Tidal interaction of a small black hole in the field of a large Kerr black hole

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The rates at which the mass and angular momentum of a small black hole change as a result of a tidal interaction with a much larger black hole are calculated to leading order in the small mass ratio. The small black hole is either rotating or nonrotating, and it moves on a circular orbit in the equatorial plane of the large Kerr black hole. The orbits are fully relativistic, and the rates are computed to all orders in the orbital velocity \( V \leq V_{\text{isco}} \), which is limited only by the size of the innermost stable circular orbit. We show that as \( V \to V_{\text{isco}} \), the rates take on a limiting value that depends only on \( V_{\text{isco}} \) and not on the spin parameter of the large black hole.

I. INTRODUCTION AND SUMMARY

The dynamics of tidally deformed black holes has been the subject of vigorous investigation in the last several years [1, 2, 3, 4, 5, 6, 7, 8, 9, 10]. Most of this work was devoted to a description of the tidal heating and torquing of a black hole within the context of a small-hole/slow-motion approximation, in which the calculations can be carried out analytically. The tidal heating of a black hole refers to the change of its mass that occurs as a result of the tidal interaction; tidal torquing refers to the change of its angular momentum. These changes can be significant [11, 12] in realistic astrophysical situations, and the effect must be taken into account in calculations of gravitational waves emitted by binary systems involving black holes.

The small-hole/slow-motion approximation is characterized by the condition \( m/R \ll 1 \), which states that \( m \), the mass of the black hole, must be small compared with \( R \), the local radius of curvature of the external spacetime in which the black hole moves. For example, suppose that the black hole is moving on a circular orbit of radius \( r \) in the gravitational field of another body of mass \( M \). Then \( R^{-1} \) is of the order of the hole’s angular velocity, and we have

\[
\frac{m}{R} \sim \frac{m}{m + M} V^3, \quad V = \sqrt{\frac{m + M}{r}}, \tag{1}
\]

where \( V \) is the hole’s orbital velocity. One way to make this ratio small is to let \( m/M \ll 1 \); then \( m/R \) will be small irrespective of the magnitude of \( V \). This is the small-hole approximation, which allows the small black hole to move at relativistic speeds in the strong gravitational field of the external body. Another way is to let \( V \ll 1 \); then \( m/R \) will be small for all mass ratios. This is the slow-motion approximation, which allows the slowly-moving black hole to have a mass comparable to (or even much larger than) \( M \).

This paper is concerned with the small-hole approximation. We take the small black hole to move on a circular orbit in the equatorial plane of a large Kerr black hole, and we apply the formalism developed in Ref. [2] to calculate the rates at which the tidal interaction changes the mass and angular momentum of the small black hole.

The large black hole possesses a mass \( M \) and an angular momentum \( J = \Xi M^2 \), where \( \Xi \) is the dimensionless Kerr parameter, whose magnitude is limited by \( |\Xi| \leq 1 \); we adopt the convention that \( \Xi \) is positive when the orbit of the small black hole is corotating with the large black hole, while \( \Xi \) is negative when the orbit is counter-rotating.

When the small black hole is nonrotating, we find that its mass and angular momentum change according to

\[
\frac{dm}{dv} = \frac{32}{5} \left( \frac{m}{M} \right)^6 V^{18} \Gamma_S, \quad \tag{2a}
\]

\[
\frac{dj}{dv} = \frac{32}{5} sgn(\Xi) \left( \frac{m}{M} \right)^6 V^{15} \Gamma_S, \quad \tag{2b}
\]

where \( v \) is advanced time, \( V := \sqrt{M/r} \) an orbital-velocity parameter (defined in terms of the Boyer-Lindquist orbital radius \( r \)), and

\[
\Gamma_S = \frac{(1 - 2V^2 + \Xi^2 V^4)(1 - V^2 - 2\Xi V^3 + 2\Xi^2 V^4)}{(1 - 3V^2 + 2\Xi V^3)^2} \tag{3}
\]

is a relativistic factor that approaches unity in the slow-motion limit \( V \ll 1 \). Equations (2) are valid to all orders in \( V \), but are are accurate only to leading order in the small mass ratio; they neglect terms of fractional order \( m/M \ll 1 \).

