ON EXTENSION OF PLURICANONICAL FORMS
DEFINED ON THE CENTRAL FIBER OF
A KÄHLER FAMILY

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To Bo Berndtsson,
On the Occasion of His 70th Birthday

1. Introduction

Let \( p: X \to D \) be a holomorphic family of smooth, \( n \)-dimensional compact manifolds whose central fiber -denoted by \( X \)- is assumed to be Kähler. We denote by \( K_X \) and \( K_X \) be the canonical bundles of \( X \) and \( X \), respectively. Let 

\[ s \in H^0(X, mK_X) \]

be a section of the pluricanonical bundle of \( X \), where \( m \) is a positive integer. Let \( \Sigma \) be the divisor associated to \( s \) and let \( I \subset O_X \) be the multiplier ideal sheaf corresponding to the \( \mathbb{Q} \)-divisor \( \frac{m - 1}{m} \Sigma \).

After Y.-T. Siu's invariance of plurigenera articles [11, 12, 9] concerning the extension of \( s \) in case of a projective family \( X \), the following very important problem is still open (despite of the avalanche of articles and crucial achievements that followed Siu’s work).

Conjecture 1.1. [12] Let \( p: X \to D \) be a family of smooth, \( n \)-dimensional compact manifolds whose central fiber is denoted by \( X \). We assume that the total space \( X \) admits a Kähler metric. Then any holomorphic pluricanonical section defined on \( X \) extends holomorphically to \( X \).

In order to state our main result, let \( k \geq 0 \) be a positive integer, and assume that \( s \) admits an extension to the \( k \)th infinitesimal nbd of \( X \) in \( X \), i.e. there exists \( s_k \) a \( C^\infty \) section of \( mK_X \) such that

\[ s_k|_X = s, \quad \partial s_k = t^{k+1} \Lambda_k \]

where \( t \) is the holomorphic function on \( X \) corresponding to \( \pi \), and \( \Lambda_k \) is a \( (n+1,1) \)-form with values in \( (m-1)K_X \). Let

\[ \lambda_k := \frac{\Lambda_k}{dt} \]

be the restriction of \( \Lambda_k \) to \( X \). We have \( \partial \lambda_k = 0 \), and then the following holds.

Theorem 1.2. Let \( p: X \to D \) be a smooth holomorphic family whose central fiber \( X \) is Kähler. Let \( s \in H^0(X, mK_X) \) be a pluricanonical section defined on \( X \). We assume that the image of the cohomology class defined by \( \lambda_k \) via the morphism

\[ H^1(X, mK_X) \to H^1(X, O_X(mK_X) \otimes O_X/I) \]

equals zero. Then \( \lambda_k \) is \( \partial \)-exact.

As a consequence of this result we obtain.

Corollary 1.3. Let \( p: X \to D \) be a smooth holomorphic family whose central fiber \( X \) is Kähler, and let \( s \in H^0(X, mK_X) \) be a pluricanonical section defined on \( X \). We assume that the set of zeros of the ideal \( I \) is discrete. Then \( s \) admits a holomorphic extension to \( X \).

In other words, the section \( s \) can be extended holomorphically to \( X \) provided that its zero-divisor is not too singular, as in Theorem 1.2.
For example, if we assume that we have
\[(1.3.1) \int_X \frac{1}{|s|^2} dV_\omega < \infty,\]
then the hypothesis of Theorem 1.2 above are satisfied.

Among the articles dedicated to Conjecture 1.1, we first mention here [5] and [6], due to M. Levine. Actually we obtain a new proof of the main result in [5] (we will present two type of arguments for it, cf. subsections 5.2 and 5.4, respectively).

**Theorem 1.4.** [5] Let \( s \in H^0(X, mK_X) \) be a pluricanonical section defined on the central fiber. We assume that the set of zeroes \((s = 0) \subset X\) of \( s \) is non-singular. Then \( s \) admits an extension to \( X \).

It turns out that the singularities of the zero-set \((s = 0)\) of the section \( s \) represent a serious obstruction to its extension, at least when it comes to the methods we develop in this article. Nevertheless, we obtain the following statement concerning the second order extension of \( s \).

**Theorem 1.5.** Let \( s \in H^0(X, mK_X) \) be a pluricanonical section defined on the central fiber. We assume that there exists a non-singular vector field \( \Xi \) on the total space \( X \) such that
\[(1.5.1) dp(\Xi) = \frac{\partial}{\partial t}, \quad \sup_X \frac{|\partial \Xi|_E^2}{\log^2 |s|^2} \leq C,\]
where \( \omega_E \) is a metric with Poincaré singularities along the set \((s = 0)\). Then there exists a section \( s_2 \in C^\infty(X, mK_X) \) such that \( s_2|_X = s \) and
\[(1.5.2) \quad \partial s_2 = t^3 \Lambda_2,\]
where \( \Lambda_2 \) belongs to the space \( C^0_{\infty}(X, mK_X) \).

These results are based on a few techniques that we next discuss. Let \( (L, h_L) \) be a Hermitian line bundle on \( X \), endowed with a metric with analytic singularities. We assume that the curvature current \( \Theta(L, h_L) \geq 0 \) is semi-positive. Let \( v \) be an \( L \)-valued form of type \((n - 1, 1)\), such that \( D^\partial v \) is \( \partial \)-closed. We state next some results concerning the \( \partial \)-equation
\[(1.5.3) \quad \partial \bar{u} = D^\partial v.\]
Indeed \((1.5.3)\) plays a key role in the proof of the aforementioned results. We assume that \( X \) is endowed with a Poincaré-type metric \( \omega_E \), with poles along the singular locus of \( h_L \). By using the \( L^2 \) theory, we establish a few results about \((1.5.3)\) in Section 3, cf. Theorem 3.2 and Theorem 3.9. It may be that the results we get are far from optimal, but quite unclear what the most general statement in this direction should be. In any case, we obtain the following.

**Theorem 1.6.** Assume moreover that \( v \) is \( L^2 \) with respect to a Poincaré-type metric \( \omega_E \) on \( X \). Then the equation \((1.5.3)\) admits a solution \( u \) such that
\[(1.6.1) \quad \int_X |u|^2 \omega e^{-\varphi_L} \leq \int_X |v|^2 \omega e^{-\varphi_L} dV_{\omega_E}.\]

The new aspect of Theorem 1.6 is that we do not assume that \( D^\partial v \) belongs to \( L^2 \).

It is a very interesting question to decide weather the equation \((1.5.3)\) can be solved with estimates involving an incomplete metric \( \omega \) on \( X \setminus (h_L = \infty) \). Let \( \omega \) be non-singular Kähler metric on \( X \). In this direction we obtain the following result.

**Theorem 1.7.** Let \((L, h_L)\) be a Hermitian line bundle, such that \( \Theta(L, h_L) \geq 0 \) is semi-positive. Let \( v \) be an \( L \)-valued form of type \((n - 1, 1)\), such that \( D^\partial v \) is \( \partial \)-closed. We assume that the following hypothesis are satisfied.

1. The metric \( h_L \) has analytic singularities, and let \( Z \) be the support of the set \((h_L = \infty)\).
2. The integrals
\[\int_X |v|^2 \omega e^{-\varphi_L} dV_\omega, \quad \int_X |D^\partial v|^2 \omega e^{-\varphi_L} dV_\omega\]
are convergent.
(3) There exists a complete metric $\omega_Z$ on $X \setminus Z$ such that we have
\[ \int_X |v|^2 e^{-\varphi_L} dV_{\omega_Z} < \infty. \]
Then the equation (1.5.3) has a solution $u$ such that
\[ \int_X |u|^2 e^{-\varphi_L} < \int_X |v|^2 e^{-\varphi_L} dV_{\omega_Z}. \]

Remark 1.8. For applications, it would be very important to remove the hypothesis (3) in Theorem 1.7. However, we are not sure whether the statement is still correct...

Finally, we highlight next a few aspects of the proof of Theorem 1.2 which seem interesting to us. Assume that the section $s$ admits a $C^\infty$ extension, say $s_k$, such that
\[ \bar{\partial}s_k = t_k^{k+1} \Lambda_k \]
for some $k \geq 0$ and form $\Lambda_k \in C_{n+1,1}^\infty(X, (m-1)K_X)$ whose restriction to $X$ is $\bar{\partial}$-closed. The first step is to show that there exist forms $\alpha_k$ and $\beta_k$ which are smooth in the complement of the support of $\Sigma$, and such that
\[ \Lambda_k \bigg|_X = \bar{\partial}\alpha_k + D'(\beta_k) \]
holds pointwise in the complement of $s = 0$, where $D'$ is the $(1,0)$-part of a Chern-type connection on $L := (m-1)K_X$ induced by the given section $s = s_k|_X$. The equality (1.8.2) holds in general, i.e. without any assumptions concerning the ideal $J$. But the problem is that the $L$-valued forms $\alpha_k$ and $\beta_k$ have singularities of type $\Phi^{s_k+1}$, where $\Phi$ is smooth. We therefore find ourselves in a rather strange situation, in which the LHS of (1.8.2) is non-singular but the primitives appearing in the RHS are meromorphic.

The main observation, upon which the proof relies, is that we have
\[ \lim_{\varepsilon \to 0} \int_X \mu \varepsilon \bar{\partial}\alpha_k \wedge Fe^{-\varphi_L} = 0, \quad \lim_{\varepsilon \to 0} \int_X \mu \varepsilon D'(\beta_k) \wedge Fe^{-\varphi_L} = 0 \]
where $\varphi_L$ is the metric on $L$ induced by $s$, and $F$ is an $L$-valued $L^2$ holomorphic form of type $(n-1,0)$. From (1.8.3) it follows that $\Lambda_k$ is $\bar{\partial}$-exact, and we conclude by the Hodge theory results we obtain in Section 3: it is mainly at this point that the additional hypothesis of Theorem 1.2 are needed.

Other than the aforementioned references, the articles [8], [7] by J. Noguchi, K. Liu, S. Rao and X. Wan contain important ideas in connection with our work here. Recent and exciting contributions in the direction of Conjecture 1.1 are due to J.-P. Demailly in [3] as well as to S. Rao and I.-H. Tsai in [10].

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2. FIRST ORDER DIFFERENTIAL OPERATORS

In this subsection we will recall a few facts from differential geometry of line bundles. We take this opportunity to fix some notations as well.

2.1. Connection induced by a smooth section.

Notation 2.1. In the setting of our article, let $L \to X$ be a holomorphic line bundle endowed with a connection whose $(0,1)$-component is given by the $\bar{\partial}$ operator,

\begin{equation}
\nabla = D_X + \bar{\partial}.
\end{equation}

Since no confusion is likely, we will use the same symbol to denote the induced operator on the space of smooth, $L$-valued $(p,q)$-forms,

\begin{equation}
D_X^t : C^\infty (X, \Omega_X^{p,q} \otimes L) \to C^\infty (X, \Omega_X^{p+1,q} \otimes L).
\end{equation}

Construction 2.2. Let $L \to X$ be a holomorphic line bundle, and let $\tilde{s}$ be a smooth section of $L$, with vanishing locus $Z \subseteq X$. Assume that the open cover $(U_i)_i$ trivialises the bundle $L$ and choose trivialisations, $L|_{U_i} \cong \mathcal{O}_{U_i}$. The section $\tilde{s}$ will therefore give rise to smooth functions $\tilde{s}_i$ on $U_i$. Set $U_i^o := U_i \setminus Z$.

Given any index $i$ we can now define a differential operator $D_X^i : \mathcal{O}_{U_i^o} \to \Omega^1_{U_i^o}$ on $U_i^o$ as follows,

\begin{equation}
D_X^i : \mathcal{O}_{U_i^o} \to \Omega^1_{U_i^o}, \quad \sigma \mapsto D'_i(\sigma_i)dt + \sum_{\alpha} D'_{\alpha}(\sigma_i)dz_i^\alpha
\end{equation}

where

\begin{align*}
D'_i(\sigma_i) &:= \partial_t \sigma_i - \frac{\partial \tilde{s}_i}{\tilde{s}_i} \sigma_i, \\
D'_{\alpha}(\sigma_i) &:= \partial_{\alpha} \sigma_i - \frac{\partial \tilde{s}_i}{\tilde{s}_i} \sigma_i.
\end{align*}

Using the trivialisations chosen above, we can view $D'_i$ as differential operators $D'_i : L|_{U_i^o} \to L|_{U_i^o} \otimes \Omega^1_{U_i^o}$ on $U_i^o$. One computes without much pain that these differential operators glue to give a globally defined operator

\begin{equation}
D'_X : L \to L \otimes \Omega^1_X
\end{equation}

on $X \setminus Z$. In particular, we obtain a connection $\nabla := D'_X + \bar{\partial}$ on $L|_{X \setminus Z}$.

Lemma 2.3. Setting as in Construction 2.2. Then, we have the following graded commutator identity

\begin{equation}
[D'_X, \bar{\partial}] = -\bar{\partial} \left( \frac{\partial \tilde{s}}{\tilde{s}} \right) \wedge
\end{equation}

Proof. Direct computation that we skip. We note that the symbol $\bar{\partial} \left( \frac{\partial \tilde{s}}{\tilde{s}} \right)$ which appears in (2.3.1) is a global $(1,1)$ form on the complement of the set $\tilde{s} = 0$, as we now explain. The global section $\tilde{s}$ corresponds to local smooth functions denoted by $\tilde{s}_i$, such that we have

\begin{equation}
\tilde{s}_i = g_{ij} \tilde{s}_j
\end{equation}

on overlapping subsets of $X$. Then we have

\begin{equation}
\frac{\partial \tilde{s}_i}{\tilde{s}_i} = \frac{\partial g_{ij}}{g_{ij}} \frac{\tilde{s}_j}{\tilde{s}_i} + \frac{\partial \tilde{s}_j}{\tilde{s}_i} \frac{g_{ij}}{\tilde{s}_i}
\end{equation}
and since $\tilde{\bar{\partial}} \left( \frac{\partial g_{ij}}{g_{ij}} \right) = 0$, we obtain
\begin{equation}
(2.3.4) \quad \tilde{\bar{\partial}} \left( \frac{\partial s_i}{s_j} \right) = \tilde{\bar{\partial}} \left( \frac{\partial \tilde{s}_i}{\tilde{s}_j} \right)
\end{equation}
which proves our claim. We note that if $\tilde{s}$ is holomorphic, then the $(1,1)$-form is simply zero on $X \setminus Z$. \hfill \Box

Remark 2.4. We note that in general, the differential operator $\nabla$ defined in (2.1.1) does not coincide with the Chern connection induced by the metric corresponding to $\tilde{s}$. This is of course the case if $\tilde{s}$ is holomorphic.

Remark 2.5. As in the usual case, a smooth section $\tilde{s}$ of the bundle $mL$ induces a connection on $L$ itself by a slight modification of the construction above that is to say, by multiplication with a constant, cf. next subsection.

