Abstract—We analyse a non-cooperative strategic game among two ride-hailing platforms, each of which is modeled as a two-sided queueing system, where drivers (with a certain patience level) are assumed to arrive according to a Poisson process at a fixed rate, while the arrival process of passengers is split across the two providers based on QoS considerations. We also consider two monopolistic scenarios: (i) each platform has half the market share, and (ii) the platforms merge into a single entity, serving the entire passenger base using their combined driver resources. The key novelty of our formulation is that the total market share is fixed across the platforms. The game thus captures the competition among the platforms over market share, which is modeled using two different Quality of Service (QoS) metrics: (i) probability of driver availability, and (ii) probability that an arriving passenger takes a ride. The objective of the platforms is to maximize the profit generated from matching drivers and passengers.

In each of the above settings, we analyse the equilibria associated with the game. Interestingly, under the second QoS metric, we show that for a certain range of parameters, no Nash equilibrium exists. Instead, we demonstrate a new solution concept called an equilibrium cycle. Our results highlight the interplay between competition, cooperation, passenger-side price sensitivity, and passenger/driver arrival rates.

Index Terms—BCMP queueing network, ride-hailing platforms, two-sided queues, Wardrop equilibrium, Nash equilibrium, cooperation

I. INTRODUCTION

There has been a significant interest in ride-hailing platforms like Uber, Lyft and Ola in recent years. These platforms have queues on two sides (passengers in need of a ride on one side and drivers on the other). Such matching platforms have attained considerable scale. For example, the OLA company that provides a ride-hailing service in India has more than 1.5 million drivers across 250 cities. Passengers also often have the option of using a competing ride-hailing platform. Indeed, the friction associated with switching platforms is negligible from the standpoint of passengers. Thus, platforms must price their rides based on the interplay between price sensitivity on the passenger side, their own goal of revenue maximization, and also the crucial division of market share between competitors.

While there has been considerable interest in the performance evaluation of two-sided matching platforms in recent times (we provide a review of the related literature later in this section), most studies do not consider the impact of competition between platforms. This paper seeks to fill this gap. Specifically, we analyse a non-cooperative game between two ride-hailing platforms, where the platforms must compete for market share. The platforms compete via their pricing policies, the passenger base being both impatient as well as price sensitive. Additionally, for benchmarking, we also consider two monopolistic scenarios: (i) each platform has half the market share, and these market shares are insensitive to the pricing strategies of either platform, and (ii) the platforms merge into a single entity, serving the entire passenger base using their combined driver resources.

Under the assumption of symmetric platforms, a single (geographical) zone of operation, and static (not state-dependent) pricing, we characterize the equilibria that emerge in the above scenarios, by approximating the payoff functions along a certain scaling regime (where driver impatience is diminishing). Comparing the different equilibria that emerge in these settings, our results highlight the impact of competition/cooperation between platforms, passenger price sensitivity, and passenger/driver arrival rates.

Our main contributions are as follows:

• We model each platform as a BCMP network ([15]), admitting a product form stationary distribution. This modeling approach is quite powerful, and allows for state-dependent pricing and driver impatience (it also permits probabilistic routing across multiple zones, though this feature is not used in our analysis here).

• We characterize the optimal pricing strategy of the platform under the monopolistic scenarios described above.

• We model market share bifurcation between the platforms in the form of a Wardrop equilibrium ([6]) as in [1], where the passenger base splits in manner that seeks to equalize a certain QoS metric across the platforms. We consider two reasonable QoS metrics: the stationary probability of driver unavailability, and the stationary probability that an arriving passenger is not served (this latter metric also captures the scenario where the customer declines the ride based on the high price quoted by the platform).

• Under each of the above passenger-side QoS metrics, we characterize the equilibria of the non-cooperative game between the providers. Interestingly, in the case of the second QoS metric, we show that under certain conditions, no Nash equilibrium exists. Instead, we demonstrate an equilibrium cycle, where each platform
has the incentive to set prices within a certain interval, though at each price (action) pair, at least one platform has the incentive to deviate to a different price within the same interval.

- Finally, we compare the equilibria under the different settings to highlight the impact of competition and cooperation between the platforms.

The remainder of this paper is organised as follows. After a brief survey of the related literature below, we describe our system model in Section II and analyse the monopolistic setting in Section III. Competition under the driver availability QoS metric is analysed in Section IV while the service availability metric (depends on the driver availability as well as the price quoted by the platform) is considered in Section V. Finally, we report numerical experiments in Section VI and conclude in Section VII. Throughout the paper, references to the appendix point to the appendix of [1].

Literature Review

Two-sided queues have been considered in [2], [7]-[10], where both customers and servers arrive over time, and wait to be matched. Under a suitable fluid scaled limit of this system, [8] showed that static pricing is sufficient to optimize the objective function, and dynamic pricing only helps in improving the robustness of the system. The BCMP model in [2] is almost similar to our model except the fact that we consider an open system (with a varying number of drivers and passengers) with the possibility of drivers abandoning (and re-joining) the system. [7] considers a two-sided queueing system (with multiple zones and customer types) under joint pricing and matching controls. However, the possibility of the drivers re-joining the system has not been taken into consideration.

