Random diophantine equations of additive type
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I. Introduction.

In this memoir, we investigate the solubility of diagonal diophantine equations

(1.1) \[ a_1x_1^k + a_2x_2^k + \ldots + a_sx_s^k = 0, \]

and the distribution of their solutions. This is a theme that has received much interest in the past (see Vaughan [19], Vaughan and Wooley [20], Heath-Brown [8], Swinnerton-Dyer [16] and the extensive bibliographies in [19, 20]). Our main concern is with the validity of the Hasse principle, and with a bound for the smallest non-zero solution in integers whenever such a solution exists. The approach is of a statistical nature. Very roughly speaking, we shall show that whenever \( s > 4k \) and the vector \( \mathbf{a} = (a_1, \ldots, a_s) \in \mathbb{Z}^s \) is chosen at random, then almost surely the Hasse principle holds for (1.1), and if there are solutions in integers, not all zero, then there is one with \( |x| \ll |\mathbf{a}|^{2/(s-2k-2)} \). Here and later, we write \( |x| = \max |x_j| \).

We now set the scene to describe our results in precise form. To avoid trivialities, suppose throughout that \( k \in \mathbb{N}, k \geq 2 \), and that \( a_j \in \mathbb{Z}\{0\} \). Since (1.1) has the trivial solution \( x = 0 \), it will be convenient to describe the equation (1.1) as soluble over a given field if there exists a solution in that field other than the trivial one. If (1.1) is soluble over \( \mathbb{R} \) and over \( \mathbb{Q}_p \) for all primes \( p \), then (1.1) is called locally soluble. We denote by \( \mathcal{C} = \mathcal{C}(k, s) \) the set of all \( \mathbf{a} \) with \( a_j \in \mathbb{Z}\{0\} \) for which (1.1) is locally soluble. Note that whenever (1.1) is soluble over \( \mathbb{Q} \), then \( \mathbf{a} \in \mathcal{C} \). The inverse conclusion is known as the Hasse principle for the equation (1.1). Recall that when \( k = 2 \), then the Hasse principle holds for any \( s \), as a special case of the Hasse-Minkowski theorem.

Whenever \( s > 2k \), a formal use of the Hardy-Littlewood method leads one to expect an asymptotic formula for the number \( \varrho_n(B) \) of solutions of (1.1) in integers \( x_j \) within the box \( |x| \leq B \). This takes the shape

\[
\varrho_n(B) = B^{s-k}J_\mathbf{a} \prod_p \chi_p(\mathbf{a}) + o(B^{s-k}) \quad (B \to \infty)
\]

where

\[
\chi_p(\mathbf{a}) = \lim_{h \to \infty} p^{h(1-s)} \# \{ 1 \leq x_j \leq p^h : a_1x_1^k + \ldots + a_sx_s^k \equiv 0 \bmod p^h \}
\]

is a measure for the density of the solutions of (1.1) in \( \mathbb{Q}_p \), and similarly, \( J_\mathbf{a} \) is related to the surface area of the real solutions of (1.1) within the box \([-1, 1]^s\). A precise definition of \( J_\mathbf{a} \) is given in (3.7) below.

As we shall see later, a condition milder than the current hypothesis \( s > 2k \) suffices to confirm that the limits (1.3) exist for all primes \( p \), and that the Euler product

\[
\mathcal{S}_\mathbf{a} = \prod_p \chi_p(\mathbf{a})
\]

is absolutely convergent. Moreover, an application of Hensel’s lemma shows that \( \chi_p(\mathbf{a}) \) is positive if and only if (1.1) is soluble in \( \mathbb{Q}_p \). Likewise, one finds that \( J_\mathbf{a} \) is positive if and only if (1.1) is soluble over \( \mathbb{R} \). It follows that (1.1) is locally soluble if and only if

\[
J_\mathbf{a} \mathcal{S}_\mathbf{a} > 0.
\]
Consequently, if (1.2) holds, then the equation (1.1) obeys the Hasse principle.

The validity of (1.2), and hence of the Hasse principle for the underlying diophantine equations, is regarded to be a safe conjecture in the range $s > 2k$, and in the special case $k = 2$, $s > 4$ rigorous proofs of (1.2) are available by various methods (see chapter 2 of [19] for one approach). When $k = 3$, the formula (1.2) is known to hold whenever $s > 8$ (implicit in Vaughan [18]), and the Hasse principle holds for $s > 7$ (Baker [1]). For larger $k$, much less is known. The asymptotic formula (1.2) has been established when $s \geq k^2 \log k \left(1 + o(1)\right)$, and the Hasse principle may be verified when $s \geq k \log k \left(1 + o(1)\right)$, see Ford [10] and Wooley [21]. Although these results fall short of the expected one by a factor of $\log k$, at least, with respect to the number of variables, it seems difficult to establish (1.2) on average over $a$ when $s$ is significantly smaller than in the aforementioned work of Ford. However, one may choose $B$ as a suitable function of $|a|$, say $B = |a|^\theta$, and then investigate whether (1.5) holds for almost all $a$. This approach is successful whenever $s > 4k$ and $\theta$ is approximately as large as $2/(s - 2k)$, and suffices to confirm the conclusions alluded to in the introductory paragraph.

The principal step is contained in the following mean value theorem. Before this is formulated, recall that $a$ is reserved for integral vectors with non-zero entries; this convention applies within the summation below, and elsewhere in this paper. Also, when $s$ is a natural number, let $\hat{s}$ denote the largest even integer strictly smaller than $s$.

**Theorem 1.1.** Let $k \geq 3$ and $s > 4k$. Then there is a positive number $\delta$ such that whenever $A, B$ are real numbers satisfying
\begin{equation}
1 \leq 2B^k \leq A \leq B^{(\hat{s} - 2k)/2},
\end{equation}
one has
\begin{equation}
\sum_{|a| \leq A} \left| \varrho_a(B) - J_a \S_a B^{s-k} \right| \ll A^{s-1-\delta} B^{s-k}.
\end{equation}

This theorem actually remains valid when $k = 2$, but the proof we give below needs some adjustments. We have excluded $k = 2$ from the discussion mainly because in that particular case one can say more, by different methods. Hence, from now on, we assume throughout that $k \geq 3$.

As a simple corollary, we note that subject to the conditions in Theorem 1.1, the number of $a$ with $|a| \leq A$ for which the inequality
\begin{equation}
|\varrho_a(B) - J_a \S_a B^{s-k}| > |a|^{-1} B^{s-k-\delta}
\end{equation}
holds, does not exceed $O(A^{s-\frac{3}{2}\delta})$. To deduce the Hasse principle for those $a$ where (1.7) fails, one needs a lower bound for $J_a \S_a$ whenever this number is non-zero. When $k$ is odd, (1.1) is soluble over $\mathbb{R}$, and one may show that
\begin{equation}
J_a \gg |a|^{-1}
\end{equation}
holds for all $a$. When $k$ is even, (1.1) is soluble over $\mathbb{R}$ if and only if the $a_j$ are not all of the same sign, and if this is the case, then again (1.8) holds. These facts will be demonstrated in §3. For the “singular product” we have the following result.

**Theorem 1.2.** Let $s \geq k + 3$, and let $\eta$ be a positive number. Then there exists a positive number $\gamma$ such that
\begin{equation}
\#\{|a| \leq A : 0 < \S_a < A^{-\eta}\} \ll A^{s-\gamma}.
\end{equation}
We are ready to derive the main result. Let $s > 4k$, and let $\delta$ be the positive number supplied by Theorem 1.1. Suppose that $a \in \mathcal{C}(k, s)$ satisfies $\frac{1}{2}A < |a| \leq A$, and choose $B = A^{2/(s-2k)}$ in accordance with (1.9). In Theorem 1.2, we take $\eta = \delta/(s-2k)$ so that $A^\eta = B^{s/2}$. If $a$ is not counted in Theorem 1.2, then $\mathcal{S}_a \geq A^{-\eta}$, and if $a$ also violates (1.7), then by (1.8) one has

$$
\rho_a(B) \geq J_a \mathcal{S}_a B^{s-k} - |a|^{-1} B^{s-k-\delta} \gg B^{s-k} A^{-1-\eta} \gg A^{1-\eta}.
$$

It follows that (1.4) has an integral solution with $0 < |x| < |a|^{2/(s-2k)}$, for these choices of $a$. The remaining $a \in \mathcal{C}(k, s)$ with $\frac{1}{2}A < |a| \leq A$ are counted in (1.7) or in Theorem 1.2. Therefore, there are at most $O(A^{s-\min(k, \eta)})$ such $a$. We now sum for $A$ over powers of 2 to conclude as follows.

