NIELSEN EQUIVALENCE IN FUCHSIAN GROUPS

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ABSTRACT. In this paper we give a complete classification of minimal generating systems in a very general class of Fuchsian groups $G$. This class includes for example any $G$ which has at least seven non-conjugate cyclic subgroups of order $\gamma_i \geq 3$. In particular, the well known problematic cases where $G$ has characteristic exponents $\gamma_i = 2$ are not excluded.

We classify generating systems up to Nielsen equivalence; this notion is strongly related to Heegaard splittings of 3-manifolds. The results of this paper provide in particular the tools for a rather general extension of previous work of the authors and others, on the isotopy classification of such splittings in Seifert fibered spaces.

1. Introduction

A Fuchsian group is a group that acts properly discontinuously and cocompactly by isometries on the hyperbolic plane $\mathbb{H}^2$. If $G$ preserves the orientation of $\mathbb{H}^2$, then it is a discrete subgroup of $\text{PSL}_2(\mathbb{R})$, and it has a presentation

$$G = \langle s_1, \ldots, s_\ell, a_1, b_1, \ldots, a_g, b_g \mid s_1^{\gamma_1}, \ldots, s_\ell^{\gamma_\ell}, s_1 s_2 \ldots s_\ell, \prod_{j=1}^g [a_j, b_j] \rangle,$$

with $\gamma_i \geq 2$ for all $i = 1, \ldots, \ell$. The isomorphism type of such a Fuchsian group is determined by the family of characteristic exponents $\gamma_i$ and by the genus $g \geq 0$ of $G$. A presentation for Fuchsian groups with glide reflections is given below in (8.2).

Fuchsian groups play a central role in both, hyperbolic geometry and in low-dimensional topology (see [2], [16], [33]). Most prominently, all prime 3-dimensional manifolds were shown by Thurston to have a natural geometric structure, with eight possible geometries. For six of these geometries, the corresponding 3-manifolds are Seifert fibered spaces. Orientable Seifert fibered spaces with an orientable base space have fundamental groups that are central extensions of Fuchsian groups $G$ as in (1.1) above (see [31]).

It is well known that the presence of characteristic exponents $\gamma_i = 2$ in (1.1) often creates serious problems, in the sense that otherwise well working arguments fail in this case. For example, if $g = 0$ and all but one exponent satisfy $\gamma_i = 2$, then the rank of $G$ (= the minimal number of generators) can unexpectedly drop by 1, leading to an intriguing phenomenon in the corresponding Seifert fibered space (see [3], [25]). The most important achievement of this paper is that in our main result, Theorem 1.2 stated below, exponents $\gamma_i = 2$ are not excluded, and even the difficult case where the number of such $\gamma_i$ is odd, is dealt with.

In order to simplify the presentation and concentrate on the crucial issues, we treat in the main body of this paper only the orientable case, and we assume $g = 0$. The general case, including the possibility of orientation reversing isometries of $\mathbb{H}^2$, can subsequently be deduced directly from this special case, see Section 8.

Date: December 9, 2020.

2010 Mathematics Subject Classification. Primary 57M99, 20H10.

Key words and phrases. Nielsen equivalence, Fuchsian groups, Fox derivatives, generating systems, Heegaard splittings.
From now on let $G$ be a group with presentation

$$G = \langle s_1, \ldots, s_\ell \mid s_1^{\gamma_1}, \ldots, s_\ell^{\gamma_\ell}, s_1 s_2 \ldots s_\ell \rangle,$$

with $\gamma_i \geq 2$ for $i = 1, \ldots, \ell$, and with $\ell \geq 3$. Let $m$ denote the number of standard generators $s_i$ with exponent $\gamma_i \geq 3$, and let $n = \ell - m$ denote the number of those $s_i$ with exponent $\gamma_i = 2$. Note that the indexing of the generators is immaterial for the isomorphism type of $G$, as any permutation of the $s_i$ can be obtained through iteratively replacing some $s_i$ by

$$s_i' = s_{i-1}^{-1}s_is_{i-1}, \text{ which yields } s_{i-1}s_i' = s_is_{i-1}. \quad (1.3)$$

The group $G$ can be generated by $\ell - 1$ elements, and G. Rosenberger has shown in [28], [29] (compare also [26], [27], [37], [38]) that for $\ell \geq 4$ and $m \geq 3$ any minimal generating system can be transformed by a sequence of elementary Nielsen operations (see Definition 2.4 below) into a generating system of the following type:

**Definition 1.1.** A family $\mathcal{U}$ of elements in $G$ is called a standard generating system of $G$ if

$$\mathcal{U} = (s_1^{u_1}, \ldots, s_{j-1}^{u_{j-1}}, s_j^{u_j+1}, \ldots, s_\ell^{u_\ell}),$$

with $\gcd(u_i, \gamma_i) = 1$ for all $i \in \{1, \ldots, j-1, j+1, \ldots, \ell\}$.

The main goal of this paper is to present a complete proof of the following:

**Theorem 1.2.** Let $G$ be a group as in (1.2), and let $m$ be the number of $\gamma_i \geq 3$. If the number $n = \ell - m$ of exponents $\gamma_i = 2$ is even, we assume that $m \geq 5$. If $n$ is odd, assume $m \geq 7$.

Assume that $\mathcal{U}$ as in (1.4) and

$$\mathcal{V} = (s_1^{v_1}, \ldots, s_{k-1}^{v_{k-1}}, s_k^{v_k+1}, \ldots, s_\ell^{v_\ell})$$

are two given standard generating systems of $G$. Define $u_j = v_k = 1$. Then $\mathcal{U}$ and $\mathcal{V}$ are Nielsen equivalent if and only if

$$u_i = \pm v_i \mod \gamma_i \quad \text{for all} \quad i = 1, \ldots, \ell.$$  

**Remark 1.3.** (1) From the proof presented in this paper it follows that the conclusion of Theorem 1.2 is valid under weaker assumptions than those stated for $m$ and $n$: It suffices that $G$ is “non-exceptional” as in Definition 4.2 below (which is a bit too cumbersome to be stated here).

(2) A direct generalization of Theorem 1.2 to orientable Fuchsian groups with genus $g \geq 1$ and to non-orientable Fuchsian groups is given below in Corollary 8.1. The methods presented in the subsequent sections are actually stable enough to include an application to more general 1-relator quotients of free products of cyclic groups, see Theorem 8.2.

(3) An extension of our methods beyond what is presented in this paper, in order to make the set of exceptional groups $G$ even smaller, is possible, but the technical effort becomes increasingly bigger if $m$ and the seize of the exponents $\gamma_i$ becomes smaller.

(4) It is known that for very special choices of $m$ and $n$, the conclusion of Theorem 1.2 fails (for instance take $G$ as in [25]). Other candidates for such a failure have been proposed for example by G. Rosenberger. The precise determination of all Fuchsian groups with standard generating systems that admit more Nielsen equivalences than expected from our result above remains unresolved and seems to be a very difficult problem.

The study of Nielsen equivalence for generating systems of groups has a long history: It has been a central theme in combinatorial group theory since the 1950’s, for example in the context of non-tame automorphisms of groups (see e.g. [23]). Even with Gromov’s paradigm change towards geometric group theory in the 1990’s, its relevance has not decreased (see e.g. [7], [12], [14], [15], [17], [24], [34] or, more classically, [25], [27], [38]). In fact, it has also spread into other branches of mathematics [4], [13], [18], [32], as well as to computer science [1], [36].
Among the various natural reasons to investigate Nielsen equivalence of generating systems, one of the most important ones comes from the study of compact 3-dimensional manifolds $M^3$: Every Heegaard splitting of $M^3$ determines two generating systems of $G = \pi_1(M^3)$ up to Nielsen equivalence, and an isotopy of the splitting preserves the Nielsen equivalence classes. Indeed, the latter are the most telling and also most useful invariants of such splittings, and in the majority of cases non-isotopic Heegaard splittings are distinguished by these invariants.

The authors of this paper have in previous work (see [19], [20] and [21]) developed the fundamentals of the method used here, and set up a K-theoretic invariant $N(G)$ to distinguish minimal generating systems in arbitrary groups (see Remark 6.7 below). This has led, by work of the second author with J. Schultens (see [22] and [30]), to a classification of minimal genus Heegaard splittings in a large class of Seifert fibered spaces, excluding, however, those where the underlying Fuchsian groups have characteristic exponents $\gamma = 2$.

In the present paper, instead of employing the powerful $N(G)$ machinery, we only need “Jacobian matrices” defined via Fox derivatives over $\mathbb{Z}G$ (see Section 2), as well as a special evaluation technique, via “cyclic-faithful” representations of $G$ in $\text{Sl}_2(\mathbb{C})$ (see Sections 3 and 4). Our final calculations take place in $2 \times 2$-matrices over a group ring of a cyclic group with coefficients in $\mathbb{C}$ (see Section 5), and various cases have to be considered that stretch over a number of pages (Sections 6 and 7).

This paper is in many ways a continuation of our previous work [19], [20] and [21]. For the convenience of the reader, however, we present here a self-contained exposition, and we also make a special effort to organize the (non-trivial) computational parts of the paper into “compartments” where they can be checked independently from the presentation of our main arguments.

We’d also like to point the reader’s attention to recent work [8] of Edison Dutra, as well as to the upcoming papers [9] by Dutra and [10] by Dutra-Weidmann on related questions.

Acknowledgements: The authors would like to thank the referee for several very valuable comments which helped to improve and correct an earlier version of this paper. Furthermore, we would like to thank Gerhard Rosenberger for straightening out some of our more classical references. We would also like to thank Wendy Sandler as well as David Kohel for advice about some intricacies of the English language.

2. Preliminaries

In this section we briefly review the notions of Fox derivatives and Nielsen equivalence.

2.1. Fox derivatives. The notion of Fox derivatives was developed by R. Fox in [11]. For a modern exposition see [5].

**Definition 2.1.** Let $X = (X_1, \ldots, X_n)$ be a basis of a free group $F_n$. Then the $i$-th Fox derivative with respect to $X$ is a $\mathbb{Z}$-linear map

$$\partial/\partial X_i : \mathbb{Z}F_n \rightarrow \mathbb{Z}F_n, \; W \mapsto \partial W/\partial X_i,$$

which satisfies (where $\delta_{i,j}$ denotes the Kronecker-delta)

1. $\partial X_j/\partial X_i = \delta_{i,j}$ for any $j \in \{1, \ldots, n\}$, and
2. $\partial(U \cdot V)/\partial X_i = \partial U/\partial X_i + U \cdot \partial V/\partial X_i$ for any $U, V \in F_n$.

The maps $\partial/\partial X_i$ are characterized by these two properties and the assumed $\mathbb{Z}$-linearity. Note that the notation $\partial/\partial X_i$ is slightly misleading, as the map $\partial/\partial X_i$ depends not only on $X_i$, but also on the choice of the other $X_j$ from the given basis $X$. 
Fox derivatives have many natural uses in algebra and topology, and they turn out to be fairly easy to handle. For example, for any \( i \in \{1, \ldots, n\} \) one can immediately derive from (1) and (2) above the following facts: For the neutral element \( 1 \in F_n \) one has \( \partial 1/\partial X_i = 0 \), and for any \( W \in F(X) \) the formula
\[
\partial W^{-1}/\partial X_i = -W^{-1} \cdot \partial W/\partial X_i.
\]
Furthermore, for any \( V, W \in F_n \) the equality
\[
\partial(WV^{-1})/\partial X_i = W \partial V/\partial X_i + (1 - WVW^{-1}) \partial W/\partial X_i.
\]
is satisfied.

Let \( Y = (Y_1, \ldots, Y_m) \) be a second basis of \( F_n \). Then for any element \( W \in F_n \) we have the chain rule:
\[
\partial W/\partial X_i = \sum_{k=1}^{n} (\partial W/\partial Y_k \cdot \partial Y_k/\partial X_i)
\]
Hence the \( n \)-tuple \( (\partial W/\partial X_i)_{i=1, \ldots, n} \) is the matrix product of the \((1 \times n)\)-matrix \( (\partial W/\partial Y_k)_{k=1, \ldots, n} \) with the Jacobian matrix
\[
\partial Y/\partial X = (\partial Y_k/\partial X_i)_{k,i=1, \ldots, n}
\]
over the group ring \( \mathbb{Z}F_n \). This matrix is invertible over \( \mathbb{Z}F_n \): From property (1) in Definition 2.1 and a direct application of the chain rule one obtains \( \partial X/\partial Y \cdot \partial Y/\partial X = \partial X/\partial X = I_n \) (where \( I_n \) denotes the \( n \times n \) identity matrix).

