Corrigendum: Quantum Bayesian rule for weak measurements of qubits in superconducting circuit QED (2014 New J. Phys. 16 123047)

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We noted an error in the integrand in equation (18). That is, the phase factor $\Phi_2(t_m)$ in the Bayesian rule for the off-diagonal element should be constructed as

$$
\Phi_2(t_m) = -\int_0^{t_m} \sqrt{F_{ba}(t)} \xi(t) \, dt.
$$

In this result, instead of $\xi(t)$ as proposed previously, we amend it here by $\tilde{I}_\psi(t)$ which is determined as follows. First, within the ‘polaron’ transformation scheme, the output current reads $I_{\psi}(t) = -\sqrt{\Gamma_{ci}}(\sigma_{+}) + \sqrt{\kappa} |\mu| \cos(\theta_{\mu} - \phi) + \xi(t)$, where $\mu = \alpha_{c}(t) + \alpha_{g}(t) \equiv |\mu|e^{i\theta_{\mu}}$. Second, the ensemble-average of $\rho_{\psi}(t_m)$ over the stochastic ‘field’ $\xi(t)$, or equivalently over the stochastic output current $I_{\psi}(t)$, should be consistent with the result by averaging equation (14). Therefore, instead of extracting a formal solution from equation (14) as $\Phi_2(t_m) = -\int_0^{t_m} \sqrt{F_{ba}(t)} \xi(t) \, dt$, we need to replace $\xi(t)$ with $\tilde{I}_\psi(t) = I_{\psi}(t) - I_{\psi}(t)$, where $I_{\psi}(t) = \sqrt{\kappa} |\mu| \cos(\theta_{\mu} - \phi)$ thus $\tilde{I}_\psi(t) = -\sqrt{\Gamma_{ci}}(\sigma_{+}) + \xi(t)$. Accordingly, in the suggested approximation of equation (19), the average current $I(t_m)$ is replaced by $I(t_m) = \frac{1}{t_m} \int_0^{t_m} \tilde{I}_\psi(t) \, dt$.

We may elaborate further the above correction idea by taking the bad-cavity limit. In this case, all the rates in equation (14) are time-independent. Ensemble-averaging equation (14) over the stochastic Wiener variable $\xi(t)$ would vanish the last two stochastic terms and lead thus to $\rho_{\psi}(t_m) \sim e^{-\Gamma_{ci} t_m}$, i.e., the overall dephasing factor. Now let us show how this result can be recovered by ensemble-averaging the conditional state based on the Bayesian rule. First, consider the bare Bayesian rule of equation (10). Ensemble-averaging the stochastic integrated current $I_{m}$, we obtain $\bar{\rho}_{\psi}(t_m) \sim e^{-\Gamma_{ci} t_m/2}$. Then, inserting $e^{-\Phi_1(t_m)}$ and making the ‘joint’ ensemble-average, we have $\rho_{\psi}(t_m) \sim e^{-\Gamma_{ci} t_m/2}$. Finally, multiplying this ensemble-average dephasing factor by $D(t_m)$, i.e., the purity degradation factor given by equation (13), we recover the overall dephasing factor $e^{-\Gamma_{ci} t_m}$ by noting that $\Gamma_{ci} + F_{ba} \equiv t_m$.

For the $(I, Q)$ two quadrature measurement, the phase factor $\Phi_2(t_m)$ of equation (26) is correct. However, in order to be in similar form of the single quadrature result, we propose to reexpress equation (26) as

$$
\Phi_2(t_m) = -\int_0^{t_m} \sqrt{\Gamma_{m}(t)}/2 \, Q_{m}(t) \, dt,
$$

where $Q_{m}(t) = Q_{m}(t) - \sqrt{\kappa} |\mu(t)| \sin \theta_{\mu}$, while $Q_{m}(t) = I_{\psi}(t)|_{\omega=\pi/2} = \sqrt{\kappa} |\mu(t)| \sin \theta_{\mu} + \xi_{z}(t)$.

At last, we mention that all the numerical results presented in the article are correct, because of the assumed initial state $|\psi(t)\rangle = |e\rangle + |g\rangle \rangle / \sqrt{2}$ which holds $\langle \sigma_{z} \rangle = 0$. For initial states with $\langle \sigma_{z} \rangle \neq 0$, the above correction for the single quadrature measurement will affect the results. That is, if using the previous $\Phi_2(t_m) = -\int_0^{t_m} \sqrt{F_{ba}(t)} \xi(t) \, dt$ (not replacing $\xi(t)$ by $\tilde{I}_\psi(t)$), there would be a more phase factor, $\exp [i \int_0^{t_m} \, dt \sqrt{F_{ba}(t)} \xi(t) \langle \sigma_{+} \rangle]$, which, in the case $\langle \sigma_{z} \rangle \neq 0$, will affect both the quantum trajectories and the ensemble-averaged result. However, as carefully explained above, the self-consistence-required correct result should not have this extra phase factor.

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Quantum Bayesian rule for weak measurements of qubits in superconducting circuit QED

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Abstract
Compared with the quantum trajectory equation (QTE), the quantum Bayesian approach has the advantage of being more efficient to infer a quantum state under monitoring, based on the integrated output of measurements. For weak measurement of qubits in circuit quantum electrodynamics (cQED), properly accounting for the measurement backaction effects within the Bayesian framework is an important problem of current interest. Elegant work towards this task was carried out by Korotkov in ‘bad-cavity’ and weak-response limits (Korotkov 2011 Quantum Bayesian approach to circuit QED measurement (arXiv:1111.4016)). In the present work, based on insights from the cavity-field states (dynamics) and the help of an effective QTE, we generalize the results of Korotkov to more general system parameters. The obtained Bayesian rule is in full agreement with Korotkov’s result in limiting cases and as well holds satisfactory accuracy in non-limiting cases in comparison with the QTE simulations. We expect the proposed Bayesian rule to be useful for future cQED measurement and control experiments.

Keywords: quantum measurement, Bayesian rule, circuit QED
1. Introduction

The circuit quantum electrodynamics (cQED) setup [1–3] is widely regarded as a promising solid-state architecture for quantum computing and quantum information processing. In the early stage, this setup is also an excellent platform for quantum measurement and control studies [4–18]. Particular examples include: the experimental test of the Leggett–Garg inequality [6], the measurement of weak values [7], the quantum back-action effect of weak measurements [10, 11], and quantum feedback control experiments [16–18]. In these studies, quantum measurements play a central role, i.e., in the dispersive regime [19, 20], a dyne-type quadrature measurement of the cavity field can reveal the qubit state information [6–10, 16–18].

In this context, rather than strong projective measurement, more interesting is the type of weak measurement [21–23] whose experimental realization is an extremely attractive subject [10, 11]. In particular, this type of monitoring on quantum state is an essential prerequisite for measurement-based feedback control of quantum systems [23]. For continuous weak measurements the quantum trajectory theory [23], as broadly applied in quantum optics and quantum control problems, is most popular. The quantum trajectory theory can also address the solid-state charge qubit measurements by mesoscopic quantum-point-contact and single-electron-transistor detectors [24, 25]. For this setup, an equivalent scheme known as the quantum Bayesian approach was proposed [26] and exploited for applications. For cQED, which is analogous to the conventional optical cavity-QED, the quantum trajectory approach seems the most natural choice [12–14, 19, 20]. Despite this, in a recent study [27], Korotkov developed a promising quantum Bayesian approach in the ‘bad-cavity’ and weak-response limits. Owing to its competitive efficiency and advantage of accounting for realistic imperfections, this approach has been employed in recent experiments on quantum measurement [11] and feedback control [16].

