New exact solutions of nonlinear variants of the RLW, the PHI-four and Boussinesq equations based on modified extended direct algebraic method

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Abstract

By means of modified extended direct algebraic method (MEDA) the multiple exact complex solutions of some different kinds of nonlinear partial differential equations are presented and implemented in a computer algebraic system. New complex solutions for nonlinear equations such as the variant of the RLW equation, the variant of the PHI-four equation and the variant Boussinesq equations are obtained.

Keywords: RLW equation, PHI-four equation, Boussinesq equations, nonlinear partial differential equations, the MEDA method.

1 Introduction

It is well known that the nonlinear physical phenomena are related to nonlinear partial differential equations which are involved in many fields such as physics, chemistry, mechanics, etc. As mathematical models of the phenomena, the investigation of exact solutions of the partial differential equations will help one to understand these phenomena better. In recent years, many powerful methods to construct exact solutions of nonlinear partial differential equations have been established and developed. Among these are variational iteration method [3, 1, 5, 2, 6, 4], tanh function method [7, 8], modified extended tanh function method [10, 12, 14, 9, 11], sine-cosine method [16, 17], Exp-method [18], inverse scattering method [19], Hirota’s bilinear method [20], the homogeneous balance method [21], the Riccati expansion method with constant coefficients [23, 22]. The regularized long wave (RLW) equation is an important nonlinear wave equation. Solitary

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waves are wave packet or pulses, which propagate in nonlinear dispersive media. Due to dynamical balance between the non-linear and dispersive effects these waves retain a stable wave form. The regularized long wave (RLW) equation is an alternative description of non-linear dispersive waves to the more usual Kortewege-de vries (KDV) equation [24]. Numerical solutions based on finite difference techniques [25, 26], Rung-Kutta method [27] and Galerkin’s method [28] have been given. Alexander and Morris [28] constructed a global trial function mainly from cubic splines. Gardner and Gardner [29], using the Galerkin’s method and cubic B-spline as element shape function to construct an implicit finite element solution. The least squares method using linear space-time finite elements used to solve the RLW equation [30], Soliman and Raslan [31] solved the RLW equation by using collocation method using quadratic B-spline at the mid point. Soliman and Hussien [32] used the collocation method with septic spline to solve the RLW equation. Soliman [2] using the finite difference method with the similarity solution of the partial differential equations to obtain the numerical scheme for RLW equation this approach eliminate the difficult associated to the boundary. in the end Soliman [2] using the variational iteration method to obtain the exact solutions of the gneralized RLW. The PHI-four equation is considered as a particular form of the Klein-Gordon equation that model phenomenon in particle physics where kink and anti-kink solitary waves interact [34]. Recently, the direct algebraic method and symbolic computation have been suggested to obtain the exact complex solutions of nonlinear partial differential equations [37, 35, 36]. The aim of this paper is to extend the modified extended direct algebraic method MEDA method to solve three different types of nonlinear differential equations such as the variant of the RLW, the variant of the PHI-four and variant Boussinesq equations [12, 11].

2 Modified extended direct algebraic method

To illustrate the basic concepts of the modified extended direct algebraic (MEDA) method. We consider a given PDE in two independent variables given by

\[ F(u, u_x, u_t, u_{xx}, \ldots) = 0 \]  

We first consider its travelling solutions \( u(x, t) = u(z) \), \( z = i(x + c t) \) or \( z = i(x - c t) \), \( i = \sqrt{-1} \), then Eq. (1) becomes an ordinary differential equation

\[ H(u, i u', -i c u', -u'', \ldots) = 0 \]  

where \( u' = \frac{du}{dz} \). In order to seek the solutions of Eq. (1), we introduce the
following ansatze
\[ u(z) = a_0 + \sum_{j=1}^{M} (a_j \phi^j + b_j \phi^{-j}) \]  
(3)

\[ \phi' = b + \phi^2 \]  
(4)

where \( b \) is a parameter to be determined, \( \phi = \phi(z), \phi' = \frac{d\phi}{dz} \). The parameter \( M \) can be found by balancing the highest-order derivative term with the nonlinear terms [38]. Substituting (3) into (2) with (4) will yield a system of algebraic equations with respect to \( a_j, b_j, b \) and \( c \) (where \( j = 1...M \)) because all the coefficients of \( \phi^j \) have to vanish. We can then determine \( a_0, a_j, b_j, b \), and \( c \). Eq. (4) has the general solutions:

