FULLY NONLINEAR ELLIPTIC EQUATIONS
WITH NON-STRICLTY CONVEX GRADIENT CONSTRAINTS,
AND FULLY NONLINEAR DOUBLE OBSTACLE PROBLEMS

MOHAMMAD SAFDARI

Abstract. We prove the optimal $W^{2,\infty}$ regularity for fully nonlinear elliptic equations with convex gradient constraints. We do not assume any regularity about the constraints; so the constraints need not be $C^1$ or strictly convex. We also show that the optimal regularity holds up to the boundary. Our approach is to show that these elliptic equations with gradient constraints are related to some fully nonlinear double obstacle problems. Then we prove the optimal $W^{2,\infty}$ regularity for the double obstacle problems.

1. Introduction

The study of elliptic equations with gradient constraints was initiated by Evans [14] when he considered the problem

$$\max\{Lu - f, |Du| - g\} = 0,$$

where $L$ is a linear elliptic operator of the form

$$Lu = -a_{ij}D_{ij}^2u + b_iD_iu + cu.$$

Equations of this type stem from dynamic programming in a wide class of stochastic singular control problems. Evans proved $W^{2,p}_{loc}$ regularity for $u$. He also obtained the optimal $W^{2,\infty}_{loc}$ regularity under the additional assumption that $a_{ij}$ are constant. Wiegner [43] removed this additional assumption and obtained $W^{2,\infty}_{loc}$ regularity in general. Later, Ishii and Koike [25] allowed the gradient constraint to be more general, and proved global $W^{2,\infty}$ regularity. We also mention that Shreve and Soner [38, 39] considered similar problems with special structure, and proved the existence of classical solutions.

Yamada [45] allowed the differential operator to be more general, and considered the problem

$$\max_{1 \leq k \leq N} \{L_k u - f_k, |Du| - g\} = 0,$$

where each $L_k$ is a linear elliptic operator. Yamada proved the existence of a solution in $W^{2,\infty}_{loc}$. Recently, there has been new interest in these types of problems. Hynd [20] considered problems with more general gradient constraints of the form

$$\max_{1 \leq k \leq N} \{L_k u - f_k, \tilde{H}(Du)\} = 0,$$
where $\tilde{H}$ is a convex function. He proved $W_{\text{loc}}^{2,\infty}$ regularity when $\tilde{H}$ is strictly convex. Finally, Hynd and Mawi [22] studied fully nonlinear elliptic equations with strictly convex gradient constraints of the form

$$\max\{F(x, D^2 u) - f, \tilde{H}(Du)\} = 0.$$ 

Here $F(x, D^2 u)$ is a fully nonlinear elliptic operator. Hynd and Mawi obtained $W_{\text{loc}}^{2,p} \cap W^{1,\infty}$ regularity in general, and $W_{\text{loc}}^{2,\infty}$ regularity when $F$ does not depend on $x$. Let us also mention that Hynd [19, 21] considered eigenvalue problems for equations with gradient constraints too.

Closely related to the above problems are variational problems with gradient constraints. An important example among them is the famous elastic-plastic torsion problem, which is the problem of minimizing the functional

$$\int_U \frac{1}{2} |Dv|^2 - v \, dx$$

over the set

$$W_{B_1} := \{v \in W^{1,2}_0(U) : |Dv| \leq 1 \text{ a.e.}\}.$$ 

Here $U$ is a bounded open set in $\mathbb{R}^n$. This problem is equivalent to finding $u \in W_{B_1}$ that satisfies the variational inequality

$$\int_U Du \cdot D(v - u) - (v - u) \, dx \geq 0 \quad \text{for every } v \in W_{B_1}.$$ 

An interesting property of variational problems with gradient constraints is that under mild conditions they are equivalent to double obstacle problems. For example, $u$, the minimizer of

$$J[v] := \int_U G(Dv) + g(v) \, dx$$

over $W_{B_1}$, also satisfies $-d \leq u \leq d$, and

$$\begin{cases}
-D_i(D_i G(Du)) + g'(u) = 0 & \text{in } \{-d < u < d\}, \\
-D_i(D_i G(Du)) + g'(u) \leq 0 & \text{a.e. on } \{u = d\}, \\
-D_i(D_i G(Du)) + g'(u) \geq 0 & \text{a.e. on } \{u = -d\},
\end{cases}$$

where $d$ is the Euclidean distance to $\partial U$; see for example [33, 34].

Brezis and Stampacchia [3] proved the $W^{2,p}$ regularity for the elastic-plastic torsion problem. Caffarelli and Rivière [5] obtained its optimal $W_{\text{loc}}^{2,\infty}$ regularity. Gerhardt [16] proved $W^{2,p}$ regularity for the solution of a quasilinear variational inequality subject to the same constraint as in the elastic-plastic torsion problem. Jensen [26] proved $W^{2,p}$ regularity for the solution of a linear variational inequality subject to a $C^2$ strictly convex gradient constraint.
Choe and Shim [8, 9] proved $C^{1,\alpha}$ regularity for the solution to a quasilinear variational inequality subject to a $C^2$ strictly convex gradient constraint, and allowed the operator to be degenerate of the $p$-Laplacian type.

Variational problems with gradient constraints have also seen new developments in recent years. By using infinite dimensional duality, Giuffrè et al. [18] studied the Lagrange multipliers of quasilinear variational inequalities subject to the same constraint as in the elastic-plastic torsion problem. De Silva and Savin [13] investigated the minimizers of some functionals subject to gradient constraints, arising in the study of random surfaces. In their work, the functionals are allowed to have certain kinds of singularities. Also, the constraints are given by convex polygons; so they are not strictly convex. They showed that in two dimensions, the minimizer is $C^1$ away from the obstacles. Choe and Souksomvang [10] generalized the regularity results of [8, 9] by allowing more general constraints.

In [31–34] we have studied the regularity and the free boundary of several classes of variational problems with gradient constraints. Our goal was to understand the behavior of these problems when the constraint is not strictly convex; and we have been able to obtain the optimal $W^{2,\infty}$ regularity for them. This has been partly motivated by the above-mentioned problem about random surfaces. There is also similar interests in elliptic equations with gradient constraints which are not strictly convex. These problems emerge in the study of some stochastic singular control problems appearing in financial models with transaction costs; see for example [2, 30].

In this paper, we obtain a link between double obstacle problems and elliptic equations with gradient constraints. This link has been well known in the case where the double obstacle problem reduces to an obstacle problem. However, we will show that there is still a connection between the two problems in the general case. This connection allows us to obtain the optimal $W^{2,\infty}$ regularity for fully nonlinear elliptic equations which do not depend explicitly on $x$, and are subject to non-strictly convex gradient constraints. It also paves the way for studying more general elliptic equations with such constraints. In this approach, we will also study fully nonlinear double obstacle problems with singular obstacles, and we will obtain the optimal $W^{2,\infty}$ regularity for them. These types of singular obstacles have not studied before, to the best of author’s knowledge. However, see [1, 27] for some recent works on double obstacle problems.

Let us introduce the problem in more detail. Let $K$ be a compact convex subset of $\mathbb{R}^n$ whose interior contains the origin. We recall from convex analysis (see [37]) that the gauge function of $K$ is the convex function

$$(1.2) \quad H_K(x) := \inf\{\lambda > 0 : x \in \lambda K\}.$$ 

The gauge function $H_K$ is subadditive and positively 1-homogeneous, so it looks like a norm on $\mathbb{R}^n$, except that $H_K(-x)$ is not necessarily the same as $H_K(x)$. Note that as $K$ is closed, $K = \{H_K \leq 1\}$; and as $K$ has nonempty interior, $\partial K = \{H_K = 1\}$. 


Another notion is that of the **polar** of $K$

\begin{equation}
K^\circ := \{ x : \langle x, y \rangle \leq 1 \text{ for all } y \in K \},
\end{equation}

where $\langle , \rangle$ is the standard inner product on $\mathbb{R}^n$. $K^\circ$, too, is a compact convex set containing the origin as an interior point.

Let $U \subset \mathbb{R}^n$ be a bounded open set with Lipschitz boundary. Let

\begin{equation}
W_{K^\circ, \varphi} = W_{K^\circ, \varphi}(U) := \{ v \in W^{1,2}(U) : Dv \in K^\circ \text{ a.e., } v = \varphi \text{ on } \partial U \}.
\end{equation}

Here $\varphi : \mathbb{R}^n \to \mathbb{R}$ is a continuous function, and the equality of $v, \varphi$ on $\partial U$ is in the sense of trace. In order to ensure that $W_{K^\circ, \varphi}$ is nonempty we assume that

\begin{equation}
-H_K(y - x) \leq \varphi(x) - \varphi(y) \leq H_K(x - y),
\end{equation}

for all $x, y \in \mathbb{R}^n$. Then by Lemma 2.1 of [41] this property implies that $\varphi$ is Lipschitz and $D\varphi \in K^\circ$ a.e.; so $\varphi \in W_{K^\circ, \varphi}$.

Also let

\begin{equation}
W_{\bar{\rho}, \rho} = W_{\bar{\rho}, \rho}(U) := \{ v \in W^{1,2}(U) : -\bar{\rho} \leq v \leq \rho \text{ a.e., } v = \varphi \text{ on } \partial U \},
\end{equation}

where the obstacles are

\begin{equation}
\rho(x) = \rho_{K, \varphi}(x; U) := \min_{y \in \partial U} [H_K(x - y) + \varphi(y)],
\end{equation}

\begin{equation}
\bar{\rho}(x) = \bar{\rho}_{K, \varphi}(x; U) := \min_{y \in \partial U} [H_K(y - x) - \varphi(y)].
\end{equation}

It is well known (see [28, Section 5.3]) that $\rho$ is the unique viscosity solution of the Hamilton-Jacobi equation

\begin{equation}
\begin{cases}
H_K(Dv) = 1 & \text{ in } U, \\
v = \varphi & \text{ on } \partial U.
\end{cases}
\end{equation}

Now, note that $-K$ is also a compact convex set whose interior contains the origin. We also have $\bar{\rho}_{K, \varphi} = \rho_{-K, -\varphi}$, since $H_{-K}(\cdot) = H_K(- \cdot)$. Thus we have a similar characterization for $\bar{\rho}$ too.

**Notation.** To simplify the notation, we will use the following conventions

\[ H := H_K, \quad H^\circ := H_{K^\circ}, \quad \bar{H} := H_{-K}. \]

Thus in particular we have $\bar{H}(x) = H(-x)$.

In [34] we have shown that $-\bar{\rho} \leq \rho$, and

\begin{equation}
-H_K(x - y) \leq \rho(y) - \rho(x) \leq H(y - x),
\end{equation}

for all $x, y \in \mathbb{R}^n$. The above inequality also holds if we replace $\rho, H$ with $\bar{\rho}, \bar{H}$. Thus in particular, $\rho, \bar{\rho}$ are Lipschitz continuous. We have also shown that $-\bar{\rho}, \rho \in W_{K^\circ, \varphi}(U)$, and

\[ W_{K^\circ, \varphi}(U) \subset W_{\bar{\rho}, \rho}(U). \]
In addition, we showed that $u$, the minimizer of the functional $J$ over $W_{K^o,\varphi}$, is also the minimizer of $J$ over $W_{\bar{\rho},\rho}$. We also proved that under appropriate assumptions $u$ belongs to $W^{2,\infty}$, without requiring any smoothness or strict convexity about the gradient constraint $K^o$.

Motivated by the double obstacle problems arising from variational problems, we are going to study the fully nonlinear double obstacle problem

$$\begin{cases}
F[u] = 0 & \text{a.e. in } \{-\bar{\rho} < u < \rho\}, \\
F[u] \leq 0 & \text{a.e. on } \{u = \rho\}, \\
F[u] \geq 0 & \text{a.e. on } \{u = -\bar{\rho}\},
\end{cases}$$

and employ it to better understand elliptic equations with gradient constraints. Here we have used the convention

$$F[u] := F(x, u, Du, D^2u).$$

**Theorem 1.** Suppose $F$ does not depend on $x$, and satisfies Assumptions A1,2,3. Also suppose $\partial U$ is $C^{\alpha}$ for some $\alpha > 0$; and $\varphi$ is $C^{\alpha}$, and satisfies the assumption (*) in Theorem 3. In addition, suppose there is $v \in C^0(\bar{U}) \cap W^{2,n}_{\text{loc}}(U) \cap W_{\bar{\rho},\rho}(U)$ that satisfies $F[v] \leq 0$ a.e.. Then there is $u \in W^{2,\infty}(U)$ that satisfies the elliptic equation with gradient constraint

$$\begin{cases}
\max\{F(u, Du, D^2u), H^o(Du) - 1\} = 0 & \text{a.e. in } U, \\
u = \varphi & \text{on } \partial U.
\end{cases}$$

**Remark.** Note that if the above equation with gradient constraint has a solution then we must have a subsolution ($F \leq 0$) inside $W_{K^o,\varphi} \subset W_{\bar{\rho},\rho}$. Thus the existence of $v$ is a natural requirement. In particular, note that this requirement is weaker than a corresponding condition in [22], which requires the existence of a “strict” subsolution of (1.11) in $C^2$.