Construction of the quantum Bayesian approach is largely based on the classical Bayes formula. For the diagonal elements of the qubit, the Bayes formula works perfectly; however, it does not work for the off-diagonal elements. One proceeds then by a purity consideration [26], together with some additional physical insights [27]. In this work, rather than using such types of consideration, we would like to fulfill a similar task by employing the quantum trajectory equation (QTE) approach. In order to gain necessary insights, our analysis will pay particular attention to the nature of the cavity field under continuous quadrature monitoring. This treatment permits us to avoid the bad-cavity and weak-response assumptions [27], thus making the obtained quantum Bayesian rule applicable to more general setup parameters.

The paper is organized as follows. We begin in section 2 with a brief description of cQED and the optical QTE, before identifying the nature of the cavity state conditioned on weak measurements in section 3. We then construct in section 4 the quantum Bayesian rule for single-quadrature measurements and in section 5 for two-quadrature measurements. In section 6, numerical results and comparisons are presented, for both the limiting and non-limiting cases. Finally, we summarize the work with remarks in section 7. In addition, two appendices are provided, for the analytic solution of the cavity field and an equivalence proof of two expressions of the purity degradation factor.
2. Model and QTE

Let us consider the simplest cQED setup with only one superconducting qubit in the resonator cavity [1]. In this setup, the central section of a superconducting coplanar waveguide plays the role of an optical cavity, and the superconducting qubit the role of an (artificial) atom. The superconducting qubit is coupled to the one-dimensional transmission line (1DTL) cavity which acts as a simple harmonic oscillator. Therefore the qubit, the 1DTL cavity, and their mutual coupling can be well described by the Jaynes–Cummings Hamiltonian. Moreover, we consider the setup in a dispersive regime [1–3], i.e., with the detuning between the cavity frequency (ωr) and qubit energy (ωq), \( \Delta = \omega_r - \omega_q \), much larger than the coupling strength \( g \). In this limit and in the rotating frame with the microwave driving frequency \( \omega_m \), the system can be described by an effective Hamiltonian [1, 2]

\[
H_{\text{eff}} = \Delta_r a^\dagger a + \frac{\tilde{\omega}_q}{2} \sigma_z + \chi a^\dagger a \sigma_z + \left( \epsilon_m^+ a + \epsilon_m a^\dagger \right),
\]

where \( \Delta_r = \omega_r - \omega_m \) (for resonant drive, \( \Delta_r = 0 \)), and \( \tilde{\omega}_q = \omega_q + \chi \) with \( \chi = g^2/\Delta \) being a dispersive shift to the qubit energy and cavity frequency. In equation (1), \( a^\dagger \) (a) and \( \sigma_z \) are respectively the creation (annihilation) operator of cavity photon and the quasi-spin operator (Pauli matrix) for the qubit. \( \epsilon_m \) is the microwave drive amplitude to the cavity.

For measurements, in this work we will first consider the single quadrature \( I_\phi \) measurement in detail, then convert the obtained results to the \( (I, Q) \) two-quadrature measurement. For the single quadrature homodyne measurement, one actually measures \( \hat{I}_\phi = \frac{1}{2}(ae^{-i\phi} + a^\dagger e^{i\phi}) \), where \( \phi \) is the local oscillator (LO) phase [23]. The measurement output can be expressed as

\[
I_\phi(t) = \sqrt{\kappa} \langle ae^{-i\phi} + a^\dagger e^{i\phi} \rangle_{\theta(t)} + \xi(t),
\]

where \( \kappa \) is the damping rate of the cavity photon and \( \xi(t) \) a Gaussian white noise originating from the stochastic quantum-jump, which satisfies the ensemble-average properties of \( E[\xi(t)] = 0 \) and \( E[\xi(t)\xi(t')] = \delta(t-t') \). The quantum average \( \langle \cdots \rangle_{\theta(t)} \) is defined by \( \langle \cdots \rangle_{\theta(t)} = \text{Tr} \left[ \langle \cdots \rangle_{\theta(t)} \right] \), with \( \theta(t) \) the qubit-cavity conditional state given by the QTE [23]:

\[
\theta = -i[H_{\text{eff}}, \theta] + \kappa D[a] \theta + \sqrt{\kappa} \mathcal{H} \left[ ae^{-i\phi} \right] \xi(t),
\]

where \( D[a] \theta = a\theta a^\dagger - \frac{1}{2} \{ a^\dagger a, \theta \} \) and \( \mathcal{H}[\bullet] \theta = (\bullet) \theta + \theta (\bullet)^\dagger - \text{Tr}[\{ (\bullet) + (\bullet)^\dagger \} \theta] \theta \).

3. Cavity state conditioned on weak measurements

To get necessary insights for constructing a quantum Bayesian rule for the qubit state, it is crucial to identify the nature of the cavity state conditioned on the quadrature outcomes. First, we notice that, for a specific qubit state \( |g \rangle \) or \( |e \rangle \), the interplay of measurement drive and cavity loss would lead to the formation of a coherent state \( |\alpha_g(t) \rangle \) or \( |\alpha_e(t) \rangle \) for the cavity field, with \( \alpha_{g(e)}(t) \) determined by the following equations:
\[
\dot{a}_e(t) = -i e_m - \chi \dot{a}_e(t) - \kappa a_e(t)/2, \\
\dot{a}_g(t) = -i e_m - \chi \dot{a}_g(t) - \kappa a_g(t)/2.
\]

Notice also that this result is associated with an ensemble average over the stochastic leakage of photons. In the stationary limit, the coherent-state parameter reads \( \alpha_{ge(e)} = -i e_m/(\Delta_{e,g(e)} + \kappa/2) \), where \( \Delta_{e,g(e)} = (\alpha_p - \alpha_m) \mp \chi \). The transient solution is also available (but with a lengthy expression, see appendix A).

As a heuristic discussion for measurement principle, let us first consider a simpler model of qubit measurement by another two-state meter (e.g., a spin), which are prepared in an entangled initial state \( |c_g \rangle \otimes |\uparrow \rangle + |c_e \rangle \otimes |\downarrow \rangle \). Here \( |\uparrow \rangle \) and \( |\downarrow \rangle \) are the meter basis states (in the \( \sigma_z \) representation). Then, let us consider a projective measurement on the spin in a different (e.g., \( x \)) direction. A specific result, for instance ‘+1’ in the \( x \)-direction, would project the joint state onto \( (c_g d_1 |g\rangle + c_e d_1 |e\rangle) \otimes |\uparrow \rangle \), where \( d_1 = -1 \langle \uparrow \uparrow | \uparrow \rangle \) and \( d_1 = -1 \langle \uparrow \downarrow | \uparrow \rangle \). Since the measurement basis \( |\uparrow \rangle \langle \uparrow | \downarrow \rangle \) is not parallel to \( |\uparrow \rangle \langle \downarrow | \downarrow \rangle \), the strong projective measurement on the meter does not collapse the qubit state onto \( |g\rangle \) or \( |e\rangle \). To the qubit state, this falls into the category of weak measurements.

