An integrable structure related with tridiagonal algebras

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Abstract

The standard generators of tridiagonal algebras, recently introduced by Terwilliger, are shown to generate a new (in)finite family of mutually commuting operators which extends the Dolan-Grady construction. The involution property relies on the tridiagonal algebraic structure associated with a deformation parameter $q$. Representations are shown to be generated from a class of quadratic algebras, namely the reflection equations. The spectral problem is briefly discussed. Finally, related massive quantum integrable models are shown to be superintegrable.

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1 Introduction

In the Liouville-Arnold sense, a classical Hamiltonian system with $N$ degrees of freedom is called completely integrable if there exists a set of $N$ functionally independent globally defined integrals of motion in involution, i.e. mutually commuting regarding to the Poisson bracket. It is said to be superintegrable if it admits more than $N$ integrals of motion. By analogy, a quantum system with $N$ degrees of freedom is completely integrable if there exists a set of $N - 1$ operators, say $\mathcal{H}_n$, $n = 1, ..., N - 1$, together with the Hamiltonian $\mathcal{H}$ which are mutually commuting. If there exists $p$—additional operators $\mathcal{I}_m$, $m = 1, ..., p$ where $1 \leq p \leq N$ such that

$$[\mathcal{H}_n, \mathcal{H}_m] = 0 \quad \text{and} \quad [\mathcal{H}, \mathcal{I}_m] = 0$$

for all $n, m$, this quantum system is said to be superintegrable. Although not necessary for superintegrability, the additional operators $\mathcal{I}_m$ can be in involution too i.e. $[\mathcal{I}_n, \mathcal{I}_m] = 0$.

The purpose of this paper is to introduce a new (in)finite set of mutually commuting quantities, whose involution property will be indentified with the defining relations for the tridiagonal algebras recently introduced and studied by Terwilliger [1] (see also [2], [3]). We will also show that the underlying structure is closely related with $U_q^{1/2}(\widehat{sl}_2)$, and coincides with the Dolan-Grady one [4] for $q = 1$. As a consequence, we will show that related quantum integrable models (open spin chains, sine-Gordon field theory,...) enjoy superintegrability.

This paper is organized as follows. In Section 2, after recalling some definitions we propose a new realization of tridiagonal pairs in terms of $U_q^{1/2}(\widehat{sl}_2)$ generators. In particular, the defining relations of the tridiagonal algebra are shown to be invariant under the coproduct homomorphism of $U_q^{1/2}(\widehat{sl}_2)$. Based on the tridiagonal algebraic structure, we propose a finite set of conserved quantities $\mathcal{I}_n$ in involution. In Section 3, an alternative construction of the conserved quantities $\mathcal{I}_n$ is described. Their generating function is expressed in terms of general solutions of a class of quadratic algebras, namely the reflection equations. In Section 4, we argue that the corresponding spectral problem is related with a system of $N$ partial $q$—difference equations. For $N = 1$, the eigenvalue is expressed in terms of solutions of Bethe equations and enjoys a remarkable symmetry property. Examples of massive quantum integrable models with (in)finite degrees of freedom are considered in Section 5, where the quantities $\mathcal{I}_n$ are explicitly identified. Concluding remarks follow in the last Section.

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2 Tridiagonal algebraic structure and involution

Tridiagonal algebras have been introduced and studied in [2][3][4], where they first appeared in the context of $P$– and $Q$–polynomial association schemes. We will here consider a slightly simpler form of the tridiagonal algebraic structure, which nevertheless possesses all interesting features for further analysis. For more informations about these algebras, we report the reader to [4] where precise definitions can be found. Let us consider the tridiagonal (associative) algebra $T$ with unity generated by two operators (called standard generators) acting on a vector space $V$, say $A : V \rightarrow V$ and $A^* : V \rightarrow V$, subjects to the tridiagonal relations

\[
[A, A^2 A^* + A^* A^2 - (q + q^{-1})A A^* - \rho A^*] = 0,
\]

\[
[A^*, A^2 A + AA^*2 - (q + q^{-1})A^* A^* - \rho A] = 0
\]

(2)

where $q$ is a deformation parameter (assumed to be not a root of unity) and $\rho$ is a fixed scalar. According to [4, Theorem 3.10] $A, A^*$ is a tridiagonal (TD) pair as defined in [4, Definition 1.1], which complete classification is still an open problem. Among the known examples of TD pairs, one finds a subset such that $A, A^*$ have eigenspaces of dimension one. These are called Leonard pairs, classified in [5]. In particular, they satisfy (for details, see [6]) the so-called Askey-Wilson (AW) relations first introduced by Zhedanov [7]. Other examples of TD pairs can be found in [1][8]: for $\rho$ details, see [6].

Furthermore, we have found that the slightly more general TD pair $\tilde{A}$ satisfy the tridiagonal relations (2). To do that we introduce five generators $Q_\pm, \overline{Q}_\pm$ and $H$ subjects to the algebraic relations

\[
q^{-1/2}Q_\pm \overline{Q}_\pm - q^{1/2}\overline{Q}_\pm Q_\pm = 0,
\]

\[
q^{1/2}Q_\pm \overline{Q}_\pm - q^{-1/2}\overline{Q}_\pm Q_\pm = \frac{q^{\pm 2H} - 1}{q^{1/2} - q^{-1/2}},
\]

\[
q^H Q_\pm = q^{\pm \epsilon} Q_\pm q^H,
\]

\[
q^H \overline{Q}_\pm = q^{\pm \epsilon} \overline{Q}_\pm q^H
\]

and the $q$–Serre relations

\[
Q_\pm^3 Q_\mp - (1 + q + q^{-1})Q_\pm^2 Q_\mp Q_\pm + (1 + q + q^{-1})Q_\pm Q_\mp^2 Q_\pm - Q_\mp Q_\pm^3 = 0,
\]

\[
\overline{Q}_\pm^3 \overline{Q}_\mp - (1 + q + q^{-1})\overline{Q}_\pm^2 \overline{Q}_\mp \overline{Q}_\pm + (1 + q + q^{-1})\overline{Q}_\pm \overline{Q}_\mp^2 \overline{Q}_\pm - \overline{Q}_\mp \overline{Q}_\pm^3 = 0.
\]