(I) If \( b < 0 \)
\[ \phi = -\sqrt{-b} \tanh (\sqrt{-b} z), \quad \text{or} \quad \phi = -\sqrt{-b} \coth (\sqrt{-b} z) \]

it depends on initial conditions.

(II) If \( b > 0 \)
\[ \phi = \sqrt{b} \tan (\sqrt{b} z), \quad \text{or} \quad \phi = -\sqrt{b} \cot (\sqrt{b} z) \]

it depends on initial conditions.

(III) If \( b = 0 \)
\[ \phi = \frac{-1}{z} \]  
(5)

Substituting the results into (3), then we obtain the exact travelling wave solutions of Eq. (1). To illustrate the procedure, three examples related to variant of the RLW, variant of the PHI-four and variant Boussinesq equations are given in the following.

3 Applications

3.1 Variant of the regularized long-wave (RLW) equation

Let us first consider the nonlinear variant of the regularized long wave equation which has the form [11]
\[ u_t + \alpha u_x - \lambda (u^n)_x + \beta (u^n)_{ext} = 0 \]  
(6)

where \( \alpha, \lambda, \) and \( \beta \) are arbitrary constants. In order to solve Eq. (6) by the MEDA method, we use the wave transformation \( u(x,t) = U(z) \) with
wave complex variable \( z = i(x - ct) \), Eq. (6) takes the form of an ordinary differential equation as

\[(\alpha - c) U' - \lambda (U^n)' + \beta c (U^n)''' = 0 \tag{7}\]

Integrating Eq. (7) once with respect to \( z \) and setting the constant of integration to be zero, we obtain

\[(\alpha - c) U - \lambda U^n + \beta c (U^n)'' = 0 \tag{8}\]

Or equivalently

\[(\alpha - c) U - \lambda U^n + \beta c n U^{n-1}U'' + \beta c n (n - 1) U^{n-2} (U')^2 = 0 \tag{9}\]

Balancing the order of \( U^n \) with the order of \( U^{n-1} U'' \) in Eq. (9), we find

\[M = -\frac{2}{n-1}.\]

To get a closed form analytic solution, the parameter \( M \) should be an integer. A transformation formula \( U = V - \frac{1}{n-1} \) should be used to obtain this analytic solution. So Eq. (9) takes the form

\[(\alpha - c)(n-1)^2 V^3 - \lambda (n-1)^2 V^2 - \beta c n (n-1) V V'' + \beta c n (2n-1)(V')^2 = 0 \tag{10}\]

Balancing the order of \( V^3 \) with the order of \( V V'' \) in Eq. (10), gives \( M = 2 \). So the solution takes the form

\[V(z) = a_0 + a_1 \phi(z) + a_2 \phi(z)^2 + b_1 \phi(z)^{-1} + b_2 \phi(z)^{-2} \tag{11}\]

Inserting Eq. (11) into Eq. (10) and making use of Eq. (4), using the Maple Package, we get a system of algebraic equations, for \( a_0, a_1, a_2, b_1, b_2 \) and \( b \). We solve the obtained system of algebraic equations give the following three cases:

Case (I): consider, \( A = (-n\lambda - \lambda + 2\alpha n a_0) \) then, \( a_1 = 0 \), \( a_2 = 0 \), \( b_1 = 0 \), \( b_2 = \frac{a_0^2 \lambda (n^2 - 2n + 1)}{2n \beta A} \), \( c = \frac{A}{2na_0} \), \( b = \frac{a_0 \lambda (n^2 - 2n + 1)}{2n \beta A} \), with \( a_0 \) being an arbitrary constant.

