Inverse Scattering Transform for the Toda Hierarchy with Quasi–Periodic Background

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INVERSE SCATTERING TRANSFORM FOR THE TODA HIERARCHY WITH QUASI-PERIODIC BACKGROUND

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Abstract. We provide a rigorous treatment of the inverse scattering transform for the entire Toda hierarchy in the case of a quasi-periodic finite-gap background solution.

1. Introduction

Since the seminal work of Gardner et al. in 1967 the inverse scattering transform is one of the main tools for solving completely integrable wave equations and numerous articles have been devoted to this subject since then. However, with only a few exceptions, only the case of short range perturbations of a constant background solution has been investigated. The first to consider a non-constant background seem to be Kuznetsov and A.V. Mikhailov, who informally treat the Korteweg-de Vries equation with the Weierstraß elliptic function as stationary background solution. Formulas for arbitrary periodic background were given by Firsova in . However, the discontinuous behavior at the band edges is not investigated there and neither is the question when the time dependent scattering data satisfy the hypothesis necessary for the Gel’fand-Levitan-Marchenko theory (see, e.g., for the constant background case). Moreover, the normalization chosen there blows up when the eigenvalues lie at certain positions in the gaps.

Our aim in the present paper is to provide a rigorous treatment of the inverse scattering transform for the Toda hierarchy in the case of quasi-periodic finite-gap background solutions (including the case of arbitrary periodic solutions). We include a discussion of the problems arising from the poles of the Baker-Akhiezer functions at the Dirichlet eigenvalues (which are absent in the constant background case). In the constant background case this was first done for the Toda equation by Flaschka in on an informal level. The first rigorous treatment for the entire Toda hierarchy was given by one of us in (see also , ). For further results on solutions of the Toda lattice with nontrivial spatial asymptotics see .

After introducing the Toda hierarchy in Section we will first show that a solution will stay close to a given background solution in Section . This result implies that a short range perturbation of a quasi-periodic finite-gap solution will stay short range for all time and it shows that the time dependent scattering data satisfy the hypothesis necessary for the Gel’fand-Levitan-Marchenko theory. This result constitutes the main technical ingredient for the inverse scattering transform. In Section we review some necessary facts on quasi-periodic finite-gap solutions.

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and in Section 3 we compute the time dependence of the scattering data and discuss its dynamics. Finally we establish the conserved quantities of the Toda hierarchy in this context.

2. The Toda hierarchy

In this section we introduce the Toda hierarchy using the standard Lax formalism (2.1). We first review some basic facts from [2] (see also [14]).

We will only consider bounded solutions and hence require

**Hypothesis H.2.1.** Suppose $a(t), b(t)$ satisfy

\[
 a(t) \in \ell^\infty(\mathbb{Z}, \mathbb{R}), \quad b(t) \in \ell^\infty(\mathbb{Z}, \mathbb{R}), \quad a(n, t) \neq 0, \quad (n, t) \in \mathbb{Z} \times \mathbb{R},
\]

and let $t \mapsto (a(t), b(t))$ be differentiable in $\ell^\infty(\mathbb{Z}) \oplus \ell^\infty(\mathbb{Z})$.

Associated with $a(t), b(t)$ is a Jacobi operator

\[
 H(t) : \ell^2(\mathbb{Z}) \to \ell^2(\mathbb{Z}), \quad f \mapsto \tau(t)f,
\]

where

\[
 \tau(t)f(n) = a(n, t)f(n+1) + a(n-1, t)f(n-1) + b(n, t)f(n)
\]

and $\ell^2(\mathbb{Z})$ denotes the Hilbert space of square summable (complex-valued) sequences over $\mathbb{Z}$. Moreover, choose constants $c_0 = 1$, $c_j$, $1 \leq j \leq r$, $c_{r+1} = 0$, set

\[
 g_j(n, t) = \sum_{\ell=0}^{j} c_{j-\ell} \langle \delta_n, H(t)^\ell \delta_n \rangle,
\]

