Boundedness in a fully parabolic attraction-repulsion chemotaxis system with nonlinear diffusion and signal-dependent sensitivity

Yutaro Chiyo, Tomomi Yokota∗†

Department of Mathematics, Tokyo University of Science
1-3, Kagurazaka, Shinjuku-ku, Tokyo 162-8601, Japan

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Abstract. This paper deals with the quasilinear fully parabolic attraction-repulsion chemotaxis system

\[
\begin{aligned}
    u_t &= \nabla \cdot (D(u)\nabla u) - \nabla \cdot (G(u)\chi(v)\nabla v) + \nabla \cdot (H(u)\xi(w)\nabla w), \quad x \in \Omega, \ t > 0, \\
    v_t &= d_1\Delta v + \alpha u - \beta v, \quad x \in \Omega, \ t > 0, \\
    w_t &= d_2\Delta w + \gamma u - \delta w, \quad x \in \Omega, \ t > 0,
\end{aligned}
\]

under homogeneous Neumann boundary conditions and initial conditions, where \(\Omega \subset \mathbb{R}^n (n \geq 1)\) is a bounded domain with smooth boundary, \(d_1, d_2, \alpha, \beta, \gamma, \delta > 0\) are constants. Also, the diffusivity \(D\), the density-dependent sensitivities \(G, H\) fulfill \(D(s) = a_0(s+1)^{m-1}\) with \(a_0 > 0\) and \(m \in \mathbb{R}\); \(0 \leq G(s) \leq b_0(s+1)^{q-1}\) with \(b_0 > 0\) and \(q < \min\{2, m+1\}\); \(0 \leq H(s) \leq c_0(s+1)^{r-1}\) with \(c_0 > 0\) and \(r < \min\{2, m+1\}\), and the signal-dependent sensitivities \(\chi, \xi\) satisfy \(0 \leq \chi(s) \leq \frac{\chi_0}{s^{k_1}}\) with \(\chi_0 > 0\) and \(k_1 > 1\); \(0 \leq \xi(s) \leq \frac{\xi_0}{s^{k_2}}\) with \(\xi_0 > 0\) and \(k_2 > 1\). Global existence and boundedness in the case that \(w = 0\) were proved by Ding (J. Math. Anal. Appl.; 2018;461;1260–1270) and Jia–Yang (J. Math. Anal. Appl.; 2019;475;139–153). However, there is no work on the above fully parabolic attraction-repulsion chemotaxis system with nonlinear diffusion and signal-dependent sensitivity. This paper develops global existence and boundedness of classical solutions to the above system by introducing a new test function.

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∗Corresponding author.
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E-mail: ycnewssz@gmail.com, yokota@rs.tus.ac.jp
1. Introduction

In this paper we consider the fully parabolic attraction-repulsion chemotaxis system with nonlinear diffusion and signal-dependent sensitivity,

\[
\begin{cases}
    u_t = \nabla \cdot (D(u)\nabla u) - \nabla \cdot (G(u)\chi(v)\nabla v) + \nabla \cdot (H(u)\xi(w)\nabla w), & x \in \Omega, \ t > 0, \\
    v_t = d_1 \Delta v + \alpha u - \beta v, & x \in \Omega, \ t > 0, \\
    w_t = d_2 \Delta w + \gamma u - \delta w, & x \in \Omega, \ t > 0, \\
    \nabla u \cdot \nu = \nabla v \cdot \nu = \nabla w \cdot \nu = 0, & x \in \partial \Omega, \ t > 0, \\
    u(x, 0) = u_0(x), \ v(x, 0) = v_0(x), \ w(x, 0) = w_0(x), & x \in \Omega,
\end{cases}
\]

where \( \Omega \subset \mathbb{R}^n \ (n \geq 1) \) is a bounded domain with smooth boundary \( \partial \Omega \); \( \nu \) is the outward normal vector to \( \partial \Omega \); \( d_1, d_2, \alpha, \beta, \gamma, \delta \) are positive constants; \( D, G, H, \chi, \xi \geq 0 \) are known functions of which the typical examples are given by \( D(u) = (u+1)^{m-1}, G(u) = (u+1)^{q-1}, H(u) = (u+1)^{r-1}, \chi(v) = \frac{1}{v^{\frac{1}{m}}}, \xi(w) = \frac{1}{w^{\frac{1}{\gamma}}}, \) where \( m, q, r \in \mathbb{R}, k_1, k_2 > 1 \). The initial data \( u_0, v_0, w_0 \) are supposed to satisfy that

\[
\begin{align*}
    u_0 &\in C^0(\overline{\Omega}), \quad u_0 \geq 0 \text{ in } \overline{\Omega} \quad \text{and} \quad u_0 \neq 0, \quad (1.2) \\
v_0 &\in W^{1,\infty}(\Omega) \quad \text{and} \quad v_0 > 0 \text{ in } \overline{\Omega}, \quad (1.3) \\
w_0 &\in W^{1,\infty}(\Omega) \quad \text{and} \quad w_0 > 0 \text{ in } \overline{\Omega}. \quad (1.4)
\end{align*}
\]

In the system (1.1), the function \( u \) shows the cell density, and the functions \( v \) and \( w \) represent the concentrations of attractive and repulsive chemical substances, respectively. The system (1.1) is a generalization of the original attraction-repulsion chemotaxis system

\[
\begin{cases}
    u_t = \Delta u - \chi \nabla \cdot (u \nabla v) + \xi \nabla \cdot (u \nabla w), \\
    v_t = d_1 \Delta v + \alpha u - \beta v, \\
    w_t = d_2 \Delta w + \gamma u - \delta w
\end{cases}
\]

with \( \chi, \xi, d_1, d_2, \alpha, \beta, \gamma, \delta > 0 \), which was introduced by Luca et al. \([24]\) to describe the aggregation of microglial cells observed in Alzheimer’s disease. The first mathematical work on this model was given by Tao–Wang \([27]\) as will be described later. We can also refer to Jin–Wang \([17]\) for the modeling and mathematical works on this model. On the other hand, the system (1.1) is one of the chemotaxis models proposed by Keller–Segel \([18]\) (for the variations with comprehensive studies, see Hillen–Painter \([13]\), Bellomo–Belloquid–Tao–Winkler \([3]\) and Arumugam–Tyagi \([2]\)). Here chemotaxis is the property such that a species reacts on some chemical substance and moves towards or moves away from this substance. In this paper we are especially interested in the case of nonlinear diffusion and signal-dependent sensitivity; note that a quasilinear generalization of Keller–Segel systems such as (1.5) was proposed by Painter–Hillen \([26]\) to show the quorum effect in the chemotactic process. In particular, when the system has signal-dependent sensitivity, it is biologically meaningful in the Weber–Fechner law and it seems to be a mathematically challenging problem to study whether the solution remains bounded.
We first focus on the Keller–Segel system with signal-dependent sensitivity,

\[
\begin{aligned}
  u_t &= \Delta u - \nabla \cdot (u \chi(v) \nabla v), \\
  v_t &= \Delta v - v + u,
\end{aligned}
\]

where \( \chi \) is a function. In this case global existence and boundedness were studied in [1, 6, 8, 9, 12, 19, 20, 25, 30, 31, 32]. More precisely, when \( 0 < \chi(s) \leq \frac{x_0}{(1+as)^k} \) for all \( s > 0 \) with some \( x_0 > 0, a > 0, k > 1 \), Winkler [30] derived global existence and boundedness. When \( \chi(s) = \frac{x_0}{s} \) and \( n \geq 2 \), Winkler [31] showed global existence of classical solutions for \( \chi_0 < \sqrt{\frac{2}{n}} \), and global existence of weak solutions for \( \chi_0 < \sqrt{\frac{n+2}{3n-4}} \). On the other hand, when \( 0 < \chi(s) \leq \frac{\Lambda}{s} \) for all \( s > 0 \) with some \( \chi_0 > 0 \), \( k > 1 \) and \( n \geq 2 \), global existence and boundedness were obtained in [12]. Also, Fujie [6] proved boundedness under the condition that \( \chi(s) = \frac{\Lambda}{s} \) and \( 0 < \chi_0 < \sqrt{\frac{2}{n}} \). Moreover, in the two-dimensional setting, Lankeit [19] established boundedness if \( \Omega \) is a convex domain and \( \chi(s) = \frac{\Lambda}{s} \) for all \( \chi_0 \in (0, \chi_0') \) with some \( \chi_0' > 1 \). The condition for \( \chi_0 \) was relaxed by Lankeit–Winkler [20] in a novel type of generalized solution framework. When \( 0 \leq \chi(s) \leq \frac{x_0}{(b+s)^k} \) for all \( s > 0 \) with some small \( \chi_0 > 0 \) and \( b \geq 0, k \geq 1 \), boundedness of classical solutions was presented in [25]. Ahn [1] improved the smallness condition for \( \chi_0 \) assumed in [25], and showed stabilization (see also [32]). In the case that \( \chi \) is a general function, global existence and boundedness of classical solutions were obtained in Fujie–Senba [8, 9]. Particularly, in the two-dimensional setting, under the condition that \( \chi > 0 \) fulfills \( \lim_{s \to \infty} \chi(s) = 0 \), boundedness for small \( \tau > 0 \) was shown in [8]. Moreover, when \( \tau > 0 \) is sufficiently large, and \( \chi \) satisfies that \( \lim_{s \to \infty} \chi(s) = 0 \) if \( n = 2 \) and \( \limsup_{s \to \infty} s \chi(s) < \frac{n}{n-2} \) if \( n \geq 3 \), boundedness was proved in [9].

We next review the quasilinear Keller–Segel system with signal-dependent sensitivity,

\[
\begin{aligned}
  u_t &= \nabla \cdot (D(u) \nabla u) - \nabla \cdot (G(u) \chi(v) \nabla v), \\
  v_t &= \Delta v - v + u,
\end{aligned}
\]

where \( D, G, \chi \) are functions. In the case that \( \chi(s) \equiv 1 \) and \( \Omega \) is a convex domain, global existence and boundedness were showed in Tao–Winkler [29] under the condition that \( K_0(s+1)^{m-1} \leq D(s) \leq K_1(s+1)^{M-1} \) for all \( s \geq 0 \) with some \( K_0, K_1 > 0, m, M \geq 1 \) and \( \frac{G(s)}{D(s)} \leq K(s+1)^a \) for all \( s \geq 0 \) with some \( K > 0, a < \frac{2}{n} \); note that the convexity of \( \Omega \) was removed by [14]. On the other hand, when \( \chi(s) = \frac{1}{s} \), under the condition that \( K_0(s+1)^{m-1} \leq D(s) \leq K_1(s+1)^{M-1} \) for all \( s \geq 0 \) with \( K_0, K_1 > 0, m, M \in \mathbb{R} \) and \( \frac{G(s)}{D(s)} \leq K(s+1)^a \) for all \( s \geq 0 \) with some \( K > 0, a < \frac{2}{n} \), global existence and boundedness were established in [7]. However, the optimality of the condition \( a < \frac{2}{n} \) was remained as an open problem. After that, by introducing a fractional type of test function, Ding [5] solved the open problem, that is, proved global existence and boundedness under the condition that \( a < 1 \) and \( m \leq 1 \). Moreover, the problem in the case \( m > 1 \) was solved by Jia–Yang [15] under a differential condition. These mean that the signal-dependent sensitivity benefits global existence and boundedness of solutions.
We now turn our eyes to the quasilinear attraction-repulsion chemotaxis system

\[
\begin{aligned}
&u_t = \nabla \cdot (D(u)\nabla u) - \nabla \cdot (G(u)\chi(v)\nabla v) + \nabla \cdot (H(u)\xi(w)\nabla w), \\
&\tau_1 v_t = \Delta v + \alpha u - \beta v, \\
&\tau_2 w_t = \Delta w + \gamma u - \delta w,
\end{aligned}
\]

where \(\alpha, \beta, \gamma, \delta > 0\), \(\tau_1, \tau_2 \geq 0\) and \(D, G, H, \chi, \xi\) are functions. In the case that \(D, \chi, \xi\) are constants and \(G(s) = s, H(s) = s\) as well as \(\tau_1 = \tau_2 = 0\), Tao–Wang [27] established the first result on global existence and boundedness under the condition \(\chi\alpha - \xi\gamma < 0\); moreover, the authors proved finite-time blow-up by assuming \(\chi\alpha - \xi\gamma > 0\), \(\beta = \delta\) and \(\int_\Omega u_0 \geq \frac{8a}{\chi\alpha - \xi\gamma}\) in the two-dimensional setting. Also, in the case that \(D, \chi, \xi\) are constants and \(G(s) = s, H(s) = s\) as well as \(\tau_1 = \tau_2 = 1\), Jin–Wang [17] derived global existence and boundedness, and stabilization under the condition \(\frac{\chi\alpha}{\xi\gamma} \geq C\) with some \(C > 0\) in the two-dimensional setting. In addition, \(D, \chi, \xi\) are constants and \(\tau_1 = \tau_2 = 1\), global existence and boundedness were studied in [10, 16, 23]; note that the transformation \(z := \chi v - \xi w\) is effective in this case. In the case that \(\chi, \xi\) are constants, \(\tau_1 = 1, \tau_2 = 0\) and \(n \geq 2\), Lin–Mu–Gao [22] proved global existence and boundedness under the condition that \(D(s) > 0\) for all \(s \geq 0\), \(D(s) \geq \frac{a}{s^k}\) for all \(s > 0\) and all \(k < \frac{2}{n} - 1\) with some \(a > 0\) as well as \(G(s) = s, H(s) = s\). Also, Li–Mu–Lin–Wang [21] established global existence and boundedness under the following two conditions:

(i) \(\tau_1 = \tau_2 = 0, G(s) = s, H(s) = s\) and \(D(s) \geq as^{b-1}\) for all \(s > 0\) with some \(a \in (0, 1]\), \(b > 2 - \frac{2}{n}\) as well as \(0 < \chi(s) \leq \frac{\chi_0}{s^k}, 0 < \xi(s) \leq \frac{\xi_0}{s^k}\) for all \(s > 0\) with some \(\chi_0, \xi_0 > 0\), \(k \geq 1, \ell \geq 2\).

