ALGORITHMS FOR BERNSTEIN–SATO POLYNOMIALS AND MULTIPLIER IDEALS

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Abstract. The Bernstein–Sato polynomial (or global $b$-function) is an important invariant in singularity theory, which can be computed using symbolic methods in the theory of $D$-modules. After providing a survey of known algorithms for computing the global $b$-function, we develop a new method to compute the local $b$-function for a single polynomial. We then develop algorithms that compute generalized Bernstein–Sato polynomials of Budur–Mustaţă–Saito and Shibuta for an arbitrary polynomial ideal. These lead to computations of log canonical thresholds, jumping coefficients, and multiplier ideals. Our algorithm for multiplier ideals simplifies that of Shibuta and shares a common subroutine with our local $b$-function algorithm. The algorithms we present have been implemented in the $D$-modules package of the computer algebra system Macaulay2.

1. Introduction

The multiplier ideals of an algebraic variety carry essential information about its singularities and have proven themselves a powerful tool in algebraic geometry. However, they are notoriously difficult to compute; nice descriptions are known only for very special families of varieties, such as monomial ideals and hyperplane arrangements \[10, 19, 35, 27\]. To briefly recall the definition of this invariant, let $X = \mathbb{C}^n$ with coordinates $x = x_1, \ldots, x_n$. For an ideal $\langle f \rangle = \langle f_1, \ldots, f_r \rangle \subseteq \mathbb{C}[x]$ and a nonnegative rational number $c$, the multiplier ideal of $f$ with coefficient $c$ is

$$\mathcal{J}(f^c) = \left\{ h \in \mathbb{C}[x] \left| \frac{|h|^2}{(\sum |f_i|^2)^c} \text{ is locally integrable} \right. \right\}.$$ 

It follows from this definition that $\mathcal{J}(f^c) \supseteq \mathcal{J}(f^d)$ for $c \leq d$ and $\mathcal{J}(f^0) = \mathbb{C}[x]$ is trivial. The (global) jumping coefficients of $f$ are a discrete sequence of rational numbers $\xi_i = \xi_i(f)$ with $0 = \xi_0 < \xi_1 < \xi_2 < \cdots$ satisfying the property that $\mathcal{J}(f^c)$ is constant exactly for $c \in [\xi_i, \xi_{i+1})$. In particular, the log canonical threshold of $f$ is $\xi_1$, denoted by $\text{lc}(f)$. This is the least rational number $c$ for which $\mathcal{J}(f^c)$ is nontrivial. The multiplier ideal $\mathcal{J}(f^c)$ measures the singularities of the variety of $f$ in $X$; smaller multiplier ideals (and lower log canonical threshold) correspond to worse singularities. For an equivalent algebro-geometric definition and an introduction to this invariant, we refer the reader to \[13, 14\].

In this paper we develop an algorithm for computing multiplier ideals and jumping coefficients by way of an even finer invariant, Bernstein–Sato polynomials, or $b$-functions. The
results of Budur et al. [6] provide other applications for our Bernstein–Sato algorithms, including multiplier ideal membership tests, an algorithm to compute jumping coefficients, and a test to determine if a complete intersection has at most rational singularities.

The first $b$-function we consider, the *global Bernstein–Sato polynomial* of a hypersurface, was introduced independently by Bernstein [4] and Sato [29]. This univariate polynomial plays a central role in the theory of $D$-modules (or algebraic analysis), which was founded by, amongst others, Kashiwara [11] and Malgrange [17]. Moreover, the jumping coefficients of $f$ that lie in the interval $(0, 1]$ are roots of its global Bernstein–Sato polynomial [7]; however, this $b$-function contains more information. Its roots need not be jumping coefficients, even if they are between 0 and 1 (see Example 6.1).

The Bernstein–Sato polynomial was recently generalized by Budur et al. [6] to arbitrary varieties. The maximal root of this *generalized Bernstein–Sato polynomial* provides a multiplier ideal membership test. Shibuta defined another generalization to compute explicit generating sets for multiplier ideals [32]. Our multiplier ideal algorithm employs the $b$-functions of Shibuta, which we call the $m$-*generalized Bernstein–Sato polynomial*. However, it circumvents primary decomposition and one elimination step through a syzygetic technique (see Algorithms 4.5 and 3.2). The correctness of our results relies heavily on the use of $V$-filtrations, as developed by Kashiwara and Malgrange [12, 18].

$D$-module computations are made possible by Gröbner bases techniques in the Weyl algebra. The computation of the Bernstein–Sato polynomial was pioneered by Oaku in [24]. His algorithm was one of the first algorithms in algebraic analysis, many of which are outlined in the book by Saito et al. [28]. The computation of the *local Bernstein–Sato polynomial* was first addressed in the early work of Oaku [24], as well as the recent work of Nakayama [20], Nishiyama and Noro [21], and Schulze [30, 31]. Bahloul and Oaku [2] address the computation of local Bernstein–Sato ideals that generalize Bernstein–Sato polynomials. In this article we provide our version of the local algorithm for Bernstein–Sato polynomials, part of which is vital to our approach to computation of multiplier ideals.

There are several implementations of algorithms for global and local $b$-functions in kan/sm1 [33], Risa/Asir [23], and Singular [9]. One can find a comparison of performance in [15]. All of the algorithms in this article have been implemented and can be found in the $D$-modules package [16] of the computer algebra system Macaulay2 [8].

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**Outline.** Section 2 surveys the known approaches for computing the global Bernstein–Sato polynomial, highlighting an algorithm of Noro [22]. In Section 3 we present an algorithm for computing the local Bernstein–Sato polynomial. Algorithms for the generalized Bernstein–Sato polynomial for an arbitrary variety, as introduced by Budur et al. [6], are discussed in Section 4 along with their applications. Based on the methods of Section 3, Section 5 considers the $m$-generalized Bernstein–Sato polynomial of Shibuta [32] and contains our algorithms for multiplier ideals.
2. Global Bernstein–Sato polynomials

Let $K$ be a field of characteristic zero, and set $X = K^n$ and $Y = X \times K$ with coordinates $(x)$ and $(x, t)$, respectively. We consider the $n$-th Weyl algebra $D_X = K \langle x, \partial \rangle$ with generators $x_1, \ldots, x_n$ and $\partial x_1, \ldots, \partial x_n$, as well as $D_Y = K \langle x, \partial x, t, \partial t \rangle$, the Weyl algebra on $Y$. Define an action of $D_Y$ on $N_f := K[x][f^{-1}, s]f^s$ as follows: $x_i$ and $\partial x_i$ act naturally for $i = 1, \ldots, n$, and
\[
t \cdot h(x, s)f^s = h(x, s + 1)f^s \quad \text{and} \quad \partial_t \cdot h(x, s)f^s = -sh(x, s - 1)f^{-1}f^s,
\]
where $h \in K[x][f^{-1}, s]$.

