ORIENTED KAUFFMAN DIAGRAMS AND UNIVERSAL QUANTUM GROUPS

YAMAGAMI SHIGERU

Department of Mathematics and Informatics
Ibaraki University
Mito, 310-8512, JAPAN

Abstract. We study the tensor category of oriented Kauffman diagrams and determine fiber functors on them as well as the associated Hopf algebras.

1. Introduction

In our previous paper [8, 9], we determined fiber functors on Temperley-Lieb categories with the help of universality property on rigidity and classification results on bilinear forms. The associated Hopf algebra as a consequence of Tannaka-Krein duality is identified with the algebraic quantum group of Dubois-Violette and Launer.

When C*-structure is entailed, unitary fiber functors are also classified with the associated compact quantum groups isomorphic to the universal quantum groups of orthogonal type due to Wang [4] and Banica [1]. There is another related class of compact quantum groups, called the universal quantum group of unitary type, investigated by the same authors.

Here we shall study an oriented version of Temperley-Lieb categories and obtain an analogous classification of fiber functors on them. Though we have failed in identifying the associated algebraic quantum groups, when restricted to the unitary case, they turn out to be the universal quantum groups of unitary type, thus revealing geometric structures behind them.

2. Oriented Kauffman Diagrams

Let $D$ be a Kauffman diagram of type $(m, n)$ ([5, 6]), i.e., $D$ is the isotopy class of planar strings in a rectangle with the strings having $m$ terminal points on the upper bounding line and $n$ terminal points on the lower bounding line. Thus $D$ contains $(m + n)/2$ strings in
total and there are $2^{(m+n)/2}$ possibilities in the choice of orientations of strings in $D$. A Kauffman diagram with a specific orientation is called an **oriented Kauffman diagram**. For an explicit description of orientation, we assign one of symbols $X$ and $X^*$ to each point so that $X$ (resp. $X^*$) indicates starting (resp. ending) for upper bounding points and ending (resp. starting) for lower bounding points. By this coding, an orientation for $D$ produces two words $(X_1, X_2, \ldots, X_m)$ and $(X'_1, X'_2, \ldots, X'_n)$ of two letters $\{X, X^*\}$ corresponding to upper and lower sequences of vertices. Clearly the set of such words is identified with the free product monoid $N \ast N$ ($N = \{1, 2, \ldots\}$ being the additive monoid of natural numbers). For an element $w \in N \ast N$, we use the notation $X^w$ to stand for the corresponding word. Let $w, w' \in N \ast N$ be defined by

$$X^w = (X_1, \ldots, X_m), \quad X^{w'} = (X'_1, \ldots, X'_n).$$

The pair $(w, w')$ is called the type of an oriented Kauffman diagram. Given words $w, w' \in N \ast N$, let $K_{w,w'}$ be the set of oriented Kauffman diagrams of type $(w, w')$ and $\mathbb{C}[K_{w,w}]$ be the free complex vector space generated by the set $K_{w,w'}$ ($\mathbb{C}[0] = \{0\}$ by definition). Let $|w|_+$ (resp. $|w|_-$) be the number of $X$’s (resp. $X^*$’s) in $X^w$. Then the set $K_{w,w'}$ is non-empty if and only if both of $|w|_+ + |w'|_-$ and $|w|_- + |w'|_+$ are even numbers, which can be seen by an easy induction argument.

Just as in the unoriented case, we introduce a (strict) tensor category $\mathcal{O}_{d_L,d_R}$ parametrized by $d_L, d_R \in 2^X$:

(i) Objects are exactly the symbols $\{X^w\}$ with $w \in N \ast N$.

(ii) Hom-sets are set to be $\text{Hom}(X^w, X^{w''}) = \mathbb{C}[K_{w,w}]$ and the operation of composition $\text{Hom}(X^{w'}, X^{w''}) \times \text{Hom}(X^w, X^{w'}) \to \text{Hom}(X^w, X^{w''})$ is defined by the concatenation of planar strings with clockwise (resp. anticlockwise) loops replaced by $d_R$ (resp. $d_L$).

(iii) The tensor product for morphisms is the linear extension of the horizontal juxtaposition of diagrams in $K_{w,w'}$.

Notice that the unit object $I$ of $\mathcal{O}_{d_L,d_R}$ is associated to the empty word, i.e., $I = X^0$.

The tensor category obtained in this way is rigid as in the case of Temperley-Lieb categories and bears a kind of universality on rigidity.

Let $\epsilon_X : X \otimes X^* \to I$ and $\epsilon_{X^*} : X^* \otimes X \to I$ be basic arcs in $\mathcal{O}_{d_L,d_R}$ with $\delta_X : I \to X^* \otimes X$ and $\delta_{X^*} : I \to X \otimes X^*$ the associated copairings.

**Lemma 2.1.** Any (strictly) monoidal functor $F$ of $\mathcal{O}_{d_L,d_R}$ into a tensor category $\mathcal{T}$ is uniquely determined by the choice

$$F(\epsilon_X) : F(X) \otimes F(X^*) \to I, \quad F(\epsilon_{X^*}) : F(X^*) \otimes F(X) \to I.$$
of rigidity pairings which satisfy the identities
\[ F(\epsilon_X)F(\delta_X^*) = d_L 1_I, \quad F(\epsilon_X^*)F(\delta_X) = d_R 1_I. \]

Note here that \( F(\delta_X) \) and \( F(\delta_X^*) \) are characterized as the rigidity co-
parings associated with \( F(\epsilon_X) \) and \( F(\epsilon_X^*) \) respectively.

