SPATIALLY QUASI-PERIODIC GRAVITY-CAPILLARY WATER WAVES OF INFINITE DEPTH

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Abstract. We formulate the two-dimensional gravity-capillary water wave equations in a spatially quasi-periodic setting and present a numerical study of traveling waves and more general solutions of the initial value problem. The former are a generalization of the classical Wilton ripple problem. We adopt a conformal mapping formulation and employ a quasi-periodic version of the Hilbert transform to determine the normal velocity of the free surface. We compute traveling waves in a nonlinear least-squares framework using a variant of the Levenberg-Marquardt method. We propose four methods for time-stepping the initial value problem, two explicit Runge-Kutta (ERK) methods and two exponential time-differencing (ETD) schemes. The latter approach makes use of the small-scale decomposition to eliminate stiffness due to surface tension. We investigate various properties of quasi-periodic traveling waves, including Fourier resonances and the dependence of wave speed and surface tension on the amplitude parameters that describe a two-parameter family of waves. We also present an example of a periodic wave profile containing vertical tangent lines that is set in motion with a quasi-periodic velocity potential that causes some of the waves to overturn and others to flatten out as time evolves.

1. Introduction

Traveling water waves have long played a central role in the field of fluid mechanics. Spatially periodic traveling waves, dating back to Stokes [15,38], have been studied extensively [8,25,26,28,29,33,34,39]. However, little research has been done on spatially quasi-periodic water waves, traveling or otherwise, in spite of their abundance in integrable model water wave equations such as the Korteweg-de Vries equation and the nonlinear Schrödinger equation. On the theoretical side, Bridges and Dias [11] used a spatial Hamiltonian structure to construct weakly nonlinear approximations of spatially quasi-periodic traveling gravity-capillary waves for two special cases: deep water and shallow water. The existence of such waves in the fully nonlinear setting is still an open problem. In this paper, we numerically demonstrate their existence, explore their properties, and also consider the general initial value problem.

To motivate our work, recall the dispersion relation for linearized traveling gravity-capillary waves in deep water:

\[ c^2 = g k^{-1} + \tau k. \]  

(1.1)

Here \( c \) is the phase speed, \( k \) is the wave number, \( g \) is the acceleration due to gravity and \( \tau \) is the coefficient of surface tension. Notice that \( c = \sqrt{(g/k) + \tau k} \) has a positive minimum, denoted by \( c_{\text{crit}} \). For any fixed phase speed \( c > c_{\text{crit}} \), there are two distinct positive wave numbers satisfying the dispersion relation (1.1), denoted \( k_1 \) and \( k_2 \). Any traveling solution

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of the linearized problem with this speed can be expressed as a superposition of waves with these wave numbers. If $k_1$ and $k_2$ are rationally related, the motion is spatially periodic and corresponds to the well-known Wilton ripples \cite{2,40,44}. However, if $k_1$ and $k_2$ are irrationally related, the motion will be spatially quasi-periodic.

Recently, Berti et al \cite{7,10} have proved the existence of small-amplitude temporally quasi-periodic gravity-capillary waves using Nash-Moser theory. They show that solutions of the linearized standing water wave problem can be combined and perturbed to obtain temporally quasi-periodic solutions of the nonlinear problem. Following the same philosophy, we look for spatially quasi-periodic solutions of the traveling water wave equations that are perturbations of solutions of the linearized problem. The velocity potential can be eliminated from the Euler equations when looking for traveling solutions, so our goal is to study traveling waves with height functions of the form

\begin{equation}
\eta(\alpha) = \tilde{\eta}(k_1 \alpha, k_2 \alpha), \quad \tilde{\eta}(\alpha_1, \alpha_2) = \sum_{(j_1, j_2) \in \mathbb{Z}^2} \hat{\eta}_{j_1, j_2} e^{i(j_1 \alpha_1 + j_2 \alpha_2)}.
\end{equation}

Here $\tilde{\eta}$ is real-valued and defined on the torus $\mathbb{T}^2 = \mathbb{R}^2 / 2\pi \mathbb{Z}^2$, and $\alpha$ parametrizes the free surface in such a way that the fluid domain is the image of the lower half-space $\{w = \alpha + i\beta : \beta < 0\}$ under a conformal map $z(w)$ whose imaginary part on the upper boundary is $\text{Im}\{z|_{\beta=0}\} = \eta$. The leading term here is $\eta_{\text{lin}}(\alpha) = 2 \text{Re}\{\hat{\eta}_{1,0} e^{ik_1 \alpha} + \hat{\eta}_{0,1} e^{ik_2 \alpha}\}$, which will be a solution of the linearized problem.

Unlike \cite{11}, as noted above, we use a conformal mapping formulation \cite{13,18,30} of the gravity-capillary water wave problem. In particular, we introduce a Hilbert transform for quasi-periodic functions to compute the normal velocity and maintain a conformal parametrization of the free surface. This leads us to a numerical method to compute the time evolution of solutions of the Euler equations from arbitrary quasi-periodic initial data. We present two variants of the method, one in a high-order explicit Runge-Kutta framework and one in an exponential time-differencing framework. The former is suitable for the case of zero or small surface tension while the latter makes use of the small-scale decomposition \cite{23,24} to eliminate stiffness due to surface tension. We present a convergence study of the methods as well as a large-scale computation of an overturning quasi-periodic wave with over 30 million degrees of freedom evolved over 5400 timesteps to maintain double-precision accuracy.

We formulate the traveling wave computation as a nonlinear least-squares problem and use the Levenberg-Marquardt method to search for solutions. This approach builds on the overdetermined shooting methods developed by Wilkening et al \cite{4,5,20,37,43} to compute standing waves and other time-periodic solutions. Specifically, we fix the ratio $k_2/k_1$, denoted by $k$, and solve simultaneously for the phase speed $c$, the coefficient of surface tension $\tau$, and the unknown Fourier modes $\hat{\eta}_{j_1, j_2}$ in (1.2) subject to the constraint that $\hat{\eta}_{1,0}$ and $\hat{\eta}_{0,1}$ have prescribed values. One could instead prescribe $\tau$ and $\hat{\eta}_{1,0}$, say, and solve for $\hat{\eta}_{0,1}$ with the other unknown Fourier modes. In this paper, we only present numerical results for the case when $k = 1/\sqrt{2}$. But the numerical methods we develop are general and can be used to compute solutions with different values of $k$. In a subsequent paper, we will extend our results to the case of finite-depth water waves and analyze the stability of solutions.

We include three appendices that cover various technical aspects of spatially quasi-periodic water waves that arose in the course of this work. In Appendix A, we prove a
theorem establishing sufficient conditions for an analytic function $z(w)$ to map the lower half-plane topologically onto a semi-infinite region bounded above by a parametrized curve and for $|1/|z_w||$ to be uniformly bounded. In Appendix B, we study families of quasi-periodic solutions obtained by introducing phases in the reconstruction formula for extracting 1d quasi-periodic functions from periodic functions on a torus. This enables us to prove that if all the solutions in the family are single-valued and have no vertical tangent lines, the solutions are also quasi-periodic in the original graph-based formulation of the Euler equations. We also present a simple procedure for computing the change of variables from the conformal representation to the graph representation. Finally, in Appendix C, we study the dynamics of traveling waves in conformal space for various choices of a free parameter in the equations of motion that controls the tangential velocity of the surface parametrization. We show that the waves maintain a permanent form but generally travel at a non-uniform speed in conformal space as they evolve.

2. Mathematical Formulation

In this section, we review the governing equations for gravity-capillary waves in physical space and their conformal mapping formulation. We then extend the formulation to allow for spatially quasi-periodic solutions. For simplicity, we initially assume the wave profile $\eta(x, t)$ remains single-valued. This assumption is relaxed when discussing the conformal formulation, and an example of a wave in which some of the peaks overturn as time advances is presented in Section 3.

2.1. Governing Equations in Physical Space. Gravity-capillary waves of infinite depth are governed by the two-dimensional free-surface Euler equations:

$$\begin{align*}
\eta(x, 0) &= \eta_0(x), \quad \varphi(x, 0) = \varphi_0(x), \quad t = 0, \\
\Phi_{xx} + \Phi_{yy} &= 0, \quad -\infty < y < \eta(x, t), \\
\Phi_y &\to 0, \quad y \to -\infty, \\
\Phi &= \varphi, \quad y = \eta(x, t), \\
\eta_t &= \Phi_y \eta_t - \frac{1}{2} \Phi_x^2 - \frac{1}{2} \Phi_y^2 - g\eta + \tau \frac{\eta_{xx}}{(1 + \eta_x^2)^{3/2}} + C(t), \quad y = \eta(x, t),
\end{align*}$$

where $x$ is the horizontal coordinate, $y$ is the vertical coordinate, $t$ is the time, $\Phi(x, y, t)$ is the velocity potential, $\eta(x, t)$ is the free surface elevation,

$$\varphi_t = \Phi_y \eta_t - \frac{1}{2} \Phi_x^2 - \frac{1}{2} \Phi_y^2 - g\eta + \tau \frac{\eta_{xx}}{(1 + \eta_x^2)^{3/2}} + C(t), \quad y = \eta(x, t),$$

is the boundary value of the velocity potential on the free surface, $g$ is the vertical acceleration due to gravity and $\tau$ is the coefficient of surface tension. The function $C(t)$ in the Bernoulli condition (2.4) is an arbitrary integration constant that is allowed to depend on time but not space.

For waves of permanent form traveling with constant speed $c$, $\eta(x, t)$ and $\varphi(x, t)$ satisfy

$$\begin{align*}
\eta(x, t) &= \eta_0(x - ct), \quad \varphi(x, t) = \varphi_0(x - ct).
\end{align*}$$
Hence,
\begin{equation}
\eta_t = -c \eta_x, \quad \varphi_t = -c \varphi_x.
\end{equation}

Combining (2.3), (2.4) and (2.7), we obtain the governing equations for traveling waves,
\begin{equation}
\eta_x(c - \Phi_x) + \Phi_y = 0, \quad y = \eta(x, t),
\end{equation}
\begin{equation}
\frac{1}{2}(\Phi_x^2 + \Phi_y^2) - c \Phi_x + g \eta - \tau \frac{\eta_{xx}}{(1 + \eta_x^2)^{3/2}} = C, \quad y = \eta(x, t),
\end{equation}
also subject to (2.2). Solving these equations when \( t = 0 \) gives \( \eta_0(x) \) and \( \varphi_0(x) \) in (2.6), which can be interpreted as initial conditions for (2.1)–(2.4) such that the solution is (2.6). Alternatively, one can pose the problem in a frame of reference moving at speed \( c \) along with the wave. The equations of motion in this co-moving frame are \( \tilde{\eta}_t = 0, \Phi_t = 0, \) where \( \tilde{\eta}(\tilde{x}, t) = \eta(\tilde{x} + ct, t), \Phi(\tilde{x}, y, t) = \Phi(\tilde{x} + ct, y, t) - c \tilde{x}, \tilde{x} = x - ct. \) The Euler equations (2.1)–(2.4) are invariant under this Galilean transformation, and (2.8) becomes
\begin{equation}
-\tilde{\eta}_x \Phi_x + \Phi_y = 0, \quad y = \tilde{\eta}(\tilde{x}),
\end{equation}
\begin{equation}
\frac{1}{2}(\Phi_x^2 + \Phi_y^2) + g \tilde{\eta} - \tau \frac{\tilde{\eta}_{xx}}{(1 + \tilde{\eta}_x^2)^{3/2}} = C, \quad y = \tilde{\eta}(\tilde{x}).
\end{equation}
The second equation of (2.9) is just the steady Bernoulli equation. We prefer to work in the lab frame at \( t = 0 \) since that is the frame in which the velocity potential is periodic or quasi-periodic in the horizontal direction without a separate background flow term, but the two viewpoints are equivalent. Note that the background flow term \( -c \tilde{x} \) is responsible for the additional terms involving \( c \) in (2.8) and a change in \( C \) by \( c^2/2. \) This integration constant is arbitrary; we will make a specific choice of \( C \) in (2.48) and (2.50) below.

2.2. The Quasi-Periodic Hilbert Transform. We find that a conformal mapping representation of the free surface greatly simplifies the solution of the Laplace equation for the velocity potential in the quasi-periodic setting. In this section, we establish the properties of the Hilbert transform that will be needed to study quasi-periodic water waves in a conformal mapping framework.

A quasi-periodic, real analytic function \( u(\alpha) \) is a function of the form
\begin{equation}
u(\alpha) = \hat{u}(k \alpha), \quad \hat{u}(\alpha) = \sum_{j \in \mathbb{Z}^d} \hat{u}_j e^{i(j, \alpha)}, \quad \alpha \in \mathbb{R}, \quad \alpha, k \in \mathbb{R}^d,
\end{equation}
where \( \langle \cdot, \cdot \rangle \) denotes the standard inner product in \( \mathbb{R}^d \) and \( \hat{u} \) is a periodic, real analytic function defined on the \( n \)-dimensional torus
\begin{equation}
\mathbb{T}^d := [0, 2\pi)^d.
\end{equation}
Entries of the vector \( k \) are required to be linearly independent over \( \mathbb{Z} \). Since \( \hat{u} \) is real-valued, \( \hat{u}_{-j} = \overline{\hat{u}_j} \). Analyticity is equivalent to the requirement that there exist positive numbers \( M \) and \( \sigma \) such that \( |\hat{u}_j| \leq Me^{-\sigma|j|} \), i.e. that the Fourier modes \( \hat{u}_j \) decay exponentially as \( |j| \to \infty. \) We define projection operators \( P \) and \( P_0 \) that act on \( u \) and \( \tilde{u} \) via
\begin{equation}
P = \text{id} - P_0, \quad P_0[u] = P_0[\tilde{u}] = \hat{u}_0 = \frac{1}{(2\pi)^d} \int_{\mathbb{T}^d} \hat{u}(\alpha) \, d\alpha_1 \ldots d\alpha_d.
\end{equation}
Note that \( P \) projects onto the space of zero-mean functions while \( P_0 \) returns the mean value, viewed as a constant function on \( \mathbb{R} \) or \( \mathbb{T}^d \).
Given \( u(\alpha) \) as above, the most general bounded analytic function \( f(w) \) in the lower half-plane whose real part agrees with \( u \) on the real axis has the form

\[
(2.13) \quad f(w) = \hat{u}_0 + i\hat{\theta}_0 + \sum_{\langle j, k \rangle < 0} 2\hat{u}_j e^{i\langle j, k \rangle w}, \quad (w = \alpha + i\beta, \beta < 0)
\]

where \( \hat{\theta}_0 \in \mathbb{R} \) and the sum is over all \( j \in \mathbb{Z}^d \) satisfying \( \langle j, k \rangle < 0 \). The imaginary part of \( f(z) \) on the real axis is given by

\[
(2.14) \quad v(\alpha) = \hat{\vartheta}(k\alpha), \quad \hat{\vartheta}(\alpha) = \sum_{j \in \mathbb{Z}^d} \hat{\vartheta}_j e^{i\langle j, \alpha \rangle}, \quad \hat{\vartheta}_j = i \text{sgn}(\langle j, k \rangle) \hat{u}_j, \quad (j \neq 0)
\]

where \( \text{sgn}(q) \in \{1, 0, -1\} \) depending on whether \( q > 0, q = 0 \) or \( q < 0 \), respectively. Similarly, given \( v(\alpha) \) and requiring \( (\text{Im} f)|_\mathbb{R} = v \) yields (2.13) with \( \hat{u}_j \) replaced by \( i\hat{\vartheta}_j \). We introduce a quasi-periodic Hilbert transform to compute \( v \) from \( u \) or \( u \) from \( v \),

\[
(2.15) \quad v = \hat{\vartheta}_0 - H[u], \quad u = \hat{u}_0 + H[v],
\]

where the constant \( \hat{\vartheta}_0 = P_0[v] \) or \( \hat{u}_0 = P_0[u] \) is a free parameter when computing \( v \) or \( u \), respectively. \( H \) returns the “zero-mean” solution, i.e. \( P_0H[u] = 0 \). Uniqueness of the bounded extension from \( u \) or \( v \) to \( f \) up to the additive constant \( i\hat{\vartheta}_0 \) or \( \hat{u}_0 \) follows from the well-known result that the only bounded solution of the Laplace equation on a half-space satisfying homogeneous Dirichlet boundary conditions is identically zero [6].

**Definition 2.1.** The Hilbert transform of a quasi-periodic, analytic function \( u(\alpha) \) of the form (2.10) is defined to be

\[
(2.16) \quad H[u](\alpha) = \sum_{j \in \mathbb{Z}^d} (-i) \text{sgn}(\langle j, k \rangle) \hat{u}_j e^{i\langle j, k \rangle} \alpha.
\]

This agrees with the standard definition [19] of the Hilbert transform as a Cauchy principal value integral:

\[
(2.17) \quad H[u](\alpha) = \frac{1}{\pi} \text{PV} \int_{-\infty}^{\infty} \frac{u(\xi)}{\alpha - \xi} \, d\xi.
\]

Indeed, it is easy to show that for functions of the form \( u(\alpha) = e^{i\rho\alpha} \) with \( \rho \) real, the integral in (2.17) gives \( H[u](\alpha) = -i \text{sgn}(\rho) e^{i\rho\alpha} \). For extensions to the upper half-plane, the sum in (2.13) is over \( \langle j, k \rangle > 0 \), the last formula in (2.14) becomes \( \hat{\vartheta}_j = -i \text{sgn}(\langle j, k \rangle) \hat{u}_j \), and the signs in front of \( H[u] \) and \( H[v] \) in (2.15) are reversed.