**Theorem 1.3.** Let $s > 4k$. Then, there is a positive number $\theta$ such that the number of $a \in \mathcal{C}(k, s)$ for which the equation (1.1) has no integral solution in the range $0 < |x| \leq |a|^{2/(s-2k)}$, does not exceed $O(A^{s-\theta})$.

Browning and Dietmann [4] have recently shown that whenever $s \geq 4$, then

$$
\# \{a \in \mathcal{C}(k, s) : |a| \leq A \} \gg A^s,
$$

so that the estimate in Theorem 1.3 is indeed a non-trivial one. In particular, it follows that when $s > 4k$, then for almost all $a \in \mathcal{C}(k, s)$, the equation (1.1) is soluble over $\mathbb{Q}$. Since the Hasse principle may fail for $a \in \mathcal{C}(k, s)$ only, this implies that the Hasse principle holds for almost all $a \in \mathcal{C}(k, s)$, but also for almost all $a \in \mathbb{Z}^s$ whenever $s > 4k$. Finally, in the same range for $s$, Theorem 1.3 implies that for almost all $a$ for which (1.1) has non-trivial integral solutions, there exists a solution with $0 < |x| \leq |a|^{2/(s-2k)}$. This last corollary is rather remarkable, in particular since the upper bound on the size of the solution is quite small, and not too far from the best possible one, as the following result shows.

**Theorem 1.4.** Let $s > 2k$, and let $\eta > 0$. Then, there exists a number $c = c(k, s, \eta) > 0$ such that the number of $a$ with $|a| \leq A$ which (1.1) admits an integral solution in the range $0 < |x| \leq c|a|^{1/(s-k)}$, does not exceed $\eta A^s$.

One should compare this with the lower bound (1.9): even among the locally soluble equations (1.1), those that have an integral solution with $0 < |x| < c|a|^{1/(s-k)}$ form a thin set, at least when $c$ is small. It follows that the exponent $2/(s-2k)$ that occurs in Theorem 1.3 cannot be replaced by a number smaller than $1/(s-k)$.

An estimate for the smallest non-trivial solution of an additive diophantine equation is of considerable importance in diophantine analysis, also for applications in diophantine approximation; see Schmidt [13] for a prominent example and Birch [2] for further comments. There are some bounds of this type available in the literature (eg. Pitman [11]), most notably by Schmidt [14] [15]. In this context, it is worth recalling that when $s > k^2$ then the equation (1.1) is soluble over $\mathbb{Q}_p$, for all primes $p$ (Davenport and Lewis [9]). When $k$ is odd, we then expect that (1.1) is soluble over $\mathbb{Q}$, and Schmidt [14] has shown that for any $\varepsilon > 0$ there exists $s_0(k, \varepsilon)$ such that whenever $s \geq s_0$ then any equation (1.1) has an integer solution with $0 < |x| < |a|^\varepsilon$. The number $s_0(k, \varepsilon)$ is effectively computable, but Schmidt’s method only yields poor bounds (see Hwang [9] for a discussion of this matter). When $k$ is even and $s > k^2$, then (1.1) is locally soluble provided only that the $a_j$ are not all of the same sign. In this situation Schmidt [13] demonstrated that there
still is some \( s_0(k, \varepsilon) \) such that whenever at least \( s_0(k, \varepsilon) \) of the \( a_j \) are positive, and at least \( s_0(k, \varepsilon) \) are negative, then the equation (1.1) is soluble in integers with

\[
0 < |x| \leq |\mathbf{a}|^{1/k + \varepsilon};
\]

see also Schlickewei [12] when \( k = 2 \). Schmidt’s result is essentially best possible: if \( a \leq b \) are coprime natural numbers, and \( k \) is even, then any nontrivial solution of

\[
(1.10)
\]

must have \( b|x_1^k + \ldots + x_t^k| \), whence \( |x| \geq (b/s)^{1/k} \). Thus, there are equations (1.1) where the smallest solution is as large as \( |\mathbf{a}|^{1/k} \), even when \( s \) is very large. However, in Theorem 1.3 the exponent \( 2/(s - 2k) \) is smaller than \( 1/k \). It follows that at least when \( k \) is even, the exceptional set for \( \mathbf{a} \) that is estimated in Theorem 1.3, is non-empty. On the other hand, Theorem 1.3 tell us that examples such as (1.10) where the smallest integer solution is large, must be sparse.

We are not aware of any previous attempts to examine additive diophantine equations on average, save for the recent dissertation of Breyer [3]. There, an estimate is obtained that is roughly equivalent to a variant of Theorem 1.1 in which \( B \approx A^{1/k} \), and where the sum over \( \mathbf{a} \) is restricted to a rather unnaturally defined, but reasonably dense subset of \( \mathbb{Z}^s \). In particular, Breyer’s estimates are not of strength sufficient to derive the Hasse principle for almost all equations (1.1) with \( \mathbf{a} \in \mathcal{C}(k, s) \), even when \( s \) is much larger than \( 4k \). Yet, our analysis in section 2 has certain features in common with Breyer’s work, most notably the use of lattice point counts to treat a certain auxiliary equation. The method could be described as an attempt to exchange the roles of coefficients and variables in (1.1). It is a pure counting device, we cannot describe the exceptional sets beyond bounds on their cardinality. We postpone a detailed description of our methods until they are needed in the course of the argument, but remark that the ideas developed herein can be refined further, and may be applied to related problems as well. With more work and a different use of the geometry of numbers, we may advance into the range \( 3k \leq s \leq 4k \). Perhaps more importantly, one may derive results similar to those announced as Theorem 1.3 for the class of general forms of a given degree. Details must be deferred to sequels of this paper.

**Notation.** Our notation is standard, or is otherwise explained within the text. Vectors are typeset in bold, and have dimension \( s \) unless indicated otherwise. The symbol \( \mathbf{a} \) is reserved for tuples \((a_1, \ldots, a_s)\) with non-zero integers \( a_j \). We use \((x_1; \ldots; x_s)\) to denote the greatest common divisor of the integers \( x_j \). The exponential \( \exp(2\pi i \alpha) \) is abbreviated to \( e(\alpha) \). Finally, we apply the familiar \( \varepsilon \)-convention: whenever \( \varepsilon \) occurs in a statement, it is asserted that the statement is valid for any positive real number \( \varepsilon \). Implicit constants in Landau’s or Vinogradov’s symbols are allowed to depend on \( \varepsilon \) in such circumstances.

**II. Applications of the geometry of numbers**

**2.1. An elementary upper bound estimate.** Our first goal is the demonstration of Theorem 1.4. The following lattice point count is the main ingredient.

**Lemma 2.1.** Let \( \mathbf{c} \in \mathbb{Z}^s \) be a primitive vector. Then, for any \( X \geq |\mathbf{c}| \), one has

\[
\#\{x \in \mathbb{Z}^s : |x| \leq X, c_1x_1 + \ldots + c_sx_s = 0\} \ll |\mathbf{c}|^{-1}X^{s-1}.
\]

**Proof.** See Heath-Brown [7], Lemma 1, for example.
By symmetry, it suffices to sum over all \( x \). Since
\[
\varXi(A, B) < d
\]
then the choice \( B = a \) is sufficiently small, Lemma 2.2 supplies the inequality \( \varXi(A, B) \) is counted by \( \varXi(A, B, CA) \). Let \( \varXi(A, B) \) denote the number of all \( a \) with \( |a| \leq A \) for which the equation \( \varXi(A, B) \) has an integral solution with \( 0 < |x| \leq B \). We proceed to derive an upper bound for \( \varXi(A, B) \). Note that whenever \( a \) is counted by \( \varXi(A, B) \), then \( \varXi(B, A) - 1 \geq 1 \).

