GLOBAL $\dot{H}^1 \cap \dot{H}^{-1}$ SOLUTIONS TO A LOGARITHMICALLY
REGULARIZED 2D EULER EQUATION

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Abstract. We construct global $\dot{H}^1 \cap \dot{H}^{-1}$ solutions to a logarithmically mod-
ified 2D Euler vorticity equation. Our main tool is a new logarithm interpolation inequality which exploits the $L^\infty-$conservation of the vorticity.

1. Introduction

The usual 2D Euler equation takes the form

$$\begin{cases}
\partial_t u + (u \cdot \nabla)u + \nabla p = 0, & (t, x) \in \mathbb{R} \times \mathbb{R}^2, \\
\nabla \cdot u = 0, \\
u \big|_{t=0} = u_0,
\end{cases}$$

(1.1)

where $u = (u_1, u_2)$ denotes the velocity and $p$ is the pressure. Introduce the vorticity function $\omega = -\partial_2 u_1 + \partial_1 u_2$. Then in vorticity formulation we have the equation

$$\begin{cases}
\partial_t \omega + u \cdot \nabla \omega = 0, & (t, x) \in \mathbb{R} \times \mathbb{R}^2, \\
u = \nabla^\perp \psi = (-\partial_2 \psi, \partial_1 \psi), & \Delta \psi = \omega, \\
\omega \big|_{t=0} = \omega_0.
\end{cases}$$

(1.2)

Under some suitable regularity assumptions, the second equations in (1.2) can be written as a single equation

$$u = \Delta^{-1} \nabla^\perp \omega,$$

(1.3)

which is the usual Biot–Savart law. We can then rewrite (1.2) more compactly as

$$\partial_t \omega + \Delta^{-1} \nabla^\perp \omega \cdot \nabla \omega = 0.$$

It is well-known that the system (1.1) is globally wellposed in $H^s(\mathbb{R}^2)$ for any $s > 2$. See, for instance, [4, 1]. On the other hand the wellposedness in the borderline space $H^2(\mathbb{R}^2)$ remains unknown. In a similar vein one can consider the wellposedness problem for the vorticity equation (1.2) in the borderline Sobolev spaces. In this case since $\omega = O(\nabla u)$ it is tempting to think that local wellposedness holds in $H^s(\mathbb{R}^2)$ for any $s > 1$. However we should point out that this is not the case due to some low frequency issues introduced by the Biot–Savart relation $u = \Delta^{-1} \nabla^\perp \omega$. In particular under the mere assumption $\omega \in H^s$ the standard contraction argument no longer applies within the pure Lebesgue space framework (see Remark [1, 2] below for more details). To rectify this some amount of negative Sobolev regularity needs to be imposed on the vorticity. For example one can prove wellposedness to (1.2) in the space $\dot{H}^s \cap \dot{H}^{-1}$ or $\dot{H}^s \cap L^p$ for some $s > 1, 1 < p < 2$. Note that by (1.3) the requirement $\omega \in \dot{H}^{-1} \cap \dot{H}^1$ is equivalent to the requirement

$$\omega \in \dot{H}^s \cap \dot{H}^{-1},$$

(1.4)

which is the standard assumption for local existence of solutions to the 3D Navier–Stokes equations.
Thus for the vorticity equation the borderline space should be the space $H^{-1} \cap \dot{H}^1$.

In this paper we consider the following generalized 2D Euler vorticity equation:

$$
\begin{align*}
\partial_t \omega + u \cdot \nabla \omega &= 0, \quad (t, x) \in \mathbb{R} \times \mathbb{R}^2, \\
 u &= \nabla^\perp \psi, \quad \Delta \psi = T_\gamma \omega, \\
 \omega \big|_{t=0} &= \omega_0.
\end{align*}
$$

Here $T_\gamma = T_\gamma(|\nabla|)$ is a Fourier multiplier operator defined by

$$
\hat{T_\gamma \omega}(\xi) = \frac{1}{\log^\gamma(|\xi| + 10)} \hat{\omega}(\xi)
$$

and $\gamma > 0$ is a parameter. This operator introduces some additional logarithmic smoothing of the velocity field through the second equation in (1.4). The system (1.4) is a model case considered in a recent paper by Chae and Wu [2]. Among other results, they obtained the local wellposedness of (1.4) with initial data in the borderline Sobolev spaces when $\gamma > 1/2$. The corresponding global wellposedness remains unknown unless some additional conditions are imposed on the initial data.