Using the standard definitions [9], it is an exercise to show that $Q_\pm, \overline{Q}_\pm$ and $H$ actually generate the quantum affine Kac-Moody algebra $U_{q^{1/2}}(\hat{sl}_2)$. Based on the results of [10], we propose the following TD pair:

\[
A = \frac{1}{c_0} Q_+ + \overline{Q}_- \quad \text{and} \quad A^* = Q_- + \frac{1}{c_0} \overline{Q}_+,
\]

(4)

together with the scalar

\[
\rho = \frac{(q^{1/2} - q^{-1/2})^2}{c_0}.
\]

(5)

It is straightforward to check using the defining relations above [4] that [4] satisfy the tridiagonal relations [4]. Furthermore, we have found that the slightly more general TD pair $\tilde{A} = A + \epsilon_+ q^H, \tilde{A}^* = A^* + \epsilon_- q^{-H}$ also satisfies [4] for any values of the parameters $\epsilon_\pm$. This is not surprising as we will see later on.

The Hopf algebraic structure of $U_{q^{1/2}}(\hat{sl}_2)$ can now be used to construct a whole family of TD pairs. Indeed, the coproduct $\Delta : U_{q^{1/2}}(\hat{sl}_2) \rightarrow U_{q^{1/2}}(\hat{sl}_2) \times U_{q^{1/2}}(\hat{sl}_2)$ associated with [9] is such that

\[
\Delta(Q_\pm) = Q_\pm \otimes I + q^{\pm H} \otimes Q_\pm,
\]

\[
\Delta(Q_\pm^*) = \overline{Q}_\pm \otimes I + q^{\mp H} \otimes \overline{Q}_\pm,
\]

\[
\Delta(q^H) = q^H \otimes q^H.
\]

(6)

This realization is different from the one proposed by Terwilliger and Ito in [8] in which case $\rho = 0$, i.e. [4] reduce to $q$–Serre relations.
More generally, one defines the $N$–coproduct $\Delta^{(N)} : U_{q^{1/2}}(sl_2) \longrightarrow U_{q^{1/2}}(sl_2) \otimes \cdots \otimes U_{q^{1/2}}(sl_2)$ as

$$\Delta^{(N)} \equiv (id \times \cdots \times id \times \Delta) \circ \Delta^{(N-1)}$$

for $N \geq 3$ with $\Delta^{(2)} \equiv \Delta$, $\Delta^{(1)} \equiv id$. The opposite $N$–coproduct $\Delta'^{(N)}$ is similarly defined with $\Delta' \equiv \sigma \circ \Delta$ where the permutation map $\sigma(x \otimes y) = y \otimes x$ for all $x, y \in U_{q^{1/2}}(sl_2)$ is used. Due to the homomorphism property of the coproduct (or its opposite) $\Delta(xy) = \Delta(x)\Delta(y)$ the algebraic relations (6) are invariant under the action of $\Delta$ (or $\Delta'$). Consequently, the family of TD pairs (acting on tensor product of irreducible representations) defined as

$$A^{(N)} = \Delta^{(N)}(A) \quad \text{and} \quad A^*^{(N)} = \Delta^{(N)}(A^*) \quad \text{for} \quad N \geq 1$$

with (4) satisfies the tridiagonal relations\(^3\) (2) with (5). It is interesting to notice that the TD pair $\hat{A}, \hat{A}^*$ can also be generalized as in (3). The corresponding tridiagonal algebra $\hat{T}^{(N)}$ is a (left) coideal subalgebra [11] of $U_{q^{1/2}}(sl_2)$. Indeed, one has $\Delta(a^{(N)}) \in U_{q^{1/2}}(sl_2) \otimes \hat{T}^{(N)}$ for all $a \in \{I, \hat{A}, \hat{A}^*\}$. As a result, this class of coideal subalgebras closes under the tridiagonal algebraic relations (2).

An interesting problem is whether the tridiagonal algebraic relations (2) might provide the condition of existence of a finite set of mutually commuting quantities. Having in mind the construction of Dolan and Grady (2) associated with (2) for $q = 1$, whose generators can be expressed in terms of the loop algebra of $sl_2$ [12, 13], this problem seems rather natural and solvable. Inspired by (3), [10] we propose the following finite set of $N$ quantities in involution, denoted $I_{2n-1}^{(N)}$ with $n = 1, \ldots, N$, generated by the TD pair $A^{(N)}, A^*^{(N)}$ (3) with (4):

$$I_1^{(N)} = \kappa A^{(N)} + \kappa^* A^*^{(N)}, \quad I_3^{(N)} = \kappa \left( [A^{(N)}, A^*^{(N)}]_q, A^{(N)}_q + \rho A^*^{(N)}_q \right) + \kappa^* \left( [A^*^{(N)}, A^{(N)}]_q, A^*^{(N)}_q + \rho A^{(N)}_q \right), \quad I_5^{(N)} = \cdots ,$$

where the $q$–commutator $[X, Y]_q = q^{1/2}XY - q^{-1/2}YX$ has been introduced and $\kappa, \kappa^*$ are arbitrary parameters. As shown in the first part of this Section, the TD pair $A^{(N)}, A^*^{(N)}$ satisfies the tridiagonal algebraic relations (2). Then, it is an exercise to check, for instance, that $[I_1^{(N)}, I_3^{(N)}] = 0$ for any $N$. Note that the special case $\kappa = \kappa^*$ was considered in [10]. For general values of $N$, it is not difficult by analogy with (3) to guess the explicit form of the $N$ independent quantities for $n = 3, \ldots, N$ (higher order polynomials in the TD pair) which are mutually commuting. However, due the $q$–commutator the proof of mutual commutativity of the corresponding quantities becomes quickly rather complicated. Consequently, in next Section we propose an alternative method to derive these quantities.

