Turbulence properties and global regularity of a modified Navier-Stokes equation

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Abstract

We introduce a modification of the Navier-Stokes equation that has the remarkable property of possessing an infinite number of conserved quantities in the inviscid limit. This new equation is studied numerically and turbulence properties are analyzed concerning energy spectra and scaling of structure functions. The dissipative structures arising in this new equation are curled vortex sheets contrary to vortex tubes arising in Navier-Stokes turbulence. The numerically calculated scaling of structure functions is compared with a phenomenological model based on the She-Lévéque approach.

Finally, for this equation we demonstrate global well-posedness for sufficiently smooth initial conditions in the periodic case and in $\mathbb{R}^3$. The key feature is the availability of an additional estimate which shows that the $L^4$-norm of the velocity field remains finite.

Keywords: PDEs, turbulence, Navier-Stokes equation, Burgers equation, regularity

1. Introduction

In this paper, we introduce a new equation which is a hybrid of the Navier-Stokes equation and the Burgers equation. This equation possesses a nonlocal part in the nonlinearity analogous to the pressure term in the Navier-Stokes equation. This new equation has the remarkable feature of possessing an infinite number of conserved quantities in the inviscid limit.

Turbulence properties of this equation are analyzed using numerical simulations. We calculate energy spectra and scaling of higher order structure functions. A key observation from the numerical simulations of this modified Navier-Stokes equation is that the most dissipative structures consist of curled vortex sheets contrary to vortex tubes in conventional Navier-Stokes turbulence. Using this information, a She-Lévéque type model is derived and compared to the numerically obtained scaling of higher order structure functions.
In addition, for this new equation we can show existence and regularity for $H^1$ initial conditions of arbitrary size. This will be carried out in $\mathbb{R}^3$ and periodic domains in $\mathbb{R}^3$. The simple modification of the nonlinearity makes the proof of global solutions possible, insofar as an additional estimate is available showing that the solution remains finite in $L^p$, $2 < p < \infty$. With $p = 4$, this is then coupled with standard estimates for the $H^1$-norm to complete the proof.

The outline of this paper is as follows: in section 2 we motivate and introduce our new equation. Turbulence statistics and phenomenological modeling is considered in section 3. Section 4 contains the proof of existence of global solutions. We finish with remarks on possible further consequences of the existence of an infinite number of conserved quantities.

2. Model equation

We consider a three-dimensional domain $\Omega$ which shall be either $\mathbb{R}^3$ or a bounded cube in $\mathbb{R}^3$ with periodic boundary conditions. Let $P = 1 - \Delta^{-1} \nabla \otimes \nabla$ be the Leray-Hopf projection operator (with periodic boundary conditions when $\Omega$ is bounded):

$$P[P[u]] = P[u] \quad \nabla \cdot P[u] = 0.$$  \hfill (1)

The usual incompressible Navier-Stokes equation

$$\frac{\partial}{\partial t} v + v \cdot \nabla v + \nabla p = \nu \Delta v + f \quad \nabla \cdot v = 0$$

can be written with the projection operator $P$

$$\frac{\partial}{\partial t} v + P[v \cdot \nabla v] = \nu \Delta v + P[f] \quad \nabla \cdot v = 0$$

such that no explicit pressure term is present in the equation.

We can rewrite the Navier-Stokes equation without the incompressibility constraint in the form

$$\frac{\partial}{\partial t} u + P[u] \cdot \nabla P[u] = \nu \Delta u + f,$$  \hfill (2)

where the solution of the Navier-Stokes equation can be recovered by taking $v = P[u]$.

The equation (2) can be compared with Burgers equation whose structure is formally similar:

$$\frac{\partial}{\partial t} u + u \cdot \nabla u = \nu \Delta u + f.$$  \hfill (3)

For equation (3) the nonlinearity is purely local, whereas for equation (2) the nonlinear interaction involves the nonlocal projection.