Conversely, given rigidity pairings \( \epsilon_Y : Y \otimes Y^* \to I \) and \( \epsilon_Y^* : Y^* \otimes Y \to I \) in a tensor category \( T \), satisfying the relations
\[ \epsilon_Y \delta_Y^* = d_L 1_I, \quad \epsilon_Y^* \delta_Y = d_R 1_I, \]
there exists a monoidal functor \( F : \mathcal{O}_{d_L,d_R} \to T \) such that \( F(X) = Y \),
\( F(X^*) = Y^* \), \( F(\epsilon_X) = \epsilon_Y \) and \( F(\epsilon_X^*) = \epsilon_Y^* \).

Although the category \( \mathcal{O}_{d_L,d_R} \) has apparently two parameters, one freedom of them is superficial as we shall see below.

**Lemma 2.2.** Let \( X^w \) (\( w \in \mathbb{N} * \mathbb{N} \)) be an object of \( \mathcal{O}_{d_L,d_R} \). Then \( \dim \text{End}(X^w) = 1 \) if and only if \( X^w = X^n \) or \( X^w = (X^*)^n \) for some \( n \in \mathbb{N} \).

**Proof.** In fact, \( \text{End}(X \otimes X^*) \) and \( \text{End}(X^* \otimes X) \) are two-dimensional,
which are included in \( \text{End}(X^w) \) unless \( X^w = X^n \) or \( X^w = (X^*)^n \). \( \square \)

Let \( F : \mathcal{O}_{d_L,d_R} \to \mathcal{O}_{d_L',d_R'} \) be an equivalence of tensor categories, i.e.,
\( F \) is an essentially surjective and fully faithful tensor functor. By the
above lemma, we then have \( F(X) = X^n \) or \( F(X) = (X^*)^n \) for some
integer \( n \geq 1 \). The case \( n \geq 2 \), however, contradicts with the essential
surjectivity of \( F \) because it implies \( X \not\cong F(X^w) \) for any \( w \in \mathbb{N} * \mathbb{N} \).
Thus we have alternatives \( F(X) = X \) or \( F(X) = X^* \).

The operation of \( F \) on morphisms is then determined by the effect
on basic arcs \( \epsilon_X \) and \( \epsilon_X^* \):
\[ F(\epsilon_X) = \lambda \epsilon_X, \quad F(\epsilon_X^*) = \mu \epsilon_X^* \quad (F(\delta_X) = \lambda^{-1} \delta_X, \quad F(\delta_X^*) = \mu^{-1} \delta_X^*) \]
or
\[ F(\epsilon_X) = \lambda \epsilon_X^*, \quad F(\epsilon_X^*) = \mu \epsilon_X \quad (F(\delta_X) = \lambda^{-1} \delta_X^*, \quad F(\delta_X^*) = \mu^{-1} \delta_X). \]

From the obvious equality
\[ F(\epsilon_X \delta_X^*) = \epsilon_X \delta_X^*, \quad F(\epsilon_X^* \delta_X) = \epsilon_X^* \delta_X, \]
we have
\[ d_L' = \lambda^{-1} \mu d_L, \quad d_R' = \lambda \mu^{-1} d_R \]
in the case \( F(X) = X \) and
\[ d_L' = \lambda \mu^{-1} d_R, \quad d_R' = \lambda^{-1} \mu d_L \]
in the case \( F(X) = X^* \).
Conversely, these relations ensure the existence of an equivalence functor $F$ by the generating property of basic arcs (Lemma 2.1). In this way, we have proved the following:

**Proposition 2.3.** We have

$$\mathcal{O}_{d_L,d_R} \cong \mathcal{O}_{d_L',d_R'} \iff d_Ld_R = d'_Ld'_R.$$ In particular, if we write $\mathcal{O}_d = \mathcal{O}_{d,d}$, $\mathcal{O}_{d_L,d_R} \cong \mathcal{O}_d$ for the choice $d = \pm \sqrt{d_Ld_R}$, with the condition $\mathcal{O}_d \cong \mathcal{O}_{d'}$ equivalent to $d = \pm d'$. Moreover self-equivalences of $\mathcal{O}_d$ are given by

$$F(\epsilon_X) = \lambda \epsilon_X, \quad F(\epsilon_X^*) = \lambda \epsilon_X^*$$
or

$$F(\epsilon_X) = \lambda \epsilon_X^*, \quad F(\epsilon_X^*) = \lambda \epsilon_X$$

with $\lambda \in \mathbb{C}^\times$.

In what follows, we shall concentrate on the category $\mathcal{O}_d = \mathcal{O}_{d,d}$ without loss of generality. As in the case of Temperley-Lieb category, we then have the natural operation of duality on the category $\mathcal{O}_d$: if we introduce the antimultiplicative involution $*$ on the monoid $\mathbb{N} * \mathbb{N}$ by switching the role of two submonoids $\mathbb{N}$, then $X^w^*$ gives a dual object of $X^w$ for $w \in \mathbb{N} * \mathbb{N}$ with respect to the obvious pairings and copairings by multiple arcs. The associated operation of transposed maps is given by rotating diagrams by an angle of $\pi$.

