A SEMIGROUP CHARACTERIZATION OF
WELL-POSED LINEAR CONTROL SYSTEMS

M. BOMBIERI, K.-J. ENGEL

Abstract. We study the well-posedness of a linear control system $\Sigma(A, B, C, D)$ with unbounded control and observation operators. To this end we associate to our system an operator matrix $A$ on a product space $X^p$ and call it $p$-well-posed if $A$ generates a strongly continuous semigroup on $X^p$. Our approach is based on the Laplace transform and Fourier multipliers. The results generalize and complement those of [4], [24] and are illustrated by a heat equation with boundary control and point observation.

1. Introduction

In this paper we investigate the well-posedness of linear control systems of the form

$$\Sigma(A, B, C, D) \begin{cases} \dot{x}(t) = Ax(t) + Bu(t), & t \geq 0, \\ y(t) = Cx(t) + Du(t), & t \geq 0, \\ x(0) = x_0. \end{cases}$$

The operators $A, B, C, D$ are linear and defined on Banach spaces $X, Y$ and $U$, called state-, observation- and control space, respectively, and satisfy the following hypotheses:

- $A : D(A) \subset X \to X$, called the state operator, is the generator of a $C_0$-semigroup,
- $B \in \mathcal{L}(U, X_{-1})$ is the control operator,
- $C \in \mathcal{L}(X_1, Y)$ is the observation operator,
- $D \in \mathcal{L}(U, Y)$ is the feedthrough operator.

For the motivation, concrete examples and a systematic treatment of such systems we refer to [5], [16], [17], [25], [27] and the references therein. Moreover, in Section 6 we illustrate our results considering a heat equation with boundary control and point observation.

Generalizing an idea of Grabowski and Callier [12], see also Engel [10], we associate to our system an operator matrix $(A, D(A))$ defined on an appropriate product space $X^p$ depending on $p \geq 1$. We then call $\Sigma(A, B, C, D)$ $p$-well posed if this operator matrix generates a $C_0$-semigroup $T = (T(t))_{t \geq 0}$ on $X^p$.

In other words, $\Sigma(A, B, C, D)$ is well-posed if the Cauchy problem

$$\begin{cases} \dot{x}(t) = Ax(t), & t \geq 0, \\ x(0) = x_0 \end{cases}$$

is well-posed on $X^p$ in the sense of Hadamard (see [11, Sect. II.6]).

It turns out that this definition of well-posedness leads to the concept of $p$-admissibility of the control operator $B$ and the observation operator $C$ as studied, e.g., by Staffans and Weiss, see [30], [29], [27], [34], [24].

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1For the definition of the inter- and extrapolation spaces $X_1, X_{-1}$ see below.
We mention that the semigroup $T$ generated by $A$ already appears in [34], [24], [23] where it is called the “Lax-Phillips semigroup”.

To carry out the program sketched above we start from the generator $(A,D(A))$ of a semigroup $(T(t))_{t \geq 0}$ on a Banach space $(X,\|\cdot\|)$. We then consider the associated abstract Sobolev spaces (see [11, Sect. II.5]) defined by

- $X_1 := (D(A),\|\cdot\|_A)$, where $\|\cdot\|_A$ is the graph norm given by $\|x\|_A := \|x\| + \|Ax\|$,
- $X_{-1} := (X,\|\cdot\|_{-1})$, where $\|x\|_{-1} := \|R(\lambda,A)x\|$ for $x \in X$ and some fixed $\lambda \in \rho(A)$.

Then $(T(t))_{t \geq 0}$ uniquely extends to the extrapolated semigroup $(T_{-1}(t))_{t \geq 0} \subset \mathcal{L}(X_{-1})$ with generator $(A_{-1},D(A_{-1}))$ where $D(A_{-1}) = X$.

For the observation operator $C$ we define as in [30, Sect. 4] its Lebesgue extension $C_L : D(C_L) \subset X \to Y$ by

$$D(C_L) := \left\{ x \in X : \lim_{t \searrow 0} C \frac{1}{t} \int_0^t T(s)x \, ds \text{ exists} \right\},$$

$$C_Lx := \lim_{t \searrow 0} C \frac{1}{t} \int_0^t T(s)x \, ds \text{ for all } x \in D(C_L).$$

Now the following holds, see [30, Prop. 4.3].

**Proposition 1.1.** The space $D(C_L)$ endowed with the norm

$$\|x\|_L := \|x\|_X + \sup_{t \in (0,1]} \left\| C \frac{1}{t} \int_0^t T(s)x \, ds \right\|_X, \quad x \in D(C_L)$$

is a Banach space. Moreover, the embeddings $X_1 \overset{\mathcal{L}}{\hookrightarrow} D(C_L) \overset{\mathcal{L}}{\hookrightarrow} X$ are continuous and $C_L \in \mathcal{L}(D(C_L),Y)$.

To proceed we need the following stability and compatibility conditions. The latter relates the operators $A$, $B$ and $C$, cf. [18, Sect. II.A]. For more information and several equivalent conditions see [34, Thm.5.8].

**Assumption 1.2.** If not stated otherwise, in the sequel we always make the following hypotheses.

(i) The semigroup $(T(t))_{t \geq 0}$ is uniformly exponentially stable, i.e., there exist $K \geq 1$ and $\omega < 0$ such that

$$\|T(t)\| \leq Ke^{\omega t} \quad \text{for all } t \geq 0.$$  

(ii) The system $\Sigma(A,B,C,D)$ is compatible (or regular), i.e., for some $\lambda \in \rho(A)$ we have

$$\text{rg}(R(\lambda,A_{-1})B) \subset D(C_L).$$

While assumption (i) is made only for convenience and to simplify the presentation (cf. also Remark 5.6), assumption (ii) is essential and cannot be omitted. Note that if the inclusion (1.3) holds for some $\lambda \in \rho(A)$ then by the resolvent equation it holds for all $\lambda \in \rho(A)$. Moreover, the closed graph theorem and Proposition 1.1 then imply that

$$C_LR(\lambda,A_{-1})B \in \mathcal{L}(U,Y) \quad \text{for all } \lambda \in \rho(A).$$

We close this introduction with a brief outline of this work. In Section 2 we introduce the operator matrix $A$ from (1.1) on the space $X^0$ and compute its resolvent $R(\lambda,A)$. In Section 3 we show how the concept of admissibility for the observation operator $C$, the control operator $B$ and the pair $(B,C)$ is related to the existence of strongly continuous operator families having as Laplace transforms the entries of $R(\lambda,A)$. Section 4 is dedicated to the characterization of admissible pairs in terms of a resolvent condition which leads to so-called Fourier multipliers. In Section 5 we summarize our results from the previous section and give several characterizations of the generator property of $A$, i.e., of the well-posedness of $\Sigma(A,B,C,D)$. In the final Section 6 we illustrate our results and show the well-posedness of a controlled heat equation.
2. The Operator Matrix \( A \)

In this section we define the operator matrix \( A \) appearing in (1.1) which governs the control system \( \Sigma(A,B,C,D) \). To this end we first fix some \( 1 \leq p < \infty \). Then we introduce

- the space \( E_p^1 := L^p((-\infty,0],Y) \) of possible observations,
- the space \( E_p^2 := L^p([0,\infty),U) \) of possible controls, and
- the extended state space \( \mathcal{X}^p = E_1^p \times X \times E_2^p \).

On \( \mathcal{X}^p \) (equipped with an arbitrary product norm) we define the operator matrix

\[
A := \begin{pmatrix} \frac{d}{ds} & 0 & 0 \\ 0 & A_{-1} & B\delta_0 \\ 0 & 0 & \frac{d}{ds} \end{pmatrix},
\]

(2.1)

\[
D(A) := \{ \begin{pmatrix} \frac{d}{ds} y \\ u \end{pmatrix} \in \mathcal{E} : A_{-1}x + Bu(0) \in X, \ y(0) = C_Lx + Du(0) \},
\]

(2.2)

where \( \delta_0 : W^{1,p}([0,\infty),U) \subset E_2^p \to U \) denotes the point evaluation given by \( \delta_0 u := u(0) \) and

\[
\mathcal{E} := W^{1,p}((-\infty,0],Y) \times D(C_L) \times W^{1,p}([0,\infty),U).
\]

Note that there is a close relation between this operator matrix and the system \( \Sigma(A,B,C,D) \). In fact, on the second row of (2.1) we can recognize the first equation of the system \( \Sigma(A,B,C,D) \), while in the definition (2.2) of the domain of \( A \) the output equation of \( \Sigma(A,B,C,D) \) appears as a boundary condition. In Section 5 we will return to the relation between the matrix \( A \) and the system \( \Sigma(A,B,C,D) \).

As already mentioned in the introduction we define the well-posedness of \( \Sigma(A,B,C,D) \) in terms of the operator matrix \( A \).

**Definition 2.1.** The system \( \Sigma(A,B,C,D) \) is called \( p \)-well-posed if the operator matrix \( A \) in (2.1), (2.2) generates a \( C_0 \)-semigroup on \( \mathcal{X}^p \).

In order to characterize the generator property of \( A \) in terms of its entries, we follow ideas developed in [9] for \( 2 \times 2 \)-matrices. To do so we introduce some more notation.

First we consider the operators

- \( D_1 := \frac{d}{ds} : D(D_1) \subset E_1^p \to E_1^p \) with domain \( D(D_1) := W^{1,p}_0((-\infty,0],Y) := \{ y \in W^{1,p}_0((-\infty,0],Y) : y(0) = 0 \} \),
- \( D_2 := \frac{d}{ds} : D(D_2) \subset E_2^p \to E_2^p \) with domain \( D(D_2) := W^{1,p}([0,\infty),U) \).

Note that \( (D_1,D(D_1)) \) generates the left shift semigroup \( S_1 = (S_1(t))_{t \geq 0} \) on \( E_1^p \) given by

\[
(S_1(t)y)(s) := \begin{cases} y(t+s) & \text{if } t+s \leq 0, \\ 0 & \text{if } t+s > 0, \end{cases}
\]

while \( (D_2,D(D_2)) \) generates the left shift semigroup \( S_2 = (S_2(t))_{t \geq 0} \) on \( E_2^p \), see [11, Sect. II.2.b] for more details.

Next, for \( \lambda \in \mathbb{C} \) with \( \text{Re} \lambda > 0 \) we consider \( \varepsilon_{\lambda} \in L^p((-\infty,0],Y) \) defined by \( \varepsilon_{\lambda}(s) := e^{s\lambda} \). Then for an operator \( Q : D(Q) \subset V \to Y \) (where \( V = X \) or \( V = E_2^p \)) we define

\[
\varepsilon_{\lambda} \otimes Q : D(Q) \subset V \to L^p((-\infty,0],Y), \quad ((\varepsilon_{\lambda} \otimes Q)x)(s) := \varepsilon_{\lambda}(s) \cdot Qx = e^{s\lambda} \cdot Qx.
\]

We are now able to represent the matrix \( \lambda - A \) as follows.