(ii) \(\tau_1 = 1, \tau_2 = 0, G(s) = s, H(s) = s^r\) with \(r \geq 2\) and \(D(s) \geq as^{b-1}\) for all \(s > 0\) with some \(a > 0, b > r - \frac{2}{n}\) as well as \(\chi(s) \equiv \chi_0, \xi(s) = \frac{\xi_0}{s}\) with some \(\chi_0, \xi_0 > 0\).

In these literatures, the results were successfully obtained by using estimates for \(v, w\) from below and by reducing the case that \(\chi, \xi\) are constants. On the other hand, in the case of linear diffusion and normal sensitivity that \(D(s) \equiv 1, G(s) = s, H(s) = s\), global existence and boundedness in the system with \(\tau_1 = \tau_2 = 1\) were proved in [4] by the method using a test function defined as a combination of an exponential function and integrals of \(\chi, \xi\). However, since the proof in [4] strongly depends on \(|(u + 1)^{m-1}\nabla u|^2 = (u + 1)^{m-1}|\nabla u|^2\) and boundedness of classical solutions to (1.1) under some conditions, independent of the dimension \(n\), for algebraic growth or decay orders among \(D, G, H\).
In order to state the main theorem, we introduce conditions for the diffusivity $D$, the density-dependent sensitivities $G, H$ and the signal-dependent sensitivities $\chi, \xi$. We suppose that the functions $D, G, H$ satisfy
\begin{align}
D &\in C^2([0, \infty)), \quad D(s) = a_0(s + 1)^{m-1} \quad (a_0 > 0, ~ m \in \mathbb{R}), \\
G &\in C^2([0, \infty)), \quad 0 \leq G(s) \leq b_0(s + 1)^{q-1} \quad (b_0 > 0, ~ q < \min\{2, m + 1\}), \\
H &\in C^2([0, \infty)), \quad 0 \leq H(s) \leq c_0(s + 1)^{r-1} \quad (c_0 > 0, ~ r < \min\{2, m + 1\}),
\end{align}
and assume that the functions $\chi, \xi$ fulfill
\begin{align}
\chi &\in C^{1+\vartheta_1}_{\text{loc}}((0, \infty)) \quad (0 < \vartheta_1 < 1), \quad 0 \leq \chi(s) \leq \frac{\chi_0}{s^{k_1}} \quad (\chi_0 > 0, ~ k_1 > 1), \\
\xi &\in C^{1+\vartheta_2}_{\text{loc}}((0, \infty)) \quad (0 < \vartheta_2 < 1), \quad 0 \leq \xi(s) \leq \frac{\xi_0}{s^{k_2}} \quad (\xi_0 > 0, ~ k_2 > 1).
\end{align}
Then the main result reads as follows.

**Theorem 1.1.** Let $\Omega \subset \mathbb{R}^n \ (n \geq 1)$ be a bounded domain with smooth boundary and let $d_1, d_2, \alpha, \beta, \gamma, \delta > 0$. Suppose that $D, G, H, \chi, \xi$ fulfill (1.6)–(1.10). Then for all $(u_0, v_0, w_0)$ satisfying (1.2)–(1.4) there exists a unique triplet $(u, v, w)$ of nonnegative functions
\[ u, v, w \in C^0(\overline{\Omega} \times [0, \infty)) \cap C^{2,1}(\overline{\Omega} \times (0, \infty)), \]
which solves (1.1) in the classical sense, and is bounded in the sense that
\[ \|u(\cdot, t)\|_{L^\infty(\Omega)} \leq C \]
for all $t > 0$ with some $C > 0$.

**Corollary 1.2.** Let $\xi = 0$. Suppose that $D, G, \chi$ fulfill (1.6), (1.7), (1.9), respectively. Then for all $(u_0, v_0)$ satisfying (1.2) and (1.3) there exists a unique classical solution $(u, v)$ which is bounded.

**Remark 1.1.** The above corollary improves a previous result. Indeed, the condition $q < \frac{5-m}{2} \ (m > 1)$ in [15] is relaxed to $q < \min\{2, m + 1\} \ (m \in \mathbb{R})$.

The strategy for the proof of Theorem 1.1 is to show $L^p$-boundedness of $u$. In the case that $D(s) \equiv 1, G(s) = s, H(s) = s$, $L^p$-estimate for $u$ was established in [4] by deriving
\[ \frac{d}{dt} \int_\Omega u^p f(v, w) \leq c_1 \int_\Omega u^p f(v, w) - c_2 \left( \int_\Omega u^p f(v, w) \right)^{1+\vartheta} \]
for some constants $c_1, c_2, \vartheta > 0$ and some function $f: \mathbb{R}^2 \to \mathbb{R}$. Unfortunately, due to the nonlinearity of $D, G, H$, this method does not work in (1.1). So, in this paper, we shift our method to that in [5] with the use of $\frac{d}{dt} \int_\Omega \frac{(u+1)^{p-m+1}}{u^{2k_1+\sigma_1-1}}$ with suitable $\sigma_1 > 0$. However, even if $\frac{d}{dt} \int_\Omega \frac{(u+1)^{p-m+1}}{u^{2k_2+\sigma_2-1}} (\sigma_2 > 0)$ is added, the parallel method does not work, because some terms with the product of $v, w$ appear in the denominator. More precisely, combining...
\begin{align}
\frac{d}{dt} \int_{\Omega} (u + 1)^{p-m+1}, \quad \frac{d}{dt} \int_{\Omega} (u+1)^{p-m+1}, \quad \frac{d}{dt} \int_{\Omega} \frac{(u+1)^{p-m+1}}{w^{2k_1 + \sigma_1} - 2}, \quad \frac{L}{2d_1} \frac{d}{dt} \int_{\Omega} v^2 \quad \text{and} \quad \frac{M}{2d_2} \frac{d}{dt} \int_{\Omega} w^2, \quad \text{we have several good terms such as}
\end{align}

\begin{align}
-c_3 \int_{\Omega} (u + 1)^{p-m-2} \nabla u^2, \quad -c_4 \int_{\Omega} \frac{(u+1)^{p-m-1}}{v^{2k_1 + \sigma_1}} |\nabla v|^2, \quad -L \int_{\Omega} |\nabla v|^2 \quad (1.11)
\end{align}

with some \(c_3, c_4 > 0\) and sufficiently large \(L > 0\), and a lot of terms such as

\begin{align}
I_1 := \int_{\Omega} \frac{(u+1)^{p+q-m-2}}{v^{3k_1 + \sigma_1 - 2}} |\nabla u| |\nabla v|, \quad I_2 := \int_{\Omega} \frac{(u+1)^{p+q-\eta-3}}{v^{2k_1 + \sigma_2 - 2}} |\nabla u| |\nabla v|.
\end{align}

Using the estimate \(v(x, t) \geq \mu_1\) with some \(\mu_1 > 0\) (see (2.1)), we can estimate \(I_1\) as

\begin{align}
I_1 \leq \varepsilon_1 \int_{\Omega} (u + 1)^{p-2} |\nabla u|^2 + \varepsilon_2 \int_{\Omega} \frac{(u+1)^{p-m+1}}{v^{2k_1 + \sigma_1}} |\nabla v|^2 + \varepsilon_5 \int_{\Omega} |\nabla v|^2 \quad (1.12)
\end{align}

with small \(\varepsilon_1, \varepsilon_2 > 0\) and some \(\varepsilon_5 > 0\), and hence all terms on the right-hand side of this inequality can be dominated by the good terms in (1.11). On the other hand, using the estimate \(w(x, t) \geq \mu_2\) with some \(\mu_2 > 0\) (see (2.2)), we can similarly estimate \(I_2\) as

\begin{align}
I_2 \leq \varepsilon_3 \int_{\Omega} (u + 1)^{p-2} |\nabla u|^2 + \varepsilon_4 \int_{\Omega} \frac{(u+1)^{p-m+1}}{v^{2k_1}} |\nabla v|^2. \quad (1.13)
\end{align}

However, the second term on the right-hand side cannot be estimated by the good terms in (1.11), because \(1/v^{2k_1} = \frac{\sigma_1}{v^{2k_1 + \sigma_1}}\) cannot be estimated by \(1/v^{2k_1 + \sigma_1}\) due to the lack of the upper estimate for \(v\). Thus we will overcome this difficulty by introducing a new test function with the product of \(v, w\) in the denominator, that is, \(\int_{\Omega} \frac{(u+1)^{p-\eta}}{v^{2k_1 + \sigma_3 - 2} w^{2k_2 + \sigma_4 - 2}}\), where \(\sigma_3, \sigma_4 < 0,\) and \(\eta > 0\) will be fixed later.

This paper is organized as follows. In Section 2 we collect some preliminary facts about local existence in (1.1), the lower bounds for \(v, w\), and the weighted Young inequality which will be employed frequently later. In Section 3 we mainly derive two differential inequalities needed to prove global existence and boundedness (Theorem 1.1).

### 2. Preliminaries

We first introduce a reasonable result on local existence of classical solutions to (1.1), which can be proved by standard arguments based on the contraction mapping principle (see e.g., [28] for nonlinear diffusion; [11] for signal-dependent sensitivity).

**Lemma 2.1.** Let \(\Omega \subset \mathbb{R}^n (n \geq 1)\) be a bounded domain with smooth boundary and let \(d_1, d_2, \alpha, \beta, \gamma, \delta > 0\). Assume that \(D, G, H, \chi, \xi\) satisfy (1.6)–(1.10). Then for all \((u_0, v_0, w_0)\) fulfilling (1.2)–(1.4) there exists \(T_{\max} \in (0, \infty)\) such that (1.1) possesses a unique classical solution \((u, v, w)\) such that

\begin{align}
(u, v, w) \in C^0(\bar{\Omega} \times [0, T_{\max})) \cap C^{2,1}(\bar{\Omega} \times (0, T_{\max})).
\end{align}

Moreover,

\begin{align}
\text{if } T_{\max} < \infty, \quad \text{then either } \limsup_{t \uparrow T_{\max}} \|u(\cdot, t)\|_{L^\infty(\Omega)} = \infty \quad \text{or} \quad \liminf_{t \uparrow T_{\max}} \inf_{x \in \Omega} v(\cdot, t) = 0,
\end{align}

and \(\int_{\Omega} u(\cdot, t) = \int_{\Omega} u_0\) for all \(t \in (0, T_{\max})\).
In the following we suppose that $\Omega \subset \mathbb{R}^n$ ($n \geq 1$) is a bounded domain with smooth boundary, $d_1, d_2, \alpha, \beta, \gamma, \delta > 0$ and $D, G, H, \chi, \xi$ fulfill (1.6)–(1.10) as well as $(u_0, v_0, w_0)$ satisfies (1.2)–(1.4). Then we denote by $(u, v, w)$ the local classical solution of (1.1) given in Lemma 2.1 and by $T_{\text{max}}$ its maximal existence time. We next present the result on the lower bounds for $v, w$, which was obtained in [6, Lemma 2.2] (see also [25, Lemma 2.1 and Remark 2.2]).

**Lemma 2.2.** Assume that $(u, v, w)$ is the local classical solution of (1.1). Then there exist constants $\mu_1, \mu_2 > 0$ such that

$$\inf_{x \in \Omega} v(x, t) \geq \mu_1,$$

$$\inf_{x \in \Omega} w(x, t) \geq \mu_2$$

for all $t \in (0, T_{\text{max}})$.

We finally recall the following weighted Young inequality which will be used frequently later.

**Lemma 2.3.** Let $p, q \in (1, \infty)$ satisfy $\frac{1}{p} + \frac{1}{q} = 1$. Then for all $a, b \geq 0$ and all $\varepsilon > 0$, the inequality

$$ab \leq \varepsilon^p \frac{a^p}{p} + \frac{1}{\varepsilon^q} \frac{b^q}{q}$$

holds.

### 3. Proof of Theorem 1.1

In this section we mainly derive two differential inequalities which lead to $L^p$-estimate for $u$. The first one is given by the following lemma.

**Lemma 3.1.** Assume that

$$p > \max\{1, m, 2(m - q + 1), 2(m - r + 1)\},$$

$$\eta < \min\{2(m - q + 1), 2(m - r + 1)\}.$$  

Then for all $\varepsilon_{01}, \varepsilon_{02} > 0$ there exist constants $C_1 = C_1(b_0, p, q, m, k_1, \mu_1, \varepsilon_{01}) > 0$ and $C_2 = C_2(c_0, p, r, m, k_2, \mu_2, \varepsilon_{02}) > 0$ such that

$$\frac{d}{dt} \int_{\Omega} (u + 1)^{p-m+1} + \frac{a_0(p-m)(p-m+1)}{2} \int_{\Omega} (u + 1)^{p-2} |\nabla u|^2$$

$$\leq \varepsilon_{01} \int_{\Omega} \frac{|\nabla v|^2}{v^{k_1+\sigma_1}} (u + 1)^{p-\eta} + \varepsilon_{02} \int_{\Omega} \frac{|\nabla w|^2}{w^{2k_2+\sigma_2}} (u + 1)^{p-\eta}$$

$$+ C_1 \int_{\Omega} |\nabla v|^2 + C_2 \int_{\Omega} |\nabla w|^2$$

for all $t \in (0, T_{\text{max}})$, where $\sigma_1 := \frac{2k_1(2m - 2q - \eta + 2)}{p - 2(m - q + 1)} > 0$ and $\sigma_2 := \frac{2k_2(2m - 2r - \eta + 2)}{p - 2(m - r + 1)} > 0$.  