Now consider the qubit measurement in cQED. The qubit-cavity state is initially prepared as \( |\Psi(0)\rangle = (c_g |g\rangle \otimes |\uparrow \rangle + c_e |e\rangle \otimes |\downarrow \rangle) \) (\( \alpha_0 = 0 \) if the cavity field is the vacuum). If one is faithfully tracking the emitted photon by continuous homodyne measurement, the subsequent time-dependent state can be expressed as

\[
|\Psi(t)\rangle = c_g(t) |g\rangle \otimes |\tilde{a}_g(t)\rangle + c_e(t) |e\rangle \otimes |\tilde{a}_e(t)\rangle.
\]

Here, instead of \( \langle \tilde{a}_{ge(e)}(t) \rangle \), we denote the respective cavity state as \( |\tilde{a}_{ge(e)}(t)\rangle \), indicating a lack of ensemble average. Based on the quantum measurement theory [23], this state is a stochastic and quantum pure state. That is, both the superposition coefficients \( c_{ge(e)}(t) \) and the cavity states \( |\tilde{a}_{ge(e)}(t)\rangle \) are stochastic, depending on the random outputs of measurement.

Now, consider a further weak measurement on this state, with an \( x \)-direction field has collapsed onto a unique eigenstate \( |\tilde{a}_{ge(e)}(t)\rangle \), indicating that this is not parallel to \( |\tilde{a}_{ge(e)}(t)\rangle \). The essential feature of this result is that the cavity field has collapsed onto a unique eigenstate \( |\Psi_m(t+\tau)\rangle \) of the quadrature operator, conditioned on the measurement record \( I_m \). However, we will show that this is not true.

As an explicit demonstration, we performed a direct simulation of equation (3) for continuous homodyne measurements, starting with a superposition qubit state and a vacuum cavity state. Based on the conditional joint qubit-plus-cavity state \( \varrho(t) \), the cavity states \( |\tilde{a}_g(t)\rangle \) and \( |\tilde{a}_e(t)\rangle \) in equation (5) can be extracted from \( \varrho(t) \), respectively, in terms of density matrix \( \varrho_{gg}(t) = \langle g\varrho(t)|g\rangle/\text{Tr}[|g\varrho(t)|g\rangle] \) and \( \varrho_{ee}(t) = \langle e\varrho(t)|e\rangle/\text{Tr}[|e\varrho(t)|e\rangle] \). Here \( \text{Tr} \{ \cdots \} \) is over the cavity degrees of freedom. Using a \( J \)-function representation, in figure 1 we plot the difference of \( \varrho_{gg}(t) \) and \( \varrho_{ee}(t) \) at a given time. We find that this type of coexistence of distinct coherent states will persist along the whole continuous weak measurement process. This result indicates that the weak measurement (with an outcome \( I_m \) during \( (t, t+\tau) \) does not collapse.
the cavity fields $\hat{a}_{g(e)}(t)$ onto any common eigenstate $|\psi_m\rangle$. This differs substantially from the simple two-state-meter example discussed earlier.

We may contrast the state structure revealed here with the assumption in [10]. In the supplementary materials (in section 2.1.2, before equation (6)), the following statement was made for the two-quadrature measurement: the measurement process, with quadrature outcomes $(I_m, Q_m)$, would project the cavity modes onto a unique eigenstate. While the final result obtained there seems correct, it might be of interest to further clarify their treatment.

More careful inspection reveals that the states $\rho_{gg}(t)$ and $\rho_{ee}(t)$ are very close to the coherent states $|\alpha_g(t)\rangle$ and $|\alpha_e(t)\rangle$, respectively. As we will see below, this identification can help us to determine the purity degradation factor in the Bayesian rule. Here, one may understand this essential result as follows. Rather than a direct point-process detection for the outgoing photon, the homodyne-type quadrature measurement is relatively soft to the cavity field. It does not drastically alter the number of cavity photons. The continuous quadrature measurement is mainly updating our knowledge about the superposition components, say, the coefficients $c_g(t)$ and $c_e(t)$ in equation (5), but not collapsing the cavity states $\rho_{gg}(t)$ and $\rho_{ee}(t)$ onto a common eigenstate $|\psi_m\rangle$. This is the so-called informational evolution [28, 29]. However, around $\alpha_g(t)$ and $\alpha_e(t)$, the cavity field does exist stochastic fluctuations, which will result in a stochastic phase factor to the qubit off-diagonal elements—as we will see in the following sections.

Finally, we remark that this cQED is an interesting way to understand the puzzle of Schrödinger’s cat. That is, the cavity state $|\hat{a}_{g(e)}(t)\rangle$ corresponds to the macroscopic state (‘dead’ or ‘alive’) of the cat, while the continuum of states outside the cavity corresponds to the infinite number of microscopic states of the cat. It is right these infinite number of microscopic degrees of freedom that destroy the coherence of the superposed entangled state.

4. Bayesian rule for single quadrature measurement

The conditional evolution of the qubit-plus-cavity under continuous measurements is well captured by equation (3), however, a full simulation of this equation is time-consuming and almost intractable in practice (e.g., in the quantum feedback control experiment [16]). More efficient method is using a quantum Bayesian rule to update the qubit state, merely based on the
measurement record $I_m$ over certain finite time interval $t_m$. For the sake of clarity, below we present our construction procedures in order by three steps (addressed by three subsections).

4.1. Bare Bayesian rule

To perform a Bayesian inference based on $I_m$, which is defined by $I_m = \frac{1}{t_m} \int_0^{t_m} \text{d}t \varphi(t)$ and collected in experiment from the quadrature measurement records, we need to know in advance the distribution $P_g(e)(I_m)$ associated with the qubit state $|g(e)\rangle$. As given in appendix A, a simple analysis shows that the distribution is Gaussian:

$$P_g(e)(I_m) = \frac{1}{\sqrt{2\pi D}} \exp \left[-\frac{(I_m - \tilde{I}_g(e))^2}{2D}\right],$$

with $D = 1/t_m$ being the variance. The average quadrature outcome, $\tilde{I}_g(e)$, is given by

$$\tilde{I}_g(e) = \frac{1}{t_m} \int_0^{t_m} \text{d}t (2\sqrt{\kappa}) \text{Re} \left[\alpha_{g(e)}(t)e^{-i\varphi}\right],$$

where $\alpha_{g(e)}(t)$ is the cavity field discussed in section 3, with explicit solution presented in appendix A.

With the knowledge of $P_g(I_m)$ and $P_e(I_m)$, using the classical Bayes formula one can determine $\langle \epsilon_g(t_m)\rangle^2$ and $\langle \epsilon_e(t_m)\rangle^2$, in equation (5). Obviously, they coincide with the diagonal elements of the reduced density matrix of the qubit state. Therefore, we have

$$\rho_{gg}(t_m) = \rho_{gg}(0) P_g(I_m) / \mathcal{N},$$
$$\rho_{ee}(t_m) = \rho_{ee}(0) P_e(I_m) / \mathcal{N},$$

where $\mathcal{N} = \rho_{gg}(0) P_g(I_m) + \rho_{ee}(0) P_e(I_m)$. One can examine that equation (9) is in full agreement with the QTE simulation, as expected.

