QUANTUM GEOMETRIC LANGLANDS CORRESPONDENCE IN
POSITIVE CHARACTERISTIC: THE $GL_N$ CASE

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Abstract. Let $C$ be a smooth connected projective curve of genus $> 1$ over an algebraically closed field $k$ of characteristic $p > 0$, and $c \in k \setminus \mathbb{F}_p$. Let $\text{Bun}_N$ be the stack of rank $N$ vector bundles on $C$ and $\mathcal{L}_{\text{det}}$ the line bundle on $\text{Bun}_N$ given by determinant of derived global sections. We construct an equivalence of derived categories of modules for certain localizations of twisted crystalline differential operator algebras $D_{\text{Bun}_N, c\mathcal{L}_{\text{det}}}$ and $D_{\text{Bun}_N, c^{-1}\mathcal{L}_{\text{det}}}$. The first step of the argument is the same as that of [2] for the non-quantum case: based on the Azumaya property of crystalline differential operators, the equivalence is constructed as a twisted version of Fourier–Mukai transform on the Hitchin fibration. However, there are some new ingredients. Along the way we introduce a generalization of $p$-curvature for line bundles with non-flat connections, and construct a Liouville vector field on the space of de Rham local systems on $C$.

1. Introduction

Fix an algebraic curve $C$ and a reductive group $G$. The geometric Langlands correspondence (GLC) is a conjectural equivalence of derived categories between $\mathcal{D}$-modules on the moduli space $\text{Bun}_G$ of $G$-bundles on $C$ and quasi-coherent sheaves on the moduli space $\text{Loc}_{L^G}$ of local systems for the Langlands dual group $L^G$. It has a classical (commutative) limit which amounts to the derived equivalence of Fourier–Mukai type between Hitchin fibrations for $G$ and $L^G$. The latter is a fibration $T^* \text{Bun}_G \to \mathcal{B}$ over an affine space with generic fibers being abelian varieties (or a little more general commutative group stacks).

In [2], a characteristic $p$ version of GLC is established. Namely, the setup of crystalline (i.e. without divided powers) $\mathcal{D}$-modules in positive characteristic is considered. In this setting, the category of $\mathcal{D}$-modules does not get far from its classical limit: it is described by a $\mathbb{G}_m$-gerbe on the Frobenius twist of the cotangent bundle. So the GLC becomes a twisted version of its classical limit. Based on this reasoning, the GLC is constructed “generically” for the case of general linear group $G = GL_N$.

In this paper, we apply the same technique to the quantum version of GLC. This deformation of GLC has the same classical limit, but now both sides are “quantized.” So in characteristic $p$ we get (generically) a twisted version of the same Fourier–Mukai transform. However, the proof that the twistings on both sides are interchanged by the Fourier–Mukai transform is more complicated than in the case of usual GLC, and contains several new ingredients. Also, we restrict to the case of irrational parameter $c$ because there is a certain degeneration happening at rational $c$. 

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First, we need a description of the category of modules for a twisted differential operator (TDO) algebra. The center of a TDO was already described in [3], but the description of the corresponding gerbe presented here seems to be new. A convenient language for this turns out to be that of extended curvature—an invariant of a line bundle with a (not necessarily flat) connection taking values in a canonical coherent sheaf $\mathcal{F}$ on the Frobenius twist of the variety. This is a generalization of the $p$-curvature of flat connections. Just as with the usual $p$-curvature, every section of $\mathcal{F}$ defines a gerbe whose splittings correspond to connections with that extended curvature. Now, if $L$ is a line bundle and $c \in k \setminus \mathbb{F}_p$ then the gerbe describing $D_L c$ corresponds to $c \cdot \alpha_L$ where $\alpha_L$ is the extended curvature of the pullback of $L$ to the associated twisted cotangent, equipped with the canonical ("universal") connection.

We then apply this to the determinant bundle on $\text{Bun}$ whose corresponding twisted cotangent is identified with $\text{Loc}$. So, to construct the desired equivalence, we have to split a gerbe on the fiber product of dual $p$-Hitchin fibrations $\text{Loc} \to B(1)$. (Although for $\text{GL}_N$ the $p$-Hitchin fibrations are the same, we use differently scaled projections to the Hitchin base.) This is done by constructing an explicit line bundle with connection on this fiber product. The problem then reduces to proving certain equality involving $\tilde{\theta} = \alpha_L$ for $L$ being the determinant bundle on $\text{Bun}$, and the torsor structure on the $p$-Hitchin fibration.

We prove this property for another section $\tilde{\theta}_0$ of $\mathcal{F}_{\text{Loc}}$ and then show that $\tilde{\theta} = \tilde{\theta}_0$. The section $\tilde{\theta}_0$ is constructed from a vector field $\xi_0$ on $\text{Loc}$ lifting the differential of the standard $G_m$-action on the Hitchin base. This vector field comes from an action of fiberwise dilations of $T^*C(1)$ on the gerbe describing $\mathcal{D}$-modules. Such structure is not unique (it depends on the choice of lifting of $C$ to the 2nd Witt vectors of $k$), however the corresponding vector fields $\xi_0$ all differ by Hamiltonian vector fields and give rise to the same $\tilde{\theta}_0$.

This version of the text contains only a sketch of some of the arguments (and some proofs are missing). A more detailed exposition will be presented in the next versions of the paper.

1.1. Quantum geometric Langlands conjecture. Let $C$ be a smooth irreducible projective curve of genus $g > 1$ over an algebraically closed field $k$ of characteristic 0 and $G$ a reductive algebraic group. We denote by $\text{Bun}_G = \text{Bun}_G(C)$ the moduli stack of $G$-bundles on $C$. The quantum geometric Langlands correspondence is a conjectural equivalence between certain derived categories of twisted $\mathcal{D}$-modules on $\text{Bun}_G$ and $\text{Bun}_L$ where $L$ denotes the Langlands dual group. The twistings should correspond to invariant bilinear forms on the Lie algebras of $G$ and $L$ that induce dual forms on the Cartan subalgebras (up to the shift by the critical level). When one of the forms tends to 0 the other tends to infinity, which corresponds to degeneration of the TDO algebra into a commutative algebra of functions on a twisted cotangent bundle to $\text{Bun}_G$. This shows that the quantum geometric Langlands is a deformation of the classical geometric Langlands, which is an equivalence between the category of (certain) $\mathcal{D}$-modules on $\text{Bun}_G$ and the category of (certain) quasi-coherent sheaves on the stack $\text{Loc}_L$ of $L$-local systems on $C$.

We will be interested in the case $G = \text{GL}_N$—the general linear group. In this case, we think of the quantum Langlands correspondence as the equivalence $\mathcal{D}_{\text{Bun}_N, \mathcal{L}_{\text{det}}^{-1/\epsilon}} \cong \mathcal{D}_{\text{Bun}_N, \mathcal{L}_{\text{det}}^{1/\epsilon}}$ where $\mathcal{L}_{\text{det}}$ is the determinant line bundle
on \( \text{Bun}_N = \text{Bun}_G = \text{Bun}_{\mathcal{L}} \) given by \((\mathcal{L}_{\text{det}})_b = \det \Gamma(C, \mathcal{E}_b)\) for any \( b \in \text{Bun}_N \) where \( \mathcal{E}_b \) denotes the rank \( N \) vector bundle corresponding to \( b \). (There is a subtle question of what kind of \( \mathcal{D} \)-modules one should consider, but we’ll ignore it for now.)

1.2. The characteristic \( p \) case: classical story. In [2], R. Bezrukavnikov and A. Braverman established a version of the classical geometric Langlands correspondence for “crystalline” \( \mathcal{D} \)-modules over a field \( k \) of characteristic \( p > 0 \). Recall that, for a smooth scheme \( X \) over \( k \), the sheaf \( \mathcal{D}_X \) of crystalline differential operators is defined as the universal enveloping algebra of the Lie algebroid \( T_X \) of vector fields on \( X \). The main tool for studying modules over such algebras is their Azumaya property (see [3]). Namely, \( \mathcal{D}_X \) turns out to be isomorphic to (the pushforward to \( X \) of) an Azumaya algebra \( \tilde{\mathcal{D}}_X \) on \( T^*X^{(1)} \) — the cotangent bundle to the Frobenius twist of \( X \). This allows one to identify the category of \( \mathcal{D} \)-modules on \( X \) with the category of coherent sheaves on a \( \mathbb{G}_m \)-gerbe on \( T^*X^{(1)} \).

This observation is generalized in [2] to the case of (a certain class of) algebraic stacks. Namely, for an irreducible smooth Artin stack \( \mathcal{Y} \) over \( k \) with \( \dim T^*\mathcal{Y} = 2 \dim \mathcal{Y} \) (i.e. \( \mathcal{Y} \) is good in the sense of [1]), they construct a sheaf \( \mathcal{D}_{\mathcal{Y}} \) of algebras on \( T^*\mathcal{Y}^{(1)} \) with properties similar to the Azumaya algebra \( \mathcal{D}_X \) described above. The pushforward of \( \mathcal{D}_{\mathcal{Y}} \) to \( \mathcal{Y}^{(1)} \) is isomorphic to \( \text{Fr}_{\mathcal{Y}^{(1)}} \mathcal{D}_{\mathcal{Y}} \) where \( \mathcal{D}_{\mathcal{Y}} \) is the sheaf of differential operators as defined in [1]. Moreover, the restriction of \( \mathcal{D}_{\mathcal{Y}} \) to the maximal open smooth Deligne–Mumford substack of \( T^*\mathcal{Y}^{(1)} \) is an Azumaya algebra.

The stack \( \text{Bun}_N \) is almost “good,” namely, it locally looks like product of a good stack and \( B\mathbb{G}_m \). So one can apply the above construction to \( \mathcal{Y} = \text{Bun}_N \) to get an Azumaya algebra on \( T^* \text{Bun}_N^{(1)} = \text{Higgs}^{(1)} \). The latter stack is the total space of the Hitchin fibration \( \chi^{(1)} : \text{Higgs}^{(1)} \to \mathcal{B}^{(1)} \) whose generic fibers are Picard stacks of (spectral) curves. On the dual side, one has the “\( p \)-Hitchin” map \( \text{Loc} \to \mathcal{B}^{(1)} \) given by \( p \)-curvature. Generic fibers of this map are torsors over the same Picard stacks, and each point of such a torsor (which corresponds to \( G \)-local system on \( C \) with given spectral curve) gives a splitting of \( \mathcal{D}_{\text{Bun}} \) on the corresponding fiber of \( \chi^{(1)} \). This splitting defines a Hecke eigensheaf corresponding to the local system. The geometric Langlands is thus realized as a twisted version of Fourier–Mukai transform.

1.3. Quantum story. In this paper, the same ideas are applied to quantum geometric Langlands correspondence. To that end, we generalize the above Azumaya algebra construction to the case of twisted differential operators. The only TDO algebras we will encounter are of the form \( \mathcal{D}_{\mathcal{L}} \) where \( \mathcal{L} \) is a line bundle and \( c \in k \) (and external tensor products of such). In this case, the situation is essentially analogous to the non-twisted case, except that the Azumaya algebra will now live on the twisted cotangent bundle, where the twisting is given by \((c^p - c)\) times the Chern class of \( \mathcal{L}^{(1)} \) (cf. [3]). We will only consider the case of irrational \( c \) (i.e. \( c \not\in \mathbb{F}_p \)): in this case one can identify this twisted cotangent bundle with the Frobenius twist of the space \( T^*_C X \) of \( 0 \)-jets of connections on \( \mathcal{L} \). This is discussed in [2.3].

It is not hard to extend it to the stack case using the above-mentioned results from [2] for usual \( \mathcal{D} \)-modules on stacks. Thus, for a line bundle \( \mathcal{L} \) on a good stack \( \mathcal{Y} \), one gets an Azumaya algebra \( \mathcal{D}_{\mathcal{Y}, \mathcal{L}} \) on the smooth part of \( (T^*_C \mathcal{Y})^{(1)} \). (For a discussion of twisted cotangent bundles to stacks, see [A.1].)
1.3.1. Main result. We apply this to the determinant bundle $L_{\det}$ on $\text{Bun}$. One can check (see Appendix A) that the corresponding twisted cotangent is identified with the moduli space $\text{Loc}_{c/2}$ of rank $N$ bundles on $C$ endowed with an action of the TDO algebra $D_{c/2}$. In fact, we can identify $\text{Loc}_{c/2}$ with Loc by tensoring bundles with $\mathcal{O}^{(p-1)/2}$. Thus, both sides of the quantum Langlands are described (again, generically over the Hitchin base) by certain gerbes on $(\text{Loc}^{(1)})$. Here $\text{Loc}^{(1)} = \text{Loc} \times \mathbb{G}(1) \mathcal{B}^{(1)}$ and $\mathcal{B}^{(1)} \subset \mathcal{B}$ is the open part parametrizing smooth spectral curves. Using the $p$-Hitchin map as above (this time to $\mathcal{B}^{(2)}$), we see that these gerbes live on two torsors over the relative Picard stack mentioned above. So we get again two “twisted versions” of the derived category of coherent sheaves on this Picard stack. In contrast to the classical (non-quantum) case, however, we have both “torsor” and “gerby” twists on each side. These two kinds of twists are interchanged by Fourier–Mukai duality.

In other words, we prove the following:

\textbf{Theorem 1.} There is an equivalence between bounded derived categories of modules for $D_c = D_{\text{Bun},c/2}((\text{Loc}^{(1)})$ and $D_{-1/c} = D_{\text{Bun},c/2}((\text{Loc}^{(1)})$. The corresponding kernel is a splitting of $\mathcal{D} \otimes D_{c}^{op} \sim \mathcal{D} \otimes D_{-1/c}$ on the fiber product $(\text{Loc}^{(1)}) \times \mathbb{G}(1) \mathcal{B}^{(1)}$ (Loc$^{(1)}$) where the projection from the second factor to $\mathcal{B}^{(2)}$ is modified by the action of $\mathbb{G} \in \mathbb{G}_m$. If we choose, locally on $\mathcal{B}^{(2)}$, a trivialization of the torsor $\text{Loc}^{(1)} \rightarrow \mathcal{B}^{(2)}$, then there are splittings of $\mathcal{D}_c$ and $D_{-1/c}$ such that the equivalence is identified with the Fourier–Mukai transform on the Picard stack $\text{Pic}(\mathcal{B}^{(1)}/\mathcal{B}^{(2)})$. (Here $\mathcal{B}^{(1)} \subset T^*C \times \mathcal{B}^{(1)}$ is the universal spectral curve.)

Note that, although the underlying spaces of the torsors are the same on both sides (namely $(\text{Loc}^{(1)})$), in order to make the duality work, one has to normalize the projection to $\mathcal{B}^{(2)}$ differently. This can also be guessed by considering what happens at rational $c$ (including $c = 0, \infty$).

1.3.2. Extended curvature. So, all we need to check is that the torsors with gerbes corresponding to $D_c$ and $D_{-1/c}$ are interchanged by Fourier–Mukai duality. For that purpose we need a description of gerbes attached to TDO algebras. Recall that in the non-twisted case, the splittings of $D_X$ on an open subset $U^{(1)} \subset T^*X^{(1)}$ correspond to line bundles on $U$ with flat connection of $p$-curvature equal to the canonical 1-form on $T^*X^{(1)}$.

To extend this description to the TDO case, we introduce a generalization of the notion of $p$-curvature to non-flat connections. For a line bundle $L$ with connection $\nabla$ on a smooth variety $X$, we define in [2, A] a section $\text{curv}(L, \nabla)$ (called the extended curvature) of the quotient sheaf $\mathcal{F}_X$ of $\Omega_X^2$ by locally exact forms. This sheaf maps to $\Omega_X^2$ via the de Rham differential; this map carries $\text{curv}(L, \nabla)$ to the usual curvature. On the other hand, for flat connections, $\text{curv}(L, \nabla)$ is a section of closed module exact forms, which corresponds to the $p$-curvature of $\nabla$ under Cartier isomorphism. This construction also allows, starting from a section $\alpha \in \mathcal{F}_X$ (such as a section which sometimes be referred to as a generalized one-form), to define a $\mathbb{G}_m$-gerbe on $X^{(1)}$: its splittings correspond to line bundles with connection whose extended curvature is equal to $\alpha$.

