Unitary realization of wavefunction collapse

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Received Day Month Year
Revised Day Month Year

Recent quantum reconstruction projects demand pure unitary time evolution which
seems to contradict the collapse postulate. Inspired by Zurek’s environment assisted
invariance idea, a natural unitary realization of wavefunction collapse is proposed using
Grothendieck group construction for the tensor product commutative monoid.

Keywords: the measurement problem; Grothendieck group construction; quantum me-
chanics reconstruction.

1. Introduction

In standard quantum mechanics textbooks quantum mechanics is presented as obey-
ing two kinds of time evolution: a unitary time evolution in between preparation
and measurement, and a non-unitary collapse of the wavefunction after measure-
ment. From an epistemic point of view, the reduction of the wavepacket is viewed
as information update but ontic interpretations propose different solutions to the
measurement problem.

In this paper we will propose a new framework for solving the measurement
problem which is fully unitary, observer independent, and in full agreement with
the epistemic interpretation. A unitary explanation of the non-unitary collapse
looks like a contradiction in terms but we will show that the standard quantum
mechanics formulation is ignoring a critical mathematical structure. This is not
unlike how writing \( \textit{ict} \) in the special theory of relativity is a consequence of ignoring
the metric tensor. The missing mathematical ingredient will turn out to be the so-
called Grothendieck group\( \mathbb{I} \).

2. The problem of non-unitary time evolution

At first sight, the epistemic approach is free of problems and the quantum Bayesian
interpretation has a self-consistent explanation to all challenges. Therefore it seems
misguided to find any fault with non-unitary time evolution for the case when
it can be understood as information update. However, the challenge comes from
a completely unexpected direction: the efforts to reconstruct quantum mechanics from first principles. Inspired by a groundbreaking result in the 70s, two very similar category theory approaches were recently independently proposed: Refs. [3][4]. The core idea is that of system composition using the tensor product. This works because imposing the physical principle of the invariance of the laws of nature under composition constraints the allowed algebraic structures. From this, quantum mechanics can be recovered and the proof demands very early on the so-called Leibniz identity in the Jordan-Lie algebraic formulation of quantum mechanics. The problem is that in state space this corresponds to unitarity. Citing Landsman here is a dictionary of correspondence between pure states and algebras of observables:

| Pure state space | Algebra of observables |
|------------------|------------------------|
| transition probabilities | Jordan product |
| Poisson structure | Poisson bracket |
| unitarity | Leibniz rule |

Therefore rejecting unitarity breaks the Leibniz identity (see Appendix A). But this is a very serious problem under any circumstances including the case where this can be interpreted as information update. What category theory derivation of quantum mechanics shows is that any violations of Leibniz identities completely destroy the entire formalism of quantum mechanics: one cannot speak of hermitean operators or Hilbert spaces anymore, before or after measurement. Hence the problem is fatal, and not a mere quantum mechanics interpretation disagreement. The consequence is that the collapse postulate breaks the logical consistency of the quantum mechanics postulates when it is understood as non-unitary time evolution.

However, a universal property in category theory (the Grothendieck group construction[1]) naturally presents itself and this is realized using a pure quantum mechanics effect: an equivalence relationship. This will allow creating an inverse operation for the tensor product which naturally realizes the collapse effect. The price we pay is that we will work with more than one Hilbert space representation. What looks like non-unitary collapse in a single Hilbert space representation is actually unitary evolution in Hilbert subspaces representations we need to consider and the information update is realized by a jump in representation.

3. Grothendick construction

Quantum mechanics can be recovered from the existence of the tensor product and the tensor product obeys all properties of a commutative monoid. A standard mathematical procedure in category theory is constructing a group from a monoid. This procedure is called the Grothendieck group construction and is the fundamental construction of K-theory.
The simplest way to understand it is seeing it at work constructing the integer numbers \( \mathbb{Z} \) given the natural numbers \( \mathbb{N} \) and the addition operation \(+\). By itself, this is not enough, otherwise there would be no additional properties of \( \mathbb{Z} \) compared with \( \mathbb{N} \) so there must be an additional ingredient. What is needed is an equivalence relationship which uses only the monoid operation \(+\). For example in this case we can see that \( 3 + 7 = 4 + 6 \) and this can be made into an equivalence relationship in a unique way. Any equivalence relationship obeys the reflexivity, symmetry, and transitivity properties. From \( 3 + 7 = 4 + 6 \) we see that we need at least four elements which reduces to two in case of reflexivity: \( a + b = a + b \). Thus we are led to form a Cartesian product of the original monoid with itself: \( \mathbb{N} \times \mathbb{N} \) and then define an abelian group as this Cartesian product up to an equivalence relationship \( \sim \): \( \mathbb{Z} = (\mathbb{N} \times \mathbb{N}) / \sim \). Then a positive element \( p \in \mathbb{Z} \) is understood as the following equivalent collection of pairs: \( \{(p, 0), (p+1, 1), \cdots \} \) and a negative element \( n \in \mathbb{Z} \) is understood as the following equivalent collection of pairs: \( \{(0, n), (1, n+1), \cdots \} \). Because of the transitivity property of the equivalence, the definition of the equivalence relation is a bit more complex. We call two pairs \((a, b)\) and \((c, d)\) equivalent if \( a + d + k = b + c + k \) for an arbitrary \( k \).

As a concrete example, the number seven is represented as: \( 7 \equiv (7, 0) \sim (8, 1) \sim (9, 2) \sim \cdots \) and negative two is represented as: \(-2 \equiv (0, 2) \sim (1, 3) \sim (2, 4) \sim \cdots \). The usual arithmetic follows: \( 7 - 2 = 7 + (-2) = (7, 0) + (0, 2) = (7, 2) \sim (5, 0) \equiv 5 \).

