Higher Moments of Heavy Quark Correlators in the Low Energy Limit at $\mathcal{O}(\alpha_s^2)$

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Abstract

We present the first 30 moments of the low energy expansions of the vector, axial-vector, scalar and pseudo-scalar heavy quark correlation functions at three-loop order, including the singlet contribution which appears for the first time at three loops. In addition we compare the behavior of the moments for large $n$ with the prediction from threshold calculations.

Key words: Heavy quarks

1 Introduction

Correlators of two currents $j_{[\mu]}(x)$, where $j_{[\mu]}(x)$ is the scalar $\bar{\psi}\psi$, pseudo-scalar $i\bar{\psi}\gamma_5\psi$, vector $\bar{\psi}\gamma_{\mu}\psi$, or axial-vector $\bar{\psi}\gamma_{\mu}\gamma_{5}\psi$ current, are important for many interesting phenomenological purposes. Their low energy expansions, in particular, are of interest for applications based on sum rules [1,2] like the determination of the charm and bottom quark mass from $R(s)$ [3,4].

In three-loop approximation, i.e. $\mathcal{O}(\alpha_s^2)$, the moments of the Taylor expansion for the four correlators were evaluated up to $(q^2)^8$ in [5] using recursion algorithms originally suggested in [6]. The series was evaluated in four-loop approximation up to the first physical moment $(q^2)^1$ [7,8] employing the Laporta algorithm [9]. The three-loop singlet contributions have been calculated in [10] up to the seventh and eighth moments for the axial-vector and the scalar and pseudo-scalar correlators, respectively.

Recently, using a completely different technique based on master differential
equations [11,12], the three-loop vector correlator has been calculated up to \((q^2)^{30}\) [13].

Higher moments of the pseudo-scalar correlator might be used to determine the impact of non-perturbative effects in the determination of quark masses from lattice calculations [14].

In this work we calculate the low energy expansions for the four types of correlators in order \(\alpha_s^2\) up to \((q^2)^{30}\), including the logarithms which arise from the singlet contribution, given by double triangle diagrams.

This paper is organized as follows: In Section 2 we briefly present the methods used in the calculation, paying special attention to the singlet parts. The results are presented in Section 3 where we also compare the explicitly calculated moments with the behavior for large powers of \(q^2\) derived from the threshold region. Section 4 contains a brief summary and conclusions.

2 Definitions and Methods

The polarization functions are defined by

\[
(-q^2 g_{\mu\nu} + q_\mu q_\nu) \Pi^\delta(q^2) + q_\mu q_\nu \Pi^\delta_L(q^2) = i \int dx e^{iqx} \langle 0|Tj^\delta_\mu(x)j^\delta_\nu(0)|0 \rangle \quad \text{for} \quad \delta = v, a,
\]

\[
q^2 \Pi^\delta(q^2) = i \int dx e^{iqx} \langle 0|Tj^\delta(x)j^\delta(0)|0 \rangle \quad \text{for} \quad \delta = s, p,
\]

with the currents

\[
j^\nu_\mu = \bar{\psi} \gamma_\mu \psi, \quad j^a_\mu = \bar{\psi} \gamma_\mu \gamma_5 \psi, \quad j^s = \bar{\psi} \psi, \quad j^p = i \bar{\psi} \gamma_5 \psi.
\]

In the low energy limit the polarization functions can be written as a series in

\[
z = \frac{q^2}{4m^2}, \quad \text{where} \quad m \text{ is the mass of the heavy quark,}
\]

\[
\Pi^\delta(q^2) = \frac{3}{16\pi^2} \sum_{n>0} C^\delta_n z^n,
\]

with
\[ C_\delta^n = C_\delta^{(0),n} + \frac{\alpha_s}{\pi} C_F C_\delta^{(1),n} + \left( \frac{\alpha_s}{\pi} \right)^2 C_\delta^{(2),n} + \ldots, \]  

\[ C_\delta^{(2),n} = C_F^2 C_{\delta, A,n} + C_F C_A C_{\delta, NA,n} + C_T F_m C_{\delta, l,n} + C_F T_{F,n} C_{\delta, S,n}. \]  

The singlet contributions \( C_{\delta, S,n}^{(2)} \) arise from the diagrams shown in Fig. 2. It has to be noted that these are not simple power series in \( z \) but contain logarithms of the form \( \log(-q^2/m^2) \). This logarithmic dependence on \( q^2 \) reflects the presence of massless cuts in the diagrams.

The expansion in the limit \( q^2 \to 0 \) can be achieved in two different ways. One possibility is to first expand the propagators and later reduce the resulting massive tadpoles to master integrals. For each order in the \( z \)-expansion two more powers of the propagators arise. For large \( n \) this makes the reduction to master integrals through the reduction formalisms of Broadhurst or Laporta very cumbersome. The second possibility is the reduction of the full propagators to master integrals. These master integrals have then to be known at least as an expansion in the external momentum which can be calculated efficiently to high orders. The price to pay is the increased difficulty in obtaining the necessary reduction to master integrals. In this paper we choose the latter approach, which we present in more detail in the following.

In the first step, the diagrams are generated with QGRAF [15]. The occurring topologies are identified with the help of \( \text{q2e and exp} \) [16,17]. The integrals are reduced to scalar propagator-type integrals of the form

\[ P(q^2, a_1, \ldots, a_9) = \int \frac{[dk_1][dk_2][dk_3]}{D_1^{a_1} D_2^{a_2} \ldots D_9^{a_9}}. \]

using the computer algebra program FORM [18]. The denominators are given by \( D_i = (l - m_i^2 + i\epsilon) \), where \( l \) is a linear combination of the loop momenta \( k_i \) and the external momentum \( q \), and \( m_i \in \{0, m\} \) is the mass of the corresponding propagator. Only one non-zero mass is taken into account for the heaviest quark, lighter quarks are treated as massless. The powers \( a_i \) of the denominators are integers. Irreducible scalar products are expressed as propagators with negative powers. The integrals are dimensionally regularized in space-time dimension \( d = 4 - 2\epsilon \).

In a next step, Integration-by-Parts [19] and Laporta’s algorithm [9,20] are used to reduce all needed integrals of the form (7) to master integrals. This reduction is done with the program Crusher [21]. Crusher uses GiNaC [22] for algebraic manipulations and Fermat [23] for the simplification of the intermediate expressions using a special interface [24]. As a result of the reduction one finds a total of 55 master integrals.
The master integrals have to be expanded for small external momenta around $q^2 = 0$. To achieve this, we use the scaling equation

$$\left( q^2 \frac{\partial}{\partial q^2} + m^2 \frac{\partial}{\partial m^2} - \hat{D} \right) M_i(q^2) = 0$$

(8)

for the master integrals $M_i$. $\hat{D}$ applied to $M_i$ gives the mass dimension of $M_i$. Carrying out the mass derivative produces integrals with additional powers of propagators, that are again mapped to master integrals. This gives a system of coupled inhomogeneous linear differential equations in $q^2$. For the non-singlet part, the system is solved by a Taylor series

$$M_i(q^2) = \sum_{k=0}^{\infty} M_i^{(k)}(q^2)^k,$$

(9)

where the coefficients $M_i^{(k)}$ have to be calculated. Diagrammatically, the coefficients are tadpole diagrams. As boundary conditions, the three-loop master tadpoles depicted in Fig. 1 (a)-(c) are chosen. The $M_i^{(k)}$ can be expressed as linear combinations of these. For the singlet part, non-integer powers of $q^2$ have to be taken into account:

$$M_i(q^2) = \sum_k \left( M_i^{(k)} + M_i^{(k)}(q^2)^{-\epsilon} + M_i^{(k)}(q^2)^{-2\epsilon} \right) (q^2)^k.$$

(10)

In this case, the boundary conditions for the coefficients $M_i^{(k)}$ and $M_i^{(k)}$ are the diagrams depicted in Fig. 1 (d) and (e), (f), respectively. Generation and solution of the system are implemented in a Mathematica program. With our setup we can calculate the $q^2$ expansion to almost arbitrary depths. In this work we limit ourselves to the first 30 moments. This way of calculating expansions of integrals was proposed in [11,12] and applied to the vector current in [13].

![Fig. 1. Diagrams, which are chosen as boundary conditions for the expansion of the propagator-type master integrals in $q^2$. The diagrams (d) – (f) contribute only to the singlet part.](image)

To perform the renormalization we need the strong coupling $\alpha_s$ [25] and the mass renormalization constant at two-loop order [26,27,28].

For the pseudo-scalar and axial-vector currents we have to pay special attention to the treatment of $\gamma_5$. While in the non-singlet contributions $\gamma_5$ can
simply be considered as anticommutating this is not possible for the singlet contributions. In the latter case the $\gamma_5$ matrix cannot easily be removed from the problem and has to be treated properly to avoid problems with dimensional regularization. Therefore we follow the prescription proposed by Larin [29] and use the definition $\gamma_5 = \frac{i}{4!} \epsilon_{\mu\nu\rho\sigma} \gamma^\mu \gamma^\nu \gamma^\rho \gamma^\sigma$. Since the $\epsilon$-tensor is intrinsically a four-dimensional object it can not be used in dimensional regularization and has to be taken out of the calculation until the renormalization has been performed. Taking the $\epsilon$-tensors out of the calculation the tensor structure of the pseudo-scalar and axial-vector correlators can be cast into the form

$$
\Pi_{\mu\nu\rho\sigma}^{\rho} = \Pi_{1\mu\nu\rho\sigma}^{\rho} + \Pi_{2\mu\nu\rho\sigma}^{\rho},
$$

$$
\Pi_{\mu\nu\rho\sigma}^{a} = \Pi_{1\mu\nu\rho\sigma}^{a} + \Pi_{2\mu\nu\rho\sigma}^{a},
$$

where $[\ldots]$ denotes total antisymmetrization. After renormalization these expressions can be multiplied with the $\epsilon$-tensors to obtain the final result. The correlators are then given by

$$
\Pi^p = \Pi_1^p + \Pi_2^p,
$$

$$
\Pi^a = \Pi_1^a + \Pi_2^a,
$$

$$
\Pi_L^a = \Pi_2^a.
$$

The pseudo-scalar and the longitudinal part of the axial-vector correlator are connected through a Ward identity.

$$
q^2 \Pi^a_L(q^2) = 4m^2 (\Pi^p(q^2) - q^2 (\partial \Pi^p(q^2) / \partial q^2)|_{q^2=0}).
$$

In order to retain this Ward identity for the singlet contribution it is necessary to cancel the axial-vector anomaly. For this reason we computed the singlet part $\Pi_{\mu\nu,S}$ of the axial-vector correlator for an isospin doublet $(\psi, \chi)$, where $\psi$ is taken to be heavy and $\chi$ light. The diagrams contributing to the full singlet part are depicted in Fig. 2. Note that the completely massless diagram does only contribute to the leading moment.

