REPRESENTATIONS AND HOMOLOGICAL INVARIANTS OF GENERALIZED BOUND PATH ALGEBRAS

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Abstract. We are interested to study the representations of a generalized bound path algebra in terms of the representations of the algebras used in its construction. From the description of projective and injective modules, we shall look at some of its homological invariants.

It is a well-established fact that any finitely generated basic algebra $A$ over an algebraically closed field $k$ can be seen as the quotient of a path algebra, that is, $A \cong kQ/I$, where $Q$ is a quiver and $I$ is an admissible ideal of $kQ$ (see for instance [1, 2]). In [5], Coelho and Liu studied a generalization of such construction. There, it is assigned an algebra to each vertex of a given quiver $Q$ instead of just assigning the base field. The multiplication in such a generalization will be given not only by the concatenation of paths of the quiver but also by those of the algebras associated with the vertices.

More specifically, let $\Gamma$ denote a quiver and $\mathcal{A} = \{A_i : i \in \Gamma_0\}$ denote a family of basic algebras of finite dimension over an algebraically closed field $k$ indexed by the set $\Gamma_0$ of the vertices of $\Gamma$. Consider also a set of relations $I$ on the paths of $\Gamma$. To such a data we assigned a generalized bound path algebra $\Lambda = k(\Gamma, \mathcal{A}, I)$ with a natural multiplication (see preliminaries for details).

In [5], where it is considered the particular case when $I = 0$, the main interest was more of ring-theoretic nature, but clearly, such a construction can be also very useful from the representation theory point of view. In [3], we start our work in this direction for the general case. Observe that any algebra $A$ can be naturally realized as a generalized bound path algebra in two ways. Firstly, Gabriel’s theorem gives a description as the usual quotient of a path algebra. But also, $A$ can be seen by using a quiver with a sole vertex and no arrows and the algebra itself assigned to it. Since for most algebras, these are the only possibilities, one can wonder for which algebras it is possible to describe them as generalized bound path algebras in a different way from these two above (we call it a non-trivial simplification of $A$). Such a description could be useful once one aims to look at properties of a given algebra from smaller ones. We deal with this problem in [3].

Here, our focus is on the representations of a generalized bound path algebra. When $I = 0$, this has been considered in [3] and we shall generalize their results here (Theorem 2.3). Then, after having described the representations of the projective, injective and simple modules, we shall investigated some homological invariants.

The first section is devoted to the preliminaries needed along the paper. In Section 2 we prove the above mentioned theorem which describes the representations.

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of a given generalized bound path algebra. Section 3 is devoted to the description of the projective, injective and simple modules while in Section 4 we concentrate in homological relations between the algebras \( A_i \) and the whole algebra. Finally, in Section 5 we obtain some consequences of the main results.

1. Preliminaries

We shall here recall some basic notions and establish some notations needed along the paper. We indicate the books [1,2] where details on Representation Theory can be found. For an algebra, we shall mean an associative and unitary basic algebra of finite dimension over an algebraically closed field \( k \). Unless otherwise stated, the modules considered here are right modules.

1.1. Quivers and path algebras. A quiver \( Q \) is given by \((Q_0,Q_1,s,e)\) where \( Q_0 \) is the set of vertices and \( Q_1 \) is the set of arrows and \( s:Q_1 \to Q_0 \) are functions which indicate, for each arrow \( \alpha \in Q_1 \), the starting vertex \( s(\alpha) \in Q_0 \) of \( \alpha \) and the ending vertex \( e(\alpha) \in Q_0 \) of \( \alpha \). Naturally, given a quiver \( Q \) one can assign a path algebra \( kQ \) with a \( k \)-basis given by all paths of \( Q \) and multiplication on that basis defined by concatenation. Even when \( Q \) is finite (that is, when \( Q_0, Q_1 \) are finite sets), the corresponding algebra could not be finite dimensional. However, a well-known result established by Gabriel states that given an algebra \( A \), there exists a finite quiver \( Q \) and a set of relations on the paths of \( Q \) which generates an admissible ideal \( I \) such that \( A \cong kQ/I \) (see [1] for details).

Along this paper we will assume that the quivers are finite.

1.2. Generalized path algebras. We shall now recall the definition of a generalized path algebra given in [5].

Let \( \Gamma = (\Gamma_0,\Gamma_1,s,e) \) be a quiver and \( \mathcal{A} = (A_i)_{i \in \Gamma_0} \) be a family of algebras, indexed by \( \Gamma_0 \). An \( \mathcal{A} \)-path of length \( n \) over \( \Gamma \) is defined as follows. If \( n = 0 \), it is just an element of \( \bigcup_{i \in \Gamma_0} A_i \), and, if \( n > 0 \), it is a sequence of the form

\[
a_1\beta_1a_2\ldots a_n\beta_na_{n+1}
\]

where \( \beta_1 \ldots \beta_n \) is an ordinary path over \( \Gamma \), \( a_i \in A_{\beta_i(\beta_{i+1})} \) if \( i \leq n \), and \( a_{n+1} \in A_{\beta_n(\beta_1)} \). Denote by \( k[\Gamma,\mathcal{A}] \) the \( k \)-vector space spanned by all \( \mathcal{A} \)-paths over \( \Gamma \). We shall give it an algebraic structure as follows.

Firstly, consider the quotient vector space \( k(\Gamma,\mathcal{A}) = k[\Gamma,\mathcal{A}]/M \), where \( M \) is the subspace generated by all elements of the form

\[
(a_1\beta_1\ldots\beta_{j-1}(a_j^1 + \ldots + a_j^m)\beta_ja_{j+1}\ldots \beta_na_{n+1}) - \sum_{l=1}^{m}(a_1\beta_1\ldots\beta_{j-1}a_j^l\beta_j\ldots \beta_na_{n+1})
\]

or, for \( \lambda \in k \),

\[
(a_1\beta_1\ldots\beta_{j-1}(\lambda a_j)\beta_ja_{j+1}\ldots \beta_na_{n+1}) - \lambda(a_1\beta_1\ldots\beta_{j-1}a_j\beta_ja_{j+1}\ldots \beta_na_{n+1})
\]

Now, consider the multiplication in \( k(\Gamma,\mathcal{A}) \) induced by the multiplications of the \( A_i \)'s and by Composition of paths. Namely, it is defined by linearity and the following rule:

\[
(a_1\beta_1\ldots\beta_na_{n+1})(b_1\gamma_1\ldots \gamma_mb_{m+1}) = a_1\beta_1\ldots \beta_n(a_{n+1}b_1)\gamma_1\ldots \gamma_mb_{m+1}
\]

if \( e(\beta_n) = s(\gamma_1) \), and

\[
(a_1\beta_1\ldots\beta_na_{n+1})(b_1\gamma_1\ldots \gamma_mb_{m+1}) = 0
\]
otherwise.

With this multiplication, we call \( k(\Gamma, \mathcal{A}) \) the **generalized path algebra** over \( \Gamma \) and \( \mathcal{A} \).

**Remark 1.1.** It should be easy to see that the ordinary path algebras are a particular case of generalized path algebras, simply by taking \( A_i = k \) for every \( i \in \Gamma_0 \).

Note that the generalized path algebra \( k(\Gamma, \mathcal{A}) \) is an associative algebra. And since we are assuming the quivers to be finite, it also has an identity element, which is equal to \( \sum_{i \in \Gamma_0} 1_A_i \). Finally, it is easy to observe that \( k(\Gamma, \mathcal{A}) \) is finite-dimensional over \( k \) if and only if so are the algebras \( A_i \) and if \( \Gamma \) is acyclic.

**Remark 1.2.** As observed in [5], if \( k(\Gamma, \mathcal{A}) \) is a generalized path algebra as defined above, then it is a tensor algebra: if \( A = \prod_{i \in \Gamma_0} A_i \) is the product of the algebras in \( \mathcal{A} \), then there is an \((A_A - A_A)\)-bimodule \( M_A \) such that \( k(\Gamma, \mathcal{A}) \cong T(A_A, M_A) \).

1.3. **Generalized bound path algebras** (gbp-algebras). Following [3], we shall extend the definition of generalized path algebras to allow them to have relations. In doing so, these algebras will be called **generalized bound path algebras** or gbp-algebras to abbreviate. As observed in [3], the idea of taking the quotient of a generalized path algebra by an ideal of relations has already been studied by Li Fang (see [6] for example). However, the concept dealt with in [3] and here is slightly different.

Observe that if \( A_i \in \mathcal{A} \), then, as explained in Subsection 1.1, there is a quiver \( \Sigma_i \) such that \( A_i \cong k\Sigma_i / \Omega_i \) where \( \Omega_i \) is an admissible ideal of \( k\Sigma_i \). Let now \( I \) be a finite set of relations over \( \Gamma \) which generates an admissible ideal in \( k\Gamma \). Consider the ideal \( (A(I)) \) generated by the following subset of \( k(\Gamma, \mathcal{A}) \):

\[
A(I) = \left\{ \sum_{i=1}^t \lambda_i \beta_{i1} \gamma_{i1} \beta_{i2} \cdots \gamma_{i(m_i-1)} \beta_{im_i} : \sum_{i=1}^t \lambda_i \beta_{i1} \cdots \beta_{im_i} \text{ is a relation in } I \text{ and } \gamma_{ij} \text{ is a path in } \Sigma_e(\beta_{ij}) \right\}
\]

The quotient \( k(\Gamma, \mathcal{A}) / (A(I)) \) is said to be a **generalized bound path algebra** (gbp-algebra). To simplify the notation, we may also write \( k(\Gamma, \mathcal{A}) / (A(I)) = k(\Gamma, \mathcal{A}, I) \). When the context is clear, we may denote the set \( A(I) \) simply by \( I \).

1.4. **Notations.** We are going to use the following notation in this article: \( \Gamma \) will always be an acyclic quiver, \( \mathcal{A} = \{A_i : i \in \Gamma_0\} \) will denote a family of basic algebras of finite dimension over an algebraically closed field \( k \), and \( I \) will be a set of relations in \( \Gamma \) generating an admissible ideal in the path algebra \( k\Gamma \). We will also denote by \( \Lambda = k(\Gamma, \mathcal{A}, I) \) the generalized bound path algebra (gbp-algebra) obtained from these objects. Also, \( A_A \) will denote the product algebra \( \prod_{i \in \Gamma_0} A_i \). For the purpose of simplifying notation, we are also going to denote the identity element of the algebras \( A_i \) by \( 1_i \) instead of \( 1_{A_i} \).

2. **Representations**

The aim of this section is to prove Theorem 2.3 below, which is an extension of Theorem 2.4 from [8]. As already mentioned above, this result will be of key importance here, and sometimes we will be using it without further clarification.
Based on [S], we start by defining what are generalized representations. However, before this we need to do a remark about the notation used here:

**Remark 2.1.** Generally speaking, if $A$ is an algebra and $M$ is a vector space, an action of $A$ over $M$ which turns $M$ into an $A$-module is equivalent to a homomorphism of algebras $\phi : A \to \text{End}_k M$. (This correspondence is given by $\phi(a)(m) = m.a$ for all $a \in A$ and $m \in M$). That way, if we understand this correspondence as being canonical, then, at least in the concepts to be treated below, an element $a$ of $A$ denoted either the element itself or $\phi(a)$, which is the endomorphism given by right translation through: $m \mapsto m.a$ for all $m \in M$. This shall be done in order to simplify the notations.

**Definition 2.2.** Let $\Lambda = k(\Gamma, A, I)$ be a generalized bound path algebra.

(a) A representation of $\Lambda$ is given by $((M_i)_{i \in \Gamma_0}, (M_\alpha)_{\alpha \in \Gamma_1})$ where

(i) for every $i \in \Gamma_0$, $M_i$ is an $A_i$-module;

(ii) for every arrow $\alpha \in \Gamma_1$, $M_\alpha : M_{s(\alpha)} \to M_{t(\alpha)}$ is a $k$-linear transformation.

(iii) it satisfies any relation $\gamma$ of $I$. That is, if $\gamma = \sum_{i=1}^t \lambda_i \alpha_{i1} \alpha_{i2} \ldots \alpha_{it}$ is a relation in $I$ with $\lambda_i \in k$ and $\alpha_{ij} \in \Gamma_1$, then

$$\sum_{i=1}^t \lambda_i M_{\alpha_{i1}} \circ \gamma_{i1} \circ \ldots \circ M_{\alpha_{ij}} \circ \gamma_{ij} \circ M_{\alpha_{it}} = 0$$

for every choice of paths $\gamma_{ij}$ over $\Sigma_{s(\alpha_{ij})}$, with $1 \leq i \leq t$, $2 \leq j \leq n_i$.

