MONOTONICITY PRINCIPLE IN TOMOGRAPHY OF NONLINEAR CONDUCTING MATERIALS

ANTONIO CORBO ESPOSITO, LUISA FAELLA, GIANPAOLO PISCITELLI, RAVI PRAKASH, ANTONELLO TAMBURRINO

Abstract. We treat an inverse electrical conductivity problem which deals with the reconstruction of nonlinear electrical conductivity starting from boundary measurements in steady currents operations. In this framework, a key role is played by the Monotonicity Principle, which establishes a monotonic relation connecting the unknown material property to the (measured) Dirichlet-to-Neumann operator (DtN). Monotonicity Principles are the foundation for a class of non-iterative and real-time imaging methods and algorithms.

In this article, we prove that the Monotonicity Principle for the Dirichlet Energy in nonlinear problems holds under mild assumptions. Then, we show that apart from linear and $p$-Laplacian cases, it is impossible to transfer this Monotonicity result from the Dirichlet Energy to the DtN operator. To overcome this issue, we introduce a new boundary operator, identified as an Average Dirichlet-to-Neumann operator.

Keywords: Inverse electrical conductivity problem, Nonlinearity, Monotonicity Principle, Average Dirichlet-to-Neumann operator.

MSC 2010: 35J60, 74G75.

1. INTRODUCTION

In this paper, we derive the Monotonicity Principle for an inverse conductivity problem modeled by a fully nonlinear variant of the Calderón problem. Specifically, we treat the problem of retrieving the nonlinear electrical conductivity $\sigma$ starting from boundary measurements for stationary cases (steady currents). More precisely, we consider a fully nonlinear problem where the constitutive relationship is local, isotropic and memoryless:

$$\mathbf{J}(x) = \sigma(x, |\mathbf{E}(x)|)\mathbf{E}(x) \quad \forall x \in \Omega,$$

(1.1)

1Dipartimento di Ingegneria Elettrica e dell’Informazione “M. Scarano”, Università degli Studi di Cassino e del Lazio Meridionale, Via G. Di Biasio n. 43, 03043 Cassino (FR), Italy.

2Departamento de Matemática, Facultad de Ciencias Físicas y Matemáticas, Universidad de Concepción, Avenida Esteban Iturra s/n, Bairro Universitario, Casilla 160 C, Concepción, Chile.

3Department of Electrical and Computer Engineering, Michigan State University, East Lansing, MI-48824, USA.

Email: antonio.corboesposito@unicas.it, luisa.faella@unicas.it, gianpaolo.piscitelli@unicas.it, rprakash@udec.cl, antonello.tamburrino@unicas.it (corresponding author).
where $\sigma$ is the nonlinear electrical conductivity, $\mathbf{J}$ the electric current density, $\mathbf{E}$ the electric field and $\Omega \subset \mathbb{R}^n$, $n \geq 2$, is an open bounded domain with Lipschitz boundary. $\Omega$ represents the region occupied by the conducting material.

In steady currents operations, the electric field can be expressed through the electrical scalar potential $u \in W^{1,p}(\Omega)$ as $\mathbf{E}(x) = -\nabla u(x)$, where $p$ depends on the behaviour of $\sigma$. The electric scalar potential $u$ solves the steady current problem:

$$
\begin{aligned}
\text{div} \left( \sigma(x, |\nabla u(x)|) \nabla u(x) \right) &= 0 \quad \text{in } \Omega \\
u(x) &= f(x) \quad \text{on } \partial \Omega,
\end{aligned}
$$

(1.2)

where $f \in X_\sigma$ is the applied boundary potential, with $X_\sigma$ being an appropriate abstract trace space (see Section 2). The existence of a solution is guaranteed under suitable assumptions on $\sigma$ (see Section 2).

Nonlinear electrical conductivities can be found in semiconducting and ceramic materials (see [1]). Among the applications, cable termination in high voltage (HV) and medium voltage (MV) systems [2, 3] are well studied examples. Nonlinear electrical conductivities is also found in superconductors, key materials for applications like energy storage, magnetic levitation systems, superconducting magnets (nuclear fusion devices, nuclear magnetic resonance) and high-frequency radio technology [4, 5]. In all these applications, one can expect the need for nondestructive evaluation and/or imaging. For example, the Authors of [6, 7, 8] treated the nondestructive testing in the presence of superconductors. Nonlinear models for the electrical conductivity can be found also in the area of biological tissues (see [9]). For instance, [10] proved that nonlinear models fit the experimental data better than linear models.

We stress that problem (1.2) can be applied without any relevant modification to other physical settings. For instance, in the framework of electromagnetism, both nonlinear electrostatic and nonlinear magnetostatic phenomena can be modeled in the form of (1.2). The unknown material property is the dielectric permittivity for electrostatic case whereas the magnetic permeability for magnetostatic one. In the first case the constitutive relationship is $\mathbf{D}(x) = \varepsilon(x, |\mathbf{E}(x)|)\mathbf{E}(x)$ (see [11] and references therein, see [12]), whereas in the second case $\mathbf{B}(x) = \mu(x, |\mathbf{H}(x)|)\mathbf{H}(x)$ (see [13]).

In this article, we develop a theoretical contribution to the field of inverse problems with nonlinear constitutive relationships. From a general perspective, as quoted in [14], “... the mathematical analysis for inverse problems governed by nonlinear Maxwell’s equations is still in the early stages of development.”. One can expect that as new methods and algorithms will be available, the demand for nondestructive evaluation and imaging of nonlinear materials will eventually and significantly arise.

\footnote{In magnetostatic, it is possible to introduce a magnetic scalar potential in simply connected and source free regions.}
In this framework, a key role is played by the (nonlinear) Dirichlet-to-Neumann (DtN) operator, mapping the boundary voltage $f$ to the current flux at the boundary

$$
\Lambda_\sigma : f \in X_\sigma \rightarrow -J \cdot \hat{n}|_{\partial \Omega} = \sigma \partial_n u|_{\partial \Omega} \in X_\sigma'
$$

where $u$ is the solution of (1.2) with boundary data $f$, $X'_\sigma$ is the dual of $X_\sigma$ and $\hat{n}$ denotes the outer unit normal on $\partial \Omega$.

The goal of this paper is to provide a “tool”, the Monotonicity Principle, to reconstruct the nonlinear electrical conductivity $\sigma$ starting from the knowledge of the boundary data $\Lambda_\sigma$.

The Monotonicity Principle Method (MPM) is an imaging method which relies on a monotone relation connecting the unknown material property to the measured DtN or its inverse. In the linear case, MPM states that

$$\sigma \leq \tau \implies \Lambda_\sigma \leq \Lambda_\tau. \tag{1.3}$$

where $\sigma$ and $\tau$ are two electrical conductivities defined in $\Omega$, $\Lambda_\sigma$ and $\Lambda_\tau$ are the corresponding DtN operators. In equation (1.3), $\sigma \leq \tau$ is understood in the almost everywhere sense in $\Omega$, and $\Lambda_\sigma \leq \Lambda_\tau$ means that $\Lambda_\sigma - \Lambda_\tau$ is negative semidefinite. Monotonicity relation (1.3) shows that a pointwise increase of the electrical conductivity leads to “greater” boundary data.

Monotonicity (1.3) is the basis to develop non-iterative and real-time reconstruction methods and algorithms [15, 16, 17]. MPM has been mainly applied to shape reconstruction problems for detecting the shape of anomalies in a given background. In this specialization, the method determines if a test inclusion is part of the anomaly or not by a simple test. Indeed, if $T$ is a “test” anomaly and $V$ is the unknown anomaly, corresponding to the electrical conductivities given by $\sigma_T$ and $\sigma_V$, respectively, (1.3) implies

$$T \subseteq V \implies \Lambda_{\sigma_T} \succeq \Lambda_{\sigma_V}, \tag{1.4}$$

where we have assumed that both anomalies have electrical conductivity smaller than that of the background. Equation (1.4) corresponds to

$$\Lambda_{\sigma_T} \preceq \Lambda_{\sigma_V} \implies T \equiv V. \tag{1.5}$$

Therefore, from the knowledge of the boundary DtN operators, we can infer if the prescribed test anomaly $T$ is contained or not in the unknown anomaly $V$. By repeating the test in (1.5) for various sets $T$, we can reconstruct an approximation of the shape of the unknown anomaly $V$.

According to our awareness, the first evidence of a monotone property (1.3) for the linear case appeared in [18]. Then its relevance to the field of Inverse Problems was first recognized in [15], where Tamburrino and Rubinacci proposed (1.5) to establish a new imaging method. Specifically, they (i) proved the equivalent of (1.3) but for a real-world system made by a finite number of electrodes, (ii) proposed and numerically tested the imaging method based on (1.5) and (iii) extended (1.3) to perfectly conducting or insulating anomalies.
Surprisingly, Monotonicity Principles appear to be a general feature which can be found in many problems governed by PDEs of different nature. Indeed, despite originally found for elliptic PDEs arising from static problems (as, for instance, Electrical Resistance, Capacitance or Inductance Tomography) [15, 17, 19, 20, 21], it was also found for elliptic PDEs but arising from quasi-static problems (as, for instance, Eddy Current Tomography) [16, 17, 22].

Parabolic PDEs (for instance, pulsed Eddy Current Tomography) have been treated in [23, 24, 25, 26, 27]. Specifically, it was proved a Monotonicity Principle for the time constants of the natural modes.

Monotonicity of the transmission eigenvalues for the Helmholtz equation was analyzed in [28]. Other Monotonicity Principles for problems governed by the Helmholtz equation were developed in [29, 30] for bounded domains and in [31, 32] for unbounded domains. Monotonicity was also applied to crack detection for the Helmholtz equation in [33].

Monotonicity for linear elasticity was introduced in [34].

A special feature of MPM is that it provides rigorous upper and lower bounds to the unknown, even in the presence of noise, under proper hypothesis [35].

The limiting case of perfectly insulating anomalies has been treated in [15, 36] and the case of perfectly conducting anomalies in [15].

The concept of regularization for MPM was introduced in [37, 38]. This is relevant because MPM is not based upon the minimization of an objective function, where regularization can be easily introduced by means of penalty terms. Also, Monotonicity has been used as regularizer in [39].

A first experimental validation of MPM for Eddy Current Tomography can be found in [40].

Additional numerical aspects have been studied in [38, 41, 42, 43, 44]. The stability of the method has been treated and proved in [21, 34, 36, 38, 43, 44, 45, 46]. Numerical evidence can be found in [23], whereas experimental evidence in [40].

Theoretical applications of the Monotonicity can be found in the proof of uniqueness results [17, 48, 49].

