In the Kerr geometry, we calculate various surfaces of constant curvature invariants. These extend well beyond the Kerr horizon, and we argue that they might be of observational significance in connection with non-minimally coupled matter fields. Moreover, we demonstrate that the principal null directions of the Kerr geometry can be obtained by projections involving either the Bel–Robinson or the Kummer tensor. We conjecture that this is also possible in more general settings.

Let us start by defining the two Cartan invariants \[ \mathcal{E} := -\frac{1}{2} C_{0101} = mr \frac{r^2 - 3a^2 \cos^2 \theta}{(r^2 + a^2 \cos^2 \theta)^{\frac{3}{2}}}, \]
\[ \mathcal{B} := -\frac{1}{2} C_{0123} = ma \cos \theta \frac{3r^2 - a^2 \cos^2 \theta}{(r^2 + a^2 \cos^2 \theta)^{\frac{3}{2}}}, \] where hatted indices refer to the orthonormal coframe \( \hat{\mu} \). These quantities capture the mass (“gravitational charge”) and angular momentum (“gravitational current”) of the Kerr solution. They are ubiquitous in analytic expressions for different (pseudo-)scalar invariants.

Focusing the discussion somewhat, let us now introduce the curvature invariants that will be examined in this work. The Kretschmann scalar \( K \) and Chern–Pontryagin pseudoscalar \( \mathcal{P} \) take the values \[ K := C_{abcd} C^{abcd} = 48(\mathcal{E}^2 - \mathcal{B}^2), \]
\[ \mathcal{P} := (\ast C)_{abcd} C^{abcd} = -96\mathcal{E}\mathcal{B}, \] where \((\ast C)_{ijkl}\) is the left dual of the Weyl tensor. Following Ref. [12] we consider the additional Karlhede–Lindström–Aman invariants \[ K' := (\nabla_a C_{bcde}) (\nabla^a C^{bcde}), \]
\[ \mathcal{P}' := [\nabla_a (\ast C)_{bcde}] (\nabla^a C^{bcde}). \] They vanish on the Kerr ergosphere (and hence at the Schwarzschild horizon, for \( a = 0 \)), but they are not very useful in locating the horizon of the Kerr solution. Their explicit form can be found in Ref. [12]. By contrast, the following invariant vanishes on the Kerr horizon [12, 14]:
\[ H := \ast [dK \wedge dP \wedge \ast (dK \wedge dP)] = 16 \times 12^2 \times m^6 a^2 \cos^2 \theta (r^2 - 2mr + a^2), \] where \( a \) is the angular momentum of the Kerr black hole.

We close by defining the Bel–Robinson tensor [15, 16] as well as the vacuum Kummer tensor [17, 18]:
\[ B_{ijkl} := C_{abck} C_{ij}^{ab} + (\ast C)_{ijak} (\ast C)_{k}^{ab}, \]
\[ K_{ijkl} := -C_{iajb} C_{abcd} C_{kld}. \]
The Bel–Robinson tensor is related to the notion of superenergy, since its full contraction with any timelike vector is positive, $B_{abcd}u^a u^b u^c u^d \geq 0$ [19]. The Kummer tensor can be introduced by analogy with electromagnetism. In Ref. [17], it was suggested that this tensor may encode specific properties of gravitational waves. An important class of vector fields, the principal null directions (PNDs) of the Weyl tensor, are the principal null directions (PNDs) of the spacetime. The Bel–Robinson tensor and the Kummer tensor admit the following invariants:

$$B = B_{abcd} B^{abcd} = 4 \times 12^2 \times (B^2 + E^2)^2,$$
$$S = K^{ab}_{\ ab} = 48 E (3B^2 - E^2),$$
$$A = K_{abcd}^\alpha = 24 B (3E^2 - B^2).$$

Now we can define invariant curvature surfaces by setting

$$f(K, \mathcal{P}, K', \mathcal{P}', H, B, S, A) = \text{const},$$

where $f$ is some polynomial function. Of course, one may consider the opposite case where just one of the invariants is a constant value, as can be seen in Fig. 2. These invariants define surfaces that may extend well outside the horizon and can take relatively complicated shapes.

One may ask: what is a special value for these invariants to take? Since the Kerr solution can be written as $ds^2 = n^2 d\tau^2$, where the new line element $d\tau^2$ only depends on the dimensionless parameter $\alpha := a/m$, there is no intrinsic length scale other than $m$. In other words, if all distances are measured in terms of $m$, the dimensionless parameter $\alpha$ no longer provides a length scale related to the rotation parameter.