\[V(x, t) = a_0(1 + \cot^2\left(\frac{i}{2} \sqrt{\frac{2a_0 \lambda (n^2 - 2n + 1)}{n \beta A}} (x - \frac{A}{2na_0} t)) \tag{12}\]

so, the travelling wave solution is given by

\[u(x, t) = (a_0(1 + \cot^2\left(\frac{i}{2} \sqrt{\frac{a_0 \lambda (n^2 - 2n + 1)}{n \beta A}} (x - \frac{1}{2} \frac{A}{na_0} t)) \right))^\frac{1}{2} \tag{13}\]
Case (II): consider, \(E = \frac{(n^2a_2-2n a_2+2 \beta n^2+2 \beta n+a_2)}{a}\), then, \(a_0 = \frac{1}{4} \lambda E\), \(a_1 = 0, \ b_1 = 0, \ b_2 = 0, \ b = \frac{1}{4} \lambda E\), \(c = \frac{a_2(n^2-2n+1)}{E}\), with \(a_2\) being an arbitrary constant, then

\[
V(x, t) = \frac{1}{4} \lambda E \beta n^2 (1 - \tan^2 \left(\frac{i}{2} \sqrt{\frac{E \beta}{\lambda n^2 a_2}} (x - \frac{a_2(n^2-2n+1)}{E} t)\right)) (14)
\]

so, the travelling wave solution is given by

\[
u(x, t) = \left(\frac{1}{16} \lambda E n^2 \beta \frac{1}{\lambda} \right) (1 - \tan^2 \left(\frac{i}{4} \sqrt{\frac{E \beta}{\lambda n^2 a_2}} (x - \frac{a_2(n^2-2n+1)}{E} t)\right)) n - 1 (15)
\]

Case (III) : \(a_0 = \frac{1}{8} \lambda E\), \(a_1 = 0, \ b_1 = 0, \ b_2 = \frac{1}{256} \lambda^2 E\), \(b = \frac{1}{16} \lambda E\), \(c = \frac{a_2(n^2-2n+1)}{E}\) with \(a_2\) being an arbitrary constant, then

\[
V(x, t) = \frac{1}{8} \beta n^2 \frac{E}{\lambda} \left(1 - \frac{1}{2} \tan^2 \left(\frac{i}{2} \sqrt{\frac{E \lambda}{\beta n^2 a_2}} (x - \frac{a_2(n^2-2n+1)}{E} t)\right)\right) \left(1 - \cot^2 \left(\frac{i}{4} \sqrt{\frac{E \lambda}{\beta n^2 a_2}} (x - \frac{a_2(n^2-2n+1)}{E} t)\right)\right) (16)
\]

so, the travelling wave solution is given by

\[
u(x, t) = \left(\frac{1}{16} \lambda E n^2 \beta \frac{1}{\lambda} \right) (1 - \frac{1}{2} \tan^2 \left(\frac{i}{4} \sqrt{\frac{E \lambda}{\beta n^2 a_2}} (x - \frac{a_2(n^2-2n+1)}{E} t)\right)) n - 1 (17)
\]

### 3.2 Variant of the PHI-four equation

A second important example is the Variant of the PHI-four equation \[11\], which can be written as

\[
u_{tt} - \alpha u_{xx} - \lambda u + \beta u^n = 0 \quad n > 0 (18)
\]

where \(\alpha, \lambda,\) and \(\beta\) are arbitrary constants. In order to solve Eq. \[18\] by the MEDA method, we use the wave transformation \(u(x, t) = U(z)\) with wave variable \(z = i(x-ct)\) Eq. \[18\] takes the form of an ordinary differential equation

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\[ c^2 U'' - \alpha U'' - \lambda U + \beta U^n = 0 \]  \hspace{1cm} (19)