2.2. Lie derivative and commutation relations. Let
\begin{equation}
(2.5.1) \quad s \in H^0(X, mK_X)
\end{equation}
be a holomorphic section of the pluricanonical bundle, where $m \geq 1$ is a positive integer.

Let $\tilde{s}$ be arbitrary smooth extension of the section $s$. As already hinted, we can use the section $\tilde{s}$ in order to define a connection on the bundle $L := (m-1)K_X/D$ restricted to the complement of the set of zeros of $\tilde{s}$. The local differential operators corresponding to $D'_X$ are given by
\begin{equation}
(2.5.2) \quad D'_t(\sigma) := \partial_t \sigma - \frac{m-1}{m} \frac{\partial_\tilde{s}_i}{\tilde{s}_i} \sigma, \quad D'_\alpha(\sigma) := \partial_\alpha \sigma - \frac{m-1}{m} \frac{\partial_\tilde{s}_i}{\tilde{s}_i} \sigma.
\end{equation}
where $\partial_t := \partial/\partial t$ and $\partial_\alpha := \partial/\partial z_\alpha$ for each $\alpha = 1, \ldots, n$. In (2.5.2) the symbol $\sigma$ represents a local section of the bundle $(m-1)K_X/D$. The sum
\begin{equation}
D'_t(\sigma) dt + \sum_\alpha D'_\alpha(\sigma) dz_\alpha^n
\end{equation}
corresponds to a global connection of $(1,0)$ type on $L := (m-1)K_X$. If $\tilde{s}$ would have been holomorphic, then this is nothing but the Chern connection; in any case, this is well (i.e. globally) defined. It is clear that we have
\begin{equation}
(2.5.3) \quad D'_X \circ D'_X = 0.
\end{equation}

We consider next a vector field $\Xi$ on the total space $X$ which projects into $\partial/\partial t$ on $D$. It can be written in local coordinates as follows
\begin{equation}
(2.5.4) \quad \Xi|_{U_i} = \frac{\partial}{\partial t} + \sum_{\alpha=1}^n v^\alpha_i \frac{\partial}{\partial z_\alpha}
\end{equation}
where $v^\alpha_i$ for $i = 1, \ldots, n$ are smooth functions.

Our vector field induces a Lie derivative operator $\mathcal{L}_\Xi$ as follows
\begin{equation}
(2.5.5) \quad \mathcal{L}_\Xi(\sigma) := D'_X(\sigma|\Xi) \sigma
\end{equation}
where $\sigma$ is any $(n+1,q)$-form with values in $L$. The result $\mathcal{L}_\Xi(\sigma)$ is a form of the same type as $\sigma$, and it will play an important role in what follows.

Let $i_\Xi$ be the operator of degree $(-1,0)$ given by the contraction with the vector field $\Xi$. Since we have $\mathcal{L}_\Xi = [D'_X, i_\Xi]$ on $(n+1,q)$-differential forms, the following Jacobi identity
\begin{equation}
[\bar{\partial}, \mathcal{L}_\Xi] + [D'_X, [i_\Xi, \bar{\partial}]] + [i_\Xi, [\bar{\partial}, D'_X]] = 0
\end{equation}
holds true, and therefore we obtain the next formula over $(n+1,q)$-forms
\begin{equation}
(2.5.6) \quad [\bar{\partial}, \mathcal{L}_\Xi] = -D'_X \circ (\bar{\partial} \Xi) - [i_\Xi, [\bar{\partial}, D'_X]].
\end{equation}

Our next statement is playing an important role in the “algebra” part of the proof of our main results.
Lemma 2.6. Let \( \rho \) be a \((n - 1,1)\)-form with values in \( L \) on \( X \). Then we have the equality
\[
\mathcal{L}_\Xi (D'_X (dt \wedge \rho)) = D'_X (dt \wedge (\Xi | D'_X | (dt \wedge \rho)))
\]
on \( X \).

Proof. The argument is quite clear; we start with the left hand side and we remark that we have
\[
D'_X (dt \wedge \rho) = -dt \wedge D'_X (\rho)
\]
and so
\[
\Xi | (D'_X (dt \wedge \rho)) = -D'_X (\rho) + dt \wedge (\Xi | D'_X (\rho))
\]
since the contraction with the vector field \( \Xi \) is a derivation. The LHS of \( (2.6.1) \) is therefore equal to
\[
- dt \wedge D'_X (\Xi | D'_X (\rho)).
\]
For the RHS, we have
\[
\Xi | (dt \wedge \rho) = \rho - dt \wedge (\Xi | \rho)
\]
so
\[
D'_X (\Xi | (dt \wedge \rho)) = D'_X (\rho) + dt \wedge D'_X (\Xi | \rho).
\]
A contraction with \( \Xi \) gives
\[
\Xi | (D'_X (\Xi | (dt \wedge \rho))) \equiv \Xi | (D'_X (\rho) + D'_X (\Xi | \rho))
\]
modulo a term in \( dt \wedge \cdot \), so we have
\[
dt \wedge (\Xi | (D'_X (\Xi | (dt \wedge \rho)))) = dt \wedge (\Xi | (D'_X (\rho))) + dt \wedge D'_X (\Xi | \rho)
\]
and a further derivative of \( (2.6.8) \) shows that the LHS of \( (2.6.1) \) equals
\[
- dt \wedge D'_X (\Xi | D'_X (\rho)),
\]
so our statement is proved. In the argument just finished, we have used many times the fact that
\( D'_X \circ D'_X = 0. \)

3. A Few Results from \( L^2 \) Hodge Theory

Let \( X \) be a \( n \)-dimensional compact Kähler manifold, and let \( (L, h_L) \) be a line bundle endowed with a (singular) metric \( h_L = e^{-\varphi_L} \) satisfying the following hypothesis:

(a) \( h_L \) has log poles, i.e. its local weights can be written as follows
\[
\varphi_L \equiv \sum a_i \log |f_i|^2
\]
modulo a smooth function, where \( a_i \) are positive real numbers and \( f_i \) are holomorphic functions.

(b) The curvature current \( \Theta_{h_L} (L) \) corresponding to the metric \( h_L \) is equal to zero in the complement of the set \( (h_L = \infty) \).

We consider a modification \( \pi : \hat{X} \to X \) of \( X \) such that the support of the singularities of \( \varphi_L \circ \pi \) is a simple normal crossing divisor \( E \). As usual, we can construct \( \pi \) such that its restriction to \( \hat{X} \setminus E \) is an biholomorphism. We write
\[
\varphi_L \circ \pi |_{\Omega} \equiv \sum_{\alpha=1}^{p} e_\alpha \log |z_\alpha|^2
\]
modulo a smooth function. Here \( \Omega \subset \hat{X} \) is a coordinate subset, and \( (z_\alpha)_{\alpha=1,\ldots,n} \) are coordinates such that \( E \cap \Omega = (z_1 \ldots z_p = 0) \).

Let \( \hat{\omega}_E \) be a complete Kähler metric on \( \hat{X} \setminus E \), with Poincaré singularities along \( E \), and let
\[
\omega_E := \pi_*(\hat{\omega}_E)
\]
be the direct image metric. We note that in this way \((X_0, \omega_E)\) becomes a complete Kähler manifold, where \( X_0 := X \setminus (h_L = \infty) \).
The following statement is well-known, but we recall the precise version we need here. We denote by $s$ the section of a line bundle for which the support of its zero divisor coincides with the support of $(h_L = \infty)$.

**Lemma 3.1.** There exist a family of smooth functions $(\mu_\varepsilon)_{\varepsilon > 0}$ with the following properties.

(a) For each $\varepsilon > 0$, the function $\mu_\varepsilon$ has compact support in $X_0$, and $0 \leq \mu_\varepsilon \leq 1$. (b) The sets $(\mu_\varepsilon = 1)$ are providing an exhaustion of $X_0$.

(c) There exists a positive constant $C > 0$ such that we have

$$\sup_{X_0} \left( |\partial \mu_\varepsilon|^2_\omega + |\partial \mu_\varepsilon|^2_{\omega_X} \right) \leq C.$$ 

**Proof.** It is enough to obtain the corresponding statement on $\hat{X}$, so that the divisor $E$ is snc. Then we take

$$\mu_\varepsilon = \rho_\varepsilon \left( \log \log \frac{1}{|s|} \right)$$

where $\rho_\varepsilon$ is equal to 1 on the interval $[1, \varepsilon^{-1}]$ and is equal to zero on $[1 + \varepsilon^{-1}, \infty]$. Also, we denote by $\|s\|$ the inverse image of the norm of the section $s$ with respect to an arbitrary, smooth metric.

Actually we can also impose the condition that

$$\max_{j=1, \ldots, N} \sup_{\hat{X}_+} |\rho_\varepsilon^{(j)}| < C_N < \infty,$$

for any positive integer $N$. The properties (a)-(c) are then verified by a simple computation. In what follows we will use this statement for

$L := (m - 1)K_X$, \hspace{1cm} $\varphi_L := \frac{m-1}{m} \log |f|^2$

and $s$ the pluricanonical section we have to extend (the holomorphic function $f$ above corresponds locally to $s$).

In this context we have the following statement, which is a slight generalisation of the usual result in $L^2$ Hodge theory.

**Theorem 3.2.** Consider a line bundle $(L, h_L) \to X$ endowed with a metric $h_L$ such that the requirements (a) and (b) above are satisfied, as well as the corresponding complete Kähler manifold $(X_0, \omega_E)$, cf. (3.0.2).

Then the following assertions are true.

(i) We have the following Hodge decomposition

$$L^2_{n,1}(X_0, L) = \mathcal{H}_{n,1}(X_0, L) \oplus \operatorname{Im} \partial \oplus \operatorname{Im} \partial^*$$

(ii) Let $u$ be a $\partial$-closed $L^2$-form of type $(n, 1)$ on $X_0$, and assume that we have $u = D'w$, where $w \in L^2_{n-1,1}(X_0, L)$. Then, there exists $v \in L^2_{n,0}(X_0, L)$ such that $u = \partial v$.

The proof of Theorem 3.2, which we give below, makes use of the following statement which is the $\partial$-version of the Poincaré inequality established in [1].

**Theorem 3.3.** Let $p \leq n$ be an integer. There exists a positive constant $C > 0$ such that the following inequality holds

$$\int_{X_0} |u|^2_{\omega_E} e^{-\varphi_L} d\omega_E \leq C \int_{X_0} |\partial u|^2_{\omega_E} e^{-\varphi_L} d\omega_E$$

for any $L$-valued form $u$ of type $(p, 0)$ which belongs to the domain of $\partial$ and which is orthogonal to the space of $L^2$ harmonic $(p, 0)$-forms.

**Proof of Theorem 3.3.** The first observation is that we can assume from the very beginning that the polar set $h_L = \infty$ has snc support. Indeed, via the map $\pi$ the hypothesis and the conclusion of our Poincaré inequality transform as follows.

- The form

$$\hat{u} = \pi^* u$$

on $\hat{X}$ is $\pi^* L$-valued and $L^2$ with respect to $\hat{\omega}_E$ and $\pi^* h_L$. 

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For any smooth function \((3.5.1)\)
\[ \varepsilon > 0 \]
now two things can happen: either we need the following auxiliary statement.

Proof. We present next the arguments for 3.5. We claim that the following inequalities hold
\[ (3.5.3) \]
\[ |\varepsilon-x|^{2} e^{-\varphi_{L}} dV_{E} \leq \int_{U} |\partial f|_{\omega_{E}}^{2} e^{-\varphi_{L}} dV_{E} \]
where \(C\) is a positive numerical constant.

The arguments in [1], Lemma 1.10 correspond to the case \(p = 0\) and \(\varphi_{L} = 0\). For the general case we need the following auxiliary statement.

Theorem 3.3. Let \((\Omega_{1})_{j=1,...,N}\) be a finite union of coordinate sets of \(X\) covering \(E\), and let \(U\) be any open subset in \(X\) and \(Y\) in values in \((L,h_{L})\). Then we have
\[ (3.5.3) \]
\[ \frac{1}{C} \int_{U} |x|^{2} e^{-\varphi_{L}} dV_{E} \leq \int_{U} |\partial f|_{\omega_{E}}^{2} e^{-\varphi_{L}} dV_{E} \]
where \(C\) is a positive numerical constant.

We present the arguments for 3.5.

Proof. Let \(x_{\delta} < 1\) be a positive real number whose expression will be determined in what follows. Let \(v : [0,1] \rightarrow \mathbb{C}\) be a smooth function whose support is contained in the interval \([\varepsilon,x_{\delta}]\) for some positive, small enough \(\varepsilon\).

We claim that the following inequalities hold
\[ (3.5.3) \]
\[ \int_{0}^{1} |v|^{2} t^{k} dt \leq C \int_{0}^{1} |v'|^{2} t^{k+2} \log^{2} t dt \]
as well as
\[ (3.5.4) \]
\[ \int_{0}^{1} |v|^{2} t^{k} \frac{dt}{\log^{2} t} \leq C \int_{0}^{1} |v'|^{2} t^{k+2} dt \]
where \(k\) is any element of the set \(\delta + \mathbb{Z}\), and \(C\) is a numerical constant (in particular independent of \(v, \varepsilon, k, \ldots\)).

We first check that \(3.5.3\) holds true: we have
\[ (3.5.5) \]
\[ - \int_{0}^{1} |v|^{2} t^{k+1} \log^{2} d \left( \frac{1}{\log t} \right) = \int_{0}^{1} \frac{d}{\log t} dt \left( v^{2} t^{k+1} \log^{2} t \right) dt \]
and this equals
\[ (3.5.6) \]
\[ 2 \int_{0}^{1} (\pi v' + v v') t^{k+1} \log^{2} t dt + \int_{0}^{1} \frac{t^{k} v^{2}}{\log t} \left( (k+1) \log^{2} t + 2 \log t \right) dt. \]
Now two things can happen: either \(k + 1 = 0\), and then the second term in \(3.5.6\) is equal to twice the quantity we are interested in, or \(k + 1 \neq 0\). Note that in the latter case it is possible to fix \(x_{\delta}\) uniformly with respect to \(k\) so that
\[ (3.5.7) \]
\[ |(k+1) \log^{2} t + 2 \log t| \geq 2 |\log t| \]
for any $t \in [0, x_3]$ precisely because $k \in \delta + \mathbb{Z}$.

As a consequence, we have
\begin{equation}
|\int_0^1 (\nabla v + v\nabla) t^{k+1} \log t dt| \geq \int_0^1 t^k v^2 dt.
\end{equation}

for any $v$ of compact support in $[0, x_3]$. Moreover, we have
\begin{equation}
\left|\int_0^1 (\nabla v + v\nabla) t^{k+1} \log t dt\right|^2 \leq \int_0^1 |v'|^2 t^{k+2} \log^2 t dt \int_0^1 |v|^2 t^k dt.
\end{equation}

by Cauchy-Schwarz, so the inequality (3.5.3) is settled.