Our main result is the following

**Theorem 1.1 (Global wellposedness).** Let $\gamma \geq 3/2$. Assume the initial data $\omega_0 \in H^1(\mathbb{R}^2) \cap \dot{H}^{-1}(\mathbb{R}^2)$. Then there exists a unique corresponding global solution $\omega$ to (1.4) in the space $C([0, \infty), \dot{H}^1(\mathbb{R}^2) \cap \dot{H}^{-1}(\mathbb{R}^2)) \cap C^1([0, \infty), L^2)$.

**Remark 1.2.** We stress that the negative regularity assumption $\omega_0 \in \dot{H}^{-1}(\mathbb{R}^2)$ is essentially needed in Theorem 1.1. In particular it cannot be replaced by $\omega_0 \in L^2(\mathbb{R}^2)$. This is due to a subtle technical issue arising from the contraction argument in the construction of local solutions. To see it, one can consider the task of proving the uniqueness of solutions in the space $C_0([0, \infty), H^1(\mathbb{R}^2) \cap H^{-1}(\mathbb{R}^2)) \cap C^1([0, \infty), L^2)$. Let $\omega_1, \omega_2 \in C_0([0, \infty), H^1(\mathbb{R}^2) \cap H^{-1}(\mathbb{R}^2))$ be two solutions with the same initial data $\omega_0$. Set $\tilde{\omega} = \omega_1 - \omega_2$. Then $\tilde{\omega}$ satisfies the difference equation

$$
\partial_t \tilde{\omega} = -\Delta^{-1} \nabla^\perp T_\gamma \tilde{\omega} \cdot \nabla \omega_1 - \Delta^{-1} \nabla^\perp T_\gamma \omega_2 \cdot \nabla \omega
$$

with zero initial data. To complete the proof of uniqueness one needs to compute the $L^2$-norm of $\tilde{\omega}$ and run a Gronwall in time argument using (1.5). Whilst the second term on the RHS of (1.5) can be easily handled using integration by parts, there is a difficulty in controlling the first term. Namely the advection velocity $\Delta^{-1} \nabla^\perp T_\gamma \tilde{\omega}$ scales like $|\nabla|^{-1} \tilde{\omega}$ in the low frequency regime and we cannot put it in any Lebesgue space using only the assumption $\tilde{\omega} \in H^1$. This is the main reason why we need to introduce some amount of negative regularity on $\omega$. Of course, we can also use the space $\dot{H}^{-\delta}$ for some $0 < \delta \leq 1$ and same results can be proved. However we shall not pursue this generality here.

**Remark 1.3.** Theorem 1.1 also holds in the periodic boundary condition case. In that situation we will consider zero mean periodic flows and the $\dot{H}^1$ regularity is enough to close the estimates. It is possible to generalize our analysis to the critical Sobolev space $W^{1,p}_*(\mathbb{R}^2) \cap W^{-1,p}(\mathbb{R}^2)$ for any $1 < p < \infty$. However we shall not pursue this issue here.

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1. The same problem will appear in the contraction argument.
Remark 1.4. It remains a very interesting question whether the condition $\gamma \geq 3/2$ in Theorem 1.1 can be relaxed. In our argument, this condition is essentially used in the proof of Lemma 2.4.

Notations and Preliminaries.

- For any two quantities $X$ and $Y$, we denote $X \lesssim Y$ if $X \leq CY$ for some harmless constant $C > 0$. Similarly $X \gtrsim Y$ if $X \geq CY$ for some $C > 0$.
- We denote $X \sim Y$ if $X \lesssim Y$ and $Y \lesssim X$. We shall write $X \lesssim Z_1, Z_2, \ldots, Z_k$ if $X \leq CY$ and the constant $C$ depends on the quantities $(Z_1, \ldots, Z_k)$.
- Similarly we define $Z_1, \ldots, Z_k$ and $\sim Z_1, \ldots, Z_k$.
- For any $f$ on $\mathbb{R}^d$, we denote the Fourier transform of $f$ has
  \[ (\mathcal{F}f)(\xi) = \hat{f}(\xi) = \int_{\mathbb{R}^d} f(x)e^{-i\xi \cdot x} \, dx. \]

