MODULUS OF CONTINUITY OF WEAK SOLUTIONS TO A CLASS OF SINGULAR ELLIPTIC EQUATIONS

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Abstract. In this paper we study the modulus of continuity of weak solutions to a singular elliptic equation in the plane under very weak assumption on the integrability of the elliptic coefficients. Our investigation reveals that the modulus of continuity can be described by the reciprocal of the logarithmic function raised to a power. However, the power can be arbitrarily large. This is in sharp contrast with a result by J. Onninen and X. Zhong (Ann. Scuola Norm. Sup. Pisa Cl. Sci., 6(2007), 103–116) for a degenerate elliptic equation in the plane, in which the power must be suitably small.

1. Introduction

In this paper we investigate the modulus of continuity of a weak solution to the Dirichlet boundary value problem

\begin{align}
-\text{div} \left[(I + \mathbf{m} \otimes \mathbf{m}) \nabla p\right] &= S \text{ in } \Omega, \\
p &= 0 \text{ on } \partial \Omega
\end{align}

under the following assumptions:

(H1) \( \Omega \) is a bounded domain in \( \mathbb{R}^2 \) with Lipschitz boundary \( \partial \Omega \);
(H2) \( S \in L^q(\Omega) \) for some \( q > 1 \);
(H3) \( \mathbf{m} = (m_1, m_2) \) is a vector-valued function in \( (L^4(\Omega))^2 \).

Recall that the outer product \( \mathbf{m} \otimes \mathbf{m} \) is the matrix given by

\[ \mathbf{m} \otimes \mathbf{m} = \mathbf{m}^T \mathbf{m}. \]

Thus,

\[ \mathbf{m} \otimes \mathbf{m} \nabla p = (\mathbf{m} \cdot \nabla p) \mathbf{m}. \]

We say that \( p \) is a weak solution to (1.1)-(1.2) if:

(D1) \( p \in W_0^{1,2}(\Omega), \ \mathbf{m} \cdot \nabla p \in L^2(\Omega); \)
(D2) for each \( \zeta \in W_0^{1,2}(\Omega) \) with \( \mathbf{m} \cdot \nabla \zeta \in L^2(\Omega) \) one has

\begin{equation}
\int_\Omega \left[\nabla p \nabla \zeta + (\mathbf{m} \cdot \nabla p)(\mathbf{m} \cdot \nabla \zeta)\right] dx = \int_\Omega S(x) \zeta dx.
\end{equation}

Note from Theorem 7.15 in [8] that the test function \( \zeta \) in (D2) satisfies

\[ \int_\Omega e^{c_0|\zeta|^2} dx < \infty \]

for some positive number \( c_0 \).

Thus, each integral in (1.3) is well-defined.

2020 Mathematics Subject Classification. 35D30, 35B65, 35J15, 35J75, 35J47.

Key words and phrases. Modulus of continuity of weak solutions, singular elliptic equations, the Stummel-Kato class of functions, the De Giorgi iteration scheme.

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The outer product of two vectors appears in many mathematical models. Our situation here is
directly related to the analysis of biological transport networks [9, 11, 12]. We summarize known
results concerning (1.1)-(1.2) in the following

**Proposition 1.1.** Let (H1)-(H3) Hold. Then there is a unique weak solution p to (1.1)-(1.2).
Moreover,

(C1) there is a positive number \( c = c(\Omega) \) such that
\[
\|p\|_{\infty, \Omega} \leq c\|S\|_{q, \Omega}, \quad \text{and}
\]
(C2) for each \( x_0 \in \Omega \) and each \( R \in (0, \inf\{1, \text{dist}(x_0, \partial \Omega)\}) \) we can find a positive constant \( c \)
with the property
\[
\text{osc}_{B_{r}(x_0)} p \equiv \sup_{B_{r}(x_0)} p - \inf_{B_{r}(x_0)} p \leq \frac{c}{\ln^{\frac{1}{2}} \frac{R}{r}} \quad \text{for all } r \in (0, R],
\]
where \( B_{r}(x_0) \) is the ball centered at \( x_0 \) with radius \( r \).

The existence of a unique weak solution can be inferred from a result in [19]. The conclusion
(C1) is true even for the space dimension \( N > 2 \) [14]. A suitable modification of the proof in
[14] can be applied to the case \( N = 2 \) (see Claim 2.4 below). The continuity property (C2) was
established in [20].

To describe our results here, we introduce some notations first. We say that \( f \in K_{2}(\Omega) \), the
Stummel-Kato class of functions [11, 13], if \( f \) is a measurable function on \( \Omega \) and
\[
(1.4) \quad \eta(f; \Omega; r) \equiv \sup_{y \in \Omega} \int_{B_{r}(y)} |f(x)| \chi_{\Omega} \ln |x - y| \, dx \to 0 \quad \text{as } r \to 0^{+},
\]
while \( f \in K_{2}^{\text{loc}}(\Omega) \) means that \( \lim_{r \to 0} \eta(f; \Omega_{1}; r) = 0 \) for each bounded subdomain \( \Omega_{1} \) of \( \Omega \) with
\( \Omega_{1} \subset \Omega \). Note that in (1.4) we have used \( y \in \Omega \) instead of \( y \in \mathbb{R}^{2} \) as was done in [13]. Our definition
here seems to be more suitable for PDE applications. As usual, the letter \( c \) or \( c_{i}, i = 0, 1, \ldots, \) will
be used to represent a generic positive constant.

Our main result is:

**Theorem 1.2.** Let (H1)-(H3) hold and \( p \) be the weak solution to (1.1)-(1.2).

(C3) Then for each \( y \in \Omega, \ell > 0 \) there is a positive constant \( c = c(\ell, \text{dist}(y, \partial \Omega)) \) such that
\[
(1.5) \quad \text{osc}_{B_{r}(y)} p \leq \frac{c}{\ln^{\frac{1}{2}} \frac{R}{r}} + cr^{\frac{(2 - N)}{2}} \quad \text{for all } r \in (0, R],
\]
where \( R \in (0, \inf\{1, \text{dist}(y, \partial \Omega)\}) \);

(C4) If we further assume that
\[
(1.6) \quad f \in (\text{BMO}(\Omega))^{2} [5],
\]
then
\[
(1.7) \quad \text{div}(A\nabla u) = 0 \quad \text{in } \Omega \subset \mathbb{R}^{N}, \quad N \geq 2.
\]

Here the entries of the coefficient matrix \( A = A(x) \) are measurable functions, satisfying
\[
\lambda(x)|\xi|^{2} \leq A(x)\xi \cdot \xi \leq \Lambda(x)|\xi|^{2}
\]
for some non-negative measurable functions \( \lambda(x), \Lambda(x) \), a.e. \( x \in \Omega \), and each \( \xi \in \mathbb{R}^{N} \). Obviously, if
\( \Lambda(x) \) is bounded above, we can replace it with a positive constant, and we can do the same with
\( \lambda(x) \) if it is bounded away from 0 below. With this in mind, we say that (1.7) is singular if \( \lambda \) is a
constant while $\Lambda$ is not, and degenerate if $\Lambda$ is a constant and $\lambda$ is not. De Giorgi proposed several open problems [3] about the equation, one of which was stated as follows:

**Conjecture 1.3** (De Giorgi). If (1.7) is singular with $\lambda = 1$, the space dimension $N \geq 3$, and

\[ \int_\Omega e^{\Lambda(x)} \, dx < \infty, \]

then weak solutions of (1.7) are continuous.

This conjecture remains open. When $A = I + m \otimes m$ for some $m \in (L^2(\Omega))^N$ then we have

\[ |\xi|^2 \leq A(x)\xi \cdot \xi = |\xi|^2 + (m \cdot \xi)^2 \leq (1 + |m|^2)|\xi|^2 \quad \text{for a.e } x \in \Omega \text{ and each } \xi \in \mathbb{R}^N. \]

One way of gaining a condition like (1.8) is to assume that $N = 2$ and $m \in (W^{1,2}_0(\Omega))^2$. Then (1.8) is a consequence of Theorem 7.15 in [8]. However, the case $N = 2$ is not included in the above conjecture because the bare continuity in this case is rather trivial. We study the modulus of continuity when our integrability condition on $\Lambda(x) = 1 + |m|^2$ is much weaker.