The arguments for (3.5.4) are completely similar: we have
\begin{equation}
\int_0^1 |v|^2 t^k \frac{dt}{\log^2 t} = \int_0^1 |v|^2 \frac{t^{k+1}}{\log^2 t} d\log t
\end{equation}

which equals
\begin{equation}
\int_0^1 \log t \frac{d}{dt} \left( |v|^2 \frac{t^{k+1}}{\log^2 t} \right) dt = \int_0^1 |v|^2 t^k \left( 2 + (k+1) \log \frac{1}{t} \right) dt - \int_0^1 \frac{t^{k+1}}{\log t} \left( v\nabla \nabla v \right) dt.
\end{equation}

and two things can happen. If $k + 1 \geq 0$, then we have $2 + (k+1) \log \frac{1}{t} \geq 2$ for any $t \in [0, 1]$ and
\begin{equation}
(k+1) \log \frac{1}{t} + 2 \leq \frac{1}{2}
\end{equation}

for any $t \in [0, x_3]$ by imposing a supplementary (but uniform) condition on $x_3$ if necessary, and we proceed exactly as we did for (3.5.3).

Therefore, we obtain (3.5.3) and (3.5.4) for any $k \in \delta + \mathbb{Z}$.

Lemma 3.5 follows from this: we consider the Fourier series
\begin{equation}
f = \sum_{k \in \mathbb{Z}} a_k(t) e^{i \sqrt{-1} \theta}
\end{equation}
of $f$, and then we have
\begin{equation}
\frac{\partial v}{\partial z} = \sum_{k \in \mathbb{Z}} \left( a_k'(t) - \frac{k}{t} a_k(t) \right) e^{i \sqrt{-1} \theta}.
\end{equation}

The identity
\begin{equation}
t^k \frac{d}{dt} \left( \frac{a_k}{t^k} \right) = a_k'(t) - \frac{k}{t} a_k(t)
\end{equation}

reduces the proof of our statement to inequalities of type (3.5.3) and (3.5.4) which are already established. In conclusion, Lemma 3.5 is completely established.

We now return to the proof of Proposition 3.4. We consider the restriction of the form $\tau$ to some co-ordinate open set $\Omega$ whose intersection with $E$ is of type $z_1 \ldots z_r = 0$. It can be written as sum of forms of type
\begin{equation}
\tau_I := f_I dz_I
\end{equation}

and assume that $I \cap \{1, \ldots, r\} = \{1, \ldots, p\}$ for some $p$ or $I \cap \{1, \ldots, r\} = \emptyset$.

In the first case we have
\begin{equation}
\frac{||\tau_I||^2 e^{-\varphi_L} dV_g}{d\lambda} = \frac{|f_I|^2 e^{-\varphi_L}}{\prod_{k=p+1}^{q} |z_k|^2 \log^2 |z_k|^2}
\end{equation}

where $e^{-\varphi_L} = \frac{1}{\prod_{k=1}^{q} |z_k|^2 \log |z_k|^2}$, and in the second case this is the same with $p = 0$. In 3.5.17 we denote by $g$ the model Poincaré metric
\[ \sum_{i=1}^{r} \sqrt{-1} dz_i \wedge d\bar{z}_i + \sum_{i=r+1}^{n} \sqrt{-1} dz_i \wedge d\bar{z}_i. \]
Now for the $\bar{\partial} \tau_j$ we have
\begin{equation}
\frac{|\bar{\partial} \tau_j|^2 e^{-\varphi_L} dV_{\omega_E}}{d\lambda} = \sum_{i=1}^r \frac{\left| \frac{\partial f_i}{\partial z_i} \right|^2}{\prod_{\alpha=p+1}^\infty |z_\alpha|^2} \frac{|z_i|^2 \log^2 |z_i|^2 e^{-\varphi_L}}{\prod_{\alpha=p+1}^\infty |z_\alpha|^2 |z_i|^2} \left( \prod_{\alpha=p+1}^\infty |z_\alpha|^2 \right)
\end{equation}
\begin{equation}
+ \sum_{i=r+1}^n \frac{\left| \frac{\partial f_i}{\partial z_i} \right|^2}{\prod_{\alpha=p+1}^\infty |z_\alpha|^2} \frac{e^{-\varphi_L}}{\prod_{\alpha=p+1}^\infty |z_\alpha|^2 |z_i|^2} \left( \prod_{\alpha=p+1}^\infty |z_\alpha|^2 \right)
\end{equation}
and the observation is that for every $\xi \in \Omega$ there exist some index $j \in \{1, \ldots, r\}$ for which the support condition we require in Lemma 3.3 is satisfied for
\begin{equation}
f(z) := f_j(\xi_1, \ldots, \xi_{j-1}, z, \xi_{j+1}, \ldots, \xi_n).
\end{equation}
We use (3.5.3) or (3.5.4) according to the case where $j \geq p$ in (3.5.18) or not. Proposition 3.4 is therefore completely proved.

The constant $C$ in (3.4.1) only depends on the distortion between the model Poincaré metric on $\Omega_j$ with singularities on $E$ and the global metric $\omega_{x_j}$ restricted to $\Omega_j$. Another important observation is that by using the cut-off function $\mu_{x_j}$ in Lemma 3.1, we infer that (3.4.1) holds in fact for any $L^2$-bounded form with compact support in $U$.

The rest of the proof follows the arguments in [1], proof of Lemma 1.10: if a positive constant as in (3.3.1) does not exists, then we obtain a sequence $u_j$ of $L^2$-valued forms of type $(p, 0)$ orthogonal to the space of holomorphic forms such that
\begin{equation}
\int_X |u_j|^2 e^{-\varphi_L} dV_{\omega_E} = 1, \quad \lim_{i} \int_X |\bar{\partial} u_i|^2 e^{-\varphi_L} dV_{\omega_E} = 0
\end{equation}
It follows that the weak limit $u_\infty$ of $(u_j)$ is holomorphic. On the other hand, each $u_i$ is perpendicular to the space of holomorphic forms, so it follows that $u_\infty$ is equal to zero. Moreover, by Bochner formula combined with Proposition 3.4 we can assume that $u_i$ converges weakly in the Sobolev space $W^1$, so strongly in $L^2$ to zero. In particular we have
\begin{equation}
|u_i|_K \to 0
\end{equation}
in $L^2$ for any compact subset $K \subset X \setminus E$.

The last step in the proof is to notice that the considerations above contradict the fact that the $L^2$ norm of each $u_i$ is equal to one. This is not quite immediate, but is precisely as the end of the proof of Lemma 1.10 in [1], so we will not reproduce it here. The idea is however very clear: we decompose each $u_j = \chi u_j + (1 - \chi) u_j$ where $\chi$ is a cutoff function which is equal to 1 in $U$ and whose norm of the corresponding gradient is small. Then the $L^2$ norm of $\chi u_j$ is small by (3.4.1) and (3.5.20). The $L^2$ norm of $(1 - \chi) u_j$ is equally small by (3.5.21), and this is how we reach a contradiction.

We have the following direct consequence of Proposition 3.3.

**Corollary 3.6.** There exists a positive constant $C > 0$ such that the following holds true. Let $v$ be a $L^2$-valued form of type $(n, 2)$. We assume that $\omega$ is $L^2$, and orthogonal to the kernel of the operator $\bar{\partial}^*$. Then we have
\begin{equation}
\int_{X_0} |\bar{\partial}^* v|^2 e^{-\varphi_L} dV_{\omega_E} \leq C \int_{X_0} |\bar{\partial}^* v|^2 e^{-\varphi_L} dV_{\omega_E}.
\end{equation}
Indeed, the inequality (3.6.1) follows immediately from the observation that the Hodge star $u := *v$ is orthogonal to the space of holomorphic forms (here we use again the fact that the curvature of $(L, h_L)$ equals zero when restricted to $X_0$). This can be seen as follows: let $F$ be a $L^2$-holomorphic $(n-2)$-form. Then we have
\[ D^* F = 0 \]
and therefore $D^*(\star F) = 0$, in other words the Hodge dual $\star F$ of $F$ belongs to the kernel of $\bar{\partial}^*$. We thus have
\[ \int_{X_0} \langle u, F \rangle e^{-\varphi_L} dV_{\omega_E} = \int_{X_0} \langle v, \star F \rangle e^{-\varphi_L} dV_{\omega_E} = 0. \]
Note that we have
\begin{equation}
\int_{X_0} |\partial u|^2 e^{-\varphi_L} dV_{\omega_E} = \int_{X_0} |D^* v|^2 e^{-\varphi_L} dV_{\omega_E} \leq \int_{X_0} |\partial^* v|^2 e^{-\varphi_L} dV_{\omega_E}
\end{equation}
where the inequality in (3.6.2) is obtained by Bochner formula. This proves the corollary by applying Proposition 3.3.

Proof of Theorem 3.2. This statement is almost contained in [2, chapter VIII, pages 367-370]. Indeed, in the context of complete manifolds one has the following decomposition
\begin{equation}
L^2_{\omega_E}(X_0, L) = \mathcal{H}_{n,1}(X_0, L) \oplus \text{Im } \partial \oplus \text{Im } \partial^*.
\end{equation}
We also know (see loc. cit.) that the adjoints \(\partial^*\) and \(D^*\) in the sense of von Neumann coincide with the formal adjoints of \(\partial\) and \(D^*\) respectively.

It remains to show that the ranges of the \(\partial, \partial^*\)-operators are closed with respect to the \(L^2\) topology. In our set-up, this is a consequence of the particular shape of the metric \(\omega_E\) at infinity (i.e. near the support of \(\pi(E)\)). The Poincaré inequality \(3.3.1\) of Proposition 3.3 implies that the image of \(\partial\) is closed. The inequality \(3.6.1\) implies that image of \(\partial^*\) is closed.

The assertion \((ii)\) is an easy consequences of \((i)\), together with standard considerations. \(\square\)

Remark 3.7. Actually the first point of Theorem 3.2 holds provided that we only impose the condition \(\Theta(L, h_L) \geq 0\). We will come back to this matter in a forthcoming paper.

In the setting of the current section, the harmonic \((n,1)\)-forms have the following properties.

Lemma 3.8. We assume that \((X_\nu, \omega_E)\) and \((L, h_L)\) have the properties \((a)\) and \((b)\) (at the beginning of this section). Let \(\alpha\) be a \(\Delta''\)-harmonic \(L^2\) form of type \((n,1)\) with values in \(L\). We denote by \(F\) its Hodge dual. Then the following hold.

1. We have \(\partial F = 0\) and \(D' F = 0\).
2. Let \(\Omega \subset X\) be a coordinate open set, and let \(f\) be any holomorphic function defined on \(\Omega\). We assume that the support of the set of zeroes of \(f\) is contained in \((h_L = \infty)\). Then the form \(F \wedge \frac{df}{f}\) is non-singular on \(X\) (but it is not clear whether this form is \(L^2\) with respect to \(e^{-\varphi_L}\) or not).
3. Let \(\Omega \subset X\) be a coordinate open set, and let \(g\) be any holomorphic function defined on \(\Omega\). Then we have
\[\int_{\Omega} |F \wedge dg|^2 e^{-\varphi_L} < \infty.\]

Proof. Note that given any \(\Delta''\)-harmonic \((n,1)\)-form \(\alpha\), by Bochner formula combined with the fact that the curvature of \((L, h_L)\) is semi-positive, we have
\begin{equation}
D''\alpha = 0 \quad \text{on } X \setminus (h_L = \infty).
\end{equation}
So if we write \(\alpha = \omega_E \wedge F\), it follows that
\begin{equation}
F \in H^0(X, \Omega^{n-1}_X \otimes K^{\otimes(m-1)}_X), \quad \int_X |F|^2_{\omega_E} e^{-\varphi_L} dV_{\omega_E} < \infty.
\end{equation}
thanks to the property \(3.8.1\). Moreover, as \(D''\alpha = 0\) it turns out that we have \(D' F = 0\) as well. We obtain thus \(1\) and \(3\). Note that \(3.8.2\) is valid in a more general setting, cf. [3, 14].

As for \(2\), by our choice of the metric \(\omega_E\), we show first that we have
\begin{equation}
|df|_{\omega_E}^2 \sim |f|^2 \log^2 |f|.
\end{equation}
Indeed, assume that via the bimeromorphism \(\pi : \tilde{X} \rightarrow X\) the function \(f\) corresponds locally to the function \(\prod_{i=1}^n z_i^{p_i}\) (notations as in the beginning of this section). Then the metric \(\omega_E\) is quasi-isometric to the Poincaré metric with singularities along \(z_1 \ldots z_r = 0\). The \(\pi\)-inverse image of \(\frac{df}{f}\) is
We are using again Bochner equality and we infer that

\[ \frac{\partial f}{L} f^2 \simeq \sum p_i^2 \log^2 |z_i|^2 \]

and the RHS of this quantity is the same as \( \log^2 |f|^2 \).

Therefore we infer

\[
\int_X |F \wedge \frac{df}{f}|^2 \frac{e^{-\varphi_L}}{\log^2 |f|} \leq C \int_X |F|^2 e^{-\varphi_L} d\omega_X.
\]

Since the RHS of (3.8.4) is finite and \( \varphi_L \) has non zero Lelong number over any component of \( \text{Div}(f) \), it follows that \( F \wedge df \) vanishes along \( \text{Div}(f) \). \qed

Consider the equation

\[
\tilde{\partial} u = D'w + \Theta(L, h_L) \wedge \tau
\]

where \( \tau, w \) and \( D'w \) are \( L^2 \), and \( D'w + \Theta(L, h_L) \wedge \tau \) are \( \tilde{\partial} \)-closed. We show next that one can solve it in a fair general context, and moreover obtain \( L^2 \) estimates for the solution with minimal norm.

We refer to [13, Thm A.5] for a similar argument in the non-singular case.

**Theorem 3.9.** Let \( (L, h_L) \) be a holomorphic line bundle on \( X \) with a possible singular metric \( h_L \) with analytic singularities along a subvarieties \( Z \subset X \) and whose curvature current is semi-positive.

Consider a complete Kähler metric \( (w, \omega_E) \). Let \( w \) be an \( L \)-valued \( (n - 1, 1) \)-form on \( X \) such that \( w \) and \( D'w \) are in \( L^2 \). Let also \( \tau \) be an \( L \)-valued \( (n - 1, 0) \)-form on \( X \) such that \( \tau \) and \( \Theta(L, h_L) \wedge \tau \) are \( L^2 \). Then there exists a solution \( u \) of the equation (3.8.5) such that

\[
\int_X |u|^2 e^{-\varphi_L} \leq \int_X |w|_{\omega_E}^2 e^{-\varphi_L} d\omega_E - \int_X \langle \Theta(L, h_L), \Lambda_{\omega_E} \rangle \tau, \tau \rangle_{\omega_E} e^{-\varphi_L} d\omega_E.
\]

**Proof.** Let \( \xi \) be a smooth \( (n, 1) \)-form of compact support in \( X \) such that \( \xi = \xi_1 + \xi_2 \) be its decomposition according to \( \text{Ker}(\tilde{\partial}) \) and its orthogonal.