  The inverse Fourier transform of any $g$ is given by
  \[ (\mathcal{F}^{-1}g)(x) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} g(\xi)e^{ix\cdot\xi} \, d\xi. \]
- For any $1 \leq p \leq \infty$ we use $\|f\|_p$, $\|f\|_{L^p(\mathbb{R}^d)}$, or $\|f\|_{L^p_\xi(\mathbb{R}^d)}$ to denote the Lebesgue norm on $\mathbb{R}^d$. The Sobolev space $H^s(\mathbb{R}^d)$ is defined in the usual way as the completion of $C^\infty_c(\mathbb{R}^d)$ functions under the norm $\|f\|_{H^s} = \|f\|_2 + \|
abla f\|_2$. For any $s \in \mathbb{R}$, we define the homogeneous Sobolev norm
  \[ \|f\|_{H^s} = \left( \int_{\mathbb{R}^d} |\xi|^{2s}|\hat{f}(\xi)|^2 \, d\xi \right)^{\frac{1}{2}}. \]

  For any integer $n \geq 0$ and any open set $U \subset \mathbb{R}^d$, we use the notation $C^n(U)$ to denote functions on $U$ whose $n^{th}$ derivatives are all continuous.
- We will need to use the Littlewood–Paley frequency projection operators. Let $\varphi(\xi)$ be a smooth bump function supported in the ball $|\xi| \leq 2$ and equal to one on the ball $|\xi| \leq 1$. For each dyadic number $N \in 2\mathbb{Z}$ we define the Littlewood–Paley operators
  \[ \dot{P}_N f(\xi) := \varphi(\xi/N)\hat{f}(\xi), \]
  \[ P_{<N} f(\xi) := [1 - \varphi(\xi/N)]\hat{f}(\xi), \]
  \[ \dot{P}_N f(\xi) := [\varphi(\xi/N) - \varphi(2\xi/N)]\hat{f}(\xi). \]

  Similarly we can define $P_{<N}$, $P_{\geq N}$, and $P_{M<\leq N} := P_{\leq N} - P_{\leq M}$, whenever $M$ and $N$ are dyadic numbers.
- We recall the following Bernstein estimates: for any $1 \leq p \leq q \leq \infty$ and dyadic $N > 0$,
  \[ \|P_N f\|_{L^q(\mathbb{R}^d)} \lesssim_d N^{d(\frac{1}{q} - \frac{1}{p})}\|f\|_{L^p(\mathbb{R}^d)}. \]

  Similar inequalities also hold when $P_N$ is replaced by $P_{<N}$ or $P_{\leq N}$.

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2. THE PROOF

We begin with the following simple variant of the inequality (1.6). The main example in mind is the Fourier multiplier

\[ m(\xi) = \frac{1}{\log^\gamma(|\xi| + 10)}. \]

It is not difficult to check that \( m \) satisfies the bound (2.1) below with \( \tilde{m}(N) = \log^{-\gamma}(\frac{N}{8} + 10) \).

**Lemma 2.1.** Let \( m \in C^{d+1}(\mathbb{R}^d \setminus \{0\}) \) and such that for any dyadic \( N > 0 \), there is a constant \( \tilde{m}(N) \) so that

\[ \sup_{N/8 \leq |\xi| \leq 8N} \left| \partial^\alpha \xi m(\xi) \right| \lesssim d \tilde{m}(N) |\alpha|, \quad \forall |\alpha| \leq d + 1. \] (2.1)

Let \( T_m \) be the associated Fourier multiplier operator defined by

\[ \hat{T_m}f(\xi) = m(\xi)\hat{f}(\xi). \]

Then for any dyadic \( N > 0 \), \( 1 \leq q \leq \infty \), we have

\[ \|T_m P_N f\|_q \lesssim_d \tilde{m}(N)\|P_N f\|_q. \]

**Proof of Lemma 2.1.** By inserting a fattened cut-off if necessary we only need to prove

\[ \|T_m P_N f\|_q \lesssim_d \tilde{m}(N)\|f\|_q. \]

By a scaling argument, it suffices to show that the kernel

\[ K(x) = \int_{\mathbb{R}^d} m(N\xi)\phi(\xi)e^{i\xi \cdot x}d\xi \]

is in \( L^1(\mathbb{R}^d) \). Here \( \phi(\xi) = \varphi(\xi) - \varphi(2\xi) \) and \( \varphi \) is the same function used in the definition of the Littlewood–Paley projection operators. Note that \( \phi \) is supported on \( |\xi| \sim 1 \). By (2.1), easy to check that

\[ \max_{|\alpha| \leq d+1} \sup_{\xi \in \mathbb{R}^d} \left| \partial^\alpha \xi \left( m(N\xi)\phi(\xi) \right) \right| \lesssim_d \tilde{m}(N). \]