Regarding the off-diagonal element $\rho_{ge}(t_m)$, the situation is subtle. No classical rule applies here. Following [26], based on purity consideration, we preliminarily have

$$\tilde{\rho}_{ge}(t_m) = \rho_{ge}(0) e^{-i\varphi t_m} \sqrt{P_g(I_m)P_e(I_m)} / \mathcal{N},$$

to approximate $\rho_{ge}(t_m)$. However, as shown in figure 2(a), this result differs considerably from the exact one from a direct simulation of equation (3).

4.2. Purity degradation factor

For further corrections to $\tilde{\rho}_{ge}(t_m)$, let us look back to equation (5). Based on the joint-state structure, we propose to amend equation (10) by a purity degradation factor as

$$\tilde{\rho}_{ge}(t_m) = \rho_{ge}(t_m) \left|\langle \alpha_e(t_m) | \alpha_g(t_m) \rangle\right|. $$

The physical meaning of this correction factor is an account of the purity degradation, after partly averaging an entangled state. More specifically, as shown in figure 1 and discussed earlier around it, we know that the cavity state evolves along $|\alpha_{g(e)}(t)\rangle$, with some tiny stochastic fluctuations. Here, as a first step, we introduce the purity degradation factor, $D(t_m) \equiv 1|\langle \alpha_e(t_m) | \alpha_g(t_m) \rangle|^2$, to characterize the decrease of the qubit coherence owing to averaging the cavity state, while studying later the fluctuation effects on the qubit. Using the
property of coherent state, an explicit result can be obtained as
\[
D(t_m) = \exp\left\{ -\frac{1}{2} \left[ |\alpha_c(t_m)|^2 + |\alpha_g(t_m)|^2 \right] + \text{Re} \left[ \alpha_c(t_m) \alpha_g^*(t_m) \right] \right\}.
\] (12)
Moreover, as proved in appendix B, this result is precisely equivalent to the following one by a more sophisticated analysis based on the QTE:
\[
D(t_m) = \exp \left\{ -\int_0^{t_m} dt \left[ \Gamma_d(t) - \Gamma_m(t)/2 \right] \right\},
\] (13)
where \( \Gamma_d(t) \) is a time-dependent overall decoherence rate of the qubit caused by measurement, and \( \Gamma_m(t) \) is the measurement (unraveling) rate. This identification reveals a deep connection between the two very different treatments, and provides an additional evidence for the validity of the purity degradation factor.

4.3. Effects of dynamic and stochastic fluctuations of the cavity field

Equation (5) and figure 1 jointly indicate the qualitative feature of the cavity state and guide our construction for the quantum Bayesian rule. However, in order to quantify the fluctuation effects of the cavity field, more sophisticated skill is useful. Let us return to the QTE. Based on equation (3), via a qubit-state-dependent displacement transformation, it is possible to eliminate...
the cavity degrees of freedom from this equation [19]. Below we continue our correction to \( \rho_{ge}(t_m) \) based on the achievement of this technique. The transformed QTE reads [19]

\[
\dot{\rho} = -i\frac{\vec{\alpha}_q + B(t)}{2}[\sigma_z, \rho] + \frac{\Gamma_d(t)}{2} D[\sigma_z] \rho - \sqrt{\Gamma_{cl}(t)} \mathcal{M}[\sigma_z] \rho \xi(t) + i\sqrt{\Gamma_{ba}(t)} [\sigma_z, \rho] \xi(t). \tag{14}
\]

In this result, the effective magnetic field \( B(t) = 2\chi \text{ Re}[\alpha_g(t)\alpha_e(t)^*] \), describes a generalized ac-Stark shift of the qubit energy, as a consequence of dynamic fluctuations of the cavity field owing to dispersive coupling to the qubit. The superoperator is defined by \( \mathcal{M}[\sigma_z] \rho \equiv \sigma_4 \rho + \rho \sigma_4 \rangle \langle -\sigma_4 \rangle \rho \), with \( \langle \sigma_4 \rangle = \text{Tr}[\sigma_4 \rho] \), an average over the reduced density matrix of qubit. Additionally

\[
\Gamma_d(t) = 2\chi \text{ Im} \left[ \alpha_g(t) \alpha_e(t)^* \right],
\]

\[
\Gamma_{cl}(t) = \kappa |\beta(t)|^2 \cos^2(\varphi - \theta_\beta),
\]

\[
\Gamma_{ba}(t) = \kappa |\beta(t)|^2 \sin^2(\varphi - \theta_\beta), \tag{15}
\]

characterize, respectively, the ensemble-average dephasing, single measurement information-gain, and back-action rates. Moreover, the sum of \( \Gamma_{cl} \) and \( \Gamma_{ba} \)

\[
\Gamma_m(t) = \Gamma_{cl}(t) + \Gamma_{ba}(t) = \kappa |\beta(t)|^2, \tag{16}
\]

gives the measurement rate [19]. In the above rates, we have used the definition \( \beta(t) = \alpha_e(t) - \alpha_g(t) \equiv |\beta(t)| e^{i\beta_\beta}. \)

Returning to equation (14), one may notice two unitary terms on the rhs: the first term, involving \( B(t) \); and the last with an effective stochastic field, \(-\sqrt{\Gamma_{ba}(t)} \xi(t)\). These two terms properly characterize the cavity-field-fluctuation effects on the qubit and can be used to amend equation (11) further. Based on this observation, we thus complete our correction to the off-diagonal element as follows:

\[
\rho_{ge}(t_m) = \tilde{\rho}_{ge}(t_m) \left\{ \langle \alpha_e(t_m)|\alpha_g(t_m) \rangle \right\} \exp \left\{ -i \left[ \Phi_1(t_m) + \Phi_2(t_m) \right] \right\}, \tag{17}
\]

with the two additional phase factors

\[
\Phi_1(t_m) = \int_0^{t_m} B(t) \, dt, \\
\Phi_2(t_m) = -\int_0^{t_m} \sqrt{\Gamma_{ba}(t)} \xi(t) \, dt. \tag{18}
\]

Equations (17) and (18), together with (9) and (10), constitute the quantum Bayesian rule we propose for qubit state under single-quadrature weak measurements.

To implement the proposed Bayesian rule in real experiments, one can directly use the analytic solutions of \( \alpha_g(t) \) and \( \alpha_e(t) \), given in appendix A, and compute all the relevant quantities including \( \langle \alpha_e(t_m)|\alpha_g(t_m) \rangle \) and \( \Phi_1(t_m) \). Provided the necessary setup parameters are determined (as discussed in further detail below), all these calculations can be fulfilled in advance. One can then update the qubit state based on the integrated output record \( I_m \) over time \( t_m \). In this context, we notice that \( \Phi_2(t_m) \) does not exactly depend on the integrated measurement record. While it is possible to evaluate this phase factor using the continuous outcomes (but not performing the unwanted continuous state-estimate along time), we propose
here a simpler scheme to approximate it. From figure 1 and the related discussion, we know that $I_{bh}(t)$ is a slowly-varying function of time, and in particular it is insensitive to the randomness of the continuous measurement. Thus, we propose the following approximation:

$$\Phi_2(t_m) \simeq -\sqrt{I_{bh}(t_m)} \left[ I_m - \bar{I}(t_m) \right],$$

where the expected output quadrature during $(0, t_m)$ is given by

$$\bar{I}(t_m) = \left| c_x(0) \right|^2 I_x(t_m) + \left| c_x(0) \right|^2 I_x(t_m).$$

In this way, the difference between $I_m$ and $\bar{I}(t_m)$, which is experimentally accessible, accounts for the accumulation of the Wiener increments. As we will demonstrate, this approximation can work well and could be valuable for future feedback control experiments.