\[
 h_j(n, t) = 2a(n, t) \sum_{\ell=0}^{j} c_{j-\ell} \langle \delta_{n+1}, H(t)^\ell \delta_n \rangle + c_{j+1},
\]

and consider the Lax operator

\[
 P_{2r+2}(t) = -H(t)^{r+1} + \sum_{j=0}^{r} (2a(t)g_j(t)S^+ - h_j(t))H(t)^{r-j} + g_{r+1}(t),
\]

where $S^+f(n) = f(n+1)$. Restricting to the two-dimensional nullspace $\text{Ker}(\tau(t) - z)$, $z \in \mathbb{C}$, of $\tau(t) - z$, we have the following representation of $P_{2r+2}(t)$

\[
 P_{2r+2}(t) \bigg|_{\text{Ker}(\tau(t) - z)} = 2a(t)G_r(z, t)S^+ - H_{r+1}(z, t),
\]

where $G_r(z, n, t)$ and $H_{r+1}(z, n, t)$ are monic polynomials in $z$ of the type

\[
 G_r(z, n, t) = \sum_{j=0}^{r} z^j g_{r-j}(n, t),
\]

\[
 H_{r+1}(z, n, t) = z^{r+1} + \sum_{j=0}^{r} z^j h_{r-j}(n, t) - g_{r+1}(n, t).
\]

A straightforward computation shows that the Lax equation

\[
 \frac{d}{dt}H(t) - [P_{2r+2}(t), H(t)] = 0, \quad t \in \mathbb{R},
\]
is equivalent to
\[
\begin{align*}
\text{TL}_r(a(t),b(t))_1 &= \dot{a}(t) - a(t)\left(g_{r+1}^r(t) - g_{r-1}^r(t)\right) = 0, \\
\text{TL}_r(a(t),b(t))_2 &= \dot{b}(t) - \left(h_{r+1}(t) - h_{r-1}^r(t)\right) = 0,
\end{align*}
\]
where the dot denotes a derivative with respect to \(t\) and \(f^\pm(n) = f(n \pm 1)\). Varying \(r \in \mathbb{N}_0\) yields the Toda hierarchy \(\text{TL}_r(a, b) = (\text{TL}_r(a, b)_1, \text{TL}_r(a, b)_2) = 0\).

Finally, we recall that the Lax equation (2.7) implies existence of a unitary propagator \(U_r(t, s)\) such that the family of operators \(H(t), t \in \mathbb{R}\), are unitarily equivalent, \(H(t) = U_r(t, s)H(s)U_r(s, t)\).

### 3. The initial value problem

First of all we recall the basic existence and uniqueness theorem for the Toda hierarchy (see, e.g., [12], [13], or [14, Section 12.2]).

**Theorem 3.1.** Suppose \((a_0, b_0) \in M = \ell^\infty(\mathbb{Z}) \oplus \ell^\infty(\mathbb{Z})\). Then there exists a unique integral curve \(t \mapsto (a(t), b(t))\) in \(C^\infty(\mathbb{R}, M)\) of the Toda equations, that is, \(\text{TL}_r(a(t), b(t)) = 0\), such that \((a(0), b(0)) = (a_0, b_0)\).

In [12] it was shown that solutions which are asymptotically close to the constant solution at the initial time stay close for all time. Our first aim is to extend this result to include perturbations of quasi-periodic finite-gap solutions. In fact, we will even be a bit more general.

**Lemma 3.2.** Suppose \(a(n, t), b(n, t)\) and \(\bar{a}(n, t), \bar{b}(n, t)\) are two arbitrary bounded solutions of the Toda system satisfying (3.1) for one \(t_0 \in \mathbb{R}\), then (3.1) holds for all \(t \in \mathbb{R}\), that is,

\[
\sum_{n \in \mathbb{Z}} w(n) \left(|a(n, t) - \bar{a}(n, t)| + |b(n, t) - \bar{b}(n, t)|\right) < \infty,
\]

where \(w(n) > 0\).