To see the truncation of the hierarchy at the order $N$ let us focus on the smallest values of $N$. For $N = 1$, using the evaluation homomorphism $\pi_v : U_{q^{1/2}}(sl_2) \longrightarrow U_{q^{1/2}}(sl_2)$ ($v \in \mathbb{C}$ is sometimes called the spectral parameter)

$$\pi_v[Q_{\pm}] = vS_{\pm}q^{\pm s_3/2} \quad \text{and} \quad \pi_v[Q_{\pm}]^{-1} = v^{-1}S_{\pm}q^{-s_3/2} \quad \pi_v[H] = s_3 ,$$

where $\{S_{\pm}, s_3\}$ are the fundamental generators of the quantum algebra $U_{q^{1/2}}(sl_2)$ satisfying the relations $[s_3, S_{\pm}] = \pm S_{\pm}$ and $[S_+, S_-] = (q^{s_3} - q^{-s_3})/(q^{1/2} - q^{-1/2})$, one finds that $A^{(1)}, A^*^{(1)}$ satisfy the AW relations\(^4\)

$$A^{(1)2}A^{(1)*} + A^{(1)1}A^{(1)*} = (q + q^{-1})A^{(1)1}A^{(1)*} = \rho A^{*1} + \omega A^{(1)} ,$$

$$A^{*12}A^{(1)} + A^{*11}A^{(1)} = (q + q^{-1})A^{*(11)}A^{(1)} = \rho A^{(1)*} + \omega A^{*(1)}$$

with (5) and

$$\omega = -(v^2q^{-1/2} + v^{-2}q^{1/2})\eta^3/c_0 .$$

\(^3\)Note that an other family of TD pairs can similarly be obtained using instead the opposite coproduct $\Delta'$.

\(^4\)Here we use the notations of (4) for $\rho = \rho^*, \gamma = \gamma^* = 0, \eta = \eta^* = 0.$
Here, \( w^{(j)} = (q^{j+1/2} + q^{-j-1/2}) \) denotes the eigenvalue of the Casimir operator of \( U_{q^{1/2}(sl_2)} \) in the spin\( -j \) representation. In other words, the TD pair \( A^{(1)}, A^*(1) \) is a Leonard pair (see [9] for details). Consequently, one can not construct an independent quantity trilinear in \( A^{(1)}, A^*(1) \) in the spirit of [11], due to [11]. Apart from the Casimir operator of the AW algebra \( \mathfrak{h} \) (quartic in the Leonard pair), there are no other independent quantities commuting with \( T^{(1)}_n \) directly built from the TD pair.

For \( N = 2 \), we have checked explicitly that the TD pair \( A^{(2)}, A^*(2) \) does not satisfy the AW relations [11]. As before, we expect that an independent quantity \( T^{(2)}_2 \) of fifth order in \( A^{(2)}, A^*(2) \) does not exist. As will be shown in the next section, this fact relies on the existence of algebraic relations of fifth order in the TD pair. More generally, there exists a set of algebraic relations generalizing the AW ones responsible of the truncation of this integrable hierarchy. In the next Section we will propose a general procedure to derive all \( T^{(N)}_{2n-1} \) for \( n = 1, \ldots, N \), all algebraic relations generalizing [11] and the involution property of the charges \( T^{(N)}_{2n-1} \). It is based on the generalized quantum inverse scattering approach.

## 3 Generating function

In the spirit of the quantum inverse scattering method, we would like to construct an object which would provide the generating function of the above mentioned mutually commuting quantities \( T^{(N)}_{2n-1} \), \( n = 1, \ldots, N \). Following the results of [10], let us consider the following quadratic algebra (called reflection equation) which was introduced by Cherednik [14]:

\[
R(u/v) (K(u) \otimes I) R(uv) (I \otimes K(v)) = (I \otimes K(v)) R(uv) (K(u) \otimes I) R(u/v) .
\]

This equation arises, for instance, in the context of quantum integrable systems with boundaries [15]. We report the reader to the literature on the subject for more details. For our purpose, we restrict our attention to the trigonometric \( R \)-matrix \( R(u) \) which solves the Yang-Baxter equation. In the spin\( -\frac{1}{2} \) representation of \( U_{q^{1/2}(sl_2)} \), it reads

\[
R(u) = \sum_{i,j \in \{0,3,\pm\}} \omega_{ij}(u) \sigma_i \otimes \sigma_j ,
\]

where

\[
\omega_{00}(u) = \frac{1}{2} (q^{1/2} + 1)(u - q^{-1/2}u^{-1}) , \quad \omega_{33}(u) = \frac{1}{2} (q^{1/2} - 1)(u + q^{-1/2}u^{-1}) ,
\]

\[
\omega_{+-}(u) = \omega_{-+}(u) = q^{1/2} - q^{-1/2}
\]