**Remark.** Note that we are not assuming any regularity about $\partial K$ or $\partial K^o$. In particular, $H^o$, which defines the gradient constraint, need not be $C^1$ or strictly convex. Furthermore, note that any convex gradient constraint which does not depend on $x, u$, and specifies a bounded region containing a neighborhood of the origin, can be written in the form $H^o - 1$ for some $K$.

**Proof.** By Theorem 3 there is $u \in W^{2,\infty}(U)$ that satisfies the double obstacle problem (1.10). Then Theorem 2 implies that $u$ must also satisfy the above elliptic equation with gradient constraint.

In contrast to the regularity result of [22], the main difference of our result is that we do not require the gradient constraint to be strictly convex. However, we do not allow $F$ to depend on $x$ (although we allow dependence on $u, Du$). This is mainly because we need the full power of the maximum principle for $Du$ on several occasions, at which mere estimates of $|Du|$ are not sufficient. Another difference is that here we obtain optimal regularity up to
the boundary in addition to local regularity. We should mention that our technique, even in case of local regularity, is inherently global. Because we use the behavior of the obstacles at $\partial U$ in a crucial way. In particular we employ Lemma 3 which is a monotonicity property for $D^2\rho, D^2\bar{\rho}$.

Now let us state our main assumptions about $F$. In the following, $\mathcal{S}^{n \times n}$ denotes the space of symmetric $n \times n$ real matrices.

**Assumption 1.** The function $F(x, z, p, M) : \bar{U} \times \mathbb{R} \times \mathbb{R}^n \times \mathcal{S}^{n \times n} \to \mathbb{R}$ is a $C^1$ function that satisfies

(a) $F$ is uniformly elliptic, i.e. there are constants $\Lambda \geq \lambda > 0$ such that

$$-\Lambda \text{tr}(N) \leq F(x, z, p, M + N) - F(x, z, p, M) \leq -\lambda \text{tr}(N),$$

for all $x \in \bar{U}$, $z \in \mathbb{R}$, $p \in \mathbb{R}^n$, and $M, N \in \mathcal{S}^{n \times n}$ with $N \geq 0$.

(b) For every $K > 0$ there is $c_1 = c_1(K) > 0$ such that

$$F(x, z, p, 0) \leq c_1(1 + |p|^2),$$

$$|F_x|, |F_z| \leq c_1(1 + |p|^2 + |M|),$$

$$|F_p| \leq c_1(1 + |p| + |M|),$$

for all $x \in \bar{U}$, $z \leq K$, $p \in \mathbb{R}^n$, and $M \in \mathcal{S}^{n \times n}$.

(c) $F$ is an increasing function of $z$ for each fixed $(x, p, M)$, i.e. $F_z \geq 0$.

(d) $F$ is a convex function of $M$.

(e) We have

$$F(x, z, p, 0) \text{sign}(z) \geq -c_3(1 + |p|)$$

for all $x \in \bar{U}$, $z \in \mathbb{R}$, $p \in \mathbb{R}^n$, and some constant $c_3 > 0$.

**Assumption 2.** Suppose that $F$ is $C^2$, and for every $K > 0$ there is $c_2 = c_2(K) > 0$ such that

$$|F_{xx}|, |F_{xz}|, |F_{zz}|, |F_{px}|, |F_{pz}|, |F_{zx}|, |F_{zz}| \leq c_2(1 + |M|),$$

for all $x \in \bar{U}$, $|z|, |p| \leq K$, and $M \in \mathcal{S}^{n \times n}$.

**Lemma 1.** Suppose $F$ satisfies Assumption 1, and $u \in C^0(\bar{U}) \cap W_{1,1}^{2,n}(U) \cap W_{\rho,\rho}(U)$ is a solution of the double obstacle problem (1.10). Also suppose that $v \in C^0(\bar{U}) \cap W_{1,1}^{2,n}(U) \cap W_{\rho,\rho}(U)$ satisfies $F[v] \leq 0$ a.e. Then we have

$$v \leq u.$$

As a result we get

$$F[u] = 0 \quad \text{a.e. in } \{-\rho \leq u < \rho\},$$

$$F[u] \leq 0 \quad \text{a.e. on } \{u = \rho\}.$$
Remark. In fact, this lemma is still true if we replace \( \rho, -\bar{\rho} \) by any other upper and lower obstacles which agree on \( \partial U \). We can also replace the 0 on the right hand side by some measurable function \( f \). In addition, parts (d),(e) of Assumption \( \square \) are not needed here.

Proof. Let \( w := v - u \). Then on the open set \( V := \{ u < \rho \} \) we have

\[
0 \geq F[v] - F[u] = \int_0^1 \frac{d}{dt}(F[u + tw])dt = a_{ij}D_{ij}w + b_i D_i w + cw,
\]

where \( a_{ij} := \int_0^1 F_{ij}[u + tw]dt \), \( b_i := \int_0^1 F_{pi}[u + tw]dt \), and \( c := \int_0^1 F_z[u + tw]dt \). Note that on \( \partial V \) we have \( u = \rho \geq v \), so \( w \leq 0 \). Hence by Aleksandrov-Bakelman-Pucci maximum principle (Theorem 9.1 of \(^{[17]}\)) we get

\[
\sup_v w = \sup_{\partial V} w^+ = 0.
\]

Thus \( v - u \leq 0 \) as desired. Finally note that when \( u = -\bar{\rho} \) we have \( u \geq v \geq -\bar{\rho} \), thus \( v = -\bar{\rho} \) too. Therefore we have \( F[u] = F[v] \leq 0 \) a.e. on \( \{ u = -\bar{\rho} \} \). Hence we must have \( F[u] = 0 \) a.e. on \( \{ u = -\bar{\rho} \} \). \( \square \)

**Theorem 2.** Suppose \( F \) does not depend on \( x \), and satisfies Assumptions \(^{[12]}\). Also suppose \( \partial U \) is \( C^1 \), and there is \( v \in C^0(\overline{U}) \cap W^{2,n}_{\text{loc}}(U) \cap W_{\rho,\bar{\rho}}^1(U) \) that satisfies \( F[v] \leq 0 \) a.e.. Let \( u \in C^1(\overline{U}) \cap W^{2,n}_{\text{loc}}(U) \cap W_{\rho,\bar{\rho}}(U) \) be a solution of the double obstacle problem \(^{[1]}\). Then \( u \) also satisfies the elliptic equation with gradient constraint \(^{[11]}\).

Proof. By Lemma \( \square \) we know that

\[
\begin{cases}
F[u] = 0 & \text{a.e. in } \{-\bar{\rho} \leq u < \rho\}, \\
F[u] \leq 0 & \text{a.e. on } \{ u = \rho \}.
\end{cases}
\]

Hence we have \( F[u] \leq 0 \). Also, on \( \{ u = \rho \} \) we have \( Du = D\rho \), since \( u - \rho \) attains its maximum there. But we know that \( H^o(D\rho) = 1 \) a.e. (see \(^{[13]}\)). Therefore when \( H^o(Du) < 1 \) we must have \( F[u] = 0 \) a.e.. Thus we only need to show that \( H^o(Du) < 1 \) a.e. in \( V = \{ u < \rho \} \).

Now note that for any ball \( B \subset V \) there is a \( C^{2,\alpha}(B) \cap C^0(\overline{B}) \) solution of

\[
\begin{cases}
F[w] = 0 & \text{in } B, \\
w = u & \text{on } \partial B,
\end{cases}
\]

as shown in \(^{[12]}\). However, similarly to the proof of Lemma \( \square \) we can show that due to the Aleksandrov-Bakelman-Pucci maximum principle we have \( w = u \) on \( B \). Therefore \( u \) is \( C^{2,\alpha} \) inside \( V \). Thus by Lemma 17.16 of \(^{[17]}\) we have \( u \in C^{3,\alpha}(V) \). Let \( \xi \in \mathbb{R}^n \) be a vector with \( H(\xi) = 1 \), and differentiate the equation \( F[u] = 0 \) to obtain

\[
F_z[u]D_\xi u + F_{pi}[u]D_i D_\xi u + F_{M_{ij}}[u]D_{ij}D_\xi u = 0.
\]

Now on \( \partial V \cap U \) we have \( Du = D\rho \), so \( D_\xi u = D_\xi \rho \leq 1 \) due to \(^{[24]}\). Also on \( \partial V \cap \partial U \) we have \( u = \varphi = \rho \). Therefore \( D_\xi u = D_\xi \rho \leq 1 \) if \( \xi \) is tangent to \( \partial U \). Finally, since \(-\bar{\rho} \leq u \leq \rho \)
in $U$, when $\xi$ is not tangent to $\partial U$ we must have $D_\xi u \leq D_\xi \rho \leq 1$ or $D_\xi u \leq -D_\xi \bar{\rho} \leq 1$. Hence by the maximum principle we have $D_\xi u \leq 1$ in $V$. Thus by (2.2) we get $H(Du) \leq 1$ in $V$, as desired.

We will later need the following additional assumption about $F$ to make sure that $W^{2,p}$ estimates hold for the solutions of the equation $F[u] = 0$. However, any other assumption that gives us the $W^{2,p}$ estimates can also be used instead.

**Assumption 3.** For every $x \in U$ we have $F(x,0,0,0) = 0$. Also, $F$ is uniformly elliptic and Lipschitz, i.e. there are constants $c_4, c_5 > 0$ such that

$$P^-(M - N) - c_4|p - q| - c_5|z - w| \leq F(x,z,p,M) - F(x,w,q,N) \leq P^+(M - N) + c_4|p - q| + c_5|z - w|$$

(1.14)

for all $z, w \in \mathbb{R}$, $p, q \in \mathbb{R}^n$, and $M, N \in S^{n \times n}$. Here $\mathcal{P}^\pm$ are the Pucci operators

$$\mathcal{P}^-(M) := \inf_{\lambda I \leq A \leq \Lambda} \text{tr}(AM), \quad \mathcal{P}^+(M) := \sup_{\lambda I \leq A \leq \Lambda} \text{tr}(AM),$$

and $\lambda, \Lambda > 0$ are the same as in Assumption 1.

The rest of this paper is organized as follows. In Section 2 we review some well-known facts about the regularity of $K$, and its relation to the regularity of $K^\circ, H,H^\circ$. Then we consider the function $\rho$ more carefully. We will review the formulas for the derivatives of $\rho$ that we have obtained in [34], especially the novel explicit formula (2.16) for $D^2\rho$. To the best of author’s knowledge, formulas of this kind have not appeared in the literature before, except for the simple case where $\rho$ is the Euclidean distance to the boundary. (Although, some special two dimensional cases also appeared in our earlier works [31, 35].) One of the main applications of the formula (2.16) for $D^2\rho$ is in the relation (2.17) for characterizing the set of singularities of $\rho$. Another important application is in Lemma 3 which implies that $D^2\rho$ attains its maximum on $\partial U$. This interesting property is actually a consequence of a more general property of the solutions to Hamilton-Jacobi equations (remember that $\rho$ is the viscosity solution of the Hamilton-Jacobi equation (1.8)). This little-known monotonicity property is investigated in [34]; but we included a brief account at the end of Section 2 for reader’s convenience.

In Section 3 we prove the regularity result for double obstacle problem (1.10), aka Theorem 3. Before stating the theorem, let us review some well-known facts from convex analysis. Consider a compact convex set $K$. Let $x \in \partial K$, and $v \in \mathbb{R}^n - \{0\}$. We say the hyperplane

$$\Gamma_{x,v} := \{x + y : \langle y, v \rangle = 0\}$$

(1.15)

is a supporting hyperplane of $K$ at $x$ if $K \subset \{x + y : \langle y, v \rangle \leq 0\}$. In this case we say $v$ is an outer normal vector of $K$ at $x$. The normal cone of $K$ at $x$ is the closed convex cone

$$N(K,x) := \{0\} \cup \{v \in \mathbb{R}^n - \{0\} : v \text{ is an outer normal vector of } K \text{ at } x\}.$$  

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It is easy to see that when $\partial K$ is $C^1$ we have

$$ N(K, x) = \{tDH(x) : t \geq 0\}. $$

For more details see [37, Sections 1.3 and 2.2].

**Theorem 3.** Suppose $F$ does not depend on $x$, and satisfies Assumptions 1, 2, 3. Also suppose $\partial U$ is $C^{2,\alpha}$ for some $\alpha > 0$. In addition, suppose that $\varphi$ is $C^{2,\alpha}$, and satisfies

\begin{equation}
(*) \quad H^\circ(D\varphi) \leq 1; \text{ and if for some } y \in \partial U \text{ we have } H^\circ(D\varphi(y)) = 1 \text{ then we must have } \\
\langle v, \nu(y) \rangle \neq 0,
\end{equation}

for every nonzero $v \in N(K^\circ, D\varphi(y))$.

Then there is $u \in W^{2,\infty}(U) \cap W_{\rho,\rho}(U)$ that satisfies the double obstacle problem (1.10).

**Remark.** Note that we are not assuming any regularity about $\partial K$ or $\partial K^\circ$; so the obstacles can be highly irregular. Also note that if $H^\circ(D\varphi) < 1$ then we do not need to impose any other restriction on $\varphi$. It is also obvious that if $H^\circ(D\varphi) \leq 1$ then we can approximate $\varphi$ with functions that satisfy $H^\circ(D\cdot) < 1$. So, intuitively, most admissible boundary conditions $\varphi$ satisfy the conditions of the theorem.

