CUBICS, INTEGRABLE SYSTEMS, AND
CALABI-YAU THREEFOLDS

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To Professor Hirzebruch, on his 65th birthday

§0. Introduction

In this work we construct an analytically completely integrable Hamiltonian system which is canonically associated to any family of Calabi-Yau threefolds. The base of this system is a moduli space of gauged Calabi-Yaus in the family, and the fibers are Deligne cohomology groups (or intermediate Jacobians) of the threefolds. This system has several interesting properties: the multivalued sections obtained as Abel-Jacobi images, or “normal functions”, of a family of curves on the generic variety of the family, are always Lagrangian; the natural affine coordinates on the base, which are used in the mirror correspondence, arise as action variables for the integrable system; and the Yukawa cubic, expressing the infinitesimal variation of Hodge structure in the family, is essentially equivalent to the symplectic structure on the total space.

We begin our study by exploring this equivalence in a much more general context. The general question is as follows: Given a family of abelian varieties, when is there a symplectic form on the total space with respect to which the abelian varieties are Lagrangian? This turns out to involve a symmetry condition on the partials of the period map for the family. Equivalently, this amounts to the existence of a field of cubics on the tangent bundle to the base, such that the differential of the period map at each point is given by contraction with the cubic. We give several versions of this ‘cubic condition’ in §1. For most of the examples and applications, we refer the reader to the survey [DM]. The one example which we work out in detail here is the integrable system of a Calabi-Yau family, where the ‘cubic’ is Yukawa’s.

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The last section is entirely speculative. We muse on the role which the CY integrable system might play in the mirror conjecture: we would like to separate the conjecture into a formal part, which deals with a ‘mirror transform’ for objects consisting of an integrable system together with certain Lagrangian multisectons and other lists of data, and a geometric, Torelli-like part, saying that a CY family can be recovered from its integrable system. The attempt to read off all relevant data (= the partition functions for the A and B models) directly from the system leads to some Hodge theoretic questions which we cannot solve, concerning the information encoded in a normal function and especially in its infinitesimal invariant.

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§1. The cubic condition.

Our goal in this section is to describe the conditions on a given family \( \pi : \mathcal{X} \to \mathcal{U} \) of abelian varieties (or complex tori) for there to exist, locally over \( \mathcal{U} \), an algebraic (or analytic) symplectic structure \( \sigma \) on \( \mathcal{X} \) such that \( \pi \) is a Lagrangian fibration with respect to \( \sigma \). (In this situation we say that the family admits a Lagrangian structure.) What we want is a closed, non-degenerate holomorphic 2-form \( \sigma \) on \( \mathcal{X} \) such that the fibers \( X_u := \pi^{-1}(u) \) are Lagrangian (= maximal isotropic) subvarieties with respect to \( \sigma \).

By way of motivation, let us review the corresponding questions for \( C^\infty \) objects. Any \( C^\infty \) family \( \pi : \mathcal{X} \to \mathcal{U} \) of real tori is locally trivial, i.e. locally (over \( \mathcal{U} \)) isomorphic to \( \mathcal{U} \times (S^1)^n \). In particular, a Lagrangian structure exists whenever \( \dim \mathcal{U} = n = \dim (\text{fibers}) \). Given a \( C^\infty \) symplectic manifold \( (\mathcal{X}, \sigma) \) with Lagrangian fibration \( \pi : \mathcal{X} \to \mathcal{U} \), there is a natural affine structure on the fibers \( X_u \) [AG,2.42]; so if these fibers are compact, they must be tori. In this case the base \( \mathcal{U} \) also acquires a natural integral-affine structure, i.e. an atlas of coordinate systems which differ on intersections by a combination of arbitrary translations and \( \mathbb{Z} \)-linear transformations. To get these coordinates, we choose a basis \( \gamma_1(u), \ldots, \gamma_n(u) \) for the integral homology \( H_1(X_u, \mathbb{Z}) \), varying smoothly with \( u \) in an open subset \( \mathcal{U}^0 \) of \( \mathcal{U} \), as well as a 1-form \( \tau \) on \( \pi^{-1}(\mathcal{U}^0) \) satisfying \( d\tau = \sigma \). (These always exist.) We then let

\[
(1) \quad t_i(u) := \int_{\gamma_i(u)} \tau, \quad i = 1, \ldots, n.
\]

The freedom of changing \( \tau \) (by a closed 1-form) and the \( \gamma_i \) (by the action of \( GL_n(\mathbb{Z}) \)) then allows the \( t_i \) to change by a translation and \( \mathbb{Z} \)-linear transformation, respectively. These \( t_i \) are called action variables. They define
Hamiltonian vector fields which are tangent to the $X_u$ and integrate to give the angle variables

$$(\varphi_1, \cdots, \varphi_n) : X_u \sim R^n/(2\pi Z)^n.$$ 

The latter give the affine structure on the fibers.

A completely integrable Hamiltonian system (CIHS) on the $2n$-dimensional symplectic manifold $(\mathcal{X}, \sigma)$ is an ODE (given, say, by a vector field $v$) which possesses $n$ independent integrals $u_1, \cdots, u_n$ in involution (i.e. they Poisson-commute). This translates immediately to a Lagrangian fibration $\pi : \mathcal{X} \to R^n$ (where $\pi = (u_1, \cdots, u_n)$), and the flow is linearized on the level tori $X_u$, i.e. $v$ is tangent to the fibers, and $v|_{X_u}$ is a constant linear combination of the $\partial/\partial \varphi_i$ with coefficients which depend smoothly on $u$.

The corresponding algebraic object is an algebraically completely integrable Hamiltonian system (ACIHS): we want $\mathcal{X}$ to be a $2n$-dimensional non-singular complex algebraic variety, $\sigma$ a closed, non-degenerate holomorphic $(2, 0)$-form on $\mathcal{X}$, $\pi : \mathcal{X} \to U$ a morphism whose fibers $X_u$ are abelian varieties which are Lagrangian subvarieties of $(\mathcal{X}, \sigma)$, and $v$ a Hamiltonian vector field for some algebraic (Hamiltonian) function on $U$. Such an ODE can be solved more-or-less explicitly, in terms of theta functions. Numerous examples, both in mathematics and in physics, are well-known; we refer to [DKN] for a survey, and to [DM] for some recent examples. What we are asking here is, essentially, to decide when a given family of abelian varieties underlies an ACIHS.

The first new feature of the algebraic situation is that a family of abelian varieties need not be locally trivial. Rather, it is specified by its classifying map

$q : U \to A_g,$

where $A_g$ is the moduli space of abelian varieties of dimension $g$ and of a given polarization type. The given family $\pi : \mathcal{X} \to U$ is recovered as the pullback via $q$ of the universal abelian variety $X_g \to A_g$. (The latter, of course, does not quite exist, but it exists locally away from abelian varieties with excess automorphisms, and also globally on a finite “level” branched cover of $A_g$.)

Let $V \to U$ be the vertical bundle determined by $\pi : \mathcal{X} \to U$. Its fiber over $u \in U$ can be identified with the tangent space $T_0X_u$, or with the universal cover of $X_u$. Deformation theory identifies the tangent space to $A_g$ at a non-singular point $[X] \in A_g$ with $\text{Sym}^2V$, where $V := T_0X$. We may thus write the differential of the classifying map as

$$(2) \quad dq : TU \to \text{Sym}^2V.$$ 

If the family $\pi : \mathcal{X} \to U$ admits a Lagrangian structure then the sym-
plectic form $\sigma$ induces an isomorphism
\begin{equation}
  i = \sigma^{-1} : \mathcal{V}^* \overset{\sim}{\rightarrow} T\mathcal{U}.
\end{equation}

We refine our question by considering pairs consisting of the family $\pi : \mathcal{X} \rightarrow \mathcal{U}$ together with the isomorphism $i$, and asking for the existence of $\sigma$ which is compatible with both. Our first answer is:

**Theorem 1. (Infinitesimal cubic condition)** Consider a family $\pi : \mathcal{X} \rightarrow \mathcal{U}$ of polarized abelian varieties with classifying map $q : \mathcal{U} \rightarrow A_g$ and vertical bundle $\mathcal{V}$, together with an isomorphism (3). Then there exists a non-degenerate holomorphic 2-form $\sigma$ on $\mathcal{X}$ for which $\pi$ is Lagrangian and which induces the given isomorphism $i$, if and only if the composition of (2) and (3):
\begin{equation}
  dq \circ i \in \text{Hom}(\mathcal{V}^*, \text{Sym}^2 \mathcal{V}) = \Gamma(\mathcal{V} \otimes \text{Sym}^2 \mathcal{V})
\end{equation}
comes from a cubic $c \in \Gamma(\text{Sym}^3 \mathcal{V})$. In this case, there is a unique $\sigma$ for which the 0-section of $\pi$ is Lagrangian.