### 3. Semisimplicity Analysis

In the tensor category $\mathcal{O}_d$, the semisimplicity analysis, together with the Jones-Wenzl recursive formula, works for objects of alternating tensor products $X \otimes X^* \otimes X \otimes \ldots$ exactly as in the case of Temperley-Lieb categories ([8]). In particular, all of the alternating algebras

$$\text{End}(X \otimes X^* \otimes X \otimes \ldots), \quad \text{End}(X^* \otimes X \otimes X^* \otimes \ldots)$$

are semisimple if and only if $d = q + q^{-1}$ with $q^2$ not a proper root of unity. Moreover, under the assumption of semisimplicity on alternating algebras $\text{End}(X \otimes X^* \otimes X \otimes \ldots)$, we can inductively define simple objects $\{X_n,Y_n\}_{n \geq 1}$ in the idempotent-completion $\overline{\mathcal{O}}_d$ of $\mathcal{O}_d$ so that $X_n$ and $Y_n$ are the new stuffs in $\widehat{X} \otimes X^* \otimes \ldots$ and $\widehat{X^*} \otimes X \otimes \ldots$ respectively, i.e., $X_n = f_n(X \otimes X^* \otimes \ldots)$ with $f_n$ the $n$-th Jones-Wenzl idempotent and similarly for $Y_n$.

To a word $w \in \mathbb{N} * \mathbb{N}$, we associate a subobject $X_w$ of $X^w$ in $\overline{\mathcal{O}}_d$ by replacing maximal alternating subwords in $w$ with the highest part.
just defined. (Note that $X^*_w = X_{w^*}$ and the transposed $^t f_n$ is again a Jones-Wenzl idempotent.)

**Example 3.1.** For the choice

$$X^w = (X)(X X^* X X^*)(X^* X X^* X X^*)(X^*)(X^*),$$

we have $X_w = X_1 \otimes X_4 \otimes Y_5 \otimes Y_1 \otimes Y_1$.

**Lemma 3.2.** For a non-empty word $w \in \mathbb{N}^* \mathbb{N}^*$, $\text{Hom}(X_w, I) = \{0\}$.

**Proof.** Let $w = w_1 \ldots w_l$ be the factorization into maximal alternating parts and write $X_{w_j} = P_j X_{w_j}$ with $P_j$ the Jones-Wenzl projection to the highest part of the alternating tensor product $X_{w_j}^w$. We shall show that any diagram $D$ in $K_{w, \emptyset} \subset \text{Hom}(X_w, I)$ annihilates $P_1 \otimes \cdots \otimes P_l$ by an induction on the word length $|w| = \sum_j |w_j| \ (|w_j| = |w_j^+| + |w_j^-|)$.

First observe that, if $D$ contains an arc connecting two vertices in some $w_j$, then $P_j$ annihilates the arc (regarded as a pairing morphism) by the highest assumption (old stuffs being killed by $P_j$). Thus, we need to deal with $D$ having no such arcs. Then any vertex inside $w_1$ should be joined to a vertex in $w_2 \ldots w_l$. The right-end vertex $v_1$ of $w_1$, however, cannot be connected to any vertex in $w_2$ because of parity mismatch. So $v_1$ is joined to a vertex $v_k$ in $w_k$ with $k \geq 3$. Now write $w_k = w'_k w''_k$, where $w''_k$ is the subword of $w_k$ starting at the vertex $v_k$ until the end of $w_k$ and $w'_k$ is the complement to $w''_k$. We then have the factorization $D = D''(1_{X_{w_1}} \otimes D' \otimes 1_{X_{w^*}})$ with $w'' = w''_k w_{k+1} \ldots w_l$ and $D' \in K_{w_2 \ldots w_{k-1} w'_k}$ (see Fig. 1).

On the other hand, the Clebsh-Gordan fusion rule on alternating parts ensures the relation $P_k = (P_k' \otimes P_k'')P_k$ ($P_k'$ and $P_k''$ being the Jones-Wenzl idempotents for $w'_k$ and $w''_k$ respectively) and hence we see

$$D(P_1 \otimes \cdots \otimes P_l) = D''(1_{X_{w_1}} \otimes D'(P_2 \otimes \cdots \otimes P_{k-1} \otimes P_k') \otimes 1_{X_{w^*}})(P_1 \otimes \cdots \otimes P_l),$$

which vanishes by the induction hypothesis $D'(P_2 \otimes \cdots \otimes P_{k-1} \otimes P_k') = 0$. \hfill $\square$

![Figure 1](image_url)
Corollary 3.3. We have \( \text{Hom}(X_w, X_{w'}) = \{0\} \) for \( w \neq w' \in \mathbb{N} \times \mathbb{N} \) and \( \text{Hom}(X_w, X_w) = \mathbb{C} 1_{X_w}. \)

Proof. Decompose \( w = w_1 \ldots w_l \) and \( w' = w'_1 \ldots w'_l \) as before. Then \( \text{Hom}(X_w, X_{w'}) \cong \text{Hom}(X_{w_1} \otimes \cdots \otimes X_{w_l}, I) \) by rigidity. If this vector space is non-trivial, the product \( w_l(w'_j)^* \) should interact at the contact point and we have the decomposition of the form
\[
X_{w_l} \otimes X_{(w'_j)^*} \cong X_{\tilde{w}_1} \oplus \cdots \oplus X_{\tilde{w}_m}
\]
according to the Clebsh-Gordan rule.