Let \( U = (x_1, \ldots, x_n) \) a generating system for a group \( G \). Then for the free group \( F(X) \) over a family \( X = (X_1, \ldots, X_n) \) of formal symbols \( X_i \) there is a canonical surjection
\[
p_U : F(X) \to G, \; X_i \mapsto x_i.
\]
Any element \( w \in G \) can be written as a “word” in \( U \), i.e.
\[
w = \omega_1 \omega_2 \ldots \omega_r \quad \text{with} \quad \omega_i \in \{x_1^{\pm 1}, \ldots, x_r^{\pm 1}\}.
\]
Lifting each \( x_i \) to \( X_i \) determines an element \( W \in F(X) \) so that
\[
p_U(W) = w.
\]
The \( n \) Fox derivatives \( \partial W/\partial X_i \), when mapped into \( ZG \) via the ring homomorphisms \( ZF(X) \to ZG \) induced by \( p_U \) (and hence also denoted by \( p_U \)), give rise to an \( n \)-tuple
\[
\partial w/\partial U = (p_U(\partial W/\partial X_1), \ldots, p_U(\partial W/\partial X_n))
\]
Any other word \( w^* = \omega_1^* \ldots \omega_r^* \) as in (2.6), which describes the same element
\[
w = w^* \quad \text{in} \quad G,
\]
gives rise to a second lift \( W^* \in F(X) \), which differs from \( W \) by an element \( R = W^*W^{-1} \in \ker p_U \). If furthermore \( \ker p_U \) is normally generated by the elements of a set \( R = \{R_1, \ldots, R_m\} \), we have
\[
W^* = (V_1R_{j_1}V_1^{-1} \ldots V_qR_{j_q}V_q^{-1})W
\]
for suitable \( V_h \in F(X) \), \( R_{j_h} \in R \) and \( \varepsilon_h = \pm 1 \). Hence we derive, from property (2) of Definition 2.1 and from formula (2.2), that
\[
p_U(\partial W^*/\partial X_i) = p_U \left( \sum_{j=1}^m \left( \sum_{\{h|j_h=j\}} \varepsilon_h V_h \right) \cdot \partial R_j/\partial X_i \right) + p_U(\partial W/\partial X_i)
\]
for any \( i \in \{1, \ldots, n\} \). As a consequence, we obtain from (2.7) that
\[
\partial w^*/\partial U = \partial w/\partial U + L,
\]
where each entry of the \( n \)-tuple \( L \) is of the same type as the first term in the sum on the right hand side of equality (2.8). We formalize this observation as follows:

**Definition-Remark 2.2.** For any group \( G \) and any generating system \( \mathcal{U} = (x_1, \ldots, x_n) \) of \( G \) consider the canonical surjection

\[
p_\mathcal{U} : F(X_1, \ldots, X_n) \to G, \quad X_i \mapsto x_i.
\]

(1) A matrix \( B \in M_n(ZG) \) is called a *correction matrix* if every coefficient of \( B \) is contained in the left ideal \( I^\ell_\mathcal{U} \) of \( ZG \) which is generated by the Fox derivative images \( p_\mathcal{U}(\partial R/\partial X_i) \), for any \( R \in \ker p_\mathcal{U} \) and \( X_i \) from \( X = (X_1, \ldots, X_n) \).

(2) If \( \mathcal{R} = \{R_1, \ldots, R_m\} \) is a set of normal generators of \( \ker p_\mathcal{U} \), then \( I^\ell_\mathcal{U} \) is generated as a left ideal in \( ZG \) by all \( p_\mathcal{U}(\partial R_j/\partial X_i) \) with \( R_j \in \mathcal{R} \) and \( X_i \) from \( X \). This is the content of (2.8), for the case \( W = 1 \in F(X) \).

Now (2.9) implies directly:

**Proposition 2.3.** Let \( \mathcal{U} = (x_1, \ldots, x_n) \) be a generating system of a group \( G \). Consider a second generating system \( (y_1, \ldots, y_m) \) of \( G \), and assume that each \( y_k \) is expressed as a word \( w_k \) in \( \mathcal{U} \). Then the collection \( \mathcal{W} = (w_1, \ldots, w_n) \) determines a “Jacobian matrix” \( \partial \mathcal{W}/\partial \mathcal{U} = (a_{k,i})_{k,i} \in M_n(ZG) \), where \( a_{k,i} = p_\mathcal{U}(\partial w_k/\partial X_i) \) (so that the \( k \)-th line of \( \partial \mathcal{W}/\partial \mathcal{U} \) is defined as in (2.7), with \( w_k \) replacing \( w \)).

Let \( \mathcal{W}^* \) be a second collection of such words \( w_k^* \) for each \( y_k \). Then there is a correction matrix \( B \) such that the two Jacobian matrices associated to \( \mathcal{W} \) and \( \mathcal{W}^* \) satisfy:

\[
\partial \mathcal{W}^*/\partial \mathcal{U} = \partial \mathcal{W}/\partial \mathcal{U} + B
\]

\( \square \)

Note that, contrary to \( \partial Y/\partial X \) in (2.4), the more general Jacobian matrix \( \partial \mathcal{W}/\partial \mathcal{U} \) in the above proposition is in general *not* invertible over \( ZG \). An example is given by the generator \( y = x^5 \) of the cyclic group \( G = \langle x \mid x^5 \rangle \). For \( \mathcal{U} = (x) \) and \( \mathcal{W} = (x^2) \) the matrix \( \partial \mathcal{W}/\partial \mathcal{U} \) is the \((1 \times 1)\)-matrix with coefficient \( 1 + x \), which is not invertible in \( ZG \).

### 2.2. Nielsen equivalence.

**Definition 2.4.** Let \( G \) be a group, let \( n \in \mathbb{N} \), and let \( \mathcal{U} = (x_1, \ldots, x_n) \) be an \( n \)-tuple of elements from \( G \). Then an *elementary Nielsen operation* on \( \mathcal{U} \) is given by one of the following:

- (1) a permutation of the \( x_i \),
  - (2) replace \( x_i \) by \( x_ix_j \) or by \( x_jx_i \), for \( j \neq i \), while all other members of \( \mathcal{U} \) stay unchanged, or
  - (3) replace \( x_i \) by \( x_i^{-1} \), while all other members of \( \mathcal{U} \) stay unchanged.

A finite sequence of elementary Nielsen operations is sometimes called a *Nielsen operation*, and two families \( \mathcal{U} \) and \( \mathcal{U}' \) are *Nielsen equivalent* if they can be derived from each other by Nielsen operations.

**Remark 2.5.** Let \( f : G \to H \) be a group homomorphism, let \( \mathcal{U} \) and \( \mathcal{U}' \) be families of elements in \( G \), and denote by \( f(\mathcal{U}) \) and \( f(\mathcal{U}') \) the families of their \( f \)-images in \( H \). If \( \mathcal{U} \) and \( \mathcal{U}' \) are Nielsen equivalent, then so are \( f(\mathcal{U}) \) and \( f(\mathcal{U}') \). This is an immediate consequence of Definition 2.3.

Nielsen operations have been introduced by J. Nielsen in the 20’s of the last century, as analogues of elementary row operations on integer matrices. He could then show that bases for a free group \( F_n \) have the property described in the following theorem, in analogy to what is well known for bases of free abelian groups \( \mathbb{Z}^n \):
Theorem 2.6. Let \( X = (X_1, \ldots, X_n) \) and \( Y = (Y_1, \ldots, Y_n) \) be two bases of a free group \( F_n \). Then there exists a finite sequence of elementary Nielsen operations that transform \( X \) into \( Y \).

Conversely, if \( X \) is a basis of \( F_n \) and \( Y \) derives from \( X \) by a finite sequence of Nielsen operations, then \( Y \) is also a basis of \( F_n \).

Contrary to rings like \( \mathbb{Z} \) or \( \mathbb{R}[X] \), for non-commutative groups \( G \) the units (= multiplicatively invertible elements) in \( \mathbb{Z}G \) may in general be quite complicated. However, within the multiplicative group of units in \( \mathbb{Z}G \) there is always the subgroup of trivial units, given by

\[
T_G = \{ \pm g \mid g \in G \}.
\]

Definition 2.7. For any group \( G \) we say that a square matrix \( M \) with entries in \( \mathbb{Z}G \) is called a generalized elementary matrix over \( \mathbb{Z}G \). If \( M \) satisfies one of the following:

1. \( M \) is a permutation matrix,
2. \( M \) differs from the identity matrix only in a single off-diagonal coefficient, or
3. \( M \) is a diagonal matrix with trivial units on the diagonal.

We will refer to these matrices as elementary \( \mathbb{Z}G \)-matrices.

Proposition 2.8. Let \( X = (X_1, \ldots, X_n) \) and \( Y = (Y_1, \ldots, Y_n) \) be two bases of a free group \( F_n \). Then the Jacobian matrix

\[
\partial Y/\partial X = (\partial Y_j/\partial X_i)_{j,i}
\]

is a product of elementary \( \mathbb{Z}F_n \)-matrices.

Proof. If \( Y \) is derived from \( X \) by a single elementary Nielsen operation, the claimed statement follows from a direct computation based on (1) and (2) in Definition 2.1. The full claim is thus an immediate consequence of Theorem 2.6 and the fact that Fox derivatives satisfy the chain rule, see (2.3).

Combining Proposition 2.8 with Proposition 2.3 gives immediately the main criterion used in this paper to detect Nielsen inequivalent generating systems in an arbitrary finitely generated group \( G \):

Proposition 2.9. Let \( U = (x_1, \ldots, x_n) \) and \( V = (y_1, \ldots, y_n) \) be two Nielsen equivalent generating systems of a group \( G \). For any family of expressions

\[
y_1 = w_1, \ldots, y_n = w_n
\]

of the \( y_k \) as words \( w_k \) in the generators \( x_i \), and their canonical lifts \( W_k \in F(X) \) under the surjection \( p_U : F(X) \to G, X_i \mapsto x_i \), consider the Jacobian matrix \( \partial V/\partial U = (p_U(\partial W_k/\partial X_i))_{k,i} \). Then there is a correction matrix \( B \in \mathbb{M}_n(\mathbb{Z}G) \) as in Definition 2.2 such that the sum

\[
\partial V/\partial U + B
\]

is a product of elementary \( \mathbb{Z}G \)-matrices.

This proposition is particularly useful in combination with a suitable homomorphism of the group ring \( \mathbb{Z}G \) into a matrix ring. For instance, let \( \zeta_5 \) be a 5-th root of unity, and for the cyclic group \( G = \langle x \mid x^5 \rangle \) consider the homomorphism \( \eta : \mathbb{Z}G \to \mathbb{C} \) given by \( x \mapsto \zeta_5 \). Then one has \( \partial x^5/\partial x = 1 + x + \ldots + x^4 \), which gives \( \eta(\partial x^5/\partial x) = 1 + \zeta_5 + \ldots + \zeta_5^4 = 0 \in \mathbb{C} \).

As a consequence, any correction matrix \( B \) is mapped by \( \eta \) to the zero-matrix, so that the image \( \eta(\partial V/\partial U) \) of the Jacobian matrix \( \partial V/\partial U \) as in Proposition 2.9 depends only on the generators \( y_k \) in \( G \) and not on their lifts \( W_k \) to \( F(X) \). As a direct application of Proposition 2.9 we obtain that the generating system \( V \) which consists of the generator \( y = x^2 \) is not Nielsen equivalent to the generating system \( U = (x) \), since the element \( \eta(\partial V/\partial U) = 1 + \zeta_5 \) is not a product of \( \eta \)-images of elementary \( \mathbb{Z}G \)-matrices (of seize \((1 \times 1)\)), as the latter are all equal a power of \( \pm \zeta_5 \).
Of course, we knew ahead of these considerations that $x^2$ and $x$ are not Nielsen equivalent in the cyclic group $G$, but with very little effort the above reasoning is extended to the free product $G \ast \mathbb{Z}$ or even $G \ast F_n$, where Nielsen equivalence is a less obvious issue.

For the purpose of this paper homomorphisms of $\mathbb{Z}G$ into commutative rings $A$ are not sufficient; instead, we need to consider homomorphisms $\eta$ of $\mathbb{Z}G$ into matrix rings $\mathbb{M}_m(A)$. By a slight abuse of notation we denote by $\eta$ also the induced map from $\mathbb{M}_n(\mathbb{Z}G)$ to $\mathbb{M}_m \cdot n(A)$, to obtain:

**Corollary 2.10.** Let $G, U$ and $V$ be as in Proposition 2.9. Let $A$ be a commutative ring, and for some integer $m \geq 1$ let $\eta : \mathbb{Z}G \to \mathbb{M}_m(A)$ denote a ring homomorphism which satisfies $\det \eta(g) = \det \eta(-g) = 1$ for all $g \in G$.

Then there exists a “correction term” $b \in I^A_{U}$ such that

$$\det \eta(\partial V/\partial U) + b = 1,$$

where $I^A_{U}$ is the ideal in $A$ generated by all coefficients of the $\eta$-image of any Fox derivative matrix $p_U(\partial R_j/\partial X_i)$, for any set of normal generators $R_j$ of $\ker(p_U : F(X) \to G)$.