(b) We say that a representation $((M_i)_{i \in \Gamma_0}, (M_\alpha)_{\alpha \in \Gamma_1})$ of $\Lambda$ is finitely generated if each of the $A_i$-modules $M_i$ is finitely generated.

(c) Let $M = ((M_i)_{i \in \Gamma_0}, (M_\alpha)_{\alpha \in \Gamma_1})$ and $N = ((N_i)_{i \in \Gamma_0}, (N_\alpha)_{\alpha \in \Gamma_1})$ be representations of $\Lambda$. A morphism of representations $f : M \to N$ is given by a tuple $f = (f_i)_{i \in \Gamma_0}$, such that, for every $i \in \Gamma_0$, $f_i : M_i \to N_i$ is a morphism of $A_i$-modules; and such that, for every arrow $\alpha : i \to j \in \Gamma_1$, it holds that $f_j M_\alpha = N_\alpha f_i$, that is, the following diagram commutes:

\[
\begin{array}{ccc}
M_i & \xrightarrow{M_\alpha} & M_j \\
| & f_i | & | \\
N_i & \xleftarrow{N_\alpha} & N_j
\end{array}
\]

We shall denote by $\text{Rep}_k(\Gamma, A, I)$ (or $\text{rep}_k(\Gamma, A, I)$, respectively) the category of the representations (or finitely generated representations) of the algebra $k(\Gamma, A, I)$.

The next step will be to establish the promised equivalence between $k(\Gamma, A, I)$-representations and $\Lambda$-modules, thus generalizing the well-known result of Gabriel for representations and also Theorem 2.4 from [S], where the equivalence was established only in the case $I = \emptyset$. The construction of the functors $F$ and $G$ is essentially the same of the original proof, but, for completeness, we will repeat it here.

**Theorem 2.3** (compare with [S], Theorem 2.4). There is a a $k$-linear equivalence

$$F : \text{Rep}_k(\Gamma, A, I) \to \text{Mod} k(\Gamma, A, I)$$

which restricts to an equivalence

$$F : \text{rep}_k(\Gamma, A, I) \to \text{mod} k(\Gamma, A, I)$$
Proof. For a given representation $M = ((M_i)_{i \in \Gamma_0}, (M_{α})_{α \in \Gamma_1})$ in $\text{Rep}_k(Γ, Δ, I)$, define

$$F(M) = \bigoplus_{i \in \Gamma_0} M_i$$

which will be an object in $\text{Mod} \ k(Γ, Δ, I)$.

It remains to define the action of $Δ$ over $F(M)$ in such a way that $F(M)$ is indeed an object in $\text{Mod} \ k(Γ, Δ, I)$. This is equivalent to constructing a homomorphism of algebras $Φ : Δ \to \text{End} \ F(M)$. The idea is to use the universal property of tensor algebras (see [8], Lemma 2.1). Let $A_Δ$ and $M_Δ$ be as in Remark 1.2.

First we define a homomorphism of algebras

$$φ_0 : A_Δ \to \text{End}_k F(M)$$

given by

$$φ_0(a_i)((x_t)_{t \in Γ_0}) = (δ_{i,t} x_t a_i)_{t \in Γ_0}$$

for all $i \in Γ_0$, for all $a_i \in A_i$, and all $(x_t)_{t \in Γ_0} \in F(M)$, where $δ_{i,t}$ is a Kronecker’s delta. We also define a morphism of $(A_Δ - A_Δ)$-bimodules

$$φ_1 : M_Δ \to \text{End}_k F(M)$$

as follows: for every $Δ$-path of length 1 $a_i a_j$, where $α : i \to j$ is an arrow of $Γ$, $a_i \in A_i$, $a_j \in A_j$, and for every tuple $(x_t)_{t \in Γ_0} \in F(M)$, define

$$φ_1(a_i a_j)((x_t)_{t \in Γ_0}) = (δ_{i,t} M_{α}(x_t a_i a_j))_{t \in Γ_0}$$

Now, since $k(Γ, Δ) = T(A_Δ, M_Δ)$, by the universal property of tensor algebras ([8], Lemma 2.1), there is a homomorphism of algebras

$$φ : k(Γ, Δ) \to \text{End}_k F(M)$$

uniquely determined by the property that $φ|_{A_Δ} = φ_0$ and $φ|_{M_Δ} = φ_1$. This shows that $F(M)$ is a $k(Γ, Δ)$-module. In order to show that $F(M)$ is a module over $Δ = k(Γ, Δ, I)$, is suffices to show that $φ(I) = 0$, because then, due to the Homomorphism Theorem, $φ$ induces a homomorphism of algebras $Φ : k(Γ, Δ)/I \to \text{End}_k F(M)$.

Therefore let us verify that $φ(I) = 0$. Let $ρ = \sum_{r=1}^t λ_r α_{r,0} \ldots α_{r,n_r}$ be a relation in $I$, where $λ_r \in k$ and the sequences $α_{r,0} \ldots α_{r,n_r}$ be paths over $Γ$ that start and end at the same vertex. And let, for every $1 ≤ r ≤ t$ and $2 ≤ j ≤ n_r$, $γ_{r,j}$ be a path over $Σ_{s(α_{r,j})}$. Then:

$$φ(\sum_{r=1}^t λ_r α_{r,0} \overline{γ_{r,2}} α_{r,2} \ldots \overline{γ_{r,n_r}} α_{r,n_r})$$

$$= \sum_{r=1}^t λ_r φ(α_{r,0} \overline{γ_{r,2}} α_{r,2} \ldots \overline{γ_{r,n_r}} α_{r,n_r})$$

$$= \sum_{r=1}^t λ_r t_e(α_{r,n_r}) \circ M_{α_{r,n_r}} \circ \overline{γ_{r,n_r}} \circ \ldots \circ M_{α_{r,2}} \circ \overline{γ_{r,2}} \circ M_{α_{r,1}} \circ π_{s(α_{r,1})}$$

$$= t_e(α_{r,n_r}) \circ \left( \sum_{r=1}^t λ_r M_{α_{r,n_r}} \circ \overline{γ_{r,n_r}} \circ \ldots \circ M_{α_{r,2}} \circ \overline{γ_{r,2}} \circ M_{α_{r,1}} \right) \circ π_{s(α_{r,1})}$$

$$= 0$$
where $i$ and $\pi$ denote respectively canonical inclusions and projections, and the last equality above holds because $M$ satisfies $\rho$. We need to see how $F$ acts on morphisms.

Let $f = (f_i)_{i \in \Gamma_0} : M \to N$ be a morphism of representations, where $M = ((M_i)_{i \in \Gamma_0}, (\alpha_i)_{i \in \Gamma_1})$ and $N = ((N_i)_{i \in \Gamma_0}, (\gamma_i)_{i \in \Gamma_1})$ are representations satisfying $I$. Then each $f_i : M_i \to N_i$ is a morphism of $A_i$-modules, and thus we may define a linear map

$$F(f) : F(M) = \bigoplus_{i \in \Gamma_0} M_i \to F(N) = \bigoplus_{j \in \Gamma_0} N_j$$

by establishing that the $(i,j)$-th coordinate of $F(f)$ is $\delta_{ij}f_i$. It can be shown that $F(f)$ is a morphism of $\Lambda$-modules and that $F$ defined as such is indeed a functor.

Now we will define that which will be the quasi-inverse functor of $F$:

$$G : \text{Mod } k(\Gamma, \mathcal{A}) \to \text{Rep}_k(\Gamma, \mathcal{A})$$

Let $M$ be a module over $\Lambda$. We need to define a $k(\Gamma, \mathcal{A})$-representation $G(M) = ((M_i)_{i \in \Gamma_0}, (\phi_i)_{i \in \Gamma_1})$ which satisfies $I$.

- For each $i \in \Gamma_0$, $M_i$ is defined by $M_i = M \cdot 1_i$, which is clearly an $A_i$-module.
- For each arrow $\alpha : i \to j \in \Gamma_1$, define the $k$-linear map $M_\alpha : M_i \to M_j$ given by $\phi_i(m) = m \cdot \alpha$.

To show that $G(M)$ thus defined satisfies $I$, let $\rho = \sum_{r=1}^t \lambda_r \alpha_{r1} \ldots \alpha_{rn_r}$ be a relation in $I$, where $\lambda_r \in k$ and the sequences $\alpha_{r1} \ldots \alpha_{rn_r}$ are paths over $\Gamma$ that start and end at the same vertex. Also let, for each $1 \leq r \leq t$ and $2 \leq j \leq n_r$, $\gamma_{rj}$ be a path over $\Sigma_{s(\alpha_{rj})}$. Then, for $m \in M_{s(\alpha_{r1})}$,

$$\left(\sum_{r=1}^t \lambda_r M_{\alpha_{r1}} \circ \gamma_{r1} \circ \ldots \circ M_{\alpha_{rn_r}} \circ \gamma_{r2} \circ M_{\alpha_{r1}}\right)(m)$$

$$= \left(\sum_{r=1}^t \lambda_r M_{\alpha_{r1}} \circ \gamma_{r1} \circ \ldots \circ M_{\alpha_{rn_r}} \circ \gamma_{r2}\right)(m\alpha_{r1})$$

$$= \left(\sum_{r=1}^t \lambda_r M_{\alpha_{r1}} \circ \gamma_{r1} \circ \ldots \circ M_{\alpha_{rn_r}}\right)(m\alpha_{r1}\gamma_{r2})$$

$$= \ldots = \sum_{r=1}^t \lambda_r m\alpha_{r1}\gamma_{r2} \ldots \gamma_{rn_r}\alpha_{rn_r}$$

$$= m \left(\sum_{r=1}^t \lambda_r \alpha_{r1} \gamma_{r2} \ldots \gamma_{rn_r}\alpha_{rn_r}\right)$$

$$= 0$$

The last equality above holds because the expression that multiplies $m$ is equal to $0$ in $\Lambda$. We have thus shown that $G(M)$ is an object in $\text{Rep}_k(\Gamma, \mathcal{A}, I)$.

Let $g : M \to N$ be a morphism in $\text{Mod } \Lambda$. We will define its image under $G$:

$$G(g) = (G(g)_i)_{i \in \Gamma_0}$$

$$G(g)_i : M_i \to N_i, \quad G(g)_i = g|_{M_i}$$

It is immediately verified that $G(g)_i$ is well-defined and is a morphism of $A_i$-modules for every $i \in \Gamma_0$. Let us show that $G(g)$ is a morphism of representations. Let
\(\alpha : i \rightarrow j\) be an arrow in \(\Gamma\). Then, for every \(m \in \mathcal{M}\), \(G(g) \circ M_\alpha(m \cdot 1_i) = G(g) \circ (m \alpha) = g(m)\alpha = G(g) \circ (m \cdot 1_i)\). Therefore \(G(g) \circ M_\alpha = N_\alpha \circ G(g)\), that means to say that the following diagram comutes:

\[
\begin{array}{ccc}
M_i & \xrightarrow{M_\alpha} & M_j \\
G(g) \downarrow & & \downarrow G(g) \\
N_i & \xrightarrow{N_\alpha} & N_j
\end{array}
\]

Therefore \(G(g)\) is a morphism of representations. It is straightforward to show \(G\) defined this way is a functor. It is also directly verified that:

- \(F\) and \(G\) are quasi-inverse functors and are therefore equivalences.
- \(F\) maps finitely generated representations to finitely generated modules, while \(G\) does the opposite. Thus the restrictions of these functors to these subcategories are still quasi-inverse equivalences.