It is well known that when $\Lambda(x) \leq c\lambda(x)$ for some $c > 0$ and $\lambda(x)$ is an $A_2$ weight [18] weak solutions of (1.7) are Hölder continuous no matter what the space dimensions are [10]. The maximum Hölder exponent was investigated in [16], under the assumptions that $N=2$ and $A$ is bounded and uniformly elliptic. If (1.7) is degenerate with $\Lambda = 1$, $N = 2$, and

\[ K \equiv \int_\Omega e^{\sqrt[N]{\Lambda(x)}} \, dx < \infty \quad \text{for some } \gamma > 1, \]

a result of [15] asserts that for each $\beta \in (0, \gamma - 1)$ there is a $c$ such that

\[ \operatorname{osc}_{B_r(x_0)} u \leq c \left( \int_\Omega A \nabla u \cdot \nabla u \, dx \right)^{\frac{1}{2}} \frac{1}{\ln^2 \frac{K}{\pi r^2}} \quad \text{for } r \text{ sufficiently small}. \]

In view of this, our result (C3) is really surprising because the power $\ell$ in (1.5) can be arbitrarily large and the entries of $A$ are only square-integrable under (H3). It is natural for us to make the following

**Conjecture 1.4.** Let (H1)-(H4) hold. Then the weak solution $p$ to (1.1) - (1.2) is Hölder continuous.

We point out that (H4) is not enough to guarantee (1.8) because $|m|^2$ may not belong to $\text{BMO}(\Omega)$. If the preceding conjecture were true, we would immediately be able to find applications for it in the mathematical analysis of the biological network formation model [22]. Unfortunately, we have not been able to prove or disprove the conjecture. However, we do have

**Theorem 1.5.** Let (H1)-(H4) hold. Then whenever $x_0 \in \Omega$ is such that

\[ \sup_{r>0} |m_{x_0,r}| < \infty, \]

where

\[ m_{x_0,r} = \frac{1}{|B_r(x_0)|} \int m \, dx = \int_{B_r(x_0)} m \, dx. \]

we can find two positive numbers $c, \alpha$ with

\[ \operatorname{osc}_{B_r(x_0)} p \leq cr^\alpha \quad \text{for } r \text{ sufficiently small}. \]

This is in essence a partial regularity result [21]. The so-called singular set

\[ S \equiv \{ x_0 \in \Omega : \limsup_{r \to 0} |m_{x_0,r}| = \infty \} \]

can be estimated in terms of the Hausdorff measure [7].
In the relationship between the continuity of \( p \) and \( |\nabla p|^2 \in K^{1}_{2}(\Omega) \) was investigated. It turns out that the former implies the latter when \( A \) is bounded and uniformly elliptic. Here, due to the singularity in our problem, the continuity of \( p \) is not enough to guarantee (1.6). As we shall see below, we must have \( \ell > 3 \) in (1.5) to ensure that (1.6) holds. Of course, the most interesting result associated with the space \( K^{1}_{2}(\Omega) \) is the following

**Lemma 1.6.** There is a constant \( c \) such that

\[
\int_{B_{r}(y)} |F|v^{2}dx \leq c\eta(F;B_{r}(x_{0});r)\int_{B_{r}(x_{0})} |\nabla v|^{2}dx \text{ for each } v \in W^{1,2}_{0}(B_{r}(x_{0})).
\]

This lemma is not really new, and it is only slightly different from Lemma 1.1 in [6], which has had many applications in the study of partial differential equations [6, 13]. We include the lemma here in the hope that this version of the inequality may be more suitable for applications in certain cases [1, 22].

We refer the reader to [22] for an application of (C3), (C4), and Lemma 1.6.

The rest of the paper is organized as follows. The proof of Theorem 1.2 is given in Section 2. It relies on a decomposition of the logarithmic function. Theorem 1.5 and Lemma 1.6 will be established in Section 3.

**2. PROOF OF THEOREM 1.2**

In this section we offer the proof of Theorem 1.2. The proof is divided into several lemmas.

**Lemma 2.1.** Let (H1)-(H3) hold and \( p \) be the weak solution to (1.1)-(1.2). Then there is a positive constant \( c \) such that

\[
\int_{B_{r}(y)} F \ln |x - y||dx \leq (\ln 2 + |\ln r|) \int_{B_{r}(y)} Fdx + c \int_{B_{r}(y)} |\nabla p|^{2}dx
\]

\[
+ c \left( \int_{B_{r}(y)} |m|^{4}dx \right)^{1/2} \int_{B_{r}(y)} |\nabla p|^{2}dx + cr^{2(\frac{2q-1}{q})} \|S(x)\|_{q,B_{r}(y)}
\]

for each \( y \in \Omega \) and \( r \in (0, \text{dist}(y, \partial \Omega)) \).

**Proof.** Let \( y, r \) be given as in the lemma. Pick a smooth function \( \zeta(x) \) on \( \mathbb{R}^{N} \) such that \( \zeta(x) = 1 \) on \( B_{1}(0) \), \( 0 \leq \zeta(x) \leq 1 \) on \( \mathbb{R}^{N} \), and \( \zeta(x) = 0 \) when \( |x| \geq 2 \). Set

\[
l(x) = - \sum_{k=1}^{\infty} \zeta^{2} \left( \frac{2^{k}(y - x)}{r} \right) .
\]

Then \( \ln \frac{|y - x|}{r} - l(x) \ln 2 \in L^{\infty}(B_{r}(y)) \). Indeed, for each \( x \in B_{r}(y) \setminus \{y\} \) there must exist a \( j \in \{1, 2, \ldots \} \) such that

\[
\frac{1}{2^{j}} \leq \frac{|y - x|}{r} < \frac{1}{2^{j-1}}.
\]

We easily verify from the properties of \( \zeta \) that

\[
\zeta^{2} \left( \frac{2^{k}(y - x)}{r} \right) = \begin{cases} 0 & \text{if } k > j, \\ 1 & \text{if } k \leq j - 1. \end{cases}
\]

That is, the series in (2.2) is actually a finite sum and

\[
-l(x) = j - 1 + \zeta^{2} \left( \frac{2^{j}(y - x)}{r} \right) .
\]

It immediately follows that

\[
j - 1 \leq -l(x) \leq j, \quad -j \ln 2 \leq \ln \frac{|y - x|}{r} \leq -(j - 1) \ln 2.
\]
Hence,
\begin{align}
\ln \left| \frac{y-x}{r} \right| - l(x) \ln 2 & \leq \ln \left| \frac{y-x}{r} \right| + j \ln 2 \leq \ln 2, \\
\ln \left| \frac{y-x}{r} \right| - l(x) \ln 2 & \geq \ln \left| \frac{y-x}{r} \right| + (j-1) \ln 2 \geq -\ln 2.
\end{align}

That is,
\begin{align}
\ln \left| \frac{y-x}{r} \right| - l(x) \ln 2 & \leq \ln \left| \frac{y-x}{r} \right| + \left( j - \frac{1}{2} \right) \ln 2 \\
\ln \left| \frac{y-x}{r} \right| - l(x) \ln 2 & \geq -\ln 2.
\end{align}

We would like to remark that this type of decomposition of the logarithmic function is often found in the study of real variable Hardy spaces [17].

Let
\[ A_j = B_{r/2^j}(y) \setminus B_{r/2^j}(y). \]

Then
\[ B_r(y) = \bigcup_{j=1}^{\infty} A_j. \]

Denote by \( p_{A_j} \) the average of \( p \) over \( A_j \), i.e.,
\[ p_{A_j} = \frac{1}{|A_j|} \int_{A_j} p \, dx. \]

Subsequently, we can infer from the Sobolev-Poincaré inequality ([8], p.174) that for each \( s \geq 2 \) there is a positive number \( c \) such that
\begin{align}
\left( \int_{A_j} |p - p_{A_j}|^s \, dx \right)^{\frac{1}{s}} & \leq \frac{cr}{2^j} \left( \int_{A_j} |\nabla p|^{2s} \, dx \right)^{\frac{1}{2s}} \leq \frac{cr}{2^j} \left( \int_{A_j} |\nabla p|^2 \, dx \right)^{\frac{1}{2}}.
\end{align}

With this in mind, we use \( (p - p_{A_j}) \zeta^2 \left( \frac{2^j(y-x)}{r} \right) \) as a test function in (1.1) to derive
\begin{align}
\int_{B_{r/2^j}(y)} (|\nabla p|^2 + (\mathbf{m} \cdot \nabla p)^2) \zeta^2 \left( \frac{2^j(y-x)}{r} \right) \, dx \\
= \frac{2^{j+1}}{r} \int_{A_j} \zeta \left( \frac{2^j(y-x)}{r} \right) \nabla p \cdot \nabla \zeta \left( \frac{2^j(y-x)}{r} \right) (p - p_{A_j}) \, dx \\
+ \frac{2^{j+1}}{r} \int_{A_j} \zeta \left( \frac{2^j(y-x)}{r} \right) (\mathbf{m} \cdot \nabla p) \mathbf{m} \cdot \nabla \zeta \left( \frac{2^j(y-x)}{r} \right) (p - p_{A_j}) \, dx \\
+ \int_{B_{r/2^j}(y)} S(y)(p - p_{A_j}) \zeta^2 \left( \frac{2^j(y-x)}{r} \right) \, dx.
\end{align}

