposed recently [5, 6].

This approach is based on a very elegant idea coming from some unique properties of the Chern-Simons action in three dimensions [5, 6]. Let us consider electro-magnetic fields in 3+1 dimentionions with the Maxwell action and additional Chern-Simons action given on 3-dimentional integral on the boundary surface $S$:

$$S = -\frac{1}{4} \int d^4x \, F_{\mu\nu} F^{\mu\nu} - \frac{\lambda}{2} \int d^3s \, \varepsilon^{\sigma\mu\nu\rho} n_\sigma A_\mu(x) F_{\nu\rho}(x),$$

where $\varepsilon^{\sigma\mu\nu\rho}$ is the Levi-Civita tensor and $n_\sigma$ is the normal vector to the boundary surface $S$, $\lambda$ is a real parameter.

Let us consider now the simplest form of boundary surface $S$, namely two parallel infinite planes placed at the distance $R$ from each other. The Chern-Simons formulation of this canonical Casimir problem was studied analytically in series of works [10, 11]. We will use this analytical answer for the fitting of our numerical data.

In the case of plane form of the boundary surface $S$ the Chern-Simons action in [11] has the
following form:

\[ S_{CS} = \frac{\lambda}{2} \int (\delta(x_3) - \delta(x_3 - R)) \varepsilon^{3\mu\nu\rho} A_\mu(x) F_{\nu\rho}(x) d^4x. \]  \hspace{1cm} (2)

where, in our formulation of this Casimir problem, normal vectors to the planes are turned in the opposite directions. This choice of the normal vector orientation corresponds to our renormalization procedure based on the connection between open and closed Casimir problems.

If the parameter \( \lambda \) is small, electro-magnetic fields obviously don’t feel any boundary and are free. What happens if the parameter \( \lambda \) becomes large and tends to infinity and fields dynamics on the boundary surface \( S \) is determined by Chern-Simons action? Let us consider the equation of motion obtained from the action (2):

\[ \Box A^\mu + \lambda (\delta(x_3) - \delta(x_3 - R)) \varepsilon^{3\sigma\nu\rho} A_\sigma \partial_\nu A_\rho = 0. \]  \hspace{1cm} (3)

At \( \lambda \to \infty \), it is easy to obtain from (3) a corresponding boundary conditions on the surface \( S \):

\[ E_n|_S = 0, \quad H_\parallel|_S = 0, \]  \hspace{1cm} (4)

where \( E_n \) and \( H_\parallel \) are normal and longitudinal components of electric and magnetic fields correspondingly. These conditions mean the nulling of the energy flux of the electromagnetic field through the surface.

This phenomenon corresponds to the well-known property of the Chern-Simons theory in 3-dimension, namely to topological vortex (strings) generation [7, 8]. As it was shown by Witten [9], Chern-Simons theory, which is a quantum field theory in three dimensions, is exactly solvable by using nonperturbative methods and the topological vortices play a leading role in this process. It was shown that the partition function of this theory depends on the topology of the surface \( S \) and gauge group only, and tends to the zero if \( \lambda \) tends to infinity. There are no propagation modes in this case and if the boundary surface \( S \) is closed, the gauge fields inside and outside this surface become separated from each other by the topological vortex fields on the surface. In the case of the finite value of \( \lambda \), we have a non-trivial interaction between topological vortices and electro-magnetic waves [8]. For our practical purposes it is enough to
know that the parameter \( \lambda \) plays the role of the effective regulator of the penetrability of our boundary surface for electro-magnetic wave.