Hence, on exchanging the order of summation,
\[
\varXi(A, B) \leq \sum_{|a| \leq A} (\varXi(B, A) - 1) = \sum_{0 < |x| \leq B} \#\{|a| \leq A : a_1 x_1^k + \ldots + a_s x_s^k = 0\}
\]
Whenever \( B^k \leq A \), Lemma 2.1 supplies the estimate
\[
\varXi(A, B) \ll \sum_{0 < |x| \leq B} A^{s-1} x_1^k \ldots x_s^k.
\]
By symmetry, it suffices to sum over all \( x \) with \( x_1 = |x| \). We sort the remaining sum according to \( d = (x_1, x_2, \ldots, x_s) \). Then \( d|x_j \) for all \( j \), and we infer that
\[
\varXi(A, B) \ll A^{s-1} \sum_{1 \leq x_1 \leq B} \sum_{d|x_1} \left( \frac{d}{x_1} \right)^k \left( \sum_{y \leq x_1 \atop d|y} 1 \right)^{s-1}.
\]
Since \( x_1/d \geq 1 \) holds for all \( d|x_1 \), it follows that
\[
\varXi(A, B) \ll A^{s-1} \sum_{1 \leq x \leq B} \sum_{d|x} \left( \frac{x}{d} \right)^{s-1-k}.
\]
In particular, this confirms the following.

**Lemma 2.2.** Let \( s \geq k + 3 \), and suppose that \( B^k \leq A \). Then
\[
\varXi(A, B) \ll A^{s-1} B^{s-k}.
\]

The proof of Theorem 1.4 is now straightforward. When \( s > 2k \) and \( 0 < C \leq 1 \), then the choice \( B = CA^{1/(s-k)} \) is admissible in Lemma 2.2. Let \( \eta > 0 \). Then, if \( C \) is sufficiently small, Lemma 2.2 supplies the inequality \( \varXi(A, CA^{1/(s-k)}) \ll \eta A^s \). If \( a \) is a vector such that \( |a| \leq A \) and \( \varXi(A, CA^{1/(s-k)}) \) has an integral solution with \( 0 < |x| < C|a|^{1/(s-k)} \), then \( a \) is also counted by \( \varXi(A, CA^{1/(s-k)}) \), and Theorem 1.4 follows.

**2.2. Another auxiliary mean value estimate.** Our next task is the derivation of an estimate for the number of solutions of a certain symmetric diophantine equation. The result will be one of the cornerstones in the proof of Theorem 1.1. We begin with an examination of a congruence related to \( k \)-th powers.

**Lemma 2.3.** The number of pairs \( (x, y) \in \mathbb{Z}^2 \) with \( |x| \leq B, |y| \leq B \) and \( x^k \equiv y^k \mod d \) does not exceed \( O(B^{1+\epsilon} + B^{2+\epsilon} d^{-2/k}) \).

**Proof.** Pairs with \( xy = 0 \) contribute \( O(B) \). We sort the remaining pairs according to the value of \( e = (x, y) \), and write \( x = a x_0, y = e y_0 \). The congruence implies \( e^k | d \), and then reduces to \( x_0^k \equiv y_0^k \mod d e^{-k} \) with \( 1 \leq |x_0| \leq B/e, 1 \leq |y_0| \leq B/e \) and \( (x_0, y_0) = 1 \). There are \( 2B/e \) choices for \( y_0 \), and since we have now assured that \( (x_0, d e^{-k}) = 1 \), the theory of \( k \)-th power residues and a divisor function estimate yield the bound \( O(1 + B^{1+\epsilon} d^{-1} e^{k-1}) \) for the number of choices for \( x_0 \), for any admissible choice of \( y_0 \). It follows that the number in question does not exceed
\[
O(B + B^{2+\epsilon} \sum_{e^k | d} d^{-1} e^{-k-2} + B \sum_{e^k | d} e^{-1}),
\]
which confirms the lemma.

Now let $t$ be a natural number, and let $V_t(A, B)$ denote the number of solutions of the equation

$$
\sum_{j=1}^{t} a_j(x_j^k - y_j^k) = 0
$$

in integers $a_j, x_j, y_j$ constrained to

$$
0 < |a_j| \leq A, \ |x_j| \leq B, \ |y_j| \leq B, \ x_j^k \neq y_j^k.
$$

**Lemma 2.4.** Let $t \geq 2$, and suppose that $A \geq 2B^k \geq 1$. Then

$$
V_t(A, B) \ll A^{t-1}(B^{t+1} + B^{2t-k})
$$

**Proof.** We have $|x_j^k - y_j^k| \leq 2B^k \leq A$. Therefore, by Lemma 2.1,

$$
V_t(A, B) \ll A^{t-1} \sum_{|x_j| \leq B} \sum_{|y_j| \leq B} \sum_{x_j^k \neq y_j^k} \max_{1 \leq j \leq t} |x_j^k - y_j^k|.
$$

By symmetry, it suffices to estimate the portion of the remaining sum where $|x_j| \leq x_1, |y_j| \leq x_1$ for all $j$. Then $x_1 > 0$, and we deduce that

$$
V_t(A, B) \ll A^{t-1} \sum_{1 \leq x_1 \leq B} \sum_{|y_1| \leq x_1} \sum_{x_1^k \neq y_1^k} \sum_{2 \leq j \leq t} (x_1^k - y_1^k, \ldots, x_1^k - y_1^k).
$$

For any pair $x_1, y_1$ with $x_1^k \neq y_1^k$, the inner sum will now be sorted according to the value of $d = (x_1^k - y_1^k, \ldots, x_1^k - y_1^k)$. Then $d|x_j^k - y_j^k$ for all $j = 1, \ldots, t$. Therefore, by Lemma 2.3,

$$
V_t(A, B) \ll A^{t-1} \sum_{1 \leq x_1 \leq B} \sum_{|y_1| \leq x_1} \sum_{x_1^k \neq y_1^k} \sum_{2 \leq j \leq t} \frac{d}{x_1^k - y_1^k} \sum_{|x_j| \leq x_1} \sum_{|y_j| \leq x_1} \sum_{x_j \equiv y_j \mod d} \frac{1}{x_1^k - y_1^k}.
$$

Here we apply the trivial inequality $(\xi + \eta)^{t-1} \ll \xi^{t-1} + \eta^{t-1}$ that is valid for non-negative reals $\xi, \eta$, and note that a standard divisor argument yields

$$
\sum_{1 \leq x_1 \leq B} \sum_{|y_1| \leq x_1} \sum_{x_1^k \neq y_1^k} \frac{d x_1^{t-1}}{x_1^k - y_1^k} \ll B^\varepsilon \sum_{1 \leq x_1 \leq B} x_1^t \ll B^{t+1+\varepsilon}
$$

so that we now deduce that

$$
V_t(A, B) \ll A^{t-1} \left( B^{t+1+\varepsilon} + B^\varepsilon T_t(B) \right)
$$

where $T_t(B)$ is the sum of the divisors of $B$ up to $B$. This completes the proof of Lemma 2.4.
Then, by a divisor function estimate, 

\[ \Upsilon_t(B) = \sum_{1 \leq x \leq B} \sum_{|y| \leq x} \sum_{d | x^k - y^k} \frac{x^{2t-2} \varepsilon}{x^k - y^k}. \] 

We proceed with examining two cases separately. First suppose that \( 2(t-1) \geq k \). Then, by a divisor function estimate, 

\[ \Upsilon_t(B) \ll B^\varepsilon \sum_{1 \leq x \leq B} \sum_{y^k \neq x^k} \frac{x^{2t-2}}{x^k - y^k}. \]

When \( k \) is even, we group together the two terms \( \pm y \), and then put \( h = x - y \). For \( y \geq 0 \), we have 

\[ x^k - y^k = h(x^{k-1} + \ldots + y^{k-1}) \geq h x^{k-1} \]

whence 

\[ \Upsilon_t(B) \ll B^\varepsilon \sum_{1 \leq x \leq B} \sum_{0 \leq y < x} \frac{x^{2t-2}}{x^k - y^k} \ll B^{2t-k+2\varepsilon}. \]

When \( k \) is odd, then we first consider the terms with \( 0 \leq y < x \). Then, we may argue as in the case where \( k \) is even, and we find that these pairs \((x, y)\) contribute \( O(B^{2t-k+2\varepsilon}) \) to \( \Upsilon_t(B) \). The remaining terms, with \(-x \leq y < 0\), are even simpler to control. Since \( k \) is odd, we have \( x^k - y^k \geq x^k \), and so, 

\[ \sum_{1 \leq x \leq B} \sum_{-x \leq y < 0} \frac{x^{2t-2}}{x^k - y^k} \leq \sum_{1 \leq x \leq B} x^{2t-1-k} \ll B^{2t-k+\varepsilon}. \]

It follows that \( \Upsilon_t(B) \ll B^{2t-k+\varepsilon} \) holds in all cases, and by (2.3), we have now shown that whenever \( 2(t-1) \geq k \), one has 

\[ V_t(A, B) \ll A^{t-1}(B^{t+1+\varepsilon} + B^{2t-k+\varepsilon}), \]

as required.