and \( \sigma_j \) are Pauli matrices, \( \sigma_{\pm} = (\sigma_1 \pm i\sigma_2)/2 \). As shown in [15], the solution of [13] is not unique: a family of solutions of the reflection equation [13] can be obtained from an “initial” solution, say \( K^{(0)}(u) \), using the so-called “dressing” procedure. Indeed, given the Lax operator \( L(u) \) which satisfies the Yang-Baxter algebra

\[
R(u/v) (L(u) \otimes L(v)) = (L(v) \otimes L(u)) R(u/v)
\]

then one obtains a family of “dressed” reflection matrix for any parameter \( k \in \mathbb{C} \)

\[
K^{(N)}(u) = L_N(uk) \cdots L_1(uk) K^{(0)}(u) L_1(uk^{-1}) \cdots L_N(uk^{-1})
\]

acting on the quantum space \( V^{(N)} = \bigotimes_{j=1}^N V_j \) which also solves [13]. Following the analysis of [15], let us then introduce the elementary solution of the “dual” reflection equation\(^5\) given by

\[
K_+(u) = \begin{pmatrix} 
\frac{u q^{1/2}}{u - q^{-1/2}} & 0 \\
0 & \frac{u q^{1/2}}{u - q^{-1/2}} 
\end{pmatrix} ,
\]

\( ^5\)This equation is obtained from [13] by changing \( u \rightarrow u^{-1}, v \rightarrow v^{-1} \) and \( K(u) \) in its transpose.
where $\kappa^* \equiv \kappa^{-1}$ for any $\kappa \in \mathbb{C}$. Together with (16), it constitutes a basic element for the construction of mutually commuting quantities. Indeed, for any values of the spectral parameters $u, v$

$$[t^{(N)}(u), t^{(N)}(v)] = 0 \quad \text{for} \quad t^{(N)}(u) = tr_0 \{ K_+(u)K^{(N)}(u) \}$$

(18)

where $tr_0$ denotes the trace over the two-dimensional auxiliary space. We refer the reader to (15) for details. We are going to show that, given $N$ and the choice $K^{(0)}(u) \equiv (\sigma_+ / c_0 + \sigma_-) / (q^{1/2} - q^{-1/2})$, the set of $N$ mutually commuting quantities $\mathcal{T}_{2n-1}^{(N)}$, $n = 1, ..., N$ described in previous Section are generated from (15). In the following, we introduce a second spectral parameter $v = kq^{1/4}$ and we take the fundamental solution of the Yang-Baxter algebra ($L$–operator) in terms of $U_{2,2}(sl_2)$ generators:

$$L(u) = \left( \begin{array}{c}
q^{1/2} - u^{-1} q^{-1/2} \frac{S_+}{q^{1/2} - q^{-1/2}} \\
q^{1/2} - q^{-1/2} S_-
\end{array} \right) .$$

(19)

Let us assume

$$K^{(N)}(u) = \sum_{j \in \{0,3,\pm\}} \sigma_j \otimes Q_j^{(N)}(u)$$

(20)

where $Q_j^{(N)}(u) \in \mathcal{F}un(u; A^{(N)}, A^{*\,(N)})$ are Laurent polynomials of degree $-2N \leq d \leq 2N$ in the spectral parameter $u$, the TD pair $A^{(N)}, A^{*\,(N)}$ being subject to the tridiagonal relations (2). For $N = 1$, using the evaluation homomorphism (10) the result for $K^{(1)}(u)$ coincides with the - called “dynamical” - solution of the reflection equation proposed in (10) (see also (16)). Explicitly, in terms of the Leonard pair $A^{(1)}, A^{*\,(1)}$ it is given by (20) with

$$\begin{align*}
\Omega_0^{(1)}(u) + \Omega_3^{(1)}(u) &= uq^{1/2}A^{(1)} - u^{-1}q^{-1/2}A^{*\,(1)} , \\
\Omega_0^{(1)}(u) - \Omega_3^{(1)}(u) &= uq^{1/2}A^{*\,(1)} - u^{-1}q^{-1/2}A^{(1)} , \\
\Omega_+^{(1)}(u) &= q^{1/2}u^2 + q^{-1/2}u^{-2} + \omega \frac{[A^{*\,(1)}, A^{(1)}]}{c_0(q^{1/2} - q^{-1/2})^2} + \frac{\omega}{(q - q^{-1})} , \\
\Omega_-^{(1)}(u) &= q^{1/2}u^2 + q^{-1/2}u^{-2} + \frac{c_0[A^{(1)}, A^{*\,(1)}]}{q^{1/2} + q^{-1/2} + c_0\omega} \frac{\omega}{(q - q^{-1})} .
\end{align*}$$

(21)

From this expression, we deduce immediately $t^{(N=1)} = (q u^2 - q^{-1} u^{-2}) \mathcal{I}_1^{(1)}$ with

$$\mathcal{I}_1^{(1)} = \kappa A^{(1)} + \kappa^* A^{*\,(1)} ,$$

(22)

where the evaluation homomorphism (10) has been used. In particular, $\mathcal{I}_1^{(1)}$ is linear in the generators $A^{(1)}, A^{*\,(1)}$, as anticipated in the previous Section. There are no higher quantities $\mathcal{I}_n^{(2n-1)}$ for $n \geq 2$ generated in the expansion, a phenomena which relies on the existence of the AW relations (11) which follow from the reflection equation (10).