**Proof.**

The short exact sequence of sheaves on $\mathcal{X}$:
\begin{equation}
  0 \rightarrow \pi^* \mathcal{V} \rightarrow T\mathcal{X} \rightarrow \pi^* T\mathcal{U} \rightarrow 0
\end{equation}
determines a subsheaf $\mathcal{F}$ of $\Lambda^2 T\mathcal{X}$ which fits in the exact sequences:
\begin{equation}
  0 \rightarrow \mathcal{F} \rightarrow \Lambda^2 T\mathcal{X} \rightarrow \pi^* \Lambda^2 T\mathcal{U} \rightarrow 0
\end{equation}
\begin{equation}
  0 \rightarrow \pi^* \Lambda^2 \mathcal{V} \rightarrow \mathcal{F} \rightarrow \pi^* (\mathcal{V} \otimes T\mathcal{U}) \rightarrow 0.
\end{equation}

We are looking for a 2-vector $\sigma^{-1}$, a section of $\Lambda^2 T\mathcal{X}$, for which the fibers of $\pi$ are isotropic; this means that $\sigma^{-1}$ goes to 0 in $\pi^* \Lambda^2 T\mathcal{U}$, so it comes from a section of $\mathcal{F}$. The question is therefore whether the given
\begin{equation}
  i \in H^0(\mathcal{U}, \mathcal{V} \otimes T\mathcal{U}) \subset H^0(\mathcal{X}, \pi^* (\mathcal{V} \otimes T\mathcal{U}))
\end{equation}
is in the image of $H^0(\mathcal{X}, \mathcal{F})$. Locally in $\mathcal{U}$, this happens if and only if $i$ goes to 0 under the coboundary map
\begin{equation}
  \pi_* \pi^* (\mathcal{V} \otimes T\mathcal{U}) \rightarrow R^1 \pi_* \pi^* \Lambda^2 \mathcal{V}
\end{equation}
\begin{equation}
  \mathcal{V} \otimes T\mathcal{U} \rightarrow \Lambda^2 \mathcal{V} \otimes \mathcal{V}.
\end{equation}

The bottom map factors as $\beta \circ (1 \otimes dq)$, where
\begin{equation}
  1 \otimes dq : \mathcal{V} \otimes T\mathcal{U} \rightarrow \mathcal{V} \otimes \text{Sym}^2 \mathcal{V}
\end{equation}
comes from the differential of the classifying map \( q \), and \( \beta \) is part of the (exact) Koszul complex

\[
0 \to \text{Sym}^3 \mathcal{V} \xrightarrow{\alpha} \mathcal{V} \otimes \text{Sym}^2 \mathcal{V} \xrightarrow{\beta} \Lambda^2 \mathcal{V} \otimes \mathcal{V} \to \cdots.
\]

We conclude that the desired 2-vector \( \sigma^{-1} \) exists locally if and only if

\[
dq \circ i = (1 \otimes dq)(i) \in \mathcal{V} \otimes \text{Sym}^2 \mathcal{V}
\]

comes, via \( \alpha \), from a section of \( \text{Sym}^3 \mathcal{V} \). The global existence of \( \sigma \) will follow by patching local solutions, once we know uniqueness. For this, let \( \sigma_1, \sigma_2 \) be two local solutions. Then \( \sigma_1 - \sigma_2 \) induces the 0-map from \( T\mathcal{U} \) to \( \mathcal{V}^\ast \), hence it is the pullback of a 2-form \( \varphi \) on \( \mathcal{U} \). But restricting to the 0-section of \( \pi \) shows \( \varphi = 0 \), so \( \sigma_1 = \sigma_2 \) as required.

Q.E.D

We view this theorem as an infinitesimal answer to our problem:

If we replace \( \mathcal{U} \) by its first-order germ at the point \( u \in \mathcal{U} \), then the theorem provides a necessary and sufficient condition for existence of a Lagrangian structure for \( \pi : \mathcal{X} \to \mathcal{U} \). In general, it provides a non-degenerate 2-form \( \sigma \), but an additional condition is needed for \( \sigma \) to be closed. For this, it is convenient to replace the moduli space \( \mathcal{A}_g \) by the Siegel upper half space:

\[
H_g := \{ g \times g \text{ symmetric complex matrices } Z \text{ with } \text{im}(Z) > 0 \}.
\]

Over \( H_g \) there is (for each polarization type) a universal (marked) abelian variety \( \mathcal{X}_g \to H_g \), whose fiber over \( Z \in H_g \) is \( V/L \), where \( V \approx \mathbb{C}^g \) is a fixed complex vector space, and \( L \) is the lattice generated by the \( 2g \) columns of the \( g \times 2g \) matrix \((I,Z)\). (Here \( I \) is the \( g \times g \) identity matrix, when the polarization is principal, and in general it is the \( g \times g \) integral diagonal matrix whose entries are the elementary divisors of the polarization.) Any family \( \pi : \mathcal{X} \to \mathcal{U} \) of abelian varieties which are marked (i.e. endowed with a continuously varying symplectic basis of the fiber homologies \( H_1(\mathcal{X}_u, Z) \), \( u \in \mathcal{U} \)) determines a period map

\[
p : \mathcal{U} \to H_g,
\]

and is recovered as the pullback via \( p \) of the universal family \( \mathcal{X}_g \). (The classifying map \( q \), from \( \mathcal{U} \) to \( \mathcal{A}_g \), is obtained by composition with the quotient map \( H_g \to \mathcal{A}_g \).)

Let us consider the complex analogue of the action-angle coordinates and the corresponding affine structures. A symplectic family \( \pi : \mathcal{X} \to \mathcal{U} \) of bare abelian varieties, given by the classifying map \( q : \mathcal{U} \to \mathcal{A}_g \), determines locally on \( \mathcal{U} \) a set of \( 2g \) action variables, given by formula (1). To get action coordinates, we need to choose \( g \) of these. Now the polarization on the fibers
$X_u$ gives a symplectic structure on $H_1(X_u, \mathbb{Z})$, so the natural thing to do is to choose a Lagrangian basis $\gamma_1(u), \ldots, \gamma_g(u)$, i.e. a basis of a continuously varying Lagrangian subspace. On the other hand, the choice of a marking of the family $\pi$, i.e. a lifting of $q$ to a period map $p : \mathcal{U} \rightarrow \mathbb{H}_g$, is equivalent to the choice of a dual pair $\gamma_1, \ldots, \gamma_g$ and $\gamma_{g+1}, \ldots, \gamma_{2g}$ of Lagrangian bases. Combining this with (1), we get:

**Lemma 1.** Let $\pi : \mathcal{X} \rightarrow \mathcal{U}$ be a family of abelian varieties with Lagrangian structure. The choice of Lagrangian basis $\gamma_1, \ldots, \gamma_g$ of the fiber homologies determines an affine structure on $\mathcal{U}$. The choice of a marking $\gamma_1, \ldots, \gamma_{2g}$ (or of a period map $p : \mathcal{U} \rightarrow \mathbb{H}_g$) determines a pair of affine structures. The affine coordinates $\{u_i\}, \{t_i\}$ corresponding to the Lagrangian bases $\gamma_1, \ldots, \gamma_g$ and $\gamma_{g+1}, \ldots, \gamma_{2g}$, respectively, are related by:

$$dt_i = \sum p_{ij}(u)du_j.$$  

We note that, even with these choices, the angle coordinates $\varphi_i$ are still multivalued, i.e. they can be defined only locally. Their differentials $d\varphi_i$ are more intrinsic: they correspond symplectically to straight flows (with respect to the affine structure) on $\mathcal{U}$. The $\varphi_i$ can be uniquely determined on the vertical bundle $\mathcal{V}$ (= the fiberwise universal cover) by setting $\varphi_i = 0$ on the 0-section of $\mathcal{V}$.

We fix once and for all the vector space $V \approx \mathbb{C}^g$. Any marked family $\pi : \mathcal{X} \rightarrow \mathcal{U}$ of abelian varieties then comes with a trivialization of its vertical bundle,

$$\mathcal{V} \approx V \otimes \mathcal{O}_U.$$  

We identify $\mathbb{H}_g$ with an open subset of $\text{Sym}^2 V$, so the differential of the period map becomes

$$dp : T\mathcal{U} \rightarrow \text{Sym}^2 V \otimes \mathcal{O}_U.$$  

The affine structure on $\mathcal{U}$, determined by $\gamma_1, \ldots, \gamma_g$ and a Lagrangian structure, is then given by an isomorphism

$$\alpha : V^* \otimes \mathcal{O}_U \cong T\mathcal{U}.$$  

**Theorem 2.** (Global Cubic Condition) Consider a family $\pi : \mathcal{X} \rightarrow \mathcal{U}$ of marked abelian varieties, with period map $p : \mathcal{U} \rightarrow \mathbb{H}_g$, where $\mathcal{U}$ has an affine structure $\alpha : V^* \otimes \mathcal{O}_U \cong T\mathcal{U}$. Then $\pi$ admits a Lagrangian structure which induces $\alpha$ if and only if the composition

$$dp \circ \alpha \in \text{Hom}(V^* \otimes \mathcal{O}_U, \text{Sym}^2 V \otimes \mathcal{O}_U) = \Gamma(V \otimes \text{Sym}^2 V \otimes \mathcal{O}_U)$$

comes from a cubic $c \in \Gamma(\text{Sym}^3 V \otimes \mathcal{O}_U)$.
Proof.

Note that the trivialization (4) takes $\alpha$ to the isomorphism $i$ of (3), and $dp$ goes to $dq$ of (2). The “only if” therefore follows from Theorem 1.

