Holographic Complexity Growth Rate in a dual FLRW Universe

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Abstract

In this paper, under the large $R$ limit, using the complexity-volume duality, we investigate the holographic complexity growth rate of a field state defined on the universe located at an asymptotical AdS boundary in Gauss-Bonnet gravity and massive gravity, respectively. For the Gauss-Bonnet gravity case, its growth behavior of the state mainly presents three kinds of contributions: the first one viewed as a coupling correlation term is finite and has to do with a conserved charge, and the second one is from the spatial volume of the universe, in addition, the third one relates to the curvature of the horizon in the Gauss-Bonnet-AdS black hole, whose behavior is very like the standard Einstein case. For massive gravity case, except the leading divergent term still obeying the growth rate of the spatial volume of the Universe without the graviton mass effect, other terms present some interesting novel behaviors: beside the conserved charge $E$, the graviton mass term also provides its contribution to the finite term; and the third divergent term is determined by the spatial curvature of its horizon $k$ and the graviton mass effect; furthermore, the graviton mass effect can be completely responsible for the second divergent term as a new additional term saturating an area law.

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I. INTRODUCTION

Anti-de Sitter/Conformal field theory (AdS/CFT) correspondence [1–4] is currently established as a valuable prescription to approach the understanding of the quantum gravity. Remarkably, the fascinating idea is that, by mapping physical degrees a strong coupled quantum system to dual gravity theory in a higher dimensional bulk space, a difficult problem is usually transformed into a tractable one. There has been extensively investigated in the modern theoretical physics over the last decades years. Specially, when it comes to the quantum information theory in the context of AdS/CFT correspondence in recent years, one famous topic in this direction is the holographic entanglement entropy proposed by Ryu and Takayanagi[5], which asserts that its quantum entanglement entropy of a conformal field theory in the subregion on the boundary can be described equivalently by the minimal area of a bulk codimension two surface anchored at the boundaries of the subregion. This indeed reveals the dual relationship between quantum information theory defined on boundary and gravity in bulk.

In the holographic context of a thermo-field double state (TFD state)on the boundary theory being dual to a eternal black hole [6], it has told us that the entanglement entropy can not capture all the information for the full time evolution of an AdS wormhole[7]. As a result, an another refined information quantity, namely complexity, has been proposed to measure some cases which entanglement entropy fails to describe holographically, such as the growth behaviors of wormhole after the thermal equilibrium. The concept of complexity in a discrete system is that the minimum numbers of quantum gates are required to produce a certain state from a reference state in quantum information theory.

Recently, the definitions of complexity from both quantum field theory and holographic dual viewpoints have been attracted many attentions. Although many works on the aspect of field theory have been made [8–18], a unique and consistent definition is still missing. While, from the holographic perspective, there are two potential prescription-s to realize the complexity, such as complexity-volume (CV)duality [19] and complexity-action(CA)duality[20, 21]. Since that, a large great of progresses attempting to better understand complexity from holographic dual point have also appeared in [22–57].

Intriguingly, the investigations on the holographic complexity can be generalized to s-tates on the dynamical boundary backgrounds, such as an asymptotic Friedman-Lemaitre-Robertson-Walker(FLRW) cosmology boundary[58] which is associated with a foliation of
the geometry in bulk. It might provide us an interesting prototype to understand the non-perturbative behaviors of cosmology. In Ref.[59, 60], the FLRW Universe on a conformal boundary can be derived from a class of AdS black holes in bulk and the growth behavior of holographic complexity of a field defined on FRLW Universe is also investigated in [61], where it has been shown that there are mainly three parts of contributions to the growth rate, namely, the first one comes from the coupling correlation between a field (or an operator) on left boundary and one on right boundary, the second one is from the rate of the spatial volume of the corresponding dual Universe and the third one is from the constant spatial curvature of the horizon in the Schwarzschild-AdS background.