By using the \( L^2 \)-assumptions, we have

\[
\int_{X \setminus Z} \langle D'w + \Theta(L, h_L) \wedge \tau, \xi \rangle_{\omega_E} e^{-\varphi_L} d\omega_E = \int_{X \setminus Z} \langle D'w + \Theta(L, h_L) \wedge \tau, \xi_1 \rangle_{\omega_E} e^{-\varphi_L} d\omega_E.
\]

Then the semipositive curvature condition and the Bochner equality imply that

\[
\int_{X \setminus Z} \langle D'w, \xi_1 \rangle_{\omega_E} e^{-\varphi_L} d\omega_E \leq \int_X |w|_{\omega_E}^2 e^{-\varphi_L} d\omega_E - \int_X |\xi_1|_{\omega_E}^2 e^{-\varphi_L} d\omega_E - \int_X |\partial^* \xi_1|_{\omega_E}^2 e^{-\varphi_L} d\omega_E.
\]

For the term containing the curvature of \( (L, h_L) \) we are using Cauchy inequality and we infer that

\[
\int_{X \setminus Z} \langle \Theta(L, h_L), \Lambda_{\omega_E} \rangle \tau, \tau \rangle_{\omega_E} e^{-\varphi_L} d\omega_E \leq \int_X \langle \Theta(L, h_L), \Lambda_{\omega_E} \rangle \xi_1, \xi_1 \rangle_{\omega_E} e^{-\varphi_L} d\omega_E.
\]

We are using again Bochner equality and we infer that

\[
\int_{X \setminus Z} \langle \Theta(L, h_L) \wedge \tau, \xi_1 \rangle_{\omega_E} e^{-\varphi_L} d\omega_E \leq \int_X \langle \Theta(L, h_L), \Lambda_{\omega_E} \rangle \xi_1, \xi_1 \rangle_{\omega_E} e^{-\varphi_L} d\omega_E.
\]
is smaller than
\begin{equation}
\int_X \langle -[\Theta(L, h_L), L\omega_E] \tau, \tau \rangle \omega_E e^{-\varphi_L} dV_{\omega_E} \cdot \int_X \left| \partial^* \xi \right|_{\omega_E}^2 e^{-\varphi_L} dV_{\omega_E}.
\end{equation}
Then the theorem is proved by the standard $L^2$-estimate argument. 

4. Proof of the main result: algebra

Let $s_i$ be an arbitrary holomorphic extension of $s_{|U_i \cap X}$. Then we have
\begin{equation}
s_idz_i \wedge dt \otimes e_i = s_j dz_j \wedge dt \otimes e_j + t \Lambda_i dz_i \wedge dt \otimes e_j.
\end{equation}
on overlapping coordinate sets $U_i \cap U_j$. Hence we interpret $s$ as a top form on $X$ with values in $L := \omega_{X/D}^{\otimes (m-1)}|_X$. In \eqref{4.0.1} the symbol $e_i$ stands for $dz_i^{\otimes (m-1)}$, the local frame of $\omega_{X/D}^{\otimes (m-1)}$.

We can reformulate this data as follows: there exists a smooth section $s_1$ of the bundle $K_X + L$ such that
\begin{equation}
\partial s_1 = t \Lambda_1
\end{equation}
on $X$.

Consider next the Lie derivative operator $L_{\Xi}$ associated to a vector field $\Xi$ such that $dp(\Xi) = \partial \mathcal{L}_{\Xi}$.

If we apply $L_{\Xi}$ to \eqref{4.0.2} on the RHS we obtain
\begin{equation}
\Lambda_1 + t \mathcal{L}_{\Xi}(\Lambda_1)
\end{equation}
By the commutation relation \eqref{2.5.6}, the LHS of \eqref{4.0.2} becomes
\begin{equation}
\partial (\mathcal{L}_{\Xi}(s_1)) + D_{\Xi}^t (\partial |s_1|) + [\Xi, \partial, D_{\Xi}]|s_1|
\end{equation}
Modulo a factor divisible with $t$, the last term of \eqref{4.0.4} is equal to $\frac{m-1}{m} \Lambda_1$ on $X \setminus Z$. Therefore we get
\begin{equation}
\frac{1}{m} \Lambda_1 \equiv \partial (\mathcal{L}_{\Xi}(s_1)) + D_{\Xi}^t (\partial |s_1|) \quad \text{on } X \setminus Z
\end{equation}
modulo a forms which is divisible by $t$ and which is non-singular on $X \setminus Z$.

Another interesting observation is that the form $\partial |s_1|$ can be written as follows
\begin{equation}
\partial |s_1| = dt \wedge \rho_1
\end{equation}
given the shape of the vector field $\Xi$, cf. \eqref{2.5.4}. We have therefore obtained the first step of the next statement.

Lemma 4.1. Let $s_k$ be a smooth section of the bundle $K_{X/\Delta} + L$, such that
\begin{equation}
\partial s_k = t^{k+1} \Lambda_k
\end{equation}
for some $(n+1, 1)$-form $\Lambda_k$. We assume that the connection on $L = (m-1)K_X$ is induced by the section $s_k$. Then we can find the forms $\alpha_k$ and $\beta_k$ such that we have
\begin{equation}
c_k \Lambda_k \equiv \partial \alpha_k + D_{\Xi}^t (dt \wedge \beta_k) \quad \text{on } X \setminus Z
\end{equation}
modulo the ideal generated by $t$. Moreover, the forms $\alpha_k$ and $\beta_k$ are smooth in the complement of the set $s_k = 0$ and $c_k$ is a positive constant.

Proof. This is obtained as follows: we take the Lie derivative in \eqref{4.1.2} and use the commutation relation \eqref{2.5.9}. The result of this first operation is that we have
\begin{equation}
\frac{k+1}{m} t^k \Lambda_k \equiv \partial (\mathcal{L}_{\Xi}(s_k)) + D_{\Xi}^t (\partial |s_k|) \quad \text{on } X \setminus Z
\end{equation}
modulo the ideal generated by $t^{k+1}$. In order to start the inductive process which will prove our statement, we rewrite \eqref{4.1.3} as follows
\begin{equation}
\frac{k+1}{m} t^k \Lambda_k \equiv \partial u_1 + D_{\Xi}^t (dt \wedge v_1) \quad \text{on } X \setminus Z,
\end{equation}
modulo \( t^{k+1} \). Here we use the notations
\[
(4.1.5) \quad u_1 := \mathcal{L}_\Xi(s_k), \quad dt \wedge v_1 := \partial \Xi | s_k.
\]
We show next that if we apply the operator \( \mathcal{L}_\Xi \) to the RHS of (4.1.5) the result is an expression of a similar type. Indeed, we have
\[
(4.1.6) \quad \mathcal{L}_\Xi(\partial u_1) = \partial (\mathcal{L}_\Xi(u_1)) + D'_X(\partial \Xi | u_1) \quad \text{on } X \setminus Z
\]
modulo the curvature term which is of order \( t^k \): this is one higher than we have to take into account, so we just drop it.

Also, we have
\[
(4.1.7) \quad \mathcal{L}_\Xi(D'_X(dt \wedge v_1)) = D'_X \left( \Xi | D'_X(\Xi | (dt \wedge v_1)) \right)
\]
thanks to Lemma 2.6, so summing up we get
\[
(4.1.8) \quad \frac{k(k+1)}{m} t^{k-1} \Lambda_k \equiv \partial u_2 + D'_X(dt \wedge v_2),
\]
modulo \( t^k \), where we use the notations
\[
(4.1.9) \quad u_2 := \mathcal{L}_\Xi(u_1), \quad dt \wedge v_2 := \partial \Xi | u_1 + dt \wedge \left( \Xi | D'_X(\Xi | (dt \wedge v_1)) \right)
\]
Our statement follows by induction on \( k \)–and moreover we have
\[
(4.1.10) \quad c_k := \frac{(k+1)!}{m}
\]
as we see by taking successive derivatives of (4.1.8). \( \square \)

Remark 4.2. The relations (4.1.9) give the explicit process of constructing \( \alpha_k \) and \( \beta_k \). They will play an important role in the analysis of the singularities of these forms (this is required by Theorem 3.2).

5. Proof of the main results: analysis

In this section we use the results established in 3 and 4 in order to establish our main statements.

We start by a simple observation, and then we will revisit a remarkable theorem by M. Levine in the spirit of the current paper.

5.1. A remark. The first result that we show is that one can always extend the section \( s \) defined on \( X \) to the first infinitesimal neighbourhood. This is established in the paper [5] by different methods.

We are in the context of (4.0.2) and let \( \lambda_1 := \mathcal{L}_\Xi \mathcal{L}_\Xi | X \) be the restriction on the central fiber. By applying the Lie derivative \( \mathcal{L}_\Xi \) to (4.0.2), as we have seen in the section 4, this can be rewritten as in (4.0.5). The restriction of this expression to the central fiber gives
\[
(5.0.1) \quad \frac{1}{m} \lambda_1 = \partial a_1 + D'_X(b_1)
\]
on \( X \setminus Z \), where we use the (abuse of...) notation
\[
(5.0.2) \quad a_1 := \mathcal{L}_\Xi(s_1) | X, \quad b_1 := \rho_1 | X
\]
and \( \rho_1 \) is the form already appearing in Lemma 4.1.

The form \( D'_X(b_1) \) is certainly closed on \( X \setminus Z \). In order to apply Theorem 3.2 we have to check the \( L^2 \) hypothesis. We recall that here the data is
\[
(5.0.3) \quad L := (m-1)K_X, \quad \varphi_L := \frac{m-1}{m} \log |s_i|^2
\]
where \( s_i \) is the local expression of of the section \( s \) of \( mK_X \).

Given the explicit expression of \( a_1 \) and \( b_1 \), the \( L^2 \) hypothesis in (ii) of Theorem 3.2 are easy to check. “For punishment”, we will give the details for \( b_1 \) and its derivative.
With respect to the local coordinates in (5.0.1), we have
\begin{equation}
(5.0.4) \quad b_1|_U \approx s_i \sum_{\alpha} \bar{\partial} z_i^\alpha \wedge \partial w_i^\alpha
\end{equation}
and this is clearly \( L^2 \) with respect to the weight in (5.0.3). The symbol \( \approx \) is (5.0.4) means that the restriction of \( b_1 \) to \( U \) equals the RHS with respect to the coordinates \( (z_i) \).

Moreover, the fact that \( D_X^p(b_1) \) is equally \( L^2 \) boils down to the convergence of the integral
\begin{equation}
(5.0.5) \quad \int_{U|_i} \frac{|df|^2}{|f|^2} d\lambda
\end{equation}
where \( f \) is a holomorphic function defined in the open set of \( \mathbb{C}^n \) containing \( U \). This in turn is quickly verified by a change of variables formula.

In conclusion, we have
\begin{equation}
(5.0.6) \quad \lambda_1 = \bar{\partial} \gamma_1 \quad \text{on } X.
\end{equation}

Now we can construct the 2-jet extension as follows. Let \( \cup U_i \) be a Stein cover of \( Y \) and \( \{ \phi_i \} \) be a partition of unity. On every \( U_i \), we can thus find a \( \gamma_{1,i} \) such that \( \Lambda_1|_{U_i} = \bar{\partial} \gamma_{1,i} \) and \( \frac{\partial}{\partial \phi_i} |_{U_i \cap X} = \gamma_1 \). Therefore \( \gamma_{1,i} - \gamma_{1,j} = t \cdot \gamma_{i,j} \) since the difference is holomorphic. Set \( s_2 := s_1 - t \sum \phi_i \gamma_{1,i} \). Then \( s_2 \) is a smooth section of the bundle \( K_X + L \) such that
\begin{equation}
(5.0.7) \quad s_2|_X = s, \quad \bar{\partial} s_2 = t^2 \Lambda_2.
\end{equation}

Remark 5.1. This first step in the extension of the section \( s \) is somehow misleading, i.e. too simple in some sense. Some of the real difficulties one has to deal with are appearing during the extension to the second infinitesimal nbd, see the subsection 5.3 below.

5.2. Extension of sections whose zero set is non-singular. In order to extend our section \( s \) to the first infinitesimal neighbourhood, we have used the Lie derivative with respect to an arbitrary vector field \( \Xi \). For higher order extension, this does not seem to be possible because of the singularities of the operator \( D_X^p \). In the case of a section \( s \) whose zero set \( (s = 0) \subset X \) is non-singular (treated in [5], this is done as follows.

We assume that the extension \( s_k \) to the \( k \)-th infinitesimal neighbourhood has already been constructed. For each trivialising open set \( \Omega_i \subset Y \) together with fixed coordinates functions \( (z_i^\alpha)_{\alpha=1,\ldots,n} \) we denote by
\begin{equation}
(5.1.1) \quad f_i
\end{equation}
the holomorphic function corresponding to the \( k \)-th-jet of \( s_k \). The set \( (f_k = 0) \subset \Omega_i \) is non-singular and transversal to the central fiber \( X \). Then we define a new set of local coordinates
\begin{equation}
(5.1.2) \quad (t, w_1^1, \ldots, w_1^n)
\end{equation}
on \( \Omega_i \) such that \( w_1^1 = f_i \) here we use the fact that at each point of \( \Omega_i \) we can find an index \( \alpha \) such that the form
\begin{equation}
(5.1.3) \quad df_i \wedge dt \wedge d\bar{z}_1^\alpha \neq 0
\end{equation}
is non-vanishing at the said point. Strictly speaking we have shrink eventually the set \( \Omega_i \); however, given that the map \( p : Y \to \mathbb{D} \) is proper we can assume that the coordinates \( (5.1.2) \) are defined on \( \Omega_i \) itself.

Therefore the equality
\begin{equation}
(5.1.4) \quad w_1^1 = g_{ij}(t, w_j) \cdot w_j^1 + t^{k+1} \tau_{ij}(t, w_j)
\end{equation}
is valid on the overlapping subsets \( \Omega_i \cap \Omega_j \), where \( (g_{ij}) \) are the transition functions for the bundle \( K_X + L \).

We introduce next the following vector field
\begin{equation}
(5.1.5) \quad \Xi_k := \sum \theta_i \frac{\partial}{\partial t}|_{\Omega_i}
\end{equation}
corresponding to the covering \( (\Omega_i, (t, w_i)) \).
By the transition relation (5.1.4), we have
\[(5.1.6)\]
\[\Xi_k|_{\Omega_t} = \frac{\partial}{\partial t} \bigg|_{\Omega_t} + \left( a_i w_i + b_i t^k \right) \frac{\partial}{\partial w_i} + \sum_{\alpha \geq 2} v^\alpha_i \frac{\partial}{\partial w_i},\]
where \(a_i, b_i\) and \(v^\alpha_i\) are smooth. We notice that the vector field \(\Xi_k\) have the following important properties.