An extension of this two-sided queue model with multiple types of servers and customers has also been considered in [9], [11]-[13], where an additional bipartite matching decision has to be made. In [11], limiting results of matching rates between certain customer and server types with FCFS scheduling have been analyzed. Without pricing, the optimal matching to minimize the queueing cost has been analyzed in [13] under a suitably scaled large system limit. Most of these analyses are presented using a large system scaling regime.

Queueing network models typically assume that the arrival process is exogenous. Contrasting system models have been considered in [3]-[4] where the passenger arrival stream is split according to the well-known Wardrop Equilibrium, WE (6). The authors in [5] consider that the users choose the queue that will minimize their own expected delay. In [4], the single-sided queueing system with multiple service providers has been considered; the objective of each of these providers is to maximize their market share. Further, the possibility of cooperation has been considered.

In contrast with the prior literature on two-sided queues, the important aspect of competition between platforms has not, to be best of our knowledge, been explored before.

II. Model and Preliminaries

In this section, we describe our model for the interaction between two competing ride-hailing platforms, and state some preliminary results.

A. Passenger arrivals

We consider a system with a set $\mathcal{N} = \{1, 2\}$ of independent ride-hailing platforms, each of which is modeled as a single-zone two-sided queueing system that matches drivers and passengers. Passengers arrive into the system as per a Poisson process with a rate $\Lambda$. This aggregate arrival process gets split between the two platforms based on QoS considerations (details in Subsection II-D below), such that the passenger arrival process seen by platform $i$ is a Poisson process with rate $\lambda_i$, where $\sum_{i=1}^{\mathcal{N}} \lambda_i = \Lambda$.

When a passenger arrives into platform $i$, if there are no waiting drivers available, the passenger is immediately lost. On the other hand, if there are one or more waiting drivers, the passenger is quoted a price $\phi_i(n_i) \in [0, \phi_h]$, where $n_i$ denotes the number of waiting drivers on platform $i$, and $\phi_h$ denotes the maximum price the platform can charge. The customer accepts this price (and immediately begins her ride) with probability $f(\phi_i(n_i))$; with probability $1 - f(\phi_i(n_i))$, the passenger rejects the offer and leaves the system (without taking the ride). Note that the function $f$ captures the price sensitivity of the passenger base. We make the following assumptions on the function $f$.

A.1 $f(\cdot)$ is a strictly concave, strictly decreasing and differentiable function.

A.2 $0 < f(\phi) \leq 1$ for all $\phi \in [0, \phi_h]$ and $f(0) = 1$.

The following extension of the inverse of $f$ will be used in the statements of our results:

\[
    f^{-1}(x) := \begin{cases} 
        \phi_h & \text{for } 0 \leq x < f(\phi_h), \\
        \phi & \text{s.t. } f(\phi) = x & \text{for } f(\phi_h) \leq x \leq 1, \\
        0 & \text{for } x > 1.
    \end{cases}
\]

B. Driver behavior

Each platform has a pool of dedicated drivers. Dedicated drivers of platform $i \in \mathcal{N}$ arrive into the system according to a Poisson process of rate $\eta_i$. The drivers wait in an FCFS queue to serve arriving passengers. Recall that the number of waiting drivers in this queue on platform $i$ is denoted by $n_i$.

When an arriving passenger is matched with the head of the line driver, a ride commences and is assumed to have a duration that is exponentially distributed with rate $\nu_i$. At the end of this ride, the driver rejoins the queue of waiting drivers (and therefore becomes available for another ride) with probability $p_i$; with probability $1 - p_i$, the driver leaves the system.

Additionally, we also model driver abandonment from the waiting queue. Specifically, each waiting driver independently abandons after an exponentially distributed duration of rate $\beta_i$. This abandonment might capture a local (off platform) ride taken up by the driver, or simply a break triggered by impatience. The duration of this ride/break post abandonment is
also assumed to be exponentially distributed with rate $v_i$ (if drivers accept a ride, off-platform, it is reasonable to assume that the distribution of the duration of this ride is the same as that of rides matched on the platform, given that these rides are in the same zone); at the end of this duration, the driver rejoins the queue with probability $p_i$ and leaves the system altogether with probability $1 - p_i$. Figure 1 presents a pictorial depiction of our system model.

C. BCMP modeling of each platform

Under the aforementioned model, each platform can be described via a continuous time (Markovian) BCMP network [15]. Formally, the state of platform $i$ at time $t$ is given by the tuple $Z_{i,t} = (N_{i,w}, R_{i,b})$, where $N_{i,w}$ is the number of waiting drivers, and $R_{i,b}$ is the number of drivers that are in the system but unavailable to be matched with passengers (because of being in a ride, or on a break). Realizations of the state of platform $i$ are represented as $(n_i, r_i)$, and state space corresponding to platform $i$ is given by $S_i = \{(n_i, r_i) : n_i, r_i \in \mathbb{Z}_+\}$.

