THE LOVÁSZ-CHERKASSKY THEOREM IN COUNTABLE GRAPHS

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Abstract. Lovász and Cherkassky discovered in the 1970s independently that if $G$ is a finite graph with a given set $T$ of terminal vertices such that $G$ is inner Eulerian, then the maximal number of edge-disjoint paths connecting distinct vertices in $T$ is

$$\sum_{t \in T} \lambda(t, T - t)$$

where $\lambda$ is the local edge-connectivity function. The optimality of a system of edge-disjoint $T$-paths in the Lovász-Cherkassky theorem is witnessed by the existence of certain cuts by Menger’s theorem. The infinite generalisation of Menger’s theorem by Aharoni and Berger (earlier known as the Erdős-Menger Conjecture) together with the characterization of infinite Eulerian graphs due to Nash-Williams makes it possible to generalize the theorem for infinite graphs in a structural way. The aim of this paper is to formulate this generalisation and prove it for countable graphs.

1. Introduction

There are several deep results and conjectures in infinite combinatorics whose restriction to finite structures is a well-known classical theorem. For example [5, 3] by Aharoni is known as Hall’s and König’s theorem when only finite graphs are considered and it is based on the results [7, 8, 9] by Aharoni, Nash-Williams and Shelah. The finite case of the Aharoni-Berger theorem [6] (earlier known as the Erdős-Menger Conjecture) is known as Menger’s theorem and the Matroid Intersection Conjecture [10] by Nash-Williams (which is only settled in the countable case [20]) extends the Matroid Intersection Theorem [15] of Edmonds.

There are several common aspects of the problems above. For example, assuming the finiteness of the involved structures simplifies the proof significantly. Indeed, the deletion of a cleverly chosen edge gives rise to an inductive argument as well as the application of an “augmenting path”. In contrast to the finite case, the deletion of a single element of an infinite set does not decrease its size, furthermore, an infinite sequence of iterative augmentations may fail to give a well-defined “limit object”. Another similarity between these statements is that they express a certain “complementary slackness” condition between suitable primal and dual objects: a matching $M$ in $G = (A, B, E)$ and a vertex-cover $C$ consisting of a single vertex from each $e \in M$; a disjoint path-system $P$ between $A$ and $B$ in $G = (V, E)$ with $A, B \subseteq V$ and an $AB$-separation $S \subseteq V$ consisting of a choice of a single vertex from each $P \in P$; a common independent set $I$ of matroids $M_0$ and $M_1$ and a bipartition $E = E_0 \sqcup E_1$ of their common edge set such that $E_i \cap I$ spans $E_i$ in $M_i$ for $i \in \{0, 1\}$. Alternative characterizations of “primal optimality” can be given through the concept of strong maximality. Let us call an element $X$ of a set family $\mathcal{X}$ strongly maximal.

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in $\mathcal{X}$ if $|Y \setminus X| \leq |X \setminus Y|$ for every $Y \in \mathcal{X}$. Note that if $\mathcal{X}$ has only finite elements, then ‘strongly maximal’ means ‘maximum size’, however, in general having maximum size is a much weaker property than strong maximality. It is known in the three problems we mentioned that the strong maximality of a matching/disjoint path system/common independent set is equivalent with the existence of a vertex-cover/separation/bipartition such that the corresponding complementary slackness conditions are satisfied.

The starting point of our investigation is the following result obtained by Lovász and Cherkassky independently in the 1970s:

**Theorem 1.1** (Lovász-Cherkassky theorem, [24, 12]). Let $G$ be a finite graph and let $T \subseteq V(G)$ such that $G$ is inner Eulerian (i.e. $d_G(v)$ is even for every $v \in V(G) \setminus T$). Then the maximal number of pairwise edge-disjoint $T$-paths is

$$\frac{1}{2} \sum_{t \in T} \lambda_G(t, T - t),$$

where $\lambda_G(t, T - t)$ stands for the maximal number of pairwise edge-disjoint paths between $t$ and $T - t$.

The literal extension of Theorem 1.1 to infinite graphs fails. Indeed, let $G = (V, E)$ be the graph we obtain from the star $K_{1,3}$ by attaching a one-way infinite path to its central vertex (see Figure 1). We define $T$ to be the set of vertices of degree one. Then we have only even degrees in $V \setminus T$ and the maximal number of edge-disjoint $T$-paths is 1 although

$$\frac{1}{2} \sum_{t \in T} \lambda(t, T - t) = \frac{3}{2}.$$

![Figure 1](image-url)

**Figure 1.** The failure of the literal infinite generalisation of the Lovász-Cherkassky theorem. Elements of $T$ are black.

The reason of this discrepancy is that after allowing $G$ to be infinite the condition “$G$ is Eulerian” (i.e. $E(G)$ can be partitioned into edge-disjoint cycles) is no longer equivalent with the property that $G$ has only even degrees. Indeed, in the two-way infinite path each degree is 2 but it is obviously not Eulerian. On the other hand, graphs with infinite degrees can be easily Eulerian. The characterization of infinite Eulerian graphs due to Nash-Williams is one of the fundamental theorems in infinite graph theory:

**Theorem 1.2** (Nash-Williams, [27, p. 235, Theorem 3]). A (possibly infinite) graph is Eulerian if and only if it does not contain an odd cut.\(^2\)

Simpler proofs for Theorem 1.2 were given by L. Soukup ([29, Theorem 5.1 ] ) and Thomassen [30] while its analogue for directed graphs (conjectured by Thomassen) was settled affirmatively in [19]. Theorem 1.2 indicates that the condition “for every $v \in V \setminus T$:

\[^1\]A $T$-path is a path connecting distinct vertices in $T$ without having internal vertex in $T$.

\[^2\]Infinite cardinals considered neither odd nor even.
d(v) is even” should be replaced by “for every \(X \subseteq V \setminus T:\ d(X)\) is not odd” in order to allow infinite graphs. Note that in finite graphs the former condition is equivalent to the statement that ‘contracting \(T\) results in an Eulerian graph’ and by Theorem 1.2 the latter condition is equivalent to the same but for graphs of any size.

The literal adaptation of the formula 
\[
\frac{1}{2} \sum_{t \in T} \lambda(t, T - t)
\]
is also not really fruitful in the presence of infinite quantities. Consider for example the graph \((\{u, v\}, E)\) where \(E\) consists of \(\aleph_0\) parallel edges between \(u\) and \(v\). Then any infinite \(P \subseteq E\), considered as a set of paths of length one, has the same size \(\aleph_0\). It demonstrates that cardinality is an overly rough measure in the presence of infinite quantities and urges us to focus on combinatorial instead of quantitative properties of an optimal path-system in Theorem 1.1.

