Weak non-linearities and cluster states

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We propose a scalable approach to building cluster states of matter qubits using coherent states of light. Recent work on the subject relies on the use of single photonic qubits in the measurement process. These schemes have a low initial success probability and low detector efficiencies cause a serious blowup in resources. In contrast, our approach uses continuous variables and highly efficient measurements. We present a two-qubit scheme, with a simple homodyne measurement system yielding an entangling operation with success probability 1/2. Then we extend this to a three-qubit interaction, increasing this probability to 3/4. We discuss the important issues of the overhead cost and the time scaling, showing how these can be vastly improved with access to this new probability range.

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I. INTRODUCTION

The intriguing idea of one-way or cluster state quantum computing was initially developed by Briegel and Raussendorf [1]. They showed that a two-dimensional array of qubits, entangled in a particular way (through Conditional Phase gates), combined with single qubit operations, feed forward and measurements are sufficient for universal quantum computation. All the required interactions are already contained inside the system, and the computation proceeds through a series of local measurements (with classical feed forward), efficiently simulating quantum circuits. In effect, the logical gates are prepared off-line and imprinted onto the qubits as they are transmitted through the cluster.

This approach was quickly applied [2, 3, 4] to linear optics quantum computing [5, 6] (and references therein), both having been experimentally demonstrated [7, 8]. It pushes the problem with the probabilistic nature of 2-qubit gates into the off-line preparation of the cluster [2, 3]. In this context it was shown that simple parity gates are sufficient for building the required states. These schemes are then bounded by the single beam-splitter success probability of 1/2 and in fact this initial probability is far reduced when the single photon detection in- eficiencies are taken into account. Supplementing the linear optical approaches with weak nonlinearities [9, 10, 11, 12, 13] allows for the construction of parity gates with significantly higher success probabilities (near unity in some cases). A core issue however with photonic qubits is their ‘flying’ nature and the storage requirements this mandates.

A natural way around this issue is to move to solid state or condensed matter qubits and use single photons for communication between them. Many proposals make use of single photons to effectively mediate interactions between matter qubits [14, 15, 16, 17, 18, 19]. Having interacted with them, the photons then interact with each other in a linear op-
interaction Hamiltonian of the form \[27, 28]:

\[ H_{int} = \hbar \chi \sigma_z a^\dagger a. \] (1)

where \( a (a^\dagger) \) refers to the annihilation (creation) operator of an electromagnetic field mode in a cavity and the matter qubit is described using the conventional Pauli operators, with the computational basis being given by the eigenstates of \( \sigma_z = |0\rangle \langle 0| - |1\rangle \langle 1| \), with \(|0\rangle \equiv | z_0 \rangle \) and \(|1\rangle \equiv | z_1 \rangle \). The atom-light coupling strength is determined via the parameter \( \chi = g^2 / \Delta \), where \( 2g \) is the vacuum Rabi splitting for the dipole transition and \( \Delta \) is the detuning between the dipole transition and the cavity field. The interaction \( H_{int} \) applied for a time \( t \) generates a conditional phase-rotation \( e^{i\theta} \) (with \( \theta = \chi t \)) on the field mode dependent upon the state of the matter qubit. We call this a conditional rotation and it is very similar to the cross-Kerr interaction between photons. This time dependent interaction implicitly requires a pulsed probe.

Now the interaction in (1) forms the basis for an entangling operation. A two-qubit gate has been proposed \[13\] based on controlled bus rotations and a subsequent measurement. The probe field coherent state \( |\alpha\rangle \) interacts with both qubits, so an initial state of the system \( |\Psi_i\rangle = 1/2 (|00\rangle + |01\rangle + |10\rangle + |11\rangle)|\alpha\rangle \) evolves to

\[ |\Psi_f\rangle = \frac{1}{2} (|00\rangle|\alpha e^{2i\theta}\rangle + |11\rangle|\alpha e^{-2i\theta}\rangle) + \frac{1}{2} (|01\rangle + |10\rangle)|\alpha\rangle. \] (2)