Other than soft-field Tomography and Nondestructive Testing, MPM has been applied to homogenization of materials [50] and concrete rebars inspection [51, 52].

Monotonicity was combined with frequency-difference and ultrasound modulated Electrical Impedance Tomography measurements, to reduce the impact of modelling errors such those arising from electrode positions and the shape of the imaging domain [53].

The converse of Monotonicity [1,4], under the assumption that the unknown anomaly consists of union of non-contractible sets, was proved by Harrach and Ullrich in [21]. This result is relevant because it states that MPM gives exact reconstruction for (union of) contractible anomalies, at least in the ideal setting when the measured boundary data is the DtN operator. Unfortunately, this is
not the case for practical systems made of a finite number of electrodes, where implication (1.4) holds but not its converse (see [13]).

Monotonicity, but in a different form, has been treated in [54].

Eventually, it is worth noting that imaging methods based on Monotonicity Principle fall in the class of non-iterative imaging methods. Colton and Kirsch introduced the first non-iterative approach named Linear Sampling Method (LSM) [55] followed by the Factorization Method (FM) proposed by Kirsch [56]. Ikehata proposed the Enclosure Method [57, 58] and Devaney applied MUSIC (MUltilpe SIGnal Classification), a well known algorithm in signal processing, as imaging method [59].

A special case for equation (1.1) is when \( \sigma(x, |E(x)|) = \theta(x)|E(x)|^{p-2} \). Then the relationship between the electrical current density \( J \) and the electric field \( E \) can be written as

\[
J(x) = \theta(x)|E(x)|^{p-2}E(x),
\]

where \( \theta \in L^\infty(\Omega) \) and \( \theta(x) \geq c_0 \) a.e. in \( \Omega \) for some positive constant \( c_0 \). This leads to the study of a steady current problem involving the \( p \)-Laplacian. Here, briefly, we give an overview of the main and most recent results concerning the nonlinear \( p \)-Laplace type model. The inverse problem of Calderón was initially posed in the setting of the \( p \)-conductivity equation by Salo and Zhong [60] and, then, studied in [61, 62] also. In [60], the authors proved boundary determination results under proper regularity assumptions on the conductivity and boundary of the domain. Moreover, they showed that to investigate Calderón-type problems for equations with weak non-linearities (see [63, 64]), one can use the Gâteaux derivative of the DtN operator at constant boundary values. The method does not work for \( p \)-Laplace equation as proved in the Appendix of [60]. For these Calderón-type problems, the Monotonicity inequality was proved in [61] (see Lemma 2.1). Here the authors study the enclosure method for non linear equation. The enclosure method introduced by Ikehata uses complex geometrical optics (CGO) solution in place of point sources (see [57, 58]).

The authors in [65] have treated a nonlinear problem (linear plus a nonlinear term) which is a particular case of the model proposed in this article.

In [62], the author proved a boundary uniqueness result for the first order normal derivative of the conductivity. In [64], the authors extended the weak DtN operator to conductivities that include regions of zero or infinite conductivity.

The extension from the isotropic case of [61] to the anisotropic one has been proposed in [67]. The authors used the Monotonicity inequality to prove injectivity for the DtN operator under a Monotonicity assumption (see Theorem 2.1 and Lemma 2.2). Their results show injectivity in two dimensions for Lipschitz conductivities. In higher dimension cases, one of the conductivities is required to be close to a constant.

In [68], a Calderón problem for nonlinear \( p \)-Laplacian type equations is studied. In this paper, the authors show that Monotonicity based shape reconstruction
methods ([15, 17]) work in the \( p \)-Laplacian case which allow them to find the complex hull of the inclusion without any regularity or interface jump assumption. As a matter of fact, any regularity on jump properties for the inclusion is not required and they obtain this subset using both the Monotonicity and enclosure method. For properties of DtN operator with \( \theta = 1 \) in [16], we refer to Hauer [69].

The original contribution of this paper consists in deriving a Monotonicity Principle in the fully nonlinear case. This result is not at all a trivial development of the previous ones. Indeed, we prove that, in general, the Dirichlet Energy \( \mathbb{F}_\sigma (u^1) \) is the quantity being monotone with respect to the electrical conductivity (Theorem 4.1), that is:

\[
\sigma_1 \leq \sigma_2 \implies \mathbb{F}_{\sigma_1}(u_1^1) \leq \mathbb{F}_{\sigma_2}(u_2^1),
\]

where \( u_1^1 \) and \( u_2^1 \) are the solutions of (1.2) corresponding to \( \sigma_1 \) and \( \sigma_2 \), respectively, with \( f \) being the applied boundary voltage. In (1.7), \( \sigma_1 \leq \sigma_2 \) means that \( \sigma_1(x,E) \leq \sigma_2(x,E) \) for a.e. \( x \in \Omega \) and \( \forall E > 0 \).

Moreover, we show that Monotonicity can be easily transferred to the boundary DtN operator, but only for linear and \( p \)-Laplacian problems. In these cases, we demonstrate that the DtN power product \( \langle \Lambda_\sigma(f), f \rangle \) is proportional to the Dirichlet Energy and, hence, the Monotonicity for the boundary DtN operator follows.

When the nonlinearity is more general, for instance, of polynomial type, the Dirichlet Energy is monotone with respect to the constitutive relationship but it is not proportional to the power product \( \langle \Lambda_\sigma(f), f \rangle \) (see Subsection 4.2). Therefore, Monotonicity cannot be transferred from an “internal” quantity such as the Dirichlet Energy to boundary data like \( \Lambda_\sigma \). This is a major issue since in solving inverse problem, we do not have any access to internal quantities like the Dirichlet Energy, instead, we have access to data which can only be measured from the boundary. Therefore, for the general nonlinear case, we need to “transfer” the Monotonicity Principle from the Dirichlet Energy (internal quantity) to another proper boundary operator other than \( \Lambda_\sigma \).

Specifically, we introduce a new nonlinear boundary operator \( \overline{\Lambda}_\sigma \) which is monotonic with respect to the nonlinear material property. We name \( \overline{\Lambda}_\sigma \) as the Average DtN Operator which is defined as

\[
\overline{\Lambda}_\sigma : f \in X_\sigma \mapsto \overline{\Lambda}_\sigma(f) = \int_0^1 \Lambda_\sigma(\alpha f) \, d\alpha \in X_\sigma'.
\]

Our main result is the Monotonicity Principle for the operator \( \overline{\Lambda}_\sigma \). To be more precise, we prove

\[
\sigma_1 \leq \sigma_2 \implies \overline{\Lambda}_{\sigma_1} \leq \overline{\Lambda}_{\sigma_2},
\]

where \( \overline{\Lambda}_{\sigma_1} \leq \overline{\Lambda}_{\sigma_2} \) means \( \langle \overline{\Lambda}_{\sigma_1}(f), f \rangle \leq \langle \overline{\Lambda}_{\sigma_2}(f), f \rangle \) for any \( f \in X_\sigma \).

The key factor in achieving this result is Theorem 4.2, which deals with the transfer of Dirichlet Energy to the power product for the Average DtN operator.
More precisely, we prove

\[(F_\sigma \circ U_\sigma)(f) = \langle \Lambda_\sigma (f), f \rangle \quad \forall f \in X_\circ, \quad (1.8)\]

where operator $U_\sigma$ maps the boundary data $f$ to the corresponding solution $u^f$ of

\[ (1.2), \quad i.e.\]

\[ U_\sigma : f \in X_\circ \rightarrow u^f \in W^{1,p}(\Omega). \]

The proof of (1.8) is based on a fundamental result obtained in Proposition 3.4. Specifically, we prove that the Gâteaux derivative operator of $F_\sigma \circ U_\sigma$, with respect to the boundary data $f$, is equal to the DtN operator $\Lambda_\sigma$, i.e.

\[ d(F_\sigma \circ U_\sigma) = \Lambda_\sigma. \quad (1.9) \]

To identify the key differences between our problem and the $p$-Laplacian / linear ($p = 2$) cases, it is worth noting that (1.8) is replaced by

\[(F_\sigma \circ U_\sigma)(f) = p^{-1} \langle \Lambda_\sigma (f), f \rangle \quad \forall f \in X_\circ,\]

whereas (1.9) remains unchanged.

The paper is organized as follows: in Section 2 we describe the problem together with the preliminaries required for its analysis; in Section 3 we study the behaviour of the solution of Problem (1.2) and of the Dirichlet Energy with respect a variation of the boundary data; in Section 4 we prove the main result and, eventually, in Section 5 we provide some important conclusions.

2. Foundations of the problem

Throughout this paper, $\Omega$ is the region occupied by the conducting material. We assume $\Omega \subset \mathbb{R}^n$, $n \geq 2$, to be an open bounded domain with Lipschitz boundary. We denote by $\hat{n}$ the outer unit normal defined on $\partial \Omega$, by $\langle \cdot, \cdot \rangle$ the integral scalar product on $\partial \Omega$ and by $V$ and $S$ the $n$-dimensional and the $(n - 1)$-dimensional Hausdorff measure, respectively. Moreover, we denote

\[ L^\infty_+(\Omega) := \{ \theta \in L^\infty(\Omega) \mid \theta \geq c_0 \text{ a.e. in } \Omega, \text{ for some positive constant } c_0 \}. \]

Then, for $1 < p < +\infty$, $W^{1,p}_0(\Omega)$ is the closure set of $C_0^1(\Omega)$ with respect to the $W^{1,p}$-norm.

Furthermore, the applied boundary voltage $f$ belongs to the abstract trace space

\[ X = W^{1,p}(\Omega)/W^{1,p}_0(\Omega) \approx B^{1-\frac{1}{p}}_p(\partial \Omega), \]

that is also a Besov space (refer to [70, Chap. 17], [69, App.]). In the sequel, by a little abuse of notation, we write $f \in X$ meaning that $f$ is a representative of an $X$-equivalence class, i.e. $f \in [a]_X$ where $[a]_X \in X$. We indicate that $X_\circ$ is the set of elements in $X$ with zero average on $\partial \Omega$ with respect to the measure $S$. 
2.1. The physical problem. Let \( \sigma \) be a function representing the nonlinear electrical conductivity, i.e. \( J(x) = \sigma(x, E(x))E(x) \) where \( E \) and \( J \) are the electric field and the electrical current density, respectively. In addition, let \( E \) and \( J \) be the magnitude of \( E \) and \( J \), respectively.