We proceed to estimate each term on the right hand side. First, by (2.5), we have
\begin{align}
\frac{2^{j+1}}{r} \int_{A_j} \zeta \left( \frac{2^j(y-x)}{r} \right) \nabla p \cdot \nabla \zeta \left( \frac{2^j(y-x)}{r} \right) (p - p_{A_j}) \, dx \\
\leq \frac{\|
abla \zeta \|_{\infty, B_2(0)} \| p \|_{L^\infty(\mathbb{R}^n)}}{r} \left( \int_{A_j} |\nabla p|^2 \, dx \right)^{\frac{1}{2}} \left( \int_{A_j} |p - p_{A_j}|^2 \, dx \right)^{\frac{1}{2}} \leq c \int_{A_j} |\nabla p|^2 \, dx.
\end{align}
Similarly, for each \( \varepsilon > 0 \) we calculate

\[
\frac{2^{j+1}}{r} \int_{A_j} \zeta \left( \frac{2^j (y - x)}{r} \right) \left( \mathbf{m} \cdot \nabla p \right) \left( \mathbf{m} \cdot \nabla \zeta \left( \frac{2^j (y - x)}{r} \right) (p - p_{A_j}) \right) dx
\]

\[
\leq \varepsilon \int_{B_{r/2^j}(y)} (\mathbf{m} \cdot \nabla p)^2 \zeta^2 \left( \frac{2^j (y - x)}{r} \right) dx + \frac{\| \nabla \zeta \|^2_{L^2, B_{2r}(0)} 2^{j+1}}{\varepsilon r^2} \int_{A_j} |\mathbf{m}|^2 |p - p_{A_j}|^2 dx
\]

\[
\leq \varepsilon \int_{B_{r/2^j}(y)} (\mathbf{m} \cdot \nabla p)^2 \zeta^2 \left( \frac{2^j (y - x)}{r} \right) dx + \frac{c 2^{j+1}}{\varepsilon r^2} \left( \int_{A_j} |\mathbf{m}|^4 dx \right)^{1/2} \left( \int_{A_j} |p - p_{A_j}|^4 dx \right)^{1/2}
\]

\[
\leq \varepsilon \int_{B_{r/2^j}(y)} (\mathbf{m} \cdot \nabla p)^2 \zeta^2 \left( \frac{2^j (y - x)}{r} \right) dx + \frac{c}{\varepsilon} \left( \int_{B_r(y)} |\mathbf{m}|^4 dx \right)^{1/2} \int_{A_j} \| \nabla p \|^2 dx.
\]

Finally,

\[
\int_{B_{r/2^j}(y)} S(x)(p - p_{A_j}) \zeta^2 \left( \frac{2^j (y - x)}{r} \right) dx \leq c \| S(x) \|_{q, B_r(y)} \left( \frac{r}{2^{j+1}} \right)^{2(q-1)/q}.
\]

Substitute the preceding three estimates into (2.6) and choose \( \varepsilon \) suitably small in the resulting inequality to obtain

\[
\int_{B_{r/2^j}(y)} F \zeta^2 \left( \frac{2^j (y - x)}{r} \right) dx = \int_{B_{r/2^j}(y)} (|\nabla p|^2 + (\mathbf{m} \cdot \nabla p)^2) \zeta^2 \left( \frac{2^j (y - x)}{r} \right) dx
\]

\[
\leq c \int_{A_j} |\nabla p|^2 dx + c \left( \int_{B_r(y)} |\mathbf{m}|^4 dx \right)^{1/2} \int_{A_j} \| \nabla p \|^2 dx
\]

\[
+ c \left( \int_{B_r(y)} |\mathbf{m}|^4 dx \right)^{1/2} \left( \int_{A_j} |\nabla p|^2 dx \right)
\]

\[
(2.7)
\]

With this and (2.3) in mind, we derive

\[
\int_{B_r(y)} F |\ln |x - y|| dx \leq \int_{B_r(y)} F |\ln |x - y| - \ln r - l(x) \ln 2| dx
\]

\[
+ |\ln r| \int_{B_r(y)} F dx + 2 \int_{B_r(y)} F |l(x)| dx
\]

\[
\leq (\ln 2 + |\ln r|) \int_{B_r(y)} F dx + 2 \sum_{j=1}^{\infty} \zeta^2 \left( \frac{2^j (y - x)}{r} \right) dx
\]

\[
\leq (\ln 2 + |\ln r|) \int_{B_r(y)} F dx + c \int_{B_r(y)} |\nabla p|^2 dx
\]

\[
+ c \left( \int_{B_r(y)} |\mathbf{m}|^4 dx \right)^{1/2} \int_{B_r(y)} |\nabla p|^2 dx + cr^{2(q-1)/q} \| S(x) \|_{q, B_r(y)}.
\]

The lemma follows. \qed
Lemma 2.2. Let the assumptions of the preceding lemma hold. Then there is a positive constant $c$ such that

$$\int_{B_r(y)} F \ln^2 |x - y| dx \leq c(1 + \ln^2 r) \int_{B_r(y)} F dx + c \int_{B_r(y)} |\ln |x - y|| \nabla p|^2 dx$$

$$+ c \left( \int_{B_r(y)} |m|^4 dx \right)^{1/2} \int_{B_r(y)} |\ln |x - y|| \nabla p|^2 dx$$

$$+ c(1 + |\ln r|) \left( \int_{B_r(y)} |m|^4 dx \right)^{1/2} \int_{B_r(y)} |\nabla p|^2 dx$$

$$(2.8) + c(1 + |\ln r|) r^{(2-2(q-1))} \sigma q^{-1} \|S(x)\|_{q,B_r(y)}. $$

for each $y \in \Omega$ and $r \in (0, \text{dist}(y, \partial \Omega))$.

Proof. We see from (2.4) that

$$\int_{B_r(y)} F |\ln |x - y||^2 dx \leq 2 \int_{B_r(y)} F |\ln |x - y| - \ln r - l(x) \ln 2|^2 dx$$

$$+ 2 \ln^2 r \int_{B_r(y)} F dx + 4 \ln^2 2 \int_{B_r(y)} F |l(x)|^2 dx$$

$$(2.9) \leq (2 \ln^2 2 + 4 \ln^2 r) \int_{B_r(y)} F dx + 4 \ln^2 2 \int_{B_r(y)} F |l(x)|^2 dx.$$

It is easy to verify that

$$\zeta^2 \left( \frac{2^i(y - x)}{r} \right) \zeta^2 \left( \frac{2^i(y - x)}{r} \right) = \zeta^2 \left( \frac{2^i(y - x)}{r} \right) \quad \text{whenever } i < j.$$

Consequently,

$$|l(x)|^2 = \sum_{i=1}^{\infty} \zeta^2 \left( \frac{2^i(y - x)}{r} \right) \sum_{j=1}^{\infty} \zeta^2 \left( \frac{2^j(y - x)}{r} \right)$$

$$(2.10) = \sum_{j=1}^{\infty} \zeta^4 \left( \frac{2^j(y - x)}{r} \right) + 2 \sum_{j=1}^{\infty} (j-1) \zeta^2 \left( \frac{2^j(y - x)}{r} \right).$$

We can infer from the proof of (2.7) that

$$\int_{B_{r/2}(y)} (|\nabla p|^2 + (m \cdot \nabla p)^2) \zeta^4 \left( \frac{2^j(y - x)}{r} \right) dx$$

$$(2.11) \leq c \int_{A_j} |\nabla p|^2 dx + c \left( \int_{B_r(y)} |m|^4 dx \right)^{1/2} \int_{A_j} |\nabla p|^2 dx + c \|S(x)\|_{q,B_r(y)} r^{(2(q-1))} \left( \frac{r}{2j-1} \right)^{\frac{2(q-1)}{q}}.$$
Use \((p - p_{A_j})(j - 1)\zeta^2 \left(\frac{2^j(y - x)}{r}\right)\) as a test function in (1.1) to derive

\[
\int_{B_{\frac{2^j - 1}{r}}(y)} \left(\|\nabla p\|^2 + (m \cdot \nabla p)^2\right) (j - 1)\zeta^2 \left(\frac{2^j(y - x)}{r}\right) dx
\]

\[
= \frac{2^{j+1}}{r} \int_{A_j} (j - 1)\zeta \left(\frac{2^j(y - x)}{r}\right) \nabla p \cdot \nabla \zeta \left(\frac{2^j(y - x)}{r}\right) (p - p_{A_j}) dx
\]

\[
+ \frac{2^{j+1}}{r} \int_{A_j} (j - 1)\zeta \left(\frac{2^j(y - x)}{r}\right) (m \cdot \nabla p) m \cdot \nabla \zeta \left(\frac{2^j(y - x)}{r}\right) (p - p_{A_j}) dx
\]

\[
+ \int_{B_{\frac{2^j}{r}}(y)} S(y)(p - p_{A_j})(j - 1)\zeta^2 \left(\frac{2^j(y - x)}{r}\right) dx.
\]

Recall from (2.3) that

\[
j - 1 \leq -\frac{1}{\ln 2} \ln \frac{|x - y|}{r}.
\]

Keep this in mind to deduce for each \(\varepsilon > 0\) that

\[
\frac{2^{j+1}}{r} \int_{A_j} (j - 1)\zeta \left(\frac{2^j(y - x)}{r}\right) \nabla p \cdot \nabla \zeta \left(\frac{2^j(y - x)}{r}\right) (p - p_{A_j}) dx
\]

\[
\leq \varepsilon \int_{B_{\frac{2^j - 1}{r}}(y)} \|\nabla p\|^2 (j - 1)\zeta^2 \left(\frac{2^j(y - x)}{r}\right) dx + \frac{\|\nabla \zeta\|^2_\infty \cdot B_{2}(0) 4^{j+1}(j - 1)}{r^2 \varepsilon} \int_{A_j} |p - p_{A_j}|^2 dx
\]

\[
\leq \varepsilon \int_{B_{\frac{2^j - 1}{r}}(y)} \|\nabla p\|^2 (j - 1)\zeta^2 \left(\frac{2^j(y - x)}{r}\right) dx + \frac{c}{\varepsilon} \int_{A_j} (j - 1)|\nabla p|^2 dx
\]

\[
\leq \varepsilon \int_{B_{\frac{2^j - 1}{r}}(y)} \|\nabla p\|^2 (j - 1)\zeta^2 \left(\frac{2^j(y - x)}{r}\right) dx + \frac{c}{\varepsilon} \int_{A_j} |\ln |x - y| - \ln r||^2 \|\nabla p\|^2 dx.
\]