Now, the pullback of any line bundle $L$ to its associated twisted cotangent $T^*_c X$ acquires a canonical connection. If $\alpha_L$ denotes the extended curvature of this
connection, the gerbe on \((\tilde{T}_X^*X)^{(1)}\) corresponding to the Azumaya algebra \(\tilde{D}_c\) for \(c \in k \setminus \mathbb{F}_p\) is obtained from the above construction applied to \(c\alpha_c\).

1.3.3. The Poincaré bundle. Then we construct an explicit kernel of the equivalence (an analogue of the Poincaré bundle). This is a line bundle with connection on the fiber product of two copies of \(\text{Loc}^0\) over the Hitchin base (see formula (13)). The construction is similar to that of Poincaré bundle on the square of the Picard stack of a curve:

\[
\text{Poincaré}(\mathcal{L}, \mathcal{L}') = \det R\Gamma(\mathcal{L} \otimes \mathcal{L}') \otimes (\det R\Gamma(\mathcal{L}) \otimes \det R\Gamma(\mathcal{L}'))^{\otimes -1}.
\]

Namely, the determinant bundle on the Picard stack gets replaced by the determinant bundle on \(\text{Loc}^0\) with “tautological” connection (the same one that is used to describe the gerbe on \((\text{Loc}^0)^{(1)}\)), while the role of tensor product of line bundles is played by the addition map on the fibers of the \(p\)-Hitchin map:

\[
\text{Loc}^0_1 \times_{\mathcal{G}(1)} \text{Loc}_c^0 \rightarrow \text{Loc}^0_{1+c}.
\]

Here subscripts indicate scaling of the projection to the Hitchin base. The fiber of \(\text{Loc}^0_1\) classifies splittings on the spectral curve of the gerbe corresponding to the canonical 1-form on \(T^*C^{(1)}\) multiplied by \(c\). This map can then be thought of as “tensoring over the spectral curve.”

The main difficulty is then to check that this bundle with connection has the correct \(p\)-curvature. This reduces to a certain linear equality on the extended curvatures (formula (15)). This formula can be interpreted as a kind of additivity of the generalized one-form \(c^{-1}\tilde{\theta}\) on \(\text{Loc}^0_c\) with respect to the addition maps above, where \(\tilde{\theta}\) denotes the extended curvature of the tautological connection on the determinant bundle.

1.3.4. Antiderivative of the symplectic form on Loc. In [31] we construct another generalized one-form \(\tilde{\theta}_0\) on \(\text{Loc}^0\) (actually on the maximal smooth part of Loc) whose image in \(\Omega^2\) coincides with that of \(\tilde{\theta}\) (both are equal to the symplectic form on \(\text{Loc}^0\)) but whose behavior with respect to the \(p\)-Hitchin map is more controllable. We prove the additivity property for it, and then show that \(\tilde{\theta} = \tilde{\theta}_0\). In fact, \(\tilde{\theta}_0\) lifts to an actual antiderivative \(\theta_0\) of the symplectic form. Such antiderivatives correspond bijectively to Liouville vector fields. We construct such a vector field using an equivariant structure of the gerbe on \(T^*C^{(1)}\) under the Euler vector field. Such structures correspond to liftings of \(C\) to the 2nd Witt vectors of \(k\). Since \(\tilde{\theta} - \tilde{\theta}_0\) is closed, it corresponds by Cartier to a 1-form \(\beta_0\) on \((\text{Loc}^0)^{(1)}\) and we have to prove that it is 0.

The definition of \(\text{Loc}_c\) above makes sense for all \(c \in k\); in particular, for \(c = 0\) it gives \(\text{Loc}_0 = \text{Higgs}^{(1)}\). On \(\text{Higgs}^{(1)}\) we have the canonical 1-form \(\theta_{\text{Higgs}}^{(1)}\) (as on a cotangent bundle). We prove that both \(\tilde{\theta}\) and \(\theta_0\) are compatible with \(\theta_{\text{Higgs}}^{(1)}\) with respect to the action map

\[
\text{Loc}^0_0 \times_{\mathcal{G}(1)} \text{Loc}^0 \rightarrow \text{Loc}^0.
\]

In the beginning of Section [3] we prove this for \(\tilde{\theta}\). It is enough to prove it on the image of the Abel-Jacobi map in Higgs, which in turn reduces to studying how the determinant bundle (with connection) on \(\text{Loc}^0\) changes when we twist the local system by a point of its \(p\)-spectral curve. For \(\theta_0\) this property this is proved as part
of the additivity for $\theta_0$. (In fact, the additive family of 1-forms on $\text{Loc}_c^0$ constructed in [3.1] specializes to $\theta_0$ for $c = 1$ and to $\theta^{(1)}_{\text{Higgs}}$ for $c = 0$.)

From this we conclude that $\beta_0$ descends to the Hitchin base. On the other hand, in [3.2] we study the behavior of $\beta_0$ with respect to the projection $\text{Loc} \to \text{Bun}$. First, by a degree estimate we show that the restriction to the fibers of this projection have constant coefficients. Then, a global argument shows that in fact this restriction must be 0. The fibers of the two projections $\text{Loc} \to \mathcal{B}^{(1)}$ and $\text{Loc} \to \text{Bun}$ are generically transversal (at least, we know how to prove this for one of the components of $\text{Loc}$ assuming $C$ is ordinary), which gives the desired equality $\beta_0 = 0$.

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Table of notation:

| Symbol | Description |
|--------|-------------|
| $k$    | an algebraically closed field of characteristic $p > 0$ |
| $G$    | the general linear group $\text{GL}(N)$ |
| $C$    | a complete smooth algebraic curve over $k$ |
| $\text{Bun}$ | the moduli stack of $G$-bundles on $C$ |
| $\mathcal{B}$ | the Hitchin base (the affine space $\bigoplus_{i=1}^N H^0(C, \omega_C^i)$) |
| $\mathcal{B}^0 \subset \mathcal{B}$ | the open part classifying smooth spectral curves. |
| $\text{Higgs}$ | the total space of the Hitchin fibration, the moduli stack of Higgs bundles, $\text{Higgs} = T^* \text{Bun}$ |
| $\tilde{\mathcal{C}}$ | “universal spectral curve,” $\tilde{\mathcal{C}} \subset T^*C \times \mathcal{B}$ |

2. Generalities

2.1. Frobenius morphisms and twists. For any scheme $S$ of characteristic $p$ (i.e., such that $p\mathcal{O}_S = 0$) the absolute Frobenius $\text{Fr}_{S/\mathbb{F}_p}: S \to S$ is defined as $\text{id}_S$ on the topological space and $\text{Fr}_{S/\mathbb{F}_p}^\#(f) = f^p$ on functions. For any $S$-scheme $X \to S$ one constructs a commutative diagram

\[
\begin{array}{ccc}
X & \xrightarrow{\pi} & \pi^{-1}(S) \\
\downarrow{\text{Fr}_{X/S}} & & \downarrow{\pi} \\
X^{(S)} & \xrightarrow{\pi^{(S)}} & \pi^{-1}(S) \\
\end{array}
\]

where the square is Cartesian. We call the $S$-scheme $X^{(S)} \xrightarrow{\pi^{(S)}} S$ the relative Frobenius twist of $X$ over $S$, and call $\text{Fr}_{X/S}$ the relative Frobenius morphism. We will denote by $\bullet^{(S)}$ the pullback by $\text{Fr}_{S/\mathbb{F}_p}$ or $W_{X/S}$. In the case $S = \text{Spec} \ k$ we will drop “$S$” and write $\text{Fr}_X$ and $X^{(1)}$ instead of $\text{Fr}_{X/S}$ and $X^{(S)}$. The $k$'th iterate of $\bullet^{(1)}$ will be denoted $\bullet^{(k)}$. 
2.2. $\lambda$-connections. Recall that a $\lambda$-connection on a vector bundle $E$ on a smooth variety $X$ is a $k$-linear morphism of sheaves $\nabla: \mathcal{E} \to \Omega^1 \otimes_\mathcal{O} \mathcal{E}$ such that

$$\forall f \in \mathcal{O} \quad \forall s \in \mathcal{E} \quad \nabla(fs) = f \cdot \nabla s + \lambda \cdot df \otimes s$$

where $\mathcal{E}$ is the sheaf of sections of $E$.

Define the curvature of a $\lambda$-connection $\nabla$ to be the section $F_{\nabla}$ of $\Omega^2 \otimes \mathcal{E}$ corresponding to the $\mathcal{O}$-linear map $\nabla^2: \mathcal{E} \to \Omega^2 \otimes \mathcal{E}$ where $\nabla$ is extended to $\Omega^* \otimes \mathcal{E}$ by the following “Leibnitz rule”:

$$\nabla(\omega \otimes s) = (-1)^{\deg \omega} \omega \wedge \nabla s + \lambda \cdot d\omega \otimes s.$$ 

An alternative definition of $F_{\nabla}$ is that for any $\xi, \eta \in \text{Vect}$ we must have

$$F_{\nabla}(\xi, \eta) = [\nabla_\xi, \nabla_\eta] - \lambda \cdot \nabla_{[\xi, \eta]}.$$

If $F_{\nabla} = 0$, we say that $\nabla$ is flat.

For $\lambda \neq 0$ if $\nabla$ is a $\lambda$-connection then $\lambda^{-1} \nabla$ is a connection, and vice versa. In this case, the curvature of a $\lambda$-connection can be expressed in terms of the ordinary curvature: $F_{\nabla} = \lambda^2 F_{\nabla -1}$. The case $\lambda = 0$ can be thought of as a limit when $\lambda \to 0$. In particular if $\nabla$ is trivialized, and $\nabla = \lambda d + \theta$ then $F_{\nabla} = \lambda d\theta + \theta \wedge \theta$.

Vector bundles with a flat $\lambda$-connection correspond to $\mathcal{O}$-flat $\mathcal{O}$-coherent modules over the algebra $\mathcal{D}_\lambda$ which is the universal enveloping algebra of the Lie algebroid $\text{Vect}_\lambda = \text{Vect}$ over $\mathcal{O}$ with rescaled commutator: $[\xi, \eta]_\lambda = \lambda [\xi, \eta]$. There is an inclusion $\mathcal{O}_{T^*X(1)} \to Z(\mathcal{D}_\lambda)$ (the center of $\mathcal{D}_\lambda$) which is an isomorphism for $\lambda \neq 0$. It is given by $f^{(1)} \mapsto f^\theta$, $\xi^{(1)} \mapsto \xi^p - \lambda \xi^{p-1} \xi^{[p]}$ where $\xi$ in the LHS is regarded as a fiberwise linear function on $T^*X$, and $\xi$ in the RHS is the corresponding element in $\mathcal{D}_\lambda$. For $\lambda = 0$ the inclusion is just the Frobenius map $Fr^*: \mathcal{O}_{T^*X(1)} \to \mathcal{O}_{T^*X}$.

We can then define the $p$-spectral variety of a $\lambda$-connection $\nabla$ on a vector bundle $E$ as the support of the corresponding $\mathcal{D}_\lambda$-module regarded as an $\mathcal{O}_{T^*X(1)}$-module. By the $p$-curvature map of $\nabla$ we will mean the map $\text{curv}_p(\nabla): \mathcal{E} \to \mathcal{E} \otimes \text{Fr}^*_X \Omega^1_{X(1)}$ coming from the action of $\mathcal{O}_{T^*X(1)}$ on $\mathcal{E}$. For $\lambda \neq 0$ it is related to the ordinary $p$-curvature by $\text{curv}_p(\nabla) = \lambda^p \text{curv}_p(\lambda^{-1} \nabla)$ (here $\lambda^{-1} \nabla$ is a usual connection).

For a line bundle $\mathcal{L}$ on $X$ and $\lambda \in k$, define a torsor $\mathcal{C}on\mathcal{L}_\lambda(\mathcal{L})$ over $T^*X$ whose sections are $\lambda$-connections on $\mathcal{L}$.

Remark 1. As a variety, $\mathcal{C}on\mathcal{L}_\lambda(\mathcal{L})$ is isomorphic to $\mathcal{H}^2_k(X) = \mathcal{C}on\mathcal{L}(\mathcal{L}) := \mathcal{C}on\mathcal{L}_1(\mathcal{L})$ for $\lambda \neq 0$ and to $T^*X$ for $\lambda = 0$.

2.3. Twisted differential operators. If $X$ is a smooth algebraic variety and $\mathcal{L}$ is a line bundle on it, we define a sheaf of algebras $\mathcal{D}_{\mathcal{L}}(X)$ for any $c \in k$ as follows: For any local trivialization $\phi: \mathcal{L}|_U \sim \mathcal{O}|_U$ of $\mathcal{L}$ on an open set $U$, we have a canonical isomorphism $\alpha_\phi: \mathcal{D}_{\mathcal{L}}(U) \sim \mathcal{D}(U)$, and if $\phi'$ on $U'$ is another trivialization then the gluing isomorphism $\alpha_{\phi'} \circ \alpha^{-1}_\phi$ is given by

$$\begin{cases} f \mapsto f & \text{for } f \in \mathcal{O}, \\ \xi \mapsto \xi + c\xi(h)/h & \text{for } \xi \in \text{Vect} \end{cases}$$

where $h \in \mathcal{O}^*(U \cap U')$ is s. t. $(\phi')^{-1} \circ \phi: \mathcal{O}(U \cap U') \to \mathcal{O}(U \cap U')$ is given by multiplication by $h$. 
**Proposition 2.** Denote by $Y$ the relative spectrum of $\mathcal{D}_{\mathcal{L}^1} : Y = \text{Spec}_X \mathcal{D}_{\mathcal{L}^1}$. Then $Y$ is canonically isomorphic to $\mathcal{C}_{\text{Conn}_{c-[c]}(\mathcal{L}^{(1)})}$ (as an $X$-scheme). Moreover, $\mathcal{D}_{\mathcal{L}^1}$ is a pushforward of an Azumaya algebra $\mathcal{D}_{\mathcal{L}^1}$ on $Y$.

2.3.1. TDO with “Planck’s constant”. Suppose $X, \mathcal{L}$ are as in 2.3. Define the $k[c, h]$-algebra $\mathcal{D}_{c, h}^\mathcal{L}(X)$ as follows. Let $\pi : \tilde{X} \to X$ be the principal $\mathbb{G}_m$-bundle corresponding to $\mathcal{L}$. Denote by $\mathcal{D}_h(\tilde{X})$ the algebra of “differential operators with parameter,” that is, the algebra

$$\mathcal{D}_h(\tilde{X}) := \bigoplus_{n \geq 0} \mathcal{D}^{\leq n}(\tilde{X})$$

over $k[h]$ where the inclusion $k[h] \hookrightarrow \mathcal{D}_h(\tilde{X})$ is given by $h \mapsto 1 \in \mathcal{D}^{\leq 1}(\tilde{X})$. Here we introduce its TDO analog. For $\xi \in \text{Vect}_{\tilde{X}}$ let $\xi$ be the corresponding element in $\mathcal{D}^{\leq 1} \subset \mathcal{D}_h$. Let $\text{Eu}$ be the Euler vector field on $\tilde{X}$ (the differential of the $\mathbb{G}_m$-action). Now set

$$\mathcal{D}_{c, h}^\mathcal{L}(X) := (\pi_* \mathcal{D}_h(\tilde{X}))^{\mathbb{G}_m}.$$ 

This is a $k[c, h]$-algebra via $h \mapsto h \in \mathcal{D}_h(\tilde{X})$, $c \mapsto \text{Eu}$. Note also that $\pi_* \mathcal{D}_h(\tilde{X})$ has two gradings: one comes from the definition of $\mathcal{D}_h$ as a direct sum, and the other comes from the $\mathbb{G}_m$-action on $\tilde{X}$. But on the $\mathbb{G}_m$-invariant part, we have only one grading (the first one), and with respect to this grading $c = \deg h = 1$. The algebra $\mathcal{D}_{c, h}^\mathcal{L}$ being graded implies that it carries an action of $\mathbb{G}_m$ and, in particular, if $\mathcal{D}_{c_0, h_0}^\mathcal{L}$ denotes the specialization $c \mapsto c_0$, $h \mapsto h_0$ of the algebra $\mathcal{D}_{c, h}^\mathcal{L}$ (where $c_0, h_0 \in k$) then

$$\mathcal{D}_{c_0, h_0}^\mathcal{L} \cong \mathcal{D}_{c_0, h_0}^\mathcal{L}$$

for any $\lambda \in k^\times$.