The analytical answer of Casimir energy per unit area for two planes is given by Green function method is the following [10, 11]:

\[
E_{\text{Cas}} = -\frac{\pi^2}{720 R^3} f(\lambda),
\]

where function \( f(\lambda) : \lim_{\lambda \to \infty} f(\lambda) = 1 \) (Fig. 1) can be written as:

\[
f(\lambda) = \frac{90}{\pi^4} \text{Li}_4 \left( \frac{\lambda^2}{\lambda^2 + 1} \right).
\]

here the polylogarithm function \( \text{Li}_4(x) \) is defined as

\[
\text{Li}_4(x) = \sum_{k=1}^{\infty} \frac{x^k}{k^4} = -\frac{1}{2} \int_0^\infty k^2 \ln(1 - xe^{-k}) dk.
\]

In our paper we will study the Casimir energy per unit area behavior at small \( \lambda \), which is the following:

\[
E_{\text{Cas}} = -\frac{\lambda^2}{8\pi^2 R^3} + O(\lambda^4)
\]

3 Wilson ”bag” and numerical lattice simulation of Maxwell-Chern-Simons theory

In our paper we use the four-dimensional hyper-cubical lattice and the simplest form of the action for the U(1) gauge fields (so-called ”Wilson action”):

\[
S_W = \beta \sum_x \sum_{\mu\nu} (1 - \text{Re} U_{p,x,\mu\nu}),
\]

where the link and plaquette variables are defined as:

\[
U_{l,x,\mu} = e^{igA_\mu(x)\alpha},
\]

\[
U_{p,x,\mu\nu} = U_{l,x,\mu} U_{l,x+\hat{\mu},\nu} U_{l,x+\hat{\nu},\mu} U_{l,x,\nu}^\dagger.
\]

Here \( \alpha \) is the step of the lattice and the lattice parameter \( \beta = 1/g^2 \). Physical quantities are calculated in the lattice formalism by means of field configuration averaging, where the field configurations (the set of all link variables) are generated with the statistical weight \( e^{-S_W} \).

We have clarified in previous sections that additional the Chern-Simons action describes Casimir effect. In order to find a lattice description of Casimir interaction between boundary surfaces let us consider Wilson loop, which describes the interaction of charged particles. Wilson loop can be written in QED as

\[
W_C = e^{ig \oint C A_\mu dx^\mu} = e^{i \int J_\mu A^\mu dx}. \tag{7}
\]

The exponent in (7) is the additional term to the action. This term describes the interaction of the field \( A_\mu \) with the current \( J_\mu(x) = g \oint C \delta(x - \xi) d\xi_\mu \) of charged particle. Configuration
averaging of Wilson loop $\langle W(R, T) \rangle$ (where R and T are dimensions of the loop) converges in the limit $T \to \infty$ to:

$$\langle W(R, T) \rangle \to C e^{-V(R)T},$$

where $V(R)$ is the energy of interaction between charged particles. The same method can be used for calculation of Casimir energy by means of Chern-Simons action.

Analogously to the description of charged particles interaction by 1D integral along Wilson loop, we will describe the Casimir interaction of surfaces by corresponding 3D integral. The first problem is that for stationary objects the action (2) is an integral from $t = -\infty$ to $t = \infty$, so by analogy to Wilson loop, we should enclose the surface of the integration in t-direction. The integration surface for two planes is shown in Fig. 2. This closing procedure can be performed both for plane surfaces and for any curved surface in 3-dimensional space. As the result of this procedure, so-called Wilson bag [3, 4] can be obtained. It can be written as

$$e^{i\lambda \int \Sigma \varepsilon_{\mu\nu\rho\sigma} A^{\nu} F^{\rho\sigma} dS^\mu},$$

where $\Sigma$ is closed 3-dimensional surface in 4-dimensional space-time. The final conclusion is that Wilson bag is (by analogy to Wilson loop) observable quantity which gives us Casimir energy of the objects, defined by the surface of integration. For two planes we will calculate the following object:

$$W_{Bag}(R, T) = e^{i\lambda S(R, T)},$$

where

$$S(R, T) = \int_0^T dt \int \int dx dy dz (\delta(z - R) - \delta(z)) \varepsilon_{3\nu\rho\sigma} A^{\nu} F^{\rho\sigma} +$$

$$+ \int_0^R dz \int \int dx dy dt (\delta(t - T) - \delta(t)) \varepsilon_{0\nu\rho\sigma} A^{\nu} F^{\rho\sigma}.$$