Let $\sigma = -\partial_t$. For a polynomial $f \in K[x]$, the global Bernstein–Sato polynomial of $f$, denoted $b_f$, is the monic polynomial $b(s) \in K[s]$ of minimal degree satisfying the equation
\[
b(\sigma)f^s = Pf^s
\]
for some $P \in D_X(\sigma)$.

There is an alternate definition for the global Bernstein–Sato polynomial in terms of $V$-filtrations. To provide this, we denote by $V^*D_Y$ the $V$-filtration of $D_Y$ along $X$, where $V^mD_Y$ is $D_X$-generated by the set $\{\mu\partial x_i - \nu \mid \mu - \nu \geq m\}$. Let $i_f : X \to Y$ defined by $i_f(x) = (x, f(x))$ be the graph of $f$. The $D$-module direct image of $K[x]_\sigma$ along $i_f$ is the module
\[
M_f := (i_f)_*K[x] \cong K[x] \otimes_K K(\partial_t)
\]
with actions of a vector field $\xi$ on $X$ and $t$,
\[
\xi(p \otimes \partial_t^\nu) = \xi p \otimes \partial_t^{\nu - 1} - (\xi f)p \otimes \partial_t^{\nu + 1} \quad \text{and} \quad t \cdot (p \otimes \partial_t^\nu) = fp \otimes \partial_t^{\nu - 1} - \nu p \otimes \partial_t^{\nu - 1},
\]
providing a $D_Y$-module structure. Notice that there is a canonical embedding of $M_f$ into $N_f$, where $s$ is identified with $-\partial_t$.

With $\delta = 1 \otimes 1 \in M_f$, the global Bernstein–Sato polynomial $b_f$ is equal to the minimal polynomial of the action of $\sigma$ on the module $(V^0D_Y)\delta/(V^1D_Y)\delta$. We now survey three ways of computing this $b$-function.

2.1. By way of an annihilator. The global Bernstein–Sato polynomial $b_f(s)$ is the minimal polynomial of $\sigma := -\partial_t$ modulo $\Ann_{D_X[\sigma]} f^s + D_X[\sigma]f^s$, where $f^s \in N_f$. By the next result, this annihilator can be computed from the left $D_Y$-ideal
\[
I_f = \langle t - f, \partial_t + \frac{\partial f}{\partial x_1}, \partial_t, \ldots, \partial_n + \frac{\partial f}{\partial x_n}, \partial_t \rangle.
\]

**Theorem 2.1.** [28 Theorem 5.3.4] The ideal $\Ann_{D[s]} f^s$ equals the image of $I_f \cap D[\sigma]$ under the substitution $\sigma \mapsto s$.

2.2. By way of an initial ideal. This method makes use of $w = (0, 1) \in \mathbb{R}^n \times \mathbb{R}$, the elimination weight vector for $X$ in $Y$.

**Theorem 2.2.** Let $b(x, s)$ be nonzero in the polynomial ring $K[x, s]$. Then $b(x, \sigma) \in (\in_{(-w, w)}I_f) \cap K[x, \sigma]$ if and only if there exists $Q \in D[s]$ satisfying the functional equation $Qf^{s+1} = b(x, s)f^s$. In particular,
\[
\langle b_f(\sigma) \rangle = \in_{(-w, w)}I_f \cap K[\sigma].
\]
Proof. The action of $t$ on $N_f$ is multiplication by $f$, hence, the existence of the functional equation is equivalent to $b(x, s) \in I_f + V^1D_Y$. The result now follows from Theorem 2.1 which identifies $s$ with $\sigma$. \qed

The following algorithm provides a more economical way to compute the global $b$-function using linear algebra. By establishing a nontrivial $K$-linear dependency between normal forms $NF_G(s^i)$ with respect to a Gröbner basis $G$ of $\text{in}_{(-w,w)}I_f$, where $0 \leq i \leq d$ and $d$ is taken as small as possible, this algorithm bypasses elimination of $\partial_1, \ldots, \partial_n$. This trick was used for the first time by Noro in [22], where a modular method to speed up $b$-function computations is provided as well. We include the following algorithm for the convenience of the reader as a similar syzygetic approach will be used in Algorithms 3.2, 4.5, and 5.12. Note that the coefficients of the output are, in fact, rational, since the roots of a $b$-function are rational [11].

\textbf{Algorithm 2.3.} \quad b = \text{globalBFuncton}(f, P)

Input: a polynomial \( f \in K[x]. \)

Output: polynomial \( b \in \mathbb{Q}[s] \) is the Bernstein–Sato polynomial of \( f \).

\( G \leftarrow \text{Gröbner basis of } \text{in}_{(-w,w)}I_f. \)

\( d \leftarrow 0. \)

repeat

\( d \leftarrow d + 1 \)

until \( \exists (c_0, \ldots, c_d) \in \mathbb{Q}^{d+1} \) such that \( c_d = 1 \) and

\[
\sum_{i=0}^{d} c_i \text{NF}_G(s^i) = 0.
\]

return \( \sum_{i=0}^{d} c_i s^i. \)

This approach can be exploited in a more general setting to compute the intersection of a left ideal with a subring generated by one element as shown in [1].

2.3. By way of Briançon–Maisonobe. This approach, which is laid out in [5], computes the annihilator of $f^s$ in an algebra of solvable type similar to, but different from, the Weyl algebra. This path has been explored by Castro-Jiménez and Ucha [36] and implemented in Singular [9] with a performance analysis given by Levandovskyy and Morales in [15] and recent improvements outlined in [1].

3. Local Bernstein–Sato polynomials

In this section, we provide an algorithm to compute the local Bernstein–Sato polynomial of $f$ at a prime ideal of $K[x]$, which is defined by replacing the use of $D_X$ in (2.1) by its appropriate localization. Algorithms 3.1 and 3.2 use Theorem 2.2 to compute an ideal $E_b \subset K[x]$ that describes the locus of points where the $b$-function does not divide the given $b \in \mathbb{Q}[s]$.