The specialization $c \mapsto 0$ gives the algebra $\mathcal{D}_h$ defined above, and in particular, $\mathcal{D}_{0, 0}^\mathcal{L} = (\text{pr}_X)_* \mathcal{O}_{T^* X}$ where $\text{pr}_X : T^* X \to X$. Furthermore, it is not hard to show that $\mathcal{D}_{c_0, h_0}^\mathcal{L} = (\text{pr}_X)_* \mathcal{O}_{\text{Conn}_{c_0}^\mathcal{L}}$ (where $\text{pr}_X$ is again the appropriate projection to $X$). One can also check that specialization $h \mapsto 1$ recovers the algebra $\mathcal{D}_{\mathcal{L}^1}$ from 2.3. Taking the isomorphism 1 into account, we can summarize:

$$\mathcal{D}_{c_0, h_0}^\mathcal{L} \cong \begin{cases} 
(\text{pr}_X)_* \mathcal{O}_{T^* X} & \text{if } c_0 = h_0 = 0; \\
(\text{pr}_X')_* \mathcal{O}_{\text{Conn}^\mathcal{L}} & \text{if } c_0 \neq 0, \ h_0 = 0; \\
\mathcal{D}_{\mathcal{L}^1/c_0, h_0} & \text{if } h_0 \neq 0.
\end{cases}$$

The following theorem is a generalization of Proposition 2.

**Theorem 2.**

1. The center of the algebra $\mathcal{D}_{c, h}^\mathcal{L}(X)$ is canonically isomorphic to $\mathcal{O}_X$ where $\mathcal{X} = \mathcal{C}_{\text{Conn}_{c-[c]}(\mathcal{L}^{(1)})}$. 

2. Moreover, if $c_0, h_0 \in k$, the specialization $c \mapsto c_0$, $h \mapsto h_0$ induces a map $\mathcal{O}_{\mathcal{X}_{c_0, h_0}} \to Z(\mathcal{D}_{c_0, h_0})^\mathcal{L}$ which is an isomorphism if and only if $h_0 \neq 0$, in which case $\mathcal{D}_{c_0, h_0}^\mathcal{L}$ is an Azumaya algebra over $\mathcal{X}_{c_0, h_0}$.

3. The isomorphism 1 is compatible with the $\mathbb{G}_m$-action on $\mathcal{X}$ given by scaling connections by $\lambda^p$. 

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1 $\mathcal{X}$ is a scheme over $k[c, h]$. One should extend the definition of $\mathcal{C}_{\text{Conn}_{\lambda}}$ to the case of families over an arbitrary scheme in order to make sense of the definition of $\mathcal{X}$. 

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2.3.2. Central reductions. Suppose \( X, \mathcal{L} \) are as above, \( c, h \in k, h \neq 0 \). Then any \((c^p - ch^{p-1})\)-connection \( \nabla_0 \) on \( \mathcal{L}^{(1)} \) gives a section of the bundle \( \mathcal{Z}_{c,h} \) over \( X^{(1)} \) defined above (we’ve changed notation from \( c_0, h_0 \) to \( c, h \)). Denote by \( \mathcal{D}_{c,h}^{\mathcal{L}, \nabla_0} \) the pullback of \( \mathcal{D}_{c,h} \) to this section. It is an Azumaya algebra on \( X^{(1)} \). If \( h = 1 \) and \( c \notin \mathbb{F}_p \) then \( \mathcal{O} \)-coherent modules over \( \mathcal{D}_{c,h}^{\mathcal{L}, \nabla_0} \) correspond to pairs \((\mathcal{E}, \nabla)\) where \( \mathcal{E} \) is a vector bundle on \( X \) and \( \nabla \) is a connection on \( \mathcal{E} \) such that

\[
F_{\mathcal{V}} = c \cdot F_{\nabla_{\mathcal{V}}};
\]

\[
\nabla^p_\xi - \nabla^{(p)}_{\xi[\mathcal{V}]} = c \cdot \left( (\nabla'_1)^p - (\nabla'_0)^{p}(\nabla'_0)^{p} \right)
\]

for any \( \xi \in \text{Vect} \), where \( \nabla_0 \) is a (usual) connection on \( X \) such that \( \nabla_0 = (c^p - c)\nabla_0^{(1)} \). (The operators in the RHS are always multiplication by a function \( F \) (resp. a two-form), and we want the LHS to be multiplication by the same function (resp. two-form), though on a different bundle.)

More generally, for arbitrary \( c, h \in k \), to describe what \( \mathcal{D}_{c,h}^{\mathcal{L}, \nabla_0} \)-modules are, choose any connection \( \nabla_1 \) on \( \mathcal{L} \). Then \( \mathcal{D}_{c,h}^{\mathcal{L}, \nabla_0} \)-modules \((\nabla_0, \nabla)\) is a \((c^p - ch^{p-1})\)-connection) corresponds to pairs \((\mathcal{E}, \nabla)\) where \( \mathcal{E} \) is a bundle (or a quasi-coherent sheaf) and \( \nabla \) is an \( h \)-connection on \( \mathcal{E} \) satisfying

\[
F_{\mathcal{V}} = ch \cdot F_{\nabla_{\mathcal{V}}};
\]

\[
\nabla^p_\xi - h^{p-1}\nabla^{(p)}_{\xi[\mathcal{V}]} = ch^{p-1} \cdot \left( (\nabla'_1)^p - (\nabla'_0)^{p}(\nabla'_0)^{p} \right) - Fr_X^*(\alpha, \xi^{(1)});
\]

for any \( \xi \in \text{Vect} \), where \( \alpha = \nabla_0 - (c^p - ch^{p-1})\nabla^{(1)}_1 \) is a 1-form on \( X^{(1)} \). The formulas from previous paragraph can be obtained by substitution \( \alpha = 0 \). If \( \nabla_1 \) is replaced by \( \nabla_1 + \beta \) then \( \nabla \) is replaced by \( \nabla + c\beta \). If \( \nabla_1 \) is flat then the RHS of \( \xi \) can be expressed as \( Fr_X^*(\alpha, \xi^{(1)}) \) where \( \alpha = \text{curv}_p(\nabla_1) \), so the condition \( \xi \) is equivalent to the connection \( \nabla_1 \) being flat and having \( p \)-curvature \( ch^{p-1}\theta - \alpha \). In particular, if

\[
\nabla_0 = (c^p - ch^{p-1})\nabla_1^{(1)} + ch^{p-1}\theta
\]

then \( ch^{p-1}\theta - \alpha = 0 \) and therefore the algebra \( \mathcal{D}_{c,h}^{\mathcal{L}, \nabla_0} \) canonically splits.

We will denote by \( \mathcal{D}_{\mathcal{L}, \nabla_0} \) the specialization of \( \mathcal{D}_{c,h}^{\mathcal{L}, \nabla_0} \) at \( h = 1 \). It is possible to reformulate the conditions \( (2) \) and \( (3) \) in a more invariant way using the following construction (we will do it only for \( h = 1 \)).

2.4. Extended curvature. Let \( X \) be a smooth variety. Define a (coherent) sheaf \( \mathcal{F}_X \) of \( \mathcal{O}_X \)-modules (= coherent sheaf on \( X^{(1)} \)) by the exact sequence

\[
0 \to \mathcal{O}_{X^{(1)}}^{p} \xrightarrow{Fr^*} \mathcal{O}_X \xrightarrow{d} \Omega_{X}^{1} \xrightarrow{\delta} \mathcal{F}_X \to 0.
\]

Then we also have an exact sequence

\[
0 \to \Omega_{X^{(1)}}^{1} \to \mathcal{F}_X \xrightarrow{Q} \Omega_{X,cl}^{2} \xrightarrow{C} \Omega_{X^{(1)}}^{2} \to 0
\]

where \( P \) is induced by (the inverse of) Cartier isomorphism \( \text{Coker}(\mathcal{O}_X \xrightarrow{d} \Omega_{X,cl}^{2}) \cong \Omega_{X^{(1)}}^{1} \), \( Q \) is induced by \( d: \Omega_{X}^{1} \to \Omega_{X,cl}^{2} \) and \( C \) is the Cartier operation. It is immediate from the definition that

\[
\delta(\omega) = P(C(\omega)) \quad \text{for} \quad \omega \in \Omega_{X,cl}^{1}.
\]

\( \text{which is a pullback from} \ X^{(1)} \)
If we define $\kappa : \Omega^*_X \to \mathcal{F}_X$ by setting $\kappa(\omega) = \mathcal{P}(\omega(1)) - \delta(\omega)$ then we will have an exact sequence
\begin{equation}
0 \to (\mathcal{O}^*_X)^p \to \mathcal{O}^*_X \xrightarrow{d \log} \Omega^*_X \xrightarrow{\kappa} \mathcal{F}_X \to 0.
\end{equation}
(Unlike (4) and (5), (7) is not $k$-linear.)

Now, if $(\mathcal{L}, \nabla)$ is a line bundle with connection, we define its extended curvature $\tilde{\text{curv}}(\mathcal{L}, \nabla) \in \Gamma(X, \mathcal{F}_X)$ to be locally given by
\[ \tilde{\text{curv}}(\mathcal{L}, \nabla) = \kappa(\omega) \quad \text{if} \quad (\mathcal{L}, \nabla) \sim (\mathcal{O}, d + \omega). \]
It is clear from (7) that this is independent of the trivialization, and that $Q(\tilde{\text{curv}}(\mathcal{L}, \nabla)) = F_{\nabla}$ — the usual curvature of $\nabla$.

**Proposition 3.** Splittings of the algebra $D_{\mathcal{L}^*, \nabla(1)}$ correspond canonically to line bundles on $X$ with connection $(\mathcal{L}', \nabla')$ such that $\tilde{\text{curv}}(\mathcal{L}', \nabla') = c \cdot \tilde{\text{curv}}(\mathcal{L}, \nabla)$.

### 2.5. Connections on determinant bundles.

We already said before that we identify the twisted cotangent bundle corresponding to the determinant line bundle on $\text{Bun}_G$. For that (and for later use) we’ll need some facts about connections on determinants. So let $\pi : X \to S$ be a smooth projective morphism of relative dimension 1, and let $\mathcal{E}$ be a vector bundle (or an $S$-flat coherent sheaf) on $X$. We are interested in the line bundle $\text{det} \mathcal{R}\pi_* \mathcal{E}$.

Define the sheaf of relative differentials $\Omega^*_{X/S} = \omega_{X/S}$ as usual. By an “$S$-relative connection” on $\mathcal{E}$ we mean a map $\mathcal{E} \to \Omega^*_{X/S} \otimes \mathcal{E}$ satisfying the usual Leibnitz rule.

**Proposition 4.** Suppose $X$ is a trivial family, i.e. $X = X_0 \times S$ for some curve $X_0$.

1. Let $\mathcal{E}$ be as above, and $\nabla$ an $S$-relative connection on $\mathcal{E} \otimes \omega^{1/2}_{X/S}$. Then there is a canonical connection on the line bundle $\text{det} \mathcal{R}\pi_* \mathcal{E}$.

2. Let $\mathcal{E}', \mathcal{E}'', \mathcal{E}''$ be three sheaves equipped with the data of point (1) and we have an exact sequence
\[ 0 \to \mathcal{E}' \to \mathcal{E} \to \mathcal{E}'' \to 0 \]
compatible with connections. Then the corresponding isomorphism $\text{det} \mathcal{R}\pi_* \mathcal{E} \cong \text{det} \mathcal{R}\pi_* \mathcal{E}' \otimes \text{det} \mathcal{R}\pi_* \mathcal{E}''$ is also compatible with connections.

### 3. Plan of the proof of Theorem 1

**Proposition 5.** Suppose $X$ is a variety (or a stack) and $\pi : A \to X$, $\pi^\vee : A^\vee \to X$ are dual families of abelian varieties. Suppose we have a torsor $\mathcal{T} \to X$ for $A \to X$ and a gerbe $\mathcal{G}$ on $\mathcal{T}$ with a fixed degree of the splitting. Then there is a canonical dual torsor $\mathcal{T}^\vee$ over $A^\vee$ and a gerbe $\mathcal{G}^\vee$ on it such that the derived categories of coherent modules over $\mathcal{G}$ and $\mathcal{G}^\vee$ are equivalent.

**Idea of proof.** When the torsor $\mathcal{T}$ and the gerbe $\mathcal{G}$ are trivial, we can take $\mathcal{T}^\vee$ and $\mathcal{G}^\vee$ to be trivial, and use (the in-families version of) the usual Fourier–Mukai transform. In general choose a (say, étale) cover $\tilde{X}$ of $X$ on which $\mathcal{T}$ and $\mathcal{G}$ trivialize so that we can apply the trivial case to get an equivalence over $\tilde{X}$, and then use properties of the Fourier–Mukai transform (namely, that it interchanges shifts along $A$ with twists by line bundles) to descend it to $X$. \qed
We will apply the above proposition to \(X = (\mathfrak{g}^0)^{(2)}, A = A^\vee = (\text{Higgs}^0)^{(2)}\),
\(\mathcal{G} = \text{Conn}_{\mathfrak{g}^0}(\mathcal{L}^{(1)}_{\det}) = \text{Conn}(\mathcal{L}^{(1)}_{\det}) = (\text{Loc}_{\mathfrak{g}^0}^0)^{(1)}\) (since \(c \notin \mathbb{F}_p\)) and the
gerbe \(\mathcal{G}\) corresponding to the Azumaya algebra \(D_{\mathcal{G}}(\text{Bun})\). We need to prove
that \(\mathcal{G}^\vee\) is also isomorphic to \(\text{Loc}_{\mathfrak{g}^0}^0/2\) and \(\mathcal{G}^\vee\) corresponds to the Azumaya algebra
\(D_{\mathcal{L}^0/2}(\text{Bun})\).