4. Equivalence relationship from swap symmetry

What we now seek is to find a natural equivalence relationship which will allow us to construct the inverse operation of system composition of a quantum systems using the tensor product \( \otimes \). In other words we want to construct \( \otimes^{-1} \). In quantum mechanics there is such an operation, the partial trace, which is used to ignore additional Hilbert spaces in a composite quantum system. But this is unable to preserve unitarity during collapse and we need to use the Grothendieck group construction to obtain a different inverse operation which can be utilized to construct the collapse inside the Hilbert space of an indivisible quantum system. The main difference between the partial trace and our inverse is that we need a distinguished (preferred) basis as well and we need to use this basis inside the definition of the equivalence relationship. The inverse operation we seek works not across quantum systems, but inside subspaces corresponding to projection operators. However those subspaces are Hilbert spaces in their own right as well but we will use the word subspace to avoid potential confusion.

The sole characteristic of a finite dimensional Hilbert space is its dimensionality and the physics is encoded by the operator algebras. In the algebraic formulation of quantum mechanics, the operator algebras are primary, and the Hilbert space is a derived secondary concept by the Gelfand-Naimark-Segal (GNS) construction. In a given Hilbert space basis, the dimension of the subspace for state representation is given by the dynamical range of the operator algebras, and right after measurement...
a subsequent measurement confirms the prior value. Hence right after measurement we can obtain a Hilbert space representation in a subspace of lower dimensionality corresponding to the collapsed wavefunction. This does not change the physics because we did not change anything in the operator algebras. However, we do change the GNS representation, which is nothing but information update in an observer-independent fashion.

If we recall how an integer corresponds to an infinite collection of Cartesian pairs of natural numbers, the equivalent picture in quantum mechanics is an infinite collection of Cartesian pairs of Hilbert subspaces where the wavefunctions are linked by an equivalence relationship. The price we pay to restore unitarity during collapse is to consider this infinite collection of Cartesian pairs of Hilbert (sub)spaces. Can we identify some natural equivalence relationship in quantum mechanics which would naturally make this infinite wavefunction collection be equivalent with the usual wavefunction?

In quantum mechanics, there is a swap symmetry known as envariance which suggests how to introduce the nontrivial equivalence relationship we seek. The original name is derived from the role of the environment, but the environment is just a label in this case with no particular meaning attached.

Citing Zurek, here is the definition of envariance:

“When a state $|\psi_{SE}\rangle$ of a pair system $S, E$ can be transformed by $U_S = u_S \otimes 1_E$ acting solely on $S$, $U_S |\psi_{SE}\rangle = (u_S \otimes 1_E) |\psi_{SE}\rangle = |\eta_{SE}\rangle$ but the effect of $U_S$ can be undone by acting solely on $E$ with an appropriately chosen $U_E = 1_S \otimes u_E$: $U_E |\eta_{SE}\rangle = (1_S \otimes u_E) |\eta_{SE}\rangle = |\psi_{SE}\rangle$ ”.

Inspired by this we can introduce an equivalence relation which will allow us to define the Grothendick group of the composability abelian monoid with the group operation the tensor product. The equivalence should also have a clear physical interpretation. We are allowed to introduce unitary transformations in our definition because they preserve angles and transition probabilities. There is freedom on where to place those unitary transformations in the definition, and the choice below is due to the desire to obtain the most straightforward physical interpretation. Here is how we proceed:

Suppose we have two wavefunctions $|\psi_p\rangle, |\phi_p\rangle \in H_p$ corresponding to a physical system, and another two wavefunctions $|\psi_n\rangle, |\phi_n\rangle \in H_n$ corresponding to another physical system, where $H_p, H_n$ are Hilbert spaces. Suppose also that $|\psi_p\rangle \otimes |\psi_n\rangle$ has the same biorthogonal decomposition base as $|\phi_p\rangle \otimes |\phi_n\rangle$ and in this base $m, n, p, q$ are the subspace dimensionality for $|\psi_p\rangle, |\psi_n\rangle, |\phi_p\rangle, |\phi_n\rangle$, respectively.

We call two pairs of a Cartesian product of wavefunctions equivalent:

$$(|\psi_p\rangle, |\psi_n\rangle) \sim (|\phi_p\rangle, |\phi_n\rangle)$$

if the energy level of the composite system $|\psi_p\rangle \otimes |\psi_n\rangle$ is equal with the energy level of $|\phi_p\rangle \otimes |\phi_n\rangle$ and if there are unitary transformations $U_p$ acting on the left element $(|\psi\rangle, \cdot)$ and unitary transformations $U_n$ acting on the right element $(\cdot, |\psi\rangle)$.
such that:

\[(U_p |\psi\rangle_p) \otimes |\phi\rangle_n = |\phi\rangle_p \otimes (U_n |\psi\rangle_n)\] (2)

subject to the constraint that:

\[m + q + k = p + n + k\] (3)

Under those conditions the pair \((|\psi\rangle_p, |\psi\rangle_n)\) is physically indistinguishable from \((|\phi\rangle_p, |\phi\rangle_n)\). Moreover one can prove the reflexivity, symmetry, and transitivity properties of this equivalence relationship (see Appendix B). The proof of symmetry and transitivity does not constrain the generality of those unitary transformations, but the most general possible unitaries one can use in establishing reflexivity comes from the Schmidt decomposition and we can call those Schmidt unitaries. We could have placed the \(U_n\) unitaries to act on \(|\phi\rangle_n\) for example and demanded envariance in the biorthogonal decomposition base and this would not have changed the mathematical equivalence. However the physical advantage of the definition above is that the \(U_p\) and \(U_n\) unitaries can now be understood as transforming one possible experimental outcome into another.