\textbf{Algorithm 3.1.} \quad E_b = \text{exceptionalLocusB}(f, b)

Input: a polynomial \( f \in K[x], \) a polynomial \( b \in \mathbb{Q}[s]. \)
Output: \( E_b \subset K[x] \) such that \( \forall \ P \in \text{Spec} \ K[x], \)
\[ b_{f,P} \mid b \iff E_b \not\subset P. \]

\[ G \leftarrow \text{generators of } \text{in}_{(-w,w)}I_f \cap K[x,s], \text{ where } s = -\partial_t. \]
return \( \text{exceptionalLocusCore}(G,b). \)

The following subroutine computes \( K[x]-\text{syzygies} \) between the elements of the form \( s^ig \)
of \( s \)-degree at most \( \deg b \) and \( b \) itself. It returns the projection of the syzygies onto the component corresponding to \( b \).

Algorithm 3.2. \( E_b = \text{exceptionalLocusCore}(f,b) \)
Input: \( G \subset K[x,s], \) a polynomial \( b \in Q[s]. \)
Output: \( E_b \subset K[x]. \)
\[ G_1 \leftarrow \{ \text{a Gröbner basis of } (G) \text{ w.r.t. a monomial order eliminating } s \}. \]
\[ d \leftarrow \deg b. \]
\[ G_2 \leftarrow \{ s^ig \mid g \in G_1, \ i + \deg_s g \leq d \}. \]
\[ S \leftarrow \ker \phi \text{ where } \]
\[ \phi : K[x]_{|G_2|+1} \to \bigoplus_{i=0}^{d} K[x]s^i \]
maps \( e_i, \) for \( i = 1, \ldots, |G_2|, \) to the elements of \( G_2 \) and \( e_{|G_2|+1} \) to \( b. \)
return projection of \( S \subset K[x]_{|G_2|+1} \) onto the last coordinate.

The computation of syzygies in line 4 and projection in line 5 of Algorithm 3.2 may be combined within one efficient Gröbner basis computation.

of correctness of Algorithms 3.1 and 3.2. The local Bernstein–Sato polynomial \( b_{f,P} \) at \( P \in \text{Spec} \ K[x] \) divides the given \( b \in Q[s] \) if and only if
\[ Q'f^{s+1} = b_{f,P}f^s, \text{ for some } Q' \in K[x]_P \otimes D[s] \]
\[ \iff Qf^{s+1} = hbf^s, \text{ for some } Q \in D[s], h \in K[x] \setminus P. \]
For \( h \in K[x], \)
\[ Qf^{s+1} = hbf^s, \text{ for some } Q \in D[s] \]
\[ \iff hb \in \text{in}_{(-w,w)}I_f \cap K[x,s] \quad \text{(by Theorem 2.2)} \]
\[ \iff h \text{ is the last coordinate of a syzygy in the module produced by line 4} \]
\[ \iff h \in E_b. \]
This proves that \( b_{f,P} \mid b \iff E_b \not\subset P. \)

Remark 3.3. [Particulars of Algorithm 3.1] In order to compute generators of \( \text{in}_{(-w,w)}I_f, \)
one may apply the homogenized Weyl algebra technique (for example, see [28, Algorithm 1.2.5]). Then to compute generators of \( \text{in}_{(-w,w)}I_f \cap K[x]\langle t, \partial_t \rangle, \) eliminate \( \partial_x \) and
apply the map $\psi$ defined as follows: for a $(-w, w)$-homogeneous $h \in K[x](t, \partial_t)$ with $\deg_{(-w, w)} h = d$,

$$\psi(h) = \begin{cases} t^d h, & \text{if } d \geq 0, \\ \partial_t^{-d} h, & \text{if } d < 0. \end{cases}$$

This is the most expensive step of the algorithm.

We are now prepared to compute the local Bernstein–Sato polynomial of $f$ at a prime ideal $P \subset K[x]$. Its correctness follows from that of its subroutine, Algorithm 3.1.

**Algorithm 3.4.** $b = \text{localBFunction}(f, P)$

**Input:** a polynomial $f \in K[x]$, a prime ideal $P \subset K[x]$.

**Output:** $b \in \mathbb{Q}[s]$, the local Bernstein–Sato polynomial of $f$ at $P$.

```plaintext
b ← bf. \{global b-function\}
for $r \in b^{-1}(0)$ do
    while $(s - r) \mid b$ do
        $b' ← b/(s - r)$.
        if exceptionalLocusB($f, b'$) $\subset P$ then
            break the while loop.
        else
            $b ← b'$.
        end if
    end while
end for
return $b$.
```

**Remark 3.5.** Algorithm 3.1 can also be used to compute the stratification of Spec $K[x]$ according to local $b$-function. Below are the key steps in this procedure.

1. Compute the global $b$-function $bf$.
2. For all roots $c \in b^{-1}(0)$ compute

   $$E_{c, i} = \text{exceptionalLocusB}(bf/(s - c)\mu_c^{-i}),$$

   where $i \geq 0$ and is at most the multiplicity $\mu_c$ of the root $c$ in $bf$.
3. The stratum of $b = \prod_{c \in b^{-1}(0)}(s - c)^{i_c}$, a divisor of $bf$, is

   $$V \left( \bigcap_{c \in b^{-1}(0), i_c > 0} E_{c, i_c - 1} \right) \setminus \left( \bigcup_{c \in b^{-1}(0)} V(E_{c, i_c}) \right).$$

This approach is similar to that in the recent work [21] of Nishiyama and Noro, which offers a more detailed treatment.

4. **Generalized Bernstein–Sato polynomials**

4.1. **Definitions.** For polynomials $f = f_1, \ldots, f_r \in K[x]$, let $f^s = \prod_{i=1}^r f_i^{s_i}$ and $Y = K^n \times K^r$ with coordinates $(x, t)$. Define an action of $D_Y = K(x, t, \partial_x, \partial_t)$ on $N_f :=$
$K[x][f^{-1}, s]f^s$ as follows: $x_i$ and $\partial x_i$, for $i = 1, \ldots, n$, act naturally and

$$t_j \cdot h(x, s_1, \ldots, s_j, \ldots, s_r)f^s = h(x, s_1, \ldots, s_j + 1, \ldots, s_r)f^s,$$

$$\partial t_j \cdot h(x, s_1, \ldots, s_j, \ldots, s_r)f^s = -s_j h(x, s_1, \ldots, s_j - 1, \ldots, s_r)f_j^{-1}f^s,$$

for $j = 1, \ldots, r$ and $h \in K[x][f^{-1}, s]$.