**Remark 2.2.** As with \( P \) and \( P_0 \), there is an analogous operator on \( L^2(\mathbb{T}^d) \) such that \( H[u](\alpha) = H[\hat{u}](k\alpha) \). The formula is

\[
(2.18) \quad H[\hat{u}](\alpha) = \sum_{j \in \mathbb{Z}^d} (-i) \text{sgn}(\langle j, k \rangle) \hat{u}_j e^{i\langle j, \alpha \rangle}.
\]

If necessary for clarity, one can also write \( H_k[\hat{u}] \) to emphasize the dependence of \( H \) on \( k \). \( H \) commutes with the shift operator \( S_{\theta} \hat{u} \rangle(\alpha) = \hat{u}(\alpha + \theta) \), so if \( \hat{\vartheta} = \hat{\vartheta}_0 - H[\hat{u}] \) and \( \hat{u}_0 = P_0[\hat{u}] \), then \( v(\alpha; \theta) = \hat{\vartheta}(k\alpha + \theta) \) is related to \( u(\alpha; \theta) = \hat{u}(k\alpha + \theta) \) by (2.15). Also, if \( f(z) \) in (2.13) is the bounded analytic extension of \( u + iv \) (with \( \theta = 0 \)) to the lower half-plane, we have

\[
(2.19) \quad f(w) = \hat{f}(k\alpha, \beta), \quad (w = \alpha + i\beta, \beta < 0),
\]
where \( \tilde{f}(\alpha, \beta) = \hat{u}_0 + i \hat{v}_0 + \sum_{j,k} \hat{u}_j e^{-ijk} \hat{v}_k e^{jk} \) is periodic in \( \alpha \) for fixed \( \beta > 0 \). The bounded analytic extension of \([u(\alpha, \theta) + iv(\alpha, \theta)]\) to the lower half-plane is then given by \( f(\alpha + i\beta; \theta) = \tilde{f}(k\alpha + \theta, \beta) \).

2.3. The Conformal Mapping. We consider a time dependent conformal mapping that maps the conformal domain

\begin{equation}
\mathbb{C}^- = \{ \alpha + i\beta : \alpha \in \mathbb{R}, \beta < 0 \}
\end{equation}

to the fluid domain

\begin{equation}
\Omega_f(t) = \{ x + iy : x \in \mathbb{R}, y < \eta(x,t) \}.
\end{equation}

This conformal mapping, denoted by \( z(w, t) \), extends continuously to \( \overline{\mathbb{C}^-} \) and maps the real line \( \beta = 0 \) to the free surface

\begin{equation}
\Gamma(t) = \{ x + iy : y = \eta(x, t) \}.
\end{equation}

We express \( z(w, t) \) as

\begin{equation}
z(w, t) = x(w, t) + iy(w, t), \quad (w = \alpha + i\beta).
\end{equation}

We also introduce the notation \( \zeta = z|_{\beta=0}, \xi = x|_{\beta=0} \) and \( \eta = y|_{\beta=0} \) so that the free surface is parametrized by

\begin{equation}
\zeta(\alpha, t) = \xi(\alpha, t) + i\eta(\alpha, t), \quad (\alpha \in \mathbb{R}).
\end{equation}

This allows us to denote a generic field point in the physical fluid by \( z = x + iy \) while simultaneously discussing points \( \zeta = \xi + i\eta \) on the free surface. To avoid ambiguity, we will henceforth denote the free surface elevation function from the previous section by \( \eta^{\text{phys}}(x, t) \). Thus,

\begin{equation}
\eta(\alpha, t) = \eta^{\text{phys}}(\zeta(\alpha, t), t), \quad \eta_\alpha = \eta^{\text{phys}}_x \xi_\alpha, \quad \eta_t = \eta^{\text{phys}}_x \xi_t + \eta^{\text{phys}}_t.
\end{equation}

The parametrization (2.24) is more general than (2.25) in that it allows for overturning waves. In deriving the equations of motion for \( \zeta(\alpha, t) \) and \( \varphi(\alpha, t) \) in Sections 2.5 and 2.6 below, we will indicate the modifications necessary to handle the case of overturning waves. In particular, as discussed in Appendix A, \( \Gamma(t) \) is defined in this case as the image of \( \zeta(\cdot, t) \), which is assumed to be injective on \( \mathbb{R} \) and \( \Omega_f(t) \) can be obtained from \( \Gamma(t) \) using a linear fractional transformation and the Jordan curve theorem.

The conformal map is required to remain a bounded distance from the identity map in the lower half-plane. Specifically, we require that

\begin{equation}
|z(w, t) - w| \leq M(t) \quad (w = \alpha + i\beta, \beta \leq 0),
\end{equation}

where \( M(t) \) is a uniform bound that could vary in time. The Cauchy integral formula implies that \( |z_w - 1| \leq M(t)/|\beta| \), so at any fixed time,

\begin{equation}
z_w \to 1 \quad \text{as} \quad \beta \to -\infty.
\end{equation}

Our goal is to investigate the case when the free surface is quasi-periodic in \( \alpha \). This differs from conformal mappings discussed in [17,19,32,45], where it is assumed to be periodic.
In the present work, \( \eta \) is assumed to have two spatial quasi-periods, i.e. at any time it has the form (2.10) with \( d = 2 \) and \( k = [k_1, k_2] \). Since \( k_1 \) and \( k_2 \) are irrationally related, we assume without loss of generality that \( k_1 = 1 \) and \( k_2 = k \), where \( k \) is irrational,

\[
(2.28) \quad \eta(\alpha, t) = \tilde{\eta}(\alpha, k\alpha, t), \quad \tilde{\eta}(\alpha_1, \alpha_2, t) = \sum_{j_1, j_2 \in \mathbb{Z}} \hat{\eta}_{j_1, j_2}(t)e^{i(j_1\alpha_1 + j_2\alpha_2)}.
\]

Here \( \hat{\eta}_{-j_1, -j_2}(t) = \hat{\eta}_{j_1, j_2}(t) \) since \( \tilde{\eta}(\alpha_1, \alpha_2, t) \) is real-valued. Since \( \omega \mapsto [z(\omega, t) - \omega] \) is bounded and analytic on \( \mathbb{C}^+ \) and its imaginary part agrees with \( \eta \) on the real axis, there is a real number \( x_0 \) (possibly depending on time) such that

\[
(2.29) \quad \xi(\alpha, t) = \alpha + x_0(t) + H[\eta](\alpha, t).
\]

Using (2.27) and \( z_w|_{\beta = 0} = \eta_\alpha \) or differentiating (2.29) gives

\[
(2.30) \quad \xi_\alpha(\alpha, t) = 1 + H[\eta_\alpha](\alpha, t).
\]

We use a tilde to denote the periodic functions on the torus that correspond to the quasi-periodic parts of \( \xi, \zeta \) and \( z \),

\[
(2.31) \quad \tilde{\xi}(\alpha, t) = \alpha + \tilde{\xi}(\alpha, k\alpha, t), \quad \tilde{\zeta}(\alpha, t) = \alpha + \tilde{\zeta}(\alpha, k\alpha, t), \\
\quad z(\alpha + i\beta, t) = (\alpha + i\beta) + \tilde{z}(\alpha, k\alpha, \beta, t), \quad (\beta \leq 0).
\]

Specifically, \( \tilde{\xi} = x_0(t) + H[\tilde{\eta}] \) in the sense of Remark 2.2, \( \tilde{\zeta} = \tilde{\xi} + i\tilde{\eta} \), and

\[
(2.32) \quad \tilde{z}(\alpha_1, \alpha_2, \beta, t) = x_0(t) + i\tilde{\eta}_{0,0}(t) + \sum_{j_1 + j_2 \leq 0} \left( 2i\tilde{\eta}_{j_1, j_2}(t)e^{-(j_1\alpha_1 + j_2\alpha_2)} \right)e^{i(j_1\alpha_1 + j_2\alpha_2)}.
\]

While the mean surface height remains constant in physical space, \( \tilde{\eta}_{0,0}(t) \) generally varies in time. Since the modes \( \hat{\eta}_{j_1, j_2} \) are assumed to decay exponentially, there is a uniform bound \( M(t) \) such that \( |\tilde{z}(\alpha_1, \alpha_2, \beta, t)| \leq M(t) \) for \( (\alpha_1, \alpha_2) \in \mathbb{T}^2 \) and \( \beta \leq 0 \). In Appendix A we show that as long as the free surface \( \tilde{\zeta}(\alpha, t) \) does not self-intersect at a given time \( t \), the mapping \( w \mapsto z(w, t) \) is an analytic isomorphism of the lower half-plane onto the fluid region.

### 2.4. The Complex Velocity Potential

Let \( \Phi^{\text{phys}}(x, y, t) \) denote the velocity potential in physical space from Section 2.1 above and let \( W^{\text{phys}}(x + iy, t) = \Phi^{\text{phys}}(x, y, t) + i\Psi^{\text{phys}}(x, y, t) \) be the complex velocity potential, where \( \Psi^{\text{phys}} \) is the stream function. Using the conformal mapping (2.23), we pull back these functions to the lower half-plane and define

\[
W(w, t) = \Phi(\alpha, \beta, t) + i\Psi(\alpha, \beta, t) = W^{\text{phys}}(z(w, t), t), \quad (w = \alpha + i\beta).
\]

We also define \( \varphi = \Phi|_{\beta = 0} \) and \( \psi = \Psi|_{\beta = 0} \) and use (2.5) and (2.25) to obtain

\[
(2.33) \quad \varphi(\alpha, t) = \varphi^{\text{phys}}(\xi(\alpha, t), t), \quad \psi(\alpha, t) = \psi^{\text{phys}}(\zeta(\alpha, t), t),
\]

where \( \varphi^{\text{phys}}(x, t) = \Psi^{\text{phys}}(x, \eta^{\text{phys}}(x, t), t) \). We assume \( \varphi \) is quasi-periodic with the same quasi-periods as \( \eta \),

\[
\varphi(\alpha, t) = \tilde{\varphi}(\alpha, k\alpha, t), \quad \tilde{\varphi}(\alpha_1, \alpha_2, t) = \sum_{j_1, j_2 \in \mathbb{Z}} \hat{\varphi}_{j_1, j_2}(t)e^{i(j_1\alpha_1 + j_2\alpha_2)}.
\]

The fluid velocity \( \nabla \varphi^{\text{phys}}(x, y, t) \) is assumed to decay to zero as \( y \to -\infty \) (since we work in the lab frame). From (2.27) and (2.35) below, \( dW/d\omega \to 0 \) as \( \beta \to -\infty \). Thus, \( \psi_\alpha = -H[\varphi_\alpha] \).

Writing this as \( \partial_\alpha[\psi + H\varphi] = 0 \), we conclude that

\[
(2.34) \quad \psi(\alpha, t) = -H[\varphi](\alpha, t).
\]
Here we have set the integration constant to zero and assumed $P_0[\phi] = \phi_{0,0}(t) = 0$ and $P_0[\psi] = \psi_{0,0}(t) = 0$, which is allowed since $\Phi$ and $\Psi$ can be modified by additive constants (or functions of time only) without affecting the fluid motion.

2.5. **Governing Equations in Conformal Space.** Before specializing to traveling waves, we discuss the time evolution of water waves with arbitrary spatially quasi-periodic initial conditions. Loosely following [13,17,45], we present a derivation of the conformal formulation of the governing equations for gravity-capillary waves and show how to handle quasi-periodic solutions. From the chain rule,

$$
\frac{dW}{dw} = \frac{dW^{\text{phys}}}{dz} \cdot \frac{dz}{dw} = \Phi^{\text{phys}}_x + i\Psi^{\text{phys}}_x = \frac{\Phi_a + i\Psi_a}{x_a + iy_a}.
$$

Evaluating (2.35) on the free surface gives

$$
\Phi^{\text{phys}}_x = \frac{\Phi_a \xi_a + \psi_\alpha \eta_\alpha}{J}, \quad \Phi^{\text{phys}}_y = -\psi_\alpha \xi_a - \psi_\alpha \xi_a, \quad J = \xi_a^2 + \eta_a^2.
$$

Using (2.25) and (2.36) in (2.3) and multiplying by $\xi_a$, we obtain

$$
\eta_\alpha \xi_a - \eta_\alpha \xi_a = -\psi_\alpha.
$$

This states that the normal velocity of the free surface is equal to the normal velocity of the fluid, $\hat{n} \cdot (\xi_t, \eta_t) = \hat{n} \cdot \nabla \Phi^{\text{phys}},$ where $\hat{n} = (-\eta_\alpha, \xi_\alpha)/\sqrt{J}$. This can also be obtained by tracking a fluid particle $x_p(t) + iy_p(t) = \xi(\alpha_p(t), t)$ on the free surface. We have $\dot{x}_p = \xi_a \dot{\alpha}_p + \xi_t = \Phi_x$ and $\dot{y}_p = \eta_a \dot{\alpha}_p + \xi_t = \Phi_y$, which leads to (2.37) after eliminating $\dot{\alpha}_p$. This argument does not assume the free surface is a graph, i.e. (2.37) is also valid for overturning waves.

Next we define a new function,

$$
q := \frac{\xi_t}{\xi_a} = \frac{(\xi_t \xi_a + \eta_\alpha \eta_a) + i(\eta_\alpha \xi_a - \xi_t \eta_a)}{J} = \frac{(\xi_t \xi_a + \eta_\alpha \eta_a) - i\psi_\alpha}{J}.
$$

Since $q$ is quasi-periodic in $\alpha$ and extends analytically to the lower half-plane via $z_t/z_a$, the real and imaginary part of $q$ can be related by the Hilbert transform. Here we have assumed that $z_t/z_a$ is bounded, which will be justified below. Thus,

$$
\frac{\xi_t \xi_a + \eta_\alpha \eta_a}{J} = -H \left[ \frac{\psi_\alpha}{J} \right] + C_1,
$$

where $C_1$ is an arbitrary integration constant that may depend on time but not space. Let $\hat{s} = (\xi_a, \eta_a)/\sqrt{J}$ denote the unit tangent vector to the curve. Equation (2.39) prescribes the tangential velocity $\hat{s} \cdot (\xi_t, \eta_t)$ of points on the curve in terms of the normal velocity in order to maintain a conformal parametrization. Note that the tangent velocity of the curve differs from that of the underlying fluid particles. This is similar in spirit to a method of Hou, Lowengrub and Shelley [23,24], who proposed a tangential velocity that maintains a uniform parametrization of the curve (rather than a conformal one); see also [3]. Combining (2.37) and (2.39), we obtain the kinematic boundary conditions in conformal space,

$$
\left( \begin{array}{c} \xi_t \\ \eta_t \end{array} \right) = \left( \begin{array}{cc} \xi_a & -\eta_a \\ \eta_a & \xi_a \end{array} \right) \left( \begin{array}{c} \xi_a \\ \eta_a \end{array} \right) \left( -H \left[ \frac{\psi_\alpha}{J} \right] + C_1 \right).$$

The right-hand side can be interpreted as complex multiplication of $z_a$ with $z_t/z_a$. Since both functions are analytic in the lower half-plane, their product is, too. Thus, $\xi_t$ is related
to $\eta_0$ via the Hilbert transform (up to a constant). The constant is determined by comparing (2.29) with (2.40), which gives

$$\frac{dx_0}{dt} = P_0 \left[ \xi_a \left( -H \left[ \frac{\psi_a}{J} \right] + C_1 \right) + \frac{\eta_a \psi_a}{J} \right]. \tag{2.41}$$

The three most natural choices of $C_1$ are

1. $C_1 = 0$ : evolve $x_0(t)$ via (2.41),
2. $C_1 = P_0 \left[ \xi_a H[\psi_a/J] - \eta_a \psi_a/J \right]$ : $x_0(t) = 0$, $\xi(0, t) = 0$.
3. $C_1 = [H[\psi_a/J] - \eta_a \psi_a/(\xi_a J)]_{a=0}$ : $\xi(0, t) = 0$.

In options (b) and (c), the evolution equation ensures that $dx_0/dt = 0$ and $\xi(0, t) = 0$, respectively, so we have assumed the initial conditions satisfy $x_0(0) = 0$ or $\xi(0, 0) = 0$. Option (c) amounts to setting $x_0(t) = -H[\eta](0, t)$ in (2.29). This arguably leads to the most natural parametrization, but would have a problem if the vertical part of an overturning wave crosses $\alpha = 0$. Indeed, such a crossing would lead to $\xi_a(0, t) = 0$ at some time $t$ in the denominator of (2.42c). For overturning waves, we recommend option (b).

Next we evaluate the Bernoulli equation $\Phi_t^{\text{phys}} + \frac{1}{2} |\nabla \Phi^{\text{phys}}|^2 + \frac{p}{\rho} + g y = C_2$ at the free surface to obtain an evolution equation for $\varphi(\alpha, t)$. Here $C_2$ is an arbitrary integration constant that may depend on time but not space. The pressure at the free surface is determined by the Laplace-Young condition, $p = p_0 - \rho \tau \kappa$, where $\kappa$ is the curvature, $\rho \tau$ is the surface tension, and $p_0$ is a constant that can be absorbed into $C_2$ and set to zero. From (2.35) or (2.36), we know $|\nabla \Phi^{\text{phys}}|^2 = (\varphi_{\alpha}^2 + \varphi_{\beta}^2)/J$ on the free surface. Finally, differentiating $\varphi(\alpha, t) = \Phi^{\text{phys}}(\xi(\alpha, t), \eta(\alpha, t), t)$ and using (2.36) and (2.40), we obtain

$$\varphi_t = (\Phi_x^{\text{phys}}, \Phi_y^{\text{phys}}) \left( \frac{\xi_a}{\eta_a} - \frac{\eta_a}{\xi_a} \right) \left( -H \left[ \frac{\psi_a}{J} \right] + C_1 \right) - \frac{\varphi_{\alpha}^2 + \varphi_{\beta}^2}{2J} - \frac{g \eta + \tau \kappa}{\rho} + C_2. \tag{2.43}$$

We choose $C_2$ so that $P_0[\varphi_t] = 0$. In conclusion, we obtain the following governing equations for spatially quasi-periodic gravity-capillary waves in conformal space

$$\xi_a = 1 + H[\eta_a], \quad \psi = -H[\varphi], \quad J = \xi_a^2 + \eta_a^2, \quad \chi = \frac{\psi_{\alpha}}{J},$$

choose $C_1$, e.g. as in (2.42), compute $\frac{dx_0}{dt}$ in (2.41) if necessary,

$$\eta_t = -\eta_a H[\chi] - \xi_a \chi + C_1 \eta_a, \quad \kappa = \frac{\xi_a \eta_{aa} - \eta_a \xi_{aa}}{J^{3/2}},$$

$$\varphi_t = P \left[ \frac{\varphi_{\alpha}^2 - \varphi_{\beta}^2}{2J} - \varphi_a H[\chi] + C_1 \varphi_a - g \eta + \tau \kappa \right]. \tag{2.44}$$

Note that these equations govern the evolution of $x_0$, $\eta$ and $\varphi$, which determine the state of the system. The functions $\xi$, $\psi$, $J$, $\chi$ and $\kappa$ are determined at any moment by $x_0$, $\eta$ and $\varphi$ through the auxiliary equations in (2.44). We emphasize that $C_1$ can be chosen arbitrarily as long as $dx_0/dt$ satisfies (2.41). The special cases (2.42b) and (2.42c) lead to nice formulas for $x_0(t)$ without having to evolve (2.41) numerically.