5. Wavefunction collapse and the Cartesian pairs

Now we have all the ingredients needed to explore if we can understand the wavefunction collapse in a fully unitary way. First, we can observe that Grothendieck group construction is an universal property. This means its uniqueness is proven by category theory and also the nature of its elements is irrelevant. As such we can follow along an integer number example to build the intuition about the Hilbert space case. Suppose we have a positive number, say 7. Also suppose we have the original wavefunction \(|\psi\rangle\). When we do arithmetic with the number 7, we do not use only 7, but with a class of Cartesian pairs which are equivalent: \(7 \equiv (7, 0) \sim (8, 1) \sim (9, 2) \sim \cdots\). Similarly, when we assign a wavefunction \(|\psi\rangle\) to a physical system we are talking about a class of Cartesian pairs which are equivalent: \(|\psi\rangle \equiv (|\psi\rangle, 0) \sim (|\psi_1\rangle, |\phi_1\rangle) \sim (|\psi_2\rangle, |\phi_2\rangle) \sim \cdots\). In the integer case, the second element corresponds to negative numbers: \(7 = 7 - 0 = 8 - 1 = 9 - 2 = \cdots\). But what physical significance can we attach to the second element of the Cartesian pair?

Let us propose the hypothesis that the first element of the Cartesian product corresponds to the quantum system, and the second element is the measurement device playing the role of an “observer”:

\(|\psi\rangle, \text{detector is ready}\rangle \sim (|\text{collapsed } \psi\rangle, \text{detector registering an outcome}\rangle)\).

Would this equivalence solve the measurement problem? This looks promising, but the obvious problem is that quantum mechanics is probabilistic and there could be several measurement outcomes. If:
Florin Moldoveanu

\(|\psi\rangle, \text{detector is ready} \sim (|\psi_A\rangle, |\text{detector registering } A\rangle)\)
we also have:
\(|\psi\rangle, \text{detector is ready} \sim (|\psi_B\rangle, |\text{detector registering } B\rangle)\)
which by transitivity implies:
\(|\psi_A\rangle, |\text{detector registering } A\rangle \sim (|\psi_B\rangle, |\text{detector registering } B\rangle)\)

Does this equivalence make physical sense? First we see that from the measuring perspective the collapsed wavefunction \(|\psi_A\rangle\) carries no additional information than the detector wavefunction \(|\text{detector registering } A\rangle\). Suppose I prepare the quantum system in a particular collapsed state and I ask you to determine what that state is. There is no instrument whatsoever you can construct to make such a determination. If such an instrument were possible then we can tell apart each Cartesian pair from another. Hence the physical meaning of the mathematical equivalence of the Cartesian pairs corresponding to the measurement outcomes is that one cannot determine the outcome before measurement. As a side note, in classical mechanics there is no such an equivalence possible because classical physics is deterministic and not probabilistic.

Coming back to the measurement problem, at this point we could take the route of the many worlds interpretation and note the correlation between the collapsed wavefunction and the measurement device state. Then we can say that each Cartesian pair corresponds to distinct worlds. However, there is something more we can do in the Grothendieck approach: one of the Cartesian pairs can become distinguished. This happens when the equivalence relationship no longer holds for one of the pairs.

If we look back at the equivalence definition, there are two mechanisms which can break the equivalence resulting in the experimental outcome. First, the quantum system and the measurement device could be in an unstable equilibrium and time evolution will result in a changed energy state for one of the pairs. As an example of this, we can look at ground state of a symmetric double-well Hamiltonian which is exponentially sensitive to tiny perturbations of the potential as \(\hbar \to 0\). In Landsman and Reuvers proposal for solving the measurement problem infinitesimal perturbations grow resulting in the collapsed wavefunction, but in our approach the exponential growth is an unnecessary additional step. Second, the unitary transformations may no longer exist after the local interaction between the quantum system and the measurement device takes place. An example of this kind is Mott’s problem. Mott asked the question why spherically symmetric wavefunctions result in straight line particle tracks in cloud chambers. The answer is that the Schrödinger equation operates in configuration space and once the first ionization happens the next one must occur on a line determined by the origin of the spherically symmetric wavefunction and the first ionization location. The same argument applies for all subsequent ionizations. The only thing remaining to be explained is the very first ionization. After the ionization takes place, the wavefunction of the free electron bears witness of the interaction, and its existence cannot be undone.
by unitary transformations of the particle and the ionized atom.

The process of breaking the equivalence relationship by unstable equilibrium is similar with spontaneous symmetry breaking mechanism, and we can call it “spontaneous equivalence breaking”. But if the breakup of the equivalence relationship is responsible for generating the measurement outcome, can we erase a measurement by restoring the equivalence relationship? To answer this question, first we need to establish that it is possible to affect the equivalence relationship by acting on the measurement device. To prove this we will review the quantum eraser experiment and explore it in our Cartesian pair framework.

5.1. Quantum eraser, equivalence, and contextuality

The setup of the quantum eraser experiment is as follows: a laser pumps a beta barium borate crystal producing two entangled photons by parametric down conversion. One beam which will call $s$ is passed through a double slit generating an interference pattern. The other beam which will call $p$ is passed through a polarizer and then it is detected. For the double slit we place quarter wave plates in front of the slit in such a way that they generate left and right circular polarizations. Those polarizations act as “which way” markers destroying the interference pattern because the left and right circular polarization are orthogonal. The interference pattern is recovered if we place a ±45° degree polarizer in the path of the $p$ beam. Let us now present the mathematical details of quantum eraser. To this aim we will follow the notation of Ref. [11]. We call $|x\rangle$ and $|y\rangle$ a horizontally and vertically polarized photon respectively. $|+\rangle$ and $|−\rangle$ are +45° and −45° polarized photons, and $|L\rangle$ and $|R\rangle$ are left, and right circularly polarized photons respectively.

