REPRESENTATION THEOREMS FOR INDEFINITE QUADRATIC FORMS
REVISITED

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ABSTRACT. The first and second representation theorems for sign-indefinite, not necessarily semi-bounded quadratic forms are revisited. New straightforward proofs of these theorems are given. A number of necessary and sufficient conditions ensuring the second representation theorem to hold is proved. A new simple and explicit example of a self-adjoint operator for which the second representation theorem does not hold is also provided.

1. INTRODUCTION

In this work we revisit the representation theorems for sign-indefinite, not necessarily semi-bounded symmetric sesquilinear forms. Let $\mathcal{H}$ be a complex Hilbert space with the inner product $\langle \cdot, \cdot \rangle$. We will be dealing with the class of forms given by

$$b[x, y] = \langle A^{1/2}x, HA^{1/2}y \rangle, \quad x, y \in \text{Dom}[b] = \text{Dom}(A^{1/2}),$$

where $A$ is a positive definite self-adjoint operator in a Hilbert space $\mathcal{H}$, and $H$ is a bounded, not necessarily positive, self-adjoint operator in $\mathcal{H}$. In the context of perturbation theory, such forms arise naturally, when the initial (in general sign-indefinite) form $\langle A^{1/2}x, JAA^{1/2}y \rangle$, with $J$ a self-adjoint involution commuting with $A$, is perturbed by a form $v$ satisfying the upper bound

$$|v[x, y]| \leq \beta |\langle A^{1/2}x, A^{1/2}y \rangle|, \quad x, y \in \text{Dom}(A^{1/2}),$$

for some $\beta > 0$, so that the sesquilinear form $b[x, y] := \langle A^{1/2}x, JAA^{1/2}y \rangle + v[x, y]$ can be transformed into the expression given by (1.1) for some self-adjoint bounded operator $H$. In particular, Dirac-Coulomb operators fit into this scheme [23]. We also mention an alternative approach to Dirac-like operators developed recently by Esteban and Loss in [6].

In this setting, in the framework of a unified approach, we provide new straightforward proofs of the following two assertions (Theorems 2.3 and 2.10, respectively):

(i) If $H$ has a bounded inverse, then there is a unique self-adjoint boundedly invertible operator $B$ with $\text{Dom}(B) \subset \text{Dom}[b]$ associated with the form $b$, that is,

$$b[x, y] = \langle x, By \rangle \quad \text{for all } x \in \text{Dom}[b], \quad y \in \text{Dom}(B) \subset \text{Dom}[b].$$

(ii) If, in addition, the domains of $|B|^{1/2}$ and $A^{1/2}$ agree, that is,

$$\text{Dom}(|B|^{1/2}) = \text{Dom}(A^{1/2}),$$

then the form $b$ is represented by $B$ in the sense that

$$b[x, y] = \langle |B|^{1/2}x, \text{sign}(B)|B|^{1/2}y \rangle \quad \text{for all } x, y \in \text{Dom}[b] = \text{Dom}(|B|^{1/2}),$$

with $\text{sign}(B)$ the sign of the operator $B$.

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Representations (1.3) and (1.5) are natural generalizations to the case of indefinite forms of the First and Second Representation Theorems for semi-bounded sesquilinear forms, Theorems VI.2.1 and VI.2.23 in [15], respectively. We remark that the existence and uniqueness of the pseudo-Friedrichs extension for symmetric operators [15, Section VI.4], [7, Section IV.4] is a particular case of this result.

A proof of the First Representation Theorem (i) for indefinite forms can be found in [23, Theorem 2.1]. Results equivalent to (1.3) have been obtained by McIntosh in [21], [22] where, in particular, the notion of a closedness to the case of indefinite forms has been extended. It is worth mentioning that the form $b$ given by (1.1) is closed in that sense (see Remark 2.8).

The Second Representation Theorem (ii) for indefinite forms, is originally also due to McIntosh [21], [22]. He also established that the form-domain stability criterion (1.4) is equivalent to the requirement that $\text{sign}(B)$ leaves $\text{Dom}(A^{1/2})$ invariant. We remark that if $B$ is a semi-bounded operator, the condition (1.4) holds automatically (cf. [15, Theorem VI.2.23]).

New proofs of the Representation Theorems (i) and (ii) given in the present work are straightforward and based on functional-analytic ideas similar to those used to prove The Representation Theorems in the semi-bounded case (cf. [15, Section VI.2]). Related results, in particular those concerned with the so-called quasi-definite matrices and operators are contained in [24] and, quite recently, in [25].

Our new results related to the Representation Theorems (i) and (ii) are as follows. As a consequence of (i), we prove the First Representation Theorem for block operator matrices defined as quadratic forms, provided that the diagonal part of the matrix has a bounded inverse and that the off-diagonal form perturbation is relatively bounded with respect to a closed positive definite form determined by the diagonal entries of the matrix (Theorem 2.5 below). This result provides a far reaching generalization of the one obtained previously by Konstantinov and Mennicken in [17].

In this context, we also revisit the Lax-Milgram theory for coercive closed forms (cf., [7, Theorem IV.1.2]) and show that the coercivity hypothesis yields the representation (1.1) (Theorem 2.7 below).

With regard to the Second Representation Theorem (ii), we obtain a number of new necessary and sufficient conditions for coincidence of the domains $\text{Dom}(|B|^{1/2})$ and $\text{Dom}(A^{1/2})$ (Theorem 3.2 below). Answering a question raised by A. McIntosh in [22], we also provide a simple and explicit example (Example 2.11) of a form $b$, that is 0-closed in the sense of McIntosh (see Remark 2.8), but not represented by its associated operator $B$. Consequently, the Second Representation Theorem does not hold if the condition (1.4) is violated. In particular, we show that the $A$-form boundedness of the operator $B$ does not yield that of its absolute value $|B|$.

An alternative approach to the Representation Theorems (i) and (ii) for indefinite sesquilinear forms has been developed in [10], [11], [12] by Fleige, Hassi, and de Snoo in the framework of the Krein space theory. Their results extended the list of criteria equivalent to (1.4). In particular, it has been proven that the condition (1.4) holds if and only if infinity is not a singular critical point (see [20] for a discussion of this notion) for the range restriction $B_a$ in the Krein space $(\mathcal{K}, [\cdot, \cdot])$ of the operator $B$. Here $\mathcal{K} := \text{Dom}(A^{1/2})$, $\text{Dom}(B_a) = [x, y] := b[x, y]$ is an indefinite inner product on $\mathcal{K}$, and

\begin{equation}
B_a = B|_{\mathcal{K}} \quad \text{on} \; \text{Dom}(B_a) = \{ x \in \text{Dom}(B) \subset \mathcal{K} \mid Bx \in \mathcal{K} \}.
\end{equation}

A number of necessary and sufficient conditions for the regularity of the critical point infinity has been discovered by Ćurgus in [5]. The existence of operators in a Krein space with a singular critical point at infinity is established in [8], [9] and [26]. We remark that, by Proposition 5.3 in [11], the existence of such operators implies the existence of a Hilbert space $\mathcal{H}$ and a symmetric sesquilinear form $b$ on it such that the condition (1.4) does not hold for the associated (by the First Representation
Theorem) operator $B$ (see Example 5.4 in [11]). In this context, as a by-product of our considerations, the range restriction $B_0$ of the operator $B$ in the Krein space $(\mathcal{K}, [\cdot, \cdot])$ constructed in Example 2.11 below provides a new fairly simple example of an operator having infinity as a critical singular point (see Remark 2.12).