(i) Its coefficients are smooth.

(ii) The projection \(dp(\Xi_k)\) equals \(\frac{\partial}{\partial t}\).

(iii) Modulo the ideal generated by \(t\), it is tangent to the set \(f_i = 0\).

The connection we are using on the bundle \((m_0 - 1)K_X\) is induced by the \(C^\infty\) section \(s_k\). This means that with respect to our local coordinates in (5.1.2) we have
\[\Xi_k|_{\Omega_t} = \frac{\partial}{\partial t} \bigg|_{\Omega_t} \equiv -\partial_t \bigg|_{\Omega_t},\]
and their iterations. Locally we can write \(s_k = f \sigma\), where \(f\) is the function given by the expression (5.1.7), and \(\sigma\) is a top form with values in \((m - 1)K_X\). We have
\[\Xi_k|_{\Omega_t} = \Xi_k(f) \sigma + f \Xi_k \sigma\]
in which the second term has a -potentially- singular component
\[\Xi_k(f) \sigma = \frac{m - 1}{m} \Xi_k(f) \sigma = \frac{m - 1}{m} \Xi_k(f) \sigma.\]

Lemma 5.2. Let \(k \geq 0\) be a positive integer, and assume that the extension \(s_k\) such that
\[(5.2.1)\]
\[\tilde{\partial} s_k = t^{k+1} \Lambda_k\]
has been already constructed. We use the vector field \(\Xi_k\) introduced in (5.1.5) and consider the Lie derivative \(\mathcal{L}_{\Xi_k}\) induced by it. Lemma 4.1 provides us with forms \(\alpha_k\) and \(\beta_k\) such that
\[(5.2.2)\]
\[\Lambda_k = \tilde{\partial} \alpha_k + D_X^k (dt \wedge \beta_k)\]
modulo the ideal generated by \(t\). Then the restrictions \(\alpha_k|_X\) and \(\beta_k|_X\) are smooth.

Before explaining the proof, we note that Lemma 5.2 combined with Theorem 5.2 show that the restriction
\[\Lambda_k|_X \frac{dt}{X}\]
is \(\tilde{\partial}\)-exact. Thus we can extend \(s\) one step further, given that there exist forms \(\mu_k\) and \(\Lambda_{k+1}\) of type \((n + 1, 0)\) and \((n + 1, 1)\) respectively, such that
\[(5.2.4)\]
\[\Lambda_k = \tilde{\partial} \mu_k + t \Lambda_{k+1}.\]

By combining (5.2.1) and (5.2.4) we infer the existence of \(s_{k+1}\) such that
\[(5.2.5)\]
\[\tilde{\partial} s_{k+1} = t^{k+2} \Lambda_{k+1}.\]
We can repeat this procedure inductively, showing that the section \(s\) admits an extension to the infinitesimal nbd of an arbitrary order. The formal arguments in [5] are implying that \(s\) extends to a topological nbd of \(X\) in \(X\).

We turn now to the proof of Lemma 5.2.

Proof. Considering the preparation we have done in the previous sections, the arguments which follow should be clear: we will proceed by induction, by using the fact that for each \(k\) we have the relations (4.1.9).

To this end, it would be helpful to remark that we have the following while computing the obstruction to extend the section modulo \(t^{k+2}\), we have to deal with quantities as
\[\mathcal{L}_{\Xi_k} s_k, \quad D_X^k \left( dt \wedge \left( \Xi|_X (D_X^k (\Xi|_X \sim s_k)) \right) \right) \]
and their iterations. Locally we can write \(s_k = f \sigma\), where \(f\) is the function given by the expression like (5.1.7), and \(\sigma\) is a top form with values in \((m - 1)K_X\). We have
\[(5.2.7)\]
\[\mathcal{L}_{\Xi_k} s_k = \Xi(f) \sigma + f \mathcal{L}_{\Xi_k} \sigma\]
in which the second term has a -potentially- singular component
\[f \mathcal{L}_{\Xi_k} \sigma = - \frac{m - 1}{m} f \wedge (\Xi_k|_X) \sigma = - \frac{m - 1}{m} \Xi_k(f) \sigma.\]
The point here is that the vector field $\Xi_k$ was constructed in such a way that $\Xi_k(f)$ is a *multiple* of $f$, plus some power of $t$. More precisely, we have
\begin{equation}
\Xi_k \cdot (f, t^{k+1}) \subset (f, t^k)
\end{equation}
where we remark that the ideal $(f, t^{k+1})$ is in fact globally defined on $X$.

Therefore we can take as many times the Lie derivative as the power of $t$ allows, the result will still be of the same type. We show next that the same is true for the second term in (5.2.6).

Indeed we have
\begin{equation}
D'_X(\Xi_k [\partial \Xi_k | s_k]) = \partial f \wedge \Xi_k [\partial \Xi_k | \sigma] + f D'_X(\Xi_k [\partial \Xi_k | \sigma])
\end{equation}
and a further contraction with $\Xi$ gives
\begin{equation}
\Xi_k [\partial f \wedge \Xi_k | (\partial \Xi_k | \sigma)] = \Xi_k (f) \cdot \Xi_k [\partial \Xi_k | \sigma]
\end{equation}
for the first term in (5.2.10). The singular part of the second one is
\begin{equation}
f \cdot \frac{df}{f} \wedge \Xi_k [\partial \Xi_k | \sigma]
\end{equation}
and when we contract it with $\Xi$ the result will be the RHS of (5.2.11).

In conclusion, after the first derivative of the relation (4.1.1) the forms $u_2$ and $v_2$ of (4.1.9) have their coefficients in the ideal $(f, t^k)$. We can therefore iterate this procedure, and obtain the conclusion. \hfill \square

**Remark 5.3.** We note that at each step $k$ we choose a vector field $\Xi_k$ adapted to the corresponding extension $s_k$.

**Remark 5.4.** If the zero set $(s = 0)$ of our initial section is singular, then one can still construct a vector field adapted to it as in the proof just finished, but the difference is that the new $\Xi$ will be singular along $s = 0, ds = 0$.

### 5.3. Extension to the second infinitesimal neighbourhood

We have already mentioned that the techniques we are developing in this paper are allowing us -in some particular cases- to extend the section $s$ to the second infinitesimal nbd of the central fiber. We present the arguments in this section.

Thanks to the subsection 5.1, we can always find a 1-order extension $s_2$, i.e.
$s_2|_X = s, \quad \bar{\partial}s_2 = t^2 \Lambda_2.$

The main observation is that if we write
\begin{equation}
\Lambda_2 = \bar{\partial}\alpha_2 + D'_X(dt \wedge \rho_2),
\end{equation}
as in Lemma 5.2, then the restriction $\rho_2|_X$ is automatically in $L^2$ under the assumption of Theorem 1.5. Here the properties (1.5.1) of the vector field $\Xi$ used in order to define the Lie derivative. This is the content of our next statement.

**Lemma 5.5.** Under the hypothesis of Theorem 1.5 we have
\begin{equation}
\int_X |\rho_2|^2 e^{-\rho_2} dV_{\omega_E} < \infty.
\end{equation}

**Proof.** This will be done by an explicit evaluation of $\rho_2$. Our first claim is that it is enough to show the convergence of the integrals
\begin{equation}
\int_{(\mathbb{C}^n, 0)} |f \bar{d}z^\alpha \wedge \bar{\partial}u|^2 e| f|^2 \omega_E dV_{\omega_E}
\end{equation}
and
\begin{equation}
\int_{(\mathbb{C}^n, 0)} |\partial f \wedge \bar{d}z^\alpha \wedge \bar{\partial}u|^2 e| f|^2 \omega_E dV_{\omega_E},
\end{equation}
as well as
\begin{equation}
\int_{(\mathbb{C}^n, 0)} |\partial f|^2 |\partial \Xi|^2 e| f|^2 \omega_E d\lambda.
\end{equation}
This is a consequence of formula (5.5.9), the computations are as follows. We have

\[(5.5.5)\quad \partial \Xi | L_\Xi(s) \simeq \frac{1}{m} \Xi(f) \cdot \partial \Xi | (dt \wedge dz)\]

modulo a term divisible by \(f\). On the other hand, we have

\[(5.5.6)\quad D'(\Xi | (\partial \Xi | s)) \simeq \frac{1}{m} \partial f \wedge (\Xi | (\partial \Xi | (dt \wedge dz)))\]

again modulo a multiple of \(f\). A further contraction with \(\Xi\) gives

\[(5.5.7)\quad \Xi | (D'(\Xi | (\partial \Xi | s)) \simeq \frac{1}{m} \Xi(f) \cdot \Xi | (\partial \Xi | (dt \wedge dz))).\]

Finally, we apply \(dt \wedge \cdot\) in (5.5.7), and what we get is the same as the RHS of (5.5.5). Now given the formula which computes \(dt \wedge \rho_2\) our claim follows.

Coming back to the integrals above, for (5.5.2) things are clear because the volume of \((X, \omega_E)\) is finite. Modulo the blow-up map \(\pi\) in Section 3, the integral (5.5.3) reduces to the evaluation of the next quantity

\[(5.5.8)\quad \sum_{\beta_{\alpha} = 1}^p \int_{(C^n, p) \setminus \beta_{\alpha}} |z_\alpha|^{2 \delta_\alpha} \left| \frac{dz_\beta}{z_\beta} \right|^2 dz \omega_E\]

where \(\delta_\alpha > 0\) are positive rational numbers. The convergence of (5.5.8) follows, given that we know the singularities of the metric \(\omega_E\).

For the expression (5.5.4) we use the fact that after the first order extension cf. Section 5.1 we have

\[(5.5.9)\quad \int_{(C^n, p) \setminus \beta_{\alpha}} |\partial f|^2 \frac{d\lambda}{|f|^{2 m - n}} < \infty.\]

But then it follows that we also have

\[(5.5.10)\quad \int_{(C^n, p) \setminus \beta_{\alpha}} |\partial f|^2 \frac{d\lambda}{|f|^{2 m - n + \varepsilon}} < \infty\]

for some positive real \(\varepsilon_0 > 0\). This is enough to absorb the term arising from \(|\partial \Xi|\omega_E^2\), because of the hypothesis (5.5.1).

In conclusion, we find ourselves in the following situation i.e. the setting of Theorem 1.6: let \((L, h_L)\) be a holomorphic line bundle on \(X\) and the possible singular metric \(h_L\) is of analytic singularities. Let \(Z\) be the singular locus of \(h_L\) and let \(\omega_E\) be a Poincaré type metric on \(X\) with poles along the \(Z\). We have a \(L\)-valued \(L^2\) form \(\rho\) of type \((n - 1, 1)\) on \(X\), such that

\[(5.5.11)\quad \tau := D_X^\rho\]

is \(\bar{\partial}\) closed. If \(\tau\) would be \(L^2\), then we can apply directly Theorem 5.9 and conclude that \(\tau\) is \(\bar{\partial}\)-exact. However, we do not pose this information, and we will follow a different path.

We define the linear form \(T_\rho\) by the formula

\[(5.5.12)\quad T_\rho(\psi) := \int_X (\rho \cdot (D_X^\rho)^* \psi) e^{-\xi_2} \omega_E dV_{\omega_E}\]

where \(\psi\) is a \(L\)-valued smooth form of \((n, 1)\)-type with compact support in \(X \setminus Z\).

The current \(T_\rho\) has the following properties.

**Lemma 5.6.** Let \(\psi\) be a test form as above, and consider the decomposition

\[(5.6.1)\quad \psi = \xi_1 + \xi_2\]

according to Ker(\(\bar{\partial}\)) and its orthogonal. Then \(T_\rho(\xi_2) = 0\).

**Proof.** Note that \(T_\rho(\xi_2)\) is indeed well-defined, thanks to Friedrichs lemma (cf. e.g. [2]): the \(L^2\) form \(\xi_2\) belongs to the domain of \(\bar{\partial}\), hence it is a limit in graph norm of smooth forms \((\xi_2, k)\) with compact support. It follows that we have

\[(5.6.2)\quad (D_X^\rho)^*(\xi_2, k) \to (D_X^\rho)^*(\xi_2)\]

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in $L^2$ as $k \to \infty$. This is a consequence of Bochner formula

$$\int_X |(D_X^*)^*(\xi_2, k \xi_2)|^2 dV \leq \int_X |\bar{\partial}(\xi_2, k \xi_2)|^2 dV + \int_X |\bar{\partial}^*(\xi_2, k \xi_2)|^2 dV$$

(here semi-positive curvature is sufficient) and by Friedrichs lemma the RHS of (5.6.3) tends to zero as $k \to \infty$.

It follows that $T_\rho(\xi_2)$ is well-defined and moreover we have

$$T_\rho(\xi_2) = \int_X \langle \rho, (D_X^*)^*(\xi_2) \rangle \omega_k e^{-\varphi_L}.$$  

Thanks to the $L^2$ hypothesis on $\rho$, we have

$$\int_X \langle \rho, (D_X^*)^*(\xi_2) \rangle \omega_k e^{-\varphi_L} = \lim_{\varepsilon \to 0} \int_X \langle \rho, (D_X^*)^*(\mu_\varepsilon \xi_2) \rangle \omega_k e^{-\varphi_L}$$

where $(\mu_\varepsilon)_{\varepsilon > 0}$ is the family of cut-off functions adapted to Poincaré metric, whose construction was recalled in Lemma 3.1.

Indeed, (5.6.5) follows since we have

$$D^*(\mu_\varepsilon \xi_2) = \mu_\varepsilon D^*(\xi_2) + \bar{\partial} \mu_\varepsilon \xi_2$$

and again by the $L^2$ condition, we have

$$\lim_{\varepsilon \to 0} \int_X \langle \rho, D^*(\mu_\varepsilon \xi_2) \rangle \omega_k e^{-\varphi_L} = \lim_{\varepsilon \to 0} \int_X \langle \rho, \mu_\varepsilon D^*(\xi_2) \rangle \omega_k e^{-\varphi_L}.$$  

Therefore, we infer that

$$\int_X \langle \rho, D^*(\xi_2) \rangle \omega_k e^{-\varphi_L} = \lim_{\varepsilon \to 0} \int_X \langle \rho, \mu_\varepsilon D^*(\xi_2) \rangle \omega_k e^{-\varphi_L}.$$  

On the other hand, $\xi_2 = \bar{\partial}^* \lambda$ for some $L^2$ form $\lambda$ which can be assumed to be $\bar{\partial}$-closed and then we write

$$\int_X \langle \mu_\varepsilon D'(\rho), \xi_2 \rangle \omega_k e^{-\varphi_L} = \int_X \langle \bar{\partial} \mu_\varepsilon \wedge D'(\rho), \lambda \rangle \omega_k e^{-\varphi_L}$$

because $D'(\rho)$ is $\bar{\partial}$-closed. The RHS term in (5.6.9) is equal to

$$\int_X \langle D'(\bar{\partial} \mu_\varepsilon \wedge \rho), \lambda \rangle \omega_k e^{-\varphi_L} + \int_X \langle \bar{\partial} \bar{\partial} \mu_\varepsilon \wedge \rho, \lambda \rangle \omega_k e^{-\varphi_L}$$

up to a sign. The second term of (5.6.10) tends to zero as $\varepsilon \to 0$, and so does the first one, because by Bochner formula we have

$$\int_X |D^*(\lambda)|^2 dV \leq \int_X |\bar{\partial} \lambda|^2 dV = \int_X |\xi_2|^2 dV$$

(because we assume that $\bar{\partial} \lambda = 0$), and the RHS of (5.6.11) is convergent. Note again that here the semi-positivity of the curvature is enough, given that $\lambda$ is of type $(n, 2)$.

