Multiscale Generative Models: Improving Performance of a Generative Model Using Feedback from Other Dependent Generative Models

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Abstract

Realistic fine-grained multi-agent simulation of real-world complex systems is crucial for many downstream tasks such as reinforcement learning. Recent work has used generative models (GANs in particular) for providing high-fidelity simulation of real-world systems. However, such generative models are often monolithic and miss out on modeling the interaction in multi-agent systems. In this work, we take a first step towards building multiple interacting generative models (GANs) that reflect the interaction in real world. We build and analyze a hierarchical set-up where a higher-level GAN is conditioned on the output of multiple lower-level GANs. We present a technique of using feedback from the higher-level GAN to improve performance of lower-level GANs. We mathematically characterize the conditions under which our technique is impactful, including understanding the transfer learning nature of our set-up. We present three distinct experiments on synthetic data, time series data, and image domain, revealing the wide applicability of our technique.

Introduction

Realistic simulations are an important component of training a reinforcement learning based decision-making system and transferring the policy to the real world. While typical video game-based simulations are, by design, realistic, the same is very difficult to claim when the problem domain is a real-world multi-agent system such as stock markets and transportation. Indeed, recent works have used generative models to design high-fidelity simulations of a real-world, e.g., prices offered in multiple regions (apart from other random factors) determining the total demand of electricity. We repurpose the discriminator (or critic in WGANs) as a conditional model that generates data conditioned on data from the lower-level GANs, mirroring agent interaction in the real world. We particularly note that our hierarchy is among different GAN instances (with distinct generative tasks), unlike hierarchy among multiple generators or discriminators for a single generative task that has been explored in the literature (Hoang et al. 2018).

Our first contribution is an architecture of how to utilize the feedback from a higher-level conditional GAN to improve the performance of a lower-level GAN. The higher-level GAN takes as input the output of (possibly multiple) lower-level GANs, mirroring agent interaction in the real world, e.g., prices offered in multiple regions (apart from other random factors) determining the total demand of electricity. We repurpose the discriminator (or critic in WGANs) of the higher-level model to provide an additional feedback (as an additional loss term) to the lower-level generator.

As our second contribution, we provide a mathematical interpretation of the interacting GAN set-up, thereby characterizing that the coupling between the higher-level and the lower-level models determines whether the additional feedback improves the performance of the lower-level GANs or not. We identify two scenarios where our set-up can be especially helpful. One is where there is a small amount of data available per agent at the lower-level and another where data is missing in some biased manner for the lower-level agents. Both these scenarios are often encountered in the real-world data, for example, in the use case of modeling electricity market, the amount of demand data for a new consumer might be small and the feedback from a higher-level model about such consumer’s expected behaviors helps in building a more realistic behavioral model. Moreover, the case of small amount of data can also be viewed as a few-shot generation problem.

Finally, we provide three distinct sets of experiments revealing the fundamental and broadly applicable nature of
our technique. The first experiment with synthetic data explores the various conditions under which our scheme provides benefits. Our second experiment with electric market time series data from Europe explores a set-up where electricity prices from three countries (plus random factors) determine demands, where we show improved generated price time series from one of the lower-level data generator using our approach. In our third experiment, we demonstrate the applicability of our work for a few-shot learning problem where the higher-level CycleGAN is a horse to zebra converter trained using ImageNet, and a single lower-level GAN generates miniature horse images using feedback from the CycleGAN and small amount of data. Overall, we believe that this work provides a basic ingredient of and a pathway for building fine-grained high-fidelity multi-agent simulations. All missing proofs are in appendix.

Related Work
Generative Models: There is a huge body of work on generative modeling, with prominent ones such as VAEs (Kingma and Welling 2013) and GANs (Goodfellow et al. 2014). The closest work to ours is a set-up in federated learning where multiple parties own data from a small part of the full space and the aim is to learn one generator over the full space representing the overall distribution. Different approaches have been proposed to tackle this, where multiple discriminators at multiple parties train a single generator (Hardy, Le Merrer, and Sericola 2019) or multiple generators at different parties share generated data (Ferdowsi and Saad 2020) or network parameters (Rasouli, Sun, and Rajagopal 2020) to learn the full distribution. Our motivation, set-up, and approach is very different. Our motivation is fine-grained multi-agent simulation where we exploit the learned knowledge of the real world interaction (embedded in the higher level conditional GAN) to improve a lower level agent’s generative model. Dependency among agents is crucial for our work, whereas these other approaches have independent parties.

Other non-federated learning works also employ multiple generators or discriminators to improve learning stability (Hoang et al. 2018) or prevent mode collapse (Ghosh et al. 2018). Pei et al. (2018) proposed multi-class discriminators structure, which is used to improve the classification performance cross multiple domains. [Liu and Tuzel 2016] proposed to learn the joint distribution of multi-domain images by using multiple pairs of GANs. The network of GANs share a subset of parameters, in order to learn the joint distribution without paired data. Again, our main insight is to exploit dependencies among agents which is not considered in these work.

Few-shot Image Generation: Few-shot image generation aims to generate new and diverse examples whilst preventing overfitting to the few training data. Since GANs frequently suffer from memorization or instability in the low-data regime (e.g., less than 1,000 examples), most prior work follows a few-shot adaptation technique, which starts with a source model, pretrained on a huge dataset, and adapts it to a target domain. They either choose to update the parameters of the source model directly with different forms of regularization (Wang et al. 2018; Mo, Cho, and Shin 2020) or embed a small set of additional parameters into the source model (Noguchi and Harada 2019; Wang et al. 2020; Robb et al. 2020). Besides these, some works also exploit data augmentation to alleviate overfitting (Karras et al. 2020; Zhao et al. 2020) but this approach struggles to perform well under extreme low-shot setting (e.g., 10 images) (Ojha et al. 2021). Different from these work, our method exploits abundant data that is dependent on the target domain data and provides effective feedback to the GAN that performs adaptation (lower level GAN), leading to higher quality generation results. Note that our MGM does not have any specific requirement for the lower level GAN so it can benefit a bunch of existing work.

Preliminaries and Problem Formulation
Although our approach is independent of the type of generative model used, to explain our main solution approach concretely, we adopt a well-known generative model called WGAN, which is briefly introduced below. Every WGAN has a generator network \( G \) and a critic network \( C_w \), where \( \theta, w \) are weights of the networks. The generator generates samples \( y = G_\theta(z, c) \in Y \) from a learned conditional distribution \( P_y(z \mid c) \), where \( z \) is some standard noise (e.g., Gaussian) and \( c \in C \) is a condition. The goal is to learn the true distribution \( P_{y \mid c} \).

In an unconditional WGAN, the critic simply outputs an estimate of the Wasserstein-1 distance, \( W(P^r, P^g) \), between the real and generated distributions, given samples from each. For the conditional case, the condition \( c \) forms part of the input \( (y, c) \) for the critic. Let \( P^g_{y \mid c} \) denote the distribution over \( Y \times C \) with \( c \sim P^r_c \) and \( y \sim P^g_{y \mid c} \), and similarly \( P^g_{y \mid c} \) denotes the distribution with \( c \sim P^r_c \) and \( y \sim P^g_{y \mid c} \). With the condition as input, the critic estimates the distance between joint distributions, \( W(P^r_{y \mid c}, P^g_{y \mid c}) \). The expected loss for the generator is

\[
E_{(y,c) \sim P^g_{y \mid c}}[C_w(G_\theta(z, c))] - E_{(y,c) \sim P^g_{y \mid c}}[C_w(y, c)] + 
\lambda J_{C_w^P_{y \mid c}}
\]

where \( J \) is the gradient penalty regularizer in WGAN-GP (Gulrajani et al. 2017) with weight \( \lambda \). The actual loss optimized is an empirical version of the above expectations, estimated from data samples. A subtle implementation issue (unaddressed in prior work) arises with the \( J \) term when the condition is continuous; we discuss how we address this in the appendix.