**Proposition 2.2.** Let \( \text{Re} \lambda > 0 \). Then

\[
\lambda - A = \begin{pmatrix} \lambda - D_1 & 0 & 0 \\ 0 & \lambda - A & 0 \\ 0 & 0 & \lambda - D_2 \end{pmatrix} =: A_{\lambda} \cdot (I - \varepsilon_{\lambda} \otimes Q) \cdot \begin{pmatrix} I & -\varepsilon_{\lambda} \otimes C_L & -\varepsilon_{\lambda} \otimes D\delta_0 \\ 0 & I & -R(\lambda, A_{-1})B\delta_0 \\ 0 & 0 & I \end{pmatrix},
\]

(2.3)

where \( D(A_{\lambda}) = D(D_1) \times D(A) \times D(D_2) \) and \( D(\varepsilon_{\lambda}) = E_1^p \times D(C_L) \times W^{1,p}([0,\infty),U) \).
Proof. A simple computation shows that $D(A\lambda (Id - K)) := \{x \in D(K) : (Id - K)x \in D(A\lambda)\}$ coincides with $D(A)$ and that $(\lambda - A)x = A\lambda (Id - K)x$ for all $x \in D(A)$. □

Using the above representation of $\lambda - A$ it is easy to find an explicit representation for the resolvent $R(\lambda, A)$ of $A$. To this end we denote by $\mathcal{L}$ the Laplace transform, i.e., for $\text{Re} \lambda > 0$ and $u \in E^p_2 = L^p([0, \infty), U)$ we define

$$(\mathcal{L}u)(\lambda) := \mathcal{L}_\lambda u := \hat{u}(\lambda) := \int_0^{+\infty} e^{-\lambda r}u(r) \, dr.$$ 

Corollary 2.3. For $\lambda \in \mathbb{C}$ with $\text{Re} \lambda > 0$ we have $\lambda \in \rho(A)$ and

$$(2.4) \quad R(\lambda, A) = \begin{pmatrix} (R(\lambda, A_1) & \varepsilon_\lambda \otimes CR(\lambda, A) & \varepsilon_\lambda \otimes CL \lambda \otimes D \lambda + \varepsilon_\lambda \otimes D \lambda \\ 0 & R(\lambda, A) & 0 \\ 0 & 0 & R(\lambda, A_2) \end{pmatrix}.$$ 

Proof. Note that $\text{Re} \lambda > 0$ implies $\lambda \in \rho(D_1) \cap \rho(A) \cap \rho(D_2)$. Using (2.3) we then obtain

$$(R(\lambda, A) = \begin{pmatrix} Id & \varepsilon_\lambda \otimes CL & \varepsilon_\lambda \otimes CL \lambda \otimes D \lambda \\ 0 & Id & R(\lambda, A_1)B \delta_0 + \varepsilon_\lambda \otimes D \lambda \\ 0 & 0 & Id \end{pmatrix} \cdot \begin{pmatrix} R(\lambda, A_1) & 0 & 0 \\ 0 & R(\lambda, A) & 0 \\ 0 & 0 & R(\lambda, A_2) \end{pmatrix}.$$ 

Since

$$\delta_0 R(\lambda, A_2)u = \delta_0 \left( e^{\lambda t} \int_0^{+\infty} e^{-\lambda r}u(r) \, dr \right) = \mathcal{L}_\lambda u$$

for all $u \in L^p([0, \infty), U)$ this implies (2.4). □

3. Characterization of Admissibility in the Time Domain

In this section we study the possible entries of a semigroup generated by the operator matrix $A$. As we will see this leads to the concept of admissibility for the observation operator $C$, the control operator $B$ and the pair $(B, C)$. Our approach is based on the Laplace transform which relates a semigroup to the resolvent of its generator. More precisely, we use the following result, see [2, Thm. 3.1.7].

Lemma 3.1. Let $(S(t))_{t \geq 0} \subset \mathcal{L}(X)$ be an exponentially bounded and strongly continuous operator family on a Banach space $X$. Then the following assertions are equivalent.

(a) There exists an operator $\mathcal{D} : D(\mathcal{D}) \subset X \to X$ and some $\lambda_0 \in \mathbb{R}$ such that $(\lambda_0, +\infty) \subset \rho(\mathcal{D})$ and

$$\mathcal{L}(S(t)) = R(\lambda, \mathcal{D})x \quad \text{for all } \lambda > \lambda_0 \text{ and all } x \in X.$$ 

(b) $(S(t))_{t \geq 0}$ is a $C_0$-semigroup.

Moreover, in this case $\mathcal{D}$ coincides with the generator of $(S(t))_{t \geq 0}$.

Recall that in Corollary 2.3 we already computed the resolvent of $A$. The idea is now to define (at least on dense subspaces) operator families $(T_{jk}(t))_{t \geq 0}$ for $j, k = 1, 2, 3$ such that their Laplace transforms coincide with $^2 [R(\lambda, A)]_{jk}$ (on these subspaces). Hence, if $A$ is the generator of a $C_0$-semigroup $(\mathcal{T}(t))_{t \geq 0}$ these operator families $(T_{jk}(t))_{t \geq 0}$ must have (by denseness unique) bounded, strongly continuous extensions. Indeed, by the uniqueness theorem for the Laplace transform (see [2, Thm. 1.7.3]), they are the only possible entries of $\mathcal{T}(t)$. On the other hand, if these operator families $(T_{jk}(t))_{t \geq 0}$ have bounded, strongly continuous extensions, then their Laplace transforms give the entries of the resolvent of $R(\lambda, A)$, hence by Lemma 3.1 the matrix $A$ is a generator.

\footnote{Here $[M]_{jk}$ indicates the $jk$-th entry $m_{jk}$ of the matrix $M = (m_{jk})_{3 \times 3}$.}
This idea works without problems for all entries of $R(\lambda,A)$ and $(\mathcal{T}(t))_{t \geq 0}$ below and on the diagonal. More precisely if $A$ is a generator then the generated semigroup has necessarily the form

$$\mathcal{T}(t) = \begin{pmatrix} S_1(t) & * & * \\ 0 & T(t) & * \\ 0 & 0 & S_2(t) \end{pmatrix}.$$ 

Therefore, we only have to consider the remaining three entries. This will be done in the following subsections.

3.1. The Entry $T_{12}(t)$ and Admissible Observation Operators. For $t \geq 0$ we define the operators

$$T_{12}(t) : D(A) \subset X \to E^p_1,$$

$$(T_{12}(t)x)(s) := \mathbb{1}_{[-t,0]}(s)CT(t+s)x, \quad s \in \mathbb{R}_-.$$

We first verify some basic properties of this operator family.

**Lemma 3.2.** For every $x \in D(A)$ the function $T_{12}(\cdot)x : \mathbb{R}_+ \to E^p_1$ is well-defined, continuous and bounded.

**Proof.** Since $x \in D(A)$ we can write

$$(T_{12}(t)x)(s) = CA^{-1}\cdot \mathbb{1}_{[-t,0]}(s)T(t+s)Ax,$$

where $CA^{-1} \in \mathcal{L}(X,Y)$. Hence to prove the claims it suffices to consider the simpler function $g : \mathbb{R}_+ \to L^p((\infty,0],X)$ defined by $g(t) := \mathbb{1}_{[-t,0]}(\cdot)T(t+\cdot)z$ where $z := Ax \in X$. By assumption $(T(t))_{t \geq 0}$ is exponentially stable, thus we get

$$\|g(t)\|_{L^p((\infty,0],X)} = \int_{-t}^0 \|T(t+s)z\|^p_X ds$$

$$\leq \frac{K}{a^p}(e^{at} - 1) \cdot \|z\|^p_X$$

$$\leq \frac{K}{a^p} \cdot \|z\|^p_X \quad \text{for all } t \geq 0.$$ 

This proves that $g$ is well-defined and bounded. To show its continuity let $0 \leq r < t$. Then

$$\|g(t) - g(r)\|_{L^p((\infty,0],X)} = \|\mathbb{1}_{[-t,0]}(\cdot)T(t+\cdot)z - \mathbb{1}_{[-r,0]}(\cdot)T(r+\cdot)z\|_{L^p((\infty,0],X)}$$

$$\leq \left( \int_{-t}^{-r} \|T(t+s)z\|^p_X ds \right)^{\frac{1}{p}} + \left( \int_{-r}^0 \|T(t+s) - T(r+s)\|_{L^p} ds \right)^{\frac{1}{p}}$$

$$\leq (t - r)^{\frac{1}{p}} K \cdot \|z\|_X + r^{\frac{1}{p}} K \|T(t-r) - Id\|_{L^p} \|z\|_X \to 0 \quad \text{as } t - r \to 0,$$

where again we used Assumption 1.2.(i). $\square$

By the previous result we can Laplace transform $T_{12}(\cdot)$.

**Lemma 3.3.** For every $x \in D(A)$ and $\lambda \in \mathbb{C}$ with $\text{Re} \lambda > 0$ we have

$$\mathcal{L}(T_{12}(\cdot)x)(\lambda) = \varepsilon_{\lambda} \otimes CR(\lambda,A)x = [R(\lambda,A)]_{12}x.$$

**Proof.** For $x \in D(A)$, $\lambda \in \mathbb{C}$ with $\text{Re} \lambda > 0$ and $s \in (-\infty,0]$ we obtain

$$\left( \mathcal{L}(T_{12}(\cdot)x)(\lambda) \right)(s) = CA^{-1} \int_{-\infty}^s e^{-\lambda t} \mathbb{1}_{[-t,0]}(s)T(t+s)Ax dt$$

$$= CA^{-1} \int_{-s}^\infty e^{-\lambda t}T(t+s)Ax dt$$

$$= e^{\lambda s}CA^{-1} \int_{0}^\infty e^{-\lambda t}T(t)Ax dt$$

$$= (\varepsilon_{\lambda} \otimes CR(\lambda,A)x)(s).$$

$\square$

We proceed by introducing the following well-known notion from control theory (see, e.g., [30]) which is closely related to the entry $T_{12}(t)$.
Definition 3.4. The observation operator $C \in \mathcal{L}(X_1, Y)$ is called $p$-admissible (with respect to $A$) if there exists $t_0 > 0$ and a constant $M \geq 0$ such that
\[
\int_0^{t_0} \|CT(s)x\|_Y^p \, ds \leq M\|x\|_X^p \quad \text{for all } x \in D(A).
\]

Remark 3.5. Since for $t \geq 0$ and $x \in D(A)$ we have
\[
\|T_{12}(t)x\|_{E^p_t}^p = \int_0^t \|CT(t+s)x\|_Y^p \, ds = \int_0^t \|CT(s)x\|_Y^p \, ds
\]
the observation operator $C \in \mathcal{L}(X_1, Y)$ is $p$-admissible if and only if $T_{12}(t) : D(A) \subset X \to E^p_t$ has a bounded extension in $\mathcal{L}(X, E^p_t)$ for some $t > 0$. Moreover we note that for $C \in \mathcal{L}(X_1, Y)$ the condition to be a $p$-admissible observation operator gets stronger with growing $p \geq 1$.

Next we give different characterizations of admissibility for observation operators where we have to distinguish the cases $p > 1$ and $p = 1$.

Lemma 3.6. Let $p > 1$. Then the operator $C$ is $p$-admissible if and only if for every $x \in X$ we have $T(\cdot)x \in L^p([0, t_0], D(C_L))$.