Thus, if the unit object \( I \) does not appear inside \( X_{w_l} \otimes X_{(w'_j)^*} \), we know
\[
\text{Hom}(X_w \otimes (X_{w'})^*, I) = \bigoplus_{j=1}^m \text{Hom}(X_{w_1 \ldots w_{l-1} \tilde{w}_j(w'_j)^* \ldots (w'_l)^*}, I) = \{0\}.
\]
When \( I \) is contained, \( w_l = w'_l \) and the unit object appears exactly once in \( X_{w_l} \otimes X_{(w'_j)^*} \) (say, \( X_{\tilde{w}_1} = I \) and \( X_{\tilde{w}_j} \not= I \) for \( j \geq 2 \)) and we get
\[
\text{Hom}(X_w \otimes (X_{w'})^*, I) \cong \bigoplus_{j=1}^m \text{Hom}(X_{w_1 \ldots w_{l-1} \tilde{w}_j} \otimes X_{(w'_1)^* \ldots (w'_{l-1})^*}, I)
\]
\[
= \text{Hom}(X_{w_1 \ldots w_{l-1} \tilde{w}_j} \otimes X_{(w'_1)^* \ldots (w'_{l-1})^*}, I)
\]
\[
\cong \text{Hom}(X_{w_1 \ldots w_{l-1}}, X_{(w'_1)^* \ldots (w'_{l-1})^*}).
\]
Now the induction argument is applied to see \( w = w' \) and
\[
\text{End}(X_{w_1 \ldots w_l}) \cong \text{End}(X_{w_1 \ldots w_{l-1}}) \cong \mathbb{C}.
\]

\( \square \)

Proposition 3.4. The tensor category \( \mathcal{O}_d \) is semisimple if and only if \( d = q + q^{-1} \) with \( q^2 \) not a proper root of unit. In this case, \( \{X_w\}_{w \in \mathbb{N} \times \mathbb{N}} \) gives a representative set of simple objects and the fusion rule is given by the following recipe: Let \( w = w_1 \ldots w_l \) and \( w' = w'_1 \ldots w'_l \) be the decompositions into maximally alternating parts.

(i) If \( w_lw'_1 \) matches in the parity, the tensor product \( X_{w_l} \otimes X_{w'_1} \) decomposes according to the Clebsh-Gordan rule, otherwise \( X_w \otimes X_{w'} = X_{w_1 w'_1} \) remains simple.

(ii) Let \( X_{w_1} \otimes X_{w'_1} \cong X_{w_1} \oplus \cdots \oplus X_{w_k} \). If \( I \not= U_j \) for \( 1 \leq j \leq k, \)
\[
X_w \otimes X_{w'} \cong \bigoplus_{j=1}^k X_{\tilde{w}_j}
\]
with \( \tilde{w}_j = w_1 \ldots w_{l-1} w'_j w'_2 \ldots w'_l \in \mathbb{N} \times \mathbb{N} \) gives an irreducible decomposition.
If \( X_{u_1} \not\cong I \) (and hence \( X_{u_j} \not\cong I \) for \( j \geq 2 \)),
\[
X_w \otimes X_{w'} \cong (X_{w_1 \ldots w_{l-1}} \otimes X_{w'_2 \ldots w'_l}) \oplus X_{\tilde{w}_2} \oplus \cdots \oplus X_{\tilde{w}_k}
\]
and we are reduced to the decomposition of \( X_{w_1 \ldots w_{l-1}} \otimes X_{w'_2 \ldots w'_l} \).

**Remark.** If we restrict ourselves to the unitary (i.e., \( C^* \)-) case, then the above formula produces the fusion rule in [2, Theorem 1] via Proposition 5.3 below.

### 4. Positivity Condition

We shall here investigate possible \( * \)-structures on \( \mathcal{O}_d \). Assume that there is a (compatible) \( * \)-structure on \( \mathcal{O}_d \). Then we should have
\[
(\epsilon_X)^* = c_X \delta_X^*, \quad (\epsilon_{X^*})^* = c_{X^*} \delta_X
\]
with \( c_X, c_{X^*} \in \mathbb{C}^\times \).

To preserve the rigidity, we should have
\[
1_X = 1_X^* = \left( (1_X \otimes \epsilon_X)(\delta_X \otimes 1_X) \right)^* = c_X \cdot (\delta_X^* \otimes 1_X)(1_X \otimes \delta_X)
\]
and
\[
1_{X^*} = (1_{X^*})^* = \left( (1_{X^*} \otimes \epsilon_X)(\delta_X \otimes 1_X) \right)^* = c_X \cdot (\delta_X^* \otimes 1_X)(1_X \otimes \delta_X^*)
\]
which are equivalent to
\[
(\delta_X)^* = c_X^{-1} \epsilon_X, \quad (\delta_{X^*})^* = c_{X^*}^{-1} \epsilon_X.
\]

To preserve trace (loop) values, we should have
\[
\overline{d} = (\epsilon_X \delta_{X^*})^* = c_X c_{X^*}^{-1} \epsilon_X \delta_{X^*} = c_X c_{X^*}^{-1} d,
\]
\[
\overline{d} = (\epsilon_{X^*} \delta_X)^* = c_{X^*}^{-1} c_X \epsilon_{X^*} \delta_X = c_{X^*}^{-1} c_X d,
\]
i.e.,
\[
c_X = \pm c_{X^*} \quad \text{and} \quad d = \pm \overline{d}.
\]