Clearly then \( x^\alpha K(x) \in L^\infty(\mathbb{R}^d) \) for any \( |\alpha| \leq d + 1 \). Therefore \( K \in L^1 \) and the desired inequality follows from Young’s inequality. \( \square \)

**Lemma 2.2.** For any \( f \in H^1(\mathbb{R}^2) \), we have

\[ \|f\|_p \leq C \cdot \sqrt{p}\|f\|_{H^1}, \quad \forall 2 \leq p < \infty, \] (2.2)

where \( C > 0 \) is an absolute constant.

**Proof of Lemma 2.2.** By Bernstein, obviously

\[ \|P_{<1} f\|_2 \lesssim \|f\|_2. \]
For the non-low frequency piece, we have

\[ \| P_{\geq 1} f \|_p \leq \sum_{j=0}^{\infty} \| P_{2^j} f \|_p \]
\[ \lesssim \sum_{j=0}^{\infty} 2^{-2j/\gamma} 2^j \| P_{2^j} f \|_2 \]
\[ \lesssim \left( \sum_{j=0}^{\infty} 2^{-\frac{2j}{\gamma}} \right)^{\frac{1}{2}} \| f \|_{\dot{H}^1} \]
\[ \lesssim \sqrt{p} \| f \|_{\dot{H}^1}. \]

Remark 2.3. The constant \( \sqrt{p} \) in the inequality (2.2) is essentially sharp up to some logarithm factors (in terms of the dependence on \( p \)). To see this we consider a radial function \( f_p(x) = f_p(r) \) (we abuse slightly the notation here) defined by

\[ f_p(r) = \begin{cases} \sqrt{p}, & r < e^{-p}; \\ \sqrt{-\log r}, & e^{-p} \leq r \leq e^{-1}; \\ \psi(r), & r \geq e^{-1}, \end{cases} \]

where \( \psi \) is a smooth compactly supported function such that \( \psi(e^{-1}) = 1 \). Then easy to calculate that \( \| f_p \|_2 \lesssim 1 \) and \( \| f_p \|_{\dot{H}^1} \lesssim \sqrt{\log p} \). On the other hand \( \| f_p \|_p \gtrsim \sqrt{p} \) so the sharp constant must be \( \gtrsim \sqrt{p}/\log p \).

Below is the key lemma in our proof of Theorem 1.1.

Lemma 2.4. Let \( \gamma \geq \frac{3}{2} \). Then for any \( f \in H^1(\mathbb{R}^2) \), we have

\[ \left\| \left( \nabla \Delta^{-1} \nabla \perp \log^{-\gamma}(|\nabla| + 10) \right) f \right\|_\infty \leq C_1 \cdot \log(\| f \|_{\dot{H}^1} + e) \sup_{2 \leq p < \infty} \frac{\| f \|_p}{\sqrt{p}}, \]  
(2.3)

where \( C_1 \) is an absolute constant.

Remark 2.5. As will become clear from the proof below, one can replace the operator \( \nabla \Delta^{-1} \nabla \perp \) by any Riesz type operator. By Lemma 2.2 we have

\[ \sup_{2 \leq p < \infty} \frac{\| f \|_p}{\sqrt{p}} \lesssim \| f \|_{H^1} \]

so that the RHS of (2.3) is well defined.

Proof of Lemma 2.4. Denote \( T f = \left( \nabla \Delta^{-1} \nabla \perp \log^{-\gamma}(|\nabla| + 10) \right) f \). By Bernstein’s inequality, we have

\[ \| TP_{\leq 2} f \|_\infty \lesssim \| TP_{\leq 2} f \|_2 \lesssim \| f \|_2 \leq \text{RHS of (2.3)}. \]

We only need to control the non-low frequency part of \( f \). Let \( N \) be a dyadic number whose value will be specified later. Now split \( f \) into low and high frequencies. By
Lemma [2.1] we have
\[ \|TP_{\geq f}\|_{\infty} \lesssim \sum_{j=2}^{N} \frac{1}{j^{\gamma}} \|P_{\geq j}f\|_{\infty} + \sum_{j=N+1}^{\infty} \frac{1}{j^{\gamma}} \|P_{\geq j}f\|_{\infty} \]
\[ \lesssim \sum_{j=2}^{N} \frac{1}{j^{\gamma}} \|f\|_{q_j} \cdot 2^{q_j} + \sum_{j=N+1}^{\infty} \frac{1}{j^{\gamma}} \cdot 2^{j} \|P_{\geq j}f\|_{2}. \]

