THE METRIC APPROXIMATION PROPERTY AND LIPSCHITZ-FREE SPACES OVER SUBSETS OF $\mathbb{R}^N$

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Abstract. We prove that for certain subsets $M \subseteq \mathbb{R}^N$, $N \geq 1$, the Lipschitz-free space $\mathcal{F}(M)$ has the metric approximation property (MAP), with respect to an $\ell_2$ norm on $\mathbb{R}^N$. In particular, $\mathcal{F}(M)$ has the MAP whenever $M$ is a finite-dimensional compact convex set. This should be compared with a recent result of Godefroy and Ozawa, who showed that there exists a compact convex subset $M$ of a separable Banach space, for which $\mathcal{F}(M)$ fails the approximation property.

1. Introduction and main results

Given a metric space $(M, d)$, a function $f : M \to \mathbb{R}$ is Lipschitz if

$$\text{Lip}(f) := \sup \left\{ \frac{|f(x) - f(y)|}{d(x, y)} : x \neq y \right\},$$

is finite. Evidently, the Lipschitz constant $\text{Lip}(f)$ of $f$ depends on $d$. A pointed metric space is simply a metric space with a distinguished point $x_0$. Given such a metric space $(M, d)$, we denote by $\text{Lip}_0(M)$ the set of all Lipschitz functions as above which also vanish at $x_0$. This set becomes a Banach space when endowed with the norm defined by $\|f\| = \text{Lip}(f)$. Let $\delta(x), x \in M$, denote the evaluation functional on $\text{Lip}_0(M)$ given by $\langle f, \delta(x) \rangle = f(x)$ for $f \in \text{Lip}_0(M)$. The Lipschitz-free space (or simply free space) $\mathcal{F}(M)$ is defined to be the norm-closed linear span of $\{\delta(x) : x \in M\} \subseteq \text{Lip}_0(M)^*$. The Dirac map $\delta$ is an isometric embedding of $M$ into $\mathcal{F}(M)$. If $M$ is a Banach space, then $\delta$ is non-linear with a linear left inverse given by a barycentre map (see [9, Proposition 2.1 and Lemma 2.4]). It turns out that the dual space $\mathcal{F}(M)^*$ of $\mathcal{F}(M)$ is linearly isometric to $\text{Lip}_0(M)$. Moreover, on bounded subsets of $\text{Lip}_0(M)$, the weak$^*$-topology induced by $\mathcal{F}(M)$ agrees with the topology of pointwise convergence. The precise location of the distinguished point $x_0$ within $M$ is not particularly important for us; given two distinguished points $x_0$ and $x_1$, the map $f \mapsto f - f(x_1)1_M$ is a dual linear isometry between the corresponding spaces of Lipschitz functions. Spaces of Lipschitz functions and their preduals (referred to as Arens-Eells spaces) are studied in the book [17] by Weaver. An introduction to the theory

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Lipschitz-free spaces are related to the study of Lipschitz isomorphism classes of spaces. Indeed, they constitute a tool for abstract linearization of Lipschitz maps in the following sense. If we use the above Dirac map $\delta$ to identify metric spaces $M$ and $N$ with subsets of the corresponding Lipschitz-free spaces $F(M)$ and $F(N)$, respectively, then any Lipschitz map $L$ from the metric space $M$ into the metric space $N$ has an extension to a continuous linear map $\hat{L}$ from $F(M)$ into $F(N)$ which preserves the Lipschitz constant (see [17] or [9, Lemma 2.2]). Moreover, if $N$ is a Banach space, then by composing $\hat{L}$ with the barycentre map we obtain an extension of $L$ to a continuous linear map from $F(M)$ into $N$. So Lipschitz-free spaces can be employed to transfer non-linear problems to a linear setting. Moreover, in [9, Theorem 3.1], Godefroy and Kalton establish the so-called isometric lifting property for separable Banach spaces. As stated in [9, Corollary 3.3], this implies that if a separable Banach space $X$ is isometric to a subset of a Banach space $Y$, then $X$ is already linearly isometric to a subspace of $Y$. This assertion fails in the non-separable case: if $X$ is non-separable and weakly compactly generated, then $X$ does not embed linearly into $F(X)$ (see [9, Section 4]).

Despite their straightforward definition, the linear structure of Lipschitz-free spaces is relatively difficult to analyse and has not been thoroughly described yet. Elucidating the properties of the class of Lipschitz-free spaces has been the topic of recent research and several interesting results have been obtained. The linear isometry between $\text{Lip}_0(\mathbb{R})$ and $L_\infty$, furnished by differentiability almost everywhere, yields a predual linear isometry between $F(\mathbb{R})$ and $L_1$. On the other hand, $F(\mathbb{R}^2)$ is not linearly isomorphic to any subspace of $L_1$, as Naor and Schechtman showed in [16] by the discretization of an argument due to Kislyakov [14]. Finally, the metric spaces whose Lipschitz-free space is linearly isometric to a subspace of $L_1$ were characterized by Godard in [8, Theorem 4.2] as metric spaces isometrically embeddable into an $\mathbb{R}$-tree. In [3, Theorem 3.4], Dalet proved that the Lipschitz-free space over a proper ultrametric space is linearly isometric to the dual of a space which is linearly isomorphic to $c_0$. Recently, Cúth and Doucha managed to relax the assumption and show that the Lipschitz-free space over a separable ultrametric space is linearly isomorphic to $\ell_1$ [2, Theorem 2]. One of the main results of the recent work of Kaufmann states that the Lipschitz-free space $F(X)$ over a Banach space $X$ is linearly isomorphic to $(\sum_{n=1}^{\infty} F(X))_{\ell_1}$ [13, Theorem 3.1]. This yields an analogue of Pelczyński’s decomposition method [13, Corollary 3.2] and enables us to find a class of separable metric spaces whose Lipschitz-free spaces are linearly isomorphic to $F(c_0)$ [13, Corollary 3.4]. This class contains in particular all $C(K)$ spaces where $K$ is an infinite compact metric space, thus this result provides another way of obtaining examples, first exhibited by Dutrieux and Ferenczi, of non-Lipschitz isomorphic Banach spaces having linearly isomorphic Lipschitz-free spaces [4, Theorem 5].

In this note we concentrate on approximation properties enjoyed by certain Lipschitz-free spaces. Recall that a Banach space $X$ has the approximation property (AP), or the $\lambda$-bounded approximation property ($\lambda$-BAP), if the identity operator on $X$ lies in the closure
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of the set of bounded, or uniformly $\lambda$-bounded, finite-rank operators on $X$, respectively, where closure is taken with respect to the topology of uniform convergence on norm-compact subsets of $X$. If $\lambda$ can be taken to be unity then $X$ is said to have the metric approximation property (MAP). A Banach space has the bounded approximation property (BAP) if it has the $\lambda$-BAP for some $\lambda$. In the case of the BAP the closure above can be taken with respect to the strong operator topology.

Godefroy and Kalton in [9, Theorem 5.3] proved that a Banach space $X$ has the $\lambda$-BAP if and only if $\mathcal{F}(X)$ has the $\lambda$-BAP. In view of the aforementioned linearization of Lipschitz maps via Lipschitz-free spaces [9, Lemma 2.2], it follows that the BAP is stable under Lipschitz isomorphisms between Banach spaces. By Godefroy and Ozawa [10, Theorem 4], every separable Banach space $X$ is linearly isometric to a 1-complemented subspace of $\mathcal{F}(K)$, where $K \subseteq X$ is closed, convex and generates $X$. Applying this result to a separable Banach space failing the AP constructed by Enflo [5, Theorem 1] yields a convex norm-compact metric space $K$ such that $\mathcal{F}(K)$ also fails the AP [10, Corollary 5]. However, if $M$ is a countable proper metric space, or a proper ultrametric space, then $\mathcal{F}(M)$ has the MAP [3]. The Lipschitz-free space over the Urysohn space has the MAP too, according to Fonf and Wojtaszczyk [7, Theorem 2.1]. Lately in [2, Theorem 1], Cúth and Doucha even built monotone Schauder bases in Lipschitz-free spaces over separable ultrametric spaces.