\(\square\)

**Example 2.4.** In this example we will illustrate Theorem 2.3 above. Let \(A\) be the path algebra given by the quiver

\[
\begin{array}{ccc}
1 & \xrightarrow{\alpha} & 2 \\
& \alpha & \\
& \alpha & \beta \\
2 & \xrightarrow{\gamma} & 3
\end{array}
\]

bound by \(\gamma^n = 0\), where \(n > 1\). Then consider the gbp-algebra \(\Lambda = k(\Gamma, \mathcal{A}, I)\), where \(\Gamma\) is the quiver below:

\[
\begin{array}{ccc}
1 & \xrightarrow{\alpha} & 2 \\
& \alpha & \\
& \alpha & \beta \\
2 & \xrightarrow{\gamma} & 3
\end{array}
\]

And where \(\mathcal{A} = \{A_1, A_2, A_3\}\), with \(A_1 = A_3 = k\), \(A_2 = A\), and \(I = (\alpha \beta)\). More simply, \(\Lambda\) is the gbp-algebra given by

\[
\begin{array}{ccc}
1 & \xrightarrow{\alpha} & A \\
& \alpha & \\
& \alpha & \beta \\
A & \xrightarrow{\gamma} & k
\end{array}
\]

bound by \(\alpha \beta = 0\). Using the proof of Theorem 2.3 we are going to calculate the representation associated with the projective \(\Lambda\)-module \(P = 1_{A_1} \Lambda\).

We have that \(P_1 = P.1_{A_1} = 1_{A_1} \Lambda.1_{A_1} = (1_{A_1})\) is the \(k\)-vector space spanned by \(1_{A_1}\). Moreover, \(P_2 = P.1_{A_2} = 1_{A_1} \Lambda.1_{A_2} = (\alpha, \alpha \gamma, \ldots, \alpha \gamma^{n-1})\), which is a right \(\mathcal{A}\)-module easily seen to be isomorphic to the regular \(\mathcal{A}\)-module \(A\). And also \(P_3 = P.1_{A_3} = 1_{A_1} \Lambda.1_{A_3} = 0\) since \(I = (\alpha \beta)\) and thus every \(\mathcal{A}\)-path of the form \(\alpha \gamma^i \beta\) for \(i \geq 0\) is identified with 0 in \(\Lambda\).

Now we have that \(P_\alpha\) is given by right multiplication by \(\alpha\), so it maps the single element of the basis of \(P_1\), which is \(1_{A_1}\), to \(1_{A_1} \cdot \alpha = \alpha \cdot 1_{P_2}\).

If we identify \(P_2 \cong A\) and consider the \(k\)-basis \(\{1, \gamma, \ldots, \gamma^{n-1}\}\) for \(A\), we may conclude that the representation associated with the \(\Lambda\)-module \(P\) is the following:

\[
P : \quad k \begin{bmatrix} 1 & 0 & \ldots & 0 \end{bmatrix}^T \rightarrow A \rightarrow 0
\]

Having obtained the equivalence in Theorem 2.3 as a tool, we are in conditions to study, over the course of the following sections, the representations associated to simple, projective and injective modules over a gbp-algebra, thus generalizing the well-known description that is done for ordinary path algebras.
2.1. Opposite algebra. The aim of this subsection is to obtain some useful lemmas involving opposite algebras, opposite quivers and the duality functor. Again we refer to [2] for the definition of these concepts. For a quiver $\Gamma$, denote by $\Gamma^{op}$ its opposite quiver (that is, the quiver with the same vertices of $\Gamma$ and with all its arrows reversed). For a set $I$ of relations in $\Gamma$, $I^{op}$ will denote the set of relations in $\Gamma^{op}$ obtained through inversion of the arrows in $I$. Also, if $\mathcal{A} = \{A_i : i \in \Gamma_0\}$ is a family of algebras, denote by $\mathcal{A}^{op} = \{A_i^{op} : i \in \Gamma_0\}$ the set where $A_i^{op}$ is the opposite algebra of $A_i$. With these notations, we have the following:

Proposition 2.5. If $\Lambda = k(\Gamma, \mathcal{A}, I)$ is a gbp-algebra, then $\Lambda^{op} \cong k(\Gamma^{op}, \mathcal{A}^{op}, I^{op})$.

Proof. As recalled in the preliminaries, the generalized path algebra $k(\Gamma, \mathcal{A})$ is a quotient of a vector space denoted as $k[\Gamma, \mathcal{A}]$ by a subspace generated by linearity relations. Let us then use the following auxiliary notation: $k(\Gamma, \mathcal{A}) \cong k[\Gamma, \mathcal{A}]/\sim$. In order to avoid confusion, let us also denote the equivalence class (relatively to $\sim$) of an $\mathcal{A}$-path $x$ by $[x]$. With these notations we can define a $k$-linear map

$$\overline{\phi} : k[\Gamma, \mathcal{A}] \rightarrow k(\Gamma^{op}, \mathcal{A}^{op})$$

by defining it in the $k$-basis of $k[\Gamma, \mathcal{A}]$:

$$\overline{\phi}(a_0 \beta_1 a_1 \ldots a_{r-1} \beta_r a_r) = [a_r \beta_r a_{r-1} \ldots a_1 \beta_1 a_0]$$

for each $\mathcal{A}$-path $a_0 \beta_1 a_1 \ldots a_{r-1} \beta_r a_r$. Then we must show that $\sim \subseteq \ker \overline{\phi}$. Indeed:

$$\overline{\phi}(a_0 \beta_1 a_1 \ldots (a_i^1 + \ldots + a_i^s) \ldots a_{r-1} \beta_r a_r - \sum_{j=1}^{s} a_0 \beta_1 a_1 \ldots a_i^j \ldots a_{r-1} \beta_r a_r) =$$

$$= \overline{\phi}(a_0 \beta_1 a_1 \ldots (a_i^1 + \ldots + a_i^s) \ldots a_{r-1} \beta_r a_r) - \sum_{j=1}^{s} \overline{\phi}(a_0 \beta_1 a_1 \ldots a_i^j \ldots a_{r-1} \beta_r a_r) =$$

$$= [a_r \beta_r a_{r-1} \ldots (a_i^1 + \ldots + a_i^s) \ldots a_1 \beta_1 a_0] - \sum_{j=1}^{s} [a_r \beta_r a_{r-1} \ldots a_i^j \ldots a_1 \beta_1 a_0] = 0$$

and, for $\lambda \in k$,

$$\overline{\phi}(a_0 \beta_1 a_1 \ldots \lambda a_i \ldots a_{r-1} \beta_r a_r - \lambda(a_0 \beta_1 a_1 \ldots a_i \ldots a_{r-1} \beta_r a_r)) =$$

$$= \overline{\phi}(a_0 \beta_1 a_1 \ldots a_i \ldots a_{r-1} \beta_r a_r) - \lambda \overline{\phi}(a_0 \beta_1 a_1 \ldots a_i \ldots a_{r-1} \beta_r a_r) =$$

$$= [a_r \beta_r a_{r-1} \ldots \lambda a_i \ldots a_1 \beta_1 a_0] - \lambda[a_r \beta_r a_{r-1} \ldots a_i \ldots a_1 \beta_1 a_0] = 0$$

We have just shown that there is a $k$-linear map

$$\phi : k(\Gamma, \mathcal{A}) \rightarrow k(\Gamma^{op}, \mathcal{A}^{op})$$

that satisfies

$$\phi([a_0 \beta_1 a_1 \ldots a_{r-1} \beta_r a_r]) = [a_r \beta_r a_{r-1} \ldots a_1 \beta_1 a_0]$$

It is easy to see that $\phi$ is bijective. To conclude the first part of the statement, it remains to show that $\phi$ is an anti-homomorphism of algebras. It is easy to see that $\phi$ preserves the identity element. We will thus show that it antipreserves multiplication. Let $a = [a_0 \beta_1 a_1 \ldots a_{r-1} \beta_r a_r]$ and $b = [b_0 \gamma_1 b_1 \ldots b_{s-1} \gamma_s b_s]$ be the
classes of two $A$-paths. If $e(\beta_r) \neq s(\gamma_1)$, it is straightforward to show that $\phi(ab) = 0 = \phi(b)\phi(a)$. So suppose that $e(\beta_r) = s(\gamma_1)$. In this case,

$$\phi(ab) = \phi([a_0\beta_1a_1 \ldots a_{r-1}\beta ra_r][b_0\gamma_1b_1 \ldots b_{s-1}\gamma sb_s])$$

$$= \phi([a_0\beta_1a_1 \ldots a_{r-1}\beta ra_r(b_r)\gamma_1b_1 \ldots b_{s-1}\gamma sb_s])$$

$$= [b_s\gamma sb_{s-1} \ldots b_1\gamma_1(b_r\beta ra_r)\gamma_1b_1 \ldots a_1\beta_1a_0]$$

$$= [b_s\gamma sb_{s-1} \ldots b_1\gamma_1(b_0\alpha_0r)\beta ra_r \ldots a_1\beta_1a_0]$$

$$= [b_s\gamma sb_{s-1} \ldots b_1\gamma_1[b_r\beta ra_r \ldots a_1\beta_1a_0a_0]$$

$$= \phi([b_0\gamma b_1 \ldots b_{s-1}\gamma sb_s])\phi([a_0\beta_1a_1 \ldots a_{r-1}\beta ra_r]) = \phi(b)\phi(a)$$

This proves that $k(\Gamma, A)$ is anti-isomorphic to $k(\Gamma^op, A^op)$ via $\phi$, which is the same to say that $k(\Gamma, A)^op$ is isomorphic to $k(\Gamma^op, A^op)$. To conclude the proof, we realize that the map $\phi$ defined above satisfies $\phi(I) = I^op$, and the statement follows directly.

\[\square\]

2.2. Duality. We now use the results of the previous subsection to dualize the representations of the gbp-algebra $\Lambda$. Denote by $D = \text{Hom}_k(-, k)$ the duality functor.

**Proposition 2.6.** Let $\Lambda = k(\Gamma, A, I)$ be a gbp-algebra. If $((M_i)_{i \in I_\alpha}, (\phi_\alpha)_{\alpha \in \Gamma_1})$ is the representation of the $\Lambda$-module $M$, then the representation of the $\Lambda^op$-module $DM$ is isomorphic to $(D(M_i)_{i \in I_\alpha}, D(\phi_\alpha)_{\alpha \in \Gamma_1})$.

**Proof.** We need to show that the representations $((\text{Hom}_{\Lambda^op}(M_i)_{i \in I_\alpha}, (\text{Hom}_{\Lambda^op}(\phi_\alpha)_{\alpha \in \Gamma_1})$ and $(D(M_i)_{i \in I_\alpha}, D(\phi_\alpha)_{\alpha \in \Gamma_1})$ are isomorphic. It is useful to recall how the quasi-inverse equivalences $F$ and $G$ discussed in the proof of Theorem 2.3 were like. Let $i \in \Gamma_0$.

First of all, note that

$$DM = \text{Hom}_k(M, k), \text{ thus } (DM)_i = 1_i(\text{Hom}_k(M, k))$$

$$D(M_i) = \text{Hom}_k(M_i, k) = \text{Hom}_k(M \cdot 1_i, k)$$

We can define

$$f_i : 1_i \text{Hom}_k(M, k) \to \text{Hom}_k(M \cdot 1_i, k)$$

$$1_i \cdot g \mapsto g|_{M \cdot 1_i}$$

We shall see that $f_i$ is an isomorphism. It is clear that it is well-defined and $k$-linear.

To show that $f_i$ is a morphism of $A_i^op$-modules, let $g \in \text{Hom}_k(M, k)$, $a \in A_i^op$ and $x \in M \cdot 1_i$. Then

$$f_i(a \cdot 1_i g)(x) = (a \cdot g)|_{M \cdot 1_i}(x) = (a \cdot g)(x) = g(ax) = g(xa_{1_i}) =$$

$$= g|_{M \cdot 1_i}(xa) = f_i(1_i g)(xa) = (a \cdot f_i(1_i g))(x)$$

which implies that $f_i(a \cdot 1_i g) = a \cdot f_i(1_i g)$, as required.

Now, to see that $f_i$ is injective, suppose $f_i(1_i g) = 0$. Then $(1_i g)(x) = 0$ for every $x \in M \cdot 1_i$ and so $(1_i \cdot g)(x) = (1_i \cdot g)(x \cdot 1_i) = 0$ for every $x \in M$. In particular, $1_i \cdot g = 0$, which shows our claim.