**Proof.** Without loss of generality we assume that \(t_0 = 0\). Consider the expression

\[
\| (a(t), b(t)) \|_* = \sum_{n \in \mathbb{Z}} w(n) \left(|a(n, t) - \bar{a}(n, t)| + |b(n, t) - \bar{b}(n, t)|\right)
\]

which remains finite at least for small \(t\) since there is a local solution of the Toda system with respect to \(\| \cdot \|_*\). We claim that the following estimate holds

\[
\sum_{n \in \mathbb{Z}} w(n)|g_r(n, t) - \bar{g}_r(n, t)| \leq C_r\| (a(t), b(t)) \|_*,
\]

\[
\sum_{n \in \mathbb{Z}} w(n)|h_r(n, t) - \bar{h}_r(n, t)| \leq C_r\| (a(t), b(t)) \|_*,
\]

where \(C_r = C(\|H(0)\|, \|\dot{H}(0)\|)\) is a positive constant due to [14]. Let us prove (3.2) by induction on \(r\). It suffices to consider the case where \(c_j = 0, 1 \leq j \leq r\), since all involved sums are finite. In this case [14, Lemma 6.4] shows that \(g_j(n, t)\),
\( h_j(n, t) \) can be recursively computed from \( g_0(n, t) = 1, h_0(n, t) = 0 \) via

\[
g_{j+1}(n, t) = \frac{1}{2} (h_j(n, t) + h_j(n - 1, t)) + b(n, t) g_j(n, t),
\]

\[
h_{j+1}(n, t) = 2a(n, t)^2 \sum_{l=0}^{j} g_{j-l}(n, t) g_l(n + 1, t) - \frac{1}{2} \sum_{l=0}^{j} h_{j-l}(n, t) h_l(n, t).
\]

Hence

\[
g_1(n, t) - \bar{g}_1(n, t) = b(n, t) - \bar{b}(n, t), \quad h_1(n, t) - \bar{h}_1(n, t) = 2(a(n, t)^2 - \bar{a}(n, t)^2),
\]

and

\[
g_{r+1}(n, t) - \bar{g}_{r+1}(n, t) = (b(n, t) - \bar{b}(n, t)) g_r(n, t) + \bar{b}(n, t) (g_r(n, t) - \bar{g}_r(n, t)) + \frac{1}{2} (h_r(n, t) - \bar{h}_r(n, t) + h_r(n - 1, t) - \bar{h}_r(n - 1, t)),
\]

\[
h_{r+1}(n, t) - \bar{h}_{r+1}(n, t) = \frac{1}{2} \sum_{l=0}^{r} (h_{r-l}(n, t) h_l(n, t) - \bar{h}_{r-l}(n, t) \bar{h}_l(n, t)) + 2a(n, t)^2 \sum_{l=0}^{r} \bar{g}_{r-l}(n, t) \bar{g}_l(n + 1, t).
\]

Since both sequences \( a, b \) and \( \bar{a}, \bar{b} \) are solutions of the Toda system, \( \bar{a} \) yields

\[
\dot{a}(t) - \bar{a}(t) = a(t) (\bar{g}_{r+1}(t) - g_{r+1}(t)) - \bar{a}(t) (\bar{g}_{r+1}(t) - g_{r+1}(t)) = (a(t) - \bar{a}(t)) (\bar{g}_{r+1}(t) - g_{r+1}(t)) - a(t) (\bar{g}_{r+1}(t) - g_{r+1}(t) + g_{r+1}(t) - \bar{g}_{r+1}(t))
\]

and thus

\[
|a(n, t) - \bar{a}(n, t)| \leq |a(n, 0) - \bar{a}(n, 0)| + C_1 \int_0^t |a(n, s) - \bar{a}(n, s)| ds
\]

\[
+ \|H(0)\| \int_0^t (|g_{r+1}(n + 1, s) - \bar{g}_{r+1}(n + 1, s)| + |g_{r+1}(n, s) - \bar{g}_{r+1}(n, s)|) ds,
\]

\[
|b(n, t) - \bar{b}(n, t)| \leq |b(n, 0) - \bar{b}(n, 0)|
\]

\[
+ \int_0^t (|h_{r+1}(n, s) - \bar{h}_{r+1}(n, s)| + |h_{r+1}(n, s) - \bar{h}_{r+1}(n, s)|) ds.
\]

