First quantized pair interactions

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The annihilation and creation operators of Quantum Field Theory presuppose a causality condition and so the theory cannot represent macroscopic entanglement. The multiple-particle parametrized Dirac wave equation can represent entanglement without recourse to a causality condition. It is shown here that the parametrized formalism can also represent pair annihilation and creation.

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I. INTRODUCTION

The parametrized Dirac wave equation is a first-quantized formalism for quantum electrodynamics [1–4]. Interactions are semiclassical, that is, electromagnetic interaction potentials are sustained by Møller currents [4–6]. Perturbation theory yields the one-loop corrections and the axial anomaly obtained originally with Quantum Field Theory (QFT) [1]. The Bethe-Salpeter equation for bound states [4, 8] is obtained from the parametrized formalism without further conjecture [4]. Unlike QFT [9, 12], a causality condition is not essential to the parametrized formalism. It is the causality condition which renders QFT almost incapable of representing quantum entanglement: in the massive vacuum state, entanglement decays exponentially with space-like separation measured in Compton wavelengths [12]. The parametrized formalism provides a simple and unrestricted representation of entanglement in spacetime. The formalism also yields the spin-statistics connection, again without presupposing a causality condition, by simple extension of the nonrelativistic first-quantized proof of Jabs [14]. The fundamental criticism of the standard Dirac formalism [15, 16], and by implication the parametrized formalism, must however be answered. If an electron cannot scatter into states of negative energy without lower bound, then the negative energy states must be identified as different particles, namely, positrons. How, then, in a particle–conserving formalism can pair annihilation such as

\[ e^- + e^+ \rightarrow \gamma + \gamma \] (1)

be represented? The symbol \( t \) above the arrow indicates that coordinate time \( t \) is increasing from left to right. It is shown here that the ostensibly single-particle scattering amplitude obtained [3] from the Dirac equation is in fact an amplitude for the two-particle parametrized Dirac equation. Creation of a muon pair \( \mu^- \mu^+ \) subsequent to \( e^- e^+ \) annihilation is similarly shown to be a scattering event for the four-particle equation. The argument is supported with a minimum of detail. The solution structure for the parametrized Dirac equation is very close to that of the standard equation [3, 8], and comprehensive detail is available elsewhere [4].

II. THE PARAMETRIZED DIRAC WAVE EQUATION

For a single spin–1/2 particle the parametrized Dirac wavefunction is a four–spinor \( \psi(x, \tau) \). The event \( x \) is in \( \mathbb{R}^4 \), while the parameter \( \tau \) is an independent variable in \( \mathbb{R} \). The event \( x \) is also denoted by \( x^\mu \) having indices \( \mu = 0, 1, 2, 3 \), with \( x^0 = ct \) where \( c \) is the speed of light and \( t \) is coordinate time. The Lorentz metric \( g^{\mu\nu} \) on \( \mathbb{R}^4 \) has signature \((-++++)\). The position \( x \) is denoted by \( x^j \) having indices \( j = 1, 2, 3 \). Thus \( x = (ct, x) \).

The parametrized Dirac wave equation for \( \psi \) is

\[ \frac{\hbar}{i} \frac{\partial}{\partial \tau} \psi + \gamma^\mu \left( \frac{\hbar}{c} \frac{\partial}{\partial x^\mu} - \frac{e}{c} A_\mu \right) \psi = 0 \] (2)