**Remark.** Let us further elaborate on the restrictions imposed on $D\varphi$, and present a geometric interpretation for it. As we will explain in Subsection 2.2, there is $\lambda > 0$ such that $\mu := D\varphi + \lambda \nu$ satisfies $H^\circ(\mu) = 1$. In addition, for a point $y \in \partial U$, $DH^\circ(\mu)$ is the direction along which lie the points in $U$ that have $y$ as their $\rho$-closest point, i.e. points that satisfy $\rho(\cdot) = H(\cdot - y) + \varphi(y)$. Note that we also have $DH^\circ(\mu) \in N(K^\circ, \mu)$. Now when $H^\circ(D\varphi) = 1$, $D\varphi$ plays the role of $\mu$. And $v \in N(K^\circ, D\varphi)$ plays the role of $DH^\circ(\mu)$. Hence we need to impose the conditions of the theorem in order to be sure that there is a direction along which we can enter $U$ and hit the points whose $\rho$-closest point is $y$.

The idea of the proof of the above theorem is to approximate $K^\circ$ with smoother convex sets. Then, as it is common in the study of the regularity of PDEs, we have to find uniform bounds for the various norms of the approximations $u_k$ to $u$. Here, among other estimations, we will use the fact that the second derivative of the approximations $\rho_k$ to $\rho$ attain their maximums on $\partial U$. We will also use our detailed knowledge of the set of singularities of $\rho_k$ to show that $u_k$ does not touch $\rho_k$ at its singularities (see Proposition 1). Let us finally mention that in order to get the optimal $W^{2,\infty}$ regularity we used the result of Figalli and Shahgholian [15], and its generalizations by Indrei and Minne [23, 24].

At the end, in Appendix A we obtain a standard regularity result for double obstacle problems, which we have used in the article. Here the obstacles are more regular. We also allowed $F$ to explicitly depend on $x$. The penalization method employed in the appendix is classical, but to the best of author’s knowledge the results have not appeared elsewhere. Nevertheless, we include the proofs here for completeness.
2. Preliminaries

First let us introduce some more notation.

(1) \( d(x) := \min_{y \in \partial U} |x - y| \) : the Euclidean distance to \( \partial U \).

(2) \([x, y], [x, y[, [x, y[, ]x, y] : the closed, open, and half-open line segments with endpoints \( x, y \).

(3) We will use the convention of summing over repeated indices.

Remember that a strong solution of a second order equation is a \( W^{2,p} \) function that satisfies the equation a.e. We will also use the notion of viscosity solution, so we are going to review its definition.

**Definition 1.** A continuous function \( u \) is a viscosity solution of \( F[u] = 0 \) if, whenever \( \phi \) is a \( C^2 \) function and \( u - \phi \) has a local maximum at \( x_0 \) we have

\[
F(x_0, u(x_0), D\phi(x_0), D^2\phi(x_0)) \leq 0,
\]

and whenever \( \psi \) is a \( C^2 \) function and \( u - \psi \) has a local minimum at \( x_0 \) we have

\[
F(x_0, u(x_0), D\psi(x_0), D^2\psi(x_0)) \geq 0.
\]

Next let us introduce the following terminology for the solutions of the double obstacle problem (1.10). (The notation is motivated by the physical properties of the elastic-plastic torsion problem, in which \( E \) stands for the elastic region, and \( P \) stands for the plastic region.)

**Definition 2.** Let

\[
P^+ := \{ x \in U : u(x) = \rho(x) \}, \quad P^- := \{ x \in U : u(x) = -\bar{\rho}(x) \}.
\]

Then \( P := P^+ \cup P^- \) is called the coincidence set; and

\[
E := \{ x \in U : -\bar{\rho}(x) < u(x) < \rho(x) \}
\]

is called the non-coincidence set. We also define the free boundary to be \( \partial E \cap U \).

2.1. Regularity of the gauge function. Recall that the gauge function \( H \) satisfies

\[
H(rx) = rH(x), \quad H(x + y) \leq H(x) + H(y),
\]

for all \( x, y \in \mathbb{R}^n \) and \( r \geq 0 \). Also, note that as \( B_c(0) \subseteq K \subseteq B_C(0) \) for some \( C \geq c > 0 \), we have

\[
\frac{1}{C}|x| \leq H(x) \leq \frac{1}{c}|x|,
\]

for all \( x \in \mathbb{R}^n \).

It is well known that for all \( x, y \in \mathbb{R}^n \), we have

(2.1) \( \langle x, y \rangle \leq H(x)H^o(y). \)
In fact, more is true and we have

$$H^\circ(y) = \max_{x \neq 0} \frac{\langle x, y \rangle}{H(x)}.$$  

(2.2)

For a proof of this, see page 54 of [37].

It is easy to see that the strict convexity of $K$ (which means that $\partial K$ does not contain any line segment) is equivalent to the strict convexity of $H$. By homogeneity of $H$, the latter is equivalent to

$$H(x + y) < H(x) + H(y)$$

when $x \neq cy$ and $y \neq cx$ for any $c \geq 0$.

Suppose that $\partial K$ is $C^{k,\alpha}$ ($k \geq 1$, $0 \leq \alpha \leq 1$). Then $H$ is $C^{k,\alpha}$ on $\mathbb{R}^n - \{0\}$ (see for example [34]). Conversely, note that as $\partial K = \{H = 1\}$ and $DH \neq 0$ by (2.3), $\partial K$ is as smooth as $H$. Suppose in addition that $K$ is strictly convex. Then $H$ is strictly convex too. By Remark 1.7.14 and Theorem 2.2.4 of [37], $K^\circ$ is also strictly convex and its boundary is $C^1$. Therefore $H^\circ$ is strictly convex, and it is $C^1$ on $\mathbb{R}^n - \{0\}$. Furthermore, by Corollary 1.7.3 of [37], for $x \neq 0$ we have

$$DH(x) \in \partial K^\circ, \quad DH^\circ(x) \in \partial K,$$

or equivalently

$$H^\circ(DH) = 1, \quad H(DH^\circ) = 1.$$

In particular $DH, DH^\circ$ are nonzero on $\mathbb{R}^n - \{0\}$.

Now assume that $k \geq 2$, and the principal curvatures of $\partial K$ are positive everywhere. Then $K$ is strictly convex. We can also show that $H^\circ$ is $C^{k,\alpha}$ on $\mathbb{R}^n - \{0\}$. To see this, let $n_K : \partial K \rightarrow S^{n-1}$ be the Gauss map, i.e. $n_K(y)$ is the outward unit normal to $\partial K$ at $y$. Then $n_K$ is $C^{k-1,\alpha}$ and its derivative is an isomorphism at the points with positive principal curvatures, i.e. everywhere. Hence $n_K$ is locally invertible with a $C^{k-1,\alpha}$ inverse $n_K^{-1}$, around any point of $S^{n-1}$. Now note that as it is well known, $H^\circ$ equals the support function of $K$, i.e.

$$H^\circ(x) = \sup\{\langle x, y \rangle : y \in K\}.$$  

Thus as shown in page 115 of [37], for $x \neq 0$ we have

$$DH^\circ(x) = n_K^{-1}(\frac{x}{|x|}).$$

Which gives the desired result. As a consequence, $\partial K^\circ$ is $C^{k,\alpha}$ too. Furthermore, as shown on page 120 of [37], the principal curvatures of $\partial K^\circ$ are also positive everywhere.

Let us recall a few more properties of $H, H^\circ$. Since they are positively 1-homogeneous, $DH, DH^\circ$ are positively 0-homogeneous, and $D^2H, D^2H^\circ$ are positively $(-1)$-homogeneous,
\[ H(tx) = tH(x), \quad DH(tx) = DH(x), \quad D^2H(tx) = \frac{1}{t}D^2H(x), \]
\[ H^\circ(tx) = tH^\circ(x), \quad DH^\circ(tx) = DH^\circ(x), \quad D^2H^\circ(tx) = \frac{1}{t}D^2H^\circ(x), \]
\[ \text{for } x \neq 0 \text{ and } t > 0. \]
As a result, using Euler’s theorem on homogeneous functions we get
\[ \langle DH(x), x \rangle = H(x), \quad D^2H(x)x = 0, \]
\[ \langle DH^\circ(x), x \rangle = H^\circ(x), \quad D^2H^\circ(x)x = 0, \]
\[ \text{for } x \neq 0. \]
Here \( D^2H(x)x \) is the action of the matrix \( D^2H(x) \) on the vector \( x \).

Finally let us mention that by Corollary 2.5.2 of [37], when \( x \neq 0 \) the eigenvalues of \( D^2H(x) \) are all positive except for one 0. We have a similar characterization of the eigenvalues of \( D^2H^\circ(x) \).

2.2. Regularity of the obstacles. Next let us consider the obstacles \( \rho, -\bar{\rho} \), and review some of their properties. All the results of this subsection are proved in [34].

Definition 3. When \( \rho(x) = H(x - y) + \phi(y) \) for some \( y \in \partial U \), we call \( y \) a \( \rho \)-closest point to \( x \) on \( \partial U \). Similarly, when \( \bar{\rho}(x) = H(y - x) - \phi(y) \) for some \( y \in \partial U \), we call \( y \) a \( \bar{\rho} \)-closest point to \( x \) on \( \partial U \).

Lemma 2. Suppose \( y \) is one of the \( \rho \)-closest points on \( \partial U \) to \( x \in U \). Then
\( \text{(a) } y \) is a \( \rho \)-closest point on \( \partial U \) to every point of \( ]x, y[ \). Therefore \( \rho \) varies linearly along the line segment \( ]x, y[ \).
\( \text{(b) If in addition, for all } x \neq y \in \mathbb{R}^n \text{ we have} \)
\[ -\gamma(y - x) < \varphi(x) - \varphi(y) < \gamma(x - y), \]
then we also have \( ]x, y[ \subset U \).
\( \text{(c) If in addition } H \text{ is strictly convex, and the strict Lipschitz property (2.6) for } \varphi \text{ holds, then } y \text{ is the unique } \rho \text{-closest point on } \partial U \text{ to the points of } ]x, y[. \)

Next, we generalize the notion of ridge introduced by Ting [40], and Caffarelli and Friedman [4]. Intuitively, the \( \rho \)-ridge is the set of singularities of \( \rho \).

Definition 4. The \( \rho \)-ridge of \( U \) is the set of all points \( x \in U \) where \( \rho(x) \) is not \( C^{1,1} \) in any neighborhood of \( x \). We denote it by
\[ R_\rho. \]
We have shown that when $H$ is strictly convex and the strict Lipschitz property (2.6) for $\varphi$ holds, the points with more than one $\rho$-closest point on $\partial U$ belong to $\rho$-ridge, since $\rho$ is not differentiable at them. This subset of the $\rho$-ridge is denoted by $R_{\rho,0}$.

Similarly we define $R_{\bar{\rho}}, R_{\bar{\rho},0}$.

We know that $\rho, \bar{\rho}$ are Lipschitz functions. We want to characterize the set over which they are more regular. In order to do that, we need to impose some additional restrictions on $K, U$ and $\varphi$.

**Assumption 4.** Suppose that $k \geq 2$ is an integer, and $0 \leq \alpha \leq 1$. We assume that
(a) $K \subset \mathbb{R}^n$ is a compact convex set whose interior contains the origin. In addition, $\partial K$ is $C^{k,\alpha}$, and its principal curvatures are positive at every point.
(b) $U \subset \mathbb{R}^n$ is a bounded open set, and $\partial U$ is $C^{k,\alpha}$.
(c) $\varphi : \mathbb{R}^n \to \mathbb{R}$ is a $C^{k,\alpha}$ function, such that $H^o(D\varphi) < 1$.

**Remark.** As shown in Subsection 2.1 the above assumption implies that $K, H$ are strictly convex. In addition, $K^o, H^o$ are strictly convex, and $\partial K^o, H^o$ are also $C^{k,\alpha}$. Furthermore, the principal curvatures of $\partial K^o$ are also positive at every point. Similar conclusions obviously hold for $-K, -\varphi$ and $( -K)^o = -K^o$ too. Hence in the sequel, whenever we prove a property for $\rho$, it holds for $\bar{\rho}$ too.

Let $\nu$ be the inward unit normal to $\partial U$. Then for every $y \in \partial U$ there is a unique scalar $\lambda(y) > 0$ such that
\[
H^o(D\varphi(y) + \lambda(y)\nu(y)) = 1.
\]
We set
\[
\mu(y) := D\varphi(y) + \lambda(y)\nu(y).
\]
We also set
\[
X := \frac{1}{\langle DH^o(\mu), \nu \rangle} DH^o(\mu) \otimes \nu,
\]
where $a \otimes b$ is the rank 1 matrix whose action on a vector $z$ is $\langle z, b \rangle a$. Let $x \in U$, and suppose $y$ is one of the $\rho$-closest points to $x$ on $\partial U$. Then we have
\[
\frac{x - y}{H(x - y)} = DH^o(\mu(y)).
\]
Or equivalently
\[
x = y + (\rho(x) - \varphi(y)) DH^o(\mu(y)).
\]
Also, $\rho$ is differentiable at $x$ if and only if $x \in U - R_{\rho,0}$. And in that case we have
\[
D\rho(x) = \mu(y),
\]
where \( y \) is the unique \( \rho \)-closest point to \( x \) on \( \partial U \).

