ON THE PROP AGATION OF REGULARITY FOR SOLUTIONS OF
THE DISPERSION GENERALIZED BENJAMIN-ONO EQUATION

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To my parents

ABSTRACT. In this paper we study some properties of propagation of regularity of
solutions of the dispersive generalized Benjamin-Ono (BO) equation. This model
defines a family of dispersive equations, that can be seen as a dispersive interpo-
lation between Benjamin-Ono equation and Korteweg-de Vries (KdV) equation.

Recently, it has been showed that solutions of the KdV equation and Benjamin-
Ono equation, satisfy the following property: if the initial data has some pre-
scribed regularity on the right hand side of the real line, then this regularity is
propagated with infinite speed by the flow solution.

In this case the nonlocal term present in the dispersive generalized Benjamin-
Ono equation is more challenging that the one in BO equation. To deal with this
a new approach is needed. The new ingredient is to combine commutator expan-
sions into the weighted energy estimate. This allow us to obtain the property of
propagation and explicitly the smoothing effect.

CONTENTS

1. Introduction 2
2. Notation 7
3. Preliminary 7
3.1. Commutator Expansions 11
4. The Linear Problem. 13
4.1. The Nonlinear Problem 14
5. Proof of Theorem A 18
5.1. A priori estimates 18
5.2. Uniqueness 21
5.3. Existence of Solutions 22
5.4. Continuity of the Flow 25
6. Proof of Theorem B 26
7. Acknowledgments 56
References 56

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1. Introduction

The aim of this work is to study some special regularity properties of solutions to the initial value problem (IVP) associated to the dispersion generalized Benjamin-Ono equation

\[
\begin{aligned}
\partial_t u - D_x^{\alpha+1} \partial_x u + u \partial_x u &= 0, \quad x, t \in \mathbb{R}, \quad 0 < \alpha < 1, \\
u(x, 0) &= u_0(x),
\end{aligned}
\]

where \( D_x^s \) denotes the homogeneous derivative of order \( s \in \mathbb{R} \),

\[
D_x^s = (-\partial_x^2)^{s/2}
\]

thus \( D_x^s f = c_s \left( |\xi|^s \hat{f}(\xi) \right)^\wedge, \)

which in its polar form is decomposed as \( D_x^s = (\mathcal{H} \partial_x)^s \), where \( \mathcal{H} \) denotes the Hilbert transform,

\[
\mathcal{H} f(x) = \lim_{\varepsilon \to 0^+} \frac{1}{\pi} \int_{|y| \geq \varepsilon} \frac{f(x - y)}{y} \, dy = (-i \text{sgn}(\xi) \hat{f}(\xi))^\wedge(x).
\]

These equations model vorticity waves in the coastal zone, see [36] and references therein.

Our starting point is a property established by Isaza, Linares and Ponce [18] concerning the solutions of the IVP associated to the \( k \)–generalized KdV equation

\[
\begin{aligned}
\partial_t u + \partial_x^2 u + u^k \partial_x u &= 0, \quad x, t \in \mathbb{R}, \quad k \in \mathbb{N}, \\
u(x, 0) &= u_0(x).
\end{aligned}
\]

It was shown in [18] that the unidirectional dispersion of the \( k \)–generalized KdV equation entails the following propagation of regularity phenomena.

**Theorem 1.3 ([18]).** If \( u_0 \in H^{3/4+} (\mathbb{R}) \) and for some \( l \in \mathbb{Z} \), \( l \geq 1 \) and \( x_0 \in \mathbb{R} \)

\[
\left\| \partial_x^l u_0 \right\|_{L^2((x_0, \infty))}^2 = \int_{x_0}^{\infty} \left| \partial_x^l u_0(x) \right|^2 \, dx < \infty,
\]

then the solution of the IVP associated to (1.2) satisfies that for any \( \nu > 0 \) and \( \varepsilon > 0 \)

\[
\sup_{0 \leq t \leq T} \int_{x_0 + \varepsilon - \nu t}^{x_0 + \varepsilon + \nu t} \left( \partial_x^l u \right)^2 (x, t) \, dx < c,
\]

for \( j = 0, 1, 2, \ldots, l \) with \( c = c \left( \| u_0 \|_{H^{3/4+} (\mathbb{R})}; \| \partial_x^l u_0 \|_{L^2((x_0, \infty))}; \nu; \varepsilon; T \right) \). In particular, for all \( t \in (0, T] \), the restriction of \( u(\cdot, t) \) to any interval \( (x_0, \infty) \) belongs to \( H^j ((x_0, \infty)) \).

Moreover, for any \( \nu > 0 \), \( \varepsilon > 0 \) and \( R > 0 \)

\[
\int_0^T \int_{x_0 + \varepsilon - \nu t}^{x_0 + \varepsilon + \nu t} \left( \partial_x^{l+1} u \right)^2 (x, t) \, dx \, dt < c,
\]

with \( c = c \left( \| u_0 \|_{H^{3/4+} (\mathbb{R})}; \| \partial_x^l u_0 \|_{L^2((x_0, \infty))}; \nu; \varepsilon; R; T \right) \).

The proof of Theorem 1.3 is based on weighted energy estimates. In detail, the iterative process in the induction argument is based in a property discovered
originally by T. Kato [20] in the context of the KdV equation. More precisely, he showed that solution of the KdV equation satisfies

\[ \int_0^T \int_{-R}^R (\partial_x u)^2 (x, t) \, dx \, dt \leq c \left( R; T; \| u_0 \|_{L^2} \right), \]

being this the fundamental fact in his proof of existence of the global weak solutions of (1.2), for \( k = 1 \) and initial data in \( L^2(\mathbb{R}) \).

This result was also obtained for the Benjamin-Ono equation [19] but it does not follow as the KdV case because of the presence of the Hilbert transform.

Later on, Kenig et al. [23] extended the results in Theorem 1.3 to the case when the local regularity of the initial data \( u_0 \) in (1.4) is measured with a fractional in-
dices. The scope to this case is quite more involved, and its proof is mainly based in weighted energy estimates combined with techniques involving pseudo-differential operators and singular integrals. The property described in Theorem 1.3 is intrin-
sic to suitable solutions of some nonlinear dispersive models (see also [35]). In the context of 2D models, analogous results for the Kadomtsev-Petviashvili II equation [17] and Zakharov-Kuznetsov [33] equations were proved.

Before state our main result we will give an overview of the local well-posedness of the IVP (1.1).

Following [20] we have that the initial value problem IVP (1.1) is locally well-
posed (LWP) in the Banach space \( X \) if for every initial condition \( u_0 \in X \), there exists \( T > 0 \) and a unique solution \( u(t) \) satisfying

\[ u \in C \left( [0, T) : X \right) \cap A_T \]

where \( A_T \) is an auxiliary function space.

Moreover, the solution map \( u_0 \mapsto u \), is continuous from \( X \) into the class (1.7). If \( T \) can be taken arbitrarily large, one says that the IVP (1.1) is globally well-posed (GWP) in the space \( X \).

It is natural to study the IVP (1.1) in the Sobolev space

\[ H^s(\mathbb{R}) = \left( 1 - \partial_x^2 \right)^{-s/2} L^2(\mathbb{R}), \quad s \in \mathbb{R}. \]

There exist remarkable differences between the KdV (1.2) and the IVP (1.1). In case of KdV e.g. it posses infinite conserved quantities, define a Hamiltonian system, have multi-soliton solutions and is a completely integrable system by the inverse scattering method [8], [10].

Instead, in the case of the IVP (1.1) there is no integrability, but three conserved quantities (see [39]), specifically

\[ I[u](t) = \int_{\mathbb{R}} u \, dx, \quad M[u](t) = \int_{\mathbb{R}} u^2 \, dx, \]

\[ H[u](t) = \frac{1}{2} \int_{\mathbb{R}} \left| D_x u \right|^2 \, dx - \frac{1}{6} \int_{\mathbb{R}} u^3 \, dx, \]

are satisfied at least for smooth solutions.

Another property in which these two models differ, resides in the fact that one can obtain a local existence theory for the KdV equation in \( H^s(\mathbb{R}) \), based on the contraction principle. On the contrary, this cannot be done in the case of the (IVP) (1.1). This is a consequence of the fact that dispersion is not enough to deal with the nonlinear term. In this direction, Molinet, Saut and Tzvetkov [36] showed that for
$0 \leq \alpha < 1$ the IVP (1.1) with the assumption $u_0 \in H^s(\mathbb{R})$ is not enough to prove local well-posedness by using fixed point arguments or Picard iteration method.

Nevertheless, Herr, Ionescu, Kenig and Koch [16] show that the IVP (1.1) is globally well-posed in the space of the real-valued $L^2(\mathbb{R})$—functions, by using a renormalization method to control the strong low-high frequency interactions. It is not clear that this theory can be used to establish our main result. We need to have a local theory obtained by using energy estimates plus dispersive properties of the smooth solutions.

In the first step, we obtain the following a priori estimate for solutions of IVP (1.1)

$$
\|u\|_{L^\infty_T H^s} \lesssim \|u_0\|_{H^s} e^{c\|\partial_x u\|_{L^1_T L^\infty}},
$$

part of this estimate is based on the Kato-Ponce commutator estimate [22].

The inequality above reads as follows: in order to the solution $u$ abide in the Sobolev space $H^s(\mathbb{R})$, continuously in time, we require to control the term $\|\partial_x u\|_{L^1_T L^\infty}$.

First, we use Kenig, Ponce and Vega in [28] results concerning oscillatory integrals, in order to obtain the classical Strichartz estimates associated to the group $S(t) = e^{itD_x^{\alpha+1} \partial_x}$, corresponding to the linear part of the equation in (1.1).

In second place, the technique introduced by Koch and Tzvetkov [30] related to refined Strichartz estimate are fundamentals in our analysis. Specifically, their method is mainly based in a decomposition of the time interval in small pieces whose length depends on the spatial frequencies of the solution. This approach allowed to Koch and Tzvetkov to prove local well-posedness, for the Benjamin-Ono equation in $H^{5/4+} (\mathbb{R})$. Succeeding, Kenig and Koenig [24] enhanced this estimate, which led to prove local well-posedness for the Benjamin-Ono equation in $H^{9/8+} (\mathbb{R})$.

Several issues arise when handling the nonlinear part of the equation in (1.1), nevertheless, following the work of Kenig, Ponce and Vega [25], we manage the loss of derivatives by means of combination of the local smoothing effect and a maximal function estimate of the group $S(t) = e^{itD_x^{\alpha+1} \partial_x}$.

These observations lead us to present our first result.

**Theorem A.** Let $0 < \alpha < 1$. Set $s(\alpha) = \frac{9}{8} - \frac{3\alpha}{8}$ and assume that $s > s(\alpha)$. Then, for any $u_0 \in H^s(\mathbb{R})$, there exists a positive time $T = T \left( \|u_0\|_{H^s(\mathbb{R})} \right) > 0$ and a unique solution $u$ satisfying (1.1) such that

$$
(1.8) \quad u \in C \left( [0, T] : H^s(\mathbb{R}) \right) \quad \text{and} \quad \partial_x u \in L^1 \left( [0, T] : L^\infty(\mathbb{R}) \right).
$$

Moreover, for any $r > 0$, the map $u_0 \mapsto u(t)$ is continuous from the ball

$$
\left\{ u_0 \in H^s(\mathbb{R}) : \|u_0\|_{H^s(\mathbb{R})} \right\}
$$

to $C \left( [0, T] : H^s(\mathbb{R}) \right)$.

Theorem A is the base result to describe the propagation of regularity phenomena. As we mentioned above the propagation of regularity phenomena is satisfied by the BO and KdV equations respectively. These two models correspond to particular cases of the IVP (1.1), specifically by taking $\alpha = 0$ and $\alpha = 1$.

A question that arises naturally is to determine whether the propagation of regularity phenomena is satisfied for a model with an intermediate dispersion between these two models mentioned above.
Our main result give answer to this problem and it is summarized in the following:

**Theorem B.** Let \( u_0 \in H^s(\mathbb{R}) \) with \( s = \frac{3-\alpha}{2} \), and \( u = u(x,t) \), be the corresponding solution of the IVP (1.1) provided by Theorem A.

If for some \( x_0 \in \mathbb{R} \) and for some \( m \in \mathbb{Z}^+ \), \( m \geq 2 \),

\[
\partial_x^m u_0 \in L^2(\{x \geq x_0\}),
\]
then for any \( v > 0, T > 0, \epsilon > 0 \) and \( \tau > \epsilon \)

\[
\begin{align*}
&\sup_{0 \leq t \leq T} \int_{x_0 + \epsilon - vt}^{\infty} (\partial_x^j u)^2(x,t) \, dx + \int_0^T \int_{x_0 + \epsilon - vt}^{x_0 + \tau - vt} \left( D_x^{\frac{\alpha+1}{2}} \partial_x^j u \right)^2(x,t) \, dx \, dt \\
&\quad + \int_0^T \int_{x_0 + \epsilon - vt}^{x_0 + \tau - vt} \left( D_x^{\frac{\alpha+1}{2}} \mathcal{H} \partial_x^j u \right)^2(x,t) \, dx \, dt \leq c
\end{align*}
\]
for \( j = 1, 2, \ldots, m \) with \( c = c \left( T; \epsilon; \alpha; u_0 \right) \).

If in addition to (1.9) there exists \( x_0 \in \mathbb{R}^+ \)

\[
D_x^{\frac{\alpha+1}{2}} \partial_x^m u_0 \in L^2(\{x > x_0\})
\]
then for any \( v \geq 0, \epsilon > 0 \) and \( \tau > \epsilon \)

\[
\sup_{0 \leq t \leq T} \int_{x_0 + \epsilon - vt}^{\infty} \left( D_x^{\frac{\alpha+1}{2}} \partial_x^m u \right)^2(x,t) \, dx + \int_0^T \int_{x_0 + \epsilon - vt}^{x_0 + \tau - vt} \left( \partial_x^{m+1} u \right)^2(x,t) \, dx \, dt \\
\quad + \int_0^T \int_{x_0 + \epsilon - vt}^{x_0 + \tau - vt} \left( \partial_x^{m+1} \mathcal{H} u \right)^2(x,t) \, dx \, dt \leq c
\]
with \( c = c \left( T; \epsilon; \alpha; u_0 \right) \).

Although the argument of the proof of Theorem B follows in spirit that of KdV i.e. an induction process combined with weighted energy estimates. The presence of the non-local operator \( D_x^{\frac{\alpha+1}{2}} \partial_x \), in the term providing the dispersion, makes the proof much harder. More precisely, two difficulties appear, in the first place and the most important is obtain explicitly the Kato smoothing effect as in [20], that as in the proof of Theorem 1.3 is fundamental.

In contrast to KdV equation, the gain of the local smoothing in solutions of the dispersive generalized Benjamin-Ono equation is just \( \frac{\alpha+1}{2} \) derivatives, so as occurs in the case of the Benjamin-Ono equation [19], the iterative argument in the induction process is carried out in two steps, one for positive integers \( m \) and another one for \( m + \frac{1-\alpha}{2} \) derivative.

In the case of the BO equation [19], the authors obtain the smoothing effect basing their analysis on several commutator estimates, such as the extension of the first Calderon’s commutator for the Hilbert transform [2]. However, their method of proof do not allow them obtain explicitly the local smoothing as in [20].

The advantage of our method is that it allows obtain explicitly the smoothing effect for any \( \alpha \in (0,1) \) in the IVP (1.1). Roughly, we rewrite the term modeling the dispersive part of the equation in (1.1), in terms of an expression involving
\[ \left[ \mathcal{H} D_x^{\alpha+2}, \chi^2 \right] \]. At this point, we incorporate Ginibre and Velo \[14\] results about commutator decomposition. This, allows us to obtain explicitly the smoothing effect as in \[20\], at every step of the induction process in the energy estimate. Besides, this approach allow us to study the propagation of regularity phenomena in models where the dispersion is lower in comparison with that of IVP \((1.1)\). We address this issue in a forthcoming work, specifically we study the propagation of regularity phenomena in real solutions of the model

\[ \partial_t u - D_x^\alpha \partial_x u + u \partial_x u = 0, \quad x, t \in \mathbb{R}, \quad 0 < \alpha < 1. \]

As a direct consequence of the Theorem \(B\) one has that for an appropriate class of initial data, the singularity of the solution travels with infinity speed to the left as time evolves. Also, the time reversibility property implies that the solution cannot have had some regularity in the past.

Concerning the nonlinear part of IVP \((1.1)\) into the weighted energy estimate, several issues arises. Nevertheless, following Kenig et al. \[23\] approach, combined with the works of Kato-Ponce \[22\], and the recent work D. Li \[31\] on the generalization of several commutators estimate, allow us to overcome these difficulties.

**Remark 1.13.**

(I) It will be clear from our proof that the requirement on the initial data, this is \(u_0 \in H^{\frac{3}{2}+\alpha}(\mathbb{R})\) in Theorem \(B\) can be lowered to \(H^{\frac{9}{8}+\alpha}(\mathbb{R})\).

(II) Also it is worth highlighting that the proof of Theorem \(B\) can be extended to solutions of the the IVP

\[ \begin{cases} \partial_t u - D_x^{\alpha+1} \partial_x u + u^k \partial_x u = 0, & x, t \in \mathbb{R}, \quad 0 < \alpha < 1, \quad k \in \mathbb{Z}^+, \\ u(x, 0) = u_0(x). \end{cases} \]  

(1.14)

(III) The results in Theorem \(B\) still holds for solutions of the defocussing generalized dispersive Benjamin-Ono equation

\[ \begin{cases} \partial_t u - D_x^{\alpha+1} \partial_x u - u \partial_x u = 0, & x, t \in \mathbb{R}, \quad 0 < \alpha < 1, \\ u(x, 0) = u_0(x). \end{cases} \]

This can be seen applying Theorem \(B\) to the function \(v(x, t) = u(-x, -t)\), where \(u(x, t)\) is a solution of \((1.1)\). In short, Theorem \(B\) remains valid, backward in time for initial data \(u_0\), satisfying \((1.9)\) and \((1.11)\).

Next, we present some immediate consequences of Theorem \(B\).

**Corollary 1.15.** Let \(u \in C\left([-T, T] : H^{\frac{3}{2}+\alpha}(\mathbb{R})\right)\) be a solution of the equation in \((1.1)\) described by Theorem \(B\). If there exist \(n, m \in \mathbb{Z}^+\) with \(m \leq n\) such that for some \(\tau_1, \tau_2 \in \mathbb{R}\) with \(\tau_1 < \tau_2\)

\[ \int_{\tau_1}^{\tau_2} |\partial_x^n u_0(x)|^2 \, dx < \infty \quad \text{but} \quad \partial_x^m u_0 \notin L^2((\tau_1, \infty)), \]

then for any \(t \in (0, T)\) and any \(\nu > 0\) and \(\epsilon > 0\)

\[ \int_{\tau_2 + \epsilon - \nu t}^{\tau_2} |\partial_x^n u(x, t)|^2 \, dx < \infty, \]

and for any \(t \in (-T, 0)\) and any \(\tau_3 \in \mathbb{R}\)

\[ \int_{\tau_3}^{\tau} |\partial_x^n u(x, t)|^2 \, dx = \infty. \]
The rest of the paper is organized as follows: in the section 2 we fix the notation to be used throughout the document. Section 3 contains a brief summary of commutators estimates involving fractional derivatives. The section 4 deals with the local well-posedness. Finally, the section 5 is devoted to the proof of Theorem B.

2. Notation

The following notation will be used extensively throughout this article. The operator \( f^s = (1 - \partial_x^2)^s/2 \) denotes the Bessel potentials of order \(-s\).

For \( 1 \leq p < \infty \), \( L^p(\mathbb{R}) \) is the usual Lebesgue space with the norm \( \| \cdot \|_{L^p} \), besides for \( s \in \mathbb{R} \), we consider the Sobolev space \( H^s(\mathbb{R}) \) is defined via its usual norm \( \| f \|_{H^s} = \| f^s \|_{L^2} \). In this context, we define \( H^\infty(\mathbb{R}) = \bigcap_{s > 0} H^s(\mathbb{R}) \).

Let \( f = f(x,t) \) be a function defined for \( x \in \mathbb{R} \) and \( t \) in the time interval \([0,T]\), with \( T > 0 \) or in the hole line \( \mathbb{R} \). Then if \( A \) denotes any of the spaces defined above, we define the spaces \( L^p_T A_x \) and \( L^p_x A_t \) by the norms

\[
\| f \|_{L^p_T A_x} = \left( \int_0^T \| f(\cdot, t) \|_A^p \, dt \right)^{1/p}
\]

and

\[
\| f \|_{L^p_x A_t} = \left( \int_0^T \| f(t, \cdot) \|_A^p \, dt \right)^{1/p},
\]

for \( 1 \leq p < \infty \) with the natural modification in the case \( p = \infty \). Moreover, we use similar definitions for the mixed spaces \( L^p_T L^q_x \) and \( L^p_x L^q_t \) with \( 1 \leq p, q \leq \infty \).

For two quantities \( A \) and \( B \), we denote \( A \lesssim B \) if \( A \leq cB \) for some constant \( c > 0 \). Similarly, \( A \gtrsim B \) if \( A \geq cB \) for some \( c > 0 \). We denote \( A \sim B \) if \( A \lesssim B \) and \( B \lesssim A \). The dependence of the constant \( c \) on other parameters or constants are usually clear from the context and we will often suppress this dependence whenever possible.

For a real number \( a \) we will denote by \( a^+ \) instead of \( a + \varepsilon \), whenever \( \varepsilon \) is a positive number whose value is small enough.

3. Preliminary

In this section, we state several inequalities to be used in the next sections.

First, we have an extension of the Calderon commutator theorem [7] established by B. Bajšanski et al. [2].

**Theorem 3.1.** For any \( p \in (1, \infty) \) and any \( l, m \in \mathbb{Z}^+ \cup \{0\} \) there exists \( c = c(p; l; m) > 0 \) such that

\[
\left\| \mathcal{C}_x^l \left[ \mathcal{H}; \psi \right] \mathcal{C}_x^m f \right\|_{L^p} \leq c \| \mathcal{C}_x^{m+1} \psi \|_{L^\infty} \| f \|_{L^p}.
\]

For a different proof see [9] Lemma 3.1.

In our analysis the Leibniz rule for fractional derivatives, established in [15, 22, 26] will be crucial. Even though most of these estimates are valid in several dimensions, we will restrict our attention to the one-dimensional case.

**Lemma 3.3.** For \( s > 0 \), \( p \in [1, \infty) \)

\[
\| D^s (f g) \|_{L^p} \lesssim \| f \|_{L^p_{l_1}} \| D^s g \|_{L^p_{l_2}} + \| g \|_{L^p_{l_3}} \| D^s f \|_{L^p_{l_4}}
\]
with
\[
\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p_3} + \frac{1}{p_4}, \quad p_j \in (1, \infty), \quad j = 1, 2, 3, 4.
\]

Also, we will state the fractional Leibniz rule proved by Kenig, Ponce and Vega [25].

**Lemma 3.5.** Let \( s = s_1 + s_2 \in (0, 1) \) with \( s_1, s_2 \in (0, s) \), and \( p, p_1, p_2 \in (1, \infty) \) satisfy
\[
\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}.
\]

Then,
\[
\|D^s(fg) - fD^s g - gD^s f\|_{L^p} \lesssim \|D^{s_1} f\|_{L^{p_1}} \|D^{s_2} g\|_{L^{p_2}}.
\]
Moreover, the case \( s_2 = 0 \) and \( p_2 = \infty \) is allowed.

A natural question about Lemma 3.5 is to investigate the possible generalization of the estimate (3.6) when \( s \gg 1 \). The answer to this question was given recently by D.Li [31], where he establishes new fractional Leibniz rules for the nonlocal operator \( D^s \), \( s > 0 \), and related ones, including various end-point situations.

**Theorem 3.7. Case 1:** \( 1 \leq p < \infty \).

Let \( s > 0 \) and \( 1 < p < \infty \). Then for any \( s_1, s_2 \geq 0 \) with \( s = s_1 + s_2 \), and any \( f, g \in \mathcal{S} (\mathbb{R}) \), the following hold:

1. If \( 1 < p_1, p_2 < \infty \) with \( \frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} \), then
\[
\left\| D^s(fg) - \sum \frac{1}{\alpha!} \partial_x^\alpha f D^s \partial_x^\alpha g - \sum \frac{1}{\beta!} \partial_x^\beta g D^s \partial_x^\beta f \right\|_{L^p} \lesssim \|D^{s_1} f\|_{L^{p_1}} \|D^{s_2} g\|_{L^{p_2}}.
\]

2. If \( p_1 = p, p_2 = \infty \), then
\[
\left\| D^s(fg) - \sum \frac{1}{\alpha!} \partial_x^\alpha f D^s \partial_x^\alpha g - \sum \frac{1}{\beta!} \partial_x^\beta g D^s \partial_x^\beta f \right\|_{L^p} \lesssim \|D^{s_1} f\|_{L^p} \|D^{s_2} g\|_{\text{BMO}},
\]
where \( \| \cdot \|_{\text{BMO}} \) denotes the norm in the BMO space\(^1\).