First, we prove that \(\mathcal{G}^\vee \cong (\mathcal{L}^0/2)^{(1)}\). This is equivalent to the existence of an
“action” on the gerbe \(\mathcal{G}\) of the gerbe \(\mathcal{G}_1\) on \(A = (\text{Higgs}^0)^{(2)}\) where \(\mathcal{G}_1\) is dual to the
torsor \(\mathcal{G}^\vee \cong (\mathcal{L}^0/2)^{(1)} \rightarrow (\mathcal{B}^{0})^{(2)}\) with the projection rescaled by \(c^0\). To describe
what we mean by an action of \(\mathcal{G}_1\) on \(\mathcal{G}\) consider the graph of action of the group
scheme \((\text{Higgs}^0)^{(1)}\) (over \((\mathcal{B}^{0})^{(1)}\) on \(\mathcal{L}^0/2\) — it is a subvariety (or substack) \(\Gamma\)
in \(\mathcal{L}^0/2 \times \mathfrak{g}^0 \mathcal{L}^0/2 \times \mathfrak{g}^0(\text{Higgs}^0)^{(1)}\) isomorphic to \(\mathcal{L}^0/2 \times \mathfrak{g}^0(\text{Higgs}^0)^{(1)}\). From
this graph we have three projections \(pr_1, pr_2, pr_3\) to the factors. An action of \(\mathcal{G}_1\) on
\(\mathcal{G}\) is by definition a splitting of \(pr_1^{(1)*}\mathcal{G} \otimes pr_2^{(1)*}\mathcal{G}^{-1} \otimes pr_3^{(1)*}\mathcal{G}\) satisfying a cocycle
condition. Now note that the gerbe \(\mathcal{G}\) corresponds to the algebra \(D_{\mathcal{L}^0/2, \nabla}\) where \(\mathcal{L}^0\)
is the pullback of the determinant line bundle, and \(\nabla\) is the canonical ("universal")
connection on this pullback. One can also show that \(\mathcal{G}_1\) is given by the Azumaya
algebra \(D_{\mathcal{G}, \theta^0}(\text{Higgs}^0)\). Now the statement follows from the results of 2.3.1 and
the following proposition (whose statement does not depend on \(c\)).

**Proposition 6.** The line bundle with connection
\[ S := pr_1^*(\mathcal{L}^0, \nabla) \otimes pr_2^*(\mathcal{L}^0, \nabla) \otimes \mathcal{O}(d + \theta^{(1)}) \]
on \(\Gamma\) is flat and has p-curvature \(pr_3^*\theta^{(2)}\).

If we define \(\tilde{\theta} = \text{curv}(\mathcal{L}^0, \nabla)\) then Proposition 6 implies the following identity:
\[(8) \quad pr_1^* \tilde{\theta} - pr_2^* \tilde{\theta} = P(pr_3^* \theta^{(2)}) - \kappa(pr_3^* \theta^{(1)}) = \delta(pr_3^* \theta^{(1)})\]
where \(P, \kappa, \delta\) are defined in 2.4.

For the proof of this proposition we will need the following lemma which will also be useful later.

**Lemma 7.** Let \(\text{Pic}(\mathcal{C})\) denote the Picard stack of a smooth projective curve \(\mathcal{C}\). Let
\(pr_1, a : \text{Pic}(\mathcal{C}) \times \mathcal{C} \rightarrow \text{Pic}(\mathcal{C})\) be the projection and addition maps, respectively (a is obtained from the Abel–Jacobi map). If \(\mathcal{L}\) is a line bundle on \(\text{Pic}(\mathcal{C})\) such that
\(pr_1^* \mathcal{L} \cong a^* \mathcal{L}\) then \(\mathcal{L}\) is trivial.

We will prove that the restriction of the line bundle from Proposition 6 to the preimage \(\Gamma_{\text{AJ}}\) in \(\Gamma\) of the image of Abel–Jacobi map in \(\text{Higgs}^0 = \text{Pic}_{\mathfrak{g}^0}(\mathfrak{g}^0)\)
under \(pr_3\) is flat and has p-curvature \(pr_3^* \theta^{(2)}\). This will follow from an alternative
description of this bundle with connection. Namely, we have an equivalence \(\mathcal{D}_{\mathcal{C}}-\text{mod} \sim \mathcal{D}_{\mathcal{T}, \mathcal{C}}-\mathfrak{g}\text{-mod}\) (\(\theta\) is the canonical form on \(\mathcal{T}^*\mathcal{C}\)). Moreover,
the “in-families” version of this is true. So if \(\mathfrak{g}\) is any stack then the category of \(\mathfrak{g}^{(1)}\)-families of \(\mathcal{D}_{\mathcal{C}}\)-modules” (i.e. quasi-coherent sheaves on \(\mathcal{C} \times \mathfrak{g}^{(1)}\)
equipped with a connection along \(\mathcal{C}\)) is equivalent to the category of \(\mathcal{D}_{\mathcal{T} \times \mathcal{C}, \mathfrak{g}}\)-modules. If we replace \(\mathcal{C}\) by \(\mathcal{C}^{(1)}\) and \(\mathfrak{g}\) by \(\mathfrak{g}^{0}\) then we have the universal
bundle with connection (along \(\mathcal{C}^{(1)}\)) on \(\mathcal{C}^{(1)} \times \mathcal{L}^{(1)}\), so applying the equivalence
gives a \(\mathcal{D}\)-module on \(\mathcal{T}^*\mathcal{C}^{(1)} \times \mathcal{L}^{0}\) with p-curvature \(pr_3^* \theta^{(1)}\). It is supported
on \((\mathfrak{g}^{0})^{(1)} \times (\mathfrak{g}^{0})^{(1)} \mathcal{L}^{0} \cong \Gamma_{\text{AJ}}\) and therefore corresponds to a \(\mathcal{D}_{\text{AJ}, \mathfrak{g}^{0}}-\mathfrak{g}^{(2)}\)-module
which we denote by \((\mathcal{L}, \nabla)_{\text{univ}}\).
Lemma 8. The restriction to \( \Gamma_{AJ} \) of the line bundle \( S \) from Proposition [6] is isomorphic to \( (L, \nabla)_{\text{univ}} \) where we identify \( \text{Loc} \) with \( \text{Loc}^0_{\omega,1/2} \) via twisting by \( \omega_C^{(p-1)/2} \).

Proof. We will first construct an isomorphism of line bundles, and then prove that it is compatible with connections. From the definition of \( \Gamma_{AJ} \) it is easy to see that a point \( \gamma \in \Gamma_{AJ} \) corresponds to a pair of rank \( N \) bundles with \( \omega^{1/2} \)-connections \((E_1, \nabla_1), (E_2, \nabla_2)\) that fit into a short exact sequence of \( D_{C,\omega^{1/2}} \)-modules

\[
0 \to E_2 \to E_1 \to F \to 0
\]

where \( F \cong \delta_{x,\xi} \) is an irreducible \( D_{C,\omega^{1/2}} \)-module corresponding to a point \((x, \xi) \in T^*C(1)\). Now recall that the fiber of \( S \) at \( \gamma \) is given by \( \det \Gamma(E_1) \otimes (\det \Gamma(E_2)) \otimes 1 \cong \det \Gamma(F) \). If \((\bar{x}, \bar{\xi}) \in T^*C \) is such that \((x, \xi) = (\bar{x}, \bar{\xi})(1)\) then \( F \) has a filtration for which \( \text{gr} F \cong F_{\bar{x}} \otimes \bigoplus_{i=0}^{p-1} \omega_{C,\bar{x}}^{\otimes i} \). Therefore \( \det \Gamma(F) = F_{\bar{x}}^{\otimes p} \otimes \omega_{\bar{x}}^{\otimes (p\bar{\xi})} \). Let \( F' = F \otimes_{D_{C}} \omega_{C}^{(1-p)/2} \) — it is an irreducible \( D_{C} \)-module supported at \( \bar{x} \). Then we can rewrite \( S_{\gamma} = \det \Gamma(F) = (F')_{\bar{x}}^{\otimes p} \). Let \( \tilde{C} \subset T^*C(1) \) be the spectral curve of \( E_1 \) (same as that of \( E_2 \)). So \( \tilde{C} = E \times_{\text{univ}} \{E_1\} \). Let \( L_i \) \((i = 1, 2) \) be \( D_{T^*C,\theta} \)-modules corresponding to \( E_i \). They are supported on the Frobenius neighborhood of \( C \) in \( T^*C \), and \( L_1/L_2 \) is the irreducible corresponding to \( F' \), which is supported at \((\bar{x}, \bar{\xi})\). From the definition of the equivalence \( D_{C}-\text{mod} \sim D_{T^*C,\theta}-\text{mod} \) we see that \( F'_{\bar{x}} = (L_1/L_2)(\bar{x}, \bar{\xi}) = (L_1)(\bar{x}, \bar{\xi}) \). Unraveling the definition of \( L_{\text{univ}} \), one can see that its fiber at \( \gamma \) is identified with

\[
(L_{\text{univ}})_{\gamma} = (L_{1}(1))_{(x,\xi)} = (L_{1})_{(\bar{x},\bar{\xi})}^{\otimes p} = (F')_{\bar{x}}^{\otimes p} = S_{\gamma}.
\]

Saying this for \( \gamma \) being an \( S \)-point for arbitrary scheme \( S \), we can thus prove that \( L_{\text{univ}} \) and \( S \) become isomorphic after pullback to \( \Gamma_{AJ} \times_{C(1)} C \). To prove that they are isomorphic on \( \Gamma_{AJ} \), we’ll use the following

Claim 9. Let \( \pi: T \to S \) be a smooth morphism of relative dimension 1. Let \( \phi: T \to T := T^{(1)} \times_{S(1)} S \) be the relative Frobenius map, and \( L \) a line bundle on \( T \). Then there is a canonical isomorphism

\[
\det R\phi_* L \cong L^{(S)} \otimes (\Omega^{1}_{T/S})^{(p-1)/2}
\]

where \( L^{(S)} \) is the “relative Frobenius twist” of \( L \): it is the pullback of \( L^{(1)} \) along the map \( T \to T^{(1)} \).

Now suppose we are given an \( S \)-point \( \gamma \) of \( \Gamma \). It corresponds to a sequence \([10]\) of \((O_S \boxtimes \mathcal{D}_{C,\omega^{1/2}})\)-modules where \( E_1, E_2 \) are rank \( N \) vector bundles on \( S \times C \), and \( F \) is a line bundle on \( \text{supp} \mathcal{F} = S \times C \) \( C \subset S \times C \) for a certain \( x: S \to C(1) \). Just as before, we have \( S_{\gamma} = \det R\pi_* F \) where \( \pi \) is the projection \( S \times C \to S \).

Now note that \( F = E_1|_{\text{supp} \mathcal{F}} \). Let \( \Gamma_x \subset S \times C(1) \) be the graph of \( x \). Then \( R\pi_* F = ((\text{id}_S \times F_{C})_{\Gamma_1})|_{\Gamma_x} \). Now, Claim 9 gives

\[
S_{\gamma} = \det R\pi_* F = (\det (\text{id}_S \times F_{C}))|_{\Gamma_x} = ((E_1)^{(S)} \otimes (O_S \boxtimes \omega_{C}^{(p-1)/2}))|_{\Gamma_x} = (F^{(S)})_{x} \otimes O_S x^{*} \omega_{C}^{(p-1)/2}.
\]

We claim that the right-hand side is canonically identified with \( (L_{\text{univ}})_{\gamma} \).

To prove that this isomorphism is compatible with connections, we will need the following statement:
Claim 10 (cf. Proposition 29). Suppose that \( T = C \times S \to S \) is a trivial family of smooth curves and \( s: S \to T^{(S)} \) is a section of \( T^{(S)} \to S \). Let \((\mathcal{F}, \nabla_{\mathcal{F}})\) be a coherent sheaf on \( T \) with an \( S \)-relative \( \omega_{T/S}^{1/2} \)-connection and assume that \( \mathcal{F} \) is a pushforward of a line bundle \( \mathcal{L} \) on the closed subset \( Z = S \times_{T^{(S)}} T \). Let \( \nabla_{\det} \) be the connection on \( \mathcal{L}' := \det R\pi_* \mathcal{F} \) given by Proposition 3. Let \( w: S \to Z \) be the \((\kappa\text{-semilinear}) map such that \( \iota_Z \circ w = W_{T/S} \circ s \). Then \( \nabla_{\mathcal{F}} \) gives rise to a connection \( \nabla_w \) on \( \mathcal{L}'' := w^* (\mathcal{L} \otimes \omega_{Z/S}^{(p-1)/2}) \).

The two line bundles with connection are related by
\[
(\mathcal{L}', \nabla_{\det}) \cong (\mathcal{L}'', \nabla_w - \hat{s}^* \theta)
\]
where \( \hat{s}: T^*(T^{(S)}/S) = T^* C^{(1)} \times S \) is the section of \( T^*(T^{(S)}/S) \to S \) whose image is the \( p \)-support of \((\mathcal{F}, \nabla_{\mathcal{F}})\) and \( \theta \) is the pullback of the canonical 1-form on \( T^* C^{(1)} \) under the projection \( T^* C^{(1)} \times S \to T^* C^{(1)} \). \hfill \( \square \)

Proposition 6 (at least, restricted to \( \Gamma_{AB} \)) is an immediate consequence of Lemma 8.

Denote by \( \mathcal{T} = \text{Loc}^0_{\omega_{1/2}, c} \to (\mathcal{B}^0)^{(1)} \) the torsor over \((\text{Higgs}^{(1)}) \to (\mathcal{B}^0)^{(1)} \) pulled back from the standard torsor \( \text{Loc}^0_{\omega_{1/2}} \to (\mathcal{B}^0)^{(1)} \) under the action of \( c \in \kappa^* \) on \((\mathcal{B}^0)^{(1)} \).

Now, to prove Theorem 4, we need to construct a line bundle \((\mathcal{L}, \nabla)_{\ker} \) with connection on \( \mathcal{T}_1 \times (\mathcal{B}^0)^{(1)} \mathcal{T}_c \) satisfying
\[
(12) \quad \text{curv}((\mathcal{L}, \nabla)_{\ker}) = c \cdot \text{curv}(\text{pr}_1^* (L, \nabla)_{\det}) + c^{-1} \cdot \text{curv}(\text{pr}_2^* (L, \nabla)_{\det}).
\]
The sought-for bundle with connection will be given by
\[
(13) \quad (\mathcal{L}, \nabla)_{\ker} := a^* (L_{\det}, \nabla) \otimes \text{pr}_1^* (L_{\det}^{\otimes -1}, \nabla^*) \otimes \text{pr}_2^* (L_{\det}^{\otimes -1}, \nabla^*)
\]
where \((L_{\det}, \nabla)\) is the universal line bundle with connection on \( \text{Loc} \), and \( a \) is the “addition” map \( \mathcal{T}_1 \times (\mathcal{B}^0)^{(1)} \mathcal{T}_c \to \mathcal{T}_{1+c} \) (one can check that the torsor \( \mathcal{T}_{1+c} \) is the sum of the torsors \( \mathcal{T}_1 \) and \( \mathcal{T}_c \)). Define \( \hat{\theta} = \text{curv}(L, \nabla)_{\det} \). Then substituting (13) in (12) yields
\[
\text{LHS} - \text{RHS} = \text{curv}((\mathcal{L}, \nabla)_{\ker}) - c \cdot \text{pr}_1^* \hat{\theta} - c^{-1} \cdot \text{pr}_2^* \hat{\theta}
\]
\[
= a^* \hat{\theta} - \text{pr}_1^* \hat{\theta} - \text{pr}_2^* \hat{\theta} - c \cdot \text{pr}_1^* \hat{\theta} - c^{-1} \cdot \text{pr}_2^* \hat{\theta}
\]
\[
= a^* \hat{\theta} - (1 + c) \cdot \text{pr}_1^* \hat{\theta} - (1 + c^{-1}) \cdot \text{pr}_2^* \hat{\theta}.
\]
So we need to show that
\[
(14) \quad (1 + c)^{-1} a^* \hat{\theta} = \text{pr}_1^* \hat{\theta} + c^{-1} \cdot \text{pr}_2^* \hat{\theta}.
\]
To prove formula (14), we proceed as follows. Let \( \tilde{\alpha} = (1 + c)^{-1} a^* \hat{\theta} - \text{pr}_1^* \hat{\theta} - c^{-1} \cdot \text{pr}_2^* \hat{\theta} \); we want to prove \( \tilde{\alpha} = 0 \). Consider the two projections \( \text{pr}_{1,3}, \text{pr}_{2,3}: \mathcal{T}_1 \times (\mathcal{B}^0)^{(1)} \mathcal{T}_c \to \mathcal{T}_1 \times (\mathcal{B}^0)^{(1)} \mathcal{T}_c \). As a first step, we prove that the difference between two pullbacks \( \text{pr}_{1,3}^* \tilde{\alpha} - \text{pr}_{2,3}^* \tilde{\alpha} = 0 \). Let \( \text{pr}_{i}^* \) \( (i = 1, 2, 3) \) be the projection from \( \mathcal{T}_1 \times (\mathcal{B}^0)^{(1)} \mathcal{T}_c \) to the \( i \)th factor, and \( a_{i,3} = a \circ \text{pr}_{i,3} \). We also have a “difference” map \( s: \mathcal{T}_1 \times (\mathcal{B}^0)^{(1)} \mathcal{T}_1 \to (\text{Higgs}^0)^{(1)} \); denote \( s_{1,2} = s \circ \text{pr}_{1,2} \).
Now we calculate
\[
\text{pr}_{1,3}^* \tilde{\alpha} - \text{pr}_{2,3}^* \tilde{\alpha} = (1 + c)^{-1} a_{1,3}^* \tilde{\theta} - \text{pr}_1^* \tilde{\theta} - c^{-1} \text{pr}_2^* \tilde{\theta}
\]
- \left[ (1 + c)^{-1} a_{2,3}^* \tilde{\theta} - \text{pr}_2^* \tilde{\theta} - c^{-1} \text{pr}_3^* \tilde{\theta} \right]
= (1 + c)^{-1} (a_{1,3}^* \tilde{\theta} - a_{2,3}^* \tilde{\theta}) - (\text{pr}_1^* \tilde{\theta} - \text{pr}_2^* \tilde{\theta})
= (1 + c)^{-1} \delta(s_{1,2}^* (\delta(1))) - \delta(s_{1,2}^* \theta(1)) = 0,
\]
where in the last line we used formula (8).