When the small black hole is rapidly rotating (in the sense that its own angular velocity is much larger than the orbital angular velocity, as discussed in Sec. IX of Ref. [2]), we find instead that

\[
\frac{dm}{dv} = -\frac{8}{5} sgn(\Xi) \left( \frac{m}{M} \right)^5 \chi(1 + 3\chi^2) V^{15} \Gamma_K, \quad \tag{4a}
\]

\[
\frac{dj}{dv} = -\frac{8}{5} \frac{m^5}{M^7} \chi(1 + 3\chi^2) V^{12} \Gamma_K, \quad \tag{4b}
\]

where \( \chi := j/m^2 \) is the dimensionless Kerr parameter of the small black hole, and

\[
\Gamma_K = \frac{1 - 2V^2 + \Xi^2 V^4}{(1 - 3V^2 + 2\Xi V^3)^2} \times \left( 1 - \frac{4 + 27\chi^2}{4 + 12\chi^2} V^2 - \frac{4 - 3\chi^2}{2 + 6\chi^2} \Xi V^3 + \frac{8 + 9\chi^2}{4 + 12\chi^2} \Xi^2 V^4 \right) \tag{5}
\]
is the appropriate relativistic factor. It is assumed that the spin of the small black hole is either aligned (\(\chi > 0\)) or anti-aligned (\(\chi < 0\)) with the spin of the large black hole. Equations (4) are valid to all orders in \(V\), but are accurate only to leading order in the small mass ratio; they neglect terms of fractional order \(m/M \ll 1\).

The results displayed in Eqs. (2) and (4) generalize those presented in Secs. VIII I and IX G of Ref. [2], which apply when the large black hole is nonrotating (\(\Xi = 0\)). From Eqs. (2) we observe that when the small black hole is nonrotating, its mass always increases, while its angular momentum increases (decreases) when the orbit is corotating (counter-rotating). From Eqs. (1) we observe that when the small black hole is rapidly rotating and its spin is aligned with that of the large black hole, its mass decreases (increases) when the orbit is corotating (counter-rotating), while its angular momentum always decreases; the signs reverse when the spins are anti-aligned.

These are the behaviors that are expected for a system in rigid rotation. In such cases (see Sec. IX F of Ref. [2]) we must have that \(\frac{dm}{dv} = \Omega^I (\Omega^I - \Omega_H) K\) and \(\frac{dj}{dv} = (\Omega^I - \Omega_H) K\), where \(\Omega^I\) is the angular frequency of the tidal fields as measured in the moving frame of the small black hole, \(\Omega_H\) is the angular velocity of the small black hole, and \(K\) is a quantity — defined by Eq. (9.46) of Ref. [2] — that is constructed from the tidal fields. According to our conventions, the signs of \(\Omega^I\) and \(\Omega_H\) are measured relative to the orientation of the spin of the large black hole, so that \(\Omega^I > 0\) when the orbit is corotating and \(\Omega_H > 0\) when the black-hole spins are aligned. The nonrotating case corresponds to \(\Omega^I > \Omega_H\), while the rapidly-rotating case corresponds to \(\Omega^I \ll \Omega_H\).

The relativistic factors are plotted in Figs. 1 and 2 for selected values of \(\Xi\), the Kerr parameter of the large black hole. For each curve the velocity parameter \(V\) ranges from 0 to the maximum value \(V_{isco}\), that corresponds to the innermost stable circular orbit. We notice the interesting phenomenon that each relativistic factor takes on a unique limiting value when \(V \rightarrow V_{isco}\). The limiting values are

\[
\begin{align*}
\Gamma_S & \rightarrow \frac{20}{9}, \\
\Gamma_K & \rightarrow \frac{54 + 9\chi^2}{9 + 3\chi^2}.
\end{align*}
\]

Making the substitutions in Eqs. (2) and (4) reveals that in this limit, \(\frac{dm}{dv}\) and \(\frac{dj}{dv}\) can be expressed entirely in terms of \(V_{isco}\), and no longer depend on \(\Xi\).

The scaling of \(\frac{dm}{dv}\) as \((m/M)^6\) in the case of a nonrotating black hole, and as \((m/M)^5\) in the case of a rapidly rotating black hole, implies that the heating and torquing of a small black hole by tidal fields produced by a much larger black hole are always insignificant. The effect will never modify the motion of the small black hole in a substantial manner, in spite of the fact that the motion takes place in the deep relativistic field of the large black hole. This circumstance might have produced large values for the relativistic factors \(\Gamma_S\) and \(\Gamma_K\), but our calculations show instead that they are bounded by the numbers displayed in Eqs. (6). The interest of our work, therefore, is not in the fact that the effect might lead to interesting observational consequences. It does not. It is instead in the fact that in the small-hole approximation, the tidal heating and torquing of the small black hole can be computed to all orders in the velocity parameter \(V\), which is permitted to approach the speed of light; and the exercise reveals that relativistic effects cannot compensate for the smallness of \(m/M\).