It remains to see that $f_i$ is surjective. Let $h \in \text{Hom}_k(M \cdot 1_i, k)$. We know that $M \cong \oplus_{j \in \Gamma_0} M \cdot 1_j$. We can thus define a $k$-linear transformation $g \in \text{Hom}_k(M, k)$, $g : \oplus_{j \in \Gamma_0} M \cdot 1_j \to k$, $g = (\delta_{ji} h)_{j \in \Gamma_0}$, where $\delta_{ji}$ is a Kronecker’s delta. Then, if $x \in M \cdot 1_i$, $f_i(1_i \cdot g)(x) = g|_{M \cdot 1_i}(x) = h(x)$. Thus $f_i(1_i \cdot g) = h$. This concludes
the proof that \( f_i \) is an isomorphism of \( A_i \)-modules. The next step is to show the commutativity of the diagram

\[
\begin{array}{ccc}
(DM)_j & \xrightarrow{(DM)_i} & (DM)_i \\
\downarrow f_j & & \downarrow f_i \\
D(M_j) & \xrightarrow{D(\phi_\alpha)} & D(M_i)
\end{array}
\]

For that, let \( g \in \text{Hom}_k(M,k) \) and \( e_i \in M \). Then:

\[
(f_i \circ (DM)_\alpha)(1_j.g)(x.1_i) = f_i((DM)_\alpha(1_j.g))(x.1_i) = f_i(1_iag)(x.1_i) = (ag)|_{M,1}(x.1_i) = g(x\alpha) = g|_{M,1_j}(x\alpha_1 j) = g|_{M,1_j}(\phi_\alpha(x.1_i)) = D(\phi_\alpha)(g|_{M,1_j})(x.1_i) = D(\phi_\alpha)(f_j(1_i.g))(x.1_i)
\]

Hence \( f_i \circ (DM)_\alpha \) = \( (D(\phi_\alpha) \circ f_j)(1_i.g)(x.1_i) \) as was required. The fact that \( DM \) satisfies \( I^{op} \) if and only if \( M \) satisfies \( I \) follows easily from the fact that \( D \) is a fully faithful and dense \( k \)-linear functor. \( \square \)

3. Realizing an \( A_i \)-module as a \( \Lambda \)-module

Let \( i \in \Gamma_0 \), and let \( M_{A_i} \) be a (right) \( A_i \)-module. In this section we shall see three ways of viewing \( M \) as a \( \Lambda \)-module. The first one is quite natural, while the second one essentially relies on the well-known technique of extension of scalars. Dualizing such a construction, we get a third way. It will be interesting to dedicate different notations for each of the three.

3.1. The inclusion functors. Given an \( A_i \)-module \( M \), define the \( \Lambda \)-representation \( I(M) = (M_j)_{j \in \Gamma_0}, (\phi_\alpha)_{\alpha \in \Gamma_1} \) given by

\[
M_j = \begin{cases} 
M & \text{if } j = i \\
0 & \text{if } j \neq i
\end{cases} \quad \text{and} \quad \phi_\alpha = 0 \quad \text{for all } \alpha \in \Gamma_1.
\]

Clearly, because of Theorem 2.3, \( I(M) \) yields a \( \Lambda \)-module, and, since \( I(M) \) and \( M \) have the same underlying vector space, we may, by abuse of notation, denote \( I(M) = M \).

Actually, for every vertex \( i \) we have a functor \( I_i : \text{mod} \ A_i \to \text{mod} \ \Lambda \) which we shall call inclusion functor. (We might even denote it simply by \( I \) if it is clear what vertex we are talking about). We have just defined its image on objects, and its image on morphisms is defined obviously. It is also easy to see why \( I \) is called an inclusion functor, because it is covariant and fully faithful.

From now on, unless stated or denoted otherwise, we will always be assuming that we are seeing \( M \) as an \( \Lambda \)-module in this way.

Remark 3.1. It is not difficult to see that simple \( A_i \)-modules viewed as \( \Lambda \)-modules are also simple. Conversely, any simple \( \Lambda \)-module is of this kind. This follows from a counting argument (see [5] for example). So, the simple \( \Lambda \)-modules can be determined easily.

\[\text{Reference:}\] [5]
3.2. Cones. We shall now see another way to view an $A_i$-module $M$ as a $\Lambda$-module.

Here again, let $k(\Gamma, A) = T(A, M) = M_A$ as in Remark 3.2. Clearly, $M$ is also an $A_i$-module (using the action $m \cdot (a_j)_{j \in \Gamma} = m \cdot a_i$ for each $m \in M$ and $(a_j)_{j \in \Gamma} \in A_A$).

Since $\Lambda$ is equal to the quotient $k(\Gamma, A)/I$, and $M_A$ is an $(A_A - A_A)$-bimodule, $\Lambda$ is also an $(A_A - A_A)$-bimodule that contains $A_A$ as a subalgebra. Therefore it makes sense to consider the extension of scalars of $M$ to $\Lambda$. We shall denote it by $C_i(M) = M \otimes_{A_A} \Lambda$. Just emphasizing, since $\Lambda$ is a right $\Lambda$-module, $C_i(M)$ is a right $\Lambda$-module too.

**Definition 3.2.** $C_i(M)$ is called **cone** over $M$.

The reason why we call it a cone is because of the shape that the representation of $C_i(M)$ has, as it will be more transparent after the description that will be done here later.

**Proposition 3.3.** If $M$ and $N$ are $A_1$-modules, then $C_i(M \oplus N) \cong C_i(M) \oplus C_i(N)$.

**Proof.** Just observe that

$$C_i(M \oplus N) = (M \oplus N) \otimes_{A_A} \Lambda \cong (M \otimes_{A_A} \Lambda) \oplus (N \otimes_{A_A} \Lambda) = C_i(M) \oplus C_i(N).$$

$\square$

**Remark 3.4.** Since we are assuming $\Gamma$ to be acyclic, it will be useful to remark that

$$C_i(M) = \left\{ \sum_{\gamma = \gamma_1 \cdots \gamma_r \text{ in a path in } \Gamma} m^\gamma \otimes \gamma_1 a_1^{(\gamma_1)} \cdots \gamma_r a_r^{(\gamma_r)} : m^\gamma \in M, a_i^{(\gamma_i)} \in A_i(\gamma_i) \right\}.$$

This equality follows by observing that $C_i(M) = M \otimes_{A_A} \Lambda = M \cdot 1_i \otimes_{A_A} \Lambda = M \otimes_{A_A} 1_i \cdot \Lambda$.

The next goal of this subsection is to describe the representation associated to the cone $C_i(M)$ of an $A_i$-module $M$.

Let $((M_j)_{j \in \Gamma_0}, (\phi_{\alpha})_{\alpha \in \Gamma_1})$ denote the representation of $M$. For each $l \in \Gamma_0$, let $\{a_1^l, \ldots, a_{\dim_k A_i}^l\}$ denote a $k$-basis of $A_i$. Also, let $\{m_1, \ldots, m_{\dim_k M}\}$ be a $k$-basis of $M$.

**Proposition 3.5.** With the notations above, it holds that $M_i = M$, and if $j \in \Gamma_0$ is different from $i$, then $M_j$ is isomorphic to the free $A_j$-module having as basis the set of equivalence classes of the formal sequences of the form

$$m_p a_i^{(\gamma_2)} \cdots a_i^{(\gamma_r)} \gamma_r$$

where $\gamma_1 \ldots \gamma_r$ is a path from $i$ to $j$, $1 \leq p \leq \dim_k M$ and $1 \leq i_l \leq \dim_k A_i(\gamma_l)$ for every $1 < l \leq r$.

Moreover, if $\alpha : j \rightarrow j'$ is an arrow, then $\phi_{\alpha}$ is the only linear transformation that satisfies

$$\phi_{\alpha} (m_p a_i^{(\gamma_2)} \cdots a_i^{(\gamma_r)} \alpha) = m_p a_i^{(\gamma_2)} \cdots a_i^{(\gamma_r)} a_{i_{r+1}}^{(\gamma_r)} \gamma_r \alpha.$$

**Proof.** The key idea here is to recall the equivalence $G$ constructed in the proof of Theorem [2.3] By Remark 3.4 above, and by the fact that $\Gamma$ is acyclic,

$$M_i = C_i(M) \cdot 1_i \cong \left\{ \sum_{\gamma : l = i} m^\gamma : m^\gamma \in M \right\} = \{m : m \in M\} = M.$$
For $j \neq i$, we have that

$$M_j = C_i(M) \cdot 1_j = \left\{ \sum_{\gamma = \gamma_1 \ldots \gamma_r; i \rightarrow j} m^\gamma \otimes \gamma_1 a_2^\gamma \gamma_2 \ldots a_r^\gamma \gamma_r a_{r+1}^\gamma : m^\gamma \in M, a_l^\gamma \in A_{s(\gamma_l)} \forall 1 \leq l \leq r, \text{ and } a_{r+1}^\gamma \in A_j \right\}$$

Since $\{a_1^j, \ldots, a_{\dim_k A_i}^j\}$ is a $k$-basis of $A_i$ and $\{m_1, \ldots, m_{\dim_k M}\}$ is a $k$-basis of $M$, the above expression equals to

$$\text{span}_k \{m_p \otimes \gamma_1 a_2^\gamma \gamma_2 \ldots a_r^\gamma \gamma_r a_{r+1}^\gamma : \gamma_1 \ldots \gamma_r \text{ is a path } i \rightarrow j, \ 1 \leq p \leq \dim_k M, 1 \leq i_t \leq \dim_k A_{s(\gamma_t)} \forall 1 \leq l \leq r, \text{ and } a_{r+1}^\gamma \in A_j \}$$

(3.1)

If one denotes $\{\theta_1, \ldots, \theta_{n_j}\} = \{m_p \otimes \gamma_1 a_2^\gamma \gamma_2 \ldots a_r^\gamma \gamma_r \gamma_{r+1}^\gamma : \gamma_1 \ldots \gamma_r \text{ is a path } i \rightarrow j, \ 1 \leq p \leq \dim_k M, 1 \leq i_t \leq \dim_k A_{s(\gamma_t)} \forall 1 \leq l \leq r, \text{ and } a_{r+1}^\gamma \in A_j \}$, then the expression [3.1] is equal to

$$\text{span}_k \{\theta a : 1 \leq l \leq n_j, a \in A_j\}.$$ 

An easy calculation shows that it is isomorphic to the free $A_j$-module having as basis $\{\theta_1, \ldots, \theta_{n_j}\}$, as we wanted to prove.

Let $\alpha : j \rightarrow j'$ be an arrow in $\Gamma_1$. Again, by Theorem 2.3, $\phi_\alpha : M_j \rightarrow M_{j'}$ is given by

$$\phi_\alpha : C_i(M) \cdot 1_j \rightarrow C_i(M) \cdot 1_{j'}$$

$$m1_j \mapsto m\alpha$$

with $m \in C_i(M)$. Therefore $\phi_\alpha$ has the form given in the statement, concluding the proof.

\[\square\]

**Remark 3.6.** If $I = 0$, then it is easier to see how the representation of $C_i(M)$ looks like: it holds that $M_i = M$, and if $j \neq i$, $M_j \cong A_{n_j}$, where

$$n_j = \sum_{\gamma;i = i_0 \rightarrow i_1 \ldots \rightarrow i_{r+1} = j \text{ is a path } i \rightarrow j} (\dim_k M). (\dim_k A_{i_1}). \ldots. (\dim_k A_{i_r})$$

In particular, if there is no path going from $i$ to $j$, $M_j = 0$.

We finish this subsection with the following result.

**Proposition 3.7.** Given $i \in \Gamma_0$, the cone functor $C_i : \text{mod } A_i \rightarrow \text{mod } \Lambda$ is exact.

**Proof.** By definition, $C_i \cong I_i(-) \otimes_{A_i} \Lambda$. Since the inclusion functor $I_i : \text{mod } A_i \rightarrow \text{mod } \Lambda$ is easily seen to be exact and a tensor product $- \otimes_{A_i} \Lambda$ is always right exact, $C_i$ is right exact. Our work here is to prove that $C_i$ maps monomorphisms to monomorphisms, because then $C_i$ will also be left exact and thus exact, concluding the proof. So let $f : M \rightarrow N$ be a monomorphism between $A_i$-modules. Then it is sufficient to fix $j \in \Gamma_0$ and prove that $(C_i(f))_j : (C_i(M))_j \rightarrow (C_i(N))_j$ is a monomorphism of $A_j$-modules.

If there is no path $i \sim j$ in $\Gamma$, then we know that $(C_i(f))_j$ will be a zero map between two zero modules and thus a monomorphism. So we may suppose that there are paths of the form $i \sim j$ in $\Gamma$.