Proof. We first introduce the following operators and spaces referring to the setting of Lemma A.1
\[
\tilde{Q} : X \to L^p([0, t_0], X), \quad \tilde{Q}x := T(\cdot)x,
\]
\[
Q : D(A) \subset X \to L^p([0, t_0], D(C_L)), \quad Qx := T(\cdot)x,
\]
\[
D := D(A), \quad V := X, \quad W := L^p([0, t_0], D(C_L)) \quad \text{and} \quad Z := L^p([0, t_0], X).
\]
Here by Proposition 1.1
\[
D(C_L) \text{ is a Banach space and for } x \in D(A) \text{ we have } Qx \in C([0, t_0], X_1) \subset L^p([0, t_0], D(C_L)).
\]

We now show that if $C$ is $p$-admissible, then there exists a constant $\bar{M} \geq 0$ such that
\[
\left(\int_0^{t_0} \|T(s)x\|_X^p \, ds\right)^\frac{1}{p} \leq \bar{M}\|x\|_X \quad \text{for all } x \in D(A).
\]
To do so we recall that for a function $f \in L^1_{loc}(\mathbb{R})$ its Maximal Function (cf. [26, Sect.I.1]) is defined by
\[
(Mf)(s) := \sup_{t > 0} \frac{1}{2t} \int_{s-t}^{s+t} |f(r)| \, dr.
\]
Then the Hardy–Littlewood Maximal Theorem (see [26, Thm.I.1.1]) asserts that $Mf \in L^p(\mathbb{R})$ for $f \in L^p(\mathbb{R})$ with $1 < p \leq \infty$ and that there exists a constant $C_p$ depending only on $p$ such that
\[
\|Mf\|_{L^p(\mathbb{R})} \leq C_p \|f\|_{L^p(\mathbb{R})}.
\]
Using this for
\[
f(r) := \begin{cases} \|CT(r)x\|_Y & \text{if } 0 \leq r \leq t_0, \\ 0 & \text{else}, \end{cases}
\]
where $x \in D(A)$, we obtain
\[
\left(\int_0^{t_0} \|T(s)x\|_X^p \, ds\right)^\frac{1}{p} \leq \left(\int_0^{t_0} \|T(s)x\|_X^p \, ds\right)^\frac{1}{p} + \left(\int_0^{t_0} \sup_{t \in [0,1]} \left\|\frac{1}{t} \int_0^{t} \|CT(r)x\|_Y^p \, dr\right\|_Y^p \, ds\right)^\frac{1}{p}
\]
\[
\leq K \left(\int_0^{t_0} \|x\|_X + \left(\int_0^{t_0} \sup_{t \in [0,1]} \frac{1}{t} \int_0^{t} \|CT(r)x\|_Y^p \, dr\right)^\frac{1}{p} \, ds\right)^\frac{1}{p}
\]
\[
\leq K \|x\|_X + 2\|Mf\|_{L^p(\mathbb{R})}
\]
\[
\leq K \|x\|_X + 2C_p M \|x\|_X.
\]
Here we used that the semigroup $(T(t))_{t \geq 0}$ is bounded by a constant $K \geq 1$, the Hardy-Littlewood Maximal Theorem and the fact that the observation operator $C$ is $p$-admissible. This shows (3.1) for $\bar{M} := K \|x\|_X + 2C_p M \|x\|_X$. 


It thus follows that if $C$ is $p$-admissible, then condition (a) of Lemma A.1 is satisfied and we conclude that $\tilde{Q} \in \mathcal{L}(X, L^p([0,t_0], D(C_L)))$, i.e., $T(\bullet)x \in L^p([0,t_0], D(C_L))$ for every $x \in X$. In particular, for every $x \in X$ this implies $T(r)x \in D(C_L)$ for almost all $r \in [0,t_0]$.

Conversely, if $T(\bullet)x \in L^p([0,t_0], D(C_L))$ for every $x \in X$ then $\text{rg}(\tilde{Q}) \in L^p([0,t_0], D(C_L))$. Thus condition (b) of Lemma A.1 is satisfied and we conclude $\tilde{Q} \in \mathcal{L}(X, L^p([0,t_0], D(C_L)))$. From Proposition 1.1 it then follows $C_L\tilde{Q} \in \mathcal{L}(X, L^p([0,t_0], Y))$ and therefore $C$ is $p$-admissible. \hfill \square

**Remark 3.7.** If $C$ is $p$-admissible for some $p > 1$ then the previous result together with the semigroup property imply that for all $x \in X$ we have $\text{rg}(T(t)x) \subset D(C_L)$ for almost all $t \geq 0$.

As we will see next the range condition in the previous remark holds also in the case $p = 1$ (see also [30, Theorem 4.5]).

**Lemma 3.8.** If $C$ is $1$-admissible and $x \in X$, then $T(t)x \in D(C_L)$ for almost all $t \geq 0$.

**Proof.** If $C$ is $1$-admissible, then the map $Q : D(A) \to L^1([0,t_0], Y)$ given by $Qx = CT(\bullet)x$ has a bounded continuous extension $\tilde{Q}$ on all of $X$. Furthermore for all $x \in D(A)$, $t \geq 0$ and $r > 0$ we have

$$
\frac{1}{r} \int_t^{t+r} (Qx)(s) \, ds = C\frac{1}{r} \int_0^r T(t+s) \, x \, ds.
$$

Since both sides depend continuously on $x$, the equality holds for every $x \in X$. Letting $r \to 0$, it follows that $T(t)x \in D(C_L)$ if and only if $\tilde{Q}x$ has a Lebesgue point in $t$. Hence by the Lebesgue differentiation theorem (see [6, Thm.II.2.9]) it follows that $T(t)x \in D(C_L)$ for almost all $t \geq 0$. \hfill \square

Finally we prove the following result which is closely related to [30, Prop. 2.3]. Here we need again Assumption 1.2.(i).

**Lemma 3.9.** If the observation operator $C$ is $p$-admissible, then there exists $M_C \geq 0$ such that

$$
\int_0^t \|CT(s)x\|^p_Y \, ds \leq M_C \|x\|^p_X \quad \text{for all } x \in D(A), \; t \geq 0.
$$

**Proof.** If $C$ is $p$-admissible, there exists $t_0 > 0$ and $M > 0$ such that

$$
\int_0^{t_0} \|CT(s)x\|^p_Y \, ds \leq M \|x\|^p_X \quad \text{for all } x \in D(A). \text{ For } t \leq t_0 \text{ it is clear that}
$$

$$
\int_0^t \|CT(s)x\|^p_Y \, ds \leq \int_0^{t_0} \|CT(s)x\|^p_Y \, ds \leq M \|x\|^p_X \quad \text{for all } x \in D(A). \text{ For } t > t_0 \text{ we can write } t = nt_0 + r \text{ where } n \in \mathbb{N} \text{ and } 0 \leq r < t_0. \text{ Using (1.2) we then obtain}
$$

$$
\int_0^t \|CT(s)x\|^p_Y \, ds \leq \sum_{k=0}^n \int_{kt_0}^{(k+1)t_0} \|CT(s)x\|^p_Y \, ds
$$

$$
= \sum_{k=0}^n \int_0^{t_0} \|CT(s)T(kt_0)x\|^p_Y \, ds
$$

$$
\leq M \sum_{k=0}^n \|T(kt_0)x\|^p_X
$$

$$
\leq MK^p \frac{1}{1-e^{-p\omega t_0}} \|x\|^p_X \quad \text{for all } x \in D(A).
$$

Choosing $M_C := M + MK^p \frac{1}{1-e^{-p\omega t_0}}$, we obtain (3.2). This concludes the proof. \hfill \square

By combining the previous results we obtain the main outcome of this subsection.
Corollary 3.10. If $A$ is a generator, then $C$ is a $p$-admissible control operator. Conversely, if $C$ is a $p$-admissible control operator, then for every $t \geq 0$ the operator $T_{12}(t) : D(A) \subset X \to E^p_t$ has a (unique) bounded extension $\mathcal{C}(t) := T_{12}(t) \in \mathcal{L}(X, E^p_t)$. Moreover, $(\mathcal{C}(t))_{t \geq 0}$ is strongly continuous and

\begin{equation}
(3.3) \quad \mathcal{C}(t)x = \mathbb{1}_{[-1,0]}(\cdot)C_LT(t + \cdot)x \quad \text{for every } x \in X.
\end{equation}

Proof. If $A$ is the generator of a $C_0$-semigroup $(\mathcal{T}(t))_{t \geq 0}$, then by Lemma 3.3 and the uniqueness of the Laplace transform (see [2, Prop. 1.7.3]) we obtain that $[\mathcal{T}(t)]_{12} = T_{12}(t)x$ for all $t \geq 0$ and $x \in D(A)$. Since $[\mathcal{T}(t)]_{12} \in \mathcal{L}(X, E^p_t)$, Remark 3.5 then implies that $C$ is $p$-admissible. Conversely assume that $C$ is $p$-admissible. Then by Remark 3.5 and Lemma 3.9 each operator $T_{12}(t) : D(A) \subset X \to E^p_t$ has a (unique) extension $\mathcal{C}(t) \in \mathcal{L}(X, E^p_t)$. Since by Lemma 3.2 the map $t \mapsto \mathcal{C}(t)x$ is continuous for every $x \in D(A)$, by a standard density argument (cf. [11, Lem. I.5.2]), $(\mathcal{C}(t))_{t \geq 0}$ is strongly continuous. Finally, using Lemma 3.8 we obtain (3.3). \hfill $\square$

3.2. The Entry $T_{23}(t)$ and Admissible Control Operators. We proceed using the same scheme as in the previous subsection and define for $t \geq 0$ the operators

\begin{align*}
T_{23}(t) : E^p_2 \to X_{-1}, \\
T_{23}(t)u := \int_0^t T_{-1}(t - r)Bu(r) \, dr \quad \text{for } u \in E^p_2.
\end{align*}

Again we first verify some basic properties of this operator family.

Lemma 3.11. For every $u \in E^p_2$ the function $T_{23}(\cdot)u : \mathbb{R}_+ \to X_{-1}$ is continuous and bounded.

Proof. This follows from [2, Prop. 1.3.5.(b)] on the continuity and boundedness of convolutions. \hfill $\square$

By the previous result we can consider the Laplace transform of $T_{23}(\cdot)u$ in $X_{-1}$.

Lemma 3.12. For every $\lambda \in \mathbb{C}$ with $\text{Re} \lambda > 0$ and every $u \in E^p_2$ we have

\begin{equation}
\mathcal{L}(T_{23}(\cdot)u)(\lambda) = R(\lambda, A_{-1})B\hat{u}(\lambda) = [R(\lambda, A)]_{23}u.
\end{equation}

Proof. For $u \in E^p_2$ we obtain by Fubini’s theorem (see [2, Thm. 1.1.9])

\begin{align*}
\mathcal{L}(T_{23}(\cdot)u)(\lambda) & = \int_0^\infty \int_0^t e^{-\lambda T_{-1}(t - r)}Bu(r) \, dr \, dt \\
& = \int_0^\infty \int_r^\infty e^{-\lambda T_{-1}(t - r)}Bu(r) \, dt \, dr \\
& = \int_0^\infty \int_0^\infty e^{-\lambda(t+r)}T_{-1}(t)Bu(r) \, dt \, dr \\
& = R(\lambda, A_{-1})B\hat{u}(\lambda). \hfill \square
\end{align*}

Next we recall the following well-known notion from control theory (see, e.g., [29]) which is closely related to the entry $T_{23}(t)$.