A natural hybrid of these two equations leads a new model equation involving a compressible velocity field $u$ that is convected by its solenoidal part $P[u]$: $u = P[u]$: $u = P[u]$:

$$\frac{\partial}{\partial t} u + P[u] \cdot \nabla u = \nu \Delta u + f.$$  \hfill (4)
More accurately this means: The convection of the velocity field \( \mathbf{u} \) is local in position space, but the projection operator is local in Fourier space and thus shares this mixture of local and non-local interactions with the original Navier-Stokes equation.

3. Turbulence statistics

By construction the presented model equation is an intermediate step between the Navier-Stokes and Burgers equation, which in turn differ significantly in their dynamical evolution and turbulent behavior. In Navier-Stokes turbulence, on the one hand, the most dissipative structures are vortex filaments, while for Burgers equation shocks dominate the turbulent flow. It is of obvious interest in how far our model equation bridges between those, which structures are the most dominant for turbulent flows and how these structures influence the turbulence statistics. We therefore extend the She-Lévêque reasoning, which describes Navier-Stokes and Burgers turbulence well, to our model equation and test it against numerical simulations by comparing the scaling exponents of the structure functions.

Numerical simulations are carried out with a second-order in space finite difference scheme with a strongly stable third-order Runge-Kutta time integration with resolutions up to \( 512^3 \). The initial conditions were chosen as Orszag-Tang-like (see [1]) large-scale perturbations:

\[
\begin{align*}
    u_x &= A \left(-2 \sin(2y) + \sin(z) + 2 \cos(2y) + \cos(z)\right) \\
    u_y &= A \left(-2 \sin(x) + \sin(z) + 2 \cos(x) + \cos(z)\right) \\
    u_z &= A \left(\sin(x) + \sin(y) - 2 \cos(2x) + \cos(y)\right).
\end{align*}
\]

For simplicity and comparability both velocity and its solenoidal projection are set to equal values. The physical domain stretches from \(-\pi\) to \(\pi\); the above defined conditions, thus, are both large-scale perturbations and periodic. All hydrodynamical models will be simulated in comparison, using these initial conditions. We consider only decaying turbulence without external forces. For the parameters of all performed runs see Table I which shows the numerical value of the quantities at the time of maximum enstrophy \( t = t_E \).

Figure 1 shows the decay of kinetic energy for the considered hydrodynamical models in comparison. The tendency of Burgers turbulence to form shocks and the dissipative nature of these structures lead to a faster energy decay compared to the Navier-Stokes equation. The new model equation exhibits a less violent form of dissipation, its energy decay lies in between the others. The difference in turbulence development is identified in a more precise way when comparing the time \( t_E \) of maximum enstrophy \( E = \int_\Omega \omega^2 dx \). As Figure 1 (right) indicates, the enstrophy of Burgers turbulence reaches its peak significantly faster than for the Navier-Stokes equation, in which vortex filaments dominate the turbulent flow. The proposed model equation ranges between them. This hints at the development of coherent structures at a timescale slower than shock-formation.
Table 1: Parameters of the numerical simulations. Number of collocation points $N^3$; grid spacing $dx$; time of fully developed turbulence $t_E$; viscosity $\nu$; root-mean-square velocity $v_{rms} = \sqrt{2/3E_{kin}}$; mean energy dissipation rate $\varepsilon$; integral scale $L = (2/3E_{kin})^{3/2}/\varepsilon$; dissipation length scale $\eta = (\nu^3/\varepsilon)^{1/4}$; Taylor-Reynolds number $R_{\lambda} = \sqrt{15v_{rms}L/\nu}$; all taken at the time of maximum enstrophy $t = t_E$.