In conclusion, the $T_\rho(\xi_2) = 0$.\qed

Now we are ready to prove Theorem 1.5 and Theorem 1.6.

**Proof of Theorem 1.6.** Let $\psi$ be an $L$-valued smooth form of $(n, 1)$-type with compact support in $X \setminus Z$. Consider the Hodge decomposition (5.6.11); then we have

$$T_\psi(\psi) = T_\psi(\xi_1)$$

by Lemma 5.6. Then it follows by Cauchy-Schwarz inequality combined with Bochner formula and the usual $L^2$ theory that there exists a $u$ such that

$$T_\psi(\psi) = \int_X \langle u, \bar{\partial}^* \psi \rangle e^{-\varphi_L}$$

in other words we infer that

$$D_X'(u) = \bar{\partial} u.$$  

\qed
Proof of Theorem 1.5. Our aim is to prove that the \( \Lambda^2|_X \) in (5.4.1) is \( \bar{\partial} \)-exact, which is equivalent to prove that the \( \partial \)-closed form \( D_X^* (\rho_2|_X) \) is \( \bar{\partial} \)-exact. It is a direct consequence of Lemma 5.6 and Theorem 1.6. □

We prove next Theorem 1.7. Actually the motivation for this result is that the form \( \rho_2|_X \) is automatically \( L^2 \) if we are using a non-singular metric on \( X \).

Proof of Theorem 1.7. The main technical difficulty here is that the metric \( \omega \) on \( X \setminus Z \) is not complete. Usually this is bypassed by using a sequence of complete metrics

\[
\omega_\varepsilon := \omega + \varepsilon \omega_Z
\]

and invoke the usual arguments in \( L^2 \) theory. This works perfectly for forms of type \( (n, q) \) (because in this case, a monotonicity argument can be used) but in our case the form \( v \) is of type \( (n-1, 1) \), hence in the absence of hypothesis (3), it is not necessarily \( L^2 \) with respect to \( \omega_\varepsilon \) above. We proceed as follows.

We recall that the weight \( \varphi_L \) of the metric \( h_L \) is assumed to have log poles along \( Z \). Let \( \xi \) be an \( L \)-valued \((n, 1)\)-form whose support is contained in \( X \setminus Z \). We have to evaluate the quantity

\[
\int_X \langle D^* v, \xi \rangle e^{-\varphi_L} dV_{\omega}. \tag{5.6.16}
\]

The form \( \xi \) can be written as

\[
\xi = \xi_1 + \xi_2
\]

according to Ker(\( \bar{\partial} \)) and its orthogonal. Since \( D^* v \) is \( L^2 \) and \( \bar{\partial} \)-closed, we have

\[
\int_X \langle D^* v, \xi_2 \rangle e^{-\varphi_L} dV_{\omega} = 0. \tag{5.6.18}
\]

So (5.6.16) equals the expression

\[
\int_X \langle D^* v, \xi_1 \rangle e^{-\varphi_L} dV_{\omega}. \tag{5.6.19}
\]

Note that we have

\[
\int_X \langle D^* v, \xi_1 \rangle e^{-\varphi_L} dV_{\omega} = \lim_{\varepsilon \to 0} \int_X \langle D^* v, \xi_{1, \varepsilon} \rangle e^{-\varphi_L} dV_{\omega}. \tag{5.6.20}
\]

In (5.6.20), the notations are as follows. The metric \( \omega_\varepsilon \) was introduced in (5.6.15), and \( \xi_{1, \varepsilon} \) is the orthogonal projection of \( \xi \) on the Ker(\( \bar{\partial} \)) with respect to \( (X \setminus Z, \omega_\varepsilon) \) and \((L, h_L)\), respectively. The equality (5.6.20) follows from the fact that \( \xi_{1, \varepsilon} \to \xi_1 \) uniformly on the compact sets of \( X \setminus Z \).

It is at this point that we have to use the hypothesis (3): as a consequence of it, we can write

\[
\int_X \langle D^* v, \xi_{1, \varepsilon} \rangle e^{-\varphi_L} dV_{\omega} = \int_X \langle v, D^* (\xi_{1, \varepsilon}) \rangle e^{-\varphi_L} dV_{\omega}. \tag{5.6.21}
\]

Moreover we have

\[
\lim_{\varepsilon \to 0} \int_X |v|^2_{\omega_{\varepsilon}} dV_{\omega_{\varepsilon}} = \int_X |v|^2_{\omega} dV_{\omega} \tag{5.6.22}
\]

because if we write in coordinates

\[
v = \sum v_{p\bar{q}} dz_p \wedge d\bar{z}_q,
\]

then we have

\[
|v|^2_{\omega_{\varepsilon}} dV_{\omega_{\varepsilon}} = \sum |v_{p\bar{q}}|^2 \frac{1 + \varepsilon \lambda_p}{1 + \varepsilon \lambda_q} dz \wedge d\bar{z}
\]

and the trivial inequality \( \frac{1 + \varepsilon \lambda_p}{1 + \varepsilon \lambda_q} < 1 + \frac{\lambda_p}{\lambda_q} \) implies that \( |v|^2_{\omega_{\varepsilon}} dV_{\omega_{\varepsilon}} \leq |v|^2_{\omega} dV_{\omega} + |v|^2_{\omega_{\varepsilon}} dV_{\omega_{\varepsilon}} \). This allows us to use dominated convergence theorem and infer (5.6.22).

The \( L^2 \) norm of the form \( D^* (\xi_{1, \varepsilon}) \) is smaller than

\[
\int_X |\bar{\partial}^* (\xi_{1, \varepsilon})|^2 e^{-\varphi_L} dV_{\omega} = \int_X |\bar{\partial}^* \xi|^2 e^{-\varphi_L} dV_{\omega}. \tag{5.6.25}
\]
and since we assume from the beginning that the support of the form $\xi$ is contained in $X \setminus Z$, the limit as $\varepsilon \to 0$ of the RHS of (5.6.25) is precisely
\[
\int_X |\partial^*\xi|^2 e^{-\varphi_L} dV_{\omega^e},
\]
and we are done. \hfill \Box

5.4. **Proof of Theorem 1.2.** The plan for this subsection is as follows. We start with a general discussion about the method we will use and reduce the problem to a local question. Then we discuss a particular case of Theorem 1.2 whose only raison d'etre is to illustrate the main ideas in the proof. A complete argument is given afterwards.

5.4.1. **Preliminaries and some reductions.** To start with, we consider the obstruction forms $\Lambda_k$ in Lemma 4.1. We first show that under the hypothesis of Theorem 1.2 we can modify the extension $s_k$ in such a way that the equality of type (4.1.1) is still satisfied and moreover the resulting $\Lambda_k$ is $L^2$ when restricted to the central fiber $X$.

Indeed, let
\[
\lambda_k := \frac{\Lambda_k}{dt}|_X
\]
be the restriction of $\Lambda_k$ to $X$. Then we have $\bar{\partial}\lambda_k = 0$ and the image of its cohomology class via the projection morphism
\[
H^1(X, mK_X) \to H^1(X, \mathcal{O}_X(mK_X) \otimes \mathcal{O}_X/\mathcal{I})
\]
is equal to zero. It follows that there exist a $C^\infty$ form $a_k$ of type $(n, 0)$ with values in $(m - 1)K_X$, as well as a $C^\infty$ form $b_k$ of type $(n, 1)$ with values the sheaf $K_X^{\otimes (m-1)} \otimes \mathcal{I}$ so that
\[
\lambda_k = b_k + \bar{\partial}a_k.
\]
Let $A_k$ and $B_k$ be smooth extensions of $dt \wedge a_k$ and $dt \wedge b_k$, respectively to $X$. We define
\[
\tilde{s}_k := s_k - t^{k+1}A_k
\]
and then we get
\[
\bar{\partial}\tilde{s}_k = t^{k+1}\Lambda_k
\]
where $\tilde{\Lambda}_k$ is smooth and its restriction to $X$ obviously satisfies
\[
\tilde{\lambda}_k = b_k
\]
and therefore its coefficients are in $\mathcal{I} \otimes C^\infty$.

Summing up and changing the notations, we can assume that the restriction $\lambda_k$ of $\Lambda_k$ to $X$ verifies
\[
\int_X |\lambda_k|^2 e^{-\varphi_L} dV_{\omega^e} < +\infty.
\]
and moreover it is $\bar{\partial}$-closed. Here $L := (m - 1)K_X$ and $\varphi_L = \frac{m-1}{m} \log |s|^2$ is the weight of the singular metric $h_L$ on $L$. This follows from what we’ve just explained, together with the fact that there exists a non-singular metric $\omega$ on $X$ such that $\omega_E \geq \omega$.

In general, let $\lambda$ be a $\bar{\partial}$-closed form of type $(n, 1)$ with values in $L = (m - 1)K_X$, which is $L^2$ with respect to $h_L$ and $\omega_E$. Thanks to Theorem 3.2 we have a $L^2$- decomposition
\[
\lambda = u + \bar{\partial}v
\]
where $u$ is harmonic. From Lemma 3.8 it follows that $\lambda$ is $\bar{\partial}$-exact if and only if
\[
\int_X \lambda \wedge Fe^{-\varphi_L} = 0
\]
for any $L^2$ holomorphic form $F$ of type $(n - 1, 0)$ with values in $L$.

Coming back to our problem, recall that we have the following equality
\[
\lambda_k = \bar{\partial}a_k + D_X^t \beta_k \quad \text{on} \ X \setminus Z.
\]
Moreover, the poles of the forms $\alpha_k$ and $\beta_k$ when restricted to $X$ are very roughly estimated as follows
\begin{equation}
\alpha_k = \frac{1}{s^{k+1}} \mathcal{O}(1), \quad \beta_k = \frac{1}{s^k} \mathcal{O}(1)
\end{equation}
where $\mathcal{O}(1)$ in both expressions represent smooth forms. The fact that (5.6.37) holds true is a consequence of the inductive formula in the proof of Lemma 3.1. Here we are simply using a smooth vector field $\Xi$, no particular requests.

Let $(\theta_i)_{i \in I}$ be an arbitrary partition of unit, then we have
\begin{equation}
\lambda_k = \sum_{i \in I} \bar{\partial}(\theta_i \alpha_k) + \sum_{i \in I} D_X(\theta_i \beta_k) \quad \text{on } X \setminus Z.
\end{equation}
The equality (5.6.38) is equivalent to showing that
\begin{equation}
\lim_{\varepsilon \to 0} \int_X \mu_{\varepsilon} \lambda_k \wedge \overline{\mathcal{F}} e^{-\varphi_{\varepsilon}} = 0,
\end{equation}
where $(\mu_{\varepsilon})_{\varepsilon > 0}$ is the family of cut-off functions in Lemma 3.1. Note that for this to hold true in general (i.e. without the assumption concerning the multiplier ideal sheaf of $h_L$) it is sufficient that $\lambda_k$ is an $L^2$ form.

By Stokes, we have to show that the integrals
\begin{equation}
\int_X \theta \bar{\partial} \mu_{\varepsilon} \wedge \alpha_k \wedge \overline{\mathcal{F}} e^{-\varphi}, \quad \int_X \theta \bar{\partial} \mu_{\varepsilon} \wedge \beta_k \wedge \overline{\mathcal{F}} e^{-\varphi}
\end{equation}
are converging to zero, where $\varphi := \frac{m-1}{m} \log |f|^2$ and $f$ is the local expression of our section $s$ and $\theta$ a partition function. If we can show that (5.6.40) tends to 0, then $\lambda_k$ is $\bar{\partial}$-exact and the theorem is proved.

5.4.2. A (very) particular case. To illustrate the reason for which (5.6.40) converges to 0, we discuss in this subsection a particular case. The complete argument will be detailed afterwards.

Concerning the second quantity in (5.6.40), note that we have
\begin{equation}
\theta \beta_k \wedge \frac{ds}{s} \wedge \overline{\mathcal{F}} e^{-\varphi} = \sum_{\rho=1}^n \Psi_{\rho, k} \overline{\mathcal{F}} \wedge \frac{dw_{\rho}}{|f|^{2-q_{\rho}}} \quad \text{on } \Omega.
\end{equation}
for some smooth forms $\Psi_{\rho, k}$ defined on some small open set $\Omega$, whose support is compact in $\Omega$.

Let $\pi : \tilde{X} \to X$ be a modification such that the inverse of the divisor of $s$ becomes snc.

Assume that we can find a smooth metric $h_0$ on $K_X + L = mK_X$ such that the inverse image of the local weights $\varphi \circ \pi$ are independent on the $z_1, \ldots, z_r$, where the $(z_i)_{i=1}^r$ are coordinates such that $f \circ \pi = z_1^{p_1} \cdots z_r^{p_r}$.

In such case we argue as follows. The measure
\begin{equation}
\pi^* \left( \frac{|F \wedge dw_{\rho}|^2}{|f|^{2-2q_{\rho}}} \right)
\end{equation}
has finite mass, by the third point of Lemma 6.8. Therefore with respect to the coordinates $(z_i)$ the integral (5.6.41) becomes
\begin{equation}
\pi^* \left( \theta \beta_k \wedge \frac{ds}{s} \wedge \overline{\mathcal{F}} e^{-\varphi} \right) \Big|_{\Omega} = \frac{\hat{\Psi}_k}{\prod_{i=1}^r z_i^{q_i}} \prod_{i=1}^r |z_i|^{2-2\delta_i} \frac{d\lambda(z)}{\prod_{i=1}^r z_i^{2-2\delta_i}},
\end{equation}
where the $\delta_i$ are strictly positive reals and $q_i$ are positive integers.