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2. Representation Theorems

Hypothesis 2.1. Assume that $A$ and $H$ are self-adjoint operators in the Hilbert space $\mathcal{H}$. Suppose that

(i) $\inf \text{spec}(A) > 0$;

(ii) $H$ is bounded and has a bounded inverse;

(iii) the open interval $(h_-, h_+)$ is a maximal spectral gap of the operator $H$ containing 0.

The following lemma introduces a self-adjoint operator naturally associated with the operators $A$ and $H$ from Hypothesis 2.1.

Lemma 2.2. Assume Hypothesis 2.1. Then the operator

$$B := A^{1/2} H A^{1/2}$$

on the domain

$$\text{Dom}(B) = \{ x \in \text{Dom}(A^{1/2}) \mid H A^{1/2} x \in \text{Dom}(A^{1/2}) \}$$

is self-adjoint with a bounded inverse.

Proof. Introducing the bounded self-adjoint operator

$$S := A^{-1/2} H^{-1} A^{-1/2},$$

we observe that $S$ has a trivial kernel and, hence, its inverse is a self-adjoint operator. It remains to note that $S^{-1} = B$. \qed

2.1. The First Representation Theorem. Under Hypothesis 2.1 consider the symmetric sesquilinear form $b$ on $\text{Dom}[b] = \text{Dom}(A^{1/2})$ defined by

(2.2) $b[x, y] = \langle A^{1/2} x, H A^{1/2} y \rangle, \quad x, y \in \text{Dom}[b] = \text{Dom}(A^{1/2})$.

Theorem 2.3. Assume Hypothesis 2.1 and suppose that $b$ is a symmetric sesquilinear form given by (2.2).

Then the operator $B$ referred to in Lemma 2.2 is a unique self-adjoint operator associated with the form $b$ in the sense that

(2.3) $b[x, y] = \langle x, By \rangle$ for all $x \in \text{Dom}[b], \quad y \in \text{Dom}(B) \subset \text{Dom}[b]$,

with $\text{Dom}(B)$ a core for $A^{1/2}$. Moreover, the open interval $(\alpha h_-, \alpha h_+)$, with $\alpha = \min \text{spec}(A)$, belongs to the resolvent of set $B$. 
Proof. By Lemma 2.2 the operator \( B = A^{1/2} H A^{1/2} \) on
\[
\text{Dom}(B) = \{ x \in \text{Dom}(A^{1/2}) \mid H A^{1/2} x \in \text{Dom}(A^{1/2}) \}
\]
is self-adjoint. It follows that
\[
b[x, y] = \langle A^{1/2} x, H A^{1/2} y \rangle = \langle x, A^{1/2} H A^{1/2} y \rangle = \langle x, By \rangle
\]
for all \( x \in \text{Dom}(A^{1/2}), y \in \text{Dom}(B) \), thereby proving the representation (2.3).

To prove that \( \text{Dom}(B) \) is a core for \( A^{1/2} \) we assume that \( \langle y, A^{1/2} x \rangle_{\mathcal{H}} = 0 \) for some \( y \in \mathcal{H} \) and for all \( x \in \text{Dom}(B) \). Since \( \text{Dom}(B) = \text{Ran}(A^{-1/2} H^{-1} A^{-1/2}) \) one arrives at the conclusion that \( \langle y, H^{-1} A^{-1/2} z \rangle = 0 \) for all \( z \in \mathcal{H} \). Thus, \( y = 0 \), since \( \text{Ran}(A^{-1/2}) = \text{Dom}(A^{1/2}) \) is dense in \( \mathcal{H} \) and \( H \) is an isomorphism. Hence, \( \text{Dom}(B) \) a core for \( A^{1/2} \).

Now we turn to the proof of the uniqueness. Assume that there exists a self-adjoint operator \( B' \) with \( \text{Dom}(B') \subset \text{Dom}[b] \) such that
\[
\langle x, B'y \rangle = b[x, y] \quad \text{for all} \quad x \in \text{Dom}(b), \ y \in \text{Dom}(B').
\]
Then
\[
\langle x, B'y \rangle = b[x, y] = \overline{b[y, x]} = \langle y, Bx \rangle = \langle Bx, y \rangle
\]
holds for all \( x \in \text{Dom}(B) \) and \( y \in \text{Dom}(B') \) which means that \( B' = B^* \). Since \( B \) is self-adjoint, we get \( B' = B \).

To complete the proof of the theorem it remains to show that the open interval \((\alpha h_-, \alpha h_+) \supseteq 0 \) belongs to the resolvent set of the operator \( B \). To this end we consider a family of shifted quadratic forms
\[
b_{\lambda}[x, y] := b[x, y] - \lambda \langle x, y \rangle = \langle A^{1/2} x, (H - \lambda A^{-1})A^{1/2} x \rangle, \quad \lambda \in (\alpha h_-, \alpha h_+),
\]
with \( \text{Dom}[b_{\lambda}] = \text{Dom}[b] \). Observe that \( H_{\lambda} := H - \lambda A^{-1} \) is bounded and has a bounded inverse if \( \lambda \in (\alpha h_-, \alpha h_+) \). Indeed, the second resolvent identity implies that
\[
H_{\lambda}^{-1} = H^{-1} + \lambda H^{-1} A^{-1/2} (I - \lambda A^{-1/2} H^{-1} A^{-1/2})^{-1} A^{-1/2} H^{-1}
\]
holds as long as \( I - \lambda A^{-1/2} H^{-1} A^{-1/2} \) is boundedly invertible. If the open interval \((h_-, h_+) \supseteq 0 \) belongs to the resolvent set of the operator \( H \), then
\[
h_{-}^{-1} I \leq H^{-1} \leq h_{+}^{-1} I.
\]
Hence, we obtain the following bounds:
\[
\langle x, A^{-1/2} H^{-1} A^{-1/2} x \rangle = \langle A^{-1/2} x, H^{-1} A^{-1/2} x \rangle \leq h_{+}^{-1} ||A^{-1/2} x||^2 \leq \frac{1}{\alpha h_+}
\]
and
\[
-\langle x, A^{-1/2} H^{-1} A^{-1/2} x \rangle = -\langle A^{-1/2} x, H^{-1} A^{-1/2} x \rangle \leq -h_{-}^{-1} ||A^{-1/2} x||^2 \leq -\frac{1}{\alpha h_-}.
\]
Combining these bounds we arrive at the following two-sided operator inequality
\[
\left( 1 + \frac{\lambda}{\alpha h_-} \right) I \leq I - \lambda A^{-1/2} H^{-1} A^{-1/2} \leq \left( 1 - \frac{\lambda}{\alpha h_+} \right) I,
\]
which shows that the operator \( I - \lambda A^{-1/2} H^{-1} A^{-1/2} \) is boundedly invertible whenever \( \lambda \in (\alpha h_-, \alpha h_+) \).

By the preceding arguments, there is a unique self-adjoint boundedly invertible operator \( B_{\lambda} \) with \( \text{Dom}(B_{\lambda}) \subset \text{Dom}[b] \) such that
\[
\langle x, B_{\lambda} y \rangle = b_{\lambda}[x, y] \quad \text{for all} \quad x \in \text{Dom}[b], \ y \in \text{Dom}(B_{\lambda}).
\]
Clearly, \( B_{\lambda} = B - \lambda I, \lambda \in (\alpha h_-, \alpha h_+) \), and, hence, the interval \((\alpha h_-, \alpha h_+) \) belongs to the resolvent set of the operator \( B \), for \( B_{\lambda} \) with \( \lambda \in (\alpha h_-, \alpha h_+) \) has a bounded inverse. This completes the proof. \( \square \)
Remark 2.4. Note that the operator $B$ referred to in Theorem 2.3 is $A$-form bounded, that is,

$A^{-1/2}BA^{-1/2} = H \in B(\mathcal{S})$,  

where the bar denotes the closure of the operator $A^{-1/2}BA^{-1/2}$ defined on $D := A^{1/2}(\text{Dom}(B))$.