In deriving (2.44) from (2.37) and (2.43), we had to assume $z_t/z_a$ remains bounded in the lower half-plane. Conditions that ensure the boundedness of $1/z_a$ are given in
Appendix A. We note that $z_t/z_\alpha$ is automatically bounded in the converse direction, where (2.37) and (2.43) are derived from (2.44). In more detail, when solving (2.44), $z_t/z_\alpha$ is constructed first, before $z_t$, as the bounded extension of the quasi-periodic function with imaginary part $(-\tilde{\psi}_\alpha/\tilde{f})$ to the lower half-plane. Equation (2.40) then defines $z_t$ as the product of this function by $z_\alpha$, which is also bounded since $\xi_\alpha = 1 + H[\eta_\alpha]$. Thus, the first component of each side of (2.40) is related to the corresponding second component by the Hilbert transform, up to a constant. Since the second components are equal (i.e. the $\eta_t$ equation holds), the $\xi_t$ equation also holds — the constants are accounted for by (2.41). Left-multiplying (2.40) by the row vector $[-\eta_\alpha, \xi_\alpha]$ gives the kinematic condition (2.37), as required.

Equations (2.44) break down if $J$ becomes zero somewhere on the curve. Such a singularity would arise, for example, if the wave profile were to form a corner in finite time. To our knowledge, it remains an open question whether the free-surface Euler equations can form such a corner. Often we wish to verify that a given curve $\zeta(\alpha, t) = \xi(\alpha, t) + i\eta(\alpha, t)$ and velocity potential $\varphi(\alpha, t)$ satisfy the conformal version of the water wave equations. We say that $(\zeta, \varphi)$ satisfy (2.44) if $\zeta$ and $\eta$ remain conformally related via (2.29), which determines $x_0(t)$, and if $x_0, \eta$ and $\varphi$ satisfy (2.44) with $C_1(t)$ obtained from (2.41) using $P_0[\xi_\alpha] = 1$. As noted above, these equations imply the kinematic condition (2.37) and Bernoulli equation (2.43). If necessary, one should replace the given $\varphi(\alpha, t)$ by $P[\varphi(\cdot, t)]$ before checking that (2.44) is satisfied.

Remark 2.3. Equations (2.44) can be interpreted as an evolution equation for the functions $\tilde{\zeta}(\alpha_1, \alpha_2, t)$ and $\tilde{\varphi}(\alpha_1, \alpha_2, t)$ on the torus $T^2$. The $\alpha$-derivatives are replaced by the directional derivatives $[\partial_{\alpha_1} + k\partial_{\alpha_2}]$ and, as noted in Remark 2.2 above, the Hilbert transform becomes a two-dimensional Fourier multiplier operator with symbol $(-i)\text{sgn}(j_1 + j_2k)$. The pseudo-spectral method we propose in Section 3.1 below is based on this representation. Equation (2.42c) becomes

\[
(2.45) \quad C_1 = \left[H \left( \frac{\tilde{\psi}_\alpha}{\tilde{f}} \right) - \frac{\tilde{\eta}_\alpha \tilde{\psi}_\alpha}{(1 + \tilde{\xi}_\alpha)^2} \right]_{(\alpha_1, \alpha_2) = (0, 0)}, \quad \tilde{\xi}(0, 0, t) = 0,
\]

where $\tilde{f} = (1 + \tilde{\xi}_\alpha)^2 + \tilde{\eta}_\alpha^2$. Note that $\xi_\alpha$ in (2.44) is replaced by

\[
(2.46) \quad \tilde{\xi}_\alpha = 1 + \tilde{\xi}_\alpha,
\]

which is the one place this notation becomes awkward. Using (2.29) and (2.31), $\zeta$ is completely determined by $x_0(t)$ and $\eta$, so only these have to be evolved — the formula for $\tilde{\xi}_t$ in (2.40) is redundant as long as (2.41) is satisfied. If both components of $\tilde{\zeta}(\alpha_1, \alpha_2, t)$ are given, we say that $(\tilde{\zeta}, \tilde{\varphi})$ satisfy the torus version of (2.44) if there is a continuously differentiable function $x_0(t)$ such that $\tilde{\xi} = x_0(t) + H[\tilde{\eta}]$ and if $x_0, \tilde{\eta}$ and $\tilde{\varphi}$ satisfy the torus version of (2.44) with $C_1 = dx_0/dt + P_0\left[(1 + \tilde{\xi}_\alpha)H[\tilde{\psi}_\alpha/\tilde{f}] - \tilde{\eta}_\alpha \tilde{\psi}_\alpha/\tilde{f}\right]$.

We show in Appendix B that solving the torus version of (2.44) yields a three-parameter family of one-dimensional solutions of the form

\[
(2.47) \quad \begin{cases} 
\zeta(\alpha, t; \theta_1, \theta_2, \delta) = \alpha + \delta + \tilde{\zeta}(\theta_1 + \alpha, \theta_2 + k\alpha, t), & (\alpha \in \mathbb{R}, \ t \geq 0) \\
\varphi(\alpha, t; \theta_1, \theta_2) = \tilde{\varphi}(\theta_1 + \alpha, \theta_2 + k\alpha, t), & (\theta_1, \theta_2, \delta \in \mathbb{R})
\end{cases}
\]
We also show that if all the waves in this family are single-valued and have no vertical
tangent lines, there is a corresponding family of solutions of the Euler equations in
the original graph-based formulation of (2.1)–(2.4) that are quasi-periodic in physical space. A
precise statement is given in Theorem B.2.

2.6. Governing Equations for Traveling Waves in Conformal Space. Starting from (2.8)
and (2.36), we follow the same procedure as above to obtain
\[
\begin{align*}
\xi_a &= 1 + H[\eta_a], \quad \psi = -H[\varphi], \\
J &= \xi_a^2 + \eta_a^2, \quad c\eta_a - \psi_a = 0,
\end{align*}
\]

\[\frac{1}{2j}(\varphi_a^2 + \psi_a^2) - \frac{c}{j}(\varphi_a\xi_a + \psi_a\eta_a) + g\eta - \tau\kappa = C.\]

Here \(c\) is the wave speed and the integration constant \(C\) will be chosen to project out the
mean of the left-hand side. Note that \(\psi_a\) and \(\varphi_a\) can be expressed in terms of \(\eta_a\) and \(\xi_a\),
\[\psi_a = c\eta_a, \quad \varphi_a = H[\psi_a] = cH[\eta_a] = c(\xi_a - 1).\]

The last equation in (2.48) then gives
\[P\left[\frac{1}{2j}(c^2(\xi_a - 1)^2 + \xi_a^2) - \frac{c}{j}(c(\xi_a - 1)\xi_a + c\eta_a^2) + g\eta - \tau\kappa\right] = 0.\]

Simplifying this expression, (2.48) is reduced to the problem of finding \(\eta(a)\) such that
\[\begin{align*}
\xi_a &= 1 + H[\eta_a], \\
J &= \xi_a^2 + \eta_a^2,
\end{align*}\]
\[\kappa = \frac{\xi_a\eta_{aa} - \eta_a\xi_{aa}}{f^{3/2}}, \quad P\left[\frac{c^2}{2j} + g\eta - \tau\kappa\right] = 0.\]

The last of these equations can also be derived directly from the steady Bernoulli equation
(2.9). In more detail, in a frame moving with the traveling wave \(\tilde{\zeta} = z - ct\), the complex
velocity potential picks up a background flow term, \(\tilde{W}^{\mathrm{phys}}(\tilde{\zeta}, t) = W^{\mathrm{phys}}(\tilde{\zeta} + ct, t) - c\tilde{\zeta}\), and becomes time-independent. We drop \(t\) in the notation and define \(\tilde{W}(w) = W^{\mathrm{phys}}(\tilde{z}(w))\),
where \(\tilde{z}(w) = z(w, 0)\) conformally maps the lower half-plane onto the fluid region of
this stationary problem. We assume \(W^{\mathrm{phys}}(\tilde{z}(\alpha), 0)\) is quasi-periodic with exponentially
decaying mode amplitudes, so \(|\tilde{W}(w) + cw| \leq |W^{\mathrm{phys}}(\tilde{z}(w), 0)| + c|\tilde{z}(w) - w|\) is bounded in
the lower half-plane. Since the stream function \(\Im W^{\mathrm{phys}}(\tilde{z})\) is constant on the free surface,
we may assume \(\Im \tilde{W}(\alpha) = 0\) for \(\alpha \in \mathbb{R}\). The function \(\Im \tilde{W}(w + cw)\) is then bounded and
harmonic in the lower half-plane and satisfies homogeneous Dirichlet boundary conditions
on the real line, so it is zero [6]. Up to an additive real constant, we then have \(\tilde{W}(w) = -cw\).
Thus, \(|\tilde{\nabla}\tilde{\Phi}^{\mathrm{phys}}|^2 = |\tilde{W}'(w)/\tilde{z}'(w)|^2 = c^2/\mathcal{J}\). The steady Bernoulli equation remains valid
for overturning waves, so this derivation applies in that case as well. Also, although we
derived \(\psi_a = c\eta_a\) from \(\eta_x^{\mathrm{phys}}(c - \Phi_x^{\mathrm{phys}}) + \Phi_y^{\mathrm{phys}} = 0\), one can instead start from the kinematic
condition \(-\psi_a/\sqrt{\mathcal{J}} = \bar{z}_t \cdot \hat{n}\), which does not assume the wave profile is a graph. (Here we
have identified \(\bar{z}_t\) with the vector \((\xi_t, \eta_t)\) in \(\mathbb{R}^2\)). Since the wave travels at constant speed \(c\)
in physical space, there is a reparametrization \(\beta(a, t)\) such that \(\bar{z}(a, t) = \xi(\beta(a, t), 0) + ct\).
Since \(\xi_a\) is tangent to the curve, the normal velocity is simply \(\bar{z}_t \cdot \hat{n} = (c, 0) \cdot \hat{n} = -c\eta_a/\sqrt{\mathcal{J}}\),
where we used \(\hat{n} = (-\eta_a, \xi_a)/\sqrt{\mathcal{J}}\). The result \(\psi_a = c\eta_a\) follows.

In the quasi-periodic traveling wave problem, we seek a solution of (2.50) of the form
(2.28), except that \(\eta\) and its Fourier modes will not depend on time. Like the initial value
problem, (2.50) can be interpreted as a nonlinear system of equations for \(\bar{\eta}(\alpha_1, \alpha_2)\) defined
on $\mathbb{T}^2$, where the $\alpha$-derivatives are replaced by $[\partial_{a_1} + k\partial_{a_2}]$ and the Hilbert transform is replaced by the two-dimensional version described in Remark 2.2. Without loss of generality, we assume

$$
(2.51) \quad \hat{\eta}_{0,0} = 0.
$$

We also assume that $\hat{\eta}$ is an even, real function of $(\alpha_1, \alpha_2)$ on $\mathbb{T}^2$. Hence, in our setup, the Fourier modes of $\hat{\eta}$ satisfy

$$
(2.52) \quad \hat{\eta}_{-j_1,-j_2} = \hat{\eta}_{j_1,j_2}, \quad \hat{\eta}_{-j_1,-j_2} = \hat{\eta}_{j_1,j_2}, \quad (j_1, j_2) \in \mathbb{Z}^2.
$$

This implies that all the Fourier modes $\hat{\eta}_{j_1,j_2}$ are real, and causes $\eta(\alpha) = \hat{\eta}(\alpha, k\alpha)$ to be even as well, which is compatible with the symmetry of (2.50). However, as in Appendix B, there is a larger family of quasi-periodic traveling solutions embedded in this solution, namely

$$
(2.53) \quad \eta(\alpha; \theta) = \hat{\eta}(\alpha, \theta + k\alpha).
$$

As in (B.4) and (B.5) in Appendix B, two values of $\theta$ lead to equivalent solutions (up to $\alpha$-reparametrization) if they differ by $2\pi(n_1 k + n_2)$ for some integers $n_1$ and $n_2$. In general, $\eta(\alpha - a_0; \theta)$ will not be an even function of $\alpha$ for any choice of $a_0$ unless $\theta = 2\pi(n_1 k + n_2)$ for some integers $n_1$ and $n_2$. In the periodic case, symmetry breaking traveling water waves have been found by Zufiria [46], though most of the literature is devoted to periodic traveling waves with even symmetry.

Equations (2.50) were derived from the requirement that their solutions travel at a constant speed in physical space. In Appendix C, we consider their evolution in conformal space under (2.44) for various choices of $C_1$. The 1d waves maintain a permanent form as functions of $\alpha$ that travel at a generally non-uniform speed, and the torus version of the waves maintain a permanent two-dimensional form that travels through the torus in the $(1, k)$ direction at a speed that generally varies in time. A particular choice of $C_1$ causes $\hat{\eta}$ and $\hat{\phi}$ to remain stationary in time, though it is not the choice (2.45) in which $\hat{\xi}(0,0,t) = 0$.

2.7. Linear theory of quasi-periodic traveling waves. Linearizing (2.50) around the trivial solution $\eta(\alpha) = 0$, we obtain,

$$
(2.54) \quad c^2 H[\delta \eta_{\alpha}] - g \delta \eta + \tau \delta \eta_{\alpha\alpha} = 0.
$$

Here $\delta \eta$ denotes the variation of $\eta$. Substituting (2.28) into (2.54), we obtain a resonance relation for the Fourier modes of $\delta \eta$:

$$
(2.55) \quad c^2 |j_1 + j_2 k| - g - \tau (j_1 + j_2 k)^2 = 0, \quad (j_1, j_2) \in \mathbb{Z}^2.
$$

Note that $j_1 + j_2 k$, which appears in the exponent of (2.28), plays the role of $k$ in the dispersion relation (1.1). In the numerical scheme, we assume that both of the base modes $\hat{\eta}_{1,0}$, $\hat{\eta}_{0,1}$ are nonzero. (If either is zero, there is another family of periodic solutions bifurcating from the quasi-periodic family of interest here.) Setting $(j_1, j_2)$ to $(1, 0)$ and $(0, 1)$, respectively, gives the first-order resonance conditions

$$
(2.56) \quad c^2 - g - \tau = 0, \quad c^2 k - g - \tau k^2 = 0.
$$

These are dimensionless equations, where the wave number $k_1$ of the first wave has been set to 1, and $k_2 = k_1 k$. For right-moving waves, we then have $c = \sqrt{g + \tau}$ and $k = g/\tau$. Any superposition of waves with wave numbers 1 and $k$ traveling with speed $c$ will solve the linearized problem (2.54).
3. Numerical Methods for Spatially Quasi-Periodic Water Waves

In this section, we describe a time-stepping strategy for evolving water waves with spatially quasi-periodic initial conditions. We also formulate the quasi-periodic traveling wave problem as a nonlinear least-squares problem and use a variant of the Levenberg-Marquardt method to compute solutions. Our numerical results are presented and discussed at the end of the section.

3.1. A pseudo-spectral method. The evolution equations (2.44) for $\eta$ and $\phi$ are nonlinear and involve computing derivatives, antiderivatives and Hilbert transforms of quasi-periodic functions. Let $f$ denote one of these functions (e.g. $\eta$, $\phi$ or $\chi$) and let $\hat{f}$ denote the corresponding periodic function on the torus,

\[ f(\alpha) = \hat{f}(\alpha, k\alpha), \quad \hat{f}(\alpha_1, \alpha_2) = \sum_{j_1, j_2 \in \mathbb{Z}} \hat{f}_{j_1, j_2} e^{i(j_1 \alpha_1 + j_2 \alpha_2)}, \quad (\alpha_1, \alpha_2) \in \mathbb{T}^2. \]

The functions $f_\alpha$ and $H[f]$ then correspond to

\[ \hat{f}_\alpha(\alpha_1, \alpha_2) = \sum_{j_1, j_2 \in \mathbb{Z}} i(j_1 + j_2) \hat{f}_{j_1, j_2} e^{i(j_1 \alpha_1 + j_2 \alpha_2)}, \]

\[ \hat{H}[f](\alpha_1, \alpha_2) = \sum_{j_1, j_2 \in \mathbb{Z}} (-i)\text{sgn}(j_1 + j_2) \hat{f}_{j_1, j_2} e^{i(j_1 \alpha_1 + j_2 \alpha_2)}. \]

We propose a pseudo-spectral method in which each such $f$ that arises in the formulas (2.44) is represented by the values of $\hat{f}$ at $M_1 \times M_2$ equidistant gridpoints on the torus $\mathbb{T}^2$,

\[ \hat{f}_{m_1, m_2} = \hat{f}(2\pi m_1/M_1, 2\pi m_2/M_2), \quad (0 \leq m_1 < M_1, 0 \leq m_2 < M_2). \]

We visualize a 90° rotation between the matrix $\hat{F}$ holding the entries $\hat{f}_{m_1, m_2}$ and the collocation points in the torus. The columns of the matrix correspond to horizontal slices of gridpoints while the rows of the matrix correspond to vertical slices indexed from bottom to top. The nonlinear operations in (2.44) consist of products, powers and division; they are carried out pointwise on the grid. Derivatives and the Hilbert transform are computed in Fourier space via (3.2). To plot the solution, we also need to compute an antiderivative to get $\xi$ from $f = \xi_\alpha$. This involves dividing $\hat{f}_{j_1, j_2}$ by $i(j_1 + j_2)k$ when $(j_1, j_2) \neq (0, 0)$ and adjusting the $(0, 0)$ mode to obtain $\xi(0, t) = 0$.

Since the functions $f$ that arise in the computation are real-valued, we use the real-to-complex (‘r2c’) version of the two-dimensional discrete Fourier transform. The ‘r2c’ transform of a one-dimensional array of length $M$ (assumed even) is given by

\[ \hat{g}_j \leftarrow \left\{ \hat{g}_m \right\}_{m=0}^{M-1} \rightarrow \left\{ \hat{g}_j \right\}_{j=0}^{M/2}, \quad \hat{g}_j = \frac{1}{M} \sum_{m=0}^{M-1} g_m e^{-2\pi i jm/M}. \]

In practice, the $\hat{g}_j$ are computed simultaneously in $O(M \log M)$ time rather than by this formula. The fully complex (‘c2c’) transform of this (real) data would give additional values $\hat{g}_j$ with $M/2 + 1 \leq j \leq M - 1$. These extra entries are actually aliased values of negative-index modes; they are redundant due to $\hat{g}_j = \hat{g}_{j-M} = \hat{g}_{M-j}$. Since the imaginary components of $\hat{g}_0$ and $\hat{g}_{M/2}$ are zero, the number of real degrees of freedom on both sides of (3.4) is $M$. The Nyquist mode $\hat{g}_{M/2}$ requires special attention. Setting $\hat{g}_{M/2} = 1$ and the other modes to zero yields $g_m = \cos(\pi m) = (-1)^m$. The derivative and Hilbert transform
of this mode are taken to be zero since they would involve evaluating \( \sin(M\alpha/2) \) at the gridpoints \( \alpha_m = 2\pi m/M \).