Then, if $\{m_1, \ldots, m_r\}$ is a $k$-basis of $M$, the set $\{f(m_1), \ldots, f(m_r)\} \subset N$ will be linearly independent. Therefore, if we denote $f(m_l) = n_l$ for every $l$, we can
complete this set to a $k$-basis of $N$: \{ $n_1, \ldots , n_r, \ldots , n_s$ \}. Also, for every vertex $l$, let \{ $a_1^l, a_2^l, \ldots , a_n^l$ \} be a $k$-basis of $A_l$.

Let $\gamma: i = l_0 \rightarrow l_1 \rightarrow \ldots \rightarrow l_t = j$ be a path between $i$ and $j$ in $\Gamma$. Then we denote

\[
\theta_{\gamma,h,i_1,\.\.\.i_{t-1}} = m_h \otimes \gamma_1 a_{i_1}^l \gamma_2 \ldots \gamma_{t-1} a_{i_{t-1}}^l \in C_i(M)
\]

\[
\zeta_{\gamma,h,i_1,\.\.\.i_{t-1}} = n_h \otimes \gamma_1 a_{i_1}^l \gamma_2 \ldots \gamma_{t-1} a_{i_{t-1}}^l \in C_i(N)
\]

And we note that

\[
C_i(f)(\theta_{\gamma,h,i_1,\.\.\.i_{t-1}}) = (f \otimes 1A)(m_h \otimes \gamma_1 a_{i_1}^l \gamma_2 \ldots \gamma_{t-1} a_{i_{t-1}}^l)
\]

\[
= f(m_h) \otimes \gamma_1 a_{i_1}^l \gamma_2 \ldots \gamma_{t-1} a_{i_{t-1}}^l
\]

\[
= n_h \otimes \gamma_1 a_{i_1}^l \gamma_2 \ldots \gamma_{t-1} a_{i_{t-1}}^l = \zeta_{\gamma,h,i_1,\.\.\.i_{t-1}}
\]

By Proposition 3.3, we know that $(C_i(M))_j$ is the free $A_j$-module generated by the $\theta_{\gamma,h,i_1,\.\.\.i_{t-1}}$, while $(C_i(N))_j$ is the free $A_j$-module generated by the $\zeta_{\gamma,h,i_1,\.\.\.i_{t-1}}$. So $(C_i(f))_j$, which is a restriction of $C_i(f)$, is a morphism that takes a basis of $(C_i(M))_j$ to a subset of a basis of $(C_i(N))_j$. Therefore, it must be a monomorphism, concluding the proof. \hfill \Box

3.3. **Dual cones.** We now dualize the notion of cone.

**Definition 3.8.** Let $i \in \Gamma_0$, and let $M$ be an $A_i$-module. Then $D(M)$ is an $A_i^{\text{op}}$-module, and therefore the cone $C_i(DM)$ is a $\Lambda^{\text{op}}$-module. Finally, $D(C_i(DM))$ is a $\Lambda$-module, which we call **dual cone** of $M$. We shall use the notation $C_i^*(M) \equiv D(C_i(DM))$.

**Proposition 3.9.** Given two $A_i$-modules $M$ and $N$, $C_i^*(M \oplus N) \cong C_i^*(M) \oplus C_i^*(N)$.

**Proof.** This follows because the duality functor preserves direct sums and because $C_i$ also preserves direct sums due to Proposition 3.3. \hfill \Box

**Example 3.10.** Let us give an example to illustrate the differences between the three ways of realizing an $A_i$-module as a $\Lambda$-module seen in this section.

Let $A$ and $B$ be two finite dimensional algebras over the base field $k$. Suppose that $A$ has dimension 2 over $k$ and that $B$ has dimension 3. Consider the gp-algebra $\Lambda$ given below:

\[
\begin{array}{ccc}
B & \rightarrow & A \\
\downarrow & & \downarrow \\
B & \rightarrow & k & \rightarrow & A & \rightarrow & B
\end{array}
\]

bound by $\alpha \beta = 0$. Let $x$ be the vertex of the quiver above to which $k$ was assigned. If we consider $k^4$ as a $\Lambda$-module via the inclusion functor relative to $x$, its representation will be

\[
\begin{array}{ccc}
0 & \rightarrow & 0 \\
\downarrow & & \downarrow \\
0 & \rightarrow & k^4 & \rightarrow & 0 & \rightarrow & 0
\end{array}
\]
By using Proposition 3.5 above, one concludes that the representation of $C_x(k^4)$, which is the cone of $k^4$, will be

$$
\begin{array}{ccc}
B^4 & \longrightarrow & A^{12} \\
\uparrow & & \uparrow \\
0 & \longrightarrow & k^4 & \longrightarrow & A^4 & \longrightarrow & 0
\end{array}
$$

The bottom right vertex needs to be assigned with 0 as a consequence of the existence of the relation $\alpha \beta = 0$. Note how the representation of $C_x(k^4)$ resembles a cone whose vertex is $x$ and whose basis is the set of vertices which are the end of non-zero paths starting at $x$. This is to complement our previous remark explaining why we are calling the functor $C_x$ a cone. Finally, the dual cone $C_x^*(k^4)$ of $k^4$ will be given by

$$
\begin{array}{ccc}
0 & \longrightarrow & 0 \\
\uparrow & & \uparrow \\
B^4 & \longrightarrow & k^4 & \longrightarrow & 0 & \longrightarrow & 0
\end{array}
$$

Remark 3.11. We gave above a description of the representations associated with cones. That is, we already know how to calculate cones. Thanks to Proposition 2.6, calculating dual cones will not present a difficulty any bigger: given an $A_i$-module $M$, we calculate the cone of $DM$ over $(\Gamma^{op}, A^{op}, I^{op})$ and then obtain the dual cone of $M$ over $(\Gamma, A, I)$ using Proposition 2.6. This proposition tells us that what we need to do is to take the duals of the modules in each vertex and take the transpose linear transformation in each arrow, which, practically, is done by transposing matrices. We shall yield examples of this in Subsection 4.2.

4. PROJECTIVE AND INJECTIVE REPRESENTATIONS

We shall now apply the results of the previous subsection to describe the indecomposable projective and injective $\Lambda$-modules. We remark that [9] contains a description of projective modules over generalized path algebras, although here we manage to extend this to the context of gbp-algebras.

4.1. Projective representations. We start with the following result.

**Proposition 4.1.** If $P$ is a projective $A_i$-module, then $C_i(P)$ is a projective $\Lambda$-module.

**Proof.** Let $g : M \rightarrow N$ be an epimorphism of $\Lambda$-modules. Since $\Lambda$ is a projective $\Lambda$-module,

$$
\text{Hom}_\Lambda(\Lambda, g) : \text{Hom}_\Lambda(\Lambda, M) \rightarrow \text{Hom}_\Lambda(\Lambda, N)
$$

is an epimorphism. Since $A_i = 1_i A_A$ and $1_i$ is an idempotent element of $A_A$, $A_i$ is a projective $A_A$-module. By hypothesis, $P$ is a direct summand of some $A_i$-module of the form $A_A^m$, with $m \in \mathbb{N}$, and thus also $P$ is projective as an $A_A$-module. It follows that

$$
\text{Hom}_{A_A}(P, \text{Hom}_\Lambda(\Lambda, g)) : \text{Hom}_{A_A}(P, \text{Hom}_\Lambda(\Lambda, M)) \rightarrow \text{Hom}_{A_A}(P, \text{Hom}_\Lambda(\Lambda, N))
$$

is an epimorphism. Finally, by the Adjunction Theorem,

$$
\text{Hom}_\Lambda(P \otimes_{A_A} \Lambda, g) : \text{Hom}_\Lambda(P \otimes_{A_A} \Lambda, M) \rightarrow \text{Hom}_\Lambda(P \otimes_{A_A} \Lambda, N)
$$
is an epimorphism. This proves that $P \otimes_{A \Lambda} \Lambda$ is a projective $\Lambda$-module. \qed

Now, for each $i \in \Gamma_0$, let $E_i = \{e_{i1}, \ldots, e_{is_i}\}$ be a complete set of primitive idempotent and pairwise orthogonal elements in $A_i$. Then every indecomposable projective $A_i$-module is isomorphic to $P_{ij}^i \cong e_{ij}A_i$ for some $1 \leq j \leq s_i$. Moreover, $E = \{e_{ij} : i \in \Gamma_0, 1 \leq j \leq s_i\}$ is a complete set of primitive idempotent and pairwise orthogonal elements in $\Lambda$. Therefore every indecomposable projective $\Lambda$-module is isomorphic to $P(i, j) \cong e_{ij}\Lambda$ for a certain pair of indexes $i \in \Gamma_0$ and $1 \leq j \leq s_i$.

**Proposition 4.2.** For each $i \in \Gamma_0$ and $1 \leq j \leq s_i$, $P(i, j) = C_i(B_i^j)$.

**Proof.** Using Remark 3.4, we have that

$$
C_i(B_i^j) = \left\{ \sum_{\gamma = \gamma_1 \cdots \gamma_l \in \Gamma \atop s(\gamma_1) = i} m\gamma \otimes \gamma_1a_{\gamma_1}(\ldots)\gamma_l a_{\gamma_l}(\ldots) : m\gamma \in B_i^j, a_{\gamma_i}(\ldots) \in A_{s(\gamma_l)} \right\}
$$

$$
= \left\{ \sum_{\gamma = \gamma_1 \cdots \gamma_l \in \Gamma \atop s(\gamma_1) = i} e_{ij}a_{\gamma_1}(\ldots)\gamma_l a_{\gamma_l}(\ldots) : a_i(\ldots) \in A_i, a_{\gamma_i}(\ldots) \in A_{s(\gamma_l)} \right\}
$$

$$
= e_{ij}\Lambda = P(i, j)
$$

\qed

Thanks to the last proposition and Proposition 3.5, we are now able to calculate the representations associated to projective indecomposable modules. The following proposition reflects the particular case of this construction when $I = 0$, i.e., when there are no relations:

**Proposition 4.3.** Suppose $I = 0$. Let $P(i, j) = ((M_i)_{i \in \Gamma_0}, (\phi_\alpha)_{\alpha \in \Gamma_0})$ be the representation associated to $P(i, j)$. Then, for $l \in \Gamma_0$.

If $l = i$, then $M_l = M_i = P_i^i$.

If $l \neq i$, denote

$$
n_l = \sum_{\gamma : i = i_0 \rightarrow i_1 \rightarrow \ldots \rightarrow i_r = l} (\dim_k P_i^i)(\dim_k A_{i_1}) \ldots (\dim_k A_{i_{r-1}})
$$

where $\gamma$ runs through all possible paths $i \rightsquigarrow l$. Then $M_l \cong (A_l)^{n_l}$ as $A_l$-modules. In particular, if there are no paths $i \rightsquigarrow l$, then $M_l = 0$.

In practical examples, however, difficulties may arise either because the matrices of the $k$-linear transformations denoted above as $\phi_\alpha$ can be too big, or, given their dependence on the choice of a $k$-basis of the algebras $A_i$ or of $P_i^i$, there could be some confusion. To avoid that, it is convenient to make use of block matrices. We shall give further details of this in the remark and example below.

**Remark 4.4.** Let $V$ be a $k$-vector space of dimension 1 and fixed basis $\{v\}$ and let $A$ be a $k$-algebra. Then there is a linear map that shall be treated as canonical from now on: it is defined as $\mu : V \rightarrow A$, $\mu(\lambda \cdot v) = \lambda \cdot 1_A$, where $\lambda \in k$. Although the vector space $V$ may vary, the letter $\mu$ will always be used for such a map.
Example 4.5. Let $A$ be the path algebra given by the quiver below:

$$
\begin{array}{c}
1 \\
\longrightarrow \\
2
\end{array}
$$

Then there are two indecomposable projective $A$-modules, namely,

$$
P_1 : \begin{array}{c}
k \\
\longrightarrow \\
\kappa
\end{array}
$$

Now let $\Lambda$ be the GPA given by

$$
\begin{array}{c}
A \\
\longrightarrow \\
A
\end{array}
$$

According to the discussions above, there are exactly 4 indecomposable projective $\Lambda$-modules, which are:

$$
P(1, 1) : P_1 \xrightarrow{\begin{bmatrix} \mu & 0 \\ 0 & \mu \end{bmatrix}} A^2
$$

$$
P(1, 2) : P_2 \xrightarrow{\mu} A
$$

$$
P(2, 1) : 0 \longrightarrow P_1
$$

$$
P(2, 2) : 0 \longrightarrow P_2
$$

We also have conditions to describe the representations associated to radicals of the projective modules, as expressed in the proposition below:

Proposition 4.6. With the same notations as before, let $i \in \Gamma_0$ and $1 \leq j \leq s_i$. Denote $P(i, j) = ((M_l)_{l \in \Gamma_0}, (\phi_\alpha)_{\alpha \in \Gamma_1})$. Then the radical of $P(i, j)$ is given by the representation $\text{rad} P(i, j) = ((N_l)_{l \in \Gamma_0}, (\psi_\alpha)_{\alpha \in \Gamma_1})$, where $N_l = \text{rad} P_l^1, N_i = M_i$ for each $l \in \Gamma_0$ with $l \neq i$, and for each $\alpha \in \Gamma_1$, $\psi_\alpha = \phi_\alpha|_{M_{s(\alpha)}}$.