On the one hand, in the context of AdS/CFT, higher-order curvature corrections in the bulk provide some prescriptions to explore more general holographic CFTs than those defined by Einstein gravity, as pointed by [36]. On the other hand, the graviton mass which breaks diffeomorphism symmetry in massive gravity theory also gives some intriguing properties to holographically characterize the behaviors of dual fields distinguishing clearly from the case in Einstein gravity. For examples, in the framework of dual fluid theory, the Kovtun-Son-Starinets bound, namely, the shear viscosity to entropy density, can be violated in both Gauss-Bonnet gravity and massive gravity, as mentioned by [68–71], which was thought to be saturated in Einstein gravity case.

Then, motivated by the recent work in Ref.[61], for Gauss-Bonnet gravity and massive gravity backgrounds, respectively, we would like to ask whether, for each case, the holographic complexity behavior of a field state defined on cosmology boundary will exhibit some fascinating behaviors or not. We expect that there will be some interesting results for the growth behaviors in these gravity theories.

The paper is organized as follows: in Section II, we simply review the previous work in literature, where, with the help of the Eddington-Finkelstein coordinates, the FLRW universe metric can be derived from a large class of static asymptotically AdS black hole spacetimes. In Section III, in the Gauss-Bonnet geometry, according to the CV conjecture, under the large $R_m$ limit, we shall analytically calculate the holographic complexity growth rate of the field state in the dual universe. In section IV, for a massive gravity case, we, in a parallel way, investigate the relevant complexity behaviors. The last section gives the relevant conclusions and discussions.
II. THE METRIC

In this section, we briefly review how to derive the FLRW spacetime from an AdS black hole background \[59, 60\]. First of all, a large class of static asymptotically $AdS_{d+1}$ black hole is described by the metric

$$ds^2 = -f(r)dt^2 + f(r)^{-1}dr^2 + \Sigma(r)^2 h_{ij}dx^i dx^j \tag{1}$$

where $h_{ij}dx^i dx^j$ is the line element of the co-dimension two maximally symmetric subspace with spatial curvature $k$ which can take spherical $k = 1$, planar $k = 0$, or hyperbolic $k = -1$ topologies, respectively. The blackening factor $f(r)$ and the function $\Sigma(r)$, which usually can be determined by Equation of motion in gravity, are required to naturally saturate the assumption that, at the large $r$ limit, $f(r) \sim \frac{r^2}{L^2}$ and $\Sigma(r) \sim \frac{r}{L}$ with AdS curvature radius $L$. In order to conveniently explore holographic complexity behavior of the field living in dual FLRW Universe embedded into an AdS black hole background, one can introduce the Eddington-Finkelstein coordinates via

$$\nu = t + r^*(r), \quad dr^* = \frac{dr}{f(r)} \tag{2}$$

such that one can write the metric (1) in the form

$$ds^2 = -f(r)d\nu^2 + 2d\nu dr + \Sigma(r)^2 h_{ij}dx^i dx^j. \tag{3}$$

In the following, we shall present a foliation of the black hole spacetime (3) in such a way that the corresponding conformal boundary can take the form of FLRW spacetime. For this aim, we need to introduce new time coordinate $V$, $d\nu = \frac{dV}{a(V)}$, and the new radial coordinate $R = \frac{r}{a(V)}$, where $a(V)$ explained as the cosmological evolving factor is a positive function in terms of $V$, in order to reexpress the metric in the form:

$$ds^2 = 2dV dR - \left[\frac{f}{a^2(V)} - 2\frac{\dot{a}}{a}\right]dV^2 + \Sigma(Ra)^2 h_{ij}dx^i dx^j. \tag{4}$$

If taking the large $r_*(or \, R)$ limit, then $f(Ra) \sim \frac{(Ra)^2}{L^2}$ and $\Sigma(Ra) \sim \frac{Ra}{L}$ are obtained. Therefore, the above line element (4) in bulk can be approximately replaced by

$$ds^2 \sim 2dV dR + \frac{R^2}{L^2}[-dV^2 + a^2 h_{ij}dx^i dx^j] \tag{5}$$

as a result, the new conformal boundary at $R \to \infty$ has precisely the desired FLRW universe refereed as the cosmological boundary with spatial curvature $k$. It’s worth noting that it is
not the same as the ordinate AdS boundary at \( r \to \infty \) in which there is a static boundary spacetime.