In addition, for every \( y \in \partial U \) there is an open ball \( B_r(y) \) such that \( \rho \) is \( C^{k,\alpha} \) on \( \overline{U} \cap B_r(y) \). Furthermore, \( y \) is the \( \rho \)-closest point to some points in \( U \), and we have

\[
D\rho(y) = \mu(y).
\]

We also have

\[
D^2\rho(y) = (I - X^T)(D^2\varphi(y) + \lambda(y)D^2d(y))(I - X),
\]

where \( I \) is the identity matrix, \( d \) is the Euclidean distance to \( \partial U \), and \( X \) is given by (2.9).

**Remark.** As a consequence, \( R_\rho \) has a positive distance from \( \partial U \).

Let \( x \in U - R_{\rho,0} \), and let \( y \) be the unique \( \rho \)-closest point to \( x \) on \( \partial U \). Let

\[
W = W(y) := -D^2H(\mu(y))D^2\rho(y),
\]

\[
Q = Q(x) := I - (\rho(x) - \varphi(y))W,
\]

where \( I \) is the identity matrix. If \( \det Q \neq 0 \) then \( \rho \) is \( C^{k,\alpha} \) on a neighborhood of \( x \). In addition we have

\[
D^2\rho(x) = D^2\rho(y)Q(x)^{-1}.
\]

In addition we have

\[
x \in R_\rho \text{ if and only if } \det Q(x) = 0.
\]

**Remark.** When \( \varphi = 0 \), the function \( \rho \) is the distance to \( \partial U \) with respect to the Minkowski distance defined by \( H \). So this case has a geometric interpretation. An interesting fact is that in this case the eigenvalues of \( W \) coincide with the notion of curvature of \( \partial U \) with respect to some Finsler structure. For the details see [12].

**Lemma 3.** Suppose the Assumption 4 holds. Let \( x \in U - R_\rho \), and let \( y \) be the unique \( \rho \)-closest point to \( x \) on \( \partial U \). Then we have

\[
D^2_{\xi\xi}\rho(x) \leq D^2_{\xi\xi}\rho(y),
\]

for every \( \xi \in \mathbb{R}^n \).

As we mentioned in the introduction, the above monotonicity property is true because \( \rho \) satisfies the Hamilton-Jacobi equation (1.8), and the segment \( [x, y] \) is the characteristic curve associated to it. Let us review the general case of the monotonicity property below.

**Monotonicity of the second derivative of the solutions to Hamilton-Jacobi equations:** Suppose \( v \) satisfies the equation \( \bar{H}(x, v, Dv) = 0 \), where \( \bar{H}(x, z, p) \) is a convex function in all of its arguments. Let \( x(s) \) be a characteristic curve of the equation. Then we have \( \dot{x} = D_\rho \bar{H} \). Let us assume that \( v \) is \( C^3 \) on a neighborhood of the image of \( x(s) \). Let

\[
q(s) := D^2_{\xi\xi}v(x(s)) = \xi_i\xi_jD^2_{ij}v,
\]

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for some vector $\xi$. Then we have
\[
\dot{q} = \xi_i \xi_j D^3_{ijk} v^k = \xi_i \xi_j D^3_{ijk} v^k D_{pk} \tilde{H}.
\]

On the other hand, if we differentiate the equation we get
\[
D_{x_i} \tilde{H} + D_{x_j} \tilde{H} D_{ij} v + D_{pk} \tilde{H} D^2_{ik} v = 0.
\]

And if we differentiate one more time we get
\[
\begin{align*}
D^2_{x_i x_j} \tilde{H} &+ D^2_{x_k x_p} \tilde{H} D^4_{ik} v + D^2_{x_k x_p} \tilde{H} D^4_{jk} v + D^2_{x_k x_p} \tilde{H} D^4_{ij} v \\
&+ D^2_{x_k x_p} \tilde{H} D^4_{jk} v + D^2_{x_k x_p} \tilde{H} D^4_{ij} v + D^2_{x_k x_p} \tilde{H} D^4_{ik} v \\
&+ D^2_{x_k x_p} \tilde{H} D^4_{jk} v + D^2_{x_k x_p} \tilde{H} D^4_{ij} v + D^2_{x_k x_p} \tilde{H} D^4_{ik} v = 0.
\end{align*}
\]

Now if we multiply the above expression by $\xi_i \xi_j$, and sum over $i, j$, we obtain the following Riccati type equation
\[
(2.18) \quad \dot{q} = - \begin{bmatrix} \xi^T \quad \xi \quad \xi T \end{bmatrix} D^2 v \begin{bmatrix} D^2_{xx} \tilde{H} & D^2_{xz} \tilde{H} & D^2_{xp} \tilde{H} \\
D^2_{zx} \tilde{H} & D^2_{zz} \tilde{H} & D^2_{zp} \tilde{H} \\
D^2_{px} \tilde{H} & D^2_{pz} \tilde{H} & D^2_{pp} \tilde{H} \end{bmatrix} \begin{bmatrix} \xi \\
D^2 v \xi \end{bmatrix} - D_z \tilde{H} q
\]
\[
= -\eta^T D^2 \tilde{H} \eta - D_z \tilde{H} q,
\]
where $\eta := \begin{bmatrix} \xi^T \quad \xi \quad \xi T D^2 v \end{bmatrix}^T$. Hence we have $\dot{q} \leq -D_z \tilde{H} q$, since $\tilde{H}$ is convex. Thus by Gronwall’s inequality we obtain
\[
q(s) \leq q(0) e^{- \int_0^s D_z \tilde{H} dr}.
\]

In particular when $D_z \tilde{H} \geq 0$ we have
\[
D^2_{\xi \xi} v(x(s)) = q(s) \leq q(0) = D^2_{\xi \xi} v(x(0)),
\]
as desired.

3. Proof of Theorem 3

In this section we prove Theorem 3 i.e. we will prove that the double obstacle problem (1.10) has a solution $u$ in $W^{2,\infty}$, without assuming any regularity about $K$. To this end, first we need to prove Proposition 1 which says that when $\partial K$ is smooth enough, $u$ does not touch the obstacles $\rho, -\bar{\rho}$ at their singularities. Before that, we need a few preliminary results. Throughout this section we assume that $F$ does not depend on $x$, and satisfies Assumptions 12. We also assume that $\partial U$ is $C^1$, and except in the proof of Theorem 3 we assume that $u \in C^1(\overline{U}) \cap W^{2,\infty}_{\text{loc}}(U) \cap W^2_{\rho,\hat{\rho}}(U)$ is a solution of the double obstacle problem (1.10). Let $E, P^\pm$ be the non-coincidence and coincidence sets of $u$.

Lemma 4. We have
\[
\mathcal{H}(Du) \leq 1.
\]

Proof. The proof is exactly the same as in Theorem 2. We only need to consider the set $\{-\bar{\rho} < u < \rho\}$ instead of $\{u < \rho\}$. $\square$
Proof. Note that $[x,y] \subset U$ by Lemma 2. Suppose $x \in P^-$; the other case is similar. We have

$$u(x) = -\bar{\rho}(x) = -H(y - x) + \varphi(y).$$

Let $w := u - (-\bar{\rho}) \geq 0$, and $\xi := \frac{y - x}{H(y - x)} = -\frac{x - y}{H(x - y)}$. Then $\bar{\rho}$ varies linearly along the segment $[x,y]$, since $y$ is a $\bar{\rho}$-closest point to the points of the segment. So we have $D_\xi(-\bar{\rho}) = D_{-\xi}\bar{\rho} = 1$ along the segment. Note that we do not assume the differentiability of $\bar{\rho}$; and $D_{-\xi}\bar{\rho}$ is just the derivative of the restriction of $\bar{\rho}$ to the segment $[x,y]$. Now since

$$D_\xi u = (Du, \xi) \leq H^\circ(Du)H(\xi) \leq 1,$$

we have $D_\xi w \leq 0$ along $[x,y]$. Thus as $w(x) = w(y) = 0$, and $w$ is continuous on the closed segment $[x,y]$, we must have $w \equiv 0$ on $[x,y]$. Therefore $u = -\bar{\rho}$ along the segment as desired. \qed

Proposition 1. Suppose the Assumption holds, and $u \in W^{2,\infty}_\text{loc}(U)$. Then we have

$$R_\rho \cap P^+ = \emptyset, \quad R_{\bar{\rho}} \cap P^- = \emptyset.$$

Proof. Note that due to Assumption 4, the strict Lipschitz property (2.6) for $\varphi$ holds, and $H$ is strictly convex. First let us show that $R_{\bar{\rho},0} \cap P^- = \emptyset$; the other case is similar. Suppose to the contrary that $x \in R_{\rho,0} \cap P^-$. Then there are at least two distinct points $y, z \in \partial U$ such that

$$\bar{\rho}(x) = H(y - x) - \varphi(y) = H(z - x) - \varphi(z).$$

Now by Lemma 2, we have $[x,y], [x,z] \subset P^-$. In other words, $u = -\bar{\rho}$ on both of these segments. Therefore by Lemma 2, $u$ varies linearly on both of these segments. Hence we get

$$\langle Du(x), \frac{y - x}{H(y - x)} \rangle = 1 = \langle Du(x), \frac{z - x}{H(z - x)} \rangle.$$

However since $H$ is strictly convex, this contradicts the fact that $H^\circ(Du(x)) \leq 1$. So we only need to show that $R_\rho - R_{\rho,0}, R_{\bar{\rho}} - R_{\bar{\rho},0}$ do not intersect $P^+, P^-$ respectively. Suppose to the contrary that there is a point $x \in U$ which belongs to $(R_\rho - R_{\rho,0}) \cap P^+$; the other case is similar. Let $y$ be the unique $\rho$-closest point to $x$ on $\partial U$. Then we must have $\det Q(x) = 0$, where $Q$ is given by (2.13). Let $z \in [x,y]$. Then by Lemma 2, we have $z \in U$, and $y$ is the unique $\rho$-closest point on $\partial U$ to $z$. In addition, as proved in [34], we have $\det Q(z) \neq 0$. Hence $\rho$ is $C^{k,\alpha}$ on a neighborhood of the line segment $[x,y]$. We call this neighborhood $V$. In the proof of Theorem 4 of [34], it has been shown that there is a vector $\xi$ with $|\xi| = 1$, which is not parallel to the segment $[x,y]$, such that

$$D_\xi^2\rho(z) \rightarrow -\infty \quad \text{as } z \rightarrow x.$$

Here $z$ converges to $x$ along the segment $[x,y]$. 

Lemma 5. Suppose that the strict Lipschitz property (2.6) for $\varphi$ holds. Then if $x \in P^+$, and $y$ is a $\bar{\rho}$-closest point on $\partial U$ to $x$, we have $[x,y] \subset P^+$. Similarly, if $x \in P^-$, and $y$ is a $\rho$-closest point on $\partial U$ to $x$, we have $[x,y] \subset P^-$.
Now since \( x \in P^+ \) we have \( u(x) = \rho(x) \). Hence by lemma \[5\] we have \([x, y] \subseteq P^+\). Thus \( u(z) = \rho(z) \) for every \( z \in [x, y] \). Also remember that \( u \leq \rho \) everywhere, since \( u \in W_{\rho, \rho'} \). Hence \( \rho - u \) is a \( C^1 \) function on \( V \), which attains its maximum, 0, on \([x, y]\). Thus \( Du = D\rho \) on the segment \([x, y]\). Next we claim that for any \( z \in [x, y] \) there are points \( z_i := z + \varepsilon_i \xi \) in \( V \) converging to \( z \), at which we have

\[
D_{\xi} u(z_i) \leq D_{\xi} \rho(z_i).
\]

Since otherwise we would have \( D_{\xi} u > D_{\xi} \rho \) on a segment of the form \([z, z + r\xi]\), for some small \( r > 0 \). But as \( u(z) = \rho(z) \) and \( Du(z) = D\rho(z) \), this implies that \( u > \rho \) on \([z, z + r\xi]\); which is a contradiction. Thus we get the desired. As a consequence we have

\[
D_{\xi} u(z_i) - D_{\xi} u(z) \leq D_{\xi} \rho(z_i) - D_{\xi} \rho(z).
\]

By applying the mean value theorem to the restriction of \( \rho \) to the segment \([z, z_i] \), we get

\[
D_{\xi} u(z_i) - D_{\xi} u(z) \leq |z_i - z| D_{\xi}^2 \rho(w_i),
\]

for some \( w_i \in [z, z_i] \).

On the other hand, \( u \) is a \( W^{2,\infty} \) function on a neighborhood of \( x \) by our assumption. Consequently there is \( C > 0 \) such that

\[
-C \leq \frac{D_{\xi} u(z_i) - D_{\xi} u(z)}{|z_i - z|},
\]

for distinct \( z, z_i \) sufficiently close to \( x \). Now let \( z \in [x, y] \) be close enough to \( x \) so that \( D_{\xi}^2 \rho(z) < -3C \), which is possible due to \[3.1\]. Then let \( z_i = z + \varepsilon_i \xi \) be close enough to \( z \) so that we have \( D_{\xi}^2 \rho(w_i) < -2C \), which is possible due to the continuity of \( D^2 \rho \) on \( V \). But this is in contradiction with \[3.2\] and \[3.3\]. \[\square\]

We do not use the next proposition directly in the proof of Theorem \[3\] however, it completes our understanding of the relation between double obstacle problems and gradient constraints. The proposition says that \( u \) hits the gradient constraint, i.e. \( H^\circ(Du) = 1 \), exactly when it hits one of the obstacles \(-\bar{\rho}, \rho\).