### 3 Results and asymptotic behavior for large $n$

The numerical results for the first 30 moments of the various currents are listed in Appendix A. The results are given in both the $\overline{\text{MS}}$ and the onshell scheme since both schemes have their own range of applications. For convenience we set the renormalization scale $\mu = m$. Since the longitudinal part of

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1. While we agree in general with the results given in [13], we disagree with several single digits.
Fig. 2. Diagrams for the singlet contribution. The first diagram contributes to scalar, pseudo-scalar and axial-vector correlators, while the latter two are only taken into account for the axial-vector to cancel the anomaly. Solid and dashed lines denote massive and massless lines, respectively.

the axial-vector correlator can be easily obtained from the pseudo-scalar one by use of the Ward identity (16) we give only the results for the transversal part in this case.

The results for the singlet contributions to the scalar, pseudo-scalar and axial-vector correlators can be split into a constant and a logarithmic part proportional to \( \log(-\frac{q^2}{m^2}) \). These are denoted by \( C_{S,n}^{(2),\delta}[1] \) and \( C_{S,n}^{(2),\delta}[L] \), respectively. In the case of the axial-vector correlator the singlet part includes contributions from heavy-light diagrams as explained in the previous section.

The corresponding analytical expressions for moments 9-12 in the \( \overline{\text{MS}} \) scheme can be found in the Appendix B. The analytical results for all calculated moments both in the \( \overline{\text{MS}} \) and onshell scheme including all logarithms are available in computer readable form from http://www-ttp.particle.uni-karlsruhe.de/Progdata/ttp07/ttp07-32.

For large expansion depths, the moments can be compared to the asymptotic behavior, which follows from the threshold behavior of heavy quark production from \( e^+e^- \).

\[
C_{k,n} = \frac{4}{9} \int_{4m^2}^{\infty} \frac{ds}{s} R_k(s) \left( \frac{4m^2}{s} \right)^n = \frac{4}{9} \int_{0}^{1} d(\beta^2) R_k(\beta) (1 - \beta^2)^{n-1}, \tag{17}
\]

where

\[
k = A, NA, l, F \quad \text{and} \quad \beta = \sqrt{1 - \frac{4m^2}{s}}. \tag{18}
\]

In view of the smooth behavior of the calculated moments, we expect good agreement for \( n \sim 30 \). Large \( n \) expansions can be found in ref. [30] and derived from [31,5,32] by using eq. (17). A power \( \beta^j \) of the velocity in the \( R \)-ratio \( R_k(\beta) \) corresponds to a power \( n^{-1-j/2} \) of the large \( n \) expansion.
\[ C_{A,n}^{(2),v} = \frac{\pi^{9/2}}{6} n^{-1/2} - 4 \pi^2 n^{-1} + \frac{\sqrt{\pi}}{144} \left( 23\pi^4 + 8\pi^2 (6H_n + \frac{1}{2}) - 47 + 36 \log(2) \right) n^{-3/2} + O(n^{-2}), \]
\[ C_{N_A,n}^{(2),v} = \frac{\pi^2}{36} \left( 33H_n + 31 - 66 \log(2) \right) n^{-1} + \frac{\sqrt{\pi}}{72} \left( \pi^2 (36H_n + \frac{1}{2}) + 107 - 120 \log(2) \right) n^{-3/2} + O(n^{-2}), \]
\[ C_{l,n}^{(2),v} = -\frac{\pi^2}{9} (3H_n + 5 - 6 \log(2)) n^{-1} + \frac{11\sqrt{\pi}}{9} n^{-3/2} + O(n^{-2}), \]
\[ C_{F,n}^{(2),v} = -4\frac{\sqrt{\pi}}{9} (\pi^2 - 11) n^{-3/2} + \frac{\sqrt{\pi}}{54} (33\pi^2 - 344) n^{-5/2} + O(n^{-7/2}), \]

where \( H_n \) is the harmonic number given by

\[ H_n = \sum_{i=1}^{n} \frac{1}{i} \]  

for integer arguments or its generalization

\[ H_n = \gamma_E + \psi_0(n + 1) \]  

for non-integer arguments, where \( \gamma_E \) is Euler’s constant and \( \psi_0 \) is the digamma function. The asymptotic behavior is in both cases given by

\[ H_n \approx \gamma_E + \log(n) + O(n^{-1}) . \]  

The comparison is shown for the vector current separately for each color factor in Fig. 3-6. The behavior of the leading order approximation is worst for the \( C_{A,n}^{(2),v} \) contribution since the series starts with \( \mathcal{O}(n^{-1/2}) \) compared with at least \( \mathcal{O}(n^{-1}) \) for the other color factors. Comparing identical orders in the \( n \) expansion of the various color structures shows roughly the same degree of convergence, but the \( C_{A,n}^{(2),v} \) is still penalized by the bad leading approximation. Overall we find good agreement between the large \( n \) expansion and the analytical moments. Also at two-loops this procedure shows nearly perfect agreement if one includes two terms in the large \( n \) expansion as can be seen in Fig. 7.

4 Conclusion

We calculated the low-energy expansion of the heavy quark correlator for the scalar, pseudo-scalar, vector and axial-vector current up to \( \left( \frac{q^2}{m_h^2} \right)^{30} \) in three-loop approximation. For the scalar, pseudo-scalar and axial-vector currents
Fig. 3. Analytical moments compared to the large $n$ expansion to different depths for the $C_{A,n}^{(2,v)}$ part.

Fig. 4. Analytical moments compared to the large $n$ expansion to different depths for the $C_{NA,n}^{(2,v)}$ part.
Fig. 5. Analytical moments compared to the large $n$ expansion to different depths for the $C_{l,n}^{(2),v}$ part.

Fig. 6. Analytical moments compared to the large $n$ expansion to different depths for the $C_{F,n}^{(2),v}$ part.
we included the singlet contributions arising from double triangle diagrams. Furthermore, for the vector current we compared the large \( n \) behavior of the moments with the asymptotic form derived from threshold behavior. Taking the lowest two or three terms of the \( 1/n \) expansion into account, we find good agreement between the asymptotic formulae and the explicit results.

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Fig. 7. Analytical moments compared to the large \( n \) expansion to different depths for the \( C_n^{(1),\nu} \) part.
A Numerical results

| n  | $C_n^{(0),v}$ | $C_n^{(1),v}$ | $C_n^{(2),v}$ | $C_{A,n}^{(2),v}$ | $C_{N,n}^{(2),v}$ | $C_{F,n}^{(2),v}$ |
|----|----------------|----------------|----------------|-------------------|-------------------|-------------------|
| 1  | 1.067          | 1.916          | 0.1541         | -0.006971         | 0.9934            | 0.3956            |
| 2  | 0.4571         | 0.8322         | 0.3025         | 0.2215            | 0.6824            | -0.01696          |
| 3  | 0.2709         | 0.3895         | 0.2711         | -0.01532          | 0.6433            | -0.1026           |
| 4  | 0.1847         | 0.1523         | 0.2785         | -0.2218           | 0.6372            | -0.1301           |
| 5  | 0.1364         | 0.007970       | 0.3424         | -0.3775           | 0.6362            | -0.1396           |
| 6  | 0.1061         | -0.08688       | 0.4535         | -0.4936           | 0.6351            | -0.1421           |
| 7  | 0.08558        | -0.1525        | 0.6007         | -0.5813           | 0.6329            | -0.1415           |
| 8  | 0.07094        | -0.1995        | 0.7750         | -0.6484           | 0.6297            | -0.1396           |
| 9  | 0.06006        | -0.2342        | 0.9694         | -0.7004           | 0.6257            | -0.1369           |
| 10 | 0.05170        | -0.2602        | 1.179          | -0.7412           | 0.6211            | -0.1340           |
| 11 | 0.04512        | -0.2801        | 1.400          | -0.7734           | 0.6162            | -0.1310           |
| 12 | 0.03983        | -0.2954        | 1.629          | -0.7991           | 0.6110            | -0.1281           |
| 13 | 0.03550        | -0.3073        | 1.864          | -0.8195           | 0.6056            | -0.1252           |
| 14 | 0.03190        | -0.3166        | 2.104          | -0.8359           | 0.6002            | -0.1225           |
| 15 | 0.02887        | -0.3239        | 2.347          | -0.8490           | 0.5948            | -0.1198           |
| 16 | 0.02629        | -0.3296        | 2.592          | -0.8595           | 0.5894            | -0.1173           |
| 17 | 0.02408        | -0.3339        | 2.838          | -0.8677           | 0.5841            | -0.1149           |
| 18 | 0.02216        | -0.3373        | 3.085          | -0.8742           | 0.5788            | -0.1126           |
| 19 | 0.02048        | -0.3398        | 3.333          | -0.8793           | 0.5737            | -0.1105           |
| 20 | 0.01900        | -0.3417        | 3.581          | -0.8830           | 0.5686            | -0.1084           |
| 21 | 0.01770        | -0.3430        | 3.828          | -0.8858           | 0.5636            | -0.1065           |
| 22 | 0.01653        | -0.3438        | 4.074          | -0.8877           | 0.5587            | -0.1046           |
| 23 | 0.01549        | -0.3443        | 4.319          | -0.8888           | 0.5540            | -0.1028           |
| 24 | 0.01455        | -0.3444        | 4.564          | -0.8894           | 0.5493            | -0.1011           |
| 25 | 0.01371        | -0.3443        | 4.807          | -0.8894           | 0.5448            | -0.09953          |
| 26 | 0.01294        | -0.3439        | 5.049          | -0.8889           | 0.5404            | -0.09798          |
| 27 | 0.01224        | -0.3434        | 5.290          | -0.8881           | 0.5360            | -0.09650          |
| 28 | 0.01161        | -0.3427        | 5.529          | -0.8870           | 0.5318            | -0.09509          |
| 29 | 0.01102        | -0.3419        | 5.767          | -0.8855           | 0.5277            | -0.09373          |
| 30 | 0.01049        | -0.3410        | 6.003          | -0.8839           | 0.5237            | -0.09243          |