Definition 3.13. The control operator $B \in \mathcal{L}(U, X_{-1})$ is called $p$-admissible (with respect to $A$) if there exists $t_0 > 0$ such that $\text{rg}(T_{23}(t_0)) \subset X$.

Remark 3.14. Note that in any case $T_{23}(t_0) \in \mathcal{L}(E^p_2, X_{-1})$. Thus if $B$ is $p$-admissible the closed graph theorem implies that $T_{23}(t_0) \in \mathcal{L}(E^p_2, X)$. On the other hand, if $u \in W^{1,p}(\mathbb{R}_+, U)$ then using integration by parts we obtain

\begin{equation}
\int_0^{t_0} T_{-1}(t_0 - r)Bu(r) \, dr = A_{-1}^{-1} \left( T(t_0)Bu(0) - Bu(t_0) + \int_0^{t_0} T_{-1}(t_0 - r)Bu'(r) \, dr \right) \in X.
\end{equation}
Since \( W^{1,p}([0,\infty), U) \) is dense in \( L^p([0,\infty), U) \) this shows that the operator \( B \) is \( p \)-admissible if and only if there exists \( t_0 > 0 \) and a constant \( M \geq 0 \) such that
\[
\left\| \int_0^{t_0} T_{-1}(t_0 - r)Bu(r) \, dr \right\|_X \leq M\|u\|_{L^p([0,\infty), U)} \quad \text{for all} \ u \in W^{1,p}([0,\infty), U).
\]
Moreover we note that for an operator \( B \in \mathcal{L}(U, X_{-1}) \) the condition to be a \( p \)-admissible control operator gets weaker with growing \( p \geq 1 \).

Analogously to Lemma 3.9 we have the following result which is closely related to [29, Prop. 2.5]. Here we need again Assumption 1.2.(i).

**Lemma 3.15.** If the control operator \( B \) is \( p \)-admissible, then there exists \( M_B \geq 0 \) such that
\[
(3.4) \quad \left\| \int_0^t T_{-1}(t - r)Bu(r) \, dr \right\|_X \leq M_B\|u\|_{L^p([0,\infty), U)} \quad \text{for all} \ u \in L^p([0,\infty), U), \ t \geq 0.
\]

**Proof.** By assumption there exists \( t_0 > 0 \) and \( M > 0 \) such that
\[
\left\| \int_0^{t_0} T_{-1}(t_0 - r)Bu(r) \, dr \right\|_X \leq M\|u\|_{L^p([0,\infty), U)} \quad \text{for all} \ u \in L^p([0,\infty), U).
\]

For \( 0 \leq t \leq t_0 \) we denote by \( u_{t_0-t} \) the translated function
\[
(3.5) \quad u_{t_0-t}(s) := \begin{cases} 0 & \text{if } 0 \leq s < t_0 - t, \\ u(s - t_0 + t) & \text{if } s \geq t_0 - t.
\end{cases}
\]

Then \( u \in L^p([0,\infty), U) \) and \( \|u\|_{L^p([0,\infty), U)} = \|u_{t_0-t}\|_{L^p([0,\infty), U)}. \) Moreover
\[
\int_0^t T_{-1}(t - r)Bu(r) \, dr = \int_0^{t_0} T_{-1}(t_0 - r)Bu_{t_0-t}(r) \, dr \in X.
\]

This implies
\[
(3.6) \quad \left\| \int_0^t T_{-1}(t - r)Bu(r) \, dr \right\|_X = \left\| \int_0^{t_0} T_{-1}(t_0 - r)Bu_{t_0-t}(r) \, dr \right\|_X \leq M\|u\|_{L^p([0,\infty), U)} \quad \text{for all} \ u \in L^p([0,\infty), U).
\]

For \( t \geq t_0 \) we write \( t = nt_0 + s \) for \( n \in \mathbb{N} \) and \( s \in [0, t_0) \). Then we obtain
\[
\int_0^t T_{-1}(t - r)Bu(r) \, dr = \int_0^s T_{-1}(nt_0 + s - r)Bu(r) \, dr + \int_s^{nt_0+s} T_{-1}(nt_0 + s - r)Bu(r) \, dr =: L_1 + L_2.
\]

We consider the sum terms of the sum separately. For the first one we get \( L_1 \in X \) and
\[
(3.7) \quad \|L_1\|_X \leq \left\|T(nt_0)\right\| \cdot \left\| \int_0^s T_{-1}(s - r)Bu(r) \, dr \right\|_X \leq KM\|u\|_{L^p([0,\infty), U)}.
\]

Here we used again that \( \{T(t)\}_{t \geq 0} \) is bounded and (3.6). For the second term we obtain
\[
L_2 = \sum_{k=0}^{n-1} \int_{kt_0}^{(k+1)t_0} T_{-1}(nt_0 - r)Bu(r + s) \, dr
\]
\[
= \sum_{k=0}^{n-1} T((n - (k + 1))t_0) \cdot \int_0^{t_0} T_{-1}(t_0 - r)Bu(r + s + kt_0) \, dr \in X.
\]

Moreover, using (1.2) and that \( B \) is a \( p \)-admissible control operator this gives the estimates
\[
(3.8) \quad \|L_2\|_X \leq K \sum_{k=0}^{n-1} e^{\omega(n-k-1)t_0} \cdot M\|u\|_{L^p([0,\infty), U)} \leq \frac{KM}{1 - e^{-\omega t_0}}\|u\|_{L^p([0,\infty), U)}.
\]

Summing up (3.7) and (3.8) we obtain (3.4) for \( M_B := MK + \frac{MK}{1-e^{-\omega t_0}}. \)

By combing the previous results we obtain the main statement of this subsection which corresponds to Corollary 3.10.
Corollary 3.16. If $A$ is a generator, then $B$ is a $p$-admissible control operator. Conversely, if $B$ is a $p$-admissible control operator then for every $t \geq 0$ we have $\text{rg}(T_{23}(t)) \subset X$ and $\mathcal{B}(t) := T_{23}(t) \in \mathcal{L}(E^p_2, X)$. Moreover, the family $\{\mathcal{B}(t)\}_{t \geq 0}$ is strongly continuous and uniformly bounded.

Proof. If $A$ is the generator of a $C_0$-semigroup $(\mathcal{T}(t))_{t \geq 0}$, then by Lemma 3.12 and the uniqueness of the Laplace transform we obtain that $T_{23}(t)u = [\mathcal{T}(t)]_2^t u \in X$ for all $t \geq 0$ and $u \in E^p_2$. This implies $\text{rg}(T_{23}(t)) \subset X$, thus $B$ is $p$-admissible and $T_{23} \in \mathcal{L}(E^p_2, X)$.

Conversely, if $B$ is a $p$-admissible control operator, then using Remark 3.14 and Lemma 3.15 we conclude that $\text{rg}(T_{23}(t)) \subset X$, hence by the closed graph theorem $\mathcal{B}(t) \in \mathcal{L}(E^p_2, X)$ for every $t \geq 0$. To show that $\{\mathcal{B}(t)\}_{t \geq 0}$ is strongly continuous let $0 \leq r \leq t$ and $u \in L^p([0, \infty), U)$. Then

$$\|B(t)u - B(r)u\|_X = \|B(t)(u - ut-r)\|_X \leq \|B(t)\| \cdot \|u - ut-r\|_{L^p([0,\infty), U)} \leq MB \|u - ut-r\|_{L^p([0,\infty), U)},$$

where $ut-r$ is defined as in (3.5). Since the shift on $L^p([0, \infty), U)$ is strongly continuous, we have

$$\lim_{|t-r| \to 0} \|u - ut-r\|_{L^p([0,\infty), U)} = 0$$

and the assertion follows. □

3.3. The Entry $T_{13}(t)$ and Admissible Pairs of Operators. We proceed as in the previous two subsections and start by defining for $t \geq 0$ the operators\footnote{We use the notation $T_{13}(t) = T_{13}^D(t)$ to indicate the dependence of this entry on $D \in \mathcal{L}(U, Y)$.}

$$T_{13}^D(t) : W^{2,p}_0([0, +\infty), U) \subset E^p_2 \to E^p_1,$$

$$(T_{13}^D(t)u)(s) := \mathbb{1}_{[-t, 0]}(s) \left( C_{\|\cdot\|_{Y}} \int_0^{t+s} T_u(t + s - r) Bu(r) \, dr + Du(t + s) \right), \quad s \in \mathbb{R},$$

where

$$W^{2,p}_0([0, +\infty), U) := \left\{ u \in W^{2,p}([0, \infty), U) : u(0) = u'(0) = 0 \right\}.$$

As before we first verify some basic properties of this operator family.

Lemma 3.17. For every $u \in W^{2,p}_0([0, +\infty), U)$ the function $T_{13}^D(\cdot)u : \mathbb{R}_+ \to E^p_1$ is well-defined, continuous and bounded.

Proof. We first consider the term involving $D \in \mathcal{L}(U, Y)$. For this it suffices to look at the function $g : \mathbb{R}_+ \to E^p_1$ defined by $g(t) := \mathbb{1}_{[-t, 0]}(\cdot)v(t + \cdot)$ where $v := Du \in W^{2,p}_0([0, +\infty), Y)$. Since

$$\|g(t)\|_{E^p_1} = \int_{-t}^0 \|v(t + s)\|_Y^p \, ds \leq \|v\|_{L^p([0,\infty), Y)}$$

for all $t \geq 0$, $g$ is well-defined and bounded. To show its continuity take $0 \leq r \leq t$. Then

$$\|g(t) - g(r)\|_{E^p_1} = \|\mathbb{1}_{[-t, 0]}(\cdot)v(t + \cdot) - \mathbb{1}_{[-r, 0]}(\cdot)v(r + \cdot)\|_{L^p([0,\infty), Y)} \leq \left( \int_{-t}^{-r} \|v(t + s)\|_Y^p \, ds \right)^\frac{1}{p} + \left( \int_{-r}^0 \|v(t + s) - v(r + s)\|_Y^p \, ds \right)^\frac{1}{p}$$

$$= \left( \int_{0}^{t-r} \|v(s)\|_Y^p \, ds \right)^\frac{1}{p} + \left( \int_{0}^{r} \|v(t - r + s) - v(s)\|_Y^p \, ds \right)^\frac{1}{p} \to 0 \quad \text{as} \quad t - r \to 0,$$

where the convergence of the first term follows from the dominated convergence theorem while the second term converges due to the strong continuity of the left-shift semigroup on $L^p([0, \infty), Y)$. 