By the same token,

\[
\frac{2^{j+1}}{r} \int_{A_j} (j - 1)\zeta \left(\frac{2^j(y - x)}{r}\right) (m \cdot \nabla p) m \cdot \nabla \zeta \left(\frac{2^j(y - x)}{r}\right) (p - p_{A_j}) dx
\]

\[
\leq \varepsilon \int_{B_{\frac{2^j - 1}{r}}(y)} (m \cdot \nabla p)^2 (j - 1)\zeta^2 \left(\frac{2^j(y - x)}{r}\right) dx
\]

\[
+ \frac{c 4^{j}(j - 1)}{\varepsilon r^2} \left(\int_{A_j} |m|^4 dx\right)^{\frac{1}{2}} \left(\int_{A_j} |p - p_{A_j}|^4 dx\right)^{\frac{1}{2}}
\]

\[
\leq \varepsilon \int_{B_{\frac{2^j - 1}{r}}(y)} (m \cdot \nabla p)^2 (j - 1)\zeta^2 \left(\frac{2^j(y - x)}{r}\right) dx
\]

\[
+ \frac{c}{\varepsilon} \left(\int_{B_{r}(y)} |m|^4 dx\right)^{\frac{1}{2}} \int_{A_j} |\ln |x - y| - \ln r|| \|\nabla p\|^2 dx.
\]
The last term in (2.12) can be estimated as follows:
\[
\int_{B_{\frac{r}{2j-1}}(y)} S(x)(p - p_{A_j})(j - 1)\zeta^2 \left(\frac{2^j(y - x)}{r}\right) \, dx
\]
\[
\leq c \int_{B_{\frac{r}{2j-1}}(x)} |S(x)| |\ln |x - y| - \ln r| \, dx
\]
\[
\leq c |\ln r||S(x)||_{q,B_r(y)} \left(\frac{r}{2j-1}\right)^{\frac{2(q-1)}{q}} + c ||S(x)||_{q,B_r(y)} \left(\frac{r}{2j-1}\right)^{\frac{(2-\sigma)(q-1)}{q}}, \sigma \in (0, 2).
\]
Collect the preceding three estimates in (2.12) to deduce
\[
\int_{B_{\frac{r}{2j-1}}(y)} (|\nabla p|^2 + (m \cdot \nabla p)^2) (j - 1)\zeta^2 \left(\frac{2^j(y - x)}{r}\right) \, dx
\]
\[
\leq c \int_{A_j} |\ln |x - y|| |\nabla p|^2 \, dx + c \left(\int_{B_{r}(y)} |m|^4 \, dx\right)^{\frac{1}{2}} \int_{A_j} |\ln |x - y|| |\nabla p|^2 \, dx
\]
\[
+ c |\ln r| \int_{A_j} |\nabla p|^2 \, dx + c |\ln r| \left(\int_{B_{r}(y)} |m|^4 \, dx\right)^{\frac{1}{2}} \int_{A_j} |\nabla p|^2 \, dx
\]
\[
+ c(1 + |\ln r|) |\nabla p|^2 \, dx + c(1 + |\ln r|) \left(\int_{B_{r}(y)} |m|^4 \, dx\right)^{\frac{1}{2}} \int_{B_{r}(y)} |\nabla p|^2 \, dx
\]
Combining this with (2.11) and (2.10) yields
\[
\int_{B_r(y)} F |l|^2(x) \, dx
\]
\[
\leq \int_{B_r(y)} F \sum_{j=1}^{\infty} \zeta^4 \left(\frac{2^j(y - x)}{r}\right) \, dx + 2 \int_{B_r(y)} F \sum_{j=1}^{\infty} (j - 1)\zeta^2 \left(\frac{2^j(y - x)}{r}\right) \, dx
\]
\[
\leq c \int_{B_r(y)} |\ln |x - y|| |\nabla p|^2 \, dx + c \left(\int_{B_{r}(y)} |m|^4 \, dx\right)^{\frac{1}{2}} \int_{B_{r}(y)} |\ln |x - y|| |\nabla p|^2 \, dx
\]
\[
+ c(1 + |\ln r|) \int_{B_{r}(y)} |\nabla p|^2 \, dx + c(1 + |\ln r|) \left(\int_{B_{r}(y)} |m|^4 \, dx\right)^{\frac{1}{2}} \int_{B_{r}(y)} |\nabla p|^2 \, dx
\]
\[
+ c(1 + |\ln r|) r^{(2-\sigma)(q-1)/q} ||S(x)||_{q,B_r(y)}.
\]
Use this in (2.9) to obtain (2.8). The proof is complete. \(\square\)

We can also verify that
\[
|l(x)|^3 = \sum_{j=1}^{\infty} \zeta^6 \left(\frac{2^j(y - x)}{r}\right) + 2 \sum_{j=1}^{\infty} (j - 1)\zeta^4 \left(\frac{2^j(y - x)}{r}\right) + \sum_{j=1}^{\infty} (j - 1)\zeta^2 \left(\frac{2^j(y - x)}{r}\right)
\]
\[
+ \sum_{j=1}^{\infty} (j - 1)^2\zeta^2 \left(\frac{2^j(y - x)}{r}\right) + \sum_{j=2}^{\infty} \frac{(j - 1)(j - 2)}{2}\zeta^2 \left(\frac{2^j(y - x)}{r}\right).
\]

It is not difficult to deduce that for each positive integer \(n\) we have
\[
|l(x)|^n = \sum_{j=1}^{\infty} \zeta^{2n} \left(\frac{2^j(y - x)}{r}\right) + \text{lower order terms}.
\]
Our earlier proof indicates that

\[ \int_{B_r(y)} F |\ln |x-y||^{n-1} dx < \infty \implies \int_{B_r(y)} F |\ln |x-y||^n dx < \infty. \]

In summary, we have:

**Lemma 2.3.** Let the assumptions of the preceding lemma hold. For each positive integer \( n, y \in \Omega, r \in (0, \text{dist}(y, \partial\Omega)) \) there is a positive number \( c > 0 \) such that

\[ \int_{B_r(y)} F |\ln |x-y||^n dx \leq c. \]  \hspace{1cm} (2.13)

We are ready to prove (C3).

**Proof of (C3).** Let \( y \in \Omega, R \in (0, \min\{1, \text{dist}(y, \partial\Omega)\}) \) be given. For each \( r \in (0, R] \), we consider the boundary value problem

\[ -\text{div} [(I + m \otimes m) \nabla p_1] = S(x) \quad \text{in} \quad B_r(y), \]  \hspace{1cm} (2.14)

\[ p_1 = 0 \quad \text{on} \quad \partial B_r(y). \]  \hspace{1cm} (2.15)

Even though the elliptic coefficients in (2.14) may not be bounded, we can easily infer from the proof of Lemma 2.3 in [19] that this problem has a unique solution \( p_1 = p_1(x, t) \) in the sense of (D1)-(D2).

**Claim 2.4.** For each \( q > 1 \) there is a positive number \( c = c(N, q) \) such that

\[ \text{ess sup}_{B_r(y)} |p_1| \leq c r^{2(q-1)/q} \left( \int_{B_r(y)} |S(x)|^q dx \right)^{1/q}. \]  \hspace{1cm} (2.16)

**Proof.** This result was established in [14] for \( N > 2 \). The proof is similar for \( N = 2 \). For completeness, we offer a proof here. We employ the classical Moser-Nash-De Giorgi type of arguments. Without loss of generality, we assume

\[ \text{ess sup}_{B_r(y)} p_1 = \text{ess sup}_{B_r(y)} |p_1|. \]

Let \( \kappa \) be a positive number to be determined. Write

\[ \kappa_n = \kappa - \frac{\kappa}{2^n}, \quad A_n = \{ x \in B_r(y) : p_1(x, t) \geq \kappa_n \}, \quad n = 0, 1, 2, \ldots. \]

Use \((p_1 - \kappa_n)^+\) as a test function in (2.14) to deduce

\[ \int_{B_r(y)} |\nabla (p_1 - \kappa_n)^+|^2 dx + \int_{B_r(y)} (m \cdot \nabla (p_1 - \kappa_n)^+)^2 dx = \int_{B_r(y)} S(x)(p_1 - \kappa_n)^+ dx. \]  \hspace{1cm} (2.17)

Here we have used the fact that

\[ \nabla p_1 \chi_{A_n} = \nabla (p_1 - \kappa_n)^+. \]

By Poincaré’s inequality,

\[ \int_{B_r(y)} S(x)(p_1 - \kappa_n)^+ dx \leq \left( \int_{B_r(y)} |S(x)|^q dx \right)^{1/q} \left( \int_{B_r(y)} [(p_1 - \kappa_n)^+]^{q/2 - 1} dx \right)^{2/(2q - 2)} \leq c \|S(x)\|_{q, B_r(y)} \left( \int_{B_r(y)} |\nabla (p_1 - \kappa_n)^+|^{2/q} dx \right)^{2q/2q - 2} \leq c \|S(x)\|_{q, B_r(y)} \left( \int_{B_r(y)} |\nabla (p_1 - \kappa_n)^+|^2 dx \right)^{1/2} |A_n|^{q-1/q}. \]
Use this in (2.17) to obtain
\[
\left( \int_{B_r(y)} \left| \nabla (p_1 - \kappa_n) \right|^2 dx \right)^{\frac{1}{2}} \leq c \|S(x)\|_{q,B_r(y)} |A_n|^{\frac{q-1}{q}}.
\]