3. If \( p_1 = \infty, p_2 = p \), then
\[
\left\| D^s(fg) - \sum \frac{1}{\alpha!} \partial_x^\alpha f D^s \partial_x^\alpha g - \sum \frac{1}{\beta!} \partial_x^\beta g D^s \partial_x^\beta f \right\|_{L^p} \lesssim \|D^{s_1} f\|_{\text{BMO}} \|D^{s_2} g\|_{L^p}.
\]

\(^1\)For any \( f \in L^1_{\text{loc}} (\mathbb{R}^n) \), the BMO semi-norm is given by
\[
\|f\|_{\text{BMO}} = \sup_{Q} \frac{1}{|Q|} \int_Q |f(y) - (f)_Q| \, dy,
\]
where \((f)_Q\) is the average of \( f \) on \( Q \), and the supreme is taken over all cubes \( Q \) in \( \mathbb{R}^n \).
Lemma 3.13. The operator $D^{s,a}$ is defined via Fourier transform\(^2\)

\[
\hat{D^{s,a}g}(\xi) = \hat{D^{s,a}}(\xi)\widehat{g}(\xi),
\]
\[
\hat{D^{s,a}}(\xi) = i^{-a}\partial_x^a (|\xi|^a).
\]

Case 2: \(\frac{1}{2} < p \leq 1\).

If \(\frac{1}{2} < p \leq 1\), \(s > \frac{1}{p} - 1\) or \(s \in 2\mathbb{N}\), then for any \(1 < p_1, p_2 < \infty\) with \(\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}\), any \(s_1, s_2 \geq 0\) with \(s_1 + s_2 = s\),

\[
\|D^s(fg) - \sum_{\alpha \leq s_1} \frac{1}{\alpha!} \partial_x^\alpha f D^{s,a} g - \sum_{\beta \leq s_2} \frac{1}{\beta!} \partial_x^\beta g D^{s,a} f\|_{L^p} \lesssim \|D^{s_1}f\|_{L^{p_1}} \|D^{s_2}g\|_{L^{p_2}}.
\]

Remark 3.9. As usual empty summation (such as \(\sum_{0 \leq \alpha < 0}\)) is defined as zero.

Proof. For a detailed proof of this Theorem and related results, see [31]. \(\square\)

Next we have the following commutator estimates involving non-homogeneous fractional derivatives, established by Kato and Ponce.

Lemma 3.10 ([22]). Let \(s > 0\) and \(p, p_2, p_3 \in (1, \infty)\) and \(p_1, p_4 \in (1, \infty)\) be such that

\[
\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p_3} + \frac{1}{p_4}.
\]

Then,

\[
\|J_s^s f g\|_{L^p} \lesssim \|\partial_x f\|_{L^{p_1}} \|J^{s-1} g\|_{L^{p_2}} + \|J^s f\|_{L^{p_3}} \|g\|_{L^{p_4}}
\]

and

\[
\|J_s^s (fg)\|_{L^p} \lesssim \|J^s f\|_{L^{p_1}} \|g\|_{L^{p_2}} + \|J^s g\|_{L^{p_3}} \|f\|_{L^{p_4}}.
\]

There are many other reformulations and generalizations of the Kato-Ponce commutator inequalities (cf. [3] and the references therein). Recently D. Li [31], has obtained a family of refined Kato-Ponce type inequalities for the operator $D^s$. In particular he showed that

Lemma 3.13. Let \(1 < p < \infty\). Let \(1 < p_1, p_2, p_3, p_4 \leq \infty\) satisfy

\[
\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p_3} + \frac{1}{p_4}.
\]

Therefore,

(a) If \(0 < s \leq 1\), then

\[
\|D^s(fg) - fD^sg\|_{L^p} \lesssim \|D^{s-1} \partial_x f\|_{L^{p_1}} \|g\|_{L^{p_2}}.
\]

(b) If \(s > 1\), then

\[
\|D^s(fg) - fD^sg\|_{L^p} \lesssim \|D^{s-1} \partial_x f\|_{L^{p_1}} \|g\|_{L^{p_2}} + \|\partial_x f\|_{L^{p_3}} \|D^{s-1} g\|_{L^{p_4}}.
\]

For a more detailed exposition on these estimates see section 5 in [31].

In addition, we have the following inequality of Gagliardo-Nirenberg type:

\(^2\)The precise form of the Fourier transform does not matter.
Lemma 3.15. Let \(1 < q, p < \infty, 1 < r \leq \infty\) and \(0 < \alpha < \beta\). Then,
\[
\|D^\alpha f\|_{L^p} \lesssim c\|f\|_{L^r}^{1-\theta}\|D^\beta f\|_{L^q}^\theta
\]
with
\[
\frac{1}{p} - \alpha = (1 - \theta)\frac{1}{r} + \theta\left(\frac{1}{q} - \beta\right), \quad \theta \in [\alpha/\beta, 1].
\]

Proof. See [4] chapter 4.

Now, we present a result that will help us to establish the propagation of regularity of solutions of (1.1). A previous result was proved by Kenig et al. (c.f [23], Corollary 2.1) using the fact that \(J'(r \in \mathbb{R})\) can be seen as a pseudo-differential operator. Thus, this approach allows to obtain an expression for \(J'\) in terms of a convolution with a certain kernel \(k(x, y)\) which enjoys some properties on localized regions in \(\mathbb{R}\). In fact, this is known as the singular integral realization of a pseudo-differential operator, whose proof can be found in [44] Chapter 4.

The estimate we consider here involves the non-local operator \(D^s\) instead of \(J^s\).

Lemma 3.16. Let \(m \in \mathbb{Z}^+\) and \(s \geq 0\). If \(f \in L^2(\mathbb{R})\) and \(g \in L^p(\mathbb{R}), 2 \leq p \leq \infty\), with
\[
\text{dist} \left(\text{supp}(f), \text{supp}(g)\right) \geq \delta > 0.
\]
Then
\[
\|g \partial_x^{\alpha} D^s f\|_{L^p} \lesssim \|g\|_{L^p} \|f\|_{L^2}.
\]

Proof. Let \(f, g\) be functions in the Schwartz class satisfying (3.17).
Notice that
\[
g(x) \left(D_x^s \partial_x^m f\right)(x) = \frac{g(x)}{(2\pi)^{1/2}} \int_{\mathbb{R}} e^{ix\xi} |\xi|^s \partial_x^m f(\xi) \, d\xi = \frac{g(x)}{(2\pi)^{1/2}} \int_{\mathbb{R}} |\xi|^s (\tau_{-x} \partial_x^m f)(\xi) \, d\xi.
\]
(3.18)

where \(\tau_h\) is the translation operator.\(^3\)

Moreover, the last expression in (3.18) defines a tempered distribution for \(s\) in a suitable class, that will be specified later. Indeed, for \(z \in \mathbb{C}\) with \(-1 < \text{Re}(z) < 0\)
\[
\frac{1}{(2\pi)^{1/2}} \int_{\mathbb{R}} |\xi|^z (\tau_{-x} \partial_x^m \varphi)(\xi) \, d\xi = c(z) \int_{\mathbb{R}} \left(\frac{\tau_{-x} \partial_x^m \varphi}{|y|^{1+z}}\right)(y) \, dy, \quad \forall \varphi \in \mathcal{S}(\mathbb{R})
\]
(3.19)

with \(c(z)\) is independent of \(\varphi\). In fact, evaluating \(\varphi = e^{-x^2/2}\) in (3.19) yields
\[
c(z) = \frac{2^{z} \Gamma\left(\frac{z+1}{2}\right)}{\pi^{1/2} \Gamma\left(-\frac{z}{2}\right)}.
\]

Thus, for every \(\varphi \in \mathcal{S}(\mathbb{R})\) the right hand side in (3.19) defines a meromorphic function for every test function, which can be extended analytically to a wider range of complex numbers \(z\)’s, specifically \(z\) with \(\text{Im}(z) = 0\) and \(\text{Re}(z) = s > 0\) that is the case that attains us. By an abuse of notation, we will denote the meromorphic extension and the original as the same.

\(^3\) For \(h \in \mathbb{R}\) the translation operator \(\tau_h\) is defined as \((\tau_h f)(x) = f(x - h)\).
Thus, combining (3.17), (3.18) and (3.19) it follows that
\[ g(x) \left( D_x^s \partial_x^m f \right)(x) = c(s) \int_{\mathbb{R}} \frac{g(y) \left( \tau_y \partial_x^m f \right)(y)}{|y|^{1+s}} \, dy \]
\[ = c(s) g(x) \left( f * \frac{\mathbf{1}_{\{|y| \geq \delta\}}}{|y|^{s+m+1}} \right)(x) \]
Notice that the kernel in the integral expression is not anymore singular due to the condition (3.17). In fact, in the particular case that \( m \) is even, we obtain after apply integration by parts
\[ g(x) \left( D_x^s \partial_x^m f \right)(x) = c(s, m) g(x) \left( f * \frac{\mathbf{1}_{\{|y| \geq \delta\}}}{|y|^{s+m+1}} \right)(x) \]
and in the case \( m \) being odd
\[ g(x) \left( D_x^s \partial_x^m f \right)(x) = c(s, m) g(x) \left( f * \frac{y \mathbf{1}_{\{|y| \geq \delta\}}}{|y|^{s+m+2}} \right)(x). \]
Finally, in both cases combining Young’s inequality and Hölder’s inequality one gets
\[ \| g \partial_x^m D_x^s f \|_{L^2} \lesssim \| g \|_{L^p} \| f \|_{L^2} \left\| \frac{\mathbf{1}_{\{|y| \geq \delta\}}}{|y|^{s+m+1}} \right\|_{L^r} \]
\[ \lesssim \| g \|_{L^p} \| f \|_{L^2} \]
where the index \( p \) satisfies \( \frac{1}{2} = \frac{1}{p} + \frac{1}{r} \), which clearly implies \( p \in [2, \infty) \), as was required.

Further, in the paper we will use extensively some results about commutator additionally to those presented in previous section. Next, we will study the smoothing effect for solutions of the dispersive generalized Benjamin-Ono equation (1.1) following Kato’s ideas [20].

3.1. Commutator Expansions. In this section we present several new main tools obtained by Ginibre and Velo [13], [14] which will be the cornerstone in the proof of Theorem B. They include commutator expansions together with their estimates. The basic problem is to handle the non-local operator \( D^s_x \) for non-integer \( s \) and in particular to obtain representations of its commutator with multiplication operators by functions that exhibit as much locality as possible.

Let \( a = 2\mu + 1 > 1 \), let \( n \) be a non-negative integer and \( h \) be a smooth function with suitable decay at infinity, for instance with \( h' \in C^\infty_0(\mathbb{R}) \).

We define the operator
\[ R_n(a) = [HD^s_x; h] - \frac{1}{2} (P_n(a) - HP_n(a)H), \]
\[ P_n(a) = a \sum_{0 \leq j \leq n} c_{2j+1} (-1)^j 4^{-j} D^{\mu-j} \left( h^{(2j+1)} D^{\mu-j} \right) \]
where
\[ c_1 = 1, \quad c_{2j+1} = \frac{1}{(2j+1)!} \prod_{0 < k < j} (a^2 - (2k + 1)^2) \quad \text{and} \quad H = -\mathcal{H}. \]
It was shown in [13] that the operator $R_n(a)$ can be represented in terms of anti-commutators\footnote{For any two operators $P$ and $Q$ we denote the anti-commutator by $[P; Q]_+ = PQ + QP$.} as follows

$$(3.22) \quad R_n(a) = \frac{1}{2} ([H; Q_n(a)]_+ + [D^{\alpha}; [H; h]]_+),$$

where the operator $Q_n(a)$ is represented in the Fourier space variables by the integral kernel

$$(3.23) \quad Q_n(a) \longrightarrow (2\pi)^{\frac{\alpha}{2}} \hat{h}(\xi - \xi')|\xi - \xi'|^{\frac{\alpha}{2}} 2aq_n(a, t).$$

Based on (3.22) and (3.23), Ginibre and Velo [14] obtain the following properties of boundedness and compactness of the operator $R_n(a)$.

**Proposition 3.24.** Let $n$ be a non-negative integer, $a \geq 1$, and $\sigma \geq 0$, be such that

$$(3.25) \quad 2n + 1 \leq a + 2\sigma \leq 2n + 3.$$ Then

(a) The operator $D^\sigma R_n(a)D^\sigma$ is bounded in $L^2$ with norm

$$(3.26) \quad \|D^\sigma R_n(a)D^\sigma f\|_{L^2} \leq C(2\pi)^{-1/2} \|D^{\alpha + 2\sigma} h\|_{L^2_1} \|f\|_{L^2}.$$ If $a \geq 2n + 1$, one can take $C = 1$.

(b) Assume in addition that

$$2n + 1 \leq a + 2\sigma < 2n + 3.$$ Then the operator $D^\sigma R_n(a)D^\sigma$ is compact in $L^2(\mathbb{R})$.

**Proof.** See Proposition 2.2 in [14].

In fact the Proposition 3.24 is a generalization of a previous result, where the derivatives of operator $R_n(a)$ are not considered (cf. Proposition 1 in [13]). The estimmative (3.26) yields the following identity of localization of derivatives.

**Lemma 3.27.** Assume $0 < \alpha < 1$. Let be $\varphi \in C^\infty(\mathbb{R})$ with $\varphi' \in C^\infty_0(\mathbb{R})$.

Then,

$$(3.28) \quad \int_{\mathbb{R}} \varphi f D^{\alpha + 1} \xi f \, dx = \left(\frac{\alpha + 2}{4}\right) \int_{\mathbb{R}} \left(\left|D^{\alpha + 1} f\right|^2 + \left|D^{\alpha + 1} H f\right|^2\right) \varphi' \, dx$$

$$+ \frac{1}{2} \int_{\mathbb{R}} f R_1(\alpha + 2) f \, dx.$$ 

**Proof.** The proof follows the ideas presented in Proposition 2.12 in [34].
4. The Linear Problem.

The aim of this section is to obtain Strichartz estimates associated to solutions of the IVP (1.1).

First, consider the linear problem

\[
\begin{cases}
\partial_t u - D_x^a \partial_x u = 0, & x, t \in \mathbb{R}, \ 0 < a < 1, \\
u(x, 0) = u_0(x),
\end{cases}
\]

whose solution is given by

\[
u(x, t) = S(t)u_0 = \left(e^{it|\xi|^{a+1}} \hat{u}_0\right) \hat{.}
\]

We begin studying estimates of the unitary group obtained in (4.2).

**Proposition 4.3.** Assume that \(0 < a < 1\). Let \(q, p\) satisfy \(2q + 1/p = 1/2\) with \(2 \leq p \leq \infty\).

Then

\[
\left\|D_x^a S(t)u_0\right\|_{L_t^q L_x^p} \lesssim \|u_0\|_{L_x^2}
\]

for all \(u_0 \in L^2(\mathbb{R})\).

**Proof.** The proof follows as an application on Theorem 2.1 in [28].

**Remark 4.5.** Notice that the condition in \(p\) implies \(q \in [4, \infty]\), which in one of the extremal cases \((p, q) = (\infty, 4)\) yields

\[
\left\|D_x^a S(t)u_0\right\|_{L_t^4 L_x^\infty} \lesssim \|u_0\|_{L_x^2}
\]

which shows the gain of \(\frac{a}{4}\) derivatives globally in time for solutions of (4.1).

**Lemma 4.6.** Assume that \(0 < a < 1\). Let \(\varphi_k\) be a \(C^\infty(\mathbb{R})\) function supported in the interval \(\left[2^{k-1}, 2^{k+1}\right]\) where \(k \in \mathbb{Z}^+\). Then, the function \(H_k^a\) defined as

\[
H_k^a(x) = \begin{cases}
2^k & \text{if } |x| \leq 1, \\
2^k |x|^{-\frac{1}{2}} & \text{if } 1 \leq |x| \leq c2^{k(a+1)}, \\
(1 + x^2)^{-1} & \text{if } |x| > c2^{k(a+1)}
\end{cases}
\]

satisfies

\[
\int_{-\infty}^{\infty} e^{it|\xi|^{a+1} + \lambda^2} \psi_k(\xi) d\xi \lesssim H_k^a(x)
\]

for \(|t| \leq 2\), where the constant \(c\) does not depends on \(t\) nor \(k\).

Moreover, we have that

\[
\sum_{l=-\infty}^{\infty} H_k^a(|l|) \lesssim 2^{k \left(\frac{a+1}{2}\right)}.
\]

**Proof.** The proof of estimate (4.7) is given in Proposition 2.6 [27] and it uses arguments of localization and the classical Van der Corput’s Lemma. Meanwhile, (4.8) follows exactly that of Lemma 2.6 in [34].
Theorem 4.9. Assume $0 < \alpha < 1$. Let $s > \frac{1}{2}$. Then,

$$
\|S(t)u_0\|_{L^\infty_t L^\infty_x([-1,1])} \lesssim \left( \sum_{j=-\infty}^{\infty} \sup_{|t| \leq T} \sup_{j \leq x < j+1} |S(t)u_0(x)|^2 \right)^{1/2} \lesssim \|u_0\|_{H^s}
$$

for any $u_0 \in H^s(\mathbb{R})$.

Proof. See Theorem 2.7 in [27]. □

Next, we recall a maximal function estimate proved by Kenig, Ponce and Vega [27].

Corollary 4.10. Assume that $0 < \alpha < 1$. Then, for any $s > \frac{1}{2}$ and any $\eta > \frac{1}{4}$

$$
\left( \sum_{j=-\infty}^{\infty} \sup_{|t| \leq T} \sup_{j \leq x < j+1} |S(t)v_0|^2 \right)^{1/2} \lesssim (1 + T)^{\eta} \|v_0\|_{H^s}.
$$

Proof. See Corollary 2.8 in [27]. □

4.1. The Nonlinear Problem. This section is devoted to study general properties of solutions of the non-linear problem

\begin{equation}
\begin{cases}
\partial_t u - D_x^{\alpha+1} \partial_x u + uu_x = 0, & x, t \in \mathbb{R}, 0 < \alpha < 1, \\
u(x,0) = u_0(x).
\end{cases}
\end{equation}

We begin this section stating the following local existence theorem proved by Kato [21] and Saut,Temam [40].

Theorem 4.12. (1) For any $u_0 \in H^s(\mathbb{R})$ with $s > \frac{3}{2}$ there exists a unique solution $u$ to (4.11) in the class $C([-T,T]:H^s(\mathbb{R}))$ with $T = T(\|u_0\|_{H^s}) > 0$.

(2) For any $T' < T$ there exists a neighborhood $V$ of $u_0$ in $H^s(\mathbb{R})$ such that the map $\tilde{u}_0 \mapsto \tilde{u}(t)$ from $V$ into $C([-T,T]:H^s(\mathbb{R}))$ is continuous.

(3) If $u_0 \in H^{s'}(\mathbb{R})$ with $s' > s$, then the time of existence $T$ can be taken to depend only on $\|u_0\|_{H^{s'}}$.

Our first goal will be obtain some energy estimates satisfied by smooth solutions of the IVP (4.11).

We firstly present a result that arises as a consequence of commutator estimates.

Lemma 4.13. Suppose that $0 < \alpha < 1$. Let $u \in C([-T,T]:H^\infty(\mathbb{R}))$ be a smooth solution of (4.11). If $s > 0$ is given, then

\begin{equation}
\|u\|_{L^\infty_t H^s} \lesssim \|u_0\|_{H^s} e^{\|\partial_x u\|_{L^1_t L^\infty_x}}.
\end{equation}

Proof. Let $s > 0$. By a standard energy estimate argument we have that

$$
\frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}} (f_x^2 u)^2 \, dx + \int_{\mathbb{R}} \left[ f_x^2 u \right] \partial_x u f_x^2 u \, dx + \int_{\mathbb{R}} u f_x^2 u f_x^2 u \, dx = 0.
$$

Hence integration by parts, Gronwall’s inequality and commutator estimate (3.11) lead to (4.14). □

Remark 4.15. In view of the energy estimate (4.14), the key point to obtain a priori estimates in $H^s_x(\mathbb{R})$ is to control $\|\partial_x u\|_{L^1_t L^\infty_x}$ at the $H^s_x(\mathbb{R})$ level.
Additionally to this estimate, we will present the smoothing effect provided by solutions of dispersive generalized Benjamin-Ono equation. In fact, the smoothing effect was first observed by Kato in the context of the Korteweg-de Vries equation (see [20]). Following Kato’s approach joint with the commutator expansions, we present a result proved by Kenig-Ponce-Vega [27] (see Lemma 5.1).

**Proposition 4.16.** Let \( \psi \) denote a non-decreasing smooth function such that \( \text{supp} \psi' \subset (-1,2) \) and \( \psi'\big|_{[0,1]} = 1 \). For \( j \in \mathbb{Z} \), we define \( \psi_j(\cdot) = \psi(\cdot - j) \).

Let \( u \in C([0,T]: H^s(\mathbb{R})) \) be a real smooth solution of (1.1) with \( 0 < \alpha < 1 \). Assume also, that \( s \geq 0 \) and \( r > 1/2 \).

Then,

\[
\left( \int_0^T \int_{\mathbb{R}} \left( \left| D_x^{\alpha+1} u(x,t) \right|^2 + \left| D_x^{\alpha+1} \mathcal{H} u(x,t) \right|^2 \right) \psi_j(x) \, dx \, dt \right)^{1/2} 
\leq \left( 1 + T + \| \partial_x u \|_{L^1_t L^\infty_x} + T \| u \|_{L^\infty_t H^s_x} \right)^{1/2} \| u \|_{L^\infty_t H^s_x}.
\]

**Remark 4.18.** Using a similar argument as the employed in the proof of Proposition 4.16 is possible obtain the smoothing effect provided by DGBO, but instead localized over any interval symmetric on the real line. This can be achieved replacing the function \( \psi_j \) by \( \psi_R \), where \( R > 0 \) and \( \psi_R \) denotes a nondecreasing smooth function such that \( \psi_R' \) is supported in \((-2R,2R)\) with \( \psi_R' = 1 \) on \([-R,R]\).

In addition to the smoothing effect presented above, we will need the following localized version of the \( H^s(\mathbb{R}) \)-norm. For this propose we will consider a cutoff function \( \psi \), with the same characteristics that in Proposition 4.16.

**Proposition 4.19.** Let \( s \geq 0 \). Then, for any \( f \in H^s(\mathbb{R}) \)

\[
\| f \|_{H^s(\mathbb{R})} \sim \left( \sum_{j=-\infty}^{\infty} \| f \psi_j' \|_{H^s(\mathbb{R})}^2 \right)^{1/2}.
\]

Hence our first goal in establishing the Local well-posedness of (4.11), will start off in obtain Strichartz estimates associated to solutions of

\[
\partial_t u - D_x^{1+a} \partial_x u = F.
\]

**Proposition 4.21.** Assume that \( 0 < \alpha < 1 \), \( T > 0 \) and \( \sigma \in [0,1] \). Let \( u \) be a smooth solution to (4.20) defined on the time interval \([0,T]\). Then there exist \( 0 \leq \mu_1, \mu_2 < 1/2 \) such that

\[
\| \partial_x u \|_{L^2_t L^\infty_x} \lesssim T^{\mu_1} \| J^{1-\frac{\alpha}{2} + \frac{T}{4} + \epsilon} u \|_{L^\infty_t H^\sigma_x} + T^{\mu_2} \| J^{1-\frac{\alpha}{2} - \frac{T}{4} + \epsilon} F \|_{L^2_t L^\infty_x}
\]

for any \( \epsilon > 0 \).

**Remark 4.23.** The optimal choice in the parameters present in the estimate (4.22) corresponds to \( \sigma = \frac{1-\alpha}{2} \). Indeed, as is pointed out by Kenig and Koenig in the case of the Benjamin-Ono equation (case \( \alpha = 0 \)) (see Remarks in Proposition 2.8 [24]) given a linear estimate of the form

\[
\| \partial_x u \|_{L^2_t L^\infty_x} \lesssim T^{\mu_1} \| J^\alpha u \|_{L^\infty_t H^\sigma_x} + T^{\mu_2} \| J^\beta F \|_{L^2_t L^\infty_x}
\]
the idea is to apply the smoothing effect (4.17) and absorb as many as derivatives as possible of the function $F$. Concerning to our case, the approach requires the choice $a = b + \frac{1-\alpha}{2}$; this particular choice, $\sigma = \frac{1-\alpha}{2}$, in the estimate (4.22) provides the regularity $s > 9/8 - 3\alpha/8$ in Theorem A.

**Proof.** Let $f = \sum_k f_k$ denote the Littlewood-Paley decomposition of a function $f$. More precisely we choose functions $\eta, \chi \in C^\infty(\mathbb{R})$ with $\text{supp}(\eta) \subseteq \{\xi : 1/2 < |\xi| < 2\}$ and $\text{supp}(\chi) \subseteq \{\xi : |\xi| < 2\}$, such that

$$\sum_{k=1}^\infty \eta(\xi/2^k) + \chi(\xi) = 1$$

and $f_k = P_k(f)$, where $\widehat{(P_0 f)}(\xi) = \chi(\xi) \hat{f}(\xi)$ and $\widehat{(P_k f)}(\xi) = \eta(\xi/2^k) \hat{f}(\xi)$ for all $k \geq 1$.