Next we consider $T_{13}^0(\cdot)u$. Note that for $u \in W_{0}^{2,p}([0, +\infty), U)$ using twice integration by parts and the compatibility condition (1.4) it follows that for $t + s \geq 0$

$$\int_0^{t+s} T_{-1}(t + s - r)Bu(r) \, dr = -A_{-1}^{-1}\left(Bu(t + s) - \int_0^{t+s} T_{-1}(t + s - r)Bu'(r) \, dr\right)$$

(3.9)

Hence for all $s \in \mathbb{R}_-$ and $t \in \mathbb{R}_+$ the term $(T_{13}^0(t)u)(s)$ is well-defined. Moreover, by the same argument as before for the function $g$ it follows that the functions

$$g_1 : \mathbb{R}_+ \to E^p_1,$$

$$g_1(t) := \mathbb{I}_{[-t,0]}(\cdot)C_LA_{-1}^{-1}Bu(t + \cdot),$$

$$g_2 : \mathbb{R}_+ \to E^p_1,$$

$$g_2(t) := \mathbb{I}_{[-t,0]}(\cdot)C_LA_{-1}^{-2}Bu'(t + \cdot)$$

are bounded and continuous. Since $C_LA_{-1}^{-1}$ is bounded, to finish the proof it suffices to prove that for $v := A_{-1}^{-2}Bu' \in L^p([0, +\infty), X)$ the function

$$g_3 : \mathbb{R}_+ \to L^p((\infty, 0], X),$$

$$g_3(t) := \mathbb{I}_{[-t,0]}(\cdot)\int_0^{t+s} T(t + \cdot - r)v(r) \, dr$$

is well-defined, continuous and bounded. Applying Young’s inequality (see [2, Prop.1.3.5.(a)]) to the convolution $T * v$ we get

$$\|g_3(t)\|_{L^p((\infty, 0], X)} = \left(\int_0^t \left\|\int_0^{t+s} T(t + s - r)v(r) \, dr\right\|_{X}^p \, ds\right)^{\frac{1}{p}}$$

$$\leq \|T \ast v\|_{L^p([0, +\infty), X)} \|v\|_{L^p([0, +\infty), X)} < +\infty,$$

where in the last step we used (1.2). This proves that $g_3$ is well-defined and bounded. To show its continuity take $0 \leq t_0 \leq t_1$. Then for $h := t_1 - t_0$ we obtain

$$\|g_3(t_1) - g_3(t_0)\|_{L^p((\infty, 0], X)} = \int_{-t_1}^{-t_0} \left\|\int_0^{t_1+s} T(t_1 + s - r)v(r) \, dr\right\|_{X}^p \, ds$$

$$+ \int_{-t_0}^{0} \left\|\int_0^{t_1+s} T(t_1 + s - r)v(r) \, dr - \int_0^{t_0+s} T(t_0 + s - r)v(r) \, dr\right\|_{X}^p \, ds$$

$$= \int_0^{h} \left\|\int_0^{\infty} T(s - r)v(r) \, dr\right\|_{X}^p \, ds$$

$$+ \int_0^{t_0} \left\|\int_0^{t_1+s} T(s + h - r)v(r) \, dr - \int_0^{t_0+s} T(s - r)v(r) \, dr\right\|_{X}^p \, ds$$

$$= L^p_1 + L^p_2.$$

Moreover, by [2, Prop. 1.3.4] the convolution $T * v : \mathbb{R}_+ \to X$ is continuous, hence it is uniformly continuous on the compact interval $[0, t_0]$. Thus

$$L^p_2 = \int_0^{t_0} \left\|\int_0^{t_1+s} (T * v)(s + h) - (T * v)(s)\right\|_{X}^p \, ds \to 0 \quad \text{as } h = t_1 - t_0 \to 0.$$ 

Furthermore, using the dominated convergence theorem

$$L^p_1 \to 0 \quad \text{as } h = t_1 - t_0 \to 0.$$ 

(3.12)

Summing up (3.11) and (3.12) we complete the proof.

By the previous result we can Laplace transform $T_{23}(\cdot)u$ for $u \in W_{0}^{2,p}([0, +\infty), U)$. To do so we need the following simple result.
Lemma 3.18. Let \( v \in L^p([0, +\infty), X) \). Then the convolution \( f := T * v \) is bounded and continuous. Hence for \( \Re \lambda > 0 \) its Laplace transform exists and is given by

\[
\hat{f}(\lambda) = R(\lambda, A) \hat{v}(\lambda).
\]

If, in addition, \( v \in L^1([0, +\infty), X) \) then the same holds for \( \Re \lambda \geq 0 \).

Proof. Boundedness of \( f \) follows as in (3.10) while its continuity is shown in [2, Prop. 1.3.4]. Now take \( \Re \lambda > 0 \). Using Assumption 1.2, the integral

\[
\int_0^{+\infty} \int_0^{+\infty} e^{-\Re \lambda(t+r)} \|T(t)v(r)\| \, dt \, dr \leq K \int_0^{+\infty} \int_0^{+\infty} e^{-\Re \lambda(t+r)} e^{\omega t} \|v(r)\| \, dt \, dr < +\infty
\]

is finite. Hence we can use Fubini’s theorem (see [2, Thm. 1.1.9]) to conclude that

\[
\hat{f}(\lambda) = \int_0^{+\infty} e^{-\lambda t} \int_0^t T(t-r)v(r) \, dr \, dt
\]

\[
= \int_0^{+\infty} \int_r^{+\infty} e^{-\lambda t} T(t-r)v(r) \, dt \, dr
\]

\[
= \int_0^{+\infty} \int_0^{+\infty} e^{-\lambda(t+r)} T(t)v(r) \, dt \, dr
\]

\[
= R(\lambda, A) \hat{v}(\lambda).
\]

Now assume that \( v \in L^1([0, +\infty), X) \). Then by Young’s inequality [2, Prop. 1.3.5.(a)] we obtain \( f \in L^1([0, +\infty), X) \). Hence (3.13) still holds for \( \Re \lambda = 0 \) and the claim follows as before. \( \square \)

Lemma 3.19. For every \( u \in W^{2, p}_0([0, +\infty), U) \) and \( \lambda \in \mathbb{C} \) with \( \Re \lambda > 0 \) we have

\[
\mathcal{L}(T^{\varepsilon}_{t_0} \ast \vdash u)(\lambda) = \varepsilon \lambda C_L R(\lambda, A_{-1}) B\hat{u}(\lambda) + \varepsilon \lambda \otimes D\hat{u}(\lambda) = [R(\lambda, A)]_{t_0} u.
\]

Proof. Let \( u \in W^{2, p}_0([0, +\infty), U) \) and \( \Re \lambda > 0 \). Then for \( s \in (-\infty, 0] \) we get using [22, Thm. 4.2]

\[
\mathcal{L}(T^{\varepsilon}_{t_0} \ast \vdash u)(\lambda)(s) = \int_{-\varepsilon - s}^{+\varepsilon - s} e^{-\lambda t} \mathbb{I}_{[-\varepsilon, 0]}(s) C_L \int_0^{t+s} T_{-1}(t+s-r) B\hat{u}(r) \, dr \, dt
\]

\[
+ \int_{-\varepsilon - s}^{+\varepsilon - s} e^{-\lambda t} \mathbb{I}_{[-\varepsilon, 0]}(s) D\hat{u}(t+s) \, dt =: L_1 + L_2.
\]

We compute the two terms of the sum separately. For \( L_2 \) we obtain

\[
L_2 = \int_{-\varepsilon - s}^{+\varepsilon - s} e^{-\lambda t} D\hat{u}(t+s) \, dt = e^{\lambda t} D \int_0^{+\varepsilon - s} e^{-\lambda s} \hat{u}(s) \, ds.
\]

Using (3.9) (for \( s = 0 \), Lemma 3.18 and [2, Cor. 1.6.6]), which states that \( \hat{\nu}(\lambda) = \lambda \hat{\nu}(\lambda) - \nu(0) \) for \( v \in W^{1, p}_{13}([0, +\infty), X) \), for the first term we obtain

\[
L_1 = \int_{-\varepsilon - s}^{+\varepsilon - s} e^{-\lambda t} C_L \int_0^{t+s} T_{-1}(t+s-r) B\hat{u}(r) \, dr \, dt
\]

\[
= \int_0^{+\varepsilon - s} e^{-\lambda t} C_L \int_0^{t+s} T_{-1}(t+r) B\hat{u}(r) \, dr \, dt
\]

\[
= e^{\lambda s} \int_0^{+\varepsilon - s} e^{-\lambda t} \left( -C_L A_{-1}^{-1} B\hat{u}(t) - C_L A_{-1}^{-1} B\hat{u}'(t) + C_L A_{-1}^{-1} \int_0^t T(t-r) A_{-1}^{-1} B\hat{u}(r) \, dr \right) dt
\]

\[
= e^{\lambda s} C_L A_{-1}^{-1} \left( -Id - \lambda A_{-1}^{-1} + \lambda^2 R(\lambda, A) A_{-1}^{-1} \right) B\hat{u}(\lambda)
\]

\[
= \varepsilon \lambda(s) C_L R(\lambda, A_{-1}) B\hat{u}(\lambda).
\]

Summing up this gives (3.14) and completes the proof. \( \square \)

We proceed by introducing the following notion closely related to the entry \( T^{\varepsilon}_{t_0}(t) \).
Definition 3.20. The pair \((B,C) \in \mathcal{L}(U,X_{-1}) \times \mathcal{L}(X_1,Y)\) is called \(p\)-admissible (with respect to \(A\)) if there exists \(t_0 > 0\) and a constant \(M \geq 0\) such that

\[
(\int_0^{t_0} \left\| CT \int_0^s T_1(s-r)Bu(r) dr \right\|_Y^p ds \leq M \| u \|_{L_p([0,\infty),U)}^p \text{ for all } u \in W_0^{2,p}([0,\infty),U).
\]

Remark 3.21. Note that, since for \(t \geq 0\) and \(u \in W_0^{2,p}([0,\infty),U)\) we have

\[
\| T_0^t(t)u \|_{E_{1}^p}^p = \int_{-\infty}^t \left\| CT \int_0^{t+s} T_1(t+s-r)Bu(r) dr \right\|_Y^p ds = \int_0^t \left\| CT \int_0^t T_1(s-r)Bu(r) dr \right\|_Y^p ds,
\]

the pair \((B,C)\) is \(p\)-admissible if and only if \(T_0^t(t) : W_0^{2,p}([0,\infty),U) \subset X \to E^p_1\) has a (since \(W_0^{2,p}([0,\infty),U)\) is dense in \(E^p_1\) necessarily unique) extension in \(\mathcal{L}(E^p_2,E^p_1)\) for some \(t > 0\). Since \(D \in \mathcal{L}(U,Y)\) this again is equivalent to the fact that \(T_0^t(t) : W_0^{2,p}([0,\infty),U) \subset X \to E^p_1\) has a bounded extension for some \(t > 0\).

Recall that we assume the semigroup \((T(t))_{t \geq 0}\) to be exponentially stable. This implies the following result which is analogous to Lemmas 3.9 and 3.15, and closely related to [23, Thm.2.5.4.(ii)] [31, Prop. 2.1].

Lemma 3.22. Let \(B \in \mathcal{L}(U,X_{-1})\) be a \(p\)-admissible control operator and \(C \in \mathcal{L}(X_1,Y)\) be a \(p\)-admissible observation operator. Then the pair \((B,C)\) is \(p\)-admissible if and only if there exists \(M_{BC} \geq 0\) such that

\[
\left( \int_0^t \left\| CT \int_0^s T_1(s-r)Bu(r) dr \right\|_Y^p ds \right)^{\frac{1}{p}} \leq \sum_{k=0}^{n} \left( \int_0^t \left\| CT \int_0^{k+s} T_1(s-r)Bu(r) dr \right\|_Y^p ds \right)^{\frac{1}{p}} \leq \sum_{k=0}^{n} \left( \int_0^t \left\| CT \int_0^{k+s} T_1(s+r)Bu(r) dr \right\|_Y^p ds \right)^{\frac{1}{p}}.
\]

(3.16)

for all \(u \in W_0^{2,p}([0,\infty),U), t \geq 0\).