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Proof. Straightforward calculations, integration by parts and (1.6)–(1.10) yield that
\[
\frac{d}{dt} \int_{\Omega} (u + 1)^{p-m+1}
\]
\[
= (p - m + 1) \int_{\Omega} (u + 1)^{p-m} \nabla \cdot [D(u) \nabla u - G(u) \chi(v) \nabla v + H(u) \xi(w) \nabla w]
\]
\[
= - (p - m)(p - m + 1) \int_{\Omega} D(u)(u + 1)^{p-m-1} |\nabla u|^2 
\]
\[
+ (p - m)(p - m + 1) \int_{\Omega} G(u)(u + 1)^{p-m-1} \chi(v) \nabla u \cdot \nabla v 
\]
\[
- (p - m)(p - m + 1) \int_{\Omega} H(u)(u + 1)^{p-m-1} \xi(w) \nabla u \cdot \nabla w 
\]
\[
\leq - a_0(p - m)(p - m + 1) \int_{\Omega} (u + 1)^{p-2} |\nabla u|^2 
\]
\[
+ b_0(p - m)(p - m + 1) \int_{\Omega} (u + 1)^{p+q-m-2} \frac{\chi_0}{v^{k_1}} |\nabla u||\nabla v| 
\]
\[
+ c_0(p - m)(p - m + 1) \int_{\Omega} (u + 1)^{p+r-m-2} \frac{\xi_0}{w^{k_2}} |\nabla u||\nabla w|,\tag{3.4}
\]
where we used the fact \( p > m \) (see (3.1)). We now estimate the second and third terms on the rightmost summand of (3.4). Using Lemma 2.3, we have
\[
\int_{\Omega} (u + 1)^{p+q-m-2} \frac{\chi_0}{v^{k_1}} |\nabla u||\nabla v|
\]
\[
= \int \frac{1}{2} \left( \frac{a_0}{b_0} \right)^{\frac{1}{2}} (u + 1)^{p-2} |\nabla u| \cdot 2\chi_0 \left( \frac{b_0}{a_0} \right)^{\frac{1}{2}} \frac{|\nabla v|}{v^{k_1}} (u + 1)^{p-2(m-q+1)}
\]
\[
\leq \frac{a_0}{4b_0} \int_{\Omega} (u + 1)^{p-2} |\nabla u|^2 + \frac{b_0\chi_0}{a_0} \int_{\Omega} \frac{|\nabla v|^2}{v^{2k_1}} (u + 1)^{p-2(m-q+1)}.\tag{3.5}
\]
Since \( \theta := \frac{p-n}{p-2(m-q+1)} > 1 \) due to (3.2), it follows from Lemma 2.3 and (2.1) that for all \( \varepsilon_{01} > 0 \),
\[
\frac{b_0\chi_0^2}{a_0} \int_{\Omega} \frac{|\nabla v|^2}{v^{2k_1}} (u + 1)^{p-2(m-q+1)}
\]
\[
= \frac{b_0\chi_0^2}{a_0} \int_{\Omega} \frac{|\nabla v|^2}{v^{2\frac{k_1+\sigma_1}{q}}}(u + 1)^{p-2(m-q+1)} \cdot \frac{|\nabla v|^2}{v^{2k_1 - 2\frac{k_1+\sigma_1}{q}}}
\]
\[
\leq \varepsilon_{01} \int_{\Omega} \frac{|\nabla v|^2}{v^{2k_1+\sigma_1}} (u + 1)^{p-n} + c_1 \int_{\Omega} |\nabla v|^2 \tag{3.6}
\]
holds with \( \sigma_1 := \frac{2k_1(2m-2q-n-2)}{p-2(m-q+1)} > 0 \) and \( c_1 = c_1(p, q, m, k_1, \mu_1, \varepsilon_{01}) > 0. \) Thus, combining (3.5) and (3.6), we see that
\[
\int_{\Omega} (u + 1)^{p+q-m-2} \frac{\chi_0}{v^{k_1}} |\nabla u||\nabla v|
\]
\[
\leq \frac{a_0}{4b_0} \int_{\Omega} (u + 1)^{p-2} |\nabla u|^2 + \varepsilon_{01} \int_{\Omega} \frac{|\nabla v|^2}{v^{2k_1+\sigma_1}} (u + 1)^{p-n} + c_1 \int_{\Omega} |\nabla v|^2. \tag{3.7}
\]
Similarly, we obtain that for all $\tilde{e}_{02} > 0$,
\[
\int_{\Omega} (u + 1)^{p + r - m - 2} \frac{\xi_0}{u^{k_2}} |\nabla u| |\nabla w|
\leq \frac{a_0}{4c_0} \int_{\Omega} (u + 1)^{p - 2} |\nabla u|^2 + \tilde{e}_{02} \int_{\Omega} \frac{|\nabla w|^2}{u^{2k_2 + \sigma_2}} (u + 1)^{p - \eta} + c_2 \int_{\Omega} |\nabla w|^2 \tag{3.8}
\]
holds with $\sigma_2 := \frac{2k_2(2m - 2\eta + 2)}{p - 2(m - r + 1)} > 0$ and $c_2 = c_2(p, r, m, k_2, \mu_2, \tilde{e}_{02}) > 0$. Hence a combination of $(3.4)$, $(3.7)$ and $(3.8)$ yields that
\[
\frac{d}{dt} \int_{\Omega} (u + 1)^{p - m + 1} \leq -a_0(p - m)(p - m + 1) \int_{\Omega} (u + 1)^{p - 2}|\nabla u|^2 + \frac{a_0(p - m)(p - m + 1)}{4} \int_{\Omega} (u + 1)^{p - 2}|\nabla u|^2
\]
\[
+ b_0(p - m)(p - m + 1) \tilde{e}_{01} \int_{\Omega} \frac{|\nabla v|^2}{v^{2k_1 + \sigma_1}} (u + 1)^{p - \eta}
\]
\[
+ b_0(p - m)(p - m + 1)c_1 \int_{\Omega} |\nabla v|^2
\]
\[
+ \frac{a_0(p - m)(p - m + 1)}{4} \int_{\Omega} (u + 1)^{p - 2}|\nabla u|^2
\]
\[
+ c_0(p - m)(p - m + 1) \tilde{e}_{02} \int_{\Omega} \frac{|\nabla w|^2}{u^{2k_2 + \sigma_2}} (u + 1)^{p - \eta}
\]
\[
+ c_0(p - m)(p - m + 1)c_2 \int_{\Omega} |\nabla w|^2
\]
\[
= -\frac{a_0(p - m)(p - m + 1)}{2} \int_{\Omega} (u + 1)^{p - 2}|\nabla u|^2
\]
\[
+ b_0(p - m)(p - m + 1) \tilde{e}_{01} \int_{\Omega} \frac{|\nabla v|^2}{v^{2k_1 + \sigma_1}} (u + 1)^{p - \eta}
\]
\[
+ c_0(p - m)(p - m + 1) \tilde{e}_{02} \int_{\Omega} \frac{|\nabla w|^2}{u^{2k_2 + \sigma_2}} (u + 1)^{p - \eta}
\]
\[
+ c_3 \int_{\Omega} |\nabla v|^2 + c_4 \int_{\Omega} |\nabla w|^2
\]
with $c_3 := b_0(p - m)(p - m + 1)c_1 > 0$ and $c_4 := c_0(p - m)(p - m + 1)c_2 > 0$. Therefore we have
\[
\frac{d}{dt} \int_{\Omega} (u + 1)^{p - m + 1} + \frac{a_0(p - m)(p - m + 1)}{2} \int_{\Omega} (u + 1)^{p - 2}|\nabla u|^2
\]
\[
\leq b_0(p - m)(p - m + 1) \tilde{e}_{01} \int_{\Omega} \frac{|\nabla v|^2}{v^{2k_1 + \sigma_1}} (u + 1)^{p - \eta}
\]
\[
+ c_0(p - m)(p - m + 1) \tilde{e}_{02} \int_{\Omega} \frac{|\nabla w|^2}{u^{2k_2 + \sigma_2}} (u + 1)^{p - \eta}
\]
\[
+ c_3 \int_{\Omega} |\nabla v|^2 + c_4 \int_{\Omega} |\nabla w|^2,
\]
which leads to $(3.3)$ due to arbitrariness of $\tilde{e}_{01}, \tilde{e}_{02} > 0$. \boxed{□}
The second inequality to be shown is given by the following lemma.

**Lemma 3.2.** Assume that $q < \min\{2, m + 1\}$, $r < \min\{2, m + 1\}$, $k_1 > 1$, $k_2 > 1$, $2(1 - k_1) < \sigma_3 < 0$, $2(1 - k_2) < \sigma_4 < 0$ and

$$p > \max\left\{1, m, 2(m - q + 1), 2(m - r + 1), \frac{2[(m - 1)(2k_1 + \sigma_1 - 1) + (m - \eta - 1)]}{2k_1 + \sigma_1 - 2} (i \in \{1, 3\}), \frac{2[(m - 1)(2k_2 + \sigma_2 - 1) + (m - \eta - 1)]}{2k_2 + \sigma_2 - 2} (j \in \{2, 4\}) \right\} \quad (3.9)$$

as well as

$$\max\{2(m - 1), 0\} < \eta < \min\{2(m - q + 1), 2(m - r + 1)\}, \quad (3.10)$$

where $\sigma_1, \sigma_2$ in (3.9) are defined in Lemma 3.1; note that existence of $\eta$ satisfying (3.10) is guaranteed by $q, r < \min\{2, m + 1\}$. Then there exist constants $\varepsilon_{03} > 0$ and $C_k > 0$ ($k \in \{3, 4, 5, 6, 7, 8\}$) such that

$$\varepsilon_{03} \frac{d}{dt} \int_{\Omega} \frac{(u + 1)^{p - \eta}}{v^{2k_1 + \sigma_1 - 2}w^{2k_2 + \sigma_2 - 2}} + \frac{d}{dt} \int_{\Omega} \frac{(u + 1)^{p - \eta}}{v^{2k_1 + \sigma_1 - 2}} + \frac{d}{dt} \int_{\Omega} \frac{(u + 1)^{p - \eta}}{w^{2k_2 + \sigma_2 - 2}} + C_3 \int_{\Omega} \frac{|\nabla v|^2}{v^{2k_1 + \sigma_1}} (u + 1)^{p - \eta} + C_4 \int_{\Omega} \frac{|\nabla w|^2}{w^{2k_2 + \sigma_2}} (u + 1)^{p - \eta}$$

$$\leq \frac{a_0 (p - m)(p - m + 1)}{4} \int_{\Omega} (u + 1)^{p - 2} |\nabla u|^2 + C_5 \int_{\Omega} |\nabla v|^2 + C_6 \int_{\Omega} |\nabla w|^2 + C_7 \int_{\Omega} (u + 1)^p + C_8$$

(3.11)

for all $t \in (0, T_{max})$.

**Proof.** We first estimate $\frac{d}{dt} \int_{\Omega} \frac{(u + 1)^{p - \eta}}{v^{2k_1 + \sigma_1 - 2}w^{2k_2 + \sigma_2 - 2}}$. Using the equations in (1.1), we have

$$\frac{d}{dt} \int_{\Omega} \frac{(u + 1)^{p - \eta}}{v^{2k_1 + \sigma_1 - 2}w^{2k_2 + \sigma_2 - 2}}$$

$$= (p - \eta) \int_{\Omega} \frac{(u + 1)^{p - \eta - 1}}{v^{2k_1 + \sigma_1 - 2}w^{2k_2 + \sigma_2 - 2}} u_t$$

$$- (2k_1 + \sigma_3 - 2) \int_{\Omega} \frac{(u + 1)^{p - \eta}}{v^{2k_1 + \sigma_1 - 2}w^{2k_2 + \sigma_2 - 2}} v_t$$

$$- (2k_2 + \sigma_4 - 2) \int_{\Omega} \frac{(u + 1)^{p - \eta}}{v^{2k_1 + \sigma_1 - 2}w^{2k_2 + \sigma_2 - 2}} w_t$$

$$= (p - \eta) \int_{\Omega} \frac{(u + 1)^{p - \eta - 1}}{v^{2k_1 + \sigma_1 - 2}w^{2k_2 + \sigma_2 - 2}} \nabla \cdot [D(u)\nabla u - G(u)\chi(v)\nabla v + H(u)\xi(w)\nabla w]$$

$$- (2k_1 + \sigma_3 - 2) \int_{\Omega} \frac{(u + 1)^{p - \eta}}{v^{2k_1 + \sigma_1 - 2}w^{2k_2 + \sigma_2 - 2}} (d_1 \Delta v + \alpha u - \beta v)$$

$$- (2k_2 + \sigma_4 - 2) \int_{\Omega} \frac{(u + 1)^{p - \eta}}{v^{2k_1 + \sigma_1 - 2}w^{2k_2 + \sigma_2 - 2}} (d_2 \Delta w + \gamma u - \delta w)$$

$$=: (p - \eta) J_1 + (2k_1 + \sigma_3 - 2) J_2 + (2k_2 + \sigma_4 - 2) J_3. \quad (3.12)$$
As to the first term $J_1$, integrating by parts leads to

$$J_1 = -\int_\Omega \nabla \left( \frac{(u + 1)^{p-\eta-1}}{v^{2k_1+\sigma_3-2}w^{2k_2+\sigma_4-2}} \right) \cdot [D(u)\nabla u - G(u)\chi(v)\nabla v + H(u)\xi(w)\nabla w]$$

$$= -(p - \eta - 1) \int_\Omega \frac{(u + 1)^{p-\eta-2}\nabla u}{v^{2k_1+\sigma_3-2}w^{2k_2+\sigma_4-2}} \cdot [D(u)\nabla u - G(u)\chi(v)\nabla v + H(u)\xi(w)\nabla w]$$

$$+ (2k_1 + \sigma_3 - 2) \int_\Omega \frac{(u + 1)^{p-\eta-1}\nabla v}{v^{2k_1+\sigma_3-1}w^{2k_2+\sigma_4-2}} \cdot [D(u)\nabla u - G(u)\chi(v)\nabla v + H(u)\xi(w)\nabla w]$$

$$+ (2k_2 + \sigma_4 - 2) \int_\Omega \frac{(u + 1)^{p-\eta-1}\nabla w}{v^{2k_1+\sigma_3-2}w^{2k_2+\sigma_4-1}} \cdot [D(u)\nabla u - G(u)\chi(v)\nabla v + H(u)\xi(w)\nabla w]$$

$$= -(p - \eta - 1) \int_\Omega \frac{(u + 1)^{p-\eta-2}}{v^{2k_1+\sigma_3-2}w^{2k_2+\sigma_4-2}} [D(u)|\nabla u|^2 - G(u)\chi(v)\nabla u \cdot \nabla v$$