Fix $\varepsilon > 0$. Let $p > \frac{1}{\varepsilon}$. By Sobolev embedding and Littlewood-Paley Theorem it follows that

$$\|f\|_{L^p} \lesssim \|f\|_{L^p_x} \lesssim \left( \sum_{k=0}^\infty \|f^k_{P}\|_{L^p_x}^{1/2} \right)^{1/2} \lesssim \sum_{k=0}^\infty \|f^k_{P}\|_{L^p_x}^{1/2} \lesssim \left( \sum_{k=0}^\infty \|f^k_{P}\|_{L^p_x}^2 \right)^{1/2}.$$ 

Therefore, to obtain (4.22) it enough to prove that for $p > 2$

$$\|\partial_x u_k\|_{L^q_t L^p_x} \lesssim \left\| D_x^{-\frac{\alpha}{2} + \frac{\alpha}{q} + \frac{\alpha}{p}} u_k \right\|_{L^q_t L^p_x} + \left\| D_x^{-\frac{\alpha}{2} - \frac{\alpha}{q} + \frac{\alpha}{p}} F_k \right\|_{L^q_t L^p_x}, \quad k \geq 1.$$ 

The estimate for the case $k = 0$ follows using H"older’s inequality and (4.4). For such reason we fix $k \geq 1$, and at these level of frequencies we have that

$$\partial_t u_k - D_x^{\alpha+1} \partial_x u_k = F_k.$$ 

Consider a partition of the interval $[0, T] = \bigcup_{j \in J} I_j$ where $I_j = [a_j, b_j]$, and $T = b_j$ for some $j$. Indeed, we choose a quantity $\sim 2^{k\alpha} T^{1-\mu}$ of intervals, with length $|I_j| \sim 2^{-k\alpha} T^{\mu}$, where $\mu$ is a positive number to be fixed.

Let $q$ be such that

$$\frac{2}{q} + \frac{1}{p} = \frac{1}{2}.$$

Using that $u$ solves the integral equation

$$u(t) = S(t)u_0 + \int_0^t S(t-t') F(t') \, dt',$$
we deduce that
\[
\|\partial_x u_k\|_{L^2_t L^r_x} \leq \left( \sum_{j \in J} \|\partial_x u_k\|_{L^2_t L^r_x}^{2} \right)^{1/2}
\]
\[
\leq \left( T^{\mu - 2 + \frac{1}{r}} \right) \left( \sum_{j \in J} \|\partial_x u_k\|_{L^r_x}^{2} \right)^{1/2}
\]
\[
\leq \left( T^{\mu - 2 + \frac{1}{r}} \right) \left( \sum_{j \in J} \|\partial_x u_k\|_{L^r_x}^{2} \right)^{1/2}
\]
\[
\leq \left( T^{\mu - 2 + \frac{1}{r}} \right) \left( \sum_{j \in J} \|S(t - a_j)\partial_x u_k(a_j)\|_{L^r_x}^{2} + \left\| \int_{a_j}^t S(t - s)\partial_x F_k(s)ds \right\|_{L^r_x}^{2} \right)^{1/2}.
\]
In this sense, it follows from (4.4) that
\[
\|\partial_x u_k\|_{L^2_t L^r_x} \leq \left( T^{\mu - 2 + \frac{1}{r}} \right) \left( \sum_{j \in J} \left\{ \left( \sum_{j \in J} \|D_x^{\frac{1}{q}} \partial_x u_k\|_{L^r_x}^{2} \right)^{1/2} + \left( \sum_{j \in J} \int_{a_j}^t \left\| D_x^{\frac{1}{q}} \partial_x F_k(t) \right\|_{L^r_x}^{2} dt \right)^{1/2} \right) \right)
\]
\[
\leq \left( T^{\mu - 2 + \frac{1}{r}} \right) \left( T^{1 - \mu - 2 + \frac{1}{r}} \right) \|D_x^{\frac{1}{q}} \partial_x u_k\|_{L^r_x} + \left( \sum_{j \in J} \int_{a_j}^t \left\| D_x^{\frac{1}{q}} \partial_x F_k(t) \right\|_{L^r_x}^{2} dt \right)^{1/2}
\]
\[
\leq T^{1/2 - \mu/r} \|D_x^{\frac{1}{q} + \frac{r}{q}} \partial_x u_k\|_{L^r_x} + T^{\mu(1 - 1/q)} \|D_x^{\frac{1}{q} + \frac{r}{q} - \sigma} \partial_x F_k\|_{L^r_x}.
\]
Since, \(1 - \frac{\alpha q}{q} + \frac{\alpha}{q} = 1 - \frac{\alpha q}{q} + \frac{\alpha q}{2p} + \frac{\alpha q}{2p} - \sigma = 1 - \frac{\alpha q}{q} - \frac{3\alpha q}{4} + \frac{\alpha q}{2p}\). We recall that \(\varepsilon > \frac{1}{p}, \sigma \in [0, 1]\) and \(\alpha \in (0, 1)\), then \(\varepsilon + \frac{\alpha - \sigma}{2p} > \frac{\alpha q + \sigma q + \varepsilon}{2p} > 0\). Next, we choose \(\mu_1 = \frac{1}{2} - \frac{\mu}{q}, \mu_2 = \mu(1 \frac{1}{q})\) with the particular choice \(\mu = 1/2\).

Gathering the inequalities above follows the proposition.

Now we turn our attention to the proof of Theorem A. Our starting point will be the energy estimate (4.13), that as was remarked above, the key point is to establish a priori control of \(\|\partial_x u\|_{L^1_t L^r_x}\).
5. Proof of Theorem A

5.1. A priori estimates. First notice that by scaling, it is enough to deal with small initial data in the $H^s$-norm. Indeed, if $u(x,t)$ is a solution of (1.1) defined on a time interval $[0,T]$, for some positive time $T$, then for all $\lambda > 0$, $u(\lambda x, \lambda^2 t) = \lambda^{1+\alpha} u(\lambda x, \lambda^{2+\alpha} t)$ is also solution with initial data $u_{0,\lambda}(x) = \lambda^{1+\alpha} u_0(\lambda x)$, and time interval $[0,T/\lambda^{2+\alpha}]$.

For any $\delta > 0$, we define $B_\delta(0)$ as the ball with center at the origin in $H^s(\mathbb{R})$ and radius $\delta$.

Since
\[
\|u_{0,\lambda}\|_{L^2_x} = \lambda^{1/2}\|u_0\|_{L^2_x} \quad \text{and} \quad \|D_x^s u_{0,\lambda}\|_{L^2_x} = \lambda^{1/2+s}\|D_x^s u_0\|_{L^2_x},
\]
then
\[
\|u_{0,\lambda}\|_{H^s_x} \lesssim \lambda^{1/2}(1 + \lambda^s)\|u_0\|_{H^s_x},
\]
so we can force $u_\lambda(\cdot,0)$ to belong to the ball $B_\delta(0)$ by choosing the parameter $\lambda$ with the condition
\[
\lambda \sim \min \left\{ \delta^{1/2}, \|u_0\|_{H^s_x}^{1/2+s}, 1 \right\}.
\]

Thus, the existence and uniqueness of a solution to (1.1) on the time interval $[0,1]$ for small initial data $\|u_0\|_{H^s_x}$ will ensure the existence and uniqueness of a solution to (1.1) for arbitrary large initial data on a time interval $[0,T]$ with
\[
T \sim \min \left\{ 1, \|u_0\|_{H^s_x}^{2(1+\alpha)} \right\}.
\]

Thus, without loss of generality we will assume that $T \leq 1$, and that
\[
\Lambda := \|u_0\|_{L^2_x} + \|D_x^s u_0\|_{L^2_x} \leq \delta,
\]
where $\delta$ is a small positive number to be fixed later.

We fix $s$ such that $s(\alpha) = \frac{9}{8} - \frac{3\alpha}{8} < s < \frac{3}{2} - \frac{\alpha}{2}$ and set $\epsilon = s - s(\alpha) > 0$.

Next, taking $\sigma = \frac{1-2s}{2} > 0$, $F = -u\hat{\sigma}_x u$ in (4.22) together with (4.14) yields
\[
(5.1) \quad \frac{\|\hat{\sigma}_x u\|_{L^2_t L^\infty_x}}{L^2_t L^\infty_x} \\
\lesssim T^{\mu_1} \|f u\|_{L^2_t L^\infty_x} + T^{\mu_2} \left\| J^{1/2-\frac{s}{2}+\epsilon} (u\hat{\sigma}_x u) \right\|_{L^2_t L^2_x} \\
\lesssim T^{\mu_1} \left( \|u_0\|_{L^2_x} + \|D_x^s u\|_{L^\infty_t L^2_x} \right) + T^{\mu_2} \left( \|u\hat{\sigma}_x u\|_{L^2_t L^2_x} + \left\| D_x^{s+\frac{1-2s}{4}} (u\hat{\sigma}_x u) \right\|_{L^2_t L^2_x} \right) \\
\lesssim \Lambda + \Lambda e^{|\hat{\sigma}_x u|_{L^\infty_t L^2_x}} + \|\hat{\sigma}_x u\|_{L^2_t L^\infty_x} \|u_0\|_{L^2_x} + \left\| D_x^{s+\frac{1-2s}{4}} (u\hat{\sigma}_x u) \right\|_{L^2_t L^2_x} \\
\lesssim \Lambda + \Lambda e^{|\hat{\sigma}_x u|_{L^\infty_t L^2_x}} + \left\| D_x^{s+\frac{1-2s}{4}} (u\hat{\sigma}_x u) \right\|_{L^2_t L^2_x}.
In summary, after gathering the estimates (5.3) joint with the energy estimate (4.14) as follows

\[
\left\| D_x^{s+\frac{\alpha-1}{2}} (u \partial_x u) \right\|_{L^2_t L^2_x} \lesssim \left\| u D_x^{s+\frac{\alpha-1}{2}} \partial_x u \right\|_{L^2_t L^2_x} + \left\| \partial_x u(t) \right\|_{L^\infty_x} \left\| D_x^{s+\frac{\alpha-1}{2}} u(t) \right\|_{L^2_t L^2_x} \\
\lesssim \left\| u D_x^{s+\frac{\alpha-1}{2}} \partial_x u \right\|_{L^2_t L^2_x} + \left\| \partial_x u \right\|_{L^1_t L^\infty_x} \left\| D_x^{s+\frac{\alpha-1}{2}} u \right\|_{L^2_t L^2_x} \\
\lesssim \left\| u D_x^{s+\frac{\alpha-1}{2}} \partial_x u \right\|_{L^2_t L^2_x} + \Lambda \left\| \partial_x u \right\|_{L^1_t L^\infty_x} \left\| u \right\|_{L^2_t H_x^s} \\
\lesssim \left\| u D_x^{s+\frac{\alpha-1}{2}} \partial_x u \right\|_{L^2_t L^2_x} + \Lambda \left\| \partial_x u \right\|_{L^1_t L^\infty_x} \left\| u \right\|_{L^2_t H_x^s} .
\]

To handle the first term in the right hand side above, we incorporate Kato’s smoothing effect estimate obtained in (4.17) in the following way

\[
\left\| u D_x^{s+\frac{\alpha-1}{2}} \partial_x u \right\|_{L^2_t L^2_x} \lesssim \left( \sum_{j=-\infty}^{\infty} \int_0^T \left\| u(t) \right\|_{L^\infty_x} \left( \left\| D_x^{s+\frac{\alpha-1}{2}} \mathcal{H} u(t) \right\|_{L^2_t L^2_x} \right)^2 \right)^{1/2} \\
\lesssim \left( \sum_{j=-\infty}^{\infty} \left\| u \right\|_{L_t^\infty L_x^\infty}^2 \right)^{1/2} \sup_{j \in \mathbb{Z}} \left\| D_x^{s+\frac{\alpha-1}{2}} \mathcal{H} u \right\|_{L^1_t L^2_x} \\
\lesssim \left( \sum_{j=-\infty}^{\infty} \left\| u \right\|_{L_t^\infty L_x^\infty}^2 \right)^{1/2} \left( 1 + \Lambda + \left\| \partial_x u \right\|_{L^1_t L^\infty_x} \right) \Lambda e^{c \left\| \partial_x u \right\|_{L^1_t L^\infty_x}} \\
\lesssim \left( \sum_{j=-\infty}^{\infty} \left\| u \right\|_{L_t^\infty L_x^\infty}^2 \right)^{1/2} \left( 1 + \Lambda \right) \Lambda e^{c \left\| \partial_x u \right\|_{L^1_t L^\infty_x}}.
\]

In summary, after gathering the estimates (5.1)-(5.3) yields

\[
\left\| \partial_x u \right\|_{L^1_t L^\infty_x} \lesssim \Lambda (1 + \Lambda) e^{c \left\| \partial_x u \right\|_{L^1_t L^\infty_x}} \left( \sum_{j=-\infty}^{\infty} \left\| u \right\|_{L_t^\infty L_x^\infty}^2 \right)^{1/2} \\
+ \Lambda + \Lambda e^{c \left\| \partial_x u \right\|_{L^1_t L^\infty_x}}.
\]

Since \( u \) is a solution to (4.11), then by Duhamel’s formula it follows that

\[
u(t) = S(t) u_0 - \int_0^t S(t-s) (u \partial_x u) (s) \, ds
\]

where \( S(t) = e^{tD_x^{s+1} \partial_x} \).
Now, we fix $\eta > 0$ such that $\eta < \frac{1 + \alpha}{8}$; this choice implies that $\eta + \frac{1}{2} < s + \frac{\alpha - 1}{2}$. Hence, Sobolev’s embedding, Hölder’s inequality and Corollary 4.10 produce

\[(5.5) \quad \left( \sum_{j=-\infty}^{\infty} \| u \|_{L^2_T L^\infty_x \{ j \}}^2 \right)^{1/2} \leq \left( \sum_{j=-\infty}^{\infty} \left\| \sum_{j=-\infty}^{\infty} \int_0^t S(t-s)(u\partial_x u)(s) \, ds \right\|_{L^2_T L^\infty_x \{ j \}}^2 \right)^{1/2} \ \leq (1 + T)\Lambda + (1 + T) \| u\partial_x u \|_{L^1_T L^2_x} \ \leq \Lambda + \Lambda \| \partial_x u \|_{L^2_T L^\infty_x} \ + \| u\partial_x u \|_{L^1_T L^2_x} + \| u\partial_x u \|_{L^1_T L^2_x} \ .
\]

Employing an argument similar to the one applied in (5.2) and (5.4) it is possible to bound the last term in the right hand side as follows

\[(5.6) \quad \| D_x^{\eta + 1/2}(u\partial_x u) \|_{L^2_T L^2_x} \leq \left( \sum_{j=-\infty}^{\infty} \| u \|_{L^2_T L^\infty_x \{ j \}}^2 \right)^{1/2} \Lambda(\Lambda + 1) e^{c\| \partial_x u \|_{L^2_T L^2_x}} + \Lambda e^{c\| \partial_x u \|_{L^2_T L^2_x}} .
\]

Next, we define

\[\phi(T) = \int_0^T \| \partial_x u(s) \|_{L^2_x}^2 \, ds + \left( \sum_{j=-\infty}^{\infty} \| u \|_{L^2_T L^\infty_x \{ j \}}^2 \right)^{1/2} \]

which is a continuous, non-decreasing function of $T$.

From obtained in (5.4), (5.5) and (5.6) follows that

\[\phi(T) \leq \Lambda(\Lambda + 1)\phi(T) e^{c\phi(T)} + \Lambda e^{c\phi(T)} \phi(T) + \Lambda e^{c\phi(T)} + \Lambda + \Lambda \phi(T) .
\]

Now, if we suppose that $\Lambda \leq \delta \leq 1$ we obtain

\[\phi(T) \leq c\Lambda + c\Lambda e^{c\phi(T)}
\]

for some constant $c > 0$.

To complete the proof we will show that there exists $\delta > 0$, such that $\Lambda \leq \delta$, then $\phi(1) \leq A$, for some constant $A > 0$.

To do this, we define the function

\[(5.7) \quad \Psi(x, y) = x - cy - cy e^{cx} .
\]

First notice that $\Psi(0, 0) = 0$ and $\partial_x \Psi(0, 0) = 1$. Then the Implicit Function Theorem asserts that there exists $\delta > 0$, and a smooth function $\zeta(y)$ such that $\zeta(0) = 0$, and $\Psi(\zeta(y), y) = 0$ for $|y| \leq \delta$. 

Notice that the condition $\Psi(\xi(y), y) = 0$ implies that $\xi(y) > 0$ for $y > 0$. Moreover, since $\partial_x \Psi(0, 0) = 1$, then the function $\Psi(\cdot, y)$ is increasing close to $\xi(y)$, whenever $\delta$ is chosen sufficiently small.

Let us suppose that $\Lambda \leq \delta$, and set $\lambda = \xi(\Lambda)$. Then, combining interpolation and Proposition 4.19 we obtain

$$\phi(0) = \left( \sum_{j=-\infty}^{\infty} \sup_{x \in [j, j+1]} |u(x, 0)| \right)^{1/2} \lesssim \|u_0\|_{H^p(\mathbb{R})} \leq c_1 \|u_0\|_{L^2} + c_1 \|D_{\xi}^{1/2} u_0\|_{L^2}$$

where we take $c > c_1$.

Therefore

$$\phi(0) \leq c_1 \Lambda < c \Lambda + c \Lambda e^{\xi(\Lambda)} = \lambda.$$ 

Suppose that $\phi(T) > \lambda$ for some $T \in (0, 1)$ and define

$$T_0 = \inf \{ T \in (0, 1) \mid \phi(T) > \lambda \}.$$ 

Hence, $T_0 > 0$ and $\phi(T_0) = \lambda$, besides, there exists a decreasing sequence $\{T_n\}_{n \geq 1}$ converging to $T_0$ such that $\phi(T_n) > \lambda$. In addition, notice that (5.7) implies $\Psi(\phi(T), \Lambda)$ for all $T \in [0, 1]$.

Since the function $\Psi(\cdot, \Lambda)$ is increasing near $\lambda$ it implies $\Psi(\phi(T_n), \Lambda) > \Psi(\phi(T_0), \Lambda) = \Psi(\lambda, \Lambda) = \Psi(\xi(\Lambda), \Lambda)$ for $n$ sufficiently large.

This is a contradiction with the fact that $\phi(T) > \lambda$. So we conclude $\phi(T) \leq A$ for all $T \in (0, 1)$, as was claimed. Thus, $\phi(1) \leq A$.

In conclusion we have proved that

$$\phi(T) = \int_0^T |\partial_x u(s)|_{L^2}^2 \, ds + \left( \sum_{j=-\infty}^{\infty} \|u_j\|_{L^2}^2 \right)^{1/2} \lesssim \|u_0\|_{H^2}, \quad \forall T \in [0, 1].$$

### 5.2. Uniqueness

This subsection is devoted to prove the uniqueness of solutions to the IVP (1.1).

Let $u(t)$ and $v(t)$ be two solutions to the equation in (1.1) with initial conditions $u(0) = u_0$ and $v(0) = v_0$, respectively.

Set

$$K = \max \left\{ \|\partial_x u\|_{L^1_x L^2_t}, \|\partial_x v\|_{L^1_x L^2_t} \right\}.$$ 

We define $w = u - v$. Then $w$ satisfies the IVP

$$\begin{cases} 
\partial_t w - D_{\xi}^{\alpha+1} \partial_x w + w \partial_x v + u \partial_x w = 0, \quad x, t \in \mathbb{R}, \ 0 < \alpha < 1, \\
|w|_{t=0} = u_0 - v_0.
\end{cases}$$

After multiply the equation in (5.9) by $w$ and integrate in the $x-$ variable we obtain

$$\frac{d}{dt} \|w(t)\|_{L^2_x}^2 \lesssim \|w(t)\|_{L^2_x}^2 \left( \|\partial_x u(t)\|_{L^2_t} + \|\partial_x v(t)\|_{L^2_t} \right)$$

which holds for all $t \in [0, T]$. Applying Gronwall’s inequality in (5.10) yields

$$\|w(t)\|_{L^2_x}^2 \lesssim \|u_0 - v_0\|_{L^2_x}^2 e^{\left( \|\partial_x u\|_{L^1_x L^2_t} + \|\partial_x v\|_{L^1_x L^2_t} \right)}$$

$$\lesssim e^{cK} \|u_0 - v_0\|_{L^2_x}^2$$
which clearly proves the uniqueness.

Remark 5.12. In fact, the estimate (5.11) besides granting the uniqueness in the $L_x^2$-space, it also provides the $L^2$—Lipschitz bound of the flow.

5.3. Existence of Solutions. To establish the existence of local solutions of (1.1) we will follow the approach of Bona and Smith [5] (Section 4). So, we will recall some results about approximation of solutions.

Let $\rho \in S(\mathbb{R})$ such that

(i) $\int_\mathbb{R} \rho(x) \, dx = 1$,

(ii) $\int_\mathbb{R} x^k \rho(x) \, dx = 0$ for $k \in \mathbb{Z}^+$ with $0 \leq k \leq \lfloor s \rfloor + 1$.

Let $\rho_\epsilon(x) = e^{-1} \rho(e^{-1} x)$ for any $\epsilon > 0$.

Lemma 5.13. Let $q \geq 0$, $\phi \in H^q(\mathbb{R})$ and for $\epsilon > 0$, $\phi_\epsilon = \rho_\epsilon \ast \phi$. Then,

\[
\|\phi_\epsilon\|_{H^{q+r}} \lesssim e^{-r} \|\phi\|_{H^q} \quad \text{for all} \quad r \geq 0
\]

and

\[
\|\phi - \phi_\epsilon\|_{H^{q-r}} = o(\epsilon^q) \quad \text{as} \quad \epsilon \to 0, \quad \text{for all} \quad p \in [0, q].
\]

Proof. See section 4 in [5].

We will consider $u_0 \in H^s(\mathbb{R})$. As was done previously, we can always assume that $\|H_0\|_{H^s} < \delta$ for some $\delta > 0$. Then, as was proved in section 5.1, it follows that for $\epsilon > 0$, the solution $u_\epsilon$ is defined in the interval $[0, 1]$ and satisfies the a priori estimate (5.8).

So, in this direction we fix $u_0 \in H^s(\mathbb{R})$ with $s > 9/8 - 3 \alpha/8$, $0 < \alpha < 1$ and the regularized initial data, this is $u_{0,\epsilon} = \rho_\epsilon \ast u_0$.

Since $u_{0,\epsilon} \in H^\infty(\mathbb{R})$, we can warranty by means of Theorem 4.12, that for any $\epsilon > 0$, there exists a unique solution $u_\epsilon \in C(\mathbb{R}; H^\infty(\mathbb{R}))$ satisfying the condition $u_\epsilon(x, 0) = u_{0,\epsilon}(x)$, $x \in \mathbb{R}$. The following lemmas summarize some properties associated to the smooth solutions.

Lemma 5.16. Let $\epsilon > 0$. Then,

\[
\|u_{0,\epsilon}\|_{H^s_x} \lesssim \|u_0\|_{H^s_x},
\]

\[
\|D_x^\alpha \partial_x u_\epsilon\|_{L^2_x L^\infty_t} \lesssim \|u_{0,\epsilon}\|_{H^s_x} \lesssim e^{-s} \|u_0\|_{H^s_x}.
\]

\[
\|D_x^{\alpha + \frac{\alpha - 1}{2}} \partial_x u_\epsilon\|_{L^2_x L^\infty_t} \lesssim \|u_{0,\epsilon}\|_{H^{s+\frac{\alpha - 1}{2}}_x} \lesssim e^{-s} \|u_0\|_{H^s_x}.
\]

Proof. By the sake of brevity we will omit the proof of these estimates. In fact, these one’s are obtained by similar arguments as the ones used in section 5.1 and the estimate (5.14).

These estimates are the basis to prove the convergence of the smooth solutions when $\epsilon, \epsilon' \to 0$.