In the following sections, using the Complexity-Volume conjecture, we focus on investigating the holographic complexity growth rate of the TFD state defined on boundary FLRW universe for the AdS black hole background in the Gauss-Bonnet gravity and massive gravity, respectively. In particular, we shall analytically compute their time dependence of complexity and explore the asymptotic growth rate how to relate a conserved quantity in bulk and the geometrical quantities in the boundary cosmology. Note that the evolution of dual state depends on two times \( t_L \) and \( t_R \) denoting the left and right boundary times, respectively. Without loss of generality, we will adopt the symmetric configuration times with \( t_L = t_R \) as shown in[22].

### III. THE CASE FOR THE NEUTRAL GAUSS-BONNET BLACK HOLE

In this section, according to the analysis in literature[61], we use the CV conjecture to explore in detail the behavior of holographic complexity growth rate with respect to the dual cosmological boundary time in the Gauss-Bonnet black hole background. The CV conjecture, being related to the size of an Einstein-Rosen bridge (ERB) to the computational complexity of the dual quantum field on the boundary, suggests that the complexity is dual to the volume of an extremal codimension-one bulk surface anchored at the time slice in the boundary on which the state is defined,

\[
\mathcal{C}_V = \frac{\max[V]}{G\ell},
\]

where \( \ell \) is some additional length scale associated with the bulk geometry, which is usually chosen to be equal to the effective AdS radius \( L_e \) or the AdS radius \( L \), as shown in following.

As for the trick, we mainly consult the procedure proposed by [22, 37, 61].

Let us start by the Einstein-Gauss-Bonnet action consisting of a cosmological constant and the simplest generalizations of Einstein gravity [62], which is

\[
S = \frac{1}{16\pi G} \int d^{d+1}x \sqrt{-g} \left[ R + \frac{(d-1)(d-2)}{L^2} + \alpha (R_{\mu\nu\gamma\delta}R^{\mu\nu\gamma\delta} - 4R_{\mu\nu}R^{\mu\nu} + R^2) \right]
\]

the metric following from the above action is given in the following form

\[
ds^2 = -f(r)dt^2 + f(r)^{-1}dr^2 + \frac{r^2}{L_e^2} h_{ij}dx^i dx^j
\]
where

\[ f(r) = k + \frac{r^2}{2\hat{\alpha}}[1 - \sqrt{1 + 4\hat{\alpha}(\frac{\hat{M}}{r^d} - \frac{1}{L^2})}] \]  

(9)

where \( \hat{M} = \frac{16\pi G M}{(d-1)M_{\text{Pl},d-1}} \) and \( \hat{\alpha} = \alpha(d - 2)(d - 3) \). The notation \( \alpha \) is the coupling constant of the Gauss-Bonnet term with dimension \((\text{length})^2\) and the effective AdS radius \( L_e \) is obtained by shifting the usual \( L \) due to the presence of \( \hat{\alpha} \), namely \( L_e^2 = \frac{L^2}{2}(1 + \sqrt{1 - \frac{4\hat{\alpha}}{L^2}}) \). Here, we have used the effective AdS radius. Note that the metric with the effective AdS radius has some differences from the solution in [37, 62], but is consistent with the one in [60].

