AN ISOPERIMETRIC RESULT ON HIGH-DIMENSIONAL SPHERES

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Abstract. We consider an extremal problem for subsets of high-dimensional spheres that can be thought of as an extension of the classical isoperimetric problem on the sphere. Let \( A \) be a subset of the \((m-1)\)-dimensional sphere \( S^{m-1} \), and let \( y \in S^{m-1} \) be a randomly chosen point on the sphere. What is the measure of the intersection of the \( t \)-neighborhood of the point \( y \) with the subset \( A \)? We show that with high probability this intersection is approximately as large as the intersection that would occur with high probability if \( A \) were a spherical cap of the same measure.

1. Introduction

Let \( S^{m-1} \subseteq \mathbb{R}^m \) denote the \((m-1)\)-sphere of radius \( R \), i.e.,
\[
S^{m-1} = \{ z \in \mathbb{R}^m : \|z\| = R \},
\]
equipped with the rotation invariant Haar measure \( \mu \) that is normalized such that
\[
\mu(S^{m-1}) = \frac{2\pi^{\frac{m}{2}}}{\Gamma(\frac{m}{2})} R^{m-1}.
\]
This normalization corresponds to the usual surface area. Let \( \mathbb{P}(A) \) denote the probability of a set \( A \) (that we will always assume to be measurable) with respect to the corresponding Haar probability measure that is normalized such that \( \mathbb{P}(S^{m-1}) = 1 \). A spherical cap of angle \( \theta \) and pole \( z_0 \) is defined as a ball on \( S^{m-1} \) using the geodesic metric \( \angle(z, y) = \arccos(\langle z/R, y/R \rangle) \), i.e.,
\[
\text{Cap}(z_0, \theta) = \{ z \in S^{m-1} : \angle(z_0, z) \leq \theta \}.
\]
We say that a set \( A \subseteq S^{m-1} \) has effective angle \( \theta \) if \( \mu(A) = \mu(C) \) with \( C = \text{Cap}(z_0, \theta) \) for some \( z_0 \in S^{m-1} \).

The classical isoperimetric inequality on the sphere implies that among all sets on the sphere with a given measure, the spherical cap has the smallest boundary, or more generally the smallest neighborhood \([4,5]\). This is formalized as follows:

Proposition 1 (Isoperimetric Inequality). For any arbitrary set \( A \subseteq S^{m-1} \) such that \( \mu(A) = \mu(C) \), where \( C = \text{Cap}(z_0, \theta) \subseteq S^{m-1} \) is a spherical cap, it holds that
\[
\mu(A_t) \geq \mu(C_t), \quad \forall t \geq 0,
\]
where \( A_t \) is the \( t \)-neighborhood of \( A \) defined as
\[
A_t = \left\{ z \in S^{m-1} : \min_{z' \in A} \angle(z, z') \leq t \right\},
\]
and similarly
\[
C_t = \left\{ z \in S^{m-1} : \min_{z' \in C} \angle(z, z') \leq t \right\} = \text{Cap}(z_0, \theta + t).
\]

Another basic geometric phenomenon, this time occurring only in high-dimensions, is concentration of measure. On a high-dimensional sphere this manifests as most of the measure of the sphere concentrating around any equator. This is captured by the following proposition [4].

**Proposition 2 (Measure Concentration).** Given any \( \epsilon > 0 \), there exists an \( M(\epsilon) \) such that for any \( m \geq M(\epsilon) \) and any \( z \in S^{m-1} \),
\[
P(\angle(z, Y) \in [\pi/2 - \epsilon, \pi/2 + \epsilon]) \geq 1 - \epsilon,
\]
where \( Y \in S^{m-1} \) is distributed according to the Haar probability measure.