For each \(s > 1\) we have
\[
\frac{\kappa}{2n+1} |A_{n+1}|^{\frac{1}{s}} \leq \left( \int_{B_r(y)} \left[ (p_1 - \kappa_n) \right]^s dx \right)^{\frac{1}{s}} \leq c \left( \int_{B_r(y)} \left| \nabla (p_1 - \kappa_n) \right|^\frac{2s}{s+2} dx \right)^{\frac{s+2}{2s}} \leq c \left( \int_{B_r(y)} \left| \nabla (p_1 - \kappa_n) \right|^2 dx \right)^{\frac{1}{2}} r^{\frac{2s}{s}} \leq cr^{\frac{2s}{s}} \|S(x)\|_{q,B_r(y)} |A_n|^{\frac{q-1}{q}},
\]
from whence follows
\[
|A_{n+1}| \leq cr^{s} \|S(x)\|_{q,B_r(y)} |A_n|^{\frac{q-1}{q}}.
\]

Now we pick \(s\) so large that
\[
\alpha \equiv \frac{(q-1)s}{q} - 1 = \frac{s(q-1) - q}{q} > 0.
\]

We are in a position to apply Lemma 4.1 in ([4], p. 12). To this end, we choose \(\kappa\) so large that
\[
A_0 \leq |B_r(y)| \leq c \left( \frac{\kappa^s}{r^2 \|S(x)\|_{q,B_r(y)}^s} \right)^{\frac{q}{s(q-1) - q}}.
\]

Then
\[
p_1 \leq \kappa \text{ on } B_r(y).
\]

It is enough for us to take
\[
\kappa = cr^{\frac{2(q-1)}{q}} \|S(x)\|_{q,B_r(y)}.
\]

The lemma follows. The proof is complete. \(\square\)

Obviously, \(p_0 \equiv p - p_1\) satisfies
\[
-\text{div} \left[ (I + m \otimes m) \nabla p_0 \right] = 0 \quad \text{in } B_r(y),
\]
\[
p_0(x,t) = p(x,t) \quad \text{on } \partial B_r(y)
\]
in the sense of (D1)-(D2) with an obvious modification to the boundary condition. Obviously, the weak maximum principle still holds, from which it follows that
\[
\text{osc}_{B_r(y)} p_0 \leq \text{osc}_{\partial B_r(y)} p_0 = \text{osc}_{\partial B_r(y)} p \leq \int_{\partial B_r(y)} |\nabla p| ds.
\]
(Here we may assume that \(p\) is a smooth function because \(p\) can be viewed as the limit of a sequence of smooth functions.) For each \(\ell > 1\) and \(\tau \in (0, R)\) we multiply through the above inequality by
Integrate the above inequality over \([\tau, R]\) to derive

\[
\int_\tau^R \frac{|\ln r|^{\ell}}{r} \text{osc } p_0 dr \leq \int_\tau^R \frac{|\ln r|^{\ell}}{r} \int_{\partial B_r(y)} |\nabla p| ds dr
\]

\[
\leq \int_{B_R(y) \setminus B_{\tau}(y)} \frac{|\nabla p||\ln |x - y||^{\ell}}{|x - y|} dx
\]

\[
\leq \left( \int_{B_R(y)} |\nabla p|^2 |\ln |x - y||^{3\ell} dx \right)^{\frac{1}{2}} \left( \int_{B_R(y) \setminus B_{\tau}(y)} \frac{1}{|\ln |x - y||^{\ell}} dx \right)^{\frac{1}{2}}
\]

\[
\leq c(R, \ell) \left( \int_0^R \frac{1}{|\ln r|^\ell} dr \right)^{\frac{1}{2}} \leq c(R, \ell).
\]

Here we have employed (2.13). Observe that \(\text{osc } p_0\) is an increasing function of \(r\). With this in mind, we obtain

\[
\text{osc } p_0 \leq \frac{\int_\tau^R \frac{|\ln r|^{\ell}}{r} \text{osc } p_0 dr}{\int_\tau^R \frac{|\ln r|^{\ell}}{r} dr} \leq c(R, \ell) \frac{c(R, \ell)}{\ln^\ell + \frac{R}{\tau}}.
\]

Here we have used the fact that \(\ln R < 0\). The above inequality together with (2.16) implies

\[
\text{osc } p \leq \text{osc } p_0 + \text{osc } p_1 \leq c(R, \ell) + c\tau \frac{2q(q-1)}{q} \quad \text{for } \tau \in (0, R).
\]

This finishes the proof of (C3).

**Lemma 2.5.** Let \(x_0 \in \Omega\) be given and define

\[
d(x_0) = \text{dist}(x_0, \partial \Omega).
\]

If \(m \in (\text{BMO}(\Omega))^N\), i.e.,

\[
\|m\|_{\text{BMO}(\Omega)} \equiv \sup_{B_r(y) \subset \Omega} \int |m - m_{y,r}| dx < \infty,
\]

then there exists a positive number \(c = c(\|m\|_{\text{BMO}(\Omega)}, d(x_0))\) such that

\[
|m_{x_0, \rho}| \leq c \ln \left( \frac{d(x_0)}{\rho} \right) + c \quad \text{for each } \rho \in (0, d(x_0)).
\]

**Proof.** We follow the proof of Proposition 1.2 in ([7], p. 68). For each \(0 < \rho \leq r \leq d(x_0)\), we have

\[
|m_{x_0, r} - m_{x_0, \rho}| \leq |m_{x_0, r} - m(x)| + |m(x) - m_{x_0, \rho}|.
\]

Integrate the above inequality over \(B_{\rho}(x_0)\) to derive

\[
|m_{x_0, r} - m_{x_0, \rho}| \leq \left( \frac{r}{\rho} \right)^2 \int_{B_{\rho}(x_0)} |m_{x_0, r} - m| dx + \int_{B_{\rho}(x_0)} |m - m_{x_0, \rho}| dx \leq c \left( \frac{r}{\rho} \right)^2.
\]

In particular, take \(r = \frac{d(x_0)}{2^i}, \rho = \frac{d(x_0)}{2^{i+1}}\), where \(i \in \{0, 1, \cdots\}\). Then we have

\[
|m_{x_0, \frac{d(x_0)}{2^i}} - m_{x_0, \frac{d(x_0)}{2^{i+1}}}| \leq c.
\]
For each \( \rho \in (0, d(x_0)) \) there is an \( i \in \{0, 1, \cdots \} \) such that
\[
\frac{d(x_0)}{2^{i+1}} \leq \rho < \frac{d(x_0)}{2^i}.
\]
Subsequently,
\[
|m_{x_0, \rho}| \leq \left| m_{x_0, \frac{d(x_0)}{2^i}} \right| + \left| m_{x_0, \rho} - m_{x_0, \frac{d(x_0)}{2^i}} \right|
\]
\[
\leq \left| m_{x_0, \frac{d(x_0)}{2^i}} - m_{x_0, d(x_0)} \right| + \left| m_{x_0, d(x_0)} \right| + c
\]
\[
\leq \sum_{j=1}^{i} \left| m_{x_0, \frac{d(x_0)}{2^j}} - m_{x_0, \frac{d(x_0)}{2^{j-1}}} \right| + \left| m_{x_0, d(x_0)} \right| + c
\]
\[
\leq c(i + 1) + \left| m_{x_0, d(x_0)} \right|.
\]
We easily see from (2.18) that
\[
i \leq \frac{1}{\ln 2} \ln \left( \frac{d(x_0)}{\rho} \right).
\]
Using this in (2.19) yields the desired result. \( \square \)

We are in a position to prove (C4).