To conveniently implement our goals, according to the previous process [22, 61], one needs to introduce the Eddington-Finkelstein coordinates, so that the line element (8) becomes

\[ ds^2 = -f(r)d\nu^2 + 2d\nu dr + \frac{r^2}{L_e^2}h_{ij}dx^i dx^j. \]  

(10)

One can easily check that the dual boundary cosmology can be derived from the Gauss-Bonnet-AdS black hole, whose result is just the form of Eq.(4). In order to calculate the time dependence of holographic complexity in Gauss-Bonnet gravity, we need to embed a surface possessing the same maximal symmetry as the horizon does, i.e. the surface is independent of the coordinates \( x^i \). As a result, the surface can be described via parameterizing equations \( \nu = \nu(\lambda) \) and \( r = r(\lambda) \) in the parameter \( \lambda \). Then, basing on the method [22, 61], its volume is calculated in the following form,

\[ V = 2\Omega_{k,d-1}L_e^{d-1}W \]  

(11)

\[ W = \int d\lambda \left( \frac{r}{L_e} \right)^{d-1} \sqrt{-f(r)\nu'^2 + 2\nu'r'} \equiv \int d\Lambda \mathcal{L} \]  

(12)

where the primes indicate the derivatives with respect to \( \lambda \). Here we have used \( \Omega_{k,d-1} \) to represent the unit spatial volume in the codimension-two subspace. Since the above integrand \( \mathcal{L} \) is not explicitly dependent on \( \nu \), we obtain a conserved quantity \( E \) written as

\[ E = -\frac{\partial \mathcal{L}}{\partial \nu'} = \left( \frac{r}{L_e} \right)^{d-1} \frac{f \nu' - r'}{\sqrt{-f\nu'^2 + 2\nu'r'}} \]  

(13)

we shall refer to it as the energy. Since Eq.(12) is reparametrization invariant, we can be free to choose parameter \( \lambda \) to keep the radial volume element fixed, namely,

\[ \left( \frac{r}{L_e} \right)^{d-1} \sqrt{-f(r)\nu'^2 + 2\nu'r'} = 1, \]  

(14)
such that Equations (12) and (13) can be rewritten as

\[ W = \int_{r_{\text{min}}}^{r_{\text{max}}} dr \left( \frac{r}{L_e} \right)^{2(d-1)} \frac{1}{\sqrt{f \left( \frac{r}{L_e} \right)^2 + E^2}} \] (15)

\[ E = \left( \frac{r}{L_e} \right)^{2(d-1)} \left( f \nu' - r' \right) \] (16)

It is easy to find that Eq. (16) has an alternative form,

\[ r' = \sqrt{f \left( \frac{L_e}{r} \right)^{2(d-1)} + E^2 \left( \frac{L_e}{r} \right)^4} \] (17)

Here, we are assuming a symmetric configuration with \( t_L = t_R \) on boundaries, as a result, the point at \( r_{\text{min}} \), as a minimal radius, should be a turning point of the surface, and then the derivative \( r' \) would vanish. Therefore, the minimal radius is determined by

\[ f(r_{\text{min}}) \left( \frac{r_{\text{min}}}{L_e} \right)^{2(d-1)} + E^2 = 0. \] (18)

As shown in literature [22], the turning point is behind the horizon and hence we have \( f(r_{\text{min}}) < 0, r' = 0 \) and \( \nu' > 0 \). Thus, we can deduce a useful result that \( E < 0 \) by calculating Eq. (16) at the turning point. Making the use of Eqs. (16) and (17), one has

\[ t_R + r^*(r_{\text{max}}) - r^*(r_{\text{min}}) = \int_{r_{\text{min}}}^{r_{\text{max}}} dr \left[ \frac{E}{f \sqrt{f \left( \frac{L_e}{r} \right)^2 + E^2}} + \frac{1}{f} \right]. \] (19)