Proof. Let $N = ((N_l)_{l \in \Gamma_0}, (\psi_\alpha)_{\alpha \in \Gamma_1})$. Note that $N$ satisfies $I$ because $M$ satisfies it. We wish to prove that $N = \text{rad} P(i, j)$. Note that, if $l \neq i$, $N_l = M_l$, so $M_l/N_l = 0$. Moreover, $M_i = P_i^1$ and $N_i = \text{rad} P_i^1$, thus $M_i/N_i = P_i^1/\text{rad} P_i^1$. This implies that $P(i, j)/N$ is isomorphic to the $A_i$-module $P_i^1/\text{rad} P_i^1$ realized as a $\Lambda$-module. Since $P_i^1$ is an indecomposable projective $A_i$-module, $P_i^1/\text{rad} P_i^1$ is a simple $A_i$-module, and it is also simple when seen as a $\Lambda$-module, according to Remark 3.1. This means that $P(i, j)/N$ is a simple $\Lambda$-module. We have thus proved that $N$ is a maximal submodule of $P(i, j)$, and since $P(i, j)$ is indecomposable projective, it has a unique maximal submodule, which is $\text{rad} P(i, j)$. This concludes the proof that $N = \text{rad} P(i, j)$. □

Example 4.7. We continue Example 4.5 above to apply Proposition 4.6 and thus obtain the radical of the 4 projective modules seen above. Thus we have:

$$
\text{rad} P(1, 1) : \begin{array}{c}
\text{rad} P_1 \\
\longrightarrow \\
A^2
\end{array}
$$

$$
\text{rad} P(1, 2) : \begin{array}{c}
0 \\
\longrightarrow \\
A
\end{array}
$$

$$
\text{rad} P(2, 1) : 0 \longrightarrow \text{rad} P_1
$$

$$
\text{rad} P(2, 2) : 0 \longrightarrow 0
$$
4.2. **Injective representations.** In this subsection we shall give a description of the representations associated with indecomposable injective modules. As we shall see, the injective modules will be particular cases of dual cones, in an analogy with the projective modules, which were particular cases of cones, as we saw in Subsection 4.1.

**Proposition 4.8.** For \( i \in \Gamma_0 \), if \( I \) is an injective \( A_i \)-module, then \( C_i^*(I) \) is an injective \( \Lambda \)-module.

**Proof.** Since \( I \) is an injective \( A_i \)-module and \( D \) is a duality, \( DI \) is a projective \( A_i^\text{op} \)-module. Because of Proposition 4.11, \( C_i(DI) \) is a projective \( \Lambda^\text{op} \)-module, and again since \( D \) is a duality, \( C_i^*(I) = D(C_i(DI)) \) is an injective \( \Lambda \)-module.

For each \( i \in \Gamma_0 \), let \( E_i = \{ e_{i1}, \ldots, e_{in_i} \} \) be a complete set of primitive idempotent and pairwise orthogonal elements in \( A_i \). If \( D : \text{mod} \, A_i^\text{op} \to \text{mod} \, A_i \) is the duality functor, then its well-known that a complete set of isomorphism classes of indecomposable injective \( A_i \)-modules is given by \( I_i^1 = D(A_i e_{i1}), \ldots, I_i^n_i = D(A_i e_{in_i}) \).

On the other hand, if \( E = \{ \overline{e_{ij}} : i \in \Gamma_0, 1 \leq j \leq s_i \} \), then \( E \) is a complete set of primitive idempotent and pairwise orthogonal elements in \( \Lambda \). This means that a complete set of isomorphism classes of indecomposable injective \( \Lambda \)-modules is given by \( \{ I(i,j) : i \in \Gamma_0, 1 \leq j \leq s_i \} \), where \( I(i,j) \equiv D(\Lambda e_{ij}) \).

**Proposition 4.9.** With the notations above, \( C_i^*(I_i^1) \cong I(i,j) \).

**Proof.**

\[ C_i^*(I_i^1) = D(C_i(D(I_i^1))) = D(C_i(D(D(A_i e_{ij})))) \cong D(C_i(A_i e_{ij})) = D(\Lambda e_{ij}) = I(i,j) \]

where the penultimate equality follows from Proposition 4.2.

Proposition 4.9 gives us a complete description of the indecomposable injective \( \Lambda \)-modules. In order to calculate these modules in practical examples, we need to combine this description with Remark 3.11 above.

The particular case of when there are no relations is expressed in the following proposition, which is dual to Proposition 4.12 above:

**Proposition 4.10.** Suppose \( I = 0 \). Let \( I(i,j) = ((M_\alpha)_{\alpha \in \Gamma_0}, (\phi_\alpha)_{\alpha \in \Gamma_0}) \) be the representation associated to \( I(i,j) \). Then, for \( l \in \Gamma_0 \),

- If \( l = i \), then \( M_l = M_i = I_i^1 \).
- If \( l \neq i \), denote \( n_l = \sum_{\gamma : l=0 \to i_1 \to \ldots \to i_n = i} (\dim_k A_{i_1}) \cdots (\dim_k A_{i_{n-1}})(\dim_k I_i^1) \).

where \( \gamma \) runs through all possible paths \( l \to i \). Then \( M_l \cong (A_i^*)^{n_l} \) as \( A_i \)-modules, where we denote \( A_i^* = D(A_i) \) for brevity. In particular, if there are no paths \( l \to i \), then \( M_l = 0 \).

**Example 4.11.** Let \( A \) be the path algebra given by the quiver

\[
\begin{array}{c}
1  \\
\end{array}
\begin{array}{c}
\rightarrow \end{array}
\begin{array}{c}
2
\end{array}
\]

Then there are 2 indecomposable injective \( A \)-modules, namely,

\[
I_1 : \xymatrix{ k & k \\ & id & k }
\quad I_2 : \xymatrix{ 0 & k }
\]
Now let $\Lambda$ be the generalized path algebra given by

$$ A \leftarrow \leftarrow A $$

We want to calculate the indecomposable injective $\Lambda$-modules. According to the discussions above, we first calculate the indecomposable projective modules over the following generalized path algebra:

$$ A^{op} \rightarrow \rightarrow A^{op} $$

and we note that $A^{op}$ is the path algebra over the following quiver:

$$ 1 \rightarrow \rightarrow 2 $$

In our case, this calculation was already done in Example 4.5. Therefore it remains only to apply Proposition 2.6. Thus the indecomposable injective $\Lambda$-modules are:

$$ I(1, 1) : I_1 \leftarrow \leftarrow \left[ \begin{array}{cc} D(\mu) & 0 \\ 0 & D(\mu) \end{array} \right] (A^*)^2 \leftarrow \leftarrow I(1, 2) : I_2 \leftarrow \left[ \begin{array}{c} D(\mu) \end{array} \right] A^* $$

$$ I(2, 1) : 0 \leftarrow \leftarrow I_1 \leftarrow \leftarrow I(2, 2) : 0 \leftarrow \leftarrow I_2 $$

5. Homological dimensions

In this section we use the previous description of projective and injective modules to obtain some results concerning the projective and injective dimensions of modules over a gbp-algebra. This will also yield a formula for calculating the global dimension of a gbp-algebra.

We still keep the notation that has been used in the previous sections: $\Gamma$ an acyclic quiver, $A = \{A_1, \ldots, A_n\}$ a collection of algebras, $I$ is a set of relations such that $(I)$ is admissible and $\Lambda = k(\Gamma, A, I)$.

5.1. First case. Let $M$ be an $A_i$-module. As observed in Subsection 3.1, we shall identify $I(M)$ with $M$.

Lemma 5.1. Let $i \in \Gamma_0$ and let $M$ be an $A_i$-module. Then $\text{pd}_A M \geq \text{pd}_A M$.

Proof. There is nothing to show if $\text{pd}_A M = \infty$. So, assume $M$ has finite projective dimension over $\Lambda$ and let

$$ 0 \rightarrow P_m \rightarrow \rightarrow \ldots \rightarrow P_1 \rightarrow P_0 \rightarrow I(M) \rightarrow 0 $$

be a minimal projective resolution of $I(M)$. In particular, $\text{pd}_A M = m$. Since a projective resolution is in particular an exact sequence, this yields an exact sequence of $A_i$-modules at the $i$-th component:

$$ 0 \rightarrow (P_m)_i \rightarrow \rightarrow \ldots \rightarrow (P_1)_i \rightarrow (P_0)_i \rightarrow M \rightarrow 0 $$

It follows from the description of the projective modules that every component of a projective representation is projective (indeed, the $i$-th component is a direct sum of indecomposable projective modules over $A_i$, copies of $A_i$, or zeros). Thus the
exact sequence above is a projective resolution of $M$ over $A_i$. This implies that $\text{pd}_{A_i} M \leq m = \text{pd}_\Lambda M$.

**Corollary 5.2.** $\text{gl.dim} \Lambda \geq \max\{\text{gl.dim} A_1, \ldots, \text{gl.dim} A_n\}$.

**Proof.** This is an easy consequence of Lemma 5.1.

In general, the inverse inequality of Lemma 5.1 does not hold. However, in the particular case in which $i$ is a sink vertex, the inverse inequality does hold, and it is useful to highlight this in the result below, whose demonstration is fairly immediate.

**Lemma 5.3.** If $i$ is a sink vertex of $\Gamma$, and if $M$ is an $A_i$-module, then $\text{pd}_\Lambda M = \text{pd}_{A_i} M$.

**Proof.** This proof is direct: every projective resolution of $M$ over $A_i$ is easily seen to yield a projective resolution of $M$ over $\Lambda$ with the same length, because $i$ is a sink.

Also noteworthy is that Lemma 5.1 has a dual version for injective dimensions:

**Lemma 5.4.** Let $i \in \Gamma_0$ and let $M$ be an $A_i$-module. Then $\text{id}_\Lambda M \geq \text{id}_{A_i} M$.

**5.2. Cones.** The next result, which relates the projective dimension of a module over $A_i$ with the projective dimension of its cone, is a direct consequence of Proposition 3.7.

**Lemma 5.5.** Given $i \in \Gamma_0$ and $M$ an $A_i$-module, $\text{pd}_{A_i} M = \text{pd}_\Lambda C_i(M)$.

**Proof.** Let

$$
0 \rightarrow P_m \rightarrow \cdots \rightarrow P_1 \rightarrow P_0 \rightarrow M \rightarrow 0
$$

be a minimal projective resolution of $M$ in $\text{mod } A_i$. Thus $m = \text{pd}_{A_i} M$. Applying the functor $C_i$, we have

$$
0 \rightarrow C_i(P_m) \rightarrow \cdots \rightarrow C_i(P_1) \rightarrow C_i(P_0) \rightarrow C_i(M) \rightarrow 0
$$

Because of Proposition 5.3.7, this sequence is exact. Moreover, because of Proposition 4.7, every term except possibly for $C_i(M)$ is known to be projective. So this is a projective resolution in $\text{mod } \Lambda$, proving that $\text{pd}_\Lambda C_i(M) \leq \text{pd}_{A_i} M$. Since the $i$-th component of $C_i(M)$ is $M$, we know from Proposition 5.1 that the inverse inequality also holds.

**5.3. General case.** Having studied the projective and injective dimension of modules which are inclusion or cones of $A_i$-modules, we turn our attention to general $\Lambda$-representations.

**Definition 5.6.** Let $M = ((M_i)_{i \in \Gamma_0}, (\phi_\alpha)_{\alpha \in \Gamma_1})$ be a representation over $k(\Gamma, \mathcal{A}, I)$. The **support** of $M$ is defined as the set of vertices $\text{supp } M = \{i \in \Gamma_0 : M_i \neq 0\}$.

