On the Concept of Frequency in Signal Processing

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Abstract

Frequency is a central concept in Mathematics, Physics, and Signal Processing. It is the main tool for describing the oscillatory behavior of signals, which is usually argued to be the manifestation of some of their key features, depending on their nature. For instance, this is the case of Electroencephalographic signals. Hence, frequency is substantially present in the most common methodologies for analyzing signals, as the Fourier Analysis or the Time-Frequency Analysis.

However, in spite of its importance as a keystone in Signal Processing, and its seemingly simple meaning, its mathematical foundation is not as straightforward as it may seem at first glance. A naive interpretation of the different mathematical concepts modelling frequency can be misleading, as their actual meanings essentially differ from the intuitive notion which are supposed to represent.

In our opinion, this circumstance should be taken into account in order to develop appropriate signal analyzing and processing tools in some applications. In the current text we discuss this topic, with the main goal to draw the attention of the mathematical and engineering community to this point, often overlooked.

Keywords — Frequency, Time-Frequency Analysis, Fourier Analysis, Signal processing.

Motivation

The concept of “frequency” is almost ubiquitous in Signal Processing and the many disciplines which make extensive use of it. Among them, the Bioengineering, in which frame the processing and analysis of bioelectric signals is a main task. Particularly, in Neurology and Cardiology, respectively, the appropriate analysis of Electroencephalographic (EEG) and Electrocardiographic (ECG) signals is essential in order to understand the brain and heart electrophysiology ([39], [30], [33], [32], [17], [10]).

Specially interesting is the case of Neurophysiology. In spite of there is not yet total consensus in the clinicians and researchers scientific community about the specific signal patterns associated to healthy states or each illness, and how they can be trustfully used in Clinical Neurophysiology, there is ample evidence that this kind of biometric signals play an important role (and will play an increased one) in assessing the physiological function in the brain ([23], [2]).

EEG signals in a healthy subject, apart from noise and artifacts, and many other considerations, are supposed to be the superposition or juxtaposition of brain rhythms ([39], [32]). From the point of view of Neuroscience, brain rhythms are the manifestation of the collective and synchronous activity of millions of neurons, and depend on many factors. However, they are characterized by frequency and amplitude ranges ([32]). Clinical experts are trained on visual inspection of EEG, mainly based on examining the oscillations of the signal, frequently by counting the zero crossings ([30], [32], [33]). Simplifying, oscillations are outlined by the ups and downs of the signal, where duration and height between them determine the frequency and amplitude, respectively.

In the absence of a rigorous definition (in a mathematical sense), ignoring the neurophysiological aspects and from a mathematical or signal processing point of view, brain rhythms are oscillatory signals which possess a close-to-uniform oscillation rate, which fluctuates inside a specific range, called frequency band (typically known as δ, θ, α, β, γ rhythms). Moreover, there is not even yet complete consensus about the limits of the frequency bands ([2]).
Consequently, from the perspective of Signal Processing, the basic goal is to decompose signals into rhythms, or at least being able to analyze these “components” or differently “oscillatory-behaved” manifestations. Because of the lack of firmly established and consensual definitions, this turns to be a hard task, far from being completely understood. In view of the basic (or “naïve”) description of rhythms, it is not surprising that frequency-based analyses are in the heart of this kind of analysis, and many techniques and procedures giving place to automatized analytical algorithms have been developed to process EEG signals (39, 30, 33, 32, 10, 29).

One question arises: how these fuzzy notions could be mathematically modeled? Well thought-out, this is a bewildering and challenging question. The customary answer follows the following paradigm: the current signal is a relation between the independent variable, let say time, and the main magnitude (in the EEG case, voltage difference). But there is another, the spectral representation of the signal (depending on the chosen technique), which is another function which relates to each frequency the amplitude corresponding to the basic oscillation at this frequency; that is, the strength at which this harmonic participates or takes part in the composition of the signal. Clearly, the first description is elementary; it is completely linked to the perception or measurement. However, the second one is less intuitive, and depends strongly on the definition of frequency itself and the analysis procedure performed.

According to this description, rhythms need to be understood in terms of the chosen spectral representation and the corresponding concept of frequency. However, it is clear that any performed analysis should respect the neuroscientists paradigm, in terms of oscillations, and the subsequent components in which signals are split should be meaningful for the specialists. Namely, the used concept of frequency, in every case, should not deviate a lot from this perspective.

