A topological metric in 2+1-dimensions

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Real-valued triplet of scalar fields as source gives rise to a metric which tilts the scalar, not the light cone, in 2+1-dimensions. The topological metric is static, regular and characterized by an integer $\kappa = \pm 1, \pm 2, \ldots$. The problem is formulated as a harmonic map of Riemannian manifolds in which the integer $\kappa$ equals to the degree of the map.

Keywords: 2+1-dimensions; Exact solution; Triplet of scalar fields

The topic of metrical kinks has a long history in general relativity [1] which declined recently toward oblivion. On the other hand in a broader sense interest in topological aspects in non-linear field theory, for a number of reasons, remains ever alive. Although these emerge mostly in flat $3 + 1$-dimensional spacetime with the advent of higher/lower dimensions the same topological concepts may find applications in these cases as well.

The aim of this note is to revisit this subject in 2 + 1-dimensions. Motivation for this lies in part by the discovery of a cosmological black hole [2] which became a center of attraction in this particular dimension. Does a topological metric also make a black hole? The answer to this question turns negative, at least in our present study. The derived metric is sourced by a triplet of scalar fields $\phi^a (r, \theta)$, $(a = 1, 2, 3)$ which tilts the light cone, leading to closed timelike curves. Our notation stands as follows: $R$ is the Ricci scalar, $A(r)$, $B(r)$ and $r$ are functions of $r$, $\beta (\theta)$ is a function of $\theta$ and $\lambda(\theta)$ is a Lagrange multiplier. $\phi^a (r, \theta)$ transforms under the symmetry group $O(3)$ and satisfies

$$\phi^a \phi^a = 1.$$  \hspace{1cm} (4)

Variational principle yields the field equation

$$\Box \phi^a = \lambda \phi^a$$  \hspace{1cm} (5)

and the constraint condition (4).

In the sequel we shall make the choice

$$\beta (\theta) = \kappa \theta$$  \hspace{1cm} (6)

with $\kappa = \pm (integer)$ for uniqueness condition. This reduces the action effectively, modulo the time sector, to

$$I = \int r dr \sqrt{\frac{A}{B}} \left( R - \frac{1}{2} B \alpha'^2 - \frac{\kappa^2}{2 r^2} \sin^2 \alpha \right)$$  \hspace{1cm} (7)

in which a prime stands for $\frac{d}{dr}$. With the energy-momentum tensor

$$T^\nu_\mu = \frac{1}{2} \left( \partial_\mu \phi^a \right) \left( \partial^\nu \phi^a \right) - \frac{1}{2} \left( \nabla \phi^a \right)^2 \delta^\nu_\mu$$  \hspace{1cm} (8)

variation with respect to $\alpha (r)$ and Einstein equations

$$G^\mu_\nu = T^\mu_\nu$$  \hspace{1cm} (9)

we obtain the following equations

$$\left( r \sqrt{ABA} \right)' = \frac{\kappa^2}{2 r} \frac{A}{B} \sin 2 \alpha$$ \hspace{1cm} (10)

$$-\frac{2 B'}{r} = B \alpha'^2 + \frac{\kappa^2}{r^2} \sin^2 \alpha$$ \hspace{1cm} (11)

$$2BA' = B \alpha'^2 - \frac{\kappa^2}{r^2} \sin^2 \alpha$$ \hspace{1cm} (12)

$$2A'' - \frac{A'^2}{A} + \frac{A' B'}{B} = - \frac{A}{B} \left( B \alpha'^2 - \frac{\kappa^2}{r^2} \sin^2 \alpha \right).$$ \hspace{1cm} (13)

This system of differential equations admits a number of particular solutions. As an example we give the following.

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1. A black hole solution

This is obtained by

$$\alpha = \pi \over 2$$

(14)

$$A(r) = B(r) = C_0 - {\kappa^2 \over 2} \ln r$$

(15)

where $C_0$ is an integration constant that can be interpreted as mass. The scalar field triplet takes the form

$$\phi^\alpha(\theta) = \begin{pmatrix} \cos \kappa \theta \\ \sin \kappa \theta \\ 0 \end{pmatrix}$$

(16)

which is effectively a doublet of scalars. Ricci scalar of this solution reads

$$R = {\kappa^2 \over 2r^2}$$

(17)

which is singular at $r = 0$. Event horizon $r_h$ of the resulting black hole is

$$r_h = e^{2C_0/\kappa^2}$$

(18)

so that it is characterized by the index $\kappa$. Similarly the Hawking temperature also is stamped by the integer $\kappa^2$. Clearly this is a different situation from the BTZ black hole [2], where the parameter, i.e. cosmological constant (and electric charge) are not integers.