Stationary currents are governed by:

\[
\text{curl } E(x) = 0 \text{ in } \Omega, \quad \int_{x}^{y} E \cdot \mathbf{\hat{t}} \, d\mathbf{\ell} = f(x) - f(y) \quad \forall x, y \in \partial \Omega; \tag{2.1}
\]

\[
\text{div } J(x) = 0 \text{ in } \Omega, \tag{2.2}
\]

\[
J(x) = \sigma(x, E(x)) \cdot E(x) \text{ in } \Omega, \tag{2.3}
\]

where \( f \) is the applied boundary voltage. Equations (2.1) and (2.2) come from Maxwell equations for stationary models. We stress that equations (2.1) and (2.2) have to be meant in weak sense and

\[
\text{curl } E \in H_{\text{curl}}(\Omega) = \{ w \in L^2(\Omega; \mathbb{R}^3) \mid \text{curl}(w) \in L^2(\Omega) \},
\]

\[
J \in H_{\text{div}}(\Omega) = \{ w \in L^2(\Omega; \mathbb{R}^3) \mid \text{div}(w) \in L^2(\Omega) \}.
\]

The curvilinear integral appearing in (2.1) is well defined for \( E \) in \( H_{\text{curl}}(\Omega) \) and holds for any \( C^1 \)-curve in \( \Omega \) with extrema \( \bar{x} \) and \( \bar{y} \in \partial \Omega \) (see \[72\]). Though problem (2.1)-(2.3) is defined in \( \mathbb{R}^3 \), it is important to observe that all the forthcoming results hold in any dimension \( n \geq 1 \), once the scalar potential \( u \) has been introduced.

2.2. The Electrical Conductivity. The well-posedness of the forward problem in Hadamard sense (see Section 2.4 below) is the minimal requirement to formulate the associated inverse problem. This objective is guaranteed by the following assumptions on \( \sigma : \overline{\Omega} \times [0, +\infty[ \rightarrow \mathbb{R}^n \):

(H1) \( x \in \overline{\Omega} \mapsto \sigma(x, E) \) is measurable \( \forall E \geq 0 \);

(H2) \( E \in [0, +\infty[ \mapsto \sigma(x, E)E \) is strictly increasing for a.e. \( x \in \overline{\Omega} \);

(H3) \( E \in [0, +\infty[ \mapsto \sigma(x, E) \) is in \( C([0, +\infty[) \) for a.e. \( x \in \overline{\Omega} \);

Moreover, for a fixed \( p \geq 2 \), we have that:

(H4) there exist three positive constants \( \sigma_2 \geq \sigma_1 \) and \( E_0 > 0 \) such that:

\[
\sigma_1 \left( \frac{E}{E_0} \right)^{p-2} \leq \sigma(x, E) \leq \sigma_2 \max \left\{ 1, \left( \frac{E}{E_0} \right)^{p-2} \right\} \quad \text{for a.e. } x \in \overline{\Omega} \text{ and } \forall E > 0;
\]

(H5) there exists \( c > 0 \) such that:

\[
(\sigma(x, E_2)E_2 - \sigma(x, E_1)E_1) \cdot (E_2 - E_1) \geq c|E_2 - E_1|^p \quad \text{for a.e. } x \in \overline{\Omega} \text{ and } \forall E_1, E_2 \in \mathbb{R}^n.
\]

Parameter \( p \) appearing in (H4) and (H5) is related to \( W^{1,p}(\Omega) \), the functional space to which the scalar potential belongs. Assumption (H5) is a generalization of a known vector inequality \[73\] Sec.12, eq (1)]. Moreover, it is also used in \[14\] eq. (3.4) for the particular case of \( p = 2 \). We remark that (H3) does not ask for the
continuous dependence of conductivity upon the space variable $x$. In other terms, one can treat materials with abrupt discontinuity of the electrical conductivity.

We stress that assumptions (H1)-(H5) are quite general (see also Figure 1). Indeed, our objective is to generalize the $p$-Laplacian and linear cases (see (2.12) and (2.13) below). For instance, polynomial electrical conductivities

$$\sigma(x, E) = \sum_{k=0}^{N} \theta_k(x) E^k \text{ for a.e. } x \in \overline{\Omega} \text{ and } \forall E > 0$$

obviously satisfy (H1)-(H3). Moreover, polynomial (2.4) satisfies (H4) with $p = N + 2$ and $\sigma_1 = c_N$, where $c_N > 0$ is the essential infimum of $\theta_N$ as in the definition of $L^+_N(\Omega)$, and $\sigma_2$ is the maximum among the essential supremums of $\theta_k$, $k = 0, ..., N$. Hypothesis (H5) is nothing but a generalization of the standard inequality (see [73, Sec.12, eq (1)])

$$(E^k_2 \cdot E - E^k_1 \cdot E_1) \cdot (E_2 - E_1) \geq \frac{1}{2^{k+1}} |E_2 - E_1|^{k+2} \quad \forall k \geq 0.$$  (2.5)

Indeed, by multiplying (2.5) with $\theta_k(x)$ and by summing from 0 to $N$, we have\[ (\sigma(x, E_2) \cdot E_2 - \sigma(x, E_1) \cdot E_1) \cdot (E_2 - E_1) \geq \frac{\theta_N(x)}{2^{N+1}} |E_2 - E_1|^{N+2} \geq \frac{c_N}{2^{N+1}} |E_2 - E_1|^{p}, \]

where $c_N > 0$ is the essential infimum of $\theta_N$, as aforementioned. In other terms, (2.4) satisfies (H5).

![Figure 1](image)

**Figure 1.** Impact of (H4) on the constitutive relationship in terms of electrical conductivity (a) and current density (b). Solid lines corresponds to the upper and lower constraints to either $\sigma$ or $J$.

In the literature, there are many practical examples with constitutive relationships satisfying assumptions (H1)-(H5).
The monomial case, corresponding to the $p$-Laplacian, for some applicable situations is referred in [60, 61, 62, 66, 74]. Polynomial constitutive relationships can be found in nonlinear inverse problems for incompressible hyperelastic materials [75] and piezoelectric materials [76]. Other examples of nonlinear constitutive relationships for conductive materials complying hypotheses (H1)-(H5) are given in [2, 3, 77].

As a final remark, we mention some cases which cannot be treated in our framework. These cases are those of exponential constitutive relationships (see [1, 78, 10]), the $p$-Laplacian for $1 < p < 2$ (see [79, 80] and reference therein) and fractional Laplacian [81].

2.3. Scalar potential and Dirichlet Energy functional. In terms of the electrical scalar potential, that is $E_{p,q}(x) = -\nabla u(x)$ with $u \in W^{1,p}(\Omega)$, the Ohm’s law (2.3) is

$$J(x) = -\sigma(x,|\nabla u(x)|) \nabla u(x).$$

(2.6)

Therefore, by (2.1), (2.2) and (2.6), the electrical potential $u$ solves the steady current problem:

$$\begin{cases}
\text{div} \left( \sigma(x,|\nabla u(x)|) \nabla u(x) \right) = 0 \text{ in } \Omega \\
u(x) = f(x) \text{ on } \partial \Omega.
\end{cases}$$

(2.7)

Here $u$ satisfies the boundary condition in the sense that $u - f \in W^{1,p}_0(\Omega)$ and we write $u|_{\partial \Omega} = f$. Specifically, problem (2.7) is meant in the weak sense, that is

$$\int_{\Omega} \sigma(x,|\nabla u(x)|) \nabla u(x) \cdot \nabla \varphi(x) \, dx = 0 \quad \forall \varphi \in W^{1,p}_0(\Omega).$$

The solution $u$ of (2.7) is variationally characterized as

$$\arg \min \{ \mathcal{F}_\sigma(u) : u \in W^{1,p}(\Omega), \ u|_{\partial \Omega} = f \}. \quad (2.8)$$

In (2.8), functional $\mathcal{F}_\sigma(u)$ is the Dirichlet Energy

$$\mathcal{F}_\sigma(u) = \int_{\Omega} Q_\sigma(x,|\nabla u(x)|) \, dx \quad (2.9)$$

and $Q_\sigma$ is the Dirichlet Energy density

$$Q_\sigma(x, E) = \int_0^E \sigma(x, \xi) \, d\xi \quad \text{for a.e. } x \in \overline{\Omega} \text{ and } \forall E \geq 0. \quad (2.10)$$

2.4. Well-posedness of the Forward Problem. To obtain the existence of the solution of the forward problem (2.8), we use (H1), the left inequality in (H4) and a simpler version of (H2) with $\sigma(x, E)E$ being weakly increasing. The proof follows from standard direct methods of calculus of variations (see e.g. [82 Sec.s 4,5], [83]). Indeed, the convexity and coercivity of $Q_\sigma$ with respect to $E$ are the key factors.
Uniqueness of the solution follows from (H2) that corresponds to the strict convexity of $Q_\sigma$ with respect to $E$.

The continuity of the solution of the forward problem with respect to the boundary data $f$ in a suitable norm follows from Lemma 3.1.

Eventually, we stress that problems, as in (2.8), have been broadly studied in various fields of mathematics (see e.g. [84, 85, 86] and reference therein).

2.5. **Connection among $\sigma$, $J_\sigma$ and $Q_\sigma$.** The Ohm’s law (2.3) is isotropic and local, i.e. $J$ is parallel to $E$ and $J_\sigma$ depends on the position $x$ and on the magnitude of the electric field $E$ at the same location $x$.

By (2.6), the Dirichlet Energy density can be also written as

$$Q_\sigma (x, E) = \int_0^E J_\sigma(x, \xi) \, d\xi \quad \text{for a.e. } x \in \overline{\Omega} \text{ and } \forall E > 0. \quad (2.11)$$

Relation (2.3) gives the electrical current density as

$$J_\sigma(x, E) = \sigma(x, E)E \quad \text{for a.e. } x \in \overline{\Omega} \text{ and } \forall E > 0.$$

Moreover, relation (2.10) gives the electrical conductivity as

$$\sigma (x, E) = E^{-1} J_\sigma(x, E) = E^{-1} \partial_E Q_\sigma (x, E) \quad \text{for a.e. } x \in \overline{\Omega} \text{ and } \forall E > 0.$$
The electrical conductivity \( \sigma(x, E) \) is the secant line to the graph of the function \( J_\sigma(x, E(x)) \) and \( Q_\sigma(x, E(x)) \) is the area of the sub-graph of \( J_\sigma(x, E(x)) \). For a geometric interpretation, see Figure 2.