**Proof of (C4).** Let \( y \in \Omega, r > 0 \) be such that \( B_{2r}(y) \subset \Omega \). Then for each \( x_0 \in B_r(y) \) and \( \rho \in (0, \frac{r}{2}) \), we have \( B_{2\rho}(x_0) \subset \Omega \). Let \( \rho \) be so chosen. Select a cutoff function \( \zeta \in C_0^\infty (B_{2\rho}(x_0)) \) the properties
\[
\zeta = 1 \text{ on } B_{\rho}(x_0), 0 \leq \zeta \leq 1 \text{ on } B_{2\rho}(x_0), \text{ and } |\nabla \zeta| \leq \frac{c}{\rho} \text{ on } B_{2\rho}(x_0).
\]
Use \( (p - p_{x_0, 2\rho})\zeta^2 \) as a test function in (1.1) to obtain
\[
\int_{B_{2\rho}(x_0)} |\nabla p|^2 \zeta^2 dx + \int_{B_{2\rho}(x_0)} (m \cdot \nabla p)^2 \zeta^2 dx
\]
\[
= -\int_{B_{2\rho}(x_0)} \nabla p \cdot \nabla \zeta (p - p_{x_0, 2\rho}) dx - \int_{B_{2\rho}(x_0)} (m \cdot \nabla p) m \cdot \nabla \zeta (p - p_{x_0, 2\rho}) dx
\]
\[
+ \int_{B_{2\rho}(x_0)} S(x)(p - p_{x_0, 2\rho}) \zeta^2 dx
\]
\[
\leq \frac{1}{2} \int_{B_{2\rho}(x_0)} (|\nabla |^2 + (m \cdot \nabla p)^2) \zeta^2 dx + \frac{c}{\rho^2} \int_{B_{2\rho}(x_0)} (p - p_{x_0, 2\rho})^2 dx
\]
\[
+ \frac{c}{\rho^2} \int_{B_{2\rho}(x_0)} |m|^2 (p - p_{x_0, 2\rho})^2 dx + \int_{B_{2\rho}(x_0)} S(x)(p - p_{x_0, 2\rho}) \zeta^2 dx,
\]
from whence follows
\[
\int_{B_{\rho}(x_0)} F dx = \int_{B_{\rho}(x_0)} |\nabla p|^2 dx + \int_{B_{\rho}(x_0)} (m \cdot \nabla p)^2 dx
\]
\[
\leq c \left( \text{osc}_{B_{2\rho}(x_0)} p \right)^2 + \frac{c}{\rho^2} \int_{B_{2\rho}(x_0)} |m - m_{x_0, 2\rho}|^2 (p - p_{x_0, 2\rho})^2 dx
\]
\[
+ cm_{x_0, 2\rho}^2 \left( \text{osc}_{B_{2\rho}(x_0)} p \right)^2 + c\|S\|_{L^q(B_{2\rho}(x_0))} \rho \frac{2(q-1)}{q} \text{osc}_{B_{2\rho}(x_0)} p
\]
\[
\leq c \left( 1 + (\ln \rho + 1)^2 \right) \left( \text{osc}_{B_{2\rho}(x_0)} p \right)^2 + c\rho \frac{2(q-1)}{q}.
\]

(2.20)
We derive from (2.1), (2.20), and (C3) that
\[ \sup_{B_r(x_0) \subset \Omega} \left( \int_{B_r(x_0)} |m - m_{x_0,r}|^s dx \right)^{\frac{1}{s}} \leq c \|m\|_{BMO(\Omega)}. \]

We derive from (2.1), (2.20), and (C3) that
\[
\begin{align*}
\int_{B_r(x_0)} F \chi_{B_r(y)} \ln |x - x_0| dx & \leq \int_{B_r(x_0)} F \ln |x - x_0| dx \\
& \leq (\ln 2 + |\ln \rho|) \int_{B_r(x_0)} F dx + c \int_{B_r(x_0)} |\nabla p|^2 dx \\
& \quad + c \left( \int_{B_r(x_0)} |m|^4 dx \right)^{\frac{1}{2}} \int_{B_r(x_0)} |\nabla p|^2 dx + c^{\frac{2(q-1)}{q}} \|S(x)\|_{q, B_r(x_0)} \\
& \leq (c + |\ln \rho|) \int_{B_r(x_0)} F dx + c^{\frac{2(q-1)}{q}} \\
& \leq c \left( 1 + (|\ln \rho| + 1)^2 \right) (c + |\ln \rho|) \left( \osc_{B_r(x_0)} p \right)^2 + c(c + |\ln \rho|) \rho^{\frac{2(q-1)}{q}}
\end{align*}
\]

provided that \( \ell > \frac{3}{2} \). The proof is complete. \( \square \)

3. PROOF OF THEOREM 1.5

In this section we prove the last two results in the introduction.

**Proof of Theorem 1.5.** For each \( \rho \in (0, 1) \) and \( x_0 \in \Omega \) such that \( B_{2\rho}(x_0) \subset \Omega \), we define
\[
M_{x_0,r} = \sup_{B_r(x_0)} p, \quad m_{x_0,r} = \inf_{B_r(x_0)} p, \quad \omega_{x_0,r} = \osc_{B_r(x_0)} p = M_{x_0,r} - m_{x_0,r}.
\]

Let \( q \) be given as in (H1). Then we can pick a number
\[
\delta \in \left( 0, \frac{2q - 2}{q} \right).
\]

Subsequently, set
\[
\bar{k}(r) = r^\delta \|S\|_{q, B_r(x_0)}.
\]

Either
\[
\left| B_r(x_0) \cap \left\{ p(x) \leq m_{x_0,r} + \frac{\omega_{x_0,r}}{2} \right\} \right| \geq \frac{1}{2} |B_r(x_0)|, \quad \text{or}
\]
\[
\left| B_r(x_0) \cap \left\{ p(x) > m_{x_0,r} + \frac{\omega_{x_0,r}}{2} \right\} \right| \geq \frac{1}{2} |B_r(x_0)|.
\]

If (3.2) is true, we consider the function
\[
w = \ln \left( \frac{\omega_{x_0,r} + \bar{k}(r)}{2(M_{x_0,r} - p) + \bar{k}(r)} \right).
\]
If (3.3) holds, we take
\[ w = \ln \left( \frac{\omega_{x_0,r} + k(r)}{2(p - m_{x_0,r}) + k(r)} \right). \]
For definiteness, we assume the first case (3.2). Then we easily verify that \( w \) satisfies the equation
\[ -\text{div} [(I + m \otimes m) \nabla w] + (I + m \otimes m) \nabla w \cdot \nabla w \]
(3.4)
\[ = \frac{2S}{2(M_{x_0,r} - p) + k(r)} \text{ in } B_r(x_0). \]
The rest of the proof is divided into several claims.

**Claim 3.1.** There is a \( \delta_0 \in (0,1) \) such that
\[ \int_{B(1-\delta_0,r)(x_0)} (w^+)^2 \, dx \leq c \int_{B_r(x_0)} |m|^2 \, dx + cr^{2-\frac{p}{q}-\delta} + c. \]

**Proof.** Select \( \delta_0 \in (0,1) \) as below. Let \( \theta(\eta) \) be a smooth decreasing function on \([0,r]\) with the properties
\[ \theta(\eta) = 1 \text{ for } \eta \leq (1-\delta_0)r, \theta(r) = 0, \text{ and } |\theta'(\eta)| \leq \frac{c}{r_0} \text{ on } [0,r]. \]
We easily see that
\[ |B_r(x_0) \setminus B_{(1-\delta_0)r}(x_0)| \leq N\delta_0 |B_r(x_0)|. \]
We can pick \( \delta_0 \) so that
(3.5) \[ |\{ \theta(|x-x_0|) = 1 \} \cap \{ x \in B_r, w^+(x,t) = 0 \} | \geq c_0 |B_r(x_0)| \text{ some } c_0 > 0. \]
To see this, take
\[ N\delta_0 = \frac{1}{16}. \]
Note that
\[ w^+(x,t) = 0 \iff x \in B_r(x_0) \cap \{ w(x,t) \leq m_{x_0,r} + \frac{\omega_{x_0,r}}{2} \}. \]
Combining this with (3.2) yields (3.5) with \( c_0 = \frac{7}{16} \). Now we are in a position to apply Proposition 2.1 in (4), p.5. Upon doing so, we obtain
(3.6) \[ \int_{B_r(x_0)} \theta^2(|x-x_0|)(w^+)^2 \, dx \leq cr^2 \int_{B_r(x_0)} \theta^2(|x-x_0|) |\nabla w|^2 \, dx. \]
Use \( \theta^2 \) as a test function in (3.4) to derive
\[ \int_{B_r(x_0)} (|\nabla w|^2 + (m \cdot \nabla w)^2) \theta^2 \, dx \]
\[ = -2 \int_{B_r(x_0)} \nabla w \cdot \nabla \theta \, dx - 2 \int_{B_r(x_0)} (m \cdot \nabla w) m \cdot \nabla \theta \, dx \]
\[ + \int_{B_r(x_0)} \frac{2S\theta^2}{2(M_{x_0,r} - p) + k(r)} \, dx. \]
(3.7)
We easily see that
\[ \frac{1}{2(M_{x_0,r} - p) + r^\delta \|S\|_{q,B_r(x_0)}} \leq \frac{1}{r^\delta \|S\|_{q,B_r(x_0)}}. \]
We shall always use this inequality when dealing with the last term in (3.4). Indeed, with this in mind, we can derive from (3.7) that
\[ \int_{B_r(x_0)} |\nabla w|^2 \theta^2 \, dx \leq c + \frac{c}{r^2} \int_{B_r(x_0)} |m|^2 \, dx + cr^{2-\frac{p}{q}-\delta}. \]
This together with (3.6) yields
\[
\int_{B_{(1-\delta_0) \cdot x_0}} (w^+)^2 dx \leq \int_{B_r(x_0)} \theta^2 (w^+)^2 dx
\]
\[
\leq cr^2 \int_{B_r(x_0)} \theta^2 |\nabla w|^2 dx
\]
\[
\leq cr^2 + cr^2 \int_{B_r(x_0)} m^2 dx + cr^{4 - \frac{2}{q} - \delta}.
\]

The lemma follows. \(\square\)

**Claim 3.2.** We have
\[
\sup_{B_{\frac{r}{2}}(x_0)} w \leq c \left( \frac{\int_{B_r(x_0)} (w^+)^2 dx}{r} \right)^\frac{1}{2} + \frac{1}{r} \|m\|^2_{q, B_r(x_0)} + r^{2 - \frac{2}{q} - \frac{1}{s}}.
\]