Let us focus now on the spaces $\mathbb{R}^N$ and their subsets. To prove the aforementioned equivalence between $X$ having the $\lambda$-BAP and $\mathcal{F}(X)$ having the $\lambda$-BAP, Godefroy and Kalton first show that $\mathcal{F}(\mathbb{R}^N)$ has the MAP with respect to any norm on $\mathbb{R}^N$ [9, Proposition 5.1]. In fact, $\mathcal{F}(\mathbb{R}^N)$ has a finite-dimensional Schauder decomposition [1], which is monotone when considered with respect to the $\ell_1$-norm [15, Theorem 3.1]. This result was extended in [11, Theorem 3.1], where Schauder bases of $\mathcal{F}(\ell_1^N)$ and $\mathcal{F}(\ell_1)$ were found. In [15, Proposition 2.3] it is shown that there is a universal constant $C$, such that $\mathcal{F}(M)$, where $M \subseteq \mathbb{R}^N$ is arbitrary, has the $C\sqrt{N}$-BAP with respect to the Euclidean norm on $\mathbb{R}^N$. One of the applications of Kaufmann’s main result asserts that if $M \subseteq \mathbb{R}^N$ has non-empty interior, then $\mathcal{F}(M)$ is linearly isomorphic to $\mathcal{F}(\mathbb{R}^N)$ [13, Corollary 3.5]. Consequently, when combined with [11, Theorem 3.1], such a $\mathcal{F}(M)$ admits a Schauder basis (having a basis constant which depends on the dimension $N$, as far as the present authors are aware).

Our aim is to show that for certain subsets $M \subseteq \mathbb{R}^N$, the space $\mathcal{F}(M)$ has the MAP with respect to any norm on $\mathbb{R}^N$. The following theorem is our main result.

**Theorem 1.1.** Let $N \geq 1$ and consider $\mathbb{R}^N$ equipped with some norm $\| \cdot \|$. Let a compact set $M \subseteq \mathbb{R}^N$ have the property that given $\xi > 0$, there exists a set $\hat{M} \subseteq \mathbb{R}^N$ and a Lipschitz map $\Psi : \hat{M} \to M$, such that $M \subseteq \text{int}(\hat{M})$, $\text{Lip}(\Psi) \leq 1 + \xi$ and $\|x - \Psi(x)\| \leq \xi$ for all $x \in \hat{M}$. Then the Lipschitz-free space $\mathcal{F}(M)$ has the MAP.

The proof of this theorem can be found in Section 4 and relies on statements from Section 3. The methods we use depend on the geometry of $M$ and, in particular, that of its boundary $\partial M$. In Section 2 we establish a sufficient condition on $\partial M$ for $M$ to satisfy the hypotheses of Theorem 1.1 and thus for $\mathcal{F}(M)$ to admit the MAP with respect to any norm on $\mathbb{R}^N$.

Theorem 1.1 yields the following corollary, which we prove after Corollary 2.4 below.
Corollary 1.2. Let $N \geq 1$ and let $M \subseteq \mathbb{R}^N$ be compact and convex. Then $\mathcal{F}(M)$ has the MAP with respect to any norm on $\mathbb{R}^N$.

The reader should compare this result to [10, Corollary 5] mentioned above, which asserts the existence of a compact convex subset $M$ of an infinite-dimensional separable Banach space, whose Lipschitz-free space $\mathcal{F}(M)$ fails the approximation property.

We do not know if $\mathcal{F}(M)$ has the MAP for all subsets $M \subseteq \mathbb{R}^N$.

2. Locally downwards closed sets

In this section we introduce a class of subsets of $\mathbb{R}^N$ satisfying the assumptions of Theorem 1.1. Given $M \subseteq \mathbb{R}^N$, let $\text{int}(M)$ denote the interior of $M$.

Definition 2.1. Let $N \geq 1$ and $M \subseteq \mathbb{R}^N$. Given open $U \subseteq \mathbb{R}^N$ and $u \in \mathbb{R}^N$, we shall say that $M$ is downwards closed relative to $U$ and $u$ if it is closed and $y - tu \in \text{int}(M)$ whenever $y \in U \cap M$, $t > 0$ and $y - tu \in U$. In addition, we will say that $M$ is locally downwards closed if, for every $x \in \mathbb{R}^N$, there is an open set $U \ni x$ and a vector $u \neq 0$, such that $M$ is downwards closed relative to $U$ and $u$.

It is clear that this notion does not depend on the choice of norm on $\mathbb{R}^N$. To test a set to see if it is locally downwards closed, it is only necessary to check the condition at points of the boundary $\partial M$: if $x \in \text{int}(M)$ or $x \in \mathbb{R}^N \setminus M$, then $M$ is downwards closed with respect to $\text{int}(M)$ and $\mathbb{R}^N \setminus M$, respectively, and any non-zero $u$. Thus, local downwards closure is a regularity condition on $\partial M$. It is designed to mimic the notion that, locally, the boundary is the graph of a continuous function (subject to a suitable change of coordinates), without having to mention any functions in the definition.

Certainly, any closed convex subset of $\mathbb{R}^N$ having non-empty interior is locally downwards closed, as the next proposition shows. Given $x, y \in \mathbb{R}^N$ and $s > 0$, let $[x, y]$ denote the straight line segment between $x$ and $y$ and let $B(x, s)$ and $U(x, s)$ be the closed and open balls in $\mathbb{R}^N$ having centre $x$ and radius $s$ with respect to the Euclidean norm $\|\cdot\|_2$, respectively.

Proposition 2.2. Let $M \subseteq \mathbb{R}^N$, $N \geq 1$, be closed and imagine that the set

$$M_0 = \{ x \in M : [x, y] \subseteq M \text{ for all } y \in M \},$$

contains an interior point $w$. Then $M$ is locally downwards closed.

Proof. Fix $w$ and $r > 0$ such that $U(w, r) \subseteq M_0$. Given $x \in M \setminus \{w\}$, set $u = x - w$ and $r' = \min\{r, \frac{1}{2}\|u\|_2\}$. Given $y \in U(x, r') \cap M$, set $s = r' - \|y - x\|_2 > 0$. Evidently, $U(y + w - x, s) \subseteq U(w, r) \subseteq M_0$, and thus $\text{conv}(U(y + w - x, s) \cup \{y\}) \subseteq M$. In particular, if $0 < t \leq 1$, then $y - tu = y + t(w - x) \in \text{int}(M)$. If $t > 1$ then $y - tu \notin U(x, r')$, because

$$\|x - (y - tu)\|_2 \geq t\|u\|_2 - \|y - x\|_2 > 2r' - r' = r'. \quad \square$$

In Proposition 2.3 below we show that a compact locally downwards closed set $M \subseteq \mathbb{R}^N$ satisfies the hypotheses of Theorem 1.1.
Proposition 2.3. Let $M \subseteq \mathbb{R}^N$, $N \geq 1$, be a compact and locally downwards closed set and let $\xi > 0$. Then for any norm $\| \cdot \|$ on $\mathbb{R}^N$ there exists a set $\hat{M} \subseteq \mathbb{R}^N$ and a Lipschitz map $\Psi : \hat{M} \rightarrow M$, such that $M \subseteq \text{int}(\hat{M})$, $\text{Lip}(\Psi) \leq 1 + \xi$ and $\| x - \Psi(x) \| \leq \xi$ for all $x \in M$.

Most of Section 2 is concerned with proving Proposition 2.3. The next corollary is obtained as an easy consequence.

Corollary 2.4. Let $M \subseteq \mathbb{R}^N$ be a compact and locally downwards closed set. Then $\mathcal{F}(M)$ has the MAP with respect to any norm on $\mathbb{R}^N$.