After exiting the beta barium borate crystal, the two photons are entangled in a Bell state:

$$|Ψ⟩ = \frac{1}{\sqrt{2}}(|x⟩_s|y⟩_p + |y⟩_s|x⟩_p)$$ (4)

which after encountering the double slit becomes:

$$|Ψ⟩ = \frac{1}{\sqrt{2}}(|ψ_1⟩ + |ψ_2⟩)$$ (5)

where

$$|ψ_1⟩ = \frac{1}{\sqrt{2}}(|x⟩_{s1}|y⟩_p + |y⟩_{s1}|x⟩_p)$$ (6)

$$|ψ_2⟩ = \frac{1}{\sqrt{2}}(|x⟩_{s2}|y⟩_p + |y⟩_{s2}|x⟩_p)$$ (7)

Adding the quarter wave plates transforms the state to:
Using the relationships:

\[ |x⟩ = \frac{1}{\sqrt{2}}(|+⟩ + |--⟩) \]  
\[ |y⟩ = \frac{1}{\sqrt{2}}(|+⟩ - |--⟩) \]  
\[ |R⟩ = \frac{1-i}{2}(|+⟩ + i|--⟩) \]  
\[ |L⟩ = \frac{1-i}{2}(|i+⟩ + |--⟩) \]

we can rewrite \(|Ψ⟩\) as:

\[ |Ψ⟩ = \frac{1}{2}(|+⟩_s1 - i|--⟩_s2)|+_p⟩ + i(|−⟩_s1 + i|−⟩_s2)|−⟩_p \]  

and the usual quantum erasure discussion follows from it. What are we interested however is to identify the photon as a measuring apparatus \(|M⟩\) able to detect the “which way” information \(11\). The original Bell state is of the form:

\[ |Ψ⟩ = \frac{1}{\sqrt{2}}(|φ_1(r)|M_1⟩ + |φ_2(r)|M_2⟩) \]

and measurement of the \(x\) or \(y\) polarization reduces \(|Ψ⟩\) to the appropriate state for passing through one or the other slit. In the quantum eraser scenario however, when we place the quarter wave plates, the point is to erase and recover the interference pattern and not to detect the “which way” information. But we can still borrow the photon designation as a measurement device and attempt to construct the Grothendieck Cartesian pairs and the corresponding equivalence relationships. We want to discuss the two cases: no-interference when we place a vertical or horizontal polarizer in front of the photon beam detector, and interference when we place the \(±45°\) polarizers.

In the no-interference case we can identify from above:

\[ |φ_1⟩ = \frac{1}{\sqrt{2}}(|L⟩_s1 + |R⟩_s2) \]  


\[ (8) \]
\[ (9) \]
\[ (10) \]
\[ (11) \]
\[ (12) \]
\[ (13) \]
\[ (14) \]
\[ (15) \]
\[ (16) \]
The equivalence is defined by the unitaries $U_p$, and $U_n$ such that:

$$U_p(|\phi_1\rangle|M_2\rangle) = |\phi_2\rangle U_n(|M_1\rangle)$$ (20)

Such pairs of unitaries exist. For example they can be Pauli-Z gates. Now we can consider the interference case and we make the following identifications:

$$|\phi_1\rangle = \frac{1}{\sqrt{2}}(|+\rangle_{s1} + i|+\rangle_{s2})$$ (21)

$$|\phi_2\rangle = \frac{1}{\sqrt{2}}(|-\rangle_{s1} - |-\rangle_{s2})$$ (22)

$$|M_1\rangle = |+\rangle_p$$ (23)

$$|M_2\rangle = |-\rangle_p$$ (24)

In this case the equivalence is given by the unitaries $V_p$, and $V_n$ such that:

$$V_p(|\phi_1\rangle|M_2\rangle) = |\phi_2\rangle V_n(|M_1\rangle)$$ (25)

Here $V_p$ could be the negative of Pauli-Y gate, and $V_n$ the Pauli-X gate. Therefore each of the two cases have their own equivalence class. But do we have an equivalence relationship across the two cases?

If we pick any index one wavefunction from one setup with any index two wavefunction from the other setup, then there are no $U_p$ unitaries such that the equivalence holds. For example there is no $U_p$ such that: $U_p(|+\rangle_{s1} + i|+\rangle_{s2}) = |L\rangle_{s1} + |R\rangle_{s2}$ because this relationship has to hold both for $s_1 = s_2$ and $s_1 \neq s_2$ and one precludes the other. As such the equivalence classes are disjoint in the two scenarios. This means that the two scenarios: interference and no-interference are distinct and swapping circular and linear polarizers in the path of the “measurement” photon changes the interference pattern.

What this analysis shows is that the unitaries needed to establish the equivalence relationship depend on the configuration of the measurement device. In other words, they are contextual and are affected by the experimenter’s decision.

Coming back to the question: “Can we erase a measurement by restoring the equivalence relationship?”, after the s photon passes the double slit but did not yet arrive at the detector, changing the measurement context changes the unitaries in
the equivalence relationship and does affect the outcome. However, after an outcome is recorded and the information about it is deposited in the environment, to undo the measurement all of the outcome information have to be restored and this is not possible in a controlled way. All one can do is affect the statistics of events.