**Proposition 2.** Suppose that the strict Lipschitz property \[2.6\] for \( \varphi \) holds, and \( H \) is strictly convex. Then we have

\[
P = \{ x \in U : H^\circ(Du(x)) = 1 \}, \quad E = \{ x \in U : H^\circ(Du(x)) < 1 \}.
\]

**Proof.** First suppose \( x \in P^- \); the case of \( P^+ \) is similar. Then we have

\[
u(x) = -\bar{\rho}(x) = -H(y - x) + \varphi(y),
\]

for some \( y \in \partial U \). Thus by Lemma \[5\] \( u = -\bar{\rho} \) along the segment \([x, y]\). We also know that \( \bar{\rho} \) varies linearly along the segment \([x, y]\), since \( y \) is a \( \bar{\rho} \)-closest point to the points of the segment. Hence we have \( D_{\xi} u(x) = 1 \) for \( \xi := \frac{y - x}{H(y - x)} \). Therefore \( H^\circ(Du(x)) \) cannot be less than 1 due to the equation \[2.2\].
Next, assume that $H^\circ(Du(x)) = 1$. Then by (2.2), there is $\tilde{\xi}$ with $H(\tilde{\xi}) = 1$ such that $D\xi u(x) = 1$. Suppose to the contrary that $x \in E$, i.e. $-\bar{\rho}(x) < u(x) < \rho(x)$. As shown in the proof of Theorem 2 we know that $D\xi u$ is $C^{2,\alpha}$ in $E$ and satisfies the elliptic equation

$$F_\xi[u]D\xi u + F_p[u]D_1D\xi u + F_{M_{ij}}[u]D_{ij}^2D\xi u = 0.$$ 

On the other hand on $\overline{U}$ we have

$$D\xi u = \langle Du, \tilde{\xi} \rangle \leq H^\circ(Du)H(\tilde{\xi}) \leq 1.$$

Let $E_1$ be the connected component of $E$ that contains $x$. Then the strong maximum principle implies that $D\xi u \equiv 1$ over $E_1$.

Now consider the line passing through $x$ in the $\tilde{\xi}$ direction, and suppose it intersects $\partial E_1$ for the first time in $y := x - \tau\tilde{\xi}$ for some $\tau > 0$. If $y \in \partial U$, then for $t > 0$ we have

$$\frac{d}{dt}[u(y + t\tilde{\xi})] = D\xi u(y + t\tilde{\xi}) = 1 = \frac{d}{dt}[H(\tilde{\xi})] = \frac{d}{dt}[H(y + t\tilde{\xi} - y)].$$

Thus as $u(y) = \varphi(y)$, we get $u(x) = u(y + \tau\tilde{\xi}) = H(x - y) + \varphi(y) \geq \rho(x)$; which is a contradiction. Now if $y \not\in U$, then as it also belongs to $\partial E$ we have $y \in P$. If $u(y) = \rho(y) = H(y - \tilde{y}) + \varphi(\tilde{y})$ for some $\tilde{y} \in \partial U$, similarly to the above we obtain

$$u(x) = H(x - y) + u(y) = H(x - y) + H(y - \tilde{y}) + \varphi(\tilde{y}) \geq H(x - \tilde{y}) + \varphi(\tilde{y}) \geq \rho(x),$$

which is again a contradiction.

On the other hand, if $u(y) = -\bar{\rho}(y) = -H(\tilde{y} - y) + \varphi(\tilde{y})$ for some $\tilde{y} \in \partial U$, then by Lemma 3 we have $u = -\bar{\rho}$ on the segment $[y, \tilde{y}]$; and consequently $D\xi u(y) = 1$, where $\hat{\xi} := \frac{\tilde{y} - y}{H(\tilde{y} - y)}$. Since $H$ is strictly convex we must have $\hat{\xi} = \tilde{\xi}$. Therefore $x, y, \tilde{y}$ are collinear, and $x, \tilde{y}$ are on the same side of $y$. But $\tilde{y}$ cannot belong to $[y, x] \subset E_1 \subset E \subset U$. Hence we must have $x \in [y, \tilde{y}] \subset P^-$, which means $u(x) = -\bar{\rho}(x)$; and this is a contradiction.

Before presenting the proof of Theorem 3 let us note that the assumptions of the theorem also hold when we replace $K, \varphi, K^\circ$ by $-K, -\varphi$ and $(-K)^{\circ} = -K^\circ$. In particular notice that if $D\varphi \in \partial K^\circ$, i.e. if $H^\circ(D\varphi(y)) = 1$, then we have $-D\varphi \in -\partial K^\circ = \partial(-K^\circ)$; and vice versa. In addition, it is easy to see that

$$v \in N(K^\circ, D\varphi(y)) \iff -v \in N(-K^\circ, -D\varphi(y)).$$

So as a result, $\rho, \bar{\rho}$ will have the same properties.

**Proof of Theorem 3**. As it is well known, a compact convex set with nonempty interior can be approximated, in the Hausdorff metric, by a shrinking sequence of compact convex sets with nonempty interior which have smooth boundaries with positive curvature (see for
example [36]). We apply this result to $K^\circ$. Thus there is a sequence $K_k^\circ$ of compact convex sets, that have smooth boundaries with positive curvature, and

$$K_{k+1}^\circ \subset \text{int}(K_k^\circ), \quad K^\circ = \bigcap K_k^\circ.$$  

Notice that we can take the approximations of $K^\circ$ to be the polar of other convex sets, because the double polar of a compact convex set with 0 in its interior is itself. Also note that $K_k$‘s are strictly convex compact sets with 0 in their interior, which have smooth boundaries with positive curvature. Furthermore we have $K = (K^\circ)^\circ \supset K_{k+1} \supset K_k$. For the proof of these facts see [37, Sections 1.6, 1.7 and 2.5].

To simplify the notation we use $H_k, H_k, \rho_k, \bar{\rho}_k$ instead of $H_{K_k}, H_{K_k^\circ}, \rho_{K_k}, \varphi, \bar{\rho}_{K_k}, \varphi$, respectively. Note that $K_k, U, \varphi$ satisfy the Assumption 4. In particular we have $H_k^\circ(D\varphi) < 1$, since $D\varphi \in K^\circ \subset \text{int}(K_k^\circ)$. Hence as we have shown in [34], $\rho_k, \bar{\rho}_k$ satisfy the Assumption 5. In addition, they are $C^{2,\alpha}$ on a neighborhood of $\partial U$. Thus by Theorem 4 in the appendix, there is $u_k \in W_{\rho_k, \bar{\rho}_k}(U) \cap W^{2,\infty}(U)$ that satisfies the double obstacle problem

$$
\begin{align*}
F[u_k] &= 0 \quad \text{a.e. in } \{-\bar{\rho}_k < u_k < \rho_k\}, \\
F[u_k] &\leq 0 \quad \text{a.e. on } \{u_k = \rho_k\}, \\
F[u_k] &\geq 0 \quad \text{a.e. on } \{u_k = -\bar{\rho}_k\}.
\end{align*}
$$

Therefore the lemmas and propositions of this section, especially Proposition 4 hold for each $u_k$. (This is our only use of the assumptions that $F$ is $C^2$ and does not depend on $x$. In the rest of the proof, we do not use these assumptions directly.) Also we know that

$$(3.4) \quad -\bar{\rho}_1 \leq -\bar{\rho}_k \leq u_k \leq \rho_k \leq \rho_1.$$  

Note that $\rho_k \leq \rho_1$ and $\bar{\rho}_k \leq \bar{\rho}_1$, since $H_k \leq H_1$ due to $K_k \supset K_1$.

We divide the rest of this proof into four parts. In Part I we derive the uniform bound (3.5). In Part II we show that $u$ is a $W^{2,p}$ solution of (1.10). In Part III we show that $u$ is in $W^{2,\infty}_{\text{loc}}$. And in Part IV we prove that the regularity of $u$ holds up to the boundary.

PART I:

Let $R_k$ be the $\rho_k$-ridge, and let $E_k, P_k^\pm$ be the non-coincidence and coincidence sets of $u_k$. Let us show that

$$(3.5) \quad \|F[u_k]\|_{L^\infty(U)} = \|F(u_k, Du_k, D^2 u_k)\|_{L^\infty(U)} \leq C,$$

for some $C$ independent of $k$. To see this, note that on $E_k$ we have $F[u_k] = 0$. So the desired bound trivially holds on $E_k$. Next consider $P_k^+$. We have

$$F[u_k] \leq 0 \quad \text{a.e. on } P_k^+.$$  

Thus we have an upper bound for $F[u_k]$ on $P_k^+$, independently of $k$.

On the other hand, since $P_k^+$ does not intersect $R_k$ due to Proposition 4 $\rho_k$ is at least $C^2$ on $P_k^+$. Now as $u_k = \rho_k$ on $P_k^+$, for a.e. $x \in P_k^+$ we have $Du_k(x) = D\rho_k(x)$ and
\(D^2 u_k(x) = D^2 \rho_k(x)\). On the other hand, by Lemma 3 we know that \(D^2 \rho_k(x) \leq D^2 \rho_k(y)\), where \(y\) is the \(\rho_k\)-closest point on \(\partial U\) to \(x\). Hence by the ellipticity of \(F\) we have
\[
F(u_k(x), Du_k(x), D^2 u_k(x)) = F(\rho_k(x), D\rho_k(x), D^2 \rho_k(x)) \geq F(\rho_k(x), D\rho_k(x), D^2 \rho_k(y)).
\]

Now note that \(\rho_k\) is uniformly bounded due to (3.4), and \(D\rho_k\) is uniformly bounded since \(D\rho_k \in K^*_k \subset K^*_1\). Thus in order to show that \(F[u_k]\) has a uniform lower bound on \(P^+_k\), we only need to show that \(D^2 \rho_k\) is bounded on \(\partial U\) independently of \(k\). This has been proved in the proof of Theorem 5 of [34]. Here, the part (*) of the assumptions of the theorem is needed. Similarly, we can show that \(F[u_k]\) is bounded on \(P^-_k\), independently of \(k\). Hence we obtain the desired bound (3.5).

**PART II:**

Now let \(f_k := F(u_k, Du_k, D^2 u_k)\). Then \(u_k\) is a strong solution to the fully nonlinear elliptic equation
\[
F(u_k, Du_k, D^2 u_k) = f_k, \quad u_k|_{\partial U} = \varphi.
\]

Thus by \(W^{2,p}\) estimates for fully nonlinear elliptic equations (see for example Theorem 4.5 of [44]) we have
\[
\|u_k\|_{W^{2,p}(U)} \leq C\left(\|f_k\|_{L^p(U)} + \|\varphi\|_{C^2(\overline{U})} + \|u_k\|_{L^\infty(U)}\right)
\]
for some constant \(C\) independent of \(k\).

Therefore \(u_k\) is a bounded sequence in \(W^{2,p}(U)\) due to (3.5) and (3.4). Consequently for every \(\bar{\alpha} < 1\), \(\|u_k\|_{C^{1,\alpha}(\overline{U})}\) is bounded independently of \(k\), because \(\partial U\) is \(C^2\). Hence there is a subsequence of \(u_k\), which we still denote by \(u_k\), that is strongly convergent in \(C^1(\overline{U})\), and weakly convergent in \(W^{2,p}(U)\). We call the limit \(u\). Note that \(u\) belongs to \(W^{2,p}(U)\) for every \(p < \infty\). Furthermore we have \(u \in W_{p,\rho}\) because of (3.4), and the fact that \(\rho_k, \bar{\rho}_k\) uniformly converge to \(\rho, \bar{\rho}\) respectively. Now note that \(u_k\) is a strong solution of the equation
\[
\max\{\min\{F[u_k], u_k + \bar{\rho}_k\}, u_k - \rho_k\} = 0.
\]

Hence \(u_k\) is also a viscosity solution of the above equation (see [29]). Therefore \(u\) is a viscosity solution of the equation
\[
\max\{\min\{F[u], u + \bar{\rho}\}, u - \rho\} = 0,
\]
due to the stability of viscosity solutions (see [11]).

Let us show that \(u\) is also a strong solution of the equation (3.7). We know that for a.e. \(x_0 \in U\) we have
\[
u(x_0 + h) = u(x_0) + \langle Du(x_0), h \rangle + \frac{1}{2} \langle D^2 u(x_0)h, h \rangle + o(|h|^2),
\]
for small $h \in \mathbb{R}^n$ (see for example Proposition 2.2 of [6]). Now let
$$
\phi(h) = u(x_0) + \langle Du(x_0), h \rangle + \frac{1}{2}((D^2u(x_0) + \varepsilon I)h, h),
$$
for some $\varepsilon > 0$. Then $\phi$ is a $C^2$ function and $u - \phi$ has a local maximum at $x_0 \in U$. Hence at $x_0$ we must have
$$
\max\{\min\{F(u, D\phi, D^2\phi), u + \bar{\rho}\}, u - \rho\} \leq 0.
$$
Thus at $x_0$ we have
$$
\max\{\min\{F(u, Du, D^2u + \varepsilon I), u + \bar{\rho}\}, u - \rho\} \leq 0.
$$
Therefore by sending $\varepsilon \to 0$ we get $\max\{\min\{F[u], u + \bar{\rho}\}, u - \rho\} \leq 0$ due to the continuity of $F$. Similarly we can show that $\max\{\min\{F[u], u + \bar{\rho}\}, u - \rho\} \geq 0$. Thus $u$ is a strong solution of (3.7) as desired. However, this means that $u$ satisfies the double obstacle problem (1.10).