Proof. The result follows directly from Theorem 1.1 and Proposition 2.3. □

Corollary 2.4 allows us to prove Corollary 1.2.

Proof of Corollary 1.2 Without loss of generality we can assume that $0 \in M$. If $X = \text{span}(M)$, then $M$ has non-empty interior relative to $X$. Working now in $X$, the result follows from Proposition 2.2 and Corollary 2.4. □

To prove Proposition 2.3 we first state and prove a few lemmas. Hereafter, we shall fix $N \geq 1$ and some norm $\| \cdot \|$ on $\mathbb{R}^N$. Unless otherwise stated, all Lipschitz constants are taken with respect to $\| \cdot \|$. The symbol $\| \cdot \|_p, p \in [1, \infty]$, stands for the $\ell_p$-norm on $\mathbb{R}^N$. We will have need of a constant $K > 0$ satisfying $\frac{1}{K} \| \cdot \| \leq \| \cdot \|_1, \| \cdot \|_2 \leq K \| \cdot \|$.

Lemma 2.5. Let $U \subseteq \mathbb{R}^N$ be a convex open set, $k \geq 1$ and let $M \subseteq \mathbb{R}^N$ be downwards closed relative to $U$ and $u_1, \ldots, u_k \in \mathbb{R}^N$. Fix $x \in U \setminus \text{int}(M)$ and $t_i > 0$, $1 \leq i \leq k$, such that $x_i = x + t_i u_i \in U \setminus M$. Then $M \cap \text{conv}(x_1, \ldots, x_k)$ is empty.

Proof. We proceed by induction on $k$. Let $y = \sum_{i=1}^k \lambda_i x_i = x + \sum_{i=1}^k \lambda_i t_i u_i \in \text{conv}(x_1, \ldots, x_k) \subseteq U$, where $\lambda_i \geq 0$ and $\sum_{i=1}^k \lambda_i = 1$. If $k = 1$ then $y = x \notin M$. Now suppose that $k > 1$ and that the statement holds for $k - 1$. We may assume that $\lambda_{k-1}, \lambda_k > 0$. Consider

$$z = y - \lambda_k t_k u_k = \left( \sum_{i=1}^{k-2} \lambda_i x_i \right) + (\lambda_{k-1} + \lambda_k) \tilde{x} \in \text{conv}(x_1, \ldots, x_{k-2}, \tilde{x}),$$

where $\tilde{x} = x + \lambda_{k-1} (\lambda_{k-1} + \lambda_k)^{-1} t_{k-1} u_{k-1}$. From the convexity of $U$ and downwards closure it follows that $\tilde{x} \in U \setminus M$. So, by inductive hypothesis, $z \notin M$. Hence $y \notin M$, again by downwards closure. □

In the construction of $\hat{M}$ and $\Psi$ we will make use of a few auxiliary functions that are perturbations of the identity, both in a Lipschitz and uniform sense. Fix $\theta > 1$, $x \in \mathbb{R}^N$, $r > 0$ and $u \in \mathbb{R}^N$, $\| u \|_2 = 1$. We define the map $T_{\theta, x, r, u} : \mathbb{R}^N \rightarrow \mathbb{R}^N$ by

$$T_{\theta, x, r, u}(y) = y + (\theta - 1)((y - x) \cdot u + r) u,$$  \hspace{1cm} (1)

where $\cdot$ denotes the scalar product. For a map $T : E \rightarrow \mathbb{R}^N$, where $E \subseteq \mathbb{R}^N$, we set

$$\| T \|_\infty = \sup \{ \| T(y) \| : y \in E \}.$$
Lemma 2.6. Let \( \theta, x, r, u \) and \( T_{\theta,x,r,u} \) be as above. Imagine that \( E \subseteq \mathbb{R}^N \) is bounded, and set \( P = \sup \{ \| y - x \| : y \in E \} < \infty \). Then \( \text{Lip}(T_{\theta,x,r,u} - I) \leq K^2(\theta - 1) \) and \( \|(T_{\theta,x,r,u} - I)|_E\|_\infty \leq K(KP + r)(\theta - 1) \).

Proof. Let \( y, z \in \mathbb{R}^N \). Then
\[
\|(T_{\theta,x,r,u} - I)z - (T_{\theta,x,r,u} - I)\|y\| = (\theta - 1)\|(z - x) \cdot u + r - (y - x) \cdot u + r\|u\|
\leq K(\theta - 1)\|z - y\|u\|u\|
\leq K(\theta - 1)\|z - y\|u\|u\|\|u\|
\leq K(\theta - 1)\|z - y\|u\|u\|\|u\|
\]
If \( y \in E \) then
\[
\|(T_{\theta,x,r,u} - I)\|y\| = (\theta - 1)\|(y - x) \cdot u + r\|u\|
\leq K(\theta - 1)\|y - x\|u\|u\| + r\|u\|
\leq K(KP + r)(\theta - 1).
\]

Next, we require a lemma about using partitions of unity to glue together Lipschitz functions. If these functions are sufficiently close to the identity map, both in a Lipschitz and uniform sense, then the resulting map is close in both senses as well.

Lemma 2.7. Let \( \xi > 0 \), \( U \) an open subset of a normed space \( X \), \( (U_j)_{j=1}^k \) an open cover of \( U \), a partition of unity \( f_j : U \rightarrow [0,1], 1 \leq j \leq k \), subordinated to the cover and consisting of \( H \)-Lipschitz functions, and functions \( \psi_j : U \rightarrow X \) such that
\[
\text{Lip}(\psi_j - I), \|\psi_j - I\|_\infty \leq \xi,
\]
\( 1 \leq j \leq k \). Then the function \( \psi : U \rightarrow X \), defined by
\[
\psi(x) = \sum_{j=1}^k f_j(x)\psi_j(x),
\]
(2)
satisfies \( \text{Lip}(\psi - I) \leq (1 + HK)\xi \) and \( \|\psi - I\|_\infty \leq \xi \).

Proof. Let \( x, y \in U \). Then
\[
\|(\psi - I)y - (\psi - I)x\| = \left\| \sum_{j=1}^k f_j(y)((\psi_j(y) - y) - (\psi_j(x) - x)) + (f_j(y) - f_j(x))\psi_j(x) \right\|
\leq \sum_{j=1}^k f_j(y)\|(\psi_j(y) - y) - (\psi_j(x) - x)\|
+ \left\| \sum_{j=1}^k (f_j(y) - f_j(x))(\psi_j(x) - x) \right\|
\]
\[
\xi \|y - x\| + \sum_{j=1}^{k} H \|y - x\| \xi = (1 + Hk)\xi \|y - x\|. 
\]

The other inequality follows easily. \(\square\)

Of course, if Lip(\(\psi - I\)) \(\leq \xi < 1\), then Lip(\(\psi\)) \(\leq 1 + \xi\), and \(\psi^{-1}\) exists and satisfies Lip(\(\psi^{-1}\)) \(\leq 1/(1 - \xi)\), because

\[
\|\psi(y) - \psi(x)\| = \|y - x - ((\psi(x) - x) - (\psi(y) - y))\|
\geq \|y - x\| - \|(\psi(x) - x) - (\psi(y) - y)\|
\geq (1 - \xi) \|y - x\|.
\]

Now we are in a position to prove Proposition 2.3.

Proof of Proposition 2.3. Initially, we make the assumption that \(M\) is connected, in addition to being compact and locally downwards closed. Once we have dealt with the connected case, we show how this assumption can be removed.

For each \(x \in \partial M\), let \(r_x \in (0, 1)\) and \(u_x \in \mathbb{R}^N\), \(\|u_x\|_2 = 1\), such that \(M\) is downwards closed with respect to \(U(x, 2r_x)\) and \(u_x\). Let \(x_1, \ldots, x_k \in \partial M\) such that \((U(x_i, r_i))_{i=1}^k\) is a cover of \(\partial M\), where \(r_i = r_{x_i}\). Set \(U_i = U(x_i, r_i),\ 1 \leq i \leq k\), and \(U_{k+1} = \text{int}(M)\).