5. Bayesian rule for two-quadrature measurement

In practice, rather than the $I_\varphi$ single-quadrature measurement, the so-called $(I,Q)$ two-quadrature measurement is another choice [10]. Among the various realizations [23], it can be implemented as follows (in terms of a quantum optics language, for convenience). Using a beam-splitter, the outgoing microwave field leaked from the cavity, $\sqrt{\kappa}a$, is split into two branches with amplitudes $a_1 = \sqrt{\kappa/2}a$ and $a_2 = i\sqrt{\kappa/2}a$. Then, perform single-quadrature homodyne measurement on each branch for $a_1$ and $a_2$, choosing the LO phases as $\varphi = 0$ and $\pi$, respectively. One can prove that, for the cavity field, this realizes an $I$-quadrature measurement in the first branch and a $Q$-quadrature measurement in the second branch, with measurement outputs described by $I_m(t) = \sqrt{\kappa/2} \langle a + a^\dagger \rangle_{\psi(t)} + \xi_1(t)$, and $Q_m(t) = \sqrt{\kappa/2} \langle -ia + ia^\dagger \rangle_{\psi(t)} + \xi_2(t)$. Conditioned on this type of $(I,Q)$ two-quadrature measurements, the evolution of the qubit-cavity joint state follows the QTE:

$$\dot{\rho} = -i[H_{\text{eff}}, \rho] + \kappa D[a]\rho + \frac{\kappa}{2} \mathcal{H}[a] \rho \xi_1(t) + \frac{\kappa}{2} \mathcal{H}[-ia] \rho \xi_2(t).$$

To the entire qubit-plus-cavity state, the last two terms fully unravel the measurement with no information loss. Since only the output of the first branch (leading to the third term) reveals the qubit state information, the information-gain rate is just half of the optimal single quadrature measurement. However, provided we keep track of the $Q$-quadrature output in the second branch, the qubit state can be maintained in high purity. In a similar manner to the single-quadrature measurement, applying the qubit-state dependent displacement transformation to the cavity field within the two-quadrature measurement framework also yields an effective QTE for the qubit state alone:

$$\dot{\rho} = -i\frac{\partial_q + B(t)}{2} \tau_{\sigma_z, \rho} + \frac{\Gamma_d(t)}{2} - D[\sigma_z] \rho - \sqrt{I_m(t)/2} \mathcal{M}[\sigma_z] \rho \xi_1(t)$$

$$+ \frac{\sqrt{I_m(t)/2}}{2} \tau_{\sigma_z, \rho} \xi_2(t).$$

Here, $B(t)$ and $\Gamma_d(t)$ are the same as in the single quadrature measurement, see equations (14) and (15). Comparing equation (22) with equations (14) and (15) indicates that the $I$-quadrature measurement of the first branch is associated with $\Gamma^{(i)}_\psi(t) = (\kappa/2)\beta(t)^2 = I_m(t)/2$ and
\( \Gamma_{ba}^{(1)}(t) = 0 \); while conversely, the \( Q \)-quadrature measurement of the second branch is associated with \( \Gamma_{ba}^{(2)}(t) = 0 \) and \( \Gamma_{ba}^{(2)}(t) = (\kappa/2)|\beta(t)|^2 = \Gamma_m(t)/2 \).

Following the same procedures as for the single-quadrature measurement, we can construct a quantum Bayesian rule for the two-quadrature measurement. First, the integrated output distribution of the two-quadrature measurement is:

\[
P_{g(e)}(I_m, Q_m) = \left( \frac{1}{2\pi D} \right) \exp \left[ -\left( I_m - \bar{I}_g(e) \right)^2/(2D) \right] \exp \left[ -\left( Q_m - \bar{Q}_g(e) \right)^2/(2D) \right],
\]

with \( D = 1/t_m \) the same as in the single-quadrature measurement. Respectively, the average of each quadrature output is given as

\[
I_{g(e)} = \frac{1}{t_m} \int_0^{t_m} dt \bar{I}_g(e)(t)
\]

with \( \bar{I}_g(e)(t) = \sqrt{2}\kappa \text{ Re}[\alpha_g(e)(t)] \); and

\[
Q_{g(e)} = \frac{1}{t_m} \int_0^{t_m} dt \bar{Q}_g(e)(t)
\]

with \( \bar{Q}_g(e)(t) = \sqrt{2}\kappa \text{ Im}[\alpha_g(e)(t)] \). As pointed out previously, tuning \( \Delta_r = 0 \) and with an initial vacuum cavity state, the imaginary parts of \( \alpha_g(t) \) and \( \alpha_e(t) \) are equal. This means that the \( Q \)-quadrature output \( Q_m \) does not provide qubit state information. However, in an ideal case, tracking this output simultaneously with the \( I \)-quadrature outcome \( (I_m, Q_m) \) can maintain the joint state of the qubit-plus-cavity in a pure state. So, with respect to the optimal single-quadrature measurement \( (\varphi = 0) \), the \( (I, Q) \) two-quadrature measurement reduces the signal \( |\bar{I}_e(t_m) - \bar{I}_g(t_m)| \) by a factor of \( 1/\sqrt{2} \).

With the knowledge of \( P_{g(e)}(I_m, Q_m) \), the diagonal elements of the qubit state, \( \rho_{gg} \) and \( \rho_{ee} \), can be determined straightforwardly using the Bayesian rule equation (9) for their informational evolution conditioned on the outcome \( (I_m, Q_m) \). For the off-diagonal elements, the quantum Bayesian rule is the same as equation (17), with the phase factor \( \Phi_1(t_m) \) unchanged but \( \Phi_2(t_m) \) now given by

\[
\Phi_2(t_m) = -\int_0^{t_m} \sqrt{I_m(t)/2} \xi_2(t) dt.
\]

Importantly, unlike equation (19) for the single-quadrature measurement, the phase factor \( \Phi_2(t_m) \) can be determined without invoking any approximations. To elaborate on this issue, we note that, for an arbitrary homodyne measurement (with LO phase \( \varphi \)), the output current can be expressed as [19]: \( I_{\varphi}(t) = -\sqrt{\Gamma_{ei}} \langle \sigma_z \rangle + \sqrt{\kappa} |\mu| \cos(\theta_\mu - \varphi) + \xi(t) \), with \( \Gamma_{ei} \) the same as in equation (15), and \( \mu(t) \) defined by \( \mu = \alpha_e(t) + \alpha_g(t) \equiv |\mu|e^{i\theta_\mu} \). For measurements under resonant driving \( (\Delta_r = 0) \), one can prove that \( \theta_\mu = \pi/2 \). Then, applying this result to the second branch \( Q \)-quadrature measurement, we have \( Q_m(t) = \sqrt{\kappa} |\mu(t)| + \xi_2(t) \), since \( \varphi = \pi/2 \) and \( \Gamma_{ei}^{(2)}(t) = 0 \). Therefore, one can separate \( \xi_2(t) \) from the experimental outcome \( Q_m(t) \) and evaluate equation (26) with no approximation.
6. Numerical results and discussions

6.1. Effects of corrections

We show the correction effects in figure 2 and demonstrate the proposed Bayesian rule by comparison with the exact results from simulation of equation (3), in (a) for single-quadrature and (b) for two-quadrature measurements. For the case of the single-quadrature measurement, we have used equation (19) to approximate \( \Phi_2(t_m) \) for the purpose of revealing the quality of the rule using only the integrated quadrature. For both types of measurements, we find that the proposed quantum Bayesian rule can give reliable estimate for the qubit state.