Having said that, it is clear that the used concept of frequency is critical in this framework. But, what do we exactly mean/understand by “frequency” in Signal Processing? How is it related to oscillations? Is it successful or adequate for answering the key questions in Neuroscience, or other disciplines?

**Classical Fourier analysis**

Undoubtedly, Signal Processing borrowed the concept of frequency from Physics and Mathematical Analysis, specifically from Fourier Analysis. The sinusoidal basic functions are given by the relations \( t \mapsto \sin(2\pi\omega t) \) or \( \cos(2\pi\omega t) \), or in its complex form, \( t \mapsto e^{2\pi i\omega t} \). Here \( i \) is the imaginary unit, and the real numbers \( t, \omega \in \mathbb{R} \) represent time (in seconds) and frequency (in hertz), respectively. The physical interpretation is the angular frequency or angular speed (but measured in hertz) in the corresponding rotary motion, and gives the rate of change of the phase, measuring the angular displacement in cycles per unit time. Thinking in terms of oscillations, frequency measures the speed of this oscillatory behaviour. The larger \( |\omega| \) is, the more cycles are traveled per unit of time. Actually, \( |\omega| \) is the number of cycles performed per unit of time itself. Note that this interpretation (in a strict sense) is naturally restricted to pure sinusoidal signals.

In Harmonic Analysis, trigonometric polynomials (linear combinations of sinusoidal functions) are studied, and more general functions are sought to be represented and developed somehow in terms of basic sinusoidal functions. These representatives are defined by the set of frequencies involved, and the corresponding amplitudes accompanying the basic sinusoidal components, in the form of sequences of coefficients or functions, depending on the nature of the frequency set (discrete or continuous). Apart from the selected set of frequencies, these amplitudes, which are the response in frequency of the signal, or spectral representation of the function, define the representation and are sought to characterize the signal.

Hence, the representation process is split into two ones: the analysis, which consists in obtaining the representative given by the corresponding amplitudes, and the synthesis, which consists in recovering the signal from its representative, both performed by suitable operations. These representations usually take the form of discrete or continuous expansions, depending on the nature of the frequency set, and series or integrals respectively appear in the definition of such representations.

Basically, classical Fourier Analysis consists of two main areas: the Fourier Series, and the Fourier Transform (we refer to 13, 20, 17 for details).

The Fourier Series theory deals with periodic functions (signals), and provide an effective way for analyzing and synthesizing them. Here, the most remarkable result is the Plancherel theorem: the suitably normalized trigonometric system \( \{ e^{\pm 2\pi ik / (b-a)} / \sqrt{b-a} : k \in \mathbb{Z} \} \) is an orthonormal basis of the Lebesgue space of locally square summable \((b-a)\)-periodic functions, where \( \mathbb{Z} \) is the set of integers. Consequently, any such periodic function \( f \) writes as the corresponding Fourier series, where the information of \( f \)
is encoded into its Fourier coefficients which, due to the orthonormality, are given by the following expression,

\[ \tilde{f}(k) = \int_a^b f(t) e^{-2\pi i k t/(b-a)} \sqrt{b-a} \, dt \quad (k \in \mathbb{Z}), \]

and satisfy the Parseval identity:

\[ \int_a^b |f(t)|^2 \, dt = \sum_{k \in \mathbb{Z}} |\tilde{f}(k)|^2. \]

Analogously, the Fourier Transform deals with functions defined on the whole real line \( \mathbb{R} \). For an integrable function \( f \), the Fourier transform is given by:

\[ \tilde{f}(\xi) = \int_{-\infty}^{\infty} f(t) e^{-2\pi i \xi t} \, dt \quad (\xi \in \mathbb{R}), \]

It turns to be a unitary operator defined in the Lebesgue space of square summable functions \( L^2(\mathbb{R}) \), satisfying the Plancherel identity,

\[ \int_{-\infty}^{\infty} |f(t)|^2 \, dt = \int_{-\infty}^{\infty} |\tilde{f}(\xi)|^2 \, d\xi. \]

Note that in the first case, the set of frequencies is \( \mathbb{Z} \), we represent periodic functions, the analysis process consists on computing the sequence of Fourier coefficients, and the synthesis process consists on summing the Fourier series of the function. In the second case, the set of frequencies is \( \mathbb{R} \), we represent functions in \( L^2(\mathbb{R}) \), and the analysis and synthesis processes are performed by the Fourier and inverse Fourier transform, respectively.