It remains to investigate the situation where \( 2(t-1) < k \). Here, a divisor function estimate applied within (2.3) yields 

\[ \Upsilon_t(B) \ll \sum_{1 \leq x \leq B} x^{2t-2} \sum_{|y| \leq x} (x^k - y^k)^{2(t-1)/k}. \]

When \( k \) is even, we manipulate this sum much as in the previous case, and find that 

\[ \Upsilon_t(B) \ll \sum_{1 \leq x \leq B} x^{2t-2} \sum_{0 \leq y < x} (x^k - y^k)^{2(t-1)/k} \ll \sum_{1 \leq x \leq B} x^{2t-2-2k(t-1)/k} \sum_{1 \leq h \leq x} h^{2(t-1)/k} \ll B^{2+\varepsilon}. \]

A similar computation yields the same result when \( k \) is odd. Therefore, when \( 2(t-1) < k \), we now deduce from (2.3) that 

\[ V_t(A, B) \ll A^{t-1}(B^{t+1+\varepsilon} + B^{2+\varepsilon}) \ll A^{t-1} B^{t+1+\varepsilon}. \]
This confirms the claim in Lemma 2.4.

Now let \( U_t(A, B) \) denote the number of solutions of (2.1) in integers \( a_j, x_j, y_j \) satisfying
\[
0 < |a_j| \leq A, \quad |x_j| \leq B, \quad |y_j| \leq B.
\]
For any solution counted by \( U_t(A, B) \), let \( r \) be the number of \( j \in \{1, 2, \ldots, t\} \) with \( x_j^k \neq y_j^k \). The contribution to \( U_t(A, B) \) made by solutions with \( r = 0 \) is obviously no larger than \( O(A^t B^t) \). By symmetry, we now deduce that
\[
U_t(A, B) \ll A^t B^t + \sum_{r=1}^{t} (AB)^{t-r} V_r(A, B).
\]

The definition of \( V_1(A, B) \) implies that \( V_1(A, B) = 0 \). We now suppose that \( A \geq 2B^k \geq 1 \), apply Lemma 2.4 to bound \( V_r(A, B) \) for \( 2 \leq r \leq t \), and then deduce the following estimate.

**Theorem 2.5.** Let \( t \geq 2 \). Then, for real numbers \( A, B \) with \( A \geq 2B^k \geq 1 \), one has
\[
U_t(A, B) \ll (AB)^t + A^{t-1} B^{2t-k+\varepsilon}.
\]

### III. Local solubility

#### 3.1. The singular integral

Local solubility of additive equations has been investigated by Davenport and Lewis [6], and by Davenport [5]. The analytic condition (1.5) for local solubility is implicit in [6]. Unfortunately, these prominent references are insufficient for our purposes. A lower bound for \( J_a \mathcal{S}_a \) in terms of \( |a| \) is needed whenever this product is non-zero, at least for almost all \( a \). An estimate of this type is supplied in this section.

We begin with the singular integral. Most of our work is routine, so we shall be brief. When \( \beta \in \mathbb{R}, B > 0 \), let
\[
(3.1) \quad v(\beta, B) = \int_{-B}^{B} e(\beta \xi^k) d\xi.
\]
A partial integration readily confirms the bound
\[
(3.2) \quad v(\beta, B) \ll B(1 + B^k|\beta|)^{-1/k}
\]
whence whenever \( s > k \) one has
\[
(3.3) \quad \int_{-\infty}^{\infty} |v(\beta, B)|^s d\beta \ll B^{s-k}.
\]
We also see that for \( s > k \) and \( a \in (\mathbb{Z}\setminus\{0\})^s \), the integral
\[
(3.4) \quad J_a(B) = \int_{-\infty}^{\infty} v(a_1 \beta, B) \ldots v(a_s \beta, B) d\beta
\]
converges absolutely. By Hölder’s inequality and (3.3),
\[
\int_{-\infty}^{\infty} |v(a_1 \beta, B) \ldots v(a_s \beta, B)| d\beta
\]
\[
\leq \prod_{j=1}^{s} \left( \int_{-\infty}^{\infty} |v(a_j \beta, B)|^s d\beta \right)^{1/s} \ll |a_1 \ldots a_s|^{-1/s} B^{s-k}.
\]
In particular, it follows that

\begin{equation}
J_n(B) \ll |a_1 \ldots a_s|^{-1/k} B^{s-k}.
\end{equation}

The integral \(J_n(B)\) arises naturally as the singular integral in our application of the circle method in section 4. The dependence on \(B\) can be made more explicit. By (3.1), one has \(v(\beta, B) = BV(\beta B^k, 1)\). Now substitute \(\beta\) for \(\beta B^k\) in (3.4) to infer that

\begin{equation}
J_n(B) = B^{s-k} J_n
\end{equation}

where \(J_n = J_n(1)\) is the number that occurs in (1.2), and in Theorem 1.1.

It remains to establish a lower bound for \(J_n\). The argument depends on the parity of \(k\), and we shall begin with the case when \(k\) is even. Throughout, we suppose that

\begin{equation}
|a_s| \geq |a_j| \quad (1 \leq j < s).
\end{equation}

Define \(\sigma_j = a_j/|a_j| \in \{1, -1\}\). Then, by (3.1),

\[
v(a_j, \beta, 1) = 2 \int_0^1 e(a_j \beta^k) d\xi = \frac{2}{k} |a_j|^{-1/k} \int_0^{|a_j|} \eta^{(1-k)/k} e(\sigma_j \eta) d\eta.
\]

Let \(\mathfrak{A} = [0, |a_1|] \times \ldots \times [0, |a_s|]\), and define the linear form \(\tau\) through the equation

\begin{equation}
\sigma_s \tau = \sigma_1 \eta_1 + \ldots + \sigma_s \eta_s.
\end{equation}

Then, we may rewrite (3.3) as

\[J_n = \left(\frac{2}{k}\right)^s |a_1 \ldots a_s|^{-1/k} \int_{-\infty}^\infty \int_{\mathfrak{A}} (\eta_1 \ldots \eta_s)^{(1-k)/k} e(\sigma_s \tau) d\eta d\beta.
\]

Now substitute \(\tau\) for \(\eta_s\) in the innermost integral. Then, by Fubini’s theorem and (3.1),

\[
\int_{\mathfrak{A}} (\eta_1 \ldots \eta_s)^{(1-k)/k} e(\sigma_s \tau) d\eta = \int_{-\infty}^\infty E(\tau) e(\sigma_s \tau) d\tau
\]

where

\begin{equation}
E(\tau) = \int_{\mathfrak{C}(\tau)} (\eta_1 \ldots \eta_{s-1})^{|(1-k)/k} d(\eta_1, \ldots, \eta_{s-1}),
\end{equation}

in which \(\eta_s\) is the linear form defined implicitly by (3.9), and where \(\mathfrak{C}(\tau)\) is the set of all \((\eta_1, \ldots, \eta_{s-1})\) satisfying the inequalities

\[
0 \leq \eta_j \leq |a_j| \quad (1 \leq j < s),
\]

\[
0 \leq \tau - \sigma_s \sigma_1 \eta_1 - \sigma_s \sigma_2 \eta_2 - \ldots - \sigma_s \sigma_{s-1} \eta_{s-1} \leq |a_s|.
\]

It transpires that \(E\) is a non-negative continuous function with compact support, and that for \(\tau\) near 0, this function is of bounded variation. Therefore, by Fourier’s integral theorem,

\[
\lim_{N \to \infty} \int_{-N}^N \int_{-\infty}^\infty E(\tau) e(\sigma_s \tau) d\tau d\beta = E(0),
\]

and we infer that

\begin{equation}
J_n = \left(\frac{2}{k}\right)^s |a_1 \ldots a_s|^{-1/k} E(0).
\end{equation}

In particular, it follows that \(J_n \geq 0\). Also, when all \(a_j\) have the same sign, then \(\mathfrak{C}(0) = \{0\}\), and (3.11) yields \(J_n = 0\).
Now suppose that not all the \( a_j \) are of the same sign. First, consider the situation where \( \sigma_1 = \ldots = \sigma_{s-1} \). Then we have \( \sigma_\sigma \sigma_j = -1 \) (\( 1 \leq j < s \)). By (3.8), we see that the set of \( (\eta_1, \ldots, \eta_{s-1}) \) defined by
\[
\frac{|a_j|}{2s} \leq \eta_j \leq \frac{|a_j|}{s} \quad (1 \leq j < s)
\]
is contained in \( E(0) \), and its measure is bounded below by \( (2s)^{-s}|a_1a_2 \ldots a_{s-1}| \). By (3.10), we now deduce that
\[
E(0) \gg |a_1 \ldots a_{s-1}|^{1/k} |a_s|^{(1-k)/k},
\]
and (3.11) then implies the bound \( J_\alpha \gg |a_s|^{-1} = |a|^{-1} \).