The above measure concentration result combined with the isoperimetric inequality immediately yields the following result:

**Proposition 3 (Blowing-up Lemma).** Let \( A \subseteq S^{m-1} \) be an arbitrary set with effective angle \( \theta > 0 \). Then for any \( \epsilon > 0 \) and \( m \) sufficiently large,
\[
P(A_{\pi/2 - \theta + \epsilon}) \geq 1 - \epsilon.
\]

An almost equivalent way to state the blowing-up lemma from Proposition 3 is the following: let \( A \subseteq S^{m-1} \) be an arbitrary set with effective angle \( \theta > 0 \), then for any \( \epsilon > 0 \) and sufficiently large \( m \),
\[
P(\mu(A \cap \text{Cap}(Y, \pi/2 - \theta + \epsilon)) > 0) > 1 - \epsilon,
\]
where \( Y \) is distributed according to the normalized Haar measure on \( S^{m-1} \). In words, if we take a \( y \) uniformly at random on the sphere and draw a spherical cap of angle slightly larger than \( \pi/2 - \theta \) around it, this cap will intersect the set \( A \) with high probability. This statement is almost equivalent to (1.3) since the \( y \)'s for which the intersection is not empty lie in the \( \pi/2 - \theta + \epsilon \)-neighborhood of \( A \). Note that this statement would trivially follow from measure concentration on the sphere (Proposition 2) if \( A \) were known to be a spherical cap, and it holds for any \( A \) due to the isoperimetric inequality in Proposition 1.

Our main result is the following generalization of (1.3).

**Theorem 1.** Given any \( \epsilon > 0 \), there exists an \( M(\epsilon) \) such that for any \( m > M(\epsilon) \) the following is true. Let \( A \subseteq S^{m-1} \) be any arbitrary set with effective angle \( \theta > 0 \), and let \( V = \mu(\text{Cap}(z_0, \theta) \cap \text{Cap}(y_0, \omega)) \) where \( z_0, y_0 \in S^{m-1} \) with \( \angle(z_0, y_0) = \pi/2 \) and \( \theta + \omega > \pi/2 \). Then
\[
P(\mu(A \cap \text{Cap}(Y, \omega + \epsilon)) > (1 - \epsilon)V) \geq 1 - \epsilon,
\]
where \( Y \) is a random vector on \( S^{m-1} \) distributed according to the normalized Haar measure.
If \( A \) itself is a cap then the statement of Theorem 1 is straightforward and follows from the fact that \( y \) will be concentrated around the equator at angle \( \pi/2 \) from the pole of \( A \) (Proposition 2). Therefore, as \( m \) gets large, the intersection of the two spherical caps will be given by \( V \) for almost all \( y \)'s. The statement however is much stronger than this and holds for any arbitrary set \( A \), analogous to the isoperimetric result in (1.3). It states that no matter what the set \( A \) is, if we take a random point on the sphere and draw a cap of angle slightly larger than \( \omega \) for \( \omega > \pi/2 - \theta \), then with high probability the intersection of the cap with the set \( A \) would be at least as large as the intersection we would get if \( A \) were a spherical cap.

The authors first encountered this problem while working on a geometric framework for proving capacity bounds in information theory. A version of this result on a spherical shell with nonzero thickness was used in [1] to resolve the Gaussian case of an open problem from [10], and some partial results for a discrete version on the Hamming sphere were used in [2].

When viewed as a concentration of measure result, it might seem as though Theorem 1 could be proved with standard methods such as the concentration of Lipschitz functions [5]. We could not find a way to make this work, since the Lipschitz constant for the function \( y \mapsto \mu(A \cap \text{Cap}(y, \omega + \epsilon)) \) can increase exponentially in the dimension \( m \). Even if one is only interested in the exponential order of the intersection measure, \( \log(\mu(A \cap \text{Cap}(y, \omega + \epsilon))) \) is equal to \(-\infty\) when the sets are disjoint, so log composed with this function is not even continuous. However, if instead of the “hard” intersection measure that can be viewed as the indicator function \( 1_A \) convolved with the kernel \( 1_{\text{Cap}(y, \omega + \epsilon)} \), we are interested in the “soft” intersection measure that is \( 1_A \) convolved with some smooth kernel, then these techniques could possibly be applied. In particular, \( \sqrt{\frac{1}{m} \log P_t 1_A} \) exhibits the correct order of Lipschitz continuity where \( P_t \) denotes the heat semigroup over time \( t \) on the sphere. Once concentration of \( \frac{1}{m} \log P_t 1_A \) has been established, a rearrangement theorem such as Theorem 4.1 from [9] could be used to establish that the smallest value it can concentrate around is given when \( A \) is a spherical cap.