Since $u_\epsilon$ satisfies

\[
\begin{cases}
\partial_t u_\epsilon - D_x^{\alpha + 1} \partial_x u_\epsilon + u_\epsilon \partial_x u_\epsilon = 0, \quad x, t \in \mathbb{R}, \quad 0 < \alpha < 1, \\
u_{0,\epsilon}|_{t=0} = u_{0,\epsilon}.
\end{cases}
\]
Let $\varepsilon > \varepsilon' > 0$. We set $v = u_\varepsilon - u_{\varepsilon'}$. Then $v$ satisfies the IVP

$$\begin{cases}
\partial_t v - D_x^{1+\varepsilon} \partial_x v + v \partial_x u_\varepsilon + u_{\varepsilon'} \partial_x v = 0, \quad x, t \in \mathbb{R}, \\
(\partial_t v)(x, 0) = u_{0,\varepsilon}(x) - u_{0,\varepsilon'}(x).
\end{cases} \tag{5.21}$$

We will prove that the sequence $\{u_\varepsilon\}_{\varepsilon > 0}$ is a Cauchy sequence in the space $C([0, 1] : H^s(\mathbb{R}))$. In this direction, we notice that arguing as in section 5.2 we obtain by Gronwall’s inequality, and (5.15) with $q = s$, $p = 0$

$$\|v\|_{L_t^p L_x^q} \leq e^{sK} \|u_{0,\varepsilon} - u_{0,\varepsilon'}\|_{L_t^p L_x^q} = o(\varepsilon^s). \tag{5.22}$$

Moreover, combining (5.22) and Lemma 3.15 it follows that

$$\|v\|_{L_t^p H_x^s} \lesssim \|v\|_{L_t^p H_x^s} \|v\|_{L_t^p L_x^q} \leq o(\varepsilon^s - r) \quad \text{for all} \quad 0 \leq r < s. \tag{5.23}$$

To prove that $\{u_\varepsilon\}_{\varepsilon > 0}$ is a Cauchy sequence in $C([0, 1] : H^s(\mathbb{R}))$, it remains to show the following:

**Proposition 5.24.** Assume that $0 < \alpha < 1$ and $\frac{9}{8} - \frac{3\varepsilon}{8} < s < \frac{3}{2} - \frac{\varepsilon}{2}$. Let $v$ be the solution of (5.21). Then, there exists a time $T_1 = T_1(\|u_0\|_{H^s})$, with $0 < T_1 < T \leq 1$ such that

$$\|v\|_{L_t^p H_x^s} \leq \max \left\{ \|v\|_{L_t^p H_x^s} , \|\partial_x v\|_{L_t^p L_x^q} \right\}_{\varepsilon \to 0} \to 0.$$ 

**Proof:** A standard energy estimate shows that

$$\frac{1}{2} \frac{d}{dt} \|v\|_{H_x^s}^2 + \int_{\mathbb{R}} J^s (v \partial_x u_\varepsilon) J^s v \, dx + \int_{\mathbb{R}} J^s (u_{\varepsilon'} \partial_x v) J^s v \, dx = 0. \tag{5.25}$$

The second term in the left side can be handled by using Lemma 3.3

$$\int_{\mathbb{R}} J^s (v \partial_x u_\varepsilon) J^s v \, dx \lesssim \left( \|v\|_{L_t^p H_x^s} \|\partial_x u_\varepsilon\|_{L_x^q} + \|D_x^s v\|_{L_x^p} \|\partial_x u_\varepsilon\|_{L_x^p} \right. \tag{5.26}
\left. + \|D_x^s \partial_x u_\varepsilon\|_{L_x^q} \|v\|_{L_x^q} \right) \|v\|_{H_x^s}.$$

On the other hand, the estimate (3.11) applied in the third term in the left hand side of (5.25) yields

$$\int_{\mathbb{R}} J^s (u_{\varepsilon'} \partial_x v) J^s v \, dx \lesssim \|\partial_x u_{\varepsilon'}\|_{L_x^q} \|v\|_{H_x^s}^2 + \|u_{\varepsilon'}\|_{H_x^s} \|\partial_x v\|_{L_x^q} \|v\|_{H_x^s}. \tag{5.27}$$

Thus, after collecting the estimate above

$$\frac{d}{dt} \|v\|_{H_x^s} \lesssim \left( \|\partial_x u_\varepsilon\|_{L_x^q} + \|\partial_x u_\varepsilon\|_{L_x^q} \right) \|v\|_{H_x^s} \tag{5.28}$$
\hfill \frac{1}{2} \|\partial_x u_\varepsilon\|_{L_x^q} + \|u_{\varepsilon'}\|_{H_x^s} \|\partial_x v\|_{L_x^q}$

Hence, by Gronwall’s inequality and (5.8) one gets that

$$\|v\|_{L_t^p H_x^s} \lesssim e^{T^{1/2} \left( \|\partial_x u_\varepsilon\|_{L_x^q} + \|\partial_x u_{\varepsilon'}\|_{L_x^q} \right)} \left( T^{1/2} \|u_0\|_{H_x^s} \|\partial_x v\|_{L_t^p L_x^q} + f(\varepsilon, \varepsilon') \right).$$
where
\[ f(e, e') = T^{1/2} \|v\|_{L^p_T L^2_x} D_x^s \partial_x u_e \|_{L^2_T L^2_x} + \|u_{0,e} - u_{0,e'}\|_{H^s_x} \rightarrow 0. \]

To finish with the estimates it is necessary to establish control on the term involving the partial derivatives of \(v\). So, as was done previously, the estimate \((4.22)\) will be our starting point, in this case with the particular choice of parameters \(\sigma = \frac{1-\alpha}{2}, \epsilon = s - s(\alpha)\), and the function \(F = -v \partial_x u_e - u_{e'} \partial_x v\).

In this way
\[
(5.29) \quad \|\partial_x v\|_{L^2_t L^2_x} \lesssim T^{\mu_1} \|v\|_{L^p_{t} H^s_x} + T^{\mu_2} \|f^{s + \frac{\alpha + 1}{2}} (v \partial_x u_e)\|_{L^2_t L^2_x} + T^{\mu_3} \|f^{s + \frac{\alpha + 1}{2}} (u_{e'} \partial_x v)\|_{L^2_t L^2_x}
\]
where \(\mu_1, \mu_2 \in (0, 1/2)\).

Since \(0 < s + \frac{1-s}{2} < 1\), we apply inequality \((3.6)\) to obtain
\[
(5.30) \quad \|f^{s + \frac{\alpha + 1}{2}} (v \partial_x u_e)\|_{L^2_t L^2_x} \lesssim \|v\|_{L^p_{t} H^s_x} \|\partial_x u_e\|_{L^2_t L^2_x} + \|D_x^{s + \frac{\alpha + 1}{2}} \partial_x u_e\|_{L^2_t L^2_x} \|v\|_{L^p_{t} L^\infty_x},
\]
and
\[
(5.31) \quad \|f^{s + \frac{\alpha + 1}{2}} (u_{e'} \partial_x v)\|_{L^2_t L^2_x} \lesssim \|u_{e'}\|_{L^p_{t} H^s_x} \|\partial_x v\|_{L^2_t L^2_x} + \|D_x^{s + \frac{\alpha + 1}{2}} \partial_x v\|_{L^2_t L^2_x} \|u_{e'}\|_{L^p_{t} L^\infty_x} + \|u_{e'} D_x^{s + \frac{\alpha + 1}{2}} \partial_x v\|_{L^2_t L^2_x},
\]
where the last term on the right hand side is handled as follows
\[
(5.32) \quad \|u_{e'} D_x^{s + \frac{\alpha + 1}{2}} \partial_x v\|_{L^2_t L^2_x} = \left( \sum_{j=-\infty}^{\infty} \|u_{e'} D_x^{s + \frac{\alpha + 1}{2}} \mathcal{H} v\|_{L^2_t L^2_{\{j+1\}}}^2 \right)^{1/2} \lesssim \left( \sum_{j=-\infty}^{\infty} \|u_{e'}\|_{L^p_{t} L^\infty_{\{j+1\}}}^2 \right)^{1/2} \sup_{j \in \mathbb{Z}} \|D_x^{s + \frac{\alpha + 1}{2}} \mathcal{H} v\|_{L^2_t L^2_{\{j+1\}}} \lesssim (1 + T)^{\eta} \|u_0\|_{H^s_x} \sup_{j \in \mathbb{Z}} \|D_x^{s + \frac{\alpha + 1}{2}} \mathcal{H} v\|_{L^2_t L^2_{\{j+1\}}} \text{ for } \eta > 1/2,
\]
being the last inequality a consequence of Duhamel’s formula and Corollary 4.10.

Since \(v\) is a solution of the equation in \((5.21)\), then we will compare the Kato smoothing effect satisfied by this solution with the last term on the right hand side of \((5.32)\).
In this way, by using an approach similar to the one in Proposition 4.16 it follows that

\[(5.33) \sup_{j \in \mathbb{Z}} \left| D_x^{s + \frac{1}{2}} \mathcal{H} v \right|_{L^2_x L^2_t} \lesssim \left( 1 + T + \| \partial_x u_e \|_{L^1_T L^2_x} + \| \partial_x u_{e'} \|_{L^1_T L^2_x} \right)^{1/2} \| v \|_{L^2_T H^s_x} \]

\[+ \| D_x^{s+\frac{1}{2}} \partial_x u_e \|_{L^2_T L^2_x} \| v \|_{L^2_T L^2_x} \]

Gathering the bounds obtained in (5.29) and (5.30)- (5.33) it follows that

\[\| \partial_x v \|_{L^2_T L^2_x} \]

\[\lesssim \left( T^\mu_1 + T^\mu_2 \| u_0 \|_{H^s_x} + \left( 1 + T + T^{1/2} \| u_0 \|_{H^s_x} \right)^{1/2} T^\mu_2 (1 + T)^{\eta} \| u_0 \|_{H^s_x} \right) \| v \|_{L^2_T H^s_x} \]

\[+ T^\mu_2 \| u_0 \|_{H^s_x} \| \partial_x v \|_{L^2_T L^2_x} + h(\epsilon, \epsilon') \]

where

\[h(\epsilon, \epsilon') = T^\mu_2 \left| D_x^{s+\frac{1}{2}} \partial_x u_e \right| \| v \|_{L^2_T L^2_x} + T^\mu_2 \left| D_x^{s+\frac{1}{2}} \partial_x u_e \right| \| v \|_{L^2_T L^2_x} \to 0.\]

in view of (5.18), (5.19), and (5.22).

By gathering the estimates above the proposition follows. \(\square\)

As was mentioned before the Proposition 5.24 is the basis to conclude that the sequence \(\{ u_\epsilon - u_{1,\epsilon} \}_{\epsilon > 0}\) defines a Cauchy sequence in the space \(L^2_x ([0, T] : H^s (\mathbb{R}))\). In fact, by using a weak compactness argument it is possible to show that there exists a function \(u\) such that

\[(5.34) \| u - u_\epsilon \|_{L^2_T H^s_x} \to 0.\]

In fact, from this we deduce that \(u\) is a solution of the IVP (1.1) in the distributional sense satisfying the condition (1.8).

5.4. **Continuity of the Flow.** To finish with the proof of Theorem A it remains to prove the continuous dependence of the solution upon the initial data.

Let \(u_0 \in H^s (\mathbb{R})\), with \(\frac{3}{2} < s < 2\). By using a scaling argument, we can always assume that \(\| u_0 \|_{H^s_x} \leq \delta\), being \(\delta\) a small positive number. In fact, in section 5.3, it was proved that the corresponding solution \(u\) to (1.1) is defined in the interval \([0, 1]\) and belongs to \(u \in C([0, 1] : H^s (\mathbb{R}))\).

Thus, the continuous dependence upon the initial data comes down to prove that given \(\nu > 0\), there exists \(\theta > 0\), such that for any \(v_0 \in H^s (\mathbb{R})\) with \(\| v_0 - u_0 \|_{H^s_x} < \nu\), then the solution \(v \in C([0, 1] : H^s (\mathbb{R}))\) of (1.1) satisfies \(\| v - u \|_{L^2_T H^s_x} < \theta\).

To achieve this goal we regularize the initial data \(u_0\) and \(v_0\), by defining \(v_{0, \epsilon} = \rho_{\epsilon} * u_0\) and \(u_{0, \epsilon} = \rho_{\epsilon} * u_0\), where \(\rho_{\epsilon}\) is the mollifier defined in section 5.3. Next, we consider the solutions \(u_{\epsilon, \epsilon}, v_{\epsilon} \in C([0, 1] : H^s (\mathbb{R}))\) associated to the initial data \(u_{0, \epsilon}\) and \(v_{0, \epsilon}\) respectively.

Then, it follows from the triangle inequality that

\[(5.35) \| u - v \|_{L^2_T H^s_x} \leq \| u - u_{\epsilon} \|_{L^2_T H^s_x} + \| u_{\epsilon} - v_{\epsilon} \|_{L^2_T H^s_x} + \| v - v_{\epsilon} \|_{L^2_T H^s_x}.\]

Because, in view of (5.34) we can choose \(\epsilon_0 > 0\) sufficiently small such that

\[(5.36) \| u - u_{\epsilon_0} \|_{L^2_T H^s_x} + \| v - v_{\epsilon_0} \|_{L^2_T H^s_x} < 2\theta / 3.\]
Next, we proceed to estimate the second term on the right hand side of (5.35). For such purpose we consider the regularization of the elements $u_{0,e_0}$ and $v_{0,e_0}$, specifically by defining for $\mu > 0$, the elements $u_{0,e_0}^\mu = u_{0,e_0} * \rho_\mu$ and $v_{0,e_0}^\mu = v_{0,e_0} * \rho_\mu$.

The function $u_{0,e_0}^\mu$, respectively $v_{0,e_0}^\mu$, is a solution of the equation in (1.1) with initial conditions $u_{0,e_0}$, respectively $v_{0,e_0}$.

Since $u_{0,e_0}$ and $v_{0,e_0}$ are elements in $H^2(\mathbb{R})$ by virtue of (5.14) it follows that

\begin{equation}
\| u_{0,e_0}^\mu \|_{H_2^2} \lesssim \epsilon^{s-2} \| u_0 \|_{H_2^2} \lesssim \epsilon_0^{s-2} \delta^2 \tag{5.37}
\end{equation}

and

\begin{equation}
\| v_{0,e_0}^\mu \|_{H_2^2} \lesssim \| u_{0,e_0} - v_{0,e_0} \|_{H_2^2} + \| u_{0,e_0} \|_{H_2^2} \lesssim \epsilon_0^{s-2} \nu + \epsilon_0^{s-2} \delta. \tag{5.38}
\end{equation}

Thus, by choosing $\delta(\epsilon_0), \nu(\epsilon_0) > 0$ small enough, we obtain by the existence theory of solutions that $\{u_{0,e_0}^\mu\}_{\mu > 0}$ and $\{v_{0,e_0}^\mu\}_{\mu > 0}$ converge to $u_{0,e_0}$ and $v_{0,e_0}$, respectively in $L^\infty([0,1]: H^s(\mathbb{R}))$ as $\mu$ goes to zero.

The convergence in $L^\infty([0,1]: H^s(\mathbb{R}))$ implies that there exists $\mu_0 > 0$ small enough, such that

\begin{equation}
\| u_{0,e_0} - v_{0,e_0} \|_{L_1^\infty H_2^1} \lesssim \| u_{0,e_0} - u_{0,e_0}^{\mu_0} \|_{L_1^\infty H_2^1} + \| v_{0,e_0} - v_{0,e_0}^{\mu_0} \|_{L_1^\infty H_2^1} + \| u_{0,e_0}^{\mu_0} - v_{0,e_0}^{\mu_0} \|_{L_1^\infty H_2^1}
\end{align*}

\begin{equation}
\lesssim \theta/6 + \| u_{0,e_0}^{\mu_0} - v_{0,e_0}^{\mu_0} \|_{L_1^\infty H_2^1} \tag{5.39}
\end{equation}

however, by (5.14) it follows that

\begin{equation}
\| u_{0,e_0} - v_{0,e_0} \|_{L_1^\infty H_2^2} < \nu. \tag{5.40}
\end{equation}

Thus, by applying Theorem 4.12, the continuity of the flow solution for initial data in $H^2(\mathbb{R})$ implies that for $\nu > 0$ small enough

\begin{equation}
\| u_{0,e_0}^{\mu_0} - v_{0,e_0}^{\mu_0} \|_{L_1^\infty H_2^2} < \theta/6. \tag{5.41}
\end{equation}

Finally, combining (5.36), (5.39)) and (5.41) it follows the continuous dependence upon the initial data.

This finish the proof of Theorem A.

\[\square\]

6. PROOF OF THEOREM B

The aim of this section is to prove Theorem B. To achieve this goal is necessary to take into account two important aspects of our analysis. First, the ambient space, that in our case is the Sobolev space where the theorem is valid together with the properties satisfied by the real solutions of the dispersive generalized Benjamin-Ono equation. In second place, the auxiliaries weights functions involved in the energy estimates that we will describe in detail.

The following is a summary of the local well-posedness and Kato’s smoothing effect presented in the previous sections.

**Theorem C.** If $u_0 \in H^s(\mathbb{R})$, $s \geq \frac{3-\alpha}{2}$, $\alpha \in (0,1)$, then there exist a positive time $T = T(\|u_0\|_{H^s}) > 0$ and a unique solution of the IVP (1.1) such that

(a) $u \in C([-T, T]: H^s(\mathbb{R}))$,

(b) $\partial_x u \in L^1([-T, T]: L^\infty(\mathbb{R}))$, \ (Strichartz),
(c) **Smoothing effect:** for $R > 0$,

\[
\int_{-T}^{T} \int_{-R}^{R} \left( \left| \partial_x D_x^{r+\frac{5+1}{2}} u \right|^2 + \left| \mathcal{H}_x D_x^{r+\frac{5+1}{2}} u \right|^2 \right) \, dx \, dt \leq C
\]

with $r \in \left( \frac{2-3\delta}{8}, \delta \right]$ and $C = C(\alpha; R; T; \|u_0\|_{H^\delta_x}) > 0$.

Since we have set the Sobolev space where we will work, the next step is the description of the cutoff functions to be used in the proof.

In this part we consider families of cutoff functions that will be used systematically in the proof of Theorem B. This collection of weights functions were constructed originally in [18] and [23] in the proof of Theorem 1.3 and Theorem ?? respectively.

More precisely, for $\epsilon > 0$ and $b \geq 5\epsilon$ define the families of functions

\[
\chi_{\epsilon,b}, \ \phi_{\epsilon,b}, \ \widetilde{\phi}_{\epsilon,b}, \ \psi_{\epsilon}, \ \eta_{\epsilon,b} \in C^\infty(\mathbb{R})
\]

satisfying the following properties:

1. $\chi'_{\epsilon,b} \geq 0$,

2. $\chi_{\epsilon,b}(x) = \begin{cases} 0, & x \leq \epsilon \\ 1, & x \geq b \end{cases}$,

3. $\text{supp}(\chi_{\epsilon,b}) \subseteq [\epsilon, \infty)$;

4. $\chi'_{\epsilon,b}(x) \geq \frac{1}{10(b-\epsilon)} \mathbb{1}_{[2\epsilon,b-2\epsilon]}(x)$,

5. $\text{supp}(\chi'_{\epsilon,b}) \subseteq [\epsilon, b]$;

6. There exists real numbers $c_j$ such that

\[
\left| \chi_{\epsilon,b}^{(j)}(x) \right| \leq c_j \chi'_{\epsilon,b+\epsilon}(x), \ \forall x \in \mathbb{R}, \ j \in \mathbb{Z}^+.
\]

7. For $x \in (3\epsilon, \infty)$

\[
\chi_{\epsilon,b}(x) \geq \frac{1}{2} \frac{\epsilon}{b-3\epsilon}.
\]

8. For $x \in \mathbb{R}$

\[
\chi'_{\epsilon,b+\epsilon}(x) \leq \frac{\epsilon}{b-3\epsilon}.
\]

9. Given $\epsilon > 0$ and $b \geq 5\epsilon$ there exist $c_1, c_2 > 0$ such that

\[
\chi'_{\epsilon,b}(x) \leq c_1 \chi'_{\epsilon/3,b+\epsilon}(x) \chi_{\epsilon/3,b+\epsilon}(x),
\]

\[
\chi_{\epsilon,b}(x) \leq c_2 \chi_{\epsilon/5,b}(x).
\]

10. For $\epsilon > 0$ given and $b \geq 5\epsilon$, we define the function

\[
\eta_{\epsilon,b} = \sqrt{\chi_{\epsilon,b} \chi'_{\epsilon,b}}.
\]

11. $\text{supp}(\phi_{\epsilon,b}), \ \text{supp}(\widetilde{\phi}_{\epsilon,b}) \subseteq [\epsilon/4, b]$,

12. $\phi_{\epsilon}(x) = \phi_{\epsilon,b}(x) = 1, \ \forall x \in [\epsilon/2, \epsilon]$,

13. $\text{supp}(\psi_{\epsilon}) \subseteq (-\infty, \epsilon/2]$,

14. for $x \in \mathbb{R}$

\[
\chi_{\epsilon,b}(x) + \phi_{\epsilon,b}(x) + \psi_{\epsilon}(x) = 1,
\]

and

\[
\chi_{\epsilon,b}^2(x) + \phi_{\epsilon,b}^2(x) + \psi_{\epsilon}(x) = 1.
\]
The family \( \{ \chi_{e,b} : e > 0, b \geq 5e \} \) is constructed as follows: let \( \rho \in C_0^\infty(\mathbb{R}) \), \( \rho(x) \geq 0 \), even, with \( \text{supp}(\rho) \subseteq (-1,1) \) and \( \|\rho\|_{L^1} = 1 \).

Then defining
\[
\nu_{e,b}(x) = \begin{cases} 
0, & x \leq 2e, \\
\frac{x}{2e} - \frac{2e}{b-3e}, & 2e \leq x \leq b - \epsilon, \\
1, & x \geq b - \epsilon,
\end{cases}
\]
and
\[
\chi_{e,b}(x) = \rho_\epsilon * \nu_{e,b}(x)
\]
where \( \rho_\epsilon(x) = \epsilon^{-1}\rho(x/\epsilon) \).

Now that it has been described all the required estimates and tools necessary, we present the proof of our main result.

**Proof of Theorem B.** Since the argument is translation invariant, without loss of generality we will consider the case \( x_0 = 0 \).

First, we will describe the formal calculations assuming as much as regularity as possible, later we provide the justification using a limiting process.

The proof will be established by induction, however in every step of induction we will subdivide every case in two steps, due to the non-local nature of the operator involving the dispersive part in the equation in (1.1).

**Case j = 1**

**Step 1.**

First we apply one spatial derivative to the equation in (1.1), after that we multiply by \( \partial_x u(x,t)\chi_{e,b}^2(x + vt) \), and finally we integrate in the \( x \)-variable to obtain the identity
\[
\frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}} (\partial_x u)^2 \chi_{e,b}^2 \, dx - \frac{3}{2} \int_{\mathbb{R}} (\partial_x u)^2 (\chi_{e,b})' \, dx - \int_{\mathbb{R}} (\partial_x D_x^{n+1}\partial_x u) \partial_x u \chi_{e,b}^2 \, dx
\]
\[
+ \int_{\mathbb{R}} \partial_x (u \partial_x u) \partial_x u \chi_{e,b}^2 \, dx = 0.
\]

\[\text{§.1} \quad \text{Combining the local theory we obtain the following}
\]
\[
\int_0^T |A_1(t)| \, dt \leq \frac{3}{2} \int_0^T (\partial_x u)^2 (\chi_{e,b})' \, dx \, dt \lesssim \|u\|_{L_t^\infty H_x^{\frac{3-n}{2}}}.
\]

\[\text{§.2} \quad \text{Integration by parts and Plancherel's identity allow us rewrite the term } A_2 \text{ as follows}
\]
\[
A_2(t) = \int_{\mathbb{R}} \partial_x u \left[ D_x^{n+1}\partial_x \chi_{e,b}^2 \right] \partial_x u \, dx - \int_{\mathbb{R}} \partial_x u D_x^{1+n} \partial_x (\chi_{e,b}^2 \partial_x u) \, dx
\]
\[
= \int_{\mathbb{R}} \partial_x u \left[ D_x^{n+1}\partial_x \chi_{e,b}^2 \right] \partial_x u \, dx + \int_{\mathbb{R}} \chi_{e,b}^2 \partial_x u \partial_x D_x^{1+n} \partial_x u \, dx
\]
\[
= \int_{\mathbb{R}} \partial_x u \left[ D_x^{n+1}\partial_x \chi_{e,b}^2 \right] \partial_x u \, dx - A_2(t).
\]
Consequently,

\begin{equation}
A_2(t) = \frac{1}{2} \int_{\mathbb{R}} \partial_x u \left[ D_x^{a+1} \partial_x \lambda_{c,b}^2 \right] \partial_x u \, dx = -\frac{1}{2} \int_{\mathbb{R}} \partial_x u \left[ \mathcal{H} D_x^{\alpha+2} \lambda_{c,b}^2 \right] \partial_x u \, dx.
\end{equation}

Since \(\alpha + 2 > 1\), we have by (3.20) that the commutator \( [\mathcal{H} D_x^{\alpha+2}, \lambda_{c,b}^2] \) can be decomposed as

\begin{equation}
[\mathcal{H} D_x^{\alpha+2}, \lambda_{c,b}^2] = -\frac{1}{2} P_n(\alpha + 2) + \frac{1}{2} \mathcal{H} P_n(\alpha + 2) \mathcal{H} - R_n(\alpha + 2)
\end{equation}

for some positive integer \(n\), that will be fixed later.

Inserting (6.3) into (6.2)

\begin{align*}
A_2(t) &= \frac{1}{2} \int_{\mathbb{R}} \partial_x u R_n(\alpha + 2) \partial_x u \, dx + \frac{1}{4} \int_{\mathbb{R}} \partial_x u P_n(\alpha + 2) \partial_x u \, dx \\
&\quad - \frac{1}{4} \int_{\mathbb{R}} \partial_x u \mathcal{H} P_n(\alpha + 2) \mathcal{H} \partial_x u \, dx \\
&= A_{2,1}(t) + A_{2,2}(t) + A_{2,3}(t).
\end{align*}

Now, we proceed to fix the value of \(n\) present in the terms \(A_{2,1}, A_{2,2}\) and \(A_{2,3}\), according to a determinate condition.

First, notice that

\begin{align*}
A_{2,1}(t) &= \frac{1}{2} \int_{\mathbb{R}} D_x \mathcal{H} u R_n(\alpha + 2) D_x \mathcal{H} u \, dx = \frac{1}{2} \int_{\mathbb{R}} \mathcal{H} u D_x R_n(\alpha + 2) D_x \mathcal{H} u \, dx.
\end{align*}

Then we fix \(n\) such that \(2n + 1 \leq a + 2\sigma \leq 2n + 3\), that according to the case we are studying \((j = 1)\), corresponds to \(a = \alpha + 2\) and \(\sigma = 1\). This produces \(n = 1\).

For this \(n\) in particular we have by Proposition 3.24 that \(R_1(\alpha + 2)\) maps \(L_x^2\) into \(L_t^1\).