With $\sigma = -(\sum_{i=1}^r \partial t_i)$, the generalized Bernstein–Sato polynomial $b_{f,g}$ of $f$ at $g \in K[x]$ is the monic polynomial $b \in \mathbb{C}[s]$ of the lowest degree for which there exist $P_k \in D_X(\partial t, t_j | 1 \leq i, j \leq r)$ for $k = 1, \ldots, r$ such that

\begin{equation}
(4.1) \quad b(\sigma)gf^s = \sum_{k=1}^r P_k g f_k f^s.
\end{equation}

**Remark 4.1.** When $r = 1$, the generalized Bernstein–Sato polynomial $b_{f,1} = b_f$ is the global Bernstein–Sato polynomial of $f = f_1$ discussed in Section 2.

There is again an equivalent definition of $b_{f,g}$ by way of the $V$-filtration. To state this, let $V^* D_Y$ denote the $V$-filtration of $D_Y$ along $X$, where $V^m D_Y$ is $D_X$-generated by the set $\{t^\mu \partial t^\nu | \mu - \nu \geq m\}$. The following statement may be taken as the definition of the $V$-filtration on $K[x]$.

**Theorem 4.2.** [3, Theorem 1] For $c \in \mathbb{Q}$ and sufficiently small $\epsilon > 0$, $\mathcal{J}(f^c) = V^{c+\epsilon} K[x]$ and $V^c K[x] = \mathcal{J}(f^{c-\epsilon})$.

Consider the graph of $f$, the map $i_f : X \rightarrow Y$ defined by $i_f(x) = (x, f_1(x), \ldots, f_r(x))$. We denote the $D$-module direct image of $K[x]$ along $i_f$ by

\begin{equation}
(4.2) \quad M_f := (i_f)_+ K[x] \cong K[x] \otimes_K K<\partial t>.
\end{equation}

This module carries a $D_Y$-module structure, where the action of a vector field $\xi$ on $X$ and that of $t_j$ are given by

$$\xi(p \otimes \partial t) = \xi p \otimes \partial t_i^\nu - \sum_{i=1}^r (\xi f_i)p \otimes \partial t_i^{\nu+e_j},$$

and

$$t_j \cdot (p \otimes \partial t) = f_j p \otimes \partial t - \nu_j p \otimes \partial t^{\nu-e_j},$$

where $\partial t^\nu = \prod_{i=1}^r \partial t_i^{\nu_i}$ for $\nu = (\nu_1, \ldots, \nu_r) \in \mathbb{N}$ and $e_j$ is the element of $\mathbb{N}^r$ with $j$-th component equal to 1 and all others equal to 0.

Further, $M_f$ admits a $V$-filtration with

$$V^m M_f = \sum_{\nu \in \mathbb{N}^r} (V^{m+\|\nu\|} K[x]) \otimes \partial t^\nu.$$

For a polynomial $g \in K[x]$ so that $g \otimes 1 \in M_f$, $b_{f,g}$ is equal to the monic minimal polynomial of the action of $\sigma$ on

$$M_{f,g} := \frac{(V^0 D_Y)(g \otimes 1)}{(V^1 D_Y)(g \otimes 1)}.$$

**Remark 4.3.** There is a canonical embedding of $M_f$ into $N_f$, where $s_i$ is identified with $-\partial t_i$. In particular, for a natural number $m$, the image of $(V^m D_Y)(1 \otimes 1)$ under this embedding is contained in $(V^0 D_Y)(f)^m f^s \subseteq N_f$. 
4.2. **Algorithms.** To compute the generalized Bernstein–Sato polynomial, we define the left $D_Y$-ideal

$$I_f = \langle t_i - f_i \mid 1 \leq i \leq r \rangle + \langle \partial_{x_j} + \sum_{i=1}^{r} \frac{\partial f_i}{\partial t_j} \partial_{x_i} \mid 1 \leq j \leq n \rangle$$

that appears in the following multivariate analog of Theorem 2.1. Recall that $\sigma = -(\sum_{i=1}^{r} \partial_{t_i} t_i)$.

**Theorem 4.4.** The ideal $I_f$ is equal to $\text{Ann}_{D_Y} f^s$. Furthermore, the ideal $\text{Ann}_{D_X[s]} f^s$ equals the image of $I_f \cap D_X[\sigma]$ under the substitution $\sigma \mapsto s$.

We now provide two subroutines in our computations of Bernstein–Sato polynomials and multiplier ideals. The first finds the left side of a functional equation of the form (4.1) without an expensive elimination step. The second finds the homogenization of a $D_Y$-ideal with respect to the weight vector $(-w,w)$, where $w = (0,1) \in \mathbb{R}^n \times \mathbb{R}^r$ determines an elimination term order for $X$ in $Y$.

**Algorithm 4.5.** $b = \text{linearAlgebraTrick}(g,G)$

**Input:** generators $G$ of an ideal $I \subset D_Y$,

a polynomial $g \in K[x]$,

such that there is $b \in K[s]$ with $b(\sigma)g \in I$.

**Output:** $b$, the monic polynomial of minimal degree such that $b(\sigma)g \in I$.

$B \leftarrow \{\text{a Gröbner basis of } D_Y G\}$.

d $\leftarrow 0$.

repeat

d $\leftarrow d + 1$

until $\exists (c_0, \ldots, c_d) \in K^{d+1}$ such that $c_d = 1$ and

$$\sum_{i=0}^{d} c_i \text{NF}_B(\sigma^i g) = 0.$$ 

return $\sum_{i=0}^{d} c_i s^i$.

**Algorithm 4.6.** $G^* = \text{starIdeal}(G,w)$

**Input:** generators $G$ of an ideal $J \subset D_Y$,

a weight vector $w \in \mathbb{Z}^{n+r}$.

**Output:** $G^* \subset \text{gr}_{(-w,w)} D_Y \cong D_Y$, a set of generators of the ideal $J^*$ of $(-w,w)$-homogeneous elements of $J$.

$G^h \leftarrow$ generators $G$ homogenized w.r.t. a weight $(-w,w)$; $G^h \subset D_Y[h]$ with a homogenizing variable $h$ of weight 1.

$B \leftarrow \{\text{a Gröbner basis of } (G^h, hu - 1) \subset D_Y[h,u] \text{ w.r.t. a monomial order eliminating } \{h,u\}\}$.

return $B \cap D_Y$.