Table A.1
Moments for the vector correlator in the $\overline{\text{MS}}$ scheme
| n  | $C_n^{(0),v}$ | $C_n^{(1),v}$ | $C_n^{(2),v}$ | $C_{N\Lambda,n}^{(2),v}$ | $C_{L,n}^{(2),v}$ | $C_{F,n}^{(2),v}$ |
|----|----------------|----------------|----------------|----------------------|------------------|------------------|
| 1  | 1.067          | 4.049          | 5.075          | 7.098                | -2.339           | 0.7270           |
| 2  | 0.4571         | 2.661          | 6.393          | 6.311                | -2.174           | 0.2671           |
| 3  | 0.2709         | 2.015          | 6.689          | 5.398                | -1.896           | 0.1499           |
| 4  | 0.1847         | 1.630          | 6.685          | 4.699                | -1.671           | 0.09947          |
| 5  | 0.1364         | 1.372          | 6.574          | 4.165                | -1.494           | 0.07230          |
| 6  | 0.1061         | 1.186          | 6.426          | 3.746                | -1.353           | 0.05566          |
| 7  | 0.08558        | 1.046          | 6.267          | 3.409                | -1.239           | 0.04459          |
| 8  | 0.07094        | 0.9356         | 6.108          | 3.132                | -1.143           | 0.03677          |
| 9  | 0.06006        | 0.8469         | 5.955          | 2.900                | -1.063           | 0.03100          |
| 10 | 0.05170        | 0.7738         | 5.809          | 2.702                | -0.9941          | 0.02661          |
| 11 | 0.04512        | 0.7126         | 5.671          | 2.532                | -0.9345          | 0.02317          |
| 12 | 0.03983        | 0.6605         | 5.542          | 2.384                | -0.8822          | 0.02041          |
| 13 | 0.03550        | 0.6156         | 5.420          | 2.254                | -0.8361          | 0.01816          |
| 14 | 0.03190        | 0.5765         | 5.305          | 2.139                | -0.7950          | 0.01629          |
| 15 | 0.02887        | 0.5422         | 5.198          | 2.035                | -0.7581          | 0.01473          |
| 16 | 0.02629        | 0.5118         | 5.096          | 1.943                | -0.7248          | 0.01340          |
| 17 | 0.02408        | 0.4847         | 5.000          | 1.858                | -0.6946          | 0.01226          |
| 18 | 0.02216        | 0.4603         | 4.910          | 1.782                | -0.6671          | 0.01127          |
| 19 | 0.02048        | 0.4383         | 4.824          | 1.712                | -0.6419          | 0.01041          |
| 20 | 0.01900        | 0.4184         | 4.743          | 1.648                | -0.6187          | 0.009654         |
| 21 | 0.01770        | 0.4002         | 4.665          | 1.589                | -0.5973          | 0.008984         |
| 22 | 0.01653        | 0.3836         | 4.592          | 1.535                | -0.5775          | 0.008389         |
| 23 | 0.01549        | 0.3683         | 4.522          | 1.484                | -0.5590          | 0.007857         |
| 24 | 0.01455        | 0.3541         | 4.455          | 1.437                | -0.5419          | 0.007379         |
| 25 | 0.01371        | 0.3411         | 4.392          | 1.393                | -0.5258          | 0.006947         |
| 26 | 0.01294        | 0.3290         | 4.331          | 1.352                | -0.5108          | 0.006556         |
| 27 | 0.01224        | 0.3177         | 4.272          | 1.314                | -0.4967          | 0.006201         |
| 28 | 0.01161        | 0.3072         | 4.216          | 1.277                | -0.4834          | 0.005876         |
| 29 | 0.01102        | 0.2974         | 4.162          | 1.243                | -0.4709          | 0.005579         |
| 30 | 0.01049        | 0.2881         | 4.111          | 1.211                | -0.4590          | 0.005306         |

Table A.2
Moments for the vector correlator in the onshell scheme
| n  | $C^{(0),a}_n$ | $C^{(1),a}_n$ | $C^{(2),a}_{AN_n}$ | $C^{(2),a}_{N_n}$ | $C^{(2),a}_{L_n}$ | $C^{(2),a}_{F_n}$ | $C^{(2),a}_{S_n}$ | $C^{(2),a}_{S_n} [L]$ |
|----|-------------|-------------|----------------|----------------|----------------|-------------|-------------|----------------|
| 1  | 0.53330     | 0.6346      | 0.5003         | -0.8386        | 0.6198         | 0.1775      | 1.970       | -0.6914       |
| 2  | 0.15240     | 0.1062      | 0.3025         | -0.3370        | 0.2883         | -0.0296     | 0.3832      | -0.1867       |
| 3  | 0.06772     | -0.009570   | 0.2271         | -0.2631        | 0.2034         | -0.04074    | 0.1447      | -0.08050      |
| 4  | 0.03694     | -0.04310    | 0.2071         | -0.2263        | 0.1602         | -0.03578    | 0.07290     | -0.04309      |
| 5  | 0.02273     | -0.05255    | 0.2079         | -0.1987        | 0.1323         | -0.02992    | 0.04308     | -0.02618      |
| 6  | 0.01516     | -0.05367    | 0.2160         | -0.1764        | 0.1124         | -0.02505    | 0.02813     | -0.01729      |
| 7  | 0.01070     | -0.05172    | 0.2261         | -0.1578        | 0.09739        | -0.02121    | 0.01966     | -0.01212      |
| 8  | 0.007883    | -0.04869    | 0.2363         | -0.1422        | 0.08563        | -0.01820    | 0.01443     | -0.008880     |
| 9  | 0.006006    | -0.04540    | 0.2456         | -0.1290        | 0.07617        | -0.01580    | 0.01098     | -0.006736     |
| 10 | 0.004700    | -0.04219    | 0.2539         | -0.1177        | 0.06841        | -0.01388    | 0.008609    | -0.005252     |
| 11 | 0.003760    | -0.03919    | 0.2610         | -0.1079        | 0.06193        | -0.01230    | 0.006905    | -0.004189     |
| 12 | 0.003064    | -0.03644    | 0.2671         | -0.09949       | 0.05646        | -0.01099    | 0.005646    | -0.003404     |
| 13 | 0.002536    | -0.03394    | 0.2723         | -0.09209       | 0.05177        | -0.009902   | 0.004690    | -0.002811     |
| 14 | 0.002127    | -0.03168    | 0.2767         | -0.08558       | 0.04772        | -0.008976   | 0.003949    | -0.002353     |
| 15 | 0.001804    | -0.02963    | 0.2803         | -0.07982       | 0.04420        | -0.008185   | 0.003364    | -0.001993     |
| 16 | 0.001547    | -0.02779    | 0.2833         | -0.07469       | 0.04109        | -0.007502   | 0.002895    | -0.001706     |
| 17 | 0.001338    | -0.02612    | 0.2858         | -0.07010       | 0.03835        | -0.006909   | 0.002514    | -0.001473     |
| 18 | 0.001166    | -0.02460    | 0.2879         | -0.06597       | 0.03591        | -0.006389   | 0.002200    | -0.001283     |
| 19 | 0.001024    | -0.02322    | 0.2895         | -0.06224       | 0.03373        | -0.005931   | 0.001938    | -0.001125     |
| 20 | 0.0009049   | -0.02196    | 0.2909         | -0.05886       | 0.03176        | -0.005525   | 0.001719    | -0.0009933    |
| 21 | 0.0008043   | -0.02081    | 0.2919         | -0.05578       | 0.02999        | -0.005162   | 0.001533    | -0.0008821    |
| 22 | 0.0007188   | -0.01975    | 0.2926         | -0.05296       | 0.02837        | -0.004838   | 0.001375    | -0.0007876    |
| 23 | 0.0006454   | -0.01878    | 0.2932         | -0.05039       | 0.02691        | -0.004546   | 0.001238    | -0.0007067    |
| 24 | 0.0005821   | -0.01789    | 0.2935         | -0.04802       | 0.02556        | -0.004282   | 0.001120    | -0.0006370    |
| 25 | 0.0005272   | -0.01706    | 0.2937         | -0.04583       | 0.02433        | -0.004043   | 0.001017    | -0.0005765    |
| 26 | 0.0004793   | -0.01630    | 0.2937         | -0.04381       | 0.02320        | -0.003825   | 0.0009275   | -0.0005238    |
| 27 | 0.0004372   | -0.01559    | 0.2936         | -0.04194       | 0.02216        | -0.003626   | 0.0008484   | -0.0004776    |
| 28 | 0.0004002   | -0.01493    | 0.2934         | -0.04020       | 0.02119        | -0.003444   | 0.0007785   | -0.0004369    |
| 29 | 0.0003674   | -0.01432    | 0.2931         | -0.03858       | 0.02029        | -0.003276   | 0.0007164   | -0.0004009    |
| 30 | 0.0003382   | -0.01375    | 0.2927         | -0.03707       | 0.01946        | -0.003121   | 0.0006611   | -0.0003689    |