The BCMP model for each platform consists of two ‘service stations’: (i) service station 1 (SS1) is the queue of waiting drivers (the occupancy of this queue is the first dimension of the state), and (ii) service station 2 (SS2) is the ‘queue’ of drivers on a ride/break (the occupancy of this queue is the second dimension of the state). SS1 is modeled as a single server queue having a state-dependent service rate. Specifically, the (state dependent) departure rate from SS1 is $\lambda_i f(\phi_i(n_i)) + n_i \beta_i$. On the other hand, SS2 is modeled as an infinite server queue, having exponential service times of rate $v_i$. SS1 sees exogenous Poisson arrivals at rate $\eta_i$, and departures from SS1 become arrivals into SS2. Finally, departures from SS2 exit the system with probability $1 - p_i$, and join SS1 with probability $p_i$; see Figure 1.

Next, we describe the steady state distribution corresponding to the above BCMP network (associated with platform $i$). Define $e_i := \frac{p_i}{1 - p_i}$, which is easily seen to be the effective driver arrival rate seen by the service stations SS1 and SS2. The following lemma follows from [15] (see Sections 3.2 and 5 therein).

**Lemma 1.** The steady state probability of state $s = (n_i, r_i) \in S_i$ is given by:

$$
\pi_i(s) = C_i \left[ \prod_{n=1}^{\nu} \left( \frac{e_i^n}{r_i^n} \right) \right] \left[ \left( \frac{e_i}{v_i} \right)^r \left( \frac{1}{r_i!} \right) \right].
$$

$$
C_i^{-1} = \sum_{n=0}^{\infty} \left[ \prod_{n=1}^{\nu} \left( \frac{e_i^n}{v_i^n} \right) \right] \exp \left( \frac{e_i}{v_i} \right).
$$

Here, $C_i$ is the normalizing constant (with the convention that $\Pi_0^k (-) = 1$ when $a > k$).

D. Passenger split across platforms: Wardrop Equilibrium

We model the split of the aggregate passenger arrival rate $\Lambda$ into the system across the two platforms (recall that platform $i$ sees passenger arrivals as per a Poisson process with rate $\lambda_i$) as a Wardrop equilibrium (WE) based on the Quality of Service (QoS). In particular, we consider two different QoS metrics in our analysis. However, before describing these formally, we first define the Wardrop split of the passenger arrival rate in terms of a generic QoS metric $Q$.

Let $Q_i(x)$ denote the QoS of platform $i$ when the passengers arrive at rate $x$. Note that $Q_i$ will, in general, also depend on the pricing policy $\phi_i := (\phi_i(n), n \in \mathbb{N})$ employed by platform $i$, though this dependence is suppressed for simplicity. The Wardrop split $(\lambda_1, \lambda_2)$ under the price policy $\phi = (\phi_1, \phi_2)$ is then defined as:

$$
\lambda_1(\phi) \in \arg\min_{\lambda \in [0,\Lambda]} (Q_1(\lambda) - Q_2(\Lambda - \lambda))^2, \quad \lambda_2(\phi) = \Lambda - \lambda_1(\phi).
$$

We address the uniqueness of the Wardrop split in Lemma 2 below. Having defined the Wardrop split in generic terms, we now define the passenger QoS metrics we consider. Note that passengers can leave the system without taking a ride either because of driver unavailability, or because of the price quoted was too high.

Let $D_i$ be the stationary probability of zero waiting drivers on platform $i$. From Lemma 1

$$
D_i = \sum_{r_i=0}^{\infty} \pi_i((0, r_i)) = \left( \sum_{n=0}^{\nu} \left[ \prod_{n=1}^{\nu} \left( \frac{e_i^n}{v_i^n} \right) \right] \right)^{-1}.
$$

By PASTA, it follows that $D_i$ is also the long-run fraction of passengers who find no available driver upon arrival into platform $i$ (and thus leave the system).

Let $B_i$ be the long-run fraction of passengers, who leave platform $i$ without taking a ride (due to driver unavailability or a high price). Using PASTA again,

$$
B_i = D_i + \sum_{n_i=1}^{\infty} \sum_{r_i=0}^{\infty} (1 - f(\phi_i(n_i))) \pi_i(n_i, r_i).
$$

We use $D_i$ and $B_i$ as our QoS metrics; note that these are both functions of $\lambda_i$. Our analysis for the case $Q_i = D_i$ is presented in Section IV while the case $Q_i = B_i$ is addressed in Section V. Next, we establish existence and uniqueness of the Wardrop equilibrium under these metrics (proof in Appendix A); we first prove existence and uniqueness under a certain generic assumption (Assumption A.0 below), and then show that our metrics of interest satisfy the assumption.
A.0 The QoS function $Q_i$ for each $i$ is continuous and strictly monotone in $\lambda_i$; both the functions are either increasing or decreasing.

**Lemma 2.** Consider any QoS metric satisfying Assumption A.0. Given any price policy $\phi = (\phi_1, \phi_2)$, there exists a unique Wardrop Equilibrium $(\lambda_1(\phi), \lambda_2(\phi))$.