Or equivalently

\[- \lambda U + \beta U^n - (c^2 - \alpha) U'' = 0 \]  \hspace{1cm} (20)

Balancing the order of \( U^n \) with the order of \( U'' \) in Eq. (20), we find \( M = \frac{2}{n-1} \). To obtain a closed form of the analytic solution we use a transformation formula \( U = V \frac{1}{n-1} \), that transforms Eq. (20) to

\[- \lambda (n-1)^2 V^2 + \beta (n-1)^2 V^3 + (\alpha - c^2)(n-1)VV'' + (\alpha - c^2)(2-n)(V')^2 = 0 \]  \hspace{1cm} (21)

Balancing the order of \( V^3 \) with the order of \( VV'' \) in Eq. (21), we find \( M = 2 \). So the solution takes the form

\[ V(z) = a_0 + a_1 \phi(z) + a_2 \phi(z)^2 + b_1 \phi(z)^{-1} + b_2 \phi(z)^{-2} \]  \hspace{1cm} (22)

Substituting Eq. (22) into Eq. (21) and making use of Eq. (4), we obtain a system of algebraic equations, for \( a_0, a_1, a_2, b_1, b_2 \), and \( b \). We solve the obtained system of algebraic equations give the following three cases:

Case (I): \( a_0 = \frac{\lambda(n+1)}{2\beta}, \ a_1 = 0, \ a_2 = 0, \ b_1 = 0, \ b_2 = \frac{(n+1)(n^2-2n+1)}{8\beta(c^2-\alpha)} \), \( b = \frac{\lambda(n^2-2n+1)}{4(c^2-\alpha)} \) with \( c \) being an arbitrary constant. then

\[ V(x,t) = \frac{\lambda(n+1)}{2\beta} + \frac{\lambda(n+1)}{2\beta} \cot^2 \left( i \frac{1}{2} \sqrt{\frac{\lambda(n^2-2n+1)}{c^2-\alpha}} (x - ct) \right) \]  \hspace{1cm} (23)

so, the travelling wave solution is given by

\[ u(x,t) = \left( \frac{\lambda(n+1)}{2\beta} + \frac{\lambda(n+1)}{2\beta} \cot^2 \left( i \frac{1}{2} \sqrt{\frac{\lambda(n^2-2n+1)}{c^2-\alpha}} (x - ct) \right) \right)^{\frac{1}{n-1}} \]  \hspace{1cm} (24)

Case (II): \( a_0 = \frac{\lambda(n+1)}{2\beta}, \ a_1 = 0, \ a_2 = \frac{2(-\alpha n + c^2 n - \alpha c^2)}{\beta(n^2-2n+1)}, \ b_1 = 0, b_2 = 0, \)

\( b = \frac{\lambda(n^2-2n+1)}{4(c^2-\alpha)} \), with \( c \) being an arbitrary constant, then

\[ V(x,t) = \frac{1}{2} \frac{\lambda(n+1)}{\beta} + \frac{1}{2} \frac{\lambda(-\alpha n + c^2 n - \alpha c^2)}{\beta(c^2-\alpha)} \cdot \tan^2 \left( i \frac{1}{2} \sqrt{\frac{\lambda(n^2-2n+1)}{c^2-\alpha}} (x - ct) \right) \]  \hspace{1cm} (25)
so, the travelling wave solution is given by

\[ u(x, t) = \left( \frac{1}{2} \frac{\lambda (n+1)}{\beta} + \frac{1}{2} \frac{\lambda (-\alpha n + c^2 n - \alpha + c^2)}{\beta (c^2 - \alpha)} \right) \tan^2 \left( \frac{i}{2} \sqrt{\frac{\lambda (n^2 - 2n + 1)}{c^2 - \alpha}} (x - c t) \right) \frac{1}{n-1} \]  

(26)