$$+ H(u)\xi(w)\nabla u \cdot \nabla w]$$

$$+ (2k_1 + \sigma_3 - 2) \int_\Omega \frac{(u + 1)^{p-\eta-1}}{v^{2k_1+\sigma_3-1}w^{2k_2+\sigma_4-2}} [D(u)\nabla u \cdot \nabla v - G(u)\chi(v)|\nabla v|^2$$

$$+ H(u)\xi(w)\nabla v \cdot \nabla w]$$

$$+ (2k_2 + \sigma_4 - 2) \int_\Omega \frac{(u + 1)^{p-\eta-1}}{v^{2k_1+\sigma_3-2}w^{2k_2+\sigma_4-1}} [D(u)\nabla u \cdot \nabla w - G(u)\chi(v)\nabla v \cdot \nabla w$$

$$+ H(u)\xi(w)|\nabla w|^2],$$

and then using (1.6)–(1.8) yields

$$J_1 \leq -a_0 (p - \eta - 1) \int_\Omega \frac{(u + 1)^{p+m-\eta-3}}{v^{2k_1+\sigma_3-2}w^{2k_2+\sigma_4-2}} |\nabla u|^2$$

$$+ b_0 \chi_0 (p - \eta - 1) \int_\Omega \frac{(u + 1)^{p+q-\eta-3}}{v^{3k_1+\sigma_3-2}w^{2k_2+\sigma_4-2}} |\nabla u||\nabla v|$$

$$+ c_0 \xi_0 (p - \eta - 1) \int_\Omega \frac{(u + 1)^{p+r-\eta-3}}{v^{2k_1+\sigma_3-2}w^{3k_2+\sigma_4-2}} |\nabla u||\nabla w|$$

$$+ a_0 (2k_1 + \sigma_3 - 2) \int_\Omega \frac{(u + 1)^{p+m-n-2}}{v^{2k_1+\sigma_3-1}w^{2k_2+\sigma_4-2}} |\nabla u||\nabla v|$$

$$- (2k_1 + \sigma_3 - 2) \int_\Omega \frac{(u + 1)^{p-\eta-1}}{v^{2k_1+\sigma_3-1}w^{2k_2+\sigma_4-2}} G(u)\chi(v)|\nabla v|^2$$

$$+ c_0 \xi_0 (2k_1 + \sigma_3 - 2) \int_\Omega \frac{(u + 1)^{p+r-\eta-2}}{v^{2k_1+\sigma_3-1}w^{3k_2+\sigma_4-2}} |\nabla v||\nabla w|$$

$$+ a_0 (2k_2 + \sigma_4 - 2) \int_\Omega \frac{(u + 1)^{p+m-n-2}}{v^{2k_1+\sigma_3-2}w^{2k_2+\sigma_4-1}} |\nabla u||\nabla w|$$

$$+ b_0 \chi_0 (2k_2 + \sigma_4 - 2) \int_\Omega \frac{(u + 1)^{p+q-\eta-2}}{v^{3k_1+\sigma_3-2}w^{2k_2+\sigma_4-1}} |\nabla v||\nabla w|$$

$$+ c_0 \xi_0 (2k_2 + \sigma_4 - 2) \int_\Omega \frac{(u + 1)^{p+r-\eta-2}}{v^{2k_1+\sigma_3-2}w^{3k_2+\sigma_4-1}} |\nabla w|^2.$$
As to the second term $J_2$ and third term $J_3$, due to integration by parts and straightforward calculations, we infer

\[
J_2 = d_1 \int_{\Omega} \nabla \left( \frac{(u + 1)^{p-\eta}}{v^{2k_1+\sigma_3-1}w^{2k_2+\sigma_4-2}} \right) \cdot \nabla v \\
- \alpha \int_{\Omega} \frac{u(u + 1)^{p-\eta}}{v^{2k_1+\sigma_3-1}w^{2k_2+\sigma_4-2}} + \beta \int_{\Omega} \frac{(u + 1)^{p-\eta}}{v^{2k_1+\sigma_3-2}w^{2k_2+\sigma_4-2}} \\
= d_1(p - \eta) \int_{\Omega} \frac{(u + 1)^{p-\eta-1}}{v^{2k_1+\sigma_3-1}w^{2k_2+\sigma_4-2}} \nabla u \cdot \nabla v \\
- d_1(2k_1 + \sigma_3 - 1) \int_{\Omega} \frac{(u + 1)^{p-\eta}}{v^{2k_1+\sigma_3}w^{2k_2+\sigma_4-2}} |\nabla v|^2 \\
- d_1(2k_2 + \sigma_4 - 2) \int_{\Omega} \frac{(u + 1)^{p-\eta}}{v^{2k_1+\sigma_3-1}w^{2k_2+\sigma_4-1}} \nabla v \cdot \nabla w \\
- \alpha \int_{\Omega} \frac{u(u + 1)^{p-\eta}}{v^{2k_1+\sigma_3}w^{2k_2+\sigma_4-2}} + \beta \int_{\Omega} \frac{(u + 1)^{p-\eta}}{v^{2k_1+\sigma_3-2}w^{2k_2+\sigma_4-2}} \\
\leq d_1(p - \eta) \int_{\Omega} \frac{(u + 1)^{p-\eta-1}}{v^{2k_1+\sigma_3-1}w^{2k_2+\sigma_4-2}} |\nabla u||\nabla v| \\
- d_1(2k_1 + \sigma_3 - 1) \int_{\Omega} \frac{(u + 1)^{p-\eta}}{v^{2k_1+\sigma_3}w^{2k_2+\sigma_4-2}} |\nabla v|^2 \\
+ d_1(2k_2 + \sigma_4 - 2) \int_{\Omega} \frac{(u + 1)^{p-\eta}}{v^{2k_1+\sigma_3-1}w^{2k_2+\sigma_4-1}} |\nabla v||\nabla w| \\
- \alpha \int_{\Omega} \frac{u(u + 1)^{p-\eta}}{v^{2k_1+\sigma_3}w^{2k_2+\sigma_4-2}} + \beta \int_{\Omega} \frac{(u + 1)^{p-\eta}}{v^{2k_1+\sigma_3-2}w^{2k_2+\sigma_4-2}}
\]

and

\[
J_3 = d_2(p - \eta) \int_{\Omega} \frac{(u + 1)^{p-\eta-1}}{v^{2k_1+\sigma_3-2}w^{2k_2+\sigma_4-1}} \nabla u \cdot \nabla w \\
- d_2(2k_1 + \sigma_3 - 2) \int_{\Omega} \frac{(u + 1)^{p-\eta}}{v^{2k_1+\sigma_3-1}w^{2k_2+\sigma_4-1}} \nabla v \cdot \nabla w \\
- d_2(2k_2 + \sigma_4 - 1) \int_{\Omega} \frac{(u + 1)^{p-\eta}}{v^{2k_1+\sigma_3-2}w^{2k_2+\sigma_4-1}} |\nabla w|^2 \\
- \gamma \int_{\Omega} \frac{u(u + 1)^{p-\eta}}{v^{2k_1+\sigma_3-2}w^{2k_2+\sigma_4-1}} + \delta \int_{\Omega} \frac{(u + 1)^{p-\eta}}{v^{2k_1+\sigma_3-2}w^{2k_2+\sigma_4-2}} \\
\leq d_2(p - \eta) \int_{\Omega} \frac{(u + 1)^{p-\eta-1}}{v^{2k_1+\sigma_3-2}w^{2k_2+\sigma_4-1}} |\nabla u||\nabla w| \\
+ d_2(2k_1 + \sigma_3 - 2) \int_{\Omega} \frac{(u + 1)^{p-\eta}}{v^{2k_1+\sigma_3-1}w^{2k_2+\sigma_4-1}} |\nabla v||\nabla w| \\
- d_2(2k_2 + \sigma_4 - 1) \int_{\Omega} \frac{(u + 1)^{p-\eta}}{v^{2k_1+\sigma_3-2}w^{2k_2+\sigma_4-1}} |\nabla w|^2 \\
- \gamma \int_{\Omega} \frac{u(u + 1)^{p-\eta}}{v^{2k_1+\sigma_3-2}w^{2k_2+\sigma_4-1}} + \delta \int_{\Omega} \frac{(u + 1)^{p-\eta}}{v^{2k_1+\sigma_3-2}w^{2k_2+\sigma_4-2}}.
\]

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Combining (3.12) and the above estimates for $J_1, J_2, J_3$, we obtain

$$
\frac{d}{dt} \int_{\Omega} \frac{(u + 1)^{p-\eta \cdot 3}}{v^{2k_1+\sigma_3-2}w^{2k_2+\sigma_4-2}} |\nabla u|^2 + A_2 \int_{\Omega} \frac{(u + 1)^{p+q-\eta \cdot 3}}{v^{3k_1+\sigma_3-2}w^{2k_2+\sigma_4-2}} |\nabla u||\nabla v| \\
+ A_3 \int_{\Omega} \frac{(u + 1)^{p+r-\eta \cdot 3}}{v^{2k_1+\sigma_3-2}w^{3k_2+\sigma_4-2}} |\nabla u|||\nabla w| + A_4 \int_{\Omega} \frac{(u + 1)^{p+m-\eta \cdot 2}}{v^{2k_1+\sigma_3-1}w^{2k_2+\sigma_4-2}} |\nabla u|| \nabla w| \\
- A_5 \int_{\Omega} \frac{(u + 1)^{p-\eta \cdot 1}}{v^{2k_1+\sigma_3-1}w^{2k_2+\sigma_4-2}} G(u)\chi(v)|\nabla v|^2 + A_6 \int_{\Omega} \frac{(u + 1)^{p+r-\eta \cdot 2}}{v^{2k_1+\sigma_3-2}w^{3k_2+\sigma_4-2}} |\nabla v||\nabla w| \\
+ A_7 \int_{\Omega} \frac{(u + 1)^{p+m-\eta \cdot 2}}{v^{2k_1+\sigma_3-2}w^{2k_2+\sigma_4-1}} |\nabla u||\nabla w| + A_8 \int_{\Omega} \frac{(u + 1)^{p+q-\eta \cdot 2}}{v^{3k_1+\sigma_3-2}w^{2k_2+\sigma_4-1}} |\nabla v||\nabla w| \\
+ A_9 \int_{\Omega} \frac{(u + 1)^{p+r-\eta \cdot 2}}{v^{2k_1+\sigma_3-2}w^{3k_2+\sigma_4-1}} |\nabla w|^2 \\
+ A_{10} \int_{\Omega} \frac{(u + 1)^{p-\eta \cdot 1}}{v^{2k_1+\sigma_3-1}w^{2k_2+\sigma_4-2}} |\nabla u||\nabla v| - A_{11} \int_{\Omega} \frac{(u + 1)^{p-\eta}}{v^{2k_1+\sigma_3-1}w^{2k_2+\sigma_4-2}} |\nabla v|^2 \\
+ A_{12} \int_{\Omega} \frac{(u + 1)^{p-\eta}}{v^{2k_1+\sigma_3-1}w^{2k_2+\sigma_4-1}} |\nabla v||\nabla w| - A_{13} \int_{\Omega} \frac{u(u + 1)^{p-\eta}}{v^{2k_1+\sigma_3-1}w^{2k_2+\sigma_4-2}} \\
+ A_{14} \int_{\Omega} \frac{(u + 1)^{p-\eta}}{v^{2k_1+\sigma_3-2}w^{2k_2+\sigma_4-2}} \\
+ A_{15} \int_{\Omega} \frac{(u + 1)^{p-\eta \cdot 1}}{v^{2k_1+\sigma_3-2}w^{2k_2+\sigma_4-2}} |\nabla u||\nabla w| + A_{16} \int_{\Omega} \frac{(u + 1)^{p-\eta}}{v^{2k_1+\sigma_3-1}w^{2k_2+\sigma_4-1}} |\nabla v||\nabla w| \\
- A_{17} \int_{\Omega} \frac{(u + 1)^{p-\eta}}{v^{2k_1+\sigma_3-2}w^{2k_2+\sigma_4-2}} |\nabla w|^2 - A_{18} \int_{\Omega} \frac{u(u + 1)^{p-\eta}}{v^{2k_1+\sigma_3-2}w^{2k_2+\sigma_4-1}} \\
+ A_{19} \int_{\Omega} \frac{(u + 1)^{p-\eta}}{v^{2k_1+\sigma_3-2}w^{2k_2+\sigma_4-2}},
$$

where

- $A_1 := a_0(p - \eta - 1)(p - \eta)$,
- $A_2 := b_0\chi_0(p - \eta - 1)(p - \eta)$,
- $A_3 := c_0\xi_0(p - \eta - 1)(p - \eta)$,
- $A_4 := a_0(p - \eta)(2k_1 + \sigma_3 - 2)$,
- $A_5 := (p - \eta)(2k_1 + \sigma_3 - 2)$,
- $A_6 := c_0\xi_0(p - \eta)(2k_1 + \sigma_3 - 2)$,
- $A_7 := a_0(p - \eta)(2k_2 + \sigma_4 - 2)$,
- $A_8 := b_0\chi_0(p - \eta)(2k_2 + \sigma_4 - 2)$,
- $A_9 := c_0\xi_0(p - \eta)(2k_2 + \sigma_4 - 2)$,
- $A_{10} := d_1(p - \eta)(2k_1 + \sigma_3 - 2)$,
- $A_{11} := d_1(2k_1 + \sigma_3 - 2)(2k_1 + \sigma_3 - 1)$,
- $A_{12} := d_1(2k_1 + \sigma_3 - 2)(2k_2 + \sigma_4 - 2)$,
- $A_{13} := \alpha(2k_1 + \sigma_3 - 2)$,
- $A_{14} := \beta(2k_1 + \sigma_3 - 2)$,
- $A_{15} := d_2(p - \eta)(2k_2 + \sigma_4 - 2)$,
- $A_{16} := d_2(2k_1 + \sigma_3 - 2)(2k_2 + \sigma_4 - 2)$,
- $A_{17} := d_2(2k_2 + \sigma_4 - 2)(2k_2 + \sigma_4 - 1)$,
- $A_{18} := \gamma(2k_2 + \sigma_4 - 2)$,
- $A_{19} := \delta(2k_2 + \sigma_4 - 2)$.
Similarly, we can derive an estimate for \( \frac{d}{dt} \int_{\Omega} \frac{(u+1)^{p-\eta}}{v^{2k_1+\sigma_1-2}} \), that is,