**Proof.** It suffices to apply $\eta$ to the sum $\partial V/\partial U + B$ from Proposition 2.9 and to consider the determinant of the resulting $(m \cdot n \times m \cdot n)$-matrix over $A$. Since $B$ is a correction matrix in $\mathbb{M}_n(\mathbb{Z}G)$, its coefficients are all contained in the left ideal $I^A_{U}$ of $\mathbb{Z}G$ that is generated by the Fox derivative images $p_U(\partial R_j/\partial X_i)$. Hence the determinant $\det \eta(\partial V/\partial U) + B$ is a sum of $\det \eta(\partial V/\partial U)$ with terms that are all products which contain at least one of the coefficients of the $\eta$-image of some of the $p_U(\partial R_j/\partial X_i)$ as factor, so that they all belong to $I^A_{U}$.

The claim then is a direct consequence of the assumption that any trivial unit $\pm g$ of $\mathbb{Z}G$ has as $\eta$-image the value $1 \in A$, since from Proposition 2.9 we know that $\partial V/\partial U + B$ is the product of elementary $\mathbb{Z}G$-matrices, and (according to the description in Definition 2.7) any such matrix has as $\eta$-image a matrix with the same determinant as the $\eta$-image of some trivial unit. \(\square\)

### 3. Cyclic-faithful representations

Let $G$ be a group as in (1.2), i.e.

$$G = \langle s_1, \ldots, s_\ell | s_1^{\gamma_1}, \ldots, s_\ell^{\gamma_\ell}, s_1 s_2 \ldots s_\ell \rangle,$$

with $\gamma_i \geq 2$ for $i = 1, \ldots, \ell$, and with $\ell \geq 3$. If in addition $G$ satisfies

\begin{equation}
\sum_{i=1}^\ell \frac{1}{\gamma_i} < \ell - 2
\end{equation}

then it is a Fuchsian group. Thus there is a faithful representation

\begin{equation}
\rho_0 : G \to P \text{Sl}_2(\mathbb{C}).
\end{equation}

It is well known (see [6] and [16], pp. 181–193) that $\rho_0$ lifts to a faithful representation

\begin{equation}
\rho : G \to \text{Sl}_2(\mathbb{C})
\end{equation}

if and only if all exponents $\gamma_i$ in (1.2) are odd. Furthermore, every standard generator $s_i$ of $G$ is mapped by $\rho$, up to conjugation in $\text{Sl}_2(\mathbb{C})$, to a matrix of type

\begin{equation}
M(\xi_i) = \begin{bmatrix}
\xi_i & 0 & 0 \\
0 & \xi_i^{-1} & 0 \\
7 & \xi_i^{-1} & 0
\end{bmatrix},
\end{equation}
where \( \zeta_i \in \mathbb{C} \) is a primitive \( \gamma_i \)-th root of unity. Matrices such as \( M(\zeta_i) \) will be called primitive \( \gamma_i \)-matrices.

In this paper we will use representations in \( SL_2(\mathbb{C}) \) which are slightly more general in that they need not be faithful on all of \( G \):

**Definition 3.1.** For any \( G \) as in (1.2) a representation \( \rho : G \to SL_2(\mathbb{C}) \) will be called cyclic-faithful if \( \rho \) maps every standard generator \( s_i \) to a conjugate of a primitive \( \gamma_i \)-matrix.

**Remark 3.2.** Regarding Definition 3.1 we note:

1. The terminology “cyclic-faithful” is justified, since the defining property of \( \rho \) is equivalent to requiring that \( \rho \) is faithful when restricted to the cyclic subgroup generated by any of the standard generators.
2. Let \( \rho_0 : G \to PSL_2(\mathbb{C}) \) be faithful, and consider for every generator \( s_i \) both lifts of \( \rho_0(s_i) \) in \( SL_2(\mathbb{C}) \). If \( \gamma_i \) is odd, then precisely one of these two lifts will have order \( \gamma_i \), while the other has order \( 2\gamma_i \). If \( \gamma_i \) is even, then both lifts will have order \( 2\gamma_i \).
3. For the special case \( \gamma_i = 2 \) we recall that one has \( \zeta_i = -1 \) and thus \( M(\zeta_i) = M(-1) = -I_2 \), where as before \( I_2 \) denotes the \( 2 \times 2 \) identity matrix. Indeed, \( M(-1) = -I_2 \) is the only matrix in \( SL_2(\mathbb{C}) \) which has order two.

In order to find cyclic-faithful representations of \( G \) it is useful to introduce a certain canonical quotient of \( G \). Since there are two similar such quotients, we will introduce them here together, so that the reader will avoid confusion later on.

**Definition 3.3.** Let \( G \) be as in (1.2).

1. Set \( \gamma_i' = \frac{\gamma_i}{2} \) if \( \gamma_i \) is even and \( \gamma_i' = \gamma_i \) if \( \gamma_i \) is odd. Define the full 2-quotient:

\[
G^* = G/\langle\langle\{s_i^{\gamma_i'} \mid i = 1, \ldots, \ell\}\rangle\rangle
\]

2. The canonical 4-quotient of \( G \) is given by

\[
G^# = G/\langle\langle\{s_i^{\gamma_i} \mid i = 1, \ldots, \ell\}\rangle\rangle,
\]

where we set \( \gamma_i' = \frac{\gamma_i}{2} \) if \( \gamma_i \) is even, but not divisible by 4 nor equal to 2, and otherwise we set \( \gamma_i' = \gamma_i \).

**Remark 3.4.** We note that the full 2-quotient \( G^* \) is in general generated by fewer elements than \( G \), since any \( s_j \) which in \( G \) has order \( \gamma_j = 2 \) will be trivial in \( G^* \).

The canonical 4-quotient \( G^# \), on the other hand, will in almost all cases\(^1\) be of the same rank as \( G \). It has the useful property that any generator \( s_i \) is mapped in \( G^# \) to an element of order which is either odd, equal to 2, or divisible by 4. Furthermore \( G^# \) is “stable” in the sense that \( (G^#)^# = G^# \).

**Remark 3.5.** In order to find a cyclic-faithful representation \( \rho \) of a Fuchsian group \( G \) as in (1.2), our strategy is to first pass to the quotient \( G^* \), then use a faithful representation \( \rho_0 \) of this quotient group in \( PSL_2(\mathbb{C}) \), and finally define the images \( \rho(s_i) \) as suitable lifts of \( \rho_0(s_i) \). According to Remark 3.2 (2), if properly chosen, these lifts \( \rho(s_i) \) are all conjugates of primitive \( \gamma_i \)-matrices, where \( \gamma_i \) is the original exponent of \( s_i \) in \( G \). There are, however, three obstructions to overcome, when attempting this procedure:

1. The quotient group \( G^* \) may not be Fuchsian. Hence, in order to ensure the existence of \( \rho_0 \) as above, one has to verify the inequality

\[\text{1} \] The only exceptions occur if \( G^# \) is one of the groups studied in [3] and [25], which have already been mentioned in the introduction.
for $\gamma'_i$ as defined in Definition 3.3 (1), and $m$ equal to the number of standard generators $s_i$ with exponent $\gamma_i \geq 3$.

(2) For the generators $s_j$ of order $\gamma_j = 2$ the above “lifting trick” doesn’t work: As noted already in Remark 3.2 (3), the only element of $SL_2(\mathbb{C})$ of order 2 is the matrix $-I_2$, which is also equal to $M(\zeta_i)$ as in (3.4). Hence any cyclic-faithful representation of $G$ must satisfy the equality

$$\rho(s_j) = -I_2,$$

for any $s_j$ with exponent $\gamma_j = 2$.

(3) Even if (3.5) and (3.6) above are satisfied, it may still be that the product relation $s_1 s_2 \ldots s_\ell = 1$ does not hold for the chosen $\rho$-images of the $s_i$. If, however, one has

$$\rho(s_1) \rho(s_2) \ldots \rho(s_\ell) = I_2,$$

then the above definition of the $\rho(s_i)$ defines a representation $\rho : G \to SL_2(\mathbb{C})$ which is cyclic-faithful.

In the following section several methods which ensure the existence of such cyclic-faithful representations $\rho$ are presented. It turns out that satisfying equality (3.7), in the case where $n$ is odd, is surprisingly tricky.

4. Exceptional Fuchsian groups

We start this section by listing conditions on groups $G$ as in (1.2) which ensure the existence of a cyclic-faithful representation of $G$ into $SL_2(\mathbb{C})$. We then define “exceptional” Fuchsian groups, and show that any non-exceptional $G$ satisfies one of these conditions.

Proposition 4.1. Let $G$ be as in (1.2). Let $n$ denote the number of exponents $\gamma_j = 2$, and let $m$ denote the number of exponents $\gamma_i \geq 3$. Assume that one of the following conditions is satisfied:

(1) There is at least one exponent $\gamma_i$ which is divisible by 4. Furthermore the inequality

$$\sum_{\{i | \gamma_i \geq 3\}} \frac{1}{\gamma'_i} < m - 2,$$

is satisfied, where, as in Definition 3.3 we set $\gamma'_i = \frac{\gamma_i}{2}$ if $\gamma_i$ is even and $\gamma'_i = \gamma_i$ if $\gamma_i$ is odd.

(2) Every exponent $\gamma_i \neq 2$ is odd, and the number $n \geq 0$ of exponents $\gamma_j = 2$ is even. Assume furthermore that

(i) $m \geq 4$, or

(ii) $m = 3$, and there is at least one $\gamma_i \geq 5$.

(3) Every exponent $\gamma_i \neq 2$ is odd, the number $n$ is odd and $m \geq 6$.

(4) Every exponent $\gamma_i \neq 2$ is odd, the number $n$ is odd and one of the following is true:

(i) $m = 5$, and there is some $\gamma_i \geq 7$.

(ii) $m = 5$, and there are at least two $\gamma_i, \gamma'_i \geq 5$.

(iii) $m = 4$, and there are at least two $\gamma_i, \gamma'_i \geq 7$.

(iv) $m = 4$, and all four exponents satisfy $\gamma_1, \gamma_2, \gamma_3, \gamma_4 \geq 5$.  

Then there is a cyclic-faithful representation 
\[ \rho : G \to \text{Sl}_2(\mathbb{C}). \]

Proof. Consider the four cases in order:

Case (1): Proceed exactly as in Remark 3.5. Assumption (4.1) ensures (see (3.1)) the existence of a faithful representation \( \rho_0 : G^* \to \text{PSl}_2(\mathbb{C}) \) of the full 2-quotient \( G^* = G/(\langle s_i^\gamma_i \mid i = 1, \ldots, \ell \rangle) \). By assumption one of the \( s_i \) has order \( \gamma_i \) in \( G \), where \( \gamma_i \) is divisible by 4; thus both lifts of \( \rho_0(s_i) \) to \( \text{Sl}_2(\mathbb{C}) \) have order \( \gamma_i \) (see Remark 3.2 (2)). Hence the right choice of \( \rho(s_i) \) ensures that (3.7) is satisfied, and thus \( \rho \) is a cyclic-faithful representation of \( G \).

Case (2): Proceed again as in Remark 3.5. The assumption that all \( \gamma_i \geq 3 \) are odd implies \( \gamma'_i = \gamma_i \) for all such \( \gamma_i \). Hence the assumptions (i) or (ii) ensure that inequality (4.1) is satisfied and thus \( \rho_0 \) exists. As all \( \gamma_i \geq 3 \) are assumed to be odd, \( \rho_0 \) lifts to a faithful representation of \( G^* \) in \( \text{Sl}_2(\mathbb{C}) \), see (3.3).

In this case the number \( n \) of generators \( s_j \) of order \( \gamma_j = 2 \) is even. We proceed as in step (2) of Remark 3.2 and extend the above lift of \( \rho_0 \) by the equalities (3.5) to all generators of \( G \). Since \( n \) is even, the product relation (3.7) is satisfied, so that we obtain the desired cyclic-faithful representation \( \rho \).

Case (3): Proceed first as in Case (2). The assumption \( m \geq 6 \) ensures that inequality (4.1) holds and hence \( \rho_0 \) exists as before. However, since in this case \( n \) is odd, the product relation (3.7) fails by a factor of \(-1\).

In order to deal with this problem we introduce the following “trick”: Using the assumption that \( m \geq 6 \), we partition the generators \( s_i \geq 3 \) into two sets \( s_1, \ldots, s_r \) and \( s_{r+1}, \ldots, s_m \) in such a way that both, \( r \geq 3 \) and \( m - r \geq 3 \), hold. Set \( s_0 = (s_1 s_2 \ldots s_r)^{-1} \) and consider the group

\[ G_1 = \langle s_0, \ldots, s_r \mid s_1^{\gamma_1}, \ldots, s_r^{\gamma_r}, s_0, s_0 s_1 s_2 \ldots s_r \rangle \]

Note that the assumptions \( m \geq 3 \) and \( \gamma'_i = \gamma_i \) for all \( \gamma_i \geq 3 \) (assumed to be odd) ensure that \( G_1 \) satisfies the conditions given in Case (1). Thus \( G_1 \) admits a cyclic-faithful representation \( \rho_1 : G_1 \to \text{Sl}_2(\mathbb{C}) \).