PART III:

Finally let us show that $u$ belongs to $W^{2,\infty}(U)$. We start by showing that $u$ belongs to $W^{2,\infty}_{loc}(U)$. But first we need to prove that $D^2u_k$ is bounded on $P_k$ independently of $k$. To see this, consider $P_k^+$; the other case is similar. We know that for a.e. $x \in P_k^+$ we have $D^2u_k(x) = D^2\rho_k(x)$, due to Proposition 4. Also, as we mentioned in Part I of the proof, $D^2\rho_k$ is bounded on $\partial U$ independently of $k$. Hence by Lemma 5 when $y$ is the $\rho_k$-closest point on $\partial U$ to $x \in P_k^+$ we have
$$
(3.8) \quad D^2u_k(x) = D^2\rho_k(x) \leq D^2\rho_k(y) \leq \tilde{C}I,
$$
for some $\tilde{C}$ independent of $k$. Thus $\tilde{C} I - D^2u_k \geq 0$ a.e. on $P_k^+$. Therefore by the uniform ellipticity of $F$ we have
$$
-\Lambda \text{tr}(\tilde{C} I - D^2u_k) \leq F(u_k, Du_k, D^2u_k + \tilde{C} I - D^2u_k) - F(u_k, Du_k, D^2u_k) \leq -\lambda \text{tr}(\tilde{C} I - D^2u_k).
$$
However, we know that $F(u_k, Du_k, D^2u_k)$ is uniformly bounded due to (3.5), and $F(u_k, Du_k, \tilde{C} I)$ is bounded due to the uniform boundedness of $u_k, Du_k$ (remember that $u_k$ is strongly convergent in $C^1$). Therefore $\text{tr}(\tilde{C} I - D^2u_k) = n\tilde{C} - \Delta u_k$ is uniformly bounded. Now let $\xi, \xi_1, \ldots, \xi_{n-1}$ be an orthonormal basis of $\mathbb{R}^n$. Then by (3.8) we have
$$
D^2_{\xi\xi}u_k = \Delta u_k - \sum_{j=1}^{n-1} D^2_{\xi_j,\xi_j}u_k \geq \Delta u_k - (n - 1)\tilde{C}.
$$
Hence $D^2u_k$ is also bounded below on $P_k^+$ independently of $k$. The case of $P_k^-$ can be treated similarly.

Now let $x_0 \in U$, and suppose that $B_r(x_0) \subset U$. Set $v_k(y) := u_k(x_0 + ry)$ for $y \in B_1(0)$. Let
$$
\tilde{F}(z, p, M) := F(z, \frac{1}{r}p, \frac{1}{r^2}M) - F(z, \frac{1}{r}p, 0).
$$
Then by (1.10), and the arguments of the above paragraph, we have
\[
\begin{cases}
\tilde{F}(v_k, Du_k, D^2 u_k) = \tilde{f}_k & \text{a.e. in } B_1(0) \cap \Omega_k, \\
|D^2 u_k| \leq C & \text{a.e. in } B_1(0) - \Omega_k,
\end{cases}
\]
for some $C$ independent of $k$. Here $\Omega_k := \{ y \in B_1(0) : u_k(x_0 + ry) \in E_k \}$, and
\[
\tilde{f}_k := -F(v_k, \frac{1}{r}Du_k, 0).
\]
Next recall that $\|u_k\|_{W^{2,n}(B_r(x_0))}$ is bounded independently of $k$ due to (3.6), (3.5). Therefore $\|v_k\|_{W^{2,n}(B_r(0))}$ is bounded independently of $k$ too. Also note that $\|\tilde{f}_k\|_{C^2(B_r(0))}$ are bounded independently of $k$, since $\|u_k\|_{C^{1,\alpha}(\overline{U})}$ is bounded independently of $k$. Thus we can apply the result of [23] to deduce that
\[
|D^2 v_k| \leq C \quad \text{a.e. in } B_{\frac{1}{2}}(0),
\]
for some $\tilde{C}$ independent of $k$. Therefore
\[
|D^2 u_k| \leq \tilde{C} \quad \text{a.e. in } B_{\frac{1}{2}}(x_0),
\]
for some $\tilde{C}$ independent of $k$. Hence $u_k$ is a bounded sequence in $W^{2,\infty}(B_{\frac{1}{2}}(x_0))$. Therefore a subsequence of them converges weakly star in $W^{2,\infty}(B_{\frac{1}{2}}(x_0))$. But the limit must be $u$; so we get $u \in W^{2,\infty}(B_{\frac{1}{2}}(x_0))$, as desired.

**PART IV:**

Next let $x_0 \in \partial U$. Let $\Phi$ be a $C^{2,\alpha}$ change of coordinates on a neighborhood of $x_0$, that flattens $\partial U$ around $x_0$. More specifically, we assume that $\Phi : x \mapsto y$ maps a neighborhood of $x_0$ onto a neighborhood of 0 that contains $\overline{B}_1(0)$, and the $\Phi$-image of $U$, $\partial U$ lie respectively in the half-space $\{ y_n > 0 \}$ and on the plane $\{ y_n = 0 \}$. Let $\Psi$ be the inverse of $\Phi$. Then we have $y = \Phi(x)$ and $x = \Psi(y)$. Let $B_1^+ := B_1(0) \cap \{ y_n > 0 \}$ and $B_1^- := B_1(0) \cap \{ y_n = 0 \}$. Now set
\[
\hat{u}_k(y) := u_k(\Psi(y)) - \varphi(\Psi(y)) = u_k(x) - \varphi(x).
\]
It is obvious that $\hat{u}_k = 0$ on $B_1^-$. We also have $\hat{u}_k \in W^{2,n}(B_1^+) \cap C^{1}(\overline{B_1^+})$ (see [17, Section 7.3]). In addition we have
\[
\begin{align*}
D\hat{u}_k(y) &= (Du_k(x) - D\varphi(x))D\Psi(y), \\
D^2\hat{u}_k(y) &= (D^2u_k(x) - D^2\varphi(x))D\Psi(y)D\Psi(y) + (Du_k(x) - D\varphi(x))D^2\Psi(y).
\end{align*}
\]
Therefore we get
\[
\|\hat{u}_k\|_{W^{2,n}(B_1^+)} \leq C(\|u_k\|_{W^{2,n}(U)} + \|\varphi\|_{C^2(\overline{U})}),
\]
for some $C$ independent of $k$. Hence $\|\hat{u}_k\|_{W^{2,n}(B_1^+)}$ is bounded independently of $k$, due to (3.6), (3.5).
Now let
\[ \hat{F}(y, z, p, M) := F(z + \varphi, pD\Phi + D\varphi, MD\Phi + D^2\varphi + pD^2\Phi) - F(z + \varphi, pD\Phi + D\varphi, D^2\varphi + pD^2\Phi), \]
where \( \varphi, \Phi \) are computed at \( x = \Psi(y) \). Note that by differentiating the equality \( \Psi \circ \Phi = \text{id} \) we get \( D\Psi D\Phi = I \), and \( D\Psi D^2\Phi D\Psi + D^2\Psi D\Phi = 0 \). Hence by (3.9) we can easily check that
\[
(3.10) \quad \hat{F}[\tilde{u}_k] = F[u_k] - F(u_k, Du_k, D^2\varphi - (Du_k - D\varphi)D\Psi D^2\Phi).
\]
It is also easy to see that \( \hat{F} \) is uniformly elliptic, Holder continuous, and convex in \( M \); and satisfies \( \hat{F}(y, z, p, 0) = 0 \).

Let \( \Omega_k := \{ y \in B_1^+ : \Psi(y) \in E_k \} \). Then \( D^2\tilde{u}_k \) is bounded on \( B_1^+ - \Omega_k := \{ y \in B_1^+ : \Psi(y) \in P_k \} \) independently of \( k \) due to (3.9); because \( D^2u_k \) is bounded on \( P_k \) independently of \( k \), and \( Du_k \) is bounded independently of \( k \). Therefore by (1.10) and (3.10) we have
\[
\begin{align*}
\hat{F}[\tilde{u}_k] &= \hat{f}_k \quad \text{a.e. in } B_1^+ \cap \Omega_k, \\
|D^2\tilde{u}_k| &\leq C \quad \text{a.e. in } B_1^+ - \Omega_k, \\
u &= 0 \quad \text{on } B_1',
\end{align*}
\]
for some \( C \) independent of \( k \). Here
\[
\hat{f}_k := -F(u_k, Du_k, D^2\varphi - (Du_k - D\varphi)D\Psi D^2\Phi).
\]
Note that \( \hat{f}_k \in C^{\alpha_0}(\overline{B_1^+}) \) for some \( \alpha_0 > 0 \), and \( \| \hat{f}_k \|_{C^{\alpha_0}(\overline{B_1^+})} \) is bounded independently of \( k \); since \( \| u_k \|_{C^{1,\alpha}(\overline{\Omega})} \) is bounded independently of \( k \), for every \( \alpha < 1 \). Hence as shown in [23, 24] we get
\[
|D^2\tilde{u}_k| \leq \tilde{C} \quad \text{a.e. in } \overline{B_1^+}(0) \cap \{ y_n > 0 \},
\]
for some \( \tilde{C} \) independent of \( k \). Thus
\[
|D^2u_k| \leq \tilde{C} \quad \text{a.e. in } B_r(x_0) \cap U,
\]
for some \( r > 0 \) and some \( \tilde{C} \) independent of \( k \); because we can compute the derivatives of \( u_k \) in terms of the derivatives of \( \tilde{u}_k \) similarly to (3.9).

Hence \( u_k \) is a bounded sequence in \( W^{2,\infty}(B_r(x_0) \cap U) \). Therefore a subsequence of them converges weakly star in \( W^{2,\infty}(B_r(x_0) \cap U) \). But the limit must be \( u \); so we get \( u \in W^{2,\infty}(B_r(x_0) \cap U) \). Finally note that we can cover \( \partial U \) with finitely many open balls of the form \( B_r(x_0) \) for \( x_0 \in \partial U \), over which \( u \) is \( W^{2,\infty} \). Also, there is an open subset of \( U \) whose union with these balls cover \( U \), and over it \( u \) is \( W^{2,\infty} \) too. Thus we can conclude that \( u \in W^{2,\infty}(U) \), as desired. \( \square \)
APPENDIX A. FULLY NONLINEAR DOUBLE OBSTACLE PROBLEMS

In this appendix we are going to study the general double obstacle problem

\[
\begin{align*}
F[u] &= 0 \quad \text{a.e. in } \{\psi^- < u < \psi^+\}, \\
F[u] &\leq 0 \quad \text{a.e. on } \{u = \psi^+\}, \\
F[u] &\geq 0 \quad \text{a.e. on } \{u = \psi^-\},
\end{align*}
\]  

(A.1)

where \( u \) belongs to

\[
W_{\psi \pm} := \{v \in W^{1,2}(U) : \psi^- \leq v \leq \psi^+ \text{ a.e.}\}.
\]

Here we allow \( F \) to also depend on \( x \). We also let the obstacles to be more general than \( \rho, -\bar{\rho} \), but we require their weak second derivatives to have one-sided bounds. We show that the solution \( u \) has the optimal \( W^{2,\infty} \) regularity. This result has been used in the proof of Theorem \( \ref{thm:main} \). Most of the methods employed in this section are classical and well known, but to the best of author’s knowledge the results have not appeared elsewhere. Especially since the results are about the double obstacle problem, and there are far fewer works on this problem compared to the obstacle problem. Nevertheless, we include the proofs here for completeness. First let us state our assumptions about the obstacles \( \psi^\pm \).

**Assumption 5.** We assume that \( \psi^\pm : \mathbb{R}^n \to \mathbb{R} \) are Lipschitz functions which satisfy

(a) For every \( x, y \in \mathbb{R}^n \) we have

\[
|\psi^\pm(x) - \psi^\pm(y)| \leq C_1|x - y|.
\]

(b) \( \psi^+ = \psi^- \) on \( \partial U \), and for all \( x \notin \partial U \) we have

\[
0 < \psi^+(x) - \psi^-(x) \leq 2C_1d(x),
\]

(A.2)

where \( d \) is the Euclidean distance to \( \partial U \).

(c) We have

\[
\pm \nabla^2_{h,\xi} \psi^\pm(x) := \pm \frac{\psi^+(x + h\xi) + \psi^-(x - h\xi) - 2\psi^\pm(x)}{h^2} \leq \frac{C_2}{d(x) - h},
\]

(A.3)

for some \( C_2 > 0 \), and every nonzero \( x, \xi \in \mathbb{R}^n \) with \( |\xi| \leq 1 \), and every \( 0 < h < d(x) \).

**Remark.** As we have seen in [34], when \( \partial K \) is \( C^2 \), and \( \varphi \) satisfies the strict Lipschitz property \( (2.6) \), then \( \rho, -\bar{\rho} \) satisfy the above assumption.