For $N = 2$, the quantities $\mathcal{I}_2^{(2)}, \mathcal{I}_3^{(2)}$ can also be exhibited in the expansion of $K^{(2)}(u)$. To show that, we first use eqs. (4), (6) and (8) to derive

$$[A^{(2)}, [A^{(2)}, A^{*\,(2)}]] = [A^{(1)}, [A^{(1)}, A^{*\,(1)}]]q^{-1} \otimes 1 + q^h \otimes [A^{(1)}, [A^{(1)}, A^{*\,(1)}]]q^{-1} +$$

$$+ \frac{(1 - q^{-1})}{c_0} Q_+ + (1 - q) Q_1 + q^h \otimes [A^{(1)}, [A^{(1)}, A^{*\,(1)}]]q^{-1} +$$

$$+ \frac{(1 - q^{-1})}{c_0} Q_+ + (1 - q^{-1}) Q_1 + q^h \otimes [A^{(1)}, [A^{(1)}, A^{*\,(1)}]]q^{-1} +$$

$$- \frac{(q - q^{-1})}{c_0} (Q_+ Q_1 - q^{-1} Q_1 Q_+) q^{-1} \otimes [A^{(1)}, [A^{(1)}, A^{*\,(1)}]]q^{-1} +$$

$$+ \frac{(1 - q^{-1})}{c_0} (Q_+ Q_1 + Q_1 Q_+ + (1 - q^{-1})(Q_+ + Q_1) q^{-1} \otimes [A^{(1)}, [A^{(1)}, A^{*\,(1)}]]q^{-1} ,$$

(23)
and similarly for \([A^{(2)}, [A^{(2)}, A^{(2)}]]_q\). Exchanging \(q \rightarrow q^{-1}\), \(A^{(1)} \leftrightarrow A^{(1)}\), \(Q_+ \leftrightarrow Q^+_+\), and \(Q_- \leftrightarrow Q_-\) in the above expression. Using the evaluation homomorphism \((10)\), we replace the AW relations \((11)\) in the two first terms of \(\Omega\). A straightforward calculation finally shows that the diagonal entries of \(K^{(2)}(u)\) can be written in terms of \((2\)). We obtain for instance

\[
\Omega^{(2)}_0(u) + \Omega^{(2)}_3(u) = (qu^3 + qu^{1/2}f(v) + u^{-1})A^{(2)} - (q^{-1}u^{-3} + u^{-1}q^{-1/2}f(v) + u)A^{(2)} \nonumber
\]

\[
+ uq^{1/2} \frac{c_0}{q^{1/2} + q^{-1/2}} \left( -[A^{(2)}, [A^{(2)}, A^{(2)}]]_q \right) + \frac{(q^{1/2} + q^{-1/2})^2}{c_0} A^{(2)} \nonumber
\]

\[
- u^{-1}q^{-1/2} \frac{c_0}{q^{1/2} + q^{-1/2}} \left( -[A^{(2)}, [A^{(2)}, A^{(2)}]]_q \right) + \frac{(q^{1/2} + q^{-1/2})^2}{c_0} A^{(2)} ,
\]

where

\[
f(v) = - \frac{2(v^2q^{-1/2} + v^{-2}q^{1/2})w^{(j)}}{q^{1/2} + q^{-1/2}} .
\]

The expression for \(\Omega^{(2)}_0(u) - \Omega^{(2)}_3(u)\) is derived similarly, and corresponds to the substitution \(A^{(2)} \leftrightarrow A^{(2)}\) in the above equation. The mutually commuting quantities can now be read off from the coefficients of the expansion in the spectral parameter \(u\) of \((13)\). Taking the trace over the auxiliary space of \((20)\) for \(N = 2\) together with the expression for \(\Omega^{(2)}\), we find

\[
t^{(2)}(u) = (q^{3/2}u^4 - q^{-3/2}u^{-4} + C(u^2, u^{-2}; v))I^{(2)}_1 + \frac{c_0(2q^2 - 2u^2 - 2)}{q^{1/2} + q^{-1/2}} I^{(2)}_3,
\]

where \(I^{(2)}_1, I^{(2)}_3\) coincide exactly with \((17)\) for \(N = 2\) and the Laurent polynomial \(C(u, u^{-1}; v)\) directly follows from \((24)\). The property \((18)\) implies that the two quantities \(I^{(2)}_1, I^{(2)}_3\) are mutually commuting. This provides an alternative derivation to the one proposed in previous Section which is solely based on the fact that \(A^{(2)}, A^{(2)}\) satisfy the tridiagonal algebraic relations \((2)\).

This technical procedure can be clearly applied for general values of \(N\). In this case, it is important to notice that the leading terms in the asymptotic \(u \rightarrow \infty\) yields to

\[
u^{2N} q^{(1-N)/2} K^{(N)}(u)|_{u \rightarrow \infty} = \left( \frac{q^{1/2}uA^{(N)}}{q^{1/2}uA^{(N)}}, \frac{q^{1/2}u^2}{uA^{(N)}}, \frac{q^{1/2}u^2}{uA^{(N)}}, \frac{q^{1/2}u^2}{uA^{(N)}} \right) + O(u^{-2}) .
\]

Similarly, the asymptotic \(u \rightarrow -\infty\) gives an analogous expression exchanging \(A^{(N)} \leftrightarrow A^{(N)}\). We immediately recognize the TD pair defined in \((5)\) which obeys \((2)\) as shown in previous Section. Then, the mutually commuting quantities \((19)\) are obtained from the expansion of \(t^{(N)}(u)\) in the spectral parameter \(u\) which can be formally written

\[
t^{(N)}(u) = \sum_{n=1}^{N} C^{(N)}_{2n-1}(u^2; u^{-2n}, ...; u^2, u^{-2}) I^{(N)}_{2n-1},
\]

where \(C^{(N)}_{2n-1}\) are Laurent polynomials in \(u, v\). According to \((13)\), they are in involution i.e. \([I^{(N)}_{2n-1}, I^{(N)}_{2m-1}] = 0\) for any \(n, m \geq 1\). Such property is actually encoded in the tridiagonal relations \((2)\). It should be stressed that the above approach does not depend on the choice of a representation for \((10)\). Consequently, finite, infinite dimensional or cyclic representations of TD pairs and related quantities like \(I^{(N)}_{2n-1}\) can be easily obtained.