Indeed, $A^{-1/2}BA^{-1/2}$ defined on $D$ coincides with $H|_D$. Since $H$ is an isomorphism, from the representation

$A^{-1/2}BA^{-1/2} = A^{-1/2}A^{-1/2}H^{-1}A^{-1/2} = H^{-1}A^{-1/2}$

it follows that the set $D$ is dense in $\mathcal{S}$. Therefore, (2.4) holds.

2.2. Applications to the case of off-diagonal form perturbations.

Theorem 2.5. Let $a$ be a positive definite closed symmetric sesquilinear form on $\text{Dom}[a]$ in a Hilbert space $H$ with the greatest lower bound $\alpha > 0$ and $v$ a symmetric $a$-bounded form on $\text{Dom}[v] \supset \text{Dom}[a]$, that is,

$v = \sup_{0 \neq x \in \text{Dom}[a]} \frac{|v[x]|}{a[x]} < \infty$.

Let $A$ be a strictly positive self-adjoint operator associated with the closed form $a$ and $J$ a self-adjoint involution commuting with $A$.

Assume, in addition, that the form $v$ is off-diagonal with respect to the orthogonal decomposition

$\mathcal{S} = \mathcal{S}_+ \oplus \mathcal{S}_-$ with $\mathcal{S}_\pm = \text{Ran}(I \pm J)\mathcal{S}$

in the sense that

$v[Jx, y] = -v[x, Jy], \quad x, y \in \text{Dom}[a]$.

On $\text{Dom}[b] := \text{Dom}[a]$ introduce the symmetric form

$b[x, y] = a[x, Jy] + v[x, y], \quad x, y \in \text{Dom}[b]$.

Then there is a unique self-adjoint operator $B$ in $\mathcal{S}$ such that $\text{Dom}(B) \subset \text{Dom}[b]$ and

$b[x, y] = \langle x, By \rangle$ for all $x \in \text{Dom}[b], y \in \text{Dom}(B)$.

Moreover, the operator $B$ is boundedly invertible and the open interval $(-\alpha, \alpha) \ni 0$ belongs to its resolvent set.

Proof. Due to the hypothesis (2.5), from the definition of the form $b$ it follows that the sesquilinear form

$b[x, y] := b[|A|^{-\frac{1}{2}}x, |A|^{-\frac{1}{2}}y]$  

with $\text{Dom}[b] = \mathcal{S}$ is bounded and symmetric. Denote by $H$ the bounded self-adjoint operator associated with the form $b$.

Since the form $v$ is off-diagonal, the operator $H$ can be represented as the following $2 \times 2$ block operator matrix

$H = \begin{pmatrix} I & T \\ T^* & -I \end{pmatrix}, \quad T \in B(\mathcal{S}_-, \mathcal{S}_+)$,

with respect to the orthogonal decomposition $\mathcal{S} = \mathcal{S}_+ \oplus \mathcal{S}_-$.

It is well known (see, e.g., [18, Lemma 1.1] or [19, Remark 2.8]) that $H$ has a bounded inverse and, moreover, the open interval $(-1, 1)$ belongs to the resolvent set of $H$. Thus, the operators $A$ and $H$ satisfy Hypothesis 2.1 and the claim follows by applying Theorem 2.3.
Remark 2.6. Denote by \( m_\pm \) the greatest lower bound of the form \( \alpha \) on the subspace \( \mathcal{H}_\pm \) (note that \( \alpha = \min\{m_-, m_+\} \)). If \( m_- \neq m_+ \) one can state that not only the interval \((-\alpha, \alpha)\) but also the open interval \((-m_-, m_+) \ni 0 \) belongs to the resolvent set of the operator \( B \).

Indeed, let

\[
\alpha_{\mu}[x, y] - \mu \langle x, J y \rangle, \quad x, y \in \operatorname{Dom}[\alpha], \quad \mu \in (-m_-, m_+).
\]

Then the form \( \alpha_{\mu} \) is closed and positive definite with

\[
v_{\mu} = \sup_{0 \neq x \in \operatorname{Dom}[\alpha]} \frac{|v[x]|}{\alpha_{\mu}[x]} < \infty.
\]

It is easy to see that the operator \( B - \mu I \) is associated with the form

\[
b_{\mu}[x, y] = \alpha_{\mu}[x, J y] + v[x, y], \quad x, y \in \operatorname{Dom}[b].
\]

Applying Theorem 2.5 one concludes that \( B - \mu I \) has a bounded inverse for \( \mu \in (-m_-, m_+) \) and hence the open interval \((-m_-, m_+) \ni 0 \) belongs to the resolvent set of the operator \( B \).

2.3. A representation theorem for coercive forms. Our next result shows how the self-adjoint operators \( A \) and \( H \) satisfying Hypothesis 2.1 naturally arise in the context of perturbation theory.

Theorem 2.7. Assume that \( \alpha \) is a positive definite closed symmetric sesquilinear form. Let \( A \) be the associated self-adjoint operator. Suppose that \( b \) is a symmetric \( \alpha \)-bounded coercive sesquilinear form on \( \operatorname{Dom}[b] = \operatorname{Dom}[\alpha] \), that is,

\[
|b[x, y]| \leq \beta \alpha[x, y] \quad \text{for all} \quad x, y \in \operatorname{Dom}[b],
\]

and

\[
|b[x, x]| \geq \alpha \alpha[x, x] \quad \text{for all} \quad x \in \operatorname{Dom}[b],
\]

for some \( 0 < \alpha \leq \beta \).

Then there is a unique bounded and boundedly invertible self-adjoint operator \( H \) such that the form \( b \) admits the representation

\[
b[x, y] = (A^{1/2} x, H A^{1/2} y), \quad x, y \in \operatorname{Dom}[b] = \operatorname{Dom}(A^{1/2}).
\]

Proof. From (2.8) it follows that one has the representation

\[
b[x, y] = \alpha[x, M y] = (A^{1/2} x, A^{1/2} M y), \quad x, y \in \operatorname{Dom}(A^{1/2}),
\]

for some bounded self-adjoint operator \( M \) on the Hilbert space \( \mathcal{H}_\alpha := \operatorname{Dom}[\alpha] \) equipped with the inner product \( \alpha[\cdot, \cdot] \). On the other hand, from (2.9) one concludes that \( \|M^{-1}\| \leq 1/\alpha \) (this is a special case of the classical Lax-Milgram lemma, see, e.g., [7, Theorem IV.1.2]).

By Lemma VI.3.1 in [15] there exists a bounded self-adjoint operator \( H \) in \( \mathcal{H} \) such that

\[
b[x, y] = (A^{1/2} x, H A^{1/2} y), \quad x, y \in \operatorname{Dom}(A^{1/2}).
\]

Comparing (2.11) and (2.12) yields the equality

\[
H A^{1/2} y = A^{1/2} M y \quad \text{for all} \quad y \in \operatorname{Dom}(A^{1/2})
\]

and, therefore,

\[
H = A^{1/2} M A^{-1/2} \quad \text{on} \quad \mathcal{H}.
\]

Since \( M \) has a bounded inverse in \( \mathcal{H}_\alpha \), it is an isomorphism of \( \mathcal{H}_\alpha \). It remains to note that \( A^{-1/2} \) maps \( \mathcal{H} \) onto \( \mathcal{H}_\alpha \) isomorphically and, therefore, \( H \) is an isomorphism of \( \mathcal{H} \) and hence the self-adjoint operator \( H \) has a bounded inverse.