Here we quickly observe that the probe field has evolved into three potentially distinct states and appropriate measurements can project our two qubits into a number of interesting states. At this stage we can choose from different types of measurements on the probe beam. The first and simplest option we have is to perform a homodyne measurement of some field quadrature \( X(\phi) = (\alpha e^{i\phi} + \alpha e^{-i\phi}) \) which for a sufficiently strong local oscillator (compared to the signal strength) implements a projective measurement \( |x(\phi)\rangle \langle x(\phi)| \) on the probe state \[29\]. The key advantages with homodyne measurement, at least in the optical regime are that it is highly efficient (99% plus \[30\]) and is a standard tool of continuous variable experimentalists. The simplest homodyne measurement to perform is the momentum \((P = X(\pi/2))\) quadrature. In this case the measurement probability distribution has three peaks with the overlap error between them given by \( P_{err} = 1/2 \text{erfc}(\alpha \sin \theta / \sqrt{2}) \). As long as \( \alpha \theta \sim \pi \) this overlap error is small \((< 10^{-3})\) and the peaks are well separated. If our \( P \) quadrature measurement projects us onto the central peak \( |\alpha\rangle \), our two matter qubits are conditioned into the entangled state \((|01\rangle + |10\rangle)/\sqrt{2}\). This occurs with a probability of 1/2. Detecting either of the other two side peaks will project the qubits to the known product states \(|00\rangle \) or \(|11\rangle \). The probability of entangling the two qubits is interesting in that we have already reached the limits of conventional linear optical implementations. When realistic detector efficiencies \((\eta \sim 70\%)\) are taken into account, their optimal success probability of 1/2 decreases dramatically (proportional to \( \eta \) or \( \eta^2 \) depending on the implementation) and so the probability of the operation succeeding is now significantly less than 1/2. In contrast homodyne measurements are highly efficient and so our success probability will remain very close to 1/2. This limit may be fundamental to the linear optical schemes but in our case we can exceed it by changing the nature of our measurement. In principle we could achieve a near deterministic gate if we measured the the position quadrature \((X = X(0))\), however the requirements to ensure the distinguishability of the probe beam states are much more severe. We could also in principle use a photon number measurement after displacing the probe beam \[13\], but we would fall back into the issues affecting the linear optical schemes. By restricting ourselves to \( P = X(\pi/2) \) quadrature measurements and single interactions between the qubits and the probe, we are opting for the most robust weak-nonlinear approach so far proposed.

Within the same framework of conditional rotations and \( P \) measurements, one can envisage three qubits interacting with a single probe beam. GHZ states are for instance one particularly useful state \[3\]. One way of projecting the qubits onto GHZ-type states is to vary the strength of the interactions between the qubits and the probe beam \[12\]. Let us represent a rotation of the coherent probe beam by \( R(\theta \sigma_z) = \exp(i\theta \sigma_z a a^\dagger) \). The sequence \( R(\theta \sigma_z) R(\theta \sigma_z) R(-2\theta \sigma_z) |\alpha\rangle \) which we depict in Fig (1) will give the optimal paths and end points in phase space. The peak centered on the origin will then correspond to the GHZ state \(|000\rangle + |111\rangle\sqrt{2} \) (after being detected). This will happen with a probability of 1/4 (all qubits started in perfect superpositions). Next the two peaks having been rotated through \( \pm 2\theta \) will correspond to the qubit states \(|01\rangle_{1,2} + |10\rangle_{1,2}\rangle|1\rangle_3/\sqrt{2} \) and \(|01\rangle_{1,2} + |10\rangle_{1,2}\rangle|0\rangle_3/\sqrt{2} \) respectively. Now in both of these possible outcomes we obtain the same Bell state on qubits 1 and 2, disentangled from qubit 3. So overall we obtain a GHZ state with probability of 1/4 and a Bell state with probability of 1/2 (on two qubits of our choice), heralded by the probe beam \( P \) quadrature measurement outcome. The other two outcomes project the qubits to known product states. Consequently, if all we want to do is entangle a pair of qubits, we can now do this with a probability of 3/4.