In figure 2(a) each correction is presented individually for the single-quadrature measurement. Since the individual effect of each correction term is similar, only the total result is shown for the two-quadrature case illustrated in figure 2(b). Whereas the consequences of the phase factors are dramatic, one may notice that in this plot the correction from the purity-degradation-factor is very weak (almost negligible). However, its physical meaning is clear. It characterizes the intrinsic purity of the qubit state imposed by the cQED measurement in the ideal case. Actually, the qubit-state purity associated with figure 2 is about 0.97, but not unity. Changing the parameters, one can make this correction effect more prominent, as to be shown in section 6.3.

6.2. Limiting cases

We now consider the three correction factors in limiting cases, making in particular a connection with the work by Korotkov [27]. First, for the purity degradation factor \( l\langle \alpha_e(t_m)|\alpha_g(t_m)\rangle l \), based on the solution in appendix A we obtain, in steady state:

\[
D = \left| \langle \bar{\alpha}_e | \bar{\alpha}_g \rangle \right| = \exp \left[ -\frac{2\epsilon_m \kappa}{(\chi^2 + \kappa^2/4)} \right]. \tag{27}
\]

Further, in the bad-cavity and weak-response limit, \( \kappa \gg \chi \), the result simplifies to \( l\langle \bar{\alpha}_e | \bar{\alpha}_g \rangle l = \exp [-8\bar{n}(\chi/\kappa)^2] \), where \( \bar{n} = |a_0|^2 \) with \( a_0 = -i\epsilon_m/(\chi^2) \). Accordingly, only in the limit \( \kappa \gg \chi \) and with small \( \bar{n} \) (cavity photon number), can the purity factor be approximated to unity, implying a pure state of qubit under the quadrature measurement. Otherwise, this factor should be taken into account. In particular, this implies that for the not very bad cavity and with \( \bar{n} \) not very small, the \( D \) factor cannot be treated as unity. For instance, as to be seen below in figure 4(a), the \( D \) factor will reduce to a value lower than 0.8 for \( \chi = 0.1\kappa \) and \( \bar{n} \approx 4 \). We note that this factor was not addressed in [27]. But in a recent report [30] the similar \( D \) factors were included in the concurrence calculation for a two-qubit cQED system (see equation (12) in the supplementary information of [30]).

Let us now consider \( \Phi_1(t_m) \). The key quantity associated is the effective magnetic field \( B(t) = 2\chi \text{ Re} [a_g(t)\alpha_e^*(t)] \). Physically speaking, it describes a generalized (time dependent) ac-Stark effect. From the analytic solution in appendix A, we formally reexpress \( \alpha_{g(e)}(t) = a(t) + ib(t) \), which leads to \( \alpha_g\alpha_e^* = -(a^2 - b^2) + 2iab \). We see that the imaginary part (2ab) determines the dephasing rate \( \Gamma_d \), and the real part \( (b^2 - a^2) \) affects the qubit energy. Further, in steady state, we have
\[ B = \frac{2\chi \epsilon_m^2}{d^2 + \kappa^2 \chi^2 / d^2}, \]  
\( d^2 \equiv \Delta_r^2 - \chi^2 + \kappa^2 / 4. \) Again, in the bad-cavity and weak-response limit, the result reduces to \( B \approx 2\chi \epsilon_m. \) This is the standard ac-Stark shift to the qubit energy.

Finally, let us consider \( \Phi_2(t_m). \) Based on the steady-state solution of the cavity fields, we obtain the back-action rate as

\[ \Gamma_{ba} = \kappa - \frac{4\epsilon_m^2 \chi^2}{\left( \chi^2 + \kappa^2 / 4 \right)^2} \sin^2 \varphi. \]

Moreover, in the limit \( \kappa \gg \chi, \) the result is further simplified as \( \Gamma_{ba} \approx 16\bar{n} \epsilon_m (\chi / \kappa)^2 \sin^2 \varphi. \)

Substituting this result into the expression of \( \Phi_2(t_m), \) we find that we recover the ‘realistic’-back-action induced phase factor in the bad-cavity and weak-response limit, obtained in [27] by using photon-caused qubit rotation considerations. We note also that, in the context of \((I,Q)\) two-quadrature measurements, the state-update rule constructed in [10] (in the supplementary materials) contains as well this same factor in the same limiting case. However, while our approach can derive theirs, it seems to be an open problem how to use their approaches to derive some results here.

6.3. Non-limiting cases

In this subsection we present some numerical results beyond the ‘bad-cavity’ and weak-response limits, and compare with the Bayesian rule constructed in [27]. Cases that violate the restrictive limits can be the following: (i) the quality factor of the cavity is relatively high, as required in quantum information processing in order to employ the cavity photon as a data-bus; (ii) the qubit-cavity coupling (\( \chi \) in the dispersive regime) is strong, which is required for quantum information processing and would violate the weak-response assumption; and (iii) the average cavity photon number \( \bar{n} \) is not tiny, which has the advantage of enhancing the measurement signal to overcome the noise from the amplifiers and circuits.

For the sake of brevity, we denote the Bayesian rule proposed in [27] by BR-I, the rule constructed in the present work by BR-II, and the one involving the approximate \( \Phi_2 \) of equation (19) by BR-II’. In figure 3 we display results for both the single- and two-quadrature measurements outside the ‘bad-cavity’ and weak-response limits. Clearly, we find that in this case the purity-degradation factor, \( D(t) = |\langle \alpha_e(t) | \alpha_e(t) \rangle|, \) is reduced to values obviously lower than unity, and for both measurements BR-II fits the (exact) QTE results (off-diagonal element of the qubit state) better than BR-I. For single-quadrature measurement in this non-limiting case, we find the use of the approximate phase factor \( \Phi_2 \) causes some deviation from the precise one, as revealed in figure 3(c). However, combining with the other two corrections \((D \text{ and } \Phi_1)\), BR-II’ can give reasonable results as shown in figure 3(e).

6.4. Experimental issues

In order to implement the Bayesian rule proposed in sections 4 and 5 in experiments, the key quantities to be fixed are the cavity fields \( \alpha_e(t) \) and \( \alpha_e(t). \) Viewing the analytic solutions in appendix A, the cavity damping rate \( \kappa \) and the dispersive coupling \( \chi \) should be determined in advance. For a given detuning \( \Delta_r \) and driving amplitude \( \epsilon_m, \) these two parameters can be
extracted from the steady-state mean values of the quadrature measurements, \( \bar{I}_g \) and \( \bar{I}_e \), which are related to \( \bar{a}_g \) and \( \bar{a}_e \). With these extracted parameters at hand, one can accordingly implement the proposed Bayesian rule, with the cavity in an initial vacuum or certain known steady state (e.g., |\( \bar{a}_g \rangle \rangle_\text{g} \rangle\rangle_\text{e}\rangle). In experiments, one also needs to properly account for unavoidable measurement inefficiencies, in particular circuit and amplifier noises. This important issue has been addressed by Wiseman et al. in a series of papers [31–33], where the so-called realistic QTE has been developed. However, the resultant equation would be difficult to use in practice. Within the framework of the Bayesian approach, it seems simpler to address this issue [34], since accounting for the extra noise only corresponds to a more Bayesian inferring. As a result, the effect of the extra noise requires us to partially average the ideal-measurement-result conditioned state. We will incorporate our present quantum Bayesian approach with this type of treatment in a separate work.