The proof is complete. \( \square \)
Remark 2.8. According to McIntosh [21] a possibly sign-indefinite sesquilinear form that admits the representation $b[x, y] = a[x, My]$, with $M$ an isomorphism of $\mathcal{F}_a$, is called $0$-closed. We refer to Theorem 2.3 in [21] to emphasize the role of $0$-closed forms that they play in the context of the First Representation Theorem.

Remark 2.9. An alternative notion of closedness of indefinite quadratic forms in the Krein space setting has been introduced in [10], [11], [12] by Fleige, Hassi, and de Snoo. The sesquilinear form $b$ referred to in Theorem 2.7 is closed in this sense as well. In other words, $(\text{Dom}(A^{1/2}), b[\cdot, \cdot])$ is a Krein space continuously embedded in $\mathcal{F}_a$.

Indeed, since the inner products on $\mathcal{F}_a$, referred to in Theorem 2.7 is closed in this sense as well. In other words, $(\text{Dom}(A^{1/2}), b[\cdot, \cdot])$ is a Krein space continuously embedded in $\mathcal{F}_a$ for all $x, y \in \mathcal{F}_a$. Since

$$
\langle x, Jy \rangle_a = |\langle A^{1/2}x, A^{1/2}A^{-1/2}\text{sign}(H)A^{1/2}x \rangle| \leq \|x\|_a \|y\|_a
$$

holds for all $x, y \in \mathcal{F}_a$. Since

$$
\langle x, Jx \rangle_H = \langle A^{1/2}x, |H|A^{1/2}x \rangle
$$
is real, $J$ is self-adjoint in $\mathcal{F}_a$ with respect to the inner product $\langle \cdot, \cdot \rangle_H$. Observing the equality

$$
b[x, y] = \langle A^{1/2}x, HA^{1/2}y \rangle = \langle x, Jy \rangle_H, \quad x, y \in \mathcal{F}_a,
$$
one concludes that $(\text{Dom}(A^{1/2}), b[\cdot, \cdot])$ is a Krein space.

2.4. The Second Representation Theorem. Our next goal is to prove the Second Representation Theorem (ii) for sign-indefinite forms, originally due to McIntosh [21], that shows that not only the operator $B$ is associated with the form $b$ (Theorem 2.3), but also that the form $b$ is represented by the operator $B$, provided that the form domain stability condition

$$
(2.13) \quad \text{Dom}(A^{1/2}) = \text{Dom}(|B|^{1/2})
$$
holds.

More precisely, we have the following result.

Theorem 2.10. Assume hypotheses of Theorem 2.3 and let $B$ be the operator referred to therein. If $\text{Dom}(A^{1/2}) = \text{Dom}(|B|^{1/2})$, then

$$
(2.14) \quad b[x, y] = \langle |B|^{1/2}x, \text{sign}(B)|B|^{1/2}y \rangle \quad \text{for all} \quad x, y \in \text{Dom}[b] = \text{Dom}(|B|^{1/2}).
$$

Proof. From Theorem 2.3 it follows that

$$
b[x, y] = \langle x, By \rangle
$$

for all $x \in \text{Dom}(|B|^{1/2})$, $y \in \text{Dom}(B)$, which yields

$$
b[x, y] = \langle |B|^{1/2}Jx, |B|^{1/2}y \rangle
$$

for all $x \in \text{Dom}(|B|^{1/2})$, $y \in \text{Dom}(B)$, where we introduced the notation $J = \text{sign} B$.

To complete the proof it remains to show that (2.15) holds for all $x, y \in \text{Dom}(|B|^{1/2})$. To this end we fix $x \in \text{Dom}(|B|^{1/2})$ and consider two linear functionals

$$
\ell_1(y) := b[x, y],
$$
$$
\ell_2(y) := \langle |B|^{1/2}Jx, |B|^{1/2}y \rangle
$$
defined on $\text{Dom}(A^{1/2}) \equiv \text{Dom}(|B|^{1/2})$. For the form $b$ is $a$-bounded, the functional $\ell_1$ is continuous on $\mathcal{F}_a$, Since $\text{Dom}(A^{1/2}) = \text{Dom}(|B|^{1/2})$, by the closed graph theorem the operator $A^{1/2}$ is $|B|^{1/2}$-bounded and $|B|^{1/2}$ is $A^{1/2}$-bounded. Therefore the norms on $\mathcal{F}_a$

$$
|x|_a := \|\langle A^{1/2}x \rangle \| \quad \text{and} \quad |x|_b := \||B|^{1/2}x\||
$$
are equivalent. The functional \( \ell_2 \) is continuous on \( \text{Dom}(|B|^{1/2}) \) in the topology of the norm \( | \cdot |_b \)
and, thus, it is continuous on \( \mathcal{D}_A \). Since \( \text{Dom}(B) \) is a core for \( |B|^{1/2} \), it follows that \( \text{Dom}(B) \)
is dense in \( \mathcal{D}_A \). Hence, since by (2.15) the functionals \( \ell_1 \) and \( \ell_2 \) agree on the set \( \text{Dom}(B) \) dense
in \( \mathcal{D}_A \), it follows that \( \ell_1 = \ell_2 \) on \( \mathcal{D}_A \). □

Under the form domain stability condition (2.13), Theorem 2.10 combined with Theorem 2.3 establishes a one-to-one correspondence between the symmetric forms of the type (2.2) and the associated self-adjoint operators \( B \) given by (2.1).

The following example provides a pair of self-adjoint operators \( A \) and \( H \) satisfying Hypothesis 2.1 such that the form domain stability condition (2.13) required in the hypothesis of Theorem 2.10 does not hold.

**Example 2.11.** In the Hilbert space \( \mathcal{H} = \bigoplus_{k \in \mathbb{N}} \mathbb{C}^2 \) consider the self-adjoint operator

\[
A = \bigoplus_{k \in \mathbb{N}} \begin{pmatrix} 1 & 0 \\ 0 & k^2 \end{pmatrix} \quad \text{on} \quad \text{Dom}(A) = \ell^{2,0}(\mathbb{N}) \oplus \ell^{2,1}(\mathbb{N}),
\]

where \( \ell^{2,p}(\mathbb{N}) \) denotes the space of sequences \( \{a_k\}_{k=1}^{\infty} \) such that \( \sum_{k \in \mathbb{N}} k^p |a_k|^2 < \infty \), and the bounded self-adjoint operator \( H \) given by

\[
H = \bigoplus_{k \in \mathbb{N}} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.
\]

A simple computation shows that the operator \( B = A^{1/2} H A^{1/2} \) associated with the form (2.2) admits the representation

\[
B = \bigoplus_{k \in \mathbb{N}} \begin{pmatrix} 0 & k \\ k & 0 \end{pmatrix} \quad \text{on} \quad \text{Dom}(B) = \ell^{2,2}(\mathbb{N}) \oplus \ell^{2,2}(\mathbb{N}),
\]

with

\[ \text{spec}(B) = \mathbb{Z} \setminus \{0\}, \]

and, therefore,

\[ |B| = \bigoplus_{k \in \mathbb{N}} \begin{pmatrix} k & 0 \\ 0 & k \end{pmatrix} \quad \text{on} \quad \text{Dom}(|B|) = \text{Dom}(B). \]