The two-dimensional ‘r2c’ transform can be computed by applying one-dimensional ‘r2c’ transforms in the \( x \)-direction (i.e. to the columns of \( \tilde{F} \)) followed by one-dimensional ‘c2r’ transforms in the \( y \)-direction (i.e. to the rows of \( \tilde{F} \)):

\[
\tilde{f}_{j_1,j_2} = \frac{1}{M_2} \sum_{m_2=0}^{M_2-1} \left( \frac{1}{M_1} \sum_{m_1=0}^{M_1-1} f_{m_1,m_2} e^{-2\pi ij_1m_1/M_1} \right) e^{-2\pi ij_2m_2/M_2}, \quad \begin{cases} 0 \leq j_1 \leq M_1/2 \\ -M_2/2 < j_2 \leq M_2/2 \end{cases}.
\]

The ‘r2c’ routine in the FFTW library actually returns the index range \( 0 \leq j_2 < M_2 \), but we use \( \tilde{f}_{j_1,j_2} - \tilde{f}_{j_1,M_2} = \tilde{f}_{j_1,j_2} \) to de-alias the Fourier modes and map the indices \( j_2 > M_2/2 \) to their correct negative values. The missing entries with \( -M_1/2 < j_1 < 0 \) are determined implicitly by

\[
\tilde{f}_{-j_1,-j_2} = \tilde{f}_{j_1,j_2}.
\]

This imposes additional constrains on the computed Fourier modes, namely

\[
\text{Im}\{\tilde{f}_{0,0}\} = 0, \quad \text{Im}\{\tilde{f}_{M_1/2,0}\} = 0, \quad \text{Im}\{\tilde{f}_{0,M_2/2}\} = 0, \quad \text{Im}\{\tilde{f}_{M_1/2,M_2/2}\} = 0,
\]

\[
\tilde{f}_{0,-j_2} = \tilde{f}_{0,j_2}, \quad \tilde{f}_{M_1/2,-j_2} = \tilde{f}_{M_1/2,j_2}, \quad (1 \leq j_2 \leq M_2/2 - 1),
\]

where we also used \( \tilde{f}_{-M_1/2,j_2} = \tilde{f}_{M_1/2,j_2} \). This reduces the number of real degrees of freedom in the complex \( (M_1/2 + 1) \times M_2 \) array of Fourier modes to \( M_1M_2 \). When computing \( \tilde{f}_a \) and \( H[f] \) via (3.2), the Nyquist modes with \( j_1 = M_1/2 \) or \( j_2 = M_2/2 \) are set to zero. Otherwise the formulas (3.2) respect the constraints (3.7) and the ‘c2r’ transform reconstructs real-valued functions \( \tilde{f}_a \) and \( H[f] \) from their Fourier modes.

The evolution equations (2.44) are not stiff when the surface tension parameter is small or vanishes, but become moderately stiff for larger values of \( \tau \). We find that the 5th and 8th order explicit Runge-Kutta methods of Dormand and Prince [21] work well for smaller values of \( \tau \), and exponential time-differencing (ETD) methods [9, 12, 14, 27, 42] work well generally. In the ETD framework, we follow the basic idea of the small-scale decomposition for removing stiffness from interfacial flows [23, 24] and write the equations in the form

\[
\begin{pmatrix} \eta \frac{\partial \eta}{\partial t} \\ \varphi \frac{\partial \varphi}{\partial t} \end{pmatrix} = \begin{pmatrix} \eta \frac{\partial \eta}{\partial \varphi} + \mathcal{N} \\ -\eta \frac{\partial \eta}{\partial \varphi} \end{pmatrix} = \begin{pmatrix} L \eta + \mathcal{N} \\ \mathcal{N} \end{pmatrix} = \begin{pmatrix} 0 & H\partial_a \\ -\left(\frac{\partial}{\partial \varphi} - \varphi \right) \partial_a \end{pmatrix},
\]

where \( P \) is the projection in (2.12), \( H \) is the Hilbert transform in (2.16), and

\[
\mathcal{N} = \begin{pmatrix} \frac{\partial^2}{\partial \varphi^2} - \eta \frac{\partial}{\partial \varphi} H[\chi] - \left( \xi_a \chi - \psi_a \right) + C_1 \eta_a \\ (\varphi \frac{\partial^2}{\partial \varphi^2} - \varphi \partial_a H[\chi] + C_1 \varphi_a + \tau (\kappa - \eta a)) \end{pmatrix}.
\]

Note that \( \mathcal{N} \) is obtained by subtracting the terms included in \( L \) from (2.44), e.g. \( H\partial_a \varphi = -\psi_a \). The eigenvalues of \( L \) are \( \pm i \sqrt{|j_1 + j_2|} \left[ \sqrt{g + \tau (j_1 + j_2)^2} \right] \), so the leading source of stiffness is dispersive. This 3/2 power growth rate of the eigenvalues of the leading dispersive term with respect to wave number is typical of interfacial fluid flows with surface tension [23, 24]. For stiffer problems such as the Benjamin-Ono and Korteweg-de Vries equations, the growth rate is faster (quadratic and cubic, respectively) and it becomes essential to use a semi-implicit or exponential time-differencing scheme to avoid severe
timestepping restrictions. Here it is less critical, but still useful. Further details on how to implement (3.8) and (3.9) in the ETD framework will be given elsewhere [12].

In both the explicit Runge-Kutta and ETD methods, as explained above, the functions evolved in time are \( \hat{\eta}(x_1, x_2, t) \) and \( \hat{\phi}(x_1, x_2, t) \), sampled on the uniform \( M_1 \times M_2 \) grid covering \( \mathbb{T}^2 \). At the end of each timestep, our code can (optionally) apply a 36th order filter \([22,23]\) with Fourier multiplier \( \exp \left(-36 \left( (2j_1/M_1)^{36} + (2j_2/M_2)^{36} \right) \right) \). In all the computations reported below, we used the same number of gridpoints in the \( x \) and \( y \)-directions, \( M_1 = M_2 = M \). The filter has almost no effect on these results since the Fourier modes decay to machine precision before the filter deviates appreciably from 1.

3.2. A nonlinear least-squares problem for quasi-periodic traveling waves. In [43], an overdetermined shooting algorithm based on the Levenberg-Marquardt method [35] was proposed for computing standing water waves accurately and efficiently. Here we adapt this method to compute quasi-periodic traveling waves instead of standing waves. We first formulate the problem in a nonlinear least-squares framework. We consider \( \tau, c^2 \) (which we denote as \( b \)) and \( \eta \) as unknowns in (2.50) and define the residual function

\[
R[\tau, b, \eta] := P \left[ \frac{b}{2j} + g\eta - \tau \tilde{\kappa} \right].
\]

Here, \( \hat{\eta} \) represents the Fourier modes of \( \eta \), which are assumed real via (2.52); \( J \) and \( \kappa \) depend on \( \eta \) through the auxiliary equations of (2.50); and a tilde indicates that the function is represented on the torus, \( \mathbb{T}^2 \), as in (3.1). We also define the objective function

\[
F[\tau, b, \eta] := \frac{1}{8\pi} \int_{\mathbb{T}^2} R^2[\tau, b, \eta] \, d\alpha_1 \, d\alpha_2.
\]

Note that solving (2.50) is equivalent to finding a zero of the objective function \( F[\tau, b, \eta] \). The parameter \( k \) in (3.1) is taken to be a fixed, irrational number when searching for zeros of \( F \).

In the numerical computation, we truncate the problem to finite dimensions by varying only the leading Fourier modes \( \hat{\eta}_{j_1,j_2} \) with \( 0 \leq |j_1|, |j_2| \leq N/2 \). We evaluate the residual \( R \) (and compute the Fourier transforms) on an \( M \times M \) grid, where \( M > N \). The resulting nonlinear least squares problem is overdetermined because we zero-pad the Fourier modes \( \hat{\eta}_{j_1,j_2} \) when \( |j_1| \) or \( |j_2| \) is larger than \( N/2 \). Assuming the \( \hat{\eta}_{j_1,j_2} \) are real (i.e. that \( \eta \) is even) also reduces the number of unknowns relative to the number of equations, which are enumerated by the \( M^2 \) gridpoints without exploiting symmetry. According to the linear theory of Section 2.7, we fix the two base Fourier modes \( \hat{\eta}_{1,0} \) and \( \hat{\eta}_{0,1} \) at nonzero amplitudes; these amplitudes are chosen independently. It might seem more natural to prescribe \( \tau \) and \( \hat{\eta}_{1,0} \) and solve for \( \hat{\eta}_{0,1} \) along with \( b \) and the other unknown Fourier modes of \( \eta \). However, linearization about the flat state leads to \( \partial R/\partial \tau = 0 \) (since \( \eta \equiv 0 \Rightarrow \kappa \equiv 0 \)). This prevents the use of the implicit function theorem to solve the system in terms of \( \tau \) and \( \hat{\eta}_{1,0} \) and would also cause difficulties for the numerical solver. Note that by (2.56) above, to linear order we have \( \tau = g/k \) and \( c = \sqrt{g + g/k} \). Variations in \( \tau \) and \( c \) enter at higher order when the two amplitude parameters \( \hat{\eta}_{1,0} \) and \( \hat{\eta}_{0,1} \) are perturbed from 0.

The Levenberg-Marquardt solver requires a linear ordering of the unknowns. We enumerate the \( \hat{\eta}_{j_1,j_2} \) so that lower-frequency modes appear first. As the “shell index” \( s \) ranges from 1 to \( N/2 \), we enumerate all the index pairs \( (j_1, j_2) \) with \( \max(|j_1|, |j_2|) = s \) before increasing \( s \). Within shell \( s \), we proceed clockwise, along straight lines through the lattice,
from \((0, s)\) to \((s, s)\) to \((s, -s)\) to \((1, -s)\). The other Fourier modes are known from (2.51) and (2.52). Shell \(s\) contains \(4s\) index pairs, so the total number of independent modes \(\hat{\eta}_{1, j_2}\) with \(\max(|j_1|, |j_2|) \leq N/2\) is \(\sum_{s=1}^{N/2} 4s = N(N/2 + 1)\). We replace \(\hat{\eta}_{1,0}\) by \(\tau\) and \(\hat{\eta}_{0,1}\) by \(b\) in the list of unknowns to avoid additional shuffling of the variables when the prescribed base modes are removed from the list. Eventually there are \(N(N/2 + 1)\) parameters to compute:

\[
p_{1} = \tau, \quad p_{2} = \hat{\eta}_{1,1}, \quad p_{3} = b, \quad p_{4} = \hat{\eta}_{1,-1}, \quad p_{5} = \hat{\eta}_{0,2}, \quad \ldots, \quad p_{N(N/2+1)} = \hat{\eta}_{1,-N/2}.
\]

The objective function \(F\) is evaluated numerically by the trapezoidal rule approximation over \(\mathbb{T}^2\), which is spectrally accurate:

\[
f(p) = \frac{1}{2} r(p)^{T} r(p) \approx F[\tau, b, \hat{\eta}]
\]

\[
r_{m}(p) = R[\tau, b, \eta](\alpha_{m_1}, \alpha_{m_2})/M, \quad \left( m = 1 + m_1 + Mm_2 \right)
\]

\[
0 \leq m_1 < M.
\]

The parameters \(p_{j}\) are chosen to minimize \(f(p)\) using the Levenberg-Marquardt method [35,43]. The method requires a Jacobian matrix \((\partial r_{m}/\partial p_{j})_{m,j}\), which we compute by solving the following variational equations:

\[
\delta \xi_{\alpha} = H[\delta \eta_{\alpha}], \quad \delta \hat{\eta} = \delta \xi_{\alpha} \delta \xi_{\alpha} + \eta_{\alpha} \delta \eta_{\alpha},
\]

\[
\delta \kappa = \frac{3}{2} \frac{\delta J}{J} + \frac{1}{J^{3/2}} \left( \delta \xi_{\alpha} \eta_{a\alpha} + \xi_{\alpha} \delta \eta_{a\alpha} - \delta \eta_{a\alpha} \xi_{\alpha} - \eta_{\alpha} \delta \xi_{a\alpha} \right),
\]

\[
\delta R[\tau, b, \hat{\eta}] = P \left[ \frac{\delta b}{2J} - \frac{1}{2J^2} b \tilde{\delta} \kappa + g \tilde{\delta} \eta - \delta \tau \kappa - \tau \tilde{\delta} \kappa \right].
\]

We then have \(\partial r_{m}/\partial p_{j} = \delta R[\tau, b, \hat{\eta}](\alpha_{m_1}, \alpha_{m_2})\), where \(m = 1 + m_1 + Mm_2\) and the \(j\)th column of the Jacobian corresponds to setting the perturbation \(\delta \tau\), \(\delta b\) or \(\delta \hat{\eta}_{1, j_2}\) corresponding to \(p_{j}\) in (3.12) to 1 and the others to 0.

Like Newton’s method, the Levenberg-Marquardt method generates a sequence of approximate solutions \(p^{(0)}, p^{(1)}\), etc., which terminate when the residual drops below the desired tolerance or fails to decrease sufficiently. If no other solutions have been computed, we use the solution of the linearized problem as an initial guess:

\[
\eta^{(0)}(\alpha) = \hat{\eta}_{1,0}(e^{i\alpha} + e^{-i\alpha}) + \hat{\eta}_{0,1}(e^{i\alpha} + e^{-i\alpha}),
\]

\[
\tau^{(0)} = g/k, \quad b^{(0)} = g + \tau^{(0)}.
\]

After computing one or two small-amplitude solutions, we use numerical continuation to increase the amplitude beyond the applicability of linear theory. In the present work, we hold the ratio \(\gamma = \hat{\eta}_{1,0}/\hat{\eta}_{0,1}\) constant to explore one-dimensional slices (or paths) through the two-dimensional family of quasi-periodic traveling waves. We find that linear extrapolation from the previous two solutions on a path works well as the starting guess for the next Levenberg-Marquardt solve. Details of our Levenberg-Marquardt implementation, including stopping criteria and a strategy for delaying the re-computation of the Jacobian, are given in [43].

4. Numerical Results

In this section, we compute spatially quasi-periodic solutions of the gravity-capillary wave problems (2.44) and (2.50) with \(k = 1/\sqrt{2}\) and \(g\) normalized to 1. Section 4.1 is
devoted to the computation of quasi-periodic traveling waves while Section 4.2 is devoted
to solving the initial value problem, first to validate the traveling wave computation and
then to study more complex dynamics in which some of the waves overturn.

4.1. Quasi-Periodic Traveling Waves. We now present detailed numerical results of solutions of (2.50) on 3 continuation paths corresponding to \( \gamma \in \{5, 1, 0.2\} \), where \( \gamma = \hat{\eta}_{1,0}/\hat{\eta}_{0,1} \) is the amplitude ratio of the prescribed base modes. In each case, we vary the larger of \( \hat{\eta}_{1,0} \) and \( \hat{\eta}_{0,1} \) from 0.001 to 0.01 in increments of 0.001. The initial guess for the first two solutions on each path are obtained using the linear approximation (3.15), which by (3.12) corresponds to

\[
(4.1) \quad p_1^{(0)} = \tau^{(0)} = \sqrt{2}, \quad p_3^{(0)} = b^{(0)} = 1 + \sqrt{2}, \quad p_j^{(0)} = 0, \quad j \notin \{1, 3\}.
\]

As noted already, the amplitudes \( \hat{\eta}_{1,0} \) and \( \hat{\eta}_{0,1} \) are prescribed, so not included among the unknowns. The initial guess for the remaining 8 solutions on each continuation path are obtained from linear extrapolation from the previous two computed solutions. In all cases, we use \( M = 60 \) for the grid size and \( N = 48 \) for the Fourier cutoff in each dimension.

Figure 1 shows the initial conditions \( \eta \) and \( \varphi \) for the last solution on each continuation path (with \( \max\{\hat{\eta}_{1,0}, \hat{\eta}_{0,1}\} = 0.01 \)). Panels (a), (b) and (c) correspond to \( \gamma = 5, 1, \) and 0.2, respectively. The solution in all three cases is quasi-periodic, i.e. \( \eta \) and \( \varphi \) never exactly repeat themselves; we plot the solution from \( x = 0 \) to \( x = 36\pi \) as a representative snapshot. For these three solutions, the objective function \( f \) in (3.13) was minimized to \( 6.05 \times 10^{-28}, \)

\[
9.28 \times 10^{-28} \quad \text{and} \quad 4.25 \times 10^{-28},
\]

respectively, with similar or smaller values for lower-amplitude solutions on each path. The number of Jacobian evaluations in the Levenberg-Marquardt method for each of the 30 solutions computed on these paths never exceeded 5, and is typically 3 or 4. In our computations, \( \eta \) and \( \varphi \) are represented by \( \tilde{\eta}(\alpha_1, \alpha_2) \) and \( \tilde{\varphi}(\alpha_1, \alpha_2) \), which are defined on the torus \( \mathbb{T}^2 \). In Figure 2, we show contour plots of \( \tilde{\eta}(\alpha_1, \alpha_2) \) and \( \tilde{\varphi}(\alpha_1, \alpha_2) \) corresponding to the final solution on each path. Following the dashed lines through \( \mathbb{T}^2 \) in Figure 2 leads to the plots in Figure 1. The even symmetry of \( \tilde{\eta} \) and odd symmetry of \( \tilde{\varphi} \) in Figure 2 are compatible with the symmetry of (2.50).