In a finite graph a system \(P\) of edge-disjoint \(T\)-paths has 
\[
\frac{1}{2} \sum_{t \in T} \lambda(t, T - t)
\]
elements if and only if \(P\) contains \(\lambda(t, T - t)\) paths between \(t\) and \(T - t\) for each \(t \in T\). By Menger’s theorem it is equivalent to the statement that for every \(t \in T\) one can choose exactly one edge from each \(P \in P\) having \(t\) as an end-vertex in such a way that the resulting edge set \(C\) is a cut separating \(t\) from \(T - t\). Now we are ready to state our main results:

**Theorem 1.3.** Let \(G\) be a graph and let \(T \subseteq V(G)\) be countable such that there is no \(X \subseteq V(G) \setminus T\) where \(d_G(X)\) is an odd natural number. Then there exists a system \(P\) of edge-disjoint \(T\)-paths such that for every \(t \in T\): one can choose exactly one edge from each \(P \in P\) having \(t\) as an end-vertex in such a way that the resulting edge set \(C\) is a cut separating \(t\) from \(T - t\).

We also prove the following closely related theorem.

**Theorem 1.4.** Let \(G\) be a graph and let \(T \subseteq V(G)\) be countable such that there is no \(X \subseteq V(G) \setminus T\) where \(d_G(X)\) is an odd natural number. Assume that for each \(t \in T\) there is a system \(P_t\) of edge-disjoint \(T\)-paths covering all the edges incident with \(t\). Then there exists a system \(P\) of edge-disjoint \(T\)-paths covering all the edges incident with any \(t \in T\).

We strongly believe that the countability of \(T\) can be omitted in the theorems above. However, based on the experience with the similar problems mentioned earlier, we suspect that the proof is significantly harder.

Mader gave in [25] a minimax theorem about the maximal number of edge-disjoint \(T\)-paths in arbitrary (i.e. not necessarily inner Eulerian) finite graphs. It can be considered as a generalisation of Theorem 1.1. The structural and algorithmic aspects of the problem have been a subject of interest ever since (see for example [28], [23], [11] and [18]) as well the analogous theorems considering vertex-disjoint [17] and internally vertex-disjoint [25] paths.

**Conjecture 1.5.** Let \(G\) be a graph and let \(T \subseteq V(G)\). Then there exists a strongly maximal system \(P\) of edge-disjoint/vertex-disjoint/internally vertex-disjoint \(T\)-paths in \(G\).

We conjecture that the path-systems \(P\) in Conjecture 1.5 can be characterized in the way that it extends the corresponding minimax theorem to infinite graphs based on complementary slackness conditions. We discuss the details in Section 5. Before we turn to the proof of our main results in Section 4, we need to introduce some notation and
We extend the definitions above for disconnected graphs \( C \) and \( F \) and first always formulate immediately that variant even if historically other version was proved.

Furthermore, through replacing undirected edges by back and forth directed ones the splitting edges by a new vertex and blowing up vertices to a highly connected vertex sets as well as the two undirected variants which can be shown by simple techniques like or edge-disjoint paths. In all of these theorems the two directed variants are equivalent have four versions depending on if the graph is directed and if we consider vertex-disjoint.

More formally \( F \) connected component \( \delta \) say, then we write simply \( B \) if \( C = \delta(X) \) and \( v \in X \). We call \( \delta(X) \) an AB-cut if \( A \subseteq X \) and \( B \cap X = \emptyset \) or the other way around. If \( A \) and \( B \) are singletons, \( A = \{s\} \) and \( B = \{t\} \) say, then we write simply st-cut instead of \( \{s\}\{t\}\)-cut. In a connected graph \( G \), a cut \( \delta(X) \) is \( \subseteq \)-minimal if and only if the induced subgraphs \( G[X] \) and \( G[V \setminus X] \) are connected. We extend the definitions above for disconnected graphs \( G \) and cuts \( C \) living in a single connected component \( M \) by considering \( C \) as a cut in \( G[M] \). For a \( U \subseteq V \) and a family \( \mathcal{F} = \{X_u : u \in U\} \) of pairwise disjoint subsets of \( V \) with \( X_u \cap U = \{u\} \), we define the graph \( G/\mathcal{F} \) obtained from \( G \) by contracting \( X_u \) to \( u \) for \( u \in U \) and deleting the resulting loops.

More formally \( V(G/\mathcal{F}) := (V \setminus \bigcup \mathcal{F}) \cup U \), \( E(G/\mathcal{F}) := E \setminus \{e \in E : (\exists u \in U)I(e) \subseteq X_u\} \) and \( I(G/\mathcal{F})(e) := \{i_\mathcal{F}(u), i_\mathcal{F}(v)\} \) where \( I(e) = \{u,v\} \) and

\[
i_\mathcal{F}(v) = \begin{cases} v & \text{if } v \notin \bigcup \mathcal{F} \\ u & \text{if } u \in X_u. \end{cases}
\]

3. Preliminaries

Menger’s theorem and the other connectivity-related results that we recall in this section have four versions depending on if the graph is directed and if we consider vertex-disjoint or edge-disjoint paths. In all of these theorems the two directed variants are equivalent as well as the two undirected variants which can be shown by simple techniques like splitting edges by a new vertex and blowing up vertices to a highly connected vertex sets. Furthermore, through replacing undirected edges by back and forth directed ones the undirected vertex-disjoint version can be reduced to the directed one.

In this paper we deal only with undirected graphs and edge-disjoint paths so let us always formulate immediately that variant even if historically other version was proved first.

Let a connected graph \( G = (V, E) \) and distinct \( s, t \in V \) be fixed. For \( \subseteq \)-minimal st-cuts \( C \) and \( D \) we write \( C \preceq D \) if the s-side of the cut \( C \) is a subset of the s-side of \( D \). Note
that the $\subseteq$-minimal $st$-cuts with $\preceq$ form a complete lattice. For a finite $G$ the optimal (minimal-sized) $st$-cuts form a distributive sublattice (see [16]) of it. In general graphs the size of the cut is an overly rough measure for optimality. A structural infinite generalisation of the class of “optimal” $st$-cuts is provided by the Aharoni-Berger theorem:

**Theorem 3.1** (Aharoni and Berger, [6]). Let $G$ be a (possibly infinite) graph and let $s,t \in V(G)$ be distinct. Then there is a system $\mathcal{P}$ of edge-disjoint $st$-paths and an $st$-cut $C$ which is orthogonal to $\mathcal{P}$, i.e. $C$ consists of choosing exactly one edge from each path in $\mathcal{P}$.