Table A.3
Moments for the axial-vector correlator in the $\overline{\text{MS}}$ scheme
| n  | $C_n^{(0),a}$ | $C_n^{(1),a}$ | $C_n^{(2),a}$ | $C_{A,n}^{(2),a}$ | $C_{1,n}^{(2),a}$ | $C_{F,n}^{(2),a}$ | $C_{S,n}^{(2),a}$ [1] | $C_{S,n}^{(2),a}$ [L] |
|----|--------------|--------------|--------------|----------------|----------------|----------------|----------------|----------------|
| 1  | 0.5333       | 1.701        | 2.314        | 2.714          | -1.046         | 0.3433         | 1.970          | -0.6914        |
| 2  | 0.1524       | 0.7158       | 1.648        | 1.693          | -0.6638        | 0.06509        | 0.3832         | -0.1867        |
| 3  | 0.06772      | 0.3968       | 1.190        | 1.090          | -0.4313        | 0.02238        | 0.1447         | -0.08050       |
| 4  | 0.03694      | 0.2524       | 0.8998       | 0.7579         | -0.3015        | 0.01013        | 0.07290        | -0.04309       |
| 5  | 0.02273      | 0.1748       | 0.7077       | 0.5583         | -0.2228        | 0.005396       | 0.04308        | -0.02618       |
| 6  | 0.01516      | 0.1282       | 0.5741       | 0.4293         | -0.1717        | 0.003202       | 0.02813        | -0.01729       |
| 7  | 0.01070      | 0.09805      | 0.4772       | 0.3409         | -0.1366        | 0.002052       | 0.01966        | -0.01212       |
| 8  | 0.007883     | 0.07743      | 0.4045       | 0.2778         | -0.1114        | 0.001393       | 0.01443        | -0.008880      |
| 9  | 0.006006     | 0.06270      | 0.3484       | 0.2310         | -0.09269       | 0.0009894      | 0.01098        | -0.006736      |
| 10 | 0.004700     | 0.05181      | 0.3040       | 0.1954         | -0.07843       | 0.0007283      | 0.008609       | -0.005252      |
| 11 | 0.003760     | 0.04353      | 0.2683       | 0.1675         | -0.06728       | 0.0005520      | 0.006905       | -0.004189      |
| 12 | 0.003064     | 0.03709      | 0.2390       | 0.1454         | -0.05840       | 0.0004287      | 0.005646       | -0.003404      |
| 13 | 0.002536     | 0.03199      | 0.2147       | 0.1275         | -0.05120       | 0.0003399      | 0.004690       | -0.002811      |
| 14 | 0.002127     | 0.02787      | 0.1942       | 0.1127         | -0.04529       | 0.0002742      | 0.003949       | -0.002353      |
| 15 | 0.001804     | 0.02450      | 0.1768       | 0.1005         | -0.04036       | 0.0002246      | 0.003364       | -0.001993      |
| 16 | 0.001547     | 0.02170      | 0.1618       | 0.09013        | -0.03621       | 0.0001864      | 0.002895       | -0.001706      |
| 17 | 0.001338     | 0.01936      | 0.1488       | 0.08136        | -0.03269       | 0.0001566      | 0.002514       | -0.001473      |
| 18 | 0.001166     | 0.01738      | 0.1375       | 0.07384        | -0.02967       | 0.0001329      | 0.002200       | -0.001283      |
| 19 | 0.001024     | 0.01569      | 0.1275       | 0.06734        | -0.02705       | 0.0001138      | 0.001938       | -0.001125      |
| 20 | 0.0009049    | 0.01424      | 0.1187       | 0.06168        | -0.02478       | 0.00009826     | 0.001719       | -0.0009933     |
| 21 | 0.0008043    | 0.01297      | 0.1109       | 0.05672        | -0.02278       | 0.00008549     | 0.001533       | -0.0008821     |
| 22 | 0.0007188    | 0.01187      | 0.1038       | 0.05236        | -0.02103       | 0.00007488     | 0.001375       | -0.0007876     |
| 23 | 0.0006454    | 0.01091      | 0.09753      | 0.04849        | -0.01947       | 0.00006599     | 0.001238       | -0.0007067     |
| 24 | 0.0005821    | 0.01005      | 0.09183      | 0.04504        | -0.01808       | 0.00005849     | 0.001120       | -0.0006370     |
| 25 | 0.0005272    | 0.009298     | 0.08668      | 0.04196        | -0.01684       | 0.00005211     | 0.001017       | -0.0005765     |
| 26 | 0.0004793    | 0.008624     | 0.08198      | 0.03919        | -0.01573       | 0.00004664     | 0.0009275      | -0.0005238     |
| 27 | 0.0004372    | 0.008021     | 0.07770      | 0.03669        | -0.01473       | 0.00004194     | 0.0008484      | -0.0004776     |
| 28 | 0.0004002    | 0.007479     | 0.07377      | 0.03443        | -0.01382       | 0.00003786     | 0.0007785      | -0.0004369     |
| 29 | 0.0003674    | 0.006991     | 0.07017      | 0.03238        | -0.01299       | 0.00003431     | 0.0007164      | -0.0004009     |
| 30 | 0.0003382    | 0.006548     | 0.06685      | 0.03051        | -0.01224       | 0.00003120     | 0.0006611      | -0.0003689     |

Table A.4
Moments for the axial-vector correlator in the onshell scheme
| n  | \( C_{\sigma_n}^{(0)} \) | \( C_{\sigma_n}^{(1)} \) | \( C_{A_n}^{(2)} \) | \( C_{N\sigma_n}^{(2)} \) | \( C_{I_{\sigma_n}}^{(2)} \) | \( C_{F_{\sigma_n}}^{(2)} \) | \( C_{S_{\sigma_n}}^{(2)} \) | \( C_{S_{\sigma_n}}^{(2)} \) \([L]\) |
|---|---|---|---|---|---|---|---|---|
| 1  | 0.8000  | 0.4519  | 0.03484  | -2.511  | 0.8815  | 2.353  | -0.4444  |
| 2  | 0.2286  | 0.3194  | 0.5536   | -0.6269 | 0.3550  | 0.1201 | 0.8228   |
| 3  | 0.1016  | 0.1152  | 0.4516   | -0.3544 | 0.2345  | -0.01803 | 0.1923  |
| 4  | 0.05541 | 0.02460 | 0.3448   | -0.2797 | 0.1857  | -0.02277 | 0.04566 |
| 5  | 0.03410 | -0.01630| 0.2876   | -0.2439 | 0.1570  | -0.02406 | 0.1134  |
| 6  | 0.02273 | -0.03517| 0.2635   | -0.2193 | 0.1366  | -0.02428 | 0.07267 |
| 7  | 0.01605 | -0.04364| 0.2577   | -0.1995 | 0.1210  | -0.02277 | 0.04566 |
| 8  | 0.01182 | -0.04691| 0.2614   | -0.1827 | 0.1083  | -0.02083 | 0.03546 |
| 9  | 0.009009| -0.04752| 0.2698   | -0.1681 | 0.09788 | -0.01890 | 0.02635 |
| 10 | 0.007050| -0.04752| 0.2803   | -0.1553 | 0.08908 | -0.01713 | 0.02019 |
| 11 | 0.005640| -0.04524| 0.2913   | -0.1440 | 0.08156 | -0.01556 | 0.01586 |
| 12 | 0.004596| -0.04342| 0.3021   | -0.1339 | 0.07508 | -0.01417 | 0.01273 |
| 13 | 0.003803| -0.04146| 0.3124   | -0.1250 | 0.06943 | -0.01296 | 0.01040 |
| 14 | 0.003190| -0.03949| 0.3219   | -0.1170 | 0.06448 | -0.01190 | 0.008624 |
| 15 | 0.002707| -0.03756| 0.3305   | -0.1098 | 0.06010 | -0.01096 | 0.007246 |
| 16 | 0.002320| -0.03571| 0.3383   | -0.1033 | 0.05621 | -0.01014 | 0.006159 |
| 17 | 0.002006| -0.03397| 0.3454   | -0.09739| 0.05273 | -0.009409 | 0.005287 |
| 18 | 0.001749| -0.03232| 0.3516   | -0.09205| 0.04960 | -0.008759 | 0.004580 |
| 19 | 0.001536| -0.03078| 0.3572   | -0.08719| 0.04678 | -0.008178 | 0.003998 |
| 20 | 0.001357| -0.02934| 0.3622   | -0.08274| 0.04422 | -0.007656 | 0.003516 |
| 21 | 0.001206| -0.02799| 0.3665   | -0.07866| 0.04189 | -0.007187 | 0.003111 |
| 22 | 0.001078| -0.02674| 0.3704   | -0.07491| 0.03976 | -0.006763 | 0.002769 |
| 23 | 0.0009681| -0.02556| 0.3738   | -0.07145| 0.03781 | -0.006378 | 0.002478 |
| 24 | 0.0008732| -0.02447| 0.3767   | -0.06826| 0.03602 | -0.006027 | 0.002228 |
| 25 | 0.0007908| -0.02344| 0.3793   | -0.06530| 0.03437 | -0.005707 | 0.002012 |
| 26 | 0.0007189| -0.02249| 0.3816   | -0.06255| 0.03285 | -0.005415 | 0.001825 |
| 27 | 0.0006559| -0.02159| 0.3835   | -0.05999| 0.03144 | -0.005146 | 0.001661 |
| 28 | 0.0006003| -0.02075| 0.3852   | -0.05761| 0.03012 | -0.004898 | 0.001517 |
| 29 | 0.0005511| -0.01996| 0.3866   | -0.05538| 0.02890 | -0.004669 | 0.001390 |
| 30 | 0.0005073| -0.01922| 0.3878   | -0.05329| 0.02776 | -0.004458 | 0.001278 |