Define \(U = \bigcup_{i=1}^{k+1} U_i \supseteq M\), and let \(f_i : U \longrightarrow \mathbb{R}^N,\ 1 \leq i \leq k + 1\), be a partition of unity subordinated to \((U_i)_{i=1}^{k+1}\), such that each \(f_i\) is \(H\)-Lipschitz, for some large enough \(H\).

Select \(w \in \text{int}(M)\) and \(\varepsilon \in (0, \min\{1, \xi\})\) such that \(B(w, \varepsilon) \subseteq M\). Let

\[
s = \min \left\{ r_j/2r_i : 1 \leq i, j \leq k \right\},
\]

fix

\[
\theta = \min \left\{ 1 + \frac{1}{2}K^{-2}(1 + (k + 1)H)^{-1}\varepsilon, 1 + K^{-1}(K \text{diam}(U) + 1)^{-1}\varepsilon, 1 + s \right\},
\]

let \(\psi_i = T_{\theta, x_i, r_i, u_i},\ 1 \leq i \leq k\), where \(u_i = u_{x_i}\) and \(T_{\theta, x_i, r_i, u_i}\) is as in (11), and set \(\psi_{k+1} = I\). From Lemma 2.6 and the fact that \(r_i \leq 1\) for all \(i\), we know that

\[
\text{Lip}(\psi_i - I) \leq K^2(\theta - 1) \quad \text{and} \quad \|\psi_i - I\|_\infty \leq K(K \text{diam}(U) + 1)(\theta - 1).
\]

If \(\psi : U \longrightarrow \mathbb{R}^N\) is the map defined in (2) (in Lemma 2.7), then (3) and (4) yield

\[
\text{Lip}(\psi - I) \leq K^2(1 + (k + 1)H)(\theta - 1) \leq \frac{1}{2}\varepsilon < 1,
\]

and

\[
\|\psi - I\|_\infty \leq K(K \text{diam}(U) + 1)(\theta - 1) \leq \varepsilon.
\]

From above, we know therefore that \(\psi^{-1}\) exists on \(\psi(U)\),

\[
\text{Lip}(\psi^{-1}) \leq \frac{1}{1 - \frac{1}{2}\varepsilon} \leq 1 + \varepsilon \leq 1 + \xi \quad \text{and} \quad \|\psi^{-1} - I\|_\infty \leq \varepsilon \leq \xi.
\]

According to Brouwer’s Theorem of Invariance of Domain, \(\psi(U)\) is open in \(\mathbb{R}^N\). This implies that \(\text{int}(\psi(M))\) is the same, relative to both \(\psi(U)\) and \(\mathbb{R}^N\), and likewise for \(\partial \psi(M)\), so we can use the terms without fear of ambiguity (of course, the same applies to \(\text{int}(M)\) and \(\partial M\), relative to \(U\) and \(\mathbb{R}^N\)). We would like to show that \(M \subseteq \text{int}(\psi(M))\).
First, we show that $M \cap \partial \psi(M)$ is empty. Since $\psi$ is a homeomorphism of $U$ onto $\psi(U)$, we know that $\partial \psi(M) = \psi(\partial M)$. Let $x \in \partial M$ and set $I = \{i \leq k + 1 : x \in U_i\}$. Of course, $I \subseteq \{1, \ldots, k\}$ because $U_{k+1} = \text{int}(M)$. It follows that

$$
\psi(x) = \sum_{i=1}^{k+1} f_i(x) \psi_i(x) = \sum_{i \in I} f_i(x) \psi_i(x),
$$

where $\sum_{i \in I} f_i(x) = 1$ and

$$
\psi_i(x) = x + (\theta - 1)((x - x_i) \cdot u_i + r_i) u_i.
$$

Given $i \in I$, we have $0 < (x - x_i) \cdot u_i + r_i < 2r_i$, because $n - \|x - x_i\|_2 < r_i$. Consequently,

$$
0 < \|\psi_i(x) - x\|_2 \leq 2r_i(\theta - 1) \leq 2r_i s \leq r_j,
$$

whenever $i \in I$ and $1 \leq j \leq k$, by (3).

Set $V = \bigcap_{i \in I} U(x_i, 2r_i)$. Of course, $x \in \bigcap_{i \in I} U(x_i, r_i) \subseteq V,$ and by (6), $\psi_i(x) \in V$ whenever $i \in I$ as well. Since $M$ is downwards closed relative to $U(x_i, 2r_i)$ and $u_i$, we must have $\psi_i(x) \notin M$, because $x \notin \text{int}(M)$. Thus, from Lemma 2.5 we see that

$$
\psi(x) = \sum_{i \in I} f_i(x) \psi_i(x) \notin M.
$$

In particular, $M \cap \partial \psi(M) = M \cap \psi(\partial M)$ is empty.

Since $M \cap \partial \psi(M)$ is empty, we can write $M$ as the union of two disjoint sets $M \cap \text{int}(\psi(M))$ and $M \setminus \psi(M)$, which are both open in $M$. Since $\psi(w) \in B(w, \varepsilon) \subseteq M$, we have $\psi(w) \in \psi(\text{int}(M)) \cap M = \text{int}(\psi(M)) \cap M$. Therefore, by the connectedness of $M$, we know that $M = M \cap \text{int}(\psi(M)) \subseteq \text{int}(\psi(M))$, as claimed. We complete the proof in the connected case by setting $\hat{M} = \psi(M)$ and $\hat{\Psi} = \psi^{-1}$, and considering (5).

We approach the general case by showing that a compact and locally downwards closed set decomposes into finitely many connected components, each one locally downwards closed. Then we apply what we have done above to each component and glue the results together.

To prove that $M$ has just finitely many connected components, we begin by showing that if $x \in M$, then there exists an open set $V \ni x$ such that $M \cap V$ is connected. Indeed, pick an open Euclidean ball $U$ having centre $x$ and $u \neq 0$ such that $M$ is downwards closed relative to $U$ and $u$, fix any $t > 0$ such that $x - tu \in U$, and then fix $r > 0$ such that $U(x - tu, r) \subseteq M \cap U$, which exists by virtue of downwards closure. We claim that $M \cap V$ is path-connected, where $V$ is the open set

$$
V = \{ y \in \mathbb{R}^N : \|y - (x - su)\|_2 < r \text{ for some } s \in [0, t] \} \subseteq U.
$$

Indeed, if $y \in M$ and $\|y - (x - su)\|_2 < r$ for some $s \in [0, t]$, then local downwards closure guarantees that $[y, y - (t - s)u] \subseteq M \cap V$, and as $y - (t - s)u \in U(x - tu, r) \subseteq M \cap V$, path-connectivity is evident.

Now imagine, for a contradiction, that $M$ possesses infinitely many connected components. Extract a sequence $(x_n) \subseteq M$ such that each $x_n$ belongs to a different component, and let $x \in M$ be a limit point of this sequence. From above, there exists an open set
Given \( \| x - y \| \geq \alpha \) whenever \( x \) and \( y \) lie in distinct components. Select \( \xi > 0 \), set \( \xi' = \min \{ \frac{1}{2} \xi \alpha, \frac{1}{4} \alpha \} \). Using the result above for connected sets, take \( \tilde{M}_i \subseteq \mathbb{R}^N \) and maps \( \Psi_i : \tilde{M}_i \to M_i \), such that \( M_i \subseteq \text{int}(\tilde{M}_i) \), \( \text{Lip}(\Psi_i) \leq 1 + \xi' \) and \( \| x - \Psi_i(x) \| \leq \xi' \) whenever \( x \in \tilde{M}_i \) and \( 1 \leq i \leq p \). The \( \tilde{M}_i \) are pairwise disjoint, because the existence of \( x \in \tilde{M}_i \cap \tilde{M}_j, i \neq j \), would imply that

\[
\alpha \leq \| \Psi_i(x) - \Psi_j(x) \| \leq \| \Psi_i(x) - x \| + \| x - \Psi_j(x) \| \leq \frac{1}{2} \alpha.
\]