We say that the $st$-cut $C$ in Theorem 3.1 is an Erdős-Menger $st$-cut and we let $\mathcal{C}(s,t)$ be the set of such cuts.

**Theorem 3.2** (J. [22]). $(\mathcal{C}(s,t), \preceq)$ is a complete lattice, although usually not a sublattice of all the minimal $st$-cuts.

Finally we introduce two more classes $\mathcal{C}^-(s,t)$ and $\mathcal{C}^+(s,t)$ of minimal $st$-cuts with $\mathcal{C}^-(s,t) \cap \mathcal{C}^+(s,t) = \mathcal{C}(s,t)$ and $\mathcal{C}^+(s,t) := \mathcal{C}^-(t,s)$. Let $\mathcal{C}^-(s,t)$ consist of those minimal $st$-cuts $C$ for which there is a system $\mathcal{W}$ of pairwise edge-disjoint paths starting at $s$ and having $C$ as the set of last edges (considering the paths directed away from $s$). Such a $\mathcal{W}$ is called an $st$-wave and plays an important role in the proof of Theorem 3.1. The cut defined as the last edges of a paths-system $\mathcal{W}$ is denoted by $C_{\mathcal{W}}$. If $\delta(s)$ is a wave (considering the edges as paths of length one), then we call it the trivial $st$-wave.

**Lemma 3.3** ([22, Lemma 3.8]). $(\mathcal{C}^-(s,t), \preceq)$ is a complete lattice and a sup-sublattice of all the minimal $st$-cuts. After the contraction of the $s$-side of its largest element to $s$, there is exactly one wave in the resulting system, namely the trivial one.

We call an $st$-wave $\mathcal{W}$ large if $C_{\mathcal{W}}$ is the largest element of $\mathcal{C}^-(s,t)$. Note that if there is no non-trivial $st$-wave, then $\delta(s)$ must be an Erdős-Menger $st$-cut because $\mathcal{C}(s,t) \subseteq \mathcal{C}^-(s,t) \subseteq \{\delta(s)\}$ and the left side is nonempty by Theorem [6]. This leads to the following conclusion:

**Corollary 3.4.** If there is no non-trivial $st$-wave, then there is a system $\mathcal{P}$ of edge-disjoint $st$-paths covering $\delta(s)$ and hence $\mathcal{C} = \{\delta(s)\}$.

**Theorem 3.5** (Diestel and Thomassen, [14]). Assume that $G$ is a (possibly infinite) graph, $s,t \in V(G)$ are distinct, furthermore, $\mathcal{P}$ and $\mathcal{Q}$ are systems of edge-disjoint $st$-paths. Then there exists a system $\mathcal{R}$ of edge-disjoint $st$-paths such that $\delta_{\mathcal{R}}(s) \supseteq \delta_{\mathcal{P}}(s)$ and $\delta_{\mathcal{R}}(t) \supseteq \delta_{\mathcal{Q}}(t)$.

Let $\mathcal{P}$ be a system of edge-disjoint $st$-paths and let $\mathcal{W}$ be a large $st$-wave. By contracting the $t$-side of $C_{\mathcal{W}}$ to $t$ and applying Theorem 3.5 with the $st$-paths obtained from $\mathcal{W}$ and from the initial segments of the paths in $\mathcal{P}$ we conclude:

**Corollary 3.6.** Let $\mathcal{P}$ be a system of edge-disjoint $st$-paths. Then there is a large $st$-wave $\mathcal{W}$ with $\delta_{\mathcal{W}}(s) \supseteq \delta_{\mathcal{P}}(s)$.

Finally, we will make use of the following classical lemma (see Lemma 3.3.2 and 3.3.3 in [13]):
Lemma 3.7 (Augmenting path lemma). Assume that $G$ is a (possibly infinite) graph, $s, t \in V(G)$ are distinct and $P$ is a system of edge-disjoint $st$-paths in $G$. Then either there exists an $st$-cut $C$ orthogonal to $P$ (i.e. $P$ is as in Theorem 3.1) or there is another system $Q$ of edge-disjoint $st$-paths for which $\delta_Q(s) \supset \delta_P(s)$ with $|\delta_Q(s) \setminus \delta_P(s)| = 1$ and $\delta_Q(t) \supset \delta_P(t)$ with $|\delta_Q(t) \setminus \delta_P(t)| = 1$.

All the definitions and results in the section remain valid (but might sound less natural) if $s$ and $t$ are not vertices but disjoint vertex sets.

4. The proof of the main result

We start by giving a short outline of the proof. In the first two subsections we apply relatively simple techniques in order to reduce Theorem 1.3 to Theorem 1.4 and cut the latter problem into countable sub-problems. The third subsection is devoted to the proof of the reduced problem, namely the countable case of Theorem 1.4. The core of that proof is to show that for every given $e \in \bigcup_{t \in T} \delta(t)$ there is a path $P$ through $e$ such that $G - E(P)$ maintains the premise of Theorem 1.4. If $G$ is countable, then one can simply use this recursively to build the desired path-system.

Proof of Theorem 1.3. We will use only that $\{t \in T : d(t) > 1\}$ is countable instead of the countability of the whole $T$. As a first step we reduce Theorem 1.3 to the following theorem.

Theorem 4.1. Let $G$ be a graph and let $T \subseteq V(G)$ be such that $d(t) \leq 1$ for all but countably many $t \in T$ and there is no $X \subseteq V(G) \setminus T$ where $d(X)$ is an odd natural number. Assume that for each $t \in T$ there is a system $P_t$ of edge-disjoint $T$-paths covering $\delta(t)$. Then there exists a system $P$ of edge-disjoint $T$-paths covering $\bigcup_{t \in T} \delta(t)$.

For $s \in T$ we will write shortly $s$-wave instead of $s(T - s)$-wave. Recall, it is a system $W$ of pairwise edge-disjoint paths starting at $s$ such that the set $C_W$ of the last edges of the paths is a minimal cut between $s$ and $T - s$.

4.1. Elimination of waves. We will call shortly the condition about the existence of the path-system $P_t$ in Theorem 4.1 the linkability condition for $t$ (w.r.t. $G$ and $T$) and we refer to the conjunction of these for $t \in T$ as the linkability condition. First we define a process that we call wave elimination. We may assume that $G$ is connected otherwise we define the elimination process component-wise. Let $T' \subseteq T$ be given. We pick an arbitrary enumeration $T' = \{t_\xi : \xi < \kappa\}$ and define by transfinite recursion $G_\xi$ for $\xi \leq \kappa$. Let $G_0 := G$. If $G_\xi$ is already defined then let $W_{t_\xi}$ be a large $t_\xi$-wave with respect to $G_\xi$ and $T$ (exists by Lemma 3.3). We obtain $G_{\xi+1}$ by contracting the $t_\xi$-side of the cut $C_{W_{t_\xi}}$ in $G_\xi$ to $t_\xi$ (see Figure 2). If $\xi$ is a limit ordinal then we obtain $G_\xi$ by doing all the previous contractions simultaneously. The recursion is done.