Let \( \eta \) be the standard mollifier. Then we define

\[
\psi^+_{\varepsilon}(x) := (\eta \ast \psi^+)(x) := \int_{|y| \leq \varepsilon} \eta(y)\psi^+(x - y) \, dy,
\]

\[
\psi^-_{\varepsilon}(x) := (\eta \ast \psi^-)(x) + \delta_{\varepsilon},
\]

(A.4)
where $3C_1\varepsilon < \delta_\varepsilon < 4C_1\varepsilon$ is chosen such that $\partial \{ \psi^-_\varepsilon < \psi^+_\varepsilon \}$ is $C^\infty$, which is possible by Sard’s Theorem. Note that since $\psi^\pm$ are defined on all of $\mathbb{R}^n$, $\psi^\pm_\varepsilon$ are smooth functions on $\mathbb{R}^n$. Also

$$|\psi^+_\varepsilon(x) - \psi^+(x)| \leq \int_{|y|\leq \varepsilon} \eta_\varepsilon(y)|\psi^+(x-y) - \psi^+(x)| dy \leq \int_{|y|\leq \varepsilon} C_1|y|\eta_\varepsilon(y) dy \leq C_1\varepsilon.$$

Similarly we have

$$2C_1\varepsilon < \psi^-_\varepsilon - \psi^- < 5C_1\varepsilon.$$

Now, let

(A.5) $U_\varepsilon := \{ x \in U : \psi^-_\varepsilon(x) < \psi^+_\varepsilon(x) \}.$

Then we have

(A.6) $\{ x \in U : \psi^+(x) - \psi^-(x) > 5C_1\varepsilon \} \subset U_\varepsilon \subset \{ x \in U \setminus \psi^-_\varepsilon(x) \leq \psi^+_\varepsilon(x) \} \subset \{ x \in U : d(x) > \varepsilon \}.$

To see this note that $\psi^-_\varepsilon(x) \leq \psi^+_\varepsilon(x)$ implies that

$$3C_1\varepsilon < \delta_\varepsilon \leq (\psi^+ - \psi^-) * \eta_\varepsilon \leq \psi^+ - \psi^- + C_1\varepsilon \leq 2C_1d(x) + C_1\varepsilon.$$

Hence $d(x) > \varepsilon$. On the other hand, if $\psi^-_\varepsilon(x) \geq \psi^+_\varepsilon(x)$ then

$$4C_1\varepsilon > \delta_\varepsilon \geq (\psi^+ - \psi^-) * \eta_\varepsilon \geq \psi^+ - \psi^- - C_1\varepsilon.$$

Thus $\psi^+(x) - \psi^-(x) < 5C_1\varepsilon$. Hence $\psi^+(x) - \psi^-(x) > 5C_1\varepsilon$ implies $\psi^-_\varepsilon(x) < \psi^+_\varepsilon(x)$, as desired.

Remark. The above inclusions show that $\overline{U}_\varepsilon \subset U$, and

(A.7) $U = \bigcup_{\varepsilon > 0} U_\varepsilon$;

since by (A.2) we know that $\psi^+ - \psi^- > 0$ on $U$. In addition, remember that we have chosen $\delta_\varepsilon$ so that $\partial U_\varepsilon$ is $C^\infty$. Furthermore, for every $\varepsilon$ there is $\bar{\varepsilon}$ such that

(A.8) $U_\varepsilon \subset \{ d > \varepsilon \} \subset \{ \psi^+ - \psi^- > 5C_1\bar{\varepsilon} \} \subset U_\varepsilon$.

Because otherwise for every $j$ there is $x_j \in U$ such that $d(x_j) > \varepsilon$, while $\psi^+(x_j) - \psi^-(x_j) \leq \frac{1}{j}$. But due to the compactness we can assume that $x_j \to x \in \overline{U}$. Then by continuity we must have $\psi^+(x) - \psi^-(x) = 0$ and $d(x) \geq \varepsilon$. Now by (A.2), $\psi^+(x) - \psi^-(x) = 0$ implies that $x \in \partial U$, which contradicts the fact that $d(x) \geq \varepsilon$.

Lemma 6. Suppose that Assumption 5 holds. Then we have

(A.9) $|\nabla \psi^\pm_\varepsilon| \leq C_1$.

Furthermore, for any unit vector $\xi$, and every $x \in U$ with $d(x) > \varepsilon$ we have

(A.10) $\pm D^2_{\xi\xi} \psi^\pm_\varepsilon(x) \leq \frac{C_2}{d(x) - \varepsilon}$.
where \(d\) is the Euclidean distance to \(\partial U\).

Proof. To show the first part, note that \(\psi^\pm\) are Lipschitz functions and \(|D\psi^\pm| \leq C_1\) a.e.. Thus we have

\[
|D\psi^\pm(x)| \leq \int_{|y| \leq \varepsilon} |\eta_\varepsilon(y)| D\psi^\pm(x - y) dy
\]

\[
= \int_{|y| \leq \varepsilon} \eta_\varepsilon(y) |D\psi^\pm(x - y)| dy \leq C_1 \int_{|y| \leq \varepsilon} \eta_\varepsilon(y) dy = C_1.
\]

Next, suppose \(d(x) > h + \varepsilon\), and \(|\xi| = 1\). Then due to the Lipschitz continuity of \(d\), for \(|y| \leq \varepsilon\) we have

\[
d(x - y) \geq d(x) - |y| \geq d(x) - \varepsilon > h.
\]

Hence by (A.3) we get

\[
\pm \mathcal{D}^2_{h,\xi} \psi^\pm(x) = \pm \int_{|y| \leq \varepsilon} \eta_\varepsilon(y) \mathcal{D}^2_{h,\xi} \psi^\pm(x - y) dy
\]

\[
\leq \int_{|y| \leq \varepsilon} \eta_\varepsilon(y) \frac{C_2}{d(x) - \varepsilon - h} dy
\]

\[
\leq \int_{|y| \leq \varepsilon} \eta_\varepsilon(y) \frac{C_2}{d(x) - \varepsilon - h} dy = \frac{C_2}{d(x) - \varepsilon - h}.
\]

Let \(h \to 0^+\). Then for \(x \in U\) with \(d(x) > \varepsilon\) we get

\[
\pm D^2_{\xi\xi} \psi^\pm(x) \leq \frac{C_2}{d(x) - \varepsilon},
\]

as desired. \(\square\)

Now consider the double obstacle problem

\[
\begin{align*}
F[u_\varepsilon] &= 0 \quad \text{a.e. in } \{\psi^-_\varepsilon < u_\varepsilon < \psi^+_\varepsilon\}, \\
F[u_\varepsilon] &\leq 0 \quad \text{a.e. on } \{u_\varepsilon = \psi^+_\varepsilon\}, \\
F[u_\varepsilon] &\geq 0 \quad \text{a.e. on } \{u_\varepsilon = \psi^-_\varepsilon\},
\end{align*}
\]

(A.11)

where \(u_\varepsilon\) belongs to \(W_{\psi^-_\varepsilon} := \{v \in W^{1,2}(U_\varepsilon) : \psi^-_\varepsilon \leq v \leq \psi^+_\varepsilon \text{ a.e.}\}\).

**Lemma 7.** Suppose \(F\) satisfies Assumptions (1), (3). Also, suppose \(\psi^\pm\) satisfy Assumption (5). Then the double obstacle problem (A.11) has a solution \(u_\varepsilon\), and for every \(p < \infty\) we have

\[
u_\varepsilon \in W^{2,p}(U_\varepsilon).
\]
Proof. Fix $\varepsilon > 0$. For $\delta > 0$, let $\beta_{\delta}$ be a smooth increasing function that vanishes on $(-\infty, 0]$, and equals $\frac{1}{\delta}t$ for $t \geq \delta$. Then the equation

$$\begin{cases} F(x, u_{\varepsilon,\delta}, D u_{\varepsilon,\delta}, D^2 u_{\varepsilon,\delta}) - \beta_{\delta}(\psi_{\varepsilon}^- - u_{\varepsilon,\delta}) + \beta_{\delta}(u_{\varepsilon,\delta} - \psi_{\varepsilon}^+) = 0, \\ u_{\varepsilon,\delta} = \psi_{\varepsilon}^+ \text{ on } \partial U_{\varepsilon}, \end{cases}$$

(A.12)

has a unique solution in $C^{2,\alpha}(\overline{U}_{\varepsilon})$ (see for example Theorem 7.4 of [7]). To simplify the notation we set

$$\tilde{u} = u_{\varepsilon,\delta}, \quad \beta = \beta_{\delta}.$$ 

First let us show that $\tilde{u}$ is uniformly bounded independently of $\delta$. Suppose $C^+$ is a positive constant larger than the maximum of $|\psi_{\varepsilon}^+| + 1$ on $\overline{U}_{\varepsilon}$. Now if we apply the above differential operator to the constant function whose value is $C^+$ we obtain

$$F(x, C^+, 0, 0) - \beta(\psi_{\varepsilon}^- - C^+) + \beta(C^+ - \psi_{\varepsilon}^+) = F(x, C^+, 0, 0) + \frac{C^+ - \psi_{\varepsilon}^+}{\delta}.$$ 

This last expression is positive for $\delta$ small enough, since $F(x, C^+, 0, 0)$ is bounded on $\overline{U}_{\varepsilon}$. Therefore by the comparison principle we have $\tilde{u} \leq C^+$. We can similarly show that $\tilde{u} \geq -C^+$. Hence for small enough $\delta$ we have

$$-C^+ \leq \tilde{u} \leq C^+.$$

Now let us show that

$$\|\beta(\pm(\tilde{u} - \psi_{\varepsilon}^+))\|_{L^\infty(U_{\varepsilon})} \leq C,$$

where $C$ is independent of $\delta$. Note that $\beta(\pm(\tilde{u} - \psi_{\varepsilon}^+))$ is zero on $\partial U_{\varepsilon}$. So assume that $\beta(\pm(\tilde{u} - \psi_{\varepsilon}^+))$ attains its positive maximum at $x_0 \in U_{\varepsilon}$. Let us consider $\beta(\tilde{u} - \psi_{\varepsilon}^+)$; the other case is similar. Since $\beta$ is increasing, $\tilde{u} - \psi_{\varepsilon}^+$ has a positive maximum at $x_0$ too. Therefore we have

$$D \tilde{u}(x_0) = D \psi_{\varepsilon}^+(x_0), \quad D^2 \tilde{u}(x_0) \leq D^2 \psi_{\varepsilon}^+(x_0).$$

We also have $\tilde{u}(x_0) > \psi_{\varepsilon}^+(x_0) \geq \psi_{\varepsilon}^-(x_0)$. Hence by the ellipticity of $F$, and its monotonicity in $z$, at $x_0$ we have

$$F(x_0, \psi_{\varepsilon}^+, D \psi_{\varepsilon}^+), D^2 \psi_{\varepsilon}^+) \leq F(x_0, \tilde{u}, D \tilde{u}, D^2 \tilde{u})$$

$$= \beta(\psi_{\varepsilon}^- - \tilde{u}) - \beta(\tilde{u} - \psi_{\varepsilon}^+) = -\beta(\tilde{u} - \psi_{\varepsilon}^+).$$

Thus $\beta(\tilde{u} - \psi_{\varepsilon}^+) \leq -F[\psi_{\varepsilon}^+]$ at $x_0$. Therefore $\beta(\tilde{u} - \psi_{\varepsilon}^+)$ is bounded independently of $\delta$, as desired.

The bound $\beta(\pm(\tilde{u} - \psi_{\varepsilon}^+)) \leq C$, and the definition of $\beta$ imply that

$$(A.14) \quad \tilde{u} - \psi_{\varepsilon}^+ \leq \delta(C + 1), \quad \psi_{\varepsilon}^- - \tilde{u} \leq \delta(C + 1).$$

In addition, from the equation (A.12) we conclude that

$$\|F[\tilde{u}]\|_{L^\infty(U_{\varepsilon})} \leq 2C.$$

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Thus by $W^{2,p}$ estimates for fully nonlinear elliptic equations (see Theorem 4.5 of [44]) we have
\begin{equation}
(A.15) \quad \|\tilde{u}\|_{W^{2,p}(U_\epsilon)} \leq \tilde{C}(\|F[\tilde{u}]\|_{L^p(U_\epsilon)} + \|\psi_\epsilon^+\|_{C^2(\overline{U})} + \|\tilde{u}\|_{L^\infty(U)})
\end{equation}
for some constant $\tilde{C}$ independent of $\delta$. We only need to check that for a constant $\tilde{\beta}_0$, which is determined by $F,p$, we have
\[\sup_M \frac{|F(x,0,0,M) - F(x_0,0,0,M)|}{|M| + 1} \leq \tilde{\beta}_0,\]
whenever $x,x_0 \in \overline{U}$ and $|x - x_0|$ is small enough. However, this follows easily from our assumption about $|F_\epsilon|$. Therefore $\tilde{u}$ is bounded in $W^{2,p}(U_\epsilon)$ independently of $\delta$, due to the uniform boundedness of $\tilde{u}, F[\tilde{u}]$. Consequently for every $\tilde{\alpha} < 1$, $\|\tilde{u}\|_{C^{1,\tilde{\alpha}}(\overline{U})}$ is bounded independently of $\delta$, because $\partial U_\epsilon$ is $C^2$. Hence there is a sequence $\delta_j \to 0$ such that $\tilde{u}_j := u_\epsilon,\delta_j$ is strongly convergent in $C^1(\overline{U}_\epsilon)$, and weakly convergent in $W^{2,p}(U_\epsilon)$. We denote this limit by $u_\epsilon$. Note that $u_\epsilon \in W^{2,p}(U_\epsilon)$. Also note that if we let $\delta_j \to 0$ in (A.14) we get $\psi^-_\epsilon \leq u_\epsilon \leq \psi^+_\epsilon$.