For the tidal heating and torquing of the black hole to be significant, it is necessary to increase the value of the mass ratio. This can be done within the small-hole/slow-motion approximation at the expense of reducing the orbital velocity so that \(V \ll 1\). This regime was examined by Taylor and Poisson [7], who show that in this case the
dependence of $dm/dv$ upon the masses becomes

$$\left(\frac{m}{M}\right)^6 \to \frac{m^6 M^2}{(m + M)^6}$$

when the black hole is nonrotating, and

$$\left(\frac{m}{M}\right)^5 \to \frac{m^5 M^2}{(m + M)^5}$$

when the black hole is rapidly rotating. In the nonrotating case the mass-dependent factor is maximized when $m = 3M$ and is then numerically equal to 0.01112. In the rapidly-rotating case it is maximized when $m = \frac{1}{2}M$ and is then equal to 0.01483. For such mass ratios the effect is likely to be significant when $V$ becomes comparable to unity. In the regime where $m$ is much larger than $M$, $dm/de$ scales as $(M/m)^2$ in each case, and this is the same scaling that is found for the rate at which energy is radiated away by gravitational waves; this was the regime examined by Hughes [11] and Martel [12], where the effect was shown to be significant.

This concludes the presentation of our results and the discussion of their significance. In the following section of the paper we describe how the results displayed in Eqs. (2), (4), and (6) were obtained.

### II. DERIVATIONS

For the purposes of computing tidal fields, the small black hole can be thought of as a test particle moving on a circular orbit in the equatorial plane of the Kerr spacetime. The large black hole has a mass $M$ and angular momentum $J = \Xi M^2$, and the orbital radius is $r$ in Boyer-Lindquist coordinates; the orbit is corotating with the black hole when $\Xi > 0$, and it is counter-rotating when $\Xi < 0$. In place of $r$ it is useful to introduce the velocity parameter $V := \sqrt{M/r}$ to describe the orbit. In Boyer-Lindquist coordinates $(t, r, \theta, \phi)$ the velocity vector of the small black hole is $u^\alpha = \gamma(1, 0, 0, \Omega)$, with

$$\Omega = \frac{\text{sgn}(\Xi)}{M} \frac{V^3}{1 + \Xi V^3}, \quad \gamma = \frac{1 + \Xi V^3}{\sqrt{1 - 3V^2 + 2\Xi V^4}};$$

$\Omega$ is the orbital angular velocity of the small black hole. The orbital velocity $V$ is limited to the interval $0 < V \leq V_{\text{isco}}$, where $V_{\text{isco}}$ is a solution to the quartic equation

$$1 - 6V^2 + 8\Xi V^3 - 3\Xi^2 V^4 = 0.$$

This equation can also be viewed as a quadratic equation for $\Xi$, whose solution gives $\Xi$ expressed as a function of $V_{\text{isco}}$; this interpretation is useful for our purposes below.

The tidal fields created by the large black hole are represented by components of the Weyl tensor evaluated as functions of proper time $\tau$ on the world line of the small black hole. In terms of a tetrad $(u^\alpha, e^\alpha_a)$ of orthonormal vectors that are parallel-transported on the world line (with the lower-case Latin index $a = \{1, 2, 3\}$ representing a spatial label), we have the electric-type components $E_{ab} := C_{\mu\nu\rho\sigma} e^\mu_a e^\nu_b e^\rho_c e^\sigma_d$ and the magnetic-type components $B_{ab} := \frac{1}{2} \epsilon^\sigma_{ac} e^\mu_a e^\nu_b C_{\rho\sigma\mu\nu}$, where $C_{\mu\nu\rho\sigma}$ is the Weyl tensor and $\epsilon_{\mu\nu\sigma\rho}$ is the Levi-Civita tensor. The tidal moments $E_{ab}$ and $B_{ab}$ are symmetric and tracefree, in the sense that $E_{ab} = E_{ba}$ and $\delta^{ab} E_{ab} = 0$ (with similar equations holding for $B_{ab}$).