where \( e \) is the charge of the particle, \( c \) is the speed of light and \( \hbar \) is the reduced Planck’s constant. The \( \gamma^\mu \) are the four Dirac matrices, while the Maxwell electromagnetic potential \( A^\mu(x) \) is independent of the parameter \( \tau \). The summation convention is assumed with respect to repetitions of Greek indices such as \( \mu = 0, 1, 2, 3 \). The covariant and contravariant indices \( \mu, \nu, \ldots \) will be omitted wherever convenient, as in \( x = x^\mu \), \( p = p^\mu \) and \( p \cdot x = p^\mu x_\mu \). Henceforth the units are chosen such that \( c = \hbar = 1 \). The covariance of the theory with respect to the homogeneous Lorentz transformation \((x^\mu)' = \Lambda^\mu_\nu x'^\nu \) and \( \psi'(x, \tau) = S(\Lambda) \psi(\Lambda^{-1} x, \tau) \) follows for \( S(\Lambda) \) generated in the standard way [5]. No mass constant appears in [3], but masses are introduced through boundary conditions as \( \tau \to \pm \infty \). Feynman’s development of quantum electrodynamics using [22] has been reviewed by Garcia Alvarez and Gaioli [1]. The indefiniteness of the invariant bilinear form \( \bar{\psi} \psi = \psi^\dagger \gamma^0 \psi \) has impeded [17, 18] the development of the parametrized Dirac formalism as a relativistic extension of quantum mechanics.

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The plane wavefunction solutions of (2) in the absence of an electromagnetic field are of the form

\[ f_p^{(+)}(x, \tau) = \frac{u_p}{(2\pi)^2} \exp[i\chi^{(+)}], \]  

\[ f_p^{(-)}(x, \tau) = \frac{v_p}{(2\pi)^2} \exp[i\chi^{(-)}]. \]  

In (3), \( \chi^{(\pm)} = \chi^{(\pm)}(p, x, \tau) = p \cdot x \pm \varphi_p m_p \tau, \) \( \varphi_p = \text{sgn}(p^0) = p^0/E_p \) where \( E_p = |p^0| \) is the energy of the particle, and \( m_p = mp \) is the positive square root of \(-p \cdot p\) for subluminal energy-momentum \( p\). At constant phase \( \chi^{(\pm)}, dx^0/d\tau = \pm m_p/E_p \) regardless of the value of \( \varphi_p \), so \( f_p^{(+)} \) and \( f_p^{(-)} \) propagate respectively forward and backward in coordinate time \( t = x^0 \) as \( \tau \) increases. The \( 4 \times 4 \) block of amplitudes \( (u_p, v_p) \) is a basis of four Dirac spinors constructed in the standard way from the elementary basis (the unit matrix) for the rest frame where \( p = 0 \). The basis is an even function of \( p \). The orthonormality of the basis is expressed as \( \mathbf{u}_p \mathbf{u}_p = \mathbf{I}_2 \mathbf{u}_p \mathbf{v}_p = -\mathbf{I}_2 \mathbf{u}_p \mathbf{v}_p = \mathbf{v}_p \mathbf{u}_p = 0 \), where \( \mathbf{I}_2 \) is the \( 2 \times 2 \) unit matrix. Antiparticle wavefunctions \( h_p^{(-)} \) are formed from particle wavefunctions \( f_p^{(+)} \) by combining the discrete symmetries\(^4\) of (2), yielding\(^1\)

\[ h_p^{(-)}(x, \tau) = (\mathcal{T}\mathcal{C}\mathcal{P} f_p^{(+)})(-x, -\tau) = -i\gamma^\mu f_p^{(+)}(-x, -\tau) = -i\gamma^\mu f_p^{(+)}(x, \tau). \]  

Physically real particles and antiparticles are associated here with free on-mass-shell wavefunctions having positive energy and positive mass, that is, they have energy \( p^0 > 0 \) and their wavefunctions are proportional to \( \exp[+im_p\tau] \) with \( m_p \) equal to the electron mass \( m_e \). Thus a backward-propagating, positive-energy positron has the wavefunction that might otherwise be attributed to a ‘backward-propagating, negative-energy electron’. While the coordinate time \( t \) for a free positron wavefunction \( h_p^{(-)}(x, \tau) \) at constant phase does decrease as the parameter \( \tau \) increases, its spacelike coordinates \( x \) change in the same sense and at the same rates as those of its \( \mathcal{C}\mathcal{P}\mathcal{T} \) conjugate wavefunction \( f_p^{(+)}(-x, \tau) \).