| PDE       | $N$  | $\Delta x$ | $t_E$ | $\nu$  | $v_{rms}$ | $\varepsilon$ | $L$   | $\eta$ | $R_{\lambda}$ |
|-----------|-----|------------|-------|--------|-----------|--------------|-------|--------|--------------|
| Burgers   | 128 | 0.0491     | 0.461 | 0.0241 | 1.761     | 3.564        | 1.533 | 0.0445 | 40.99        |
|           | 256 | 0.0245     | 0.603 | 0.015  | 1.637     | 3.320        | 1.320 | 0.0318 | 46.49        |
|           | 512 | 0.0123     | 0.634 | 0.009  | 1.656     | 3.361        | 1.352 | 0.0216 | 61.10        |
| Model     | 128 | 0.0491     | 1.438 | 0.01   | 1.316     | 0.983        | 2.318 | 0.0318 | 67.65        |
| equation  | 256 | 0.0245     | 1.459 | 0.006  | 1.369     | 1.044        | 2.459 | 0.0213 | 91.75        |
|           | 512 | 0.0123     | 1.624 | 0.0036 | 1.370     | 1.096        | 2.344 | 0.0144 | 115.7        |
| Navier-Stokes | 128 | 0.0491     | 2.502 | 0.007  | 1.482     | 0.918        | 3.549 | 0.0247 | 106.2        |
|           | 256 | 0.0245     | 2.616 | 0.00278| 1.518     | 1.268        | 2.761 | 0.0114 | 150.4        |
|           | 512 | 0.0123     | 2.545 | 0.00110| 1.551     | 1.572        | 2.376 | 0.00539| 224.2        |

Figure 1: **Left**: Comparison of the kinetic energy for Navier-Stokes, Burgers and the proposed model equation. The steep discontinuities of a Burgers flow explain the fast energy dissipation. **Right**: Evolution of free-falling turbulence for Burgers equation, the proposed model equation and Navier-Stokes equation. The turbulent flow is fully developed at $t = t_E$, when the enstrophy $\overline{\varepsilon}$ reaches its maximum.

of Burgers equation but faster than the formation of vortex tubes for Navier-Stokes equation. Precise values for the timescales are stated in Table 1.

Figure 2 depicts a volume render of the fully developed turbulence, the snapshot in each case taken at $t = t_E$. As expected, the Burgers flow (left) is dominated by shocks and the Navier-Stokes flow (right) consists of vortex filaments. For the proposed model equation (middle), the most dominant structures are two-dimensional folded vortex-sheets.

A phenomenological description, which takes into account the most dissipative structures of the turbulent flow, is the model of She and Lévêque [2] connected to log-Poisson statistics of the local energy dissipation [3]. They
state that the scaling exponent $\zeta_p$ of the p-th structure function behaves like

$$
\zeta_p = \frac{(1 - k)p}{3} + C_0 \left( 1 - \left( \frac{C_0 - k}{C_0} \right)^\frac{2}{p} \right),
$$

(5)

where $C_0$ is the co-dimension of the most dissipative structures in the evolved flow and $k$ is the time-scaling exponent. This formula will be referred to as the She-Lévéque model even though in [2] it is applied exclusively to the Navier-Stokes equation (where $C_0 = 2$, $k = 2/3$).

The typical time $t_l$ for the evolution of discontinuities for turbulent Burgers flows scales linearly with $l$, which accounts for $k = 1$. As the shocks traveling through the domain are two-dimensional we furthermore obtain $C_0 = 1$. Inserting this into equation (5) leads to $\zeta_p = 1$. Since parts of the velocity field are continuous, for $p < 1$ the smoother regions of the velocity field are pronounced. This would equal a scaling exponent of $\zeta_p = p$ for these orders. Since this result is smaller than $\zeta_p = 1$, it is dominant for $p < 1$. Therefore, we get

$$
\zeta_p = \begin{cases} 
p & \text{for } p < 1 \\
1 & \text{for } p \geq 1
\end{cases}
$$

(6)

for Burgers equation.

| p | 1  | 2  | 4  | 5  | 6  | 7  |
|---|----|----|----|----|----|----|
| $\alpha_p$ | 0.70 | 0.95 | 1.00 | 1.00 | 0.98 | 1.00 |
| $\zeta_p$ | 1.00 | 1.00 | 1.00 | 1.00 | 1.00 | 1.00 |

Table 2: Scaling exponents $\zeta_p$ for Burgers equation. The $\alpha_p$ measured with ESS are compared to the $\zeta_p$ predicted by the She-Lévéque model.