Conversely, we start with the standard symplectic structure on the cotangent bundle $T^*U$. Via the affine structure $\alpha$ and the trivialization (4), we get a symplectic structure $\sigma$ on the vertical bundle $V$. The projection to $U$ is Lagrangian with respect to $\sigma$ as is the 0-section. We want $\sigma$ to descend to $X$. Equivalently, $\sigma$ should be invariant under translation by the section

$$s_{m,n} : u \mapsto m + p(u)n,$$

for any $m, n \in \mathbb{Z}^g$, which happens if and only if the sections $s_{m,n}$ are Lagrangian. Back on $T^*U$, we want the corresponding 1-forms on $U$ to be closed. This is always true for the $s_{m,0}$, since $\alpha$ is assumed to be an affine structure. For $s_{0,n}$, we need closedness of the $g$ 1-forms $\sum p_{ij}(u)du_j$. This amounts to equality of mixed partials, hence to the symmetry condition on $dp \circ \alpha$. We conclude that if $dp \circ \alpha$ is symmetric then $\sigma$ descends to a Lagrangian structure on $\pi$. This induces $\alpha$ by construction.

Q.E.D

Remarks.

(1) Another way to state this is as follows. Given the period map $p$ and the isomorphism $i$, we get an isomorphism

$$\alpha : V^* \otimes \mathcal{O}_U \cong TU.$$

If the cubic condition holds (with respect to $i$), we get the 2-form $\sigma$ by Theorem 1. This $\sigma$ is closed (i.e. gives a Lagrangian, or symplectic, structure) if and only if the isomorphism $\alpha$ is integrable, i.e. if and only if $\alpha$ integrates to an affine structure on $U$.

(2) In the situation of the theorem, we deduce from the equality of mixed partials that locally in $U$ there is a function $f \in \Gamma(\mathcal{O}_U)$ such that

$$\partial^2 f = p, \quad \partial^3 f = c,$$

as sections of $\text{Sym}^2 T^*U$, $\text{Sym}^3 T^*U$ respectively. Here $\partial$ denotes the total derivative with respect to the affine structure on $U$, so $\partial^k$ denotes, essentially, the $k^{th}$ term in the Taylor expansion. $f$ is unique up to adding an affine function. Conversely, any holomorphic $f$ on a complex manifold $U$ with $V^*$-affine structure determines a Lagrangian family of abelian varieties on the open subset of $U$ where $\text{Im}(\partial^2 f) > 0$.  

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(3) Much of the above extends to Lagrangian structures on families of complex tori. The new feature is that $p$ lands in $V \otimes V$, instead of $\text{Sym}^2 V$, so we can write $p = p_+ + p_-$, with $p_+$ symmetric, $p_-$ skew symmetric. The result is that in order to have a Lagrangian structure, $p_-$ must be locally constant on $U$, while $p_+$ satisfies a cubic condition as above.

In particular, everything goes through with no changes for families of polarized complex tori, regardless of the positivity of the polarization. In terms of the period matrix, we still require symmetry (Riemann’s first bilinear relation) but not positivity. This is the situation which arises in §2.

(4) Another rather straightforward extension involves replacing the symplectic structure $\sigma$ (or rather, its inverse) by a Poisson structure, given by a 2-vector $\psi \in \Gamma (\Lambda^2 T\mathcal{X})$ satisfying a closedness condition. Now we start with an inclusion $i : V^* \hookrightarrow TU$ (or $\alpha : V^* \otimes \mathcal{O}_U \hookrightarrow TU$) and we ask whether a given family $\pi : \mathcal{X} \rightarrow U$ of abelian varieties (or complex tori) admits a Poisson structure $\psi$ such that $\pi_* \psi = 0$ on $U$ and which induces the given $i$. As before, the answer amounts to a cubic condition,

$$dp \circ i \in \Gamma (\text{Sym}^3 \mathcal{V}).$$

We refer to [DM] for the details.

(5) Dually, one can ask for existence of quasisymplectic structures (= closed, but possibly degenerate, 2-forms) on the total space $\mathcal{X}$ of a family of abelian varieties, such that the family is Lagrangian, but this time inducing a given inclusion

$$j : TU \hookrightarrow V^*.$$

Again, the answer [DM] is in terms of a symmetry condition on $dp$, but of a slightly different form: the condition is that $dp \otimes j$, which lives in

$$(\text{Sym}^2 \mathcal{V} \otimes T^* U) \otimes (V^* \otimes T^* U),$$

maps under contraction to an element of $\mathcal{V} \otimes T^* U \otimes T^* U$ which is symmetric in the last two variables, i.e. it is in $\mathcal{V} \otimes \text{Sym}^2 T^* U$. (This can be contracted with $j$ again, to give an element of $\otimes^3 T^* U$. The “cubic condition”, asserting that this last tensor is in $\text{Sym}^3 T^* U$, follows from the previous symmetry condition, but is in general too weak to imply it.)

(6) Several examples and applications of the cubic conditions are discussed in [DM] and [M2]. Jacobi’s ACIHS (whose flows include the geodesic flows on ellipsoids and the Neumann flow) is particularly simple: the “cubics” turn out to be Fermat cubics in $g$ variables, $\sum_{i=1}^{g} x_i^3$. These happen to have an invariance property which can be stated as follows: the linear system of polar quadrics (generated by the $g$ quadrics $x_i^3$) is invariant under the $g$-dimensional group of rescalings, $x_i \mapsto a_i x_i$, $i = 1, \cdots, g$. This in turn translates into existence
of a \(g\)-dimensional family of two-forms on the family for which the fibration is Lagrangian, locally near each point of the base \(U\). (Only one of these, up to homothety, extends to a Lagrangian structure on the entire family.)

Jacobi’s system has been extended to higher rank by Adams, Harnad and Hurtubise [AHH] and by Beauville [B]. This system as well as Hitchin’s [H] are special cases of the spectral system constructed in [M1] and [Bn] in terms of twisted endomorphisms of vector bundles on curves. The Poisson structure on this system has a simple description in terms of the cubic condition, cf. [DM]. A further generalization to a Lagrangian structure on the space of Lagrangian sheaves on an arbitrary symplectic (or Poisson, or quasi-symplectic) variety is given in [M2]. (This includes the various spectral systems, as well as Mukai’s symplectic structure on spaces of sheaves on \(K3\) surfaces, etc.) Here too, the symplectic (respectively Poisson, quasi-symplectic) structure can be exhibited in terms of its underlying cubic, cf. [DM].

\section*{§2. The Calabi-Yau integrable system.}

In this section we construct the analytically completely integrable Hamiltonian system associated to any complete family of Calabi-Yau threefolds. We also study the sections of this system corresponding to curves on the generic Calabi-Yau.

Let \(X\) be a Calabi-Yau threefold, i.e. a three dimensional compact Kähler manifold satisfying

\[
H^0(\Omega^1_X) = H^0(\Omega^2_X) = 0
\]

\[
\omega_X \approx \mathcal{O}_X.
\]

Here \(\Omega^i_X\) is the sheaf of holomorphic \(i\)-forms on \(X\), \(\mathcal{O}_X = \Omega^0_X\) is the structure sheaf, and \(\omega_X = \Omega^3_X\) the canonical sheaf. We denote the holomorphic tangent bundle by \(T_X\). By (holomorphic) gauge we mean a non-zero section \(s \in H^0(\omega_X)\).

Hodge theory gives us the Hodge decomposition

\[
H^3(X, \mathbb{C}) = H^{3,0} \oplus H^{2,1} \oplus H^{1,2} \oplus H^{0,3}
\]

and the Hodge filtration

\[
F^p = \oplus_{i \geq p} H^{i,3-i},
\]

together with bilinear maps

\[
B_p : H^1(T_X) \times H^{p,q} \to H^{p-1,q+1}
\]

which express the variation of the Hodge structure: as \(X\) is deformed in a direction \(v \in H^1(T_X)\), its \(H^{p,q}\) moves to first order within \(F^{p-1}\) (Griffiths
transversality) and an element \( w \in H^{p,q} \) moves in the direction of \( B_p(v, w) \). The \( H^{p,q} \) can be described as spaces of harmonic \((p,q)\)-forms on \( X \) (thus giving the inclusion \( H^{p,q} \hookrightarrow H^3(X, \mathbb{C}) \)), or as the sheaf cohomology groups \( H^q(X, \Omega^p_X) \). In the case of a CY threefold \( X \), we have:

(i) \( H^{3,0} = H^0(\omega_X) \) is the one-dimensional complex vector space of holomorphic gauges.

(ii) \( H^{2,1} = H^1(\Omega^2) \) is isomorphic to \( H^1(T_X) \). More canonically, each holomorphic gauge \( s \in H^{3,0}, s \neq 0 \), determines an isomorphism

\[
\hat{s} : H^1(T_X) \xrightarrow{\sim} H^{2,1}
\]

\[
v \mapsto B_3(v, s).
\]

(iii) \( H^{1,2} \) and \( H^{0,3} \) can be identified either with the complex conjugates of \( H^{2,1} \) and \( H^{3,0} \) (with respect to the real subspace \( H^3(X, \mathbb{R}) \) of \( H^3(X, \mathbb{C}) \)), or with their dual spaces \((H^{2,1})^*, (H^{3,0})^* \) (the pairing is given by integration over \( X \)).