Finally, each of the QoS metrics $\mathcal{D}_i$ and $\mathcal{B}_i$ satisfies A.0, so long as $\beta_i > 0$.

Given the existence and uniqueness of WE, we next define the platform utility functions. This defines a non-cooperative game between the two platforms.

**E. Platform Utilities**

We treat the action of each platform $i$ to be its pricing policy $\phi_i = (\phi_i(n), n \in \mathbb{N})$. We define the utility of platform $i$ as the (almost sure) rate at which it derives revenue from matching drivers with passengers, denoted by $\mathcal{M}_i$. Note that $\mathcal{M}_i$ depends on the price (action) profile $\phi = (\phi_1, \phi_2)$ and other parameters; the pricing policy of each platform influences the other’s utility, via the Wardrop split that determines the market shares of both platforms.

In the following, we derive the matching revenue rate (often referred to simply as the matching revenue, or MR) of each platform in terms of the Wardrop split $(\lambda_1, \lambda_2)$ (proof in Appendix A).

**Lemma 3.** The matching revenue rate of platform $i$ is given by

$$\mathcal{M}_i = \sum_{s_i; n_i \neq 0} \lambda_i f(\phi_i(n_i)) \phi_i(n_i) \pi_i(s_i).$$

We conclude this section with a result characterizing the limit of the MR of any platform as $\beta \rightarrow 0$ (proof in Appendix A).

**Lemma 4.** Consider static price profile $(\phi_1, \phi_2)$. Suppose that the passenger arrival rates $\lambda_i(\beta) \rightarrow \lambda_i$ as $\beta \rightarrow 0$. Then the matching revenue $\mathcal{M}_i = \mathcal{M}_i(\phi_i; \lambda_i(\beta), \beta)$, given by Lemma 3, converges to $\mathcal{M}_i$ as $\beta \rightarrow 0$, where

$$\mathcal{M}_i(\phi) = \begin{cases} e \phi_i & \text{if } \beta_i f(\phi_i) \phi_i < 1, \\ \lambda_i f(\phi_i) \phi_i & \text{else.} \end{cases}$$

Along similar lines, we have $\mathcal{D}_i(\phi; \lambda_i(\beta), \beta) \rightarrow \mathcal{D}_i(\phi; \lambda_i)$, where

$$\mathcal{D}_i(\phi; \lambda_i) = \left(1 - \frac{e}{\lambda_i f(\phi_i)}\right)^+.$$  

By virtue of the above result, we derive approximate equilibria of the actual system when $\beta$ is small, by analysing those of a ‘limit system,’ which is obtained by letting $\beta \rightarrow 0$. Throughout, we refer the system (i.e., functions) obtained from the limit $\beta \rightarrow 0$ as the limit system. In fact, we also provide a formal justification of our conclusions from this limit system by establishing convergence of relevant optimizers and zeros, as and when required.

### III. MONOPOLY

In this section, we consider the scenario where a single platform operates alone (with no competitors), with an incoming passenger arrival rate half that of the original system, i.e., $\Lambda/2$. The remaining details (like passenger price sensitivity, driver behavior, etc.) are as described in Section II. This is equivalent to the special case of our model where the passenger arrival rate is split equally between the two platforms, independent of the platforms’ pricing policies. In this case, the non-cooperative game reduces to separate (and identical) optimization problems for the platforms to maximize their revenue rate; this optimization is analysed in this section. Note that the results in this section serve as the baseline against which we can evaluate the impact of the strategic interaction between the two competing platforms.

The platform offers a price $\phi$ to the incoming passenger, if there are waiting drivers and then the passenger accepts the price with probability $f(\phi)$. The matching revenue derived by the platform is given by Lemma 3 with $\lambda = \Lambda/2$ and the static policy $\phi(n) = \phi$ for all $n$ (see (4)):

$$\mathcal{M}(\phi; \beta) = \frac{\Lambda}{2} f(\phi) \phi(1 - \mathcal{D}(\phi; \beta)).$$

1Our results can be generalized to the case where the $e_i$ are distinct.

2Throughout our notations, we emphasize functional dependence on parameters of interest only as and when required.
This can be approximated using Lemma 3 when $\beta$ is close to zero. Formally, we consider the approximate platform utility corresponding to the limit system:

$$\tilde{\mathcal{M}}(\phi) = \begin{cases} e\phi & \text{if } \left(\frac{e}{\mathcal{F}(\phi)}\right) < 1, \\ \frac{1}{2}f(\phi) & \text{else}. \end{cases} \tag{8}$$

The optimal pricing strategy for the monopolistic platform, seeks to maximize $\tilde{\mathcal{M}}(\phi)$, is characterized as follows (proof in Appendix A).