These objects extend the concept of frequency from pure sinusoidal functions to more general signals. Now, frequency is a parameter which indexes the set of responses of the signal when it is faced against a pure oscillation, resulting the corresponding value of the amplitude in the representation: the Fourier coefficient \( \tilde{f}(k) \) or the Fourier transform value \( \tilde{f}(\xi) \), respectively.

But there is (always) a price. In this framework, the exact meaning of the concept “frequency” is hidden behind the frequency variables \( k \) and \( \xi \), respectively. It is no longer an oscillation speedometer (although it is supposed to be). This is the point we want to highlight here, which plunges into the deep nature of the greatest problem in time-frequency analyses, since it is inherent to the central concept of frequency itself.

It is easy to see some eloquent symptoms of this: one of the main tasks in signal processing is time-frequency localization, or identification of the characteristic oscillations of signals, and where they occur. First of all, it is worth noting that, in real applications, one (always) deals with discrete (actually finite) signals on a finite interval. Nevertheless, the corresponding theory for analyzing finite signals runs parallel to the continuous one.

The analysis is mostly performed by (or based on) linear functionals. Under orthogonality conditions (or similar, as in Frame Theory -for the theory of frames and Riesz bases we refer to [1]), one computes the inner product of basic harmonics on a finite window. Thus, for any \( \xi, \eta \in \mathbb{R} \) one has

\[ \int_a^b e^{2\pi i \eta t} e^{-2\pi i \xi t} \, dt = (b-a) e^{\pi i (\eta-\xi)(a+b)} \frac{\sin(\pi(\eta-\xi)(b-a))}{\pi (\eta-\xi)(b-a)}. \]

Note that this can be interpreted as the Fourier transform of a windowed pure tone, also. This implies that, computed in this way, basic tones interfere each other in the whole spectrum of frequencies, excluding a discrete sequence of pairings in harmonic consonance: \( (\eta-\xi)(b-a) \in \mathbb{Z} \). Also, this means that this Fourier transform is entirely spread over \( \mathbb{R} \), not condensed at the corresponding frequency \( \eta \). In some sense, this interference disappears when one considers the “whole signals”, supported in the whole real line, but this occurs outside the frontiers of \( L^2(\mathbb{R}) \) because of the failing of integrability. This results very inappropriate since, apart from mathematical technicalities, recall we are usually interested in studying signals on finite intervals. But it turns out that Fourier transform is designed to handle signals defined in \( \mathbb{R} \). You can restrict the support of the signal, but you don’t fool Fourier transform.

The other side of the same coin are the Fourier coefficients on an interval, where orthogonality relations are obtained at the price of neglecting the dissonant frequencies. As a consequence, Fourier series scatters the energy of a dissonant pure tone through the whole spectrum, due to the Parseval identity, giving place
Harmonic (or Fourier) analysis is a classical discipline, which has turned to be the cornerstone of many others, in pure and applied sciences. Concerning to time-frequency localization, some techniques have been developed based on these concepts, as the family of Fast Fourier Transforms (FFT) algorithms (see [12], [11], [21], [10]), or the Theory of band-limited functions and the Prolate Spheroidal Wave Functions (PSWF) (see [37], [27], [28], [34], [31]). These tools have been successfully applied in many applications. However, in spite of their remarkable properties, there are also disadvantages in their use. For instance, PSWFs are computationally challenging ([42]). Anyway, we want to focus here on the fact that all these techniques depend on the concepts of frequency just explained above, and consequently share their faults.

Going further, beyond the classical harmonic framework, much effort has been devoted to developing alternative tools and procedures for capturing the main features of a signal by simultaneously localize it in time and frequency. In other words, the point is to being able to identify the pairs of time windows and frequency bands in which characteristic oscillations of the signal occur. These different approaches can be included under the common name of Time-Frequency Analysis (see [8], [20], [4]). In contrast to classical Harmonic Analysis, Time-Frequency Analysis treats (or attempts to) the time and frequency variables equally, as primary concepts, and works with them simultaneously.