Clearly,

\[
\text{Dom}(A^{1/2}) = \ell^{2,0}(\mathbb{N}) \oplus \ell^{2,1}(\mathbb{N}) \quad \text{and} \quad \text{Dom}(|B|^{1/2}) = \ell^{2,1}(\mathbb{N}) \oplus \ell^{2,1}(\mathbb{N})
\]

and, hence,

\[
\text{Dom}(|B|^{1/2}) \neq \text{Dom}(A^{1/2}).
\]

In the particular case considered in Example 2.11 we face the following phenomenon, which apparently never happens whenever \( B \) is semi-bounded (cf. Lemma 3.6 below): The self-adjoint operator \( B \) is associated with two different sesquilinear forms, with

\[
b_1[x, y] = \langle A^{1/2}x, H A^{1/2}y \rangle, \quad x, y \in \text{Dom}[b] = \text{Dom}(A^{1/2}),
\]

and with

\[
b_2[x, y] = \langle |B|^{1/2}x, \text{sign}(B)|B|^{1/2}y \rangle, \quad x, y \in \text{Dom}[b] = \text{Dom}(|B|^{1/2}),
\]

and only one of them, the form \( b_2 \), is represented by \( B \). We will turn back to the discussion of this phenomenon in Section 4.
Lemma 3.1. Let the stability condition (2.13). Then the operator (3.1)

\[ X := A^{-1/2}B|B|^{-1/2} \quad \text{on} \quad \text{Dom}(X) = A^{1/2} \text{Dom}(B) \]

and

\[ Y := A^{1/2}|B|^{-1}A^{1/2} \quad \text{on} \quad \text{Dom}(Y) = \text{Dom}(A^{1/2}). \]

Remark 2.12. Let \( B_a \) denote the range restriction of the operator \( B \) from Example 2.11 in the Krein space \((\mathcal{K}, [\cdot, \cdot])\) with \( \mathcal{K} = \text{Dom}(A^{1/2}) \), the indefinite inner product

\[ [x, y] = \langle A^{1/2}x, H A^{1/2}y \rangle_\mathcal{K}, \]

and the fundamental symmetry

\[ J := A^{-1/2}HA^{1/2} = \bigoplus_{k \in \mathbb{N}} \begin{pmatrix} 0 & k \\ k^{-1} & 0 \end{pmatrix} \]

(cf. (1.6) and Remark 2.9). One easily verifies that

\[ \text{Dom}(B_a) = \ell^{2,4}(\mathbb{N}) \oplus \ell^{2,2}(\mathbb{N}) \]

and

\[ [x, B_ax] = \langle A^{1/2}x, H A^{1/2}B_ax \rangle_\mathcal{K} \]

\[ = \langle A^{1/2}HA^{1/2}x, A^{1/2}H A^{1/2}x \rangle_\mathcal{K} = \|Bx\|^2_\mathcal{K} \]

for all \( x \in \text{Dom}(B_a) \), that is, the operator \( B_a \) is positive with respect to the indefinite inner product \([\cdot, \cdot]_\mathcal{K}\). Since \( B \) is boundedly invertible, its range restriction \( B_a \) is boundedly invertible as well. Hence, according to Theorem 2.5 in [5], infinity is not a singular critical point of the operator \( B_a \) if and only if the norms generated by the positive definite inner products \([\cdot, J\cdot]_\mathcal{K}\) and \([\cdot, B_aB_a^{-1}\cdot]_\mathcal{K}\) on \( \text{Dom}(JB_a) = \text{Dom}(B_a) \) are equivalent. Since

\[ [x, Jy] = \langle A^{1/2}x, A^{1/2}y \rangle_\mathcal{K}, \]

\[ [x, B_aB_a^{-1}y] = \langle B^{1/2}x, B^{1/2}y \rangle_\mathcal{K} \]

hold for all \( x, y \in \text{Dom}(B_a) \), these norms are equivalent if and only if the domains \( \text{Dom}(A^{1/2}) \) and \( \text{Dom}(|B|^{1/2}) \) agree. Therefore, due to (2.17), infinity is a singular critical point of the operator \( B_a \).

3. Form-Domain Stability Criteria

The main goal of this section is to establish a number of criteria ensuring the form-domain stability condition (2.13).

The following simple function-analytic lemma plays a key role in our further considerations.

Lemma 3.1. Let \( (\mathcal{H}, \langle \cdot, \cdot \rangle) \) and \( (\mathcal{H}', \langle \cdot, \cdot \rangle') \) be Hilbert spaces. Assume that \( \mathcal{H}' \) is continuously embedded in \( \mathcal{H} \). If \( T \) is a continuous linear map from \( \mathcal{H} \) to \( \mathcal{H}' \) leaving \( \mathcal{H}' \) invariant (as a set), then the operator \( T' \) induced by \( T \) on \( \mathcal{H}' \) is continuous (in the topology of \( \mathcal{H}' \)).

Proof. By the hypothesis of the lemma the operator \( T' \) is defined on the whole of \( \mathcal{H}' \). Therefore, by the closed graph theorem it suffices to prove that \( T' \) is closed. Assume that

\[ x_n \xrightarrow{\mathcal{H}} x \quad \text{and} \quad T'x_n \xrightarrow{\mathcal{H}'} y. \]

Since the Hilbert space \( \mathcal{H}' \) is continuously embedded into \( \mathcal{H} \), one also has

\[ x_n \xrightarrow{\mathcal{H}} x \quad \text{and} \quad Tx_n \xrightarrow{\mathcal{H}} y. \]

From the continuity of \( T \) in \( \mathcal{H} \), it follows that \( T x = y \) in \( \mathcal{H} \), and, thus, \( T'x = y \) in \( \mathcal{H}' \), which proves the claim. \( \square \)

Introduce the following symmetric nonnegative operators

\[ X := A^{-1/2}|B|A^{-1/2} \quad \text{on} \quad \text{Dom}(X) = A^{1/2} \text{Dom}(B) \]

and

\[ Y := A^{1/2}|B|^{-1}A^{1/2} \quad \text{on} \quad \text{Dom}(Y) = \text{Dom}(A^{1/2}). \]
By Remark 2.4, $\text{Dom}(X) = A^{1/2} \text{Dom}(B)$ is dense in $\mathfrak{H}$. Hence, $X$ is a densely defined operator, so is $Y$, since $\text{Dom}(A^{1/2})$ is obviously a dense set.

Now we are prepared to present the main result of this section.

**Theorem 3.2.** Let the operators $A$ and $B$ be as in Theorem 2.3. Then the following conditions are equivalent

(i) $\text{Dom}(\|B\|^{1/2}) = \text{Dom}(A^{1/2})$,
(ii) $\text{Dom}(A^{1/2}) \subset \text{Dom}(\|B\|^{1/2})$,
(iii) $X = A^{-1/2}\|B\|^{1/2}$ is a bounded symmetric operator on $\text{Dom}(X) = A^{1/2} \text{Dom}(B)$,
(iii') $Y = A^{1/2}\|B\|^{-1/2}$ is a bounded symmetric operator on $\text{Dom}(Y) = \text{Dom}(A^{1/2})$,
(iv) $K := A^{1/2} \text{sign}(B) A^{-1/2}$ is a bounded involution on $\mathfrak{H}$,
(v) $\text{sign}(B) \text{Dom}(A^{1/2}) \subset \text{Dom}(A^{1/2})$.