In the special case of \( \sigma(x, E(x)) = \theta(x)E(x)^{p-2}, \ 1 < p < +\infty \), the electrical current density is given by

\[
J(x) = -\theta(x)|\nabla u(x)|^{p-2}\nabla u(x).
\]  

(2.12)

This leads to the study of a steady current problem involving the \( p \)-Laplacian. When \( \sigma \) is independent of \( E \), we have the standard linear model. More precisely,

\[
\sigma(x, E(x)) = \sigma(x) \quad \text{and consequently}
\]

\[
J(x) = -\sigma(x)\nabla u(x).
\]  

(2.13)

In this linear case, we do not need hypothesis (H3) and, since \( dE^2 = \frac{1}{2}E \, dE \), the integral \( (2.11) \) is a half of the ohmic power absorbed by the system (refer to Figure 3), related to the Joule effect:

\[
Q_\sigma(x, E(x)) = \frac{1}{2}J_\sigma(x, E(x))E(x) = \frac{1}{2}\sigma(x)E(x)^2 \quad \text{for a.e. } x \in \Omega \text{ and } \forall E(x) > 0.
\]
2.6. The DtN operator. One of the quite general ways to represent boundary measurements is by considering the Dirichlet-to-Neumann (DtN) operator
\[
\Lambda_\sigma : f \in X_\sigma \mapsto -J^f \cdot \hat{n}|_{\partial \Omega} = \sigma \hat{\nabla} u^f|_{\partial \Omega} \in X'_\sigma,
\]
where \(X'_\sigma\) is the dual space of \(X\), \(J^f\), the current density produced by the boundary data \(f\) and \(u^f\), the minimizer of (2.8). We stress that \(-J^f \cdot \hat{n}|_{\partial \Omega}\) represents the following linear functional
\[
\varphi \in X_\sigma \mapsto -\int_{\partial \Omega} \varphi(x) J^f(x) \cdot \hat{n}(x) \, dS.
\]
Summing up, the DtN operator evaluated at \(f\) is equal to:
\[
\langle \Lambda_\sigma (f) , \varphi \rangle = -\int_{\partial \Omega} \varphi(x) J^f(x) \cdot \hat{n}(x) \, dS \quad \forall \varphi \in X_\sigma.
\] (2.14)

The minus sign in the definition is because we consider passive conducting material. Specifically, \(-J^f \cdot \hat{n}\) corresponds to the current density entering the conductor through \(\partial \Omega\). It is worth noting that the injectivity of the DtN operator is guaranteed by the assumption of zero average of \(f\). Indeed, \(J^f\) is invariant up to an additive constant in \(f\).

Furthermore, by testing the DtN operator (2.14) with the minimizer \(u^f\) of (2.8) and using a divergence Theorem, we obtain the ohmic power dissipated by the conducting material:
\[
\langle \Lambda_\sigma (f) , f \rangle = \int_{\Omega} J^f(x) E(x) E(x) \, dx.
\] (2.15)

If \(\varphi \neq f\) in (2.14), we have the so-called virtual power product that plays an important role since it is equal to the Gâteaux derivative of the Dirichlet Energy \(F_\sigma\) when evaluated at the solution \(u^f\) (see Section 3).

When the nonlinear constitutive relation (2.6) holds, the DtN operator is
\[
\Lambda_\sigma : f \in X_\sigma \mapsto \sigma \left(x, |\nabla u^f|\right) \hat{\nabla} u^f \in X'_\sigma.
\]
In weak form, the DtN operator is
\[
\langle \Lambda_\sigma (f) , \varphi \rangle = \int_{\partial \Omega} \varphi(x) \sigma \left(x, |\nabla u^f(x)|\right) \hat{\nabla} u^f(x) dS \quad \forall \varphi \in X_\sigma.
\]
In the \(p\)-Laplacian (2.12) and in the linear (2.13) case, the DtN operators occupy the forms
\[
\langle \Lambda_\sigma (f) , \varphi \rangle = \int_{\partial \Omega} \varphi(x) \theta(x) |\nabla u^f(x)|^{p-2} \hat{\nabla} u(x) \, dS \quad \forall \varphi \in X_\sigma
\]
and
\[
\langle \Lambda_\sigma (f) , \varphi \rangle = \int_{\partial \Omega} \varphi(x) \sigma \left(x, |\nabla u^f|\right) \hat{\nabla} u^f(x) \, dS \quad \forall \varphi \in X_\sigma,
\]
respectively.
Moreover, the power product \((2.15)\) is proportional to the absorbed ohmic power, represented by the area of the dashed rectangle in figure [3].

3. The Gâteaux derivative of the Dirichlet Energy

A key role in the problem, we are dealing with, is played by the Gâteaux derivative of the Dirichlet Energy \(E_{\sigma}\) and by the Gâteaux derivative of the composite function \(F_{\sigma} \circ U_{\sigma}\), where

\[
U_{\sigma} : f \in X_{\sigma} \rightarrow u^{f} \in W^{1,p}(\Omega).
\]

Operator \(U_{\sigma}\) maps the boundary data \(f\) to the solution \(u^{f}\) of problem \((2.8)\).

It is worth noting that the results of this Section are the foundations of the Monotonicity Principle in Section 4. Firstly, we prove a convergence result (Lemma 3.1), regarding \(\nabla u\), i.e. the Electric field \(E\), with respect to the boundary data. Then, in Proposition 3.2 we study the Gâteaux derivative of the Dirichlet Energy. As corollary, we prove that when evaluated on a physical solution, this Gâteaux derivative is equal to the virtual power product (Corollary 3.3). Eventually, in Proposition 3.4, we prove that the Gâteaux derivative of the composite operator \(F_{\sigma} \circ U_{\sigma}\) is equal to the DtN operator \(\Lambda_{\sigma}\). This is the key result for proving the Monotonicity Principle in Section 4.

For notation and properties of Gâteaux-derivative, we refer to [87, Chap. 4]). Considering the definition of \(X_{\sigma}\), we stress that many terms appearing in this section depend only on the restriction of \(f\) on the boundary of \(\Omega\).

3.1. A convergence result. Firstly, we show the following useful convergence result.

**Lemma 3.1.** Let \(\Omega\) be an open bounded domain with Lipschitz boundary and \(f, \varphi \in X_{\sigma}\). Then

\[
\nabla u^{f+\varepsilon \varphi} \rightarrow \nabla u^{f} \text{ in } L^{p}(\Omega) \text{ as } \varepsilon \rightarrow 0,
\]

where \(u^{f+\varepsilon \varphi} \in W^{1,p}(\Omega)\) is the minimizer of \((2.8)\) corresponding to the boundary data \(f + \varepsilon \varphi\).

**Proof.** For a fixed \(|\varepsilon| < 1\), using the divergence Theorem, we have

\[
I := \int_{\Omega} \sigma(x, |\nabla u^{f+\varepsilon \varphi}(x)|) \nabla u^{f+\varepsilon \varphi}(x) \cdot (\nabla u^{f+\varepsilon \varphi}(x) - \nabla u^{f}(x)) dx = \varepsilon \int_{\partial \Omega} \varphi(x) \sigma(x, |\nabla u^{f+\varepsilon \varphi}(x)|) \partial_{n} u^{f+\varepsilon \varphi}(x) dS,
\]

\[
II := \int_{\Omega} \sigma(x, |\nabla u^{f}(x)|) \nabla u^{f}(x) \cdot (\nabla u^{f+\varepsilon \varphi}(x) - \nabla u^{f}(x)) dx = \varepsilon \int_{\partial \Omega} \varphi(x) \sigma(x, |\nabla u^{f}(x)|) \partial_{n} u^{f}(x) dS.
\]

\(^{2}\)We remind that \(f\) and \(\varphi\) are representatives of their equivalence classes in \(X_{\sigma}\).
By subtracting (3.3) from (3.2) and using assumption (H5), we get a positive constant \(c > 0\) such that

\[
I - II \geq c \|\nabla u^{f+\varepsilon \varphi} - \nabla u^f\|_p^p.
\]

(3.4)

On the other hand, from (H4), we know that \(\sigma(x, E) \leq \sigma_2 \max \left\{1, \left(\frac{E}{E_0}\right)^{p-2}\right\}\) and, therefore, by H"{o}lder inequality, we have

\[
I - II = \varepsilon \int_{\partial \Omega} \varphi(x) \left[\sigma(x, |\nabla u^{f+\varepsilon \varphi}(x)|) \partial_n u^{f+\varepsilon \varphi}(x) - \sigma(x, |\nabla u^f(x)|) \partial_n u^f(x)\right] \, dS
\]

\[
= \varepsilon \int_{\Omega} \nabla \varphi(x) \cdot \left[\sigma(x, |\nabla u^{f+\varepsilon \varphi}(x)|) \nabla u^{f+\varepsilon \varphi}(x) - \sigma(x, |\nabla u^f(x)|) \nabla u^f(x)\right] \, dx
\]

\[
\leq \varepsilon \sigma_2 \int_{\Omega} |\nabla \varphi(x)| \left[\max \left\{|\nabla u^{f+\varepsilon \varphi}(x)|, \frac{|\nabla u^{f+\varepsilon \varphi}(x)|^{p-1}}{E_0^{p-2}}\right\} + \max \left\{|\nabla u^f(x)|, \frac{|\nabla u^f(x)|^{p-1}}{E_0^{p-2}}\right\}\right] \, dx.
\]

Moreover, we have

\[
I - II \leq \varepsilon C \left[\max\{|\nabla \varphi|_2, |\nabla u^{f+\varepsilon \varphi}|_2, |\nabla \varphi|_p, |\nabla u^{f+\varepsilon \varphi}|_p^{p-1}\}\right]
\]

\[
+ \max\{|\nabla \varphi|_2, |\nabla u^f|_2, |\nabla \varphi|_p, |\nabla u^f|_p^{p-1}\}\right]
\]

\[
\leq \varepsilon C \left[\max\{|\nabla \varphi|_2, |\nabla f + \varepsilon \nabla \varphi|_2, |\nabla \varphi|_p, |\nabla f + \varepsilon \nabla \varphi|_p^{p-1}\}\right]
\]

\[
+ \max\{|\nabla \varphi|_2, |\nabla f|_2, |\nabla \varphi|_p, |\nabla f|_p^{p-1}\}\right]
\]

\[
\leq \varepsilon C \left[\max\{|\nabla \varphi|_2 (|\nabla f|_2 + |\nabla \varphi|_2), |\nabla \varphi|_p (|\nabla f|_p^{p-1} + |\nabla \varphi|_p^{p-1})\}\right]
\]

\[
+ \max\{|\nabla \varphi|_2, |\nabla f|_2, |\nabla \varphi|_p, |\nabla f|_p^{p-1}\}\right]
\]

(3.5)

The quantity in the squared bracket is finite and hence, by (3.4) and (3.5), we get

\[
|\nabla u^{f+\varepsilon \varphi} - \nabla u^f|_p^p \leq C \varepsilon,
\]

where \(C\) is a positive constant independent of \(\varepsilon\). Therefore, the conclusion follows by passing the limit \(\varepsilon \to 0\).

\[\square\]

3.2. The perturbation of the Dirichlet Energy with respect to the minimizer. Before proving the main result, we observe that (H2) gives the convexity of \(Q_\sigma(x, E)\) with respect to \(E\) and, since \(Q_\sigma(x, 0) = 0\), then \(\partial_E Q_\sigma(x, E)\) is increasing with respect to \(E > 0\) for a.e. \(x \in \Omega\). This monotonic behaviour of \(\partial_E Q_\sigma(x, E)\)
leads to the following inequalities for a.e. $x \in \Omega$:

$$
0 \leq \hat{c}E_{\sigma}(x, E_1)(E_2 - E_1) \leq Q_{\sigma}(x, E_2) - Q_{\sigma}(x, E_1) \\
\leq \hat{c}E_{\sigma}(x, E_2)(E_2 - E_1) \quad \text{for any } 0 < E_1 \leq E_2.
$$

(3.6)

Now, we study the Gâteaux-derivative of (2.9).