Many techniques have been developed with this flavor: the Short-Time Fourier Transform (STFT), in which a suitable (very localized in time and frequency) window function is used to “redefine” the Fourier Transform; the Spectrogram, the Ambiguity function, the Wigner-Ville distribution, and other Quadratic time-frequency representations, usually closely related to the STFT, which can be thought as energy densities in the time-frequency plane, which are not completely satisfactory (see [20]); and also the Gabor frames, where discrete expansions in terms of time-frequency shifts on a lattice of a suitable window function are considered. All these tools have been successfully applied to many processing tasks, as feature extraction, separation of signal components, and signal compression. See [8], [20], [4] for more details.

In some sense, the Time-Frequency techniques define a local frequency spectrum which is supposed to report the strength of each oscillation in the signal. However, in spite of their success, they borrow their concept of “frequency” from Fourier analysis. That is, the response in frequency is obtained by facing up (by integration) an averaged portion of the signal (by means of window functions) to the exponential kernel, which encodes the oscillation speed; that is, the chosen frequency. Consequently, the deficiencies of this approach are inherited from the corresponding Fourier transform ones; they cannot break out of the limits that classical Fourier Analysis imposes. It is worth noting that the frequency variable is not actually an absolute primary concept; the way it appears in each transform (and only that) accurately determines its precise meaning, not the name with which it was coined.

Some symptoms of this situation are well known actually (although the diagnosis maybe not so well). Two main principles are ubiquitous in Time-Frequency Analysis (they refer to the time and spectral representations of the signal): 1. The smoothness-decay duality: if one representation is smooth, the other one decays. 2. The uncertainty principles: the two representations cannot be simultaneously well localized.

About the first one, at first sight it seems reasonable that sudden jumps in the signal produce respective manifestations at high frequencies. But consider the following example. Let the square signal given by the sign of $\sin(2\pi t)$. How would you describe its periodicity, its oscillation, in terms of frequency? How do you think a neurophysiologist would do it? Now compute its Fourier transform and compare.

Concerning the uncertainty principles, they constitute a well studied topic (see [5], [8], [12], [20], [16], [38], [18]): the Heisenberg-Pauli-Weyl inequality, the Donoho-Stark uncertainty principle (about the
concentration of a signal and its Fourier Transform, in a wider sense: on the real line, periodic signals, discrete signals, on LCA groups. See [5], [20], [10], [35]: Lieb’s inequalities (for the STFT); the radar uncertainty principle (for the Ambiguity function); the Wigner-Ville distribution uncertainty principles, or the results on the positivity of smoothed Wigner-Ville distributions; the Balian-Low theorem and density theorems for Gabor frames, etc.

About their meaning, as far as we are concerned, the uncertainty principles make the existence of an absolute and ideal concept of instantaneous frequency (or spectrum) impossible (see [20] for a detailed discussion), in the following sense: there is no (square-summable) function concentrated on an arbitrarily small interval with Fourier transform also concentrated on an arbitrarily small band. However, in this regard, it is extremely interesting to recover the discussion in [35] (see also [30]) about the hypothesis of bandlimitedness and the mathematical modelling of “real world” signals, and the 2WT-theorem.

Nevertheless, there are exceptions. Remarkable ones are the Wilson basis and the Malvar basis (see [14], [15], [9]), or more generally the local Fourier bases. Being extremely simplistic, the basic idea is to consider simultaneously symmetric frequencies ($\xi$ and $-\xi$), replacing complex exponentials by sine and cosine functions, and overlap block families with suitable window functions (see [1], [20], [22]). This kind of bases escape from the uncertainty principle. But again, due to its structure, as orthogonal bases and $\xi$-functions, and the resulting time-frequency-amplitude/energy distribution is called the Hilbert Spectrum (see [24], [25], [29]).

Another case of interest is Wavelet theory. Wavelets have been applied with great success in Signal processing and many other disciplines. It is a fact that there are well-localized in time-frequency wavelets, very suitable for time-frequency analysis (see [13], [22], [41]). Nevertheless, not all that glitters is gold. In this case, in spite of Fourier Analysis is a fundamental tool in Wavelet theory (the fact that Fourier transform interchange translations and modulations basically explains the interplay between Wavelet and Gabor Analyses), actually the concept of frequency is in some way replaced by the concept of scale (or resolution), as modulations are replaced by dilations, in this theory. The scale spectrum grows geometrically (at least for the Discrete Wavelet Transform, the more relevant in practice), as powers of two in the classical setting. Also, the mesh refines according to scale. These circumstances induce that “dissonant” features (that is, those whose scale and position fall between the lattice ones) will be scattered through several spurious components, according to energy-preservation laws. On the other hand, there is relatively much freedom in choosing the waveform for wavelets, which do not need at all to be sinusoidal (actually, they do not are waves, but “little waves”, in virtue of the so called oscillation property $\int \psi = 0$).