By \( \|a(t), b(t)\|_* \leq \|(a(0), b(0))\|_* + C \int_0^t \|a(s), b(s)\|_* ds \), where \( C = (C_1 + 2\|H(0)\| + 2C_{r+1}) \). Finally, applying Gronwall’s inequality

\[
\|(a(t), b(t))\|_* \leq \|(a(0), b(0))\|_* \exp (Ct)
\]

finishes the proof. \( \square \)

4. Quasi-periodic finite-gap solutions

As a preparation for our next section we first need to recall some facts on quasi-periodic finite-gap solutions (again see [2] or [13]).

Let \( M \) be the Riemann surface associated with the following function

\[
R_{2g+2}^{1/2}(z) = -\prod_{j=0}^{2g+1} \sqrt{z - E_j}, \quad E_0 < E_1 < \cdots < E_{2g+1},
\]

where \( g \in \mathbb{N} \) and \( \sqrt{\cdot} \) is the standard root with branch cut along \((-\infty, 0)\). \( M \) is a compact, hyperelliptic Riemann surface of genus \( g \). A point on \( \mathbb{M} \) is denoted by
\( p = (z, \pm R_{2g+2}^{1/2}(z)) = (z, \pm), \) \( z \in \mathbb{C}, \) or \( p = \infty_\pm, \) and the projection onto \( \mathbb{C} \cup \{\infty\} \) by \( \pi(p) = z. \)

Now pick \( g \) numbers (the Dirichlet eigenvalues)

\[
(\hat{\mu}_j)^g_{j=1} = (\mu_j, \sigma_j)_{j=1}^g
\]

whose projections lie in the spectral gaps, that is, \( \mu_j \in [E_{2j-1}, E_{2j}] \). Associated with these numbers is the divisor \( D_{\hat{\mu}} \) which is one at the points \( \hat{\mu}_j \) and zero else. Using this divisor we introduce

\[
\hat{z}(p, n, t) = \hat{A}_{\mu_0}(p) - \hat{A}_{\mu_0}(D_{\hat{\mu}}) - n\hat{A}_{\infty_+}(\infty_+) + t\hat{U}_s - \hat{E}_{\mu_0} \in \mathbb{C}^g,
\]

where \( \hat{E}_{\mu_0} \) is the vector of Riemann constants, \( \hat{U}_s \) the \( b \)-periods of the Abelian differential \( \Omega_s \) defined below, and \( \hat{A}_{\mu_0} (\alpha_{\mu_0}) \) is Abel’s map (for divisors). The hat indicates that we regard it as a (single-valued) map from \( \hat{\mathbb{M}} \) (the fundamental polygon associated with \( \mathbb{M} \)) to \( \mathbb{C}^g \). We recall that the function \( \theta(z(p, n, t)) \) has precisely \( g \) zeros \( \hat{\mu}_j(n, t) \) (with \( \hat{\mu}_j(0, 0) = \mu_j \)), where \( \theta(z) \) is the Riemann theta function of \( \mathbb{M} \).

Then our solution is given by

\[
a_q(n, t)^2 = \hat{a}^2 \frac{\theta(z(n+1, t))\theta(z(n-1, t))}{\theta(z(n, t))^2},
\]

\[
b_q(n, t) = \hat{b} + \sum_{j=1}^g c_j(g) \frac{\partial}{\partial w_j} \ln \left| \frac{\theta(w+z(n, t))}{\theta(w+z(n-1, t))} \right|_{w=0}.
\]

The constants \( \hat{a}, \hat{b}, c_j(g) \) depend only on the Riemann surface (see Section 9.2).