With these conditions satisfied, the universality on rigidity (Lemma 2.1) allows us to extend the operation to the whole hom-sets by antilinearity and antimultiplicativity. The obtained map is then involutive if and only if
\[
\epsilon_X = (\epsilon_X)^{**} = \frac{c_X}{c_{X^*}} \overline{\delta_{X^*}} = \frac{c_X}{c_X} \epsilon_X, \quad \epsilon_{X^*} = (\epsilon_{X^*})^{**} = \frac{c_{X^*}}{c_X} \overline{\delta_X} = \frac{c_{X^*}}{c_{X^*}} \epsilon_{X^*},
\]
i.e., \( \overline{c_X} = c_{X^*} \).
Proposition 4.1. The tensor category $\mathcal{O}_d$ admits a compatible $^*$-structure if and only if $d^2 \in \mathbb{R}$, i.e., $d = \pm \sqrt{d}$. If this is the case, $^*$-structures are parametrized by $c \in \mathbb{C}^\times$ satisfying $cd = cd$ with the associated $^*$-operation given by

$$D^* = c^{\sharp(D)} \left( \frac{d}{d'} \right)^{l(D)} D'$$

for an oriented diagram $D$, where $D'$ is the orientation reversion of the reflection of $D$ (see Fig. 2),

$$\sharp(D) = \sharp\{\epsilon_X \text{'s and } \epsilon_{X^*} \text{'s inside } D\} - \sharp\{\delta_X \text{'s and } \delta_{X^*} \text{'s inside } D\},$$

and

$$l(D) = \sharp\{\epsilon_{X^*} \text{'s inside } D\} - \sharp\{\delta_{X^*} \text{'s inside } D\}.$$  

Moreover two $^*$-structures are equivalent if and only if $c = \lambda c'$ or $\overline{c} = \lambda c'$ for some $\lambda > 0$.

Proof. Let $c, c' \in \mathbb{C}^\times$ be parameters of $^*$-structures with the associated $^*$-structures on $\mathcal{O}_d$ denoted by $*$ and $\star$ respectively. Since monoidal automorphisms of $\mathcal{O}_d$ are of the form

$$F(\epsilon_X) = \lambda \epsilon_X, \quad F(\epsilon_{X^*}) = \lambda \epsilon_{X^*}, \quad G(\epsilon_X) = \lambda \epsilon_{X^*}, \quad G(\epsilon_{X^*}) = \lambda \epsilon_X,$$

the condition $F(D^*) = F(D)^*$ (resp. $G(D^*) = G(D)^*$) is equivalent to $c = |\lambda|^2 c'$ (resp. $\overline{c} = |\lambda|^2 c'$).

\[\begin{array}{ccc}
D = & \xymatrix{ X & X^* & X \\
\ar@/^1pc/[r] & \ar@/_1pc/[l] & \\
X & & X}
& , & \\
D' = & \xymatrix{ X & X^* & X \\
\ar@/^1pc/[r] & \ar@/_1pc/[l] & \\
X & & X}
\end{array}\]

\textbf{Figure 2.}

Theorem 4.2. The tensor category $\mathcal{O}_d$ admits a compatible $C^*$-structure if and only if $d \in \mathbb{R}^\times$ and $d^2 \geq 4$.

If this is the case, $C^*$-structure is unique up to monoidal equivalences and is given by

$$D^* = \left( \frac{d}{|d|} \right)^{\sharp(D)} D'$$

for a diagram $D$. 

Proof. We argue as in the case of Temperley-Lieb category: Since idempotents
\[ e_1 = \frac{1}{d}(\delta_X \epsilon_X) \otimes 1_X, \quad e_2 = \frac{1}{d}1_X \otimes (\delta_X \epsilon_X), \]
are related to the central decomposition of \( \text{End}(X \otimes X^*) \) and \( \text{End}(X^* \otimes X) \), we have \( e_1^* = e_1 \) and \( e_2^* = e_2 \), which are equivalent to \( c = d \) (automatically satisfied). From the relation \( e_1 e_2 e_1 = d^{-2} e_1 \), we see \( d^2 > 0 \iff d \in \mathbb{R} \) and hence \( c \in \mathbb{R}^\times \) as well. Moreover \( \epsilon_X \epsilon_X^* = cd > 0 \) implies the condition \( c/|c| = d/|d| \). Thus, up to equivalences, C*-structure (if any exists) is unique and given by the above formula for the choice \( |c| = 1 \).

By a standard argument based on Jones-Wenzl formula (cf. \[8\]), we can derive the condition \( d^2 \geq 4 \) from the positivity of the *-structure.