**Proof.** We employ the classical De Giorgi iteration scheme. For this purpose, we define
\[
r_n = \frac{r}{2} + \frac{r}{2^{1+n}}, \quad n = 0, 1, \cdots.
\]
Select a sequence of smooth functions \(\zeta_n(x)\) so that
\[
\zeta_n(x) = 1 \text{ on } B_n(x_0), \quad \zeta_n(x) = 0 \text{ outside } B_{n-1}(x_0), \quad |\nabla \zeta_n| \leq \frac{c^2}{r}, \quad 0 \leq \zeta_n \leq 1.
\]
Choose \(k > 0\) as below. Set
\[
k_n = k - \frac{k}{2n+1} \quad \text{for } n = 0, 1, \cdots.
\]
Use \((w - k_{n+1})^+ \zeta_{n+1}^2\) as a test function in (3.4) and make use of the fact that
\[
\nabla w = \nabla (w - k_{n+1})^+ \text{ on } \{w \geq k_{n+1}\}
\]
to derive
\[
\int_{B_{r_n}(x_0)} \left( |\nabla (w - k_{n+1})^+|^2 + [m \cdot \nabla (w - k_{n+1})^+]^2 \right) \zeta_{n+1}^2 dx
\]
\[
+ \int_{B_{r_n}(x_0)} (I + m \otimes m) \nabla (w - k_{n+1})^+ \cdot \nabla (w - k_{n+1})^+ (w - k_{n+1})^+ \zeta_{n+1}^2 dx
\]
\[
= -2 \int_{B_{r_n}(x_0)} \nabla (w - k_{n+1})^+ \cdot \nabla \zeta_{n+1} \zeta_{n+1} (w - k_{n+1})^+ dx
\]
\[
-2 \int_{B_{r_n}(x_0)} m \cdot \nabla (w - k_{n+1})^+ m \cdot \nabla \zeta_{n+1} \zeta_{n+1} (w - k_{n+1})^+ dx
\]
\[
+ \int_{B_{r_n}(x_0)} \frac{2S(w - k_{n+1})^+ \zeta_{n+1}^2}{2(M_{x_0} - p) + r^\delta \|S\|_{q, B_r(x_0)}} dx.
\]

Fix \(s \geq \max\{2, \frac{q}{q-1}\}\) so that
\[
\frac{s}{s-1} \leq q \quad \text{and} \quad \frac{2s}{s+2} \geq 1.
\]
We can use the Sobolev inequality to estimate the last integral in (3.8) as follows:

\[
\int_{B_{r_n}(x_0)} \frac{2S(w - k_{n+1})^+ \zeta_{n+1}^2}{2(M_{2r_n} + r^\delta \| S \|_{q,B_{r_n}(x_0)})} \, dx
\]

\[
\leq \frac{2}{r^\delta \| S \|_{q,B_{r_n}(x_0)}} \| S \|_{r_n,B_{r_n}(x_0)} \| (w - k_{n+1})^+ \zeta_{n+1} \|_{s,B_{r_n}(x_0)}
\]

\[
\leq \frac{2}{r^\delta} |B_{r_n}(x_0) \cap \{ w \geq k_{n+1} \}| \frac{4 - 1}{2} \| \nabla [(w - k_{n+1})^+ \zeta_{n+1}] \|_{2,B_{r_n}(x_0)}
\]

\[
\leq \frac{2}{r^\delta} |B_{r_n}(x_0) \cap \{ w \geq k_{n+1} \}| \frac{4 - 1}{2} \| \nabla [(w - k_{n+1})^+ \zeta_{n+1}] \|_{2,B_{r_n}(x_0)}
\]

Here we have used the fact that \( |B_{r_n}(x_0) \cap \{ w \geq k_{n+1} \}| \leq c \chi^2 \). Substitute this into (3.8), choose \( \varepsilon \) suitably small in the resulting equation, and thereby obtain

\[
\int_{B_{r_n}(x_0)} |\nabla [(w - k_{n+1})^+ \zeta_{n+1}]|^2 \, dx
\]

\[
\leq \frac{c4^n}{r} y_n + \frac{c4^n}{r^2} \int_{B_{r_n}(x_0)} |m|^2 [(w - k_{n+1})^+]^2 \, dx
\]

\[
+ c |B_{r_n}(x_0) \cap \{ w \geq k_{n+1} \}|^{2 - \frac{2}{q}}.
\]

where

\[
y_n = \left( \int_{B_{r_n}(x_0)} [(w - k_n)^+]^4 \, dx \right)^{\frac{1}{5}}.
\]

It is easy to see that

\[
y_n \geq \left( \int_{B_{r_n}(x_0) \cap \{ w > k_{n+1} \}} [(w - k_n)^+]^4 \, dx \right)^{\frac{1}{5}} \geq \frac{k^2}{4n+2} |B_{r_n}(x_0) \cap \{ w > k_{n+1} \}|^{\frac{1}{2}}.
\]

Let \( s \) be given as before. We derive from the Sobolev inequality that

\[
y_{n+1} \leq \int_{B_{r_n}(x_0)} [(w - k_{n+1})^+ \zeta_{n+1}]^2 \, dx
\]

\[
\leq \left( \int_{B_{r_n}(x_0)} [(w - k_{n+1})^+ \zeta_{n+1}]^{2s} \, dx \right)^{\frac{1}{s}} |B_{r_n}(x_0) \cap \{ w > k_{n+1} \}|^{\frac{s-1}{s}}
\]

\[
\leq c \left( \int_{B_{r_n}(x_0)} |\nabla [(w - k_{n+1})^+ \zeta_{n+1}]|^{2s} \, dx \right)^{\frac{1}{s}} |B_{r_n}(x_0) \cap \{ w > k_{n+1} \}|^{\frac{s-1}{s}}
\]

\[
\leq c \int_{B_{r_n}(x_0)} |\nabla [(w - k_{n+1})^+ \zeta_{n+1}]|^2 \, dx |B_{r_n}(x_0) \cap \{ w > k_{n+1} \}|
\]

\[
\leq \frac{c4^n}{r^2} y_n |B_{r_n}(x_0) \cap \{ w > k_{n+1} \}|
\]

\[
+ \frac{c4^n}{r^2} \int_{B_{r_n}(x_0)} |m|^2 [(w - k_{n+1})^+]^2 \, dx |B_{r_n}(x_0) \cap \{ w > k_{n+1} \}|
\]

\[
+ c |B_{r_n}(x_0) \cap \{ w \geq k_{n+1} \}|^{3 - \frac{2}{q}}.
\]
Set
\[ \sigma = \min \left\{ 2 - \delta - \frac{2}{q}, \frac{1}{2} \right\} > 0 \text{ due to (3.1)}. \]

Then we obtain from (3.9) that
\[
\begin{align*}
y_n |B_{r_n}(x_0) \cap \{ w > k_{n+1} \}| &= y_n |B_{r_n}(x_0) \cap \{ w > k_{n+1} \}|^{1-\sigma+\sigma} \\
&\leq c r^2(1-\sigma) \left( \frac{4^{n+2} y_n}{k^2} \right)^\sigma y_n \leq \frac{c4^\sigma r^2}{r^2 \sigma k^{2\sigma}} y_n^{1+\sigma}.
\end{align*}
\]

Similarly,
\[
\begin{align*}
\int_{B_{r_n}(x_0)} |m|^2 [(w - k_{n+1})^+]^2 dx |B_{r_n}(x_0) \cap \{ w > k_{n+1} \}|
&\leq \|m\|_{L^4(B_r(x_0))}^2 \left( \int_{B_{r_n}(x_0)} [(w - k_{n+1})^+]^4 dx \right)^\frac{1}{2} |B_{r_n}(x_0) \cap \{ w > k_{n+1} \}|^{\frac{1}{2}+\sigma + \frac{1}{2}-\sigma} \\
&\leq \text{ess sup}_{B_r(y)} w^+ \|m\|_{L^4(B_r(x_0))}^2 \frac{1}{r^{1-2\sigma}} \left( \frac{4^{n+2} y_n}{k^2} \right)^\sigma \\
&\leq \frac{c4^{\frac{1}{2}+\sigma} r^2}{r^{2\sigma} k^{2\sigma}} y_n^{1+\sigma}.
\end{align*}
\]

Here we have assumed that
\[ (3.11) \]
\[ k \geq r^2 - \frac{\delta - \frac{2}{q}}{q}. \]

As for the last term in (3.10), we have
\[
\begin{align*}
|B_{r_n}(x_0) \cap \{ w \geq k_{n+1} \}|^{1+\sigma+2-\delta-\frac{2}{q}-\sigma}
&\leq c r^4 - 2\delta - \frac{2}{q}-2\sigma \left( \frac{4^{n+2} y_n}{k^2} \right)^{1+\sigma} \leq \frac{c4^{(1+\sigma)n}}{r^{2\sigma} k^{2\sigma}} y_n^{1+\sigma}.
\end{align*}
\]

In this case, we have taken
\[ (3.12) \]
\[ k \geq r^2 - \frac{\delta - \frac{2}{q}}{q}. \]

Collecting these estimates in (3.10) yields
\[ y_{n+1} \leq \frac{c4^{\frac{1}{2}+\sigma} r^2}{r^{2\sigma} k^{2\sigma}} y_n^{1+\sigma}. \]