**Proposition 3.2.** Let $\Omega$ be a bounded connected domain with Lipschitz boundary and $u, \varphi \in W^{1,p}(\Omega)$. Then

$$
dF_{\sigma}(u; \varphi) = F_{\sigma}^p(u)\varphi = \int_{\Omega} \sigma(x, |\nabla u(x)|) \nabla u(x) \cdot \nabla \varphi(x) \, dx.
$$

(3.7)

**Proof.** For any $\varepsilon \in \mathbb{R}$, we have

$$
F_{\sigma}(u + \varepsilon \varphi) = \int_{\Omega} Q_{\sigma}(x, |\nabla u(x)| + \varepsilon |\nabla \varphi(x)|) \, dx.
$$

(3.8)

The magnitude of the integrand function can be bounded easily from above. Indeed, by (3.6), we have

$$
\left| \frac{Q_{\sigma}(x, |\nabla u(x)| + \varepsilon |\nabla \varphi(x)|) - Q_{\sigma}(x, |\nabla u(x)|)}{\varepsilon} \right| \\
\leq \frac{1}{|\varepsilon|} \sigma(x, |\nabla u(x)| + |\varepsilon \nabla \varphi(x)|) (|\nabla u(x)| + |\varepsilon \nabla \varphi(x)|)|\nabla \varphi(x)|
$$

(3.9)

for any $|\varepsilon| < 1$. By assumption (H4), the last term in (3.9) is a $L^1$ function and hence, by Lebesgue Dominated Convergence Theorem, we can pass to the limit in (3.8), as $\varepsilon \to 0$. Then, by assumption (H2), we have that $Q_{\sigma}(x, \cdot)$ is in $C^1([0, +\infty])$ for a.e $x \in \Omega$ and hence

$$
\frac{Q_{\sigma}(x, |\nabla u(x)| + \varepsilon |\nabla \varphi(x)|) - Q_{\sigma}(x, |\nabla u(x)|)}{\varepsilon} \\
= \sigma(x, \xi_{\varepsilon} \varepsilon^{\frac{1}{p}} |\nabla u(x)| + \varepsilon |\nabla \varphi(x)|) |\nabla \varphi(x)|
$$

(3.10)

with

$$
\xi_{\varepsilon} \in \min\{ |\nabla u(x)|, |\nabla u + \varepsilon |\nabla \varphi(x)| \}, \max\{ |\nabla u(x)|, |\nabla u + \varepsilon |\nabla \varphi(x)| \}
$$

(3.11)

where the dependence of $\xi_{\varepsilon}$ upon $x$ is apparent.
The last term in (3.10) is equal to
\[
\begin{cases}
\sigma (x, \xi) \xi \frac{2\nabla u(x) \cdot \nabla \varphi(x) + \varepsilon |\nabla \varphi(x)|^2}{|\nabla u(x)| + |\nabla u(x) + \varepsilon \nabla \varphi(x)|}, & \text{if } |\nabla u(x)| \neq 0, \\
\sigma (x, \xi) \xi \text{sign}(\varepsilon)|\nabla \varphi(x)|, & \text{otherwise}.
\end{cases}
\] (3.12)

Hence, since (3.11) holds, \(\xi \rightarrow |\nabla u(x)|\) as \(\varepsilon \rightarrow 0\) for a.e. \(x \in \Omega\). We pass to the limit as \(\varepsilon \rightarrow 0\) in (3.12), (3.10) and (3.8) to obtain (3.7).

Proposition 3.2 implies the following corollary by replacing \(u\) with \(u^f\), the minimizer of (2.8).

Corollary 3.3. Let \(\Omega\) be a bounded connected domain with Lipschitz boundary, \(f \in X_\sigma\) and \(\varphi \in W^{1,p}(\Omega)\). Then
\[
d\mathcal{F}_\sigma(u^f; \varphi) = \mathcal{F}'_\sigma(u^f) \varphi = \langle \Lambda_\sigma(f), \varphi \rangle,
\] (3.13)
where \(u^f\) is the minimizer of \(2.8\) corresponding to the boundary data \(f\).

It is worth noting that also the Gâteaux derivative \(d\mathcal{F}_\sigma(u^f; \varphi)\) appearing in (3.13) depends only on the restriction of \(\varphi\) on the boundary of \(\Omega\).

3.3. The perturbation of \(\mathcal{F}_\sigma\) with respect to the boundary values. In this section, we analyze the Gâteaux derivative operator for the composition of the Dirichlet Energy functional \(\mathcal{F}_\sigma\) and the operator \(U_\sigma\), defined in (3.1).

Proposition 3.4. Let \(\Omega\) be a bounded connected domain with Lipschitz boundary and \(f \in X_\sigma\). Then \(d(\mathcal{F}_\sigma \circ U_\sigma) = (\mathcal{F}_\sigma \circ U_\sigma)' = \Lambda_\sigma\), i.e.
\[
d(\mathcal{F}_\sigma \circ U_\sigma)(f; \varphi) = (\mathcal{F}_\sigma \circ U_\sigma)'(f) \varphi = \langle \Lambda_\sigma(f), \varphi \rangle \quad \forall \varphi \in X_\sigma.
\] (3.14)

Proof. For sake of simplicity, we set
\[
\mathcal{G}_\sigma(f) := (\mathcal{F}_\sigma \circ U_\sigma)(f) = \mathcal{F}_\sigma(u^f).
\]
For any \(\varepsilon > 0\), we consider
\[
\mathcal{G}_\sigma(f + \varepsilon \varphi) = \int_\Omega Q_\sigma(x, |\nabla u^f + \varepsilon \nabla \varphi(x)|) \, dx \leq \int_\Omega Q_\sigma(x, |\nabla u^f(x) + \varepsilon \nabla \varphi(x)|) \, dx.
\]
Let us consider the incremental ratio \([\mathcal{G}_\sigma(f + \varepsilon \varphi) - \mathcal{G}_\sigma(f)] / \varepsilon\). On one hand, we have
\[
\frac{\mathcal{G}_\sigma(f + \varepsilon \varphi) - \mathcal{G}_\sigma(f)}{\varepsilon} \leq \frac{1}{\varepsilon} \int_\Omega Q_\sigma(x, |\nabla u^f + \varepsilon \nabla \varphi(x)|) - Q_\sigma(x, |\nabla u^f(x)|) \, dx.
\] (3.15)
The second term in (3.15) is equal to the second term in (3.8), therefore we can pass to the limit and we have

$$\limsup_{\varepsilon \to 0} \frac{G_\sigma(f + \varepsilon \varphi) - G_\sigma(f)}{\varepsilon} \leq \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \int_\Omega Q_\sigma(x, |\nabla u^\varepsilon(x) + \varepsilon \nabla \varphi(x)|) - Q_\sigma(x, |\nabla u^\varepsilon(x)|) \, dx$$

$$= \int_\Omega \sigma(x, |\nabla u^\varepsilon(x)|) \nabla u^\varepsilon(x) \cdot \nabla \varphi(x) \, dx.$$ \hspace{1cm} (3.16)

On the other hand, we have

$$\frac{G_\sigma(f + \varepsilon \varphi) - G_\sigma(f)}{\varepsilon} \geq \frac{1}{\varepsilon} \int_\Omega Q_\sigma(x, |\nabla u^{f + \varepsilon \varphi}(x)|) - Q_\sigma(x, |\nabla u^{f + \varepsilon \varphi}(x) - \varepsilon \nabla \varphi(x)|) \, dx.$$ \hspace{1cm} (3.17)

In Lemma 3.1, we showed that $\nabla u^{f + \varepsilon \varphi} \to \nabla u^f$ strongly in $L^p(\Omega)$ when $\varepsilon \to 0$. Now, let us consider a sequence $\{\varepsilon_j\}_{j \in \mathbb{N}}$, such that $\varepsilon_j \to 0$ and

$$\liminf_{\varepsilon \to 0^+} \frac{G_\sigma(f + \varepsilon \varphi) - G_\sigma(f)}{\varepsilon} = \lim_{j \to +\infty} \frac{G_\sigma(f + \varepsilon_j \varphi) - G_\sigma(f)}{\varepsilon_j}.$$ \hspace{1cm} (3.18)

By standard arguments, we can say that there exists a subsequence $\{\varepsilon_{j_k}\}_{k \in \mathbb{N}}$ such that $\nabla u^{f + \varepsilon_{j_k} \varphi} \to \nabla u^f$ a.e. in $\Omega$ and there exists a measurable real function $\psi$, defined on $\Omega$, such that

$$|\nabla u^{f + \varepsilon_{j_k} \varphi}| \leq \psi, \quad \int_\Omega |\psi(x)|^p \, dx < +\infty.$$  