The cardinal $d_{G_{\kappa}}(X)$ for $X \subseteq V(G_\kappa) \setminus T$ cannot be an odd natural number because $d_{G_{\kappa}}(X) = d_G(X)$ and $G$ was inner Eulerian w.r.t. $T$. Furthermore, Corollary 3.4 ensures that for $\xi < \kappa$ there is no non-trivial $t_\xi$-wave in $G_{\xi+1}$. Since any $t_\xi$-wave in $G_\kappa$ is corresponding to a $t_\xi$-wave in $G_{\xi+1}$, it follows that for each $t \in T'$ there is only the trivial $t$-wave in $G_\kappa$. By taking $T' := T$, this is (more than) enough to guarantee the
linkability condition at Theorem 4.1 (see Corollary 3.4). Therefore $G_\kappa$ satisfies the premise of Theorem 4.1 and hence assuming Theorem 4.1 we may conclude that there is a system $P$ of $T$-paths in $G_\kappa$ covering $\bigcup_{t \in T} \delta_{G_\kappa}(t)$. By using the waves $W_\xi$, the system $P$ can be extended to a system $Q$ of $T$-paths in $G_\kappa$ where the $t_\xi(T - t_\xi)$-cut $C_{W_\xi}$ is orthogonal to $Q_{t_\xi} := \{Q \in Q : t_\xi \in V(Q)\}$. Therefore $Q$ satisfies the requirements of Theorem 1.3.

Figure 2. The contracted vertex sets during the wave elimination

4.2. Reduction to countable graphs. In the next reduction we show that it is enough to restrict our attention to countable graphs in the proof of Theorem 4.1. First of all, we may assume without loss of generality that $T$ does not span any edges. Indeed, otherwise we consider the graph $G'$ obtained from $G$ via the deletion of those edges. Then we pick a path-system $P'$ by applying this special case of Theorem 4.1 with $G'$ and $T$. Finally, we obtain $P$ by extending $P'$ with the deleted edges as $T$-paths of length one.

By applying some basic elementary submodel-type arguments we cut $E$ into countable pieces each of them satisfying both the inner Eulerian and the linkability condition w.r.t. $T$. The contraction of $T$ to an arbitrary $t$ results in an Eulerian graph $G/T$ by Theorem 1.2 thus we can take a partition $O$ of $E(G/T) = E$ into (edge sets of) $G/T$-cycles. These are cycles and $T$-paths in $G$. Let $T' := \{t \in T : d(t) > 1\}$. For $t \in T'$ let $P_t$ be a system of $T$-paths witnessing the linkability condition for $t$ and let $\mathcal{E} := \{E(P) : (\exists t \in T') P \in P_t\}$. We define a closure operation $c$ on $2^E$ in the following way. Intuitively we want to close a set $F_0 \subseteq E$ under the property that if it shares an edge with some $O \in O$ or $E(P) \in \mathcal{E}$, then it contains it completely. Formally let $c(F_0) := \bigcup_{n \in \mathbb{N}} F_n$ where

$$F_{n+1} := F_n \cup \bigcup \{O \in O : F_n \cap O \neq \emptyset\} \cup \bigcup \{E(P) \in \mathcal{E} : F_n \cap E(P) \neq \emptyset\}.$$

We call an $F$ $c$-closed if $c(F) = F$. We claim that $c$ satisfies the following properties:

(1) The family of $c$-closed sets forms a complete Boolean algebra with respect to the usual $\cup$ and $\cap$;

(2) If $F$ is countable then so is $c(F)$;

(3) If $F$ is $c$-closed, then $(V, F, I \upharpoonright F)$ and $T$ satisfy the premise of Theorem 4.1.

Indeed, property (1) follows directly from the construction and (2) holds because of the assumption $|T'| \leq \aleph_0$ and the fact that each edge $e$ is used by at most one path in $P_t$.
for every fixed \( t \) and \( e \) is contained in a unique element of \( \mathcal{O} \). The ‘inner Eulerian’ and linkability for \( t \in T' \) in condition (3) are ensured by \( F \) not subdividing any \( O \) and \( E(P) \) respectively. Recall that \( d(t) \leq 1 \) for \( t \in T \setminus T' \) by definition. Preservation of the linkability for these \( t \) is “automatic”:

**Lemma 4.2.** If \( H \) is an inner Eulerian graph w.r.t. \( T \subseteq V(H) \), then the linkability condition holds for all \( t \in T \) with \( d(t) \leq 1 \).

**Proof.** \( E(H) \) can be partitioned into the edge sets of cycles and \( T \)-paths. If \( d(t) = 1 \), then the unique edge incident with \( t \) cannot be in a cycle so must be in a \( T \)-path. \( \square \)

In order to reduce Theorem 4.1 to countable graphs, it is enough to partition \( E \) into countable \( c \)-closed sets \( F_\xi \). Indeed, then \( G_\xi := (V, F_\xi, I \setminus F_\xi) \) is countable (apart from isolated vertices) and satisfies the premise of Theorem 4.1 with \( T \) by property (3). Hence by applying the countable case of Theorem 4.1, we can take a system \( P_\xi \) of \( T \)-paths in \( G_\xi \) covering the edges \( \bigcup_{\xi \in T} \delta c_\xi(t) \). Finally, \( \bigcup_\xi P_\xi \) is as desired.

Suppose that the pairwise disjoint countable \( c \)-closed sets \( \{F_\xi : \xi < \alpha\} \) are already defined for some ordinal \( \alpha \). Then \( E \setminus \bigcup_{\xi < \alpha} F_\xi \) is \( c \)-closed by property (1). If it is empty then we are done. Otherwise let \( F_\alpha := c(\{e\}) \) for an arbitrary \( e \in E \setminus \bigcup_{\xi < \alpha} F_\xi \), which is countable by property (2). The recursion is done.

4.3. **The proof of Theorem 4.1.** We will make use of the following simple observation.

**Observation 4.3.** The deletion of the edges of a \( T \)-path preserves the condition that there is no \( X \subseteq V \setminus T \) with \( d(X) \) odd.