Problem Formulation: Let there be \( n \) agents, where the agent in \( i^{th} \) position produces data distributed in the space \( X_i \subset \mathbb{R}^{d_i} \) for some positive integer dimension \( d_i \). Corresponding to position \( i \), let \( P_i \) be a probability distribution and the agent generates \( x_i \in X_i \) distributed as \( x_i \sim P_i \). Denote the data generated by all the agents as a single vector \( x = (x_1, \ldots, x_n) \). These lower-level agents generate data that feeds into a higher-level agent, which we call as mixer.

\[1\] As a heuristic, generated sample is mixed with real sample in WGAN-GP; we ignore this in writing for ease of presentation.
Our goal is to obtain a fine-grained model of the agents as well as the mixer. With sufficient data, building multiple GAN models for the mixer and every agent is not very hard. However, the data for some lower-level agents are often limited or even corrupted; e.g., when an agent has recently moved into the system. For example, this could happen with a new electricity supplier in the electricity market, or a new trader in a stock market, or even a new image generation task with low or biased amount of data. Thus, we assume one of the $n$ agents, indexed by $i_0$ has newly arrived and replaces the prior agent at index $i_0$. This leads to the technical problem that we solve in this paper.

**Problem Statement:** Our goal is to utilize the higher-level mixer GAN to improve upon the generative model of a lower-level agent GAN, namely one that is newly arrived.

We believe that the above is tractable since the mixer model is trained on and relies on the output distribution of every kind of agent, and hence encodes information about the agent-level model within its own model. This information can be used to guide the training of an agent-level GAN.

**Examples:** We provide two possible real world practical applications of a solver of the aforementioned problem. This is apart from the practical few-shot learning experiment we present later. First, a biased data scenario. Consider three transportation network companies (TNCs), such as Uber, providing online ride-hailing services in a city with multiple districts. They would like to model the average behavior of drivers in different districts to help with optimized vehicle dispatching. This can be done by training a generative model for every district (each district as an agent) if a TNC has all the data. However, a TNC only has part of the drivers’ data, that is, a biased dataset. E.g., 30% drivers work for TNC-1, while 40% and 30% work for TNC-2 and TNC-3 respectively. Due to the competitive relationship, TNCs do not share data with each other. They agree on providing their data to a third party, where all the data is used to pre-train a mixer that models a city-level output (e.g., total supply). When a TNC trains an agent, the agent can obtain extra feedback by feeding its own generator’s output to the third part (mixer) and receiving the gradient feedback (as explained in the approach section). During the process, the third party does not share any data with a TNC.

Next, consider a low data scenario. Continuing from the above example, consider a new TNC (a fourth TNC) that starts to provide service in the city. As it is a new TNC, it has very limited data. In this case, the fourth TNC can also improve its own model by the feedback from the mixer.

**Approach**

Our approach, which we call MGM, to the problem stated above relies upon the architecture in Fig. 1. We choose GANs as our generative model (different types of GANs for different problems). As shown in the architecture, the mixer GAN is a conditional GAN in which the conditions are the output of the agent GANs. We use the notation: $x_j \rightarrow y_r$ as a vector in $x_{\neq i_0} X_j$ and $(x_{i_0}, y_{i_0})$ denotes the full vector with the $i_0$ component.

The $i^{th}$ agent generator takes as input noise $z_i$ to produce a sample $x_i = G_{\theta_i}(z_i)$ from its learned distribution. The mixer generator takes as input (1) noise $z_{mix}$ and (2) the output $x = \langle x_1, \ldots, x_n \rangle$ of the agent generators as a condition to produce a sample $y = G_{\theta_{mix}}(z_{mix}, x)$ from its learned distribution. The conditional mixer GAN is trained in the traditional manner using available data for all the agents. We assume that there is plenty of data to train the mixer GAN effectively. That is, the learned mixer generator $G_{mix}$ models the distribution $P_{y|r}$ accurately.

We focus on the training of one newly arrived agent $i_0$, which replaces the prior agent $i_0$. The position $i_0$ has a true average distribution $P_{r|y}$, this new arrival happens after the mixer GAN is trained. $\hat{D}^r \subset Y$ is the real output data corresponding to the mixer. We also have datasets for every agent $j \neq i_0$, given by $D^r_j \subset X_j$ and $D^r_{i_0} \subset X_{i_0}$. In particular, $D^r_{i_0}$ has low amount of or biased data that prevents learning the true distribution $P_{r|y}$ just using $D^r_{i_0}$.

Our technical innovation is in training generator of agent $i_0$. The framework for this is shown in Algorithm 1. The agent trainer has access to the datasets stated above, $D^r_j, D^r_{i_0}$ for all agents $j$, as well as the pre-trained $G_{mix}$ of the mixer. The agent $i_0$ generator is trained using an empirical version $L_i$ (i.e., sample average estimate) of its own expected loss $L_{i_0}$. and also using an empirical version $L_{f}$ of the feedback $L_{f}$ from the mixer critic (line 16-17). There are two ways to combine these loss terms: (1) form one combined loss as $\alpha L_i + (1 - \alpha)L_{f}$ where $\alpha \in (0, 1)$ is a tunable hyperparameter and (2) train with $L\alpha$ for some iterations and then

![Figure 1: Architecture of the proposed MGM model.](Image)
The empirical estimates for $\mathcal{L}_s$, $\mathcal{L}_f$ are obtained using samples in line 14-15. Additionally, $\mathcal{L}_s$, $\mathcal{L}_f$ depend on the agent level critic $C_{\theta_{i0}}$ and mix level critic $C_{\theta_{mix}}$ respectively (see example below). $C_{\theta_{i0}}$ and $C_{\theta_{mix}}$ are trained with the empirical form $W_s$ and $W_f$ of the expected loss $W_s$ and $W_f$, respectively (line 10 and 12). These empirical losses $W_s$ and $W_f$ are built from the samples from line 5-9.

Concrete Example: We use the WGAN-GP loss terms here. With WGAN-GP, the loss $L_a$ and $L_f$ would be $-E_{x_{i0} \sim P_{\theta_{i0}}}[C_{\theta_{i0}}(x_{i0})] + E_{x_{i0} \sim P_{\theta_{i0}}}[C_{\theta_{mix}}(x_{i0})]$, respectively. $W_s$ is the standard WGAN-GP loss

$$E_{x_{i0} \sim P_{\theta_{i0}}}[C_{\theta_{i0}}(x_{i0})]$$

In the above expressions, all expectations are replaced by averages over samples to form the counterpart empirical losses. These samples are obtained as follows: $D^r_{ij}$ are samples from $P_{\theta_{i0}}$ (line 9) and samples from $P_{\theta_{j0}}$ are generated by (1) forming $x_i$ by obtaining $x_j$ from $D^r_{ij}$ for $j \neq 0$, $x_0 \sim P_{\theta_{i0}}$, $y \sim P_{\theta_{j0}}$. Then, $W_f$ is

$$E_{y \sim P_{\theta_{j0}}}[C_{\theta_{mix}}(y)] - E_{y \sim P_{\theta_{j0}}}[C_{\theta_{mix}}(y)] + \lambda_{mix}J_{C_{\theta_{mix}}, P_{\theta_{j0}}}$

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We reiterate that Algorithm 1 is a framework, and in the above paragraph we provide a possible instantiation with WGAN loss. The framework can be instantiated with CycleGAN and StyleGAN as well as a time series generator COTGAN. These are used in the experiments, and we provide the detailed losses in the appendix due to space constraint.

Mathematical Characterization

We derive mathematical result for WGAN. As the high level idea of all GANs are similar, we believe that the high level insight obtained at the end of sub-section hold for any type of GAN. From the theory of WGAN [Arjovsky, Chintala, and Bottou 2017], the loss term $\mathcal{L}_f$ estimates the Wasserstein-1 distance between $P_{\theta_{j0}}$ and $P_{\theta_{i0}}$. As these two distributions differ only in the way $x_{i0}$ is generated, it is intuitive that $\mathcal{L}_f$ provides meaningful feedback. However, there are many interesting conclusions and special cases that our mathematical analysis reveals, which we show next.