Below are two algorithms that are simplified versions of Shibuta’s algorithms for the generalized Bernstein–Sato polynomial. In the first, we use a module $D_Y[s]$, where the new variable $s$ commutes with all variables in $D_Y$. 

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**Algorithm 4.5.** $b = \text{linearAlgebraTrick}(g,G)$

**Input:** generators $G$ of an ideal $I \subset D_Y$,

a polynomial $g \in K[x]$,

such that there is $b \in K[s]$ with $b(\sigma)g \in I$.

**Output:** $b$, the monic polynomial of minimal degree such that $b(\sigma)g \in I$.

$B \leftarrow \{\text{a Gröbner basis of } D_Y G\}$.

d $\leftarrow 0$.

repeat

d $\leftarrow d + 1$

until $\exists (c_0, \ldots, c_d) \in K^{d+1}$ such that $c_d = 1$ and

$$\sum_{i=0}^{d} c_i \text{NF}_B(\sigma^i g) = 0.$$ 

return $\sum_{i=0}^{d} c_i s^i$.

**Algorithm 4.6.** $G^* = \text{starIdeal}(G,w)$

**Input:** generators $G$ of an ideal $J \subset D_Y$,

a weight vector $w \in \mathbb{Z}^{n+r}$.

**Output:** $G^* \subset \text{gr}_{(-w,w)} D_Y \cong D_Y$, a set of generators of the ideal $J^*$ of $(-w,w)$-homogeneous elements of $J$.

$G^h \leftarrow$ generators $G$ homogenized w.r.t. a weight $(-w,w)$; $G^h \subset D_Y[h]$ with a homogenizing variable $h$ of weight 1.

$B \leftarrow \{\text{a Gröbner basis of } (G^h, hu - 1) \subset D_Y[h,u] \text{ w.r.t. a monomial order eliminating } \{h,u\}\}$.

return $B \cap D_Y$. 

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Algorithm 4.7. $b_{f,g} = \text{generalB}(f, g, \text{StarIdeal})$

Input: $f = \{f_1, \ldots, f_r\} \subset K[x], g \in K[x]$.

Output: $b_{f,g}$, the generalized Bernstein–Sato polynomial of $f$ at $g$.

$G_1 \leftarrow \{t_j - f_j \mid j = 1, \ldots, r\} \cup \{\partial x_i + \sum_{j=1}^{r} \frac{\partial f_j}{\partial x_i} \partial t_j \mid i = 1, \ldots, n\}$.

$G_2 \leftarrow \text{starIdeal}(G_1, w) \cup \langle gf_i \mid 1 \leq i \leq r \rangle \cup \{s - \sigma \} \subset D_{Y}[s]$, where $w$ assigns weight 1 to all $\partial t_j$ and 0 to all $\partial x_i$.

return linearAlgebraTrick($G_2$).

Algorithm 4.8. $b_{f,g} = \text{generalB}(f, g, \text{InitialIdeal})$

Input: $f = \{f_1, \ldots, f_r\} \subset K[x], g \in K[x]$.

Output: $b_{f,g}$, the generalized Bernstein–Sato polynomial of $f$ at $g$.

$G_1 \leftarrow \{t_j - f_j \mid j = 1, \ldots, r\} \cup \{\partial x_i + \sum_{j=1}^{r} \frac{\partial f_j}{\partial x_i} \partial t_j \mid i = 1, \ldots, n\}$.

$G_2 \leftarrow G_1 \cap D_{Y} \cdot g$.

$G_3 \leftarrow \text{generators of in}_{(-w,w)}(G_2)$, where $w$ assigns weight 1 to all $\partial t_j$ and 0 to all $\partial x_i$.

return linearAlgebraTrick($G_3$).

Their correctness follows from [32] Theorems 3.4 and 3.5].

Remark 4.9. According to the experiments in [15], a modification of Algorithm 4.6 that uses elimination involving one less additional variable exhibits better performance. Our current implementation does not take advantage of this.

4.3. Applications. The study of the generalized Bernstein–Sato polynomial in [6] yields several applications of our algorithms, which we mention here. Each has been implemented in Macaulay2.

We begin with a result that shows that comparison with the roots of $b_{f,g}(s)$ provides a membership test for $\mathcal{J}(f^c)$ for any positive rational number $c$.

Proposition 4.10. [6, Corollary 2] Let $g \in K[x]$ and fix a positive rational number $c$. Then $g \in \mathcal{J}(f^c)$ if and only if $c$ is strictly less than all roots of $b_{f,g}(-s)$.

When $f$ defines a complete intersection, Algorithms 4.7 and 4.8 provide tests to determine if $Z$ has at most rational singularities.

Theorem 4.11. [6, Theorem 4] Suppose that $Z$ is a complete intersection of codimension $r$ in $Y$ defined by $f = f_1, \ldots, f_r$. Then $Z$ has at most rational singularities if and only if $\text{lct}(f) = r$ and has multiplicity one as a root of $b_{f,-}(s)$.

To compute a local version of the generalized Bernstein–Sato polynomial, we need the following analog of Theorem 2.2

Theorem 4.12. Let $b(x, s)$ be a nonzero polynomial in $K[x, s]$. Then the polynomial $b(x, \sigma) \in \text{in}_{(-w,w)}I_f \cap K[x, \sigma]$ if and only if there exist $Q_k \in D[s]$ s.t. $\sum_{k=1}^{r} Q_k f_k f^s = b(x, s) f^s$.

Proof. This follows by the same argument as that of Theorem 2.2.

Remark 4.13. In light of Theorem 4.12, the strategy in Section 3 yields a computation of the local version of the generalized Bernstein–Sato polynomial. The only significant
difference comes from the lack of an analogue to the map $\psi$ of Remark 3.3. However, it is still possible to compute $\text{in}_{(-w,w)}I_f \cap K[x, \sigma]$ by adjoining one more variable $s$ to the algebra and $s - \sigma$ to the ideal and eliminating $t$ and $\partial_t$. In case of the hypersurface this is a more expensive strategy than the one described in Remark 3.3.

5. Multiplier ideals via $m$-generalized Bernstein–Sato polynomials

For this section, we retain the notation of Section 4 and discuss Shibuta’s $m$-generalized Bernstein–Sato polynomials. These are defined using the $V$-filtration of $D_Y$ along $X$, but they also possess an equational definition. In contrast to the generalized Bernstein–Sato polynomials of Section 4, this generalization allows us to simultaneously consider families of polynomials $K[x]$, yielding a method to compute multiplier ideals.