Introduce

\[
\phi_q(p, n, t) = C(n, t) \frac{\theta(z(p, n+1, t))}{\theta(z(p, n, t))} \exp \left( \int_{p_0}^p \omega_{\infty_+, \infty_-} \right),
\]

\[
\psi_q(p, n, t) = C(n, 0, t) \frac{\theta(z(p, n, t))}{\theta(z(p, 0, 0))} \exp \left( n \int_{p_0}^p \omega_{\infty_+, \infty_-} + t \int_{p_0}^p \Omega_s \right),
\]

where \( C(n, t), C(n, 0, t) \) are real-valued,

\[
C(n, t)^2 = \frac{\theta(z(n-1, t))}{\theta(z(n+1, t))}, \quad C(n, 0, t)^2 = \frac{\theta(z(0, 0))\theta(z(-1, 0))}{\theta(z(n, t))\theta(z(n-1, t))},
\]

and the sign of \( C(n, t) \) is opposite to that of \( a_q(n, t) \). \( \omega_{\infty_+, \infty_-} \) is the Abelian differential of the third kind with poles at \( \infty_+ \) respectively \( \infty_- \) and \( \Omega_s \) is an Abelian differential of the second kind with poles at \( \infty_+ \) respectively \( \infty_- \) whose Laurent expansion is given by the coefficients \( (j + 1)\hat{c}_{s-j} \) associated with \( \hat{T}_s \) (see Sections 13.1, 13.2). Then

\[
\tau_q(t)\psi_q(p, n, t) = \pi(p)\psi_q(p, n, t),
\]

\[
\frac{d}{dt}\psi_q(p, n, t) = 2a_q(n, t)\hat{G}_s(p, n, t)\psi_q(p, n+1, t) - \hat{H}_{s+1}(p, n, t)\psi_q(p, n, t)
\]

\[
= \hat{P}_{q,2s+2}(t)\psi_q(p, n, t)(n).
\]
It is well known that the spectrum of $H_q(t)$ is time independent and consists of $g + 1$ bands

\[(4.8) \quad \sigma(H_q) = \bigcup_{j=0}^{g} [E_{2j}, E_{2j+1}].\]

For further information and proofs we refer to [14, Chapter 9].

Next let us renormalize the Baker-Akhiezer function

\[
\tilde{\psi}_q(p, n, t) = \frac{\psi_q(p, n, t)}{\psi_q(p, 0, t)}
\]

such that $\tilde{\psi}_q(p, 0, t) = 1$ and let us abbreviate

\[(4.9) \quad \exp(\alpha_s(p, t)) = \psi_q(p, 0, t) = C(0, 0, t) \frac{\theta(z(p, 0, t))}{\theta(z(p, 0, 0))} \exp\left(t \int_{P_0}^p \Omega_s \right).\]

One can show ([14], (13.49))

\[
\exp(\alpha_s(p, t)) = \exp\left(\int_0^1 (2a_q(0, x)\hat{G}_q(p, 0, x)\phi_q(p, 0, x) - \hat{H}_{s+1}(p, 0, x))dx\right)
\]

\[(4.10) \quad \frac{G_g(p, 0, t)}{G_g(p, 0, 0)} \exp\left(\frac{1}{2} \int_0^t \frac{\hat{G}_q(p, 0, x)}{G_g(p, 0, x)} dx\right),\]

where

\[(4.11) \quad G_g(z, n, t) = \prod_{j=1}^{g} (z - \mu_j(n, t)).\]

However, let us emphasize that the integrals in (4.10) are only well-defined until we hit the first pole of the integrand. After that time they have to be interpreted as principal values (compare also Remark 5.4 (i)), respectively (4.9) has to be used.

In addition, observe

\[
\exp(\alpha_{s,+}(z, t) + \alpha_{s,-}(z, t)) = \frac{G_g(z, 0, t)}{G_g(z, 0, 0)},
\]

\[(4.12) \quad \exp(\alpha_{s,+}(z, t) - \alpha_{s,-}(z, t)) = \exp\left(2\frac{1}{2} \int_0^t \frac{\hat{G}_q(z, 0, x)}{G_g(z, 0, x)} dx\right).
\]

For $\lambda \in \sigma(H_q)$ we have

\[(4.13) \quad \alpha_{s,\pm}(\lambda, t) = \alpha_{s,\mp}(\lambda, t),\]

but note that $\alpha_{s,\mp}(\lambda, t) \neq \mp \alpha_{s,\pm}(\lambda, t)$ unless $g = 0$.