**Background on Wasserstein-1 metric**: Let $P_{\theta_{i0}}$, $P_{\theta_{j0}}$ be the set of distributions (also called couplings) whose marginal distributions are $P_{\theta_{i0}}$ and $P_{\theta_{j0}}$, i.e., $\int_A d\pi(x_i, x_j) = P_{\theta_{j0}}(A)$. We work in Euclidean spaces and standard Lebesgue measure. Probability distribution refers to the probability measure function.

**Algorithm 1: MGM Template**

1. **Inputs**: Pre-trained $G_{mix}$, datasets $D^r_s$, $D^r_f \forall j$,
2. $N(0, I)$ is standard normal dist., $U(\cdot)$ is uniform dist.
3. while $\theta_{i0}$ has not converged do
4. **for m times do**
5. Sample $\{z_k\}_{k=1}^K$, $\{z^*_k\}_{k=1}^K \sim N(0, I)$
6. Sample $\{x^r_{j,k}\}_{k=1}^K \sim U(D^r_j)$
7. $x^g_{i0,k} = G_{\theta_{i0}}(z_k), x^g_{0,k} = (x^r_{i0,k}, x^g_{i0,k})$
8. $y^g_k = G_{mix}(x^g_{i0,k})$
9. Sample $\{y^r_k\}_{k=1}^K \sim U(D^r_f)$
10. Form loss $W_a$ using $\{x^g_k, y^r_k\}_{k=1}^K$
11. Update $\theta_{i0}$ using $\nabla_{\theta_{i0}} W_a$
12. Form loss $W_f$ using $\{y^r_k, y^g_k\}_{k=1}^K$
13. Update $\theta_{mix}$ using $\nabla_{\theta_{mix}} W_f$
14. Sample $\{z_k\}_{k=1}^K$, $\{z^*_k\}_{k=1}^K \sim N(0, I)$
15. Sample $\{x^r_{j,k}\}_{k=1}^K \sim U(D^r_f)$
16. Form loss $L_a$ using $G_{\theta_{i0}}(z_k), C_{\theta_{i0}}$
17. Form loss $L_f$ using $G_{\theta_{i0}}(z_k), C_{\theta_{i0}}$
18. Update $\theta_{i0}$ using $\nabla_{\theta_{i0}} W_a$ and $\nabla_{\theta_{i0}} L_f$

and $\int_{A \times X_j} d\pi(x_i, x_j) = \mathcal{P}_i(A)$ for any event $A$ in the appropriate $\sigma$-algebra, for any $\pi \in P_{\theta_{j0}, P_{\theta_{i0}}}$. The Wasserstein-1 metric is:

$$W(P_{\theta_{i0}}, P_{\theta_{j0}}) = \inf_{\pi \in P_{\theta_{i0}, P_{\theta_{j0}}}} \int_{X_i \times X_j} d(x_i, x_j) \pi(x_i, x_j)$$

(1)

It is known that the Wasserstein-1 metric is also equal to

$$\sup_{f \in C_b(X_i, \mathbb{R})} \int_{X_i \times X_j} f(x_i) d\mathcal{P}_{\theta_{i0}}(x_i) + \int_{X_j} g(x_j) d\mathcal{P}_{\theta_{j0}}(x_j)$$

where $C_b(X)$ is the space of bounded and continuous real valued functions on $X$. The functions $f$ and $g$ are also the dual functions [Villani 2009], as they correspond to dual variables of the infinite linear program that Eq. 1 represents. It is known that the minimizer of Eq. 1 and maximizer functions of the dual form exist [Villani 2009].

**Analysis of $\mathcal{L}_f$**: We are given the system generator that represents the distribution $P_{\theta_{j0}}$, with probability density function given by $P_{\theta_{i0}}(\cdot)$. We also have true distributions $P_{\theta_{i0}}$ for all $j$. The data that we are interested in generating is from $P_{\theta_{i0}}$, and the corresponding agent level generator generates data from $P_{\theta_{j0}}$. We aim to have $P_{\theta_{i0}}$ close to $P_{\theta_{i0}}$. Towards this end, we repurpose the critic of the mixer GAN. In training, the critic is fed real data that is distributed according to the density $P_{\theta_{j0}}(y) = \int_{X_i} P_{\theta_{i0}}(y|x_i) P_{\theta_{j0}}(x_i) dx_i$ where $P_{\theta_{j0}}(y|x) = P_{\theta_{j0}}(y|x) \times P_{\theta_{i0}}(x_i)$. The mix critic is additionally fed generated data that is distributed according to the density $P_{\theta_{j0}}^g(y) = \int_{X_i} P_{\theta_{j0}}^g(y|x_i) P_{\theta_{i0}}(x_i) \times \int_{X_j} P_{\theta_{j0}}(x_j) dx_j$. Thus, the repurposed mixer critic estimates the Wasserstein-1 dis-
tance between $\mathbb{P}_y^r(\cdot)$ and $\mathbb{P}_y^g(\cdot)$ given as

$$W(\mathbb{P}_y^r, \mathbb{P}_y^g) = \inf_{\pi \in \Pi_{\mathbb{P}_y^r, \mathbb{P}_y^g}} \int_Y d(y, y') \, d\pi(y, y')$$

In order to analyze the above feedback from the mixer to the agent generator, we define some terms. Consider

$$W(\mathbb{P}_y^r|x_i, \mathbb{P}_y^g|x_i') = \inf_{\pi \in \Pi_{\mathbb{P}_y^r, \mathbb{P}_y^g|x_i, x_i'}} \int_Y d(y, y') \, d\pi(y, y')$$

Dual functions $f^*_x$ and $g^*_x$ exist here such that $f^*_x(y) + g^*_x(y) \leq d(y, y')$ for all $y, y'$ as well as for any $x, x'$ and

$$W(\mathbb{P}_y^r|x, \mathbb{P}_y^g|x') = \int_Y f^*_x(y) \mathbb{P}_y^r(y) \, dy + \int_Y g^*_x(y) \mathbb{P}_y^g(y) \, dy$$

Let $\pi^*_d(x_i, x_i')$ denote the distribution (or coupling) that achieves the minimum for the Wasserstein-1 distance below between the same distributions $\mathbb{P}_i^r$ and $\mathbb{P}_i^g$

$$\pi^*_d = \arg \inf_{\pi \in \Pi_{\mathbb{P}_i^r, \mathbb{P}_i^g}} \int_{X_i \times X_i} d(x_i, x_i') \, d\pi(x_i, x_i')$$

Note that $\pi^*$ takes $d$ as a subscript, i.e., it depends on the distance metric $d$. We denote this explicitly since we later deal with different distance metrics. We will also use subscript $d$ for $W(\cdot, \cdot)$, if not obvious. $\otimes_{x \neq x_i} \mathbb{P}_y^r$ denotes the joint probability distribution (product measure) over $X_i \times X_j$.

The next result provides a mathematical formulation of the Wasserstein-1 distance feedback provided from the mixer critic.

**Theorem 1.** Assume that (1) all spaces $X_j$’s, $Y$ are compact, (2) all probability density functions are continuous, bounded and $> 0$ over its domain, and (3) $f^*_x(y), g^*_x(y)$ are jointly continuous in $x, y$. Then, the following holds:

$$W(\mathbb{P}_y^r, \mathbb{P}_y^g) = \int_{X_i \times X_i} E_{x_i \sim \otimes_{x \neq x_i} \mathbb{P}_y^r} \left[ \int_{X_i \times X_i} f^*_x(y) \mathbb{P}_y^r(y) \, dy + \int_{X_i \times X_i} g^*_x(y) \mathbb{P}_y^g(y) \, dy \right] \, d\pi^*_d(x_i, x_i'),$$

where $d$ is any metric on $X_i$.