Define \( \hat{M} = \bigcup_{i=1}^p \tilde{M}_i \) and \( \Psi : \hat{M} \to M \) by \( \Psi(x) = \Psi_i(x) \) whenever \( x \in \tilde{M}_i \). Certainly, \( \| x - \Psi(x) \| \leq \xi \). Moreover, given \( x, y \in M \), either they are in the same \( M_i \), giving \( \| \Psi(x) - \Psi(y) \| = \| \Psi_i(x) - \Psi_i(y) \| \leq (1 + \xi) \| x - y \| \), or they are in distinct \( \tilde{M}_i \) and \( \tilde{M}_j \), respectively, whence

\[
\| \Psi(x) - \Psi(y) \| \leq \| \Psi_i(x) - x \| + \| \Psi_j(y) - y \| + \| x - y \| \leq (1 + \xi) \| x - y \|. \tag{7}
\]

3. Preparatory lemmas

In order to prove Theorem 1.1, we will need to demonstrate the existence of certain operators on \( F(M) \). However, we shall work mostly with dual operators on the dual space \( \text{Lip}_0(M) \), because in our opinion \( \text{Lip}_0(M) \) is a more `concrete' space than \( F(M) \) and, as a consequence, the dual operators can be defined and described more easily.

In this section, we prove three lemmas that will be used in the proof of the main result. Lemma 3.1 makes use of small Lipschitz perturbations of the identity to map Lipschitz functions on \( M \) to Lipschitz functions on a slightly enlarged set, without changing the Lipschitz constants very much. Lemma 3.2 concerns the convolution of Lipschitz functions to make them smooth, and Lemma 3.3 addresses the problem of approximating certain smooth Lipschitz functions by `coordinatewise affine interpolation', again without increasing the Lipschitz constants by very much.

Given \( M \subseteq \mathbb{R}^N \) and \( r > 0 \), we define the open set

\[
M(r) = \{ x \in \mathbb{R}^N : d_2(x, \mathbb{R}^N \setminus M) > r \},
\]

where \( d_2(\cdot, E) \) denotes distance to the set \( E \) with respect to the Euclidean norm \( \| \cdot \|_2 \).

**Lemma 3.1.** Let a compact set \( M \subseteq \mathbb{R}^N \) have the property that given \( \xi > 0 \), there exists a set \( \hat{M} \subseteq \mathbb{R}^N \) and a Lipschitz map \( \Psi : \hat{M} \to M \), such that \( M \subseteq \text{int}(\hat{M}) \), \( \text{Lip}(\Psi) \leq 1 + \xi \) and \( \| x - \Psi(x) \| \leq \xi \) for all \( x \in \hat{M} \). Then, given \( \epsilon > 0 \), there is \( \hat{M} \subseteq \mathbb{R}^N \) such that \( M \subseteq \hat{M}(r) \) for some \( r > 0 \), and there is a dual operator \( Q : \text{Lip}_0(M) \to \text{Lip}_0(\hat{M}) \) (where \( M \) and \( \hat{M} \) share the same distinguished point \( x_0 \)), such that \( \| Q \| \leq 1 + \epsilon \) and

\[
|f(x) - Qf(x)| \leq \epsilon \text{Lip}(f),
\]

whenever \( x \in M \).
Proof. Let \( \varepsilon > 0 \). Let \( \xi \in (0, \frac{1}{2} \varepsilon) \), take \( \hat{M} \) and \( \Psi \) from the hypotheses, and define \( Qf = f \circ \Psi - f(\Psi(x_0))1_{\hat{M}} \). Evidently, \( Q : \text{Lip}_0(M) \rightarrow \text{Lip}_0(\hat{M}), \|Q\| \leq 1 + \varepsilon \), and \( Q \) has predual \( Q_* \) given by \( Q_* \delta_x = \delta_{\Psi(x)} - \delta_{\Psi(x_0)}, x \in \hat{M} \). By compactness and the fact that \( M \subseteq \text{int}(\hat{M}) \), there exists \( r > 0 \) such that \( M \subseteq \hat{M}(r) \). If \( x \in M \) then we estimate
\[
|f(x) - Qf(x)| = |f(x) - f(\Psi(x)) + f(\Psi(x_0)) - f(x_0)|
\leq |f(x) - f(\Psi(x))| + |f(\Psi(x_0)) - f(x_0)|
\leq (\|x - \Psi(x)\| + \|\Psi(x_0) - x_0\|) \text{Lip}(f)
\leq \varepsilon \text{Lip}(f).
\]

We move on to Lemma 3.2. Following [6, pp. 629], we define \( \eta : \mathbb{R}^N \rightarrow [0, \infty) \) by
\[
\eta(x) = \begin{cases}
A \exp \left( \frac{1}{\|x\|_2 - 1} \right) & \text{if } \|x\|_2 < 1, \\
0 & \text{if } \|x\|_2 \geq 1,
\end{cases}
\]
where the constant \( A > 0 \) is chosen so that \( \int_{\mathbb{R}^N} \eta(x) \, dx = 1 \). Next, for each \( s > 0 \), we put
\[
\eta_s(x) = \frac{1}{s^N} \eta \left( \frac{x}{s} \right).
\]
Then the function \( \eta_s \) lies in \( C^\infty(\mathbb{R}^N) \) and satisfies \( \int_{\mathbb{R}^N} \eta_s(x) \, dx = 1 \) and \( \text{supp}(\eta_s) \subseteq B(0, s) \).

Consider a bounded set \( M \subseteq \mathbb{R}^N \) having non-empty interior, and distinguished point \( x_0 \in \text{int}(M) \). Fix \( r > 0 \) small enough so that \( x_0 \in M(r) \), where \( M(r) \) is as in (7). For a locally integrable map \( f : M \rightarrow \mathbb{R} \) and \( x \in M(r) \), define
\[
f_r(x) = (\eta_r * f)(x) = \int_M \eta_r(x - y)f(y) \, dy = \int_{B(0,r)} \eta_r(y)f(x - y) \, dy.
\]
Finally, set
\[
S_r(f) = f_r - f_r(x_0)1_{M(r)}, \quad (8)
\]
on \( M(r) \).

Given a function \( g : \mathbb{R}^N \rightarrow \mathbb{R} \), we denote by \( Dg(x) \) its total derivative at \( x \), should it exist. We shall regard \( Dg(x) \) both as a functional on \( \mathbb{R}^N \) and as an \( n \)-tuple in \( \mathbb{R}^N \), via the usual identification.

**Lemma 3.2.** In the above setting, the mapping \( S_r \) is a dual operator from \( \text{Lip}_0(M) \) to \( \text{Lip}_0(M(r)) \) (where \( M \) and \( M(r) \) have the same distinguished point \( x_0 \)) and satisfies \( \|S_r\| \leq 1 \) and \( S_r(\text{Lip}_0(M)) \subseteq C^\infty(M(r)) \). Moreover, for every \( \varepsilon > 0 \), there exists \( \delta > 0 \) having the property that for every \( f \in \text{Lip}_0(M), x \in M(r) \) and \( h \in \mathbb{R}^N \), \( \|h\| \leq \delta \), such that \( x + h \in M(r) \), we have
\[
|S_r(f)(x + h) - S_r(f)(x) - DS_r(f)(x)[h]| \leq \varepsilon \text{Lip}(f) \|h\|.
\]
Finally, for every \( f \in \text{Lip}_0(M) \) and \( x \in M(r) \),
\[
|S_r(f)(x) - f(x)| \leq 2 \text{Lip}(f)Kr,
\]
(9)
where $K$ is as in Section 2.