5. INVERSE SCATTERING TRANSFORM

Let $a(n, t), b(n, t)$ be a solution of the Toda system satisfying

\[(5.1) \quad \sum_{n \in \mathbb{Z}} |n| \left(|a(n, t) - a_q(n, t)| + |b(n, t) - b_q(n, t)| \right) < \infty\]

for one (and hence for any) $t_0 \in \mathbb{R}$. In [14] (see also [14]) we develop scattering theory for the Jacobi operator $H$ associated with $a(n), b(n)$. Jost solutions, transmission and reflection coefficients depend now on an additional parameter $t \in \mathbb{R}$. More precisely,

\[(5.2) \quad \sigma(H(t)) = \sigma(H), \quad \sigma_{ess}(H) = \sigma(H_q), \quad \sigma_p(H) = \{\rho_j\}_{j=1}^{q} \subseteq \mathbb{R}\setminus \sigma(H_q),\]
where \( q \in \mathbb{N} \) is finite. The Jost solutions \( \psi_{\pm}(z, n, t) \) are normalized such that
\[
\tilde{\psi}_{\pm}(z, n, t) = \tilde{\psi}_{q, \pm}(z, n, t) \left( 1 + o(1) \right) \quad \text{as} \quad n \to \pm\infty.
\]

Transmission \( T(\lambda, t) \) and reflection \( R_{\pm}(\lambda, t) \) coefficients are defined via the scattering relations
\[
(5.3) \quad T(\lambda, t)\tilde{\psi}_{\pm}(\lambda, n, t) = \tilde{\psi}_{\mp}(\lambda, n, t) + R_{\mp}(\lambda, t)\tilde{\psi}_{\mp}(\lambda, n, t), \quad \lambda \in \sigma(H_q),
\]
and the norming constants \( \gamma_{\pm, j}(t) \) corresponding to \( \rho_j \in \sigma_p(H) \) are given by
\[
(5.4) \quad \gamma_{\pm, j}(t)^{-1} = \sum_{n \in \mathbb{Z}} |\tilde{\psi}_{\pm}(\rho_j, n, t)|^2.
\]

To avoid the poles of the Baker-Akhiezer function, we will assume that none of the eigenvalues \( \rho_j \) coincides with a Dirichlet eigenvalue \( \mu_k(0, 0) \). This can be done without loss of generality by shifting the initial time \( t_0 = 0 \) if necessary.

Remark 5.1. Due to this assumption there is no need to remove these poles for the definition of \( \gamma_{\pm, j} \), as we did in [3]. Since the Dirichlet eigenvalues rotate in their gap, the factor needed to remove the poles would only unnecessarily complicate the time evolution of the norming constants. Moreover, these factors would eventually cancel in the Gel’fand-Levitan-Marchenko equation, which is the only interesting object from the inverse spectral point of view in the first place. See Remark 5.4 below.

Lemma 5.2. Let \((a(t), b(t))\) be a solution of the Toda hierarchy such that \( \psi_{\pm}(z, n, t) \) holds. The functions
\[
(5.5) \quad \psi_{\pm}(z, n, t) = \exp(\alpha_{s, \pm}(z, t))\tilde{\psi}_{\pm}(z, n, t)
\]
satisfy
\[
(5.6) \quad H(t)\psi_{\pm}(z, n, t) = z\psi_{\pm}(z, n, t), \quad \frac{d}{dt}\psi_{\pm}(z, n, t) = \hat{P}_{2s+2}(t)\psi_{\pm}(z, n, t).
\]

Proof. We proceed as in [12, Theorem 3.2]. The Jost solutions \( \psi_{\pm}(z, n, t) \) are continuously differentiable with respect to \( t \) by the same arguments as for \( z \) (compare [3, Theorem 4.2]) and the derivatives are equal to the derivatives of the Baker-Akhiezer functions as \( n \to \pm\infty \).