On the other hand, if this is assumed, the Temperley-Lieb category \( K_d \) admits the (unique) C*-structure. Since \( O_d \) is a subcategory of \( K_d \), the *-structure of \( O_d \) meets the positivity and we are done. \( \square \)

5. Fiber Functors

From the universality property, any fiber functor \( \Phi : O_d \rightarrow \text{Vec} \) is determined by non-degenerate bilinear forms \( \Phi(\epsilon_X) \) and \( \Phi(\epsilon_X^*) \). Set \( V = \Phi(X), W = \Phi(X^*), E = \Phi(\epsilon_X) : V \otimes W \rightarrow \mathbb{C} \) and \( F = \Phi(\epsilon_X^*) : W \otimes V \rightarrow \mathbb{C} \). Let \( \delta_E \in W \otimes V \) and \( \delta_F \in V \otimes W \) be the associated vector: Given a basis \( \{v_i\} \) of \( V \) and \( \{w_j\} \) of \( W \),
\[ \delta_E = \sum c_{ji} w_j \otimes v_i, \quad \delta_F = \sum d_{ij} v_i \otimes w_j, \]
with \( \{c_{ji}\} \) and \( \{d_{ij}\} \) defined by
\[ \sum_j E(v_k \otimes w_j)c_{ji} = \delta_{ki}, \quad \sum_i F(w_k \otimes v_i)d_{ij} = \delta_{kj}, \]
give the covectors, whence they satisfy
\[ F(\delta_E) = \sum_{ij} c_{ji} F(w_j \otimes v_i) = d = E(\delta_F) = \sum_{ij} d_{ij} E(v_i \otimes w_j). \]

Conversely, given invertible matrices \( A = (a_{ij}) \) with \( a_{ij} = E(v_i \otimes w_j) \) and \( B = (b_{ji}) \) with \( b_{ji} = F(w_j \otimes v_i) \) satisfying
\[ \text{trace}(tBA^{-1}) = d = \text{trace}(tAB^{-1}), \]
we obtain a fiber functor.

Two fiber functors \( \Phi \) and \( \Phi' \) are naturally equivalent if and only if we can find isomorphisms of vector spaces
\[ \Phi(X) \rightarrow \Phi'(X) = V', \quad \Phi(X^*) \rightarrow \Phi'(X^*) = W'. \]
such that the diagrams
\[ V \otimes W \longrightarrow V' \otimes W' \quad W \otimes V \longrightarrow W' \otimes V' \]
\[ \begin{array}{cccc}
E & \quad & E' & \\
\downarrow & & \downarrow & \\
C & & C & \\
\end{array}, \quad \begin{array}{cccc}
F & \quad & F' & \\
\downarrow & & \downarrow & \\
C & & C & \\
\end{array} \]
commute. This means that $A$ and $B$ can be replaced by
\[ t^T AS, \quad t^SBT \]
with $S$ and $T$ invertible square matrices. Thus we may choose $A = 1$ and there remains the gauge freedom of $t^T S = 1$, i.e.,
\[ B \mapsto T^{-1} BT. \]

**Theorem 5.1.** Fiber functors on $\mathcal{O}_d$ is completely classified by similarity orbits in
\[ \{ B \in GL(n, \mathbb{C}); \text{trace}(B) = d = \text{trace}(B^{-1}) \}. \]

For the description of the associated algebraic group (or a Hopf algebra $H$), we restore the matrix $A$ and choose bases $\{ v_i \}$ in $V$ and $\{ w_j \}$ in $W$. Then $H$ is generated, as an algebra, by $2n^2$ elements $\{ v_{ij} \}$ and $\{ w_{kl} \}$ with the defining relations
\[
\sum_{i,k} a_{ik} v_{ij} w_{kl} = a_{jl} 1, \quad \sum_{j,i} c_{ji} v_{kj} w_{li} = c_{kl} 1, \\
\sum_{i,k} b_{ik} w_{ij} v_{kl} = b_{jl} 1, \quad \sum_{j,i} d_{ji} w_{kj} v_{li} = d_{kl} 1
\]
corresponding to basic morphisms $V \otimes W \xrightarrow{E} C \xleftarrow{\delta_E} W \otimes V$ and $W \otimes V \xrightarrow{F} C \xleftarrow{\delta_F} V \otimes W$. The comultiplication is given by
\[
\Delta(v_{ij}) = \sum_k v_{ik} \otimes v_{kj}, \quad \Delta(w_{kl}) = \sum_i w_{ki} \otimes w_{il}.
\]

Next let $\Phi : \mathcal{O}_d \rightarrow \mathcal{H}ilb$ be a unitary fiber functor. Here $\mathcal{O}_d (d^2 \geq 4)$ is furnished with the $^*$-structure $D^* = (d/|d|)^{\mathcal{H}ilb} D'$ and $\mathcal{H}ilb$ denotes the $C^*$-tensor category of finite-dimensional Hilbert spaces. The condition $\Phi(D)^* = \Phi(D^*)$ for any $D$, i.e.,
\[
\Phi(\epsilon_X)^* = \frac{d}{|d|} \Phi(\delta_X), \quad \Phi(\epsilon_X)^* = \frac{d}{|d|} \Phi(\delta_X)
\]
is equivalent to the relation
\[
B = \frac{d}{|d|} A^{-1}
\]
with the trace-value condition given by
\[
\text{trace}(BB^*) = |d| = \text{trace}((BB^*)^{-1}).
\]

As in the case of non-unitary fiber functors, the gauge freedom is given by
\[
A \mapsto t^TAS, \quad B \mapsto t^SBT
\]
with $S, T$ unitary matrices. Note that the condition $\overline{A} = (d/|d|)B^{-1}$ as well as the trace-value relation is preserved under this transformation. Thus the orbit space for invertible matrices $B$ is identified with
\[
U(n)\backslash \text{GL}(n, \mathbb{C})/U(n) \cong \{\text{positive definite matrices}\}/\text{unitary similarity}.
\]

**Theorem 5.2.** (cf. [7, Theorem 1.1], [3, Theorem 6.2]) Unitary fiber functors on the $C^*$-tensor category $\mathcal{O}_d$ are parametrized by positive eigenvalue lists $\{\mu_1, \ldots, \mu_n\}$ satisfying
\[
\sum_j \mu_j^2 = |d| = \sum_j \mu_j^{-2}.
\]

We shall identify the associated compact quantum group with the universal quantum group of unitary type in [1, 4].