**Proof.** Let $f \in \text{Lip}_0(M)$. Obviously, $S_r(f)(x_0) = 0$. For $x, y \in M(r)$, we have

$$|S_r(f)(x) - S_r(f)(y)| = |f_r(x) - f_r(y)|$$

$$= \left| \int_{B(0,r)} \eta_r(z)(f(x - z) - f(y - z)) \, dz \right|$$

$$\leq \int_{B(0,r)} \eta_r(z) \text{Lip}(f) \|x - y\| \, dz$$

$$= \text{Lip}(f) \|x - y\|. \quad (11)$$

So $S_r$ is a well-defined mapping from $\text{Lip}_0(M)$ to $\text{Lip}_0(M(r))$. Furthermore, it is clearly linear and, by (11), bounded with $\|S_r\| \leq 1$.

Since we can identify compactly supported Borel measures on $M$ with elements of $\mathcal{F}(M)$, we can see that the predual operator $(S_r)_*$ can be defined by writing $d((S_r)_* \delta_x) = (\eta_r(x - y) - \eta_r(x_0 - y)) \, dy, \ x \in M(r)$.

The fact that $S_r(f) \in C^\infty(M(r))$ follows from [8, Appendix C.4, Theorem 6 (i)]. For $x \in M(r)$ and $h \in \mathbb{R}^N$, let

$$L(x)[h] = \int_M D\eta_r(x - y)[h] f(y) \, dy = \int_{B(0,r)} D\eta_r(y)[h] f(x - y) \, dy.$$ 

Clearly $L(x)$ is a bounded linear functional on $\mathbb{R}^N$. We will show that $L(x)$ is the derivative of $S_r(f)$ at $x$ and that the differentiability is uniform in the sense of [9]. Indeed, let $h \in \mathbb{R}^N$ be such that $x + h \in M(r)$. Then, by [12, Corollary 4.99],

$$|S_r(f)(x + h) - S_r(f)(x) - L(x)[h]|$$

$$= |f_r(x + h) - f_r(x) - L(x)[h]|$$

$$= \left| \int_M (\eta_r(x + h - y) - \eta_r(x - y) - D\eta_r(x - y)[h]) f(y) \, dy \right|$$

$$\leq \int_M |(\eta_r(x + h - y) - \eta_r(x - y) - D\eta_r(x - y)[h]) f(y)| \, dy$$

$$\leq A(M) \text{Lip}(f) \omega_{D\eta_r}(\|h\|) \|h\|,$$

where $A(M) > 0$ is a constant depending only on $M$, and $\omega_{D\eta_r}$ is the modulus of continuity of $D\eta_r$. Hence $DS_r(f)(x) = L(x)$ and (9) holds whenever $\delta > 0$ is chosen to satisfy $\omega_{D\eta_r}(\delta) \leq \frac{\varepsilon}{A(M)}$.

To conclude, for any $x \in M(r)$,

$$|f_r(x) - f(x)| = \left| \int_{B(0,r)} \eta_r(y)(f(x - y) - f(x)) \, dy \right|$$

$$\leq \int_{B(0,r)} \eta_r(y)|f(x - y) - f(x)| \, dy$$

$$\leq \text{Lip}(f) Kr.$$
Thus, for every $x \in M(r)$, we obtain
\[ |S_r(f)(x) - f(x)| = |f_r(x) - f_r(x_0) - f(x) + f(x_0)| \leq 2 \text{Lip}(f)K r, \]
which yields (10).

Finally we address Lemma 3.3 and the approximation of smooth Lipschitz functions by coordinatewise affine functions. Fix $w \in \mathbb{R}^N$. We define a closed hypercube $C \subseteq \mathbb{R}^N$ having edge length $\delta > 0$ and vertices $v_\gamma \in \mathbb{R}^N$, $\gamma \in \{0,1\}^N$, given by $v_\gamma = w + \delta \gamma$. We will write $V_C$ for the set of all vertices of $C$.

Imagine that $f$ is a real-valued function whose domain of definition includes the set $V_C$. We define the interpolation function $\Lambda(f, C)$ on $\mathbb{R}^N$ by
\[ \Lambda(f, C)(x) = \sum_{\gamma \in \{0,1\}^N} \left( \prod_{i=1}^{N} \left( 1 - \gamma_i + (-1)^{i+1} \frac{x_i - w_i}{\delta} \right) \right) f(v_\gamma). \] (12)

This is the same $\Lambda(f, C)$ as defined in [15, Section 3.1] and [11, Section 2.1], except that in those cases the function is defined inductively, rather than by means of an explicit formula. This function is coordinatewise affine, i.e., $t \mapsto \Lambda(f, C)(x_1, \ldots, x_{i-1}, t, x_{i+1}, \ldots, x_N)$ is affine whenever $1 \leq i \leq N$. Of course, $\Lambda(f, C)$ agrees with $f$ on the vertices of $C$ and, moreover, it is the only coordinatewise affine function to do so. In the following lemma, we estimate the Lipschitz constant of $\Lambda(f, C)$ on $C$, given a certain uniform differentiability assumption on $f$. Below, the sequence $(e_i)_{i=1}^N$ denotes the standard unit vector basis of $\mathbb{R}^N$.

**Lemma 3.3.** Let $\varepsilon > 0$, $U \subseteq \mathbb{R}^N$ be open and let $f : U \subseteq \mathbb{R}^N \to \mathbb{R}$ be a differentiable Lipschitz function. Moreover, suppose there exists $\delta > 0$ such that for each $x \in U$ and each $h \in \mathbb{R}^N$ with $\|h\| \leq K \delta$ and $x + h \in U$, we have
\[ |f(x + h) - f(x) - Df(x)[h]| \leq \varepsilon \|h\|. \] (13)

Then, given a hypercube $C \subseteq U$ as above, having edge length $\delta$, and $\Lambda(f, C)$ as in (12), we have
\[ \text{Lip}(\Lambda(f, C)|_C) \leq K^2 \varepsilon + \text{Lip}(f). \] (14)

In addition, for every $x \in C$,
\[ |\Lambda(f, C)(x) - f(x)| \leq \sqrt{N} K \text{Lip}(f) \delta. \] (15)

**Proof.** Fix a hypercube $C \subseteq U$ having edge length $\delta$ and let $z \in \text{int}(C)$. We take $j \in \{1, \ldots, N\}$ and compute the $j$-th partial derivative of the function $\Lambda(f, C)$ at $z$. By (12), we have
\[ \frac{\partial \Lambda(f, C)}{\partial x_j}(z) = \sum_{\gamma \in \{0,1\}^N} \left( \prod_{i=1}^{N} \left( 1 - \gamma_i + (-1)^{i+1} \frac{z_i - w_i}{\delta} \right) \right) \frac{(-1)^{\gamma_j+1}}{\delta} f(v_\gamma) \]
Thus, by (13),

\[
\left| f(v_γ + δe_j) - f(v_γ) \right| = \frac{f(v_γ + δe_j) - f(v_γ)}{δ} = \left( 1 - \frac{z_j - w_j}{δ} \right) \frac{f(v_γ + δe_j) - f(v_γ)}{δ} + \frac{z_j - w_j}{δ} \frac{f(v_γ + δe_j) - f(v_γ)}{δ} = \left( 1 - γ_j + (-1)^{γ_j + 1} \frac{z_j - w_j}{δ} \right) \frac{f(v_γ + (-1)^{γ_j} δe_j) - f(v_γ)}{(-1)^{γ_j} δ} + \left( 1 - \tilde{γ}_j + (-1)^{\tilde{γ}_j + 1} \frac{z_j - w_j}{δ} \right) \frac{f(v_γ + (-1)^{\tilde{γ}_j} δe_j) - f(v_γ)}{(-1)^{\tilde{γ}_j} δ},
\]

where \( \tilde{γ} \in \{0,1\}^N \) satisfies \( \tilde{γ}_j = 1 \) and \( \tilde{γ}_i = γ_i \) for \( i \neq j \). Hence

\[
\frac{∂Λ(γ, C)}{∂x_j}(z) = \sum_{γ \in \{0,1\}^N} \left( \prod_{i=1}^N \left( 1 - γ_i + (-1)^{γ_i + 1} \frac{z_i - w_i}{δ} \right) \right) \frac{f(v_γ + (-1)^{γ_j} δe_j) - f(v_γ)}{(-1)^{γ_j} δ}.
\]