**Theorem 1.** The optimizer of the $\tilde{\mathcal{M}}$ in the limit system (given by (8)) is,

$$\phi^* = \max\{\phi^*, \bar{\phi}\}, \text{ where } \phi := f^{-1}\left(\frac{2e}{\Lambda}\right), \text{ and } \phi^* = \arg \max_{\phi} \mathcal{P}(\phi), \text{ with, } \mathcal{P}(\phi) = f(\phi)\phi. \tag{9}$$

Moreover, for any sequence of optimal prices corresponding to a given sequence $\beta_n \to 0$, there exists a sub-sequence that converges to the unique optimal price of the limit system.

Theorem 1 characterizes the optimal policy (price) of the monopolistic platform in the limit system, and also justifies this approximation when $\beta$ is small. Importantly, the optimal price depends only on the price sensitivity function $f$ and the ratio $e/\Lambda$. Specifically, when $e/\Lambda \geq f(\phi^*)/2$ (i.e., when the passenger arrival rate is ‘small’ relative to the driver arrival rate), the optimal price equals $\phi^*$. However, when $e/\Lambda < f(\phi^*)/2$ (i.e., when the passenger arrival rate is ‘large’ relative to the driver arrival rate), the optimal price is higher and is given by $f^{-1}\left(\frac{2e}{\Lambda}\right)$; see Figures 4 and 5.

In the following sections, we will find it instructive to compare the equilibrium pricing policy of each platform to the monopolistic optimal policy to shed light on the impact of passenger-side churn on platforms’ pricing policies.

**IV. DUOPOLY DRIVEN BY DRIVER AVAILABILITY**

We now return to the platform duopoly model introduced in Section II. Specifically, in this section, we assume that the passengers are primarily sensitive to the non-availability of drivers, i.e., to the probabilities $\{\mathcal{D}_i\}$. Accordingly, the QoS metric $Q_i$ is taken to be $\mathcal{D}_i$, in order to define the Wardrop split of passenger arrival rate across the two platforms.

This results in a non-cooperative game between the two platforms. We now seek to characterize the Nash equilibria of this game in the limit system. We begin with a characterization of the WE (proof in Appendix A).

**Lemma 5.** For a given price vector $\phi$, $\beta > 0$ and passenger response function $f$, the WE under the QoS metric $\mathcal{D}_i$ is given by,

$$(\hat{\lambda}_i(\phi), \hat{\lambda}_2(\phi)) = \left(\frac{\Lambda f(\phi_2)}{f(\phi_1) + f(\phi_2)}, \frac{\Lambda f(\phi_1)}{f(\phi_1) + f(\phi_2)}\right).$$

The above lemma provides the unique WE for any given price-vector/strategy profile $\phi$, and for any $\beta > 0$ (i.e., we have not needed to appeal to the limit system). Indeed, note that the Wardrop split is insensitive to the value of $\beta$ under QoS metric $\mathcal{D}_i$. Now, the utility of player $i$, i.e., its matching revenue $\mathcal{M}_i(\phi) = \mathcal{M}_i(\phi; \hat{\lambda}_i(\phi))$, can be approximated by $\tilde{\mathcal{M}}(\phi)$ in the limit system; this approximation is obtained after replacing $\lambda_i$ by $\hat{\lambda}_i(\phi)$ of Lemma 5. We now derive the NE of this ‘limit game’ (proof in Appendix A).

**Theorem 2.** Define $d(\phi) := 2f(\phi) + f'(\phi)$. Then

i) $(\phi_1, \phi_2)$ is a Nash equilibrium (NE) if and only if $d(\phi) \leq 6$;

ii) If $d(\phi) > 0$ and $d(\phi_1) \leq 6$, then $(\phi_0^1, \phi_0^2)$ is a NE, where $d(\phi_0^i) = 0$;

iii) If not the first two cases, then $(\phi_0^1, \phi_0^2)$ is a NE.

Theorem 2 can be interpreted as follows. Given our assumptions on the price sensitivity function $f$, it is easy to see that $d$ is strictly decreasing. Case (iii) above arises when $d(\phi_0^1) > 0$; this means passengers are (relatively) price-insensitive. In this case, the NE corresponds to both platforms charging the maximum permissible price. On the other hand, Cases (i) and (ii) arise when $d(\phi_0^1) \leq 0$, i.e., when passengers are (relatively) price-sensitive. In particular, Case (i) arises when $e/\Lambda$ is small (i.e., the passenger arrival rate is large relative to the rate at which drivers are available); in this case, the symmetric NE coincides with the monopoly price. Indeed, when passengers are ‘abundant,’ the platforms no longer need to compete with one another for market share. On the other hand, Case (ii) arises when $e/\Lambda$ is large (i.e., the passenger arrival rate is small relative to the rate at which drivers are available); in this case, the platforms do compete for market share, and the equilibrium price differs from the monopoly pricing strategy. Overall, the equilibrium price decreases as a function of $e/\Lambda$; in other words, when the rate at which passengers enter the system decreases, the equilibrium price quoted by the platforms also decreases. We compare the values of the equilibrium prices here to the monopolistic optimal price (characterized in Section II in Section VI).

Finally, a (partial) justification of the equilibria under the limit system can be provided as follows. The best response (BR) of player $i$ against any given $\phi^i$ of the other player, converges to the corresponding BR in limit system in the same sense (along a sub-sequence) as in Theorem 1; the proof follows using exactly the same argument as that in Theorem 1.