Hence,

\begin{align*}
A_{2,1}(t) &\lesssim \|\mathcal{H} u(t)\|_{L_x^2}^2 \left\| D_x^{\frac{\alpha+1}{2}} \lambda_{c,b}^2 \right\|_{L_t^1} = c \|u_0\|_{L_x^2}^2 \left\| D_x^{\frac{\alpha+1}{2}} \lambda_{c,b}^2 \right\|_{L_t^1},
\end{align*}

which after integrating in time yields

\begin{align*}
\int_0^T |A_{2,1}(t)| \, dt &\lesssim \|u_0\|_{L_x^2}^2 \sup_{0 \leq t \leq T} \left\| D_x^{\frac{\alpha+1}{2}} \lambda_{c,b}^2 \right\|_{L_t^1}.
\end{align*}

Next, we turn our attention to \(A_{2,2}\). Replacing \(P_1(\alpha + 2)\) into \(A_{2,2}\)

\begin{align*}
A_{2,2}(t) &= \left( \frac{\alpha + 2}{4} \right) \int_{\mathbb{R}} \left( D_x^{\frac{\alpha+1}{2}} \partial_x u \right)^2 (\lambda_{c,b}^2)' \, dx \\
&\quad - c_3 \left( \frac{\alpha + 2}{16} \right) \int_{\mathbb{R}} \left( D_x^{\frac{\alpha+1}{2}} \mathcal{H} u \right)^2 (\lambda_{c,b}^2)'' \, dx \\
&= A_{2,2,1}(t) + A_{2,2,2}(t).
\end{align*}

We shall underline that \(A_{2,2,1}(t)\) is positive, besides it represents explicitly the smoothing effect for the case \(j = 1\).
Regarding $A_{2,2,2}$, the local theory combined with interpolation leads to

\[
\int_0^T |A_{2,2,2}(t)| \, dt \lesssim \|u\|_{L^\infty_T H_x^{\frac 32}}.
\]

After replacing (3.21) into $A_{2,3}$ and using the fact that Hilbert transform is skew-symmetric

\[
A_{2,3}(t) = \left( \frac{\alpha + 2}{4} \right) \int_{\mathbb{R}} \left( D_1^{\alpha} u \right)^2 \chi_{b}^2 \, dx
\]

\[
- c_3 \left( \frac{\alpha + 2}{16} \right) \int_{\mathbb{R}} \left( H D_1^{\alpha} u \right)^2 \chi_{b}^2 \, dx
\]

\[
= A_{2,3,1}(t) + A_{2,3,2}(t).
\]

Notice that the term $A_{2,3,1}$ is positive and represents the smoothing effect. In contrast, the term $A_{2,3,2}$ is estimated as we did with $A_{2,2,2}$ in (6.4). So, after integration in the time variable

\[
\int_0^T |A_{2,3,2}(t)| \, dt \lesssim \|u\|_{L^\infty_T H_x^{\frac 32}}.
\]

Finally, after apply integration by parts

\[
A_3(t) = \int_{\mathbb{R}} \partial_x (u_\partial_x u) \partial_x u \chi_{b}^2 \, dx
\]

\[
= \frac{1}{2} \int_{\mathbb{R}} \partial_x u (\partial_x u)^2 \chi_{b}^2 \, dx - \frac{1}{2} \int_{\mathbb{R}} u (\partial_x u)^2 \chi_{b}^2 \, dx
\]

\[
= A_{3,1}(t) + A_{3,2}(t).
\]

On one hand,

\[
|A_{3,1}(t)| \lesssim \|\partial_x u(t)\|_{L^\infty_T} \int_{\mathbb{R}} (\partial_x u)^2 \chi_{b}^2 \, dx,
\]

where the integral expression on the right-hand side is the quantity to be estimated by means of Gronwall’s inequality.

On the other hand,

\[
|A_{3,2}(t)| \lesssim \|u(t)\|_{L^\infty_T} \int_0^T (\partial_x u)^2 \chi_{b}^2 \, dx.
\]

By Sobolev embedding we have after integrating in time

\[
\int_0^T |A_{3,2}(t)| \, dt \lesssim \left( \sup_{0 \leq t \leq T} \|u(t)\|_{H_x^{1,1}} \right) \int_0^T (\partial_x u)^2 \chi_{b}^2 \, dx \, dt \lesssim \epsilon
\]

Since $\|\partial_x u\|_{L^1_T L^\infty_T} < \infty$. Then after gathering all estimates above and apply Gronwall’s inequality we obtain

\[
\sup_{0 \leq t \leq T} \int_{\mathbb{R}} (\partial_x u)^2 \chi_{b}^2 \, dx + \int_0^T \int_{\mathbb{R}} \left( D_1^{\alpha \frac{\alpha + 1}{2}} u \right)^2 \chi_{b}^2 \, dx \, dt
\]

\[
+ \int_0^T \int_{\mathbb{R}} \left( D_1^{\alpha \frac{\alpha + 1}{2}} u \right)^2 \chi_{b}^2 \, dx \, dt \leq \epsilon^*_{1,1}
\]

(6.5)
where $c_{1,1}^n = c_{1,1}^n \left( \alpha; c; T; \|u_0\|^{\frac{3}{2-n}}_{L^2}; \|\partial_x u_0 \chi_{c,b}\|_{L^2_x} \right) > 0$, for any $c > 0$, $b \geq 5c$ and $c > 0$.

This estimate finish the step 1 corresponding to the case $j = 1$.

The local smoothing effect obtained above is just $\frac{1}{2} + \frac{\alpha}{2}$ derivative (see [19]). So, the iterative argument is carried out in two steps, the first step for positive integers $m$ and the second one for $m + \frac{1}{2} + \frac{\alpha}{2}$.

**Step 2.**

After apply the operator $D^{\frac{1+\alpha}{2}}_x \partial_x$ to the equation in (1.1) and multiply the resulting by $D^{\frac{1+\alpha}{2}}_x \partial_x u \chi_{c,b}^2(x + vt)$ one gets

$$
\begin{align*}
& D^{\frac{1+\alpha}{2}}_x \partial_x \partial_t u D^{\frac{1+\alpha}{2}}_x \partial_x u \chi_{c,b}^2 - D^{\frac{1+\alpha}{2}}_x \partial_x \chi_{c,b}^{1+\alpha} \partial_x u D^{\frac{1+\alpha}{2}}_x \partial_x u \chi_{c,b}^2 \\
& + D^{\frac{1+\alpha}{2}}_x \partial_x (u \partial_x u) D^{\frac{1+\alpha}{2}}_x \partial_x u \chi_{c,b}^2 = 0,
\end{align*}
$$

which after integrate in the spatial variable it becomes

$$
\begin{align*}
& \frac{1}{2} \frac{d}{dt} \int_R (D^{\frac{1+\alpha}{2}}_x \partial_x u)^2 \chi_{c,b}^2 \, dx - \int_R \left( D^{\frac{1+\alpha}{2}}_x \partial_x u \right)^2 (\chi_{c,b}^2)' \, dx \\
& - \int_R \left( D^{\frac{1+\alpha}{2}}_x \partial_x \chi_{c,b}^{1+\alpha} \partial_x u \right) D^{\frac{1+\alpha}{2}}_x \partial_x u \chi_{c,b}^2 \, dx \\
& + \int_R D^{\frac{1+\alpha}{2}}_x \partial_x (u \partial_x u) D^{\frac{1+\alpha}{2}}_x \partial_x u \chi_{c,b}^2 \, dx = 0.
\end{align*}
$$

§.1 First observe that by the local theory

$$
\int_0^T |A_1(t)| \, dt \lesssim \|u\| \int_0^T \int_R \left( D^{\frac{1+\alpha}{2}}_x \partial_x u \right)^2 (\chi_{c,b}^2)' \, dx \, dt
$$

§.2 Concerning to the term $A_2$ integration by parts and Plancherel’s identity yields

$$
A_2(t) = \int_R D^{\frac{1+\alpha}{2}}_x \partial_x u \left[ D^{\frac{1+\alpha}{2}}_x \partial_x \chi_{c,b}^2 \right] D^{\frac{1+\alpha}{2}}_x \partial_x u \, dx - A_2(t).
$$

As a consequence

$$
(6.6) \quad A_2(t) = -\frac{1}{2} \int_R D^{\frac{1+\alpha}{2} + 1}_x \partial_x u \left[ \mathcal{H} D^{\frac{1+\alpha}{2}}_x \chi_{c,b}^2 \right] D^{\frac{1+\alpha}{2} + 1}_x \partial_x u \, dx.
$$
Since $2 + \alpha > 1$, we have by (3.20) that the commutator $[\mathcal{H}D_x^{a+2}; \chi_{\epsilon,b}^2]$ can be decomposed as

\begin{equation}
[\mathcal{H}D_x^{a+2}; \chi_{\epsilon,b}^2] + \frac{1}{2} P_n(\alpha + 2) + R_n(\alpha + 2) = \frac{1}{2} \mathcal{H}P_n(\alpha + 2) \mathcal{H}
\end{equation}

for some positive integer $n$ that as in the previous cases it will be fixed suitably.

Replacing (6.7) into (6.6)

\[
A_2(t) = \frac{1}{2} \int_R D_x^{3-a} \mathcal{H} u \left( R_n(\alpha + 2) D_x^{3-a} \mathcal{H} u \right) \, dx
+ \frac{1}{4} \int_R D_x^{3-a} \mathcal{H} u \left( P_n(\alpha + 2) D_x^{3-a} \mathcal{H} u \right) \, dx
- \frac{1}{4} \int_R D_x^{3-a} \mathcal{H} u \left( \mathcal{H}P_n(\alpha + 2) \mathcal{H} D_x^{3-a} \mathcal{H} u \right) \, dx
= A_{2,1}(t) + A_{2,2}(t) + A_{2,3}(t).
\]

Fixing the value of $n$ present in the terms $A_{2,1}, A_{2,2}$ and $A_{2,3}$ requires an argument almost similar to that one used in step 1. First, we deal with $A_{2,1}$ where a simple computation produces

\[
A_{2,1}(t) = \frac{1}{2} \int_R \mathcal{H} u \left( D_x^{3-a} R_n(\alpha + 2) D_x^{3-a} \mathcal{H} u \right) \, dx.
\]

We fix $n \in \mathbb{Z}^+$ in such a way

\[2n + 1 \leq a + 2 \sigma \leq 2n + 3\]

where $a = \alpha + 2$ and $\sigma = \frac{3-a}{2}$ in order to obtain obtain $n = 1$ or $n = 2$. For the sake of simplicity we choose $n = 1$.

Hence, by construction $R_1(\alpha + 2)$ satisfies the hypothesis of Proposition 3.24, and

\[
|A_{2,1}(t)| \lesssim \|\mathcal{H} u(t)\|_{L^\frac{6}{5}_\chi} \left\| D_x^{5}(\chi_{\epsilon,b}^2) \right\|_{L^1_\chi} \lesssim \|u_0\|_{L^\frac{6}{5}_\chi} \left\| D_x^{5}(\chi_{\epsilon,b}^2) \right\|_{L^1_\chi}.
\]

Thus

\[
\int_0^T |A_{2,1}(t)| \, dt \lesssim \|u_0\|_{L^\frac{6}{5}_\chi} \sup_{0 \leq t \leq T} \left\| D_x^{5}(\chi_{\epsilon,b}^2) \right\|_{L^1_\chi}.
\]

Next, after replacing $P_1(\alpha + 2)$ in $A_{2,2}$

\[
A_{2,2}(t) = \left( \frac{\alpha + 2}{4} \right) \int_R (\mathcal{H} \partial_x^2 u)^2 (\chi_{\epsilon,b}^2)^{\prime} \, dx - c_3 \left( \frac{\alpha + 2}{16} \right) \int_R (\partial_x u)^2 (\chi_{\epsilon,b}^2)^{\prime \prime} \, dx
= A_{2,2,1}(t) + A_{2,2,2}(t).
\]

The smoothing effect corresponds to the term $A_{2,2,1}$ and it will be bounded after integrating in time. In contrast, to bound $A_{2,2,2}$ is required only the local theory, in fact

\[
\int_0^T |A_{2,2,2}(t)| \, dt \lesssim \|u\|_{L^{\frac{6}{5}}_\chi H_x^{\frac{3-a}{2}}}.
\]
Concerning the term $A_{2,3}$ we have after replacing $P_1(\alpha + 2)$ and using the properties of the Hilbert transform that

$$A_{2,3}(t) = \left(\frac{\alpha + 2}{4}\right) \int \left(\partial_x^2 u\right)^2 \left(\partial_x e_b^2\right) \, dx - c_3 \left(\frac{\alpha + 2}{16}\right) \int \left(\partial_x u\right)^2 \left(\partial_x e_b^4\right) \, dx$$

$$= A_{2,3,1}(t) + A_{2,3,2}(t).$$

As before, $A_{2,3,1} \geq 0$ and represents the smoothing effect. Besides, the local theory and interpolation yields

$$\int_0^T |A_{2,3,2}(t)| \, dt \lesssim \|u\|_{L^1_T H^{\frac{1}{2}-\epsilon}}.$$

§3 It only remains to handle the term $A_3$.

Since

$$D_x^{\frac{1}{2}+\epsilon} \partial_x (u \partial_x u) \chi_{e,b}$$

$$= - D_x^{\frac{1}{2}+\epsilon} \partial_x \left(\chi_{e,b} \right) \left( u \partial_x u \right) + D_x^{\frac{1}{2}+\epsilon} \partial_x \left( \chi_{e,b} u \partial_x u \right)$$

$$= - \frac{1}{2} \left\{ D_x^{\frac{1}{2}+\epsilon} \partial_x \left( \chi_{e,b} u \right) \partial_x \left( u \right) \right\} + \left\{ D_x^{\frac{1}{2}+\epsilon} \partial_x \left( u \chi_{e,b} \right) \partial_x \left( u \right) \right\}$$

$$+ \left\{ D_x^{\frac{1}{2}+\epsilon} \partial_x \left( \chi_{e,b} u \right) \partial_x \left( u \right) \right\}$$

$$+ \left\{ D_x^{\frac{1}{2}+\epsilon} \partial_x \left( u \chi_{e,b} \right) \partial_x \left( u \right) \right\}$$

(6.8)

$$= \sim A_{3,1}(t) + A_{3,2}(t) + \sim A_{3,3}(t) + \sim A_{3,4}(t) + \sim A_{3,5}(t) + \sim A_{3,6}(t) + \sim A_{3,7}(t).$$

First, we rewrite $\sim A_{3,1}$ as follows

$$\sim A_{3,1}(t) = c_3 \mathcal{H} \left[ D_x^{\frac{1}{2}+\epsilon} \partial_x \left( u \chi_{e,b} \right) \partial_x \left( u \chi_{e,b} \right) \right] + c_3 \left[ \mathcal{H} \chi_{e,b} \right] D_x^{\frac{1}{2}+\epsilon} \partial_x \left( \chi_{e,b} u \right)^2,$$

where $c_3$ denotes a non-null constant. Next, combining (3.4), (3.14) and Lemma 3.15 one gets

$$\|\sim A_{3,1}(t)\|_{L^2_T} \lesssim \left\| D_x^{\frac{1}{2}+\epsilon} \left( u \chi_{e,b} \right) \right\|_{L^2_T} \|u\|_{L^\infty_T} + \|u_0\|_{L^2_T} \|u\|_{L^\infty_T}$$

and

$$\|\sim A_{3,2}(t)\|_{L^2_T} \lesssim \left\| D_x^{\frac{1}{2}+\epsilon} \left( u \phi_{e,b} \right) \right\|_{L^2_T} \|u\|_{L^\infty_T} + \|u_0\|_{L^2_T} \|u\|_{L^\infty_T}.$$

Next, we recall that by construction

$$\text{dist} \left( \text{supp} \left( \chi_{e,b} \right), \text{supp} \left( \psi_{e} \right) \right) \geq \frac{\epsilon}{2},$$

so, by Lemma 3.16

$$\|\sim A_{3,3}(t)\|_{L^2_T} = \left\| D_x^{\frac{1}{2}+\epsilon} \partial_x \left( \psi_{e,b} u \right)^2 \right\|_{L^2_T} \lesssim \|u_0\|_{L^2_T} \|u\|_{L^\infty_T}.$$
Rewriting
\[ \widetilde{A}_{3A}(t) = c \mathcal{H} \left[ D_{x}^{1 + \frac{1-a}{2}} \chi_{e,b} \right] \partial_{x} (u_{X_{e,b}}) - c \left[ \mathcal{H}; u_{X_{e,b}} \right] \partial_{x} D_{x}^{1 + \frac{1-a}{2}} (u_{X_{e,b}}) \]
for some non-null constant \( c \).

Thus, by the commutator estimates (3.2) and (3.13)
\[ \| \widetilde{A}_{3A}(t) \|_{L_{t}^{2}} \lesssim \| \partial_{x} (u_{X_{e,b}}) \|_{L_{t}^{\infty}} \| D_{x}^{1 + \frac{1-a}{2}} (u_{X_{e,b}}) \|_{L_{t}^{2}}. \]

Applying the same procedure to \( \widetilde{A}_{3S} \) yields
\[ \| \widetilde{A}_{3S}(t) \|_{L_{t}^{2}} \lesssim \| \partial_{x} (u_{X_{e,b}}) \|_{L_{t}^{\infty}} \| D_{x}^{1 + \frac{1-a}{2}} (u_{X_{e,b}}) \|_{L_{t}^{2}} + \| \partial_{x} (u\phi_{e,b}) \|_{L_{t}^{\infty}} \| D_{x}^{1 + \frac{1-a}{2}} (u\phi_{e,b}) \|_{L_{t}^{2}}. \]

Since the supports of \( \chi_{e,b} \) and \( \psi_{e} \) are separated we obtain by Lemma 3.16
\[ \| \widetilde{A}_{3S}(t) \|_{L_{t}^{2}} = \left\| u_{X_{e,b}} \partial_{x}^{2} D_{x}^{1 + \frac{1-a}{2}} (u\psi_{e,b}) \right\|_{L_{t}^{2}} \lesssim \| u_{0} \|_{L_{t}^{2}} \| u \|_{L_{t}^{\infty}}. \]

To finish with the estimates above we use the relation
\[ \chi_{e,b}(x) + \phi_{e,b}(x) + \psi_{e}(x) = 1 \quad \forall x \in \mathbb{R}. \]

Then
\[ D_{x}^{1 + \frac{1-a}{2}} (u_{X_{e,b}}) = D_{x}^{1 + \frac{1-a}{2}} u_{X_{e,b}} + \left[ D_{x}^{1 + \frac{1-a}{2}} \chi_{e,b} \right] \left( u_{X_{e,b}} + u\phi_{e,b} + u\psi_{e} \right) = I_{1} + I_{2} + I_{3} + I_{4}. \]

Notice that \( I_{1} \) is the quantity to estimate. In contrast, \( I_{2} \) and \( I_{3} \) can be handled by Lemma 3.13 combined with the local theory. Meanwhile \( I_{3} \) can be bounded by using Lemma 3.16.

We notice that the gain of regularity obtained in the step 1 implies that \( \| D_{x}^{1 + \frac{1-a}{2}} (u\phi_{e,b}) \|_{L_{t}^{2}} < \infty \). To show this we use Theorem 3.7 and Hölder’s inequality as follows
\[
\left\| D_{x}^{1 + \frac{1-a}{2}} (u\phi_{e,b}) \right\|_{L_{t}^{2}} \lesssim \left\| u \right\|_{L_{t}^{2}} \left\| D_{x}^{1 + \frac{1-a}{2}} \phi_{e,b} \right\|_{L_{t}^{2}} + \sum_{\beta \leq 1} \frac{1}{\beta !} \epsilon_{x}^{\beta} \phi_{e,b} D_{x}^{\epsilon_{b}} u \right\|_{L_{t}^{2}} 
\lesssim \left\| u_{0} \right\|_{L_{t}^{2}}^{1/2} \left\| u \right\|_{L_{t}^{\infty}}^{1/2} + \left\| \phi_{e,b} D_{x}^{1 + \frac{1-a}{2}} u \right\|_{L_{t}^{2}} + \left\| \partial_{x} \phi_{e,b} \mathcal{H} D_{x}^{\epsilon_{b}} u \right\|_{L_{t}^{2}} 
\lesssim \left\| u_{0} \right\|_{L_{t}^{2}} + \left\| u \right\|_{L_{t}^{\infty}} + \left\| 1_{[\epsilon/4, \epsilon]} \right\|_{L_{t}^{2}} \left\| D_{x}^{1 + \frac{1-a}{2}} u \right\|_{L_{t}^{2}} + \left\| 1_{[\epsilon/4, \epsilon]} \right\|_{L_{t}^{2}} \left\| \mathcal{H} D_{x}^{\epsilon_{b}} u \right\|_{L_{t}^{2}}. 
\]

The second term on the right hand side after integrate in time is controlled by using Sobolev’s embedding. Meanwhile, the third term can be handled after integrate in time and use (6.5) with \((e, b) = (e/24, b + 7e/24)\).
The fourth term in the right hand side can be bounded combining the local theory and interpolation.

Hence, after integration in time

\begin{equation}
\left\| D_x^{1+\frac{1-a}{2}} (u\phi_{c,b}) \right\|_{L^2_x L^2_t} < \infty,
\end{equation}

which clearly implies \( \left\| D_x^{1+\frac{1-a}{2}} (u\phi_{c,b}) \right\|_{L^2_x L^2_t} < \infty \), as was required. Analogously, can be handled \( \left\| D_x^{1+\frac{1-a}{2}} (u\phi_{c,b}) \right\|_{L^2_x} \).

Finally,

\[ A_{3,7}(t) = -\frac{1}{2} \int_{\mathbb{R}} \partial_x u \chi_{c,b}^2 \left( D_x^{1-a} \partial_x u \right)^2 \, dx - \frac{1}{2} \int_{\mathbb{R}} u (\chi_{c,b}^2)' \left( D_x^{1-a} \partial_x u \right)^2 \, dx \]

\[ = A_{3,7,1}(t) + A_{3,7,2}(t). \]

Since

\[ |A_{3,7,1}(t)| \lesssim \| \partial_x u(t) \|_{L^\infty_x} \int_{\mathbb{R}} \left( D_x^{1-a} \partial_x u \right)^2 \chi_{c,b}^2 \, dx, \]

being the last integral the quantity to be estimated by means of Gronwall’s inequality, and by the local theory \( \| \partial_x u \|_{L^1_x L^\infty_t} < \infty \).

Sobolev’s embedding led us to

\[ \int_0^T |A_{3,7,2}(t)| \, dt \lesssim \left( \sup_{0 \leq t \leq T} \| u(t) \|_{H^{(a)\ast}'} \right) \int_0^T \int_{\mathbb{R}} \chi_{c,b} \chi_{c,b}' \left( D_x^{1-a} \partial_x u \right)^2 \, dx \, dt \]

Gathering all the information corresponding to this step combined with Gronwall’s inequality yields

\begin{equation}
\sup_{0 \leq t \leq T} \int_{\mathbb{R}} \left( D_x^{1-a} \partial_x u \right)^2 \chi_{c,b}^2 (x + vt) \, dx + \int_0^T \int_{\mathbb{R}} \left( \partial_x^2 u \right)^2 (\chi_{c,b}^2)' \, dx \, dt \leq c_{1,2}^* \]

\[ + \int_0^T \int_{\mathbb{R}} (\mathcal{H} \partial_x^2 u)^2 (\chi_{c,b}^2)' \, dx \, dt \leq c_{1,2}^* \]

with \( c_{1,2}^* = c_{1,2}^* (a; \epsilon; T; \nu; \| u_0 \|_{H^{(a)\ast}}; \| D_x^{1-a} \partial_x u_0 \chi_{c,b} \|_{L^2_t}) \) for any \( \epsilon > 0, b \geq 5 \epsilon \) and \( \nu > 0 \).

This finishes the step two corresponding to the case \( j = 1 \) in the induction process.

Next, we present the case \( j = 2 \), to show how we proceed in the case \( j \) even.

**Case j = 2**

**Step 1.**

First we apply two spatial derivatives to the equation in (1.1), after that we multiply by \( \partial_x^2 u(x, t) \chi_{c,b}^2 (x + vt) \), and finally we integrate in the \( x \)-variable to obtain the


\[ \frac{1}{2} \frac{d}{dt} \left( \int_{\mathbb{R}} (\partial_x^2 u)^2 \chi_{e,b}^2 \, dx \right) - \frac{3}{2} \int_{\mathbb{R}} (\partial_x^2 u)^2 (\chi_{e,b}^2)' \, dx - \int_{\mathbb{R}} \left( \partial_x^2 D_x^{(a+1)} \partial_x u \right) \partial_x u \chi_{e,b}^2 \, dx \]

\[ + \int_{\mathbb{R}} \partial_x^2 (u \partial_x u) \partial_x u \chi_{e,b}^2 \, dx = 0. \]

Similarly as was done in the previous steps we first proceed to estimate \( A_1 \).

\( \text{§.1 By (6.11) it follows that} \]

\[ \int_0^T |A_1(t)| \, dt \leq \int_0^T \int_{\mathbb{R}} (\partial_x^2 u)^2 (\chi_{e,b}^2)' \, dx \, dt \leq c_{1,2}^* . \]

\( \text{§2. To extract information from the term} \ A_2 \ \text{we use integration by parts and Plancherel’s identity to obtain} \]

\[ A_2(t) = \int_{\mathbb{R}} \partial_x^2 u \left[ D_x^{a+1} \partial_x \chi_{e,b}^2 \right] \partial_x u \, dx - A_2(t). \]

Consequently,

\[ A_2(t) = 1 \int_{\mathbb{R}} \partial_x^2 u \left[ D_x^{a+1} \partial_x \chi_{e,b}^2 \right] \partial_x^2 u \, dx = - \frac{1}{2} \int_{\mathbb{R}} \partial_x^2 u \left[ \mathcal{H} D_x^{a+2} \chi_{e,b}^2 \right] \partial_x^2 u \, dx. \]

Although this stage of the process is related to the one performed in step 1 (for \( j = 1 \)), we will use again the commutator expansion in (3.20), taking into account in this case that \( a = \alpha + 2 > 1 \) and \( n \) is a non-negative integer whose value will be fixed later.

Then,

\[ A_2(t) = \frac{1}{2} \int_{\mathbb{R}} \partial_x^2 u \left[ D_x^{a+1} \partial_x \chi_{e,b}^2 \right] \partial_x^2 u \, dx + \frac{1}{4} \int_{\mathbb{R}} \partial_x^2 u \left[ \mathcal{H} D_x^{a+2} \chi_{e,b}^2 \right] \partial_x^2 u \, dx \]

\[ = A_{2,1}(t) + A_{2,2}(t) + A_{2,3}(t). \]

Essentially, the key term which allows us to fix the value of \( n \), correspond to \( A_{2,1} \).