Now, we consider the integrand function in (3.17). By (3.6) and (H4), we have

$$\left| \frac{Q_\sigma(x, |\nabla u^{f + \varepsilon_{j_k} \varphi}(x)|)}{\varepsilon_{j_k}} - Q_\sigma(x, |\nabla u^{f + \varepsilon_{j_k} \varphi}(x) - \varepsilon_{j_k} \nabla \varphi(x)|) \right|$$

$$\leq \sigma(x, |\nabla u^{f + \varepsilon_{j_k} \varphi}(x)|) + |\varepsilon_{j_k} \nabla \varphi(x)||\nabla u^{f + \varepsilon_{j_k} \varphi}(x)| + |\varepsilon_{j_k} \nabla \varphi(x)||\nabla \varphi(x)|$$

$$\leq C \max\left\{\tilde{C}, (|\nabla u^{f + \varepsilon_{j_k} \varphi}(x)| + |\varepsilon_{j_k} \nabla \varphi(x)|)^{p-2}(|\nabla u^{f + \varepsilon_{j_k} \varphi}(x)| + |\varepsilon_{j_k} \nabla \varphi(x)||\nabla \varphi(x)|^2\right\}$$

$$\leq C \max\left\{\tilde{C}, (|\nabla \varphi(x)|^2, (|\nabla \varphi(x)|^2)^p\right\},$$

for a.e. $x \in \Omega$ and for any $\varepsilon < 1$. In the above expression $C$ and $\tilde{C} > 0$ are appropriate positive constants independent of $\varepsilon_{j_k}$.
Now, recalling that \( Q_\sigma(x, \cdot) \) is in \( C^1([0, +\infty]) \) for a.e. \( x \in \Omega \), the integrand function in (3.17) can be written as

\[
Q_\sigma(x, |\nabla u^{l+\varepsilon\varphi}(x)|) - Q_\sigma(x, |\nabla u^{l+\varepsilon\varphi}(x) - \varepsilon \nabla \varphi(x)|)
\]

\[
\begin{aligned}
&= \sigma(x, \xi') \xi' |\nabla u^{l+\varepsilon\varphi}(x)| - |\nabla u^{l+\varepsilon\varphi}(x) - \varepsilon \nabla \varphi(x)| \varepsilon
\end{aligned}
\]

with

\[
\xi' \in \min\{|\nabla u^{l+\varepsilon\varphi}(x)|, |\nabla u^{l+\varepsilon\varphi}(x) - \varepsilon \nabla \varphi(x)|\},
\]

\[
\max\{|\nabla u^{l+\varepsilon\varphi}(x)|, |\nabla u^{l+\varepsilon\varphi}(x) - \varepsilon \nabla \varphi(x)|\},
\]

where the dependence of \( \xi' \) upon \( x \) is apparent.

The conclusion follows by observing that (3.16) and (3.21) imply (3.14).

In this section, we demonstrate a Monotonicity Principle for the Dirichlet Energy (Theorem 4.1), more precisely,

\[
\sigma_1 \leq \sigma_2 \implies F_{\sigma_1}(u_1) \leq F_{\sigma_2}(u_2) \quad \forall f \in X_\sigma,
\]

4. Monotonicity Principle

In this section, we demonstrate a Monotonicity Principle for the Dirichlet Energy (Theorem 4.1), more precisely,
where $u^f_i$ is the minimizer of (2.8) with $\sigma = \sigma_i$, for $i = 1, 2$ and $f$ is the applied boundary voltage. We recall that, both in the $p$-Laplacian and in the linear cases, the Dirichlet Energy is proportional to the power product $\langle \Lambda_\sigma(f), f \rangle$ and, therefore, in these cases, Monotonicity Principle for the DtN operator easily follows from (4.1).

In our nonlinear case (2.6), the Dirichlet Energy is not proportional to the power product $\langle \Lambda_\sigma(f), f \rangle$. Rather, the power product $\langle \Lambda_\sigma(f), f \rangle$ is equal to the Gâteaux derivative $d\langle \mathcal{F}_\sigma \circ \mathcal{U}_\sigma \rangle(f; f)$ of the composite mapping $f \rightarrow \mathcal{F}_\sigma(u^f)$ (see Proposition 3.4). It is a fundamental difference between our analysis and previous ones. Then, starting from this latter equality, we relate the Dirichlet Energy to boundary data through the fundamental relation (see Theorem 4.2)

$$\mathcal{F}_\sigma(u^f) = \langle \overline{\Lambda}_\sigma(f), f \rangle \quad \forall f \in X_\sigma.$$

The new operator $\overline{\Lambda}_\sigma$, we call Average DtN, is defined as

$$\overline{\Lambda}_\sigma : f \in X_\sigma \mapsto \int_0^1 \Lambda_\sigma(\alpha f) \, d\alpha \in X_\sigma',$$

where

$$\langle \overline{\Lambda}_\sigma(f), \varphi \rangle = \int_0^1 \langle \Lambda_\sigma(\alpha f), \varphi \rangle \, d\alpha \quad \forall \varphi \in X_\sigma.$$

Operator $\overline{\Lambda}_\sigma$ gives the average flown of the electrical current density through $\partial \Omega$ for an applied boundary potential of the type $\alpha f$, for $\alpha \in [0, 1]$. This is the key development for transferring the Monotonicity of the Dirichlet Energy to the boundary data and, in particular, the new operator $\overline{\Lambda}_\sigma$ replaces for nonlinear problems.

Eventually, in Theorem 4.3, we prove the Monotonicity Principle for operator $\overline{\Lambda}_\sigma$, i.e.

$$\sigma_1 \leq \sigma_2 \implies \langle \overline{\Lambda}_{\sigma_1}(f), f \rangle \leq \langle \overline{\Lambda}_{\sigma_2}(f), f \rangle \quad \forall f \in X_\sigma,$$

where we recall that $\sigma_1 \leq \sigma_2$ means

$$\sigma_1(x, E) \leq \sigma_2(x, E) \quad \text{for a.e. } x \in \overline{\Omega} \text{ and } \forall E > 0. \quad (4.3)$$

We remark that Monotonicity Principle involves the knowledge of $\langle \overline{\Lambda}_\sigma(f), f \rangle = \int_0^1 \langle \Lambda_\sigma(\alpha f), f \rangle \, d\alpha$. From a physical standpoint, quantity $\langle \Lambda_\sigma(\alpha f), \alpha f \rangle$ is nothing but the electrical power $P(\alpha f)$ absorbed by the conductor when the boundary potential is $\alpha f$ (see Section 2.6). Therefore, key quantity $\langle \overline{\Lambda}_\sigma(f), f \rangle$ is equal to $\int_0^1 \alpha^{-1} P(\alpha f) \, d\alpha$, that is, a weighted integral of the ohmic power dissipated in the conductor.

4.1. **Monotonicity Principle for the Dirichlet Energy.** Firstly, we state Monotonicity Principle for the Dirichlet Energy in the nonlinear case. This includes the $p$-Laplacian (2.12) and linear (2.13) cases, where the proof is apparently simpler.
Theorem 4.1. Let $\Omega$ be an open bounded domain in $\mathbb{R}^n$ with Lipschitz boundary and $\sigma_1, \sigma_2$ satisfying (H1), (H2), (H3), (H4), (H5). Then,

$$\sigma_1 \leq \sigma_2 \implies \mathbb{F}_{\sigma_1}(u^I_1) \leq \mathbb{F}_{\sigma_2}(u^I_2) \quad \forall f \in X_0,$$

where $\sigma_1 \leq \sigma_2$ is meant in the sense (4.3) and $u^I_i$ is the minimizer of (2.8) with $\sigma = \sigma_i$, for $i = 1, 2$.

Proof. Since $u^I_2$ is an admissible function for problem (2.8) with $\sigma = \sigma_1$, we have

$$\mathbb{F}_{\sigma_1}(u^I_1) \leq \mathbb{F}_{\sigma_2}(u^I_2) = \int_{\Omega} Q_{\sigma_2}(x, |\nabla u^I_2(x)|) \ dx$$

$$= \int_{\Omega} \int_0^{\nabla u^I_2(x)} \sigma_2(x, \xi) \xi \ d\xi \ dx \leq \int_{\Omega} \int_0^{\nabla u^I_2(x)} \sigma_1(x, \xi) \xi \ d\xi \ dx = \mathbb{F}_{\sigma_1}(u^I_2),$$

where the second inequality follows from the assumption $\sigma_1 \leq \sigma_2$. \qed

4.2. Connection between Dirichlet Energy and DtN operator. The motivation for our research is in generalizing the Monotonicity Principle from linear and $p$-Laplacian to more general cases. The first step in this direction is to study the polynomial type nonlinear constitutive relationship. For this, let us consider

$$\sigma(x, E) = \sum_{k=0}^N \theta_k(x) E^k$$

for a.e. $x \in \overline{\Omega}$ and $\forall E > 0$,

where either $\theta_k \in L^\infty_+^{*}(\Omega)$ or $\theta_k \equiv 0$ in $\overline{\Omega}$ and $N \in \mathbb{N}$. This polynomial type nonlinearity leads to minimization problem (2.8) where $p = N + 2$, $u \in W^{1,N+2}(\Omega)$, $u|_{\partial \Omega} = f \in X_0$ and

$$\mathbb{F}_{\sigma}(u) = \sum_{k=0}^N \int_{\Omega} \left( \frac{\theta_k(x)}{k+2} |\nabla u(x)|^{k+2} \right) dx.$$

Furthermore, we have

$$\langle \Lambda_{\sigma}(f), f \rangle = \sum_{k=0}^N \int_{\Omega} \theta_k(x) |\nabla u(x)|^{k+2} dx \quad \forall f \in X_0,$$

and, consequently,

$$\mathbb{F}_{\sigma}(u^I) \neq \langle \Lambda_{\sigma}(f), f \rangle \text{ in } X_0.$$

This is a major issue because it prevents to transfer the monotonic connection of electrical conductivity with the boundary data.

Here, we investigate the issue which derives the proportionality between Dirichlet Energy and the power product (4.4). This proportionality holds only for some special case of nonlinearity. For a general nonlinear constitutive relationship satisfying (H1)-(H5), the Dirichlet Energy (2.9) and the ohmic power (2.15) are proportional if and only if there exists $c > 0$ such that

$$\langle \Lambda_{\sigma}(f), f \rangle = c \mathbb{F}_{\sigma}(u^I) \quad \forall f \in X_0,$$

(4.5)
that is
\[
\int_{\Omega} \left[ J_\sigma(x, E(x)) E(x) - c \int_0^{E(x)} J_\sigma(x, \xi) \, d\xi \right] \, dx = 0, \quad \forall f \in X_\circ. \tag{4.6}
\]
A sufficient condition for (4.6) is
\[
J_\sigma(\cdot, E) = c \int_0^{E} J_\sigma(\cdot, \xi) \, d\xi;
\]
that is
\[
J_\sigma'(\cdot, E) = (c - 1) J_\sigma(\cdot, E).
\]
This latter ODE gives
\[
J_\sigma(\cdot, E) = a E^{c-1}, \quad \text{with} \quad a \in \mathbb{R}
\]
and \( \sigma(\cdot, E) = a E^{c-2} \). Hence the proportionality (4.5) is not expected for general nonlinear constitutive relations except for monomial type nonlinearities.