Table A.5
Moments for the scalar correlator in the \( \overline{\text{MS}} \) scheme
| n  | $C^{(0),s}_n$ | $C^{(1),s}_n$ | $C^{(2),s}_{A,n}$ | $C^{(2),s}_{N A,n}$ | $C^{(2),s}_{F,n}$ | $C^{(2),s}_{S,n}$ | $C^{(2),s}_{S,n}[1]$ | $C^{(2),s}_{S,n}[L]$ |
|----|-------------|-------------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|
| 1  | 0.8000      | 0.4519      | -2.511          | 0.8815          | 0.7186          | 2.353           | -0.4444         |                  |
| 2  | 0.2286      | 0.7765      | 1.426           | 0.8955          | -0.3591         | 0.1911          | 0.8228          | -0.2074         |
| 3  | 0.1016      | 0.5215      | 1.526           | 0.9989          | -0.4002         | 0.07144         | 0.3660          | -0.1089         |
| 4  | 0.05541     | 0.3571      | 1.327           | 0.8275          | -0.3336         | 0.03362         | 0.1923          | -0.06377        |
| 5  | 0.03410     | 0.2565      | 1.115           | 0.6646          | -0.2691         | 0.01832         | 0.1134          | -0.04057        |
| 6  | 0.02273     | 0.1922      | 0.9371          | 0.5378          | -0.2185         | 0.01103         | 0.07267         | -0.02746        |
| 7  | 0.01605     | 0.1489      | 0.7952          | 0.4417          | -0.1798         | 0.007146        | 0.04956         | -0.01950        |
| 8  | 0.01182     | 0.1186      | 0.6824          | 0.3685          | -0.1502         | 0.004890        | 0.03546         | -0.01438        |
| 9  | 0.009009    | 0.09662     | 0.5921          | 0.3119          | -0.1273         | 0.003494        | 0.02635         | -0.01093        |
| 10 | 0.007050    | 0.08017     | 0.5191          | 0.2673          | -0.1092         | 0.002584        | 0.02019         | -0.008525       |
| 11 | 0.005640    | 0.06756     | 0.4592          | 0.2317          | -0.09464        | 0.001967        | 0.01586         | -0.006789       |
| 12 | 0.004596    | 0.05769     | 0.4096          | 0.2028          | -0.08285        | 0.001532        | 0.01273         | -0.005505       |
| 13 | 0.003803    | 0.04982     | 0.3681          | 0.1790          | -0.07315        | 0.001218        | 0.01040         | -0.004532       |
| 14 | 0.003190    | 0.04345     | 0.3329          | 0.1592          | -0.06508        | 0.0009850       | 0.008624        | -0.003782       |
| 15 | 0.002707    | 0.03823     | 0.3028          | 0.1426          | -0.05828        | 0.0008084       | 0.007246        | -0.003193       |
| 16 | 0.002320    | 0.03388     | 0.2769          | 0.1285          | -0.05251        | 0.0006722       | 0.006159        | -0.002723       |
| 17 | 0.002006    | 0.03024     | 0.2544          | 0.1164          | -0.04757        | 0.0005654       | 0.005287        | -0.002344       |
| 18 | 0.001749    | 0.02715     | 0.2347          | 0.1060          | -0.04330        | 0.0004804       | 0.004580        | -0.002034       |
| 19 | 0.001536    | 0.02451     | 0.2174          | 0.09695         | -0.03959        | 0.0004119       | 0.003998        | -0.001778       |
| 20 | 0.001357    | 0.02224     | 0.2020          | 0.08903         | -0.03635        | 0.0003561       | 0.003516        | -0.001565       |
| 21 | 0.001206    | 0.02027     | 0.1884          | 0.08206         | -0.03349        | 0.0003101       | 0.003111        | -0.001385       |
| 22 | 0.001078    | 0.01855     | 0.1762          | 0.07589         | -0.03097        | 0.0002719       | 0.002769        | -0.001233       |
| 23 | 0.0009681   | 0.01703     | 0.1653          | 0.07041         | -0.02872        | 0.0002398       | 0.002478        | -0.001103       |
| 24 | 0.0008732   | 0.01570     | 0.1554          | 0.06551         | -0.02672        | 0.0002127       | 0.002228        | -0.0009917      |
| 25 | 0.0007908   | 0.01451     | 0.1464          | 0.06112         | -0.02492        | 0.0001896       | 0.002012        | -0.0008952      |
| 26 | 0.0007189   | 0.01346     | 0.1383          | 0.05716         | -0.02330        | 0.0001698       | 0.001825        | -0.0008113      |
| 27 | 0.0006559   | 0.01252     | 0.1309          | 0.05359         | -0.02184        | 0.0001528       | 0.001661        | -0.0007379      |
| 28 | 0.0006003   | 0.01167     | 0.1241          | 0.05035         | -0.02051        | 0.0001380       | 0.001517        | -0.0006734      |
| 29 | 0.0005511   | 0.01090     | 0.1179          | 0.04739         | -0.01930        | 0.0001251       | 0.001390        | -0.0006165      |
| 30 | 0.0005073   | 0.01021     | 0.1122          | 0.04470         | -0.01820        | 0.0001138       | 0.001278        | -0.0005661      |

Table A.6
Moments for the scalar correlator in the onshell scheme
| n  | $C_n^{(0),p}$ | $C_n^{(1),p}$ | $C_n^{(2),p}$ | $C_{N_A,n}^{(2),p}$ | $C_{I_n}^{(2),p}$ | $C_{F,n}^{(2),p}$ | $C_{S,n}^{(2),p}$ [1] | $C_{S,n}^{(2),p}$ [L] |
|----|---------------|---------------|---------------|-----------------|-----------------|-----------------|--------------------|--------------------|
| 1  | 1.333         | 2.333         | 2.712         | -1.858          | 0.9259          | 1.311           | 5.629              | -1.000             |
| 2  | 0.5333        | 1.548         | 3.399         | 0.02615         | 0.4346          | 0.3307          | 3.065              | -0.6667            |
| 3  | 0.3048        | 0.9088        | 2.797         | 0.04435         | 0.4017          | 0.06951         | 1.952              | -0.4667            |
| 4  | 0.2032        | 0.5346        | 2.198         | -0.1204         | 0.4294          | -0.02758        | 1.375              | -0.3471            |
| 5  | 0.1478        | 0.3010        | 1.774         | -0.2777         | 0.4600          | -0.07127        | 1.034              | -0.2703            |
| 6  | 0.1137        | 0.1458        | 1.506         | -0.4046         | 0.4846          | -0.09305        | 0.8132             | -0.2180            |
| 7  | 0.09093       | 0.03751       | 1.358         | -0.5039         | 0.5029          | -0.1044         | 0.6612             | -0.1804            |
| 8  | 0.07488       | -0.04085      | 1.300         | -0.5813         | 0.5162          | -0.1104         | 0.5512             | -0.1525            |
| 9  | 0.06306       | -0.09923      | 1.310         | -0.6422         | 0.5256          | -0.1133         | 0.4686             | -0.1311            |
| 10 | 0.05405       | -0.1437       | 1.369         | -0.6903         | 0.5320          | -0.1144         | 0.4048             | -0.1142            |
| 11 | 0.04700       | -0.1783       | 1.467         | -0.7286         | 0.5362          | -0.1145         | 0.3542             | -0.1007            |
| 12 | 0.04136       | -0.2055       | 1.595         | -0.7594         | 0.5386          | -0.1139         | 0.3134             | -0.08963           |
| 13 | 0.03677       | -0.2271       | 1.745         | -0.7842         | 0.5398          | -0.1128         | 0.2798             | -0.08044           |
| 14 | 0.03296       | -0.2446       | 1.913         | -0.8042         | 0.5400          | -0.1116         | 0.2519             | -0.07273           |
| 15 | 0.02977       | -0.2587       | 2.095         | -0.8205         | 0.5394          | -0.1101         | 0.2283             | -0.06617           |
| 16 | 0.02707       | -0.2702       | 2.288         | -0.8336         | 0.5383          | -0.1086         | 0.2082             | -0.06054           |
| 17 | 0.02475       | -0.2796       | 2.490         | -0.8442         | 0.5367          | -0.1071         | 0.1908             | -0.05566           |
| 18 | 0.02274       | -0.2874       | 2.699         | -0.8527         | 0.5347          | -0.1056         | 0.1758             | -0.05141           |
| 19 | 0.02099       | -0.2937       | 2.913         | -0.8595         | 0.5324          | -0.1040         | 0.1626             | -0.04767           |
| 20 | 0.01945       | -0.2989       | 3.131         | -0.8649         | 0.5299          | -0.1025         | 0.1510             | -0.04436           |
| 21 | 0.01810       | -0.3032       | 3.353         | -0.8690         | 0.5273          | -0.1010         | 0.1407             | -0.04142           |
| 22 | 0.01689       | -0.3067       | 3.577         | -0.8721         | 0.5246          | -0.09956        | 0.1315             | -0.03879           |
| 23 | 0.01581       | -0.3095       | 3.803         | -0.8744         | 0.5217          | -0.09814        | 0.1233             | -0.03642           |
| 24 | 0.01484       | -0.3118       | 4.030         | -0.8759         | 0.5188          | -0.09677        | 0.1159             | -0.03429           |
| 25 | 0.01397       | -0.3136       | 4.258         | -0.8768         | 0.5159          | -0.09544        | 0.1092             | -0.03236           |
| 26 | 0.01318       | -0.3150       | 4.487         | -0.8771         | 0.5129          | -0.09415        | 0.1032             | -0.03060           |
| 27 | 0.01246       | -0.3160       | 4.715         | -0.8770         | 0.5099          | -0.09291        | 0.09765            | -0.02899           |
| 28 | 0.01181       | -0.3168       | 4.944         | -0.8765         | 0.5069          | -0.09170        | 0.09260            | -0.02752           |
| 29 | 0.01121       | -0.3173       | 5.173         | -0.8757         | 0.5039          | -0.09053        | 0.08797            | -0.02618           |
| 30 | 0.01065       | -0.3176       | 5.401         | -0.8746         | 0.5009          | -0.08940        | 0.08372            | -0.02493           |