Table 2 shows the measurements of a direct numerical simulation with $512^3$ grid points. Here, $\alpha_p$ is the data obtained via ESS and $\zeta_p$ is the prediction of equation (6). The visualization of this result is shown in figure 3 on the left.
Especially for high order of $p$ ($p > 3$) the scaling exponent agrees with the prediction, yet smears out for smaller $p$ (see also [4]).

Similar to Burgers equation the introduced model equation allows for shocks to develop, since no incompressibility condition is stated. Nevertheless, energy decays solely via the dissipative term. Because of this, the hypotheses of Kolmogorov may be adapted to Euler-Burgers equation. The Richardson cascade as well as the properties of the energy spectrum and the scaling behavior of structure functions should agree with the conclusions of Kolmogorov and the She-Lévéque model. On the other hand, the structures that evolve seem to be significantly different from the vortex filaments known from Navier-Stokes.

Figure 2 (middle) shows the most dissipative structures of fully developed turbulent flow. The norm of the vorticity visualizes two-dimensional folded vortex sheets as the structures that correspond to the vortex tubes of Navier-Stokes. This suggest $C_0 = 1$. The time-scaling exponent $t_1$ for the introduced equation is estimated as $k = 2/3$, with the same reasoning as for Navier-Stokes. Thus,

$$\zeta_p = \frac{p}{9} + \left(1 - \left(\frac{1}{3}\right)^{\frac{3}{p}}\right)$$

is the prediction for the scaling exponents proposed by the She-Lévéque model.

Table 3: Scaling exponents $\zeta_p$ for the new model equation. The $\alpha_p$ for $P\mathbf{u}$ measured with ESS are compared to the $\zeta_p$ predicted by the She-Lévéque model.

| $p$ | $\alpha_p(P\mathbf{u})$ | $\zeta_p$ |
|-----|----------------|----------|
| 1   | 0.40          | 0.418    |
| 2   | 0.73          | 0.751    |
| 4   | 1.22          | 1.213    |
| 5   | 1.39          | 1.395    |
| 6   | 1.55          | 1.556    |
| 7   | 1.69          | 1.701    |

Table 3 features the results measured via ESS for the scaling exponents of $P\mathbf{u}$. As can be seen in figure 3 (right) the numerical data agrees very well with the prediction of the She-Lévéque model. The solenoidal field is consistent with theory from low orders of $p$ up to the highest order that was measured.
4. Global solutions

In this section, we show global regularity for equation (4) for suitable initial data without any size restrictions. For this, we proof the remarkable property, that this equation possesses an infinite number of conserved quantities in the inviscid limit. Especially, the finiteness of the $L^4$-norm of the velocity field coupled with standard estimates for the $H^1$-norm enables us to show global regularity.

The problem of whether the three-dimensional incompressible Navier-Stokes equations can develop a finite time singularity from smooth initial conditions or if it has global solutions remains unresolved (see [5–7] and the references therein). The answer to this important question is recognized as one of the Millennium prize problems [8, 9].

Despite the complexity of the topic, a lot of progress has been made on this field in the past. For the two-dimensional case, global-in-time existence of unique weak and strong solutions is well-known (see [5, 6]). In three dimensions weak solutions are known to exist globally in time. For strong solutions, existence and uniqueness is known for a short interval of time which depends continuously on the initial data [10]. Many results published in the past, starting with [11], provide criteria for the global regularity of solutions via conditions applied to the velocity field [12, 13] or components thereof [14], the vorticity [15], its direction [16] or to the pressure field [17, 18].