![FIG. 1: Schematic diagram (a) of a three qubit entangling operation. In (b) the possible probe trajectories caused by the three conditional rotations. There are five different end-states. Upon measurement, three of these will project the qubits to entangled states of interest.](image-url)
This method can be extended to larger numbers of qubits, but the 3-qubit case minimizes the ratio of operation time over success probability. We shall use this result in the remainder of the paper, observing how current work on the generation of cluster states is simply inadequate for probabilities exceeding 1/2. Until now strategies have been said to be scalable if the resources don’t scale exponentially with the size of the cluster (in general they will scale sub-exponentially). This is a purely theoretical notion which bares little relation to the practical scalability we obtain in our approach.

We stress that although the 3-qubit operation is a probabilistic entangling operation with different outcomes, these outcomes are heralded by the measurement of the bus and so the operation is a very useful entangling primitive for the construction of cluster states. For example, applying it to join two sections of cluster with a third ancillary qubit works with a new dangling bond (probability 1/2), or joined clusters and two new dangling bonds (probability 1/4). Applying the operation to join three sections of cluster gives (heralded) outcomes of two sections joined and a new dangling bond (probability 1/2), or all three sections joined and two new dangling bonds (probability 1/4). All these outcomes contribute to cluster state construction.

III. SCALING

Now let’s turn to the issue of building up linear cluster states (chains). In order to efficiently grow a chain with probabilistic gates, one needs to first inefficiently build small chains exceeding a critical length \( L_c = 1 + 2(1 - p)/p \) and then try joining them to the main one. This critical length varies between different entangling operations. If an actual conditional phase gate can be immediately implemented, then \( L_c = 2(1 - p)/p \) for example. Or if this logical gate requires the qubits from the cluster to interact directly (non-distributive approach) then \( L_c = 4(1 - p)/p \). Starting from this, and adopting a ‘divide and conquer’ approach to building these minimal chains, scaling relations are obtained for the average number of entangling operations required and the average time taken, to build a chain of length \( L \). Using our 2-qubit gate \((L_c = 3)\) and these scaling relations we obtain \( N[L] = 12L - 38 \). This is already the limiting scenario for simple single photon applications. In the repeat-until-success method \([21]\), for a failure probability of 0.6 (and equal success and insurance probabilities, on all results), the scaling is \( N[L] = 185L - 1115 \) and for a failure probability of 0.4 it becomes \( N[L] \simeq 16.6L - 47.7 \). Now if we switch to our 3-qubit gate, then \( L_c < 2 \) and our minimal chain is now simply a 2-qubit cluster (locally equivalent to a Bell state) yielding \( N[L] = 8L - 44/3 \). This is a vast improvement over previous proposals. For the two-qubit entangling gate, we essentially stand at the same point as the photonic cluster state approaches. Optimizing the resources boils down to finding the optimal strategies in combining elements of cluster states. This is a very complex task, which Gross et al. \([32]\) analyzed in great detail.

For higher probabilities however, this critical length insuring average growth is no longer existent. All previously derived strategies become trivial within this probability range. Additional scalable approaches such as sequential adding are at hand and we shall go over the obvious ones. From previous works on generating cluster states \([20, 31]\), we know that the simplest way to grow short chains with probabilistic gates is through a ‘divide and conquer’ approach. It also turns out to be much quicker than a sequential adding, as we allow for many gates to operate in parallel. This technique links up chains of equal length on each round, and discards the chains which failed to do so.