Proof. If the pair \((B,C)\) is \(p\)-admissible, then we can suppose without loss of generality that in (3.15) we have \(t_0 = 1\). Then it is clear that (3.15) also holds for \(t_0\) replaced by some \(0 < t < 1\). In particular it follows that for every \(0 < t < 1\) the operator \(T_0^t(t)\) has a (unique) extension \(\mathcal{F}(t) \in \mathcal{L}(L^p([0,\infty),U),L^p((0,\infty)-,Y))\).

Now to prove (3.16) it suffices to show that it holds for every \(t = n \in \mathbb{N}\). To this end we write

\[
\left( \int_0^t \left\| CT \int_0^s T_1(s-r)Bu(r) dr \right\|_Y^p ds \right)^{\frac{1}{p}} = \left( \int_0^t \left\| CT \int_0^s T_1(s-r)Bu(r) dr \right\|_Y^p ds \right)^{\frac{1}{p}} \leq \sum_{k=0}^{n} \left( \int_0^t \left\| CT \int_0^{k+s} T_1(s-r)Bu(r) dr \right\|_Y^p ds \right)^{\frac{1}{p}}.
\]

(3.17)

In order to proceed we first estimate the terms of the last sum.

\[
\left( \int_0^1 \left\| CT \int_0^{k+s} T_1(s+r)Bu(r) dr \right\|_Y^p ds \right)^{\frac{1}{p}}
\]

\[
= \left( \int_0^1 \left\| CT \int_0^{k+s} T_1(s+k-r)Bu(r) dr \right\|_Y^p ds \right)^{\frac{1}{p}} \leq \left( \int_0^1 \left\| CT \int_0^{k+s} T_1(s+k-r)Bu(r) dr \right\|_Y^p ds \right)^{\frac{1}{p}} \leq \left( \int_0^1 C T(s) \sum_{m=0}^{k-1} \left( \int_{m}^{m+1} T(k-m-1) \int_{m}^{s+k} T_1(s+r)Bu(r) dr \right) \right)^{\frac{1}{p}} \leq \left( \int_0^1 C T(s) \sum_{m=0}^{k-1} \left( \int_{m}^{s+k} T_1(s+r)Bu(r) dr \right) \right)^{\frac{1}{p}} = L_1 + L_2.
\]

We consider the two terms of this sum separately. To this end we define for \(m \in \mathbb{N}\) the operators \(P_m \in \mathcal{L}(L^p([0,\infty),U))\) by \((P_m(u))(s) := \mathbb{1}_{[0,1]}(s)u(s+m)\) for \(s \in [0,\infty)\). Then

\[
L_2 = \| F(1)PKu \|_{L_p((0,\infty),Y)} \leq M \| PKu \|_{L_p([0,\infty),U)}
\]
where we used that the pair \((B, C)\) is \(p\)-admissible. The first term of the sum can be estimated as
\[
L_1 \leq M\frac{\|}{m=0} T(k-m-1) \int_0^1 T_1(1-r)Bu(r+m)\,dr \|_X
\]
\[
\leq M\frac{\|}{m_0} K \sum_{m=0}^{k-1} e^{\omega(k-m-1)} \|B(1) P_m u\|_X
\]
\[
\leq M\frac{\|}{m_0} MBK \sum_{m=0}^{k-1} e^{\omega(k-m-1)} \|P_m u\|_{L^p([0,\infty), U)}.
\]

Here we used that \(C\) is a \(p\)-admissible observation operator, the stability condition (1.2) and that \(B\) is a \(p\)-admissible control operator. Thus introducing the notation
\[
l_m := \begin{cases} M & \text{if } m = 0 \\
M\frac{\|}{m_0} MBK e^{\omega(m-1)} & \text{if } 1 \leq m \leq n - 1
\end{cases}
\]
we obtain that for \(0 \leq k \leq n - 1\)
\[
\left( \int_0^1 \left\| C_L \int_0^{s+k} T_1(s+k-r)Bu(r)\,dr \right\|^p\,dr \right)^{\frac{1}{p}} \leq \sum_{m=0}^{k} l_{k-m} \|P_m u\|_{L^p([0,\infty), U)}.
\]
Summing up we obtain by (3.17) for arbitrary \(n \in \mathbb{N}\) and \(u \in W_0^{2,p}([0,\infty), U)\) that
\[
\left( \int_0^\infty \left\| C_L \int_0^s T_1(s-r)Bu(r)\,dr \right\|^p\,ds \right)^{\frac{1}{p}} \leq \sum_{k=0}^{n-1} \sum_{m=0}^{k} l_{k-m} \|P_m u\|_{L^p([0,\infty), U)}
\]
\[
\leq \left( \sum_{k=0}^{n-1} l_k \right) \cdot \left( \sum_{k=0}^{n-1} \|P_k u\|_{L^p([0,\infty), U)} \right)^{\frac{1}{p}}
\]
\[
\leq \left( M + M\frac{\|}{m_0} MBK \right) \cdot \|u\|_{L^p([0,\infty), U)},
\]
where in the second estimate we used Young’s inequality for the convolution of sequences. □

\textbf{Remark 3.23.} If \(B\) and \(C\) are both \(p\)-admissible then by Lemma 3.22 the pair \((B, C)\) is \(p\)-admissible if and only if the operator
\[
\tilde{F} : W_0^{2,p}([0,\infty), U) \subset L^p([0,\infty), U) \rightarrow L^p([0,\infty), Y),
\]
\[
(\tilde{F}u)(\cdot) := C_L \int_0^{\cdot} T_1(\cdot-r)Bu(r)\,dr
\]
has a bounded extension in \(L(L^p([0,\infty), U), L^p([0,\infty), Y))\).

Combining the previous results we obtain the main statement of this subsection.

\textbf{Corollary 3.24.} If \(A\) is a generator, then \((B, C)\) is \(p\)-admissible. Conversely, if \(B\) and \(C\) are all \(p\)-admissible, then for every \(t \geq 0\) the operator \(T_{13}^\uparrow(t) : W_0^{2,p}([0,\infty), U) \subset E_2^p \rightarrow E_2^p\) has a (unique) bounded extension \(F(t) := T_{13}^\uparrow(t) \in L(E_2^p, E_2^p)\) and \((F(t))_{t \geq 0}\) is strongly continuous.

\textbf{Proof.} If \(A\) is the generator of a \(C_0\)-semigroup \((T(t))_{t \geq 0}\), then by Lemma 3.19 and the uniqueness of the Laplace transform we obtain that \([T(t)]_{\lambda=1} u = T_{13}(u)\) for all \(t \geq 0\) and \(u \in W_0^{2,p}([0,\infty), U)\). Since \([T(t)]_{\lambda=1} \in L(E_2^p, E_2^p)\), Remark 3.21 then implies that the pair \((B, C)\) is \(p\)-admissible.

Conversely, if \((B, C)\) is \(p\)-admissible, then Remark 3.21 and Lemma 3.22 imply that the operator \(T_{13}^\uparrow(t) : W_0^{2,p}([0,\infty), U) \subset X \rightarrow E_2^p\) has a (unique) bounded extension for every \(t \geq 0\). We recall that by Lemma 3.17 the map \(t \mapsto T_{13}^\uparrow(t)u\) is continuous for every \(u \in W_0^{2,p}([0,\infty), U)\). Hence by a density argument, using Lemma 3.22, we conclude that \((F(t))_{t \geq 0}\) is strongly continuous. □
4. Characterization of Admissible Pairs in the Frequency Domain

The aim of this section is to characterize admissibility in the frequency domain, i.e., in terms of the entries of the resolvent $R(\lambda, A)$ of $A$. For the admissibility of the observation operator $C$ (related to the boundedness of the entry $T_{12}(t)$ of the semigroup operators $T(t) = (T_{jk}(t))_{j,k=0}^3$, cf. Subsection 3.1) and the admissibility of the control operator $B$ (related to the boundedness of $T_{23}(t)$, cf. Subsection 3.2) this problem was posed by Weiss in [32], [35] and in the sequel has been studied by various authors. We refer to [20] for a nice survey on this matter.

Here we concentrate on the entry $T_{13}(t)$ related to the admissibility of the pair $(B, C)$. Our approach is based on the concept of Fourier multipliers, cf. [1, Sect. 2.5], [3, Sect. 5.2], [15, App. E.1]. First we recall the basic definition, where $\mathcal{F}$ denotes the Fourier transform.

**Definition 4.1.** Let $V, W$ be two Banach spaces and $1 \leq p < \infty$. A function $m \in L^\infty(\mathbb{R}, L(V, W))$ is called (bounded) $L^p$-Fourier multiplier if the map

$$
v \mapsto \mathcal{F}^{-1}(m\mathcal{F}v) \quad \text{for } v \in S(\mathbb{R}, V)
$$

has a continuous extension to a bounded operator from $L^p(\mathbb{R}, V)$ to $L^p(\mathbb{R}, W)$.

Since by Assumption 1.2(i) we have $i\mathbb{R} \subset \rho(A)$ we can, using (1.4), define the map

$$
m_{13} : \mathbb{R} \to L(U, Y), \quad m_{13}(\gamma) := C_L R(i\gamma, A_{-1}) B.
$$

Now the following characterization holds.

**Proposition 4.2.** Let $B$ and $C$ be $p$-admissible control and observation operators, respectively. Then the pair $(B, C)$ is $p$-admissible if and only if $m_{13}$ is a bounded Fourier multiplier.

**Proof.** As we have seen in Remark 3.23, the pair $(B, C)$ is $p$-admissible if and only if the operator $\tilde{F}$ has a bounded extension to $L^p([0, \infty), U)$. Let $\gamma \in \mathbb{R}$ and

$$
u \in W^{2,p}_{0,c}([0, \infty), U) := \left\{ u \in W^{1,p}([0, \infty), U) : u(0) = u'(0) = 0 \text{ and } u \text{ has compact support} \right\}
$$

For such $u$ we have $u, u', u'' \in L^1([0, \infty), U)$. Hence even though $\text{Re}(i\gamma) = 0$, we can argue as in the proof of Lemma 3.19, using the second part of Lemma 3.18, to obtain

$$
L(\tilde{F}u)(i\gamma) = C_L R(i\gamma, A_{-1}) B L(u)(i\gamma) \quad \text{for all } \gamma \in \mathbb{R}.
$$

It thus follows

$$
\mathcal{F}(\tilde{F}u) = m_{13} \mathcal{F}u.
$$

Using this we conclude that $m_{13}$ is a bounded Fourier-multiplier if and only if $\tilde{F}$ has a bounded extension to $L^p([0, \infty), U)$ if and only if the pair $(B, C)$ is $p$-admissible. 