In particular, \( \rho_1 \) maps the product \( s_0^{-1} = s_1 s_2 \ldots s_r \) to a conjugate of the primitive 4-matrices \( M(i) \) or \( M(-i) \) (see equality (3.4)), so that after conjugating \( \rho_1 \) suitably in \( \text{Sl}_2(\mathbb{C}) \) we can assume \( \rho_1(s_1 s_2 \ldots s_r) = M(i) \) or \( \rho_1(s_1 s_2 \ldots s_r) = M(-i) \).

Apply the same method to the generators \( s_{r+1}, \ldots, s_m \) to obtain a group \( G_2 \) and a representation \( \rho_2 : G_2 \to \text{Sl}_2(\mathbb{C}) \) which maps the product \( (s_{r+1} \ldots s_m) \) to \( M(i) \) or to \( M(-i) \). Let \( \overline{\rho}_2 : G \to \text{Sl}_2(\mathbb{C}) \) be the representation obtained from \( \rho_2 \) through replacing, in the \( 2 \times 2 \) image matrix of any element of \( G_2 \), each coefficient by its complex conjugate.

It follows that combining \( \rho_1 \) with either \( \rho_2 \), or \( \rho_1 \) with \( \overline{\rho}_2 \), will map the product \( s_1 s_2 \ldots s_m \) to \( M(i)^2 = -I_2 \). Thus we can again proceed as in step (2) of Remark 3.2 and extend this representation by the equalities (3.6) to all generators of \( G \): Since \( n \) is odd, now the product relation (3.7) is satisfied, so that we obtain again a cyclic-faithful representation \( \rho \) as claimed.

Case (4): In order to apply the same trick as in the previous case, extra arguments are needed to ensure that the cyclic-faithful representations \( \rho_1 \) and \( \rho_2 \) exist.

In the subcases (i) and (ii), in order to define \( G_1 \) and \( G_2 \) we partition (after reordering) the generators \( s_1, \ldots, s_5 \) of order \( \gamma_i \geq 3 \) into two subsets \( \{s_1, s_2, s_3\} \) and \( \{s_4, s_5\} \). This partition is chosen so that for (i) either \( s_4 \) or \( s_5 \) has order \( \geq 7 \), and for (ii) both \( s_4 \) and \( s_5 \) have order \( \geq 5 \). It follows that, after adding a generator \( s_0 \) as in Case (3) above, both \( G_1 \) and \( G_2 \) satisfy the inequality (4.1): Indeed, for \( G_2 \) the corresponding triple sum of exponents is smaller or equal to \( \frac{1}{7} + \frac{1}{7} + \frac{1}{2} < 1 \) (for (i)) or \( \frac{1}{5} + \frac{1}{5} + \frac{1}{2} < 1 \) (for (ii)).
In the subcases (iii) and (iv) the partition is \( \{s_1, s_2\} \cup \{s_3, s_4\} \), where both sides are treated precisely as the subset \( \{s_4, s_5\} \) in subcases (i) and (ii) above.

This ensures that both resulting "partial quotient groups" \( G_1 \) and \( G_2 \) admit cyclic-faithful representations \( \rho_1 \) and \( \rho_2 \) as in Case (3) above. The rest of the proof of Case (3) applies word-by-word.

\[ \square \]

**Definition 4.2.** Let \( G \) be a group as in (1.2), i.e.

\[ G = \langle s_1, \ldots, s_\ell \mid s_1^{\gamma_1}, \ldots, s_\ell^{\gamma_\ell}, s_1 s_2 \ldots s_\ell \rangle \]

with all \( \gamma_i \geq 2 \). To simplify notation, we assume that the indexing of the generators has been adjusted (using equality (1.3)) to achieve \( \gamma_i \geq \gamma_j \) if \( i \leq j \).

Let \( m \) be the number of exponents \( \gamma_i \geq 3 \), and \( n \geq 0 \) be the number of exponents \( \gamma_j = 2 \). In this case we say \( G \) is of type

\[ (\gamma_1, \ldots, \gamma_m \mid n). \]

Furthermore we use below the following convention: for any odd integer \( k \geq 0 \) we write \( k^* \) to include both cases, \( k^* = k \) or \( k^* = 2k \). For even \( k \) the term \( k^* \) is purposefully undefined, so that the use of \( k^* \) implies in particular that \( k \) is odd.

Then \( G \) is called *exceptional* if one of the following conditions is satisfied.

(a) The number of exponents \( \gamma_i \geq 3 \) satisfies \( m \leq 3 \).

(b) \( n \) is even and \( G \) is of type \( (\gamma_1, 6, 5, 4 \mid n) \) or \( (\gamma_1, 5, 4, 3 \mid n) \).

(c) \( n \) is even and \( G \) is of type \( (\gamma_1, \gamma_2, 4, 4 \mid n) \).

(d) \( n \) is odd and \( m = 4 \).

(e) \( n \) odd and \( G \) is of type \( (s^*, t^*, p^*, q^*, 3^* \mid n) \), with \( p, q \in \{3, 5\} \).

(f) \( n \) odd and \( G \) is of type \( (p^*, q^*, 3^*, 3^*, 3^* \mid n) \), with \( q \in \{3, 5\} \).

Otherwise \( G \) is non-exceptional.

**Remark 4.3.** The following two statements follow directly from Definition 4.2.

(1) For any non-exceptional group \( G \) the canonical 4-quotient \( G^\# \) (see Definition 3.3) is also non-exceptional. Note that, in order for this statement to be true, in the Cases (e) and (f) one needs the above definition of \( k^* \).

(2) Any group \( G \) as in Theorem 1.2 is non-exceptional.

**Lemma 4.4.** Let \( G \) be a group as in (1.2) which is non-exceptional, and assume that

\[ (4.2) \quad \text{if } \gamma_i \text{ is even and } \gamma_i \neq 2, \text{ then } \gamma_i \text{ is divisible by } 4. \]

In other words, one has \( G = G^\# \). Assume also that for one of the standard generators, say \( s_h \), we have \( \gamma_h \geq 5 \). Then the following hold:

1. There is a cyclic-faithful representation \( \rho : G \to SL_2(\mathbb{C}) \).

2. For any choice of \( k \neq h \) the quotient group \( G_0 = G/\langle \langle s_h \rangle \rangle \) admits a cyclic-faithful representation \( \rho : G_0 \to SL_2(\mathbb{C}) \).

**Proof.** It follows immediately from Definition 4.2 that \( G \) being exceptional implies that \( G_0 \) is also exceptional. Hence it suffices to prove statement (2). This is done below by considering several cases and showing in each case that \( G_0 \) satisfies one of the four conditions listed in Proposition 4.1. This shows the existence of the desired cyclic-faithful representation \( \rho \).

The assumption that \( G \) is non-exceptional implies that each of the conditions (a) - (f) stipulated in Definition 4.2 is false. The negation of condition (a) implies that for \( G_0 \) the number \( m_0 \) of
exponents $\gamma_i \geq 3$ satisfies:

\[(4.3)\quad m_0 \geq 3\]

There are two cases to be distinguished:

(A) Assume that one of the $G_0$-exponents $\gamma_i$ is divisible by 4. This case splits further into three subcases:

(i) If all $\gamma_i \geq 3$ satisfy $\gamma_i \neq 4$, then we have for $\gamma_i'$ (as in Definition 4.3) that $\frac{1}{\gamma_i'} \leq \frac{1}{3}$, if $i \neq h$. From assumption (4.2) we deduce furthermore that $\gamma_h \neq 6$ and thus $\frac{1}{\gamma_h} \leq \frac{1}{9}$. Hence the inequality

\[(4.4)\quad \sum_{\{i \mid \gamma_i \geq 3, i \neq k\}} \frac{1}{\gamma_i} < m_0 - 2\]

holds, and thus all assumptions from Case (1) in Proposition 4.1 are satisfied for $G_0$.

(ii) Assume that for $G_0$ one has $m_0 \geq 4$. Then even if some $\gamma_i$ are equal to 4, we still have $\frac{1}{\gamma_i} \leq \frac{1}{2}$ for $i \neq h$, so that inequality (4.4) holds again, and we can apply the same conclusion as in subcase (i).

(iii) In the remaining case there are precisely three exponents $\gamma_i \geq 3$ in $G_0$, among which we have $\gamma_h \geq 5$, and another exponent, say $\gamma_j$, which is equal to 4. By the negation of conditions (c) and (d) in Definition 4.2 for $G$, the third exponent $\gamma_i \geq 3$ must be different from 4. Furthermore, the negation of conditions (b) and (d) rules out the possibility of $\frac{1}{4} + \frac{1}{5} + \frac{1}{8}$ on the left hand side of the above inequality (4.3). For all other cases inequality (4.4) is satisfied, so that again we have the same conclusion as in subcase (i) above.

(B) In this case all exponents $\gamma_i \geq 3$ are assumed to be odd, so that $\gamma_i' = \gamma_i$ holds for any $\gamma_i \geq 3$. There are still two more subcases to consider:

(iv) If $n$ is even, then by (4.3) the assumptions of conditions (i) or (ii) of Case (2) in Proposition 4.1 are satisfied for $G_0$, due to our hypothesis that $\gamma_h \geq 5$.

(v) If $n$ is odd, then the negation of condition (d) in Definition 4.2 for $G$, together with condition (4.3), ensure that $G_0$ has at least four exponents $\gamma_i \geq 3$. If there are precisely four such exponents, the negation of condition (e) in Definition 4.2 (for $G$) shows that $G_0$ satisfies (iii) or (iv) of Case (4) in Proposition 4.1. If there are precisely 5 such exponents in $G_0$, then (for $G$) the negation of condition (f) in Definition 4.2 shows that $G_0$ satisfies conditions (i) or (ii) of Case (4) in Proposition 4.1. Finally, if there are six or more exponents $\gamma_i \geq 3$ in $G_0$, then $G_0$ satisfies Case (3) in Proposition 4.1.

Hence in all cases the desired representation $\rho : G_0 \to SL_2(\mathbb{C})$ is provided by Proposition 4.1. \[\square\]

5. COMPUTATIONS IN A CYCLIC-GROUP RING WITH COEFFICIENTS IN $\mathbb{C}$

In this section we will prove Proposition 5.1, which plays a crucial role in the proof of Proposition 6.1 and thus of Theorem 1.2. This section can be read independently from the rest of the paper; the arguments presented here include several lengthy computations in a group ring with complex coefficients.

Throughout this section $p$ and $q$ will denote integers which satisfy

$p, q \geq 3$ and $p \mid q$,

and we also fix some primitive $q$-th root of unity $\zeta$. Let $t$ be the generator of a cyclic group $\langle t \mid t^p \rangle$ of order $p$. For any $a \in (\mathbb{Z}/q\mathbb{Z})^*$, $b \in (\mathbb{Z}/p\mathbb{Z})^*$ and $r \in \mathbb{R}$ we define the following element in the group ring $\mathbb{C}[\{t \mid t^p\}]$:

\[\Pi(a, b, r) = r(\zeta^a t^a - 1)(\zeta^{-a} t^{-a} - 1)(t^b - 1)(t^{-b} - 1)\]
We compute:

\[
\Pi(a, b, r) = r(2-\zeta^{a}t^{a} - \zeta^{-a}t^{-a})(2-t^{b} - t^{-b})
\]
\[
= r[2(2-\zeta^{a}t^{a} - \zeta^{-a}t^{-a}) - (2-\zeta^{a}t^{a} - \zeta^{-a}t^{-a})t^{b} - (2-\zeta^{a}t^{a} - \zeta^{-a}t^{-a})t^{-b}]
\]
\[
= r[(4-2\zeta^{a}t^{a} - 2\zeta^{-a}t^{-a}) - (2t^{b} - \zeta^{a}t^{a+b} - \zeta^{-a}t^{-a+b}) - (2t^{-b} - \zeta^{a}t^{a-b} - \zeta^{-a}t^{-a-b})]
\]

(5.1)

**Proposition 5.1.** Let \(a, b\) and \(r\) be as above, and let \(a', b'\) and \(r'\) be a second such triple. Then

\[
\Pi(a, b, r) = \Pi(a', b', r')
\]

implies:

\[
a = \pm a' \in \mathbb{Z}/q\mathbb{Z}
\]

**Proof.** This proof proceeds by considering various cases and subcases, where each case needs distinct careful considerations. The assumption \(p | q\) implies that \(a \in (\mathbb{Z}/q\mathbb{Z})^*\) has a canonical image \(\overline{a} \in (\mathbb{Z}/p\mathbb{Z})^*\); however, since below the context is always unambiguous, we will simplify notation and consistently write \(a\) for \(\overline{a}\).