The theory for the compressible Navier-Stokes equation is less well developed, and we will not attempt a summary here. The multi-dimensional Burgers equation [19] can be regarded as a crude simplification of this model. Global existence and uniqueness of strong solutions can be established in two and three-dimensions for suitably small initial conditions, much as with the Navier-Stokes system. Irrotational flows do possess global solutions for large data in arbitrary dimension, thanks to the Cole-Hopf transformation [20, 21]. However, there is no multidimensional weak theory because of the absence of a mechanism for energy dissipation, unlike Navier-Stokes.

The situation for this new modified Navier-Stokes like equation is rather different. In this section, the existence of global solutions is proven for the model equation (4) in a domain $\Omega$ which shall either be $\mathbb{R}^3$ or a periodic cube in $\mathbb{R}^3$.

**Theorem 1.** Let $u_0 \in H^1(\Omega)$. Let $f \in L^2_{\text{loc}}(\mathbb{R}^+, L^2(\Omega)) \cap L^1_{\text{loc}}(\mathbb{R}^+, L^4(\Omega))$. Then the initial value problem for the model equation (4) has a unique global solution $u \in C(\mathbb{R}^+, H^1(\Omega)) \cap L^2_{\text{loc}}(\mathbb{R}^+, H^2(\Omega))$.

The aim is to show that the solution remains a priori bounded in $L^\infty([0, T), H^1(\Omega)) \cap L^2([0, T), H^2(\Omega))$ for any $T > 0$, which implies its existence and uniqueness with standard arguments comparable to e.g. [3, 22]. Throughout the argument, we denote the Euclidean norm of the vector $u = \sum_i u_i e_i$ by $u = (\sum_i u_i^2)^{1/2}$. We first prove the following lemma:

**Lemma 1.** Let $u_0$, $f$, $\Omega$ be defined as above. Then the quantity $\|u(t)\|_{L^p}$ remains finite for $2 \leq p \in \mathbb{R}$. 

Proof. Taking the Euclidean inner product of (4) with $u$ yields the identity

$$\frac{1}{2} \left( \frac{\partial}{\partial t} u^2 + Pu \cdot \nabla u^2 \right) = \frac{\nu}{2} \Delta u^2 - \nu |\nabla u|^2 + f \cdot u.$$  

(8)

Integrate (8) over $\Omega$, use the fact that $Pu$ is divergence free, and then apply the Cauchy-Schwarz inequality to obtain

$$\frac{1}{2} \frac{\partial}{\partial t} \|u\|_{L^2}^2 + \nu \|\nabla u\|_{L^2}^2 \leq \|f\|_{L^2} \|u\|_{L^2}.$$  

(9)

Defining $x(t) = \frac{1}{2} \left( \|u(t)\|_{L^2}^2 + \int_0^t \nu \|\nabla u(s)\|_{L^2}^2 ds \right)$, we have that

$$x'(t) \leq \|f(t)\|_{L^2} (2x(t))^{1/2}.$$  

Upon integration, this gives the inequality

$$\|u(t)\|_{L^2}^2 + \int_0^t \nu \|\nabla u(s)\|_{L^2}^2 ds \leq \left( \|u_0\|_{L^2} + \int_0^t \|f(s)\|_{L^2} ds \right)^2.$$  

(10)

With this estimate the lemma is shown for the case $p = 2$.