An orthonormal tetrad adapted to equatorial, circular orbits of a Kerr black hole can be constructed by specializing the general work of Marck [14] to this specific case. We find

$$e^\alpha_1 = (-\mu \sin \Phi, \lambda \cos \Phi, 0, -\nu \sin \Phi), \quad (11a)$$

$$e^\alpha_2 = (\mu \cos \Phi, \lambda \sin \Phi, 0, \nu \cos \Phi), \quad (11b)$$

$$e^\alpha_3 = (0, 0, -V^2/M, 0), \quad (11c)$$

where

$$\lambda = \sqrt{1 - 2V^2 + \Xi^2 V^4},$$

$$\mu = \frac{\text{sgn}(\Xi) V(1 - 2\Xi^2 + \Xi^4)}{\sqrt{1 - 2V^2 + \Xi^2 V^4} \sqrt{1 - 3V^2 + 2\Xi V^4}},$$

$$\nu = \frac{V^2(1 - 2V^2 + \Xi^2 V^4)}{M \sqrt{1 - 2V^2 + \Xi^2 V^4} \sqrt{1 - 3V^2 + 2\Xi V^4}},$$

and

$$\Phi = \Omega^\dagger \tau, \quad \Omega^\dagger = \frac{\text{sgn}(\Xi)}{M} V^3.$$ (13)

Notice that the basis is right-handed, and that $e^\alpha_3$ points in the direction of the angular-momentum vector of the large black hole. The angular frequency of the tetrad vectors is $\Omega^\dagger$, and this differs from the orbital angular velocity $\Omega$ because (i) $\Omega^\dagger$ refers to proper time $\tau$ instead of coordinate time $t$, and (ii) the tetrad vectors undergo geodetic and Lense-Thirring precession relative to the global inertial frame.

The nonvanishing components of the tidal fields $E_{ab}$ and $B_{ab}$ are

$$1/2 (E_{11} + E_{22}) = \frac{-V^6}{2M^2} \frac{1 - 4\Xi V^3 + 3\Xi^2 V^4}{1 - 3V^2 + 2\Xi V^4} = \frac{-1}{2} E_{33},$$

$$1/2 (E_{11} - E_{22}) = \frac{-3V^6}{2M^2} \frac{1 - 2V^2 + \Xi^2 V^4 \cos 2\Phi}{1 - 3V^2 + 2\Xi V^4},$$

$$E_{12} = \frac{-3V^6}{2M^2} \frac{1 - 2V^2 + \Xi^2 V^4}{1 - 3V^2 + 2\Xi V^4} \sin 2\Phi,$$

$$B_{13} = -\frac{3\text{sgn}(\Xi) V^7}{M^2} \frac{(1 - \Xi V) \sqrt{1 - 2V^2 + \Xi^2 V^4}}{1 - 3V^2 + 2\Xi V^4} \cos \Phi,$$

$$B_{23} = -\frac{3\text{sgn}(\Xi) V^7}{M^2} \frac{(1 - \Xi V) \sqrt{1 - 2V^2 + \Xi^2 V^4}}{1 - 3V^2 + 2\Xi V^4} \sin \Phi.$$

These equations generalize Eqs. (2.107) and (2.108) of Ref. [13], which hold for $\Xi = 0$ only.
The limiting values of the tidal fields when $V \to V_{\text{isco}}$ can be simplified by solving Eq. (10) for $\Xi(V_{\text{isco}})$. Making the substitution reveals that $\frac{1}{2}(\xi_{11} + \xi_{22}) \to -M^{-2}V_{\text{isco}}^2 \xi_{11} - 2M^{-2}V_{\text{isco}}^2 \cos 2\Phi \xi_{12} - 2m^{-2}V_{\text{isco}}^2 \sin 2\Phi \xi_{13} - 2m^{-2}V_{\text{isco}}^2 \cos \Phi \xi_{23}$. When the substitution is made into Eqs. (3), $\frac{dm}{dv} = \frac{1}{4}m^5 \chi [8(1 + 3\chi^2)](E_1 + B_1) - 3(4 + 17\chi^2)(E_2 + B_2) + 15\chi^2(E_3 + B_3)$, where $E_1 := \xi_{ab}\xi^{ab}$, $E_2 := (\xi_{ab})^2$, $E_3 := (\xi_{ab}s^a s^b)^2$, and $B_1 := B_{ab}B^{ab}$, $B_2 := (B_{ab}B^{ab})^2$, $B_3 := (B_{ab}s^a s^b)^2$. When the black hole is a member of a system in rigid rotation, we also have $\frac{ddj}{dv} = \Omega^2 \frac{dj}{dv}$. Substitution of the tidal fields of Eqs. (14) gives rise to Eqs. (4).

The limiting values of the relativistic factors $\Gamma_S$ and $\Gamma_K$ when $V \to V_{\text{isco}}$ can be simplified by solving Eq. (10) for $\Xi(V_{\text{isco}})$. When the substitution is made into Eqs. (3) and (4), we find the results displayed in Eq. (6).

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