In the remaining cases, both signs occur among \( \sigma_1, \ldots, \sigma_{s-1} \). We may therefore suppose that for some \( r \) with \( 2 \leq r < s \) we have
\[
\sigma_\sigma \sigma_j = -1 \quad (1 \leq j < r), \\
\sigma_\sigma \sigma_j = 1 \quad (r \leq j < s).
\]
Take \( \tau = 0 \) in (3.9). Then \( \eta_\alpha \) is the linear form
\[
(3.12) \quad \eta_\alpha = \eta_1 + \ldots + \eta_{r-1} - \eta_r - \ldots - \eta_{s-1}.
\]
By symmetry, we may suppose that
\[
|a_1| \leq |a_2| \leq \ldots \leq |a_{r-1}|, \\
|a_r| \leq |a_{r+1}| \leq \ldots \leq |a_{s-1}|.
\]
We define \( t \) by \( t = r - 1 \) when \( |a_{r-1}| \leq |a_r| \), and otherwise as the largest \( t \) among \( r, r+1, \ldots, s-1 \) where \( |a_t| \leq |a_{r-1}| \). Now consider the set of \( (\eta_1, \ldots, \eta_{s-1}) \) defined by the inequalities
\[
\frac{|a_j|}{2s} \leq \eta_j \leq \frac{|a_j|}{s} \quad (1 \leq j \leq r-1), \\
\frac{|a_j|}{8s^2} \leq \eta_j \leq \frac{|a_j|}{4s^2} \quad (r \leq j \leq t), \\
\frac{|a_{r-1}|}{8s^2} \leq \eta_j \leq \frac{|a_{r-1}|}{4s^2} \quad (t < j < s).
\]
It is readily checked that on this set, the number \( \eta_\alpha \) defined in (3.12) satisfies the inequalities \( \frac{|a_{r-1}|}{4s^2} \leq \eta_\alpha \leq |a_{r-1}| \). Moreover, the measure of this set is \( \gg |a_1 \ldots a_t|^{1/k} |a_{r-1}|^{(t+2)/k} |a_{r-1}|^{1-k/k} \). By (3.10), it follows that
\[
E(0) \gg |a_1 \ldots a_t|^{1/k} |a_{r-1}|^{(t+2)/k} |a_{r-1}|^{1-k/k},
\]
and again one then deduces from (3.11) the bound \( J_\alpha \gg |a|^{-1} \).

Finally, we discuss the case where \( k \) is odd. The main differences in the treatment occur in the initial steps. When \( k \) is odd, one may transform (3.11) into
\[
v(\sigma_\beta, 1) = \frac{1}{k} |a_j|^{-1/k} \int_0^{\eta_j} \int_{\eta_j^{(1-k)/k}} \int_{\eta_j^{(1-k)/k}} e(\beta \eta) + e(-\beta \eta) d\eta.
\]
Let \( \sigma = (\sigma_1, \ldots, \sigma_s) \) with \( \sigma_j \in \{1, -1\} \). For any such \( \sigma \), define \( \tau \) through (3.9). Then, following through the argument used in the even case, we first arrive at the identity
\[
J_\alpha = k^{-s} |a_1 \ldots a_s|^{-1/k} \sum_{\sigma} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (\eta_1 \ldots \eta_\alpha)^{(1-k)/k} e(\sigma_\alpha \tau \beta) d\eta \eta \beta.
\]
Here the sum is over all \( 2^s \) choices of \( \sigma \). Again as before, we see that each individual summand is non-negative, and when not all of \( \sigma_1, \ldots, \sigma_s \) have the same sign, then
one finds the lower bound \( \gg |a|^{-1} \) for this summand. Thus, we now see that \( J_a \gg |a|^{-1} \) again holds, this time for any choice of \( a \).

For easy reference, we summarize the above results as a lemma.

**Lemma 3.1.** Suppose that \( s > k \). Then the singular integral \( J_a \) converges absolutely, and one has \( 0 \leq J_a \ll |a_1 a_2 \ldots a_s|^{1/s} \). Furthermore, when \( k \) is odd, or when \( k \) is even and \( a_1, \ldots, a_s \) are not all of the same sign, then \( J_a \gg |a|^{-1} \). Otherwise \( J_a = 0 \).

### 3.2. The singular series

In the introduction, we defined the classical singular series as a product of local densities. We briefly recall its representation as a series.

Though this is standard in principle, our exposition makes the dependence on the coefficients \( a \) in (1.1) as explicit as is necessary for the proof of Theorem 1.2 in the next section. Recall that \( k \geq 3 \).

For \( q \in \mathbb{N}, r \in \mathbb{Z} \) define the Gaussian sum

\[
S(q, r) = \sum_{x=1}^q e(rx^{k}/q).
\]

Let \( \kappa(q) \) be the multiplicative function that, on prime powers \( q = p^l \), is given by

\[
\kappa(p^{uk+v}) = p^{-u-1} \quad (u \geq 0, \ 2 \leq v \leq k), \quad \kappa(p^{uk+1}) = kp^{-u-1/2}.
\]

Then, as a corollary to Lemmas 4.3 and 4.4 of Vaughan [19], one has

\[
S(q, r) \ll q\kappa(q)
\]

whenever \((q; r) = 1\), and one concludes that

\[
q^{-1}S(q, r) \ll \kappa(q/(q; r))
\]

holds for all \( q \in \mathbb{N}, r \in \mathbb{Z} \). Now let

\[
T_a(q) = q^{-s} \sum_{r=1 \atop (r,q)=1}^q S(q, a_1 r) \ldots S(q, a_s r).
\]

Then, by (3.14),

\[
T_a(q) \ll q\kappa(q/(q; a_1)) \ldots \kappa(q/(q; a_s)).
\]

Moreover, by working along the proof of Lemma 2.11 of Vaughan [19], one finds that \( T_a(q) \) is a multiplicative function of \( q \). Also, one can use the definition of \( \kappa \) to confirm that whenever \( s \geq k + 2 \) then the expression on the right hand side of (3.16) may be summed over \( q \) to an absolutely convergent series. Thus, we may also sum \( T_a(q) \) over \( q \) and rewrite the series as an Euler product. This gives

\[
\sum_{q=1}^\infty T_a(q) = \prod_p \sum_{h=0}^{\infty} T_a(p^h).
\]

However, by (3.13) and (3.15), and orthogonality,

\[
\sum_{h=0}^{l} T_a(p^h) = p^{-ls} \sum_{r=1}^{p^l} S(p^l, ra_1) \ldots S(p^l, ra_s) = p^{l(1-s)} M_a(p^l)
\]

where \( M_a(p^l) \) is the number of incongruent solutions of the congruence

\[
a_1 x_1^k + \ldots + a_s x_s^k \equiv 0 \ mod \ p^l.
\]

We may take the limit for \( l \to \infty \) in (3.18) because all sums in (3.17) are convergent. This shows that the limit \( \chi_{p, s} \), as defined in (1.3), exists. In view of (3.17) and (1.4),
we may summarize our results as follows.

**Lemma 3.2.** Let \( s \geq k + 2 \). Then, for any \( a \in (\mathbb{Z} \setminus \{0\})^s \), the singular product (1.4) converges, and has the alternative representation

\[
\mathcal{S}_a = \sum_{q=1}^{\infty} T_a(q).
\]

A slight variant of the preceding argument also supplies an estimate for \( \chi_p(a) \) when \( p \) is large.