\[
\frac{d}{dt} \int_{\Omega} \frac{(u+1)^{p-\eta}}{v^{2k_1+\sigma_1-2}} \leq -a_0(p-\eta-1)(p-\eta) \int_{\Omega} \frac{(u+1)^{p+m-\eta-3}}{v^{2k_1+\sigma_1-2}} |\nabla u|^2
\]

\[
+ b_0\chi_0(p-\eta-1)(p-\eta) \int_{\Omega} \frac{(u+1)^{p+q-\eta-3}}{v^{3k_1+\sigma_1-2}} |\nabla u||\nabla v|
\]

\[
+ c_0\xi_0(p-\eta-1)(p-\eta) \int_{\Omega} \frac{(u+1)^{p+r-\eta-3}}{v^{2k_1+\sigma_1-2}w^k_2} |\nabla u||\nabla w|
\]

\[
+ a_0(p-\eta)(2k_1+\sigma_1-2) \int_{\Omega} \frac{(u+1)^{p+m-\eta-2}}{v^{2k_1+\sigma_1-1}} |\nabla u||\nabla v|
\]

\[
- (p-\eta)(2k_1+\sigma_1-2) \int_{\Omega} \frac{(u+1)^{p+n-1}}{v^{2k_1+\sigma_1-1}} G(u)\chi(v)|\nabla v|^2
\]

\[
+ c_0\xi_0(p-\eta)(2k_1+\sigma_1-2) \int_{\Omega} \frac{(u+1)^{p+r-\eta-2}}{v^{2k_1+\sigma_1-1}w^k_2} |\nabla v||\nabla w|
\]

\[
+ d_1(p-\eta)(2k_1+\sigma_1-2) \int_{\Omega} \frac{(u+1)^{p+n-1}}{v^{2k_1+\sigma_1-1}} |\nabla u||\nabla v|
\]

\[
- d_1(2k_1+\sigma_1-2)(2k_1+\sigma_1-1) \int_{\Omega} \frac{(u+1)^{p-\eta}}{v^{2k_1+\sigma_1}} |\nabla v|^2
\]

\[
- \alpha(2k_1+\sigma_1-2) \int_{\Omega} \frac{u(u+1)^{p-\eta}}{v^{2k_1+\sigma_1-1}}
\]

\[
+ \beta(2k_1+\sigma_1-2) \int_{\Omega} \frac{(u+1)^{p-\eta}}{v^{2k_1+\sigma_1-2}}.
\]  \( (3.14) \)

We next estimate \( \frac{d}{dt} \int_{\Omega} \frac{(u+1)^{p-\eta}}{w^{2k_2+\sigma_2-2}} \). Using the first and third equations in (1.1), we see that

\[
\frac{d}{dt} \int_{\Omega} \frac{(u+1)^{p-\eta}}{w^{2k_2+\sigma_2-2}}
\]

\[
= (p-\eta) \int_{\Omega} \frac{(u+1)^{p-\eta-1}}{w^{2k_2+\sigma_2-2}} u_t - (2k_2 + \sigma_2 - 2) \int_{\Omega} \frac{(u+1)^{p-\eta}}{w^{2k_2+\sigma_2-1}} w_t
\]

\[
= (p-\eta) \int_{\Omega} \frac{(u+1)^{p-\eta-1}}{w^{2k_2+\sigma_2-2}} \nabla \cdot [D(u)\nabla u - G(u)\chi(v)\nabla v + H(u)\xi(w)\nabla w]
\]

\[
- (2k_2 + \sigma_2 - 2) \int_{\Omega} \frac{(u+1)^{p-\eta}}{w^{2k_2+\sigma_2-1}} (d_2\Delta w + \gamma u - \delta w)
\]

\[
=: (p-\eta)J_4 + (2k_2 + \sigma_2 - 2)J_5.
\]  \( (3.15) \)
Estimating $J_4, J_5$ in the same way as $J_1, J_2$, respectively, we obtain

$$J_4 \leq -a_0(p - \eta - 1) \int_{\Omega} \frac{(u + 1)^{p+m-\eta-3}}{w^{2k_2 + \sigma_2 - 2}} |\nabla u|^2$$

$$+ b_0 \chi_0(p - \eta - 1) \int_{\Omega} \frac{(u + 1)^{p+q-\eta-3}}{v^{k_1} w^{2k_2 + \sigma_2 - 2}} |\nabla u||\nabla v|$$

$$+ c_0 \xi_0(p - \eta - 1) \int_{\Omega} \frac{(u + 1)^{p+r-\eta-3}}{w^{3k_2 + \sigma_2 - 2}} |\nabla u||\nabla w|$$

$$+ a_0(2k_2 + \sigma_2 - 2) \int_{\Omega} \frac{(u + 1)^{p+m-\eta-2}}{w^{2k_2 + \sigma_2 - 1}} |\nabla u||\nabla w|$$

$$+ b_0 \chi_0(2k_2 + \sigma_2 - 2) \int_{\Omega} \frac{(u + 1)^{p+q-\eta-2}}{v^{k_1} w^{2k_2 + \sigma_2 - 1}} |\nabla v||\nabla w|$$

$$+ c_0 \xi_0(2k_2 + \sigma_2 - 2) \int_{\Omega} \frac{(u + 1)^{p+r-\eta-2}}{w^{3k_2 + \sigma_2 - 1}} |\nabla w|^2$$

and

$$J_5 \leq d_2(p - \eta) \int_{\Omega} \frac{(u + 1)^{p-\eta-1}}{w^{2k_2 + \sigma_2 - 1}} |\nabla u||\nabla w| - d_2(2k_2 + \sigma_2 - 1) \int_{\Omega} \frac{(u + 1)^{p-\eta}}{w^{2k_2 + \sigma_2}} |\nabla w|^2$$

$$- \gamma \int_{\Omega} \frac{u(u + 1)^{p-\eta}}{w^{2k_2 + \sigma_2 - 1}} + \delta \int_{\Omega} \frac{(u + 1)^{p-\eta}}{w^{2k_2 + \sigma_2 - 2}}.$$

Thus a combination of (3.15) and the above estimates for $J_4, J_5$ yields that

$$\frac{d}{dt} \int_{\Omega} \frac{(u + 1)^{p-\eta}}{w^{2k_2 + \sigma_2 - 2}} \leq -a_0(p - \eta - 1)(p - \eta) \int_{\Omega} \frac{(u + 1)^{p+m-\eta-3}}{w^{2k_2 + \sigma_2 - 2}} |\nabla u|^2$$

$$+ b_0 \chi_0(p - \eta - 1)(p - \eta) \int_{\Omega} \frac{(u + 1)^{p+q-\eta-3}}{v^{k_1} w^{2k_2 + \sigma_2 - 2}} |\nabla u||\nabla v|$$

$$+ c_0 \xi_0(p - \eta - 1)(p - \eta) \int_{\Omega} \frac{(u + 1)^{p+r-\eta-3}}{w^{3k_2 + \sigma_2 - 2}} |\nabla u||\nabla w|$$

$$+ a_0(p - \eta)(2k_2 + \sigma_2 - 2) \int_{\Omega} \frac{(u + 1)^{p+m-\eta-2}}{w^{2k_2 + \sigma_2 - 1}} |\nabla u||\nabla w|$$

$$+ b_0 \chi_0(p - \eta)(2k_2 + \sigma_2 - 2) \int_{\Omega} \frac{(u + 1)^{p+q-\eta-2}}{v^{k_1} w^{2k_2 + \sigma_2 - 1}} |\nabla v||\nabla w|$$

$$+ c_0 \xi_0(p - \eta)(2k_2 + \sigma_2 - 2) \int_{\Omega} \frac{(u + 1)^{p+r-\eta-2}}{w^{3k_2 + \sigma_2 - 1}} |\nabla w|^2$$

$$+ d_2(p - \eta)(2k_2 + \sigma_2 - 2) \int_{\Omega} \frac{(u + 1)^{p-\eta-1}}{w^{2k_2 + \sigma_2 - 1}} |\nabla u||\nabla w|$$

$$- d_2(2k_2 + \sigma_2 - 2)(2k_2 + \sigma_2 - 1) \int_{\Omega} \frac{(u + 1)^{p-\eta}}{w^{2k_2 + \sigma_2}} |\nabla w|^2$$

$$- \gamma(2k_2 + \sigma_2 - 2) \int_{\Omega} \frac{u(u + 1)^{p-\eta}}{w^{2k_2 + \sigma_2 - 1}} + \delta(2k_2 + \sigma_2 - 2) \int_{\Omega} \frac{(u + 1)^{p-\eta}}{w^{2k_2 + \sigma_2 - 2}}. \quad (3.16)$$
Adding estimates (3.14) and (3.16), and moving $\int_\Omega (u+1)^{p-\eta} |\nabla v|^2$ and $\int_\Omega (u+1)^{p-\eta} |\nabla w|^2$ to the left-hand side, we have

\[
\frac{d}{dt} \int_\Omega \frac{(u+1)^{p-\eta}}{v^{2k_1+\sigma_1-2}} + \frac{d}{dt} \int_\Omega \frac{(u+1)^{p-\eta}}{w^{2k_2+\sigma_2-2}} + B_{19} \int_\Omega \frac{(u+1)^{p-\eta}}{v^{2k_1+\sigma_1}} |\nabla v|^2 + B_{20} \int_\Omega \frac{(u+1)^{p-\eta}}{w^{2k_2+\sigma_2}} |\nabla w|^2
\]

\[\leq -B_1 \int_\Omega \frac{(u+1)^{p+m-n-3}}{v^{2k_1+\sigma_1-2}} |\nabla u|^2 - B_2 \int_\Omega \frac{(u+1)^{p+m-n-3}}{w^{2k_2+\sigma_2-2}} |\nabla u|^2 + B_3 \int_\Omega \frac{(u+1)^{p+m-n-3}}{v^{2k_1+\sigma_1-1}} |\nabla u||\nabla v| + B_4 \int_\Omega \frac{(u+1)^{p+m-n-3}}{w^{2k_2+\sigma_2-1}} |\nabla u||\nabla w|
\]

\[+ B_5 \int_\Omega \frac{(u+1)^{p+a-n-3}}{v^{3k_1+\sigma_1-2}} |\nabla u||\nabla v| - B_6 \int_\Omega \frac{(u+1)^{p-\eta-1}}{v^{2k_1+\sigma_1-2}} G(u) \chi(v)|\nabla v|^2
\]

\[+ B_7 \int_\Omega \frac{(u+1)^{p+a-n-3}}{v^{3k_1+\sigma_1-2}} |\nabla u||\nabla v| + B_8 \int_\Omega \frac{(u+1)^{p+r-\eta-3}}{w^{3k_2+\sigma_2-2}} |\nabla u||\nabla w|
\]

\[+ B_9 \int_\Omega \frac{(u+1)^{p+r-n-3}}{v^{2k_1+\sigma_1-2}} |\nabla u||\nabla w| + B_{10} \int_\Omega \frac{(u+1)^{p+r-n-3}}{w^{2k_2+\sigma_2-1}} |\nabla v||\nabla w|
\]

\[+ B_{11} \int_\Omega \frac{(u+1)^{p+r-\eta-2}}{v^{k_1+\sigma_1-2}w^{k_2}} |\nabla v||\nabla w| + B_{12} \int_\Omega \frac{(u+1)^{p+r-\eta-2}}{w^{2k_2+\sigma_2-1}} |\nabla w|^2
\]

\[+ B_{13} \int_\Omega \frac{(u+1)^{p-\eta-1}}{v^{2k_1+\sigma_1-2}} |\nabla u||\nabla v| + B_{14} \int_\Omega \frac{(u+1)^{p-\eta-1}}{w^{2k_2+\sigma_2-2}} |\nabla u||\nabla w|
\]

\[B_{15} \int_\Omega \frac{u(u+1)^{p-\eta}}{v^{2k_1+\sigma_1-1}} + B_{16} \int_\Omega \frac{u(u+1)^{p-\eta}}{v^{2k_1+\sigma_1-2}} + B_{17} \int_\Omega \frac{u(u+1)^{p-\eta}}{w^{2k_2+\sigma_2-1}} + B_{18} \int_\Omega \frac{u(u+1)^{p-\eta}}{w^{2k_2+\sigma_2-2}}, \tag{3.17}
\]

where

\[
B_1 := a_0(p-\eta-1)(p-\eta), \quad B_2 := a_0(p-\eta-1)(p-\eta),
\]

\[
B_3 := a_0(p-\eta)(2k_1+\sigma_1-2), \quad B_4 := a_0(p-\eta)(2k_2+\sigma_2-2),
\]

\[
B_5 := b_0\chi_0(p-\eta-1)(p-\eta), \quad B_6 := (p-\eta)(2k_1+\sigma_1-2),
\]

\[
B_7 := b_0\chi_0(p-\eta-1)(p-\eta), \quad B_8 := c_0\xi_0(p-\eta-1)(p-\eta),
\]

\[
B_9 := c_0\pi_0(p-\eta-1)(p-\eta), \quad B_{10} := c_0\xi_0(p-\eta-1)(p-\eta),
\]

\[
B_{11} := b_0\chi_0(p-\eta)(2k_2+\sigma_2-2), \quad B_{12} := c_0\xi_0(p-\eta)(2k_2+\sigma_2-2),
\]

\[
B_{13} := d_1(p-\eta)(2k_1+\sigma_1-2), \quad B_{14} := d_2(p-\eta)(2k_2+\sigma_2-2),
\]

\[
B_{15} := \alpha(2k_1+\sigma_1-2), \quad B_{16} := \beta(2k_1+\sigma_1-2),
\]

\[
B_{17} := \gamma(2k_2+\sigma_2-2), \quad B_{18} := \delta(2k_2+\sigma_2-2),
\]

\[
B_{19} := d_1(2k_1+\sigma_1-2)(2k_1+\sigma_1-1), \quad B_{20} := d_2(2k_2+\sigma_2-2)(2k_2+\sigma_2-1).
\]