**Case(III):** Consider, 
\[ D = \sqrt{\frac{2 \lambda n - \lambda n^2 - \lambda - 16}{b}}, \quad a_0 = \frac{1}{4} \frac{\lambda (n+1)}{\beta}, \quad a_1 = 0, \quad a_2 = \frac{1}{8} \frac{(n+1) \lambda}{\beta b}, \quad b_1 = 0, \quad b_2 = \frac{1}{8} \frac{(n+1) \lambda b}{\beta}, \quad c = \frac{i}{4} D \] 

with \( b \) being an arbitrary constant. Then,

\[ V(x, t) = \frac{\lambda (n+1)}{4 \beta} + \frac{\lambda (n+1)}{8 \beta} \tan^2 (\sqrt{-b} (x - \frac{i}{4} D t)) + \frac{\lambda (n+1)}{8 \beta} \cot^2 (\sqrt{-b} (x - \frac{i}{4} D t)) \]  

(27)

so, the travelling wave solution is given by

\[ u(x, t) = \left( \frac{\lambda (n+1)}{4 \beta} + \frac{\lambda (n+1)}{8 \beta} \tan^2 (\sqrt{-b} (x - \frac{i}{4} D t)) + \frac{\lambda (n+1)}{8 \beta} \cot^2 (\sqrt{-b} (x - \frac{i}{4} D t)) \right) \frac{1}{n-1} \]  

(28)

**Case(IV):** 
\[ a_0 = \frac{1}{4} \frac{\lambda (n+1)}{\beta}, \quad a_1 = 0, \quad a_2 = \frac{1}{8} \frac{(n+1) \lambda}{\beta b}, \quad b_1 = 0, \quad b_2 = \frac{1}{8} \frac{(n+1) \lambda b}{\beta}, \quad c = -\frac{i}{4} D \] 

with \( b \) being an arbitrary constant. Then,

\[ V(x, t) = \frac{\lambda (n+1)}{4 \beta} + \frac{\lambda (n+1)}{8 \beta} \tan^2 (\sqrt{-b} (x + \frac{i}{4} D t)) + \frac{\lambda (n+1)}{8 \beta} \cot^2 (\sqrt{-b} (x + \frac{i}{4} D t)) \]  

(29)

so, the travelling wave solution is given by

\[ u(x, t) = \left( \frac{\lambda (n+1)}{4 \beta} + \frac{\lambda (n+1)}{8 \beta} \tan^2 (\sqrt{-b} (x + \frac{i}{4} D t)) + \frac{\lambda (n+1)}{8 \beta} \cot^2 (\sqrt{-b} (x + \frac{i}{4} D t)) \right) \frac{1}{n-1} \]  

(30)

All the solutions of the equations are new.
3.3 The variant Boussinesq equations

Finally, we consider a very important example as an illustration of the modified extended direct algebraic method for solving the variant Boussinesq equations [12], we will consider the following system of equations

\[ u_t + v_x + u u_x = 0 \quad (31) \]
\[ v_t + (uv)_x + u_{xxx} = 0 \quad (32) \]

To solve the system of Eqs. (31), (32) by means of the modified extended direct algebraic method, we use the wave transformation \( u(x,t) = U(z) \) and \( v(x,t) = V(z) \) with complex wave variable \( z = i(x+\lambda t) \). Therefore, system (31), (32) is reduced to the ordinary differential equations in the form

\[ \lambda U' + V' + \frac{1}{2} (U')^2 = 0 \quad (33) \]
\[ \lambda V' + (UV)' - U''' = 0 \quad (34) \]

Integrating both equations once leads to:

\[ C_1 - \lambda U - \frac{1}{2} U^2 = V \quad (35) \]
\[ \lambda V + UV - U'' = C_2 \quad (36) \]

where \( C_1 \) and \( C_2 \) are integrating constants, so as to we find the special forms of the exact solutions, for simplicity purpose, we take \( C_1 = C_2 = 0 \) Substituting (35) into (36) gives

\[ U'' + \frac{1}{2} U^3 + \frac{3}{2} \lambda U^2 + \lambda^2 U = 0, \quad (37) \]