The amplitude ratio, \( \gamma := \hat{\eta}_{1,0}/\hat{\eta}_{0,1} \), determines the bulk shape of the solution. If \( \gamma \gg 1 \), the wave with wave number 1 will be dominant; if \( \gamma \ll 1 \), the wave with wave number \( k = 2^{-1/2} \) will be dominant; and if \( \gamma \) is close to 1, both waves together will be dominant over higher-frequency Fourier modes (at least in the regime we study here). This is demonstrated with \( \gamma = 5, 1 \) and 0.2 in panels (a), (b) and (c) of Figure 1. Panels (a) and (c) show a clear dominant mode with visible variations in the amplitude. The oscillations are faster in panel (a) than in (c) since 1 > \( k \approx 0.707 \). By contrast, in panel (b), there is no single dominant wavelength.

This can also be understood from the plots in Figure 2. In case (a), \( \gamma \gg 1 \) and the contour lines of \( \tilde{\eta} \) and \( \tilde{\varphi} \) are perturbations of sinusoidal waves depending only on \( \alpha_1 \). The unperturbed waves would have vertical contour lines. The \( \alpha_2\)-dependence of the perturbation causes local extrema to form at the crest and trough. As a result, the contour lines join to form closed curves that are elongated vertically since the dominant variation is in the \( \alpha_1 \) direction. Case (c) is similar, but the contour lines are elongated horizontally since the dominant variation is in the \( \alpha_2 \) direction. Following the dashed lines in Figure 2, a cycle of \( \alpha_1 \) is completed before a cycle of \( \alpha_2 \) (since \( k < 1 \)). In case (a), a cycle of \( \alpha_1 \) traverses the dominant variation of \( \tilde{\eta} \) and \( \tilde{\varphi} \) on the torus, whereas in case (c), this is true of \( \alpha_2 \). So
the waves in Figure 1 appear to oscillate faster in case (a) than case (c). In the intermediate case (b) with $\gamma = 1$, the contour lines of the crests and troughs are nearly circular, but not perfectly round. The amplitude of the waves in Figure 1 are largest when the dashed lines in Figure 2 pass near the extrema of $\tilde{\eta}$ and $\tilde{\phi}$, and smallest when the dashed lines pass near the zero level sets of $\tilde{\eta}$ and $\tilde{\phi}$. If the slope of the dashed lines were closer to 1 and the functions $\tilde{\eta}$ and $\tilde{\phi}$ were to remain qualitatively similar to the results of panel (b) of Figure 2, the waves would have a beating pattern with many cycles with larger amplitude followed by many cycles with smaller amplitude. The former would occur when the dashed lines pass near the diagonal from $(0, 0)$ to $(2\pi, 2\pi)$, which passes over the peaks and troughs of $\tilde{\eta}$ and $\tilde{\phi}$, while the latter would occur when the dashed lines pass near the lines connecting $(\pi, 0)$ to $(2\pi, \pi)$ and $(0, \pi)$ to $(\pi, 2\pi)$, where $\tilde{\eta}$ and $\tilde{\phi}$ are close to zero. The dashed lines would linger in each regime over many cycles if $k$ were close to 1.

Next we examine the behavior of the Fourier modes that make up these solutions. Figure 3 shows two-dimensional plots of the Fourier modes $\tilde{\eta}_{j_1,j_2}$ for the 3 cases above, with $\gamma \in \{5, 1, 0.2\}$ and $\max(\tilde{\eta}_{1,0}, \tilde{\eta}_{0,1}) = 0.01$. Only the prescribed modes and the modes
that were optimized by the solver (see \((3.12)\)) are plotted, which have indices in the range \(0 \leq j_1 \leq N/2\) and \(-N/2 \leq j_2 \leq N/2\), excluding \(j_2 \leq 0\) when \(j_1 = 0\). The other modes are determined by the symmetry of \((2.52)\) and by zero-padding \(\hat{\eta}_{j_1,0} = 0\) if \(N/2 < j_1 \leq M\) or \(N/2 < |j_2| \leq M\). We used \(N = 48\) and \(M = 60\) in all 3 calculations. One can see that the fixed Fourier modes \(\hat{\eta}_{1,0}\) and \(\hat{\eta}_{0,1}\) are the two highest-amplitude modes in all three cases. In this sense, our solutions of the nonlinear problem \((2.50)\) are small-amplitude perturbations of \((3.15)\). However, there are many active Fourier modes, so these solutions are well outside of the linear regime. Carrying out a weakly nonlinear Stokes expansion to high enough order to accurately predict all these modes would be difficult, especially considering the degeneracies that arise already in the periodic Wilton ripple problem \([40,41]\).

In panels (a), (b) and (c) of Figure 3, the modes appear to decay more slowly in one direction than in other directions. This is seen more clearly when viewed from above, as shown in panel (d) for the case of \(\gamma = 1\). (The other two cases are similar). The
Two-dimensional Fourier modes of $\tilde{\eta}$ for the $k = 1/\sqrt{2}$ solutions plotted in Figures 1 and 2. (a) $\gamma = 5$. (b,d) $\gamma = 1$. (c) $\gamma = 0.2$. In all three cases, the modes decay visibly slower along the line $j_1 + j_2k = 0$, indicating the presence of resonant mode interactions.

Figure 3. Two-dimensional Fourier modes of $\tilde{\eta}$ for the $k = 1/\sqrt{2}$ solutions plotted in Figures 1 and 2. (a) $\gamma = 5$. (b,d) $\gamma = 1$. (c) $\gamma = 0.2$. In all three cases, the modes decay visibly slower along the line $j_1 + j_2k = 0$, indicating the presence of resonant mode interactions.

direction along which the modes decay less rapidly appears to coincide with the line $\{ (j_1, j_2) : j_1 + j_2k = 0 \}$, which is plotted in red. A partial explanation is that when $j_1 + j_2k$ is close to zero, the corresponding modes $e^{i(j_1+j_2k)\alpha}$ in the expansion of $\eta(\alpha)$ in (2.28) have very long wavelength. Slowly varying perturbations lead to small changes in the residual of the water wave equations, so these modes are not strongly controlled by the governing equations (2.50). We believe this would lead to a small divisor problem [36] that would complicate a rigorous proof of existence of quasi-periodic traveling water waves.

Next we show that $\tau$ and $c$ depend nonlinearly on the amplitude of the Fourier modes $\tilde{\eta}_{1,0}$ and $\tilde{\eta}_{0,1}$. Panels (a) and (b) of Figure 4 show plots of $\tau$ and $c$ versus $\tilde{\eta}_{\text{max}} := \max(\tilde{\eta}_{1,0}, \tilde{\eta}_{0,1})$ for 9 values of $\gamma = \tilde{\eta}_{1,0}/\tilde{\eta}_{0,1}$, namely $\gamma = 0.1, 0.2, 0.5, 0.8, 1, 1.25, 2, 5, 10$. On each curve, $\tilde{\eta}_{\text{max}}$ varies from 0 to 0.01 in increments of 0.001. At small amplitude, linear theory predicts $\tau = g/k = 1.41421$ and $c = \sqrt{g(1 + 1/k)} = 1.55377$. This is represented by the black marker at $\tilde{\eta}_{\text{max}} = 0$ in each plot. For each value of $\gamma$, the curves $\tau$ and $c$ are seen to have zero slope at $\tilde{\eta}_{\text{max}} = 0$, and can be concave up or concave down depending on $\gamma$. This can be understood from the contour plots of panels (c) and (d). Both $\tau$ and $c$ appear to be even
functions of \( \hat{\eta}_{1,0} \) and \( \hat{\eta}_{0,1} \) when the other is held constant. Both plots have a saddle point at the origin, are concave down in the \( \hat{\eta}_{1,0} \) direction holding \( \hat{\eta}_{0,1} \) fixed, and are concave up in the \( \hat{\eta}_{0,1} \) direction holding \( \hat{\eta}_{1,0} \) fixed. The solid lines in the first quadrant of these plots are the slices corresponding to the values of \( \gamma \) plotted in panels (a) and (b). The concavity of the 1d plots depends on how these lines intersect the saddle in the 2d plots.

The contour plots of panels (c) and (d) of Figure 4 were made by solving (2.50) with \((\hat{\eta}_{1,0}, \hat{\eta}_{0,1})\) ranging over a uniform 26 \times 26 grid on the square \([-0.01, 0.01] \times [-0.01, 0.01]\). Using an even number of gridpoints avoids the degenerate case where \( \hat{\eta}_{1,0} \) or \( \hat{\eta}_{0,1} \) is zero. At those values, the two-dimensional family of quasi-periodic solutions meets a sheet of periodic solutions where \( \tau \) or \( c \) becomes a free parameter. Alternative techniques would be needed in these degenerate cases to determine the value of \( \tau \) or \( c \) from which a periodic traveling wave in the nonlinear regime bifurcates to a quasi-periodic wave. In panel (e), we plot the magnitude of the Chebyshev coefficients in the expansion

\[
(4.2) \quad c(\hat{\eta}_{1,0}, \hat{\eta}_{0,1}) = \sum_{m=0}^{15} \sum_{n=0}^{15} \hat{c}_{mn} T_m(100\hat{\eta}_{1,0}) T_n(100\hat{\eta}_{0,1}), \quad -0.01 \leq \hat{\eta}_{1,0}, \hat{\eta}_{0,1} \leq 0.01.
\]

This was done by evaluating \( c \) on a cartesian product of two 16-point Chebyshev-Lobatto grids over \([-0.01, 0.01]\) and using the one-dimensional Fast Fourier Transform in each direction to compute the Chebyshev modes. We see that the modes decay to machine precision by the time \( m + n \geq 10 \) or so, and only even modes \( m \) and \( n \) are active. The
plot for $|\tau_{mn}|$ is very similar, so we omit it. These plots confirm the visual observation from the contour plots that $\tau$ and $c$ are even functions of $\hat{\eta}_{1,0}$ and $\hat{\eta}_{0,1}$ when the other is held constant. In summary, over the range $-0.01 \leq \hat{\eta}_{1,0}, \hat{\eta}_{0,1} \leq 0.01$ considered here, $\tau$ and $c$ show interesting nonlinear effects that would be difficult to model using weakly nonlinear theory since polynomials of degree 10 are needed to represent $\tau$ and $c$ accurately to machine precision.

4.2. Numerical Solution of the Spatially Quasi-Periodic Initial Value Problem. In Figure 5, we plot the time evolution of $\zeta(\alpha, t)$ in the lab frame from $t = 0$ to $t = 3$ using the timestepping algorithm described in Section 3.1. The initial conditions, plotted in thick blue lines, are those of the traveling waves computed in Figures 1–3 above. The grey curves give snapshots of the solution at uniformly sampled times with $\Delta t = 0.1$. The solutions are plotted over the representative interval $0 \leq x \leq 12\pi$, though they extend in both directions to $\pm \infty$ without exactly repeating. Note that the solutions proceed without changing shape, which is verified quantitatively below. This confirms that the quasi-periodic solutions we obtained by minimizing the objective function (3.13) are indeed traveling waves under the evolution equations (2.44).

The left panel of Figure 6 shows the error in timestepping the solution of Figure 5b from $t = 0$ to $t = 3$ using the 5th and 8th order explicit Runge-Kutta methods of Dormand and Prince [21], the 4th order ETD scheme of Cox and Matthews [14,27], and the 5th order ETD scheme of Whalen, Brio and Moloney [42]. Computing errors requires an “exact” solution, and we use the numerically computed traveling wave together with its time evolution on the torus (from Corollary C.3 of Appendix C) for this purpose. In more detail, minimizing the objective function (3.13) gives the torus version of the traveling wave profile $\tilde{\eta}_0(\alpha_1, \alpha_2)$, the surface tension $\tau$, and the wave speed $c$ such that $(\hat{\eta}_{0,1})_{1,0}$ and $(\hat{\eta}_{0,0,1})_{1,0}$ have prescribed values at $t = 0$. We then compute $\tilde{\xi}_0 = H[\hat{\eta}_0]$ and $\tilde{\varphi}_0 = c \hat{\xi}_0$. From Corollary C.3, we know the time evolution of the exact solution from this initial condition is given by

$$
\zeta_{\text{exact}}(\alpha_1, \alpha_2, t) = \zeta_0(\alpha_1 - ct + A(t), \alpha_2 - kct + kA(t)) + A(t),
$$
$$
\varphi_{\text{exact}}(\alpha_1, \alpha_2, t) = \varphi_0(\alpha_1 - ct + A(t), \alpha_2 - kct + kA(t)),
$$

where $A(t) = A(-ct, -kct)$ and $A(x_1, x_2)$ is defined by (C.9). The waves do not change shape as they move through the torus, but the traveling speed in conformal space varies in time. Only $\hat{\eta}(\alpha_1, \alpha_2, t)$ and $\hat{\varphi}(\alpha_1, \alpha_2, t)$ are evolved numerically via (2.44). We use $C_1$ in (2.45) to maintain $\tilde{\xi}(0, 0, t) = 0$, so it is not necessary to explicitly evolve $x_0(t)$ in time. The error plotted in the left panel of Figure 6 is computed at time $T = 3$ using the discrete norm

$$
\text{err} = \sqrt{\|\hat{\eta} - \hat{\eta}_{\text{exact}}\|^2 + \|\hat{\varphi} - \hat{\varphi}_{\text{exact}}\|^2}, \quad \|\hat{\eta}\|^2 = \sum_{m_1, m_2} \frac{1}{M_1 M_2} \hat{\eta} \left( \frac{2\pi m_1}{M_1}, \frac{2\pi m_2}{M_2}, T \right)^2.
$$

A contour plot of $\hat{\eta}(\alpha_1, \alpha_2, T)$ at the final time $T = 3$ is shown in the right panel of Figure 6. The dashed line shows the trajectory from $t = 0$ to $t = 3$ of the wave crest that begins at $(0, 0)$ and continues along the path $\alpha_1 = ct - A(t), \alpha_2 = k[ct - A(t)]$. We use Newton’s method to solve the implicit equation (C.9) for $A(x_1, x_2)$ at each point of the pseudo-spectral grid. We then use FFTW to compute the 2d Fourier representation of $A(x_1, x_2)$, which can then be used to quickly evaluate the function at any point. It would also have been easy to compute $A(t) = A(-ct, -kct)$ directly by Newton’s method, but the Fourier approach is also very fast and gives more information about the function $A(x_1, x_2)$. In particular, the
Figure 5. Time evolution of the traveling wave profiles, $\zeta(\alpha, t)$, from $t = 0$ to $t = 3$ in the lab frame. The thick blue lines correspond to the initial conditions.

Figure 6. Error and contour plots of the quasi-periodic traveling wave of Figure 5b. (left) Accuracy of various timesteppers at the final time $T = 3$. (right) Contour plot of the exact solution on the torus at this time. The dashed line shows the trajectory of the wave crest from $t = 0$ to $t = T$. 
modes decay to machine roundoff on the grid, corroborating the assertion in Theorem B.2 that $\mathcal{A}$ is real analytic.

Next we present a spatially quasi-periodic water wave computation in which some of the waves overturn as they evolve while others flatten out. For simplicity, we set the surface tension parameter, $\tau$, to zero. We first seek spatially periodic dynamics in which the initial wave profile has a vertical tangent line that overturns when evolved forward in time and flattens out when evolved backward in time. Through trial and error, we selected the following parametric curves for the initial wave profile and velocity potential of this auxiliary periodic problem:

\begin{align}
\xi_1(\sigma) &= \sigma + \frac{3}{5} \sin \sigma - \frac{1}{5} \sin 2\sigma, \\
\eta_1(\sigma) &= -(1/2) \cos(\sigma + \pi/2.5), \\
\phi_1(\sigma) &= -(1/2) \cos(\sigma + \pi/4). \tag{4.4}
\end{align}

Note that $\xi'_1(\sigma) = 0$ when $\sigma \in \pi + 2\pi\mathbb{Z}$, and otherwise $\xi'_1(\sigma) > 0$. Thus, vertical tangent lines occur where $\xi_1(\sigma) \in \pi + 2\pi\mathbb{Z}$ and $\eta_1(\sigma) = 0.154508$. To convert (4.4) to a conformal parametrization, we search for $2\pi$-periodic functions $\eta_3(\alpha)$ and $B_3(\alpha)$ and a number $x_3$ such that

\begin{align}
\alpha + x_3 + H[\eta_3](\alpha) &= \xi_1(\alpha + B_3(\alpha)), \\
\eta_3(\alpha) &= \eta_1(\alpha + B_3(\alpha)), \\
B_3(0) &= 0. \tag{4.5}
\end{align}

First we solve $\alpha + H[\eta_3](\alpha) = \xi_1(\alpha + B_3(\alpha))$, $\eta_3(\alpha) = \eta_1(\alpha + B_3(\alpha))$ for $\eta_3(\alpha)$ and $B_3(\alpha)$ on a uniform grid with $M$ gridpoints on $[0, 2\pi]$ using Newton’s method. The Hilbert transform is computed with spectral accuracy in Fourier space. We then define $x_3$ as the solution of $x_3 + B_2(x_3) = 0$ that is smallest in magnitude. We solve this equation by a combination of root bracketing and Newton’s method; the result is $x_3 = 0.393458$. Finally, we define $B_3(\alpha) = x_3 + B_2(\alpha + x_3)$ and $\eta_3(\alpha) = \eta_2(\alpha + x_3)$, which satisfy (4.5). The initial conditions $\eta_3(\alpha)$ and $\phi_3(\alpha) = \phi_1(\alpha + B_3(\alpha))$ have the desired property that the wave overturns when evolved forward in time and flattens out when evolved backward in time. We turn this into a quasi-periodic solution by defining initial condition on the torus of the form

\begin{align}
\tilde{\eta}_0(\alpha_1, \alpha_2) &= \eta_3(\alpha_1), \\
\tilde{\phi}_0(\alpha_1, \alpha_2) &= \phi_3(\alpha_1) \cos(\alpha_2 - q), \tag{4.6}
\end{align}

where $q$ is a free parameter that we choose heuristically to be $q = 0.6k\pi = 1.3329$ in order to make the first wave crest to the right of the origin behave similarly to the periodic 1d solution. (This will be explained below).