**Proof.** The implications (i)$\Rightarrow$(ii) and (i)$\Rightarrow$(iii') are obvious.

(ii)$\Rightarrow$(iii). Since $\text{Dom}(A^{1/2}) \subset \text{Dom}(\|B\|^{1/2})$, the operator $\|B\|^{1/2} A^{-1/2}$ is bounded. Introducing the sesquilinear form

$$f[x, y] = \langle x, Xy \rangle \quad \text{on} \quad \text{Dom}[x] = \text{Dom}(X),$$

one concludes that the form $f$ can also be represented as a bounded form (since $\|B\|^{1/2} A^{-1/2}$ is bounded)

$$f[x, y] = (\|B\|^{1/2} A^{-1/2}x, \|B\|^{1/2} A^{-1/2}y).$$

Therefore, the sesquilinear form $f$ is associated with a bounded operator and, therefore, the closure of $X$ is a bounded operator defined on the whole Hilbert space $\mathfrak{H}$.

(iii')$\Rightarrow$(iii'). Arguing as above one shows that the operator $A^{1/2} \|B\|^{-1/2}$ is bounded and therefore the form

$$\eta[x, y] = \langle x, Yy \rangle = (A^{1/2} \|B\|^{-1/2}x, A^{1/2} \|B\|^{-1/2}y) \quad \text{on} \quad \text{Dom}[\eta] = \text{Dom}(Y),$$

is a bounded form. Hence, the closure of $Y$ is a bounded operator defined on the whole Hilbert space $\mathfrak{H}$.

(iii)$\Rightarrow$(iv). Note that the operator $K$ on its natural domain

$$\text{Dom}(K) = \{ x \in \mathfrak{H} | \text{sign}(B) A^{-1/2}x \in \text{Dom}(A^{1/2}) \}$$

is obviously closed. Moreover, it is also clear that

$$\text{Dom}(X) = A^{1/2} \text{Dom}(B) \subset \text{Dom}(K).$$

Since for any $x \in \text{Dom}(X)$ one gets that

$$H^{-1}X x = H^{-1} A^{-1/2} \|B\| A^{-1/2} x = A^{1/2} A^{-1/2} H^{-1} A^{-1/2} A^{-1/2} \|B\| A^{-1/2} x$$

$$= A^{1/2} \|B\|^{-1/2} \|B\| A^{-1/2} x = A^{1/2} \text{sign}(B) A^{-1/2} x = K x$$

and both $H^{-1}$ and $X$ are bounded operators, one concludes that $K|_{\text{Dom}(X)}$ is a bounded operator. Since $K$ is closed, and $\text{Dom}(X) \subset \text{Dom}(K)$, the operator $K|_{\text{Dom}(X)}$ is closable. Since $K|_{\text{Dom}(X)}$ is bounded, the domain of its closure is a closed subspace that contains a set dense in $\mathfrak{H}$. Therefore, $K = K|_{\text{Dom}(X)}$ is a bounded involution with $\text{Dom}(K) = \mathfrak{H}$.

(iii')$\Rightarrow$(iv). For any $x \in H^{-1} \text{Dom}(Y) = H^{-1} \text{Dom}(A^{1/2})$ one gets that

$$Y H x = A^{1/2} \|B\|^{-1} A^{1/2} H x = A^{1/2} \|B\|^{-1} A^{1/2} H A^{1/2} A^{-1/2} x$$

$$= A^{1/2} \|B\|^{-1} B A^{-1/2} = A^{1/2} \text{sign}(B) A^{-1/2} x = K x.$$
Next we check that the dense set $H^{-1} \text{Dom}(A^{1/2})$ is a subset of $\text{Dom}(K)$. Indeed, if $x \in H^{-1} \text{Dom}(A^{1/2})$, then $x = H^{-1} A^{-1/2} y$ for some $y \in \mathfrak{H}$. Hence,

$$\text{sign}(B)A^{-1/2} x = \text{sign}(B)A^{-1/2} H^{-1} A^{-1/2} y = \text{sign}(B)B^{-1} y \in \text{Dom}(B) \subset \text{Dom}(A^{1/2}),$$

and, therefore, $x \in \text{Dom}(K)$. Now, to conclude that $K$ is a bounded involution it remains to argue as in the proof of the implication (iii)$\Rightarrow$(iv).

(iv)$\Rightarrow$(v). From $\text{Dom}(K) = \mathfrak{H}$ it follows that $\text{sign}(B)$ leaves $\text{Dom}(A^{1/2})$ invariant.

(v)$\Rightarrow$(i). We start with a particular case of positive $H$. Consider the positive definite sesquilinear form

$$b[x, y] := \langle A^{1/2} x, H A^{1/2} y \rangle$$

defined on $\text{Dom}[b] = \text{Dom}(A^{1/2})$. Since $H$ is positive, one can represent the form $b$ as

$$b[x, y] = \langle H^{1/2} A^{1/2} x, H^{1/2} A^{1/2} y \rangle.$$

For $H^{1/2} A^{1/2}$ is closed, the form $b$ is closed.

By the definition of the operator $B$ (cf. Lemma 2.2)

$$b[x, y] = \langle x, A^{1/2} H A^{1/2} y \rangle = \langle x, By \rangle$$

for all $x \in \text{Dom}[b] \equiv \text{Dom}(A^{1/2})$ and $y \in \text{Dom}(B)$. By Lemma 2.2 the operator $B$ is self-adjoint. Therefore, the operator $B$ is associated with the form $b$. The second representation theorem for positive definite sesquilinear forms [15] yields $\text{Dom}[b] = \text{Dom}(B^{1/2})$, which proves the claim.

We turn to the case when $H$ is not necessarily positive.

Set for brevity $J := \text{sign}(B)$. Denote by $J_{\mathfrak{H}}$ the operator on $\mathfrak{H}$ induced by $J$. Since $J^2 = I$, by Lemma 3.1 the operator $J_{\mathfrak{H}}$ is a bounded involution, not necessarily unitary.

This observation allows to conclude that

$$K = A^{1/2} J A^{-1/2}$$

is a bounded involution in the Hilbert space $\mathfrak{H}$. To complete the proof of the lemma one notes that

$$|B| = BJ = A^{1/2} HA^{1/2}.$$  

Since $|B| \geq 0$, one immediately verifies that the bounded operator $HK$ is nonnegative. Since both $H$ and $K$ are Hilbert space isomorphisms, the self-adjoint operator $HK$ has a bounded inverse. Since the case of positively definite $H$ has already been discussed, we arrive at the conclusion that $\text{Dom}(|B|^{1/2}) = \text{Dom}(A^{1/2})$. \hfill $\square$

Remark 3.3. We note that the equivalence of (i) and (iv) has been established by McIntosh in [22, Lemma 2.5].

Remark 3.4. If the domains $\text{Dom}(|B|^{1/2})$ and $\text{Dom}(A^{1/2})$ agree, then the sesquilinear form $|b|$ associated with the positive operator $|B|$ can be represented as

$$|b|[x, y] = \langle (HK)^{1/2} A^{1/2} x, (HK)^{1/2} A^{1/2} y \rangle, \quad x, y \in \text{Dom}(|B|^{1/2}) = \text{Dom}(A^{1/2})$$

and, therefore, along with (3.4) one obtains the factorization

$$|B| = (HK)^{1/2} A^{1/2} (HK)^{1/2} A^{1/2}.$$
3.1. **Sufficient criteria.** The following lemma provides several sufficient (but not necessary) criteria for the form-domain stability condition to hold.