Thanks to (6) we immediately see that the second quantity in (5.6.40) tends to zero, regardless to the size of the multiplicities $q_i$. Indeed, locally it can be written as
\begin{equation}
\int_{(\mathbb{C}^n, 0)} \frac{\hat{\Psi}_k(z)}{\prod_{i=1}^r z_i^{q_i}} \rho_k'(\log r(z', z'')) \prod_{i=1}^r |z_i|^{2-2\delta_i} \frac{d\lambda(z)}{\prod_{i=1}^r |z_i|^{2-2\delta_i}}.
\end{equation}
where \( z' := (z_1, \ldots, z_r), z'' := (z_{r+1}, \ldots, z_n) \) and

\[
    r(|z'|, z'') := \prod_{i=1}^{r} |z_i^m| \cdot e^{-\gamma(z')}
\]

Since \( \Psi_k \) is smooth, \([5.6.43]\) amounts to the evaluation of

\[
    \int_{(C^0, 0)} g(z'') \prod_{i=1}^{n} z_i^{m_i} e^{-\gamma(z'')} \frac{\rho_k \left( \log r(|z'|, z'') \right)}{\prod_{i=1}^{n} |z_i|^{2\gamma_i}} d\lambda(z)
\]

where \( g \) is a smooth function only depending of the variables \( z'' \).

The integral \([5.6.44]\) is equal to zero -for any \( \epsilon > 0 \) - unless we have

\[
    m_i - q_i = m_\gamma \geq 0.
\]

But for such exponents the limit as \( \epsilon \to 0 \) of \([5.6.43]\) is equal to zero, because the pole of order \( q_i \) disappears, and \( \delta_i > 0 \).

Dealing with the first term in \([5.6.49]\) is a bit trickier, and we postpone this until the last section.

### 5.4.3. Taylor expansions

In general, i.e. in the absence of the assumption \( (\bigcirc) \), our arguments are virtually the same, modulo a few technicalities that we discuss next.

We will use Taylor formula with integral reminder. In order to illustrate the type of statement we are after, we first start with a simple case.

Let \( \phi \) be a smooth function defined on \( \mathbb{R} \), and consider

\[
    f(x_1, x_2) := \phi(x_1 + x_2)
\]

a function defined say in a ball centered at the origin in \( \mathbb{R}^2 \).

The Taylor expansion of \( f \) with respect to \( x_1 \) reads as follows

\[
    f(x_1, x_2) = \phi(x_2) + x_1 \phi'(x_2) + \cdots + \frac{x_1^{N}}{N!} \phi^{(N)}(x_2) + \frac{x_1^{N+1}}{N!} \int_{0}^{1} \phi^{(N+1)}(tx_1 + x_2)(1-t)^N dt.
\]

Next we expand each of the functions involved in \([5.6.47]\) and \([5.6.48]\) with respect to \( x_2 \), and we have:

\[
    f(x_1, x_2) = \sum_{0 \leq m_1, m_2 \leq N} \frac{x_1^{m_1} x_2^{m_2}}{m_1! m_2!} \phi^{(m_1 + m_2)}(0) + \sum_{0 \leq m_1 \leq N} \frac{x_1^{m_1} x_2^{N+1}}{N! m_1!} \int_{0}^{1} \phi^{(m_1 + N + 1)}(tx_2)(1-t)^N dt
\]

\[
    + \sum_{0 \leq m_2 \leq N} \frac{x_1^{N+1} x_2^{m_2}}{N! m_2!} \int_{0}^{1} \phi^{(N+1 + m_2)}(tx_1)(1-t)^N dt
\]

\[
    + \frac{x_1^{N+1} x_2^{N+1}}{N!} \int_{0}^{1} \int_{0}^{1} \phi^{(2N+2)}(t_1x_1 + t_2x_2)(1-t_1)(1-t_2)^{N} dt_1 dt_2.
\]

Now we remark that in the expressions \([5.6.49]-[5.6.52]\) we have three type of terms, according to the exponents of \( x_1 \) and \( x_2 \), respectively.

1. If both exponents \( m_1 \) and \( m_2 \) are smaller than \( N \), then the function \( \phi^{(m_1 + m_2)}(0) \) is constant with respect to both variables.
2. If, say, \( m_1 \leq N \) and \( m_2 = N + 1 \), then the function in question is only depending on \( x_2 \). The same is of course true if the roles of \( x_1 \) and \( x_2 \) are exchanged.
3. If both exponents \( m_1 \) and \( m_2 \) are equal to \( N + 1 \), then the function in \([5.6.52]\) is smooth (and in general, depending on both variables).
This is not limited to the case of two variables, and the result is the same: the function which multiplies the monomial

\[(5.6.53)\]

\[ x_1^{m_1} \cdots x_r^{m_r} \]

is independent on \( x_i \) if we have \( m_i \leq N \), for each \( i = 1, \ldots, r \)

Coming back to our problem, we will use a variation of this type of considerations in order to obtain an expansion which is adapted to our context.

Assume that the section \( s \) corresponds to the monomial \( z_1^{p_1} \cdots z_p^{p_r} \) when restricted to \( \Omega \)

\[(5.6.54)\]

\[ s|_\Omega \simeq z_1^{p_1} \cdots z_p^{p_r} \]

In what follows we will denote it simply by \( s \), for notation’s convenience.

The function we want to expand is

\[(5.6.55)\]

\[ \rho_{\varepsilon} \left( \log \log \frac{1}{|s|^{2e^{-\phi(z)}}} \right). \]

To this end, remark that we have

\[(5.6.56)\]

\[ \log \frac{1}{|s|^{2e^{-\phi(z)}}} = \log \frac{1}{|s|^{2e^{-\phi_1(z)}}} + z_1 \phi_1(z) + z_1 \phi_{\varepsilon}(z) \]

for some functions \( \phi_1, \phi_{\varepsilon} \), where \( \phi_1(z) := \phi(0, z_2, \ldots, z_n) \).

We consider a function \( \tau_1 \) defined by the formula

\[(5.6.57)\]

\[ \tau_1(z) := \log \left( 1 + \frac{z_1 \phi_1(z) + z_1 \phi_{\varepsilon}(z)}{\log |s|^{2e^{-\phi_1(z)}}} \right), \]

and then we have

\[(5.6.58)\]

\[ \log \log \frac{1}{|s|^{2e^{-\phi(z)}}} = \log \log \frac{1}{|s|^{2e^{-\phi_1(z)}}} + \tau_1(z). \]

In order to simplify the writing, we set the following notations, valid throughout the current subsection.

**Conventions.**

- **We will systematically denote by** \( a_{1-p} \) **any smooth function independent of the set of variables** \( z_1, \ldots, z_p \) **and their conjugates.**

- **We use the same notation e.g.** \( a, b, \phi, \ldots \) **for functions which are not necessarily identical, but they share similar properties, which will be clearly specified.**

Then we have

\[(5.6.59)\]

\[ \mu_{\varepsilon}(z) = \rho_{\varepsilon} \left( \log \log \frac{1}{|s|^{2e^{-\phi_1(z)}}} \right) \]

\[(5.6.60)\]

\[ + \rho_{\varepsilon}' \left( \log \log \frac{1}{|s|^{2e^{-\phi_1(z)}}} \right) \tau_1(z) + \ldots \]

\[(5.6.61)\]

\[ + \rho_{\varepsilon}^{(N)} \left( \log \log \frac{1}{|s|^{2e^{-\phi_1(z)}}} \right) \frac{\tau_1(z)}{N!} \]

\[(5.6.62)\]

\[ + \frac{1}{N!} \tau_1(z)^{N+1} \int_0^1 \rho_{\varepsilon}^{(N+1)} \left( \tau_1(z) + \log \log \frac{1}{|s|^{2e^{-\phi_1(z)}}} \right) (1-t)^N dt. \]

We consider next the quantity

\[(5.6.63)\]

\[ \frac{z_1 \phi_1(z) + z_1 \phi_{\varepsilon}(z)}{\log |s|^{2e^{-\phi_1(z)}}} \]

involved in the expression of the function \( \tau_1 \). It can be rewritten as

\[(5.6.64)\]

\[ \frac{z_1 \phi_1(z) + z_1 \phi_{\varepsilon}(z)}{\log |s|^{2e^{-\phi_1(z)}}} = z_1 a_1(z) + z_2 a_2(z) + \ldots + z_r a_r(z) + z_{r+1} \phi_{\varepsilon}(z) \]

where \( a_i, \phi_{\varepsilon} \) are smooth, and \( z'' := (z_{r+1}, \ldots, z_n) \).
In conclusion, we have

\[(5.6.65)\]
\[\tau_1(z) = \Psi_1(z_1, w_1, \ldots, z_r, w_r)\]

where

\[(5.6.66)\]
\[w_i := \frac{z_i}{\log |s|^{2e^{-\phi_1(z)}}}\]

for \(i = 1, \ldots, r\), and \(\Psi_1\) is smooth and belonging to the ideal generated by \(w_1, w_1\).

Next we iterate this: in the expressions \((5.6.59)-(5.6.62)\), we use the fact that

\[(5.6.67)\]
\[\log |s|^{2e^{-\phi_1(z)}} = \log |s|^{2e^{-\phi_1(z)}} + z_2 \phi_1(z) + \tau_2\phi_1(z)\]

where \(\phi_1\) and \(\phi_1\) are independent of \(z_1\).

We define

\[(5.6.68)\]
\[\tau_2(z) := \log \left(1 + \frac{z_2 \phi_1(z) + \tau_2\phi_1(z)}{\log |s|^{2e^{-\phi_1(z)}}}\right),\]

and consider the Taylor expansion of \(\rho_\varepsilon\) and its derivatives up to order \(N\) in \((5.6.59)-(5.6.60)\).

For \((5.6.69)\), we do the following. First, we remark that we can write

\[(5.6.69)\]
\[\tau_1(z) = a_2(z, w) + b(z, w)\]

where the notations in \((5.6.69)\) are as indicated below.

- As above, we have \(w_i := \frac{z_i}{\log |s|^{2e^{-\phi_1(z)}}}\).
- The function \(a_2\) is smooth, and it only depends on \(z_1, w_1, z_2, \ldots, z_n\).
- The function \(b\) belongs to the ideal generated by \(\tau_2, \tau_2\).

By \((5.6.67)\), we have

\[\log \log \frac{1}{|s|^{2e^{-\phi_1(z)}}} = \log \log \frac{1}{|s|^{2e^{-\phi_1(z)}}} + b(z, w)\]

with the same properties as in the third bullet above.

Then the function inside \((5.6.62)\) can be written as follows

\[(5.6.70)\]
\[\tau_1(z) + \log \log \frac{1}{|s|^{2e^{-\phi_1(z)}}} = a_2(z, w) + \log \log \frac{1}{|s|^{2e^{-\phi_1(z)}}} + b_0(z, w) + b_1(z, w),\]

and we expand the function \(\rho_\varepsilon^{(N+1)}\) in \((5.6.62)\) at

\[ta_2(z, w) + \log \log \frac{1}{|s|^{2e^{-\phi_1(z)}}}\]

After this second step, we see that the function \(\mu_\varepsilon\) can be written as sum of terms of the following type.

\[(5.6.71)\]
\[\tau_1^{m_1} \tau_2^{m_2} \rho_\varepsilon^{(m)} \left( \log \log \frac{1}{|s|^{2e^{-\phi_1(z)}}} \right)\]

where \(m := m_1 + m_2\), and both \(m_1\) are smaller or equal than \(N\),

\[(5.6.72)\]
\[\tau_1^{m_1} \tau_2^{N+1} \int_0^1 \rho_\varepsilon^{(N+1+m_1)} \left( \tau_2(z) + \log \log \frac{1}{|s|^{2e^{-\phi_1(z)}}} \right)(1-t)^N dt\]

where \(m_1 \leq N + 1\) together with

\[(5.6.73)\]
\[\tau_1^{N+1} \tau_2^{m_2} \int_0^1 \rho_\varepsilon^{(N+1+m_2)} \left( ta_2(z) + \log \log \frac{1}{|s|^{2e^{-\phi_1(z)}}} \right)(1-t)^N dt\]

(here \(\tau_2\) is not the same as in \((5.6.72)\), but it belongs to the ideal generated by \(z_2\) and \(\tau_2\), so we are using the same notation).
Finally, the last term is
\[ \int_0^1 \int_0^1 \rho_{\varepsilon}^{(2N+2)}(t_1 b_1 + t_2 b_2 + t_1 t_2 b_3 + \log \log \frac{1}{|s|^2 e^{-\phi_{12}(z)}}) (1-t)^N dt. \]
multiplied with \( \tau_1^{N+1} \tau_2^{N+1} \), where \((1-t)^N dt := (1-t_1)^N (1-t_2)^N dt_1 dt_2 \).

The conclusion is that the resulting expression has one “special” term, namely
\[ \rho_{\varepsilon} \left( \log \log \frac{1}{|s|^2 e^{-\phi_{12}(z)}} \right) \]
and for the others the support of the function in (5.6.71) is contained in the domain
\[ \frac{1}{\varepsilon} = C < \log \log \frac{1}{|s|^2 e^{-\phi_{12}(z)}} < \frac{1}{\varepsilon} + C. \]

Moreover, the function
\[ z \to \rho_{\varepsilon}^{(m)} \left( \log \log \frac{1}{|s|^2 e^{-\phi_{12}(z)}} \right) \]
in (5.6.71) only depends on \(|z_1|, |z_2|, z_3, \ldots, z_n\). The functions involved is the other expressions have similar properties, i.e. the one in (5.6.72) is independent of \( z_1 \), the one in (5.6.73) is independent of \( z_2 \). Note that the function in (5.6.74) depends -in general- on the full set of variables, but its support is contained in a domain like (5.6.76).

We emphasise all this because integrals of the type
\[ \int_{\Omega} \tau_1^{m_1} \tau_2^{m_2} \rho_{\varepsilon}^{(m)} \left( \log \log \frac{1}{|s|^2 e^{-\phi_{12}(z)}} \right) \Psi(z, w) \frac{d\lambda(z)}{|z_1|^{2-2\delta_1} |z_2|^{2-2\delta_2}} \]
are converging to zero as \( \varepsilon \to 0 \) for any \( m = m_1 + m_2 \geq 1 \), for any \( q_1, q_2 \) positive integers, and \( \delta_i > 0 \) positive reals. In (5.6.75) we denote by \( \Psi \) a smooth function.

Anyway, we iterate this procedure, and we obtain the following statement.