The core of our proof is the repeated application of the following claim:

**Claim 4.4.** Let \( G \) be a graph and let \( T \subseteq V(G) \) be such that \( G \) is inner Eulerian w.r.t. \( T \) (i.e. there is no \( X \subseteq V(G) \setminus T \) where \( d(X) \) is an odd natural number). Assume that for each \( t \in T \) there is a system \( P_t \) of edge-disjoint \( T \)-paths covering \( \delta(t) \). Then for every \( t \in T \) and \( e \in \delta(t) \) there exists a \( T \)-path \( P \) through \( e \) such that \( G - E(P) \) satisfies the linkability condition (and remains inner Eulerian w.r.t. \( T \)).

Indeed, we only need to prove Theorem 4.1 for countable \( G \) as discussed in the previous subsection. Assuming Claim 4.4, a system of \( T \)-paths covering \( \bigcup_{t \in T} \delta(t) \) can be constructed by a straightforward recursion.

**Proof: [Proof of Claim 4.4]** First we give a proof in the special case where there is some \( s \in T \) such that \( d(t) \leq 1 \) for all \( t \in T \setminus s \). Let us fix a system \( P_s \) of edge-disjoint paths between \( s \) and \( T \setminus s \) covering \( \delta(s) \).

For \( e \in \delta(s) \), we simply take the unique \( P \in P_s \) through \( e \). By Observation 4.3, the graph \( G - E(P) \) is still inner Eulerian w.r.t. \( T \). By Lemma 4.2 it is enough to check that the linkability condition is preserved for \( s \) but it is obviously true witnessed by \( P_s \setminus \{P\} \).

Suppose now that \( e \in \delta(t) \) for a \( t \in T \setminus s \). If \( t \) is an end-vertex of some \( P \in P_s \), then we take \( P \) and argue as in the previous paragraph. If it is not the case, then either we replace \( P_s \) by another \( P'_s \) where \( t \) is an end-vertex of some \( P \in P'_s \) or choose \( P \) to be edge-disjoint from \( P_s \). To do so, let \( Q \) be an arbitrary path between \( t \) and \( T \setminus t \). If \( E(Q) \cap E(P_s) = \emptyset \),
then we take $P := Q$ and the linkability condition holds for $s$ since $P$ lives in $G - E(P)$. If $E(Q) \cap E(P) \neq \emptyset$, then let $v \in V(Q) \cap V(P)$ be the first common vertex while going along $Q$ from $t$. Let $P' \in \mathcal{P}_s$ be such that $v \in V(P')$. We get $\mathcal{P}'_s$ by replacing $P'$ in $\mathcal{P}_s$ with the path $P$ we obtain by unifying the initial segment of $P'$ from $s$ to $v$ with the initial segment of $Q$ from $t$ to $v$.

Applying this iteratively we conclude:

**Corollary 4.5.** Theorem 4.1 holds whenever there is an $s \in T$ such that $d(t) \leq 1$ for every $t \in T - s$.

**Proposition 4.6.** Assume that $G = (V, E, I)$ is an inner Eulerian graph w.r.t. $T \subseteq V$ and there is an $s \in T$ such that there is no non-trivial $s$-wave. Then for every $f, h \in E$, the linkability condition holds for $s$ in $G - f - h$.

**Proof.** We may assume without loss of generality that $G$ is connected, since only the component containing $s$ is relevant. Since deletion of edges in $\delta(s)$ makes the linkability for $s$ a weaker requirement, we can also assume that $f, h \in E \setminus \delta(s)$. If $G$ is finite and $X \subseteq V$ with $X \cap T = \{s\}$, then $d(s)$ and $d(X)$ must have the same parity because $d(v)$ is even for $v \in V - s$. This observation of Lovász led immediately to the justification of Proposition 4.6 for finite graphs. Indeed, on the one hand, $d(s) < d(X)$ if $\{s\} \subsetneq X \subseteq V \setminus (T - s)$, since $\delta(s)$ is the only Erdős-Menger $s(T - s)$-cut by assumption. On the other hand, the same parity of $d(s)$ and $d(X)$ ensures $d(s) + 2 \leq d(X)$. The proof of Proposition 4.6 for infinite graphs is more involved and we need some preparation.

For a graph $H$ and distinct $s, t \in V(H)$, we call an Erdős-Menger $st$-cut $C$ $s$-tight if there is system $\mathcal{P}$ of edge-disjoint paths in $H$ between $s$ and $t$ covering $\delta_H(s)$ and every such a path-system is orthogonal to $C$.

**Lemma 4.7.** Assume that $H$ is a graph, $s, t \in V(H)$ are distinct and there is a system $\mathcal{P}$ of edge-disjoint paths in $H$ between $s$ and $t$ covering $\delta_H(s)$ and there is an $e \in E(H) \setminus \delta_H(s)$ such that $e \in E(\mathcal{P})$ for every such path-system. Then there exists an $s$-tight Erdős-Menger $st$-cut $C$ containing $e$.

**Proof.** We may assume that $H$ is connected, since otherwise we consider the component containing $s$ and $t$. Let $\mathcal{P}$ and $e$ be as in the lemma. Then there is a unique $P_e \in \mathcal{P}$ through $e$. If $H - e$ is disconnected, then we must have $\mathcal{P} = \{P_e\}$ and the cut $C := \{e\}$ is as desired. Suppose that $H - e$ is connected. Let $D$ be the $\preceq$-smallest Erdős-Menger $st$-cut in $H - e$ (see Lemma 3.2). We are going to prove that $C := D + e$ is as desired. To do so, it is enough to show that $Q := \mathcal{P} \setminus \{P_e\}$ is orthogonal to $D$. Indeed, if this holds, then $e$ must connect the two parts of the cut $D$ in $H - e$ and therefore $D + e$ is an $st$-cut in $H$ and $\mathcal{P}$ is orthogonal to it.

Suppose for a contradiction that $Q$ is not orthogonal to $D$. 
Let $H'$ be the graph we get by contracting the $t$-side of $D$ to $t$ in $H$. Then $D = \delta_{H' - e}(t)$ and it is the only element of $\mathcal{C}_{H' - e}(s, t)$ since it is the smallest but also the largest one. We apply the Augmenting path lemma 3.7 in $H' - e$ with $s, t$ and the set $Q'$ of $st$-paths in $H' - e$ given by the initial segments of the paths in $Q$. The augmentation must be successful, since otherwise it would give a $D' \in \mathcal{C}_{H' - e}(s, t)$ with $D' \neq D$. Indeed, $D \setminus E(Q') \neq \emptyset$ by the indirect assumption but $D' \subseteq E(Q')$ according to Lemma 3.7. The successful augmentation provides a system $Q''$ of edge-disjoint $st$-paths in $H' - e$ covering $\delta_{H' - e}(s)$. Indeed, there is a unique $e' \in \delta_{H' - e}(s)$ which is uncovered by $Q'$, namely the first edge of $P_e$, but the Augmenting path lemma 3.7 ensures $\delta_{Q''}(s) \subset \delta_{Q''}(s)$. Since $D \in \mathcal{C}_{H - e}(s, t)$, the paths in $Q''$ can be forward extended in $H$ to obtain a system of edge-disjoint $st$-paths in $H - e$ covering $\delta_{H}(s)$ contradicting the obligatory usage of $e$ in the assumption of the lemma. \hfill $\square$