5. Well-Posed Linear Control Systems and the Lax-Phillips Semigroup

We now sum up the findings of Subsections 3.1–3.3 to obtain our main result. For a linear control system $\Sigma(A, B, C, D)$ verifying Assumption 1.2 the following holds.

**Theorem 5.1.** The system $\Sigma(A, B, C, D)$ is $p$-well-posed on $\mathcal{X}^p$, i.e., $A$ is the generator of a $C_0$-semigroup $T = (T(t))_{t \geq 0}$ on $\mathcal{X}^p$, if and only if $B$ is a $p$-admissible control operator, $C$ is a $p$-admissible observation operator and also the pair $(B, C)$ is $p$-admissible. In this case the semigroup $T$ is given by

$$
T(t) = \begin{pmatrix}
S_1(t) & \tilde{C}(t) & \tilde{F}(t) \\
0 & T(t) & B(t) \\
0 & 0 & S_2(t)
\end{pmatrix} \quad \text{for all } t \geq 0.
$$

\[\text{By } \delta(\mathbb{R}, V) \text{ we denote the set of all Schwartz functions with values in the Banach space } V.\]
Proof. If \( A \) is a generator on \( \mathcal{X}^p \) then \( C, B \) and \( (B, C) \) are \( p \)-admissible by Corollaries 3.10, 3.16 and 3.24, respectively. Conversely, if \( C, B \) and \( (B, C) \) are \( p \)-admissible, again by Corollaries 3.10, 3.16 and 3.24 we can define a strongly continuous operator family \( (\mathcal{T}(t))_{t \geq 0} \) by (5.1). Using Lemmas 3.3, 3.12 and 3.19 it follows that
\[
\mathcal{L}(\mathcal{T}(\lambda))(\lambda) = R(\lambda, A)
\]
hence by Lemma 3.1, \( \mathcal{T} \) is a \( C_0 \)-semigroup with generator \( A \).
\( \square \)

Combining Proposition 4.2 with Theorem 5.1 we obtain a second characterization.

**Corollary 5.2.** The system \( \Sigma(A, B, C, D) \) is \( p \)-well-posed on \( \mathcal{X}^p \) if and only if \( B, C \) are \( p \)-admissible control and observation operators, respectively, and \( m_{13} \) is a bounded Fourier multiplier.

As a corollary we characterize the 2-well-posedness of the system \( \Sigma(A, B, C, D) \) in case all the spaces \( X, Y \) and \( U \) are Hilbert spaces. Using the Plancharel Theorem (see [2, Thm.1.8.2]) one can first prove the following.

**Lemma 5.3.** Let \( V, W \) be two Hilbert spaces, then every \( m \in L^\infty(\mathbb{R}, \mathcal{L}(V, W)) \) is a (bounded) \( L^2 \)-Fourier multiplier.

Combining Corollary 5.2 and Lemma 5.3 we immediately obtain our next result.

**Corollary 5.4.** Let \( X, Y \) and \( U \) be Hilbert spaces. Then the system \( \Sigma(A, B, C, D) \) is 2-well-posed if and only if \( B, C \) are 2-admissible and \( m_{13} = C_L R(i \lambda, A_{-1}) B \in L^\infty(\mathbb{R}, \mathcal{L}(U, Y)) \).

**Remark 5.5.** The semigroup \( (\mathcal{T}(t))_{t \geq 0} \) in (5.1) already appears in Staffans and Weiss [24, Prop. 6.2] and there is called the Lax-Phillips semigroup (of index 0) referring to the paper [14] by Lax and Phillips.

This semigroup describes the solutions of the well-posed system \( \Sigma(A, B, C, D) \) as follows. For \( x = (y(\bullet), x, u(\bullet))^T \in \mathcal{X}^p \)
- the first component of \( \mathcal{T}(\bullet)x \) gives the past output,
- the second component of \( \mathcal{T}(\bullet)x \) represents the present state,
- the third component of \( \mathcal{T}(\bullet)x \) can be interpreted as the future input
of the system.

**Remark 5.6.** If the semigroup \( (T(t))_{t \geq 0} \) generated by the state operator \( A \) is not exponentially stable as supposed in Assumption 1.2 (i) (i.e., if the growth bound \( \omega_0(A) \geq 0 \), cf. [11, Def. I.5.6]) we choose \( \lambda_0 > \omega_0(A) \). Then for the rescaled generator \( A - \lambda_0 \) we obtain \( \omega_0(A - \lambda_0) < 0 \). Moreover, on the product space \( \mathcal{X}^p \) we introduce the operator matrix \( A^{\lambda_0} \) associated to the control problem \( \Sigma(A - \lambda_0, B, C, D) \). This operator can be written as
\[
A^{\lambda_0} = A - \lambda_0 P_2 \quad \text{for} \quad P_2 := \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \in \mathcal{L}(\mathcal{X}^p).
\]
If there exists \( \lambda \in \rho(A) \) such that \( \text{rg}(R(\lambda, A_{-1})B) \subseteq D(C_L) \) then this holds for every \( \lambda \in \rho(A) \). Hence \( \text{rg}(R(\mu, A_{-1})B) \subseteq D(C_L) \) for every \( \mu \in \rho(A - \lambda_0) \). This shows that \( A \) satisfies the compatibility assumption (1.3) if and only if \( A - \lambda_0 \) does.

Moreover we have the following result.

**Theorem 5.7.** Let \( \lambda_0 \in \rho(A) \). Then the following are equivalent.

(a) \( A \) is the generator of a \( C_0 \)-semigroup on \( \mathcal{X}^p \),
(b) \( A^{\lambda_0} \) is the generator of a \( C_0 \)-semigroup on \( \mathcal{X}^p \),
(c) \( B, C \) and the pair \( (B, C) \) are \( p \)-admissible with respect to \( A - \lambda_0 \),
(d) \( B \) and \( C \) are \( p \)-admissible with respect to \( A - \lambda_0 \) (or \( A \)) and \( m_{13}^{\lambda_0} := C_L R(\lambda_0 + i \lambda, A_{-1})B \) is a bounded Fourier-multiplier.


Proof. (a) $\iff$ (b). Since $A$ and $A^\lambda_0$ differ only by a bounded operator this equivalence holds by the bounded perturbation theorem, cf. [11, Thm.III.1.3].

(b) $\iff$ (c). This equivalence holds by Theorem 5.1.

(c) $\iff$ (d). It is clear that $B$ and $C$ are $p$-admissible with respect to $A - \lambda_0$ if and only if they are $p$-admissible with respect to $A$. By Theorem 4.2 the pair $(B, C)$ is $p$-admissible with respect to $A - \lambda_0$ if and only if $m_{\lambda_0}^p = C_L R(i \cdot, A_{-1} - \lambda_0) B = C_L R(\lambda_0 + i \cdot, A_{-1}) B$ is a bounded Fourier-multiplier. □

6. Example: A Heat Equation with Boundary Control and Point Observation

To illustrate our results we consider a metal bar of length $\pi$ modeled as a segment $[0, \pi]$. Our aim is to control its temperature by putting controls $u_0(t)$ and $u_1(t)$ at the edges 0 and $\pi$. Moreover, we observe the system by measuring its temperature at the center $\pi/2 \in [0, \pi]$.

As state space we choose the Hilbert space $X = L^2[0, \pi]$ and consider the state function $x(s, t)$ representing the temperature in the point $s \in [0, \pi]$ at time $t \geq 0$.

If we start from the temperature profile $x_0 \in X$, the time evolution of our system can be described by a heat equation with boundary control and a point observation, more precisely by

$$
\begin{align*}
\frac{\partial x(s, t)}{\partial t} - \frac{\partial^2 x(s, t)}{\partial s^2}, & \quad t \geq 0, \ s \in [0, \pi], \\
x(s, 0) = x_0(s), & \quad s \in [0, \pi], \\
\frac{\partial x}{\partial s}(0, t) = u_0(t), & \quad t \geq 0, \\
\frac{\partial x}{\partial s}(\pi, t) = u_1(t), & \quad t \geq 0, \\
y(t) = x\left(\frac{\pi}{2}, t\right), & \quad t \geq 0.
\end{align*}
$$

(6.1)

Here the boundary conditions in $s = 0$ and $s = \pi$ involving $u_0(\bullet)$ and $u_1(\bullet)$ describe the heat exchange between the ends of the bar and the environment.

In order to write (6.1) as a linear control system of the form $\Sigma(A, B, C, D)$ we use the approach for boundary control problems developed in [8, Sect. 2]. To this end we define the following operators and spaces.

- The maximal system operator

  $$
  A_m := \frac{d^2}{ds^2} \quad \text{with domain} \quad D(A_m) := W^{2, 2}[0, \pi] \subset X = L^2[0, \pi];
  $$

- the boundary space $\partial X := C^2$ and the boundary operator\(^5\)

  $$
  Q : D(A_m) \to \partial X, \quad Qf := (f'(0), f'(')),
  $$

- the control space $U := C^2$ and the control operator $\bar{B} := Id \in \mathcal{L}(U, \partial X)$;

- the observation space $Y := C$ and the observation operator\(^6\) $C := \delta_{\pi/2}$.

With this notation (6.1) can be rewritten as an abstract Boundary Control System

$$
\begin{align*}
\dot{x}(t) & = A_m x(t), \quad t \geq 0, \\
Qx(t) & = \bar{B} u(t), \quad t \geq 0, \\
y(t) & = C x(t), \quad t \geq 0, \\
x(0) & = x_0.
\end{align*}
$$

(aBCS)

\(^5\)Here $[D(A_m)]$ indicates the space $D(A_m)$ endowed with the graph norm $\|\cdot\|_{A_m}$.