The Griffiths intermediate Jacobian of \( X \) is

\[
J(X) := H^3(X, \mathbb{C})/(F^2H^3 + H^3(X, \mathbb{Z})) \approx (F^2H^3)^*/\lambda(H_3(X, \mathbb{Z}))
\]

where \( H_3(X, \mathbb{Z}) \) is identified with a lattice in \((F^2H^3)^* \) by the map \( \lambda : H_3(X, \mathbb{Z}) \hookrightarrow (F^2H^3)^* \) which is the composition of integration

\[
H_3(X, \mathbb{Z}) \to H^3(X, \mathbb{C})^*
\]

\[
\gamma \mapsto \int_\gamma
\]

and projection

\[
H^3(X, \mathbb{C})^* \to (F^2H^3)^*.
\]

\( J(X) \) is a complex torus, but in general not an abelian variety: it satisfies Riemann’s first bilinear relation, but not the second. It has the important property that it depends holomorphically on \( X \). This means that given a holomorphic family \( \chi : \mathcal{X} \to \mathcal{M} \) of compact Kähler manifolds \( X_t, t \in \mathcal{M} \), the intermediate Jacobians \( J(X_t) \) fit together naturally to form a holomorphic family \( \pi : J \to \mathcal{M} \), called the relative intermediate Jacobian of the family \( \chi \).

The theorem of Bogomolov, Tian and Todorov [Bo,Ti,To] says that each moduli space \( \mathcal{M} \) of Calabi-Yau’s is smooth and unobstructed, i.e. the Kodaira-Spencer map gives an isomorphism

\[
T_{[X]}\mathcal{M} \xrightarrow{\sim} H^1(T_X)
\]

for every CY \( X \). We say that a family \( \chi : \mathcal{X} \to \mathcal{M} \) of CYs \( X_t, t \in \mathcal{M} \), is \textit{complete} if its classifying map \( \mathcal{M} \to \mathcal{M} \) is a local isomorphism. It follows
that $\mathcal{M}$ is smooth with tangent space $H^1(T_{X_t})$ at $t$. Typically such families might consist of all CYs in some open subset of $\mathcal{M}$, together with some “level” structure. We fatten the family by adding the holomorphic gauge: let $\rho: \tilde{\mathcal{M}} \to \mathcal{M}$ be the $\mathbb{C}^*$-bundle over $\mathcal{M}$ underlying the line bundle $\chi_* \omega_{X/\mathcal{M}}$, so a point of $\tilde{\mathcal{M}}$ is given by a pair $(X, s)$ with $X \in \mathcal{M}$ and $s \in H^0(\omega_X) \setminus \{0\}$.

There is a pullback family $\tilde{\chi}: \tilde{\mathcal{X}} \to \tilde{\mathcal{M}}$, where $\tilde{\mathcal{X}} := \mathcal{X} \times_\mathcal{M} \tilde{\mathcal{M}}$. We let $\pi: \mathcal{J} \to \mathcal{M}$ and $\tilde{\pi}: \tilde{\mathcal{J}} \to \tilde{\mathcal{M}}$ denote the relative intermediate Jacobians of $\chi, \tilde{\chi}$ respectively.

**Theorem 3.** $\tilde{\pi}: \tilde{\mathcal{J}} \to \tilde{\mathcal{M}}$ is an analytically completely integrable Hamiltonian system.

**Proof.**

**Step I.** We claim that there is a canonical isomorphism

$$i^{-1}: T_{(X, s)} \tilde{\mathcal{M}} \approx F^2 H^3(X, \mathbb{C}).$$

Indeed, $\rho: \tilde{\mathcal{M}} \to \mathcal{M}$ gives a short exact sequence:

$$0 \to T_{(X, s)}(\tilde{\mathcal{M}}/\mathcal{M}) \to T_{(X, s)} \tilde{\mathcal{M}} \to T_X \mathcal{M} \to 0,$$

in which the subspace can be identified with $H^0(\omega_X)$, and the quotient with $H^1(T_X)$, hence via $s$ with $H^1(\Omega^3_X)$. We need to identify this sequence naturally with the one defining $F^2 H^3$:

$$0 \to H^0(\omega_X) \to F^2 H^3(X, \mathbb{C}) \to H^1(\Omega^2_X) \to 0.$$ 

In other words, we are claiming that the extension data in the two sequences match, globally over $\tilde{\mathcal{M}}$.

For this, it suffices to construct a natural map from $T_{(X, s)} \tilde{\mathcal{M}}$ to $F^2 H^3(X, \mathbb{C})$, which induces the identity on the subspace and quotient.

One way to obtain this map is to note that the bundle $R^3 \tilde{\chi}_* \mathbb{C}$ has the tautological section $s$, which is actually in the subbundle $F^3 R^3 \tilde{\chi}_* \mathbb{C}$. The Gauss-Manin connection applied to this section maps $T_{(X, s)} \tilde{\mathcal{M}}$ to $H^3(X, \mathbb{C})$, but by Griffiths transversability it lands in $F^2 H^3$. This map is easily seen to have the required properties. For future use we want an explicit description of this map. We do this using Dolbeault cohomology.

Think of a 1-parameter family $(X_t, s_t) \in \tilde{\mathcal{M}}$, depending on the parameter $t$, as a fixed $C^\infty$ manifold $X$ on which there is given a 1-parameter family of complex structures, specified by their $\bar{\partial}$-operators $\bar{\partial}_t$, and also a family $s_t$ of $C^\infty$ 3-forms on $X$ such that $s_t$ is of type $(3, 0)$ with respect to $\bar{\partial}_t$. Since the $s_t$ are now on a fixed underlying $X$, we can expand with respect to $t$:

$$s_t = s_0 + ta \quad (\text{mod. } t^2).$$
Griffiths’ transversality now says that the derivative, $a$, is in $F^2H^3$. It clearly depends only on first-order data, i.e. on the tangent vector to $\tilde{M}$ along $(X_t, st)$ at $t = 0$. This produces the map

$$T_{(X,s)}\tilde{M} \to F^2H^3(X, C)$$

with the desired properties.

**Step II.** We want a non-degenerate 2-form $\sigma$ on $\tilde{J}$ making $\pi$ Lagrangian. By Theorem 1, this is equivalent to finding a cubic $\tilde{c}$ on $T\tilde{M}$ satisfying $\tilde{c} = d\tilde{p} \circ i$, where $\tilde{p}$ is the period map for $\tilde{X} \to \tilde{M}$ and $i$ is the inverse of (5). Since $\tilde{p}$ factors through the period map $p$ for $\chi: X \to M$, it is reasonable to look for a cubic $c$ in $\rho^*\text{Sym}^3T^*M \subset \text{Sym}^3T^*\tilde{M}$, satisfying $\tilde{c} = dp \circ i$. But this is precisely where the Yukawa cubic of our CY family (also known as the Bryant-Griffiths cubic, cf. [BG]) lives. Abstractly, this is cup product, followed by a “rescaling” by the gauge squared, and integration,

$$c: \text{Sym}^3H^1(T_X) \cup H^3(\Lambda^3T_X) = H^3(\omega_X^{-1}) \to H^3(\omega_X) \to \mathbb{C}. $$

Hodge theoretically, this is the third iterate of the derivative of the period map (or of the infinitesimal variation of Hodge structure):

$$B_1 \circ B_2 \circ B_3: \text{Sym}^3H^1(T_X) \times H^{3,0} \to H^{0,3},$$

where the 1-dimensional spaces $H^{3,0}$ and $H^{0,3} \cong (H^{3,0})^*$ are trivialized by means of $s$. The equality $|c = dp \circ i$ follows immediately from this interpretation, cf. [CGGH].

**Step III.** We still need to check that the 2-form we constructed is closed. We could construct an affine structure on $\tilde{M}$ and use Theorem 2. Instead, we verify the closedness directly, and obtain the affine structure as corollary. We do this by checking that the standard symplectic form on $T^*\tilde{M}$ is invariant under translation by the image of $\lambda: H_3(\tilde{X}/\tilde{M}, Z) \hookrightarrow (F^2H^3)^*$, hence it descends to a symplectic form on $\tilde{J}$.

Since the question is local on $\tilde{M}$, we may assume that the bundle of lattices $R^3\chi, Z$ on $M$ is trivial. Let $X_0$ be the CY fiber over a base point $0 \in M$. The choice of $\gamma_0 \in H_3(X_0, Z)$ determines a family $\gamma = (\gamma_t)_{t \in \mathbb{M}}, \gamma_t \in H_3(X_t, Z)$, and these are taken by $\lambda$ to $F^2H^3(X_t, C)^*$. By the identification $i$ of (5), we end up with a section

$$\int_\gamma \in \Gamma(\tilde{M}, T^*\tilde{M}).$$

We need to show that (the image of) this section is Lagrangian, with respect to the standard symplectic form on $T^*\tilde{M}$. Equivalently, we need to show
that the 1-form \( \xi \) on \( \tilde{\mathcal{M}} \) which corresponds to the section \( \int_\gamma \) of \( T^* \tilde{\mathcal{M}} \) is closed.