**V. DUOPOLY DRIVEN BY PRICE & DRIVER AVAILABILITY**

Contrary to the previous section, we now consider the scenario where passengers are sensitive to the overall likelihood of finding a ride on arrival; note that the ride may not materialise either due to driver unavailability, or due to the price being prohibitive. Formally, we choose the QoS metric to be the combined blocking probability $\{\mathcal{B}_i\}$. Thus Wardrop split ensures that the difference between the combined blocking probabilities $\{\mathcal{B}_i\}$ of the two platforms is minimized.

The combined blocking probability of platform $i$ when it uses static price policy $\phi$ and when passengers arrive at rate $\hat{\lambda}_i$ is given by (see (3)),

$$\mathcal{B}_i(\phi_1) = \mathcal{B}_i + (1 - f(\phi_i))(1 - \mathcal{D}_i) = \mathcal{B}_i f(\phi_1) + (1 - f(\phi_i)), \tag{10}$$
which can be approximated by considering the limit system as follows (using Lemma 4),

$$
\hat{R}_i(\phi_i; \lambda_i) = \begin{cases} 
1 - \frac{\phi_i}{\lambda_i} & \text{if } \left( \frac{\phi_i}{\lambda_i(\phi_i)} \right) < 1, \\
1 - f(\phi_i) & \text{else.}
\end{cases}
$$

(11)

As before, the matching revenue of platform $i$ after WE split, when the two platforms operate with static price policies $\phi$, equals $\mathcal{M}(\phi) = \mathcal{M}_i(\phi, \lambda_i(\phi))$, where $\lambda_i(\phi)$ is now defined using QoS $\mathcal{B}$. As in the previous sections, we derive approximate platform utilities when $\beta \to 0$. However, we require a second level of approximation in this case; we first approximate the WE and then the corresponding MR to obtain the utility functions in the limit system (proof in Appendix A).

**Theorem 3.** Fix $\phi_i$. Let the (unique) WE be represented by $W(\beta) = \lambda_i(\phi; \beta)$ for any $\beta > 0$. For $\beta = 0$, define $W(0) = \hat{\lambda}_i(\phi)$ as per Table 2. Then the mapping $\beta \mapsto W(\beta)$ is continuous on the interval $[0, \infty)$. Further, the same is the case with the matching revenue function $\beta \mapsto \mathcal{M}(\phi; \beta)$, when we set $\mathcal{M}_i(\phi; 0) := \mathcal{M}_i(\phi)$, where $\mathcal{M}_i$ is defined in Table 1.

| $\phi_i < \phi$ | $\phi_i = [\phi, \phi]$ | $\phi_i > \phi$ |
|-----------------|-----------------|-----------------|
| $\lambda_i(\phi)$ | $\frac{1}{\lambda_i}$ | $\lambda_i(\phi)$ |
| $\mathcal{M}_i(\phi)$ | $\phi_i$ | $\min(\lambda_i(\phi), e\phi_i)$ |
| $\mathcal{M}_i(\phi)$ | $\phi_i$ | $\phi_i$ |

**TABLE 1**

LIMIT SYSTEM UNDER QoS $\mathcal{B}$, $\phi = f^{-1}(\frac{\phi}{\lambda_i})$, $\hat{\phi} = f^{-1}(\frac{\phi}{\lambda_i})$ AND $m(\phi) = (\Lambda f(\phi) - e)\phi$.

By virtue of the above theorem, the revenue rate (MR) of the actual system can be approximated by that of the limit system as defined in Table 1 when $\beta$ is close to zero. Accordingly, in the remainder of this section, we analyse the equilibria in the limit system. We begin with deriving the conditions under which classical Nash Equilibrium (NE) exists (proof in Appendix A).

**Theorem 4.** Assume $e < \Lambda$. Then $(\phi, \phi)$ is a NE if $\phi_i^* \leq \phi \leq \phi_0$, where $\phi_i^*$ is the unique maximizer of $m(\phi) = (\Lambda f(\phi) - e)\phi$ over $[0, \phi_0]$.

The conditions of Theorem 4 are satisfied when $e/\Lambda$ is small, i.e., passengers are ‘abundant’ relative to drivers. In this case, a unique (pure) Nash equilibrium exists, and this symmetric equilibrium price coincides with the monopoly pricing strategy; a similar observation was made in Section IV under a different passenger QoS metric. Interestingly however, when $e/\Lambda$ is large, there are significant contrasts with the results in Section IV. The most important is that a Nash equilibrium does not exist (this is because of discontinuities in the platform payoff functions, as shown in Figure 2). Instead, for a range of values of $e/\Lambda$, we demonstrate an *equilibrium cycle*, which is defined next.