### 4.3. The power product for the average DtN operator.

We prove that the Dirichlet Energy (2.9) is transferred to a boundary measurement involving \( \Lambda_\sigma \), i.e. to the power product \( \langle \Lambda_\sigma(f), f \rangle \).

**Theorem 4.2.** Let \( \Omega \) be an open bounded domain in \( \mathbb{R}^n \) with Lipschitz boundary and \( \sigma \) satisfying (H1), (H2), (H3), (H4), (H5). Then
\[
\mathbb{F}_\sigma(u^f) = \langle \Lambda_\sigma(f), f \rangle \quad \forall f \in X_\circ,
\]
where \( u^f \) is the minimizer of (2.8).

**Proof.** For \( \alpha \in [0, 1] \), we set
\[
g(\alpha) := (\mathbb{F}_\sigma \circ \mathbb{U}_\sigma)(\alpha f).
\]
By Proposition 3.4, since \( \mathbb{F}_\sigma \circ \mathbb{U}_\sigma \) is Gâteaux-differentiable, \( g \) is differentiable. By replacing \( f \) and \( \varphi \) with \( \alpha f \) and \( f \), we have
\[
g'(\alpha) = d(\mathbb{F}_\sigma \circ \mathbb{U}_\sigma)(\alpha f; f) = \langle \Lambda_\sigma(\alpha f), f \rangle.
\]
Eventually, by integrating over the interval \( [0, 1] \), we obtain
\[
\mathbb{F}_\sigma(u^f) = (\mathbb{F}_\sigma \circ \mathbb{U}_\sigma)(f) = g(1) = \int_0^1 g'(\alpha) \, d\alpha = \int_0^1 \langle \Lambda_\sigma(\alpha f), f \rangle \, d\alpha = \langle \Lambda_\sigma(f), f \rangle.
\]
\( \square \)

### 4.4. Monotonicity Principle for \( \overline{\Lambda}_\sigma \).

The main result of this section is to derive Monotonicity Principle for the operator (4.2).

**Theorem 4.3.** Let \( \Omega \) be an open bounded domain in \( \mathbb{R}^n \) with Lipschitz boundary and \( \sigma_1, \sigma_2 \) satisfying (H1), (H2), (H3), (H4), (H5). Then
\[
\sigma_1 \leq \sigma_2 \quad \Rightarrow \quad \langle \overline{\Lambda}_{\sigma_1}(f), f \rangle \leq \langle \overline{\Lambda}_{\sigma_2}(f), f \rangle \quad \forall f \in X_\circ, \tag{4.7}
\]
where \( \sigma_1 \leq \sigma_2 \) is meant in the sense (4.3).
Proof. Let \( u^f_1 \) be the unique solution of problem (2.8) with \( \sigma = \sigma_i, i = 1, 2 \), and boundary value \( f \). By Theorems 4.1 and 4.2 we have
\[
\langle \Lambda_{\sigma_1} (f), f \rangle = F_{\sigma_1}(u^f_1) \leq F_{\sigma_2}(u^f_2) = \langle \Lambda_{\sigma_2} (f), f \rangle
\]
that proves (4.7). \( \square \)

If the constitutive relationship \( J_{\sigma} = J_{\sigma}(x, E) \) is strictly decreasing with respect to \( E \), then the Monotonicity is reversed, i.e.
\[
\sigma_1 \leq \sigma_2 \implies \langle \Lambda_{\sigma_1} (f), f \rangle \geq \langle \Lambda_{\sigma_2} (f), f \rangle \quad \forall f \in X_\sigma.
\]

We stress that Theorem 4.3 generalizes both the \( p \)-Laplacian and the linear cases. Indeed, in such cases, we have
\[
\langle \Lambda_{\sigma_1} (f), f \rangle = p \langle \Lambda_{\sigma_1} (f), f \rangle \leq p \langle \Lambda_{\sigma_2} (f), f \rangle = \langle \Lambda_{\sigma_2} (f), f \rangle \quad \forall f \in X_\sigma.
\]

5. Conclusions

In this paper, we proved Monotonicity Principle for inverse electrical conductivity problems where the conductor is modeled by a fully nonlinear constitutive relationship. Namely, we proved Monotonicity Principle where the “standard” DtN operator \( \Lambda_{\sigma} \), required for \( p \)-Laplacian and linear cases, is replaced by the Average DtN operator \( \overline{\Lambda}_{\sigma} \). Specifically, we unveiled the “mechanics” of Monotonicity by first recognizing that \( F_{\sigma} \circ U_{\sigma} \) is monotonic with respect to the material property \( \sigma \), even in the nonlinear case and, then, by transferring this Monotonicity to the new boundary operator \( \overline{\Lambda}_{\sigma} \). This is a relevant result because, apart from linear and \( p \)-Laplacian cases, it is impossible to transfer the Monotonicity from \( F_{\sigma} \circ U_{\sigma} \) to the DtN operator \( \Lambda_{\sigma} \). This result is based on the fundamental relation \( d(F_{\sigma} \circ U_{\sigma}) = \Lambda_{\sigma} \) proved in Proposition 3.4. During this analysis, we proved two general results: (i) the virtual power product \( \langle \Lambda_{\sigma} (f), \varphi \rangle \) is equal to \( dF_{\sigma}(u^f; \varphi) \), that is the Gâteaux derivative of the Dirichlet Energy \( F_{\sigma} \), evaluated at the solution \( u^f \) in the direction of \( f \), and (ii) the power product \( \langle \Lambda_{\sigma} (f), f \rangle \) is equal to \( d(F_{\sigma} \circ U_{\sigma})(f; f) \), that is, the Gâteaux derivative of the composed function \( F_{\sigma} \circ U_{\sigma} \), evaluated at \( f \) in the direction of \( f \).

Future work will deal with the adaptation of Monotonicity Principle in an inversion or imaging method for nonlinear materials.

Acknowledgements

This work has been partially supported by the MiUR-Progetto Dipartimenti di eccellenza grant “Sistemi distribuiti intelligenti” of Dipartimento di Ingegneria Elettrica e dell’Informazione “M. Scarano”, by MiSE project “SUMMa: Smart Urban Mobility Management” and by GNAMPA of INdAM.
REFERENCES

[1] Bueno P R, Varela J A and Longo E 2008 SnO2, ZnO and related polycrystalline compound semiconductors: an overview and review on the voltage-dependent resistance (non-ohmic) feature Journal of the European Ceramic Society 28 505–529

[2] Boucher S, Hassanzadeh M, Ananthakumar S and Metz R 2018 Interest of nonlinear zno/silicone composite materials in cable termination Material Sci & Eng 2 83–88

[3] Lupo G, Miano G, Tucci V and Vitelli M 1996 Field distribution in cable terminations from a quasi-static approximation of the Maxwell equations IEEE Transactions on Dielectrics and Electrical Insulation 3 399–409

[4] Scédé P 2015 Applied superconductivity: handbook on devices and applications (John Wiley & Sons)

[5] Krabbes G, Fuchs G, Canders W R, May H and Palka R 2006 High temperature superconductor bulk materials: fundamentals, processing, properties control, application aspects (John Wiley & Sons)

[6] Lee S Y, Viswanathan V, Huckans J, Matthews J and Wellstood F 2005 Nde of defects in superconducting wires using squid microscopy IEEE transactions on applied superconductivity 15 707–710

[7] Takahashi Y, Suwa T, Nabara Y, Ozeki H, Hemmi T, Nunoya Y, Isono T, Matsui K, Kawano K, Oshikiri M et al. 2014 Non-destructive examination of jacket sections for ITER central solenoid conductors IEEE Transactions on Applied Superconductivity 25 1–4

[8] Amoros J, Carrera M and Granados X 2012 An effective model for fast computation of current distribution in operating hts tapes from magnetic field measurements in non-destructive testing Superconductor Science and Technology 25 104005

[9] Foster K and Schwan H 1989 Dielectric properties of tissues and biological materials: A critical review, crc biomed Eng 17 25–104

[10] Corovic S, Lackovic I, Sustaric P, Sustar T, Rodic T and Miklavcic D 2013 Modeling of electric field distribution in tissues during electroporation Biomedical engineering online 12 16

[11] Miga S, Dec J and Kleemann W 2011 Non-linear dielectric response of ferroelectrics, relaxors and dipolar glasses Ferroelectrics-Characterization and Modeling

[12] Yarali E, Baniasadi M, Bodaghi M and Baghani M 2020 3d constitutive modelling of electromagneto-visco-hyperelastic elastomers: a semi-analytical solution for cylinders under large torsion-extension deformation Smart Materials and Structures

[13] Bozorth R M 1993 Ferromagnetism (Harvard)

[14] Lam K F and Yousept I 2020 Consistency of a phase field regularisation for an inverse problem governed by a quasilinear maxwell system Inverse Problems 36 045011

[15] Tamburrino A and Rubinacci G 2002 A new non-iterative inversion method for electrical resistance tomography Inverse Problems 18 1809–1829

[16] Tamburrino A and Rubinacci G 2006 Fast methods for quantitative eddy-current tomography of conductive materials IEEE Transactions on Magnetics 42 2017–2028

[17] Tamburrino A 2006 Monotonicity based imaging methods for elliptic and parabolic inverse problems Journal of Inverse and Ill-posed Problems 14 633–642

[18] Gisser D, Isaacson D and Newell J 1990 Electric current computed tomography and eigenvalues SIAM Journal on Applied Mathematics 50 1623–1634

[19] Calvano F, Rubinacci G and Tamburrino A 2012 Fast methods for shape reconstruction in electrical resistance tomography NDT and E International 46 32–40
[20] Tamburrino A, Rubinacci G, Soleimani M and Lionheart W 2003 Non iterative inversion method for electrical resistance, capacitance and inductance tomography for two phase materials 3rd World Congress on Industrial Process Tomography pp 233–238

[21] Harrach B and Ullrich M 2013 Monotonicity-based shape reconstruction in electrical impedance tomography SIAM Journal on Mathematical Analysis 45 3382–3403

[22] Tamburrino A, Ventre S and Rubinacci G 2010 Recent developments of a monotonicity imaging method for magnetic induction tomography in the small skin-depth regime Inverse Problems 26 074016

[23] Su Z, Ventre S, Udpa L and Tamburrino A 2017 Monotonicity based imaging method for time-domain eddy current problems Inverse Problems 33 125007

[24] Tamburrino A, Su Z, Ventre S, Udpa L and Udpa S 2016 Monotonicity based imaging method in time domain eddy current testing Studies in Applied Electromagnetics and Mechanics 41 1–8