Since the only Erdős-Menger $s(T - s)$-cut is $\delta(s)$ (see Corollary 3.4) and $f \notin \delta(s)$, Lemma 4.7 (applied in $G/(T - s)$) ensures that there is a system $P_s$ of edge-disjoint paths in $G - f$ between $s$ and $T - s$ covering $\delta(s)$. Suppose for a contradiction that such a path-system cannot be found in $G - f - h$. By applying Lemma 4.7 again this time with $G - f$ and $h$, we obtain an $s$-tight Erdős-Menger $s(T - s)$-cut $C$ in $G - f$ containing $h$. Let $S$ be the $s$-side of the cut $C$. Then $\delta_{G - f - h}(S) = C$ and we must have $f \in \delta_G(S)$ since otherwise the initial segments of the paths in $P_s$ up to their unique edge in $C$ would form a non-trivial $s$-wave with respect to $G$ and $T$. Thus $f, h \in \delta(s) \setminus \delta(s)$. We define $G'$ by extending $G[S]$ with new vertices $\{t_e : e \in \delta_G(S)\}$ and with the edges $\delta(S)$ where an $e \in \delta(S)$ keeps its original end-vertex in $S$ and gets $t_e$ as the other end-vertex. Let $T' := \{s\} \cup \{t_e : e \in \delta_G(S)\}$.
For $X \subseteq V(G') \setminus T'$, the cardinal $d_{G'}(X)$ cannot be an odd number because $d_{G'}(X) = d_{G}(X)$ by construction. Moreover, the linkability condition with respect to $G'$ and $T'$ holds, since for $s$ it is witnessed by the initial segments of the paths in $P_s$ while the connectivity of $G'$ guarantees it for the vertices in $T' - s$. Thus the premise of Theorem 4.1 are satisfied, furthermore, every vertex in $T'$ except possibly $s$ has degree 1. By applying Corollary 4.5 to $G'$ and $T'$, we can take a system $Q$ of $T'$-paths in $G'$ covering all the edges $\cup_{e \in T'} \delta_{G'}(e)$.

It cannot happen that all $Q \in Q$ have $s$ as an end-vertex because then $Q$ would provide a non-trivial s-wave with respect to $G$ and $T$ (where $f \notin \delta(s)$ is used to ensure ‘non-trivial’). Let $Q_{s,s} \in Q$ be a path with $f \notin V(Q_{s,s})$ and let us denote the end-vertices of $Q_{s,s}$ by $t_{e_0}$ and $t_{e_1}$.

**Lemma 4.8.** Every system $R'$ of edge-disjoint $T'$-paths in $G'$ covering $\delta(s)$ and avoiding $t_f$ must use all the vertices $\{t_e : e \in C\}$.

**Proof.** Suppose for a contradiction that $R'$ is a counterexample. We may assume that each path in $R'$ starts at $s$ since otherwise we remove the rest. Let $R$ be the path-system in $G - f$ corresponding to $R'$. Then $R$ is a system of edge-disjoint paths starting at $s$ and having exactly their last edges in $C$ such that $C \setminus E(R) \neq \emptyset$. Since $C$ is an Erdős-Menger $s(T-s)$-cut in $G - f$, the paths in $R$ can be forward extended to obtain a system $R^+$ of edge-disjoint $s(T - s)$-paths with $C \cap E(R^+) = C \cap E(R)$. Then $R^+$ also covers $\delta(s)$ and $C \setminus E(R^+) \neq \emptyset$ contradicting the $s$-tightness of $C$ in $G - f$. □

There must be a $Q_f \in Q$ with $t_f \in V(Q_f)$ since otherwise $R' = Q - Q_{s,s}$ contradicts Lemma 4.8. We claim that the other end-vertex of $Q_f$ must be $s$, thus in particular $Q_f \neq Q_{s,s}$ and hence $f \notin \{e_0, e_1\}$. Indeed, since otherwise the system $Q_s := \{Q \in Q : s \in V(Q)\}$ of edge-disjoint $T'$-paths in $G'$ covers $\delta(s)$ using neither $t_f$ nor the other end-vertex of $Q_f$ which contradicts Lemma 4.8. Now we consider the path-system $Q_s \setminus \{Q_f\}$. It covers all but one edges in $\delta(s)$ and avoids $e_0$ and $e_1$. We apply the Augmenting path lemma 3.7 in $G'$ with $Q_s \setminus \{Q_f\}, s$ and $\{t_e : e \in C\}$. If the augmentation is successful, then the resulting path-system covers $\delta(s)$ and at least one of $e_0$ and $e_1$ is still unused contradicting Lemma 4.8. Thus the Augmenting path lemma ensures that we can pick a single edge from each path in $Q_s \setminus \{Q_f\}$ such that the resulting edge set $C'$ separates $s$ and $\{t_e : e \in C\}$ in $G'$. We take the initial segments of the paths in $P_s$ until the first meeting with $C'$ and continue them forward using the terminal segments of the corresponding paths from $Q_s \setminus \{Q_f\}$ to obtain a set of $T'$-paths in $G'$ covering $\delta(s)$ without using $t_{e_0}, t_{e_1}$ and $t_f$, which contradicts Lemma 4.8. □

Now we can finish the proof of Claim 4.4. Suppose for a contradiction that $G, T, s \in T$ and $e_0 \in \delta(s)$ form a counterexample and $P_s = \{P_e : e \in \delta(s)\}$ is a system of edge-disjoint $T$-paths with $e \in E(P_s)$. We assume that $G, T, s, e_0$ and $P_s$ have been chosen to minimize $|E(P_{e_0})|$ among the possible options. We apply wave elimination with $T - s$ choosing each wave according to Corollary 3.6. Consider the resulting $G'$ and observe that $G', T, s, e_0$ must be also a counterexample (see Figure 2). For $e \in \delta(s)$, let $P'_e$ be the longest initial segment of $P_e$ from $s$ that lives in $G'$. Then we must have $P_{e_0} = P'_{e_0}$ since otherwise $P'_s = \{P'_e : e \in \delta(s)\}$ and $|E(P'_{e_0})| < |E(P_{e_0})|$ contradicts the choice of $G, T, s$ and $e_0$. 
Note that $|E(P_{e_0})| \geq 2$ because if $P_{e_0}$ consisted of the single edge $e_0$, then $P := P_{e_0}$ would satisfy Claim 4.4 for $G', s$ and $e_0$.