Here, we have used a fact that, at the innermost point or turning point, there is \( t = 0 \) due to the symmetry. Meanwhile, the above equation both sides multiply with the conserved quantity \( E \), we obtain

\[ W = \int_{r_{\text{min}}}^{r_{\text{max}}} dr \sqrt{f \left( \frac{r}{L_e} \right)^{2(d-1)} + E^2 + E - E[t_R + r^*(r_{\text{max}}) - r^*(r_{\text{min}})]}. \] (20)

Now, we turn to considering the growth behavior for time dependent holographic complexity in the Gauss-Bonnet theory. Hence, adopting the new time coordinate \( V \) and radial coordinate \( R \), and employing the chain rule of differentiation,

\[ \frac{\partial W(V_R, R_{\text{max}})}{\partial V_R} = \frac{\partial W}{\partial t} \frac{\partial t}{\partial V_R} + \frac{\partial W}{\partial r_{\text{max}}} \frac{\partial r_{\text{max}}}{\partial V_R}. \] (21)
the partial derivatives can be given as

\[ \frac{\partial W}{\partial t_R} = -E \]
\[ \frac{\partial t_R}{\partial V_R} = \frac{1}{a(V_R)} - \frac{R_{\text{max}} \dot{a}(V_R)}{f(R_{\text{max}} a)} \]
\[ \frac{\partial W}{\partial r_{\text{max}}} = \sqrt{\frac{f(r_{\text{max}})(r_{\text{max}}^{2d-1} + E^2)}{f(r_{\text{max}})}} \]
\[ \frac{\partial r_{\text{max}}}{\partial V_R} = R_{\text{max}} \dot{a}(V_R) \]

where the dots denote the derivatives with respect to \( V \). Therefore, utilizing above results, we arrive at

\[ \frac{\partial W(V_R, R_{\text{max}})}{\partial V_R} = -E \left[ \frac{1}{a(V_R)} - \frac{R_{\text{max}} \dot{a}(V_R)}{f(R_{\text{max}} a)} \right] + \sqrt{\frac{f(R_{\text{max}} a(V_R))(R_{\text{max}} a(V_R))^{2(d-1)} + E^2}{f(R_{\text{max}} a(V_R))}} R_{\text{max}} \dot{a}(V_R). \]

So far, the analysis is quite general. From now on, we focus on the growth rate of holographic complexity in \( d + 1 = 5 \) dimensional Gauss-Bonnet gravity. Taking the limit \( R_{\text{max}} \to \infty \), and keeping \( V_R \) fixed, we come to the conclusion that the growth rate with respect to the cosmology boundary time is given by

\[ \frac{\partial C}{\partial V_R} = \frac{1}{G L_c} \frac{\partial V}{\partial V_R} \]
\[ = -\frac{2\Omega_{k,3} L_c^2}{G} \frac{E}{a(V_R)} + \frac{2\Omega_{k,3}}{G} \left[ \frac{R_{\text{max}}^3}{3} \frac{d(a^3(V_R))}{dV_R} - \frac{A}{2} R_{\text{max}} \dot{a}(V_R) + \ldots \right] \]

where \( A = \frac{2 \tilde{\alpha}}{1 - \sqrt{1 - \frac{2}{L^2}}} \). Here we have used \( \ell = L_c \). To summarize, taking the large \( R_{\text{max}} \) condition and keeping the time \( V_R \) fixed, for holographic complexity of the field on the FLRW universe being dual to a five dimensional Gauss-Bonnet-AdS black hole, we come to a conclusion that its growth behavior includes the first term, as a finite term, mainly relating to the conserved quantity \( E \) and the second term (the leading divergent term) being proportional to the rate of the spatial volume in the dual FLRW universe as well as the third term (the sub-leading divergent term) coming from the codimension-two spatial constant curvature, whose behaviors are very like the one presented in literature [61]. When the parameter \( \tilde{\alpha} \to 0 \), its growth rate in Gauss-Bonnet-AdS black hole reduces to the result in Schwarzschild-AdS case.
IV. THE CASE FOR THE NEUTRAL MASSIVE BLACK HOLE