In the context of higher success probabilities this approach can be extended to growing large chains in the aim of saving time. The corresponding average number of entangling operations becomes:

\[
N_{dc}[L] = \frac{(2/p)^{\log_2(L - 1)} - 1}{2 - p}.
\]

From the initial strategy we reach a value linear in \( L \):

\[
N[L] = \frac{(2/p)^L - 1 - 2(1 - p)/p - 1 - 1/p}{1 - 2(1 - p)/p} = 1/p,
\]

and a sequential adding yields:

\[
N_{seq}[L] = (L - 1)/(2p - 1).
\]

Obviously the latter represents a considerable saving, as can be verified in Fig. \([2]\). Though the divide and conquer method doesn’t scale linearly, up till lengths of 250 qubits, it requires less entangling operations than the initial scheme (which in fact is a full recycling approach). This is due to the fact that the probabilities we are dealing with are significantly higher than in previous proposals, which were undertaken in two steps, the building of minimal elements and then their merging, in order to be scalable. If we look at the qubit resources however, the less recycling we do, the more qubits we waste in the process. But as the success probability of the gate increases, the recycling strategies all converge with the no-recycling strategy (in terms of qubit resources), this being particularly noticeable for success probabilities higher than 1/2.

We can also compare the time scaling of these various strategies, in units of time \( t \) corresponding to a single measurement. For the complete divide and conquer scheme we simply have:

\[
T_{dc}[L] = t(1 + \log_2(L - 1))
\]

and for the initial scheme:

\[
T[L] = \left(1 + \log_2\left(\frac{L - L_c}{L_0 - L_c}\right)\right) \frac{t}{p}.
\]

For the sequential adding, the cumulative time obeys \( T_{L+1} = T_L + t/p \), and the general form for \( T \) becomes:

\[
T_{seq}[L] = t(L - 1)/p.
\]

The first two approaches have a logarithmic dependence on the length \( L \), however \( T_{dc} \) is significantly lower as might have
The cluster state comprises of active regions in which it is being built or measured in the computation (both can be undertaken simultaneously) and regions in which the qubits are simply waiting. Now this waiting can be minimized in the building itself, through the appropriate protocols, and in the measurement process. That is, the cluster can be built only a few layers in advance, so that the qubits have less waiting to do, between the building and the actual measurement. In any case, there will be some waiting. Therefore the lowest decoherence realization would be preferred, but it may not be the easiest to manipulate. Thus we may envisage having two different physical realizations constituting the cluster state. For example, we could use single electron spins initially in building the cluster. Once the links are made between one site and its nearest neighbors, the qubit could be switched into a nuclear spin state which has a significantly longer coherence time, via a swap operation or some other coherent write and read actions. Most of the waiting would be done in the long-lived state, before being swapped again for the readout [33, 34]. This follows the same scenario as using a second physical system to mediate the interaction and make the measurements, in distributed quantum computing. In the present proposal, we use a continuous variable bus and homodyne measurements to generate the links. This physical system shows itself to be very efficient in this application. Then, for example, electron spins or superconducting charge qubits could be the matter realization interacting with the bus and serving for the final readout. These systems provide the useful static aspect required, they interact well with the mediating bus and ensure good single qubit measurements. Finally a low decoherence realization such as nuclear spin could be envisaged, mainly as a storage medium. The swapping or write and read procedure should have a high fidelity for this storage to be beneficial. On the whole, we see that optimization will depend directly on the physical realization(s) we have chosen to work with. For systems with long dephasing times we would give priority to sequential adding approaches, as we have some freedom in the time scaling and thus we can make significant savings in resources. But for realizations with short dephasing times, we would probably want to divide the task up and run operations in parallel, in order to accelerate the fabrication of the cluster state, at the expense of extra resources.

V. CONCLUSION

In summary we have shown how the concept of the quantum bus can be adapted to efficiently generating cluster states of matter qubits. We can straightforwardly gain access to entangling probabilities higher than 1/2, removing the need to break up the building process into inefficient and efficient parts. This opens up a new class of strategies, for which the resource consumption and the time scaling are consequently vastly improved. Clearly, within this class, detailed strategies can be envisaged and they will depend on the chosen physical realization and the levels of decoherence present.

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