**Definition 1. Equilibrium Cycle:** A closed interval $[a, b]$ is called an equilibrium cycle if:

$(i)$ for any $i \in \mathcal{N}$, for $\phi_i \in [a, b]$, there exists $\phi_i \in [a, b]$ such that,

$$\mathcal{M}_i(\phi_i; \phi, \phi_0) > \mathcal{M}_i(\phi_i, \phi_i) \forall \phi_i \in [0, \phi_0] \setminus [a, b]$$

$(ii)$ for any price vector $\phi \in [a, b]^2$, there exists $i \in \mathcal{N}$ and $\phi_i^* \in [a, b]$ such that $\mathcal{M}_i(\phi_i^*, \phi, \phi_0) > \mathcal{M}_i(\phi_i^*)$ and

$$\mathcal{M}_i(\phi_i^*, \phi, \phi_0) > \mathcal{M}_i(\phi_i^*, \phi_i) \forall \phi_i \in [0, \phi_0] \setminus [a, b]$$

$(iii)$ No subset of $[a, b]$ satisfies the above two conditions.

The first condition above establishes the ‘stability’ of the interval $[a, b]$; if any player has action in this interval, the other player is also incentivized to play an action in the same interval (this choice dominating any action outside the interval). The second condition establishes the ‘cyclicity’ of the same interval; if both players play any actions within the interval $[a, b]$, at least one player has an incentive to deviate to a different action within the same interval (this deviation also dominating any action outside the interval). The last condition ensures that no subset of an equilibrium cycle is an equilibrium cycle. Intuitively, an equilibrium cycle can be interpreted (in dynamic terms) as an interval in which platforms’ actions oscillate indefinitely.

We now show that our system admits an equilibrium cycle $[\phi_L^*, \phi_U^*]$ defined as follows (recall that $\phi_L^*$ is defined in Theorem 4):

$$[\phi_L^*, \phi_U^*] = \left[ \left( \frac{\Lambda f(\phi_i^*)}{e} - 1 \right) \phi_i^*, \phi_i^* \right] = \left[ \frac{m(\phi_i^*)}{e}, \phi_i^* \right].$$

**Theorem 5.** Assume $e < \Lambda$. If $\phi \lneq \phi_0 \leq \min\{\phi_i, \hat{\phi}\}$, then $[\phi_L^*, \phi_U^*]$ is an equilibrium cycle.

The proof of Theorem 5 is in Appendix A. The conditions of Theorem 5 are satisfied by a range of values of $e/\Lambda$ that exceed those satisfying the conditions of Theorem 4. In this range, there is no (pure or mixed) Nash equilibrium between the platforms. Instead, competition drives the platforms’ actions to ‘oscillate’ over the equilibrium cycle. We provide numerical illustrations of the equilibrium cycle in Section VI.

For the remaining range of parameters (values of $e/\Lambda$ even larger than those that satisfy the conditions of Theorem 5),
there is neither a NE nor an equilibrium cycle. The market share is small, and the competition is high, so the platforms are driven to offer prices close to zero. However, it can be shown that \((0,0)\) is not a NE. Instead, we show that \((\delta, \delta)\) is an \(\epsilon\)-NE for suitably small values of \(\delta\) and \(\epsilon\), suggesting that players choose a price ‘close to zero’ in such a competitive environment (proof in Appendix A).

**Theorem 6.** Consider the case with \(\epsilon \geq \Lambda\). For any \(\epsilon > 0\), choose \(0 < \delta \leq \phi_0\) such that \(\sup_{\phi \leq 2} A f(\phi) \phi < \epsilon\). Then \((\delta, \delta)\) is an \(\epsilon\)-Nash equilibrium. ■

Jointly, Theorems 4-6 characterize the equilibrium behavior in the limit system. We compare these equilibria with those derived in Section IV as well as the monopolistic optimal pricing in Section VI in Section VI.

**VI. COOPERATION AND COMPARISONS**

We now study the scenario in which the platforms seek to operate together. They combine their individual driver databases and attempt to serve the passengers together. The analysis of this scenario is exactly similar to that corresponding to the monopoly scenario (see Section III); by Theorem 1, the optimal price policy for limit combined system equals \(\phi^* = \max\{\phi_1^*, \phi_2^*, \phi\}\).

**Cooperation and monopoly:** We immediately have an interesting observation about the comparison between monopoly (the platforms operate independently without interfering with each other) and cooperation (they operate together): the ratio \(\epsilon/\Lambda\) remains the same and thus the optimal price (and hence the optimal matching revenue per platform) remains the same for both the systems. This is because the value \(\phi^*_p\) depends solely upon the passenger response function \(f\), while \(\phi^*\) depends only upon the ratio \(\epsilon/\Lambda\) and \(f\).

The above is the case with small \(\beta\). In other words, when drivers are willing to wait for sufficiently long periods (as is usually the case with many practical scenarios), the usual economies of scale that we observe in queuing systems resulting from two or more systems operating together is not seen with the double-sided queues. However, the same is not the case when \(\beta\) is sufficiently large. We consider an example in Figure 3, where we numerically compute the optimal values of the MR using Lemma 3 and 4 for the cases with \(\beta > 0\). We now observe that cooperation makes a difference when \(\beta\) is large. The platforms derive better optimal prices and matching revenue when they operate together. However, the difference in optimal price across the two configurations is only about 4.6% even for \(\beta = 1\). On the other hand, the difference in matching revenues is more; it is around 17% for \(\beta = 0.5\). Thus, one can conclude that cooperation does not significantly improve the revenue of the two platforms unless the drivers are highly impatient.