So, for the total derivative of the function \( Λ(γ, C) \) at \( z \) we obtain

\[
DΛ(γ, C)(z) = \sum_{γ \in \{0,1\}^N} \left( \prod_{i=1}^N \left( 1 - γ_i + (-1)^{γ_i + 1} \frac{z_i - w_i}{δ} \right) \right) \frac{f(v_γ + (-1)^{γ_j} δe_j) - f(v_γ)}{(-1)^{γ_j} δ}.
\]

Let \( |||·||| \) denote the dual of \( ||·|| \). For every \( γ \in \{0,1\}^N \), we have

\[
|||\left( \frac{f(v_γ + (-1)^{γ_j} δe_j) - f(v_γ)}{(-1)^{γ_j} δ} \right)_{j=1}^N \right||| \leq \left| \left( \frac{f(v_γ + (-1)^{γ_j} δe_j) - f(v_γ) - Df(v_γ)[(-1)^{γ_j} δe_j]}{(-1)^{γ_j} δ} \right)_{j=1}^N \right| + ||Df(v_γ)|| ||| f(v_γ + (-1)^{γ_j} δe_j) - f(v_γ) ||| \leq K δ \left( f(v_γ + (-1)^{γ_j} δe_j) - f(v_γ) - Df(v_γ)[(-1)^{γ_j} δe_j] \right)_{j=1}^N ||| + Lip(f).
\]

Thus, by [13],

\[
|||\left( \frac{f(v_γ + (-1)^{γ_j} δe_j) - f(v_γ)}{(-1)^{γ_j} δ} \right)_{j=1}^N \right||| \leq \frac{K δ}{δ} \cdot ε |||(-1)^{γ_j} δe_j|| + Lip(f) \leq K^2 ε + Lip(f).
\]
To conclude, since $D\Lambda(f, C)(z)$ lies in the convex hull of the set
\[ \left\{ \frac{\left( f(v_\gamma + (-1)^\gamma \delta e_j) - f(v_\gamma) \right)}{(-1)^\gamma \delta} \right\}_{j=1}^N : \gamma \in \{0, 1\}^N \right\}, \]
we have
\[ \|D\Lambda(f, C)(z)\| \leq K^2 \varepsilon + \text{Lip}(f). \]

Therefore $\text{Lip}(\Lambda(f, C)|_C) \leq K^2 \varepsilon + \text{Lip}(f)$.

Finally, if $x \in C$, then
\[
|\Lambda(f, C)(x) - f(x)| = \left| \sum_{\gamma \in \{0, 1\}^N} \left( \prod_{i=1}^N \left( 1 - \gamma_i + (-1)^{\gamma_i+1} \frac{x_i - w_i}{\delta} \right) \right) f(v_\gamma) - f(x) \right|
\leq \sum_{\gamma \in \{0, 1\}^N} \left( \prod_{i=1}^N \left( 1 - \gamma_i + (-1)^{\gamma_i+1} \frac{x_i - w_i}{\delta} \right) \right) |f(v_\gamma) - f(x)|
\leq \sum_{\gamma \in \{0, 1\}^N} \left( \prod_{i=1}^N \left( 1 - \gamma_i + (-1)^{\gamma_i+1} \frac{x_i - w_i}{\delta} \right) \right) \|v_\gamma - x\| \text{Lip}(f).
\]
As $\|v_\gamma - x\| \leq K \|v_\gamma - x\|_2 \leq \sqrt{NK} \delta$, we have
\[ |\Lambda(f, C)(x) - f(x)| \leq \sqrt{NK} \text{Lip}(f) \delta. \]
\[
\] \hfill □

\textbf{Remark 3.4.} The interpolation result [15, Lemma 3.2] follows quickly from the proof above. In $\mathbb{R}^N$, $\mathbb{R}^N$ is considered only with respect to $\|\cdot\|_1$. If $\|\cdot\| = \|\cdot\|_1$, then for each $\gamma \in \{0, 1\}^N$ we get
\[
\left\| \left( \frac{f(v_\gamma + (-1)^\gamma \delta e_j) - f(v_\gamma)}{(-1)^\gamma \delta} \right) \right\|_{N} ^\infty = \left\| \left( \frac{f(v_\gamma + (-1)^\gamma \delta e_j) - f(v_\gamma)}{(-1)^\gamma \delta} \right) \right\|_{N} ^\infty \leq \text{Lip}(f).
\]
Consequently, $\|D\Lambda(f, C)(z)\| \leq \text{Lip}(f)$ and $\text{Lip}(\Lambda(f, C)|_C) \leq \text{Lip}(f)$.

\textbf{4. The proof of Theorem 1.1}

The proof of Theorem 1.1 combines the processes spelled out in Lemmas 3.1–3.3: first, we approximate a given Lipschitz function by another Lipschitz function on a slightly larger domain, then we apply a convolution to produce a smooth Lipschitz function, and finally we approximate this smooth function by a number of locally coordinatewise affine functions.

\textbf{Proof of Theorem 1.1.} We will build a sequence $(T_n)_{n=1}^\infty$ of finite-rank dual operators on $\text{Lip}_0(M)$ such that $\|T_n\| \leq 1 + n^{-1}$ for all $n \in \mathbb{N}$ and $T_n(f)(x) \to f(x)$ uniformly, simultaneously in $x \in M$ and $f \in B_{\text{Lip}_0(M)}$. Once we have done this, we indicate why this yields the MAP at the end of the proof.
Let $n \in \mathbb{N}$. By Lemma 3.1, there is $\hat{M}_n \subseteq \mathbb{R}^N$ and
\[
r_n \in \left(0, \frac{1}{n+1}\right),
\]
such that $M \subseteq \hat{M}_n((\frac{1}{2}L + 2)r_n)$ (where $L$ is as in Section 2 and $\hat{M}_n((\frac{1}{2}L + 2)r_n)$ is as in (17)), and there exists a bounded linear operator $Q_n : \text{Lip}_0(M) \to \text{Lip}_0(\hat{M}_n)$ satisfying $\|Q_n\| \leq 1 + (3n)^{-1}$ and
\[
|f(x) - Q_n(f)(x)| \leq n^{-1}\text{Lip}(f),
\]
whenever $f \in \text{Lip}_0(M)$ and $x \in M$. Next, we press the smoothing operator $S_{r_n} : \text{Lip}_0(\hat{M}_n) \to \text{Lip}_0(\hat{M}_n(r_n))$ defined in (8) into service. By Lemma 3.1, $S_{r_n}(\text{Lip}_0(\hat{M}_n)) \subseteq C^\infty(\hat{M}_n(r_n))$, and there exists
\[
\delta_n \in \left(0, \frac{r_n}{2\sqrt{N}K(3n + 1)}\right),
\]
such that for every $g \in \text{Lip}_0(\hat{M}_n)$, every $x \in \hat{M}_n(r_n)$ and every $h \in \mathbb{R}^N$, $\|h\| \leq \delta_n$, satisfying $x + h \in \hat{M}_n(r_n)$, we have
\[
|S_{r_n}(g)(x + h) - S_{r_n}(g)(x) - DS_{r_n}(g)(x)[h]| \leq \frac{1}{3nK^2}\text{Lip}(g)\|h\|.
\]
Consider the cover $C_n$ of $\hat{M}_n(2r_n)$ by hypercubes of edge length $\delta_n$, determined by the mesh $\mathcal{Z}_n = \{x_0 + \delta_n\zeta, \zeta \in \mathbb{Z}^N\}$. In other words,
\[
C_n = \left\{C \subseteq \mathbb{R}^N : C \text{ is a hypercube, } V_C = C \cap \mathcal{Z}_n \text{ and } C \cap \hat{M}_n(2r_n) \neq \emptyset\right\}
\]
(here we recall that $V_C$ is the set of all vertices of $C$). According to (18), we have $\bigcup C_n \subseteq \hat{M}_n(r_n)$. Define $V_n = \bigcup_{C \in C_n} V_C$.