The results are summarized in Figures 7 and 8. Panel (a) of Figure 7 shows snapshots of the solution at $t = (\ell/6)T$ for $0 \leq \ell \leq 6$ over the range $0 \leq \xi(\alpha) \leq 16\pi$, where $T = 0.225$. The initial wave profile, $\tilde{z}_0(\alpha) = \xi_0(\alpha) + i\eta_0(\alpha)$ with $\xi_0(\alpha) = \tilde{\eta}_0(\alpha, k\alpha)$, is plotted with a thick blue line; it is $2\pi$-periodic and has vertical tangent lines where $\xi_0(\alpha) \in \pi + 2\pi\mathbb{Z}$ and $\eta_0(\alpha) = 0.154508$. The final time at $t = T$ is plotted with a thick black line, and the intermediate times are plotted with thin black lines. Panel (b) zooms in on the first wave in panel (a), which overturns as the wave crest moves up and right while the wave trough moves down and left, as indicated by the blue arrows. This is very similar (by design) to the evolution of the auxiliary periodic wave with initial conditions $\eta_3(\alpha)$, $\phi_3(\alpha)$. Panels (c) and (d) zoom in on two other waves from panel (a) that flatten out rather than overturn over the time window $0 \leq t \leq T$. Panel (e) shows a wave that overturns due to the wave trough moving down and left faster than the wave crest moves down and left. Panel (f) shows the evolution of the velocity potential $\phi(\alpha, t)$ over $0 \leq t \leq T$. Unlike $\eta_0(\alpha)$, the initial velocity potential $\phi_0(\alpha) = \tilde{\phi}_0(\alpha, k\alpha)$ is not $2\pi$-periodic due to the factor of $\cos(\alpha_2 - q)$ in (4.6).
SPATIALLY QUASI-PERIODIC WATER WAVES

Figure 7. Snapshots in time of a spatially quasi-periodic water wave with a periodic initial wave profile with vertical tangent lines at $\xi = \pi + 2\pi n$, $n \in \mathbb{Z}$. A quasi-periodic initial velocity potential causes some of the waves to overturn for $t > 0$ and others to flatten out. Panels (a) and (f) show $\eta(\alpha, t)$ and $\phi(\alpha, t)$ versus $\xi(\alpha, t)$ over $0 \leq x \leq 16\pi$ and $0 \leq t \leq T = 0.225$. Panels (b)–(e) show the results of panel (a) in more detail. The blue arrows show the direction of travel of the waves.

Panels (a) and (d) of Figure 8 show surface plots of $\tilde{\eta}(\alpha_1, \alpha_2, T)$ and $\tilde{\phi}(\alpha_1, \alpha_2, T)$ at the final time computed, $T = 0.225$. The corresponding contour plots are shown in panels (b) and (c). Initially, $\tilde{\eta}(\alpha_1, \alpha_2, 0)$ depends only on $\alpha_1$; however, by $t = T$, the dependence on $\alpha_2$ is clearly visible. Although the waves overturn in some places when $\eta(\alpha, t) = \tilde{\eta}(\alpha, k\alpha, t)$ is plotted parametrically versus $\xi(\alpha, t)$ with $t > 0$ held fixed, both $\tilde{\eta}$ and $\tilde{\phi}$ are single-valued functions of $\alpha_1$ and $\alpha_2$ at all times. Nevertheless, throughout the evolution, $\tilde{\eta}(\alpha_1, \alpha_2, t)$ has a steep dropoff over a narrow range of values of $\alpha_1$. Initially, $\tilde{\eta}_0(\alpha_1, \alpha_2) = \eta_3(\alpha_1) = \eta_1(\alpha_1 + B_3(\alpha_1))$ and the rapid dropoff occurs for $\alpha_1$ near the solution of $\alpha_1 + B_3(\alpha_1) = \pi$ (since the vertical tangent line occurs at $\xi(\sigma) + i\eta(\sigma)$ with $\sigma = \pi$). Using Newton’s method, we find that this occurs at $\alpha_1 = 0.634096\pi$. A plot shows that $\eta_1(\alpha)$ decreases rapidly by more than half its crest to trough height over the narrow range $0.6\pi \leq \alpha \leq 0.667\pi$. At later times,
\( \tilde{\eta}(\alpha_1, \alpha_2, t) \) continues to drop off rapidly when \( \alpha_1 \) traverses this narrow range in spite of the dependence on \( \alpha_2 \). This can be seen in panel (b) of Figure 8, where there is a high clustering of nearly vertical contour lines separating the yellow-orange region from the blue region. \( \tilde{\phi}(\alpha_1, \alpha_2, t) \) also varies rapidly with respect to \( \alpha_1 \) over this narrow window.

Many gridpoints are needed to resolve these rapid variations with spectral accuracy. Although \( \xi(\sigma) \), \( \eta(\sigma) \) and \( \varphi(\sigma) \) involve only a few nonzero Fourier modes, conformal reparametrization via (4.5) vastly increases the Fourier content of the initial condition. We had to use 4096 gridpoints in the computation of \( \tilde{\eta}_3(\alpha) \) and \( \tilde{\phi}_3(\alpha) \) in order for their Fourier modes to decay to double-precision accuracy. We evolved (2.44) on a 4096 \( \times \) 4096 spatial grid using the 8th order explicit Runge-Kutta method described in Section 3.1. The calculation involved 5400 timesteps from \( t = 0 \) to \( t = T = 0.225 \), which took 2.5 days on 12 threads running on a server with two 3.0 GHz Intel Xeon Gold 6136 processors. Additional threads had little effect on the running time as the FFT calculations require a lot of data movement relative to the number of floating point operations involved. Panel (e) of Figure 8 shows the \( \ell^2 \) average of the Fourier modes in shell \( s \) of the 2d Fourier modes \( \tilde{\eta}_{j_1,j_2} \) for \( 0 \leq s \leq 2048 \). Recall from Section 3.2 that shell \( s \) contains the modes \((j_1, j_2)\) along straight lines through the lattice from \((0, s)\) to \((s, s)\) to \((s, -s)\) to \((1, -s)\), and for \( s \geq 1 \) there are \( 4s \) such modes. We see that the modes continue to decay at an exponential rate with respect to \( s \), but the rate slows as time increases. The rapid dropoff in the mode amplitudes for \( s \geq 1536 \) is due to the Fourier filter. At the final time \( t = T = 0.225 \), the modes still decay by 12 orders of magnitude...
magnitude from \( s = 1 \) to \( s = 1536 \), so we believe the solution is correct to 10-12 digits. A finer grid would be required to maintain this accuracy over longer times.

The rationale for setting \( q = 0.6k\pi \) in (4.6) is that \( \cos(\alpha_2 - q) \approx 1 \) where the characteristic line \( (\alpha, k\alpha) \) crosses the dropoff in the torus near \( \alpha_1 = 0.6\pi \) for the first time. Locally, \( \varphi_0(\alpha) = \tilde{\varphi}_0(\alpha, k\alpha) \) is close to \( \varphi_3(\alpha) \), the initial condition of the auxiliary periodic problem, so we expect the quasi-periodic wave to evolve similarly to the periodic wave near \( x = \pi \) for a short time. (Here \( z = x + iy \) describes physical space). This is indeed what happens in panel (b) of Figure 7. Advancing \( \alpha \) from 0.6\( \pi \) to 10.6\( \pi \) causes the characteristic line \( (\alpha, k\alpha) \) to cross a periodic image of the dropoff at \( \alpha_2 = 10.6k\pi \), where \( \cos(\alpha_2 - q) = -0.9752 \approx -1 \). Locally, \( \varphi_0(\alpha) \) is close to \(-\varphi_3(\alpha)\), the initial condition of the time-reversed auxiliary periodic problem. We thus expect the quasi-periodic wave to evolve similarly to the time-reversed periodic wave near \( x = 11\pi \), and this is indeed what happens in panel (d) of Figure 7. (Recall that \( \xi_0(0.634096\pi) = \pi \), so \( \xi_0(10.634096\pi) = 11\pi \)). At most wave peaks, the velocity potential of the quasi-periodic solution is not closely related to that of the periodic auxiliary problem since the cosine factor is not near a relative maximum or minimum, where it is flat. As a result, the wave peaks of the quasi-periodic solution evolve in many different ways as \( \alpha \) varies over the real line.

5. Conclusion

In this work, we have formulated the two-dimensional, infinite depth gravity-capillary water wave problem in a spatially quasi-periodic, conformal mapping framework. We have studied solutions of the initial value problem in which some of the wave crests overturn as time evolves while others flatten out, and have numerically demonstrated the existence of traveling solutions, which are a quasi-periodic generalizations of Wilton's ripples. For the initial value problem, we developed two timestepping strategies, an explicit Runge-Kutta approach and an exponential time differencing method. For traveling waves, we adapted an overdetermined nonlinear least squares technique introduced in [43] for a different problem. The value of \( k \) and the amplitudes of two base Fourier modes \( \tilde{\eta}_{1,0} \) and \( \tilde{\eta}_{0,1} \) are fixed while \( \tau, c \) and the other Fourier modes of \( \eta \) are varied to search for solutions of (2.50). The solutions we obtain are perturbations of solutions (3.15) of the linear problem. The results we presented correspond to solutions with \( \alpha = 2.50 \). The solutions we obtain are perturbations of solutions (3.15) of the linear problem.

The question of what happens in our framework if \( k \) is rational is interesting. We believe the initial value problem could still be solved, though in that case solving the torus version of the equations is equivalent to simultaneously computing a family of 1d solutions on a periodic domain. Families of 1d waves corresponding to a single 2d periodic wave on the torus are discussed in Appendix B below. If \( k = q/p \) with \( p \) and \( q \) relatively prime integers, the waves in this family all have period \( 2\pi p \). The traveling wave problem becomes degenerate if \( k \) is rational — solutions of the torus version of (2.50) may still exist (we do not know), but if so, they are not unique. Indeed, if \( k = q/p \) as above and \( \tilde{\eta}_1 \) solves the torus version of (2.50), then for any \( 2\pi \)-periodic, real analytic function \( a_0(r) \),

\[
\tilde{\eta}_2 \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix} = \tilde{\eta}_1 \left( \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix} - \begin{pmatrix} p \\ q \end{pmatrix} a_0(-q\alpha_1 + p\alpha_2) \right)
\]
Another degeneracy is that the modes in the strip torus along characteristic lines are related by a simple reparametrization,

$$
(5.2) \quad \eta_2(\alpha; \theta) = \tilde{\eta}_2\left(\frac{\alpha}{\theta + k\alpha}\right) = \tilde{\eta}_1\left(\frac{\alpha - p\alpha_0(p\theta)}{\theta + k\alpha - q\alpha_0(p\theta)}\right) = \eta_1(\alpha - p\alpha_0(p\theta); \theta).
$$

Another degeneracy is that the modes $\tilde{\eta}_{j_1,j_2}$ of a solution of (2.50) with $j_1 + k j_2 = 0$ and $(j_1, j_2) \neq (0, 0)$ can be modified arbitrarily (maintaining $\tilde{\eta}_{-j_1,-j_2} = \tilde{\eta}_{j_1,j_2}$) to obtain additional solutions of (2.50). These modes are plane waves that only affect the 1d functions passing through the torus along characteristic lines by an additive constant. The resonance phenomenon observed in the Fourier modes in Figure 3 is presumably a small-divisor phenomenon [36] in the irrational case related to this degeneracy. If solutions for rational $k$ exist, a natural open question is whether they can be selected to fit together continuously with solutions for nearby irrational wave numbers. In floating point arithmetic, irrational wave numbers are approximated by rational ones. We did not encounter difficulties with this, presumably because the above degeneracies are not visible with the grid resolution used. More work is needed to understand this rigorously.

Our results show that the amplitude ratio $\gamma$ plays an important role in determining the shapes of solutions; the quasi-periodic features of solutions are most evident when $\gamma \approx 1$. In the future, we plan to study the behavior of different perturbation families, e.g. fixing the amplitudes of different base Fourier modes in (2.55) such as $\tilde{\eta}_{1,0}$ and $\tilde{\eta}_{1,1}$. We also aim to use this methodology to compute spatially quasi-periodic traveling gravity-capillary waves of finite depth, to compute the time evolution of solutions of the finite depth quasi-periodic initial value problem, and to study the stability of spatially quasi-periodic water waves along the lines of what has been done for periodic traveling waves [16, 31] and Wilton ripples [40].

Appendix A. Conformal Mappings of the Lower Half-Plane

In this section we discuss sufficient conditions for an analytic function $z(w)$ to map the lower half-plane topologically onto a semi-infinite region bounded above by a parametrized curve. We will prove the following theorem, and a corollary concerning boundedness of $1/|z(w)|$ when the functions are quasi-periodic:

**Theorem A.1.** Suppose $\varepsilon > 0$ and $z(w)$ is analytic on the half-plane $C_{\varepsilon}^- = \{w : \text{Im } w < \varepsilon\}$. Suppose there is a constant $M > 0$ such that $|z(w) - w| \leq M$ for $w \in C_{\varepsilon}^-$, and that the restriction $\zeta = z|_{\mathbb{R}}$ is injective. Then the curve $\zeta(\alpha)$ separates the complex plane into two regions, and $z(w)$ is an analytic isomorphism of the lower half-plane onto the region below the curve $\zeta(\alpha)$.

**Proof.** We do not assume $\zeta(\alpha) = \xi(\alpha) + i\eta(\alpha)$ is a graph — only that it does not self-intersect. We first need to show that $\Gamma = \{\xi(\alpha) : \alpha \in \mathbb{R}\}$ separates the complex plane into precisely two regions. (In the graph case, this is obvious.) Let $A = 2M$ and consider the linear fractional transformation

$$
(A.1) \quad T(z) = \frac{z + Ai}{z - Ai}, \quad T^{-1}(\lambda) = Ai\frac{\lambda - 1}{\lambda + 1}.
$$

Note that $T$ maps the real line to the unit circle and $T^{-1}(e^{i\theta}) = A\tan(\theta/2)$. Let $g(\theta) = T \circ \zeta \circ T^{-1}(e^{i\theta})$. Since $\zeta(\alpha)$ lies inside a closed ball of radius $M$ centered at $\alpha$, $\zeta(\alpha)$ remains in the strip $\text{Im } z \in [-M, M]$ and approaches complex $\infty$ as $\alpha \to \pm \infty$. Since $T(\infty) = -1$, $\zeta$ is
g becomes continuous on \([-\pi, \pi]\) if we define \(g(\pm \pi) = -1\). Since \(T\) is bijective and \(\zeta\) is injective, \(g\) is a Jordan curve and separates the complex plane into two regions. The curve \(g(\theta)\) takes values in the set \(\{\lambda : |\lambda + 1/3| \geq 2/3, |\lambda - 1| \leq 2\}\), which is the image of the strip \(\text{Im} z \in [-M, M]\) under \(T\). An argument similar to Lemma 2 of section 4.2.1 of [1] shows that if \(|\lambda + 1/3| < 2/3\), then \(\lambda\) is inside the Jordan curve. In particular, if \(\text{Im} z < -M\), then \(T(z)\) is inside the curve. We conclude that there is a well-defined “fluid” region \(\Omega_f\) that is mapped topologically by \(T\) to the inside of the Jordan curve, and a “vacuum” region \(\Omega_v\) that is mapped topologically to the outside of the Jordan curve.

Next we show that \(z(w)\) is univalent and maps the lower half-plane onto the fluid region. Consider the path \(\gamma_a\) in the \(w\)-plane that traverses the boundary \(\partial S_a\) of the half-disk \(S_a = \{w : |w| < a, \text{Im} w < 0\}\). Let \(\Gamma = \{\zeta(\alpha) : \alpha \in \mathbb{R}\}\) and suppose \(z_0 \in \Omega_f \cup \Omega_v = \mathbb{C} \setminus \Gamma\). If \(a > |z_0| + M\), then \(z(w) - z_0\) has no zeros on \(\gamma_a\). Indeed, \(w \in \gamma_a\) requires \(\text{Im} w = 0\) or \(|w| = a\). Assuming \(z(w) = z_0\) would require \(z_0 \in \Gamma\) or \(|z(w)| > a - M > |z_0|\), a contradiction in either case. We can therefore define the winding number of \(\Gamma_a = z(\gamma_a)\) around \(z_0\).

\[
(A.2) \quad n(\Gamma_a, z_0) = \int_{\gamma_a} \frac{dz}{z - z_0} = \int_{\gamma_a} \frac{z'(w)dw}{z(w) - z_0}, \quad (a > |z_0| + M).
\]

Since \(n(\Gamma_a, z_0)\) counts the number of solutions of \(z(w) = z_0\) inside \(S_a\) and all solutions in the lower half-plane belong to \(S_a\) as soon as \(a > |z_0| + M\), \(n(\Gamma_a, z_0)\) is a non-negative integer that gives the number of solutions of \(z(w) = z_0\) in the lower half-plane. It is independent of \(a\) once \(a > |z_0| + M\).

We decompose \(n(\Gamma_a, z_0) = n_2(z_0, a) - n_1(z_0, a)\), where

\[
(A.3) \quad n_1(z_0, a) = \frac{1}{2\pi i} \int_{-a}^{a} \frac{\zeta'(\alpha)d\alpha}{\zeta(\alpha) - z_0}
\]

and \(n_2(z_0, a)\) is given in \((A.5)\) below. Let \(n_1(z_0) = \lim_{a \to \infty} n_1(z_0, a)\). We will show that

\[
(A.4) \quad n_1(z_0) = \begin{cases} 
-1/2, & z_0 \in \Omega_f, \\
1/2, & z_0 \in \Omega_v.
\end{cases}
\]

First, if \(z_0 = -iA\), where \(A > M\), then \(\text{Im}\{\zeta(\alpha) - z_0\} > 0\) for \(\alpha \in \mathbb{R}\). Thus, \(\zeta(\alpha) - z_0\) does not cross the principal branch cut of the logarithm. As a result,

\[
n_1(-iA, a) = \frac{1}{2\pi i} \left[ \ln \left| \frac{\zeta(a) + iA}{\zeta(-a) + iA} \right| + i \text{Arg} (\zeta(a) + iA) - i \text{Arg} (\zeta(-a) + iA) \right].
\]

Since \(|\zeta(\alpha) - a| \leq M < A\), as \(a \to \infty\) we obtain

\[
n_1(-iA) = \frac{\ln(1) + i0 - i\pi}{2\pi i} = -\frac{1}{2}.
\]

A similar argument shows that \(n_1(iA) = 1/2\). Let \([z_0, z_1]\) denote the line segment in \(\mathbb{C}\) connecting two points \(z_0\) and \(z_1\), and suppose this line segment does not intersect the curve \(\zeta(\alpha)\). We claim that if the limit defining \(n_1(z_0)\) exists, the limit defining \(n_1(z_1)\) also exists, and \(n_1(z_1) = n_1(z_0)\). First note that

\[
n_1(z_1, a) = n_1(z_0, a) + \frac{1}{2\pi i} \int_{-a}^{a} \left[ \frac{\zeta'(\alpha)}{\zeta(\alpha) - z_1} - \frac{\zeta'(\alpha)}{\zeta(\alpha) - z_0} \right] d\alpha.
\]
Since $\zeta(\alpha)$ does not cross $[z_0, z_1]$, $\frac{\zeta(\alpha) - z_1}{\zeta(\alpha) - z_0}$ does not cross the negative real axis or 0. Thus,
\[
n_1(z_1, a) = n_1(z_0, a) + \left[ \log \frac{\zeta(\alpha) - z_1}{\zeta(\alpha) - z_0} \right]_{a = -a}^{a = a}.
\]
Taking the limit as $a \to \infty$ gives $n_1(z_1) = n_1(z_0) + 0$, as claimed. Every point of $\Omega_f$ is connected to $-iA$ by a polygonal path that remains inside $\Omega_f$, and every point in $\Omega_v$ is connected to $iA$ by a polygonal path that remains inside $\Omega_v$. The result (A.4) follows.