**Cooperation and competition:** Next, we consider competition between the two platforms. We evaluate and compare the equilibrium pricing and the corresponding matching revenues across the different passenger QoS models considered in Sections IV and V and also the monopoly (and the cooperation) setting. This comparison is performed for different choices of the price sensitivity function \(f\).

We begin with linear response function \(f(\phi) = 1 - \alpha \phi\) for some \(\alpha < 1/\phi_0\); this satisfies A.1-2. After simple algebra, one can compute the following quantities of Theorems 1-6 that define various equilibria as below:

\[
\phi^*_p = \frac{1}{\sqrt{a}}, \quad \phi = \left(\frac{a}{1 - \frac{\epsilon}{\Lambda}}\right), \quad \phi^*_p = \left(\frac{1}{a} - \frac{\epsilon}{\Lambda}\right); \\
\phi^*_a = \frac{a}{\sqrt{a}}, \quad \phi^*_u = \left(\frac{1}{\sqrt{a}}\right) \left(\frac{\Lambda}{\epsilon} - 1\right) \sqrt{1 - \frac{\epsilon}{\Lambda}}.
\]

We also consider a non-linear response function, \(f(\phi) = 1 - (a \phi)^2\) for some \(a < 1/\phi_0\) that satisfies A.1-2. For this function, the respective quantities are:

\[
\phi^*_p = \frac{1}{\sqrt{a}}, \quad \phi = \left(\frac{a}{1 - \frac{\epsilon}{\Lambda}}\right), \quad \phi^*_p = \left(\frac{1}{a} - \frac{\epsilon}{\Lambda}\right), \\
\phi^*_a = \frac{a}{\sqrt{a}}, \quad \phi^*_u = \left(\frac{1}{\sqrt{a}}\right) \left(\frac{\Lambda}{\epsilon} - 1\right) \sqrt{1 - \frac{\epsilon}{\Lambda}}.
\]

The immediate observation, which is strikingly visible from the above two sets of expressions (which is also seen from Theorems 1-6) is that the equilibrium prices, as well as the normalized MR (MR at \(\Lambda = 1\), depend only upon the arrival-ratio \(\epsilon/\Lambda\) and response function \(f\) and nothing else (the ratio is like the well-known load-factor of the queuing systems).

To derive further insights, we plot two case studies, one with a linear response function in Fig. 4 and the one with a square response function in Fig. 5. When the arrivals-ratio \(\epsilon/\Lambda\) is small, the system is governed by the effects of the overwhelming number of passengers, and all the equilibria are highly sensitive to the above ratio. We observe this phenomenon for both the response functions (left sub-figures of Fig. 4 and Fig. 5). When the arrivals-ratio is large, the equilibrium price for monopoly (and hence cooperation) as well as the duopoly-QoS-\(\beta\) is insensitive to further increase in arrivals-ratio. However, for the duopoly driven by \(\beta\), we observe the existence of an equilibrium cycle (EC). At each such value of arrivals-ratio \(\epsilon/\Lambda\), we have a vertical line representing the EC.

We also capture the equilibrium MR for monopoly and duopoly (driven by \(\beta\)) scenarios in the right sub-figures. For the case with EC, we plot the MR of a platform when it prices at one end point of the EC, while the opponent offers at the other end point (the end points would not matter as seen from proof of Theorem 5).

We have several observations: a) interestingly, the performance of the platforms is unaffected by the existence of the competition for the cases with abundant passengers (or with \(\epsilon/\Lambda\) small); b) the price and the normalized MR of monopoly (hence cooperation) equals that corresponding to either of the two duopoly cases (up to \(\epsilon/\Lambda \leq 0.3\) for the linear case and up to \(\epsilon/\Lambda \leq 0.45\) for the square \(f\), in right sub-figures); c) however the observation is very different when there is a scarcity of passengers; cooperation significantly improves the MR of both the platforms for larger values of \(\epsilon/\Lambda\).

In all, the price of anarchy is large when the system operates with large arrival ratios. At the same time, the same is negligible when it operates at medium or low arrival rates.
is in line with the observation that many results in queuing systems depend just upon load factor; b) the same dependency also holds for the normalized matching revenue (per unit passenger rate) derived at various equilibria; c) when the drivers are willing to wait for a reasonably long time, the platforms will not benefit from cooperation unless there is a competition over the market share; d) when the passengers are sensitive to the offered prices as well as the availability of drivers, there are scenarios when a Nash equilibrium does not exist. In this case, we defined a new concept called an equilibrium cycle and established the existence of the same.

This paper motivates future work in several directions: multiple zones, competition in the presence of dynamic pricing, drivers and/or passengers who opportunistically hop between platforms, etc. We believe the BCMP-style modeling approach we have adopted is amenable to these extensions.

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