2. Rearrangement on the Sphere

Our main tool for proving Theorem 1 for arbitrary \( A \) is the symmetric decreasing rearrangement of functions on the sphere, along with a version of the Riesz rearrangement inequality on the sphere due to Baernstein and Taylor [8].

For any measurable function \( f : S^{m-1} \to \mathbb{R} \) and pole \( z_0 \), the symmetric decreasing rearrangement of \( f \) about \( z_0 \) is defined to be the function \( f^* : S^{m-1} \to \mathbb{R} \) such that \( f^*(y) \) depends only on the angle \( \angle(y, z_0) \), is nonincreasing in \( \angle(y, z_0) \), and has super-level sets of the same size as \( f \), i.e.

\[
\mu(\{ y : f^*(y) > d \}) = \mu(\{ y : f(y) > d \})
\]

for all \( d \). The function \( f^* \) is unique except for on sets of measure zero.

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1Special thanks to Ramon van Handel for pointing out that concentration of Lipschitz functions can indeed be used in this “soft” intersection case.
One important special case is when the function \( f = 1_A \) is the characteristic function for a subset \( A \). The function \( 1_A \) is just the function such that

\[
1_A(y) = \begin{cases} 
1 & y \in A \\
0 & \text{otherwise.}
\end{cases}
\]

In this case, \( 1_A^* \) is equal to the characteristic function associated with a spherical cap of the same size as \( A \). In other words, if \( A^* \) is a spherical cap about the pole \( z_0 \) such that \( \mu(A^*) = \mu(A) \), then \( 1_A^* = 1_{A^*} \).

**Theorem 2** (Baernstein and Taylor [8]). Let \( K \) be a nondecreasing bounded measurable function on the interval \([-1, 1]\). Then for all functions \( f, g \in L^1(S^{m-1}) \),

\[
\int_{S^{m-1}} \left( \int_{S^{m-1}} f(z)K(\langle z/R, y/R \rangle) \, dz \right) g(y) \, dy \\
\leq \int_{S^{m-1}} \left( \int_{S^{m-1}} f^*(z)K(\langle z/R, y/R \rangle) \, dz \right) g^*(y) \, dy.
\]

**3. Proof of Theorem 1**

In order to prove Theorem 1, we will apply Theorem 2 by choosing \( f, K \) such that the inner integral

\[
\int_{S^{m-1}} f(z)K(\langle z/R, y/R \rangle) \, dz = \mu(A \cap \text{Cap}(y, \omega + \epsilon)).
\]

To do this, given an arbitrary set \( A \), we set \( f = 1_A \) and

\[
K(\cos \alpha) = \begin{cases} 
1 & 0 \leq \alpha \leq \omega + \epsilon \\
0 & \omega + \epsilon < \alpha \leq \pi.
\end{cases}
\]

Note that \( K \) is nondecreasing, bounded, and measurable. Furthermore, the product \( f(z)K(\langle z/R, y/R \rangle) \) is one precisely when \( z \in A \) and \( \angle(z, y) \leq \omega + \epsilon \), and it is zero otherwise. Thus the integral

\[
(3.1) \quad \psi(y) = \int_{S^{m-1}} f(z)K(\langle z/R, y/R \rangle) \, dz
\]

is exactly the measure of the set \( A \cap \text{Cap}(y, \omega + \epsilon) \).