Table A.7
Moments for the pseudo-scalar correlator in the $\overline{\text{MS}}$ scheme

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| $n$ | $C_n^{(0),p}$ | $C_n^{(1),p}$ | $C_{A,n}^{(2),p}$ | $C_{N A,n}^{(2),p}$ | $C_{L,n}^{(2),p}$ | $C_{F,n}^{(2),p}$ | $C_{S,n}^{(2),p}$ | $C_{S,n}^{(2),p} [1]$ | $C_{S,n}^{(2),p} [L]$ |
|-----|----------------|----------------|------------------|---------------------|-----------------|----------------|----------------|------------------|----------------|
| 1   | 1.333          | 2.333          | 2.712            | -1.858              | 0.9259          | 1.311         | 5.629          | -1.000           |                 |
| 2   | 0.5333         | 2.615          | 7.040            | 3.578               | -1.232          | 0.4964        | 3.065          | -0.6667          |                 |
| 3   | 0.3048         | 2.128          | 8.273            | 4.104               | -1.502          | 0.2589        | 1.952          | -0.4667          |                 |
| 4   | 0.2032         | 1.754          | 8.466            | 3.939               | -1.475          | 0.1618        | 1.375          | -0.3471          |                 |
| 5   | 0.1478         | 1.483          | 8.331            | 3.659               | -1.387          | 0.1124        | 1.034          | -0.2703          |                 |
| 6   | 0.1137         | 1.282          | 8.090            | 3.381               | -1.291          | 0.08353       | 0.8132         | -0.2180          |                 |
| 7   | 0.09093        | 1.129          | 7.821            | 3.130               | -1.202          | 0.06509       | 0.6612         | -0.1804          |                 |
| 8   | 0.07488        | 1.008          | 7.554            | 2.910               | -1.121          | 0.05249       | 0.5512         | -0.1525          |                 |
| 9   | 0.06306        | 0.9097         | 7.300            | 2.718               | -1.050          | 0.04347       | 0.4686         | -0.1311          |                 |
| 10  | 0.05405        | 0.8292         | 7.062            | 2.550               | -0.9878         | 0.03674       | 0.4048         | -0.1142          |                 |
| 11  | 0.04700        | 0.7618         | 6.842            | 2.402               | -0.9322         | 0.03157       | 0.3542         | -0.1007          |                 |
| 12  | 0.04136        | 0.7045         | 6.639            | 2.271               | -0.8828         | 0.02751       | 0.3134         | -0.08963         |                 |
| 13  | 0.03677        | 0.6553         | 6.451            | 2.154               | -0.8385         | 0.02424       | 0.2798         | -0.08044         |                 |
| 14  | 0.03296        | 0.6125         | 6.276            | 2.050               | -0.7987         | 0.02157       | 0.2519         | -0.07273         |                 |
| 15  | 0.02977        | 0.5749         | 6.115            | 1.956               | -0.7627         | 0.01936       | 0.2283         | -0.06617         |                 |
| 16  | 0.02707        | 0.5418         | 5.965            | 1.871               | -0.7301         | 0.01749       | 0.2082         | -0.06054         |                 |
| 17  | 0.02475        | 0.5122         | 5.825            | 1.793               | -0.7003         | 0.01591       | 0.1908         | -0.05566         |                 |
| 18  | 0.02274        | 0.4858         | 5.694            | 1.722               | -0.6730         | 0.01455       | 0.1758         | -0.05141         |                 |
| 19  | 0.02099        | 0.4619         | 5.571            | 1.657               | -0.6480         | 0.01338       | 0.1626         | -0.04767         |                 |
| 20  | 0.01945        | 0.4403         | 5.456            | 1.597               | -0.6248         | 0.01235       | 0.1510         | -0.04436         |                 |
| 21  | 0.01810        | 0.4207         | 5.348            | 1.542               | -0.6034         | 0.01145       | 0.1407         | -0.04142         |                 |
| 22  | 0.01689        | 0.4027         | 5.246            | 1.490               | -0.5836         | 0.01065       | 0.1315         | -0.03879         |                 |
| 23  | 0.01581        | 0.3862         | 5.150            | 1.443               | -0.5651         | 0.009943      | 0.1233         | -0.03642         |                 |
| 24  | 0.01484        | 0.3710         | 5.059            | 1.398               | -0.5478         | 0.009309      | 0.1159         | -0.03429         |                 |
| 25  | 0.01397        | 0.3570         | 4.972            | 1.357               | -0.5317         | 0.008739      | 0.1092         | -0.03236         |                 |
| 26  | 0.01318        | 0.3440         | 4.890            | 1.318               | -0.5166         | 0.008225      | 0.1032         | -0.03060         |                 |
| 27  | 0.01246        | 0.3320         | 4.812            | 1.281               | -0.5023         | 0.007760      | 0.09765        | -0.02899         |                 |
| 28  | 0.01181        | 0.3207         | 4.737            | 1.247               | -0.4889         | 0.007337      | 0.09260        | -0.02752         |                 |
| 29  | 0.01121        | 0.3102         | 4.666            | 1.214               | -0.4763         | 0.006951      | 0.08797        | -0.02618         |                 |
| 30  | 0.01065        | 0.3004         | 4.598            | 1.183               | -0.4643         | 0.006597      | 0.08372        | -0.02493         |                 |

Table A.8
Moments for the pseudo-scalar correlator in the onshell scheme
B Analytical Results

In this appendix we present the analytical results for the moments 9 to 12 of the various currents in the $\overline{\text{MS}}$-scheme. The expression for the moments up to 30 can be obtained from http://www-ttp.particle.uni-karlsruhe.de/Progdata/ttp07/ttp07-32 in computer readable form in the $\overline{\text{MS}}$ and onshell scheme. In the following expressions we use $L = \log \mu^2/m^2$ and $L_{-q^2} = \log(-q^2/m^2)$. The singlet contributions $C_{S,n}^{(2),\delta}$ are always given separately.

B.1 Scalar Current

$$\sum_{n=9}^{12} C_{n}^{(0),s} z^n = \frac{131072}{14549535} z^9 + \frac{262144}{37182145} z^{10} + \frac{1048576}{185910725} z^{11} + \frac{2097152}{456326325} z^{12}$$

$$\sum_{n=9}^{12} C_{n}^{(1),s} z^n = \left( -\frac{4979043746471936}{104786039514531375} - \frac{524288}{4849845} L \right) z^9$$
$$+ \left( -\frac{2888036521295872}{61796895098313375} - \frac{3538944}{37182145} L \right) z^{10}$$
$$+ \left( -\frac{38132976301932544}{842822916656645925} - \frac{3145728}{37182145} L \right) z^{11}$$
$$+ \left( -\frac{33183292389464276992}{764287144877276645625} - \frac{11534336}{152108775} L \right) z^{12}$$

$$C_{9}^{(2),s} = \left( \frac{1048576}{1616615} L^2 + \frac{2510802269650944}{34928679838177125} \right) L$$
$$+ \frac{7374653970894133873833421163704138849}{577139180201947480857772032000} - \frac{340959238671049404767011}{320780855251304448} \zeta_3 C_F^2$$
$$+ C_A \left( -\frac{65536}{1322685} L^2 - \frac{5406868428560384}{28578010776690375} \right) L$$
$$+ \frac{102915497024490763942656449854922612500879}{4375292125110963852382769745920000} - \frac{1345087895588279678743561}{6873875469670809600} \zeta_3 C_F$$
\[
\begin{align*}
+ T_F \left( \frac{262144}{14549535} n_h + \frac{262144}{14549535} n_t \right) L^2 + \left( \frac{14418875171495936}{314358118543594125} n_h \right) \\
+ \frac{14418875171495936}{314358118543594125} n_t L - \frac{370939064258803430677307}{48861314420492468240000} n_h \\
+ \frac{12790934874367656891392}{130678984236690621356625} n_t + \frac{125624633262191}{2040968459059200} n_h \zeta_3 \right) C_F \\
\end{align*}
\]

\[
C^{(2),s}_{10} = \left( \frac{23887872}{37182145} L^2 + \frac{1743551373172736}{2288773892530125} L \right) \\
+ \frac{1181283682867330857434847115794497652295927361}{18741992087247811295709473066188800000} \\
- \frac{1856910365195457777240839095367}{35414206419744011059200} \zeta_3 \right) C_F^2 \\
+ C_A \left( - \frac{147456}{3380195} L^2 - \frac{2883103404986368}{16853698663176375} L \right) \\
+ \frac{19485093544688649029885182650128118344232413}{20183683786266873703071740225126400000} \\
- \frac{738743868672823822714945309}{919849517395948339200} \zeta_3 \right) C_F \\
+ T_F \left( \frac{589824}{37182145} n_h + \frac{589824}{37182145} n_t \right) L^2 + \left( \frac{7789487453519872}{185390685294940125} n_h \right) \\
+ \frac{7789487453519872}{185390685294940125} n_t L - \frac{4979656088382493508774203867}{66263521631983019092869120000} n_h \\
+ \frac{48054627678191205275648}{53946965287453334318375} n_t + \frac{332499096340061}{544258255749120} n_h \zeta_3 \right) C_F \\
\]