We provide further explanation of the above mathematical result. The following result aids in the explanation:

**Lemma 1.** Define $M_{x-i_0}(\cdot)$ as the function $M_{x-i_0}(x_{i_0}) = \mathbb{P}_y^r(x_{i_0}, x_{i_0})$, and assume all conditions in Theorem 1 hold. If $M_{x-i_0}(\cdot)$ is an injection for all $x_i$, then

$$d_0(x_{i_0}, x'_{i_0}) = E_{x_i \sim \otimes_{x \neq x_i} \mathbb{P}_y^r} \left[ W(\mathbb{P}_y^r|x_{i_0}, \mathbb{P}_y^r|x_{i_0}, x'_{i_0}) \right]$$

is a distance metric on $X_{i_0}$. If in the above case, $M_{x-i_0}(\cdot)$ is not an injection then $d_0$ is a pseudo-metrics.

Using the above result, if $M_{x-i_0}(\cdot)$ is an injection $\forall x_{i_0}$ then we can check that by choosing $d = d_0$ in Thm 1 we get

$$W(\mathbb{P}_y^r, \mathbb{P}_y^g) = W_{d_0}(\mathbb{P}_i^r, \mathbb{P}_i^g)$$

In words, under the injective assumption, the feedback $\mathcal{L}_f$ from the mixer critic is the Wasserstein-1 distance between real and fake distributions of position $i_0$ but measured in a different underlying distance metric $d_0$ on $X_{i_0}$ induced by the mapping $M_{x-i_0}(\cdot)$. When $M_{x-i_0}(\cdot)$ is not an injection, then it induces a pseudo-metric on $X_{i_0}$. A pseudo-metric satisfies all properties of a distance metric except that two distinct points may have distance 0. This provides less fine-grained feedback with a push to match probability density only over those points with $d_0 > 0$, and in the extreme case when $M_{x-i_0}(\cdot)$ is a constant function then $W(\mathbb{P}_y^r|x_{i_0}, \mathbb{P}_y^r|x_{i_0}, x'_{i_0})$ is zero for any $x_{i_0}, x'_{i_0}$ and thus the feedback $\mathcal{L}_f$ from the mixer critic is zero. Indeed, $M_{x-i_0}(\cdot)$ constant is the case when the mixer output distribution $\mathbb{P}_y^r$ does not depend on $x_{i_0}$ and hence the mixer model does not encode any useful information about agent $i_0$.

At a high level, how the mixer depends on the agent $i_0$ (captured in $M_{x-i_0}(\cdot)$) determines the nature and magnitude of the feedback. This feedback combined with the agent $i_0$’s own loss still aims to bring $\mathbb{P}_i^r$ closer to $\mathbb{P}_i^g$.

**Experiments**

We showcase the effectiveness of MGM on three different domains: (1) synthetic data modeling relationship between a mixer and two agents, (2) time series generation task, and (3) few-shot image generation that reveals the transfer learning nature of the MGM set-up. We use different specialized GANs for each domain.

For the synthetic and image data, we ran the experiments on a server (Intel(R) Xeon(R) Gold 5218R CPU, 2x Quad [https://github.com/cameron-chen/mgm](https://github.com/cameron-chen/mgm)) code and data are available at this url.
RTX 6000, 128GB RAM) on the GPU. For time series data, we ran the experiments on a cloud instance (14 vCPUs, 56GB memory, 8x Tesla K80) on the GPU. For all experiments, we use Adam optimizer with a constant learning rate tuned between 1e-5 and 1e-3 by optuna (Akiba et al. 2019).

**Evaluation:** Our evaluation methodology has the following common approach: we first train a mixer GAN (or use pre-trained one) using full data available. We train the agent \(i_0\) GAN using low or biased data only and additionally with feedback from the mixer. We then compare the performance of agent \(i_0\) with and without the feedback from the mixer. The evaluation metrics vary by domain as described below.

**Synthetic Data**

We first show our results on synthetic data to explore various conditions under which the MGM provides benefits. The synthetic data scenario models a system where the output of the mixer is linearly dependent on the outputs of two agents. Agent 1 acts as the new arrival in this experiment.

**Experimental Setup:** We generate 128,000 samples for all agents, output of agent 1, \(x_1 \in \mathbb{R}^2\), is drawn from Swiss roll (see Fig. 2 (1)), and output of agent 2, \(x_2 \in \mathbb{R}^2\), is drawn from a 2D Gaussian distribution. The data for the mixer is a linear combination of agent outputs, \(y = \beta x_1 + (1-\beta)x_2\), where the parameter \(\beta \in [0, 1]\) controls the dependence of \(y\) on \(x_1\). We use Wasserstein-1 distance, computed by POT package (Flamary et al. 2021), to evaluate the model.

We use unconditional WGAN-GP to model the agents and conditional WGAN-GP to model the mixer. Each WGAN-GP consists of a generator network with 3 fully-connected layers of 512 nodes with Leaky ReLU activations and a critic network of the same structure as the generator except for input and output layer. We choose the weight of gradient penalty term \(\lambda_{\text{gp}} = 0.1\) for agent GAN and \(\lambda_{\text{mix}} = 1\) for mixer GAN based on a grid search. This experiment used the combined loss \(\alpha L_{\text{adv}} + (1-\alpha) L_{\text{f}}\), thus, the MGM trained model is denoted as WGAN-GP-C in Table 1.

**Biased data:** We remove the data of agent 1 from the top right area (see Fig 2 (1)) either completely (bias-100) or 90% (bias-90). We train with a minibatch size of 256. Then, the model is evaluated against a test set of size 2,000 that contains data in all area.

**Low data:** We sample random subsets of the training data for agent 1, of size 32 and 64. We train with a minibatch size of 32 or 64, depending on the size of the training data. Then, same as the biased data scenario, the model is evaluated with a test set of size 2,000.

**Results Discussion:** We visualize the trained distribution for bias-90 and low-32 (\(\beta = 0.7\)) in Fig. 2 (2) and 2 (3) respectively. We report quantitative results in Table 1. The results show that tighter coupling (higher \(\beta\)) results in better performance gain of agent 1’s generator. This is consistent with the inference derived from our theoretical results (text after Lemma 1). We observe that agent 1’s generator improves more in the case where the learned distribution by a single GAN is further away from the true distribution. On average, low-32 improves 8.0% compared with 4.4% for low-64, while bias-100 improves 15.7% compared with 9.0% for bias-90, where a single WGAN-GP (baseline) performs worse for low-32 and bias-100. More results obtained on multiple different random low datasets can be found in appendix.

**Real-World Time Series Data**

Prior work has investigated synthesizing time series data of a single agent via GANs (Xu et al. 2020; Yoon, Jarrett, and van der Schaar 2019; Esteban, Hyland, and Rätsch 2017). We model the time series data of an agent who is interacting in an electricity market. The European electricity system enables electricity exchange among countries. The demand for the electricity in one country is naturally affected by the electricity prices in its neighboring countries within Europe because of the cross-border electricity trade. For example, the demand for electricity in Spain is affected by the electricity prices in France, Portugal, and Spain itself.

We model the price in France as agent 1, the prices in Spain and Portugal as agents 2 and 3 respectively, and the demand in Spain as the mixer. The intuition is that the dependency between the demand for electricity in Spain and the price in France is able to provide extra feedback for agent 1 and hence help with better modeling of agent 1.

**Experimental Setup:** We use the electric price and demand data publicly shared by Red Eléctrica de España, which operates the national electricity grid in Spain (Electrical 2021). The data contains the day-ahead prices in Spain and neighboring countries, and the total electricity demand in Spain. We use the data between 5 Nov, 2018 and 13 July, 2019 aggregated over every hour and split by days. We use the difference between autocorrelation metric (ACE) (Xu et al. 2020) to evaluate the output.