### 4 Related spectral problem

In the context of quantum integrable systems (see next Section), an important problem is the diagonalization of \((24)\) which leads to study the spectral problem associated with \((10)\) that we shall briefly discuss now. We write it as

\[
I^{(N)}_{2n-1} \Psi^{(N)}(z_1, ..., z_N) = \Delta^{(N)}_{2n-1} \Psi^{(N)}(z_1, ..., z_N) , \quad \text{for} \quad n = 1, ..., N ,
\]

\[
(28)
\]
where $\Psi_n^{(N)}(z_1, \ldots, z_N)$ denote the eigenfunctions. For $q$ not a root of unity and general values $N$, we expect that this problem can be “algebraized”\textsuperscript{6} i.e. part of the spectrum corresponds to multivariable polynomial eigenfunctions. Here we do not attempt to study in details the corresponding system of $N$ partial $q$–difference equations, which goes beyond the scope of this paper. Instead, let us focus on the case $N = 1$ which can be solved easily following [17] (see also [10]). Expressed in the weight basis, the generators of $U_{q^{1/2}}(sl_2)$ leave invariant the linear space (of $2j + 1$ dimension) of polynomials $F(z)$ of degree $2j$. The lowest/highest weights are identified with $F_0 = 1$ and $F_{2j} = z^{2j}$, respectively. One has

$$
q^{k_s/2}F(z) = q^{-j/2}F(q^{-k/2}z),
$$

$$
S_+F(z) = -\frac{z}{(q^{1/2} - q^{-1/2})(q^{-j}F(q^{1/2}z) - q^jF(q^{-1/2}z))},
$$

$$
S_-F(z) = \frac{1}{z(q^{1/2} - q^{-1/2})(F(q^{1/2}z) - F(q^{-1/2}z))}.
$$

In this basis of single-variable polynomials, the spectral problem for (9) with $j$ to which corresponds exactly 2

$$
a^{(1)}(z; v)\Psi(qz) + d^{(1)}(z; v)\Psi(q^{-1}z) - v^{(1)}(z; v)\Psi(z) = \Lambda^{(1)}_1(\Psi(z)).
$$

Indeed, we should keep in mind that - at this special value of $N$ - all the higher charges can be written in terms of the first one due to the AW relations [11]. In (29), the coefficients are given by

$$
a^{(1)}(z; v) = \frac{c_0v^{-1}q^{-j/2}z^{-1} - \kappa v^{-3j/2}z}{c_0(q^{1/2} - q^{-1/2})},
$$

$$
d^{(1)}(z; v) = \frac{\kappa^*v^{-1}q^{3j/2}z - c_0\kappa^*vq^{j/2}z^{-1}}{c_0(q^{1/2} - q^{-1/2})},
$$

$$
v^{(1)}(z; v) = \frac{(c_0v^{-1}q^{-j/2} - c_0\kappa^*vq^{j/2}z^{-1} + (\kappa^*v^{-1}q^{-j/2} - \kappa vq^{j/2})z}{c_0(q^{1/2} - q^{-1/2})}
$$

and

$$
\Psi(z) = \prod_{m=1}^M (z - z_m),
$$

where $z_m$ denote the roots of the polynomial. Dividing (29) by (31), the l.h.s. of (29) gives a meromorphic function of $z$ and the r.h.s. becomes a constant. Singularities of the l.h.s must be cancelled. They are located at $z = 0, \infty$ and $z = z_m$. For $z = 0$, (29) vanishes identically. For $z = \infty$ and general values of $\kappa, \kappa^*$ on gets the constraint $M = 2j$. For $z = z_m$, one finds the following system of Bethe equations:

$$
a(z_l) = \frac{M}{M} \prod_{m=1, m \neq l}^{2j} \frac{q^{-1}z_l - z_m}{q^{-1}z_l - z_m} \quad \text{for} \quad l = 1, \ldots, 2j
$$

(32)

to which corresponds exactly $2j + 1$ polynomial eigenfunctions. Comparing the constant terms of both sides in (29) one obtains

$$
\Lambda^{(1)}_1 = -\left(\kappa^*v^{-1}q^{-j/2+1/2} + \kappa vq^{j/2-1/2}\right) \sum_{m=1}^{2j} \frac{z_m}{c_0}.
$$

(33)

Let us also mention an other way to solve the spectral problem (29) for $N = 1$, which exhibits an interesting link with Askey-Wilson $q$–orthogonal polynomials. These Laurent polynomials are symmetric in the variable $y$ and defined by

$$
P_n(y) = 4\Phi_3 \left( q^{-n}, abcdq^{n-1}, ay, ay^{-1}; q, q \right)
$$

(34)

\textsuperscript{6}We use the definition of [12].
where \( \Phi_3 \) denotes the basic \( q \)--hypergeometric function, \( n \) is an integer and \( a, b, c, d \) are arbitrary parameters. In particular, the zeros of the AW polynomials are determined by the system of Bethe equations \(^1\)

\[
\frac{(y_k - a)(y_k - b)(y_k - c)(y_k - d)}{(ay_k - 1)(by_k - 1)(cy_k - 1)(dy_k - 1)} = \prod_{l=1, l \neq k}^{n} \frac{(qy_k - y_l)(qy_ky_l - 1)}{(y_k - qy_l)(y_ky_l - q)} \quad \text{for} \quad k = 1, \ldots, n .
\]