Consider the function \( g : \tilde{\mathcal{M}} \to \mathbb{C} \) given by
\[
g(X, s) := \int_\gamma s.
\]
We claim \( \xi = dg \). Indeed, if \( a \in T_{(X,s)}\tilde{\mathcal{M}} \) is the tangent vector to the 1-parameter family \((X_t, s_t)\) then we have, as in Step I:
\[
s_t = s_0 + ta \pmod{t^2}.
\]
and therefore
\[
< dg, a > = (\partial/\partial t)|_{t=0} g(X_t, s_t)
= (\partial/\partial t)|_{t=0} \int_\gamma s_t
= \int_\gamma a = < \xi, a >.
\]

Q.E.D

Remark.

In order to obtain the Lagrangian structure, we did not need to specify the “level” of the moduli space \( \mathcal{M} \). In order to have an ACIHS, we need in addition to the Lagrangian structure also a set of global Hamiltonians, i.e. a set of \( 1 + h^{2,1} \) independent functions on \( \tilde{\mathcal{M}} \). For this, we choose the level structure to include, at least, the choice of a Lagrangian basis for the \( H_3(X, \mathbb{Z}) \), say \( \gamma_0, \ldots, \gamma_{h^{2,1}} \). As our Hamiltonians we then take
\[
\tilde{t}_i := \int_{\gamma_i} s \quad i = 0, \ldots, h^{2,1}.
\]
We will see another interpretation for these functions shortly.

Lemma 2. The symplectic form \( \sigma \) on \( \tilde{\mathcal{J}} \) is exact.

Proof.

For very general reasons, the symplectic form \( \tilde{\sigma} \) on \( T^* \tilde{\mathcal{M}} \) is exact: it equals \( d\tilde{\tau} \), where \( \tilde{\tau} \) is the action 1-form on \( T^* \tilde{\mathcal{M}} \). (If \( q_i \) are coordinates on \( \tilde{\mathcal{M}} \) and \( p_i \) the corresponding linear coordinates on the fibers, then \( \tilde{\tau} \) is given by \( \sum q_i dq_i \) and \( \tilde{\sigma} \) by \( \sum dp_i \wedge dq_i \).) To obtain \( \sigma \), we identified \( T^* \tilde{\mathcal{M}} \) with the vertical bundle \( \mathcal{V} \) of \( \tilde{\mathcal{J}} \) (whose fiber at \((X, s)\) is \( F^2H^3(X, \mathbb{C}) \)), and observed that \( \tilde{\sigma} \) was invariant under translation by locally constant integral cycles \( \gamma \), hence it descended to \( \sigma \) on \( \tilde{\mathcal{J}} \).
A first guess for an antidifferential $\tau$ of $\sigma$ might be simply $\tilde{\tau}$ (pulled back by $i$, but we suppress this), but $\tilde{\tau}$ is not invariant under translations: if the cycle $\gamma$ corresponds, as in Step III of the proof of Theorem 3, to a 1-form $\xi$ on $\tilde{M}$, then translating by $\gamma$ changes $\tilde{\tau}$ by addition of $\pi^*\xi$, where $\pi : V \rightarrow \tilde{M}$ is the projection. To correct this discrepancy, we consider the tautological function $f \in \Gamma(O_V)$ whose value at a point $(X, s, v)$ with $(X, s) \in \tilde{M}$ and $v \in F^2H^3(X)^*$ is given by

\[
f(X, s, v) := v(s).
\]

Now $f$ is fiber-linear, so $df$ is constant on fibers, and therefore translation by $\gamma$ changes $df$ by $\pi^*$ of a 1-form on $\tilde{M}$, and this 1-form is immediately seen to be $\xi$. We conclude that $\tilde{\tau} - df$ is a global 1-form on $V$ which is invariant under translation by each $\gamma$, hence descends to a 1-form $\tau$ on $\tilde{J}$ which satisfies $d\tau = \sigma$, as required.

Q.E.D

Remarks.

(1) Another way to see the exactness of $\sigma$ is to note that it is the symplectification of a contact structure $\kappa$ on $J$. In general, a contact manifold $(J, \kappa)$ determines an exact symplectic structure on the natural $C^*$-bundle $\tilde{J}$ over $J$. Conversely, according to [AG], page 78, a symplectic structure $\sigma$ on a manifold $\tilde{J}$ with a $C^*$-action $\rho$ is the symplectification of a contact structure on the quotient $J$ if and only if $\sigma$ is homogeneous of degree 1 with respect to $\rho$. In our case, there are two independent $C^*$-actions on the total space of $\tilde{V} \approx T^*\tilde{M}$: the $C^*$-action on $\tilde{M}$ lifts to an action $\rho'$ on $T^*\tilde{M}$, and there is also the action $\rho''$ which commutes with the projection to $\tilde{M}$ and is linear on the fibers. The symplectic form $\tilde{\sigma}$ is homogeneous of weight 0 with respect to $\rho'$ and of weight 1 with respect to $\rho''$. Neither of these actions descends to $\tilde{J}$, but their product $\rho := \rho' \otimes \rho''$ does. The symplectic form $\sigma$ on $\tilde{J}$ is homogeneous of weight 1 with respect to $\rho$, so it is the symplectification of a contact structure on $J$, and in particular it is exact.

In the notation of the proof, the actions are given, for $t \in C^*$, by:

\[
\begin{align*}
\rho' : (X, s, v) &\mapsto (X, ts, t^{-1}v) \\
\rho'' : (X, s, v) &\mapsto (X, s, tv) \\
\rho : (X, s, v) &\mapsto (X, ts, v).
\end{align*}
\]

The corresponding vector fields on $\tilde{V}$ are taken by $\tilde{\sigma}$ to the 1-forms $-df, \tilde{\tau}$, and $\tau = \tilde{\tau} - df$, respectively. Of these, only $\tau$ descends to a closed 1-form on $\tilde{J}$. (Another description of this contact structure is in Remark (2) to Theorem 4.)
(2) The tautological relative 3-form $s$ on $\tilde{\mathcal{X}}$ over $\tilde{\mathcal{M}}$ induces a relative 1-form on $\tilde{\mathcal{J}}$ over $\tilde{\mathcal{M}}$. We note that our 1-form $\tau$ on $\tilde{\mathcal{J}}$ is a lift of this tautological relative 1-form.

**Lemma 3.** The Hamiltonians $\tilde{t}_i = \int_{\gamma_i} s$ (cf. remark after Theorem 3) are equal (up to sign) to the action variables for $\tilde{\mathcal{J}}$, given by (1).

**Proof.**

The 3-cycle $\gamma_i$ in $X$ determines a 1-cycle, say $\Gamma_i$, in $J(X)$. The action variables (1) are given as $f_{\Gamma_i} \tau$. By the proof of Lemma 2, $\tau$ pulls back to $\tilde{\tau} - df$, where $f$ is given by (6), and $\tilde{\tau}$ vanishes on fibers of $\pi$. Therefore:

$$\int_{\Gamma_i} \tau = -\int_{\Gamma_i} df = -\int_{\gamma_i} s.$$  

Q.E.D

Let $\mathcal{A}$ be the group of 1-cycles in $X$ (i.e. the free abelian group generated by all curves), and $\mathcal{A}_0$ the subgroup of 1-cycles homologous to 0. There is a well-defined *Abel-Jacobi map*:

$$\nu : \mathcal{A}_0 \to J(X).$$

If the cycle $C$ is the boundary of a 3-chain $\Gamma$, then $\nu(C)$ is the image in $J(X)$ of $f_{\Gamma}$, considered as an element of $H^3(X, C)$. The image is independent of the choice of $\Gamma$. (This is the analogue for 1-cycles of the natural map from divisors homologous to 0 to $Pic^0(X)$.)

Given a family $\chi : \mathcal{X} \to \mathcal{M}$ of threefolds and a relative 1-cycle $C \to \mathcal{M}$ in $\mathcal{A}_0(\mathcal{X}/\mathcal{M})$ (i.e. for each $t \in \mathcal{M}$, $C_t$ is a 1-cycle homologous to 0 in $X_t$), there is a corresponding section

$$\nu(C) \in H^0(\mathcal{M}, J(\mathcal{X}/\mathcal{M}))$$

whose value at $t \in \mathcal{M}$ is $\nu(C_t)$ in $J(X_t)$. This is called the *normal function* of the relative cycle $C$.

Abstractly, a normal function is defined as a section $\nu$ of $J$ whose local liftings to sections $\bar{\nu}$ of $R^3\chi_* C = \mathcal{H}^3$ satisfy the differential equation

$$\nabla \bar{\nu} \in F^1 \mathcal{H}^3,$$

or equivalently

$$(\nabla \bar{\nu}, s) = 0,$$

(7)
where $\nabla$ is the Gauss-Manin connection on $H^3$. This condition is independent of the choice of $\varpi$, by Griffiths transversality. It is automatically satisfied by the Abel-Jacobi image of a family of relative 1-cycles, cf. [G2].

More generally, we can consider multivalued 1-cycles and normal functions: a map $B \to \mathcal{M}$ gives a pullback family of CYs over $B$, and we can consider a family of relative, null-homologous 1-cycles in this pullback; this determines a normal function over $B$, and we can consider its image (or that of an abstract normal function over $B$) in $\mathcal{J}$. We refer to such subvarieties (no longer sections) of $\mathcal{J}$ as multivalued normal functions.