**Case 1:** First consider the special case \(p = 3\). Then for any \(q \in \mathbb{N}\) so that \(p | q\) the conditions \(a \in (\mathbb{Z}/q\mathbb{Z})^*\) and \(b \in (\mathbb{Z}/p\mathbb{Z})^*\) imply that \(b = a \pm 1 \mod 3\), or \(b = -a = \pm 1 \mod 3\). In both cases we obtain, for \(\epsilon = \pm 1\):

\[
\Pi(a, b, r) = r(\zeta^{a}t^{a} - 1)(\zeta^{-a}t^{-a} - 1)(t^{b} - 1)(t^{-b} - 1) =
\]
\[
r(4 + \zeta^{a} + \zeta^{-a} + r(-2 - 2\zeta^{a} + \zeta^{-a})t^{\epsilon} + r(-2 - 2\zeta^{-a} + \zeta^{a})t^{-\epsilon}
\]
Hence, if \(D\) is the coefficient of the monomial \(t^{0}\), we have the equality

\[
r \Re \zeta^{a} = \frac{1}{2}(D - 4r),
\]
and for the other two coefficients \(E\) and \(F\) we get

\[
r \Im \zeta^{a} = \pm \frac{1}{3}(E - F).
\]

This gives:

\[
r^{2} = \left(\frac{1}{2}(D - 4r)\right)^{2} + \left(\frac{1}{3}(E - F)\right)^{2}
\]

Furthermore, notice that \(4 + \zeta^{a} + \zeta^{-a} > 0\) for any value of \(a\), so that \(r\) and \(D\) have the same sign. Hence we can derive the values of \(\Re \zeta^{a}\) and of \(\pm \Im \zeta^{a}\) from \(\Pi(a, b, r)\), and thus also the value of \(\zeta^{\pm a}\). This shows that \(\Pi(a, b, r) = \Pi(a', b', r')\) implies \(a = \pm a' \in \mathbb{Z}/q\mathbb{Z}\).

**Case 2:** Assume from now on that \(p, q \geq 4\). Consider the case where in the expression \(\Pi(a, b, r) = \Pi(a', b', r')\) the 9 “t-monomials” in the sum (5.1), interpreted as “polynomial” in \(t\), all have distinct exponents. In other words, the nine exponents

\[
(5.2)\quad 0, a, -a, b, -b, a+b, a-b, -a+b, -a-b
\]

define pairwise distinct elements of \(\mathbb{Z}/p\mathbb{Z}\).

Hence \(4r\) is the only term in the sum \(\Pi = \Pi(a, b, r)\) with \(t\)-exponent equal to 0. Similarly \(4r'\) is the only such term in \(\Pi' = \Pi(a', b', r')\). It follows that \(r = r'\). Thus, after dividing both \(\Pi\) and \(\Pi'\) by \(r\), we see that \(\zeta^{a}\) and \(\zeta^{-a}\) are the only coefficients of any \(t\)-monomial with modulus 1 in \(\Pi\), and similarly for \(\Pi'\). Hence in this case as well we can deduce that \(a = \pm a' \in \mathbb{Z}/q\mathbb{Z}\).

**Case 3:** The remaining case is more complicated and will be dealt with by splitting it into various subcases. First observe that \(a \neq -a\) and \(b \neq -b\) follows from the assumptions \(p \geq 4\) and \(\gcd(a, p) = \gcd(b, p) = 1\). By the same argument we deduce that the only cases, where two or more of the nine \(t\)-exponents, listed above in (5.2), can agree, are given by:

1. \(a = \pm b \in \mathbb{Z}/p\mathbb{Z}\), or
(2) \(a = \pm 2b \in \mathbb{Z}/p\mathbb{Z}\), or
(3) \(b = \pm 2a \in \mathbb{Z}/p\mathbb{Z}\), or
(4) \(2a = \pm 2b \in \mathbb{Z}/p\mathbb{Z}\).

We examine now these cases separately:

(a) Assume first that assumption (e) holds for \(a\), \(b\), \(r\). As in the previous case, by comparing the constant terms we deduce that \(\zeta \equiv p\). Hence one has \(p \geq 8\). We calculate, again for both cases:

\[ \Pi(a, b, r) = \]
\[ r[(4 + \zeta^{-a} + \zeta^a) + (-2\zeta^a - 2)t^a + (-2\zeta^{-a} - 2)t^{-a} + \zeta^a t^{2a} + \zeta^{-a} t^{-2a}] \]

(b) Assume \(b = 2a \in \mathbb{Z}/p\mathbb{Z}\) or \(b = -2a \in \mathbb{Z}/p\mathbb{Z}\), and assume \(p = 5\), which yields \(a = -2b \in \mathbb{Z}/p\mathbb{Z}\) or \(a = 2b \in \mathbb{Z}/p\mathbb{Z}\) respectively. We calculate, again for both cases:

\[ \Pi(a, b, r) = \]
\[ r[(4 + (-2\zeta^a + \zeta^{-a})t^a + (-2\zeta^{-a} + \zeta^a)t^{-a} - 2\zeta^{2a} + \zeta^a t^{3a} - 2t^{-2a} + \zeta^{-a} t^{-3a}] \]

(c) Assume \(b = 2a \in \mathbb{Z}/p\mathbb{Z}\) or \(b = -2a \in \mathbb{Z}/p\mathbb{Z}\), and assume \(p \neq 5\), which yields \(a \neq \pm 2b \in \mathbb{Z}/p\mathbb{Z}\). We calculate, again for both cases:

\[ \Pi(a, b, r) = \]
\[ r[4 + (-2\zeta^a + \zeta^{-a})t^a + (-2\zeta^{-a} + \zeta^a)t^{-a} - 2\zeta^{2a} + \zeta^a t^{3a} - 2t^{-2a} + \zeta^{-a} t^{-3a}] \]

(d) Assume \(a = 2b \in \mathbb{Z}/p\mathbb{Z}\), or \(a = -2b \in \mathbb{Z}/p\mathbb{Z}\). If \(p = 5\) then we deduce that we are back in case (b) above. Thus we can assume \(p \neq 5\), which yields \(b \neq -2a = -4b \in \mathbb{Z}/p\mathbb{Z}\) or \(b \neq 2a = -4b \in \mathbb{Z}/p\mathbb{Z}\) respectively. We calculate, again for both cases:

\[ \Pi(a, b, r) = \]
\[ r[4 + (\zeta^{2b} - 2)b^b + (\zeta^{-2b} - 2)t^{-b} - 2\zeta^{2b}t^{2b} - 2\zeta^{-2b}t^{-2b} + \zeta^{2b}t^{3b} + \zeta^{-2b}t^{-3b}] \]

(e) Assume \(2a = 2b \in \mathbb{Z}/p\mathbb{Z}\), or \(2a = -2b \in \mathbb{Z}/p\mathbb{Z}\). We can assume that \(a \neq \pm b \in \mathbb{Z}/p\mathbb{Z}\), as otherwise we are back in case (a). We deduce that \(p\) is even, i.e.

\[ p = 2p' \]

for some integer \(p' \geq 1\). It follows that \(a = \pm b\) modulo \(p'\), and thus \(b = \pm a + p' \in \mathbb{Z}/p\mathbb{Z}\). As a consequence, from \(a \neq \pm b \in \mathbb{Z}/p\mathbb{Z}\) and \(\gcd(a, p) = \gcd(b, p) = 1\) we deduce that \(p \neq 4\) and \(p \neq 6\), and hence \(p \geq 8\). We calculate for both, \(b = a + p' \in \mathbb{Z}/p\mathbb{Z}\) or \(b = -a + p' \in \mathbb{Z}/p\mathbb{Z}\) that

\[ \Pi(a, b, r) = \]
\[ r[(4 - 2\zeta^a t^a - 2\zeta^{-a} t^{-a}) + (\zeta^{-a} p'^{-2a} - 2t^{-a} + (\zeta^a + \zeta^{-a}) p'^{-a} - 2t^{-a} + \zeta^a p'^{2a})]. \]

We note that from \(p \geq 8\) it follows that all 8 terms in this “polynomial” have distinct \(t\)-exponents.

In order to finish this Case 3, we now need to consider the other triple \(a', b', r'\); a priori it may not fall into the same cases (a) - (e) as the triple \(a, b, r\) considered above.

(A) Assume first that assumption (e) holds for \(a, b, r\). As this is the only case where in the expression \(\Pi = \Pi(a, b, r)\) there are precisely 8 distinct terms, it follows that the other triple \(a', b', r'\) must also be in case (e). This implies, by comparing the constant terms, that \(r = r'\). Hence in \(\frac{1}{r} \Pi = \frac{1}{r} \Pi'\) the only non-real coefficients with modulus 1 are equal to \(\zeta^{\pm a}\). Thus we obtain \(a = \pm a' \in \mathbb{Z}/q\mathbb{Z}\).

(B) Assume next that assumptions (c) or (d) hold for \(a, b, r\). Then \(p = 5\) is excluded, and \(p \neq 4\) and \(p \neq 6\) follow from \(a = \pm 2b\) or \(b = \pm 2a\) and the assumption that both, \(a\) and \(b\), are relatively prime to \(p\). Hence one has \(p \geq 7\), which implies that all 7 terms in the expression of \(\Pi = \Pi(a, b, r)\) in the cases (c) and (d) must be distinct. It follows that the other triple \(a', b', r'\) must also be in cases (c) or (d). As in the previous case, by comparing the constant terms we deduce \(r = r'\). And
similarly, in $\frac{1}{r} \Pi = \frac{1}{r} \Pi'$ the only coefficients with modulus 1 are equal to $\zeta^{\pm a}$, thus showing $a = \pm a'$ in $\mathbb{Z}/q\mathbb{Z}$.

(C) We can now assume that both triples $a, b, r$ and $a', b', r'$ are as in cases (a) or (b) above. If $p \neq 4$, then for both, (a) and (b), it follows from the fact that $p \neq 3$ that all 5 terms in the expression of $\Pi = \Pi'$ are distinct. In case (a) there are two terms with non-zero $t$-exponent, which have the property that their coefficients $E$ and $F$ satisfy $F + 2E \in \mathbb{R}$. This is not true for case (b). Thus, either both triples are in case (a), or both are in case (b).

In the first case we notice that for any choice of coefficients $E$ and $F$ with $F + 2E \in \mathbb{R}$ one has $\frac{E}{|E|} = \zeta^{\pm a}$, which again yields $a = \pm a'$ in $\mathbb{Z}/q\mathbb{Z}$. In the second case we can again compute the value $r = r'$ from the only term in $\Pi = \Pi'$ with zero $t$-exponent. Further, since $p = 5$ and gcd$(a, p) = 1$, one has:

$$\text{Re}(-2 + \zeta^{\pm a}) < \text{Re}(-2\zeta^a + \zeta^{-a}) = \text{Re}(-2\zeta^{-a} + \zeta^a) < 4$$

Hence it suffices to consider the coefficient $E$ of $\Pi = \Pi'$ with the smallest real part to observe that $\frac{E+2}{F'} = \zeta^{\pm a} = \zeta^{\pm a'}$, which gives again $a = \pm a'$ in $\mathbb{Z}/q\mathbb{Z}$.

(D) It remains to consider the case where both triples $a, b, r$ and $a', b', r'$ are as in cases (a) or (b) above, and in addition we have $p = 4$. The latter, however, contradicts the assumption $p = 5$ in (b), so that in fact both triples belong to case (a). From $p = 4$ we obtain $t^{2a} = t^{-2a} = t^2$ so that we have precisely 4 terms:

$$\Pi = \Pi' = E + Ft + Gt^2 + Ht^3$$

Now note that $E - G = 4r$, so that one computes $1 - \frac{2F}{E-G} = \zeta^{\pm a} = \zeta^{\pm a'}$, which gives once more $a = \pm a'$ in $\mathbb{Z}/q\mathbb{Z}$.

6. Proof of Theorem 1.2

In this section we give a proof of Theorem 1.2. The proof given below applies also to the more general statement (see Remark 1.3), obtained by weakening the assumptions on $n$ and $m$ stated in in the second paragraph of Theorem 1.2 to simply assuming that $G$ is non-exceptional.

The crucial ingredient in this proof is Proposition 6.1. The proof of this proposition is preceded by a sequence of simplifications and by three technical lemmas. The proofs of the latter are deferred to the next section.

Proof of Theorem 1.2. We first show that the “if” direction in the statement of Theorem 1.2 is a direct consequence of the concept of “Nielsen equivalence”, see Definition 2.4.

Assume $j = k$, i.e. the indices $j$ and $k$ of the “missing generators” $s_j$ for $U$ and $s_k$ for $V$ are identical. Then the assumption

$$u_i = \pm v_i \mod \gamma_i$$

implies that $U$ and $V$ are the same up to inversion of some generators, which is one of the allowed operations within a Nielsen equivalence class.

Assume $j \neq k$. By assumption we have $1 = u_j = v_j$, so that $s_j$ is part of the family $V$. We apply to the generating system $V$ the operation which replaces $s_j$ first by $s_j^{-1}$, and then the latter by $s_k = (s_{k-1}^{-1} \ldots s_1^{-1} \ldots s_{j+1}^{-1}) \cdot s_j^{-1} \cdot (s_{j-1}^{-1} \ldots s_{k+1}^{-1})$, if $k < j$. For $j < k$ use the analogous operation. Each replacements are all Nielsen operations, and the result is a standard generating system $V'$ with the same “missing generator” as $U$. Hence the arguments for the above treated case $j = k$ apply, to conclude that $U$ is Nielsen equivalent to $V'$ and hence to $V$.