Let $f_0 \in E(P_{e_0})$ be the edge right after $e_0$ in $P_{e_0}$. We replace in $G'$ the edges $e_0$ and $f_0$ by a single new edge $h_0$ connecting $s$ and the end-vertex of $f_0$ that is not shared with $e_0$ (splitting technique by Lovász from [24]). Let $P_{h_0}$ be the path in the resulting graph $G''$ with $E(P_{h_0}) = E(P_{e_0}) - e_0 - f_0 + h_0$ and let us define $P'' := P' - P_{e_0} + P_{h_0}$. For $X \subseteq V \setminus T$ the quantities $d_{G'}(X)$ and $d_{G''}(X)$ are either both infinite or they have the same parity, thus $G''$ is also inner Eulerian w.r.t. $T$. The linkability condition for $s$ in $G''$ is witnessed by $P''$. Let $t \in T - s$ be arbitrary. The linkability condition for $t$ holds in $G' - e_0 - f_0$ by Proposition 4.6, moreover, if $h_0 \in \delta_{G''}(t)$, then $h_0$ is an edge between $s$ and $t$ and hence a $T$-path itself. Thus the linkability condition holds in $G''$. Note that $G''$, $T$, $s$ and $h_0$ cannot be a counterexample for Claim 4.4 because $|E(P_{h_0})| = |E(P_{e_0})| - 1$. Therefore we can pick some $T$-path $P''$ in $G''$ through $h_0$ such that the linkability condition holds in $G'' - E(P'')$.

Let us take then a $T$-path $P''$ in $G'$ through $e_0$ with $E(P') \subseteq E(P'') - h_0 + e_0 + f_0$. Since $G'' - E(P'')$ is a subgraph of $G' - E(P')$ and $\delta_{G'' - E(P'')} = \delta_{G' - E(P')} = \delta_{G'' - E(P'')}$, the linkability condition in $G'' - E(P'')$ implies the linkability in $G' - E(P')$. This contradicts the fact that $G', T, s$ and $e_0$ form a counterexample for Claim 4.4.

\[ \Box \]

5. Open questions

First of all, we expect that in Theorem 1.3 the restriction about the size of $T$ can be completely omitted:

**Conjecture 5.1.** Let $G$ be a graph and let $T \subseteq V(G)$ such that there is no $X \subseteq V(G) \setminus T$ where $d_G(X)$ is an odd natural number. Then there exists a system $\mathcal{P}$ of edge-disjoint $T$-paths such that for every $t \in T$: one can choose exactly one edge from each $P \in \mathcal{P}$ having $t$ as an end-vertex in such a way that the resulting edge set $C$ is a cut separating $t$ and $T - t$.

We conjectured already in the Introduction (Conjecture 1.5) the existence of strongly maximal systems of $T$-paths with different concepts of disjointness. We believe that strong maximality can be characterized by the existence of a certain dual object reflecting the corresponding classical theorems of Gallai [17] and Mader [25, 26].

5.1. Edge-disjoint $T$-paths in not necessarily inner Eulerian graphs. Let $G$ be a graph and let $T \subseteq V(G)$. A $T$-partition is a family $\mathcal{A} = \{X_t : t \in T\}$ of pairwise disjoint subsets of $V(G)$ such that $X_t \cap T = \{t\}$. If $G$ is finite, then we call a component $Y$ of $G - \bigcup \mathcal{A}$ obstructive if $d(Y)$ is odd. Let $o(G, \mathcal{A})$ be the number of the obstructive components.

**Theorem 5.2** (Mader, [25]). Let $G$ be a finite graph and let $T \subseteq V(G)$. Then the maximal number of pairwise edge-disjoint $T$-paths is

\[
\min \left\{ \frac{1}{2} \left( \sum_{t \in T} d(X_t) - o(G, \mathcal{A}) \right) : \mathcal{A} \text{ is a } T\text{-partition} \right\}.
\]

Let us define $E(\mathcal{A}) := \bigcup_{t \in T} \delta(X_t)$. In Theorem 5.2, for a system $\mathcal{P}$ of edge-disjoint $T$-paths and a $T$-partition $\mathcal{A}$ we have equality if and only if the following conditions hold:
**Condition 5.3** (complementary slackness).

1. Each $P \in \mathcal{P}$ uses either only a single edge from $E(A)$ (which must connect two vertex sets in $A$) or two edges incident with a component of $G - \bigcup A$.
2. For each component $Y$ of $G - \bigcup A$, the path-system $\mathcal{P}$ uses all but at most one edge from $\delta(Y)$.

**Conjecture 5.4.** Let $G$ be a (possibly infinite) graph and let $T \subseteq V(G)$. Then there exists a system $\mathcal{P}$ of edge-disjoint $T$-paths such that there is a $T$-partition $A$ satisfying Condition 5.3.

Although the Lovász-Cherkassky theorem 1.1 is a special case of Mader’s edge-disjoint $T$-path theorem 5.2, Conjecture 5.4 does not seem to imply Conjecture 5.1. This (together with the behaviour of $T$-joins, see [21, Theorem 2]) motivates to formulate a stronger conjecture based on the extension of the concept of obstructive components.

For a possibly infinite graph $G$, we define a component $Y$ of $G - \bigcup A$ to be *obstructive* if after the contraction of $V(G) \setminus Y$ to a vertex $v$ the resulting graph $H$ does not contain a set of pairwise edge-disjoint cycles covering $\delta_H(v)$. This extends our previous definition of obstructive. Indeed, on the one hand, if $d_G(Y)$ is odd, then $d_H(v)$ is the same odd number and hence $\delta_H(v)$ cannot be covered by edge-disjoint cycles. On the other hand, if $d(Y)$ is even, then finding the desired cycles is equivalent to finding a $J$-join in the connected graph $G[Y]$ where $J$ consists of those $u \in Y$ for which there are odd number of edges between $u$ and $v$ in $H$.

**Condition 5.5.**

1. Each $P \in \mathcal{P}$ uses either only a single edge from $E(A)$ (which must connect two vertex sets in $A$) or two edges incident with a component of $G - \bigcup A$.
2. The path-system $\mathcal{P}$ uses all the edges $E(A)$ except one from $\delta(Y)$ for each obstructive component $Y$.

Note that if $G$ is inner Eulerian, then there cannot be any obstructive components (regardless of the choice of $A$) and therefore by replacing Condition 5.3 with Condition 5.5 in Conjecture 5.4 it will imply Conjecture 5.1. We also point out that for finite graphs Conditions 5.3 and 5.5 are equivalent because if $d(Y)$ is even, then $P$ cannot miss exactly one edge from $\delta(Y)$.