\[
C^{(2),s}_{11} = \left( \frac{4718592}{7436429} L^2 + \frac{8933862862782464}{11237638888755279} L \right) \\
+ \frac{7404966335674218474752347314237559396292812077443}{24289621745073163439239477093780684800000} \\
- \frac{61139862184252373166516579786803}{24107416151913040737536} \zeta_3 \right) C_F^2 + C_A \left( - \frac{131072}{3380195} L^2 \right)
\]
\[
\begin{align*}
C_{12}^{(2),s} &= \left( \frac{31719424}{50702925} L^2 + \frac{19006459960269111296}{2316021651143265625} \right) \cdot \frac{L}{L} \\
&+ \frac{188473078860503880648883122776543550015991250654673059}{129919789020675223866838798646389820620800000} \\
&- \frac{23090275323409880205366729096347}{19132869961835725455360} \cdot \frac{C_F^2}{\zeta_3} + C_A \left( - \frac{15859712}{456326325} L^2 \right) \\
&- \frac{29590089938919374848}{208441948602893630625} L \\
&+ \frac{20886533237898744255234695965842642441637712801424593}{129919789020675223866838798646389820620800000} \\
&- \frac{41752924406364782322130747883}{312191351358624890880} \frac{C_F}{\zeta_3} + T_F \left( \frac{5767168}{456326325} n_h + \frac{5767168}{456326325} n_t \right) L^2 \\
&+ \left( \frac{81479567700203732992}{2292861434631829936875} n_h + \frac{81479567700203732992}{2292861434631829936875} n_t \right) L \\
&- \frac{88884847785166765220997026873}{12059960937020909479021798400} n_h + \frac{41147533572114438909275987968}{548058254961784643538602765625} n_t \\
&+ \frac{77358608492779}{128642860449792} n_h \zeta_3 \right) C_F
\end{align*}
\]

\[
C_{S,9}^{(2),s} = \frac{-28771023505735526373476647}{1454362410400965092966400000} + \frac{1436000001918689}{3741775508275200} \zeta_3 - \frac{289369094656}{26467422856875} L_{-q^2}
\]
\[ C_{S,10}^{(2),s} = \frac{-1090589835840240580931001947}{57980581427985141706260480000} + \frac{48559274491381}{1496710203310080} \zeta_3 - \frac{3471392768}{407191120875} L^{-q^2} \]

\[ C_{S,11}^{(2),s} = \frac{-115469790307862708811970912146409}{6584738671613416573296590192640000} + \frac{433705924964423}{156086670679080960} \zeta_3 \]

\[ - \frac{225667578742784}{33237789623663625} L^{-q^2} \]

\[ C_{S,12}^{(2),s} = \frac{-639705925147065881595158754516867}{3950843202968049943779541155840000} + \frac{6759944716431528}{28095600722234572800} \zeta_3 \]

\[ - \frac{2744536517378048}{498566844354954375} L^{-q^2} \]

### B.2 Pseudo-Scalar Current

\[ \sum_{n=9}^{12} C_{n}^{(0),p} z^n = \frac{131072}{2078505} z^9 + \frac{262144}{4849845} z^{10} + \frac{1048576}{22309287} z^{11} + \frac{2097152}{50702925} z^{12} \]

\[ \sum_{n=9}^{12} C_{n}^{(1),p} z^n = \left( -\frac{49751393148928}{501368610117375} - \frac{524288}{692835} L \right) z^9 \]

\[ + \left( -\frac{557796221206528}{3880964426464125} - \frac{1179648}{1616615} L \right) z^{10} \]

\[ + \left( -\frac{4091600265773056}{22953132465087825} - \frac{5242880}{7436429} L \right) z^{11} \]

\[ + \left( -\frac{35891639960928256}{174694204543377519} - \frac{11534336}{16900975} L \right) z^{12} \]

\[ C_{9}^{(2),p} = + C_F T_R \left( \frac{24213861812657343488}{46071664047183800025} n_l - \frac{23936615221766899739377}{13951233275688824416000} n_h \right) \]

\[ + \frac{386477922523}{289910292480} n_h \zeta_3 + \frac{365917995900928}{1504105830352125} L(n_l + n_h) + \frac{262144}{2078505} L^2(n_l + n_h) \]
\[
C_{10}^{(2)} = C_F C_A \left( - \frac{2774476486497511252284870305976724044391}{96593822739271767841057013760000} \right)
\]
\[
+ \frac{1870074094390219597606687}{7855857679623782400} \zeta_3 - \frac{151838577682432}{136736893668375} L - \frac{65536}{18955} L^2 \right) 
\]
\[
+ C_F^2 \left( - \frac{18248707238521284112217374676684189}{1201215870545458762878480000} \right) 
\]
\[
+ \frac{143085894538571009718455041}{11321677244163686400} \zeta_3 + \frac{3728972041094632}{167122870039125} L + \frac{1048576}{230945} L^2 \right) 
\]
\[
C_{11}^{(2)} = C_F T_R \left( \frac{18023374701238546946048}{33879736653956827759125} n_l - \frac{1835086571234207640548161}{941510679624652161024000} n_h 
\]
\[
+ \frac{46149597022417}{30236569763840} n_h \zeta_3 + \frac{2917754077462528}{11642893279392375} L(n_l + n_h) + \frac{196608}{1616615} L^2(n_l + n_h) \right) 
\]
\[
+ C_F C_A \left( - \frac{1539378756641920703135316484755839467328419}{1333422361938579459773796502732800000} \right) 
\]
\[
+ \frac{12803248135021178751056537}{1333115246028236800} \zeta_3 - \frac{1179975928287232}{105844843581125} L - \frac{49152}{146965} L^2 \right) 
\]
\[
+ C_F^2 \left( - \frac{27324314428644557274188086091814971713169}{369369075329246387748974100480000} \right) 
\]
\[
+ \frac{135367797910726154580166718023}{219964015029465907200} \zeta_3 + \frac{423117757374464}{143739423202375} L + \frac{7962624}{1616615} L^2 \right) 
\]
\[
C_{11}^{(2)} = C_F T_R \left( \frac{3760136484063862872174592}{701310548736906346138875} n_l - \frac{14436891835130136916846548977}{626352163198301909286912000} n_h 
\]
\[
+ \frac{186926938565003}{108851651149824} n_h \zeta_3 + \frac{17577073730093056}{6883939735263475} L(n_l + n_h) + \frac{2621440}{22309287} L^2(n_l + n_h) \right) 
\]
\[
+ C_F C_A \left( - \frac{2196670741691390865449882336983942060570342639}{472298200598644844651878721267957760000} \right) 
\]
\[
+ \frac{5140924281488464972357892859}{1328671525172480934400} \zeta_3 - \frac{6968767912075264}{6259945217751225} L - \frac{655360}{2028117} L^2 \right) 
\]
\[
+ C_F^2 \left( - \frac{19892582218578011106338578696247144162810751553}{5622597626174343388712841919856640000} \right) 
\]
\[ C_{12}^{(2),p} = C_F T_R \left( \frac{23616507902150100407548239872}{4384466039694277148308822125} n_l - \frac{43084675796375645372731027301693}{17917656249288208362711810048000} n_h \right) \]
\[ + \frac{3922441614750337}{2058285767196672} n_l \zeta_3 + \frac{676219888715104256}{2620413068150662785} L(n_l + n_h) + \frac{5767168}{50702925} L^2(n_l + n_h) \]
\[ + C_F C_A \left( -\frac{700347946825814813249178402086293107823942203323}{3741689923795446447364957401016026833879040000} \right) \]
\[ + \frac{77778913328384051652725192513}{4995061621737998254080} \zeta_3 - \frac{1319456473035685888}{11910968491593921275} L - \frac{15859712}{50702925} L^2 \]
\[ + C_F \left( -\frac{226747327707094876433590459813785417415724639080638329}{136021881772409715259741071725171834880000} \right) \]
\[ + \frac{68634887121309888808884620697226071}{48214832303826028147507200} \zeta_3 + \frac{572835921739218944}{132344094351043575} L + \frac{95158272}{16900975} L^2 \]

\[ C_{S,9}^{(2),p} = \frac{27352144660490483390633}{174887254737970790400000} + \frac{2454162037509}{9448928051200} \zeta_3 - \frac{263495168}{2010133125} L_{-q^2} \]

\[ C_{S,10}^{(2),p} = \frac{99214339418590131445021}{783494901226109140992000} + \frac{46177240370863}{199561360441344} \zeta_3 - \frac{732934144}{641634935} L_{-q^2} \]

\[ C_{S,11}^{(2),p} = \frac{2787996822340736999889074473}{26799913193379798833115955200} + \frac{53552348130925}{257285720899584} \zeta_3 - \frac{21320382464}{211739382855} L_{-q^2} \]

\[ C_{S,12}^{(2),p} = \frac{14641282147919288479322671871}{169262609642398729472311296000} + \frac{561179696637195}{2973079441506304} \zeta_3 - \frac{1212441755648}{13527793904625} L_{-q^2} \]