6. The measurement problem solution proposal

We can now introduce the new framework for solving the measurement problem using the Grothendieck group construction approach. To this aim we will describe what happens during a simple case of position measurement of an electron. Suppose initially the electron is localized at the source and starts propagating towards a photographic plate. When the electron reaches the plate, it can locally interact with any particles in the plate and this corresponds to a set of Cartesian pairs. Let us pick two pairs for illustration purposes: (electron at the plate in position A, scintillation at position A) \sim (electron at the plate in position B, scintillation at position B). As long as the interaction between the electron and the plate did not take place, the situation is reversible and we can find unitaries which would transform \(|\text{electron at the plate in position A}\rangle\) into \(|\text{electron at the plate in position B}\rangle\) and \(|\text{scintillation at position A}\rangle\) into \(|\text{scintillation at position B}\rangle\). After the interaction takes place at say point A, we have an irreversible outcome selecting one distinguished pair. The irreversibility come from the existence of the generated photon which is not reabsorbed by the photographic plate at the place of emission.

Mathematically, the collapse corresponds to changing the operator algebras representation and is not a dynamic process and does not violate the Leibniz identity. The spontaneous equivalence breaking is a unitary process in the context of the Cartesian pairs which is equivalent with a non-unitary process in the context of a single Hilbert space representation. The measurement outcome comes from breaking the equivalence relationship and if the equivalence is re-established, the measurement is undone. An example of this are the vacuum polarization effects where a photon splits into a virtual electron positron pair that later on recombines forming a Feynman diagram loop. Here we consider the electron to be the system and the positron to be the measurement device. If we exclude virtual particles from consideration however, all measurements are irreversible. This is because quantum mechanics is inherently probabilistic due to non-commutativity of operators. After the interaction between the system and the measurement device takes place, to restore the equivalence we need to be able to control individual quantum outcomes and this is impossible.

Now we can discuss the usual way the measurement process is presented and analyze it in the proposed approach. Let us denote the measurement device as \(|M\rangle\). We can call ready state of the measurement device \(|M_0\rangle\), the pointer “A” state as \(|M_A\rangle\), and the pointer “B” state as \(|M_B\rangle\). Since measurement of a prior measured state does not affect the state during the subsequent interaction with the measurement device, the standard explanation demands we have: \(|\psi_A\rangle \otimes |M_0\rangle \rightarrow \)
\(|\psi_A\rangle \otimes |M_A\rangle\). Similarly \(|\psi_B\rangle \otimes |M_0\rangle \rightarrow |\psi_B\rangle \otimes |M_B\rangle\) and by superposition: \((|\psi_A\rangle + |\psi_B\rangle) \otimes |M_0\rangle \rightarrow |\psi_A\rangle \otimes |M_A\rangle + |\psi_B\rangle \otimes |M_B\rangle\).

Let us compare this with our prior quantum eraser scenario where we called one of the entangled photons a “measurement device”. While the original Bell state is of the form \(|\psi_A\rangle \otimes |M_A\rangle + |\psi_B\rangle \otimes |M_B\rangle\) here we lack the assumptions: \(|\psi_A\rangle \otimes |M_0\rangle \rightarrow |\psi_A\rangle \otimes |M_A\rangle\) and \(|\psi_B\rangle \otimes |M_0\rangle \rightarrow |\psi_B\rangle \otimes |M_B\rangle\). This is important to point out because we want to show that the standard argument for the measurement problem is mathematically incorrect.

The key criticism is that there is no such thing as \(|\psi_A\rangle \otimes |M_0\rangle \rightarrow |\psi_A\rangle \otimes |M_A\rangle\) or \(|\psi_B\rangle \otimes |M_0\rangle \rightarrow |\psi_B\rangle \otimes |M_B\rangle\) understood as unitary evolution. The reason this is presented as unitary evolution is because of two requirements: a repeated measurement does not change the state, and in between two consecutive measurements there is only unitary evolution. However if we look at say \(|\psi_A\rangle \otimes |M_0\rangle \rightarrow |\psi_A\rangle \otimes |M_A\rangle\) we notice something odd: there is no back reaction on \(|\psi_A\rangle\) from the interaction with the measurement device. In the algebraic formulation of quantum mechanics one encounters two products: the commutator and the Jordan product. We can call them \(\alpha\) and \(\sigma\) respectively. Mathematically \(\sigma\) defines the algebra of observables (Hermitian operators), and \(\alpha\) the algebra of generators (anti-Hermitian operators), the one-to-one correspondence between observables and generators is known as “dynamic correspondence” and it is intimately linked with Noether’s theorem. The commutator for a bipartite system responsible for the time evolution has the following form:

\[
(A_1 \otimes A_2)\alpha_{12}(B_1 \otimes B_2) = (A_1\alpha B_1) \otimes (A_2\sigma B_2) + (A_1\sigma B_1) \otimes (A_2\alpha B_2)
\] (26)

where \(A_1, B_1\) are operators in the Hilbert space of system 1 (the quantum system), and \(A_2, B_2\) are operators in the Hilbert space of system 2 (the measurement device). If the interaction Hamiltonian is \(h_{12} = h_1 \otimes h_2\), the time evolution in the Heisenberg picture for the operators in system one is as follows:

\[
(\dot{A}_1 \otimes I) = (h_1 \otimes h_2)\alpha_{12}(A_1 \otimes I) = (h_1\alpha A_1) \otimes (h_2\sigma I) + (h_1\sigma A_1) \otimes (h_2\alpha I)
\] (27)

Because the product \(\alpha\) is skew-symmetric we have \((h_2\alpha I) = 0\) which demands:

\[
\dot{A}_1 = (h_1\alpha A_1)h_2
\] (28)

Since \((h_1\alpha A_1)\) cannot be zero because the quantum system does evolve in time, the only possibility to get \(A_1\) to be constant is for \(h_2\) to be zero. But then there is no coupling with the measurement device whatsoever.