**Theorem 4.** For the symplectic structure on $\tilde{\mathcal{J}}$ constructed in Theorem 3, the image $\nu(\mathcal{A}_0(\tilde{X}/\tilde{\mathcal{M}}))$ is isotropic. In other words, the normal function of any (multivalued) family of null-homologous 1-cycles on Calabi-Yau threefolds gives an isotropic subvariety of $\tilde{\mathcal{J}}$.

**Proof.**

This is a variation on Step III in the proof of Theorem 3. Say we are given a family $\mathcal{C} \to \mathcal{B}$ of null-homologous 1-cycles on gauged Calabi-Yaus; this means that there is a map $t : \mathcal{B} \to \tilde{\mathcal{M}}$ such that for $b \in \mathcal{B}$, the fiber $C_b$ is a null-homologous 1-cycle in $X_{t(b)}$. The Abel-Jacobi map $\nu : \mathcal{B} \to \tilde{\mathcal{J}}$ is then a lift of $t$, and we claim that its image is isotropic. Locally in $\mathcal{B}$ we choose 3-chains $\Gamma_b$ such that $\partial \Gamma_b = C_b$; this determines local lifts $\nu : \mathcal{B} \to H^3$ and $\tilde{\nu} : \mathcal{B} \to T^*\tilde{\mathcal{M}}$ of $\nu$. Let $\tilde{\tau}$ be the action 1-form on $T^*\tilde{\mathcal{M}}$, so $d\tilde{\tau}$ is the symplectic form. We need to show that $\xi := \tilde{\nu}^*\tilde{\tau}$ is a closed 1-form on $\mathcal{B}$. At $b \in \mathcal{B}$, $\xi$ is given by $\int_{\Gamma_b}$. The new feature here is that instead of 3-cycles $\gamma_b \in H_3(X_b, \mathbb{Z})$, we have the 3-chains $\Gamma_b$, with variable boundary $C_b$.

As before, we think (locally in $\tilde{\mathcal{M}}$) of $X$ as being a fixed $C^\infty$ manifold, with variable complex structure $\overline{\partial}_t$ and 3-form $s_t$, $t \in \tilde{\mathcal{M}}$. We consider the function defined locally on $\mathcal{B}$:

$$g(b) := \int_{\Gamma_b} s_t(b);$$

we claim that $\xi = dg$. This time, both the integrand and the chain in $\int_{\Gamma_b} s_t(b)$ depend on $b$. Let $v$ denote an appropriate normal vector to (the support of) $C_b$ along $\Gamma_b$. (For details, see [Gr2]). We then have:

$$\frac{\partial}{\partial b} g(b) = \int_{\Gamma_b} \frac{\partial s_t(b)}{\partial b} + \int_{C_b} v| s_{t(b)}. $$

But $s_{t(b)}$ is of type $(3,0)$ with respect to the complex structure $\overline{\partial}_{t(b)}$, so the contraction $v| s_{t(b)}$ is of type $(2,0)$, regardless of the type of $v$. On the other
hand, the (Poincaré dual of) $C_b$ is of type $(2, 2)$ with respect to this complex structure, so the second term vanishes identically. As in the proof of Theorem 3, we then conclude that $dg = \xi$, as required.

Q.E.D

Remarks. The proof shows that Theorem 4 can be strengthened in two directions:

(1) Let $\tau = \tilde{\tau} - df$ be the 1-form on $\tilde{J}$ constructed in Lemma 2, satisfying $d\tau = \sigma$. (The function $f$ is given by formula (6).) Then each normal function $\nu$ gives an integral manifold of $\tau$, i.e. $\nu^*\tau = 0$. (This of course implies $\nu^*\sigma = 0$.) Indeed, in the notation of the proof we have $\nu^*\tilde{\tau} = \xi$ and $\nu^*f = g$, so what we show is exactly the vanishing of $\nu^*\tau = \xi - dg$. If we consider instead normal functions for $J$ over $M$ (suppressing the gauge), we see that they are integral manifolds for the contact structure on $J$ given in Remark 1 after Lemma 2.

(2) The result of Theorem 4 holds for any normal function, not only for those coming from cycles. In fact, the condition $\nu^*\tau = 0$ is equivalent to Griffiths’ differential equation for a normal function. In this general setting we define the function $g$ on $B$ as the inner product

$$g := (\nu, s)$$

of the sections $\nu$ (= the lift of $\nu$) and $s$ (= the $(3, 0)$-form). Its value depends only on the partial lift $\tilde{\nu} : B \to T^*\tilde{M}$; we have $g = \tilde{\nu}^*f$. The crucial formula (8) becomes:

$$dg = (\nu, \nabla s) + (\nabla \nu, s).$$

So we have $\nu^*\tau = \tilde{\nu}^*\tilde{\tau} - dg = -(\nabla \nu, s)$, and the vanishing of $\nu^*\tau$ is indeed equivalent to the differential equation (7) of a normal function.

We can define the contact structure on $J$, and hence the symplectic structure on $\tilde{J}$, in terms of normal functions: The contact distribution is given at a point $(X_0, \nu_0) \in J$ as the subspace of $T_{(X_0,\nu_0)}J$ spanned by differentials of all normal functions $\nu$ defined near $X_0$. If we choose one such $\nu$, we obtain a splitting

$$T_{(X_0,\nu_0)}J \approx F^2H^3(X_0)^* \oplus T_{X_0}\mathcal{M},$$

in terms of which the contact distribution is

$$H^{2,1}(X_0)^* \oplus T_{X_0}\mathcal{M},$$

and this subspace is independent of the choice of $\nu$, by (7).

(3) An example of a “multi-valued” normal function is given by the inclusion $\nu : B \hookrightarrow \tilde{J}$ of the total space $B$ of a family of abelian subvarieties,
$\mathcal{B} \to S$ (where $S$ comes with a map $i : S \to \tilde{\mathcal{M}}$, and the fiber $B_s$ is an abelian subvariety of the torus $J(X_i(s)))$. We conclude:

**Corollary.** The total space of any family of abelian subvarieties of $\tilde{\mathcal{J}}$ is isotropic.

The generalized Hodge conjecture says that $\nu(\mathcal{B})$ is actually contained in $\nu(\mathcal{A}_0(\tilde{\mathcal{X}}/\tilde{\mathcal{M}}))$, but this is not known. At first sight it also seems plausible that a maximal $\nu(\mathcal{B})$, as well as each component of $\nu(\mathcal{A}_0(\tilde{\mathcal{X}}/\tilde{\mathcal{M}}))$, should actually be Lagrangian; there are, however, counterexamples to this.

The intermediate Jacobian is the connected component of the origin in a larger group $D(X)$, the Deligne cohomology group of $X$, cf. [EZ]. This fits in the exact sequence

$$0 \to J(X) \to D(X) \xrightarrow{\varphi} H^{2,2}(X, \mathbb{Z}) \to 0,$$

where $H^{2,2}(X, \mathbb{Z}) := H^{2,2}(X, \mathbb{C}) \cap H^4(X, \mathbb{Z})$. For a Calabi-Yau, this is the same as $H^4(X, \mathbb{Z})$. (The relation of $D$ to $J$ is analogous to that of $Pic$ to $Pic^0$.) The Abel-Jacobi map $\mathcal{A}_0(X) \to J(X)$ extends to an Abel-Jacobi (aka cycle-class) map,

$$\nu : \mathcal{A}(X) \to D(X),$$

such that $\varphi \circ \nu : \mathcal{A}(X) \to H^{2,2}(X, \mathbb{Z})$ sends a cycle to its fundamental class. Everything works well for families: a family $\mathcal{X} \to \mathcal{M}$ determines a relative Deligne group $D(\mathcal{X}/\mathcal{M})$ whose fiber over $t \in \mathcal{M}$ is $D(X_t)$, and there is an Abel-Jacobi map

$$\nu : \mathcal{A}(\mathcal{X}/\mathcal{M}) \to D(\mathcal{X}/\mathcal{M})$$

which fiber-by-fiber agrees with the previous $\nu$.

**Theorem 5.** Let $\mathcal{D} = D(\mathcal{X}/\mathcal{M})$ be the relative Deligne group of a complete family of CY threefolds, let $\tilde{\mathcal{D}} = D(\tilde{\mathcal{X}}/\tilde{\mathcal{M}})$ be the relative Deligne group of the gauged family, and let $\mathcal{J}, \tilde{\mathcal{J}}$ be the corresponding relative Jacobians. Then there is a natural contact structure $\kappa$ on $\mathcal{D}$ with symplectification $\sigma = d\tau$ on $\tilde{\mathcal{D}}$ with the following properties:

(a) $\sigma, \tau$ and $\kappa$ restrict to the previously constructed structures on $\tilde{\mathcal{J}}$ and $\mathcal{J}$.

(b) The fibration $\tilde{\mathcal{D}} \to \tilde{\mathcal{M}}$ is Lagrangian.

(c) All normal functions $\nu(\mathcal{A}(\tilde{\mathcal{X}}/\tilde{\mathcal{M}}))$ are integral manifolds for $\tau$ (hence isotropic for $\sigma$), and normal functions $\nu(\mathcal{A}(\mathcal{X}/\mathcal{M}))$ are integral manifolds for $\kappa$. 

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Proof.