**B.3 Vector Current**

\[ \sum_{n=9}^{12} C_n^{(0),v} z^n = \frac{524288}{8729721} z^9 + \frac{524288}{10140585} z^{10} + \frac{8388608}{185910725} z^{11} + \frac{4194304}{105306075} z^{12} \]
\[
\sum_{n=9}^{12} C_n^{(1),a} z^n = \left( -\frac{2103270735183872}{8981660529816975} - \frac{262144}{323323} L \right) z^9
\]
\[+ \left( -\frac{57014372204019712}{219098082621292875} - \frac{524288}{676039} L \right) z^{10}
\]
\[+ \left( -\frac{13456105329705877504}{4804906249428817725} - \frac{12582912}{16900975} L \right) z^{11}
\]
\[+ \left( -\frac{249307596877594624}{843894528756654375} - \frac{8388608}{11700675} L \right) z^{12}
\]
\[
C_9^{(2),v} = \left( 1769472 \frac{L^2}{323323} + \frac{1422485887860736}{332654093696925} L \right)
\]
\[+ \frac{38142720052428585982747934296145120691697}{26317546617208805127114404659200000}
\]
\[+ \frac{4641207108486463789099278533}{384937026301565337600} \zeta_3 \right) C_F^2 + C_A \left( -\frac{32768}{88179} L^2 - \frac{3201458214330368}{249543780859175} L \right)
\]
\[+ \frac{4800616671655745922224606179869582464114657}{21001402200532626491437294918041600000} \zeta_3 \right) C_F
\]
\[+ T_R \left( \left( \frac{131072}{969969} n_h + \frac{131072}{969969} n_l \right) L^2 + \left( \frac{817173794127872}{26944981589450925} n_h \right) L - \frac{750522973653044996167293}{2893884388042736664576000} n_h \right)
\]
\[+ \frac{817173794127872}{26944981589450925} n_l \right) L - \frac{750522973653044996167293}{2893884388042736664576000} n_h \right)
\]
\[+ \frac{1716980884029425246777344}{2744258668970503048489125} n_l + \frac{15641077379375}{7653631721472} n_l \zeta_3 \right) C_F
\]
\[
C_{10}^{(2),v} = \left( \frac{3932160}{676039} L^2 + \frac{72590094055309312}{14606538841419525} L \right)
\]
\[+ \frac{541634465543720570056096925789635688755714501}{790677791180767039037433949798400000}
\]
\[+ \frac{3503779464918218459698976337}{61482997256500019200} \zeta_3 \right) C_F^2
\]
\[+ C_A \left( -\frac{720896}{2028117} L^2 - \frac{76685205430140928}{59754022533079875} L \right)
\]
\[
C_{11}^{(2),v} = \left( \frac{103809024}{16900975} L^2 + \frac{1643667546316865536}{291157007572295865} L \right)
- \frac{1070887924864138198352946851863557684684339508412651}{335196780082009654615047838941734502400000}
+ \frac{96108341407518837292239482548466517}{36161124227869521110630400} \zeta_3 \right) C_F^2
+ C_A \left( - \frac{5767168}{16900975} L^2 - \frac{16505631011785080832}{1310206340753313925} L \right)
- \frac{10142587654701589960628097458296025022529793100251069}{2806267442846584835523718050762020125409280000}
+ \frac{1150547909207069411531224161671}{382657399236714509107200} \zeta_3 \right) C_F + T_R \left( \left( \frac{2097152}{16900975} n_h + \frac{2097152}{16900975} n_l \right) L^2
+ \left( \frac{4326180666433656064}{144122718748286453175} n_h + \frac{4326180666433656064}{144122718748286453175} n_l \right) L \right)
- \frac{8514227027168542081460662817}{27996337889512825566737203200} n_h + \frac{849053414353143539189776384}{1377975041046772818039915525} n_l
+ \frac{428230928483939}{17683933118464} n_h \zeta_3 \right) C_F

C_{12}^{(2),v} = \left( \frac{8388608}{1300075} L^2 + \frac{591048256550862848}{93766058750739375} L \right)
\[
\sum_{n=9}^{12} C_n^{(0),a} z^n = \frac{262144}{43648605} z^9 + \frac{524288}{111546435} z^{10} + \frac{2097152}{557732175} z^{11} + \frac{4194304}{1368978975} z^{12}
\]

\[
\sum_{n=9}^{12} C_n^{(1),a} z^n = \left( -\frac{14273234676170752}{314358118543594125} - \frac{131072}{1616615} \right) z^9 + \left( -\frac{101689470330109952}{2410078908834221625} - \frac{524288}{7436429} \right) z^{10} + \left( -\frac{14155800052957184}{361209821424276825} - \frac{1048576}{16900975} \right) z^{11} + \left( -\frac{8354812412930228224}{2292861434631829936875} - \frac{8388608}{152108775} \right) z^{12}
\]

\[C_9^{(2),a} = C_F T_R \left( \frac{209029528206436443461632}{2744258668970503048489125} n_h - \frac{721707504396631265775619523}{1296460205843146025730048000} \right) n_t \]
\[
\begin{align*}
&+ \frac{1102011173546563}{2449162150871040} n_h \zeta_3 + \frac{35512855382474752}{943074355630782375} L(n_l + n_h) + \frac{65536}{4849845} L^2(n_l + n_h) \\
&+ C_F C_A \left( \frac{20448218956035278169815363567041306877706357}{3360224352085220238629967186886656000} \right) \\
&- \frac{668136450098900377328556313}{1319784090176795443200} \zeta_3 - \frac{12933050525913088}{85734032330071125} L - \frac{16384}{440895} L^2 \\
&+ C_F^2 \left( \frac{217311353924672252345666774738607455908207}{5540536129938695816234611507200000} \right) \\
&- \frac{4567332086889583465526764451}{139977100473296486400} \zeta_3 + \frac{8434594159026176}{11642893297932375} L + \frac{884736}{1616165} L^2
\end{align*}
\]

\[
C_{10,a}^{(2)} = C_F T_R \left( \frac{287854153300932624846848}{420786329241438007683325} n_l - \frac{209290713314001851931555887}{38228954787682511015116800} n_h \\
+ \frac{181196559131413}{408193691811840} n_h \zeta_3 + \frac{243286941705469952}{7230236726502664875} L(n_l + n_h) + \frac{262144}{22309287} L^2(n_l + n_h) \right)
\]

\[
\begin{align*}
&+ C_F C_A \left( \frac{8649742991474532665515591016625431163403791601}{35422365044898363348890904095095683200000} \right) \\
&- \frac{44535077782003477491448862123}{21923080164603435417600} \zeta_3 - \frac{87853979961663488}{657294247863878625} L - \frac{65536}{2028117} L^2 \\
&+ C_F^2 \left( \frac{3633687108507123867092545337354886401661662813}{1946283793675734249939066066456576000000} \right) \\
&- \frac{35752565032549034445184262371577}{230192341728336071884800} \zeta_3 + \frac{117265192181399552}{160671927255614775} L + \frac{3932160}{7436429} L^2
\end{align*}
\]

\[
C_{11,a}^{(2)} = C_F T_R \left( \frac{157211774390308655582420992}{2538375075612476243757739125} n_l - \frac{175561738935918272688195117821}{32577556816887651568566923600} n_h \\
+ \frac{99187789191302813}{22641143439163392} n_h \zeta_3 + \frac{32831051515756544}{1083629464278304375} L(n_l + n_h) + \frac{524288}{50702925} L^2(n_l + n_h) \right)
\]

\[
\begin{align*}
&+ C_F C_A \left( \frac{1157200807257694296198427311596640234150785132824303}{118158629172487782548367075821558742122496000} \right) \\
&- \frac{261883214808207508025819129858143}{32143221535884018765004800} \zeta_3 - \frac{619633587273728}{5184829972597275} L - \frac{1441792}{50702925} L^2
\end{align*}
\]
\[
C_{12}^{(2),a} = C_F T_R \left( \frac{92827398267438355820629508906}{16441746746885353930615808296875} n_l - \frac{104450315270262159531899064503}{196742892149046993786639482880} n_h \right) \\
+ \frac{5341389426809173}{12349714603180032} n_h \zeta + \frac{18891850399998132224}{6875843038954989160625} L(n_l + n_h) \\
+ \frac{4194304}{456326325} L^2(n_l + n_h)
\]

\[
C_{S,9}^{(2),a} = \left( \frac{14691137807907519650408236687148333789997689179014563981}{3741689923795446447364957401016026883387994000000} \right) \\
- \frac{81811578538634207243886625532379}{250466661318576769574400} \zeta_3 - \frac{67346421666621587456}{625325845808680891875} L - \frac{11534336}{456326325} L^2 \\
- \frac{325873646497863020834421263009179937}{964296646076520562950144000} \zeta_3 \\
+ \frac{1864081349501562388848}{254762381625758881875} L + \frac{8388608}{16900975} L^2
\]

\[
C_{S,10}^{(2),a} = \left( \frac{15199793691885953196394940736545376607752983761243261719}{3741689923795446447364957401016026883387994000000} \right) \\
- \frac{81811578538634207243886625532379}{250466661318576769574400} \zeta_3 - \frac{67346421666621587456}{625325845808680891875} L - \frac{11534336}{456326325} L^2 \\
- \frac{325873646497863020834421263009179937}{964296646076520562950144000} \zeta_3 \\
+ \frac{1864081349501562388848}{254762381625758881875} L + \frac{8388608}{16900975} L^2
\]

\[
C_{S,11}^{(2),a} = \left( \frac{15199793691885953196394940736545376607752983761243261719}{3741689923795446447364957401016026883387994000000} \right) \\
- \frac{81811578538634207243886625532379}{250466661318576769574400} \zeta_3 - \frac{67346421666621587456}{625325845808680891875} L - \frac{11534336}{456326325} L^2 \\
- \frac{325873646497863020834421263009179937}{964296646076520562950144000} \zeta_3 \\
+ \frac{1864081349501562388848}{254762381625758881875} L + \frac{8388608}{16900975} L^2
\]
\[ C_{S,12}^{(2),a} = - \frac{547363507378582633984776869035596987576523}{436094396712755996196381981470399397888000000} + \frac{26173581704692859}{1728960044445204480} \zeta_3 \]

\[ - \frac{466514873679872}{137042701251080625} L_{-q^2} \]

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