Next consider the contribution of the semicircular part of $\gamma_\alpha$ in (A.2), namely
\[
(A.5) \quad n_2(z_0, a) = \frac{1}{2\pi i} \int_{-\pi}^{0} \frac{z'(ae^{i\theta})(i\alpha e^{i\theta})}{z(ae^{i\theta}) - z_0} d\theta.
\]
Since $|z(w) - w| \leq M$, the Cauchy integral formula gives $|z'(\alpha + i\beta) - 1| \leq M/(\varepsilon - \beta)$ for $\beta \in (-\infty, \varepsilon)$. For $a > 2(M + |z_0|)$,
\[
|z(ae^{i\theta}) - z_0| \geq (a - M - |z_0|) > a/2,
\]
thus the modulus of the integrand in (A.5) is bounded uniformly by $2(1 + M/\varepsilon)$ for large $a$. For fixed $\theta \in (-\pi, 0)$, $z'(ae^{i\theta}) \to 1$ and $|z(ae^{i\theta}) - z_0|/a \to e^{i\theta}$, so the integrand approaches $i$ pointwise on the interior of the integration interval as $a \to \infty$. By the dominated convergence theorem, $\lim_{a \to \infty} n_2(z_0, a) = 1/2$. Combining these results, we find that $\lim_{a \to \infty} n(\Gamma_\alpha, z_0) = 1/2 - n_1(z_0)$. But since $n(\Gamma_\alpha, z_0)$ is constant for $a > |z_0| + M$, we conclude that
\[
(A.6) \quad n(\Gamma_\alpha, z_0) = \begin{cases} 1, & z_0 \in \Omega_f \\ 0, & z_0 \in \Omega_v \end{cases}, \quad (a > |z_0| + M).
\]
This shows that when solving the equation $z(w) = z_0$, if $z_0 \in \Omega_f$, there is precisely one solution $w_0$ in the lower half-plane, and if $z_0 \in \Omega_v$, there are no solutions $w_0$ in the lower half-plane. Since $z(w)$ is an open mapping, it cannot map a point in the lower half-plane to the boundary $\Gamma$, since a nearby point would then have to be mapped to $\Omega_v$. It follows that $z(w)$ is a 1-1 mapping of $\{\Re w < 0\}$ onto $\Omega_f$. It is then a standard result that $z'(w)$ has no zeros in the lower half-plane and the inverse function $w(z)$ exists and is analytic on $\Omega_f$. □

Example A.2. The function $z(w) = 2w^2/(2w-i)$ satisfies $|z(w) - w| = |iw|/|2w-i|$. Writing $w = \alpha + i\beta$, we have $|z(w) - w|^2 = (\alpha^2 + \beta^2)/(4\alpha^2 + (2\beta - 1)^2)$. If $\beta \in (-\infty, 1/4]$, then $(2\beta - 1)^2 = 4\beta^2 - 4\beta + 1 \geq 4\beta^2$ and $|z(w) - w|^2 \leq 1/4$. Moreover, $\zeta(\alpha) = 2\alpha^2(2\alpha + i)/(4\alpha^2 + 1)$ is injective in spite of a cusp at the origin (where $\zeta'(0) = 0$). The hypotheses of Theorem A.1 are satisfied with $\varepsilon = 1/4$ and $M = 1/2$, so $z$ maps the half-plane $C^-$ conformally onto the region below the curve $\zeta(\alpha)$, in spite of the cusp. We will usually assume $\zeta'(\alpha) \neq 0$ for $\alpha \in \mathbb{R}$ so that the curve is smooth.

Corollary A.3. Suppose $k > 0$ is irrational, $\tilde{\eta}(\alpha_1, \alpha_2) = \sum_{(j_1, j_2) \in \mathbb{Z}^2} \tilde{\eta}_{j_1, j_2} e^{i(j_1\alpha_1 + j_2\alpha_2)}$, and there exist constants $C$ and $\varepsilon > 0$ such that
\[
(A.7) \quad \tilde{\eta}_{-j_1, -j_2} = \tilde{\eta}_{j_1, j_2}, \quad |\tilde{\eta}_{j_1, j_2}| \leq Ce^{-3\varepsilon K \max(|j_1|, |j_2|)} \quad (j_1, j_2) \in \mathbb{Z}^2,
\]
where $K = \max(k, 1)$. Let $x_0$ be real and define $\tilde{\xi} = x_0 + H[\tilde{\eta}]$, $\tilde{\zeta} = \tilde{\xi} + i\tilde{\eta}$ and
\[
(A.8) \quad \tilde{z}(\alpha_1, \alpha_2, \beta) = x_0 + i\tilde{\eta}_{0, 0} + \sum_{j_1 + j_2 k < 0} 2i\tilde{\eta}_{j_1, j_2} e^{-(j_1 + j_2 k)\beta} e^{i(j_1\alpha_1 + j_2\alpha_2)}, \quad (\beta < \varepsilon),
\]
where the sum is over all integer pairs \((j_1, j_2)\) satisfying the inequality. Suppose also that for each fixed \(\theta \in [0, 2\pi]\), the function \(\alpha \mapsto \zeta(\alpha; \theta) = \alpha + \bar{\zeta}(\alpha, \theta + k\alpha)\) is injective from \(\mathbb{R}\) to \(\mathbb{C}\) and \(\zeta_\alpha(\alpha; \theta) \neq 0\) for \(\alpha \in \mathbb{R}\). Then for each \(\theta \in \mathbb{R}\), the curve \(\zeta(\alpha; \theta)\) separates the complex plane into two regions and

\[
(A.9) \quad z(\alpha + i\beta; \theta) = (\alpha + i\beta) + \bar{z}(\alpha, \theta + k\alpha, \beta), \quad (\beta < \varepsilon)
\]

is an analytic isomorphism of the lower half-plane onto the region below \(\zeta(\alpha; \theta)\). Moreover, there is a constant \(\delta > 0\) such that \(|z_w(w; \theta)| \geq \delta\) for \(\text{Im } w \leq 0\) and \(\theta \in \mathbb{R}\).

**Proof.** First we confirm that \(z(w; \theta)\) and \(\zeta(w; \theta)\) satisfy the hypotheses of Theorem A.1. The formula

\[
(A.10) \quad z(w; \theta) = w + x_0 + i\hat{\eta}_{0,0} + \sum_{j_1 + j_2k < 0} \left(2i\hat{\eta}_{j_1,j_2}e^{ij_2\theta}\right) e^{i(j_1+j_2)w}
\]

expresses \(z(w; \theta)\) as a uniformly convergent series of analytic functions on the region \(\text{Im } w < \varepsilon\), so it is analytic in this region. This follows from the inequalities

\[
0 < -(j_1 + j_2k) \leq |j_1| + |j_2|k \leq 2K\max(|j_1|, |j_2|),
\]

\[
-(j_1 + j_2k)\beta \leq -(j_1 + j_2k)\varepsilon \leq 2\varepsilon K\max(|j_1|, |j_2|), \quad (-\infty < \beta \leq \varepsilon)
\]

\[
\left|\left(2i\hat{\eta}_{j_1,j_2}e^{ij_2\theta}\right) e^{i(j_1+j_2)w}\right| \leq 2Ce^{-\varepsilon K\max(|j_1|, |j_2|)}, \quad (\text{Im } w \leq \varepsilon)
\]

and the fact that for each non-negative integer \(s\), there are \(4s\) index pairs \((j_1, j_2)\) in the shell \(\max(|j_1|, |j_2|) = s\) and satisfying \(j_1 + j_2k < 0\):

\[
(A.12) \quad \sum_{j_1 + j_2k < 0} \left|\left(2i\hat{\eta}_{j_1,j_2}e^{ij_2\theta}\right) e^{i(j_1+j_2)w}\right| \leq \sum_{s=1}^{\infty} (2C)(4s)e^{-\varepsilon Ks} < \infty, \quad (\text{Im } w \leq \varepsilon).
\]

This also implies that there is a bound \(M\) such that \(|z(w; \theta) - w| \leq M\) for \(\text{Im } w \leq \varepsilon\) and \(\theta \in \mathbb{R}\). Let \(\xi(\alpha; \theta) = \alpha + \hat{\xi}(\alpha, \theta + k\alpha)\) and \(\eta(\alpha; \theta) = \hat{\eta}(\alpha, \theta + k\alpha)\) denote the real and imaginary parts of \(\zeta(\alpha; \theta)\). Setting \(w = \alpha \in \mathbb{R}\) in (A.10) and taking real and imaginary parts confirms that \(z(w; \theta)|_{w=\alpha} = \zeta(\alpha; \theta)\). By assumption, \(\zeta(\alpha; \theta)\) is injective, so Theorem A.1 implies that \(z(w; \theta)\) is an analytic isomorphism of the lower half-plane onto the region below the curve \(\zeta(\alpha; \theta)\). Differentiating (A.10) term by term [1] shows that \(z_w(\alpha + i\beta; \theta) = F(\alpha, \theta + k\alpha, \beta)\), where

\[
(A.13) \quad F(\alpha_1, \alpha_2, \beta) = 1 - \sum_{j_1 + j_2k < 0} 2(j_1 + j_2k)\hat{\eta}_{j_1,j_2}e^{-(j_1+j_2k)\beta}e^{i(j_1\alpha_1+j_2\alpha_2)}.
\]

We claim that \(F(\alpha_1, \alpha_2, \beta) \to 1\) uniformly in \((\alpha_1, \alpha_2)\) as \(\beta \to -\infty\). Indeed, arguing as in (2.27), we see that \(|F(\alpha_1, \alpha_2, \beta) - 1| = |z_w(\alpha_1 + i\beta; \alpha_2 - k\alpha_1) - 1| \leq M/(\varepsilon - \beta)\). Thus, for \(\beta \leq -B\) with \(B = 2M\), \(|F(\alpha_1, \alpha_2, \beta)| \geq 1/2\). Since \(|F(\alpha_1, \alpha_2, \beta)|\) is continuous, it achieves its minimum over \((\alpha_1, \alpha_2) \in \mathbb{T}^2\) and \(-B \leq \beta \leq 0\). Denote this minimum by \(\delta\). If \(\delta\) were zero, there would exist \(\alpha_1, \alpha_2\) and \(\beta \leq 0\) such that \(F(\alpha_1, \alpha_2, \beta) = 0\). But then \(z_w(\alpha_1 + i\beta; \theta) = F(\alpha_1, \theta + k\alpha_1, \beta) = 0\) with \(\theta = \alpha_2 - k\alpha_1\). The case \(\beta = 0\) is ruled out by the assumption that \(\zeta_\alpha(\alpha; \theta) \neq 0\) while \(\beta < 0\) contradicts \(z(w; \theta)\) being 1-1 on \(\mathbb{C}^-\). So \(\delta > 0\) and \(|F(\alpha_1, \alpha_2, \beta)| \geq \min(\delta, 1/2)\) for all \(\beta \leq 0\). Decreasing \(\delta\) to 1/2 if necessary gives the desired lower bound \(|z_w(w; \theta)| \geq \delta\). \(\square\)
Appendix B. Quasi-Periodic Families of Solutions

In this appendix we explore the effect of introducing phases in the reconstruction of one-dimensional quasi-periodic solutions of (2.44) from solutions of the torus version of these equations. This ultimately makes it possible to show that if all the solutions in the family are single-valued and have no vertical tangent lines, the corresponding solutions of the original graph-based formulation (2.1)–(2.4) of the Euler equations are quasi-periodic in physical space.

Theorem B.1. The solution pair \((\zeta, \varphi)\) on the torus represents an infinite family of quasi-periodic solutions on \(\mathbb{R}\) given by

\[
\begin{align*}
\zeta(\alpha, t; \theta_1, \theta_2, \delta) &= \alpha + \delta + \zeta(\theta_1 + \alpha, \theta_2 + k\alpha, t), \\
\varphi(\alpha, t; \theta_1, \theta_2) &= \varphi(\theta_1 + \alpha, \theta_2 + k\alpha, t),
\end{align*}
\]

\(\alpha \in \mathbb{R}, \ t \geq 0, \ \theta_1, \theta_2, \delta \in \mathbb{R}\).

Proof. We claim that by solving (2.44) throughout \(\mathbb{T}^2\) in the sense of Remark 2.3, any one-dimensional (1d) slice of the form (B.1) will satisfy the kinematic condition (2.37) and the Bernoulli equation (2.43). Let us freeze \(\theta_1, \theta_2\) and \(\delta\) and drop them from the notation on the left-hand side of (B.1). Consider substituting \(\eta = \text{Im } \zeta\) and \(\varphi\) from (B.1) into (2.44), and let \(u(\alpha) = \tilde{u}(\theta_1 + \alpha, \theta_2 + k\alpha)\) represent the input of any \(\alpha\)-derivative or Hilbert transform in an intermediate calculation. Both \(\eta\) and \(\varphi\) are of this form. By Remark 2.2, \(H[u](\alpha) = H[\tilde{u}](\theta_1 + \alpha, \theta_2 + k\alpha)\), and clearly \(u'(\alpha) = (\tilde{\partial}_\alpha + k\tilde{\partial}_{\alpha_2})\tilde{u}(\theta_1 + \alpha, \theta_2 + k\alpha)\), so the output retains this form. We conclude that computing (2.44) on the torus gives the same results for \(\tilde{\eta}\) and \(\tilde{\varphi}\) when evaluated at \((\theta_1 + \alpha, \theta_2 + k\alpha)\) as the 1d calculations of \(\eta\) and \(\varphi\) when evaluated at \(\alpha\). Since \(\tilde{\xi}(\cdot, t) = x_0(t) + H[\tilde{\eta}(\cdot, t)]\) on \(\mathbb{T}^2\),

\[
\xi(\alpha, t) = \alpha + \delta + x_0(t) + H[\eta(\cdot, t)](\alpha),
\]

which follows from (B.1) and \(H[\eta(\cdot, t)](\alpha) = H[\tilde{\eta}(\cdot, t)](\theta_1 + \alpha, \theta_2 + k\alpha)\). Thus, computing \(\tilde{\xi}_\alpha = 1 + H[\eta_\alpha]\) in (2.44) gives the same result as just differentiating \(\xi\) from (B.1) and (B.2).

In the 1d problem, the right-hand side of (2.40) represents complex multiplication of \(z_\alpha\) with a bounded analytic function (namely \(z_t/z_\alpha\)) whose imaginary part equals \(-\chi\) on the real axis; thus, in (2.40), \(\tilde{\xi}\) differs from \(H[\eta]\) by a constant. This constant is determined by comparing \(\tilde{\xi}\) in (2.40) with \(\xi\) from (B.2), which leads to the same formula (2.41) for \(dx_0/dt\) that is used in the torus calculation. Here we note that a phase shift does not affect the mean of a periodic function on the torus, i.e. \(P_0[S_\alpha \tilde{u}] = P_0[\tilde{u}]\) where \(S_\alpha[\tilde{u}](\alpha) = \tilde{u}(\alpha + \theta)\).

We have assumed that in the 1d calculation, \(C_1\) is chosen to agree with that of the torus calculation. Since \(C_1\) only affects the tangential velocity of the interface parametrization, it can be specified arbitrarily. Left-multiplying (2.40) by \((-\eta_\alpha, \tilde{\xi}_\alpha)\) eliminates \(C_1\) and yields the kinematic condition (2.37). Since the Bernoulli equation (2.43) holds on the torus, it also holds in the 1d calculation, as claimed. \(\Box\)

For each solution in the family (B.1), there are many others that represent identical dynamics up to a spatial phase shift or \(\alpha\)-reparametrization. Changing \(\delta\) merely shifts the solution in physical space. In fact, \(\delta\) does not appear in the equations of motion (2.40) — it is only used to reconstruct the curve via (B.2). The relations

\[
\begin{align*}
\zeta(\alpha + \alpha_0, t; \theta_1, \theta_2, \delta) &= \zeta(\alpha, t; \theta_1 + \alpha_0, \theta_2 + k\alpha_0, \delta + \alpha_0), \\
\varphi(\alpha + \alpha_0, t; \theta_1, \theta_2) &= \varphi(\alpha, t; \theta_1 + \alpha_0, \theta_2 + k\alpha_0),
\end{align*}
\]

(B.3)
show that shifting $\alpha$ by $\alpha_0$ leads to another solution already in the family. This shift reparametrizes the curve but has no effect on its evolution in physical space. If we identify two solutions that differ only by a spatial phase shift or $\alpha$-reparametrization, the parameters $(\theta_1, \theta_2, \delta)$ become identified with $(0, \theta_2 - k\theta_1, 0)$. Every solution is therefore equivalent to one of the form

$$
\zeta(\alpha, t : 0, \theta, 0) = \alpha + \tilde{\zeta}(\alpha, \theta + k\alpha, t),
\phi(\alpha, t : 0, \theta) = \tilde{\phi}(\alpha, \theta + k\alpha, t)
$$

$$
\alpha \in \mathbb{R}, \ t \geq 0, \ \theta \in [0, 2\pi).
$$

Within this smaller family, two values of $\theta$ lead to equivalent solutions if they differ by $2\pi(n_1k + n_2)$ for some integers $n_1$ and $n_2$. This equivalence is due to solutions “wrapping around” the torus with a spatial shift,

$$
\zeta(\alpha + 2\pi n_1, t : 0, \theta, 0) = \zeta(\alpha, t : 0, \theta + 2\pi(n_1k + n_2), 2\pi n_1), \ (\alpha \in [0, 2\pi), \ n_1 \in \mathbb{Z}).
$$

Here $n_2$ is chosen so that $0 \leq [\theta + 2\pi(n_1k + n_2)] < 2\pi$ and we used periodicity of $\zeta(\alpha, t : \theta_1, \theta_2, \delta)$ with respect to $\theta_1$ and $\theta_2$. It usually suffices to restrict attention to $\alpha \in [0, 2\pi)$ by making use of (B.5). One exception is determining whether the curve self-intersects. In that case it is easier to work directly with (B.4) since it is conceivable that $\zeta(\alpha, t) = \zeta(\alpha, t)$ with $|\alpha_2 - \alpha_1|$ as large as $2M$, where $M$ is a bound on $|\tilde{\zeta}|$ over $\mathbb{T}^2$.