The only way out of this mathematical problem is to realize that the valid transition \(|\psi_A\rangle \otimes |M_0\rangle \rightarrow |\psi_A\rangle \otimes |M_A\rangle\) represents not a unitary time evolution, but a change in representation (a collapse). However changes in representation do not satisfy the superposition principle. As such \((|\psi_A\rangle + |\psi_B\rangle) \otimes |M_0\rangle \rightarrow |\psi_A\rangle \otimes |M_A\rangle\).
\[ |M_A\rangle + |\psi_B\rangle \otimes |M_B\rangle \] (known as measurement of the first kind, or premeasurement) is mathematical nonsense.

The proper way to express what is going on is as follows: construct the Cartesian pairs \((|\psi_0\rangle, |M_0\rangle) \sim (|\psi_A\rangle, |M_A\rangle) \sim (|\psi_B\rangle, |M_B\rangle)\) where \(|\psi_0\rangle\) could be any linear combination of \(|\psi_A\rangle\) and \(|\psi_B\rangle\). It is informative to look at the dimensions of the Hilbert (sub)spaces involved. Recall that the Cartesian pairs share a preferred basis from the biorthogonal decomposition theorem. This basis is unique when the modulus of the coefficients in the linear combination are all distinct. In the following we will use the same \(m,n,p,q\) notation as in Eq. (3).

Suppose the quantum system is a spin \(1/2\) particle which means that \(m = 2\). Also suppose that the measurement device can measure the spin. Since \(|M_A\rangle\) and \(|M_B\rangle\) corresponding to the two measurement outcomes are orthogonal, the smallest allowed dimension for the Hilbert space of the measurement device is \(n = 2\) but the argument works in general. Let us pick \(n = 2\) for illustration. If outcome “A” is revealed by the experiment, the wavefunction of the measurement device is \(|M_A\rangle\) and \(q = 1\). From Eq. (3) similar with the Cartesian pairs in the case of integer numbers, we have that:

\[
(m,n) \sim (p,q)
\]

which means that \(m + q = p + n\). In our spin example \((2,2) \sim (p,1)\) implies that \(p = 1\) which shows that \(|\psi_0\rangle\) is projected on the \(|\psi_A\rangle\) subspace which must be of dimension one. In other words \(|\psi_0\rangle\) “collapsed” into \(|\psi_A\rangle\). But we have seen that \(|\psi_A\rangle\) is actually a distinct mathematical object and the collapse mechanism is not a dynamic process. Hence \((|\psi_A\rangle + |\psi_B\rangle) \otimes |M_0\rangle \rightarrow |\psi_A\rangle \otimes |M_A\rangle + |\psi_B\rangle \otimes |M_B\rangle\) is not correct. The mathematically incorrect von Neumann measurement of the first kind is the basis of how the measurement problem is usually presented\cite{12} as a “trilemma”:

- the wave-function of a system is complete,
- the wave-function always evolves in accord with a linear dynamical equation,
- measurements have determinate outcomes.

where any two items contradict the third one. For example Bohmian interpretation violates the first item, spontaneous collapse the second item, and many worlds violates the third one. We see that implicit in the first item is the assumption of a single representation for the pair: (system, observer/measurement device). Once we remove this mathematically invalid assumption which cannot explain the transition \(|\psi_A\rangle \otimes |M_0\rangle \rightarrow |\psi_A\rangle \otimes |M_A\rangle\), and introduce the equivalent representations of the system and measurement device, the “trilemma” vanishes. The measurement process is a lifting of the representation degeneracy through equivalence breaking.
7. Conclusion

Just like subtraction in the case of integer numbers, collapse is a convenience shorthand if we do not want to be bothered by the Grothendieck group construction. In the case of the abelian group of integer numbers, we do not have two distinct operations $+$ and $-$, like the distinct operations of addition and multiplication in a field. Similarly, in the quantum case we do not have two distinct time evolutions: unitary evolution and collapse whose interplay generates the measurement problem. There is only unitary evolution and the Cartesian pairs of Hilbert (sub)spaces which arise out of the GNS and the Grothendieck group constructions.

Once a Cartesian pair becomes distinguished by spontaneous equivalence breaking an outcome is registered and the rest is usually only an amplification effect. It is the configuration of the measurement device which determines the potential outcomes, and the role of the experimentalists is only to pick (by free will) what to measure. In this sense we have an “observer-independent” solution to the measurement problem which depends on the measurement apparatus acting as a witness.

To complete the new approach of solving the measurement problem we need derive the Born rule using spontaneous equivalence breaking. Essential in establishing the equivalence relation was the swap symmetry. We are naturally led on a similar mathematical path with Zurek’s environment assisted invariance derivation of Born rule but as of now the problem is still open. Recently Zurek’s environment was criticized for circularity. That criticism is based on Zurek’s assumption of distinguishable environmental subsystems and we do not use any of this in our approach. We do not use a system-environment swap symmetry but a measurement outcome swap symmetry. If in Zurek’s case the environmental subsystems are distinguished, in our case the measurement device outcomes are distinguished. But this is justified by the very definition of a measurement device.

Appendix A

Here we present the relationship between Leibniz identity and unitarity. Both of them originate from the physical principle of the invariance of the laws of nature under time evolution. We will show how each can be obtained from the other.