It is readily verified that \( \mathcal{T}\mathcal{C}\mathcal{P} \) conjugation commutes with not only the vector interaction in (2), but with all the invariant linear interactions (scalar, pseudoscalar, vector, axial vector and tensor)\(^5\). In particular, the Moller currents and hence the semiclassical potentials transform as vectors.

### III. PAIR ANNIHILATION

An incident free wavefunction \( \phi_i \) scatters off an electromagnetic field as an open final free wavefunction \( \phi_f \), with scattering amplitude\(^4\)\(^6\)

\[ S_{fi} = (\pm) \lim_{\tau \rightarrow \pm \infty} \int d^4x \, \phi_f(x, \tau)(\omega_+ \phi_i)(x, \tau) \]  

where \( \omega_+ \) is the forward Møller operator\(^4\) for (2). Consistent with the normalization of the spinor basis, the leading sign in (4) is \( (+) \) for a final wavefunction that is forward-propagating and \( (-) \) for one that is backward-propagating. There is a nonvanishing amplitude for the mathematical contingency of a forward-propagating, positive-energy wavefunction scattering off two free photons and into a backward-propagating, negative-energy wavefunction. To leading order, the nontrivial and nonvanishing amplitude is

\[ S_{fi} = -e^2 \int d^3w \int d^3w' \times (\hbar f^{(-)}(w) A(x) I^0_+(w - w') B(x') f^{(+)}_f), \]  

where \( \hbar f^{(-)} = h_p^{(-)}(p = p_f) \) and \( f^{(+)}_f = f^{(+)}_f(p = p_i) \), while \( w = (x, \tau) \). The Feynman slashed notation is \( A = \gamma^\mu A_\mu \) and \( B = \gamma^\mu B_\mu \), where the classical potentials \( A^\mu \) and \( B^\mu \) represent the two free photons. Finally, \( I^0_+ \) is the free forward influence function for positive-energy electrons and protons\(^4\). For clarity, the spins of the incident and final wavefunctions have not been stipulated and the additive amplitude with photons exchanged\(^5\) is suppressed. The integrals with respect to \( \tau \) reduce to delta functions that enforce equality of the incident and final masses \( m(p_i) \) and \( m(p_f) \) respectively. Otherwise the parameterized scattering amplitude\(^5\) is the same as the standard Dirac amplitude\(^3\). Identifying \( h_f^{(-)} \), which again is the \( \mathcal{T}\mathcal{C}\mathcal{P} \) conjugate of the final ‘negative-energy electron’ \( f_f^{(+)} \) as the parameter \( \tau \rightarrow +\infty \), with a positive-energy positron injected as the coordinate time \( t \rightarrow -\infty \) yields observed cross-sections for pair annihilation. The interaction is now expressed as

\[ e^- + \gamma \rightarrow e^+ + \gamma. \]  

In (5) it is the parameter \( \tau \) which increases from left to right. The standard first-quantized Dirac theory does not support more than one particle, much less the loss or gain of a particle. What becomes of the incident electron wave function in (5)? From whence came the final positron wave function? The issue is resolved by analyzing the influence function or internal line in (5) as

\[ I^0_+(w - w') = i \int d^4p \int d^4q \, \delta^4(p - q) \]

\[ \times \left\{ \theta(\tau - \tau') \theta(p^0) \theta(q^0) - \theta(\tau' - \tau) \theta(-p^0) \theta(-q^0) \right\} \]

\[ \times \left[ f^{(+)}_p(w) \overline{f}^{(+)}_q(w') - h^{(-)}_p(w) \overline{h}^{(-)}_q(w') \right]. \]  