The “only if” statement of Theorem 1.2 follows from Proposition 6.1 stated below: Since the generators in the presentation (1.2) of $G$ can be permuted by formula (1.3), we may restrict our attention to the first standard generator $s_1$, and then repeat the argument for the other $\ell - 1$ standard generators.
It has been already verified in Remark 4.3 (2) that the assumption used in Proposition 6.1, that $G$ is non-exceptional, is weaker than the assumptions $m \geq 5$ for even $n$, and $m \geq 7$ if $m$ is odd, from Theorem 1.2.

**Proposition 6.1.** Let $G$ be a group with a presentation as in (1.2), and assume that $G$ is non-exceptional (see Definition 4.2). Let $U$ and $V$ be two standard generating systems for $G$ as defined in Definition 1.1. If $U$ and $V$ are Nielsen equivalent, then one has $$u_1 = \pm v_1 \mod \gamma_1.$$  

Before the proof of this proposition is presented, we go through some preliminary considerations. First note that without loss of generality the hypotheses on $G$ can be strengthened slightly by making use of the following observation:

**Lemma 6.2.** Consider the special case where for every standard generator $s_i$ of $G$ the exponent $\gamma_i$ satisfies the following: If $\gamma_i$ is even and $\gamma_i \neq 2$, then $\gamma_i$ is divisible by 4 (in other words: $G = G^\#$).

Then proving Proposition 6.1 for any $G$ as in this special case implies Proposition 6.1 in full generality.

**Proof.** By assumption, $G$ has a presentation as in (1.2). For every even exponent $\gamma_i \neq 2$ which is not divisible by 4 define a new exponent $\hat{\gamma}_i = \frac{\gamma_i}{2}$ (which is an odd integer), and consider the canonical 4-quotient group $$G^\# = G/\langle\langle \{\hat{s}_i \} \rangle\rangle,$$
as in Definition 3.3 (2). We know from Remark 4.3 (1) that, if $G$ is non-exceptional, then so is $G^\#$.

Assume that the generating systems $U$ and $V$ are Nielsen equivalent. Then (see Remark 2.5) their images in the quotient group $G^\#$ are also Nielsen equivalent. If Proposition 6.1 holds for $G^\#$, then we know that $u_1 = \pm v_1$ modulo $\hat{\gamma}_1$. If $\gamma_1 \neq \hat{\gamma}_1$, then modulo $\gamma_1$ we have:

$$u_1 = \pm v_1 \mod \gamma_1.$$

However, since $\hat{\gamma}_1$ is odd, it follows that for any integer $k$ at most one of $k$ or $k + \hat{\gamma}_1$ can be relatively prime to $\gamma_1 = 2\hat{\gamma}_1$. Since by Definition 1.1 both, $u_1$ and $v_1$, are assumed to be relatively prime to $\gamma_1$, we deduce that $$u_1 = \pm v_1 \mod \gamma_1.$$  

Next, observe that in Proposition 6.1 we can assume

(6.1) \quad \gamma_1 \geq 5

since for $\gamma_1 \leq 4$ the conclusion of the proposition becomes trivial.

Furthermore, in the special case that $\gamma_1$ is relative prime to all other $\gamma_i \geq 3$, the proof of Proposition 6.1 becomes much simpler, as will be seen below. The complementary case, though, poses several problems, which are dealt with now, using the work already done in the previous sections.

We thus assume from now on that $\gamma_1$ is not relatively prime to some other $\gamma_i \geq 3$. Then we can assume further from the commutator equality (1.3) that $i = 2$, and from the extra hypothesis $G = G^\#$, achieved in Lemma 6.2 that some integer $p \geq 3$ is a common divisor of $\gamma_1$ and $\gamma_2$.

From Lemma 4.4 (2) we know that there exists a cyclic-faithful representation $$\eta'_1 : G_1 = G/\langle\langle s_2 \rangle\rangle \to SL_2(\mathbb{C}).$$
In particular, every generator $s_i$ other than $s_2$ is mapped by $\eta'_1$, up to conjugation in $SL_2(\mathbb{C})$, to a primitive $\gamma_i$-matrix

$$(6.2) \quad M(\zeta_i) = \begin{bmatrix} \zeta_i & 0 \\ 0 & \zeta_i^{-1} \end{bmatrix}. $$

More specifically, after possibly conjugating $\eta'_1$ in $SL_2(\mathbb{C})$, we can require that $\eta'_1(s_1) = M(\zeta_1)$, while for $i \geq 3$ we only require that $\eta'_1(s_i)$ and $M(\zeta_i)$ agree up to conjugation in $SL_2(\mathbb{C})$.

Let $\eta_1 : G \to SL_2(\mathbb{C})$ be the composition of the quotient map $G \to G/\langle \langle s_2 \rangle \rangle$ with $\eta'_1$. Consider now the quotient homomorphism

$$\eta_2 : G \to \langle \langle t \mid t^p \rangle \rangle, \ s_1 \mapsto t, \ s_2 \mapsto t^{-1}, \ s_i \mapsto 1 \ (i \geq 3)$$

and combine the maps $\eta_1$ and $\eta_2$ to obtain a homomorphism

$$\eta : G \to SL_2(\mathbb{C}[\langle \langle t \mid t^p \rangle \rangle])$$

given by $\eta(s_i) = \eta_1(s_i)$ for $i \geq 3$, and by

$$\eta(s_1) = \begin{bmatrix} \zeta_1 t & 0 \\ 0 & \zeta_1^{-1} t^{-1} \end{bmatrix} \quad \text{and} \quad \eta(s_2) = \begin{bmatrix} t^{-1} & 0 \\ 0 & t \end{bmatrix}. $$

Consider now the generating system

$$U = (x_1 = s_1^{u_1}, \ldots, x_{j-1} = s_{j-1}^{u_{j-1}}, x_j = s_j^{u_j}, \ldots, x_{\ell} = s_{\ell}^{u_{\ell}}), $$

and recall that for each generator $x_i$ the exponent $u_i$ is relatively prime to $\gamma_i$ (and that this holds also for the formally introduced exponent $u_2 = 1$). Thus we can pick an integer $z_i \in \mathbb{Z}$ with

$$z_i \cdot u_i = 1 \mod \gamma_i, $$

and obtain in $G$ the equalities

$$s_i = x_i^{z_i} \quad \text{for} \quad x_i = s_i^{u_i}. $$

For a family $X = (X_1, \ldots, X_{j-1}, X_{j+1}, \ldots, X_{\ell})$ of formal symbols $X_i$ consider as in (2.5) the free group $F(X)$ and the canonical surjection

$$p_U : F(X) \to G, \ X_i \mapsto x_i = s_i^{u_i}. $$

For any second generating system $W = (w_1, \ldots, w_{\ell-1})$ of $G$ each element $w_h$ can be written as a word in the $x_i^{\pm 1}$ with $i \neq j$, and hence we obtain a family $W$ of elements $W_h \in F(X)$ which satisfy

$$p_U(W_h) = w_h. $$

As has been discussed in Section 2 (see Proposition 2.3), compute the $(\ell - 1) \times (\ell - 1)$-matrix $\partial W/\partial X$ of Fox derivatives $\partial W_h/\partial X_i \in \mathbb{Z}F(X)$. We denote by $\partial W/\partial U$ the associated Jacobian matrix in the matrix ring $M_{(\ell - 1) \times (\ell - 1)}(\mathbb{Z}G)$, i.e. $\partial W/\partial U$ is the image of $\partial W/\partial X$ under the map induced by $p_U$.

In order to apply the method from Corollary 2.10 we now pass to the image of $\partial W/\partial U$ under the above defined “mixed” representation $\eta$, and compute the determinant $D(W, U)$ of the resulting matrix $M(W, U) = \eta(\partial W/\partial U) \in SL_{2\ell - 2}(\mathbb{C}[\langle \langle t \mid t^p \rangle \rangle])$. As a final step, we multiply $D(W, U)$ by the product $\Pi(u_1, u_2, 1)$, where we use the expression

$$\Pi(a, b, r) := r(\zeta_1^a t^a - 1)(\zeta_1^{-1} t^{-a} - 1)(t^b - 1)(t^{-b} - 1)$$

defined in Section 3 with parameters specified to $a = u_1, b = u_2$ and $r = 1$.

**Remark 6.3.** We should alert the reader that the above introduced notation $D(W, U)$ is slightly misleading, since the value of this determinant may well depend not just on $U$ and $W$, but also on the chosen lifts $W_h$ of the elements $w_h \in W$. However, Lemma 6.4 below “repairs” this lapsus, which mainly serves to avoid adding further extra notation.

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We now state three lemmas which will be proved in the next section. We then show that combining these three lemmas and applying Proposition 6.1 yields, without much ado, the statement of Proposition 6.1.

**Lemma 6.4.** The product

\[ \Pi(u_1, u_2, 1) D(W, U) = (\zeta_1^{u_1} t^{u_1} - 1)(\zeta_1^{-u_1} t^{-u_1} - 1)(t^{u_2} - 1)(t^{-u_2} - 1) \det(\eta(\partial W / \partial U)) \]

depends only on the families \( W \) and \( U \) in \( G \), and not on the particular choice of the words \( W_h \) in the free group \( F(X) \) which represent via \([6,4]\) the elements of the generating system \( W \).

**Lemma 6.5.** If \( U \) and \( W \) are Nielsen equivalent, then one obtains:

\[ \Pi(u_1, u_2, 1) D(W, U) = \Pi(u_1, u_2, 1) \]

**Lemma 6.6.** Given generating systems \( U \) and \( V \) of \( G \) as in Definition 1.7 one computes

\[ \Pi(u_1, u_2, 1) D(V, U) = \Pi(v_1, v_2, r) , \]

for some value \( r \in \mathbb{R} \).

**Proof of Proposition 6.1.** (1) First consider the case, treated above, where \( \gamma_1 \) is not relatively prime to some other \( \gamma_i \geq 3 \). As shown above, we can assume \( i = 2 \) and \( p = \gcd(\gamma_1, \gamma_2) \geq 3 \).

While (as pointed out in Remark 6.3) the determinant \( D(V, U) \) may well depend on the choice of the lifts under the map \( p_U \) of the elements \( v_i \in V \), it follows from Lemma 6.4 that the product \( \Pi(u_1, u_2, 1) D(V, U) \) is a true invariant of the two generating systems \( U \) and \( V \) of \( G \). Hence combining Lemmas 6.5 and 6.6 allows us to conclude, for Nielsen equivalent generating systems \( U \) and \( V \) as in Proposition 6.1 that \( \Pi(v_1, v_2, r) = \Pi(u_1, u_2, 1) \). Now apply Proposition 5.1 for \( q = \gamma_1, (a, b, r) = (v_1, v_2, r) \) and \( (a', b', r') = (u_1, u_2, 1) \), to directly obtain the conclusion of Proposition 6.1.

(2) Let us now assume that \( \gamma_1 \) is relatively prime to all other \( \gamma_i \geq 3 \).

Then \( G \) has, by Lemma 6.2 and Lemma 6.4 (1), a representation in \( \text{Sl}_2(\mathbb{C}) \) which is faithful on every cyclic subgroup that is generated by one of the generators \( s_i \). But then Proposition 6.1 is a direct consequence of what has been shown in previous work of the authors, see [20], Lemma 1.9. Indeed, all the arguments used in Lemma 1.9 of [20] are based on the fact that, under the conditions given in this lemma, there is a cyclic-faithful representation of \( G \) in \( \text{Sl}_2(\mathbb{C}) \). \( \square \)

**Remark 6.7.** In their previous work [21] the authors have defined the Nielsen torsion \( N(V, U) \), for any minimal generating systems \( U \) and \( V \) of a finitely generated group \( G \). This torsion invariant depends only on the Nielsen equivalence classes of \( U \) and \( V \), and it is based on the same Fox derivative approach as used here.

The invariant \( N(V, U) \) is an element in the first \( K \)-group \( K_1(\mathbb{Z}G/I_G) \) over the quotient of the group ring \( \mathbb{Z}G \) modulo the Fox ideal \( I_G \). Here \( I_G \) is the two-sided ideal generated by the \( p_U \)-images of the Fox derivatives \( \partial R / \partial X_i \), for any \( R \in \ker(p_U : F(X) \rightarrow G) \) and any element \( X_i \) of \( X = (X_1, \ldots, X_n) \). More precisely, \( N(V, X) \) lies in the quotient (called \( N(G) \)) of \( K_1(\mathbb{Z}G/I_G) \) modulo the subgroup \( T_G \) of all trivial units, i.e. all elements given by \( \pm g \) for any \( g \in G \).

A careful analysis of the proof of Proposition 6.1 presented in this section reveals that, for any two standard generating systems \( U, V \) of a non-exceptional Fuchsian group \( G \), one has actually

\[ N(V, U) \neq 1 , \]

if the family of exponents for \( U \) and \( V \) do not satisfy the condition \( u_i = \pm v_i \) modulo \( \gamma_i \), for all \( i = 1, \ldots, \ell \).