We will use test functions \( g \) that are also characteristic functions. Let \( g = 1_C \) where \( C = \{ y : \psi(y) > d \} \) for some \( d \) (i.e. \( C \) is a super-level set). For a fixed measure \( \mu(C) \), the left-hand side of the inequality from Theorem 2 will be maximized by this choice of \( C \). With this choice we have the following equality:

\[
\int_{S^{m-1}} \psi(y)1_C(y) \, dy = \int_{S^{m-1}} \psi^*(y)1_C^*(y) \, dy = \int_{C^*} \psi^*(y) \, dy.
\]
This follows from the layer-cake decomposition for any non-negative and measurable function $\psi$ in that

$$\int_{S^{m-1}} \psi(y) 1_{C}(y) dy = \int_{C} \psi(y) dy$$

$$= \int_{C} \int_{0}^{\infty} 1_{\{\psi(y) > t\}} dt dy$$

$$= \int_{0}^{\infty} \int_{C} 1_{\{\psi(y) > t\}} dy dt$$

$$= \int_{0}^{d} \int_{C} 1_{\{\psi(y) > t\}} dy dt + \int_{d}^{\infty} \int_{C} 1_{\{\psi(y) > t\}} dy dt$$

$$= \int_{0}^{d} \int_{C} 1_{\{\psi(y) > t\}} dy dt + \int_{d}^{\infty} \int_{C} 1_{\{\psi(y) > t\}} dy dt$$

$$= \int_{C} \psi^{*}(y) dy .$$

(3.2)

Using this equality and our choices for $f, g, K$ above we will rewrite the inequality from Theorem 2 as

$$\int_{C^{*}} \psi^{*}(y) dy \leq \int_{C^{*}} \tilde{\psi}(y) dy$$

where

$$\tilde{\psi}(y) = \int_{S^{m-1}} f^{*}(z) K ((z, R, y/R)) dz .$$

Note that $\tilde{\psi}(y)$ is exactly $\mu(A^{*} \cap \text{Cap}(y, \omega + \epsilon)).$

Note that both $\psi^{*}(y)$ and $\tilde{\psi}(y)$ are spherically symmetric. More concretely, they both depend only on the angle $\angle(y, z_{0})$, so in an abuse of notation we will write $\tilde{\psi}(\alpha)$ and $\psi^{*}(\alpha)$ where $\alpha = \angle(y, z_{0}).$

For convenience we will define a measure $\nu$ by

$$d\nu(\phi) = A_{m-2}(R \sin \phi) Rd\phi$$

where $A_{m}(R)$ denotes the Haar measure of the $m$-sphere with radius $R$. We do this so that an integral like

$$\int_{S^{m-1}} \psi^{*} dy = \int_{0}^{\pi} \psi^{*}(\phi) A_{m-2}(R \sin \phi) Rd\phi$$

can be expressed as

$$\int_{0}^{\pi} \psi^{*} d\nu .$$

We are now ready to prove Theorem 1. Proposition 2 implies that for any $0 < \epsilon < 1$, there exists an $M(\epsilon)$ such that for $m > M(\epsilon)$ we have

$$\mathbb{P} (\angle(z_{0}, y) \in [\pi/2 - \epsilon, \pi/2 + \epsilon]) \geq 1 - \frac{\epsilon^{2}}{2} .$$

(3.4)