Recall that a system $\mathcal{P}$ of edge-disjoint/vertex-disjoint/internally vertex-disjoint $T$-paths is called *strongly maximal* if $|Q \setminus \mathcal{P}| \leq |\mathcal{P} \setminus Q|$ for every edge-disjoint/vertex-disjoint/internally vertex-disjoint system $Q$ of $T$-paths.

**Conjecture 5.6.** Let $G$ be a (possibly infinite) graph and let $T \subseteq V(G)$. Then for a system $\mathcal{P}$ of edge-disjoint $T$-paths the following statements are equivalent:

(i) $\mathcal{P}$ is a strongly maximal system of edge-disjoint $T$-paths.

(ii) There exists a $T$-partition $A$ satisfying Condition 5.3 with $\mathcal{P}$.

(iii) There exists a $T$-partition $A$ satisfying Condition 5.5 with $\mathcal{P}$.

Notice that $(iii) \implies (ii) \implies (i)$. Indeed, the implication $(iii) \implies (ii)$ is trivial. Assuming $(ii)$, $\mathcal{P}$ must be an inclusion-wise maximal system of edge-disjoint $T$-paths. If
If $|P \setminus Q| = \kappa \geq \aleph_0$, then $|E(P \setminus Q)| = \kappa$ and since each $P \in Q \setminus P$ must contain an edge from $E(P \setminus Q)$, we obtain $|Q \setminus P| \leq \kappa$. If $|P \setminus Q| = k \in \mathbb{N}$, then let $G' := G - E(P \cap Q)$. Then $d_{G'}(Y)$ is finite for every component of $G' - \bigcup A$ and for all of but finitely many $Y$ it is 0, moreover,

$$\frac{1}{2} \left( \sum_{t \in T} d_{G'}(X_t) - o(G', A) \right) = k,$$

from which $|Q \setminus P| \leq k$ follows. Thus $P$ is strongly maximal. Hence for establishing Conjecture 5.6 it is sufficient to prove $(i) \implies (iii)$.

5.2. Vertex-disjoint $T$-paths. If $T = V(G)$, then a vertex-disjoint system of $T$-paths is a matching. Infinite matching theory was intensively investigated and is well-understood (see the survey [2]). The existence of a strongly maximal matching first in countable and then in arbitrary graphs was proven by Aharoni (see [1, 4]) together with the following theorem:

**Theorem 5.7** (Aharoni, [2, Theorem 5.2]). In every (possibly infinite) graph $G = (V,E)$ there is a matching $M \subseteq E$ such that there is an $X \subseteq V$ with the following properties:

1. For each component $Y$ of $G - X$, the edges in $M$ spanned by $Y$ cover all but at most one vertex of $Y$.
2. The vertices in $X$ are covered by $M$ in such a way that $X$ does not span any edge in $M$.
3. $G[Y]$ is factor-critical\(^3\) whenever $Y$ is a component of $G - X$ for which $M$ does not contain a perfect matching of $G[Y]$.
4. Let $\Pi(G,X)$ be the bipartite graph whose vertex classes are $X$ and the set $\mathcal{Y}$ of the factor-critical components of $G - X$, furthermore, $xY$ is an edge if $x$ has a neighbour in $Y$ in $G$. Then for every $Y \in \mathcal{Y}$ there is a matching in $\Pi(G,X)$ covering $X$ while avoiding vertex $Y$.

**Remark 5.8.**

- Properties (1) and (2) in Theorem 5.7 are already sufficient to ensure the strong maximality of the matching $M$.
- For every strongly maximal matching $M$ there is an $X$ satisfying (1)-(4).
- Property (4) was originally not mentioned by Aharoni but it can be obtained easily by applying for example [7, Lemma 3.6].
- If there is a matching $M$ for which $V(M)$ is $\subseteq$-maximal (which is always the case in countable graphs), then the set $X$ in Theorem 5.7 is unique.

By omitting the assumption of $T = V(G)$ we leave matching theory and formulate an infinite generalisation of Gallai’s theorem [17]:

**Conjecture 5.9.** Let $G = (V,E)$ be a (possibly infinite) graph and let $T \subseteq V$. Then there exists a system $\mathcal{P}$ of vertex-disjoint $T$-paths such that there is an $X \subseteq V$ with the following properties:

\(^3\)A graph is factor-critical if it does not admit a perfect matching but after deleting any vertex the resulting graph does.
(1) For each component $Y$ of $G - X$, the paths $\{ P \in \mathcal{P} : V(P) \subseteq Y \}$ cover all but at most one vertex of $T \cap Y$.

(2) $X \subseteq V(\mathcal{P})$ where $|V(P) \cap X| \leq 1$ for every $P \in \mathcal{P}$.

A minimax formula for the maximal number of internally vertex-disjoint $T$-paths was given by Mader in [26]. We expect the following generalisation based on the complementary slackness conditions to be true:

**Conjecture 5.10.** Let $G = (V, E)$ be a (possibly infinite) graph and let $T \subseteq V$. Then there exists a system $\mathcal{P}$ of internally vertex-disjoint $T$-paths such that there is an $X \subseteq V \setminus T$ and a partition $\mathcal{Y}$ of $V \setminus (T \cup X)$ with the following properties:

(0) After the deletion of the vertex set $X$ and the edges of the subgraphs $G[Y]$ for $Y \in \mathcal{Y}$ the resulting graph does not contain any $T$-path.

Let $B_Y := \{ v \in Y : v$ has a neighbour in $V \setminus (X \cup Y) \}$ for $Y \in \mathcal{Y}$.

(1) The paths in $\mathcal{P}$ cover $X$ and all but at most one vertex of $B_Y$ for every $Y \in \mathcal{Y}$.

(2) For every $P \in \mathcal{P}$ either $|V(P) \cap X| = 1$ and $|V(P) \cap B_Y| \leq 1$ for every $Y \in \mathcal{Y}$ or $|V(P) \cap X| = 0$ and there is a unique $Y_P \in \mathcal{Y}$ with $|V(P) \cap B_{Y_P}| = 2$ while $|V(P) \cap B_Y| \leq 1$ for $Y \in \mathcal{Y} \setminus \{Y_P\}$.

(3) For every $Y \in \mathcal{Y}$ there is at most one $P \in \mathcal{P}$ with $|V(P) \cap B_Y| = 1$.

**Conjecture 5.11.** The systems of $T$-paths described in Conjecture 5.9 (Conjecture 5.10) are exactly the strongly maximal systems of vertex-disjoint (internally vertex-disjoint) $T$-paths.

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