\(^6\)By $\delta_{\pi/2}$ we indicate the point evaluation in $\pi/2$. 
We note that
- the operator $A \subset A_m$ with domain
  \[ D(A) := \ker(Q) = \{ h \in W^{2,2}[0,\pi] : f'(0) = f'(\pi) = 0 \} \]
  is the generator of a $C_0$-semigroup $(T(t))_{t \geq 0}$ on $X$, and its spectrum is given by $\sigma(A) = \{-n^2 : n \in \mathbb{N}\}$ (see [11, Sect. II.3.30]);
- the boundary operator $Q$ is surjective,
  i.e., the Main Assumptions 2.3 in [8] are satisfied. In order to use the abstract theory for boundary control systems developed in [8, Sect. 2] we need the Dirichlet operator
  \[ Q_\lambda = (Q_{\ker(\lambda-A_m)})^{-1} : \partial X \to \ker(\lambda-A_m) \]
  which by [8, Lem. 2.4.(iii)] exists for every $\lambda \in \rho(A)$.
Since\footnote{With $\text{span}\{f, g\}$ we denote the linear vector space generated by $f$ and $g$.} $\ker(\lambda-A_m) = \text{span}\{\cosh(\sqrt{\lambda} \cdot), \cosh(\sqrt{\lambda}(\pi - \cdot))\}$, a simple computation shows that
  \[ Q_\lambda = (q_0(\cdot), q_1(\cdot)), \]
  where for $s \in [0, \pi]$
  \[ q_0(s) := -\frac{\cosh(\sqrt{\lambda}(\pi - s))}{\sqrt{\lambda}\sinh(\sqrt{\lambda}\pi)}, \quad q_1(s) := \frac{\cosh(\sqrt{\lambda}s)}{\sqrt{\lambda}\sinh(\sqrt{\lambda}\pi)} \]
  Let $B_\lambda := Q_\lambda \tilde{B} = Q_\lambda$. Then by [8, Sect. 2] the system (aBCS) is equivalent to $\Sigma(A, B, C, D)$ for the operators
  \[ B := (\lambda - A_1)Q_\lambda \in \mathcal{L}(U, X_1), \]
  \[ C := \delta_\varphi \in \mathcal{L}(X_1, Y), \]
  \[ D := 0 \in \mathcal{L}(U, Y). \]
In order to prove 2-well-posedness of the system $\Sigma(A, B, C, 0)$ we transform it into an isomorphic problem on $\ell^2$.
To this end we first note that $A$ is self-adjoint and has compact resolvent. Hence its normalized eigenvectors given by
  \[ e_n(s) = \sqrt{\frac{2}{\pi}} \cos(ns), \quad \text{where} \quad w_n = \begin{cases} 1 & \text{if } n = 0, \\ 2 & \text{if } n \geq 1 \end{cases} \]
form an orthonormal basis of $X$. Using this basis we define the surjective isometry
  \[ J : X \to \ell^2, \quad Jf := (f, e_n)_{n \in \mathbb{N}}, \]
which associates to a function $f \in X$ the sequence of its Fourier coefficients relatively to $(e_n)_{n \in \mathbb{N}}$.
Next we put $z(t) := Jx(t)$. Then the system $\Sigma(A, B, C, 0)$ transforms to
  \[ \Sigma(JAJ^{-1}, JB, CJ^{-1}, 0) = \Sigma(JAJ^{-1}, J(\lambda - A_1)Q_\lambda, \delta_\varphi J^{-1}, 0). \]
In particular, the differential operator $A$ transforms into the multiplication operator
  \[ JAJ^{-1} =: M_\alpha =: M : D(M) \subset \ell^2 \to \ell^2 \]
where $\alpha = (-n^2)_{n \in \mathbb{N}}$ and
  \[ D(M) = \{(a_n)_{n \in \mathbb{N}} \in \ell^2 : (-n^2a_n)_{n \in \mathbb{N}} \in \ell^2 \}. \]
This gives for $\lambda > 0$ the extrapolation space
  \[ X^M_\lambda = \{(a_n)_{n \in \mathbb{N}} \in \mathbb{C}^\mathbb{N} : \frac{a_n}{\lambda + n^2} \in \ell^2 \}. \]
Moreover, the Dirichlet operator $Q_\lambda$ transforms into the operator
  \[ JQ_\lambda = \left( \frac{\sqrt{w_n/\pi}}{\lambda + n^2}, \frac{(-1)^n \sqrt{w_n/\pi}}{\lambda + n^2} \right)_{n \in \mathbb{N}} \]
Thus the control operator $B$ transforms into
\begin{equation}
(6.2) \quad b := J(\lambda - A_{-1})Q_\lambda = (\lambda - M)JQ_\lambda = \left( -\sqrt{\frac{u_n}{\pi}} \right)_{n \in \mathbb{N}}, \left( (-1)^n \sqrt{\frac{u_n}{\pi}} \right)_{n \in \mathbb{N}},
\end{equation}
while the observation operator $C$ transforms into the operator
\begin{equation}
(6.3) \quad c := CJ^{-1} = \left( e_n(\frac{\pi}{2}) \right)_{n \in \mathbb{N}}
\end{equation}
where
\[e_n(\frac{\pi}{2}) = \begin{cases} 
0 & \text{if } n \text{ is odd,} \\
(-1)^{\frac{n}{2}} \sqrt{\frac{\pi}{n}} & \text{if } n \text{ is even.}
\end{cases}\]
Summing up, the Control System (6.1) is isometrically isomorphic to
\begin{equation}
(6.4) \quad \begin{cases}
\dot{z}(t) = Mz(t) + bu(t), & t \geq 0, \\
y(t) = cz(t), & t \geq 0, \\
z(0) = z_0,
\end{cases}
\end{equation}
where $z(t) := Jx(t) \in L^2$ and $z_0 := Jx_0$.

Our aim is now to prove the 2-well-posedness of the system $\Sigma(\lambda, b, c, 0)$ in (6.4). Since $\omega_0(A) = \omega_0(M) = 0$ we consider in the sequel $M - 1$ instead of $M$, cf. Remark 5.6 and Theorem 5.7.

First we verify the compatibility condition (1.3).

**Lemma 6.1.** For every $\gamma \in \mathbb{R}$ we have
\begin{equation}
(6.5) \quad \text{rg} \left( R(1 + i\gamma, M_{-1})b \right) \subset D(cL).
\end{equation}
Moreover, $m_{13}(\bullet) := cL R(1 + i \cdot, M_{-1})b \in L^\infty(\mathbb{R}, \mathcal{L}(U, Y)) = L^\infty(\mathbb{R}, \mathcal{L}(C^2, C))$.

**Proof.** Let $u := \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \in U = C^2$ and $\gamma \in \mathbb{R}$. Then it follows
\begin{equation}
R(1 + i\gamma, M_{-1})bu = \left( \frac{1}{1 + n^2 + i\gamma} \left( -\sqrt{\frac{u_n}{\pi}} u_1 + (-1)^n \sqrt{\frac{u_n}{\pi}} u_2 \right) \right)_{n \in \mathbb{N}} =: (r_n)_{n \in \mathbb{N}}.
\end{equation}
Since
\[\left| e_n(\frac{\pi}{2}) \cdot r_n \right| \leq \frac{4}{(1 + n^2)\pi} \cdot (|u_1| + |u_2|) \quad \text{for all } n \in \mathbb{N}, \gamma \in \mathbb{R}\]
the series
\[\sum_{n=0}^{\infty} e_n(\frac{\pi}{2}) \cdot r_n\]
converges. By [30, Prop. 7.2] this implies (6.5) and
\[|cL R(1 + i\gamma, M_{-1})bu| \leq \frac{4\sqrt{2}}{\pi} \sum_{n=0}^{\infty} \frac{1}{1 + n^2} \cdot ||u||_2 \quad \text{for all } \gamma \in \mathbb{R}, u \in U.
\]
Since this implies that $m_{13}(\bullet)$ is bounded the proof is complete. \hfill $\square$

Next we verify the 2-admissibility of the operators $c$ and $b$. To this end we denote by $(S(t))_{t \geq 0}$ the semigroup generated by $M - 1$.

**Proposition 6.2.** The observation operator $c$ is 2-admissible with respect to $M - 1$.

**Proof.** Let $t_0 > 0$ and $z = (z_n)_{n \in \mathbb{N}} \in D(M)$. Then by the Cauchy–Schwarz inequality we obtain
\begin{align*}
\int_0^{t_0} |cS(s)z|^2 \, ds &= \int_0^{t_0} \left( \sum_{n=0}^{\infty} e_n(\frac{\pi}{2}) e^{-(1 + n^2)s} z_n \right)^2 \, ds \\
&\leq \frac{2}{\pi} \sum_{n=0}^{\infty} \int_0^{t_0} e^{-2(1 + n^2)s} \, ds \cdot \sum_{n=0}^{\infty} |z_n|^2 \\
&\leq \frac{1}{\pi} \sum_{n=0}^{\infty} \frac{1}{1 + n^2} \cdot ||z||^2_{L^2},
\end{align*}
hence by definition $c$ is an admissible observation operator. \hfill $\square$
Proposition 6.3. The control operator \( b = (b_1, b_2) \) is 2-admissible with respect to \( M - 1 \).

Proof. Clearly \( b \) is 2-admissible if and only if \( b_1, b_2 : \mathbb{C} \to X_{-1} \) are both 2-admissible. Let \( t_0 > 0 \) and \( u \in L^2[0, +\infty) \). Then by Young’s inequality (cf. [2, Prop. 1.3.5.(a)]) we obtain for \( i = 1, 2 \)

\[
\left\| \int_0^{t_0} S_{-1}(t_0 - r)b_i u(r) \, dr \right\|_2^2 \leq \frac{2}{\pi} \sum_{n=0}^{+\infty} \left( \int_0^{t_0} e^{-(1+n^2)(t_0-r)} |u(r)| \, dr \right)^2 \leq \frac{2}{\pi} \sum_{n=0}^{+\infty} \left( \int_0^{+\infty} e^{-2(1+n^2)r} \, dr \right)^2 \cdot \left( \int_0^{+\infty} |u(r)|^2 \, dr \right)^2 = \frac{1}{\pi} \sum_{n=0}^{+\infty} \frac{1}{1 + n^2} \cdot \| u \|^2_{L^2[0, +\infty]},
\]

hence by definition \( b_i \) is an admissible control operator. \( \square \)

Remark 6.4. For multiplication semigroups and finite dimensional observation/control spaces there exists a characterization for the admissability of an observation/control operator via a Carleson measure criteria. For the details we refer to [27, Thm. 5.3.2] and [19, Cor. 2.5], [28, Thm. 1.2], respectively.

Finally, from Lemmas 4.2, 5.3 and 6.1 we obtain the following.

Corollary 6.5. The pair \((b, c)\) is 2-admissible.

Summing up we obtain by Theorem 5.7 the main result of this section.

Corollary 6.6. The system \( \Sigma(M, b, c, 0) \), hence also the Heat Equation (6.1), is 2-well-posed.

Appendix A.

We used several times the following simple result which relates the existence of a bounded extension of a densely defined operator to a range condition. It allowed us to characterize admissibility by a range condition or, alternatively, by a boundedness condition on a dense set.

Lemma A.1. Let \( V, W, Z \) be arbitrary Banach spaces, \( D \subset V \) be a dense subspace and assume that \( W \hookrightarrow Z \) is continuously embedded. Then for linear operators \( \tilde{Q} : V \to Z, \quad Q \in \mathcal{L}(V, Z) \),

\[
Q : D \subset V \to W,
\]

the following assertions are equivalent.

(a) There exists \( M \geq 0 \) such that \( \| Qv \|_W \leq M \cdot \| v \|_V \) for all \( v \in D \)

and \( \tilde{Q} \) is the unique bounded extension of \( Q \).

(b) \( Q = \tilde{Q}|_D \) and \( \text{rg}(\tilde{Q}) \subset W \).

In this case, \( \tilde{Q} \in \mathcal{L}(V, W) \).

Proof. (a) \( \Rightarrow \) (b). It is clear that \( Q \) has a unique bounded extension \( \tilde{Q} \in \mathcal{L}(V, W) \). Since \( \tilde{Q} = Q \) it follows \( \text{rg}(\tilde{Q}) \subset W \).

(b) \( \Rightarrow \) (a). Since \( \text{rg}(\tilde{Q}) \subset W \) the closed graph theorem implies that \( \tilde{Q} \in \mathcal{L}(V, W) \). Hence there exists a constant \( M \geq 0 \) such that for all \( v \in D \)

\[
\| Qv \|_W = \| \tilde{Q}v \|_W \leq M \| v \|_V.
\]

Moreover, \( \tilde{Q} \) is the unique bounded extension of \( Q \). As claimed in this case \( \tilde{Q} \in \mathcal{L}(V, W) \). \( \square \)
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