For each of 249 days, agent 1 outputs the price every hour in France, \(x_1 \in \mathbb{R}^{23}\), while other agents output the price every hour in Spain and Portugal, \(x_2, x_3 \in \mathbb{R}^{24}\). The mixer takes as input the prices in the three countries and outputs the total demand in Spain, \(y \in \mathbb{R}^{24}\). We use COT-GAN (Xu et al. 2020) to model the agents and its conditional version to model the mixer, using a minibatch size of 32, over 20,000 iterations. For other settings, we follow COT-GAN defaults.

As stated in preliminaries, when pre-training the mixer, the critic of the conditional COT-GAN should take as input the condition as well. However, we find that such pre-trained mixer performs poorly. Instead, we try an alternate pre-training where we do not feed the condition into the critic, thus, the critic measures the distance between the marginal distribution of the mixer output and real data (discussed more in appendix); we denote the corresponding MGM model by COT-GAN-CM (C for combined loss, M for marginal distribution). We denote the former MGM model by COT-GAN-CJ (J for joint distribution). Empirical results show that COT-GAN-CM works better than COT-GAN-CJ.

**Biased data:** Modeling missing data in a biased manner is tricky in time series data. We take the simple approach of dropping the second half of the time series sequences, after which we have 124 time series sequences (call this Half) from Nov 2018 to Feb 2019, that may exhibit seasonality.

**Low data:** We randomly sample 64 and 124 time series sequences (out of the 249) once for agent 1 and treat these...
the few-shot image generation problem is to generate images with very limited data, e.g., 10 images. Many works have proposed to solve this problem by adapting a pre-trained GAN-based model to a new target domain (Mo, Cho, and Shin 2020; Ojha et al. 2021; Li et al. 2020b; Wang et al. 2020). As low data is one of the scenarios in which our approach provides benefits, we explore few-shot image generation in a MGM set-up.

Results Discussion: The results are summarized in Table 2. We observe the two MGM models consistently outperform the baseline and COT-GAN-CM outperforms COT-GAN-CJ most of the times. The results provide evidence that our MGM approach works for time series data. Additional supporting results are in the appendix.

Few-Shot Image Generation

The few-shot image generation problem is to generate images with very limited data, e.g., 10 images. Many works have proposed to solve this problem by adapting a pre-trained GAN-based model to a new target domain (Mo, Cho, and Shin 2020; Ojha et al. 2021; Li et al. 2020b; Wang et al. 2020). As low data is one of the scenarios in which our approach provides benefits, we explore few-shot image generation problem in a MGM set-up.

We consider a single agent and a mixer. The agent generates images of the miniature horse, while the mixer is a converter from horse to zebra images. The intuition here is that the mixer can provide feedback about the high-level concept of a horse structure, such that it mitigates the overfitting of the agent due to limited training data.

Experiment Setup: We collected images of miniature horses from the search engine yandex.com by keyword “miniature horse”. All the images were resized to $256 \times 256$. After processing, the dataset contains 1061 images, 34 of them forming training set, the remainder forming test set.

Results Discussion: The quantitative results in Table 3 with 5, 10, 30 shot for FSGAN and FreezeD show that the MGM approach improves the baseline FSGAN and FreezeD by FSGAN-A and FreezeD-A respectively.

We follow the defaults from FSGAN, using minibatch size of 16, stopping at iteration 1250 and choosing the better of the model at iteration 1000 or 1250. We evaluate the model in the same manner as Ojha et al. (2021).

Results Discussion: The quantitative results in Table 3 show some clear cases of improved quality of images in Fig. 5.

Ablation Study

To further investigate how the mixer improves the agent MGM model, we perform an ablation study for synthetic and image data. For synthetic data, instead of combined loss, we use FID (Heusel et al. 2017) score to evaluate the output.

The agent model produces images of miniature horse, $x$. The mixer takes input $x$ to produce zebra images $y$, where $x, y \in \mathbb{R}^{256 \times 256 \times 3}$. We select two models, FSGAN (Robb et al. 2020) and FreezeD (Mo, Cho, and Shim 2020) as the agent model; both of these fine tune a pre-trained StyleGAN2 (Karras et al. 2020) to the miniature horse domain.

We select the prior pre-trained CycleGAN (Zhu et al. 2017) (horse to zebra) as the mixer model. Recall that we train agent with $L_f$ followed by $L_a$ (alternate updating) in this experiment. We thus denote the MGM version of FSGAN and FreezeD by FSGAN-A and FreezeD-A respectively.

We perform an ablation study for synthetic and image data, instead of combined loss, we use FID score to evaluate the output.

The dataset will be released publicly. We employ the widely used FID (Heusel et al. 2017) score to evaluate the output.

The agent model produces images of miniature horse, $x$. The mixer takes input $x$ to produce zebra images $y$, where $x, y \in \mathbb{R}^{256 \times 256 \times 3}$. We select two models, FSGAN (Robb et al. 2020) and FreezeD (Mo, Cho, and Shim 2020) as the agent model; both of these fine tune a pre-trained StyleGAN2 (Karras et al. 2020) to the miniature horse domain.

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We follow the defaults from FSGAN, using minibatch size of 16, stopping at iteration 1250 and choosing the better of the model at iteration 1000 or 1250. We evaluate the model in the same manner as Ojha et al. (2021).

Results Discussion: The quantitative results in Table 3 show some clear cases of improved quality of images in Fig. 5.

Ablation Study

To further investigate how the mixer improves the agent MGM model, we perform an ablation study for synthetic and image data. For synthetic data, instead of combined loss, we train the agent generator with $L_a$ and $L_f$ alternately. We denote this alternate by WGAN-GP-A and show results in Table 4. We observe that the alternate updating strategy has less improvement than the combined loss (in Table 1) in all cases, except low-64 with $\beta = 0.7$. This reveals combined loss strategy is more effective for synthetic data. For image data, we perform ablation by using combined loss. We denote this alternate by FSGAN-C and FreezeD-C, and present results in Table 5. The results show that the alternate updating strategy is more effective on image data. Also, we found the training of FreezeD-C to be unstable, which is reflected in the results.
Scenario Baseline (WGAN-GP) WGAN-GP-A over $\beta \in [0, 1]$  
| Scenario | 0 | 0.1 | 0.3 | 0.5 | 0.7 | 1.0 |
|----------|---|----|----|----|----|----|
| Bias-100 | 0.705(0.009) | 0.735(0.008) | 0.664(0.018) | 0.622(0.018) | 0.603(0.018) | 0.591(0.018) | 0.488(0.008) |
| Low-64   | 0.582(0.010) | 0.571(0.007) | 0.571(0.008) | 0.575(0.005) | 0.559(0.006) | 0.551(0.004) | 0.551(0.007) |

Table 4: Ablation study, MGM vs baseline in Wasserstein-1 distance (over 16 runs) for 2 scenarios and varying $\beta$.

| 5-Shot | 10-Shot | 30-Shot |
|--------|---------|---------|
| FSGAN  | 60.01(0.47) | 49.85(0.71) | **47.64(0.69)** |
| FSGAN-C| 55.79(0.82) | 49.19(0.56) | 49.30(0.73) |
| FreezeD| 78.07(0.95) | **53.69(0.38)** | **47.94(0.87)** |
| FreezeD-C| 67.50(1.68) | 66.03(1.01) | 69.45(0.45) |

Table 5: Ablation study, FID scores (↓) where MGM uses single combined loss. Std dev. are computed across 5 runs.

Figure 3: Source is the pretrained Style-GAN2, FSGAN is tuned Source, FSGAN-A is MGM tuned FSGAN. Our MGM approach (FSGAN-A) produces more natural images w.r.t. horse body, tail and clearer background.

**Conclusion**

In this paper, we demonstrate that we can significantly improve the performance of generative models by incorporating real-world interactions. We presented and analyzed our MGM approach both theoretically and empirically, and we show that our approach is particularly effective when data available is small. A possible future work could be to explore the approach with generative models other than GANs.

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More Concrete Examples of the MGM Framework

As stated in the main paper, the framework in Algorithm [1] can be instantiated with other GANs other than the WGAN presented in the main paper. Here we present instantiations with COT-GAN (as agent and mixer), and StyleGAN2 as agent and CycleGAN as mixer.