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2 The evaluation methods employed in [20] are quite different from the techniques described explicitly in this and the previous sections. As a consequence, a quick inclusion of the methods from [20], for the benefit of completeness of the presentation here, doesn’t quite seem possible.
This is in fact a stronger statement than the one given in Theorem 1.2, since there are pairs of minimal generating systems (in different groups \( G \)) which are known to be not Nielsen equivalent, but have trivial \( N \)-torsion. Since \( N \)-torsion behaves functorially (see Theorem I (iv) of [21]), this can be used to exhibit inequivalent generating systems in certain quotients of \( G \), while in general Nielsen inequivalence is not preserved when passing even to mild quotients of a group.

7. Proof of three lemmas

It remains to prove Lemmas 6.4, 6.5 and 6.6. We will use the notation and terminology introduced in the previous section.

Proof of Lemma 6.4. Any second set \( W^* \subset F(X) \) of lifts of the elements in \( W \) under the map \( p_U \) gives rise to a second “Jacobian matrix” \( M(W^*, U) \in SL_{2\ell-2}(\mathbb{C}[\langle t \mid t^p \rangle]) \) analog to \( M(W, U) \). It satisfies (see Proposition 2.3)\n
\[
M(W^*, U) = M(W, U) + A,
\]

where the matrix \( A \) has the property that each row is given by the \( \eta \)-image of some \((\ell - 1)\)-tuple

\[
(\partial R/\partial X_1, \ldots, \partial R/\partial X_{j-1}, \partial R/\partial X_{j+1}, \ldots, \partial R/\partial X_{\ell}),
\]

with \( R \in \ker(p_U : F(X) \to G) \).

In particular, if \( \ker p_U \) is normally generated by elements \( R_1, \ldots, R_t \), then each coefficient of \( A \) is the \( \eta \)-image of a sum of \( \mathbb{Z}G \)-left-multiples of \( p_U(\partial R_s/\partial X_i) \), with \( i \in \{1, \ldots, j - 1, j + 1, \ldots, \ell\} \) and \( s \in \{1, \ldots, t\} \). As a consequence (see Corollary 2.10), the determinant \( D(W^*, U) \), analogously defined as the determinant \( D(W, U) \) before Remark 6.3, satisfies the equality

\[
D(W^*, U) = D(W, U) + B,
\]

where \( B \in \mathbb{C}[\langle t \mid t^p \rangle] \) is a sum of products which all contain, as factor, a coefficient of one of the above \((2 \times 2)\)-matrices \((\eta \circ p_U)(\partial R_s/\partial X_i)\). Hence the claim of Lemma 6.4 follows if we prove

\[
(7.1) \quad \Pi(u_1, u_2, 1) b = 0
\]

for any such coefficient \( b \).

Observe (by performing a suitable sequence of Tietze operations on the presentation (1.2) of \( G \)) that the kernel of the surjection \( p_U : F(X) \to G, X_i \mapsto x_i = s_i^{u_i} \) is normally generated by the elements

\[
X_i^{\gamma_i} \quad \text{for all} \quad i \in \{1, \ldots, j - 1, j + 1, \ldots, \ell\},
\]

together with the relator

\[
R_0 = (X_{j+1}^{\gamma_{j+1}} \ldots X_j^{\gamma_j} X_1^{\gamma_1} \ldots X_{j-1}^{\gamma_{j-1}})^{\gamma_j},
\]

for the exponents \( z_i \) as defined in (5.3).

Now \( \partial X_i^{\gamma_i}/\partial X_h = 0 \) for \( h \neq i \), and \( \partial X_i^{\gamma_i}/\partial X_i = 1 + X_i + \cdots + X_i^{\gamma_i-1} \). For \( i \geq 3 \) the \((\eta \circ p_U)\)-image of \( X_i \) is conjugate to the matrix \( M(\zeta_i) \) as in (6.2), so that since

\[
(7.2) \quad 1 + \zeta_i + \zeta_i^2 + \cdots + \zeta_i^{\gamma_i-1} = 0
\]

we have \((\eta \circ p_U)(\partial X_i^{\gamma_i}/\partial X_i) = I_2 + M(\zeta_i^2) + \cdots + M(\zeta_i^{\gamma_i-1}) = 0\). Note that this argument is also true for the special case \( \gamma_i = 2 \), see Remark 6.3 (3).

For \( i = 2 \) the matrix \((\eta \circ p_U)(\partial X_2^{\gamma_2}/\partial X_2)\) is conjugate to

\[
\begin{bmatrix}
\Sigma_0 & 0 \\
0 & \Sigma_1
\end{bmatrix}
\]
with \( \Sigma_0 = 1 + t^{-u_2} + (t^{-u_2})^2 + \ldots + (t^{-u_2})^{\gamma_2 - 1} = 1 + t + \cdots + t^{\gamma_2 - 1} \). Since \( p \) is a divisor of \( \gamma_2 \), we have \((t^{u_2} - 1)\Sigma_0 = 0\), and thus

\[
\Pi(u_1, u_2, 1) \Sigma_0 = 0.
\]

The analogous calculations shows \( \Pi(u_1, u_2, 1) \Sigma_1 = 0 \).

For \( i = 1 \) the situation is similar: One obtains

\[
(\eta \circ p_U)(\partial X_1^{\gamma_1}/\partial X_1) = \begin{bmatrix} \Sigma'_0 & 0 \\ 0 & \Sigma'_1 \end{bmatrix}
\]

with \( \Sigma'_0 = 1 + \zeta_1 t^{u_1} + (\zeta_1 t^{u_1})^2 + \ldots + (\zeta_1 t^{u_1})^{\gamma_1 - 1} \), which gives

\[
(\zeta_1 t^{u_1} - 1)\Sigma'_0 = \Pi(u_1, u_2, 1) \Sigma'_0 = 0.
\]

For \( \Sigma'_1 \) the computations are essentially the same.

It remains to check the relator \( R_0 = (X_{j+1}^{z_{j+1}} \ldots X_{\ell}^{z_\ell} X_1^{z_1} \ldots X_{j-1}^{z_{j-1}})^{\gamma_j} \). Use the chain rule for Fox-derivatives (see (2.3)) and the abbreviation \( X_j = X_{j+1}^{z_{j+1}} \ldots X_{\ell}^{z_\ell} X_1^{z_1} \ldots X_{j-1}^{z_{j-1}} \) (recalling \( i \neq j \)), to compute

\[
\partial R_0/\partial X_i = (1 + X_j + X_j^2 + \ldots + X_j^{\gamma_j - 1}) \partial X_j/\partial X_i.
\]

However, \( p_U \) maps \( X_j \) to \( s_j^{-1} \), which in turn is mapped by \( \eta \) to a conjugate of the matrix \( M(\zeta_j^{-1}) \). Hence we are now able to apply the same argument as above for the relators \( X_i^{\gamma_i} \), as follows:

For the case \( j \geq 3 \) one computes directly, from equality (7.2) with \( j \) replacing \( i \), that (7.3) gives (for any index \( i \neq j \))

\[
(\eta \circ p_U)(\partial R_0/\partial X_i) = 0.
\]

For \( j = 1 \) or \( j = 2 \) every coefficient of \( (\eta \circ p_U)(\partial R_0/\partial X_i) \) is the sum of products each of which contains as factor one of the terms \( \Sigma_0, \Sigma_1, \Sigma'_0 \) or \( \Sigma'_1 \) defined above. In this case we have shown already that multiplication with \( \Pi(u_1, u_2, 1) \) annihilates each such sum.

Thus the equality (7.1) holds for any coefficient \( b \) as desired, and hence the claim stated in Lemma 6.4 is proved.

**Proof of Lemma 6.5** For any generating system \( W \) of \( G \) we know from Lemma 6.4 that the left hand side of the equality claimed in Lemma 6.5 doesn’t depend on the choice of the lift \( W \) of \( W \) under map \( p_U : F(X) \to G \).

By Theorem 2.6 we can use the assumption that \( W \) is Nielsen equivalent to \( U \) to pick such a lift \( W \subset F(X) \) which is a basis of \( F(X) \). It follows (see Proposition 2.8) that the matrix \( \partial W/\partial U \) is a product of generalized elementary \( \mathbb{Z}G \)-matrices. Hence \( D(W, U) \) is the product of the determinants of the \( \eta \)-images of these elementary matrices, and thus a product of terms of type

\[
det(\eta(\pm g)) \quad \text{with} \quad g \in G.
\]

However, from the definition of \( \eta \) in section 6 we compute directly that \( \det(\eta(s_i)) = \det(\eta(-s_i)) = 1 \) for any \( i \in \{1, \ldots, \ell\} \). This proves the claim of Lemma 6.5.

**Proof of Lemma 6.6** Consider the generating system

\[
V = (y_1 = s_1^{v_1}, \ldots, y_{k-1} = s_{k-1}^{v_{k-1}}, y_{k+1} = s_{k+1}^{v_{k+1}}, \ldots, y_{\ell} = s_{\ell}^{v_{\ell}})
\]

for \( G \). For \( h \notin \{j, k\} \) define the element \( Y_h = X_h^{z_h} \in F(X) \), where \( X_h \subset X \), and the \( z_h \) are given in (6.3). Recall the formal definitions \( u_j = v_k = 1 \) and set \( y_k = s_k \).

Compute now (recalling \( x_i = s_i^{u_i} \))

\[
s_j = (s_{j+1} \ldots s_{\ell} s_1 \ldots s_{j-1})^{-1} = (x_{j+1}^{z_{j+1}} \ldots x_{\ell}^{z_\ell} x_1^{z_1} \ldots x_{j-1}^{z_{j-1}})^{-1}
\]
and set
\[ Y_0 = (X_{j+1}^{z_{j+1}} \cdots X_{j}^{z_{j}} X_1^{z_1} \cdots X_{j-1}^{z_{j-1}})^{-1} \]
as well as
\[ Y_k = Y_0^{s_k} = (X_{j+1}^{z_{j+1}} \cdots X_{j}^{z_{j}} X_1^{z_1} \cdots X_{j-1}^{z_{j-1}})^{-s_k}. \]
This gives:
\[ p_d(Y_h) = y_h \quad \text{for} \quad h \notin \{j, k\}, \quad \text{and} \quad p_d(Y_0) = s_j, \quad p_d(Y_k) = y_j. \]
Now compute the \((\ell - 1) \times (\ell - 1)\)-matrix \(\partial Y/\partial X\) of Fox derivatives \(\partial Y_h/\partial X_i\) with \(h, i \neq j\), and denote by \(\partial V/\partial U\) its image in the matrix ring \(M_{(\ell-1) \times (\ell-1)}(\mathbb{Z}G)\), under the map induced by \(p_d\).

In order to understand the matrix \(\partial V/\partial U\), compute for \(h \notin \{j, k\}\) the Fox derivatives
\[ \partial Y_h/\partial X_h = 1 + X_h + X_h^2 + \cdots + X_h^{z_h} u_h^{-1} \]
and
\[ \partial Y_h/\partial X_i = 0 \quad \text{for} \quad i \neq h, k. \]
Furthermore, using the formula (2.1) we obtain:
\[ \partial Y_k/\partial X_i = (1 + Y_0 + Y_0^2 + \cdots + Y_0^{s_k-1}) \partial Y_0/\partial X_i \]
\[ = - (1 + Y_0 + Y_0^2 + \cdots + Y_0^{s_k-1}) Y_0 \partial (X_{j+1}^{z_{j+1}} \cdots X_j^{z_j} X_1^{z_1} \cdots X_{j-1}^{z_{j-1}})/(\partial X_i) \]
\[ = -(1 + Y_0 + Y_0^2 + \cdots + Y_0^{s_k-1}) (X_{j-1}^{z_{j-1}} \cdots X_1^{z_1}) \partial X_{j-1}^{z_{j-1}} \partial X_i. \]
It follows that for \(j = k\) the matrix \(\partial V/\partial U = (p_d(\partial Y_h/\partial X_i)_{h,j \in \{1, \ldots, j-1, j+1, \ldots, \ell\}})\) is a diagonal matrix, while for \(j \neq k\) it differs from a diagonal matrix only in the line with index \(h = k\). In both cases, if we now apply the representation \(\eta\) to obtain the matrix \(M(V, U)\), then its determinant \(D(V, U)\) is the product of the determinants of the \((2 \times 2)\)-diagonal blocks \(M_h\) of \(M(V, U)\). Hence the equality claimed in Lemma 6.6 is equivalent to proving the following equality:
\[ [(\zeta_1^{s_1} t^{u_1} - 1)(\zeta_1^{s_1} t^{-u_1} - 1)(t^{u_2} - 1)(t^{-u_2} - 1)] \cdot (\det M_1 \cdot \ldots \cdot \det M_{j-1} \cdot \det M_{j+1} \cdot \ldots \cdot \det M_{\ell}) \]
\[ = r(\zeta_1^{s_1} t^{u_1} - 1)(\zeta_1^{s_1} t^{-u_1} - 1)(t^{u_2} - 1)(t^{-u_2} - 1) \]
for some \(r \in \mathbb{R}\).

In order to prove (7.6) we now evaluate the \((2 \times 2)\)-matrix \(M_h = \eta(\partial Y_h/\partial X_h)\) in the various possible cases for the indices \(j, k, h\), where we keep in mind that one always has \(h \neq j\).