In the section, we are going to study the counterpart in massive black hole in a parallel way. First, let us review briefly the solution from massive gravity in five dimensional space-time. And then, we shall demonstrate how the growth behavior of holographic complexity of the field in the FLRW universe links the conserved quantity in bulk and some geometric quantities on boundary cosmology by holographic dual, and what role the graviton mass term can play on this growth rate. First, let us write down the neutral massive Einstein action consisting of the Ricci scalar, cosmological constant term and graviton mass terms\([63–67]\), which can be expressed as

\[
S = \int d^5x \sqrt{-g} \left\{ \frac{1}{2\kappa^2} (R - 2\Lambda) + \frac{m^2}{2\kappa^2} (c_1 u_1 + c_2 u_2 + c_3 u_3 + c_4 u_4) \right\},
\]

where

\[
\begin{align*}
    u_1 &= tr K, \\
    u_2 &= (tr K)^2 - tr (K^2), \\
    u_3 &= (tr K)^3 - 3tr K tr (K^2) + 2tr (K^3), \\
    u_4 &= (tr K)^4 - 6tr (K^2) (tr K)^2 + 8tr (K^3) tr K + 3(tr (K^2))^2 - 6tr (K^4)
\end{align*}
\]

\(c_1, c_2, c_3\) and \(c_4\) are negative constants, but \(c_0\) is a positive constant; \(\kappa^2 = 8\pi G\), and the metric \(K^\mu_\nu\) is defined by \(K^\mu_\nu = \sqrt{g^\alpha_\gamma} f_{\alpha\gamma}\). It tells us that the graviton is allowed to obtain its mass \(m^2\) by the reference metric coupling the bulk metric to break differemorphism symmetry. Here following the ansatz in \([66]\), the reference metric without dynamical behavior is chosen as \(f_{\mu\nu} = diag(0, 0, c_0^2 h_{ij})\). Its solution following from the above action is given in static coordinate as

\[
ds^2 = -f(r) dt^2 + \frac{dr^2}{f} + \frac{r^2}{L^2} h_{ij} dx^i dx^j
\]

where

\[
f(r) = k + \frac{r^2}{L^2} - \frac{m_0}{r^2} + \frac{c_0 c_1 m^2}{3} r + \frac{c_0^2 c_2 m^2}{r} + \frac{2c_0^3 c_3 m^2}{r^2} + \frac{2c_0^4 c_4 m^2}{r^2}
\]

\(^1\) Here, If requiring the graviton mass \(m^2 \in (0, \frac{12c_2 - 12c_2 \sqrt{1 + \frac{L^2 c_1^2}{c_0^2}}}{L^2 c_1^2})\), then the single horizon appears, as a consequence, it shares the similar Penrose diagram with the neutral AdS-black hole.
where $m_0$ is related to the mass parameter of black hole in massive gravity in five dimensional geometric configuration, namely

$$M = \frac{3\Omega_3 m_0}{16\pi G} \quad (32)$$

similarly, when taking the large $r_{\text{max}} = R_{\text{max}}a$ limit, the solution (30) can give rise to the form of metric (5). Next, using the symmetric configuration times with $t_L = t_R$ as stated above, we shall concentrate on time dependent complexity in five dimensional manifold and evaluate the volume of Einstein-Rosen Bridge at a specific boundary time, its mathematical relation is similarly given by

$$V = 2\Omega_3 L^3 W \quad (33)$$

$$W = \int_{\lambda_{\text{min}}}^{\lambda_{\text{max}}} \left( \frac{r}{L} \right)^3 \sqrt{ - f(r) \nu'^2 + 2\nu' r' d\lambda } \quad (34)$$

Repeating the work in a similar way, it is not hard to find that

$$W = \int_{r_{\text{min}}}^{r_{\text{max}}} dr \sqrt{\left( \frac{r}{L} \right)^6 f(r) + \frac{E^2}{f} } - E [ t_R + r^*(r_{\text{max}}) - r^*(r_{\text{min}}) ]. \quad (35)$$

Here, the physical meaning of $E$ is also a conserved charge on the ERB in such system.