where \( \theta \) is the Heaviside unit step function. The second-order amplitude\(^5\) is thereby analyzed as a weighted sum of products of first-order amplitudes, for (i) a physical electron \( f^{(+)}_i \) scattering into a virtual (off mass shell) electron \( f^{(+)}_q \) of positive mass \( m_q \), and a virtual electron \( f^{(+)}_p \) of positive mass \( m_p \) scattering into a physical positron \( h^{(-)}_f \), or (ii) a physical electron \( f^{(+)}_i \) scattering
into a virtual positron $h_{q}^{-}(\gamma_{e})$ of positive mass $m_{q}$, and a virtual positron $h_{p}^{-}(\gamma_{e})$ of positive mass $m_{p}$ scattering into a physical positron $h_{f}^{-}(\gamma_{e})$. In case (i) the virtual energies $p^{0}$ and $q^{0}$ are positive, while in case (ii) they are negative. The minus sign preceding the second tensor product in (7) is consistent with the normalization of the spinor basis. The other two scattering contingencies $-h_{p}^{(+)}h_{q}^{-}(\gamma_{e})$ and $+h_{p}^{-}h_{q}^{(+)}(\gamma_{e})$ are kinematically excluded. Indicating the virtual particles with braces $\{\ldots\}$, in case (i) above the interaction is now

$$e^{-} + \{e^{-}\} + \gamma \rightarrow \{e^{-}\} + e^{+} + \gamma,$$  \hspace{1cm} \text{(8)}

while in case (ii) it is

$$e^{-} + \{e^{+}\} + \gamma \rightarrow \{e^{+}\} + e^{+} + \gamma.$$  \hspace{1cm} \text{(9)}

If the double integral with respect to $p$ and $q$ is uniformly weighted then the single-particle amplitudes vanish.

It is a textbook exercise, taking into account the kinematic constraints, to verify that the amplitude products which are the integrands on the right hand side of (9) modulo (7) are precisely those arising from the two-particle parametrized Dirac wave equation:

$$\frac{1}{i} \frac{\partial}{\partial \tau} \Psi + \not \chi(x) \otimes \lambda, \Psi + \lambda \otimes \not \chi(y) \Psi = 0,$$  \hspace{1cm} \text{(10)}

where $\Psi(x, y, \tau)$ is the two-particle wavefunction, while $\pi^{\mu}(x) = (1/\hbar) \partial / \partial x_{\mu} - e_{1} A^{\mu}(x)$ and $\pi^{\mu}(y) = (1/\hbar) \partial / \partial y_{\mu} - e_{2} A^{\mu}(y)$. Tensor products of the single-particle free wavefunctions form a basis for the two-particle free wavefunctions. A nonseparable two-particle wavefunction provides an elementary and unrestricted representation of entanglement. The number of single-particle wavefunctions is conserved by the two-particle scattering event analyzed with (7): there are two incident wavefunctions, one physical and one virtual, and there are two final wavefunctions, one virtual and one physical.

### IV. PAIR CREATION

It is observed that an electron and a positron can annihilate, followed by the creation of a muon and an antimuon. In the sense of $t$ increasing the interaction is

$$e^{-} + e^{+} \rightarrow \mu^{-} + \mu^{+}.$$  \hspace{1cm} \text{(11)}

The interaction is mediated by a virtual photon. In terms of $\tau$ increasing the interaction is

$$e^{-} + \mu^{+} \rightarrow e^{+} + \mu^{-}.$$  \hspace{1cm} \text{(12)}

The amplitude for the interaction may be constructed semi-classically using (2). Consider the scattering of the antimuon into a muon. The leading-order nontrivial amplitude is

$$S_{fi}^{(1)} = i e \int dt \int d^{4} x f_{j}^{(\gamma_{e})}(x, \tau) A(x) h_{j}^{-}(x, \tau).$$  \hspace{1cm} \text{(13)}

The primed subscripts $f'$ and $i'$ indicate that the wavefunctions have the energy-momenta of the final antimuon and incident muon respectively. The scattering potential $A^{\lambda}$ is the (virtual) photon owing to the electron-positron current. That is,

$$A^{\lambda}(x) = e \int d^{4} y D^{\lambda \nu}(x - y) J_{\nu}(y)$$  \hspace{1cm} \text{(14)}

where $D^{\lambda \nu}$ is the standard influence function for a massless vector boson, while $J^{\nu}$ is the Møller current owing to the electron and positron. To leading order,

$$J^{\lambda}(y) = \int d \sigma h_{f}^{-}(y, \sigma) \gamma^{\lambda} f_{i}^{(\gamma_{e})}(y, \sigma).$$  \hspace{1cm} \text{(15)}

Note that the current has been concatenated with respect to the parameter, that is, integrated for all $\sigma$.