By balancing \( U'' \) with \( U^3 \) in eq. (37), we find \( M = 1 \). So the solutions take the form

\[ U(z) = a_0 + a_1 \phi(z) + b_1 \phi(z)^{-1} \quad (38) \]

\[ V(z) = -\lambda \left[ a_0 + a_1 \phi(z) + b_1 \phi(z)^{-1} \right] - \frac{1}{2} \left[ a_0 + a_1 \phi(z) + b_1 \phi(z)^{-1} \right]^2 \quad (39) \]

Inserting Eqs. (35), (47) into Eq. (36), making use of Eq. (41), and by using Maple Package, we get a system of algebraic equations, for \( a_0, a_1, b_1, \lambda \), and \( b \). We solve the obtained algebraic system of equations by Maple Package and select four cases of solutions as:
Case (I): $a_1 = 2i$, $b_1 = 0$, $b = -\frac{1}{4} a_0^2$, $\lambda = -a_0$, with $a_0$ being an arbitrary constant, the complex wave solutions are:

$$u(x, t) = a_0(1 + i \tan(\frac{ia}{2}(x - a_0t))) \quad (40)$$

$$v(x, t) = a_0(a_0(1 + i \tan(\frac{ia}{2}(x - a_0t)))) - \frac{1}{2}(a_0(1 + i \tan(\frac{ia}{2}(x - a_0t))))^2 \quad (41)$$

Case (II): $a_1 = -2i$, $b_1 = 0$, $b = -\frac{1}{4} a_0^2$, $\lambda = -a_0$, with $a_0$ being an arbitrary constant, the complex wave solutions are:

$$u(x, t) = a_0(1 - i \tan(\frac{ia}{2}(x - a_0t))) \quad (42)$$

$$v(x, t) = a_0(a_0(1 - i \tan(\frac{ia}{2}(x - a_0t)))) - \frac{1}{2}(a_0(1 - i \tan(\frac{ia}{2}(x - a_0t))))^2 \quad (43)$$

Case (III): $a_1 = 2i$, $b_1 = -\frac{1}{4} a_0^2 i$, $b = -\frac{1}{8} a_0^2$, $\lambda = -a_0$ with $a_0$ being an arbitrary constant, the complex wave solutions are

$$u(x, t) = a_0 + \frac{\sqrt{2} a_0}{2} (\tan(\frac{\sqrt{2} a_0}{4}(x - a_0 t)) + \cot(\frac{\sqrt{2} a_0}{4}(x - a_0 t))) \quad (44)$$

$$v(x, t) = a_0(a_0 + \frac{\sqrt{2} a_0}{2}(\tan(\frac{\sqrt{2} a_0}{4}(x - a_0 t)) + \cot(\frac{\sqrt{2} a_0}{4}(x - a_0 t))))$$

$$- \frac{1}{2}(a_0(a_0 + \frac{\sqrt{2} a_0}{2}(\tan(\frac{\sqrt{2} a_0}{4}(x - a_0 t)) + \cot(\frac{\sqrt{2} a_0}{4}(x - a_0 t))))^2 \quad (45)$$

Case (IV): $a_1 = -2i$, $b_1 = \frac{1}{4} a_0^2 i$, $b = \frac{1}{8} a_0^2$, $\lambda = -a_0$ with $a_0$ being an arbitrary constant, the complex wave solution are

$$u(x, t) = a_0 - \frac{\sqrt{2} a_0}{2} (\tan(\frac{\sqrt{2} a_0}{4}(x - a_0 t)) + \cot(\frac{\sqrt{2} a_0}{4}(x - a_0 t))) \quad (46)$$

$$v(x, t) = a_0(a_0 - \frac{\sqrt{2} a_0}{2}(\tan(\frac{\sqrt{2} a_0}{4}(x - a_0 t)) + \cot(\frac{\sqrt{2} a_0}{4}(x - a_0 t))))$$

$$- \frac{1}{2}(a_0 - \frac{\sqrt{2} a_0}{2}(\tan(\frac{\sqrt{2} a_0}{4}(x - a_0 t)) + \cot(\frac{\sqrt{2} a_0}{4}(x - a_0 t))))^2 \quad (47)$$