For \( z \in \rho(H) \), the solution \( u_{\pm}(z, n, t) \) of [10] with initial condition \( \psi_{\pm}(z, n, 0) \in \ell_1^2(\mathbb{Z}) \) remains square summable near \( \pm\infty \) for all \( t \in \mathbb{R} \) (see [18] or [11, Lemma 12.16]), that is, \( u_{\pm}(z, n, t) = C_{\pm}(t)\psi_{\pm}(z, n, t) \). Letting \( n \to \pm\infty \) we see \( C_{\pm}(t) = 1 \). The general result for all \( z \in \mathbb{C} \) now follows from continuity.

This implies

\[
(5.7) \quad T(\lambda, t) = T(\lambda, 0), \quad R_{\pm}(\lambda, t) = R_{\pm}(\lambda, 0) \exp(\pm(\alpha_{s, +}(z, t) - \alpha_{s, -}(z, t))), \quad \gamma_{\pm, j}(t) = \gamma_{\pm, j}(0) \exp(2\alpha_{s, \pm}(\rho_j, t)), \quad 1 \leq j \leq q.
\]
Proof. Since \( \alpha_{s,\pm}(\lambda, t) = \alpha_{s,\mp}(\lambda, t) \) by (5.8), we have

\[
T(\lambda, t) = \frac{W(\tilde{\psi}_-(\lambda, t), \psi_-(-\lambda, t))}{W(\psi_-(-\lambda, t), \psi_+(-\lambda, t))} = \exp(\alpha_{s,-}(\lambda, t)) \exp(\alpha_{s,+}(\lambda, t)) \frac{W(\psi_-(-\lambda, t), \psi_-(-\lambda, t))}{W(\psi_-(-\lambda, t), \psi_+(-\lambda, t))} = T(\lambda, 0).
\]

Here we used that the Wronskian \( W_n(f, g) = a(n)(f(n)g(n + 1) - f(n + 1)g(n)) \) of two solutions satisfying \( \gamma \) does not depend on \( n \) or \( t \) (see [16, Lemma 12.15]). Similarly,

\[
R_\pm(\lambda, t) = \frac{\exp(\alpha_{s,\pm}(\lambda, t)) \exp(\alpha_{s,\mp}(\lambda, t)) W(\psi_\mp(\lambda, t), \psi_\mp(\lambda, t))}{\exp(\alpha_{s,\mp}(\lambda, t)) \exp(\alpha_{s,\pm}(\lambda, t)) W(\psi_\pm(\lambda, t), \psi_\pm(\lambda, t))}.
\]

The time dependence of \( \gamma_{\pm,j}(t) \) follows from \( \|\psi_\pm(\rho_j, t)\| = \|\tilde{U}_s(t, 0)\psi_\pm(\rho_j, 0)\| = \|\psi_\pm(\rho_j, 0)\| \).

A few remarks concerning this theorem are in order:

**Remark 5.4.** (i) First of all note that the time dependence of \( R_\pm(\lambda, t) \) is not continuous at the band edges (except the lowest and highest). In fact, the integrand in \( \text{(5.8)} \) has a pole whenever \( \mu_j(0, t) \) hits a band edge. Suppose that \( \mu_j(0, t) \) gets close to \( E_{2j} \) at \( t = t_1 \), then (assuming \( G_s(E_{2j}, 0, t_1) \neq 0 \) for simplicity)

\[
\mu_j(0, t) = E_{2j} + \nu_j^2(t - t_1)^2 + O(t - t_1)^3, \quad \nu_j = \frac{G_s(E_{2j}, 0, t_1) \prod_{l \neq 2j} \sqrt{E_{2j} - E_l}}{\prod_{l \neq 2j} (\mu_j(0, t_1) - \mu_l(0, t_1))}.
\]