And in the limit $T \to \infty$:

$$\langle W_{Bag}(R, T) \rangle \to C e^{-E_{cas}(R)T}.$$

The second problem is to rewrite of the Wilson bag in terms of lattice objects (links and plaquettes). Using the expansion of link variable:

$$U_{t,x,\mu} \approx 1 + igA_\mu(x)a - \frac{1}{2}g^2a^2A^2_\mu + O(g^3)$$
Figure 3: $E_{\text{cas}}(R)$ at $\beta = 8.0$ and $\lambda = 0.0005$.

(here $a$ is the step of the lattice) and plaquette variable:

$$U_{p,x,\mu\nu} \approx 1 + igF_{\mu\nu}(x)a^2 - \frac{1}{2}g^2a^4F_{\mu\nu}^2 + O(g^3),$$

we can construct the following expression:

$$(1 - \text{Re}(U_{p,x,\mu\nu}U_{l,x,\rho})) - (1 - \text{Re}U_{p,x,\mu\nu}) - (1 - \text{Re}U_{l,x,\rho}).$$

(9)

It can be easily proved that expansion of (9) by $g$ is:

$$g^2a^3F_{\mu\nu}A_{\rho} + O(g^4).$$

(10)

In the main order it gives us one term in the density of Chern-Simons action multiplied by $g^2$. After this, we replace integration by sum, and multiple this sum by $\beta$ in order to eliminate $g^2$ in the last expression. So we obtain a computable expression for Wilson bag.

Let us consider some results of numerical simulations. You can see on the Fig. 3 the dependence of the Casimir energy $E_{\text{cas}}$ on $R$ (the distance between planes).

The parameter $\beta$ is equal to 8.0 and $\lambda = 0.0005$ for this calculation. The size of the lattice is $32^4$. Unfortunately, we can not see here $E_{\text{cas}}(R)$ dependence like $R^{-3}$. All what we can see is a distinction of the first point from the other ones which form the plateau. The dynamics in the plateau can not be revealed due to numerical errors. Nevertheless, let us make a suggestion that this difference of the first point from the plateau is a manifestation of the dependence of Casimir energy on the $R$ as $R^{-3}$ (this suggestion follows obviously from dimensional reasons). So we have done a fitting of this point sequence by the function

$$E_{\text{cas}} = C_1 + \frac{C_2}{R^3}.$$ 

The values of coefficients $C_1$ and $C_2$ are given on the Fig. 3. Our aim is, of course, the coefficient $C_2$. $C_1$ is an analog of something like self energy of charged particles. Such self energy can be
Figure 4: Dependence of $C_2$ on the lattice size.

Figure 5: Dependence of $C_2$ on $\lambda$. 
obtained in Wilson loop calculation. We have performed the same calculations for different values of \( \beta \). The situation is qualitatively the same.

Now let us consider the physical meaning of the coefficient \( C_2 \). A full Casimir energy of the interaction between two planes have been obtained in these calculations. This energy is expressed in units \( a^{-1} \), where \( a \) is the lattice step. If we take into account that the area of the plane is equal to \((aN)^2\), where \( N = 32 \) is the size of the lattice, we can write the physical (dimensional) value of the Casimir energy density:

\[
E_{\text{Cas,phys}} = \frac{1}{a} \frac{1}{R^3} \frac{1}{(aN)^2} = \frac{C_2 N^{-2}}{(Ra)^3}.
\]

After comparison between this formula and (5) we can conclude that \( C_2 N^{-2} \) should be equal to coefficient before \( R^{-3} \) in (5) or in (6) (\( \lambda \) is small in our calculations by reasons discussed later). And this coincidence is rather good. For example, for the calculation with \( \beta = 8 \) the expression \( C_2 N^{-2} = -3.27 \times 10^{-9} \pm 1.1 \times 10^{-10} \) and the theoretical value of this quantity for \( \lambda = 0.0005 \) is \(-3.17 \times 10^{-9}\).