All the solutions of the variant Boussinesq equations are new.
4 Conclusions

In this paper, the MEDA method has been successfully applied to find the solution for three nonlinear partial differential equations such as the the variant of the RLW equation, the variant of the PHI-four equation, and the variant Boussinesq equations. The modified extended direct algebraic method is used to find a new complex travelling wave solutions. The results show that the modified extended direct algebraic method is a powerful mathematical tool to solve the the variant of the RLW equation, the variant of the PHI-four equation, and the variant Boussinesq equations it is also a promising method to solve other nonlinear equations.

References

[1] M. A. Abdou, A. A. Soliman, Variational iteration method for solving Burgers and coupled Burger equations, Journal of Computational and Applied Mathematics, 181 (2005) 245–251.

[2] A. A. Soliman, Numerical simulation of the generalized regularized long wave equation by He’s variational iteration method, Mathematics and Computers in Simulation, 70 -2 (2005) 119–124.

[3] M. A. Abdou , A. A. Soliman, New applications of Variational iteration method, Physica D, 211, (2005) 1–8.

[4] A. A. Soliman, M. A. Abdou, Numerical Solutions of nonlinear evolution equations using variational iteration method, Journal of computational and Applied Mathematics, 207-1, (2007) 111–120.

[5] M. T. Darvishi, F. Khani, A. A. Soliman, The numerical simulation for stiff systems of ordinary differential equations, Computer and Mathematics with Applications, 54, 7-8 (2007) 1055–1063.

[6] A. A. Soliman, On the solution of two-dimensional coupled Burgers’ equations by variational iteration method, Solitons & Fractals Chaos, In Press, (2007)

[7] E. Fan, Soliton solutions for a generalized Hirota-Satsuma coupled KdV equation and a coupled MKdV equation, Phys. Lett. A, 282, (2001) 18–22.

[8] E. J. Parker and B. R. Duffy, An automated tanh-function method for finding solitary wave solutions to nonlinear evolution equations, A Comput. Phys. Commun., (1996) 288–300.

[9] A. A. Soliman, The modified extended tanh-function method for solving Burgers type equations, Physica A, 361-2, (2006) 394–404.
[10] M. A. Abdou, A. A. Soliman, Modified extended tanh-function method and its application on nonlinear physical equation, Physics Letters A, 353, (2006) 487–492.

[11] A. A. Soliman, Exact travelling wave solution of nonlinear variants of the RLW and the PHI-four equations, Physics Letters A, 368-5, (2007) 383–390.

[12] A. H. A. Ali, A. A. Soliman, New Exact Solutions of some Nonlinear Partial Differential Equations, International Journal of Nonlinear Science, 4-3, (2007) 1–11.

[13] A. A. Soliman, A numerical simulation and explicit solutions of KdV-Burgers and Lax-s seventh-order KdV Equations, Chaos Solitons and fractals, 29, (2006) 294–302.

[14] S. A. El-Wakil, S. K. El-Labany, M. A. Zahran, R. Sabr, Modified extended tanh-function method and its applications to nonlinear equations, Appl. Math. & Comput, 161, (2005) 403–412.

[15] S. A. El-Wakil, S. K. El-Labany, M. A. Zahran, R. Sabry, Modified extended tanh-function method for solving nonlinear partial differential equations, Phys. Lett. A, 299, (2002) 179–188.

[16] A. H. A. Ali, A. A. Soliman, K. R. Raslan, Soliton Solution for nonlinear partial differential equations by Cosine-function method, Physics Letters A, 368, (2007) 299–304.