2. The topological solution

Our system of equations (10-13) admits a solution with the choice $A(r) = 1$. Accordingly, Eq. (10) reduces to the Sine-Gordon equation

$$(2\alpha)_{uu} = \sin (2\alpha)$$

(19)

where

$$e^{2u} = \left( {r_0 \over r} \right)^2 - 1$$

(20)

in which $r_0$ is a constant that will be set $r_0 = 1$. In the new variable

$$\rho = \tanh^{-1} r.$$  

(21)

the solution for $\alpha(\rho)$ and $B(\rho)$ become

$$\alpha(\rho) = 2 \tan^{-1} \left( {1 \over \sinh \rho} \right)$$

(22)

$$B(\rho) = \frac{\kappa^2}{\cosh^2 \rho}$$

(23)

so that the resulting 2+1-dimensional line element takes the form

$$ds^2 = -dt^2 + \frac{d\rho^2}{\kappa^2} + (\tanh \rho)^2 d\theta^2.$$  

(24)

This represents a regular, non-black hole, static spacetime. The non-zero geometrical quantities are

$${\text{Ricci scalar}}: R = \frac{4\kappa^2}{\cosh^2 \rho}$$

(25)

$${\text{Kretschmann scalar}}: K = \frac{1}{2} R^2$$

(26)

and the energy-momentum tensor is

$$T^\nu_{\mu} = - \frac{R}{2} \delta^0_{\mu} \delta^\nu_r.$$  

(28)

As a result our triplet of scalar fields take the form

$$\phi^\alpha(\rho, \theta) = \begin{pmatrix} \cos \kappa \theta \\ \sin \kappa \theta \\ \frac{\sinh \rho}{\cosh^2 \rho - 1} \end{pmatrix}.$$  

(29)

This leads for $\rho = 0$, to

$$\phi^\alpha(0, \theta) = \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix}$$

(30)

while for $\rho \to \infty$ we have

$$\phi^\alpha(\infty, \theta) = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$  

(31)

It is observed that between $0 \leq \rho < \infty$ the angle $\alpha(\rho)$ shifts from $-1$ to $+1$, which amounts to the case of one-kink. It should also be remarked that ‘kink’ herein is used in the sense of flip of the $\phi^3$ component of the triplet, not in the sense of light cone tilt. The energy density of the kink is maximum at $\rho = 0$, which decays asymptotically whose energy $E_\kappa$ is

$$E_\kappa = \int_0^\infty \int_0^{2\pi} (-T^t_t) \sqrt{-g} d\rho d\theta = 4\pi |\kappa|.$$  

(32)

We wish to add, for completeness that the $\alpha(r)$ equation can be described as a harmonic map, between two Riemannian manifolds $M$ and $M'$ [3]

$$f^A : M \to M'$$

(33)

which are defined by

$$M' : ds'^2 = da^2 + \sin^2 \alpha d\beta^2$$

$$= g'^A_B df^A df^B, \ (A, B = 1, 2)$$

(34)
\[ M : ds^2 = \frac{d\rho^2}{\kappa^2} + \tanh^2 \rho d\theta^2 \]  
\[ = g_{ab} dx^a dx^b, \quad (a, b = 1, 2). \]  
(35)

The energy functional of the map is defined by

\[ E(f^A) = \int g'_{AB} \frac{df^A}{dx^a} \frac{df^B}{dx^b} g^{ab} \sqrt{g} d^2 x. \]  
(36)

which yields, upon variation the equation for \( \alpha(\rho) \). Note that in this map we consider a priori that \( \alpha = \alpha(\rho) \) and \( \beta = \beta(\theta) \). The degree of harmonic map \( (d) \) is defined in an orthonormal frame \( \{x^i\} \) by

\[ d = \frac{1}{2\pi} \int d^2 x \sin \alpha \left| \frac{\partial (\alpha, \beta)}{\partial (x_1, x_2)} \right| = \kappa \]  
(37)

which equals to the topological charge \[3\].

Although the maps in the original work of Eells and Sampson [4] were considered between unit spheres (in particular \( S^2 \to S^2 \)) in the present problem our map is from \( R^2 \to S^2 \). Unfortunately the non-compact and singular manifolds of general relativity create serious handicaps which prevented a wider application of the concept of degrees of the maps once they are formulated in harmonic forms.

In conclusion we comment that topological properties of field theory were well-defined in a flat space background. Due to the singular and non-compact manifolds of general relativity these concepts found no simple applications in a curved spacetime. In this note we have shown that at least in the \( 2+1 \)-dimensional spacetime the problem can be overcome. The source of our metric is provided by a triplet of scalar fields which may find applications as multiplets of scalar fields in higher-dimensions. It has been shown that the triplet source gives rise to other solutions, such as black holes, besides the topological metric. The technical problems such as the non-linear superposition of Sine-Gordon solutions in a curved space leading to the 'multi-kink' metric remains to be seen.

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