**Definition 5.1.** Let $K$ of polynomials $K[x, \sigma]$ by adjoining one more variable $s$ to the algebra and $s - \sigma$ to the ideal and eliminating $t$ and $\partial_t$. In case of the hypersurface this is a more expensive strategy than the one described in Remark 3.3.

For this section, we retain the notation of Section 4 and discuss Shibuta’s $m$-generalized Bernstein–Sato polynomials. These are defined using the $V$-filtration of $D_Y$ along $X$, but they also possess an equational definition. In contrast to the generalized Bernstein–Sato polynomials of Section 4, this generalization allows us to simultaneously consider families of polynomials $K[x]$, yielding a method to compute multiplier ideals.

**Definition 5.1.** Let $\overline{M_f}^{(m)} := (V^0 D_Y)\delta/(V^m D_Y)\delta$ with $\delta = 1 \otimes 1 \in M_f \cong K[x] \otimes_K K(\partial_t)$. Define the $m$-generalized Bernstein–Sato polynomial $b_{f,g}^{(m)}$ to be the monic minimal polynomial of the action of $\sigma := -(\sum_{i=1}^r \partial_i t_i)$ on

$$\overline{M_f}^{(m)} := (V^0 D_Y)g \otimes 1 \subseteq \overline{M_f}^{(m)}.$$

**Remark 5.2.** Since $M_f$ is $V$-filtered, the polynomial $b_{f,g}^{(m)}$ is nonzero and its roots are rational.

**Proposition 5.3.** The $m$-generalized Bernstein–Sato polynomial $b_{f,g}^{(m)}$ is equal to the monic polynomial $b(s)$ of minimal degree in $K[s]$ such that there exist $P_k \in D_X(-\partial_i t_j \mid 1 \leq i, j \leq r)$ and $h_k \in \mathfrak{f}^m$ such that in $N_f$ there is an equality

$$b(\sigma)gf^s = \sum_{k=1}^r P_k h_k f^s.$$  

(5.1)

**Proof.** By the embedding in Remark 3.3, the existence of such an equation is equivalent to the existence of $Q_k \in D_X(-\partial_i t_j \mid 1 \leq i, j \leq r)$ and $\mu(k) \in \mathbb{N}^r$ with $|\mu(k)| \geq m$ such that in $M_f$, $b(\sigma) \cdot (g \otimes 1) = \sum_{k=1}^r Q_k t^{\mu(k)} \cdot (1 \otimes 1).$  

**Remark 5.4.** Since $(V^0 D_Y)g \otimes 1 \subseteq \overline{M_f}^{(1)}$ is a quotient of $\overline{M_f,g}$, the generalized Bernstein–Sato polynomial $b_{f,g}$ is a multiple of the $m$-generalized Bernstein–Sato polynomial $b_{f,g}^{(1)}$. When $g$ is a unit, the equality $b_{f,g} = b_{f,g}^{(1)}$ holds, as seen easily by comparing (4.11) and (5.1). However, this equality does not hold in general.

**Example 5.5.** When $n = 3$ and $f = \sum_{i=1}^3 x_i^2$, we have

$$b_{f,x_1}(s) = (s + 1)(s + 5/2) \quad \text{and} \quad b_{f,x_1}^{(1)}(s) = s + 1.$$  

In particular, $b_{f,x_1}^{(1)}$ strictly divides $b_{f,x_1}$.

Proposition 5.3 translates into the following algorithm.

**Algorithm 5.6.** $b_{f,g}^{(m)} = \text{generalB}(f, g, m)$
**Corollary 5.8.** For any positive integer $m$, the minimal root of $b_f^{(m)}(-s)$ is equal to the log-canonical threshold $\text{lct}(f)$ of $\langle f \rangle \subseteq K[x]$. Further, the jumping coefficients of $\langle f \rangle$ within the interval $[\text{lct}(f), \text{lct}(f) + m]$ are all roots of $b_f^{(m)}(-s)$.

**5.2. Computing multiplier ideals.** Here we present an algorithm to compute multiplier ideals that simplifies the method of Shibuta [32]. In particular, significant improvement is achieved bypassing the primary decomposition computations required by Shibuta’s method.

For a positive integer $m$, define the $K[x, \sigma]$-ideal

$$J_f(m) = (I_f^* + D_Y \cdot \langle f \rangle^m) \cap K[x, \sigma],$$

where $I_f^* \subseteq D_Y$ is the ideal of the $(-w, w)$-homogeneous elements of $I_f$. This ideal is closely related to the $m$-generalized Bernstein–Sato polynomials.
Lemma 5.9. For $g \in K[x]$, the $m$-generalized Bernstein–Sato polynomial $b_{f,g}^{(m)}$ is equal to the monic polynomial $b(s) \in K[s]$ of minimal degree such that
\[
\langle b(\sigma) \rangle = (J_f(m) : g) \cap K[\sigma].
\]

Proof. By (5.1), $b_{f,g}^{(m)}$ is the monic polynomial $b(s) \in K[s]$ of minimal degree such that
\[
b(\sigma)g \in I_f + D_X(-\partial_i t_j \mid 1 \leq i, j \leq r) \cdot \langle f \rangle^m.
\]
Since $b(\sigma)g$ is $(-w,w)$-homogeneous, we obtain (5.3). □

Theorem 5.10. Let $J_f(m) = \bigcap_{i=1}^l q_i$ be a primary decomposition with $q_i \cap K[\sigma] = (\sigma + c(i))^\kappa(i)$ for some positive integer $\kappa(i)$. Then for $c < \text{lct}(f) + m$,
\[
J(f^c) = \bigcap_{j:c(j) \geq c} (q_j \cap K[x]).
\]

Proof. We see from (5.1) that $b_{f,g}^{(m)}(s)$ is the monic polynomial $b(s)$ of minimal degree such that there exist some $P_k \in D_X(-\partial_i t_j \mid 1 \leq i, j \leq r)$ and $h_k \in \langle f \rangle^m$ such that $(b(\sigma)g - \sum_k P_k h_k) \in I_f$. Equivalently,
\[
b(\sigma)g \in \big( I_f + D_Y \cdot \langle f \rangle^m \big) \cap K[x, \sigma].
\]
The theorem now follows from Lemma 5.9 □

The following is based on methodology used in the computation of the local $b$-function and, in particular, employs Algorithm 32. Its correctness follows immediately from Theorem 5.10 and the results of Section 3.