As shown in \(^3\) (see also \(^4\) for finite dimensional representations), the Leonard pair \( A^{(1)}, A^{* (1)} \) can be written in this basis. Introducing the so-called Askey-Wilson second order \( q \)--difference operator

\[
\mathcal{D} = \xi(y)(\tau - I) + \xi(y^{-1})(\tau^{-1} - I) + (1 + abcdq^{-1})I
\]

where \( \tau(y) = qy \) and

\[
\xi(y) = \frac{(1 - ay)(1 - by)(1 - cy)(1 - dy)}{(1 - y^2)(1 - qy^2)},
\]

one can define the representation \( \pi' \) such that \(^3\)

\[
\pi'[A^{(1)}] \sim (y + y^{-1})P_n(y) \quad \text{and} \quad \pi'[A^{* (1)}] \sim \mathcal{D}P_n(y) .
\]

Using the explicit form \(^5\) in \(^6\), setting \( abcd = q, c_0 = 1 \) and rescaling the parameters \( \kappa, \kappa^* \) one obtains a second order \( q \)--difference equation of the form

\[
\kappa^*\xi(y)P_n(qy) + \kappa^*\xi(y^{-1})P_n(q^{-1}y) - (\kappa^*(\xi(y) + \xi(y^{-1}) - \kappa(y + y^{-1}) - 2))P_n(y) = \Lambda_1^{(1)}P_n(y) .
\]

Similarly to the case discussed above, the eigenvalues \( \Lambda_1^{(1)} \) can be written in terms of the zeros \( y_1, y_l^{-1} \) of the AW polynomials determined by \(^7\).

As both representations are available, let us consider the first (finite dimensional representation) \(^8\). Then, it is worth important to notice that the Bethe equations \(^9\) as well as the eigenvalue \( \Lambda_1^{(1)} \) are invariant under the symmetry transformation

\[
q \leftrightarrow q^{-1}, \quad v \leftrightarrow v^{-1}, \quad \kappa \leftrightarrow \kappa^* .
\]

Although we will not discuss this remarkable symmetry for general values of \( N \), we expect it will be preserved for higher values of \( N \) too.

## 5 Examples

For several quantum completely integrable models, it is not difficult to construct non-local integrals of motion in terms of \( I_{2n-1}^{(N)} \). Below, we list some examples (see also \(^1\)).

### Integrable (XXZ) spin chain models

In the context of Sklyanin formalism, integrable boundary conditions for a system of finite size are encoded in the representations of the \( K \)--matrix \( K^{(N)}(u) \). From previous analysis, the problem is reduced to classify all possible (tensor product) representations of \( A^{(N)}, A^{* (N)} \) associated with the tridiagonal algebra \(^2\). For \( q \) not a root of unity, the representations \(^1\) and \(^2\) are proposed.

All known examples of integrable boundary conditions can be indeed seen as special cases of \(^2\). For instance, let us denote \( \pi^{(j)} \) the spin--\( j \) irreps. of \( U_{q^{1/2}}(sl_2) \) with \(^1\). For \( N = 1 \) and the trivial representation we can define \( \pi^{(j=0)}[A^{(1)}] = \epsilon_+ \), \( \pi^{(j=0)}[A^{* (1)}] = \epsilon_- \). Then \( (id \times \pi^{(j=0)})[K^{(1)}(u)] \) coincides with the non-diagonal solution of the reflection equation for the open XXZ spin chain with \( M \) sites and non-diagonal boundary conditions found in \(^2\). The method of deriving the “boundary quantum group” generators \(^2\) proposed in \(^2\) was recently applied to this model \(^2\). It is then easy to see that the so-called “boundary quantum group” is actually the tridiagonal algebra \(^2\), the boundary generators being nothing but a TD pair of the form \( \tilde{A}^{(M)}, \tilde{A}^{* (M)} \) in the spin--\( \frac{1}{2} \) representation. According to Section 2, this model possesses \( M \) mutually conserved quantities given by
for $\kappa = \kappa^* = 1$. Interestingly, the corresponding spectral problem gives a system of $M$-partial $q$–difference equations. For $N = 2$, the $K$–matrix coincides with the one derived in $[10]$. One can then construct a XXZ spin chain coupled with a quantum mechanical system $[22]$ associated with $K^{(2)}(u)$, in which case the dynamical boundary conditions are associated with the AW relations. For general values of $N$, and any number of sites $M$, it becomes clear that one obtains a XXZ open spin chain coupled with $N$–type dynamical boundary conditions taking $K^{(M+N)}(u)$ in $[15]$. Obviously, the examples above do not exhaust all possibilities. Indeed, starting from a different representation for the monodromy matrix satisfying $[14]$ one gets new quantum integrable models which share the same underlying symmetry algebra $[24]$. Consequently, from a general point of view this family of systems with $N$ degrees of freedom possesses

- Local integrals of motion derived from the expansion $\ln (t(\exp(\lambda))) = \sum_n \mathcal{H}_n \lambda^n$;
- Non-local integrals of motion derived from the expansion $[27]$.

Applying the first relation in $[18]$, it follows that they are mutually commuting. The existence of such integrable structure of the form $[9]$, besides the known local integrals of motion, allows us to say that these models are superintegrable.