Finally, we mention that so far there have been a few experiments involving the quantum Bayesian rule in the bad-cavity and weak-response regime [10, 11, 16]. In the feedback control experiment [16], the phase-sensitive detection scheme corresponds to a case where the phase factor \( \Phi_2(t_m) \) in equation (17) vanishes. Moreover, in the bad-cavity and weak-response limit, the factor \( \Phi_1 \) reduces to \( 2\chi \bar{n}_m \) and the purity degradation factor is about unity. In two recent experiments [10, 11], however, the phase factor \( e^{-i\Phi_2(t_m)} \) is present and was demonstrated with

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**Figure 3.** Comparison of the Bayesian rules against the exact QTE in the non-limiting case, for both single-quadrature (with \( \varphi = \pi/4 \)) and two-quadrature measurements. In (a) and (b) we plot the common purity-degradation factor \( D(t) = |\langle \alpha_e(t) | \alpha_s(t) \rangle| \) and phase factor \( \Phi_1 \), while the results in (c) and (e) correspond to the single-quadrature measurement, and the results in (d) and (f) the two-quadrature measurement. BR-I and BR-II denote the Bayesian rules proposed in [27] and in the present work, whereas BR-II involves the approximation of equation (19). We assume the qubit in initial state \(|\psi\rangle = (|e\rangle + |g\rangle)/\sqrt{2} \) and the setup parameters \( \Delta_e = 0 \), \( \chi = 0.1\kappa \) and \( \epsilon_m = \kappa \). These parameters indicate the cavity photon number \( \bar{n} \approx 4 \).
satisfactory accuracies. In view of these remarkable advances in cQED measurements, further experiments to demonstrate the quantum Bayesian rule in more general cases would be of great interest.

7. Concluding remarks

To summarize, we have constructed a quantum Bayesian rule for weak measurements of qubits in cQED. Our construction was guided by a microscopic analysis of the cavity field and the cavity-photon-eliminated effective QTE. This type of treatment provided a different route from Korotkov’s method in [27]. The present work, for both the single- and two-quadrature measurements, generalizes the results in [27] from ‘bad-cavity’ and weak-response limits to more general conditions. Numerical comparisons with the direct QTE simulations show that the proposed rule can work with high accuracy even in non-limiting cases.

We would like to remark that, originally, the Bayesian approach was not constructed or integrated from QTE [26, 27]. Their mathematical connection is also not very straightforward. For infinitesimal short-time evolution, their equivalence can be proved. However, for longer time state update, there exist slight numerical differences, despite the reasonable agreements observed in figures 2 and 3. Rather than an exact derivation from QTE, we would like to regard the Bayesian rule as a construction. In particular, the cavity-photon-eliminated effective QTE contains unusual stochastic unitary term. As an ansatz, we inserted (integrated) it into the Bayesian dynamics of the off-diagonal elements and revealed an interesting connection with the results of [27] in limiting cases.

It has become clear that, in addition to the QTE, the empirical Bayesian formulas are very useful in experiments [10, 11, 16]. In connection with the experiment of [10], a POVM-type formalism was constructed for qubit state update in cQED, which is actually equivalent to the result obtained in [27]. Also, both results work in the ‘bad-cavity’ and weak response limits. Their minor difference is: the (POVM) measurement operator (i.e., \( M_{\text{Q}_0} \) in equation (6) in the supplementary materials of [10]), avoids the purity consideration in constructing the Bayesian rule and contains the phase factors discussed in [27] and in our present work. Finally, we mention that at the final review stage of this work we were recommended to become aware of the new publication [30], in which similar purity degradation factor was included in the concurrence calculation for a two-qubit cQED system (see equation (12) in the supplementary information). However, it seems that in this work the Bayesian rule proposed in [27], particularly the ‘realistic’ back-action phase factor (\( \Phi_2 \) denoted in our present work) was not involved. There, the QTE approach was also used to compare with the experimental quantum trajectories, see, equations (14) and (22) in the supplementary information. Viewing these remarkable efforts and progress, we expect our proposed Bayesian rule to be useful, for both theoretical interests and future cQED measurement and control experiments.

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1 Reference [30] appeared slightly later than arXiv:1404.3870, while the preliminary manuscript of arXiv:1404.3870 was completed about half a year earlier and was communicated with one of the authors of [30].
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Appendix A. Cavity fields and measurement principle

The interplay of external driving and cavity damping would evolve the cavity field as an optical coherent state, with the time-dependent coherence parameter determined by

\[
\hat{a}_g(t) = -i\epsilon_m - i(\Delta_r - \chi) a_g(t) - \kappa a_g(t)/2,
\]

\[
\hat{a}_e(t) = -i\epsilon_m - i(\Delta_r + \chi) a_e(t) - \kappa a_e(t)/2 .
\] (A.1)

Here, corresponding to the qubit state \(|g(e)\rangle\), the frequency of the cavity has a dispersive shift \(\pm \chi\). Moreover, analytically solving these two equations yields

\[
\begin{align*}
\alpha_g(t) &= \alpha_g \left[ 1 - e^{-i\Delta_r^{(+)} t - \kappa t/2} \right] + \alpha_0 e^{-i\Delta_r^{(-)} t - \kappa t/2}, \\
\alpha_e(t) &= \alpha_e \left[ 1 - e^{-i\Delta_r^{(+)} t - \kappa t/2} \right] + \alpha_0 e^{-i\Delta_r^{(-)} t - \kappa t/2}. 
\end{align*}
\] (A.2)

For simplicity we introduce \(\Delta_r^{(+)} = \Delta_r \pm \chi\). In this solution, \(\alpha_0\) is the initial cavity field before the dispersive measurement, which is zero if one starts the measurement with a vacuum state. Also, \(\alpha_g(e) = -i\epsilon_m/[i\Delta_r^{(+)} + \kappa/2]\), are the respective steady-state fields.

Based on the above solutions, one can understand the basic principle of quadrature measurement which is employed to extract the information of the qubit state. For a single-quadrature homodyne measurement, the observable (measurement operator) is

\[
\tilde{I}_g = \frac{1}{2} (a e^{-i\varphi} + a^\dagger e^{i\varphi}),
\]

with \(\varphi\) the local oscillator (lo) phase. Corresponding to the qubit state \(|g(e)\rangle\), the average quadrature output is

\[
\tilde{I}_{g(e)}(t) = 2 \sqrt{\kappa} \operatorname{Re} \left[ \alpha_{g(e)}(t) e^{-i\varphi} \right].
\] (A.3)

Denoting

\[
\alpha_e(t) - \alpha_g(t) = |\beta(t)| e^{i\varphi_0},
\] (A.4)

we have

\[
\tilde{I}_e(t) - \tilde{I}_g(t) = 2 \sqrt{\kappa} |\beta(t)| \cos \left( \theta_\beta - \varphi \right).
\] (A.5)

From this result, it becomes clear that the optimal qubit-state information can be inferred if we tune the LO phase to \(\varphi = \theta_\beta\). In contrast, if we tune \(\varphi = \theta_\beta + \pi/2\), the quadrature measurement does not reveal any information for the qubit-state \([35]\). More specifically, starting the measurement with an empty cavity and choosing a resonant measurement driving

\[^2\text{In this case the continuous measurement does not collapse the qubit state. In contrast, it maintains the joint state of the qubit-plus-cavity in a pure superposition state, and the reduced qubit state with high purity. The interesting point here is the following: whether a photon has leaked or not from the cavity, depends on the means of the subsequent detection, but is not an objective event. This issue has been highlighted in [35].} \]
(Δr = ωr − ωm = 0), one can easily prove that θβ = 0. This means that, choosing ϕ = 0, one can achieve maximal information for the qubit state, while no qubit state information can be inferred if choosing ϕ = π/2.