Choosing \( q_j = j \) and using the fact that \( \gamma \geq \frac{3}{2} \), we have
\[ \|TP_{\geq f}\|_{\infty} \lesssim \sum_{j=2}^{N} \frac{1}{j^{\gamma-\frac{3}{2}}} \|f\|_{j} + \left( \sum_{j=N+1}^{\infty} \frac{1}{j^{2\gamma}} \right)^{\frac{1}{2}} \cdot \|f\|_{H^{1}} \]
\[ \lesssim \log N \cdot ( \sup_{2 \leq p < \infty} \frac{\|f\|_{L^{p}}}{\sqrt{p}} ) + N^{-1} \|f\|_{H^{1}}. \]

Now choose \( N \) such that \( N/2 < \|f\|_{H^{1}} + \epsilon \leq N \). The desired inequality (2.3) follows.

We are now ready to complete the

**Proof of Theorem 1.1.** For the sake of completeness, we first sketch the proof of local existence and uniqueness. Start with uniqueness. Let \( T \) be two solutions to (1.4) with the same initial data \( \omega_{0} \). The difference \( \tilde{\omega} = \omega_{1} - \omega_{2} \) then satisfies the equation
\[ \partial_{t} \tilde{\omega} = -\Delta^{-1} \nabla^{\perp} T_{\gamma} \tilde{\omega} \cdot \nabla \omega_{1} - \Delta^{-1} \nabla^{\perp} T_{\gamma} \omega_{2} \cdot \nabla \tilde{\omega} \]
with zero initial data. For \( L^{2} \)-norm, we compute
\[ \partial_{t}(\|\tilde{\omega}\|_{2}^{2}) \lesssim \|\Delta^{-1} \nabla^{\perp} T_{\gamma} \tilde{\omega}\|_{\infty} \|\nabla \omega_{1}\|_{2} \|\tilde{\omega}\|_{2} \]
\[ \lesssim \|\nabla^{-1} \tilde{\omega}\|_{2} \cdot \|\nabla \omega_{1}\|_{2} \cdot \|\tilde{\omega}\|_{2} \]
\[ + \|\nabla \omega_{1}\|_{2} \|\tilde{\omega}\|_{2}^{2}. \] (2.4)

For the \( H^{-1} \)-norm, we have
\[ \partial_{t}(\|\tilde{\omega}\|_{H^{-1}}^{2}) \lesssim \left| \int \left( |\nabla|^{-1} \nabla \cdot (\Delta^{-1} \nabla^{\perp} T_{\gamma} \omega_{1}) \right) |\nabla|^{-1} \tilde{\omega} \right| \right. \]
\[ + \left| \int \left( |\nabla|^{-1} \nabla \cdot (\Delta^{-1} \nabla^{\perp} T_{\gamma} \omega_{2}) \right) |\nabla|^{-1} \tilde{\omega} \right| \]
\[ \lesssim \|\Delta^{-1} \nabla^{\perp} T_{\gamma} \tilde{\omega}\|_{\infty} \|\omega_{1}\|_{2} \|\tilde{\omega}\|_{H^{-1}} \]
\[ + \|\Delta^{-1} \nabla^{\perp} T_{\gamma} \omega_{2}\|_{\infty} \cdot \|\tilde{\omega}\|_{2} \cdot \|\tilde{\omega}\|_{H^{-1}} \]
\[ \lesssim \|\omega_{1}\|_{2} \left( \|\tilde{\omega}\|_{H^{-1}}^{2} + \|\omega_{2}\|_{H^{-1}} \right) \]
\[ + \|\omega_{2}\|_{H^{1} \cap H^{-1}} \|\tilde{\omega}\|_{2} \cdot \|\tilde{\omega}\|_{H^{-1}}. \] (2.5)

Adding together (2.4) and (2.5), we get
\[ \partial_{t}(\|\tilde{\omega}\|_{2}^{2} + \|\tilde{\omega}\|_{H^{-1}}^{2}) \lesssim \|\omega_{1}\|_{2} \|\omega_{2}\|_{H^{-1}}^{2} + \|\tilde{\omega}\|_{H^{-1}}^{2}. \]