The following theorem shows that $\eta^{\text{phys}}(x, t : \theta_1, \theta_2, \delta)$ and $\phi^{\text{phys}}(x, t : \theta_1, \theta_2, \delta)$ can be computed easily from $\zeta(\alpha, t : \theta_1, \theta_2, \delta)$ and $\phi(\alpha, t : \theta_1, \theta_2)$ if all of the waves in the family (B.4) are single-valued and have no vertical tangent lines, and that $\eta^{\text{phys}}$ and $\phi^{\text{phys}}$ are quasi-periodic functions of $x$.

**Theorem B.2.** Fix $t \geq 0$ and suppose $\xi_\alpha(\alpha, t : 0, \theta, 0) > 0$ for $\alpha \in [0, 2\pi)$ and $\theta \in [0, 2\pi)$. Then there is a periodic, real analytic function $A(x_1, x_2, t)$ defined on $\mathbb{T}^2$ satisfying

$$
A(x_1, x_2, t) + \tilde{\xi}(x_1 + A(x_1, x_2, t), x_2 + kA(x_1, x_2, t), t) = 0, \ (x_1, x_2) \in \mathbb{T}^2.
$$

Given $\theta \in [0, 2\pi)$, the change of variables $\alpha = x + A(x, \theta + kx, t)$ satisfies

$$
\xi(\alpha, t : 0, \theta, 0) = \alpha + \tilde{\xi}(\alpha, \theta + k\alpha, t) = x, \ (x \in \mathbb{R}).
$$

This allows us to express solutions in the family (B.4) as functions of $x$ and $t$,

$$
\eta^{\text{phys}}(x, t : 0, \theta, 0) = \eta(\alpha, t : 0, \theta, 0), \ (\alpha = x + A(x, \theta + kx, t)).
\phi^{\text{phys}}(x, t : 0, \theta, 0) = \phi(\alpha, t : 0, \theta),
$$

These functions are real analytic, quasi-periodic functions of $x$ in the sense that

$$
\eta^{\text{phys}}(x, t : 0, \theta, 0) = \tilde{\eta}^{\text{phys}}(x, \theta + kx, t)
\phi^{\text{phys}}(x, t : 0, \theta, 0) = \tilde{\phi}^{\text{phys}}(x, \theta + kx, t)
$$

with

$$
\tilde{\eta}^{\text{phys}}(x_1, x_2, t) = \tilde{\eta}(x_1 + A(x_1, x_2, t), x_2 + kA(x_1, x_2, t), t),
\tilde{\phi}^{\text{phys}}(x_1, x_2, t) = \tilde{\phi}(x_1 + A(x_1, x_2, t), x_2 + kA(x_1, x_2, t), t).
$$

More generally, one can define

$$
\eta^{\text{phys}}(x, t : \theta_1, \theta_2, \delta) = \eta(\alpha, t : \theta_1, \theta_2, \delta) = \tilde{\eta}^{\text{phys}}(\theta_1 + x - \delta, \theta_2 + k(x - \delta), t)
\phi^{\text{phys}}(x, t : \theta_1, \theta_2, \delta) = \phi(\alpha, t : \theta_1, \theta_2) = \tilde{\phi}^{\text{phys}}(\theta_1 + x - \delta, \theta_2 + k(x - \delta), t)
$$
with \( \alpha = x - \delta + \mathcal{A}(\theta_1 + x - \delta, \theta_2 + k(x - \delta), t) \) to express \( \zeta(\alpha, t; \theta_1, \theta_2, \delta) \) as a graph and \( \varphi(\alpha, t; \theta_1, \theta_2) \) as a function of \( x \).

**Proof.** Since \( \xi_0(\alpha, t; 0, \theta, 0) \) is continuous and \( 2\pi \)-periodic in \( \alpha \) and \( \theta \), there is an \( \varepsilon > 0 \) such that \( 1 + [c_{\alpha_1} + k\zeta_{\alpha_2}]\|\xi(\alpha_1, \alpha_2, t)\| \geq \varepsilon \) for \( (\alpha_1, \alpha_2) \in \mathbb{T}^2 \). Let \( M(t) \) be a bound on \( \|\xi(\alpha_1, \alpha_2, t)\| \) over \( \mathbb{T}^2 \). Then for fixed \( (x_1, x_2) \), the function \( g(\alpha) = g(\alpha; x_1, x_2) = \alpha + \tilde{\xi}(x_1 + \alpha, x_2 + k\alpha, t) \) is strictly monotonically increasing (as \( g'(\alpha) \geq \varepsilon \)) and satisfies \( g(-M) \leq 0 \) and \( g(M) \geq 0 \). Thus, we can define \( \mathcal{A}(x_1, x_2, t) \) as the unique solution of \( g(\alpha) = 0 \), and \( |\mathcal{A}(x_1, x_2, t)| \leq M \). If \( n_1 \) and \( n_2 \) are integers and \( y_1 = x_1 + 2\pi n_1 \) and \( y_2 = x_2 + 2\pi n_2 \), then \( \alpha = \mathcal{A}(x_1, x_2) \) satisfies \( \alpha + \tilde{\xi}(y_1 + \alpha, y_2 + k\alpha, t) \), so \( \mathcal{A}(y_1, y_2, t) = \mathcal{A}(x_1, x_2, t) \). It is also real analytic, which follows from the implicit function theorem, noting that \( g(\alpha; x_1, x_2) \) is real analytic in all three variables and \( \frac{\partial g}{\partial \alpha} \) is never zero.

The first equality of (B.7) is the definition of \( \bar{\xi}(\alpha, t; 0, \theta, 0) \) and the second follows from (B.6) when \( \alpha = x + \mathcal{A}(x, \theta + kx, t) \) is substituted into \( \alpha + \tilde{\xi}(\alpha, \theta + k\alpha, t) \). From (B.8) and (B.4) we have

\[
\eta^{\text{phys}}(x, t; 0, \theta, 0) = \tilde{\eta}(\alpha, \theta + k\alpha, t),
\]

\[
\varphi^{\text{phys}}(x, t; 0, \theta, 0) = \tilde{\varphi}(\alpha, \theta + k\alpha, t),
\]

where \( \alpha = x + \mathcal{A}(x, \theta + kx, t) \). Comparing (B.12) with (B.10) gives (B.9). Equation (B.11) is derived similarly using (B.1), (B.6) and (B.10) to check that

\[
\xi(\alpha, t; \theta_1, \theta_2, \delta) = x,
\]

\[
\eta(\alpha, t; \theta_1, \theta_2, \delta) = \eta^{\text{phys}}(\theta_1 + x - \delta, \theta_2 + k(x - \delta), t),
\]

\[
\varphi(\alpha, t; \theta_1, \theta_2) = \varphi^{\text{phys}}(\theta_1 + x - \delta, \theta_2 + k(x - \delta), t)
\]

when \( \alpha = x - \delta + \mathcal{A}(\theta_1 + x - \delta, \theta_2 + k(x - \delta), t) \).

**Appendix C. Dynamics of Traveling Waves in Conformal Space**

In this section we study the dynamics of the traveling waves computed in Section 2.6 under the evolution equations (2.44) for various choices of \( C_1 \). We show that the waves maintain a permanent form but generally travel at a non-uniform speed in conformal space. We start by showing that there is a choice of \( C_1 \) for which \( \eta \) and \( \varphi \) remain stationary in time. We then show how \( C_1 \) changes when the waves are phase shifted by \( \alpha_0(t) \), and how to determine \( \alpha_0(t) \) so that \( C_1 \) takes the value in (2.45). The evolution of the torus version of (2.50) under (2.44) is also worked out.

**Theorem C.1.** Suppose \( \tilde{\eta}_0(\alpha_1, \alpha_2) \) satisfies the torus version of (2.50) as well as the assumptions in Corollary A.3. Define \( \tilde{\xi}_0 = H[\tilde{\eta}_0], \xi_0 = \tilde{\xi}_0 + i\tilde{\eta}_0 \) and \( \varphi_0 = c\tilde{\xi}_0 \). Let \( \eta_0(\alpha; \theta) = \tilde{\eta}_0(\alpha, \theta + k\alpha), \varphi_0(\alpha; \theta) = \tilde{\varphi}_0(\alpha, \theta + k\alpha), \xi_0(\alpha; \theta) = \alpha + \tilde{\xi}_0(\alpha, \theta + k\alpha) \) and \( \zeta_0 = \xi_0 + i\tilde{\eta}_0 \). Suppose that for each \( \theta \in [0, 2\pi) \), \( \alpha \mapsto \zeta_0(\alpha; \theta) \) is injective, i.e. none of the curves in the family (2.53) self-intersect. Then for each \( \theta \in \mathbb{R} \),

\[
(\zeta(t; \theta) = \zeta_0(\alpha; \theta) + ct, \quad \varphi(t; \theta) = \varphi_0(\alpha; \theta)
\]

satisfy (2.44) with \( C_1 = cP_0[\xi_0]/f \).

**Proof.** We have assumed the initial reconstruction of \( \xi \) from \( \eta \) yields \( \xi_0(\alpha, 0; \theta) = \xi_0(\alpha; \theta) \), so \( x_0(0) = 0 \) in (2.29). We need to show that \( \eta_1 = 0, \varphi_1 = 0 \) and \( dx_0/dt = c \) in (2.44), from which it follows that \( \xi(t; \theta) = \xi_0(\alpha; \theta) + ct \). Since \( \tilde{\xi}_0 = H[\tilde{\eta}_0] \) and none of the
curves in the family (2.53) self-intersect, Theorem A.1 and Corollary A.3 in Appendix A show that the holomorphic extension from $\zeta_0(\alpha; \theta)$ to $z_0(w; \theta)$ is an analytic isomorphism of the lower half-plane to the fluid region, and $1/|z_{0,w}|$ is uniformly bounded. In (2.44), we define $\xi_\alpha = 1 + H[\eta_\alpha], \psi = -H[\varphi], J = \xi_\alpha^2 + \eta_\alpha^2$ and $\chi = \psi_\alpha/j$. This formula for $\xi_\alpha$ gives the same result as differentiating $\xi(\alpha, t; \theta)$ in (C.1) with respect to $\alpha$. From $\varphi_0 = c \xi_0$ and $\hat{\eta}_{0,0} = 0$, we have $\chi = c \eta_\alpha/J$. The extension of $\zeta(\alpha, t; \theta)$ to the lower half-plane is $z(w, t; \theta) = [z_0(w; \theta) + ct]$. We have not yet established that $\zeta(\alpha, t; \theta)$ solves (2.44), but we know $z_t/z_w$ is bounded in the lower half-plane, so there is a $C_1$ such that

\[
(-H \chi + C_1) = \frac{1}{J} \begin{pmatrix} \xi_\alpha & \eta_\alpha \\ -\eta_\alpha & \xi_\alpha \end{pmatrix} \begin{pmatrix} c \\ 0 \end{pmatrix},
\]

where the right-hand side represents complex division of $z_t$ by $z_w$. Since $P_0 H \chi = 0$, we learn from (C.2) that $C_1 = c P_0[\xi_\alpha/j]$. But $\xi_t$ and $\eta_t$ in (2.40) are obtained by multiplying (C.2) by $[\xi_\alpha, -\eta_\alpha; \eta_\alpha, \xi_\alpha]$, which gives $\xi_c = c, \eta_t = 0$. Equation (2.41) is then $dx_0/dt = P_0[\xi_t] = c$.

Finally, using $\chi = c \eta_\alpha/j, H \chi = C_1 - c \xi_\alpha/j, \varphi_\alpha = c(\xi_\alpha - 1)$ and $\psi_\alpha = c \eta_\alpha$ in (2.44) gives

\[
\varphi_t = \frac{\psi_\alpha^2 - \varphi_\alpha^2}{2} - \varphi_\alpha H[\chi] + C_1 \varphi_\alpha - g \eta + \tau \kappa
\]

\[
= \frac{c^2 \eta_\alpha^2 - c^2(\xi_\alpha^2 - 2 \xi_\alpha + 1)}{2} + c \frac{c(\xi_\alpha - 1) \xi_\alpha}{J} - g \eta + \tau \kappa
\]

\[
= \frac{c^2}{2} (J - 1) - g \eta + \tau \kappa
= \frac{c^2}{2} - g \eta + \tau \kappa = 0,
\]

where we used (2.50) in the last step. \hfill \Box

**Corollary C.2.** Suppose $\zeta_0(\alpha_1, \alpha_2), \varphi_0(\alpha_1, \alpha_2), \zeta_0(\alpha; \theta)$ and $\varphi_0(\alpha; \theta)$ satisfy the hypotheses of Theorem C.1 and $\alpha_0(t)$ is any continuously differentiable, real-valued function. Then

\[
\zeta(\alpha, t; \theta) = \zeta_0(\alpha - \alpha_0(t); \theta) + ct, \quad \varphi(\alpha, t) = \varphi_0(\alpha - \alpha_0(t); \theta)
\]

are solutions of (2.44) with $C_1 = c P_0[\xi_\alpha/j] - \alpha'_0(t)$. The corresponding solutions of the torus version of (2.44) for this choice of $C_1$ are

\[
\tilde{\zeta}(\alpha_1, \alpha_2, t) = \tilde{\zeta}_0(\alpha_1 - \alpha_0(t), \alpha_2 - k \alpha_0(t)) + ct - \alpha_0(t),
\]

\[
\tilde{\varphi}(\alpha_1, \alpha_2, t) = \tilde{\varphi}_0(\alpha_1 - \alpha_0(t), \alpha_2 - k \alpha_0(t)).
\]

**Proof.** Since $\partial_\alpha$ and $H$ commute with $\alpha$-translations, substitution of $\eta_0(\alpha - \alpha_0(t); \theta)$ and $\varphi_0(\alpha - \alpha_0(t); \theta)$ in the right-hand sides of (2.44) without changing $C_1$ would still lead to $\eta_t = 0, \varphi_t = 0$ and $dx_0/dt = c$, and (2.40) would still give $\xi_t = c$. Including $-\alpha'_0(t)$ in $C_1$ leads instead to $\eta_t = -\alpha'_0(t) \eta_\alpha$ and $\varphi_t = -\alpha'_0(t) \varphi_\alpha$ in (2.44) and $\xi_t = c - \alpha'_0(t) \xi_\alpha$ in (2.40), which are satisfied by (C.4). It also leads to $dx_0/dt = [c - \alpha'_0(t)]$ in (2.41), which keeps the reconstruction of $\xi$ from $\eta$ via (2.29) consistent with the evolution equation for $\xi_t$.

The functions in (C.4) and (C.5) are related by

\[
\zeta(\alpha, t; \theta) = \alpha + \tilde{\zeta}(\alpha, \theta + k \alpha, t), \quad \varphi(\alpha, t; \theta) = \varphi(\alpha, \theta + k \alpha, t).
\]

Applying the 1d version of (2.44) to (C.6) is equivalent to applying the torus version of (2.44) to (C.5) and evaluating at $(\alpha, \theta + k \alpha, t)$. Since (C.4) satisfies the 1d version of (2.44) and every point $(\alpha_1, \alpha_2) \in T^2$ can be written as $(\alpha, \theta + k \alpha)$ for some $\alpha$ and $\theta$, (C.5) satisfies the torus version. \hfill \Box
Corollary C.3. Suppose \( \xi_0(\alpha_1, \alpha_2), \phi_0(\alpha_1, \alpha_2), \zeta_0(\alpha; \theta) \) and \( \varphi_0(\alpha; \theta) \) satisfy the hypotheses of Theorem C.1 and \( \xi_{0,a}(\alpha; \theta) > 0 \) for \( \alpha \in [0, 2\pi) \) and \( \theta \in [0, 2\pi) \). Then if \( C_1 \) is chosen as in (2.45) to maintain \( \xi(0,0, t) = 0 \), the solution of the torus version of (2.44) with initial conditions

\[
(C.7) \quad \xi(\alpha_1, \alpha_2, 0) = \xi_0(\alpha_1, \alpha_2), \quad \phi(\alpha_1, \alpha_2, 0) = \phi_0(\alpha_1, \alpha_2)
\]

has the form (C.5) with

\[
(C.8) \quad \alpha_0(t) = ct - \mathcal{A}(-ct, -kct),
\]

where \( \mathcal{A}(x_1, x_2) \) is defined implicitly by

\[
(C.9) \quad \mathcal{A}(x_1, x_2) + \xi_0(x_1 + \mathcal{A}(x_1, x_2), x_2 + k\mathcal{A}(x_1, x_2)) = 0, \quad (x_1, x_2) \in \mathbb{T}^2.
\]

Proof. The assumption that \( \xi_{0,a}(\alpha; \theta) \) is positive ensures that all the waves in the family \( \zeta_0(\alpha; \theta) \) are single-valued and have no vertical tangent lines. The proof of Theorem B.2 shows that there is a unique function \( \mathcal{A}(x_1, x_2) \) satisfying (C.9) and that it is real analytic and periodic. We seek a solution of the form (C.5) satisfying \( \xi(0,0, t) = 0 \),

\[
(C.10) \quad \xi(0,0, t) = \xi_0(-\alpha_0(t), -k\alpha_0(t)) + ct - \alpha_0(t)
\]

\[
= [ct - \alpha_0(t)] + \xi_0(-ct + [ct - \alpha_0(t)], -kct + k[ct - \alpha_0(t)]) = 0.
\]

Comparing with (C.9), we find that \( [ct - \alpha_0(t)] = \mathcal{A}(-ct, -kct) \), which is (C.8). Since \( \tilde{\eta}_0(\alpha_1, \alpha_2) \) is even, \( \xi_0 = H[\tilde{\eta}_0] \) is odd and \( \mathcal{A}(0, 0) = 0 \). Thus, \( \alpha_0(0) = 0 \) and the initial conditions (C.7) are satisfied. Since \( \xi(0,0, t) = 0 \), \( C_1 \) satisfies (2.45). \( \square \)

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