The formula (10) implies that \( \tilde{\alpha} = \text{pr}_2^* \tilde{\alpha}' \) for some \( \tilde{\alpha}' \in \Gamma(\mathcal{F}, \mathcal{F}_X) \). Similarly, we can show that \( \tilde{\alpha} = \text{pr}_1^* \tilde{\alpha}'' \) for \( \tilde{\alpha}'' \in \Gamma(\mathcal{F}, \mathcal{F}_X) \). It follows that \( \tilde{\alpha} = \text{pr}_i(\mathcal{D}^0)^* \tilde{\beta} \) for some \( \tilde{\beta} \in \Gamma(\mathcal{D}^0, \mathcal{F}_{\mathcal{D}^0}) \). So we need to show that \( \tilde{\beta} = 0 \).

3.1. Alternative construction of \( \tilde{\theta} \). Let \( \Omega_{\text{Loc}} \) be the canonical symplectic form on the smooth part \( \text{Loc}^m \) of Loc, so that \( \Omega_{\text{Loc}} = F_{\text{Loc}} \). We will construct a 1-form \( \theta_0 \) on \( \text{Loc}^m \) such that \( \Omega_{\text{Loc}} = d\theta_0 \) and \( \delta(\theta_0) = \theta \) (where \( \delta \) is defined in \( 2.4 \)). Since \( \Omega_{\text{Loc}} \) is symplectic, constructing \( \theta_0 \) satisfying the first condition is equivalent to constructing a vector field \( \xi_0 \) which is Liouville, i.e. \( L_{\xi_0} \Omega_{\text{Loc}} = \Omega_{\text{Loc}} \) where \( L \) denotes the Lie derivative.

First, we need another interpretation of the form \( \Omega_{\text{Loc}} \). Note that for a \( \mathcal{G}_n \)-gerbe \( \mathcal{G} \) on a smooth variety \( X \), and two \( \mathcal{G} \)-modules \( \mathcal{M} \) and \( \mathcal{N} \) with proper support, we have the following version of Serre duality: \( \text{RHom}(\mathcal{M}, \mathcal{N})^* \cong \text{RHom}(\mathcal{N}, \mathcal{M} \otimes \omega_X)[\dim X] \) where \( \omega_X \) is the sheaf of top degree differential forms on \( X \). In particular, if \( X \) is a symplectic surface and \( \mathcal{M} = \mathcal{N} \), we have a nondegenerate (in fact, antisymmetric) bilinear form on the space \( \text{Ext}^1(\mathcal{M}, \mathcal{M}) \) which is identified with the tangent space at \( \mathcal{M} \) to the moduli space \( \mathcal{M}_{X, \mathcal{G}} \) of \( \mathcal{G} \)-modules on \( X \), so we get a non-degenerate 2-form on this moduli space. One can use properties of the Serre duality to show that this form is closed.

**Lemma 11.** The form \( \Omega_{\text{Loc}} \) coincides with the one just described, where we put \( X = T^*C^{(1)} \), and \( \mathcal{G} = \mathcal{G}_{\mathcal{D}} \) is the gerbe corresponding to the Azumaya algebra \( \mathcal{D}_C \).

**Proof sketch.** It is known that (in any characteristic) the category of \( \mathcal{D} \)-modules on a smooth variety \( Z \) admits a Serre functor \( \mathcal{S}_Z \) which, moreover, is canonically isomorphic to the shift by \( 2 \dim Z \). The same is true for \( \mathcal{D}' \)-modules where \( \mathcal{D}' \) is any twisted differential operator algebra. We will need the case \( \mathcal{D}' = \mathcal{D}_{C, \omega_{1/2}} \). The lemma will follow from the following two statements:

- The curvature of \( (\mathcal{L}, \nabla)_{\text{det}} \) coincides with the 2-form constructed from this isomorphism \( \mathcal{S}_C \cong \mathcal{G}_\mathcal{D} \).
- The composite equivalence \( \mathcal{D}_{C, \omega_{1/2}} \otimes_{\omega_{(1-\rho)/2}} \mathcal{D}_{\text{C-mod}} \sim \mathcal{G}_\theta \text{-mod} \) is compatible with the trivializations of Serre functors.

The first statement makes sense in any characteristic and should be well known. As for the second statement, the difference between two trivializations is an invertible function on \( T^*C^{(1)} \), therefore a constant. With a little more work, one can show that this constant is equal to 1.

In the situation above, consider the open substack \( \mathcal{M}_{X, \mathcal{G}}^0 \subset \mathcal{M}_{X, \mathcal{G}} \) consisting of modules which (locally after splitting \( \mathcal{G} \)) look like (pushforward of) a line bundle on a smooth curve in \( X \). Denote by \( \mathcal{D}_{X}^0 \) the moduli space of proper smooth curves in \( X \), and let \( \mathcal{C}_X^0 \subset \mathcal{D}_{X}^0 \times X \rightarrow \mathcal{D}_{X}^0 \) be the universal family of curves. Then we
have a map $\chi: \mathcal{M}_{X, G}^0 \to \mathcal{P}_X^0$ given by taking supports, which presents $\mathcal{M}_{X, G}^0$ as a torsor for the relative Picard stack $\text{Pic}(\mathcal{P}_X^0 / \mathcal{P}_X)$. One can show that this defines an integrable system (i.e., that the fibers are Lagrangian).

Denote $\mathcal{D} = \text{Spec}(k[\varepsilon]/\varepsilon^2)$. A vector field on $\text{Loc}^{\text{sm}}$ is the same as an automorphism of $\text{Loc}^{\text{sm}} \times \mathcal{D}$ over $\mathcal{D}$ which is identity on $\text{Loc}^{\text{sm}} \subset \text{Loc}^{\text{sm}} \times \mathcal{D}$. Let $h$ be the automorphism of $T^*C(1) \times \mathcal{D}$ corresponding to the Euler vector field on $T^*C(1)$. Since $H^2(T^*C(1), \mathcal{O}) = 0$, there exists a (non-unique) equivalence

$$\Phi: \text{pr}_1^* \mathcal{G}_D \sim h^* \text{pr}_1^* \mathcal{G}_D$$

which is identity on $T^*C(1) \times \mathcal{D}$. Now, if we have a $\mathcal{D}$-module $\mathcal{M}$ on $\mathcal{C}$, let $\mathcal{M}'$ be the corresponding $\mathcal{G}_D$-module, and let $\mathcal{M}' = \Phi^{-1}h^*(\mathcal{M} \boxtimes \mathcal{O}_D) - $ this is a $\text{pr}_1^* \mathcal{G}_D$-module on $T^*C(1) \times \mathcal{D}$, and denote by $\mathcal{M}$ the corresponding $\mathcal{D}_C \boxtimes \mathcal{O}_D$-module. By construction, $\mathcal{M}/\varepsilon \mathcal{M} \cong \mathcal{M}$, so $\mathcal{M}$ defines a tangent vector to $\text{Loc}^{\text{sm}}$ at $\mathcal{M}$. This way we get a vector field on $\text{Loc}^{\text{sm}}$ which is the desired field $\xi_0$. Denote $\theta_0 = \iota_0 \Omega_{\text{Loc}}$.

**Proposition 12.** The vector field $\xi_0$ is Liouville. Equivalently, $d\theta_0 = \Omega_{\text{Loc}}$.

**Proof sketch.** The proposition follows from the functoriality of the Serre duality. Namely, if we have a symplectic surface $(X, \Omega)$ with a $\mathcal{G}_m$-gerbe $\mathcal{G}$ and an automorphism $\phi$ of the pair $(X, \mathcal{G})$ such that $\phi^* \Omega = \Omega$ for some $\lambda \in k^*$, then the corresponding automorphism $\phi$ of the moduli space $\mathcal{M}_C$ of (coherent, properly supported) $\mathcal{G}$-modules will satisfy $\phi^* \Omega_{\mathcal{M}_C} = \lambda \Omega_{\mathcal{M}_C}$ where $\Omega_{\mathcal{M}_C}$ is the symplectic form constructed by the Serre duality. One can also formulate an in-families version of this statement. In particular, if we take $X = T^*C(1)$, $\mathcal{G} = \mathcal{G}_0$ and the $\mathcal{D}$-family of automorphisms given by $(h, \Phi)$, then the equality $h^* \Omega_{T^*C(1)} = (1 + \varepsilon) \Omega_{T^*C(1)}$ implies that $h^* \Omega_{\text{Loc}} = (1 + \varepsilon) \Omega_{\text{Loc}}$ which means (since $h = 1 + \varepsilon \xi_0$ by definition) that $L_{\xi_0} \Omega_{\text{Loc}} = \Omega_{\text{Loc}}$. 

**Proposition 13.** The class of $\xi_0$ modulo Hamiltonian vector fields does not depend on the choice of $\Phi$.

**Proof sketch.** Suppose we have two equivalences $\Phi_1, \Phi_2: \text{pr}_1^* \mathcal{G}_D \sim h^* \text{pr}_1^* \mathcal{G}_D$. Then they differ by an auto-equivalence $\Phi_1^{-1} \circ \Phi_2$ of $\text{pr}_1^* \mathcal{G}_D$ which corresponds to an element $\phi \in H^1(T^*C(1), \mathcal{O})$. From any such element we can construct a function $f^\phi_0$ on $\mathcal{B}(1)$ as follows. If a point $b \in \mathcal{B}(1)$ corresponds to a smooth spectral curve $\mathcal{C} \subset T^*C(1)$ (i.e., if $b \in \mathcal{B}^{(1)}$) then we just put $f^\phi_0(b) = \langle \phi|_{\mathcal{C}}, \theta_{\mathcal{C}} \rangle$ where $\langle \cdot, \cdot \rangle$ denotes the Serre duality pairing. (It can also be defined for $b \notin \mathcal{B}^{(1)}$. ) One can check that the pullback $f_\theta$ of $f^\phi_0$ to $\text{Loc}^{\text{sm}}$ satisfies $H_{f_\theta} = -\xi_0 - \xi_0$ where $\xi_0$ and $\xi_0$ are the $\xi_0$’s corresponding to $\Phi_1$ and $\Phi_2$, respectively.

For a smooth variety $X$ and a 1-form $\theta$ on $X^{(1)}$ denote by $\mathcal{G}_\theta$ the $\mathcal{G}_m$-gerbe on $X^{(1)}$ corresponding to the Azumaya algebra $\mathcal{D}_\theta$. [Need in families over $\mathcal{D}$.] In particular, the gerbe $\mathcal{G}_D$ on $T^*C(1)$ is equivalent to $\mathcal{G}_\theta$ where $\theta$ is the canonical 1-form on $T^*C$. One easily checks that $\mathcal{G}_\theta$ depends additively on $\theta$ in the sense that $\mathcal{G}_{\theta_1 + \theta_2} = \mathcal{G}_{\theta_1} \cdot \mathcal{G}_{\theta_2}$. Therefore on $T^*C(1) \times \mathcal{D}$ we have

$$\text{pr}_1^* \mathcal{G}_D^{-1} \cdot h^* \text{pr}_1^* \mathcal{G}_D \sim \mathcal{G}_{h \cdot \theta - \theta} = \mathcal{G}_{e \cdot \theta} = \mathcal{G}_{e \cdot \theta} \sim e^* \text{pr}_1^* \mathcal{G}_D$$

where $\theta$ here is considered as a relative 1-form on $T^*C(1) \times \mathcal{D}$ over $\mathcal{D}$, and $e: T^*C(1) \times \mathcal{D} \to T^*C(1) \times \mathcal{D}$ is given by fiberwise multiplication by $\varepsilon$. Now, $e$ factors through the 1st infinitesimal neighborhood $Z_1$ of the zero section in $T^*C(1)$. Therefore the
above equivalence \( \Phi \) can be constructed from any trivialization \( \Psi \) of \( \mathcal{G}_D \) on \( Z_1 \) which coincides with the canonical trivialization on the zero section. We assume from now on that \( \Phi \) is obtained in this way.

Note that, given \( \Psi \), one can construct a family of equivalences

\[
\Phi_c : \text{pr}_1^* \mathcal{G}_{\theta} \sim h^* \text{pr}_1^* \mathcal{G}_{\theta} \tag{17}
\]

parametrized by \( c \in k \) (we just have to replace \( e \) by fiberwise dilation by \( ce \)). One can check that \( \Phi_c \) is additive in \( c \), i.e. compatible with the equivalence \( \mathcal{G}_{1+c^e \theta} \sim \mathcal{G}_{\theta} \cdot \mathcal{G}_{c^e \theta} \) and, if \( c \in k^* \), \( \Phi_c \) is obtained from \( \Phi \) by conjugation with dilation by \( c \) on \( T^* C(1) \).

**Proposition 14.** If the equivalence \( \Phi \) is from the class just described then, in the notation of \([15]\) (for any \( c \in k \setminus F_p \)), we have

\[
(1 + c)^{-1} a^* \theta_0 = \text{pr}_1^* \theta_0 + c^{-1} \text{pr}_2^* \theta_0. \tag{18}
\]

**Proof sketch.** First we’ll show that

\[
(1 + c)^{-1} a^* \Omega_{\text{Loc}} = \text{pr}_1^* \Omega_{\text{Loc}} + c^{-1} \text{pr}_2^* \Omega_{\text{Loc}}. \tag{19}
\]

This will follow from a more general statement:

**Lemma 15.** Let \( X \) be a symplectic surface, \( \mathcal{G}, \mathcal{G}', \mathcal{G}'' \) three \( \mathbb{G}_m \)-gerbes on it, and suppose we are given an equivalence \( \mathcal{G} \sim \mathcal{G}' \cdot \mathcal{G}'' \). Consider the corresponding \( \text{Pic}(\mathcal{E}_X^0/\mathcal{B}_X^0) \)-torsors \( \mathcal{T} = \mathcal{M}_X^0, \mathcal{T}' = \mathcal{M}_X^{0, \mathcal{G}'} \), \( \mathcal{T}'' = \mathcal{M}_X^{0, \mathcal{G}''} \) endowed with symplectic structures. Clearly, \( \mathcal{T} \) is identified with the sum of torsors \( \mathcal{T}' \) and \( \mathcal{T}'' \), so we can define the graph of addition \( \Gamma \subset \mathcal{T} \times \mathcal{T}' \times \mathcal{T}'' \). Then \( \Gamma \) is a Lagrangian subvariety in \( (\mathcal{T} \times \mathcal{T}' \times \mathcal{T}'') \). It is a well-known fact that this graph is Lagrangian (at least for the usual Hitchin fibration, i.e. for \( X = T^* C(1) \)).