As long as \( t < t_1 \) we have

\[
\lim_{\lambda \to E_{2j}} (\alpha_{s,\pm}(\lambda, t) - \alpha_{s,\mp}(\lambda, t)) = 0.
\]

But at \( t = t_1 \) we obtain (setting \( \lambda = E_{2j} + \varepsilon^2 \) and using \( \nu_j = (-1)^{g-j}i|\nu_j| \))

\[
\lim_{\lambda \to E_{2j}} (\alpha_{s,\pm}(\lambda, t) - \alpha_{s,\mp}(\lambda, t)) = \lim_{\lambda \to E_{2j}} 2R_{2g+2}^{1/2}(\lambda) \int_0^{t_1} \frac{G_s(\lambda, 0, x)}{G_s(\lambda, 0, x)} \, dx = \lim_{\varepsilon \to 0} 2 \int_0^{t_1} \frac{\nu_j^2(x) - t_1^2}{\varepsilon^2 - \nu_j^2(x) - t_1^2} \, dx = 2(-1)^{g-j}i \lim_{\varepsilon \to 0} \arctan \left( \frac{t_1|\nu_j|}{\varepsilon} \right) = (-1)^{g-j}i \pi.
\]

This is in agreement with the conditions

\[
\begin{align*}
\lim_{\lambda \to E_l} R_{2g+2}^{1/2}(\lambda) \frac{R_{\pm}(\lambda, t)+1}{T(\lambda)} &= 0, \quad E_l \neq \mu_j(0, t) \\
\lim_{\lambda \to E_l} R_{2g+2}^{1/2}(\lambda) \frac{R_{\pm}(\lambda, t)-1}{T(\lambda)} &= 0, \quad E_l = \mu_j(0, t)
\end{align*}
\]

found in [3].

(ii) The norming constants \( \gamma_{\pm,j}(t) \) will vanish whenever \( (\rho_j, \pm) = \mu_k(0, t) \). This is important, since the norming constants must cancel the pole of \( \tilde{\psi}_{q,\pm}(\rho_j, n, t) \) at these points such that the Gel’fand-Levitan-Marchenko equation remains well-defined.

In fact, these poles come from the normalization \( \tilde{\psi}_{q,\pm}(z, 0, t) = 1 \) and this normalization is natural in the constant background case since it renders \( \tilde{\psi}_{q,\pm}(z, n, t) \) time-independent. In the (quasi-)periodic background case this is no longer true and it seems more natural to allow a time-dependent normalization and replace
and then solve the Gel’fand-Levitan-Marchenko (GLM) equation to obtain (5.12)

The time dependence of the kernel of the Gel’fand-Levitan-Marchenko
Theorem 5.5.

The kernel of the GLM equation which follows from Theorem 5.3.

as in [3]. In this respect the most important ingredient is the time dependence of
the initial condition, we can compute

for the initial condition, we can compute

\( R_{2g+2}(\lambda,0) \) = 1 \( \int_{\lambda \in \pi(H_e)} R_{2g+2}(\lambda,0,\lambda,0,0) d\lambda \)

Finally note that since the transmission coefficient is time independent we obtain
Corollary 5.6. Let \((a(t), b(t))\) be a solution of the Toda hierarchy such that \((5.1)\) holds. Then the quantities

\begin{equation}
A = \prod_{j=-\infty}^{\infty} \frac{a(j, t)}{a_q(j, t)}
\end{equation}

and \(\tau_j = \text{tr}(H^j(t) - H_q(t)^j)\), that is,

\[
\tau_1 = \sum_{n \in \mathbb{Z}} b(n, t) - b_q(n, t)
\]
\[
\tau_2 = \sum_{n \in \mathbb{Z}} 2(a(n, t)^2 - a_q(n, t)^2) + (b(n, t)^2 - b_q(n, t)^2)
\]

etc., are conserved quantities.

Proof. This follows from the fact that the transmission coefficient is the perturbation determinant (in the sense of Krein) of the pair \(H_q\) and \(H\) (see [11]). \(\square\)

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