Adding (3.17) and (3.13) multiplied by $\varepsilon_{03} > 0$ and dropping the nine terms containing $A_i, B_j$ ($i \in \{1, 5, 13, 18\}, j \in \{1, 2, 6, 15, 17\}$), we can see that the following inequality
holds:

\[
\varepsilon_{03} \int_{\Omega} \frac{(u + 1)^{p - \eta}}{v^{2k_1 + \sigma_3 - 2} w^{2k_2 + \sigma_4 - 2}} + \frac{d}{dt} \int_{\Omega} \frac{(u + 1)^{p - \eta}}{v^{2k_1 + \sigma_1 - 2}} + \frac{d}{dt} \int_{\Omega} \frac{(u + 1)^{p - \eta}}{w^{2k_2 + \sigma_2 - 2}} + B_{19} \int_{\Omega} \frac{(u + 1)^{p - \eta}}{v^{2k_1 + \sigma_1}} |\nabla v|^2 \leq \varepsilon_{03} A_{17} \int_{\Omega} \frac{(u + 1)^{p - \eta}}{v^{2k_1 + \sigma_3 - 2} w^{2k_2 + \sigma_4}} |\nabla w|^2
\]

\[
\leq \varepsilon_{03} A_{11} \int_{\Omega} \frac{(u + 1)^{p + m - \eta - 2}}{v^{2k_1 + \sigma_3 - 1} w^{2k_2 + \sigma_4 - 2}} |\nabla u||\nabla v| + B_{20} \int_{\Omega} \frac{(u + 1)^{p + m - \eta - 2}}{w^{2k_2 + \sigma_2 - 2}} |\nabla u||\nabla v|
\]

\[
\leq \varepsilon_{03} A_{4} \int_{\Omega} \frac{(u + 1)^{p + m - \eta - 2}}{v^{2k_1 + \sigma_3 - 1} w^{2k_2 + \sigma_4 - 2}} |\nabla u||\nabla v| + B_{3} \int_{\Omega} \frac{(u + 1)^{p + m - \eta - 2}}{v^{2k_1 + \sigma_1 - 1}} |\nabla u||\nabla v|
\]

\[
+ \varepsilon_{03} A_{7} \int_{\Omega} \frac{(u + 1)^{p + m - \eta - 2}}{v^{2k_1 + \sigma_3 - 2} w^{2k_2 + \sigma_4 - 2}} |\nabla u||\nabla v| + B_{4} \int_{\Omega} \frac{(u + 1)^{p + m - \eta - 2}}{w^{2k_2 + \sigma_2 - 1}} |\nabla u||\nabla v|
\]

\[
+ \varepsilon_{03} A_{2} \int_{\Omega} \frac{(u + 1)^{p + q - \eta - 3}}{v^{3k_1 + \sigma_2 - 2} w^{2k_2 + \sigma_4 - 2}} |\nabla u||\nabla v| + B_{5} \int_{\Omega} \frac{(u + 1)^{p + q - \eta - 3}}{v^{3k_1 + \sigma_1 - 2}} |\nabla u||\nabla v|
\]

\[
+ B_{7} \int_{\Omega} \frac{(u + 1)^{p + q - \eta - 3}}{v^{2k_1 + \sigma_2 - 1} w^{2k_2 + \sigma_4 - 2}} |\nabla u||\nabla v| + B_{8} \int_{\Omega} \frac{(u + 1)^{p + q - \eta - 3}}{w^{2k_2 + \sigma_2 - 2}} |\nabla u||\nabla v|
\]

\[
+ \varepsilon_{03} A_{3} \int_{\Omega} \frac{(u + 1)^{p + r - \eta - 3}}{v^{2k_1 + \sigma_3 - 2} w^{2k_2 + \sigma_4 - 2}} |\nabla u||\nabla v| + B_{9} \int_{\Omega} \frac{(u + 1)^{p + r - \eta - 3}}{v^{2k_1 + \sigma_1 - 2} w^{2k_2}} |\nabla u||\nabla v|
\]

\[
+ \varepsilon_{03} A_{6} \int_{\Omega} \frac{(u + 1)^{p + r - \eta - 2}}{v^{2k_1 + \sigma_3 - 1} w^{2k_2 + \sigma_4 - 2}} |\nabla u||\nabla v| + \varepsilon_{03} A_{8} \int_{\Omega} \frac{(u + 1)^{p + q - \eta - 2}}{v^{3k_1 + \sigma_3 - 2} w^{2k_2 + \sigma_4 - 1}} |\nabla v||\nabla w|
\]

\[
+ B_{10} \int_{\Omega} \frac{(u + 1)^{p + r - \eta - 2}}{v^{2k_1 + \sigma_3 - 2} w^{2k_2 + \sigma_4 - 2}} |\nabla v||\nabla w| + B_{11} \int_{\Omega} \frac{(u + 1)^{p + r - \eta - 2}}{v^{2k_1 + \sigma_1 - 2} w^{2k_2}} |\nabla v||\nabla w|
\]

\[
+ \varepsilon_{03} A_{9} \int_{\Omega} \frac{(u + 1)^{p + r - \eta - 2}}{v^{2k_1 + \sigma_3 - 2} w^{2k_2 + \sigma_4 - 2}} |\nabla w|^2 + B_{12} \int_{\Omega} \frac{(u + 1)^{p + r - \eta - 2}}{w^{2k_2 + \sigma_2 - 2}} |\nabla w|^2
\]

\[
+ \varepsilon_{03} A_{10} \int_{\Omega} \frac{(u + 1)^{p - \eta - 1}}{v^{2k_1 + \sigma_3 - 1} w^{2k_2 + \sigma_4 - 2}} |\nabla u||\nabla v| + B_{13} \int_{\Omega} \frac{(u + 1)^{p - \eta - 1}}{v^{2k_1 + \sigma_1 - 1}} |\nabla u||\nabla v|
\]

\[
+ \varepsilon_{03} A_{15} \int_{\Omega} \frac{(u + 1)^{p - \eta - 1}}{v^{2k_1 + \sigma_3 - 2} w^{2k_2 + \sigma_4 - 1}} |\nabla u||\nabla w| + B_{14} \int_{\Omega} \frac{(u + 1)^{p - \eta - 1}}{w^{2k_2 + \sigma_2 - 2}} |\nabla u||\nabla w|
\]

\[
+ \varepsilon_{03} (A_{12} + A_{16}) \int_{\Omega} \frac{(u + 1)^{p - \eta}}{v^{2k_1 + \sigma_3 - 1} w^{2k_2 + \sigma_4 - 1}} |\nabla v||\nabla w|
\]

\[
+ \varepsilon_{03} A_{14} \int_{\Omega} \frac{(u + 1)^{p - \eta}}{v^{2k_1 + \sigma_3 - 2} w^{2k_2 + \sigma_4 - 2}} + B_{16} \int_{\Omega} \frac{(u + 1)^{p - \eta}}{v^{2k_1 + \sigma_1 - 2}}
\]

\[
+ \varepsilon_{03} A_{19} \int_{\Omega} \frac{(u + 1)^{p - \eta}}{v^{2k_1 + \sigma_3 - 2} w^{2k_2 + \sigma_4 - 2}} + B_{18} \int_{\Omega} \frac{(u + 1)^{p - \eta}}{w^{2k_2 + \sigma_2 - 2}}.
\]

(3.18)
We now estimate the twenty-five terms $I_1 - I_{25}$ by dividing it into the four steps. Here we note from (2.1), (2.2) and the condition $q, r < 2$ that $I_5, I_6, I_8, I_9$ can be estimated by $I_{17}, I_{18}, I_{20}, I_{19}$, and that $I_{11}, I_{12}$ can be estimated by $I_{21}$, respectively.

**Step 1.** We estimate the fourteen terms containing $|\nabla u|$ (i.e., $I_1 - I_{10}$, $I_{17} - I_{20}$) so that the integral $\int_{\Omega} (u + 1)^{p-2}|\nabla u|^2$ appears. For instance, as to an estimate for $I_1$, for all $\varepsilon_1 > 0$, it can be obtained upon Lemma 2.3 that

$$I_1 = \varepsilon_{03} A_4 \int_{\Omega} (u + 1)^{p-2} |\nabla u| \cdot \frac{(u + 1)^{p+2m-2\eta-2}}{v^{2k_1+\sigma_3-1}w^{2k_2+\sigma_4-2}}|\nabla v|$$

$$\leq \varepsilon_{03} A_4 \cdot \frac{\varepsilon_1}{2} \int_{\Omega} (u + 1)^{p-2} |\nabla u|^2 + \varepsilon_{03} A_4 \cdot \frac{1}{2\varepsilon_1} \int_{\Omega} \frac{(u + 1)^{p+2m-2\eta-2}}{v^{4k_1+2\sigma_3-2}w^{4k_2+2\sigma_4-4}}|\nabla v|^2$$

$$\leq \frac{\varepsilon_{03} A_4'}{2} \cdot \varepsilon_1 \int_{\Omega} (u + 1)^{p-2} |\nabla u|^2 + \varepsilon_{03} A_4' \cdot \frac{1}{2} \cdot \frac{\varepsilon_1}{\varepsilon_1^{-1}} \int_{\Omega} \frac{(u + 1)^{p-\eta}}{v^{4k_1+2\sigma_3-2}\theta_1 w^{4k_2+2\sigma_4-4}}|\nabla v|^2$$

with $\theta_1 := \frac{\varepsilon_1^{-1}}{p+2m-2\eta-2} > 1$ by the condition $\eta > 2(m-1)$ (see (3.10)). Here we can check that $(4k_1 + 2\sigma_3 - 2)\theta_1 \geq 2k_1 + \sigma_3$ and $(4k_2 + 2\sigma_4 - 4)\theta_1 \geq 2k_2 + \sigma_4 - 2$. Indeed, as to the former, a simple calculation and the assumption $\sigma_3 > 2(1 - k_1)$ (see Lemma 3.2) as well as (3.9) yield that

$$(4k_1 + 2\sigma_3 - 2)\theta_1 - (2k_1 + \sigma_3)$$

$$= (2k_1 + \sigma_3 - 1) \left( 2\theta_1 - 1 - \frac{1}{2k_1 + \sigma_3 - 1} \right)$$

$$= \frac{2k_1 + \sigma_3 - 1}{p + 2m - 2\eta - 2} \left( 2p - 2\eta - (p + 2m - 2\eta - 2) - \frac{p + 2m - 2\eta - 2}{2k_1 + \sigma_3 - 1} \right)$$

$$= \frac{2k_1 + \sigma_3 - 1}{p + 2m - 2\eta - 2} \left( 2k_1 + \sigma_3 - 1 - \frac{2k_1 + \sigma_3 - 2}{2k_1 + \sigma_3 - 1} \right)$$

$$= \frac{2k_1 + \sigma_3 - 2}{p + 2m - 2\eta - 2} \left( 2(p + 2m - 2\eta - 2) - \frac{2(m - \eta - 1)}{2k_1 + \sigma_3 - 1} \right)$$

$$> 0.$$
Proceeding similarly as the above estimate, we also obtain that for all $\varepsilon_i > 0$ there exist constants $\theta_i > 1$ ($i \in \{2, 3, 4, 7, 10, 17, 18, 19, 20\}$) such that

\[
I_2 \leq \frac{B_3}{2} \cdot \varepsilon_2 \int_{\Omega} (u + 1)^{p-2} |\nabla u|^2 + \frac{B_4}{2} \cdot \varepsilon_2^{\theta_2-1} \int_{\Omega} \frac{(u + 1)^{p-\eta}}{v^{2k_1+\sigma_1}} |\nabla v|^2 \\
+ \frac{B''_5}{2} \cdot \varepsilon_2^{\theta_5-1} - 1 \int_{\Omega} |\nabla v|^2, 
\]

(3.21)

\[
I_3 \leq \frac{\varepsilon_3 A_7}{2} \cdot \varepsilon_3 \int_{\Omega} (u + 1)^{p-2} |\nabla u|^2 + \frac{\varepsilon_3 A_8}{2} \cdot \varepsilon_3^{\theta_3-1} \int_{\Omega} \frac{(u + 1)^{p-\eta}}{v^{2k_1+\sigma_1-2w^{2k_2+\sigma_2}}}|\nabla w|^2 \\
+ \frac{\varepsilon_3 B''_7}{2} \cdot \varepsilon_3^{\theta_7-1} - 1 \int_{\Omega} |\nabla w|^2, 
\]

(3.22)

\[
I_4 \leq \frac{B_3}{2} \cdot \varepsilon_4 \int_{\Omega} (u + 1)^{p-2} |\nabla u|^2 + \frac{B_5}{2} \cdot \varepsilon_4^{\theta_4-1} \int_{\Omega} \frac{(u + 1)^{p-\eta}}{w^{2k_2+\sigma_2}} |\nabla w|^2 \\
+ \frac{B''_6}{2} \cdot \varepsilon_4^{\theta_6-1} - 1 \int_{\Omega} |\nabla w|^2, 
\]

(3.23)

\[
I_7 \leq \frac{B_3}{2} \cdot \varepsilon_7 \int_{\Omega} (u + 1)^{p-2} |\nabla u|^2 + \frac{B_7}{2} \cdot \varepsilon_7^{\theta_7-1} \int_{\Omega} \frac{(u + 1)^{p-\eta}}{v^{2k_1+\sigma_1-2w^{2k_2+\sigma_2}}}|\nabla v|^2 \\
+ \frac{B''_8}{2} \cdot \varepsilon_7^{\theta_8-1} - 1 \int_{\Omega} |\nabla v|^2, 
\]