**Proof sketch.** Suppose we want to prove that \( \Gamma \) is Lagrangian in a formal neighborhood of some point \( \gamma \in \Gamma \), and let \( \tilde{C} \subset X \) be the corresponding spectral curve. Clearly one can replace \( X \) by the formal neighborhood \( \tilde{C} \) of \( C \) in \( X \). Since \( H^2(\tilde{C}, \mathcal{O}) = 0 \), we can trivialize \( \mathcal{G} \) on \( \tilde{C} \). So we can assume that \( \mathcal{G} \) is trivial. Then \( \Gamma \) is the graph of addition on \( \text{Pic}(\mathcal{E}_X^0/\mathcal{B}_X^0) \). It is a well-known fact that this graph is Lagrangian (at least for the usual Hitchin fibration, i.e. for \( X = T^* C(1) \)).

Note that for \( X = T^* C(1) \), \( \mathcal{G} = \mathcal{G}_{\theta} \) the torsor \( \mathcal{M}_X^0, \mathcal{G} \) is identified with \( \mathcal{T}_c \) from the previous subsection. Moreover, under this identification, we have \( c^{-1} \Omega_{\text{Loc}} = \Omega_{\mathcal{M}_X^0, \mathcal{G}} \). Therefore, formula \(19\) follows from Lemma \(15\) applied to \( \mathcal{G} = \mathcal{G}_{(1+c)^e}, \mathcal{G}' = \mathcal{G}_\theta, \mathcal{G}'' = \mathcal{G}_{c^e} \).

Denote by \( \xi_{0,c} \) the vector field on \( \mathcal{T}_c \) obtained from \( \xi_0 \) under the identification \( \mathcal{T}_c \cong \text{Loc}^0 \). Then, in order to prove Proposition \(15\) we need to prove that the vector field \( \eta_c = \text{pr}_1^* \xi_{0,1+c} + \text{pr}_2^* \xi_{0,1} + \text{pr}_3^* \xi_{0,c} \) on \( \mathcal{T}_{1+c} \times \mathcal{T}_1 \times \mathcal{T}_c \) preserves \( \Gamma \) (because \( \Gamma \) is Lagrangian, and this vector field corresponds to the 1-form \((1+c)^{-1} \text{pr}_1^* \theta_0 - \text{pr}_2^* \theta_0 - c^{-1} \text{pr}_3^* \theta_0 \)). To see this, note that \( \xi_{0,c} \) can be obtained the same way as \( \xi_0 \) with \( \Phi \) replaced by \( \Phi_c \) from \(17\). Now, \( \eta_c \) comes from an infinitesimal automorphism of the quadruple \((X, \mathcal{G}_\theta, \mathcal{G}_{c^e}, \mathcal{G}_{(1+c)^e})\) given by \((h, \Phi_1, \Phi_c, \Phi_{1+c})\). Additivity of \( \Phi_c \) in \( c \) implies that this automorphism is compatible with the equivalence \( \mathcal{G}_\theta \cdot \mathcal{G}_{c^e} \sim \mathcal{G}_{(1+c)^e} \). Therefore \( \eta_c \) preserves \( \Gamma \), which is what we want.

**Lemma 16.** The extended curvature \( \dot{\theta} \) of the determinant line bundle with connection \((\mathcal{L}, \nabla)_{\text{det}}\) on \( \text{Loc}^m \) is equal to \( \delta(\theta_0) \).
Since $\delta$ is $k$-linear and compatible with pullbacks, Lemma 16 implies formula (13) and therefore Theorem 1.

Here we prove the following partial result:

**Proposition 17.** We have $\hat{\theta} - \delta(\theta_0) = P(\chi(1)^*\beta'_0)$ where $\chi'$ is the map $\text{Loc}^{\text{sm}} \to \mathcal{B}(1)$, $\delta$ and $P$ are defined in (12) and $\beta'_0$ is some form on $\mathcal{B}(1)$.

**Proof sketch.** Denote $\hat{\alpha}_0 = \hat{\theta} - \delta(\theta_0)$. We have already mentioned above that $Q(\delta(\theta_0)) = d\theta_0 = \Omega_{\text{Loc}} = Q(\hat{\alpha})$. So $Q(\hat{\alpha}_0) = 0$, which means that $\hat{\alpha}_0 = P(\alpha_0)$ for some $\alpha_0 \in \Gamma(\mathcal{B}(1), \Omega^1_{\mathcal{B}(1)})$. We want to prove that $\alpha_0 = \chi(1)^*\beta'_0$ for some $\beta'_0$.

We’ll show that $\alpha_0|_{\text{Loc}^{\text{sm}}}$ is a pullback of some 1-form $\beta_0$ on $\mathcal{B}(1)$. Using the properties of $\chi'$, we can then deduce that $\beta_0$ extends to the whole $\mathcal{B}(1)$ since $\chi(1)^*\beta'_0$ extends to $\text{Loc}^{\text{sm}}$.

Now let $\Gamma$ be the graph of addition in $Higgs^{0(1)} \times_{\mathcal{B}(1)} \text{Loc}^{0} \times_{\mathcal{B}(1)} \text{Loc}^{0}$. The argument of the proof of Proposition 18 applied to $c = 0$ shows that on $\Gamma$ we have $pr^*_1 \gamma^{(1)} + pr^*_2 \theta_0 = pr^*_2 \theta_0$ where $\theta$ is the canonical 1-form on Higgs $= T^* \text{Bun}$. (We use that the vector field $\xi_{0,0}$ on Higgs coincides with the Euler vector field, and therefore $\theta_{0,0} = \theta$.) Applying $\delta$ yields $\delta(pr^*_1 \gamma^{(1)}) + \delta(pr^*_2 \theta_0) = \delta(pr^*_2 \theta_0)$. On the other hand, we know from (8) that $\delta(pr^*_1 \gamma^{(1)}) + pr^*_2 \gamma^{(1)} = pr^*_2 \gamma$, so subtracting the previous equation, we get $pr^*_2\gamma^{(1)}(\alpha_0) = pr^*_2\gamma$, so $\alpha_0|_{\text{Loc}^{\text{sm}}}$ is a pullback of some $\beta'_0 \in \Gamma(\mathcal{B}(1), \mathcal{F}_{\mathcal{B}(1)})$. Since $\alpha_0 \in \text{Im}(P)$, we must have $\beta'_0 \in \text{Im}(P)$, so $\beta'_0 = P(\beta_0)$ for some $\beta_0$. \hfill $\square$

### 3.2. Proof of Lemma 16

Denote by $\text{Bun}^{[d]}$ the connected component of $\text{Bun}$ consisting of vector bundles of degree $d$ and $\text{Loc}^{[d]}$, its preimage in $\text{Loc}$. Note that $\text{Loc}^{[d]}$ is nonempty only for $d$ divisible by $p$.

We can deduce Lemma 16 from the following statement:

**Lemma 18.** For generic curve $C$ the fibers of the maps $q: \text{Loc} \to \text{Bun}$ and $\chi: \text{Loc} \to \mathcal{B}(1)$ are transversal generically on $\text{Loc}^{[d]}$.

Namely, we will show the following:

**Proposition 19.** The 1-form $\beta_0 = \chi(1)^*\beta'_0$ (where $\beta'_0$ is defined in Proposition 17) vanishes on $q^{-1}(b)$ if $b \in \text{Bun}$ is such that $q$ is smooth over $b$.

Clearly, together with Lemma 18, this implies the desired equality $\beta'_0 = 0$.

Consider the stack $\text{Loc}$ over $\mathbb{A}^1$ whose fiber over $\lambda \in \mathbb{A}^1(k) = k$ is the stack $\text{Loc}_\lambda$ of rank $N$ bundles on $C$ with $\lambda$-connection. The stack $\text{Loc}$ has a canonical $\mathbb{G}_m$-action lifting the dilation action on $\mathbb{A}^1$. Consequently, we have an isomorphism $\text{Loc} \times_{\mathbb{A}^1} \mathbb{G}_m \cong \text{Loc} \times \mathbb{G}_m$. Let $t$ be the coordinate function on $\mathbb{A}^1$. Denote by $\text{Loc}^{\text{sm}}$ the smooth part of $\text{Loc}$ and by $\text{Loc}^{-}$ the maximal open subset in $\text{Loc}$ smooth over $\mathbb{A}^1$. The stack $\text{Loc}$ defines a filtration on functions, differential forms, etc. on $\text{Loc}$ (and on open subsets thereof): a form $\eta$ on $\text{Loc}$ belongs to the $k$’th filtered piece iff the pullback of $\eta$ to $\text{Loc} \times \mathbb{G}_m \hookrightarrow \text{Loc}$ has pole of order not greater than $k$ along $\text{Higgs} = \text{Loc} \times_{\mathbb{A}^1} \{0\}$. This filtration is compatible with the de Rham differential. Similarly, we get a filtration on $\Gamma(\mathcal{F}_{\text{Loc}^{\text{sm}}} = \Gamma(\mathcal{O}_{\text{Loc}^{\text{sm}}}/d\mathcal{O}_{\text{Loc}^{\text{sm}}})$. All these filtrations will be denoted by $\mathcal{F}^*$. 

For example, there is a relative symplectic form on $\widetilde{\text{Loc}}^{\text{sm}}/\mathbb{A}^1$ of weight 1 with respect to the $\mathbb{G}_m$-action. Its restriction to the fibers over 0, 1 $\in \mathbb{A}^1$ are the standard
symplectic forms on Higgs\textsuperscript{sm} and Loc\textsuperscript{sm}. This means that $\Omega_{\text{Loc}} \in F^1\Omega^2(\text{Loc}\textsuperscript{sm})$. It is also straightforward to check that
\begin{equation}
\tilde{\theta} \in F^p\mathcal{F}(\text{Loc}\textsuperscript{sm}).
\end{equation}

**Lemma 20.** We have $\theta_0 \in F^{p+1}\Omega^1(\text{Loc}\textsuperscript{sm})$. Equivalently, $\xi_0 \in F^p\mathcal{T}(\text{Loc}\textsuperscript{sm})$.

**Proof.** We need to prove that $t^p\xi_0$ extends to $\tilde{\text{Loc}}\textsuperscript{sm}$. Recall that the value of $\xi_0$ at a point corresponding to a bundle with connection (or $\mathcal{O}$-coherent $D$-module) $\mathcal{M} = (\mathcal{E}, \nabla)$ is given by the infinitesimal deformation $\tilde{\mathcal{M}} = (\tilde{\mathcal{E}}, \tilde{\nabla})$ of $\mathcal{M}$ constructed in Section 3.1.

We will think of $\tilde{\mathcal{M}}$ as an extension of $\mathcal{M}$ by itself. Recall that the construction of $\tilde{\mathcal{M}}$ uses the splitting of the gerbe $\mathcal{G}$ on the 1st infinitesimal neighborhood of zero section in $T^*\mathcal{C}^{(1)}$. This splitting can be thought of as an extension of $D$-modules $0 \to \mathcal{T}_C^{(1)} = \mathcal{T}_C^{\otimes p} \to \mathcal{O}_0 \to \mathcal{O}_C \to 0$. Denote by $v \in \text{Ext}_D(\mathcal{O}_C, \mathcal{T}_C^{(1)})$ the class of this extension. We will also need the $p$-curvature of $\mathcal{M}$ thought of as a map of $D$-modules $\text{curv}_p(\mathcal{M}) : \mathcal{M} \otimes \mathcal{T}_C^{(1)} \to \mathcal{M}$. By unwinding the definition of $\tilde{\mathcal{M}}$, it is not hard to check the following:

**Claim 21.** The class of $\tilde{\mathcal{M}}$ in $\text{Ext}_D^1(\mathcal{M}, \mathcal{M})$ is given by
\begin{equation}
\text{class}(\tilde{\mathcal{M}}) = \text{curv}_p(\mathcal{M}) \cdot (\text{id}\mathcal{M} \otimes v)
\end{equation}
where $\cdot$ denotes the composition $\text{Hom}_D(\mathcal{M} \otimes \mathcal{T}_C^{(1)}, \mathcal{M}) \otimes \text{Ext}_D^1(\mathcal{M}, \mathcal{M} \otimes \mathcal{T}_C^{(1)}) \to \text{Ext}_D^1(\mathcal{M}, \mathcal{M})$. More precisely, the exact sequence $0 \to \mathcal{M} \to \tilde{\mathcal{M}} \to \mathcal{M} \to 0$ is canonically isomorphic to the pullback of $\mathcal{M} \otimes (0 \to \mathcal{T}_C^{(1)} \to \mathcal{O}_0 \to \mathcal{O}_C \to 0)$ by $\text{curv}_p(\mathcal{M})$.

In order to construct the vector field $\tilde{\xi}_0$ on $\tilde{\text{Loc}}\textsuperscript{sm}$ extending $t^p\xi_0$, recall the notion of $p$-curvature of a $\lambda$-connection from Section 2.2. It allows to extend the above construction of $\tilde{\mathcal{M}}$ to $\lambda$-connections to get the desired vector field $\tilde{\xi}_0$. (We just need to multiply the connection on $\mathcal{M}_0$ by $\lambda$ and use tensor product of $\lambda$-connections.)

**Proof of Proposition 19.** From Lemma 20 and formula (20) we see that $P(\beta_0) = \tilde{\theta} - \delta(\theta_0) \in F^{p+1}\mathcal{F}(\text{Loc}\textsuperscript{sm})$. So we must have
\begin{equation}
\beta_0 \in F^1\Omega^1((\text{Loc}\textsuperscript{sm})^{(1)})
\end{equation}
(otherwise $P(\beta_0) \otimes 1_{\mathcal{O}(\mathbb{G}_m)}$ extended to $\tilde{\text{Loc}}\textsuperscript{sm}$ would have a pole of order $\geq 2p > p+1$ along Higgs). Let $\mathcal{V}$ be the open part of Higgs given by $\mathcal{V} := \text{Higgs}^{\text{sm}([2])} = \text{Loc}^{\text{sm}} \times_{\mathbb{A}^1} \{0\}$. Then we get a 1-form $\beta_1$ on $\mathcal{V}^{(1)}$ obtained by extending the (relative) 1-form $t\beta_0$ from $(\text{Loc}^{\text{sm}})^{(1)} \times \mathbb{G}_m$ to $(\tilde{\text{Loc}}^{\text{sm}})^{(1)}$ and then restricting to $\mathcal{V}^{(1)}$. The form $\beta_1$ has weight 1 with respect to the $\mathbb{G}_m$-action on $\mathcal{V}^{(1)} \subset \text{Higgs}^{(1)}$. For $b \in \text{Bun}$ as in the statement of Proposition 19 (21) implies that $\beta_0|_{\text{Loc}_b}$ is a translation-invariant form on $\text{Loc}_b$, and the restriction of $\beta_1$ to the fiber Higgs\textsuperscript{b} of Higgs at $b$ is the corresponding translation-invariant form on Higgs\textsuperscript{b} (recall that Loc\textsuperscript{b} is an affine space over the vector space Higgs\textsuperscript{b}).