**Lemma 3.3.** Let \( s \geq k + 2 \). Then there is a real number \( c = c(k, s) \) such that for any choice of \( a_1, \ldots, a_s \in \mathbb{Z} \setminus \{0\} \) for which at least \( k + 2 \) of the \( a_j \) are not divisible by \( p \), one has \( |\chi_p(a) - 1| \leq cp^{-2} \).

**Proof.** We begin with (3.18), and note that \( T_a(1) = 1 \). Then

\[
p^{l(1-s)}M_a(p^l) - 1 = \sum_{h=1}^{l} T_a(p^h).
\]

One has \( \kappa(q) \leq k \) for any prime power \( q \). Hence, by (3.10), and since \( k + 2 \) of the \( a_j \) are coprime to \( p \), one finds that \( |T_a(p^h)| \leq k^s \kappa(p^h)^{k+2}p^h \). Consequently, a short calculation based on the definition of \( \kappa \) reveals that

\[
|p^{l(1-s)}M_a(p^l) - 1| \leq k^s \sum_{h=1}^{l} \kappa(p^h)^{k+2}p^h \leq k^{s+k+2}p^{-2}.
\]

The lemma follows on considering the limit \( l \to \infty \).

**3.3. Proof of Theorem 1.2.** Throughout, we suppose that \( s \geq k + 3 \). For \( a \in (\mathbb{Z} \setminus \{0\})^s \), let \( S(a) \) denote the set of all primes that divide at least two of the integers \( a_j \). Lemma 3.3 may then be applied to all primes \( p \notin S(a) \), and we deduce that there exists a number \( C = C(k, s) > 0 \) such that the inequalities

\[
\frac{1}{2} \leq \prod_{p \notin S(a)} \prod_{p > C} \chi_p(a) \leq 2
\]

hold for all \( a \). It will be convenient to write

\[
P(a) = S(a) \cup \{p : p \leq C\};
\]

this set contains all primes not covered by (3.19). For a prime \( p \in P(a) \), let

\[
l(p) = \max\{l : p^l | a_j \text{ for some } j\},
\]

and then define the numbers

\[
P(a) = \prod_{p \in P(a)} p, \quad P_0(a) = \prod_{p \in S(a) \cup \{p : p > C\}} p, \quad P^\dagger(a) = \prod_{p \in P(a)} p^{l(p)}.
\]

For later use, we note that

\[
P_0(a) | P(a), \quad P(a) | HP_0(a)
\]

in which we wrote

\[
H = \prod_{p \leq C} p.
\]
Now fix a number $\delta > 0$, to be determined later, and consider the sets
\begin{align}
(3.20) \quad & \mathcal{A}_1 = \{ |a| \leq A : P(a) > A^\delta \}, \\
(3.21) \quad & \mathcal{A}_2 = \{ |a| \leq A : P(a) \leq A^\delta, P^1(a) > A^{2\delta} \}
\end{align}
It transpires that the set $\mathcal{A}_1 \cup \mathcal{A}_2$ contains all $a$ where the singular series is likely to be smallish. Fortunately, $\mathcal{A}_1$ and $\mathcal{A}_2$ are defined by divisibility constraints that are related to convergent sieves, so one expects $\mathcal{A}_1, \mathcal{A}_2$ to be thin sets. This is indeed the case, as we shall now show.

We begin by counting elements of $\mathcal{A}_1$. For a natural number $d$, let $\mathcal{A}_1(d) = \{ a \in \mathcal{A}_1 : P_0(a) = d \}$. If there is some $a \in \mathcal{A}_1(d)$, then by the definition of $S(a)$, we have $d^2 | a_1 a_2 \ldots a_s$, whence $d \leq A^{s/2}$. On the other hand, $A^\delta < P(a) \leq HP_0(a) \leq H d$. This shows that
\[
\# \mathcal{A}_1 = \sum_{A^\delta/H < d \leq A^{s/2}} \# \mathcal{A}_1(d) \leq \sum_{A^\delta/H < d \leq A^{s/2}} \# \{ |a| \leq A : d^2 | a_1 a_2 \ldots a_s \}.
\]
By a standard divisor argument, we may conclude that
\begin{equation}
(3.22) \quad \# \mathcal{A}_1 \ll A^{s+\varepsilon} \sum_{A^\delta/H < d \leq A^{s/2}} d^{-2} \ll A^{s-\delta+\varepsilon}.
\end{equation}
The estimation of $\# \mathcal{A}_2$ proceeds along the same lines, but we will have to bound the number of integers with small square-free kernel. When $n$ is a natural number, let

\[ n^* = \prod_{p \mid n} p \]
denote its squarefree kernel. One then has the following simple bound (Tenenbaum \cite{17}, Theorem II.1.12).

**Lemma 3.4.** Let $\nu \geq 1$ be a real number. Then,
\[
\# \{ n \leq X^\nu : n^* \leq X \} \ll X^{1+\varepsilon}.
\]
For $d \in \mathbb{N}$, let $\mathcal{A}_2(d) = \{ a \in \mathcal{A}_2 : P^1(a) = d \}$. Since we have $P^1(a) | a_1 a_2 \ldots a_s$, we must have
\[
A^{2\delta} < d \leq A^\delta
\]
whenever $\mathcal{A}_2(d)$ is non-empty. Moreover, $P^1(a)$ is the square-free kernel of $P^1(a)$, so that $d^* \leq P^\delta$. This yields the bound
\[
\# \mathcal{A}_2 = \sum_{A^{2\delta} < d \leq A^\delta} \# \mathcal{A}_2(d) \leq \sum_{A^{2\delta} < d \leq A^\delta} \# \{ |a| \leq A : d | a_1 a_2 \ldots a_s \}.
\]
The divisor argument used within the estimation of $\# \mathcal{A}_1$ also applies here, and gives
\[
\# \mathcal{A}_2 \ll A^{s+\varepsilon} \sum_{A^{2\delta} < d \leq A^\delta} \frac{1}{d} \ll A^{s-2\delta+\varepsilon} \sum_{d \leq A^\delta} 1.
\]
By Lemma 3.4, it follows that
\begin{equation}
(3.23) \quad \# \mathcal{A}_2 \ll A^{s-\delta+\varepsilon}.
\end{equation}

We are ready to establish Theorem 1.2. It will suffice to find a lower bound for $S(a)$ for those $|a| \leq A$ where $S(a) > 0$ and $a \notin \mathcal{A}_1 \cup \mathcal{A}_2$. Let $p \in P(a)$. We have $\chi_p(a) > 0$, whence \[\ref{13}\] is soluble in $\mathbb{Q}_p$. By homogeneity, there is then a solution
\( x \in \mathbb{Z}_p \) of \( \{1,2\} \) with \( p \nmid x \). In particular, for any \( h \in \mathbb{N} \), we can find integers \( y_1, \ldots, y_s \) that are not all divisible by \( p \), and satisfy the congruence

\[
(3.24) \quad a_1y_1^k + \ldots + a_s y_s^k \equiv 0 \mod p^h.
\]

It will be convenient to rearrange indices to assure that \( p \nmid y_1 \). Let \( \nu(p) \) be defined by \( p^{\nu(p)} \| k \), and recall that a \( k \)-th power residue \( \mod p^{\nu(p)+2} \) is also a \( k \)-th power residue modulo \( p^\nu \), for any \( \nu \geq \nu(p)+2 \). We choose \( h = l(p)+\nu(p)+2 \) in \( (3.24) \), and define \( e \) by \( p^e \| a_1 \). For \( l > h \), choose numbers \( x_j \), for \( 2 \leq j \leq s \), with \( 1 \leq x_j \leq p^l \) and \( x_j \equiv y_j \mod p^h \). Then, by \( (3.24) \),

\[
-\frac{a_1}{p^e} y_1^k = \frac{a_2 x_2^k + \ldots + a_s x_s^k}{p^e} \mod p^{h-e},
\]

and we have \( e \leq l(p) \), whence \( h - e \geq \nu(p) + 2 \). Thus, for any choice of \( x_2, \ldots, x_s \) as above, there is a number \( x_1 \) with

\[
a_1 x_1^k + \ldots + a_s x_s^k \equiv 0 \mod p^l.
\]

Counting the number of possibilities for \( x_2, \ldots, x_s \) yields \( M_n(p^l) \geq p^{(s-1)(l-h)} \), and consequently,

\[
\chi_p(a) \geq p^{(1-s)h}.
\]