Let $2 \leq n \in \mathbb{R}$ and multiply the identity (6) by $u^{2n}$:

$$\frac{1}{2n} \left( \frac{\partial}{\partial t} u^{2n} + Pu \cdot \nabla u^{2n} \right) = \nu \left( \frac{\partial}{\partial x_j} (u^{2(n-1)} \frac{\partial}{\partial x_j} u^2) - \frac{4(n-1)}{n} |\nabla u^n|^2 \right) - \nu u^{2(n-1)} |\nabla u|^2 + u^{2(n-1)} f \cdot u.$$  

Integrate this over $\Omega$ and apply Hölder's inequality:

$$\frac{1}{2n} \frac{\partial}{\partial t} \|u\|_{L^{2n}}^2 + \int \left( \frac{2\nu(n-1)}{n^2} |\nabla u^n|^2 + \nu u^{2(n-1)} |\nabla u|^2 \right) dx \leq \|f\|_{L^{2n}} \|u\|_{L^{2n}}^{2n-1}.$$  

If we let

$$y(t) = \frac{1}{2n} \|u(t)\|_{L^{2n}}^2 + \int_0^t \int \left( \frac{2\nu(n-1)}{n^2} |\nabla u^n(s)|^2 + \nu u^{2(n-1)}(s) |\nabla u(s)|^2 \right) dx ds,$$

then we obtain

$$y'(t) \leq \|f(t)\|_{L^{2n}} (2n y(t))^{\frac{2n-1}{2}}.$$  

This leads to the estimate

$$\|u(t)\|_{L^{2n}}^2 + 2n \int_0^t \int \left( \frac{2\nu(n-1)}{n^2} |\nabla u^n(s)|^2 + \nu u^{2(n-1)}(s) |\nabla u(s)|^2 \right) dx ds \leq \left( \|u_0\|_{L^{2n}} + \int_0^t \|f(s)\|_{L^{2n}} ds \right)^{2n}.$$  

(11)
Remark: This key argument fails for the case of the Navier-Stokes equation. At the same time, this estimate establishes an infinite number of conserved quantities in the unforced inviscid case.

Proof of Theorem 1. Take the $L^2$-inner product of (4) with $\Delta u$ and integrate by parts:

$$\frac{1}{2} \frac{\partial}{\partial t} \|\nabla u\|_{L^2}^2 + \nu \|\Delta u\|_{L^2}^2 = \int_{\Omega} \left( (P \cdot \nabla u) \cdot \Delta u \right) \, d\Omega + \int_{\Omega} f \cdot \Delta u \, d\Omega . \tag{i}$$

The forcing term (ii) has the bound

$$\int_{\Omega} f \cdot \Delta u \, d\Omega \leq \|f\|_{L^2} \|\Delta u\|_{L^2} \leq \frac{\nu}{4} \|\Delta u\|_{L^2}^2 + \frac{1}{\nu} \|f\|_{L^2}^2 .$$

The nonlinear term (i) is estimated as follows:

$$\int_{\Omega} (P \cdot \nabla u) \cdot \Delta u \, d\Omega = - \int \frac{\partial}{\partial x_k} u_i \frac{\partial}{\partial x_k} (P_j \frac{\partial}{\partial x_j} u_i) \, d\Omega \tag{ii}$$

$$= - \int \frac{\partial}{\partial x_k} u_i \left( (P_j \frac{\partial}{\partial x_j} \nabla u^2 + \frac{\partial}{\partial x_k} u_i \frac{\partial}{\partial x_j} (P_j u_i)) \right) \, d\Omega \tag{13}$$

$$= \int \frac{\partial}{\partial x_j} \frac{\partial}{\partial x_j} (P_j u_i) \, d\Omega \leq \|\nabla^2 u\|_{L^2} \|\nabla^2 P u\|_{L^2} \|u\|_{L^2} .$$

The second norm above is handled by interpolation. We first note that

$$\|\nabla^2 P u\|_{L^2} \leq \|\nabla P u\|_{L^6}^{3/4} \|\nabla^2 u\|_{L^2}^{1/4}. \tag{12}$$