Indeed, after some integration by parts

\[ A_{2,1}(t) = \frac{1}{2} \int_{\mathbb{R}} u \partial_x^2 R_n(\alpha + 2) \partial_x^2 u \, dx \]

\[ = \frac{1}{2} \int_{\mathbb{R}} u \partial_x^2 R_n(\alpha + 2) \partial_x^2 u \, dx. \]

We fix \( n \) such that it satisfies

\[ 2n + 1 \leq a + 2 \sigma \leq 2n + 3. \]

In this case with \( a = \alpha + 2 > 1 \) and \( \sigma = 2 \), we obtain \( n = 2 \).
Hence by construction the Proposition 3.24 guarantees that $D_x^2 R_2(\alpha + 2) D_x^2$ is bounded in $L_x^2$. Therefore

$$|A_{2,1}(t)| \lesssim \|u(t)\|_{L_1^3}^2 \|D_x^2 R_2(\alpha + 2) D_x^2 u\|_{L_x^1} = c \|u_0\|_{L_x^3}^2 \|D_x^{\alpha + 6} \chi_{c,b}\|_{L_x^1}.$$ 

Since we fixed $n = 2$, we proceed to handle the contribution coming from $A_{2,2}$ and $A_{2,3}$.

Next, $A_{2,2}(t)$

$$= \left(\frac{\alpha + 2}{4}\right) \int_{\mathbb{R}} \left(D_x^{\alpha + 4} c_x^2 u\right)^2 \left(\chi_{c,b}\right) \, dx - c_3 \left(\frac{\alpha + 2}{16}\right) \int_{\mathbb{R}} \left(D_x^{1 + \frac{\alpha + 4}{2}} u\right)^2 \left(\chi_{c,b}\right)^{(3)} \, dx$$

$$+ c_5 \left(\frac{\alpha + 2}{64}\right) \int_{\mathbb{R}} \left(D_x^{\alpha + 4} u\right)^2 \left(\chi_{c,b}\right)^{(5)} \, dx$$

$$= A_{2,2,1}(t) + A_{2,2,2}(t) + A_{2,2,3}(t).$$

Notice that $A_{2,2,1} \geq 0$ represents the smoothing effect.

We recall that

$$\left|\chi_{c,b}^{(j)}(x)\right| \lesssim \chi_{c,b}^{(j)}(x) \quad \forall x \in \mathbb{R}, \ j \in \mathbb{Z}^+,$$

then

$$\int_0^T |A_{2,2,2}(t)| \, dt \lesssim \int_0^T \int_{\mathbb{R}} \left(D_x^{1 + \frac{\alpha + 4}{2}} u\right)^2 \chi_{c/3,b+\epsilon} \, dx \, dt.$$ 

Taking $(c, b) = (e/9, b + 10e/9)$ in (6.5) combined with the properties of the cutoff function we have

$$\int_0^T |A_{2,2,2}(t)| \, dt \lesssim c_{1,1}^*.$$ 

To finish the terms that make $A_2$ we proceed to estimate $A_{2,2,3}$.

As usual the low regularity is controlled by interpolation and the local theory. Therefore

$$\int_0^T |A_{2,2,3}(t)| \, dt \lesssim \|u\|_{L_{\infty}^3 L_{\infty}^{\frac{1+\epsilon}{2}}}.$$ 

Next, $A_{2,3}(t)$

$$= \left(\frac{\alpha + 2}{4}\right) \int_{\mathbb{R}} \left(D_x^{\alpha + 4} c_x^2 u\right)^2 \left(\chi_{c,b}\right) \, dx - c_3 \left(\frac{\alpha + 2}{16}\right) \int_{\mathbb{R}} \left(D_x^{\alpha + 4} c_x u\right)^2 \left(\chi_{c,b}\right)^{(3)} \, dx$$

$$+ c_5 \left(\frac{\alpha + 2}{64}\right) \int_{\mathbb{R}} \left(D_x^{\alpha + 4} u\right)^2 \left(\chi_{c,b}\right)^{(5)} \, dx$$

$$= A_{2,3,1}(t) + A_{2,3,2}(t) + A_{2,3,3}(t).$$

$A_{2,3,1}$ is positive and it will provide the smoothing effect after being integrated in time.

The terms $A_{2,3,2}$ and $A_{2,3,3}$ can be handled exactly in the same way that were treated $A_{2,2,2}$ and $A_{2,2,3}$ respectively, so we will omit the proof.
Finally,
\[
A_3(t) = 3 \int_{\mathbb{R}} \partial_x u (\partial_x^2 u)^2 \chi_{\epsilon,b}^2 \, dx + \int_{\mathbb{R}} u \partial_x \partial_x^3 u \partial_x^2 u \chi_{\epsilon,b}^2 \, dx
\]
\[
= \frac{5}{2} \int_{\mathbb{R}} \partial_x u (\partial_x^2 u)^2 \chi_{\epsilon,b}^2 \, dx - \frac{1}{2} \int_{\mathbb{R}} u (\partial_x^2 u)^2 (\chi_{\epsilon,b}^2)' \, dx
\]
\[
= A_{3,1}(t) + A_{3,2}(t).
\]

First,
\[
|A_{3,1}(t)| \lesssim \| \partial_x u(t) \|_{L^\infty} \int_{\mathbb{R}} (\partial_x^2 u)^2 \chi_{\epsilon,b}^2 \, dx,
\]
by the local theory \( \partial_x u \in L^1([0, T]; L^\infty(\mathbb{R})) \) (see Theorem C-(b)); and the integral expression is the quantity we want estimate.

Next,
\[
|A_{3,2}(t)| \lesssim \| u(t) \|_{L^\infty} \int_{\mathbb{R}} (\partial_x^2 u)^2 (\chi_{\epsilon,b}^2)' \, dx.
\]

After apply the Sobolev embedding and integrate in the time variable we obtain
\[
\int_0^T |A_{3,2}(t)| \, dt \lesssim \left( \sup_{0 \leq t \leq T} \| u(t) \|_{H^{\frac{3}{2}}(\mathbb{R})} \right) \int_0^T \int_{\mathbb{R}} (\partial_x^2 u)^2 (\chi_{\epsilon,b}^2)' \, dx \, dt,
\]
and the integral term in the right hand side was estimated previously in (6.13).

Thus, after grouping all the terms and apply Gronwall’s inequality we obtain
\[
\sup_{0 \leq t \leq T} \int_{\mathbb{R}} (\partial_x^2 u)^2 \chi_{\epsilon,b}^2 \, dx + \int_0^T \int_{\mathbb{R}} (D^{\frac{\alpha+1}{2}} \partial_x^2 u)^2 (\chi_{\epsilon,b}^2)' \, dx \, dt
\]
\[
+ \int_0^T \int_{\mathbb{R}} (D^{\frac{\alpha+1}{2}} \partial_x^2 u)^2 (\chi_{\epsilon,b}^2)' \, dx \, dt \leq c_{2,1}^t
\]
where \( c_{2,1}^t = c_{2,1}^t(\alpha; \epsilon; T; v; \| u_0 \|_{H^{\frac{3}{2}}}; \| \partial_x^2 u_0 \chi_{\epsilon,b} \|_{L^2}) \), for any \( \epsilon > 0, b \geq 5\epsilon \) and \( v > 0 \).

Step 2.

From equation in (1.1) one gets after applying the operator \( D^{\frac{1+\alpha}{2}} \partial_x^2 u \) and multiplying the result by \( D^{\frac{1+\alpha}{2}} \partial_x^2 u \chi_{\epsilon,b}^2 (x + vt) \),
\[
D^{\frac{1+\alpha}{2}} \partial_x^2 u D^{\frac{1+\alpha}{2}} \partial_x^2 u \chi_{\epsilon,b}^2 - D^{\frac{1+\alpha}{2}} \partial_x^2 u D^{1+\alpha} \partial_x u D^{\frac{1+\alpha}{2}} \partial_x^2 u \chi_{\epsilon,b}^2
\]
\[
+ D^{\frac{1+\alpha}{2}} \partial_x^2 (u \partial_x u) D^{\frac{1+\alpha}{2}} \partial_x^2 u \chi_{\epsilon,b}^2 = 0
\]
which after integration in the spatial variable it becomes

\[
\frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}} \left( D_x^{\frac{1}{2} + \alpha} \partial_x^2 u \right)^2 \chi_{\varepsilon, b}^2 \, dx - \frac{v}{2} \int_{\mathbb{R}} \left( D_x^{\frac{1}{2} + \alpha} \partial_x^2 u \right)^2 (\chi_{\varepsilon, b}^2)' \, dx
\]

\[
- \int_{\mathbb{R}} \left( D_x^{\frac{1}{2} + \alpha} \partial_x^2 D_x^{1+a} \partial_x u \right) \left( D_x^{\frac{1}{2} + \alpha} \partial_x^2 u \right) \chi_{\varepsilon, b}^2 \, dx
\]

\[
+ \int_{\mathbb{R}} \left( D_x^{\frac{1}{2} + \alpha} \partial_x^2 (u \partial_x u) \right) \left( D_x^{\frac{1}{2} + \alpha} \partial_x^2 u \right) \chi_{\varepsilon, b}^2 \, dx = 0.
\]

To estimate $A_1$ we will use different techniques to the ones implemented to bound $A_1$ in the previous step. The main difficulty we have to face is to deal with the non-local character of the operator $D_x^s$ for $s \in \mathbb{R}^+ \setminus 2\mathbb{N}$, the case $s \in 2\mathbb{N}$ is less complicated because $D_x^s$ becomes local, so we can integrate by parts.

The strategy to solve this issue will be the following. In (6.17) we proved that $u$ has a gain of $\frac{\alpha+1}{2}$ derivatives (local) which in total sum $2 + \frac{\alpha+1}{2}$. This suggests that if we can find an appropriated channel where we can localize the smoothing effect, we shall be able to recover all the local derivatives $r$ with $r \leq 2 + \frac{\alpha+1}{2}$.

Henceforth we will employ recurrently a technique of localization of commutator used by Kenig, Linares, Ponce and Vega [23] in the study of propagation of regularity (fractional) for solutions of the k-generalized KdV equation. Indeed, the idea consists in constructing an appropriate system of smooth partition of unit, localizing the regions where is available the information obtained in the previous cases.

We recall that for $\varepsilon > 0$ and $b \geq 5\varepsilon$

(6.18) \[ \eta_{\varepsilon, b} = \sqrt{\chi_{\varepsilon, b} \chi_{\varepsilon, b}'} \quad \text{and} \quad \chi_{\varepsilon, b} + \phi_{\varepsilon, b} + \psi_{\varepsilon} = 1. \]

\section{Claim}

(6.19) \[ \left\| D_x^{\frac{1}{2} + \alpha} \partial_x^2 (u \eta_{\varepsilon, b}) \right\|_{L^2_x L^2_t} < \infty. \]
Combining the commutator estimate (3.14), (6.18), Hölder’s inequality and (6.17) yields

\[
\left\| \frac{D^2}{D^2_x} \varphi_2^2(u\eta_{e,b}) \right\|_{L^1_T L^2_x} \leq \left\| \frac{D^2}{D^2_x} \psi_{e,b} \right\|_{L^1_T L^2_x} + \left\| \frac{D^2}{D^2_x} (u\varphi_{e,b} + u\eta_{e,b}) \right\|_{L^1_T L^2_x} \lesssim (c^*_{2,1})^2 + \left\| \frac{D^2}{D^2_x} (u\varphi_{e,b}) \right\|_{L^1_T L^2_x} + \left\| \frac{D^2}{D^2_x} (u\eta_{e,b}) \right\|_{L^1_T L^2_x} + \left\| u_0 \right\|_{L^2_x}.
\]

(6.20)

Since \( \varphi_{e,b} = 1 \) on the support of \( \varphi_{e,b} \), then

\[ \varphi_{e,b}(x) \varphi_{e,b}(x) = \varphi_{e,b}(x) \quad \forall x \in \mathbb{R}. \]

Thus, combining Lemma 3.15 and Young’s inequality we obtain

\[
\left\| \frac{D^2}{D^2_x} (u\varphi_{e,b}) \right\|_{L^1_T L^2_x} \lesssim \left\| \varphi_{2,1}^2 (u\varphi_{e,b}) \right\|_{L^1_T L^2_x} \left\| u_0 \right\|_{L^2_x} \lesssim \left\| \varphi_{2,1}^2 (u\varphi_{e,b}) \right\|_{L^1_T L^2_x} + \left\| u_0 \right\|_{L^2_x} = c \left\| \varphi_{2,1}^2 u\varphi_{e,b} + 2\varphi_{2,1}^2 u\varphi_{e,b} \right\|_{L^1_T L^2_x} + \left\| u_0 \right\|_{L^2_x} \lesssim \left\| \varphi_{2,1}^2 u\varphi_{e,b} \right\|_{L^1_T L^2_x} + \left\| u_0 \right\|_{L^2_x}.
\]

(6.21)

Then, by an application of (6.17) adapted to every case yields

\[
B_1 \lesssim \left\| \varphi_{2,1}^2 u\varphi_{e,b} \right\|_{L^1_T L^2_x} + \left\| \varphi_{2,1}^2 u\varphi_{e,b} \right\|_{L^1_T L^2_x} + \left\| u_0 \right\|_{L^2_x} \lesssim c^*_{2,1} + c^*_{2,1} + \left\| u_0 \right\|_{L^2_x}.
\]

(6.22)

Notice that \( B_2 \) was estimated in the case \( j = 1 \), step 2 see (6.10), so we will omit the proof. Next, we recall that by construction

\[ \text{dist}(\text{supp}(\eta_{e,b}), \text{supp}(\psi_e)) \geq \frac{\varepsilon}{2}. \]

Hence by Lemma 3.16

\[
B_3 = \left\| \eta_{e,b} \frac{D^2}{D^2_x} (u\psi_e) \right\|_{L^1_T L^2_x} \lesssim \left\| \eta_{e,b} \right\|_{L^1_T L^2_x} \left\| u_0 \right\|_{L^2_x}.
\]

(6.23)

The claim follows gathering the calculations above.

At this point we have proved that locally in the interval \([e, b]\) there exists \( 2 + \frac{\alpha+1}{2} \) derivatives. By Lemma 3.15 we get

\[
\left\| \frac{D^2}{D^2_x} (u\eta_{e,b}) \right\|_{L^1_T L^2_x} \lesssim \left\| \frac{D^2}{D^2_x} (u\eta_{e,b}) \right\|_{L^1_T L^2_x} + \left\| u_0 \right\|_{L^2_x} < \infty.
\]
As before
\[ D_x^{2+\frac{1+\alpha}{2}} u\eta_{e,b} = D_x^{2+\frac{1+\alpha}{2}} (u\eta_{e,b}) - \left[ D_x^{2+\frac{1+\alpha}{2}}; \eta_{e,b} \right] (u\chi_{e,b} + u\phi_{e,b} + u\psi_e). \]

The argument used in the proof of the claim yields
\[ \left\| D_x^{2+\frac{1+\alpha}{2}} u\eta_{e,b} \right\|_{L^2_x L^2_t} < \infty. \]

Therefore,
\[ \int_0^T |A_1(t)| \, dt \leq |v| \int_0^T \left( D_x^{\frac{1+\alpha}{2}} \partial_x^2 u \right)^2 (\chi_{e,b}^2)' \, dx \, dt \]
\[ \lesssim \left\| D_x^{2+\frac{1+\alpha}{2}} u\eta_{e,b} \right\|_{L^2_x L^2_t}^2 < \infty. \]

§.2 Now we focus our attention in the term \( A_2 \). Notice that after integration by parts and Plancherel's identity
\[ A_2(t) = \int_{\mathbb{R}} D_x^{\frac{1+\alpha}{2}} \partial_x^2 u \left[ D_x^{1+\alpha} \partial_x \chi_{e,b}^2 \right] D_x^\alpha \partial_x^2 u \, dx \]
\[ - \int_{\mathbb{R}} \left( D_x^{\frac{1+\alpha}{2}} \partial_x^2 u \right) D_x^{1+\alpha} \partial_x \left( \chi_{e,b}^2 D_x^{\frac{1+\alpha}{2}} \partial_x^2 u \right) \, dx \]
\[ = \int_{\mathbb{R}} D_x^{\frac{1+\alpha}{2}} \partial_x^2 u \left[ D_x^{1+\alpha} \partial_x \chi_{e,b}^2 \right] D_x^{\frac{1+\alpha}{2}} \partial_x^2 u \, dx - A_2(t). \]

Consequently
\[ A_2(t) = \frac{1}{2} \int_{\mathbb{R}} D_x^{\frac{1+\alpha}{2}} \partial_x^2 u \left[ D_x^{1+\alpha} \partial_x \chi_{e,b}^2 \right] D_x^\alpha \partial_x^2 u \, dx \]
\[ = - \frac{1}{2} \int_{\mathbb{R}} D_x^{\frac{1+\alpha}{2}} u \left[ \mathcal{H} D_x^{2+\alpha} \chi_{e,b}^2 \right] D_x^\alpha u \, dx. \]

The procedure to decompose the commutator will be almost similar to the introduced in the previous step, the main difference relies on the fact that the quantity of derivatives is higher in comparison with the step 1.

Concerning this, we notice that \( 2 + \alpha > 1 \) and by (3.20) the commutator \([\mathcal{H} D_x^{\alpha+2}; \chi_{e,b}^2]\) can be decomposed as
\[ [\mathcal{H} D_x^{\alpha+2}; \chi_{e,b}^2] + \frac{1}{2} p_n(\alpha + 2) + r_n(\alpha + 2) = \frac{1}{2} \mathcal{H} p_n(\alpha + 2) \mathcal{H} \]
for some positive integer \( n \). We shall fix the value of \( n \) satisfying a suitable condition.
Replacing (6.26) into (6.25) produces

\begin{equation}
A_2(t) = \frac{1}{2} \int_{\mathbb{R}} D_{x}^{5-a} u \left( R_n(\alpha + 2) D_{x}^{5-a} u \right) dx + \frac{1}{4} \int_{\mathbb{R}} D_{x}^{5-a} u \left( P_n(\alpha + 2) D_{x}^{5-a} u \right) dx
- \frac{1}{4} \int_{\mathbb{R}} D_{x}^{5-a} u \left( \mathcal{H} P_n(\alpha + 2) \mathcal{H} D_{x}^{5-a} u \right) dx
= A_{2,1}(t) + A_{2,2}(t) + A_{2,3}(t).
\end{equation}

Now we proceed to fix the value of \( n \) present in \( A_{2,1}, A_{2,2} \) and \( A_{2,3} \).

First we deal with the term that determines the value \( n \) in the decomposition associated to \( A_2 \). In this case it corresponds to \( A_{2,1} \).

Applying Plancherel’s identity, \( A_{2,1} \) becomes

\[ A_{2,1}(t) = \frac{1}{2} \int_{\mathbb{R}} u \left( D_{x}^{5-a} R_n(\alpha + 2) D_{x}^{5-a} u \right) dx. \]

We fix \( n \) such that it satisfies (3.25) i.e.,

\[ 2n + 1 \leq a + 2\sigma \leq 2n + 3 \]

with \( a = \alpha + 2 \) and \( \sigma = \frac{5-a}{2} \), which produces \( n = 2 \) or \( n = 3 \). Nevertheless, for the sake of simplicity we take \( n = 2 \).

Hence, by construction \( R_2(\alpha + 2) \) is bounded in \( L^2_x \) (see Proposition 3.24).

Thus,

\[ \int_0^T |A_{2,1}(t)| dt \leq c \int_0^T \| u(t) \|_{L^2_x} \| D_x^2(\overline{\chi_{e,b}^2}(\cdot + vt)) \|_{L^1_x} dt \]
\[ \lesssim \| u_0 \|_{L^2_x} \sup_{0 \leq t \leq T} \| D_x^2(\overline{\chi_{e,b}^2}) \|_{L^1_x}. \]

Since we have fixed \( n = 2 \), we obtain after replace \( P_2(\alpha + 2) \) into \( A_{2,2} \)

\begin{align*}
A_{2,2}(t) &= \left( \frac{\alpha + 2}{4} \right) \left( \int_{\mathbb{R}} \mathcal{H} \partial_x^3 u \right)^2 \left( \chi_{e,b}^2 \right)^{'} dx - c_3 \left( \frac{\alpha + 2}{16} \right) \left( \int_{\mathbb{R}} \partial_x^2 u \right)^2 \left( \chi_{e,b}^2 \right)^{(3)} dx
+ c_5 \left( \frac{\alpha + 2}{64} \right) \int_{\mathbb{R}} \mathcal{H} \partial_x u \left( \chi_{e,b}^2 \right)^{(5)} dx \\
&= A_{2,2,1}(t) + A_{2,2,2}(t) + A_{2,2,3}(t).
\end{align*}

We underline that \( A_{2,2,1} \) is positive and represents the smoothing effect.

On the other hand, by (6.11) with \((\epsilon, b) = (\epsilon / 5, \epsilon)\) we have

\begin{equation}
\left( \int_0^T |A_{2,2,2}(t)| dt \right) c \int_{\mathbb{R}} \left( \partial_x^2 u \right)^2 \left( \chi_{5,\epsilon}^2 \right)^{'''} dx dt \\
\lesssim \sup_{0 \leq t \leq T} \int_{\mathbb{R}} \left( \partial_x^2 u \right)^2 \left( \chi_{5,\epsilon}^2 \right)^{''} dx \lesssim c_1^*. 
\end{equation}

Next, by the local theory

\begin{equation}
\int_0^T |A_{2,2,3}(t)| dt \lesssim \| u \|_{L^\infty T^\frac{5-a}{2}}. 
\end{equation}
After replacing $P_2(\alpha + 2)$ into $A_{2,3}$, and using the fact that Hilbert transform is skew adjoint

$$A_{2,3}(t) = \left(\frac{\alpha + 2}{4}\right) \int_{\mathbb{R}} (\partial_x^3 u)^2 (\chi_{e,b}^2) \, dx - c_3 \left(\frac{\alpha + 2}{16}\right) \int_{\mathbb{R}} (\mathcal{H} \partial_x^2 u)^2 (\chi_{e,b}^2) \, dx$$

$$+ c_5 \left(\frac{\alpha + 2}{64}\right) \int_{\mathbb{R}} (\partial_x u)^2 (\chi_{e,b})^{(5)} \, dx$$

$$= A_{2,3,1}(t) + A_{2,3,2}(t) + A_{2,3,3}(t).$$

Notice that $A_{2,3,1} \geq 0$ and it represents the smoothing effect. However, $A_{2,3,2}$ can be handled if we take $(e, b) = (e/5, e)$ in (6.5) as follows

$$A_{2,3,3}(t) = \int_{\mathbb{R}} (\partial_x^2 u)^2 (\chi_{e,b}^2) \, dx \lesssim \int_{\mathbb{R}} (\partial_x u)^2 (\chi_{e,b}^2) \, dx,$$

thus,

$$\int_0^T |A_{2,3,3}(t)| \, dt \lesssim \sup_{0 \leq t \leq T} \int_{\mathbb{R}} (\partial_x u)^2 (\chi_{e,b}^2) \, dx \lesssim c_1^*. $$

To finish the estimate of $A_2$ only remains to bound $A_{2,3,2}$. To do this we recall that

$$|\chi_{e,b}(x)| \lesssim \chi_{e/3,b+\epsilon}(x), \quad \forall x \in \mathbb{R}, j \in \mathbb{Z}^+,$$

that joint with the property (9) of $\chi_{e,b}$ yields

$$\int_0^T \int_{\mathbb{R}} (\mathcal{H} \partial_x^2 u)^2 \chi_{e/3,b-i-\epsilon} \, dx \, dt \lesssim \int_0^T \int_{\mathbb{R}} (\mathcal{H} \partial_x^2 u)^2 \chi_{e/9,b+10\epsilon/9\chi_{e/9,b+10\epsilon/9}} \, dx \, dt$$

$$\lesssim c_1^*,$$

where the last inequality is obtained taking $(e, b) = (e/9, b + 10\epsilon/9)$ in (6.11). The term $A_{2,3,3}$ can be handled by interpolation and the local theory.