Given $f \in \text{Lip}_0(M)$ and $x \in M$, set
\[
T_n(f)(x) = \Lambda(S_{r_n}(Q_n(f)), C)(x),
\]
whenever $x \in C \in C_n$. Observe that the definition of the $\Lambda$ functions ensures that if $x \in C \cap C'$ and $C, C' \in C_n$, then
\[
\Lambda(S_{r_n}(Q_n(f)), C)(x) = \Lambda(S_{r_n}(Q_n(f)), C')(x).
\]
Therefore $T_n$ is well-defined.

We will show that $T_n$ has the required properties. Fix $f \in B_{\text{Lip}_0(M)}$. To begin with,
\[
T_n(f)(x_0) = S_{r_n}(Q_n(f))(x_0),
\]
because $x_0 \in V_n$. Therefore $T_n(f)(x_0) = 0$. Let $x, y \in M$. Recall that $\|S_{r_n}\| \leq 1$ and $\|Q_n\| \leq 1 + (3n)^{-1}$. If $\|x - y\| \geq r_n$, then
\[
|T_n(f)(x) - T_n(f)(y)|
\]
\[
\leq |S_{r_n}(Q_n(f))(x) - S_{r_n}(Q_n(f))(y)| + 2\sqrt{N}K(1 + (3n)^{-1})\delta_n
\]
by (13)
\[
\leq |S_{r_n}(Q_n(f))(x) - S_{r_n}(Q_n(f))(y)| + (3n)^{-1}r_n
\]
by (18).
On the other hand, if \( \|x - y\| \leq r_n \), then \( z \in [x, y] \) implies \( d_2(z, M) \leq d_2(z, \{x, y\}) \leq \frac{1}{2} \|x - y\|_2 \leq \frac{1}{2} K r_n \), so as \( M \subseteq \hat{M}_n(\frac{1}{2} K + 2) r_n \), the line segment \([x, y]\) lies entirely inside \( \hat{M}_n(2r_n) \subseteq \bigcup \mathcal{C}_n \subseteq \hat{M}_n(r_n) \). Thus, by partitioning \([x, y]\) with respect to the hypercubes through which it passes, the estimate of the Lipschitz constants of the interpolation functions on hypercubes (14), which follows from (19) by Lemma 3.3, yields
\[
|T_n(f)(x) - T_n(f)(y)| \leq ((3n)^{-1} \text{Lip}(Q_n(f)) + \text{Lip}(S_{r_n}(Q_n(f)))) \|x - y\|
\leq (1 + (3n)^{-1})^2 \|x - y\| \leq (1 + n^{-1}) \|x - y\|.
\]

This thus conclude that \( T_n \) is a well-defined mapping on \( \text{Lip}_0(M) \). Moreover, it is obviously a linear operator and \( \|T_n\| \leq 1 + n^{-1} \).

Given \( x \in V_n \), denote by \( \phi_x : \bigcup \mathcal{C}_n \rightarrow \mathbb{R} \) the unique Lipschitz function that is coordinatewise affine on each \( C \in \mathcal{C}_n \) and satisfies \( \phi_x | V_n = 1_{\{x\}} | V_n \). Since \( T_n(\text{Lip}_0(M)) \subseteq \text{span} \{ \phi_x | M : x \in V_n \} \), the operator \( T_n \) is of finite rank.

As stated in Lemmas 3.1 and 3.2, the operators \( Q_n \) and \( S_{r_n} \) are both dual operators. From the definition of the interpolation formula (12), it is easy to see that \( T_n \) is also a dual operator.

Finally, by combining (13), (10) and (17), we get
\[
|T_n(f)(x) - f(x)| \leq |T_n(f)(x) - S_{r_n}(Q_n(f))(x)| + |S_{r_n}(Q_n(f))(x) - Q_n(f)(x)|
+ |Q_n(f)(x) - f(x)|
\leq \sqrt{N} K (1 + (3n)^{-1}) \delta_n + 2K (1 + (3n)^{-1}) r_n + n^{-1},
\]
for \( f \in B_{\text{Lip}_0(M)} \) and \( x \in M \). Then the choice of \( \delta_n \) and \( r_n \) (see (15) and (16), respectively) gives
\[
|T_n(f)(x) - f(x)| \leq \frac{1}{6n(n + 1)} + (2K + 1)n^{-1}.
\]
(20)

Thus \( T_n(f)(x) \rightarrow f(x) \) uniformly, both in \( x \in M \) and \( f \in B_{\text{Lip}_0(M)} \).

Now we explain why this means that \( \mathcal{F}(M) \) has the MAP. Since \( T_n \) is a dual operator, by (20), the finite-rank predual operator \( (T_n)_* \) on \( \mathcal{F}(M) \) satisfies
\[
\|(T_n)_* \delta_x - \delta_x\| \rightarrow 0
\]
for all \( x \in M \). Consequently, \( (T_n)_* \) converges to the identity operator in the strong operator topology. Since \( \|(T_n)_*\| \rightarrow 1 \), we deduce that \( \mathcal{F}(M) \) has the MAP. \( \square \)

References

[1] L. Borel-Mathurin, Approximation properties and non-linear geometry of Banach spaces, Houston J. of Math. 38 (4) (2012) 1135–1148.
[2] M. Cúth and M. Doucha, Lipschitz-free spaces over ultrametric spaces, preprint [http://arxiv.org/abs/1411.2434] .
[3] A. Dalet, Free spaces over some proper metric spaces, preprint [http://arxiv.org/abs/1404.3939] .
[4] Y. Dutrieux and V. Ferenczi, *The Lipschitz free Banach spaces of C(K)-spaces*, Proc. Amer. Math. Soc., 134 (4) (2005) 1039–1044.
[5] P. Enflo, *A counterexample to the approximation problem in Banach spaces*, Acta Math. 130 (1973), 309–317.
[6] L. C. Evans, *Partial differential equations*, American Mathematical Society, Providence, 1998.
[7] V. P. Fonf and P. Wojtaszczyk, *Properties of the Holmes space*, Topology Appl. 155 (14) (2008), 1627–1633.
[8] A. Godard, *Tree metrics and their Lipschitz-free spaces*, Proc. Amer. Math. Soc. 138 (12) (2010) 4311–4320.
[9] G. Godefroy and N. Kalton, *Lipschitz-free Banach spaces*, Studia Math. 159 (1) (2003) 121–141.
[10] G. Godefroy and N. Ozawa, *Free Banach spaces and the approximation properties*, Proc. Amer. Math. Soc. 142 (5) (2014) 1681–1687.
[11] P. Hájek and E. Pernecká, *On Schauder bases in Lipschitz-free spaces*, J. Math. Anal. Appl. 416 (2) (2014) 629–646.
[12] P. Hájek and M. Johanis, *Smooth analysis in Banach spaces*, Walter de Gruyter GmbH, Berlin, 2014.
[13] P. L. Kaufmann, *Products of Lipschitz-free spaces and applications*, preprint [http://arxiv.org/abs/1403.6605](http://arxiv.org/abs/1403.6605).
[14] S.V. Kislyakov, *Sobolev imbedding operators and the nonisomorphism of certain Banach spaces*, Funct. Anal. Appl. 9 (4) (1975) 290–294.
[15] G. Lancien and E. Pernecká, *Approximation properties and Schauder decompositions in Lipschitz-free spaces*, J. Funct. Anal. 264 (2013), 2323–2334.
[16] A. Naor and G. Schechtman, *Planar earthmover is not in L₁*, SIAM J. Comput. 37 (3) (2007) 804–826.
[17] N. Weaver, *Lipschitz Algebras*, World Sci., 1999.

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