Quantum mechanics reconstruction (and unitarity) from Leibniz

Quantum mechanics is usually discussed in the Hilbert state space formalism. The underlying algebraic structure is that of a C* algebra and one recovers the state space from it by the GNS construction. The C* algebra can be decomposed into the symmetric Jordan algebra of observables (Hermitean operators) $\sigma$, and the skew-symmetric Lie algebra of generators (anti-Hermitean operators) $\alpha$. The regular complex number multiplication of the operators is a combination of the two products: $\sigma - \frac{i\hbar}{2\alpha}$. In Hilbert space, the concrete realizations of those products are:
\[ A \sigma B = \{A, B\} = \frac{1}{2} (AB + BA) \]  
\[(A.1)\]

and
\[ A \alpha B = \frac{i}{\hbar} [A, B] = \frac{i}{\hbar} (AB - BA) \]  
\[(A.2)\]

Those products are bilinear maps, and by the universal property of the tensor product one can construct linear algebras. As such any physical requirements we introduce for the tensor product have algebraic consequences. This is the starting point of the reconstruction of quantum mechanics from physical principle in the category theory formalism.

This reconstruction approach is based on two physical principles:

- Invariance of the laws of nature under time evolution,
- Invariance of the laws of nature under system composition.

Suppose \( T \) is a time translation operator. For any algebraic product \( \circ \), invariance under time evolution should preserve the algebraic product:

\[ T(A \circ B) = T(A) \circ T(B) \]  
\[(A.3)\]

In the infinitesimal case \( T \) can be written as: \( T = I + \epsilon D \) and \( D \) obeys:

\[ D(A \circ B) = D(A) \circ B + A \circ D(B) \]  
\[(A.4)\]

One can show that \( D \) can be expressed as a product \( \alpha \): \( D(A) = H \alpha A \) where the product \( \alpha \) obeys the Leibniz identity:

\[ A \alpha (B \circ C) = (A \alpha B) \circ B + B \circ (A \alpha C) \]  
\[(A.5)\]

Then invariance of the laws of nature under composition demands the existence of a secondary product \( \sigma \) and all algebraic properties of the Lie, Jordan, and C*-algebras follow. Next by GNS construction one obtains the usual Hilbert space formulation of quantum mechanics. The time evolution of operators in the Heisenberg picture \( \dot{\tilde{A}} = \frac{i}{\hbar} [H, \tilde{A}] \) (which comes from the starting point of the reconstruction: \( D(A) = H \alpha A \) and Leibniz identity) is transformed into the Schrödinger equation in the Schrödinger picture. Unitarity follows from the Schrödinger equation.

\textbf{Leibniz from unitarity}

To see the inverse implication we need to go in depth in the Jordan-Lie algebraic formulation of quantum mechanics. The algebraic formalism is a unifying framework for both quantum and classical mechanics and the only difference is that the
observables bipartite product $\sigma_{12}$ has an additional element in quantum mechanics: $-\alpha_1 \otimes \alpha_2$. This additional element prevents the Bell locality factorization (the observable bipartite product $\sigma_{12}$ can no longer be factorized in terms of $\sigma_1$ and $\sigma_2$ and this prevents in turn the state factorization) and makes possible superposition and continuous transitions between pure states. In the following we will closely follow Landsman monograph on the mathematical structure of classical and quantum mechanics.\(^5\) The reader should consult this reference for the detailed proofs of the statements below.

Classical mechanics is described by a Poisson algebra while quantum mechanics is described by a Jordan-Lie algebra. If we add norm properties to a Jordan-Lie algebra, we get a J L B (Jordan, Lie, Banach) algebra which is the real part of a C* algebra. The C* algebra is the algebra of bounded operators on some Hilbert space arising out of GNS construction. The pure state space $\mathcal{P}(\mathfrak{A})$ of a C* algebra $\mathfrak{A}$ is a Poisson space with transition probability. Unitarity means that the Hamiltonian flow of the states generated by a given observable $\psi \mapsto \psi(t)$ preserves the transition probability $p$:

$$p(\psi(t), \phi(t)) = p(\psi, \phi) \quad (A.6)$$

This definition is more general than the usual definition of unitarity in a Hilbert space, because it works for both the Hilbert space representation and for the state space of the C* algebra. The definition also applies to the classical case. If the classical case there is no superposition and the transition probabilities are trivial: $p(\psi, \phi) = \delta_{\psi\phi}$.

To derive Leibniz identity from unitarity one can proceed in two steps. First the algebra of observables $\mathfrak{A}_R$ is recovered from the pure state space. Then the Hamiltonian flow $\psi \mapsto \psi(t)$ defines a Jordan homomorphism and the derivative with respect to time of the homomorphism property yields the Leibniz identity.

Given a transition probability space we can define linear combinations of transition probabilities and this defines a real vector space $\mathfrak{A}_R(\mathcal{P})$. In the quantum mechanics case the elements of this vector space have a spectral decomposition $A = \sum_j \lambda_j p_{e_j}$ which allows the definition of a squaring map: $A^2 = \sum_j \lambda_j^2 p_{e_j}$. This in turn is used to define the Jordan product:

$$A \sigma B = \frac{1}{4}((A + B)^2 - (A - B)^2) \quad (A.7)$$

With the sup norm and the product $\sigma$, $\mathfrak{A}_R(\mathcal{P})$ now becomes a J B (Jordan Banach) algebra and the first step is complete: starting from the pure state space equipped with transition probability we arrived at the Jordan algebra of observables. In the classical mechanics case the Jordan product is simply the regular function multiplication.