Algorithm 5.11. $J(f^c) = \text{multiplierIdeal}(f, c)$

Input: $f = \{f_1, \ldots, f_r\} \subset K[x]$, $c \in \mathbb{Q}$.

Output: $J(f^c)$, the multiplier ideal of $f$ with coefficient $c$.

G_1 \leftarrow \{t_j - f_j \mid j = 1, \ldots, r\} \cup \{\partial_{x_i} + \sum_{j=1}^r \frac{\partial f}{\partial x_i} \partial_{t_j} \mid i = 1, \ldots, n\}.

m \leftarrow \lceil c - \text{lct}(f) \rceil.

\textbf{if} $c - \text{lct}(f)$ is integer and $\geq 1$ \textbf{then}

$m \leftarrow m + 1$

\textbf{end if}

G_2 \leftarrow \text{starIdeal}(G_1, w) \cup \{f^\alpha \mid \alpha \in \mathbb{N}^r, |\alpha| = m\} \cup \{s - \sigma\} \subset D_Y[s]$, where $w$ assigns weight 1 to all $\partial t_j$ and 0 to all $\partial x_i$.

B \leftarrow \{a \text{ a Gröbner basis of } G_2 \text{ w.r.t. an order eliminating } \{\partial_x, t, \partial_t\}\} \cap K[x, s].

b \leftarrow \text{generalB}(f, 1, m). \{ \text{The computation of } b_{f,1}^{(m)} \text{ may make use of } B. \}

b' \leftarrow \text{product of factors } (s - c')^{\alpha(c')} \text{ of } b \text{ over all roots } c' \text{ of } b_{f,1}^{(m)} \text{ such that } -c' > c, \text{ where } \alpha(c') \text{ equals the multiplicity of the root } c'.

\textbf{return} \text{ exceptionalLocusCore}(B, b').

As noted in 32, Remark 4.6.ii], $J(f^c) = \langle f \rangle \cdot J(f^{c-1})$ when $c$ is at least equal to the analytic spread $\lambda(f)$ of $f$. (The analytic spread of $\langle f \rangle$ is the least number of generators for an ideal $I$ such that $\langle f \rangle$ is integral over $I$.) Hence, to find generators for any multiplier ideal of $f$, it is enough to compute $J_f(m)$ for one $m \geq \lambda(f) - \text{lct}(f)$. 
When it is known that the multiplier ideal \( J(f^c) \) is 0-dimensional, it is possible to bypass the elimination step (line 7 of Algorithm 5.11) in the following fashion. For a fixed monomial ordering \( \geq \) on \( K[x] \), we know that there are finitely many standard monomials (monomials not in the initial ideal \( \text{in}_\geq J(f^c) \)). Let \( b' \in \mathbb{Q}[s] \) be the polynomial produced by lines \( 8 \) and \( 9 \) of the above algorithm. A basis for the \( K \)-linear relations amongst \( \{ x^\alpha b'(\sigma) \mid |\alpha| \leq d \} \) modulo \( J_f(m) \) gives a basis \( P_d \) for the \( K \)-space of polynomials in \( J(f^c) \) up to degree \( d \). By starting with \( d = 0 \) and incrementing \( d \) until all monomials of degree \( d \) belong to \( \text{in}_\geq \langle P_{d-1} \rangle \), we obtain \( \langle P_d \rangle = J(f^c) \) upon termination.

**Algorithm 5.12.** \( J(f^c) = \text{multiplierIdealLA}(f, c, d_{max}) \)

**Input:** \( f = \{ f_1, \ldots, f_r \} \subset K[x], \ c \in \mathbb{Q}, \ d_{max} \in \mathbb{N} \).

**Output:** the multiplier ideal \( J(f^c) \subset K[x] \), when it is generated in degrees at most \( d_{max} \).

\[
G_1 \leftarrow \{ t_j - f_j \mid j = 1, \ldots, r \} \cup \{ \partial_{x_i} + \sum_{j=1}^r \frac{\partial f_j}{\partial x_i} \partial_{t_j} \mid i = 1, \ldots, n \}.
\]

\[
m \leftarrow \lceil \max\{ c - \text{lct}(f), 1 \} \rceil.
\]

*if* \( c - \text{lct}(f) \) is an integer and \( \geq 1 \) *then*

\[
m \leftarrow m + 1
\]

*end if*

\[
G_2 \leftarrow \text{starIdeal}(G_1, w) \cup \{ f^\alpha \mid \alpha \in \mathbb{N}^r, |\alpha| = m \} \subset D_Y, \text{ where } w \text{ assigns weight 1 to all } \partial_{t_j} \text{ and 0 to all } \partial_{x_i}.
\]

\[
B \leftarrow \{ \text{ a Gröbner basis of } G_2 \text{ w.r.t. any monomial order } \}.
\]

\[
b \leftarrow \text{generalB}(f, 1, m).
\]

\[
b' \leftarrow \text{product of factors } (s - c')^{\alpha(c')} \text{ of } b \text{ over all roots } c' \text{ of } b_j^{(m)} \text{ such that } -c' > c, \text{ where } \alpha(c') \text{ equals the multiplicity of the root } c'.
\]

\[
d \leftarrow -1; \ P \leftarrow \emptyset \subset K[x] \text{ with } \geq \text{ that respects degree}.
\]

*while* \( P = \emptyset \) or \( (\text{in}_\geq \langle P \rangle \text{ does not contain all monomials of degree } d \text{ and } d < d_{max}) \) *do*

\[
d \leftarrow d + 1,
\]

\[
A \leftarrow \{ \alpha \mid |\alpha| \leq d, \ x^\alpha \notin \text{in}_\geq \langle P \rangle \}.
\]

*Find a basis \( Q \) for the \( K \)-syzygies \( (q_{\alpha})_{\alpha \in A} \) such that*

\[
\sum_{\alpha \in A} q_{\alpha} \text{NF}_B(x^\alpha b(\sigma)) = 0.
\]

\[
P \leftarrow P \cup \{ \sum_{\alpha \in A} q_{\alpha} x^\alpha \mid (q_{\alpha}) \in Q \}.
\]

*end while*

*return* \( \langle P \rangle \).