Moreover, for a single realization of quadrature measurement during (0, tm), we introduce the mean integrated quadrature output \( I_m = \frac{1}{t_m} \int_0^{t_m} dt I(t) \), where the continuous outcome is described by \( I(t) = \sqrt{K} \langle a e^{-i\phi} + a^* e^{i\phi} \rangle_{\psi(t)} + \xi(t) \). Corresponding to qubit state \( \lg(e) \), the distribution \( P_{\lg(e)}(I_m) \) of the integrated quadrature \( I_m \) is Gaussian, \( P_{\lg(e)}(I_m) = \frac{1}{\sqrt{2\pi D}} \exp\left\{-(I_m - \bar{I}_{\lg(e)})^2/(2D)\right\} \). Here, \( \bar{I}_{\lg(e)} \) is the average quadrature outcome, given by \( \bar{I}_{\lg(e)} = \frac{1}{t_m} \int_0^{t_m} dt \bar{I}_{\lg(e)}(t) \). The variance \( D \) can be analytically determined as follows. Consider the expression of the homodyne current. Noting that the first term describes the average current, the deviation of the individual quadrature output \( I_m \) from the averaged \( \bar{I}_{\lg(e)} \) is thus caused by the second term \( \xi(t) \). Denoting \( I(t) \equiv \xi(t) \), from definition we have

\[
\langle \bar{I}^2 \rangle = \int P(I) \bar{I}^2 = D. \tag{A.6}
\]

On the other hand, via a direct calculation,

\[
\langle \bar{I}^2 \rangle = \frac{1}{t_m^2} \int_0^{t_m} dt_1 \int_0^{t_m} dt_2 \langle \xi(t_1) \xi(t_2) \rangle = \frac{1}{t_m}, \tag{A.7}
\]

we thus obtain \( D = 1/t_m \).

Appendix B. Revisit the purity degradation factor

Based on the transformed equation (14), in this appendix we revisit the purity degradation factor \( \Gamma \). First, we notice that the last term of equation (14) does not only play the usual role of unitary evolution, but also has an effect of unraveling the qubit state. This can be clearly seen by setting \( \phi = \frac{\pi}{2} \). In this case, \( \Gamma = 0 \), then one may expect that the r.h.s. second Lindblad term will gradually completely dephase the qubit state. However, interestingly, the last term will prevent this, owing to its certain unraveling ability. This feature is in agreement with the Bayesian rule for the case of identical \( \lg(e) \) and \( \lg(m) \). Both approaches predict that the qubit will be maintained in a superposition state with high purity. Even better, we can combine the last two terms into a single unraveling term: \( \sqrt{\Gamma_m(t)}/2H[\Lambda \sigma_z] \rho(t) \xi(t) \), where \( \Gamma_m(t) = \Gamma_{ci}(t) + \Gamma_{m}(t) = \kappa |b(t)|^2 \) (the measurement rate) is independent of the choice of \( \phi \), and \( \Lambda = \cos(\phi - \theta_\beta) - i \sin(\phi - \theta_\beta) \) depends on \( \phi \). We may interpret this result as follows: \( \Gamma_m(t) \) determines the unraveling extent; and \( \Lambda \sigma_z \) describes the unraveling (measuring) means. We propose then a dephasing factor of the form \( \exp\left\{ -\int_0^{t_m} dt \left[ \Gamma_d(t) - \Gamma_m(t)/2 \right] \right\} \).

Below, we prove that the two expressions of the purity degradation factor for the qubit state are identical, i.e.,

\[
\exp \left\{ -\int_0^t dt' \left[ \Gamma_d(t') - \Gamma_m(t')/2 \right] \right\} = \left| \langle \alpha_g(t) | \alpha_e(t) \rangle \right|. \tag{B.1}
\]
This is equivalent to proving the following:

\[
- \int_0^t dt' \left[ \Gamma_d(t') - \Gamma_m(t')/2 \right] = -\frac{1}{2} \left[ |\alpha_e(t)|^2 + |\alpha_g(t)|^2 \right] + \text{Re} \left[ \alpha_e(t)\alpha_g^*(t) \right].
\]  

(B.2)

In obtaining this result, the property of coherent states has been used. Under the conditions \( \Delta_r = 0 \) and \( \alpha_e(0) = \alpha_g(0) = 0 \), more explicitly we reexpress the solution of equation (A.1) as

\[
\alpha_e(t) = \frac{i e_m}{i \chi + \kappa/2} \left[ e^{-(i\chi+\kappa/2)t} - 1 \right],
\]

\[
\alpha_g(t) = \frac{i e_m}{-i \chi + \kappa/2} \left[ e^{i(i\chi-\kappa/2)t} - 1 \right].
\]  

(B.3)

Substituting these two expressions into the lhs of equation (B.2) gives

\[
- \int_0^t dt' \left[ \Gamma_d(t') - \Gamma_m(t')/2 \right] = -2\chi \int_0^t dt' \text{Im} \left[ \alpha_g(t')\alpha_e^*(t') \right] + \frac{\kappa}{2} \int_0^t dt' \left| \alpha_e(t') - \alpha_g(t') \right|^2 = \frac{\kappa}{2} \int_0^t dt' \left[ \alpha_e(t')\alpha_e^*(t') + \alpha_g(t')\alpha_g^*(t') \right] + (i\chi - \kappa/2) \int_0^t dt' \alpha_e^*(t')\alpha_g(t') - (i\chi + \kappa/2) \int_0^t dt' \alpha_e(t')\alpha_g^*(t').
\]  

(B.4)

Further evaluation yields:

\[
- \int_0^t dt' \left[ \Gamma_d(t') - \Gamma_m(t')/2 \right] = -\frac{e_m^2}{\chi^2} \left[ e^{-\kappa t} + 1 - e^{(i\chi-\kappa/2)t} - e^{(-i\chi-\kappa/2)t} \right] + \frac{e_m^2}{2(i\chi - \kappa/2)^2} \left[ e^{2(i\chi-\kappa/2)t} + 1 - 2e^{(i\chi-\kappa/2)t} \right] + \frac{e_m^2}{2(-i\chi - \kappa/2)^2} \left[ e^{2(-i\chi-\kappa/2)t} + 1 - 2e^{(-i\chi-\kappa/2)t} \right]
\]

\[= -\frac{1}{2} \left( |\alpha_e(t)|^2 + |\alpha_g(t)|^2 \right) + \text{Re} \left[ \alpha_e(t)\alpha_g^*(t) \right].
\]  

(B.5)

showing the validity of equation (B.2) and thus of equation (B.1).

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