For the second step, we use the Poisson structure and the Hamiltonian flow: $\psi \mapsto \psi(t)$. For each element $A$ of $\mathfrak{A}_R(\mathcal{P})$ (corresponding to an operator in a Hilbert
space by the GNS construction) we can define a one-parameter map $\beta_t$ given by $\beta_t(A) : \psi \mapsto A(\psi(t))$. Then we have: $\beta_t(A\sigma B) = \beta_t(A)\sigma\beta_t(B)$ because $\beta_t(A^2) = \beta_t(A)^2$. As such $\beta_t$ is a Jordan homomorphism similar with Eq. A.3.

From the Hamiltonian flow we have:

$$\frac{dA}{dt}(\psi(t)) = \{ h, A \}(\psi(t))$$

(A.8)

We take the time derivative of the isomorphism $\beta_t(A\sigma B) = \beta_t(A)\sigma\beta_t(B)$ and by using Eq. A.8, we obtain the Leibniz identity the same way Eqs. A.4 and A.5 were obtained from Eq. A.3.

We need to point a notation difference between what we and Landsman call Leibniz rule. In Landsman’s case the Leibniz identity defines only how the Poisson bracket $\alpha$ acts on the observable algebra $\sigma$ and the proof was outlined above. In our case Leibniz identity applies to all algebraic products. However the only other algebraic product is the Poisson bracket itself and because its skew-symmetry the Leibniz identity becomes the Jacobi identity. The Jacobi identity is assumed by the structure of a Poisson space and we do not need to derive it from unitarity. Both classical and quantum mechanics poses the Poisson space structure and there is a subtle point to be made. The Jordan-Lie abstract formulation of quantum mechanics has two realizations: the usual Hilbert space formulation where $\alpha$ is the commutator and a phase space formulation. However even in the Hilbert space formulation we have a symplectic structure and we can think about quantum mechanics as a constrained classical mechanics: the Hamiltonian flows preserve an additional mathematical structure: a metric. Under normal flat space conditions, when combined with the symplectic structure, the metric gives rise to a complex projective space.

Appendix B

In this appendix we establish the usual properties of an equivalence relationship (reflexivity, symmetry, and transitivity). We need to establish those properties for both the wavefunctions obeying Eq. A.2 and their subspace dimensionality obeying Eq. A.4 but the proof for the dimensionality part is trivial and will be skipped. Still, transitivity proof for Eq. A.3 demands the existence of a wavefunction $|\xi\rangle$ and another unitary operator $U_\xi$ in a generalization of Eq. A.3 to:

$$(U_p|\psi\rangle_p) \otimes |\phi\rangle_n \otimes |\xi\rangle = |\phi\rangle_p \otimes (U_n|\psi\rangle_n) \otimes (U_\xi|\xi\rangle)$$

(B.1)

but this only introduces notation complexity of no physical significance and will be omitted.
Reflexivity

To prove reflexivity we need to show that: $(|a\rangle, |b\rangle) \sim (|a\rangle, |b\rangle)$ This means that exists unitaries $U_p$ and $U_n$ such that

$$(U_p |a\rangle) \otimes |b\rangle = |a\rangle \otimes (U_n |b\rangle)$$

(B.2)

The most general unitaries, like in Zurek’s envariance case, come from Schmidt decomposition:

**Proof.** For any $U_p = \sum_{k=1}^{N} e^{i\phi_k} |a_k\rangle \langle a_k|$ we have $U_n = \sum_{k=1}^{N} e^{-i(\phi_k+2\pi l_k)} |b_k\rangle \langle b_k|$ where $|a\rangle \otimes |b\rangle = \sum_{i=1}^{N} \lambda_i |a_i\rangle |b_i\rangle$ and $l_k$ arbitrary natural numbers.

Symmetry

Suppose that $(|a\rangle, |b\rangle) \sim (|c\rangle, |d\rangle)$. This means that exists unitaries $U_p$ and $U_n$ such that:

$$(U_p |a\rangle) \otimes |d\rangle = |b\rangle \otimes (U_n |c\rangle)$$

(B.3)

To prove symmetry we need to show that $(|c\rangle, |d\rangle) \sim (|a\rangle, |b\rangle)$ is true as well. This means that there are $V_p$ and $V_n$ unitaries such that:

$$(V_p |c\rangle) \otimes |b\rangle = |a\rangle \otimes (V_n |d\rangle)$$

(B.4)

**Proof.** From Eq. (B.3) we see by inspection that $V_p = U_n$ and $V_n = U_p$.

Transitivity

For transitivity, we need to show that if $(|a\rangle, |b\rangle) \sim (|c\rangle, |d\rangle)$ and $(|c\rangle, |d\rangle) \sim (|e\rangle, |f\rangle)$ then $(|a\rangle, |b\rangle) \sim (|e\rangle, |f\rangle)$

**Proof.** From the first equivalence there exists $U_p$ and $U_n$ unitaries such that:

$$(U_p |a\rangle) \otimes |d\rangle = |c\rangle \otimes (U_n |b\rangle)$$

(B.5)

and from the second equivalence there exists $V_p$ and $V_n$ unitaries such that:

$$(V_p |c\rangle) \otimes |f\rangle = |e\rangle \otimes (V_n |d\rangle)$$

(B.6)

We need to show that exists $W_p$ and $W_n$ unitaries such that:

$$(W_p |a\rangle) \otimes |f\rangle = |e\rangle \otimes (W_n |b\rangle)$$

(B.7)

Again by inspection we see that $W_p = V_p U_p$ and $W_n = V_n U_n$. Those operators are indeed unitary. For example: $W_p^\dagger W_p = (V_p U_p)^\dagger (V_p U_p) = U_p^\dagger V_p^\dagger V_p U_p = 1$
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