We may combine this with \( (3.19) \) to infer that

\[
(3.25) \quad \mathcal{S}_a \geq \frac{1}{2} \prod_{p \in \mathcal{P}(a)} p^{(1-s)h}.
\]

In this product, we first consider primes \( p \in \mathcal{P}(a) \) where \( l(p) = 0 \). Then \( p \nmid a_1a_2 \ldots a_s \), and the definition of \( \mathcal{P}(a) \) implies that \( p \leq C \). Also, since \( \nu(p) \leq k \), we have \( h \leq k + 2 \) so that

\[
\prod_{p \in \mathcal{P}(a) \atop l(p) = 0} p^{(1-s)h} \geq \prod_{p \leq C} p^{(1-s)(k+2)} \geq H^{(1-s)(k+2)}.
\]

Next, consider \( p \in \mathcal{P}(a) \) with \( l(p) \geq 1 \). Then, much as before, \( h \leq k + 2 + l(p) \leq l(p)(k + 3) \). Hence,

\[
\prod_{p \in \mathcal{P}(a) \atop l(p) \geq 1} p^{(1-s)h} \geq P^l(a)^{(1-s)(k+3)}.
\]

However, since \( a \notin \mathcal{A}_1 \cup \mathcal{A}_2 \), we have \( P^l(a) \leq A^{2\delta} \), so that we now deduce from \( (3.25) \) that

\[
(3.26) \quad \mathcal{S}_a \gg A^{2\delta(1-s)(k+3)}.
\]

The synthesis is straightforward. Let \( \gamma > 0 \). Then choose \( \delta = \gamma / (8(s-1)(k+3)) \), and suppose that \( A \) is large. Then \( (3.26) \) implies that \( \mathcal{S}_a \gg A^{-\gamma} \). If that fails, then \( \mathcal{S}_a = 0 \), or else \( a \in \mathcal{A}_1 \cup \mathcal{A}_2 \). The estimates \( (3.22) \) and \( (3.23) \) imply Theorem 1.2.

### 3.4. An auxiliary upper bound

We close this section with a succession of lemmata that involve the function \( \kappa \), and that will provide an upper bound for \( \mathcal{S}_a \) on average. The results will be relevant for the application of the circle method in the next section.
Lemma 3.5. One has
\[ \sum_{d \mid q} d \kappa(d) \ll q^{1+\varepsilon} \kappa(q). \]

Proof. Let \( p \) be a prime, and suppose that \( 0 \leq j \leq l \). Then, an inspection of the definition of \( \kappa \) readily reveals that the crude inequality \( p^j \kappa(p^j) \leq kp^j \kappa(p^j) \) holds. Consequently, one also has
\[ \sum_{d \mid p^l} d \kappa(d) = \sum_{j=1}^{l} p^j \kappa(p^j) \leq k(l+1) \kappa(p^j). \]
By multiplicativity, this implies the bound
\[ \sum_{d \mid q} d \kappa(d) \leq q \kappa(q) \prod_{p^j \parallel q} k(l+1), \]
which is more than required.

Lemma 3.6. Uniformly for \( q \in \mathbb{N} \) and \( A \geq 1 \), one has
\[ \sum_{1 \leq a \leq A} \kappa(q/(q;a)) \ll Aq \kappa(q). \]

Proof. We sort the \( a \) according to the value of \( d = (q;a) \). Then
\[
\sum_{1 \leq a \leq A} \kappa(q/(q;a)) = \sum_{d \mid q} \kappa(q/d) \sum_{1 \leq a \leq A \atop (a;q)=d} 1 \\
\leq A \sum_{d \mid q} d^{-1} \kappa(q/d) = \frac{A}{q} \sum_{d \mid q} d \kappa(d).
\]
The lemma now follows by appeal to Lemma 3.5.

Lemma 3.7. Let \( s \geq k+2 \). Then
\[ \sum_{|a| \leq A} \sum_{q=1}^{\infty} (a_1 a_2 \ldots a_s)^{-1/s} q^{1+1/(2k)} \kappa(q/(q; a_1)) \ldots \kappa(q/(q; a_s)) \ll A^{s-1}. \]

Proof. The terms to be summed are non-negative. Thus, we may take the sum over \( a \) first. This then factorizes, and by Lemma 3.6 and partial summation, the left hand side in Lemma 3.7 is seen not to exceed
\[ A^{s-1} \sum_{q=1}^{\infty} q^{1+1/(2k)} \kappa(q)^s. \]
The remaining sum converges for \( s \geq k+2 \), as one readily confirms by considering the corresponding Euler product. The lemma follows.

We now apply the last estimate to the singular series. Let \( T_a(q) \) be as in (3.15). When \( Q \geq 1 \), define the tail of \( \mathcal{G}_a \) as
\[ (3.27) \quad \mathcal{G}_a(Q) = \sum_{q \geq Q} T_a(q) \]
which is certainly convergent for \( s \geq k+2 \); compare Lemma 3.2. Also, note that \( \mathcal{G}_a = \mathcal{G}_a(1) \).
Lemma 3.8. Let \( s \geq k + 2 \). Then, uniformly in \( A \geq 1, Q \geq 1 \), one has
\[
\sum_{|a| \leq A} (a_1 a_2 \ldots a_s)^{-1/s} |\mathcal{G}_a(Q)| \ll A^{s-1} Q^{-1/(2k)}.
\]

Proof. By (3.16),
\[
|\mathcal{G}_a(Q)| \leq Q^{-1/(2k)} \sum_{q=1}^{\infty} q^{1/(2k)} |T_a(q)|
\]
\[
\leq Q^{-1/(2k)} \sum_{q=1}^{\infty} q^{1+1/(2k)} \kappa(q/(q; a_1)) \ldots \kappa(q/(q; a_s)),
\]
and the lemma follows from Lemma 3.7.

IV. The circle method

4.1. Preparatory steps. In this section, we establish Theorem 1.1. The argument is largely standard, save for the ingredients to be imported from the previous sections of this memoir.

We employ the following notational convention throughout this section: if \( h : \mathbb{R} \to \mathbb{C} \) is a function, and \( a \in \mathbb{Z}^s \), then we define
\[
h_a(\alpha) = h(a_1 \alpha) h(a_2 \alpha) \ldots h(a_s \alpha).
\]
As is common in problems of an additive nature, the Weyl sum
\[
f(\alpha) = \sum_{|x| \leq B} e(\alpha x^k)
\]
is prominently featured in the argument to follow, because by orthogonality, one has
\[
\varrho_a(B) = \int_0^1 f_a(\alpha) d\alpha.
\]
The circle method will be applied to the integral in (4.3). With applications in mind that go well beyond those in the current communication, we shall treat the “major arcs” under very mild conditions on \( A, B \), and for the range \( s \geq k + 2 \).

Let \( A \geq 1, B \geq 1 \), and fix a real number \( \eta > 0 \). Then put \( Q = B^\eta \). Let \( \mathfrak{M} \) denote the union of the intervals
\[
|\alpha - \frac{r}{q}| \leq \frac{Q}{AB^k}
\]
with \( 1 \leq r \leq q < Q \), and \((r, q) = 1\). When \( \eta \leq \frac{1}{3} \), these intervals are pairwise disjoint, and we write \( m = [Q/(AB^k), 1 + Q/(AB^k)] \setminus \mathfrak{M} \). When \( \mathfrak{A} \) is one of \( \mathfrak{M} \) or \( m \), let
\[
\varrho_a(B, \mathfrak{A}) = \int_{\mathfrak{A}} f_a(\alpha) d\alpha
\]
and note that
\[
\varrho_a(B) = \varrho_a(B, \mathfrak{M}) + \varrho_a(B, m).
\]
4.2. The major arc analysis. In this section we make heavy use of the results in Vaughan’s book [19] on the subject. He works with the Weyl sum
\[ g(\alpha) = \sum_{1 \leq x \leq B} e(\alpha x^k) \]
that is related with our \( f \) through the formulae
\[ f(\alpha) = 1 + 2g(\alpha) \quad (k \text{ even}), \quad f(\alpha) = 1 + g(\alpha) + g(-\alpha) \quad (k \text{ odd}). \]
Thus, in particular, Theorem 4.1 of [19] yields the following.