**Proposition 5.7.** The function \( \mu_{\varepsilon}(\cdot) \) admits a Taylor expansion whose finite number of terms are of the following type
\[ \tau_1^{m_1} (z, w) \ldots \tau_r^{m_r} (z, w) \Phi_{\varepsilon, m}(z, w) \]
where:

(a) the integer \( r \) is given by (5.6.32), and \( \max(m_i) \leq N + 1; \)
(b) for each \( i = 1, \ldots, r \) we have \( w_i := \frac{z_i}{e^{\phi(z')}} \), where \( z'' := (z_{r+1}, \ldots, z_n) \);
(c) the functions \( \tau_i \) are smooth, and they belong to the ideal generated by \( w_i \) and \( \Phi_{\varepsilon, m} \);
(d) the function \( \Phi_{\varepsilon, m}(z, w) \) only depends on \(|z_i|\) and \(|w_i|\) if the corresponding exponent \( m_i \) is smaller or equal to \( N \), for all \( i = 1, \ldots, r \);
(e) if \( m_1 = \cdots = m_r = 0 \), then we have
\[ \Phi_{\varepsilon, 0}(z, w) = \rho_{\varepsilon} \left( \log \log \frac{e^{\phi(z'')}}{\prod |z_i|^2} \right); \]
(f) if \( m_1 + \cdots m_r \geq 1 \), then the support of the function \( \Phi_{\varepsilon, m}(z, w) \) is contained in the set (5.6.70).
5.4.4. End of the proof of Theorem 5.3. In the absence of the over-simplifying assumption \((\Diamond)\), we consider the \((n, 1)\)-form \(\pi^* \Lambda_k\) with values in \(L := (m - 1) \pi^* K_X\). We still have a decomposition
\[(5.7.2)\]
\[\pi^* \lambda_k = \partial \tilde{\alpha}_k + D' \tilde{\beta}_k\]
where the forms \(\tilde{\alpha}_k\) and \(\tilde{\beta}_k\) are the \(\pi\)-inverse image of \(\alpha_k\) and \(\beta_k\), respectively. Moreover, we have an estimate for the pole order of \(\tilde{\alpha}_k\) and \(\tilde{\beta}_k\) similar to \((5.6.37)\).

We consider a holomorphic section \(\hat{F} \in H^0(\tilde{X}, \Omega^{n-1}) \otimes \pi^* L)\) such that
\[(5.7.3)\]
\[\int_{\tilde{X}} |\hat{F}|^2 \omega_{\tilde{X}, \pi^*} dV_{\tilde{X}} < +\infty,\]
where we recall that \(\omega_{\tilde{X}}\) is a metric Poincaré type singularities along the support of the divisor associated to the inverse image of \(s\) (here we are back to our initial notation, so that \(s\) is the section to be extended). It is clear that there exist a section \(F \in H^0(X, \Omega^{n-1} \otimes L)\) such that
\[(5.7.4)\]
\[\hat{F} = \pi^* F,\]
and thus we have
\[(5.7.5)\]
\[\hat{F} \wedge \frac{D' \hat{s}}{s} = \pi^* \left( F \wedge \frac{D' \hat{s}}{s} \right)\]
where here \(D'\) is the \((1, 0)\) derivation for sections of \(mK_X\).

In what follows, we place ourselves on a ball centered at the origin of \(\mathbb{C}^n\) and we assume that the section \(s\) corresponds to the function \(z_1^{p_1} \ldots z_r^{p_r}\). As in Lemma 5.8 for every \(i = 1, \ldots, n\) we have
\[(5.7.6)\]
\[\int_{\Omega} |\hat{F} \wedge dz_i|^2 \omega_{\tilde{X}, \pi^*} dV_{\tilde{X}} \leq \int_{\Omega} |\hat{F}|^2 \omega_{\tilde{X}, \pi^*} dV_{\tilde{X}} < +\infty\]
We infer that all the coefficients of \(\hat{F}\) are divisible by
\[(5.7.7)\]
\[\prod_{i=1}^r z_i^{[p_i(1-1/m)].}\]
As discussed in \((5.6.40)\), we have to show that the expressions
\[(5.7.8)\]
\[\int_{(\mathbb{C}^n, 0)} \mu_\varepsilon (\log \prod_{j \neq i} x_j^{p_i - 1}) D'(\theta_1 \pi^* \beta_k) \wedge \overline{F} e^{-\varphi_{L^0}}\]
\[\int_{\tilde{X}} \mu_\varepsilon D'(\theta_1 \pi^* \beta_k) \wedge \overline{F} e^{-\varphi_{L^0}}\]
are converging to zero as \(\varepsilon \to 0\).

The only difference with respect to the section \(5.4.2\) is that we need the Taylor expansion argument for the function \(\mu_\varepsilon\). Moreover, the fact that the local coefficients of \(\hat{F}\) are divisible with the \((5.7.7)\) is crucial. We start with the term corresponding to \(m_i = 0\) for all \(i = 1, \ldots, r\).

By point \((e)\) of Proposition 5.7, we have to analyse the quantities
\[(5.7.9)\]
\[\int_{(\mathbb{C}^n, 0)} \rho_\varepsilon \left( \log \prod_{j \neq i} x_j^{p_i - 1} \right) D'(\theta_1 \pi^* \beta_k) \wedge \overline{F} e^{-\varphi_{L^0}}\]
and
\[(5.7.10)\]
\[\int_{(\mathbb{C}^n, 0)} \rho_\varepsilon \left( \log \prod_{j \neq i} x_j^{p_i - 1} \right) \overline{\partial}(\theta_1 \pi^* \alpha_k) \wedge \overline{F} e^{-\varphi_{L^0}}\]
respectively.

We start with \((5.7.9)\): integration by parts together with the fact that \(\overline{\partial} \hat{F} = 0\) shows that this quantity is equivalent with a sum of expressions of the following type
\[(5.7.11)\]
\[\int_{(\mathbb{C}^n, 0)} \rho_\varepsilon(z) \frac{dz_i}{\prod_{j \neq i} x_j^{p_i - 1}} \sum_{j=1}^r \frac{\Phi_k \pi^* \Lambda_j}{\prod_{i=1}^r z_i^{p_i(1-1/m)}} \wedge \frac{\Phi_k \pi^* \Lambda_j}{\prod_{i=1}^r z_i^{p_i(1-1/m)}} \wedge \overline{F} \wedge dz_i\]
plus a similar integral which is less singular, with \( \frac{dz_j}{z_j} \) replaced by the derivative of \( \varphi(z') \). The notations in (5.7.11) are as follows: \( \Phi_{k,\gamma} \) is a smooth \((n - 1, 0)\) form for each \( k, \gamma \) and the \( \tau_i \) are positive integers. Moreover, we denote by \( \rho_{\varepsilon}(z) := \rho_{\varepsilon}\left(\log \log e^{\varphi(z')}\right) \).

Since the coefficients of \( \hat{F} \) are divisible by (5.7.10), the integrals in (5.7.11) can be rewritten as

\[
\int_{(\mathbb{C}^n, 0)} \rho_{\varepsilon}(z) \Psi(w) \prod_{i=1}^{r} z_i^\varepsilon \prod_{i=1}^{r} \left| z_i - 2\varepsilon \log \prod_{i=1}^{r} z_i^\varepsilon \right|^2 \, d\lambda(z)
\]

where we recall that \( z' := (z_{r+1}, \ldots, z_n) \). We denote by \( \Psi \) a smooth function, and the \( b_i \) in (5.7.12) are positive integers. Now as \( \varepsilon \to 0 \) the expression (5.7.12) clearly tends to zero, as we have already explained.

Next, we are considering (5.7.10). The limit as \( \varepsilon \to 0 \) in (5.7.10) is a bit more subtle to analyse because of the fact that \( \bar{\partial}_s \wedge \hat{F} \) might not be \( L^2 \) with respect to the weight \( e^{-\varphi_L \circ \pi} \). We proceed as follows.

In order to simplify the writing, we assume in what follows that \( r = 2 \) and we have

\[
\theta_i \pi^* \alpha_k := \frac{\Phi}{z_1^\varepsilon z_2^\varepsilon} \, dz
\]

where \( \tau_i \) are positive integers, \( \Phi \) is a smooth function with support is a coordinate set \( \Omega \subset X \) and \( z_1, \ldots, z_n \) are coordinates on \( \Omega \) such that \( s|_\Omega = z_1^{p_1} z_2^{p_2} \) for some \( p_1, p_2 \geq 1 \).

The quantity we have to evaluate is

\[
\lim_{\varepsilon \to 0} \int_{\Omega} \mu_\varepsilon \hat{\partial}(\theta_i \pi^* \alpha_k) \wedge \overline{\xi} e^{-\varphi_L}
\]

where \( (\mu_\varepsilon)_{\varepsilon > 0} \) are our truncation functions, and \( \xi \) is a holomorphic \((n - 1, 0)\) form with values in \( \mathcal{L} = (m_0 - 1) \pi^* K_X \).

Prior to using the \( \hat{\partial} \) for the integration by parts, we make a few adjustments, based on the following observation

\[
D_X' \left( \frac{\Phi}{z_1^{\gamma_1 - 1} z_2^{\gamma_2}} \right) = \frac{\Phi}{z_1^{\gamma_1 - 1} z_2^{\gamma_2}} \, dz - c(\gamma) \frac{\Phi}{z_1^{\gamma_1 - 1} z_2^{\gamma_2}} \, dz
\]

where \( c(\gamma) := \gamma - 1 + \frac{m-1}{m} \) and \( \Phi_{z_1} \) is the partial derivative of \( \Phi \) with respect to \( z_1 \). In the formula (5.7.15) the connection \( D_X' \) on the bundle \((m - 1) \pi^* (K_X)\) is induced by a metric whose local expression of the weight \( \varphi_L \circ \pi \) reads as

\[
\varphi_L \circ \pi = \frac{m-1}{m} \sum_{j=1,2} p_j \log |z_j|^2,
\]

which explains the reason why it holds true.

Assuming that \( c(\tau) \neq 0 \) the equality (5.7.15) can be re-written as

\[
\theta_i \pi^* \alpha_k = \frac{\Phi_{z_1}}{z_1^{\gamma_1 - 1} z_2^{\gamma_2}} \, dz - D_X' \left( \frac{\Phi}{z_1^{\gamma_1 - 1} z_2^{\gamma_2}} \right)
\]

and so the form \( \alpha_k \) can be written as the \( D_X' \) of something plus a less singular form. The corresponding part of (5.7.14) involving \( D_X' \) is

\[
\int_{\Omega} \mu_\varepsilon \hat{\partial}(\delta_k) \wedge \overline{\xi} e^{-\varphi_L \circ \pi}
\]

where \( \delta_k := \frac{\Phi}{z_1^{\gamma_1 - 1} z_2^{\gamma_2}} \). The operators \( \hat{\partial} \) and \( D_X' \) are commuting, and the support of \( \delta_k \) is still contained in \( \Omega \). Therefore (5.7.17) becomes

\[
\int_{\Omega} \hat{\partial}(\mu_\varepsilon) \wedge \hat{\partial}(\delta_k) \wedge \overline{\xi} e^{-\varphi_L \circ \pi}
\]

which we handle exactly as we did for the term (5.7.9): it converges to zero as \( \varepsilon \to 0 \).
The other term we have to deal with in (5.7.14) is

\[
(5.7.19) \quad \int_{\Omega} \mu_{\varepsilon} \tilde{\partial}(\gamma_k) \wedge \xi e^{-\varphi_{\varepsilon,\pi}}
\]

where \(\gamma_k := \frac{\Phi_{z_{1}}}{z_{1}^{\tau_{1}} - \frac{1}{2}} dz_{1}\), and the order of the pole is smaller than before. We can repeat this procedure as long as the successive constants \(c(\tau)\) above are not zero, it does not matter if the new \(\tau\) is positive or negative.

If \(c(\tau) = 0\), then multiplicity \(p_1\) is divisible by \(m\), and that the form \(\gamma_k\) in (5.7.19) is equal to

\[
(5.7.20) \quad \gamma_k = \frac{z_{1}^{q_{1}-1}}{z_{1}^2} \Phi dz
\]

where \(q_1 := p_1 \frac{m - 1}{m}\) is a positive integer. In this case we have

\[
(5.7.21) \quad D_{X} \left( z_{1}^{q_{1}} \log |z_{1}|^2 \frac{\Phi}{z_{2}^2} dz_{1} \right) = z_{1}^{q_{1}-1} \frac{\Phi}{z_{2}^2} dz + z_{1}^{q_{1}} \log |z_{1}|^2 \frac{\Phi}{z_{2}^2} dz
\]

and we then restart this procedure with respect to \(z_2\).

In the end, modulo terms whose limit as \(\varepsilon \to 0\) is zero we have

\[
(5.7.22) \quad \int_{\Omega} \tilde{\partial}(\mu_{\varepsilon}) \wedge \tau_k \wedge \xi e^{-\varphi_{\varepsilon,\pi}}
\]

where \(\tau_k := z_{1}^{q_{1}} z_{2}^{\delta_{1}} \log |z_{1}|^2 \log |z_{2}|^2 \Phi dz\). These terms induced by Stokes formula are now "friendly", because we have

\[
(5.7.23) \quad \int_{\Omega} \mu'_{\varepsilon}(z) \log \frac{1}{|z_{2}|^2} \left| \frac{\xi \wedge dz_{1}}{|z_{1}|^2} \right| \frac{d\lambda(z)}{|z_{1}|} \to 0
\]

as \(\varepsilon \to 0\), and another one corresponding to \(z_2\).

The above discussion applies to the case \(p_1\) and \(p_2\) are both divisible by the degree \(m\) of the pluricanonical form. If for example this is not the case for \(p_2\), then it means that the corresponding constant \(c(\tau)\) is never zero, so successive integration by parts will completely eliminate the pole with respect to \(z_2\). In other words, this case is easier to treat. In conclusion, (5.7.9) and (5.7.10) converges to 0 when \(\varepsilon \to 0\).

The considerations above concern the term in the expansion of the global \(\mu_{\varepsilon}\) in Lemma 5.7 with \(m_1 = \cdots = m_r = 0\). For the remaining ones, we do not integrate by parts but simply use the properties of \(\Phi_{\varepsilon,m}\). Prior to this, we have the following observation.

Let \(\psi\) be a local \((n,1)\)-form defined on some \(\Omega_{1}\). With respect to the aforementioned coordinates we have

\[
(5.7.24) \quad \psi \wedge \hat{F}_{e^{-\varphi_{\varepsilon,\pi}}} = \frac{\Psi(z)}{\prod_{i=1}^{N} z_{1}^{N+i}} \frac{d\lambda(z)}{|z_{1}|^2 - 2\delta_{1}}
\]

where \(\Psi\) is a smooth, local function, \(q_i\) are positive integers and \(\delta_{1}\) are strictly positive rational numbers.

Next, we have to evaluate e.g. expressions like

\[
(5.7.25) \quad \int_{(C^{n},0)} \tau_{m_{1}}^{m_{1}}(z,w) \cdots \tau_{m_{r}}^{m_{r}}(z,w) \Phi_{\varepsilon,m}(z,w) \frac{\Psi(z)}{\prod_{i=1}^{N} z_{1}^{N+i}} \wedge \hat{F}_{e^{-\varphi_{\varepsilon,\pi}}}
\]

which thanks to (5.7.24) can be written as

\[
(5.7.26) \quad \int_{(C^{n},0)} \tau_{m_{1}}^{m_{1}}(z,w) \cdots \tau_{m_{r}}^{m_{r}}(z,w) \Phi_{\varepsilon,m}(z,w) \frac{\Psi(z)}{\prod_{i=1}^{N} z_{1}^{N+i}} \frac{d\lambda(z)}{|z_{1}|^2 - 2\delta_{1}}
\]

for some positive \(N\). We use the Proposition 5.7 and infer that all the relevant expressions are tending to zero. Theorem 5.2 is thus proved.
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