The amplitude (13) again raises the issue for first-quantized formalism: a positive-energy physical antimuon is being scattered into a different particle, namely, a positive-energy physical muon. There is of course an identical amplitude for the physical electron being scattered into a physical positron by the potential sustained by the antimuon-muon current. The issue is resolved in two stages. First, the boson influence function is analyzed as a sum of products. Second, the number of vertices is doubled from two to four by introducing two trivial vertices. At each vertex there is an on-shell wavefunction for a physical particle and an off-shell wavefunction for a virtual particle. There is also a virtual photon at each of the two original vertices, but none at either of the two trivial vertices. The amplitude is then precisely of the form arising from the four-particle parametrized Dirac equation. Some details now follow.

First, the concatenation in (11) is postponed, and the massless vector boson influence function is replaced with the influence function for a massive vector boson. The spectral representation of the influence function becomes

$$D^{\lambda \nu}(k, \omega) = \frac{G^{\lambda \nu}(k, \omega)}{k \cdot k + \omega^{2}},$$  \hspace{1cm} \text{(16)}

where $k$ is the four-vector wavenumber, while $\omega$ is the frequency with respect to the parameter $\tau$ as in $\exp[i \omega \tau]$. The numerator is variously expressed as

$$G^{\lambda \nu}(k, \omega) = g^{\lambda \nu} + k^{\lambda} k^{\nu}/\omega^{2} = \sum_{j=1}^{3} \varepsilon_{j}^{\lambda} \varepsilon_{j}^{\nu},$$  \hspace{1cm} \text{(17)}

where $\varepsilon_{j}^{\lambda}(k)$ is a polarization amplitude such that $k_{j} \varepsilon_{j}^{\mu} = 0$ for $j = 1, 2, 3$. It is evident that the boson influence function is an integral of products of virtual, $\tau$-dependent ‘preMaxwell’ photons, or plane waves that are off the ‘shell’ where $k \cdot k = -\omega^{2}$. Second, the incident antimuon in (13) is expressed as

$$h_{j}^{-}(x, \tau) = \frac{1}{i} \int d^{4} z \{\theta(\tau - \rho) \Gamma^{0}_{+}(x - z, \tau - \rho) - \theta(\rho - \tau) \Gamma^{0}_{-}(x - z, \tau - \rho)\} h_{j}^{-}(z, \rho).$$  \hspace{1cm} \text{(18)}
The fermion influence function $\Gamma^0$ has an analysis similar to (7). The expression (18) can accordingly be interpreted as the trivial scattering of the incident physical antimuon $\bar{h}_i$ into a virtual particle. The other virtual particle (in the tensor product within either influence function) is incident to the physical vertex at $(x, \tau)$. The incident virtual particle scatters off the virtual photon and into the final physical muon $f_{\mu}^{(+)}$.

The analysis of the boson influence function, the introduction of trivial vertices and the analysis of the fermion influence function convert both the $e^-e^+$ side and the $\mu^-\mu^+$ side of the single-particle ansatz (13)–(15) to instances of (4). Then (13)–(15) is expressed as an amplitude for a multiple-particle parametrized Dirac equation, without any physical particle being converted by scattering into a different physical particle. The expression is verbose, but the ansatz is equivalent and provides a convenient short-hand.

V. SUMMARY

Particles and antiparticles are different positive-energy particles, identified respectively with the forward and backward propagating free solutions of the parametrized Dirac equation. Annihilation and creation of physical (on-shell) particles do not involve annihilation or creation of wavefunctions. Rather, in both interactions either real or virtual photons scatter incident physical wavefunctions off mass shell, and incident virtual wavefunctions onto mass shell.

Spacetime entanglement and $T\mathcal{CP}$ invariance are simple consequences of the parametrized Dirac equation. The proof of $T\mathcal{CP}$ invariance in Quantum Field Theory requires a causality condition [23], but none is required here.

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