(3.24)

\[
I_{10} \leq \frac{B_3}{2} \cdot \varepsilon_{10} \int_{\Omega} (u + 1)^{p-2} |\nabla u|^2 + \frac{B_9}{2} \cdot \varepsilon_{10}^{\theta_{10}-1} \int_{\Omega} \frac{(u + 1)^{p-\eta}}{v^{2k_1+\sigma_1-2w^{2k_2+\sigma_2}}}|\nabla w|^2 \\
+ \frac{B''_{11}}{2} \cdot \varepsilon_{10}^{\theta_{11}-1} - 1 \int_{\Omega} |\nabla w|^2, 
\]

(3.25)

\[
I_{17} \leq \frac{\varepsilon_{17} A_{10}}{2} \cdot \varepsilon_{17} \int_{\Omega} (u + 1)^{p-2} |\nabla u|^2 + \frac{\varepsilon_{17} A_{11}}{2} \cdot \varepsilon_{17}^{\theta_{17}-1} \int_{\Omega} \frac{(u + 1)^{p-\eta}}{v^{2k_1+\sigma_1-2w^{2k_2+\sigma_2}}}|\nabla v|^2 \\
+ \frac{\varepsilon_{17} B''_{12}}{2} \cdot \varepsilon_{17}^{\theta_{12}-1} - 1 \int_{\Omega} |\nabla v|^2, 
\]

(3.26)

\[
I_{18} \leq \frac{B_{13}}{2} \cdot \varepsilon_{18} \int_{\Omega} (u + 1)^{p-2} |\nabla u|^2 + \frac{B_{13}}{2} \cdot \varepsilon_{18}^{\theta_{18}-1} \int_{\Omega} \frac{(u + 1)^{p-\eta}}{v^{2k_1+\sigma_1}} |\nabla v|^2 \\
+ \frac{B''_{19}}{2} \cdot \varepsilon_{18}^{\theta_{19}-1} - 1 \int_{\Omega} |\nabla v|^2, 
\]

(3.27)

\[
I_{19} \leq \frac{\varepsilon_{19} A_{14}}{2} \cdot \varepsilon_{19} \int_{\Omega} (u + 1)^{p-2} |\nabla u|^2 + \frac{\varepsilon_{19} A_{15}}{2} \cdot \varepsilon_{19}^{\theta_{19}-1} \int_{\Omega} \frac{(u + 1)^{p-\eta}}{v^{2k_1+\sigma_1-2w^{2k_2+\sigma_2}}}|\nabla w|^2 \\
+ \frac{\varepsilon_{19} B''_{20}}{2} \cdot \varepsilon_{19}^{\theta_{20}-1} - 1 \int_{\Omega} |\nabla w|^2, 
\]

(3.28)

\[
I_{20} \leq \frac{B_{14}}{2} \cdot \varepsilon_{20} \int_{\Omega} (u + 1)^{p-2} |\nabla u|^2 + \frac{B_{14}}{2} \cdot \varepsilon_{20}^{\theta_{20}-1} \int_{\Omega} \frac{(u + 1)^{p-\eta}}{w^{2k_2+\sigma_2}} |\nabla w|^2 \\
+ \frac{B''_{21}}{2} \cdot \varepsilon_{20}^{\theta_{21}-1} - 1 \int_{\Omega} |\nabla w|^2. 
\]

(3.29)

**Step 2.** We estimate the five terms containing $|\nabla v||\nabla w|$ (i.e., $I_{11}$, $I_{12}$, $I_{21}$). Here we can omit estimates for $I_{11}$, $I_{12}$ as mentioned above. As to an estimate for $I_{13}$, we see that
for all $\varepsilon_{13} > 0$,

$$I_{13} = B_{10} \int_{\Omega} \frac{(u+1)^{\frac{p-r}{2}}}{v^{2k_1+2sigma_1-\sigma_2}} |\nabla v| \cdot \frac{(u+1)^{\frac{p-q}{2}}}{w^{2k_2+sigma_4}} |\nabla w|$$

$$\leq B_{10} \cdot \frac{1}{2\varepsilon_{13}} \int_{\Omega} \frac{(u+1)^{p+2r-q-4}}{v^{2k_1+2sigma_1-\sigma_4} w^{2k_2+sigma_4}} |\nabla v|^2 + B_{10} \cdot \frac{\varepsilon_{13}}{2} \int_{\Omega} \frac{(u+1)^{p-q}}{v^{2k_1+sigma_3-2} w^{2k_2+sigma_4}} |\nabla w|^2$$

$$\leq B_{11} \cdot \frac{1}{2\varepsilon_{13}} \int_{\Omega} \frac{(u+1)^{p-q}}{v^{2k_1+2sigma_1} w^{sigma_4}} |\nabla v|^2 + B_{11}'' \cdot \frac{1}{2\varepsilon_{13}} \int_{\Omega} \frac{(u+1)^{p-q}}{v^{2k_1+sigma_3} w^{2k_2+sigma_4}} |\nabla w|^2$$

$$= B_{12} \cdot \frac{1}{2\varepsilon_{13}} \int_{\Omega} \frac{(u+1)^{p-q}}{v^{2k_1+sigma_3} w^{2k_2+sigma_4}} |\nabla v|^2 + B_{12}'' \cdot \frac{1}{2\varepsilon_{13}} \int_{\Omega} \frac{(u+1)^{p-q}}{v^{2k_1+sigma_3} w^{2k_2+sigma_4}} |\nabla w|^2$$

(3.30)

holds, where $\theta_{13} := \frac{p-q}{p+2r-q-4} > 1$ by the condition $r < 2$. Here we used the facts that $(2k_1 + 2sigma_1 - sigma_3) \theta_{13} \geq 2k_1 + sigma_1$ and $-sigma_4 \theta_{13} > 0$ due to $\theta_{13} > 1, \sigma_1 > 0, \sigma_3 < 0$ and $\sigma_4 < 0$, respectively. Similarly, we establish that for all $\varepsilon_{14} > 0$ there exists $\theta_{14} > 1$ such that

$$I_{14} \leq B_{14} \cdot \frac{1}{2\varepsilon_{14}} \int_{\Omega} \frac{(u+1)^{p-q}}{v^{2k_1+sigma_4} w^{2k_2+sigma_2}} |\nabla v|^2$$

$$+ B_{14}'' \cdot \frac{1}{2\varepsilon_{14}} \int_{\Omega} \frac{(u+1)^{p-q}}{w^{2k_2+sigma_2}} |\nabla w|^2$$

(3.31)

Also, we can derive that for all $\varepsilon_{21} > 0$,

$$I_{21} = \varepsilon_{03} A_{12} \cdot \frac{1}{2} \int_{\Omega} \frac{(u+1)^{p-q}}{v^{2k_1+sigma_1} w^{2k_2+sigma_2}} |\nabla v|^2$$

$$\leq \varepsilon_{03} A_{12} \cdot \frac{1}{2} \int_{\Omega} \frac{(u+1)^{p-q}}{v^{2k_1+sigma_1} w^{2k_2+sigma_2}} |\nabla v|^2$$

(3.32)

holds, since $2k_1 - sigma_1 + 2sigma_2 - 2 \geq 0$ and $4k_2 + 2sigma_4 - 2 \geq 2k_2 + sigma_2$ due to $sigma_3 > 2(1 - k_1)$ and $sigma_4 > 2(1 - k_2)$, respectively.

**Step 3.** We estimate the two terms containing $|\nabla w|^2$ (i.e., $I_{15}, I_{16}$). As to an estimate for $I_{15}$, we deduce that for all $\varepsilon_{15} > 0$,

$$I_{15} \leq \varepsilon_{03} A_9 \cdot \frac{1}{2} \int_{\Omega} \frac{(u+1)^{p-q}}{v^{2k_1+sigma_3-2} w^{2k_2+sigma_4-1}} |\nabla v|^2$$

$$\leq \varepsilon_{03} A_9 \cdot \frac{1}{2} \int_{\Omega} \frac{(u+1)^{p-q}}{v^{2k_1+sigma_3-2} w^{2k_2+sigma_4-1}} |\nabla v|^2$$

(3.33)

holds, where $\theta_{15} := \frac{p-q}{p+2r-q-2} > 1$ by the condition $r < 2$. Proceeding similarly as the above estimate we obtain that for all $\varepsilon_{16} > 0$ there exists $\theta_{16} > 1$ such that

$$I_{16} \leq B_{12} \cdot \frac{1}{2\varepsilon_{16}} \int_{\Omega} \frac{(u+1)^{p-q}}{w^{2k_2+sigma_2}} |\nabla v|^2 + B_{12}'' \cdot \frac{1}{2\varepsilon_{16}} \int_{\Omega} \frac{(u+1)^{p-q}}{w^{2k_2+sigma_2}} |\nabla w|^2$$

(3.34)
Step 4. We estimate the four terms which do not contain $|\nabla u|, |\nabla v|, |\nabla w|$ (i.e., $I_{22} - I_{25}$) by $\int_{\Omega}(u + 1)^p$. Indeed, it suffices to note that
\[
\int_{\Omega}(u + 1)^{p-\frac{\eta}{p}} \leq \frac{p-\frac{\eta}{p}}{p} \int_{\Omega}(u + 1)^p + \frac{\eta}{p},
\] (3.35)
which leads to the required estimates.

Thus, in view of Steps 1–4, the estimates for the twenty-five terms $I_1 - I_{25}$ in (3.18) are complete. We finally derive (3.11). Combining (3.18) and (3.20)–(3.35), we have
\[
\begin{align*}
&\varepsilon_{03}^3 d\int_{\Omega} \frac{(u + 1)^{p-\frac{\eta}{p}}}{\theta_{2k_1+\sigma_2}}|\nabla u|^{\frac{\eta}{p}} + d\int_{\Omega} \frac{(u + 1)^{p-\frac{\eta}{p}}}{\theta_{2k_1+\sigma_2}}|\nabla v|^{\frac{\eta}{p}} + d\int_{\Omega} \frac{(u + 1)^{p-\frac{\eta}{p}}}{\theta_{2k_2+\sigma_2}}|\nabla w|^{\frac{\eta}{p}} \\
&+ B_{19} \int_{\Omega} \frac{(u + 1)^{p-\frac{\eta}{p}}}{\theta_{2k_1+\sigma_2}}|\nabla u|^{\frac{\eta}{p}} + B_{20} \int_{\Omega} \frac{(u + 1)^{p-\frac{\eta}{p}}}{\theta_{2k_2+\sigma_2}}|\nabla w|^{\frac{\eta}{p}} \\
&+ \varepsilon_{03}A_{11} \int_{\Omega} \frac{(u + 1)^{p-\frac{\eta}{p}}}{\theta_{2k_1+\sigma_2}}|\nabla v|^{\frac{\eta}{p}} + \varepsilon_{03}A_{17} \int_{\Omega} \frac{(u + 1)^{p-\frac{\eta}{p}}}{\theta_{2k_2+\sigma_2}}|\nabla w|^{\frac{\eta}{p}} \\
&\leq c_1 \int_{\Omega} \frac{(u + 1)^{p-\frac{\eta}{p}}}{\theta_{2k_1+\sigma_2}}|\nabla u|^{\frac{\eta}{p}} + c_2 \int_{\Omega} \frac{(u + 1)^{p-\frac{\eta}{p}}}{\theta_{2k_2+\sigma_2}}|\nabla w|^{\frac{\eta}{p}} \\
&+ c_3 \int_{\Omega} \frac{(u + 1)^{p-\frac{\eta}{p}}}{\theta_{2k_2+\sigma_2}}|\nabla v|^{\frac{\eta}{p}} + c_4 \int_{\Omega} \frac{(u + 1)^{p-\frac{\eta}{p}}}{\theta_{2k_2+\sigma_2}}|\nabla w|^{\frac{\eta}{p}} \\
&+ c_5 \int_{\Omega} \frac{(u + 1)^{p-\frac{\eta}{p}}}{\theta_{2k_2+\sigma_2}}|\nabla u|^{\frac{\eta}{p}} + c_6 \int_{\Omega} \frac{(u + 1)^{p-\frac{\eta}{p}}}{\theta_{2k_2+\sigma_2}}|\nabla w|^{\frac{\eta}{p}} + c_7 \int_{\Omega} \frac{(u + 1)^{p-\frac{\eta}{p}}}{\theta_{2k_2+\sigma_2}}|\nabla w|^{\frac{\eta}{p}} + c_8 \int_{\Omega} (u + 1)^p + c_9,
\end{align*}
\]
where
\[
\begin{align*}
c_1 &:= \frac{1}{2} (\varepsilon_{03}A_{12} + B_{3}\varepsilon_{2}^{\theta_{2}-1} + B_{10}\varepsilon_{13}^{\theta_{1}-1} + B_{18}\varepsilon_{18}^{\theta_{1}-1}), \\
c_2 &:= \frac{1}{2} (\varepsilon_{03}A_{12} + B_{4}\varepsilon_{14}^{\theta_{1}-1} + B_{11}\varepsilon_{16}^{\theta_{1}-1} + B_{12}\varepsilon_{16}^{\theta_{1}-1} + B_{14}\varepsilon_{20}^{\theta_{1}-1}), \\
c_3 &:= \frac{1}{2} (\varepsilon_{03}A_{14}^{\frac{\eta}{p}} - \varepsilon_{03}A_{10}^{\frac{\eta}{p}} + B_{11}^{\varepsilon_{16}^{\theta_{1}-1}} + B_{11}^{\varepsilon_{14}^{\theta_{1}-1}}), \\
c_4 &:= \frac{1}{2} (\varepsilon_{03}A_{16}^{\frac{\eta}{p}} - \varepsilon_{03}A_{15}^{\frac{\eta}{p}} + \varepsilon_{03}A_{15}^{\frac{\eta}{p}} - \varepsilon_{03}A_{16}^{\frac{\eta}{p}} - B_{9}^{\varepsilon_{10}^{\theta_{1}-1}} + B_{10}^{\varepsilon_{13}^{\theta_{1}-1}}), \\
c_5 &:= \frac{1}{2} (\varepsilon_{03}A_{14}^{\frac{\eta}{p}} + \varepsilon_{03}A_{7}^{\frac{\eta}{p}} - \varepsilon_{03}A_{10}^{\frac{\eta}{p}} - B_{13}^{\varepsilon_{18}^{\theta_{1}-1}} + B_{14}^{\varepsilon_{20}^{\theta_{1}-1}}), \\
c_6 &:= \frac{1}{2} (\varepsilon_{03}A_{14}^{\frac{\eta}{p}} - \varepsilon_{03}A_{10}^{\frac{\eta}{p}} - B_{13}^{\varepsilon_{18}^{\theta_{1}-1}} - B_{14}^{\varepsilon_{20}^{\theta_{1}-1}}), \\
c_7 &:= \frac{1}{2} (\varepsilon_{03}A_{16}^{\frac{\eta}{p}} - \varepsilon_{03}A_{15}^{\frac{\eta}{p}} - \varepsilon_{03}A_{15}^{\frac{\eta}{p}} - \varepsilon_{03}A_{16}^{\frac{\eta}{p}} - B_{9}^{\varepsilon_{10}^{\theta_{1}-1}} + B_{10}^{\varepsilon_{13}^{\theta_{1}-1}}), \\
c_8 &:= \frac{p-\frac{\eta}{p}}{p} (\varepsilon_{03}A_{14}^{\frac{\eta}{p}} + \varepsilon_{03}A_{19}^{\frac{\eta}{p}} + B_{16}^{\varepsilon_{18}^{\theta_{1}-1}}), \\
c_9 &:= \frac{\eta}{p} (\varepsilon_{03}A_{14}^{\frac{\eta}{p}} + \varepsilon_{03}A_{19}^{\frac{\eta}{p}} + B_{16}^{\varepsilon_{18}^{\theta_{1}-1}}).
\end{align*}
\]
Consequently, by choosing $\varepsilon_{03}, \varepsilon_i > 0 \ (i \in \{1, 2, 3, 4, 7, 10, 13, 14, 15, 16, 17, 18, 19, 20\})$ satisfying $B_{19} - c_1 > 0$, $B_{20} - c_2 > 0$, $\varepsilon_{03}A_{11} = c_3$, $\varepsilon_{03}A_{17} = c_4$ and $c_5 = \frac{a_0(p - m)(p - m + 1)}{4}$, we derive the differential inequality (3.11).