**A** Assume \(h \geq 3\) and \(j \geq 3\):

For the case \(h \neq k\) we observe from (7.4) that \(M_h\) is conjugate to a diagonal matrix with complex-conjugate terms in the diagonal. Thus we have
\[ \det M_h \in \mathbb{R}, \]
so that its value doesn’t effect the equality claimed in (7.6).

For the case that \(h = k\) we obtain from (7.5) that \(\det M_h\) is the product
\[ \det M_h = I \cdot II \cdot III \]
where:
\[ I = \det(-\eta(1 + s_j + \cdots s_j^{v_j-1})), \]
\[ II = \det(\eta(s_j^{-1} \cdots s_1^{-1} s_k^{-1} \cdots s_i^{-1})), \]
and
\[ III = \det(\eta(1 + s_h^{u_h} + \cdots + s_h^{u_h})). \]
Independently of the choice of the indices the determinant of type II is always contained in $\mathbb{R}$. The same is true for the determinants of type I and III, as long as we assume, as in the present case (A), that $j \geq 3$ and $h \geq 3$. In case (B) below the product decomposition (7.7) of $\det M_h$ is still true, but the factors I or III will take on non-real values.

(B) Assume $h \leq 2$ or $j \leq 2$:

Case (B) will be split below into 8 subcases (a) - (h). In each of them we will apply an argument similar to the one that has already been used in the proof of Lemma 6.4. In order to simplify the exposition, we use, for any integer $q \geq 1$, the notation

$$
\Sigma_q = 1 + \zeta_1 t^{u_1} + \ldots + \zeta_1^{q-1} (t^{u_1})^{q-1} \in \mathbb{C}(t \mid t^p)
$$

and observe that

$$
(\zeta_1^{u_1} t^{u_1} - 1) \Sigma_q = (\zeta_1^{q u_1} t^{q u_1} - 1) \quad \text{and} \quad (t^{u_2} - 1) \Sigma_q' = (t^{q u_2} - 1).
$$

Now the eight remaining cases are considered:

(a) $h = 1$ and $k \neq 1$: One has $\det M_1 = \Sigma_{2v_1} \cdot \Sigma_{2v_1}$ (where $\Sigma_q$ denotes the complex-conjugate of $\Sigma_q$). Hence (7.8) gives:

$$
(\zeta_1^{u_1} t^{u_1} - 1)(\zeta_1^{-v_1} t^{-u_1} - 1) \det M_1 =
(\zeta_1^{v_1} t^{v_1} - 1)(\zeta_1^{-v_1} t^{-v_1} - 1) = (\zeta_1^{v_1} t^{v_1} - 1)(\zeta_1^{-v_1} t^{-v_1} - 1)
$$

(b) $h = 2$ and $k \neq 2$: One has $\det M_2 = \Sigma'_{2v_2} \cdot \Sigma'_{2v_2}$. Hence (7.8) gives:

$$
(t^{u_2} - 1)(t^{-u_2} - 1) \det M_2 =
(t^{2 v_2 u_2} - 1)(t^{-2 v_2 u_2} - 1) = (t^{v_2} - 1)(t^{-v_2} - 1)
$$

(c) $h = 1$, $k = 1$ and $j \geq 3$: In this case the determinant decomposition (7.7) of $\det M_1$ has real factor I but non-real factor III, which gives $\det M_1 = r_0 \Sigma_{2v_1} \Sigma_{2v_1}$ for some $r_0 \in \mathbb{R}$. Hence (7.8) gives:

$$
(\zeta_1^{u_1} t^{u_1} - 1)(\zeta_1^{-v_1} t^{-v_1} - 1) \det M_1 =
\det M_1 =
$$

(d) $h = 2$, $k = 2$ and $j \geq 3$: Again, the determinant decomposition (7.7) of $\det M_2$ has real factor I but non-real factor III. We obtain $\det M_2 = r_0 \Sigma'_{2v_2} \Sigma'_{2v_2}$ for some $r_0 \in \mathbb{R}$. Hence (7.8) gives:

$$
(t^{v_2} - 1)(t^{-v_2} - 1) \det M_2 =
r_0 (t^{2 v_2 u_2} - 1)(t^{-2 v_2 u_2} - 1) = (t^{v_2} - 1)(t^{-v_2} - 1)
$$

(e) $h = k \geq 3$ and $j = 1$: Here the determinant decomposition (7.7) of $\det M_h$ has real factor III but non-real factor I. Compute that $\det M_h = r_0 \Sigma_{2v_1} \Sigma_{2v_1}$ for some $r_0 \in \mathbb{R}$. Hence (7.8) gives (recalling the formal convention $u_1 = 1$):

$$
(\zeta_1^{u_1} t^{u_1} - 1)(\zeta_1^{-v_1} t^{-u_1} - 1) \det M_h =
\det M_h =
$$

(f) $h = k \geq 3$ and $j = 2$: Here again, the determinant decomposition (7.7) of $\det M_h$ has real factor III but non-real factor I. Compute $\det M_h = r_0 \Sigma_{2v_2} \Sigma_{2v_2}$ for some $r_0 \in \mathbb{R}$. Recalling the formal convention $u_2 = 1$ we deduce from (7.8):

$$
(t^{u_2} - 1)(t^{-u_2} - 1) \det M_h =
$$
\[ r_0(t^{2v_2u_2} - 1)(t^{-2v_2u_2} - 1) = r_0(t^{v_2} - 1)(t^{-v_2} - 1) \]

(g) \( h = k = 2 \) and \( j = 1 \): In this case in the determinant decomposition \((7.7)\) of \( \det M_2 \) both factors \( I \) and \( III \) are non-real. Compute \( \det M_2 = r_0 \sum_{z_1v_1} \sum_{z_2v_2} \Sigma'_{z_2v_2} \Sigma'_{z_2v_2} \) for some \( r_0 \in \mathbb{R} \). Hence \((7.8)\) gives (recalling again \( u_1 = 1 \)):

\[ \det M_2 = r_0(\zeta_{u_1}^{v_1} - 1)(\zeta_{-u_1}^{v_1} - 1)(t^{v_2} - 1)(t^{-v_2} - 1) \]

(h) \( h = k = 1 \) and \( j = 2 \): Here too, in the determinant decomposition \((7.7)\) of \( \det M_1 \), both factors \( I \) and \( III \) are non-real. Compute \( \det M_1 = r_0 \sum_{z_1v_1} \sum_{z_2v_2} \Sigma'_{z_2v_2} \Sigma'_{z_2v_2} \) for some \( r_0 \in \mathbb{R} \). Hence \((7.8)\) gives (for \( u_2 = 1 \) as before):

\[ \det M_1 = r_0(\zeta_{v_1}^{u_1} - 1)(\zeta_{-v_1}^{u_1} - 1)(t^{v_2} - 1)(t^{-v_2} - 1) \]

In order to finish the proof we have to “paste together” these calculations and verify, for each possibility of the values for the indices \( j \) and \( k \), that the equality \((7.6)\) is satisfied.

To guide the reader through the various combinations, the arguments needed in each case are assembled into the following table:

| k = 1 | k = 2 | k ≥ 3 |
|-------|-------|-------|
| j = 1 | (b)   | (b) & (c) |
| j = 2 | (h)   | (a) & (f) |
| j ≥ 3 | (b) & (c) | (a) & (d) | (a) & (b) |

Each of the nine cases is easily verified, where for the first two cases in the diagonal we also use the formal conventions \( u_j = v_k = 1 \). This completes the proof. \( \square \)
8. Generalizations

In this short final section we discuss how the results from the previous sections generalize to Fuchsian groups \( G \) with associated quotient orbifold that is topologically a closed surface with handles or crosscaps. In the orientable case the presentation given in (1.2) becomes

\[
\langle s_1, \ldots, s_\ell, d_1, \ldots, d_q \mid s_1^{\gamma_1}, \ldots, s_\ell^{\gamma_\ell}, s_1 \ldots s_\ell \prod_{j=1}^q [a_j, b_j] \rangle
\]

with \( \ell \geq 1, q \geq 1 \) and all exponents \( \gamma_k \geq 2 \).

If the orbifold associated to \( G \) is non-orientable, then there is at least one crosscap, and the corresponding presentation for \( G \) is

\[
\langle s_1, \ldots, s_\ell, c_1, \ldots, c_h \mid s_1^{\gamma_1}, \ldots, s_\ell^{\gamma_\ell}, s_1 \ldots s_\ell c_1^2 \ldots c_h^2 \rangle
\]

with \( \ell \geq 1, h \geq 2 \) or \( \ell \geq 2, h \geq 1 \), and all exponents \( \gamma_k \geq 2 \).

Consider, as before, standard generating systems \( U^* \) and \( V^* \) of \( G \), which are obtained from \( U \) and \( V \) as in Theorem 1.2 by

\[
U^* = U \cup \{a_1, b_1, \ldots, a_g, b_g\} \quad \text{and} \quad V^* = V \cup \{a_1, b_1, \ldots, a_g, b_g\}
\]

in the orientable case, and by

\[
U^* = U \cup \{c_1, \ldots, c_h\} \quad \text{and} \quad V^* = V \cup \{c_1, \ldots, c_h\}
\]

in the non-orientable case. We then obtain:

**Corollary 8.1.** Let \( G \) be a group with presentation (8.1) or (8.2), and let \( U^* \) and \( V^* \) be as defined above. In the orientable case assume that \( m \geq 5 \) if \( n \) is even, and that \( m \geq 7 \) if \( n \) is odd. In the non-orientable \( n \) and \( m \) must satisfy the same conditions, but with \( n \) replaced by \( n + h \).

Then \( U^* \) and \( V^* \) are Nielsen equivalent if and only if \( u_i = \pm v_i \) modulo \( \gamma_i \), for all \( i = 1, \ldots, \ell \).

**Proof.** In the non-orientable case (8.2) we quotient \( G \) to a group with presentation as in (1.2), by adding the relators \( c_1^2, \ldots, c_h^2 \). For the “only if” direction we then use the observation (see Remark 2.5) that Nielsen equivalence is preserved when passing to a quotient group, while for the “if” direction the same proof as given for Theorem 1.2 applies.

In the orientable case we use the same proof as given in the previous sections: We extend the evaluation representations

\[
\eta: \mathbb{Z}G \to \text{Sl}_2(\mathbb{Z}[t \mid t^p])
\]

from section 6 by mapping every \( a_k \) and every \( b_k \) to the unit matrix \( I_2 \). This extension method has already been used in our previous paper [20], and all needed details are given there. \( \square \)

Alternatively to the quote given at the end of the last proof, one can also derive the argument directly from the material presented in the previous sections. This leads indeed to a much stronger statement, which we will sketch now:

Consider any group \( G \) with presentation

\[
\langle s_1, \ldots, s_\ell, d_1, \ldots, d_q \mid s_1^{\gamma_1}, \ldots, s_\ell^{\gamma_\ell}, s_1 \ldots s_\ell W \rangle,
\]

for an arbitrary element \( W \in F(d_1, \ldots, d_q) \). Define generating systems

\[
U^* = U \cup \{d_1, \ldots, d_q\} \quad \text{and} \quad V^* = V \cup \{d_1, \ldots, d_q\},
\]

where \( U \) and \( V \) are standard generating systems of the quotient group

\[
G_0 = G/\langle\langle\{d_1, \ldots, d_q\}\rangle\rangle.
\]

This group is clearly of type (1.2) as considered in the previous sections. We note that if \( G_0 \) is non-exceptional, then the cyclic-faithful representation \( \eta: G_0 \to \text{Sl}_2(\mathbb{C}) \) given by Lemma 4.3 (under
the hypotheses stated there) lifts to a representation \( \eta^* : G \to Sl_2(\mathbb{C}) \), where every generator \( d_k \) is mapped to the identity matrix \( I_2 \).

As a consequence, all the arguments from the previous sections apply to \( G \) as well, in particular the crucial argument in section 7 (proof of Lemma 6.4): The \( \eta^* \)-image of the Fox derivatives \( \partial R_0/\partial d_k \) vanish, independently of the choice of the element \( W \in F(d_1, \ldots, d_q) \). This is because the formula (7.3) also holds for the generators \( d_k \), so that \( \partial R_0/\partial d_k \) contains the factor

\[
1 + s_j + s_j^2 + \ldots + s_j^{\gamma_j - 1}
\]

which is mapped by \( \eta^* \) to 0. Hence only minor adaptations in the proof of Theorem 1.2 are needed to give the following:

**Theorem 8.2.** Let \( G \) be a group with presentation (8.3), and assume that the above quotient group \( G_0 \) is non-exceptional.

Then the generating systems \( U^* \) and \( V^* \) as in (8.4) are Nielsen equivalent if and only if \( u_i = \pm v_i \mod \gamma_i \), for all \( i = 1, \ldots, \ell \).

In fact, by restricting the choice of \( W \) slightly, one can do even better, in that also many exceptional groups \( G_0 \) satisfy the conclusion of the above theorem. The details, however, will be provided elsewhere.

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