The next step is the analysis of \( C_2 \) dependences on the size of the lattice \( N \) and on the parameter \( \lambda \). The first one is presented on the Fig. 4. We can see here that \( C_2 \) is proportional to \( N^2 \) with high accuracy. So the calculated Casimir energy of two planes is really proportional to theirs area.

The dependence \( C_2(\lambda) \) is shown on Fig. 5. In accordance with (6), the calculated coefficient at \( R^{-3} \) is proportional to \( \lambda^2 \) for small \( \lambda \). Unfortunately, present calculation methods can not give values of \( C_2 \) for large \( \lambda \) and we have in fact only small \( \lambda \) limit. This problem appears because the number in the exponent (8) becomes too big and the errors arise for calculations with large \( \lambda \). We are studying this problem and hope that some methods of solution can be suggested.

![Figure 6: Dependence of \( C_2 \) on \( \beta \).](image)

The last point is the consideration of the continuous limit of our calculations. There are two aspects of this problem. First, when we calculate Wilson bag, in fact, only the special
quantity being approximately equal to Chern-Simons action is calculated \( \langle \rangle \). Expansions of all quantities have been done by powers of \( g \). So the smaller is \( g \), the closer is our lattice Wilson bag quantity \( n \) to its first term in expansion \( \langle \rangle \). Thereby calculations in the limit \( g \to 0 \) or \( \beta \to \infty \) are the first stage in continuous limit. This stage is shown in Fig. 6 for \( C_2 \) and in Fig. 7 for \( C_1 \). Additional line at Fig. 6 is the theoretical value of \( C_2 \) for \( \lambda = 0.0005 \). As we can see in the limit \( \beta \to \infty \) the coefficient \( C_2 \) becomes stable around the right analytical value. This stability can be considered as an additional argument for a suggestion that \( C_2 \) has the real physical meaning. As for \( C_1 \), it hasn’t any continuous limit due to permanent growth with increasing of \( \beta \).

The second stage is ”infinite lattice volume” limit \( N \to \infty \). But our method is not very sensitive to the lattice size for the plane form of the boundaries. Even for rather small lattices (see Fig. 4) the value of the coefficient before \( R^{-3} \) is close to the right quantity. There is one peculiarity in this continuous limit procedure and we think that this moment should be emphasized. The lattice step \( a \) disappears from our considerations because the final result of our calculation is dimensionless coefficient at \( R^{-3} \) in expression for Casimir energy per unit area. So the value of \( a \) is not important for us and instead of limit \( a \to 0 \) we have limit \( N \to \infty \).

**Conclusions**

In this paper we have proposed the numerical method for the Casimir energy calculation based on the lattice simulations of QED. We have combined two ideas: the generation of the boundary conditions by means of the additional Chern-Simons boundary action and the lattice ”Wilson bag” concept (the lattice presentation of the closed 3-dimensions surface in Euclidian 4-dimensions space). This combination is in fact a lattice definition of the respective quantum observable for the Casimir energy and for the Casimir interaction between surfaces. We have tested our method in the simplest case of the Casimir interaction between two plane surfaces and have achieved a good agreement with the analytical results for this problem.
A big advantage of our method is its universality. This method can be applied for following calculations:

- Investigation of the Casimir interaction between surfaces of complicated shapes which are interesting from the experimental point of view.

- Calculation of thermal corrections to the Casimir force. (Unfortunately, this very important problem is very difficult for any analytical investigations in physically interesting cases.)

- Study of Casimir effects for non-abelian gauge fields and for fermions which is very essential for phenomenological models in heavy-ion collisions and for the models of low-energy hadron states [12, 13].

These problems are under consideration now.

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