[17] C. T. Yan, A simple transformation for nonlinear waves, Phys. Lett. A., 224, (1996) 77–84.

[18] M. A. Abdou, A. A. Soliman, S. T. El-Basyony, New application of Exp-function method for improved Boussinesq equation, Physics Letters A, 369 5-6 (2007) 469–475.

[19] C. S. Gardner, J. M. Green, M. D. Kruskal, R. M. Miura, Method for Solving the Korteweg-deVries Equation, Phys. Rev. Lett., 19, (1967) 1095–1097.

[20] R. Hirota, Exact Solution of the Korteweg-de Vries Equation for Multiple Collisions of Solitons, Phys. Rev. Lett., 27, (1971) 1192–1194.

[21] M. L. Wang, Exact solutions for a compound KdV-Burgers equation, Phys. Lett. A,213, (1996) 279–287.

[22] Z. Y. Yan, H. Q. Zhang, New explicit solitary wave solutions and periodic wave solutions for Whitham-Broer-Kaup equation in shallow water, Phys. Lett. A, 285, (2001) 355–362.

[23] Z. Y. Yan, New explicit travelling wave solutions for two new integrable coupled nonlinear evolution equations, Phys. Lett. A, 292, (2001) 100–106.
[24] D. H. Peregrine, *Calculations of the development of an undular bore*, J. Fluid Mech, 25, (1966) 321–330.
[25] Kh. O. Abdullove, H. Bogalubsky, V. G. Makhankov, *One more example of inelastic soliton interaction* Phys. Lett. A, 56, (1976) 427–428.
[26] J. C. Eilbek, G. R. McGuire, *Numerical study of the regularized long wave equation II. Interaction of solitary wave*, J. Comp. Phys., 23, (1977) 63–73.
[27] J. L. Bona, W. G. Pritchard, L. R. Scott, *Numerical scheme for a model of nonlinear dispersive waves*, J. Comp. Phys., 60, (1985) 167–176.
[28] M. E. Alexander, J. H. Morris, *Galerkin method for some model equation for nonlinear dispersive waves*, J. Comp. Phys., 30, (1979) 428–451.
[29] L. R. Gardner, G. A. Gardner, *Solitary waves of the regularized long wave equation*, J. Comp. Phys., 91, (1990) 441–459.
[30] L. R. Gardner, G. A. Gardner, A. Dogan, *A least-squares finite element scheme for RLW equation*, Commun. Numer. Meth. Eng., 12, (1996) 795–804.
[31] A. A. Soliman, K. R. Raslan, *Collocation method using quadratic B-spline for the RLW equation*, Int. J. Comput. Math., 78, (2001) 399–412.
[32] A. A. Soliman, M. H. Hussien, *Collocation Solution for RLW Equation with Septic Spline*, Appl. Math. Comp., 161, (2005) 623–636.
[33] A. A. Soliman, *Numerical Scheme based on Similarity Reductions for the regularized Long Wave Equation*, Int. J. Comput. Math., 81, (2004) 1281–1288.
[34] Benjamin RT, Bona JL, Mahony JJ, *Model equations for long waves in dispersive systems*, Philos Trans Roy Soc London, 272, (1972) 47–78.
[35] H. Zhang, *A direct algebraic method applied to obtain complex solutions of some nonlinear partial differential equations*, Chaos, Solitons & Fractals, In Press, (2007)
[36] H. Zhang, *New exact travelling wave solutions to the complex coupled KdV equations and modified KdV equation*, Communications in Nonlinear Science and Numerical Simulation, In Press, (2007)
[37] A. A. Soliman, *The modified extended direct algebraic method for solving nonlinear partial differential equations*, International Journal of Nonlinear Science, Accepted, (2008)
[38] ML Wang, *Solitary wave solutions for variant Boussinesq equations*, Physics Letters A, 199, (1995) 169–172.