According to Lemma 4.1 in [4], p.12, if we pick \( k \) so large that
\[ y_0 \leq \frac{r^2 k^2}{c^{\frac{1}{2}+\sigma} 4^{\frac{1}{2}+\sigma}}, \]
then
\[ \sup_{B_{\frac{r}{2}}(x_0)} w \leq k. \]

Recall that
\[ y_0 = \int_{B_r(x_0)} \left( \left( w - \frac{k}{2} \right)^+ \right)^2 dx \leq \int_{B_r(x_0)} (w^+)^2 dx. \]
This together with (3.11) and (3.12) implies that it is enough for us to take

\[ k = c \left( \frac{1}{B_{r}(x_0)} \left( w^+ \right)^2 \, dx \right)^{\frac{1}{2}} + \frac{1}{r} \left\| m \right\|_{L^2 B_r(x_0)}^2 + r^{2-\delta-\frac{2}{q}}. \]

This gives the desired result. \( \Box \)

We are ready to complete the proof of Theorem 1.5. We estimate from (2.21) that

\[ \frac{1}{r} \left\| m \right\|_{L^2 B_r(x_0)}^2 \leq c \left( \frac{1}{B_{r}(x_0)} \left| m \right|^4 \, dx \right)^{\frac{1}{2}} \]

\[ \leq c \left( \frac{1}{B_{r}(x_0)} \left| m - m_{x_0,r} \right|^4 \, dx \right)^{\frac{1}{2}} + c \left| m_{x_0,r} \right|^2 \]

\[ \leq c \left| m_{x_0,r} \right|^2 + c. \]

Similarly,

\[ \int_{B_r(x_0)} \left| m \right|^2 \, dx \leq c \left| m_{x_0,r} \right|^2 + c. \]

We conclude from Claims 3.1 and 3.2 that

\[ \sup_{B_{\frac{r}{2}}(x_0)} w \leq c \left( \frac{1}{B_{\frac{r}{2}}(x_0)} w^+ \, dx \right)^{\frac{1}{2}} + \frac{1}{r} \left\| m \right\|_{L^2 B_r(x_0)}^2 + c r^{2-\delta-\frac{2}{q}} \]

\[ \leq c \left( \frac{1}{B_{r}(x_0)} \left| m \right|^2 \, dx \right)^{\frac{1}{2}} + c r^{1-\frac{\delta}{q}} + c + \frac{1}{r} \left\| m \right\|_{L^2 B_r(x_0)}^2 + c r^{2-\delta-\frac{2}{q}} \]

\[ \leq \left| m_{x_0,r} \right|^2 + c \leq c. \]

The last step is due to (1.9). By the definition of \( w \),

\[ \omega_{x_0,r} + \overline{k}(r) \leq e^c \left[ 2(M_{x_0,r} - p) + \overline{k}(r) \right] \text{ on } B_{\frac{r}{2}}(x_0), \]

from whence follows

\[ 2e^c M_{x_0,\frac{r}{2}} \leq (2e^c - 1) \omega_{x_0,r} + 2e^c m_{x_0,r} \]

\[ + (e^c - 1) \overline{k}(r). \]

Subtract \( 2e^c m_{x_0,\frac{r}{2}} \) from both sides of the above inequality and make use of the fact that \( m_{x_0,r} \) is a decreasing function of \( r \) to deduce.

\[ \omega_{x_0,\frac{r}{2}} \leq \frac{2e^c - 1}{2e^c} \omega_{x_0,r} + \frac{e^c - 1}{2e^c} \overline{k}(r). \]

Obviously, recall the definition of \( \overline{k}(r) \) to get

\[ \omega_{x_0,\frac{r}{2}} \leq \frac{2e^c - 1}{2e^c} \omega_{x_0,r} + r^\delta \left\| S \right\|_{L^q B_r(x_0)} \leq \frac{2e^c - 1}{2e^c} \omega_{x_0,r} + cr^\delta. \]

It is very important to note that the two coefficients on the right-hand side of the above inequality can be made independent of \( r \) for \( r \) small. This enables us to invoke Lemma 8.23 in (S), p.201) to get

\[ \omega_{x_0,r} \leq c \rho^\alpha \] for some \( c > 0, \alpha > 0 \) and \( \rho \) sufficiently small.

The proof is complete. \( \Box \)

Finally, we give the proof of Lemma 1.6.
Proof of Lemma 1.6. We can offer a much simpler proof than the one in [6] due to our boundary condition. Take
\[ 0 < r \leq \frac{1}{2} \]
so that
\[ \ln |x - y| \leq 0 \quad \text{whenever} \quad x, y \in B_r(x_0). \]
Consider the problem
\[
\begin{align*}
-\Delta \psi &= F \quad \text{in} \ B_r(x_0), \\
\psi &= 0 \quad \text{on} \ \partial B_r(x_0).
\end{align*}
\]
Claim 3.3. There exists a constant \( c \) such that
\[ \sup_{B_r(x_0)} |\psi| \leq c\eta(F; B_r(x_0); r), \]
where \( \eta(F; B_r(x_0); r) \) is given as in (1.4).
Proof. We can write \( F = F^+ - F^- \).
Then solve the following two problems
\[
\begin{align*}
-\Delta \psi_1 &= F^+ \quad \text{in} \ B_r(x_0), \\
\psi_1 &= 0 \quad \text{on} \ \partial B_r(x_0)
\end{align*}
\]
and
\[
\begin{align*}
-\Delta \psi_2 &= F^- \quad \text{in} \ B_r(x_0), \\
\psi_2 &= 0 \quad \text{on} \ \partial B_r(x_0).
\end{align*}
\]
Clearly,
\[ \psi = \psi_1 - \psi_2. \]
The non-negative function
\[ -\frac{1}{2\pi} \int_{B_r(x_0)} F^+ \ln |x - y| dy \]
satisfies the equation (3.16). The comparison principle asserts
\[ 0 \leq \psi_1 \leq -\frac{1}{2\pi} \int_{B_r(x_0)} F^+ \ln |x - y| dy \quad \text{on} \ B_r(x_0). \]
Similarly,
\[ 0 \leq \psi_2 \leq -\frac{1}{2\pi} \int_{B_r(x_0)} F^- \ln |x - y| dy \quad \text{on} \ B_r(x_0). \]
The preceding inequalities yield the claim. \( \square \)

The rest of the proof here largely follows [6]. Without loss of generality, assume \( F \geq 0. \)
Let \( \psi \) be given as in (3.13)-(3.14). For \( u \in W_{0}^{1,2}(B_r(x_0)) \cap L^\infty(B_r(x_0)), \) we calculate
\[
\int_{B_r(x_0)} Fu^2 dx = -\int_{B_r(x_0)} \Delta \psi u^2 dx
= 2 \int_{B_r(x_0)} u \nabla \psi \cdot \nabla u dx
\]
\[ \leq 2 \left( \int_{B_r(x_0)} |\nabla u|^2 dx \right)^{\frac{1}{2}} \left( \int_{B_r(x_0)} u^2 |\nabla \psi|^2 dx \right)^{\frac{1}{2}}. \]
(3.17)
It is easy to verify

\[ |\nabla \psi|^2 = \frac{1}{2} \Delta \psi^2 - \psi \Delta \psi = \frac{1}{2} \Delta \psi^2 - \psi F. \]

This together with (3.15) implies

\[
\int_{B_r(x_0)} u^2|\nabla \psi|^2 \, dx = \frac{1}{2} \int_{B_r(x_0)} u^2 \Delta \psi^2 \, dx - \int_{B_r(x_0)} \psi F u^2 \, dx
\]

\[
\leq \frac{1}{2} \int_{B_r(x_0)} \nabla u^2 \cdot \nabla \psi^2 \, dx + \|\psi\|_{\infty, B_r(x_0)} \int_{B_r(x_0)} F u^2 \, dx
\]

\[
\leq \frac{1}{2} \int_{B_r(x_0)} u^2|\nabla \psi|^2 \, dx + 2 \int_{B_r(x_0)} \psi^2 |\nabla u|^2 \, dx
\]

\[+ c\eta(F; B_r(x_0); r) \int_{B_r(x_0)} F u^2 \, dx,\]

from whence follows

\[
\int_{B_r(x_0)} u^2|\nabla \psi|^2 \, dx \leq c\eta^2(r) \int_{B_r(x_0)} |\nabla u|^2 \, dx + c\eta(F; B_r(x_0); r) \int_{B_r(x_0)} F u^2 \, dx.
\]

Plug this into (3.17) to derive

\[
\int_{B_r(x_0)} |F|^2 u^2 \, dx \leq c\eta(F; B_r(x_0); r) \int_{B_r(x_0)} |\nabla u|^2 \, dx
\]

\[+ c \left( \eta(F; B_r(x_0); r) \int_{B_r(x_0)} F u^2 \, dx \right)^{\frac{1}{2}} \left( \int_{B_r(x_0)} |\nabla u|^2 \, dx \right)^{\frac{1}{2}}\]

\[
\leq \frac{1}{2} \int_{B_r(x_0)} |F|^2 u^2 \, dx + c\eta(F; B_r(x_0); r) \int_{B_r(x_0)} |\nabla u|^2 \, dx.
\]

The lemma follows. \(\square\)

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