Real-world time series generation: The authors of the COT-GAN work proposed to generate time-series data by solving a causal optimal transport problem (Xu et al. [2020]). The pipeline is (1) obtain the cost from the source distribution to the target distribution with the causality constraint, (2) compute a regularized distance between distributions using samples and use of the Sinkhorn algorithm (effect of the regularization is controlled by weight ε), (3) form the loss term by the regularized distances of last step.

We first introduce some notation. There are two discriminator networks, denoted by \( h_{w_1} : X \rightarrow \mathbb{R}^K \) and \( M_{w_2} : X \rightarrow \mathbb{R}^K \) where \( K \) is a constant, and let \( w = (w_1, w_2) \). Then, the authors define and compute the cost \( c^\epsilon_w (x, \hat{x}) \) by the equation

\[
c(x, \hat{x}) = \sum_{k=1}^K \sum_{t=1}^T h^k_{w_1,t}(\hat{x}) \Delta_{t+1} M^k_{w_2}(x)
\]

where \( c(\cdot, \cdot) \) is the Euclidean distance between the examples of \( x \) and that of \( \hat{x} \), \( T \) is the length of the time series sequence, \( \Delta_{t+1} M^k_{w_2}(\cdot) \) represents \( M^k_{w_2,t+1}(\cdot) - M^k_{w_2,t}(\cdot) \). The cost \( c^\epsilon_w (x, \hat{x}) \) represents the underlying distance between time two series in the space of time series. For more details of the form of the cost, refer to the paper (Xu et al. [2020]).

The authors also propose a regularized COT distance between causal couplings over time-series (Eq. 3 in [Xu et al. 2020]), which is an expectation. The empirical version of the regularized distance \( W_{c,\epsilon}(x, x) \) can be approximated by the Sinkhorn algorithm, where \( x \) is an empirical distribution formed from samples of a mini-batch.

However, the authors show that the simple application of Sinkhorn algorithm with standard Sinkhorn divergence provides a poor estimate of the regularized distance. Hence, the authors defined a mixed Sinkhorn divergence, which uses two sets of samples for each of real data and generated data as a variance reduction technique. Mixed Sinkhorn divergence \( \hat{W}_{c,\epsilon}^{\text{mix}}(x, x', \hat{x}_\theta, \hat{x}'_\theta) \) is given as:

\[
\hat{W}_{c,\epsilon}^{\text{mix}}(x, x', \hat{x}_\theta, \hat{x}'_\theta) = W_{c,\epsilon}(x, \hat{x}_\theta) + W_{c,\epsilon}(x', \hat{x}'_\theta) - W_{c,\epsilon}(x, x') - W_{c,\epsilon}(\hat{x}_\theta, \hat{x}'_\theta)
\]

where \( \hat{x}_\theta \) is the empirical distribution from one mini-batch generated by the generator, \( \hat{x}'_\theta \) is the empirical distribution from another mini-batch generated by the generator, \( x' \) is the empirical distribution from one mini-batch of real data and \( x' \) is the empirical distribution from another mini-batch of real data. (Note that this algorithm requires two mini-batches to compute \( \hat{W}_{c,\epsilon}^{\text{mix}}(x, x', \hat{x}_\theta, \hat{x}'_\theta) \). When updating the discriminator network, the author introduced a regularization term \( p_{M_{\epsilon,\theta}}(x) \).

We utilize the COT-GAN in our MGM set-up as follows: \( L_a \) and \( L_f \) is \( \hat{W}_{c,\epsilon}^{\text{mix}}(x, x', \hat{x}_\theta, \hat{x}'_\theta) \) and \( -L_a + \lambda p_{M_{\epsilon,\theta}}(x) \) respectively. Then, \( W_a \) is

\[
-L_a + \lambda p_{M_{\epsilon,\theta}}(x)
\]

and \( W_f \) is

\[
-L_f + \lambda p_{M_{\epsilon,\theta}}(y)
\]

Few-shot image generation: We use StyleGAN2 as the agent and CycleGAN as the mixer. The \( L_f \) is \( \frac{1}{2} E_{(x,c) \sim P_x}[\|C_{\text{mix}}(x) - c\|^2] \). \( W_f \) is the LSGNA loss (Mao et al. [2017]).

\[
\frac{1}{2} E_{y \sim P_y} [\|C_{\text{mix}}(y) - b\|^2] + \frac{1}{2} E_{y \sim P_y} [\|C_{\text{mix}}(y) - a\|^2]
\]

where we use \( b = c = 1, a = 0 \).

For expected loss \( L_a \), the author of StyleGAN2 introduced a new regularizer, path length regularization, denoted by \( J_{G_\theta}^{PL} \). The whole expected loss \( L_a \) is

\[
- E_{x \sim P_{x_0}} [\log C_{\text{mix}}(x_0)] + \lambda J_{G_\theta}^{PL}
\]

\( W_a \) is the standard GAN loss with R1 regularization, denoted by \( R \).

\[
- E_{x \sim P_{x_0}} [\log C_{\text{mix}}(x_0)] - E_{x \sim P_{x_0}} [\log (1 - C_{\text{mix}}(x_0))]
\]

+ \( \lambda R_{C_{\text{mix}}}(x_0) \)

Note these two models are pre-trained models. We use the online implementation. The loss terms thus follow that in their implementation. In different GANs, people denote the discriminator net by \( D_w, C_w \) or \( F_w \) depending on the tasks. For ease of presentation, we use \( C_w \) in all cases.

More Experiment Results of Synthetic Data

Due to the randomness of sampling low datasets, the results vary for different low datasets. However, MGM consistently shows effective improvement compared with the baseline. We show results on different low datasets of size 32 and size 64 in Table [A.4].

Training of the Mixer for COT-GAN

One of the requirements to make the MGM approach work is a good pre-trained mixer generator. Generally, we can obtain such a mixer generator by the manner introduced in preliminaries. However, for real world time series data, we observed poor performance of the conditional COT-GAN when the critic of the conditional COT-GAN takes as input the condition.

First, we argue why the mixed Sinkhorn divergence \( \hat{W}_{c,\epsilon}^{\text{mix}}(x, x', \hat{x}_\theta, \hat{x}'_\theta) \) is not the correct term to use for a conditional version. Here every sample in a mini-batch is a condition time-series followed by the continuation time-series (for fake one also the condition is from real data and then the generator produces fake continuation). Note that when samples are generated for the different mini-batches

---

1. StyleGAN2: https://github.com/NVlabs/stylegan2
2. CycleGAN: https://github.com/XHUJOY/CycleGAN-tensorflow
x, x', x^\prime_y, \hat{x}_y^\prime \text{ then the conditions within real mini-batches } x \text{ and } x' \text{ are different and also within fake mini-batches } x_0 \text{ and } \hat{x}_0^\prime. \text{ Comparing these terms with different conditions is meaningless as the conditional distribution is dependent on the condition. Thus, mixed Sinkhorn divergence does not apply here.}

As a consequence we train by not feeding conditions into the critic, which still learns quite well. We explain this in easier set-up using WGAN and the notation in preliminaries. Feeding conditions into the critic measures the Wassertein-1 distance between a joint distributions, \( W(\overrightarrow{P}_{y,c}^{\text{joint}}, \overrightarrow{P}_{y,c}^{\text{joint}}) \), which provides strong supervision to the generator, while not feeding conditions into the critic measures the distance between marginal distributions, \( W(\overrightarrow{P}_y, \overrightarrow{P}_y) \), marginalized over condition c. When we learn a conditional generator by minimizing \( W(\overrightarrow{P}_y, \overrightarrow{P}_y) \) with batch optimization, the generator can ignore the relationship between c and y. However, the nature of mini-batch optimization prevents the generator from ignoring this relation. In a mini-batch, the empirical marginal distribution of y is different among mini-batches (due to limited size of mini-batches) and this highly depends on the corresponding conditions (which are not fed to critic but fed to generator). The generator can do best by taking into account c to minimize the distance between the real mini-batch distribution and the generated mini-batch distribution, especially for continuous sample-condition pair (c, y) as is the case of our real time series experiment.