Applying the chain rule of differentiation and the new coordinates above, we can easily find

$$\frac{\partial W (V_R, R_{\text{max}})}{\partial V_R} = \frac{\partial W}{\partial t_R} \frac{\partial t_R}{\partial V_R} + \frac{\partial W}{\partial r_{\text{max}}} \frac{\partial r_{\text{max}}}{\partial V_R}. \quad (36)$$

The partial derivatives can be exactly given as

$$\frac{\partial W}{\partial t_R} = -E \quad (37)$$

$$\frac{\partial t_R}{\partial V_R} = \frac{1}{a(V_R)} - \frac{R_{\text{max}} \dot{a}(V_R)}{f(R_{\text{max}} a)} \quad (38)$$

$$\frac{\partial W}{\partial r_{\text{max}}} = \sqrt{ f(r_{\text{max}})(\frac{r_{\text{max}}}{L})^6 + \frac{E^2}{f(r_{\text{max}})} } \quad (39)$$

$$\frac{\partial r_{\text{max}}}{\partial V_R} = R_{\text{max}} \dot{a}(V_R). \quad (40)$$

Hence, plugging the above partial derivatives relations into Eq.(35), one naturally arrives at

$$\frac{\partial W (V_R, R_{\text{max}})}{\partial V_R} = -E \left[ \frac{1}{a(V_R)} - \frac{R_{\text{max}} \dot{a}(V_R)}{f(R_{\text{max}} a)} \right] + \sqrt{ f(R_{\text{max}} a(V_R))(\frac{R_{\text{max}} a(V_R)}{L})^6 + \frac{E^2}{f(R_{\text{max}} a(V_R))} } R_{\text{max}} \dot{a}(V_R). \quad (41)$$
From the above equation, after taking the large $R_{\text{max}} \equiv R_m$ expansion and keeping the time $V_R$ fixed, it can approximately reduce to

$$\frac{\partial C_V}{\partial V_R} = \frac{2\Omega_{k,3}}{G} \left[ -EL^2 + \dot{a}(V_R)(\frac{3}{4}AB - \frac{C}{2} - \frac{5}{16}A^3) + \frac{1}{3}R_m^3 \frac{da^3}{a(V_R)} - \frac{A}{4}R_m^2 \frac{da^2}{dV_R} + R_m \dot{a} \left( \frac{3A^2}{8} - \frac{B}{2} \right) + \ldots \right]$$

(42)

where the constants in the above result are $A = 2c_1 c_2 L^2$, $B = L^2 (k + c_0^2 c_2 m^2)$, and $C = 2c_0^3 c_3 m^2 L^2$, respectively. We have used $\ell = L$ for this case. So far, under the same conditions, we have captured the holographic complexity growth behaviors for the TFD state on a dual FLRW universe in massive gravity. The first divergent term being proportional to the growth of the spatial volume of the Universe on the boundary is still given, which is in agreement with the relevant result from the last model. In contrast with the relevant results in the previous model or the ones in [61], we find that there are some new interesting phenomenons in the other terms due to the graviton mass effect. The new intriguing phenomenons are that the finite term consists of, the usual conserved charge $E$ and the novel contribution from graviton mass effect; and the third divergent term contains both the spatial curvature of the horizon $k$ and graviton mass effect; furthermore, to be more interesting, the second divergent term is totally caused by the rate of the area of the dual Universe on account of the graviton mass effect. This, In this sense, means that the graviton mass effect plays a vital role on the growth behaviors of the conformal field state defined on the dual FLRW Universe.