Denoting by Eu the differential of the $\mathbb{G}_m$-action on $\mathcal{V}$, we get a function $F = \iota_{\text{Eu}} \beta_1$ on $\mathcal{V}^{(1)}$ of $\mathbb{G}_m$-weight 1. The restriction of $F$ to Higgs\textsuperscript{b} is a linear function whose differential is $\beta_1|_{\text{Higgs}_b}$. Since the projection $\mathcal{V} \to \mathcal{B}$ is proper over $\mathcal{B}^0$, the restriction of $F$ to each component of Higgs must be a pullback of a function $F'$.
on \(\mathcal{B}\). This function must also have degree 1 with respect to the standard \(\mathbb{G}_m\)-action on \(\mathcal{B}\). We want to show that \(F' = 0\) and hence \(F = 0\). This will imply that \(\beta_1|_{Higgs} = 0\) and therefore \(\beta_0|_{Loc} = 0\).

Since we know that \(\beta_0\) descends to \(\mathcal{B}\), the function \(F'\) does not depend on the choice of connected component of Higgs. Now consider the Serre duality involution \(\sigma\) on \(Loc_{\omega_{1/2}}\). Via the identification \(Loc_{\omega_{1/2}} \sim \rightarrow Loc\) given by \(M \mapsto M \otimes \omega \otimes (p^{-1}/2)\) it corresponds to an involution on \(Loc\) given by \(\mathcal{M} \mapsto \mathcal{M}^\vee \otimes \omega^{\otimes p}\) which we will also denote by \(\sigma\). It is easy to see that the determinant line bundle \((L, \nabla)\) is invariant under \(\sigma\), hence so is its curvature \(\Omega_{Loc}\) and extended curvature \(\tilde{\theta}\). The vector field \(\xi_0\) can also be shown to be invariant under \(\sigma\).

So the 1-form \(\theta_0\) is \(\sigma\)-invariant as well. Recalling that \(P(\beta_0) = \tilde{\theta} - \delta(\theta_0)\), we see that \(\beta_0\) must also be \(\sigma\)-invariant. Thus for the function \(F\) we get that it is invariant under an analogous involution on Higgs. But then for \(F'\) it means that it should be invariant under the action of \(-1 \in \mathbb{G}_m\) on \(\mathcal{B}\), whereas in the preceding paragraph we saw it is \emph{anti-}invariant under the same element. The desired equality \(F' = 0\) follows. \(\square\)

Appendix A. Twisted cotangent bundle to the moduli stack of coherent sheaves

The main goal of this appendix is to show that the twisted cotangent bundle to the stack of coherent sheaves on a smooth projective curve is identified with the stack of half-form twisted coherent \(\mathcal{D}\)-modules on that curve. This is a characteristic-independent statement, except that we have to assume that the characteristic is not 2. In fact, we prove it for arbitrary base scheme \(S\) defined over \(\mathbb{Z}[1/2]\).

We will understand all derived categories in the higher-categorical (i.e. \((\infty, 1)\) or DG) sense, so that it makes sense to talk about homotopies between (1-)morphisms. (In fact, the \((2, 1)\)-categorical level would suffice: all our morphism spaces will be 1-groupoids.)

A.1. Twisted cotangent bundles to stacks. Let \(S\) be a (Noetherian?) scheme and \(\mathcal{X} \to S\) a smooth Artin stack over \(S\). For an \(R\)-point \(x\) of \(\mathcal{X}\) where \(R\) is a commutative ring, consider the groupoid \(\mathcal{T}_{\mathcal{X}/S, x}\) of all dotted arrows in the diagram

\[
\begin{array}{ccc}
\text{Spec } R & \xrightarrow{x} & \mathcal{X} \\
\downarrow \iota & & \downarrow \pi \\
\mathcal{D}_R & \xrightarrow{\pi_{\text{op}}} & S
\end{array}
\]

where \(\mathcal{D}_R = \text{Spec}(R[\varepsilon]/\varepsilon^2)\) and \(\iota: \text{Spec } R \to \mathcal{D}_R, p: \mathcal{D}_R \to \text{Spec } R\) are the natural morphisms. This is an \(R\)-linear Picard groupoid, so it corresponds to a 2-step complex of \(R\)-modules \(T^*_{\mathcal{X}/S, x}\) living in degrees 0 and \(-1\). This complex is perfect, compatible with derived base change and satisfies descent, and therefore defines a perfect object \(T^*_{\mathcal{X}/S} \in \mathcal{D}^b(\mathcal{X})\). It is called the \emph{tangent complex} of \(\mathcal{X}\) over \(S\). The \emph{cotangent stack} is then defined as

\[
T^*(\mathcal{X}/S) := \text{Spec }_{\mathcal{X}}(\text{Sym}_{\mathcal{O}_{\mathcal{X}}} \mathcal{H}^0(T^*_{\mathcal{X}/S}))
\]

where \(\mathcal{H}^0\) is taken with respect to the natural t-structure on \(\mathcal{D}^b(\mathcal{X})\).
Now let $\mathcal{L}$ be a line bundle on $\mathscr{X}$ and $\mathscr{P} \to \mathscr{X}$ the corresponding principle $\mathbb{G}_m$-bundle. Then $\mathcal{T}_{\mathscr{P}/\mathscr{S}}$ is a $\mathbb{G}_m$-equivariant complex and therefore descends to a complex $\mathcal{T}_{\mathscr{X}/\mathscr{S},\mathcal{L}}^\bullet$ on $\mathscr{X}$ which fits into an exact triangle

$$ \mathcal{O}_\mathscr{X} \overset{i}{\to} \mathcal{T}_{\mathscr{X}/\mathscr{S},\mathcal{L}}^\bullet \to \mathcal{T}_{\mathscr{X}/\mathscr{S}}^\bullet \overset{\delta}{\to} \mathcal{O}_\mathscr{X}[1] \tag{23} $$

We will sometimes refer to $\mathcal{T}_{\mathscr{X}/\mathscr{S},\mathcal{L}}^\bullet$ as the extended tangent complex. Now define the twisted cotangent stack $\mathcal{T}_\mathcal{L}^\bullet(\mathscr{X}/\mathscr{S})$ as

$$ \mathcal{T}_\mathcal{L}^\bullet(\mathscr{X}/\mathscr{S}) := \text{Spec}_\mathscr{X}(\text{Sym}_{\mathcal{O}_\mathscr{X}} \mathcal{H}^0(\mathcal{T}_{\mathscr{X}/\mathscr{S},\mathcal{L}}^\bullet)) \times_{\mathbb{A}^1_{\mathscr{S}}} \{1\}_{\mathscr{S}} \tag{24} $$

where the morphism from the first factor to $\mathbb{A}^1_{\mathscr{S}}$ is induced by $i$.

**Remark 2.** There are several alternative interpretations of the stack $\mathcal{T}_\mathcal{L}^\bullet(\mathscr{X}/\mathscr{S})$:

1. It is the spectrum of the quotient of $\text{Sym} \mathcal{H}^0(\mathcal{T}_{\mathscr{X}/\mathscr{S}}^\bullet)$ by the ideal generated by $1 - i(1_{\mathcal{O}_\mathscr{X}})$.
2. Its $R$-points lying over $x$: $\text{Spec } R \to \mathscr{X}$ are given by “splittings” of the pullback under $x$ of triangle (23).
3. These splittings are the same as null-homotopies of $\delta_x$.

There is a closed substack $\mathscr{Z} \subset \mathscr{X}$ consisting of points $x \in \mathscr{X}$ where $i$ acts non-trivially on local cohomology. Then the map $\mathcal{T}_\mathcal{L}^\bullet(\mathscr{X}/\mathscr{S}) \to \mathscr{X}$ factors through $\mathscr{Z}$ and over $\mathscr{Z}$ it looks like a torsor for $T^*(\mathscr{X}/\mathscr{S})$.

Another way to define the complex $\mathcal{T}_{\mathscr{X}/\mathscr{S},\mathcal{L}}^\bullet$ is as follows. For an $R$-point $x$ of $\mathscr{X}$ let $B\mathbb{G}_a(R)$ denote the classifying groupoid for the group $\mathbb{G}_a(R) = (R,+)$. We will define a functor $\delta'_x: \mathcal{T}_{\mathscr{X}/\mathscr{S},x} \to B\mathbb{G}_a(R)$. Namely, for any $\tilde{x}: \mathcal{D}_R \to \mathscr{X}$ as in (22), we set $\delta'_x(\tilde{x})$ to be the torsor of isomorphisms $\tilde{x}^*\mathcal{L} \sim p^*x^*\mathcal{L}$ whose restriction to $\text{Spec } R \subset \mathcal{D}_R$ is $\text{id}_{x^*\mathcal{L}}$. The action of $\mathbb{G}_a(R)$ on $\delta'_x(\tilde{x})$ is given by the composition $\mathbb{G}_a(R) \to 1 + \varepsilon R \subset R[\varepsilon]/\varepsilon^2 = \mathcal{O}(\mathcal{D}_R) \to \text{End}(\tilde{x}^*\mathcal{L})$. The map $\delta'_x$ corresponds to a map $\delta_x: \mathcal{T}_{\mathscr{X}/\mathscr{S},x}^\bullet \to R[1]$. The maps $\delta_x$ for all $R$ and $x$ glue to a map $\delta: \mathcal{T}_{\mathscr{X}/\mathscr{S}}^\bullet \to \mathcal{O}_\mathscr{X}[1]$. Then $\mathcal{T}_{\mathscr{X}/\mathscr{S},\mathcal{L}}^\bullet$ can be reconstructed as “the” cone of $\delta$.

Below we write $\delta^\mathcal{L}$ instead of $\delta$ to show explicitly the dependence of $\delta$ on $\mathcal{L}$.

We will need the following lemma whose straightforward proof is omitted.

**Lemma 22.** Let $f: \mathscr{Y} \to \mathscr{X}$ be a morphism of smooth Artin stacks and $\mathcal{L}$ a line bundle on $\mathscr{Y}$. Then we have a commutative diagram

$$ \begin{array}{ccc} \mathcal{T}_{\mathscr{Y}/\mathscr{S}}^\bullet & \overset{\delta^\mathcal{L}}{\longrightarrow} & \mathcal{O}_{\mathscr{Y}}[1] \\ \downarrow f^* \mathcal{T}_{\mathscr{X}/\mathscr{S}}^\bullet & \mathcal{O}_{\mathscr{Y}}[1] \downarrow \mathcal{O}_{\mathscr{Y}}[1] \end{array} \tag{25} $$

A.2. The stack $\text{Coh}(C)$ and its twisted cotangent bundle. Now let $C \to S$ be a smooth proper family of algebraic curves. We work with schemes over $S$ throughout, so we will often drop “$S$” from the notation writing $\mathcal{T}_C$, $T^*\mathscr{X}$, $\times$, etc. instead of $\mathcal{T}_{\mathscr{X}/S}$, $T^*(\mathscr{X}/S)$, $\times_S$, etc. If $R$ is a ring, and $s$: $\text{Spec } R \to S$ is

\[ \begin{array}{|c|} \hline \text{In order to reconstruct } \mathcal{T}_{\mathscr{X}/\mathscr{S},\mathcal{L}}^\bullet \text{ canonically, one needs to understand the derived categories in the } (\infty,1)-\text{categorical (or DG) sense.} \hline \end{array} \]
an \(R\)-point of \(S\), we will denote by \(C_s\) or \(C_R\) the base change of \(C\) by \(s\), that is, \(C_s = C \times_{Spec \, R} Spec \, s\).

We consider the stack \(\text{Coh}(C)\) of coherent sheaves on fibers of \(C\). Its groupoid of \(R\)-points is given by

\[
\text{Coh}(C)(R) = \left\{ (s, \mathcal{F}) \mid \begin{array}{c} s \in S(R); \\ \mathcal{F} \text{ is an } S'\text{-flat coherent sheaf on } C_s \end{array} \right\}.
\]

Below we abbreviate \(\text{Coh} = \text{Coh}(C)\). Consider the determinant bundle \(L_{\text{det}}\) on \(\text{Coh}\). Its fiber at an \(R\)-point \(x\) of \(\text{Coh}\) corresponding to a coherent sheaf \(\mathcal{F}_x\) on \(C_R\) is given by

\[
(L_{\text{det}})_x = \det R \Gamma(C_R, \mathcal{F}_x).
\]

From now on we assume that \((*) 2\) is invertible on \(S\), i.e. \(2 \cdot 1_{\mathcal{O}(S)} \in \mathcal{O}(S)^\times\).

We will be interested in the twisted cotangent stack corresponding to \(L_{\text{det}}\).

Namely we will prove the following.

**Theorem 23.** There is a canonical isomorphism of stacks over \(\text{Coh}\):

\[
\tilde{T}^*_x \text{Coh} \cong \text{Conn}_{1/2}^{\text{coh}}
\]

where \(\text{Conn}_{1/2}^{\text{coh}}\) is the moduli stack of \(\mathcal{O}_C\)-coherent modules over the algebra \(\mathcal{D}_{C, \omega^{1/2}}\) of differential operators in \(\omega^{1/2}_C\).

Denote \(\tilde{T}^*_{\text{Coh}, L_{\text{det}}}\) by \(\tilde{T}^*_x \text{Coh}\). Now it follows from the formula \((24)\) that the datum of an \(R\)-point of \(\tilde{T}^*_x \text{Coh}\) lying over \(x \in \text{Coh}(R)\) is equivalent to the datum of a nilhomotopy of the map

\[
\delta_x : T_{\text{Coh}, x} \to R.
\]

We can therefore reduce the study of \(\tilde{T}^*_x \text{Coh}\) to the study of \(\delta_x\).

It is known\(^4\) that the tangent complex \(T_{\text{Coh}, x}\) to \(\text{Coh}\) at a point \(x \in \text{Coh}(R)\) is canonically isomorphic to \((\text{REnd} \mathcal{F}_x)[1] \in D^b(\text{R-mod})\). So we have to study the map \(\delta_x[-1] : \text{REnd} \mathcal{F}_x \to R\). Using the Serre duality, we get a map

\[
\alpha_x : \mathcal{F}_x \to \mathcal{F}_x \otimes \omega_{C_R/R}[1].
\]

The map \(\alpha_x\) corresponds to an extension

\[
(26) 0 \to \mathcal{F}_x \otimes \omega_{C_R/R} \to \Phi_s(\mathcal{F}_x) \to \mathcal{F}_x \to 0.
\]

It is clear that, for any \(R\) and a map \(s : Spec \, R \to S\), the assignment \(\mathcal{F}_x \mapsto \Phi_s(\mathcal{F}_x)\) defines a functor from the groupoid of \(R\)-flat coherent sheaves on \(C_R\) to itself compatible with base-change of \(R\). In other words, we have a morphism of stacks \(\Phi : \text{Coh} \to \text{Coh}\).

We will show that \(\Phi_s\) extends to non-invertible morphisms of sheaves and, moreover, has the following description:

**Proposition 24.** Let \(\Delta_C^{(2)}\) be the 2nd infinitesimal neighborhood of the diagonal \(\Delta_C \subset C \times C\) and \(p, q : \Delta_C^{(2)} \to C\) the restriction of the two projections \(C \times C \to C\).

\(^4\)At least for the open part of \(\text{Coh}\) parametrizing vector bundles; but the same proof works for all of \(\text{Coh}\).
Denote also by $p_R$, $q_R$ their base-change by $s$: $\text{Spec } R \to S$. We have a canonical isomorphism
\[(27) \quad \Phi_s(\mathcal{F}) \cong q_{R*}(p_R^*\mathcal{F} \otimes (s \times \text{id}_{\Delta_C^{(2)}})^*(p^*\omega_C \otimes q^*\omega_C^{-1})^{\otimes 1/2})\]
where $\otimes 1/2$ denotes the canonical square root of the line bundle on $\Delta_C^{(2)}$ which is trivial on $\Delta_C$. (It exists and is essentially unique due to the assumption $\text{(F).}$)

Denote by $\tilde{\Phi}_s(\mathcal{F})$ the right-hand side of (27).