Overall, we pre-train the mixer generator of COT-GAN by not feeding conditions into the critic. We empirically observed the better performance of this approach against the approach that uses condition with mixed Sinkhorn divergence (as we argued, mixed Sinkhorn divergence is meaningless to start with in conditional case). This empirical observation is shown on Table A.2.

As the work (Xu et al. 2020) shows that standard Sinkhorn divergence also provides poor estimates, thus, the design of a conditional COT-GAN where the critic explicitly takes as input the condition is an open question. In any case, these results actually provide more support for our MGM approach. The result in Table A.2 and Table 2 together supports our claim that a good mixer generator is critical for the MGM approach.

| Scenario Seed | Baseline (WGAN-GP) | WGAN-GP-C over J \( \in [0, 1] \) |
|---------------|---------------------|-----------------------------------|
| Low-32 6128   | 0.660(0.010)        | 0.010 0.013 0.588(0.018) 0.531(0.008) |
| Low-32 968    | 0.607(0.009)        | 0.592(0.007) 0.584(0.007) 0.538(0.004) 0.539(0.009) |
| Low-32 1025   | 0.572(0.007)        | 0.536(0.003) 0.536(0.004) 0.510(0.004) |
| Low-64 6160   | 0.558(0.009)        | 0.518(0.006) 0.538(0.006) 0.528(0.011) |
| Low-64 3786   | 0.552(0.007)        | 0.519(0.006) 0.538(0.009) 0.500(0.010) |
| Low-64 3304   | 0.545(0.005)        | 0.531(0.006) 0.531(0.004) 0.518(0.007) |

Table A.1: MGM vs baseline in Wasserstein-1 distance (mean and std. dev. of 16 runs) for different low datasets and varying J.

Continuous Condition for Conditional WGAN-GP

The gradient regularizer \( J_{\text{non}} \) of non-conditional WGAN-GP (Gulrajani et al. 2017) is

\[
\mathbb{E}_{y \sim \overrightarrow{P}_y} \left[ (\|\nabla_y C(y)\|_2 - 1)^2 \right]
\]

where \( \overrightarrow{P}_y \) is a mixture of data distribution \( \overrightarrow{P}_y \) and the generated distribution \( \overrightarrow{P}_y \). The \( J_{\text{non}} \) is a way to enforce the Lipschitz constraint by penalizing the gradient w.r.t. \( y \sim \overrightarrow{P}_y \).

As we state in preliminaries, the critic of the conditional WGAN-GP estimates the distance between the joint distributions, \( W(\overrightarrow{P}_{y,c}^{\text{joint}}, \overrightarrow{P}_{y,c}^{\text{joint}}) \). Theoretically, the regularizer should penalize the gradient w.r.t. \( (y, c) \sim \overrightarrow{P}_{y,c}^{\text{joint}} \). However, we did not find any implementation of such a version online.

A variation of \( J_{\text{disc}} \) in WGAN-GP (Zheng et al. 2020) conditioned on the discrete conditions, e.g. labels, is

\[
\mathbb{E}_{(y, c) \sim \overrightarrow{P}_{y,c}^{\text{joint}}} \left[ (\|\nabla_y C(y, c)\|_2 - 1)^2 \right]
\]

They ignore the condition \( c \) when they compute \( J_{\text{disc}} \) as discrete labels have no gradients.

In the experiment on synthetic data, the conditions are continuous. So we use this penalty term \( J_{\text{cont}} \)

\[
\mathbb{E}_{(y, c) \sim \overrightarrow{P}_{y,c}^{\text{joint}}} \left[ (\|\nabla_y C(y, c)\|_2 - 1)^2 \right]
\]

And we find \( J_{\text{cont}} \) works better than \( J_{\text{disc}} \) in the case of continuous conditions.

Table A.2: We compare the conditional COT-GAN with and without feeding conditions into the critic, denoted by CMIX-COT-GAN-J and CMIX-COT-GAN-M respectively. The results are in the ACE (\( \frac{J}{J} \) metric (avg (std. dev.) and maximum over 100 runs with a batch size of 64) between the real time series and the generated time series given the same condition in each batch. Bold shows best results.
Also, the notation

\[ p \]

is implicit since we later deal with different distance metrics.

Note that the usage will be clear from the arguments of \( g \).

By definition, \( f \) and \( g \) are such that

\[ f(x) = \sup_{y \in Y} f^*(x,y) \]

\[ g(x) = \inf_{y \in Y} g^*(x,y) \]

for any measurable function \( h \). This can be proved by showing that for the event \( E \) defined as \( \{ x \neq x' \} \) we must have \( \pi^*_x(x,x')(E) = 0 \) in order to obtain \( 0 = \int_{X \times X} d(x,x') d\pi^*_d(x,x') \).

Very similarly, let \( \pi^*_d(x_{i_0},x'_{i_0}) \) denote the joint distribution that achieves the min for the Wasserstein-1 distance below between the different densities \( P_{i_0}^r \) and \( P_{i_0}^g \).

\[ \pi^*_d = \arg \inf \pi \in \Pi_{i_0} (x_{i_0},x'_{i_0}) \int_{X \times X} d(x,y) d\pi(x,y) \]

Again by definition of couplings, \( \int_{X \times X} d\pi^*_d(x_{i_0},x'_{i_0}) = \int_A P_{i_0}^r(x_{i_0}) d\pi_{i_0}^r(x_{i_0}) \) and \( \int_{A \times X} d\pi_{i_0}^r(x_{i_0},x'_{i_0}) = \int_A P_{i_0}^g(x_{i_0}) d\pi_{i_0}^g(x_{i_0}) \) (here we used the assumption that density exists).

Further, let \( \pi^*_d(x_{i_0},x'_{i_0}) \) denote the joint distribution that achieves the min for the Wasserstein-1 distance below between \( P_{y|x}^r \) and \( P_{y|x'}^r \) for any fixed \( x, x' \).

\[ W(P_{y|x}^r, P_{y|x'}^r) = \pi \in \Pi_{y|x} \int_{Y \times Y} d(y,y') d\pi(y,y') \]

Similar functions \( f^*_d \) and \( g^*_d \) (as for the earlier cases of \((x) \) exists here such that \( f^*_d(y) + g^*_d(y) = d(y,y') \) for all \( y, y' \) as well as for any \( x, x' \).

Then, define the probability distribution

\[ \pi^*(y,y')(E) = \int_{X \times X} 1_E d\pi^*_d(x,y) d\pi^*_d(x',y) \] for any event \( E \) in the appropriate \( \sigma \)-algebra (\( 1_E \) is the indicator of \( E \)). We will define \( d_0 \) later, which for now is any arbitrary distance metric. We want to show that the distribution \( \pi^*(y,y') \) is the minimizer for

\[ W(P_y^r, P_y^g) = \pi \in \Pi_{y|x} \int_{Y \times Y} d(y,y') d\pi(y,y') \]