The constant $M(\epsilon)$ is determined only by the concentration of measure phenomenon cited above, and it does not depend on any parameters in the problem other than $\epsilon$. From now on, let us restrict our attention to dimensions $m > M(\epsilon)$. Due to the triangle inequality for the geodesic metric, for $y$ such that $\angle(z_{0}, y) \in [\pi/2 - \epsilon, \pi/2 + \epsilon]$ we have

$$A^{*} \cap \text{Cap}(y_{0}, \omega) \subseteq A^{*} \cap \text{Cap}(y, \omega + \epsilon)$$
where $y_0$ is such that $\angle(z_0, y_0) = \pi/2$. Therefore,

$$
\tilde{\psi}(\angle(z_0, y)) = \mu(A^* \cap \text{Cap}(y, \omega + \epsilon)) \geq V
$$

for all $y$ such that $\angle(z_0, y) \in [\pi/2 - \epsilon, \pi/2 + \epsilon]$ and

$$
\mathbb{P} \left( \tilde{\psi}(Y) \geq V \right) = \mathbb{P} \left( \mu(A^* \cap \text{Cap}(Y, \omega + \epsilon)) \geq V \right)
\geq 1 - \frac{\epsilon^2}{2}
\geq 1 - \frac{\epsilon}{2}.
$$

(3.6)

To prove the lemma, we need to show that

$$
\mathbb{P} \left( \psi^*(Y) > (1 - \epsilon)V \right) = \mathbb{P} \left( \mu(A \cap \text{Cap}(Y, \omega + \epsilon)) > (1 - \epsilon)V \right) \geq 1 - \epsilon
$$

for any arbitrary set $A \subset \mathbb{S}^{m-1}$. Recall that by the definition of a decreasing symmetric rearrangement, we have

$$
\mathbb{P} \left( \psi^*(Y) > d \right) = \mathbb{P} \left( \psi(Y) > d \right)
$$

for any threshold $d$ and this implies

$$
\mathbb{P} \left( \psi^*(Y) \leq (1 - \epsilon)V \right) = \mathbb{P} \left( \psi(Y) \leq (1 - \epsilon)V \right) \cdot
$$

Therefore, the desired statement in (3.7) can be equivalently written as

$$
\mathbb{P} \left( \psi^*(Y) \leq (1 - \epsilon)V \right) \leq \epsilon.
$$

(3.8)

Turning to proving (3.9), recall that by the definition of a decreasing symmetric rearrangement, $\psi^*(\alpha)$ is nonincreasing over the interval $0 \leq \alpha \leq \pi$. Let $\beta$ be the smallest value such that $\psi^*(\beta) = (1 - \epsilon)V$, or more explicitly,

$$
\beta = \inf \{\alpha | \psi^*(\alpha) \leq (1 - \epsilon)V \}.
$$

If $\beta \geq \pi/2 + \epsilon$, then (3.9) would follow trivially from (3.4) and the fact that $\psi^*(\alpha)$ would be greater than $(1 - \epsilon)V$ for all $0 < \alpha < \pi/2 + \epsilon$. We will therefore assume that $0 < \beta < \pi/2 + \epsilon$. It remains to show that even if this is the case, we have (3.9).

By the definition of $\beta$ and the fact that $\psi^*$ is nonincreasing,

$$
\mathbb{P} \left( \psi^*(Y) \leq (1 - \epsilon)V \right) = \frac{1}{A_{m-1}(R)} \int_{\beta}^{\pi} d\nu
\geq \frac{1}{A_{m-1}(R)} \int_{\beta}^{\max \{\beta, \pi/2 - \epsilon\}} d\nu + \frac{1}{A_{m-1}(R)} \int_{\max \{\beta, \pi/2 - \epsilon\}}^{\pi/2 + \epsilon} d\nu
\geq \frac{1}{A_{m-1}(R)} \int_{\beta}^{\pi}/2 \cdot
$$

To bound the first and third terms of (3.10) note that

$$
\frac{1}{A_{m-1}(R)} \int_{\beta}^{\max \{\beta, \pi/2 - \epsilon\}} d\nu + \frac{1}{A_{m-1}(R)} \int_{\pi/2 + \epsilon}^{\pi} d\nu \leq \frac{\epsilon^2}{2}
\leq \frac{\epsilon}{2}
$$

(3.12)
as a consequence of (3.1). In order to bound the second term, we establish the following chain of (in)equality which will be justified below.