**Lemma 3.6.** Assume hypotheses of Theorem 2.3 and let $B$ be the operator referred to therein. If one of the following conditions

(i) the operator $H$ maps $\text{Dom}(A^{1/2})$ onto itself;

(ii) the operator $H$ is strictly positive, that is, $H > 0$;

(iii) the operator $B$ is semi-bounded;

hold, then

$$\text{Dom}(|B|^{1/2}) = \text{Dom}(A^{1/2}).$$  \hfill (3.5)

**Proof.** (i). Since $\text{Dom}(A^{1/2})$ is $H$-invariant, and

$$\text{Dom}(B) = \{ x \in \text{Dom}(A^{1/2}) \mid H A^{1/2} x \in \text{Dom}(A^{1/2}) \}$$

one concludes that $\text{Dom}(B) = \text{Dom}(A)$. By the Heinz inequality (cf. [14, Theorem 3], [16, Theorem IV.1.11], [2, Ch. 10, Section 4]), this implies (3.5).

(ii). If $H$ is strictly positive, then the operator $B$ is nonnegative and, therefore, $B$ is also strictly positive by the First Representation Theorem. Thus, $\text{sign}(B) = I$ and condition (iv) of Theorem 3.2 is trivially fulfilled and, hence, (3.5) holds.

(iii). Assume, for definiteness, that the operator $B$ is semi-bounded from below, and, hence, $B + \beta I \geq 0$ for $\beta > |\inf \text{spec}(B)|$. Therefore, for those $\beta$, one gets that

$$B + \beta I = A^{1/2} H A^{1/2} + \beta I = A^{1/2} (H + \beta A^{-1}) A^{1/2}$$

and, moreover, $H + \beta A^{-1} \geq 0$.

Since $H$, by hypothesis, has a bounded inverse, by the Birman-Schwinger principle $H + \beta A^{-1}$ has a bounded inverse if and only if the operator $I + \beta A^{-1/2} H^{-1} A^{-1/2} = I + \beta B^{-1}$ does, which is the case for $\beta > |\inf \text{spec}(B)|$. Thus, $H + \beta A^{-1}$ is strictly positive and by (ii) one obtains that

$$\text{Dom} |B + \beta I|^{1/2} = \text{Dom}(A^{1/2}).$$

It remains to remark that

$$\text{Dom} |B + \beta I|^{1/2} = \text{Dom}(B + \beta I)^{1/2} = \text{Dom}(|B|^{1/2})$$

and the claim follows. \hfill \square

3.2. **The form domain stability in pictures.** Given a not necessarily semibounded self-adjoint operator $A$, introduce the Sobolev-like scale of spaces

$$\mathcal{H}^s_A = \text{Dom}(|A|^s/2), \quad s \geq 0,$$

equipped with the graph norm of $|A|$, with a natural convention that $\mathcal{H}^0_A = \mathcal{H}$, the underlying Hilbert space.

We remark that if self-adjoint operators $A$ and $B$ have bounded inverses, and $\mathcal{H}^s_A = \mathcal{H}^s_B$ for some $s > 0$, then, by the Heinz inequality $\mathcal{H}^t_A = \mathcal{H}^t_B$ for all $0 \leq t \leq s$. In particular, under hypothesis (i) of Lemma 3.6, the domains $\text{Dom}(A)$ and $\text{Dom}(B)$ of the operators $A$ and $B$ coincide. That is, $\mathcal{H}^1_A = \mathcal{H}^1_B$, and, therefore, the form domain stability condition $\mathcal{H}^1_A = \mathcal{H}^1_B$ holds automatically.

The diagram depicted in Fig. 1 illustrates the case.

Under Hypothesis 2.1 the perturbation may change the domain of $A$, so that $\text{Dom}(A) \neq \text{Dom}(B)$. However, the form domain stability condition may still hold. For instance, it is the case when the operator $B$ is semibounded. A typical diagram is presented in Fig. 2.
Fig. 1. The Sobolev-like scale of spaces for the operators $A$ and $B$. $H^2_A = H^2_B$ and, hence, the form domain stability condition $H^1_A = H^1_B$ holds.

Fig. 2. The Sobolev-like scale of spaces for the operators $A$ and $B$. $H^2_A \neq H^2_B$ but the form-domain stability condition $H^1_A = H^1_B$ still holds.

**Remark 3.7.** If $\text{Dom}(A) \neq \text{Dom}(B)$, then any of the possibilities $\text{Dom}(B) \subset \text{Dom}(A)$ and $\text{Dom}(B) \triangle \text{Dom}(A) \neq \emptyset$ may occur. Indeed, in the Hilbert space $\mathcal{H} = \bigoplus_{k \in \mathbb{N}} \mathbb{C}^2$ consider the self-adjoint operator $A$ defined in (2.16). Let

$$H = \begin{pmatrix} 1 & 1 \\ 1 & -q \end{pmatrix} \quad \text{with} \quad q = 0 \quad \text{or} \quad q = 1$$

and $B = A^{1/2}HA^{1/2}$ defined on its natural domain. It is straightforward to verify that

$$\text{Dom}(B) = \ell^2,2(\mathbb{N}) \oplus \ell^2,4(\mathbb{N}) \subset \text{Dom}(A) \quad \text{if} \quad q = 1$$

and

$$\text{Dom}(B) = \ell^2,2(\mathbb{N}) \oplus \ell^2,4(\mathbb{N}) \not\subset \text{Dom}(A) \quad \text{if} \quad q = 0.$$  

In the second case we, obviously, have $\text{Dom}(B) \triangle \text{Dom}(A) \neq \emptyset$.

Revisiting Example 2.11, one can illustrate the statement of Theorem 3.2 as follows. By direct computations one easily checks that
(a) the sets $\text{Dom}(A^{1/2})$ and $\text{Dom}(|B|^{1/2})$ are in general position, that is, the symmetric difference $\text{Dom}(A^{1/2}) \triangle \text{Dom}(|B|^{1/2})$ is a non-empty set and, hence, (i), (ii), and (ii$'$) do not hold,

(b) the operators

$$X = \bigoplus_{k \in \mathbb{N}} \begin{pmatrix} k & 0 \\ 0 & k^{-1} \end{pmatrix} \quad \text{and} \quad Y = \bigoplus_{k \in \mathbb{N}} \begin{pmatrix} k^{-1} & 0 \\ 0 & k \end{pmatrix}$$

are, obviously, unbounded. Hence (iii) and (iii$'$) do not hold,

(c) the involution

$$K = \bigoplus_{k \in \mathbb{N}} \begin{pmatrix} 0 & k^{-1} \\ k & 0 \end{pmatrix}$$

is obviously unbounded so that (iv) does not hold, and, finally,

(d) $\text{sign}(B) \text{Dom}(A^{1/2})$ is not a subset of $\text{Dom}(A^{1/2})$ and, hence, (v) does not hold.

It is also worth mentioning that the $A$-form boundedness of the self-adjoint operator $B$ guaranteed under Hypothesis 2.1 by Remark 2.4 does not imply the $A$-form boundedness of $|B|$ in general (cf. Theorem 3.2 (iii)) which seems to be a bit unexpected.

The corresponding (typical) diagram illustrating “counter” example 2.11 is depicted in Fig. 3.

**Fig. 3.** The Sobolev-like scale of spaces for the operators $A$ and $B$. The form-domain stability condition $\mathcal{H}_A^1 = \mathcal{H}_B^1$ does not hold and the domains $\text{Dom}(A^{1/2})$ and $\text{Dom}(|B|^{1/2})$ are in general position in accordance with Theorem 3.2 (i), (ii) and (ii$'$), that is, $\mathcal{H}_A^1 \triangle \mathcal{H}_B^1 \neq \emptyset$.