Each component of $D$ (respectively $\tilde{D}$) is locally isomorphic to $J$ (respectively $\tilde{J}$). In order to extend $\sigma, \tau, \kappa$ we need a collection of local sections of $D$ (and $\tilde{D}$) with the properties:

(1) For each $t \in M$ and each component of $D(X_t)$, there is a local section defined near $t$ and passing through this component.

(2) The difference of any two local sections in the same component is an integral manifold for $\sigma, \tau, \kappa$.

Indeed, such a collection of sections determines a collection of local isomorphisms of components of $D$ (respectively $\tilde{D}$) with $J$ (respectively $\tilde{J}$), so we define $\sigma, \tau, \kappa$ as pullbacks of the corresponding structures on $\tilde{J}$ and $J$. This is well-defined by (2), and covers $\tilde{D}$ and $D$, by (1). Now such a collection is given, in our case, by the normal functions of all 1-cycles, $\nu(A(X/M))$: property (1) follows from the Lefschetz theorem on $(1, 1)$-classes, and property (2) amounts to our Theorem 4 and the remarks following it.

Q.E.D

As an immediate corollary, we get the following result of Griffiths [G1]. (There it is proved only for quintic threefolds, but the argument works in general, once we know the unobstructedness of moduli [Bo,Ti,To].)

Corollary. On a generic CY, algebraic equivalence of 1-cycles implies $AJ$-equivalence; in other words, the Abel-Jacobi map is constant on any connected family of 1-cycles.

Proof.

The $AJ$ image of the family is isotropic, by Theorem 5, and dominates $\tilde{M}$, hence it is Lagrangian and meets the generic fiber discretely, so the $AJ$ map is locally constant on the connected family, hence constant.

Q.E.D

Note. These results have analogues for Calabi-Yaus of odd dimensions $n = 2k + 1 \geq 3$: The universal $k^{th}$ intermediate (or middle-dimensional) Jacobian $\tilde{J}^k(\tilde{X}/\tilde{M})$, as well as the corresponding relative Deligne cohomology, have
natural quasi-symplectic structures, i.e. closed 2-forms which in general will be degenerate. The fibers of the projections of $\tilde{J}$ or $\tilde{D}$ to $\tilde{M}$ are Lagrangian, i.e. maximal isotropic with respect to the 2-form (but of dimension greater than $\dim \tilde{M}$, because of the degeneracy). All normal functions, or Abel-Jacobi images of families of $k$-cycles, are isotropic. For the details, see [DM].

§ 3. Mirrors.

To a given family $X \to M$ of Calabi-Yaus, there are two ways of associating an ($N = 2$, $c = 9$) (super) conformal field theory (CFT). Physicists refer to these as the A-model and B-model theories. A key quantity in the CFT is its partition function, which is a cubic form on a vector space $V$, depending on some auxiliary parameters $q$.

In the A-model, one takes $V := H^{1,1}(X) = H^2(X, \mathbb{C})$, where $X$ is a general CY in the given family. On it there is the cubic form $\text{Topo}$, giving the topological cup product. For each $k \in H_2(X, \mathbb{Z}) \approx H^2(X, \mathbb{Z})^\ast$ (we replace $\mathbb{Z}$-homology or cohomology by its image in the $\mathbb{Q}$-(co)homology, i.e. we ignore torsion classes), we think of $k^3$ as a cubic form on $V$. The partition function is then:

$$T_{\text{Topo}} + \sum_{k,d} n_k k^3 q^{dk}.$$ (9)

In the summation, $k$ runs over $H^2(X, \mathbb{Z})^\ast$ and $d$ over positive integers. The coefficient $n_k$ is the “number of rational curves” of fundamental class $k$ in $X$. This number is conjectured, but not known, to be finite. The multi-parameter $q$ is defined by $q := e^{2\pi i t}$, where $t$ is a set of linear coordinates on $H^2(X, \mathbb{C})$. Intrinsically, we define $q^{dk}$ as the function given at $t \in H^2(X, \mathbb{C})$ by:

$$q^{dk} := e^{2\pi i d \langle t, k \rangle}.$$ (10)

The sum is expected to converge for $t$ in $K$, the complexified Kähler cone of $X$, i.e. when $\text{Im}(t) \in H^2(X, \mathbb{R})$ is in the Kähler cone.

In the B-model, $V$ is $H^{2,1}(X)$, and the partition function is a normalized version of the Yukawa cubic, so its coefficients are functions on $\tilde{M}$. The Yukawa cubic is defined only up to a scalar factor which is determined by a choice of holomorphic gauge $s \in H^{3,0}(X)$. In order to get the coefficients in the partition function to be functions on $\tilde{M}$ (rather than $\tilde{M}$), we need a section of $\tilde{M} \to M$. According to [Mo], one chooses a maximally unipotent degeneration of $X$; this determines a vanishing cycle

$$\gamma_0 \in H_3(X, \mathbb{Z}),$$
and $s$ is normalized by the condition
\[
\int_{\gamma_0} s_0 = 1.
\]

Physicists believe [Gep] that either the $A$-model or $B$-model constructions set up bijections between CFTs and CY families. This, together with the existence of an internal symmetry of CFT which interchanges $A$ and $B$ variables, led [Di,LVW] to the mirror conjecture, which says that each CY family $\mathcal{X} \to \mathcal{M}$, perhaps with some extra data, determines another family $\mathcal{X}' \to \mathcal{M}'$ such that the $A$-model CFT of the former is isomorphic to the $B$-model CFT of the latter, and vice versa.

It is not clear (to us) what the precise mathematical formulation of this conjecture should be. What is clear, though, is that the conjecture involves the existence of an isomorphism, called the mirror map.

(11)
\[
t : \mathcal{M} \to K'
\]
where $\mathcal{M}$ is (an open subset of ?) the moduli space of Calabi-Yaus (with extra structure ?) in the first family, and $K'$ is the complexified Kähler cone of an $X'$ in the second family. This should satisfy:

(a) $dt : H^1(T_X) \cong H^{1,1}(X')$ is an isomorphism. (In particular, the dimensions agree, so that the Hodge diamonds of $X, X'$ are obtained from each other by a $90^\circ$ turn).

(b) The partition functions correspond through $t$, i.e.:
\[
Yukawa_{(X,s_0)} = (\text{Topo}_{X'} + \sum_{k \in H^2(X',\mathbb{Z})} n_k(X')k^3q^{dk}) \circ dt,
\]
where both sides are interpreted as sections of $\text{Sym}^3T_{_{\mathcal{M}^*}}$.

In [CdOGP], this conjectural equality of partition functions was used to obtain some spectacular predictions for numbers of rational curves of arbitrary degrees in quintic hypersurfaces in $\mathbb{P}^4$. These predictions have since been extended to many other CY families. We refer to [Mo] for more details of the conjecture and the evidence for it.

In the remainder of this section we wish to speculate on the role which the ACIHS on relative Deligne cohomology $\tilde{D}$ might play in the mirror story. A couple of relationships are evident:

(i) Both of the relevant dimensions, $h^{1,1}$ and $h^{2,1}$, occur in $\tilde{D}$, as the rank of the discrete part and (one less than) the dimension of the continuous
part (of either base or fibers), respectively. This suggests that the relation between the two ACIHS attached to mirror-symmetric CY families might be expressible directly as some sort of Fourier transform.

(ii) The mirror map, \( t \) of (11), as constructed in [CdOGP], [Mo], etc., is given by

\[ t_i = \frac{\tilde{t}_i}{\tilde{t}_0}, \quad i = 1, \cdots, h^{2,1}, \]

where \( \tilde{t}_i \) are the action coordinates for the ACIHS, for an appropriate choice of Lagrangian basis \( \gamma_0, \cdots, \gamma_{h^{2,1}} \) in terms of a maximally unipotent degeneration. (This follows immediately from the construction in [Mo] and our Lemma 3.)

(iii) We also recall from [Mo] that mirror symmetry is expected to hold for “most” Calabi-Yaus, but probably not all; a notorious collection of probable exceptions is provided by Calabi-Yaus which are rigid, i.e. \( h^{2,1} = 0 \). Now a somewhat analogous situation occurs for the Torelli and Schottky problems, of recovering a variety or family of varieties from its Hodge structure: this is known to be true, at least generically, for many classes of varieties, but various counterexamples exist. It is conceivable that all failures of mirror symmetry could be explained by failures of Schottky or Torelli results, i.e. that there is still some mirror data, but it does not come from a geometric object, namely a CY family.

The above observations, and some others which will be discussed below, suggest that we might try to split the mirror conjecture roughly into three parts:

(1) Axiomatize the integrable system associated to a CY family, by listing a collection of cohomological and symplectic data which can be attached to each family, and which satisfies certain axioms.

(2) Describe a formal mirror transform on abstract systems, which interchanges the continuous and discrete parts.

(3) Study the validity of Torelli- and Schottky-type results, which say that a CY-family can be uniquely recovered from its system, and describe which abstract systems actually arise from geometry.

A weaker version of this last step, likely to hold in greater generality, would be:

(3’) Extract from an abstract system enough data to reconstruct either the A- or B-model partition functions. (Ideally, we should be able to extract both partition functions from the system, and the formal transform of step (2) should interchange them.)

An abstract system should consist at least of the following data:
An analytically completely integrable Hamiltonian system $\tilde{D} \to \tilde{M}$, which is an extension of another ACIHS $\tilde{J} \to \tilde{M}$ by a lattice $H^{1,1}(\mathbb{Z})$.