Notice that with \( d_{max} = \infty \) the algorithm terminates in case \( \text{dim } J(f^c) = 0 \). It also can be used to provide a \( K \)-basis of the up-to-degree-\( d_{max} \) part of an ideal of any dimension.
6. Examples

We have tested our implementation on the problems in [32]. In addition, this section provides examples from other sources with the theoretically known Bernstein–Sato polynomials, log-canonical thresholds, jumping numbers, and/or multiplier ideals; below is the output of our algorithms on several of them.

The authors would like to thank Zach Teitler for suggesting interesting examples, some of which are beyond the reach of our current implementation. We also thank Takafumi Shibuta for sharing his script (written in risa/asir [23]), which is the only other existing software for computing multiplier ideals.

A note on how to access Macaulay2 scripts generating examples, including the ones in this paper and some unsolved challenges, is posted at [3] along with other useful links.

Example 6.1. When \( f = x^5 + y^4 + x^3y^2 \), Saito observed that not all roots of \( b_f(-s) \) are jumping coefficients [27, Example 4.10]. The roots of \( b_f(-s) \) within the interval \((0, 1] \) are

\[
\frac{9}{20}, \frac{11}{20}, \frac{13}{20}, \frac{7}{10}, \frac{17}{20}, \frac{9}{10}, \frac{19}{20}, 1.
\]

However, \( \frac{11}{20} \) is not a jumping coefficient of \( f \). This can be seen in \( J_f(1) \) from Theorem 5.10, which has, among others, the primary components \( \langle s + \frac{9}{20}, y, x \rangle \) and \( \langle s + \frac{11}{20}, y, x \rangle \). In fact,

\[
J(f^c) = \begin{cases} 
\mathbb{C}[x, y] & \text{if } 0 \leq c < \frac{9}{20}, \\
\langle x, y \rangle & \text{if } \frac{9}{20} \leq c < \frac{13}{20}, \\
\langle x^2, y \rangle & \text{if } \frac{13}{20} \leq c < \frac{7}{10}, \\
\langle x^2, xy, y^2 \rangle & \text{if } \frac{7}{10} \leq c < \frac{17}{20}, \\
\langle x^2, xy, y^2 \rangle & \text{if } \frac{17}{20} \leq c < \frac{19}{20}, \\
\langle x^2, x^2y, y^2 \rangle & \text{if } \frac{19}{20} \leq c < 1,
\end{cases}
\]

and \( J(f^c) = \langle f \rangle \cdot J(f^{c-1}) \) for all \( c \geq 1 \).

Example 6.2. We compute Bernstein–Sato polynomials to verify examples corresponding to [34, Example 7.1]. The \( \mathbb{C}[x, y, z] \)-ideal

\[
\langle f \rangle = \langle x - z, y - z \rangle \cap \langle 3x - z, y - 2z \rangle \cap \langle 5y - x, z \rangle
\]

defining three non-collinear points in \( \mathbb{P}^2 \) has

\[
b_f(s) = (s + \frac{3}{2})(s + 2)^2.
\]

In particular, its log canonical threshold is \( \frac{3}{2} \). The multiplier ideals in this case are

\[
J(f^c) = \begin{cases} 
\mathbb{C}[x, y, z] & \text{if } 0 \leq c < \frac{3}{2}, \\
\langle x, y, z \rangle & \text{if } \frac{3}{2} \leq c < 2,
\end{cases}
\]

and \( J(f^c) = \langle f \rangle \cdot J(f^{c-1}) \) for all \( c \geq 2 \). On the other hand, the \( \mathbb{C}[x, y, z] \)-ideal

\[
\langle g \rangle = \langle y, z \rangle \cap \langle x - 2z, y - z \rangle \cap \langle 2x - 3z, y - z \rangle
\]
defines three collinear points in \( \mathbb{P}^2 \). Since
\[
   b_g(s) = (s + \frac{5}{3})(s + 2)(s + \frac{7}{3}),
\]
the log canonical threshold of \( g \) is \( \frac{5}{3} \). Here the multiplier ideals are
\[
   J(g^c) = \begin{cases} 
   \mathbb{C}[x, y, z] & \text{if } 0 \leq c < \frac{5}{3}, \\
   \langle x, y, z \rangle & \text{if } \frac{5}{3} \leq c < 2,
\end{cases}
\]
and \( J(g^c) = \langle g \rangle \cdot J(g^{c-1}) \) for all \( c \geq 2 \). Thus, as Teitler points out, although \( g \) defines a more special set than \( f \), it yields a less singular variety.

**Example 6.3.** Consider \( f = (x^2 - y^2)(x^2 - z^2)(y^2 - z^2)z \), the defining equation for a nongeneric hyperplane arrangement. Saito showed that \( \frac{5}{7} \) is a root of \( b_f(-s) \) but not a jumping coefficient \([26, 5.5]\). We verified this, obtaining the root 1 of \( b_f(-s) \) with multiplicity 3, as well the following roots of multiplicity 1 (including \( \frac{5}{7} \)):
\[
   \frac{3}{7}, \frac{4}{7}, \frac{2}{3}, \frac{5}{7}, \frac{6}{7}, \frac{8}{7}, \frac{9}{7}, \frac{4}{3}, \frac{10}{7}, \frac{11}{7}.
\]
Further,
\[
   J(f^c) = \begin{cases} 
   \mathbb{C}[x, y, z] & \text{if } 0 \leq c < \frac{3}{7}, \\
   \langle x, y, z \rangle & \text{if } \frac{3}{7} \leq c < \frac{4}{7}, \\
   \langle x, y, z \rangle^2 & \text{if } \frac{4}{7} \leq c < \frac{2}{3}, \\
   \langle z, x \rangle \cap \langle z, y \rangle \cap \langle y + z, x + z \rangle \cap \langle y + z, x - z \rangle & \text{if } \frac{2}{3} \leq c < \frac{6}{7}, \\
   \langle z, x \rangle \cap \langle z, y \rangle \cap \langle y + z, x + z \rangle \cap \langle y + z, x - z \rangle & \text{if } \frac{6}{7} \leq c < 1,
\end{cases}
\]
and \( J(f^c) = \langle f \rangle \cdot J(f^{c-1}) \) for all \( c \geq 1 \).

All examples in this section involve multiplier ideals of low dimension. In our experience, Algorithm 5.12 for a multiplier ideal of positive yet low dimension with a large value for \( d_{\text{max}} \) runs significantly faster than Algorithm 5.11. This is due to the avoidance of an expensive elimination step.

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