Finally we turn our attention to $A_3$. We start rewriting the nonlinear part as follows

$$D_{x}^{\frac{1-a}{2}} \partial_x^3 (u \partial_x u) \chi_{e,b}$$

$$= - \left[ D_{x}^{\frac{1-a}{2}} \partial_x^2 \chi_{e,b} \right] (u \partial_x u) + D_{x}^{\frac{1-a}{2}} \partial_x^2 (\chi_{e,b} u \partial_x u)$$

$$= - \frac{1}{2} \left[ D_{x}^{\frac{1-a}{2}} \partial_x^2 \chi_{e,b} \right] \partial_x (u^2) + \left[ D_{x}^{\frac{1-a}{2}} \partial_x^2 \chi_{e,b} \right] \partial_x u + u \chi_{e,b} D_{x}^{\frac{1-a}{2}} \partial_x^3 u$$

$$= - \frac{1}{2} \left[ D_{x}^{\frac{1-a}{2}} \partial_x^2 \chi_{e,b} \right] \partial_x \left( (u \chi_{e,b})^2 + (u \psi_{e,b})^2 + (\psi_{e} u^2) \right)$$

$$+ \left[ D_{x}^{\frac{1-a}{2}} \partial_x^2 \chi_{e,b} \right] \partial_x \left( (u \chi_{e,b}) + (u \psi_{e,b}) + (u \psi_{e}) \right)$$

$$= \widetilde{A}_{3,1}(t) + \widetilde{A}_{3,2}(t) + \widetilde{A}_{3,3}(t) + \widetilde{A}_{3,4}(t) + \widetilde{A}_{3,5}(t) + \widetilde{A}_{3,6}(t) + \widetilde{A}_{3,7}(t).$$
Hence, after replacing (6.29) into $A_3$ and apply Hölder’s inequality

$$A_3(t) = \sum_{1 \leq m \leq 6} \int_{\mathbb{R}} \widetilde{A}_{3,m}(t) D_x^{\frac{1}{2}} \partial_x^2 u \chi_{e,b} \, dx + \int_{\mathbb{R}} \widetilde{A}_{3,7}(t) D_x^{\frac{1}{2}} \partial_x^2 u \chi_{e,b} \, dx$$

$$\leq \sum_{1 \leq m \leq 6} \| \widetilde{A}_{3,m}(t) \|_{L_x^2} \left\| D_x^{2 + \frac{1}{2}} u(t) \chi_{e,b}(\cdot + vt) \right\|_{L_x^2} + \int_{\mathbb{R}} \widetilde{A}_{3,7}(t) D_x^{\frac{1}{2}} \partial_x^2 u \chi_{e,b} \, dx$$

$$= \left\| D_x^{2 + \frac{1}{2}} u(t) \chi_{e,b}(\cdot + vt) \right\|_{L_x^2} \sum_{1 \leq m \leq 6} \| \widetilde{A}_{3,m}(t) \|_{L_x^2} + \int_{\mathbb{R}} \widetilde{A}_{3,7}(t) D_x^{\frac{1}{2}} \partial_x^2 u \chi_{e,b} \, dx$$

$$= \left\| D_x^{2 + \frac{1}{2}} u(t) \chi_{e,b}(\cdot + vt) \right\|_{L_x^2} \sum_{1 \leq m \leq 6} A_{3,m}(t) + A_{3,7}(t)$$

Notice that the first factor in the right hand side is the quantity to be estimated by Gronwall’s inequality. So, we shall focus on establish control in the remaining terms.

First, combining (3.4), (3.14) and Lemma 3.15 one gets that

$$\| \widetilde{A}_{3,1}(t) \|_{L_x^2} = \left\| \left[ D_x^{2 + \frac{1}{2}} ; \chi_{e,b} \right] \partial_x ((u \chi_{e,b})^2) \right\|_{L_x^2}$$

$$\lesssim \left\| D_x^{2 + \frac{1}{2}} (u \chi_{e,b}) \right\|_{L_x^2} \| u \|_{L_x^2} + \| u_0 \|_{L_x^2} \| u \|_{L_x^2},$$

and

$$\| \widetilde{A}_{3,2}(t) \|_{L_x^2} = \left\| \left[ D_x^{2 + \frac{1}{2}} ; \chi_{e,b} \right] \partial_x ((u \phi_{e,b})^2) \right\|_{L_x^2}$$

$$\lesssim \left\| D_x^{2 + \frac{1}{2}} (u \phi_{e,b}) \right\|_{L_x^2} \| u \|_{L_x^2} + \| u_0 \|_{L_x^2} \| u \|_{L_x^2}.$$
Meanwhile,

\[
\|\widetilde{A}_{3,5}(t)\|_{L^2_x} = \left\| \left[ D^{2+\frac{1-a}{2}}_x u\chi_{e,b} \right] \partial_x (u\psi_{e,b}) \right\|_{L^2_x} \\
\lesssim \|\partial_x (u\chi_{e,b})\|_{L^2_x} \left\| D^{2+\frac{1-a}{2}}_x (u\psi_{e,b}) \right\|_{L^2_x} + \|\partial_x (u\psi_{e,b})\|_{L^2_x} \left\| D^{2+\frac{1-a}{2}}_x (u\chi_{e,b}) \right\|_{L^2_x}.
\]

Next, we recall that by construction

\[
\text{dist} (\text{supp} (\chi_{e,b}), \text{supp} (\psi_e)) \geq \frac{\varepsilon}{2}.
\]

Thus by Lemma 3.16

\[
\|\widetilde{A}_{3,6}(t)\|_{L^2_x} = \left\| u\chi_{e,b} \partial_x D^{2+\frac{1-a}{2}}_x (u\psi_e) \right\|_{L^2_x} \\
\lesssim \|u_0\|_{L^2_x} \|u\|_{L^2_t}.
\]

To complete the estimates in (6.31)-(6.32) only remains to bound \( \left\| D^{2+\frac{1-a}{2}}_x (u\chi_{e,b}) \right\|_{L^2_x} \), \( \left\| D^{2+\frac{1-a}{2}}_x (u\phi_{e,b}) \right\|_{L^2_x} \), and \( \left\| D^{2+\frac{1-a}{2}}_x (u\psi_{e,b}) \right\|_{L^2_x} \).

For the first term we proceed by writing

\[
D^{2+\frac{1-a}{2}}_x (u\chi_{e,b}) = D^{2+\frac{1-a}{2}}_x u\chi_{e,b} + \left[ D^{2+\frac{1-a}{2}}_x \chi_{e,b} \right] (u\chi_{e,b} + u\phi_{e,b} + u\psi_e) = I_1 + I_2 + I_3 + I_4.
\]

Notice that \( \|I_1\|_{L^2_x} \) is the quantity to be estimated by Gronwall's inequality. Meanwhile, \( \|I_2\|_{L^2_x} \), \( \|I_3\|_{L^2_x} \) and \( \|I_4\|_{L^2_x} \) were estimated previously in the case \( j = 1 \), step 2.
Next, we focus on estimate the term \(\|D^{2+\frac{1+\alpha}{2}}_x \left( u \phi_{e,b} \right) \|_{L^2_x}\) which will be treated by means of Hölder’s inequality and Theorem 3.7 as follows

\[
\|D^{2+\frac{1+\alpha}{2}}_x \left( u \phi_{e,b} \right) \|_{L^2_x}
\]

\[\lesssim \|u\|_{L^4_x} \left\| D^{2+\frac{1+\alpha}{2}}_x \phi_{e,b} \right\|_{L^4_x} + \left\| \sum_{\beta \leq 2} \frac{1}{\beta!} c^\beta_x \phi_{e,b} D^\beta_x u \right\|_{L^2_x} \]

\[\lesssim \|u_0\|_{L^2_x} \left\| u^{1/2} \right\|_{L^{\infty}_x} + \left\| \chi_{e/4,b \geq e/4} D^{2+\frac{1+\alpha}{2}}_x u \right\|_{L^2_x} + \left\| \chi_{e/4,b \geq e/4} \frac{D^{1+\alpha}}{\partial x} \nabla u \right\|_{L^2_x} \]

\[\lesssim \|u_0\|_{L^2_x} \left\| u^{1/2} \right\|_{L^{\infty}_x} + \left\| \chi'_{e/8,b ^{+} + e/4} D^{2+\frac{1+\alpha}{2}}_x u \right\|_{L^2_x} + \left\| \chi'_{e/8,b ^{+} + e/4} \frac{D^{1+\alpha}}{\partial x} \nabla u \right\|_{L^2_x} \]

\[\lesssim \|u_0\|_{L^2_x} \left\| u^{1/2} \right\|_{L^{\infty}_x} + \left\| \eta_{e/24,b ^{+} + 7e/24} D^{2+\frac{1+\alpha}{2}}_x u \right\|_{L^2_x} + \left\| \eta_{e/24,b ^{+} + 7e/24} \frac{D^{1+\alpha}}{\partial x} \nabla u \right\|_{L^2_x} \]

\[\lesssim \|u_0\|_{L^2_x} \left\| u^{1/2} \right\|_{L^{\infty}_x} + \left\| \eta_{24,b ^{+} + 7e/24} D^{2+\frac{1+\alpha}{2}}_x u \right\|_{L^2_x} + \left\| \eta_{24,b ^{+} + 7e/24} \frac{D^{1+\alpha}}{\partial x} \nabla u \right\|_{L^2_x} \]

After integrate in time, the second and third term on the right hand side can be estimated taking \((e,b) = (e/24,b ^{+} + 7e/24)\) in (6.17) and (6.5) respectively. Hence, after integrate in time follows by interpolation that \(\|D^{2+\frac{1+\alpha}{2}}_x \left( u \phi_{e,b} \right) \|_{L^2_x} \) is finite.

Analogously can be bounded \(\|D^{2+\frac{1+\alpha}{2}}_x \left( u \phi_{e,b} \right) \|_{L^2_x} \).

§.3 Finally, after apply integration by parts

\[A_{3,7}(t) = \int_{\mathbb{R}} \tilde{A}_{3,7}(t) D^{\frac{1+\alpha}{2}}_x \partial^2_x \chi_{e,b} \, dx \]

\[= -\frac{1}{2} \int_{\mathbb{R}} \partial_x u \chi^2_{e,b} \left( D^{\frac{1+\alpha}{2}}_x \partial^2_x u \right)^2 \, dx - \int_{\mathbb{R}} u \chi_{e,b} \chi'_{e,b} \left( D^{\frac{1+\alpha}{2}}_x \partial^2_x u \right)^2 \, dx \]

\[= A_{3,7,1}(t) + A_{3,7,2}(t). \]

First,

\[|A_{3,7,1}(t)| \lesssim \|\partial_x u(t)\|_{L^{\infty}_x} \int_{\mathbb{R}} \left( D^{\frac{1+\alpha}{2}}_x \partial^2_x u \right)^2 \chi^2_{e,b} \, dx, \]

where the last integral is the quantity that will be estimated using Gronwall’s inequality, and the other factor will be controlled after integration in time.
After integration in time and Sobolev's embedding it follows that

\[
\int_0^T |A_{3,7,2}(t)| \, dt \lesssim \int_0^T \int \left( D_x^{\frac{1+\alpha}{2}} \partial_x^2 u \right)^2 \, dx \, dt
\]

\[
\lesssim \left( \sup_{0 \leq t \leq T} \| u(t) \|_{H^{\alpha+1}_x(\mathbb{R})} \right) \int_0^T \int \left( D_x^{\frac{1+\alpha}{2}} \partial_x^2 u \right)^2 (\Lambda_{\epsilon, b})' \, dx \, dt
\]

and the last term was already estimated in (6.24).

Thus, after collecting all the information in this step and applying Gronwall's inequality together with hypothesis (1.11), we obtain

\[
\sup_{0 \leq t \leq T} \int_0^T \int (D_x^{\frac{1+\alpha}{2}} \partial_x^2 u)^2 \Lambda_{\epsilon, b}^2 \, dx \, dt + \int_0^T \int (\partial_x^3 u)^2 \Lambda_{\epsilon, b}^2 \, dx \, dt \leq c_{2,2}^2
\]

where \( c_{2,2}^2 = c_{2,2}^2 \left( \alpha; \epsilon; T; v; \| u_0 \|_{H_x^{\alpha+1}}; \| D_x^{\frac{1+\alpha}{2}} \partial_x^2 u_0 \|_{L_2^2} \right) \) for any \( \epsilon > 0, b > 5\epsilon \) and \( v > 0 \).

According to the induction argument we shall assume that (1.12) holds for \( j \leq m \) with \( j \in \mathbb{Z} \) and \( j \geq 2 \), i.e.

\[
(6.33)
\]

\[
\sup_{0 \leq t \leq T} \int_0^T \int (\partial_x^j u)^2 \Lambda_{\epsilon, b}^2 \, dx \, dt + \int_0^T \int \left( D_x^{\frac{1+\alpha}{2}} \partial_x^j u \right)^2 (\Lambda_{\epsilon, b})' \, dx \, dt
\]

\[
+ \int_0^T \int \left( \mathcal{H}D_x^{\frac{1+\alpha}{2}} \partial_x^j u \right)^2 \Lambda_{\epsilon, b}^2 \, dx \, dt \leq c_{j,1}^n
\]

for \( j = 1, 2, \ldots, m \) with \( m \geq 1 \), for any \( \epsilon > 0, b > 5\epsilon \) \( v > 0 \).

**Step 2**

We will assume \( j \) an even integer. The case where \( j \) is odd follows by an argument similar to the case \( j = 1 \).

By an analogous reasoning to one employed in the case \( j = 2 \) it follows that

\[
D_x^{\frac{1+\alpha}{2}} \partial_x^j u D_x^{\frac{1+\alpha}{2}} \partial_x^2 u \Lambda_{\epsilon, b}^2 = D_x^{\frac{1+\alpha}{2}} \partial_x^j D_x^{1+\alpha} \partial_x u D_x^{\frac{1+\alpha}{2}} \partial_x^2 u \Lambda_{\epsilon, b}^2
\]

\[
+ D_x^{\frac{1+\alpha}{2}} \partial_x^j (u \partial_x u) D_x^{\frac{1+\alpha}{2}} \partial_x^2 u \Lambda_{\epsilon, b}^2 = 0
\]
which after integration in time yields the identity

\[
\frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}} \left( D_x^{1-a} \partial_x^j u \right)^2 \chi_{e,b}^2 \, dx - \frac{\nu}{2} \int_{\mathbb{R}} \left( D_x^{1-a} \partial_x^j u \right)^2 \left( \chi_{e,b}^2 \right)' \, dx
\]

\[A_1(t)\]

\[
- \int_{\mathbb{R}} \left( D_x^{1-a} \partial_x^j D_x^{1+a} \partial_x u \right) \left( D_x^{1-a} \partial_x^j u \chi_{e,b}^2 \right) \, dx \quad A_2(t)
\]

\[+ \int_{\mathbb{R}} D_x^{1-a} \partial_x^j (u \partial_x u) \left( D_x^{1-a} \partial_x^j u \chi_{e,b}^2 \right) \, dx = 0. \quad A_3(t)
\]

§.1 We claim that

\[
\left\| D_x^{j+\frac{1+a}{2}} (u \eta_{e,b}) \right\|_{L_t^2 L_x^2} < \infty.
\]

We proceed as in the case \(j = 2\). A combination of the commutator estimate (3.14), (6.18), Hölder’s inequality and (6.33) yields

\[
\left\| D_x^{j+\frac{1+a}{2}} (u \eta_{e,b}) \right\|_{L_t^2 L_x^2} \leq \left\| D_x^{j+\frac{1+a}{2}} u \eta_{e,b} \right\|_{L_t^2 L_x^2} + \left\| D_x^{j+\frac{1+a}{2}} \eta_{e,b} \right\|_{L_t^2 L_x^2} \left\| u \chi_{e,b} + u \psi_{e,b} + u \eta_{e,b} \right\|_{L_t^2 L_x^2} \approx (c_{j,1}^e)^2 + \left\| D_x^{j+1+\frac{1+a}{2}} (u \chi_{e,b}) \right\|_{L_t^2 L_x^2} + \left\| u_0 \right\|_{L_x^2} + \left\| D_x^{j+1+\frac{1+a}{2}} (u \psi_{e,b}) \right\|_{L_t^2 L_x^2} + \left\| \eta_{e,b} D_x^{j+\frac{1+a}{2}} (u \psi_{e}) \right\|_{L_t^2 L_x^2}.
\]

Since \(\chi_{e/5e} = 1\) on the support of \(\chi_{e,b}\) then

\[
\chi_{e,b}(x) \chi_{e/5e}(x) = \chi_{e,b}(x) \quad \forall x \in \mathbb{R}.
\]
Combining Lemma 3.15 and Young’s inequality

\( (6.37) \)

\[
\left\| D_x^{j + \frac{a+1}{2}} (uX_{e,b}) \right\|_{L_t^2} \lesssim \left\| \partial_x^j (uX_{e,b}) \right\|_{L_t^2}^{2j+1} \left\| u \right\|_{L_t^{6}}^\frac{1}{2} + \left\| u \right\|_{L_t^{6}}^\frac{1}{2} \]

\[
\lesssim \left\| \partial_x^j (uX_{e,b}) \right\|_{L_t^2} + \left\| u \right\|_{L_t^{6}}^\frac{1}{2} \]

\[
\lesssim \sum_{0 \leq k \leq j} \gamma_{j,k} k^2 \left\| uX_{e,b} \right\|_{L_t^2}^{(j-k)} + \left\| u \right\|_{L_t^{6}}^\frac{1}{2} \]

\[
\lesssim \left\| \partial_x^j uX_{e,b} \right\|_{L_t^2}^2 + \sum_{2 \leq k \leq j-1} \gamma_{j,k} \left\| uX_{e,b} \right\|_{L_t^2}^{(j-k)} \left\| \partial_x^k uX_{e,b} \right\|_{L_t^2}^\frac{1}{2} \]

\[
+ \left\| u \right\|_{L_t^6} \frac{1}{2} + \left\| u \right\|_{L_t^{6}}^{\frac{1}{2}} \]

Hence, taking \((e, b) = (e/5, e)\) in \((6.33)\) yields

\( (6.38) \)

\[
B_1 \lesssim c_{j,1} + \sum_{2 \leq k \leq j-1} \gamma_{j,k} c_{k,1} + \left\| u \right\|_{L_t^6} \frac{1}{2} + \left\| u \right\|_{L_t^6}^{\frac{1}{2}} \]

\( B_2 \) can be estimated as in the step 2 of the case \( j - 1 \), so is bounded by the induction hypothesis.

Next, since

\[
\text{dist} \left( \text{supp} \left( \eta_{e,b} \right), \text{supp} \left( \psi_e \right) \right) \geq \frac{\epsilon}{2} \]

we have by Lemma 3.16

\[
\left\| \eta_{e,b} D_x^{j + \frac{a+1}{2}} (u \psi_e) \right\|_{L_t^2} = \left\| \eta_{e,b} D_x^{j + \frac{a+1}{2}} (u \psi_e) \right\|_{L_t^2} \lesssim \left\| \eta_{e,b}^{\cdot \cdot \cdot} \right\|_{L_t^2} \left\| u \right\|_{L_t^{2}}. \]

Gathering the estimates above follows the claim 1.

We have proved that locally in the interval \([e, b]\) there exists \( j + \frac{a+1}{2} \) derivatives.

So, by Lemma 3.15 we obtain

\[
\left\| D_x^{j + \frac{a+1}{2}} (u \eta_{e,b}) \right\|_{L_t^2} \lesssim \left\| D_x^{j + \frac{a+1}{2}} (u \eta_{e,b}) \right\|_{L_t^2} + \left\| u \right\|_{L_t^{2}} \]

then, as before

\[
D_x^{j + \frac{a+1}{2}} u \eta_{e,b} = c_j D_x^{j + \frac{a+1}{2}} (u \eta_{e,b}) - c_j \left[ D_x^{j + \frac{a+1}{2}} (u \eta_{e,b}) (uX_{e,b} + u\psi_{e,b} + u\psi_e), \right. \]

where \( c_j \) is a constant depending only on \( j \).

Hence, if we proceed as in the proof of claim 1 we have

\( (6.39) \)

\[
\left\| D_x^{j + \frac{a+1}{2}} u \eta_{e,b} \right\|_{L_t^2} < \infty. \]

Therefore

\[
\int_0^T |A_1(t)| \, dt = \left\| D_x^{j + \frac{a+1}{2}} u \eta_{e,b} \right\|_{L_t^2}^2 < \infty. \]
To handle the term $A_2$ we use the same procedure as in the previous steps. First,

$$A_2(t) = \int_{\mathbb{R}} D_x^{1/\nu} \partial_x^j u \left[ D_x^{1+\sigma} \partial_x^2 \chi_{\ell,b}^2 \right] D_x^{1/\nu} \partial_x^j u \, dx - A_2(t)$$

and

$$A_2(t) = -\frac{1}{2} \int_{\mathbb{R}} D_x^{2j+1} u \left[ \mathcal{H} D_x^{2+\alpha} \chi_{\ell,b}^2 \right] D_x^{2j+1} u \, dx. \quad (6.40)$$

Since

$$[\mathcal{H} D_x^{\alpha+2} \chi_{\ell,b}^2] + \frac{1}{2} P_n(\alpha+2) + R_n(\alpha+2) = \frac{1}{2} \mathcal{H} P_n(\alpha+2) \mathcal{H} \quad (6.41)$$

for some positive integer $n$. Replacing (6.41) into (6.40) produces

$$A_2(t) = \frac{1}{2} \int_{\mathbb{R}} D_x^{2j+1} u \left( R_n(\alpha+2) D_x^{2j+1} u \right) \, dx$$

$$+ \frac{1}{4} \int_{\mathbb{R}} D_x^{2j+1} u \left( P_n(\alpha+2) D_x^{2j+1} u \right) \, dx$$

$$- \frac{1}{4} \int_{\mathbb{R}} D_x^{2j+1} u \left( \mathcal{H} P_n(\alpha+2) \mathcal{H} D_x^{2j+1} u \right) \, dx$$

$$= A_{2,1}(t) + A_{2,2}(t) + A_{2,3}(t). \quad (6.42)$$

As above we deal first with the crucial term in the decomposition associated to $A_2$, that is $A_{2,1}$.

Applying Plancherel’s identity yields

$$A_{2,1}(t) = \frac{1}{2} \int_{\mathbb{R}} u \left( D_x^{2j+1} R_n(\alpha+2) D_x^{2j+1} u \right) \, dx.$$

We fix $n$ such that (3.25) is satisfied. In this case we have to take $a = \alpha + 2$ and $\sigma = \frac{2j+1-\alpha}{2}$, to get $n = j$. As occurs in the previous cases it is possible for $n = j+1$.

Thus, by construction $R_j(\alpha+2)$ is bounded in $L^2_x$ (see Proposition 3.24). Then

$$|A_{2,1}(t)| \lesssim \|u_0\|_{L^2_x} \left\| D_x^{2j+3} \chi_{\ell,b}^2 \right\|_{L^1_t},$$

and

$$\int_0^T |A_{2,1}(t)| \, dt \lesssim \|u_0\|_{L^2_x} \sup_{0 \leq t \leq T} \left\| D_x^{2j+3} \chi_{\ell,b}^2 \right\|_{L^1_t}.$$
Replacing $P_j(\alpha + 2)$ into $A_{2,2}$,

$$A_{2,2}(t) = \left( \frac{\alpha + 2}{4} \right) \int_{\mathbb{R}} \left( \mathcal{H} \partial_x^{j+1} u \right)^2 \left( \chi_{\varepsilon,b}^2 \right)' \, dx$$

$$+ \left( \frac{\alpha + 2}{2} \right) \sum_{l=1}^{j} c_{2l+1} (-1)^{l} 4^{-l} \int_{\mathbb{R}} \left( D_x^{j-l+1} u \right)^2 \left( \chi_{\varepsilon,b}^2 \right)^{(2l+1)} \, dx$$

$$= A_{2,2,1}(t) + \sum_{l=1}^{j-1} A_{2,2,l}(t) + A_{2,2,j}(t).$$

Note that $A_{2,2,1}$ is positive and it gives the smoothing effect after integration in time, and $A_{2,2,j}$ is bounded by using the local theory. To handle the remainder terms we recall that by construction

$$\left| (\chi_{\varepsilon,b}^{(j)})(x) \right| \lesssim \chi_{\varepsilon/3,b+\epsilon}(x) \lesssim \chi_{\varepsilon/9,b+10\epsilon/9}(x) \chi_{\varepsilon/9,b+10\epsilon/9}(x)$$

for $x \in \mathbb{R}, j \in \mathbb{Z}^+$.

So that, for $j > 2$

$$\int_0^T |A_{2,2,l}(t)| \, dt \lesssim \int_0^T \int_{\mathbb{R}} \left( D_x^{j-l+1} u \right)^2 \chi_{\varepsilon/3,b+\epsilon} \, dx \, dt$$

$$\lesssim \int_0^T \int_{\mathbb{R}} \left( D_x^{j-l+1} u \right)^2 \chi_{\varepsilon/9,b+10\epsilon/9} \chi_{\varepsilon/9,b+10\epsilon/9} \, dx \, dt,$$

thus if we apply (6.33) with $(e/9, b + 4\epsilon/3)$ instead of $(e,b)$ we obtain

$$\int_0^T \int_{\mathbb{R}} \left( D_x^{j-l+1} u \right)^2 \chi_{\varepsilon/9,b+10\epsilon/9} \chi_{\varepsilon/9,b+10\epsilon/9} \, dx \, dt \leq c^*_{l,2}$$

for $l = 1, 2, \ldots, j - 1$.

Meanwhile,

$$A_{2,3}(t) = \left( \frac{\alpha + 2}{4} \right) \int_{\mathbb{R}} \left( \phi_x^{j+1} u \right)^2 \left( \chi_{\varepsilon,b}^2 \right)' \, dx$$

$$+ \left( \frac{\alpha + 2}{4} \right) \sum_{l=1}^{j} c_{2l+1} (-1)^{l} 4^{-l} \int_{\mathbb{R}} \left( \mathcal{H} D_x^{j-l+1} u \right)^2 \left( \chi_{\varepsilon,b}^2 \right)^{(2l+1)} \, dx$$

$$= A_{2,3,1}(t) + \sum_{l=1}^{j-1} A_{2,3,l}(t) + A_{2,3,j}(t)$$

As we can see $A_{2,3,1} \geq 0$ and it represents the smoothing effect. Besides, applying a similar argument to the employed in (6.43)-(6.45) is possible to bound the remainders terms in (6.45). Anyway,

$$\int_0^T |A_{2,3,l}(t)| \, dt \lesssim c^*_{l,2} \quad 1 \leq l \leq j - 1.$$
§.3 Only remains to estimate $A_3$ to finish this step.