**Proposition 5.7.** Suppose $\Gamma$ is acyclic and let $M$ be a $\Lambda$-module. Then

$$\text{pd}_\Lambda M \leq \max_{j \in \Gamma_0}\{\text{pd}_\Lambda M_j\}$$

**Proof.** We proceed by induction on $|\text{supp } M|$. If $|\text{supp } M| = 1$ then $M$ has only one non-zero component, say the $i$-th component, and it is clear that $\text{pd}_\Lambda M \leq \text{pd}_\Lambda M_i$. Suppose $|\text{supp } M| > 1$ and that the statement holds for representations whose support is smaller than that of $M$. Then, since $\Gamma$ is acyclic, there is at least one
vertex \( i \in \Gamma_0 \) which is a source in the subquiver determined by \( \text{supp} M \).

We consider the following representations:

\[
N = ((N_j)_{j \in \Gamma_0}, (\psi_\alpha)_{\alpha \in \Gamma_1}) \quad \text{given by} \quad N_i = M_i, N_j = 0 \text{ if } j \neq i, \psi_\alpha = 0; \quad \text{and}
\]

\[
Q = ((Q_j)_{j \in \Gamma_0}, (\rho_\alpha)_{\alpha \in \Gamma_1}) \quad \text{given by} \quad Q_i = 0, Q_j = M_j \text{ if } j \neq i, \rho_\alpha = \phi_\alpha|_{Q_j(\alpha)}
\]

We have that \( N \) satisfies \( I \) (because it has support of size 1), and it is easy to see that \( Q \) also satisfies \( I \), because \( M \) does.

We also consider two morphisms of representations \( f = (f_j)_{j \in \Gamma_0} : Q \to M \), and \( g = (g_j)_{j \in \Gamma_0} : M \to N \), given by:

\[
f_j : Q_j \to M_j, f_i = 0, f_j = id_{M_j} \text{ if } j \neq i; \quad \text{and} \quad g_j : M_j \to N_j, g_i = id_{M_i}, g_j = 0 \text{ if } j \neq i
\]

It is directly verified that these are in fact morphisms of representations. Thus we have a short exact sequence of representations:

\[
0 \to Q \overset{f}{\longrightarrow} M \overset{g}{\longrightarrow} N \to 0
\]

It is indeed exact because the \( i \)-th component is

\[
0 \to 0 \to M_i \overset{id_{M_i}}{\longrightarrow} M_i \to 0
\]

and for \( j \neq i \), the \( j \)-component is

\[
0 \to M_j \overset{id_{M_j}}{\longrightarrow} M_j \to 0 \to 0
\]

and these clearly are exact sequences.

We obtain that \( \text{pd}_A M \leq \max\{\text{pd}_A Q, \text{pd}_A N\} \). Note that \( |\text{supp} Q| = n - 1 \) and \( |\text{supp} N| = 1 \). Therefore, by the induction hypothesis,

\[
\text{pd}_A N = \text{pd}_A N_i = \text{pd}_A M_i
\]

\[
\text{pd}_A Q \leq \max_{j \in \Gamma_0}\{\text{pd}_A Q_j\} = \max_{j \neq i}\{\text{pd}_A Q_j\} = \max_{j \neq i}\{\text{pd}_A M_j\}
\]

Assembling the pieces together, we conclude that \( \text{pd}_A M \leq \max\{\text{pd}_A N, \text{pd}_A Q\} \leq \max_{j \in \Gamma_0}\{\text{pd}_A M_j\} \), as we wanted to prove. \( \square \)

**Remark 5.8.** We will adopt the following notation from here on: if \( i \) is a source vertex of \( \Gamma \), then \( \Gamma \setminus \{i\} \) shall denote the quiver obtained from \( \Gamma \) by deleting the vertex \( i \) and the arrows starting at \( i \). Moreover, if \( \Gamma \) is equipped with a set of relations \( I \), \( \Gamma \setminus \{i\} \) will be the set obtained from \( I \) by excluding the relations starting at \( i \). Also, since \( \Gamma \) is acyclic, we can assume that \( \Gamma_0 = \{1, \ldots, n\} \) and that \( i \) is a source vertex of \( \Gamma \setminus \{1, \ldots, i - 1\} \) for every \( i \).

**Remark 5.9.** In the results below, besides from assuming \( \Gamma \) acyclic, we will also need to do an additional assumption on the ordinary path algebra \( k\Gamma/I \): let \( j \) be a vertex of \( \Gamma \), and let \( S_j \) denote the simple \( k\Gamma/I \)-module over \( j \). Then we suppose that, for every \( i \in \Gamma_0 \),

\[
\text{pd}_{k\Gamma/I} S_i \geq \max\{\text{pd}_{k\Gamma/I} S_j : j \text{ is a successor of } i\} + 1
\]

every time we have \( I \setminus \{1, \ldots, i\} \neq I \setminus \{1, \ldots, i - 1\} \) (i.e., every time there are relations starting at \( i \)).
Lemma 5.10. Let \( i \in \Gamma_0 \), \( M \) be an \( \Lambda \)-module, and let \((P,g)\) be its projective cover. Then there is an exact sequence of \( \Lambda \)-modules:

\[
0 \longrightarrow C_i(\text{Ker } g) \oplus L \longrightarrow C_i(P) \longrightarrow M \longrightarrow 0
\]

where \( L \) is a \( \Lambda \)-module such that

\[
\text{supp } L \subseteq \{ j \in \Gamma_0 : j \neq i \text{ and there is a path } i \rightsquigarrow j \}
\]

Moreover,

1. \( L_j \) is free for every vertex \( j \), and
2. If \( i \in \Gamma_0 \) is such that \( I \setminus \{ 1, \ldots, i \} = I \setminus \{ 1, \ldots, i-1 \} \), then \( L \) is projective over \( \Lambda \).

Proof. We have already discussed (Proposition 3.5 and Remark 3.6), that \((C_i(P))_i = P\), thus we can define a morphism of representations \( g' : C_i(P) \rightarrow M \) by establishing that \( g'_j = g \) and that \( g'_j = 0 \) for \( j \neq i \). We want to show that \( \text{Ker } g' = C_i(\text{Ker } g) \oplus L \), where \( L \) satisfies the conditions in the statement.

Let \( \{ p_1, \ldots, p_r \} \) be a \( k \)-basis of \( \text{Ker } g \) and complete it to a \( k \)-basis \( \{ p_1, \ldots, p_r, \ldots, p_s \} \) of \( P \). Also let, for every \( j \in \Gamma_0 \), \( \{ a_{i_1}^1, \ldots, a_{i_s}^j \} \) be a \( k \)-basis of \( A_j \).

As in the proof of Proposition 3.7 for every path \( \gamma : i \rightsquigarrow j \) in \( \Gamma \), denote

\[
\theta_{\gamma, h, i_1, \ldots, i_s} = p_h \otimes \gamma_1 a_{i_1}^{e(\gamma_1)} \gamma_2 a_{i_2}^{e(\gamma_2)} \cdots \gamma_s a_{i_s}^{e(\gamma_s)}
\]

Remember that \( g' \) was defined as a morphism of representations, and by using Theorem 2.3 it corresponds to a morphism of \( \Lambda \)-modules. Therefore, by the proof of that theorem:

\[
g'(\theta_{\gamma, h, i_1, \ldots, i_s}) = g'(p_h \otimes \gamma_1 a_{i_1}^{e(\gamma_1)} \gamma_2 a_{i_2}^{e(\gamma_2)} \cdots \gamma_s a_{i_s}^{e(\gamma_s)}) = g(p_h)
\]

So \( \theta_{\gamma, h, i_1, \ldots, i_s} \notin \text{Ker } g' \) if and only if \( \gamma \) is the zero-length path \( e_i \) and \( r < h \leq s \). Thus we can write

\[
\text{Ker } g' = (\theta_{e_i, h} : 1 \leq h \leq r) + (\theta_{\gamma, h, i_1, \ldots, i_s} : l(\gamma) > 0) = (\theta_{\gamma, h, i_1, \ldots, i_s} : 1 \leq h \leq r) \oplus (\theta_{\gamma, h, i_1, \ldots, i_s} : l(\gamma) > 0 \text{ and } r < h \leq s)
\]

where \( L = (\theta_{\gamma, h, i_1, \ldots, i_s} : l(\gamma) > 0 \text{ and } r < h \leq s) \). Since the generators of \( L \) involve only paths of length strictly greater than zero, the only components of \( L \) that are non-zero are the ones over the successors of \( i \), except for \( i \) itself. Therefore the condition in the statement about the support of \( L \) is satisfied. It remains to prove the other two assertions in the statement.

1. The components of \( L \) are all free:

   Fix \( j \in \Gamma_0 \). If \( j = i \) or if \( j \) is not a successor of \( i \), then \( L_j = 0 \), so we may suppose this is not the case. Again remembering the proof of Theorem 2.3

   \[
   L_j = L.1_j = (\theta_{\gamma, h, i_1, \ldots, i_s} : \gamma : i \rightsquigarrow j \text{ and } r < h \leq s) = (p_h \otimes \gamma_1 a_{i_1}^{e(\gamma_1)} \gamma_2 a_{i_2}^{e(\gamma_2)} \cdots \gamma_s a_{i_s}^{e(\gamma_s)} : \gamma : i \rightsquigarrow j \text{ and } r < h \leq s)
   \]
So $L_i$ is isomorphic to the free $A_j$-module whose basis is the set of all possible $p_h \otimes \gamma_1 a_{i_1}^e(\gamma_1) \gamma_2 a_{i_2}^e(\gamma_2) \ldots \gamma_t$. In particular, $L_i$ is free over $A_j$.

(2) If $I \setminus \{1, \ldots, i\} = I \setminus \{1, \ldots, i-1\}$, then $L$ is projective over $\Lambda$.

Let $i^*$ denote the set of immediate successors of $i$. Since, by hypothesis, there are no relations starting at $i$, we can write:

\[
L \doteq (\theta_{\gamma,h,i_1,\ldots,i_t} : l(\gamma) > 0 \text{ and } r < h \leq s)
\]

\[
= (p_h \otimes \gamma_1 a_{i_1}^e(\gamma_1) \gamma_2 a_{i_2}^e(\gamma_2) \ldots \gamma_t a_{i_t}^e(\gamma_t) : l(\gamma) > 0 \text{ and } r < h \leq s)
\]

\[
= (p_h \otimes \gamma_1 a_{i_1}^e(\gamma_1) \otimes \gamma_2 a_{i_2}^e(\gamma_2) \ldots \gamma_t a_{i_t}^e(\gamma_t) : l(\gamma) > 0 \text{ and } r < h \leq s)
\]

\[
\cong \left(a_{i_1}^e(\gamma_1) \otimes \gamma_2 a_{i_2}^e(\gamma_2) \ldots \gamma_t a_{i_t}^e(\gamma_t) : l(\gamma) > 0\right)^{s-r}
\]

\[
\cong \bigoplus_{i' \in i^*} C_i(A_{i'})^{s-r}
\]

Since $A_{i'}$ is projective over $A_i$, then $C_i(A_{i'})$ is projective over $\Lambda$ by Proposition 4.1. We have thus shown that $L$ is isomorphic to a direct sum of projective $\Lambda$-modules, and therefore it is also projective, concluding the proof.

\[
\square
\]

**Theorem 5.11.** Let $\Lambda = k(\Gamma, A, I)$ be a ghp-algebra, with $k\Gamma/I$ satisfying the conditions in Remark 5.5, and with $A = \{A_1, \ldots, A_n\}$. Then, for every representation $M$ over $\Lambda$,

\[
\text{pd}_{\Lambda} M \leq \max_{i \in \supp M} \{\text{pd}_{A_i} M_i, \text{pd}_{k\Gamma/I} S_i\}
\]

**Proof.** The proof is done by induction. First, suppose $\supp M = \{m\}$, where $m$ is a sink vertex of $\Gamma_0$. Then we know from Lemma 5.8 that $\text{pd}_{\Lambda} M = \text{pd}_{A_m} M_m$. Since $m$ is a sink vertex, the simple $k\Gamma/I$-module $S_m$ is projective, and thus it holds that $\text{pd}_{\Lambda} M = \max\{\text{pd}_{A_m} M_m, \text{pd}_{k\Gamma/I} S_m\}$.