For this, we use the notation and the construction in [9]. Choose orthonormal bases $\xi = \{v_i\}$ for $V$ and $\eta = \{w_k\}$ for $W$ with the basis $\xi^* = \{v_i^*\}$ of $W$ defined by the relation $E(v_i^* \otimes v_i) = \delta_{ij}$, i.e.,
\[
\sum_k (w_k|v_i^*) b_{kj} = \delta_{ij}.
\]
Let $\{u_{ij}^\xi\}$ and $\{u_{kl}^\eta\}$ be the associated generators. Then, from the covariance condition,
\[
\sum_l (u_l|v_i^*) u_{kl}^\eta = \sum_j (w_k|v_j^*) u_{ji}^\xi
\]
and the *-relation $(u_{ij}^\xi)^* = u_{ij}^\xi^*$, we see that
\[
u_\eta(t^tB^{-1}) = (t^tB^{-1})u_{ij}^\xi^*
\]
in the matrix notation for $u_\eta^\xi$ and $u_{ij}^\xi^*$. Thus the associated Hopf *-algebra $H$ is generated by $u = (u_{ij}^\xi)$, which constitutes a unitary matrix by the orthonormality of the basis $\xi = \{v_j\}$.

From the universality on rigidity, the defining relations are given by the covariance conditions for the morphisms $\Phi(\epsilon_X), \Phi(\epsilon_{X^*}), \Phi(\delta_X)$ and
\Phi(\delta_X) = \sum (\varepsilon_X(v_i \otimes w_k)u_{j,l}^{i,k}) = \sum (\varepsilon_X^+(w_k \otimes v_i)u_{i,j,k}^{\eta\xi})
\sum u_{i,j,l,k}^{\xi\eta}(v_j \otimes w_l \Phi(\delta_X)) = (v_i \otimes w_k \Phi(\delta_X^+))1_H,
\sum u_{i,j,l,k}^{\xi\eta}(v_j \otimes w_l \Phi(\delta_X^+)) = (v_i \otimes w_k \Phi(\delta_X^+))1_H.

From the multiplication relations
\begin{align*}
u_{i,j}^{\xi\eta}u_{k,l} = u_{i,j}^{\xi}u_{k,l}^{\eta},
u_{i,j}^{\eta\xi} = u_{k,l}^{\eta}u_{i,j}^{\xi},
\end{align*}
the condition is equivalent to
\begin{align*}
A &= t^\xi u_{i,j}^{\xi}u_{k,l}^{\eta},
B &= t^\eta u_{i,j}^{\eta}u_{k,l}^{\xi},
\end{align*}
and therefore to the conditions
\begin{align*}
u^{\xi\eta} &= 1 = u^{\xi*}u,
u(AA^*)^{-1}u = (AA^*)^{-1},
\end{align*}
Thus the associated compact quantum group is identified with $A_u(AA^*)$ in [4].

**Proposition 5.3.** The compact quantum group associated to the unitary fiber functor is naturally isomorphic to $A_u(AA^*)$ in [4], i.e., $A_u(A^*)$ in [1].

**Remark.** In [3], unitary fiber functors are classified for representation categories of the compact quantum group $A_u(F)$ based on representation theory of [2]. Since these representation categories are exactly $O_d$ as tensor categories by the above proposition, we get an access to some results in [3] in an elementary way.

**Appendix A. Faithfulness**

We shall here check the automatic faithfulness of relevant functors.

**Proposition A.1.** Fiber functors on the Temperley-Lieb categories are faithful whenever the fundamental vector space $V$ has dimension two or more.

Fiber functors on $O_d$ are faithful if the fundamental vector spaces $V$ and $W$ have dimension two or more.
Proof. Since Frobenius transforms are isomorphisms, we need to show that the family \( \{ \Phi(D); D \in K_{2n,0} \} \) is linearly independent for any \( n \geq 1 \), where \( \Phi : \mathcal{T}L_d \to \text{Vec} \) denotes a fiber functor such that the fundamental vector space \( V = \Phi(X) \) has dimension two or more.

Given \( 1 \leq k \leq n \), let \( \{ x_1, \ldots, x_k \} \) be the first \( k \)-vertices (counting from the left end) for diagrams in \( K_{2n,0} \) and set \( D_{k,n} = \{ D \in K_{2n,0}; \text{there are no arcs connecting } x_i \text{ and } x_j \text{ in } D \} \).

Note that \( D_{n,n} \subset D_{n-1,n} \subset \cdots \subset D_{2,n} \subset D_{1,n} = K_{2n,0} \) and \( D_{n,n} \) consists of a single diagram.