(ii) An indefinite ("Lorentzian") polarization on the torus fibers of $\tilde{J} \to \tilde{M}$ and a (symmetric, cubic, $\mathbb{Z}$-valued) intersection product on $H^{1,1}(\mathbb{Z})$.

(iii) A $\mathbb{C}^*$-action on $\tilde{D}$ lifting an action of $\mathbb{C}^*$ on $\tilde{M}$.

(iv) A 1-form $\tau$ on $\tilde{D}$, homogeneous of weight 1 with respect to the $\mathbb{C}^*$-action, satisfying $d\tau = \sigma$ (= the symplectic form on $\tilde{D}$).

Some further pieces of data may also need to be specified, such as the choice of a maximally unipotent degeneration. The above data lets us recover the variation of weight-3 Hodge structures over $\tilde{M}$, and the family of all normal functions: these are the integral manifolds for $\tau$. A key question which we cannot answer is whether the collection of normal functions coming from 1-cycles is determined by the above data, or needs to be specified explicitly.

These data certainly seem rich enough that one expects to recover from them much of the underlying CY geometry. The hope, which of course we are nowhere near realizing, is that when these data satisfy some appropriate axioms they can be used to generate a mirror system of the same type. If this is so, one must be able to extract both the $A$- and $B$-model data (i.e. the two partition functions) from a given system. Let us discuss the various ingredients.

For the $B$-model, we need two essential pieces: the Yukawa cubic $c$ and the affine structure on $\tilde{M}$. Now the Yukawa cubic is determined by the symplectic form on $\tilde{J}$ and the polarization on the fibers, by Theorem 1. The affine structure used to convert $c$ to a cubic on the fixed vector space $V$ is determined by the coordinates $\tilde{t}_i$, which by Lemma 3 are the action variables, so they too are determined by $\tilde{J}$. This of course depends on the discrete choice of a Lagrangian basis for $H_3(X,\mathbb{Z})$; we do not see how to avoid this choice, so it must either be built into the axioms, or be deducible somehow from knowing the collection of all normal functions (and in particular, their monodromy).

For the $A$-model it seems plausible that the relevant data can be obtained, but this leads us to Hodge-theoretic questions which we cannot (yet?) resolve. We need the topological intersection form Topo, which is built into the axioms. The essential problem is to recover the coefficients $n_k$. Each rational curve on the generic CY in the family gives a (finitely multivalued) normal function in $\tilde{D}$, and each homologous pair gives one in $\tilde{J}$. The problem is that not all normal functions arise in this way. We are led to a series of problems.
(i) Which normal functions in $\mathcal{D}$ arise from effective curves?

(ii) Which of these arise from irreducible curves?

(iii) Which of these arise from rational curves?

We note that a given normal function $\nu$ might arise from many different curves. The simplest example is that of complete intersection curves: a given family of such curves is parametrized by a (usually positive dimensional) projective space on each CY in the family, and all such curves have the same Abel-Jacobi image since they are rationally equivalent. In fact, the Corollary to Theorem 5 says that all curves in any connected family on the generic CY give rise to the same $\nu$.

The above remark suggests that we need not only decide which normal functions $\nu$ come from rational curves, but to determine a multiplicity or weight of $\nu$ accounting for all its realizations in terms of rational curves. Now this seems rather special; one can just as well hope for a weight function $w_\nu$ counting all curves, of all genera, corresponding to $\nu$, through an expansion such as $w_\nu = \sum N_{\nu,g} \lambda^{g-1}$, where $\lambda$ is a dummy variable and the coefficients $N_{\nu,g}$ account for the curves.

A master partition function $\mathcal{F}$, accounting for curves of all genera on a CY, has recently been proposed in [BCOV]. It can be written:

$$\mathcal{F} = \sum \frac{F_g}{\lambda^{g-1}} = \sum \frac{N_{k,g} q^k}{\lambda^{g-1}} = \sum w_k q^k,$$

where the $q^k$ are as in (10), the $F_g$ are the genus-$g$ partition functions (they depend on the $q^k$), and the coefficients $N_{k,g}$ are certain combinations of the “numbers” $n_{k,g}$ of curves of genus $g$ and fundamental class $k$. [A rigid non-singular curve of class $k$ and genus $g$ contributes a term of $1 \cdot \lambda^{g-1}$, but also makes contributions to higher-genus terms, because of multiple (branched) coverings. The situation is more complicated for continuous families of curves, where the basic contribution involves the Euler characteristic of the base, but there are other terms corresponding, e.g., to singular curves in the family.] This $\mathcal{F}$ is a function of the variables $q$ and $\lambda$. Via the mirror correspondence (11), it is carried to a function on $\tilde{\mathcal{M}}$, whose leading term is the Yukawa coupling. The genus-0 partition function $F_0$ is recovered as a residue of $\mathcal{F}$ in the $\lambda$ direction.

In order to recover the partition function $F_0$, or $\mathcal{F}$ for that matter, we need a refinement of the BCOV coefficients: to each Lagrangian normal function $\nu$ we want to associate a weight function

$$w_\nu = \sum N_{\nu,g} \lambda^{g-1}$$
such that for each nef class $k$ we have

$$w_k = \sum_{\{\nu | \varphi(\nu) = k\}} w_{\nu},$$

or

$$N_{k,g} = \sum_{\{\nu | \varphi(\nu) = k\}} N_{\nu,g},$$

or equivalently:

$$\mathcal{F} = \sum N_{\nu,g} q^{\varphi(\nu)} \lambda^{g-1}.$$ 

The point is that we want a formula for the $w_{\nu}$ in terms of $\nu$ alone, without knowledge of the actual curves on $X$ whose Abel-Jacobi image is $\nu$.

We do not know whether such a formula exists. But if it does, it is likely to involve only the infinitesimal invariant $\delta \nu$, associated to a normal function by Griffiths [G2] and improved by Green [Gr1]. This invariant lives in a certain Koszul cohomology group. It vanishes if and only if $\nu$ is locally constant, i.e. can be lifted locally to a constant section $\overline{\nu}$ of the Gauss-Manin local system $R^3\chi_\ast \mathbb{C}$ of the family $\tilde{\chi} : \tilde{\mathcal{X}} \to \tilde{\mathcal{M}}$.

The reason is that the sum, over all $\nu$ in $\varphi^{-1}(k)$, is too big. For example, when $k = 0$, any torsion element in $J(X)$ (or in $H^3(X, \mathbb{Q}/\mathbb{Z})$) extends to a finitely valued Lagrangian multisection of $\tilde{J}$, but there is no reason to expect any contribution of these torsion sections to the curve count. By continuity, one expects no contribution from any locally constant normal function, so $w_{\nu}$ should depend only on $\delta \nu$.

In our case, of normal functions on CY threefolds, we can describe the Koszul cohomology group in which $\delta \nu$ lives explicitly in terms of the Yukawa cubic. In general, $\delta \nu$ is obtained by choosing a lift $\overline{\nu}$, differentiating it with respect to the Gauss-Manin connection $\nabla$, and taking the part independent of choices. In our case, $\nabla \overline{\nu}$ lives in

$$\text{Hom}(T_{X,s} \tilde{\mathcal{M}}, H^3/F^2H^3)$$

which is canonically identified with $\otimes^2(F^2)^\ast$. The differential equation (7) satisfied by any normal function asserts that $\overline{\nu}$ lands in

$$\text{Hom}(T_{X,s} \tilde{\mathcal{M}}, F^1H^3/F^2H^3)$$

which is $(F^2)^\ast \otimes (H^{2,1})^\ast$. By Remark (2) to Theorem 4, any normal function is Lagrangian. This means that $\nabla \overline{\nu}$ is symmetric, so we are down to

$$\text{Sym}^2(H^{2,1})^\ast,$$

or the space of quadrics on $V = H^{2,1}$. The ambiguity arising from a change of the lift $\overline{\nu}$ amounts to those quadrics which are partial derivatives, in some
direction, of the Yukawa cubic $c \in \text{Sym}^3(H^{2,1})^*$. Summarizing, the Koszul complex in our case can be identified with the complex

$$(H^{2,1})^{\text{Yukawa}} \xrightarrow{\otimes^2} (H^{2,1})^* \xrightarrow{\wedge^2} (H^{2,1})^*.$$ 

We conclude that $\delta \nu$ is in $R^2$, where $R^d$ is the $d^{th}$ graded piece of the Jacobian ring of $c$: $R^\bullet := S^\bullet / \mathcal{J}^\bullet$, where $S^\bullet = \text{Sym}^\bullet(V^*)$ and $\mathcal{J}^\bullet \subset S^\bullet$ is the Jacobian ideal, generated by the partials of $c$. The problem of reconstructing the curves on $X$ with given normal function $\nu$ from the invariant $\delta \nu$ may therefore have some resemblance to the variational Torelli problem, where certain parts of $R$ are given and one needs to recover the underlying variety, as in [D]. It is known how to extract quite a bit of information from the invariants $\delta \nu$ in some situations, e.g. [G2], [Gr1], [CP]. The question is whether, on a generic Calabi-Yau threefold $X$, one can recover the collection of all curves from their $\delta \nu$ invariants.

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