A simple Gronwall in time argument then yields \( \tilde{\omega} = 0 \).
Therefore the \( \dot{A} \) log-Gronwall in time argument then yields that
\[
\| \omega \|_{C([0,T], H^1 \cap H^{-1})} < M_1 < \infty.
\]
where \( P_{\leq N} \) is the usual Littlewood–Paley operator. By an ODE argument in
Banach spaces it is easy to check that there exists a unique solution \( \omega^{(N)} \) in
\( C^0([0,T], H^1 \cap H^{-1}) \). Moreover there exists \( T_0 = T_0(\|\omega_0\|_{H^1 \cap H^{-1}}) > 0 \) such that
\[
\sup_{N \geq 0} \| \omega^{(N)} \|_{L^p([0,T], H^1 \cap H^{-1})} \leq M_1 < \infty.
\]
By using a calculation similar to (2.4) – (2.6), it is not difficult to check that \( \omega^{(N)} \)
forms a Cauchy sequence in \( C([0,T_0], L^2 \cap \dot{H}^{-1}) \) and hence admits a unique limit
point \( \omega \). One can then use norm continuity along with weak continuity to show
\( \omega \in C([0,T_0], H^1 \cap \dot{H}^{-1}) \) is the desired local solution. By using (1.4) it is easy to
check that \( \dot{\partial}_t \omega \in C^0([0,T_0], L^2_x) \) and hence \( \omega \in C^0(T_0) \).
Finally we need to show that the local solution \( \omega \) can be continued for all time.
For this, it suffices to control the \( \dot{H}^1 \cap \dot{H}^{-1} \) norm of \( \omega \).

By (1.4), we have for any \( 2 \leq p < \infty \),
\[
\| \omega(t) \|_p \leq \| \omega_0 \|_p, \quad \forall t \geq 0.
\]
By (2.6), Lemma 2.2 and Lemma 2.4 we get
\[
\| \nabla \Delta^{-1} \nabla^\perp T_\gamma \omega(t) \|_\infty \lesssim \log(\| \omega \|_{H^1}) \cdot \sup_{2 \leq p < \infty} \frac{\| \omega_0 \|_p}{\sqrt{p}} \lesssim \log(\| \omega \|_{H^1}) \cdot \| \omega_0 \|_{H^1}.
\]
By (2.6) and an argument similar to (2.4) (one can just take \( \omega_2 = 0 \)), we have
\[
\dot{\partial}_t \left( \| \omega \|_{H^{-1}} \right) \lesssim \| \omega_0 \|_2 \cdot \| \omega \|_{H^{-1}}^2 + \| \omega_0 \|_2^2 \cdot \| \omega \|_{H^{-1}}.
\]
Therefore the \( \dot{H}^{-1} \)-norm of \( \omega \) is controlled for all time.

On the other hand, by (2.7), we have
\[
\dot{\partial}_t \left( \| \omega \|_{H^1} \right) \lesssim \| \nabla \Delta^{-1} \nabla^\perp T_\gamma \omega \|_\infty \cdot \| \omega \|_{H^1} \lesssim \| \omega_0 \|_{H^1} \cdot \log(\| \omega \|_{H^1} + \| \omega_0 \|_2 + e) \cdot \| \omega \|_{H^1}^2.
\]
A log-Gronwall in time argument then yields that \( \| \omega(t) \|_{H^1} \) is bounded for all \( t > 0 \).
This completes the proof the theorem. \( \square \)

\footnote{One can also use a slightly different iteration scheme: \( \dot{\partial}_t \omega^{(k)} + \Delta^{-1} \nabla^\perp T_\gamma \omega^{(k-1)} \cdot \nabla \omega^{(k)} = 0 \). cf. [2].}
REFERENCES

[1] A. Bertozzi and A. Majda, Vorticity and incompressible flow, Cambridge Texts in Applied Mathematics, 27. Cambridge University Press, Cambridge, 2002.

[2] D. Chae and J. Wu, Logarithmically regularized inviscid models in the borderline Sobolev spaces, J. Math. Phys. 53 (2012), no. 11, 115601, 15 pp.

[3] H. Dong and D. Li, On a one-dimensional α-patch model with nonlocal drift and fractional dissipation. [arXiv:1207.0987]

[4] T. Kato, Remarks on the Euler and Navier-Stokes equations in $\mathbb{R}^2$, Proc. Sympos. Pure Math., 45, Part 2, Amer. Math. Soc., Providence, RI, 1986.

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