Since \( \dim V \geq 2 \), we can find a vector \( 0 \neq v \in V \) such that \( F(\epsilon)(v \otimes v) = 0 \) as a solution of a quadratic equation, for which we shall show that the family \( \mathcal{E}_{k,n} = \{ \Phi(D)(v^{\otimes k} \otimes \cdot) \in \mathbb{C}; D \in D_{k,n} \} \) is linearly independent by an induction on \( (k, n) \).

Given \( 1 \leq k \leq n \), assume the linear independence of \( \mathcal{E}_{k',n'} \) for \( 1 \leq k' < n \) and for \( k' > k, n' = n \). We shall prove that the family \( \mathcal{E}_{k,n} \) is linearly independent. Suppose that

\[
\sum_{D \in D_{k,n}} c_D \Phi(D)(v^{\otimes k} \otimes \cdot) = 0.
\]

Performing the evaluation by the vector \( v \) one step further, we have

\[
\sum_{D \in D_{k,n}} c_D \Phi(D)(v^{\otimes (k+1)} \otimes \cdot) = 0
\]

in \( (V^{\otimes (2n-k-1)})^* \). Since \( \Phi(D) \) is killed by this evaluation for \( D \in D_{k,n} \setminus D_{k+1,n} \), the summation can be restricted to \( D_{k+1,n} \) and the induction hypothesis ensures \( c_D = 0 \) for \( D \in D_{k+1,n} \).

By the non-degeneracy of the bilinear form \( \Phi(\epsilon) \), we can find a vector \( v' \in V \) satisfying \( \Phi(\epsilon)(v \otimes v') = 1 \). Evaluating by \( v' \) in the starting equation, we then have

\[
\sum_{D \in D_{k,n} \setminus D_{k+1,n}} c_D \Phi(D)(v^{\otimes k} \otimes v' \otimes \cdot) = 0.
\]

Since \( D_{k,n} \setminus D_{k+1,n} \) consists of diagrams which contain the arc connecting \( x_k \) and \( x_{k+1} \) (Fig. 3), we have the natural bijection \( D_{k,n} \setminus D_{k+1,n} \ni D \mapsto D' \in D_{k-1,n-1} \) and then the last equation takes the form

\[
\sum_{D' \in D_{k-1,n-1}} c_D \Phi(D')(v^{\otimes (k-1)} \otimes \cdot) = 0.
\]
Again by the induction hypothesis, \( c_D = 0 \) for \( D \in D_{k,n} \setminus D_{k+1,n} \) and we are done for the Temperley-Lieb case.

For a fiber functor \( \Phi : \mathcal{O}_d \to \text{Vec} \), we normalize the bilinear forms \( \Phi(\epsilon_X) : V \otimes W \to \mathbb{C} \) and \( \Phi(\epsilon_Y) : W \otimes V \to \mathbb{C} \) so that the associated matrices \( A \) and \( B \) are upper-triangular. Then, for the choice \( v = (1, 0, \ldots, 0) \) and \( w = (0, 1, 0, \ldots, 0) \), we see \( \Phi(\epsilon_X)(v \otimes w) = \Phi(\epsilon_Y)(w \otimes v) = 0. \)

Now, given an object \( X^\omega \) of \( \mathcal{O}_d \), we can take the tensor product of \( v \) and \( w \) according to the arrangement of \( X \) and \( Y = X^\ast \) in \( X^\omega \) up to the \( k \)-th factor. By using these as probing vectors, we can repeat the above argument to conclude the independence of the family \( \{ \Phi(D); D \in K_{\omega,0} \} \).

\begin{figure}[h]
\centering
\begin{tikzpicture}
\draw[step=1cm,gray,very thin] (0,0) grid (3,3);
\node at (0.5,0.5) {1};
\node at (2.5,0.5) {k - 1};
\node at (3.5,0.5) {k + 1};
\draw[dashed] (0.5,0.5) -- (2.5,2.5);
\draw[dashed] (0.5,0.5) -- (0.5,2.5);
\draw[dashed] (2.5,0.5) -- (2.5,2.5);
\draw[dashed] (3.5,0.5) -- (3.5,2.5);
\end{tikzpicture}
\caption{}
\end{figure}

\[ \square \]

References

[1] T. Banica, Théorie des représentations du groupe quantique compact libre \( O(n) \), \textit{C. R. Acad. Sci. Paris}, 322(1996), 241–244.
[2] , Le groupe quantique compact libre \( U(n) \), \textit{Commun. Math.Phys.}, 190(1997), 143–172.
[3] J. Bichon, A. De Rijdt and S. Vaes, Ergodic coactions with large multiplicity and monoidal equivalence of quantum groups, \textit{Commun. Math.Phys.}, 262(2006), 703–728.
[4] A. van Daele and S.Z. Wang, Universal quantum groups, \textit{Intern. J. Math.}, 7(1996), 255–264.
[5] L.H. Kauffman, State models and the Jones polynomial, \textit{Topology}, 26(1987), 395–407.
[6] An invariant of regular isotopy, \textit{Trans. Amer. Math. Soc.}, 318(1990), 417–471.
[7] S. Wang, Structure and isomorphism classification of compact quantum groups \( A_u(Q) \) and \( B_u(Q) \), \textit{J. Operator Theory}, 48(2002), 573–583.
[8] S. Yamagami, A categorical and diagrammatical approach to Temperley-Lieb algebras, \texttt{math.QA/0405267}.
[9] Fiber functors on Temperley-Lieb categories, \texttt{math.QA/0405517}.