Proof of Theorem 1.1. Setting $\varepsilon_{01} := C_3$ and $\varepsilon_{02} := C_4$ in Lemma 3.1, where $C_3, C_4 > 0$ are constants appearing in Lemma 3.2, we know that

$$\frac{d}{dt} \left( \int_{\Omega} (u + 1)^{p-m+1} + \int_{\Omega} \frac{(u + 1)^{p-n}}{v^{2k_1+\sigma_1-2}} + \int_{\Omega} \frac{(u + 1)^{p-n}}{w^{2k_2+\sigma_2-2}} + \varepsilon_{03} \int_{\Omega} \frac{(u + 1)^{p-n}}{v^{2k_1+\sigma_3-2}w^{2k_2+\sigma_4-2}} \right)
+ a_0(p - m)(p - m + 1) \int_{\Omega} (u + 1)^{p-2}|\nabla u|^2
\leq c_1 \int_{\Omega} |\nabla v|^2 + c_2 \int_{\Omega} |\nabla w|^2 + c_3 \int_{\Omega} (u + 1)^p + c_4$$

(3.36)

for all $t \in (0, T_{\text{max}})$ with some $c_1, c_2, c_3, c_4 > 0$, $\varepsilon_{03} > 0$. Also, multiplying the second and third equations in (1.1) by $v$ and $w$, respectively, and integrating them over $\Omega$, we have

$$\frac{d}{dt} \int_{\Omega} v^2 + 2d_1 \int_{\Omega} |\nabla v|^2 \leq \gamma \int_{\Omega} u^2 - \beta \int_{\Omega} v^2,$$

(3.37)

$$\frac{d}{dt} \int_{\Omega} w^2 + 2d_2 \int_{\Omega} |\nabla w|^2 \leq \frac{\gamma^2}{\delta} \int_{\Omega} u^2 - \delta \int_{\Omega} w^2.$$

(3.38)

Multiplying (3.37) and (3.38) by $\frac{c_1}{2d_1}$ and $\frac{c_2}{2d_2}$, respectively, and adding them to (3.36), we obtain

$$\frac{d}{dt} \left( \int_{\Omega} (u + 1)^{p-m+1} + \int_{\Omega} \frac{(u + 1)^{p-n}}{v^{2k_1+\sigma_1-2}} + \int_{\Omega} \frac{(u + 1)^{p-n}}{w^{2k_2+\sigma_2-2}} + \varepsilon_{03} \int_{\Omega} \frac{(u + 1)^{p-n}}{v^{2k_1+\sigma_3-2}w^{2k_2+\sigma_4-2}} \right)
+ \frac{c_1}{2d_1} \int_{\Omega} v^2 + \frac{c_2}{2d_2} \int_{\Omega} w^2
+ a_0(p - m)(p - m + 1) \int_{\Omega} (u + 1)^{p-2}|\nabla u|^2
\leq c_3 \int_{\Omega} (u + 1)^p + \left( \frac{c_1 \alpha^2}{2d_1 \beta} + \frac{c_2 \gamma^2}{2d_2 \delta} \right) \int_{\Omega} u^2 - \frac{c_1 \beta}{2d_1} \int_{\Omega} v^2 - \frac{c_2 \delta}{2d_2} \int_{\Omega} w^2 + c_4$$

(3.39)

for all $t \in (0, T_{\text{max}})$. By virtue of the Gagliardo–Nirenberg inequality, we see that

$$c_3 \int_{\Omega} (u + 1)^p = c_3 \|(u + 1)^\frac{p}{2}\|_{L^2(\Omega)}^{\frac{p}{2}}
\leq c_5 \left( \|\nabla (u + 1)^\frac{p}{2}\|_{L^2(\Omega)}^{\theta_1} \|(u + 1)^\frac{p}{2}\|_{L^p(\Omega)}^{1-\theta_1} + \|(u + 1)^\frac{p}{2}\|_{L^p(\Omega)} \right)^2$$

with some $c_5 > 0$ and $\theta_1 := \frac{pn-n}{pn+2-n} \in (0, 1)$. Here, noting from the first equation in (1.1) that the mass conservation $\int_\Omega u(\cdot, t) = \int_\Omega u$ holds for all $t \in (0, T_{\text{max}})$ and using Young’s inequality, we derive

$$c_3 \int_{\Omega} (u + 1)^p \leq c_6 \left( \|\nabla (u + 1)^\frac{p}{2}\|_{L^2(\Omega)} + 1 \right)^2
\leq \frac{a_0(p - m)(p - m + 1)}{16} \int_{\Omega} (u + 1)^{p-2}|\nabla u|^2 + c_7$$

(3.40)
with some \( c_6, c_7 > 0 \). Also, recalling the lower estimates (2.1) and (2.2), we infer from (3.40) that

\[
\int_{\Omega} \frac{(u+1)^{p-\eta}}{v^{2k_1+\sigma_1-2}} + \int_{\Omega} \frac{(u+1)^{p-\eta}}{w^{2k_2+\sigma_2-2}} + \varepsilon_0 \int_{\Omega} \frac{(u+1)^{p-\eta}}{v^{2k_1+\sigma_1-2}w^{2k_2+\sigma_2-2}} \\
\leq \left( \frac{1}{\mu_1^{2k_1+\sigma_1-2}} + \frac{1}{\mu_2^{2k_2+\sigma_2-2}} + \frac{\varepsilon_0}{\mu_1^{2k_1+\sigma_1-2}\mu_2^{2k_2+\sigma_2-2}} \right) \int_{\Omega} (u+1)^{p-\eta} \\
\leq c_3 \int_{\Omega} (u+1)^p + c_8 \\
\leq \frac{a_0(p-m)(p-m+1)}{16} \int_{\Omega} (u+1)^{p-2} |\nabla u|^2 + c_9
\]

(3.41)

with some \( c_8, c_9 > 0 \). Moreover, we derive from the relation \( u^2 \leq (u+1)^p \) and (3.40) that

\[
\left( \frac{c_1 \alpha_2^2 + c_2 \gamma^2}{2d_1 \beta} + \frac{c_2 \gamma^2}{2d_2 \delta} \right) \int_{\Omega} u^2 \leq \frac{a_0(p-m)(p-m+1)}{16} \int_{\Omega} (u+1)^{p-2} |\nabla u|^2 + c_{10}
\]

(3.42)

with some \( c_{10} > 0 \). Collecting (3.40)–(3.42) in (3.39), we establish

\[
\frac{d}{dt} \left( \int_{\Omega} (u+1)^{p-m+1} + \int_{\Omega} \frac{(u+1)^{p-\eta}}{v^{2k_1+\sigma_1-2}} + \int_{\Omega} \frac{(u+1)^{p-\eta}}{w^{2k_2+\sigma_2-2}} + \varepsilon_0 \int_{\Omega} \frac{(u+1)^{p-\eta}}{v^{2k_1+\sigma_1-2}w^{2k_2+\sigma_2-2}} \right) \\
+ \frac{c_1}{2d_1} \int_{\Omega} u^2 + \frac{c_2}{2d_2} \int_{\Omega} w^2 \\
+ \frac{a_0(p-m)(p-m+1)}{16} \int_{\Omega} (u+1)^{p-2} |\nabla u|^2 + \int_{\Omega} \frac{(u+1)^{p-\eta}}{v^{2k_1+\sigma_1-2}} + \int_{\Omega} \frac{(u+1)^{p-\eta}}{w^{2k_2+\sigma_2-2}} \\
+ \varepsilon_0 \int_{\Omega} \frac{(u+1)^{p-\eta}}{v^{2k_1+\sigma_1-2}w^{2k_2+\sigma_2-2}} + \frac{c_1 \beta}{2d_1} \int_{\Omega} u^2 + \frac{c_2 \delta}{2d_2} \int_{\Omega} w^2 \\
\leq c_{11}
\]

(3.43)

for all \( t \in (0, T_{\max}) \) with some \( c_{11} > 0 \). Here we estimate the term \( \int_{\Omega} (u+1)^{p-2} |\nabla u|^2 \). Again by the Gagliardo–Nirenberg inequality, we have

\[
\int_{\Omega} (u+1)^{p-m+1} = \|(u+1)^{\frac{\theta_2}{L}}\|_{L^{\frac{p}{p-m+1}}(\Omega)}^{2(p-m+1)} \\
\leq c_{12} \left( \|\nabla (u+1)^{\frac{\theta_2}{L}}\|_{L^2(\Omega)}^{\theta_2} \| (u+1)^{\frac{\theta_2}{L}} \|_{L^{\frac{2}{\theta_2}}(\Omega)}^{1-\theta_2} + \| (u+1)^{\frac{\theta_2}{L}} \|_{L^{\frac{2}{\theta_2}}(\Omega)} \right)^{2(p-m+1)}
\]

with some \( c_{12} > 0 \) and \( \theta_2 := \frac{(p-m)pn}{(p-m+1)(pn+2-n)} \in (0, 1) \) for sufficiently large \( p \) fulfilling (3.9). This together with the mass conservation yields

\[
\int_{\Omega} (u+1)^{p-m+1} \leq c_{13} \left( \|\nabla (u+1)^{\frac{\theta_2}{L}}\|_{L^2(\Omega)}^{\theta_2} + 1 \right)^{\frac{2(p-m+1)}{p}}
\]

with some \( c_{13} > 0 \) and hence

\[
c_{14} \left( \int_{\Omega} (u+1)^{p-m+1} \right)^{\theta_3} \leq \frac{a_0(p-m)(p-m+1)}{16} \int_{\Omega} (u+1)^{p-2} |\nabla u|^2 + c_{15}
\]

(3.44)
with some $c_{14}, c_{15} > 0$ and $\theta_3 := \frac{p + 2 - n}{p - m + n} > 0$. Combining (3.44) with (3.43), we obtain

$$
\frac{d}{dt} \left( \int_\Omega (u + 1)^{p-m+1} + \int_\Omega \frac{(u + 1)^{p-\eta}}{v^{2k_1+\sigma_1-1}} + \int_\Omega \frac{(u + 1)^{p-\eta}}{w^{2k_2+\sigma_2-1}} + \varepsilon_0 \int_\Omega \frac{(u + 1)^{p-\eta}}{v^{2k_1+\sigma_3-2}w^{2k_2+\sigma_4-2}} \right)
+ \frac{c_1}{2d_1} \int_\Omega v^2 + \frac{c_2}{2d_2} \int_\Omega w^2)
\leq c_{16}$$

(3.45)

for all $t \in (0, T_{\text{max}})$ with some $c_{16} > 0$. Putting

$$
y(t) := \int_\Omega (u(\cdot, t) + 1)^{p-m+1} + \int_\Omega \frac{(u(\cdot, t) + 1)^{p-\eta}}{v^{2k_1+\sigma_1-2}(\cdot, t)} + \int_\Omega \frac{(u(\cdot, t) + 1)^{p-\eta}}{w^{2k_2+\sigma_2-2}(\cdot, t)}
+ \varepsilon_0 \int_\Omega \frac{(u(\cdot, t) + 1)^{p-\eta}}{v^{2k_1+\sigma_3-2}w^{2k_2+\sigma_4-2}(\cdot, t)} + \frac{c_1}{2d_1} \int_\Omega v^2(\cdot, t) + \frac{c_2}{2d_2} \int_\Omega w^2(\cdot, t)$$

for $t > 0$, we see from (3.45) that

$$
y'(t) + c_{17}y^\kappa(t) \leq c_{18}
$$

for all $t \in (0, T_{\text{max}})$ with some $c_{17}, c_{18} > 0$ and $\kappa := \min\{\theta, 1\}$. Thus we have

$$
\sup_{t \in [0, T_{\text{max}}]} \int_\Omega (u(\cdot, t) + 1)^{p-m+1} < \infty
$$

for sufficiently large $p$ satisfying (3.9). This yields $\sup_{t \in [0, T_{\text{max}}]} \|u(\cdot, t)\|_{L^\infty(\Omega)} < \infty$ (see [29, Lemma A.1]) which leads to $T_{\text{max}} = \infty$. Therefore we arrive at the conclusion. \( \square \)

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