And last but not least, we should mention the Instantaneous Frequency. This concept has a long history, but its use and indeed its own nature have been quite controversial, in spite of it has been successfully applied in many situations, specially for the analysis of nonstationary signals. Instantaneous frequency is a magnitude which is supposed to measure a local time-varying frequency, following the spectral frequency peak. It plays the role of a variable frequency in an amplitude-phase representation (the analytic signal) which locally best fits the signal, and can be also related to the mean frequency (see [2], [5], [4]).

However, the whole spectrum of a signal cannot be represented in general by a single number, so the instantaneous frequency is just appropriate for monocomponent signals (single oscillations). Consequently, multicomponent signals need to be decomposed. Many algorithms have been proposed for that. One of the most successful is the Empirical Mode Decomposition, which is an algorithm for decomposing a signal into a finite set of oscillatory signals, called Intrinsic Mode Functions (IMF), which are (argued and supposed to be) monocomponent and thus suitable to possess a meaningful instantaneous frequency (via the Hilbert Transform and the analytic signal procedure). This is the so called Hilbert-Huang transform, and the resulting time-frequency-amplitude/energy distribution is called the Hilbert Spectrum (see [24], [25], [29]).

The Hilbert spectrum seems to settle a more appropriate notion of frequency than previous methodologies. However, in spite of their success, these tools have also counterparts. The definition and physical interpretation of the notions of instantaneous frequency, as well as the amplitude-phase decomposition, the monocomponent signal, etc. are far from being completely clarified, and doubts persist. In addition, the instantaneous frequency at each IMF is not homogeneous (it does not need to remain into a narrow band), so apparently different vibrating modes could seem to coexist at each IMF. In other words, depending on the context, the resulting IMF are not assured to be meaningful. Furthermore, the
mathematical foundation of all these data-driven techniques turn to be very challenging, and most of the confidence they generate comes from numerical experiments and particular data analyses. There are also technical problems to overcome (25).

Conclusion

Frequency is a fundamental concept in Science. As it is thought that the celebrated inventor Nikola Tesla once said, "if you want to find the secrets of the universe, think in terms of energy, frequency and vibration".

However, it is not an easy task to rigorously “define” what frequency means, or even should mean. Some attempts have firmly been established in the context of Fourier Analysis, with great success and diffusion. However, some problems arise with these definitions in the Time-Frequency Analysis, as the uncertainty principles, which seem to be counterintuitive from a naive perspective (see the musical score metaphor in [5], [20]), or quite natural, from the perspective of Quantum Mechanics. Other attempts, as the instantaneous frequency, are quite promising, but need a more rigorous ground.

In this regard, it is interesting to recover the discussion in [35] (see also [36]) about the mathematical modelling of “real world” signals. Slepian distinguishes two constitutive components in quantitative physical sciences: Facet A, for the real world of observations and measurements, and Facet B, for the abstract world of mathematical modelling of Facet A. According to Slepian ([35]), "as usually used [...], the words “bandlimited,” “start,” “stop,” and even “frequency” describe secondary constructs from Facet B of our field. They are abstractions we have introduced into our paper and pencil game for our convenience in working with the model. They require precise specification of the signals in the model at times in the infinitely remote past and in the infinitely distant future. These notions have no meaningful counterpart in Facet A". Moreover, in [36] he also writes that "it is senseless to ask if real signals are bandlimited, or timelimited. Verification requires real measurements at arbitrarily high frequencies or at arbitrarily remote or future times, experiments that can never be carried out. The notions of bandlimitedness or timelimitedness belong to the engineer’s model, not the real world [...]. As it suits him, he can assume in his model either that his signals are timelimited, or that they are bandlimited, or neither. But he should take care that the deductions he makes from his model about the real world do not depend sensitively on which assumption he has made ".

This is definitely an old topic, but still in vogue. In our humble opinion, there is a real need for generating new strategies which could be able to fill the previously explained gaps between theory and practice, between real problems and mathematical modelling answers. Consequently, there are many developments to come in this area, which will produce more appropriate and fruitful notions of frequency, susceptible to be applied in practice in many disciplines, as EEG Analysis.

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