Of course, one special value still exists: zero. Suppose we consider the dynamics of a non-minimally coupled matter field in the vicinity of a black hole. Under some assumptions on the Lagrangian, the non-minimally coupled curvature expressions can serve as an effective potential for the matter field. Therefore, in a WKB approximation where the field dynamics are fast compared to the gravitational dynamics, the field can condense in the minimum of its potential. Notably, in some circumstances, the expression of this minimum may have the structure of Eq. (8), see Fig. 2 for a few zero-curvature surfaces around the Kerr black hole. Again, they may extend far outside the horizon.

II. PROJECTIVE SURFACES AND PRINCIPAL NULL DIRECTIONS

Suppose we have a vector field at our disposal: then, we may consider scalar invariants formed by contractions of expressions in the curvature and that vector field:

$$[f a_1 \ldots a_p (C_{ijk})] n^{a_1} \ldots n^{a_p} = \text{const},$$

An important class of vector fields, $n^i$, intrinsic to a given spacetime, are the principal null directions (PNDs) of the Weyl tensor, defined by $n_i (C)_{ijkl} n^k n^l n^m n_n = 0$ with $n^2 = 0$ [20]. The algebraic multiplicity of the corresponding eigenvalue problem then defines the Petrov type of the given spacetime at each point. The Kerr spacetime is of type D, and the PNDs are

$$n^i = (\frac{\rho}{\Delta}, \pm 1, 0, \frac{1}{\Delta}), \ n^\mu = \sqrt{\frac{\rho^2}{\Delta}} (1, \pm 1, 0, 0).$$

Here, $n^\mu$ denotes the components of the PNDs with respect the coframe $d\mu$. According to Eq. (6) in Bel’s work [15], there is an equivalent way to formulate this eigenvalue problem for Petrov type D spacetimes (what Bel calls type IIIb). For the Kerr solution it reads

$$C^\alpha_{\ \beta} n^\alpha n^\beta = -2 E n^\alpha n^\nu,$$
$$\ast C)_{\ \beta} n^\alpha n^\beta = +2 S n^\alpha n^\nu.$$  (11)

It is straightforward to check that the above implies

$$B_{abcd} n^a n^b n^c n^d = 0, \ K_{abcd} n^a n^b n^c n^d = 0. \ (12)$$

Is the converse also true? Interestingly, inserting instead the general null vector $n^\mu = (\pm \sqrt{v_1^2 + v_2^2 + v_3^2}, v_1, v_2, v_3)$ into the left-hand side of Eq. (12) implies

$$m^2 (v_2^2 + v_3^2)^2 = 0, \ m^2 E (v_2^2 + v_3^2)^2 = 0,$$

respectively. For $m \neq 0$ and $E \neq 0$, the unique solution is $v_2 = v_3 = 0$ for any $v_1$. Hence either the Bel–Robinson surface or the Kummer surface imply the PNDs of the Kerr spacetime. Due to the algebraic nature of this proof, it seems plausible to us that this result may hold for general type D spacetimes. It remains to be seen whether these concepts can be generalized to different Petrov types.

III. CONCLUSIONS

Invariant curvature surfaces and projective surfaces seem to play an important role in the study of the Kerr geometry, both for experimental reasons (non-minimally coupled matter fields) as well as for theoretical considerations (Petrov classification). More work is necessary to extend our conclusions beyond the Kerr metric to general Petrov type D solutions and perhaps to other algebraically special spacetimes as well.

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Figure 1: (Colors online.) Various invariants for the Kerr black hole with $m = 1$, $a = 0.8$, plotted in the plane $y = 0$. The $z$ axis is the axis of rotation, and $r^2 = x^2 + y^2 + z^2$. Note that all scalars are reflection symmetric with respect to $z \to -z$, whereas all pseudoscalars are antisymmetric. The inner and outer ergosphere are indicated by dashed green, whereas the inner and outer horizon correspond to the solid green lines.

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Figure 2: (Colors online.) Surfaces of vanishing Kretschmann scalar (orange) and vanishing Pontryagin pseudoscalar (green) visualized for Kerr black holes of different parameters, and \( r^2 = x^2 + y^2 + z^2 \). The inner and outer ergosphere, as well as the inner and outer horizon, are plotted in black. We excluded the interior of the inner horizon from the plots in order to improve visibility of the structures close to the horizon.

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