In particular, this implies the initial step of the induction: if $\supp M = \{n\}$, $\text{pd}_{\Lambda} M = \max\{\text{pd}_{A_n} M_n, \text{pd}_{k\Gamma/I} S_n\}$.

Now suppose that $\supp M \subseteq \{i, \ldots, n\}$ and that the statement is valid for representations whose support is contained in $\{i+1, \ldots, n\}$. Initially we are going to study the projective dimension of $M_i$ over $\Lambda$. If $i$ is a sink vertex, we have already seen that $\text{pd}_{\Lambda} M_i = \max\{\text{pd}_{A_i} M_i, \text{pd}_{k\Gamma/I} S_i\}$, so suppose $i$ is not a sink vertex.

Let $(P, g)$ be a projective cover of $M_i$ over $A_i$. Then, because of Lemma 5.10, there is an exact sequence in mod $\Lambda$:

\[
0 \longrightarrow C_i(\Ker g) \oplus L \longrightarrow C_i(P) \longrightarrow M_i \longrightarrow 0
\]

where $L$ satisfies the conditions given in the statement of the cited lemma. From this exact sequence, we deduce that

\[
\text{pd}_{\Lambda} M_i \leq \max\{\text{pd}_{A_i} C_i(P), \text{pd}_{A_i} (C_i(\Ker g) \oplus L) + 1\}
\]

Since $P$ is projective over $A_i$, Proposition 4.1 implies that $\text{pd}_{A_i} (C_i(P)) = 0$. Thus

\[
\text{pd}_{\Lambda} M_i \leq \text{pd}_{A_i} (C_i(\Ker g) \oplus L) + 1 \leq \max\{\text{pd}_{A_i} C_i(\Ker g), \text{pd}_{A_i} L\} + 1
\]
Using Corollary \textit{5.5} we have

(5.1) \[ \text{pd}_A M_i \leq \max\{\text{pd}_A M_i, \text{pd}_L M_i\} + 1 \]

Now we divide our analysis in cases:

- Case 1: \( \text{pd}_A \text{Ker} g \geq \text{pd}_L M_i \)
  
  In this case, Equation (5.1) implies that \( \text{pd}_A M_i \leq \text{pd}_A \text{Ker} g + 1 = \text{pd}_A M_i \), because \((P, g)\) is the projective cover of \( M_i \).

- Case 2: \( \text{pd}_A \text{Ker} g \leq \text{pd}_L M_i \)
  
  Now, from Equation (5.1), \( \text{pd}_A M_i \leq \text{pd}_L M_i + 1 \).

  - Case 2.1: \( I \setminus \{1, \ldots, i\} \neq I \setminus \{1, \ldots, i-1\} \)
    
    From Lemma \textit{5.10}, \( \text{pd}_A L_j = 0 \) for every \( j \), and since the support of \( L \) is contained in \( \{i+1, \ldots, n\} \), by the induction hypothesis and the hypothesis on \( k\Gamma/I \), \( \text{pd}_A L_i \leq \max\{\text{pd}_L S_i, \ldots, \text{pd}_L S_n\} \leq \text{pd}_{k\Gamma/I} S_i - 1 \). Then \( \text{pd}_A M_i \leq \text{pd}_A L + 1 \leq \text{pd}_{k\Gamma/I} S_i - 1 + 1 = \text{pd}_{k\Gamma/I} S_i \).

Putting together all cases discussed above, we conclude that

\[ \text{pd}_A M_i \leq \max\{\text{pd}_A M_i, \text{pd}_{k\Gamma/I} S_i\} \]

Now, using Proposition \textit{5.7} we have that

\[ \text{pd}_A M \leq \max_{j \geq i} \{\text{pd}_A M_j, \text{pd}_{k\Gamma/I} S_j\} \]

which proves the theorem. \( \square \)

**Corollary 5.12.** Let \( \Lambda = k(\Gamma, A, I) \) be a \( gbp \)-algebra as in Theorem \textit{5.11}. Then, for every \( j \in \Gamma_0 \), \( \text{gl.dim} A_j \leq \text{gl.dim} \Lambda \), and the following inequality holds:

\[ \text{gl.dim} \Lambda \leq \max_{j \in \Gamma_0} \{\text{gl.dim} \frac{k\Gamma}{I}, \text{gl.dim} A_j\} \]

**Proof.** This is just a consequence from Lemma \textit{5.1} and Theorem \textit{5.11} \( \square \)

**Corollary 5.13.** Let \( \Lambda = k(\Gamma, A, I) \) be a \( gbp \)-algebra as in Theorem \textit{5.11} and let \( M \) be a representation over \( \Lambda \). Then

\[ \text{id}_A M = \max_{i \in \text{supp } M} \{\text{id}_A M_i, \text{id}_{k\Gamma/I} S_i\} \]

where \( S_i \) denotes the simple \( k\Gamma/I \)-module over the vertex \( i \).

**Proof.** The idea is to use Theorem \textit{5.11} and the fact that the duality functor anti-preserves homological properties. Again, let \( D = \text{Hom}(-, k) \) denote the duality functor. Let \( S'_i \) denote the simple \( k\Gamma^{op}/I^{op} \)-module over the vertex \( i \). Thus:
id_{A} M = \text{pd}_{A^{\text{op}}} D M \\
= \max_{i \in \text{supp} M} \{ \text{pd}_{A^{\text{op}}} (DM)_{i}, \text{pd}_{k^{\text{op}}/I^{\text{op}}} S'_{i} \} \quad \text{(because } D \text{ is a duality)} \\
= \max_{i \in \text{supp} M} \{ \text{pd}_{A^{\text{op}}} D (M_{i}), \text{pd}_{k^{\text{op}}/I^{\text{op}}} S'_{i} \} \quad \text{(Thm. 5.11 and Lemma 2.5)} \\
= \max_{i \in \text{supp} M} \{ \text{pd}_{A^{\text{op}}} D (M_{i}), \text{pd}_{k^{\text{op}}/I^{\text{op}}} D (S_{i}) \} \quad \text{(Proposition 2.6)} \\
= \max_{i \in \text{supp} M} \{ \text{id}_{A^{\text{op}}} M_{i}, \text{id}_{k^{\text{op}}/I} S_{i} \} \quad \text{(because } D \text{ is a duality)} \\
\square

6. Some consequences

This final section is devoted to present some immediate consequences of the main results discussed in the previous section.

6.1. Shod and quasitilted algebras. The first consequence is an application to the study of shod and quasitilted algebras. Quasitilted algebras were introduced in [7] as a generalization of the concept of tilting objects to abelian categories. We shall, however, use a characterisation of quasitilted algebras, also proven in [7], which suits better our purpose here. The shod algebras were then introduced in [4] in order to generalize the concept of quasitilted. We refer to these papers for more details.

Definition 6.1. Let $A$ be an algebra. We say that $A$ is a shod algebra if, for every indecomposable $A$-module $M$, either $\text{pd}_{A} M \leq 1$ or $\text{id}_{A} M \leq 1$. If, besides being shod, $A$ has global dimension of at most two, we say that $A$ is quasitilted.

Our next result allows us to produce a quasitilted or shod generalized path algebra from other algebras. It is worth mentioning that it is not intended as a complete description of which generalized (bound) path algebras are quasitilted or shod. Before stating it, please note that every hereditary algebra is quasitilted, and thus also shod.

Proposition 6.2. Let $\Lambda = k(\Gamma, A)$ be a generalized path algebra, with $\Gamma$ acyclic. Suppose that $A_{j}$ is hereditary for every $j \in \Gamma_{0}$, except possibly for a single vertex $i \in \Gamma_{0}$. Then:

(a) If $A_{i}$ is shod, then $\Lambda$ is shod.

(b) If $A_{i}$ is quasitilted, then $\Lambda$ is quasitilted.

Proof. (a) Let $M = ((M_{j})_{j \in \Gamma_{0}}, (\phi_{j})_{j \in \Gamma_{0}})$ be an indecomposable representation over $\Lambda$. Since $\Gamma$ is acyclic, we infer that the algebra $k\Gamma$ is hereditary and so every simple module over it will have projective and injective dimension of at most one. Observe also that, since $A_{j}$ is hereditary for $j \neq i$, we also have $\text{pd}_{A_{j}} M_{j} \leq 1$ and $\text{id}_{A_{j}} M_{j} \leq 1$ if $j \neq i$.

Now, since $A_{i}$ is shod, either $\text{pd}_{A_{i}} M_{i} \leq 1$ or $\text{id}_{A_{i}} M_{i} \leq 1$. In the former case, from Theorem 5.11 we have that $\text{pd}_{A} M \leq \max_{j \in \Gamma_{0}} \{ \text{pd}_{A_{j}} M_{j}, \text{pd}_{k^{\text{op}}/I} S_{j} \} \leq 1$, and in the latter, using Corollary 5.13 in an analogous manner, one obtains that $\text{id}_{A} M \leq 1$.

Thus $\Lambda$ is shod.

(b) Since $A_{i}$ is quasitilted, it is shod and from the last item we get that $\Lambda$ is shod. It remains to prove that $\text{gl.dim} \Lambda \leq 2$. We apply Corollary 5.12.
max_{j \in \Gamma_0} \{k\Gamma, \text{gl.dim } A_j\} \leq 2$, because $A_i$ is quasitilted and the other algebras are hereditary. 

**Example 6.3.** This example will show that the converse of proposition above could not hold. Let $A$ be the bound path algebra over the quiver

$$
\begin{array}{c}
1 \overset{\alpha}{\rightarrow} 2 \overset{\beta}{\rightarrow} 3
\end{array}
$$

bound by $\alpha\beta = 0$, and let $\Lambda$ be the generalized path algebra given by

$$
\begin{array}{c}
1 \overset{\alpha}{\rightarrow} k \overset{}{\rightarrow} A \overset{}{\rightarrow} A
\end{array}
$$

We have that, with this setting, $\Lambda$ does not satisfy the hypothesis from the last proposition: there is more than one vertex upon which the algebra is quasitilted and non-hereditary.

However, using [8], Theorem 3.3 or [3], Theorem 3.9, we see that $\Lambda$ is isomorphic to the bound path algebra over the quiver

$$
\begin{array}{c}
1 \overset{\alpha}{\downarrow} \overset{\gamma}{\downarrow} \overset{}{\downarrow} 4 \overset{\beta}{\downarrow} \overset{\delta}{\downarrow} 6
\end{array}
$$

bound by $\alpha\beta = \gamma\delta = 0$. Then it is easy to see that $\Lambda$ is a quasitilted algebra. The same example shows that the converse of the above proposition also does not hold for shod algebras.

### 6.2. Finitistic dimension.

Now we proceed to another application, this time for finitistic dimensions.

**Definition 6.4.** Let $A$ be an algebra. The **finitistic dimension** of $A$ is given by:

$$
\text{fin.dim } A = \sup \{ \text{pd}_A M : M \text{ is an } A\text{-module of finite projective dimension} \}
$$

A still open conjecture, called the **Finitistic Conjecture**, states that every algebra has finite finitistic dimension.

**Proposition 6.5.** Let $\Lambda = k(\Gamma, A, I)$ be a gbp-algebra, with $k\Gamma/I$ satisfying the conditions in Remark 5.9. Then

$$
\text{fin.dim } \Lambda = \max_{i \in \Gamma_0} \left\{ \text{gl.dim } \frac{k\Gamma}{I}, \text{fin.dim } A_i \right\}
$$

In particular, if the bound path algebra $k\Gamma/I$ has finite global dimension and the Finitistic Conjecture holds for every $A_i$, then it holds for $\Lambda$.

**Proof.** Let $M = (M_i)_{i \in \Gamma_0}, (\phi_\alpha)_{\alpha \in \Gamma_1}$ be a representation of finite projective dimension over $\Lambda$. From Lemma 5.11, for every $i \in \Gamma_0$, $\text{pd}_{A_i} M_i \leq \text{pd}_A M_i$ and thus $\text{pd}_A M_i \leq \text{fin.dim } A_i$. Using Theorem 5.11, $\text{pd}_A M \leq \max_{i \in \Gamma_0} \{ \text{pd}_{k\Gamma/I} S_i, \text{pd}_{A_i} M_i \} \leq \max_{i \in \Gamma_0} \{ \text{gl.dim } k\Gamma/I, \text{fin.dim } A_i \}$. Since $M$ is arbitrary, the statement follows. \qed
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