[25] Tamburrino A, Su Z, Lei N, Udpa L and Udpa S 2015 The monotonicity imaging method for pect Studies in Applied Electromagnetics and Mechanics 40 159–166

[26] Su Z, Udpa L, Giovinco G, Ventre S and Tamburrino A 2017 Monotonicity principle in pulsed eddy current testing and its application to defect sizing 2017 International Applied Computational Electromagnetics Society Symposium - Italy, ACES 2017

[27] Tamburrino A, Su Z, Ventre S, Udpa L and Udpa S 2016 Monotonicity based imaging method in time domain eddy current testing Electromagnetic Nondestructive Evaluation (XIX) vol 41 pp 1–8

[28] Tamburrino A, Barbato L, Colton D and Monk P 2015 Imaging of dielectric objects via monotonicity of the transmission eigenvalues abstracts book of the 12th International Conference on Mathematical and Numerical Aspects of Wave Propagation (Germany) pp 99–100

[29] Harrach B, Pohjola V and Salo M 2019 Dimension bounds in monotonicity methods for the helmholtz equation SIAM Journal on Mathematical Analysis 51 2995–3019

[30] Harrach B, Pohjola V and Salo M 2019 Monotonicity and local uniqueness for the helmholtz equation Analysis & PDE 12 1741–1771

[31] Griesmaier R and Harrach B 2018 Monotonicity in inverse medium scattering on unbounded domains SIAM Journal on Applied Mathematics 78 2533–2557

[32] Albicker A and Griesmaier R 2020 Monotonicity in inverse obstacle scattering on unbounded domains Inverse Problems 36 085014

[33] Daimon T, Furuya T and Saiin R 2020 The monotonicity method for the inverse crack scattering problem Inverse Problems in Science and Engineering 1–12

[34] Eberle S and Harrach B 2020 Shape reconstruction in linear elasticity: Standard and linearized monotonicity method arXiv preprint arXiv:2003.02598

[35] Tamburrino A, Vento A, Ventre S and Maffucci A 2016 Monotonicity imaging method for flaw detection in aeronautical applications Studies in Applied Electromagnetics and Mechanics 41 284–292

[36] Caudiani V, Dardé J, Garde H and Hyvönen N 2019 Monotonicity-based reconstruction of extreme inclusions in electrical impedance tomography arXiv preprint arXiv:1909.12110

[37] Rubinacci G, Tamburrino A and Ventre S 2006 Regularization and numerical optimization of a fast eddy current imaging method IEEE Transactions on Magnetics 42 1179–1182

[38] Garde H and Staboulis S 2017 Convergence and regularization for monotonicity-based shape reconstruction in electrical impedance tomography Numerische Mathematik 135 1221–1251

[39] Harrach B and Minh M N 2016 Enhancing residual-based techniques with shape reconstruction features in electrical impedance tomography Inverse Problems 32 125002
[40] Tamburrino A, Calvano F, Ventre S and Rubinacci G 2012 Non-iterative imaging method for experimental data inversion in eddy current tomography *NDT and E International* **47** 26–34

[41] Harrach B and Ullrich M 2015 Resolution guarantees in electrical impedance tomography *IEEE transactions on medical imaging* **34** 1513–1521

[42] Harrach B and Minh M N 2018 Monotonicity-based regularization for phantom experiment data in electrical impedance tomography *New trends in parameter identification for mathematical models* (Springer) pp 107–120

[43] Garde H 2018 Comparison of linear and non-linear monotonicity-based shape reconstruction using exact matrix characterizations *Inverse Problems in Science and Engineering* **26** 33–50

[44] Garde H and Staboulis S 2019 The regularized monotonicity method: Detecting irregular indefinite inclusions *Inverse Problems & Imaging* **13** 93

[45] Harrach B and Seo J K 2010 Exact shape-reconstruction by one-step linearization in electrical impedance tomography *SIAM Journal on Mathematical Analysis* **42** 1505–1518

[46] Eberle S, Harrach B, Meftahi H and Rezgui T 2020 Lipschitz stability estimate and reconstruction of lamé parameters in linear elasticity *Inverse Problems in Science and Engineering* 1–22

[47] Harrach B 2009 On uniqueness in diffuse optical tomography *Inverse problems* **25** 055010

[48] Harrach B 2012 Simultaneous determination of the diffusion and absorption coefficient from boundary data *Inverse Problems & Imaging* **6** 663

[49] Harrach B and Ullrich M 2017 Local uniqueness for an inverse boundary value problem with partial data *Proceedings of the American Mathematical Society* **145** 1087–1095

[50] Maffucci A, Vento A, Ventre S and Tamburrino A 2016 A novel technique for evaluating the effective permittivity of inhomogeneous interconnects based on the monotonicity property *IEEE Transactions on Components, Packaging and Manufacturing Technology* **6** 1417–1427

[51] De Magistris M, Morozov M, Rubinacci G, Tamburrino A and Ventre S 2007 Electromagnetic inspection of concrete rebars *COMPEL - The International Journal for Computation and Mathematics in Electrical and Electronic Engineering* **26** 389–398

[52] Rubinacci G, Tamburrino A and Ventre S 2007 Concrete rebars inspection by eddy current testing *International Journal of Applied Electromagnetics and Mechanics* **25** 333–339

[53] Harrach B, Lee E and Ullrich M 2015 Combining frequency-difference and ultrasound modulated electrical impedance tomography *Inverse Problems* **31** 095003

[54] Alessandri G 1989 Remark on a paper by bellout and Friedman *Boll. Un. Mat. Ital. A (7)* **3** 243–249

[55] Colton D and Kirsch A 1996 A simple method for solving inverse scattering problems in the resonance region *Inverse Problems* **12** 383–393

[56] Kirsch A 1998 Characterization of the shape of a scattering obstacle using the spectral data of the far field operator *Inverse Problems* **14** 1489–1512

[57] Ikehata M 1999 How to draw a picture of an unknown inclusion from boundary measurements. two mathematical inversion algorithms *Journal of Inverse and Ill-Posed Problems* **7** 255–272

[58] Ikehata M 2000 On reconstruction in the inverse conductivity problem with one measurement *Inverse Problems* **16** 785–793

[59] Devaney A J Super-resolution processing of multi-static data using time reversal and music northeastern University preprint

[60] Salo M and Zhong X 2012 An inverse problem for the p-laplacian: Boundary determination *SIAM Journal on Mathematical Analysis* **44** 2474–2495
[61] Brander T, Kar M and Salo M 2015 Enclosure method for the p-laplace equation Inverse Problems 31 045001
[62] Brander T 2016 Calderón problem for the p-laplacian: First order derivative of conductivity on the boundary Proceedings of the American Mathematical Society 144 177–189
[63] Sun Z 2004 Inverse boundary value problems for a class of semilinear elliptic equations Advances in Applied Mathematics 32 791 – 800 ISSN 0196-8858
[64] Sun Z 2005 Anisotropic inverse problems for quasilinear elliptic equations Journal of Physics: Conference Series 12 156–164
[65] Cărstea C I and Kar M 2020 Recovery of coefficients for a weighted p-laplacian perturbed by a linear second order term arXiv preprint arXiv:2001.01436
[66] Brander T, Ilmavirta J and Kar M 2018 Superconductive and insulating inclusions for linear and non-linear conductivity equations Inverse Problems and Imaging 12 91–123
[67] Guo C, Kar M and Salo M 2016 Inverse problems for p-laplace type equations under monotonicity assumptions Rendiconti dell’Istituto di Matematica dell’Università di Trieste 48 79–99
[68] Brander T, Harrach B, Kar M and Salo M 2018 Monotonicity and enclosure methods for the p-laplace equation SIAM Journal on Applied Mathematics 78 742–758
[69] Hauer D 2015 The p-dirichlet-to-neumann operator with applications to elliptic and parabolic problems Journal of Differential Equations 259 3615–3655
[70] Jerison D and Kenig C 1995 The inhomogeneous dirichlet problem in lipschitz domains Journal of Functional Analysis 130 161–219
[71] Leoni G 2017 A First Course in Sobolev Spaces (American Mathematical Society)
[72] Bossavit A 1998 Computational electromagnetism: variational formulations, complementarity, edge elements (Academic Press)
[73] Lindqvist P 2017 Notes on the p-laplace equation (second edition) Report/University of Jyväskylä, Department of Mathematics and Statistics 161 1–106
[74] Gorb Y and Novikov A 2012 Blow-up of solutions to a p-laplace equation Multiscale Modeling & Simulation 10 727–743
[75] Ferreira E R, Oberai A A and Barbone P E 2012 Uniqueness of the elastography inverse problem for incompressible nonlinear planar hyperelasticity Inverse problems 28 065008
[76] Nakamura G, Watanabe M and Kaltenbacher B 2009 On the identification of a coefficient function in a nonlinear wave equation Inverse problems 25 035007
[77] Donzel L, Greuter F and Christen T 2011 Nonlinear resistive electric field grading part 2: Materials and applications IEEE Electrical Insulation Magazine 27 18–29
[78] Zha J W, Zhang Z M, Zhao K, Zheng X Q and Li S T 2012 Prominent nonlinear electrical conduction characteristic in t-znow/ptfe composites with low threshold field IEEE Transactions on Dielectrics and Electrical Insulation 19 567–573
[79] DiBenedetto E 2012 Degenerate parabolic equations (Springer Science & Business Media)
[80] Yuan H, Xu X, Gao W, Lian S and Cao C 2005 Extinction and positivity for the evolution p-laplacian equation with l1 initial value Journal of mathematical analysis and applications 310 328–337
[81] Bueno-Orovio A, Kay D, Grau V, Rodriguez B and Burage K 2014 Fractional diffusion models of cardiac electrical propagation: role of structural heterogeneity in dispersion of repolarization Journal of the Royal Society Interface 11 20140352
[82] Giusti E 2003 Direct methods in the calculus of variations (World Scientific)
[83] Dacorogna B 2014 Introduction to the Calculus of Variations (World Scientific Publishing Company)
[84] Della Pietra F, Gavitone N and Piscitelli G 2019 On the second dirichlet eigenvalue of some nonlinear anisotropic elliptic operators Bulletin des Sciences Mathématiques 155 10–32
[85] Della Pietra F, Gavitone N and Piscitelli G 2017 A sharp weighted anisotropic Poincaré inequality for convex domains *Comptes Rendus Mathematique* **355** 748–752

[86] Piscitelli G 2016 A nonlocal anisotropic eigenvalue problem *Differential Integral Equations* **29** 1001–1020

[87] Zeidler E 1988 *Nonlinear Functional Analysis and Its Applications* (Berlin, Heidelberg: Springer-Verlag)