### 4. On a Converse to the First Representation Theorem

In the semibounded case there is a one-to-one correspondence between the closed symmetric forms and the associated self-adjoint operators. For non-semibounded case the situation may be quite different and examples of an operator associated with infinitely many sesquilinear forms naturally arise.

To illustrate this phenomenon we assume the following hypothesis.

**Hypothesis 4.1.** Assume that the (separable) Hilbert space $\mathcal{H}$ admits an orthogonal decomposition $\mathcal{H} = \mathcal{H}_0 \oplus \mathcal{H}_1$ such that the subspaces $\mathcal{H}_0$ and $\mathcal{H}_1$ have infinite dimension. Suppose that $D : \mathcal{H}_1 \to \mathcal{H}_0$ is a closed densely defined operator. Assume, in addition, that $D$ has
a bounded inverse and let $D = U|D|$ be the polar decomposition of the operator $D$ with a unitary $U : \mathcal{H}_0 \to \mathcal{H}_1$.

Given $\mu \in [0, 1]$, under Hypothesis 4.1 introduce the self-adjoint positive definite operator matrix

$$A_\mu = \begin{pmatrix} |D|^{2-2\mu} & 0 \\ 0 & |D^*|^{2\mu} \end{pmatrix}$$

on $\text{Dom}(A_\mu) = \text{Dom}(|D|^{2-2\mu}) \oplus \text{Dom}(|D^*|^{2\mu})$ and the self-adjoint bounded involution on $\mathcal{H}_0 \oplus \mathcal{H}_1$

$$H = \begin{pmatrix} 0 & U \\ U^* & 0 \end{pmatrix}.$$

Clearly, the pair $A_\mu$ and $H$ satisfy Hypothesis 2.1 and, therefore, the sesquilinear symmetric form

$$(4.1) \quad b_\mu[x, y] := \langle A_\mu^{1/2} x, H A_\mu^{1/2} y \rangle_{\mathcal{H}}, \quad x, y \in \text{Dom}[b_\mu] = \text{Dom}(A_\mu^{1/2}),$$

is a 0-closed form in the sense of McIntosh (cf. Remark 2.8). Since

$$\text{Dom}(A_\mu^{1/2}) \neq \text{Dom}(A_\nu^{1/2}), \quad \mu \neq \nu,$$

the 0-closed forms $b_\mu$ are defined on different domains and therefore $b_\mu \neq b_\nu$ whenever $\mu$ and $\nu$ from the interval $[0, 1]$ are different.

By the First Representation Theorem there exists a unique self-adjoint operator associated with the form $b_\mu$. However, our next result shows that this operator does not depend on $\mu$ and, therefore, there exist infinitely many 0-closed forms the self-adjoint operator is associated with.

**Proposition 4.2.** Assume Hypothesis 4.1. Then

(i) The block operator matrix

$$(4.2) \quad B := \begin{pmatrix} 0 & D \\ D^* & 0 \end{pmatrix}$$

defined on its natural domain $\text{Dom}(B) = \text{Dom}(D^*) \oplus \text{Dom}(D)$ is a self-adjoint operator.

(ii) For any $\mu \in [0, 1]$ the operator $B$ is associated with the form $b_\mu$ given by (4.1).

(iii) The form $b_\mu$ is represented by the operator $B$ if and only if $\mu = 1/2$.

**Proof.** Under Hypothesis 4.1 let

$$D = U|D| \quad \text{on} \quad \text{Dom}(D) = \text{Dom}(|D|)$$

be the polar decomposition of $D$ (cf. [15, Sect. VI.2.7]). Recall that

$$D^* = U^*|D^*| \quad \text{on} \quad \text{Dom}(D^*) = \text{Dom}(|D^*|).$$

By a result in [13, Theorem 2.7], for any $\mu \in [0, 1]$ the operators $D$ and $D^*$ can be represented as the products

$$(4.3) \quad D = |D^*|^{1-\mu} U |D|^{1-\mu} \quad \text{on} \quad \text{Dom}(D)$$

and

$$(4.4) \quad D^* = |D|^{1-\mu} U^* |D^*|^\mu \quad \text{on} \quad \text{Dom}(D^*).$$

Therefore, the operator matrix

$$B = \begin{pmatrix} 0 & D \\ D^* & 0 \end{pmatrix} \quad \text{on} \quad \text{Dom}(B) = \text{Dom}(D^*) \oplus \text{Dom}(D)$$
admits the factorization
\[ B = \begin{pmatrix} |D|^{1-\mu} & 0 \\ 0 & |D^*|^\mu \end{pmatrix} \begin{pmatrix} 0 & U \\ U^* & 0 \end{pmatrix} \begin{pmatrix} |D|^{1-\mu} & 0 \\ 0 & |D^*|^\mu \end{pmatrix}. \]

By Lemma 2.2 the operator
\[ B_\mu = A_{\mu}^{1/2} H A_{\mu}^{1/2} \]
defined on its natural maximal domain
\[ \text{Dom}(B_\mu) = \{ x \in \text{Dom}(A_{\mu}^{1/2}) | H A_{\mu}^{1/2} x \in \text{Dom}(A_{\mu}^{1/2}) \} \]
is a self-adjoint operator with a bounded inverse.

It remains to notice that due to (4.3) and (4.4), \( \text{Dom}(B) = \text{Dom}(B_\mu) \) and, therefore,
\[ B = B_\mu \quad \text{for all } \mu \in [0, 1]. \]

By Theorem 2.3 the self-adjoint operator \( B_\mu \) is associated with the form \( b_\mu \), so does \( B \) which proves (i) and (ii).

(iii). The claim follows from Theorem 3.2 (i) and Theorem 2.10. \( \square \)

We illustrate the statement of Proposition 4.2 on the classical example of the free Dirac operator of Quantum Mechanics.

**Example 4.3.** Let \( \mathcal{H}_0 \cong \mathcal{H}_1 = L^2(\mathbb{R}^3; \mathbb{C}^2) \). Consider the free Dirac operator defined on its natural domain as a block operator matrix
\[ B := \begin{pmatrix} I & \partial \\ \partial^* & -I \end{pmatrix}, \]
where \( \partial = i \vec{\sigma} \cdot \vec{\nabla} \) and \( \vec{\sigma} = (\sigma_1, \sigma_2, \sigma_3) \), the Pauli matrices,
\[ \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \]

The “pathology” we dealt with in Proposition 4.2, already occurs for the Dirac operator. Indeed, it is well known that the Dirac operator \( B \) is self-adjoint on its natural domain and that \( B \) has absolutely continuous spectrum of infinite multiplicity filling in the set \((-\infty, -1] \cup [1, \infty)\). By the Spectral Theorem, the operator \( B \) is unitarily equivalent to the block operator matrix
\[ \tilde{B} = \begin{pmatrix} 0 & M \\ \overline{M^*} & 0 \end{pmatrix}, \]
where \( M \) is the multiplication operator by the independent variable in the Hilbert space
\[ \mathcal{L} = L^2((1, \infty); \mathcal{F}') \]
of vector-valued functions with values in an infinite dimensional (separable) Hilbert space \( \mathcal{F}' \).

Since by Proposition 4.2, the operator \( \tilde{B} \) is associated with infinitely many \( 0 \)-closed sesquilinear symmetric forms, so does the Dirac operator \( B \).

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