Now when $\Omega = \mathbb{R}^3$, the Sobolev embedding theorem gives

$$\|\nabla P u\|_{L^6} \leq C \|\nabla^2 P u\|_{L^2} .$$

When $\Omega$ is a periodic domain, the norm on the right must be replaced by $\|\nabla P u\|_{H^1}$. However, since $\nabla P u$ has zero mean, this is bounded again by $C \|\nabla^2 P u\|_{L^2}$, by the Poincaré inequality. Therefore, (12) holds in both cases. Using the facts that the operator $P$ commutes with derivatives and that it is a projection in $L^2$, we have that

$$\|\nabla^2 u\|_{L^2} \leq \|\nabla u\|_{L^2} \quad \text{and} \quad \|\nabla^2 P u\|_{L^2} \leq \|\nabla^2 u\|_{L^2} .$$

Next, we use integration by parts to obtain the simple ellipticity identity

$$\|\nabla^2 u\|_{L^2}^2 = \int_{\Omega} \frac{\partial}{\partial x_j} \frac{\partial}{\partial x_j} u_i \frac{\partial}{\partial x_k} \frac{\partial}{\partial x_k} u_i \, d\Omega \tag{13}$$

$$= \int_{\Omega} \frac{\partial}{\partial x_j} \frac{\partial}{\partial x_j} u_i \frac{\partial}{\partial x_k} \frac{\partial}{\partial x_k} u_i \, d\Omega$$

$$= \|\Delta u\|_{L^2}^2 .$$
Combining these observations with Young’s inequality, we conclude that the nonlinear term (i) is bounded by

\[ C \| \Delta u \|_{L^2}^{7/4} \| \nabla u \|_{L^2}^{1/4} \| u \|_{L^4} \leq \frac{\nu}{4} \| \Delta u \|_{L^2}^2 + \frac{C}{\nu^7} \| \nabla u \|_{L^2}^2 \| u \|_{L^4}^8. \]

Altogether, we get the inequality

\[ \frac{\partial}{\partial t} \| \nabla u \|_{L^2}^2 + \nu \| \Delta u \|_{L^2}^2 \leq \frac{C}{\nu^7} \| \nabla u \|_{L^2}^2 \| u \|_{L^4}^8 + C \nu \| f \|_{L^2}. \]

Using Gronwall’s inequality, we find that

\[ \| \nabla u \|_{L^2}^2 + \nu \int_0^t \| \Delta u(s) \|_{L^2}^2 \, ds \leq \| \nabla u_0 \|_{L^2}^2 \exp \left( \int_0^t \nu \| u(s) \|_{L^4}^8 \, ds \right) \| f(s) \|_{L^2}^2 \, ds \quad (14) \]

Combining (10) and Lemma 1 with \( p = 4 \), and (14), we see that the quantity

\[ \| u(t) \|_{H^1}^2 + \int_0^t \nu | \| \nabla u(s) \|_{L^2}^2 + \| \Delta u(s) \|_{L^2}^2 | \, ds \]

remains finite. However, by (13) and the fact that \( L^\infty_{\text{loc}}(\mathbb{R}^+, L^2(\Omega)) \subset L^2_{\text{loc}}(\mathbb{R}^+, L^2(\Omega)) \) we have that

\[ \| u(t) \|_{H^1}^2 + \int_0^t \nu \| u(s) \|_{H^2(\Omega)}^2 \, ds \]

also remains finite.

5. Final remarks

In this paper, a modified Navier-Stokes equation is presented. Its dynamics and turbulent behavior are studied in terms of the scaling properties of its structure functions. The most dissipative structures are identified as vortex sheets of co-dimension 1, which allows us to compare the numerically measured scaling exponents to a modified phenomenologically based She-Lévéque approach.

Furthermore, we prove the existence of global solutions for this equation. A remarkable consequence of this proof is the existence of an infinite number of conserved quantities \( \| u \|_{L^p} \) in the ideal (non-viscous) case without forcing. This property is not only responsible for the existence of global solutions but should show up in the statistics of intermittent turbulent fluctuations. Work in this direction is in progress.

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