\begin{equation}
\frac{1}{A_{m-1}(R)} \int_{\frac{\pi}{2^{+}\epsilon}}^{\pi} d\nu \geq \frac{1}{(1-\epsilon)VA_{m-1}(R)} \int_{\frac{\pi}{2^{+}\epsilon}}^{\pi} (\psi^* - \bar{\psi}) d\nu
\end{equation}

\begin{equation}
= \frac{1}{(1-\epsilon)VA_{m-1}(R)} \int_{0}^{\frac{\pi}{2^{+}\epsilon}} (\bar{\psi} - \psi^*) d\nu
\end{equation}

\begin{equation}
\geq \frac{1}{(1-\epsilon)VA_{m-1}(R)} \int_{\beta}^{\frac{\pi}{2^{+}\epsilon}} (\bar{\psi} - \psi^*) d\nu
\end{equation}

\begin{equation}
\geq \frac{\epsilon}{(1-\epsilon)A_{m-1}(R)} \int_{\max\{\beta, \frac{\pi}{2^{+}\epsilon} - \epsilon\}}^{\frac{\pi}{2^{+}\epsilon}} d\nu
\end{equation}

\begin{equation}
\geq \frac{\epsilon}{A_{m-1}(R)} \int_{\max\{\beta, \frac{\pi}{2^{+}\epsilon} - \epsilon\}}^{\frac{\pi}{2^{+}\epsilon}} d\nu
\end{equation}

Combining (3.17) with (3.1) reveals that the second term in (3.10) is also bounded by \(\epsilon/2\), therefore

\[ P(\psi^*(Y) \leq (1-\epsilon)V) \]

must be bounded by \(\epsilon\) which implies the lemma.

The first inequality (3.13) is a consequence of the fact that over the range of the integral, \(\psi^*\) is less than or equal to \((1-\epsilon)V\) and \(\bar{\psi}\) is non-negative. The equality in (3.14) follows from

\[ \int_{0}^{\pi} \psi^* d\nu = \int_{0}^{\pi} \bar{\psi} d\nu , \]

which is itself a consequence of (3.2) with \(C = S^{m-1}\) and

\[ \int_{S^{m-1}} \psi(y) dy = \int_{S^{m-1}} \int_{S^{m-1}} f(z) K((z/R, y/R)) dz dy \]

\[ = \int \int K((y, z)) dy f(z) dz \]

\[ = \mu(Cap(y, \omega)) f(z) dz \]

\[ = \mu(Cap(y, \omega)) \mu(A) \]

\[ = \int \mu(Cap(y, \omega)) f^*(z) dz \]

\[ = \int \int f^*(z) K((z/R, y/R)) dz dy = \int_{S^{m-1}}^{\frac{\pi}{2^{+}}} \bar{\psi}(y) dy . \]

Next we have (3.15) which is due to the rearrangement inequality (3.3) when \(C\) is the super-level set \(\{y \mid \psi(y) > (1-\epsilon)V\}\). By the definition of a symmetric decreasing rearrangement, \(\mu(\{y \mid \psi(y) > (1-\epsilon)V\}) = \mu(\{y \mid \psi^*(y) > (1-\epsilon)V\})\), and the set on the right-hand side is an open or closed spherical cap of angle \(\beta\). Thus \(C^*\) is a spherical cap with angle \(\beta\) and the rearrangement inequality (3.3) gives

\[ \int_{0}^{\beta} \psi^* d\nu \leq \int_{0}^{\beta} \bar{\psi} d\nu . \]

Finally, for the inequality (3.16), we first replace the lower integral limit with \(\max\{\beta, \pi/2 - \epsilon\} \geq \beta\). Then \(\psi \geq V\) over the range of the integral due to (3.5).
Additionally, $\psi^* \leq (1 - \epsilon)V$ over the range of the integral, and the inequality follows.

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