**Lemma 4.1.** Let \( \alpha \in \mathbb{R}, \ r \in \mathbb{Z}, \ q \in \mathbb{N} \) and \( a \in \mathbb{Z} \) with \( a \neq 0 \). Then
\[ f(a\alpha) = q^{-1}S(q, ar)v(a(r/q)) + O(q^{1/2+\varepsilon}(1 + |aB^k|a - r/q)^{1/2}). \]
Here, and throughout the rest of this section, we define \( v(\beta) = v(\beta, B) \) through (3.1). When \( |a| \leq A \), and \( a \in \mathfrak{M} \) is in the interval \( [1, 5] \), we find that
\[ f(a\alpha) = q^{-1}S(q, ar)v(a(r/q)) + O(Q^2). \]
This we use with \( a = a_j \) and multiply together. Then
\[ f_n(a) = q^{-s}S(q, a_1r)\ldots S(q, a_r) v_n(a - r/q) + O(Q^2B^{s-1}). \]
Now integrate over \( \mathfrak{M} \), and recall the definition of the latter. By (4.5) and (3.15), we then arrive at
\[ \varrho_n(B, \mathfrak{M}) = \sum_{q < Q} T_n(q) \int_{-Q/AB^k}^{Q/AB^k} v_n(\beta) d\beta + O(Q^5B^{s-1-k}A^{-1}). \]
Here, we complete the sum over \( q \) to the singular series, and the integral over \( \beta \) to the singular integral. On writing
\[ \int_{-Q/AB^k}^{Q/AB^k} v_n(\beta) d\beta = J_n(B) + E_n, \]
we may recall (3.27) to infer that
\[ \varrho_n(B, \mathfrak{M}) = (\mathfrak{G}_n - \mathfrak{G}_n(Q))(J_n(B) + E_n) + O(Q^5B^{s-1-k}A^{-1}), \]
and hence that
\[ \varrho_n(B, \mathfrak{M}) - \mathfrak{G}_nJ_n(B) \ll |\mathfrak{G}_n(Q)||J_n(B) + E_n| + \mathfrak{G}_n|E_n| + Q^5B^{s-1-k}A^{-1}. \]
On the left hand side, we may invoke (4.7). On the right hand side, we observe that by (4.7) and (3.5), one has \( J_n(B) + E_n \ll (a_1\ldots a_s)^{-1/s}B^{s-1-k} \). Hence, we may sum over \( a \) and apply Lemma 3.8, provided only that \( s \geq k + 2 \), as we now assume. Then
\[ \sum_{|a| \leq A} |\varrho_n(B, \mathfrak{M}) - \mathfrak{G}_nJ_nB^{s-k}| \ll A^{s-1}B^{s-k}Q^{-1/(2k)} + Q^5B^{s-k-1}A^{s-1}. \]
where
\[ \sum = \sum_{|a| \leq A} \mathfrak{G}_n|E_n|. \]
Now, by (4.7) followed by an application of Hölder’s inequality,
\[ |E_n| \leq \int_{|\beta| \geq Q/(AB^k)} |v_n(\beta)| d\beta \leq \prod_{j=1}^s \left( \int_{|\beta| \geq Q/(AB^k)} |v(a_j\beta)|^sd\beta \right)^{1/s}. \]
However, whenever $0 < |a| \leq A$, then by (3.2),
\[
\int_{Q/(AB^k)} |v(a\beta)|^s d\beta = \frac{1}{|a|} \int_{Q|a|/(AB^k)} |v(\beta)|^s d\beta \lesssim |a|^{s-1} B^{s-k}(1 + |a|A^{-1})^{-s/k},
\]
which produces the estimate
\[
|E_a| \ll |a_1 a_2 \ldots a_s|^{-1/s} B^{s-k} \prod_{j=1}^s (1 + |a_j|A^{-1})^{-1/k}.
\]
Hence, provided only that $A \geq Q$, Lemma 3.8 combined with a dyadic dissection argument for $|a|$, shows that
\[
\Sigma \ll Q^{-1/k} A^{s-1} B^{s-k}.
\]
We finally choose $\eta = \frac{1}{6}$, and then by (4.8), conclude as follows.

**Lemma 4.2.** Let $A \geq 1, B \geq 1$ and $Q = B^{1/6}$. Then, whenever $s \geq k + 2$, one has
\[
\sum_{|a| \leq A} |g_a(B, \mathfrak{M}) - \mathfrak{G}_a J_a B^{s-k}| \ll A^{s-1} B^{s-k-1/(12k)}.
\]

**4.3. The minor arcs.** We begin the endgame with a variant of Weyl’s inequality.

**Lemma 4.3.** Let $A \geq 1, B \geq 1$, and suppose that $r \in \mathbb{Z}$ and $q \in \mathbb{N}$ are coprime with $|\alpha - (r/q)| \leq q^{-2}$. Then
\[
\sum_{0 < |a| \leq A} |f(a\alpha)|^{2k-1} \ll AB^{2k-1} \left( \frac{1}{q} + \frac{1}{B} + \frac{q}{AB^k} \right) (ABq)^\varepsilon.
\]
This is well known, but we give a brief sketch for completeness. Write $K = 2^{k-1}$. Then, as an intermediate step towards the ordinary form of Weyl’s inequality, one has
\[
|f(\beta)|^K \ll B^{K-1} + B^{K-k+\varepsilon} \sum_{1 \leq h \leq 2^k B^{k-1}} \min(B, \|h\beta\|^{-1})
\]
where $\|\beta\|$ denotes the distance of $\beta$ to the nearest integer; compare the arguments underpinning Lemma 2.4 of Vaughan [19]. Now choose $\beta = a\alpha$ and sum over $a$. A divisor function argument then yields
\[
\sum_{0 < |a| \leq A} |f(a\alpha)|^K \ll AB^{K-1} + B^{K-k}(AB)^\varepsilon \sum_{h \leq AB^{k-1}} \min(B, \|h\beta\|^{-1}),
\]
and Lemma 4.3 follows from Lemma 2.2 of Vaughan [19].

Now let $\alpha \in \mathfrak{m}$. By Dirichlet’s theorem on diophantine approximations, there are $r \in \mathbb{Z}, q \in \mathbb{N}$ with $q \leq Q^{-1} AB^k$ and
\[
|q\alpha - r| \leq Q(AB^k)^{-1}.
\]
But $\alpha \notin \mathfrak{M}$, whence $q > Q$. Lemma 4.3 in conjunction with Hölder’s inequality now yields
\[
\sup_{\alpha \in \mathfrak{m}} \sum_{0 < |a| \leq A} |f(a\alpha)| \ll (AB)^{1+\varepsilon} Q^{-2^{1-k}}.
\]
We now apply this estimate to establish the following.
Lemma 4.4. Let $s \in \mathbb{N}$, $s = 2t + u$ with $t \in \mathbb{N}$, $u = 1$ or $2$. Then there is a number $\delta > 0$ such that whenever $1 \leq B^k \leq A \leq B^{t-k}$ holds, then
\[
\sum_{|a| \leq A} |g_a(B, m)| \ll A^{s-1}B^{s-k-\delta}.
\]

Proof. By (4.5), one has
\[
\sum_{|a| \leq A} |g_a(B, m)| \leq \int_m^1 \left( \sum_{0 < |a| \leq A} |f(a\alpha)| \right)^s d\alpha.
\]
Moreover, by Cauchy’s inequality and orthogonality,
\[
\int_0^1 \left( \sum_{0 < |a| \leq A} |f(a\alpha)| \right)^{2t} d\alpha \leq (2A + 1)^t \int_0^1 \left( \sum_{0 < |a| \leq A} |f(a\alpha)|^2 \right)^t d\alpha \leq (2A + 1)^t U_t(A, B).
\]
On combining the last two inequalities with (4.10) and Theorem 2.5, we deduce that
\[
(4.11) \quad \sum_{|a| \leq A} |g_a(B, m)| \ll A^sB^{t+u-\delta} + A^{s-1}B^{s-k-\delta}
\]
where any $0 < \delta < \frac{1}{6}2^{1-k}$ is admissible. Note that the condition that $B^k \leq A$ is required in Theorem 2.5, whereas the inequality $A \leq B^{t-k}$ makes the second term on the right of (4.11) the dominating one. This establishes the lemma.

Theorem 1.1 is also available: one has $\hat{s} = 2t$, and the theorem follows on combining (4.6) with Lemma 4.2 and Lemma 4.4.

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