**Proposition 24**: Implies Theorem 23 According to Remark 2, point 3, for any $x$: $\text{Spec } R \to \text{Coh}$ the elements of $\text{Hom}_{\text{Coh}}(\text{Spec } R, T^*_s \mathcal{L}_{\text{det}} \text{Coh})$ correspond bijectively to null-homotopies of the map $\delta_x$. By the Serre duality isomorphism, these are the same as null-homotopies of $\alpha_x$ and therefore the same as splittings of (26). Now, using (27), it is not hard to see that these splittings correspond to $(s \times \text{id}_C)^*\mathcal{D}_{C,\omega_1/2}$-module structures on $\mathcal{F}_x$.

**Lemma 25.** For any $s$: $\text{Spec } R \to S$ the map of groupoids $\Phi_s: \text{Coh}(R) \to \text{Coh}(R)$ extends to a self-functor of the category of $R$-flat coherent sheaves on $C \times_s \text{Spec } R$. In other words, $\Phi_s$ extends to non-invertible morphisms.

**Proof.** Let $\text{pr}_{1,2}: \text{Coh} \times \text{Coh} \Rightarrow \text{Coh}$ be the projections and $\Sigma: \text{Coh} \times \text{Coh} \to \text{Coh}$ the map classifying direct sum of coherent sheaves. By the properties of determinant, we have a canonical isomorphism
\[(28) \quad \text{pr}_1^*\mathcal{L}_{\text{det}} \otimes \text{pr}_2^*\mathcal{L}_{\text{det}} \cong \Sigma^*\mathcal{L}_{\text{det}}.\]

Applying Lemma 22 to $\mathcal{L}_{\text{det}}$ and $\Sigma$ yields a commutative diagram on the left:

\[
\begin{array}{ccc}
\mathcal{T}_{\text{Coh }} \otimes \text{Coh} & \xrightarrow{\delta_{s}\mathcal{L}_{\text{det}}} & \mathcal{O}_{\text{Coh } \times \text{Coh}}[1] \\
\downarrow \Sigma^* & & \downarrow \\
\Sigma^*\mathcal{T}_{\text{Coh }} \otimes \text{Coh} & \xrightarrow{\delta_{s}\mathcal{L}_{\text{det}}} & \mathcal{O}_{\text{Coh } \times \text{Coh}}[1]
\end{array}
\]

Pulling it back under some $(x, y)$: $\text{Spec } R \to \text{Coh} \times \text{Coh},$ we obtain the diagram on the right (boxed). From (28) we see that the top arrow of this diagram is equal (canonically homotopic to $\delta_{s}\mathcal{L}_{\text{det}}\mathcal{p}_1 + \delta_{s}\mathcal{L}_{\text{det}}\mathcal{p}_2$ where $\mathcal{p}_1, \mathcal{p}_2$ are projections to the summands. Note also that the left vertical arrow is given by direct sum of (derived) endomorphisms.

Now if we apply the Serre duality to the boxed diagram, we get
\[\alpha_{\Sigma(x, y)} = \alpha_x + \alpha_y: \mathcal{F}_x \oplus \mathcal{F}_y \to (\mathcal{F}_x \oplus \mathcal{F}_y) \otimes \omega_{R/R}[1].\]
(Note that the ‘equals’ sign here again really means ‘canonically homotopic.’) In other words, we have an isomorphism
\[\Phi_s(\mathcal{F}_x \oplus \mathcal{F}_y) \cong \Phi_s(\mathcal{F}_x) \oplus \Phi_s(\mathcal{F}_y).\]

We already know that $\Phi_s$ is functorial with respect to isomorphisms, so $\Phi_s(\mathcal{F}_x \oplus \mathcal{F}_y)$ is acted on by the automorphism group of $\mathcal{F}_x \oplus \mathcal{F}_y$. Moreover, since $\Phi$ is compatible with base change, we have an action of the group-scheme over $R$ given by $R' \mapsto \text{Aut}((\mathcal{F}_x \oplus \mathcal{F}_y) \otimes_R R')$. In particular, consider the automorphism $a_f = \begin{pmatrix} \text{id}_{\mathcal{F}_x} & 0 \\ f & \text{id}_{\mathcal{F}_y} \end{pmatrix}$ for some $f: \mathcal{F}_x \to \mathcal{F}_y$. It can be shown that $\Phi(a_f)$ has the form
Lemma 26. Let $\mathcal{L}$ be a line bundle on $\text{Spec } R$ and $p$ an $R$-point of $C$. Let $\gamma_p = (p, \id_{\text{Spec } R}) : \text{Spec } R \to C_R = C \times \text{Spec } R$ be the embedding of the graph of $p$. Then Proposition 24 holds for $\mathcal{L}$.

Proof. Let $x : \text{Spec } R \to \text{Coh}$ be the point corresponding to $\mathcal{F}$ and $\Gamma_p = \gamma_p(\text{Spec } R) = \text{supp } \mathcal{F} \subset C_R$. First we will show that the two extensions of $\mathcal{F} \otimes \omega$ by $\mathcal{F}$ in the two sides of (27) have the same class in $\text{Ext}^1(\mathcal{F}, \mathcal{F} \otimes \omega)$. Note that for the “relative skyscraper” as in the lemma, this $R$-module is identified with $R$. We claim that the classes of both extensions are equal to 1 under this identification. For the RHS of (27) we have pushforward of a line bundle on the 2nd infinitesimal neighborhood of $\Gamma_p$ whose restriction to $\Gamma_p$ is $\mathcal{L}$. Now the statement can be easily seen from the construction of the identification.

For the LHS we are interested in the image of $\alpha_x$ under the projection $\text{RHom}(\mathcal{F}, \mathcal{F} \otimes \omega[1]) \to \text{Ext}^1(\mathcal{F}, \mathcal{F} \otimes \omega)$.

By Serre duality, this map is dual to $\text{End}^\bullet(\mathcal{F})[1] \cong H^{-1}(\mathcal{T}_{\text{Coh},x}[1] \to (\text{REnd} \mathcal{F})[1] \cong \mathcal{T}_{\text{Coh},x}$. Therefore we have to study the restriction of $\delta_x$ to the $(-1)$st cohomology of $\mathcal{T}_{\text{Coh},x}$. This is responsible for the action of infinitesimal automorphisms of $x$ on $(\mathcal{L}_{\text{det}})_x$. The group scheme of automorphisms of $x$ is identified with $G_m \times C$, and it acts on $(\mathcal{L}_{\text{det}})_x$ via the tautological character. This means that $\delta_x$ restricted to $H^{-1}(\mathcal{T}_{\text{Coh},x}[1] \cong (\text{End} \mathcal{F})[1] \cong R[1]$ is the identity. Now the statement about the class in $\text{Ext}^1$ follows from the fact that the Serre duality pairing of $\text{End} \mathcal{F}$ and $\text{Ext}^1(\mathcal{F}, \mathcal{F} \otimes \omega)$ is equal to 1.

Thus we showed that the two extensions in (27) are non-canonically isomorphic.

The set of isomorphisms is a torsor for the group $\mathcal{R}$ which is compatible with base change. So we get a $G_a$-torsor $\mathcal{A}$ on the open piece $\text{Coh}[1] \cong C \times B\mathbb{G}_m$ of $\text{Coh}$ classifying length 1 sheaves, and we need to construct a canonical trivialization of $\mathcal{A}$.

Consider the involution $\sigma$ on $\text{Coh}$ given by Serre duality: $\mathcal{F} \Rightarrow \mathcal{F}^\vee \otimes \omega$. We have $\sigma^*\mathcal{L}_{\text{det}} \cong \mathcal{L}_{\text{det}}$. One can check that the isomorphism $\text{RHom}(\mathcal{F}_x, \mathcal{F}_x \otimes \omega) \cong (\mathcal{T}_{\text{Coh},x})^\vee \cong (\mathcal{T}_{\text{Coh},\sigma(x)})^\vee \cong \text{RHom}(\mathcal{F}_{\sigma(x)}, \mathcal{F}_{\sigma(x)} \otimes \omega)$ is given by taking dual morphism. Hence the morphism $\delta_\sigma$ is dual to $\delta_{\sigma(x)}$, and therefore $\Phi_\sigma(\mathcal{F}_x) \cong (\Phi_\sigma(\mathcal{F}_{\sigma(x)}))^\vee$.

Restricting to $\text{Coh}[1]$, for the torsor $\mathcal{A}$ we get $(\sigma|_{\text{Coh}[1]})^* \mathcal{A} \cong -\mathcal{A}$. On the other hand, any $G_a$-torsor on $\text{Coh}[1] \cong C \times B\mathbb{G}_m$ descends to a torsor $\tilde{\mathcal{A}}$ on $C$. We get $\tilde{\mathcal{A}} \cong -\mathcal{A}$ and thus $\tilde{\mathcal{A}}$ is trivial due to (24), so we are done.

Lemma 27. Suppose $x$ is an $R$-point of $\text{Coh}$ corresponding to a line bundle $\mathcal{L}$ on $C_R$. Then Proposition 24 holds for $x$.

Proof. Consider the $S$-scheme $S' := C_R$ and let $\Delta : C_R \to C_{S'} = C \times_S C \times_S \text{Spec } R$ be the diagonal embedding. Let $\mathcal{L}_{S'}$ be the line bundle on $C_{S'}$ obtained from $\mathcal{L}$ by base-change along $S' \to \text{Spec } R$. Now consider the map of coherent sheaves on $C_{S'}$:

$$f : \mathcal{L}_{S'} \to \Delta_* \mathcal{L} = \Delta_* \Delta^* \mathcal{L}_{S'}.$$
According to Lemma \[25\] we have a map
\[
\Phi_S(f) : \text{pr}_1^* \Phi_s(\mathcal{L}) \to \Phi_S(\Delta_s \mathcal{L})
\]
(we used that \(\Phi\) is compatible with base-change), and therefore, by adjunction, a map
\[
g : \Phi_s(\mathcal{L}) \to \text{pr}_1^* \Phi_S(\Delta_s \mathcal{L}).
\]
By Lemma \[26\] we have \(\Phi_S(\Delta_s \mathcal{L}) \cong \tilde{\Phi}_S(\Delta_s \mathcal{L})\). Also, from the definition of \(\tilde{\Phi}\) one can see that \(\text{pr}_1^* \Phi_S(\Delta_s \mathcal{L}) \cong \tilde{\Phi}_s(\mathcal{L})\). So \(g\) is a map \(\Phi_s(\mathcal{L}) \to \tilde{\Phi}_s(\mathcal{L})\). By construction, \(g\) is a map between extensions of \(\mathcal{L}\) by \(\mathcal{L} \otimes \omega\), so it gives the desired isomorphism.

**Proof of Proposition \[24\] (sketch).** Locally on \(\text{Spec} R\), we can find a resolution of a coherent sheaf \(\mathcal{F}\) on \(C_R\) by direct sums of line bundles. Any two such resolutions are related by a sequence of homotopy equivalences. Thus, due to the functoriality of \(\Phi\) (Lemma \[25\]), we can reduce Proposition \[24\] to Lemma \[27\].

### A.3. The determinant bundle with connection on \(\text{Conn}_{1/2}^{\text{coh}}\)

For a smooth stack \(\mathcal{X}\) and a line bundle \(\mathcal{L}\) on \(\mathcal{X}\), the pullback of \(\mathcal{L}\) to \(\tilde{T}^*_C \mathcal{X}\) carries a canonical connection. Applying this observation to \(\mathcal{X} = \text{Coh}(C)\) and \(\mathcal{L} = L_{\text{det}}\), and taking Theorem \[23\] into account, we get a connection \(\nabla_{\text{det}}\) on the pullback \(L'_{\text{det}}\) of \(L_{\text{det}}\) to \(\text{Conn}_{1/2}^{\text{coh}}\). In this subsection we state some properties of \(L'_{\text{det}}\) and \(\nabla_{\text{det}}\).

Suppose we have an \(S\)-scheme \(S'\), and a short exact sequence of \(S'\)-families of coherent \(\mathcal{D}_{1/2}\)-modules on \(C\), i.e. of \(S'\)-flat coherent sheaves on \(C_{S'}\):

\[
0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0
\]

with compatible (relative) \(\omega^{1/2}\)-connections. Let \(x, x', x'' \in \text{Coh}(C)(S')\) be the corresponding points of \(\text{Coh}\). Then we have the following relation between the pullbacks of the determinant bundle:

\[
(L_{\text{det}})_x \cong (L_{\text{det}})_{x'} \otimes (L_{\text{det}})_{x''}.
\]

Now from the connection \(\nabla_{\text{det}}\) we get connections on both sides of the above isomorphism.

**Lemma 28.** The isomorphism \[29\] is compatible with connections.

### A.3.1. The case of characteristic \(p\)

Now suppose that \(S\) is a scheme in characteristic \(p\), i.e. for a prime \(p\) we have \(pO_S = 0\). Then there is a component \(\mathcal{J}\) of \(\text{Conn}_{1/2}^{\text{coh}}\) classifying irreducible \(\mathcal{D}\)-modules. We identify \(\text{Conn}_{1/2}^{\text{coh}}\) with \(\text{Conn}^{\text{coh}}\) by \(\cdot \otimes \omega^{(p-1)/2}\). Recall the Azumaya algebra \(\mathcal{D}_{C/S}\) on \(T^*(C^{(S)}/S)\) whose pushforward to \(C^{(S)}\) is isomorphic to \(\text{Fr}_{C/S} \mathcal{D}_{C/S}\). For an \(S\)-scheme \(S'\), the \(S'\)-points of \(\mathcal{J}\) correspond to pairs \((y, \mathcal{E})\) where \(y\) is an \(S'\)-point of \(T^*C^{(S)}\), and \(\mathcal{E}\) is a splitting of \(y^* \mathcal{D}_{C/S}\). In other words, \(\mathcal{J}\) is isomorphic to the \(G_m\)-gerbe \(G_\theta\) on \(T^*C^{(1)}\) where \(\theta\) is the canonical 1-form on \(T^*C\). We want to describe the restriction of \((L'_{\text{det}}, \nabla_{\text{det}})\) to \(\mathcal{J}\).

Because of the equivalence \(\mathcal{D}_{C/\text{mod}} \sim \mathcal{D}_{T^*C, \theta/\text{mod}}\), the gerbe \(G_\theta\) also classifies the irreducible modules over \(\mathcal{D}_{T^*C, \theta}\). Therefore we can consider the universal object: this is a coherent sheaf on \(\mathcal{J} \times T^*C\) with connection in the \(T^*C\)-direction and with support given by \(\mathcal{J} \times T^*C^{(1)}\). Now if we apply the relative Frobenius twist over \(\mathcal{J}\) to this sheaf, the resulting sheaf on \(\mathcal{J} \times T^*C^{(1)}\) will have connection along both factors. So we get a \(\mathcal{D}\)-module on \(\mathcal{J} \times T^*C^{(1)}\) supported on the Frobenius twist.
neighborhood of the “diagonal” or, more precisely, of the graph of $v: \mathcal{I} \to T^* C^{(1)}$. The restriction of this $\mathcal{D}$-module to the graph itself is a line bundle on $\mathcal{I}$ with connection which we will denote by $(\mathcal{L}_{\text{univ}}, \nabla_{\text{univ}})$.

**Proposition 29.** There is a canonical isomorphism of line bundles with connections on $\mathcal{I}$:

$$\mathcal{L}\det, \nabla\det \cong (\mathcal{L}_{\text{univ}}, \nabla_{\text{univ}} - v^* \theta)$$

**References**

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[3] R. Bezrukavnikov, I. Mirkovic, D. Rumynin, *Localization of modules for a semi-simple Lie algebra in prime characteristic*, [arxiv:math.NT/0205144](http://arxiv.org/abs/math.NT/0205144)