Consider

\[ \int_{Y \times Y} d(y,y') d\pi^*(y,y') \]
By definition, this is same as
\[ \int_{Y \times Y'} d(y,y') \int_X d \pi_{x,y'}(x,y') d \pi_{x_0}(x_{i_0},x'_{j_0}) \prod_{j \neq i_0} d \pi^*(x_j,x'_j) \]
This is same as (using Fubini’s theorem) \(\text{(Billingsley 2008)}\)
\[ \int_{Y \times Y' \times X} \left[ d(y,y') d \pi_{x,y'}(x,y') \right] d \pi_{x_0}(x_{i_0},x'_{j_0}) \prod_{j \neq i_0} d \pi^*(x_j,x'_j) \]
Grouping the first two terms that have \(y, y'\) and integrating over \(Y \times Y'\) (using Fubini’s theorem) we get the above term is same as
\[ \int_X W(\mathbb{P}^y_{|x}, \mathbb{P}^{y'}_{|x'}) d \pi_{x_0}(x_{i_0},x'_{j_0}) \prod_{j \neq i_0} d \pi^*(x_j,x'_j) \]
Using dual representation of the Wasserstein distance this is same as
\[ \int_Y \int_Y f_{x,y}(y) P^y_{|x}(y|x) d y + \int_Y g_{x,y}(y) P^{y'}_{|x'}(y|x') d y \]
\[ d \pi_{x_0}(x_{i_0},x'_{j_0}) \prod_{j \neq i_0} d \pi^*(x_j,x'_j) \]
As \(\pi^*\)’s are couplings, we have the fact that
\[ \int_X \mathbb{P}^y_{|x}(x_{i_0}) \prod_{j \neq i_0} d \pi^*(x_j,x'_j) = \int_X \mathbb{P}^y_{|x}(x_{i_0}) \prod_{j \neq i_0} d \pi^*(x_j,x'_j) \]
\[ = \int_X P^y_{i_0}(x_{i_0}) \prod_{j \neq i_0} P_j(x_j) d x \]
\[ = \int_X P^y_{i_0}(x_{i_0}) \prod_{j \neq i_0} P_j(x_j) d x \]
Since \(X'\) is same as \(X\), and plugging back in
\[ = \int_X \int_Y f_{x,y}(y) P^y_{|x}(y|x) P^y_{i_0}(x_{i_0}) \prod_{j \neq i_0} P_j(x_j) d y d x + \]
\[ + \int_X \int_Y g_{x,y}(y) P^{y'}_{|x'}(y|x') P^y_{i_0}(x_{i_0}) \prod_{j \neq i_0} P_j(x_j) d y' d x' \]
\[ = \int_X \int_Y f_{x,y}(y) P^y_{|x}(y|x) P^y_{i_0}(x_{i_0}) \prod_{j \neq i_0} P_j(x_j) d y + \]
\[ + \int_X \int_Y g_{x,y}(y) P^{y'}_{|x'}(y|x') P^y_{i_0}(x_{i_0}) \prod_{j \neq i_0} P_j(x_j) d y' \]
Define \(F(y) = \int_X f_{x,y}(y) P^y_{|x}(y|x) d x / P^y_{i_0}(y)\) and \(G(y) = \int_X g_{x,y}(y) P^y_{|x}(y|x) d x / P^y_{i_0}(y)\) for \(y \in A\) and \(y \in B\) respectively. First, by assumption \(P^y_{i_0}(y) > 0\) for all \(y \in Y\) and \(P^y_{i_0}(\cdot)\) is continuous.
We claim that \(\int_X f_{x,y}(y) P^y_{|x}(y|x) d x\) is also continuous in \(y\). First, \(f_{x,y}(y) P^y_{|x}(y|x)\) is jointly continuous in \(x, y\) and all densities are continuous. As the space \(X \times Y\) is compact, \(f_{x,y}(y) P^y_{|x}(y|x)\) is also bounded by Weierstrass extreme value theorem \(\text{(Rudin 1986)}\). Then, define \(h_n(x) = f_{x,y}(y) P^y_{|x}(y|x)\) for some sequence \(y_n\) converging to \(y\) and clearly \(h_n(x)\) is continuous and bounded for every \(n\). Then, we can apply dominated convergence theorem \(\text{(Billingsley 2008)}\) to get
\[ \lim_{y_n \to y} \int_X h_n(x) d x = \int_X \lim_{y_n \to y} h_n(x) d x \]
The above yields
\[ \lim_{y_n \to y} \int_X h_n(x) d x = \int_X f_{x,y}(y) P^y_{|x}(y|x) d x \]
which proves our claim.
Next, since \(F, G\) are continuous in \(y\) and \(Y\) is compact, we have \(F, G\) are bounded by Weierstrass extreme value theorem. Hence, \(F, G \in \mathcal{C}_0(Y)\) Thus, Equation \(3\) can be written as
\[ \int_Y F(y) P^y_{|x}(y|x) d y + \int_Y G(y) P^y_{|x'}(y|x') d y \]
Overall, we proved that
\[ \int_Y \int_{\mathbb{P}^y_{|x}} d \pi_{x_0}(x_{i_0},x'_{j_0}) \prod_{j \neq i_0} d \pi^*(x_j,x'_j) \]
Using Equation \(2\) we can infer that
\[ \int_X \int_X W(\mathbb{P}^y_{|x}, \mathbb{P}^{y'}_{|x'}) d \pi_{x_0}(x_{i_0},x'_{j_0}) \prod_{j \neq i_0} d \pi^*(x_j,x'_j) \]
is same as
\[ \int_{X_0 \times X_0} \mathbb{E}_{x_0 \sim \mathbb{P}^y_{|x}} \mathbb{P}^y_{|x} d \pi_{x_0}(x_{i_0},x'_{j_0}) \prod_{j \neq i_0} d \pi^*(x_j,x'_j) \]
The above was proven without any assumption on the metric \(d_0\) (or even on the metric for \(Y\) or \(X_j\)’s), hence it holds for any metric.

\[ \text{Proof of Lemma 1} \]
\[ \text{Proof. } \]
We first claim that if \(M_{x_{-i_0}}(\cdot)\) is an injection (one-to-one) for some \(x_{-i_0}\), then
\[ \hat{d}(x_{i_0},x'_{j_0}) = \mathbb{P}^y(x_{i_0},x'_{j_0}) \]
is a distance metric on \(X_{i_0}\) for that \(x_{-i_0}\). It is a known result that if \(M : X \to Y\) is an injection and \(d_Y\) is a distance metric on \(Y\) then \(d\) defined as \(d(x_1,x_2) = d_Y(M(x_1), M(x_2))\)
is a metric on $X$. Also, if $M$ is not an injection then $d_Y$ is a pseudo-metric. This directly proves the claim above about $d$, where the Wasserstein-1 metric is the analog to $d_Y$ here.

For the second claim, some parts follow easily: (1) $d_0(x_{i_0}, x'_{i_0}) = d_0(x'_{i_0}, x_{i_0})$ is by definition, (2) triangle inequality follows from monotonicity of expectations and the triangle inequality of the term inside the expectation, and (3) $d_0(x_{i_0}, x_{i_0}) = 0$ is by definition. This already shows the pseudo-metric nature of $d_0$. The only tricky part is showing that $d_0(x_{i_0}, x'_{i_0}) = 0$ implies $x_{i_0} = x'_{i_0}$ in case of the injection. For this, let $d_0(x_{i_0}, x'_{i_0}) = 0$ which is same as

$$E_{x_{i_0} \sim \otimes j \neq i_0 P_j}[W(P^r_y(\langle x_{i_0}, x\rangle), P^r_y(\langle x_{i_0}, x'_{i_0} \rangle)) = 0$$

This implies that $W(P^r_y(\langle x_{i_0}, x\rangle), P^r_y(\langle x_{i_0}, x'_{i_0} \rangle)) = 0$ almost everywhere w.r.t. the measure $\otimes j \neq i_0 P_j$. For contradiction, assume $x_{i_0} \neq x'_{i_0}$. By assumption of injection, $P^r_y(\langle x_{i_0}, x\rangle)$ and $P^r_y(\langle x_{i_0}, x'_{i_0} \rangle)$ are different distributions. Hence, $W(P^r_y(\langle x_{i_0}, x\rangle), P^r_y(\langle x_{i_0}, x'_{i_0} \rangle)) > 0$ for all $x_{i_0}$. But this contradicts the inferred result that $W(P^r_y(\langle x_{i_0}, x\rangle), P^r_y(\langle x_{i_0}, x'_{i_0} \rangle)) = 0$ almost everywhere w.r.t. the measure $\otimes j \neq i_0 P_j$. Hence, we must have $x_{i_0} = x'_{i_0}$.

This proves all properties required to make $d_0$ a distance metric on $X_{i_0}$ in case of the injection.

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