V. CONCLUSIONS AND DISCUSSIONS

In this paper, inspired by the recent work[61], using the complexity-volume duality and the prescription on the time dependence of holographic complexity [22], we have holographically computed the growth behaviors of complexity for a field in the boundary cosmology being dual to Gauss-Bonnet gravity and massive gravity in bulk, respectively.

In the framework context of Gauss-Bonnet gravity, under the large $R_m$ condition, we have found a conclusion that the change behavior of holographic complexity for the field defined in the dual Universe located at an asymptotic AdS boundary is mainly governed by a finite term relating to the conserved charge $E$ and the leading divergent term being proportional to the rate of the spatial volume in the dual FLRW universe as well as the sub-leading divergent term coming from the contribution of the codimension-two spatial
constant curvature, whose behaviors are very like the one presented in literature [61].

In contrast to the relevant result in the Gauss-Bonnet gravity or the one in [61], for the massive gravity case, except the first divergent term obeying the growth rate of the spatial volume of the Universe located at an asymptotic AdS-boundary, there are some new remarkable results to be observed in the other terms due to the graviton mass effect. We have demonstrated that, under the same conditions, the new intriguing results are that, beside the conserved charge $E$, the some graviton mass effect also contributes the finite term; and the third divergent term is determined by the spatial curvature of the horizon $k$ and graviton mass effect; furthermore, to be more interesting and surprising, the graviton mass effect can be completely responsible for the second divergent term as an new additional term saturating an area law. Thus, they allow us to distinguish clearly from the results from the Schwarzschild-AdS case.

According to the information of the above results, our models have explicitly exhibited some universal complexity growth behaviors. Firstly, the finite term is inversely proportional to the cosmological factor, which can be interpreted as a coupling correlation term between two localized operators at the left and right boundaries [61], respectively. Secondly, the leading divergent term in each case obeys a volume law without the modified effects, which at the qualitative level, is quite consistent with the definition of complexity from field theory landscape. Lastly, our results also contain the spatial curvatures of its horizon for each model. These may also have implicitly suggested that the nonperturbative evolution properties of the field in the FLRW universe can be implemented in such holographic dual, as mentioned in literatures [60, 61].

The behavior of the coupling correlation term can be interpreted in the following. After preparing the TFD state at $t = 0$, one can embed two operators $O_L(x)$ and $O_R(x)$ into left and right boundaries, respectively. As time evolves, besides the primordial their relation, both of them would correlate with many other operators (or dofs) on the left and right sides. Thus, the original correlation between $O_L$ and $O_R$ is naturally diluted by these dofs. As the dual universe expands or contracts, the spreading of the operator therein will decrease or increase, respectively. So, these can account for the behaviors of the finite terms from our results (27) and (42). The similar physical picture was pointed out in [61].

The power law divergent terms here are consistent with general divergence structure of holographic complexity given by evaluating a geometric quantity which extends to the asymptotic AdS boundary such that the holographic complexity is divergent. Then we can
see that these results have some geometric interpretations as follows. Firstly, the leading divergence for each case in our paper can be expressed as the volume of boundary cosmology because the bulk calculations evaluate the volume of an extremal surface extending to the asymptotic boundary. It is noted that, from the field theory definition of complexity, the leading divergent behavior of complexity is indeed the volume law, as emphasised in literature [8, 61]. Secondly, the subsequent divergent term in Gauss-Bonnet gravity is clearly different from massive gravity case, e.g. the sub-leading term (the linear divergent term) in our Gauss-Bonnet case should come from the integrals of curvature invariants over this time slice on boundary, while the sub-leading one in massive gravity can be found to satisfy an area law, which the graviton mass effect totally governs, and the linear divergent term is related to the integrals consisting of the curvature invariants and the graviton mass terms, whose structure is rather similar in nature to the divergence structure found in [67].

Furthermore, applying the CA conjecture in the holographic context, one will attempt to explore such data of holographic complexity in the above each case and desire to capture the similar results in future.

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