- **Integrable field theory:** Although the construction of Section 3 can not be directly applied to the case $N \to \infty$, it is however possible to identify the conserved quantities $[9]$ in known quantum integrable models. For instance, as shown in $[23]$ the sine-Gordon model has a $U_q(\hat{sl}_2)$ symmetry generated by non-local conserved charges (up to an overall scalar factor) $Q_\pm, \bar{Q}_\pm$ satisfying the defining relations $[9]$. In this model, the particle spectrum consists of soliton/anti-soliton and breathers. Usually, each particle is associated with an asymptotic scattering state $|\theta; m\rangle$ with rapidity $\theta$ and topological charge $2m$ where $-j \leq m \leq j$. In this representation, the charges act as $[(14)\ becoming $v = \exp((2/\beta^2 - 1)\theta)$ with coupling constant $\beta^2$. On a general asymptotic $p$–particle states, they act as $\Delta^{(p)}(Q_\pm), \Delta^{(p)}(\bar{Q}_\pm)$ $[23]$. Following $[11]$ and the results of Section 2, the sine-Gordon model or its boundary version (for $c_0 = 1$) $[21]$ possess $N \to \infty$ non-local conserved charges of form $[9]$ in involution. The corresponding spectral problem is, in general, rather complicated. However, acting on a $p$–particle state the spectral problem associated with $[9]$ truncates to the one associated with the conserved charges $\mathcal{I}_{2n-1}^{(\infty)}$, $n = 1, ..., p$. In particular, for a single $1$–particle state the corresponding eigenstates are expressed in terms of Askey-Wilson $q$-orthogonal polynomials.

- **The limit $q \to 1$ and the Onsager algebra:** For $q = 1$ and $c_0 = 1$, the structure $[9]$ becomes isomorphic to the Dolan-Grady one $[4]$, with a trivial coproduct. In the homogeneous gradation of $[4]$, it is easy to show that the loop algebra $\hat{sl}_2$ occurs explicitly as $A \to E_+ + E_-$ and $A^* \to tE_+ + t^{-1}E_-, t \in \mathbb{C}$, with $[E_+, E_-] = 2H$, $[H, E_\pm] = \pm E_\pm$. In terms of the Onsager algebra $[25]$ generators $A_n, G_n$, $n \in \mathbb{Z}$ one has $[12] A_m = 2t^m S_+ + 2t^{-m}S_-$ which gives the identification $A \to A_0/2$ and $A^* \to A_1/2$. It follows that the AW relations $[11]$ in this limit become

\[
\frac{1}{8}[A_0, [A_0, A_1]] = A_1 - A_{-1} \quad \text{and} \quad \frac{1}{8}[A_1, [A_1, A_0]] = A_0 - A_2.
\]

(40)

Higher order relations are similarly generated. Also, in this limit it is easy to show that the mutually commuting quantities $[9]$ coincide with the well-known ones $[20]$. As we mentioned in the previous Section, for $N = 1$ the eigenfunctions for $[23]$ are related with the Askey-Wilson $q$–orthogonal polynomials $[44]$. In the limit $q = 1$, these eigenfunctions reduce to their “classical” analogue, i.e. Wilson, Jacobi, Racah,.., polynomials. In this context, it is probably not surprising that the Jacobi polynomials arise $[27]$ in the study of the spectrum of the chiral Potts model $[28]$ at the superintegrable point $[20]$. Finally, let us point out that the integrable models discussed above (XXZ, sine-Gordon) at this special point $q = 1$ enjoy the Onsager symmetry algebra.

6 Concluding remarks

In this paper, we have proposed a generalization of the Dolan-Grady construction $[4]$ based on the tridiagonal algebraic relations $[2]$. This construction was possible due to the realization of the TD pair in terms of tensor products of $U_{q^{1/2}}(\hat{sl}_2)$, which plays a crucial role in the analysis. It was argued that the corresponding finite set of mutually commuting quantities can be either derived by imposing the algebraic structure $[2]$, or generated from solutions $[10]$ of the reflection equation. Finally, the tridiagonal algebraic symmetry has been exhibited in
various examples of quantum integrable models with finite $N$ (or infinite) degrees of freedom, which are known to possess already $N$ local conserved charges. Having identified the fundamental generators $A^{(N)}, A^{*(N)}$ and the corresponding $N$ (or infinite) conserved quantities to non-local charges \( g \) in involution, we conclude that these models are superintegrable.

The next step is the explicit construction of the corresponding $q$–Onsager symmetry algebra in the spirit of \([30, 26]\) that we leave for further investigation. Let us mention that beyond the explicit construction of the mutually commuting quantities $T_{2n-1}^{(N)}$ with $1 \leq n \leq N$, a new (finite) set of algebraic relations generalizing the Askey-Wilson ones \([11]\) can be derived directly from \([13]\). Given $N$, these relations are responsible of the truncation of the integrable hierarchy i.e. any quantity $T_{2n-1}^{(N)}$ for $n \geq N+1$ can be expressed in terms of $T_{2n}^{(N)}$ with $n \leq N$. For $N = 2$, it is not difficult to obtain such algebraic relations satisfied by $A^{(2)}, A^{*(2)}$ using the explicit form of the off-diagonal entries of $K^{(2)}(u)$. We intend to discuss these relations, as well as the related spectral problem which arises in the classification of multivariable $q$–orthogonal polynomials separately.

To conclude, as the Dolan-Grady relations arise explicitly in various problems \([29, 31, 32]\) we believe our construction will provide a new tool in order to derive exact results in massive quantum integrable models. In this direction, it would be interesting to find the extension of the Bazhanov-Lukyanov-Zamolodchikov program \([33]\).

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