Finally, let us show that $u_\epsilon$ satisfies the double obstacle problem (A.11). It suffices to show that $u_\epsilon$ satisfies
\begin{equation}
(A.16) \quad \max\{\min\{F[u_\epsilon, u_\epsilon - \psi^-_\epsilon], u_\epsilon - \psi^+_\epsilon\} = 0.
\end{equation}
First let us show that $u_\epsilon$ is a viscosity solution of the above equation. Suppose $\phi$ is a $C^2$ function and $u_\epsilon - \phi$ has a local maximum at $x_0 \in U$. We can assume that $u_\epsilon - \phi$ has a strict local maximum at $x_0$ without loss of generality, since we can approximate $\phi$ with $\phi + \epsilon |x - x_0|^2$. We must show that at $x_0$ we have
\begin{equation}
(A.17) \quad \max\{\min\{F(x_0, u_\epsilon, D\phi, D^2\phi), u_\epsilon - \psi^-_\epsilon\}, u_\epsilon - \psi^+_\epsilon\} \leq 0.
\end{equation}
Now we know that $u_j - \phi$ has a local maximum at a point $x_j$ where $x_j \to x_0$; because $u_j$ uniformly converges to $u_\epsilon$. Hence we have
\[Du_j(x_j) = D\phi(x_j), \quad D^2u_j(x_j) \leq D^2\phi(x_j).
\]
We also know that $\psi^-_\epsilon \leq u_\epsilon \leq \psi^+_\epsilon$. If $\psi^-_\epsilon(x_0) = u_\epsilon(x_0)$ then (A.17) holds trivially. So suppose $\psi^-_\epsilon(x_0) < u_\epsilon(x_0)$. Then for large $j$ we have $\psi^-_\epsilon(x_j) < u_j(x_j)$. Hence by ellipticity of $F$ and equation (A.12), at $x_j$ we have
\[F(x_j, u_j, D\phi, D^2\phi) \leq F(x_j, u_j, Du_j, D^2u_j) = \beta_{\delta_j}(\psi^-_\epsilon - u_j) - \beta_{\delta_j}(u_j - \psi^+_\epsilon) = -\beta_{\delta_j}(u_j - \psi^+_\epsilon) \leq 0.
\]
Thus (A.17) holds in this case too. Similarly, we can show that when $\psi$ is a $C^2$ function and $u_\epsilon - \psi$ has a local minimum at $x_0$, we have
\[\max\{\min\{F(x_0, u_\epsilon, D\psi, D^2\psi), u_\epsilon - \psi^-_\epsilon\}, u_\epsilon - \psi^+_\epsilon\} \geq 0.
\]
Therefore \( u_\varepsilon \) is a viscosity solution of equation (A.16). Hence, as we have shown in Part II of the proof of Theorem 3, \( u_\varepsilon \) is also a strong solution of (A.16); so it satisfies the double obstacle problem (A.11) as desired.

**Theorem 4.** Suppose \( F \) satisfies Assumptions \( \text{[7][8]} \). Also, suppose \( \psi^\pm \) satisfy Assumption \( \text{[8]} \). Then the double obstacle problem (A.11) has a solution \( u \), and we have

\[
u \in W^{2,\infty}(U).
\]

Furthermore, if \( \partial U \) is \( C^{2,\alpha} \) for some \( \alpha > 0 \), and \( \psi^\pm \) are \( C^{2,\alpha} \) on a neighborhood of \( \partial U \) in \( \overline{U} \), we have

\[
u \in W^{2,\infty}(U).
\]

**Proof.** Let \( u_\varepsilon \) be as in Lemma \( \text{[7]} \). Let us first show that

\[
(A.18) \quad |F[u_\varepsilon]| \leq C + \frac{C}{d - \varepsilon}, \quad \text{a.e. on } U_\varepsilon,
\]

where \( d \) is the Euclidean distance to \( \partial U \), and \( C \) is independent of \( \varepsilon \). (Note that by (A.6) we have \( U_\varepsilon \subset \{d > \varepsilon\} \). To see this, note that in the open set \( \{\psi^-_\varepsilon < u_\varepsilon < \psi^+_\varepsilon\} \) we have \( F[u_\varepsilon] = 0 \); so the desired bound holds trivially. Next consider the set \( \{u_\varepsilon = \psi^+_\varepsilon\} \). By (A.11) we have \( F[u_\varepsilon] \leq 0 \) a.e. on \( \{u_\varepsilon = \psi^+_\varepsilon\} \). On the other hand, since both \( u_\varepsilon, \psi^\pm_\varepsilon \) are twice weakly differentiable, for a.e. \( x \in \{u_\varepsilon = \psi^+_\varepsilon\} \) we have \( Du_\varepsilon(x) = D\psi^+_\varepsilon(x) \) and \( D^2u_\varepsilon(x) = D^2\psi^+_\varepsilon(x) \). Hence by the ellipticity of \( F \) and the bound (A.10) for \( D^2\psi^+_\varepsilon \) we have

\[
F(x, u_\varepsilon(x), Du_\varepsilon(x), D^2u_\varepsilon(x)) = F(x, \psi^+_\varepsilon(x), Du^+_\varepsilon(x), D^2\psi^+_\varepsilon(x)) \geq F(x, \psi^+_\varepsilon(x), Du^+_\varepsilon(x), \frac{C_\alpha}{d(x) - \varepsilon} I) \geq F(x, \psi^+_\varepsilon(x), Du^+_\varepsilon(x), 0) - \frac{\alpha C_\alpha}{d(x) - \varepsilon} \geq -C - \frac{C}{d(x) - \varepsilon}.
\]

Note that \( D\psi^+_\varepsilon \) is uniformly bounded by (A.9). We can similarly show that \( F[u_\varepsilon] \) has the desired bound on \( \{u_\varepsilon = \psi^-_\varepsilon\} \).

Now, we choose a decreasing sequence \( \varepsilon_k \to 0 \) such that \( \overline{U}_{\varepsilon_k} \subset U_{\varepsilon_{k+1}} \) (this is possible by (A.8)). For convenience we use \( U_k, u_k, \psi_\varepsilon^\pm \) instead of \( U_{\varepsilon_k}, u_{\varepsilon_k}, \psi_{\varepsilon_k}^\pm \). Consider the sequence \( u_k|_{U_2} \) for \( k > 2 \). By (A.18), (A.6) we have

\[
\|F[u_k]\|_{L^\infty(U_2)} \leq C,
\]

for some \( C \) independent of \( k \). Thus by interior \( W^{2,p} \) estimates for fully nonlinear elliptic equations (see Theorem 4.2 of \([14]\)), and the proof of Lemma \( \text{[7]} \) we have

\[
\|u_k\|_{W^{2,p(U_1)}} \leq \bar{C} \left( \|F[u_k]\|_{L^p(U_2)} + \|u_k\|_{L^\infty(U_2)} \right)
\]

for some constant \( \bar{C} \) independent of \( k \). Therefore \( u_k \) is bounded in \( W^{2,p(U_1)} \). Consequently for every \( \bar{\alpha} < 1, \|u_k\|_{C^{1,\alpha}(\overline{U_1})} \) is bounded independently of \( k \), because \( \partial U_1 \) is \( C^2 \).

Therefore there is a subsequence of \( u_k \)'s, which we denote by \( u_{k_1} \), that weakly converges in \( W^{2,p(U_1)} \) to a function \( \tilde{u}_1 \). In addition, we can assume that \( u_{k_1}, Du_{k_1} \) uniformly converge to
\[ \hat{u}_1, D\hat{u}_1. \] Now we can repeat this process with \( u_{k_1}|_{U_2} \) and get a function \( \hat{u}_2 \) in \( W^{2,p}(U_2) \), which agrees with \( \hat{u}_1 \) on \( U_1 \). Continuing this way with subsequences \( u_{k_l} \) for each positive integer \( l \), we can finally construct a \( C^1 \) function \( u \) in \( W^{2,p}_{\text{loc}}(U) \) (note that \( U = \bigcup U_k \) by (A.7)). It is obvious that \( \hat{u}_k \leq u \leq \hat{u}_k \), since \( \hat{u}_k \leq u \) for every \( k \).

Let us show that \( u \) satisfies the double obstacle problem (A.1). It suffices to show that \( u \) satisfies

\[
\max\{\min\{F[u], u - \psi^-\}, u - \psi^+\} = 0.
\]

Similarly to the proof of Lemma 7, we can show that \( u \) is a viscosity solution of the above equation. Then, as we have shown in Part II of the proof of Theorem 3, it follows that \( u \) is also a strong solution of the above equation; so it satisfies the double obstacle problem (A.1) as desired.

Next, similarly to Part III of the proof of Theorem 3 by utilizing the bounds (A.10) for \( D^2\psi^\pm_k \) and (A.18) for \( F[u_k] \), we can show that \( D^2u_k \) is bounded on \( \{u_k = \psi^\pm_k\} \) independently of \( k \). Then we can apply the result of [23] to deduce that \( D^2u_k \) is locally bounded independently of \( k \), and conclude that \( u \) belongs to \( W^{2,\infty}_{\text{loc}}(U) \).

Finally, suppose that \( \partial U \) is \( C^{2,\alpha} \), and \( \psi^\pm \) are \( C^{2,\alpha} \) on \( \overline{U} \cap \{d < 3r\} \). Let \( 0 \leq \zeta \leq 1 \) be a \( C^\infty \) function which equals 1 on \( U \cap \{d < r\} \) and equals 0 on \( U \cap \{d > 2r\} \). Let \( \eta_\varepsilon \) be the standard mollifier, and set

\[
\hat{\psi}^\pm_\varepsilon := \zeta \psi^\pm + (1 - \zeta)(\eta_\varepsilon \ast \psi^\pm),
\]

for \( \varepsilon \) small enough. Note that \( \hat{\psi}^\pm_\varepsilon \) are \( C^{2,\alpha} \) on \( \overline{U} \), and agree on \( \partial U \). Also, \( \hat{\psi}^\pm_\varepsilon \) uniformly converges to \( \psi^\pm \) as \( \varepsilon \to 0 \). It is obvious that \( \hat{\psi}^-_\varepsilon = \psi^- < \psi^+ = \hat{\psi}^+_\varepsilon \) on \( U \cap \{d < r\} \). Now on \( U \cap \{d \geq \frac{1}{2}r\} \) we have \( \psi^+ - \psi^- \geq c > 0 \). Hence if \( d(x) \geq r \) we get

\[
\eta_\varepsilon \ast \psi^+_\varepsilon(x) - \eta_\varepsilon \ast \psi^-_\varepsilon(x) = \int_{|y| \leq \varepsilon} \eta_\varepsilon(y)[\psi^+_\varepsilon(x - y) - \psi^-_\varepsilon(x - y)] \, dy \geq c \int_{|y| \leq \varepsilon} \eta_\varepsilon(y) \, dy = c.
\]

Therefore we have

\[
\hat{\psi}^+_\varepsilon - \hat{\psi}^-_\varepsilon := \zeta(\psi^+_\varepsilon - \psi^-_\varepsilon) + (1 - \zeta)(\psi^+_\varepsilon - \psi^-_\varepsilon) \geq c(\zeta + 1 - \zeta) = c > 0.
\]

Thus we have \( \hat{\psi}^-_\varepsilon < \hat{\psi}^+_\varepsilon \) on \( U \). In addition, note that around \( \partial U \), \( D^2\hat{\psi}^\pm_\varepsilon = D^2\psi^\pm_\varepsilon \) are bounded. Thus similarly to Lemma 7, we can show that for any unit vector \( \xi \) and every \( x \in U \) we have

(A.19) \[ |D\hat{\psi}^\pm_\varepsilon| \leq C, \quad \pm D^2_{\xi \xi}\hat{\psi}^\pm_\varepsilon(x) \leq C, \]

for some \( C \) independent of \( \varepsilon \).

Now we can repeat the construction of \( u_\varepsilon \) with \( \hat{\psi}^\pm_\varepsilon \) instead of \( \psi^\pm_\varepsilon \). Note that in this case we have \( U_\varepsilon = U \) for every \( \varepsilon \). Also, if we use the bound (A.19) instead of (A.10) in the first
paragraph of the proof of this theorem, we can conclude that
\[ |F[u_\varepsilon]| \leq \tilde{C}, \quad \text{a.e. on } U, \]
for some \( \tilde{C} \) independent of \( \varepsilon \). Hence we can deduce that \( \|u_\varepsilon\|_{W^{2,p}(U)} \) is uniformly bounded. Thus a subsequence of \( u_\varepsilon \) converges to a function \( u \). Then we can repeat Parts II-IV of the proof of Theorem 3 to show that \( u \) satisfies the double obstacle problem (A.1), and we have \( u \in W^{2,\infty}(U) \) as desired. \( \square \)

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