(6.46)

$$D_x^{\frac{1-a}{2}} \partial_x^j (u \partial_x u) \chi_{e,b}$$

$$= - \left[ D_x^{\frac{1-a}{2}} \partial_x^j (u \partial_x u) \right] \chi_{e,b} + D_x^{\frac{1-a}{2}} \partial_x^j \left( \chi_{e,b} u \partial_x u \right)$$

$$= - \frac{1}{2} \left[ D_x^{\frac{1-a}{2}} \partial_x^j \chi_{e,b} \right] \partial_x(u^2) + \left[ D_x^{\frac{1-a}{2}} \partial_x^j \chi_{e,b} u \right] \partial_x u + u \chi_{e,b} D_x^{\frac{1-a}{2}} \partial_x^j (\partial_x u)$$

$$= - \frac{1}{2} \left[ D_x^{\frac{1-a}{2}} \partial_x^j \chi_{e,b} \right] \partial_x( (u \chi_{e,b})^2 + (u \phi_{e,b})^2 + (\psi_{e,b})^2 )$$

$$+ \left[ D_x^{\frac{1-a}{2}} \partial_x^j \chi_{e,b} \right] \partial_x( (u \chi_{e,b}) + (u \phi_{e,b}) + (\psi_{e,b})) + u \chi_{e,b} D_x^{\frac{1-a}{2}} \partial_x^j (\partial_x u)$$

$$= \tilde{A}_{3,1}(t) + \tilde{A}_{3,2}(t) + \tilde{A}_{3,3}(t) + \tilde{A}_{3,4}(t) + \tilde{A}_{3,5}(t) + \tilde{A}_{3,6}(t) + \tilde{A}_{3,7}(t).$$

Replacing (6.46) into $A_3$ and apply Hölder’s inequality

$$A_3(t)$$

$$= \sum_{1 \leq k \leq 6} \int_{\mathbb{R}} \tilde{A}_{3,k}(t) D_x^{\frac{1-a}{2}} \partial_x^j u \chi_{e,b} \, dx + \int_{\mathbb{R}} \tilde{A}_{3,7}(t) D_x^{\frac{1-a}{2}} \partial_x^j u \chi_{e,b} \, dx$$

$$\leq \sum_{1 \leq k \leq 6} \| \tilde{A}_{3,k}(t) \|_{L^2} \| D_x^{\frac{1-a}{2}} u(t) \chi_{e,b} (\cdot + vt) \|_{L^2} + \int_{\mathbb{R}} \tilde{A}_{3,7}(t) D_x^{\frac{1-a}{2}} \partial_x^j u \chi_{e,b} \, dx.$$

$$= \left\| D_x^{\frac{1-a}{2}} u(t) \chi_{e,b} (\cdot + vt) \right\|_{L^2} \sum_{1 \leq k \leq 6} \| \tilde{A}_{3,k}(t) \|_{L^2} + \int_{\mathbb{R}} \tilde{A}_{3,7}(t) D_x^{\frac{1-a}{2}} \partial_x^j u \chi_{e,b} \, dx$$

$$= \left\| D_x^{\frac{1-a}{2}} u(t) \chi_{e,b} (\cdot + vt) \right\|_{L^2} \sum_{1 \leq m \leq 6} A_{3,k}(t) + A_{3,7}(t).$$

The first factor on the right hand side is the quantity to be estimated.

We will start by estimating the easiest term.

$$A_{3,7}(t) = - \frac{1}{2} \int_{\mathbb{R}} \partial_x u \chi_{e,b}^2 \left( D_x^{\frac{1-a}{2}} \partial_x^j u \right)^2 \, dx - \int_{\mathbb{R}} u \chi_{e,b} \chi_{e,b}^2 \left( D_x^{\frac{1-a}{2}} \partial_x^j u \right)^2 \, dx$$

$$= A_{3,7,1}(t) + A_{3,7,2}(t).$$

We have that

$$|A_{3,7,1}(t)| \lesssim \| \partial_x u(t) \|_{L^2} \int_{\mathbb{R}} \left( D_x^{\frac{1-a}{2}} \partial_x^j u \right)^2 \chi_{e,b}^2 \, dx,$$

where the last integral is the quantity that we want to estimate, and the another factor will be controlled after integration in time.

After integration in time and Sobolev’s embedding

$$\int_0^T |A_{3,7,2}(t)| \, dt \lesssim \int_0^T \int_{\mathbb{R}} u \chi_{e,b}^2 \left( D_x^{\frac{1-a}{2}} \partial_x^j u \right) \, dx \, dt$$

$$\lesssim \left( \sup_{0 \leq t \leq T} \| u(t) \|_{H^{\frac{a}{2}}(\mathbb{R})} \right) \int_0^T \left( D_x^{\frac{1-a}{2}} \partial_x^j u \right)^2 \chi_{e,b}^2 \, dx \, dt.$$
where the integral expression on the right hand side was already estimated in (6.39).

To handle the contribution coming from $A_{3,1}$ and $A_{3,2}$ we apply a combination of (3.4), (3.14) and Lemma 3.15 to obtain

$$\|A_{3,1}(t)\|_{L^3_x} = \left\| \left[ D_x^{j + \frac{1}{2}} \chi_{e,b} \right] \partial_x \left( (u\chi_{e,b})^2 \right) \right\|_{L^3_x} \lesssim \left\| D_x^{j + \frac{1}{2}} (u\chi_{e,b}) \right\|_{L^3_x} \|u\|_{L^\infty_x} + \|u_0\|_{L^3_x} \|u\|_{L^\infty_x}$$

(6.47)

and

$$\|A_{3,2}(t)\|_{L^3_x} = \left\| \left[ D_x^{j + \frac{1}{2}} \chi_{e,b} \right] \partial_x \left( (u\phi_{e,b})^2 \right) \right\|_{L^3_x} \lesssim \left\| D_x^{j + \frac{1}{2}} (u\phi_{e,b}) \right\|_{L^3_x} \|u\|_{L^\infty_x} + \|u_0\|_{L^3_x} \|u\|_{L^\infty_x}.$$  

The condition on the supports of $\chi_{e,b}$ and $\phi_e$ combined with Lemma 3.16 implies

$$\|A_{3,3}(t)\|_{L^3_x} \lesssim \|u_0\|_{L^3_x} \|u\|_{L^\infty_x}.$$  

By using (3.2) and (3.14)

$$\|A_{3,4}(t)\|_{L^3_x} \lesssim \left\| \partial_x (u\chi_{e,b}) \right\|_{L^\infty_x} \left\| D_x^{j + \frac{1}{2}} (u\chi_{e,b}) \right\|_{L^3_x}$$

and

$$\|A_{3,5}(t)\|_{L^3_x} \lesssim \left\| \partial_x (u\chi_{e,b}) \right\|_{L^\infty_x} \left\| D_x^{j + \frac{1}{2}} (u\phi_{e,b}) \right\|_{L^3_x} + \left\| \partial_x (u\phi_{e,b}) \right\|_{L^\infty_x} \left\| D_x^{j + \frac{1}{2}} (u\chi_{e,b}) \right\|_{L^3_x}.$$  

An application of Lemma 3.16 leads to

$$\|A_{3,6}(t)\|_{L^3_x} = \left\| u\chi_{e,b} \partial_x D_x^{j + \frac{1}{2}} (u\phi_{e,b}) \right\|_{L^3_x} \lesssim \|u_0\|_{L^3_x} \|u\|_{L^\infty_x}.$$  

(6.48)

To complete the estimate in (6.47)-(6.48) we write

$$\chi_{e,b}(x) + \phi_{e,b}(x) + \psi_e(x) = 1 \quad \forall x \in \mathbb{R};$$

then

$$D_x^{j + \frac{1}{2}} (u\chi_{e,b}) = D_x^{j + \frac{1}{2}} u\chi_{e,b} + \left[ D_x^{j + \frac{1}{2}} \chi_{e,b} \right] (u\chi_{e,b} + u\phi_{e,b} + u\psi_e)$$

$$= I_1 + I_2 + I_3 + I_4.$$  

Notice that $\|I_1\|_{L^3_x}$ is the quantity to be estimated. In contrast, $I_4$ is handled by using Lemma 3.16. In regards to $\|I_2\|_{L^3_x}$ and $\|I_3\|_{L^3_x}$ the Lemma 3.13 combined with the local theory, and the step 2 corresponding to the case $j - 1$ produce the required bounds.
By Theorem 3.7 and Hölder’s inequality

\[
\left\| D_x^{j + \frac{1}{4}} (u \phi_{e,b}) \right\|_{L_x^\infty L_t^2} \lesssim \|u\|_{L_x^{1/2} L_t^{1/2}} \left\| D_x^{j + \frac{1}{4}} \phi_{e,b} \right\|_{L_x^4 L_t^4} + \left\| \mathbf{1}_{\mathcal{B}^l_j} c_\beta \phi_{e,b} D_x^{j - \beta + \frac{3}{4}} u \right\|_{L_x^3 L_t^3} + \sum_{\beta \in \mathcal{Q}_2(j), \beta \neq j} \left\| \mathbf{1}_{\mathcal{B}^l_j} c_\beta \phi_{e,b} \mathcal{H} D_x^{j - \beta + \frac{3}{4}} u \right\|_{L_x^3 L_t^3} \]
\]

(6.49)

where \( \mathcal{Q}_1(j), \mathcal{Q}_2(j) \) denotes odd integers and even integers in \( \{0, 1, \ldots, j\} \) respectively.

To estimate the second term in (6.49), note that \( c_\beta \phi_{e,b} \) is supported in \([\mathbf{e}/4, b]\) then

\[
\sum_{\beta \in \mathcal{Q}_1(j)} \frac{1}{\beta!} \left\| \mathbf{1}_{[\mathbf{e}/8,b]} D_x^{j - \beta + \frac{3}{4}} u \right\|_{L_x^3 L_t^3} \lesssim \sum_{\beta \in \mathcal{Q}_1(j)} \frac{1}{\beta!} \left\| \mathbf{1}_{[\mathbf{e}/8,b]} \left( \chi_{\mathbf{e}/8,b+\mathbf{e}/4} \right)^{1/2} D_x^{j - \beta + \frac{3}{4}} u \right\|_{L_x^3 L_t^3}
\]

\[
\lesssim \sum_{\beta \in \mathcal{Q}_1(j)} \frac{1}{\beta!} \left\| \eta_{\mathbf{e}/24,b+7\mathbf{e}/24} D_x^{j - \beta + \frac{3}{4}} u \right\|_{L_x^3 L_t^3}
\]

Hence, after integrate in time and apply (6.33) with \((\mathbf{e}, b) = (\mathbf{e}/24, b + 7\mathbf{e}/24)\) we obtain

\[
\sum_{\beta \in \mathcal{Q}_1(j)} \frac{1}{\beta!} \left\| \eta_{\mathbf{e}/24,b+7\mathbf{e}/24} D_x^{j - \beta + \frac{3}{4}} u \right\|_{L_x^3 L_t^3} \lesssim \sum_{\beta \in \mathcal{Q}_1(j)} (c_{-\beta,1})^{1/2} < \infty
\]

by the induction hypothesis.

Analogously, we can handle the third term in (6.49)

\[
\sum_{\beta \in \mathcal{Q}_2(j), \beta \neq j} \frac{1}{\beta!} \left\| \mathbf{1}_{\mathcal{B}^l_j} c_\beta \phi_{e,b} \mathcal{H} D_x^{j - \beta + \frac{3}{4}} u \right\|_{L_x^3 L_t^3} \lesssim \sum_{\beta \in \mathcal{Q}_2(j), \beta \neq j} (c_{-\beta,1})^{1/2} + \|u\|_{L_{x}^\infty H_{x}^{1/2}} < \infty.
\]

Therefore, after integrate in time and apply Hölder’s inequality we have

\[
\left\| D_x^{j + \frac{1}{4}} (u \phi_{e,b}) \right\|_{L_x^3 L_t^3} < \infty.
\]

Next, by interpolation and Young’s inequality

\[
D_x^{j + \frac{1}{4}} (u \phi_{e,b}) \lesssim D_x^{j + \frac{1}{4}} (u \phi_{e,b}) + \|u_0\|_{L_x^3 L_t^3} < \infty.
\]

If we apply (6.49)-(6.50) then

\[
\left\| D_x^{j + \frac{1}{4}} (u \phi_{e,b}) \right\|_{L_x^3 L_t^3} < \infty.
\]
Finally, after collecting all information and apply Gronwall’s inequality we obtain

$$
\sup_{0 \leq t \leq T} \int_{\mathbb{R}} \left( \frac{1}{\alpha^2} \partial_x^2 u \right)^2 \chi_{\varepsilon,b}^2 \, dx + \int_0^T \int_{\mathbb{R}} \left( \partial_x^{\frac{1}{2} + \frac{1}{4}} u \right)^2 (\chi_{\varepsilon,b}^2)' \, dx \, dt
\quad + \int_0^T \int_{\mathbb{R}} \left( \mathcal{H} \partial_x^{\frac{1}{2} + \frac{1}{4}} u \right)^2 (\chi_{\varepsilon,b}^2)' \, dx \, dt \leq c_{*,j2}^\varepsilon
$$

where $c_{*,j2}^\varepsilon = c_{*,j2}^\varepsilon \left( \alpha; \varepsilon; T; \varnothing; \|u_0\|_{H_{\varepsilon}^{\frac{1}{2}}} \|\partial_x^m u_0 \chi_{\varepsilon,b}\|_{L_2^\varepsilon} \right)$ for any $\varepsilon > 0$, $b \geq 5\varepsilon$ and $\varnothing > 0$.

This finishes the induction process.

To justify the previous estimates we shall follow the following argument of regular-ization. For arbitrary initial data $u_0 \in H^s(\mathbb{R})$ $s > \frac{3}{2}$, we consider the regularized initial data $u_0^\mu = \rho \mu * u_0$ with $\rho \in C_0^\infty(\mathbb{R})$, supp $\rho \subset (-1,1)$, $\rho \geq 0$, $\|\rho\|_{L^1} = 1$ and

$$
\rho \mu(x) = \mu^{-1} \rho(x/\mu), \quad \text{for } \mu > 0.
$$

The solution $u^\mu$ of the IVP (1.1) corresponding to the smoothed data $u_0^\mu = \rho \mu * u_0$, satisfies

$$
u^\mu \in C([0,T] : H^\infty(\mathbb{R})),
$$

we shall remark that the time of existence is independent of $\mu$.

Therefore, the smoothness of $u^\mu$ allows us to conclude that

$$
\sup_{0 \leq t \leq T} \int_{\mathbb{R}} \left( \partial_x^m u^\mu \right)^2 \chi_{\varepsilon,b}^2 \, dx + \int_0^T \int_{\mathbb{R}} \left( \partial_x^{m + \frac{1}{2} + \frac{1}{4}} u^\mu \right)^2 (\chi_{\varepsilon,b}^2)' \, dx \, dt
\quad + \int_0^T \int_{\mathbb{R}} \left( \mathcal{H} \partial_x^{m + \frac{1}{2} + \frac{1}{4}} u^\mu \right)^2 (\chi_{\varepsilon,b}^2)' \, dx \, dt \leq c^*
$$

where $c^* = c^* \left( \alpha; \varepsilon; T; \varnothing; \|u_0\|_{H_{\varepsilon}^{\frac{1}{2}}} \|\partial_x^m u_0 \chi_{\varepsilon,b}\|_{L_2^\varepsilon} \right)$. In fact our next task is to prove that the constant $c^*$ is independent of the parameter $\mu$.

The independence from the parameter $\mu > 0$ can be reached first noticing that

$$
\|u_0^\mu\|_{H_{\varepsilon}^{\frac{1}{2}}} \leq \|u_0\|_{H_{\varepsilon}^{\frac{1}{2}}} \|\partial_x^m u_0 \chi_{\varepsilon,b}\|_{L_2^\varepsilon} \|\rho \mu\|_{L_1^\varepsilon} = \|u_0\|_{H_{\varepsilon}^{\frac{1}{2}}} \|\partial_x^m u_0 \chi_{\varepsilon,b}\|_{L_2^\varepsilon}.
$$

Next, since $\chi_{\varepsilon,b}(x) = 0$ for $x \leq \varepsilon$, then restricting $\mu \in (0,\varepsilon)$ it follows by Young’s inequality

$$
\int_{\varepsilon}^{\infty} (\partial_x^m u_0^\mu)^2 \, dx = \int_{\varepsilon}^{\infty} \left( \rho \mu * \partial_x^m u_0 \|_{[0,\infty)} \right)^2 \, dx
\quad \leq \|\rho \mu\|_{L_1^\varepsilon} \|\partial_x^m u_0 \|_{L_2^\varepsilon((0,\infty))}
\quad = \|\partial_x^m u_0 \|_{L_2^\varepsilon((0,\infty))}.
$$

Using the continuous dependence of the solution upon the data we have that

$$
\sup_{t \in [0,T]} \|u^\mu(t) - u(t)\|_{H_{\varepsilon}^{\frac{1}{2}}} \rightarrow 0,
$$
Combining this fact with the independence of the constant $c^*$ from the parameter $\mu$, weak compactness and Fatou’s Lemma, the theorem holds for all $u_0 \in H^s(\mathbb{R})$, $s > \frac{3 - \alpha}{2}$.

**Remark 6.51.** The proof of Theorem B remains valid for the defocusing dispersive generalized Benjamin-Ono equation

\[
\begin{cases}
\partial_t u - D_x^{\alpha+1} \partial_x u - u \partial_x u = 0, & x, t \in \mathbb{R}, 0 < \alpha < 1, \\
(u(x, 0) = u_0(x)).
\end{cases}
\]

In this direction, the propagation of regularity holds for $u(-x, -t)$, being $u(x, t)$ a solution of (1.1). In other words, this means that for initial data satisfying the conditions (1.9) and (1.11) on the left hand side of the real line, the Theorem B remains valid backward in time.

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References

[1] L. Abdelouhab, J.L. Bona, M. Felland, J.-C Saut, *Nonlocal models for nonlinear dispersive waves*, Physica D 40 (1989) 360-392.
[2] B. Bajšanski, R. Coifman, *On singular integrals*, in: Proc. Sympos. Pure Math. Chicago, Amer.Math. Soc. Providence, RI, 1966, pp.1-17.
[3] Á. Bényi, T. Oh, *Smoothing of commutators for a Hörmander class of bilinear pseudodifferential operators*, J. Fourier Anal. Appl. 20 (2014), no.2, 282-300.
[4] J. Berg, J. Löfström, *Interpolation Spaces*, Springer-Verlag, 1976.
[5] J. L. Bona, R. Smith, *The initial value problem for the Korteweg-de Vries equation*, Philos. Trans.R. Soc. London. Ser. A, 278 (1975), pp.555-601.
[6] J. Bourgain and D. Li, *On an endpoint Kato-Ponce inequality*, Differential Integral Equations 11(12)27 (2014c), 1037-1072.
[7] A.P. Calderon, *Commutators of singular integral operators*, Proc. Natl. Acad. Sci. USA 53 (1965).
[8] R. Coifman, V.Wickerhauser, *The scattering transform for the Benjamin-Ono equation*, Inverse problems, 6(1990), 825-861.
[9] L. Dawson, H. McGahagan and G. Ponce, *On the decay properties of solutions to a class of Schrödinger equations*, Proc. Amer. Math. Soc. 136 (2008), no.6, 2081-2090.
[10] A.S. Fokas, M. Ablowitz, *The inverse scattering transform for the Benjamin-Ono equation-a pivot to multidimensional problems*, Stud. Appl. Math. 68 (1984), 1-10.
[11] G. Folland, *Introduction to Partial Differential Equations second edition*, Princeton University Press, 1995.
[12] G. Fonseca, F.Linares, G. Ponce, *The IVP for the dispersion generalized Benjamin-Ono equation in weighted Sobolev spaces*, Ann. Inst. H. Poincaré Anal. Non Linéaire 30 (2013).
[13] J. Ginibre and G. Velo, *Commutator expansions and smoothing properties of generalized Benjamin-Ono equations*, Ann. Inst. H. Poincaré Phys. Théorique, 51 (1989), pp.221-229.
[14] J. Ginibre and G. Velo, *Smoothing properties and existence of solutions for the generalized Benjamin-Ono equations*, J. Differential Equations, 93 (1991), pp.150-212.
[15] L. Grafakos, O. Seungly, *The Kato-Ponce Inequality*, Comm. PDE., 39, Issue 6, 1128-1157, (2014).
[16] S. Herr, A.D. Ionescu, C.E.Kenig, H. Koch, *A para-differential renormalization technique for nonlinear dispersive equations*, Comm. Partial Differential Equations 35, no. 10, 1827-1875 (2010).
DISPERSION GENERALIZED BENJAMIN-ONO EQUATION

[17] P. Isaza, F. Linares, G. Ponce, *On the propagation of regularity of solutions of the Kadomtsev-Petviashvili equation*, SIAM J. Math. Anal., 48, no. 2, 1006-1024, 2016.

[18] P. Isaza, F. Linares, G. Ponce, *On the propagation of regularity and decay of solutions to the $k-$generalized Korteweg-de Vries equation*, Comm. Partial Differential Equations 40 (2015), pp 1336-1364.

[19] P. Isaza, F. Linares, G. Ponce, *On the propagation of regularities in solutions of the Benjamin-Ono equation*, J. Funct. Anal. (270) (2016) pp 976-1000.

[20] T. Kato, *On the Cauchy problem for the (generalized) Korteweg-de Vries equations*, Advances in Mathematics Supplementary Studies, Stud. Math. 8 (1983) 93-128.

[21] T. Kato, *Quasilinear equations of evolution, with applications to partial differential equations*, Lectures Notes in Math., vol 448, Springer Verlag, Berlin and New York, 1975, pp. 27-50.

[22] T. Kato, G. Ponce, *Commutator estimates and the Euler and Navier-Stokes equations*, Comm. Pure Appl. Math 41 (1988) 891-907.

[23] C.E. Kenig, F. Linares, G. Ponce, L. Vega, *On the regularity of solutions to the k-generalized Korteweg-de Vries equation*, (2016). arXiv:1606.03715v2.

[24] C.E. Kenig, K.D. Koenig, *On the local well-posedness of the Benjamin-Ono and modified Benjamin-Ono equations*, Math.Res.Lett., 10 (2003), pp.879-895.

[25] C.E. Kenig, G. Ponce, L. Vega, *Well-posedness and scattering results for the generalized Korteweg-de Vries equation via contraction principle*. Comm. Pure Appl. Math 46 (1993) 527-620.

[26] C. Kenig, G. Ponce, L. Vega, *On the generalized Benjamin-Ono equation*, Trans. Amer. Math.Soc. 342 (1994), pp.155-172.

[27] C.E. Kenig, G. Ponce, and L. Vega, *Well-posedness of the initial value problem for the Korteweg-de Vries equation*, J.Amer. Math. Soc., 4 (1991), pp.323-346.

[28] C. Kenig, G. Ponce, L. Vega, *Oscillatory integrals and regularity of dispersive equations*, Indiana Univ. Math.J.,40 (1991), pp.33-69.

[29] C.E. Kenig, G. Ponce, L. Vega, *A bilinear estimate with applications to the KdV equation*, J.Amer. Math. Soc 9 (1996) 573-603.

[30] H. Koch, N. Tzvetkov, *Local well-posedness of the Benjamin-Ono equation in $H^s(R)$*, Int. Math. Res. Not., 14 (2003), 1449-1464.

[31] D. Li, *On Kato-Ponce and fractional Leibniz*, (2016) arXiv:1609.01780v2.

[32] F. Linares, G. Ponce, *Introduction to Nonlinear Dispersive Equations second edition*, Springer, New York, (2015).

[33] Linares,F. Ponce, G On special regularity properties of solutions of the Zakharov-Kuznetsov equation, Commun. Pure Appl. Anal. 17 (2018), no. 4, 1561?1572.

[34] F. Linares, D. Pilod and J.C. Saut, *Dispersive perturbations of Burgers and hyperbolic equations I: Local theory*, SIAM J.Math. Anal. Vol 46, No 2, pp.1505-1537.

[35] F. Linares, G. Ponce, D. Smith, *On the regularity of solutions to a class of nonlinear dispersive equations*, Math. Ann. 369, no. 1-2, 797-837, 2017.

[36] L. Molinet, J.-C. Saut, N. Tzvetkov, *Ill-posedness issues for the Benjamin-Ono and related equations*, SIAM J.Math. Anal., 33 (2001), 982-988.

[37] C. Muscalu, W. Schlag, *Classical multilinear harmonic analysis. Vol II*. Cambridge Studies in Advanced Mathematics, 138. Cambridge University Press, Cambridge, 2013.

[38] G. Ponce, *On the global well-posedness of the Benjamin-Ono equation*, Differential Integral Equations 4 (1991) 527-542.

[39] A. Sidi, C. Sulem, and P.-L. Sulem, *On the long time behavior of a generalized KdV equation*, Acta Appl. Math. 7 (1986), pp.35-47.

[40] J.C. Saut and R.Temam, *Remarks on the Korteweg-de Vries equation*, Israel J. Math.24 (1976),pp.78-87.

[41] J.-C.Saut, *Sur quelques généralisations de l’équations de Korteweg-de Vries*, J.Math.Pures Appl.58 (1979) 21-61.

[42] J. Segata, D. Smith, *Propagation of Regularity and Persistence of Decay for Fifth Order dispersive Models*, J.Dyn. Diff Equat (2017) 29:701-736.

[43] E.M. Stein, *Singular Integrals and Differentiability Properties of Functions*, Princeton University Press, Princeton, (1970).

[44] E.M. Stein, *Harmonic analysis real-variable methods, orthogonality, and oscillatory integrals*, Princeton University Press, Princeton, N.J. (1993).
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